Integrating CAD and Numerical Analysis: ’Dirty Geometry’ handling using the Finite Cell Method

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Abstract

This paper proposes a computational methodology for the integration of Computer Aided Design (CAD) and the Finite Cell Method (FCM) for models with “dirty geometries”. FCM, being a fictitious domain approach based on higher order finite elements, embeds the physical model into a fictitious domain, which can be discretized without having to take into account the boundary of the physical domain. The true geometry is captured by a precise numerical integration of elements cut by the boundary. Thus, an effective Point Membership Classification algorithm that determines the inside-outside state of an integration point with respect to the physical domain is a core operation in FCM. To treat also “dirty geometries”, i.e. imprecise or flawed geometric models, a combination of a segment-triangle intersection algorithm and a flood fill algorithm being insensitive to most CAD model flaws is proposed to identify the affiliation of the integration points. The present method thus allows direct computations on geometrically and topologically flawed models. The potential and merit for practical applications of the proposed method is demonstrated by several numerical examples.

Keywords: Computer-Aided Design, Dirty geometry, Finite Cell Method, Flood Fill, Point Membership Classification, Flawed geometry, Geometrical flaws, Topological Flaws

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1 Introduction

Product development in the scope of Computer Aided Engineering (CAE) typically involves Computer Aided Design (CAD) and numerical analyses. The life cycle of almost every complex mechanical product starts with the creation of a CAD model which is then converted into a suitable format for downstream CAE applications such as Finite Element Analysis, Rapid Prototyping, or automated manufacturing. However, a truly smooth transition from a geometric to a computational model is still challenging. This is especially the case for numerical simulations like the Finite Element Method. Very often, complex and time-consuming model preparation and pre-processing steps are necessary to obtain a decent numerical model that is suitable for analysis purposes. For complex CAD models, this transition process can take up to 80% of the overall analysis time [1].

For this reason, various alternative numerical approaches have been developed which seek to avoid or shorten this costly transition process (e.g. meshing). Isogeometric analysis (IGA) as the most prominent example aims at easing the transition from CAD to computational analysis by using the same spline basis functions for geometric modeling and numerical simulation [1, 2]. In a related earlier approach Cirak and Scott [3] presented an integrated design process based on Subdivision Surfaces. Kagan and Fischer [4] used B-spline finite elements in an effort to join design and analysis.

However, independent of the respective numerical approach, flaws may appear in the CAD model during the design-analysis cycle – such as double entities, gaps, overlaps, intersections, and slivers – as shown in Fig. 1. They are mainly due to data loss while the model is exchanged between different CAD and/or CAE systems, to inappropriate operations by the designer, or to approximation steps resulting in incompatible geometries. These model flaws, also called dirty topologies or dirty geometries, may be extremely small or even unapparent. While they are of no particular importance to a CAD engineer they may, however, cause serious problems for structural analyses. In the best case, they merely generate excessively fine meshes in some regions which are not relevant to structural analysis (such as slivers) but drive up computational time unnecessarily. In the worst case, computations fail completely because no finite element mesh can be created. This is due to the fact that neither classical finite element approaches nor the newly developed methods mentioned above are designed to handle dirty topologies and geometries. Thus, extra effort is necessary to repair, heal, or reconstruct the model into an analysis-suitable geometry [5], even if the affected region is not of special interest to the structural analyst.

Solid CAD modeling systems mainly rely on two different representation techniques: Boundary Representation (B-Rep) and Constructive Solid Geometry (CSG) [6] which is often extended to a so-called procedural modeling. In CSG, a volume is described explicitly, whereas in B-Rep it is described implicitly via its surfaces. Consequently, B-Rep models provide direct and easy access to the explicit boundaries. However, B-Rep models are, unlike CSG models, not necessarily valid, meaning that it might in some cases not be possible to determine whether a point lies inside or outside (Point Membership Classification). A novel representation technique – V-Rep (volumetric representation) – was recently proposed by Elber et al. [7]. V-Reps are constructed of volumetric, non-singular B-Spline primitives, thus, providing both an explicit volume and explicit surface description. Within this paper, we focus on flawed or 'dirty' B-Rep models. The most direct way to address CAD model flaws is to heal or repair the model before
meshing. The healing process involves identifying the type of model errors and fixing them individually. Butlin and Stops [8] listed topological and geometrical inconsistencies. The geometrical inconsistencies relate to their positions in space, while the topological inconsistencies relate to the connections or relationships among entities. Gu et al. [9] presented a visual catalog of potential flaws. Petersson and Chand [10] developed a suite of tools for the preparation of CAD geometries that are imported from IGES files and stored in the boundary representation for mesh generation; the algorithm can identify gross flaws and remove them automatically. Yang et al. [11] classified topological and geometrical flaws in CAD models and proposed a procedural method to verify 19 flaw types in STEP format and 12 types in the IGES format. Yang and Han [5] conducted a case study to investigate the typical nature of CAD model flaws. They reported the classification and frequency of each of the six most common error types that significantly increase the lead times, and they proposed a repair method based on the design history. Healing methods act either on the CAD model or on the mesh [5]. According to their approach, these methods can be classified into surface [12], volumetric [13] and hybrid [14] types. Surface-based geometry repair methods perform local modifications merging and fixing incorrect surface patches. Volumetric techniques are used to reconstruct a new global shape without flaws. However, this approach typically leads to information loss, especially at sharp features such as kinks. Hybrid methods combine the advantages of local surface healing and global volumetric healing. To this end, flaws are detected and a volumetric reconstruction is performed only in their vicinity. These methods have been used for CAD models that are represented in typical B-Rep formats (e.g. STEP and IGES) as well as for polygonal meshes [15].

Although healing and repair methods have been applied successfully in recent years, healing can still be very labor intensive and time consuming in the scope of product development. In
addition, it can not be guaranteed that all flaws are actually detected. As a remedy, mesh generation techniques have been developed which have the potential to generate meshes from flawed geometric models. In this line of research, Wang and Srinivasan [16] proposed an adaptive Cartesian mesh generation method. Herein, the computational grid is created inside the domain, which then connects to the boundary. Another technique – the Cartesian shrink-wrapping technique – was presented in [17] to generate triangular surface meshes automatically for 3D flawed geometries without healing. However, to generate a mesh, an initial watertight shell (called wrapper surface) needs to be constructed. Another line of research proposed by Gasparini et al. [18] is an approach to analyze geometrically imperfect models based on a geometrically adaptive integration technique that uses different model representations, i.e., space decomposition, B-Rep, and distance fields. This approach relies on a method that was first introduced by Kantorovich [19] and that has recently been commercialized [20]. Furthermore, this approach requires computation of a well-defined distance function to the boundaries – which is non-trivial for dirty geometries, as the orientation of boundary surfaces might be incorrect or the location of the boundaries is anticipated incorrectly, e.g., due to spurious entities, or intersections.

