Yes, Topology Matters in Decentralized Optimization: Refined Convergence and Topology Learning under Heterogeneous Data

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Abstract

One of the key challenges in federated and decentralized learning is to design algorithms that efficiently deal with highly heterogeneous data distributions across agents. In this paper, we revisit the analysis of Decentralized Stochastic Gradient Descent algorithm (D-SGD), a popular decentralized learning algorithm, under data heterogeneity. We exhibit the key role played by a new quantity, that we call neighborhood heterogeneity, on the convergence rate of D-SGD. Unlike prior work, neighborhood heterogeneity is measured at the level of the neighborhood of an agent in the graph topology. By coupling the topology and the heterogeneity of the agents’ distributions, our analysis sheds light on the poorly understood interplay between these two concepts in decentralized learning. We then argue that neighborhood heterogeneity provides a natural criterion to learn sparse data-dependent topologies that reduce (and can even eliminate) the otherwise detrimental effect of data heterogeneity on the convergence time of D-SGD. For the important case of classification with label skew, we formulate the problem of learning such a good topology as a tractable optimization problem that we solve with a Frank-Wolfe algorithm. Our approach provides a principled way to design a sparse topology that balances the number of iterations and the per-iteration communication costs of D-SGD under data heterogeneity.

1 Introduction

Decentralized learning methods have recently gained a lot of attention. They allow learning from data stored locally by several agents (nodes) without exchanging raw data, in line with the increasing demand for more privacy-preserving algorithms [41, 19]. In this setup, each agent collects its own data, which then reflects its own behaviors or production patterns. Therefore, the common consideration is that local datasets are not identically distributed across the agents: in many use-cases, they are in fact highly heterogeneous. Data heterogeneity is thus widely regarded as one of the key challenges of decentralized learning [19]. It has notably been studied in standard federated learning [31, 20, 24, 16], where a (trustworthy) central server orchestrates the learning process, acting as a link between the different nodes and allowing the global aggregation of locally updated model parameters. This traditional server-client paradigm however comes with some disadvantages. In particular, the central server is a single point of failure which makes the system vulnerable to breakdowns and attacks [39], and can create a communication bottleneck when the number of nodes is large [26]. Fully decentralized versions of federated learning algorithms, where the nodes directly communicate with each other along the edges of a graph, have been recently studied as a useful alternative [9, 26, 39, 36, 2, 22, 21, 42]. Full decentralization however comes with its own challenges, such as understanding the impact of the communication topology [33] and data heterogeneity [36, 21, 3], as well as handling communication costs [40, 22, 46].

Fully decentralized optimization algorithms, such as the celebrated Decentralized SGD (D-SGD) [26, 27, 21, 42, 23], operate on a graph representing the communication topology, i.e. which pairs of nodes exchange information with each other. The connectivity of the topology has a contrasting effect on the convergence time of fully decentralized algorithms [40]. On the one hand, using a well-connected
topology can approach the convergence rate of centralized algorithms. On the other hand, the more connected the topology, the more interactions between nodes, and thus the higher the per-iteration communication costs. Choosing a good topology for fully decentralized machine learning is therefore an important question, and remains a largely open problem in the presence of data heterogeneity.

Until recently, the impact of the graph topology on the convergence of decentralized algorithms was believed to be entirely characterized by its spectral gap. A larger spectral gap indicating better connectivity and thus faster convergence, the best choice of topology is then the one that maximizes the spectral gap while respecting some communication constraints [5, 40, 30, 42]. Focusing solely on the connectivity of the topology, independently of the data, has however shown to be restrictive. In particular, recent studies suggest an interplay between the topology and the data distribution across nodes. In the homogeneous setting (i.e., when data is distributed nearly identically across nodes), the work of [33] demonstrated theoretically that “topology does not matter”: they show that even with a small spectral gap, it is still possible to achieve convergence rates of the same order as centralized algorithms, in line with previous empirical observations [26, 27]. In the heterogeneous setting however, the empirical work of [3] on classification problems with label skew shows that the choice of topology has a large influence, and provides a heuristic approach to design a relatively sparse but data-dependent topology for which D-SGD converges as fast as in the centralized setting. However, their work remains entirely empirical and needs to be theoretically understood.

In this work, we seek to fill the theoretical gap that currently exists on these questions. To this aim, our first contribution is a refined convergence analysis of D-SGD which introduces a new quantity, which we call neighborhood heterogeneity, that couples the graph topology and the data heterogeneity. Neighborhood heterogeneity essentially measures the expected distance between gradients in the neighborhood of a node in the topology and the global gradients. Our results demonstrate that the impact of the topology on the speed of convergence of D-SGD does not only depend on its connectivity, but also on its capacity to compensate the heterogeneity of local data distributions at the neighborhood level thanks to appropriate connections between nodes. If the data distributions are homogeneous, our analysis recovers the results of [33] mentioned above. More importantly, it demonstrates that the topology does matter as it can be used to mitigate the effects of data heterogeneity. In fact, our new perspective allows to get rid of the restrictive assumption of bounded heterogeneity used in previous work on decentralized learning under heterogeneous data (see e.g., [26, 27, 36, 1, 40, 21, 42]).

Our second contribution deals with the problem of learning a good data-dependent graph topology, going beyond prior work which focused only on the spectral gap and the communication costs. For this purpose, we show that neighborhood heterogeneity can be used as a natural objective. We present a concrete case where this criterion can be effectively optimized in practice: a classification model where data heterogeneity comes from a difference in the proportions of labels observed at each node (known as label distribution skew [19, 16, 3]). We solve the resulting problem using a Frank-Wolfe algorithm [14, 18], allowing to track the quality of the learned topology as new edges are added in a greedy manner. This last property also highlights the trade-off between the convergence rate of D-SGD under data heterogeneity and its per-iteration communication costs, which can be efficiently controlled with the proposed algorithm. In particular, our results imply that we can approximately minimize neighborhood heterogeneity up to a fixed additive error with a topology whose maximum degree is constant in the number of nodes. To the best of our knowledge, our work is the first to learn the graph topology for decentralized learning in a way that is data-dependent, controls communication costs, and is theoretically justified.

The paper is organized as follows. In Section 2, we introduce the problem setup, present the standard D-SGD algorithm, and recall classic hypotheses required for our analysis. In Section 3, we present neighborhood heterogeneity, discuss its relation with quantities studied in previous work, and derive a new convergence rate for D-SGD. Section 4 presents our approach for learning the graph topology in the realistic statistical learning framework of label skew. We discuss some related work in Section 5, and conclude with promising lines for future research in Section 6.
2 Preliminaries

2.1 Problem Setting

In decentralized and federated learning, a set of $n \in \mathbb{N}^*$ agents (nodes) with their own data distribution seek to collaborate in order to solve a global consensus problem. Formally, the agents aim to learn a global parameter $\theta \in \mathbb{R}^d$ so as to optimize the following global objective [26]:

$$f^* \triangleq \min_{\theta \in \mathbb{R}^d} \left[ f(\theta) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \right],$$

where each $f_i : \mathbb{R}^d \to \mathbb{R}$ is the local objective function associated to node $i \in \{1, \ldots, n\}$ and

$$f_i(\theta) \triangleq \mathbb{E}_{Z_i \sim D_i} [F_i(\theta; Z_i)].$$

In the previous equation, $Z_i$ is a random vector drawn from the data distribution $D_i$ of agent $i$, having support over a space $\Omega_i$, and $F_i : \mathbb{R}^d \times \Omega_i \to \mathbb{R}$ is the pointwise loss function (assumed to be differentiable in its first argument). This generic formulation covers a large spectrum of problems and in particular includes standard risk minimization problems found in statistical learning. The empirical version of such risk minimization problems can be obtained by defining $D_i$ as a discrete uniform distribution over a finite training set $(Z^1_i, \ldots, Z^m_i)$. In this specific case, we have $f_i(\theta) = \frac{1}{m_i} \sum_{j=1}^{m_i} F_i(\theta; Z^j_i)$.

Importantly, we do not make any assumption on the distributions $D_i$ so that they can be arbitrarily different. Our framework therefore covers strongly heterogeneous settings, which is a key requirement in decentralized and federated learning applications [19]. From an optimization point of view, this means that the local optima $\theta^*_i \triangleq \arg \min_{\theta} f_i(\theta)$ can be far away from the global optima $\theta^*$ of (1).

In order to collaboratively optimize (1) in a fully decentralized manner, the agents are allowed to communicate with each other over a directed graph. The graph topology is represented by a matrix $W \in [0, 1]^{n \times n}$, where $W_{ij} > 0$ means that agent $i$ can exchange messages with agent $j$ (with the value of the weight indicating the importance that $i$ gives to the information from $j$), while $W_{ij} = 0$ (no edge) means that node $i$ and $j$ are not allowed to communicate with each other.

The choice of topology (i.e., of matrix $W$) affects the trade-off between the convergence rate of decentralized optimization algorithms and the communication costs. Indeed, more edges imply higher communication costs but often faster convergence. Communication costs are often regarded as proportional to the maximum (in or out)-degrees of nodes in the topology, representing the maximum (incoming or outcoming) load of a node [26]:

$$d_{\text{max}}^{\text{in}}(W) = \max_i \sum_{j=1}^{n} [W_{ji} > 0], \quad d_{\text{max}}^{\text{out}}(W) = \max_i \sum_{j=1}^{n} [W_{ij} > 0].$$

From this perspective, the star topology induced by server-based federated learning yields high communication costs, as the in-degree and out-degree of the server node are both equal to $n$.

2.2 Decentralized SGD

Decentralized Stochastic Gradient Descent (D-SGD) [26, 21] is a popular fully decentralized algorithm for solving problems of the form (1). As mentioned above, such algorithms operate on a graph topology represented by the matrix $W$. D-SGD requires that $W$ is a mixing matrix.

**Definition 1.** (Consensus matrix) A matrix $W \in [0, 1]^{n \times n}$ is a mixing matrix if it is doubly stochastic, i.e. $W1 = 1$ and $1^T W = 1^T$. 


For sake of generality, we consider a setting where the mixing matrix may change at each iteration [21]. On the other hand, we assume that the mixing matrices are deterministic, which may require synchronization between nodes. Although not considered here for simplicity, extensions to asynchronous versions are possible [27, 21].

D-SGD is summarized in Algorithm 1. Each iteration $t = 0, \ldots, T - 1$ consists of two steps. First, each node $i = 1, \ldots, n$ updates its local estimate $\theta_i^{(t)}$ by performing a standard stochastic gradient step. This step requires the computation of $\nabla F_i(\theta_i^{(t)}, Z_i^{(t)})$, the stochastic gradient of $F_i$ evaluated at $\theta_i^{(t)}$ with $Z_i^{(t)}$ sampled from $\mathcal{D}_i$. Then, each node aggregates its current parameter value with its neighbors according to the mixing matrix $W^{(t)}$ (neighborhood averaging step).

2.3 General Assumptions

We now present the assumptions that are needed for our analysis. They have been extensively considered in decentralized learning [6, 34, 26, 36, 1, 25, 23, 42].

**Assumption 1.** (Convexity) Each local objective function $f_i(\cdot) = \mathbb{E}_{Z_i \sim \mathcal{D}_i}[F_i(\cdot; Z_i)]$ is convex.

**Assumption 2.** ($L$-smoothness) Each local function $F_i : \mathbb{R}^d \times \Omega_i \to \mathbb{R}$ is differentiable and there exists a constant $L > 0$ such that for any $Z \in \Omega_i$, $\theta, \tilde{\theta} \in \mathbb{R}^d$ we have:

$$
\| \nabla F_i(\theta, Z) - \nabla F_i(\tilde{\theta}, Z) \| \leq L \| \theta - \tilde{\theta} \|.
$$

(4)

**Assumption 3.** (Bounded variance) For any node $i \in [1, \ldots, n]$, there exists a constant $\sigma_i^2 > 0$ such that for any $\theta \in \mathbb{R}^d$, we have:

$$
\mathbb{E}_{Z \sim \mathcal{D}_i} \left[ \| \nabla F_i(\theta, Z) - \nabla f_i(\theta) \|_2^2 \right] \leq \sigma_i^2.
$$

(5)

This assumption ensures that the local stochastic gradients of each agent $i$ have uniformly bounded variance. Note that if the stochastic gradients are built using mini-batches of samples from $\mathcal{D}_i$, the variance $\sigma_i$ of the stochastic gradients can be made arbitrarily small by using large batch sizes. In particular, when $\mathcal{D}_i$ is a distribution over a (finite) training set, using full batches imply $\sigma_i^2 = 0$.

