A proof of a support theorem for stochastic wave equations in Hölder norm with some general noises

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Abstract: In this paper we characterize the topological support in Hölder norm of the law of the solution to a stochastic wave equation with three-dimensional space variable is proved. This note is a continuation of [9] and [10]. The result is a consequence of an approximation theorem, in the convergence of probability, for a sequence of evolution equations driven by a family of regularizations of the driving noise. We extend two previous results on this subject. The first extension is that we cover the case of multiplicative noise and non-zero initial conditions. The second extension is related to the covariance function associated to the noise, here we follow the approach of Hu, Huang and Nualart and ask conditions in terms the of the mean Hölder continuity of such covariance function.

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1. Introduction

This paper is an extension and (in some sense) also continuation of [9] and [10] where we prove a characterization of the topological support in Hölder norm for the law of the solution of a stochastic wave equation. The main difference in the method used here, with respect to the one used in [9] and [10], is very technical and it is described in the Remark 2.1 below. This fact was noticed in [12] and here we have used very often; it allow us to get very good estimates for several quantities which leave us to prove Hölder continuity of the solution of the stochastic wave equation as in [12].

Consider the stochastic partial differential equation (SPDE)

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta \right) u(t, x) = \varsigma(u(t, x)) \dot{M}(t, x) + b(u(t, x)),
\]

\[
u_0(x) \quad \frac{\partial}{\partial t} u(0, x) = \tilde{v}_0(x),
\]

where \(\Delta\) denotes the Laplacian on \(\mathbb{R}^3\), \(T > 0\) is fixed, \(t \in (0, T]\) and \(x \in \mathbb{R}^3\). The non-linear terms and the initial conditions are defined by functions \(\varsigma, b : \mathbb{R} \to \mathbb{R}\) and \(\nu_0, \tilde{v}_0 : \mathbb{R}^3 \to \mathbb{R}\), respectively.
For the definition of the noise we follow [12]. The notation $\dot{M}(t, x)$ refers to the formal derivative of a Gaussian random field $M$ white in the time variable and with a correlation in the space variable given by some function. More specifically,

$$E\left(\dot{M}(t, x)\dot{M}(s, y)\right) = \delta_0(t-s)f(x-y),$$

(2)

where $\delta_0$ denotes the delta Dirac measure and $f$ is a non-negative, non-negative definite function which is a tempered distribution on $\mathbb{R}^3$ and then $f$ is locally integrable. We know that in this case $f$ is the Fourier transform of a non-negative tempered measure $\mu \in \mathbb{R}^3$ which is called the spectral measure of $f$. That is, for all $\varphi$ belonging to the space $\mathcal{S}(\mathbb{R}^3)$ of rapidly decreasing $C^\infty$ functions we have

$$\int_{\mathbb{R}^3} f(x)\varphi(x)dx = \int_{\mathbb{R}^3} \mathcal{F}\varphi(\xi)\mu(d\xi),$$

(3)

and there is an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^3} (1 + |\xi|^2)^{-m}\mu(d\xi) < +\infty$$

(4)

where $\mathcal{F}\varphi$ is the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^3)$:

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^3} \varphi(x)e^{-i\xi \cdot x}dx$$

Our basic assumption on $f$ is

$$\int_{|x| \leq 1} \frac{f(x)}{|x|}dx < +\infty$$

(5)

It is well-known (see for instance [5] or [7]) that this is equivalent to

$$\int_{|x| \leq 1} \frac{\mu(d\xi)}{1 + |\xi|^2} < +\infty,$$

(6)

and since we are in $\mathbb{R}^3$, (5) is satisfied if there is a $\kappa < 2$ such that in a neighborhood of 0, $f(x) \leq C|x|^{-\kappa}$.

Let $G(t)$ be the fundamental solution to the wave equation in dimension three, $G(t, dx) = \frac{1}{4\pi t^2} \sigma_1(dx)$, where $\sigma_1(x)$ denotes the uniform surface measure on the sphere of radius $t > 0$ with total mass $4\pi t^2$.

We consider a random field solution to the SPDE (1), which means a real-valued adapted (with respect to the natural filtration generated by the Gaussian process $M$) stochastic process $\{u(t, x), (t, x) \in (0, T] \times \mathbb{R}^3\}$ satisfying

$$u(t, x) = X^0(t, x) + \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y)\varphi(u(s, y))M(ds, dy)$$

$$+ \int_0^t [G(t-s, \cdot) \ast b(u(s, \cdot))] (x)ds.$$  

(7)
Here
\[ X^0(t, x) = [G(t) \ast \tilde{v}_0](x) + \left[ \frac{d}{dt}G(t) \ast v_0 \right](x), \]  
(8)
and the symbol “\ast” denotes the convolution in the spatial argument.

The stochastic integral (also termed stochastic convolution) in (7) is defined as a stochastic integral with respect to a sequence of independent standard Brownian motions \( \{W_j(s)\}_{j \in \mathbb{N}} \), as follows. Let \( \mathcal{H} \) be the Hilbert space defined by the completion of \( \mathcal{S}(\mathbb{R}^3) \), endowed with the semi-inner product

\[ \langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \mu(\xi) \mathcal{F} \varphi(\xi) \mathcal{F} \psi(\xi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(y) \psi(x)f(x-y)dydx, \]

where \( \mu \) is the spectral measure of \( f \). Then

\[ \int_0^t \int_{\mathbb{R}^3} G(t-s, y) \zeta(u(s, y))M(ds, dy) \]
\[ := \sum_{j \in \mathbb{N}} \int_0^t \langle G(t-s, x-\cdot) \zeta(u(s, \cdot)), e_j \rangle_{\mathcal{H}} W_j(ds), \]
(9)
where \( (e_j)_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^3) \) is a complete orthonormal basis of \( \mathcal{H} \).

Assume that \( \varphi \in \mathcal{H} \) is a signed measure with finite total variation. Then, by applying [13, Theorem 5.2] (see also [15, Lemma 12.12] for the case of probability measures with compact support) and a polarization argument on the positive and negative parts of \( \varphi \), we obtain

\[ \|\varphi\|^2_{\mathcal{H}} = C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(dx) \varphi(dy)f(x-y) = C \int_{\mathbb{R}^3} \mu(\xi) \mathcal{F} \varphi(\xi). \]
(10)

For \( t_0 \in [0, T], K \subset \mathbb{R}^3 \) compact and \( \rho \in (0, 1) \), we denote by \( \mathcal{C}^\rho([t_0, T] \times K) \) the space of real functions \( g \) such that \( \|g\|_{\rho,t_0,K} < \infty \), where

\[ \|g\|_{\rho,t_0,K} := \sup_{(t,x) \in [t_0,T] \times K} |g(t,x)| + \sup_{(t,s),(t,x) \in [t_0,T] \times K, (t,x) \neq (t,\tilde{x})} \frac{|g(t,x) - g(t,\tilde{x})|}{(|t-t| + |x-\tilde{x}|)^\rho}. \]

Let \( 0 < \rho' < \rho \) and \( \mathcal{E}^{\rho'}([t_0, T] \times K) \) be the space of Hölder continuous functions \( g \) of degree \( \rho' \) such that

\[ O_g(\delta) := \sup_{|t-s|+|x-y|<\delta} \frac{|g(t,x) - g(s,y)|}{(|t-s| + |x-y|)^{\rho'}} \to 0, \text{ if } \delta \to 0. \]
(11)
The space \( \mathcal{E}^{\rho'}([t_0, T] \times K) \) endowed with the norm \( \| \cdot \|_{\rho',t_0,K} \) is a Polish space and the embedding \( \mathcal{C}^\rho([t_0, T] \times K) \subset \mathcal{E}^{\rho'}([t_0, T] \times K) \) is compact.

Assume that the functions \( \zeta \) and \( b \) are Lipschitz continuous and the initial conditions \( v_0, \tilde{v}_0 \) satisfy the assumption a) from Hypothesis 1 below. When \( f \) is the Riesz kernel, i.e. \( f(x) = |x|^{-\beta} \), with \( \beta \in [0, 2] \), Theorem 4.11 in [8] along with [7, Proposition 2.6] give the existence of a random field solution to (7)
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with sample paths in $C^\rho([0,T] \times K)$, with $\rho \in \left(0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2}\right)$. In our case one can prove the existence and uniqueness of the solution to (1) in the same way as in Theorem 4.3 in [7]; indeed, in the appendix we sketch a proof of such result (see Teorema 5.1).

For any $t \in (0,T]$, let $H_T = L^2([0,t]; H)$. Fix $h \in H_T$ and consider the deterministic evolution equation

$$\Phi^h(t,x) = X^0(t,x) + \langle G(t-\cdot, x-\cdot)\varsigma(\Phi^h(\cdot, \cdot)), h \rangle_{H_t} + \int_0^t ds \left[G(t-s, \cdot) (\Phi^h(s, \cdot)) \right](x).$$

The main objective in [9] is to prove that, in the particular case $v_0 = \tilde{v}_0 = 0$, the topological support of the law of the solution to (7) in the space $E^\rho([t_0,T] \times K)$ with $\rho \in \left(0, 2-\beta \right)$ is the closure in the Hölder norm of the set $\{\Phi^h, h \in H_T\}$, for any $t_0 > 0$. (see [9, Theorem 3.1]). In [10] they obtained a similar result for the case of non zero initial conditions but restricted to the case of the function $\varsigma$ being a linear function, this is called the affine case. Their approach is based on the fractional Sobolev imbedding theorem and the Fourier transform technique.

The aim of this paper is to prove a extension of this result allowing non null initial conditions $v_0$, $\tilde{v}_0$, with a multiplicative noise (i.e. we will let $\varsigma$ to be globally Lipschitz) and the covariance functional will satisfy the Hypothesis 1.(b) and Hypothesis 2.(a),(c) (defined below). Notice that with these assumptions we will cover the case of the Riesz Kernel (see [12] for more details). The theorem is stated below.

We will use the following two sets of hypotheses (see [12]).

**Hypothesis 1.**

(a) $v_0 \in C^2(\mathbb{R}^3)$, $v_0$, $\nabla v_0$ are bounded and $\Delta v_0$ and $\bar{v}_0$ are Hölder continuous of orders $\gamma_1$ and $\gamma_2$, respectively, $\gamma_1, \gamma_2 \in [0,1]$.

(b) The function $f$ satisfies condition (5) and for some $\gamma \in [0,1]$ and $\gamma' \in [0,2]$ we have for all $w \in \mathbb{R}^3$

$$\int_{|z| \leq 2T} \frac{|f(z+w) - f(z)|}{|z|} dz \leq C|w|^{\gamma} \quad (13)$$

and

$$\int_{|z| \leq 2T} \frac{|f(z+w) - 2f(z) + f(z-w)|}{|z|} dz \leq C|w|^{\gamma'} \quad (14)$$

For the Hölder continuity in time we will use, in addition, the following set of hypothesis. Let $S^2$ denote the unit sphere in $\mathbb{R}^3$ and $\sigma(d\xi)$ the uniform measure on it.

**Hypothesis 2.**

(a) For some $0 < \nu \leq 1$,

$$\int_{|z| \leq h} \frac{f(z)}{|z|} dz \leq C|h|^\nu \quad \text{for any } 0 < h \leq 2T \quad (15)$$
For some $0 < \kappa_1 \leq 1$ and for any $q \geq 2$ and $t \in (0, T]$ we have

$$\mathbb{E}\left(|u(t, x) - u(t, y)|^q 1_{L_n(t)} \right) \leq C|x - y|^q \kappa_1. \quad (16)$$

where $L_n(t)$ is defined in (32).

(c) Let $\xi$ and $\eta$ unit vectors in $\mathbb{R}^3$ and $0 < h \leq 1$. We have

$$\int_0^T \int_{S^2} \int_{S^2} \left| f(s(\xi + \eta) + h(\xi + \eta)) - f(s(\xi + \eta)) \right| \sigma(d\xi) \sigma(d\eta) ds \leq C|h|^{\rho_1}, \quad (17)$$

for some $\rho_1 \in [0, 1]$, and

$$\int_0^T \int_{S^2} \int_{S^2} \left| f(s(\xi + \eta) + h(\xi + \eta) - f(s(\xi + \eta) + h\xi) - f(s(\xi + \eta) + h\eta) + f(s(\xi + \eta)) \right| s^2 \sigma(d\xi) \sigma(d\eta) ds \leq C|h|^{\rho_2}$$

for some $\rho_2 \in [0, 2]$.

Notice that when we prove the results, first we will prove the Hölder continuity in space and after that we will prove the Hölder continuity in time, so at the point we use (b) of the hypothesis 2 we will have established it.

**Theorem 1.1.** Assume that

(1) the functions $\zeta$ and $b$ are Lipschitz continuous;

(2) Assume Hypothesis 1 and Hypothesis 2 hold.

Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Then the topological support of the law of the solution to (7) in the space $\mathcal{E}^\rho([t_0, T] \times K)$ with $\rho \in \left[0, \min\left(\gamma_1, \gamma_2, \gamma, \frac{\gamma}{2}, \frac{\gamma_1 + \kappa}{2}, \frac{\gamma_2 + \kappa}{2}\right)\right]$ is the closure in the Hölder norm $\| \cdot \|_{\rho, t_0, K}$ of the set $\{ \Phi^h, h \in \mathcal{H}_T \}$, where $\Phi^h$ is given in (12) and $\kappa \in \left[0, \min\left(\gamma_1, \gamma_2, \gamma, \frac{\gamma}{2}\right)\right]$.

After the seminal paper [19], an extensive literature on support theorems for stochastic differential equations appeared (see for example, [1], [11], [17], and references herein). The analysis of the uniqueness of invariant measures is one of the motivations for the characterization of the support of stochastic evolution equations (see [9, Section 1] for more details).

As in [9, 10], Theorem 1.1 will be a corollary of a general result on approximations of Equation (7) by a sequence of SPDEs obtained by smoothing the noise $M$. The precise statement, given in Theorem 2.2, provides a Wong-Zakai type theorem in Hölder norm. The method relies on [1], further developed and used in [2], [11], [16], [17], [18]. We refer the reader to [9, Section 1] for a detailed description of the method for the proof of support theorems based on approximations.

As in [9] (see also [10]) we apply the approximation method of [17] to obtain a characterization of the topological support of the law of $u$ (the solution to (7)) in the Hölder norm $\| \cdot \|_{\rho, t_0, K}$. The core of the work consists of an approximation
result for a family of equations more general than equation (7) by a sequence of pathwise evolution equations obtained by a smooth approximation of the driving process $M$. In finite dimensions, the celebrated Wong–Zakai approximations for diffusions in the supremum norm could be considered as an analogue. However there are two substantial differences, first the type of equation we consider in this paper is much more complex, and moreover we deal with a stronger topology.

For the sake of completeness, we give a brief description of the procedure of [16] in the particular context of this work, and refer the reader to [16] for further details.

Let $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mu})$ be the canonical space of a standard real-valued Brownian motion on $[0, T]$. In the sequel, the reference probability space will be $(\Omega, \mathcal{G}, \mathbb{P}) := (\tilde{\Omega}^\mathbb{N}, \tilde{\mathcal{G}}^\mathbb{N}, \tilde{\mu}^\mathbb{N})$. By the preceding identification of $M$ with $(W_j, j \in \mathbb{N})$, this is the canonical probability space of $M$.

Assume that there exists a measurable mapping $\xi_1 : L^2([0, T]; \ell^2) \to \mathcal{C}^\rho([t_0, T] \times K)$, and a sequence $w^n : \Omega \to L^2([0, T]; \ell^2)$ such that for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\{\|u - \xi_1(w^n)\|_{\rho, t_0, K} > \varepsilon\} = 0.$$  \hfill (19)

Then $\text{supp}(u \circ \mathbb{P}^{-1}) \subset \overline{\xi_1(L^2([0, T]; \ell^2))}$, where the closure refers to the Hölder norm $\| \cdot \|_{\rho, t_0, K}$.

Next, we assume that there exists a mapping $\xi_2 : L^2([0, T]; \ell^2) \to \mathcal{C}^\rho([t_0, T] \times K)$ and for any $h \in L^2([0, T]; \ell^2)$, we suppose that there exist a sequence $T_n^h : \Omega \to \Omega$ of measurable transformations such that, for any $n \geq 1$, the probability $\mathbb{P} \circ (T_n^h)^{-1}$ is absolutely continuous with respect to $\mathbb{P}$ and, for any $h \in L^2([0, T]; \ell^2)$, $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\{\|u(T_n^h) - \xi_2(h)\|_{\rho, t_0, K} < \varepsilon\} > 0.$$  \hfill (20)

Then $\text{supp}(u \circ \mathbb{P}^{-1}) \supset \overline{\xi_2(L^2([0, T]; \ell^2))}$.

For any $h \in L^2([0, T]; \ell^2)$ (or equivalently, $h \in \mathcal{H}_T$), recall the deterministic evolution equation $\Phi^h(t, x)$ given by (12) and similarly as for $u$, the mapping $(t, x) \in [t_0, T] \times K \mapsto \Phi^h(t, x)$ belongs to $\mathcal{C}^\rho([t_0, T] \times K)$.

Let $\xi_1(h) = \xi_2(h) = \Phi^h$, and $(w^n)_{n \geq 1}$ be given by (22). From (23) and the isometric representation of $\mathcal{H}_T$, we see that $w^n : \Omega \to L^2([0, T]; \ell^2)$. Given $h \in L^2([0, T]; \ell^2)$, we define

$$T_n^h(\omega) = \omega + h - w^n.$$  \hfill (21)

By Girsanov’s theorem, the probability $\mathbb{P} \circ (T_n^h)^{-1}$ is absolutely continuous with respect to $\mathbb{P}$.

According to (19), (20), the final objective is to prove that

$$\lim_{n \to \infty} \Phi^{w_n} = u, \quad \lim_{n \to \infty} u(T_n^h) = \Phi^h,$$

in probability and with the Hölder norm $\| \cdot \|_{\rho, t_0, T}$. Then, by the preceding discussion we infer that the support of the law of $u$ in the Hölder norm is the
closure of the set of functions \( \{ \Phi^h, h \in \mathcal{H}_T \} \) (see Theorem 3.1 for the rigorous statement). Notice that the characterization of the support does not depend on the approximating sequence \((w^n)_{n \in \mathbb{N}}\).

