STOCHASTIC EVOLUTION EQUATIONS WITH SINGULAR DRIFT AND
GRADIENT NOISE VIA CURVATURE AND COMMUTATION
CONDITIONS

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Abstract. We prove existence and uniqueness of solutions to a nonlinear stochastic evolution equation on the d-dimensional torus with singular p-Laplace-type or total variation flow-type drift with general sublinear doubling nonlinearities and Gaussian gradient Stratonovich noise with $C^1$-vector field coefficients. Assuming a weak defective commutator bound and a curvature-dimension condition, the well-posedness result is obtained in a stochastic variational inequality setup by using resolvent and Dirichlet form methods and an approximative Itô-formula.

1. Introduction

We shall study the nonlinear stochastic partial differential equation (nonlinear SPDE) with singular drift and gradient-type multiplicative Stratonovich noise

$$dX_t = \text{div}(a^*\phi(a\nabla X_t))\,dt + \langle b\nabla X_t, \circ\,dW_t \rangle, \quad t \in [0,T],$$

$$X_0 = x.$$ (1.1)

In order to eliminate boundary or curvature effects from the underlying space, we shall consider the SPDE on $T^d = \mathbb{R}^d/\mathbb{Z}^d$. Here, $a, b : T^d \to \mathbb{R}^{d \times d}$ are $C^1$-coefficient fields (where $a^*$ denotes the adjoint of $a$) and $\{W_t\}_{t \geq 0}$ is an $\mathbb{R}^d$-valued Wiener process. In this work, we shall prove well-posedness for initial data $x \in L^2(T^d)$ assuming some linear growth conditions on $\phi : \mathbb{R}^d \to \mathbb{R}$ (we also consider the multi-valued case $\phi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$) and a geometric curvature-dimension condition on the Riemannian metric $g_a := (a^*a)^{-1}$ and a (defective) commutator estimate for the Riemannian metric $g_b := (b^*b)^{-1}$.

The drift operator $u \mapsto \text{div}(a^*\phi(a\nabla u))$ is a distorted $p$-Laplace-type operator, when $\phi(\zeta) = |\zeta|^{p-2}\zeta$, $p \in [1,2]$, which reduces to the linear diffusion operator $u \mapsto \text{div}(a^*a\nabla u)$ for $p = 2$ and includes multi-valued examples as the total variation flow operator $u \mapsto \text{div}(\text{sgn}(\nabla u))$ for
\( p = 1, a = 1 \), which is also called 1-Laplace. The equation is perturbed by independent Brownian motions \( \{W_i^t\}_{t \geq 0}, 1 \leq i \leq d \), driven by a non-degenerate family of \( C^1 \)-vector fields \( b_i, 1 \leq i \leq d \), which act in normal direction.

Equations of the type (1.1), have previously been studied in [6] (for the case \( p > 1 \)), in [16] (for the case of a domain with symmetries and Neumann boundary conditions), in [38] (for Dirichlet boundary conditions), and in [7] (by an approach with weak solutions in the sense of distributions). As the coefficient field of the noise term is given by an unbounded operator in space, the Itô-analogue of (1.1) is ill-posed in general. The equation is discussed in [11, Section 7] as an example for an approach via a transformation by a group of random multipliers. Note that our approach does not rely on a transformation of (1.1) to a random PDE — rather than that, we obtain the unique solutions in terms of (stochastic) variational inequalities by a multi-step approximation procedure and a perturbation argument which relies on a weak defective commutator bound formulated in terms of Dirichlet forms, see [21, 35] for this notion. The conditions are discussed in Section 2 below. In particular, we need a curvature bound related to heat kernel estimates and lower bounds for Ricci curvature in order to derive a priori estimates for the singular equation. The higher order a priori estimates, which we need for the approximation procedure in the proof of the main result, are then derived from the defective commutator bounds. In [49], some previous results and the rough idea, which this work is based on, have been proposed. In the future, equations involving more general Dirichlet operators, e.g. of nonlocal type, could be considered.

As the drift term in the SPDE (1.1) is singular, it is a known issue that solutions to the SPDE do not satisfy an Itô-equation in general (on these lines, see e.g. [24]), when, for instance, there is no Sobolev embedding for the energy of the drift available (as is e.g. in [33] for \( p > 1 \sqrt{\frac{d+2}{d}} \)). See [37] for a new approach under rather minimal assumptions. In our situation, the Itô-Stratonovich correction and the special linear structure of the noise adds some regularity to the equation, however, even for initial data in \( H^1(\mathbb{T}^d) \) and for \( p \approx 1 \), to the best of our knowledge, the (limit) solutions to (1.1) can merely be characterized in terms of stochastic variational inequalities (SVI), see e.g. [10]. Among others, SVI-solutions to stochastic evolution equations have also been discussed in [12, 22, 23, 25, 12]. As seen in [26], the SVI-approach is quite robust under perturbations of the convex-subpotential-type drift with respect to the Mosco-topology. In this work, we prove existence and uniqueness for initial data in \( L^2(\mathbb{T}^d) \), see our main result Theorem 4.1 in Section 4 below.

Let us point out that we generalize the previously known results on solutions to (1.1) in the following aspects. As we assume periodic boundary conditions, we are able to dispense with the requirement of the boundary to be assumed to be \( C^0 \) and coefficient fields to be assumed to be divergence-free, perpendicular to the boundary and \( C^2 \). We do merely require that the driving vector fields \( b_i, 1 \leq i \leq d \) are \( C^1 \) and pointwise linearly independent — we do not need to assume commutation here, as is done e.g. in the classical linear SPDE case in [17,18]. The introduction of the deformed diffusivity is new and may possibly be applied to more general compact Riemannian manifolds than the torus in the future. Apart from the geometric deformation, we also include more general nonlinearities (in the spirit of [24, Section 7]), extending the results of [16,26] from homogeneous nonlinearities to ones satisfying a doubling condition, see Example 2.1 below. In fact, the nonlinearities might become multi-valued so that equation (1.1) and its Itô-Stratonovich corrected form (2.1) below are actually given by the more general subpotential equation (2.2) below, which involves the relaxed convex potentials. In the sequel, all of our analysis is generally referring to equation (2.2) below rather than to the formal equation (1.1).

Possible future topics for equations of this type include ergodicity and uniqueness of invariant measures for Markov semigroups (compare also [31,20,25,31] for additive noise), stability under perturbations of the drift or noise (cf. [15,26]) and regularity (see e.g. [13,14]).
Notation. Denote by $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ the standard flat torus of dimension $d \geq 1$, equipped with the $d$-dimensional Lebesgue measure $d\xi$ on its Borel $\sigma$-algebra $\mathcal{B}(\mathbb{T}^d)$. On $\mathbb{R}^d$, the Euclidean norm and inner product are denoted by $| \cdot |$ and $\langle \cdot, \cdot \rangle$ respectively. For $\alpha \in \mathbb{R}$, we denote the linear operator $u \mapsto \alpha u$ (on some vector space) simply by $\alpha$. Denote $H := L^2(\mathbb{T}^d)$, $S := H^1(\mathbb{T}^d)$. Let $S^*$ denote the topological dual of $S$. Note that the embedding $S \hookrightarrow H$ is dense and compact. Let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form of the Laplace-Beltrami operator $L := \Delta := \text{div} (\nabla \cdot)$ on $H$, that is, $D(\mathcal{E}) := S$ and

$$\mathcal{E}(u, v) := \int_{\mathbb{T}^d} (\nabla u, \nabla v) \, d\xi, \quad u, v \in D(\mathcal{E}),$$

see [21,35]. Let $(G_\alpha)_{\alpha > 0}$ be the resolvent of $L$, i.e., for $\alpha > 0$, $G_\alpha u := (\alpha - L)^{-1}u$, $u \in H$. For convenience, we shall also introduce the alternative resolvent $J_\delta u := (1 - \delta L)^{-1}u$, where $u \in H$ and $\delta > 0$. Clearly, $J_\delta = \frac{1}{\delta}G_{1/\delta}$ for every $\delta > 0$. The resolvent $J_\delta$, when considered both as a map from $H$ to $H$ or as a map from $S$ to $S$, is a contraction. Denote the negative of the Yosida-approximation of $L$ by $L^{(\delta)} u := LJ_\delta u = \frac{1}{\delta}(J_\delta - 1)u$, $u \in H$. $L^{(\delta)} : H \rightarrow H$ is a negative definite bounded operator with operator norm equal to $\frac{1}{\delta}$. Furthermore, for $\alpha > 0$, $u, v \in D(\mathcal{E})$, let $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \langle \alpha (u, v) \rangle_H$ be inner products for $S$ with the property that $\mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}$ are mutually equivalent for $\alpha_1, \alpha_2 > 0$. For $\beta > 0$, define also approximate forms $\mathcal{E}^{(2)}(u, v) := \beta (u - \beta G_\delta u, v)_H$, $u, v \in H$, see e.g. [35] Chapter I, p. 20. Set also $\mathcal{E}_\alpha^{(2)}(u, v) := \mathcal{E}^{(2)}(u, v) + \langle \alpha (u, v) \rangle_H$, $\alpha, \beta > 0$, $u, v \in H$. Denote by $(P_t)_{t \geq 0}$ the heat semigroup associated to $L$.

Organization of the paper. In Section 2 we shall discuss the different parts of and ingredients for equation (1.1) in several subsections, where also the main hypotheses of this work are stated and discussed. In Section 3 we shall introduce and discuss the concept of so-called stochastic variational inequality (SVI) solutions to our equation. Section 4 which is subdivided into several subsections, contains the statement and the proof of our main result, including the a priori estimates, the passage to the limit of the approximating equations, the existence and the uniqueness result. In Appendix A we recall the conditions from [24] which are needed to guarantee the existence of approximating solutions to equation (1.1).

2. Ingredients for the equation

Let $\{W_t\}_{t \geq 0}$ be a Wiener process on $\mathbb{R}^d$, modeled on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ that satisfies the usual conditions.

Consider the following Itô-SPDE in $H$,

\begin{equation}
\begin{aligned}
&dx_t = \text{div}(a^* \phi(a \nabla X_t)) \, dt + \frac{1}{2} L^b X_t \, dt + \langle b \nabla X_t, dW_t \rangle, \\
&X_0 = x.
\end{aligned}
\end{equation}

(2.1)

Equation (2.1) is the formal analogue to equation (11.1), after passing over to an (merely formal) Itô-Stratonovich correction, see e.g. [29,30] and compare with the previous works [10,11,14,39]. In the sequel, we shall rather investigate equation (2.1). In this section, we will introduce and discuss the following “ingredients” for the equation, subdivided into the following subsections.

In [2.1]: The nonlinearity $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ (the single-valued case) or $\phi : \mathbb{R}^d \rightarrow 2^{\mathbb{R}^d}$ (the multi-valued case), and the assumptions on $\phi$.

In [2.2]: the coefficient fields $a : T^d \rightarrow \mathbb{R}^{d \times d}$ and $b : T^d \rightarrow \mathbb{R}^{d \times d}$, and the basic assumptions on them as well as the operator $L^b$.

In [2.3]: the curvature dimension conditions for $a$ and weak defective commutation conditions for $a$ and $b$.

In [2.4]: the resulting commutator estimates for $a$ and $b$. 

