MST in $O(1)$ Rounds of the Congested Clique

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Abstract

We present a distributed randomized algorithm finding Minimum Spanning Tree (MST) of a given graph in $O(1)$ rounds, with high probability, in the Congested Clique model. The input graph in the Congested Clique model is a graph of $n$ nodes, where each node initially knows only its incident edges. The communication graph is a clique with limited edge bandwidth: each two nodes (not necessarily neighbours in the input graph) can exchange $O(\log n)$ bits.

As in previous works, the key part of the MST algorithm is an efficient Connected Components (CC) algorithm. However, unlike the former approaches, we do not aim at simulating the standard Boruvka algorithm, at least at initial stages of the CC algorithm. Instead, we develop a new technique which combines connected components of sample sparse subgraphs of the input graph in order to accelerate the process of uncovering connected components of the original input graph. More specifically, we develop a sparsification technique which reduces an initial CC problem in $O(1)$ rounds to its two restricted instances. The former instance has a graph with maximal degree $O(\log \log n)$ as the input – here our sample-combining technique helps. In the latter instance, a partition of the input graph into $O(n/\log \log n)$ connected components is known. This gives an opportunity to apply previous algorithms to determine connected components in $O(1)$ rounds.

Our result improves over previous $O(\log^* n)$ algorithm of Ghaffari et al. [PODC 2016], $O(\log \log \log n)$ algorithm of Hegeman et al. [PODC 2015] and the $O(\log \log n)$ algorithm of Lotker et al. [SPAA 2003; SICOMP 2005]. It also determines $\Theta(1)$ round complexity in the congested clique for MST, as well as other graph problems, including bipartiteness, cut verification, $s$-$t$ connectivity, cycle containment.

Keywords: congested clique, connected components, minimum spanning tree, randomized algorithms, broadcast, unicast
1 Introduction

The congested clique model of distributed computation attracted much attention in algorithmic community in recent years. The underlying network is a clique and in each round, a message of size $O(\log n)$ bits can be sent in each direction across each edge of the network. Initially, each node knows its incident edges in the input graph $G(V, E)$. Thus, unlike the classical CONGEST model [16], the communication graph connects each pair of nodes, even if they are not neighbours in the input graph.

The main algorithm-theoretic motivation of the model is to understand the role of congestion in distributed computing. The well-known LOCAL model of distributed computing ignores congestion by allowing unlimited size of transmitted messages [16, 13] and focuses on locality. The CONGEST model on the other hand takes congestion into account by limiting the size of transmitted messages. Simultaneously, locality plays an important role as well in the CONGEST model, since direct communication is possible only between neighbors of the input graph. The congested clique is considered as a complement, which focuses merely on congestion.

Some variants and complexity measures for the congested clique have applications to efficiency of algorithms in other models adjusted to current computing challenges such as $k$-machine big data model [10] or MapReduce [6, 9].

1.1 Related work

The general congested clique model as well as its limited variant called broadcast congested clique were studied in several papers, e.g., [14, 5, 3, 2, 12, 15]. The recent Lenzen’s [12] constant time routing and sorting algorithm in the unicast congested clique exhibited strength of this model and triggered a new wave of the interest in the model.

Besides general interest in the congested clique, specific attention has been given to MST and connectivity. Lotker et al. [14] designed a $O(\log \log n)$ round deterministic algorithm for MST in the unicast model. An alternative solution of the same complexity has been presented recently [11]. The best known randomized solution for MST works in $O(\log^* n)$ rounds [4], improving the recent $O(\log \log \log n)$ bound [5]. The result from [5] uses a concept of linear graph sketches [1], while the authors of [4] introduce new sketches, which are sensitive to the degrees of nodes and adjusted to the congested clique model. Reduction of the number of transmitted messages in the MST algorithms was studied in [17]. The tradeoff between the number of rounds and the size of messages has also been studied recently for various problems. In contrast to the general (unicast) congested clique, no sub-logarithmic round algorithm is known for MST in the broadcast congested clique, while the first sub-logarithmic solution for the CC problem has been obtained only recently [7].

In [3], a simulation of powerful classes of bounded-depth circuits in the congested clique is presented, which points out to the power of the congested clique and explains difficulty in obtaining lower bounds.

An extreme scenario that the algorithm consists of one rounds in which each node can send only one message has also been considered. As shown in [11], connectivity can be solved with public random bits in this model, provided nodes can send messages of size $\Theta(\log^3 n)$.

1.2 Our result

The main result of the paper determines $O(1)$ round complexity of the MST problem in the congested clique.

**Theorem 1.** There is a randomized algorithm in the Congested Clique model that computes a minimum spanning tree in $O(1)$ rounds, with high probability.

Using standard reductions of some graph problems to the connectivity problem, we establish $O(1)$ round complexity of several graph problems in the congested clique model.

**Corollary 1.** There are randomized distributed algorithms that solve the following verification problems in the Congested Clique model in $O(1)$ rounds with high probability: bipartiteness verification, cut verification, s-t connectivity and cycle containment.
1.3 Preliminaries

In this section we provide some terminology and tools for design of distributed algorithms in the congested clique. Given a natural number \( p \), \([p]\) denotes the set \( \{1, 2, \ldots, p\} \).

We use the following Lenzen’s routing result in our algorithms.

Lemma 1. \([13]\) Assume that each node of the congested clique is given a set of \( O(n) \) messages with fixed destination nodes. Moreover, each node is the destination of \( O(n) \) messages from other nodes. Then, it is possible to deliver all messages in \( O(1) \) rounds of the congested clique.

Efficient congested clique algorithms often make use of auxiliary nodes. Below, we formulate this opportunity to facilitate design of algorithms.

Lemma 2. Let \( A \) be a congested clique algorithm which, except of the nodes \( u_1, \ldots, u_n \) corresponding to the input graph, uses \( O(n) \) auxiliary nodes \( v_1, v_2, \ldots \) such that the auxiliary nodes do not have initially any knowledge of the input graph on the nodes \( u_1, \ldots, u_n \). Then, each round of \( A \) might be simulated in \( O(1) \) rounds in the standard congested clique model, without auxiliary nodes.

Proof. Assume that there are at most \( cn \) auxiliary nodes, for constant \( c \in \mathbb{N} \). We can assign the set of \( c \) auxiliary nodes \( V_j = \{v_{(j-1)c+1}, \ldots, v_{jc}\} \) to \( u_j \) for each \( j \in [n] \) and assign internal IDs in the range \([c]\) to the elements of \( V_j \). Then, a round of an original algorithm with auxiliary nodes is simulated in \( c^2 \) actual rounds indexed by \((a,b)\) with \( (a,b) \in [c]^2 \). The \( a \)th auxiliary nodes assigned to \( u_j \) transmit messages addressed to the \( b \)th auxiliary nodes in the rounds indexed \((a,b)\). \(\square\)

We consider randomized algorithms in which a computational unit in each node of the input network can use private random bits in its computation. We say that some event holds with high probability (whp) for an algorithm \( A \) running on an input of size \( n \) if this event holds with probability \( 1 - 1/n^c \) for a given constant \( c \). We require here that the constant \( c \) can be chosen arbitrarily large, without changing asymptotic complexity of the considered algorithm.

Graph terminology

If not stated otherwise, we consider undirected graphs. Thus, in particular, an edge \((u,v)\) appears in the graph iff \((v,u)\) appears in that graph as well. For a node \( v \in V \) of a graph \( G(V,E) \), \( N(v) \) denotes the set of neighbours of \( v \) in \( G \) and \( d(v) \) denotes the degree of \( v \), i.e., \( d(v) = |N(v)| \). We say that a graph \( G(V,E) \) has degree \( \Delta \) if the degree of each node of \( G \) is smaller or equal to \( \Delta \).

A component of a graph \( G(V,E) \) is a connected subgraph of \( G \). That is, \( C \subseteq V \) corresponds to a component of \( G \) iff the graph \( G(C,E \cap (C \times C)) \) is connected. A component \( C \) of a graph \( G(V,E) \) is growable if there are edges connecting \( C \) with \( V \setminus C \), i.e., the set \( E \cap (C \times (V \setminus C)) \) is non-empty. Otherwise, the component \( C \) is ungrowable.

For a graph \( G(V,E) \), sets \( C_1, C_2, \ldots, C_k \subseteq V \) form a partition of \( G \) into components if \( C_i \)'s are pairwise disjoint, \( \bigcup_{i \in [k]} C_i = V \), and \( C_i \) is a component of \( G(V,E) \) for each \( i \in [k] \). A partition \( C_1, \ldots, C_k \) of a graph \( G(V,E) \) into components is the complete partition if \( C_i \) is ungrowable for each \( i \in [k] \). For a fixed \( E' \subseteq E \), the complete partition of \( G(V,E') \) will be also called the complete partition with respect to \( E' \).