The structure of this paper is exactly the same as the one in [9] and it is structured as follows. The next Section 2 is devoted to a general approximation result (see [9] for the details of the approximation). Section 3 is devoted to the proof of the characterization of the support of \( u \). It is a corollary of Theorem 2.2. Section 4 is of technical character. It is devoted to establish some auxiliary results which are needed in some proofs of Section 2. In the Appendix, a theorem on existence and uniqueness of a random field solution for a quite general evolution equation is proved. It provides the rigorous setting for all the stochastic partial differential equations that appear in this paper. The section also contains a known but fundamental result used at some crucial parts of the proofs of Sections 2 and 3.

As was formulated in [2], and further developed in [18], there are two main elements in the proof of Theorem 2.2: a control on the \( L^p(\Omega) \)-increments in time and in space of the processes \( X \) and \( X_n \), independently of \( n \), and \( L^p(\Omega) \) convergence of \( X_n(t, x) \) to \( X(t, x) \), for any \((t, x)\). The precise assertions are given in Theorems 2.3 and 2.4, respectively.

For an Itô’s stochastic differential equation, smoothing the noise leads to a Stratonovich (or pathwise) type integral, and the correction term between the two kinds of integrals appears naturally in the approximating scheme. In our setting, correction terms explode and therefore they must be avoided. Instead, a control on the growth of the regularized noise is used. This method was introduced in [18] and successfully used in [9] and [10]. Here we use it too. The control is achieved by introducing a localization in \( \Omega \) (see (32)). With this method, the convergence of the approximating sequence \( X_n \) to \( X \) takes place in probability.

Throughout the paper, we shall often call different positive and finite constants by the same notation, even if they differ from one place to another.

2. Approximations of the wave equation

Consider smooth approximations of \( W \) defined as follows. Fix \( n \in \mathbb{N} \) and consider the partition of \([0, T]\) determined by \( \frac{it}{2^n}, i = 0, 1, \ldots, 2^n \). Denote by \( \Delta_i \) the interval \( \left[ \frac{it}{2^n}, \frac{(i+1)t}{2^n} \right] \) and by \( |\Delta_i| \) its length. We write \( W_j(\Delta_i) \) for the increment \( W_j\left( \frac{(i+1)t}{2^n} \right) - W_j\left( \frac{it}{2^n} \right), i = 0, \ldots, 2^n - 1, j \in \mathbb{N} \). Define differentiable approximations of \((W_j, j \in \mathbb{N})\) as follows:

\[
W^n = \left( W^n_j = \int_0 \dot{W}^n_j(s) \, ds, j \in \mathbb{N} \right),
\]

where for \( j > n, \dot{W}^n_j = 0 \), and for \( 1 \leq j \leq n, \)

\[
\dot{W}^n_j(t) = \begin{cases} 
\sum_{i=0}^{2^n-2} 2^{nT-1} W_j(\Delta_i) 1_{\Delta_{i+1}}(t) & \text{if } t \in \left[ 2^{-n}T, T \right], \\
0 & \text{if } t \in \left[ 0, 2^{-n}T \right].
\end{cases}
\]
Set
\[ w^n(t, x) = \sum_{j\in\mathbb{N}} \hat{W}_j^n(t)e_j(x). \] (22)

It is easy to check that, for any \( p \in [2, \infty[ , \)
\[ \|w^n\|_{L^p(\Omega, \mathcal{H}_T)} \leq Cn^{1/2}2^n/2. \] (23)

In particular, from (23) it follows that \( w^n \) belongs to \( \mathcal{H}_T \) a.s.

In this section, we shall consider the equations

\[
X(t, x) = X^0(t, x) + \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y)(A + B)(X(s, y))M(ds, dy) \\
+ \langle G(t - \cdot, x - \cdot)D(X(\cdot, \cdot)), h \rangle_{\mathcal{H}_t} \\
+ \int_0^t [G(t-s, \cdot) \ast b(X(s, \cdot))](x) \, ds,
\]

\[
X_n(t, x) = X^0(t, x) + \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y)A(X_n(s, y))M(ds, dy) \\
+ \langle G(t - \cdot, x - \cdot)B(X_n(\cdot, \cdot)), w^n \rangle_{\mathcal{H}_t} \\
+ \langle G(t - \cdot, x - \cdot)D(X_n(\cdot, \cdot)), h \rangle_{\mathcal{H}_t} \\
+ \int_0^t [G(t-s, \cdot) \ast b(X_n(s, \cdot))](x) \, ds,
\]

where \( n \in \mathbb{N}, h \in \mathcal{H}_T, w^n \) defined as in (22) and \( A, B, D, b : \mathbb{R} \rightarrow \mathbb{R} \).

Moreover, we also need the slight modification of these equations defined by

\[
X_n^-(t, x) = X^0(t, x) + \int_0^{t_n} \int_{\mathbb{R}^3} G(t-s, x-y)A(X_n(s, y))M(ds, dy) \\
+ \langle G(t - \cdot, x - \cdot)B(X_n(\cdot, \cdot))1_{[0, t_n]}(\cdot), w^n \rangle_{\mathcal{H}_t} \\
+ \langle G(t - \cdot, x - \cdot)D(X_n(\cdot, \cdot))1_{[0, t_n]}(\cdot), h \rangle_{\mathcal{H}_t} \\
+ \int_0^{t_n} [G(t-s, \cdot) \ast b(X_n(s, \cdot))](x) \, ds,
\]

\[
X(t, t_n, x) = X^0(t, x) + \int_{t_n}^t \int_{\mathbb{R}^3} G(t-s, x-y)(A + B)(X(s, y))M(ds, dy) \\
+ \langle G(t - \cdot, x - \cdot)D(X(\cdot, \cdot))1_{[0, t_n]}(\cdot), h \rangle_{\mathcal{H}_t} \\
+ \int_0^{t_n} [G(t-s, \cdot) \ast b(X(s, \cdot))](x) \, ds,
\]

where for any \( n \in \mathbb{N}, t \in [0, T], t_n = \max\{t_n - 2^{-n}T, 0\} \), with
\[
L_n = \max\{k2^{-n}T, k = 1, \ldots, 2^n - 1 : k2^{-n}T \leq t\}. \] (28)

We will consider the following assumption during all paper.
Hypothesis 3. The coefficients $A, B, D, b : \mathbb{R} \to \mathbb{R}$ are globally Lipschitz continuous.

Notice that equation (25) is more general than (24) and (7). In Theorem 5.1, we prove a result on existence and uniqueness of a random field solution to a class of SPDEs which applies to equation (25).

Remark 2.1. In opposition to Remark A.2 in [9], here we have not the translation invariance of the moments. Following [12], we have the following facts that will be used at several points in the proofs of the results of the following section.

For any $z \in \mathbb{R}^3$, set $z_1 = x - z$ and $z_2 = y - z$ then trivially $z_1 - z_2 = x - y$.

Note the following. First, we see that

$$\sup_{z \in \mathbb{R}^3} \mathbb{E}\left( |X(t, x - z) - X(t, y - z)|^p \right) = \sup_{z_1 - z_2 = x - y} \mathbb{E}\left( |X(t, z_1) - X(t, z_2)|^p \right).$$

Secondly, from $z_1 = x - z$ and $z_2 = y - z$ we have $x = z_1 + z$ and $y = z_2 + z$, then

$$\mathbb{E}\left( |X(t, x) - X(t, y)|^p \right) = \mathbb{E}\left( |X(t, z_1 + z) - X(t, z_2 + z)|^p \right)$$

and again $x - y = z_1 - z_2$ then

$$\sup_{z \in \mathbb{R}^3} \mathbb{E}\left( |X(t, z_1 + z) - X(t, z_2 + z)|^p \right) = \sup_{x - y = z_1 - z_2} \mathbb{E}\left( |X(t, x) - X(t, y)|^p \right),$$

for any $x, y, z_1, z_2 \in \mathbb{R}^3$ and any $p \in [1, \infty]$. Consequently, a similar property also holds for $X_n^x(t, *)$ and $X_n(t, n, *)$ defined in (26), (27), respectively.

The aim of this section is to prove the following theorem.

Theorem 2.2. We assume (Hypothesis 1) and Hypothesis 2 holds. Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Set $\gamma, \gamma_1, \gamma_2, \rho_1, \nu \in [0, 1], \gamma', \rho_2 \in [0, 2]$ and $\kappa \in [0, \min\left(\gamma_1, \gamma_2, \gamma, \gamma', \frac{\rho_1}{2}, \frac{\rho_2}{2}\right)]$. Then for any $\rho \in [0, \min\left(\gamma_1, \gamma_2, \gamma, \gamma', \frac{\rho_1}{2}, \frac{\rho_2}{2}\right)]$ and $\lambda > 0$,

$$\lim_{n \to \infty} \mathbb{P}\left( \|X_n - X\|_{\rho, t_0, K} > \lambda \right) = 0. \quad (31)$$

The convergence (31) will be proved through several steps. The main ingredients are local $L^p$ estimates of increments of $X_n$ and $X$, in time and in space, and a local $L^p$ convergence of the sequence $X_n(t, x)$ to $X(t, x)$.

Let us describe the localization procedure (see [18]). Fix $\alpha > 0$. For any integer $n \geq 1$ and $t \in [0, T]$, define

$$L_n(t) = \left\{ \sup_{1 \leq j \leq n} \sup_{0 \leq i \leq \lfloor 2^n t \lfloor^{t - 1} + 1} |W_j(\Delta_i)| \leq \alpha n^{1/2} 2^{-n/2} \right\}, \quad (32)$$

where $\alpha > (2 \ln 2)^{1/2}$. Notice that the sets $L_n(t)$ decrease with $t \geq 0$. Moreover, in [18], Lemma 2.1, it is proved that $\lim_{n \to \infty} \mathbb{P}(L_n(t)^c) = 0$. 
It is easy to check that
\[ \| w^n(t, \ast) \|_{L^n} \leq C n^{3/2} 2^{n/2} . \] (33)
Moreover, for any \( 0 \leq t \leq t' \leq T \)
\[ \| w^n 1_{L_n(t')} 1_{[t, t']} \|_{L^n} \leq C n^{3/2} 2^{n/2} |t' - t|^{1/2} . \]
In particular, if \([t, t'] \subset \Delta_i\) for some \( i = 0, \ldots, 2^n - 1 \), then
\[ \| w^n 1_{L_n(t')} 1_{[t, t']} \|_{L^n} \leq C n^{3/2} . \] (34)

As has been announced in the Introduction, the proof of Theorem 2.2 will follow from Theorems 2.3 and 2.4 below. We denote by \( \| \cdot \|_p \) the \( L^p(\Omega) \) norm.

**Theorem 2.3.** We assume (Hypothesis 1) and Hypothesis 2 holds. Fix \( t_0 \in ]0, T[ \) and a compact subset \( K \subset \mathbb{R}^3 \). Let \( t_0 \leq t \leq \bar{t} \leq T, \ x, \bar{x} \in K \). Set \( \gamma, \gamma_1, \gamma_2, \rho, \nu \in ]0, 1[ \) and \( \kappa \in ]0, \min \left( \gamma_1, \gamma_2, \gamma, \gamma_2 / 2, \gamma_1 / 2, \nu + 1, \nu, \beta_2 / 2 \right) [ \). Then, for any \( p \in [1, \infty) \) and any \( \rho \in ]0, \min \left( \gamma_1, \gamma_2, \gamma, \gamma_2 / 2, \gamma_1 / 2, \nu + 1, \nu, \beta_2 / 2 \right) [ \), there exists a positive constant \( C \) such that
\[ \sup_{n \geq 1} \left\| X_n(t, x) - X_n(\bar{t}, \bar{x}) \right\|_p \leq C \left( |\bar{t} - t| + |\bar{x} - x| \right)^\rho . \] (35)

**Theorem 2.4.** The assumptions are the same as in Theorem 2.3. Fix \( t \in [t_0, T], \ x \in \mathbb{R}^3 \). Then, for any \( p \in [1, \infty) \)
\[ \lim_{n \to \infty} \sup_{t \in [t_0, T], \ x \in K(t)} \left\| X_n(t, x) - X(t, x) \right\|_p = 0 , \] (36)
where for \( t \in [0, T], \)
\[ K(t) = \{ x \in \mathbb{R}^3 : d(x, K) \leq T - t \} , \]
and \( d \) denotes the Euclidean distance.

The proof of Theorem 2.3 is carried out through two steps. First, we shall consider \( t = \bar{t} \) and obtain (35), uniformly in \( t \in [t_0, T] \). Using this, we will consider \( x = \bar{x} \) and establish (35), uniformly in \( x \in K \). We devote the next two subsections to the proof of these results.

### 2.1. Increments in space

Throughout this section, we fix \( t_0 \in ]0, T[ \) and a compact set \( K \subset \mathbb{R}^3 \). The objective is to prove the following proposition.
Proposition 2.5. Suppose that Hypothesis 1 holds. Fix \( t \in [t_0, T] \) and \( x, \bar{x} \in K \). Then, for any \( p \in [1, \infty) \) and \( \rho \in \left]0, \min\left(\gamma_1, \gamma_2, \gamma, \frac{1}{4}\right)\right] \), there exists a finite constant \( C \) such that

\[
\sup_{n \geq 0} \sup_{t \in [t_0, T]} \left\| (X_n(t, x) - X_n(t, \bar{x}))1_{L_n(t)} \right\|_p \leq C|x - \bar{x}|^\rho. \tag{37}
\]

In the next lemma, we give an abstract result that will be used throughout the proofs. We start by introducing some notation.

For a function \( f : \mathbb{R}^3 \to \mathbb{R} \), we set

\[
\begin{align*}
Df(u, x) &= f(u + x) - f(u), \\
\tilde{D}^2 f(u, x, y) &= f(u + x + y) - f(u + y) - f(u + x) + f(u), \\
D^2 f(u, x) &= \tilde{D}^2 f(u, x, x) = f(u - x) - 2f(u) + f(u + x).
\end{align*}
\]

Lemma 2.6. Consider a sequence of predictable stochastic processes \( \{Z_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\} \), \( n \in \mathbb{N} \), such that, for any \( p \in [2, \infty] \),

\[
\sup_n \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \mathbb{E}(\left| Z_n(t, x) \right|^p) < C, \tag{38}
\]

for some finite constant \( C \). For any \( t \in [0, T] \), \( x, \bar{x} \in \mathbb{R}^3 \), we define

\[
I_n(t, x, \bar{x}) := \int_0^t ds \left\| Z_n(s, *) [G(t - s, x - *) - G(t - s, \bar{x} - *)] \right\|_H^2.
\]

Then, for any \( p \in [2, \infty] \),

\[
\mathbb{E}(\left| I_n(t, x, \bar{x}) \right|^{p/2}) \leq C \left\{ |x - \bar{x}|^{\gamma p} + |x - \bar{x}|^{\gamma' p/2} \right. \\
+ \left. \int_0^t ds \left( \sup_{z_1, z_2 \neq x - \bar{x}} \mathbb{E}(\left| Z_n(s, z_1) - Z_n(s, z_2) \right|^p) \right)^{p/2} \right\}
\]

where \( \gamma \in [0, 1] \) and \( \gamma' \in [0, 2] \).

