L^p-NORM BOUNDS FOR AUTOMORPHIC FORMS VIA SPECTRAL RECIPROCITY

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ABSTRACT. Let \( g \) be a Hecke–Maaß cusp form on the modular surface \( \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \), namely an \( L^2 \)-normalised nonconstant Laplacian eigenfunction on \( \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \) that is additionally a joint eigenfunction of every Hecke operator. We prove the \( L^p \)-norm bound \( \|g\|_4 \ll \lambda_g^{3/304+\epsilon} \), where \( \lambda_g \) denotes the Laplacian eigenvalue of \( g \), which improves upon Sogge’s \( L^p \)-norm bound \( \|g\|_4 \ll \lambda_g^{1/16} \) for Laplacian eigenfunctions on a compact Riemann surface by more than a six-fold power-saving. Interpolating with the sup-norm bound \( \|g\|_\infty \ll \lambda_g^{5/344+\epsilon} \) due to Iwaniec and Sarnak, this yields \( L^p \)-norm bounds for Hecke–Maaß cusp forms that are power-saving improvements on Sogge’s bounds for all \( p > 2 \). Our paper marks the first improvement of Sogge’s result on the modular surface. Furthermore, these methods yield for compact arithmetic surfaces the best \( L^4 \)-norm bound to date.

Via the Watson–Ichino triple product formula, bounds for the \( L^p \)-norm of \( g \) are reduced to bounds for certain mixed moments of \( L \)-functions. We bound these using two forms of spectral reciprocity: identities between two different moments of central values of \( L \)-functions. The first is a form of GL3 × GL2 \( \leftrightarrow \) GL4 × GL4 spectral reciprocity, which relates a GL2 moment of GL3 × GL2 Rankin–Selberg \( L \)-functions to a GL3 moment of GL3 × GL4 Rankin–Selberg \( L \)-functions; this can be seen as a cuspidal analogue of Motohashi’s formula relating the fourth moment of the Riemann zeta function to the third moment of central values of Hecke \( L \)-functions. The second is a form of GL4 × GL2 \( \leftrightarrow \) GL4 × GL2 spectral reciprocity, which is a cuspidal analogue of a formula of Kuznetsov for the fourth moment of central values of Hecke \( L \)-functions.

1. INTRODUCTION

1.1. \( L^p \)-Norm Bounds for Hecke–Maaß Cusp Forms. A fundamental problem in analysis is understanding the distribution of mass of Laplacian eigenfunctions via bounds for their \( L^p \)-norms in terms of the size of their Laplacian eigenvalue. We study this problem for arithmetic Laplacian eigenfunctions on the modular surface \( \Gamma \backslash \mathbb{H} \), where \( \mathbb{H} := \{ z = x + iy \in \mathbb{C} : y > 0 \} \) is the upper half-plane upon which the modular group \( \Gamma := \text{SL}_2(\mathbb{Z}) \) acts via Möbius transformations.

Let \( g \) be a Hecke–Maaß cusp form on \( \Gamma \backslash \mathbb{H} \), namely a nonconstant Laplacian eigenfunction lying in the discrete spectrum of the Laplacian on \( \Gamma \backslash \mathbb{H} \) that is additionally a joint eigenfunction of every Hecke operator\(^1\). Thus \( \Delta g = \lambda_g g \), where \( \Delta := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) denotes the Laplace–Beltrami operator on \( \Gamma \backslash \mathbb{H} \) and \( \lambda_g \in (0, \infty) \) is the Laplacian eigenvalue of \( g \). We scale \( g \) to be \( L^2 \)-normalised with respect to the probability Haar measure \( \frac{3}{\pi} \frac{dx \, dy}{y^2} \) on \( \Gamma \backslash \mathbb{H} \). The main result of this paper is the following bound for the \( L^4 \)-norm of a Hecke–Maaß cusp form \( g \) on \( \Gamma \backslash \mathbb{H} \) in terms of \( \lambda_g \).

**Theorem 1.1.** Let \( g \) be a Hecke–Maaß cusp form on \( \Gamma \backslash \mathbb{H} \) of Laplacian eigenvalue \( \lambda_g \). Then

\[
\|g\|_4 := \left( \int_{\Gamma \backslash \mathbb{H}} |g(z)|^4 \frac{3 \, dx \, dy}{\pi y^2} \right)^{1/4} \ll \epsilon \lambda_g^{3/304+\epsilon}.
\]

\(^\dagger\)Since the Laplacian commutes with each Hecke operator, every nonconstant Laplacian eigenfunction on \( \Gamma \backslash \mathbb{H} \) is a linear combination of Hecke–Maaß cusp forms. Moreover, this Hecke assumption ought to be automatic since the discrete spectrum of the Laplacian on \( \Gamma \backslash \mathbb{H} \) is expected to be simple \([\text{Ste94}]\).
Interpolating between the $L^2$-norm normalisation $\|g\|_2 = 1$ and the $L^\infty$-norm bound $\|g\|_\infty \ll \varepsilon \lambda_g^{5/24+\varepsilon}$ of Iwaniec and Sarnak [IS95, Theorem 0.1] via the log-convexity of $L^p$-norms, we deduce the following $L^p$-norm bounds for a Hecke–Maaß cusp form $g$.

**Corollary 1.2.** Let $g$ be a Hecke–Maaß cusp form on $\Gamma \backslash \mathbb{H}$ of Laplacian eigenvalue $\lambda_g$. Then for $p \in [2, \infty]$, we have that $\|g\|_p \ll \lambda_g^{\delta(p)+\varepsilon}$, where

$$
\delta(p) = \begin{cases} 
\frac{3}{152} - \frac{3}{76p} & \text{for } 2 \leq p \leq 4, \\
\frac{5}{24} - \frac{181}{228p} & \text{for } 4 \leq p \leq \infty.
\end{cases}
$$

The method of proof of Theorem 1.1 is quite general and applies to Hecke–Maaß cusp forms on arithmetic surfaces other than the modular surface, leading to the following result.

**Theorem 1.4.** Let $g$ be squarefree and fixed and let $\Gamma'$ either be the Hecke congruence subgroup $\Gamma_0(q)$ or the congruence subgroup $\Gamma^D$ corresponding to the norm one units of a maximal order of an indefinite quaternion division algebra $D$ over $\mathbb{Q}$ of discriminant $q$. Let $g$ be a Hecke–Maaß newform on $\Gamma' \backslash \mathbb{H}$ of Laplacian eigenvalue $\lambda_g$. Then

$$
\|g\|_4 \ll \varepsilon \lambda_g^{\frac{3}{152}+\varepsilon}.
$$

We sketch in Section 12.1 how the method of proof of Theorem 1.1 extends to yield Theorem 1.4.

After this paper was written, Ki announced a proof of the essentially sharp upper bound $\|g\|_4 \ll \varepsilon \lambda_g^\varepsilon$ for a Hecke–Maaß cusp form $g$ on $\Gamma \backslash \mathbb{H}$ [Ki23, Theorem 2]. The proof is via completely different methods: instead of relating $\|g\|_4^2$ to moments of $L$-functions, as we do, Ki uses the Fourier–Whittaker expansion of $g$ over a Siegel set. This method would potentially extend to Hecke–Maaß newforms on arithmetic surfaces other than the modular surface for which there exists a Fourier–Whittaker expansion. However, no such Fourier–Whittaker expansion exists for Hecke–Maaß newforms on a compact congruence arithmetic surface arising from a quaternion division algebra; nonetheless, our method remains valid in this setting.

We sketch the method of proof of Theorem 1.1 in Section 1.3: broadly speaking, we relate $\|g\|_4^2$ to a mixed moment of central values of $L$-functions via Parseval’s identity and the Watson–Ichino triple product formula and then proceed to bound this moment. To experts, it may come as no surprise that current conventional machinery in the analytic theory of automorphic forms (approximate functional equations, the Kuznetsov and Petersson formulæ, the Voronoï summation formulæ, spectral large sieve inequalities, etc.) leads to some nontrivial $L^4$-norm bound for Hecke–Maaß cusp forms. In this paper, we do not simply push such methods to their limit. The novelty of our method is the development and implementation, for the first time, of spectral reciprocity identities for moments of $L$-functions in the context of the $L^4$-norm problem. This opens up new avenues of approach that would otherwise be completely unavailable if one were working with approximate functional equations and other standard techniques. Moreover, as we discuss in Section 1.4, the usage of these spectral reciprocity formulæ cannot be substituted with the method of approximate functional equations without majorly weakening the result.

### 1.2. Related Results.

#### 1.2.1. $L^p$-Norm Bounds for Laplacian Eigenfunctions

Theorem 1.1 and Corollary 1.2 fall under the umbrella of a large swathe of results concerning $L^p$-norm bounds for Laplacian eigenfunctions on manifolds. The fundamental result in this area is due to Sogge [Sog88], who has shown for $p \in [2, \infty]$ the $L^p$-norm bounds $\|g\|_p \ll \lambda_g^{\delta(n,p)}$, where

$$
\delta(n, p) = \begin{cases} 
n\frac{n-1}{8} - \frac{n-1}{4p} & \text{for } 2 \leq p \leq \frac{2(n+1)}{n-1}, \\
n\frac{n-1}{4} - \frac{n}{2p} & \text{for } \frac{2(n+1)}{n-1} \leq p \leq \infty,
\end{cases}
$$
for an $L^2$-normalised Laplacian eigenfunction $g$ with Laplacian eigenvalue $\lambda_g$ on a compact $n$-dimensional Riemannian manifold $M$. These bounds are sharp on the $n$-sphere $S^n$ and should be thought of as the convexity bounds for $L^p$-norms; thus Corollary 1.2 should be viewed as giving subconvex $L^p$-norm bounds on $\Gamma \backslash \mathbb{H}$ for all $p > 2$.

![Figure 1. A comparison of the exponent $\delta(p)$ given by (1.3) to Sogge’s exponent $\delta(2, 1/p)$ given by (1.5).](image)

Logarithmic improvements to Sogge’s $L^p$-norm bounds have been shown under various geometric assumptions on the underlying manifold, such as nonpositive sectional curvature [BS18, BS19, CG20, HT15]. Furthermore, power-saving improvements to Sogge’s bounds are known for certain manifolds: Zygmund [Zyg74] proved that $\|g\|_p \ll 1$ for $2 \leq p \leq 4$ with $M = \mathbb{T}^2$, while for $M = \mathbb{T}^n$, Bourgain and Demeter [BD15] (cf. [Dem20, Theorem 13.12]) proved more generally the improved bounds $\|g\|_p \ll_{\varepsilon} \lambda_g^{\delta'(n,p) + \varepsilon}$ with $\delta'(n, p) = 0$ for $2 \leq p \leq \frac{2(n+1)}{n-1}$ and $n \geq 3$ and $\delta'(n, p) = \frac{n-2}{4} - \frac{n}{2p}$ for $p \geq \frac{2(n-1)}{n-3}$ and $n \geq 4$. Finally, Marshall proved power-saving improvements to Sogge’s bounds for Laplacian eigenfunctions on certain compact locally symmetric spaces that are additionally eigenfunctions of the full ring of invariant differential operators [Mar16b, Theorem 1.1].

1.2.2. The Iwaniec–Sarnak Conjecture. For negatively curved surfaces, Iwaniec and Sarnak have conjectured that Sogge’s $L^p$-norm bounds fall well shy of the truth.

**Conjecture 1.6** (Iwaniec–Sarnak [Sar03, Conjecture 4]). Let $M$ be a negatively curved surface and let $K \subseteq M$ be compact. Then for all $p \in [2, \infty]$, 

$$\|g|_K\|_p \ll_{\varepsilon, K} \lambda_g^\varepsilon.$$ 

This conjecture is quite strong: if $M$ is a compact arithmetic hyperbolic surface arising from a quaternion division algebra over $\mathbb{Q}$, then the bound $\|g\|_\infty \ll_{\varepsilon} \lambda_g^\varepsilon$ implies the generalised Lindelöf hypothesis for certain $L$-functions, since Hecke–Maaß cusp forms evaluated at distinguished points are essentially equal to central values of $L$-functions by Waldspurger’s formula [Wal85]. If true, Conjecture 1.6 is essentially sharp [Mil10, Theorem 1]; moreover, the assumption that $M$ be a surface is necessary, since there are compact Riemannian manifolds with negative sectional curvature of dimension greater than 2 for which there exist subsequences of Laplacian eigenfunctions whose $L^\infty$-norms exhibit power growth [Mil11, Theorem 1].

Towards this conjecture, the only direct unconditional progress that has been made up until now is for Hecke–Maaß cusp forms on arithmetic hyperbolic surfaces, for which Iwaniec and Sarnak have proven the sup-norm bound $\|g\|_\infty \ll_{\varepsilon} \lambda_g^{5/24 + \varepsilon}$ [IS95, Theorem 0.1], while
Marshall has proven the $L^4$-norm bound $\|g\|_4 \ll \lambda_g^{3/56}$ [Mar16a, Corollary 1.2] for compact arithmetic hyperbolic surfaces arising from quaternion division algebras. The $L^4$-norm bound with exponent $\delta(4) = 3/304$ obtained in Theorem 1.1 for $\Gamma \backslash \mathbb{H}$ and in Theorem 1.4 for $\Gamma_0(q) \backslash \mathbb{H}$ gives more than a six-fold improvement on the exponent $\delta(2, 4) = 1/16$ of Sogge’s $L^4$-norm bound for compact surfaces and more than a five-fold improvement on the exponent $3/56$ of Marshall’s $L^4$-norm bound for compact arithmetic hyperbolic surfaces arising from quaternion division algebras.

1.2.3. Conditional Improvements. We highlight that the strengths of Theorems 1.1 and 1.4 lie in the fact that these are unconditional power-saving improvements upon Sogge’s bound, as these bounds can be greatly strengthened conditionally. Watson observed that under the assumption of the generalised Lindelöf hypothesis for $GL_3 \times GL_2$ Rankin–Selberg $L$-functions and $GL_2$ standard $L$-functions, we have the almost sharp upper bound $\|g\|_4 \ll \lambda_g^4$ [Wat08, Corollary 2].

Buttcane and the second author improved this to the asymptotic formula $\|g\|_4^4 \sim 3$ under the same assumption [BuK17, Theorem 1.1], in accordance with the random wave conjecture.

Asymptotics for the $L^4$-norm are known unconditionally for certain distinguished sparse (in particular, density zero) subsequences of automorphic forms with additional arithmetic structure, namely for dihedral Hecke–Maaß cusp forms [HK20, Theorem 1.9] and (in a regularised form) for Eisenstein series [DK20, Theorem 1.1]. We explain why these strong results are possible unconditionally yet Theorems 1.1 and 1.4 fall short of such an asymptotic formula in Remark 6.18. In Sections 12.2 and 12.3, we discuss how Theorems 1.1 and 1.4 may be improved under various conditional assumptions.

1.2.4. Weight-Aspect and Level-Aspect Generalisations. There are natural generalisations of these $L^p$-norm bounds to more general families of automorphic forms rather than just Hecke–Maaß cusp forms on the modular surface $\Gamma \backslash \mathbb{H}$. In particular, one can instead study $L^p$-norm bounds for holomorphic modular forms (so that the Laplacian is replaced by the weight $k$ Laplacian with $k$ varying); moreover, one can investigate $L^p$-norm bounds in the level aspect (so that the underlying orbifold $\Gamma_0(q) \backslash \mathbb{H}$ is varying); finally, one can study this in both aspects simultaneously. We direct the reader to [HS20, Sah17, Xia07] and the references therein for results on $L^\infty$-norm bounds, [Blo13, BKY13, BuK15, DK20, Hum18, HK20, Kha14, KhSt20, Liu15, Luo14, Spi03] for results on $L^4$-norm bounds, and [Mar16c] for $L^p$-norm bounds.

Notably, Blomer, the second author, and Young have proven that a holomorphic Hecke cusp form $G$ of weight $k \in 2\mathbb{N}$ satisfies $\|y^{k/2}G\|_4 \ll k^{1/2+\varepsilon}$ [BKY13, Theorem 1.1]; as $k^2$ is the analogue of the Laplacian eigenvalue $\lambda_g$, this should be thought of as a Weyl-strength subconvex improvement upon the convexity bound $\|y^{k/2}G\|_4 \ll k^{1/8+\varepsilon}$. Under the assumption of the generalised Riemann hypothesis for $GL_3 \times GL_2$ Rankin–Selberg $L$-functions and $GL_2$ standard $L$-functions, Zenz has shown the sharp upper bound $\|y^{k/2}G\|_4 \ll 1$ [Zen23, Theorem 1.1], while it is conjectured that the asymptotic formula $\|y^{k/2}G\|_4^4 \sim 2$ holds [BKY13, Conjecture 1.2].

1.3. Method of Proof.

1.3.1. Reduction to Mixed Moments of $L$-Functions. The initial manoeuvres of our proof of Theorem 1.1 follow a well-trodden path pioneered by Sarnak [Sar03, p. 461]: we spectrally expand $\|g\|_4^4 = \langle|g|^2, |g|^2 \rangle$ and spectrally expand this via Parseval’s theorem for $L^2(\Gamma \backslash \mathbb{H})$. The resulting spectral expansion involves a sum over Hecke–Maaß cusp forms $f$ of terms of the form $\langle|g|^2, f\rangle^2$ and an integral over $t \in \mathbb{R}$ of terms of the form $\langle|g|^2, E(\cdot, 1/2 + it)^2\rangle$, where $E(z, s)$ denotes the real analytic Eisenstein series. We then invoke the Watson–Ichino triple product formula, namely

\[\|g\|_4^4 = \langle|g|^2, |g|^2 \rangle\]
For the bulk range \([0, t_g^{1/α}]\), we have the bounds
\[
\sum_{f \in \mathcal{B}_0 \atop t_f \leq t_g^{1/α}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad } g \otimes f \right)}{L(1, \text{ad } f)L(1, \text{ad } g)^2} H(t_f) + \frac{1}{2\pi} \int_{|t| \leq t_g^{1/α}} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad } g \right)}{\zeta(1 + 2it)L(1, \text{ad } g)} \right|^2 H(t) \, dt \ll_{ε} t_g^{3+ε}.
\]

(2) For the bulk range \([t_g^{1/α}, 2t_g - t_g^{1/α}]\), there exists a continuous function \(c(α)\) of \(α\) satisfying \(\lim_{α \to 0} c(α) = 0\) such that
\[
\sum_{f \in \mathcal{B}_0 \atop t_g^{1/α} \leq t_f \leq 2t_g - t_g^{1/α}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad } g \otimes f \right)}{L(1, \text{ad } f)L(1, \text{ad } g)^2} H(t_f) + \frac{1}{2π} \int_{t_g^{1/α} \leq |t| \leq 2t_g - t_g^{1/α}} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad } g \right)}{\zeta(1 + 2it)L(1, \text{ad } g)} \right|^2 H(t) \, dt \ll_{ε} t_g^{c(α)+ε}.
\]

(3) For the short transition range \([2t_g - t_g^{1/α}, 2t_g]\), we have that
\[
\sum_{f \in \mathcal{B}_0 \atop 2t_g - t_g^{1/α} \leq t_f \leq 2t_g} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad } g \otimes f \right)}{L(1, \text{ad } f)L(1, \text{ad } g)^2} H(t_f) \ll_{ε} t_g^{1/2+ε}.
\]
Thus the generalised Lindelöf hypothesis and the Weyl law combine to give the conditional bound (1.10) for the short initial range, (1.12) for the bulk range, (1.13) for the short transition ranges, which merely recovers Sogge’s large sieve. Unfortunately, this only yields the bounds $O(t_g^\alpha)$, $O(t_g^{\alpha/2+\varepsilon})$ for the short initial and short transition ranges, however, are far from immediate and require several new ideas. Buttcane and the second author [BuK17] showed that with further effort, one can obtain the conditional asymptotic formula $2 + o(1)$ for the bulk range.

Without appealing to the generalised Lindelöf hypothesis, such strong bounds are no longer easily attained. Nonetheless, we shall show that the bound (1.13) for the tail range is readily achieved due to the fact that $H(t)$ decays exponentially for $|t| \geq 2t_g$; moreover, the requisite bound (1.11) for the bulk range can be deduced with a modicum of effort from earlier work of Buttcane and the second author [BuK17]. The bounds (1.10) and (1.12) for the short initial and short transition ranges, however, are far from immediate and require several new ideas.

A standard approach to bound the mixed moments of $L$-functions in these ranges would be to apply the Cauchy–Schwarz inequality to separate the $L$-functions, write these $L$-functions in terms of Dirichlet polynomials via the approximate functional equation, and apply the spectral large sieve. Unfortunately, this only yields the bounds $O(t_g^{1/2})$ for the short initial and short transition ranges, which merely recovers Sogge’s $L^2$-norm bound. To improve upon these weaker bounds, we prove new forms of spectral reciprocity: identities between two different moments of central values of $L$-functions. We apply these with a mix of other ideas, as we describe below.

### 1.4. Spectral Reciprocity Formuale.

#### 1.4.1. GL$_3 \times$ GL$_2 \rightsquigarrow$ GL$_4 \times$ GL$_1$ Spectral Reciprocity.

By an appropriate application of Hölder’s inequality, we are led to the problem of determining bounds for the moments of $L$-functions

$$
\sum_{f \in \mathcal{B}_0} \frac{L \left( \frac{1}{2}, \text{ad } g \otimes f \right)}{L(1, \text{ad } f)} h(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{L \left( \frac{1}{2} + it, \text{ad } g \right)}{\zeta(1 + 2it)} \right|^2 h(t) \, dt
$$

with $h(t)$ an appropriately chosen test function, such as a smooth approximation of the indicator function of a dyadic interval $[-2T, -T] \cup [T, 2T]$.

The analogous moment with $g$ replaced by an Eisenstein series is

$$
\sum_{f \in \mathcal{B}_0} \frac{L \left( \frac{1}{2}, f \right)^3}{L(1, \text{ad } f)} h(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\zeta \left( \frac{1}{2} + it \right)^3}{\zeta(1 + 2it)} \right|^2 h(t) \, dt.
$$
Via work of Motohashi [Mot97, Theorem 4.2], given a sufficiently well-behaved test function $h$, there exists a corresponding transform $\mathcal{H}$ such that there is an exact equality between the moment (1.16) and the sum of a main term dependent on $h$ together with a dual moment

$$\int_{-\infty}^{\infty} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \mathcal{H}(t) \, dt.$$ 

For example, if one chooses $h(t)$ in (1.16) to localise to an interval of the form $[-T - U, -T] \cup [T, T + U]$ with $1 \leq U \leq T$, then with some effort one can show that $\mathcal{H}(t)$ is essentially localised to $|t| \leq T/U$, where it is of size $\approx U$.

In Theorem 3.1, we prove a cuspidal analogue of Motohashi’s formula, namely that given a sufficiently well-behaved test function $h$, there exists a corresponding transform $\mathcal{H}$ (given as an explicit integral transform in (3.5)), such that the moment (1.15) is exactly equal to the sum of a main term dependent on $h$ together with a dual moment

$$\int_{-\infty}^{\infty} L \left( \frac{1}{2} + it, \text{ad} \, g \right) \zeta \left( \frac{1}{2} - it \right) \mathcal{H}(t) \, dt. \quad (1.17)$$

1.4.2. $GL_4 \times GL_2 \rightsquigarrow GL_4 \times GL_2$ Spectral Reciprocity. We additionally prove a new form of spectral reciprocity for the mixed moment

$$\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right)}{L(1, \text{ad} \, f)} \mathcal{H}(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \zeta \left( \frac{1}{2} + it \right) \frac{L \left( \frac{1}{2} + it, \text{ad} \, g \right)}{\zeta(1 + 2it)} \right|^2 h(t) \, dt \quad (1.18)$$

with $h(t)$ an appropriately chosen test function, such as a smooth approximation of the indicator function of a dyadic interval $[-2T, -T] \cup [T, 2T]$. The analogous moment with $g$ replaced by an Eisenstein series is

$$\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right)^4}{L(1, \text{ad} \, f)} \mathcal{H}(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\zeta \left( \frac{1}{2} + it \right)^4}{\zeta(1 + 2it)} \right|^2 h(t) \, dt. \quad (1.19)$$

Given a sufficiently well-behaved test function $h$, there exists a corresponding transform $\tilde{h}$ such that there is an exact equality between the moment (1.19) and the sum of a main term dependent on $h$ together with a dual moment

$$\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right)^4}{L(1, \text{ad} \, f)} \tilde{h}(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\zeta \left( \frac{1}{2} + it \right)^4}{\zeta(1 + 2it)} \right|^2 \tilde{h}(t) \, dt$$

as well as an additional dual moment of the same form involving a sum over holomorphic cusp forms. Such an identity is due to Kuznetsov (in an incomplete form in [Kuz89] and completed as a transform in (4.3)) such that the mixed moment (1.18) is equal to the sum of a main term dependent on $h$ together with a dual moment

$$\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right)}{L(1, \text{ad} \, f)} \tilde{h}(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} \, g \right)}{\zeta(1 + 2it)} \right|^2 \tilde{h}(t) \, dt$$

as well as an additional dual moment of the same form involving a sum over holomorphic cusp forms.

\footnote{Motohashi’s formulation of this identity involves first specifying $\mathcal{H}$ and then determining $h$ as a transform involving $\mathcal{H}$. This process can be reversed; see, for example, work of Motohashi [Mot99] and of Nelson [Nel19b].}
1.4.3. **Applications of Spectral Reciprocity for the Short Initial Range.** We apply these forms of spectral reciprocity to prove the bound (1.10) for the short initial range. By a dyadic subdivision, we are led to bounding the mixed moment

\[
L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right) \left( \frac{1}{2}, \text{ad} f \right) \frac{1}{2\pi} \int_{T \leq |t| \leq T} \left| \zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} g \right) \frac{\zeta(1 + 2it)}{\zeta(1 + 2it)} \right|^2 dt
\]

with \( T \leq t_g^{-\alpha} \).

For \( T \leq t_g^{3/19} \), so that the mixed moment (1.20) is particularly short, we are unable to do better than simply applying the Cauchy–Schwarz inequality and the spectral large sieve. For \( t_g^{1/2} \leq T \leq t_g^{16/19} \), on the other hand, we obtain improved bounds for (1.20) via Hölder’s inequality and an application of GL\(_3 \times GL_2 \rightleftharpoons GL_4 \times GL_1 \) spectral reciprocity to bound

\[
L \left( \frac{1}{2}, \text{ad} g \otimes f \right) \left( \frac{1}{2}, \text{ad} f \right) \frac{1}{2\pi} \int_{T \leq |t| \leq T} \left| \frac{L \left( \frac{1}{2} + it, \text{ad} g \right)}{\zeta(1 + 2it)} \right|^2 dt,
\]

which crucially relies on the nonnegativity of \( L(1/2, \text{ad} g \otimes f) \) [Lap03, Theorem 1.1]. After choosing our test function \( h \) to approximate the indicator function of \([-2T, -T] \cup [T, 2T]\), we are left with determining the behaviour of the corresponding transform \( \mathcal{H} \). This is no easy task: unlike most previous work involving similar spectral reciprocity formulae (such as [LNQ23]), we require hybrid bounds with explicit dependence not only on the dyadic parameter \( T \) but additionally on the spectral parameter \( t_g \), which is further complicated by the fact that \( T < t_g \), so that our moment is short relative to the conductor of the \( L \)-function \( L(1/2, \text{ad} g \otimes f) \). Instead of being of size \( \approx T \) and localised to \( |t| \leq 1 \), as is the case for \( t_g \) *bounded* relative to \( T \), we find that \( \mathcal{H}(t) \) is essentially localised to \( |t| \leq t_g^2/T^2 \), where it is of size \( \approx T^2/t_g \). We subsequently bound the dual mixed moment (1.17) of \( L(1/2 + it, \text{ad} g) \zeta(1/2 - it) \) via the Cauchy–Schwarz inequality and the Montgomery–Vaughan mean value theorem for Dirichlet polynomials.

Finally, for the remaining ranges \( t_g^{3/19} \leq T \leq t_g^{1/2} \) and \( t_g^{16/19} \leq T \leq t_g^{1-\alpha} \), we obtain strongest bounds for (1.20) by using \( GL_4 \times GL_2 \rightleftharpoons GL_4 \times GL_2 \) spectral reciprocity. We once more choose \( h \) to approximate the indicator function of \([-2T, -T] \cup [T, 2T]\) and determine the behaviour of the corresponding transform \( \tilde{h} \). Again, this is a cumbersome task due to the hybrid bounds required, unlike other previous applications of spectral reciprocity (such as [BLM19]). Instead of being of size \( \approx T \) and localised to \( |t| \leq 1 \), as is the case for \( t_g \) *bounded* relative to \( T \), we find that \( \tilde{h}(t) \) is essentially localised to \( |t| \leq t_g/T \), where it is of size \( \approx T^2/t_g \). Thus we treat the range \( t_g^{16/19} \leq T \leq t_g^{1-\alpha} \) by using \( GL_4 \times GL_2 \rightleftharpoons GL_4 \times GL_2 \) spectral reciprocity to reduce the problem to the range \( T \leq t_g^{3/19} \), which we previously treated by the spectral large sieve. Similarly, the range \( t_g^{3/19} \leq T \leq t_g^{1/2} \) is treated by using \( GL_4 \times GL_2 \rightleftharpoons GL_4 \times GL_2 \) spectral reciprocity to reduce the problem to the range \( t_g^{1/2} \leq T \leq t_g^{16/19} \), which we previously treated via \( GL_3 \times GL_2 \rightleftharpoons GL_4 \times GL_1 \) spectral reciprocity. Note that our treatment of this latter range is highly unusual and seemingly counter-intuitive, since we move from a relatively short moment to a *longer* moment, and yet this process nonetheless yields improved bounds.

It is crucial to note that this last step *cannot* be achieved without using \( GL_4 \times GL_2 \rightleftharpoons GL_4 \times GL_2 \) in its form as an *exact* identity of two moments of \( L \)-functions. Indeed, by modifying the proof to use approximate functional equations this spectral reciprocity formula may instead be proven in the form of an *approximate* identity of moments of \( L \)-functions, where each moment involves Dirichlet polynomials in place of \( L \)-functions. Such an approximate identity, however, is insufficient for our applications. We make vital use of the nonnegativity of the \( L \)-function \( L(1/2, \text{ad} g \otimes f) \) appearing in the dual moment, for this allows us to use Hölder’s inequality with exponents that lead to *odd* moments of these central \( L \)-values. With Dirichlet polynomials, we no longer have nonnegativity, and so this avenue of approach is closed to us; in particular, we
would only be able to prove strictly weaker $L^1$-norm bounds for Hecke–Maaß cusp forms via this alternative method using approximate functional equations.

1.4.4. Applications of Spectral Reciprocity for the Short Transition Range. To prove the bound (1.12) for the short transition range, we must work over shorter intervals, namely with mixed moments of the form

\[
\sum_{f \in B_0 \atop |T-U| \leq |t| \leq |T+U|} \frac{L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, \text{ad} g \otimes f\right)}{L(1, \text{ad} f)} + \frac{1}{2\pi} \int_{T-U \leq |t| \leq T+U} \left| \frac{\zeta\left(\frac{1}{2} + it\right) L\left(\frac{1}{2} + it, \text{ad} g\right)}{\zeta(1+2it)} \right|^2 dt
\]

with $2t_g - t_g^{1-\alpha} \leq T < 2t_g$ and $U = t_g - T/2 + 1$. This is the conductor-dropping range: the conductor of $L(1/2, f)L(1/2, \text{ad} g \otimes f)$ in this range is $\approx t_g^6(1 + 2t_g - t_f)^2$, which decreases as $t_f$ approaches $2t_g$.

