Scale Invariance in disordered systems: the example of the Random Field Ising Model

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We show by numerical simulations that the correlation function of the random field Ising model (RFIM) in the critical region in three dimensions has very strong fluctuations and that in a finite volume the correlation length is not self-averaging. This is due to the formation of a bound state in the underlying field theory. We argue that this non-perturbative phenomenon is not particular to the RFIM in 3-d. It is generic for disordered systems in two dimensions and may also happen in other three dimensional disordered systems.

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Ferromagnetic systems in a random fields (RFIM) have a puzzling behaviour. It is already known that perturbative renormalization group (PRG) leads to the prediction of dimensional reduction that is known to give wrong results for the RFIM, and this although RFIM is, together with branched polymers [3], the only case where PRG can be analyzed to all orders of perturbation theory [1,4–8]. Several attempts were made to explain this discrepancy [4–8].

One suggested possibility is the appearance of a non perturbative phenomenon (i.e. the formation of a bound state in the underlying field theory) which invalidates the perturbative analysis. There are several consequences of this. It is not any more obvious that naive scaling arguments apply in the scaling region as it is usually assumed, or if and how they should be modified. We expect also these non perturbative phenomena, related to the formation of bound states, to invalidate the predictions of perturbative renormalization group for the critical behaviour of disordered systems (calculation of critical exponents, proof of universality classes etc.) A new examination of these questions is required.

We claim that the appearance of bound states is not peculiar to the 3-dimensional RFIM (what is peculiar to the RFIM is the possibility to perform the very large scale simulations needed in order to observe these non perturbative phenomena). Kardar et al. [3] have already noticed that averaging over the disorder generates attractive interactions among replicas that produce bound states in several two dimensional systems. In the formation of bound states there is competition between the strength of the attractive forces and the size of the available phase space. In two dimensions phase space is small and even a very small attraction wins, so bound states are generic. In three dimensions formation of bound states depends on the strength of the attraction. In the case of the RFIM Brézin and De Dominicis [3] have already found that an instability appears in the kernel of the Bethe-Salpeter equation, leading to the formation of a bound state in less than 4 dimensions. Our results confirm their analysis.

The relevance of the attractive interaction among replicas can also be seen in a case where the interaction between replicas is repulsive. This is the case of diluted branched polymers (or lattice animals). There are not bound states in that case and perturbative renormalization group, i.e. dimensional reduction, has to work. Indeed dimensional reduction has recently been proven rigorously in this case [20].

In this paper we address the question of the existence of bound states and of the validity of the usual scaling laws by studying the 3-d RFIM by extensive numerical simulations at zero temperature. As many previous authors [11–15,17] we have taken advantage of its equivalence at zero temperature with the maximum flow problem in a graph [11], for which a very fast polynomial algorithm is known [18]. The simulations mentioned above provided evidence of an infinite correlation length at the phase transition through a finite size scaling analysis, but no direct measurement of a correlation length was performed. In the present paper we perform direct measurements of correlation lengths. We will see that the correlation functions have very strong fluctuations. We will show by a scaling analysis that these fluctuations are so important that in any finite volume the correlation length in the critical region is not self-averaging (i.e. it is strongly sample dependent).

Let us remind that the Hamiltonian of the RFIM is of the form $H = -J \sum_{<i,j>} \sigma_i \sigma_j - \sum_i h_i \sigma_i$. As usually $\sum_{<i,j>}$ runs over neighbouring sites of a cubic lattice with coordinates $x, y, z = 1, \cdots, L$ and $h_i$ are independent random Gaussian variables with variance $\bar{h}_i^2 = 1$. It is well known that there is a zero temperature phase transition in this model. For $J > J_c$ the spins are ferromagnetically ordered, while for $J < J_c$ we get a disordered phase where a large number of spins are locally aligned with the random external field.
In order to measure correlation length at zero temperature we proceed as follows. We consider cubic lattices with linear size \( L \). For every sample of the random field, we considered two copies of the system with different boundary conditions. For the first copy \( \sigma \) all the spins at the plane \( x = 1 \) are set to plus one. For the other copy \( \tau \) they are set to minus one. We let free boundary conditions for both copies at the other end of the lattice, i.e. \( x = L \). We impose periodic boundary conditions on the two perpendicular directions. For each value of the ferromagnetic coupling \( J \) we find the two ground states \( \sigma_1 \) and \( \tau_1 \). For every sample \( s \) of the random field we measure the distance \( d_s(x) = \frac{1}{2}(1 - q_s(x)) \) where the overlap is given by

