Phase transition of AdS black holes in 4D EGB gravity coupled to nonlinear electrodynamics

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Einstein-Gauss-Bonnet (EGB) gravity is an outcome of quadratic curvature corrections to the Einstein-Hilbert gravity action in the form of a Gauss-Bonnet (GB) term in $D > 4$ dimensions and EGB gravity is topologically invariant in 4D. Recently a number of ways have been proposed for regularizing, a $D \rightarrow 4$ limit of EGB, for nontrivial gravitational dynamics in 4D. Motivated by the importance of anti-de Sitter gravity/conformal field theory correspondence (AdS/CFT), we analyze black holes with AdS asymptotic to regularized 4D EGB gravity coupled to the nonlinear electrodynamics (NED) field. For a static spherical symmetric ansatz the field equations are solved exactly for a NED Lagrangian - namely NED charged AdS black holes in 4D EGB gravity which retains several known solutions. Owing to the NED charge corrected EGB black holes, the thermodynamic quantities are also modified and the entropy does not obey the usual area law. We calculate the heat capacity and Helmholtz free energy, in terms of horizon radii, to investigate both local and global thermodynamic stability of black holes. We observe a secondary Hawking-Page transition between the smaller thermally favored black hole and thermal AdS space. Our results show that the behavior of Hawking’s evaporation abruptly halts at smaller radii regime such that the black holes do have a thermodynamically stable remnant with vanishing temperature.

I. INTRODUCTION

Lovelock theories [1], with higher order curvature terms, are generalizations of Einstein’s general relativity (GR) to higher dimensions (HD), but the field equations are not of more than second derivatives of the metric functions. Hence Lovelock theories are free from several problems that affect other higher derivative gravity theories in which the equations of motion are fourth order or higher, and linear perturbations modes reveal the existence of ghost instabilities. In the second order Lovelock theory or Einstein-Gauss-Bonnet (EGB) gravity the action [1] is supplemented with the quadratic curvature, namely the Gauss-Bonnet (GB) [2] term apart from the cosmological constant ($\Lambda$) and the Ricci scalar ($R$). This special case of EGB gravity has received noteworthy attention for the reason that the GB action naturally appears in the low energy of heterotic string theory [3]. The spherically symmetric static solution in the EGB theory was originally discovered by Boulware and Deser [4], thereby generalizing the $D$-dimensional Schwarzschild-Tangherlini black hole [5]. The Boulware and Deser black hole solution was extended to the charged counterpart by Wiltshire [6] and the thermodynamics of black hole was also analysed [6–8]. A series of subsequent interesting works analysed black hole solutions in EGB gravity [9] for various sources [10–15], including those coupled to the nonlinear electrodynamics (NED) fields [16, 17], and also in asymptotic AdS spacetime [18] due to the Hawking-Page type transitions. However the EGB black holes in asymptotically AdS background have the noteworthy characteristic of being thermodynamically favoured for higher temperatures [17, 18] as commonly termed Hawking-Page type transitions [19]. Due to the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, AdS black holes have become more significant to investigate [17, 18].

EGB gravity allows us to explore several conceptual issues in a broader setup; however in 4D the GB term is a topological invariant as its variation is a total derivative with no local dynamics and the theory becomes GR. One requires $D \geq 5$ for non-trivial gravitational dynamics. Glavan and Lin [20] proposed a 4D EGB gravity by rescaling the GB term in $D$-dimensional spacetime, as $\alpha/(D-4)$ bypasses conditions of Lovelock’s theorem [21]. Further they considered the limit $D = 4$ at the level of the field equations so that the GB term does makes a nontrivial contribution.
to local dynamics at least for the case of spherical symmetry. This non-trivial theory will be referred to as the $4D$ EGB theory, which admits spherically symmetric black hole solutions [20], generalizing the Schwarzschild black holes, and has the repulsive nature of gravity at short distances. We refer to the process of obtaining a nontrivial $4D$ EGB gravity as regularization, which was originally considered by Tomozawa [22] with finite one-loop quantum corrections to Einstein gravity, and he also found the spherically symmetric black hole solution. Later this was also done by Cognola et al. [23] within a classical Lagrangian approach. Incidentally the identical spherically symmetric black hole solutions [20, 22, 23] have also been obtained in the semi-classical Einstein equations with conformal anomaly [24] and also in the $4D$ non-relativistic Horava-Lifshitz theory of gravity [25].