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2.1. The nonlinearity $\phi$. To be even more precise, instead of (2.1), we shall study the following Itô-SPDE (stochastic evolution equation / inclusion):

$$
\begin{align*}
\frac{dX_t}{dt} & \in -\partial\Psi(X_t)dt + \frac{1}{2}L^bX_t dt + \langle b \nabla X_t, dW_t \rangle, \quad t \in (0,T], \\
X_0 & = x.
\end{align*}
$$

where $\partial\Psi$ denotes the subdifferential of $\Psi$, which, in turn, is defined to be the lower semi-continuous (l.s.c.) envelope of the map

$$
\Psi(u) := \inf \left\{ \liminf_{n \to \infty} \tilde{\Psi}(u_n) \mid u_n \to u \in L^2(\mathbb{T}^d) \text{ strongly} \right\},
$$

see e.g. [3] for further details.

Let us assume that there exist constants $C,K > 0$ and a map $\theta : [0,\infty) \to [0,\infty)$ such that

$$
\begin{align*}
\text{(N)} \quad \psi(\zeta) & = \theta(|\zeta|), \quad \text{for all } \zeta \in \mathbb{R}^d \text{ and } \theta \text{ is convex, continuous, and satisfies } \theta(0) = 0, \\
& \quad \lim_{r \to \infty} \theta(r) = \infty, \quad \theta(2r) \leq K\theta(r) \quad \text{for } r \geq 0.
\end{align*}
$$

Note that the doubling condition $\theta(2r) \leq K\theta(r)$, $r \geq 0$ is also known as $\Delta_2$-condition in the literature, see e.g. [11]. Our assumptions on $\theta$ shall be made precise further below.

**Example 2.1.** Condition (N) is e.g. satisfied for the following choices of $\psi$. For $\zeta \in \mathbb{R}^d$, let

- **$p$-Laplace, total variation flow**: $\psi(\zeta) := \frac{1}{p}||\zeta||^p$, $p \in [1,2]$,
- **Logarithmic diffusion**: $\psi(\zeta) := (1 + |\zeta|)\log(1 + |\zeta|) - |\zeta|$, 
- **Minimal surface flow**: $\psi(\zeta) := \sqrt{1 + |\zeta|^2}$,
- **Curve shortening flow**: $\psi(\zeta) := |\zeta| \arctan(|\zeta|) - \frac{1}{2} \log((|\zeta|^2 + 1).$

**Remark 2.2.** Note that e.g. for $\psi(\zeta) = |\zeta|$, the subdifferential $\phi = \partial\psi$ becomes multi-valued, and the l.s.c. envelope $\Psi$ is a Radon measure on the space of functions of bounded variation, see [3] for details. At this point it is enough to recall that the above l.s.c. envelope exists in $L^2(\mathbb{T}^d)$.

We continue with two technical lemmas needed later.

**Lemma 2.3.** Assume that (N) holds. Then

$$
\langle \eta, \zeta \rangle \leq K\psi(\zeta) \quad \forall \eta \in \phi(\zeta) \forall \zeta \in \mathbb{R}^d.
$$

**Proof.** By (N) and [13] Chapter II, Example 8.A], there exists a function $\theta'_+ : [0,\infty) \to [0,\infty)$ such that $\theta'_+$ is nondecreasing, right continuous and satisfies

$$
\theta(s) = \int_0^s \theta'_+(r) \, dr, \quad s \geq 0.
$$

Also, for $r \geq 0$, $\theta'_+(r) \geq \sup_{s \in [\theta(r)]} |s|$. Using (N) again, for $s \geq 0$,

$$
K\theta(s) \geq \theta(2s) = \int_0^{2s} \theta'_+(r) \, dr \geq \int_s^{2s} \theta'_+(r) \, dr \geq s\theta'_+(s).
$$

However, it is easy to see that from $\psi = \theta(|\cdot|)$, it follows that $\langle \eta, \zeta \rangle \leq |\zeta|\theta'_+(|\zeta|)$, for all $\zeta \in \mathbb{R}^d$ and $\eta \in \phi(\zeta)$, cf. [13] Chapter II, Proposition 8.6]. Hence the claim follows. □

1 The inclusion sign is to be understood as an equality whenever the r.h.s. is single-valued.

2 The subdifferential $\partial F : H \to 2^H$ of a convex, lower semi-continuous, proper map $F : H \to [0,\infty]$ is defined by $y \in \partial F(x)$, $x, y \in H$, whenever $(y, z - x, H) \leq F(z) - F(x)$ for every $z \in H$, see e.g. [10].
For the statement of the following lemma and for use further below, we shall introduce the notion of the so-called Moreau-Yosida approximation \( \psi^\lambda \) of \( \psi \), that is, the family of continuous convex functions defined by the following variational formula

\[
\psi^\lambda(\zeta) := \inf_{\eta \in \mathbb{R}^d} \left[ \psi(\eta) + \frac{1}{2\lambda} |\zeta - \eta|^2 \right], \quad \zeta \in \mathbb{R}^d, \; \lambda > 0,
\]

see [2, p. 266] or [9, p. 97] for further details.

\textbf{Lemma 2.4.} Assume that \( (N) \) holds and let \( \psi^\lambda, \lambda > 0 \) be the Moreau-Yosida approximation of \( \psi \). Then, there exists a constant \( C > 0 \) (not depending on \( \lambda \)) such that

\[
|\psi(\zeta) - \psi^\lambda(\zeta)| \leq C\lambda(1 + \psi(\zeta)) \quad \forall \zeta \in \mathbb{R}^d.
\]

\textbf{Proof.} As above, denote \( \phi = \partial \psi \). By the arguments preceding [26, Eq. (A.4) in Appendix A], we get in this slightly more general situation that

\[
|\psi(\zeta) - \psi^\lambda(\zeta)| \leq \lambda \sup_{\eta \in \phi(\zeta)} |\eta|^2 \quad \forall \zeta \in \mathbb{R}^d.
\]

By [24, Proof of Proposition 7.1] and \( (N) \), there exists a constant \( C > 0 \), such that

\[
|\eta|^2 \leq C(1 + \langle \eta, \zeta \rangle) \quad \forall \eta \in \phi(\zeta) \quad \zeta \in \mathbb{R}^d.
\]

Again, by \( (N) \) and by \textbf{Lemma 2.3} there exists another constant \( C > 0 \) such that

\[
(\eta, \zeta) \leq C|\psi(\zeta)| \quad \forall \eta \in \phi(\zeta) \quad \forall \zeta \in \mathbb{R}^d.
\]

Combining these inequalities finishes the proof. \( \square \)

2.2. The coefficient fields \( a \) and \( b \). Let \( a : T^d \to \mathbb{R}^{d \times d} \) such that each row \( a_i, 1 \leq i \leq d \), of \( a \) satisfies

\[
(2.4) \quad a_i \in C^1(T^d; \mathbb{R}^d) \quad \text{for every } 1 \leq i \leq d.
\]

\( L^a \) denotes the Dirichlet operator associated to the Dirichlet form

\[
\mathcal{A}(u, v) := \int_{T^d} \langle a\nabla u, a\nabla v \rangle \, d\xi, \quad u, v \in H^1(T^d),
\]

where \( az \) denotes the application of matrix-multiplication for \( z \in \mathbb{R}^d \). For smooth functions \( u \in D(L^a) \cap C^\infty(T^d) \), by assumption \( (2.4) \), we have that \( L^a u = \text{div}(a^*a\nabla u) \), where \( a^* \) denotes the matrix-adjoint of \( a \).

Let us further assume that there exists a constant \( \kappa = \kappa_0 > 0 \) such that

\[
(\text{E}) \quad \text{Assume that (2.4) holds for } a, \text{ and assume that } |a(\xi)|^2 \geq \kappa|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \text{ and all } \xi \in T^d.
\]

\textbf{Remark 2.5.} Note that \( (\text{E}) \) implies that \( D(\mathcal{A}) = D(\mathcal{E}) = S \) and that \( -L^a \) is uniformly elliptic. On these lines, see also [35].

\textbf{Lemma 2.6.} Assume that \( (2.4) \) holds. Then condition \( (\text{E}) \) is equivalent to \( a_1(\xi), \ldots, a_d(\xi) \) being linearly independent in \( \mathbb{R}^d \) for each \( \xi \in T^d \).

\textbf{Proof.} First note that \( a^*a \) is precisely the Gram matrix of the vectors \( a_1, \ldots, a_d \). Consider the statement of \( a_1, \ldots, a_d \) being pointwise linearly independent. It is well-known that this is equivalent to the Gram matrix being positive definite, see e.g. [31, Chapter 10], which in turn is equivalent to \( (\text{E}) \) with \( \kappa := \min_{\xi \in T^d} \kappa(\xi), \kappa(\xi) \) being the smallest eigenvalue of \( a^*(\xi)a(\xi) \) (all eigenvalues are strictly positive and real). By continuous dependence of the eigenvalues of \( a^*a \) on \( \xi \in T^d \), and since \( T^d \) is compact, we argue by contradiction to see that \( \kappa > 0 \). \( \square \)
Denote the associated Dirichlet operators by $L^a$, $L^b$, respectively with Dirichlet forms $\mathcal{A}$, $\mathcal{B}$ respectively, where

$$\mathcal{B}(u, v) := \int_{\mathbb{T}^d} (b \nabla u, b \nabla v) \, d\xi, \quad u, v \in H^1(\mathbb{T}^d).$$

Assume the following for the coefficient fields of equation (2.1):

(M) Assume that $a : \mathbb{T}^d \to \mathbb{R}^{d \times d}$ and $b : \mathbb{T}^d \to \mathbb{R}^{d \times d}$ in equation (2.1) both satisfy (E).

We also shall assume here that the resolvent $J^a_\delta$, $\delta > 0$ leaves the domain $D(L^b)$ of $L^b$ invariant and that $L^b J^a_\delta z \to L^b z$ weakly in $H$ as $\delta \to 0$ for any $z \in D(L^b)$.

Remark 2.7. Note that the second part of (M) certainly holds when $a = b$. By Lemma 2.16 below, the second part of (M) also follows from the defective commutation property (2.8) below, which in turn implies the weak commutator condition (R) below. As for the main result, conditions (M) and (R) are assumed, condition (2.8) (and the first part of (M)) may be assumed instead.

2.3. Curvature-dimension (CD) conditions. Set $M := \mathbb{T}^d$. Assume (M) and set $g_a := (a^* a)^{-1}$, $g_b := (b^* b)^{-1}$. Note that $(M, g_a)$ and $(M, g_b)$ are Riemannian manifolds which are quasi-isometric\[1\] to $M$, when equipped with the flat metric. We write $M^a$ or $M^b$ if we want to emphasize the choice of the metric. We denote the volume measure on $M^a$ by $dv := \sqrt{\text{det} g_a} \, d\xi$.

Note that $(M^a, g_a)$, equipped with the Lebesgue measure, is thus a weighted manifold $(M^a, g_a, d\xi)$ with density $p_a := \sqrt{\text{det}(a^* a)}$ with respect to $v$. Note that by \[28\] Exercise 3.12, $L^a = \Delta^a$ on the weighted manifold $(M^a, g_a, d\xi)$, where $\Delta^a$ denotes the weighted Laplace-Beltrami operator of $M^a$. All of this paragraph applies analogously to $M^b$ and the weighted manifold $(M^b, g_b, d\xi)$.

Definition 2.8. Let $\Lambda^a := \{ f \in S : L^a f \in S \}$. We say that $(M^a, g_a, d\xi)$ satisfies a Bakry-Émery curvature-dimension condition $BE(K, \infty)$ if there exists $K \in \mathbb{R}$ with

$$L^a |a \nabla f|^2 - 2(a \nabla f, a \nabla L^a f) \geq \frac{K}{2} |a \nabla f|^2, \quad \forall f \in \Lambda^a.$$ (BE)

Theorem 2.9. Let $(P^a_t)_{t \geq 0}$ be the heat semigroup associated to $\mathcal{A}$. Then the following condition is equivalent to condition $BE(K, \infty)$.