Proof. Set \( w = x - \bar{x} \). First, we notice that \( I_n(t, x, \bar{x}) \) is the second order moment of the stochastic integral

\[
\int_0^t \int_{\mathbb{R}^3} Z_n(s, y) [G(t - s, x - y) - G(t - s, \bar{x} - y)] M(ds, dy).
\]

We write \( I_n(t, x, \bar{x}) \) using (10). This yields

\[
I_n(t, x, \bar{x}) = C \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Z_n(s, u) Z_n(s, v) [G(t - s, x - du) - G(t - s, \bar{x} - du)] \\
\times [G(t - s, x - dv) - G(t - s, \bar{x} - dv)] f(u - v).
\]
Then, as in [8], pages 19–20, we see that, by decomposing this expression into the sum of four integrals, by applying a change of variables and rearranging terms, we have

\[ I_n(t, x, \bar{x}) = C \sum_{i=1}^{4} J_i^{t}(x, \bar{x}), \]

where, for \(i = 1, \ldots, 4,\)

\[ J_i^{t}(x, \bar{x}) = \int_{0}^{t} ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du)G(s, dv)h_i(x, t, s, u, v) \]

with

\[ h_1(x, \bar{x}; t, s, u, v) = f(\bar{x} - x + v - u)\left[Z_n(t - s, x - u) - Z_n(t - s, \bar{x} - u)\right] \times \left[Z_n(t - s, x - v) - Z_n(t - s, \bar{x} - v)\right], \]

\[ h_2(x, \bar{x}; t, s, u, v) = Df(v - u, x - \bar{x})Z_n(t - s, x - u) \times \left[Z_n(t - s, x - v) - Z_n(t - s, \bar{x} - v)\right], \]

\[ h_3(x, \bar{x}; t, s, u, v) = Df(v - u, \bar{x} - x)Z_n(t - s, \bar{x} - v) \times \left[Z_n(t - s, x - u) - Z_n(t - s, \bar{x} - u)\right], \]

\[ h_4(x, \bar{x}; t, s, u, v) = -D^2f(v - u, x - \bar{x})Z_n(t - s, x - u)Z_n(t - s, x - v). \]

Fix \(p \in [2, \infty[.\) It holds that

\[ \mathbb{E}\left(|I_n(t, x, \bar{x})|^{p/2}\right) \leq C \sum_{i=1}^{4} \mathbb{E}(|J_i^{t}(x, \bar{x})|^{p/2}). \quad (40) \]

The next purpose is to obtain estimates for each term on the right hand-side of (40). Let

\[ \mu_1(x, \bar{x}) = \sup_{s \in [0, T]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du)G(s, dv)f(\bar{x} - x + v - u). \]

By Lemma 7.1 in [12] and (5) we have that

\[ \sup_{x, \bar{x}} \mu_1(x, \bar{x}) < \infty \]

Hence using firstly Hölder’s inequality and then Cauchy–Schwarz’s inequality.
Indeed, this follows from Lemma 7.1 in \[ \text{The following property holds: there exists a positive finite constant } C \]

along with remark 2.1, we see that

\[
\mathbb{E} \left( |J^t_1(x, \bar{x})|^{p/2} \right) \\
\leq \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) f(\bar{x} - x + v - u) \right)^{(p/2) - 1} \\
\times \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) f(\bar{x} - x + v - u) \\
\times \mathbb{E} \left( |Z_n(t - s, x - u) - Z_n(t - s, \bar{x} - u)| \right. \\
\times \left. |Z_n(t - s, x - v) - Z_n(t - s, \bar{x} - v)| \right)^{p/2} \\
\leq C \sup_{x, \bar{x}} \mu_1(x, \bar{x})^{p/2} \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E} \left( |Z_n(t - s, z_1) - Z_n(t - s, z_2)|^p \right) \\
\leq C \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E} \left( |Z_n(s, z_1) - Z_n(s, z_2)|^p \right). \quad (41)
\]

Set

\[
\mu_2(x, \bar{x}) = \sup_{s \in [0, T]} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \right| \quad (42)
\]

The following property holds: there exists a positive finite constant \( C \) such that

\[
\mu_2(x, \bar{x}) \leq C |x - \bar{x}|^\gamma, \quad \gamma \in \left] 0, 1 \right].
\]

Indeed, this follows from Lemma 7.1 in [12] and (b) in Hypothesis 1:

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \leq C \int_{|z| \leq 2T} \frac{|f(z + w) - f(z)|}{|z|} \\
\leq C |w|^\gamma = C |x - \bar{x}|^\gamma
\]

Now, using the inequality \( ab \leq \frac{a^2 + b^2}{2} \) we write

\[
\mathbb{E} \left( |J^t_2(x, \bar{x})|^{p/2} \right) \leq C \mathbb{E} \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \\
\times |Z_n(t - s, x - u) - Z_n(t - s, \bar{x} - u)| \right)^{p/2} \\
\leq C \mathbb{E} \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \\
\times |w|^{\gamma} |Z_n(t - s, x - u)|^2 \right)^{p/2} \\
+ C \mathbb{E} \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \\
\times |w|^{-\gamma} |Z_n(t - s, x - v) - Z_n(t - s, \bar{x} - v)|^2 \right)^{p/2} \leq C \left( \mathbb{E} \left[ |J^t_{2,1}(x, \bar{x})|^{p/2} \right] + \mathbb{E} \left[ |J^t_{2,2}(x, \bar{x})|^{p/2} \right] \right).
\]
Using Hölder’s and (38) along with the bound for \(\mu_2\), we have

\[
\mathbb{E}(J_{2,1}^t(x, \bar{x}) | p/2) \leq C|w|^{\gamma p/2} \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \right)^{p/2}
\]

\[
\times \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}(|Z_n(t, x)|^p)
\]

\[
\leq C|w|^{\gamma p/2} \left( \sup_{s \in [0,T]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \right)^{p/2}
\]

\[
\leq C|x - \bar{x}|^{\gamma p/2} |x - \bar{x}|^{\gamma p/2} = C|x - \bar{x}|^p
\]

with \(\gamma \in [0, 1]\).

Using Hölder’s inequality, Remark 2.1 and the bound for \(\mu_2\), we have

\[
\mathbb{E}(J_{2,2}^t(x, \bar{x}) | p/2) \leq C|w|^{-\gamma p/2} \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \right)^{-\frac{p}{2}}
\]

\[
\times \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(|Z_n(t, z_1) - Z_n(t, z_2)|^p)
\]

\[
\times \sup_{s \in [0,T]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \right)^{p/2}
\]

\[
\leq C|w|^{-\gamma p/2} \left( \sup_{s \in [0,T]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du) G(s, dv) |Df(v - u, x - \bar{x})| \right)^{p/2}
\]

\[
\times \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(|Z_n(t, z_1) - Z_n(t, z_2)|^p)
\]

\[
\leq C|w|^{-\gamma p/2} |w|^{\gamma p/2} \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(|Z_n(t, z_1) - Z_n(t, z_2)|^p)
\]

\[
\leq C \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(|Z_n(t, z_1) - Z_n(t, z_2)|^p)
\]

Thus,

\[
\mathbb{E}(J_{1,2}^t(x, \bar{x}) | p/2) \leq C \left( |x - \bar{x}|^p + \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(|Z_n(t, z_1) - Z_n(t, z_2)|^p) \right)
\]

with \(\gamma \in [0, 1]\).

Similarly,

\[
\mathbb{E}(J_{1,3}^t(x, \bar{x}) | p/2) \leq C \left( |x - \bar{x}|^p + \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(|Z_n(t, z_1) - Z_n(t, z_2)|^p) \right)
\]

with \(\gamma \in [0, 1]\).
Hölder’s and Cauchy–Schwarz’s inequalities, along with (38) and (14), imply
\[
\mathbb{E}(\lvert J_t^\varepsilon(x, \bar{x}) \rvert^{p/2}) \leq C \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du)G(s, dv) \lvert D^2 f(v - u, x - \bar{x}) \rvert \right)^{(p/2) - 1}
\times \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du)G(s, dv) \lvert D^2 f(v - u, x - \bar{x}) \rvert
\times \mathbb{E}(\lvert Z_n(t - s, x - u)Z_n(t - s, \bar{x} - v) \rvert^{p/2})
\leq C \left( \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(s, du)G(s, dv) \lvert D^2 f(v - u, x - \bar{x}) \rvert \right)^{p/2}
\times \sup_{n} \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \mathbb{E}(\lvert Z_n(t, x) \rvert^p)
\leq C \left( \int_0^t ds \int_{\lvert z \rvert \leq 2T} \frac{\lvert f(z + w) - 2f(z) - f(z - w) \rvert}{\lvert z \rvert} ds \right)^{p/2}
\leq C \lvert x - \bar{x} \rvert^{\gamma/p^2}.
\] (45)

From (40), (41), (43), (44) and (45), we obtain (39).

For any \( t \in [t_0, T] \), \( x, \bar{x} \in K \), \( p \in [1, \infty] \) set \( w = x - \bar{x} \) and
\[
\varphi^0_{n,p}(t, x, \bar{x}) = \sup_{z_1 - z_2 = w} \mathbb{E}(\lvert X_n(t, z_1) - X_n(t, z_2) \rvert^p 1_{L_n(t)}),
\varphi^-_{n,p}(t, x, \bar{x}) = \sup_{z_1 - z_2 = w} \mathbb{E}(\lvert X^-_n(t, z_1) - X^-_n(t, z_2) \rvert^p 1_{L_n(t)}),
\varphi_{n,p}(t, x, \bar{x}) = \varphi^0_{n,p}(t, x, \bar{x}) + \varphi^-_{n,p}(t, x, \bar{x}).
\]

Proposition 2.7 is a consequence of the following assertion.

**Proposition 2.7.** The hypotheses are the same as in Proposition 2.5. Fix \( t \in [t_0, T] \), \( x, \bar{x} \in K \). Then, for any \( p \in [1, \infty], \rho \in [0, \min(\gamma, \gamma_1, \gamma_2, \frac{\gamma}{2}, \frac{\gamma'}{2})] \),
\[
\sup_{n \geq 0} \varphi_{n,p}(t, x, \bar{x}) \leq C \lvert x - \bar{x} \rvert^\rho p.
\] (46)

The proof of this proposition relies on the next lemma and Gronwall’s lemma.

**Lemma 2.8.** We assume the same hypotheses as in Proposition 2.5. For any \( n \geq 1, t \in [t_0, T], x, \bar{x} \in K, p \in [2, \infty], \) there exists a finite constant \( C \) (not depending on \( n \)) such that
\[
\varphi_{n,p}(t, x, \bar{x}) \leq C \left[ f_n + \lvert x - \bar{x} \rvert^{\gamma p} + \lvert x - \bar{x} \rvert^{\gamma'p/2} + \lvert x - \bar{x} \rvert^{\gamma_1 p} + \lvert x - \bar{x} \rvert^{\gamma_2 p} + \int_0^t ds (\varphi_{n,p}(s, x, \bar{x})) \right].
\] (47)

where \( (f_n, n \geq 1) \) is a sequence of positive real numbers which converges to zero as \( n \to \infty \), \( \gamma, \gamma_1, \gamma_2 \in [0, 1], \gamma' \in [0, 2] \).
We postpone the proof of this lemma to the end of this section.

Proof. (of Proposition 2.7).
We will prove this proposition by contradiction. Suppose that the Lemma 2.8 does not imply the Proposition 2.7. It means that (47) does not imply (46). In such case, there exists some \( m \in \mathbb{N} \) and \( x, \bar{x} \in K \) such that (47) is satisfied but (46) (without the \( sup_n \)) does not. But, since \( f_n \) is a bounded sequence, there exists \( C_0 > 0 \) (\( C_0 \) depending only on \( x, \bar{x} \)) such that

\[
f_m \leq C_0 |x - \bar{x}|^{\alpha p},
\]

where \( \alpha = \min(\gamma, \gamma_1, \gamma_2, \frac{\gamma'}{2}) \). This inequality and (47) leave us

\[
\varphi_{m,p}(t, x, \bar{x}) \leq C \left[ |x - \bar{x}|^{\gamma p} + |x - \bar{x}|^{\gamma' p'/2} + |x - \bar{x}|^{\gamma_1 p} + |x - \bar{x}|^{\gamma_2 p} + \int_0^t ds \varphi_{m,p}(s, x, \bar{x}) \right].
\]

Then by Gronwall’s lemma we have

\[
\varphi_{m,p}(t, x, \bar{x}) \leq C \left[ |x - \bar{x}|^{\gamma p} + |x - \bar{x}|^{\gamma' p'/2} + |x - \bar{x}|^{\gamma_1 p} + |x - \bar{x}|^{\gamma_2 p} \right].
\]

With the constant \( C \) not depending on \( m \). We recall that \( \gamma, \gamma_1, \gamma_2 \in [0, 1] \) and \( \gamma' \in [0, 2] \). Therefore, (48) implies

\[
\varphi_{m,p}(t, x, \bar{x}) \leq C |x - \bar{x}|^{\rho p},
\]

with \( \rho \in [0, \min(\gamma, \gamma_1, \gamma_2, \frac{\gamma'}{2})] \).

This is a contradiction, thus (47) does imply (46). This ends the proof of Proposition 2.7.

Proof. (of Lemma 2.8).
Fix \( \rho \in [2, \infty[ \) and set \( w = x - \bar{x} \). From (25), we have the following:

\[
\varphi_{n,p}^0(t, x, \bar{x}) := E \left( |X_n(t, x) - X_n(t, \bar{x})|^{p} 1_{L_n(t)} \right) \leq C \sum_{i=0}^{6} R_i(t, x, \bar{x}),
\]
with

\[ R_n^0(t, x, \bar{x}) = |X^0(t, x) - X^0(t, \bar{x})|^p \]

\[ R_n^1(t, x, \bar{x}) = \mathbb{E} \left( \left. \left( \int_0^t \int_{\mathbb{R}^3} [G(t-s, x-y) - G(t-s, \bar{x} - y)] A(X_n(s, y)) M(ds, dy) \right) \right| 1_{L_n(t)} \right)^p \]

\[ R_n^2(t, x, \bar{x}) = \mathbb{E} \left( \left. \left( \int_0^t \int_{\mathbb{R}^3} [G(t-s, x-y) - G(t-s, \bar{x} - y)] B(X_n(s, y)) M(ds, dy) \right) \right| 1_{L_n(t)} \right)^p \]

\[ R_n^3(t, x, \bar{x}) = \mathbb{E} \left( \left. \left( \int_0^t \int_{\mathbb{R}^3} [G(t-s, x-y) - G(t-s, \bar{x} - y)] B(X_n(s, y)) M(ds, dy) \right) \right| 1_{L_n(t)} \right)^p \]

\[ R_n^4(t, x, \bar{x}) = \mathbb{E} \left( \left. \left( \int_0^t \int_{\mathbb{R}^3} [G(t-s, x-y) - G(t-s, \bar{x} - y)] B(X_n(s, y)) M(ds, dy) \right) \right| 1_{L_n(t)} \right)^p \]

It is well known that with the Hypothesis 1. (a) we have that (see for instance [12, theorem 3.1])

\[ R_n^0(t, x, \bar{x}) \leq C_1|w|^\gamma_1 p + C_2|w|^\gamma_2 p \]

(50)

for \( \gamma_1, \gamma_2 \in [0, 1] \).

Using Burkholder's inequality and then Plancherel's identity, we have

\[ R_n^1(t, x, \bar{x}) = \mathbb{E} \left( \left. \left( \int_0^t \int_{\mathbb{R}^3} [G(t-s, x-y) - G(t-s, \bar{x} - y)] A(X_n(s, y)) M(ds, dy) \right) \right| 1_{L_n(t)} \right)^p \]

\[ = \mathbb{E} \left( \left. \left( \sum_{j \in \mathbb{N}} \int_0^t \left< [G(t-s, x-\cdot) - G(t-s, \bar{x} - \cdot)] A(X_n(s, \cdot)), e_k(\cdot) \right> \right) \right| 1_{L_n(t)} \right)^p \]

\[ \leq C \mathbb{E} \left( \left. \left( \int_0^t ds \sum_{j \in \mathbb{N}} \left< [G(t-s, x-\cdot) - G(t-s, \bar{x} - \cdot)] A(X_n(s, \cdot)), e_k(\cdot) \right> \right| 1_{L_n(t)} \right)^{p/2} \]

\[ = C \mathbb{E} \left( \left. \left( \int_0^t ds \left[ \left[ G(t-s, x-\cdot) - G(t-s, \bar{x} - \cdot) \right] A(X_n(s, \cdot)) \right] \right| 1_{L_n(t)} \right)^{p/2} \]

(51)

The process \( \{ Z_n(t, x) = A(X_n(t, x)) 1_{L_n(t)}(t, x) \in [0, T] \times \mathbb{R}^3 \} \) satisfies the assumption (38). Indeed, this is a consequence of the linear growth of \( A \) and (120). Then, by applying Lemma 2.6 and using the Lipschitz continuity of \( A \), we obtain

\[ R_n^1(t, x, \bar{x}) \leq C \left\{ |x - \bar{x}|^{\gamma p} + |x - \bar{x}|^{\gamma' p/2} \right. \]

\[ + \int_0^t ds \left[ \sup_{z_1 - z_2 = w} \mathbb{E} \left( \left| X_n(s, z_1) - X_n(s, z_2) \right|^p 1_{L_n(s)} \right) \right] \]

(52)
with $\gamma \in [0, 1]$ and $\gamma' \in [0, 2]$.

For a given function $\rho : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ and $t \in [0, T]$, let $\tau_n$ be the operator defined by

$$\tau_n(\rho) = \rho(\{s + 2^{-n}\} \land t, x).$$

(53)

Let $\mathcal{E}_n$ be the closed subspace of $\mathcal{H}_T$ generated by the orthonormal system of functions

$$2^n T^{-1} \Delta_i(\cdot) \otimes e_j(\cdot), \quad i = 0, \ldots, 2^n - 1, \quad j = 1, \ldots, n,$$

and denote by $\pi_n$ the orthogonal projection on $\mathcal{E}_n$. Notice that $\pi_n \circ \tau_n$ is a bounded operator on $\mathcal{H}_T$, uniformly in $n$.

Since $X_n^-(s, \cdot)$ is $\mathcal{F}_{n-1}$-measurable, by using the definition of $w_n$ we easily see that

$$R_n^2(t, x, \bar{x}) = \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^3} \left( \pi_n \circ \tau_n \right) \left( \left[ G(t - s, x - \cdot) - G(t - s, \bar{x} - \cdot) \right] \times B(X_n^-(s, \cdot))(s, y) M(ds, dy) \right)^p \right) \mathbb{1}_{L_n(s)}. \tag{54}$$

By Burkholder’s inequality and the properties of the operator $\pi_n \circ \tau_n$, this last expression is bounded up to a constant by

$$\mathbb{E} \left( \int_0^t ds \left\| \left[ G(t - s, x - \cdot) - G(t - s, \bar{x} - \cdot) \right] B(X_n^-(s, \cdot)) \right\|_{\mathcal{H}} \mathbb{1}_{L_n(s)} \right)^{p/2}.$$

The properties of the function $B$ along with (120) imply that the process $\{Z_n(t, x) := B(X_n^-(t, x))\mathbb{1}_{L_n(t)}(t, x) \in [0, T] \times \mathbb{R}^3\}$ satisfies the hypotheses of Lemma 2.6. This yields

$$R_n^2(t, x, \bar{x}) \leq C \left\{ |x - \bar{x}|^{\gamma p} + |x - \bar{x}|^{\gamma' p/2} \right. \tag{54a}
\left. + \int_0^t ds \left( \sup_{z_1 - z_2 = w} \mathbb{E} \left( \left| X_n^-(s, z_1) - X_n^-(s, z_2) \right| \mathbb{1}_{L_n(s)} \right) \right)^{\delta} \right\}.$$

where as before, $\gamma \in [0, 1]$ and $\gamma' \in [0, 2]$.

Cauchy–Schwarz’s inequality along with (33) yield

$$R_n^3(t, x, \bar{x}) \leq C \left(2^{np/2} \mathbb{E} \left( \int_0^t ds \left[ G(t - s, x - \cdot) - G(t - s, \bar{x} - \cdot) \right] \times B(X_n) - B(X_n^-) \right)_{L_n(\cdot)}^2 \right)^{p/2}.$$

Notice that an upper bound for the second factor on the right-hand side of the preceding inequality could be obtained using Lemma 2.6 with $Z_n(t, x) := [B(X_n(t, x)) - B(X_n^-(t, x))]\mathbb{1}_{L_n(t)}$. However, this would not be a good strategy
to compensate the first factor (which explodes when \( n \to \infty \)). Instead, we will try to quantify the discrepancy between \( B(X_n(t, x)) \) and \( B(X_n^-(t, x)) \). This can be achieved by transferring the increments of the Green function to increments of the process

\[
\hat{B}(X_n(t, x)) = \left[ B(X_n(t, x)) - B(X_n^-(t, x)) \right],
\]

in the same manner as we did in the proof of Lemma 2.6 (see [8], pages 19–20 for the original idea).

