THE EQUIVALENCE BETWEEN MANY-TO-ONE POLYGRAPHS AND OPETOPIC SETS

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Abstract. From the polynomial approach to the definition of opetopes of Kock et al., we derive a category of
opetopes, and show that its set-valued presheaves, or opetopic sets, are equivalent to many-to-one polygraphs. As
an immediate corollary, we establish that opetopic sets are equivalent to multitopic sets, introduced and studied
by Harnick et al, and we also address an open question of Henry.

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1. Introduction

Opetopes were originally introduced by Baez and Dolan in [BD98] as an algebraic structure to describe compo-
sitions and coherence laws in weak higher dimensional categories. They differ from other shapes (such as globular
or simplicial) by their (higher) tree structure, giving them the informal designation of “many-to-one”. Pasting
opetopes give rise to opetopes of higher dimension (it is in fact how they are defined!), and the analogy between
opetopes and cells in a free higher category starts to emerge. On the other hand, polygraphs (also called com-
putads) are higher dimensional directed graphs used to generate free higher categories by specifying generators
and the way they may be pasted together (by means of source and targets).

In this paper, we relate opetopes and polygraphs in a direct way. Namely, we define a category \( \mathcal{O} \) whose objects
are opetopes, in such a way that the category of its set-valued presheaves, or opetopic sets, is equivalent to the
category of many-to-one polygraphs. This equivalence was already known from [HMZ02, HMZ08, HMP00],
however the proof is very indirect. The recent work of Henry [Hen19] showed the category of many-to-one
polygraphs (among many others) to be a presheaf category, but left the equivalence between “opetopic plexes”

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(serving as shapes for many-to-one polygraphs in his paper) and opetopes open. We establish this in our present work.

The notion of multitope [HMP02, HMZ08] is related to that of opetope, and has been developed based on similar motivations. However the approaches used are different: opetopes are based on operads (specifically, $T_n$-operads, where $T_n$ is a certain sequence of cartesian monads) [Lei04], while multitopes are based on (symmetric) multicategories. It is known that multitopic sets are equivalent to many-to-one polygraphs [HMZ08, HMZ02], and in particular our present contribution reasserts the equivalence between multitopic and opetopic sets.

Plan. We begin by recalling elements of the theory of polynomial functors and polynomial monads section 2. This formalism is at the base of our chosen approach to opetopes, which we present in section 3. In section 4, we review some basic polygraphs theory, and pay special attention to those that are many-to-one. Finally, we state and prove the equivalence between opetopic sets and many-to-one polygraphs in section 5.

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2. Polynomial functors and polynomial monads

We survey elements of the theory of polynomial functors, trees, and monads. For more comprehensive references, see [Koc11, GK13].

2.1. Polynomial functors.

Definition 2.1 (Polynomial functor [GK13, paragraph 1.4]). A polynomial functor $P$ is a diagram in $\text{Set}$ of the form

$$ I \leftarrow^s E \rightarrow^p B \rightarrow^t J. \tag{2.2} $$

We say that $P$ is a polynomial endofunctor if $I = J$. In this case, we also say that $P$ is a polynomial functor over $I$. We say that $P$ is finitary if the fibres of $p : E \rightarrow B$ are finite sets. We will always assume polynomial functors and endofunctors to be finitary.

We use the following terminology for a polynomial functor $P$ as in equation (2.2), which is motivated by the intuition that a polynomial functor encodes a multi-sorted signature of function symbols. The elements of $B$ are called the nodes or operations of $P$, and for every node $b$, the elements of the fibre $E(b) := p^{-1}(b)$ are called the inputs of $b$. The elements of $I$ are called the input colors or input sorts of $P$, and the elements of $J$ are output colors or output sort. For every input $e$ of a node $b$, we denote its color by $s_e(b) := s(e)$.

$$ s_{e_1}(b) \quad \cdots \quad s_{e_k}(b) $$

\[ b \]

\[ t(b) \]

Definition 2.3 (Morphism of polynomial functor). A morphism $f$ from a polynomial functor $P$ over $I$ (on the first row) to a polynomial functor $P'$ over $I'$ (on the second row) is a commutative diagram of the form

$$ I \leftarrow^s E \rightarrow^p B \rightarrow^t J $$

$$ f_0 \downarrow \quad f_2 \downarrow \quad \downarrow f_1 \quad \downarrow f_0 $$

$$ I' \leftarrow^{s'} E' \rightarrow^{p'} B' \rightarrow^{t'} J' $$

where the middle square is cartesian (i.e. is a pullback square). If $P$ and $P'$ are both polynomial functors over $I$, then a morphism from $P$ to $P'$ over $I$ is a commutative diagram as above, but where $f_0$ is required to be the identity [Koc11, paragraph 0.1.3] [KJBM10, section 2.5]. Let $\text{PolyEnd}$ denote the category of polynomial functors and morphisms of polynomial functors, and $\text{PolyEnd}(I)$ the category of polynomial functors over $I$ and morphisms of polynomial functors over $I$. 
2.2. Trees. The combinatorial notion of a tree fits nicely in the framework of polynomial functors. We now state the definition of a polynomial tree, and refer the reader to [Koc11, section 1.0.3] for more details about the intuition behind it.

**Definition 2.4** (Polynomial tree [Koc11, section 1.0.3]). A polynomial functor $T$ given by

$$T_0 \leftarrow s \ T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$$

is a *polynomial tree* (or just tree) if

1. the sets $T_0$, $T_1$ and $T_2$ are finite (in particular, each node has finitely many inputs); by convention we assume $T_0 \neq \emptyset$;
2. the map $t$ is injective;
3. the map $s$ is injective, and the complement of its image $T_0 - \im s$ has a single element, called the *root*;
4. let $T_0 = T_2 + \{r\}$, with $r$ the root, and define the *walk-to-root* function $\sigma$ by $\sigma(r) = r$, and otherwise $\sigma(e) = tp(e)$; then we ask that for all $x \in T_0$, there exists $k \in \mathbb{N}$ such that $\sigma^k(x) = r$.

We call the colors of a tree its *edges* and the inputs of a node its *input edges* of that node.

Let $\Tree$ be the full subcategory of $\PolyEnd$ whose objects are trees. Note that it is the category of *symmetric* or *non-planar* trees (the automorphism group of a tree is in general non-trivial) and that its morphisms correspond to inclusions of non-planar subtrees. An *elementary tree* is a tree with at most one node, and we write $\Tree_{\text{elem}}$ for the full subcategory of $\Tree$ spanned by elementary trees.

**Definition 2.5** ($P$-tree). For $P \in \PolyEnd$, the category $\tr P$ of $P$-trees is the slice $\Tree/P$. If $f : P \to Q$ is a morphism of polynomial functors, then it induces a natural functor $f_\ast : \tr P \to \tr Q$ by postcomposition.

**Notation 2.6.** A $P$-tree $T \in \tr P$ is a morphism from a polynomial tree, which we shall denote by $(T)$, to $P$, as in $T : (T) \to P$. We point out that $(T)_1$ is the set of nodes of the $P$-tree $T$, while $T_1 : (T)_1 \to P_1$ provides a *decoration* of the nodes of $(T)$ by operations of $P$, and likewise for edges.

**Definition 2.7** (Address). Let $T \in \Tree$ be a polynomial tree and $\sigma$ be its walk-to-root function (definition 2.4). We define the *address function* $\&$ on edges inductively as follows:

1. if $r$ is the root edge, let $\& r := [\cdot]$;
2. if $i \in T_0 - \{r\}$ and if $\& \sigma(i) = [x]$, define $\& i := [xe]$, where $e \in T_2$ is the unique element such that $s(e) = i$.

Thus an address is a sequence of elements of $T_2$, enclosed by brackets. The address of a node $b \in T_1$ is simply $\& b := \& t(b)$. Note that this function is injective since $t$ is. Let $T^\ast$ denote its image, the set of *node addresses* of $T$, and let $T^1$ be the set of addresses of leaf edges, i.e., those edges not in the image of $t$.

Assume now that $U : (U) \to P$ is a $P$-tree. If $b \in (U)_1$ has address $\& b = [p]$, write $s_{[p]} U := U_1(b)$. For convenience, we let $U^\ast := (U)^\ast$, and $U^1 := (U)^1$.

**Remark 2.8.** The formalism of addresses is a useful bookkeeping syntax for the operations of grafting and substitution on trees. The syntax of addresses will extend to the category of opetopes and will allow us to give a precise description of the composition of morphisms in the category of opetopes (see definition 3.6) as well as certain constructions on opetopic sets.

**Notation 2.9.** We denote by $\tr^1 P$ the set of $P$-trees with a marked leaf, i.e. endowed with the address of one of its leaves. Similarly, we denote by $\tr^\ast P$ the set of $P$-trees with a marked node.

**Definition 2.10** (Elementary $P$-trees). Let $P$ be a polynomial endofunctor as in equation (2.2). For $i \in I$, define $I_i \in \tr P$ as having underlying tree

$$\{i\} \leftarrow \emptyset \longrightarrow \emptyset \longrightarrow \{i\},$$

along with the obvious morphism to $P$, that which maps $i$ to $i \in I$. This corresponds to a tree with no nodes and a unique edge, decorated by $i$. Define $\corollary b \in \tr P$, the *corolla* $b$, as having underlying tree

$$s(E(b)) + \{\ast\} \leftarrow s(E(b)) \longrightarrow \{b\} \longrightarrow s(E(b)) + \{\ast\},$$

where the right map sends $b$ to $\ast$, and where the morphism $\corollary b \to P$ is the identity on $s(E(b)) \subseteq I$, maps $\ast$ to $t(b) \in I$, is the identity on $E(b) \subseteq E$, and maps $b$ to $b \in B$. This corresponds to a $P$-tree with a unique node,
decorated by $b$. Observe that for $T \in \text{tr } P$, giving a morphism $I_i \to T$ is equivalent to specifying the address $[p]$ of an edge of $T$ decorated by $i$. Likewise, morphisms of the form $\mathcal{Y}_b \to T$ are in bijection with addresses of nodes of $T$ decorated by $b$.

Remark 2.13. Let $P$ be a polynomial endofunctor as in equation (2.2).

1. Let $i \in I$ be a color of $P$. Since $I_i$ does not have any nodes, the set $I_i^\bullet$ of its node addresses is empty. On the other hand, the set of its leaf addresses is $I_i^\circ = \{[1]\}$, since the unique leaf is the root edge.
2. Let $b \in B$ be an operation of $P$. Then $\mathcal{Y}_b^\circ = \{[1]\}$ since the only node is that above the root edge. For leaves, we have $\mathcal{Y}_b^\circ = \{[e] \mid e \in E(b)\}$.

Definition 2.14 (Grafting). For $S,T \in \text{tr } P$, $[l] \in S^1$ such that the leaf of $S$ at $[l]$ and the root edge of $T$ are decorated by the same $i \in I$, define the grafting $S \circ_{[l]} T$ of $S$ and $T$ on $[l]$ by the following pushout (in $\text{tr } P$):

$$
\begin{array}{ccc}
I_i & \xrightarrow{[l]} & T \\
\downarrow & \nearrow \gamma \\
S & \xrightarrow{\circ_{[l]}} & S \circ_{[l]} T.
\end{array}
$$

(2.15)

Note that if $S$ (resp. $T$) is a trivial tree, then $S \circ_{[l]} T = T$ (resp. $S$). We assume, by convention, that the grafting operator $\circ$ associates to the right. In particular $I_i \circ_{[l]} T = T$ and $S \circ_{[l]} I_i = S$.

Lemma 2.16. For $S,T \in \text{tr } P$, $[l] \in S^1$ such that the grafting $S \circ_{[l]} T$ is defined, we have

$$(S \circ [l] T)^\bullet = S^\bullet + \{[l]p \mid [p] \in T^\bullet\}, \quad (S \circ [l] T)^\circ = S^\circ - \{[l]\} + \{[l]p \mid [p] \in T^\circ\}.$$

Notation 2.17 (Total grafting). Let $T,U_1,\ldots,U_k \in \text{tr } P$, write $T^1 = \{[l_1],\ldots,[l_k]\}$, and assume the grafting $T \circ_{[l_i]} U_i$ is defined for all $i$. Then the total grafting will be denoted concisely by

$$T \bigcup_{[l_i]} U_i = (\cdots(T \circ_{[l_1]} U_1) \circ_{[l_2]} U_2) \cdots \circ_{[l_k]} U_k. \quad (2.18)$$

It is easy to see that the result does not depend on the order in which the graftings are performed.

Proposition 2.19 ([Koc11, proposition 1.1.21]). Every $P$-tree is either of the form $I_i$, for some $i \in I$, or obtained by iterated graftings of corollas (i.e. $P$-trees of the form $\mathcal{Y}_b$ for $b \in B$).

Proof. This can easily be proved by induction on the number of nodes. \hfill \Box

Remark 2.20. As a consequence of [Koc11, proposition 1.1.3], a morphism $T \to S$ of $P$-trees exhibits $T$ as a subtree of $S$ as in

$$S = U \circ_{[p]} T \bigcup_{[l]} V_{[l]}$$

where $U$ is spanned by all the edges of $S$ that are either descendant of the root edge of $T$, or incomparable to it [Koc11, paragraphs 1.0.7 and 1.1.11], and where $[l]$ ranges over $T^1$. Conversely, any such decomposition of $S$ induces a morphism $T \to S$.