There exists a constant $K \in \mathbb{R}$, such that

$$|a \nabla P^a_t f| \leq e^{-2Kt} P^a_0 |a \nabla f| \quad \forall t \geq 0 \forall f \in C^1(M).$$ (2.5)

Proof. See e.g. [15].

See [15, 27, 30] for the terminology and further results on equivalent curvature-dimension conditions as well as Ricci curvature bounds in weighted Riemannian manifolds.

Lemma 2.10. Let condition (N) hold. Let $(P^b_t)_{t \geq 0}$ be the heat semigroup associated to $\mathcal{B}$. Assume that $BE(K, \infty)$ holds for some $K \leq 0$. Let $J^b_\delta := (1 - \delta (L^a + 2K))^{-1}$, $\delta > 0$ be the resolvent associated to $\mathcal{A}_{-2K}^b := \mathcal{A} - 2K(\cdot, \cdot)_H$. Then

$$\tilde{\Psi}(J^b_\delta u) \leq \tilde{\Psi}(u)$$

for any $u \in S$ and any $\delta > 0$, where $\tilde{\Psi}$ is as in (2.3).

Proof. Compare with [24] Proof of Example 7.11. Let $(P^b_t)_{t \geq 0}$ be the $C_0$-semigroup associated to $J^b_\delta$, $\delta > 0$. Then by the Trotter product formula [33] Ch. VIII.8], $P^0_1 = e^{2Kt} P^0_1$, $t \geq 0$. We get for $u \in C^1(\mathbb{T}^d)$ and $\delta > 0$ that

$$|a \nabla J^b_\delta u| \leq |a \nabla \int_0^\infty e^{-t} P^0_t u \, dt| = |a \nabla \int_0^\infty e^{-t+2K\delta t} P^a_{\delta t} u \, dt| \leq \int_0^\infty e^{-t+2K\delta t} |a \nabla P^a_{\delta t} u| \, dt \leq \int_0^\infty e^{-t} P^a_{\delta t} |a \nabla u| \, dt = J^b_\delta |a \nabla u|.$$
where we have used Theorem [2.14.3.]. Since $J_\beta^a$, $\delta > 0$ is Markovian symmetric on $L^2(\mathbb{T}^d)$, we get by an application of [16. Theorem 3] and condition (N) that for $u \in L^2(\mathbb{T}^d)$,
\[
\int_{\mathbb{T}^d} \theta(J_\beta^a u) \, d\xi \leq \int_{\mathbb{T}^d} \theta(u) \, d\xi.
\]
Altogether,
\[
\tilde{\Psi}(J_\beta^0 u) = \int_{\mathbb{T}^d} \theta(|a \nabla J_\beta^0 u|) \, d\xi \\
\quad \leq \int_{\mathbb{T}^d} \theta(J_\beta^a |a \nabla u|) \, d\xi \leq \int_{\mathbb{T}^d} \theta(|a \nabla u|) \, d\xi = \tilde{\Psi}(u),
\]
density of $C^1(\mathbb{T}^d) \subset S$ and Lebesgue’s dominated convergence theorem completes the proof. \(\square\)

**Remark 2.11.** The above statement remains true if $BE(K, \infty)$ holds for $K > 0$. If one investigates the proof carefully, one sees that then it even holds that
\[
\tilde{\Psi}(J_\beta^a u) \leq \tilde{\Psi}(u)
\]
for any $u \in S$ and any $\delta > 0$.

**Definition 2.12** (Weak commutator condition). We say that $a$ and $b$ as above satisfy the commutator condition $R(a, b)$ whenever

\(\text{(R)}\) there exists a constant $c \in \mathbb{R}$ such that for every $\beta > 0$, we have that
\[
\beta \int_{\mathbb{T}^d} \langle \beta G_\beta^a \nabla f - \beta b \nabla G_\beta^a f, b \nabla f \rangle \, d\xi \geq c A(f, f), \quad \forall f \in S.
\]
As seen later in the proof of Theorem [14.3.1] below (where condition (R) is needed), we could allow for replacing the term $cA(f, f)$ on the l.h.s. of (R) by $cA_1(f, f)$.

**Example 2.13.** Suppose that $a = 1$ and let $b$ satisfy (E). By [16. Theorem 3.1, Proposition 3.2], (R) holds with $c = 0$, if $L^\alpha = \Delta$ and the first order operator $S \ni u \mapsto b \nabla u \in L^2(\mathbb{T}^d; \mathbb{R}^d)$ commute componentwise on some core of $L^\alpha$. This is related to the notion of so-called Killing vector fields, namely $b_i \in C^2$ and the components satisfy
\[
\partial_k b_i^l + \partial_i b_k^l = 0 \quad \forall 1 \leq i, k, l \leq d,
\]
see [16] for examples.

**Remark 2.14.** If one lets $\beta \to \infty$ in (R) for $f \in C^\infty(\mathbb{T}^d)$, one obtains,
\[
\int_{\mathbb{T}^d} [\langle L^\alpha b \nabla f, b \nabla f \rangle - \langle b \nabla f, b \nabla L^\alpha f \rangle] \, d\xi \geq c A(f, f), \quad \forall f \in C^\infty(\mathbb{T}^d),
\]
which resembles an integrated version of the Bakry-Émery curvature-dimension condition, see (BE) above, see [16.2] for similar kinds of conditions. This is further illustrated in the Proposition 2.17 below, which deepens the observation that is discussed in Example 2.13.

**Definition 2.15.** Let $R : H^1(\mathbb{T}^d) \to L^2(\mathbb{T}^d; \mathbb{R}^d)$ be a linear operator such that there exists $C > 0$ with
\[
\|Ru\|_{L^2(\mathbb{T}^d; \mathbb{R}^d)}^2 \leq C A(u, u) \quad \forall u \in S.
\]
We say that the *weak defective commutation property* holds for the operators $u \mapsto b \nabla u$ and $L^\alpha$ if there exists an operator $R$ with the above properties such that for every $\beta > 0$,
\[
b \nabla G_\beta^a u = G_\beta^a b \nabla u + G_\beta^a RG_\beta^a u \quad \forall u \in S.
\]
Note that under Assumption (E), (2.7) implies that $R$ is bounded as an operator from $S$ to $L^2(\mathbb{T}^d; \mathbb{R}^d)$. See Theorem 2.1 [47] for equivalent conditions to (2.8), as there is e.g. the statement that $b\nabla L^a = L^a b\nabla + R$ holds on some core of $L^a$.

The statement of the following lemma has already been discussed in Remark 2.7 above.

**Lemma 2.16.** Suppose that $a$ and $b$ both satisfy (E). Then the weak defective commutation property implies (M) (more precisely, the second part of the statement of (M)).

**Proof.** For $u \in D(L^b)$, $v \in S$, $\delta > 0$, consider

$$
\left| (L^b J^a_\delta u - L^b u, v)_H \right| 
\leq \int_{\mathbb{T}^d} \langle b\nabla (u - J^a_\delta u), b\nabla v \rangle \, d\xi 
\leq \int_{\mathbb{T}^d} \langle b\nabla u - J^a_\delta b\nabla u, b\nabla v \rangle \, d\xi + \delta \int_{\mathbb{T}^d} \langle J^a_\delta R J^a_\delta u, b\nabla v \rangle \, d\xi.
$$

The first term converges to zero as $\delta \to 0$. The second term is bounded for $\delta \in (0, 1)$ as follows

$$
d\delta C(R)A(J^a_\delta u, J^a_\delta v)^{1/2} \leq C(R, a, b) \|L^b u\|_H A(\delta J^a_\delta v, \delta J^a_\delta v)^{1/2} 
\leq \sqrt{d} C(R, a, b) \|L^b u\|_H \|J^a_\delta v - v\|_H^{1/2},
$$

and hence converges to zero as $\delta \to 0$. Now, as we have the bound

$$
\|L^b J^a_\delta u\|_H \leq C(R, b, a)\|L^b u\|_H (1 + \|u\|_H),
$$

by an $\varepsilon/2$-argument, we get the convergence to zero as $\delta \to 0$ for every $v \in H$. As a consequence, $L^b J^a_\delta u \rightharpoonup L^b u$ weakly in $H$ for $u \in D(L^b)$. \hfill \Box

**Proposition 2.17.** Suppose that (M) holds. If the first order operator $u \mapsto b\nabla u$ and $L^a$ have the defective commutation property in the sense of Definition 2.15, then condition $R(a, b)$ holds.

**Proof.** Let $f \in S$ and $\beta > 0$. We get from (2.8) that

$$
\langle b\nabla G^a_\beta f, b\nabla f \rangle = \langle G^a_\beta b\nabla f, b\nabla f \rangle + \langle G^a_\beta R G^a_\beta f, b\nabla f \rangle.
$$

Integrating over $\mathbb{T}^d$ and multiplying with $\beta^2$ yields

$$
\beta \int_{\mathbb{T}^d} \langle \beta G^a_\beta b\nabla f - \beta b\nabla G^a_\beta f, b\nabla f \rangle \, d\xi 
\geq - \int_{\mathbb{T}^d} \langle \beta G^a_\beta f, b\nabla f \rangle \, d\xi 
\geq - \sqrt{d} A(f, f)^{1/2} \beta \|\nabla f\|_{L^2(\mathbb{T}^d; \mathbb{R}^d)} 
\geq - \sqrt{d} A(f, f)^{1/2} \|b\nabla f\|_{L^2(\mathbb{T}^d; \mathbb{R}^d)} 
\geq - \sqrt{d} A(f, f)^{1/2} B(f, f)^{1/2} 
\geq - \sqrt{d} A(f, f)^{1/2} \|b\|_{L^\infty(\mathbb{T}^d; \mathbb{R}^d)} \|\nabla f\|_{L^2(\mathbb{T}^d; \mathbb{R}^d)} 
\geq - \sqrt{d} \|b\|_{L^\infty(\mathbb{T}^d; \mathbb{R}^d)} A(f, f)
$$

which yields $R(a, b)$. \hfill \Box
Corollary 2.18. Suppose that $a$ satisfies (E) and let $b = a$. Suppose the defective commutation of the first order operator $u \mapsto a\nabla u$ and $L^a$ in the sense of Definition 2.15. Then condition $R(a, a)$ holds.

The defective commutation property is a variant of the so-called defective intertwining property, see [47]. Note that by the Weitzenböck formula the defective intertwining property from [47] holds for instance for the Laplace-Beltrami operator $L^a$ and the so-called Hodge-de Rham-Laplace operator $\Box^a$ on $(M, g_a)$, see also [39], where related questions on comparison of these two operators occur.

Remark 2.19. Assume that (M) holds and that $A$ satisfies the following Poincaré inequality, i.e., there exists $C > 0$ such that

$$\|f\|_{L^2(T^d)}^2 \leq CA(f, f) + \left( \int_{T^d} f \, d\xi \right)^2 \quad \forall f \in S,$$

cf. [51, Example 1.1.2]. Observe that then there exists a non-constant function $f \in S$ such that $L^a f = -\lambda_1 f$, where $\lambda_1 > 0$ is the smallest non-zero eigenvalue of $-L^a$. For such an eigenfunction $f$, we get by the Poincaré inequality (2.10) that

$$\int_{T^d} [2\langle L^a b\nabla f, b \nabla f \rangle - \langle b \nabla f, b \nabla L^a f \rangle] \, d\xi = -\int_{T^d} [a\nabla b \nabla f]^2 \, d\xi + \lambda_1 \int_{T^d} |b \nabla f|^2 \, d\xi$$

(2.10)

$$\leq \lambda_1 \sum_{i=1}^d \left( \int_{T^d} \langle b_i, \nabla f \rangle \, d\xi \right)^2 = 0,$$

where, in the last step, we have used integration by parts and (E). Hence in the situation of a Poincaré inequality for $A$ and for general $b$ satisfying (E), the estimate (2.10), combined with Remark 2.14, shows that $c \leq 0$ is necessary for (R) to hold.

