Resolvent Estimates in $L^p$ for the Stokes Operator in Lipschitz Domains

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Abstract

We establish the $L^p$ resolvent estimates for the Stokes operator in Lipschitz domains in $\mathbb{R}^d$, $d \geq 3$ for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2d} + \varepsilon$. The result, in particular, implies that the Stokes operator in a three-dimensional Lipschitz domain generates a bounded analytic semigroup in $L^p$ for $(3/2) - \varepsilon < p < 3 + \varepsilon$. This gives an affirmative answer to a conjecture of M. Taylor.

1 Introduction

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Consider the Dirichlet problem for the Stokes system

$$
\begin{cases}
-\Delta u + \nabla \phi + \lambda u = f & \text{in } \Omega, \\
\text{div}(u) = 0 & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1.1)

where $\lambda \in \Sigma_\theta = \{z \in \mathbb{C} : \lambda \neq 0 \text{ and } |\arg(z)| < \pi - \theta\}$ and $\theta \in (0, \pi/2)$. It is well known that for any $f \in L^2(\Omega; \mathbb{C}^d)$, there exist a unique $u \in H^1_0(\Omega; \mathbb{C}^d)$ and $\phi \in L^2(\Omega)$, unique up to constants, solving (1.1). Moreover, the solution $u$ satisfies the estimate

$$
\|u\|_{L^2(\Omega)} \leq C|\lambda|^{-1}\|f\|_{L^2(\Omega)},
$$

(1.2)

where $C$ depends only on $\theta$.

In this paper we shall be interested in the $L^p$ resolvent estimate

$$
\|u\|_{L^p(\Omega)} \leq C_p |\lambda|^{-1}\|f\|_{L^p(\Omega)}
$$

(1.3)

for $p \neq 2$. The following is the main result of the paper.

Theorem 1.1. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. There exists $\varepsilon > 0$, depending only on $d$, $\theta$ and the Lipschitz character of $\Omega$, such that if $f \in L^2(\Omega; \mathbb{C}^d) \cap L^p(\Omega; \mathbb{C}^d)$ and

$$
|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2d} + \varepsilon,
$$

(1.4)

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then the unique solution \( u \) of (1.1) in \( H_0^1(\Omega; \mathbb{C}^d) \) satisfies the estimate

\[
\| u \|_{L^p(\Omega)} \leq \frac{C_p}{|\lambda| + r_0^2} \| f \|_{L^p(\Omega)},
\]

(1.5)

where \( r_0 = \text{diam}(\Omega) \) and \( C_p \) depends at most on \( d, p, \theta \), and the Lipschitz character of \( \Omega \).

Let \( C_{0,\sigma}^\infty(\Omega) = \{ \varphi \in C_{0,\sigma}^\infty(\Omega; \mathbb{C}^d) : \text{div}(\varphi) = 0 \in \Omega \} \) and

\[
L^p_\sigma(\Omega) = \text{the closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } L^p(\Omega; \mathbb{C}^d),
\]

(1.6)

where \( 1 < p < \infty \). Let \( \mathbb{P} = \mathbb{P}_2 \) denote the orthogonal projection from \( L^2(\Omega; \mathbb{C}^d) \) onto \( L^2_\sigma(\Omega) \). If \( \Omega \) is a \( C^1 \) domain, the operator \( \mathbb{P} \) extends to a bounded operator \( \mathbb{P}_p \) on \( L^p(\Omega; \mathbb{C}^d) \) for \( 1 < p < \infty \) (see earlier work in [13] for smooth domains). It was also proved in [11] that if \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^d \) and \( d \geq 3 \), the operator \( \mathbb{P} \) extends to a bounded operator on \( L^p(\Omega; \mathbb{C}^d) \) for \( (3/2) - \varepsilon < p < 3 + \varepsilon \); and this range of \( p \) is sharp. If \( \Omega \) is smooth, the Stokes operator \( A_p \) may be defined by \( A_p = \mathbb{P}(-\Delta) \). This definition is problematic for Lipschitz domains. Here we define the Stokes operator \( A_p \) in \( L^p_\sigma(\Omega) \) by

\[
A_p(u) = -\Delta u + \nabla \phi,
\]

(1.7)

with the domain

\[
D(A_p) = \{ u \in W^{1,p}_0(\Omega; \mathbb{C}^d) : \text{div}(u) = 0 \text{ in } \Omega \text{ and } -\Delta u + \nabla \phi \in L^p_\sigma(\Omega) \text{ for some } \phi \in L^p(\Omega) \}.
\]

(1.8)

Since \( C_{0,\sigma}^\infty(\Omega) \subset D(A_p) \), the operator \( A_p \) is densely defined in \( L^p_\sigma(\Omega) \) and \( A_p(u) = \mathbb{P}(-\Delta)u \) for \( u \in C_{0,\sigma}^\infty(\Omega) \). It is not hard to see that if \( p = 2 \), the operator \( A_2 \) is self-adjoint in \( L^2_\sigma(\Omega) \) [9] [2]. One may also show that if \( p \) satisfies the condition (1.4), then \( A_p \) is a closed operator in \( L^p_\sigma(\Omega) \) (see Remark 6.4). It follows from Theorem 1.1 and Remark 6.4 that if \( p \) satisfies (1.4) and \( \lambda \in \Sigma_\theta \),

\[
\| (A_p + \lambda)^{-1}f \|_{L^p(\Omega)} \leq C |\lambda|^{-1} \| f \|_{L^p(\Omega)}
\]

for any \( f \in C_0^\infty(\Omega; \mathbb{C}^d) \) with \( \text{div}(f) = 0 \) in \( \Omega \). As a result we obtain the following.

**Corollary 1.2.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \), \( d \geq 3 \). Then there exists \( \varepsilon > 0 \), depending only on \( d \) and the Lipschitz character of \( \Omega \), such that \( -A_p \) generates a bounded analytic semigroup in \( L^p_\sigma(\Omega) \) for \( \frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon \).

Note that by Corollary 1.2, the operator \( -A_p \) generates a bounded analytic semigroup in \( L^p_\sigma(\Omega) \) for any bounded Lipschitz domain \( \Omega \) in \( \mathbb{R}^3 \) for \( (3/2) - \varepsilon < p < 3 + \varepsilon \), where \( \varepsilon > 0 \) depends on \( \Omega \). This gives an affirmative answer to a conjecture of M. Taylor [32].

There exists an extensive literature on the study of initial boundary value problems for the Navier-Stokes equations, using the functional analytical approach introduced by Fujita and Kato in [12]. The resolvent estimates for the Stokes operator \( A \) as well as the analyticity property of the semigroup generated by \( -A \) play a fundamental role in this classical approach. It has long been known that if \( \Omega \) is a bounded \( C^2 \) domain, the resolvent estimate (1.3) holds for \( \lambda \in \Sigma_\theta \) and \( 1 < p < \infty \) (see e.g. [30] [16] [34] [7]). Consequently, the operator \( -A \)
generates a bounded analytic semigroup in $L^p$ for any $1 < p < \infty$, if $\partial \Omega$ is $C^2$. The case of nonsmooth domains is more complicated. In [8] P. Deuring constructed a three-dimensional Lipschitz domain (with a narrow reentrant corner) for which the $L^p$ resolvent estimate (1.3) fails for $p$ sufficiently large. This is somewhat unexpected. Indeed it was proved in [27] that the estimate (1.3) for $1 \leq p \leq \infty$ holds in bounded Lipschitz domains in $\mathbb{R}^3$ for any second order elliptic system with constant coefficients satisfying the Legendre-Hadamard condition (the range for $p$ is $\frac{2d}{d+3} - \varepsilon < p < \frac{2d}{d-3} + \varepsilon$, if $d \geq 4$). We mention that the analyticity of the semigroup in $L^p$ generated by the Stokes operator with Hodge boundary conditions in Lipschitz domains in $\mathbb{R}^3$ was obtained in [25] for $(3/2) - \varepsilon < p < 3 + \varepsilon$. To the best of the author’s knowledge, no positive result on the resolvent estimate (1.3) in Lipschitz domains for $p \neq 2$ was known for the Stokes operator with Dirichlet condition. The main results in this paper make it possible to study the existence of mild solutions in $L^3$ of the Navier-Stokes initial value problems in nonsmooth domains in $\mathbb{R}^3$, using the classical Fujita-Kato approach (see e.g. [17] for the case of the smooth domains, and [2] [9] [32] [24] as well as their references for related work in Lipschitz domains).

We now describe our approach to the proof of Theorem 1.1. Consider the operator $T_\lambda$ on $L^2(\Omega; \mathbb{C}^d)$, defined by $T_\lambda(f) = \lambda u$, where $u \in H_0^1(\Omega; \mathbb{C}^d)$ is the unique solution to (1.1) in $\Omega$. Note that $T_\lambda$ is bounded on $L^2(\Omega; \mathbb{C}^d)$ and $\|T_\lambda\|_{L^2 \rightarrow L^2} \leq C$. To show that $T_\lambda$ is bounded on $L^p(\Omega; \mathbb{C}^d)$ and $\|T_\lambda\|_{L^p \rightarrow L^p} \leq C_p$ for $2 < p < \frac{2d}{d-1} + \varepsilon$, we appeal to a real variable argument, which may be regarded as a refined (and dual) version of the celebrated Calderón-Zygmund Lemma. According to this argument (see Lemma 6.3), which originated from [3] and further developed in [28] [29], one only needs to establish the weak reverse Hölder estimate,

$$\left( \int_{B(x_0, r) \cap \Omega} |u|^{p_d} \right)^{1/p_d} \leq C \left( \int_{B(x_0, 2r) \cap \Omega} |u|^2 \right)^{1/2}$$

(1.10)

for $p_d = \frac{3d}{d-1}$, whenever $u \in H_0^1(\Omega; \mathbb{C}^d)$ is a (local) solution of the Stokes system

$$\begin{aligned}
-\Delta u + \nabla \phi + \lambda u &= 0, \\
\text{div}(u) &= 0
\end{aligned}$$

(1.11)
in $B(x_0, 3r) \cap \Omega$ for some $x_0 \in \overline{\Omega}$ and $0 < r < c \text{diam}(\Omega)$. Here and thereafter, we use $\overline{\text{f}}_E u = \frac{1}{|E|} \int_E u$ to denote the average of $u$ over $E$. To prove (1.10), we study the $L^2$ Dirichlet problem for (1.11) in Lipschitz domains, with $\lambda \in \Sigma_\theta$. Let $n$ denote the outward unit normal to $\partial \Omega$ and $(u)^*$ the nontangential maximal function of $u$. We will show that for any $f \in L^2(\partial \Omega; \mathbb{C}^d)$ with $\int_{\partial \Omega} f \cdot n = 0$, there exists a unique $u$ and a harmonic function $\phi$ (unique up to constants) such that $(u, \phi)$ satisfies (1.11) in $\Omega$, $(u)^* \in L^2(\partial \Omega)$ and $u = f$ in the sense of nontangential convergence. More importantly, the solution $u$ satisfies the estimate $\|(u)^*\|_{L^2(\partial \Omega)} \leq C \|f\|_{L^2(\partial \Omega)}$, where $C$ depends at most on $d, \theta$, and the Lipschitz character of $\Omega$ (see Theorem 5.3). This, together with the inequality

$$\left( \int_{\Omega} |u|^{p_d} \, dx \right)^{1/p_d} \leq C \left( \int_{\partial \Omega} |(u)^*|^2 \, d\sigma \right)^{1/2},$$

(1.12)

which holds for any continuous function $u$ in $\Omega$, leads to

$$\left( \int_{\Omega} |u|^{p_d} \, dx \right)^{1/p_d} \leq C \left( \int_{\partial \Omega} |u|^2 \, d\sigma \right)^{1/2}.$$  

(1.13)
The desired estimate (1.10) follows by applying (1.13) in the domain $B(x_0, tr) \cap \Omega$ for $t \in (1, 2)$ and then integrating the resulting inequality with respect to $t$ over the interval $(1, 2)$.