2.3. Polynomial monads.

Definition 2.21 (Polynomial monad). A polynomial monad over $I$ is a monoid in $\text{PolyEnd}(I)$. Note that a polynomial monad over $I$ is thus necessarily a cartesian monad on $\text{Set}/I$.\footnote{We recall that a monad is cartesian if its endofunctor preserves pullbacks and its unit and multiplication are cartesian natural transformations.} Let $\text{PolyMnd}(I)$ be the category of monoids in $\text{PolyEnd}(I)$. That is, $\text{PolyMnd}(I)$ is the category of polynomial monads over $I$ and morphisms of polynomial functors over $I$ that are also monad morphisms.

Definition 2.22 ($(-)^\ast$ construction). Given a polynomial endofunctor $P$ as in equation (2.2), we define a new polynomial endofunctor $P^\ast$ as

$$I \leftarrow^s \text{tr}^1 P \xrightarrow{p} \text{tr } P \xrightarrow{i} I \quad (2.23)$$
Theorem 2.24 ([Koc11, section 1.2.7], [KJB10, sections 2.7 to 2.9]). The polynomial functor $P^*$ has a canonical structure of a polynomial monad. Furthermore, the functor $(-)^*$ is left adjoint to the forgetful functor $\text{PolyEnd}(I) \to \text{PolyEnd}(I)$, and the adjunction is monadic.

Definition 2.25 (Target and readdressing map). We abuse notations and letting $(-)^*$ denote the associated monad on $\text{PolyEnd}(I)$. Let $M$ be a polynomial monad as in

$$ I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I. $$

By theorem 2.24, $M$ is a $(-)^*$-algebra, and we will write its structure map $M^* \to M$ as

$$ I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I. $$

(2.26)

For $T \in \text{tr} M$, we call $\varphi_T : T' \xrightarrow{\varphi} E(tT)$ the readdressing function of $T$, and $tT \in B$ is called the target of $T$. If we think of an element $b \in B$ as the corolla $Y_b$, then the target map $t$ “contracts” a tree to a corolla, and since the middle square is a pullback, the number of leaves is preserved. The map $\varphi_T$ establishes a decoration-preserving correspondence between the set $T'$ of leaf addresses of a tree $T$ and the elements of $E(tT)$.

Definition 2.27 (Baez–Dolan $(-)^*$ construction). Let $M$ be a polynomial monad as in equation (2.2), and define its Baez–Dolan construction $M^*$ to be

$$ B \xleftarrow{s} \text{tr}^* M \xrightarrow{p} \text{tr} M \xrightarrow{t} B $$

(2.28)

where $s$ maps an $M$-tree with a marked leaf to the decoration of that leaf, $p$ forgets the marking, and $t$ maps a $P$-tree to the decoration of its root. Remark that for $T \in \text{tr} P$ we have $p^{-1}(T) \equiv T'$. Clearly, there is an inclusion $P \hookrightarrow P^*$, mapping $b \in B$ to $Y_b \in \text{tr} P$, and $e \in E(b)$ to $[e] \in Y_b$ (see remark 2.13).

Theorem 2.29 ([KJB10, section 3.2]). If $M$ a polynomial monad, then $M^*$ has a canonical structure of a polynomial monad.

Remark 2.30 (Nested addresses). Let $M$ be a polynomial monad, and $T \in \text{tr} M^*$. Then the nodes of $T$ are decorated in $M$-trees, and its edges by operations of $M$. Assume that $U \in \text{tr} M$ decorates some node of $T$, say $U = s[p] \in T^*$. Then

1. The input edges of that node are in bijection with $U^*$. In particular, the address of those input edges are of the form $[p[q]]$, where $[q]$ ranges over $U^*$. This really motivates enclosing addresses in brackets.

2. On the other hand, the output edge of that node is decorated by $tU$ (where $t$ is defined in definition 2.25).

Notation 2.31. Let $M$ be a polynomial monad, and $T \in \text{tr} M^*$. For $[a]$ the address of an edge of $T$, let $e_{[a]} T$ be the operation of $M$ decorating that edge. Explicitly, if $[a] = []$, then $e_{[]} T := t s[[]] T$. Otherwise, $[a] = [p[q]]$ for some $[p] \in T^*$ (the node below the edge) and $[q] \in (s[p]) T^*$, and let $e_{[p[q]]} T := s[q] e_{[p]} T$.

3. Opetopes

3.1. Definition.

Definition 3.1 (The 3^n monad). Let $3^0$ be the identity polynomial monad on Set, as depicted on the left below, and let $3^n := (3^{n-1})^*$. Write $3^n$ as on right:

$$ \{\ast\} \xleftarrow{s} \{\ast\} \xrightarrow{p} \{\ast\} \xrightarrow{t} \{\ast\}, \hspace{1cm} \Omega_n \xleftarrow{s} E_{n+1} \xrightarrow{p} \Omega_{n+1} \xrightarrow{t} \Omega_n. $$

(3.2)

Definition 3.3 (Opetope). An $n$-dimensional opetope (or $n$-opetope for short) $\omega$ is simply an element of $\Omega_n$, and we write $\text{dim } \omega = n$. If $n \geq 2$, then $n$-opetopes are exactly the $3^{n-2}$-trees. In this case, an opetope $\omega \in \Omega_n$ is called degenerate if its underlying tree has no nodes (and thus consists of a unique edge), so that $\omega = 1_\varnothing$ for
some \( \phi \in \mathcal{O}_{n-2} \). We say that \( \omega \) it is an \textit{endotope} if its underlying tree has exactly one node, i.e. \( \omega = Y_{\psi} \) for some \( \psi \in \mathcal{O}_{n-1} \).

Following equation (2.26), for \( n \geq 2 \) and \( \omega \in \mathcal{O}_n \), the structure of polynomial monad \((3^{n-2})^* \rightarrow 3^{n-2}\) gives a bijection \( \varphi_{\omega} : \omega^1 \rightarrow (t \omega)^* \) between the leaves of \( \omega \) and the nodes of \( t \omega \), preserving the decoration by \((n-2)\)-opetopes.

\textbf{Example 3.4.} (1) The unique 0-opetope is denoted \( \bullet \) and called the \textit{point}.

(2) The unique 1-opetope is denoted \( \bullet \) and called the \textit{arrow}.

(3) If \( n \geq 2 \), then \( \omega \in \mathcal{O}_n \) is a \( 3^{n-2}\)-tree, i.e. a tree whose nodes are labeled in \((n-1)\)-opetopes, and edges are labeled in \((n-2)\)-opetopes. In particular, 2-opetopes are \( 3^0\)-trees, i.e. linear trees, and thus in bijection with \( \mathbb{N} \). We will refer to them as \textit{opetopic integers}, and write \( n \) for the 2-opetope having exactly \( n \) nodes.

(4) A 3-opetope is a \( 3^1\)-tree, i.e. a planar tree.

(5) A 4-opetope is a \( 3^2\)-tree. Unfolding definitions, if \( \omega : (\omega) \rightarrow 3^2 \), then nodes of \( \omega \) are decorated by elements of \( \mathcal{O}_3 \), i.e. planar trees. Further, if \( x \in (\omega)_1 \) is a node of \( \omega \), then \( \omega_2 \) exhibits a bijection between the input edges of \( x \) and the nodes of \( \omega_1(\omega)(x) \in \mathcal{O}_3 \).

\textbf{3.2. The category of opetopes.} Akin to the work of Cheng [Che03], we define a category of opetopes by means of generators and relations. The difference with the aforementioned reference is our use of polynomial opetopes (also equivalent to Leinster’s definition [Lei04, KJB10]), while Cheng uses an approach by multicategorical slicing, yielding “symmetric” opetopes.

\textbf{Lemma 3.5} (Opetopic identities). \( \mathcal{O} \in \mathcal{O}_n \) with \( n \geq 2 \).

(1) \textit{(Inner edge)} For an inner edge \([p[q]] \in \omega^* \) (the fact that \( \omega \) has an inner edge implies that it is non degenerate), we have \( ts[p[q]] \omega = s_q s_p \omega \).

(2) \textit{(Globularity 1)} If \( \omega \) is non degenerate, we have \( ts[q] \omega = tt \omega \).

(3) \textit{(Globularity 2)} If \( \omega \) is non degenerate, and \( [p[q]] \in \omega_1^1 \), we have \( s_q s_p \omega = s_{p_\omega,p[p]} t \omega \).

(4) \textit{(Degeneracy)} If \( \omega \) is degenerate, we have \( s[q] t \omega = tt \omega \).

\textbf{Proof.} \hspace{1cm} (1) \textit{(Inner edge)} By definition of a \( 3^{n-2}\)-tree.

(2) \textit{(Globularity 1 and 2)} By theorem 2.24, the monad structure on \( 3^{n-2} \) amounts to a structure map \((3^{n-2})^* \rightarrow 3^{n-2}\), which, taking the notations of definition 2.25, is written as

\[
\begin{array}{c}
\mathcal{O}_{n-2} \xleftarrow{\epsilon} 3^{n-2} \xrightarrow{p} 3^{n-2} \xrightarrow{q} \mathcal{O}_{n-2}
\end{array}
\]

The claims follow from the commutativity of the right and left square respectively.

(3) \textit{(Degeneracy)} Let \( \omega = 1_\phi \), for \( \phi \in \mathcal{O}_{n-2} \). Then, \( t t \omega = t Y_\phi = \phi \), and clearly, \( \phi = s[q] t \omega \).

\textbf{Definition 3.6} (The category \( \mathcal{O} \) of opetopes). With the identities of lemma 3.5, we define the category \( \mathcal{O} \) of opetopes by generators and relations as follows.

(1) \textit{(Object)} We set \( \text{ob} \mathcal{O} = \sum_{n \in \mathbb{N}} \mathcal{O}_n \).

(2) \textit{(Generating morphism)} Let \( \omega \in \mathcal{O}_n \) with \( n \geq 1 \). We introduce a generator \( t : t \omega \rightarrow \omega \), called the \textit{target embedding}. If \( [p] \in \omega^* \), then we introduce a generator \( s[p] : s[p] \omega \rightarrow \omega \), called a \textit{source embedding}. An \textit{elementary face embedding} is either a source or the target embedding. \[2\]

(3) \textit{(Relation)} We impose 4 relations by the following commutative squares, which just enforce the identities of lemma 3.5. Let \( \omega \in \mathcal{O}_n \) with \( n \geq 2 \).

\textbullet \hspace{0.1cm} \textit{(Inner)} For \([p[q]] \in \omega^* \) (forcing \( \omega \) to be non degenerate), the following square must commute:

\[
\begin{array}{c}
\mathbf{s}_{[p]} s_{[p]} \omega & \xrightarrow{[p]} & s_{[p]} \omega & \xrightarrow{[p]} & \omega \\
\downarrow{[p]} & & \downarrow{[p]} & & \downarrow{[p]}
\end{array}
\]

\[2\] Note if \( n \geq 2 \) and \([p] \) is an edge address of \( \omega \), then the edge operation of \( e[p] \) notation 2.31 also translates to a morphism \( e[p] : e[p] \omega \rightarrow \omega \).
• (Glob1) If $\omega$ is non degenerate, the following square must commute:

\[
\begin{array}{c}
t t \omega \\
\downarrow t \\
\downarrow t \\
t \omega \\
\end{array} \quad \begin{array}{c}
t \omega \\
\downarrow t \\
\downarrow t \\
t \omega \\
\end{array}
\]

• (Glob2) If $\omega$ is non degenerate, and for $[p[q]] \in \omega^1$, the following square must commute:

\[
\begin{array}{c}
s_{p \cdot [p[q]]} t \omega \\
\downarrow s_{[p]} \\
\downarrow t \\
s_{[p]} \omega \\
\end{array} \quad \begin{array}{c}
t \omega \\
\downarrow t \\
\downarrow t \\
t \omega \\
\end{array}
\]

• (Degen) If $\omega$ is degenerate, the following square must commute:

\[
\begin{array}{c}
t t \omega \\
\downarrow t \\
\downarrow t \\
t \omega \\
\end{array} \quad \begin{array}{c}
t \omega \\
\downarrow t \\
\downarrow t \\
t \omega \\
\end{array}
\]

Remark 3.7. Let us explain this definition a little more. Opetopes are trees whose nodes (and edges) are decorated by opetopes. The decoration is now interpreted as a geometrical feature, namely as an embedding of a lower dimensional opetope. Further, the target of an opetope, while not an intrinsic data, is also represented as an embedding. The relations can be understood as follows.

• (Inner) The inner edge at $[p[q]] \in \omega$ is decorated by the target of the decoration of the node “above” it (here $s_{[p]} \omega$), and in the $[q]$-source of the node “below” it (here $s_{[p]} \omega$). By construction, those two decorations match, and this relation makes the two corresponding embeddings $s_{[q]} s_{[p]} \omega \rightarrow \omega$ match as well. On the left is an informal diagram about $\omega$ as a tree (reversed gray triangle), and on the right is an example of pasting diagram represented by an opetope, with the relevant features of the (Inner) relation colored or thickened.

• (Glob1-2) If we consider the underlying tree of $\omega$ as its “geometrical source”, and the corolla $Y_{t \omega}$ as its “geometrical target”, then they should be parallel. The relation (Glob1) expresses this idea by “gluing” the root edges of $\omega$ and $Y_{t \omega}$ together, while (Glob2) glues the leaves according to $p_\omega$. 

• (Degen) If $\omega$ is a degenerate opetope, depicted as on the right, then its target should be a “loop”, i.e. its only source and its target should be glued together.

Notation 3.8. For $n \in \mathbb{N}$, let $\mathcal{O}_{\leq n}$ be the full subcategory of $\mathcal{O}$ spanned by opetopes of dimension at most $n$.