**Assumption 4.** (Mixing parameter) There exists a mixing parameter $p \in [0, 1]$ such that for any $t = 0, \ldots, T - 1$ and any matrix $M \in \mathbb{R}^{d \times n}$, we have:

$$
\| MW^{(t)} - \bar{M} \|_F^2 \leq (1 - p) \| M - \bar{M} \|_F^2,
$$

(6)

where $\| \cdot \|_F$ denotes the Frobenius norm and $\bar{M} = M \cdot \frac{1}{n} 11^T$ is the $d \times n$ matrix that contains the column-wise average of $M$ repeated $n$ times.
Equation (6) measures how well an averaging step using a mixing matrix $W$ brings an arbitrary matrix $M$ closer to its column-wise average $\bar{M}$. This general property covers a large variety of settings [21, 23]. In particular, it is always possible to take $p = 1 - \lambda_2(W^TW)$ where $\lambda_2(W^TW)$ is equal to the second largest eigenvalue of $W^TW$ [5, 33]. Recall that the spectral gap $\rho$ of $W$ corresponds to the absolute difference of the two largest eigenvalues (in magnitude) of $W$. Then, if $W$ is also symmetric, $p$ directly relates to the classical spectral gap $\rho$ of $W$ by the relation $p = 1 - (1 - \rho)^2$ [23].

Note that no assumptions have been made regarding the distributions $D_1, \ldots, D_n$ nor data heterogeneity in general. We address data heterogeneity in the next section.

3 Joint Effect of Topology and Data Heterogeneity

In this section, we present our first contribution. We introduce a new quantity, which we call neighborhood heterogeneity, that links the graph topology and the heterogeneity of data distributions $D_1, \ldots, D_n$. We then derive new convergence rates for D-SGD that depend on this quantity rather than on a simple heterogeneity bound. These convergence rates have several nice properties: (i) they hold under weaker assumptions than previous work (unbounded heterogeneity), (ii) they highlight the interplay between the topology and the data distribution across nodes, and (iii) they provide a criterion for choosing topologies not only based on their mixing properties but also based on data.

3.1 Neighborhood Heterogeneity

We start by introducing and discussing our new quantity: neighborhood heterogeneity. Given a mixing matrix $W$, neighborhood heterogeneity measures the expected distance between the aggregated gradients in the neighborhood of a node (as weighted by $W$) and the global average of gradients. In the rest of our analysis, we assume this distance to be bounded.

**Assumption 5 (Bounded neighborhood heterogeneity).** There exists a constant $\bar{\tau}^2 > 0$ such that:

$$H \triangleq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \sum_{j=1}^{n} W_{ij} \nabla f_j(\theta, Z_j) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\theta, Z_j) \right\|^2_2 \leq \bar{\tau}^2, \quad \forall \theta \in \mathbb{R}^d. \quad (7)$$

where the expectation is taken with respect to $(Z_1, \ldots, Z_n) \sim D_1 \otimes \ldots \otimes D_n$.

To better understand Assumption 5, let us decompose the neighborhood heterogeneity $H$ using a bias-variance decomposition, which leads to the following bound:

$$H \leq \frac{1}{n} \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} W_{ij} \nabla f_j(\theta) - \nabla f(\theta) \right\|^2_2 + \frac{\sigma_{\text{max}}^2}{n} \|W - \frac{1}{n} 11^T\|_F^2, \text{ with } \sigma_{\text{max}}^2 = \max_i \sigma_i^2. \quad (8)$$

This upper bound contains two terms. The first one is a bias term, related to the heterogeneity of the problem. It essentially measures how the gradients of local objective functions $f_1, \ldots, f_n$ differ from the gradient of the global objective $f$ when they are aggregated at the neighborhood level of the graph topology through $W$. The second one is a variance term which is closely related to the mixing parameter $p$ of Assumption 4. This term is always upper bounded by $\sigma_{\text{max}}^2 (1 - p)$ and lower bounded by $\frac{\sigma_{\text{max}}^2}{n} (1 - p)$ (see Proposition 3 in Appendix C).

**Comparison to classic bounded heterogeneity assumption.** In our analysis, we will use Assumption 5 in replacement of the bounded local heterogeneity condition used in previous literature [26, 27, 36, 1, 40, 21, 42]. We recall it below.
**Assumption 6** (Bounded local heterogeneity). There exists a constant $\zeta^2 > 0$ such that

$$
\frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(\theta) - \nabla f(\theta) \|^2_2 \leq \zeta^2, \quad \forall \theta \in \mathbb{R}^d.
$$

This assumption requires that the local gradients should not be too far from the global gradient. Hence, it is a fundamental property of the statistical learning problem at hand: the more heterogeneous the nodes’ distribution (and their objectives), the bigger $\zeta^2$. In contrast, neighborhood heterogeneity shifts the measure of heterogeneity to the neighborhood level in the graph topology $W$.

Crucially, Assumption 5 is more flexible than Assumption 6. More precisely, our set of assumptions (Assumptions 1-5) is less restrictive than those considered in previous works (Assumptions 1-4, 6).

To see this, we first show that our set of assumptions is implied by the latter (Proposition 1). The proof of this proposition is given in Appendix C.

**Proposition 1.** Let Assumptions 1-4 and 6 to be verified. Then Assumption 5 is satisfied with $\bar{\tau}^2 = (1 - p) (\bar{\zeta}^2 + \bar{\sigma}^2)$, where $\bar{\sigma}^2 \triangleq \frac{1}{n} \sum_i \sigma_i^2$.

We now show that Assumption 5 is strictly more general than Assumption 6 by identifying situations where the bias-variance decomposition (8) is bounded while (9) is not. As $W^{(t)}$ is doubly stochastic, the second term in Equation (8) is always bounded. We must therefore ensure that the first term can also be bounded, even if Assumption 6 is not verified. This is trivially achieved when the topology is the complete graph with uniform weights, i.e., $W = \frac{1}{n} 11^T$. Indeed, in that case $H = 0$ and thus $\bar{\tau}^2 = 0$, regardless of the heterogeneity of the distributions. More interestingly, there exists combinations of sparse topologies and data distributions for which $\bar{\tau}^2$ remains small while the local heterogeneity $\bar{\zeta}^2$ can be made arbitrary large. We give a simple example below (detailed derivations can be found in Appendix A).

**Example 1** (Two clusters and a ring topology). Let $n$ be an even number and for all $i = 1, \ldots, n$, assume $Z_i \sim D_i \triangleq \mathcal{N}(m, \tilde{\sigma}^2)$ if $i$ is odd and $Z_i \sim D_i \triangleq \mathcal{N}(-m, \tilde{\sigma}^2)$ if $i$ is even. Assume further that $\tilde{\sigma}^2 < +\infty$ and $m > 0$ can be asymptotically large. For all $i = 1, \ldots, n$ we fix $F_i(\theta, Z_i) = (\theta - Z_i)^2$ (mean estimation). Consider a ring topology that alternates between one odd node and one even node, with $W$ diagonal entries equal to $1/2$ and off-diagonal entries equal to $1/4$. Then we have $\bar{\tau}^2 = \tilde{\sigma}_i^2 = 4\tilde{\sigma}^2 < +\infty$, while $\bar{\zeta}^2 = 4m^2$ can be arbitrarily large as $m$ grows.

This example illustrates that an appropriately chosen topology, even as sparse as a ring, can control the value of $H$ and mitigate the underlying heterogeneity of the problem. In Section 4, we investigate a practical and realistic setting where we can learn a sparse topology $W$ that (approximately) minimizes the neighborhood heterogeneity $H$. Let us first validate the relevance of our new Assumption 5 by deriving a novel convergence result for D-SGD.

### 3.2 Convergence Analysis

We now present the main theoretical result of this section: a new non-asymptotic bound on the error of the iterates of D-SGD under Assumption 5. The proof of this theorem is given in Appendix B.

**Theorem 1.** Consider Algorithm 1 with mixing matrices $W^{(0)}, \ldots, W^{(T-1)}$ satisfying Assumptions 4 and 5. Assume further that Assumptions 1-3 are respected, and denote $\bar{\theta}^{(t)} \triangleq \frac{1}{n} \sum_{i=1}^{n} \theta_i^{(t)}$. For any target accuracy $\varepsilon > 0$, there exists a constant stepsize $\eta \leq \eta_{\max} = \frac{L}{\sigma L}$ such that $\frac{1}{T + 1} \sum_{t=n}^{T} \mathbb{E}(f(\bar{\theta}^{(t)}) - f^*) \leq \varepsilon$ as soon as

$$
T \geq O\left( \frac{\sigma^2}{n \varepsilon^2} + \frac{\sqrt{T \bar{\tau}}}{\rho \varepsilon^2} + \frac{L}{\rho \varepsilon} \right) r_0,
$$

(10)
where $T$ is the number of iterations and $r_0 = \|\theta^{(0)} - \theta^*\|_2^2$ is the initial distance between the initial point $\theta^{(0)}$ and the closest optimal parameter $\theta^*$ that minimizes (1). Here, $\mathcal{O}(\cdot)$ hides the numerical constants, which are explicitly provided in the proof.

Analysis and comparison to prior results. To put the above theorem into perspective, let us recall some important results of the literature. Centralized (Parallel) Stochastic Gradient Descent (C-PSGD) is equivalent to D-SGD with the mixing matrix $W = \frac{1}{N}11^T$ (complete graph). For this specific case, it has been shown that accuracy $\varepsilon$ is achieved after $T \geq \mathcal{O}(\frac{\bar{\sigma}^2}{n\varepsilon^2} + \frac{\bar{\zeta} + \bar{\sigma}}{p\varepsilon})$ iterations [11, 4, 35]. On the other hand, D-SGD (under Assumption 6 instead of Assumption 5) requires $T \geq \mathcal{O}(\frac{\bar{\sigma}^2}{n\varepsilon^2} + \sqrt{\frac{L(1-p)(\bar{\zeta} + \bar{\sigma})}{pe^3}} + \frac{L}{pe^3})$ iterations [21]. Note that asymptotically, when $\varepsilon$ is sufficiently small (or the number of iterations $T$ is sufficiently large), these rates are equivalent and dominated by the term $\frac{\bar{\sigma}^2}{n\varepsilon^2}$, which provides a linear speed up in the number of nodes $n$. Importantly, the speed at which D-SGD becomes equivalent to C-PSGD highly depends on the constants in the middle term.

The first thing to note about our rate (10) is that it is consistent with the rates recalled above. When the complete graph topology $W = \frac{1}{N}11^T$ is used at each iteration we have $\bar{\tau} = 0$ and $p = 1$, which allows us to recover the rate of C-PSGD. More generally, introducing the usually considered Assumption 6 and using Proposition 1 gives the looser bound $\mathcal{O}(\frac{\bar{\sigma}^2}{n\varepsilon^2} + \sqrt{\frac{L(1-p)(\bar{\zeta} + \bar{\sigma})}{pe^3}} + \frac{L}{pe^3})$ which is equivalent to the rate of D-SGD found in [21]. We also recover the result of [35] which states that the choice of topology (i.e., the value of $p$) does not have a big influence on the convergence when data distributions are close to uniform and large batches are used (since this makes $\bar{\tau}$ close to 0).

Focusing now on the heterogeneous setting, recall that $\bar{\tau}$ can be much smaller than $\sqrt{1-p(\bar{\zeta} + \bar{\sigma})}$ (see Section 3.1), which makes our bound sharper. This is because the topology does only not influence the convergence rate in Theorem 1 via the mixing parameter $p$, but through $\bar{\tau}$ as well. This is of particular significance in situations where communication constraints are strong so that the topology connectivity has to be low (i.e., $p$ close to 0). This makes the rates of the literature heavily impacted by data heterogeneity as $p$ can no longer compensate for it. In contrast, we can expect that a well-chosen sparse topology can achieve small $\bar{\tau}$ and thus mitigate the impact of data heterogeneity. To highlight this, we can go back to Example 1. For the chosen ring topology, we have $p = \Theta(\frac{1}{n^2})$, but the specific arrangement of nodes and the weights in $W$ still allow a small upper bound $\bar{\tau}^2$ on $H$.

Given the rate (10), the smaller $H$, the fewer iterations are needed before becoming equivalent to the rate of C-PSGD. In the next section, we investigate the problem of learning a sparse topology that minimizes (an upper bound of) the neighborhood heterogeneity $H$.

4 Learning the Topology

In this section, we propose to learn the topology (i.e., the mixing matrix $W$) so as to minimize the neighborhood heterogeneity $H$ defined in Equation (7). More precisely, we aim like to find a sparse approximate minimizer so as to control the trade-off between convergence rate and per-iteration communication complexity. Recall that the latter is typically measured by the maximum (in or out)-degrees of a node in the topology [26, 42], see Equation (3).