In this work, we present an alternative computational methodology which aims at dealing robustly with dirty topologies and geometries. At its core, it utilizes the Finite Cell Method (FCM) [21, 22], a fictitious domain method based on higher-order finite elements or trimmed tri-variate spline approximations. The FCM embeds the physical model into a fictitious domain which is then discretized by a simple, often axis-aligned grid. This grid does not have to conform to the boundary of the physical domain. Instead, the physical domain is recovered on the level of integration of element matrices and load vectors. A Point Membership Classification (PMC) test is carried out at each integration point to determine whether it lies inside or outside the physical domain. Hence, the only information needed from the CAD model is a reliable and robust PMC, which strongly reduces the geometrical and topological requirements on the validity of the geometric model. This observation allows for a new paradigm in the computational analysis: not to create an analysis-suitable model and/or to derive a mesh or distance field, but rather to directly compute on geometrically and/or topologically flawed models by a flaw-insensitive computational method. Thus, it is neither required to heal the flawed geometry nor to construct conforming meshes or distance fields. Instead, a PMC is constructed which is robust w.r.t. to a large number of model flaws. The Point Membership Classification test can then be evaluated with a certainty at least up to a geometric magnitude of the defect itself (as e.g., in the case of gaps). This is important because a subsequent computational analysis can then directly be carried out without healing. Moreover, the computational analysis may still deliver the necessary accuracy on those flawed models as their effect on the results of the computation remains local to the flaw itself. Only, if the local flaw lies directly in the region of interest it must be fixed. This is, however, only necessary to achieve higher accuracy – an analysis can be carried out either way.

The Finite Cell Method is a widely applicable method itself. While the original publications concerning the FCM treated linear elasticity in 2D and 3D [22], the scope of application was extended to various fields, such as elastoplasticity [23], constructive solid geometric models [24], topology optimization [25, 26], local enrichment for material interfaces [27], elastodynamics and wave propagation [28, 29, 30], and contact problems [31, 32]. Further developments include weakly enforced essential boundary conditions [33], local refinement schemes [34], and efficient
integration techniques \cite{35,36,37,38}. Furthermore, the concept of the FCM is independent of the underlying approximation method. It does not have to be based on p-FEM shape functions but can likewise be built on a spline-based approximation like in Isogeometric Analysis. In this case, the fictitious domain approach is an adequate method for trimming Isogeometric Analysis, as presented and analyzed e.g. in \cite{39,40,41}. In \cite{42}, an efficient method to overcome the inherent problem of bad condition numbers based on precondition is presented. Approaches very similar to the FCM have been presented more recently, like the cutFEM method \cite{43}, which builds on earlier publications of Hansbo et al. \cite{44}. Therein, small elements are explicitly stabilized by controlling the gradients across embedded boundaries connected neighboring cells in the fictitious domain. This is different to FCM where a stabilization is achieved to a certain extend by a small but non-zero stiffness in the fictitious domain.

In this contribution, the FCM is extended in order to directly simulate a CAD model with flaws. The paper is structured as follows: Section 2 provides a brief overview over geometrical and topological flaws. The basic formulation of the FCM and the requirements of a numerical simulation on flawed geometric CAD models are given in Section 3. A robust algorithm for Point Membership Classification on dirty geometries is presented in Section 4. Several numerical examples for the proposed methodology are presented and discussed in Section 5. Finally, conclusions are drawn in Section 6.

2 Dirty Topology/Geometry

In this section, we provide a very short general overview of Boundary Representation (B-Rep) models (sec. 2.1) and necessary conditions for their validity (sec. 2.2). By implication, 'dirty' geometries, or topologies are models which do not meet these requirements and are therefore mathematically invalid. To describe the wide variety of different flaws (sec. 2.3), we define mathematical operators (sec. 2.4) and apply them to a valid B-Rep model, thereby transforming a 'valid' into a 'dirty' B-Rep model (sec. 2.5). Several of these flaw operators allow introducing a control parameter $\epsilon$, indicating a geometric size of the respective flaws. Applying a sequence of flaw operators maps a flawless model to exactly one resulting flawed model. It is obvious that, given some flawed model, it is not possible to determine on which flawless model it could be based meaning that a class of equivalent flawless models can be associated to one 'dirty' model. Our conceptual approach therefore only assumes the existence of a flawless model that is expected to be 'close' to the 'dirty' one. This is used as the geometric basis for analysis. Further, it is to be noted that no explicit knowledge of this flawless model is required.

2.1 Boundary Representation Models

B-Rep objects are described by their boundaries. A model $\Omega$ can consist of several sub-domains, which all describe a separate closed volumetric body $B_i$.

$$\Omega = \{ B_i \mid i \in \{1, ..., n\} \}$$ (1)
with \( n \) being the number of volumetric bodies. For simplicity of presentation, we assume that a B-Rep model consists only of one domain \( \Omega = B \). A B-Rep body consists of topology \( T \) and geometry \( G \) [6]:

\[
B(T,G)
\] (2)

The topology \( T \) describes the relations or logical location of all entities (2.1.1), whereas the geometry \( G \) provides the physical location of points, consequently defining the actual shape of the model (2.1.2).

### 2.1.1 Topology

The topology \( T(t_\text{int}, r_\text{ext}) \) provides the logical internal \( t_\text{int} = \{ t_\text{int}_i \} \) and external relations \( r_\text{ext} = \{ r_\text{ext}_j \} \) between the topological entities \( t = \{ t_i \} \), i.e., vertices \( v_j \), edges \( e_j \), and faces \( f_k \). Thereby, each topological entity \( t_i \) has its own local, internal relation \( r_\text{int}_i \), defining how and from which underlying topological entities it is constructed. Topological entities are typically represented by sets:

\[
V = \{ \, v_i \mid i \in \{ 1, \ldots, n \} \, \}
\] (3)

\[
E = \{ \, e_i \mid i \in \{ 1, \ldots, m \} \, , \, e_i = (v_\alpha, v_\beta) \, , \, v_\alpha, v_\beta \in V \, \}
\] (4)

\[
F = \{ \, f_i \mid i \in \{ 1, \ldots, o \} \, , \, f_i = \left( e_\kappa \right)_{\kappa \in \{ \alpha, \ldots, \psi \}} \, , \, n_i \, , \, e_\kappa \in E \, \}
\] (5)

with \( n, m, o \) being the number of vertices, edges, and faces, respectively. The ordered pair of vertices \( (v_\alpha, v_\beta) \) contains the bounding vertices of an edge. The ordered tuple of edges \( (e_\kappa) \) contains the edges, which form the boundary of face \( f_i \), with its normal vector \( n_i \). In some cases, the normal vector is provided implicitly by the order of the boundary edges \( (e_\kappa) \). The external relations \( r_\text{ext} \) describe the global adjacency relations between the particular entities (e.g., which faces are neighbors to each other). There are various possible methods to represent the internal and external adjacency relations, or a combination of both, such as the winged edge model or the double connected edge list [6]. Thereby, the adjacency relations can be represented by graphs. Figure 2 shows an exemplary detail of a topology consisting of three triangles. The pure external relations can, for example, be represented by the adjacency matrix \( R_\text{FF} \) (see example (6)), whereas the adjacency matrices for faces and edges \( R_{FE} \) (see example (7)) and for edges and vertices \( R_{VE} \) (see example (8)) represent a combination of internal and external relations.
2.1.2 Geometry

