The high–energy quark–quark scattering: from Minkowskian to Euclidean theory

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Abstract

In this paper we consider some analytic properties of the high–energy quark–quark scattering amplitude, which, as is well known, can be described by the expectation value of two lightlike Wilson lines, running along the classical trajectories of the two colliding particles. We shall prove that the expectation value of two infinite Wilson lines, forming a certain hyperbolic angle in Minkowski space–time, and the expectation value of two infinite Euclidean Wilson lines, forming a certain angle in Euclidean four–space, are connected by an analytic continuation in the angular variables. This could open the possibility of evaluating the high–energy scattering amplitude directly on the lattice or using the stochastic vacuum model. The Abelian case (QED) is also discussed.
1. Introduction

There is a class of *soft* high–energy scattering processes, i.e., elastic scattering processes at high squared energies $s$ in the center of mass and small squared transferred momentum $t$ (that is $s \to \infty$ and $|t| \ll s$, let us say $|t| \leq 1 \text{ GeV}^2$), for which QCD perturbation theory cannot be safely applied, since $t$ is too small. Elaborate procedures for summing perturbative contributions have been developed [1] [2], even if the results are not able to explain the most relevant phenomena.

A non–perturbative analysis, based on QCD, of these high–energy scattering processes was performed by Nachtmann in [3]. He studied the $s$ dependence of the quark–quark (and quark–antiquark) scattering amplitude by analytical means, using a functional integral approach and an eikonal approximation to the solution of the Dirac equation in the presence of a non–Abelian external gluon field.

In a previous paper [4] we proposed an approach to high–energy quark–quark (and quark–antiquark) scattering, based on a first–quantized path–integral description of quantum field theory developed by Fradkin in the early 1960’s [5]. In this approach one obtains convenient expressions for the full and truncated–connected scalar propagators in an external (gravitational, electromagnetic, etc.) field and the eikonal approximation can be easily recovered in the relevant limit. Knowing the truncated–connected propagators, one can then extract, in the manner of Lehmann, Symanzik, and Zimmermann (LSZ), the scattering matrix elements in the framework of a functional integral approach. This method was originally adopted in [6] in order to study Planckian–energy gravitational scattering.

The high–energy quark–quark scattering amplitude comes out to be described by the expectation value of two lightlike Wilson lines, running along the classical trajectories of the two colliding particles.

In the center–of–mass reference system (c.m.s.), taking the initial trajectories of the two quarks along the $x^1$–axis, the initial four–momenta $p_1$, $p_2$ and the final four–momenta $p'_1$ and $p'_2$ are given, in the first approximation (*eikonal approximation*), by

$$p_1 \simeq p'_1 \simeq (E, E, 0_t) \quad , \quad p_2 \simeq p'_2 \simeq (E, -E, 0_t). \quad (1.1)$$

Let us indicate with $x^1_q(\tau)$ and $x^2_q(\tau)$ the classical trajectories of the two colliding particles.
Fig. 1. The space–time configuration of the two lightlike Wilson lines $W_1$ and $W_2$ entering in the expression (1.4) for the high–energy quark–quark elastic scattering amplitude.

In Minkowski space–time:

$$x_1^\mu(\tau) = z_\tau^\mu + p_1^\mu \tau, \quad x_2^\mu(\tau) = p_2^\mu \tau,$$

where $z_\tau = (0, 0, z_t)$, with $z_t = (z^2, z^3)$, is the distance between the two trajectories in the transverse plane (the coordinates $(x^0, x^1)$ are often called longitudinal coordinates). The high–energy ($s \to \infty$ and $|t| \ll s$) quark–quark scattering amplitude turns out to be controlled by the Fourier transform, with respect to the transverse coordinates $z_t$, of the expectation value of the two lightlike Wilson lines running along $x_1^\mu(\tau)$ and $x_2^\mu(\tau)$:

$$W_1(z_t) = P \exp \left[ -ig \int_{-\infty}^{+\infty} A_\mu(z_\tau + p_1 \tau) p_1^\mu d\tau \right];$$

$$W_2(0) = P \exp \left[ -ig \int_{-\infty}^{+\infty} A_\mu(p_2 \tau) p_2^\mu d\tau \right],$$

where $P$ stands for “path ordering” and $A_\mu = A_\mu^a T^a$. The space–time configuration of these two Wilson lines is shown in Fig. 1.

Explicitly indicating the color indices $(i, j, \ldots)$ and the spin indices $(\alpha, \beta, \ldots)$ of the quarks, the scattering amplitude can be written as

$$M_{fi} = \langle \psi_{i\alpha}(p_1') \psi_{k\gamma}(p_2') | M | \psi_{j\beta}(p_1) \psi_{l\delta}(p_2) \rangle \sim -\frac{i}{Z_\psi^2} \cdot \delta_{ij} \delta_{\alpha\beta} \cdot 2s \int d^2 z_t e^{iq_z z_t} \langle [W_1(z_t) - 1]_{ij} [W_2(0) - 1]_{kl} \rangle_A, $$

where $Z_\psi$ is the vacuum quark wave function and $q_z$ is the transverse momentum. The expression (1.4) captures the essence of quark–quark scattering at high energies, emphasizing the role of Wilson lines and their transverse correlations.
where \( \langle \ldots \rangle_A \) is the average, in the sense of the functional integration, over the gluon field \( A^\mu \). \( Z_\psi \) is the fermion–field renormalization constant, which can be written in the eikonal approximation as

\[
Z_\psi \simeq \frac{1}{N_c} \langle \text{Tr}[W_1(z_t)] \rangle_A = \frac{1}{N_c} \langle \text{Tr}[W_1(0)] \rangle_A = \frac{1}{N_c} \langle \text{Tr}[W_2(0)] \rangle_A .
\]

(The two last equalities come from the Poincaré invariance of the theory.)

In a perfectly analogous way one can also derive the high–energy scattering amplitude in the case of the Abelian group \( U(1) \) (QED). The resulting amplitude is equal to Eq. (1.4), with the only obvious difference being that now the lightlike Wilson lines \( W_1 \) and \( W_2 \) are functionals of the Abelian field \( A^\mu \) (so they are not matrices). Thanks to the simple form of the Abelian theory (in particular to the absence of self–interactions among the vector fields), it turns out that it is possible to explicitly evaluate (at least in the quenched approximation) the expectation value of the two Wilson lines: the details of the calculation are reported in the Appendix of Ref. [4] and one finally recovers the well–known result for the eikonal amplitude of the high–energy scattering in QED [7] [8] [9].

From Eq. (1.4) it seems that the \( s \) dependence of the scattering amplitude is all contained in the kinematic factor \( 2s \) in front of the integral. In fact we can write

\[
M_{fi} = \langle \psi_{i\alpha}(p'_1)\psi_{k\gamma}(p'_2)|M|^\psi_{j\beta}(p_1)\psi_{l\delta}(p_2) \rangle \sim -i \cdot 2s \cdot \delta_{\alpha\beta}\delta_{\gamma\delta} \cdot g_{M(ij,kl)}(t, s) ,
\]

where, apparently, the quantity

\[
g_{M(ij,kl)}(t, s) \equiv \frac{1}{Z_\psi} \int d^2z_t e^{i\mathbf{q} \cdot \mathbf{z}_t} \langle [W_1(z_t) - 1]_{ij} [W_2(0) - 1]_{kl} \rangle_A
\]

only depends on \( t = -q^2 \). Yet, as was pointed out by Verlinde and Verlinde in [10], this is not true: in fact, one can easily be convinced (for example by making a perturbative expansion) that it is a singular limit to take the Wilson lines in (1.7) exactly lightlike. As suggested in [10], one can regularize this sort of “infrared” divergence by letting each line have a small timelike component, so that they coincide with the classical trajectories for quarks with a finite mass \( m \). Therefore, one first has to evaluate the quantity

\[
g_{M(ij,kl)}(t, \beta)
\]
for two Wilson lines along the trajectories of two quarks moving with velocity $\beta$ and $-\beta$ ($0 < \beta < 1$) along the $x^1$–axis. In other words, one first considers two infinite Wilson lines forming a certain hyperbolic angle $\chi$ in Minkowski space–time. Then, to obtain the correct high–energy scattering amplitude, one has to perform the limit $\beta \to 1$, that is $\chi \to \infty$, into the expression (1.8):

$$M_{fi} = \langle \psi_{i\alpha}(p'_1)\psi_{k\gamma}(p'_2)|M|\psi_{j\beta}(p_1)\psi_{l\delta}(p_2) \rangle \sim -i \cdot 2s \cdot \delta_{\alpha\beta}\delta_{\gamma\delta} \cdot g_{M(ijkl)}(t, \beta \to 1). \quad (1.9)$$

In this way one obtains a $\ln s$ dependence of the amplitude, as expected from ordinary perturbation theory [1] [2] and as confirmed by the experiments on hadron–hadron scattering processes. In Sect. 3 we shall see how this explicitly works by evaluating the amplitude (1.7) for QCD up to the order $O(g^4_R)$ in the perturbative expansion ($g_R$ being the renormalized coupling constant).

The direct evaluation of the expectation value (1.7) is a highly non–trivial matter, as it is also strictly connected with the ultraviolet properties of Wilson–line operators [1]. Recently, in Ref. [12], it has been found that there is a correspondence between high–energy asymptotics in QCD and renormalization properties of the so–called cross singularities of Wilson lines. The asymptotic behavior of the quark–quark scattering amplitude turns out to be controlled by a $2 \times 2$ matrix of the cross anomalous dimensions of Wilson lines. An alternative non–perturbative approach for the calculation of the expectation value (1.7) has been proposed in Ref. [13]. It consists in studying the Regge regime of large energies and fixed momentum transfers as a special regime of lattice gauge theory on an asymmetric lattice, with a spacing $a_0$ in the longitudinal direction and a spacing $a_t$ in the transverse direction, in the limit $a_0/a_t \to 0$.

At the moment, the only non–perturbative numerical estimate of (1.7), which can be found in the literature, is that of Ref. [14] (where it has been generalized to the case of hadron–hadron scattering): it has been obtained in the framework of the model of the stochastic vacuum (SVM). Before its application to high–energy scattering, the SVM must be translated from Euclidean space–time, in which it is naturally formulated, to the Minkowski continuum. As is claimed in Ref. [14], the more safe way (from the point of view of the functional integration) would be the other way, i.e., to continue the scattering amplitude from the Minkowski world to the Euclidean world.

