A hadron-quark hybrid model reliable for the EoS in $\mu_B \leq 400$ MeV

Akihisa Miyahara, Masahiro Ishii, Hiroaki Kouno, and Masanobu Yahiro

1Observation Division, Chubu aviation weather service center, Japan Meteorological Agency, Tokoname 479-0881, Japan
2Department of Physics, Graduate School of Sciences, Kyushu University, Fukuoka 819-0395, Japan
3Department of Physics, Saga University, Saga 840-8502, Japan

(Dated: July 18, 2019)

We present a simple version of hadron-quark hybrid (HQH) model. The model is composed of the simple independent quark model for QGP states and an improved version of volume-excluded HRG model for hadronic states. The improved version of volume-excluded HRG model yields the pressure as a simple analytic form. The switching function from hadron states to QGP states in the present model has no chemical potential dependence. The present HQH model with the simple switching function is successful in reproducing the Polyakov loop at zero chemical potential and the EoS in $\mu_B \leq 400$ MeV.

PACS numbers: 11.30.Rd, 12.40.-y, 21.65.Qr, 25.75.Nq

I. INTRODUCTION

Lattice QCD (LQCD) provides a lot of information on hot QCD. In particular, the recent 2+1-flavor LQCD simulation [1] has confirmed that the chiral and the deconfinement transition are “crossover” at finite temperature ($T$) and zero baryon chemical potential ($\mu_B = 0$), where the continuum and thermodynamic limits were carefully taken. In general, such crossover nature means that the chiral and the deconfinement transition are not completely decoupled and the transition temperatures depends on the choice of observables. In fact, observable-dependent transition temperatures $T_c^{(O)}(\mu_B)$ have been discussed in LQCD simulations for zero and small $\mu_B$; actually, the renormalized chiral condensate $\langle \Delta \rangle = \Delta_{l,s}(T, \mu_B)$, the Polyakov loop $O = F(T, \mu_B)$, the energy density $\varepsilon = \varepsilon(T, \mu_B)$ and the trace anomaly $O = I(T, \mu_B)$ are taken in Refs. [2–9]. The definition and the determination are essential for the investigation of the presence or absence of critical endpoint (CEP) in QCD phase diagram. Particularly in Ref. [6], the LQCD data disfavors the existence of the CEP in $\mu_B/T \leq 2$ and $T/T_c^{(\Delta)}(\mu_B = 0) > 0.9$. Another important subject is to understand LQCD data on $T$ dependence of the equation of state (EoS), especially for the crossover region (100 MeV $\leq T \leq 400$ MeV). In the region, hadrons are supposed to melt into the strongly-correlated quark-gluon plasma; however, the physical interpretation of such hadron-quark transition has not been established yet. The EoS including the hadron-quark transition is necessary for the analyses of relativistic nuclear collisions. For these reasons, a lot of QCD data have been accumulated [1–12].

As a complementary approach to LQCD simulations, we can consider effective models. This approach is useful for the prediction of the transition lines, the existence and the location of the CEP, and the EoS. In fact, a lot of predictions are made for these quantities. The hadron resonance gas (HRG) model is a simple model for hadronic matter. The hadron is treated as non-interacting gas, and all hadrons listed in Particle Data Book [14] are taken into account in the model. The HRG model remarkably reproduces LQCD data on the various thermodynamic quantities in $T \leq 1.3T_c^{(\Delta)}(\mu_B = 0)$ [11], which indicates that one cannot neglect the excited hadrons even above the chiral transition temperature and hence hadrons possibly coexist with quarks and gluons. The simultaneous treatment of quarks and hadrons has been discussed for a long time. The various models have been proposed so far, such as the quark-meson model [15] and the Nambu–Jona-Lasinio (NJL) model with mesonic loops, quark-diquark coupling and chiral soliton. However, excited hadrons are absent in such models.

In our previous papers [20, 21], we proposed the hadron-quark hybrid (HQH) model in order to describe the coexistence of quarks and hadrons. In the model, the total entropy $s(T, \mu_B)$ is divided into hadron and quark pieces: Namely,

$$s(T, \mu_B) = f_{\text{H}}(T, \mu_B) s_{\text{H}}(T, \mu_B) + [1 - f_{\text{H}}(T, \mu_B)] s_{\text{Q}}(T, \mu_B),$$

where the function $s_{\text{H}}$ ($s_{\text{Q}}$) is the entropy density for hadronic matter (quark–gluon plasma). The weight function $f_{\text{H}}$ makes the occupancy of hadronic matter in the total entropy and is constrained in $0 \leq f_{\text{H}} \leq 1$. The $s(T, \mu_B)$ was determined from LQCD data on $T$ dependence of $s_{\text{LQCD}}$ and the second-order susceptibilities at $\mu_B = 0$. We apply HRG model for $s_{\text{H}}$ and independent-quark (IQ) model for $s_{\text{Q}}$. The IQ model is a simplified version of Polyakov-loop extended Nambu–Jona-Lainio (PNJL) model [22–25], that is, this model treats the coupling between the quark field and the homogeneous classical gauge field, but not the couplings between quarks. The neglect of quark-quark interactions are justified from the fact that the light-quark chiral condensate is quite small in $T \geq 220$ MeV where $s_{\text{Q}} > s_{\text{H}}$. In our previous version of HQH model [20, 21], we have confirmed that the HQH model well describes the LQCD data on the EoS for both $T \leq T_c$ and $T \geq T_c$ [16, 21].