After the regularization of $4D$ EGB theory proposed by Glavan and Lin [20], interesting measures have been taken to investigate the $4D$ EGB gravity, which includes generalizing the black hole solution [26, 27], a Vaidya-like radiating black hole in [28, 29], black holes coupled with magnetic charge [30–33], to the axially symmetric or rotating case (Kerr-like) was also addressed [34, 35], derivation of regularized field equations [36], Morris-Thorne-like wormholes [37], accretion disk around black holes [38], thermodynamics [39, 40], gravitational lensing by a black hole [41–44], and generalization to more general Lovelock gravity theories [45].

Motivated by the above arguments and extensive importance of AdS/CFT correspondence, the aim of this paper is to consider static spherically symmetric black hole solutions with AdS asymptotic to regularised $4D$ EGB gravity coupled to the NED, i.e., we obtain NED charged $4D$ EGB-AdS black holes. The metrics depend on the mass ($M$), GB coupling constant ($\alpha$), and a parameter ($k$) coming from the NED field that measures potential deviation from the $4D$ EGB black holes [20, 22, 23], and they are encompassed as special cases when the charge is switch off ($k = 0$). We also find exact expressions for the thermodynamical quantities associated with NED charged $4D$ EGB-AdS black holes, and also perform both local and global thermodynamic stability analysis.

The paper is organized as follows: Sect. II is devoted to a brief review of the gravitational field equations of regularized EGB gravity minimally coupled with the NED field in the $D \to 4$ limit, and the exact static spherically symmetric NED charged $4D$ EGB-AdS black hole solution is obtained. In Sect. III we obtain the analytical expressions for various thermodynamic quantities and discuss the effect of NED charge on the stability and phase transitions. Considering the cosmological constant as thermodynamical pressure, we investigate the P-V criticality in Sect. III C. Finally in Sect. IV we summarize the obtained results.

II. ACTION, FIELD EQUATIONS AND SOLUTION

Before we start our discussion on $4D$ EGB gravity coupled to the NED field, it is worthwhile to mention that the regularization proposed in [20, 23], is subject to debate [46–50] and a number of questions have been raised. In addition, several alternate regularizations have also been proposed [47, 51–53]. The regularization proposed in Refs. [51, 52] leads to a well defined special scalar-tensor theory, a member of the family of Horndeski gravity. Hennigar et al. [47] proposed a well defined $D \to 4$ limit of EGB gravity generalizing the previous work of Mann and Ross [54]. However, the spherically symmetric $4D$ black hole solution obtained in [20, 23] still remains valid in these regularized theories [47, 51, 53]. Hence one can say that these regularization procedures lead to exactly the same black hole solutions [20, 22, 23] at least for the case of $4D$ spherically symmetric spacetimes, but may not be valid beyond spherical symmetry [55]. Thus it turns out that the spherically symmetric solution obtained using any of these regularization methods will be the same. However, for convenience, we shall follow the regularization proposed in Ref. [20].

The action of EGB gravity, which is motivated by the heterotic string theory [2, 56], coupled to NED becomes [17]

$$\mathcal{I} = \frac{1}{2} \int d^D x \sqrt{-g} \left[ \mathcal{R} - 2\Lambda + \alpha \mathcal{L}_{GB} \right] + \mathcal{I}_{NED},$$  \hspace{1cm} (1)

which contains the Einstein-Hilbert (EH) action $\mathcal{R}$, a cosmological term, $\Lambda = -(D - 1)(D - 2)/2l^2$, and the GB quadratic curvature correction given by

$$\mathcal{L}_{GB} = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R^{\mu\nu} + R^2.$$

(2)

The GB coupling constant $\alpha > 0$ is of dimension [length$^2$], and it is related to the string scale. The matter is described by the action $\mathcal{I}_{NED}$, which for the NED reads

$$\mathcal{I}_{NED} = \int d^D x \sqrt{-g} \mathcal{L}(F),$$

(3)