Effectively minimizing neighborhood heterogeneity in the general setting appears to be challenging, as Equation (7) should hold for all $\theta \in \mathbb{R}^d$. However, considering specific statistical problems and types of data heterogeneity can lead to more practical scenarios. Below, we focus on the important case of classification with label skew, and show that Equation (7) simplifies to a more tractable quantity.
4.1 Statistical Learning with Label Skew

We consider the problem of statistical learning with label skew, an important type of data heterogeneity in federated classification problems [19, 16, 3]. In this framework, each agent $i = 1, \ldots, n$ is associated with a couple of random variables $(X_i, Y_i) \sim D_i$ where $X_i \in \mathbb{R}^d$ represents the feature vector and $Y_i \in \{1, \ldots, K\}$ the associated class label. The agents aim to learn a classifier $h_\theta : \mathbb{R}^d \rightarrow \{1, \ldots, K\}$ parameterized by $\theta \in \mathbb{R}^p$ such that $h_\theta(X_i)$ is a good predictor of $Y_i$ for all $i$. Crucially, the heterogeneity of the distributions $\{D_i\}_{i=1}^n$ comes only from a difference in the label distribution $P_i(Y)$.

Formally, this scenario can be seen as a special case of the general setting defined in Section 2 with:

(i) $\forall i \in \{1, \ldots, n\}$, $Z_i = (X_i, Y_i) \in \Omega_i$ with $\Omega_i = \mathbb{R}^d \times \{1, \ldots, K\}$ (classification problem).

(ii) $\forall i \in \{1, \ldots, n\}$, $D_i = P_i(X, Y) = P(X|Y)P_i(Y)$ (label skew).

For simplicity, we assume that all agents use the same pointwise loss function, i.e., $F_i(\theta; X, Y) = F(\theta; X, Y)$ for all $i$. For instance, $F_i$ may be the cross-entropy.

Under the above statistical framework, we can obtain an upper bound on $H$ that can effectively be minimized, as shown by the following proposition.

**Proposition 2** (Upper bound on $H$ under label skew). Consider the statistical framework defined above and assume there exists $B > 0$ such that $\forall k = 1, \ldots, K$ and $\forall \theta \in \mathbb{R}^d$:

\[
\left\| \mathbb{E}_X[\nabla F(\theta; X, Y)|Y = k] - \frac{1}{K} \sum_{k'=1}^K \mathbb{E}_X[\nabla F(\theta; X, Y)|Y = k'] \right\|_2^2 \leq B. \tag{11}
\]

Then, denoting $\pi_{jk} \triangleq P_j(Y = k)$, we can bound the neighborhood heterogeneity $H$ in Equation (7) by:

\[
H \leq \frac{KB}{n} \sum_{k=1}^K \sum_{i=1}^n \left( \sum_{j=1}^n W_{ij} \pi_{jk} - \frac{1}{n} \sum_{j=1}^n \pi_{jk} \right)^2 + \frac{\sigma_{\max}^2}{n} \left\| W - \frac{1}{n} 1_1 1_1^T \right\|_F^2. \tag{12}
\]

The proof of this proposition is provided in Appendix C: it relies on the bias-variance decomposition already used in Equation (8). Note that Equation (11) corresponds to a bounded heterogeneity hypothesis at the class level (rather than at the agent level as in Assumption 6). It requires the distance between the expected gradient in a given class $k \in \{1, \ldots, K\}$ and the average of expected gradients over all classes to be bounded.

The upper bound (12) is quadratic in $W$ and composed of two terms. The first one is a bias term due to the label skew: it will be minimal if neighborhood-level class proportions (weighted by $W$) match the global class proportions. Note that this will be trivially achieved for any choice of $W$ if the class distributions are homogeneous across nodes (i.e., if $\pi_{i,k} = \pi_{j,k}$ for all $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, K\}$). The second term is a variance term: it is easy to see that this term is minimal only if $W$ corresponds to the complete topology with uniform weights, i.e., $W = \frac{1_1 1_1^T}{n}$. In fact, this topology is also the unique global minimum of the overall quantity (12), which is equal to 0 in this case. However, as already discussed before, such a dense mixing matrix is impractical as it yields huge communication costs. In the next section, we show how the per-iteration communication complexity can be controlled while approximately minimizing the upper bound (12).

4.2 Optimization with the Frank-Wolfe Algorithm

In this section, we design an algorithm to find sparse approximate minimizers of the quantity (12). For simplicity, we focus on learning a single mixing matrix $W$ as a “pre-processing” step (before running
The algorithm is summarized in Algorithm 2. Starting from the identity matrix $\hat{W}^{(0)} = I_n$, class proportions $\Pi \in [0,1]^{n \times K}$ and hyperparameter $\lambda > 0$.

### Algorithm 2 Frank-Wolfe algorithm for learning the mixing matrix $W$

**Require:** Initialization $\hat{W}^{(0)} = I_n$, class proportions $\Pi \in [0,1]^{n \times K}$ and hyperparameter $\lambda > 0$.

for $l = 0, \ldots, L$ do

$P^{(l+1)} = \arg\min_{P \in A} \langle P, \nabla g(\hat{W}^{(l)}) \rangle$ \hspace{1cm} \text{\textgreater{} Find best permutation matrix}

$\gamma^{(l+1)} = \min_{\gamma \in [0,1]} g \left( (1 - \gamma)\hat{W}^{(l)} + \gamma P^{(l+1)} \right)$ \hspace{1cm} \text{\textgreater{} Line-search}

$\hat{W}^{(l+1)} = (1 - \gamma^{(l+1)})\hat{W}^{(l)} + \gamma^{(l+1)}P^{(l+1)}$ \hspace{1cm} \text{\textgreater{} Convex update}

end for

D-SGD), and do so in a centralized manner. Specifically, we assume that a single party (which may be one of the agents, or a third-party) has access to the class proportions $\pi_{ik} = P_i(Y = k)$ for each agent $i$ and each class $k$. In practice, since each agent has access to its local dataset, it can compute these local proportions locally and share them without sharing the local data itself.

**Optimization problem.** Our objective is to learn a mixing matrix $W$ which (approximately) minimizes the upper bound (12). Denoting the set of doubly stochastic matrices by

$$\mathcal{S} \triangleq \{ W \in [0,1]^{n \times n} : W \mathbf{1} = \mathbf{1}, \mathbf{1}^T W = \mathbf{1}^T \} ,$$

the optimization problem can be written as follows:

$$\min_{W \in \mathcal{S}} \left\{ g(W) \triangleq \frac{1}{n} \| W\Pi - \frac{11^T}{n} \Pi \|_F^2 + \frac{\lambda}{n} \| W - \frac{11^T}{n} \|_F^2 \right\} , \quad (13)$$

where $\Pi \in [0,1]^{n \times K}$ contains the class proportions $\{\pi_{ik}\}$ and $\lambda > 0$ is a hyperparameter. To exactly match (12), $\lambda$ should be equal to $\frac{\sigma_{max}}{\sqrt{|B|}}$ (thus involving quantities that are typically unknown in practice). Instead, we use $\lambda$ to control the bias-variance trade-off: the bigger $\lambda$ is, the less important the bias correction term and the more important the variance term. As discussed in Section 3.1, the variance term is an upper bound of $1 - p$ with $p$ the mixing parameter of $W$ from Assumption 4. Therefore, the hyperparameter $\lambda$ allows to tune a trade-off between the minimization of the bias due to label skew and the maximization of the mixing parameter of $W$.

**Algorithm.** As discussed before, we want to avoid the trivial (impractical) solution $W = \frac{11^T}{n}$ and instead find sparse approximations of Problem (13) so as to control the per-iteration communication complexity. We propose to do this using a Frank-Wolfe algorithm, which is known to be particularly well-suited to learn a sparse parameter over convex hulls of finite set of atoms [18]. In our case, $\mathcal{S}$ corresponds to the convex hull of the set $A$ of all permutation matrices [29, 37, 38].

The algorithm is summarized in Algorithm 2. Starting from the identity matrix $\hat{W}^{(0)} = I_n \in \mathcal{S}$, each iteration $l \geq 0$ consists of moving towards a feasible point $P^{(l+1)}$ that minimizes a linearization of $g$ at the current iterate $\hat{W}^{(l)}$. As finding $P^{(l+1)}$ is a linear problem, solving it over $\mathcal{S}$ is equivalent to solving it over $A$. Although $A$ contains $n!$ elements, the linear program corresponds to the well-known assignment problem [7, 10] and can be solved efficiently with the Hungarian algorithm in time $O(n^3)$ [29]. Note that the gradient needed to solve the assignment problem is given by

$$\nabla g(W) = \frac{2}{n} \sum_{k=1}^{K} (W\Pi_{.,k} - \Pi_{.,k}\mathbf{1}) \cdot \Pi_{.,k}^T + \frac{2}{n} \lambda \left( \frac{11^T}{n} \right) ,$$
where $\Pi_{i,k}$ corresponds to the $k$-th column of $\Pi$. The next iterate $\hat{\Pi}^{(l+1)}$ is then obtained as a convex combination of $P^{(l+1)}$ and $\hat{\Pi}^{(l)}$, and is thus guaranteed to be in $S$. The optimal combining weight is computed by line-search, which has a closed-form solution since $g$ is quadratic (see Appendix D).

Crucially, Algorithm 2 allows to control the sparsity of the final solution: since a permutation matrix contains exactly one non-zero entry in each row and each column, at most one new incoming and one new outgoing edge per node are added. As we start from the identity matrix (i.e., only self-edges), this guarantees that at the end of the $l$-th iteration, each node will have at most $l$ in-neighbors and at most $l$ out-neighbors. The per-iteration communication complexity of the learned topology can thus be directly controlled by the number of iterations of our Franck-Wolfe algorithm, inducing a trade-off with the quality of the solution. This trade-off is quantified by the following theorem, which is derived from standard convergence results for Frank-Wolfe [18] combined with a tight bound on the smoothness of $g$ in an appropriate norm (see Appendix C).

**Theorem 2.** (Bound on the objective) Consider the statistical setup presented in Section 4.1 and let $\{\hat{\Pi}^{(l)}\}_{l=1}^L$ be the sequence of mixing matrices generated by Algorithm 2. Then, at any iteration $l = 1, \ldots, L$, we have:

$$g(\hat{\Pi}^{(l)}) \leq \frac{16}{l+2} \left( \lambda + \frac{1}{n} \sum_{k=1}^K \left( \Pi_{i,k} - \Pi_{i,k} \Pi_{i,k}^T \right)^2 \right),$$

(14)

where $\|\cdot\|_2^*$ stands for the nuclear norm, i.e., the sum of singular values. Furthermore, we have $d_{\text{max}}^\text{in}(\hat{\Pi}^{(l)}) \leq l$ and $d_{\text{max}}^\text{out}(\hat{\Pi}^{(l)}) \leq l$, resulting in a per-iteration complexity bounded by $l$.

The above theorem shows that the objective $g$ decreases at a rate of $O(1/l)$ as new connections between nodes are made. Note that the constant in (14) depends on the proportions of classes across nodes. In general, we can bound (14) by $g(\hat{\Pi}^{(l)}) \leq \frac{16}{l+2} (\lambda + 1)$, which is independent from the number of nodes $n$. In some cases, (14) can be tighter than the upper-bound $g(\hat{\Pi}^{(l)}) \leq \frac{16}{l+2} (\lambda + 1)$ stated above. For instance, in the favorable case where all nodes have the same class proportions, we have $\Pi_{i,k} = \Pi_{i,k} 1$ and thus $g(\hat{\Pi}^{(l)}) \leq \frac{16}{l+2} \lambda$. Another example is when each node holds only one class (i.e., $\forall i, \exists k$ such that $\pi_{ik} = 1$) and $\frac{n}{K}$ nodes hold each class. In this case, we get $g(\hat{\Pi}^{(l)}) \leq \frac{16}{l+2} (\lambda + 1 - \frac{1}{K})$.