As another advantage of our approach, $s_{\text{LQCD}}$ automati-
\[ \frac{\partial s(T, \mu_B)}{\partial T} \bigg|_{\mu_B=0} > 0, \quad s(T, \mu_B)|_{T=\mu_B=0} = 0. \quad (2) \]

In the HRG model, the interactions between baryons (anti-baryon) are neglected, but it should be taken into account for \( \mu_B \) dependence of thermodynamic quantities. A simple way of treating volume-exclusion effects (repulsive force) \( [27] \) was suggested in Refs. \( [28, 29] \). This model is called “excluded-volume HRG (EV-HRG) model”. Furthermore, a method of treating an attractive force in addition to the repulsive force was proposed in Ref. \( [30] \). The volume-exclusion effects are included by fitting the volume \( b = 4 \cdot 4\pi r^3/3 \) \( [26] \) to either LQCD data or the core radius \( r \) of nucleon-nucleon force \( [28, 29] \). In the framework of Refs. \( [28, 30] \), the interaction between baryon and anti-baryon and the radius of meson are neglected.

In this paper, we improve the HQH model of Ref. \( [21] \), taking the EV-HRG model for the hadron piece and using the simple IQ model for the quark-gluon piece. The EV-HRG model takes yields the pressure as a simple analytic function and guarantees that the pressure is \( \mu_B \) even. We refer to the present version of HQH model as “simple HQH (sHQH) model”.

The switching function \( f_H \) is determined from \( s_{\text{LQCD}} \) at \( \mu_B = 0 \), i.e., \( f_H \) has no \( \mu_B \) dependence. The sHQH model with the switching function well accounts for the Polyakov loop at zero chemical potential and the EoS in \( \mu_B \leq 400 \text{ MeV} \), without introducing \( \mu_B \) dependence to \( f_H \), where the core radius \( r = 0.335 \text{ fm} \) is taken.

The \( \Delta s \) and the \( \Phi \) signal the chiral and the deconfinement transition, respectively. The \( \Delta s \), calculated with the HRG model becomes negative in higher \( T \) \( [5] \), whereas the corresponding LQCD result is positive. The present model has the same problem. As an interesting result of LQCD simulations in Ref. \( [5] \), the peak position of \( d\Delta s/dT \) agrees with that of \( d\varepsilon/dT \) at \( \mu_B = 0 \). In Ref. \( [7] \), furthermore, the transition line is estimated by the peak of \( d\varepsilon/dT \) for finite \( \mu_B \). Therefore, we use the peak and the half-value width of \( d\varepsilon/dT \) as a transition region in \( \mu_B-T \) plane. We guess that the transition region determined from \( \varepsilon \) is close to the chiral-transition region calculated with LQCD simulations \( [8] \). As a result, we show that both the regions are close to each other in \( \mu_B \leq 400 \text{ MeV} \).

As a deconfinement-transition region, we take the peak and the half-value width of \( d\Phi/dT \) and predict the transition region for \( \mu_T \leq 400 \text{ MeV} \). We also determine a transition line from isentropic trajectories, and show that the transition line is between the deconfinement line and the transition line determined from \( \varepsilon \).

This paper is organized as follows. In Sec. \( \text{III} \) we show the model building. Numerical results are shown in Sec \( \text{III} \). Section \( \text{IV} \) is devoted to a summary.

II. MODEL BUILDING

We improve the HQH model of Ref. \( [21] \), modifying the EV-HRG model for the hadron part and taking the IQ mode for the quark-gluon one.

For the \( 2+1 \) flavor system, we can consider the chemical potentials of \( u, d, s \) quarks by \( \mu_u, \mu_d, \mu_s \) respectively. These potentials are related to the baryon-number (\( B \)) chemical potential \( \mu_B \), the isospin (\( I \)) chemical potential \( \mu_I \) and the hypercharge (\( Y \)) chemical potential \( \mu_Y \) as

\[
\begin{align*}
\mu_B &= \mu_u + \mu_d + \mu_s, \\
\mu_I &= \mu_u - \mu_d, \\
\mu_Y &= \frac{1}{2}(\mu_u + \mu_d - 2\mu_s). \\
\end{align*}
\]

(3)

As for \( \mu_I \) and \( \mu_Y \), the right-hand side of Eq. \( (3) \) comes from the diagonal elements of the matrix representation of Cartan algebra in \( SU(3) \) group: \( \mu_I = (1, -1, 0)(\mu_u, \mu_d, \mu_s)^{\dagger} \) and \( \mu_Y = (1/2)(1, 1, -2)(\mu_u, \mu_d, \mu_s)^{\dagger} \). Equation \( (3) \) yields

\[
\begin{align*}
\mu_u &= \frac{3}{2}\mu_B + \frac{1}{2}\mu_I + \frac{1}{4}\mu_Y, \\
\mu_d &= \frac{3}{2}\mu_B - \frac{1}{2}\mu_I + \frac{1}{4}\mu_Y, \\
\mu_s &= \frac{3}{2}\mu_B - \frac{3}{4}\mu_Y. \\
\end{align*}
\]

