Existence and Non-Triviality of Equilibria in Economies with Default and Government

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Abstract
This paper attempts to accommodate the government into the Dubey, Geanakoplos, and Shubik (2005) framework in an explicit way, and proves that a non-trivial equilibrium exists. By non-trivial equilibrium, we mean an equilibrium where there exists trading in the financial markets and some private borrower defaults. The government, characterised by its spending and tax plans,\(^1\) enters our economic model by trading financial assets in order to balance its budget. Proof of existence is made by considering a generalised game à la Debreu (1952),\(^2\) and its non-triviality is obtained by using the non-differentiable optimisation theory.

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JEL Codes: D50, C65

1 Introduction
Governments play key roles in their economies. On one hand, they provide a legal system and public goods, correct market failures, maintain competition, redistribute income and stabilise the economy. On the other hand, their presence and mainly their intervention in the financial markets may have a great impact on credit, and this impact may have devastating effects like that of the recent subprime mortgage crisis in the USA.

Credit is fundamental for both the consumption and production sectors. Nevertheless, credit might turn out to be problematic when financial markets

\(^1\) Both are assumed to be exogenous like in Gale (1990).
\(^2\) Already well known in the GEI literature.

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are incomplete, and even more so if default exists. Moreover, if the default is quite considerable in the economy, lenders could become too pessimistic and reduce the credit in the whole economy. Consequently, some assets might not be traded, causing the government to play a main role in restoring the credit.

Although there is evidence of government intervention in large financial markets, as in the case of the GNMA\(^3\) mortgage program in the USA, where the government guaranteed delivery of promises from all the borrowers eligible for the program, no one included it in an explicit way, as far as we know. One exception is Dubey and Geanakoplos (2002) and Dubey, Geanakoplos, and Shubik (2005). In fact, they do it, via a trembling in the market and not in the players’ strategies, which is the norm in game theory. Such a trembling approach was carried out in order to eliminate absurd beliefs in relation to the payment rate of assets that were not being traded.

More precisely, Dubey and Geanakoplos (2002) and Dubey et al. (2005) offered refinements that used perturbations to rule out equilibria supported by “unreasonable” beliefs. Yet, their perturbations required a small set of agents (a government, perhaps) to behave in a particular, suboptimal, way: in Dubey and Geanakoplos (2002), the government entered every pool and made full deliveries; in Dubey et al. (2005), the government borrowed in every market and never defaulted.

Our main objective is to extend the Dubey et al. (2005) model to accommodate the government by maintaining great part of the framework\(^4\) of the authors above. The novel feature of the present model is that the government enters the economy in an explicit way by trading financial assets in order to balance its budget. The introduction of a government’s budget is the same as imposing a more optimistic belief regarding the repayments for some states, as one of the agents always honor their contracts. This new assumption may be similar to Dubey et al. (2005)’s clearing of pessimistic beliefs. Thus, our extension is plausible.

The government is characterised by its spending and tax plans. Both are assumed to be exogenous as in Gale (1990). The presence of government in this extended framework leads us to give an alternative approach to the equilibrium existence. Technically, the essential difference between this model and that of Dubey et al. (2005) lies in the methodology used to show the existence of

\(^{3}\) Government National Mortgage Association.

\(^{4}\) That is, assets are considered as pools; lending and borrowing plans are modelled in a separated way; the payment rate are anticipated in a rational manner by the landers, government included, etc. See Remark 1 of section 3 for more details.
equilibrium. Dubey et al. (2005) use trembling hand while we use the generalised game approach used by Debreu (1952). The latter is much simpler than the former. Moreover, it will allow us to avoid dealing with any kind of refinement of equilibrium.

Although our approach does not deal with any kind of refinement, it does not prevent the problem that the government faces when it comes to financing its deficit, as in principle the government may not be able to fund its public deficit due to the incompleteness of the financial markets. We overcome this problem by allowing the government to also purchase assets from the financial markets. This makes the government suffer default from part of private investors. These two earlier facts make asset span enlarge sufficiently so that it contains the public deficit.

Another objective of this paper is to prove that the equilibrium is non-trivial in the sense that a private borrower is trading in the financial markets and defaulting on that asset. To reach that goal, we give sufficient conditions for a private borrower to sell at least one asset in the first period. Such conditions have to do with the comparison between the desire to consume and the penalty felt by defaulting. For someone to sell at least one asset, it is sufficient that the discounted aggregate penalty felt for delivering nothing in the second period is less than the desire to consume in the first period. Instead of delivering less than its debt (defaulting), the condition is formalised in terms of desire to consume in the future and the state contingent penalty felt by defaulting.

Before explaining the organisation of the paper, we find it convenient to point out some problems that arose in regard to the existence of equilibrium. To prove the existence of equilibrium in the presence of incomplete markets, particularly when the assets are real, is very difficult because the budget correspondence fails to be lower hemi-continuous, see Hart (1975). Still, in a couple of papers, Duffie and Sahafer (1985, 1986) for example proved that the problem raised by Hart was rare. That is to say, the equilibrium always exists generically when assets are real. However, when assets are nominal or numeraire the problem raised by Hart disappears; see, e.g., Werner (1985), Duffie and Sahafer (1986), and Geanakoplos and Polemarchakis (1986). In all those models, default was

5 A government purchasing and selling assets in incomplete markets is not an assumption which belongs to us, Gale (1990) had already considered it.

6 This situation may be interpreted as a kind of “non financial crowding out” in the economy. By financial crowding out we mean a situation where either the government is the only borrower or there is no private borrowing in the economy. That is, private borrowers are not displaced from the asset markets by the government.