To conclude, recall that the bound given in Theorem 2 provides a bound on the neighborhood heterogeneity $H$ through (12). Fixing $\lambda = \frac{\pi^2}{KB}$, the upper bound (12) is exactly equal to $KB \cdot g(W)$, therefore using Theorem 2 we have

$$H \leq \frac{16}{l+2} \left( \sigma_{\text{max}}^2 + \frac{K}{n} \left( \sum_{k=1}^K (\Pi_{i,k} - \Pi_{i,k} \Pi_{i,k}^T)^2 \right) \right) \triangleq \tilde{\tau}_l \cdot \tilde{\pi}_l^2.\n
(15)$$

Therefore, if we use $\hat{\Pi}^{(l)}$ in D-SGD, we can replace $\tilde{\pi}$ by $\tilde{\pi}$ in the convergence results derived in Section 3. Interestingly, $\tilde{\tau}$ still contains a variance term and a heterogeneity term (the right-hand term in the parenthesis). Our approach provides a principled way to reduce these terms by learning the topology while controlling the per-iteration communication complexity of D-SGD. Remarkably, the fact that $g(\hat{\Pi}^{(l)})$ is independent of $n$ implies that we can find topologies that approximately optimize the convergence rate of D-SGD while keeping the communication load per node constant, thereby guaranteeing the scalability of D-SGD to a large number of nodes even in highly heterogeneous scenarios.

5 Related Work

**Algorithmic improvements to decentralized SGD.** Significant work has been devoted to the improvement of D-SGD. We can mention in particular approaches based on momentum [1, 15, 28, 45],
cross-gradient aggregations [12] and the use of bias correction (or variance reduction) techniques [36, 44, 43, 17]. Many of these schemes are able to reduce the order of the term that depends on the local heterogeneity constant $\bar{\zeta}$ in the convergence rates. However, a close inspection reveals that $\bar{\zeta}$ (or a similar local heterogeneity term measured at the optimum or the initialization) remains present in the bounds, making their convergence rates still impacted by strong heterogeneous scenarios. We stress that the above line of research is complementary to ours as it is based on modifications of the D-SGD algorithm (which typically requires additional computation and/or communication). In contrast, our work does not modify the algorithm and only relies on the choice of graph topology. We believe that our analysis can be combined with the above algorithmic improvements. We leave such extensions to future work.

Moving away from the consensus objective (1), data heterogeneity motivated fully decentralized approaches that learn a personalized parameter for each node, inspired from multi-task learning. As opposed to the consensus problem, the topology should in this case connect similar nodes [39, 2, 46, 13].

**Choosing and learning good topologies for decentralized learning.** There is a long line of research on choosing or learning topologies so as to maximize the mixing parameter $p$ (or equivalently, the spectral gap). Prior research has studied the properties of classic topologies that enable communication-efficient decentralized optimization (e.g. grid, circle, exponential graphs) [8, 32, 42]. Another line of work tries to learn the mixing matrix $W$, or more precisely the distribution over which $W$ is sampled at each iteration [5, 40]. In these approaches, the graph structure is given, and a subgraph is sampled at each iteration. The maximization of $p$ is done with respect to the sampling parameters, under communication constraints that must be satisfied. Unlike our approach, these methods simply seek to maximize the connectivity of the topology (through $p$) and do not take into account in any way the data distributions across nodes.

To the best of our knowledge, the only work that leverages the distribution of data in the design of the topology is D-cliques [3]. Focusing on classification under label skew (see Section 4.1), they propose a heuristic approach to construct a topology for D-SGD that consists of inter-connected cliques, where the proportion of classes in each clique should be as close as possible to the global proportion. This corresponds to minimizing the first term in (12) for each clique. Our approach is more flexible as it can learn more general topologies, and provides full control over the sparsity of the topology. Furthermore, our topology learning criteria is theoretically justified, while D-Cliques is only supported by empirical experiments. We think that the ideas of this paper could however pave the way for a theoretical analysis of D-Cliques.

### 6 Conclusion

This paper addressed two important open problems from the literature on decentralized and federated learning. First, thanks to our newly introduced notion of neighborhood heterogeneity, we exhibited the joint effect of the graph topology and the data heterogeneity in the convergence rate of D-SGD. Our result shows that, if chosen appropriately, the topology can compensate for the heterogeneity and allow for faster convergence. Second, we tackled the problem of learning a good topology in the presence of data heterogeneity. To the best of our knowledge, our work is the first to provide a principled and data-dependent approach to learn the topology, with an algorithm that allows to control the trade-off between the communication cost and convergence speed and D-SGD.

An immediate future work will consist in an extensive empirical study of the topologies learned with our approach. From a theoretical point of view, we plan to extend our convergence result to the non-convex setting. More generally, thanks to the neighborhood heterogeneity criterion, we believe that our work paves the way to the development of other data-dependent topology learning techniques.
Indeed, we aim to investigate different statistical models (e.g., different types of heterogeneity beyond label skew) under different knowledge assumptions (e.g., not knowing the proportions). We can also envision fully decentralized and privacy-preserving versions of topology learning algorithms.

References

[1] Mahmoud Assran, Nicolas Loizou, Nicolas Ballas, and Mike Rabbat. Stochastic gradient push for distributed deep learning. In ICML, 2019.

[2] Aurélien Bellet, Rachid Guerraoui, Mahsa Taziki, and Marc Tommasi. Personalized and private peer-to-peer machine learning. In AISTATS, 2018.

[3] Aurélien Bellet, Anne-Marie Kermarrec, and Erick Lavoie. D-cliques: Compensating noniidness in decentralized federated learning with topology. arXiv:2104.07365, 2021.

[4] Léon Bottou, Frank E Curtis, and Jorge Nocedal. Optimization methods for large-scale machine learning. Siam Review, 60(2):223–311, 2018.

[5] Stephen Boyd, Arpita Ghosh, Balaji Prabhakar, and Devavrat Shah. Randomized gossip algorithms. IEEE Transactions on Information Theory, 52(6):2508–2530, 2006.

[6] Sébastien Bubeck. Convex optimization: Algorithms and complexity. arXiv:1405.4980, 2014.

[7] Rainer Burkard, Mauro Dell’Amico, and Silvano Martello. Assignment problems: revised reprint. SIAM, 2012.

[8] Yat-Tin Chow, Wei Shi, Tianyu Wu, and Wotao Yin. Expander graph and communication-efficient decentralized optimization. In 2016 50th Asilomar Conference on Signals, Systems and Computers, pages 1715–1720. IEEE, 2016.

[9] Igor Colin, Aurélien Bellet, Joseph Salmon, and Stéphan Clémenc¸on. Gossip dual averaging for decentralized optimization of pairwise functions. In ICML, 2016.

[10] David F Crouse. On implementing 2d rectangular assignment algorithms. IEEE Transactions on Aerospace and Electronic Systems, 52(4):1679–1696, 2016.

[11] Ofer Dekel, Ran Gilad-Bachrach, Ohad Shamir, and Lin Xiao. Optimal distributed online prediction using mini-batches. Journal of Machine Learning Research, 13(1), 2012.

[12] Yasaman Esfandiari, Sin Yong Tan, Zhanhong Jiang, Aditya Bunu, Ethan Herron, Chinmay Hegde, and Soumik Sarkar. Cross-Gradient Aggregation for Decentralized Learning from Non-IID data. Technical report, arXiv:2103.02051, 2021.

[13] Mathieu Even, Laurent Massoulié, and Kevin Scaman. Sample Optimality and All-for-all Strategies in Personalized Federated and Collaborative Learning. Technical report, arXiv:2201.13097, 2022.

[14] Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. Naval Research Logistics Quarterly, 3:95–110, 1956.

[15] Hongchang Gao and Heng Huang. Periodic stochastic gradient descent with momentum for decentralized training. arXiv:2008.10435, 2020.

[16] Kevin Hsieh, Amar Phanishayee, Onur Mutlu, and Phillip B. Gibbons. The Non-IID Data Quagmire of Decentralized Machine Learning. In ICML, 2020.
[17] Kun Huang and Shi Pu. Improving the transient times for distributed stochastic gradient methods. *arXiv:2105.04851*, 2021.

[18] Martin Jaggi. Revisiting frank-wolfe: Projection-free sparse convex optimization. In *ICML*, 2013.

[19] Peter Kairouz, H Brendan McMahan, Brendan Avent, Aurélien Bellet, et al. Advances and open problems in federated learning. *Foundations and Trends® in Machine Learning*, 14(1–2):1–210, 2021.

[20] Sai Praneeth Karimireddy, Satyen Kale, Mehryar Mohri, Sashank Reddi, Sebastian Stich, and Ananda Theertha Suresh. Scaffold: Stochastic controlled averaging for federated learning. In *ICML*, 2020.

[21] Anastasia Koloskova, Nicolas Loizou, Sadra Boreiri, Martin Jaggi, and Sebastian U. Stich. A unified theory of decentralized sgd with changing topology and local updates. In *ICML*, 2020.

[22] Anastasia Koloskova, Sebastian Stich, and Martin Jaggi. Decentralized stochastic optimization and gossip algorithms with compressed communication. In *ICML*, 2019.

[23] Lingjing Kong, Tao Lin, Anastasia Koloskova, Martin Jaggi, and Sebastian Stich. Consensus control for decentralized deep learning. In *ICML*, 2021.

[24] Tian Li, Anit Kumar Sahu, Manzil Zaheer, Maziar Sanjabi, Ameet Talwalkar, and Virginia Smith. Federated optimization in heterogeneous networks. *MLSys*, 2020.

[25] Xiang Li, Wenhao Yang, Shusen Wang, and Zhihua Zhang. Communication-efficient local decentralized sgd methods. *arXiv:1910.09126*, 2019.

[26] Xiangru Lian, Ce Zhang, Huan Zhang, Cho-Jui Hsieh, Wei Zhang, and Ji Liu. Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. *NIPS*, 2017.

[27] Xiangru Lian, Wei Zhang, Ce Zhang, and Ji Liu. Asynchronous Decentralized Parallel Stochastic Gradient Descent. In *ICML*, 2018.

[28] Tao Lin, Sai Praneeth Karimireddy, Sebastian Stich, and Martin Jaggi. Quasi-global momentum: Accelerating decentralized deep learning on heterogeneous data. In *ICML*, 2021.

[29] László Lovász and Michael D Plummer. *Matching theory*, volume 367. American Mathematical Soc., 2009.

[30] Othmane Marfoq, Chuan Xu, Giovanni Neglia, and Richard Vidal. Throughput-optimal topology design for cross-silo federated learning. *NeurIPS*, 2020.

[31] Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *AISTATS*, 2017.

[32] Angelia Nedić, Alex Olshevssky, and Michael G Rabbat. Network topology and communication-computation tradeoffs in decentralized optimization. *Proceedings of the IEEE*, 106(5):953–976, 2018.

[33] Giovanni Neglia, Chuan Xu, Don Towsley, and Gianmarco Calbi. Decentralized gradient methods: does topology matter? In *AISTATS*, 2020.

[34] Lam M Nguyen, Phuong Ha Nguyen, Peter Richtárik, Katya Scheinberg, Martin Takáč, and Marten van Dijk. New convergence aspects of stochastic gradient algorithms. *Journal of Machine Learning Research*, 20:176–1, 2019.
[35] Sebastian U Stich and Sai Praneeth Karimireddy. The error-feedback framework: Better rates for sgd with delayed gradients and compressed updates. *Journal of Machine Learning Research*, 21:1–36, 2020.

[36] Hanlin Tang, Xiangru Lian, Ming Yan, Ce Zhang, and Ji Liu. D²: Decentralized training over decentralized data. In *ICML*, 2018.

[37] Ambuj Tewari, Pradeep Ravikumar, and Inderjit Dhillon. Greedy algorithms for structurally constrained high dimensional problems. *NIPS*, 2011.

[38] Víctor Valls, George Iosifidis, and Leandros Tassiulas. Birkhoff’s decomposition revisited: Sparse scheduling for high-speed circuit switches. *arXiv:2011.02752*, 2020.

[39] Paul Vanhaesebrouck, Aurélien Bellet, and Marc Tommasi. Decentralized collaborative learning of personalized models over networks. In *AISTATS*, 2017.

[40] Jianyu Wang, Anit Kumar Sahu, Zhouyi Yang, Gauri Joshi, and Soummya Kar. Matcha: Speeding up decentralized sgd via matching decomposition sampling. In *ICC*, 2019.

[41] Kang Wei, Jun Li, Ming Ding, Chuan Ma, Howard H Yang, Farhad Farokhi, Shi Jin, Tony Q S Quek, and H Vincent Poor. Federated learning with differential privacy: Algorithms and performance analysis. *IEEE Transactions on Information Forensics and Security*, 15:3454–3469, 2020.

[42] Bicheng Ying, Kun Yuan, Yiming Chen, Hanbin Hu, Pan Pan, and Wotao Yin. Exponential graph is provably efficient for decentralized deep training. *NeurIPS*, 34, 2021.

[43] Kun Yuan and Sulaiman A Alghunaim. Removing data heterogeneity influence enhances network topology dependence of decentralized sgd. *arXiv:2105.08023*, 2021.

[44] Kun Yuan, Sulaiman A Alghunaim, Bicheng Ying, and Ali H Sayed. On the influence of bias-correction on distributed stochastic optimization. *IEEE Transactions on Signal Processing*, 68:4352–4367, 2020.