Given a partition \( \mathcal{C} \) of a graph \( G(V,E) \) into components and \( v \in V \), \( C(v) \) denotes the component of \( \mathcal{C} \) containing \( v \).

An edge \((u,v)\) is incident to a component \( C \) wrt to a partition \( \mathcal{C} \) of a graph if it connects \( C \) with another component of \( \mathcal{C} \), i.e., \( C(u) \neq C(v) = C \) or \( C(v) \neq C(u) = C \).

Graph problems in the congested clique model

Graph problems in the congested clique model are considered in the following framework. The joint input to the \( n \) nodes of the network is an undirected \( n \)-node weighted graph \( G(V,E,c) \), where each node in \( V = \{u_1, \ldots, u_n\} \) corresponds to a node of the communication network and weights \( c(e) \) of edges are integers of polynomial size (i.e., each weight is a bit sequence of length \( O(\log n) \)). Each node \( u_i \) initially knows the network size \( n \), its unique ID \( i \in [n] \), the list of IDs of its neighbors in the input graph and the weights of its incident edges. Specifically, \( ID(v) = i \) for \( v = u_i \). All graph problems are considered in this paper in accordance with this definition.
Connected Components and Minimum Spanning Tree

In the paper, we consider the connected components problem (CC) and the minimum spanning tree problem (MST). A solution for the CC problem consists of the complete partition of the input graph $G(V, E)$ into connected components $C_1 \cup \cdots \cup C_k = V$, accompanied by spanning trees of all components. Our goal is to compute CC or MST of the input graph, i.e., each node should know the set of edges inducing CC/MST at the end of an execution of an algorithm.

For the purpose of fast simultaneous executions of many instances of the CC algorithms, we also consider the definition of the CC problem, where spanning trees of all components are known only to a fixed node of a network. The presented solutions usually correspond to this weaker definition. However, for a single instance of the CC/MST problem, a spanning forest can be made known to all nodes in two rounds, provided it is known to a fixed node $v$. Namely, it is sufficient that $v$ fixes roots of spanning trees of all components. Then, in the former round, $v$ sends to each $u$ the ID of the parent of $u$ in the appropriate tree. In the latter round each $u$ sends the ID of its parent to all nodes of the network.

We also consider the situation that some “initial” partition $C$ of the input graph into connected components is known at the beginning of an execution of an algorithm to a fixed node. We say that a component $C$ of a partition $C$ which is known to all/some nodes of the network is an active component if $C$ is a growable component of the original input graph $G(V, E)$. Otherwise, if $C$ is not a growable component of $G(V, E)$, $C$ is an inactive component.

Below, we make a simple observation which we use in our algorithms.

**Fact 1.** Assume that solutions of CC problem for the graphs $G(V, E_1)$ and $G(V, E_2)$ are known. Then, one can determine a solution of CC problem for $G(V, E_1 \cup E_2)$.

## 2 High-level description of our solution

In this section we describe our MST algorithm on a top level. The main technical result is an $O(1)$ round connectivity algorithm. The extension to MST is based on a known technique described in Section 4, based on $n^{1/2}$ simultaneous executions of the connectivity algorithm. Thus, the key issue in design of the MST algorithm is to guarantee that $n^{1/2}$ simultaneous executions of the connected components algorithm can be performed in $O(1)$ rounds.

The algorithm for connected components works in two phases: Sparsification Phase and Size-reduction Phase. In Sparsification Phase, we reduce the original connectivity problem to two specific instances of the CC problem (Lemma 5). (We find this reduction interesting by its own, as its idea be implemented in the much weaker broadcast congested clique model and allows to obtain new round-efficient algorithms in that model 7.) In the former instance, a partition of the input graph into $O(n/\log \log n)$ active and some non-active components is known. The connected components can be determined for such an instance in $O(1)$ rounds by the algorithm from 4. The latter instance is a graph with degree $O(\log \log n)$. Therefore, Size-reduction Phase gets a graph with degree $O(\log \log n)$ as the input. In this phase, the CC problem for such a sparse input graph is reduced to an instance of the CC problem in the case that a partition of the input graph into $O(n/\log \log n)$ active components is known. Therefore, as before, the connected components can be determined for this final instance in $O(1)$ rounds by the algorithm from 4.

**Connected Components: Sparsification Phase.** Sparsification Phase is based on a simple deterministic procedure (see Alg. 2):

- Firstly, for each node $v$, an edge $(u, v)$ connecting $v$ with its highest degree neighbour is determined and delivered to a fixed node called the coordinator.
- Then, the complete partition $C$ with respect to the set of edges delivered to the coordinator is computed. The degree of each component of $C$ is defined as the largest degree of its elements.
- Next, for each node $v$, an edge $(u, v)$ connecting $v$ with the highest degree component $C \neq C(v)$ of $C$ is determined and send to the coordinator.
• The coordinator computes the complete partition $C'$ with respect to the set of edges announced in all steps of this procedure, assigns IDs to the components of this partition. Then, the coordinator sends to each node the ID and the degree of its component. Finally, the nodes pass information obtained from the coordinator to their neighbours.

Let $C$ be a component of $C'$. We say that $C$ is awake if the degree of $C$ (i.e., the largest degree of its elements) is at most $s$, for some fixed $s \in \mathbb{N}$. Otherwise, $C$ is asleep. A node $u$ is awake (asleep, respectively) iff $u$ belongs to an awake (asleep, respectively) component. As we show, the partition $C'$ contains $O(n/s)$ awake components. Moreover, the degree of the graph induced by edges incident to nodes from asleep components is smaller than $s$. (This fact does not follow simply from the definitions, since the graph contains also neighbors of nodes from asleep components located in awake components).

Using the above properties for $s = \log \log n$ we can split the input graph $G(V, E)$ into a subgraph containing $O(n/\log \log n)$ growable components and a subgraph with degree $O(\log \log n)$. For the former subgraph, we determine connected components in $O(1)$ rounds using the algorithm from [4], based on graph sketches. The connected components of the latter subgraph are determined in Size-reduction Phase.

**Connected Components: Size-reduction Phase.** The key technical novelty in our solution is an algorithm which reduces the CC problem for a graph of degree $O(\log \log n)$ to the CC problem for a graph with $O(n/\log \log n)$ components. This algorithm is the main ingredient of Size-reduction Phase. A pseudocode of the algorithm is presented in Alg. 3. Using this reduction, we find connected components of the described above graph $G_2$, by applying the reduction and the algorithm from [1] for graphs with a known partition into $O(n/\log \log n)$ components.

In order to describe the above reduction, assume that the (upper bound on) degree of an input graph $G(V, E)$ is $\Delta = O(\log \log n)$. The idea of the reduction is to calculate simultaneously $m = n^{1/2}$ spanning forests for randomly chosen sparse subgraphs $G_i = G(V, E_i)$ of the input graph (called samples), and use the results to build a partition of $G$ into $O(n/\log \log n)$ growable components and some non-growable components. Below, we describe the reduction in more detail.

First, we build random subgraphs $G_i = G(V, E_i)$ of $G$ for $i \leq m = \sqrt{n}$ such that, for each $i \in [m]$, the set of edges $E_i$ can be collected at a single node. As a node can receive only $O(n)$ messages in a round, the size of $E_i$ should be $O(n)$ as well. To assure this property, each edge will belong to $E_i$ for each $i \in [m]$ with probability $1/\log \log n$, and the random choices for each $i \in [m]$ and each edge are independent. For each $i \in [m]$, all edges from $E_i$ are sent to a fixed node $b_i$ (the $i$th boss) and $b_i$ computes connected components of $G_i$ locally. The following obstacle arises in the process of sending edges of sample graphs to the bosses. Only one message per round might be transmitted on each edge, while a node can choose non-constant number of edges incident to itself as the elements of the $i$th sample $E_i$ for some $i \in [m]$. (Thus, all those edges should be delivered to $b_i$.) This problem is solved by the fact that, whp, the random choices defining the graphs $G_i$ require to send $O(n)$ messages and receive $O(n)$ messages by each node. (Note that the expected number of edges in $E_i$ is $O(n)$.) If this is the case, all messages can be delivered with help of Lenzen’s routing algorithm [2] (Lemma 1).

Another challenge is how to combine connected components of the graphs $G_i$ such that the number of growable components is reduced to $O(n/\log \log n)$. To this aim, each node is chosen to be a leader, independently of other nodes, with probability $1/\log \log n$. Thus, the number of leaders will be $\Theta(n/\log \log n)$ whp. Then, for each node $v$, if $v$ is in a connected component containing a leader in some graph $G_i$, information about its connection with some leader will be delivered to the coordinator. Thus, if each node is connected to some leader in the partition determined by the coordinator then we have $O(n/\log \log n)$ connected components and we are done. Certainly, we cannot get such a guarantee. However, as we show in Section 3.2, the number of nodes from growable components which are not connected to any leader will be $O(n/\log \log n)$, whp. The main part of the proof of this fact is to determine a sequence of independent random variables whose sum gives the upper bound on the number of nodes in growable components which are not connected to a leader in the final partition.