Much of the paper is devoted to the solvability of the $L^2$ Dirichlet problem for the Stokes system (1.11) in Lipschitz domains by the method of layer potentials. We point out that the case $\lambda = 0$ was studied in [10], [6], where the $L^2$ Dirichlet problem as well as two Neumann type boundary value problems with boundary data in $L^2$ for the system $\Delta u = \nabla \phi$, $\text{div}(u) = 0$ in $\Omega$ was solved by the method of layer potentials, using the Rellich type estimates $\|\partial u/\partial \nu\|_{L^2(\partial \Omega)} \approx \|\nabla_{\tan} u\|_{L^2(\partial \Omega)}$ (c.f. [18], [19], [33] for harmonic functions). Here $\partial u/\partial \nu$ is a conormal derivative and $\nabla_{\tan} u$ denotes the tangential gradient of $u$ on $\partial \Omega$. In an effort to solve the $L^2$ initial boundary value problems for the nonstationary Stokes system in Lipschitz cylinders, the Stokes system (1.11) for $\lambda = i \tau$ with $\tau \in \mathbb{R}$ was considered in [26]. One of the key observations in [26] is that in the case $|\tau| \neq 0$ (and large), the Rellich estimates involve two extra terms: $|\tau|^{1/2} \|u\|_{L^2(\partial \Omega)}$ and $|\tau||n \cdot u|_{H^{-1}(\partial \Omega)}$. While the first term $|\tau|^{1/2} \|u\|_{L^2(\partial \Omega)}$ was expected in view of the Rellich estimates for the Helmholtz equation $-\Delta + i \tau$ in [1], the second term $|\tau||n \cdot u|_{H^{-1}(\partial \Omega)}$ was not (it is this second term that makes it difficult to localize $L^2$ estimates for solutions of (1.11)). Let

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial n} - \phi n$$

(1.14)

denote a conormal derivative of $u$ for (1.11). Here we shall follow the approach in [26] and provide a complete proof of the following Rellich estimates:

$$\frac{\partial u}{\partial \nu} \bigg|_{L^2(\partial \Omega)} \approx \|\nabla_{\tan} u\|_{L^2(\partial \Omega)} + |\lambda|^{1/2} \|u\|_{L^2(\partial \Omega)} + |\lambda||n \cdot u|_{H^{-1}(\partial \Omega)},$$

(1.15)

which are uniform in $\lambda$ for $\lambda \in \Sigma_\theta$ with $|\lambda| \geq c > 0$. As in the case of Laplace’s equation [33], the desired estimate $\|(u)^*\|_{L^2(\partial \Omega)} \leq C \|u\|_{L^2(\partial \Omega)}$ follows from (1.15) by the method of layer potentials.

The rest of the paper is organized as follows. In Section 2 we establish some key estimates on the matrix of fundamental solutions $\Gamma(x; \lambda)$ for (1.11) in $\mathbb{R}^d$, with pole at the origin. In Section 3 we introduce the single and layer potentials for the system (1.11) and reduce the solvability of the $L^2$ boundary value problems to the invertibility of some integral operators on $L^2(\partial \Omega; \mathbb{C}^d)$. Section 4 is devoted to the proof of the Rellich estimates (1.15), while the $L^2$ Dirichlet problem, as well as the $L^2$ Neumann type problem associated with $\frac{\partial u}{\partial \nu}$, is solved in Section 5. Finally we give the proof of Theorem 1.1 in Section 6.

For simplicity we will assume that $\partial \Omega$ is connected in Sections 4 and 5. However, we point out that this extra connectivity assumption is not needed in Theorem 1.1 as the results from Section 5 are only used in Section 6 for the domain $B(x_0, r) \cap \Omega$. We also remark that the general approach developed in this paper should work in the case $d = 2$ as well as in the case of exterior domains. But the sharp range of $p$’s for which the resolvent estimate (1.3) holds in a three-dimensional Lipschitz or $C^1$ domain is a more challenging problem.

2 Fundamental solutions of the Stokes system

In this section we study the properties of fundamental solutions for the Stokes system (1.11). Given $\lambda = re^{i\tau} \in \Sigma_\theta$ with $0 < r < \infty$ and $-\pi + \theta < \tau < \pi - \theta$, let $k = \sqrt{r}e^{i(\pi+\tau)/2}$. Then
\[ k^2 = -\lambda \text{ and } (\theta/2) < \arg(k) < \pi - (\theta/2). \] Notice that
\[ \text{Im}(k) > \sin(\theta/2) \sqrt{|\lambda|}. \] (2.1)

A fundamental solution for the (scalar) Helmholtz equation \(-\Delta u + \lambda u = 0\) in \(\mathbb{R}^d\), with pole at the origin, is given by
\[ G(x; \lambda) = \frac{i}{4(2\pi)^{d/2}} \cdot \frac{1}{|x|^{d-2}} \cdot (k|x|)^{d-1} H^{(1)}_{d-1}(k|x|) \] (2.2)
(see e.g. [23, p.282]). The function \(H^{(1)}_{\nu}(z)\) in (2.2) is the Hankel function \(J_{\nu}(z) + iY_{\nu}(z)\), which may be written as
\[ H^{(1)}_{\nu}(z) = \frac{2^{\nu+1}e^{i(z-\nu\pi)}z^\nu}{i\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{2isz} s^{\nu-\frac{1}{2}} (1 + s)^{\nu-\frac{1}{2}} ds, \] (2.3)
if \(\nu > -(1/2)\) and \(0 < \text{arg}(z) < \pi\) (see [22, p.120]). Note that if \(d = 3\), one has a simple formula:
\[ G(x; \lambda) = \frac{e^{ik|x|}}{4\pi |x|}. \] (2.4)

**Lemma 2.1.** Let \(\lambda \in \Sigma_\theta\). Then
\[ |\nabla^\ell_x G(x; \lambda)| \leq \frac{C_\ell e^{-c \sqrt{|\lambda|}|x|}}{|x|^{d-2+\ell}} \] (2.5)
for any integer \(\ell \geq 0\), where \(c > 0\) depends only on \(\theta\) and \(C_\ell\) depends only on \(d\), \(\ell\) and \(\theta\).

**Proof.** It follows from (2.3) that
\[ |H^{(1)}_{\nu}(z)| \leq Ce^{-\text{Im}(z)} |z|^{\nu} \int_0^\infty e^{-2s\text{Im}(z)} s^{\nu-\frac{1}{2}} (1 + s)^{\nu-\frac{1}{2}} ds \] (2.6)
if \(\nu \geq (1/2)\) and \(\text{Im}(z) > 0\). In view of (2.2) this gives
\[ |G(x; \lambda)| \leq C|x|^{2-d} e^{-\text{Im}(k)|x|} \leq C|x|^{2-d} e^{-c \sqrt{|\lambda||x|}}, \] (2.7)
where we have used (2.1). Thus we have proved (2.5) for the case \(\ell = 0\). The general case may be proved inductively by using the relation
\[ \frac{d}{dz} \{z^{-\nu} H^{(1)}_{\nu}(z)\} = -z^{-\nu} H^{(1)}_{\nu+1}(z) \] (see e.g. [22, p.108]). Since \(\Delta_x G(x; \lambda) = \lambda G(x; \lambda)\) in \(\mathbb{R}^d \setminus \{0\}\), one may also establish the estimate (2.5) for \(\ell \geq 1\) inductively, using the interior estimate
\[ |\nabla^\ell w(x)| \leq Cr^{-\ell} \sup_{B(x,r)} |w| + C \max_{0 \leq j \leq \ell-1} \sup_{B(x,r)} r^{j-\ell+2} |\nabla^j f| \] (2.8)
for solutions of \(\Delta w = f\) in \(B(x, r)\). We omit the details. \(\square\)
Let \( \nu = \frac{d}{2} - 1 \). Using the series expansions for the Bessel functions \( J_\nu(z) \) and \( Y_\nu(z) \) (see e.g. [22 Chapter 5]), one may deduce the following asymptotic expansions for the function \( z^\nu H^{(1)}_\nu(z) \) in \( \{ z \in \mathbb{C} : |z| < (1/2) \) and \( z \notin (-1/2, 0) \}:

\[
\begin{align*}
z^\nu H^{(1)}_\nu(z) &= \frac{2^\nu \Gamma(\nu)}{i\pi} + \frac{i z^2 \log z + \omega z^3 + O(|z|^4 \log|z|)}{\pi} \quad \text{for some } \omega \in \mathbb{C}, \text{ if } d = 4, \quad (2.9) \\
z^\nu H^{(1)}_\nu(z) &= \frac{2^\nu \Gamma(\nu - 1)}{4\pi i} + \frac{2^\nu \Gamma(\nu - 1) z^2 + \omega z^3 + O(|z|^4 \log|z|)}{\pi} \quad \text{for some } \omega \in \mathbb{C}, \text{ if } d = 5, \quad (2.10) \\
z^\nu H^{(1)}_\nu(z) &= \frac{2^\nu \Gamma(\nu)}{i\pi} + \frac{2^\nu \Gamma(\nu - 1) z^2 + O(|z|^4 \log|z|)}{\pi} \quad \text{if } d = 6, \quad (2.11) \\
z^\nu H^{(1)}_\nu(z) &= \frac{2^\nu \Gamma(\nu - 1)}{4\pi i} + \frac{2^\nu \Gamma(\nu - 1) z^2 + O(|z|^4 \log|z|)}{\pi} \quad \text{if } d \geq 7. \quad (2.12)
\end{align*}
\]

Let \( G(x; 0) = c_d |x|^{2-d} \) denote the fundamental solution for \(-\Delta\) in \( \mathbb{R}^d \), with pole at the origin, where

\[
c_d = \frac{1}{(d-2)\omega_d} = \frac{\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{\frac{d}{2}}},
\]

and \( \omega_d = |\mathbb{S}^{d-1}|. \) Let

\[
a_d := \lim_{z \in (-1, 0)} |z|^{-1} H^{(1)}_{\frac{d}{2} - 1}(z) = \frac{2^\nu \Gamma\left(\frac{d}{2} - 1\right)}{i\pi}. \quad (2.13)
\]

Then

\[
G(x; \lambda) - G(x; 0) = \frac{i}{4(2\pi)^{\frac{d}{2}} - 1} \cdot \frac{1}{|x|^{d-2}} \left\{(k|x|)^{\frac{d}{2} - 1} H^{(1)}_{\frac{d}{2} - 1}(k|x|) - a_d\right\}. \quad (2.14)
\]

**Lemma 2.2.** Let \( \lambda \in \Sigma_\theta. \) Then

\[
|\nabla_x^\ell \{G(x; \lambda) - G(x; 0)\}| \leq C |\lambda||x|^{4-d-\ell} \quad (2.15)
\]

if \( d \geq 5 \) and \( \ell \geq 0, \) where \( C \) depends only on \( d, \ell \) and \( \theta. \) If \( d = 3 \) or \( 4, \) estimate \( (2.13) \) holds for \( \ell \geq 1. \)

**Proof.** In view of Lemma 2.1, we may assume that \( |\lambda||x|^2 < (1/2)\). Let \( w(x) = G(x; \lambda) - G(x; 0) \). Note that \( \Delta_x w = \lambda G(x; \lambda) \) in \( \mathbb{R}^d \setminus \{0\}. \) Using the interior estimate \( (2.8) \) and Lemma 2.1, we see that it suffices to prove \( (2.15) \) in two cases: (1) \( d \geq 5 \) and \( \ell = 0; \) (2) \( d = 3 \) or \( 4 \) and \( \ell = 1. \)

We first consider the case that \( d \geq 5 \) and \( \ell = 0. \) By \( (2.14) \) and the mean value theorem,

\[
|G(x; \lambda) - G(x; 0)| \leq C |x|^{2-d} \cdot |k| |x| \max_{|z| \leq |k||x|} \left| \frac{d}{dz} \left( \frac{z^{\frac{d}{2} - 1} H^{(1)}_{\frac{d}{2} - 1}(z)}{z^{\frac{d}{2} - 1} H^{(1)}_{\frac{d}{2} - 1}(z)} \right) \right|
\]

\[
= C |x|^{2-d} \cdot |k| |x| \max_{|z| \leq |k||x|} \left| \frac{d}{dz} \left( \frac{z^{\frac{d}{2} - 1} H^{(1)}_{\frac{d}{2} - 1}(z)}{z^{\frac{d}{2} - 1} H^{(1)}_{\frac{d}{2} - 1}(z)} \right) \right|, \quad (2.16)
\]
where the last equality follows from the relation
\[
\frac{d}{dz} \left\{ z^{\nu} H^{(1)}_{\nu}(z) \right\} = z^{\nu} H^{(1)}_{\nu-1}(z) \tag{2.17}
\]
(see [22, p.108]). Since \( |z^{\nu} H^{(1)}_{\nu}(z)| \leq C_\nu \) for \( \nu > 0 \) and \( |z| \leq 1 \) with \( \text{Im}(z) > 0 \), it follows from (2.16) that
\[
|G(x; \lambda) - G(0; \lambda)| \leq C|x|^{2-d} \cdot |k||x| \cdot |k||x| = C|\lambda||x|^{4-d}.
\]
Next we consider the case that \( d = 4 \) and \( \ell = 1 \). Note that by (2.9),
\[
\left| \frac{d}{dz} \left\{ z H^{(1)}_{1}(z) - a_4 \right\} \right| \leq C |z|^{-1} \tag{2.18}
\]
for any \( |z| \leq (1/2) \) with \( \text{Im}(z) > 0 \). Since
\[
G(x; \lambda) - G(x; 0) = C(z H^{(1)}_{1}(z) - a_4),
\]
where \( z = k|x| \), it follows from (2.18) that
\[
|\nabla_x \{ G(x; \lambda) - G(x; 0) \}| \leq C|\lambda||x|^{-1}.
\]
Finally, we note that the case \( d = 3 \) and \( \ell = 1 \) may be handled by a direct calculation, using the simple formula (2.3) and the observation
\[
\frac{\partial}{\partial x_j} \left\{ e^{i|k|x|} \frac{|x| - 1}{|x|} \right\} = \frac{\partial}{\partial x_j} \left\{ e^{i|k|x|} - 1 - i|k||x| \right\}.
\]
This completes the proof. \( \square \)