where the Lagrangian density $\mathcal{L}(F)$, is an arbitrary continuous function of the invariant $F = \frac{1}{4} F_{ab} F^{ab}$ and $F_{ab} = \partial_a A_b - \partial_b A_a$, with $A_a$ being the potential for the NED charge. Varying the action (1), we obtain the equations of
motion [17]
\[ G_{ab} + \alpha H_{ab} = T_{ab} \equiv 2 \left[ \frac{\partial \mathcal{L}(F)}{\partial F} F_{ac} F_b^c - g_{ab} \mathcal{L}(F) \right], \]  
(4)
\[ \nabla_a \left( \frac{\partial \mathcal{L}(F)}{\partial F} F^{ab} \right) = 0 \quad \text{and} \quad \nabla_a (\ast F^{ab}) = 0, \]  
(5)
where \( G_{ab} \) and \( H_{ab} \), respectively, are the Einstein tensor and the Lanczos tensor [2]:
\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab}, \]
\[ H_{ab} = 2 \left[ RR_{ab} - 2 R_{ac} R_b^c - 2 R^{cd} R_{acbd} + R_{acde} R_{bdce} - \frac{1}{2} g_{ab} \mathcal{L}_G \right], \]  
(6)
with energy momentum tensor [17]
\[ T_{ab} = -\frac{2}{\sqrt{-g}} \delta T_{NED}, \]
(7)
\[ T^a_b = 2 \left[ \frac{\partial \mathcal{L}(F)}{\partial F} F^{ac} F_{bc} - \delta_a^b \mathcal{L}(F) \right]. \]
(8)
We wish to obtain static spherically symmetric black hole solutions of Eq. (4). We consider the metric to be of the following form [12]
\[ ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_{D-2}^2, \]  
(9)
where \( d\Omega_{D-2} \) is the metric of a \((D - 2)\)-dimensional unit sphere and \( T_{ab} \) is the energy momentum tensor of matter that we consider as a NED as in Ref. [15], where the Lagrangian reads
\[ \mathcal{L}(F) = \beta F \exp \left[ -k g^{-\gamma} (2F)^\zeta \right], \]  
(10)
where \( \beta = (D - 2)(D - 3)/2, \gamma = (D - 3)/(D - 2), \zeta = (D - 3)/(2D - 4) \), \( q \) is magnetic charge, and \( k \) is NED parameter. The Maxwell field reads
\[ F_{\mu\nu} = 2\delta^D_{[\mu} \delta^D_{\nu]} q \sin \theta_1; \quad \text{if} \quad D = 4, \]
\[ F_{\mu\nu} = 2\delta^D_{[\mu} \delta^D_{\nu]} q^{D-3} \frac{D-3}{r^{D-4}} \sin \theta_{D-3} \left[ \prod_{j=1}^{D-4} \sin^2 \theta_j \right]; \quad \text{if} \quad D \geq 5. \]  
(11)
Eq. (5) implies that \( df = 0 \) so that we obtain
\[ q(r) 2\delta^D_{[\mu} \delta^D_{\nu]} q^{D-3} \frac{D-3}{r^{D-4}} \sin \theta_{D-3} \left[ \prod_{j=1}^{D-4} \sin^2 \theta_j \right] d\theta \wedge d\phi \wedge \ldots \wedge d\psi_{(D-2)} = 0. \]  
(12)
This leads to \( q(r) = q = \text{constant magnetic charge} \). Hence the field strength tensor \( F_{\theta\phi} \) is
\[ F_{\theta\phi} = \frac{q(r)}{r^{D-4}} \sin \theta_{D-3} \left[ \prod_{j=1}^{D-4} \sin^2 \theta_j \right], \]  
(13)
The other components of \( F_{\mu\nu} \) have negligible influence in comparison of \( F_{\theta\phi} \). Then \( F \) and \( \mathcal{L}(F) \) are, respectively, simplified as
\[ F = \frac{q^2}{2r^{2(D - 2)}} \quad \text{and} \quad \mathcal{L}(F) = \frac{\beta q^2}{2r^{2(D - 2)}} \exp \left[ -k \frac{D-3}{r^{D-3}} \right]. \]  
(14)
The energy momentum tensor can be given as
\[ T^t_t = T_r^r = \frac{(D - 2)MK}{r^{2D-4}} \exp \left( -k r^{D-3} \right) = \rho(r). \]  
(15)
The Bianchi identity $T^a_{\alpha b} = 0$ gives

$$0 = \partial_r T^r_r + \frac{1}{2} g^{00} [T^r_r - T^0_0] \partial_r g_{00} + \frac{1}{2} \sum g^{ii} [T^r_r - T^i_i] \partial_r g_{ii},$$

and the energy momentum tensor is completely specified by (15) and (17) and corresponds to an anisotropic fluid. We are interested in a static spherical solution of 4D EGB gravity coupled to NED which can be obtained by integrating the $(r, r)$ equation of (4), in the limit $D \to 4$, and the solution reads as

