Quasi-equilibrium in glassy dynamics: an algebraic view

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Abstract. We study a chain of identical glassy systems in a constrained equilibrium, where each bond of the chain is forced to remain at a preassigned distance to the previous one. We apply this description to mean-field glassy systems in the limit of a long chain where each bond is close to the previous one. We show that this construction defines a pseudo-dynamic process that in specific conditions can formally describe real relaxational dynamics for long times. In particular, in mean-field spin glass models we can recover in this way the equations of Langevin dynamics in the long time limit at the dynamical transition temperature and below. We interpret the formal identity as evidence that in these situations the configuration space is explored in a quasi-equilibrium fashion. Our general formalism, which relates dynamics to equilibrium, puts slow dynamics in a new perspective and opens the way to the computation of new dynamical quantities in glassy systems.

Keywords: cavity and replica method, spin glasses (theory), slow dynamics and ageing (theory)

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1. Introduction

Glassy dynamics is often described as a quasi-equilibrium process. Phase space exploration is depicted as a walk from a metastable state to another, the choice of which for large times is dictated by generalized equilibrium conditions. This picture has been used in the past to interpret slow dynamics for liquids and glasses in equilibrium and aging conditions [1, 2], and more recently for studying dynamical fluctuations in equilibrium terms, using constrained Boltzmann–Gibbs measures [3]. Theoretical evidence in support of this view comes from the emergence of effective temperatures in glassy dynamics [4], the coincidence of certain static and dynamic quantities in the long time limit [5] and a detailed analysis of the response properties during aging dynamics [1]. In addition, numerical simulations of simple models [2] and realistic systems [6] agree with this picture.

Despite the appeal of this picture and the many arguments that can be brought to its support, a formal description of glassy dynamics in equilibrium terms is missing, with the consequence that many quantities of dynamical interest, such as for example the entropy of the trajectories, have not been computed even in the simplest mean-field models.

In this note we would like to fill this gap by establishing a general methodology allowing us to test the dynamical quasi-equilibrium hypothesis at least at the mean-field level. To this aim we employ the method of constrained equilibrium measures [7] and consider a chain of replicas of the system under study, each one in constrained equilibrium with respect to the previous one. Such a construction was used for chains of length 2 in [7] and of length 3 in [8], as a tool to probe the configuration space of glassy systems. The generalization to an arbitrary number $L$ of bonds in the chain was sketched in [9] but no computations based on it were presented. Recently, it has been suggested that large values of $L$ might be necessary to adiabatically follow metastable states in temperature [10, 11] and clarify some anomalies encountered in the $L = 2, 3$ case. Progress has been made by Krzakala and Zdeborova in treating this problem with the cavity method for finite $L$.
We will see that the interest of this construction goes much beyond the problem of following states. Here we employ the replica method and concentrate on fully connected spin glass models, and show how the $L \to \infty$ case relates to slow glassy dynamics.

2. A Markov chain of replicas

Given a physical macroscopic system with configurations labeled by $S, S'$ etc subject to some Hamiltonian $H(S)$, and given a notion of similarity $q(S, S')$ between configurations, we consider a linear chain of $t$ copies such that:

1. the first configuration is chosen with Boltzmann probability at temperature $T_1$
   \[ P(S_1) = \frac{1}{Z_1} \exp \left( -\beta_1 H(S_1) \right), \]
2. for any integer $s > 1$, the $s + 1$th copy is drawn from the Boltzmann–Gibbs measure at temperature $T_{s+1}$ (that may or not depend on $s + 1$) with a chain constraint that $q(S_s, S_{s+1})$ is fixed to some preassigned values $\tilde{C}_c(s + 1, s)$, namely
   \[ M(S_{s+1}|S_s) = \frac{1}{Z_{s+1}(S_s)} \exp \left( -\beta_{s+1} H(S_{s+1}) \right) \delta(q(S_s, S_{s+1}) - \tilde{C}_c(s + 1, s)). \]

Notice that the conditional probability kernel $M$ for fixed $\tilde{C}_c(s + 1, s)$ mathematically defines a Markov chain, where the probability of a trajectory is given by
\[ P(S_t, S_{t-1}, \ldots, S_1) = \prod_{s=1}^{L-1} M(S_{s+1}|S_s) P(S_1). \]

Such a general chain construction was mentioned in [9], but to our knowledge it has not been employed in actual calculations or theories. We will often refer to the chain (3) as Boltzmann pseudo-dynamics and we call the variables $s$, etc, times in the following. In order to understand the physical property of the chain, we would like to study the free energy of the ‘last replica’,
\[ F(L) = -T_L \sum_{S_{L-1}, \ldots, S_1} P(S_{L-1}, S_{L-2}, \ldots, S_1) \log Z_L(S_{L-1}), \]
as a function of the number of bonds in the chain $L$. Although this free energy will be the starting point of our analysis, we concentrate in this paper on the properties of the measure (3), through mean-field analysis based on the replica method.