For the range $t_g^{1/3} \leq U \leq t_g^{1-\alpha}$, we bound (1.21) via $\text{GL}_4 \times \text{GL}_2 \rightsquigarrow \text{GL}_4 \times \text{GL}_2$ spectral reciprocity. We begin by choosing $h$ to approximate the indicator function of $[-T - U, -T + U] \cup [T - U, T + U]$ and determining the behaviour of the transform $\tilde{h}$. Once more, the hybrid nature of the problem alters the behaviour of the transform: instead of being of size $U(1 + |t|)^{-1/2}$ and localised to $|t| \leq T/U$, as is the case for $t_g$ bounded relative to $T$ and $U$, the transform $\tilde{h}(t)$ is instead localised to the much shorter range $|t| \leq (T/U)^{1/2}$, where it is of larger size $\approx U$. Since $T \approx t_g$, we have thereby reduced the problem back to the short initial range, and so we can simply appeal to our previously determined bounds for this range.

For the remaining range $1 \leq U \leq t_g^{1/3}$, we first apply Hölder’s inequality, which leaves us with the problem of bounding

\[
\sum_{f \in B_0 \atop |T-U| \leq |t| \leq |T+U|} \frac{L\left(\frac{1}{2}, \text{ad} g \otimes f\right)}{L(1, \text{ad} f)} + \frac{1}{2\pi} \int_{T-U \leq |t| \leq T+U} \left| \frac{L\left(\frac{1}{2} + it, \text{ad} g\right)}{\zeta(1+2it)} \right|^2 dt.
\]

While it is likely that we could bound this adequately using $\text{GL}_3 \times \text{GL}_2 \rightsquigarrow \text{GL}_4 \times \text{GL}_1$ spectral reciprocity, the short length of the moment (in particular, the fact that $U \leq T^{1/3}$) means that the analysis of the size and length of the transform $\mathcal{H}$ becomes significantly more challenging. Instead, we bound (1.22) via a more classical approach using approximate functional equations and the Kuznetsov formula, which is more straightforward and yields sufficiently strong bounds for our purposes.
2. Automorphic Preliminaries

2.1. Spectral Summation Formula. The central tools that we make use of are the Kuznetsov and Petersson formulæ. The former involves sums of Hecke eigenvalues $\lambda_f(n)$ of Hecke–Maaß cusp forms $f$ on $\Gamma \backslash \mathbb{H}$ over an orthonormal basis $B_0$ of such cusp forms together with the root number $\epsilon_f \in \{1, -1\}$, as well as an integral involving the Hecke eigenvalues $\lambda(n, t) := \sum_{ab=n} a^{it}b^{-it}$ of the real analytic Eisenstein series $E(z, 1/2 + it)$. The latter involves sums of Hecke eigenvalues $\lambda_f(n)$ of holomorphic Hecke cusp forms $f$ on $\Gamma \backslash \mathbb{H}$ of weight $k_f \in 2\mathbb{N}$ over an orthonormal basis $B_{\text{hol}}$ of such cusp forms. Both formulæ express sums of Hecke eigenvalues in terms of sums of Kloosterman sums weighted by Bessel functions.

**Theorem 2.1** (Kuznetsov formula [Iwa02, Theorem 9.3]). Let $\delta > 0$, and let $h^\pm(t)$ be a function that is even, holomorphic in the horizontal strip $|\Im(t)| \leq 1/2 + \delta$, and satisfies $h^\pm(t) \ll (1 + |t|)^{-2-\delta}$. Then for $m, n \in \mathbb{N}$,

\[
(2.2) \quad \sum_{f \in B_0} \epsilon_f \frac{\lambda_f(m) \lambda_f(n)}{L(1, ad_f)} h^\pm(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda(m, t) \lambda(n, t)}{\zeta(1 + 2it)\zeta(1 - 2it)} h^\pm(t) \, dt = \delta_{m, n} N^\pm h^\pm + \sum_{c=1}^{\infty} S(m, n; c) \left( \frac{\sqrt{mn}}{c} \right),
\]

where

\[
(2.3) \quad S(m, n; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e \left( \frac{md + nd\bar{d}}{c} \right),
\]

\[
(2.4) \quad N^\pm h^\pm := \int_{-\infty}^{\infty} h^\pm(r) \, d_{\spec} r, \quad (\mathcal{K}^\pm h^\pm)(x) := \int_{-\infty}^{\infty} \mathcal{J}_r^\pm(x) h^\pm(r) \, d_{\spec} r,
\]

\[
(2.5) \quad \mathcal{J}_r^+(x) := \frac{\pi i}{\sinh \pi r} \left( J_{2ir}(4\pi x) - J_{-2ir}(4\pi x) \right), \quad \mathcal{J}_r^-(x) := 4 \cosh \pi r K_{2ir}(4\pi x),
\]

\[
(2.6) \quad d_{\spec} := \frac{1}{2\pi^2} \frac{1}{\tanh \pi r} dr.
\]

Here $J_\alpha(x)$ denotes the Bessel function of the first kind and $K_\alpha(x)$ denotes the modified Bessel function of the second kind.

**Theorem 2.7** (Petersson formula [Iwa02, Theorem 9.6]). Let $h_{\text{hol}} : 2\mathbb{N} \to \mathbb{C}$ be a sequence satisfying $h_{\text{hol}}(k) \ll k^{-2-\delta}$ for some $\delta > 0$. Then for $m, n \in \mathbb{N}$,

\[
(2.8) \quad \sum_{f \in B_{\text{hol}}} \frac{\lambda_f(m) \lambda_f(n)}{L(1, ad_f)} h_{\text{hol}}(t_f) = \delta_{m, n} N_{\text{hol}} h_{\text{hol}} + \sum_{c=1}^{\infty} S(m, n; c) \left( \mathcal{K}^\text{hol} h_{\text{hol}} \right) \left( \frac{\sqrt{mn}}{c} \right).
\]

Here

\[
(2.9) \quad N_{\text{hol}} h_{\text{hol}} := \sum_{k \equiv 0 (\text{mod } 2)}^{\infty} \frac{k - 1}{2\pi^2} h_{\text{hol}}(k),
\]

\[
(2.10) \quad (\mathcal{K}^\text{hol} h_{\text{hol}})(x) := \sum_{k \equiv 2 (\text{mod } 2)}^{\infty} \frac{k - 1}{2\pi^2} \mathcal{J}_k^\text{hol}(x) h_{\text{hol}}(k), \quad \mathcal{J}_k^\text{hol}(x) := 2\pi i^{-k} J_{k-1}(4\pi x).
\]

We also use the Kuznetsov and Petersson formulæ in reverse, namely the spectral decomposition of sums of Kloosterman sums.

**Theorem 2.11** (Kloosterman summation formula [IK04, Theorem 16.5]). Let $H \in C^3((0, \infty))$ be a function satisfying $x^{\delta} \frac{d^j}{dx^j} H(x) \ll \min\{x^{1/2+\delta}, x^{-1-\delta}\}$ for $j \in \{0, 1, 2, 3\}$ and for some $\delta > 0$. Then for $m, n \in \mathbb{N}$,

\[
(2.12) \quad \sum_{f \in B_0} \frac{\epsilon_f^{1+\frac{j}{2}} \lambda_f(m) \lambda_f(n)}{L(1, ad_f)} (\mathcal{L}^\pm H)(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda(m, t) \lambda(n, t)}{\zeta(1 + 2it)\zeta(1 - 2it)} (\mathcal{L}^\pm H)(t) \, dt
\]

where $\mathcal{L}^\pm H(x)$ denotes the convolution $H(x) * K^\pm(x)$.
\[ + \delta_{\pm,+} \sum_{f \in B_{\text{hol}}} \frac{\lambda_f(m)\lambda_f(n)}{L(1, \text{ad } f)} (\mathcal{L}^{\text{hol}}H)(k_f) \]

\[ = \sum_{c=1}^{\infty} \frac{S(m, \pm n; c)}{c} H\left(\frac{\sqrt{m}n}{c}\right), \]

where

\[ (\mathcal{L}^{\pm}H)(t) := \int_0^\infty J_t^\pm(x)H(x) \frac{dx}{x}, \quad (\mathcal{L}^{\text{hol}}H)(k) := \int_0^\infty J_k^{\text{hol}}(x)H(x) \frac{dx}{x}. \]

2.2. Mellin Transforms. We record the following Mellin transforms of the functions \( J_t^\pm \) and \( J_k^{\text{hol}} \) as in (2.5) and (2.10).

**Lemma 2.14 ([BLM19, (A.7)], [BIK19b, (3.13)]).** We have that

\[ \widehat{J}_t^+(s) = \frac{\pi i(2\pi)^{-s}}{2 \sinh \pi r} \left( \frac{\Gamma\left(\frac{s}{2} + ir\right)}{\Gamma\left(1 - \frac{s}{2} + ir\right)} - \frac{\Gamma\left(\frac{s}{2} - ir\right)}{\Gamma\left(1 - \frac{s}{2} - ir\right)} \right) \]

\[ = (2\pi)^{-s} \Gamma\left(\frac{s}{2} + ir\right) \Gamma\left(\frac{s}{2} - ir\right) \cos \frac{\pi s}{2}, \]

\[ \widehat{J}_t^-(s) = \frac{\pi i(2\pi)^{-s}}{2 \tanh \pi r \cos \frac{\pi s}{2}} \left( \frac{\Gamma\left(\frac{s}{2} + ir\right)}{\Gamma\left(1 - \frac{s}{2} + ir\right)} - \frac{\Gamma\left(\frac{s}{2} - ir\right)}{\Gamma\left(1 - \frac{s}{2} - ir\right)} \right) \]

\[ = (2\pi)^{-s} \Gamma\left(\frac{s}{2} + ir\right) \Gamma\left(\frac{s}{2} - ir\right) \cosh r, \]

\[ \widehat{J}_k^{\text{hol}}(s) = \pi i^{-k}(2\pi)^{-s} \frac{\Gamma\left(\frac{s+k-1}{2}\right)}{\Gamma\left(1 - \frac{s+k}{2}\right)} \]

\[ = (2\pi)^{-s} \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s-k+1}{2}\right) \cos \frac{\pi s}{2}. \]

We additionally state bounds for these Mellin transforms and their residues.

**Corollary 2.18 ([HK20, Corollary A.27]).** The functions \( \widehat{J}_t^\pm(s) \) extend meromorphically to \( \mathbb{C} \) with simple poles at \( s = 2(\pm ir - \ell) \) for \( \ell \in \mathbb{N}_0 \), where \( \mathbb{N}_0 \) denotes the nonnegative integers. For \( s = \sigma + ir \in \mathbb{C} \) in bounded vertical strips at least a bounded distance away from \( \{2(\pm ir - \ell) : \ell \in \mathbb{N}_0\} \) and for \( r \in \mathbb{R} \),

\[ \widehat{J}_t^+(s) \ll_{\sigma} ((1 + |\tau + 2r|)(1 + |\tau - 2r|))^{\frac{s-1}{2}} \times \begin{cases} e^{-\frac{\pi}{2}(|r|-|\tau|)} & \text{if } |\tau| \leq 2|r|, \\ 1 & \text{if } |\tau| \geq 2|r|, \end{cases} \]

\[ \widehat{J}_t^-(s) \ll_{\sigma} ((1 + |\tau + 2r|)(1 + |\tau - 2r|))^{\frac{s-1}{2}} \times \begin{cases} 1 & \text{if } |\tau| \leq 2|r|, \\ e^{-\frac{\pi}{2}(|r|-2|r|)} & \text{if } |\tau| \geq 2|r|. \end{cases} \]

Moreover,

\[ \text{Res}_{s=2(\pm ir - \ell)} \widehat{J}_t^+(s) = (-1)^\ell \text{ Res}_{s=2(\pm ir - \ell)} \widehat{J}_t^-(s) \ll_{\ell} (1 + |r|)^{-\ell - \frac{1}{2}}. \]

The function \( \widehat{J}_k^{\text{hol}}(s) \) extends meromorphically to \( \mathbb{C} \) with simple poles at \( s = 1 - k - 2\ell \) for \( \ell \in \mathbb{N}_0 \). For \( s = \sigma + ir \in \mathbb{C} \) in bounded vertical strips, at least a bounded distance away from \( \{1 - k - 2\ell : \ell \in \mathbb{N}_0\} \),

\[ \widehat{J}_k^{\text{hol}}(s) \ll_{\sigma} (k + |\tau|)^{-\sigma - 1}. \]

Moreover,

\[ \text{Res}_{s=1-k-2\ell} \widehat{J}_k^{\text{hol}}(s) = \frac{(2\pi i)^{k+2\ell}}{\Gamma(k + \ell)\Gamma(\ell + 1)}. \]
2.3. **Voronoï Summation Formulæ.** Along with the Kuznetsov and Petersson formulæ, we also make use of the GL$_3$ Voronoï summation formula. This involves distinguished special functions. For these special functions, we have the following lemma, which is a straightforward consequence of the meromorphic continuation of the gamma function together with Stirling’s formula.

**Lemma 2.24.** Let $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3$ with $-1/2 < \Re(\mu_j) < 1/2$ and $\mu_1 + \mu_2 + \mu_3 = 0$, and for $s = \sigma + i\tau \in \mathbb{C}$, define

\[
\mathcal{G}_\mu^\pm(s) := \frac{1}{2} \prod_{j=1}^{3} G_0(s + \mu_j) \pm \frac{1}{2} \prod_{j=1}^{3} G_1(s + \mu_j),
\]

where

\[
G_j(s) := \frac{\Gamma(s + j)}{\Gamma(1 - s + j)} = 2(2\pi)^{-s} \Gamma(s) \times \begin{cases} \cos \frac{\pi s}{2} & \text{if } j = 0, \\ \sin \frac{\pi s}{2} & \text{if } j = 1, \end{cases}
\]

with $\Gamma(s) := \pi^{-s/2} \Gamma(s/2)$. Then $\mathcal{G}_\mu^\pm(s)$ is meromorphic on $\mathbb{C}$ with simple poles at $s = -\mu_j - \ell$ for each $\ell \in \mathbb{N}_0$. Moreover, if $s$ is a bounded distance away from such a pole, we have that

\[
\mathcal{G}_\mu^\pm(s) \ll_{\mu, \sigma} (1 + |\tau|)^{3\sigma - \frac{3}{2}}.
\]

Similarly, for $s \in \mathbb{C}$, define

\[
G^\pm(s) := \frac{1}{2} G_0(s) \mp \frac{1}{2} G_1(s) = (2\pi)^{-s} \Gamma(s) \exp \left( \pm \frac{i\pi s}{2} \right).
\]

Then $G^\pm(s)$ is meromorphic on $\mathbb{C}$ with simple poles at $s = -\ell$ with residue $(-1)^{\ell} \frac{i}{2} (2\pi)^{\ell} \ell!$ for each $\ell \in \mathbb{N}_0$. Moreover, if $s = \sigma + i\tau$ is a bounded distance away from such a pole, we have that

\[
G^\pm(s) \ll_{\sigma} (1 + |\tau|)^{\sigma - \frac{1}{2}}.
\]

We use the GL$_3$ Voronoï summation formula for Hecke–Maaß cusps forms for SL$_3(\mathbb{Z})$ (i.e. Hecke–Maaß cusp forms on $Z(\mathbb{R}) \setminus \GL_3(\mathbb{R})/\O(3)$).

**Lemma 2.29** (Voronoï Summation Formula [BLK19b, Section 4]). Given a Hecke–Maaß cusp form $F$ for SL$_3(\mathbb{Z})$ with Hecke eigenvalues $A_F(\ell, n)$, define the Voronoï series

\[
\Phi_F(c, d, \ell; w) := \sum_{n=1}^{\infty} \frac{A_F(\ell, n)}{nw} e \left( \frac{n\ell}{c} \right),
\]

where $c, \ell \in \mathbb{N}$, $d \in (\mathbb{Z}/c\mathbb{Z})^\times$, and $w = u + iv \in \mathbb{C}$. This converges absolutely for $u > 1$ and extends holomorphically to the entire complex plane. We have the functional equation

\[
\Phi_F(c, d, \ell; w) = \sum_{\pm} \mathcal{G}_{\mu_F}^\pm(1 - w) \Xi_F(c, \pm d, \ell; -w),
\]

with $\mathcal{G}_{\mu_F}$ as in (2.25) with $\mu = \mu_F$ equal to the spectral parameters of $F$, where

\[
\Xi_F(c, \pm d, \ell; -w) := c \sum_{n_1|\ell \cap n_2 = 1} \sum_{n_2|n_1} \frac{A_F(n_2, n_1)}{n_2n_1} S \left( d\ell, \pm n_2; \frac{c\ell}{n_1} \right) \left( \frac{n_2n_1^2}{c\ell} \right)^w,
\]

which converges absolutely for $u < 0$. Moreover, we have the bounds

\[
\Phi_F(c, d, \ell; w) \ll_{F, \ell} \begin{cases} (c^3(1 + |\Im(w)|^3))^{\frac{3}{2}} \max_{a|\ell} |A_F(a, 1)| & \text{if } \Re(w) > 1, \\ (c^3(1 + |\Im(w)|^3))^{\frac{1}{2}(1 - \Re(w)) + \varepsilon} \max_{a|\ell} |A_F(a, 1)| & \text{if } 0 \leq \Re(w) \leq 1, \\ (c^3(1 + |\Im(w)|^3))^{\frac{1}{2}(1 - 2\Re(w)) + \varepsilon} \max_{a|\ell} |A_F(a, 1)| & \text{if } \Re(w) < 0. \end{cases}
\]
Here we have included the weak bounds (2.33) for Φ in vertical strips, which follow from Stirling’s formula together with the Phragmén–Lindelöf convexity principle.

The GL\(_1\) analogue of the GL\(_3\) Voronoï summation formula is simply the functional equation for the Hurwitz zeta function, which we record in the following form.

**Lemma 2.34.** For \(c \in \mathbb{N}, \ d \in (\mathbb{Z}/c\mathbb{Z})^\times, \) and \(w = u + iv \in \mathbb{C}, \) let

\[
\Phi(c, d; w) := \sum_{m=1}^{\infty} \frac{e \left( \frac{dn}{c} \right)}{m^w}.
\]

This converges absolutely for \(u > 1\) and extends meromorphically to all of \(\mathbb{C}\) with a simple pole at \(w = 1\) with residue \(1\) if and only if \(c = 1.\) We have the functional equation

\[
\Phi(c, d; w) = \sum_{\pm} G^\pm (1 - w) \Xi(c, \pm d; -w),
\]

where

\[
\Xi(c, \pm d; -w) := c^{-w} \sum_{b \in \mathbb{Z}/c\mathbb{Z}} e \left( \frac{bd}{c} \right) \sum_{m=1}^{\infty} \frac{e \left( \frac{bm}{c} \right)}{m^{1-w}},
\]

which converges absolutely for \(u < 0.\) Moreover, for any \(M > 0,\) we have the bounds

\[
\Phi(c, d; w) \ll_{u, \varepsilon} \begin{cases} 
  (1 + |v|)^2 & \text{if } u > 1, \\
  c^{1-u+\varepsilon}(1 + |v|)^{\frac{1}{2}(1-u)+\varepsilon} & \text{if } 0 \leq u \leq 1, \\
  c^{1-u+\varepsilon}(1 + |v|)^{\frac{1}{2}-u+\varepsilon} & \text{if } u < 0
\end{cases}
\]

for \(c > 1,\) while for \(c = 1,\) and \(M \geq 1,\) we have the bounds

\[
\frac{w-1}{w+M} \Phi(c, d; w) \ll_{M, u, \varepsilon} \begin{cases} 
  (1 + |v|)^2 & \text{if } u > 1, \\
  (1 + |v|)^{\frac{1}{2}(1-u)+\varepsilon} & \text{if } 0 \leq u \leq 1, \\
  (1 + |v|)^{\frac{1}{2}-u+\varepsilon} & \text{if } -M < u < 0.
\end{cases}
\]

Once more, the bounds (2.37) and (2.38) for \(\Phi\) in vertical strips follow from Stirling’s formula together with the Phragmén–Lindelöf convexity principle.

### 2.4. Applications of Voronoï Summation Formulae

During the course of the proof of GL\(_3 \times\) GL\(_2 \simeq \) GL\(_3 \times\) GL\(_1\) spectral reciprocity, a certain multiple sum of GL\(_3\) Fourier coefficients twisted by Kloosterman sums arises. The following lemma states that this sum is closely related to the inverse Mellin transform of \(L(s, \hat{F}).\)

**Lemma 2.39.** Let \(F\) be a Hecke–Maass cusp form for \(\text{SL}_3(\mathbb{Z})\) and let \(\Xi_F\) be as in (2.32). For \(\ell \in \mathbb{N}\) and \(w = u + iv \in \mathbb{C}\) with \(u < -1/2,\) we have that

\[
\sum_{c | \ell} c^{2w-1} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e \left( \frac{d}{c} \right) \Xi_F \left( c, \pm d, \frac{\ell}{c}; -w \right) = \frac{1}{2\pi i} \int_{C_0} L \left( 1 - w + z, \hat{F} \right) G^+ (z) \ell^{1+z} dz,
\]

where \(G^+\) is as in (2.27) and \(C_0\) is the contour consisting of the straight lines connecting the points \(x_0 - i\infty, x_0 - i, \delta - i, \delta + i, x_0 + i,\) and \(x_0 + i\infty,\) with \(u < x_0 < -1/2\) and \(\delta > 0.\)

**Proof.** Since \(u < 0,\) we may replace \(\Xi_F(c, \pm d, \ell/c; -w)\) by its absolutely convergent expression (2.32). The left-hand side of (2.40) is therefore equal to

\[
\sum_{n_1 | \ell} \sum_{n_2 = 1}^{\infty} \delta_{F,n_1} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e \left( \frac{d}{c} \right) S \left( \frac{\ell}{n_1} \right) \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e \left( \frac{d}{c} \right)\]

upon interchanging the order of summation. By opening up the Kloosterman sum, we find that

\[
\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e \left( \frac{d}{c} \right) S \left( \frac{\ell}{n_1} \right) = \sum_{a \in (\mathbb{Z}/n_1\mathbb{Z})^\times} e \left( \frac{1}{n_1 a \ell} \right) \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e \left( \frac{(1+n_1)ad}{c} \right).
\]
where we have inflated the sum over \( a \in (\mathbb{Z}/\ell \mathbb{Z})^\times \) to run over elements of \((\mathbb{Z}/\ell \mathbb{Z})^\times\), at the cost of multiplying through by \( \varphi(\ell n)/\varphi(\ell) \), and we have used the Ramanujan sum identity

\[
(2.42) \quad \sum_{d \mid (c,n)} e\left(\frac{dn}{c}\right) = \sum_{d \mid (c,n)} \mu\left(\frac{c}{d}\right).
\]

We insert this back into (2.41) and make the change of variables \( c \mapsto cd \), so that \( c \mid \ell \) and \( d \mid \ell \).

Since \( \sum_{d \mid \ell} \mu(c) = 1 \) if \( d = \ell \) and 0 otherwise, we deduce that (2.40) is equal to

\[
\ell \sum_{n_1 \mid \ell} \sum_{n_2=1}^{\infty} A_F(n_2, n_1) \varphi\left(\frac{\ell}{n_1}\right) \sum_{a \in (\mathbb{Z}/\mathbb{Z})^\times, n_1 \mid a-1 (\text{mod } \ell)} e\left(\pm \frac{n_1 n_2 \varpi}{\ell}\right) = \ell \sum_{n=1}^{\infty} A_F(n, 1) \frac{n}{n_1 - w} e\left(\pm \frac{n}{\ell}\right),
\]

since the congruence condition \( n_1 a \equiv -1 \pmod{\ell} \) subject to the restriction \( n_1 \mid \ell \) can only be met if \( n_1 = 1 \). We now invoke the **analytic reciprocity** identity

\[
e\left(\frac{n}{\ell}\right) = \frac{1}{2\pi i} \int_{C_0} G^+(z) \left(\frac{n}{\ell}\right)^{-z} \, dz,
\]

where \( x_0 < -1/2 \), so that this integral converges absolutely by (2.28). Interchanging the order of integration and summation, which is valid so long as \( x_0 > u \), we obtain the desired identity. □

Similarly, during the course of the proof of \( \text{GL}_4 \times \text{GL}_2 \leftrightarrow \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity, a certain double sum of Voronoï series arises. The following lemma states that this sum is closely related to sums of Kloosterman sums.

**Lemma 2.43.** Let \( F \) be a Hecke–Maaß cusp form for \( \text{SL}_3(\mathbb{Z}) \), let \( \Xi_F \) be as in (2.32), and let \( \Xi \) be as in (2.36). For \( \ell \in \mathbb{N} \) and \( w_1 = u_1 + iv_1, w_2 = u_2 + iv_2 \in \mathbb{C} \) with \( u_1, u_2 < 0 \), we have that

\[
(2.44) \quad \sum_{c \mid \ell} c^{2w_1 - 1} \sum_{d \in (\mathbb{Z}/\mathbb{Z})^\times} \Xi(c, \pm d; -w_1) \Xi_F\left(c, \pm 2d; \frac{\ell}{c}; -w_2\right) = \ell^{-w_1 - w_2} \sum_{n_1 \mid \ell} \sum_{m, n_2 = 1}^{\infty} A_F(n_2, n_1) \frac{n_1}{n_2} e\left(\frac{1}{2n_1 n_2} \varpi \right) S\left(m, \mp 1 \pm 2 n_2; \frac{\ell}{n_1}\right).
\]

**Proof.** Since \( u_1, u_2 < 0 \), we may replace both Voronoï series on the left-hand side of (2.44) with their absolutely convergent expressions and interchange the order of summation and integration, which gives

\[
(2.45) \quad \ell^{-w_1} \sum_{c \mid \ell} c^{-w_1} \sum_{m=1}^{\infty} \frac{1}{m^{w_1 - w_2}} \sum_{n_1 \mid e^{w_2}} \sum_{n_2=1}^{\infty} A_F(n_2, n_1) \frac{n_1}{n_2} e\left(\frac{1}{2n_1 n_2} \varpi \right) S\left(m, \mp 1 \pm 2 n_2; \frac{\ell}{n_1}\right) 
\]

\[
\times \sum_{b \in \mathbb{Z}/\mathbb{Z}} e\left(\frac{bm}{c}\right) \sum_{d \in (\mathbb{Z}/\mathbb{Z})^\times} e\left(\pm \frac{bd}{c}\right) S\left(\frac{d\ell}{c}, \pm 2 n_2; \frac{\ell}{n_1}\right).
\]

Opening up the Kloosterman sum and using the Ramanujan sum identity (2.42), we find that

\[
\sum_{b \in \mathbb{Z}/\mathbb{Z}} e\left(\frac{bm}{c}\right) \sum_{d \in (\mathbb{Z}/\mathbb{Z})^\times} e\left(\pm \frac{bd}{c}\right) S\left(\frac{d\ell}{c}, \pm 2 n_2; \frac{\ell}{n_1}\right) 
\]

\[
= \sum_{d \mid c} \mu\left(\frac{c}{d}\right) \sum_{a \in (\mathbb{Z}/\mathbb{Z})^\times} \sum_{b \in \mathbb{Z}/\mathbb{Z}} e\left(\pm 1 \pm 2 \frac{n_1 n_2 \varpi}{\ell}\right) e\left(\frac{bm}{c}\right) 
\]

\[
= \sum_{d \mid c} d \mu\left(\frac{c}{d}\right) \sum_{a \in (\mathbb{Z}/\mathbb{Z})^\times} e\left(\mp 1 \pm 2 \frac{n_1 n_2 \varpi}{\ell}\right) \sum_{b \equiv n_1 a (\text{mod } d)} e\left(\frac{bm}{c}\right).
\]
upon making the change of variables $a \mapsto \mp_1 a$. Making the change of variables $b \mapsto n_1 a + bd$, where now $b \in \mathbb{Z}/\frac{c}{d}\mathbb{Z}$, we see that
\[
\sum_{b \in \mathbb{Z}/\mathbb{Z}} e\left(\frac{bm}{c}\right) = \begin{cases} \frac{c}{d} e\left(\frac{mn_1 a}{c}\right) & \text{if } \frac{c}{d} | m, \\ 0 & \text{otherwise}. \end{cases}
\]

We insert this identity into (2.45) and make the change of variables $c \mapsto cd$ and $m \mapsto cm$, so that $c | \frac{d}{m}$ and $d | \ell$. Since $\sum_{d | \ell} \mu(c)$ is 1 if $d = \ell$ and 0 otherwise, we obtain the desired identity. \(\square\)

## 3. GL₃ × GL₂ \(\leftrightarrow\) GL₄ × GL₁ Spectral Reciprocity

We show the following form of spectral reciprocity: a GL₂ moment of GL₃ × GL₂ Rankin–Selberg $L$-functions is equal to a main term plus a dual moment, which is a GL₁ moment of GL₄ $L$-functions that factorise as the product of GL₃ and GL₁ $L$-functions. The proof uses the Kuznetsov and Petersson formulæ and the GL₃ Voronoï summation formula in the guise of Lemma 2.39.

**Theorem 3.1.** Let $h^{\pm}(t)$ be functions that are even, holomorphic in the horizontal strip $|\Im(t)| \leq 1/2 + \delta$ for some $\delta > 0$, and satisfy $h^{\pm}(t) \ll (1 + |t|)^{-4}$, and let $h^{\text{hol}} : 2\mathbb{N} \rightarrow \mathbb{C}$ be a sequence satisfying $h^{\text{hol}}(k) \ll k^{-4}$. Suppose additionally that the functions
\[
(3.2) \quad H^{+}(x) := (\mathcal{X}^{+} h^{+})(x) + (\mathcal{X}^{\text{hol}} h^{\text{hol}})(x), \\
(3.3) \quad H^{-}(x) := (\mathcal{X}^{-} h^{-})(x),
\]
where $\mathcal{X}^{\pm}$ and $\mathcal{X}^{\text{hol}}$ are as in (2.4) and (2.10), are such that their Mellin transforms $\widehat{H}^{\pm}(s) := \int_{0}^{\infty} H^{\pm}(x) x^{s} \frac{dx}{x}$ are holomorphic in the strip $-4 < \Re(s) < 1$, in which they satisfy the bounds $\widehat{H}^{\pm}(s) \ll (1 + |\Im(s)|)^{1/(s)-4}$. Let $F$ be a self-dual Hecke–Maaß cusp form for SL₃(ℤ). Then
\[
(3.4) \quad \sum_{f \in B_0} \sum_{\pm} \frac{L\left(\frac{1}{2}, F \otimes f\right)}{L(1, ad f)} h^{\pm}(t_f) + \sum_{f \in B_{\text{hol}}} \frac{1}{2\pi} \int_{-\infty}^{\infty} L\left(\frac{1}{2} + it, F\right) L\left(\frac{1}{2} - it, F\right) \zeta(1 + 2it) \zeta(1 - 2it) \quad h^{\pm}(t) \ dt \\
+ \sum_{f \in B_{\text{hol}}} \frac{L\left(\frac{1}{2}, F \otimes f\right)}{L(1, ad f)} h^{\text{hol}}(k_f) = L(1, F) \sum_{\pm} \int_{-\infty}^{\infty} h^{\pm}(r) d_{\text{spec}} r + L(1, F) \sum_{k=4}^{\infty} \frac{k-1}{\pi} h^{\text{hol}}(k) \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} L\left(\frac{1}{2} + it, F\right) \zeta\left(\frac{1}{2} - it\right) \mathcal{H}_{\mu_F}(t) \ dt,
\]
where for $0 < \sigma < 1$,
\[
(3.5) \quad \mathcal{H}_{\mu_F}(t) := \frac{1}{2\pi i} \int_{\sigma+i\infty}^{\sigma+i\infty} \sum_{\pm_1, \pm_2} \frac{H^{\pm_1}(s) \mathcal{G}_{\mu_F}^{\pm_2} \left(\frac{1-s}{2}\right) G^{\pm_1 \pm_2} \left(\frac{s}{2} + it\right) }{s} ds
\]
with $\mathcal{G}_{\mu}^{\pm}$ as in (2.25) and $G^{\pm}$ as in (2.27).