4. Opetopic sets and many-to-one polygraphs

Polygraphs where originally introduced by Street [Str76, section 2] under the name of computad. They are to (strict) $\omega$-categories what graphs are to 1-categories: a combinatorial device that freely generates them. However, unlike graphs, the category $\mathcal{P}ol$ of polygraphs fails to be a presheaf category [CJ04] [MZ08] [Che13]. The obstruction for it to be the case is an unpleasant corollary of the exchange law: if $f$ and $g$ are endomorphisms of an identity cell, then $fg = gf$.

Nonetheless, the recent work of Henry [Hen19] characterized many subcategories of $\mathcal{P}ol$ to be presheaf categories. Among them, the category $\mathcal{P}ol^{mto}$ of many-to-one polygraphs, in which the target (or codomain) of generating cells are themselves generating cells. In this section, we relate opetopes and many-to-one polygraphs in a formal way. Namely, we construct an equivalence of categories $\mathcal{P}sh(\mathcal{O}) \xrightarrow{\text{polygraphic realization}} \mathcal{P}ol^{mto}$.

Recall from [BD86, theorem 1] that if $\mathcal{A}$ and $\mathcal{B}$ are two Cauchy-complete categories such that $\mathcal{P}sh(\mathcal{A}) \simeq \mathcal{P}sh(\mathcal{B})$, then $\mathcal{A} \simeq \mathcal{B}$ (see theorem 5.23 for more details). In particular, any Cauchy-complete category $\mathcal{A}$ that acts as a “shape theory for many-to-one polygraphs”, i.e. such that $\mathcal{P}sh(\mathcal{A}) \simeq \mathcal{P}ol^{mto}$, is equivalent to $\mathcal{O}$ (since it doesn’t have any non-identity endomorphism, it is Cauchy-complete). This shows that the geometrical intuition behind the definition of $\mathcal{O}$ (remark 3.7) is essentially the unique way to faithfully implement the combinatorics of pasting diagrams.

The fact that many-to-one polygraphs are equivalent to opetopic sets was already known from [HMP00] [HZ02] [Che04b] [HMZ08], however the proof there is indirect and spanned over multiple articles. The formalism we developed so far allows us to establish this result directly.

4.1. Strict higher categories.

Definition 4.1 ($\omega$-category). An $\omega$-category $\mathcal{C}$ (also called strict $\infty$-category) is the datum of a diagram of sets

$$
\begin{align*}
\mathcal{C}_0 & \xleftarrow{id} \mathcal{C}_1 & \cdots & \xleftarrow{id} \mathcal{C}_n & \cdots \\
& \xleftarrow{s, t} & & \xleftarrow{s, t} & \\
& & \cdots & & 
\end{align*}
$$
with composition maps \( o_k : \mathcal{C}_{n,k} \to \mathcal{C}_n \), where \( k < n \) and \( \mathcal{C}_{n,k} \) is the pullback

\[
\begin{array}{c}
\mathcal{C}_{n,k} \\
\downarrow \\
\mathcal{C}_n \\
\mathcal{C}_k \\
\end{array}
\]

such that the following conditions hold:

1. for all \( k < n \), the diagram

\[
\begin{array}{c}
\mathcal{C}_k \\
\downarrow^{s^{-k}} \\
\mathcal{C}_n \\
\end{array}
\]

with the composition map \( o_k : \mathcal{C}_{n,k} \to \mathcal{C}_n \) is a 1-category;

2. for all \( l < k < n \), the diagram

\[
\begin{array}{c}
\mathcal{C}_l \\
\downarrow^{s^{-l}} \\
\mathcal{C}_k \\
\mathcal{C}_n \\
\end{array}
\]

with the composition maps \( o_l : \mathcal{C}_{k,l} \to \mathcal{C}_k \), \( o_l : \mathcal{C}_{n,l} \to \mathcal{C}_n \) and \( o_k : \mathcal{C}_{n,k} \to \mathcal{C}_n \) is a strict 2-category.

The maps \( s \) and \( t \) are called source and target maps, respectively, and if \( x \in \mathcal{C}_k \), then \( id_x := id(x) \) is the identity cell of \( x \). Note that by definition, the following equalities hold:

\[
ssx = stx, \quad tsx = ttz, \quad sid_x = x = tid_x.
\]

The first two are called the globular identities. Still by definition, for \( 0 \leq l < k < n \) and \( w, x, y, z \in \mathcal{C}_n \) the following exchange law holds:

\[
(w \circ x) \circ (y \circ z) = (w \circ y) \circ (x \circ z),
\]

assuming both sides are well-defined. Note that a strict \( n \)-category \( \mathcal{C} \) may be viewed as an \( \omega \)-category where \( \mathcal{C}_m \) only has identities for all \( m > n \). Given an \( \omega \)-category \( \mathcal{C} \), we write \( \mathcal{C}_{\leq n} \) for the underlying strict \( n \)-category

\[
\mathcal{C}_0 \xrightarrow{s,t} \mathcal{C}_1 \xrightarrow{s,t} \cdots \xrightarrow{s,t} \mathcal{C}_n.
\]

An \( \omega \)-functor \( f : \mathcal{B} \to \mathcal{C} \) between two \( \omega \)-categories is just a sequence of maps \( f_n : \mathcal{B}_n \to \mathcal{C}_n \) that induces a \( n \)-functor \( f_{\leq n} : \mathcal{B}_{\leq n} \to \mathcal{C}_{\leq n} \) for all \( n \in \mathbb{N} \). If the context is clear, we simply write \( f \) for \( f_n : \mathcal{B}_n \to \mathcal{C}_n \).

**Definition 4.2** (Parallel cells [MÓś]). Let \( \mathcal{B} \) be a strict \( \omega \)-category, \( n \in \mathbb{N} \). Two \( n \)-cells \( x, y \in \mathcal{B}_n \) are parallel, denoted by \( x \parallel y \), if \( s x = s y \) and \( t x = t y \). By convention, 0-cells are pairwise parallel.

**Definition 4.3** (Cellular extension). Let \( \mathcal{B} \) be a strict \((n-1)\)-category. A cellular extension of \( \mathcal{B} \) consists in a set \( X \) and two maps \( s, t : X \to \mathcal{B}_{n-1} \) such that the globular identities hold, i.e. for all \( x \in X \), we have \( s x \parallel t x \). We also denote such a cellular extension by \( \mathcal{B} \xrightarrow{s,t} X \).

**Definition 4.4** (Free \( n \)-category). Let \( \mathcal{B} \) be a strict \((n-1)\)-category and \( \mathcal{B} \xrightarrow{s,t} X \) be a cellular extension of \( \mathcal{B} \). The free strict \( n \)-category generated by this cellular extension is the strict \( n \)-category \( \mathcal{B}[X] \) such that

1. as strict \((n-1)\)-categories, \( \mathcal{B} = \mathcal{B}[X]_{\leq n-1} \);
2. there is an inclusion \( X \to \mathcal{B}[X]_n \), and the following diagrams commute:

\[
\begin{array}{c}
\mathcal{D}_{n-1} \xleftarrow{s} \mathcal{B}[X]_n, \\
\mathcal{D}_{n-1} \xleftarrow{t} \mathcal{B}[X]_n;
\end{array}
\]

3. if \( \mathcal{E} \) is a strict \( n \)-category, \( f : \mathcal{B} \to \mathcal{E}_{\leq n-1} \) is an \((n-1)\)-functor, and \( f_n : X \to \mathcal{E}_n \) is a map such that for all \( x \in X \), \( f(sx) = sf_n(x) \) and \( f(tx) = tf_n(x) \), then \( f \) and \( f_n \) extend uniquely to an \( n \)-functor \( \mathcal{B}[X] \to \mathcal{E} \).
The free extension $D[X]$ always exists and is unique up to isomorphism, see [HMZ08, section 1].

For the rest of this section, let $D$ be a strict $(n - 1)$-category, $D \xrightarrow{s.t.} X$ be a cellular extension of $D$, and $E := D[X]$ be the $n$-category freely generated by the cellular extension.

**Definition 4.5** (Counting function). Define an $n$-category $N(n)$ by

$$ \{0\} \xrightarrow{s.t.} \{0\} \xrightarrow{s.t.} \cdots \xrightarrow{s.t.} \{0\} \xrightarrow{s.t.} \mathbb{N}, $$

where all compositions correspond to the addition of integers. For $x \in X$, define a counting function $\#_x : X \to \mathbb{N}$ that maps $x$ to 1, and all other elements of $X$ to 0. This extends to a $n$-functor $E \to N(n)$. Similarly, let $\# : X \to \mathbb{N}$ be the map sending all elements to 1, and extend it as $\#: E \to N(n)$.

**Definition 4.6** (Context [GM09, definition 2.1.1]). Consider another cellular extension

$$ D \xrightarrow{s.t.} (X + \{\square\}) $$

of $D$, where $s\Box$ and $t\Box$ are chosen arbitrarily. A $n$-context of $E$ is a cell $C \in D[X + \{\square\}]$, such that $\#\Box C = 1$. One may think of $C$ as a cell of $E_n$ with a “hole”, and we sometime write $C = C[\square]$. If $u \in E_n$ is parallel to $\Box$ in $D[X + \{\square\}]$, let $C[u]$, be $C[\square]$ where $\square$ has been replaced by $u$.

**Definition 4.7** (Category of contexts [GM09, definition 2.1.2]). The category $\text{ctx}_n E$ of $n$-contexts of $E$ has objects the $n$-cells of $E$, and a morphism $C : x \to y$ is an $n$-context $C = C[\square]$ such that $C[x] = y$. If $D : y \to z$ is another context, then the composite of $C$ and $D$ is $DC := D[C[\square]] : x \to z$, as instead, $D[C[x]] = D[y] = z$.

**Definition 4.8** (Primitive context). A context is primitive over a cell $y \in E_n$ if it is of the form $C : x \to y$ with $x \in X$.

### 4.2. Many-to-one polygraphs.

**Definition 4.9** (Polygraph [HMZ08, definition 7.1]). A polygraph (also called a computad) $P$ consists of a small $\omega$-category $E$ and sets $P_n \subseteq E_n$ for all $n \in \mathbb{N}$, such that $P_0 = E_0$, and such that $E_{n+1} = E_n[P_{n+1}]$, i.e. the underlying $(n + 1)$-category of $E$ is freely generated by $P_{n+1}$ over its underlying $n$-category. We usually write $P_*$ instead of $E$. A polygraph $P$ is an $n$-polygraph if $P_k = \emptyset$ whenever $k > n$. A morphism of polygraphs is an $\omega$-functor mapping generators to generators. Let $\text{Pol}$ be the category of polygraphs and morphisms between them.

**Example 4.10.** A 1-polygraph $P$ is simply a free 1-category generated by the graph $P_0 \xrightarrow{s.t.} P_1$.

**Proposition 4.11.** The category $\text{Pol}$ is cocomplete. If $F : \exists \to \text{Pol}$ is diagram, and $n \in \mathbb{N}$, then $(\text{colim}_{\exists \phi} F_i)_n \cong \text{colim}_{\exists \phi} (Fi)_n$.

**Notation 4.12.** If $P \in \text{Pol}$, we write $\text{ctx}_n P$ instead of $\text{ctx}_n P_{n-1}[P_n]$ (see definition 4.7).

**Proposition 4.13** ([GM09, proposition 2.1.3]). Let $P$ be a polygraph, and $C \in \text{ctx}_n P$. Then $C$ decomposes as

$$ C = d_n \circ \cdots \circ (d_1 \circ \Box \circ e_1) \cdots \circ e_{n-1} \circ e_n, $$

where $d_n, e_n \in P_n^*$, and for $1 \leq i < n$, $d_i$ and $e_i$ are identities of $i$-cells.

**Definition 4.14** (Whisker [GM09, paragraph 2.1.4]). Let $P$ be a polygraph. An $n$-whisker of $P$ is an $n$-context of the form $d_{n-1} \circ \cdots \circ (d_1 \circ \Box \circ e_1) \cdots \circ e_{n-1},$ where for $1 \leq i \leq n - 1$, $d_i$ and $e_i$ are identities of $i$-cells.

**Remark 4.15.** If $C$ is an $(n - 1)$-context, then by proposition 4.13, it decomposes as on the left, and induces an $n$-whisker on the right

$$ C[\Box] = d_{n-1} \circ \cdots \circ (d_1 \circ \Box \circ e_1) \cdots \circ e_{n-1}, $$

which we shall also denote by $C$.

**Proposition 4.16** ([GM09, proposition 2.1.5]). Let $P$ be a polygraph, $u \in P_n^*$, $k := \#u$, and assume $k \geq 1$. Then $u$ decomposes as

$$ u = C_1[x_1] \circ \cdots \circ C_k[x_k], $$

where all of the $C_i$’s are $n$-whiskers, and $x_1, \ldots, x_k \in P_n$. 

Definition 4.17 (Partial composition [HMZ08, definition 3.8]). Let $\mathcal{P}$ be a polygraph, $x, y \in \mathcal{P}_n^*$ be $n$-cells, and $C : t y \rightarrow s x$ be a context. The partial composition (called placed composition in [HMZ08, definition 3.8]) $x \circ_C y$ is defined as $x \circ_C y := x \circ_{n-1} C[y]$, where the notation $C[y]$ follows remark 4.15.

Lemma 4.18. With $x, y$, and $C$ as in definition 4.17, we have $s(x \circ_C y) = C[s y]$ and $t(x \circ_C y) = t x$.