Example 2.20. See [39, Example 7]. Let $d = 2$. Let $a = 1$. Denote $g_t := g(t) := \sin(2\pi t)$, $h_t := h(t) := \cos(2\pi t)$, $t \in [0, 1]$. Consider $f(t, s) := g_t + h_s$, $\xi = (t, s) \in T^2$. Clearly, $f \in C^\infty(T^2)$, $\nabla f = 2\pi(h_t, -g_s)$, and $\Delta f = -4\pi^2 f$. For a constant $\gamma > \frac{4\pi^2}{\sqrt{2}}$, let

$$b(\xi) = \begin{pmatrix} b_1(t, s) \\ b_2(t, s) \end{pmatrix} := \begin{pmatrix} g_t g_s + \gamma & h_t h_s \\ g_t h_s & -h_t g_s + \gamma \end{pmatrix}.$$ 

By a simple calculation and Lemma 2.6 we see that $b$ satisfies (E) for such $\gamma$. We have that

$$\nabla f = 2\pi h_t(g_t g_s - g_t h_s, g_t h_s + g_s^2) + \gamma \nabla f,$$

$$|\nabla f|^2 = 4\pi^2 h_t^2(g_t^2 g_s^2 + g_t^2 h_s^2 + g_s^2 h_t^2 + g_s^4) + 2\gamma |\nabla f - \gamma \nabla f, \nabla f| + \gamma^2 |\nabla f|^2,$$

$$\langle b \nabla f, b \nabla \Delta f \rangle = -4\pi^2 |\nabla f|^2,$$

$$\langle b \nabla f \rangle - \gamma \nabla f$$

and $\gamma \nabla f$ are orthogonal in $L^2(T^2; \mathbb{R}^2)$. Also, $\langle b \nabla f \rangle - \gamma \text{Hess}(f)$ and $\gamma \text{Hess}(f)$ are orthogonal in $L^2(T^2; \mathbb{R}^{2 \times 2})$. Combining these identities, we obtain

$$-\int_{T^2} |\nabla b \nabla f|^2 \, d\xi - \int_{T^2} \langle b \nabla f, b \nabla \Delta f \rangle \, d\xi = -4\pi^2 \left( \frac{11}{2} + 4\gamma^2 \right) \pi^2 - \left( \frac{3}{2} + 4\gamma^2 \right) \pi^2 = -16\pi^4,$$

This is certainly the case if $a = 1$. 


and
\[ \int_{\mathbb{T}^d} |b \nabla f|^2 \, d\xi = \left( \frac{3}{2} + 4\gamma^2 \right) \pi^2. \]

As a consequence, by Remark 2.13, if \( \alpha = 1 \) and (E) holds for \( b \) as above, it is necessary for \( R(1, b) \) to hold that \( c \leq -\frac{32}{3+8\gamma^2} \pi^2 \).

2.4. Commutator estimates. We shall prove some approximative commutator bounds needed later.

Lemma 2.21. Assume that (M) holds. Let \( u \in S \). Then, for every \( \alpha \geq 0 \), \( \beta > 0 \),
\[ A_\alpha^{(3)}(u, J^a_y \mathcal{L}_y J^a_y u) \leq \left( \frac{1}{\beta} + \alpha \right) \|b\|_{L_\infty^a}^2 \mathcal{A}(u, u). \]

Proof. Let \( \alpha \geq 0 \), \( \beta > 0 \) and \( u \in S \). Set \( y_\delta := J^a_y u \in D(L^a) \). Then, noting that \((L^a)\) and \( \sqrt{-\mathcal{L}G_\beta^a} \) commute in \( H \),
\[ A_\alpha^{(3)}(u, J^a_y \mathcal{L}_y J^a_y u) = \beta(y_\delta - \beta G^a_\beta y_\delta, \mathcal{L}_y y_\delta)_H + \alpha(y_\delta, L^a y_\delta)_H \]
\[ = \beta B(\beta G^a_\beta y_\delta - y_\delta, y_\delta) + \alpha B(y_\delta, y_\delta) \]
\[ \leq \beta \|b\|_{L_\infty^a}^2 \mathcal{A}(y_\delta, y_\delta) \|A(u, u)\|_{L_\infty^a}^2 + \alpha \mathcal{A}(u, u) \]
\[ = \beta \|b\|_{L_\infty^a}^2 \mathcal{A}(u, u) + \alpha \mathcal{A}(u, u) \]
\[ \leq \left( \frac{1}{\beta} + \alpha \right) \|b\|_{L_\infty^a}^2 \mathcal{A}(u, u), \]

where we have used that for \( u \in D(L^a) \), \( \beta > 0 \), it holds that \( \beta G^a_\beta u - u = G^a_\beta L^a u \).

Lemma 2.22. Assume that (M) holds. Suppose that (R) holds. Let \( u \in H \). Then there exists \( c \in \mathbb{R} \) such that for every \( \delta > 0 \) and every \( \alpha \geq 0 \), \( \beta > 0 \),
\[ \sum_{i=1}^{d} A_\alpha^{(3)}(\langle b_i, \nabla J^a_y u \rangle, \langle b_i, \nabla J^a_y u \rangle) + A_\alpha^{(3)}(J^a_y \mathcal{L}_y J^a_y u, u) \leq -cA(J^a_y u, J^a_y u). \]

Proof. Let \( u \in H \), let \( \delta, \beta > 0 \), \( \alpha \geq 0 \). Set \( y_\delta := J^a_y u \). Then by (R) there exists \( c \in \mathbb{R} \), such that
\[ \sum_{i=1}^{d} A_\alpha^{(3)}(\langle b_i, \nabla J^a_y u \rangle, \langle b_i, \nabla J^a_y u \rangle) + A_\alpha^{(3)}(J^a_y \mathcal{L}_y J^a_y u, u) \]
\[ = \beta B(\beta G^a_\beta y_\delta - y_\delta, y_\delta) + \sum_{i=1}^{d} (\beta G^a_\beta (\langle b_i, \nabla y_\delta \rangle), \langle b_i, \nabla y_\delta \rangle)_H \]
\[ = \beta \int_{\mathbb{T}^d} (\beta \nabla G^a_\beta y_\delta - \beta G^a_\beta b \nabla y_\delta, b \nabla y_\delta) \, d\xi \]
\[ \leq -cA(y_\delta, y_\delta). \]
Corollary 2.23. Assume that (M) holds. Suppose that (R) holds. Let \( u \in S \). Then the estimate of Lemma \[2.21\] improves to

\[
A^{(\delta)}_n(u, J^b_s L^b_j J^w_s u) \leq (-c \vee 0) A(u, u) \quad \forall \delta, \beta > 0,
\]

where \( c \in \mathbb{R} \) does neither depend on \( \delta \), nor on \( \beta \).

3. Stochastic variational inequalities (SVI)

As above, \( H = L^2(\mathbb{T}^d) \), \( S = H^1(\mathbb{T}^d) \). Let \( U := \mathbb{R}^d \). Denote by \( L_2(U, H) \) the space of linear Hilbert-Schmidt operators from \( U \) to \( H \). Let \( B : S \to L_2(U, H) \) denote the linear operator

\[
B(u) \zeta := \sum_{i=1}^d \langle b_i, \nabla u \rangle \zeta^i, \quad u \in S, \quad \zeta \in U.
\]

Remark 3.1. By assumption (M), \( B \) is bounded from \( S \) to \( L_2(U, H) \) and its norm is given by

\[
\|B(u)\|_{L_2(U, H)} = B(u, u), \quad u \in S.
\]

Definition 3.2. Let \( x \in L^2(\Omega; F_0, \mathbb{P}; H), T > 0 \). A progressively measurable\[5\] map \( X \in L^2([0, T] \times \Omega; H) \) is said to be an SVI-solution to \[1.1\] if there exists a constant \( C > 0 \) such that

(i) (Regularity)

\[
\begin{aligned}
\text{ess sup}_{t \in [0, T]} \mathbb{E}\|X_t\|^2_H + 2\mathbb{E} \int_0^T \Psi(X_s) \, ds &\leq \mathbb{E}\|x\|^2_H.
\end{aligned}
\]

(ii) (Variational inequality) For every choice of admissible test-elements \((Z_0, Z, G, P)\), that is, by definition, \( Z_0 \in L^2(\Omega, F_0, \mathbb{P}; S) \), \( Z \in L^2([0, T] \times \Omega; D(L^b) \cap S) \), \( G \in L^2([0, T] \times \Omega; H) \), \( P \in \mathcal{L}(H) \) such that \( G \) is progressively measurable, such that \( P(D(L^b)) \subset D(L^b) \) and such that the following equation holds in Itô-sense

\[
Z_t = Z_0 + \int_0^t G_s \, ds + \frac{1}{2} \int_0^t P^* L^b P Z_s \, ds + \int_0^t B P Z_s \, dW_s \quad \forall t \in [0, T],
\]

we have that

\[
\begin{aligned}
\mathbb{E}\|X_t - Z_t\|^2_H + 2\mathbb{E} \int_0^t \Psi(X_s) \, ds &\leq \mathbb{E}\|x - Z_0\|^2_H + 2\mathbb{E} \int_0^t \Psi(Z_s) \, ds \\
&\quad - 2\mathbb{E} \int_0^t \langle G_s, X_s - Z_s \rangle_H \, ds \\
&\quad - \mathbb{E} \int_0^t \langle L^b P Z_s, P X_s - X_s \rangle_H \, ds - \mathbb{E} \int_0^t \langle X_s, L^b(Z_s - P Z_s) \rangle_H \, ds
\end{aligned}
\]

for almost all \( t \in [0, T] \).

If, additionally, it holds that \( X \in C([0, T]; L^2(\Omega; H)) \), we say that \( X \) is a \((time-)continuous\) SVI-solution to \[2.2\].

Definition 3.3. We say that an \( \{\mathcal{F}_t\} \)-adapted process \( X \in L^2(\Omega, C([0, T]; H)) \cap L^2([0, T] \times \Omega; D(L^b) \cap S) \) is an \textit{analytically strong solution} to \[2.2\] with initial datum \( x \in L^2(\Omega; H) \), if there

\[5\] That is, for every \( t \in [0, T] \) the map \( X : [0, t] \times \Omega \to H \) is \( \mathcal{F}_t \)-measurable.
exists $\eta \in L^2([0,T] \times \Omega; H)$ such that $\eta$ is progressively measurable and such that $\eta \in \partial \Psi(X)$ a.e. and we have that

$$X_t = x - \int_0^t \eta_s \, ds + \frac{1}{2} \int_0^t L^b X_s \, ds + \int_0^t \langle b \nabla X_s, dW_s \rangle \quad \mathbb{P}\text{-a.s.}$$

for all $t \geq 0$.

**Lemma 3.4.** If $X$ is an analytically strong solution to (2.2), then $X$ is a time-continuous SVI-solution to (2.2).

**Proof.** Compare with [26, Proof of Remark 2.3]. Let $X$ be a strong solution to (1.1). Then by Itô’s formula and a standard localization argument, compare with Remark 3.1:

$$E \|X_t\|^2_H = E\|x\|^2_H - 2E \int_0^t \langle \eta_s, X_s \rangle_H \, ds + E \int_0^t \langle L^b X_s, X_s \rangle_H \, ds + E \int_0^t B(X_s, X_s) \, ds.$$

By the definition of the subdifferential $\partial \Psi$, we get that

$$(-\eta, X)_H = \langle \eta, 0 - X \rangle \leq \Psi(0) - \Psi(X) = \Psi(X) \, dt \circ \mathbb{P}\text{-a.e.}$$

Hence (3.1) is satisfied.