$$f_{\pm}(r) = 1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + 4\alpha \left( \frac{2M \exp(-k/r)}{r^3} - \frac{1}{l^2} \right)} \right),$$

by appropriately relating $M$ with integration constants. Solution (18) is an exact solution of the field equation (4) for stress energy tensor $T_{ab}$. The charged 4D EGB black hole solution (18), in absence of the NED field, $k = 0$, reduces to the Glavan and Lin [20] solution (5), and for $r \gg k$ it becomes

$$f_{\pm}(r) = 1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + 4\alpha \left( \frac{2M}{r^3} - \frac{q^2}{r^4} - \frac{1}{l^2} \right)} + O \left( \frac{1}{r^2} \right) \right).$$

We note that the above form is exactly that of the charged black hole of the 4D EGB gravity with a negative cosmological constant [26], if one identifies the electric charge as $q^2 = 2Mk$. Whereas, in the limit $\alpha \to 0$ or large $r$, the solution (18) behaves asymptotically as

$$f_-(r) \approx 1 - \frac{2M}{r} \exp \left( -\frac{k}{r^2} \right) + O \left( \frac{1}{r^2} \right),$$

$$f_+(r) \approx 1 + \frac{2M}{r} \exp \left( -\frac{k}{r^2} \right) - \frac{r^2}{\alpha} + O \left( \frac{1}{r^2} \right),$$

the $-$ve branch corresponds to the 4D regular AdS black hole [57], whereas the $+$ve branch does not lead to a physically meaningful solution as the opposite sign in the mass indicates instabilities of the graviton [4], and hence we shall confine ourselves to the $-$ve branch of the solution (18). Further if we take the limit $r \to \infty$ or $M = 0$ in solution (18), the $-$ve branch of the solution (18) is asymptotically flat whereas the $+$ve branch of the solution (18) is asymptotically dS (AdS) depending on the sign of $\alpha$ ($\pm$).

### A. Horizons and extremality

The solution (18) can be characterized by the mass $M$, the cosmological constant $1/l^2$, the GB coupling constant $\alpha$, and the deviation parameter $k$, which is assumed to be positive. For definiteness we call the solution (18) as the NED charged 4D EGB black hole. The horizon radii are zeros of $g^{rr} = 0$ of $f(r_h) = 0$, which implies that

$$\frac{r_h^4}{l^2} + \frac{r_h^2}{2} - 2Mr_h \exp \left( -\frac{k}{r_h} \right) + \alpha = 0.$$  

We solve the above equation numerically to find that it admits multiple roots (cf. Fig. 1) and depending on the choice of parameters $M, \alpha$, and $k$, we have two distinct positive roots $r_h = r_{\pm}$ corresponding to two horizons, namely, the smaller inner or Cauchy horizon ($r_-$), and the outer event horizon ($r_+ > r_-$). It turns out that, for a given $M$ and $\alpha$ there exists a critical $k_E$ and radius $r_E$ such that, for $r_h > 0$, $f(r_h) = 0$ admits one double zero at $r_E$ if $k = k_E$, and two simple zeros at $r_{\pm}$ if $k < k_E$ (cf. Fig. 1). These two cases therefore describe, respectively, a regular extremal black hole with degenerate Killing horizon, and a regular non-extremal black hole with both outer and inner Killing horizons. Whereas if $k > k_E$, $f(r) = 0$ has no roots or a black hole does not exist. Using similar arguments, we can also determine $\alpha_E$. It is evident from Fig. 1 that the event horizon radius $r_+$ decreases with increase in the deviation parameter ($k$) and the GB coupling constant $\alpha$. 


III. BLACK HOLE THERMODYNAMICS

It may be useful to investigate how the NED affects the thermodynamical properties of 4D EGB black holes; therefore we calculate the thermodynamic quantities associated with the NED charged 4D EGB AdS black hole. We note that the gravitational mass of a black hole, determined by \( f(r_+) = 0 \) \[12\], which reads as

\[
M_+ = \frac{r_+ \exp \left(\frac{k}{r_+}\right)}{2} \left(1 + \frac{\alpha}{r_+^2} + \frac{r_+^2}{l^2}\right),
\]