(4)

A. HRG model

For later convenience, we start with the HRG model. In the model, the pressure \( P_H \) is divided into the baryon (B) part \( P_B \), the anti-baryon (aB) part \( P_{aB} \) and the meson (M) part \( P_M \):

\[
P_H = P_B + P_{aB} + P_M \]

(5)

with

\[
P_B = \sum_{i\in B} d_i T \int \log(1 + e^{-E_{B,i} - \mu_{B,i}}/T), \\
P_{aB} = \sum_{i\in aB} d_i T \int \log(1 + e^{-E_{aB,i} + \mu_{B,i}}/T), \\
P_M = \sum_{j\in \text{Mesons}} d_j T \int \left\{ \log(1 - e^{-(E_{M,j} - \mu_{M,j})/T}) + \log(1 - e^{-(E_{M,j} + \mu_{M,j})/T}) \right\}
\]

(8)

for \( E_{B,i} = \sqrt{p^2 + m_{B,i}^2} \) and \( E_{M,j} = \sqrt{p^2 + m_{M,j}^2} \), where \( m_{B,i} (m_{M,j}) \) and \( \mu_i (\mu_j) \) is the mass and the chemical potential of the \( i \)-th baryon (\( j \)-th meson), respectively. Here we have used the shorthand notation

\[
\int \equiv \int \frac{d^3 p}{(2\pi)^3}.
\]

(9)

for the integration over 3d-momentum \( p \). In Eq. \( (3) \), all the hadrons listed in the Particle Data Table \( [14] \) are taken.
B. Modified version of EV-HRG

We first explain the EV-HRG model of Refs. [28, 30]. The pressure \( P_{\text{EV:H}} \) is obtained by

\[
P_{\text{EV:H}} = P_{\text{EV:B}} + P_{\text{EV:aB}} + P_M \tag{10}
\]

with

\[
P_{\text{EV:B}} = \sum_{i \in B} d_i T \int \log(1 + e^{-(E_{iB} - \mu_{\text{EV:B},i})/T}), \tag{11}
\]

\[
P_{\text{EV:aB}} = \sum_{i \in aB} d_i T \int \log(1 + e^{-(E_{iB} + \mu_{\text{EV:aB},i})/T}), \tag{12}
\]

Here the effective baryon and anti-baryon chemical potentials, \( \mu_{\text{EV:B},i} \) and \( \mu_{\text{EV:aB},i} \), are defined by

\[
\mu_{\text{EV:B},i}/T = \mu_{B,i}/T - \bar{b}P_{\text{EV:B}}/T^4, \tag{13}
\]

\[
\mu_{\text{EV:aB},i}/T = \mu_{B,i}/T - \bar{b}P_{\text{EV:B}}/T^4, \tag{14}
\]

where \( \bar{b} = bT^3 \) for a positive volume parameter \( b \). It is not easy to obtain \( P_{\text{EV:B}} \) and \( P_{\text{EV:aB}} \), since \( \mu_{\text{EV:B},i} \) includes \( P_{\text{EV:B}} \) (\( \mu_{\text{EV:aB},i} \), \( \mu_{\text{EV:aB}}, \) and \( P_{\text{EV:aB}} \) are obtained by solving Eqs. (11) and (12) numerically.

In QCD, the pressure is charge-conjugation even (\( \mu_B \) even). Hence the \( P_{\text{EV:H}} \) should be \( \mu_B \) even, because it is a model of explaining QCD in \( T < T_c \). However, \( \mu_{\text{EV:B},i} \) includes a \( \mu_B \)-odd term \( \mu_B \) and a \( \mu_B \)-even term \( bP_{\text{EV:B}}/T^4 \), so that \( P_{\text{EV:H}} \) is not \( \mu_B \) even.

The \( P_{\text{EV:B}} \) and \( P_{\text{EV:aB}} \) are now modified as

\[
P_{\text{mod:B}} = \sum_{i \in B} d_i T \int \log(1 + e^{-(E_{iB} - \mu_{\text{mod:B},i})/T}), \tag{15}
\]

\[
P_{\text{mod:aB}} = \sum_{i \in aB} d_i T \int \log(1 + e^{-(E_{iB} + \mu_{\text{mod:aB},i})/T}). \tag{16}
\]

with

\[
\mu_{\text{mod:B},i}/T = \mu_{B,i}/T - \bar{b}P_{\text{EV:B}}/T^4, \tag{17}
\]

\[
\mu_{\text{mod:aB},i}/T = \mu_{B,i}/T + \bar{b}P_{\text{EV:B}}/T^4, \tag{18}
\]

The sum of \( P_{\text{mod:B}} \) and \( P_{\text{mod:aB}} \) are \( \mu_B \) even, since the sum is invariant under \( \mu_B \to -\mu_B \). For this reason, we take Eqs. (15)–(18). These equations show that \( P_B \geq P_{\text{AB}} \).