**Remark 3.6.** The assumptions on the decay of $h^{\pm}(t)$, $h^{\text{hol}}(k)$, and $\widehat{H}^{\pm}(s)$ are sufficient but certainly not necessary for the identity (3.4) to hold; with more work, one can impose weaker assumptions on $h^{\pm}(t)$ and $h^{\text{hol}}(k)$.

**Theorem 3.1** is a cuspidal analogue of Motohashi’s formula, as discussed in Section 1.4.1; indeed, if $F$ is replaced by a minimal parabolic Eisenstein series, then the identity (3.4) is Motohashi’s formula (with additional degenerate terms appearing on the right-hand side due to the non-cuspidality of $F$). Motohashi’s formula has previously been generalised to allow
for character twists [BHKM20, Kan22, Pet15] as well as in the more general setting of $L$-functions over number fields [Nel19b, Wu22]; Theorem 3.1 gives a new generalisation in a different direction.

The identity (3.4) has been independently proven by Kwan [Kwa23, Theorem 1.1] via different means, albeit with more stringent conditions imposed on the triple of test functions $(h^+, h^-, h^{\text{hol}})$, which are insufficiently flexible for our desired applications\footnote{In particular, Kwan’s result only allows for the possibility that $h^{\text{hol}}(k) = h^+(t) - h^-(t) = 0$ and that $h^+(t) + h^-(t)$ is the product of $\cosh \pi t \prod \Gamma_3(\frac{1}{2} \pm it) \prod \Gamma_2(\frac{1}{2} \pm it \pm 2it_2)$ and an even function that is holomorphic in a sufficiently wide horizontal strip in which it decays exponentially. On the other hand, Kwan’s proof is valid more generally for arbitrarily Hecke–Maaß cusp forms on $\text{SL}_3(\mathbb{Z})$, not just self-dual forms. The proof that we give of the identity (3.4) also remains valid for non-self-dual forms (with an additional term appearing on the right-hand side of (3.4)), though the ensuing identity is no longer relevant for the applications that we have in mind.}. Approximate forms of the identity (3.4) (due to the usage approximate functional equations) go back to work of Li [Li11, Theorem 1.1], who showed that with a particular choice of triple $(h^+, h^-, h^{\text{hol}})$, one can prove subconvex bounds for $L(1/2, F \otimes f)$ and $L(1/2 + it, F)$. The state of the art in this regard is the pair of subconvex bounds [LNQ23, Corollary 1.2]

\begin{equation}
L \left( \frac{1}{2}, F \otimes f \right) \ll_{F, \varepsilon} t_f^{\frac{3}{2} + \varepsilon}, \quad L \left( \frac{1}{2} + it, F \right) \ll_{F, \varepsilon} (1 + |t|)^{\frac{3}{2} + \varepsilon}.
\end{equation}

The existence of the identity (3.4) answers in the affirmative a speculation of Lin, Nunes, and Qi [LNQ23, Section 1.4], for one can choose a triple of test functions $(h^+, h^-, h^{\text{hol}})$ in such a way that $h^-(t)$ localises to the interval $[T - U, T + U]$; upon determining the support and size of the transform $H_{\mu_f}(t)$, one recovers an upper bound roughly of the form

\begin{equation}
\sum_{T - U \leq t \leq T + U} \frac{L \left( \frac{1}{2}, F \otimes f \right)}{L(1, \text{ad } f)} + \frac{1}{2\pi} \int_{T - U \leq |t| \leq T + U} \frac{L \left( \frac{1}{2} + it, F \right) L \left( \frac{1}{2} - it, F \right)}{\zeta(1 + 2it)\zeta(1 - 2it)} \, dt \ll_F TU + U \int_{-\frac{T}{U}}^{\frac{T}{U}} \left| L \left( \frac{1}{2} + it, F \right) \zeta \left( \frac{1}{2} - it \right) \right| \, dt.
\end{equation}

In conjunction with the Montgomery–Vaughan mean value theorem for Dirichlet polynomials [MV74, Corollary 3] (see Lemma 3.8 below), this can be used to give an alternate proof of the subconvex bounds (3.7).

Before proceeding to the proof of Theorem 3.1, we must include the following weak bounds for the second moment of the Riemann zeta function and for the $L$-function of a Hecke–Maaß cusp form $F$ for $\text{SL}_3(\mathbb{Z})$. These bounds will be required in the proof of Theorem 3.1.

**Lemma 3.8.** We have the bounds

\begin{align}
&\int_{-\frac{2U}{T}}^{\frac{2U}{T}} |\zeta(\sigma + it)|^2 \, dt \ll \varepsilon U^{1 + \varepsilon} \quad \text{for } \sigma \geq \frac{1}{2}, \\
&\int_{-\frac{2U}{T}}^{\frac{2U}{T}} |L(\sigma + it, F)|^2 \, dt \ll_{F, \varepsilon} \begin{cases} U^{3(1 - \sigma) + \varepsilon} & \text{if } \frac{1}{2} \leq \sigma \leq \frac{2}{3}, \\
U^{1 + \varepsilon} & \text{if } \sigma \geq \frac{2}{3}, \end{cases}
\end{align}

where $F$ is a Hecke–Maaß cusp form for $\text{SL}_3(\mathbb{Z})$.

Under the assumption of the generalised Lindelöf hypothesis, the bound (3.9) is essentially optimal but (3.10) falls shy of the conjecturally optimal upper bound $O_{F, \varepsilon}(U^{1 + \varepsilon})$ when $1/2 \leq \sigma < 2/3$.

**Proof.** This follows by using the approximate functional equation [IK04, Theorem 5.3] to write $\zeta(\sigma + it)$ and $L(\sigma + it, F)$ in terms of Dirichlet polynomials and then invoking the Montgomery–Vaughan mean value theorem for Dirichlet polynomials [MV74, Corollary 3].
Proof of Theorem 3.1. Let \( w = u + iv \) be a complex variable. Given \( f \in \mathcal{B}_0 \) or \( f \in \mathcal{B}_{\text{hol}} \) with Hecke eigenvalues \( \lambda_f(n) \), the \( \GL_3 \times \GL_2 \) Rankin–Selberg \( L \)-function \( L(w, F \otimes f) \) has the Dirichlet series expansion

\[
L(w, F \otimes f) = \sum_{\ell, n=1}^{\infty} \frac{A_F(\ell, n) \lambda_f(n)}{\ell^{2w} n^w}
\]

for \( u > 1 \). Similarly, let \( E(z, 1/2 + it) \) be the real analytic Eisenstein series on \( \Gamma \setminus \mathbb{H} \) with Hecke eigenvalues \( \lambda(n, t) := \sum_{ab=n} a^{it} b^{-it} \); then for \( u > 1 \), we have the identity

\[
L(w + it, F)L(w - it, F) = \sum_{\ell, n=1}^{\infty} \frac{A_F(\ell, n) \lambda(n, t)}{\ell^{2w} n^w}.
\]

With this in mind, we initially assume that \( 5/4 < u \leq 4/3 \) and multiply the Kuznetsov and Petersson formulæ, (2.2) and (2.8), with \( m = 1 \) by \( A_F(\ell, n)\ell^{-2w} n^{-w} \), then sum over \( \ell, n \in \mathbb{N} \). Adding the Petersson formula to the sum of the same sign and opposite sign Kuznetsov formulæ, we obtain the identity

\[
L(w(1, \ad f), h^\pm(k_f)) = L(2w, F) h^\pm + L(2w, \tilde{F}) h^\text{hol}^\pm h^\text{hol}
\]

\[
\sum_{c, \ell=1}^{\infty} \frac{1}{c^{2w}} \sum_{\sigma_0 + i\sigma < \infty} \frac{H^\pm(s)}{n^{1+w}} S(1, \pm n; c) \ ds.
\]

Here \( \mathcal{N}^+ \) and \( \mathcal{N}^\text{hol} \) are as in (2.4) and (2.9), while \( S(m, n; c) \) denotes the Kloosterman sum, as in (2.3). This identity is valid for \( 2 - 2u < \sigma_0 < -1/2 \), which is a nonempty region provided that \( u > 5/4 \). We have used the Mellin inversion formula to write

\[
H^\pm(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} H^\pm(s) x^s \ ds
\]

for \( -4 < \sigma_0 < 1 \), since in this range we have the bounds

\[
H^\pm(s) \ll \sigma (1 + |\tau|)^{\sigma-4}.
\]

By the Weil bound for Kloosterman sums, the sum over \( c \) converges absolutely since \( \sigma_0 < -1/2 \), while the sum over \( n \) converges absolutely since \( \sigma_0 > 2 - 2u \); the sum over \( \ell \) converges since \( u > 1 \).

In anticipation of future simplifications, we write \( \ell' = \ell c \), relabel \( \ell' \) as \( \ell \), and open up the Kloosterman sum, so that the last term on the right-hand side of (3.11) is

\[
\sum_{c, \ell=1}^{\infty} \frac{1}{c^{2w}} \sum_{\sigma_0 + i\sigma < \infty} \frac{H^\pm(s)}{n^{1+w}} \sum_{d(2, c)} e \left( \frac{d}{c} \right) \Phi_F \left( c, \pm d, \frac{\ell}{c}; \frac{s}{2} + w \right) \ ds,
\]

where the Voronoï series \( \Phi_F \) is as in (2.30).

The left-hand side of (3.11) extends holomorphically to \( u \geq 1/2 \), since the convexity bound for \( L(w, F \otimes f) \) and \( L(w + it, F) \) together with the assumptions \( h^\pm(r) \ll (1 + |\tau|)^{-4} \) and \( h^\text{hol}(k) \ll k^{-4} \) ensure that each term on the left-hand side is absolutely convergent for all \( u \geq 1/2 \). The holomorphic extension to \( w = 1/2 \) is precisely the left-hand side of (3.4), since if \( f \in \mathcal{B}_0 \), the root number of \( L(w, F \otimes f) \) is \( \epsilon_f \), and hence \( L(1/2, F \otimes f) = 0 \) when \( \epsilon_f = -1 \).

We shall show that the right-hand side of (3.11) extends holomorphically to \( w = 1/2 \) and is equal to the right-hand side of (3.4). To begin, we shift the contour of integration of (3.13) to \( \Re(s) = \sigma_1 \) with \( -4 < \sigma_1 < -1 - 2u \), which is a nonempty region since \( u < 3/2 \); due to the bounds (2.33) for \( \Phi_F \) and (3.12) for the Mellin transforms of \( H^\pm \), the ensuing integral is
absolutely convergent. We then use the Voronoi summation formula (2.31). Via the identity (2.40), we deduce that for \( \Re(s) = \sigma_1 \), the integrand in (3.13) is equal to

\[
\sum_{\pm_1, \pm_2} H_{\mu_F}^{\pm}(s) \mathcal{H}_{\mu_F}^{\pm}(1 - \frac{s}{2} - w) \frac{1}{2\pi i} \int_{C_0} L \left( 1 - \frac{s}{2} - w + z, \tilde{F} \right) G^{\tau_{1,2}}(z) e^{\frac{z}{2} - w + z} dz
\]

with \( C_0 \) the contour defined in Lemma 2.39 such that \( \sigma_1/2 + u < x_0 < -1/2 \) and \( 0 < \delta < \sigma_1/2 + 3u - 2 \). With this replacing the integrand of (3.13) and with the contour of integration shifted to \( \Re(s) = \sigma_1 \), the resulting expression is absolutely convergent, and so we may interchange the order of summation and integration. Thus we see that (3.13) is equal to

\[
\frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \sum_{\pm_1, \pm_2} \mathcal{H}_{\mu_F}^{\pm}(s) \mathcal{H}_{\mu_F}^{\pm}(1 - \frac{s}{2} - w) \times \frac{1}{2\pi i} \int_{C_0} L \left( 1 - \frac{s}{2} - w + z, \tilde{F} \right) \zeta \left( \frac{s}{2} + 3w - z - 1 \right) G^{\tau_{1,2}}(z) dz ds.
\]

We are ensured absolute convergence of this double integral by the bounds (3.12) for the Mellin transform of \( H^{\pm} \), (2.26) for \( \mathcal{H}_{\mu_F}^{\pm}(s) \), and (2.28) for \( G^{\pm}(s) \). Thus we may make the change of variables \( z \mapsto s/2 + 3w - z - 3/2 \) and interchange the order of integration, yielding

\[
\frac{1}{2\pi i} \int_{x_1 - i\infty}^{x_1 + i\infty} \sum_{C_1} \mathcal{H}_{\mu_F}^{\pm}(s) \mathcal{H}_{\mu_F}^{\pm}(1 - \frac{s}{2} - w) G^{\tau_{1,2}}(\frac{s}{2} + 3w - z - \frac{3}{2}) ds.
\]

Here \( x_1 = -x_0 + \sigma_1/2 + 3u - 3/2 \), so that \( 1 < x_1 < 2u - 3/2 \), and \( C_1 \) is the contour consisting of the straight lines connecting the points \( \sigma_1 - i\infty, 2z - 6w + 2x_0 + 3 - 2i, 2z - 6w + 2\delta + 3 - 2i, 2z - 6w + 2\delta + 3 + 2i, 2z - 6w + 2\delta + 3 \) and \( \sigma_1 + i\infty \). Finally, we may straighten the inner contour of integration from \( C_1 \) to the vertical line \( \Re(s) = \sigma_2 \) with \( 2x_1 - 6u + 3 < \sigma_2 < 2 - 2u \).

We now begin the process of analytically continuing this expression to \( w = 1/2 \). Suppose that \( w \) lies in a compact subset \( K \) of the closed vertical strip \( 1/2 \leq \Re(w) \leq 4/3 \). Then by Stirling’s formula and (3.12), the integrand in (3.14) is meromorphic as a function of \( s \in \mathbb{C} \) with simple poles at \( s = 2z - 6w + 3 - 2\ell \) for \( \ell \in \mathbb{N}_0 \) with residues of size \( O_F(K) (1 + |y|)^{-\frac{x - 4 + \ell}{2}} \) for \( z = x + iy \), while for \( s = \sigma + i\tau \) a bounded distance away from such a pole, the integrand is

\[
O_F(x, \sigma, K) \left( (1 + |\tau|)^{-\frac{1}{2}(\sigma + 6u + 5)} (1 + |\tau - 2y|)^{-\frac{1}{2}(\sigma - 6u - 2x - 3)} \right).
\]

Thus by shifting the contour of integration of the inner integral to the left to \( \Re(s) = \sigma_3 \) with \( \sigma_3 = 2x - 6u + 3 + \alpha \) for \( 6u - 2x - 7 < \alpha < -8/3 \), which picks up residues at \( s = 2z - 6w + 3 - 2\ell \) for \( \ell \in \mathbb{N}_0 \) from the poles of \( G^{\tau_{1,2}}(s/2 + 3w - z - 3/2) \), and breaking up the integral into the three ranges \( |\tau| \leq |y|, |y| \leq |\tau| \leq 3|y|, \) and \( |\tau| \geq 3|y| \), we find that \( \mathcal{H}_{\mu_F}(w, z) \) is holomorphic as a function of \( z \in \mathbb{C} \) and satisfies the bound

\[
\mathcal{H}_{\mu_F}(w, z) \ll F, K, \alpha \left( 1 + |y| \right)^{\frac{x}{2}} + (1 + |y|)^{-x - \frac{5}{2} - 4}.
\]

Next, we observe that by the Cauchy–Schwarz inequality and the bounds (3.9) and (3.10),

\[
\int_U^{2U} \sum_{\pm} \left| L \left( 2w - x + iy - \frac{1}{2}, \tilde{F} \right) \zeta \left( \frac{1}{2} + x + iy \right) \right| dy \ll F, K, x \left\{ \begin{array}{ll} U^{1+\varepsilon} & \text{if } 0 \leq x \leq 2u - \frac{7}{6}, \\ U^{\frac{11}{4} - 3u + \frac{3\varepsilon}{2}} & \text{if } \max \left\{ 2u - \frac{7}{6}, 0 \right\} \leq x \leq 2u - 1. \end{array} \right.
\]

Thus for \( w \in K \), we may shift the outer contour to \( \Re(z) = x_2 \) with \( x_2 = 2u - 1 \), since the bounds (3.15) and (3.16) ensure that the resulting integral converges absolutely. For \( u < 3/4 \),
this introduces an additional term
\[ L\left(2w - 1, \tilde{F}\right) \mathcal{H}_{\mu_F}\left(w, \frac{1}{2}\right) \]
arising from the residue at \( z = 1/2 \) of the outer integral, since \( \zeta(1/2 + z) \) has a simple pole at \( z = 1/2 \) with residue 1. Shifting the contour of integration in (3.14) from \( \Re(s) = \sigma_2 \) with \( 1 - 2u < \sigma_2 < 2 - 2u \) to \( \Re(s) = \sigma_4 \) with \(-4 < \sigma_4 < 2 - 6u \), which picks up residues at \( s = 4 - 6w \) and \( s = 2 - 6w \) from the poles of \( G^{+1/2}(s/2 + 3w - 2) \), we see that this additional term can be written as
\[
2L(2 - 2w, F) \sum_{\pm} \overline{H}(4 - 6w) - L\left(2w - 1, \tilde{F}\right) \sum_{\pm_1, \pm_2} \frac{\overline{H}(2 - 6w)\mathcal{G}_{\mu_F}(2w)}{\pm_1, \pm_2} + L\left(2w - 1, \tilde{F}\right) \frac{1}{2\pi i} \int_{\sigma_4 - i\infty}^{\sigma_4 + i\infty} \sum_{\pm_1, \pm_2} \overline{H}(s)\mathcal{G}_{\mu_F}(1 - \frac{s}{2} - w) G^{+1/2}\left(\frac{s}{2} + 3w - 2\right) ds
\]
via the functional equation \( L(2 - 2w, F) = \sum_{\pm} \mathcal{G}_{\mu_F}(2w - 1)L(2w - 1, \tilde{F}) \). All three of these terms extend holomorphically to \( w = 1/2 \); furthermore, the holomorphic extensions to \( w = 1/2 \) of the second and third terms vanish since \( L(0, \tilde{F}) = 0 \) due to the self-duality of \( F \), while the holomorphic extension to \( w = 1/2 \) of the first term is equal to
\[
2L(1, F) \sum_{\pm} \overline{H}(1) = L(1, F) \int_{-\infty}^{\infty} h^{-}(r) \, \text{d} \sigma r + L(1, F) \sum_{k=2}^{\infty} \frac{k - 1}{2\pi^2} \, t^{-k} h^{\text{hol}}(k).
\]
Finally, the main term
\[
\frac{1}{2\pi i} \int_{x_2 - i\infty}^{x_2 + i\infty} \frac{L\left(2w - z - \frac{1}{2}, \tilde{F}\right)}{\zeta\left(\frac{1}{2} + z\right)} \mathcal{H}_{\mu_F}(w, z) \, dz
\]
extends holomorphically to \( w = 1/2 \), where it becomes
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{L\left(\frac{1}{2} + it, \tilde{F}\right)}{\zeta\left(\frac{1}{2} - it\right)} \mathcal{H}_{\mu_F}(t) \, dt
\]
with \( \mathcal{H}_{\mu_F}(t) \) as in (3.5) upon writing \( z = -it \). \( \Box \)

4. \( \text{GL}_4 \times \text{GL}_2 \leftrightarrow \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity

We show the following form of spectral reciprocity: a \( \text{GL}_2 \) moment of \( \text{GL}_4 \times \text{GL}_2 \) Rankin–Selberg \( L \)-functions is equal to a main term plus a dual moment, which is a \( \text{GL}_2 \) moment of \( \text{GL}_4 \times \text{GL}_2 \) \( L \)-functions. The proof uses the Kuznetsov and Petersson formulæ and the \( \text{GL}_3 \) Voronoï summation formula in the guise of Lemma 2.43. The archetypal version of this form of spectral reciprocity is a reciprocity formula for the fourth moment of \( L(1/2, f) \) due to Kuznetsov [Kuz89, Kuz99], though the initial proof was incomplete in parts and was subsequently completed by Motohashi [Mot03]. With the goal of proving Proposition 1.9, we prove a new form of \( \text{GL}_4 \times \text{GL}_2 \leftrightarrow \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity. In place of \( L(1/2, f)^4 \), our identity instead involves \( L(1/2, f) L(1/2, F \otimes f) \), where \( F \) is a self-dual Hecke–Maaß cusp form for \( \text{SL}_3(\mathbb{Z}) \).

**Theorem 4.1.** Let \( h^{\pm}(t) \) be functions that are even, holomorphic in the horizontal strip \(|\Im(t)| \leq 1/2 + \delta \) for some \( \delta > 0 \), and satisfy \( h^{\pm}(t) \ll (1 + |t|)^{-5} \), and let \( h^{\text{hol}} : \mathbb{N} \rightarrow \mathbb{C} \) be a sequence satisfying \( h^{\text{hol}}(k) \ll k^{-5} \). Suppose additionally that the functions \( H^{\pm} \) given by (3.2) and (3.3) are such that their Mellin transforms \( \overline{H}^{\pm}(s) := \int_{0}^{\infty} H^{\pm}(x)x^s \, dx \) are holomorphic in the strip \(-5 < \Re(s) < 1 \), in which they satisfy the bounds \( \overline{H}^{\pm}(s) \ll (1 + |\Im(s)|)^{\Re(s) - 5} \). Let \( F \) be a self-dual Hecke–Maaß cusp form for \( \text{SL}_3(\mathbb{Z}) \). Then
\[
\sum_{\pm} \sum_{f \in B_0} \frac{L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, F \otimes f\right)}{L(1, \text{ad } f)} h^{\pm}(t_f)
\]

(4.2)
Proof of Theorem 4.1. Let $L(1/2, f)$, where $F$ is a dihedral Hecke–Maaß cusp form [HK20, Proposition 7.1], with applications towards subconvexity; see in particular [AK18, BlK19a, BlK19b, Nun23, Zac19, Zac21]. These level-aspect versions have also been generalised to higher rank spectral reciprocity formulæ [HN21, M12].

Remark 4.5. The assumptions on the decay of $h^\pm(t)$, $h^{\text{hol}}(k)$, and $\overline{H}^\pm(s)$ are sufficient but certainly not necessary for the identity (4.2) to hold; with more work, one can impose weaker assumptions on $h^\pm(t)$ and $h^{\text{hol}}(k)$ for this identity to remain valid.

Theorem 4.1 is a cuspidal analogue of Kuznetsov’s formula for the fourth moment of $L(1/2, f)$, as discussed in Section 1.4.2; indeed, if $F$ is replaced by a minimal parabolic Eisenstein series, then the identity (4.2) is Kuznetsov’s formula (with additional degenerate terms appearing on the right-hand side due to the non-cuspidality of $F$). The authors have previously proven an analogue of the identity (4.2) with $F$ replaced by a maximal parabolic Eisenstein series induced from a dihedral Hecke–Maaß cusp form [HK20, Proposition 7.1], with applications towards $L^4$-norm asymptotic formulæ for dihedral Maass cusp forms. More generally, Blomer, Li, and Miller have proven a completely cuspidal version of (4.2) for the first moment of $L(1/2, G \otimes f)$, where $G$ is a Hecke–Maaß cusp form for $\text{SL}_4(\mathbb{Z})$ [BLM19, Theorem 1].

We briefly mention that there additionally exist level-aspect versions of $\text{GL}_4 \times \text{GL}_2 \sim \text{GL}_4 \times \text{GL}_2$ spectral reciprocity, which have striking applications towards subconvexity; see in particular [AK18, BlK19a, BlK19b, Nun23, Zac19, Zac21]. These level-aspect versions have also been generalised to higher rank spectral reciprocity formulæ [HN21, M12].

**Proof of Theorem 4.1.** Let $w_1 = u_1 + iv_1$ and $w_2 = u_2 + iv_2$ be complex variables. We initially assume that $5/4 < u_1 < u_2 \leq 4/3$ and multiply the Kuznetsov and Petersson formulæ, (2.2) and (2.8), by $A_F(t, n)m_{w_1}e^{-2w_2}n_{w_2}$, then sum over $\ell, m, n \in \mathbb{N}$. Adding the Petersson formulæ to the sum of the same sign and opposite sign Kuznetsov formulæ, we obtain the identity

\[
+ \sum_{f \in \mathcal{B}_0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{2} + it \right) \psi \left( \frac{1}{2} - it, F \right) L \left( \frac{1}{2} - it, F \right) h^\pm(t) \, dt
\]

where for $0 < \sigma < 1$,

\[
\tilde{h}^\pm(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \sum_{\pm, \pm, 2} \overline{H}^\pm(s) \mathcal{J}^\pm(s) G^\pm \left( \frac{1 - s}{2} \right) \mathcal{G}^\pm \left( \frac{1 - s}{2} \right) \, ds,
\]

\[
\overline{h}^{\text{hol}}(k) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \sum_{\pm, \pm, 2} \overline{H}^\pm(s) \overline{\mathcal{J}}^\pm(s) G^\pm \left( \frac{1 - s}{2} \right) \mathcal{G}^\pm \left( \frac{1 - s}{2} \right) \, ds,
\]

with $\mathcal{J}^\pm$ and $\overline{\mathcal{J}}^\pm$ as in (2.15), (2.16), and (2.17), $\mathcal{G}^\pm$ as in (2.25), and $G^\pm$ as in (2.27).
\[ L(w_1, f)L(w_2, F \otimes f) \frac{h_{h_{\text{hol}}}(k_f)}{L(1, \text{ad} f)} = \frac{L(2w_2, \tilde{F})L(w_1 + w_2, F)}{\zeta(w_1 + 3w_2)} + \frac{L(2w_2, \tilde{F})L(w_1 + w_2, F)}{\zeta(w_1 + 3w_2)} + \sum_{c, \ell = 1}^\infty \frac{1}{\ell^{2w_2} 2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \sum_{c, \ell} \sum_{m,n=1}^\infty \frac{AF(\ell, n)}{m_2 + w_1 n} A_F(s, \pm c) d(s). \]

For the diagonal terms, we have used the Hecke relations

\[ AF(\ell, n) = \sum_{d|d(\ell, n)} \mu(d) A_F \left( \frac{\ell}{d}, 1 \right) A_F \left( 1, \frac{n}{d} \right) \]

and made the change of variables \( \ell \mapsto d \ell \) and \( n \mapsto d n \) in order to see that

\[ \sum_{\ell, n=1}^\infty \frac{AF(\ell, n)}{\ell^{2w_2} n^{w_1 + w_2}} = \frac{L(2w_2, \tilde{F})L(w_1 + w_2, F)}{\zeta(w_1 + 3w_2)}. \]

The identity (4.6) is valid for \( 2 - 2u_1 < \sigma_0 < -1/2 \), which is a nonempty region provided that \( u_1 > 5/4 \). Here we have used the Mellin inversion formula to write

\[ H^\pm(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{H^\pm(s) x^{-s}}{d(s)} ds \]

for \(-5 < \sigma_0 < 1\), since in this range we have the bounds

\[ H^\pm(s) \ll_{\sigma} (1 + |\tau|)^{\sigma - 5}. \]

By the Weil bound for Kloosterman sums, the sum over \( c \) converges absolutely since \( \sigma_0 < -1/2 \), while the sums over \( m, n \) converge absolutely since \( \sigma_0 > 2 - 2u_1 > 2 - 2u_2 \); the sum over \( \ell \) converges since \( u_2 > 1 \).

In anticipation of future simplifications, we write \( \ell' = \ell \ell \), relabel \( \ell' \) as \( \ell \), and open up the Kloosterman sum, so that the last term on the right-hand side of (4.6) is

\[ \sum_{\ell = 1}^\infty \frac{1}{\ell^{2w_2} 2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \sum_{c, \ell} \Phi \left( c, d, \frac{s}{2} + w_1 \right) \Phi \left( c, \pm d, \frac{s}{2} + w_2 \right) ds, \]

where the Voronoï series \( \Phi \) and \( \Phi_F \) are as in (2.35) and (2.30).

The left-hand side of (4.6) extends holomorphically to \( w_1, w_2 \geq 1/2 \), since the convexity bounds for \( L(w_1, f) \), \( L(w_2, F \otimes f) \), \( \zeta(w_1 + it) \), and \( L(w_2 + it, F) \) together with the assumptions \( h^\pm(r) \ll (1 + |r|)^{-5} \) and \( h^\text{hol}(k) \ll k^{-5} \) ensure that the left-hand side converges for all \( w_1, w_2 \geq 1/2 \). The holomorphic extension to \( w_1 = w_2 = 1/2 \) is precisely the left-hand side of the desired identity (4.2), since if \( f \in B_0 \), the root number of \( L(w, F \otimes f) \) is \( \epsilon_f \), and hence \( L(1/2, F \otimes f) = 0 \) when \( \epsilon_f = -1 \). Note that for \( \Re(w_1) < 1 \), additionally polar divisors arise via shifting the contour in the integration over \( t \in \mathbb{R} \) in the second term of (4.6), since the integrand has poles at \( t = \pm i(1 - w_1) \). The holomorphic extension of these polar divisors vanishes when \( w_1 = w_2 = 1/2 \), however, since \( L(0, F) = 0 \) as \( F \) is self-dual.

We shall show that the right-hand side of (4.6) extends holomorphically to \( w_1 = w_2 = 1/2 \) and is equal to the right-hand side of the desired identity. To begin, we shift the contour of integration of the third term on the right-hand side of (4.6) to \( \Re(s) = \sigma_1 \) with \( 5/2 - u_1 - 3u_2 < \sigma_1 < -2u_2 \), which is a nonempty region since \( u_2 > u_1 > 5/4 \); due to the bounds (2.33), (2.37), and (2.38) the ensuing integral is absolutely convergent. The only pole that we encounter along the way is at \( s = 2(1 - w_1) \) when \( c = 1 \). For \( u_2 > u_1 \), the resulting residue is

\[ \sum_{\ell = 1}^\infty \frac{1}{\ell^{2w_2} 2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \sum_{c, \ell} \Phi \left( c, d, \frac{s}{2} + w_1 \right) \Phi \left( c, \pm d, \frac{s}{2} + w_2 \right) ds, \]
This extends holomorphically to $w_1 = w_2 = 1/2$, where it is equal to
\[
\frac{L(1, \tilde{F})L(1, F)}{\zeta(2)} \int_{-\infty}^{\infty} h_-(r) \, dr + \frac{L(1, \tilde{F})L(1, F)}{\zeta(2)} \sum_{k=2}^{\infty} \frac{k-1}{2\pi^2} i^{-k} h_{\text{hol}}(k),
\]
namely the first two terms on the right-hand side of (4.2).