The geometry $G(|g|)$ contains the geometric entities $g_i$, i.e., the points $P_i$, curves $C_i(\xi)$, and surfaces $S_i(\xi, \eta)$, which describe the actual physical location of the boundary and, thus, the shape of the geometry. Curves and surfaces are often expressed in parametric representation:

\[ P_i = (x_i, y_i, z_i)^T \]  \hspace{1cm} (9)

\[ C_i(\xi) = \begin{bmatrix} x(\xi) \\ y(\xi) \\ z(\xi) \end{bmatrix} \text{ e.g. } C_i(\xi) = \sum_{j}^{n_0_j} N_j(\xi) \cdot Q_j \]  \hspace{1cm} (10)

\[ S_i(\xi) = \begin{bmatrix} x(\xi) \\ y(\xi) \\ z(\xi) \end{bmatrix} \text{ e.g. } S_i(\xi) = \sum_{j}^{n_0_j} \sum_{k}^{n_0_k} N_j(\xi) \cdot N_k(\eta) \cdot Q_{jk} \]  \hspace{1cm} (11)

with $\xi \in \mathbb{R}$ and $\xi = (\xi, \eta) \in \mathbb{R}^2$. $N_i(\xi)$ denote shape functions (such as Lagrange or Legendre polynomials, B-Splines, NURBS, etc.) and $Q_i$ the associated (control-)points, which can, depending on the curve description, coincide with the geometrical points $P_i$.

Analogous to the topology, the geometry $G$ can be represented by sets:

\[ P = \{ P_i \mid i = \{1, \ldots, n\} \} \]  \hspace{1cm} (12)

\[ C = \{ C_i \mid i = \{1, \ldots, 2 \cdot m\} \} \]  \hspace{1cm} (13)

\[ S = \{ S_i \mid i = \{1, \ldots, o\} \} \]  \hspace{1cm} (14)

where the number of points and surfaces equals the number of vertices $n$ and surfaces $o$, respectively. A special case are curves, where at each edge two adjoined faces meet, whose underlying surfaces have each their own boundary curves. Consequently, the number of curves is $2 \cdot m$. 

Figure 2: Example topology with $n = 5$ vertices, $m = 7$ edges, and $o = 3$ faces.
2.1.3 Minimal B-Rep and the STL format

The most commonly used B-Rep exchange format between CAD and analysis is STL (STereoLithography, or more expressive Standard Tessellation Language). STL can be interpreted as a minimal B-Rep format, as it provides only the least amount of necessary information. Additionally, no explicit separation between topology and geometry is made. STL consists of independent triangles, which are defined by their three corner points. As geometric information in form of point coordinates is provided explicitly only for vertices, curves and surfaces are linearly interpolated. No adjacency, or ‘consistency’ information is provided, which makes STL quite flexible – but also particularly prone to a variety of potential flaws. The relation between faces and vertices reads:

\[ F_{\text{STL}} = \{ f_i \mid i \in \{1, \ldots, o\}, f_i = (v_\alpha, v_\beta, v_\gamma), t_i \}, ~ v_\kappa \in V, |V| = 3 \cdot o \} \]  

Note that – due to the multiple definition of vertices – STL models are, strictly speaking, topologically not valid.

2.2 Conditions for valid B-Rep models

Although intuitively quite apparent, it is not straightforward to define a valid B-Rep model. Patrikalakis et al. [45] provided a definition: ”A B-Rep model is valid if its faces form an orientable 2-manifold without boundary.” From this, several requirements can be derived, some of which are also mentioned by Mäntylä [6] and Hoffmann [46].

Topology:

1. Different vertices do have different coordinates.
2. One edge is shared by exactly two faces.
3. Faces at one vertex belong to one surface, i.e. at a vertex it is possible to cycle through all adjacent faces such that all of the vertex’ edges are crossed exactly once (see Fig. 3).
4. The orientation of faces must follow Moebius’ Rule, i.e. inside and outside must be distinguishable from each other.

Geometry:

5. A curve must lie on the respective surface whose partial boundary it forms.
6. Both boundary curves at one edge must coincide.
7. Surfaces must not self-intersect. From this – and from 5 – it follows that curves do not self-intersect either.
8. Surfaces must not touch or intersect with other surfaces except at common edges.
2.3 CAD model flaws

Model flaws can originate from different sources, such as mathematical inaccuracies, data conversion problems between different software systems, mistakes by designers, different design goals, etc. The probably most famous example of mathematical inaccuracies is the ‘leaking teapot’ model, as depicted in Fig. 4. The gap between spout and body of the teapot could only be avoided by more complex spline types (see, e.g. T-splines [47]), or unreasonably high polynomial degrees. The simplification results in a non-watertight geometry, a major obstacle for the interoperability between CAD and CAE. Figures 5, 6 and 7 provide an overview over the most common topological and geometrical modeling flaws.
(a) Double vertices  
(b) Double edges  
(c) Double faces  

(d) Wrong orientations  
(e) Missing faces  

Figure 5: Topological flaws

(a) Curves at common edge do not coincide  
(b) Surface and boundary curve self-intersect  
(c) Surface intersects with another surface  

Figure 6: Geometrical flaws
2.4 Flaw operators

In the following, we will introduce several operators that perform transformations on a valid B-Rep model, allowing for a controlled imposition of different flaws. To measure the size of the flaws, we introduce an error parameter \( \varepsilon \), indicating the ‘dirtiness’ or inaccuracy of the model.

To provide an easily understandable formulation of the operators, we consider an object-oriented B-Rep data structure. Thereby, the implementation must allow a distinction between internal and external/adjacency relations. Figure 8 provides a UML diagram of a possible hierarchical implementation. For an introduction to the notation of the UML (Unified Modeling Language) see, e.g. [49]. Here, the external adjacency relations \( r^{ext} \) are realized at the faces, where the adjacent faces are stored in the field: \( \text{adjacentFaces} \). All other external adjacency relations (e.g. which edges are adjacent) can be derived from this and from the respective internal relations \( r^{int} \).

Let \( \omega^i \) be the B-Rep sub-part, or segment, which corresponds to a topological entity \( t_i \), e.g. a face, an edge, or a vertex. The segment \( \omega^i \) consists of all information that is needed to visualize \( t_i \). Hence, it must contain \( t_i \) and, recursively, all underlying sub-topologies and geometries that are related by respective internal adjacency relations \( r^{int} \) (see Fig. 9).

\[
\omega^i(T^i, G^i) \subset B
\]  

(16)

with \( T^i(\tau, \rho^{int}) \) and \( G^i(\gamma) \) being the respective topology and geometry, where \( \tau = \{\tau_i\} \) and \( \gamma = \{\gamma_i\} \) denote the sets of those topological and geometrical entities which are recursively
Figure 8: UML-diagram of a possible object-oriented B-Rep implementation
related by the internal relations $\rho^{\text{int}} = \{\rho_i^{\text{int}}\}$. Consequently, three different segments are possible: 1) vertex segments, b) edge segments, and c) face segments. As topological entities $\{\tau_i\}$, a face segment, for example, contains the face itself and all associated edges and vertices. As geometric entities $\{\gamma_j\}$, it holds the corresponding surface with its boundary curves and corner points. Additionally, all internal relations $\{\rho_i^{\text{int}}\}$ are contained, i.e. the relations among face, boundary edges, and vertices, as well as the relations to the geometric entities. Not contained are external adjacency relations, i.e. those to neighboring faces or edges.