As we stated in the introduction, a full analysis has been achieved in the cases $L = 2$ [7] and $L = 3$ [8] with the purpose of investigating the structure of metastable states and the barrier separating them (in these case the temperatures are equal along the chain) and the properties of the metastable states when cooled and the temperatures depend on $s$. It has been recently remarked that in order to explain certain anomalies found for $L = 2$ in the case of different temperatures, the general construction with arbitrary $L$ might be needed [10]. Unfortunately, the complexity of the computation strongly increases with total number of steps $L$ involved in the Markov chain. In this note we show that simple expressions can be obtained in the limit where the total number of steps in the chain goes to infinity: differences along neighboring bonds in the chain become small and time becomes continuous.
3. The replica algebra

The problem of analyzing the chain (3) can be addressed using the replica method starting from the computation of the free energy (4). In order to deal with the logarithm of the partition function in (4) and the configuration-dependent partition functions that appear in the denominators of each term of the measure (3) one needs to replicate each of the configurations $S_s$ a number of time $n(s)$, in principle different for each $s$, and consider at the end the limit $n(s) \to 0$; the free energy (4) being associated with the term of order $n(L)$ of the resulting expression. More explicitly, one substitutes $\log Z_L(S_{L-1})$ with $Z_L(S_{L-1})^{n(L)}$ and $1/Z_{s+1}(S_s)$ with $Z_{s+1}(S_s)^{n(s+1)-1}$ for $s = 1, \ldots, L-2$, sets all the $n$’s to integer values in intermediate computations and expresses the partition functions as sums over replicated configurations. We will call the replicas $S_s^a$, with the convention that replica indices associated with the time $s$ run from 1 to $n(s)$. Denoting by $Q_{a,b}(s,u) = q(S_s^a, S_u^b)$ the overlap between two replicas $S_s^a$ and $S_u^b$, the chain constraint reads

$$Q_{a,1}(s+1, s) = \tilde{C}_c(s+1, s),$$

in words, all replicas $a = 1, \ldots, n(s+1)$ at time $s+1$ are constrained to have a fixed overlap $\tilde{C}_c(s+1, s)$ with replica number 1 at time $s$. The constraint (5) can be imposed through Lagrange multipliers $\nu(s)$. In this first paper we will only deal with cases in which the constraints ‘do not make work’ and $\nu(t) = 0$, for which we will show that remarkable solutions exist. Future work it is planned to deal with the fully constrained case.

3.1. Mean-field spin Glasses

In mean-field spin glasses the free energy is obtained as a saddle point over the time-replica matrix order parameter $Q_{a,b}(s,u) = \langle q(S_s^a, S_u^b) \rangle$. As usual, we need an ansatz for the replica matrix that allows the analytic continuation to $n(s) \to 0$. In this paper we are interested in the time structure of the matrix and we stick to a replica symmetric ansatz as far as the ‘$a, b$’ indices are concerned. The generalization to replica symmetry breaking (RSB) is straightforward (see [7, 8] for the case $L = 2, 3$) and does not pose any particular problem of principle. In fact we expect RSB to be crucial in many applications and it will be studied elsewhere.

The form of the constraint (5), symmetry considerations and the experience gained in [7] and [8] suggests considering matrices $Q_{a,b}(s,u)$ that for $s \neq u$ depend only on the index $a$ if $s < u$ and depend only on $b$ if $u < s$. In the case of $s = u$ the matrix $Q_{a,b}(s,s)$ is assumed to have the usual structure. The most general replica symmetric matrix, therefore, can take the following form

$$Q_{a,b}(s,u) = C(s,u) + [\tilde{C}(s,u) - C(s,u)]\Theta_>(s-u)\delta_{b,1} + [\tilde{C}(u,s) - C(u,s)]\Theta_>(u-s)\delta_{a,1} + [\tilde{C}(u,u) - C(u,u)]\delta_{a,u}\delta_{a,b},$$

where $C(s,u)$ and $\tilde{C}(s,u)$ are symmetric functions of their arguments. Having defined $\Theta_>(s) = 1$ if $s > 0$ and zero otherwise. In Ising spherical models $\tilde{C}(u,u) = 1$. We prefer