Now we wish to reexpress the remaining Kloosterman term where $\sigma_0$ has been replaced by $\sigma_1$, with $5/2 - u_1 - 3u_2 < \sigma_1 < -2u_2$. We apply the Voronoï summation formulae to both Voronoï series, yielding
\[
\sum_{\ell=1}^{\infty} \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \sum_{\pm} H^\pm_1(s) \sum_{\pm} G^\pm_2 \left(1 - \frac{s}{2} - w_1\right) \sum_{\pm} \frac{g_{\mu,F}^\pm}{s} \left(1 - \frac{s}{2} - w_2\right) \times \sum_{c\ell} e^{s+2u_2 - 1} \sum_{d \in (\mathbb{Z}/\ell\mathbb{Z})^*} \Xi(c, \pm 2d, -\frac{s}{2} - w_1) \Xi_F(c, \pm 1 \pm 3d, \frac{s}{\ell} - \frac{s}{2} - w_2) \, ds.
\]
Inserting the identity (2.44), interchanging the order of summation and integration, and making the change of variables $\ell \mapsto \ell n_1$, we find that this is equal to
\[
\sum_{\pm} \sum_{m,n_1,n_2} A_F(n_2, n_1) \sum_{m,n_1,n_2} S(m, \pm n_2; \ell) \tilde{H}_{u_1,u_2}^\pm \left(\frac{\sqrt{mn_2}}{\ell}\right),
\]
where
\[
\tilde{H}_{u_1,u_2}^\pm(x) := \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \sum_{\pm} H^\pm_1(s) G^\pm_2 \left(1 - \frac{s}{2} - w_1\right) g_{\mu,F}^\pm \left(1 - \frac{s}{2} - w_2\right) x^{s+w_1+3w_2-2} \, ds.
\]
Here the integral converges since $\sigma_1 > -3 - u_1 - 3u_2$ via the bounds (4.8), (2.28), and (2.26), the sum over $m$ converges since $\sigma_1 < -2u_1$, the sum over $n_1$ converges since $u_1 + u_2 > 1$, the sum over $n_2$ converges since $\sigma_1 < -2u_2$, and the sum over $\ell$ converges since $\sigma_1 > 5/2 - u_1 - 3u_2$ via the Weil bound for Kloosterman sums.

Now we apply the Kloosterman summation formula, (2.12). In order to do so, we require that there exists some $\delta > 0$ such that for each $j \in \{0, 1, 2, 3\}$,
\[
x^{j} \frac{d^j}{dx^j} \tilde{H}_{u_1,u_2}^\pm(x) \ll \begin{cases} x^{\frac{1}{2} + \delta} & \text{for } x \leq 1, \\ x^{-1 - \delta} & \text{for } x \geq 1. \end{cases}
\]
This can readily be seen by differentiating under the integral sign and shifting the contour of integration to $\Re(s) = \sigma_2$ with $\sigma_2 = 5/2 + \delta - u_1 - 3u_2$ for $x \leq 1$ and shifting the contour of integration to $\Re(s) = \sigma_3$ with $\sigma_3 = 1 - \delta - u_1 - 3u_2$ for $x \geq 1$.

As in [Mot97, Proofs of Theorems 2.3 and 2.4], these conditions on $\tilde{H}_{u_1,u_2}^\pm$ imply that
\[
\tilde{h}_{u_1,u_2}^\pm(t) := (\mathcal{L} \tilde{H}_{u_1,u_2}^\pm)(t) \ll (1 + |t|)^{-\frac{1}{2} + \delta},
\]
\[
\tilde{h}_{u_1,u_2}^\text{hol}(k) := (\mathcal{L} h_{\text{hol}} \tilde{H}_{u_1,u_2}^\pm)(k) \ll k^{-\frac{1}{2} + \delta}.
\]
Via the Weyl law, this ensures the absolute convergence of the ensuing expression, and so we may interchange the order of summation in order to arrive at
\[
(4.9) \quad \sum_{\pm} \sum_{f \in B_0} \epsilon_f \int_{\mathbb{R}} \frac{L(3u_2-w_1, f)}{L(1, ad f)} L \left(\frac{w_1+u_2}{2}, \tilde{F} \otimes f\right) \tilde{h}_{u_1,u_2}^\pm(t_f) + \sum_{\pm} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta(3u_2-w_1 + it)}{\zeta(1+2it)} \zeta(1+2it) \frac{\zeta(3u_2-w_1 - it)}{\zeta(1-2it)} \tilde{h}_{u_1,u_2}^\pm(t) \, dt.
\]
It remains to holomorphically extend this expression to \( w_1 = w_2 = 1/2 \), which gives us the last three terms on the right-hand side of (4.2). Note that for \( 3\Re(w_2) - \Re(w_1) < 2 \), additionally polar divisors arise via shifting the contour in the integration over \( t \in \mathbb{R} \) in the second term of (4.9), since the integrand has poles at \( t = \pm i(1 - 3(w_2 - w_1))/2 \). The holomorphic extension of these polar divisors vanishes when \( w_1 = w_2 = 1/2 \), however, since \( L(0, F) = 0 \) as \( F \) is self-dual. We are left with showing that the functions \( \tilde{h}^\pm_{w_1,w_2}(t) \) and \( \tilde{h}^\text{hol}_{w_1,w_2}(k) \) extend holomorphically to \( w_1 = w_2 = 1/2 \) and have sufficiently rapid decay to ensure the absolute convergence of the three terms in (4.9).

We first observe that by Parseval’s identity for the Mellin transform, we have that

\[
\begin{align*}
\tilde{h}^{\pm}_{w_1,w_2}(t) &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{\pm_1,\pm_2} \hat{H}^{\pm_{\pm_1}}_{\pm_{\pm_2}}(s) \mathcal{J}_{t}^{\pm_{\pm_2}}(s + w_1 + 3w_2 - 2) \\
& \quad \times G^{\pm_{\pm_2}} \left(1 - \frac{s}{2} - w_1\right) \mathcal{G}^{\pm_{\pm_1} \pm_{\pm_2}}_{\mu_F} \left(1 - \frac{s}{2} - w_2\right) ds,
\end{align*}
\]

\[
\tilde{h}^\text{hol}_{w_1,w_2}(k) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{\pm_1,\pm_2} \hat{H}^{\pm_{\pm_1}}_{\pm_{\pm_2}}(s) \mathcal{J}_{k}^{\pm_{\pm_2}}(s + w_1 + 3w_2 - 2) \\
& \quad \times G^{\pm_{\pm_2}} \left(1 - \frac{s}{2} - w_1\right) \mathcal{G}^{\pm_{\pm_1} \pm_{\pm_2}}_{\mu_F} \left(1 - \frac{s}{2} - w_2\right) ds
\]

for \( 2 - u_1 - 3u_2 < \sigma < 2 - 2u_2 \). As a function of \( s = \sigma + i\tau \), the functions

\[
\sum_{\pm_2} G^{\pm_{\pm_2}} \left(1 - \frac{s}{2} - w_1\right) \mathcal{G}^{\pm_{\pm_1} \pm_{\pm_2}}_{\mu_F} \left(1 - \frac{s}{2} - w_2\right)
\]

extend meromorphically to the entire complex plane and are holomorphic in the left-half plane \( \sigma < 2 - 2u_2 \). By (2.28) and (2.26), we have the bounds

\[
\sum_{\pm_2} \mathcal{G}^{\pm_{\pm_2}} \left(1 - \frac{s}{2} - w_1\right) \mathcal{G}^{\pm_{\pm_1} \pm_{\pm_2}}_{\mu_F} \left(1 - \frac{s}{2} - w_2\right) \ll F,K,\sigma \left((1 + |\tau|)^{2-2\sigma-u_1-3u_2}\right)
\]

for \( w_1, w_2 \) in a compact subset \( K \) of the vertical strip \( 1/2 \leq \Re(w_1), \Re(w_2) \leq 4/3 \). Recalling the bounds (2.19), (2.20), and (2.22) for the Mellin transforms of \( \mathcal{J}^{\pm} \) and \( \mathcal{J}_{k}^{\text{hol}} \), we see that the integrands are integrable along the vertical line \( \Re(s) = \sigma \) provided that \( \sigma > -5 \). In particular, as functions of the complex variables \( w_1 = u_1 + iv_1 \) and \( w_2 = u_2 + iv_2 \), these integrals extend holomorphically to \( w_1 = w_2 = 1/2 \).

By the convexity bound and the Weyl law, in order to ensure the absolute convergence of each of the three terms in (4.9), it suffices to show that there exists some \( \delta > 0 \) such that \( \tilde{h}^{\pm}_{w_1,w_2}(t) \ll F,K \left(1 + |t|\right)^{u_1+3u_2-6-\delta} \) and that \( \tilde{h}^\text{hol}_{w_1,w_2}(k) \ll F,K \left(k^{u_1+3u_2-6-\delta}\right) \). For the former, we shift the contour of integration in (4.10) to \( \Re(s) = \sigma_4 \) with \( -5 < \sigma_4 < -u_1 - 3u_2 - 2 \). Due to the poles of \( \mathcal{J}_t(s + w_1 + 3w_2 - 2) \), this picks up residues at \( s = -w_1 - 3w_2 + 2 + 2(\pm it - \ell) \) of size \( O_{F,K}(1 + |\ell|^{(6-\delta)/2}) \) for \( \ell \in \mathbb{N}_0 \) by (2.21), (4.8), and (4.11). We then break up the ensuing integral into the three ranges \( |\tau| \leq |t|, |t| \leq |\tau| \leq 3|t|, \) and \( |\tau| \geq 3|t| \). By (2.19), (2.20), (4.8), and (4.11), the former and latter contributions are \( O_{F,K}(1 + |t|)^{-\delta/2} \), while the middle range is \( O_{F,K}(1 + |\tau|)^{-\delta/3} \). Our assumption on \( \sigma_4 \) then ensures that this decays sufficiently rapidly. The same method (using the bounds (2.22) in place of (2.19) or (2.20)) yields the desired bounds for \( \tilde{h}^\text{hol}_{w_1,w_2}(k) \).

5. Test Functions and Transforms for the Short Initial Range

Our treatment of the short initial range requires the usage of \( \text{GL}_2 \times \text{GL}_2 \rightleftharpoons \text{GL}_4 \times \text{GL}_4 \) and \( \text{GL}_4 \times \text{GL}_2 \rightleftharpoons \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity in the guises of Theorems 3.1 and 4.1. In order to apply these two forms of spectral reciprocity, we must choose triples of test functions \((h^+, h^-, h^\text{hol})\) that localise to dyadic intervals \([T, 2T]\). We subsequently bound the associated
transforms $H_{\mu_T}$ as in (3.5) and $(\tilde{h}^+, \tilde{h}^-, \tilde{h}_{\text{hol}}^+)$ as in (4.3) and (4.4). Extra care must be undertaken in producing these bounds due to the hybrid nature of the problem at hand: we must obtain bounds that are uniform in both the dyadic parameter $T$ and the spectral parameter $t_g$.

5.1. Test Functions. We define two triples of test functions $(h^+, h^-, h_{\text{hol}})$:

\begin{align}
(5.1) & \quad h^+(t) = 0, \quad h^-(t) = e^{-\pi^2 t^2} \prod_{j=1}^{M} \left( \frac{t^2 + \left( j \frac{1}{T} - \frac{1}{2} \right)^2}{T^2} \right)^{2}, \quad h_{\text{hol}}(k) = 0, \\
(5.2) & \quad h^+(t) = (\mathcal{L}^+ H^+)(t), \quad h^-(t) = 0, \quad h_{\text{hol}}(k) = (\mathcal{L}^{\text{hol}} H^+)(k).
\end{align}

Here $\mathcal{L}^+$ and $\mathcal{L}^{\text{hol}}$ are the transforms given by (2.13), while $H^+$ is chosen to be the function

\begin{equation}
(5.3) \quad H^+(x) := \sinh^{M-1} \left( \frac{1}{T} \right) (4\pi x)^M e^{-4\pi x \sinh \left( \frac{1}{T} \right)},
\end{equation}

which depends on auxiliary parameters $M \in \mathbb{N}$ and $T > 0$; for both (5.1) and (5.2), $M$ is a fixed positive integer and $T > M$. These are chosen to localise to dyadic regions: (5.1) is evidently constructed such that $h^-(t)$ localises to the intervals $[-2T, -T] \cup [T, 2T]$, while we shall presently show that (5.2) is constructed such that $i^k h_{\text{hol}}(k)$ localises to the interval $[T, 2T]$. There are of course plenty of other choices of triples of test functions that localise to dyadic intervals. In order for these test functions to be admissible for Theorems 3.1 and 4.1, however, it is necessary that the Mellin transforms of the functions $H^+(x)$ given by (3.2) and (3.3) are holomorphic in a sufficiently wide vertical strip in which they decay sufficiently rapidly. This feature is by no means automatic and is crucial behind our choices of test functions.

**Lemma 5.4.** Let $H^+$ be as in (5.3) with $M \in \mathbb{N}$ and $T > M$.

1. For $s = \sigma + i\tau$, the Mellin transform $\overline{H^+}(s)$ of $H^+$ is holomorphic for $\sigma > -M$, in which it satisfies the bound

$$\overline{H^+}(s) \ll_{\sigma, M} T^{1+\sigma} (1 + |\tau|)^{\sigma+M-\frac{1}{2}} e^{-\frac{\pi}{2} T |\tau|}.$$ 

2. The transform $(\mathcal{L}^{\text{hol}} H^+)(k)$ is such that for $k \in 2\mathbb{N}$,
   
   a. $(\mathcal{L}^{\text{hol}} H^+)(k) \ll_M (k/T)^{M-1} e^{-k/T}$,
   
   b. $i^k (\mathcal{L}^{\text{hol}} H^+)(k) > 0$ for $k > M$,
   
   c. $i^k (\mathcal{L}^{\text{hol}} H^+)(k) \sim_M 1$ for $k \approx T$.

3. The transform $(\mathcal{L}^+ H^+)(r)$ is such that for $r \in \mathbb{R} \cup i(-\frac{1}{2}, \frac{1}{2})$,

$$\mathcal{L}^+ H^+)(r) \ll \left( \frac{1 + |r|}{T} \right)^{M-1} e^{-\pi |r|}.$$

**Proof.**

1. By making the change of variables $x \mapsto (4\pi \sinh(1/T))^{-1} x$, we see that

$$\overline{H^+}(s) := \int_0^\infty H^+(x) x^{s} \frac{dx}{x} = 4\pi \left( 4\pi \sinh \left( \frac{1}{T} \right) \right)^{-s-1} \Gamma(s + M),$$

which is holomorphic for $\Re(s) > -M$. The desired bound for $\overline{H^+}(s)$ then follows from Stirling’s formula.

2. By [GR15, 6.621.1 and 6.621.4], we have that

\begin{align*}
(\mathcal{L}^{\text{hol}} H^+)(k) &= 2\pi i^{-k} (-1)^{M-1} \sinh^{M-1} \left( \frac{1}{T} \right) \left( \frac{\sech \frac{y d}{dy}}{y} \right)_{y=\frac{1}{T}}^{y=\frac{1}{T}} \left( e^{-(k-1)y} \text{sech } y \right) \\
&= 2\pi i^{-k} \tanh^{M-1} \left( \frac{1}{T} \right) \left( \frac{\sech \left( \frac{1}{T} \right)}{\Gamma(k - 1 + M) P_{M-1}^{1-k} \left( \tanh \left( \frac{1}{T} \right) \right)} \right). 
\end{align*}
where \( P_\alpha^\beta(x) \) denotes the associated Legendre function. By [GR15, 8.714.2],
\[
i^k(\mathcal{L}^{\text{hol}} H^+)(k) = 2\pi \sinh^{M-1}(1/T) e^{-k/T} \\
\times \sum_{j=0}^{M-1} (-1)^j \left( \frac{M-1}{j} \right) \frac{(M-1+j)!(k-2+M)!}{(M-1)!(k-1+j)!} 2^{-j} e^{-(j-1)/T} \cosh^{M-j-2}(1/T) .
\]
for \( k > M \), and in particular is positive in this range. Finally, by [GR15, 8.704],
\[
i^k(\mathcal{L}^{\text{hol}} H^+)(k) = 2\pi \sinh^{M-1}(1/T) e^{-k/T} \\
\times \sum_{j=0}^{M-1} (-1)^j \left( \frac{M-1}{j} \right) \frac{(M-1+j)!(k-2+M)!}{(M-1)!(k-1+j)!} 2^{-j} e^{-(j-1)/T} \cosh^{M-j-2}(1/T) .
\]
Since \((k-2+M)/(k-1+j)! \approx_M k^{M-1-j}\), the above quantity is \( O_M((k/T)^{M-1}e^{-k/T}) \) for all \( k \in 2\mathbb{N} \) and is \( \approx_M 1 \) for \( k \approx T \).

(3) Similarly, by [GR15, 6.621.1 and 6.621.4],
\[
(\mathcal{L}^{\text{hol}}(H^+))(r) = \frac{2\pi}{\sinh\pi r} (-1)^{M-1} \sinh^{M-1}(1/T) \left( \frac{d}{dy} \right)^{M-1} \left( \frac{\sinh y}{\cosh y} \right) \bigg|_{y=1/T} (\sinh y \sin(2y)) \\
= \frac{\pi i}{\sinh\pi r} \tanh^{M-1}(1/T) \sech(1/T) \sum_{\pm} \pm \Gamma(M \pm 2ir) P^{2ir}_{M-1} \left( \tanh(1/T) \right).
\]
By [GR15, 8.714.2],
\[
\sum_{\pm} \pm \Gamma(M \pm 2ir) P^{2ir}_{M-1} \left( \tanh(1/T) \right) \\
= \sum_{j=0}^{M-1} (-1)^j \left( \frac{M-1}{j} \right) \frac{(M-1+j)!(k-2+M)!}{(M-1)!(k-1+j)!} 2^{-j} e^{-(j-1)/T} \sech^{M-j-2}(1/T) \sum_{\pm} \pm e^{\mp 2ir/T} \frac{\Gamma(M \pm 2ir)}{\Gamma(1+j \pm 2ir)} .
\]
The desired bound then holds from the fact that
\[
\sum_{\pm} \pm e^{\mp 2ir/T} \frac{\Gamma(M \pm 2ir)}{\Gamma(1+j \pm 2ir)} \ll_M (1 + |r|)^{M-1-j} . \tag{5.2}
\]

5.2. \( GL_4 \times GL_1 \) Transforms. Next, we determine the behaviour of \( \mathcal{H}_{ad}(t) \) as in (3.5) with \((h^+, h^-, h^{\text{hol}})\) either of the triples of test functions (5.1) or (5.2).

**Lemma 5.5.** Let \( g \) be a Hecke–Maass cusp form on \( \Gamma \backslash \mathbb{H} \) with spectral parameter \( t_g \). Let \((h^+, h^-, h^{\text{hol}})\) be either of the triples of test functions (5.1) or (5.2) with \( M > 50 \) and \( T \leq t_g^{1-\delta} \) for some fixed \( \delta > 0 \). Then for \( F = ad g \) and \( \mathcal{H}_{ad}(t) \) as in (3.5), we have that
\[
\mathcal{H}_{ad}(t) \ll_M \begin{cases} \frac{T^2}{t^2_g} & \text{for } |t| \leq \frac{t^2_g}{72} , \\ \frac{T}{|t|^{1/2}} \left( \frac{T^2 |t|}{t^2_g} \right)^{2/3} & \text{for } |t| \geq \frac{t^2_g}{72} . \end{cases} \tag{5.6}
\]

Combining the bounds in Lemma 5.5 with the \( GL_3 \times GL_2 \rightleftarrows GL_4 \times GL_1 \) spectral reciprocity formula obtained in Theorem 3.1, we may deduce an identity roughly of the form
\[
\sum_{f \in B_{hol}^T} L \left( \frac{1}{2}, ad g \otimes f \right) L(1, ad f) \rightleftarrows \int_{T \leq |t| \leq 2T} \left| \frac{L \left( \frac{1}{2} + it, ad g \right)}{\zeta(1 + 2it)} \right|^2 dt \rightleftarrows \sum_{f \in B_{hol}^T} L \left( \frac{1}{2}, ad g \otimes f \right) L(1, ad f) \\
\approx T^2 + \frac{T^2}{t^2_g} \int_{t^2_g}^{\frac{t^2_g}{72}} L \left( \frac{1}{2} + it, ad g \right) \zeta \left( \frac{1}{2} - it \right) dt .
\]
In Proposition 6.6, we use this identity to produce upper bounds for each of the terms on the left-hand side.

Proof of Lemma 5.5. For the triple of test functions (5.1), we have that for $s = \sigma + i\tau$ with $-M/2 < \sigma < M/2$,

$$\widetilde{H}^- (s) \ll_{\sigma, M} T^{1 + \sigma} (1 + |\tau|)^{-M}$$

by [BLM19, Lemma 4], while $\widetilde{H}^+ (s) = 0$, where $H^+$ and $H^-$ are as in (3.2) and (3.3). Similarly, for the triple of test functions (5.2), the Sears–Titchmarsh inversion formula [Iwa02, Appendix B.5] implies that $(H^+ + h^+) (x) + (H^\text{hol} + h^\text{hol}) (x) = H^+ (x)$, so that

$$\widetilde{H}^+ (s) \ll_{\sigma, M} T^{1 + \sigma} (1 + |\tau|)^{\sigma + M - \frac{1}{2} e^{-\frac{3}{2} |\tau|}}$$

for $\sigma > M$ by Lemma 5.4 (1), while $\widetilde{H}^- (s) = 0$.

Provided that $\sigma$ is bounded and $s$ is a bounded distance away from the poles at $s = 1 + 2\ell$, $s = 1 + 2\ell \pm 4it_g$, and $s = -2(it + \ell)$ with $\ell \in \mathbb{N}_0$, we have by the definitions (2.25) and (2.27) and Stirling’s formula that the integrand in (3.5) satisfies the bounds

$$\sum_{\pm_1, \pm_2} \widetilde{H}^\pm_1 (s) \varphi^\pm_{\mu_F} \left( \frac{1 - s}{2} \right) G^{\mp_1 \pm_2} \left( \frac{\sigma}{2} + it \right) \ll_{\sigma, M} T^{1 + \sigma} (1 + |\tau|)^{-M} \left( (1 + |\tau + 4t_g|) (1 + |\tau|) (1 + |\tau - 4t_g|) \right)^{\frac{2}{|\tau|} (1 + |\tau + 2t|)^{\frac{2}{|\tau|}}.}

Since

$$\text{Res}_{s=1} \varphi^\pm_{\mu_F} \left( \frac{1 - s}{2} \right) \ll_{\ell} \frac{1}{t_g}, \quad \text{Res}_{s=1} \varphi^\pm_{\mu_F} \left( \frac{1 - s}{2} \right) \ll_{\ell} \frac{1}{t_g},$$

we have that

$$\text{Res}_{s=1} \sum_{\pm_1, \pm_2} \widetilde{H}^\pm_1 (s) \varphi^\pm_{\mu_F} \left( \frac{1 - s}{2} \right) G^{\mp_1 \pm_2} \left( \frac{\sigma}{2} + it \right) \ll_{\ell} \frac{T^2}{t_g}. \quad \text{Res}_{s=1} \sum_{\pm_1, \pm_2} \widetilde{H}^\pm_1 (s) \varphi^\pm_{\mu_F} \left( \frac{1 - s}{2} \right) G^{\mp_1 \pm_2} \left( \frac{\sigma}{2} + it \right) \ll_{\ell} \frac{T^2}{t_g}.$$

To bound $H_{\mu_F} (t)$ when $|t| \leq t_g^2/T^2$, we shift the contour of integration in (3.5) to $\Re(s) = 1$. Since the integrand decays rapidly due to the decay of the Mellin transform of $H^\pm_1 (x)$, the main contribution arises from the pole at $s = 1$ and from the portion of the integral for which $\tau$ is essentially bounded. In this way, we find that for $|t| \leq t_g^2/T^2$,

$$H_{\mu_F} (t) \ll_{M} \frac{T^2}{t_g}. \quad \text{Res}_{s=-2(it + \ell)} G^\pm \left( \frac{\sigma}{2} + it \right) \ll_{\ell} 1,$$

we have that for each nonnegative integer $\ell < M$,

$$\text{Res}_{s=-2(it + \ell)} \sum_{\pm_1, \pm_2} \widetilde{H}^\pm_1 (s) \varphi^\pm_{\mu_F} \left( \frac{1 - s}{2} \right) G^{\mp_1 \pm_2} \left( \frac{\sigma}{2} + it \right) \ll_{M} T^{1 - 2it} (1 + |t|)^{-M} \left( (1 + |2t_g + t|) (1 + |t|) (1 + |2t_g - t|) \right)^{\ell}.$$

To bound $H_{\mu_F} (t)$ when $|t| \geq t_g^2/T^2$, we shift the contour of integration in (3.5) to $\Re(s) = -M/4$. For the resulting integral on the line $\Re(s) = -M/4$, the integrand is negligibly small unless $\tau$ is essentially bounded, and hence for $|t| \geq t_g^2/T^2$,

$$H_{\mu_F} (t) \ll_{M} t_g \sqrt{T} \left( \frac{T^2}{t_g^2} \right)^{-\frac{M}{2}}. \quad \square$$
5.3. GL$_4 \times$ GL$_2$ Transforms. Similarly, we determine the behaviour of $(\tilde{h}^+, \tilde{h}^-, \tilde{h}^{\text{hol}})$ as in (4.3) and (4.4) with $(h^+, h^-, h^{\text{hol}})$ either of the triples of test functions (5.1) or (5.2).

Lemma 5.7. Let $g$ be a Hecke–Maass cusp form on $\Gamma \backslash \mathbb{H}$ with spectral parameter $t_g$. Let $(h^+, h^-, h^{\text{hol}})$ be either of the triples of test functions (5.1) or (5.2) for a fixed positive integer $M > 50$ and $M < T \leq 2t_g$. Then for $F = \text{ad} g$ and $(\tilde{h}^+, \tilde{h}^-, \tilde{h}^{\text{hol}})$ as in (4.3) and (4.4), we have that

$$
(5.8) \quad \tilde{h}^+(t) \ll_M T^2 \log t_g (1 + |t|)^{-\frac{1}{2}(M+1)},
$$

$$
(5.9) \quad \tilde{h}^-(t) \ll_M \begin{cases} 
T^2 \log t_g & \text{for } |t| \leq \frac{t_g}{T}, \\
T \left( \frac{|t|}{t_g} \right)^{-\frac{M}{4}} & \text{for } |t| \geq \frac{t_g}{T},
\end{cases}
$$

$$
(5.10) \quad \tilde{h}^{\text{hol}}(k) \ll_M \begin{cases} 
T^2 \log t_g & \text{for } k \leq \frac{t_g}{T}, \\
T \left( \frac{k}{t_g} \right)^{-\frac{M}{4}} & \text{for } k \geq \frac{t_g}{T}.
\end{cases}
$$

Combining the bounds in Lemma 5.7 with the GL$_4 \times$ GL$_2 \leftrightarrow$ GL$_4 \times$ GL$_2$ spectral reciprocity formula obtained in Theorem 4.1, we may deduce an identity roughly of the form

$$
\sum_{f \in B_0 \atop t_f \leq \frac{t_g}{T}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f)} + \frac{1}{2\pi} \int_{|t| \leq T} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} g \right)}{\zeta(1 + 2it)} \right|^2 dt
$$

$$
+ \sum_{f \in B_{\text{hol}} \atop t_f \leq \frac{t_g}{T}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f)}
$$

$$
\approx T^2 + T^2 \log t_g \sum_{f \in B_0 \atop t_f \leq \frac{t_g}{T}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f)} + T^2 \log t_g \int_{|t| \leq \frac{t_g}{T}} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} g \right)}{\zeta(1 + 2it)} \right|^2 dt
$$

$$
+ \frac{T^2 \log t_g}{t_g} \sum_{f \in B_{\text{hol}} \atop k_f \leq \frac{t_g}{T}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f)}.
$$

In Proposition 6.11, we use this identity to provide upper bounds for each of the terms on the left-hand side.

We first state bounds for the integrands in (4.3) and (4.4).

Lemma 5.11. Provided that $s$ is a bounded distance away from the poles at $s = 1 + 2\ell$, $s = 1 + 2\ell \pm 4it_g$, and $s = -2(\pm it + \ell)$ with $\ell \in \mathbb{N}_0$, we have that for $t \in \mathbb{R}$,

$$
(5.12) \quad \sum_{\pm 2} \mathcal{J} \left( s \right) \frac{1 - s}{2} \frac{1 - s}{2} \sigma \left( \frac{1 - s}{2} \right)
$$

$$
\ll \left( 1 + |\tau| \right)^{-\sigma} (|1 + \tau + 4t_g| + |\tau - 4t_g|)^{-\frac{1}{2} - \sigma} (|1 + \tau + 2t| + |\tau - 2t|)^{\frac{1}{2} - \sigma - \epsilon} e^{-\frac{1}{2} \Omega^\pm \left( \tau, t, t_g \right)},
$$

and for $k \in 2\mathbb{N}$,

$$
(5.13) \quad \sum_{\pm 2} \mathcal{J} \left( k \right) \frac{1 - s}{2} \frac{1 - s}{2} \sigma \left( \frac{1 - s}{2} \right)
$$

$$
\ll \left( 1 + |\tau| \right)^{-\sigma} (|1 + \tau + 4t_g| + |\tau - 4t_g|)^{-\frac{1}{2} - \sigma} (|k| + |\tau|)^{\sigma - \epsilon} e^{-\frac{1}{2} \Omega^\pm \left( \tau, t, t_g \right)},
$$

where $\Omega^\pm(u, v, w)$ is defined as

$$
\Omega^\pm(u, v, w) = \left( u \tau + v w \right)^{-\frac{1}{2} - \sigma}.
$$
where
\[
\Omega^{+,+}(\tau, t, t_g) := \begin{cases} 
2|t| & \text{if } |\tau| \leq 2 \min\{t_g, |t|\}, \\
|\tau| & \text{if } 2|t| \leq |\tau| \leq 2t_g, \\
2(2t_g + |t| - |\tau|) & \text{if } 2t_g \leq |\tau| \leq 2 \min\{2t_g, |t|\}, \\
4t_g - |\tau| & \text{if } 2 \max\{t_g, |t|\} \leq |\tau| \leq 4t_g, \\
2|t| - |\tau| & \text{if } 4t_g \leq |\tau| \leq 2|t|, \\
0 & \text{if } |\tau| \geq 2 \max\{2t_g, |t|\}, 
\end{cases}
\]
\[
\Omega^{-,+}(\tau, t, t_g) := \begin{cases} 
|\tau| - 2|t| & \text{if } 2|t| \leq |\tau| \leq 4t_g, \\
|\tau| - 4t_g & \text{if } 4t_g \leq |\tau| \leq 2|t|, \\
2(|\tau| - |t| - 2t_g) & \text{if } |\tau| \geq 2 \max\{2t_g, |t|\}, 
\end{cases}
\]
\[
\Omega^{\text{hol},+}(\tau, t_g) := \begin{cases} 
4t_g - |\tau| & \text{if } 2t_g \leq |\tau| \leq 4t_g, \\
0 & \text{if } |\tau| \geq 4t_g, 
\end{cases}
\]
while
\[
\Omega^{+,-}(\tau, t, t_g) := \begin{cases} 
2|t| - |\tau| & \text{if } |\tau| \leq 2 \min\{2t_g, |t|\}, \\
0 & \text{if } 2|t| \leq |\tau| \leq 4t_g, \\
2(|t| - 2t_g) & \text{if } 4t_g \leq |\tau| \leq 2|t|, \\
|\tau| - 4t_g & \text{if } |\tau| \geq 2 \max\{2t_g, |t|\}, \\
2(|\tau| - |t|) & \text{if } 2 \max\{t_g, |t|\} \leq |\tau| \leq 2t_g, \\
\end{cases}
\]
\[
\Omega^{-,-}(\tau, t, t_g) := \begin{cases} 
4t_g - |\tau| & \text{if } 2t_g \leq |\tau| \leq 2 \min\{2t_g, |t|\}, \\
2(2t_g - |t|) & \text{if } 2 \max\{t_g, |t|\} \leq |\tau| \leq 4t_g, \\
0 & \text{if } 4t_g \leq |\tau| \leq 2|t|, \\
|\tau| - 2|t| & \text{if } |\tau| \geq 2 \max\{2t_g, |t|\}, 
\end{cases}
\]
\[
\Omega^{\text{hol},-}(\tau, t_g) := \begin{cases} 
0 & \text{if } |\tau| \leq 4t_g, \\
|\tau| - 4t_g & \text{if } |\tau| \geq 4t_g. 
\end{cases}
\]

**Proof.** This follows from the definitions (2.25) and (2.27) of $\mathcal{G}_{\mu }^{\pm}(s)$ and $G^{\pm}(s)$, the bounds (2.19), (2.20), and (2.22) for the Mellin transforms of $\mathcal{F}^\pm_t$ and $\mathcal{F}_k^{\text{hol}}$, and Stirling’s formula. □

**Proof of Lemma 5.7.** For the triple of test functions (5.1), we have that for $s = \sigma + i\tau$ with $-M/2 < \sigma < M/2$,
\[
\widetilde{H}^-(s) \ll \sigma_M T^{1+\sigma}(1 + |\tau|)^{-M},
\]
while $\widetilde{H}^+(s) = 0$. Similarly, for the triple of test functions (5.2), we have that for $\sigma > -M$,
\[
\widetilde{H}^+(s) \ll \sigma_M T^{1+\sigma}(1 + |\tau|)^{\sigma+M - \frac{1}{2}} e^{-\frac{\pi}{2}|\tau|},
\]
while $\widetilde{H}^-(s) = 0$.