**Remark 2.3.** It is not hard to see that if \( |\lambda||x|^2 \leq (1/2) \),
\[
|G(x; \lambda) - G(x; 0)| \leq \begin{cases} 
C \sqrt{|\lambda|} & \text{if } d = 3, \\
C|\lambda| \{ |\log(|\lambda||x|^2)| + 1 \} & \text{if } d = 4. \tag{2.19}
\end{cases}
\]

We now introduce a matrix of fundamental solutions \( \Gamma(x; \lambda) = (\Gamma_{\alpha\beta}(x; \lambda))_{d \times d} \) for the Stokes system (1.11) in \( \mathbb{R}^d \), with pole at the origin, where \( \lambda \in \Sigma_0 \) and
\[
\Gamma_{\alpha\beta}(x; \lambda) = G(x; \lambda) \delta_{\alpha\beta} - \frac{1}{\lambda} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \{ G(x; \lambda) - G(x; 0) \}. \tag{2.20}
\]
Let
\[
\Phi_{\beta}(x) = -\frac{\partial}{\partial x_\beta} \{ G(x; 0) \} = \frac{x_\beta}{\omega_d |x|^d}. \tag{2.21}
\]
Using \( \Delta_x G(x; \lambda) = \lambda G(x; \lambda) \) in \( \mathbb{R}^d \setminus \{0\} \), it is easy to see that for each \( 1 \leq \beta \leq d \),
\[
\begin{cases} 
( -\Delta_x + \lambda) \Gamma_{\alpha\beta}(x; \lambda) + \frac{\partial}{\partial x_\alpha} \{ \Phi_{\beta}(x) \} = 0 & \text{for } 1 \leq \alpha \leq d, \\
\frac{\partial}{\partial x_\alpha} \{ \Gamma_{\alpha\beta}(x; \lambda) \} = 0
\end{cases} \tag{2.22}
\]
in \( \mathbb{R}^d \setminus \{0\} \), where the summation convention is used in the second equation.
Theorem 2.4. Let $\lambda \in \Sigma_\theta$. Then, for any $d \geq 3$ and $\ell \geq 0$,

$$|\nabla_x^\ell \Gamma(x; \lambda)| \leq \frac{C}{(1 + |\lambda||x|^2)|x|^{d-2+\ell}}, \quad (2.23)$$

where $C$ depends only on $d$, $\ell$ and $\theta$.

Proof. This follows easily from Lemma 2.1 if $|\lambda||x|^2 > 1$, and from Lemmas 2.1 and 2.2 if $|\lambda||x|^2 \leq 1$. \qed

If $\lambda = 0$, a matrix of fundamental solutions for (1.11) in $\mathbb{R}^d$, with pole at the origin, is given by $\Gamma(x; 0) = (\Gamma_{\alpha\beta}(x; 0))_{d \times d}$, where

$$\Gamma_{\alpha\beta}(x; 0) = \frac{1}{2\omega_d} \left\{ \frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{x_\alpha x_\beta}{|x|^d} \right\}, \quad (2.24)$$

and $\omega_d = |S^{d-1}|$ (see [21] [10]). Using

$$\frac{x_\alpha x_\beta}{|x|^d} = \frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{1}{(d-4)(d-2)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left( \frac{1}{|x|^{d-4}} \right)$$

for $d \geq 5$ or $d = 3$, we may rewrite $\Gamma_{\alpha\beta}(x; 0)$ as

$$\Gamma_{\alpha\beta}(x; 0) = G(x; 0)\delta_{\alpha\beta} + \frac{1}{2\omega_d(d-4)(d-2)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left( \frac{1}{|x|^{d-4}} \right). \quad (2.25)$$

Similarly, for $d = 4$, one has

$$\Gamma_{\alpha\beta}(x; 0) = G(x; 0)\delta_{\alpha\beta} - \frac{1}{8\pi^2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (\log |x|). \quad (2.26)$$

Theorem 2.5. Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \leq (1/2)$. Then

$$|\nabla_x \{ \Gamma(x; \lambda) - \Gamma(x; 0) \}| \leq \begin{cases} C|\lambda||x|^{3-d} & \text{if } d \geq 7 \text{ or } d = 5, \\ C|\lambda||x|^{3-d} \log(|\lambda||x|^2)| & \text{if } d = 4 \text{ or } 6, \\ C\sqrt{|\lambda||x|^{-1}} & \text{if } d = 3, \end{cases} \quad (2.27)$$

where $C$ depends only on $d$ and $\theta$.

Proof. The proof uses the asymptotic expansions (2.9)-(2.12). We first consider the case $d \neq 4$. In view of (2.25) we have

$$\Gamma_{\alpha\beta}(x; \lambda) - \Gamma_{\alpha\beta}(x; 0) = \{G(x; \lambda) - G(x; 0)\}\delta_{\alpha\beta} - \frac{1}{\lambda} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left\{ G(x; \lambda) - G(x; 0) + \frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}} \right\}.$$
If \( d = 3 \), the desired estimate may be obtained by a direct computation. If \( d \geq 5 \), the estimate \( |\nabla_x \{ G(x; \lambda) - G(x; 0) \}| \leq C|\lambda||x|^{3-d} \) is contained in Lemma 2.2. To handle the second term above for \( d \geq 5 \), we use (2.14) to obtain

\[
G(x; \lambda) - G(x; 0) = \frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}} + \frac{i}{4(2\pi)^{d/2}} \left\{ z^{d/2-1} H_{d/2-1}^{(1)}(z) - a_d - b_d z^2 \right\}
\]

where \( z = k|x| \), \( a_d \) is given by (2.13), and

\[
b_d = -\frac{2i(2\pi)^{d/2-1}}{\omega_d(d-4)(d-2)} \frac{2\pi^{-1} \Gamma(\frac{d}{2} - 2)}{4\pi i}.
\]

Note that by (2.12), if \( d \geq 7 \),

\[
|\frac{d^\ell}{dz^\ell} \left\{ z^{d/2-1} H_{d/2-1}^{(1)}(z) - a_d - b_d z^2 \right\}| \leq C|z|^{4-\ell}
\]

for \( 0 \leq \ell \leq 3 \), \( |z| < (1/2) \) and \( \text{Im}(z) > 0 \). In view of (2.28), this implies that

\[
|\nabla_x \{ \Gamma(x; \lambda) - \Gamma(x; 0) \}| \leq C|\lambda||x|^{3-d}.
\]

(2.30)

If \( d = 6 \), we may use (2.11) to obtain

\[
|\frac{d^\ell}{dz^\ell} \left\{ z^{d/2-1} H_{d/2-1}^{(1)}(z) - a_d - b_d z^2 \right\}| \leq C|z|^{4-\ell} \log |z|,
\]

(2.31)

for \( 0 \leq \ell \leq 3 \), \( |z| < (1/2) \) and \( \text{Im}(z) > 0 \). This gives

\[
|\nabla_x \{ \Gamma(x; \lambda) - \Gamma(x; 0) \}| \leq C|\lambda||x|^{3-d} \log |\lambda||x|^{2}.
\]

(2.32)

In the case of \( d = 5 \) we may write

\[
\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left\{ G(x; \lambda) - G(x; 0) + \frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}} \right\}
\]

\[
= \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left\{ \frac{i}{4(2\pi)^{d/2}} \frac{1}{|x|^3} \left[ z^{d/2} H_{d/2}^{(1)}(z) - a_5 - b_5 z^2 - wz^3 \right] \right\}
\]

for any constant \( w \in \mathbb{C} \), where \( z = k|x| \). In view of (2.10) this leads to the estimate (2.30), as in the case \( d \geq 7 \).

Finally, the case \( d = 4 \) may be treated in a similar manner. Note that by (2.20),

\[
\Gamma_{\alpha\beta}(x; \lambda) - \Gamma_{\alpha\beta}(x; 0)
\]

\[
= \{ G(x; \lambda) - G(x; 0) \} \delta_{\alpha\beta} - \frac{1}{\lambda \partial x_\alpha \partial x_\beta} \left\{ G(x; \lambda) - G(x; 0) - \frac{\lambda \log |x|}{8\pi^2} \right\}.
\]

\[
= \{ G(x; \lambda) - G(x; 0) \} \delta_{\alpha\beta} - \frac{i}{\lambda \partial x_\alpha \partial x_\beta} \left\{ \frac{1}{8\pi^2 |x|^2} \left[ z H_{1}^{(1)}(z) - a_4 - wz^2 - b_4 z^2 \log z \right] \right\},
\]
where \( z = k|x|, \) \( b_4 = i/\pi \) and \( w \in \mathbb{C} \) is an arbitrary constant. By (2.9) there exists \( w \in \mathbb{C} \) such that
\[
\left| \frac{d^\ell}{dz} \left\{ zH_1^{(1)}(z) - a_4 - wz^2 - b_4 z^2 \log z \right\} \right| \leq C|z|^{4-\ell} \log |z|
\] (2.33)
for \( 0 \leq \ell \leq 3, |z| < (1/2) \) and \( \text{Im}(z) > 0 \). This is enough to show (2.32) and thus completes the proof. \( \square \)

3 Layer potentials for the Stokes system

In this section we study the properties of the single and double layer potentials for the Stokes system (1.11). Throughout this section we will assume that \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^d, d \geq 3 \) and \( 1 < p < \infty \). Also, the summation convention will be used in the rest of the paper.

Let \( \lambda \in \Sigma_\theta \). Given \( f \in L^p(\partial\Omega; \mathbb{C}^d) \), the single layer potential \( u = S_\lambda(f) \) is defined by
\[
u_j(x) = \int_{\partial\Omega} \Gamma_{jk}(x - y; \lambda)f_k(y) \, d\sigma(y), \tag{3.1}
\]
where \( \Gamma_{jk} \) is given by (2.20). Let
\[
\phi(x) = \int_{\partial\Omega} \Phi_k(x - y)f_k(y) \, d\sigma(y), \tag{3.2}
\]
where \( \Phi_k \) is given by (2.21). It follows from (2.22) that \( (u, \phi) \) is a solution of (1.11) in \( \mathbb{R}^d \setminus \partial\Omega \).

Define
\[
T_\lambda^*(f)(P) = \sup_{t > 0} \left| \int_{\partial\Omega} \nabla_x \Gamma(P - y; \lambda)f(y) \, d\sigma(y) \right|, \tag{3.3}
\]
\[
T_\lambda(f)(P) = \text{p.v.} \int_{\partial\Omega} \nabla_x \Gamma(P - y; \lambda)f(y) \, d\sigma(y)
\]
for \( P \in \partial\Omega \).

**Lemma 3.1.** Let \( 1 < p < \infty \) and \( T_\lambda(f), T_\lambda^*(f) \) be defined by (3.3). Then \( T_\lambda(f)(P) \) exists for a.e. \( P \in \partial\Omega \) and
\[
\|T_\lambda(f)\|_{L^p(\partial\Omega)} \leq \|T_\lambda^*(f)\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}, \tag{3.4}
\]
where \( C_p \) depends only on \( d, \theta, p, \) and the Lipschitz character of \( \Omega \).

**Proof.** The lemma is known in the case \( \lambda = 0 \) [10], and is a consequence of the theorem of Coifman, McIntosh, and Meyer [4]. The case \( \lambda \in \Sigma_\theta \) follows from the case \( \lambda = 0 \), by using the estimates in Theorems 2.4 and 2.5. Indeed, if \( t^2|\lambda| \geq (1/2) \), we may use Theorem 2.4 to obtain
\[
\left| \int_{|y-P|>t} \nabla_x \Gamma(P - y; \lambda)f(y) \, d\sigma(y) \right| \leq C \int_{|P-y|>t} \frac{|f(y)| \, d\sigma(y)}{|P - y|^{d+1}} \leq C \mathcal{M}_{\partial\Omega}(f)(P),
\]
\]
Lemma 3.2. Let \( \Omega \) be given by (3.1)-(3.2) with 
\[
M_{\partial \Omega}(f) = \left\| \frac{1}{T_0(f)(P)} + C M_{\partial \Omega}(f)(P),
\right.
\]
where we have used the estimates in Theorems 2.4 and 2.5 for the last inequality. It follows that 
\[
\|T^*_\lambda(f)\|_{L^p(\partial \Omega)} \leq C_p \|f\|_{L^p(\partial \Omega)}.
\]

Finally we note that \( \nabla \{ (P - y; \lambda) - (P - y; 0) \} \) is integrable on \( \partial \Omega \). It follows that 
for \( f \in C^\infty_0(\mathbb{R}^d; \mathbb{C}^d) \), \( T_\lambda(f)(P) \) exists whenever \( T_0(f)(P) \) exists. Since \( T_0(f)(P) \) exists for a.e. \( P \in \partial \Omega \), by the boundedness of \( T^*_\lambda \) on \( L^p(\partial \Omega) \), we may conclude that \( T_\lambda(f)(P) \) exists for a.e. \( P \in \partial \Omega \), if \( f \in L^p(\partial \Omega; \mathbb{C}^d) \).

