Mixing Markovian Pauli channels: the general case

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We study the memory property of the channels obtained by convex combinations of Markovian channels that are not necessarily quantum dynamical semigroups (QDSs). In particular, we characterize the geometry of the region of (non-)Markovian channels obtained by the convex combination of the three Pauli channels, as a function of deviation from the semigroup form in a family of channels. The regions are highly convex, and interestingly, the measure of the non-Markovian region shrinks with greater deviation from the QDS structure for the considered family, underscoring the counterintuitive nature of (non-)Markovianity under channel mixing.

I. INTRODUCTION

Non-Markovian dynamics of open quantum systems [1] is an active area of research, throwing new challenges and surprises [2, 3]. The finite-time dynamics of open quantum systems are described by time-dependant completely positive trace preserving (CPTP) maps, usually referred to as quantum channels [4, 5]. Quantum non-Markovianity, unlike its classical counterpart does not have a unique definition and mathematical characterization. The two widely used approaches to study quantum non-Markovianity, are based on a deviation from CP-divisibility criterion [6, 7] and on the distinguishability of states [8].

Convex combinations of quantum channels have been actively studied recently [9–13]. In [14], we considered the problem of mixing three Pauli channels, each assumed to be a quantum dynamical semigroup (QDS), and characterized the resulting “Pauli simplex”. We showed that neither the set of non-Markovian (CP-indivisible) nor Markovian channels is convex in the Pauli simplex, and that the measure of non-Markovian channels is about 0.87. This means that the probability of resulting channel being non-Markovian is nearly 0.87 when the three Pauli channels are mixed in random proportions.

In this paper, we generalize this problem to consider the convex combinations of Pauli channels which are not QDS. For a given family of such channels, we characterize at each deviation from the
QDS form, the Pauli simplex obtained by mixing the three Pauli channels, and obtain the measure of the associated (non-)Markovian regions. The paper is organized as follows. In Section II, we present the preliminaries and discuss the convex combination of the three Markovian Pauli channels which are not QDSs. We characterize the geometry of the (non-)Markovian region obtained by mixing, and evaluate its measure in Section III. Further, the behaviour of the regions as a function of deviation of the mixing channels from QDS form is discussed. We then conclude in Sec. IV.

II. CONVEX COMBINATIONS OF CHANNELS

Consider the channel \( \Phi \) acting on a qubit, represented by the density matrix

\[
\rho = \frac{1}{2}(1 + a_i \sigma_i) = \frac{1}{2} \begin{pmatrix} 1 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1 - a_3 \end{pmatrix}.
\]

The vector \( a = (a_1, a_2, a_3) \), with \( |a| \leq 1 \), is the Bloch vector. Here, we consider Pauli channels which are unital, as defined by \( \Phi(\sigma_I) = \sigma_I \), and \( \Phi(\sigma_i) = x_i \sigma_i \), where \( \sigma_I = 1 \) and \( \sigma_i \)'s are the Pauli matrices.

Consider the Pauli \( z \) channel, \( \Phi_q^z(\rho) = (1 - q) \rho + q \sigma_z \rho \sigma_z \), where \( q \) is a decoherence parameter, which in general is time-dependent. We now choose \( q \) from the family with the functional form

\[
q = \frac{1 - \exp(-rt)}{n},
\]

with \( n \) being any positive real number greater than or equal to 2, and \( r \) being a constant. The time-local generator \( L \) of a channel \( \Phi \), is defined by \( \dot{\Phi} = L \Phi \). For the channel, \( \Phi_z \), the corresponding differential form of the channel is

\[
L(\rho) = \frac{r}{(n - 2)e^{rt} + 2}(\sigma_z \rho \sigma_z - \rho),
\]

with the time-dependence of the rate showing that the generator is no longer a semigroup. For \( n = 2 \), this corresponds to a QDS with a time-independent Lindblad generator as \( L(\rho) = \frac{r}{2}(\sigma_z \rho \sigma_z - \rho) \).

Let us now consider arbitrary convex combinations of the three Pauli channels. The general form of the three-way mixing is described by

\[
\tilde{\Phi}_s(q) = x \Phi_x^q + y \Phi_y^p + z \Phi_z^p,
\]

with \( x, y, z \geq 0 \) and \( x + y + z = 1 \) and, analogously to \( \Phi_x^p(\rho) \), we define \( \Phi_y^p(\rho) = (1 - q) \rho + q \sigma_x \rho \sigma_x \), and \( \Phi_y^p(\rho) = (1 - q) \rho + q \sigma_y \rho \sigma_y \). The set of all channels of the form Eq. (3) constitutes the Pauli
simplex, whose vertices are the Pauli channels assumed to be described by the same parameter $q$ [14].