The \( P_{\text{mod:B}} \) and \( P_{\text{mod:aB}} \) can be rewritten into

\[
\frac{P_{\text{mod:B}}}{T^4} = \sum_{i \in B} A_i \sum_{\ell = 1}^{\infty} \frac{(-1)^{\ell+1}}{\ell^2} K_2\left(\frac{\ell m_i}{T}\right) \exp\left(-\frac{\ell \mu_{\text{mod:B},i}}{T}\right), \tag{19}
\]

\[
\frac{P_{\text{mod:aB}}}{T^4} = \sum_{i \in aB} A_i \sum_{\ell = 1}^{\infty} \frac{(-1)^{\ell+1}}{\ell^2} K_2\left(\frac{\ell m_i}{T}\right) \exp\left(-\frac{\ell \mu_{\text{mod:aB},i}}{T}\right) \tag{20}
\]

for

\[
A_i = \frac{d_i}{2\pi} \left(\frac{m_i}{T}\right)^2. \tag{21}
\]

LQCD data on the EoS are available for \( T \leq 400 \text{ MeV} \) and \( \mu_B \leq 400 \text{ MeV} \) [8, 7]. We then consider this region. We consider \( P_B \), because of \( P_B \geq P_{\text{AB}} \). The \( \ell \) convergence of Eq. (19) becomes worse as \( (|\mu_B - m_i|)/T \) becomes larger; note that \( K_2(x) \) is proportional to \( \exp(-x) \) for large \( x \) and \( \mu_B - m_i \) is negative. Therefore, the convergence is worst for the smallest case \((939-400)/400 \) where \( T = \mu_B = 400 \text{ MeV} \) and \( m_N = 939 \text{ MeV} \). Taking the \( \ell = 1 \) term only is a 3 % error in Eqs. (19). In actual calculations, nucleon contribution in \( P_B \) is only 3 %, so that taking the \( \ell = 1 \) term only corresponds to 0.1 % error. We can identify \( P_B \) with its \( \ell = 1 \) term and \( P_{\text{AB}} \) with its \( \ell = 1 \) one. This approximation is called \( ^*\ell = 1 \) identification in this paper.

Using the \( \ell = 1 \) identification, we can rewrite \( P_{\text{mod:B}} \) as

\[
\frac{P_{\text{mod:B}}}{T^4} = \sum_{i \in B} A_i K_2\left(\frac{m_i}{T}\right) \exp\left(\frac{\ell \mu_{\text{mod:B},i}}{T}\right), \tag{22}
\]

Multiplying both the sides of Eq. (22) by \( \bar{b} \exp(\bar{b}P_{\text{mod:B}}/T^4) \) and using the \( \ell = 1 \) identification, one can obtain

\[
\bar{b} \frac{P_{\text{mod:B}}}{T^4} = \bar{b} \sum_{i \in B} A_i K_2\left(\frac{m_i}{T}\right) \exp\left(\frac{\ell \mu_{B,i}}{T}\right) = \bar{b} P_{\text{mod:B}}/T^4, \tag{23}
\]

Noting that the Lambert \( W(z) \) function is the inverse function of \( We^W = z \), one can get \( P_{\text{mod:B}} \) as a simple analytic function: Namely,

\[
\frac{P_{\text{mod:B}}}{T^4} = \frac{W\left(\frac{\bar{b}P_{\text{mod:B}}/T^4}{\bar{b}}\right)}{\bar{b}}. \tag{24}
\]

In the limit of \( \bar{b} \to 0 \), the \( P_{\text{mod:B}} \) tends to \( P_B \), because of \( W(z) \to z \). Parallel discussion is possible for anti-baryon.

The result is

\[
\frac{P_{\text{mod:aB}}}{T^4} = \frac{W\left(\frac{\bar{b}P_{\text{aB}}/T^4}{\bar{b}}\right)}{\bar{b}}. \tag{25}
\]

Hence the hadronic pressure becomes

\[
P_{\text{mod:H}} = P_{\text{mod:B}} + P_{\text{mod:aB}} + P_M \tag{26}
\]

with Eqs. (22) and (25). The entropy density \( s_{\text{mod:H}} \) is obtained from \( P_{\text{mod:H}} \) as

\[
s_{\text{mod:H}} = \frac{\partial P_{\text{mod:H}}}{\partial T}. \tag{27}
\]
This modified version of EV-HRG model is referred to as "modified EV-HRG model".

Figure 1 shows $T$ dependence of the total pressure $P(T)$ for $\mu_B = 0$, 400 MeV. The results of modified EV-HRG and HRG models are compared with LQCD ones [13]. In the modified EV-HRG model, we take the core radius 0.335 fm as a value of $r$, i.e., $b = 0.64$ fm$^3$. For $\mu_B = 400$ MeV (lower panel), the EV-HRG result (solid line) agrees with LQCD one [13] in $T \leq 210$ MeV, while the HRG result (dashed line) is consistent with LQCD one in $T \leq 150$ MeV. For $\mu_B = 0$ MeV (upper panel), both the EV-HRG and the HRG result are consistent with LQCD one [13] in $T \leq 210$ MeV.