Proof. We have $t(x \circ_C y) = t(x \circ_{n-1} C[y]) = t x$. On the other hand, $s(x \circ_C y) = sC[y]$. By proposition 4.13 and remark 4.15, $C[y]$ decomposes as

$$C[y] = \text{id}_{d_{n-1}} \circ \ldots \circ (\text{id}_{d_i} \circ y \circ \text{id}_{e_1}) \circ \ldots \circ \text{id}_{e_{n-1}},$$

where for $1 \leq k \leq n$, $d_k$ and $e_k$ are identities of $k$-cells. Thus,

$$sC[y] = d_{n-1} \circ \ldots \circ (d_1 \circ (s y) \circ e_1) \circ \ldots \circ e_{n-1} = C[s y].$$

Definition 4.19 (Many-to-one polygraph [HMZ08, definition 7.4]). Let $\mathcal{P} \in \text{Pol}$ be a polygraph. For $n \geq 1$, an $n$-cell $x \in \mathcal{P}_n^*$ is said many-to-one if $t x \in \mathcal{P}_{n-1}$, and we write $\mathcal{P}_{n}^{\text{into}}$ for the set of many-to-one $n$-cells of $\mathcal{P}$. By convention, all 0-cells are many-to-one. In turn, the polygraph $\mathcal{P}$ is called many-to-one (or opetopic) if all its generators are many-to-one. Let $\text{Pol}_{n}^{\text{into}}$ be the full subcategory of $\text{Pol}$ spanned by many-to-one polygraphs.

Lemma 4.20. Let $u \in \mathcal{P}_n^*$ be such that $\#u \geq 1$. By proposition 4.16, it decomposes as

$$u = C_1[x_1] \circ \ldots \circ C_k[x_k],$$

where $k:= \#u$, where all of the $C_i$’s are $n$-whiskers, and $x_1, \ldots, x_k \in \mathcal{P}_n$. Then $u$ is a many-to-one cell if and only if $C_1 = \Box$, i.e. if $C_1$ is the trivial context.

Proof. First, note that $t u = tC_1[x_1]$. Write $C_1$ as

$$C_1[\Box] = d_{n-1} \circ \ldots \circ (d_1 \circ \Box \circ e_1) \circ \ldots \circ e_{n-1},$$

where for $1 \leq i \leq n-1$, $d_i$ and $e_i$ are identities of $i$-cells (see definition 4.14). We have

$$t u = tC_1[x_1] = (t d_{n-1}) \circ \ldots \circ ((t d_1) \circ (t x_1) \circ (t e_1)) \circ \ldots \circ (t e_{n-1}).$$

Thus, $t u$ is a generator if and only if $t d_i$ and $t e_i$ are identities, for all $0 \leq i \leq n-1$. In this case, $d_i$ and $e_i$ are identity cells of $(i-1)$-cells, thus $C_1 = \Box$. Conversely, if $C_1 = \Box$, then $t u = t x_1$ is a generator since $\mathcal{P}$ is a many-to-one polygraph, thus $u$ is a many-to-one cell. $\square$

The following result comes as a polygraphic analogue to proposition 2.19.

Proposition 4.21. A many-to-one $n$-cell $u$ of $\mathcal{P}$ is of either of the following forms:

1. $\text{id}_a$ for $a \in \mathcal{P}_{n-1}$,
2. $x \in \mathcal{P}_n$,
3. $v \circ_C x = v \circ_{n-1} C[x]$, for some $v \in \mathcal{P}_{n-1}^{\text{into}}$ with $1 \leq \#v < \#u$, $x \in \mathcal{P}_n$, and $C : t x \rightarrow s v$.

Proof. If $\#u = 0$, then $u = \text{id}_a$ for some $a \in \mathcal{P}_{n-1}^*$. Further, $a = tu \in \mathcal{P}_n$. If $\#u = 1$, then by lemma 4.20, $u$ is necessarily a generator. If $\#u = k \geq 2$, then by proposition 4.16, $u$ decomposes as

$$u = C_1[x_1] \circ \ldots \circ C_k[x_k],$$

where all of the $C_i$’s are $n$-whiskers, and $x_1, \ldots, x_k \in \mathcal{P}_n$. Let

$$v := C_1[x_1] \circ \ldots \circ C_k[x_k].$$

Then $C_k$ is a context $t x_k \rightarrow s v$, and $u = v \circ_{C_k} x_k$. By lemma 4.20, and since $u$ is many-to-one, $C_1 = \Box$. By lemma 4.20 again, $v$ is many-to-one, and $\#v = k - 1 \geq 1$, finishing the proof. $\square$

Notation 4.22. For $\mathcal{P} \in \text{Pol}_{n}^{\text{into}}$, let $\text{Ctx}_{n}^{\text{into}} \mathcal{P}$ be the full subcategory of $\text{Ctx}_n \mathcal{P}$ generated by many-to-one $n$-cells. In other words, an $n$-context $C : u \rightarrow v$ is in $\text{Ctx}_{n}^{\text{into}} \mathcal{P}$ if $u, v \in \mathcal{P}_{n}^{\text{into}}$. Necessarily, such a context is itself a many-to-one cell, as $tC[\Box] = tC[u] = tu$ is a generator.
Definition 4.23. Define a polygraph \( \mathcal{T} \in \text{Pol}^{\text{into}} \) by \( \mathcal{T}_0 := \{ \bullet \} \), \( \mathcal{T}_{n+1} := \{ (u, v) \in \text{Pol}^{\text{into}} \times \mathcal{T}_n \mid u \parallel v \} \) (see definitions 4.2 and 4.19), with \( s(u, v) := u \) and \( t(u, v) := v \).

Proposition 4.24. The polygraph \( \mathcal{T} \) is terminal in \( \text{Pol}^{\text{into}} \).

**Proof.** For \( \mathcal{P} \in \text{Pol}^{\text{into}} \), we show that there exists a unique morphism \( f : \mathcal{P} \to \mathcal{T} \).

- (Existence) If \( x \in \mathcal{P}_0 \), let \( f(x) := \bullet \), and if \( x \in \mathcal{P}_n \) with \( n \geq 1 \), let \( f(x) = (f(sx), f(tx)) \). The source and target compatibility is trivial.

- (Uniqueness) Consider \( g : \mathcal{P} \to \mathcal{T} \). Necessarily \( g_0 = f_0 \) as \( \mathcal{T}_0 \) is a singleton. Let \( x \in \mathcal{P}_n \) with \( n \geq 1 \). We have

\[
 f_n(x) = (f_{n-1}(sx), f_{n-1}(tx)) = (g_{n-1}(sx), g_{n-1}(tx)) = (sg_{n-1}(x), tg_{n-1}(x)) = g_n(x)
\]

Therefore, \( f = g \). \( \square \)

**Notation 4.25.** If \( \mathcal{P} \) is a many-to-one polygraph, we write \( ! : \mathcal{P} \to \mathcal{T} \) for the terminal map.

Definition 4.26. (1) An effective category is a category \( \mathcal{C} \) equipped with a functor \( F : \mathcal{C} \to \text{Set} \). For example:

- (a) if \( \mathcal{A} \) is a small category, then \( \text{Psht}(\mathcal{A}) \) (or any subcategory thereof) is naturally an effective category with the functor \( \text{Psht}(\mathcal{A}) \to \text{Set} \) mapping a presheaf \( X \) to \( \sum_{a \in \mathcal{A}} X_a \);

- (b) \( \mathcal{P} \) (or any subcategory thereof) is an effective category, where the functor \( \mathcal{P} \to \text{Set} \) maps a polygraph \( \mathcal{P} \) to \( \sum_{n \in \mathbb{N}} \mathcal{P}_n \).

(2) A category \( \mathcal{C} \) is an effective presheaf category if it is effective, and equivalent, as an effective category\(^3\), to a presheaf category.

(3) A functor \( F : \mathcal{C} \to \text{Set} \) is family representable [CJ95, definition 2.4] if \( F \cong \sum_{i \in I} \mathcal{C}(c_i, -) \) for some family \( \{ c_i \mid i \in I \} \) of objects of \( \mathcal{C} \).

Theorem 4.27. (1) [Hen19, corollary 2.4.9] The category \( \text{Pol}^{\text{into}} \) is a good class of polygraphs [Hen19, definition 2.2.2], and in particular

- (a) it is an effective presheaf category;

- (b) for all \( n \in \mathbb{N} \), the functor \( (-)^n : \text{Pol}^{\text{into}} \to \text{Set} \) that maps a polygraph \( \mathcal{P} \in \text{Pol}^{\text{into}} \) to its set \( \mathcal{P}^n \) of \( n \)-cells is family representable, and the representing objects are called the opetopic \( n \)-polyplexes (or just \( n \)-polyplexes):

\[
\mathcal{P}^n \cong \sum_{\omega \text{ is an opetopic } n \text{-polyplex}} \text{Pol}^{\text{into}}(\omega, \mathcal{P}).
\]

(2) [Hen19, proposition 2.2.6] The opetopic \( n \)-polyplexes are in bijective correspondence with \( \mathcal{T}^*_n \). Isomorphic polyplexes are equal. If \( u \in \mathcal{T}^*_n \), let \( u \) be the associated polyplex (refer to [Hen19, section 2.3] for the precise construction). The isomorphism above can be reformulated as

\[
\mathcal{P}^n \cong \sum_{u \in \mathcal{T}^*_n} \text{Pol}^{\text{into}}(u, \mathcal{P})
\]

where the vertical morphism maps an element in the \( u \in \mathcal{T}^*_n \) component of the sum to \( u \). In other words, a cell \( v \in \mathcal{P}^n \) corresponds to a unique morphism of the form \( u \to \mathcal{P} \), and \( u \vdash v \) (see notation 4.25).

(3) [Hen19, lemma 2.4.4, corollary 2.3.13] Let \( 0 \leq k < n \), \( a \in \mathcal{T}^*_k \), and \( u, v \in \mathcal{T}^*_n \) be such that \( t^{n-k}v = a = s^{n-k}u \). Then we have natural maps \( s^{n-k} : a \to u \) and \( t^{n-k} : a \to v \), and \( u \vdash v \) is obtained as the pushout

\[
\begin{array}{ccc}
\mathcal{T}^*_n & \xrightarrow{t^v} & \mathcal{T}^*_n \\
\mathcal{T}^*_n & \xrightarrow{s^{n-k}} & \mathcal{T}^*_n \\
\end{array}
\]

\[
\begin{array}{ccc}
a & \xrightarrow{t^v} & v \\
\downarrow{s^{n-k}} & & \downarrow{s^{n-k}} \\
u & \xrightarrow{t^v} & u \vdash v.
\end{array}
\]

\(^3\)i.e. the equivalence functor commutes with the equipped functors to \( \text{Set} \).
Furthermore, the maps $\iota_u$ and $\iota_v$ are injective on $(n-1)$- and $n$-cells.

**Example 4.28.** By definition 4.23, $\mathcal{T}$ has a unique 0-cell $\bullet$, and the corresponding polyplex $\bullet$ is simply the polygraph with a single 0-generator. Indeed, $\text{Pol}_n(\bullet, -)$ maps a polygraph $\mathcal{P} \in \text{Pol}_n^\text{into}$ to its set of 0-cells $\mathcal{P}_0$.

If we write $\bullet := (\bullet, \bullet)$ for the unique 1-generator of $\mathcal{T}$, then

$$\mathcal{T}_1^* = \left\{ \text{id}_{\bullet}, \bullet, \bullet \circ \bullet, \bullet \circ \bullet \circ \bullet, \ldots \right\}.$$ 

Write $l_0 := \text{id}_{\bullet}$, and $l_k$ for the composite $\bullet \circ \bullet \ldots \circ \bullet$ of $k$ instances of $\bullet$. Then the polyplex $l_0$ is simply $\bullet$, and $l_k$ spans the free category on the linear graph with $k$ vertices $\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet$. Indeed, let $\mathcal{P} \in \text{Pol}_n^\text{into}$. Then a 1-cell $u$ of $\mathcal{P}$ is either

1. an identity of a 0-cell;
2. a sequence of $k$ composable 1-generators of $\mathcal{P}$.

If $u = l_k$ for some $a \in \mathcal{P}_0$, then it is uniquely identified (as a 1-cell) by a morphism $\overline{u} \rightarrow \mathcal{P}$ mapping the unique 0-cell of $l_0$ to $a$. If $u$ is a composite of $k$ generators, then uniquely identified (as a 1-cell) by a morphism $\overline{u} \rightarrow \mathcal{P}$.

In conclusion,

$$\mathcal{P}_1^* \cong \sum_{k \in \mathbb{N}} \text{Pol}_n^\text{into}(l_k, \mathcal{P}) = \sum_{n \in \mathbb{N}} \text{Pol}_n^\text{into}(\overline{u}, \mathcal{P}).$$

**Remark 4.29.** If $\overline{u}$ is an $n$-polyplex, then under the isomorphism of theorem 4.27 (2), the identity morphism $\overline{u} \rightarrow \overline{u}$ corresponds to an $n$-cell of $\overline{u}$, which we call its fundamental cell, and following [Hen19], denote by $u$. If $\mathcal{P}$ is a many-to-one polygraph, $v \in \mathcal{P}_n^*$, and $u = \overline{v}$, then the map $f : \overline{u} \rightarrow \mathcal{P}$ corresponding to $v$ maps the fundamental cell $u$ to $v$. Indeed, by naturality of the isomorphism of theorem 4.27 (2), the diagram on the left commutes, which results in the mappings displayed on the right:

$$\begin{array}{ccc}
\mathcal{P}_n^* & \xrightarrow{\overline{v}} & \sum_{n \in \mathbb{N}} \text{Pol}_n^\text{into}(u, \mathcal{P}) \\
\downarrow f & & \downarrow f \\
\mathcal{P}_n^* & \xrightarrow{\overline{u}} & \sum_{n \in \mathbb{N}} \text{Pol}_n^\text{into}(u, \mathcal{P}) \\
\end{array}$$

Necessarily, $f(u) = v$.