3 Notice that this replica representation of the free energy is independent of any possible presence or absence of quenched disorder in the Hamiltonian.
not to specify the value of $\tilde{C}(u,u) = 1$ at this level to simplify the analysis of functions of the replica matrix. We remark that the Markov chain structure of (3) implies that in the $n(s) \to 0$ limit the saddle point equations must have a causal structure, and the equations for $C(s,u)$ and $\tilde{C}(s,u)$ should not contain values of the functions at later times. One can see that this causality property is respected whenever $Q_{a,b}(s,u)$ and only depends on index $a$ if $s < u$ and on the index $b$ if $u < s$.

In this way, for example, the equation for $C(1,1)$ is the usual RS equation for the Edwards–Anderson parameter in a single system. We will here consider the so called ‘annealed cases’, where the solution is $C(1,1) = 0$, but the formalism in not specific to this case. It is reasonable (and consistent with the equations) in the annealed case to take $C(1,s) = 0$, a choice that we will adopt in the following. In this paper we concentrate on the structure of the saddle point equations and some physical implications of their solutions, leaving the study of the free energy to future work.

In many models (e.g. in the spherical spin glass model or in the Ising Sherrington–Kirkpatrick model near to the phase transition\(^4\) the only non-trivial operation that is needed in order to solve the saddle point equations is the product of two replica matrices. This is well studied in the case of the standard hierarchical structure, where the matrix is parametrized in terms of a function $q(x)$ [13].

We are now interested in finding out a simple expression for this product. As we have already mentioned, we want to study the limit where the discrete time $t$ goes to infinity and the Markov chain collapses onto a Markov process. To this end it is convenient to evaluate the product between two matrices $Q_A$ and $Q_B$ which have the structure (6), i.e. they are parametrized in terms of functions $C_g(s,u)$ and $\tilde{C}_g(s,u)$ with $g = A,B$ respectively.

It turns out that a good long chain limit $L \to \infty$ is obtained if one supposes that $C(s,u)$ tends to continuous function of $s/L$ and $u/L$ and, for $s \neq u$ and for $s,u \neq 1$, one has $\tilde{C}(s,u) - C(s,u) = O(1/L)$. We therefore abandon the discrete time: without causing confusion we change notation and from now on the variables $u, s, etc will denote continuous variables taking values in some interval $[0, t_{\text{max}}]$, (without necessarily normalizing the final point to $t_{\text{max}}$ to 1). We then define the function

$$T_u R(s,u) du = \Theta(s-u)(\tilde{C}(s,u) - C(s,u)),$$

which we call the response function; we will see later that this name is non-abusive.

Denoting therefore by $Q_C$ the product between $Q_A$ and $Q_B$, one finds that $Q_C$ is consistently parametrized by functions $C_C(s,u)$ and $R_C(s,u)$. In the continuous limit a careful computation\(^5\) shows that in the limit in which all the $n(s)$ go to zero, these functions verify the relations:

$$C_C(t,r) = \int_0^r ds \ C_A(t,s) T_s R_B(r,s) + \int_0^t ds \ T_s R_A(t,s) C_B(r,s)$$

$$\quad + [\tilde{C}_A(t,t) - C_A(t,t)] C_B(r,t) + C_A(t,r) [\tilde{C}_B(r,r) - C_B(r,r)]$$

$$\quad + C_A(t,0) C_B(r,0),$$

\(^4\) This is true up to the order $(T_c - T)^{10}$ in an expansion in $T_c - T$!