In this paper we try to go just that way and adapt the scattering amplitude to the Euclidean world. More explicitly, we shall prove that the expectation value of two infinite
Wilson lines, forming a certain hyperbolic angle in Minkowski space–time, and the expectation value of two infinite Euclidean Wilson lines, forming a certain angle in Euclidean four–space, are connected by an analytic continuation in the angular variables. In Sect. 2 we shall first prove this for the Abelian case (QED), by explicitly evaluating the correlation of two infinite Wilson lines both in Minkowski space–time and in Euclidean four–space, using the so–called quenched approximation. Then, in Sect. 3, we shall prove that this result can be extended also to non–Abelian gauge theories. Finally, in the last section, we shall discuss some interesting consequences (such as the re–derivation of the Regge pole model [15]) and some possible direct applications, mostly for lattice gauge theories (LGT) and the stochastic vacuum model, of this relationship of analytic continuation.

2. The Abelian case

In this section we shall discuss the Abelian case (See also Refs. [12] and [13]). The fermion–fermion electromagnetic scattering amplitude, in the high–energy limit \( s \to \infty \) and \(|t| \ll s\), can be derived following the same procedure used in Ref. [4]. The resulting amplitude is formally identical to Eq. (1.4), with the only obvious difference being that now the Wilson lightlike lines \( W_1 \) and \( W_2 \) are functions of the Abelian field \( A^\mu \) (so they are not matrices):

\[
M_{fi} = \langle \psi_\alpha(p'_1)\psi_\gamma(p'_2)|M|\psi_\beta(p_1)\psi_\delta(p_2)\rangle \\
\sim \frac{i}{E_\psi} \cdot \delta_{\alpha\delta} \delta_{\gamma\beta} \cdot 2s \int d^2z_t e^{iqz_t} \langle [W_1(z_t) - 1][W_2(0) - 1] \rangle_A .
\]

The electromagnetic lightlike Wilson lines \( W_1 \) and \( W_2 \) are defined as in (1.3), after replacing \( g \) with \( e \), the electric coupling–constant (electric charge), and the gluon field with the photon field. Thanks to the simple form of the Abelian theory (in particular to the absence of self–interactions among the vector fields), it turns out that it is possible to explicitly evaluate the expectation value of the two Wilson lines, thus finally recovering the well–known result for the eikonal amplitude of the high–energy scattering in QED (see Refs. [7], [8] and [9]). The details of the calculation are reported in Ref. [4].

The Wilson lines in (2.1) are taken exactly lightlike. We shall now let each line to have a small timelike component, so that they coincide with the classical trajectories for
fermions with a finite mass \( m \). The electromagnetic lightlike Wilson lines \( W_1 \) and \( W_2 \) are now defined as

\[
W_1(z_t) = \exp \left[ -ie \int_{-\infty}^{+\infty} A_\mu (z_t + \frac{p_1^\mu}{m} \tau) \frac{p_1^\mu}{m} d\tau \right],
\]

\[
W_2(0) = \exp \left[ -ie \int_{-\infty}^{+\infty} A_\mu (\frac{p_2^\mu}{m} \tau) \frac{p_2^\mu}{m} d\tau \right],
\]

(2.2)

where \( m \) is the mass of the fermions and \( p_1^\mu \) and \( p_2^\mu \) are the two four–momenta defining the two trajectories 1 and 2 in Minkowski space–time:

\[
X^\mu_{(1)}(\tau) = z^\mu_t + \frac{p_1^\mu}{m} \tau,
\]

\[
X^\mu_{(2)}(\tau) = \frac{p_2^\mu}{m} \tau.
\]

(2.3)

In the c.m.s. of the two particles, taking the spatial momenta \( p_1 \) and \( p_2 = -p_1 \) along the \( x^1 \)–direction, the two four–momenta \( p_1 \) and \( p_2 \) are

\[
p_1^\mu = E(1, \beta, 0, t),
\]

\[
p_2^\mu = E(1, -\beta, 0, t),
\]

(2.4)

where \( \beta \) is the velocity (in the units with \( c = 1 \)) and \( E = m/\sqrt{1 - \beta^2} \) is the energy of each particle (so that: \( s = 4E^2 \)).

We shall evaluate the expectation value \( \langle W_1(z_t)W_2(0) \rangle_A \) in the so–called \textit{quenched} approximation, where vacuum polarization effects, arising from the presence of loops of dynamical fermions, are neglected. This amounts to setting \( \text{det}(K[A]) = 1 \), where \( K[A] = i\gamma^\mu D_\mu - m \) is the fermion matrix. Thus we can write that

\[
\langle W_1(z_t)W_2(0) \rangle_A \simeq \frac{1}{Z} \int [dA] e^{iS_A} W_1(z_t)W_2(0),
\]

(2.5)

where \( S_A = -\frac{i}{4} \int d^4xF_{\mu\nu}F^{\mu\nu} \) is the action of the electromagnetic field and \( Z = \int [dA] e^{iS_A} \) is the pure–gauge partition function. We then add to the pure–gauge Lagrangian \( L_A = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \) a gauge–fixing term \( L_{GF} = -\frac{1}{2\alpha} (\partial^\mu A_\mu)^2 \) (covariant or Lorentz gauge). The expectation value (2.5) becomes, denoting \( L^F_0 = L_A + L_{GF} \),

\[
\langle W_1(z_t)W_2(0) \rangle_A \simeq \frac{1}{Z'} \int [dA] \exp \left[ i \int d^4x (L^F_0 + A_\mu J^\mu) \right],
\]

(2.6)
where $Z' = \int [dA] \exp \left( i \int d^4x L_0^F \right)$ and $J^\mu(x)$ is a four–vector source defined as

$$J^\mu(x) = J_{(1)}^\mu(x) + J_{(2)}^\mu(x) = -e[\epsilon_{(1)}^\mu(x - \beta t)\delta(x_t - z_t) + \epsilon_{(2)}^\mu(x + \beta t)\delta(x_t)] ,$$  \hspace{1cm} (2.7)$$

with $\epsilon_{(1)}^\mu = (1, \beta, 0, 0)$ and $\epsilon_{(2)}^\mu = (1, -\beta, 0, 0)$. The functional integral

$$Z_0[J] \equiv \int [dA] \exp \left[ i \int d^4x (L_0^F + A_\mu J^\mu) \right]$$  \hspace{1cm} (2.8)$$

can be evaluated with standard methods (completing the quadratic form in the exponent). One thus obtains

$$Z_0[J] = Z' \cdot \exp \left[ \frac{i}{2} \int d^4x \int d^4y J^\mu(x) D_{\mu\nu}(x - y) J^\nu(y) \right] ,$$  \hspace{1cm} (2.9)$$

where $D_{\mu\nu}$ is the free photon propagator (apart from a factor $-i$):

$$D_{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \left( g_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right) .$$  \hspace{1cm} (2.10)$$

In the following we shall choose the gauge–fixing parameter $\alpha$ equal to 1 (Feynman gauge). From Eqs. (2.6) ÷ (2.10), we derive the following expression for the expectation value (2.5) of the two Wilson lines:

$$\langle W_1(z_t) W_2(0) \rangle_A \simeq \langle W_1(z_t) \rangle_A \langle W_2(0) \rangle_A$$

$$\times \exp \left[ i \int d^4x \int d^4y J_{(1)}^\mu(x) J_{(2)\mu}(y) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \right] .$$  \hspace{1cm} (2.11)$$

Therefore, using the explicit form (2.7) of the four–vector source $J^\mu(x)$ to evaluate the double integral in Eq. (2.11), one finds that (in the quenched approximation)

$$\frac{\langle W_1(z_t) W_2(0) \rangle_A}{\langle W_1 \rangle_A} \simeq \exp \left[ -ie^2 \left( \frac{1 + \beta^2}{2\beta} \right) \int \frac{d^2k_t}{(2\pi)^2} \frac{e^{-ik_t \cdot z_t}}{k_t^2 - i\epsilon} \right] .$$  \hspace{1cm} (2.12)$$

We have made use of the Poincaré invariance of the theory to write: $\langle W_1 \rangle_A \equiv \langle W_1(z_t) \rangle_A = \langle W_1(0) \rangle_A = \langle W_2(0) \rangle_A$. We now introduce the hyperbolic angle $\psi$ [in the plane $(x^0, x^1)$]
of the trajectory 1 in Eq. (2.3), i.e., \( \Delta X^0_1 = \beta \Delta X^0_1 \), so that, if \( \Delta l \) is the line–distance \( (\Delta l^2 = (\Delta X^0_1)^2 - (\Delta X^1_1)^2 = (\Delta X^0_1)^2(1 - \beta^2)) \) we have

\[
\begin{align*}
\Delta X^0_1 &= \Delta l \cosh \psi , \\
\Delta X^1_1 &= \Delta l \sinh \psi ,
\end{align*}
\]

and, therefore, \( \beta = \tanh \psi \). With this notation, it is immediate to recognize that the \( \beta \)–dependent factor in front of Eq. (2.12) is equal to \( \coth(2\psi) \); so that

\[
\langle W_1(z_t)W_2(0) \rangle_A \langle W_1^2 \rangle_A \simeq \exp \left[ -i e^2 \coth \chi \cdot \int \frac{d^2 k_t}{(2\pi)^2} \frac{e^{-ik_t \cdot z_t}}{k_t^2 - i\varepsilon} \right],
\]

(2.14)

where \( \chi = 2\psi \) is hyperbolic angle [in the plane \( (x^0, x^1) \)] between the two trajectories 1 and 2 of the two colliding particles. The exponential in the last equation turns out to be equal to

\[
\exp \left( -i \frac{e^2}{2\pi} \coth \chi \cdot \ln |z| \right),
\]

(2.15)

where \( \Lambda \) is an infinite constant phase and is therefore physically unobservable. The origin of this infinite constant phase resides in the fact that the fermion–fermion scattering amplitude in QED has infrared (IR) divergences, due to the emission of low–energy massless vector mesons. The traditional way to handle these IR divergences is to introduce an IR cutoff in the form of a vector meson mass \( \lambda \). In this way the integral over \( k_t \) in the exponent of Eq. (2.14) is substituted by the expression

\[
\int \frac{d^2 k_t}{(2\pi)^2} \frac{e^{-ik_t \cdot z_t}}{k_t^2 + \lambda^2} \equiv \frac{1}{2\pi} K_0(\lambda |z_t|),
\]

(2.16)

where \( K_0 \) is the modified Bessel function. In the limit of small \( \lambda \) this last expression can be replaced by

\[
\frac{1}{2\pi} K_0(\lambda |z|) \sim \frac{1}{2\pi} \ln \left( \frac{1}{2} e^\gamma \lambda |z_t| \right).
\]

(2.17)

Absorbing \( \frac{1}{2} e^\gamma \) in \( \lambda \) and putting \( \Lambda = -(1/2\pi) \ln(1/2 e^\gamma) \) (so that \( \Lambda \to \infty \) when \( \lambda \to 0 \), we just obtain the expression (2.15) for the exponential in Eq. (2.14).