Since $\sum_{\pm_2} G^{\pm_2}(\frac{1-s}{2}) g^{\pm_2}_{\mu_\nu}(\frac{1-s}{2})$ has a double pole at $s = 1$ and simple poles at $s = 1 \pm 2it_g$, we have that for $|t| \leq t_g/T$ and $k \leq t_g/T$,
\[
\text{Res}_{s=1} \sum_{\pm_2} \widetilde{H}^{\pm_1}(s) \mathcal{F}_t^{\pm}(s) G^{\pm_2} \left( \frac{1-s}{2} \right) g^{\pm_1\pm_2}_{\mu_\nu} \left( \frac{1-s}{2} \right) \ll_M \frac{T^2 \log t_g}{t_g} e^{-\frac{\pi}{8}\Omega^{\pm_1}(0,t,t_g)},
\]
\[
\text{Res}_{s=1} \sum_{\pm_2} \widetilde{H}^{\pm_1}(s) \mathcal{F}_k^{\text{hol}}(s) G^{\pm_2} \left( \frac{1-s}{2} \right) g^{\pm_1\pm_2}_{\mu_\nu} \left( \frac{1-s}{2} \right) \ll_M \frac{T^2 \log t_g}{t_g} e^{-\frac{\pi}{8}\Omega^{\text{hol},\pm_1}(0,t,t_g)},
\]
To bound $e$ via (2.21), while for each nonnegative integer $\ell < M/4$, Proposition 6.1. Below, we state the bounds that one achieves via this approach.

**Bounds via the Spectral Large Sieve.** The simplest approach to bounding the mixed moment of $L$-functions (1.10) is to perform a dyadic subdivision, apply the Cauchy–Schwarz inequality, and bounding the ensuing second moments of $L$-functions via the spectral large sieve. Below, we state the bounds that one achieves via this approach.

**Proposition 6.1.**
Next, we apply the Cauchy–Schwarz inequality to the sum over $m$, relabelling, we obtain the upper bound
\[
\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, \text{ad } g \otimes f \right)^2}{L(1, \text{ad } f)}
\]
\[
\frac{1}{2\pi} \int_{T \leq |t| \leq 2T} \left| \frac{L \left( \frac{1}{2} + it, \text{ad } g \right)^2}{\zeta(1+2it)} \right|^2 dt
\]
\[
\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, \text{ad } g \otimes f \right)^2}{L(1, \text{ad } f)}
\]
\[
\left\{ \begin{array}{ll}
t_g^{2+\varepsilon} T & \text{if } T \leq 2t_g, \\
T^{3+\varepsilon} & \text{if } T \geq 2t_g.
\end{array} \right.
\]

(2) For $T \geq 1$ and $1 \leq U \leq T$, we have the bounds
\[
\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right)^2}{L(1, \text{ad } f)}
\]
\[
\frac{1}{2\pi} \int_{T-U \leq |t| \leq T+U} \left| \zeta \left( \frac{1}{2} + it \right)^2 \zeta(1+2it) \right|^2 dt
\]
\[
\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right)^2}{L(1, \text{ad } f)}
\]
\[
\ll_{\varepsilon} T^{1+\varepsilon} U.
\]

Proof. These are all consequences of the approximate functional equation [IK04, Theorem 5.3] and the spectral large sieve. We give details for (6.2) for the first term on the left-hand side; the other cases are similar. For $f \in B_0$, we have that
\[
L(s, \text{ad } g \otimes f) = \sum_{m,n=1}^{\infty} \frac{A_F(m,n)\lambda_f(n)}{m^{2s+n}}
\]
for $\Re(s) > 1$, where $F = \text{ad } g$, and the conductor of $L(1/2, \text{ad } g \otimes f)$ is $O(t_g^2 \max\{t_g^4, t_f^4\})$. Thus by writing $L(1/2, \text{ad } g \otimes f)$ as a Dirichlet polynomial via the approximate functional equation [IK04, Theorem 5.3] and applying the spectral large sieve [Mot97, Theorem 3.3], we find that the first term on the left-hand side is
\[
\ll_{\varepsilon} t_g^{\varepsilon} T^e \sup_{M^2N \leq T^{1+\varepsilon} \max\{t_g^2, T^2\}} (T^2 + N) \left( \sum_{N \leq n \leq 2N} \sum_{M \leq m \leq 2M} \frac{A_F(m,n)}{m^{\sqrt{n}}} \right)^2.
\]

Now we open up the square, yielding sums over $M \leq m_1, m_2 \leq 2M$, then interchange the order of summation and apply the Cauchy–Schwarz inequality to the sum over $N \leq n \leq 2N$. After relabelling, we obtain the upper bound
\[
\ll_{\varepsilon} t_g^{\varepsilon} T^e \sup_{M^2N \leq T^{1+\varepsilon} \max\{t_g^2, T^2\}} (T^2 + N) \left[ \sum_{M \leq m \leq 2M} \frac{1}{m} \left( \sum_{N \leq n \leq 2N} \frac{|A_F(m,n)|^2}{n} \right)^{1/2} \right]^2.
\]

Next, we apply the Cauchy–Schwarz inequality to the sum over $m$. In this way, we find that the expression above is
\[
\ll_{\varepsilon} t_g^{\varepsilon} T^e \sup_{M^2N \leq T^{1+\varepsilon} \max\{t_g^2, T^2\}} (T^2 + N) \sum_{M \leq \ell \leq 2M} \frac{1}{\ell} \sum_{M \leq m \leq 2M} \sum_{N \leq n \leq 2N} \frac{|A_F(m,n)|^2}{mn}.
\]
The sum over $\ell$ is $O_{\varepsilon}(t^\varepsilon_g L^\varepsilon)$. By Rankin’s trick and the fact that $A_F(m, n) = A_F(n, m)$, the double sum over $m$ and $n$ is

\[ O_{\varepsilon} \left( t^\varepsilon_g L^\varepsilon \min\{M, N\} \sum_{m,n=1}^{\infty} \frac{|A_F(m, n)|^2}{(m^2 n)^{1+\varepsilon}} \right). \]

This double sum over $m, n \in \mathbb{N}$ is equal to

\[ \frac{L(1 + \varepsilon, \text{ad} g \otimes \text{ad} g)}{\zeta(3 + 3\varepsilon)} = L(1 + \varepsilon, \text{sym}^4 g)L(1 + \varepsilon, \text{ad} g)\zeta(1 + \varepsilon). \]

By [Li10, Theorem 2], this is $O_{\varepsilon}(t^\varepsilon_g)$. Thus the left-hand side of (6.2) is

\[ \ll_{\varepsilon} t^\varepsilon_g L^\varepsilon \sup_{M^2 N \leq T^{1+\varepsilon} \max(t^2_g, T^2)} (T^2 + N) \min\{M, N\} \ll_{\varepsilon} \begin{cases} t^{2+\varepsilon}_g T & \text{if } T \leq 2t_g, \\ T^{3+\varepsilon} & \text{if } T \geq 2t_g. \end{cases} \]

We shall shortly use this to bound the last term on the right-hand side of (3.4).

**Lemma 6.4.** Let $g$ be a Hecke–Maaß cusp form on $\Gamma \backslash \mathbb{H}$ with spectral parameter $t_g$. Then for $U \geq 1$, we have that

\[ \int_U^{2U} \left| L \left( \frac{1}{2} + it, \text{ad} g \right) \right|^2 dt \ll_{\varepsilon} \begin{cases} t^{1+\varepsilon}_g U^{\frac{1}{2}} & \text{if } U \leq t_g, \\ U^{\frac{1}{2}+\varepsilon} & \text{if } U \geq t_g. \end{cases} \]

**Proof.** This follows by using the approximate functional equation [IK04, Theorem 5.3] to write $L(1/2 + it, \text{ad} g)$ in terms of a Dirichlet polynomial and then invoking the Montgomery–Vaughan mean value theorem for Dirichlet polynomials [MV74, Corollary 3], noting that the analytic conductor of $L(1/2 + it, \text{ad} g)$ is $O(t^2_g |1 + |t||)$ if $|t| \leq t_g$ and is $O(|t|^3)$ if $|t| \geq t_g$. \( \square \)

6.2. Bounds via Spectral Reciprocity. We now show how to obtain improved bounds for the mixed moment of $L$-functions (1.10) via spectral reciprocity. Our first step is to use Theorem 3.1 to prove bounds for the first moment of $L(1/2, \text{ad} g \otimes f)$.

**Proposition 6.6.** Let $g$ be a Hecke–Maaß cusp form on $\Gamma \backslash \mathbb{H}$ with spectral parameter $t_g$. Then for $T \geq 1$, we have that

\[ \begin{cases} \sum_{T \leq j \leq 2T} L \left( \frac{1}{2}, \text{ad} g \otimes f \right) L(1, \text{ad} f) \left( \frac{1}{2} + it, \text{ad} g \right) \left( \frac{1}{2} + 2it \right) & \text{if } T \leq t_g, \\ \frac{1}{2\pi} \int_{T \leq |t| \leq 2T} \left| L \left( \frac{1}{2} + it, \text{ad} g \right) \right|^2 dt & \text{if } t_g \leq T \leq \frac{1}{2} t_g, \\ \frac{1}{2\pi} \int_{T \leq |t| \leq 2T} \left| L \left( \frac{1}{2} + it, \text{ad} g \right) \right|^2 dt & \text{if } \frac{1}{2} t_g \leq T \leq \frac{3}{2} t_g, \\ \sum_{T \leq k \leq 2T} L \left( \frac{1}{2}, \text{ad} g \otimes f \right) L(1, \text{ad} f) \left( \frac{1}{2} + 2it \right) & \text{if } T \geq \frac{3}{2} t_g. \end{cases} \]

The summands and integrands on the left-hand side of (6.7) are nonnegative due to the fact that $L(1/2, \text{ad} g \otimes f) \geq 0$ by [Lap03, Theorem 1.1].

**Proof.** For $T \leq t_g^{1/4}$, this follows by the Cauchy–Schwarz inequality together with the bounds (6.2). For $T \geq t_g^{1/4}$, we use Theorem 3.1 with $F = \text{ad} g$, so that $\mu_F = (2it_g, 0, -2it_g)$.

We first take the triple of test functions $(h^+, h^-, h^{\text{hol}})$ given by (5.1). With this choice of test functions, the left-hand side of (3.4) provides an upper bound for the first and second terms on the left-hand side of (6.7) by positivity, as $h^+(t) = 0$ and $h^{\text{hol}}(k) = 0$, while $h^-(t) \geq 0$ for all $t \in \mathbb{R}$ and $h^-(t) \approx_M 1$ for $t \in [-2T, -T) \cup [T, 2T]$. The first term on the right-hand side of (3.4) is $O_{\varepsilon}(t^2_g T^3)$ via the bounds $L(1, \text{ad} g) \ll_{\varepsilon} t^\varepsilon_g$ and $\int_{-\infty}^\infty h^r(t) d\text{spec} r \ll T^2$, which is clear from the definitions (5.1) of $h^\pm$ and (2.6) of $d_{\text{spec}} r$. The second term is equal to zero since
Theorem 4, [Jut87, Theorem 4.7], and more generally [Ivi03, Chapter 8] together with the Weyl-strength subconvex bound for the twelfth moment of the Riemann zeta function [H-B78, Theorem 1] (see also [Iwa80, Proposition 6.8].

For the second term on the left-hand side, this bound follows from Heath-Brown’s bound for the second moments of $L(1/2 + it, \text{ad } f)$, and then apply the Cauchy–Schwarz inequality. The desired bounds then follow from the bounds (6.5) and (3.9) for the second moments of $L(1/2 + it, \text{ad } g)$ and $\zeta(1/2 + it)$.

We next take the triple of test functions $(h^+ , h^- , h^{\text{hol}})$ given by (5.2). Here it is no longer the case that the left-hand side of (3.4) consists of only nonnegative terms. Nonetheless, the first two terms on the left-hand side of (3.4) as well as the contribution from the terms in the third for which $k_f \leq M$ are $O_M, \varepsilon (1 + \varepsilon T^{1-M})$ by Lemma 5.4 (2) (a) and (3), the Cauchy–Schwarz inequality, and the bounds (6.2). The contribution from the terms in the third term on the left-hand side of (3.4) for which $k_f > M$ provides an upper bound for the third term on the left-hand side of (6.7) by positivity via Lemma 5.4 (2) (b) and (c), noting that the root number of $L(s, \text{ad } g \otimes f)$ is $it^k$, and hence $L(1/2, \text{ad } g \otimes f) = 0$ when $k_f \equiv 2 \pmod{4}$. Finally, the right-hand side of (3.4) is bounded in the same way as for the triple of test functions given by (5.1).

In our treatment of the mixed moment of $L$-functions (1.10), we shall apply Hölder’s inequality to separate the $L$-functions involved. Underlying this step is the key requirement that we have strong bounds for high moments of $L(1/2, f)$ and $\zeta(1/2 + it)$. While we could merely employ the individual Weyl-strength subconvex bounds $L(1/2, f) \ll \varepsilon (1/2 + \varepsilon)$ and $\zeta(1/2 + it) \ll \varepsilon (1 + |t|)^{1/6 + \varepsilon}$, stronger bounds hold on average; indeed, we are best served by using bounds for the twelfth moment (though fifth moment bounds would also be advantageous, as discussed in Section 12.3).

**Proposition 6.8.** For $T \geq 1$, we have that

$$
\sum_{f \in B_0 \atop T \leq k_f \leq 2T} \left( \frac{L(1/2, f)}{L(1, \text{ad } f)} \right)^{12} \ll_{\varepsilon} T^{4 + \varepsilon}.
$$

Proof. For the second term on the left-hand side, this bound follows from Heath-Brown’s bound for the twelfth moment of the Riemann zeta function [H-B78, Theorem 1] (see also [Iwa80, Theorem 4], [Jut87, Theorem 4.7], and more generally [Ivi03, Chapter 8]) together with the Weyl-strength subconvex bound $\zeta(1/2 + it) \ll \varepsilon (1 + |t|)^{1/6 + \varepsilon}$ and the classical lower bound $|\zeta(1 + 2it)| \gg 1/ \log(2 + |t|)$; alternatively, we can simply appeal to [Ivi03, Theorem 8.3]. For the first term on the left-hand side, this bound is a result of Jutila [Jut04b, Theorem 2]. The authors extended Jutila’s result to cover the same result for the third term on the left-hand side, namely for holomorphic cusp forms, in [HK23, Theorem 1.1].

**Remark 6.10.** Notably, the proofs [Jut04b, Theorem 2] and [HK23, Theorem 1.1] of the cuspidal cases of (6.9) use $GL_4 \times GL_2 \leftrightarrow GL_4 \times GL_2$ spectral reciprocity in a crucial way, as discussed in [HK23, Section 3].

We can combine the bounds attained so far to prove new bounds via $GL_4 \times GL_2 \leftrightarrow GL_4 \times GL_2$ spectral reciprocity. In this way, we can show the following bounds for mixed moments of $L$-functions in the short initial range.

**Proposition 6.11.** Let $g$ be a Hecke–Maaß cusp form on $\Gamma \backslash \mathbb{H}$ with spectral parameter $t_g$. Then for $T \geq 1$, we have that

$$
\sum_{f \in B_0 \atop T \leq k_f \leq 2T} \left( \frac{L(1/2, f)}{L(1, \text{ad } f)} \right)^{12} \ll_{\varepsilon} T^{4 + \varepsilon}.
$$
\[
\sum_{f \in \mathcal{B}_0 \atop T \leq t f \leq 2T} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \ g \otimes f \right)}{L(1, \text{ad} f)} \left( \begin{array}{c}
 t_g^{1+\varepsilon} T^\frac{3}{2} \\
 t_g^{\frac{5}{2}+\varepsilon} T^{-\frac{1}{2}} \\
 t_g^{\frac{3}{2}+\varepsilon} T^{\frac{7}{6}} \\
 t_g^{\frac{5}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{1}{2}+\varepsilon} T^\frac{1}{2} \\
 T_4^{\frac{3}{2}+\varepsilon}
\end{array} \right)
\]

(6.12)

\[
\frac{1}{2\pi} \int_{T \leq |t| \leq 2T} \left| \zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} \ g \right) \right|^2 dt \ll \varepsilon T^\frac{5}{6}
\]

\[
\sum_{f \in \mathcal{B}_0 \atop T \leq t f \leq 2T} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \ g \otimes f \right)}{L(1, \text{ad} f)} \left( \begin{array}{c}
 t_g^{1+\varepsilon} T^\frac{3}{2} \\
 t_g^{\frac{5}{2}+\varepsilon} T^{-\frac{1}{2}} \\
 t_g^{\frac{3}{2}+\varepsilon} T^{\frac{7}{6}} \\
 t_g^{\frac{5}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{1}{2}+\varepsilon} T^\frac{1}{2} \\
 T_4^{\frac{3}{2}+\varepsilon}
\end{array} \right)
\]

Proof. Our first approach is to apply the Cauchy–Schwarz inequality and use the bounds (6.2) and (6.3) from Proposition 6.1 arising from the spectral large sieve; this shows that the left-hand side of (6.12) is

\[
\ll \varepsilon \left( \begin{array}{c}
 t_g^{1+\varepsilon} T^\frac{3}{2} \\
 T_4^{\frac{3}{2}+\varepsilon}
\end{array} \right) \quad \text{if } T \leq 2t_g,
\]

(6.13)

Our second approach is to write

\[
\frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \ g \otimes f \right)}{L(1, \text{ad} f)} = \left( \frac{L \left( \frac{1}{2}, f \right)}{L(1, f)} \right)^{12} \left( \frac{L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, f)} \right)^{12} \left( \frac{L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f)} \right)^{\frac{5}{6}},
\]

apply Hölder’s inequality with exponents \((1/12, 1/12, 5/6)\), and use the bounds from Propositions 6.1, 6.6, and 6.8; that is, as well as bounds arising from the spectral large sieve, we use twelfth moment bounds and bounds from \(GL_3 \times GL_2 \sim GL_4 \times GL_1\) spectral reciprocity. This shows that the left-hand side of (6.12) is

\[
\ll \varepsilon \left( \begin{array}{c}
 t_g^{1+\varepsilon} T^\frac{3}{2} \\
 t_g^{\frac{5}{2}+\varepsilon} T^{-\frac{1}{2}} \\
 t_g^{\frac{3}{2}+\varepsilon} T^{\frac{7}{6}} \\
 t_g^{\frac{5}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{1}{2}+\varepsilon} T^\frac{1}{2} \\
 T_4^{\frac{3}{2}+\varepsilon}
\end{array} \right) \quad \text{if } T \leq 2t_g,
\]

(6.14)

These bounds improve upon the earlier bounds in the range \(T \geq t_g^{5/18}\).

Our final approach is to use Theorem 4.1, namely \(GL_4 \times GL_2 \sim GL_4 \times GL_2\) spectral reciprocity. We first take the triple of test functions \((h^+, h^-, h^{\text{hol}})\) given by (5.1). With this choice of test functions, the left-hand side of (4.2) provides an upper bound for the first and second terms on the left-hand side of (6.12) by positivity, as \(h^+(t) = 0\) and \(h^{\text{hol}}(k) = 0\), while \(h^-(t) \geq 0\) for all \(t \in \mathbb{R}\) and \(h^-(t) \approx_M 1\) for \(t \in [-2T, -T] \cup [T, 2T]\). The first term on the right-hand side of (4.2) is \(O(t_g^2 T^2)\) as \(L(1, \text{ad} g) \ll t_g^3\). The second term is equal to zero since \(h^{\text{hol}}(k) = 0\). Finally, for the third, fourth, and fifth terms, we divide the terms into dyadic ranges and use the bounds (5.8), (5.9), and (5.10) for the transforms \(h^+(t), h^-(t), \text{and } h^{\text{hol}}(t)\). We then apply the pre-existing bounds for the left-hand side of (6.12) from (6.13) and (6.14). By this method, we deduce slightly improved bounds for the first and second terms on the left-hand side of (6.12) in certain ranges, namely that the first and second terms are

\[
\ll \varepsilon \left( \begin{array}{c}
 t_g^{\frac{5}{2}+\varepsilon} T^{-\frac{1}{2}} \\
 t_g^{\frac{5}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 T_4^{\frac{3}{2}+\varepsilon}
\end{array} \right) \quad \text{if } t_g^3 \leq T \leq t_g^3,
\]

\[
\ll \varepsilon \left( \begin{array}{c}
 t_g^{\frac{5}{2}+\varepsilon} T^{-\frac{1}{2}} \\
 t_g^{\frac{5}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 T_4^{\frac{3}{2}+\varepsilon}
\end{array} \right) \quad \text{if } t_g^3 \leq T \leq t_g^3,
\]

\[
\ll \varepsilon \left( \begin{array}{c}
 t_g^{\frac{5}{2}+\varepsilon} T^{-\frac{1}{2}} \\
 t_g^{\frac{5}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 t_g^{\frac{3}{2}+\varepsilon} T^\frac{5}{6} \\
 T_4^{\frac{3}{2}+\varepsilon}
\end{array} \right) \quad \text{if } t_g^3 \leq T \leq t_g^3.
\]
To deduce analogous improved bounds for the third term on the left-hand side of (6.12), we use Theorem 4.1 with the triple of test functions \((h^+, h^-, h^{hol})\) given by (5.2). Here it is no longer the case that the left-hand side of (4.2) consists of only nonnegative terms. Nonetheless, the first two terms on the left-hand side of (4.2) as well as the contribution from the terms in the third term for which \(k_f \leq M\) are \(O_{M, \varepsilon}(t_g^{1+\varepsilon}T^{1-M})\) by Lemma 5.4 (2) (b) and (c), noting once more that the root number of \(L(s, \operatorname{ad} g \otimes f)\) is \(i^{k_f}\), and hence \(L(1/2, F \otimes f) = 0\) when \(k_f \equiv 2 \pmod{4}\). Finally, the right-hand side of (4.2) is bounded in the same way as for the triple of test functions given by (5.1).

\[\square\]

\textbf{Remark 6.15.} For fixed \(g\), the bound \(O_{\varepsilon}(T^{2+\varepsilon})\) for the first term on the left-hand side of (6.12) was previously known by work of Li [Li09, Theorem 1.1]. Bounds of this strength can be proven for the terms on the left-hand side of (6.12) when \(T\) is sufficiently large with respect to \(t_g\); in particular, the method of proof is able to yield the bounds \(O_{\varepsilon}(T^{2+\varepsilon})\) when \(T \geq 2t_g\). However, we shall not require stronger bounds in this range for our applications.

\section{6.3. Proof of Proposition 1.9 (1).} We now prove Proposition 1.9 (1), namely the bound (1.10) for the short initial range, via Proposition 6.11.

\textbf{Proof of Proposition 1.9 (1).} By the lower bound \(L(1, \operatorname{ad} g) \gg \varepsilon t_g^{-\varepsilon}\) and the asymptotic formula (1.14) for \(H(t)\), it suffices to show that

\[
\sum_{f \in \mathcal{B}_0} \sum_{t_f \leq t_g^{1-\varepsilon}} \frac{1}{t_f} \left( \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \operatorname{ad} g \otimes f \right)}{L(1, \operatorname{ad} f)} \right) \left( \frac{1}{1 + |t|} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \operatorname{ad} g \right)}{\zeta(1 + 2it)} \right|^2 dt \right) \ll \varepsilon t_g^{\frac{41}{13} + \varepsilon}.
\]

We dyadically decompose both the sum over \(f\) and the integral over \(t\), so that we are left with proving the bounds

\[
\sum_{f \in \mathcal{B}_0} \sum_{T \leq t_f \leq 2T} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \operatorname{ad} g \otimes f \right)}{L(1, \operatorname{ad} f)} \left( \frac{1}{2\pi} \int_{T \leq |t| \leq 2T} \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \operatorname{ad} g \right)}{\zeta(1 + 2it)} \right|^2 dt \right) \ll \varepsilon t_g^{\frac{41}{13} + \varepsilon} T
\]

for \(T \leq t_g^{1-\varepsilon}\). These desired bounds are a consequence of Proposition 6.11, which gives these bounds when either \(T \asymp t_g^{3/19}\) or \(T \asymp t_g^{16/19}\) and gives stronger bounds otherwise. \(\square\)

\textbf{Remark 6.17.} Obtaining the essentially optimal bound \(\|g\|_4 \ll \varepsilon t_g^\varepsilon\) for the \(L^4\)-norm would require us to improve the the bounds for the left-hand side of (6.16) to \(O_{\varepsilon}(t_g^{1+\varepsilon}T)\) for \(T \leq t_g^{1-\varepsilon}\).

Proposition 6.11 gives these bounds apart from the ranges \(T \leq t_g^{3/13}\) and \(t_g^{10/13} \leq T \leq t_g^{1-\varepsilon}\).

\textbf{Remark 6.18.} Improvements on the bounds (6.16) are known when \(g\) is either an Eisenstein series or a dihedral Hecke–Maaß cusp form, which is crucial to the \textit{unconditional} \(L^4\)-norm asymptotic formulæ that have been proven in these settings [DK20, HK20]. The key difference behind these improvements is the factorisation of the \(\text{GL}_3 \times \text{GL}_2\) Rankin–Selberg \(L\)-function into the product of \(L\)-functions of lower degree in these settings: in particular, if \(g(z) = E(z, 1/2 + it_g)\) is an Eisenstein series, then \(L(1/2, \operatorname{ad} g \otimes f) = L(1/2, f)\|L(1/2 + 2it_g, f)\|^2\). This allows for additionally flexibility in applying Hölder’s inequality in bounding (6.16); for instance, in place of \(\text{GL}_3 \times \text{GL}_2 \sim \text{GL}_4 \times \text{GL}_1\) spectral reciprocity, one can do better by instead inputting the
individual Weyl–strength subconvex bounds $L(1/2, f) \ll \epsilon t_f^{1/3 + \epsilon}$ and $\zeta(1/2 + it) \ll \epsilon (1 + |t|)^{1/6 + \epsilon}$ together with the second moment bounds

\begin{equation}
\sum_{\substack{j \leq T \leq 2T \\ j \in B_0}} \left| L\left( \frac{1}{2} + 2it, f \right) \right|^2 + \frac{1}{2\pi} \int_{T \leq |t| \leq 2T} \left| \frac{\zeta\left( \frac{1}{2} + it + 2itg \right)}{\zeta(1 + 2it)} \right|^2 \, dt \ll \epsilon \begin{cases}
t_g^{2+\epsilon} & \text{if } 1 \leq T \leq t_g^3, \\
T^{-2+\epsilon} & \text{if } t_g^3 \leq T \leq t_g.
\end{cases}
\end{equation}

due to Jutila \cite[Theorem]{Jut04a}. This gives an effective treatment of the portion $T \leq t_g^{1/2}$ of the short initial range; coupled with GL$_4 \times$ GL$_2 \leftrightarrow$ GL$_4 \times$ GL$_2$ spectral reciprocity for the remaining portion, this shows that the short initial range in the Eisenstein setting is negligibly small.

7. Bounds for Mixed Moments of $L$-functions in the Bulk Range

7.1. Proof of Proposition 1.9 (2). The proof of Proposition 1.9 (2), namely the bound (1.11) for the bulk range, follows by modifying earlier work of Buttcane and the second author \cite{BuK17}, where the asymptotic formula $2 + o(1)$ was proven for the mixed moment of $L$-functions in the bulk range (1.11) under the assumption of the generalised Lindelöf hypothesis (GLH). We explain the minor modifications required to weaken this to an unconditional upper bound.

Proof of Proposition 1.9 (2). The bound (1.11) for the bulk range follows by modifying the main result of \cite{BuK17}. There are several places in \cite{BuK17} where we must remove the assumption of the GLH, which we list below. For the sake of consistency, we use the notation in \cite{BuK17}; in particular, $g$ is replaced by $f$ and $t_g$ is replaced by $T$.

1) It is stated in \cite[Lemma 2.1]{BuK17} and proven in \cite[Section 4]{BuK17} that under GLH,

\begin{equation}
\int_{-\infty}^{\infty} \left| \left< f^2, E\left( \cdot, \frac{1}{2} + it \right) \right> \right|^2 \, dt \ll \epsilon T^{-1+\epsilon}.
\end{equation}

We do not need to bound this separately.

2) In \cite[p. 1494]{BuK17}, it is stated that under GLH,

\begin{equation}
T^{-1/2+\beta/2+\alpha+\epsilon} \sum_{T^{-1} < t_j < T^{1+\epsilon}} |H(t_j)W(t_j)| \left| \sum_{m, r \geq 1} \frac{\lambda_j(m)A_f(r, m)}{rm^{1/2}} V_2(r^2m, t_j) \right| \ll T^{-1/2+\beta/2+\alpha/2+\epsilon}.
\end{equation}

To bound this unconditionally, we use the fact that $V_2(r^2m, t_j)$ is negligibly small unless $r^2m \leq T^{2+\epsilon}(1 + 2T^2 - t_j^2)$, in which case it is $O(1)$. We then apply the Cauchy–Schwarz inequality and the spectral large sieve, as in the proof of Proposition 6.1 (1), to unconditionally obtain the weaker bound $O(T^{\beta/2+\alpha+\epsilon})$.

3) In \cite[p. 1495]{BuK17}, it is shown under GLH that the error term in \cite[(6.1)]{BuK17} is $O(T^{-1/2})$. This same bound holds unconditionally by using the fact that for $R(s_1) = R(s_2) = \epsilon$, $L(1 + 2s_2, \text{sym}^2 f) L(1 + s_1 + s_2, \text{sym}^2 f) \ll T^\epsilon$ by \cite[Theorem 1.1]{Li09}.

4) In \cite[p. 1496]{BuK17}, it is shown under GLH that in \cite[(6.2)]{BuK17} there is an error term of size $O(T^{-(1-\beta)/10+\epsilon})$ from shifting the line of integration to $R(s_1) = -\frac{1}{2}$. Instead using the convexity bound for $L(1 + s_1 + s_2, \text{sym}^2 f)$ with $R(s_1) = -\frac{1}{2}$ and $R(s_2) = \epsilon$, we get the unconditional error term $O(T^{3/10+\epsilon})$. Similarly, after shifting the line of integration to $R(s_1) = -\frac{1}{10}$, we get the unconditional error term $O(T^\epsilon)$ instead of $O(T^{-1/20})$.