\(^5\) In fact this computation can be fully automatized by implementing the matrix multiplication in algebraic manipulation software.
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\[ R_C(t,r) = \int_r^t ds R_A(t,s)R_B(s,r) \]
\[ + \beta_r R_A(t,r) [\tilde{C}_B(r,r) - C_B(r,r)] + \beta_t [\tilde{C}_A(t,t) - C_A(t,t)] R_B(t,r), \]  
\[ \tilde{C}_C(t,t) = \int_{t_0}^t ds C_A(t,s)T_s R_B(t,s) + T_s R_A(t,s)C_B(t,s) \]
\[ + \tilde{C}_A(t,t) \tilde{C}_B(t,t) - C_A(t,t) C_B(t,t) + C_A(t,0) C_B(t,0). \]  

The alert reader will recognize the similarity of equations (8)–(10) to the convolution of two functions in the supersymmetric formalism used in the Langevin relaxational dynamics starting from random initial conditions [14]. In that context one defines the supersymmetric correlation function

\[ Q(t, \theta; s, \theta') = C(t,s) + \theta T_t R(s,t) + \theta' T_s R(t,s), \]

where \( \theta \) and \( \theta' \) are (commuting) Grassmannian variables. The convolution between two functions of that type

\[ Q_C(t, \theta; s, \theta') = \int du d\theta'' Q_A(t, \theta; u, \theta'')Q_B(u, \theta''; s, \theta') \]

is still a function of the same form with

\[ C_C(t,r) = \int_0^r ds C_A(t,s)T_s R_B(r,s) + \int_0^t ds T_s R_A(t,s)C_B(r,s) \]
\[ R_C(t,r) = \int_r^t ds R_A(t,s)R_B(s,r). \]

We see that our replica product equals the supersymmetric one if \( C(t,0) = 0 \) and \( \tilde{C}(t,t) = C(t,t) \). In this respect we notice that the term containing \( C(t,0) \) comes as a consequence of choosing the first replica in equilibrium, and would disappear if for example \( T_1 \to \infty \), or if the memory of the initial condition were to be lost. As the matter of fact, if one considers a Langevin dynamical relaxation starting from an equilibrium initial condition the supersymmetric product is modified and that very additional term appears [15]. The other additional term comes from a fundamental difference between real dynamics and Boltzmann pseudo-dynamics. In real dynamics, short time scales are dominated by fast relaxation processes and one cannot say in any sense that the vicinity of a given instantaneous configurations is explored according to the Boltzmann weight. This property can only hold on large time scales. By contrast, pseudo-dynamics samples accord to Boltzmann by construction. Notice that the minimal distance between subsequent bonds in the chain is \( \tilde{C}(t,t) - C(t,t) \), which one can expect to be macroscopic. In situations where time scale separation occurs, and the fast part of the dynamics is seen as instantaneous by the slow one, the additional terms that we find in the replica product exactly coincide with the ones that couple the slow part of the dynamics to the fast one in Langevin dynamics. We will see that in fact this property implies that the resulting equations have the familiar property of invariance [16] under time reparametrizations \( t \to h(t) \) for monotonous functions \( h(t) \).

It is remarkable that the replicas algebra in the limit of continuous time and vanishing \( n(s) \) is isomorphic to the supersymmetric algebra of the dynamics. In fact, this long sought
isomorphism [17] roughly reduces to the correspondence
\[ \delta_{b,1} \, ds \rightarrow \theta \]
and to neglecting systematically terms of order \( ds^2 \).

Albeit much more complicated, a formula for the matrix product can be written for finite \( n(s) \). The resulting structure constitutes a deformation of the supersymmetric algebra, whose formal properties would be interesting to study.

Having established the product algebras for the replicas, we continue with the computation of the equations for the spherical model.

4. Spherical \( p \)-spin models

The replica analysis, as well as the study of dynamics are simplified in mean-field spherical models [18]. In these models the spins \( S_i \) verify the spherical constraint \( \sum_{i=1}^{N} S_i^2 = N \) and the Hamiltonian \( H(S) \) is a random Gaussian function of the configurations with covariance
\[ H(S)H(S') = NF(q(S, S')) , \]
where the function \( f(q) \) is in general chosen to be a polynomial with positive coefficients. Depending on the function \( f \), the static equilibrium of the model can have either one step (1RSB) or full (fRSB) replica symmetry breaking at low temperature. A well known case of 1RSB is the \( p \)-spin model for \( p \geq 3 \), where \( f(q) = q^p \) [18]. An example of fRSB is given by \( f(q) = q^2 + a q^4 \) for suitable values of \( a > 0 \) [19].