7 This comparison is made state by state.
excluded by assumption. That is, it was assumed that the agents always honour their financial obligations. Yet, in practice, the agents do not honour their commitments. A model that incorporates this phenomenon would be necessary and convenient. In that sense, Kehoe and Levine (1993, 2001) pioneer this subject; see also Alvarez and Jermann (2000). Nevertheless, in their models, there was no default in equilibrium. It was Dubey et al. (2005) who showed that the equilibrium always exists in the presence of default under the usual condition on utilities and initial endowments.

It is useful to stress that for Dubey et al. (2005) to be able to extend the GEI model to one incorporating default, they had to face many problems, especially concerning how to model default. To accomplish that, they simply allowed borrowers to deliver less than they owed, and allowed lenders to anticipate the payment rate in a rational way. Regarding the latter, they had to deal with the possibility of some assets not being traded, leading to indeterminacy of the delivery rate of those assets. They overcame this problem by refining their notion of equilibrium by excluding the lenders’ pessimistic beliefs on the delivery rates. We will not consider any type of refinement here.

The paper is organised as follows: In section 2, we describe the model of our economy with default and government. In section 3, we define equilibrium and in section 4, we state our main results. In section 5 we prove them: both existence and non-triviality. Finally, the paper ends with two short sections: concluding remarks and an appendix.

2 The Model
In this section, we set up the ingredients of our model, which includes markets, agents, etc. We end this section by defining our economy.

2.1 Time and uncertainty
There are two periods or dates: \( t = 0, 1 \). In \( t = 0 \) there is no uncertainty, while in \( t = 1 \) there is. This uncertainty is modelled by a finite set of states of nature \( S = \{1, \ldots, S\} \). Any element belonging to \( S \) is denoted by \( s \in S \).

2.2 Commodities and promises
In each period and state of the nature there are \( L \) commodities for trading. Thus, the consumption space is \( \mathbb{R}^L \times \mathbb{R}^{SL} \). Any element belonging to \( \mathbb{R}^L \times \mathbb{R}^{SL} \) is called a consumption plan and will be denoted by \( x = (x_0, \tilde{x}) \), where \( x_0 \in \mathbb{R}^L \)
and $\tilde{x} \in \mathbb{R}^{SL}$. The coordinates of $\tilde{x}$ are elements of $\mathbb{R}^{L}$ and are denoted by $x_s \in \mathbb{R}^{L}$ for all $s \in S$.

In $t = 0$, there are $J$ real assets, meaning that what they deliver are commodity bundles. Each asset $j \in J$ is represented by the promise to be delivered at $t = 1$. To be more precise, each asset $j$ promises the commodity bundle $A^j \in \mathbb{R}_+^{LS}$. Thus, in each state $s$ the asset $j$ promises $A^j_s \in \mathbb{R}_+^L$. We can then arrange all $J$ real assets in a matrix with $SL$ arrows and $J$ columns. Let us call this matrix $A$. Thus, $A \in \mathbb{R}_+^{SL \times J}$.

The value of each real asset $j$ at prices $p = (p_0, \tilde{p}) \in \mathbb{R}_+^{L(S+1)}$ is the vector $\tilde{p} A^j$ belonging to $\mathbb{R}_+^S$. Thus, the payoff matrix at prices $\tilde{p}$ is $V(\tilde{p}) \in \mathbb{R}_+^{S \times J}$. If there is no confusion, we drop the price vector $\tilde{p}$ and we write $V(\tilde{p})$ as $V \in \mathbb{R}_+^{S \times J}$. The arrows of this matrix are denoted by $p_s A_s \in \mathbb{R}_+^J$ or simply by $V_s \in \mathbb{R}_+^J$.

In general, the payoff matrix $V$ has entries $p_s A^j_s \in \mathbb{R}$ with $s \in S$ and $j \in J$. Notice that the payoff matrix depends on the price system $\tilde{p}$ or simply by $V_s \in \mathbb{R}_+^J$.

If there is no default, agents as both borrowers and lenders pay and would receive their debts and investments, respectively.

### 2.3 Defaulting: Deliveries and receipts

Lending and borrowing are modelled by two portfolios $\theta$ and $\varphi$, both belonging to $\mathbb{R}_+^J$. The prices of assets are summarised by a vector $\pi \in \mathbb{R}_+^J$.

On one hand, when there is defaulting, borrowers who issued $\varphi \in \mathbb{R}_+^J$ in $t = 0$ decided to deliver according to the matrix $D \in \mathbb{R}_+^{S \times J}$. Thus, the delivery vector is then $D \mathbf{1} \in \mathbb{R}_+^S$, where $\mathbf{1}$ is the vector of $\mathbb{R}^S$ consisting of ones in each coordinate. Therefore, the vector of deliveries for each portfolio sold $\varphi \in \mathbb{R}_+^J$ is $E(\varphi) \in \mathbb{R}_+^S$ whose $s-$ coordinate is defined by $E_s(\varphi) = \sum_{j \in J} D^j_s$, where $D^j_s$ satisfies

\[0 \leq D^j_s \leq p_s A^j_s \varphi_j, \quad \forall j, s.\]

On the other hand, due to defaulting, lenders will not receive the payoffs matrix $V$, but rather, they will receive according to the structure payment rate $k \in [0, 1]^{S \times J}$. Thus, the receipt matrix is defined by $kV \in \mathbb{R}_+^{S \times J}$, whose entries are $k^j_s V^j_s = k^j_s p_s A^j_s \in \mathbb{R}_+$. The arrows of the receipt matrix $kV$ are denoted by $k_s V_s$ or $k_s p_s A_s$, which are defined by $k_s V_s = \begin{bmatrix} k_s V^1_s & \cdots & k_s V^J_s \end{bmatrix} = \begin{bmatrix} k_s p_s A^1_s & \cdots & k_s p_s A^J_s \end{bmatrix}$.