We can now repeat the above procedure and evaluate the quantity (2.5) in the Euclidean theory. The electromagnetic lightlike Euclidean Wilson lines \( W_{E1} \) and \( W_{E2} \) are
defined as in (2.2):

\[
W_{E1}(\bar{z}_t) = \exp \left[ -ie \int_{-\infty}^{+\infty} A^{(E)}_{\mu}(\bar{z}_t + v_1 \tau) v_{1\mu} d\tau \right],
\]

\[
W_{E2}(0) = \exp \left[ -ie \int_{-\infty}^{+\infty} A^{(E)}_{\mu}(v_2 \tau) v_{2\mu} d\tau \right],
\]

(2.18)

where now \(v_{1\mu}\) and \(v_{2\mu}\) are the Euclidean four–vectors [lying in the plane \((x_1, x_4)\)] defining the two trajectories 1 and 2 in Euclidean four–space:

\[
X^{(1)}_{E\mu}(\tau) = \bar{z}_{t\mu} + v_{1\mu} \tau,
\]

\[
X^{(2)}_{E\mu}(\tau) = v_{2\mu} \tau,
\]

(2.19)

and \(\bar{z}_{t\mu} = (z_1, z_2, z_3, z_4) = (0, \bar{z}_t, 0)\). We can choose \(v_1\) and \(v_2\) normalized to 1: \(v_1^2 = v_2^2 = 1\). Moreover, due to the \(O(4)\) symmetry of the theory, we can choose the c.m.s. of the two particles, taking the spatial momenta \(v_1\) and \(v_2 = -v_1\) along the \(x_1\)–direction. The two four–momenta \(v_1\) and \(v_2\) are, therefore,

\[
v_{1\mu} = (\sin \phi, 0_t, \cos \phi),
\]

\[
v_{2\mu} = (- \sin \phi, 0_t, \cos \phi),
\]

(2.20)

where \(\phi\) is the angle formed by each trajectory with the \(x_4\)–axis. As before, we can evaluate the expectation value \(\langle W_{E1}(\bar{z}_t)W_{E2}(0) \rangle_A\) (where now \(\langle \ldots \rangle_A\) is the functional integral with respect to the gluon field \(A^{(E)}_{\mu}\) in the Euclidean theory) in the quenched approximation. Thus we have

\[
\langle W_{E1}(\bar{z}_t)W_{E2}(0) \rangle_A \simeq \frac{1}{Z_E} \int [dA^{(E)}_{\mu}] e^{-S^{(E)}_A} W_{E1}(\bar{z}_t)W_{E2}(0),
\]

(2.21)

where \(S^{(E)}_A = \frac{1}{4} \int d^4x E^{(E)}_{\mu\nu} F^{(E)}_{\mu\nu}\) is the Euclidean action of the electromagnetic field and \(Z_E = \int [dA^{(E)}_{\mu}] e^{-S^{(E)}_A}\) is the corresponding pure–gauge partition function. As usually, we add to the pure–gauge Lagrangian \(L^{(E)}_A = \frac{1}{4} F^{(E)}_{\mu\nu} F^{(E)}_{\mu\nu}\) a gauge–fixing term \(L^{(E)}_{GF} = \frac{1}{2\alpha} (\partial_{\mu} A^{(E)}_{\mu})^2\) (covariant or Lorentz gauge). The expectation value (2.21) becomes, denoting \(L^{(E)}_0 = L^{(E)}_A + L^{(E)}_{GF}\),

\[
\langle W_{E1}(\bar{z}_t)W_{E2}(0) \rangle_A \simeq \frac{1}{Z'_E} \int [dA^{(E)}] \exp \left[ - \int d^4x (L^{(E)}_0 + iA^{(E)}_{\mu} J_{E\mu}) \right],
\]

(2.22)
where $Z'_E = \int [dA^{(E)}] \exp \left( - \int d^4x E L_0^{(E)} \right)$ and $J_{E\mu}(x_E)$ is a four–vector source defined as

$$J_{E\mu}(x_E) = J_{E\mu}^{(1)}(x_E) + J_{E\mu}^{(2)}(x_E) = e [v_{1\mu} \delta(x_{E1} \cos \phi - x_{E4} \sin \phi) \delta(x_{E}) - z_t) + v_{2\mu} \delta(x_{E1} \cos \phi + x_{E4} \sin \phi) \delta(x_{E1})].$$

(2.23)

The functional integral

$$Z_0^{(E)}[J_E] \equiv \int [dA^{(E)}] \exp \left[ - \int d^4x E L_0^{(E)} + i A^{(E)}_\mu J_{E\mu} \right]$$

(2.24)

can be evaluated with standard methods (completing the quadratic form in the exponent).

One thus obtains

$$Z_0^{(E)}[J_E] = Z'_E \cdot \exp \left[ - \frac{i}{2} \int d^4x E \int d^4y E J_{E\mu}(x_E) D_{\mu\nu}^{(E)}(x_E - y_E) J_{E\nu}(y_E) \right],$$

(2.25)

where $D_{\mu\nu}^{(E)}$ is the free photon Euclidean propagator, i.e.,

$$D_{\mu\nu}^{(E)}(x_E - y_E) = \int \frac{d^4k_E}{(2\pi)^4} \frac{e^{-ik_E(x_E - y_E)}}{k_E^2} \left( \delta_{\mu\nu} - (1 - \alpha) \frac{k_{E\mu} k_{E\nu}}{k_E^2} \right).$$

(2.26)

In the following we shall choose the gauge–fixing parameter $\alpha$ equal to 1 (Feynman gauge). From Eqs. (2.22) $\div$ (2.26), we derive the following expression for the expectation value (2.21) of the two Euclidean Wilson lines (including the regulating IR cutoff in the form of a small photon mass $\lambda$, which must be put equal to zero at the end of the calculation):

$$\langle W_{E1}(\bar{z}_t)W_{E2}(0) \rangle_A \simeq \langle W_{E1}(\bar{z}_t) \rangle_A \langle W_{E2}(0) \rangle_A \times \exp \left[ - \int d^4x E \int d^4y E \int J_{E\mu}^{(1)}(x_E) J_{E\nu}^{(2)}(y_E) \int \frac{d^4k_E}{(2\pi)^4} \frac{e^{-ik_E(x_E - y_E)}}{k_E^2} \right].$$

(2.27)

Finally, making use of the explicit form (2.23) of the four–vector source $J_{E\mu}(x_E)$ to evaluate the double integral in Eq. (2.27), and using also the $O(4)$ plus translation invariance of the Euclidean theory to write: $\langle W_{E1} \rangle_A \equiv \langle W_{E1}(\bar{z}_t) \rangle_A = \langle W_{E1}(0) \rangle_A = \langle W_{E2}(0) \rangle_A$, one finds the result

$$\frac{\langle W_{E1}(\bar{z}_t)W_{E2}(0) \rangle_A}{\langle W_{E1} \rangle_A^2} \simeq \exp \left[ -e^2 \cot \theta . \int \frac{d^2k_t}{(2\pi)^2} \frac{e^{-ik_t \cdot \bar{z}_t}}{k_t^2 + \lambda^2} \right].$$

(2.28)
We have indicated with $\theta \equiv 2\phi$ the angle [in the plane $(x_1, x_4)$] between the two trajectories in Euclidean four–space. The angle $\theta$ in Eq. (2.28) is taken in the interval $[0, \pi]$: it is always possible to make such a choice by virtue of the $O(4)$ symmetry of the Euclidean theory. When comparing the two expressions (2.14) and (2.28), we immediately recognize that they are linked by the following analytic continuation in the angular variables:

$$\langle W_E(\bar{z}_1(\bar{t})) W_E(0) \rangle_A \langle W_E(\bar{z}_1(0)) W_E(0) \rangle_A \rightarrow \theta \rightarrow -i\chi, \langle W_{E1}(\bar{z}_t) W_{E2}(0) \rangle_A \langle W_{E1}(\bar{z}_t) W_{E2}(0) \rangle_A \rightarrow \chi \rightarrow \infty (i.e., \beta \rightarrow 1).$$

(2.29)

This allows to reconstruct the high–energy scattering amplitude by evaluating a correlation of infinite Wilson lines in the Euclidean world, then by continuing this quantity in the angular variable, $\theta \rightarrow -i\chi$, and finally by performing the limit $\chi \rightarrow \infty$.

In fact, from Eq. (2.1) we can write

$$M_{fi} = \langle \psi_\alpha(p'_1) \psi_\beta(p_1) | M | \psi_\gamma(p'_2) \psi_\delta(p_2) \rangle \sim s \rightarrow \infty -i \cdot 2s \cdot \delta_\alpha_\beta \delta_\gamma_\delta \cdot g_M(t, \chi \rightarrow \infty),$$

(2.30)

where the quantity $g_M(t, \chi)$ is defined as

$$g_M(t, \chi) = \int d^2z_te^{i\bar{q} \cdot \bar{z}_t} \frac{\langle [W_1(\bar{z}_t) - 1][W_2(0) - 1] \rangle_A}{\langle W_{E1}^2 \rangle_A}. $$

(2.31)

It was shown in Ref. [3] that, in the eikonal approximation, one can approximate the fermion–field renormalization constant as follows:

$$Z_\psi \simeq \langle W_1 \rangle_A. $$

(2.32)

Therefore $g_M(t, \chi \rightarrow \infty)$ is exactly the Abelian version of the asymptotic amplitude (1.7). Moreover, whenever $t$ is not exactly equal to zero, i.e., $\bar{q} \neq 0 (t = -\bar{q}^2 < 0)$, the expression (2.31) reduces to

$$g_M(t, \chi) = \int d^2z_te^{i\bar{q} \cdot \bar{z}_t} \frac{\langle W_1(\bar{z}_t) W_2(0) \rangle_A}{\langle W_{E1}^2 \rangle_A}. $$

(2.33)

Therefore $g(t, \chi)$ turns out to be the Fourier transform, with respect to the transverse coordinates $z_t$, of the quantity (2.14), which we have evaluated in the quenched approximation. In the Euclidean theory we can define the corresponding quantity

$$g_E(t, \theta) = \int d^2z_te^{i\bar{q} \cdot \bar{z}_t} \frac{\langle W_{E1}(\bar{z}_t) W_{E2}(0) \rangle_A}{\langle W_{E1}^2 \rangle_A}. $$

(2.34)
This is just the Fourier transform, with respect to the transverse coordinates $z_t$, of the quantity (2.28), which we have evaluated in the quenched approximation. Using the relation (2.29), we can derive that

$$g_E(t, \theta) \rightarrow -i \chi g_E(t, -i \chi) = g_M(t, \chi) ;$$

or:

$$g_M(t, \chi) \rightarrow i \theta g_M(t, i \theta) = g_E(t, \theta) . \quad (2.35)$$