Indeed, similarly as in (40), we obtain

\[
R_n^t(t, x, \bar{x}) \leq C n^{3p/2} 2^{np/2} \sum_{i=1}^{4} \mathbb{E}\left( |K_i^t(x, \bar{x})|^{p/2} 1_{L_n(t)} \right),
\]

where for any \( i = 1, \ldots, 4 \), \( K_i^t(x, \bar{x}) \) is given by \( J_i^t(x, \bar{x}) \) of Lemma 2.6 with \( Z_n \) replaced by \( \hat{B}(X_n) \).

With the definition of \( \hat{B}(X_n) \) given in (55), we easily get

\[
\mathbb{E}\left( |\hat{B}(X_n(s, x - y)) - \hat{B}(X_n(s, \bar{x} - y))|^{p/2} 1_{L_n(s)} \right)
\leq C \left[ \mathbb{E}\left( |X_n(s, x - y) - X_n^-(s, x - y)|^{p/2} 1_{L_n(s)} \right) + \mathbb{E}\left( |X_n(s, \bar{x} - y) - X_n^-(s, \bar{x} - y)|^{p/2} 1_{L_n(s)} \right) \right]
\leq C n^{3p/2} 2^{-np(\nu+1)/2},
\]

uniformly in \((s, x, y) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3\), where the last bound is obtained by using (121). This estimate will be applied to the study of the right-hand side of (56).

For \( i = 1 \), (41) with \( Z_n(s, y) := \hat{B}(X_n(s, y))1_{L_n(s)} \), along with (57) yields

\[
\mathbb{E}\left( |K_1^t(x, \bar{x})|^{p/2} 1_{L_n(t)} \right) \leq C n^{3p/2} 2^{-np(\nu+1)/2}.
\]

Let \( \mu_2(x, \bar{x}) \) be as in (42). Since \( x, \bar{x} \in K \), and \( K \) is bounded,

\[
\sup_{x, \bar{x} \in K} \mu_2(x, \bar{x}) \leq C,
\]

for some finite constant \( C > 0 \). Hence, (43), (44) (with the same choice of \( Z_n \) as before) together with (57) gives

\[
\mathbb{E}\left( |K_2^t(x, \bar{x})|^{p/2} 1_{L_n(t)} \right) + \mathbb{E}\left( |K_3^t(x, \bar{x})|^{p/2} 1_{L_n(t)} \right) \leq C n^{3p/2} 2^{-np(\nu+1)/2}.
\]

Proceeding as in (45), but replacing \( Z_n(s, y) \) by \( \hat{B}(X_n(s, y))1_{L_n(s)} \), we obtain

\[
\mathbb{E}\left( |K_4^t(x, \bar{x})|^{p/2} 1_{L_n(t)} \right) \leq C |x - \bar{x}|^{2p/2} \int_0^t ds \sup_{y \in \mathbb{R}^3} \mathbb{E}\left( |\hat{B}(X_n(s, y))|^{p/2} 1_{L_n(s)} \right).
\]

By the definition of \( \hat{B}(X_n) \), and applying (121), we have

\[
\sup_{(s, y) \in [0, T] \times \mathbb{R}^3} \mathbb{E}\left( |\hat{B}(X_n(s, y))|^{p/2} 1_{L_n(s)} \right) \leq C n^{3p/2} 2^{-np(\nu+1)/2}.
\]
Thus,
\[
\mathbb{E}(|R_n^3(t, \bar{x})|^{p/2} 1_{L_n(t)}) \leq C n^{3p/2 - np(v+1)/2}.
\] (60)
Putting together (56) and (58)-(60) yields
\[
R_n^3(t, x, \bar{x}) \leq C f_n,
\] (61)
where \( f_n := n^{3p/2 - np(v+1)/2} = n^{3p/2 - np/2} \). Since \( \nu \in [0,1] \), \( \lim_{n \to \infty} f_n = 0 \). Notice that each member of the sequence \( \{f_n\} \) is positive.

To estimate the term \( R_n^4(t, x, \bar{x}) \) we use first Cauchy–Schwarz’s inequality and then, by applying Lemma 2.6 with \( Z_n \) replaced by \( D(X_n) 1_{L_n} \). The Lipschitz continuity of \( D \) along with the estimate (120) ensure that assumption (38) is satisfied. We obtain
\[
R_n^4(t, x, \bar{x}) \leq \|h\|_{H^1}^p \mathbb{E}\left( \left| \int G(t-s, x-s) - G(t-s, \bar{x}-s) \right|^p \right) \left| D(X_n(t, *)) \right|_{H^1}^2 1_{L_n(t)}^{p/2}
\]
\[
\leq C \left( |x-\bar{x}|^{\gamma p} + |x-\bar{x}|^{\gamma'p/2} + \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(\left| X_n(s, z_1) - X_n(s, z_2) \right|^p 1_{L_n(s)}) \right),
\] (62)
where as before, \( \gamma \in [0,1] \) and \( \gamma' \in [0,2] \).

After having applied the change of variable \( u \mapsto x - \bar{x} + y \), we have
\[
R_n^4(t, x, \bar{x}) = \mathbb{E}\left( \int_0^t \int_{\mathbb{R}^3} G(t-s, x-dy) [b(X_n(s, y)) - b(X_n(s, y-x+\bar{x}))] ds \right) 1_{L_n(t)}^p.
\]

Applying Hölder’s inequality, we obtain
\[
R_n^4(t, x, \bar{x}) \leq \left( \int_0^t \int_{\mathbb{R}^3} G(t-s, x-dy) ds \right)^{p-1}
\]
\[
\times \int_0^t \int_{\mathbb{R}^3} G(t-s, x-dy) \mathbb{E}(\left| b(X_n(s, y)) - b(X_n(s, y-x+\bar{x})) \right|^p 1_{L_n(s)}) ds
\]
\[
\leq C \int_0^t ds \sup_{y \in \mathbb{R}^3} \mathbb{E}(\left| X_n(s, x-y) - X_n(s, \bar{x}-y) \right|^p 1_{L_n(s)}),
\]
\[
\leq C \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(\left| X_n(s, z_1) - X_n(s, z_2) \right|^p 1_{L_n(s)})
\] (63)

Bringing together the inequalities (50), (52), (54), (61), (62) and (63), yields
\[
\sup_{z_1 - z_2 = w} \mathbb{E}(\left| X_n(t, z_1) - X_n(t, z_2) \right|^p 1_{L_n(t)})
\]
\[
\leq C \left\{ f_n + |x-\bar{x}|^{\gamma p} + |x-\bar{x}|^{\gamma'p/2} + |x-\bar{x}|^{\gamma_1p} + |x-\bar{x}|^{\gamma_2p}
\]
\[
+ \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(\left| X_n(s, z_1) - X_n(s, z_2) \right|^p 1_{L_n(s)})
\]
\[
+ \int_0^t ds \sup_{z_1 - z_2 = w} \mathbb{E}(\left| X_n^-(s, z_1) - X_n^-(s, z_2) \right|^p 1_{L_n(s)}) \right\}. (64)
\]
With this, we see that \( \varphi_{0,p}^0(t, x, \bar{x}) \) is bounded by the right-hand side of (64).

Finally, we prove that the same bound holds for \( \varphi_{n,p}^{-}(t, x, \bar{x}) \) too. Indeed, for every \( i = 1, \ldots, 5 \), we consider the terms \( R_n^i(t, x, \bar{x}) \) defined in the first part of the proof, and we replace the domain of integration of the time variable \( s \) \([-0, t]\) by \([-0, t_n]\). We denote the corresponding new expressions by \( S_n^i(t, x, \bar{x}) \).

From (26), we obtain the following
\[
\varphi_{n,p}^{-}(t, x, \bar{x}) \leq C \sum_{i=1}^{5} S_n^i(t, x, \bar{x}).
\]

Since \( t_n \leq t \), it can be checked that, similarly as for \( R_n^i(t, x, \bar{x}) \), \( S_n^i(t, x, \bar{x}) \), \( i = 1, \ldots, 5 \), are bounded by (52), (54), (61), (62), (63), respectively. This ends the proof of the lemma. \( \square \)

### 2.2. Increments in time

Throughout this section, we fix \( t_0 \in [0, T] \), and a compact set \( K \subset \mathbb{R}^3 \). We shall prove the following proposition.

**Proposition 2.9.** Assume that Hypothesis 1 and Hypothesis 2 holds. Fix \( t, \bar{t} \in [t_0, T] \) and set \( \kappa \in \left(0, \min\left(\gamma_1, \gamma_2, \gamma, \frac{\gamma_1'}{2}\right) \right) \]. Then for any \( p \in [1, \infty) \) there exists a finite constant \( C \) such that
\[
\sup_n \sup_{(t, x) \in [t_0, T] \times \mathbb{R}^3} \mathbb{E}\left(|D_n(t, x)|^p\right) \leq C.
\]

There exists \( \kappa > 0 \) and for any \( x, y \in K \),
\[
\sup_n \sup_{t \in [t_0, T]} \mathbb{E}\left(|D_n(t, x) - D_n(t, y)|^p\right) \leq C|x - y|^\kappa p,
\]

where \( C \) is a finite constant and \( \rho > 0 \). Suppose in addition that hypotheses 1 and 2 (see page 4) are satisfied.
For $0 \leq t_0 \leq t \leq \bar{t} \leq T$ and $x \in K$, set

$$J_n(t, \bar{t}, x) = \int_0^t ds \| D_n(t, *) [G(\bar{t} - s, x - *) - G(t - s, x - *)] \|^2_{H}.$$  

Then, for any $p \in [2, \infty]$ there exists a finite constant $C > 0$ such that

$$\mathbb{E}(J_n(t, \bar{t}, x)^{p/2}) \leq C(|\bar{t} - t|^\gamma p),$$

with $0 < \gamma < \min(\kappa, \frac{\nu+1}{2}, \frac{\rho_1 + \kappa}{2}, \frac{\rho_2}{2}).$

Proof. First of all we notice that, as a consequence of Burkholder's inequality, the $L^p$-moment of the stochastic integral

$$\int_0^t \int_{\mathbb{R}^3} D_n(t, y) [G(\bar{t} - s, x - y) - G(t - s, x - y)] M(ds, dy),$$

is bounded up to a positive constant, by $\mathbb{E}(J_n(t, \bar{t}, x)^{p/2})$.

We write $J_n(t, \bar{t}, x)$ using (10). This gives

$$J_n(t, \bar{t}, x) = C \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} D_n(t, y) [G(\bar{t} - s, x - dy) - G(t - s, x - dy)]$$

$$\times D_n(t, z) [G(\bar{t} - s, x - dz) - G(t - s, x - dz)] f(y - z)$$

$$= C \int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} D_n(t - s, x - y) [G(\bar{t} - t + s, dy) - G(s, dy)]$$

$$\times D_n(t - s, x - z) [G(\bar{t} - t + s, dz) - G(s, dz)] f(y - z)$$

As was noted in [12] the integral with respect to the space variables $y$ and $z$ is taken in the sphere $S^2$ in the three dimensional space, this is because of the structure of the fundamental solution $G$. We denote by $\xi = \frac{y}{|y|}$ and $\eta = \frac{z}{|z|}$, moreover we denote by $\sigma(d\xi)$ and $\sigma(d\eta)$ the uniform measure on $S^2$, so

$$G(s, dy) = \frac{1}{4\pi} \sigma(d\xi)$$

$$G(s, dz) = \frac{1}{4\pi} \sigma(d\eta)$$

Denote by $h = \bar{t} - t$. Then, we do the following decomposition

$$\mathbb{E}(J_n(t, \bar{t}, x)^{p/2}) \leq C \sum_{k=1}^4 \mathbb{E}(|Q^i(t, \bar{t}, x)|^{p/2}),$$
where

\[ Q^1(t, \bar{t}, x) := \int_0^t ds \int_{S^2} \int_{S^2} (s + h)^2 f([s + h]\xi - [s + h]\eta) \]
\[ \times \left[ D_n\left(t - s, x - (s + h)\xi\right) - D_n\left(t - s, x - s\xi\right) \right] \sigma(d\xi)\sigma(d\eta) \]
\[ Q^2(t, \bar{t}, x) := \int_0^t ds \int_{S^2} \int_{S^2} \left( [s + h]^2 f\left([s + h]\xi - [s + h]\eta\right) - s(s + h)f\left(s\xi - [s + h]\eta\right) \right) \]
\[ \times D_n(t - s, x - s\xi) \left[ D_n\left(t - s, x - (s + h)\eta\right) - D_n\left(t - s, x - s\eta\right) \right] \sigma(d\xi)\sigma(d\eta) \]
\[ Q^3(t, \bar{t}, x) := \int_0^t ds \int_{S^2} \int_{S^2} \left( [s + h]^2 f\left([s + h]\xi - [s + h]\eta\right) - s(s + h)f\left(s\xi - [s + h]\eta\right) \right) \]
\[ \times D_n(t - s, x - s\eta) \left[ D_n\left(t - s, x - (s + h)\xi\right) - D_n\left(t - s, x - s\xi\right) \right] \sigma(d\xi)\sigma(d\eta) \]
\[ Q^4(t, \bar{t}, x) := \int_0^t ds \int_{S^2} \int_{S^2} \left( [s + h]^2 f\left([s + h]\xi - [s + h]\eta\right) - s(s + h)f\left(s\xi - [s + h]\eta\right) \right) \]
\[ -s(s + h)f\left([s + h]\xi - s\eta\right) + s^2 f\left(s\xi - s\eta\right) \]
\[ \times D_n(t - s, x - s\eta)D_n(t - s, x - s\xi) \sigma(d\xi)\sigma(d\eta) \]

Following the arguments of the proof of Theorem 4.1 in [12], we see that by using Hölder inequality, Cauchy-Schwarz inequality, hypothesis on $D$, Lemma
we get

\[ \mathbb{E}(|Q^1(t, \tilde{t}, x)|^{p/2}) \leq C \left( \int_0^t ds \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} (s + h)^2 f(\lfloor s + h \rfloor \xi - \lfloor s + h \rfloor \eta) \sigma(d\xi) \sigma(d\eta) \right)^{\frac{p}{2}} \]

\[ \times \int_0^t ds \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} (s + h)^2 f(\lfloor s + h \rfloor \xi - \lfloor s + h \rfloor \eta) \sigma(d\xi) \sigma(d\eta) \]

\[ \times \mathbb{E} \left[ D_n(t - s, x - (s + h)\xi) - D_n(t - s, x - s\xi) \right] \]

\[ \times \mathbb{E} \left[ D_n(t - s, x - (s + h)\eta) - D_n(t - s, x - s\eta) \right] \]

\[ \leq C \left( \int_0^t ds \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} (s + h)^2 f(\lfloor s + h \rfloor \xi - \lfloor s + h \rfloor \eta) \sigma(d\xi) \sigma(d\eta) \right)^{\frac{p}{2}} \]

\[ \times \int_0^t ds \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} (s + h)^2 f(\lfloor s + h \rfloor \xi - \lfloor s + h \rfloor \eta) \sigma(d\xi) \sigma(d\eta) \]

\[ \times \mathbb{E} \left[ D_n(t - s, x - (s + h)\xi) - D_n(t - s, x - s\xi) \right] \]

\[ \times \mathbb{E} \left[ D_n(t - s, x - (s + h)\eta) - D_n(t - s, x - s\eta) \right] \]

\[ \leq C|h|^{p \rho_\xi} |\eta|^{p \rho_\xi/2} |\xi|^{p \rho_\xi/2} \left( \int_0^t ds \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} (s + h)^2 f(\lfloor s + h \rfloor \xi - \lfloor s + h \rfloor \eta) \sigma(d\xi) \sigma(d\eta) \right)^{\frac{p}{2}} \]

\[ = C|h|^{p \rho_\xi} \left( \int_0^t ds \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} f(y - z)G(s + h, dy)G(s + h, dz) \right)^{\frac{p}{2}} \]

\[ \leq C|h|^{p \rho_\xi} \left( \int_0^t ds \int_{|z| \leq 2(s + h)} \frac{f(z)}{|z|} dz \right)^{\frac{p}{2}} \]

\[ \leq C|h|^{p \rho_\xi} = C|\tilde{t} - t|^{p \rho_\xi} \] (71)

For \( Q^2 \), by making the trivial decomposition \((s + h)^2 = s(s + h) + h(s + h)\)

we get
\[
E(|Q^2(t, t, x)|^{p/2}) \leq C \left( \int_0^t ds \int_{S^2} \int_{S^2} s(s + h) \left| f\left(\lceil s + h \rceil \xi - \lfloor s + h \rfloor \eta \right) - f\left(s \xi - \lfloor s + h \rfloor \eta \right) \right| \sigma(d\xi)\sigma(d\eta) \right)
\times \left| D_n(t-s, x - s\xi) \right|
D_n\left(t-s, x - (s + h)\eta\right) - D_n\left(t-s, x - s\eta\right) \right|^{\frac{p}{2}}
+ C \left( \int_0^t ds \int_{S^2} \int_{S^2} h(s + h) f\left(\lceil s + h \rceil \xi - \lfloor s + h \rfloor \eta \right) \sigma(d\xi)\sigma(d\eta) \right)
\times \left| D_n(t-s, x - s\xi) \right|
D_n\left(t-s, x - (s + h)\eta\right) - D_n\left(t-s, x - s\eta\right) \right|^{\frac{p}{2}}
=: Q^{2,1} + Q^{2,2}
\]
(72)

For \(Q^{2,1}\) we use Hölder inequality, Cauchy-Schwarz inequality, assumptions (66) and (67), condition (17) along with the change of variables \(\eta \mapsto -\eta\) and we obtain

\[
Q^{2,1} \leq C \left( \int_0^t ds \int_{S^2} \int_{S^2} s(s + h) \left| f\left(\lceil s + h \rceil \xi - \lfloor s + h \rfloor \eta \right) - f\left(s \xi - \lfloor s + h \rfloor \eta \right) \right| \sigma(d\xi)\sigma(d\eta) \right)^{\frac{p}{2}}
\times \left( \int_0^t ds \int_{S^2} \int_{S^2} s(s + h) \left| f\left(\lceil s + h \rceil \xi - \lfloor s + h \rfloor \eta \right) - f\left(s \xi - \lfloor s + h \rfloor \eta \right) \right| \sigma(d\xi)\sigma(d\eta) \right)^{\frac{p}{2}}
\times \left( E\left| D_n(t-s, x - s\xi) \right|^p \right)^{1/2}
\left( E\left| D_n\left(t-s, x - (s + h)\eta\right) - D_n\left(t-s, x - s\eta\right) \right|^p \right)^{1/2}
\leq C|h|^{p/2} \left( \int_0^t ds \int_{S^2} \int_{S^2} s(s + h) \left| f\left(\lceil s + h \rceil \xi - \lfloor s + h \rfloor \eta \right) - f\left(s \xi - \lfloor s + h \rfloor \eta \right) \right| \sigma(d\xi)\sigma(d\eta) \right)^{\frac{p}{2}}
\leq C|h|^{p/2} |h|^{\rho_1/2} = C|h|^{\frac{p+\rho_1}{2}}
\]
(73)

with \(\kappa, \rho_1 \in [0, 1]\).