\[
q_s(x) = \frac{1}{L^3} \sum_{y,z} \sigma(x,y,z)\tau(x,y,z)
\]

which depends on the sample \( s \); (we remind that because of the boundary conditions at \( x = 1 \), \( \sigma(1,y_0,z_0)\tau(1,y_0,z_0) = -1 \) ) \( d_s(x) \) measures the proportion of spins which have opposite values in the two ground states at distance \( x \) from the boundary. By construction \( d_s(1) = 1 \). If \( J << J_c \), \( d_s(x) \) will rapidly decrease with \( x \), while for \( J >> J_c \) it will asymptotically remain constant. It is known from lattice gauge theories [21] that one efficient way to measure correlation lengths, is to impose boundary conditions on an observable and measure the variation of the observable with the distance from the boundary. From the behaviour of \( d_s(x) \) measured numerically we are going to measure the correlation length and draw conclusions on the underlying field theory.

We simulated 1200 samples for \( L = 160 \), 2000 samples for \( L = 120 \) and 3500 samples for \( L = 80 \) for different values of the ferromagnetic coupling \( J \leq J_c \). From the previous simulations [12–15,17] it is known that \( J_c \sim 0.44 \). We found very strong sample to sample fluctuations of \( d_s(x) \), even for systems of sizes \( 160^3 \). In order to extrapolate to the infinite volume limit, we studied the behaviour of \( d^{(k)}(x) = d_s(x)^k \) for \( k = 1, 2, 3 \), (as usually the bar denotes average over the random fields), i.e. the average of the \( k \)’th power of \( d_s(x) \) over the samples.

For \( J < J_c \) we expect an exponential decay in \( x \) of \( d^{(k)}(x) \sim \exp(-m(k)x) \). The masses \( m(k) \) are inverse correlation lengths. We first fitted our data with the formula

\[
d^{(k)}(x) = \exp\left(\frac{-m(k)x}{x^{\nu(k)}}\right)\left(a(k) + \frac{b(k)}{x}\right)
\]

We expect \( m(1) \) to vanish at the critical value \( J = J_c \). According to finite size scaling \( m(1) \sim (J_c - J)^\nu f((J_c - J)L^{1/\nu}) \). We found that the scaling function \( f \) is a constant with a good approximation, i.e. a very small \( L \) dependence of \( m(1) \), and \( 1.3 \leq \nu \leq 1.9 \), compatible with previous results [14,17]. We have obtained a better determination of \( \nu \) in another set of simulations, (with different boundary conditions) that we will publish elsewhere. For a given value of \( J_c \), the statistical errors in \( \nu \) are very small. The large uncertainty is due to subdominant corrections and to the extreme dependence of \( \nu \) on the value of \( J_c \). For \( J < J_c \) in practice for \( J < 0.43 \) we found \( \alpha(1) = 0 \) in agreement with dimensional arguments, while for \( J \sim J_c \) \( \alpha(1) = 0.27 \pm 0.05 \).

In figure 1 we plot \( m(k)^{1/\nu} \) (with \( \nu = 1.5 \)) as a function of \( J \) for \( L = 160 \). Not too far from \( J_c \), \( m(1)^{1/\nu} \) is linear in \( J \) to a very good approximation, \( m(2)^{1/\nu} \) and \( m(3)^{1/\nu} \) have a more complex behaviour. What is very surprising is that for \( J > 0.425 \), \( m(2) \) and \( m(3) \) collapse.
with \(m(1)\). This is best illustrated in figure 2, where we plot the ratios \(m(2)/m(1)\) and \(m(3)/m(1)\) as function of \(J\) for \(L = 160\). Almost identical results were obtained for \(L = 120\) and \(L = 80\). For small enough ferromagnetic coupling \(m(k)/m(1) = k\), i.e. the masses are self-averaging, as expected in perturbative field theory \([19]\), while close to the critical point \(m(k)/m(1) = 1\). It is obvious that \(0 \leq d_s(x) \leq 1\) from the very definition of \(d_s(x)\). It follows that \(d^{(k)}(x) \leq d^{(1)}(x)\) and therefore \(m(k) \geq m(1)\); \(m(k) = m(1)\) amounts to the maximum possible violation of self-averaging of \(d_s(x)\). This is also illustrated in figure 3, where we plot the probability distribution (histogramme) \(H(d)\) of \(d_s(x)\) over the random field samples, for \(L = 160\), \(J = .436\) and and two values of \(x\), \(x = 55\) (dashed line) and \(x = 110\) (continuous line). Theses histogrammes separate into two peaks, one around \(d = 0\) and the other around \(d = 1\). As we vary \(x\) the shape of the histogram remains the same, only the relative height of the two peaks changes. For smaller but not too small \(J\)‘s, in the region of \(J\) where still \(m(k) \sim m(1)\), \(H(d)\) has also contributions outside the two peaks. The peak around \(d = 1\) is necessary for \(m(k) = m(1)\), otherwise \(d^{(k)}(x)\) would decay with \(x\) faster than \(d^{(1)}(x)\).