![Diagram](image)

**Figure 9:** B-Rep sub-part $\omega_f^i$, which corresponds to face $f_i$ and consists of the topology $T^{f_i}$ and the corresponding geometry $G^{f_i}$.

In the following, $\hat{a}$ denotes the object $a$ after the transformation and let $\text{dist}(a, b) = \inf\{\|a, b\|_2\}$ be the minimum euclidean distance between two objects, e.g. the distance between the closest points on two different surfaces.

1. Let $O^{\text{extract}}$ be an extraction operator, which selects for a topological entity $t_i$ the corresponding segment $\omega^{\text{hi}}$ from the body $B$.

$$O^{\text{select}}(B, t_i) \mapsto \omega^{\text{hi}}$$  \hspace{1cm} (17)

Note that the external relations are not extracted. Hence, the segment forgets about its logical location in the body.

2. Let $O^{\text{join}}$ be a join operator that adds a segment $\omega^{\text{hi}}$ to the body $B$.

$$O^{\text{join}}(B, \omega^{\text{hi}}) \mapsto \tilde{B}, \text{ where } \tilde{B} = B \cup \omega^{\text{hi}}$$  \hspace{1cm} (18)

3. Let $O^{\text{shallowCopy}}$ be a shallow copy operator that copies an arbitrary entity $a_i$. For a more detailed description of the object-oriented concept of ‘shallow’ and ‘deep’ copying see, e.g. [50]. $a_i$ can be a topological entity $t_i$, a geometrical entity $g_i$, or an internal $\rho_i^{\text{int}}$ or
external relation $r^\text{ext}_i$.

\[ O^{\text{shallowCopy}}(a_i) \mapsto \tilde{a}_i, \text{ where } \tilde{a}_i := a_i \]  

The ‘:=’ in eq. (19) is to be understood as the shallow copy assignment, according to [50]. Note that the new object is distinguishable from the old object, e.g. by an updated id, or, in the context of object-oriented programming, by a different memory address. Yet, it still uses the same references to other objects as the original segment.

4. Let $O^{\text{deepCopy}}$ be an internal deep copy operator [50] that performs a deep copy operation on a segment $\omega^k_i$. To this end, a shallow copy operation is carried out on all corresponding topological and geometrical entities, as well as the internal adjacency relations.

\[ O^{\text{deepCopy}}(\omega^k_i) \mapsto \bar{\omega}^k_i(\bar{T}^k, \bar{G}^k), \text{ with } \bar{T}^k(\bar{\tau}, \bar{\rho}^\text{int}) , \bar{\tau}_i := O^{\text{shallowCopy}}(\tau_i) , \]  

\[ \bar{\rho}^\text{int}_i := O^{\text{shallowCopy}}(\rho^\text{int}_i) \quad \forall \tau_i, \rho^\text{int}_i \in T^k_i ; \]  

and

\[ \tilde{G}^k(\tilde{\gamma}) , \tilde{\gamma}_i := O^{\text{shallowCopy}}(\gamma_i) \quad \forall \gamma_i \in G^k_i ; \]  

Note that the deep copied segment $\omega^k_i$ has no information about its logical location in $B$, i.e. it has no external adjacency relations, and that all internal relations are updated to reference the new topological and geometrical entities.

5. Let $O^{\text{delete}}$ be a deletion operator that deletes a face $f_i$ and its related geometry $g^k_i \subset G$ consisting of the underlying surface $S_i$ and the corresponding boundary curves $\{C^k_i\}$. Thereby, the characteristic size of the resulting opening must not exceed a given, e.g. user-defined minimal accuracy $\varepsilon$. Let $\delta$ be the diameter of the largest possible inscribed sphere of the surface $S_i$ to be deleted.

\[ O^{\text{delete}}(B, f_i, g^k_i) \mapsto \tilde{B}(T, G) , \text{ where } F = F \setminus f_i ; \quad G = G \setminus g^k_i , \quad \delta < \varepsilon \]  

Note that, as a deletion of an edge, or vertex would lead to an uncontrollable cascade of deletions of superior entities, only a face deletion is allowed in this context. A B-Rep model with a deleted edge or node without deletion of referencing faces would not even be readable and is not considered in our investigation.

6. Let $O^{\text{explode}}$ be an operator that removes all external relations $r^\text{ext}$ from a body $B$. This can be achieved by extracting (17), copying (20), and joining (18) all face segments $\omega_f \forall f_i \in F$. The resulting body $\tilde{B}$ is then described by independent topological sup-parts/segments $\omega_f$.

\[ O^{\text{explode}}(B) = O^{\text{join}}(B^* , O^{\text{deepCopy}}(O^{\text{extract}}(B, F))) \mapsto \tilde{B} , \text{ where } r^\text{ext} = \emptyset \]  

where $B^*$ is an empty body.
7. Let $O^{flip}$ be a topological flip operator that flips the normal $n_i$ of a face $f_i$.

   \[ O^{flip}(f_i) \mapsto \tilde{f}_i, \quad \text{where} \quad \tilde{f}_i = ((e_i, \tilde{n}_i) = -1 \cdot n_i) \quad (23) \]

8. Let $O^{move}$ be a geometric move operation that moves the point $P_i$ within the range $\varepsilon$. Additionally, all adjoined surfaces and curves are adapted consistently such that they form a 2-manifold without boundaries after the operation. This involves the following adaptions to the adjoined surfaces $S^{P_i} = \{S_i^{P_i}\}$ and curves $C^{P_i} = \{C_i^{P_i}\}$:

   - The resulting point $\tilde{P}_i$ must again lie on the altered surfaces $S^{\tilde{P}_i}$ and curves $C^{\tilde{P}_i}$.
   - All pairs of the resulting adjoined surfaces ($\tilde{S}_A^{e_k}, \tilde{S}_B^{e_k}$) must again meet at their common edge/cap $e_k$. Consequently, the two respective boundary curves ($\tilde{C}_A^{e_k}, \tilde{C}_B^{e_k}$) must coincide.