For \(Q^{2,2}\) we use Hölder inequality, Cauchy-Schwarz inequality, assumptions (66) and (67), condition (17) and we obtain
\[ Q_{2,2} \leq C|h|^{p/2} \left( \int_0^t ds \int_{S^2} \int_{S^2} (s+h) f \left( |s+h| \xi - |s+h| \eta \right) \sigma(d\xi) \sigma(d\eta) \right)^{p/2 - 1} \]

\[ \times \int_0^t ds \int_{S^2} \int_{S^2} (s+h) f \left( |s+h| \xi - |s+h| \eta \right) \sigma(d\xi) \sigma(d\eta) \]

\[ \times \left( \mathbb{E} \left| D_n(t-s, x-(s+h)\eta) - D_n(t-s, x-s\eta) \right| \right)^{1/2} \]

\[ \leq C|h|^{p/2} |h|^{p\kappa/2} |\eta|^{p\rho_1/2} \left( \int_0^t ds \int_{S^2} \int_{S^2} (s+h) f \left( |s+h| \xi - |s+h| \eta \right) \sigma(d\xi) \sigma(d\eta) \right)^{p/2} \]

\[ \leq C|h|^{p_{\kappa, \rho_1}/2} \left( \int_0^t ds \int_{|z| \leq 2(s+h)} f(z) G(s+h, dz) \right)^{p/2} \]

\[ \leq C|h|^{p_{\kappa, \rho_1}/2} \left( \int_0^t \frac{ds}{s+h} \int_{|z| \leq 2(s+h)} f(z) dz \right)^{p/2} \]

(74)

with \( \kappa, \rho_1 \in [0,1] \). Hence, by (73) and (74) we set

\[ \mathbb{E} \left( |Q^2(t, \bar{t}, x)|^{p/2} \right) \leq C|h|^{p_{\kappa, \rho_1}/2} \] (75)

with \( \kappa, \rho_1 \in [0,1] \).

Similarly,

\[ \mathbb{E} \left( |Q^3(t, \bar{t}, x)|^{p/2} \right) \leq C|h|^{p_{\kappa, \rho_1}/2}, \] (76)

where \( \kappa, \rho_1 \in [0,1] \).

By applying Hölder’s and Cauchy–Schwarz’s inequalities along with (66), we
get

$$E\left( |Q^4(t, \bar{t}, x)|^{p/2} \right) \leq C \left( \int_{0}^{t} ds \int_{S^2} \int_{S^2} (s + h)^2 f\left( [s + h] \xi - [s + h] \eta \right) - s(s + h) f(s \xi - [s + h] \eta) - s(s + h) f\left( [s + h] \xi - s \eta \right) + s^2 f\left( s \xi - s \eta \right) \left| \sigma(d \xi) \sigma(d \eta) \right| \right)^{p/2} \times \left( \int_{0}^{t} ds \int_{S^2} \int_{S^2} (s + h)^2 f\left( [s + h] \xi - [s + h] \eta \right) - s(s + h) f(s \xi - [s + h] \eta) - s(s + h) f\left( [s + h] \xi - s \eta \right) + s^2 f\left( s \xi - s \eta \right) \left| \sigma(d \xi) \sigma(d \eta) \right| \right)^{p/2} \times \left[ E|D_n(t - s, x - s \xi)|^p \right]^{1/2} \left[ E|D_n(t - s, x - s \eta)|^p \right]^{1/2} \leq C \left( Q^{4,1} + Q^{4,2} + Q^{4,3} + Q^{4,4} \right)$$

Where $Q^{4,i}$, for $i = 1, \ldots, 4$, is defined exactly as $R^i_4$ in [12, page 20]. Then, by following exactly the same procedure as in [12] we arrive to

$$E\left( |Q^4(t, \bar{t}, x)|^{p/2} \right) \leq C \left( |h|^{\rho_0} + |h|^{\rho_1 + 1} + |h|^{p + 1} + |h|^{p(1 - \varepsilon)} \right) \quad (77)$$

with $\nu, \rho_1 \in [0, 1]$ and $\rho_2 \in [0, 2]$ and where the last inequality is true for any $\varepsilon > 0$.

The inequalities (69), (75), (76), (77) imply (68) which completes the proof.  

**Proof.** (of Proposition 2.9).

Fix $0 \leq t \leq \bar{t} \leq T$, $x \in K$, $p \in [2, \infty]$, and according to (25) consider the decomposition

$$E\left( |X_n(t, x) - X_n(t, x)|^p 1_{L_n(\bar{t})} \right) \leq C \sum_{i=0}^{6} R^i_n(t, \bar{t}, x),$$
where
\[
R_0^n(t, \bar{t}, x) = |X^n(t, x) - X^0(\bar{t}, x)|^p
\]
\[
R_1^n(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^3} \left[G(\bar{t} - s, x - y) - G(t - s, x - y)\right] \times A(X_n(s, y)) M(ds, dy)\right|^p \right)^{1/L_n(t)}.
\]
\[
R_2^n(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^3} \left[G(\bar{t} - s, x - y) - G(t - s, x - y)\right] \times [B(X_n) - B(X_n^{-})](\cdot, *), w^n\right|^p \right)^{1/L_n(t)}.
\]
\[
R_3^n(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^3} \left[G(\bar{t} - s, x - y) - G(t - s, x - y)\right] \times [D(X_n) - D(X_n^{-})](\cdot, *), h\right|^p \right)^{1/L_n(t)}.
\]
\[
R_4^n(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^3} \left[G(\bar{t} - s, x - y) - G(t - s, x - y)\right] \times [b(X_n(s, y))] ds\right|^p \right)^{1/L_n(t)}.
\]

Let \( \gamma' = \min(\gamma_1, \gamma_2) \). By the assumptions on \( \Delta v_0 \) and \( \bar{v}_0 \) and by Lemma 4.9 in [8] we have
\[
R_0^n(t, \bar{t}, x) \leq C|t - \bar{t}|^{p\gamma'}
\] (78)

Similarly as for the term \( R_1^n(t, x, \bar{x}) \) in the proof of Lemma 2.8 (see (51)), we have
\[
R_1^n(t, \bar{t}, x) \leq C\mathbb{E}\left(\int_0^t ds\left[\left[G(\bar{t} - s, x - \ast) - G(t - s, x - \ast)\right] A(X_n(s, \ast))\right]^2 \right)^{p/2}.
\]

This is bounded up to a positive constant by \( R_1^{1,1}(t, \bar{t}, x) + R_1^{1,2}(t, \bar{t}, x) \), where
\[
R_1^{1,1}(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^t \|G(\bar{t} - s, x - \ast) - G(t - s, x - \ast)\|_{H^1_{\mathbb{R}^n(s)}}^2\right| ds\right)^{p/2},
\] (79)

and
\[
R_1^{1,2}(t, \bar{t}, x) = \mathbb{E}\left(\left|\int_0^t ds\left[\left[G(s, x - \ast) A(X_n(\bar{t} - s, \ast))\right]^2 \right|_{H^1_{\mathbb{R}^n(s)}}^2\right| ds\right)^{p/2},
\] (80)

Using Burkholder and then Hölder inequalities, the linear growth of \( A \) and (120), we get
\[
R_{n}^{1,1}(t, \tilde{t}, x) \leq C(t - \tilde{t})^{-1/2} \int_{0}^{t-\tilde{t}} ds \left( \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} G(s, x - dy) G(s, x - dz) f(y - z) \right)^{p/2} \\
\times \left( 1 + \sup_{(t,x)\in[0,T] \times \mathbb{R}^{3}} \mathbb{E} \left( |X_{n}(t,x)|^{p} 1_{L_{n}(t)} \right) \right) \\
\leq C(t - \tilde{t})^{-1/2} \int_{0}^{t-\tilde{t}} ds \left( \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} G(s, dy) G(s, dz) f(y - z) \right)^{p/2} \\
\leq C(t - \tilde{t})^{-1/2} \int_{0}^{t-\tilde{t}} ds \left( \int_{|z| \leq 2s} \frac{f(z)}{|z|} dz \right)^{p/2} \\
\leq C(t - \tilde{t})^{-1/2} \int_{0}^{t-\tilde{t}} s^{p/2} ds = C(t - \tilde{t})^{p+1/2}. 
\] (82)

Set \( D_{n}(t, x) = A(X_{n}(t, x)) 1_{L_{n}(t)} \). Owing to Hypothesis 1, (120) and Proposition 2.5, the conditions (66), (67) of Lemma 2.10 are satisfied with \( \kappa \in \left[ 0, \min \left( \gamma, \gamma_1, \gamma_2, \frac{\nu}{2} \right) \right] \). Thus,

\[
R_{n}^{1,2}(t, \tilde{t}, x) \leq C |t - \tilde{t}|^{p\rho}, 
\]
with \( \rho \in \left[ 0, \min \left( \gamma, \gamma_2, \frac{\nu}{2}, \frac{\kappa + \alpha}{2}, \frac{\kappa + \beta}{2} \right) \right] \).

It is easy to check that (82) and (83) imply

\[
R_{n}^{1}(t, \tilde{t}, x) \leq C |t - \tilde{t}|^{p\rho}, 
\]
with \( \rho \in \left[ 0, \min \left( \gamma, \gamma_2, \frac{\nu}{2}, \frac{\kappa + \alpha}{2}, \frac{\kappa + \beta}{2} \right) \right] \).

With the same arguments as those applied in the study of the term \( R_{n}^{2}(t, x, \tilde{x}) \) in the proof of Lemma 2.8, we have

\[
R_{n}^{2}(t, \tilde{t}, x) \leq C \mathbb{E} \left( \int_{0}^{t} ds \right) \left[ \left| G(\tilde{t} - s, x + *) - G(t - s, x - *) \right| B(X_{n}^{-}(s,*)) \right]^{2}1_{L_{n}(s)} \right)^{p/2}.
\]

This yields \( R_{n}^{2}(t, \tilde{t}, x) \leq C \left( R_{n}^{2,1}(t, \tilde{t}, x) + R_{n}^{2,2}(t, \tilde{t}, x) \right) \), where

\[
R_{n}^{2,1}(t, \tilde{t}, x) = \mathbb{E} \left( \int_{0}^{t} ds \right) \left( \left| G(\tilde{t} - s, x + *) - G(t - s, x - *) \right| B(X_{n}^{-}(s,*)) \right) \right]^{2}1_{L_{n}(s)} \right)^{p/2},
\]

\[
R_{n}^{2,2}(t, \tilde{t}, x) = \mathbb{E} \left( \int_{0}^{t-\tilde{t}} ds \right) \left( \left| G(s, x - *) B(X_{n}^{-}(s,*)) \right| \right) \right]^{2}1_{L_{n}(s)} \right)^{p/2}.
\]

The term \( R_{n}^{2,1}(t, \tilde{t}, x) \) is similar as \( R_{n}^{1,2}(t, \tilde{t}, x) \), with \( A(X_{n}) \) replaced by \( B(X_{n}^{-}) \). Hence both can be studied using the same approach. First, we see that the process \( D_{n}(t, x) := B(X_{n}^{-}(t, x)) 1_{L_{n}(t)} \) satisfies the hypothesis of Lemma 2.10 with \( \kappa \in \left[ 0, \min \left( \gamma, \gamma_1, \gamma_2, \frac{\nu}{2} \right) \right] \). In fact, this is a consequence of (120) and Proposition 2.7. Therefore, as for \( R_{n}^{1,2}(t, \tilde{t}, x) \), we have

\[
R_{n}^{2,1}(t, \tilde{t}, x) \leq C |t - \tilde{t}|^{p\rho}, 
\]
(85)
with \( \rho \in \left[0, \min \left( \gamma_1, \gamma_2, \gamma, \frac{\nu'}{2}, \frac{\nu+1}{2}, \frac{p+\nu}{2}, \frac{p}{2} \right) \right] \).

As for \( R_{n,2}^2(t, \bar{t}, x) \), it is analogous to \( R_{n,1}^1 \) with \( A(X_n) \) replaced by \( B(X_n^-) \). As in (82), we have

\[
R_{n,2}^2(t, \bar{t}, x) \leq C|\bar{t} - t|^{\nu+1}. \tag{86}
\]

Consequently, from (85), (86), we obtain

\[
R_n^2(t, \bar{t}, x) \leq C|\bar{t} - t|^\rho, \tag{87}
\]

with \( \rho \in \left[0, \min \left( \gamma_1, \gamma_2, \gamma, \frac{\nu'}{2}, \frac{\nu+1}{2}, \frac{p+\nu}{2}, \frac{p}{2} \right) \right] \).

Let \( \hat{B}(X_n(\cdot, *)) \) be defined by (55). Using Cauchy–Schwarz’s inequality and (33) we have

\[
R_n^3(t, \bar{t}, x) \leq Cn^{3p/2}2^{np/2} \left[R_{n,1}^3(t, \bar{t}, x) + R_{n,2}^3(t, \bar{t}, x)\right], \tag{88}
\]

where

\[
R_{n,1}^3(t, \bar{t}, x) = \mathbb{E}\left( \int_0^4 \|G(\bar{t} - s, x - \star) - G(t - s, x - \star)\|_{L^1}^2 \hat{B}(X_n(s, *)) \right)^{p/2},
\]

\[
R_{n,2}^3(t, \bar{t}, x) = \mathbb{E} \left( \int_0^{\bar{t} - t} \|G(s, x - \star)\hat{B}(X_n(\bar{t} - s, \star))\|_{L^1}^2 \right)^{p/2}.
\]

From (121), it follows that

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}(\|\hat{B}(X_n(t,x))\|_{L^1}^p) \leq Cn^{3p/2}2^{-np^{p+1}}. \tag{89}
\]

Let us study \( R_{n,2}^3(t, \bar{t}, x) \). This term is similar to \( R_{n,1}^1(t, \bar{t}, x) \) with \( A(X_n) \) replaced here by \( \hat{B}(X_n) \). Hence, as in (81) we have

\[
R_{n,2}^3(t, \bar{t}, x) \leq |\bar{t} - t|^{\nu+1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}(\|\hat{B}(X_n(t,x))\|_{L^1}^p) \leq C|\bar{t} - t|^{\nu+1} n^{3p/2}2^{-np^{p+1}}, \tag{90}
\]

where in the last inequality we have applied (89).

The analysis of \( R_{n,1}^1 \) relies on a variant of Lemma 2.10 where the process \( D_n \) is replaced by \( \hat{B}(X_n) \). By (89), this process satisfies a stronger assumption than (66). This fact is expected to compensate the factor \( n^{3p/2}2^{np/2} \) in (88).

As in the proof of Lemma 2.10, we consider the decomposition

\[
R_{n,1}^3(t, \bar{t}, x) \leq \sum_{k=1}^4 \mathbb{E}(\|Q^k(t, \bar{t}, x)\|_{L^1}^{p/2}),
\]

where \( Q^i(t, \bar{t}, x), i = 1, \ldots, 4 \), are defined in (69) and subsequent lines, with \( D_n := \hat{B}(X_n)1_{L_n} \).
From (89) and the triangular inequality, we obtain
\[
\mathbb{E}\left(\left|B\left(X_n \left(s, x - \bar{t} - s u\right)\right) - B\left(X_n(s, x - u)\right)\right|^p 1_{L_n(s)}\right) \leq Cn^{3p/2 - np\frac{\nu+1}{2}}.
\]
(91)

Consider the expression (70) with \(D_n = B(X_n)1_{L_n}\). The above estimate (91) yields
\[
\mathbb{E}\left(|Q^1(t, \bar{t}, x)|^{p/2} 1_{L_n(\bar{t})}\right) \leq Cn^{3p/2 - np\frac{\nu+1}{2}} \left(\int_0^t ds \int_{S^2} (s+h)^2 f([s+h]e - [s+h]\eta)\sigma(d\xi)\sigma(d\eta)\right)^{\frac{p}{2}}.
\]

This implies
\[
\mathbb{E}\left(|Q^1(t, \bar{t}, x)|^{p/2} 1_{L_n(\bar{t})}\right) \leq Cn^{3p/2 - np\frac{\nu+1}{2}}.
\]
(92)

Consider the procedure to get the expressions (73) and (74), with \(D_n = \hat{B}(X_n)1_{L_n}\). Using (55), (89), (91) and (121), we obtain
\[
\mathbb{E}\left(|Q^2(t, \bar{t}, x)|^{p/2} 1_{L_n(\bar{t})}\right) \leq Cn^{3p/2 - np\frac{\nu+1}{2}}.
\]
(93)

Similarly,
\[
\mathbb{E}\left(|Q^3(t, \bar{t}, x)|^{p/2} 1_{L_n(\bar{t})}\right) \leq Cn^{3p/2 - np\frac{\nu+1}{2}}.
\]
(94)

Let us now consider the procedure to estimate \(Q^4\) in Lemma 2.10, with \(D_n = \hat{B}(X_n)1_{L_n}\). Appealing to (89), we obtain
\[
\mathbb{E}\left(|Q^4(t, \bar{t}, x)|^{p/2} 1_{L_n(\bar{t})}\right) \leq Cn^{3p/2 - np\frac{\nu+1}{2}}.
\]
(95)

From (92)–(95) it follows that
\[
R_n^{3,1}(t, \bar{t}, x) \leq Cn^{3p/2 - np\frac{\nu+1}{2}},
\]
(96)

where \(C\) is a finite constant.