Furthermore, let $(Z_0, Z, G, P)$ be a fixed choice of admissible test-elements. We have that

$$Z_t = Z_0 + \int_0^t G_s \, ds + \frac{1}{2} \int_0^t P^* L^b P Z_s \, ds + \int_0^t B(PZ_s) \, dW_s \quad \forall t \in [0,T].$$

Consider

$$d(X - Z) = (\eta - G) dt + \frac{1}{2} (L^b X - P^* L^b P Z) \, dt + B(X - PZ) \, dW$$

and we get by Itô’s formula that for $t \in [0,T]$,

$$\|X_t - Z_t\|^2_H = \|x - Z_0\|^2_H + 2 \int_0^t \langle \eta_s - G_s, X_s - Z_s \rangle_H \, ds$$

$$+ \int_0^t \langle L^b X_s - P^* L^b P Z_s, X_s - Z_s \rangle_H \, ds$$

$$+ 2 \int_0^t \langle X_s - Z_s, B(X_s - PZ_s) \, dW_s \rangle_H$$

$$+ \int_0^t \|B(X_s - PZ_s)\|^2_{L^2(U,H)}. $$

Taking expectations yields for $t \in [0,T]$,

$$\|X_t - Z_t\|^2_H = E\|x - Z_0\|^2_H + 2E \int_0^t \langle \eta_s - G_s, X_s - Z_s \rangle_H \, ds$$

$$+ E \int_0^t \langle L^b X_s - P^* L^b P Z_s, X_s - Z_s \rangle_H \, ds$$

$$+ E \int_0^t B(X_s - PZ_s, X_s - PZ_s) \, ds$$

$$= E\|x - Z_0\|^2_H + 2E \int_0^t \langle \eta_s - G_s, X_s - Z_s \rangle_H \, ds$$

$$+ E \int_0^t B(X_s, Z_s - PZ_s) \, ds + E \int_0^t B(PZ_s, PX_s - X_s) \, ds.$$
The proof is completed by the application of symmetry of $L^b$ in $H$ and of the subdifferential property for $\eta \in \partial \Psi(X)$

$(-\eta, X - Z) \leq \Psi(Z) - \Psi(X) \ dt \otimes \mathbb{P}$-a.e.

\[ \square \]

Remark 3.5. In [16], a weaker notion for a solution for an analogue of equation (4.1) on a bounded, convex domain with Neumann boundary conditions was proposed. Translated to our situation, in [32], one would test the SVI with elements $(Z_0, Z, G)$ and $P \equiv 1$ (the identity) — making the notion a weaker one compared to the one of Definition 3.2. However, in [16, Definition 3.1], less regularity for $Z$ was demanded. Let us point out again that we are able to consider more general gradient noise $B$ here.

4. The main result and its proof

**Theorem 4.1.** Suppose that condition (N) holds for $\psi$ and that condition (M) holds. Suppose that condition (BE) holds for $a$. Suppose that condition $R(a, b)$ holds.

Then, for every initial datum $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and every $T > 0$, there exists a unique adapted time-continuous SVI-solution $X \in C([0, T]; L^2(\Omega; H))$ to the SPDE (2.2) such that

$$
\text{ess sup}_{t \in [0, T]} \mathbb{E}\|X_t - Y_t\|^2_H \leq \mathbb{E}\|x - y\|^2_H
$$

where $Y \in C([0, T]; L^2(\Omega; H))$ is the unique SVI-solution to another initial datum $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$.

The remaining part of this section is devoted to the proof of Theorem 4.1.

4.1. The approximating equation. Consider the Gelfand triple of dense and compact embeddings

$$
S \hookrightarrow H \hookrightarrow S^*.
$$

Suppose that condition (E) and condition BE($K, \infty$) holds for $a$ with $K \in \mathbb{R}$. There are two cases. If $K \leq 0$, we shall equip $S$ with the equivalent norm $(A_{1-2K})^{1/2} = (A + (1 - 2K)||\cdot||_H^2)^{1/2}$. If $K > 0$, it is sufficient to consider $A_1^{1/2}$. Note that these norms are equivalent to the standard norm of $H^1(T^d)$ by [35, Chapter I, Exercise 2.1] and assumption (E). From now on, we shall concentrate on the first case $K \leq 0$, as it is slightly more involved. We remark that in the latter case $K > 0$, we can recover the a priori bounds proved below from the first case by a standard perturbation argument.

Let us denote by $(W_t)_{t \geq 0}$ a cylindrical Wiener process in $U := \mathbb{R}^d$ for the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Consider the Itô-equation in the above Gelfand triple

$$
dx_t + A^{\lambda, \delta, \varepsilon}(X_t) dt = B^\delta(X_t) dW_t, \quad X_0 = x_n,
$$

where $x_n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; S)$, and where, from now on, whenever it seems convenient, we suppress the indices in the notation as in $X = X^{n, \lambda, \delta, \varepsilon}$. Here, we define for $\lambda, \delta, \varepsilon > 0$:

$$
A^{\lambda, \delta, \varepsilon}(x) := -\text{div}(a^* \phi^{\lambda}(a \nabla x)) - \varepsilon L^a x - \frac{1}{2} J_2^b L^b J_2^d x,
$$

writing $\phi^{\lambda} = \partial \psi^{\lambda}, \lambda > 0$, meaning that $\phi^{\lambda}$ is the Yosida-approximation of $\phi$, whereas $\psi^{\lambda}$ denotes the Moreau-Yosida approximation of $\psi$, see [2, p. 266]. Furthermore, for $\zeta = (\zeta^1, \ldots, \zeta^d) \in U$, $x \in S$, $t \in [0, T]$,

$$
B^\delta(x) \zeta := \sum_{i=1}^d \langle b_i, \nabla J_2^d x \rangle \zeta^i.
$$
Proposition 4.2. Suppose that condition (N) holds for $\psi$ and that condition (M) holds. Suppose that condition (BE) holds for $a$ with $K \leq 0$. Then $A^{\lambda,\delta,\varepsilon}$ and $B^{\delta}$ as defined in (4.2) and (4.3) respectively, satisfy conditions (A1)--(A3) in Appendix A, and (B1)--(B2) in Appendix A respectively, for $U := \mathbb{R}^d$ and all $\lambda, \delta, \varepsilon > 0$.

Proof. Fix $\lambda, \delta, \varepsilon > 0$.

(A1): The map

$$x \mapsto -\text{div}(a^\ast \phi^\lambda(a \nabla x)) - \varepsilon L^a x$$

is the maximal monotone subdifferential of the l.s.c. map

$$u \mapsto \int_{\mathbb{T}^d} \psi^\lambda(a \nabla u) d\xi + \frac{\varepsilon}{2} \int_{\mathbb{T}^d} |a \nabla u|^2 d\xi.$$ 

In this regard, see also [48, Proposition II.7.8] for the chain rule of a convex functional and bounded linear operators. The map

$$x \mapsto -\frac{1}{2} J_a^b J_a^b x$$

is linear and positive definite due to condition (M). As this map is also bounded linear, the conditions of Browder’s theorem on the maximality of the sum of monotone operators are satisfied, see [44, Theorem, pp. 75–76], and hence (A1) holds.

(A2): As $A^{\lambda,\delta,\varepsilon}$ is a subpotential operator of a convex energy which is bounded by $C(1 + \|x\|_2)$ for some positive constant $C > 0$, condition (A2) easily follows with similar arguments as in [24, Proposition 7.1].

(A3): The claim for the nonlinear part follows from Lemma 2.10 and by similar arguments as in [24, Proposition 7.1]. Compare also to [10, 16] for the Dirichlet and Neumann boundary cases respectively. Regarding the second part of the linear part, by Lemma 2.21 for every $\mu > 0$, $x \in S$,

$$- (J_a^b L_a^b x, \tilde{L}^{(\mu)} x)_H = A^{(1/\mu)}(x, J_a^b L_a^b x) \leq \left( \frac{1}{\delta} + 1 - 2K \right) \|b\|^2_\infty \kappa_a^{-1} A(x, x) \leq \left( \frac{1}{\delta} + 1 - 2K \right) \|b\|^2_\infty \kappa_a^{-1} A_{1-2K}(x, x).$$

Note that due to the renorming of $S$, $\tilde{L}^{(\mu)} := (L^a + 2K - 1)^{\mu}$, plays the role of $L^{(\mu)}$ here.

(B1): By a straightforward calculation, we get for $x \in S$,

$$\|B^\delta(x)\|_{L_2(U,H)}^2 = B(J_a^b x, J_a^b x) \leq \|b\|^2_\infty \kappa_a^{-1} A_{1-2K}(x, x).$$

(B2): By linearity, for $x, y \in S$,

$$\|B^\delta(x) - B^\delta(y)\|_{L_2(U,H)}^2 = \|B^\delta(x - y)\|_{L_2(U,H)}^2 = B(J_a^b(x - y), J_a^b(x - y)) \leq \|b\|^2_\infty \kappa_a^{-1} A^{(1/\delta)}(x - y, x - y) \leq \left( \frac{2}{\delta} + 1 \right) \|b\|^2_\infty \kappa_a^{-1} \|x - y\|_H^2.$$

compare also with [35] Chapter I, Lemma 2.11. \qed
As a consequence, (A1)–(A3), (B1)–(B2) guarantee the existence and uniqueness of limit solutions to \( \text{[4.4]} \), see Appendix A for the definition and the precise result.

4.2. **A priori estimates.** As before, consider the single-valued Itô-SPDE

\[
(4.4) \quad dX_t + A^{\lambda,\delta,\varepsilon}(X_t) \, dt = B^\delta(X_t) \, dW_t, \quad X_0 = x_n,
\]

where \( x_n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; S) \), and where, as above, we suppress the indices in the notation as in \( X = X^{n,\lambda,\delta,\varepsilon} \). The existence and uniqueness of limit solutions to \( \text{[4.4]} \) follows from \cite[Theorem 4.6]{24}, see Appendix A.

For fixed \( \delta > 0 \) and for every \( m \in \mathbb{N} \), let \( y \mapsto B^{\delta,m}(y) \) be progressively measurable maps on \( S \) such that

1. each \( B^{\delta,m} \) satisfies (B1)–(B2) with constants \( C_1 \) and \( C_2 \) not depending on \( m \),
2. each \( B^{\delta,m} \) satisfies (B3) (with constants \( C_3 = C_3(m) \) typically depending on \( m \)),
3. \( \|B^{\delta,m}(y) - B^{\delta}(y)\|_{L^2(U,H)} \to 0 \) for every \( y \in S \) as \( m \to \infty \).

The existence of such a sequence of maps can be proved e.g. by introducing an approximation step that employs standard mollifiers. Consider the sequence of Itô-processes

\[
(4.5) \quad X^m_t = x_n - \int_0^t A^{\lambda,\delta,\varepsilon}(X^m_s) \, ds + \int_0^t B^{\delta,m}(X^m_s) \, dW_s,
\]

where \( x_n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; S) \), \( m \in \mathbb{N} \). The existence of such processes is guaranteed by \cite[Theorem 4.4]{24}, see Appendix A. The following proposition is a modification of \cite[Proposition 14]{10}.