The latter condition is omitted in the case of an already broken topology, where an edge no longer has two adjoined faces/surfaces.

   \[ O^{move}(P_i, G) \mapsto \tilde{G}, \quad \text{where} \quad 0 < \text{dist}(P_i, \tilde{P}_i) < \varepsilon, \quad \text{and} \]

   \[ \exists \xi : \tilde{S}_i^{\tilde{P}_i}(\xi) = \tilde{P}_i \land \exists \zeta : \tilde{C}_i(\zeta) = \tilde{P}_i \land \forall \eta \in [\xi_0, \xi_0] \]

   \[ \text{dist}(\tilde{C}_A^{e_k}, \tilde{C}_B^{e_k}) = 0 \land (\tilde{S}_A^{e_k}, \tilde{S}_B^{e_k}) \text{ at } v_i \quad (24) \]

9. Let $O^{detach}$ be a geometrical operator that detaches two adjacent surfaces, $S_i$ and $S_j$, which meet at the edge $e_k$ (see Fig. 6a). To this end, one surface $S_i$ and its respective boundary curve $C_{S_i}^{e_k}$ at $e_k$ are changed. Again, the characteristic size of the potentially resulting opening must not exceed $\varepsilon$.

   \[ O^{detach}(G, e_k) \mapsto \tilde{G}, \quad \text{where} \quad \text{dist}(\tilde{S}_i, \tilde{C}_{S_i}^{e_k}(\xi)) = 0 , \]

   \[ 0 \leq \text{dist}(\tilde{C}_{S_i}^{e_k}, \tilde{C}_{S_j}^{e_k}(\xi)) < \varepsilon \land \forall \xi \in [\xi_0, \xi_0] \quad (25) \]

with $[\xi_0, \xi_0]$ being the respective interval on which the boundary curve is defined.

10. Let $O^{intersect}$ be a geometric operator that alters a surface $S_i(\xi)$ such that it touches or intersects with another surface $S_j(\eta)$ apart from common edges. Note that we assume that there is no intersection in the original model, according to the definition of a valid B-Rep model.

   \[ O^{intersect}(S_i(\xi)) \mapsto \tilde{S}_i(\xi), \quad \text{where} \]

   \[ \exists (\xi, \eta) : \text{dist}(\tilde{S}_i(\xi), S_j(\eta)) = 0 \land \text{dist}(\tilde{S}_i(\xi), \tilde{S}_j) > 0 \quad (26) \]

   \[ \forall \tilde{C}_k^{\tilde{S}_i} \in \Gamma^{\tilde{S}_i}, i \neq j \]

with $\Gamma^{\tilde{S}_i}$ being the set of boundary curves of $\tilde{S}_i$. 

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A special case of intersections are self-intersections:

\[ O_{\text{SelfIntersect}}(S_i(\xi)) \mapsto \tilde{S}_i(\xi), \quad \text{where} \quad \exists (\xi, \eta) : \text{dist}(\tilde{S}_i(\xi), S_i(\eta)) = 0, \quad \xi \neq \eta \quad (27) \]

2.5 Application of flaw operators

We now continue with the definition of a flawed model. To this end, we apply the flaw operators defined in section 2.4 onto a valid B-Rep model. The 'dirtiness' of the model is then defined by \( \varepsilon \). It should be mentioned that, for models that are drafted by a real-life CAD system, flaws do not necessarily originate from these operators, yet most flawed models can equivalently be created by a sequence of these operators.

Let \( B(T, G) \) be a valid flawless B-Rep body. Note that operators acting on the body is to be understood as acting on a segment \( \omega^k \), or single topological, or geometrical entity, or relation.

1. Single topological entities \( t_i \) and their corresponding segments \( \omega_{t_i} \) can be copied and added to \( B \) with a combination of the extraction (17), the deep copying (20), and the joining (18) operator:

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O_{\text{Join}}(B(T, G), O_{\text{DeepCopy}}(O_{\text{Extract}}(B(T, G), t_i))) \quad (28) \]

The resulting B-Rep model is invalid as it has multiple entities (refer to Figs. 5a, 5b, and 5c), which violates condition refcondition:VertexCoords. As an example, consider the STL format where each triangle (re-)defines its corner points. Also, multiply defined faces/surfaces appear frequently in free form CAD models, which leads to a touching/intersection of the surfaces (refer to fig. 7c).

2. Application of the deletion operator (21) on a face \( f_i \):

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O_{\text{Delete}}(B, f_i, g_{f_i}) \quad (29) \]

The deletion of a face violates condition 2 (see fig. 5e). Thereby, the size of the resulting opening restricted to be smaller than \( \varepsilon \).

3. Application of the explosion operator (22):

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O_{\text{Explode}}(B(T, G)) \quad (30) \]

Most B-Rep models are constructed from independent surfaces, which are later joined into a (hopefully) valid B-Rep model. This join operation corresponds to the inverse of the explosion operation. It is yet well known that a strict 'join'-operation is not necessarily possible (or maybe not feasible) e.g. in case of an intersection of two NURBS surfaces [48]. Also, STL models are constructed by independent triangles. Such models violate the topological conditions 1, 2, and 3. Geometrically, they can still form a closed 2-manifold without boundaries. However, these models are very prone to a variety of different flaws, as no external adjacency relations are provided explicitly.
4. Application of the flip operator (23):

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O^{flip}(B(T, G)) \]  (31)

The resulting B-Rep model does not fulfill Moebius’ Rule anymore (see condition 4). This flaw usually appears if the normal is defined implicitly by the order of the boundary edges, or in the case of STL of the corner vertices (see Fig. 5d). However, this error also appears quite frequently if the normal is given explicitly.

5. Application of the move operator (24):

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O^{move}(B(T, G)) \]  (32)

Applied on a valid B-Rep body, the move operator preserves a geometric 2-manifold, without boundary. However, the orientability can be lost (see condition 4). As an example, consider a point \( P_i \) on surface \( S_j \), which is close to surface \( S_k \) with distance \( \text{dist}(P_i, S_k) < \varepsilon \). A movement then can lead to an intersection of the two surfaces. This violates condition 8 (see 6c).

6. Application of the detach operator (25):

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O^{detach}(B(T, G)) \]  (33)

The resulting model violates condition 6. This is likely to happen at the intersection of free-form surfaces. The boundary curves would require unreasonably high polynomial degrees to perfectly coincide. Possible flaws can e.g. be openings or intersections (see Figs. 6a and 6c). As an example, consider the leaking Utah teapot.

7. Application of the intersection operator (eq. 26):

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O^{intersect}(B(T, G)) \]  (34)

The resulting model may violate condition 7 or 8. Apart from gaps, intersections frequently appear at patch boundaries as well (see Fig. 6a). Intersections can also occur if two surfaces are too close to each other. In this case, they additionally violate Moebius’ Rule 4 (see Figs. 6c and 6b). A special case are overlaps, where two surfaces touch each other (see fig. 7b).