For any finite \(x\) there is also the possibility of a superposition of exponentials in \(d^{(k)}(x)\). We have therefore tried another fit to \(d^{(2)}(x)\),

\[
d^{(2)}(x) = c_1d^{(1)}(x) + c_2(d^{(1)}(x))^2 + c_3\exp(-m'x)\]

where \(m'\) is a new masse. It turned out, particularly for intermediate values of \(J\), that this is a much better fit. We found that no superposition of exponentials is needed in the fit of \(d^{(1)}(x)\). We found that when \(c_3\) is \(\neq 0\), \(m' > 2m(1)\), i.e. this term contributes only for small \(x\).

\(d^{(1)}(x)\) has the slowest decay in \(x\) in equation \([3]\). Therefore \(c_1\) measures the violation of self-averaging of \(d\) at large \(x\). For small \(J\), \(c_1 \sim c_3 \sim 0\) i.e. the masse is self-averaging, while for \(J \sim J_c\), \(c_2 \sim c_3 \sim 0\), \(c_1 \sim 1\). In order to study the crossover between the two regimes, we tried the finite size scaling ansatz

\[
c_1 = g((J_c - J)L^{1/\nu}) = \mathcal{G}(Lm)\]

where \(m = m(1)\). In the previous equation we used the finite size scaling of \(m\), i.e. \(mL = f((J_c - J)L^{1/\nu})\) to change variables from \((J_c - J)L^{1/\nu}\) to \(mL\). As is seen in figure 3 where \(c_1\) is plotted versus \(mL\) this ansatz works rather well. In the scaling region, i.e. for small \(m\), \(c_1 \sim 1\) and \(d^{(2)}(x) \sim \exp(-mx)\), i.e. \(m(2) = m(1)\) and we observe the maximum possible violation of self-averaging, as we have seen before. Outside the scaling region (i.e. for \(m > 0.75\)) we found that for constant \(J\), \(c_1 \sim 1/L^2\) (we remind that in this region \(m\) depends on \(J\) and not on \(L\) ) and \(d^{(2)}(x) \sim \exp(-2mx)\). This means that the violation of self-averaging of \(d\) vanishes as \(1/L^2\). From the very definition of \(d_s(x)\), if \(d_s \neq 0\), \(d_s \geq 1/L^2\). It follows that \(d^{(2)}(x) \geq d(x)/L^2\), i.e. \(c_1 \sim 1/L^2\) amounts to the smallest possible violation of self-averaging. It is, we think, very remarkable that in the scaling region we cross from the smallest possible violation of self-averaging, i.e. \(c_1 \sim 1\).