**Lemma 4.30 ([Hen19, lemma 2.4.5]).** Let $n \geq 1$, $\overline{u}$ be an $n$-polyplex, and $u$ be its fundamental cell. For $a \in \overline{u}_{n-1}$, exactly one of the following two possibilities is true:

1. $a$ occurs in $su$;
2. $a$ is the target of a $n$-generator of $\overline{u}$.

Furthermore, in the second case, the $n$-generator in question is unique.

**Lemma 4.31 ([Hen19, remark 2.2.9, corollary 2.2.13]).** Let $f : \mathcal{P} \rightarrow \mathcal{Q}$ a morphism of many-to-one polygraphs, and recall from definition 4.1 that $\mathcal{P}_n^* \times_{\mathcal{Q}_n^*} \mathcal{Q}_n^*$. The following square is cartesian

$$\begin{array}{ccc}
\mathcal{P}_n^* & \xrightarrow{\iota_k} & \mathcal{P}_n^* \\
\downarrow f & & \downarrow f \\
\mathcal{Q}_n^* & \xrightarrow{\iota_k} & \mathcal{Q}_n^* \\
\end{array}$$

Consequently, for $u_1, u_2, v_1, v_2 \in \mathcal{P}_n^*$, if $u_1 \circ_k v_1 = u_2 \circ_k v_2$, $f(u_1) = f(u_2)$, and $f(v_1) = f(v_2)$, then $u_1 = u_2$ and $v_1 = v_2$.

**Proof (sketch).** Let us first consider the case $\mathcal{Q} = \mathcal{T}$, and let $u, v \in \mathcal{P}_n^*$ be such that $t^{n-k} v = s^{n-k} u = a$. In particular, pair $(u, v)$ is in $\mathcal{P}_n^*$. We have a series of correspondences

- A tuple $(u, v) \in \mathcal{P}_{n,k}^*$ theorem 4.27 (2)
- A map $\overline{u} \cdot \Pi_{n-1} v \rightarrow \mathcal{P}$ theorem 4.27 (3)
- An element $u \circ_k v \in \mathcal{P}_n^*$ with a decomposition of $!(u \circ_k v)$ as $x \circ_k y$. theorem 4.27 (2)
In other words, the following square is a pullback
\[
\begin{array}{ccc}
\mathcal{P}_{n,k}^* & \xrightarrow{\circ_k} & \mathcal{P}_n^* \\
\uparrow & & \uparrow \\
\mathcal{T}_{n,k}^* & \xrightarrow{\circ_k} & \mathcal{T}_n^* \\
\end{array}
\]

For the general case, note that in the following diagram, the lower and outer squares are cartesian, and by the pasting lemma, so is the upper one:
\[
\begin{array}{c}
\mathcal{P}_{n,k}^* \xrightarrow{\circ_k} \mathcal{P}_n^* \\
\uparrow f \quad \uparrow f \\
\mathcal{Q}_{n,k}^* \xrightarrow{\circ_k} \mathcal{Q}_n^* \\
\end{array}
\]

\[\square\]

Let \(\mathcal{P} \in \mathsf{Pol}^\mathsf{into}\) and \(v \in \mathcal{P}_{n}^\mathsf{into}\). Then \(v\) is a composition of \(n\)-generators of \(\mathcal{P}\), which are many-to-one, so intuitively, \(v\) is a “tree of \(n\)-generators”. In this section, we make this idea formal. We first define a polynomial functor \(\nabla_n\mathcal{P}\) whose operations are the \(n\)-generators of \(\mathcal{P}\) (definition 4.32), and then construct the composition \(T^\circ \in \mathcal{P}_{n}^\mathsf{into}\) of a \(\nabla_n\mathcal{P}\)-tree \(T\). In proposition 4.39, we show that this construction is bijective.

**Definition 4.32** (The \(\nabla\) construction). For \(\mathcal{P} \in \mathsf{Pol}^\mathsf{into}\) and \(n \geq 1\), let \(\nabla_n\mathcal{P}\) be the following polynomial endofunctor:
\[
\mathcal{P}_{n-1} \xleftarrow{s} \mathcal{P}_n^* \xrightarrow{p} \mathcal{P}_n \xrightarrow{t} \mathcal{P}_{n-1},
\]
where for \(x \in \mathcal{P}_n\), the fiber \(\mathcal{P}_n^*(x)\) is the set of primitive contexts over \(s\,x\), and for \(C : a \to s\,x\) in \(\mathcal{P}_n^*(x)\), \(s(C) := a\), \(p(C) := x\), and \(t\) is the target map of \(\mathcal{P}\).

**Lemma 4.33.** Let \(\mathcal{P}, \mathcal{Q} \in \mathsf{Pol}^\mathsf{into}\), and \(f : \mathcal{P} \to \mathcal{Q}\). For \(v \in \mathcal{P}_{n}^\mathsf{into}\), write \(E(v)\) for the set of primitive contexts over \(v\), and likewise for many-to-one cells of \(\mathcal{Q}\). Then \(f\) induces a bijection \(E(v) \to E(f(v))\).

**Proof.** We proceed by induction on \(v\) (see proposition 4.21).

1. If \(v\) is an identity (resp. a generator), then so is \(f(v)\), thus \(E(v)\) and \(E(f(v))\) are both empty (resp. singletons). Trivially, \(f : E(v) \to E(f(v))\) is a bijection.
2. Assume that \(v\) decomposes as \(v = w \circ C \cdot x\) with \(|w| \geq 1\) and \(x \in \mathcal{P}_n\), and \(C : t\,x \to s\,w\). Then a primitive context over \(v\) is either \(w \circ C \cdot \square\) or of the form \(D[\square] \circ C\cdot x\) for \(D \in E(w)\), and \(E(v) \cong 1 + E(w)\).

**Proposition 4.34.** Let \(f : \mathcal{P} \to \mathcal{Q}\) be a morphism of many-to-one polygraphs. For all \(n \geq 1\), it induces a morphism of polynomial functors \(\nabla_n f : \nabla_n \mathcal{P} \to \nabla_n \mathcal{Q}\), where \((\nabla_n f)_1 = f_n : \mathcal{P}_n \to \mathcal{Q}_n\).

**Proof.** Consider
\[
\begin{array}{c}
\mathcal{P}_{n-1} \xleftarrow{s} \mathcal{P}_n^* \xrightarrow{p} \mathcal{P}_n \xrightarrow{t} \mathcal{P}_{n-1} \\
\downarrow f \quad \downarrow f \quad \downarrow f \\
\mathcal{Q}_{n-1} \xleftarrow{s} \mathcal{Q}_n^* \xrightarrow{p} \mathcal{Q}_n \xrightarrow{t} \mathcal{Q}_{n-1} \\
\end{array}
\]

where \(f_n^*\) maps a context \(C : a \to s\,x\) to \(f(C) : f(a) \to f(s\,x)\). Clearly, all squares commute, and by lemma 4.33, the middle one is cartesian.

**Definition 4.35** (Composition). We define the composition operation \((-)^\circ : \mathsf{tr}\nabla_n\mathcal{P} \to \mathcal{P}_{n}^\mathsf{into}\). At the same time, we establish a bijection between \(T^1\) and the primitive contexts over \(s\,T^\circ\), where \(T \in \mathsf{tr}\nabla_n\mathcal{P}\).

1. If \(a \in \mathcal{P}_{n-1}\), then \((I_a)^\circ := \text{id}_a\). Note that the only primitive context over \(s\,\text{id}_a\) is \(\square : a \to a\), and let \(C[1] := \square\).
(2) If \( x \in \mathcal{P}_n \), then \((Y_x)^* := x \). Note that by definition of \( \nabla_n \mathcal{P} \) (definition 4.32) we have \( Y_x^! = \{ [D] \mid D \in x^* \} \) (see remark 2.13), and let \( C[D] := D \).

(3) Consider a tree of the form \( S = T \circ [l] Y_x \), with \( T \in \text{tr} \nabla_n \mathcal{P} \) having at least one node, \([l] \in T^! \) and \( x \in \mathcal{P}_n \). By induction, the leaf \([l] \) corresponds to a primitive context \( C[l] : a \rightarrow s T^\circ \), and moreover, \( a = t x \). Let \( S^\circ := T^\circ \circ C[l] \). Let \([l'] \in S^! \). If \([l'] \) is of the form \([D] \), for some \([D] \in Y_x^! \), let \( C[l'] := C[l][D] \). Otherwise, \([l'] \) is a leaf of \( T \), and so \( C[l'] \) is already defined.

If \( S = T' \circ [l] Y_{x'} \) is another decomposition of \( S \), then the fact that \( T^\circ \circ C[l] x = (T')^\circ \circ C[l'] x' \) follows from the exchange law.

**Definition 4.36** (Composition tree). A composition tree is simply a \( \nabla_n \mathcal{P} \)-tree. If \( v \in \mathcal{P}^\text{mto}_n \) and \( T \) is a composition tree such that \( T^\circ = v \), then we say that \( T \) is a composition tree of \( v \).

The following result generalizes lemma 4.31.

**Proposition 4.37.** Let \( f : \mathcal{P} \rightarrow \mathcal{Q} \) a morphism of many-to-one polygraphs. The following square is cartesian

\[
\begin{array}{ccc}
\text{tr} \nabla_n \mathcal{P} & \xrightarrow{(\cdot)^\circ} & \mathcal{P}^\text{mto} \\
\downarrow f & & \downarrow f \\
\text{tr} \nabla_n \mathcal{Q} & \xrightarrow{(\cdot)^\circ} & \mathcal{Q}^\text{mto}
\end{array}
\]

**Proof.** This amounts to showing that if \( v \in \mathcal{P}^\text{mto}_n \), then \( f \) establishes a bijective correspondence between the composition trees of \( v \) and \( f(v) \). This is clear if \( \# v \leq 1 \), so let us assume \( \# v \geq 2 \), and let \( T \) be a composition tree of \( v \). Then \( T \) decomposes as

\[ T = Y_a \bigcup_{[C]} T_C, \]

where \( a \in \mathcal{Q}_n \), \( C \) ranges over the primitive contexts over \( sa \), and \( T_C \in \text{tr} \nabla_n \mathcal{Q} \). Then, by lemma 4.31, there exists a unique \( b \in \mathcal{P}_n \) such that \( f(b) = a \), and unique \( uc \)'s such that \( f(uc) = T_C \). Further, \( f \) exhibits a bijection between the primitive contexts over \( a \) and over \( b \). Note that \( \# uc < \# v \), so by induction, there exists a unique \( S_C \in \text{tr} \nabla_n \mathcal{P} \) such that \( f(S_C) = T_C \). Finally,

\[ S := Y_b \bigcup_{[D]} S_{f(D)} \]

is the unique tree such that \( f(S) = T \).

**Lemma 4.38.** Let \( u \) be a polyplex. The fundamental cell \( u \in u_{\rightarrow} \) has at most one composition tree.

**Proof.** Assume that \( S \) and \( T \) are two composition trees of \( u \). Necessarily, \( e[l]S = t u = e[l] T \), i.e. the root edges of \( S \) and \( T \) are both decorated by the target of \( u \) (recall the notation from notation 2.31 and definition 3.6). By lemma 4.30, exactly one of the following two possibilities is true.

1. If \( tu \) occurs in \( su \), then \( u \) is an identity cell, and \( S = t u = T \). We are done.
2. Otherwise \( tu \) is the target of a unique \( n \)-generator of \( u \), say \( x \). In particular, \( s[l]S = x = s[l] T \), i.e. the root nodes of \( S \) and \( T \) are decorated by \( x \). Let \( a \) be an \((n - 1)\)-generator occurring in \( su \). It decorates an input edge \( e \) (resp. \( e' \)) of the root node of \( S \) (resp. \( T \)). If \( a \) occurs in \( su \), then by lemma 4.30 again, there is no \( n \)-generator whose target is \( a \). So in \( S \) (resp. \( T \), there cannot be a node above \( e \) (resp. \( e' \)), i.e. \( e \) (resp. \( e' \)) is a leaf. Otherwise, \( a \) is the target of a unique \( n \)-generator \( y \), and necessarily, the nodes above \( e \) and \( e' \), namely \( s[e]S \) and \( s[e']T \) are decorated by \( y \). Applying lemma 4.30 repeatedly, we show that \( S = T \).

**Proposition 4.39.** The composition map \((\cdot)^\circ : \text{tr} \nabla_n \mathcal{P} \rightarrow \mathcal{P}^\text{mto}_n \) is a bijection.