Making the path integral over quark fields leads to

$$P_Q = -U(T, \Phi, \bar{\Phi}) + 2\sum f \left[ \int \left( T \log z^+ f + T \log z^- f \right) \right], \quad (29)$$

where

$$z^+_f = 1 + 3\bar{\Phi}e^{-(E_f + \mu)/T} + 3\Phi e^{-2(E_f + \mu)/T} + e^{-3(E_f + \mu)/T}, \quad (30)$$

$$z^-_f = 1 + 3\Phi e^{-(E_f - \mu)/T} + 3\bar{\Phi} e^{-2(E_f - \mu)/T} + e^{-3(E_f - \mu)/T}, \quad (31)$$

with $E_f = \sqrt{p^2 + m_f^2}$. In Eq. (29), the vacuum term has been omitted, since the pressure calculated with LQCD simulations does not include the term. The $\bar{\Phi}$ and $\Phi$ are obtained by minimizing $\Omega_Q = -P_Q$.

The entropy density $s_Q$ is obtained from $P_Q$ as

$$s_Q = \frac{\partial P_Q}{\partial T}. \quad (32)$$

We take the Polyakov-loop potential of Ref. [21]:

$$U(T, \Phi, \bar{\Phi}) = -\frac{a(T)}{2} \Phi \bar{\Phi} + b(T) \log \left( 1 - 6\Phi \bar{\Phi} + 4(\Phi^2 + \bar{\Phi}^2) - 3(\Phi^2 \bar{\Phi}^2) \right) + a_1 \left( \frac{T_0}{T} \right)^2 + a_2 \left( \frac{T_0}{T} \right)^3, \quad (33)$$

$$b(T) = b_3 \left( \frac{T_0}{T} \right)^3. \quad (35)$$

The parameters $a_0, a_1, a_2, b_3$ and $T_0$ were fitted to $2+1$ flavor $s_{LQCD}$ in $400 \leq T \leq 500$ MeV; see Fig. 1 of Ref. [21] for the fit. The resulting values are tabulated in Table I.

| $a_0$ | $a_1$ | $a_2$ | $b_3$ | $T_0$ |
|-------|-------|-------|-------|-------|
| 2.457 | -2.47 | 15.2  | -1.75 | 270[MeV] |

**TABLE I: Parameters in the Polyakov-loop potential.**

C. Independent quark model

We have to consider QGP states in the region $T \geq 200$ MeV by using the simple IQ model, since $f_H(T) < 1$, as shown later in Fig. 2. The Lagrangian density of the IQ model is

$$\mathcal{L}_Q = \sum_f \{ \bar{q}_f \gamma^\mu D_\mu m_f q_f \} - U(T, \Phi, \bar{\Phi}), \quad (28)$$

where $m_f$ is the current mass of $f$ quark and $D_\mu = \partial_\mu - ig A_\mu^A \lambda_A \delta^A_f$ with the Gell-Mann matrix $\lambda_A$ in color space. See Refs. [24, 25] for the definition of the Polyakov loop $\Phi$ and its conjugate $\bar{\Phi}$.

D. sHQH model

The total entropy reads

$$s(T, \mu_B) = f_H(T) s_{mod:H}(T, \mu_B) + [1 - f_H(T)] s_{Q}(T, \mu_B) \quad (36)$$

in the sHQH model, where it is assumed that the $f_H(T)$ has no chemical-potential dependence. Note that $s_{mod:H}$ and $s_Q$ have chemical-potential dependence. Therefore, $f_H(T)$ is determined so as to $s = s_{LQCD}$ [13] at $\mu_B = 0$: Namely,

$$f_H(T) = \frac{s_{LQCD}(T) - s_{Q}(T)}{s_{mod:H}(T) - s_{Q}(T)}. \quad (37)$$
In Fig. 2, the \( f_H(T) \) of Eq. (37) is shown by dots with error bars. The errors come from \( s_{LQCD} \). The solid line is a fitting function for the \( f_H(T) \) of Eq. (37); in the \( \chi^2 \) fitting, the line is assumed to be 1 in \( T < 180 \) MeV. From now on, we regard the solid line as the switching function \( f_H(T) \).

The pressure \( P \) with no vacuum contribution is obtainable from \( s_{LQCD} \) of Eq. (36):

\[
P(T, \mu_B) = \int_0^T dT' s(T', \mu_B)
\] (38)

Fig. 2: \( T \) dependence of the switching function \( f_H(T) \). The dots with error bars are the \( f_H(T) \) of Eq. (37). The solid line is a fitting function for the \( f_H(T) \); see the text for the fitting.

### III. NUMERICAL RESULTS

As mentioned in Sec. I, we consider the transition region determined from with the peak and the half-value width of \( d\varepsilon(T, \mu_B)/dT \) and the deconfinement-transition region with the peak and the half-value width of \( d\Phi(T, \mu_B)/dT \).