Set \(f_n := n^{3p/2 - np\frac{\nu+1}{2}} 2^{1/2} = n^{3p/2 - np\frac{\nu}{2}}\). From (88), (90), (96), it follows that
\[
R_n^{3,1}(t, \bar{t}, x) \leq C|\bar{t} - t|^{\rho p} + Cf_n, \quad \rho \in \left[0, \frac{\nu + 1}{2}\right].
\]
(97)

By applying Cauchy–Schwarz’s inequality, we see that
\[
R_n^4(t, x, \bar{x}) \leq C\mathbb{E}\left(\int_0^t ds \left|G(t-s, x-s) - G(s, x)\right| D\left(X_n(s, *)\right)\right)^{\frac{p}{2}}.
\]

The last expression is similar as (79) with the function \(A\) replaced by \(D\). Therefore, as in (84) we obtain
\[
R_n^4(t, \bar{t}, x) \leq C|\bar{t} - t|^{\rho p},
\]
(98)
with \( \rho \in \left[ 0, \min \left( \gamma_1, \gamma_2, \gamma, \frac{\nu}{2}, \frac{\nu+1}{2}, \frac{n+1}{2}, \frac{\nu}{2} \right) \right] \).

Finally, we consider \( R_n^5(t, \bar{t}, x) \). Clearly,

\[
R_n^5(t, \bar{t}, x) \leq C \left[ R_n^{5,1}(t, \bar{t}, x) + R_n^{5,2}(t, \bar{t}, x) \right],
\]

where

\[
R_n^{5,1}(t, \bar{t}, x) := \mathbb{E} \left( \left| \int_0^t \int_{\mathbb{R}^3} \left[ G(\bar{t} - s, x - dy) - G(t - s, x - dy) \right] b_n(s, y) dy ds \right|^p 1_{L_n(t)} \right),
\]

\[
R_n^{5,2}(t, \bar{t}, x) := \mathbb{E} \left( \left| \int_0^t \int_{\mathbb{R}^3} G(\bar{t} - s, x - dy) b_n(s, y) dy ds \right|^p 1_{L_n(t)} \right).
\]

Applying the change of variable, \( y \mapsto \frac{y-x}{t-s} + x \) and \( y \mapsto \frac{y-x}{t-s} + x \), we see that

\[
R_n^{5,1}(t, \bar{t}, x) = \mathbb{E} \left( \left| T_1(t, \bar{t}, x) - T_2(t, \bar{t}, x) \right|^p 1_{L_n(t)} \right),
\]

where

\[
T_1(t, \bar{t}, x) = \int_0^t (\bar{t} - s) \int_{\mathbb{R}^3} G(1, x - dy) b_n(s, (\bar{t} - s)(y - x) + x) dy ds,
\]

\[
T_2(t, \bar{t}, x) = \int_0^t (t - s) \int_{\mathbb{R}^3} G(1, x - dy) b_n(s, (t - s)(y - x) + x) dy ds.
\]

By adding and subtracting \( t \) in \( T_1 \) we get

\[
T_1(t, \bar{t}, x) = \int_0^t (\bar{t} - t) \int_{\mathbb{R}^3} G(1, x - dy) b_n(s, (\bar{t} - s)(y - x) + x) dy ds + \int_0^t (t - s) \int_{\mathbb{R}^3} G(1, x - dy) b_n(s, (\bar{t} - s)(y - x) + x) dy ds.
\]

Then, Hölder’s inequality yields

\[
R_n^{5,1}(t, \bar{t}, x) \leq C \left| \bar{t} - t \right|^p \int_0^t \int_{\mathbb{R}^3} G(1, x - dy) \mathbb{E} \left( \left| b_n(s, (\bar{t} - s)(y - x) + x) \right| \right)^p 1_{L_n(s)} dy ds + C \int_0^t \left| t - s \right|^p \int_{\mathbb{R}^3} G(1, x - dy) \mathbb{E} \left( \left| b_n(s, (\bar{t} - s)(y - x) + x) \right| \right)^p 1_{L_n(s)} dy ds.
\]

Owing to (120), the first term on the right hand-side of the last inequality is bounded up to a constant by \(|t - \bar{t}|^p\). For the second one, we use the Hypothesis 3.
along with (37) to obtain
\[
\int_0^t |t-s|^\rho \, ds \int_{\mathbb{R}^3} G(1, x - dy) \
\times \mathbb{E}\left(\left| b \left( X_n(s, \bar{t} - s)(y - x) + x \right) - b \left( X_n(s, (t - s)(y - x) + x \right) \right|^{p} 1_{L_n(s)}(s) \right)
\leq C \int_0^t \int_{\mathbb{R}^3} G(1, x - dy) \
\times \mathbb{E}\left( \left| X_n(s, \bar{t} - s)(y - x) + x - X_n(s, (t - s)(y - x) + x) \right|^{p} 1_{L_n(s)}(s) \right)
\leq C|t - \bar{t}|^\rho,
\]
with \( \rho \in \left[ 0, \min\left( \gamma, \gamma_1, \gamma_2, \frac{\nu_1}{2} \right) \right][. \)

Hölder inequality along with (120) clearly yields
\[
R_n^{5,2}(t, \bar{t}, x) \leq C|\bar{t} - t|^{p-1} \int_0^\bar{t} \int_{\mathbb{R}^3} G(\bar{t} - s, x - dy) \mathbb{E}\left( \left| b(X_n(s, y)) \right|^{p} 1_{L_n(s)}(s) \right) \, ds
\leq C|t - \bar{t}|^p. \tag{99}
\]
Hence, we have proved that
\[
R_n^{5}(t, \bar{t}, x) \leq C|t - \bar{t}|^{\rho'}, \tag{100}
\]
where \( \rho \in \left[ 0, \min\left( \gamma, \gamma_1, \gamma_2, \frac{\nu_1}{2} \right) \right][. \)

With the inequalities (84), (87), (97), (98) and (100), we have
\[
\mathbb{E}\left( \left| X_n(\bar{t}, x) - X_n(t, x) \right|^{p} 1_{L_n(\bar{t})} \right) \leq C \left[ |\bar{t} - t|^{\rho'} + f_n \right],
\]
with \( \rho \in \left[ 0, \min\left( \gamma_1, \gamma_2, \gamma, \frac{\nu+1}{2}, \frac{\nu+n}{2}, \frac{\nu}{2} \right) \right][. \)

For a given fixed \( \bar{t} \in [t_0, T] \), we introduce the function
\[
\Psi^{\bar{t}}_{n,x,p}(t) := \mathbb{E}\left( \left| X_n(\bar{t}, x) - X_n(t, x) \right|^{p} 1_{L_n(\bar{t})} \right),
\]
for \( t_0 \leq t \leq \bar{t} \).

Notice that \( \lim_{n \to \infty} f_n = 0 \) and thus, \( \sup_n f_n \leq C \). Thus, there exists a constant \( 0 < C_0 < \infty \), such that
\[
\sup_n f_n \leq C_0 t_0 \leq C_0 \bar{t} \leq C_0 \int_0^\bar{t} ds \left[ 1 + \Psi^{\bar{t}}_{n,x,p}(s) \right].
\]
With a similar argument, there exists \( 0 < C_1 < \infty \) such that
\[
1 \leq C_1 t_0 \leq C_1 \bar{t} \leq C_1 \int_0^\bar{t} ds \left[ 1 + \Psi^{\bar{t}}_{n,x,p}(s) \right].
\]
Therefore,
\[
1 + \Psi^{\bar{t}}_{n,x,p}(t) \leq C \left\{ |\bar{t} - t|^{\rho'} + \int_0^\bar{t} ds \left[ 1 + \Psi^{\bar{t}}_{n,x,p}(s) \right] \right\}.
\]
Then, by Gronwall’s lemma,
\[ 1 + \varphi^{\tilde{t}}_{n,x,p}(t) \leq C(|\tilde{t} - t|^p), \]
where \( p \in \left] 0, \min \left( \gamma_1, \gamma_2, \gamma, \frac{\gamma'}{2}, \frac{\gamma + \gamma'}{2}, \frac{\rho + \gamma'}{2} \right) \). This finish the proof of the proposition.

\[ \square \]

### 2.3. Pointwise convergence

For the sake of completeness we will sketch the proof of the Theorem 2.4. The proofs is very similar to the Theorem 2.4 in [9] just with small changes. The main difference is the bound used, in [9] they used the bound \( 2^{-np(3-\beta)/2} \) while here we use \( 2^{-np\frac{\rho + \gamma'}{2}} \) (see (112) and (121)).

Using equations (24), (25), we write the difference \( X_n(t,x) - X(t,x) \) grouped into comparable terms in order to prove their convergence to zero.

Notice that the initial conditions vanish so we could do the following decomposition.

\[ X_n(t,x) - X(t,x) = \sum_{i=1}^{s} U^i_n(t,x), \]

where

\[
\begin{align*}
U^1_n(t,x) &= \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y) \left[ (A + B)(X_n(s,y)) - (A + B)(X(s,y)) \right] M(ds,dy), \\
U^2_n(t,x) &= \langle G(t-\cdot,x-\cdot)[D(X_n(\cdot,\cdot)) - D(X(\cdot,\cdot))], h_1 \rangle_{\mathcal{H}_1}, \\
U^3_n(t,x) &= \int_0^t ds \int_{\mathbb{R}^3} G(t-s,x-dy) \left[ b(X_n(s,y)) - b(X(s,y)) \right], \\
U^4_n(t,x) &= \langle G(t-\cdot,x-\cdot)[B(X_n(\cdot,\cdot)) - B(X(\cdot,\cdot))], w^n \rangle_{\mathcal{H}_1}, \\
U^5_n(t,x) &= \langle G(t-\cdot,x-\cdot)[B(X_n^-(\cdot,\cdot)) - B(X^-(\cdot,\cdot))], w^n \rangle_{\mathcal{H}_1}, \\
U^6_n(t,x) &= \langle G(t-\cdot,x-\cdot)B(X^-(\cdot,\cdot)), w^n \rangle_{\mathcal{H}_1}, \\
&\quad - \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y)B(X^-)(s,y) M(ds,dy), \\
U^7_n(t,x) &= \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y) \left[ B(X^-)(s,y) - B(X_n^-)(s,y) \right] M(ds,dy), \\
U^8_n(t,x) &= \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y) \left[ B(X_n^-(s,y)) - B(X_n(s,y)) \right] M(ds,dy).
\end{align*}
\]

Here, we have used the abridged notation \( X^-(\cdot,\cdot) \) for the stochastic process \( X^-(t,x) := X(t,t_n,x) \) defined in (27). Notice that, although this is not apparent in the notation \( X^-(\cdot,\cdot) \) does depend on \( n \).
Fix \( p \in [2, \infty] \) we get
\[
\mathbb{E}([X_n(t, x) - X(t, x)]^p 1_{L_n(t)}) \leq C \sum_{i=1}^{8} \mathbb{E}([U^i_n(t, x)]^p 1_{L_n(t)}).
\]

Next, we analyze the contribution of each term \( U^i_n(t, x), i = 1, \ldots, 8 \). By following the procedure in [9] we have,
\[
\mathbb{E}([U^1_n(t, x)]^p 1_{L_n(t)}) + \mathbb{E}([U^2_n(t, x)]^p 1_{L_n(t)}) + \mathbb{E}([U^3_n(t, x)]^p 1_{L_n(t)})
\leq C \int_0^t ds \left[ \sup_{y \in K(s)} \mathbb{E}([X_n(s, y) - X(s, y)]^p 1_{L_n(s)}) \right]^{01}
\]

For \( U^5_n(t, x), U^7_n(t, x) \) we follow the same procedure as in [9] and we arrive to
\[
\mathbb{E}([U^5_n(t, x) + U^7_n(t, x)]^p 1_{L_n(t)}) \leq C \int_0^t ds \left[ \sup_{y \in K(s)} \mathbb{E}([X_n(s, y) - X(s, y)]^p 1_{L_n(s)}) \right]^{01} \]
\[
+ C \int_0^t ds \left[ \sup_{y \in K(s)} \mathbb{E}([X_n(s, y) - X(s, y)]^p 1_{L_n(s)}) \right]^{01}. \tag{102}
\]

Recall that \( X^-(s, y) = X(s, s_n, y) \). By applying (112) and (121), we obtain
\[
\mathbb{E}([U^5_n(t, x) + U^7_n(t, x)]^p 1_{L_n(t)}) \leq C \int_0^t ds \left[ \sup_{y \in K(s)} \mathbb{E}([X_n(s, y) - X(s, y)]^p 1_{L_n(s)}) \right]^{01}
\]
\[
+ Cn^{3p/2 - np^{2} + 1}. \tag{102}
\]

Next, we will see that
\[
\lim_{n \to \infty} \left( \sup_{t \in [0, T]} \sup_{x \in K(t)} \mathbb{E}([U^i_n(t, x)]^p 1_{L_n(t)}) \right) = 0, \quad i = 4, 6, 8. \tag{103}
\]

Consider \( i = 4 \). Cauchy–Schwarz’ inequality along with (33) implies
\[
\mathbb{E}([U^4_n(t, x)]^p 1_{L_n(t)}) \leq Cn^{3p/2 - np^{2}/2} \left( \int_0^t ds \|G(t - s, x - s) [B(X_n) - B(X_n^ -)](s, *) 1_{L_n(s)} \|_{H^p}^2 \right)^{p/2}.
\]

Then, the Lipschitz continuity of \( B \) and (121) yield
\[
\mathbb{E}([U^4_n(t, x)]^p 1_{L_n(t)}) \leq Cn^{3p/2 - np^{2}/2} \int_0^t ds \left[ \sup_{y \in \mathbb{R}^3} \mathbb{E}([X_n(s, y) - X_n^ -(s, y)]^p 1_{L_n(s)}) \right]^{01}
\]
\[
\leq Cn^{3p/2 - np^{2}/2} \frac{np^{2} + 1}{2}.
\]
\[
\leq Cn^{3p/2 - np^{2}/2}.
\]
Clearly, \( U^1 \) applies Burkholder’s and Hölder’s inequalities, along with the contraction properties of the projection \( \pi_n \). This yields,

\[
\mathbb{E}(\|U^1_n(t,x)\|^p 1_{L_n(t)}) \leq C \int_0^t ds \sup_{y \in \mathbb{R}^3} \mathbb{E}(\|X_n^-(s,y) - X_n(s,y)\|^p 1_{L_n(s)}) \leq C n^{3p/2} 2^{-n(6/2)}
\]

where, in the last inequality we have used (121). Thus, (103) holds for \( i = 8 \).

Let us now consider the case \( i = 6 \). Define

\[
U^6_1(t,x) = \int_0^t \int_{\mathbb{R}^3} \{ \pi_n \left[ \tau_n \left[ G(t - \cdot, x - \cdot) B(X^-) \right] \right] - G(t - \cdot, x - \cdot) \tau_n \left[ B(X^-) \right] \} (s,y) M(ds,dy),
\]

\[
U^6_2(t,x) = \int_0^t \int_{\mathbb{R}^3} \pi_n \left[ G(t - \cdot, x - \cdot) \tau_n \left[ B(X^-) \right] \right] - G(t - \cdot, x - \cdot) B(X^-) (s,y) M(ds,dy),
\]

\[
U^6_3(t,x) = \int_0^t \int_{\mathbb{R}^3} \{ \pi_n \left[ G(t - \cdot, x - \cdot) B(X^-) \right] \} - G(t - s, x - y) B(X^-) (s,y) M(ds,dy).
\]

Clearly,

\[
U^6_n(t,x) = U^6_1(t,x) + U^6_2(t,x) + U^6_3(t,x).
\]

In a rather similar way to [9] we establish

\[
\mathbb{E}(\|U^6_1(t,x)\|^p 1_{L_n(t)}) \leq C g_n.
\]

where \( g_n \) is a bounded sequence that converges to zero. Thus, we have established the convergence

\[
\lim_{n \to \infty} \sup_{(t,x) \in [0,T] \times K(t)} \mathbb{E}(\|U^6_1(t,x)\|^p 1_{L_n(t)}) = 0. \tag{104}
\]

Next, we consider the term \( U^6_2(t,x) \). As usually for these type of terms, we apply Burkholder’s and then Hölder’s inequalities, along with the contraction property of the projection \( \pi_n \). This yields,

\[
\mathbb{E}(\|U^6_2(t,x)\|^p 1_{L_n(t)}) = \mathbb{E}
\]

\[
= \mathbb{E}
\]

\[
= \mathbb{E}
\]

\[
\leq C \int_0^t \sup_{x \in \mathbb{R}^3} \mathbb{E}(\|X((s + 2^{-n}) \land t, (s_n + 2^{-n}) \land t, x) - X(s, s_n, x)\|^p).
\]
Equation (24) is a particular case of equation (25). Therefore, Proposition 2.9 also holds with \( X_n \) replaced by \( X \). Then, by virtue of (112) and (65), this is bounded up to a constant by \( 2^{-np\frac{1+p}{2}} + 2^{-np} \), for some \( \rho > 0 \) Consequently,

\[
\lim_{n \to \infty} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\left( |U_n^{6,2}(t,x)|^p 1_{L_n(t)} \right) = 0. \tag{105}
\]

For \( U_n^{6,3}(t,x) \), after having applied Burkholder’s inequality we have

\[
\mathbb{E}\left( |U_n^{6,3}(t,x)|^p 1_{L_n(t)} \right) \leq C \mathbb{E}\left( \left\| (\pi_n - I_{\mathcal{H}_t}) \left[ G(t, \cdot, x - \cdot) B(\chi^{-}(\cdot, \cdot)) \right] \right\|_{\mathcal{H}_t}^p \right). \]

We want to prove that the right-hand side of this inequality tends to zero as \( n \to \infty \), uniformly in \( (t,x) \in [t_0, T] \times K(t) \). By using a similar approach as in [18], pages 906–909.