For a function in \( \Omega \), the nontangential maximal function \((u)^*\) is defined by 
\[
(u)^*(P) = \sup \{|u(x)| : x \in \Omega \text{ and } |x - P| < C \text{dist}(x, \partial \Omega)\}
\]  
(3.5)
for \( P \in \partial \Omega \), where \( C > 2 \) is a fixed and sufficiently large constant depending only on \( d \) and the Lipschitz character of \( \Omega \).

**Lemma 3.2.** Let \( 1 < p < \infty \) and \((u, \phi)\) be given by (3.1)-(3.2). Then 
\[
\|(\nabla u)^*\|_{L^p(\partial \Omega)} + \|(|(u)^*|)^{1/2}\|_{L^p(\partial \Omega)} \leq C_p \|f\|_{L^p(\partial \Omega)},
\]  
(3.6)
where \( C_p \) depends only on \( d, \theta, p, \) and the Lipschitz character of \( \Omega \).

**Proof.** The estimate \( \|\phi\|^*\|_{L^p(\partial \Omega)} \leq C_p \|f\|_{L^p(\partial \Omega)} \) is well known (see e.g. [33]). The proof for 
\( \|\nabla u\|^*\|_{L^p(\partial \Omega)} \leq C_p \|f\|_{L^p(\partial \Omega)} \) follows the same line of argument, using Lemma 3.1 and the estimate 
\( |\nabla^2_\lambda \Gamma(x; \lambda)| \leq C|x|^{-d} \). Note that 
\[
(u)^*(P) \leq C \int_{\partial \Omega} e^{-c \sqrt{|\lambda|}|P - y|} |f(y)| \, d\sigma(y)
\]
for any \( P \in \partial \Omega \). This implies that \( \|(u)^*\|_{L^p(\partial \Omega)} \leq C|\lambda|^{-1/2} \|f\|_{L^p(\partial \Omega)}. \]

**Lemma 3.3.** Let \((u, \phi)\) be given by (3.1)-(3.2) with \( f \in L^p(\partial \Omega; \mathbb{C}^d) \) and \( 1 < p < \infty \). Then 
\[
\left( \frac{\partial u}{\partial x_j} \right)_{\pm} (x) = \pm \frac{1}{2} \{ n_j(x)f_i(x) - n_i(x)n_j(x)n_k(x)f_k(x) \}
\]  
\[]^+\ \text{p.v.} \int_{\partial \Omega} \frac{\partial}{\partial x_j} \left\{ \Gamma_{ik}(x - y; \lambda) \right\} f_k(y) \, d\sigma(y),
\]  
(3.7)
\[
\phi_{\pm}(x) = \mp \frac{1}{2} n_k(x)f_k(x) + \text{p.v.} \int_{\partial \Omega} \Phi_k(x - y)f_k(y) \, d\sigma(y)
\]
for a.e. \( x \in \partial \Omega \), where the subscripts + and − indicate nontangential limits taken inside \( \Omega \) and outside \( \overline{\Omega} \), respectively.
The main goal of the next two sections is to show that the operator norms of their inverses on \( \lambda \) readily from the case given by (3.1)-(3.2) with \( \pm \) and Dirichlet problems for the Stokes system (1.11) to the invertibility of the operators (1.11), where \( \lambda \) is a bounded operator on \( L^p(\partial \Omega; \mathbb{C}^d) \). In fact the invertibility of these operators on \( \lambda \) on \( \partial \Omega \), where \( \lambda \) is a bounded operator on \( L^p(\partial \Omega; \mathbb{C}^d) \). Moreover, \( \|K_\lambda(f)\|_{L^p(\partial \Omega)} \leq C_p \|f\|_{L^p(\partial \Omega)} \), where \( C_p \) depends only on \( d, \theta, p \), and the Lipschitz character of \( \Omega \).

**Proof.** As in the case \( \lambda = 0 \), this follows readily from Lemmas 3.3 and 3.2

Next we introduce the double layer potential \( u(x) = D_\lambda(f)(x) \) for the Stokes system (1.11), where

\[
    u_j(x) = \int_{\partial \Omega} \left\{ \frac{\partial}{\partial y_j} \{\Gamma_j(y - x; \lambda)\} n_i(y) - \Phi_j(y - x) n_k(y) \right\} f_k(y) d\sigma(y). \tag{3.9}
\]

Let

\[
    \phi(x) = \frac{\partial^2}{\partial x_i \partial x_k} \int_{\partial \Omega} G(y - x; 0) n_i(y) f_k(y) d\sigma(y) \tag{3.10}
\]

\[
    + \lambda \int_{\partial \Omega} G(y - x; 0) n_k(y) f_k(y) d\sigma(y).
\]

Using (2.22) and (2.21), it is not hard to verify that \( (u, \phi) \) is a solution of (1.11) in \( \mathbb{R}^d \setminus \partial \Omega \).

**Theorem 3.5.** Let \( \lambda \in \Sigma_{\theta} \) and \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \), \( d \geq 3 \). Let \( u \) be given by (3.9) with \( f \in L^p(\partial \Omega; \mathbb{C}^d) \) and \( 1 < p < \infty \). Then (1) \( \|(u)^*\|_{L^p(\partial \Omega)} \leq C_p \|f\|_{L^p(\partial \Omega)} \), where \( C_p \) depends only on \( d, p, \theta \), and the Lipschitz character of \( \Omega \); (2)

\[
    u_+ = \left( \mp (1/2) I + K_\lambda^* \right) f, \tag{3.11}
\]

where \( K_\lambda^* \) is the adjoint of the operator \( K_\lambda \) in (3.8).

**Proof.** The estimate of \( (u)^* \) follows from Lemma 3.2 while the trace formula (3.11) follows from Lemma 3.3.

As in the case \( \lambda = 0 \), Theorems 3.4 and 3.5 reduce the solvability of the \( L^p \) Neumann and Dirichlet problems for the Stokes system (1.11) to the invertibility of the operators \( \pm (1/2) I + K_\lambda \) on \( L^p(\partial \Omega; \mathbb{C}^d) \). In fact the invertibility of these operators on \( L^p(\partial \Omega; \mathbb{C}^d) \) follow readily from the case \( \lambda = 0 \) in [10], as \( K_\lambda - K_\theta \) is compact on \( L^p(\partial \Omega; \mathbb{C}^d) \) for any \( p \). The main goal of the next two sections is to show that the operator norms of their inverses on \( L^p(\partial \Omega; \mathbb{C}^d) \) are bounded by constants independent of \( \lambda \in \Sigma_{\theta} \).
4 Rellich estimates

In this section we establish Rellich type estimates for the Stokes system (1.11). Throughout this section we will assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$ with connected boundary and $|\partial \Omega| = 1$. Recall that $\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial n} - \phi n$ and $n$ denotes the outward unit normal to $\partial \Omega$. We will use $\| \cdot \|_\partial$ to denote the norm in $L^2(\partial \Omega)$.

The goal of this section is to prove the following.

Theorem 4.1. Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0, 1)$. Let $(u, \phi)$ be a solution of (1.11) in $\Omega$. Suppose that $(\nabla u)^* \in L^2(\partial \Omega)$ and $(\phi)^* \in L^2(\partial \Omega)$. We further assume that $\nabla u, \phi$ have nontangential limits a.e. on $\partial \Omega$. Then

\[
\|\nabla u\|_\partial + \|\phi - \int_{\partial \Omega} \phi \|_\partial \leq C \left\{ \|\nabla u\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda| \|n\|_{H^{-1}(\partial \Omega)} \right\}
\]

and

\[
\|\nabla u\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda| \|n\|_{H^{-1}(\partial \Omega)} + \|\phi\|_\partial \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial,
\]

where $C$ depends only on $d$, $\tau$, $\theta$, and the Lipschitz character of $\Omega$.

We begin with two Rellich type identities for the Stokes system (1.11).

Lemma 4.2. Under the same conditions on $(u, \phi)$ as in Theorem 4.1, we have

\[
\int_{\partial \Omega} h_k n_k |\nabla u|^2 d\sigma = 2\text{Re} \int_{\partial \Omega} h_k \frac{\partial u_i}{\partial x_k} \left( \frac{\partial u_i}{\partial \nu} \right) d\sigma + \int_{\Omega} \text{div}(h) |\nabla u|^2 dx
\]

\[
- 2\text{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} dx + 2\text{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_k} dx
\]

\[
- 2\text{Re} \int_{\Omega} h_k \frac{\partial u_i}{\partial x_k} \cdot \bar{\lambda} \bar{u}_i dx,
\]

and

\[
\int_{\partial \Omega} h_k n_k |\nabla u|^2 d\sigma = 2\text{Re} \int_{\partial \Omega} h_k \frac{\partial u_i}{\partial x_j} \left\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \right\} d\sigma
\]

\[
+ 2\text{Re} \int_{\partial \Omega} h_k \bar{\phi} \left\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \right\} d\sigma - \int_{\Omega} \text{div}(h) |\nabla u|^2 dx
\]

\[
+ 2\text{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_i} dx - 2\text{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_i} dx
\]

\[
+ 2\text{Re} \int_{\Omega} h_k \frac{\partial u_i}{\partial x_k} \cdot \bar{\lambda} \bar{u}_i dx,
\]

where $h = (h_1, \ldots, h_d) \in C^1_0(\mathbb{R}^d, \mathbb{R}^d)$ and $\bar{u}$ denotes the complex conjugate of $u$.

Proof. The identities (4.3) and (4.4) follow from several applications of integration by parts, using (1.11). We refer the reader to [10] for the case $\lambda = 0$ and to [26] for the case $\lambda = i \tau$. Note that with the assumptions that $(\nabla u)^* , (p)^* \in L^2(\partial \Omega)$ and that $\nabla u, p$ have nontangential limits a.e. on $\partial \Omega$, the integration by parts may be justified by an approximation argument, as in [33] [10] [26]. We omit the details. \qed
The next lemma is needed to handle the solid integrals in (4.3) and (4.4).

**Lemma 4.3.** Under the same assumptions on \((u, \phi)\) and \(\lambda\) as in Theorem 4.1, we have

\[
\int_{\Omega} |
abla u|^2 \, dx + |\lambda| \int_{\Omega} |u|^2 \, dx \leq C \| \frac{\partial u}{\partial \nu} \|_{\partial \Omega} \| u \|_{\partial \Omega},
\]

where \(C\) depends only on \(\theta\).

**Proof.** It follows from (1.11) and integration by parts that

\[
\int_{\Omega} |
abla u|^2 \, dx + \lambda \int_{\Omega} |u|^2 \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \, d\sigma.
\]

(4.6)

By taking the real and imaginary parts of (4.6) we obtain

\[
\int_{\Omega} |
abla u|^2 \, dx + \{ \text{Re}(\lambda) + \alpha |\text{Im}(\lambda)| \} \int_{\Omega} |u|^2 \, dx \leq (1 + \alpha) \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \, d\sigma
\]

(4.7)

for any \(\alpha > 0\). Observe that there exist \(\alpha, c > 0\), depending only on \(\theta\), such that \(\text{Re}(\lambda) + \alpha |\text{Im}(\lambda)| \geq c |\lambda|\) for any \(\lambda \in \Sigma_\theta\). Hence,

\[
\int_{\Omega} |
abla u|^2 \, dx + |\lambda| \int_{\Omega} |u|^2 \, dx \leq C \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \, d\sigma,
\]

(4.8)

from which the estimate (4.5) follows by the Cauchy inequality.

We now combine (4.3) and (4.4) with the estimate (4.5).

**Lemma 4.4.** Under the same assumptions on \((u, \phi)\) and \(\lambda\) as in Theorem 4.1, we have

\[
\| \nabla u \|_{\partial \Omega} \leq C \varepsilon \| \frac{\partial u}{\partial \nu} \|_{\partial \Omega} + \varepsilon \left\{ \| \nabla u \|_{\partial \Omega} + \| \phi \|_{\partial \Omega} + \| |\lambda|^{1/2} u \|_{\partial \Omega} \right\}
\]

(4.9)

and

\[
\| \nabla u \|_{\partial \Omega} \leq C \varepsilon \left\{ \| \nabla \tan u \|_{\partial \Omega} + \| |\lambda|^{1/2} u \|_{\partial \Omega} \right\} + \varepsilon \left\{ \| \nabla u \|_{\partial \Omega} + \| \phi \|_{\partial \Omega} \right\}
\]

(4.10)

for any \(\varepsilon \in (0, 1)\), where \(C_\varepsilon\) depends only on \(d, \theta, \tau, \varepsilon, \) and the Lipschitz character of \(\Omega\).