**Minimum Spanning Tree.** As shown in [3][5], it is possible to reduce the MST problem of an input graph, to two instances of MST on graphs with $O(n^{3/2})$ edges. Then, MST for a graph with $O(n^{3/2})$ edges is reduced to $O(\sqrt{n})$ instances of the CC problem, where the set of edges in the $i$th instance is included in the
set of edges of the \((i+1)\)st instance. In Section 4, we show that our algorithm can be executed in parallel on these specific \(\sqrt{n}\) instances of the CC problem. The main challenge here is that a “naive” implementation of these parallel executions requires to send superlinear number of messages by some nodes. However, using Lenzen’s routing [12] (see Lemma 1) and the fact that the set of edges of the \((i+1)\)st instance of CC includes the set of edges of the \(i\)th instance for each \(i \in [m]\), we show that connected components of all those instances can be computed in parallel.

3 Connectivity in \(O(1)\) rounds

In this section we describe our CC algorithm which leads to the following theorem.

**Theorem 2.** There is a randomized algorithm in the Congested Clique model that computes connected components in \(O(1)\) rounds, with high probability.

The algorithm consists of two phases: Sparsification Phase and Size-reduction Phase.

In Sparsification Phase, we reduce the original problem to two specific instances of the CC problem. In the former instance, the CC problem has to be solved for a graph with degree \(O(\log \log n)\). The latter instance is equipped with additional information about a partition of the considered graph in \(O(\log \log n)\).

The pseudocode of the appropriate algorithm is presented as Alg. 2. The following lemma describes the reduction more precisely.

**Lemma 3.** There is a deterministic algorithm in the Congested Clique that reduces in \(O(1)\) rounds the CC problem for an arbitrary graph \(G\) to the instances of the CC problem for graphs \(G_A\) and \(G_B\) such that

- a partition \(C_A\) of \(G_A\) into \(O(n/\log \log n)\) active components is known to a fixed node;
- the degree of \(G_B\) is \(O(\log \log n)\).

An important building block of our solution comes from [4], where the properties of graph sketches play the key role. It is the algorithm which determines connected components of the input graph in \(O(1)\) rounds, provided an initial partition of the input graph into \(O(n/\log \log n)\) growable and some ungrowable components is known at the beginning. For further applications in a solution of the MST problem, we state a stronger result regarding several simultaneous executions of the algorithm.

**Lemma 4.** [4] There is a randomized algorithm in the Congested Clique model that computes connected components of a graph in \(O(1)\) rounds with high probability, provided that a partition of the input graph into \(O(n/\log \log n)\) growable (and some ungrowable) components is known to a fixed node at the beginning of an execution of the algorithm.

Moreover, it is possible to execute \(m = \sqrt{n}\) instances of the problem simultaneously in \(O(1)\) rounds.

**Proof.** In [4], the authors gave \(O(1)\) round procedure \(\text{ReduceCC}(x)\), reducing the number of active components from \(O(n/\log^2 x)\) to \(n/x\) in \(O(1)\) rounds, with high probability. Thus, by using \(\text{ReduceCC}\) constant number of times, it is possible to reduce the number of active components from \(n/\log \log n\) to 0 in \(O(1)\) rounds. And, if there are no active (i.e., growable) components in a partition, then that partition describes connected components of the input graph (i.e., it is the complete partition of the input graph).

Let \(GP\text{Reduction}\) denote the algorithm satisfying properties from Lemma 4. Using \(GP\text{Reduction}\), we can determine connected components of the graph \(G_A\) (described in Lemma 3) in \(O(1)\) rounds.

Thus, in order to build \(O(1)\) rounds CC algorithm, it is sufficient to solve the problem for graphs with degree \(O(\log \log n)\). This problem is addressed in Size-reduction Phase. In the following lemma, we show that the CC problem for a \(O(\log \log n)\)-degree graph can be reduced to the CC problem for a graph with \(O(n/\log \log n)\) growable components in \(O(1)\) rounds. The pseudocode of the algorithm performing this reduction is given in Alg. 3.

**Lemma 5.** There is a randomized algorithm in the Congested Clique model that reduces in \(O(1)\) rounds the CC problem for a graph with degree bounded by \(\log \log n\) to an instance of the CC problem for which a partition with \(O(n/\log \log n)\) active connected components is known to a fixed node, with high probability.
Then, the next application of the algorithm from [4] (Lemma 4) gives the final partition of the input graph into connected components, as summarized in Alg. 1.

The proofs of Lemma 3 and Lemma 5 are presented in Section 3.1 and Section 3.2 respectively. Using Lemmas 4, 3, and 5, one can show that Alg. 1 determines connected components of an input graph in \(O(1)\) rounds, with high probability. This in turn gives the proof of Theorem 2.

**Algorithm 1 ConnectedComponents**  
\(\triangleright G(V, E)\) is the input graph

| Step | Description |
|------|-------------|
| 1    | Execute Alg. 2 on the input graph for \(s = \log \log n\) |
| 2    | Let \(G_A\) and \(C_A\) be the graph and its \(O(n/\log \log n)\)-size partition determined in Alg. 2  \(\triangleright\) Lemma 3 |
| 3    | \(G_B\) ← the graph of degree \(O(\log \log n)\) determined in Alg. 2  \(\triangleright\) Lemma 3 |
| 4    | Execute Alg. GPReduction on \(G_A\), using the partition \(C_A\)  \(\triangleright\) Lemma 4 |
| 5    | Execute Alg. 3 on \(G_B\) |
| 6    | \(G'\) ← the graph obtained in Alg. 3 with its partition into \(O(n/\log \log n)\) active components  \(\triangleright\) Lemma 5 |
| 7    | Execute Alg. GPReduction on \(G'\), using its partition determined by Alg. 3  \(\triangleright\) Lemma 4 |

### 3.1 Graph sparsification

In this section we describe Sparsification Phase and prove Lemma 3. Let the *coordinator* be a fixed node of the input network. The general idea of the reduction is to build components from the edges determined in the following two stages:

- **Stage 1.** For each node \(v\), choose an edge connecting \(v\) to its neighbour with the largest degree. Then, determine the complete partition with respect to the set of chosen edges. Moreover, set the degree of each component of the obtained partition as the maximum of the degrees of its elements.

- **Stage 2.** For each node \(v\), choose an edge connecting \(v\) to a component \(C\) with the largest degree, excluding edges inside \(C(v)\), the component of \(v\). Determine the complete partition with respect to the set of edges chosen in both stages.

For \(s \leq n\), we say that components with degree at least \(s\) are *high-degree* components, while components with degree smaller than \(s\) are *low-degree* components. As we show below, the complete partition \(C\) determined by the edges chosen in Stages 1 and 2 satisfies the following conditions:

- each node of degree larger or equal to \(s\) belongs to a high-degree component of \(C\),
- each node belonging to a low-degree component of \(C\) does not have neighbors with the degree larger than \(s\),

are satisfied for each \(s \leq n\). This fact will imply Lemma 3. Moreover, we show that Stages 1 and 2 can be implemented in the congested clique model in \(O(1)\) rounds. Importantly, for further applications in the MST problem, our implementation allows for several parallel execution of of Stages 1 and 2 in \(O(1)\) rounds.

Now, we describe the algorithm in more detail. For a partition \(C\) of a graph \(G(V, E)\) into components, we use the following notations:

- \(d(C) = \max_{v \in C} d(v)\) is the *degree* of the component \(C\) of \(C\),
- \(I(v)\) is the ID of the component \(C(v)\), according to a fixed labeling of components of \(C\).

Algorithm 2 contains a pseudo-code of an implementation of the above described idea in the congested clique model. Let \(C\) be a component of a partition determined by the coordinator in step 7 of Alg. 2. Recall that a component \(C\) is *awake* iff at least one node from \(C\) is awake at the end of Alg. 2 i.e., the degree of \(C\) is at least \(s\).