**Proposition 4.3.** Suppose that condition (N) holds and that condition (M) holds and that condition (BE) holds for \( a \) with \( K \leq 0 \). Let \( \lambda, \delta, \varepsilon > 0 \). Let \( x_n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; S) \), and let \( X = X^{n,\lambda,\delta,\varepsilon} \) be a limit solution to \( \text{[4.4]} \). Then, we have that

\[
(4.6) \quad \text{ess sup}_{t \in [0,T]} \mathbb{E}\|X_t\|_H^2 + 2\mathbb{E}\int_0^T \int_\mathbb{R} \psi^\lambda(a \nabla X_s) \, d\xi \, ds + 2\varepsilon\mathbb{E}\int_0^T A(X_s, X_s) \, ds \leq \mathbb{E}\|x_n\|_H^2.
\]

**Proof.** Let \( x_n \in S \), \( m \in \mathbb{N} \). We may apply the Itô formula \cite[Theorem 4.2.5]{10} for the Gelfand triple \( S \subset H \subset S^* \) and the process \( \text{[4.5]} \). Let

\[
A^{\lambda,\delta}(x) := - \text{div}(a^* \phi^\lambda(a \nabla x)) - \frac{1}{2} J^b_\delta L^a J^b_\delta x, \quad x \in H.
\]

We get for \( t \in [0,T] \), after taking the expected value,

\[
\mathbb{E}\|X^m_t\|_H^2 - \mathbb{E}\|x_n\|_H^2 + 2\varepsilon\mathbb{E}\int_0^t A(X^m_s, X^m_s) \, ds \\
\leq \mathbb{E}\int_0^t \left[ -2 S^r_A(A^{\lambda,\delta}(X^m_s), X^m_s)_S + \|B^{\delta,m}(X^m_s)\|_{L^2(U,H)}^2 \right] \, ds \\
\leq \mathbb{E}\int_0^t \left[ -2 S^r_A(A^{\lambda,\delta}(X^m_s), X^m_s)_S + \|B^{\delta,m}(X^m_s) - B^{\delta}(X^m_s)\|_{L^2(U,H)}^2 + \|B^{\delta,m}(X^m_s)\|_{L^2(U,H)}^2 \right] \, ds \\
\leq \mathbb{E}\int_0^t \left[ -2 S^r_A(A^{\lambda,\delta}(X^m_s), X^m_s)_S + C\|X^m_s - X_s\|_H^2 + \|B^{\delta,m}(X^m_s)\|_{L^2(U,H)}^2 \right] \, ds,
\]

where \( X \) is as in \( \text{[4.4]} \) and \( C \) does not depend on \( m \). By \cite[Theorem 4.6]{24}, \( X^m \to X \) in \( L^2(\Omega; C([0,T]; H)) \) as \( m \to \infty \). Note that \( A^{\lambda,\delta} : H \to H \) is monotone, single-valued, continuous and bounded and thus by Minty’s trick (see e.g. \cite[Remark 4.1.1]{10}), \( A^{\lambda,\delta}(X^m) \to A^{\lambda,\delta}(X) \) weakly
in \( H \) as \( m \to \infty \). Hence by (i), (iii) above, by lower semi-continuity of \( A \) (see [35, Chapter I, Lemma 2.12]) and by Lebesgue’s dominated convergence theorem, we converge to the inequality

\[
\mathbb{E}\|X_t\|_H^2 - \mathbb{E}\|x_n\|_H^2 + 2\varepsilon \mathbb{E} \int_0^t A(X_s, X_s) \, ds \\
\leq \mathbb{E} \int_0^t \left[ -2 S, \langle A^{\lambda, \delta}(X_s), X_s \rangle_S + \|B^\delta(X_s)\|^2_{L^2(U, H)} \right] \, ds.
\]

Note that, as above, for \( y \in H \), we have

\[
\|B^\delta(y)\|_{L^2(U, H)}^2 = \mathcal{B}(J_\beta^o y, J_\beta^o y).
\]

On the other hand,

\[
(J_\beta^o L^a J_\beta^o y, y)_H = -\mathcal{B}(J_\beta^o y, J_\beta^o y),
\]

for all \( y \in H \). We get that

\[
\mathbb{E}\|X_t\|_H^2 - \mathbb{E}\|x_n\|_H^2 + 2\varepsilon \mathbb{E} \int_0^t A(X_s, X_s) \, ds \\
\leq -2\mathbb{E} \int_0^t (\phi^\lambda(a\nabla X_s), a\nabla X_s)_H \, ds \\
\leq -2\mathbb{E} \int_0^t \int_{\mathbb{T}^d} \psi^\lambda(a\nabla X_s) \, d\xi \, ds,
\]

which yields the claim.

\( \square \)

Let \( \beta > 0 \). Define renormed spaces \( H_\beta := L^2(\mathbb{T}^d) \) with norm \( \|u\|_{H_\beta}^2 := A_{1-2K}^{(\beta)}(u, u) \). Obviously, \( \|\cdot\|_H \leq \|\cdot\|_{H_\beta} \leq \sqrt{\beta + 1}\|\cdot\|_H \) and \( S \hookrightarrow H_\beta \hookrightarrow S^\ast \) forms a family of Gelfand triples. The following theorem is a modification of [49, Theorem 15].

**Theorem 4.4.** Suppose that condition (N) holds and that condition (M) holds and that condition (BE) holds for \( a \) with \( K \leq 0 \). Suppose that \( R(a, b) \) holds with \( c \leq 0 \). Let \( \lambda, \delta, \varepsilon > 0 \). Let \( x_n \in L^2(\Omega, \mathcal{F}_0, P; S) \), and let \( X = X^{n, \lambda, \delta, \varepsilon} \) be a limit solution to \( (1.3) \). Then, we have that

\[
(4.7) \quad \text{ess sup}_{t \in [0, T]} \mathbb{E}[A_{1-2K}(X_t, X_t)] + 2\varepsilon \int_0^T \|L^n X_t\|^2_H \, dt \leq e^{-(2\varepsilon(1-2K)+c)T} A_{1-2K}(x_n, x_n),
\]

where \( c \in \mathbb{R} \) is as in condition (R).

**Proof.** We shall apply the Itô formula [10, Theorem 4.2.5] for the Gelfand triple \( S \subset H_\beta \subset S^\ast \) and the process \((1.3)\). As in the proof of Proposition 1.13 we get for \( t \in [0, T] \), after taking the expected value,

\[
\mathbb{E}\|X_t^n\|_{H_\beta}^2 - \mathbb{E}\|x_n\|_{H_\beta}^2 \\
\leq \mathbb{E} \int_0^t \left[ -2 S, \langle A^{\lambda, \delta, \varepsilon}(X^n_s), (L^a)^{(1/\beta)} X^n_s \rangle_S + C\|X^n_s - X_s\|_{H_\beta}^2 + \|B^{\delta, m}(X_s)\|^2_{L^2(U, H_\beta)} \right] \, ds,
\]

where \( X \) is as in \( (4.1) \) and \( C \) does not depend on \( m \). Now, we argue essentially as in the proof of Proposition 1.13 and obtain that

\[
\mathbb{E}\|X_t\|_{H_\beta}^2 - \mathbb{E}\|x_n\|_{H_\beta}^2 \\
\leq \mathbb{E} \int_0^t \left[ -2 S, \langle A^{\lambda, \delta, \varepsilon}(X_s), (L^a)^{(1/\beta)} X_s \rangle_S + \|B^\delta(X_s)\|^2_{L^2(U, H_\beta)} \right] \, ds,
\]
where we have used the commutation of \((L^a)^{(1/\beta)}\) and \(L^a\) in \(H\). Note that by Lemma 2.10, 
\[
\leq \langle \text{div}(\phi^\lambda (\nabla y)), (L^a)^{(1/\beta)} y \rangle_S \leq 0 \quad \forall y \in S,
\]
as in the proof of Proposition 4.2 (A3). An application of Lemma 2.22 shows that 
\[
E \|x_t\|_\beta^2 - E \|x_n\|_\beta^2 
\leq 2\varepsilon E \int_0^t \langle (L^a)^{(1/\beta)} x_s, x_s \rangle_S ds - c \int_0^t A(J^a_x x_s, J^a_x x_s) ds.
\]
Let \(\beta \to \infty\),
\[
E A_{1-2K}(x_t, x_t) - E A_{1-2K}(x_n, x_n) 
\leq 2\varepsilon E \int_0^t A_{1-2K}(L^a x_s, x_s) ds - c \int_0^t A_{1-2K}(x_s, x_s) ds,
\]
where we use weak convergence, weak lower semi-continuity and the Mosco convergence \(\frac{M}{\beta} \to A_{1-2K}(\cdot, \cdot)\). Compare also [55, Chapter I, Lemma 2.12]. The claim follows by resolvent contraction in \(S\) and Gronwall’s lemma. \(\square\)

4.3. Passage to the limit. Let us continue with the proof of the main result Theorem 4.1. We shall construct an approximation in several steps, such that the limit is a time-continuous process which is an SVI-solution to (2.2).

Spatial rough limit \(\delta \to 0\). Fix \(\lambda, \varepsilon, n\). Let \(X^{\delta, m} = X^m\) be the unique continuous solution to (4.5). For two solutions \(X^{\delta_1, m}, X^{\delta_2, m}\), \(\delta_1, \delta_2 > 0\) with initial condition \(x_n \in L^2(\Omega, F_0, \mathbb{P}; S)\) we have that 
\[
E \|X^{\delta_1, m}_t - X^{\delta_2, m}_t\|_H^2 = -2E \int_0^t \langle A^{\delta_1, \varepsilon}(X^{\delta_1, m}_s), X^{\delta_1, m}_s - X^{\delta_2, m}_s \rangle_S ds + E \int_0^t \|B^{\delta_1, m}(X^{\delta_1, m}_s) - B^{\delta_2, m}(X^{\delta_2, m}_s)\|_{L^2(U,H)}^2 ds.
\]
Let \(m \to \infty\) and use Minty’s trick as in the proof of Proposition 4.3. We get that 
\[
E \|X^{\delta_1}_t - X^{\delta_2}_t\|_H^2 
\leq -E \int_0^t \langle J^{\delta_1}_s L^{\delta_1} J^{\delta_1}_s X^{\delta_1}_s - J^{\delta_2}_s L^{\delta_2} J^{\delta_2}_s X^{\delta_2}_s, X^{\delta_1}_s - X^{\delta_2}_s \rangle_S ds + E \int_0^t \|B^{\delta_1}(X^{\delta_1}_s) - B^{\delta_2}(X^{\delta_2}_s)\|_{L^2(U,H)}^2 ds
\]
\[
= E \int_0^t \langle B(J^{\delta_1}_s X^{\delta_1}_s, J^{\delta_1}_s X^{\delta_2}_s) + B(J^{\delta_2}_s X^{\delta_1}_s, J^{\delta_2}_s X^{\delta_2}_s) \rangle ds - 2E \int_0^t \langle J^{\delta_1}_s X^{\delta_1}_s, J^{\delta_2}_s X^{\delta_2}_s \rangle ds
\]
\[
\leq \|b\|_{\infty}^2 E \int_0^t \|J^{\delta_1}_s X^{\delta_1}_s\|_S \|J^{\delta_2}_s X^{\delta_2}_s\|_S ds + \|J^{\delta_1}_s X^{\delta_2}_s\|_S \|J^{\delta_2}_s X^{\delta_1}_s\|_S ds \leq \|b\|_{\infty}^2 \kappa^- \sum_{\alpha=1}^{1/2} E \int_0^t \left[ A_{1-2K}(X^{\delta_1}_s, X^{\delta_2}_s)^{1/2} A_{1-2K}((J^{\delta_1}_s - J^{\delta_2}_s)X^{\delta_1}_s, (J^{\delta_2}_s - J^{\delta_1}_s)X^{\delta_2}_s)^{1/2}
\right.\]
\[
\left. + A_{1-2K}(X^{\delta_2}_s, X^{\delta_1}_s)^{1/2} A_{1-2K}((J^{\delta_1}_s - J^{\delta_2}_s)X^{\delta_2}_s, (J^{\delta_2}_s - J^{\delta_1}_s)X^{\delta_1}_s)^{1/2} \right] ds.
\]