**Proof.** (Surjectivit) This clearly holds if \( n \leq 1 \). Assume \( n \geq 2 \) and let \( v \in \mathcal{P}^\text{mto}_n \). If \( v = \text{id}_a \) for some \( a \in \mathcal{P}_{n-1} \), then \( v = P_a \). If \( v \in \mathcal{P}_n \), then \( v = Y_v \). Otherwise, by proposition 4.21, \( v \) decomposes as \( w \circ C x \), where \( w \in \mathcal{P}^\text{mto}_n \), \( x \in \mathcal{P}_n \), and \( C : tx \rightarrow su \) is a context. By induction, there exist \( T \in \text{tr} \nabla_n \mathcal{P} \) such that \( T^\circ = w \). By construction, \( C \) corresponds to a unique leaf address \([l] \in T^! \). Finally, \( v = (T \circ [l] Y_x)^\circ \).
(Injectivity) Let \( v \in \mathcal{P}^\text{into}_n \), \( u := ! v \), and \( f : \mathbb{N} \rightarrow \mathcal{P}^\text{into} \) be the map associated with \( v \) (see theorem 4.27 (2)). By proposition 4.37, the following square is cartesian

\[
\begin{array}{ccc}
\text{tr} \nabla_n u & \xrightarrow{(\cdot)^v} & \mathcal{P}^\text{into}_n \\
f \downarrow & & \downarrow f \\
\text{tr} \nabla_n \mathcal{P} & \xrightarrow{(\cdot)^v} & \mathcal{P}^\text{into}_n,
\end{array}
\]

and in particular, since \( f \) maps the fundamental cell \( u \) to \( v \), it induces a bijection between the composition trees of \( u \in \mathcal{P}^\text{into}_n \) and \( v \in \mathcal{P}^\text{into}_n \). By lemma 4.38 and surjectivity proved above, \( u \) has exactly one composition tree, and thus, so does \( v \).

\[\square\]

\textbf{Remark 4.43.} Given a many-to-one cell as on the left below, corollary 4.41 decomposes it as on the right.
Given a context as on the left below, corollary 4.42 detaches \( \Box \) from the cell “below” and the cells “above” it:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {}; \\
\node (B) at (1,1) {}; \\
\node (C) at (2,0) {}; \\
\node (D) at (1,-1) {}; \\
\node (E) at (0,-2) {}; \\
\draw[->] (A) -- (B); \\
\draw[->] (B) -- (C); \\
\draw[->] (C) -- (D); \\
\draw[->] (D) -- (E); \\
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {}; \\
\node (B) at (1,1) {}; \\
\node (C) at (2,0) {}; \\
\node (D) at (1,-1) {}; \\
\node (E) at (0,-2) {}; \\
\draw[->] (A) -- (B); \\
\draw[->] (B) -- (C); \\
\draw[->] (C) -- (D); \\
\draw[->] (D) -- (E); \\
\end{tikzpicture}
\end{array}
\end{align*}
\]

**Remark 4.44.** Let \( C : x \to y \) be an \( n \)-context of \( \mathcal{P} \in \mathcal{Pol}^{\text{mto}} \) between many-to-one cells. In particular, \( C \) is a many-to-one cell in the extended category \( \mathcal{Q} \) of definition 4.6, so by virtue of corollary 4.42, it uniquely decomposes as

\[
C = z \circ D \bigcap_{E} v_E.
\]

where \( z, v_E \in \mathcal{P}_n^* \). Since \( C[x] = y \), we have

\[
y = z \circ D \bigcap_{E} v_E.
\]

Conversely, any decomposition of \( y \) of the form above induces a context \( C : x \to y \). In particular, the number of primitive contexts over \( y \) is \( \#y \), and \( \nabla_n \mathcal{P} \) is finitary.

**Definition 4.45.** We now extend \( (\cdot)^\circ \) and \( \text{ct} \) to functors between \( \text{tr} \nabla_n \mathcal{P} \) and \( \text{ctx}_{\text{mto}} \mathcal{P} \). On objects, they are respectively defined in definitions 4.35 and 4.36:

1. Let \( f : T \to S \) be a morphism in \( \text{tr} \nabla_n \mathcal{P} \). It corresponds to a decomposition of \( S \) as on the left, and let \( f^\circ \) be the context \( T^\circ \to S^\circ \) on the right:

\[
S = U \circ T \bigcap_{[p]} V_{[l]}, \\
T = U \circ T \bigcap_{[p]} V_{[l]}, \\
f^\circ := U \circ C_{[p]} \bigcap_{[l]} V_{[l]},
\]

where \([l]\) ranges over \( T^\circ \).

2. Let \( C : x \to y \) be an \( n \)-context. By corollary 4.42, it decomposes uniquely as on the left, and let \( \text{ct} C \)

\[
C[\Box] = z \circ D \bigcap_{E} t_E,
\]

\[
\text{ct} y = (\text{ct} z) \circ (\text{ct} x) \bigcap_{[l]} (\text{ct} t_E),
\]

where \( E \) ranges over all primitive contexts over \( s x \).

One readily checks the following:

**Proposition 4.46** (Composition tree duality). The functors \( (\cdot)^\circ \) and \( \text{ct} \) are mutually inverse isomorphisms of categories.

**Corollary 4.47.** For \( n \geq 2 \) and \( x \in \mathcal{P}_n \), the functor \( \text{ct} \) induces a natural bijection

\[
\mathcal{P}_n^*(x) \cong \sum_{a \in \mathcal{P}_{n-1}} (\text{tr} \nabla_{n-1} \mathcal{P})(Y_a, \text{ct} s x).
\]

**Proof.** Direct consequence of proposition 4.46. \( \Box \)

**Notation 4.48.** If \( x \in \mathcal{P}_n \) and \([p] \in (\text{ct} s x)^*\), then we write \( s_{[p]} x \) instead of \( s_{[p]} \text{ct} s x \in \mathcal{P}_{n-1} \).
5. The equivalence

We now aim at proving that the category of opetopic sets, i.e. Set-presheaves over the category \( \mathcal{O} \) defined previously, is equivalent to the category of many-to-one polygraphs \( \mathcal{Pol}^{\text{into}} \). We achieve this by first constructing the \textit{polygraphic realization} functor \( [-] : \mathcal{O} \to \mathcal{Pol}^{\text{into}} \). This functor “realizes” an opetope as a polygraph that freely implements all its tree structure by the means of adequately chosen generators in each dimension. Secondly, we consider the left Kan extension \( [-] : \mathcal{Psh}(\mathcal{O}) \to \mathcal{Pol}^{\text{into}} \) along the Yoneda embedding. This functor has a right adjoint, the “opetopic nerve” \( N : \mathcal{Pol}^{\text{into}} \to \mathcal{Psh}(\mathcal{O}) \), and we prove this adjunction to be an adjoint equivalence.

This is done using the \textit{shape function}, defined in section 5.2, which to any generator \( x \) of a many-to-one polygraph \( \mathcal{P} \) associates an opetope \( x^3 \) along with a canonical morphism \( \hat{\mathcal{P}} : [x^3] \to \mathcal{P} \).

5.1. Polygraphic realization. An opetope \( \omega \in \mathcal{O}_n \), with \( n \geq 1 \), has one target \( t_\omega \), and sources \( s_{[p]} \omega \) layed out in a tree. If the sources \( s_{[p]} \omega \) happened to be generators in some polygraph, then that tree would describe a way to compose them. With this in mind, we define a many-to-one polygraph with its boundary be the polygraph with a unique generator in dimension 0, which we denote by \( \diamond \). For \( \diamond \) the unique 0-opetope, let \( \partial[\diamond] \) be the empty polygraph, and \( |\diamond| \) be the polygraph with a unique generator in dimension 0, which we denote by \( \bullet \). For \( \bullet \) the unique 1-opetope, let \( \partial[\bullet] := \{\diamond\} + |\bullet| \), and let \( |\diamond| \) be induced by the cellular extension

\[
\partial[\bullet] |\diamond| \xleftarrow{\mathsf{s}, \mathsf{t}} \{\diamond\},
\]

where \( \mathsf{s} \) and \( \mathsf{t} \) map \( \bullet \) to distinct 0-generators. There are obvious functors \( |\diamond|, |\mathsf{t}| : |\bullet| \to |\bullet| \), mapping \( \diamond \) to \( \mathsf{s} \bullet \) and \( \mathsf{t} \bullet \), respectively.

Definition 5.1 (Low dimensional cases). For \( \diamond \) the unique 0-opetope, let \( \partial[\diamond] \) be the empty polygraph, and \( |\diamond| \) be the polygraph with a unique generator in dimension 0, which we denote by \( \bullet \). For \( \bullet \) the unique 1-opetope, let \( \partial[\bullet] := \{\diamond\} + |\bullet| \), and let \( |\diamond| \) be induced by the cellular extension

\[
\partial[\bullet] |\diamond| \xleftarrow{\mathsf{s}, \mathsf{t}} \{\diamond\},
\]

where \( \mathsf{s} \) and \( \mathsf{t} \) map \( \bullet \) to distinct 0-generators. There are obvious functors \( |\diamond|, |\mathsf{t}| : |\bullet| \to |\bullet| \), mapping \( \diamond \) to \( \mathsf{s} \bullet \) and \( \mathsf{t} \bullet \), respectively.

Definition 5.2 (Dimension 2). For the reader’s convenience, we construct \( \partial[k] \) and \( |k| \) for every opetopic integer \( k \), although this case already falls under the inductive definition (see definitions 5.4 and 5.11).

1. Let \( \partial[0] \) be the 1-polygraph given by the following coequalizer:

\[
\begin{array}{ccc}
|\bullet| & \xrightarrow{|\mathsf{s}|} & |\bullet| \\
\sim & & \Downarrow \mathsf{t}
\end{array}
\quad \partial[0].
\]

In other words, \( \partial[0] \) has one object \( x \) and one generating endomorphism \( f : x \to x \). The 2-polygraph \( |0| \) is obtained by adjoining a generating 2-cell \( \alpha : \mathsf{id}_x \to f \) to \( \partial[0] \).

2. Let \( k \geq 1 \), and consider the 1-polygraph

\[
|\bullet| \quad \begin{array}{ccc}
\mathsf{s} & \xrightarrow{|\mathsf{s}|} & |\bullet| \\
\sim & & \Downarrow \mathsf{t}
\end{array}
\]

where there are \( k \) instances of \( |\bullet| \). In other words, \( \mathcal{P} \) is generated by a chain of \( k \) composable 1-cells, which we denote by

\[
x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} x_k.
\]

Alternatively, \( \mathcal{P} \) is the polyplex \( l_k \) of example 4.28. Let \( \partial[k] \) be the obvious pushout

\[
\begin{array}{ccc}
|\bullet| & \xrightarrow{(x_0, x_k)} & \mathcal{P} \\
\Downarrow & & \Downarrow
\end{array}
\quad \partial[k],
\]

i.e., \( \partial[k] \) is \( \mathcal{P} \) with an additional generating 1-cell \( g : x_0 \to x_k \). Finally, \( |k| \) is obtained from \( \partial[k] \) by adjoining a generating 2-cell \( \alpha : f_k \cdots f_2 f_1 \to g \).
In both cases, there is a bijective correspondence between the generating cells of \( |k| \) and the objects of \( \mathbb{O}/k \), and the obvious inclusions induce functors
\[
\partial|\cdot|: \mathcal{O}_{\leq 2} \rightarrow \mathcal{P}\mathcal{O}^\text{into}.
\]

Let \( n \geq 2 \) and assume by induction that \( \partial|\cdot| \) and \( |\cdot| \) are defined on \( \mathbb{O}_{\leq n} \). Assume further that the following induction hypotheses hold (they are easily verified for \( n = 2 \)).

**Assumptions 5.3.** For all \( \psi \in \mathbb{O}_k \) with \( k < n \), the following hold:

- (PR1) for all \( j \in \mathbb{N} \), the set \( |\psi|_j \) of \( j \)-generators of \( |\psi| \) is in bijection with the set of objects of the slice \( \mathbb{O}_j/\psi \), i.e. of the form \( \left( \phi \xrightarrow{a} \psi \right) \) for \( \phi \in \mathbb{O}_j \) and \( a: \phi \rightarrow \psi \) a morphism in \( \mathbb{O} \);

- (PR2) for \( \left( \phi \xrightarrow{a} \psi \right) \) a generator of \( |\psi| \), its target is \( \left( t \phi \xrightarrow{t} \phi \xrightarrow{a} \psi \right) \);

- (PR3) for \( l \leq k \), and for \( \left( \phi \xrightarrow{a} \psi \right) \) a \( l \)-generator of \( |\psi| \), the composition tree of its source \( \text{ct} s \left( \phi \xrightarrow{a} \psi \right) \in \text{tr} \nabla_{l-1}|\psi| \) is
\[
\left( \phi \rightarrow \nabla_{l-1}|\psi| \right) [p] \mapsto \left( s[p] \phi \xrightarrow{s[p]} \phi \xrightarrow{a} \psi \right).
\]

Recall that by proposition 4.46, this completely determines \( s \left( \phi \xrightarrow{a} \psi \right) \in |\psi|_{l-1}^* \).

We now define \( \partial|\cdot| \) and \( |\cdot| \) when \( \omega \in \mathbb{O}_n \). Defining the former is easy, and done in definition 5.4. The latter is defined in definition 5.11 as generated by a cellular extension
\[
\partial|\omega| \leftarrow^{s_t} \{ \omega \}
\]
of \( \partial|\omega| \), where the target and source of the new generator are given by (PR2) and (PR3). Lastly, we check the inductive hypotheses in proposition 5.12.

**Definition 5.4** (Inductive step for \( \partial|\cdot| \)). For \( \omega \in \mathbb{O}_n \), let \( \partial|\omega| \) be the following many-to-one \( (n-1) \)-polygraph:
\[
\partial|\omega| := \underset{\dim \psi \leq n}{\text{colim}} |\psi|.
\]

For \( a: \psi \rightarrow \omega \) in \( \mathbb{O}_{\leq n}/\omega \), this colimit comes with a corresponding coprojection \( |a|: |\psi| \rightarrow \partial|\omega| \).

**Remark 5.5.** Let \( 0 \leq k < n \). By (PR1), the set of \( k \)-generators of \( \partial|\omega| \) is \( \mathbb{O}_k/\omega \).

**Lemma 5.6.** For \( \omega \in \mathbb{O}_n \), and \( j < n \), the set \( \partial|\omega|_j \) of \( j \)-generators of \( \partial|\omega| \) is the slice \( \mathbb{O}_j/\omega \).