8. Application of the copy and the move operators to a single face \( f_i \) (eqs. 20 and 24):

\[ \tilde{\omega}^f_i := O^{deepCopy}(O^{extract}(B, f_i)) \]
\[ \tilde{\omega}^f_i := O^{move}(P_j \in \tilde{\omega}^f_i, \tilde{G}^i) \]  (35)
\[ \tilde{B}(\tilde{T}, \tilde{G}) := O^{join}(B, \tilde{\omega}^f_i) \]

This chain of operations allows to create offsets and artifacts, i.e. entities which do not belong to the outer hull and lead to a violation of the conditions (3 and 2) (see Figs. 7d and 18).
9. Application of the explosion and move operators (eqs. 22 and 24):

\[ \tilde{B}(\tilde{T}, \tilde{G}) := O^{\text{explode}}(B) \]
\[ \tilde{B}(\tilde{T}, \tilde{G}) := O^{\text{move}}(\tilde{P}_i \in \tilde{G}, \tilde{G}) \]  

(36)

Starting from an exploded model, moving one or more points can lead to various common flaws – such as gaps, intersections, or overlaps (see Figs. 7a, 7c, 7b). As many B-Rep modeling tools work with exploded models, i.e. with independent surfaces, these flaws appear very commonly, particularly at patch boundaries. Also, the STL format stores a body with independent triangles.

Note that, independent of the performed operations, it is imperative for the presented method that the size of all openings and gaps is restricted to be smaller than a pre-defined \( \varepsilon \). This is required not only for each individual flaw operation but also for the resulting model after a sequence of flaw operations, e.g. a sequence of individual moves of a segment.

The resulting flawed models are invalid in a mathematical sense, which renders a subsequent conversion into a simulation model either impossible or invalid. The necessity to heal the flaws can neither be circumvented by meshing, as in the classical FEM, nor by a direct simulation as in IGA. It is, however, possible to compute ‘dirty’ models directly with an embedded domain method such as the Finite Cell Method (section 3). To this end, we construct a specially adapted Point Membership Classification test (section 4) which is blind to flaws up to a characteristic size \( \varepsilon \).

3 Finite Cell Method

The Finite Cell Method is a higher order fictitious domain approach. It offers simple meshing of potentially complex domains into a structured grid of e.g. cuboid cells without compromising the accuracy of the underlying numerical method. For completeness of this paper, the basic concepts are briefly introduced in this section. We restrict ourselves to linear elasticity – emphasizing however, that the formulation has been extended to more general partial differential equations \[51, 52, 53, 54\].

3.1 Basic formulation

In the Finite Cell Method, an n-dimensional open and bounded physical domain \( \Omega_{\text{phy}} \) is embedded in a fictitious domain \( \Omega_{\text{fict}} \) to form an extended domain \( \Omega_{e} \), as illustrated in Fig. 10 in two dimensions. The resulting domain \( \Omega_{e} \) has a simple shape which can be meshed easily, without conforming to the boundary of \( \Omega_{\text{phy}} \).
The weak form of the equilibrium equation for the extended domain \( \Omega_{\cup} \) is defined as

\[
\int_{\Omega_{\cup}} [Lv]^T \alpha C [Lu] \, d\Omega = \int_{\Omega_{\cup}} v^T \alpha f \, d\Omega + \int_{\Gamma_N} v^T \Gamma \, d\Gamma ,
\]

where \( u \) is a displacement function, \( v \) a test function, \( L \) is the linear strain operator, and \( C \) denotes the elasticity matrix of the physical domain \( \Omega_{phy} \), yet extended to \( \Omega_{\cup} \). \( f \) and \( \Gamma \) denote the body load and the prescribed tractions on the Neumann boundary, respectively. The indicator function \( \alpha \) is defined as

\[
\alpha(x) = \begin{cases} 
1 & \forall x \in \Omega_{phy} \\
10^{-q} & \forall x \in \Omega_{fict}
\end{cases}
\]

In the limiting case of \( q \to \infty \), the standard weak form for an elasticity problem on \( \Omega_{phys} \) is obtained. In practical applications, a sufficiently large \( q = 6..10 \) (see [21, 22]) is chosen, introducing a modeling error to the formulation [56], which yet stabilizes the numerical scheme and controls the conditioning number of the discrete equation system – see [42] for a detailed analysis. \( \Omega_{fict} \) is then discretized in 'finite cells' of simple shape (rectangles or cuboids). In the context of this paper, we assume for simplicity a uniform grid of finite cells, yet note that generalizations to locally refined grids have been studied extensively [57, 58].

3.2 Geometry treatment

In FCM, the physical domain \( \Omega_{phys} \) (i.e. the geometry) is recovered by the discontinuous scalar field \( \alpha \). Consequently, the complexity of the geometry is shifted from the finite elements to the integration of the element matrices and load vectors, which imposes less geometrical requirements on the model. It is in fact sufficient to provide a robust Point Membership Classification (PMC), i.e. for every point \( x \in \mathbb{R}^n \), it must be possible to decide whether it is inside or outside of \( \Omega_{phys} \). This implies that \( \Omega_{phys} \) must have a mathematically valid description. Due to the discontinuity of \( \alpha \), the integrands in cut cells need to be computed by specially constructed quadrature rules, see e.g. [35, 38, 59] for a recent overview of possible schemes. To perform a suitable integration, the domain is approximated by a space-tree \( TR_{int} \). The leaves of \( TR_{int} \) are called integration leaves \( c_{int} \). Additional information, such as explicit surface descriptions are only needed for the application of boundary conditions as well as for post-processing (see Sec. 3.3).
3.3 Boundary conditions

Neumann boundary conditions are applied according to equation (37) in an integral sense on the boundary $\Gamma_N$. Homogeneous Neumann conditions (i.e. zero traction) require no treatment, as they are automatically satisfied by setting $\alpha = 0$ or, in an approximate sense, to a small value in $\Omega_{\text{ fict}}$. Dirichlet boundary conditions are more elaborate, as the boundary of the physical model typically does not coincide with the edges/faces of the finite cell mesh. Hence, they need to be enforced in a weak sense. To this end, several methods have been adopted, such as the penalty method, Nitsche’s method, or Lagrange Multipliers \[33, 60, 61, 62\].

For the integration of Dirichlet and inhomogeneous Neumann boundary conditions, an explicit surface description is needed. This can be of poor quality. For the enforcement of Neumann boundary conditions, however, a surface without multiple faces/surfaces or large overlaps is required, as these flaws would introduce physically modified boundary conditions (i.e. additional loads, heat sources, etc.). To this end, we propose the following automatable method to convert a ‘dirty’ surface into a surface without multiple entities or overlaps:

1. Triangulate the respective surface (if not already provided, e.g. with STL).
2. Get the intersection points between the surface mesh and the element boundaries.
3. Create an element-wise point cloud from the intersection points and respective triangle corner points.
4. Perform an element-wise Delaunay triangulation on the respective point cloud.

The resulting element-wise triangular meshes are used only for integration and can consequently be independent of each other. Note that the requirements to these local surface meshes are by far less restrictive than they would be for a surface mesh as a starting point for volume mesh generation.

4 Robust Point Membership Classification for flawed CAD models

As explained in Section 3, the only geometric information required to setup the system matrices for the Finite Cell Method is an unambiguous statement about the location of a point, i.e whether it lies inside or outside of the domain of computation. Considering flawed CAD models (e.g. with undesired openings), the concept of ‘inside’ or ‘outside’ is fuzzy – at least up to the characteristic size of the flaw $\varepsilon$. In this section, we present a robust Point Membership Classification method for ‘dirty’ STL B-Rep models. The presented approach is, however, not restricted to STL models, and it can easily be extended to other boundary representations.