Thus, if a lender purchases the portfolio $\theta \in \mathbb{R}_+^J$, then the vector of receipts is denoted by $kV \theta \in \mathbb{R}_+^S$, whose $s-$ coordinate is $k_s V_s \theta := \sum_{j \in J} k^j_s p_s A^j_s \theta_j, \forall s \in S$.
2.4 Households

There are $H$ households. Each one of them is represented by a utility function $U^h: \mathbb{R}^{S(L+1)}_+ \to \mathbb{R}$, and an initial endowment $\omega^h \in \mathbb{R}^{L(S+1)}_+$. Since the model allows households to default, a utility penalty must be imposed which is proportional to the unpaid amount of the debt originating from the sale of an asset portfolio in the first period. More precisely, the payoff of each household is the map $W^h: \mathbb{R}^{S(L+1)}_+ \times \mathbb{R}_+^J \times \mathbb{R}^J_+ \times \mathbb{R}^{SL \times J}_+ \to \mathbb{R}$ defined by

$$W^h(x, \theta, \varphi, D) = U^h(x) - \sum_{j,s} \lambda_{hs}^j [p_s A_s^j \varphi_j - D_s^j]^+. $$

This payoff is composed of two parts: their pleasure $U^h(x)$, derived from the consumption of goods $x \in \mathbb{R}^{L(S+1)}_+$, and their penalty $\sum_{j,s} \lambda_{hs}^j [p_s A_s^j \varphi_j - D_s^j]^+$, which increases in proportion to unpaid debt $p_s A_s^j \varphi_j - D_s^j$. The parameter $\lambda_{hs}^j > 0$ is the penalty assumed to be strictly positive, and without losing generality it can be taken to be constant for all agents.

Each household maximises its payoff $W^h$ subject to its budget set $B^h(p, \pi, k, t)$, which is defined by the following budget and short-sale constraints:

$$p_0 x_0 + \pi (\theta - \varphi) \leq p_0 \omega^h_0 - p_0 t^h_0, \quad (1)$$

$$p_s x_s + E_s(\varphi) \leq p_s \omega^h_s + k_s V_s \theta - p_s t_s, \quad s \in S, \quad (2)$$

$$\varphi_j \leq v^h_j, \quad \forall j \in J. \quad (3)$$

The right side in (1) is the income after paying taxes, while the left side is the expenditures due to the consumption and the net investments in financial markets. The budget constraint (2) says that the expenditure for consumption and deliveries is financed by the ex post income after paying taxes. Lastly, (3) expresses that private agents faced a bound on short sales.

2.5 Government

There is also a government that consumes, taxes and trades in the financial markets. The government is further assumed not to default on its debts, but as a lender suffers default, since we are assuming private defaulting. The government is characterised by a spending plan $G = (G_0, \tilde{G}) \in \mathbb{R}^{L(S+1)}_+$ and by a tax plan $T = (T_0, \tilde{T}) \in \mathbb{R}^{L(S+1)}_+$, which it collects. Both are assumed to be exogenous like in Gale (1990). Let $t = (t^h_0, \tilde{t})_{h \in H}$ be the tax obligations of agents such that $T = \sum_{h \in H} t^h$. 
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Given the price system \((p, \pi) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^J\), the government chooses a portfolio consisting of two vectors \((\phi^g^+, \phi^g^-) \in \mathbb{R}_+^{2J}\), representing purchases and sales of assets, respectively. The objective of the government is to balance its budget (4) and (5) below. That is, the following budget constraints must be satisfied:

\[
\begin{align*}
p_0 G_0 + \pi \phi^g^+ &= \pi \phi^g^- + p_0 T_0, \quad (4) \\
p_s G_s + V_s \phi^g^- &= k_s V_s \phi^g^+ + p_s T_s, \quad s \in S. \quad (5)
\end{align*}
\]

Furthermore, the government cannot be indebted ad infinitum. That is, there exists \(v^g \in \mathbb{R}_+^J\) such that

\[
-v_j^g \leq \phi_j^g^+ - \phi_j^g^-, \quad \forall j \in J. \quad (6)
\]

Equation (4) says that both expenditures and financial investments of the government are funded by the collection of taxes and by the sale of assets (borrowing). Equation (5) says that the contingent spendings\(^8\) are financed by the return of the investment made in the first period and by the value of the collection of taxes.

Writing (4) and (5) in a matrix way one has

\[
p \square (G - T) = Q \begin{bmatrix} \phi^g^+ \\ \phi^g^- \end{bmatrix},
\]

where \(p \square (G - T)\) is a vector of \(\mathbb{R}^{1+S}\) whose \(s\)-coordinate is \(p_s (G_s - T_s)\) with \(s = 0, 1, \ldots, S\).

The matrix \(Q\) is defined to be

\[
Q = \begin{bmatrix} -\pi & \pi \\ kV & -V \end{bmatrix}_{(1+S) \times 2J},
\]

where \(V_s^j = p_s A_s^j \in \mathbb{R}_+, \text{ and } k_s^j \in [0, 1]\) is the payment rate.

Define the sub-space \(\mathcal{M} := \langle \{Q^j\}_{j=1, \ldots, 2J} \rangle\) generated by the \(2J\) columns of \(Q\) of the contingent claim space \(\mathbb{R}^{1+S}\).

We assume that \(1 + S < 2J\) and that \(Q\) has full rank, so that the dimension of \(\mathcal{M}\) is \(1 + S\). This implies that the vector of deficit of government \(p \square (G - T)\)

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\(^8\) Expenditures and payment of debts.
always belongs\(^9\) to \(\mathcal{M}\).