**Proof.** We start by choosing a vector field \(h = (h_1, \ldots, h_d) \in C^1_0(\mathbb{R}^d, \mathbb{R}^d)\) such that \(h_k n_k \geq c > 0\) on \(\partial \Omega\). In view of (4.3) this implies that

\[
\| \nabla u \|_{\partial \Omega} \leq C \varepsilon \left\{ \| \nabla u \|_{\partial \Omega} + \| \phi \|_{\partial \Omega} + \| |\lambda|^{1/2} u \|_{\partial \Omega} \right\}
\]

(4.11)

where we also used the Cauchy inequality. Since \(\Delta \phi = 0\) in \(\Omega\) and \((\phi)^* \in L^2(\partial \Omega)\), it follows from [5] that

\[
\int_{\Omega} |\phi|^2 \, dx \leq C \| (\phi)^* \|^2_{\partial \Omega} \leq C \| \phi \|_{\partial \Omega}^2.
\]

(4.12)

Also, by (4.5) and the Cauchy inequality,

\[
|\lambda| \int_{\Omega} |\nabla u| |u| \, dx \leq C \| \frac{\partial u}{\partial \nu} \|_{\partial \Omega} \| |\lambda|^{1/2} u \|_{\partial \Omega}.
\]

(4.13)
In view of (4.11), (4.12), (4.13) and (4.5), using the Cauchy inequality, we obtain

$$\|\nabla u\|_\partial^2 \leq C\|\nabla u\|_\partial \|\partial u\|_\partial + C\|\partial u\|_\partial \|\nabla \phi\|_\partial + C\|\partial u\|_\partial \|\lambda|^{1/2} u\|_\partial.$$ 

Estimate (4.9) now follows by using the Cauchy inequality with an $\varepsilon > 0$. The fact $|\lambda| \geq \tau$ is also used here to bound $\|u\|_\partial$ by $C|\lambda|^{1/2} u\|_\partial$.

To see (4.10), we first use the Rellich identity (4.4) to obtain

$$\|\nabla u\|_\partial^2 \leq C\|\nabla \tan u\|_\partial \{\|\nabla u\|_\partial + \|\nabla \phi\|_\partial\} + C\int_\Omega |\nabla u|^2 \, dx$$

$$+ C\int_\Omega |\nabla u| \, |\phi| \, dx + C|\lambda| \int_\Omega |\nabla u| \, |u| \, dx.$$ 

The desired estimate again follows from (4.14), (4.12), (4.13) and (4.5) by using the Cauchy inequality with an $\varepsilon$.

The following lemma is crucial in our approach to the $L^2$ estimates for the system (1.11) (cf. [26]).

**Lemma 4.5.** Assume that $(u, \phi)$ satisfies the same conditions as in Theorem 4.1. Then

$$\|\phi - \int_{\partial \Omega} \phi\|_\partial \leq C\{\|\nabla u\|_\partial + |\lambda|\|u \cdot n\|_{H^{-1}(\partial \Omega)}\}$$ 

(4.15)

and

$$|\lambda|\|u \cdot n\|_{H^{-1}(\partial \Omega)} \leq C\{\|\nabla \phi\|_\partial + \|\nabla u\|_\partial\},$$

(4.16)

where $C$ depends only on $d$ and the Lipschitz character of $\Omega$.

*Proof.* By approximating $\Omega$ by a sequence of Lipschitz domains from inside with uniform Lipschitz characters (see [33]), we may assume that $(u, \phi)$ satisfies the equations (1.11) in $\Omega'$ for some $\Omega'$ containing $\Omega$. Thus $\Delta u = \nabla \phi + \lambda u$ on $\partial \Omega$, and we obtain

$$\|\nabla \phi \cdot n\|_{H^{-1}(\partial \Omega)} \leq \|\Delta u \cdot n\|_{H^{-1}(\partial \Omega)} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial \Omega)}.$$ 

$$|\lambda|\|u \cdot n\|_{H^{-1}(\partial \Omega)} \leq \|\Delta u \cdot n\|_{H^{-1}(\partial \Omega)} + \|\nabla \phi \cdot n\|_{H^{-1}(\partial \Omega)}.$$ 

(4.17)

We will show that

$$\|\Delta u \cdot n\|_{H^{-1}(\partial \Omega)} \leq C\|\nabla u\|_\partial$$ 

(4.18)

and

$$c\|\phi - \int_{\partial \Omega} \phi\|_\partial \leq \|\nabla \phi \cdot n\|_{H^{-1}(\partial \Omega)} \leq C\|\phi\|_\partial.$$ 

(4.19)

Clearly, estimates (4.15) and (4.16) follow from (4.17), (4.18) and (4.19).

To see (4.18), we observe that

$$\Delta u \cdot n = n_i \frac{\partial^2 u_i}{\partial x_j^2} = \left(n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i}\right) \frac{\partial u_i}{\partial x_j},$$

(4.20)
where we have used $\text{div}(u) = 0$ in $\overline{\Omega}$. Since $(n_i \frac{\partial}{\partial x_i} - n_j \frac{\partial}{\partial x_j})$ is a tangential derivative, this gives the estimate (4.18).

The proof of (4.19) relies on the $L^2$ estimates for the Neumann and regularity problems for Laplace’s equation in Lipschitz domains. Given $g \in L^2(\partial\Omega)$ with mean value zero, let $\psi$ be a harmonic function in $\Omega$ such that $(\nabla \psi)^* \in L^2(\partial\Omega)$ and $\frac{\partial \psi}{\partial n} = g$ on $\partial\Omega$. By the Green’s identity,

$$ |\int_{\partial\Omega} \phi \ g \, d\sigma| = |\int_{\partial\Omega} \frac{\partial \phi}{\partial n} \psi \, d\sigma| \leq \left\| \frac{\partial \phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)} \left\| \psi \right\|_{H^1(\partial\Omega)} $$

$$ \leq C \left\| \frac{\partial \phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)} \left\| g \right\|_{\partial\Omega}, $$

where we have used the estimate $\left\| \psi \right\|_{H^1(\partial\Omega)} \leq C \left\| g \right\|_{\partial\Omega}$ for the $L^2$ Neumann problem [19]. By duality, this gives

$$ \left\| \phi - \int_{\partial\Omega} \phi \, d\sigma \right\|_{\partial\Omega} \leq C \left\| \frac{\partial \phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)}. $$

Similarly, given $f \in H^1(\partial\Omega)$, let $\psi$ be the harmonic function in $\Omega$ such that $(\nabla \psi)^* \in L^2(\partial\Omega)$ and $\psi = f$ on $\partial\Omega$. Note that

$$ |\int_{\partial\Omega} \frac{\partial \phi}{\partial n} \, f \, d\sigma| = |\int_{\partial\Omega} \phi \frac{\partial \psi}{\partial n} \, d\sigma| \leq \left\| \phi \right\|_{\partial\Omega} \left\| \nabla \psi \right\|_{\partial\Omega} $$

$$ \leq C \left\| \phi \right\|_{\partial\Omega} \left\| f \right\|_{H^1(\partial\Omega)}, $$

where we have used the estimate $\left\| \nabla \psi \right\|_{\partial\Omega} \leq C \left\| f \right\|_{H^1(\partial\Omega)}$ for the $L^2$ regularity problem [18]. By duality this implies that $\left\| \frac{\partial \phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)} \leq C \left\| \phi \right\|_{\partial\Omega}$.

We are now in a position to give the proof of Theorem 4.1.

**Proof of Theorem 4.1** To prove estimate (4.1), by subtracting a constant from $\phi$, we may assume that $\int_{\partial\Omega} \phi = 0$. In view of (4.15) and (4.10) we have

$$ \left\| \nabla u \right\|_{\partial\Omega} + \left\| \phi \right\|_{\partial\Omega} \leq C \left\{ \left\| \nabla u \right\|_{\partial\Omega} + |\lambda| \left\| u \cdot n \right\|_{H^{-1}(\partial\Omega)} \right\} $$

$$ \leq C \varepsilon \left\{ \left\| \nabla_{\text{tan}} u \right\|_{\partial\Omega} + |\lambda|^{1/2} \left\| u \right\|_{\partial\Omega} + |\lambda| \left\| u \cdot n \right\|_{H^{-1}(\partial\Omega)} \right\} $$

$$ + C \varepsilon \left\{ \left\| \nabla u \right\|_{\partial\Omega} + \left\| \phi \right\|_{\partial\Omega} \right\} $$

for any $\varepsilon \in (0, 1)$. By choosing $\varepsilon$ so small that $C \varepsilon < (1/2)$ we obtain the estimate (4.1).

To establish (4.2), we first use (4.16) to obtain

$$ \left\| \nabla u \right\|_{\partial\Omega} + \left\| \phi \right\|_{\partial\Omega} + |\lambda| \left\| u \cdot n \right\|_{H^{-1}(\partial\Omega)} \leq C \left\{ \left\| \nabla u \right\|_{\partial\Omega} + \left\| \phi \right\|_{\partial\Omega} \right\} \leq C \left\{ \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial\Omega} + \left\| \nabla u \right\|_{\partial\Omega} \right\}. $$

This, together with (4.9), yields that

$$ \left\| \nabla u \right\|_{\partial\Omega} + \left\| \phi \right\|_{\partial\Omega} + |\lambda| \left\| u \cdot n \right\|_{H^{-1}(\partial\Omega)} \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial\Omega} + C |\lambda|^{1/2} \left\| u \right\|_{\partial\Omega}. $$

(4.21)

To handle the term $||\lambda||^{1/2}u||_{\partial\Omega}$, we use the identity

$$ \int_{\partial\Omega} h_k n_k |u|^2 \, d\sigma = \int_{\Omega} \text{div}(h)|u|^2 \, dx + 2\text{Re} \int_{\Omega} h_k \frac{\partial u_i}{\partial x_k} \bar{u}_i \, dx. $$

(4.22)
Choose $h \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ so that $h_k n_k \geq c > 0$ on $\partial \Omega$. It follows from (4.22) that
\[
\|u\|_0^2 \leq C \int_\Omega |u|^2 \, dx + C \int_\Omega |u| \, |\nabla u| \, dx.
\] (4.23)
In view of (4.3) and the assumption $|\lambda| \geq \tau > 0$, this gives
\[
\| |\lambda|^{1/2} u \|_0^2 \leq C \| |\lambda|^{1/2} \| \frac{\partial u}{\partial \nu} \|_0 \| u \|_0.
\] (4.24)
It follows that
\[
\| |\lambda|^{1/2} u \|_0 \leq C \| \frac{\partial u}{\partial \nu} \|_0.
\] (4.25)
It is easy to deduce from (4.21) and (4.25) that
\[
\| \nabla u \|_0 + \| \phi \|_0 + \| |\lambda|^{1/2} u \|_0 + |\lambda| \| u \cdot n \|_{H^{-1}(\partial \Omega)} \leq C \| \frac{\partial u}{\partial \nu} \|_0.
\] (4.26)
This completes the proof. \hfill \Box

An careful inspection of the proof of Theorem 4.1 shows that the analogous result to that in Theorem 4.1 also hold in the exterior domain $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$. However, some decay assumptions at $\infty$ are needed to justify the use of integration by parts in the unbounded domain $\Omega_-$. In the proof of Lemmas 4.3 and 4.5. Also note that the term $\int_{\partial \Omega} \phi \, d\sigma$ should be dropped in this case. We omit the proof of the following theorem.

**Theorem 4.6.** Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0, 1)$. Let $(u, \phi)$ be a solution of (1.11) in $\Omega_-$. Suppose that $(\nabla u)^*, (\phi)^* \in L^2(\partial \Omega)$ and that $\nabla u, \phi$ have nontangential limits a.e. on $\partial \Omega$. We further assume that as $|x| \to \infty$, $|\phi(x)| + |\nabla u(x)| = O(|x|^{-d})$ and $u(x) = O(|x|^{2-d})$ if $d \geq 3$, $u(x) = o(1)$ if $d = 2$. Then
\[
\| \nabla u \|_0 + \| \phi \|_0 \leq C \{ \| \nabla_{\text{tan}} u \|_0 + |\lambda|^{1/2} \| u \|_0 + |\lambda| \| u \cdot n \|_{H^{-1}(\partial \Omega)} \}
\] (4.27)
and
\[
\| \nabla u \|_0 + |\lambda|^{1/2} \| u \|_0 + |\lambda| \| u \cdot n \|_{H^{-1}(\partial \Omega)} + \| \phi \|_0 \leq C \| \frac{\partial u}{\partial \nu} \|_0,
\] (4.28)
where $C$ depends only on $d$, $\tau$, $\theta$, and the Lipschitz character of $\Omega$.

## 5 $L^2$ Dirichlet and Neumann problems

In this section we use the method of layer potentials to solve the $L^2$ Dirichlet and Neumann problems for the Stokes system (1.11). As a consequence of the nontangential-maximal-function estimate for the $L^2$ Dirichlet problem, we also obtain a uniform $L^p$ estimate that will play a crucial role in the proof of Theorem 4.1.