**Fact 2.** The following conditions are satisfied at the end of an execution of Alg. 2: (i) there are at most \(n/s\) awake components; (ii) the degree of the graph \(G_B\) induced by edges incident to the asleep nodes is smaller than \(s\).
Indeed, if we contrary assume that C is asleep, let v \in N(v) be the element of C corresponding to the largest tuple in the set \{(d(v), ID(v)) \mid v \in C\}. Thus, d(v_{\text{max}}) \geq s. Moreover, each u \in N(v_{\text{max}}) reports the edge (u, v) in Stage 1 of the algorithm. Indeed, if we contrary assume that

\begin{itemize}
  \item d(v_{\text{max}}) < s:
    Then d(C) < s and C is asleep. This contradicts the assumption that C is awake.
  \item u \in N(v_{\text{max}}) announces (u, w) for w \neq v_{\text{max}} in Stage 1:
    Then (d(v_{\text{max}}), ID(v_{\text{max}})) < (d(w), ID(w)). Thus, v_{\text{max}} announces the edge (u', v_{\text{max}}) in Stage 2, such that (d(v_{\text{max}}), ID(v_{\text{max}})) < (d(u'), ID(u')). (It might be the case that u = u', or u \neq u' and (d(u), ID(u)) < (d(u'), ID(u'))) As a result u' \in C(v_{\text{max}}) at the end of an execution of the algorithm which contradicts the assumption that (d(v_{\text{max}}), ID(v_{\text{max}})) is the largest element of \{(d(v), ID(v)) \mid v \in C\}.
\end{itemize}

Given that d(v) \geq s, v \in C and N(v) \subseteq C, we see that the size of C is larger than s.

Now, we prove the property (ii). As the degree of all asleep nodes is smaller than s, it is sufficient to show that the degrees all neighbors of asleep nodes are smaller than s as well. Contrary, assume that a node u is asleep, v \in N(u), and d(v) \geq s. Then, u reports a node w in Stage 1 such that (d(v), ID(v)) \leq (d(w), ID(w)) which implies that d(w) \geq d(v) \geq s. This in turn implies that u and w are eventually in the same component and, by the fact that d(w) \geq s, they are both awake. This however contradicts the assumption that u is asleep.

Now, we apply Fact 2 for s = log log n to prove Lemma 3. Let G_A be the subgraph of G containing edges whose both ends are awake. By Fact 2(i), the partition determined in the algorithm contains at most n/s = O(n/\log log n) awake components. As the nodes from asleep components do not have neighbours in G_A and therefore they form inactive single-element components of G_A, there are O(n/\log log n) active (i.e., growable) components in the partition C_A of G_A (see step 11 of Alg. 2). The partition C_A is known to the coordinator. As the nodes learn components’ IDs and degrees of their neighbours in step 8 of Alg. 2, they know which edges incident to them belong to G_A and which to G_B. The graph G_B contains the edges incident to asleep nodes. By Fact 2(ii), the degree of G_B is smaller than s = log log n.
3.2 Size-reduction Phase

In this section we provide \(O(1)\) round algorithm reducing the number of active components for sparse graphs (Alg. 3). Assuming that the degree of the input graph \(G(V, E)\) is at most \(\Delta \in O(\log \log n)\), our algorithm returns a partition of the input graph into \(O\left(\frac{n}{\log \log n}\right)\) active components. Additionally, some fixed node (the coordinator) knows a spanning tree of each component in the final partition. Thus, Lemma 5 follows from the properties of the presented algorithm.

In Algorithm 3 \(C_i(u_j)\) denotes the component of the node \(u_j\) in the \(i\)th sample graph \(G_i\). Moreover, for a node \(u_i \in V\), let \(e_{(i, 1)}, \ldots, e_{(i, r)}\) for \(r \leq |N(u_i)|\) denote all edges \((u_i, u_j)\) such that \(j < i\).

The algorithm randomly selects \(m = \sqrt{n}\) random subgraphs \(G_1, \ldots, G_m\) of the input graph \(G\), called samples. Each sample will consist of \(O(n)\) edges, with high probability. We will ensure that all edges of the sample \(G_i\) are known to a fixed node called the boss \(b_i\). Therefore, for each sample, we can locally determine its connected components and its spanning forest. Finally, the results from samples are combined in order to obtain a partition of the input graph which consists of \(O(n/\log \log n)\) active components. The key challenge here is how to combine knowledge about locally available components of sample graphs such that significant progress towards establishing components of the original input graph is achieved. To this aim, we select randomly \(\Theta(n/\log \log n)\) leaders among nodes of the input network. More precisely, each node of the network assigns itself the status leader with probability \(1/\log \log n\), independently of other nodes. Thus, the number of leaders is \(\Theta(n/\log \log n)\), with high probability. Then, the idea is to build a (global) knowledge about connected components of the input graph by assigning nodes to the leaders which appear together with them in connected components of samples. More precisely, if the connected component of \(u_j\) in the \(i\)th sample contains some leader, the boss \(b_i\) will send a message to \(u_j\), containing the ID of that leader. More precisely:

- If the connected component of \(u_j\) in the \(i\)th sample contains a leader, the boss \(b_i\) determines a shortest path \(P\) connecting \(u_j\) and a leader in \(C_i(u_j)\).
- If the connected component of \(u_j\) in the \(i\)th sample does not contain a leader, the boss \(b_i\) determines a shortest path \(P\) connecting \(u_j\) and the node of \(C_i(u_j)\) with the smallest ID.

Then, \(b_i\) sends a message to \(u_j\) containing the ID of the neighbour of \(u_j\) in \(P\).

We will say that a component \(C\) is small if \(C\) is ungrowable and the size of \(C\) is at most \(s = \sqrt{\log n}\). In the following, we split nodes of the input graph into three subsets:

- \(V_\alpha\): the nodes connected to a leader in at least one sample graph,
- \(V_\beta\): the elements of small components of the input graph,
- \(V_\gamma\): the remaining nodes of the graph; thus, \(v\) belongs to \(V_\gamma\) when \(v\) is not an element of a small component of the input graph and there are no leaders in connected components of \(v\) in samples \(G_1, \ldots, G_m\).

In the analysis of Alg. 3 we show that each node from \(V_\alpha\) will belong to a component containing a leader in the final partition \(C\) determined by the coordinator.

**Proposition 1.** Assume that \(C_i(v)\) (i.e., the connected component of \(v \in V\) in \(G_i\)) for some \(i \in [m]\) contains a leader. Then, the connected component of \(v\) in the final partition \(C\) contains a leader as well.

Moreover, we show that small components of the input graph are uncovered by the coordinator with high probability, which determines the final components of nodes from \(V_\beta\).

**Proposition 2.** The following property holds with high probability for each small component \(C\) (i.e., a component of size at most \(s = \sqrt{\log n}\)) of the input graph: \(C\) is a connected component of the final partition determined by the coordinator in Alg. 3 or (at least one) leader belongs to \(C\).

While Prop. 1 and 2 concern \(V_\alpha\) and \(V_\beta\), we give an estimation of the size of \(V_\gamma\) in the following proposition.

**Proposition 3.** The number of nodes from \(V_\gamma\) is \(O(n/\log \log n)\), with high probability.

We postpone the proofs of the above propositions and show properties of Alg. 3 following from them. This in turn directly implies Lemma 5.
Algorithm 3 Reduce Components in Sparse Graph

1: \( m \leftarrow \sqrt{n} \)
2: **for** \( i \in [m] \) **do** \( b_i \leftarrow u_i \) \( \triangleright u_1, \ldots, u_m \) are the bosses
3: the coordinator \( \leftarrow u_n \) \( \triangleright \) fix the coordinator
4: **for** \( i \in [n] \) **do** \( \triangleright \) simultaneously, in one round
5: \( \quad \quad \quad \textbf{for} \ k \in \lfloor N(u_i) \rfloor \textbf{ do} \)
6: \( \quad \quad \quad \textbf{for} \ j \in \lfloor m \rfloor \textbf{ do} \)
7: \( \quad \quad \quad \quad u_i \text{ adds } e_{(i,k)} \text{ to } G_j \text{ and sends it to } b_j \text{ with probability } 1/\log \log n \quad \triangleright \text{determine } G_j \)
8: **for** each \( i \in [m] \) **do** \( b_i \) calculates a spanning forest \( F_i \) induced by received edges
9: each node, with probability \( 1/\log \log n \) becomes a leader, and announces it to all bosses \( b_i \)
10: **for** \( i \in \lfloor \sqrt{m} \rfloor \) **do**
11: \( \quad \quad \quad \textbf{for} \ j \in [n] \textbf{ do} \)
12: \( \quad \quad \quad \quad \textbf{if} \ C_i(u_j) \text{ contains a leader then} \quad \triangleright C_i(u_j): \) the component of \( u_j \) in \( G_i \)
13: \( \quad \quad \quad \quad \quad p_i(u_j) \leftarrow \text{the first node on the path from } u_j \text{ to the closest leader in } C_i(u_j) \)
14: \( \quad \quad \quad \quad b_i \text{ sends the message } (1, |C_i(u_j)|, i, p_i(u_j)) \text{ to } u_j \)
15: \( \quad \quad \quad \quad \textbf{else} \)
16: \( \quad \quad \quad \quad p_i(u_j) \leftarrow \text{the first node on the path from } u_j \text{ to the node with the smallest ID in } C_i(u_j) \)
17: \( \quad \quad \quad \quad b_i \text{ sends the message } (0, |C_i(u_j)|, i, p_i(u_j)) \text{ to } u_j \)
18: **for** \( j \in [m] \) **do**
19: \( \quad \quad \quad \text{let } (x, |C_i(u_j)|, i, p_i(u_j)) \text{ be the largest message according to the lexicographic order received by } u_j \)
20: \( \quad \quad \quad p(u_j) \leftarrow p_i(u_j) \)
21: \( \quad \quad u_j \text{ sends the edge } (u_j, p(u_j)) \text{ to the coordinator} \)
22: the coordinator computes components determined by the received edges

Lemma 6. At the end of Algorithm 3 a partition with at most \( O(n/\log \log n) \) active components is determined, the coordinator knows this partition and a spanning tree for each component of the partition.