#### A. \( T \) dependence of the Polyakov loop for \( \mu_B = 0 \sim 400 \) MeV

Figure 3 shows the Polyakov loop \( \Phi \) as a function of \( T \) for the cases of \( \mu_B = 0, 100, 200, 300, 400 \) MeV. The LQCD result is available only for \( \mu_B = 0 \) MeV [5]. In the upper panel for \( \mu_B = 0 \) MeV, the sHQH result (solid line) well reproduces LQCD one in which the continuum limit is taken. We then predict the \( \Phi \) for \( \mu_B = 100, 200, 300, 400 \) MeV in the lower panel. \( \mu_B \) dependence of \( \Phi \) is small.

#### B. Transitions

We first consider the case of \( \mu_B = 0 \). Table II shows results of sHQH model for the transition region \( T_c^\varepsilon \) determined from the peak and the half-valued width of \( d\varepsilon(T, \mu_B)/dT \) and the deconfinement-transition region \( T_c^d \) deduced from \( d\Phi(T, \mu_B)/dT \). The results are compared with LQCD data [5] on the chiral transition temperature \( T_c^{\Delta\varepsilon,LQCD} \) and the deconfinement temperature \( T_c^{\delta,LQCD} \). One can see that \( T_c^{\Delta\varepsilon,LQCD} \) is included in the region \( T_c^\varepsilon \), while \( T_c^{\delta,LQCD} \) is consistent with \( T_c^{\delta,LQCD} \).

\[
\begin{array}{cccc}
T_c^\varepsilon & T_c^{\Delta\varepsilon,LQCD} & T_c^{\delta,LQCD} \\
137–204[\text{MeV}] & 157(4)(3)[\text{MeV}] & 177–239[\text{MeV}] & 170\pm7[\text{MeV}]
\end{array}
\]

TABLE II: Comparison between lattice transition temperatures and transition regions calculated with sHQH model for \( \mu_B = 0 \).

Figure 4 shows the transition region \( T_c^\varepsilon \) determined from the peak and the half-valued width of \( d\varepsilon(T, \mu_B)/dT \) and the lattice chiral-transition region in \( \mu_B-T \) plane; the former is calculated with the sHQH model and the latter is analytic continuation of LQCD simulations from imaginary to real \( \mu \) [8]. The transition region determined from \( d\varepsilon(T, \mu_B)/dT \) is shown by a horizontal line with cross for each of \( \mu_B = 0, 100, 200, 300, 400 \) MeV; the cross is a maximum value of \( d\varepsilon/T \) and the line means the half-value width of \( d\varepsilon/T \). The red solid line is made by connecting the crosses. Meanwhile, the blue band indicates the width of the chiral-transition region extrapolated from the imaginary-\( \mu \) region [8]. The model result is consistent with the LQCD result.

Figure 5 shows the deconfinement-transition region in \( \mu_B-T \) plane. The shah result is shown by a horizontal line with cross for each of \( \mu_B = 0, 100, 200, 300, 400 \) MeV. The solid line made by connecting the points stands for the
The deconfinement-transition line is compared with the transition line determined from \( \frac{d\varepsilon}{dT} \) and is calculated with the sHQH model. The transition line (red solid line), obtained by connecting the crosses, is expressed by \( T = 172(1 - 0.03(\mu_B/172)^2) \) MeV. The blue band is the chiral-transition region determined by analytic continuation of LQCD simulations from imaginary to real \( \mu_B \) [8].

Fig. 4: The transition line determined from \( d\varepsilon/dT \) in \( \mu_B-T \) plane. The horizontal line with cross stands for the transition region determined from \( d\varepsilon/dT \) and is calculated with the sHQH model. The transition line (red solid line), obtained by connecting the crosses, is expressed by \( T = 172(1 - 0.03(\mu_B/172)^2) \) MeV. The blue band is the chiral-transition region determined by analytic continuation of LQCD simulations from imaginary to real \( \mu_B \) [8].

In Fig. 5, the solid curve is a line connecting the points at which the curvature of isentropic trajectory becomes maximum. This figure suggests that a transition line can be estimated by \( n/s \) in \( \mu_B-T \) plane. Hence, the transition calculated with \( n/s \) may be deduced from relativistic nuclear collisions. There is no evidence of attractor of isentropic trajectory in the sHQH model.

In Fig. 6, the solid curve is a line connecting the points at which the curve of trajectory becomes maximum; the resulting curve is \( T = 194(1 - 0.035(\mu_B/194)^2) \) MeV. The isentropic trajectories are shown by \( n/s = 0.010, 0.015, 0.020, 0.025 \) from left to right.

Fig. 5: Deconfinement-transition region in \( \mu_B-T \) plane. See the text for the definition of lines. The deconfinement-transition line is \( T = 201(1 - 0.0075(\mu_B/201)^2) \) MeV.

Fig. 6: Isentropic trajectories, \( n/s=\text{const.} \) in \( \mu_B-T \) plane. The solid curve is a line connecting the points at which the curve of trajectory becomes maximum; the resulting curve is \( T = 194(1 - 0.035(\mu_B/194)^2) \) MeV. The isentropic trajectories are shown by \( n/s = 0.010, 0.015, 0.020, 0.025 \) from left to right.