5) In \cite[Section 7]{BuK17}, it is shown that the continuous spectrum has a negligible contribution under GLH. However we do not need to bound this separately, since it is present in (1.11).

6) In \cite[p. 1500]{BuK17}, it is stated that the quantity in \cite[(9.7)]{BuK17} is trivially $O(T^\epsilon)$ unconditionally, which suffices for our purposes; we do not need a power-saving for this, which is given under GLH. Here $\epsilon$ is dependent on $\alpha$ and is arbitrarily small for arbitrarily small $\alpha$. 


With this choice of triple of test functions, we have that
\[ c \]

and is arbitrarily small for arbitrarily small \( \alpha \).

Remark 7.1. The presence of the bound \( O_\epsilon(t_g^{c(\alpha)+\epsilon}) \) on the right-hand side of (1.11) with \( \lim_{\alpha \to 0} c(\alpha) = 0 \) is due to the fact that the \( \epsilon \)-convention is used in [BuK17, Section 9] (see in particular [BuK17, p. 1499]). Indeed, from this usage of the \( \epsilon \)-convention, our unconditional modification of [BuK17] shows that for all \( \beta > 0 \), there exists some \( \alpha > 0 \) such that

\[
\sum_{f \in \mathcal{B}_0} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f) L(1, \text{ad} g)^2} H(t_f)
\]

\[
+ \frac{1}{2\pi} \int_{t_g^{-\alpha} \leq t_f \leq 2t_g - t_g^{-\alpha}} \left| \zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} g \right) \right|^2 H(t) \, dt \ll_\epsilon t_g^{\beta+\epsilon}.
\]

With more care, we could make the dependence of \( c(\alpha) \) on \( \alpha \) in (1.11) explicit by a more precise treatment of [BuK17, Section 9]. This, however, is not necessary for our purposes, since our bounds (1.10) and (1.12) for the short initial and short transition ranges are worse than our bound (1.11) for the bulk range.

8. Test Functions and Transforms for the Short Transition Range

Our treatment of the short transition range once more requires the usage of \( \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity in the guise of Theorem 4.1. We choose a triple of test functions \( (h^+, h^-, h^{\text{hol}}) \) that localises to short intervals \([T, T+U] \) with \( T \sim 2t_g \) and subsequently bound the associated transforms \( (\tilde{h}^+, \tilde{h}^-, \tilde{h}^{\text{hol}}) \) as in (4.3) and (4.4). Just as with our treatment of the short initial range, the hybrid nature of the problem at hand requires us to obtain bounds that are uniform in both of the dyadic parameter \( T \) and the spectral parameter \( t_g \).

8.1. Test Functions. We define the following triple of test functions \( (h^+, h^-, h^{\text{hol}}) \):

\[
h^+(t) = 0, \quad h^-(t) = \sum_{\pm} e^{-\frac{(t \pm U)^2}{2U}} \prod_{j=1}^M \left( \frac{t^2 + (j-\frac{1}{2})^2}{T^2} \right)^2, \quad h^{\text{hol}}(k) = 0.
\]

Here \( M \in \mathbb{N} \) is a fixed positive integer and \( T, U \) are auxiliary parameters for which \( M < U < T \).

With this choice of triple of test functions, we have that \( H^+(x) := (\mathcal{X}^+ h^+)(x) + (\mathcal{X}^{\text{hol}} h^{\text{hol}})(x) = 0 \), while we have the following bounds for \( H^-(x) := (\mathcal{X}^- h^-)(x) \) and its derivatives.

Lemma 8.2 ([HK20, Lemma 12.2]). For \( j \in \mathbb{N}_0 \) with \( j \leq N \), we have that

\[
x^j \frac{d^j}{dx^j} H^-(x) \ll_M \begin{cases} U \min \left\{ \left( \frac{x}{T} \right)^{M/2}, \left( \frac{x}{T} \right)^{-M/2} \right\} & \text{if } |x-T| > U \log T, \\ T \left( \frac{T}{U} \right)^j \left( 1 + \frac{|x-T|}{U} \right)^{4M} e^{-\left( \frac{x-T}{T} \right)^2} & \text{if } |x-T| \leq U \log T. \end{cases}
\]

Using Lemma 8.2 and integration by parts, we deduce the following bounds for the Mellin transform of \( H^- \).

Lemma 8.4 ([HK20, Corollary 12.10]). For \( s = \sigma + i\tau \), \( \tilde{H}^-(s) \) is holomorphic in the strip \(-M/2 < \Re(s) < M/2\) and satisfies the bounds

\[
\tilde{H}^-(s) \ll_{\sigma, M} \begin{cases} U T^\sigma & \text{for } |\tau| \leq \frac{T}{U}, \\ U T^\sigma \left( \frac{|\tau| U}{T} \right)^{-M/2} & \text{for } |\tau| \geq \frac{T}{U}. \end{cases}
\]

\[\text{For instance, with a little work, it can be shown that } c(\alpha) \leq 9\alpha/2 \text{ is admissible. With an overhaul of the methods in [BuK17], we expect that it should be possible to take } c(\alpha) \leq \alpha/2.\]
8.2. GL₄ × GL₂ Transforms. We now determine the behaviour of \((\tilde{h}^+, \tilde{h}^-, \tilde{h}^{\text{hol}})\) as in (4.3) and (4.4) with \((h^+, h^-, h^{\text{hol}})\) the triple of test functions (8.1).

Lemma 8.6. Let \(g\) be a Hecke–Maaß cusp form on \(\Gamma \backslash \mathbb{H}\) with spectral parameter \(t_g\). Fix \(\delta > 0\) and \(0 < \varepsilon < \delta/3\), and let \((h^+, h^-, h^{\text{hol}})\) be the triple of test functions (8.1) with \(M \geq 1000\) a sufficiently large positive integer dependent on \(\varepsilon\) and \(\delta\), \(2t_g - t_g^2 - \delta \leq T \leq 2t_g - t_g^{3/2} + \delta\), and \(U = t_g - \frac{T}{2} + 1\). Then for \(F = \text{ad} g\) and \((\tilde{h}^+, \tilde{h}^-, \tilde{h}^{\text{hol}})\) as in (4.3) and (4.4), we have that

\[
\tilde{h}^+(t) \ll \begin{cases} 
U^{1+\varepsilon} & \text{for } |t| \leq \left(\frac{T}{U}\right)^{1+\varepsilon}, \\
\left(|t|T\right)^{-100} & \text{for } |t| \geq \left(\frac{T}{U}\right)^{1+\varepsilon}, 
\end{cases}
\]

\[
\tilde{h}^-(t) \ll \begin{cases} 
U^{1+\varepsilon} & \text{for } |t| \leq T^\varepsilon, \\
\left(|t|T\right)^{-100} & \text{for } |t| \geq T^\varepsilon, 
\end{cases}
\]

\[
\tilde{h}^{\text{hol}}(k) \ll \begin{cases} 
U^{1+\varepsilon} & \text{for } k \leq \left(\frac{T}{U}\right)^{1+\varepsilon}, \\
(kT)^{-100} & \text{for } k \geq \left(\frac{T}{U}\right)^{1+\varepsilon}. 
\end{cases}
\]

The proof of Lemma 8.6 is rather involved. We first prove the bounds (8.8) as well as weakened forms of the bounds (8.7) and (8.9); we then refine these latter two bounds.

Throughout, we use the \(\varepsilon\)-convention: \(\varepsilon\) denotes an arbitrarily small constant whose value may change from occurrence to occurrence. The same applies for the auxiliary constant \(M\), which is an arbitrarily large constant, and an auxiliary constant \(A\), which is also an arbitrarily large constant.

8.2.1. Weak Bounds. We first prove the bounds (8.8) for \(\tilde{h}^-(t)\).

Lemma 8.10. Let \(t \in \mathbb{R}\). We have that

\[
\tilde{h}^-(t) \ll_{M, \varepsilon} \begin{cases} 
U^{1+\varepsilon} & \text{for } |t| \leq T^\varepsilon, \\
\left(|t|T\right)^{-100} & \text{for } |t| \geq T^\varepsilon, 
\end{cases}
\]

Proof. From (4.3), we have that for \(0 < \sigma < 1\),

\[
\tilde{h}^-(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{H}^{-}(s) \hat{J}^{-}(s) \sum \pm G^\pm \left(\frac{1-s}{2}\right) g_{\mu_F}^\pm \left(\frac{1-s}{2}\right) ds.
\]

First we show that the required bound for \(\tilde{h}^-(t)\) holds when \(|t| > t_g^{1000}\) by shifting the line of integration to \(\sigma = -\left\lfloor \frac{M}{1000}\right\rfloor + \frac{1}{2}\). The residues of the poles crossed are \(O_M((|t|T)^{-100})\) provided that \(M\) is sufficiently large since

\[
\text{Res}_{s=2(\pm t_0 - t)} \hat{H}^{-}(s) \hat{J}^{-}(s) \sum \pm G^\pm \left(\frac{1-s}{2}\right) g_{\mu_F}^\pm \left(\frac{1-s}{2}\right)
\ll M \frac{U}{T} \frac{M}{T} (1 + |t|)^{M + \ell - \frac{1}{2}} ((1 + |t + 4t_g|)(1 + |t - 4t_g|)) e^{-\frac{\pi}{2} \Omega^{-\varepsilon}(t,t,t_g)}
\]

On the new line of integration \(\sigma = -\left\lfloor \frac{M}{1000}\right\rfloor + \frac{1}{2}\), we apply Lemma 8.4 and Lemma 5.11 and then integrate trivially to see that the integral is \(O((|t|T)^{-100})\) for large enough \(M\). This is because the \(|\tau|^{M/2}\) term in (8.5) ensures convergence and the term \(((1 + |\tau + 2t|)(1 + |\tau - 2t|))^{\nu}\) in (5.12) gives the saving, since one of \(|\tau + 2t|\) and \(|\tau - 2t|\) is \(>> |t|\).

Now assume that \(T^\varepsilon \leq |t| \leq t_g^{1000}\). The contribution of the range \(|\tau| > T^{1+\varepsilon}/U\) is \(O_M((|t|T)^{-100})\) provided that \(M\) is sufficiently large via Lemma 8.4. The contribution of the range \(T^{\varepsilon}/2 \leq |\tau| \leq T^{1+\varepsilon}/U\) is \(O_M((|t|T)^{-100})\) since \(\Omega^{-\varepsilon}(\tau,t,t_g) \geq |\tau|\) in this range, so that the integrand in (8.11) decays exponentially. For the range \(|\tau| < T^{\varepsilon}/2\), we deform the \(|\tau| < T^{\varepsilon}/2\) segment to a contour going horizontally from \(-iT^{\varepsilon}/2\) to \(-B - iT^{\varepsilon}/2\), vertically to \(-B + iT^{\varepsilon}/2\), and horizontally to \(iT^{\varepsilon}/2\), for a large constant \(0 < B < M/4\). Since the poles of
The contribution of the sub-range $k \leq |k|$ in the remaining range $k$ for $0 \leq k < T$. Let $\tau, t, t_g$ be redefined values of $\tau, t, t_g$ for redefined values of $\epsilon$. We can crudely bound the integral (8.13) over the range $\sigma = \varepsilon$ and using Lemma 5.11, yielding the bound $O(T^{1+\varepsilon})$.

Now suppose that $|\tau| \leq T^{1+\varepsilon}/U$. The contribution of the range $|\tau| > T^{1+\varepsilon}/U$ is $O(T^{1+\varepsilon}/U)$ as shown above. We can bound the integral (8.13) over the range $\sigma = \varepsilon$ and using Lemma 5.11 to get

$$\widetilde{h}^+(t) \ll \int_{|\tau| \leq T^{1+\varepsilon}/U} U T^{\sigma}(1 + |\tau|)^{-1} d\tau + O(T^{-A}) \ll U^{1+\varepsilon}. \quad \square$$

Similarly, we prove the bounds (8.9) for $\widetilde{h}^{\text{hol}}(k)$ apart from the range $T^{1+\varepsilon}/U$. Let $k \in \mathbb{N}$. We have that

$$\widetilde{h}^{\text{hol}}(k) \ll \begin{cases} U^{1+\varepsilon} & \text{for } k \leq T^{1+\varepsilon}/U, \\ T^{1+\varepsilon} & \text{for } T^{1+\varepsilon}/U \leq k \leq T^{1+\varepsilon}/U, \\ (kT)^{100} & \text{for } k > T^{1+\varepsilon}/U. \end{cases}$$

Proof. The proof follows similar ideas, so we do not give full details. By (4.4),

$$\widetilde{h}^{\text{hol}}(k) = \frac{1}{2\pi i} \int_{\sigma-\infty}^{\sigma+\infty} H^{-}(s) \widetilde{J}^{\text{hol}}(s) \sum_{\pm} G^\pm \left( \frac{1-s}{2} \right) \frac{d}{ds} \left( \frac{1-s}{2} \right) ds,$$

for $0 < \sigma < 1$. First we show that $\widetilde{h}^{\text{hol}}(k) \ll (kT)^{100}$ for $k \leq t_g^{1000}$, by shifting the contour to the left, noting that for $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$,

$$\text{Res}_{s=1-k-2\ell} H^{-}(s) \widetilde{J}^{\text{hol}}(s) \sum_{\pm} G^\pm \left( \frac{1-s}{2} \right) \frac{d}{ds} \left( \frac{1-s}{2} \right) \ll U T^{1-k-2\ell} t_g^{k+2\ell} \left( \frac{k-1}{2\pi e} \right)^{1-k} k^{-1/2}.$$
For \( k \leq T^\varepsilon \), we trivially estimate the integral at \( \sigma = \varepsilon \) to get that \( \tilde{h}^{\text{hol}}(k) \ll U^{1+\varepsilon} \). \( \square \)

### 8.2.2. Strong Bounds for \( \tilde{h}^+(t) \)

Our goal is to prove refined estimates for \( \tilde{h}^+(t) \) in the range \( T^\varepsilon \leq |t| \leq T^{1+\varepsilon}/U \). In this range, any error term \( O(T^{-A}) \) for arbitrarily large \( A \) may also be written as \( O(|t|T^{-A}') \) for arbitrarily large \( A' \). We begin by noting that

\[
(8.15) \quad \sum_{\pm} G^\pm \left( \frac{1 - s}{2} \right) g_{\mu\nu}^\pm \left( \frac{1 - s}{2} \right) = 4(\cosh 2\pi t_g + 1)(2\pi)^{2(s-1)} \Gamma \left( \frac{1 - s}{2} \right)^2 \Gamma \left( \frac{1 - s}{2} + 2it_g \right) \Gamma \left( \frac{1 - s}{2} - 2it_g \right) \sin \frac{\pi s}{2} = 4\tilde{g}_0(1 - s)\tilde{g}_{2u_g}(1 - s) + 4\tilde{g}_0(1 - s)\tilde{g}_{2u_g}(1 - s).
\]

The latter term turns out to give a negligible contribution, while for the former term, we make use of the following asymptotic formula.

#### Lemma 8.16

For \( s = \sigma + it \) with \( |t| \leq T^{1+\varepsilon}/U \) and \( 0 < \sigma < 1 \), there exist constants \( c_{j,j_i} \) such that

\[
(8.17) \quad \tilde{J}_{2u_g}(s) = \frac{1}{2} \left( \frac{t_g}{\pi} \right)^{s-1} \left( 1 + \sum_{2 \leq j \leq M} \sum_{0 \leq j_i \leq 3j/2} c_{j,j_i} \frac{s^{j_i}}{t_g^{j_i}} \right) + O(T^{-A})
\]

for all \( A > 0 \).

**Proof.** By Stirling’s formula, there exist constants \( c_{j,j_i} \) such that

\[
\log \Gamma \left( \frac{s}{2} + 2it_g \right) + \log \Gamma \left( \frac{s}{2} - 2it_g \right) = \left( \frac{s}{2} - \frac{1}{2} + 2it_g \right) \log \left( \frac{s}{2} + 2it_g \right) + \left( \frac{s}{2} - \frac{1}{2} - 2it_g \right) \log \left( \frac{s}{2} - 2it_g \right) - s + \log 2\pi + \sum_{2 \leq j \leq M} \sum_{0 \leq j_i \leq j} c_{j,j_i} \frac{s^{j_i}}{t_g^{j_i}} + O(T^{-A}),
\]

for any \( A > 0 \), provided that \( M \) is sufficiently large with respect to \( A \). The complex logarithm satisfies the identity

\[
\log \left( \frac{s}{2} \pm 2it_g \right) = \log(\pm 2it_g) + \log \left( 1 \mp \frac{is}{4t_g} \right) = \log 2t_g \pm i \frac{\pi}{2} - \sum_{j=1}^{\infty} \frac{1}{j} \left( \frac{\pm is}{4t_g} \right)^j.
\]

It follows that

\[
(8.18) \quad \left( \frac{s}{2} - \frac{1}{2} + 2it_g \right) \log \left( \frac{s}{2} + 2it_g \right) + \left( \frac{s}{2} - \frac{1}{2} - 2it_g \right) \log \left( \frac{s}{2} - 2it_g \right) - s + \log 2\pi
\]

\[
= (s - 1) \log 2t_g - 2\pi t_g + \log 2\pi + \sum_{2 \leq j \leq M} \sum_{0 \leq j_i \leq j+1} c_{j,j_i} \frac{s^{j_i}}{t_g^{j_i}} + O(T^{-A})
\]

for some constants \( c_{j,j_i} \). We take the exponential of the right hand side of (8.18). Writing

\[
\exp \left( \sum_{2 \leq j \leq M} \sum_{0 \leq j_i \leq j+1} c_{j,j_i} \frac{s^{j_i}}{t_g^{j_i}} \right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \sum_{2 \leq j \leq M} \sum_{0 \leq j_i \leq j+1} c_{j,j_i} \frac{s^{j_i}}{t_g^{j_i}} \right)^\ell
\]

and expanding, we obtain the desired expansion (8.17) (for different values of \( c_{j,j_i} \) and \( M \)) upon recalling the definition (2.16) of \( \tilde{J}_{2u_g}(s) \). Thus we have written \( \tilde{J}_{2u_g}(s) \) as \( \frac{1}{2}(t_g/\pi)^{s-1} \) plus similar
but smaller functions. This is because the largest term in the series in (8.17), corresponding to \( j = 2 \) and \( j_1 = 3 \), is of size

\[
\frac{|s|^3}{t_3^2} \ll \frac{T^{1+\varepsilon}}{U^3} \ll T^{-3d+\varepsilon},
\]

using the assumptions \( |\tau| \ll T^{1+\varepsilon}/U \) and \( U \geq T^{1/3+\delta} \).

We require the following bounds for \( \mathcal{J}^+_t(x) \).

**Lemma 8.19** ([HM06, Proposition 9, Remark 6]). For \( t \in \mathbb{R} \), we have that

\[
\mathcal{J}^+_t(x) \ll_{\varepsilon} \begin{cases} 
(1 + |t|)^\varepsilon (x^\varepsilon + x^{-\varepsilon}) & \text{if } 0 < 4\pi x \leq 1 + 4|t|^2, \\
\frac{1}{\sqrt{x}} & \text{if } 4\pi x > 1 + 4|t|^2.
\end{cases}
\]  

**Proof.** This follows by using the integral representation of \( Y_0(x) \) found in [Wat44, p. 26] in the case \( x \geq 1 \), and by using the power series representation [GR15, 8.402, 8.403.2] in the case \( 0 < x < 10^6 \).

Now we work towards the improved estimates (8.7) for \( \tilde{h}^+(t) \).

**Lemma 8.21.** For \( T^\varepsilon \leq |t| \leq T^{1+\varepsilon}/U \), there are constants \( c_{j,j_1} \) such that

\[
\tilde{h}^+(t) = \frac{x}{t_3} \sum_{0 \leq j \leq M} \sum_{0 \leq j_1 \leq 3j/2} \frac{1}{t_3^{j_1}} \int_0^\infty \left( y \frac{d}{dy} \right)^j \left( yH^-(y) \right) Y_0 \left( \frac{4\pi^2 xy}{t_3} \right) dy + O(T^{-A})
\]

for all \( A > 0 \).

Here \( Y_0(x) \) denotes the Bessel function of the second kind.

**Proof.** Via (8.15), we may write

\[
\tilde{h}^+(t) = \int_0^\infty \mathcal{J}^+_t(x) \phi(x) \frac{dx}{x} + \frac{2}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} H^-(1-s) \mathcal{J}^+_t(1-s) \tilde{Y}_0(s) \tilde{J}_2^+(s) ds,
\]

where

\[
\phi(x) := \frac{2}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} H^-(1-s) \tilde{Y}_0(s) \tilde{J}_2^+(s) x^{1-s} ds
\]

for \( x > 0 \) and \( 1/2 < \sigma < 1 \). The second term on the right-hand side of (8.22) is negligibly small by Stirling’s formula. The bounds (8.5) for \( H^-(s) \) ensure that the integral (8.23) converges absolutely, and by (8.20), the first term on the right-hand side of (8.22) is absolutely convergent for \( t \in \mathbb{R} \) provided that \( 1/2 < \sigma < 1 \).

Next, we may restrict the range of integration in (8.23) to \( |\Im(s)| < T^{1+\varepsilon}/U \) up to a negligibly small error term, since the bounds (8.5) for \( H^-(s) \) ensure that the remaining portion of the integral is negligibly small. In the range \( |\Im(s)| < T^{1+\varepsilon}/U \), we may replace \( \tilde{J}_{2y_0}(s) \) by the expansion in (8.17) and then extend the integral back to the whole line \( \Re(s) = \sigma \) at the cost of a negligibly small error term, since once more (8.5) ensures that \( H^-(s) \) is negligibly small in the remaining portion of the integral. We are led to studying linear combinations of terms of the form

\[
\int_0^\infty H^-(y) \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{J}_0^+(s) y^{j_1} \left( \frac{\pi xy}{t_3} \right)^{1-s} ds dy
\]

for \( 1/2 < \sigma < 1 \) and \( j \geq 0 \) fixed, where \( 0 \leq j_1 \leq j \) if \( 0 \leq j \leq 1 \) and \( 0 \leq j_1 \leq 3j/2 \) if \( j > 1 \). Integrating by parts \( j_1 \) times and using the fact that

\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{J}_0^+(s) w^{-s} ds = \tilde{J}_0^+(w) = -2\pi Y_0(4\pi w),
\]
As a preliminary step, we show that, up to negligible error term, we may restrict the outer

\[ -2 \pi^2 x \int_0^\infty \frac{1}{t_g} \left( \frac{d}{dy} \right)^{j_1} (y H^{-}(y)) Y_0 \left( \frac{4 \pi^2 xy}{t_g} \right) \frac{dy}{y}. \]

Since

\[ \left| \frac{1}{t_g} \left( \frac{d}{dy} \right)^{j_1} (y H^{-}(y)) \right| \leq \frac{t_g}{t_g} U^{j_1} y H^{-}(y) \]

by (8.3), which is \( \ll \frac{1}{t_g} y H^{-}(y) \) if \( j = 1 \) and \( \ll \frac{t_g}{t_g^3} y H^{-}(y) \) if \( j \geq 2 \), we see that it suffices to consider only the case \( j = 0 \), as the other cases give rise to similar but smaller functions. We need to understand

\[ \tilde{\phi}(x) := \frac{x}{t_g} \int_0^\infty H^{-}(y) Y_0 \left( \frac{4 \pi^2 xy}{t_g} \right) dy \]

in place of \( \phi(x) \). Thus we need to estimate

\[ (L^+ \tilde{\phi})(t) := \int_0^\infty J^+_t(x) \left( \frac{x}{t_g} \right) \int_0^\infty H^{-}(y) Y_0 \left( \frac{4 \pi^2 xy}{t_g} \right) dy \frac{dx}{x} \tag{8.24} \]

As a preliminary step, we show that, up to negligible error term, we may restrict the outer integral in (8.24) to \( x \geq 2 \). This will be useful because we will be able to use of the following expression for \( Y_0(x) \).

**Lemma 8.25 ([Kha22, Lemma 2.7]).** For \( x \geq 1 \), we have that

\[ Y_0(x) = \Im \left( \frac{e^{ix}}{\sqrt{x}} W(x) \right) \tag{8.26} \]

for some smooth function \( W \) satisfying

\[ W^{(j)}(x) \ll_j x^{-j} \tag{8.27} \]

for \( j \in \mathbb{N}_0 \). For \( 0 < x < 10^6 \), we have that

\[ Y_0^{(j)}(x) \ll_{j, \epsilon} 1 + x^{-j + \epsilon} \tag{8.28} \]

for \( j \in \mathbb{N}_0 \).

We shall make use of the following bounds for the transform \( L^+ \).

**Lemma 8.29 ([BHM07, Lemma 1], [Jut99, Lemma 3]).**

1. If \( \phi(x) \) is a smooth function supported on \( 1 < X < x < 2X \), with \( \phi^{(j)}(x) \ll_j X^{-j} \) for all \( j \in \mathbb{N}_0 \), then for \( t \in \mathbb{R} \), we have that

\[ (L^+ \phi)(t) \ll \frac{1 + \log X}{1 + X} \left( 1 + X \right)^{-\ell} \]

for any \( \ell \in \mathbb{N}_0 \).

2. If \( \phi(x) = e(\pm 2x)\psi(x) \) with \( \psi(x) \) a smooth function supported on \( 1 < X < x < 2X \) satisfying \( \psi^{(j)}(x) \ll_j X^{-j} \) for all \( j \in \mathbb{N}_0 \), then for \( t \in \mathbb{R} \), we have that

\[ (L^+ \phi)(t) \ll X^{-1/2} \epsilon. \]

For \( |t| > X^{1/2 + \epsilon} \), we have that

\[ (L^+ \phi)(t) \ll_{\epsilon} (|t| + X)^{-\ell} X^\epsilon \]

for any \( \ell \in \mathbb{N}_0 \).

**Lemma 8.30.** For \( |t| > T^\epsilon \), we have that

\[ (L^+ \phi)(t) = \int_2^\infty J^+_t(x) \int_0^\infty H^{-}(y) Y_0 \left( \frac{4 \pi^2 xy}{t_g} \right) dy dx + O(T^{-A}) \tag{8.31} \]

for any \( A > 0 \).
Proof. First consider the contribution to (8.24) of the range \( x \leq T^{-100A} \) for \( A > 0 \). Inserting the bounds (8.20) and (8.28), we get that the contribution of this range is \( O(T^{-A}) \).

Now consider the range \( T^{-100A} < x < 2 \). The portion of the integral over \( y \in \mathbb{R} \) for which \( 2|\pi y - T| > U^{1+\varepsilon} \) is \( O(T^{-A}) \) by the bounds (8.3) for \( H^- \), (8.20) for \( J_t^+ \), and (8.28) for \( Y_0 \). For the remaining range \( 2|\pi y - T| < U^{1+\varepsilon} \), we have that \( H^-(y) \ll T \) by (8.3) and that
\[
\frac{d^j}{dx^j} Y_0 \left( \frac{4\pi^2 xy}{t g} \right) \ll T^e x^{-j}
\]
by (8.28). So by dividing the interval \( T^{-100A} < x < 2 \) into dyadic intervals and using Lemma 8.29 (1) with \( \ell \) sufficiently large, we get in the current range that when \( |t| > T^e \), the contribution to (8.24) is \( O(T^{-A}) \).

Next we show that up to a negligible error term, the inner integral in (8.31) can be restricted to \( |2\pi y - T| < U^{1+\varepsilon} \).

Lemma 8.32. For \( |t| > T^e \), we have that
\[
(\mathcal{L}^+ \phi)(t) = \int_2^\infty J_t^+(x) \int_{|2\pi y - T| < U^{1+\varepsilon}} \frac{H^-(y)}{t g} Y_0 \left( \frac{4\pi^2 xy}{t g} \right) dy dx + O(T^{-A})
\]
for any \( A > 0 \).

Proof. We write the integral in (8.31) as
\[
\int_2^\infty J_t^+(x) \int_0^{t/4\pi^2 x} \frac{H^-(y)}{t g} Y_0 \left( \frac{4\pi^2 xy}{t g} \right) dy + \int_2^\infty \int_{t/4\pi^2 x}^\infty \frac{H^-(y)}{t g} Y_0 \left( \frac{4\pi^2 xy}{t g} \right) dy dx.
\]
The first of these integrals is bounded, using Lemma 8.25 to estimate \( Y_0(4\pi^2 xy/t g) \), by
\[
\int_2^\infty |J_t^+(x)| \left( \int_0^{t/4\pi^2 x} \frac{|H^-(y)| \left( \frac{4\pi^2 xy}{t g} \right)^{-\varepsilon}}{t g} dy \right) dx.
\]
By inserting the bounds (8.3) for \( H^- \), integrating over \( y \), and then inserting the bound from (8.20) for \( J_t^+ \), we see that (8.34) converges. For the range \( 2|\pi y - T| \geq U^{1+\varepsilon} \), we can insert the bound (8.3) for \( H^- \) to see that the contribution of this range is \( O(T^{-A}) \) for any \( A > 0 \). The second integral is, using Lemma 8.25, the imaginary part of
\[
\int_2^\infty J_t^+(x) \int_0^{t/4\pi^2 x} \frac{H^-(y)}{t g} \left( \frac{4\pi^2 xy}{t g} \right)^{-\frac{1}{2}} e \left( \frac{2\pi xy}{t g} \right) W \left( \frac{4\pi^2 xy}{t g} \right) dy dx.
\]
By integrating by parts once, this equals
\[
\int_2^\infty J_t^+(x) \left[ \frac{t g}{i 4\pi^2 x} e \left( \frac{2\pi xy}{t g} \right) H^-(y) \left( \frac{4\pi^2 xy}{t g} \right)^{-\frac{1}{2}} W \left( \frac{4\pi^2 xy}{t g} \right) \right]_{y=\frac{t g}{4\pi^2 x}}^{y=\infty} dx
\]
\[
- \int_2^\infty J_t^+(x) \left[ \frac{t g}{i 4\pi^2 x} e \left( \frac{2\pi xy}{t g} \right) \frac{d}{dy} \frac{H^-(y)}{t g} \left( \frac{4\pi^2 xy}{t g} \right)^{-\frac{1}{2}} W \left( \frac{4\pi^2 xy}{t g} \right) \right] dy dx.
\]
By (8.20) and the rapid decay of \( H^-(y) \), these integrals converge and the contribution of \( |2\pi y - T| \geq U^{1+\varepsilon} \) is \( O(T^{-A}) \) for any \( A > 0 \).

Now we prove that \( (\mathcal{L}^+ \phi)(t) \) is in the form needed in order to apply Lemma 8.29.

Lemma 8.35. We have that
\[
(\mathcal{L}^+ \phi)(t) = U \sum_{\pm} \sum_{1 \leq i \leq T^e} X_i^{\frac{1}{2}} (\mathcal{L}^+ \phi_{i,\pm})(t)
\]
for some functions \( \phi_{i,\pm} \) of the form \( \phi_{i,\pm}(x) = \epsilon(\pm 2x) \psi_{i,\pm}(x) \) with \( \psi_{i,\pm}(x) \) supported on \( 1 \leq X_i < x < 2X_i < (T/U)^{1+\varepsilon} \) and satisfying \( \psi_{i,\pm}^{(j)}(x) \ll_j X_i^{-j} \) for all \( j \geq 0 \).
Proof. We make the substitution $y \mapsto \frac{Uy + T}{2\pi}$ in the inner integral in (8.33) and insert the identity (8.26) for $Y_0(x)$, so that the inner integral is

\begin{equation}
(8.37) \quad \frac{U}{(2\pi)^{3/2}i \sqrt{tgT}x} \sum_{\pm} \pm e(\pm 2x)e \left( \pm \frac{2(U - 1)x}{tg} \right) \times \int_{|y|<U^e} \frac{1}{\sqrt{1 + \frac{Uy}{T}}} \frac{1}{H^-} \left( \frac{Uy + T}{2\pi} \right) W \left( \frac{2\pi x(Uy + T)}{tg} \right) e \left( \pm \frac{Uxy}{tg} \right) dy,
\end{equation}

recalling that $T = 2tg - 2U + 2$. For $x \geq T^{1+\epsilon}/U$, we repeatedly integrate by parts with respect to $y$, integrating $e(\pm Uxy/tg)$ and differentiating the rest. Via the bounds (8.3) for the derivatives of $H^-$ and (8.27) for the derivatives of $W$, this shows that the portion of the outer integral in (8.33) for which $x \geq T^{1+\epsilon}/U$ is negligibly small. We obtain the desired decomposition upon applying a smooth partition of unity to split the remaining portion of the outer integral into dyadic intervals.