Set

\[
\tilde{Z}_n(t,x) = \left\| (\pi_n - I_{\mathcal{H}_t}) \left[ G(t, \cdot, x - \cdot) B(\chi^{-}(\cdot, \cdot)) \right] \right\|_{\mathcal{H}_t}. \]

Since \( \pi_n \) is a projection on the Hilbert space \( \mathcal{H}_t \), the sequence \( \{\tilde{Z}_n(t,x), n \geq 1\} \) decreases to zero as \( n \to \infty \). Assume that

\[
\mathbb{E}\left( \sup_n \| G(t, \cdot, x - \cdot) B(\chi^{-}(\cdot, \cdot)) \|_{\mathcal{H}_t}^p \right) < \infty. \tag{106}
\]

Remember that \( \chi^{-}(s,y) \) stands for \( X(s, s_n, y) \), defined in (27), and therefore it depends on \( n \). Then, by bounded convergence, this would imply \( \lim_{n \to \infty} \mathbb{E}(\tilde{Z}_n(t,x))^p = 0 \). Set \( Z_n(t,x) = \mathbb{E}(\tilde{Z}_n(t,x))^p \). Proceeding as in the proof of Lemmas 2.6, 2.10, we can check that

\[
\left| (Z_n(t,x))^{1/p} - (\tilde{Z}_n(t,x))^{1/p} \right| \leq C |t - \bar{t}| + |x - \bar{x}|^p,
\]

for some \( \rho > 0 \).

Hence, \( (Z_n) \) is a sequence of monotonically decreasing continuous functions defined on \( [0, T] \times \mathbb{R}^3 \) which converges pointwise to zero. Appealing to Dini’s theorem, we obtain

\[
\lim_{n \to \infty} \sup_{(t,x) \in [t_0, T] \times K(t)} \mathbb{E}(\tilde{Z}_n(t,x))^p = 0. \tag{107}
\]

This yields the expected result on \( U_n^{6,3} \).

To prove (106) just follow the procedure in [9, page 2205].

In order to conclude the proof, let us consider the estimates (101), (102), along with (103). We see that

\[
\mathbb{E}\left( |X_n(t,x) - X(t,x)|^p 1_{L_n(t)} \right) \leq C_1 \theta_n + C_2 \int_0^t ds \sup_{x \in K(s)} \mathbb{E}\left( |X_n(s,x) - X(s,x)|^p 1_{L_n(s)} \right),
\]

where \( (\theta_n, n \geq 1) \) is a sequence of real numbers which converges to zero as \( n \to \infty \). Applying Gronwall’s lemma, we finish the proof of the theorem. \( \square \)
2.4. Proof of Theorem 2.2

Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Let $Y_n(t,x) := X_n(t,x) - X(t,x)$ and $B_n(t) := L_n(t)$, $n \geq 1$, $(t,x) \in [t_0,T] \times K$, $p \in [1,\infty]$. From Theorems 2.3 and 2.4, we see that the conditions (P1) and (P2) of Lemma 5.3 are satisfied with $\delta = pp - 4$, for any $\rho \in \left[0,\min \left(\gamma_1,\gamma_2,\gamma,\frac{\nu+1}{2},\frac{\rho_1+\kappa}{2},\frac{\rho_2}{2}\right)\right]$, where $\kappa \in \left[0,\min \left(\gamma_1,\gamma_2,\gamma,\frac{\nu+1}{2}\right)\right]$. We infer that

$$\lim_{n \to \infty} \mathbb{E}(\|X_n - X\|^p_{\rho,t_0,K}1_{L_n(t)}) = 0,$$  \hspace{1cm} (108)

for any $p \in [1,\infty]$ and $\rho \in \left[0,\min \left(\gamma_1,\gamma_2,\gamma,\frac{\nu+1}{2},\frac{\rho_1+\kappa}{2},\frac{\rho_2}{2}\right)\right]$.

Fix $\varepsilon > 0$. Since $\lim_{n \to \infty} \mathbb{P}(L_n(t)^c) = 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\mathbb{P}(L_n(t)^c) < \varepsilon$. Then, for any $\lambda > 0$ and $n \geq N_0$,

$$\mathbb{P}(\|X_n - X\|_{\rho,t_0,K} > \lambda) \leq \varepsilon + \mathbb{P}(\|X_n - X\|_{\rho,t_0,K} > \lambda) \cap L_n(t)) \leq \varepsilon + \lambda^{-p}\mathbb{E}(\|X_n - X\|^p_{\rho,t_0,K}1_{L_n(t)}) \hspace{1cm} (\text{by (108)})$$

Since $\varepsilon > 0$ is arbitrary, this finishes the proof of the theorem. \qed

3. Support theorem

This section is devoted to the characterization of the topological support of the law of the random field solution to the stochastic wave equation (7). As has been explained in the Introduction, this is a corollary of Theorem 2.2.

**Theorem 3.1.** Assume that the functions $\zeta$ and $b$ are Lipschitz continuous. Fix $t_0 \in [0,T]$ and a compact set $K \subset \mathbb{R}^3$. Let $u = \{u(t,x), (t,x) \in [t_0,T] \times K\}$ be the random field solution to (7). Fix $\rho \in \left[0,\min \left(\gamma_1,\gamma_2,\gamma,\frac{\nu+1}{2},\frac{\rho_1+\kappa}{2},\frac{\rho_2}{2}\right)\right]$, where $\kappa \in \left[0,\min \left(\gamma_1,\gamma_2,\gamma,\frac{\nu+1}{2}\right)\right]$. Then the topological support of the law of $u$ in the space $\mathcal{C}^p([t_0,T] \times K)$ is the closure in $\mathcal{C}^p([t_0,T] \times K)$ of the set of functions $\{\Phi^h, h \in \mathcal{H}_T\}$, where $\{\Phi^h(t,x), (t,x) \in [t_0,T] \times K\}$ is the solution of (12).

Let $\{w^n, n \geq 1\}$ be the sequence of $\mathcal{H}_T$-valued random variables defined in (22). For any $h \in \mathcal{H}_T$, we consider the sequence of transformations of $\Omega$ defined in (21). As has been pointed out in Section 1, $P \circ (T_n^h)^{-1} \ll P$.

Notice also that the process $v_n(t,x) := (u \circ T_n^h)(t,x)$, $(t,x) \in [t_0,T] \times \mathbb{R}^3$, satisfies the equation

$$v_n(t,x) = \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y)\zeta(v_n(s,y))M(ds,dy)$$

$$+ \langle G(t-\cdot,x-\cdot)\zeta(v_n(\cdot,\cdot)), h-w^n \rangle_{\mathcal{H}_T} + \int_0^t ds \left[G(t-s,\cdot) \ast b(v_n(s,\cdot))\right](x).$$  \hspace{1cm} (109)
Proof. of Theorem 3.1 According to the method developed in [16] (see also [2] and Section 1 in [9] for a summary), the theorem will be a consequence of the following convergences:

\[
\lim_{n \to \infty} \mathbb{P}\{\|u - \Phi^w_n\|_{\rho,t_0,K} > \eta\} = 0, \quad (110)
\]

\[
\lim_{n \to \infty} \mathbb{P}\{\|u \circ T_n^h - \Phi^h\|_{\rho,t_0,K} > \eta\} = 0, \quad (111)
\]

where \(\eta\) is an arbitrary positive real number.

This follows from the general approximation result developed in Section 2. Indeed, consider equations (24) and (25) with the choice of coefficients \(A = D = 0, B = \varsigma\). Then the processes \(X\) and \(X_n\) coincide with \(u\) and \(\Phi^w\), respectively.

Hence, the convergence (110) follows from Theorem 2.2. Next, we consider again equations (24) and (25) with a new choice of coefficients: \(A = D = \varsigma, B = -\varsigma\). In this case, the processes \(X\) and \(X_n\) are equal to \(\Phi^h\) and \(v_n := u \circ T_n^h\), respectively. Thus, Theorem 2.2 yields (111). \(\square\)

4. Auxiliary results

The most difficult part in the proof of Theorem 2.2 consists of establishing (35). In particular, handling the contribution of the pathwise integral (with respect to \(w^n\)) requires a careful analysis of the discrepancy between this integral and the stochastic integral with respect to \(M\). This section gathers several technical results that have been applied in the analysis of such questions in the preceding Section 2.

The first statement in the next lemma provides a measure of the discrepancy between the processes \(X(t,x)\) and \(X(t,t_n,x)\) defined in (24), (27), respectively.

**Lemma 4.1.** Suppose that Hypothesis 3 is satisfied. Then for any \(p \in [1, \infty)\) and every integer \(n \geq 1\),

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \left\|X(t,x) - X(t,t_n,x)\right\|_p \leq C2^{-n\frac{p+1}{2}} \quad (112)
\]

and

\[
\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \left\|X(t,t_n,x)\right\|_p \leq C < \infty, \quad (113)
\]

where \(C\) is a positive constant not depending on \(n\).

**Proof.** Fix \(p \in [2, \infty[\). From equations (24), (27), we obtain

\[
\left\|X(t,x) - X(t,t_n,x)\right\|_p^p \leq C(1V_1(t,x) + V_2(t,x) + V_3(t,x)),
\]

0
where
\[
V_1(t, x) := \left\| \int_{t_n}^{t} \int_{\mathbb{R}^3} G(t - s, x - y)(A + B)(X(s, y))M(ds, dy) \right\|_p^p,
\]
\[
V_2(t, x) := \left\| G(t - \cdot, x - \cdot)D(X(\cdot, \cdot))1_{[t_n, t]}(\cdot) \right\|_{p^1}^p,
\]
\[
V_3(t, x) := \left\| \int_{t_n}^{t} G(t - s, \cdot) \ast b(X(s, \cdot))(x) ds \right\|_p^p.
\]

Applying first Burholder’s and then Hölder’s inequalities, we obtain

\[
V_1(t, x) \leq CE \left( \int_{t_n}^{t} ds \int_{\mathbb{R}^3} G(t - s, x - dy)G(t - s, x - dz)f(y - z) \times (A + B)(X(t, y))(A + B)(X(t, z)) \right)^{p/2}
\]
\[
= CE \left( \int_{0}^{t-t_n} ds \int_{\mathbb{R}^3} G(s, x - dy)G(s, x - dz)f(y - z) \times (A + B)(X(t - s, y))(A + B)(X(t - s, z)) \right)^{p/2}
\]
\[
\leq C(t - t_n)^{p-1} \int_{0}^{t-t_n} ds \left( \int_{\mathbb{R}^3} G(s, x - dy)G(s, x - dz)f(y - z) \right)^{p/2}
\]
\[
\times \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}(\|A + B)(X(t, x)^p)
\]
\[
\leq C(t - t_n)^{p-1} \int_{0}^{t-t_n} ds \left( \int_{\mathbb{R}^3} G(s, x - dy)G(s, x - dz)f(y - z) \right)^{p/2}
\]
\[
\times \left( 1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}(\|X(t, x)^p) \right)
\]

Applying the inequality (4), Lemma 7.1 in [12] and then a) of the hypothesis 2, we get

\[
V_1(t, x) \leq C(t - t_n)^{p-1} \int_{0}^{t-t_n} ds \left( \int_{|z| \leq 2s} \frac{f(z)}{|z|} \right)^{p/2}
\]
\[
\leq C(t - t_n)^{p-1} \int_{0}^{t-t_n} s^{p/2} ds \leq C(t - t_n)^{p \cdot \frac{p+1}{2}} \leq C2^{-np^{\frac{p+1}{2}}}
\]

For the study of \(V_2\), we apply first Cauchy–Schwarz inequality and then Hölder’s inequality. We obtain

\[
V_2(t, x) \leq \|h1_{[t_n, t]}(\cdot)\|_{p^1}^p \mathbb{E} \left( \int_{0}^{t} ds \|G(t - s, x - \cdot)D(X(s, \cdot))1_{[t_n, t]}(s)\|_{H_t}^2 \right)^{p/2}
\]

Hence, similarly as for \(V_1\) we have

\[
V_2(t, x) \leq C2^{-np^{\frac{p+1}{2}}}
\]
By applying Hölder’s inequality, we get

\[
V_3(t, x) \leq \left( \int_a^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \right)^{p-1} \int_a^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \mathbb{E}(\|b(X(s, y))\|^p)
\]

\[
\leq C \left( \int_a^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \right)^p \left( 1 + \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \mathbb{E}(\|X(s, y)\|^p) \right)
\]

\[
\leq C2^{-2np}.
\]

The condition \( \nu \in [0, 1] \) implies \( 2^{-2np} < 2^{-np\frac{\nu+1}{2}} \). Thus from the estimates on \( V_i(t, x), i = 1, 2, 3 \) (which hold uniformly on \( (t, x) \in [0, T] \times \mathbb{R}^3 \)) we obtain (112).

Finally, (113) is a consequence of the triangular inequality, (112) and (4).

The next result states an analogue of Lemma 4.1 for the stochastic processes \( X_n, X_n^- \) defined in (25), (26), respectively, this time including a localization by \( L_n \).

**Lemma 4.2.** We assume Hypothesis 3. Then for any \( p \in [2, \infty) \) and \( t \in [0, T] \),

\[
\sup_{(s, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}(\|X_n(s, y) - X_n^-(s, y)\|^p 1_{L_n(s)\}})
\]

\[
\leq Cn^{3p/2 - np\frac{\nu+1}{2}} \left[ 1 + \sup_{(s, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}(\|X_n(s, y)\|^p 1_{L_n(s)}) \right].
\]

**Proof.** Fix \( p \in [2, \infty] \) and consider the decomposition

\[
\mathbb{E}(\|X_n(t, x) - X_n^-(t, x)\|^p 1_{L_n(t)}) \leq C \sum_{i=1}^{4} T_{n,i}^E(t, x),
\]

where

\[
T_{n,1}(t, x) = \mathbb{E}\left( \left( \int_a^t ds \int_{\mathbb{R}^3} G(t - s, x - y)A(X_n(s, y))M(ds, dy) \right)^p 1_{L_n(t)} \right),
\]

\[
T_{n,2}(t, x) = \mathbb{E}\left( \left( \int_a^t ds \int_{\mathbb{R}^3} G(t - s, x - y)B(X_n(\cdot, *))1_{[t_n, t]}(\cdot, w^n)_{H_{\nu}} \right)^p 1_{L_n(t)} \right),
\]

\[
T_{n,3}(t, x) = \mathbb{E}\left( \left( \int_a^t ds \int_{\mathbb{R}^3} G(t - s, x - y)D(X_n(\cdot, *))1_{[t_n, t]}(\cdot, h)_{H_{\nu}} \right)^p 1_{L_n(t)} \right),
\]

\[
T_{n,4}(t, x) = \mathbb{E}\left( \left( \int_a^t ds \int_{\mathbb{R}^3} G(t - s, \cdot)B(X_n(s, \cdot)) \right)^p 1_{L_n(t)} \right).
\]

By the same arguments used for the analysis of \( V_1(t, x) \) in the preceding lemma, we obtain

\[
T_{n,1}(t, x) \leq C2^{-np\frac{\nu+1}{2}} \times \left[ 1 + \sup_{(s, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}(\|X_n(s, y)\|^p 1_{L_n(s)}) \right].
\]

For \( T_{n,2}(t, x) \), we first use Cauchy–Schwarz’ inequality to obtain

\[
T_{n,2}(t, x) \leq \mathbb{E}\left( \|w^n1_{[t_n, t]}1_{L_n(t)}\|_{H_{\nu}} \|G(t - \cdot, x - \cdot)B(X_n(\cdot, *))1_{[t_n, t]}(\cdot)1_{L_n(t)}\|_{H_{\nu}}^p \right).
\]
Appealing to (34), this yields

\[
T_{n,2}(t, x) \leq Cn^{3p/2} \mathbb{E} \left( \int_{t_n}^t ds \|G(t - s, x - \cdot)B(X_n(s, \cdot)) \|_{\mathcal{H}}^2 \right)^{p/2}
\]

\[
= Cn^{3p/2} \mathbb{E} \left( \int_{t_n}^t ds \int_{\mathbb{R}^3} G(t - s, x - dy)G(t - s, x - dz)f(y - z) \times B(X(t, y))B(X(t, z)) \right)^{p/2}.
\]

We can now proceed as for the term \(V_2((t, x))\) in the proof of Lemma 4.1. We obtain

\[
T_{n,2}(t, x) \leq Cn^{3p/2}2^{-np^{p+1}/2} \mathbb{E} \left( \left| X_n(s, y) \right|^p 1_{L_n(s)} \right)^{p/2}.
\]  

(117)

The difference between the terms \(T_{n,3}(t, x)\) and \(T_{n,2}(t, x)\) is that \(w^n\) in the latter is replaced by \(h\) in the former. Hence, following similar arguments as for the study of \(T_{n,2}(t, x)\), and using that \(\|h1_{[t_n, t]} 1_{L_n(t)}\|_{\mathcal{H}_X} < \infty\), we prove

\[
T_{n,3}(t, x) \leq C2^{-np^{p+1}/2} \times \mathbb{E} \left( \left| X_n(s, y) \right|^p 1_{L_n(s)} \right)^{p/2}.
\]  

(119)

Finally, we notice the similitude between \(T_{n,4}(t, x)\) and \(V_3(t, x)\) in Lemma 4.1. Proceeding as for the study of this term, we obtain

\[
T_{n,4}(t, x) \leq C \left( \int_{t_n}^t ds \int_{\mathbb{R}^3} G(t - s, x - dy) \right)^p \left[ 1 + \sup_{(s, y) \in \mathbb{R}^3} \mathbb{E} \left( \left| X_n(s, y) \right|^p 1_{L_n(s)} \right) \right]^{p/2}
\]