4.1 Point Membership Classification for valid CAD models

PMC algorithms are fundamental and extensively used operations e.g. in computer graphics, computer games, and in geoinformatics [63]. For different geometric representations, various
PMC algorithms exist. For CSG models, a point is classified against all the underlying primitives and the resulting boolean expressions (see [24]). Ray casting [63] is often used for boundary representation models. Further variants are approximation-tree-based algorithms [64], point cloud methods [65], sign of offset [66], and the swath method [67]. As the space-tree based approximation and the ray-casting are needed in the following, these aspects will be explained in more detail:

- **Ray casting**: The ray casting method is an efficient and suitable algorithm for general polytones, and it is extensively used in computational graphics, e.g. for depth maps. To classify a given point with respect to a geometric model, a ray is shot in an arbitrary direction and the intersections with the boundary are counted. The parity (even, or odd) of intersections then provides information on whether the point lies inside or outside. For flawless models, ray-casting is accurate. For flawed CAD models, however, ray casting delivers no reliable statement about the point’s domain membership, as almost all flaws influence the parity of intersections.

- **Space-tree based PMC**: For the tree-based PMC, the domain is discretized by a space-tree $T_{int}$, with leaves $c_{geo}$. Leaves intersected by the surface are marked as cut. Subsequently, a flood-fill algorithm is applied to the leaves $c_{geo}$. Starting from a seed point, whose domain membership is known, all connected leaves are marked as inside or outside, respectively. A challenge in this methodology is posed only by undesired openings or unintentional gaps. In these cases, a too fine approximation with leaves smaller than the size of the flaws would cause the flood-fill algorithm to mark the entire domain as inside or outside. Furthermore, despite its robustness against most flaws, the octree $T_{int}$ gives only a coarse step-wise approximation of the geometry.

### 4.2 General approach for flawed models

The presented PMC method combines the robustness of space-tree approximation with the accuracy of ray-casting. The general approach works as follows:

1. The CAD model is approximated by a watertight space-tree $T_{geo}$. Watertightness is imperative to ensure that the subsequent flood-fill can distinguish between inside and outside.

2. A flood fill algorithm is applied on $T_{geo}$ to mark all connected points as inside and outside, respectively. This yields a filled space tree $\hat{T}_{geo}$. Remark: For all points that are not on cut leaves, the approximation tree $\hat{T}_{geo}$ can be used as fast, efficient, and accurate PMC.

3. An additional ray-casting is only carried out for points lying inside the cut boundary leaves – in order to approximate the structure more precisely.

Step 1 and Step 3 will now be described in more detail. For a description of the well-known flood fill algorithm in step 2 we refer to, e.g. [68].
4.3 Watertight space tree approximation

To ensure that the approximation space-tree $\hat{T}_{\text{geo}}$ is watertight, the size of the smallest leaves $d_{\text{geo}}$ must not undercut the characteristic size of the largest gap/opening $\varepsilon$.

$$d_{\text{geo}} > \varepsilon_{\text{gap}}$$  \hspace{1cm} (39)

From this, it follows that the maximal partitioning depth $n_{\text{max}}$ of $\hat{T}_{\text{geo}}$ is bounded by the ratio of domain size $d_{\text{domain}}$ of the tree $\hat{T}_{\text{geo}}$ to the dimension of the gaps/openings $\varepsilon_{\text{gap}}$:

$$n_{\text{max}} < \log_2 \left( \frac{d_{\text{domain}}}{\varepsilon_{\text{gap}}} \right)$$  \hspace{1cm} (40)

This limitation allows, depending on the size of the gaps/openings only a very coarse approximation of the true geometry. Concerning all other types of considered flaws, a test using the space tree $\hat{T}_{\text{geo}}$ is robust.

Note that, generally, the space-trees $T_{\text{int}}$ and $\hat{T}_{\text{geo}}$ are distinct. While $T_{\text{int}}$ is constructed in order to numerically integrate the discontinuous element matrices for finite cells (see sec. [3]), the purpose of $\hat{T}_{\text{geo}}$ is merely to support the Point Membership Classification of the integration points.

After the surface is approximated by the space tree, the flood fill algorithm [68] can be applied to mark connected regions. Figure [12] shows the octree approximation of a simple example (fig. [11]), which has several typical flaws. The size of the opening $\varepsilon_{\text{gap}}$ allows a maximum subdivision level of $n_{\text{max}} = 7$. Hence, the ratio of the largest gap to overall size is in the range of:

$$\frac{1}{256} < \frac{\varepsilon_{\text{gap}}}{d_{\text{domain}}} < \frac{1}{128}$$  \hspace{1cm} (41)

![Diagram](image)

Figure 11: Example of an STL model with typical flaws.
Figure 12: Octree approximation of the embedded tetrahedral domain. The outer domain (blue) is separated by the cut leaves (red) from the inner domain (grey). The subdivision level is $n_{\text{max}} = 7$.

4.4 Point Membership Classification on cut leaves

The space-tree $\hat{R}_{\text{geo}}$ represents the surface only very roughly and, thus, cannot be used for a precise numerical analysis. Hence, in order to improve the representation of the boundary, an additional PMC using ray casting is carried out on cut leaves. Let us first assume that the model is flawless (see fig. 13a). Then, the ray test for any integration point in an integration leaf $c_{\text{int}}$ yields a unique result without ambiguity, independent of the direction of the ray. In case of a flawed surface, the result may be ambiguous, depending on the selected direction of the ray (Figs. 13b-f). To handle this problem, we test rays in different directions and decide ‘following the vote of the majority’. Clearly, this ‘vote’ can be wrong w.r.t. the (in general unknown) flawless model. This wrong decision results in an integration error for the computation of element matrices. In a mathematical sense, we are performing a ‘variational crime’ (see, e.g. [69]). For geometrically small flaws, the smallness of this integration error can be readily assumed – as, by construction of the two-stage PMC, it can only occur in the smallest leaf $c_{\text{geo}}$, cut by the surface.

We can even bound this error by bracketing, i.e. by solving the elasticity problem (23) – once under the assumption that all ambiguous integration points are inside, and once assuming them outside of the domain of computation (see example 5.1), thus ensuring that the approximation quality of the method is not corrupted.
Figure 13: Multiple ray casting for different flaws (red: cut leaves, grey: inside, blue: outside). In these examples d) and f) would lead to indifferent results.

5 Numerical examples

To demonstrate the accuracy and robustness of the proposed approach, three examples are presented. The first simple example serves to verify the proposed method. To this end, a plate with a hole is simulated and compared to a flawless reference solution. The complex screw in the second example proves the applicability for sophisticated, defective CAD models. Again, a flawless reference model was available. The last example is an engine bracket taken directly from engineering practice. This model is a perfect example of a flawed geometry, as many NURBS-patches do not fit together. An attempt to mesh the model showed that 337,544 triangles had a free edge, i.e. are flawed.