The replica analysis of the model shows that the replicated free energy as a function of a generic replica matrix can be written as [18]
\[ -\beta F[Q] = \frac{1}{2} \sum_{a,b,s,u} \beta_s \beta_a f(Q_{a,b}(s, u)) + \frac{1}{2} \text{Tr \log } Q - \frac{1}{2} \text{Tr } \mu(Q - C_v) , \]
where the last term is needed to enforce the constraints (5), as announced we consider the case of \( \nu(t) = 0 \), and \( \mu(t) = \mu(t) \delta_{t,0} \delta_{t,b} \), where \( \mu(t) \) enforces the spherical constraint at all times. The saddle point equations read
\[ \frac{\beta_s \beta_a}{2} f'(Q_{a,b}(s, u)) + \frac{1}{2} Q^{-1}_{a,b}(s, u) - \frac{1}{2} \mu(s) \delta_{a,b} \delta(s-u) = 0. \]

In order not to need to invert \( Q \) we can just multiply by \( Q \) and get
\[ \frac{1}{2} \sum_{u=1}^{L} \sum_{b=1}^{n(u)} \beta_s \beta_a f'(Q_{a,b}(s, u)) Q_{b,c}(u, v) + \frac{1}{2} \delta_{a,b} \delta_{s,v} - \frac{1}{2} \mu(s) Q_{a,c}(s, v) = 0. \]

Inserting (8)–(10) we get the equations:
\[ \mu(t) C(t, u) = \beta_t \int_0^u ds f'(C(t, s)) R(u, s) + \beta_t \int_0^t ds f''(C(t, s)) R(t, s) C(u, s) + \beta_t \beta_a f' C(t, s) ) C(t, u) + \beta_t \beta_a f' C(t, u) ) (1 - C(u, u)) + \beta_t \beta_a f' C(t, 0) ) C(u, 0) , \]
\[ \mu(t) R(t, u) = \beta_t \int_0^t f''(C(t, s)) R(t, s) ds R(s, u) + \beta_t f''(C(t, u) ) R(t, u) (1 - C(u, u)) + \beta_t (f' (1 - f' C(t, t))) R(t, u) , \]
\[ \frac{Q^{-1}_{a,b}(s, u) - \frac{1}{2} \mu(s) \delta_{a,b} \delta(s-u) = 0. \]

\[ \text{doi:10.1088/1742-5468/2013/02/P02003} \]
\[ \mu(t) = T_i + \beta_t^2 (f'(1) - f'(C(t, t)) C(t, t)) \\
+ \beta_t \int_0^t ds (f'(C(t, s)) R(t, s)) \\
+ \int_0^t ds (f''(C(t, s)) R(t, s) C(t, s)) + \beta_t f'(C(t, 0)) C(t, 0), \tag{19} \]

where we used the condition \( \tilde{C}(t, t) = 1 \). As announced these equation are, for constant \( \beta_t = \beta \), invariant under time reparametrization. Indeed they coincide with the equations for the slow part of the relaxation of Langevin dynamics of the \( p \)-spin model [22, 23], and some of their solutions are well known [20]–[23], [15].

In fact for 1RSB systems, non-trivial solutions have been found in two cases, describing:

1. The equilibrium alpha relaxation process for constant temperature \( T \rightarrow T_d \) (\( T_d \) is the dynamical transition temperature of the model) [21]. In this case one chooses the function \( C(t, s) \) and \( R(t, s) \) to be time translation invariant and verifying the fluctuation dissipation relation \( R(t-s) = \beta \partial C(t-s)/\partial s \).

2. The slow part of aging relaxation starting from a non-equilibrium condition, for \( T < T_d \) [22, 23]. This situation can be achieved in our formalism if we take a very high value of \( T_1 \) and later a constant temperature \( T \), and supposing loss of memory of the initial condition \( C(t, 0) = 0 \). One finds then a family of solutions of the kind \( C(t, s) = C(h(s)/h(t)) \) (if \( t > s \)), with the response verifying the modified fluctuation dissipation relation \( R(t, s) = \beta x \partial C(t, s)/\partial s \) with \( x \in [0, 1] \).

Similarly, for fRSB systems the equations admit a non-trivial aging solution with loss of memory of the initial condition, below the model’s critical temperature. The long time aging solution in spin glass systems with fRSB has been described in general in [23, 24]. One has that correlations at different times are related by ultrametricity and the ‘fluctuation dissipation’ ratio between response and correlation is a continuous function of the correlation.

In all cases the dynamics is critical. Indeed marginal stability, physically associated with vanishing of free-energy barriers, appears as a necessary condition for having non-trivial solutions where \( C(t, s) \) actually depends on time, and in the last analysis for the equivalence of slow real dynamics and Boltzmann pseudo-dynamics\(^6\).