(119)

From (115)–(119) we obtain (114).

\[ \square \]

**Lemma 4.3.** We assume Hypothesis 3. Then, for any \(p \in [1, \infty)\), there exists a finite constant \(C\) such that

\[
\sup_{n \geq 1} \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \mathbb{E} \left( \left| X_n(t, x) \right|^p + \left| X_n(t, x) \right|^p 1_{L_n(t)} \right) \leq C.
\]  

(120)

Moreover,

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \left\| \left( X_n(t, x) - X_n(t, x) \right) 1_{L_n(t)} \right\|_p \leq Cn^{3p/2}2^{-np^{p+1}/2}.
\]  

(121)

**Proof.** For \(0 \leq r \leq t\), define

\[
X_n(t, r; x) = \frac{d}{dt} \left( G(t) \ast v_0 \right)(x) + \left( G(t) \ast \overline{v}_0 \right)(x)
\]

\[
\int_0^r \int_{\mathbb{R}^3} G(t - s, x - y) A(X_n(s, y)) M(ds, dy)
\]

\[
+ \left( G(t - \cdot, x - \cdot) \ast B(X_n(\cdot, \cdot)) \right) 1_{[0, r]}(\cdot, w^n)_{\mathcal{H}_t}
\]

\[
+ \left( G(t - \cdot, x - \cdot) \ast D(X_n(\cdot, \cdot)) \right) 1_{[0, r]}(\cdot, h)_{\mathcal{H}_t} + \int_0^r G(t - s, \cdot) \ast b(X_n(s, \cdot))(x) ds.
\]
Fix $p \in [2, \infty]$ and consider the decomposition
\[
\mathbb{E}\left(|X_n(t, r; x)|^p \mathbb{1}_{L_n(t)}\right) \leq C \sum_{i=1}^{5} T_{n,i}(t, r; x),
\]
where
\[
T_{n,0}(t, r; x) = \left| \frac{d}{dt} (G(t) * v_0)(x) + (G(t) * \bar{v}_0)(x) \right|^p \]
\[
T_{n,1}(t, r; x) = \mathbb{E}\left(\left(\int_0^t \int_{\mathbb{R}^3} G(t - s, x - y) A(X_n(s, y)) M(ds, dy) \mathbb{1}_{L_n(t)}\right)^p\right),
\]
\[
T_{n,2}(t, r; x) = \mathbb{E}\left(\left(\int_0^t \int_{\mathbb{R}^3} (G(t - s, x - y) B(X_n(-, s)) \mathbb{1}_{[0, r]}(\cdot), w^n)_{H_t} M(ds, dy) \mathbb{1}_{L_n(t)}\right)^p\right),
\]
\[
T_{n,3}(t, r; x) = \mathbb{E}\left(\left(\int_0^t \int_{\mathbb{R}^3} (G(t - s, x - y) B(X_n(-, s)) - B(X_n(-, s))) \mathbb{1}_{[0, r]}(\cdot), w^n)_{H_t} M(ds, dy) \mathbb{1}_{L_n(t)}\right)^p\right),
\]
\[
T_{n,4}(t, r; x) = \mathbb{E}\left(\left(\int_0^t \int_{\mathbb{R}^3} (G(t - s, x - y) D(X_n(-, s)) \mathbb{1}_{[0, r]}(\cdot), h)_{H_t} M(ds, dy) \mathbb{1}_{L_n(t)}\right)^p\right),
\]
\[
T_{n,5}(t, r; x) = \mathbb{E}\left(\left(\int_0^t \int_{\mathbb{R}^3} G(t - s, \cdot) \ast b(X_n(s, \cdot)) M(ds, dy) \mathbb{1}_{L_n(t)}\right)^p\right).
\]

By Hypothesis on $v, \bar{v}$ and for the Theorem 4.6 in [7], we have that
\[
T_{n,0}(t, r; x) \leq C \tag{122}
\]

Similarly as for the term $V_1(t, x)$ in Lemma 4.1, we have
\[
T_{n,1}(t, r; x) \leq C \left(\int_0^t ds \left(\int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}G(t - s)(\xi)|^2 \right)^{p/2 - 1} \right) \tag{123}
\]
\[
\times \int_0^t ds \left[1 + \sup_{(\hat{s}, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}\left(|X_n(\hat{s}, y)|^p \mathbb{1}_{L_n(\hat{s})}\right) \right]
\]
\[
\times \left(\int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}G(t - s)(\xi)|^2 \right) \leq C \int_0^t ds \left[1 + \sup_{(\hat{s}, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}\left(|X_n(\hat{s}, y)|^p \mathbb{1}_{L_n(\hat{s})}\right) \right].
\]

Let $\tau_n$ and $\pi_n$ be as in the proof of Lemma 2.8 (see (53) and the successive lines). Since $X_n^-(s, y)$ is $\mathcal{F}_{n}^+$-measurable, the definition of $w^n$ implies
\[
T_{n,2}(t, r; x) = \mathbb{E}\left(\left(\int_0^t \int_{\mathbb{R}^3} (\pi_n \circ \tau_n)(G(t - s, x - y) B(X_n(-, s))) \mathbb{1}_{L_n(t)} M(ds, dy) \mathbb{1}_{L_n(s)}\right)^p\right).
\]

Then, applying Burkholder’s inequality, using the boundedness of the operator $\pi_n \circ \tau_n$, and similar arguments as for the term $T_{n,1}(t, r; x)$ we obtain
\[
T_{n,2}(t, r; x) \leq C \int_0^t ds \left[1 + \sup_{(\hat{s}, y) \in [0, t] \times \mathbb{R}^3} \mathbb{E}\left(|X_n(\hat{s}, y)|^p \mathbb{1}_{L_n(\hat{s})}\right) \right]. \tag{124}
\]
To study $T_{n,3}(t, r; x)$, we apply Cauchy–Schwarz and then Hölder’s inequality. This yields

$$T_{n,3}(t, r; x) \leq \mathbb{E}\left(\left\|w^n 1_{[0, r]} 1_{L_n(t)}\right\|_{H_t}^2 \|G(t -, x - s) [B(X_n) - B(X_n^-)](s, r) 1_{[0, r]}(\cdot) 1_{L_n(t)}\|_{H_t}^2 \right)^{p/2}$$

$$\leq C_n^{3p/2} 2^{np/2} \mathbb{E}\left(\int_0^t ds \|G(t - s, x - s) [B(X_n) - B(X_n^-)](s, r) 1_{[0, r]}(s) 1_{L_n(s)}\|_{H_t}^2 \right)^{p/2}$$

$$\leq C_n^{3p/2} 2^{np/2} \left(\int_0^r ds \int_{\mathbb{R}^3} \mu(d\xi) \mathcal{F}G(t - s) \|t\|_{L_\xi}^2(x) \right)^{p/2 - 1}$$

$$\times \int_0^r ds \sup_{(s, y) \in [0, s] \times \mathbb{R}^3} \mathbb{E}\left(\left|X_n(s, y) - X_n^-(s, y)\right|^p 1_{L_n(s)}\right)$$

$$\times \left(\int_{\mathbb{R}^3} \mu(d\xi) \mathcal{F}G(t - s) \|t\|_{L_\xi}^2(x) \right),$$

where we have used (33) and the Lipschitz continuity of the function $B$. By applying (114), we obtain

$$T_{n,3}(t, r; x) \leq C_n^{3p/2} 2^{-np/2} \int_0^r ds \left[1 + \sup_{(s, y) \in [0, s] \times \mathbb{R}^3} \mathbb{E}\left(\left|X_n(s, y)\right|^p 1_{L_n(s)}\right)\right]$$

$$\leq C_n^{3p/2} 2^{-np/2} \int_0^r ds \left[1 + \sup_{(s, y) \in [0, s] \times \mathbb{R}^3} \mathbb{E}\left(\left|X_n^-(s, y)\right|^p 1_{L_n(s)}\right)\right],$$

where in the last inequality we have used that $\sup_n \{n^{3p/2} 2^{-np/2}\} < \infty$.

We now consider $T_{n,4}(t, r; x)$. With similar arguments as those used in the analysis of $T_{n,3}(t, x)$ in Lemma 4.2, we prove

$$T_{n,4}(t, r; x) \leq C \int_0^r ds \left[1 + \sup_{(s, y) \in [0, s] \times \mathbb{R}^3} \mathbb{E}\left(\left|X_n(s, y)\right|^p 1_{L_n(s)}\right)\right]. \quad (125)$$

Finally, we notice that $T_{n,5}(t, r; x)$ is very similar to $T_{n,4}(t, x)$ in Lemma 4.2. With similar arguments as those used in the analysis of this term, we have

$$T_{n,5}(t, r; x) \leq C \int_0^r ds \left[1 + \sup_{(s, y) \in [0, s] \times \mathbb{R}^3} \mathbb{E}\left(\left|X_n(s, y)\right|^p 1_{L_n(s)}\right)\right]. \quad (126)$$

Bringing together (122), (124)–(126) yields

$$\mathbb{E}\left(\left|X_n(t, r; x)\right|^p 1_{L_n(t)}\right) \leq C \left\{1 + \int_0^r \sup_{(s, y) \in [0, s] \times \mathbb{R}^3} \mathbb{E}\left(\left\{|X_n(s, y)|^p + |X_n^-(s, y)|^p\right\} 1_{L_n(s)}\right) ds\right\}. \quad (127)$$

Notice that $X_n(t, t; x) = X_n(t, x)$. Hence, for $r := t$, (127) tells us

$$\mathbb{E}\left(\left|X_n(t, x)\right|^p 1_{L_n(t)}\right) \leq C \left\{1 + \int_0^t \sup_{(s, y) \in [0, s] \times \mathbb{R}^3} \mathbb{E}\left(\left\{|X_n(s, y)|^p + |X_n^-(s, y)|^p\right\} 1_{L_n(s)}\right) ds\right\}. \quad (128)$$
Next, take \( r := t_n \) and remember that \( X_n(t, t_n; x) = X_n^-(t, x) \). From (127), and since \( t_n \leq t \), we obtain

\[
E\left( |X_n^-(t, x)|^p 1_{L_n(t)} \right) 
\leq C \left\{ 1 + \int_0^t \sup_{(\hat{s}, y) \in [0, s] \times \mathbb{R}^3} E\left( \{ |X_n(\hat{s}, y)|^p + |X_n^-(\hat{s}, y)|^p \} 1_{L_n(\hat{s})} \right) \, ds \right\}.
\]

For \( t \in [0, T] \), set

\[
\varphi_n(t) = \sup_{(s, y) \in [0, t] \times \mathbb{R}^3} E\left( \{ |X_n(s, y)|^p + |X_n^-(s, y)|^p \} 1_{L_n(s)} \right).
\]

The inequalities (128), (129) imply \( \varphi_n(t) \leq C \{ 1 + \int_0^t \varphi_n(s) \, ds \} \). By Gronwall’s lemma, this implies (120). Finally, the inequality (121) is a consequence of (114) and (120).

\[\square\]

5. Appendix

We start this section with a theorem on existence and uniqueness of solution to a class of equations which in particular applies to (24), and therefore also to (7), and to (25). For related results, we refer the reader to [5], Theorem 13, [7], Theorem 4.3. In comparison with these references, here we state the theorem in spatial dimension \( d = 3 \), and we assume that \( G \) is the fundamental solution of the wave equation in dimension three.

**Theorem 5.1.** Let \( G \) denote the fundamental solution to the wave equation in dimension three and \( M \) a Gaussian process as given in the Introduction. Consider the stochastic evolution equation defined by

\[
Z(t, x) = \frac{d}{dt} (G(t) \ast v_0)(x) + (G(t) \ast \bar{v}_b)(x)
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y) \zeta(Z(s, y)) M(ds, dy)
\]

\[
+ (G(t - \cdot, x - \cdot) g(Z(\cdot, \cdot)), H)_{\mathcal{H}_T}
\]

\[
+ \int_0^t [G(t - s, \cdot) \ast b(Z(s, \cdot))] (x),
\]

where the functions \( \zeta, g, b : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous.

(i) Assume that \( H = \{ H_t, t \in [0, T] \} \) is an \( \mathcal{H} \)-valued predictable stochastic process such that \( C_0 := \sup_\omega \| H(\omega) \|_{\mathcal{H}_T} < \infty \).

Then, there exists a unique real-valued adapted stochastic process \( Z = \{ Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^3 \} \) satisfying (1), a.s., for all \( (t, x) \in [0, T] \times \mathbb{R}^3 \). Moreover, the process \( Z \) is continuous in \( L^2 \) and satisfies

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} E\left( |Z(t,x)|^p \right) \leq C < \infty,
\]

for any \( p \in [1, \infty[ \), where the constant \( C \) depends among others on \( C_0 \).
(ii) Assume that there exist an increasing sequence of events \( \{ \Omega_n, n \geq 1 \} \) such that \( \lim_{n \to \infty} P(\Omega_n) = 1 \), and that \( H_n = \{ H_n(t), t \in [0, T] \} \) is a sequence of \( \mathcal{H} \)-valued predictable stochastic processes such that \( C_n := \sup_\omega \| H(\omega)1_{\Omega_n}(\omega) \|_{\mathcal{H}T} < \infty \). Then, the conclusion on existence and uniqueness of solution to (1) stated in part (i) also holds.

The process \( Z \) is termed a random field solution to (1).

**Proof.** (Sketch of the proof) We start with part (i). Consider the Picard iteration scheme

\[
Z^0(t, x) = \frac{d}{dt} (G(t) * \nu_0)(x) + (G(t) * \nu_0)(x),
\]

\[
Z^{(k+1)}(t, x) = \frac{d}{dt} (G(t) * \nu_0)(x) + (G(t) * \nu_0)(x)
+ \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \zeta(Z^{(k)}(s, y)) M(ds, dy)
+ \langle G(t - \cdot, x - \cdot) g(Z^{(k)}(\cdot, \cdot)), H \rangle_{\mathcal{H}T} + \int_0^t [G(t-s, \cdot) * b(Z^{(k)}(\cdot, \cdot))](x),
\]

\( k \geq 0 \).

Fix \( p \in [2, \infty] \). First, we prove by induction on \( k \geq 0 \) that

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} E \left( |Z^k(t, x)|^p \right) \leq C < \infty,
\]

with a constant \( C \) independent of \( k \). Second, we prove that

\[
\sup_{x \in \mathbb{R}^3} E \left( |Z^{(k+1)}(t, x) - Z^{(k)}(t, x)|^p \right)
\leq C(1 + C_0) \left[ \int_0^t ds \sup_{y \in \mathbb{R}^3} E \left( |Z^{(k)}(s, y) - Z^{(k-1)}(s, y)|^p \right) \right].
\]

With this, we conclude that the sequence of processes \( \{ Z^{(k)}(t, x), (t, x) \in [0, T] \times \mathbb{R}^3 \} \), \( k \geq 0 \) converges in \( L^p(\Omega) \) as \( k \to \infty \), uniformly in \( (t, x) \in [0, T] \times \mathbb{R}^3 \). The limit is a random field that satisfies the properties of the statement. We refer the reader to [5, 7], for more details on the proof.

The proof of part (ii) is done by localizing the preceding Picard scheme using the sequence \( \{ \Omega_n, n \geq 1 \} \).

In comparison with the equation considered in [7], Theorem 4.3, (1) has the extra term \( \langle G(t - \cdot, x - \cdot) g(Z(\cdot, \cdot)), H \rangle_{\mathcal{H}T} \).

Part (i) of Theorem 5.1 can be applied to (7), (24). Therefore, we have

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} E \left( |X(t, x)|^p \right) < \infty.
\]

Let \( \Omega_n = L_n(t) \) as given in (32). The sequence \( H_n := w^n \) defined in (22) satisfies the assumptions of part (ii) of Theorem 5.1 (see (33)). Therefore the conclusion applies to the stochastic process solution of (25).
Remark 5.2. Set $Z^{(z)}(s, x) = Z(s, x+z)$, $z \in \mathbb{R}^3$. In opposition to [5], we cannot argue that the finite dimensional distributions of the process $\{Z^{(z)}(s, x), (s, x) \in [0, T] \times \mathbb{R}^3 \}$ do not depend on $z$. This is a consequence from the fact that the initial condition of the SPDE is not zero.

In the proof of Theorem 2.2, we have used the lemma below. For its proof, we refer the reader to Lemma A.2 in [18], with a trivial change on the spatial dimension ($d = 3$ in [18], while $d = 4$ in Lemma 5.3).

Lemma 5.3. Fix $[t_0, T]$ with $t_0 \geq 0$ and a compact set $K \subset \mathbb{R}^3$. Let $\{Y_n(t, x), (t, x) \in [t_0, T] \times K, n \geq 1 \}$ be a sequence of processes and $\{B_n(t), t \in [t_0, T] \} \subset \mathcal{F}$ be a sequence of adapted events which, for every $n$, decreases in $t$. Assume that for every $p \in [1, \infty[$ the following conditions hold:

(P1) There exists $\delta > 0$ and $C > 0$ such that, for any $t_0 \leq t \leq \bar{t} \leq T$, $x, \bar{x} \in K$,
\[
\sup_n \mathbb{E}\left(\left|Y_n(t, x) - Y_n(\bar{t}, \bar{x})\right|^p 1_{B_n(\bar{t})}\right) \leq C (|t - \bar{t}| + |x - \bar{x}|)^{4+\delta}.
\]

(P2) For every $(t, x) \in [t_0, T] \times K$,
\[
\lim_{n \to \infty} \mathbb{E}\left(Y_n(t, x)^p 1_{B_n(t)}\right) = 0.
\]

Then, for any $\eta \in ]0, \delta/p [\$ and any $r \in [1, p[$,
\[
\lim_{n \to \infty} \mathbb{E}\left(\|Y_n\|_{\mathcal{F}_{t_0, T}^r} 1_{B_n(T)}\right) = 0.
\]

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