3. The case of QCD at order $O(g^4 R)$

In this section we shall see that the same relationship of analytic continuation appears to be valid also for the case of a non–Abelian gauge theory: we shall prove this up to the order $O(g^4 R)$ in perturbation theory ($g_R$ being the renormalized coupling constant). Let us consider the following quantity, defined in Minkowski space–time:

$$g_M(t, p_1 \cdot p_2) = \frac{1}{Z_W^2} \int d^2 z_t e^{iq \cdot z_t} \langle [W_1(z_t) - 1]_{ij} [W_2(0) - 1]_{kl} \rangle_A , \quad (3.1)$$

where $p_1$ and $p_2$ are the four–momenta [lying (for example) in the plane $(x^0, x^1)$], which define the two Wilson lines $W_1$ and $W_2$. By virtue of the Lorentz symmetry, we can define $p_1$ and $p_2$ as in Eq. (2.4): that is, we choose, as the reference frame, the c.m.s. of the two particles, moving with speed $\beta$ and $-\beta$ along the $x^1$–direction. Of course, $g_M$ can only depend on the scalar quantities constructed with the vectors $p_1$, $p_2$ and $q = (0, 0, q)$: the only possibilities are $q^2 = -q^2 = t$ and $p_1 \cdot p_2$, because $p_1 \cdot q = p_2 \cdot q = 0$ and $p_1^2 = p_2^2 = m^2$ are fixed. In such a reference frame, we can write $p_1 \cdot p_2 = m^2 \cosh \chi$, where $\chi = 2\psi$ (with $\beta = \tanh \psi$) is the hyperbolic angle [in the plane $(x^0, x^1)$] between the two Wilson lines $W_1$ and $W_2$. The Wilson lines are defined as

$$W_1(z_t) \equiv P \exp \left[ -ig \int_{-\infty}^{+\infty} A_\mu(z_t + \frac{p_1}{m} \tau) \frac{p_1^\mu}{m} d\tau \right] ;$$

$$W_2(0) \equiv P \exp \left[ -ig \int_{-\infty}^{+\infty} A_\mu(\frac{p_2}{m}) \frac{p_2^\mu}{m} d\tau \right] . \quad (3.2)$$
Moreover, we have put, in Eq. (3.1),

\[ Z_W \equiv \frac{1}{N_c} \langle \text{Tr}[W_1(z_t)] \rangle_A = \frac{1}{N_c} \langle \text{Tr}[W_1(0)] \rangle_A = \frac{1}{N_c} \langle \text{Tr}[W_2(0)] \rangle_A . \]  

(3.3)

(The two last equalities come from the Poincaré invariance.) This is a sort of Wilson-line’s renormalization constant: as shown in Ref. [3], \( Z_W \) coincides with the fermion renormalization constant \( Z_\psi \) in the eikonal approximation. We want to explicitly evaluate the quantity (3.1) up to the order \( O(g^4_R) \) in perturbation theory. This corresponds to evaluate the Feynman diagrams in Figs. 2 and 3, where the two horizontal oriented lines represent the Wilson lines \( W_1 \) and \( W_2 \). First of all we need to expand \( Z_W \) up to the order \( O(g^2_R) \) in perturbation theory:

\[ Z_W = 1 + Z_W^{(2)} g^2_R + O(g^4_R) . \]  

(3.4)

Clearly, we do not need to consider also the \( O(g^4_R) \) piece, of the form \( Z_W^{(4)} g^4_R \), since the expectation value \( \langle \ldots \rangle_A \) in Eq. (3.1) is an object of order \( O(g^2_R) \). As will become clear in the following, we do not need to know the explicit expression for the coefficient \( Z_W^{(2)} \). Since we are interested in evaluating the quantity (3.1) up to the order \( O(g^4_R) \), we need to consider also the effects of the renormalizations of the fields and the coupling constant \( g \), up to the order \( O(g^2_R) \). Using the conventional notation, we write

\[ A^a_\mu = Z_3^{1/2} A^a_{R\mu} \; ; \; \; \; g = Z_g g_R , \]  

(3.5)

where the suffix “\( R \)” denotes the renormalized quantities. Therefore, we have that

\[ W_i(z_t) = P \exp \left[ -iZ_{1W} g_R \int_{-\infty}^{+\infty} A_{R\mu}(z_t + \frac{p_1}{m} \tau, \frac{p_1^\mu}{m} d\tau \right] , \]  

(3.6)

where the renormalization constant \( Z_{1W} \) is defined as

\[ Z_{1W} \equiv Z_g Z_3^{1/2} = 1 + Z_{1W}^{(2)} g^2_R + O(g^4_R) . \]  

(3.7)

Since we are interested in evaluating the amplitude

\[ M(t, \chi) = \int d^2 z_t e^{i q \cdot z_t} \langle [W_1(z_t) - 1]_{ij} [W_2(0) - 1]_{kl} \rangle_A , \]  

(3.8)
up to the order $O(g_4^R)$, the effects of $Z_{1W}$ are visible when we expand the Wilson lines $W_i$ only up to the first order in $g$:

$$W_i(z_t) = 1 - iZ_{1W}g_R \int_{-\infty}^{+\infty} A_{\mu}(z_t + \frac{p_1}{m}) \frac{p_1^\mu}{m} d\tau + \ldots .$$  \hspace{1cm} (3.9)$$

This corresponds to consider only the diagrams of the one–gluon–exchanged type, having the following amplitude:

$$M_{(1,1)} = Z_{1W}^2 M_{R(1,1)} = M_{R(1,1)} + (Z_{1W}^2 - 1) M_{R(1,1)} ,$$  \hspace{1cm} (3.10)$$

where $M_{R(1,1)}$ is the “renormalized” one–gluon–exchanged amplitude, obtained from $M_{(1,1)}$ by substituting the coupling constant $g$ in front and the gluon field $A^\mu$ with the corresponding renormalized quantities:

$$M_{R(1,1)} = -g_R^2 (T^a)_{ij} (T^b)_{kl} \frac{P_1^\mu P_2^\nu}{m^2} \int d^2 z d^4 k (2\pi)^4 \frac{i}{4} e^{ik(z_t + p_1^\mu m)} A_{R\mu}^a(z_t + \frac{p_1}{m}) A_{R\nu}^b(p_2^\mu \omega) A.$$  \hspace{1cm} (3.11)$$

In our notation, $M_{(i,j)}$ denotes the contribution to the amplitude $M$, defined in Eq. (3.8), obtained after expanding the Wilson lines $W_1$ up to the order $O(g^i)$ (i.e., up to the term containing $i$ gluon fields) and expanding the other Wilson line $W_2$ up to the order $O(g^j)$ (i.e., up to the term containing $j$ gluon fields). Moreover, we define: $M_{R(i,j)} = Z_{1W}^{-(i+j)} M_{(i,j)}$. If one wants to compute $M_{(1,1)}$ up to the order $O(g_4^R)$, one must proceed as follows, by virtue of Eqs. (3.10) and (3.4):

$$M_{(1,1)}|_{g_4^R} = M_{R(1,1)}|_{g_4^R} + 2Z_{1W}^2 g_R^2 \cdot M_{R(1,1)}|_{g_R^2} .$$  \hspace{1cm} (3.12)$$

The expression for $M_{R(1,1)}|_{g_R^2}$ can be immediately derived: it corresponds to the diagram shown in Fig. 2(a). In a given Lorentz gauge with a (bare) gauge parameter $\alpha$, the free gluon–field propagator is given by

$$G_{\mu\nu}^{ab}(x - y) = -i\delta_{ab} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + i\varepsilon} \left[ g_{\mu\nu} - (1 - \alpha) \frac{k_{\mu} k_{\nu}}{k^2 + i\varepsilon} \right] e^{-ik(x - y)} .$$  \hspace{1cm} (3.13)$$

One thus finds that

$$M^{(a)}(t,\chi) = M_{R(1,1)}(t,\chi)|_{g_4^R} = g_R^2 \frac{1}{t} i \coth \chi \cdot (G_1)_{ij,kl} ,$$  \hspace{1cm} (3.14)$$
where \( G_1 \) is the color factor for the one–gluon–exchanged process:

\[
G_1 \equiv T^c_{(1)} \otimes T^c_{(2)},
\]

\[
(G_1)_{ij,kl} \equiv (T^c)_{ij} (T^c)_{kl}.
\]

(3.15)

Let us observe that \( M_{R(1,1)}|_{g_R^2} \) is gauge–independent, since the gauge parameter \( \alpha \) does not appear at the right–hand–side of Eq. (3.14). The last term of Eq. (3.12) can be represented by the diagrams in Figs. 3(q) and 3(r), in which we have put a counterterm of the form \(-iZ^{(2)}_{1W} g_R^3 (T^a)_{ij}\) in one of the two vertices between the gluon line and a Wilson line. So we have that

\[
M^{(q)}(t, \chi) = M^{(r)}(t, \chi) = Z^{(2)}_{1W} g_R^4 \cdot \frac{1}{t} \coth \chi \cdot (G_1)_{ij,kl}.
\]

(3.16)

The expression for the one–gluon–exchanged renormalized amplitude up to the order \( O(g_R^4) \), i.e., \( M_{R(1,1)}|_{g_R^4} \), is given by the sum of the contributions from the diagrams shown in Figs. 2(a), 3(l) to 3(p): this last one represents the insertion of a counterterm \((Z_3 - 1)\delta_{ab}(k^\mu k^\nu - g^{\mu\nu} k^2)\) into the gluon line. In other words, one has to compute the quantity (3.11), using the renormalized gluon propagator up to the order \( O(g_R^2) \) when evaluating the expectation value \( \langle A^a_{\mu}(z_t + \frac{\not{p}}{m} \tau) A^b_{\nu}(\frac{\not{p}}{m} \omega) \rangle_A \). That is, in a given Lorentz gauge with a renormalized gauge parameter \( \alpha_R = Z_3^{-1} \alpha \):

\[
\langle A^a_{\mu}(x) A^b_{\nu}(y) \rangle_A = -i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \tilde{D}^{ab}_{R\mu\nu}(k),
\]

(3.17)

where \( \tilde{D}^{ab}_{R\mu\nu}(k) \) is given by

\[
\tilde{D}^{ab}_{R\mu\nu}(k) = Z_3^{-1} \tilde{D}^{ab}_{\mu\nu}(k) = \frac{\delta_{ab}}{k^2 + i\varepsilon} \left[ \frac{g_{\mu\nu} - k_{\mu} k_{\nu}}{k^2 + i\varepsilon} + \alpha_R \frac{k_{\mu} k_{\nu}}{k^2 + i\varepsilon} \right].
\]