The right-hand of (7) can be rewritten as \(\sum_{j=1}^{J} \phi_j^g + \sum_{j=J+1}^{2J} \phi_j^g - Q^j\). Thus, (7) is reduced to

\[
p\Box(G - T) = \sum_{j=1}^{J} \phi_j^g + \sum_{j=J+1}^{2J} \phi_j^g - Q^j.
\] (8)

### 2.6 The economy

After introducing the model, we can define the economy with private default and government. Precisely, \(\mathcal{E} = \left(\left(U^h, \omega^h\right)_{h \in H}, A, \lambda, v, (G, T)\right)\), where \((U^h, \omega^h)_{h \in H}\) represents the individuals, \((A, \lambda, v)\) the financial structure, and \((G, T)\) the government. Note that the assets consist of promises, penalties for default, and limits on short sales, \(v = \left(\left(v^h\right)_{h \in H}, v^g\right)\).

### 3 Equilibrium and its non-triviality

We begin by defining the notion of equilibrium with government and private defaulting.

**Definition 1.** An equilibrium for \(\mathcal{E}\) is an array

\[
\left[(p, \pi, k), (x^h, \theta^h, \varphi^h, D^h)_{h \in H}, (\phi^g, \phi^g -)\right]
\]

such that

1. The macro variables \((p, \pi, k)\) are such that \((p_0, \pi) \in \triangle_{+}^{L-1} \times \triangle_{+}^{L-1}; p_s \in \triangle_{+}^{L-1}, \forall s \in S\); and \(k_{s}^j \in [0, 1], \forall(s, j)\).

2. The choices are optimal. That is,

   (a) for each \(h \in H\), the choice \((x^h, \theta^h, \varphi^h, D^h)\) maximizes \(W^h\) subject to \(B^h(p, \pi, k, t)\).

   (b) The portfolio \((\phi^g, \phi^g -)\) balances the government budget. That is

   \[
p\Box(G - T) = Q\begin{bmatrix} \phi^g \cr \phi^g - \end{bmatrix}.
\]

In addition (6) holds.

\(^9\)This means that the government will always be able to fund its deficit purchasing and selling assets, although the markets are incomplete. That is, \(J < S\).
3. Markets Clear

\[ \sum_{h \in H} x^h_0 + G_0 = \sum_{h \in H} \omega^h_0 \quad \text{and} \quad \sum_{h \in H} (\theta^h - \varphi^h) + (\phi^{g^+} - \phi^{g^-}) = 0, \]

\[ \sum_{h \in H} x^h_s + G_s = \sum_{h \in H} \omega^h_s, \quad s \in S. \]

4. Payment rate \( k \in [0,1]^{S \times J} \) satisfies, for each \( j \in J \) and \( s \in S \), the following:

\[ k^j_s \left( p_s A^j_s \phi^g_{j}^{g^-} + \sum_{h \in H} p_s A^j_s \varphi^h_{j} \right) = \left( p_s A^j_s \phi^g_{j}^{g^-} + \sum_{h \in H} D^h_{sj} \right) \]

**Remark 1.** Item 2 says that the choices of all agents, including that of the government,\(^1\) are optimal. Item 3 states that all markets are clearing. Although lenders know that the government never defaults, the purchase of \( \theta_j \) is made from a pool of borrowers, government included. Thus, the delivery of the government appears on the left side of the equation of Item 4 because lenders do not identify any borrowers.

The presence of the government in the pool banishes whimsical pessimism from the lenders. This is true because our model assumes that the government will never default. So, we do not need any external agents like in Dubey et al. (2005) to boost the financial markets.

In what follows, we will give sufficient conditions for an equilibrium to exist. The following conditions guarantee the existence of an equilibrium.

H1 \( \omega^h_s - t_s > 0, \forall h, \forall s \in \{0,1,\ldots,S\} =: S^* \).

H2 \( \sum_{h \in H} \omega^h_s > 0, \forall s, l. \)

H3 \( U^h \) is continuous, concave, and strictly increasing in each of its \( S^* \times L \) variables.

**Remark 2.** H1 says that the individuals have non-zero initial endowments after being taxed by the government. H2 says that all commodities are present in the economy in all states of nature. H3 is a usual hypothesis and therefore does not require details.

\(^1\) Remember that the goal of the government is to only balance its budget.
There are many ways of defining non-triviality of equilibria. Dubey et al. (2005)’s definition is one of them. There are others also—for instance that of Araujo, Monteiro, and Páscoa (1998). If we adapted it to our context, there could be an equilibrium with trading and without default as the government could be the only borrower who is assumed not to default. Since our objective is to guarantee that in equilibrium someone is defaulting, we will abandon the latter definition and offer the following definition,\footnote{See Maldonado and Orrillo (2006) for a similar definition, but in another context.} which allows default.

**Definition 2.** An equilibrium $[(p,\pi,k),(x^h,\varphi^h,D^h)_{h\in H},\phi^g]$ is said to be non-trivial if there exists an asset $j$ and a state $s \in S$ such that

$$\sum_{h \in H} \varphi^h j \neq 0,$$

and there exists $h \in H$ such that $0 \leq D^h sj < p_s A^j s \varphi^h j$.

Definition 3 says that for equilibrium to be non-trivial, there must be trading in the financial markets at least for one asset and one state and someone must be defaulting on that asset.

### 3.1 Non-triviality of the equilibrium

As said above, we want to show that our equilibrium is non-trivial. To do that, we must offer some hypotheses that guarantee trading in the financial markets by some private borrower. We did not want private borrowers being displaced from the asset markets by the government. If we want someone to default in equilibrium, we must first ensure that some private borrower is selling some asset. That is the role that H5 plays below.

The following conditions will be assumed

**H4** $\exists h \in H, \exists s \in S: \forall j, \frac{\partial U(t)}{\partial x^s} > \lambda^j j L, \text{ where } \hat{\omega} = \sum_{h \in H} \omega^h s.$

**H5** $\exists h \in H: \forall j, \left(\max_{s \in S} \lambda^j j s \right) \sum_{s \in S} \lambda^h s j L < \frac{\partial U(t)}{\partial x^0}, \text{ where } \hat{\omega}_0 = \sum_{h \in H} \omega^h 0, \text{ and } r^j s = \frac{\nu^j}{\pi^j} \text{ with } \pi^j > 0.$

**Remark 3.** H4 states that there is at least one individual whose desire for consumption in the second period is stronger than the penalty suffered for defaulting. This happens in some state of the nature and for all assets. Finally, H5 states that there is at least one agent such that the first-period weighed penalty felt for delivering nothing for each unit borrowed is strictly lower than...
the desire of consumption in the first period. Notice that the payoff $V^j_s$ is measured in nominal terms, so that $r^j_s$, as defined in H5, is its return.