The differential form of the channel follows to be of the form

$$L(\rho) = \sum_{k=X,Y,Z} \gamma_k (\sigma_k \rho \sigma_k - \rho),$$

with the decay rates being (cf. [14])

$$\gamma_X = \left( \frac{1 - y}{1 - 2(1 - y)q} + \frac{1 - z}{1 - 2(1 - z)q} - \frac{1 - x}{1 - 2(1 - x)q} \right) \frac{\dot{q}}{2},$$

$$\gamma_Y = \left( \frac{1 - x}{1 - 2(1 - x)q} + \frac{1 - z}{1 - 2(1 - z)q} - \frac{1 - y}{1 - 2(1 - y)q} \right) \frac{\dot{q}}{2},$$

$$\gamma_Z = \left( \frac{1 - x}{1 - 2(1 - x)q} + \frac{1 - y}{1 - 2(1 - y)q} - \frac{1 - z}{1 - 2(1 - z)q} \right) \frac{\dot{q}}{2}.$$  \(5\)

The study of these rates is largely simplified because the summands that make them up have the same functional form. This can be exploited to quantify the measure of non-Markovian maps.

III. MEASURE OF (NON-)MARKOVIAN CHANNELS

It can be shown that the structure Eqs. (5) guarantees that if a given rate (say) $\gamma_Y(x, y, z = 1 - x - y, q)$ turns negative at $q = q_0 \leq \frac{1}{n}$, then it remains negative throughout the remaining range of $[q_0, \frac{1}{n}]$ [14]. To find the set of all pairs $(x, y)$ such that $\gamma_Y(x, y, q) \leq 0$ at $q = \frac{1}{n}$, we solve the equation $\gamma_Y(x, y, \frac{1}{n}) = 0$. The result is a constraint on the pairs $(x, y)$, which can be represented by expressing $x$ in terms of $y$:

$$x_Y^\pm(y) = \frac{1}{2} \times \left( \pm \sqrt{-n + y + 1}(n + y - 1) (\beta_n^+ - y) (\beta_n^- - y) \right) \frac{-y + 1}{y + (n - 1)},$$

where

$$\beta_n^\pm = \pm \sqrt{n^2 + 1 - n}. \quad (7)$$

The values $x_Y^\pm(y)$ are real only in the range $y \in [0, \beta_n^+].$

Further, the form of Eq. (6) means that for any given $y$ in the above allowed range, the values $x \in (x_Y^-, x_Y^+)$ yield $\gamma_Y < 0$, and those outside, i.e., the values $x \in [0, x_Y^+] \cup [x_Y^-, 1]$, yield $\gamma_Y \geq 0$. Thus, we determine the region $\mathcal{R}_Y$ as corresponding to these points $(x, y)$ which yield a negative $\gamma_Y$:

$$|\mathcal{R}_Y| = 2 \times \int_{y=0}^{\beta_n^+} (x_Y^+(y) - x_Y^-(y)) \, dy.$$  \(8\)
The pre-factor 2 comes from the fact that the space of \((x, y)\) does not have area 1 but instead must be normalized to \(\int_{x=0}^{1} \int_{y=0}^{1-x} dx\ dy = \frac{1}{2}\). The form of the rates Eq. (5) is such that at most only one of the three rates can be negative \[14\]. This means that regions \(R_X, R_Y\) and \(R_Z\), respectively, of points \((x, y, z)\) where \(\gamma_X, \gamma_Y\) and \(\gamma_Z\), can assume negative values within the time range \(p \in [0, \frac{1}{n}]\), is non-overlapping. Therefore, the measure, \(|\mathcal{M}|\) of the set of all non-Markovian channels in the Pauli simplex \(\mathcal{P}\), is simply \(|\mathcal{M}| = 3|R_Y|\).