and for \( k = 0 \) it reduces to that for the 4D EGB AdS black hole \[20\], to the 4D charged EGB black hole \[26\] for \( r \gg k \), and for \( \alpha \to 0 \) we obtain the mass for the regular black hole in AdS spacetime. Let us analyze the effects of NED on the \( M_+ \), which is depicted in the Fig. 2, where we have shown \( M_+ \) as a function of horizon radius \( r_+ \) for different values of \( k \) and \( \alpha \), and compared also with the neutral \((k = 0)\) 4D EGB counterpart. A minimum mass \( M_+^{\text{min}} \) for the existence of 4D EGB black holes (charged or neutral) occurs at relatively smaller radii \( r_+ \) and also \( r_+^{\text{min}} \), where the minimum mass appears, are larger in the NED charged case \((k \neq 0)\) and so is the \( M_+^{\text{min}} \) values when compared with the neutral \((k = 0)\) 4D EGB counterpart (cf. Fig. 2) \[20\]. Figure 2 infers that for \( M = M_+^{\text{min}} \), the black hole possesses a degenerate horizon at \( r_+^{\text{min}} \); whereas, for \( M > M_+^{\text{min}} \) two distinct horizons exists; nevertheless, \( r_+^{\text{min}} \) can be identified as \( r_E \). Hence the NED charged 4D EGB extremal black hole are heavier when compared to the neutral ones.
The Hawking temperature \([58]\) can also help us to understand the NED effects on the final stage of the black hole evaporation. The Hawking temperature of a black hole can be obtained using the relation \(T = \kappa/2\pi\), where \(\kappa\) is the surface gravity given by \([58, 59]\)

\[
\kappa = \left( -\frac{1}{2} \nabla_\mu \xi_\nu \nabla^\mu \xi^\nu \right)^{1/2},
\]

and \(\xi^\mu = \partial/\partial t\) is the timelike Killing vector for the black hole metric \((9)\). On using our solution \((18)\), the Hawking temperature for the NED charged 4D EGB black hole reads

\[
T_+ = \frac{1}{4\pi r_+^2 (r_+^2 + 2\alpha)} \left[ \frac{r_+^4 (3r_+ - k)}{l^2} + r_+^2 (r_+ - k) - \alpha(r_+ + k) \right].
\]

The temperature of the NED charged 4D EGB black hole \((24)\) reduces to that of the 4D charged EGB black hole \([26]\) when \(r \gg k\), 4D EGB black hole \([20]\) in the limit of \(k = 0\), 4D AdS regular black hole when \(\alpha \to 0\), and also to the Schwarzschild black hole for \(\alpha \to 0\) and \(k = 0\).

We plot the NED charged 4D EGB black hole Hawking temperature \((T_+)\) as a function of horizon radii \((r_+)\) for different values of \(k\) and fixed \(\alpha\) (cf. Fig. 3). From Fig. 3, it is evident that with decreasing \(r_+\) the Hawking temperature decreases to attain a local minimum \(T_{\text{min}}\) at horizon radii \(r_+^0\) and then grows to a local maximum \(T_{\text{max}}\) at radii \(r_+^m (r_+^m < r_+^0)\), then further steeply drops to zero temperature at the critical radius \(r_+^\text{crit}\): Colored points in Fig. 3 represent the \(T_{\text{max}}\) and \(T_{\text{min}}\) at radii \(r_+^m\) and \(r_+^0\). It turns out that the local maximum and minimum values of the Hawking temperature decreases with increase in the values of NED parameter \(k\) as well as with GB coupling constant \(\alpha\) (cf. Fig. 3). The temperature \(T_{\text{max}}\) for the NED charged 4D EGB black hole is lower when compared with analogous 4D EGB case. We note that for the extremal black holes with the degenerate horizon radius \(r_E\) (or \(r_+^\text{min}\)) the temperature vanishes, \(T_+ = 0\), thereby in the end stage of Hawking evaporation we are left with the zero temperature extremal black holes with minimal mass as a stable remnant. It is evident from Fig. 3, that the NED charged 4D EGB black holes have larger remnant size compared to the uncharged case. Furthermore, for \(T_{\text{min}} \leq T_+ \leq T_{\text{max}}\) there exists two black hole configurations at equilibrium, such that the smaller black hole, represented by the branch with \(r_+ < r_+^m\) is thermodynamically unstable, whereas the larger black hole with \(r_+ > r_+^m\) is thermodynamically stable.