The quantities \(d^{(k)}\) have a simple correspondence in field theory at finite temperature \(T\). Let us consider an infinite system at finite temperature. We study the field theoretic equivalent to the quantity \(q_s(x)\) defined in equation \([3]\) and its correlations. To the Ising spins \(\sigma\) and \(\tau\)
correspond the fields $\phi(\vec{x})$ and $\psi(\vec{x})$. Their vacuum expectation (i.e. their space average) has been substracted from $\phi(\vec{x})$ and $\psi(\vec{x})$ so that their correlation, for example $\langle \phi(\vec{x})\phi(\vec{y}) \rangle$ decay with their distance $|\vec{x} - \vec{y}|$. The connected correlation function is defined as

$$G_s(\vec{r}, \vec{r}') = \langle \phi(\vec{r})\psi(\vec{r}')\phi(\vec{r}')\psi(\vec{r}') \rangle$$

(5)

The subscript $s$ represents the dependence on the random field sample. We can also define $G^{(k)}(\vec{r} - \vec{r}')$ the average over the samples of the $k$th moment of $G_s(\vec{r}, \vec{r}')$ (averaging over the random fields restores translation invariance), $G^{(k)}(\vec{r} - \vec{r}') = (G_s(\vec{r}, \vec{r}'))^k$. These correlations $G^{(k)}(\vec{r} - \vec{r}')$ can be simply defined if, as usually, we introduce $n$ replicas (at the end $n \to 0$) to average over the random fields. In the high temperature phase and for large $|\vec{r} - \vec{r}'|$, $G^{(k)}(\vec{r} - \vec{r}') \sim \exp(-\mu(\vec{r})|\vec{r} - \vec{r}'|)$. If we sum over the “transverse” $y$ and $z$ components of the vectors $\vec{r}$ and $\vec{r}'$, $G^{(k)}(r_x - r'_x) = \int_{\vec{r}_{y},\vec{z}_{y},\vec{z}_{z}} G^{(k)}(\vec{r} - \vec{r}')$, then $G^{(k)}(r_x - r'_x) \sim \exp(-\mu(k)r_x - r'_x)$ i.e. $G^{(k)}(r_x - r'_x)$ and $G^{(k)}(\vec{r} - \vec{r}')$ have the same exponential behaviour, only the power prefactor is different. If we define

$$Q^\alpha(x) = \frac{1}{L^3} \sum_{y,z} \phi^\alpha(x, y, z)\psi^\alpha(x, y, z)$$

(6)

where $\phi$ and $\psi$ are the fields corresponding to two identical copies of the system, and the “composite” operators $A^{\alpha_1\alpha_2\cdots\alpha_k}(x) = Q^{\alpha_1}(x)Q^{\alpha_2}(x)\cdots Q^{\alpha_k}(x)$, where $\alpha_1,\alpha_2,\cdots,\alpha_k$ are $k$ replica indices all different from each other,

$$G^{(k)}(x - y) = \langle A^{\alpha_1\alpha_2\cdots\alpha_k}(x)A^{\alpha_1'\alpha_2'\cdots\alpha_k'}(y) \rangle$$

(7)

We can insert a complete set of states $r$ in equation 7

$$G^{(k)}(x - y) = \sum_r \langle A^{\alpha_1\alpha_2\cdots\alpha_k}(x)|r\rangle\langle r|A^{\alpha_1'\alpha_2'\cdots\alpha_k'}(y) \rangle$$

(8)

Let’s call $r_0(k)$ the lowest mass state giving a non zero contribution to the previous sum and $\mu(k)$ its masse. For large $|x - y|$, $G^{(k)}(x - y) \sim \exp(-\mu(k)x)$. In perturbative field theory these lowest masse states $r_0(k)$ are those created by the $2k$ fields $\phi^{\alpha_1}$, $\psi^{\alpha_1}$, $\cdots$, $\phi^{\alpha_k}$, $\psi^{\alpha_k}$, i.e., $\mu(k) = 2k\mu$, where $\mu$ is the mass of the field $\phi$. If we find $\mu(k) \neq 2k\mu$ perturbation theory breaks down.

If universality is valid along the transition line, down to zero temperature, as it is usually assumed, the previous arguments are also valid at zero temperature. First remark that for $J < J_c$ and for $x >> 1$, $q_s(x) = 1$, so the “connected” part of $q_s(x)$ is $q_s(x) = q_s(x - 1) = -2d_s(x)$. It follows from the previous that the long distance behaviour of $d^{(k)}$ and $G^{(k)}(x - y)$ are the same. We found in our simulations that, near the critical region, $m(k) \neq km(1)$, i.e. there exists a new state $r_0$, coupled to all the composite operators $A^{\alpha_1\alpha_2\cdots\alpha_k}$ (we have only measured the correlation functions for $k = 1, 2, 3$). This means that there exists a bound state and this explains why perturbation theory breaks down in the RIFM.

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