4 Main Results

The first result concerns the existence of equilibrium and the second one concerns non-triviality.

Theorem 1 (Existence). Under H1 and H3 an equilibrium for the economy always exists.

As for the existence argument, we adopt the methodology of truncated economies. We show that the sequence of truncated economies, obtained by cutting the domain of decision variables, has an equilibrium. This result is derived from Debreu’s approach applied to the generalised game associated with each truncated economy. The existence result then follows from asymptotic properties of truncated equilibria.

In order to show Theorem 1, we use the generalised game methodology used by Debreu (1952), which will be detailed in the next subsection.

Theorem 2 (Non-triviality). If the utility functions are differentiable and all assets are nominal, then under H1–H5 the equilibrium is non-trivial.

The non-triviality of the equilibrium follows from the non-differentiable optimisation theory. More precisely, under H1-H4, an equilibrium always exists from Theorem 1. Thus, each agent is maximising. Kuhn–Tucker conditions are satisfied. Hence, Theorem 2 follows by adapting the argument used in Theorem 2 by Maldonado and Orrillo (2006). Details are shown in the next section.

5 Methodology and proof of results

The methodology used to demonstrate the existence of equilibrium first consists in defining a sequence of truncated economies where the truncating concerns the set of macro variables (commodity prices, asset prices and expected delivery rates on assets) and individual choice variables (consumption, asset purchases, asset sales and deliveries on assets). Next, we associate a generalised game to each truncated economy. We prove that such a game has an equilibrium, and for $n$ large enough such an equilibrium is also an equilibrium of the truncated economy. We then show that the sequence of equilibria is uniformly bounded so that it will have a subsequence of equilibria converging to a certain limit. Finally, we prove that the limit obtained corresponds to an equilibrium of our original economy.
5.1 The truncated economy

Let us consider a sequence of truncated economies \{\mathcal{E}^n\}_{n \in \mathbb{Z}_+} defined as follows. The truncated economy will be the same as the whole economy, except that agents will maximise in truncated budget sets. The budget set of agent \(h\) in the truncated economy is defined by

\[
B^h_n(p, \pi, k) = \left\{ (x, \theta, \varphi, D) \in [0, n]^{L(S+1)+J} \times [0, v]^J \times [0, n]^{(S \times J)} : (1), (2), \text{ and (3) are satisfied.} \right\}
\]

where \(v := \sum_h v^h\) is the bound for the short sales of each individual.

5.2 The generalised game

For each positive integer \(n \in \mathbb{Z}_+\) define the following generalised game played by \(H + 1 + (S + 1) + SJ\) players. Let us denote this game by \(G^n\), which is described as follows:

1. Each agent \(h \in H\), maximises \(W(\cdot)\) in the constrained strategy set \(B^h_n(p, \pi, k)\).

2. The government chooses \((\phi^g+, \phi^g-) \in [0, n]^J \times [0, v_1]^J\) with \(v_1 := \max_j v^g_j\) in order to maximise

\[
- \left( \sum_{j=1}^{J} \phi^g_j Q_j^+ + \sum_{j=J+1}^{2J} \phi^g_j Q_j^- - p \Box (G - T) \right)^2.
\]

3. The auctioneer of the first period 0 chooses \((p_0, \pi) \in \triangle_+^{L-1} \times \triangle_+^{J-1}\) in order to maximise

\[
p_0 \left( \sum_{h \in H} (x^h_0 - \omega^h_0) + G_0 \right) + \pi \left( \sum_{h \in H} (\theta^h - \varphi^h) + (\phi^g+ - \phi^g-) \right).
\]

4. The second-period auctioneer of the state \(s\) chooses \(p_s \in \triangle_+^{L-1}\) in order to maximise

\[
p_s \left( \sum_{h \in H} (x^h_s - \omega^h_s) + G_s \right).
\]

5. Each of the remaining fictitious agents of state \(s \in S\) chooses \(k^j_s \in [0, 1]\)
in order to minimise
\[
\left\{ k_s^j \left( p_s A^j_s \phi^g_{j} - \sum_{h \in H} p_s A^j_s \phi^h_{j} \right) - \left( p_s A^j_s \phi^g_{j} + \sum_{h \in H} D^h_{nj} \right) \right\}^2.
\]

Next, we define equilibrium for $G^n$.

**Definition 3.** The array $[(p, \pi, k); (x^n h, \theta^n h, \phi^n h, D^n h)_{h \in H}, (\phi^n g^+, \phi^n g^-)]$ constitutes an equilibrium in pure strategies for $G^n$ if each player maximises on his strategy sets given the actions of their rivals.

Notice that each element of the array depends on $n$, but will be dropped for not having to accrue the notation. The following lemma guarantees the existence of an equilibrium for $G^n$.

**Lemma 1.** Under the hypotheses of Theorem 1, the generalised game $G^n$ has an equilibrium in pure strategies.

**Proof.** It follows from the equilibrium existence theorem in a generalised game of Debreu (1952).

The following lemma guarantees that the equilibrium for $G^n$ is also an equilibrium for $E^n$, as $n$ is large enough.

**Lemma 2.** There exists $n \in \mathbb{Z}_+$ such that
\[
e^n = [(p^n, \pi^n, k^n); (x^n h, \theta^n h, \phi^n h, D^n h)_{h \in H}, (\phi^n g^+, \phi^n g^-)],
\]
which is an equilibrium for $G^n$, is also a competitive equilibrium with a strictly positive price system, for $E^n$, for all $n \geq N$.