5. The SK model

The reader may wonder at this point if our findings are specific to models where only products and functions of the replica matrix elements are important. In this section we will study the Sherrington–Kirkpatrick model (with Ising spins) and show that the equivalence with long time dynamics still holds. The Hamiltonian of the model is a Gaussian function, as in the previous section, with a correlation function specified by (15) with \( f(q) = q^2/2 \).

For simplicity we will consider the case of infinite initial temperature \( T_i \rightarrow \infty \) and constant temperature \( T_s = T \) for \( s > 1 \). We will not include in the analysis a low-temperature

\(^6\) We notice that in the case of \( f(q) = q^2 \), corresponding to the spherical Sherrington–Kirkpatrick model, which is replica symmetric [25] and whose dynamical properties are similar to the ones of domain growth [26], the long time solution is not time reparametrization invariant, and equivalence with pseudo-dynamics does not hold.
equilibrium initial condition. Contrary to the p-spin case above \( T_{\text{stat}} \) this would require replica symmetry breaking, which is not the main emphasis here.

Quoting from [27], we write the replicated partition function of the model (up to irrelevant terms) as

\[
Z_{\text{Rep}} = s.p. e^{-N \beta^2/2} \sum_{\alpha < \beta} Q_{\alpha, \beta}^2 \times \zeta[Q]^N \quad \zeta[Q] = \sum_{\{S_a\}} e^{(1/2)\beta \sum_{\alpha \neq \beta} Q_{\alpha, \beta} S_a S_\beta},
\]

where s.p. denotes saddle point over the elements \( Q_{\alpha, \beta} \). In our case the indices \( \alpha, \beta \) take the form \( \alpha = (s, a) \) \( \beta = (u, b) \) with \( s, u = 1, \ldots, L, a = 1, \ldots, n(s) \) and \( b = 1, \ldots, n(u) \). The new interesting term with respect to the previous analysis is \( \zeta[Q] \), which we write as

\[
\zeta[Q] = \sum_{\{S_a(s)\}} e^{\pm 1} (1/2)\beta \sum_{s, a, b} Q_{a, b}(s, u) S_a(s) S_b(u),
\]

For small \( Q \) one can study the development in powers of \( Q \) and many results can be straightforwardly obtained by the algebra we have just derived. In the following we will derive compact expressions that go beyond such an expansion. We then substitute (6) and (7), we get for \( s > u \)

\[
\sum_{a, b} Q_{a, b}(s, u) S_a(s) S_b(u) = \left( \sum_a S_a(s) \right) C(s, u) \left( \sum_b S_b(u) \right) + \left( \sum_a S_a(s) \right) R(s, u) S_1(u) du,
\]

while for \( s = u \)

\[
\sum_{a \neq b} Q_{a, b}(s, s) S_a(s) S_b(s) = \left( \sum_a S_a(s) \right) C(s, s) \left( \sum_b S_b(s) \right).
\]

Introducing a field \( i \hat{h}(s) ds = \sum_{a=1}^{n(s)} \sigma_a(s) \) and its conjugate \( \beta h(s) \), and mixing freely discrete and continuous time notation, we rewrite

\[
\zeta[Q] = \sum_{\{S_a(s)\}} e^{\pm 1} \int \prod_u dh(u) \exp \left( -\frac{1}{2} \int ds du \beta \hat{h}(s) C(s, u) \hat{h}(u) - i \beta \int du \hat{h}(u) h(u) \right) \times \exp \left( \int ds du i \hat{h}(u) R(u, s) S_1(s) + \sum_u \beta h(u) (S_1(u) + \sum_{a=2}^{n(u)} S_a(u)) \right).
\]

Notice that the spins \( S_a(u) \) in replicas \( a > 1 \) are decoupled, and can be summed over; resulting in terms of the kind \((2 \cosh(h(u)))^{-n(u)-1} \rightarrow (2 \cosh(h(u))^{-1} \) for \( n(u) \rightarrow 0 \). The final expression is

\[
\zeta[Q] = \sum_{\{S_1(s)\}} e^{\pm 1} \int \prod_u \left( \frac{e^{\beta h(u)}}{2 \cosh(\beta h(u))} \right) \hat{h}(u) dh(u) \exp \left( -\frac{1}{2} \int ds du \beta \hat{h}(s) C(s, u) \hat{h}(u) \right) \times \exp \left( \int du \hat{h}(u) h(u) + \int ds du i \hat{h}(u) R(u, s) S_1(s) \right).
\]
Taking into account the first term in (20), one finds the self-consistency equations
\[ C(s,u) = \langle S_1(s)S_1(u) \rangle \]
and
\[ T(u)R(s,u) = \delta \langle S_1(s) \rangle / \delta h(u). \]