(3.18)

\( \Pi_R(k^2) \) is a finite function of order \( O(g_R^4) \), whose precise form depends on the renormalization scheme which has been adopted:

\[
\Pi_R(k^2) = g_R^2 F^{(2)}(k^2) + O(g_R^4).
\]

(3.19)
At this point one can derive the full expression for the amplitude $M_{R(1,1)}$:

$$M_{R(1,1)}(t, \chi) = \frac{M_{R(1,1)}(t, \chi)|_{g_R^2}}{1 + \Pi_R(t)} = g_R^2 \frac{1}{t[1 + \Pi_R(t)]} i \coth \chi \cdot (G_1)_{ij,kl} . \tag{3.20}$$

Let us observe that, differently from $M_{R(1,1)}|_{g_R^2}$, the value of $M_{R(1,1)}$ is gauge–dependent, since the gauge parameter $\alpha_R$ does appear inside $\Pi_R$ at the right–hand–side of Eq. (3.20). This is also generally true for the other diagrams in Figs. (2) and (3). Therefore, for the following calculations, we shall fix the gauge parameter $\alpha_R$ to 1 (the so–called Feynman gauge). Eq. (3.20) is the full expression for $M_{R(1,1)}$, not truncated at any perturbative order. Yet, we are only interested in the expression for $M_{R(1,1)}$ up to the order $O(g^4_R)$:

$$M_{R(1,1)}(t, \chi)|_{g_R^4} = g_R^4 \frac{1}{t[1 - g_R^2 F^{(2)}(t)]} i \coth \chi \cdot (G_1)_{ij,kl} . \tag{3.21}$$

Therefore, the contribution coming from the $O(g^4_R)$ diagrams shown in Figs. 3(l) to 3(p) is given by

$$M^{(l)}(t, \chi) + \ldots + M^{(p)}(t, \chi) = -g_R^4 \frac{F^{(2)}(t)}{t} i \coth \chi \cdot (G_1)_{ij,kl} . \tag{3.22}$$

The contribution of order $O(g^4_R)$ coming from the two Feynman diagrams shown in Figs. 2(b) and 2(c) is obtained by multiplying the two pieces of order $O(g^2)$ for each of the two terms $W_i - 1$ in Eq. (3.8): i.e., we must evaluate the contribution $M_{(2,2)}|_{g_R^2}$. The contribution $M^{(b)}$ coming from the graph in Fig. 2(b) is conventionally called ladder term, while the other contribution $M^{(c)}$, coming from the graph in Fig. 2(c), will be called cross term. These two contributions can be evaluated in perturbation theory and the final result is (in the Feynman gauge, where $\alpha_R = 1$)

$$M^{(b)}(t, \chi) + M^{(c)}(t, \chi) =$$

$$= M_{(2,2)}(t, \chi)|_{g_R^2} = M^{(G_1)}(t, \chi) \cdot (G_1)_{ij,kl} + M^{(G_2)}(t, \chi) \cdot (G_2)_{ij,kl} , \tag{3.23}$$

where $G_1$ has been already defined in Eq. (3.15) and $G_2$ is the color factor for the ladder process in Fig. 2(b), i.e.,

$$G_2 \equiv (T^a_{(1)} T^b_{(1)}) \otimes (T^a_{(2)} T^b_{(2)}) ,$$

$$(G_2)_{ij,kl} \equiv (T^a T^b)_{ij} (T^a T^b)_{kl} . \tag{3.24}$$
In writing Eq. (3.23), we have made use of the following relation for the color factors:

\[(T^a T^b)_{ij} (T^b T^a)_{kl} = (T^a T^b)_{ij} (T^a T^b)_{kl} + \frac{N_c}{2} (T^c)_{ij} (T^c)_{kl} \equiv (G_2)_{ij,kl} + \frac{N_c}{2} (G_1)_{ij,kl}, \]  

(3.25)

which can be easily recovered using the algebra of the generators $T^a$, i.e., $[T^a, T^b] = i f_{abc} T^c$; $N_c$ is the number of colors of the theory (the gauge group is $SU(N_c)$). The coefficients $M^{(G_1)}(t, \chi)$ and $M^{(G_2)}(t, \chi)$ in front of the color factors in Eq. (3.23) are found to be

\[M^{(G_1)}(t, \chi) = i N_c g_R^4 I(t) \chi \coth^2 \chi; \]
\[M^{(G_2)}(t, \chi) = -\frac{1}{2} g_R^4 I(t) \coth^2 \chi. \]  

(3.26)

In the previous expression we have adopted the conventional notation

\[I(t) = \int \frac{d^2 k_t}{(2\pi)^2} \frac{1}{k_t^2 + \lambda^2} \frac{1}{(q - k_t)^2 + \lambda^2} \]
\[= \int d^2 z_t e^{\frac{\lambda}{2} (q - k_t)^2} \left( \frac{d^2 k_t}{(2\pi)^2} \frac{e^{-i k_t \cdot z_t}}{k_t^2 + \lambda^2} \right)^2. \]  

(3.27)

(Remember that: $t = -q^2$). The quantity $\lambda$ is the usual regularizing gluon mass, used as an IR cutoff: it must be put equal to zero at the end of the calculation.

We shall now compute the contribution from the diagrams in Figs. 2(d) to 2(i). They are $O(\alpha_0^4 R)$ diagrams, obtained after expanding one of the two Wilson lines up to the order $O(\alpha_0^3)$, and the remaining one up the first order in $\alpha$. We shall denote the sum of all these contributions by $M^{(3,1)}|g_R^4 + M^{(1,3)}|g_R^4$, in agreement with the notation introduced above. One thus finds that

\[M^{(3,1)}|g_R^4 = Z_W^{(2)} g_R^2 \cdot M_{R(1,1)}|g_R^2 + \Delta M^{(3,1)}; \]
\[M^{(1,3)}|g_R^4 = Z_W^{(2)} g_R^2 \cdot M_{R(1,1)}|g_R^2 + \Delta M^{(1,3)}; \]  

(3.28)

where $\Delta M^{(3,1)}$ and $\Delta M^{(1,3)}$ are divergent quantities, whose regularized expressions depend on the adopted renormalization scheme. In the minimal subtraction (MS) renormalization scheme one finds that

\[\Delta M^{(3,1)} = \Delta M^{(1,3)} = M_{R(1,1)}|g_R^2 \frac{g_R^2}{(4\pi)^2} N_c \left[ \frac{1}{\epsilon} + B \right], \]  

(3.29)
where \( \varepsilon = (4 - D)/2 \), \( D \) being the number of space–time dimensions, and \( B \) is a finite number (as \( \varepsilon \) goes to zero). In the same renormalization scheme (MS), one also has that (always in the Feynman gauge \( \alpha_R = 1 \)):

\[
Z_{1W} = 1 + Z_{1W}^{(2)} g_R^2 + O(g_R^4) , \quad \text{with:} \quad Z_{1W}^{(2)} = -\frac{g_R^2}{(4\pi)^2} N_c \frac{1}{\varepsilon} . \quad (3.30)
\]

Therefore, from Eqs. (3.29) and (3.30), one immediately concludes that

\[
\Delta M_{(3,1)} + \Delta M_{(1,3)} + 2Z_{1W}^{(2)} g_R^2 \cdot M_{R(1,1)}|_{g_R^2} = M_{R(1,1)}|_{g_R^2} \cdot \frac{g_R^2}{(4\pi)^2} 2N_c B . \quad (3.31)
\]

In other words, the divergence contained in \( \Delta M_{(3,1)} + \Delta M_{(1,3)} \) is exactly cancelled out by the two diagrams with the counterterm \( Z_{1W}^{(2)} \), represented in Figs. 3(q) and 3(r). Finally, we have to evaluate the two diagrams in Figs. 3(j) and 3(k): in agreement with the notation introduced above, we shall denote their contribution by \( M_{(2,1)}|_{g_R^4} \) and \( M_{(1,2)}|_{g_R^4} \), respectively. However, explicit calculations show that their contribution vanishes:

\[
M_{(2,1)}|_{g_R^4} = M_{(1,2)}|_{g_R^4} = 0 . \quad (3.32)
\]

At this point we can sum up all the contributions previously evaluated in order to find the complete expression for the amplitude \( M \), defined by Eq. (3.8), up to the order \( O(g_R^4) \). We find that

\[
M(t, \chi)|_{g_R^4} = \left[ 1 + \left( 2Z_{W}^{(2)} - F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} \right) \cdot \frac{g_R}{2} \right] \cdot M_{R(1,1)}(t, \chi)|_{g_R^2} + M_{R(2,2)}(t, \chi) . \quad (3.33)
\]

Introducing here the expressions found above for \( M_{R(1,1)}|_{g_R^4} \) [see Eq. (3.14)] and for \( M_{R(2,2)}|_{g_R^4} \) [see Eqs. (3.23) and (3.26)], we finally find the following expression for \( M(t, \chi)|_{g_R^4} \):

\[
M(t, \chi)|_{g_R^4} = \frac{g_R^2}{2} \frac{1}{t} \coth \chi
\]

\[
\times \left[ 1 + \left( 2Z_{W}^{(2)} - F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} + \frac{N_c t I(t) \chi \coth \chi}{4\pi} \right) g_R^2 \right] \cdot (G_1)_{ijkl}
\]

\[
-\frac{1}{2} g_R^4 I(t) \coth^2 \chi \cdot (G_2)_{ijkl} . \quad (3.34)
\]
This expression allows us to immediately derive the quantity \( g_M(t, \chi) \), defined by Eq. (3.1), up to the order \( O(g^4_R) \). In fact, making use also of the expansion (3.30) for the renormalization constant \( Z_{1W} \), one finds that:

\[
g_M(t, \chi)|_{g^4_R} = \frac{M(t, \chi)}{Z^2_{1W}}|_{g^4_R} = M(t, \chi)|_{g^4_R} - 2Z^{(2)}_{1W} g^2_R \cdot M(t, \chi)|_{g^4_R} \]

\[
= g^2_R \frac{1}{t} \coth \chi \left[ 1 - \left( F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} + \frac{N_c}{4\pi} t I(t) \chi \coth \chi \right) g^2_R \right] \cdot (G_1)_{ij,kl} \]

\[
- \frac{1}{2} g^4_R I(t) \coth^2 \chi \cdot (G_2)_{ij,kl} .
\]

(3.35)