In Fig. 7, the transition line determined from \( \varepsilon \) is \( T = 172(1 - 0.03(\mu_B/172)^2) \), the deconfinement-transition line is \( T = 201(1 - 0.0075(\mu_B/201)^2) \), the transition line determined from \( n/s \) is \( T = 194(1 - 0.035(\mu_B/194)^2) \).

Fig. 7: Transition lines in \( \mu_B-T \) plane. The chiral-transition line determined from \( \varepsilon \) is \( T = 172(1 - 0.03(\mu_B/172)^2) \), the deconfinement-transition line is \( T = 201(1 - 0.0075(\mu_B/201)^2) \), the transition line determined from \( n/s \) is \( T = 194(1 - 0.035(\mu_B/194)^2) \).

C. The EoS

In Sec. II B, we considered the transition line determined from \( d\varepsilon/dT \), where \( \varepsilon(T, \mu_B) = sT - P + \mu_B n \).

One uses the Tolman-Oppenheimer-Volkof equations (TOV) equation to study structure of stars. The input EoS of the TOV equation is \( P \) and \( n \). In addition, the isentropic trajectories, \( n/s=\text{const.} \), are important to study relativistic nuclear collisions. Therefore, we focus on \( P, s, \varepsilon, n \).

In order to compare the present model with the previous model [21], we take the same assumption “ \( f_H(T) \) has no \( \mu_B \) dependence”, in the the previous model. The resulting switching function \( f_H^{\text{prev}}(T) \) is shifted to the left by about 10 MeV from \( f_H(T) \) in Fig. 4. The difference between the present model with \( f_H(T) \) and the previous model with \( f_H^{\text{prev}}(T) \) shows EV effects. The previous model with \( f_H^{\text{prev}}(T) \) is referred to as “HRG-HQH model” in this paper.
Figure 8: $T$ dependence of $s, P, \varepsilon$ at $\mu_B = 0$ MeV. See the text for the definition of lines. LQCD data are taken from Ref. [7].

Figure 9: $T$ dependence of $s, P, \varepsilon, n$ at $\mu_B = 100$ MeV. See the text for the definition of lines. LQCD data are taken from Ref. [7].

Figure 8 shows $T$ dependence of $s, P, \varepsilon$, at $\mu_B = 0$ MeV. The solid and dashed lines are the results of sHQH and HRG-HQH models, respectively. Seeing $s(T)$, we find that the fitting of $f_H(T)$ is good, since the sHQH result agrees with LQCD data [7]. Also for $P$ and $\varepsilon$, the sHQH model agree
with LQCD data. Comparing the results of sHQH and HRG-EV HQH models, we can find that EV effects are small for $\mu_B = 0$ MeV.

Figure 9 shows $T$ dependence of $s$, $P$, $\varepsilon$, $n$ at $\mu_B = 100$ MeV. The solid and dashed lines stand for the results of sHQH and HRG-HQH models, respectively. The $s$, $P$, $\varepsilon$, $n$ of sHQH model reproduce LQCD data [7]. Comparing the two lines, we can see that EV effects are small still for $\mu_B = 100$ MeV.

Figures 10–12 shows $T$ dependence of $s$, $P$, $\varepsilon$, $n$ for $\mu_B = 200, 300, 400$ MeV. The results of sHQH model well reproduces the LQCD data [7]. EV effects become large as $\mu_B$ increases from 200 MeV to 400 MeV.
Fig. 11: \( T \) dependence of \( s \), \( P \), \( \varepsilon \), \( n \) at \( \mu_B = 300 \) MeV. See the text for the definition of lines. LQCD are taken from Ref. [7]; note that \( n \) is deduced from \( s \), \( P \), \( \varepsilon \).

Fig. 12: \( T \) dependence of \( s \), \( P \), \( \varepsilon \), \( n \) at \( \mu_B = 400 \) MeV. See the text for the definition of lines. LQCD are taken from Ref. [7]; note that \( n \) is deduced from \( s \), \( P \), \( \varepsilon \).
IV. SUMMARY

We have improved the HQH model of Ref. [21], modifying the EV-HRG model [28, 29] for the hadron piece and using the simple IQ model for the quark-gluon piece. The modified EV-HRG model yields the baryon and antibaryon pressures as simple analytic functions of Eqs. (24)–(25), and ensures that the pressure is $\mu_B$ even.

We have determined the switching function $f_H$ from $s_{\text{LQCD}}$ at $\mu_B = 0$. The sHQ model with the switching function $f_H(T)$ well accounts for LQCD data on the Polyakov loop at $\mu_B = 0$ MeV. This makes it possible to predict the Polyakov loop for $\mu_B = 100, 200, 300, 400$ MeV. The EoS calculated with the sHQ model is successful in reproducing the corresponding LQCD data in $\mu_B \leq 400$ MeV, without introducing $\mu_B$ dependence to $f_H$, where the core radius $r = 0.335$ fm is taken. The switching function $f_H$ has also a simple form, since it has no $\mu_B$ dependence.