Throughout this section we will assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. We use $L^2_n(\partial \Omega)$ to denote the space
\[
L^2_n(\partial \Omega) := \left\{ f \in L^2(\partial \Omega; \mathbb{C}^d) : \int_{\partial \Omega} f \cdot n \, d\sigma = 0 \right\},
\] (5.1)
and $L^2_0(\partial \Omega; \mathbb{C}^d)$ the subspace of $L^2$ functions with mean value zero. Recall that $\| \cdot \|_0$ denotes the norm in $L^2(\partial \Omega)$.  

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Lemma 5.1. Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0, 1)$. Suppose that $|\partial \Omega| = 1$. Then $(1/2)I + \mathcal{K}_\lambda$ is an isomorphism on $L^2(\partial \Omega; \mathbb{C}^d)$ and

$$\|f\|_{\partial} \leq C \|(1/2)I + \mathcal{K}_\lambda\| f\|_{\partial} \quad \text{for any } f \in L^2(\partial \Omega; \mathbb{C}^d),$$

(5.2)

where $C$ depends only on $d, \theta, \tau$, and the Lipschitz character of $\Omega$.

Proof. Let $f \in L^2(\partial \Omega; \mathbb{C}^d)$ and $(u, \phi)$ be the single layer potentials, given by (3.1)-(3.2). It follows from Section 3 that $(u, \phi)$ satisfies (1.11) in $\mathbb{R}^d \setminus \partial \Omega$ and $(\nabla u)^*, (\phi)^* \in L^2(\partial \Omega)$. Moreover, $\nabla u$ and $\phi$ have nontangential limits a.e. on $\partial \Omega$, $\nabla_{\text{tan}} u_+ = \nabla_{\text{tan}} u_-$, and $\left(\frac{\partial \phi}{\partial \nu}\right)_\pm = (\pm (1/2)I + \mathcal{K}_\lambda)f$. We will show that

$$\|\nabla u_+\|_{\partial} + \|\phi_-\|_{\partial} \leq C \left\| \left(\frac{\partial u}{\partial \nu}\right)_{+} \right\|_{\partial} \quad \text{for any } f \in L^2(\partial \Omega; \mathbb{C}^d),$$

(5.3)

By the jump relation $f = (\frac{\partial u}{\partial \nu})_+ - (\frac{\partial u}{\partial \nu})_-$, we deduce from (5.3) that

$$\|f\|_{\partial} \leq \left\| \left(\frac{\partial u}{\partial \nu}\right)_{+} \right\|_{\partial} + \left\| \left(\frac{\partial u}{\partial \nu}\right)_{-} \right\|_{\partial} \leq C \left\| \left(\frac{\partial u}{\partial \nu}\right)_{+} \right\|_{\partial} = C \|(1/2)I + \mathcal{K}_\lambda\| f\|_{\partial}.$$

Estimate (5.3) is a consequence of Theorems 4.6 and 5.1. Indeed, since $|u(x)| + |\nabla u(x)| = O(|x|^{-N})$ for any $N > 0$ and $\phi(x) = O(|x|^{-\theta})$ as $|x| \to \infty$, we may apply Theorem 4.6 to obtain

$$\|\nabla u_+\|_{\partial} + \|\phi_-\|_{\partial} \leq C \left\{ \|\nabla_{\text{tan}} u_+\|_{\partial} + |\lambda|^{1/2}\|u_-\|_{\partial} + |\lambda|\|\phi\|_{H^{-1}(\partial \Omega)} \right\}$$

$$= C \left\{ \|\nabla_{\text{tan}} u_+\|_{\partial} + |\lambda|^{1/2}\|u_+\|_{\partial} + |\lambda|\|\phi\|_{H^{-1}(\partial \Omega)} \right\},$$

where we used $u_+ = u_-$ and $\nabla_{\text{tan}} u_+ = \nabla_{\text{tan}} u_-$ on $\partial \Omega$. In view of Theorem 4.1 this gives the estimate (5.3) and hence, the estimate (5.2).

Finally, if $\lambda = 0$, it was proved in [10] that as an operator on $L^2(\partial \Omega; \mathbb{R}^d)$, the null space of $(1/2)I + \mathcal{K}_0$ is of dimension one and the range is $L_0^2(\partial \Omega; \mathbb{R}^d)$. It follows that the index of $(1/2)I + \mathcal{K}_0$ is zero. The same is true if we replace $L^2(\partial \Omega; \mathbb{R}^d)$ by $L^2(\partial \Omega; \mathbb{C}^d)$. Using Theorem 2.5 it is not hard to see that the operator $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $L^2(\partial \Omega; \mathbb{C}^d)$. As a result we may deduce that the index of $(1/2)I + \mathcal{K}_\lambda$ on $L^2(\partial \Omega; \mathbb{C}^d)$ is zero for any $\lambda \in \Sigma_\theta$. Since the operator is clearly injective by (5.2), it is also surjective and hence an isomorphism.

Note that the condition $|\lambda| \geq \tau$ (hence $|\partial \Omega| = 1$) is not needed in the next lemma.

Lemma 5.2. Let $\lambda \in \Sigma_\theta$. Then $-(1/2)I + \mathcal{K}_\lambda$ is a Fredholm operator on $L^2(\partial \Omega; \mathbb{C}^d)$ with index zero, and

$$\|f\|_{\partial} \leq C \| - (1/2)I + \mathcal{K}_\lambda \| f\|_{\partial} \quad \text{for any } f \in L^2_0(\partial \Omega),$$

(5.4)

where $C$ depends only on $d, \theta$, and the Lipschitz character of $\Omega$.

Proof. By rescaling we may assume that $|\partial \Omega| = 1$. In the case $\lambda = 0$, it was proved in [10] that as an operator on $L^2(\partial \Omega; \mathbb{R}^d)$, the index of $-(1/2)I + \mathcal{K}_0$ is zero, and the estimate (5.4)
holds. Since $K_\lambda - K_0$ is compact on $L^2(\partial \Omega; \mathbb{C}^d)$, this implies that the index of $-(1/2)I + K_\lambda$ on $L^2(\partial \Omega; \mathbb{C}^d)$ is zero for any $\lambda \in \Sigma_\theta$.

To establish estimate (5.4) for any $\lambda \in \Sigma_\theta$, we first note that by Theorem 2.5
\[
\left| (K_\lambda - K_0)f(x) \right| \leq C \int_{\partial \Omega} |\nabla_x \{ \Gamma(x - y; \lambda) - \Gamma(x - y; 0) \}| |f(y)| \, d\sigma(y)
\leq C|\lambda|^{1/2} \int_{\partial \Omega} \frac{|f(y)|}{|x - y|} \, d\sigma(y),
\]
if $d = 3$. This yields that $\|(K_\lambda - K_0)f\|_{\partial} \leq C|\lambda|^{1/2}\|f\|_{\partial}$. It is easy to see that this estimate also holds for $d \geq 4$, if $|\lambda| \leq 1$. It follows that for $f \in L^2_\alpha(\partial \Omega)$,
\[
\|f\|_{\partial} \leq C\|(1/2)I + K_\lambda\|_{\partial} f\|_{\partial}
\leq C\|(1/2)I + K_\lambda\|_{\partial} + C|\lambda|^{1/2}\|f\|_{\partial}.
\]
This implies that estimate (5.4) holds for $\lambda \in \Sigma_\theta$ and $|\lambda| < \tau$, where $\tau > 0$ depends only on $d$, $\theta$ and the Lipschitz character of $\Omega$.

We will use the Rellich estimates in Section 4 to handle the case $|\lambda| \geq \tau$. The argument is similar to that in the proof of Lemma 5.1. Let $f \in L^2_\alpha(\partial \Omega)$ and $(u, \phi)$ be given by (3.1)-(3.2). By Theorems 4.1 and 4.6
\[
\|\nabla u_+\|_{\partial} + \|\phi_+ - \int_{\partial \Omega} \phi_+\|_{\partial} \leq C\{\|\nabla_{\tan} u\|_{\partial} + |\lambda|^{1/2}\|u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial \Omega)}\}
\leq C\left\| \left( \frac{\partial u}{\partial \nu} \right)_- \right\|_{\partial}.
\]
It follows that
\[
\|f\|_{\partial} \leq \left\| \left( \frac{\partial u}{\partial \nu} \right)_+ \right\|_{\partial} + \left\| \left( \frac{\partial u}{\partial \nu} \right)_- \right\|_{\partial}
\leq C\left\| \left( \frac{\partial u}{\partial \nu} \right)_- \right\|_{\partial} + C\left| \int_{\partial \Omega} \phi_+ \right|
\leq C\left| (1/2)I + K_\lambda \right| f\|_{\partial} + C\left| \int_{\partial \Omega} \phi_+ \right|.
\]
Finally, to deal with the term $\int_{\partial \Omega} \phi_+$, we note that
\[
\left( \frac{\partial u}{\partial \nu} \right)_+ \cdot n = \frac{\partial u_i}{\partial x_j} n_i n_j - \phi_+ = n_j \left( n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i} \right) u_i - \phi_+,
\]
which may be justified by taking nontangential limits inside $\Omega$. It follows that
\[
\left| \int_{\partial \Omega} \phi_+ \right| \leq \left| \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)_+ \cdot n \right| + C\left\| \nabla_{\tan} u \right\|_{\partial}
\leq \left| \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)_- \cdot n \right| + C\left\| \nabla_{\tan} u \right\|_{\partial}
\leq C\left\| \frac{\partial u}{\partial \nu} \right\|_{\partial}.
\]
where we have used the jump relation and $\int_{\partial \Omega} f \cdot n = 0$ for the second inequality and Theorem 4.6 for the third. This, together with (5.5), gives the estimate (5.4).

The next theorem establishes the solvability of the $L^2$ Neumann problem for the Stokes equation (1.11) in a bounded Lipschitz domain, with nontangential-maximal-function estimates that are uniform in $\lambda$. We note that this theorem is not needed in the proof of Theorem 1.1.

**Theorem 5.3.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. Let $\lambda \in \Sigma_\theta$ and $|\lambda| r^2 > \tau$, where $\tau \in (0, 1)$ and $r = \text{diam}(\Omega)$. Given any $g \in L^2(\partial \Omega; \mathbb{C}^d)$, there exist a unique $u$ and a harmonic function $\phi$, unique up to constants, such that $(u, \phi)$ satisfies (1.11) in $\Omega$, $(\nabla u)^* , (\phi)^* \in L^2(\partial \Omega)$, and $\frac{\partial u}{\partial n} = g$ on $\partial \Omega$ in the sense of nontangential convergence. Moreover, the solution $(u, \phi)$ satisfies

$$\| (\nabla u)^* \|_\partial + \| (\phi)^* \|_\partial + |\lambda|^{1/2} \| (u)^* \|_\partial + |\lambda| \| u \cdot n \|_{H^{-1}(\partial \Omega)} \leq C \| g \|_\partial,$$

and may be represented by a single layer potential given by (3.1)-(3.2) with $\| f \|_\partial \leq C \| g \|_\partial$, where $C$ depends only on $d, \theta, \tau$, and the Lipschitz character of $\Omega$.

**Proof.** The uniqueness follows readily from the identity (4.6). To establish the existence, we first note that by rescaling, we may assume $|\partial \Omega| = 1$. This implies that $|\lambda| \geq c \tau$, where $c > 0$ depends only on $d$ and the Lipschitz character of $\Omega$. Choose $f \in L^2(\partial \Omega; \mathbb{C}^d)$ such that $((1/2)I + K_{\lambda})f = g$. In view of Lemmas 5.1 and 3.2 as well as Theorem 4.1, the solution $(u, \phi)$ given by (3.1)-(3.2) satisfies the estimate (5.6).

The following lemma will be used to establish the uniqueness for the $L^2$ Dirichlet problem.

**Lemma 5.4.** Let $\lambda \in \Sigma_\theta$ and $(u, \phi)$ be a solution of (1.11) in $\Omega$. Suppose that $u$ has nontangential limit a.e. on $\partial \Omega$ and $(u)^* \in L^2(\partial \Omega)$. Then

$$\int_{\Omega} |u|^2 \, dx \leq C \int_{\partial \Omega} |u|^2 \, d\sigma,$$

where $C$ depends only on $d, \theta$ and $\Omega$.