Our findings have a clear interpretation: in the long chain limit they describe the equilibrium of spins with their local field as well as the adiabatic evolution of the field. In fact, we find that local spin and the field are related by
\[
P(S(t)|h(t)) = \frac{e^{h(t)S(t)}}{2 \cosh(h(t))},
\]
while the field is determined by
\[
h(t) = \eta(t) + \int_0^t ds R(t,s)S(s),
\]
where \( \eta(t) \) is a zero mean Gaussian variable with covariance \( \langle \eta(t)\eta(s) \rangle = C(t,s) \). The equations are closed by observing that in the long chain limit, where the dependence of \( h(t) \) and \( R(t,s) \) on time is slow, one can substitute \( S(s) \) in (27) with its conditional average from (26) \( m(s) = \tanh h(s) \). Our equations provide the long time limit of the Eissfeller–Opper equations for the dynamics of the SK model [28]. The same equations are the skeleton of the dynamic cavity equations in the long time limit (as discussed in reference [27], that can be derived by a direct analysis in [29]).

It is well known that in the SK model the dynamical formalism allows one to recover many equilibrium quantities related to replica symmetry breaking, such as e.g. the function \( q(x) \), which describes the statistics of pure states [13]. In dynamics this quantity is intimately related to the breakdown of the fluctuation dissipation theorem [22, 23] and the emergence of effective temperatures [4]. Our analysis unambiguously shows that their appearance is associated with quasi-equilibrium sampling of phase space.

6. Perspectives

In this paper we have formalized the notion that slow glassy dynamics consists in quasi-equilibrium exploration of configuration space. Our analysis has been achieved with mean-field models as a reference. In the past thirty years such models have been a precious guide in forming physical pictures of the glassy behavior of more realistic systems, we can then conjecture that the quasi-equilibrium description holds in general for marginally stable glassy dynamics. Indeed we think that quasi-equilibrium exploration of phase space is at the heart of the emergence modified fluctuation dissipation relations [22, 23] and effective temperatures [4], the equivalence between equilibrium and dynamics discussed in [5] and the time reparametrization invariance properties [16]. All these properties have been well verified to hold beyond the mean-field level.

The formalism we have introduced here is very general and opens new perspectives in the comprehension of glassy dynamics.

We would like to mention here a few problems where we expect it will lead to relevant progress. First of all let us quote some problems where the chain constraints do not make work, in connection with glassy slow time dynamics.

At the most basic level our formalism suggests how to obtain a sensible discretization of dynamic equations in the slow time limit. This could be useful in the context of the cavity dynamical method for spin models on sparse graphs and Bethe lattices, where a
direct dynamical approach leads to considerable technical difficulties [30]. This opens new perspectives for studying dynamical processes on networks and network dynamics [31], such as epidemic or damage spreading.

At a more fundamental level, one can tackle the task of studying dynamical quantities that cannot be computed in a direct dynamical approach. For example, the dynamical entropy, which is related to the terms of order $n(L)$ of the free energy.

The basic property that slow dynamics is a quasi-equilibrium state, suggests using our pseudo-dynamic formalism to study dynamical quantities in cases where a direct dynamic approach is problematic. For example, we argue that it enables one in principle to study slow dynamics (alpha dynamical processes) of liquids, starting from equilibrium approaches based on the replica method. A remarkable example is the HNC approximation, where one can find equations similar to mode coupling theory, but in the context of a fully consistent theory. Work is underway in this direction [32].

As far as systems with different temperatures are concerned, one can use the formalism to follow metastable states adiabatically in temperature [10]. A related question concerns the inclusion of replica symmetry breaking effects. Finally we would like to mention interpretation questions that are far from being settled, to start with the surprising appearance of the dynamical response in a purely thermodynamic setting.

The other class of problems which can be addressed is the one where the constraints are effective. The chain construction constitutes a powerful probe of configuration space. With working constraints one can study free-energy barriers among states. The properties of these barriers and whether the barriers of pseudo-dynamics are related to the ones of the real dynamics is a question to be investigated.

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