**Proof.** Follows from the induction hypothesis (PR1) and proposition 4.11.

**Corollary 5.7.** For \( \omega \in \mathbb{O}_n \) and \( 1 \leq k < n \), the polynomial functor \( \nabla_k \partial|\omega| \) is described as follows:
\[
\mathbb{O}_{k-1}/\omega \leftrightarrow^s E \rightarrow^{p} \mathbb{O}_k/\omega \rightarrow^t \mathbb{O}_{k-1}/\omega
\]
where for \( \left( \psi \xrightarrow{a} \omega \right) \in \mathbb{O}_k/\omega \),

1. the fiber \( E \left( \psi \xrightarrow{a} \omega \right) \) is simply \( \psi^* \);

2. for \( [p] \in E \left( \psi \xrightarrow{a} \omega \right) \cong \psi^* \), we have \( s[p] = \left( s[p] \psi \xrightarrow{s[p]} \psi \xrightarrow{a} \omega \right) \);

3. \( t \left( \psi \xrightarrow{a} \omega \right) = \left( t \psi \xrightarrow{1} \psi \xrightarrow{a} \omega \right) \).

**Proof.** Direct consequence of lemma 5.6 and (PR1), (PR2), and (PR3).
**Lemma 5.9.** Let $\omega \in \mathbb{O}_n$ and $1 \leq k < n$, we have a morphism $u : \nabla_k \partial|\omega| \to 3^{k-1}$

$$
\mathbb{O}_{k-1}/\omega \xleftarrow{s} E \xrightarrow{p} \mathbb{O}_k/\omega \xrightarrow{t} \mathbb{O}_{k-1}/\omega
$$

induced by the forgetful maps $\mathbb{O}_{k-1}/\omega \to \mathbb{O}_{k-1}$ and $\mathbb{O}_k/\omega \to \mathbb{O}_k$.

**Proof.** Let $\bar{\omega}$ map a node $[p] \in \omega^*$ to the cell $(s_{[p]} \omega \xrightarrow{s_{[p]}} \omega) \in \mathbb{O}_{n-1}/\omega$, and map an edge $[I]$ to the cell $(e_{[I]} \omega \xrightarrow{e_{[I]}} \omega) \in \mathbb{O}_{n-2}/\omega$ (recall the notation from notation 2.31 and definition 3.6). $\square$

**Definition 5.8.** For $\omega \in \mathbb{O}_n$ we have a morphism $u : \nabla_{\omega} \partial|\omega| \to 3^{n-2}$

$$
\mathbb{O}_{n-1}/\omega \xleftarrow{s} E \xrightarrow{p} \mathbb{O}_k/\omega \xrightarrow{t} \mathbb{O}_{n-1}/\omega
$$

where $\mathbb{O}_k/\omega$ is a monomorphism, since $\omega$ is

**Lemma 5.9.** Let $\omega \in \mathbb{O}_n = \text{tr} 3^{n-2}$. The map $\omega : \langle \omega \rangle \to 3^{n-2}$ factors through $u : \nabla_{\omega} \partial|\omega| \to 3^{n-2}$ (definition 5.8):

$$
\nabla_{\omega} \partial|\omega| \xrightarrow{u} \langle \omega \rangle \xrightarrow{w} 3^{n-2}.
$$

**Proof.** Let $\bar{\omega}$ map a node $[p] \in \omega^*$ to the cell $(s_{[p]} \omega \xrightarrow{s_{[p]}} \omega) \in \mathbb{O}_{n-1}/\omega$, and map an edge $[I]$ to the cell $(e_{[I]} \omega \xrightarrow{e_{[I]}} \omega) \in \mathbb{O}_{n-2}/\omega$ (recall the notation from notation 2.31 and definition 3.6).

**Proposition 5.10.** On the one hand, consider the tree $\nabla_{\omega} \partial|\omega|$-tree $\bar{\omega}$ of lemma 5.9, and on the other hand, recall from remark 5.5 that there is a $(n-1)$-generator $(t_\omega \xrightarrow{t} \omega)$ of $\partial|\omega|$ corresponding to the target embedding of $\omega$. Then, in $\partial|\omega|$, the composite $\bar{\omega}^\circ$ (definition 4.35) and the generator $(t_\omega \xrightarrow{t} \omega)$ are parallel.

**Proof.** If $\omega$ is degenerate, say $\omega = l_\phi$ for some $\phi \in \mathbb{O}_{n-2}$, then $\bar{\omega}^\circ = \text{id}_{(\phi \xrightarrow{l_\phi} \omega)}$ while $t_\omega \xrightarrow{t} \omega = (\text{id}_{\phi} \xrightarrow{t} \omega)$. By (Degen), those two cells are parallel.

For the rest of the proof, we assume that $\omega$ is not degenerate. First, we have

$$
t_\omega \circ \bar{\omega}^\circ = t s_{[\phi]} \bar{\omega}^\circ = t (s_{[\phi]} \omega \xrightarrow{s_{[\phi]}} \omega) = (t t_\omega \xrightarrow{t t} \omega) = t (t_\omega \xrightarrow{t} \omega).
$$

Then, in order to show that $s \circ t_\omega^\circ = s (t_\omega \xrightarrow{t} \omega)$, we show that the $(n-2)$-generators occurring on both sides are the same, and that the way to compose them is unique.

1. Generators in $s \circ t_\omega^\circ$ are of the form $(\phi \xrightarrow{[q]} \psi \xrightarrow{[p]} \omega)$, for $[p[q]] \in \omega^1$. By (Glob2), those are equal to $(\phi \xrightarrow{[p[q]]} t_\omega \xrightarrow{t} \omega)$, which are exactly the generators in the cell $s (t_\omega \xrightarrow{t} \omega)$.

2. To show that there is a unique way to compose all the $(n-2)$-generators of the form $(\phi \xrightarrow{[q]} \psi \xrightarrow{[p]} \omega)$, where $[p[q]]$ ranges over $\omega^1$, it is enough to show that no two have the same target. Assume $(\phi_i \xrightarrow{[q_i]} \psi_i \xrightarrow{[p_i]} \omega)$, with $i = 1, 2$, are $(n-2)$-generators occurring in $s \circ t_\omega^\circ$ with the same target. Consider the following diagram:

$$
\begin{array}{c}
\phi_1 \xrightarrow{[q_1]} \psi_1 \\
\phi_2 \xrightarrow{[q_2]} \psi_2 \\
\rho \end{array}
$$

where $[q_i] = p_{\omega_i}[p_{qi}] \in t_\omega^*$. The outer hexagon commutes by assumption, the two squares on the right are instances of (Glob2), and the left square commutes as $t : t_\omega \to \omega$ is a monomorphism, since $\omega$ is
non degenerate. By inspection of the opetopic identities (see definition 3.6), the only way for the left square to commute is the trivial way, i.e. \([r_1] = [r_2]\). Since \(\varrho_\omega\) is a bijection, we have \([p_1[q_1]] = [p_2[q_2]]\), thus \([p_1] = [p_2]\) and \([q_1] = [q_2]\). □

**Definition 5.11** (Inductive step for \(|−|\)). For \(ω ∈ \mathcal{O}_n\), let \(|ω|\) be the cellular extension
\[\partial|ω| \xrightarrow{s_1} \{ω\},\]
where \(t\) maps \(ω\) to the \((n−1)\)-generator \((tω \xrightarrow{t} ω)\), and where the composition tree of \(sω\) is \(\bar{ω}\) (lemma 5.9). For consistency, we also write \((ω \xrightarrow{id} ω)\) for the unique \(n\)-generator of \(|ω|\). This is well-defined by proposition 5.10, and gives a functor \(|−| : \mathcal{O}_{≤n} \to \mathcal{P}ol\).

**Proposition 5.12.** For \(ω ∈ \mathcal{O}_n\), the polygraphs \(\partial|ω|\) (definition 5.4) and \(|ω|\) (definition 5.11) satisfy the assumptions 5.3.

**Proof.**
- (PR1) For \(j < n\), by lemma 5.6, we already have \(|ω|_j = \partial|ω|_j = \mathcal{O}_j/ω\). In dimension \(n\), the only element of \(\mathcal{O}_n/ω\) is \(id : ω \to ω\), which corresponds to the unique \(n\)-generator of \(|ω|\). If \(j > n\), then both \(\mathcal{O}_j/ω\) and \(|ω|_j\) are empty.
- (PR2) and (PR3) By definition, those hypotheses hold for the unique \(n\)-generator \((ω \xrightarrow{id} ω)\) of \(|ω|\). By induction, they also hold on the other generators. □

To conclude, we have defined a functor \(|−| : \mathcal{O} \to \mathcal{P}ol^{into}\) which satisfies the assumptions 5.3 for all \(n ∈ \mathbb{N}\).

5.2. **The shape function.** In section 5.3, we will define an adjunction \(|−| : \mathcal{P}sh(\mathcal{O}) \rightleftarrows \mathcal{P}ol^{into} : N\), where \(|−| : \mathcal{P}sh(\mathcal{O}) \to \mathcal{P}ol^{into}\) is the left Kan extension of the polygraphic realization defined in section 5.1, and \(N\) its associated nerve. In order to show that this is an adjoint equivalence, we need a good understanding of \(N\). An crucial step is theorem 5.17 that establishes a bijection \(\mathcal{P}_n \to \sum_{ω ∈ \mathcal{O}_n} (N^P)_ω\), where \(P ∈ \mathcal{P}ol^{into}\) and \(n ∈ \mathbb{N}\).

Furthermore, we show that this bijection is over \(\mathcal{O}_n\):
\[
\begin{array}{ccc}
\mathcal{P}_n & \xrightarrow{z} & \sum_{ω ∈ \mathcal{O}_n} (N^P)_ω \\
\downarrow \scriptstyle{(-)^t} & & \downarrow \scriptstyle{(−)^t} \mathcal{O}_n,
\end{array}
\]

where the vertical arrow maps an element in the \(ω ∈ \mathcal{O}_n\) component to \(ω\), and where \((-)^t\) is the shape function which we define in this section.

Here is a sketch of the definition of \((-)^t : \mathcal{P}_n \to \mathcal{O}_n\). The cases \(n = 0, 1\) are trivial, since there is a unique 0-opetope and a unique 1-opetope. Assume \(n ≥ 2\), and take \(x ∈ \mathcal{P}_n\). Then the composition tree of \(sx\) is a tree whose nodes are decorated \((n−1)\)-generators, and edges by \((n−2)\)-generators. Replacing these generators by their inductively defined shapes, we obtain a tree whose nodes are decorated by \((n−1)\)-opetopes, and edges by \((n−2)\)-opetopes, a.k.a. an \(n\)-opetope, which we shall denote by \(x^t\).

\[
\begin{array}{c}
\begin{tikzpicture}
\fill[black!30!white] (0,0) circle (0.1cm);
\fill[black!30!white] (1,0) circle (0.1cm);
\fill[black!30!white] (0,1) circle (0.1cm);
\fill[black!30!white] (1,1) circle (0.1cm);
\fill[black!30!white] (0,2) circle (0.1cm);
\fill[black!30!white] (1,2) circle (0.1cm);
\fill[black!30!white] (2,0) circle (0.1cm);
\fill[black!30!white] (2,1) circle (0.1cm);
\fill[black!30!white] (2,2) circle (0.1cm);
\draw (0,0) -- (1,0);
\draw (1,0) -- (1,1);
\draw (1,1) -- (0,1);
\draw (0,1) -- (0,2);
\draw (1,1) -- (1,2);
\draw (1,2) -- (0,2);
\draw (0,2) -- (2,2);
\draw (1,2) -- (1,1);
\draw (2,0) -- (2,1);
\draw (0,1) -- (2,0);
\end{tikzpicture}
\end{array}
\end{array}
\]

The fact that \(x^t\) corresponds to the intuitive notion of “shape” of \(x\) is justified by theorem 5.17.