[45] Kun Yuan, Yiming Chen, Xinmeng Huang, Yingya Zhang, Pan Pan, Yinghui Xu, and Wotao Yin. Decentlam: Decentralized momentum sgd for large-batch deep training. In *ICCV*, 2021.

[46] Valentina Zantedeschi, Aurélien Bellet, and Marc Tommasi. Fully decentralized joint learning of personalized models and collaboration graphs. In *AISTATS*, 2020.
Appendix

A Example from Section 3.1

In this section, we detail the example mentioned in Section 3.1 by giving exact parametrization. The objective is to find an example where Assumption 6 is not verified while Assumption 5 is.

Let consider \( n \) nodes (\( n \) is even). For all \( i = 1, \ldots, n \), assume \( Z_i \sim \mathcal{N}(m, \tilde{\sigma}^2) \) if \( i \) is odd and \( Z_i \sim \mathcal{N}(-m, \tilde{\sigma}^2) \) if \( i \) is even. Assume further that \( \tilde{\sigma}^2 < +\infty \) but \( m > 0 \) can be asymptotically large. For all \( i = 1, \ldots, n \) we fix \( F_i(\theta, Z_i) = (\theta - Z_i)^2 \) (mean estimation).

Let the consensus matrix \( W \) fixed in time and associate it to a ring topology that alternates between the two Gaussian distributions. We fix the weights as follows.

\[
W_{ij} = \begin{cases} 
\frac{1}{2} & \text{if } j = i , \\
\frac{1}{4} & \text{if } j = i + 1 \text{ or } j = i - 1 , \\
0 & \text{otherwise} . 
\end{cases}
\]

With such parametrization we have \( \nabla F_i(\theta, Z_i) = 2(\theta - Z_i) \) and therefore \( \nabla f_i(\theta) = 2(\theta - m) \) if \( i \) is odd and \( \nabla f_i(\theta) = 2(\theta + m) \) if \( i \) is even. Moreover, we have the gradient (derivative) of the global objective \( \nabla f(\theta) = \frac{1}{n} \sum_i \nabla f_i(\theta) = 2\theta \) and the neighborhood averaging \( \sum_j W_{ij} \nabla f_j(\theta) = 2\theta \) for all \( i \).

Before looking at the Assumptions 5 and 6 that interest us, we verify that Assumptions 3 is true:

\[
E \left[ (\nabla F_i(\theta, Z_i) - \nabla f_i(\theta))^2 \right] = E \left[ 4 (Z_i - E Z_i)^2 \right] = 4\tilde{\sigma}^2 < \infty .
\]

Let us found a bound \( \bar{\tau}^2 \) on the Neighborhood heterogeneity. Using a bias-variance decomposition, we have:

\[
H = \frac{1}{n} \sum_{i=1}^{n} E \left( \sum_{j=1}^{n} W_{ij} \nabla F_j(\theta) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(\theta) \right)^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} W_{ij} \nabla f_j(\theta) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\theta) \right)^2 + \frac{1}{n} \sum_{i=1}^{n} E \left( \sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) (\nabla f_j(\theta) - \nabla F_j(\theta)) \right)^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (2\theta - 2\theta)^2 + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (W_{ij} - \frac{1}{n})^2 E(\nabla f_j(\theta) - \nabla F_j(\theta))^2
\]

\[
= 0 + 4\tilde{\sigma}^2 \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (W_{ij} - \frac{1}{n})^2 \leq 4\tilde{\sigma}^2 .
\]

The third equality was obtained thanks to the fact that \( E[\nabla f_j(\theta) - \nabla F_j(\theta)] = 0 \). Using the previous result, we can take \( \bar{\tau}^2 = 4\tilde{\sigma}^2 < \infty \) which proves that Assumption 5 is verified.

On the contrary, since \( m \) can be arbitrary large, Assumption 6 is not verified. Indeed,
\[ \frac{1}{n} \sum_{i=1}^{n} \left( \nabla f_i(\theta) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\theta) \right)^2 = \frac{1}{n} \sum_{i=1}^{n} (2m)^2 = 4m^2 \Rightarrow \frac{\sigma^2}{m} \to +\infty. \]

**B Proof of Theorem 1**

**B.1 organization**

Before moving to the proof, we need to re-write D-SGD into matrix notation.

Let \( \Theta(t) \triangleq (\theta_1^{(t)}, \ldots, \theta_n^{(t)}) \in \mathbb{R}^{d \times n} \) be the matrix that contains all learned parameter vectors at time \( t \). Denote by \( \nabla F(\Theta(t), Z(t)) \triangleq (\nabla F_1(\theta_1^{(t)}, Z_1^{(t)}), \ldots, \nabla F_n(\theta_n^{(t)}, Z_n^{(t)})) \in \mathbb{R}^{d \times n} \) the matrix containing all stochastic gradients at time \( t \). The update of one D-SGD step at all nodes can be written:

\[ \Theta(t+1) = \left( \Theta(t) - \eta \nabla F(\Theta(t), Z(t)) \right) W(t). \]

In the following, we denote \( \overline{\Theta}(t) \triangleq (\bar{\theta}^{(t)}_1, \ldots, \bar{\theta}^{(t)}_n) = \Theta(t) \cdot \frac{1}{n} \mathbf{1} \mathbf{1}^T \).

The proof follows the classical steps found in the literature (see e.g. [21, 33]). The main difference resides in how the consensus term \( \|\Theta(t) - \overline{\Theta}(t)\|_F^2 \) is controlled along iterations (Lemma 2). It is organized as follows:

Lemma 1 provides a descent recursion that allows to control the decreasing of the term \( \|\bar{\theta}^{(t)} - \theta^*\|^2 \). The proof closely follows the one of [21, 33].

In Lemma 2, the consensus term \( \|\Theta(t) - \overline{\Theta}(t)\|_F^2 \), which appears in the equation provided by Lemma 1, is upper-bounded. The found upper-bound exhibits the new constant \( \bar{\tau}^2 \).

Corollary 1 uses the previous lemma to bound

\[ \frac{1}{T+1} \sum_{t=0}^{T} \left\| \Theta(t) - \overline{\Theta}(t) \right\|_F^2. \]

Finally, Lemma 3 provides an upper-bound on the error term with the following form:

\[ \frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}(f(\bar{\theta}^{(t)}) - f^*) \leq 2 \left( \frac{br_0}{T+1} \right)^\frac{1}{2} + 2e^{\frac{T}{2}} \left( \frac{r_0}{T+1} \right)^{\frac{1}{2}} + \frac{dr_0}{T+1}, \]

where \( b = \frac{\sigma^2}{n}, \bar{e} = \frac{36L^2 \sigma^2}{p^2}, \bar{d} = \frac{8L}{p} \) and \( r_0 = \|\theta(0) - \theta^*\|^2 \).

To get the final rate of Theorem 1, it suffices to find \( T \) such that each term in the right-hand side of the previous equation in bounded by \( \frac{\epsilon}{3} \):

\[ 2 \left( \frac{br_0}{T+1} \right)^{\frac{1}{2}} \leq \frac{36br_0}{\bar{e}^2} \leq T + 1 \iff \frac{36\sigma^2 r_0}{n \bar{e}^2} \leq T + 1 \]

\[ 2e^{\frac{T}{2}} \left( \frac{r_0}{T+1} \right)^{\frac{1}{2}} \leq \frac{\epsilon}{3} \iff \frac{1}{2} \bar{e} \frac{e}{\epsilon} \leq T + 1 \iff \frac{36L^2 \sigma^2}{pe} \leq T + 1 \]

\[ \frac{dr_0}{T+1} \leq \frac{\epsilon}{3} \iff \frac{4L r_0}{pe} \leq T + 1 \iff \frac{24 L r_0}{pe} \leq T + 1 \]
In particular, it suffices to take
\[ T \geq \frac{36\sigma^2 r_0}{n\varepsilon^2} + \frac{89\sqrt{L^\tau r_0}}{p\varepsilon^2} + \frac{24Lr_0}{p\varepsilon^2} = \mathcal{O}\left(\frac{\sigma^2}{n\varepsilon^2} + \frac{\sqrt{L^\tau}}{p\varepsilon^2} + \frac{L}{p\varepsilon^2}\right) r_0, \]
in order to have all three terms bounded by \( \varepsilon \), and obtain the final result.

### B.2 Preliminaries and useful results

**Property 1.** (Implication of mixing matrices) Let \( W \in \mathbb{R}^{n \times n} \) be a mixing matrix and \( \Theta \) be any matrix in \( \mathbb{R}^{d \times n} \). Then, \( W \) preserves averaging:
\[
(\Theta W) \frac{11^T}{n} = \Theta \frac{11^T}{n} = \Theta \tag{16}
\]

**Property 2.** (Implication of \( L \)-smoothness and convexity)

- Under assumption 1 (convexity), we have for all \( i \in \{1, \ldots, n\} \):
  \[
  \left\langle \nabla f_i(\hat{\theta}), \hat{\theta} - \theta \right\rangle \geq f_i(\hat{\theta}) - f_i(\theta) \tag{17}
  \]

- Under assumption 2 (\( L \)-smoothness), it holds for all \( i \in \{1, \ldots, n\} \):
  \[
  F_i(\theta, Z) \leq F_i(\hat{\theta}, Z) + \left\langle \nabla F_i(\hat{\theta}, Z), \theta - \hat{\theta} \right\rangle + \frac{L}{2}\|\theta - \hat{\theta}\|_2^2, \quad \forall \theta, \hat{\theta} \in \mathbb{R}^d, Z \in \theta_i \tag{18}
  \]

  Passing the previous equation to the expectation, we also have:
  \[
  f_i(\theta) \leq f_i(\hat{\theta}) + \left\langle \nabla F_i(\hat{\theta}), \theta - \hat{\theta} \right\rangle + \frac{L}{2}\|\theta - \hat{\theta}\|_2^2, \quad \forall \theta, \hat{\theta} \in \mathbb{R}^d \tag{19}
  \]

- If we further assume that functions \( F_i \) are convex, Assumption 2 also implies \( \forall \theta, \hat{\theta} \in \mathbb{R}^d, Z \in \theta_i, \)
  \[
  \|\nabla f_i(\theta) - \nabla f_i(\hat{\theta})\|_2 \leq L\|\theta - \hat{\theta}\|_2 \tag{20}
  \]
  \[
  \|\nabla f_i(\theta) - \nabla f_i(\hat{\theta})\|_2^2 \leq 2L \left( f_i(\theta) - f_i(\hat{\theta}) - \left\langle \nabla f_i(\hat{\theta}), \theta - \hat{\theta} \right\rangle \right) \tag{21}
  \]
  \[
  \|\nabla F_i(\theta, Z) - \nabla F_i(\hat{\theta}, Z)\|_2^2 \leq 2L \left( F_i(\theta, Z) - F_i(\hat{\theta}, Z) - \left\langle \nabla F_i(\hat{\theta}, Z), \theta - \hat{\theta} \right\rangle \right) \tag{22}
  \]

These results can be found in many convex optimization books and papers, e.g. in [6].