The quark–quark scattering amplitude in the high–energy limit turns out to be, up to the order \( O(g^4_R) \),

\[
f_M(s, t)|_{g^4_R} \sim -i \cdot \delta_{\alpha \beta} \delta_{\gamma \delta} \cdot 2s \cdot g_M(t, \chi \rightarrow \infty)|_{g^4_R} \]

\[
= \delta_{\alpha \beta} \delta_{\gamma \delta} \left[ g^2_R \frac{2s}{t} \left[ 1 - \bar{\alpha}(t) \ln s \right] \cdot (G_1)_{ij,kl} + ig^4_R s I(t) \cdot (G_2)_{ij,kl} \right] ,
\]

(3.36)

where we have used the notation

\[
\bar{\alpha}(t) \equiv -\frac{N_c g^2_R}{4\pi} t I(t) = \frac{N_c g^2_R}{4\pi} q^2 I(t) .
\]

(3.37)

We have used the fact that both \( \beta \) and \( \psi \) (or equivalently \( \chi \)) are dependent on \( s \). In fact, from \( E = m/\sqrt{1 - \beta^2} \) and from \( s = 4E^2 \), one immediately finds that

\[
\beta = \sqrt{1 - \frac{4m^2}{s}} .
\]

(3.38)

By inverting this equation and using the relation \( \beta = \tanh \psi \), we derive that

\[
s = 4m^2 \cosh^2 \psi = 2m^2(\cosh \chi + 1) .
\]

(3.39)

Therefore, in the high–energy limit \( s \rightarrow \infty \) (or \( \beta \rightarrow 1 \)), the hyperbolic angle \( \chi = 2\psi \) is essentially equal to the logarithm of \( s \):

\[
\chi = 2\psi \sim \ln s .
\]

(3.40)
Moreover, $\coth \chi \sim 1$ in this limit. This is why we have been able to approximate the $O(g_R^2)$ term which multiplies $(G_1)_{ij,kl}$ as reported in Eq. (3.36). The result (3.36) is exactly what can be found by applying ordinary perturbation theory to evaluate the scattering amplitude up to the order $O(g_R^2)$ \cite{[1]} \cite{[4]}. In particular, as was pointed out in the Introduction, the $\ln s$ factor in Eq. (3.36) comes from the fact that it is really a singular limit to take the Wilson lines in (3.1) exactly on the light cone. As first predicted in \cite{[10]}, a proper regularization of these singularities give rise to the $\ln s$ dependence of the amplitude, as confirmed by the experiments on hadron–hadron scattering processes.

We want now to repeat the analogous calculation for the Euclidean theory. Let us consider, therefore, the following quantity, defined in Euclidean space–time:

$$g_E(t, v_1 \cdot v_2) = \frac{1}{Z_{EW}^{2}} \int d^2z_t e^{iq \cdot z_t} \langle [W_{E1}(\bar{z}_t) - 1]_{ij} [W_{E2}(0) - 1]_{kl} \rangle_A ,$$

(3.41)

where $\bar{z}_t = (0, z_t, 0)$ and the expectation value $\langle \ldots \rangle_A$ must be intended now as a functional integration with respect to the gauge variable $A^{(E)}_\mu$ in the Euclidean theory. The Euclidean four–vectors $v_1$ and $v_2$ [lying (for example) in the plane $(x_1, x_4)$] define the two Wilson lines $W_{E1}$ and $W_{E2}$: we can take $v_1$ and $v_2$ normalized to 1, with respect to the Euclidean scalar product (that is, $v_1^2 = v_2^2 = 1$). Clearly, $g_E$ can only depend on the scalar variables constructed using the vectors $v_1$, $v_2$ and $q_E = (0, q, 0)$: they are $q_E^2 = q^2 = -t$ and $v_1 \cdot v_2$, since $q_E \cdot v_1 = q_E \cdot v_2 = 0$ and $v_1^2 = v_2^2 = 1$ are fixed. By virtue of the $O(4)$ symmetry of the Euclidean theory, we can choose a reference frame in which $v_1$ and $v_2$ have the following values:

$$v_1 = (\sin \phi, 0_t, \cos \phi) ;
\quad v_2 = (-\sin \phi, 0_t, \cos \phi) ,$$

(3.42)

with a value of $\phi$ between 0 and $\pi/2$ (so that the angle $2\phi$ between the two trajectories is in the interval $[0, \pi]$). In such a reference frame, we can write $v_1 \cdot v_2 = \cos \theta$, where $\theta = 2\phi$ is the angle [in the plane $(x_1, x_4)$] between the two Euclidean Wilson lines $W_{E1}$ and $W_{E2}$. These last are defined as

$$W_{E1}(\bar{z}_t) \equiv P \exp \left[ -ig \int_{-\infty}^{+\infty} A^{(E)}_\mu (\bar{z}_t + v_1 \tau) v_1 \mu d\tau \right] ;
\quad W_{E2}(0) \equiv P \exp \left[ -ig \int_{-\infty}^{+\infty} A^{(E)}_\mu (v_2 \tau) v_2 \mu d\tau \right] ,$$

(3.43)
where $A_\mu^{(E)} = A_\mu^{(E)a} T^a$. Moreover, we have put

$$Z_{EW} \equiv \frac{1}{N_c} \langle \text{Tr}[W_{E1}(\bar{z}_t)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W_{E1}(0)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W_{E2}(0)] \rangle .$$  \hfill (3.44)

(The two last equalities come from the $O(4)$ plus translation invariance.) We want to explicitly evaluate the quantity (3.41) up to the order $O(g^4_R)$ in perturbation theory. Therefore, we have to evaluate the Feynman diagrams in Figs. 2 and 3, where the two horizontal oriented lines now represent the Euclidean Wilson lines $W_{E1}$ and $W_{E2}$. As before, we need to expand $Z_{EW}$ only up to the order $O(g^2_R)$ in perturbation theory:

$$Z_{EW} = 1 + Z^{(2)}_{EW} g^2_R + O(g^4_R) ,$$  \hfill (3.45)

even if, as will become clear in the following, we shall not need to know the explicit expression for the coefficient $Z^{(2)}_{EW}$. As in the previous case, we need to consider also the effect of the renormalizations of the fields and the coupling constant $g$, up to the order $O(g^2_R)$, that is:

$$A_\mu^{(E)a} = Z^{1/2}_{R\mu} A_\mu^{(E)a} ; \quad g = Z_g g_R ,$$  \hfill (3.46)

where the suffix “$R$” denotes the renormalized quantities. Therefore we have that

$$W_{Ei}(\bar{z}_t) = P \exp \left[ -i Z_{1W} g_R \int_{-\infty}^{+\infty} A^{(E)}_{R\mu}(\bar{z}_t + v_1 \tau) v_{1\mu} d\tau \right] ,$$  \hfill (3.47)

the renormalization constant $Z_{1W}$ being defined by Eq. (3.7). The renormalization constants in Eq. (3.46) are the same as those in Eq. (3.5) for the Minkowski world, if we adopt the same renormalization scheme (for example, the MS scheme) for the Euclidean theory and the Minkowskian one. In such a case, the correspondence

$$A_0(x) \to i A_4^{(E)}(x_E) , \quad A_k(x) \to A_k^{(E)}(x_E) \quad (\text{with} : x^0 \to -ix_E^4)$$  \hfill (3.48)

between the bare gluon fields in the two theories, turns into the same correspondence for the renormalized gluon fields:

$$A_{R0}(x) \to i A_{R4}^{(E)}(x_E) , \quad A_{Rk}(x) \to A_{Rk}^{(E)}(x_E) \quad (\text{with} : x^0 \to -ix_E^4) .$$  \hfill (3.49)
[As a matter of fact, the renormalization constants $Z_3$, $Z_g$, etc., are always evaluated in the Euclidean world, also when they refer to the Minkowskian one: when evaluating these constants in Minkowski (momentum) four–space, one always performs a Wick rotation to Euclidean (momentum) four–space.]

We shall evaluate the amplitude

$$E(t, \theta) = \int d^2z e^{iq \cdot z_t} \langle [W_{E1}(\bar{z}_t) - 1]_{ij} [W_{E2}(0) - 1]_{kl} \rangle_A,$$

up to the order $O(g^4 R)$. As in the previous case, the effects of $Z_{1W}$ are visible when we consider only the diagrams of the one–gluon–exchanged type, having the following amplitude:

$$E_{(1,1)} = Z_{1W}^2 E_{R(1,1)} = E_{R(1,1)} + (Z_{1W}^2 - 1) E_{R(1,1)},$$

$E_{R(1,1)}$ being the “renormalized” one–gluon–exchanged amplitude:

$$E_{R(1,1)} = -g_R^2 (T^a)_{ij} (T^b)_{kl} v_1 v_2 \int d^2 z e^{iq \cdot z_t} \int d\tau \int d\omega \langle A^{(E) a}_{R\mu}(\bar{z}_t + v_1 \tau) A^{(E)b}_{R\nu}(v_2 \omega) \rangle_A.$$

In our notation, $E_{(i,j)}$ denotes the contribution to the amplitude $E$, defined in Eq. (3.50), obtained after expanding the Euclidean Wilson line $W_{E1}$ up to the order $O(g^i)$ (i.e., up to the term containing $i$ gluon fields) and expanding the other Euclidean Wilson line $W_{E2}$ up to the order $O(g^j)$ (i.e., up to the term containing $j$ gluon fields). Moreover, we define

$$E_{R(i,j)} \equiv Z_{1W}^{-1(i+j)} E_{(i,j)}.$$

We have to compute $E_{(1,1)}$ up to the order $O(g_R^4)$, which, by virtue of Eqs. (3.51) and (3.7), is given by

$$E_{R(1,1)} \bigg|_{g_R^2} = E_{R(1,1)} \bigg|_{g_R^2} + 2 Z_{1W}^{(2)} E_{R(1,1)} \bigg|_{g_R^2}.$$

The expression for $E_{R(1,1)} \bigg|_{g_R^2}$, corresponding to the diagram shown in Fig. 2(a), can be derived using in the calculation the Euclidean free gluon–field propagator. This last, in any given Lorentz gauge with a (bare) gauge parameter $\alpha$, is given by

$$G^{(E)ab}_{\mu\nu}(x_E - y_E) = \delta_{ab} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2} \left[ \delta_{\mu\nu} - (1 - \alpha) \frac{k_E^\mu k_E^\nu}{k_E^2} \right] e^{-ik_E(x_E - y_E)}.$$