Finally, we complete the proof of the bounds (8.7) for $\tilde{h}^+(t)$.

Proof of (8.7). Due to the bounds attained for $\tilde{h}^+(t)$ in Lemma 8.12, it remains only to show that

$$
\tilde{h}^+(t) \ll_{\epsilon} \begin{cases} 
U^{1+\epsilon} & \text{if } T^\epsilon \leq |t| \leq \left( \frac{T}{T} \right)^{\frac{1}{2}+\epsilon}, \\
(|t|T)^{-100} & \text{if } \left( \frac{T}{t} \right)^{\frac{1}{2}+\epsilon} \leq |t| \leq T^{1+\epsilon}.
\end{cases}
$$

Via Lemma 8.21, it suffices to prove these bounds for $(L^+\phi)(t)$ in place of $\tilde{h}^+(t)$. The desired bounds then follow by combining the expansion (8.36) for $(L^+\phi)(t)$ together with the bounds for $(L^+\phi_{t,\pm})(t)$ given in Lemma 8.29 (2)\]

Remark 8.38. The calculation of $\tilde{h}^+(t)$ is delicate because in order to apply Lemma 8.29 (2), one needs $\phi$ to be of the form $\phi(x) = e(ax)\psi(x)$ with $a \in \{2, -2\}$, which is what we arrived at in (8.37); no other constants $a$ will suffice.

8.2.3. Strong Bounds for $\tilde{h}^{\text{hol}}(k)$. The bounds (8.9) for $\tilde{h}^{\text{hol}}(k)$ are deduced in exactly the same way.

Proof of (8.9). Due to the bounds attained for $\tilde{h}^{\text{hol}}(t)$ in Lemma 8.14, it remains only to show that

$$
\tilde{h}^{\text{hol}}(k) \ll_{\epsilon} \begin{cases} 
U^{1+\epsilon} & \text{if } T^\epsilon \leq k \leq \left( \frac{T}{T} \right)^{\frac{1}{2}+\epsilon}, \\
(kT)^{-100} & \text{if } \left( \frac{T}{T} \right)^{\frac{1}{2}+\epsilon} \leq k \leq T^{1+\epsilon}.
\end{cases}
$$

This follows by precisely the same method as for $\tilde{h}^+(t)$; an analogue of Lemma 8.21 holds for $\tilde{h}^{\text{hol}}(k)$ in place of $\tilde{h}^+(t)$ by using in place of (8.20) the bounds [HM06, Proposition 8]

$$
\mathcal{J}^{\text{hol}}_k(x) \ll \begin{cases} 
\frac{(4\pi x)^{k-1}}{2k-1\Gamma(k - \frac{1}{2})} & \text{if } 0 < 4\pi x \leq 1, \\
\frac{k}{\sqrt{x}} & \text{if } 4\pi x > 1,
\end{cases}
$$

while the analogues of Lemma 8.29 (1) and (2) hold with $L^+$ replaced by $L^{\text{hol}}$ by [BHM07, Lemma 1] and [Jut99, Remark 1].

9. Bounds for Mixed Moments of $L$-Functions in the Short Transition Range

9.1. Bounds via the Spectral Large Sieve. When bounding the mixed moment of $L$-functions (1.12) in the short transition range, we can no longer perform a dyadic subdivision, since the analytic conductors of $L(1/2, ad g\otimes f)$ and $L(1/2+it, ad g)$ exhibit conductor-dropping in this range. Instead, we must divide into shorter intervals. After an application of Hölder’s
inequality, this leads us to studying the second moment of $L(1/2, \text{ad } g \otimes f)$ in short intervals. We can bound this via the spectral large sieve.

**Proposition 9.1.** Let $g$ be a Hecke–Maaß cusp form on $\Gamma \backslash \mathbb{H}$ with spectral parameter $t_g$. For $t_g \leq T \leq 3t_g$, and $U = |t_g - \frac{T}{2}| + 1$, we have the bounds

$$\sum_{f \in B_0} \frac{L(\frac{1}{2}, \text{ad } g \otimes f)}{L(1, \text{ad } f)} \leq \sum_{f \in B_0} \frac{L(\frac{1}{2}, \text{ad } g \otimes f)}{L(1, \text{ad } f)} \ll \varepsilon T^2 U.$$

**Proof.** Just as in the proof of Proposition 6.1, this follows via the approximate functional equation and the spectral large sieve, noting that the conductor of $L(1/2, \text{ad } g \otimes f)$ and of $|L(1/2 + it, \text{ad } g)|^2$ in these ranges is $O(t_g^4 t^2)$. \qed

9.2. **Bounds via the Kuznetsov Formula.** When applying Hölder’s inequality, we also are led to the study of the first moment of $L(1/2, \text{ad } g \otimes f)$ in short intervals close to $2t_g$. While we expect that this can be achieved via GL$_3 \times$ GL$_2 \leadsto$ GL$_4 \times$ GL$_1$ spectral reciprocity, we instead approach this problem in a more traditional fashion via the Kuznetsov formula.

**Proposition 9.3.** Let $g$ be a Hecke–Maaß cusp form on $\Gamma \backslash \mathbb{H}$ with spectral parameter $t_g$. Fix $\delta > 0$. Then for $2t_g - t_g^{1/3 + \delta} \leq T \leq 2t_g - t_g^\delta$ and $U = t_g - \frac{T}{2} + 1$, we have that

$$\sum_{f \in B_0} \frac{L(\frac{1}{2}, \text{ad } g \otimes f)}{L(1, \text{ad } f)} \ll \varepsilon \begin{cases} t_g^3 \varepsilon U^{\frac{3}{2}} & \text{if } U \leq t_g^\frac{1}{2}, \\ t_g^{\frac{7}{4}} \varepsilon U^{\frac{7}{4}} & \text{if } t_g^\frac{1}{2} \leq U \leq t_g^2. \end{cases}$$

We postpone the proof of Proposition 9.3 to Section 10.

9.3. **Bounds via Spectral Reciprocity.** The application of Hölder’s inequality that we use in the proof of Proposition 1.9 (3) leads us to a short interval third moment of $L(1/2, f)$ for $f \in B_0$. For this, we have the following well-known bound, which is essentially a consequence of GL$_3 \times$ GL$_2 \leadsto$ GL$_4 \times$ GL$_1$ spectral reciprocity in the guise of Motohashi’s formula.

**Proposition 9.5** (Ivić [Ivi01, Theorem]). For $1 \leq U \leq T$, we have that

$$\sum_{f \in B_0} \frac{L(\frac{1}{2}, f)^3}{L(1, \text{ad } f)} \ll \varepsilon T^{1+\varepsilon} U.$$

It is important to note that the terms on the left-hand side of (9.6) are nonnegative, as $L(1/2, f) \geq 0$ for $f \in B_0$ by [KaSa93, Corollary 0.1].

With these collections of bounds in hand, we are able to show the following bounds for mixed moments of $L$-functions in the short transition range.

**Proposition 9.7.** Let $g$ be a Hecke–Maaß cusp form on $\Gamma \backslash \mathbb{H}$ with spectral parameter $t_g$. Fix $\alpha > 0$. Then for $2t_g - t_g^{1-\alpha} \leq T \leq 2t_g$ and $U = t_g - \frac{T}{2} + 1$, we have that
for the short transition range, via Proposition 9.7.

With this choice of test functions, the left-hand side of (4.2) provides an upper bound for the left-hand side of (9.8) is

\[ \frac{1}{2\pi} \int_{T-U \leq |t| \leq T+U} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} \, g \right)}{\zeta(1+2it)} \right|^2 \, dt \]

\[ \ll_{\varepsilon} \begin{cases} t_g^{2+\varepsilon} t_h^\frac{1}{2} & \text{if } 1 \leq U \leq t_g^{\frac{1}{3}}, \\ t_g^{3+\varepsilon} t_h^\frac{1}{2} & \text{if } t_g^{\frac{1}{3}} \leq U \leq t_g^{\frac{1}{2}}, \\ t_g^{2+\varepsilon} t_h^\frac{1}{2} & \text{if } t_g^{\frac{1}{2}} \leq U \leq t_g^{\frac{13}{39}}, \\ t_g^{2+\varepsilon} t_h^\frac{1}{2} & \text{if } t_g^{\frac{13}{39}} \leq U \leq t_g^{1-\alpha}. \end{cases} \]

Proof. Our first approach, valid for \( 1 \leq U \leq t_g^{1/3} \), is to write

\[ \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right)}{L(1, \text{ad} \, f)} = \left( \frac{L \left( \frac{1}{2}, f \right)}{L(1, \text{ad} \, f)} \right)^{\frac{1}{2}} \left( \frac{L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right)}{L(1, \text{ad} \, f)} \right)^{\frac{1}{2}}, \]

apply Hölder’s inequality with exponents \((1/3, 1/3, 1/3)\), and use the bounds from Propositions 9.1, 9.3, and 9.5; that is, as well as bounds arising from the spectral large sieve, we use third moment bounds and bounds from the Kuznetsov formula. This shows that the left-hand side of (9.8) is

\[ \ll_{\varepsilon} \begin{cases} t_g^{2+\varepsilon} U^\frac{1}{2} & \text{if } 1 \leq U \leq t_g^{\frac{1}{3}}, \\ t_g^{10+\varepsilon} U^\frac{1}{2} & \text{if } t_g^{\frac{1}{3}} \leq U \leq t_g^{\frac{1}{2}}, \end{cases} \]

Our second approach, valid for \( t_g^{1/3} \leq U \leq t_g^{1-\alpha} \), is to use Theorem 4.1, namely \( \text{GL}_4 \times \text{GL}_2 \cong \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity. We take the triple of test functions \((h^+, h^-, h^{\text{hol}})\) given by (8.1). With this choice of test functions, the left-hand side of (4.2) provides an upper bound for the left-hand side of (6.12) by positivity, as \( h^+(t) = 0 \) and \( h^{\text{hol}}(k) = 0 \), while \( h^-(t) \geq 0 \) for all \( t \in \mathbb{R} \) and \( h^-(t) \ll_M 1 \) for \( t \in [-T-U, -T+U] \cup [T-U, T+U] \). The first term on the right-hand side of (4.2) is \( O_{\varepsilon}(t_g^{1+\varepsilon}) \) as \( L(1, \text{ad} \, g) \ll_{\varepsilon} t_g^{3} \). The second term is equal to zero since \( h^{\text{hol}}(k) = 0 \). Finally, for the third, fourth, and fifth terms, we divide the terms into dyadic ranges and use the bounds (8.7), (8.8), and (8.9) for the transforms \( \widehat{h}^+(t) \), \( \widehat{h}^-(t) \), and \( \widehat{h}^{\text{hol}}(t) \). Due to the rapid decay of \( \widehat{h}^+(t) \) for \(|t| \geq (T/U)^{1/2+\varepsilon} \), \( \widehat{h}^-(t) \) for \(|t| \geq T^c \), and \( \widehat{h}^{\text{hol}}(k) \) for \( k \geq (T/U)^{1/2+\varepsilon} \), we are left with bounding the quantities

\[ \sum_{f \in \mathcal{B}_0} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right)}{L(1, \text{ad} \, f)}, \quad \frac{1}{2\pi} \int_{V \leq |t| \leq 2V} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} \, g \right)}{\zeta(1+2it)} \right|^2 \, dt, \]

\[ \sum_{f \in \mathcal{B}_0^{\text{hol}}} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right)}{L(1, \text{ad} \, f)} \]

for \( V \leq (T/U)^{1/2+\varepsilon} \), for which we may apply the bounds from Proposition 6.11. This shows that the left-hand side of (9.8) is

\[ \ll_{\varepsilon} \begin{cases} t_g^{47+\varepsilon} U & \text{if } t_g^{\frac{13}{39}} \leq U \leq t_g^{\frac{13}{39}}, \\ t_g^{7+\varepsilon} U^\frac{1}{2} & \text{if } t_g^{\frac{13}{39}} \leq U \leq t_g^{1-\alpha}. \end{cases} \]

9.4. Proof of Proposition 1.9 (3). We now prove Proposition 1.9 (3), namely the bound (1.12) for the short transition range, via Proposition 9.7.
**Proof of Proposition 1.9 (3).** By the lower bound $L(1, \text{ad } g) \gg t_g^{-\varepsilon}$ and the asymptotic formula (1.14) for $H(t)$, it suffices to show that

$$
\sum_{f \in B_0} \frac{1}{1 + (2t_g - t_f)^{1/2}} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, \text{ad } g \otimes f\right) L(1, \text{ad } f) \ll \varepsilon t_g^{30 + \varepsilon}. 
$$

We dyadically decompose both the sum over $f$ and the integral over $t$, so that we are left with proving the bounds

$$
\sum_{f \in B_0} \frac{L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, \text{ad } g \otimes f\right)}{L(1, \text{ad } f)} \ll \varepsilon t_g^{30 + \varepsilon} U^2 
$$

for $2t_g - t_g^{-1/\alpha} \leq T \leq 2t_g$ and $U = t_g - \frac{T}{2} + 1$. These desired bounds are a consequence of Proposition 9.7, which gives these bounds when $U \asymp t_g^{13/19}$ and gives stronger bounds otherwise.

**Remark 9.10.** Obtaining the essentially optimal bound $\|g\|_4 \ll \varepsilon t_g^\varepsilon$ for the $L^4$-norm would require us to improve the the bounds for the left-hand side of (9.9) to $O_\varepsilon(t_g^{3/2 + \varepsilon} U^{1/2})$ for $2t_g - t_g^{-1/\alpha} \leq T \leq 2t_g$ and $U = t_g - \frac{T}{2} + 1$.

10. **Bounds for the First Moment of $L\left(\frac{1}{2}, \text{ad } g \otimes f\right)$ via the Kuznetsov Formula**

The method of proof of Proposition 9.3 involves replacing the $L$-functions $L(1/2, \text{ad } g \otimes f)$ and $|L(1/2 + i t, \text{ad } g \otimes f)|^2$ with Dirichlet polynomials via the approximate functional equation, interchanging the order of summation, and applying the Kuznetsov formula. In this way, the problem is reduced to bounding a sum of Kloosterman sums weighted by Hecke eigenvalues.

10.1. **Approximate Functional Equations.** For $\Re(s) > 1$, we have that

$$
L(s, \text{ad } g \otimes f) = \sum_{m,n=1}^{\infty} \frac{A_F(m,n) \lambda_f(n)}{m^{2s} n^s},
$$

where $F = \text{ad } g$. This has functional equation

$$
L(s, \text{ad } g \otimes f) G(s, t_f, \epsilon_f) = \epsilon_f L(1 - s, \text{ad } g \otimes f) G(1 - s, t_f, \epsilon_f),
$$

where $\epsilon_f \in \{1, -1\}$ is the parity of $f$, and

$$
G(s, t, \epsilon) := \prod \Gamma_F \left( s + \frac{1}{2} - \frac{\epsilon}{2} \pm it \right) \Gamma_F \left( s + \frac{1}{2} - \frac{\epsilon}{2} \pm it - 2it_g \right).
$$

From this, we get the following approximate functional equation for the central $L$-values, which are known to be nonnegative. We also give a related approximate functional equation for $|L(1/2 + it, F)|^2$.

**Lemma 10.1** ([IK04, Theorem 5.3]). For $t \in \mathbb{R}$, $\epsilon \in \{1, -1\}$, $x > 0$, $\sigma > 0$ and $X \geq 1$ a fixed parameter of our choice, let

$$
V_\pm(x, t, \epsilon) := \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s^2} \left( \frac{X^{\pm 1}}{x} \right)^s \frac{G\left(\frac{1}{2} + s, t, \epsilon\right)}{G\left(\frac{1}{2}, t, \epsilon\right)} ds.
$$
We have that

\begin{equation}
(10.3) \quad L \left( \frac{1}{2}, \text{ad} g \otimes f \right) = S_+(t_f, \epsilon_f) + \epsilon_f S_-(t_f, \epsilon_f),
\end{equation}

where

\begin{equation}
S_\pm(t_f, \epsilon_f) := \sum_{m,n=1}^{\infty} \frac{A_F(m,n)\lambda_f(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}} V_\pm(m^2n, t_f, \epsilon_f).
\end{equation}

Similarly

\begin{equation}
\left| L \left( \frac{1}{2} + it, \text{ad} g \right) \right|^2 = E_+(t) + E_-(t),
\end{equation}

where

\begin{equation}
E_\pm(t) := \sum_{m,n=1}^{\infty} \frac{A_F(m,n)\lambda(n,t)}{m^{\frac{1}{2}}n^{\frac{1}{2}}} V_\pm(m^2n, t, 1).
\end{equation}

Fix \( \varepsilon > 0 \), and let \( t_g^\varepsilon \leq U \leq t_g^{1/3+\varepsilon} \). We will be interested in the cases

\begin{equation}
(10.4) \quad |t \pm 2t_g| \asymp U, \quad X = U^{1-\varepsilon}.
\end{equation}

By Stirling’s estimates and a standard contour shifting argument, we have that

\begin{equation}
(10.5) \quad V_\pm(x, t, \varepsilon) \ll \left( \frac{X^{\pm 1} U t_g^2}{x} \right)^{\sigma}.
\end{equation}

Thus the sums in the approximate functional equations (10.3) have different lengths: the sums \( S_+(t_f, \epsilon_f) \) and \( E_+(t) \) are of length about \( U^{2-\varepsilon}t_g^2 \), while the sums \( S_-(t_f, \epsilon_f) \) and \( E_-(t) \) are of length about \( U^{\varepsilon}t_g^2 \).

We will also need a version of the approximate functional equation in which the weight functions \( V_\pm(x, t, \varepsilon) \) do not depend on \( \varepsilon \).

**Lemma 10.6.** Suppose \( |t_f|, |t_f - 2t_g|, |t_f + 2t_g| > t_g^\varepsilon \). We have that

\begin{equation}
L \left( \frac{1}{2}, \text{ad} g \otimes f \right) = S_+(t_f, 1) + \epsilon_2 S_-(t_f, 1) + \delta_{\epsilon_f, \varepsilon_2} (s_+(t_f, 1) + \epsilon_f s_-(t_f, 1) + O(t_g^{50})),
\end{equation}

where

\begin{equation}
s_\pm(t_f, 1) := \sum_{m,n=1}^{\infty} \frac{A_F(m,n)\lambda_f(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}} v_\pm(m^2n, t_f, 1)
\end{equation}

with

\[ v_\pm(n^2m, t_f, 1) := \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{s \frac{X^{\pm 1}}{m^2n}} G \left( \frac{1}{2} + s, t_f, 1 \right) \frac{G \left( \frac{1}{2}, t, 1 \right)}{G \left( \frac{1}{2}, t_f, 1 \right)} P_s \left( 1 - \frac{1}{t^2} ; \frac{1}{t - 2t_g} ; \frac{1}{t + 2t_g} \right) \frac{ds}{s} \]

for \( \sigma > 0 \) and some non-constant polynomial \( P_s(x_1, x_2, x_3) \) whose coefficients depend on \( s \) and are bounded by \( t_g^{\varepsilon} \).

**Remark 10.7.** In the cases of interest (10.4) with \( t = t_f \), the sums \( s_\pm \) are of the same length as \( S_\pm \) but their weight functions are smaller by a factor of \( U^{-2+\varepsilon} \); that is, \( v_\pm(n^2m, t_f, 1) \ll U^{-2+\varepsilon} \).

**Proof.** For \( |s| < t_g^{\varepsilon} \), we have that

\[ \frac{G \left( \frac{1}{2} + s, t, -1 \right)}{G \left( \frac{1}{2}, t, 1 \right)} = \frac{G \left( \frac{1}{2} + s, t, 1 \right)}{G \left( \frac{1}{2}, t, 1 \right)} \left( 1 + P_s \left( \frac{1}{t^2} ; \frac{1}{t - 2t_g} ; \frac{1}{t + 2t_g} \right) \right) + O(t_g^{100}), \]

for \( P_s(x_1, x_2, x_3) \) as described above. This is obtained by applying Stirling’s formula to each gamma factor and using that \( G(1/2 + s, t, \varepsilon) \) is an even function of \( t, t - 2t_g, \) and \( t + 2t_g \). This, together with **Lemma 10.1**, completes the proof. \( \square \)
10.2. Bounds for the First Moment of Dirichlet Polynomials. As shorthand, we write
\[ S_\pm(t) := S_\pm(t, 1), \quad V_\pm(x, t) := V_\pm(x, t, 1). \]

The chief input in the proof of Proposition 9.3 is the following.

**Proposition 10.8.** Let
\[ t^\varepsilon_g \leq U \leq t^{1/4+\varepsilon}_g, \quad X = U^{1-\varepsilon}, \quad h(t) = \exp \left( - \left( \frac{t - 2t_g - U}{U^{1-\varepsilon}} \right)^2 \right) + \exp \left( - \left( \frac{t + 2t_g + U}{U^{1-\varepsilon}} \right)^2 \right). \]

(1) We have that
\[ \sum_{f \in B_0} \frac{S_+(t_f)}{L(1, \text{ad } f)} h(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E_+(t)}{|\zeta(1 + 2it)|^2} h(t) \, dt \ll \varepsilon \, t^{1+\varepsilon}_g U. \]

(2) If we further restrict to \( U \geq t^{1/5}_g \), then we have that
\[ \sum_{f \in B_0} \varepsilon_f \frac{S_-(t_f)}{L(1, \text{ad } f)} h(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E_-(t)}{|\zeta(1 + 2it)|^2} h(t) \, dt \ll \varepsilon \, t^{7+\varepsilon}_g U^{-7/2}. \]

The proof of Proposition 10.8, which we give in Section 10.3, proceeds via a series of steps. Taking this result for granted for the time being, we proceed to the proof of Proposition 9.3.

**Proof of Proposition 9.3.** We use Lemmata 10.1 and 10.6 to express each of the \( L \)-functions \( L(1/2, \text{ad } g \otimes f) \) and \( |L(1/2 + it)\text{, ad } g|^2 \) as Dirichlet series. The sums involving \( S_+(t_f) \) and \( E_+(t) \) are dealt with using (10.10). The sums involving \( s_\pm(t_f) \) are seen to be \( O(t_3^{2/7}gU^{-1/2}) \) by using the spectral large sieve after an application of the Cauchy–Schwarz inequality, keeping in mind Remark 10.7. In this way, we are left to deal with the sums involving \( S_-(t_f) \) and \( E_-(t) \). With \( h(t) \) as in (10.9), we deduce that
\[ \sum_{f \in B_0, 2t_g - U - U^{1-\varepsilon} \leq t_f \leq 2t_g - U + U^{1-\varepsilon}} \frac{L \left( \frac{1}{2}, \text{ad } g \otimes f \right)}{L(1, \text{ad } f)} + \frac{1}{2\pi} \int_{2t_g - U - U^{1-\varepsilon} \leq |t| \leq 2t_g - U + U^{1-\varepsilon}} \frac{|L \left( \frac{1}{2} + it, \text{ad } g \right)|^2}{\zeta(1 + 2it)} \, dt \ll \varepsilon \, t^{3+\varepsilon}_g U^{-\frac{1}{2}}. \]

Recall that \( S_-(t_f) \) and \( E_-(t) \) are sums of length \( O(t_3^{2+\varepsilon}) \). We proceed differently according to the size of \( U \). For \( U \leq t^{1/5}_g \), we apply the spectral large sieve to get the bound
\[ \sum_{f \in B_0} \varepsilon_f \frac{S_-(t_f)}{L(1, \text{ad } f)} h(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E_-(t)}{|\zeta(1 + 2it)|^2} h(t) \, dt \ll t^{3+\varepsilon}_g U^{1/2}. \]

For \( t^{1/5}_g \leq U \leq t^{1/3+\varepsilon}_g \), we apply (10.11). This yields (9.4). \( \square \)

10.3. **Proof of Proposition 10.8.** Let \( t^\varepsilon_g \leq U \leq t^{1/3+\varepsilon}_g \). By the Kuznetsov formula (2.2), we have that
\begin{align*}
\sum_{f \in B_0} \varepsilon_f \frac{S_+(t_f)}{L(1, \text{ad } f)} h(t_f) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E_+(t)}{|\zeta(1 + 2it)|^2} h(t) \, dt &= \delta_{\pm, +} \sum_{m=1}^{\infty} A_f(m, 1) \frac{E_+(t)}{m} \int_{-\infty}^{\infty} V_+(m^2, t) h(t) \, dt \\
&= \sum_{c,m,n=1}^{\infty} A_f(m, n) S(1, n; c) \frac{m V_+(m^2, t)}{c} \left( \frac{\sqrt{n}}{c} \right). 
\end{align*}
The diagonal contribution (10.12) is $O_{\varepsilon}(t_{g}^{1+\varepsilon}U)$ by bounding trivially. The main challenge is to treat the off-diagonal (10.13). We note first of all that we can crudely truncate the off-diagonal sum to $c \leq t_{g}^{1000}$, up to a negligible error term of size $O(t_{g}^{-1000})$, in a standard way by using Weil’s bound for Kloosterman sums and shifting the $t$-integral in the definition of $(\mathcal{X}^+ h V_{\pm}(m^2 n, \cdot))(\sqrt{t_{g}} t)$ to $\Im(t) = -\frac{1}{2} + \varepsilon$ or $\Im(t) = \frac{1}{2} - \varepsilon$ and then bounding absolutely, as done in [Blo12, Lemma 5].

10.3.1. The Positive Sign Case of the Off-Diagonal. We first prove the bound (10.10). The main idea is that the approximate functional equation sum is short due to conductor dropping while the spectral family we average over is relatively large. We show the following.

**Lemma 10.14.** We have that

\[(\mathcal{X}^+ h V_{+}(m^2 n, \cdot))(x) \ll t_{g}^{-500}\]

unless $x \geq U^{1-\varepsilon/3}t_{g}$.

We use this to deduce the bound (10.10).

**Proof of Proposition 10.8 (1).** We must show that the off-diagonal term (10.13) is $O_{\varepsilon}(t_{g}^{1+\varepsilon}U)$. Since we may restrict to $m^2 n \ll U^{1-\varepsilon}t_{g}^{2}$ by the decay properties (10.5) of $V_{+}(m^2 n, \cdot)(x)$, no value of $c \in \mathbb{N}$ can satisfy $\sqrt{t_{g}} t \geq U^{1-\varepsilon/3}t_{g}$ once $t_{g}$ is sufficiently large. Coupled with (10.15), this shows that the off-diagonal is negligibly small, and so the only contribution comes from the diagonal, which trivially satisfies the required bound (10.10). \hfill \Box

**Proof of Lemma 10.14.** It suffices to prove this for $h(t)$ redefined as

\[h(t) = \exp \left( - \left( \frac{t - 2t_{g} - U}{U^{1-\varepsilon}} \right)^{2} \right) \]

We require the derivative bounds

\[(10.16) \quad \frac{d^{j}}{dt^{j}} h(t) V_{\pm}(m^2 n, t) \ll_{j} (U^{1-\varepsilon})^{j} \]

for $|t - 2t_{g}| = U$ and $j \in \mathbb{N}_{0}$. Assuming this for the time being, we proceed. In the integral defining $(\mathcal{X}^+ h V_{+}(m^2 n, \cdot))(x)$ given by (2.4), we insert a smooth bump function $W(t)$ compactly supported on

\[(10.17) \quad |t - 2t_{g} - U| \leq U^{1-\varepsilon} \]

and satisfying $\|W(j)\| \ll_{j} (U^{1-\varepsilon})^{j}$ for $j \in \mathbb{N}_{0}$, since $h(t)$ decays rapidly outside this interval. Then using the identity [Wat44, p. 180]

\[\frac{J_{2u}(2\pi x)}{\cosh \pi t} - J_{-2u}(2\pi x)}{\cosh \pi t} = -2i \tanh(\pi t) \int_{-\infty}^{\infty} \cos(2\pi x \cosh \pi v - 2\pi tu) dv \]

and the fact that $\tanh \pi t = 1 + O(e^{-t})$ within the support of $W(t)$, we see that it suffices to show that

\[\int_{-\infty}^{\infty} e(x \cosh \pi v) \int_{-\infty}^{\infty} h(t) V_{+}(m^2 n, t) W(t) e(-vt) dt dv \ll t_{g}^{-100} \]

unless $x \geq U^{1-\varepsilon/3}t_{g}$. By integrating by parts multiple times in the inner integral and using (10.16), we deduce that the required bound holds for the portion of the outer integral for which $|v| \gg U^{-1+\varepsilon}$. To treat the remaining portion of the outer integral, we insert a smooth bump function $\Omega(U v)$ such that $\Omega$ is supported on $(-U^{\varepsilon}, U^{\varepsilon})$ with derivatives satisfying $\|\Omega^{(j)}\| \ll_{j} (U^{\varepsilon})^{j}$ for $j \in \mathbb{N}_{0}$. Thus it suffices to show that under the assumption (10.17), we have that

\[(10.18) \quad \int_{-\infty}^{\infty} e \left( x \cosh \frac{\pi v}{U} - \frac{vt}{U} \right) \Omega(v) dv \ll t_{g}^{-300} \]

if $x < U^{1-\varepsilon/3}t_{g}$. This follows by integration by parts after observing that the phase

\[\phi(v) := x \cosh \frac{\pi v}{U} - \frac{vt}{U} \]
satisfies
\[ |\phi'(v)| = \left| \frac{x}{U} \sinh \frac{\pi v}{U} - \frac{t}{U} \right| \gg \frac{t_g}{U}, \]
and
\[ |\phi^{(j)}(v)| \ll_j x U^{-j+\varepsilon}. \]
for \( j \geq 2 \). We apply [BKY13, Lemma 8.1] with \( R = t_g/U, Y = x, \) and \( Q = U \) being the key parameters, which tells us that the integral (10.18) is negligible provided \( R \gg t_g^\varepsilon \) and \( QRY^{-\frac{1}{4}} \gg t_g^\varepsilon \), which is the case here.

It remains to establish (10.16). It is clear that the derivatives of \( h(t) \) satisfy the bound, so we just need to consider the derivatives of \( V_\pm(m^2n,t) \). By taking \( \sigma = \varepsilon \) in (10.2) and using the rapid decay of \( e^{x^2} \), it suffices to show that
\[ \frac{d^j G(\frac{1}{2} + s, t)}{dv} G(\frac{1}{2}, t) \ll_j (U^{-1+\varepsilon})^j \]
for \( |s| \leq t_g^\varepsilon \) and \( j \in \mathbb{N}_0 \). By Stirling’s estimates (see [BlK19b, (2.4)]), for \( y \geq t_g^\varepsilon \) and \( \kappa > 0 \) a fixed constant, there exists some function \( \psi_{s,M}(y) \) satisfying \( \psi^{(j)}_{s,M}(y) \ll_j y^{-j} \) for any \( j \in \mathbb{N}_0 \) such that
\[ \frac{\Gamma_R(s + \kappa + iy)}{\Gamma_R(\kappa + iy)} = y^{\frac{j}{2}} \psi_{s,M}(y) + O_{s,M}(y^{-M}) \]
for any \( M > 0 \). This gives (10.19) (noting that the error term is the same under differentiation by Cauchy’s integral formula) since each gamma ratio \( \frac{G(1/2+s,t)}{G(1/2,t)} \) can be written as \( \frac{\Gamma_R(s+\kappa+iy)}{\Gamma_R(\kappa+iy)} \) with \( y \gg U \) under the assumption (10.17).