**Property 3.** (Norm inequalities)

- For a set of vectors \( \{a_i\}_{i=1}^n, a_i \in \mathbb{R}^d \)
  \[
  \left\| \sum_{i=1}^n a_i \right\|_2^2 \leq n \sum_{i=1}^n \|a_i\|_2^2 \tag{23}
  \]

- For two vectors \( a, b \in \mathbb{R}^d \),
  \[
  \|a + b\|_2^2 \leq (1 + \alpha)\|a\|_2^2 + (1 + \alpha^{-1})\|b\|_2^2, \quad \forall \alpha > 0 \tag{24}
  \]
B.3 Needed Lemmas

Lemma 1. (Descent Lemma) Consider the settings of Theorem 1 and let $\eta_t \leq \frac{1}{4L}$, then:

$$
\mathbb{E}_{Z_1^{(t)}, \ldots, Z_n^{(t)}} \left\| \tilde{\theta}^{(t+1)} - \theta^* \right\|^2 \leq \left\| \tilde{\theta}^{(t)} - \theta^* \right\|^2 + \frac{\eta_t^2 \bar{\sigma}^2}{n} - \eta_t \left( f(\tilde{\theta}^{(t)}) - f^* \right) + \frac{3L}{2n} \eta_t \left\| \Theta^{(t)} - \bar{\Theta}^{(t)} \right\|^2 _F \tag{25}
$$

Proof. The proof closely follows the one in [21]. Using the recursion of D-SGD and since all mixing matrices are doubly stochastic and preserve the average (Proposition 1) we have:

$$
\left\| \tilde{\theta}^{(t+1)} - \theta^* \right\|^2 = \left\| \tilde{\theta}^{(t)} - \frac{\eta_t}{n} \sum_{i=1}^n \nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)}) - \theta^* \right\|^2 \\
= \left\| \tilde{\theta}^{(t)} - \theta^* - \frac{\eta_t}{n} \sum_{i=1}^n \nabla f_i(\hat{\theta}_i^{(t)}) + \frac{\eta_t}{n} \sum_{i=1}^n \nabla f_i(\hat{\theta}_i^{(t)}) - \frac{\eta_t}{n} \sum_{i=1}^n \nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)}) \right\|^2 \\
= \left\| \tilde{\theta}^{(t)} - \theta^* - \frac{\eta_t}{n} \sum_{i=1}^n \nabla f_i(\hat{\theta}_i^{(t)}) \right\|^2 + \frac{\eta_t^2}{n} \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_i(\hat{\theta}_i^{(t)}) - \frac{1}{n} \sum_{i=1}^n \nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)}) \right\|^2 \\
+ 2 \left\langle \tilde{\theta}^{(t)} - \theta^* - \frac{\eta_t}{n} \sum_{i=1}^n \nabla f_i(\hat{\theta}_i^{(t)}), \frac{\eta_t}{n} \sum_{i=1}^n \nabla f_i(\hat{\theta}_i^{(t)}) - \frac{\eta_t}{n} \sum_{i=1}^n \nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)}) \right\rangle,
$$

where, in expectation, the last term (the inner product) is equal to 0. This comes from the fact that $\mathbb{E}_{Z_i^{(t)}}[\nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)})] = \nabla f_i(\hat{\theta}_i^{(t)})$, making the second term in the inner product equal to 0 (in expectation). We therefore need to bound the first two terms in expectation, with respect to $Z_i^{(t)} = (Z_1^{(t)}, \ldots, Z_n^{(t)})$.

The second one, can easily be bounded:

$$
\eta_t^2 \mathbb{E}_{Z_i^{(t)}} \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_i(\hat{\theta}_i^{(t)}) - \frac{1}{n} \sum_{i=1}^n \nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)}) \right\|^2 = \frac{\eta_t^2}{n^2} \mathbb{E}_{Z_i^{(t)}} \left\| \sum_{i=1}^n (\nabla f_i(\hat{\theta}_i^{(t)}) - \nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)})) \right\|^2 \\
= \frac{\eta_t^2}{n^2} \sum_{i=1}^n \mathbb{E}_{Z_i^{(t)}} \left\| \nabla f_i(\hat{\theta}_i^{(t)}) - \nabla F_i(\hat{\theta}_i^{(t)}, Z_i^{(t)}) \right\|^2 \\
\leq \frac{\eta_t^2 \bar{\sigma}^2}{n},
$$

where the second equality was obtained using the identity $\mathbb{E} \| \sum_i Y_i \|_2^2 = \sum_i \mathbb{E} Y_i \|_2^2$ when $Y_i$ are independent and $\mathbb{E} Y_i = 0$.

Now that the second term is bounded, we can move to the first one:
Combining all previous results, we get:

\[ \left\| \bar{\theta}^{(t)} - \theta^* - \frac{\eta}{n} \sum_{i=1}^{n} \nabla f_i(\theta_i^{(t)}) \right\|^2 = \left\| \bar{\theta}^{(t)} - \theta^* \right\|^2 + \eta_t^2 \left( \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{\theta}_i^{(t)}) \right) - 2 \eta_t \left( \bar{\theta}^{(t)} - \theta^*, \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\theta_i^{(t)}) \right). \]

In order to bound \( T_1 \), recall that by definition \( \frac{1}{n} \sum_i \nabla f_i(\theta^*) = 0 \), therefore:

\[ T_1 = \left\| \frac{1}{n} \sum_{i=1}^{n} (\nabla f_i(\theta_i^{(t)}) - \nabla f_i(\bar{\theta}^{(t)}) + \nabla f_i(\bar{\theta}^{(t)}) - \nabla f_i(\theta^*)) \right\|^2 \]

\[ \leq 2 \left( \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla f_i(\theta_i^{(t)}) - \nabla f_i(\bar{\theta}^{(t)}) \right\|^2 + \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla f_i(\bar{\theta}^{(t)}) - \nabla f_i(\theta^*) \right\|^2 \right) \]

\[ \leq \frac{2L^2}{n} \sum_{i=1}^{n} \left( \theta_i^{(t)} - \bar{\theta}^{(t)} \right) + \frac{4L}{n} \sum_{i=1}^{n} \left( f_i(\bar{\theta}^{(t)}) - f_i(\theta^*) - \left( \nabla f_i(\theta^*), \bar{\theta}^{(t)} - \theta^* \right) \right) \]

\[ = \frac{2L^2}{n} \sum_{i=1}^{n} \left( \theta_i^{(t)} - \bar{\theta}^{(t)} \right)^2 + \frac{4L}{n} \sum_{i=1}^{n} \left( f_i(\bar{\theta}^{(t)}) - f_i(\theta^*) \right) - 4L \left( \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\theta^*), \bar{\theta}^{(t)} - \theta^* \right) \]

\[ = \frac{2L^2}{n} \sum_{i=1}^{n} \left( \theta_i^{(t)} - \bar{\theta}^{(t)} \right)^2 + 4L \left( f(\bar{\theta}^{(t)}) - f^* \right) \]

Finally, we have to bound \( T_2 \):

\[ -T_2 = -\frac{2\eta_t}{n} \sum_{i=1}^{n} \left( \bar{\theta}^{(t)} - \theta^*, \nabla f_i(\theta_i^{(t)}) \right) \]

\[ = -\frac{2\eta_t}{n} \sum_{i=1}^{n} \left[ \left( \bar{\theta}^{(t)} - \theta_i^{(t)}, \nabla f_i(\theta_i^{(t)}) \right) + \left( \theta_i^{(t)} - \theta^*, \nabla f_i(\theta_i^{(t)}) \right) \right] \]

\[ \leq -\frac{2\eta_t}{n} \sum_{i=1}^{n} \left[ f_i(\bar{\theta}^{(t)}) - f_i(\theta_i^{(t)}) - \frac{L}{2}\left\| \bar{\theta}^{(t)} - \theta_i^{(t)} \right\|_2^2 + f_i(\theta_i^{(t)}) \right. \]

\[ = -2\eta_t \left( f(\bar{\theta}^{(t)}) - f(\theta^*) \right) + \frac{L\eta_t}{n} \sum_{i=1}^{n} \left\| \bar{\theta}^{(t)} - \theta_i^{(t)} \right\|^2 \]

\[ = -2\eta_t \left( f(\bar{\theta}^{(t)}) - f^* \right) + \frac{L\eta_t}{n} \left\| \bar{\Theta}^{(t)} - \Theta^{(t)} \right\|_F^2 \]

Combining all previous results, we get:
\[ E_{Z^{(t)}} \| \tilde{\theta}^{(t+1)} - \theta^* \|^2 \leq \| \tilde{\theta}^{(t)} - \theta^* \|^2 + \frac{\eta_t^2 \sigma^2}{n} + \frac{L\eta_t}{n} (2L\eta_t + 1) \| \Theta^{(t)} - \Theta^{(t)} \|_F^2 + 2\eta_t (2L\eta_t - 1) \left( f(\tilde{\theta}^{(t)}) - f^* \right) \]

Since, by hypothesis, \( \eta_t \leq \frac{1}{4\sigma} \), we have \( 2L\eta_t + 1 \leq \frac{3}{4} \) and \( 2L\eta_t - 1 \leq -\frac{1}{4} \), which concludes the proof. \( \square \)

**Lemma 2.** (Consensus Control) Consider the settings of Theorem 1 and let \( \eta_t \leq \frac{\sigma}{4\bar{\sigma}} \), then:

\[ E \| \Theta^{(t)} - \Theta^{(t)} \|_F^2 \leq (1 - \frac{p}{4})E \| \Theta^{(t-1)} - \Theta^{(t-1)} \|_F^2 + \frac{6n\bar{\sigma}^2}{p} \eta_t^2 \]

*Proof.* For simplicity in the following, we may denote \( \nabla F(\Theta, Z) \) by \( \nabla F(\Theta) \).

\[
E \| \Theta^{(t)} - \Theta^{(t)} \|_F^2 = E \left\| \Theta^{(t)} \left( I - \frac{11^T}{n} \right) \right\|_F^2 \\
= E \left\| \left( \Theta^{(t-1)} - \eta_t^{-1} \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \right) W^{(t-1)} \left( I - \frac{11^T}{n} \right) \right\|_F^2 \\
\overset{(16)}{=} E \left\| \left( \Theta^{(t-1)} - \eta_t^{-1} \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \right) \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 \\
\overset{(24)}{\leq} (1 + \alpha)E \left\| \Theta^{(t-1)} \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 \\
+ (1 + \alpha^{-1}) \eta_t^{-2} E \left\| \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 \\
\overset{(6)}{\leq} (1 + \alpha)(1 - p)E \left\| \Theta^{(t-1)} - \Theta^{(t-1)} \right\|_F^2 + (1 + \alpha^{-1}) \eta_t^2 T_3
\]

We now need to bound \( T_3 \):

\[
T_3 = E \left\| \left( \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) - \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) + \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \right) \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 \\
\overset{(23)}{\leq} 2E \left\| \left( \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) - \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \right) \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 \\
+ 2E \left\| \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 \\
= 2E \left\| \left( \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) - \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \right) \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 \\
+ 2 \sum_{i=1}^{n} \left( \sum_{j=1}^{n} W^{(t-1)}_{ij} \nabla F_{ij} (\tilde{\theta}^{(t-1)}, Z^{(t-1)}) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_{ij} (\theta, Z^{(t-1)}) \right) \right\|_F^2 \\
\overset{(7)}{\leq} 2E \left\| \left( \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) - \nabla F(\Theta^{(t-1)}, Z^{(t-1)}) \right) \left( W^{(t-1)} - \frac{11^T}{n} \right) \right\|_F^2 + 2n\bar{\sigma}^2
\]
Denoting $F_i(\theta_i^{(t-1)}, z_j^{(t-1)})$ by $F_i(\theta_i^{(t-1)})$ and using Assumption 4, we can bound the first term by:

\[
2(1-p)E \left\| \left( \nabla F(\Theta^{(t-1)}) - \nabla F(\overline{\Theta}^{(t-1)}) \right) \right\|^2_F \\
\leq 4(1-p) \left[ E \left\| \nabla F(\Theta^{(t-1)}) - \nabla F(\overline{\Theta}^{(t-1)}) \right\|^2_F + E \left\| \nabla F(\Theta^{(t-1)}) - \nabla F(\overline{\Theta}^{(t-1)}) \right\|^2_F \right] \\
= 4(1-p) \left[ \sum_{i=1}^n \left( E \left\| \nabla F_i(\theta_i^{(t-1)}) - \nabla F_i(\overline{\theta}^{(t-1)}) \right\|^2_2 + E \left\| \frac{1}{n} \sum_{j=1}^n \left( \nabla F_j(\theta_j^{(t-1)}) - \nabla F_j(\overline{\theta}^{(t-1)}) \right) \right\|^2_2 \right] \right) \\
\leq 4(1-p) \left[ L^2 \sum_{i=1}^n E \left\| \theta_i^{(t-1)} - \overline{\theta}^{(t-1)} \right\|^2_2 + \frac{n}{2} E \left\| \frac{1}{n} \sum_{j=1}^n \left( \nabla F_j(\theta_j^{(t-1)}) - \nabla F_j(\overline{\theta}^{(t-1)}) \right) \right\|^2_2 \right] \\
\leq 4(1-p) \left[ L^2 \sum_{i=1}^n E \left\| \theta_i^{(t-1)} - \overline{\theta}^{(t-1)} \right\|^2_2 + \sum_{j=1}^n E \left\| \nabla F_j(\theta_j^{(t-1)}) - \nabla F_j(\overline{\theta}^{(t-1)}) \right\|^2_2 \right] \\
\leq 4(1-p) \left[ L^2 \sum_{i=1}^n E \left\| \theta_i^{(t-1)} - \overline{\theta}^{(t-1)} \right\|^2_2 + L^2 \sum_{j=1}^n E \left\| \theta_j^{(t-1)} - \overline{\theta}^{(t-1)} \right\|^2_2 \right] \\
= 8(1-p)L^2E \left\| \Theta^{(t-1)} - \overline{\Theta}^{(t-1)} \right\|^2_F
\]