**Proof.** Let $e^n$ be an equilibrium in pure strategies for $G^n$. From the definition of equilibrium point, the vector $(x^n h, \theta^n h, \phi^n h, D^n h)$ is budget feasible and maximizes $W(\cdot)$ on $B^n_{nh}$. From the former claim, it follows that (1), (2) and (3), renamed here by (9), (10) and (11), are satisfied. More precisely,
\[
\begin{align*}
p^n_0 (x^n_0 + t^n_0 - \omega^n) + \pi^n (\theta^n h - \phi^n h) & \leq 0, \\
p^n_s (x^n_s + t^n_s - \omega^n) & \leq k^n_s V^n_s \theta^n h - E_s (\phi^n h), \quad s \in S, \\
\phi^n h & \leq v^n, \quad \forall n,
\end{align*}
\]
where $k^n_s V^n_s \theta^n h = \sum_{j \in J} k^n_{sj} p^n_s A^n_s \theta^n_{j}$, and $E_s (\phi^n h) = \sum_{j} D^n s h j$ with
\[
0 \leq D^n s h j \leq p^n_s A^n_s \phi^n j, \quad \forall s, \forall j.
\]
Optimal conditions of the government imply that

\[
p^n_0 G_0 + \pi^n (\phi^n^{g+} - \phi^n^{g-}) = p^n_0 T_0, \\
p^n_s G_s + p^n_s A_s \phi^n^{g-} = k_s p^n_s A_s \phi^n^{g+} + p^n_s T_s, \quad s \in S. \tag{13}
\]

Summing over \( h \) in (9) and then summing it with the first equality of (13), and using the fact that \( T^{nh} = \sum_{h \in H} t^n_{sh} \), one has

\[
p_0 \left( \sum_{h \in H} (x^n_{0h} - \omega^n_0) + G_0 \right) \\
\quad + \pi^n \left( \sum_{h \in H} (\theta^{hn} - \varphi^{nh} + (\phi^{ng+} - \phi^{ng-}) \right) \leq 0. \tag{14}
\]

Optimal conditions of fictitious agents who choose \( k^{nj}_s \) imply that

\[
k^{nj}_s \left( p^n_s A^n j\phi^{ng-} + \sum_{h \in H} p^n_s A^n s \varphi^{nh}_h \right) = \left( p^n_s A^n j\phi^{ng+} + \sum_{h \in H} D^{njhs} \right). \tag{15}
\]

Adding over \( h \) in (10) and adding over \( j \) in (15), and then summing them up with the second equality of (13) we have

\[
p^n_s \left( \sum_{h \in H} (x^n_{sh} - \omega^n_s) + G_s \right) \\
\quad \leq \sum_{j \in J} \sum_{h \in H} k^{nj}_s v^{nj}_s \left( \theta^{nh}_j - \varphi^{nh}_j + (\phi^{ng+} - \phi^{ng-}) \right). \tag{16}
\]

The optimality conditions of the first-period auctioneer’s problems and (14) imply

\[
\sum_{h \in H} (x^n_{0h} - \omega^n_0) + G_0 \leq 0, \tag{17}
\]

\[
\sum_{h \in H} (\theta^{hn} - \varphi^{nh}) + (\phi^{ng+} - \phi^{ng-}) \leq 0, \tag{18}
\]

otherwise, that auctioneer could choose other prices for which the value of the aggregate demand is greater.

Substituting (18) in (16) and then using the optimality conditions of the
second period auctioneer’s problems, we have

\[
\left( \sum_{h \in H} (x_{nh} - \omega_h) + G_s \right) \leq 0, \quad s \in S.
\]  (19)

For a large enough \( n \), one has that \( p^n_s, \forall s \in S^* \) is a strictly positive vector. Otherwise, every agent \( h \) would choose \( x_{nh} = n \), contradicting (17) and (19). In a similar way, one has that \( \pi^n \gg 0 \) for otherwise there would exist \( j \) such that \( \pi_j = 0 \), in which case the agent \( h \) would choose \( \theta_{j}^{nh} = n \), \( \varphi_{j}^{nh} = v_j^h \), \( \phi_{j}^{ng+} = n \) and \( \phi_{j}^{ng-} = v_j^g \). This would however contradict (18) for \( n \) large enough. Thus, when \( p^n_s, \forall s \in S^* \) and \( \pi^n \) are strictly positive we must have (17), (18), and (19), with equality by implyng that all the markets clear in each truncated economy \( E^n \). This ends the proof of Lemma 2.

5.3 Asymptotic Properties of the Sequence of Equilibria

Let \( n \in \mathbb{Z}_+ \) be the integer given by Lemma 2. Let \( \{e^n\}_{n \geq N} \) be the sequence of equilibria where each element of the sequence corresponds to an equilibrium of \( E^n, \forall n \geq N \).

We are going to show that the sequence of equilibria is uniformly bounded, and therefore it will have a subsequence that converges, say \( \bar{e} \). After that we will prove that the limit \( \bar{e} \) corresponds to an equilibrium of our original economy \( E \).

Lemma 3. The sequence of equilibria \( \{e^n\}_{n \geq N} \) is uniformly bounded and therefore there is a subsequence that converges.

Proof. By definition, \( e^n \) is \( \{(p^n, \pi^n, k^n); (x^{nh}, \theta^{nh}, \varphi^{nh}, D^{nh})_{h \in H}, (\phi^{ng+}, \phi^{ng-})\} \). First, we have that \( \{(p^n, \pi^n, k^n)\}_{n} \) is uniformly bounded because it belongs to the compact set \( (\Delta_{L-1}^{S+1} \times \Delta_{L-1}^{J} \times [0,1]^SJ) \). Therefore, it has a subsequence that converges, say \( (p, \pi, k) \).