Proof. Let \( C \) be the final partition determined by the coordinator. Proposition 2 implies that all small components of the input graph are also components of \( C \). Thus, they are inactive in \( C \). As there are \( \Theta(n/\log \log n) \) leaders whp, Proposition 1 implies that all nodes from \( V_\alpha \) belong to \( O(n/\log \log n) \) active components. Finally, as there are only \( O(n/\log \log n) \) nodes from \( V_\gamma \), with high probability (Prop. 3), there are at most \( O(n/\log \log n) \) components of \( C \) containing those nodes.

Finally, as the coordinator computes the final partition into connected components based on the received edges (step 21), it can also determine spanning trees of the components of this partition. \( \square \)

The remaining part of this section contains the proofs of Propositions 1, 2 and 3.

Spanning trees of small components: Proof of Prop. 2

Before the formal proof of Prop. 2 we give a general statement regarding connected subgraphs of size \( s \leq 3\sqrt{\log n} \) of the input graph \( G \). Below, we show that a spanning tree of \( G' \) will appear in some sample, with high probability.

Proposition 4. For a given set of nodes \( V' \) of size \( s \leq 3\sqrt{\log n} \), such that it is a connected set in the input graph \( G \), there is no spanning tree of \( V' \) in all samples with probability at most \( O \left( \frac{1}{n^\omega(1)} \right) \).

Proof. A spanning tree of \( V' \) consists of at most \( s - 1 \) edges. Thus, it is present in some particular random sample with probability \( \text{Prob}(\text{present}) \geq \left( \frac{1}{\log \log n} \right)^{s-1} \). Thus, with probability at most \( 1 - \text{Prob}(\text{present}) \), it is not present in some particular random sample, and is not present in all samples simultaneously with probability.
\[(1 - \text{Prob}(\text{present}))^{\sqrt{n}} \leq \left(1 - \left(\frac{1}{\log \log n}\right)^{s-1}\right)^{\sqrt{n}} = \left(1 - \left(\frac{1}{\log \log n}\right)^{s-1}\right)^{(\log \log n)^{s-1} \sqrt{n} (\log \log n)^{1-s}} \leq \left(\frac{1}{e}\right)^{\frac{\sqrt{n}}{(\log \log n)^{s-1}}} = O\left(\frac{1}{n^{o(1)}}\right)\]

Using Prop. 4, we will prove Prop. 2. Let \( C \) be a (ungrowable) connected component of the input graph \( G \) such that \( |C| \leq \sqrt{\log n} \). By Prop. 4, \( C \) has no spanning tree in all samples with probability at most \( O\left(\frac{1}{n^{o(1)}}\right) \). As there are at most \( n \) small components, by union bound, the probability that there exists a small component of the input graph which is not a component of any sample is at most

\[n \cdot O\left(\frac{1}{n^{o(1)}}\right) = O\left(\frac{1}{n^{o(1)}}\right)\]

Therefore, with probability \( 1 - O\left(\frac{1}{n^{o(1)}}\right) \), each small component of the input graph is a component of some sample. Thus, in order to prove Prop. 2, it is sufficient to show the following fact: if a small component \( C \) of the input graph is a component of some sample and no leader belongs to \( C \), then \( C \) will also be a component of the partition \( \mathbb{C} \) determined by the coordinator. Assume that \( C \) is a small component of \( G \), \( C \) is a component of a sample \( G_i \) for \( i \in [m] \) and no leader belongs to \( C \). W.l.o.g. assume that \( i \) is the largest index of a sample containing \( C \) as a component. That is, \( C \) is not a component of \( G_j \) for \( j > i \) and \( C \) is a component of \( G_i \). Let \( v_{\text{min}} \) be the node of \( C \) with the smallest ID. Then, for each \( v \in C \), the boss \( b_i \) sends \((0, |C|, i, p_i(v))\) to \( v \), where \( p_i(v) \) is the parent of \( v \) in a tree \( T \) rooted at \( v_{\text{min}} \), consisting of shortest paths between \( v_{\text{min}} \) and other elements of \( C \). The choice of \( i \) and the assumption that there are no leaders in \( C \) guarantee that, for each \( v \in C \), the message received by \( v \) from \( b_i \) is the largest message according to the lexicographic ordering among messages received by \( v \) from the bosses (see line 19). Thus, each \( v \in C \) sends \( p(v) = p_i(v) \) to the coordinator. Thanks to that fact, the coordinator learns about the described above spanning tree \( T \) of \( C \). Hence, \( C \) is a component of the partition determined by the coordinator. Therefore, the coordinator knows a spanning tree for every small connected component \( C \) of \( G \) with probability at least \( 1 - O\left(\frac{1}{n^{o(1)}}\right) \).

Connections to leaders: Proof of Prop. 1

W.l.o.g. assume that the connected component of a node \( v \) in the sample \( G_i \) for some \( i \in [m] \) contains a leader. That is, \( v \) receives a message containing 1 as the first coordinate from \( b_i \). Observe that, for such \( v \), the node \( p(v) \) also receives at least one message containing 1 as the first coordinate. Indeed, \( p(v) \) belongs to the same connected component \( C \) of \( G_i \) as \( v \). Moreover, according to our assumption, \( C \) contains a leader. Thus, \( p(v) \) also receives at least one message containing 1 as the first coordinate.

Now, consider a directed graph \( H \) with the set of nodes \( V \) and edges \((v, p(v))\) such that \( v \) received at least one message from a boss containing 1 as the first coordinate. Note that, intuitively, a directed path in \( H \) of non-zero length corresponds to a path in the final graph known to the coordinator with an increasing sequence (according to the lexicographic order) of tuples received by nodes from the bosses. Each sinks of this graph with non-zero in-degree is a leader. Thus, each directed non-zero length path ending in a sink corresponds to a path to a leader in the input graph. More formally, let \( T(v) = ([C_i(v)], i, n - d) \) where \((x_i, |C_i(v)|, i, p_i(j))\) is the largest message received by \( v \) in step 13 of Alg. 4 according to the lexicographic ordering and \( d \) is the smallest distance from \( v \) to a leader in \( C_i(v) \). Let \((u_1, u_2)\) be an edge in the graph \( H \), where \( u_2 = p(u_1) \). Then, according to the definition of \( T(v) \) and the choice of messages transmitted by the bosses, the tuple \( T(u_2) \) associated with \( u_2 \) is larger than the tuple \( T(u_1) \) associated with \( u_1 \) according to the lexicographic ordering. Thus, there are no cycles in the graph \( H \).

Observe also that the only sinks with non-zero in-degree in \( H \) are the leaders. Indeed, as long as the third coordinate of a node \( u \) with non-zero outdegree is smaller than \( n \) (i.e., \( u \) is not a leader), there is an edge from \( u \) to \( p(u) \neq u \) in \( H \).
Concluding, if a node \( v \) belongs to a component with a leader in some sample, its out-degree in the acyclic directed graph \( H \) is larger than 0. Then, there exists a path \( P \) in \( H \) from \( v \) to a leader (a sink in \( H \)). The path \( P \) is also known to the coordinator and therefore the connected component of \( v \) in the final partition \( \mathbb{C} \) contains a leader.

Leaderless nodes in large components: Proof of Prop. 3

We say that a node is bad if it does not belong to a small component nor to a component containing a leader in the final partition \( \mathbb{C} \) determined by the coordinator. That is, a node is bad iff it belongs to \( V_2 \). In order to prove Prop. 3 it is sufficient to show that there are \( O(\frac{n}{\log n}) \) bad nodes. Then, in the worst-case, bad nodes would be partitioned into \( \Theta(\frac{n}{\log n}) \) active components.