**Proof.** By approximating $\Omega$ by a sequence of smooth domains with uniform Lipschitz characters from inside, we may assume that $\Omega$ is smooth and $u, \phi$ are smooth in $\Omega$. Let $(w, \psi)$ be a solution to the system

$$\begin{cases}
-\Delta w + \lambda w + \nabla \psi = \overline{u} & \text{in } \Omega, \\
\text{div}(w) = 0 & \text{in } \Omega,
\end{cases}$$

(5.8)

where $w \in H_0^1(\Omega; \mathbb{C}^d)$ and $\psi \in H^1(\Omega)$. It follows from integration by parts and (5.8) that

$$\int_{\Omega} |u|^2 \, dx = \int_{\Omega} u \cdot \left\{ -\Delta w + \lambda w + \nabla \psi \right\} \, dx$$

$$= -\int_{\partial \Omega} u \cdot \left\{ \frac{\partial w}{\partial n} - \psi n \right\} \, d\sigma$$

$$\leq \| u \|_\partial \left\{ \| \nabla w \|_\partial + \| \psi \|_\partial \right\}.$$

(5.9)
By subtracting a constant from $\psi$, we may assume that $\int_{\partial \Omega} \psi = 0$. Note that $\Delta \psi = \text{div}(\pi) = 0$ in $\Omega$. By the proof of Lemma 4.5, this implies that

$$\|\psi\|_{\partial} \leq C \|\nabla \psi \cdot n\|_{H^{-1}(\partial \Omega)}$$

$$\leq C \left\{ \|\Delta w \cdot n\|_{H^{-1}(\partial \Omega)} + \|u \cdot n\|_{H^{-1}(\partial \Omega)} \right\}$$

(5.10)

In view of (5.9)-(5.10), we obtain

$$\int_\Omega |u|^2 \, dx \leq C \|u\|_{\partial} \|\nabla w\|_{\partial} + C \|u\|_{\partial}^2.$$  

(5.11)

As a result it suffices to show that

$$\int_{\partial \Omega} |\nabla w|^2 \, d\sigma \leq C \int_\Omega |u|^2 \, dx + C \int_{\partial \Omega} |u|^2 \, d\sigma.$$  

(5.12)

To see (5.12), we use a Rellich type identity, similar to (4.4), and the fact $w = 0$ on $\partial \Omega$, to obtain

$$\int_{\partial \Omega} |\nabla w|^2 \, d\sigma \leq C \left\{ \int_\Omega |\nabla w|^2 \, dx + \int_\Omega |\nabla w| \psi \, dx \right\}$$

$$+ |\lambda| \int_\Omega |\nabla w| \, dx + \int_\Omega |\nabla w| u \, dx \right\}.$$  

(5.13)

As in the proof of Lemma 4.3, it follows from (5.8) and integration by parts that

$$\int_\Omega |\nabla w|^2 \, dx + |\lambda| \int_\Omega |w|^2 \, dx \leq C \int_\Omega |w||u| \, dx.$$  

(5.14)

This, together with the Cauchy inequality and Poincaré inequality, gives

$$\int_\Omega |\nabla w|^2 \, dx + (1 + |\lambda|) \int_\Omega |w|^2 \, dx \leq C \frac{1}{1 + |\lambda|} \int_\Omega |u|^2 \, dx.$$  

(5.15)

Finally, using (5.13), (5.15) and the Cauchy inequality, we obtain

$$\int_{\partial \Omega} |\nabla w|^2 \, d\sigma \leq C \varepsilon \int_\Omega |u|^2 \, dx + \varepsilon \int_{\partial \Omega} |\psi|^2 \, d\sigma,$$  

(5.16)

where we also used the estimate $\|\psi\|_{L^2(\Omega)} \leq C \|\psi\|_{\partial}$. The desired estimate (5.12) now follows from (5.16) and (5.10).

**Theorem 5.5.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. Let $\lambda \in \Sigma$. Given $g \in L^2_0(\partial \Omega)$, there exist a unique $u$ and a harmonic function $\phi$, unique up to constants, such that $(u, \phi)$ satisfies (1.11) in $\Omega$, $(u)^* \in L^2(\partial \Omega)$ and $u = g$ on $\partial \Omega$ in the sense of nontangential convergence. Moreover, the solution $u$ satisfies the estimate $\|(u)^*\|_{\partial} \leq C \|g\|_{\partial}$, and may be represented by a double layer potential $D_\lambda(f)$ with $\|f\|_{\partial} \leq C \|g\|_{\partial}$, where $C$ depends only on $d, \theta$, and the Lipschitz character of $\Omega$.
Proof. The uniqueness follows directly from Lemma 5.4. We will use Lemma 5.2 to establish the existence. To this end we first note that since \(- (1/2) I + \mathcal{K}_\lambda\) is a Fredholm operator on \(L^2(\partial \Omega; \mathbb{C}^d)\) with index zero, so is its adjoint \(- (1/2) I + \mathcal{K}_\lambda^*\). Let \(u\) be the double layer potential given by (3.9), with \(f \in L^2(\partial \Omega; \mathbb{C}^d)\). Since \(\text{div}(u) = 0\) in \(\Omega\), we have \(\int_{\partial \Omega} u \cdot n = 0\). This shows that the range of \(- (1/2) I + \mathcal{K}_\lambda^*\) is contained in \(L_2^2(\partial \Omega)\). Consequently, the normal vector \(n\) is in the null space of \(- (1/2) I + \mathcal{K}_\lambda\). Moreover, by the estimate (5.4), the null space of \(- (1/2) I + \mathcal{K}_\lambda\) is the one-dimensional subspace spanned by \(n\). This in turn implies that the range of \(- (1/2) I + \mathcal{K}_\lambda^*\) is contained in \(L_2^2(\partial \Omega)\). As a result, the operator \(- (1/2) I + \mathcal{K}_\lambda^* : R(- (1/2) I + \mathcal{K}_\lambda) \rightarrow L_2^2(\partial \Omega)\) is invertible, where \(R(- (1/2) I + \mathcal{K}_\lambda)\) denotes the range of \(- (1/2) I + \mathcal{K}_\lambda\). Furthermore, by duality, we may deduce from the estimate (5.4) that

\[
\|f\|_{\partial} \leq C \|(- (1/2) I + \mathcal{K}_\lambda^*)f\|_{\partial}
\]

for any \(f \in R(- (1/2) I + \mathcal{K}_\lambda)\).

Finally, given \(g \in L_2^2(\partial \Omega)\), we choose \(f \in R(- (1/2) I + \mathcal{K}_\lambda)\) such that \((- (1/2) I + \mathcal{K}_\lambda^*)f = g\). Let \((u, \phi)\) be the double layer potential given by (3.9) - (3.10). Then \(u = g\) on \(\partial \Omega\) and \(\|(u)^*\|_{\partial} \leq C \|f\|_{\partial} \leq C \|g\|_{\partial},\) where the last inequality follows from (5.17). This completes the proof.

We end this section with a uniform \(L^p\) estimate for the \(L^2\) Dirichlet problem as well as a remark on the interior estimates.

**Theorem 5.6.** Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^d, d \geq 3\) with connected boundary. Let \(u \in H^1(\Omega; \mathbb{C}^d)\) and \(\phi \in L^2(\Omega)\). Suppose that \((u, \phi)\) satisfies the Stokes system (1.11) in \(\Omega\) for some \(\lambda \in \Sigma_\theta.\) Then

\[
\left(\int_\Omega |u|^p \, dx \right)^{1/p} \leq C \left(\int_{\partial \Omega} |u|^2 \, d\sigma \right)^{1/2},
\]

where \(p = \frac{2d}{d-1}\) and \(C\) depends only on \(d, \theta,\) and the Lipschitz character of \(\Omega.\)

**Proof.** Let \(f\) denote the trace of \(u\) on \(\partial \Omega\) and \(w\) the solution of the \(L^2\) Dirichlet problem in \(\Omega\), given by Theorem 5.5, with boundary data \(f\). Let \(\{\Omega_j\}\) be a sequence of smooth domains that approximates \(\Omega\) from inside [33, p.581]. It follows from Lemma 5.4 that

\[
\int_{\Omega_j} |u - w|^2 \, dx \leq C \int_{\partial \Omega_j} |u - w|^2 \, d\sigma,
\]

where \(C\) is independent of \(j\). Letting \(j \to \infty\) in (5.19), we may deduce that \(w = u\) in \(\Omega.\) As a result we obtain \(\|(u)^*\|_{\partial} \leq C \|u\|_{\partial}\). This, together with the inequality

\[
\left(\int_\Omega |u|^p \, dx \right)^{1/p} \leq C \left(\int_{\partial \Omega} |(u)^*|^2 \, d\sigma \right)^{1/2}
\]

for any continuous function \(u\) in \(\Omega\), where \(p = \frac{2d}{d-1}\), gives (5.18).

Estimate (5.22) is known (see e.g. [20, Remark 9.3]). We provide a proof here for the sake of completeness. By rescaling we may assume that \(\text{diam}(\Omega) = 1\). Using the observation

\[
|u(x)| \leq C \int_{\partial \Omega} \frac{(u)^*(y)}{|x - y|^{d-1}} \, d\sigma(y)
\]

for any \(x \in \Omega\).
and a duality argument, it suffices to show that
\[ \|I_1(F)\|_{L^2(\partial\Omega)} \leq C \|F\|_{L^q(\Omega)}, \]  
(5.21)
where \( q = p' = \frac{2d}{d+1} \) and
\[ I_1(F)(y) = \int_{\Omega} \frac{F(x)}{|x-y|^{d-1}} \, dx. \]

Let \( v(x) = I_1(F)(x) \). Using integration by parts we may show that
\[ \int_{\partial\Omega} |v|^2 \, d\sigma \leq C \int_{\Omega} |v|^2 \, dx + C \int_{\Omega} |v| |\nabla v| \, dx \]
(c.f. (4.22)-(4.23)). By Hölder’s inequality, this gives
\[ \|v\|_{L^2(\partial\Omega)} \leq C \left\{ \|\nabla v\|_{L^q(\Omega)} + \|v\|_{L^p(\Omega)} \right\} \leq C \|F\|_{L^q(\Omega)}, \]
where the last inequality follows from the well known \((L^q, L^p)\) bound for fractional integrals as well as the \(L^q\) bound for singular integrals [31]. This completes the proof.

\[ \square \]

**Remark 5.7.** Let \((u, \phi)\) be a solution of (1.11) in \(B(x_0, r)\). Then
\[ |\nabla^\ell u(x_0)| \leq C_{\ell} \frac{1}{r^\ell} \left( \int_{B(x_0, r)} |u|^2 \right)^{1/2} \]
(5.22)
for any \( \ell \geq 0 \), where \( C_{\ell} \) depends only on \( d, \ell \) and \( \theta \). To see (2.8), by rescaling, we may assume that \( r = 2 \). Let \( t \in (1, 2) \). By applying Theorem 5.5 in the domain \( B(x_0, t) \) and using the double layer representation, we obtain
\[ |\nabla^\ell u(x_0)|^2 \leq C_{\ell} \int_{\partial B(x_0, t)} |u|^2 \, d\sigma. \]
(5.23)
Estimate (5.22) now follows by integrating both sides of (5.23) in \( t \) over the interval \((1, 2)\).

6 Proof of Main Theorem

In this section we give the proof of Theorem 1.1. Throughout the section we will assume that \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^d \), \( d \geq 3 \). The condition that \( \partial\Omega \) is connected is not needed.

The first step is to establish a weak reverse Hölder estimate for local solutions of (1.11). Let \( \eta : \mathbb{R}^{d-1} \to \mathbb{R} \) be a Lipschitz function such that \( \eta(0) = 0 \) and \( \|\nabla \eta\|_{\infty} \leq M \). Define
\[ D(r) = \{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \eta(x') < x_d < 10d(M+1)r \}, \]
\[ I(r) = \{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } x_d = \eta(x') \} \]
for \( 0 < r < \infty \).
Lemma 6.1. Let \( u \in H^1(D(2r); \mathbb{C}^d) \) and \( \phi \in L^2(D(2r)) \). Suppose that \((u, \phi)\) satisfies the Stokes system \((1.11)\) in \( D(2r) \) and \( u = 0 \) on \( I(2r) \) for some \( 0 < r < \infty \) and \( \lambda \in \Sigma_0 \). Let \( p_d = \frac{2d}{d-1} \). Then
\[
\left( \int_{D(2r)} |u|^{pd} \right)^{1/p_d} \leq C \left( \int_{D(2r)} |u|^2 \right)^{1/2},
\]
where \( C \) depends only on \( d, M, \) and \( \theta \).

Proof. By rescaling we may assume that \( r = 1 \). Let \( t \in (1, 2) \). We apply Theorem 5.6 to \( u \) in the Lipschitz domain \( D(t) \) to obtain
\[
\left( \int_{D(t)} |u|^p \right)^{2/p} \leq C \int_{\partial D(t)} |u|^2 d\sigma,
\]
where \( p = p_d \) and \( C \) depends only on \( d, \theta \) and \( M \). Since \( u = 0 \) on \( I(2) \), this implies that
\[
\left( \int_{D(1)} |u|^p \right)^{2/p} \leq C \int_{\partial D(1) \setminus I(2)} |u|^2 d\sigma.
\]
We now integrate both sides of \((6.4)\) with respect to \( t \) over the interval \((1, 2)\). This gives
\[
\left( \int_{D(1)} |u|^p dx \right)^{2/p} \leq C \int_{D(2)} |u|^2 dx,
\]
which yields the desired estimate. \( \square \)

The next lemma is a consequence of Lemma 6.1 and its proof.