We can consider the black hole as a canonical ensemble system where the chemical potential \(\phi\), associated with the NED charge \(q\), is held fixed. We calculate the entropy \((S_+)\) of the black hole in terms of horizon radius \(r_+\). In GR the entropy satisfies the black hole’s area law \(S_+ = A/4\). The black hole behaves as a thermodynamic system; quantities associated with it must obey the first law of thermodynamics \(dM_+ = T_+ dS_+ + \phi dq\). The entropy \([59]\), for constant charge \(q\), can be obtained by integrating the first law as

\[
S_+ = \frac{A}{4} \left[ \left( 1 + \frac{k}{r_+} \right) \exp \left[ \frac{k}{r_+} \right] - \frac{k^2 + 4\alpha}{r_+} \text{Ei} \left[ \frac{k}{r_+} \right] \right],
\]

where \(\text{Ei}[k/r_+]\) is the exponential integral function. Black holes, as thermodynamic systems, should have positive entropy and the cases when it becomes negative will be excluded from our discussion. The entropy of the 4D EGB

![FIG. 3: Plot of Hawking temperature \(T_+\) vs horizon radius \(r_+\) for different values of GB coupling constant \(\alpha\) and deviation parameter \(k\). Colored points correspond to local minima and maxima of temperature.](image-url)
black hole, in the absence of NED charge \((k = 0)\), becomes

\[
S_+ = \frac{A}{4} + 2\pi\alpha \log \left(\frac{A}{A_0}\right),
\]

with \(A = 4\pi r_+^2\) and \(A_0\) is a constant \([27, 36]\). This is the standard area law known as the Bekenstein-Hawking area law for 4D EGB black holes having correction terms \([24, 36]\). However it is interesting to note that the entropy of the black hole has no effect of a AdS background and is same as in the asymptotic flat case. Further, we also notice from Eq. (25) that the thermodynamic entropy can become negative due to NED.

### A. Global stability

The Hawking-Page \([19]\) phase transition states that asymptotically AdS Schwarzschild black holes are thermally favoured when their temperatures are sufficiently high, whereas the pure AdS background spacetimes are preferred at comparatively low temperatures and there occurs a phase transition between the thermal AdS and the AdS black holes at some critical temperature. The phase transition has been widely studied for higher dimensional EGB gravity asymptotically AdS black holes \([17, 18, 60]\) and we wish to analyse this for the NED charged 4D EGB black hole via Helmholtz’s free energy. The reason for this is that even if the black hole is locally thermodynamically stable, it could be globally unstable or vice-versa \([60–62]\). The Helmholtz free energy of a black hole can be defined as \([61]\)

\[
F_+ = M_+ - T_+ S_+,
\]

which for the NED charged 4D EGB black hole reads as

\[
F_+ = \frac{1}{4\pi r_+^2 (r_+^2 + 2\alpha)} \left[ 2 \exp \left( \frac{k}{r_+} r_+^2 + 2\alpha \right) \left( r_+^4 + \frac{k^2}{r_+^2} (r_+^2 + \alpha) \right) + \left( -3r_+^5 + \frac{k^2}{r_+^2} (-r_+^2 + \alpha) + k \left( r_+^4 + \frac{k^2}{r_+^2} (r_+^2 + \alpha) \right) \right) \right.
\]

\[
\times \left( \exp \left[ \frac{k}{r_+} (k + r_+) - \frac{k^2}{r_+^2} (k^2 + 4\alpha) E_i \left( \frac{k}{r_+} \right) \right] \right].
\]

Generally, it is demonstrated that black holes with negative values of \(F_+\) are more thermodynamically stable. The Helmholtz free energy of NED charged 4D EGB black holes for various values of parameters \(k\) and \(\alpha\) is depicted in Fig. 4. As it is shown in Fig. 4, the free energy \(F_+\) for various \(k\), have local minimum and local maximum, respectively, at horizon radii \(r_+^a\) and \(r_+^b\) with \(r_+^b > r_+^a\), which can be identified as the extremal points of the Hawking temperature shown in Fig. 3 and where the specific heat capacity \(C_+\) diverges. For \(r_+ > r_+^b\), the free energy \(F_+\) is a monotonically decreasing function of \(r_+\) and becomes negative at large \(r_+\), i.e., \(F_+(r_{HP}) = 0\) such that \(F_+ > 0\) for \(r_+ < r_{HP}\) and \(F_+ < 0\) for \(r_+ > r_{HP}\). Whereas for \(r_+ < r_+^a\) the free energy \(F_+\) decreases with decreasing \(r_+\) and attains a local minimum at