The outline of the proof is as follows. Firstly, we cover all nodes from non-small components (i.e., from components of size at least \( s = \sqrt{\log n} \)) by connected sets \( V_1, V_2, \ldots \) of sizes in the range \([s, 3s]\) for \( s = \sqrt{\log n} \) (Fact 3) such that \( V_i \)'s are “almost pairwise disjoint” (a more precise definition will be provided later). Then, we associate the random 0/1 variable \( X_i \) to each set \( V_i \) such that \( X_i = 0 \) implies that no nodes from \( V_i \) are bad. (In particular, \( X_i = 0 \) holds when \( V_i \) contains a leader and at least one sample graph \( G_j \) contains a spanning tree of \( V_i \). Thus, by Prop. 1 the nodes of \( V_i \) are not bad if \( X_i = 0 \).) Importantly, the variables \( X_i \) are independent and the probabilities \( \text{Prob}(X_i = 1) \) are small. As the number of bad nodes is at most

\[
\sum_i |V_i|X_i \leq 3\sqrt{\log n} \sum_i X_i,
\]

we prove the upper bound on \( \sum_i X_i \) which ensures that the number of bad nodes is \( O(n/\log n) \) with high probability.

We start with a cover of non-small components by connected “almost pairwise disjoint” components of sizes in the range \([s, 3s]\), called an almost-partition. More precisely, we say that sets \( A_1, \ldots, A_k \) form an almost-partition of a set \( A \) iff \( \bigcup_{j=1}^k A_j = A \) and, for each \( j \in [k] \), \( A_j \) contains at most one element belonging to other sets from \( A_1, \ldots, A_k \), i.e., \( |A_j \cap \bigcup_{i \neq j} A_i| \leq 1 \). The elements of \( A \setminus \bigcup_{i \neq j} A_i \) are called unique for \( A_i \).

**Fact 3.** Let \( T \) be a tree of size at least \( s \in \mathbb{N} \). Then, there exists an almost-partition \( T_1, T_2, \ldots, T_k \) of \( T \) such that

\[
T_i \text{ is a connected subgraph of } T \text{ and } |T_i| \in [s, 3s] \text{ for each } i \in [k].
\]

**Proof.** We prove the statement of the fact inductively. If the size of \( T \) is in the interval \([s, 3s]\), the almost-partition consisting from \( T \) only satisfies the given constraints.

For the inductive step, assume that the fact holds for trees of size at most \( n \) for \( n > 3s \).

Let \( T \) be a tree of size \(|T| = n + 1\) on a set of nodes \( V \). In the following, we say that a subgraph \( T' \) of \( T \) induced by \( V' \subseteq V \) is a subtree of \( T \) iff \( T' \) and the subgraph of \( T \) induced by \( V \setminus V' \) are trees. Assume that \( T \) contains a subtree \( T' \) such that \(|T'| \geq s \) and \(|T| - |T'| \geq s\). Then, by the inductive hypothesis, there exists an almost-partition of \( T \setminus T' \) and an almost-partition of \( T' \) satisfying (1). Thus, an almost-partition of \( T' \) obtained from the almost-partitions of \( T \setminus T' \) and of \( T' \) satisfies (1) as well.

Now, assume that \( T \) does not contain a subtree \( T' \) such that \(|T'| \geq s \) and \(|T| - |T'| \geq s\). (2)

For a tree with a fixed root \( r \), \( T(u) \) denotes a subtree of \( T \) rooted at \( u \). Now, we show an auxiliary property of trees satisfying (2).

**Claim 1.** Let \( T \) be a tree satisfying (2), one can chose \( r \in T \) as the root of \( T \) such that

(a) \(|T(v_i)| < s \) for each \( i \in [k] \), where \( \{v_1, \ldots, v_k\} \) is the set of children of \( r \).

**Proof of Claim 1.** Let \( r_0 \) be an arbitrary node of a tree \( T \) which satisfies (2). If (a) is satisfied for \( r = r_0 \), we are done. Otherwise, we define the sequence \( r_0, r_1, \ldots \) of nodes such that \( r_{i+1} \) for \( i \geq 0 \) is the child of \( r_i \) in \( T \) (rooted at \( r \)) with the largest subtree. Then,

(i) \(|T(r_i)| > |T(r_{i+1})|\), because \( T(r_{i+1}) \) is a subtree of \( T(r_i) \).
(ii) if \(|T(r_1)| \geq s\), then \(|T \setminus T(r_1)| < s\), by the assumption \(2\).

The condition (i) guarantees that \(|T(r_j)| \geq s\) and \(|T(r_{j+1})| < s\) for some \(j \geq 0\). Thus, \(|T(v)| < s\) for all children of \(r_j\), since \(T(r_{j+1})\) has the largest size among subtrees rooted at children of \(r_j\). The assumption \(2\) implies also that \(|T \setminus T(r_j)| < s\). Thus, (a) is satisfied for \(T\) if the root \(r\) is equal to \(r_j\). (Proof of Claim \(1\)) \(\square\)

Using Claim \(1\) we can choose the root \(r\) of \(T\) such that

\[ s > |T(v_1)| \geq |T(v_2)| \geq \cdots \geq |T(v_k)|, \]

where \(\{v_1, \ldots, v_k\}\) is the set of children of \(r\) in \(T\) (when \(r\) is its root). Next, we split the set of trees \(T(v_1), \ldots, T(v_k)\) into subsets such that the number of nodes in each subset is in the range \([s-1, 3s-1]\). Such a splitting is possible thanks to the facts that \(|T(v_i)| < s\) for each \(i \in [k]\) and \(\sum_{i=1}^{k} |T(v_i)| \geq 3s\). Finally, by adding the node \(r\) to each subset, we obtain an almost-partition satisfying \(4\). \(\square\)

Using Fact \(4\) we will eventually prove Prop. \(3\). Let \(S_1, S_2, \ldots, S_k\) be non-small connected components of the input graph \(G\), i.e., \(|S_i| > \sqrt{\log n}\) for each \(i \in [k]\). By Fact \(3\) there exists an almost-partition of spanning trees of \(S_i\)'s into trees of sizes from the interval \([\sqrt{\log n}, 3\sqrt{\log n}]\). Let \(\mathcal{T} = \{T_1, T_2, \ldots\}\) be the set of trees equal to the union of all those almost-partitions. Observe that, according to the properties of almost-partitions, there are at least \(\sqrt{\log n} - 1\) nodes unique for \(T_i\), i.e., nodes which belong to \(T_i\) and do not belong to any other tree of the above specified almost-partitions of the components \(S_1, S_2, \ldots, S_k\). We associate random events \(A_i\) and \(B_i\) with each tree \(T_i\), where

- \(A_i\) is the event that all edges of \(T_i\) appear in at least one sample graph among \(G_1, \ldots, G_m\) in an execution of Alg. \(3\)
- \(B_i\) is the event that at least one element of the set of nodes unique for \(T_i\) has the status leader in an execution of Alg. \(3\)

Importantly, all event \(A_i\) and \(B_i\) are independent, thanks to the facts that each edge is decided to be included in each sample graph independently, the sets of edges of \(T_i\)'s are disjoint, the set of nodes unique for \(T_i\)'s are disjoint as well, and the random choices determining whether a node has a status leader are also independent.

By Prop. \(4\) the probability of \(A_i\) is \(\text{Prob}(A_i) = 1 - O\left(\frac{1}{\sqrt{n\log n}}\right)\). As \(T_i\) has at least \(s = \sqrt{\log n}\) unique nodes, the probability of \(B_i\) is at least

\[ \text{Prob}(B_i) \geq 1 - \left(1 - \frac{1}{\log \log n}\right)^{\sqrt{\log n-1}}. \]

Observe that the conjunction of the events \(A_{i,j}\) and \(B_{i,j}\) guarantees that the nodes of \(T_{S_{j,i}}\) are connected to a leader in the partition \(C\), i.e., they are not bad nodes. Let \(X_i\) be a 0/1 variable, where \(X_i = 0\) iff \(A_i\) and \(B_i\) are satisfied. Thus, \(X_i = 0\) implies that no node from \(T_i\) is bad. As the events \(A_i\) and \(B_i\) are independent, the probability that \(X_i = 0\) (implying that no node from \(T_{S_{j,i}}\) is bad) can be estimated as follows:

\[ \text{Prob}(X_i = 0) > \text{Prob}(A_i) \cdot \text{Prob}(B_i) \]
\[ > \left(1 - O\left(\frac{1}{n^{(1)}}\right)\right) \cdot \left(1 - \left(1 - \frac{1}{\log \log n}\right)^{\sqrt{\log n-1}}\right) \]
\[ > \left(1 - O\left(\frac{1}{n^{(1)}}\right)\right) \cdot \left(1 - \frac{1}{e^{(\sqrt{\log n-1})/\log \log n}}\right) \]
\[ = 1 - O\left(\frac{1}{e^{(\sqrt{\log n-1})/\log \log n}}\right). \]

Thus, \(\text{Prob}(X_i = 1) = O\left(\frac{1}{e^{(\sqrt{\log n-1})/\log \log n}}\right)\). The expected number of bad nodes is upper bounded by

\[ E \left[ \sum_i X_i \cdot |T_i| \right] = O\left(\sqrt{\log n}\right) \sum_i X_i, \quad (3) \]
since \(|T_i| \in \Theta(\sqrt{\log n})\) for each \(i\) under consideration. The expected value of the sum of the variables \(X_i\) can be estimated as

\[
E \left[ \sum_i X_i \right] = \sum_i O \left( \frac{1}{e(\sqrt{\log n} - 1)/\log \log n} \right) = O \left( \frac{n}{\sqrt{\log n}} \right) \cdot O \left( \frac{1}{e(\sqrt{\log n} - 1)/\log \log n} \right)
\]

= \(O \left( \frac{n}{\sqrt{\log n} \cdot e(\sqrt{\log n} - 1)/\log \log n} \right)\).