\(^6\)See [2] for the notion of Mosco convergence.
Note that
\[ A((J_{\delta_1}^a - J_{\delta_2}^a)u, (J_{\delta_1}^a - J_{\delta_2}^a)u) \leq \left( \|L^a J_{\delta_1}^a u\|_H + \|L^a J_{\delta_2}^a u\|_H \right)^{1/2} \left( \|J_{\delta_1}^a - J_{\delta_2}^a u\|_H \right)^{1/2} \]
\[ \leq \left( \|L^a J_{\delta_1}^a u\|_H^{1/2} + \|L^a J_{\delta_2}^a u\|_H^{1/2} \right) \left( \sqrt{\delta_1} \|J_{\delta_1} L^a u\|_H^{1/2} + \sqrt{\delta_2} \|J_{\delta_2} L^a u\|_H^{1/2} \right) \]

According to (4.6) and (4.7), we thus get an estimate
\[ \text{ess sup}_{t \in [0, T]} \mathbb{E}[X_t^{\delta_1} - X_t^{\delta_2}] \leq \frac{1}{\varepsilon} \left( \sqrt{\delta_1} + \sqrt{\delta_2} \right) C(T) \mathbb{E}[(A_{1-2K} x_n, x_n)]. \]

Thus, as \( \varepsilon > 0 \) is fixed, there exists an \( \{F_t\} \)-adapted process \( X = X^{\lambda, \varepsilon, n} \in C([0, T]; L^2(\Omega; H)) \) with \( X_0 = x_n \) such that \( X^\delta \to X \) strongly in \( L^\infty([0, T]; L^2(\Omega; H)) \) as \( \delta \to 0 \).

**Singular limit \( \lambda \to 0 \).** Fix \( \delta, \varepsilon, n \). Let \( X^{\lambda, m} = X^m \) be the unique continuous solution to (4.5). For two solutions \( X^{\lambda_1, m}, X^{\lambda_2, m}, \lambda_1, \lambda_2 > 0 \) with initial condition \( x_n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; S) \) we have that
\[ \mathbb{E}[X_t^{\lambda_1, m} - X_t^{\lambda_2, m}]^2 \leq -2 \mathbb{E} \int_0^t A^{\lambda_1, \delta, \varepsilon}(X_s^{\lambda_1, m}) - A^{\lambda_2, \delta, \varepsilon}(X_s^{\lambda_2, m}), X_s^{\lambda_1, m} - X_s^{\lambda_2, m} \rangle_H ds \]
\[ + \mathbb{E} \int_0^t \|B^{\delta, m}(X_s^{\lambda_1, m} - X_s^{\lambda_2, m})\|^2_{L^2(\mathcal{U}; H)} ds. \]

Let \( m \to \infty \) and use monotonicity and Minty’s trick as in the proof of Proposition 4.3 We get that
\[ \mathbb{E}[X_t^{\lambda_1} - X_t^{\lambda_2}]^2 \leq -2 \mathbb{E} \int_0^t A^{\lambda_1, \delta, \varepsilon}(X_s^{\lambda_1}) - A^{\lambda_2, \delta, \varepsilon}(X_s^{\lambda_2}), X_s^{\lambda_1} - X_s^{\lambda_2} \rangle_H ds \]
\[ + \mathbb{E} \int_0^t \|B^{\delta}(X_s^{\lambda_1} - X_s^{\lambda_2})\|^2_{L^2(\mathcal{U}; H)} ds. \]

Note that by (N) and (26) equation (A.6) in the appendix), we have that
\[ \langle \phi^{\lambda_1}(z_1) - \phi^{\lambda_2}(z_2), z_1 - z_2 \rangle \geq -C(\lambda_1 + \lambda_2)(1 + |z_1|^2 + |z_2|^2) \quad \forall z_1, z_2 \in \mathbb{R}^d. \]

We get that
\[ \mathbb{E}[X_t^{\lambda_1} - X_t^{\lambda_2}]^2 \leq C(\lambda_1 + \lambda_2) \mathbb{E} \int_0^t (1 + A(X_s^{\lambda_1}, X_s^{\lambda_1}) + A(X_s^{\lambda_2}, X_s^{\lambda_2})) ds. \]

Hence by (4.6) and (4.7),
\[ \text{ess sup}_{t \in [0, T]} \mathbb{E}[X_t^{\lambda_1} - X_t^{\lambda_2}]^2 \leq C(T)(\lambda_1 + \lambda_2)(\mathbb{E}[A_{1-2K} x_n, x_n]) + 1). \]

Hence there exists an \( \{F_t\} \)-adapted process \( X \in C([0, T]; L^2(\Omega; H)) \) with \( X_0 = x_n \) such that \( X^\lambda \to X \) strongly in \( L^\infty([0, T]; L^2(\Omega; H)) \) as \( \lambda \to 0 \).

**Vanishing viscosity limit \( \varepsilon \to 0 \).** In a similar manner as before, we get for two solutions \( X^{\lambda, \varepsilon_1, m}, X^{\lambda, \varepsilon_2, m}, \varepsilon_1, \varepsilon_2 > 0 \) with initial conditions \( x_1, x_2 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; S) \) by monotonicity arguments (after passing to \( m \to \infty \)) that for \( t \in [0, T] \),
\[ \mathbb{E}[X_t^{\lambda, \varepsilon_1} - X_t^{\lambda, \varepsilon_2}]^2 \leq \mathbb{E}[x_1 - x_2]^2 + C(\varepsilon_1 + \varepsilon_2) \mathbb{E} \int_0^t (A(X_s^{\lambda, \varepsilon_1}, X_s^{\lambda, \varepsilon_1}) + A(X_s^{\lambda, \varepsilon_2}, X_s^{\lambda, \varepsilon_2})) ds. \]
Now, let \( m \) be the unique continuous solution to (4.5) with initial condition \( x_n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; S) \). By Itô’s formula (compare with Lemma 3.4 and [26, Step 6 of the proof of Theorem 3.1]),

\[
\begin{aligned}
&\mathbb{E}[X_t^\varepsilon - X_t^k]^2 \\
= &\mathbb{E}[x_n - Z_0]^2_H + 2\mathbb{E}\int_0^t (\text{div}(a^\varepsilon \psi^\lambda(a^\nabla X_s))) + \varepsilon L^a X_s - G_s, X_s - Z_s)_H ds \\
&+ \mathbb{E}\int_0^t \|B^{\delta, m}(X_s) - B(PZ_s)\|_{L^2(U,H)}^2 ds + \mathbb{E}\int_0^t (J^a L^b J^b X_s - P^a L^b P Z_s, X_s - Z_s)_H ds.
\end{aligned}
\]

Recall that by Lemma [24]

\[|\psi(\zeta) - \psi^\lambda(\zeta)| \leq C\lambda(1 + \psi(\zeta)) \quad \forall \zeta \in \mathbb{R}^d.\]

Hence, by Young inequality,

\[
\begin{aligned}
\mathbb{E}[X_t^\varepsilon - X_t^k]^2_H &+ 2\mathbb{E}\int_0^t \int_{\mathbb{T}} \langle a \nabla X_s \rangle d\xi ds \\
\leq &\mathbb{E}[x_n - Z_0]^2_H + 2\mathbb{E}\int_0^t \int_{\mathbb{T}} \psi(a \nabla Z_s) d\xi ds + 2C\lambda\mathbb{E}\int_0^t (1 + \psi(a \nabla X_s)) d\xi ds \\
&- 2\mathbb{E}\int_0^t (G_s, X_s - Z_s)_H ds \\
&+ 2\mathbb{E}\int_0^t \left(\varepsilon^{4/3} \|L^a X_s\|_H^2 + \varepsilon^{2/3} \|X_s - Z_s\|_H^2\right) ds \\
&+ \mathbb{E}\int_0^t \|B^{\delta, m}(X_s) - B(PZ_s)\|_{L^2(U,H)}^2 ds + \mathbb{E}\int_0^t (J^a L^b J^b X_s - P^a L^b P Z_s, X_s - Z_s)_H ds.
\end{aligned}
\]

Now, let \( m \to \infty \). We can pass to the limit by convergence and lower-semicontinuity on the r.h.s., whereas on the l.h.s., we shall use Lebesgue’s dominated convergence theorem and the...
Now, first let $\lambda \to 0$ and then $\delta \to 0$, where we use the bound (4.7) and assumption (M), taking into account that $Z \in L^2([0,T] \times \Omega; D(L^b) \cap S)$ and that $X^\delta \to X$ converges strongly in $L^\infty([0,T]; L^2(\Omega; H))$.

Reordering terms, we get after some cancellation,

$$
\mathbb{E}\|X_t - Z_t\|^2_H + 2\mathbb{E} \int_0^t \int_{\mathbb{T}^d} \psi(a\nabla X_s) \, d\xi ds \\
\leq \mathbb{E}\|x_n - Z_0\|^2_H + 2\mathbb{E} \int_0^t \int_{\mathbb{T}^d} \psi(a\nabla Z_s) \, d\xi ds + 2C\lambda\mathbb{E} \int_0^t \int_{\mathbb{T}^d} (1 + \psi(a\nabla X_s)) \, d\xi ds \\
- 2\mathbb{E} \int_0^t (G_s, X_s - Z_s)_H \, ds \\
+ 2\mathbb{E} \int_0^t \left( \epsilon^{4/3}\|L^a X_s\|^2_H + \epsilon^{2/3}\|X_s - Z_s\|^2_H \right) \, ds \\
+ \mathbb{E} \int_0^t \|B(J^a_s X_s - PZ_s)\|^2_{L^2(\Omega; H)} \, ds + \mathbb{E} \int_0^t (J^a_s L^b J^a_s X_s - P^* L^b PZ_s, X_s - Z_s)_H \, ds.
$$

Now, first let $\lambda \to 0$ and then $\delta \to 0$, where we use the bound (4.7) and assumption (M), taking into account that $Z \in L^2([0,T] \times \Omega; D(L^b) \cap S)$ and that $X^\delta \to X$ converges strongly in $L^\infty([0,T]; L^2(\Omega; H))$.