![FIG. 4: Plot of free energy \(F_+\) vs horizon radius \(r_+\) for different values of GB coupling constant \(\alpha\) and deviation parameter \(k\). Colored points correspond to local minima and maxima of free energy.](https://example.com/figure4.png)
$r_+^a$ with $F_+(r_+^a) > 0$, and with further decreasing $r_+$ the $F_+$ starts increasing (cf. Fig. 4). The Hawking-Page first order phase transition occurs at $r_+ = r_{HP}$, where the free energy turns negative viz., $r_{HP} > r_+^b$. Thus the larger black holes, with horizon radii $r_+ > r_{HP}$, are thermodynamically globally stable. However, at very small horizon radii the Hawking temperature is negative and hence not physical for global stability. This is exactly in accordance with the Hawking-Page phase transition in GR [19]. In addition, one can find the temperature $T_+ = T_{HP}$ at which Hawking-Page phase transition happens, in terms of the horizon radius $r_+$, by solving $F(T_{HP}) = 0$. One can notice that $T_{HP} > T_{min}$ (cf. Fig. 5). Therefore for $T_+ > T_{HP}$ we find that the black hole solution is favored globally with respect to the thermal AdS background solution as $F_+ < 0$. While for $T_{min} < T_+ < T_{HP}$ the radiation in the AdS background solution is globally favored over the black hole as $F_+ > 0$. In particular, for $\alpha = 0.20$, and $k = 0.10, 0.30, 0.50$, and 0.70, the horizon radii $r_+^\alpha$ are, respectively, $r_+^a = 1.22133, 1.54263, 1.92382, 2.42639$ where $T_+$ admits a local maximum and $F_+$ attains a local minimum, and $r_+^b$ are, respectively, $r_+^b = 5.45688, 5.1966, 4.87942, 4.44678$ where $T_+$ admits a local minimum and $F_+$ attains a local maximum. Whereas the horizon radii $r_{HP}$, for $\alpha = 0.20$, and $k = 0.10, 0.30, 0.50$, and 0.70 are, respectively, $r_{HP} = 9.35535, 9.33188, 9.19912, 9.04365$ and the corresponding temperature are, respectively, $T_{HP} = 0.0305108, 0.0301342, 0.0295824, 0.0289863$. Therefore one can infer that, with increasing $k$, the horizon radii $r_{HP}$ and the corresponding Hawking temperature $T_{HP}$ for the Hawking-Page phase transition decrease. In Fig. 5, we have shown the behaviour of $F_+$ with $T_+$. The behaviour at small $r_+$ is bit unusual from GR and higher dimensional EGB which is attributed to the logarithmic correction term.

B. Local stability

Having discussed the conditions for global thermodynamical stability of a NED charged 4D EGB black hole, we turn our attention to the local thermodynamics by computing the heat capacity which informs us about the thermal stability of the black hole under temperature fluctuations. The reason to consider local stability is that even when a black hole configuration is globally stable, it could be locally unstable [11]. We use the canonical ensemble where the charge is fixed to investigate local instability. The local stability of the black hole depends on behaviour of the heat capacity $C_+$. When $C_+ > 0$ the black hole is locally thermodynamical stable and $C_+ < 0$ means it is unstable. To analyse the thermodynamic stability of the NED charged 4D EGB black hole, we calculate its heat capacity $C_+$, and

![Figure 5: Plot of free energy $F_+$ vs horizon temperature $T_+$ for different values of GB coupling constant $\alpha$ and deviation parameter $k$.](image-url)
see how the NED affects thermodynamic stability. The heat capacity of the black hole is given \[ C_+ = \frac{\partial M_+}{\partial T_+} = \left( \frac{\partial M_+}{\partial r_+} \right) \left( \frac{\partial r_+}{\partial T_+} \right). \] (28)

Substituting the values of mass and temperature from Eqs. (22) and (24) in Eq. (28), we obtain the heat capacity of the 4D AdS regular EGB black hole as

\[ C_+ = -\frac{2\pi \exp \left[ \frac{1}{r_+^2} \right] (r_+^2 + 2\alpha) \left( 3r_+^2 + l^2 (r_+^2 - \alpha) - k (r_+^4 + l^2 (r_+^2 + \alpha)) \right)}{2k \left( 2r_+^4 \alpha - l^2 (r_+^4 + 2r_+^2 \alpha + 2\alpha^2) + l^2 r_+ (r_+^4 - 5r_+^2 \alpha - 2\alpha^2) - 3r_+^5 (r_+^2 + 6\alpha) \right)}. \] (29)