A plot of the measure \(|\mathcal{M}|\) of non-Markovian channels with varying \(n\) is shown in Fig. 1. It shows that as the mixing channels move to a greater degree \(n\) away from QDS \((n = 2)\), somewhat counter-intuitively, the fraction of non-Markovianity in the corresponding Pauli simplex falls. This generalizes the result for QDS reported in \[14\].

\[
\begin{align*}
\frac{|\mathcal{M}|}{n} & \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1.0 \\
\end{align*}
\]

**FIG. 1.** (Color online) Plot of the measure of non-Markovian channels in the Pauli simplex, \(|\mathcal{M}|\) with varying \(n\). One finds that \(|\mathcal{M}|\) decreases with increasing \(n\). The case \(n = 2\) corresponds to QDS.

The natural diagrammatic depiction of the Pauli simplex as per our above analysis is in the \((x, y)\) representation, or analogously in the corresponding \((x, z)\) or \((y, z)\) representation. This is a right angle triangle (bordered by \(y = 1 - x\)). To go to a “Pauli neutral” representation, we require the linear transformation that maps a right angle triangle with vertices \(\{(0, 0), (0, 1), (1, 0)\}\) to an equilateral triangle. This is given by the matrix \(M \equiv k \begin{pmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{pmatrix}\), where \(k\) is a constant set to \(\sqrt{\frac{1}{2\sqrt{3}}}\) to ensure that the transformation is area preserving (i.e., \(\det(M) = 1\)). The Pauli simplex in this representation corresponds to the equilateral triangle \(\{(0, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (1, 0)\}\). The Markovian squeezed triangular regions \(\mathcal{M}_n\) are mapped correspondingly, as depicted in Fig. 2. Here, the equilateral triangle corresponds to a Pauli simplex for any \(n\) with the corresponding Pauli channels of the type Eq. (1).
FIG. 2. (Color online) The outermost triangle (in red) represents the Pauli simplex for a given functional form $q(n)$, with the vertices representing the three Pauli channels. The squeezed triangles represent the Markovian regions $M_n$ of Markovianity for different degrees $n$ of deviation from the QDS value of $n = 2$. We note that $M_n \subset M_{n'}$ if and only if $n < n'$.

Fig. 2 shows that as the degree $n$ of deviation from QDS form increases, the Markovian regions corresponding to a larger deviation contain those of a smaller deviation in the Pauli simplex, i.e. $M_n \subset M_{n'}$ if and only if $n < n'$. Certain points of similarity with the QDS case may be worth noting: in the case of two-channel mixing, which corresponds to any edge of the Pauli simplex, note that the result is the same as the QDS case: namely, any finite mixing leads to non-Markovianity. As in the QDS case, the form Eq. (3) automatically guarantees that for all channels in our Pauli simplex, the sum of any two decay rates is positive, implying that the channels are P-divisible.

Finally, as in the QDS case, for any $n$ neither the set of Markovian nor that of non-Markovian channels in the Pauli simplex is convex. In Figure 2, line segments or triangles connecting the “horns” of the squeezed triangle give us infinite number of examples of non-Markovian channels obtained by mixing Markovian channels. On the other hand, line segments or triangles linking the convex regions $R_j$ outside the squeezed triangles give an infinite number of examples of Markovian channels obtained by mixing non-Markovian ones.

IV. DISCUSSIONS AND CONCLUSIONS

We have studied the convex combination of Markovian Pauli non-QDS channels. The Pauli simplex obtained by the convex combination of the three Pauli channels is characterized and the measure of the associated non-Markovian regions is evaluated analytically. For the family of
channels parametrized by mixing fraction Eq. (1), the measure of the non-Markovian region in the Pauli simplex is found to decrease for mixing of channels that deviate more from the QDS structure. In other words, mixing time-dependent Markovian channels results in the production of “more” Markovian channels in comparison to mixing Markovian semigroups.

From Eq. (5), it follows that the functional form of the mixing fraction \( q = q(t) \) determines the instant \( q_0 \) at which a given channel \( \tilde{\Phi}_\star(t) \) in Eq. (3) turns non-Markovian. However, we note from the form Eq. (6) that the non-Markovian regions don’t depend on the functional form but only the value \( \frac{1}{n} \) that \( q(t) \) asymptotes to. This means, for example, that, as far as the measure of (non)-Markovian channels is concerned, for any fixed \( n \), all channels corresponding to \( q = (1 - \exp(-rt^{m_1}))/m_2/n \), with \( m_j \) being a real number greater than 1, are mutually equivalent. However, physical realization of channels with \( m_j > 1 \), \( j \in \{1, 2\} \) may not be straightforward.

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