Reordering terms, we get after some cancellation,

$$
\mathbb{E}\|X_t - Z_t\|^2_H + 2\mathbb{E} \int_0^t \int_{\mathbb{T}^d} \psi(a\nabla X_s) \, d\xi ds \\
\leq \mathbb{E}\|x_n - Z_0\|^2_H + 2\mathbb{E} \int_0^t \int_{\mathbb{T}^d} \psi(a\nabla Z_s) \, d\xi ds + 2C\lambda\mathbb{E} \int_0^t \int_{\mathbb{T}^d} (1 + \psi(a\nabla X_s)) \, d\xi ds \\
- 2\mathbb{E} \int_0^t (G_s, X_s - Z_s)_H \, ds \\
+ 2\mathbb{E} \int_0^t \left( \epsilon^{4/3}\|L^a X_s\|^2_H + \epsilon^{2/3}\|X_s - Z_s\|^2_H \right) \, ds \\
- \mathbb{E} \int_0^t (L^b PZ_s, PX_s - J^a_s X_s)_H \, ds - \mathbb{E} \int_0^t (J^a_s L^b J^a_s X_s - P^* L^b PZ_s, X_s - Z_s)_H \, ds.
$$
Now, we can let $\varepsilon \to 0$ and use the bound \([4.7]\), so that we get together with the lower semi-continuity of $\Psi$ that
\[
\mathbb{E}\|X_t - Z_t\|^2_H + 2\mathbb{E}\int_0^t \Psi(X_s) ds \\
\leq \mathbb{E}\|x_n - Z_0\|^2_H + 2\mathbb{E}\int_0^t \int_{\Omega} \psi(a \nabla Z_s) d\xi ds \\
\quad - 2\mathbb{E}\int_0^t \langle (G_s, X_s - Z_s)_H \rangle ds \\
\quad - \mathbb{E}\int_0^t \langle (L^b PZ_s, PX_s - X_s)_H \rangle ds - \mathbb{E}\int_0^t \langle (X_s, L^b (Z_s - PZ_s))_H \rangle ds.
\]
Note that, as in particular $Z \in L^2([0, T] \times \Omega; S)$, $\int_{\Omega} \psi(a \nabla Z) d\xi = \Psi(Z)$.

Passing to $n \to \infty$ and using convergence and lower semi-continuity again, yields the existence of a time-continuous and adapted SVI-solution for equation (2.2). The existence part of Theorem 4.1 is proved.

4.5. Uniqueness. Compare with \([10, 16, 22, 23, 26]\). Let $X \in L^2([0, T] \times \Omega; H)$ be any SVI solution to (2.2) with initial datum $x \in L^2(\Omega, F_0, \mathbb{P}; H)$. Let $Z_0 = y_n$, $Z = Z^{\lambda, m, \delta, \varepsilon, n} = X^{\lambda, m, \delta, \varepsilon, n}$, the strong solution to (4.5) with initial datum $y_n \in L^2(\Omega, F_0, \mathbb{P}; S)$. Let $G = G^{\lambda, m, \varepsilon, n} = \text{div}(a^* \phi^\lambda(a \nabla Z)) + \varepsilon L^a Z$ and $P = P^\delta = J_0^\delta$. Again, we omit the indices, whenever it seems convenient. By the energy estimates (4.4) and (4.7), the integrals are finite. By the definition of SVI-solutions, we get for $t \in [0, T]$ that
\[
\mathbb{E}\|X_t - Z_t\|^2_H + 2\mathbb{E}\int_0^t \Psi(X_s) ds \\
\leq \mathbb{E}\|x - y_n\|^2_H + 2\mathbb{E}\int_0^t \Psi(Z_s) ds \\
\quad - 2\mathbb{E}\int_0^t \langle (\text{div}(a^* \phi^\lambda(a \nabla Z_s)) + \varepsilon L^a Z_s, X_s - Z_s)_H \rangle ds \\
\quad - \mathbb{E}\int_0^t \langle L^b J_0^\delta Z_s, J_0^\delta X_s - X_s)_H \rangle ds - \mathbb{E}\int_0^t \langle (X_s, L^b (Z_s - J_0^\delta Z_s))_H \rangle ds.
\]
By Lemma \([2.4]\) for all $w \in S$, we have
\[-(\text{div}(a^* \phi^\lambda(a \nabla Z)), w - Z)_H + \Psi(Z) \leq \Psi(w) + C\lambda(1 + \Psi(Z)) \hspace{1cm} \mathbb{P} \otimes ds \text{ a.e.}\]
Since $\Psi$ is the lower-semicontinuous envelope of $\tilde{\Psi} = \Psi|_S$ (i.e., $\Psi$ restricted to $S$), for a.e. $(t, \omega) \in [0, T] \times \Omega$, we can choose a sequence $w^k \in S$, $k \in \mathbb{N}$ such that $w^k \to X_s(\omega)$ in $H$ and $\Psi(w^k) \to \Psi(X_s(\omega))$.
Hence,
\[-(\text{div}(a^* \phi^\lambda(a \nabla Z)), X - Z)_H + \Psi(Z) \leq \Psi(X) + C\lambda(1 + \Psi(Z)) \hspace{1cm} \mathbb{P} \otimes ds \text{ a.e.}\]
Thus, after letting \( m \to \infty \),
\[
\mathbb{E}\|X_t - Z_t\|_H^2 \leq \mathbb{E}\|x - y_n\|_H^2 + C \int_0^t \left(1 + \Psi(Z_s)\right) ds + 2 \mathbb{E} \int_0^t \left( \varepsilon^{4/3} \|L^a Z_s\|_H^2 + \varepsilon^{2/3} \|X_s - Z_s\|_H^2 \right) ds
\]
\[
- \mathbb{E} \int_0^t (L^b J^a_s Z_s, J^a_s X_s - X_s)_H ds - \mathbb{E} \int_0^t (X_s, L^b(Z_s - J^a_s Z_s))_H ds.
\]

We can take the limit \( \lambda \to 0 \) by using the bound (1.7), which is uniform in \( \lambda \) and \( \delta \), and get that
\[
\mathbb{E}\|X_t - Z_t^\delta\|_H^2 \leq \mathbb{E}\|x - y_n\|_H^2 + 2 \mathbb{E} \int_0^t \left( \varepsilon^{4/3} \|L^a Z_s^\delta\|_H^2 + \varepsilon^{2/3} \|X_s - Z_s^\delta\|_H^2 \right) ds.
\]

Due to the bound (1.7) and assumption (M), we can use the \( \mathbb{P} \otimes ds \)-a.e. strong convergence of \( J^a_s X \to X \) in \( H \) and Lebesgue’s dominated convergence theorem, the strong convergence \( Z^\delta \to Z \) in \( L^\infty([0, T]; L^2(\Omega; H)) \) and the weak convergence of \( Z^\delta \to Z \) in \( L^2([0, T]; L^2(\Omega; D(L^b) \cap S)) \) in order to let \( \delta \to 0 \) so that the above expression converges to
\[
\mathbb{E}\|X_t - Z_t\|_H^2 \leq \mathbb{E}\|x - y_n\|_H^2 + 2 \mathbb{E} \int_0^t \left( \varepsilon^{4/3} \|L^a Z_s\|_H^2 + \varepsilon^{2/3} \|X_s - Z_s\|_H^2 \right) ds.
\]

Now, for \( \varepsilon \to 0 \), using the bound (1.7) that the expression converges to
\[
\mathbb{E}\|X_t - Z_t^\varepsilon\|_H^2 \leq \mathbb{E}\|x - y_n\|_H^2,
\]
for a.e. \( t \in [0, T] \). The bound (1.1) follows by approximating initial data \( y \in L^2(\Omega, F_0, \mathbb{P}; H) \) by \( y_n \to y \), i.e., a strongly convergent sequence in \( L^2(\Omega, F_0, \mathbb{P}; H) \) as \( n \to \infty \), and using lower semi-continuity. The uniqueness part of Theorem 4.4 is proved.

**Appendix A. Existence of approximating solutions**

Let us recall the following conditions from [24], simplified with regard to the time-dependence of the drift coefficients, which is not needed here. Suppose that \( A : S \to 2^{S^*} \) satisfies the following conditions:

(A1) The map \( x \mapsto A(x) \) is maximal monotone with non-empty values.

(A2) For all \( x \in S \), for all \( y \in A(x) \):
\[
\|y\|_{S^*} \leq C \|x\|_S.
\]

(A3) For all \( x \in S \), for all \( y \in A(x) \), and for all \( \mu > 0 \):
\[
2_{S^*} \langle y, L^{(\mu)} x \rangle_S \leq C \|x\|_S^2,
\]
such that \( C \) is independent of \( \mu \), where \( L^{(\mu)} \) denotes the Yosida-approximation of \( L \).
Let $U$ be a separable Hilbert space. Denote the space of Hilbert-Schmidt operators from $U$ to $H$ by $L_2(U, H)$. Suppose that $B : [0, T] \times \Omega \times S \to L_2(U, H)$ is progressively measurable\(^7\) and that there exist constants $C_1, C_2, C_3 > 0$ such that

(A1) There is $h \in L^1([0, T] \times \Omega)$ such that
\[
\|B_t(x)\|_{L_2(U, H)}^2 \leq C_1 \|x\|_S^2 + h_t
\]
for all $t \in [0, T)$, $x \in S$ and $\omega \in \Omega$.

(B2) There is $\|B_t(x) - B_t(y)\|_{L_2(U, H)}^2 \leq C_2 \|x - y\|_H^2$
for all $t \in [0, T)$, $x, y \in S$ and $\omega \in \Omega$.

(B3) There is $\tilde{h} \in L^1([0, T] \times \Omega)$ such that
\[
\|B_t(x)\|_{L_2(U, S)}^2 \leq C_3 \|x\|_S^2 + \tilde{h}_t
\]
for all $t \in [0, T)$, $x \in S$ and $\omega \in \Omega$.

Denote by $\{W_t\}_{t \geq 0}$ a cylindrical Wiener process in $U$ for the stochastic basis $(\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{P})$.

**Definition A.1.** We say that a continuous $\{F_t\}_{t \geq 0}$-adapted stochastic process $X : [0, T] \times \Omega \to H$ is a solution to
\[
\begin{align*}
  dX_t + A(X_t) dt &\ni B_t(X_t) dW_t, \\
  X_0 &= x,
\end{align*}
\]
if $X \in L^2(\Omega; C([0, T]; H)) \cap L^2([0, T] \times \Omega; S)$ and solves the following integral equation in $S^*$
\[
X_t = x - \int_0^t \eta_s \, ds + \int_0^t B_s(X_s) \, dW_s,
\]
$\mathbb{P}$-a.s. for all $t \in [0, T]$, where $\eta \in A(X)$, $dt \otimes \mathbb{P}$-a.s.

**Theorem A.2.** Suppose that conditions (A1)–(A3), (B1)–(B3) hold. Let $x \in L^2(\Omega, F_0, \mathbb{P}; S)$. Then there exists a unique solution in the sense of the previous definition to the equation
\[
\begin{align*}
  dX_t + A(X_t) dt &\ni B_t(X_t) dW_t, \\
  X_0 &= x,
\end{align*}
\]
that satisfies
\[
\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_S^2 < \infty.
\]

**Proof.** See [24, Theorem 4.4].

**Definition A.3.** An $\{F_t\}_{t \geq 0}$-adapted stochastic process $X \in L^2(\Omega; C([0, T]; H))$ is called a limit solution to (A.1) with starting point $x \in H$ if for all approximations $x^m \in S$, $m \in \mathbb{N}$ with $\|x^m - x\|_H \to 0$ as $m \to \infty$ and all $B^m$ satisfying (B1)–(B3) and such that $B^m(y) \to B(y)$ strongly in $L^2([0, T] \times \Omega; L_2(U, H))$ for every $y \in S$, we have that $X^m \to X$ strongly in $L^2(\Omega; C([0, T]; H))$ as $m \to \infty$.

**Theorem A.4.** Suppose that conditions (A1)–(A3), (B1)–(B2) hold. Let $x \in L^2(\Omega, F_0, \mathbb{P}; H)$. Then there exists a unique limit solution in the sense of the previous definition to the equation
\[
\begin{align*}
  dX_t + A(X_t) dt &\ni B_t(X_t) dW_t, \\
  X_0 &= x.
\end{align*}
\]

**Proof.** See [24, Theorem 4.6].

---

\(^7\)That is, for every $t \in [0, T]$ the map $B : [0, t] \times \Omega \times S \to L_2(U, H)$ is $\mathcal{B}([0, t]) \otimes F_t \otimes \mathcal{B}(S)$-measurable.
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