The contribution coming from the one–gluon–exchange process pictorially represented in Fig. 2(a), comes out to be, with the notation already introduced for the color factor,

$$E^{(a)}(t, \theta) = E_{R(1,1)}(t, \theta) \bigg|_{g_R^2} = g_R^2 \frac{1}{t} \cot \theta \cdot (G_1)_{ij,kl}.$$
The last term of Eq. (3.53), represented by the diagrams in Figs. 3(q) and 3(r), is given by

\[ E(q)(t, \theta) = E(r)(t, \theta) = Z_{1W}g_R^2 \cdot E_{R(1,1)}(t, \theta)|_{g_R^2} = Z_{1W}g_R^4 \cdot \frac{1}{t} \cot \theta \cdot (G_1)_{ij,kl} \] (3.56)

The first term in Eq. (3.53), i.e., \( E_{R(1,1)}|_{g_R^4} \), is the one–gluon–exchanged renormalized amplitude up to the order \( O(g_R^4) \). It is given by the sum of the contributions from the diagrams shown in Figs. 2(a), 3(l) to 3(p). [This last one represents the insertion of a counterterm \((Z_3 - 1)\delta_{ab}(k_{E\mu}k_{E\nu} - \delta_{\mu\nu}k_E^2)\) into the gluon line.] Therefore, one has to compute the quantity (3.52) using the renormalized gluon propagator up to the order \( O(g_R^2) \) when evaluating the expectation value \( \langle A^{(E)\a}(z_t + v_1 \tau)A^{(E)\b}(v_2 \omega) \rangle_A \). That is:

\[ \langle A^{(E)\a}(x_E)A^{(E)\b}(y_E) \rangle_A = \int \frac{d^4k_E}{(2\pi)^4} e^{-ik_E(x_E-y_E)} \tilde{D}_{R\mu\nu}^{ab}(k_E). \] (3.57)

The expression for \( \tilde{D}_{R\mu\nu}^{ab}(k_E) \) can be derived from the corresponding expression (3.18) for \( \tilde{D}_{R\mu\nu}^{ab}(k) \), making use of the correspondence law (3.49) between the (renormalized) gluon field in Minkowski four–space and the (renormalized) gluon field in Euclidean four–space. \( \tilde{D}_{R\mu\nu}^{ab}(k_E) \) is obtained from \( \tilde{D}_{R\mu\nu}^{ab}(k) \) by making the replacements

\[ k^2 \rightarrow -k_E^2 \quad (\text{i.e.,} \quad k^0 \rightarrow ik_E, \quad k \rightarrow k_E); \]

\[ g_{\mu\nu} \rightarrow -\delta_{\mu\nu} \quad ; \quad k_{\mu}k_{\nu} \rightarrow k_{E\mu}k_{E\nu}. \] (3.58)

Therefore, in a Lorentz gauge with a renormalized gauge parameter \( \alpha_R = Z_3^{-1} \alpha \), \( \tilde{D}_{R\mu\nu}^{ab}(k_E) \) is given by

\[ \tilde{D}_{R\mu\nu}^{ab}(k_E) = Z_3^{-1} \tilde{D}_{\mu\nu}^{ab}(k_E) = \frac{\delta_{ab}}{k_E^2} \left[ \frac{\delta_{\mu\nu} - \frac{k_{E\mu}k_{E\nu}}{k_E^2}}{1 + \Pi_R(-k_E^2) + \alpha_R \frac{k_{E\mu}k_{E\nu}}{k_E^2}} \right], \] (3.59)

where \( \Pi_R \) is exactly the same finite function of order \( O(g_R^4) \), appearing in the expression (3.18) for the gluon propagator in Minkowski space–time:

\[ \Pi_R(-k_E^2) = g_R^2 F^{(2)}(-k_E^2) + O(g_R^4). \] (3.60)
As said before, the precise form of $\Pi_R$ depends on the renormalization scheme which has been adopted. At this point the derivation of the full expression for the amplitude $E_{R(1,1)}$ is rather immediate and gives

$$E_{R(1,1)}(t, \theta) = \frac{E_{R(1,1)}(t, \theta)|_{g_R^4}}{1 + \Pi_R(t)} = \frac{g_R^2}{t[1 + \Pi_R(t)]} \cot \theta \cdot (G_1)_{ij,kl}.$$  (3.61)

The value of $E_{R(1,1)}$ is gauge–dependent (as $M_{R(1,1)}$ was, too), since the gauge parameter $\alpha_R$ does appear inside $\Pi_R$ at the right–hand–side of Eq. (3.61). For the following calculations we shall fix the gauge parameter $\alpha_R$ to 1 (Feynman gauge), in conformity with the choice we have made in the previous case. Eq. (3.61) is the full expression for $E_{R(1,1)}$, not truncated at any perturbative order. Yet, we only need the expression for $E_{R(1,1)}$ up to the order $O(g_R^4)$:

$$E_{R(1,1)}(t, \theta)|_{g_R^4} = g_R^2 \frac{1}{t[1 - g_R^2 F^{(2)}(t)]} \cot \theta \cdot (G_1)_{ij,kl}.$$  (3.62)

Therefore, in the Euclidean theory, the contribution coming from the $O(g_R^4)$ diagrams shown in Figs. 3(l) to 3(p) is given by

$$E^{(l)}(t, \theta) + \ldots + E^{(p)}(t, \theta) = -g_R^4 \frac{F^{(2)}(t)}{t} \cot \theta \cdot (G_1)_{ij,kl}.$$  (3.63)

The contribution of order $O(g_R^4)$ coming from the two Feynman diagrams shown in Fig. 2(b) (the ladder term) and Fig. 2(c) (the cross term), i.e., the contribution $E_{(2,2)}|_{g_R^4}$, turns out to be (in the Feynman gauge $\alpha_R = 1$)

$$E^{(b)}(t, \theta) + E^{(c)}(t, \theta) =$$

$$= E_{(2,2)}(t, \theta)|_{g_R^4} = E^{(G_1)}(t, \theta) \cdot (G_1)_{ij,kl} + E^{(G_2)}(t, \theta) \cdot (G_2)_{ij,kl},$$  (3.64)

where $G_1$ and $G_2$ are two color factors defined in Eqs. (3.15) and (3.24). The coefficients $E^{(G_1)}(t, \theta)$ and $E^{(G_2)}(t, \theta)$ in front of the color factors in Eq. (3.64) are found to be

$$E^{(G_1)}(t, \theta) = \frac{N_c g_R^4}{4\pi I(t)} \cot^2 \theta;$$

$$E^{(G_2)}(t, \theta) = \frac{1}{2} g_R^4 I(t) \cot^2 \theta.$$  (3.65)
The contribution $E_{(3,1)}|_{g_R^4} + E_{(1,3)}|_{g_R^4}$ from the diagrams in Figs. 2(d) to 2(i), obtained after expanding one of the two Wilson lines up to the order $O(g^3)$, and the remaining one up to the first order in $g$, can be written as

$$E_{(3,1)}|_{g_R^4} = Z_{EW}^{(2)} g_R^2 \cdot E_{R(1,1)}|_{g_R^2} + \Delta E_{(3,1)} ,$$

$$E_{(1,3)}|_{g_R^4} = Z_{EW}^{(2)} g_R^2 \cdot E_{R(1,1)}|_{g_R^2} + \Delta E_{(1,3)} ,$$  \hspace{1cm} (3.66)

where $\Delta E_{(3,1)}$ and $\Delta E_{(1,3)}$ are divergent quantities, whose regularized expressions depend on the adopted renormalization scheme. In the MS renormalization scheme one finds that

$$\Delta E_{(3,1)} = \Delta E_{(1,3)} = E_{R(1,1)}|_{g_R^2} \cdot \frac{g_R^2}{(4\pi)^2} N_c \left[ \frac{1}{\varepsilon} + B \right] ,$$  \hspace{1cm} (3.67)

where $B$ is the same finite number (as $\varepsilon$ goes to zero), which appears in the corresponding expression (3.29) for the Minkowskian case. From Eq. (3.67) and from the expression (3.30) of $Z_{1W}$ up to the order $O(g_R^2)$, one immediately derives that

$$\Delta E_{(3,1)} + \Delta E_{(1,3)} + 2Z_{1W}^{(2)} g_R^2 \cdot E_{R(1,1)}|_{g_R^2} = E_{R(1,1)}|_{g_R^2} \cdot \frac{g_R^2}{(4\pi)^2} 2N_c B .$$  \hspace{1cm} (3.68)

As in the Minkowskian case, the divergence contained in $\Delta E_{(3,1)} + \Delta E_{(1,3)}$ is exactly cancelled out by the two diagrams with the counterterm $Z_{1W}^{(2)}$, represented in Figs. 3(q) and 3(r). Finally, one has to evaluate the contributions $E_{(2,1)}|_{g_R^4}$ and $E_{(1,2)}|_{g_R^4}$, represented by the two diagrams in Figs. 3(j) and 3(k), respectively. Again, explicit calculations show that their contribution vanishes:

$$E_{(2,1)}|_{g_R^4} = E_{(1,2)}|_{g_R^4} = 0 .$$  \hspace{1cm} (3.69)

We can now sum up all the contributions previously evaluated in order to find the complete expression for the amplitude $E$, defined by Eq. (3.50), up to the order $O(g_R^4)$:

$$E(t, \theta)|_{g_R^4} = \left[ 1 + \left( 2Z_{EW}^{(2)} - F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} \right) g_R^2 \right] \cdot E_{R(1,1)}(t, \theta)|_{g_R^2} + E_{R(2,2)}(t, \theta) .$$  \hspace{1cm} (3.70)

Introducing here the expressions found above for $E_{R(1,1)}|_{g_R^2}$ [see Eq. (3.55)] and for $E_{R(2,2)}|_{g_R^4}$ [see Eqs. (3.64) and (3.65)], we finally find the following expression for $E(t, \theta)|_{g_R^4}$:

$$E(t, \theta)|_{g_R^4} = g_R^2 \frac{1}{t} \cot \theta$$

26
\[
\begin{align*}
&\times \left[ 1 + \left( 2Z_{EW}^{(2)} - F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} + \frac{N_c t I(t)\cot\theta}{4\pi} \right) g_R^2 \right] \cdot (G_1)_{ij,kl} \\
&+ \frac{1}{2} g_R^4 I(t) \cot^2\theta \cdot (G_2)_{ij,kl}.
\end{align*}
\]

(3.71)

The quantity \( g_E(t,\theta) \), defined by Eq. (3.41), can be immediately derived up to the order \( \mathcal{O}(g_R^4) \), making use also of the expansion (3.45) for the renormalization constant \( Z_{EW} \):

\[
\begin{align*}
g_E(t,\theta)|_{g_R^4} &= \frac{E(t,\theta)}{Z_{EW}^2}|_{g_R^4} = E(t,\theta)|_{g_R^4} - 2Z_{EW}^{(2)} g_R^2 \cdot E(t,\theta)|_{g_R^4} \\
&= g_R^2 \frac{1}{t} \cot\theta \left[ 1 - \left( F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} + \frac{N_c t I(t)\cot\theta}{4\pi} \right) g_R^2 \right] \cdot (G_1)_{ij,kl} \\
&+ \frac{1}{2} g_R^4 I(t) \cot^2\theta \cdot (G_2)_{ij,kl}.
\end{align*}
\]

(3.72)

After comparing the two expressions (3.35) and (3.72) for \( g_M(t,\chi)|_{g_R^4} \) and \( g_E(t,\theta)|_{g_R^4} \), we immediately recognize that they are linked by the following analytic continuation in the angular variable:

\[
\begin{align*}
g_E(t,\theta)|_{g_R^4} \xrightarrow{\theta \rightarrow -i\chi} g_E(t,-i\chi)|_{g_R^4} = g_M(t,\chi)|_{g_R^4}; \\
or: \quad g_M(t,\chi)|_{g_R^4} \xrightarrow{\chi \rightarrow i\theta} g_M(t,i\theta)|_{g_R^4} = g_E(t,\theta)|_{g_R^4}.
\end{align*}
\]

(3.73)

This is the same relation we have already found in the preceding section for the corresponding quantities in the Abelian case. It appears to be an absolutely “natural” correspondence law. There is apparently no reason why it should not be true at higher perturbative orders: we shall therefore assume that it is valid for the “full” amplitudes, i.e., not truncated at any perturbative order. In the next section we shall discuss some interesting consequences and some possible applications of this relationship of analytic continuation.