10.3.2. The Negative Sign Case of the Off-Diagonal. We next prove the bound (10.11). In this case, the off-diagonal is not empty but it is very nearly so; that is, we shall show that we may restrict the sum over \( c \in \mathbb{N} \) to \( c \ll t_g^\varepsilon \). An analysis of the \( \mathcal{K}^- \)-transform then shows that the off-diagonal sum may be restricted to a short interval (see (10.24) and (10.27)). We would then like to bound trivially, but two problems arise that we must circumvent. First, we do not know the Ramanujan conjecture, or efficient versions of this on average over short intervals; we instead use \( L \)-functions and the spectral large sieve to obtain reasonable averaged bounds for sums of Hecke eigenvalues. Second, even if we were able to appeal to the Ramanujan conjecture, a trivial bound is insufficient in the range \( U \leq t_g^{1/3-\varepsilon} \). We get further savings by obtaining some stationary phase cancellation in the \( \mathcal{K}^- \)-transform in Lemma 10.22.

Proof of Proposition 10.8 (2). We assume that
\[ t_g^{1/2+\varepsilon} \leq U \leq t_g^{3/4+\varepsilon}. \]
It suffices to consider \( h(t) \) redefined as
\[ h(t) = \exp \left( -\left( t - 2t_g - U \right) \frac{U^{1-\varepsilon}}{U^{1-\varepsilon}} \right). \]
We must show that
\[ \sum_{c,m,n=1}^\infty \frac{A_F(m,n)}{m} \frac{S(1,n;c)}{c} \int_{-\infty}^{\infty} \cosh \pi t K_{2it} \left( \frac{4\pi \sqrt{n}}{c} \right) h(t) V_-(m^2n,t) W(t) t \tanh \pi t dt \]
is \( O_{c}(t_g^{7/4+\varepsilon} U^{-5/4}) \), where \( W(t) \) is a bump function defined exactly as in the proof of Lemma 10.14 with support as in (10.17). Using the identity [GR15, 8.432.4]
\[ \sinh \pi t K_{2it}(2\pi x) = \frac{\pi \tanh \pi t}{2} \int_{-\infty}^{\infty} \cos(2\pi x \sinh \pi v) c(tv) dv, \]
it suffices to prove this bound for
\[ t_g \sum_{c,m,n=1}^{\infty} \frac{A_F(m,n) S(1,n;c)}{\sqrt{n}} \frac{1}{c} \int_{-\infty}^{\infty} e \left( -\frac{2\pi \sqrt{n}}{c} \sin \pi v \right) \int_{-\infty}^{\infty} e(tv) h(t) V_- (m^2 n, t) W(t) \, dt \, dv. \]

The desired bound holds for the portion of the outer integral for which \(|v| \geq U^{-1+\varepsilon}\) by (10.16) and repeated integration by parts in the inner integral. To treat the remaining portion of the outer integral, we insert a smooth bump function \(\Omega(Uv)\) as defined exactly as in the proof of Lemma 10.14, so that we are left with
\[ t_g \sum_{c,m,n=1}^{\infty} \frac{A_F(m,n) S(1,n;c)}{\sqrt{n}} \frac{1}{c} \int_{-\infty}^{\infty} e \left( -\frac{2\pi \sqrt{n}}{c} \sin \pi v \right) \left( -\frac{2\pi \sqrt{v}}{c} \sin \pi v + \frac{tv}{U} \right) \Omega(v) \int_{-\infty}^{\infty} h(t) V_- (m^2 n, t) W(t) \, dt \, dv. \]

If \(\frac{\sqrt{n}}{c} \leq t_g^{1-\varepsilon}\), then the outer integral is readily seen to be negligible via repeated integration by parts using [BKY13, Lemma 8.2] with \(R = t_g/U\), \(Y = \sqrt{n}/c\), and \(Q = U\). Thus we may restrict to \(\frac{\sqrt{n}}{c} > t_g^{1-\varepsilon}\); as we may additionally restrict to \(nm^2 \ll U^\varepsilon t_g\) by (10.5), we are left with the ranges
\[ t_g^{2-\varepsilon} < n < t_g^{2+\varepsilon}, \quad 1 \leq c, m \leq t_g^{\varepsilon}. \]

A further simplification we can make is the following. Since \(U \geq t_g^{1/5+\varepsilon}\), we may take the power series expansion of \(\sinh \frac{\pi v}{c}\) and absorb the exponential of all terms in the expansion beyond the cubic term into the weight function \(\Omega(v)\). After opening up the Kloosterman sum \(S(1,n;c) = \sum_{a \in (\mathbb{Z}/c \mathbb{Z})^\times} e\left( \frac{a+n}{c} \right)\), using the Hecke relations (4.7), and making the change of variables \(n \mapsto Cn\), we are thereby left with showing that
\[ t_g \sum_{n=1}^{\infty} \frac{A_F(1,n) e\left( \frac{adn}{c} \right)}{\sqrt{n}} \Psi\left( \frac{n t_g^2}{c} \right) \int_{-\infty}^{\infty} e \left( -\frac{2\pi \sqrt{n} \sqrt{v}}{c} \sin \pi v \right) \left( \frac{\pi v}{U} + \frac{\pi^3 v^3}{3 U^3} \right) + \frac{tv}{U} \right) \Omega(v) \, dv \ll_{\varepsilon} t_g^{1+2\varepsilon} U^{-\frac{3}{2}} \]
for any \(t \asymp 2t_g\), \(1 \leq c, d \leq t_g\), and \(a \in (\mathbb{Z}/c \mathbb{Z})^\times\), where \(\Psi\) is a smooth function, compactly supported on \((t_g^{-\varepsilon}, t_g^{\varepsilon})\), with derivatives satisfying \(\|\Psi^{(j)}\| \ll_j (t_g^{2j})^j\) for any \(j \in \mathbb{N}_0\). By Mellin inversion, this expression is equal to
\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} t_g^{1+2s} \Phi_F \left( \frac{c}{(c,d)^{\gamma}}, \frac{ad}{(c,d)^{\gamma}}, 1; \frac{1}{2} + s \right) I(s) \, ds, \]
where \(\sigma > 0\), \(\Phi_F\) is the Voronoi series (2.30), and
\[ (10.20) \quad I(s) := \int_0^{\infty} \left( \int_{-\infty}^{\infty} e \left( -\frac{2\pi t_g \sqrt{adx}}{c} \left( \frac{\pi v}{U} + \frac{\pi^3 v^3}{3 U^3} \right) + \frac{tv}{U} \right) \Omega(v) \, dv \right) \Psi(x) x^{-s-1} \, dx. \]
We may shift the line of integration to \(\sigma = 0\) and repeatedly integrate by parts in (10.20) in order to restrict the line of integration to the range
\[ |s| \leq \frac{t_g^{1+\varepsilon}}{U} \]
at the cost of a negligible error term. To bound the remaining integral, we show in Lemma 10.22 that \(I(iy) \ll_{\varepsilon} t_g^{-1/2+\varepsilon} U^{-1/2}\), at which point we are left with showing that
\[ (10.21) \quad \int_{|y| \leq \frac{t_g^{1+\varepsilon}}{U}} \left| \Phi_F \left( \frac{c}{(c,d)^{\gamma}}, \frac{ad}{(c,d)^{\gamma}}, 1; \frac{1}{2} + iy \right) \right| dy \ll_{\varepsilon} t_g^{\frac{3}{2}+\varepsilon} U^{-\frac{3}{2}}. \]
Via the functional equation (2.31) for \(\Phi_F\), namely the Voronoi summation formula, we may write \(\Phi_F\) in terms of a Dirichlet polynomial of length at most \(t_g^{1+\varepsilon}|y| + 1)\). Thus (10.21) follows by the Cauchy–Schwarz inequality and the Montgomery–Vaughan mean value theorem for Dirichlet polynomials [MV74, Corollary 3].
10.3.3. A Stationary Phase Estimate. It remains to prove the following estimate, which we invoked in the proof of Proposition 10.8 (2).

Lemma 10.22. We have that

\begin{align}
(10.23) \quad \frac{\int_{t^{-\varepsilon} g}^{t^\varepsilon g}}{\int_{t^{-\varepsilon} g}^{t^\varepsilon g}} \left| e \left( \left( \frac{v}{U} - \frac{2\pi^2 t_g x}{Uc} \right) - v^3 \frac{\pi^4 t_g x}{3U^3 c} \right) \right| \Omega(v) dv \right| dx \ll t_g^{-\frac{1}{2}+\varepsilon} U^{-\frac{1}{2}}.
\end{align}

Proof. First we consider the contribution of the small values of \( v \). Let \( V_0 := (\frac{t_g}{U})^{-1/3+\varepsilon} \). The range \(|v| < V_0\) may essentially be picked out using a bump function \( \Omega_0(v/V_0) \), where \( \Omega_0 \) is smooth, compactly supported on \([-1, 1]\) and has bounded derivatives. In this range, the exponential of the cubic term in the phase may be absorbed in the weight function \( \Omega(v) \), since this exponential has derivatives bounded by powers of \( t_g \). Then by integrating by parts repeatedly in the integral

\[ \int_{-\infty}^{\infty} e \left( \left( \frac{v}{U} - \frac{2\pi^2 t_g x}{Uc} \right) - v^3 \frac{\pi^4 t_g x}{3U^3 c} \right) \Omega(v) \Omega_0 \left( \frac{v}{V_0} \right) dv, \]

we see that we may restrict to

\[ (10.24) \quad \left| \frac{v}{U} - \frac{2\pi^2 t_g x}{Uc} \right| \ll t_g^\varepsilon \frac{1}{V_0}, \]

which implies that the outer integral is restricted to an interval of size at most \( t_g^\varepsilon \frac{U}{t_g V_0} \). Using this and the fact that the inner integral is restricted to an interval of size \( O(V_0) \), by the support of \( \Omega_0 \), we bound trivially to get the bound \( O(t_g^\varepsilon \frac{U}{t_g V_0}) \) for the double integral, which is sufficient.

Now we consider the contribution of larger values of \( v \); by symmetry, it suffices to consider \( v \) positive. For

\[ (10.25) \quad \left( \frac{t_g}{U^3} \right)^{-\frac{1}{2}+\varepsilon} \leq V_1 \leq t_g^\varepsilon, \]

we insert a bump function \( \Omega_1(v/V_1) \), where \( \Omega_1 \) is smooth, compactly supported on \([1, 2]\) and has \( j \)-th derivative bounded by \( (t_g^\varepsilon)^j \). After a substitution, the inner integral is equal to

\[ (10.26) \quad V_1 \int_{-\infty}^{\infty} e(\phi(v)) \Omega_1(v) dv, \]

where \( \Omega(v V_1) \) has been absorbed into \( \Omega_1(v) \), and

\( \phi(v) := v \left( \frac{t V_1}{U} - \frac{2\pi^2 t_g x V_1}{U c} \right) - v^3 \frac{\pi^4 t_g x V_1^3}{3U^3 c} \)

is the phase, with derivatives

\[ \phi'(v) = \frac{1}{U} \left( t V_1 - \frac{2\pi^2 t_g x V_1}{c} \right) - t_g x \frac{\pi^4 V_1^3 v^2}{c}, \]

\[ \phi''(v) = - \frac{2\pi^4 V_1^3 v}{U^2 c}. \]

First, by repeated integration by parts, we see that we may restrict to

\[ (10.27) \quad \left| \frac{t V_1}{U} - \frac{2\pi^2 t_g x V_1}{U c} \right| \ll t_g^\varepsilon \frac{V_1^3}{U^3}. \]

To see this, one may use [BKY13, Lemma 8.1] with the parameters therein being \( R = \frac{t_g V_1^3}{U^3} \), \( Y = t_g \), and \( Q = \left( \frac{U}{t_g} \right)^{3/2} \). By (10.27), the outer integral is restricted to an interval of size at most \( t_g^\varepsilon \frac{U}{t_g V_1^3} \). Now we will bound (10.26) by \( V_1 |\phi''|^{-\frac{1}{2}} \approx (\frac{t_g V_1}{U})^{-\frac{1}{2}} \). This is expected by stationary phase analysis in the inner integral, but we give the details below. Once we have this bound, the double integral is seen to be bounded by \( t_g^\varepsilon \frac{U}{t_g V_1^3} V_1^2 \). By taking the maximum over \( V_1 \) in the range (10.25), we obtain the required bound in (10.23).
Case 1: $tV_1 - \frac{2\pi^2 xt_g V_1}{c} > 0$. In this range, there are two stationary points, possibly outside the interval $[1, 2]$. Let $\nu$ be the positive stationary point, so that $\nu > 0$ and $\phi'(\nu) = 0$. We split the integral in (10.26) into two integrals $I_1 + I_2$, where $I_1$ is an integral over

$$S := \left\{ v \in [1, 2] : |v - \nu| \leq \left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}} \right\},$$

and $I_2$ is an integral over $[1, 2]\setminus S$. Bounding trivially, we have $|I_1| \ll \left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}}$. For $I_2$, we integrate by parts once as follows:

$$I_2 = \int_{[1, 2]\setminus S} \frac{\phi'(v)}{\phi''(v)} e(\phi(v)) \Omega_1(v) \, dv \ll V_1 \int_{[1, 2]\setminus S} \left| \frac{\phi''(v) \Omega_1(v)}{\phi'(v)} - \frac{\phi''(v) \Omega_1(v)}{(\phi'(v))^2} \right| e(\phi(v)) \, dv + t_g V_1 \sup_{v \in [1, 2]\setminus S} \left| \frac{1}{\phi'(v)} \right|.$$

Writing $v = \nu + u$ with $|u| \geq \left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}}$, we have that for $v \in [1, 2]\setminus S$,

$$|\phi'(v)| = \left| \frac{\pi^4 xt_g V_1^3}{cU^3} (2\nu u + u^2) \right| = \left| \frac{\pi^4 xt_g V_1^3}{cU^3} (\nu + v)u \right| \gg \left| \frac{\pi^4 xt_g V_1^3}{cU^3} u \right| \gg t_g^{-\varepsilon} \left( \frac{t_g V_1^3}{U^3} \right)^{\frac{1}{2}}.$$

Thus

$$V_1 \sup_{v \in [1, 2]\setminus S} \left| \frac{1}{\phi'(v)} \right| \ll t_g^{\varepsilon} \left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}}, \quad V_1 \int_{[1, 2]\setminus S} \left| \frac{\phi''(v) \Omega_1(v)}{\phi'(v)} \right| \, dv \ll t_g^{\varepsilon} \left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}},$$

and

$$V_1 \int_{[1, 2]\setminus S} \left| \frac{\phi''(v) \Omega_1(v)}{(\phi'(v))^2} \right| \, dv \ll t_g^{\varepsilon} \left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}} \int_{\left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}}}^{2} \frac{1}{u} \, du \ll t_g^{\varepsilon} \left( \frac{t_g V_1^3}{U^3} \right)^{-\frac{1}{2}}.$$

Case 2: $tV_1 - \frac{2\pi^2 xt_g V_1}{c} \leq 0$. In this range, there is no stationary point, and we have that

$$|\phi'(v)| \geq t_g^{-\varepsilon} \left( \frac{t_g V_1^3}{U^3} \right)^{\frac{1}{2}}$$

for $v \in [1, 2]$. We integrate by parts once as in (10.28), and then the required bound follows easily by using (10.29).

\[\square\]

11. Bounds for Mixed Moments of $L$-Functions in the Tail Range

11.1. Proof of Proposition 1.9 (4). The proof of Proposition 1.9 (4), namely the bound (1.13) for the tail range, follows in a straightforward manner from bounds attained via the spectral large sieve.

Proof of Proposition 1.9 (4). By the lower bound $L(1, ad g) \gg_{\varepsilon} t_g^{-\varepsilon}$ and the asymptotic formula (1.14) for $H(t)$, it suffices to show that

$$\sum_{j \in B_0, |t_j| \geq 2t_g} e^{-\pi (t_j - 2t_g)} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, ad g \otimes f \right)}{L(1, ad f)} \ll_{\varepsilon} t_g^{\varepsilon},$$

and

$$\frac{1}{2\pi} \int_{|t| \geq 2t_g} e^{-\pi (|t| - 2t_g)} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, ad g \right)}{\zeta(1 + 2it)} \right|^2 \, dt \ll_{\varepsilon} t_g^{\varepsilon}.$$
We dyadically decompose both the sum over $f$ and the integral over $t$, so that we are left with proving the bounds

$$
\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f)} \leq \varepsilon t_g^{\frac{3}{2} + \varepsilon} U^{\frac{1}{2}} e^{3\pi U}
$$

$$
\frac{1}{2\pi} \int_{-U \leq |t| \leq U} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} g \right)}{\zeta(1 + 2it)} \right|^2 dt
$$

for $2t_g \leq T \leq 3t_g$ and $U = \frac{T}{2} + 1 - t_g$, as well as the bounds

$$
\sum_{f \in B_0} \frac{L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right)}{L(1, \text{ad} f)} \leq \varepsilon t_g^{\varepsilon T} T^2 (\log T)^{-2} e^{\pi(T - 2t_g)}
$$

$$
\frac{1}{2\pi} \int_{-U \leq |t| \leq U} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} g \right)}{\zeta(1 + 2it)} \right|^2 dt
$$

for $T \geq 3t_g$. Via the Cauchy–Schwarz inequality, the former follows (with polynomial room to spare unless $U = o(\log t_g)$) from the bounds (6.3) and (9.2) from Propositions 6.1 and 9.1 arising from the spectral large sieve, while the latter follows (with exponential room to spare) from the bounds (6.2) and (6.3).

\[\square\]

12. Extensions and Improvements

We finish by sketching how the methods in this paper extend to yield Theorem 1.4 and discussing some conditional approaches that lead to strengthenings of Theorems 1.1 and 1.4.

12.1. A Sketch of the Proof of Theorem 1.4. The method of proof of Theorem 1.1 can readily seen to extend to Hecke–Maaß newforms on $\Gamma_0(q) \setminus \mathbb{H}$. The key reason for this is that all of the tools used, such as the Watson–Ichino triple product formula and various spectral reciprocity formulæ, remain applicable in this more general setting. Moreover, all of the estimates for various moments of $L$-functions given in this paper are purely archimedean in nature, and so the same estimates hold on $\Gamma_0(q) \setminus \mathbb{H}$ (albeit with unspecified dependence on $q$). We list below the major alterations required in order to extend Theorem 1.1 in this direction.

1. Via Parseval’s identity for $L^2(\Gamma_0(q) \setminus \mathbb{H})$, for $g$ a Hecke–Maaß newform on $\Gamma_0(q) \setminus \mathbb{H}$, we express $\|g\|_a^4$ in terms of a spectral expansion of triple products of automorphic forms. Choosing an explicit orthonormal basis of cusp forms and Eisenstein series in terms of newforms and oldforms, and then applying the Watson–Ichino triple product formula, we obtain a level $q$ analogue of the identity (1.7), namely

$$
\int_{\Gamma_0(q) \setminus \mathbb{H}} |g(z)|^4 \frac{3}{\pi [\Gamma : \Gamma_0(q)]} \frac{dx \, dy}{y^2} = 1 + \sum_{q_1 q_2 = q} \sum_{f \in B_0(\Gamma_0(\Gamma_0(q_1)))} c_{f, g, q_1} L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} g \otimes f \right) \frac{L(1, \text{ad} f)L(1, \text{ad} g)}{H(t_f)}
$$

$$
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} c_{g, q_1} \left| \frac{\zeta \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, \text{ad} g \right)}{\zeta(1 + 2it)L(1, \text{ad} g)} \right|^2 dt.
$$

(Cf. [HK20, Propositions 1.13 and 1.16].) Here $B_0(\Gamma_0(q_1))$ denotes an orthonormal basis of Hecke–Maaß newforms of level $q_1$, while $c_{f, g, q_2}, c_{g, q_1}$ are local constants arising from the Watson–Ichino triple product formula that are bounded by a constant dependent only on $q$.

2. Next, we derive level $q$ analogues of the $\text{GL}_3 \times \text{GL}_2 \rightarrow \text{GL}_4 \times \text{GL}_1$ and $\text{GL}_4 \times \text{GL}_2 \rightarrow \text{GL}_4 \times \text{GL}_2$ spectral reciprocity formulæ given in Theorems 3.1 and 4.1. The methods of proof are essentially identical; the chief modifications are the usage the Kuznetsov and Petersson formulæ for $\Gamma_0(q) \setminus \mathbb{H}$ associated to $(\infty, 0)$ pair of cusps [HK20, Theorems A.16...
and A.19], which naturally introduces the root number into this formula, and the usage the GL3 Voronoi summation formula for \( \text{ad} \, g \) with \( g \) of level \( q \) [HL23].

(3) Once we have a level \( q \) analogue of Theorem 4.1, in order to prove the level \( q \) analogue of Proposition 6.11 (which in turn yields the level \( q \) analogue of Proposition 1.9 (1)), we require level \( q \) analogues of Propositions 6.1, 6.6, and 6.8. The former result is immediate since the spectral large sieve also holds for level \( q \) cusp forms, the second result follows from the level \( q \) analogue of Theorem 3.1, while the latter result follows from [HK23, Theorem 7.1].

(4) The proof of the level \( q \) analogue of Proposition 1.9 (2) is via the identical method except using the level \( q \) Kuznetsov formula (cf. [HK20, Proof of Proposition 1.21 (2)]).

(5) Finally, to prove the level \( q \) analogue of Proposition 1.9 (3), we require the level \( q \) analogue of Proposition 9.7. In turn, this requires level \( q \) analogues of Propositions 9.1, 9.3, and 9.5. The former result is again an immediate consequence of the spectral large sieve, the second result is via the same method of proof except using the level \( q \) Kuznetsov formula, and the final result follows from [AW23, Theorem 4.1].

To prove Theorem 1.4 for Hecke–Maaß newforms on \( \Gamma^D \backslash \mathbb{H} \), where \( D \) is the indefinite quaternion division algebra over \( \mathbb{Q} \) of squarefree discriminant \( q \), we begin again via Parseval’s identity for \( L^2(\Gamma^D \backslash \mathbb{H}) \) coupled with the Watson–Ichino triple product formula, which yields an appropriate analogue of the identity (1.7) of the form

\[
\int_{\Gamma^D \backslash \mathbb{H}} |g(z)|^4 \frac{3}{\pi [\Gamma : \Gamma^D]} \frac{dx \, dy}{y^2} = 1 + \sum_{f \in B_0^+(\Gamma^D)} c_{f,g,q} L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right) H(t_f).
\]

Here \( B_0^+(\Gamma^D) \) denotes an orthonormal basis of Hecke–Maaß cusp forms for \( \Gamma^D \backslash \mathbb{H} \) (which are all newforms), while \( c_{f,g,q} \) are once more local constants arising from the Watson–Ichino triple product formula that are bounded in absolute value by a constant dependent only on \( q \); note that there is no integral over \( t \in \mathbb{R} \) since the compactness of \( \Gamma^D \backslash \mathbb{H} \) means that there is no continuous spectrum of the Laplacian. Via the Jacquet–Langlands correspondence, each \( f \in B_0^+(\Gamma^D) \) corresponds bijectively with a Hecke–Maaß newform on \( \Gamma_0(q) \backslash \mathbb{H} \) with identical spectral parameter and Hecke eigenvalues. Thus we in turn have that

\[
\int_{\Gamma^D \backslash \mathbb{H}} |g(z)|^4 \frac{3}{\pi [\Gamma : \Gamma^D]} \frac{dx \, dy}{y^2} = 1 + \sum_{f \in B_0^+(\Gamma_0(q))} c_{f,g,q} L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, \text{ad} \, g \otimes f \right) H(t_f),
\]

at which point the desired result now follows by the same method sketched above for Hecke–Maaß newforms on \( \Gamma_0(q) \backslash \mathbb{H} \).

12.2. Conditional Improvements via the Generalised Lindelöf Hypothesis. As discussed in Section 1.2.3, Watson observed that the essentially optimal bound \( |g|_4 \ll_q \lambda_g^q \) follows from the generalised Lindelöf hypothesis for GL3 × GL2 Rankin–Selberg L-functions and GL2 standard L-functions. Our method of proof of Theorems 1.1 and 1.4 demonstrates that the same holds under a slightly weaker assumption.

Proposition 12.1. Let \( g \) be a Hecke–Maaß newform of Laplacian eigenvalue \( \lambda_g \) on either \( \Gamma_0(q) \backslash \mathbb{H} \) or \( \Gamma^D \backslash \mathbb{H} \), where \( q \) is squarefree and fixed and \( D \) is the indefinite quaternion division algebra over \( \mathbb{Q} \) of discriminant \( q \). Under the assumption of the generalised Lindelöf hypothesis for GL2 standard L-functions, \( ||g||_4 \ll \epsilon \nu^q \).

Sketch of proof. From Remarks 6.17 and 9.10, all that is needed is the improved bound \( O_\epsilon(t_{1/2 + \epsilon}^3 T) \) for (6.16) in the ranges \( T \leq t_g^{3/13} \) and \( t_g^{10/13} \leq T \leq t_g^{-\alpha} \) and the improved bound \( O_{\epsilon}(t_g^{3/2 + \epsilon} U^{1/2}) \) for (9.9). The former holds immediately in the range \( t_g^{10/13} \leq T \leq t_g^{-\alpha} \) by the assumption \( L(1/2, f) \ll \epsilon \nu^q t_f^\epsilon \) and \( |\zeta(1/2 + it)|^2 \ll \epsilon (1 + |t|)^\epsilon \) in conjunction with the bounds (6.7); GL4 × GL2 \( \sim \sim \) GL4 × GL2 spectral reciprocity then yields the same result in the range \( T \leq t_g^{3/13} \). The latter holds for \( U \leq t_g^{1/3} \) by the same assumption in conjunction with the bounds (9.4);
once more, \( \text{GL}_4 \times \text{GL}_2 \sim \sim \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity then yields the same result in the range \( t_g^{1/3} \leq U \leq t_g^{1-\alpha} \).

Similar conditional analogues of Proposition 12.1 also hold for the \( L^4 \)-norm of holomorphic Hecke cusp forms in the weight aspect [BKY13, Theorem 1.4] and in the level aspect [BuK15, Theorem 1.1].

We may also obtain the improved bound \( O_\varepsilon(t_g^{1+\varepsilon} T) \) for (6.16) in the ranges \( T \leq t_g^{3/13} \) and \( t_g^{10/13} \leq T \leq t_g^{1-\alpha} \) and the improved bound \( O_\varepsilon(t_g^{3/2+\varepsilon} U^{1/2}) \) for (9.9) in the range \( t_g^{1/3} \leq U \leq t_g^{1-\alpha} \) under a different assumption, namely the Lindelöf-on-average bound

\[
\text{Theorem 1.1}.
\]

We obtain the improved bound \( \int_{U}^{2U} \left| L\left( \frac{1}{2} + it, \text{ad } g \right) \right|^2 \ dt \ll_{\varepsilon} U^{1+\varepsilon} \)

uniformly for \( t_g^{34/25} \leq U \leq t_g^2 \). This conditional strengthening of (6.5) would yield the improved bound \( O_\varepsilon(t_g^{1+\varepsilon} + t_g^{7/10+\varepsilon} T^2) \) for (6.7) in the range \( T \leq t_g^{3/13} \), which would ensure the requisite bound \( O_\varepsilon(t_g^{1+\varepsilon} T) \) for (6.16) in this range. An application of \( \text{GL}_4 \times \text{GL}_2 \sim \sim \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity would then yield this same requisite bound in the range \( t_g^{10/13} \leq T \leq t_g^{1-\alpha} \) as well as the requisite bound \( O_\varepsilon(t_g^{3/2+\varepsilon} U^{1/2}) \) for (9.9) in the range \( t_g^{1/3} \leq U \leq t_g^{1-\alpha} \). This reduction to the assumption (12.2) can be thought of as a “reduction to Eisenstein observables” akin to the work of Nelson [Nel19a]. Unfortunately, while an unconditional proof of (12.2) is not inconceivably unrealistic using current technology, the best known estimates in this regard fall shy of what is required (cf. [ALM22, Pal22]).

12.3. Conditional Improvements via Fifth Moment Bounds. An alternate conditional approach to improving Theorems 1.1 and 1.4 would be to appeal to the conditional fifth moment bounds

\[
\sum_{\substack{f \in \mathcal{B}_0 \\ T \leq t_f \leq 2T}} \frac{L\left( \frac{1}{2}, f \right)^5}{L(1, \text{ad } f)} \left. \right| \frac{\zeta\left( \frac{1}{2} + it \right)^5}{\zeta\left( 1 + 2it \right)} \right| dt \ll_{\varepsilon} T^{2+\varepsilon}.
\]

\[
(12.3)
\]

Proposition 12.4. Let \( g \) be a Hecke–Maass newform of Laplacian eigenvalue \( \lambda_g \) on either \( \Gamma_0(q) \backslash \mathbb{H} \) or \( \Gamma^D \backslash \mathbb{H} \), where \( q \) is squarefree and fixed and \( D \) is the indefinite quaternion division algebra over \( \mathbb{Q} \) of discriminant \( q \). Under the assumption of (12.3), \( \|g\|_4 \ll_{\varepsilon} \lambda_g^{1/104+\varepsilon} \).

Sketch of proof. Using Hölder’s inequality with exponents \((1/5, 1/5, 3/5)\) and combining (6.2), (6.7), and the assumption (12.3), we obtain the improved bounds \( O_\varepsilon(t_g^{2/5+\varepsilon} T^{9/5}) \) for (6.16) in the range \( t_g^{14/17} \leq T \leq t_g^{11/13} \). Via \( \text{GL}_4 \times \text{GL}_2 \sim \sim \text{GL}_4 \times \text{GL}_2 \) spectral reciprocity, we similarly improve (6.16) to \( O_\varepsilon(t_g^{6/5+\varepsilon} T^{1/5}) \) in the range \( t_g^{2/13} \leq T \leq t_g^{3/17} \) and (9.9) to \( O_\varepsilon(t_g^{13/10+\varepsilon} U^{9/10}) \) in the range \( t_g^{11/17} \leq U \leq t_g^{9/13} \). These in turn imply the improved bounds \( O_\varepsilon(t_g^{1/13+\varepsilon}) \) for (1.10) and (1.12).

The second author [Kha20, Theorem 1.1] has shown that the third term on the left-hand side of (12.3) is \( O_\varepsilon(T^{2+2\alpha+\varepsilon}) \), where \( \alpha \) denotes the current best bound towards the Selberg eigenvalue conjecture; the same method yields the same bound for the first and second terms.

\footnote{In fact, we could make do with the weaker bound \( \int_{U}^{2U} |L(1/2 + it, \text{ad } g)|^2 \ dt \ll_{\varepsilon} t_g^{7/5+\varepsilon} U^{3/10} \) uniformly for \( t_g^{34/25} \leq U \leq t_g^2 \), which is a Lindelöf-on-average bound only when \( U \asymp t_g \).}
on the left-hand side of (12.3). Thus the Selberg eigenvalue conjecture implies the improved $L^4$-norm bound $\|g\|_4 \ll \epsilon^{1/104+\epsilon}$.

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