5.1 Example 1: Thick-walled plate with circular hole

As a classical benchmark for 3D problems, we choose the thick-walled plate with four circular holes [21]. The Young’s modulus is set to \( E = 10000.0 \text{N/mm}^2 \) and the Poisson’s ratio to \( \nu = 0.30 \). The plate is loaded with a surface traction \( \tilde{t}_n = 100.0 \text{N/mm}^2 \). Symmetry boundary conditions are used, allowing to simulate only a quarter of the domain (see Fig. [14]). The dimensions of the model are \( b = h = 4.0 \text{mm} \) and \( t = r = 1.0 \text{mm} \). To show the robustness of the proposed method, several flaws – namely intersections, gaps, double entities, and offsets – are introduced on the surfaces (see Figs. [15] and [16]).
The domain is discretized into $10 \times 10 \times 1$ finite cells employing integrated Legendre polynomials as basis functions. The background grid and the qualitative displacement are depicted in Figure 17. A convergence study was carried out for $p$-refinement using $p = 1...6$. To measure the accuracy of the approach, the strain energy is computed and passed on to the reference solution $u_{e,x}$, which was computed with an extensive boundary-conforming finite element analysis. The minimal size of the cells $\hat{c}_{geo}$ of the geometric tree $\hat{TR}_{geo}$ was limited by the size of the largest gap, allowing a maximum subdivision depth of $n_{max} = 5$. Note that the tree is also refined at the 'flat' surfaces of the plate. This results in $\sim 1.33$ million cells, of which $\sim 1.15$ million cells are located on the deepest level (see Fig. 18).
Figure 17: Approximation tree $\hat{TR}_{geo}$ with subdivision depth $n_{max} = 5$

Figure 18: Displacement and finite cell discretization

Figure 19 plots the strain energy for the different polynomial degrees. Note that the relative error in the strain energy can only be computed for the valid model, as this is the only possible basis to compute a reference solution.

In Figure 19, it can be seen, that both models converge to a slightly different value. This is, of course, to be expected – as both models have a slightly different shape and volume. The good convergence of the flawed model is attributed to the fact that errors due to flaws are very localized. This is an inherent property of the proposed methodology.

A detailed investigation of the flawed geometry can be carried out based on the ray-casting tests, which are carried out on the integration cells (see Section 4.4). As an example, we consider
a polynomial degree of \( p = 3 \). For the integration of the system matrices, 6 406 920 points need to be evaluated. From these, a total number of 2 225 641 points (\( \sim 35\% \)) are lying on cut cells. Typically, 12 to 18 ray-castings are carried out on each of these points. 1 503 636 points (\( \sim 23\% \)) are ambiguous, i.e. at least one ray delivers a different result compared to the majority. In 656 009 cases (\( \sim 10\% \)), a ‘vote for the majority’ is not possible, as the number of rays voting for inside and outside is equal. This large amount of is mainly due to the many double entities. To compute the upper and lower boundaries of the energy norm, two additional simulations were carried out – once with all ambiguous integration points counting as inside and once counting as outside. The energy norm for lower boundary was \( \sim 0.4\% \) lower, and for the upper boundary \( \sim 2.9\% \) larger compared to the simulation with ‘vote for the majority’. Due to the fact that the error is restricted to the smallest geometrical leaves and the ray-casting errors occur only in the vicinity of the flaws, the deviation in the strain energy norm is rather small.

### 5.2 Example 2: Screw

This example demonstrates how the algorithm performs for a more complex geometry. To this end, we consider the potentially flawed CAD model of a screw, depicted in Figure 1. The simulation was carried out on \( 10 \times 30 \times 10 \) finite cells using trivariate B-Splines of polynomial degree \( p = 3 \) and the open knot vector \( U = [0, 0, 0, 0, 1, 2, 3, 3, 3, 3] \). The partitioning depth for the integration of cut FCM cells was set to \( k = 3 \). The tip surface was loaded with a constant pressure, and the bottom surface was clamped. At the top, several flaws were introduced, resulting in gaps, overlaps, and intersections. In the detailed view in Figure 24, the free edges (i.e. edges which have only one adjoined face) are highlighted in blue.

Figure 20 shows the effect of a too fine resolution of \( \hat{TR}_{geo} \). For a subdivision depth \( n_{max} = 5 \), the flood fill algorithm marks the entire domain as outside.

![Figure 20](image)

\( (a) \ n_{max} = 5 \)
\( (b) \ n_{max} = 4 \)

Figure 20: Cut through an octree approximation \( \hat{TR}_{geo} \), with a) too small cut leaves \( c_{geo} \) (black), so that all (non cut) leaves are marked as outside (light grey). b) with one subdivision level less leaves inside (dark grey) can be detected.

A visual inspection of the displacements of the flawed and the valid model shows no difference (see Fig. 21), whereas differences around the flaws can be detected for the von Mises stresses (see Fig. 22). The stresses at the flawed model are more noisy compared to the valid model.
5.3 Example 3: Engine Brake

In 2013, a collaboration of Grab Cad and General Electric arranged a competition to find the optimal design of an engine bracket for a General Electric turbofan [70]. The submitted designs were then evaluated, and the top ten were produced using additive manufacturing. The model depicted in Figure 23 was designed by Sean Morrissey.¹

¹https://grabcad.com/sean.morrissey-1
Figure 23: General Electric design challenge for the optimal shape of a jet engine brake

In an attempt to perform a heat diffusion simulation motivated by a local heat source induced by a laser beam during additive manufacturing it turns out that 337,544 triangles have a free edge, indicating a gap/opening between the patches. 2324 triangles were oriented in the wrong direction and innumerable intersections occurred. Due to the immense amount of flaws, geometry healing – and, thus, also the meshing – is not applicable on this raw model. However, using the approach presented in this paper, we were able to immediately run a simulation without any further treatment of flaws.
Figure 24: Blue lines denote open edges: In many cases the free edges are fairly close. As can be seen in the detailed views, however, some of the gaps and openings are quite large.

We chose $18 \times 11 \times 6$ elements for the simulation. A partitioning depth for the geometry approximation tree $\overrightarrow{TR}_{geo}$ of $n_{max} = 3$ on each finite cell was applicable. A laser beam is modeled by a small heat source where a local refinement of the finite cell grid was applied. Figure 25 shows the resulting temperatures in the specimen.
Conclusions

This work presents a methodology to address challenges flawed CAD models pose to computational mechanics. Unlike other methods that rely on model reconstruction or geometry healing, the proposed approach herein allows for a numerical analysis directly on the corrupted 'dirty' geometry. Certainly, a simulation on broken geometries will inevitably lead to errors, which are yet of a similar nature to modeling errors due to a representation of a NURBS-based geometry by faceted surfaces. The size of this modeling error depends on the geometric size of the flaws, e.g. the width of a gap between patches. The influence of these errors remains local to the flaw itself. Moreover, the error can be bounded by performing bracketing simulations. Therein, the upper bound is delivered by a computation considering all ambiguous integration points to lie inside the physical domain and the lower bound is generated by considering the inverse situation. Several examples demonstrate the capability of the proposed method, showing that results of high accuracy can be obtained.
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