As \(X_i\) are independent 0–1 random variables,

\[
\sum_i X_i \in O \left( \frac{n}{\sqrt{\log n} \cdot e(\sqrt{\log n} - 1)/\log \log n} \right) \text{ with high probability}
\]

by a standard Chernoff bound. Therefore, by (3) and (4), the number of bad nodes is

\[
O(\sqrt{\log n}) \cdot O \left( \frac{n}{\sqrt{\log n} \cdot e(\sqrt{\log n} - 1)/\log \log n} \right) = O \left( \frac{n}{e(\sqrt{\log n} - 1)/\log \log n} \right) = O \left( \frac{n}{\log \log n} \right)
\]

with high probability. This fact finishes the proof of Prop. 3.

4 MST in \(O(1)\) fact rounds

In order to find MST of a given input graph, we will use the \(O(1)\) round CC algorithm and the reduction from [5]. The MST problem for a given graph can be reduced, by using the KKT random sampling [8] to two consecutive instances of MST, each for a graph with \(O(n^{3/2})\) edges. The authors of [5] observed that the MST problem for a graph with \(O(n^{3/2})\) edges can be reduced to \(\sqrt{n}\) instances of the CC problem simultaneously. In each of those \(\sqrt{n}\) instances of the CC problem, the set of neighbours of each node is a subset of the set of its neighbours in the original input graph with \(O(n^{3/2})\) edges (a nice exposition of the reduction is also given in [4]). For further references, we state these reductions more precisely.

Lemma 7. [5] Let \(G(V,E,c)\) be an instance of the MST problem. There are congested clique \(O(1)\) rounds algorithms \(A_1, A_2\) such that, with high probability,

1. \(A_1\) builds \(G_1(V,E_1,c)\) with \(O(n^{3/2})\) edges such that \(E_1 \subseteq E\) and,

2. given a minimum spanning forest of \(G_1\), \(A_2\) builds \(G_2(V,E_2,c)\) with \(O(n^{3/2})\) edges such that \(E_2 \subseteq E\) and a minimum spanning tree of \(G_2\) is also a minimum spanning tree of \(G\).

Lemma 8. [5] Let \(G(V,E,c)\) be an instance of MST, where \(|E| = O(n^{3/2})\). There is a congested clique \(O(1)\) round algorithm which reduces the MST problem for \(G\) to \(m = \sqrt{n}\) instances \(G_i(V,E_i)\) for \(i \in [m]\) of the CC problem, such that (i) \(E_1 \subseteq E_2 \subseteq \cdots \subseteq E_m = E\); (ii) each node \(v\) knows edges incident to \(v\) in \(E_i\) for each \(i \in [m]\) at the end of an execution of the algorithm.

If we show that our CC algorithm can be executed simultaneously for \(\sqrt{n}\) instances satisfying the properties from Lemma 8 in \(O(1)\) rounds, then we obtain the \(O(1)\) round randomized algorithm for MST.

As shown in Section 3 of [4], the algorithm ReduceCC applied in GPReduction (Lemma 4) can be executed in parallel for \(\sqrt{n}\) instances as above. As the algorithm GPReduction satisfying Lemma 4 consist of \(O(1)\) executions of ReduceCC (see the proof of Lemma 4), this algorithm can be executed in parallel for \(\sqrt{n}\) instances of the CC problem satisfying conditions from Lemma 8.

Therefore, in order to prove that our CC algorithm can be executed in parallel for \(\sqrt{n}\) instances described above, it is sufficient to show that Algorithm 2 and Algorithm 3 can be executed in parallel in such a way (see Alg. 1). In Sections 4.1 and 4.2 we show that it is the case. Finally, in Section 4.3 we discuss a parallel execution of the whole Alg. 1 for all those instances. This gives a \(O(1)\) round randomized algorithm determining a minimum spanning tree which proves Theorem 1.
4.1 Parallel executions of Alg. 2

In this section we show that Alg. 2 can be executed for $\sqrt{n}$ related sparse instances of the CC problem in parallel, as stated in the following lemma.

**Lemma 9.** Let $G_1(V, E_1), \ldots, G_m(V, E_m)$ for $m = \sqrt{n}$ be the input graphs in the congested clique model such that $|E_i| = O(n^{3/2})$ for each $i \in [m]$, $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_m$, and $v_j$ knows its neighbours in each of the graphs $G_1, \ldots, G_m$. Then, Alg. 2 can be executed simultaneously for $G_1, \ldots, G_m$ in $O(1)$ rounds in the following framework:

- for each $i \in [m], j \in [n]$, the node $u_j$ has assigned a node of proxy$(i, j) \in \{u_1, \ldots, u_n\}$ such that proxy$(i, j)$ works on behalf of $u_j$ in the $j$th instance of the CC problem;
- for each $j \in [n]$, the node $u_j$ works as the proxy of $O(1)$ nodes, i.e.,
  \[ \{k \mid u_j = \text{proxy}(i, k)\} = O(1). \]

The proof of Lemma 9 is presented in the remaining part of this section. As there are $m$ instances of the CC problem to solve, we can set $m$ coordinators $c_1, \ldots, c_m$ such that, e.g., the node $u_i$ acts as the coordinator $c_i$.

Steps 3, 4, 6 and 7 of Alg. 2 either do not require any communication (steps 4 and 7) or each node transmits a single message to the coordinator (steps 3 and 6). Thus, these steps can be executed in parallel in $O(1)$ rounds: instead of sending a message to one coordinator, each node can send appropriate messages to the coordinators $c_1, \ldots, c_m$ in a round for $m = \sqrt{n}$.

The main problem with parallel execution of of steps 2, 5 and 8 of Alg. 2 is that each node sends a message to all its neighbours. Thus, a node with degree $\Delta = \omega(\sqrt{n})$ needs to send

\[ m \cdot \Delta = \sqrt{n} \cdot \omega(\sqrt{n}) = \omega(n) \]

messages in $m$ parallel executions of Alg. 2 which cannot be done in $O(1)$ rounds (even with help of e.g. Lenzen’s routing), because of limited bandwidth of edges.

In order to overcome the above observed problem, we will take advantage of the fact that the number of edges in each of $m = \sqrt{n}$ instances of the problem is $O(n^{3/2})$. Thus, the overall number of messages to send in all instances is

\[ O \left( \sum_{v \in V} \sum_{i \in [m]} d_{G_i}(v) \right) = O \left( \sum_{i \in [m]} \sum_{v \in V} d_{G_i}(v) \right) = O \left( \sum_{i \in [m]} |E_i| \right) = O(\sqrt{n} \cdot n^{3/2}) = O(n^2). \]

Hence, the amount of communication fits into quadratic number of edges of the (congested) clique.

In our solution, we distribute communication load among so-called proxies. Let $d(u_j) = |\{(u_j, v) \mid (u_j, v) \in E_m\}|$ be the upper bound on the degree of $u_j$ in all graphs, due to the assumption $E_1 \subseteq \cdots \subseteq E_m$. Assume that a pool of proxy nodes $v_1, v_2, \ldots$ is available (see Lemma 2). We assign $l_j = \lfloor d(u_j)/\sqrt{n} \rfloor$ proxy nodes to $u_j$ for each $j \in [n]$. Altogether, we need

\[ \sum_{j \in [n]} \lfloor d(u_j)/\sqrt{n} \rfloor \leq n + \frac{1}{\sqrt{n}} \sum_{j \in [n]} d(u_j) = n + O \left( \frac{1}{\sqrt{n}} \cdot n^{3/2} \right) = O(n) \]

proxy nodes. The key idea is that the work of the node $u_j$ is split between its proxies such that each proxy node is responsible for simulating $u_j$ in $\min \{\sqrt{n}, \lfloor n/d(u_j) \rfloor \}$ instances of Alg. 2. In order to guarantee feasibility of a simulation of all $\sqrt{n}$ executions of Alg. 2 by the proxies in $O(1)$ rounds, we have to address the following issues:

(a) In order to simulate the nodes $\{u_1, \ldots, u_n\}$ in various instances of the CC problem, the proxies need to know the mapping between the nodes $\{u_1, \ldots, u_n\}$ and proxies simulating them in respective instances of the CC problem.