4. Concluding remarks and prospects

In the preceding section we have seen that also for a non–Abelian gauge theory it is possible to reconstruct the high–energy scattering amplitude by evaluating a correlation of
infinite Wilson lines forming a certain angle \( \theta \) in Euclidean four–space, then by continuing this quantity in the angular variable, \( \theta \to -i\chi \), where \( \chi \) is the hyperbolic angle between the two Wilson lines in Minkowski space–time, and finally by performing the limit \( \chi \to \infty \) (i.e., \( \beta \to 1 \)). In fact, the high–energy scattering amplitude is given by

\[
M_{fi} = \langle \psi_i\alpha(p'_1)\psi_k\gamma(p'_2)|M|\psi_j\beta(p_1)\psi_l\delta(p_2) \rangle 
\sim -i \cdot 2s \cdot \delta_{\alpha\beta} \delta_{\gamma\delta} \cdot g_M(t, \chi \to \infty). \tag{4.1}
\]

The quantity \( g_M(t, \chi) \), defined by Eq. (3.1) in the Minkowski world, is linked to the corresponding quantity \( g_E(t, \theta) \), defined by Eq. (3.41) in the Euclidean world, by the following analytic continuation in the angular variables:

\[
g_E(t, \theta) \xrightarrow{\theta \to -i\chi} g_E(t, -i\chi) = g_M(t, \chi) ;
\]

or:

\[
g_M(t, \chi) \xrightarrow{\chi \to i\theta} g_M(t, i\theta) = g_E(t, \theta). \tag{4.2}
\]

The important thing to note here is that the quantity \( g_E(t, \theta) \), defined in the Euclidean world, may be computed non perturbatively by well–known and well–established techniques: first of all, of course, by means of the formulation of the theory on the lattice. Also the stochastic vacuum model \([14]\), which is naturally defined for the Euclidean theory, may provide a suitable instrument to evaluate the quantity (3.41). In all cases, once one has obtained the quantity \( g_E(t, \theta) \), one still has to perform an analytic continuation in the angular variable \( \theta \to -i\chi \), and finally one has to extrapolate to the limit \( \chi \to \infty \) (i.e., \( \beta \to 1 \)). We are fully aware that this may not be an easy way. Nevertheless, interesting new results are expected along this direction. As an example, we shall show how, using this approach, one can re–derive the well–known Regge Pole Model \([15]\), but in a different way, with respect to the original derivation. First of all, we write \( g_E(t, \theta) \) in the partial–wave expansion:

\[
g_E(t, \theta) = \sum_{l=0}^{\infty} A_l(t)P_l(\cos \theta). \tag{4.3}
\]

If \( A_l(t) \) can be analytically continued to complex values of \( l \), then we can re–write Eq. (4.3) in the following way:

\[
g_E(t, \theta) = \frac{1}{2i} \int_C \frac{A_l(t)P_l(-\cos \theta)}{\sin(\pi l)} dl, \tag{4.4}
\]

where \( C \) is a contour in the complex \( l \)–plane, running anti–clockwise around the real positive \( l \)–axis and enclosing all non–negative integers, while excluding all the singularities.
of \( A_l \). Eq. (4.4) can be verified after recognizing that \( P_l(-\cos \theta) \) is an integer function of \( l \) and that the singularities enclosed by the contour \( C \) of the expression under integration in the Eq. (4.4) are simple poles at the non-negative integer values of \( l \). So the right-hand side of (4.4) is equal to the sum of the residues of the integrand in these poles and this gives exactly the right-hand side of (4.3). The “minus” sign in the argument of the Legendre function \( P_l \) into Eq. (4.4) is due to the following relation, valid for integer values of \( l \):

\[
P_l(-\cos \theta) = (-1)^l P_l(\cos \theta). \tag{4.5}
\]

Then, we can reshape the contour \( C \) into the straight line \( \Re(l) = -\frac{1}{2} \). Eq. (4.4) then becomes

\[
ge_E(t, \theta) = -\sum_{\Re(\alpha_n) > -\frac{1}{2}} \frac{\pi r_n(t)P_{\alpha_n(t)}(-\cos \theta)}{\sin(\pi \alpha_n(t))} - \frac{1}{2i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} \frac{A_l(t)P_l(-\cos \theta)}{\sin(\pi l)}dl, \tag{4.6}
\]

where \( \alpha_n(t) \) is a pole of \( A_l(t) \) in the complex \( l \)-plane and \( r_n(t) \) is the corresponding residue. We have assumed that \( A_l \) vanishes enough rapidly as \( |l| \to \infty \) in the right half-plane, so that the contribution from the infinite contour is zero. Eq. (4.6) immediately leads to the asymptotic behavior of the scattering amplitude in the limit \( s \to \infty \), with a fixed \( t \) (\(|t| \ll s\)). In fact, making use of the analytic extension (4.2) when continuing the angular variable, \( \theta \to -i\chi \), we derive that

\[
ge_M(t, \chi) = g_E(t, -i\chi)
= -\sum_{\Re(\alpha_n) > -\frac{1}{2}} \frac{\pi r_n(t)P_{\alpha_n(t)}(-\cosh \chi)}{\sin(\pi \alpha_n(t))} - \frac{1}{2i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} \frac{A_l(t)P_l(-\cosh \chi)}{\sin(\pi l)}dl. \tag{4.7}
\]

The hyperbolic angle \( \chi \) is linked to \( s \) by the relation (3.39). Therefore we can re-express \( \cosh \chi \) in terms of \( s \) in the following way:

\[
\cosh \chi = \frac{s}{2m^2} - 1. \tag{4.8}
\]

The asymptotic form of \( P_n(z) \) when \(|z| \to \infty \) is well known. It is a linear combination of \( z^\alpha \) and of \( z^{-\alpha-1} \). When \( \Re(\alpha) > -1/2 \), this last term can be neglected. Therefore, in the limit \( s \to \infty \), with a fixed \( t \) (\(|t| \ll s\)), we are left with the following expression:

\[
ge_M(t, \chi \to \infty) \sim \sum_{\Re(\alpha_n) > -\frac{1}{2}} \frac{\beta_n(t)s^\alpha_n(t)}{\sin(\pi \alpha_n(t))}, \tag{4.9}
\]
where $\beta_n(t)$ is independent on $s$ (it only depends on $t$). The integral in Eq. (4.7), usually called the background term, vanishes at least as $s^{-1/2}$. Eq. (4.9) allows to immediately extract the scattering amplitude according to Eq. (4.1):

$$M_{fi} \sim -i \cdot 2s \cdot \delta_{\alpha\beta} \delta_{\gamma\delta} \cdot g_M(t, \chi \rightarrow \infty)$$

$$\sim \delta_{\alpha\beta} \delta_{\gamma\delta} \sum_{\Re(\alpha_n) > -\frac{1}{2}} \frac{\beta_n(t) s^{1+\alpha_n(t)}}{\sin(\pi \alpha_n(t))} .$$

(4.10)

This equation gives the explicit dependence of the scattering amplitude at very high energy $s \rightarrow \infty$ and a fixed transferred momentum $t$ ($|t| \ll s$). As we can see, this amplitude comes out to be a sum of powers of $s$. If we put $\bar{\alpha}(t) = 1 + \alpha_n(t)$, where $\alpha_n(t)$ is the pole with the largest real part (at that given $t$), we can also write

$$M_{fi} \sim \delta_{\alpha\beta} \delta_{\gamma\delta} \cdot \bar{\beta}(t) s^{\bar{\alpha}(t)} .$$

(4.11)

This sort of behavior for the scattering amplitude was first proposed by Regge in [15] and $\bar{\alpha}(t)$ is often called a “Regge pole”. In the original derivation [15], the asymptotic behavior (4.11) was recovered by analytically continuing to very large imaginary values the angle between the trajectories of the two exiting particles in the $t$–channel process. Instead, in our derivation, we have analytically continued the quantity (3.41), defined in the Euclidean theory, to very large (negative) imaginary values of the angle $\theta$ between the two Euclidean Wilson lines. As in the original derivation, we have assumed that the singularities of $A_t$ are simple poles. If there are other kinds of singularities, different from simple poles, their contribution will be of a different type and, in general, also logarithmic terms (of $s$) may appear in the amplitude. Only a precise evaluation of $g_E(t, \theta)$ can reveal such behaviors (after the analytic continuation). In the preceding section this was done up to the order $O(g_R^4)$ in perturbation theory. New interesting results are expected from a non perturbative approach, for example by directly computing $g_E(t, \theta)$ on the lattice or by means of the stochastic vacuum model.

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**FIGURE CAPTIONS**

**Fig. 2.** The contributions of the type $(1,1)|_{g_R^2}$ [(a)], $(2,2)|_{g_R^4}$ [(b) and (c)], $(3,1)|_{g_R^4}$ [from (d) to (f)] and $(1,3)|_{g_R^4}$ [from (g) to (i)] to the amplitudes (3.8) and (3.50). The notation is explained in the text.

**Fig. 3.** The contributions of the type $(2,1)|_{g_R^4}$ [(j)], $(1,2)|_{g_R^4}$ [(k)], $(1,1)$ of order $O(g_R^4)$ [from (l) to (p)], plus the counterterms [(q) and (r)], for the amplitudes (3.8) and (3.50). The notation is explained in the text.