Second, (17), (18) and (19) imply that the sequence of consumption \( \{x^{nh}\} \) is bounded from above, and since it also belongs to \( \mathbb{R}^{L(S+1)} \) we conclude that such a sequence is bounded. Thus, without loss of generality, we can suppose that it converges, say, to \( x^h \). Third, since neither consumers nor government can borrow without limit, due to the bounds on short sales imposed above, the sequence of borrowing \( \{(\varphi^{nh})_{h \in H}, (\phi^{ng-})\}_{n} \) is bounded, and therefore it has a subsequence that converge, say \( \{(\varphi^h)_{h \in H}, (\phi^{g-})\} \).

Since each individual is maximising, we must have

\[
0 \leq D_{s}^{nhj} \leq p^n_s A_s \varphi_j^{nh}, \quad \forall j, \forall s.
\]
From the fact that \( p^n_s \) belongs to simplex and \( \varphi_j \leq v_j, \forall j \), it follows that
\[
0 \leq D^{nhj}_s \leq v_j ||A_s||_{\infty}, \quad \forall j, \forall s,
\]
where \( ||\cdot||_{\infty} \) is the norm of maximum. The last inequality implies that the sequence of deliveries converges along a subsequence, say, to \( D^h_s \).

From (18) and the existence of a limit on the short sales, one has that
\[
\sum_{h \in H} \theta^{nh}_j + \phi^{ng+}_j \leq \sum_{h \in H} \varphi^{nh}_j - \phi^{ng}_j \leq H v_j + v_1, \quad \forall H.
\]

The earlier inequality and the non-negativity of \( \theta^{nh}_j \) and \( \phi^{ng+}_j \) imply that the sequences of investments,\(^{12}\) \( \{(\theta^{nh}_h)_{h \in H}, (\phi^{ng+}_h)\}_n \) is bounded; and therefore, it has a subsequence that converges, say \( \{(\theta^h_h)_{h \in H}, (\phi^{g+}_h)\} \).

Without losing the generality, we can suppose that
\[
e^n \rightarrow \bar{e} = [(p, \pi, k); (x^h, \theta^h, \varphi^h, D^h)_{h \in H}, (\phi^{g+}, \phi^{g-})].
\]

### 5.4 Proof of Theorem 1

We are going to demonstrate that \( \bar{e} \) found in Lemma 1 constitutes an equilibrium for economy \( \mathcal{E} \). We will do it in three steps:

1. Markets clear: this condition follows after taking limit in (17), (18) and (19), where they all hold with equality.

2. Government balances its budget: this follows, from (13), after taking a limit when \( n \) goes to infinity.

3. The default rate is correctly anticipated: taking a limit, when \( n \) tends to infinity in (15), we obtain what we want.

4. Optimality of agents: let \( z^{nh} = (x^{nh}, \theta^{nh}, \varphi^{nh}, D^{nh}) \) and let \( z^h \) be its limit. Since \( e^n \) is an equilibrium in the truncated economy \( \mathcal{E}^n \), we have that \( z^n \) satisfies (9), (10) and (11). That is, \( z^n \) is budget-feasible. Taking a limit when \( n \) goes to infinity, the limit \( z^h \) satisfies (1), (2) and (3). We claim that \( W^h(z^h) \geq W^h(z'), \forall z' \in B^h(p, \pi, k) \). Suppose that there exists \( z^h \in B^h(p, \pi, k) \) such that \( W^h(\hat{z}) > W^h(z^h) \). Lower hemi-continuity of \( B(p, \pi, k) \) (see A) implies that there exists a sequence \( \{z^n\} \subset B^h_n(p^n, \pi^n, k^n) \) such that \( z^n \rightarrow \hat{z} \). Notice that the arguments of \( B^h_n(\cdot) \) are terms of the

---

\(^{12}\) private and public respectively.
sequence that form part of the sequence of equilibria \(\{e^n\}\) of the truncated economy. That is, \((p^n, \pi^n, k^n) \rightarrow (p, \pi, k)\). From the continuity of \(W\), it follows that there exists \(n_0 \in \mathbb{Z}_+\) such that
\[
W^h(z^n) > W^h(z^{nh}), \quad \forall n \geq n_0,
\]
contradicting the optimality of \(z^{nh}\) in the truncated economy \(E^n\). Hence, the sequence \(z^h\) is optimal in the limit economy \(E\). This fact implies that non-arbitrage necessary conditions are satisfied in the commodity markets, due to the strictly monotonicity of utility functions at the cluster point, by implying \(p = \lim_{n \to \infty} p^n \gg 0\).

### 5.5 Proof of Theorem 2

To show Theorem 2, we need the following lemmas:

**Lemma 4.** Theorem 1 implies the existence of Lagrange’s multipliers \((\alpha^h_0, \alpha^h) \in \mathbb{R}^{S+1}_+\) such that

\[
U^h(x) - U^h(x^h) \leq \sum_{s,j} \lambda^h_{sj} \left\{ p_s A_s^j(\varphi_j - \varphi^h_j) - (D^j_s - D^h_{sj}) \right\} \\
+ \alpha^h_0 \left\{ p_0(x_0 - x^h_0) + \pi(\theta - \theta^h) - \pi(\varphi - \varphi^h) \right\} \\
+ \sum_{s \in S} \left( p_s(x_s - x^h_s) + \sum_{j \in J} (D^j_s - D^h_{sj}) \right) \\
- \sum_{j \in J} k^h_j p_s A^j_s(\theta_j - \theta^h_j),
\]

\[\forall (x, \theta) \geq 0, \forall 0 \leq \varphi \leq v^h, \forall 0 \leq D^j_s \leq p_s A^j_s \varphi^j, \forall j, \forall s.\]  

**Proof.** H1–H3 imply the existence of an equilibrium, which in turn implies that for each \(h, (x^h, \theta^h, \varphi^h, D^h)\) maximises \(W^h(x, \theta^h, \varphi, D)\) subject to the budget constraints. Thus, KKT’s theorem implies Lemma 4.