Combining all previous results and fixing $\alpha = \frac{p}{2}$, we get:

\[
E \left\| \Theta^{(t)} - \overline{\Theta}^{(t)} \right\|^2_F \leq (1 + \alpha)(1-p)E \left\| \Theta^{(t-1)} - \overline{\Theta}^{(t-1)} \right\|^2_F \\
+ 8(1+\alpha^{-1})(1-p)L^2 \eta_{t-1}^2E \left\| \Theta^{(t-1)} - \overline{\Theta}^{(t-1)} \right\|^2_F + 2(1+\alpha^{-1})\eta_{t-1}^2n \bar{\tau}^2 \\
\leq (1 + \frac{p}{2})(1-p)E \left\| \Theta^{(t-1)} - \overline{\Theta}^{(t-1)} \right\|^2_F \\
\leq 1 - \frac{p}{4} \\
+ 8(1 + \frac{2}{p})(1-p)L^2 \eta_{t-1}^2E \left\| \Theta^{(t-1)} - \overline{\Theta}^{(t-1)} \right\|^2_F + 2(1 + \frac{2}{p})\eta_{t-1}^2n \bar{\tau}^2 \leq \frac{p}{4}
\]

Since by hypothesis we have $\eta_{t-1} \leq \frac{\sigma^2}{8L^2}$, we can bound the second term and get:

\[
E \left\| \Theta^{(t)} - \overline{\Theta}^{(t)} \right\|^2_F \leq \left( 1 - \frac{p}{2} + \frac{p}{4} \right)E \left\| \Theta^{(t-1)} - \overline{\Theta}^{(t-1)} \right\|^2_F + \frac{6n \bar{\tau}^2}{p} \eta_{t-1}^2 \\
= \left( 1 - \frac{p}{4} \right)E \left\| \Theta^{(t-1)} - \overline{\Theta}^{(t-1)} \right\|^2_F + \frac{6n \bar{\tau}^2}{p} \eta_{t-1}^2
\]
Corollary 1. (Consensus recursion) Let the settings of Theorem 1 and fix $\eta_t = \eta \leq \frac{p}{8L}$, we have:

$$\frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E} \left\| \bar{\Theta}^{(t)} - \bar{\Theta} \right\|_F^2 \leq \frac{24\eta^2n\tau^2}{p^2}.$$  \hspace{1cm} (27)

Proof. Unrolling the expression (26) found in Lemma 2 (Consensus Control) up to $t = 0$, we have:

$$\mathbb{E} \left\| \bar{\Theta}^{(t)} - \bar{\Theta} \right\|_F^2 \leq \left(1 - \frac{p}{4}\right)t \left\| \bar{\Theta}^{(0)} - \bar{\Theta} \right\|_F^2 + \frac{6n\tau^2}{p} \eta^2 \sum_{j=0}^{t-1} \left(1 - \frac{p}{4}\right)^j$$

$$= \frac{6n\tau^2}{p} \eta^2 \times \frac{1 - (1 - \frac{p}{4})^t}{1 - (1 - \frac{p}{4})}$$

$$\leq \frac{6n\tau^2}{p} \eta^2 \times \frac{4}{p}$$

$$= \frac{24\eta^2n\tau^2}{p^2}$$

Summing and dividing by $T + 1$, we get the final result.

□

Lemma 3. (Convergence rate with $T$) Consider the settings of Theorem 1. There exists a constant stepsize $\eta \leq \eta_{\text{max}} = \frac{p}{8L}$ such that

$$\frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}(f(\theta^{(t)}) - f^*) \leq 2 \left(\frac{br_0}{T+1}\right)^\frac{1}{2} + 2c^\frac{1}{2} \left(\frac{r_0}{T+1}\right)^\frac{1}{2} + \frac{dr_0}{T+1},$$  \hspace{1cm} (28)

where $b = \frac{\sigma^2}{n}$, $c = \frac{36L\tau^2}{p^2}$, $d = \frac{8L}{p}$ and $r_0 = \|\theta^{(0)} - \theta^*\|_2^2$. 

Proof. Using the descent lemma 1, we have:

$$\frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}(f(\theta^{(t)}) - f^*)$$

$$\leq \frac{1}{\eta(T+1)} \sum_{t=0}^{T} \left( \mathbb{E} \left\| \bar{\Theta}^{(t)} - \theta^* \right\|^2 - \mathbb{E} \left\| \bar{\Theta}^{(t+1)} - \theta^* \right\|^2 + \eta^2 \bar{\sigma}^2 \frac{n}{2n} + \frac{3L}{2n} \eta \mathbb{E} \left\| \bar{\Theta}^{(t)} - \bar{\Theta} \right\|_F^2 \right)$$

$$\leq \frac{1}{\eta(T+1)} \left\| \theta^{(0)} - \theta^* \right\|^2 + \frac{\eta \bar{\sigma}^2}{n} + \frac{3L}{2n} \frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E} \left\| \Theta^{(t)} - \bar{\Theta}^{(t)} \right\|_F^2$$

$$\leq \frac{1}{\eta(T+1)} \left\| \theta^{(0)} - \theta^* \right\|^2 + \frac{\sigma^2}{n} \eta + \frac{36L\tau^2}{p^2} \eta^2 \hspace{1cm} (27)$$

Fixing $\eta = \min \left\{ \left(\frac{r_0}{b(T+1)}\right)^\frac{1}{2}, \left(\frac{r_0}{c(T+1)}\right)^\frac{1}{2}, \frac{1}{d}\right\}$ with $b = \frac{\sigma^2}{n}$, $e = \frac{36L\tau^2}{p^2}$, $d = \frac{8L}{p}$ and $r_0 = \|\theta^{(0)} - \theta^*\|_2^2$, then applying the following Lemma 4, we obtain the final result. □

Lemma 4. (Tuning stepsize [21]) For any parameter $r_0, b, e, d \geq 0$, $T \in \mathbb{N}$, we can fix

$$\eta = \min \left\{ \left(\frac{r_0}{b(T+1)}\right)^\frac{1}{2}, \left(\frac{r_0}{c(T+1)}\right)^\frac{1}{2}, \frac{1}{d}\right\} \leq \frac{1}{d}.$$
and get
\[ \frac{r_0}{\eta(T+1)} + b\eta + e\eta^2 \leq 2 \left( \frac{br_0}{T+1} \right)^{\frac{3}{2}} + 2e\left( \frac{r_0}{T+1} \right)^{\frac{3}{2}} + \frac{d}{T+1} \]

**Proof.** The proof of this lemma can be found in [21] (Lemma 15).

\[ \square \]

### C Additional results and proofs

**Proposition 1.** Let Assumptions 1-4 and 6 to be verified. Then Assumption 5 is satisfied with \( \hat{\tau}^2 = (1 - p) (\hat{\zeta}^2 + \hat{\sigma}^2) \), where \( \hat{\sigma}^2 \triangleq \frac{1}{n} \sum_i \sigma_i^2 \).

**Proof.** Denoting \( \nabla F(\theta) = (\nabla F_1(\theta, Z_1), \ldots, \nabla F_n(\theta, Z_n)) \in \mathbb{R}^{d \times n} \), and using the relation (i) \( \mathbb{E}\|Y\|_2^2 = \mathbb{E}\|Y\|_2^2 + \mathbb{E}\|Y - \mathbb{E}Y\|_2^2 \), we have:

\[
H^{(i)} = \frac{1}{n} \mathbb{E}\nabla F(\theta) W^{(i)} - \nabla F(\theta) \|_F
\]

\[
\leq \frac{1 - p}{n} \mathbb{E}\|\nabla F(\theta) - \nabla F(\theta)\|_F^2
\]

\[
= \frac{1 - p}{n} \sum_{i=1}^n \mathbb{E}\|\nabla F_i(\theta, Z_i) - \nabla F_i(\theta, Z_i)\|_2^2
\]

\[
\leq \frac{1 - p}{n} \mathbb{E}\|\nabla f_i(\theta) - \frac{1}{n} \sum_{j=1}^n \nabla f_j(\theta)\|_2^2 + \mathbb{E}\left\| \sum_{j=1}^n (\mathbb{1}_{j=i} - \frac{1}{n}) (\nabla F_j(\theta, Z_j) - \nabla f_j(\theta)) \right\|_2^2
\]

\[
\leq (1 - p) \left( \hat{\zeta}^2 + \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left\| \sum_{j=1}^n (\mathbb{1}_{j=i} - \frac{1}{n}) (\nabla F_j(\theta, Z_j) - \nabla f_j(\theta)) \right\|_2^2 \right)
\]

Since all terms \( j \) in the norm are independent and with expectation 0, the expectation of the sum is equal to the sum of expectations and

\[
H^{(i)} \leq (1 - p) \left( \hat{\zeta}^2 + \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n (\mathbb{1}_{j=i} - \frac{1}{n})^2 \mathbb{E}\|\nabla F_j(\theta, Z_j) - \nabla f_j(\theta)\|_2^2 \right)
\]

\[
= (1 - p) \left( \hat{\zeta}^2 + \frac{1}{n} \sum_{j=1}^n \mathbb{E}\|\nabla F_j(\theta, Z_j) - \nabla f_j(\theta)\|_2^2 \sum_{i=1}^n (\mathbb{1}_{j=i} - \frac{1}{n})^2 \right)
\]

\[
\leq (1 - p) \left( \hat{\zeta}^2 + \frac{n - 1}{n} \sigma^2 \right) \leq (1 - p) \left( \hat{\zeta}^2 + \sigma^2 \right),
\]

which concludes the proof.

\[ \square \]

**Proposition 2.** (Upper-bound on \( H \)) Consider the setup defined above and assume there exists
$B > 0$ such that $\forall k = 1, \ldots, K$ and $\forall \theta \in \mathbb{R}^d$, we have

$$\left\| \mathbb{E}_X[\nabla F(\theta; X, Y)|Y = k] - \frac{1}{K} \sum_{k'}^{K} \mathbb{E}_X[\nabla F(\theta; X, Y)|Y = k'] \right\|_2^2 \leq B,$$

then we get the upper-bound:

$$H \leq \frac{KB}{n} \sum_{k=1}^{K} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} W_{ij} \pi_{jk} \right)^2 + \frac{\sigma_{\max}^2}{n} \| W - \frac{1}{n} \mathbf{1}\mathbf{1}^T \|_F^2,$$

where $\pi_{jk} \equiv P_j(Y = k)$.

**Proof.** First, observe that the local objective functions can be re-written

$$f_j(\theta) = \mathbb{E}_{(X,Y) \sim \mathcal{D}_j}[F(\theta; X, Y)]$$

$$= \sum_{k=1}^{K} P_j(Y = k) \mathbb{E}_X[F(\theta; X, Y)|Y = k]$$

$$= \sum_{k=1}^{K} \pi_{jk} \mathbb{E}_X[F(\theta; X, Y)|Y = k].$$

From (8), we have the bias-variance decomposition

$$H \leq \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} W_{ij} \nabla f_j(\theta) - \nabla f(\theta) \right)^2 + \frac{\sigma_{\max}^2}{n} \| W - \frac{1}{n} \mathbf{1}\mathbf{1}^T \|_F^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) \nabla f_j(\theta) \right)^2 + \frac{\sigma_{\max}^2}{n} \| W - \frac{1}{n} \mathbf{1}\mathbf{1}^T \|_F^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) \sum_{k=1}^{K} \pi_{jk} \mathbb{E}_X[\nabla F(\theta; X, Y)|Y = k] \right)^2 + \frac{\sigma_{\max}^2}{n} \| W - \frac{1}{n} \mathbf{1}\mathbf{1}^T \|_F^2 + T_4$$

Then, observing that $\sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) = 0$ and $\sum_{k=1}^{K} \pi_{jk} = 1$ imply

$$\sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) \sum_{k=1}^{K} \pi_{jk} \frac{1}{K} \sum_{k'=1}^{K} \mathbb{E}_X[\nabla F(\theta; X, Y)|Y = k'] = 0,$$

we can add this in the norm of the term $T_4$ defined above and get

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Finally, plugging this into the upper-bound on \( H \) found before, we get the final result.

\[ T_4 = \left\| \sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) \sum_{k=1}^{K} \pi_{jk} \left( \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k] - \frac{1}{K} \sum_{k'=1}^{K} \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k'] \right) \right\|_2^2 \\
= \left\| \sum_{k=1}^{K} \left( \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k] - \frac{1}{K} \sum_{k'=1}^{K} \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k'] \right) \sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) \pi_{jk} \right\|_2^2 \\
\leq K \sum_{k=1}^{K} \left\| \left( \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k] - \frac{1}{K} \sum_{k'=1}^{K} \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k'] \right) \sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) \pi_{jk} \right\|_2^2 \\
= K \sum_{k=1}^{K} \left\| \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k] - \frac{1}{K} \sum_{k'=1}^{K} \mathbb{E}_X [\nabla F(\theta; X, Y)|Y = k'] \right\|_2^2 \cdot \left( \sum_{j=1}^{n} (W_{ij} - \frac{1}{n}) \pi_{jk} \right)^2 \\
\leq KB \sum_{k=1}^{K} \left( \sum_{j=1}^{n} W_{ij} \pi_{jk} - \frac{1}{n} \sum_{j=1}^{n} \pi_{jk} \right)^2 \\
\leq 25 \]

Finally, plugging this into the upper-bound on \( H \) found before, we get the final result.

\[ \square \]

**Theorem 2** (Bound on the objective) Consider the statistical setup presented at the beginning of section 4 and let \( \{\tilde{W}^{(l)}\}_{l=1}^{L}, \tilde{W}^{(l)} \in \mathcal{S} \) be the sequence of mixing matrices found through the iterations of Frank-Wolfe algorithm described above. Then, at any iteration \( l = 1, \ldots, L \), we have:

\[ g(\tilde{W}^{(l)}) \leq 16 \frac{l + 1}{l + 2} \left( \lambda + \frac{1}{n} \left\| \sum_{k=1}^{K} (\Pi_{\pi,k} - \Pi_{\pi,k}1) \cdot \Pi_{\pi,k}^T \right\|_2 \right) \]

where \( \| \cdot \|_2 \) stands for the nuclear norm i.e. the sum of singular values. Bounding the second term in the parenthesis, we can obtain the looser bound

\[ g(\tilde{W}^{(l)}) \leq 16 \frac{l + 1}{l + 2} (\lambda + 1) \]

Furthermore, each node has at most \( l \) in-neighbors (respectively out-neighbors) resulting in a per-iteration complexity bounded by \( l \).