As an interesting result of LQCD simulations for $\mu_B = 0$ [5], the peak position of $d\Delta_{\text{lat}}/dT$ agrees with that of $d\varepsilon(T, \mu_B)/dT$. In LQCD simulations for finite $\mu_B$ [7], furthermore, a transition line is estimated by the peak of $d\varepsilon(T, \mu_B)/dT$. We can then guess that the transition region determined from $\varepsilon$ is close to the chiral-transition region calculated with LQCD simulations. In fact, we show that the transition region determined from $d\varepsilon(T, \mu_B)/dT$ almost agrees with the lattice result [8] on the chiral-transition region in $\mu_B \leq 400$ MeV. This may make it possible to define a chiral-transition region in $\mu_B$–$T$ plane with the peak and the half-value width of $d\varepsilon(T, \mu_B)/dT$.

As a deconfinement-transition region, we take the peak and the half-value width of $d\phi(T, \mu_B)/dT$. As for the deconfinement transition, we predict the transition region and confirm that the deconfinement-transition line is above the transition line determined from $d\varepsilon(T, \mu_B)/dT$. In sHQ model, there is no evidence of attractor of isentropic trajectory. We have also found that the transition line determined from isentropic trajectories is between the deconfinement line and the transition line determined from $d\varepsilon(T, \mu_B)/dT$. The transition determined from isentropic trajectories may be deduced from relativistic nuclear collisions.

Acknowledgments

The authors thank Junpei Sugano and Takehiro Hirakida for useful contributions. H. K. is supported by Grant-in-Aid for Scientific Research (No.17K05446) from the Japan Society for the Promotion of Science (JSPS).

[1] Y. Aoki, G. Endrödi, Z. Fodor, S. D. Katz and K. K. Szabó, Nature 443, 675 (2006).
[2] Z. Fodor and S. D. Katz, JHEP 0404, 050 (2004).
[3] Y. Aoki, A. Fodor, S. D. Katz and K. K. Szabó, Phys. Lett. B 643, 46 (2006).
[4] Y. Aoki, S. Borsanyi, S. Durr, Z. Fodor, S. D. Katz, S. Krieg and K. K. Szabó, JHEP 0906, 088 (2009). doi:10.1088/1126-6708/2009/06/088 [arXiv:0903.4155 [hep-lat]].
[5] S. Borsanyi et al. [Wuppertal-Budapest Collaboration], JHEP 1009, 073 (2010).
[6] G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo, JHEP 1104, 001 (2011).
[7] S. Borsanyi, G. Endrödi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti and K. K. Szabo, JHEP 1208, 053 (2012).
[8] R. Bellwied, S. Borsanyi, Z. Fodor, J. Gunther, S. D. Katz, C. Ratti and K. K. Szabo, Phys. Lett. B 751, 559 (2015).
[9] A. Bazavov et al., Phys. Rev. D 95, no. 5, 054504 (2017).
[10] S. Borsanyi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo, JHEP 01, 138 (2012).
[11] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg and K. K. Szabo, Phys. Lett. B 730, 99 (2014).
[12] A. Bazavov et al. [HotQCD Collaboration], Phys. Rev. D 90, 094503 (2014).
[13] S. Borsanyi et al., Nature 539, 69 (2016).
[14] K. Y. Olive et al. (Particle Data Group), Chin. Phys. C 38, 090001 (2014).
[15] D. U. Jungnickel and C. Werticher, Phys. Rev. D 53, 5142 (1996) doi:10.1103/PhysRevD.53.5142 [hep-ph/9505267].
[16] M. Asakawa, T. Hatsuda, Phys. Rev. D 55, 4488 (1997).
[17] C. Nonaka and M. Asakawa, Phys. Rev. C 71, 044904 (2005) doi:10.1103/PhysRevC.71.044904 [nucl-th/0410078].
[18] M. Albright, J. Kapusta and C. Young, Phys. Rev. C 90, 024915 (2014).
[19] M. Albright, J. Kapusta and C. Young, Phys. Rev. C 92, 044904 (2015).
[20] A. Miyahara, Y. Torigoe, H. Koono and M. Yahiho, Phys. Rev. D 94, 016003 (2016).
[21] A. Miyahara, M. Ishii, H. Koono and M. Yahiho, Int. J. Mod. Phys. A 32, no. 36, 1750205 (2017).
[22] P. N. Meisinger, and M. C. Ogilvie, Phys. Lett. B 379, 163 (1996).
[23] A. Dumitru, and R. D. Pisarski, Phys. Rev. D 66, 096003 (2002).
[24] K. Fukushima, Phys. Lett. B 591, 277 (2004); Phys. Rev. D 77, 114028 (2008).
[25] Y. Sakai, K. Kashiwa, H. Kouno, and M. Yahiho, Phys. Rev. D 77, 051901(R) (2008); Phys. Rev. D 78, 036001 (2008).
[26] L. Landau and E. Lifshitz, Statistical Physics (Pergamon, New York, 1980).
[27] H. Kouno and F. Takagi, Z. Phys. C45, 43 (1989).
[28] V. Vovchenko, D. V. Ananchishkin and M. I. Gorenstein, Phys. Rev. C 91, 024905 (2015).
[29] V. Vovchenko and H. Stoecker, J. Phys. G 44, 055103 (2017).
[30] V. Vovchenko, M. I. Gorenstein and H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017).