**Lemma 5.** Under the hypotheses of Lemma 4, the following inequalities hold:

1. \(\forall h, j, s,\)

\[
0 \leq \left( \alpha^h_s - \lambda^h_{sj} \right) \left( D^j_s - D^h_{sj} \right), \quad 0 \leq D^j_s \leq p_s A^j_s \varphi^h_j.
\]

2. \(\forall h, j,\)

\[
\alpha^h_0 \pi_j (\varphi_j - \varphi^h_j) \leq \sum_s \lambda^h_{sj} p_s A^j_s (\varphi_j - \varphi^h_j).
\]
Proof. Item 1 immediately follows after substituting $x = x^h$, $\varphi = \varphi^h$, $\theta = \theta^h$, and $D_s = D_s^h \in \mathbb{R}^d_+$, except the coordinate $j$ of the previous variables. Similarly, Item 2 follows after substituting all the variables for the optimal ones, except the coordinate $j$ of the variable $\varphi$.

Lemma 6. The following inequalities are held:

1. If $\varphi^h > 0$, then either $\alpha^h_s > \lambda^h_{sj}$ implies that $0 \leq D^h_{sj} < p_s A^i_s \varphi^h_j$, or $\lambda^h_{sj} > \alpha^h_s$ implies that $D^h_{sj} > 0$.

2. If $\pi_j \alpha^h_0 > \sum_{s \in S} \lambda^h_{sj} p_s A^i_s$ implies that $\varphi^h_j > 0$.

Proof. We show Lemma 6 in a separate way.

1. Let $D^h_{sj} = p_s A^i_s \varphi^h_j$ with $\varphi^h_j > 0$. From the first inequality of Item 1 of Lemma 5 it follows that $\alpha^h_s \leq \lambda^h_{sj}$. Consequently, $\alpha^h_s > \lambda^h_{sj}$ implies that $0 \leq D^h_{sj} < p_s A^i_s \varphi^h_j$. On the other hand, if $D^h_{sj} = 0$ with $\varphi^h_j > 0$, then the first inequality of Item 1 of Lemma 5 now implies that $\lambda^h_{sj} \leq \alpha^h_s$. Suppose the contrary. That is, $\lambda^h_{sj} > \alpha^h_s$. Using the previous inequality and putting $D^i_s = \left(p_s A^i_s \varphi^h_j\right)/2$, we have that the second factor of the first inequality of Lemma 5 is strictly positive, leading to a contradiction. Thus, $\lambda^h_{sj} > \alpha^h_s$ implies that $D^h_{sj} > 0$.

2. Suppose that $\varphi^h_j = 0$. Given that $\varphi^h_j \geq 0$, so from inequality of Item 2 of Lemma 5 one has that $\pi_j \alpha^h_0 \leq \sum_{s \in S} \lambda^h_{sj} p_s A^i_s$. However this is equivalent to stating that $\pi_j \alpha^h_0 > \sum_{s \in S} \lambda^h_{sj} p_s A^i_s$ implies $\varphi^h_j > 0$, as $\varphi^h_j \geq 0$.

In order to finish Theorem 2, first notice that the assets are nominal, so that Lemmas 4, 5 and 6 will be used in their nominal versions. Second, H4 and H5 together with the concavity of the utility function imply that

$$\alpha^h_s > \lambda^h_{sj} \quad \text{and} \quad \pi_j \alpha^h_0 > \sum_{s \in S} \lambda^h_{sj} V^j_s.$$ 

Now using Lemma 6, Theorem 2 follows.

6 Concluding remarks

This paper has shown that the inclusion of a government in an economy with incomplete financial markets where private agents are allowed to default does not destroy the existence of equilibria. We have also demonstrated that equilibria
are non-trivial. Actually, what we provided were the sufficient conditions for trading in the financial markets (some asset is being traded) and the possibility for some private agent to default on that asset.

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Appendix A

Let $IntB(p, \pi, k)$ be the “interior” of $B(p, \pi, k)$, which is defined as follows: $(x, \theta, \varphi, D) \in IntB(p, \pi, k)$ if and only if (1), (2) and (3) hold with strict inequality. The following lemma states that $IntB(p, \pi, k)$ is lower hemi-continuous (lhc) at any $(p, \pi, k) \gg 0$ provided $\omega^h - (t_0, t_{1SL}) \gg 0$, which is true because of H1.

Lemma 7. The budget correspondence of each individual $h$ is lhc at any $(p, \pi, k) \gg 0$ provided that $\omega^h - (t_0, t_{1SL}) \gg 0$.

Proof. First, $IntB(p, \pi, k)$ is a non empty set, since $x = 0, \theta = 0, \varphi = 0$ and $D_s = 0, \forall s$, satisfy (1), (2), and (3), with strict inequality.

Let $\{(p^n, \pi^n, k^n)\}$ be a sequence converging to $(p, \pi, k) \gg 0$ and be $(x, \theta, \varphi, D) \in IntB^h(p, \pi, k)$. Thus, for every sequence $\{(x^n, \theta^n, \varphi^n, D^n)\}$ converging to $(x, \theta, \varphi, D)$ belonging to $IntB^h(p, \pi, k)$ one has that for $n$ large enough $(x^n, \theta^n, \varphi^n, D^n) \in IntB^h(p^n, \pi^n, k^n)$, which implies that $IntB^h(\cdot)$ is lhc.

Remark 4. From Hildenbrand (1974, p.26, Fact 4), follows that $B^h(\cdot)$, which is the closure of $IntB^h(\cdot)$, is also lhc.