**Lemma 5.13.** If \(x, y ∈ \mathcal{T}_n\) are two parallel generators, then they are equal.

**Proof.** This is trivial if \(n = 0\), as \(\mathcal{T}\) only has one 0-generator. If \(n ≥ 1\), note that \(x = (sx, tx) = (sy, ty) = y\). □

**Proposition 5.14.** For \(x ∈ \mathcal{T}_n\) there exists a unique \(x^t ∈ \mathcal{O}_n\) such that the terminal morphism \(! : x^t \to \mathcal{T}\) maps \(x^t\) (the unique \(n\)-generator of \(|x^t|\)) to \(x\). In particular, the map \((-)^t : \mathcal{T}_n \to \mathcal{O}_n\) is a bijection.
Proof. • (Uniqueness) Assume that there exists two distinct opetopes \( \omega, \omega' \in \Omega_k \) such that \( !\omega = !\omega' \), with \( k \) minimal for this property. Then necessarily, \( k \geq 2 \). On the one hand, we have \( (\omega) = (\text{ct } s\omega) = (\text{ct } s\omega') = (\omega') \). On the other hand, for \( [p] \in \omega^* = (\omega')^* \), we have

\[
!s_{[p]}\omega = !s_{[p]}\omega = s_{[p]}!\omega = s_{[p]}!\omega' = !s_{[p]}\omega' = !s_{[p]}\omega',
\]

since \( ! \) is also \( |s_{[p]}\omega| \rightarrow |\omega| \rightarrow \mathcal{T} \). Hence, the composite \( !s_{[p]}g^h \rightarrow !g^h \rightarrow !g^h \rightarrow !g^h \rightarrow !p^h \) is isomorphic \( ⟨H⟩ \rightarrow (−)\). Consequently, \( (s_{[p]}g)^h = s_{[p]}(g^h) \) and \( (t^h g)^h = t(g^h) \), and the following displays an isomorphism \( \nabla_{n-1} \mathcal{T} \rightarrow 3^{n-2} \):

\[
\begin{array}{ccc}
\mathcal{T}_{n-2} & \xleftarrow{s} & \mathcal{T}_{n-1}^* \\
(-)^1 \downarrow & & \downarrow (-)^1 \\
\mathcal{O}_{n-2} & \xleftarrow{s} & \mathcal{O}_{n-1}^* \\
(-)^1 \downarrow & & \downarrow (-)^1 \\
\mathcal{O}_{n-2} & \xrightarrow{p} & \mathcal{O}_{n-1} \\
(-)^1 \downarrow & & \downarrow (-)^1 \\
\mathcal{O}_{n-2} & \xrightarrow{t} & \mathcal{O}_{n-2}.
\end{array}
\]

Hence, the composite \( (\text{ct } s\omega)^h \rightarrow (\text{ct } s\omega)^h \rightarrow (\text{ct } s\omega)^h \rightarrow (\text{ct } s\omega)^h \rightarrow (\text{ct } s\omega)^h \) defines an \( n \)-opetope \( x^h \) with \( (x^h) = (\text{ct } s\omega) \). We claim that \( !x^h = x \). We first show that \( !s(x^h) = s x \). We have

\[
\begin{align*}
(\text{ct } s\omega)^h & = (x^h) \\
= (\text{ct } s(x^h)) & \quad \text{by definition} \\
= (\text{ct } !s(x^h)) & \quad \text{by (PR3)}.
\end{align*}
\]

Then, for any address \([p]\) in \( (\text{ct } s\omega)^h \), we have

\[
\begin{align*}
s_{[p]}x &= !(s_{[p]}x^h) & \quad \text{by induction} \\
&= !s_{[p]}(x^h) & \quad \text{by definition of } x^h \\
&= s_{[p]}!(x^h) & \quad \text{since } s(x^h)!t(x^h) \\
&= s_{[p]}!(x^h) & \quad \text{by definition of polygraphs,}
\end{align*}
\]

and therefore, by proposition 4.46, \( s x = s!(x^h) \). Next,

\[
\begin{align*}
t!(x^h) &= !t(x^h) & \quad \text{by induction} \\
&= s x & \quad \text{since } s(x^h)!t(x^h) \\
&= !t x, & \quad \text{showed above}
\end{align*}
\]

and therefore, \( t!(x^h) = t x \). Finally, \( !(x^h) \parallel x \), and by lemma 5.13, \( !(x^h) = x \). \( \square \)

Notation 5.15. In the light of proposition 5.14, we identify \( \mathcal{T}_n \) with \( \Omega_n \). This identification is compatible with faces, i.e. \( s_{[p]} \) and \( t \). Further, \( !|\omega| \rightarrow \mathcal{T} \) maps a generator \( (\phi \rightarrow \omega) \) to \( \phi \).

Notation 5.16. For \( \mathcal{P} \in \text{Pol} \) and \( \omega \in \Omega_n \), let \( \mathcal{P}_\omega := \{ x \in \mathcal{P}_n | x^h = \omega \} \). If \( f : \mathcal{P} \rightarrow \mathcal{Q} \) is a morphism of polygraphs, then it restricts and corestricts as a map \( f : \mathcal{P}_\omega \rightarrow \mathcal{Q}_\omega \).
Theorem 5.17. For $P \in \text{Poly}^{mto}$ and $x \in P_n$, there exists a unique pair$^4$

$$(x^i, |x^i| \xrightarrow{\tilde{x}} P) \in |\overline{f}|P$$

such that $\tilde{x}_n(x^i) = x$. Further, the shape function $(-)^3 : P_n \to \Omega_n$ maps an $n$-generator $x$ to $x^i = \overline{x}$, and the map

$$\overline{(-)} : P \to \text{Poly}^{mto}(|\omega|, P)$$

is a bijection$^5$.

Proof.  

• (Uniqueness) Assume $|\omega| \to P \xrightarrow{f'} |\omega'|$ are different morphisms such that $f(\omega) = x = f'(\omega')$. Then $\overline{x} = \overline{f(\omega)} = \overline{f'(\omega')} = \overline{\omega}'$, and by proposition 5.14, $\omega = \omega'$. Let $\left( \phi \xrightarrow{a} \omega \right) \in |\omega|_k$ be such that $f(\phi \xrightarrow{a} \omega) = f'(\phi \xrightarrow{a} \omega)$, with $k$ minimal for this property. Then $k < n$ (since by assumption $f(\omega) = x = f'(\omega')$), and a factorizes as $\left( \phi \xrightarrow{j} \psi \xrightarrow{b} \omega \right)$, where $j$ is a face embedding, i.e. either $t$ or $s_{[p]}$ for some $[p] \in \omega^\bullet$. Then by assumption,

$$f(\phi \xrightarrow{a} \omega) = jf(\psi \xrightarrow{b} \omega)$$

$$= jf'(\psi \xrightarrow{b} \omega)$$

by minimality of $k$

$$= f'(\phi \xrightarrow{a} \omega),$$

a contradiction.

• (Existence) The cases $n = 0, 1$ are trivial, so assume $n \geq 2$, and that by induction, the result holds for all $k < n$. Let $x^i := \overline{x} \in \Omega_n$. We wish to construct a morphism $O[x^i] \xrightarrow{\tilde{x}} P$ having $x$ in its image. For $(x^i \xrightarrow{j} x^k)$ a face of $x^k$ (i.e. $t$ or $s_{[p]}$ for some $[p] \in (x^k)^\bullet$), we have $(\overline{j}x)^k = \psi$, so that by induction, there exists a morphism $|\psi| \xrightarrow{j x} P$ having $jx$ in its image, providing a commutative square

$$|\psi| \xrightarrow{j x} P$$

$$|\overline{x}| \xrightarrow{\overline{j}} \overline{T}.$$

To alleviate upcoming notations, write $\overline{j} := \overline{\tilde{j}} : |\psi| \to P$. Let $\left( \phi \xrightarrow{a} x^k \right) \in \Omega_n/x^k$. If $a$ is a face embedding, define $\tilde{a}$ as before. If not, then it factors through a face embedding as $a = \left( \phi \xrightarrow{j} \psi \xrightarrow{b} \omega \right)$, and let $\tilde{a} := b \cdot |j|$. Then the left square commutes, and passing to the colimit over $\Omega_n/x^k$, we obtain the right square:

$$|\phi| \xrightarrow{\tilde{a}} P$$

$$|\overline{x}| \xrightarrow{\overline{\tilde{a}}} \overline{T},$$

$$|\partial|x^k| \xrightarrow{f} P$$

$$|\overline{x}| \xrightarrow{\overline{\partial|x^k|}} \overline{T},$$

We want a diagonal filler of the right square. Since $|x^k|$ is a one-generator cellular extension of $\partial|x^k|$ (definition 5.11), it is enough to check that $f \circ s x^i = s x$, and $f \circ t x^i = t x$. The latter is clear, as $f$ extends $\tilde{t} : |tx^i| \to P$, and $f \circ t x^i = t x$ by definition. We now proceed to prove the former. First,

$^4$In [Hen19, proposition 2.2.3 (2)], $x^i$ is written $\overline{x}$ and called the universal cell (or top cell) of $x$.

$^5$In other words, the functor $\text{Poly}^{mto} \to \text{Set}$ that maps a polygraph $P$ to $P_n$ is familially representable (definition 4.26) with $\{|\omega| \mid \omega \in \Omega_n\}$ as representing family.
\[(ct \cdot x^3) = (ct \cdot x)\] since both are mapped to the same element of \(\mathcal{T}_n\). Then, for \([p]\) a node address of \(ct \cdot x^3\), we have \(f_{[p]} s_{[p]} x^3 = \overline{s_{[p]} x^3} = s_{[p]} x\). Hence \(f \cdot s x^3 = s x\).

\[\square\]

5.3. The adjoint equivalence.

**Definition 5.19** (Polygraphic realization-nerve adjunction). The polygraphic realization functor \(|-|: \mathbb{O} \to \text{Pol}^{mto}\) extends to a left adjoint

\[|\cdot| : \text{Psh}(\mathbb{O}) \rightleftarrows \text{Pol}^{mto} : N,\]

by left Kan extension of \(|\cdot|: \mathbb{O} \to \text{Pol}^{mto}\) along the Yoneda embedding \(y: \mathbb{O} \to \text{Psh}(\mathbb{O})\). Explicitly, the polygraphic realization of an opetopic set \(X \in \text{Psh}(\mathbb{O})\) can be computed with the coend on the left, while the polygraphic nerve \(NP\) of a polygraph \(P \in \text{Pol}^{mto}\) is given on the right:

\[|X| = \int^{\omega \in \mathbb{O}} X_{\omega} \times |\omega|, \quad NP = \text{Pol}^{mto}(|\cdot|, P) : \mathbb{O}^{op} \to \text{Set}.\]

**Theorem 5.20.** The unit and counit are natural isomorphisms. Consequently, the polygraphic realization-nerve adjunction of definition 5.19 is an adjoint equivalence between \(\text{Psh}(\mathbb{O})\) and \(\text{Pol}^{mto}\).

**Proof.** Let \(X \in \text{Psh}(\mathbb{O})\) and \(P \in \text{Pol}^{mto}\). Note that with the nerve functor of definition 5.19, the bijection of equation (5.18) becomes \((-\cdot) : \mathcal{O}_w \to (NP)_w\). An \(n\)-generator \(x \in P\) then corresponds to a cell \(\bar{x} \in NP_w\), where \(\omega := x^h\), in other words, the shape function partitions the set of \(n\)-generators to form an opetopic set \(NP\). In the converse direction, for \(X \in \text{Psh}(\mathbb{O})\),

\[|X|_{\omega} = \int^{\omega \in \mathbb{O}} X_{\omega} \times |\omega|_{\omega} \quad \text{by definition 5.19}\]

\[\cong \int^{\omega \in \mathbb{O}} X_{\omega} \times \sum_{\psi \in O_n} \mathbb{O}(\psi, \omega) \quad \text{see assumptions 5.3, (PR1)}\]

\[\cong \sum_{\psi \in O_n} \int^{\omega \in \mathbb{O}} X_{\omega} \times \mathbb{O}(\psi, \omega) \]

\[\cong \sum_{\psi \in O_n} X_{\psi},\]

so the set of \(n\)-generators of \(|X|\) is the set of \(n\)-cells of \(X\). In particular, \(|X|_{\omega} \cong X_{\omega}\). Finally, the unit \(\eta: X \to N|X|\) and counit \(\varepsilon: |NP| \to P\) are the following composites

\[X_{\omega} \xrightarrow{\varepsilon} |X|_{\omega} \xrightarrow{(\cdot)} (N|X|)_{\omega},\]

\[|NP|_{\omega} \xrightarrow{\varepsilon} \sum_{\omega \in O_n} |NP|_{\omega} \xrightarrow{\varepsilon} \sum_{\omega \in O_n} (NP)_{\omega} \xrightarrow{(\cdot)^{-1}} \sum_{\omega \in O_n} P_{\omega} \xrightarrow{\varepsilon} P_n,\]

which by definition are isomorphisms.

Many-to-one polygraphs have been the subject of other work [Henz02] [Henz08], and proved to be equivalent to the notion of multitopic sets. This, together with theorem 5.20, prove the following:

**Corollary 5.21.** The category \(\text{Psh}(\mathbb{O})\) of opetopic sets is equivalent to the category of multitopic sets.

An opetopic plex is an opetopic polyplex of the form \(\underline{u}\) where \(u \in \mathcal{T}_n\) (as opposed to \(\overline{T}_n\)). In [Hen19, corollary 2.4.9 and remark 2.5.1], Henry shows that \(\text{Pol}^{mto}\) is a presheaf category over some category \(\mathcal{O}\) of opetopic plexes, and asks whether they are the same as opetopes. We now answer this question positively.

**Definition 5.22** (Cauchy-complete category). An idempotent morphism \(e: a \to a\) splits if it decomposes as \(e = ir\) with \(ri = id_a\). A category is Cauchy-complete if all its idempotent morphisms split.

**Theorem 5.23** ([BD06, theorem 1]). Let \(\mathcal{A}\) and \(\mathcal{B}\) be Cauchy-complete categories. An equivalence of categories \(\text{Psh}(\mathcal{A}) \xrightarrow{\sim} \text{Psh}(\mathcal{B})\) restricts and corestricts to the representable presheaves, or in other words, to an equivalence \(\mathcal{A} \xrightarrow{\sim} \mathcal{B}\).
Corollary 5.24. The category $\mathcal{O}plex$ of opetopic plexes is equivalent to $\mathcal{O}$.

Proof. By definition, $\mathcal{O}$ is a directed category, and by [Hen19, proposition 2.2.3 (4)], so is $\mathcal{O}plex$. In particular, they are both Cauchy-complete. On the other hand, $\mathcal{P}sh(\mathcal{O}) \cong \mathcal{P}ol^{\text{into}} \cong \mathcal{P}sh(\mathcal{O}plex)$, and we conclude using theorem 5.23. $\square$

In [Pal04], Palm studies another approach to weak higher-dimensional categories, based on dendrotopic sets, and show that dendrotopic sets are equivalent to many-to-one polygraphs. Therefore,

Corollary 5.25. Opetopic sets are equivalent to dendrotopic sets.

6. Conclusion

To conclude, here is a diagram of the various approaches to the notion of pasting diagrams and “many-to-one shapes”, and how they are related throughout the literature and in this paper:

![Diagram of various approaches to pasting diagrams](image)

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