**Proof.** The proof of this theorem is directly derived with theorem 3 given below, applied with the parameters of our problem.

To prove the first inequality, we first need to find a bound on the diameter of the set of doubly stochastic matrices, denoted \( \text{diam}_{\| \cdot \|}(\mathcal{S}) \), for a certain (matrix) norm \( \| \cdot \| \). We fix this norm to be the operator norm induced by the \( \ell_2 \)-norm, denoted \( \| \cdot \|_2 \), and which is simply the maximum singular value of the matrix.

Since \( \forall W, P \in \mathcal{S} \), we have
\[ \|W - P\|_2 \leq \|W\|_2 + \|P\|_2 = 1 + 1 = 2, \]

where this comes from the fact that \( W \) and \( P \) are doubly stochastic i.e. their largest eigenvalue is 1. In the end, this shows that \( \text{diam}_{\|\cdot\|_2}(S) \leq 2. \)

Let’s now find the Lipschitz constant associated to the gradient of the objective

\[ \nabla g(W) = \frac{2}{n} \sum_{k=1}^{K} (W \Pi_{:,k} - \Pi_{:,k} 1) \cdot \Pi_{:,k}^T + \frac{2}{n} \lambda \left( W - \frac{11^T}{n} \right). \]

Recall that the dual norm \( \|\cdot\|_1^* \) of \( \|\cdot\|_1 \) is the nuclear norm i.e. the sum of the singular values.

For any \( W \) and \( P \) in \( S \), we have

\[ \|\nabla g(W) - \nabla g(P)\|_2^* = \frac{2}{n} \left\| (W - P) \left( \lambda I + \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T \right) \right\|_2^* \]

\[ \leq \frac{2}{n} \|\lambda(W - P)I\|_2^* + \frac{2}{n} \left\| (W - P) \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T \right\|_2^* \]

\[ \leq \frac{2\lambda}{n} \|W - P\|_2 \|I\|_2^* + \frac{2}{n} \left\| (W - P) \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T \right\|_2^*, \]

where the last inequality was obtained using the fact that for any real matrices \( A \) and \( B \), \( \|AB\|_2^* \leq \|A^T\| \|B\|_2^* \).

Before bounding the second term, we must observe that because \( W \) and \( P \) are doubly stochastic, \( (W - P) 1 = 0 \) and therefore, for any matrix \( A \in \mathbb{R}^{n \times n}, (W - P) A = (W - P)(A - \frac{11^T}{n} A) \).

Now, the second term can be re-written and bounded

\[ \frac{2}{n} \left\| (W - P) \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T \right\|_2^* = \frac{2}{n} \left\| (W - P) \left( \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T - \frac{11^T}{n} \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T \right) \right\|_2^* \]

\[ \leq \frac{2}{n} \|W - P\|_2 \left\| \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T - \frac{11^T}{n} \sum_{k=1}^{K} \Pi_{:,k} \Pi_{:,k}^T \right\|_2^* \]

\[ = \frac{2}{n} \|W - P\|_2 \left\| \sum_{k=1}^{K} (\Pi_{:,k} - \Pi_{:,k} 1) \cdot \Pi_{:,k} \right\|_2^*. \]

Plugging the previous result into the bound obtained upper, and since \( \|I\|_2^* = n \), we get

\[ \|\nabla g(W) - \nabla g(P)\|_2^* \leq 2 \left( \lambda + \frac{1}{n} \left\| \sum_{k=1}^{K} (\Pi_{:,k} - \Pi_{:,k} 1) \cdot \Pi_{:,k} \right\|_2^* \right) \|W - P\|_2. \]

It is now possible to apply theorem 3 with the found Lipschitz constant and diameter, i.e.:
$$g(\hat{W}^{(i)}) - g(W^*) \leq \frac{16}{l+2} \left( \lambda + \frac{1}{n} \left\| \sum_{k=1}^{K} (\Pi_{i,k} - \Pi_{i,k}^\top) \cdot \Pi_{i,k}^\top \right\|_2^* \right),$$

where $W^*$ is the optimal solution of the problem. Since we known that $W^* = \frac{11^T}{n}$ is the optimal solution, with $g(W^*) = 0$, we obtain the first inequality.

To prove the second inequality, it suffices to show that

$$\left\| \sum_{k=1}^{K} (\Pi_{i,k} - \Pi_{i,k}^\top) \cdot \Pi_{i,k}^\top \right\|_2^* \leq n.$$

$$\left| \sum_{k=1}^{K} (\Pi_{i,k} - \Pi_{i,k}^\top) \cdot \Pi_{i,k}^\top \right|_2^* = \left| \left( I - \frac{11^T}{n} \right) \sum_{k=1}^{K} \Pi_{i,k} \Pi_{i,k}^\top \right|_2^*$$

$$\leq \left| I - \frac{11^T}{n} \right|_2 \left| \sum_{k=1}^{K} \Pi_{i,k} \Pi_{i,k}^\top \right|_2^*$$

$$= \left| \sum_{k=1}^{K} \Pi_{i,k} \Pi_{i,k}^\top \right|_2^*$$

$$\leq \sum_{k=1}^{K} \left| \Pi_{i,k} \Pi_{i,k}^\top \right|_2^*$$

Because for any $k = 1, \ldots, K$, $\Pi_{i,k} \Pi_{i,k}^\top$ is a rank 1 matrix, its unique eigenvalue is $\Pi_{i,k}^\top \Pi_{i,k}$ and therefore

$$\left| \sum_{k=1}^{K} (\Pi_{i,k} - \Pi_{i,k}^\top) \cdot \Pi_{i,k}^\top \right|_2^* \leq \sum_{k=1}^{K} \left| \Pi_{i,k} \Pi_{i,k}^\top \right|_2^*$$

$$= \sum_{k=1}^{K} \Pi_{i,k}^\top \Pi_{i,k}$$

$$= \sum_{k=1}^{K} \sum_{i=1}^{n} \pi_{ik}^2$$

$$\leq \sum_{i=1}^{n} \max_k \left\{ \pi_{ik}^2 \right\} \sum_{k=1}^{K} \pi_{ik}$$

$$\leq \sum_{i=1}^{n} 1 = n,$$

which concludes the proof of the second inequality.

The last statement of the theorem follows directly from the structure of permutation matrices and the greedy nature of the algorithm.
**Theorem 3.** (Frank-Wolfe Convergence [18, 6]) Let the gradient of the objective function \( g: x \to g(x) \) be \( L \)-smooth with respect to a norm \( \| \cdot \| \) and its dual norm \( \| \cdot \|^* \):

\[
\| \nabla g(x) - \nabla g(y) \|^* \leq L \| x - y \|.
\]

If \( g \) is minimized over \( S \) using Frank-Wolfe algorithm, then for each \( l \geq 1 \), the iterates \( x^{(l)} \) satisfy

\[
g(x^{(l)}) - g(x^*) \leq 2L \text{diam}_{\| \cdot \|}(S)^2 \frac{2}{l + 2},
\]

where \( x^* \in S \) is an optimal solution of the problem and \( \text{diam}_{\| \cdot \|}(S) \) stands for the diameter of \( S \) with respect to the norm \( \| \cdot \| \).

**Proof.** The proof of this theorem is a direct combination of Theorem 1 and Lemma 7 in [18], both proved in the paper.

**Proposition 3.** (Relation between \( p \) and \( \| W - \frac{11^T}{n} \|_F^2 \)) Let \( W \) be a mixing matrix satisfying Assumption 4. Then,

\[
(1 - p) \leq \| W - \frac{11^T}{n} \|_F^2 \leq (n - 1)(1 - p).
\]

**Proof.** The upper-bound is a direct application of Assumption 4 with \( M = I \), the identity matrix of size \( n \):

\[
\| W - \frac{I1^T}{n} \|_F^2 = \| IW - I \frac{11^T}{n} \|_F^2 \overset{(6)}{=} (1 - p) \left\| I - \frac{11^T}{n} \right\|_F^2 = (1 - p)(n - 1).
\]

To show the lower-bound, denote by \( s_1(M), \ldots, s_n(M) \) the (decreasing) singular values of any square matrix \( M \in \mathbb{R}^{n \times n} \). Denote similarly \( \lambda_1(M), \ldots, \lambda_n(M) \) the eigenvalues of any symmetric square matrix \( M \in \mathbb{R}^{n \times n} \).

\[
\| W - \frac{11^T}{n} \|_F^2 = \sum_{i=1}^n s_i^2 \left( W - \frac{11^T}{n} \right) \geq s_1^2 \left( W - \frac{11^T}{n} \right)
\]

\[
= \lambda_1 \left( (W - \frac{11^T}{n})^T(W - \frac{11^T}{n}) \right)
\]

\[
= \lambda_1 \left( W^T W - \frac{11^T}{n} \right)
\]

\[
= \lambda_2(W^T W) \geq 1 - p.
\]

The last equality is obtained by noticing that \( W^T W \) is a symmetric doubly stochastic matrix. It therefore admits an eigenvalue decomposition where the largest eigenvalue 1 is associated with the eigenvector \( \frac{1}{\sqrt{n}} \mathbf{1} \). This makes \( W^T W - \frac{11^T}{n} \) having the eigenvalue 0 associated to the vector \( \frac{1}{\sqrt{n}} \mathbf{1} \) and the largest eigenvalue of \( W^T W - \frac{11^T}{n} \) becomes the second-largest eigenvalue of \( W^T W \). The final inequality comes from the fact that Assumption 4 is always true with \( p = 1 - \lambda_2(W^T W) \) which implies that the best \( p \) satisfying Assumption 4 in necessarily greater or equal to \( 1 - \lambda_2(W^T W) \). 

\( \square \)
D Closed-form for the line-search

In this section, we give the closed-form solution of the line-search problem found in the Frank-Wolfe algorithm 2. Recall that we seek to solve:

\[ \gamma^* = \arg \min_{\gamma \in [0,1]} \left\{ \hat{g}(\gamma) \triangleq g((1 - \gamma)W + \gamma P) \right\}, \]

with

\[ g(W) = \frac{1}{n} \|W \Pi - \frac{11^T}{n} \Pi\|^2_F + \frac{\lambda}{n} \|W - \frac{11^T}{n}\|^2_F. \]

The function \( g \) being quadratic, the objective \( \hat{g}(\gamma) \) is also quadratic with respect to \( \gamma \). Hence, it suffices to put the derivative \( \hat{g}' \) of \( \hat{g} \) equal to 0, and we get the closed-form solution:

\[ \gamma^* = \frac{\sum_{k=1}^{K} (\Pi_{:,k} - W \Pi_{:,k})^T (P - W) \cdot \Pi_{:,k} - \lambda \cdot \text{tr} \left( \left(W - \frac{11^T}{n}\right)^T (P - W) \right)}{\| (P - W) \Pi \|^2_F + \lambda \|P - W\|^2_F}. \]