(b) In order to simulate $u_j$ in the $j$th instance of the problem, the appropriate proxy should know the neighbors of $u_j$ in $G_j$. Thus, information about neighbours of appropriate nodes should be delivered to proxies before the actual executions of Alg. 2.
It should be possible that each proxy \( v \) of \( u_j \) is able to simulate each step of \( u_j \) in all instances of the problem in which \( v \) works on behalf of \( u_j \) in \( O(1) \) rounds.

Regarding (a), note that the values \( d(u_j) \) can be distributed to all nodes in a single round. Using this information, each node can locally compute which proxies are assigned to particular nodes of the network in consecutive instances of the problem, assuming that the proxies are assigned in the ascending order, i.e., \( v_1, \ldots, v_t \) are assigned to \( u_1, v_i, \ldots, v_i \) are assigned to \( u_2 \) and so on.

As for (b), we rely on the fact that \( E_1 \subseteq \cdots \subseteq E_m \). The node \( u_j \) encodes information about its neighbours in all sets \( E_i \) as the set \( T_j \) of tuples (\( v, l \)) for each \( (u_j, v) \in E_m \), where \( l \) is the smallest \( i \) such that \( (u_j, v) \in E_i \). (That is, \( (u_j, v) \in E_1, E_1+1, \ldots, E_m \), \( (u_j, v) \notin E_1, \ldots, E_{i-1} \).) Thus, knowing \( T_j \), it is possible to determine the neighbors of \( u_j \) in \( G_i \), for each \( i \in [m] \). The set \( T_j \) is delivered to all \( \lceil d(u_j)/\sqrt{n} \rceil \) proxies of \( u_j \) in the following way:

- **Stage 1.** The set \( T_j \) is split into \( \lceil d(u_j)/\sqrt{n} \rceil \) subsets of size \( \sqrt{n} \), each subset is delivered to a different proxy of \( u_j \).

- **Stage 2.** The subset of \( T_j \) of size \( \sqrt{n} \) delivered to a proxy of \( u_j \) in Stage 1 is delivered to all other proxies of \( u_j \).

In order to perform Stages 1 and 2 in \( O(1) \) rounds, we apply Lenzen’s routing algorithm \cite{12} (Lemma 1), which works in \( O(1) \) rounds provided each node has \( O(n) \) messages to send and \( O(n) \) messages to receive. Note that \( u_j \) has \( d(u_j) = O(n) \) messages to be transmitted and each proxy has \( O(\sqrt{n}) = O(n) \) messages to receive in Stage 1. In Stage 2, each proxy of \( u_j \) is supposed to deliver and receive

\[
O \left( \sqrt{n} \cdot \frac{d(u_j)\sqrt{n}}{\sqrt{n}} \right) = O(n)
\]
messages.

Knowing that the issues (a) and (b) are solved, we can also address (c). Thanks to the presented solution for (a), there is global knowledge which proxy nodes are responsible for particular nodes of the network in various instances of the problem and all proxies start with the knowledge of the nodes simulated by them. Thus, in each step of the algorithm, transmissions between nodes are replaced with transmissions between appropriate proxy nodes. It remains to verify whether proxy nodes are able to deliver messages on behalf of the actual nodes simulated by them in all instances of Alg. 2 simulated by them. The number of messages supposed to be sent and received by the node \( u_j \) in a step of Alg. 2 is at most \( d(u_j) \). As each proxy of \( u_j \) simulates \( u_j \) in \( \min\{\sqrt{n}, \lceil n/d(u_j) \rceil \} \) instances of Alg. 2, it is supposed to send/receive at most

\[
O \left( d(u_j) \cdot \min\{\sqrt{n}, \lceil n/d(u_j) \rceil \} \right) = O(n)
\]
messages in each round. Hence, using Lenzen’s routing \cite{12} (Lemma 1), each step of each execution might be simulated in \( O(1) \) rounds.

### 4.2 Parallel executions of Algorithm 3

In this section, we show feasibility of simulation of \( \sqrt{n} \) instances of Alg. 3 in \( O(1) \) rounds.

**Lemma 10.** Assume that \( m = \sqrt{n} \) graphs of degree \( O(\log \log n) \) are given in the congested clique model, i.e., each node knows its neighbors in each of the graphs. Then, Alg. 3 can be executed simultaneously on all those instances in \( O(1) \) rounds.

In order to prove Lemma 10, it is sufficient to analyze the number of messages which nodes send/receive in the case that they simulate \( \sqrt{n} \) instances Alg. 3 simultaneously. As Alg. 3 is executed on a graph with degree \( O(\log \log n) \), the number of edges in each instance is \( O(n \log \log n) \).

As there are \( \sqrt{n} \) bosses and one coordinator in the “original” Alg. 3, the number of bosses increases to \( \sqrt{n} \cdot \sqrt{n} = O(n) \) and the number of coordinators to \( \sqrt{n} \) when \( \sqrt{n} \) instances are executed simultaneously. In step 3 of Alg. 3 the node \( u_j \) is supposed to deliver \( |N(u_j)| \) messages to other nodes in an instance of the CC problem. Hence, \( u_j \) has \( \sqrt{n} \cdot |N(u_j)| = O(n) \) messages to send in all \( \sqrt{n} \) instances of the CC problem. As each
edge is sent to each boss independently with probability \(1/\log \log n\) and all graphs have \(O(n \log \log n)\) edges, each boss receives
\[
O \left( \frac{1}{\log \log n} \cdot n \log \log n \right) = O(n)
\]
edges with high probability, by a standard Chernoff bound. In the remaining steps of Alg. 8

- the original nodes \(u_j\) send a single message to each boss or to the coordinator;
- the bosses send a message to all nodes of the network.

As there are \(O(n)\) bosses and coordinators (each of them participating in exactly one instance of the CC problem), all \(\sqrt{n}\) executions can be performed without asymptotic slowdown of the algorithm.

4.3 Parallel executions of Algorithm 1

Using Lemma 9, we can execute step 1 of Alg. 1 in parallel for all instances. However, after such execution, the results are distributed among proxies, not available in the original nodes of the network corresponding to the nodes of the graph. It might seem that, in order to perform the remaining steps of \(m = \sqrt{n}\) instances of Alg. 1 we can collect appropriate information back at the “original” nodes and use Lemma 4 and Lemma 10. This however is not that simple, as we have to face the following obstacle. The original instances \(G_i(V, E_i)\) satisfied the relationship \(E_1 \subseteq \cdots \subseteq E_m\), which helped to pass information about neighborhoods to the proxies. After executions of Alg. 2, this “inclusion property” does not hold any more. Therefore, we continue with proxies: both executions of GPReduction as well as an execution of Alg. 8 are executed in parallel in such a way that proxies work on behalf of their “master” nodes. Thanks to the fact that each node is a proxy of \(c = O(1)\) “original” nodes only, Lemma 4 and Lemma 10 can be applied here as follows. We can simulate each round of the original executions of GPReduction and Alg. 8 in \(c^2\) rounds indexed by pairs \((a, b) \in [c]^2\). In the round \((a, b)\) each proxy acts as a transmitter (receiver, resp.) on behalf of the \(a\)th (\(b\)th resp.) node assigned to it. (See Lemma 2 for more detailed exposition of this technique.) Then, each node in each round is supposed to perform a subset of transmissions of the node “simulated” by it in the execution performed by the original nodes and therefore the parallel simulations work by Lemma 4 and Lemma 10.

Finally, when spanning forests of all \(m = \sqrt{n}\) instances of the CC problem are determined, each proxy knows its parent in the respective spanning trees. As we consider \(m = \sqrt{n}\) simulations and each node is the proxy of \(O(1)\) nodes of the input network, information about parents of nodes in the spanning trees can be delivered from the proxies to the original nodes in \(O(1)\) rounds using Lenzen’s routing algorithm [12] (Lemma 1).

5 Conclusions

In the paper, we have established \(O(1)\) round complexity for randomized algorithms solving MST in the congested clique. In contrast to recent progress on randomized algorithms for MST, the best deterministic solution has not been improved from 2003 [14]. As shown in [17], the \(O(\log^* n)\) round MST algorithm from [4] can be implemented with relatively small number of messages transmitted over an execution of an algorithm. We believe that our solution might also be optimized in a similar way. However, to obtain such a result, more refined analysis and adjustment of parameters are necessary (e.g., adjustment of the number of samples, the threshold between high-degree components and low-degree components, the size of small components in the analysis).

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