Lemma 6.2. Let \( x_0 \in \overline{\Omega} \) and \( 0 < r < c \text{diam}(\Omega). \) Let \( u \in H^1(B(x_0, 2r) \cap \Omega; \mathbb{C}^d) \) and \( \phi \in L^2(B(x_0, 2r) \cap \Omega) \). Suppose that \((u, \phi)\) satisfies the Stokes system \((1.11)\) in \( B(x_0, 2r) \cap \Omega \) and \( u = 0 \) on \( B(x_0, 2r) \cap \partial \Omega \) (if \( B(x_0, 2r) \cap \partial \Omega \neq \emptyset \)). Then
\[
\left( \int_{B(x_0, r) \cap \Omega} |u|^p \right)^{1/p} \leq C \left( \int_{B(x_0, 2r) \cap \Omega} |u|^2 \right)^{1/2},
\]
where \( p = p_d + \varepsilon \) and \( C > 0, \varepsilon > 0 \) depends only on \( d, \theta \) and the Lipschitz character of \( \Omega \).

Proof. We first point out that the estimate \((6.6)\) is a weak reverse Hölder inequality, which has the well known self-improving property (see e.g. [15, Chapter V]). As a result it suffices to prove \((6.6)\) for \( p = p_d = \frac{2d}{d-1} \). Furthermore, by a geometric consideration, we only need to establish the estimate in two cases: (1) \( x_0 \in \Omega \) and \( B(x_0, 3r) \subset \Omega \); (2) \( x_0 \in \partial \Omega \).

The first case follows readily from the interior estimate \((5.22)\). The second case concerns a boundary estimate. By translation and rotation of the coordinate system we may assume that \( x_0 = 0 \) and
\[
B(x_0, r_0) \cap \Omega = B(x_0, r_0) \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \eta(x') \},
\]
where \( r_0 = c \text{diam}(\Omega) \) and \( \eta \) is a Lipschitz function in \( \mathbb{R}^{d-1} \) such that \( \eta(0) = 0 \) and \( \|\nabla \eta\|_\infty \leq M \). By a simple covering argument it is not hard to see that estimate \((6.6)\) follows from Lemma 6.1 as well as the interior estimate in the first case. \( \square \)
The following lemma contains the real variable argument needed to complete the proof of Theorem 1.1.

**Lemma 6.3.** Let $p > 2$ and $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Suppose that (1) $T$ is a bounded sublinear operator in $L^2(\Omega; \mathbb{C}^m)$ and $\|T\|_{L^2 \to L^2} \leq C_0$; (2) there exist constant $0 < \alpha < 1$ and $N > 1$ such that for any bounded measurable $f$ with $\text{supp}(f) \subset \Omega \setminus 3B$,

$$\left( \int_{\Omega \cap B} |Tf|^p \right)^{1/p} \leq N \left\{ \left( \int_{\Omega \setminus 2B} |Tf|^2 \right)^{1/2} + \sup_{B' \subset B} \left( \int_{B'} |f|^2 \right)^{1/2} \right\},$$

where $B = B(x, r)$ is a ball with $x \in \overline{\Omega}$ and $0 < r < \alpha \text{diam}(\Omega)$. Then $T$ is bounded on $L^q(\Omega, \mathbb{C}^m)$ for any $2 < q < p$. Moreover, $\|T\|_{L^q \to L^q}$ is bounded by a constant depending at most on $d$, $m$, $\alpha$, $N$, $C_0$, $p$, $q$, and the Lipschitz character of $\Omega$.

**Proof.** The boundedness of $T$ on $L^q(\Omega, \mathbb{C}^m)$ is proved in [28, Theorem 3.3]. The statement that $\|T\|_{L^q \to L^q}$ is bounded by a constant depending at most on $d$, $m$, $\alpha$, $N$, $C_0$, $p$, $q$, and the Lipschitz character of $\Omega$ follows from the proof of Theorem 3.3 in [28].

We are now in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By rescaling we may assume that $\text{diam}(\Omega) = 1$. Let $\lambda \in \Sigma_{\theta}$. Given any $f \in L^2(\Omega; \mathbb{C}^d)$, there exist a unique $u \in H^1_0(\Omega; \mathbb{C}^d)$ and a function $\phi \in L^2(\Omega)$, unique up to constants, such that

$$\begin{cases} -\Delta u + \nabla \phi + \lambda u = f, \\ \text{div}(u) = 0 \end{cases}$$

(6.8)

in $\Omega$. Note that

$$\int_{\Omega} |\nabla u|^2 \, dx + |\lambda| \int_{\Omega} |u|^2 \, dx \leq C \int_{\Omega} |f||u| \, dx,$$

where $C$ depends only on $\theta$. By Hölder inequality as well as Poincaré inequality, this implies that

$$(|\lambda| + 1)\|u\|_{L^2(\Omega)} \leq C_0 \|f\|_{L^2(\Omega)},$$

(6.9)

where $C_0$ depends only on $d$, $\theta$ and the Lipschitz character of $\Omega$. We now define the operator $T_\lambda$ by $T_\lambda(f) = (|\lambda| + 1)u$. Clearly, $T_\lambda$ is a bounded linear operator on $L^2(\Omega; \mathbb{C}^d)$ and $\|T_\lambda\|_{L^2 \to L^2} \leq C_0$. We will use Lemma 6.3 to show that $\|T_\lambda\|_{L^q \to L^q} \leq C$ for $2 < q < p_d + \varepsilon$.

To verify the assumption (2) in Lemma 6.3 we let $B = B(x_0, r)$, where $x_0 \in \overline{\Omega}$ and $0 < r < c$. Let $f \in L^2(\Omega; \mathbb{C}^d)$ with $\text{supp}(f) \subset \Omega \setminus 3B$ and $(u, \phi)$ be the solution of (6.8) in $\Omega$. Since $-\Delta u + \nabla \phi + \lambda u = 0$, $\text{div}(u) = 0$ in $\Omega \cap 3B$, and $u \in H^1_0(\Omega; \mathbb{C}^d)$, we may apply Lemma 6.2 to obtain

$$\left( \int_{\Omega \cap B} |u|^p \right)^{1/p} \leq C \left( \int_{\Omega \setminus 2B} |u|^2 \right)^{1/2},$$

where $p = p_d + \varepsilon$. It follows that

$$\left( \int_{\Omega \cap B} |T_\lambda(f)|^p \right)^{1/p} \leq C \left( \int_{\Omega \setminus 2B} |T_\lambda(f)|^2 \right)^{1/2},$$

(6.10)
where $C$ depends only on $d$, $\theta$ and the Lipschitz character of $\Omega$. Hence, by Lemma 6.3, we may conclude that the operator $T_\lambda$ is bounded on $L^q(\Omega; \mathbb{C}^d)$ for any $2 < q < p_\lambda + \varepsilon$, and that $\|T_\lambda\|_{L^q \to L^q}$ is bounded by a constant $C_q$ depending at most on $d$, $\theta$, $q$ and the Lipschitz character of $\Omega$. In view of the definition of $T_\lambda$ we have proved that

$$\|u\|_{L^q(\Omega)} \leq \frac{C_q}{|\lambda| + 1} \|f\|_{L^q(\Omega)} \quad (6.11)$$

for any $2 \leq q < p_\lambda + \varepsilon$. By duality the estimate also holds for $(p_\lambda + \varepsilon)' < q < 2$. This completes the proof.

We end this section with a remark on the definition of the Stokes operator.

**Remark 6.4.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d, d \geq 3$. Suppose that $F \in W^{-1,p}(\Omega; \mathbb{R}^d)$, where $\frac{1}{p} - \frac{1}{2} < \frac{1}{2d} + \varepsilon$. Then there exist a unique $u \in W^{1,p}_0(\Omega; \mathbb{R}^d)$ and a function $\phi \in L^p(\Omega)$, unique up to constants, such that $-\Delta u + \nabla \phi = F$ and $\text{div}(u) = 0$ in $\Omega$. Moreover, the solution satisfies the estimate

$$\|\nabla u\|_{L^p(\Omega)} + \|\phi - \int_\Omega \phi\|_{L^p(\Omega)} \leq C_p \|F\|_{W^{-1,p}(\Omega)}, \quad (6.12)$$

where $C_p$ depends only on $d$, $p$ and the Lipschitz character of $\Omega$ (see [2] for the case $d = 3$ and [14] for $d \geq 4$). Using the estimate (6.12), it is not hard to show that the Stokes operator $A_p$ defined in the Introduction is closed in $L^p(\Omega)$ if $p$ satisfies (1.4). One may also deduce that if $u \in H^1_0(\Omega; \mathbb{C}^d)$ is a solution of (1.1) with $f \in C_0^{\infty}(\Omega)$, then $u \in D(A_p)$. It follows that $A_p(u) = f - \lambda u$ and thus $u = (A_p + \lambda)^{-1}f$.

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**References**

[1] R. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math. 111 (1989), 339–379.

[2] R. Brown and Z. Shen, *Estimates for the Stokes operator in Lipschitz domains*, Indiana Univ. Math. J. 44 (1995), 1183–1206.

[3] L. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. (1998), 1–21.

[4] R. Coifman, A. McIntosh, and Y. Meyer, *L’intégrale de Cauchy définit un opérateur borné sur $L^2$ pour les courbes lipschitziennes*, Ann. of Math. 116 (1982), 361–387.

[5] B. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. 65 (1977), 275–288.

[6] B. Dahlberg, C. Kenig, and G. Verchota, *Boundary value problems for the system of elastostatics in Lipschitz domains*, Duke Math. J. 57 (1988), no. 3, 795–818.
[7] P. Deuring, *The resolvent problem for the Stokes system in exterior domains: an elementary approach*, Math. Methods Appl. Sci. **13** (1990), 335–349.

[8] ______, *The Stokes resolvent in 3D domains with conical boundary points: non-regularity in $L^p$-spaces*, Adv. Differential Equations **6** (2001), 175–228.

[9] P. Deuring and W. von Wahl, *Strong solutions of the Navier-Stokes system on Lipschitz bounded domains*, Math. Nachr. **171** (1995), 111–148.

[10] E. Fabes, C. Kenig, and G. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J. **57** (1988), no. 3, 769–793.

[11] E. Fabes, O. Mendez, and M. Mitrea, *Boundary layers on Sobolev-Besov spaces and Poisson’s equation for the Laplacian in Lipschitz domains*, J. Funct. Anal. **159** (1998), 323–368.

[12] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[13] D. Fujiwara and H. Morimoto, *An $L_r$-theorem for the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo Sect. I-A Math. **24** (1977), 685–700.

[14] J. Geng and J. Kilty, *The $L^p$ regularity problem for the Stokes system on Lipschitz domains*, Preprint (2011).

[15] M Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Ann. of Math. Studies, vol. 105, Princeton Univ. Press, 1983.

[16] Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in $L_r$ spaces*, Math. Z. **178** (1981), 297–329.

[17] Y. Giga and T. Miyakawa, *Solutions in $L_r$ of the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal. **89** (1985), 267–281.

[18] D. Jerison and C. Kenig, *An identity with applications to harmonic measure*, Bull. Amer. Math. Soc. **2** (1980), 447–451.

[19] ______, *The Neumann problem in Lipschitz domains*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 203–207.

[20] C. Kenig, F. Lin, and Z. Shen, *Homogenization of elliptic systems with Neumann boundary conditions*, Preprint available at arXiv:1010.6114 (2010).

[21] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1963.

[22] N.N. Lebedev, *Special Functions and Their Applications*, Dover Publications, Inc., 1972, Translated and edited by R. Silverman.
[23] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge Univ. Press, 2000.

[24] M. Mitrea and S. Monniaux, *The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value problem in Lipschitz domains*, J. Funct. Anal. 254 (2008), 1522–1574.

[25] _, On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian Manifolds*, Trans. Amer. Math. Soc. 361 (2009), 3125–3157.

[26] Z. Shen, *Boundary value problems for parabolic Lame systems and a non stationary linearized system of Navier-Stokes equations in Lipschitz cylinders*, Amer. J. Math. 113 (1991), 293–373.

[27] _, Resolvent estimates in $L^p$ for elliptic systems in Lipschitz domains*, J. Funct. Anal. 133 (1995), 224–251.

[28] _, Bounds of Riesz transforms on $L^p$ spaces for second order elliptic operators*, Ann. Inst. Fourier (Grenoble) 55 (2005), 173–197.

[29] _, Necessary and sufficient conditions for the solvability of the $L^p$ Dirichlet problem on Lipschitz domains*, Math. Ann. 336 (2006), no. 3, 697–724.

[30] V.A. Solonnikov, *Estimates for solutions of nonstationary Navier-Stokes equations*, J. Soviet Math. 8 (1977), 467–529.

[31] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[32] M. Taylor, *Incompressible fluid flow on rough domains*, Semigroups of operators: theory and applications (Newport Beach, CA, 1998), Progr. Nonlinear Differential Equations Appl., vol. 42, Birkhäuser, Basel, 2000, pp. 320–334.

[33] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains*, J. Funct. Anal. 59 (1984), 572–611.

[34] W. von Wahl, *The Equations of Navier-Stokes and Abstract Parabolic Equations*, Vieweg, Braunschweig, 1985.

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