To further analyse \( C_+ \), we plot the heat capacity in Fig. 6 for different values of deviation parameter \( k \) and GB coupling constant \( \alpha \), which clearly exhibits that the heat capacity, for a given value of \( k \) and \( \alpha \), is discontinuous at the critical radii \( r_a^+ \) (cf. Fig. 6 left) and \( r_b^+ \) (cf. Fig. 6 right) with \( r_a^+ < r_b^+ \) (cf. Fig. 6). This signals a second order phase transition [19, 64]. Thus, a NED charged 4D EGB black hole is thermodynamically stable for \( r_0 < r_+ < r_a^+ \) and \( r_+ > r_b^+ \), whereas it is thermodynamically unstable for \( r_+ < r_+ < r_a^+ \) and \( r_+ < r_0 \). It is evident from Fig. 6, that the black hole undergoes a phase transition twice, firstly at \( r_a^+ \) from the smaller stable black hole to larger unstable

FIG. 6: NED Charged 4D EGB black hole specific heat \( C_+ \) vs. horizon radius \( r_+ \) for different values of GB coupling parameter \( \alpha \) and NED parameter \( k \).
black holes and then secondly at \( r_{+}^{k} \) from the smaller unstable black hole \( (r_{+}^{a} < r_{+} < r_{+}^{k}) \) to larger stable black holes \( (r_{+} > r_{+}^{k}) \). For fixed value of \( \alpha \) and increasing NED parameter \( k \), the critical radii \( r_{+}^{a} \) increase, whereas radii \( r_{+}^{k} \) decrease (cf. Fig. 6). The heat capacity \((29)\), in the limit \( \alpha \to 0 \), reduces to the value for the analogous GR case, which is the heat capacity of the asymptotically 4D regular black holes. The heat capacity Eq. \((29)\), in the absence of deviation parameter \( (k = 0) \), reduces for the 4D AdS EGB black hole \([27]\).  

a. **Black hole remnant:** We finally comment on the black hole remnant which is considered as a source for dark energy \([65]\) and also serves as one of the potential candidates to resolve the information loss puzzle \([66]\). It turns out that the extremal black hole with degenerate horizon is given by  

\[
f(r_{E}) = f'(r_{E}) = 0. \tag{30}
\]

Thus at the extremal black hole one obtains \( r_{E} = r_{-} = r_{+} \), and the temperature decreases with increasing \( r_{-} \) and finally vanishes leaving a regular double-horizon remnant with \( M = M_{t}^{\min} \) which is depicted in the Fig. 7. It means that at the late stage of Hawking evaporation the black hole attains a maximum temperature and then cools down at \( r_{E} \) with a stable remnant mass \( M_{t}^{\min} \). Thus the NED charged 4D EGB black hole shrinks to a dS-like core with a corresponding remnant of mass \( M_{t}^{\min} \). It can be seen that the temperature decreases with decreasing horizon radius and vanishes when the two horizons coincide \( T_{+} \to 0 \), \( C_{+} \to 0 \) as \( r_{-} \to r_{+} \). Hence a NED charged 4D EGB black hole has a zero temperature thermodynamically stable remnant of mass \( M_{t}^{\min} \) and size \( r_{E} \) (cf. Fig. 7). Thus, we can say that NED charged 4D EGB black hole has in general two horizons, which degenerate to one at \( M = M_{t}^{\min} \). The black holes has a phase transition where a heat capacity diverges and flips its sign; a mass decreases during evaporation, temperature vanishes at a double horizon thereby sudden halt of evaporation leaving a double-horizon remnant with \( M = M_{t}^{\min} \).  

![Plot of metric function f(r) vs horizon radius r_+ for different values of deviation parameters k and α.](image)

**FIG. 7:** The plot of metric function \( f(r) \) as the function horizon radius \( r_{+} \) for different values of deviation parameters \( k \) and \( \alpha \).  

C. **P-V criticality**

We are considering the cosmological constant in the extended phase space, where the pressure is related to \( \Lambda \) through \( P = -\Lambda /8\pi = 3(2\alpha + r_{+}^{2})T_{+} \) which leads to the interpretation of mass not only as internal energy but also as Enthalpy \( H_{+} \) of thermodynamical system \([67]\). This interpretation leads to following relation for the free energy \( F_{+} = H_{+} - T_{+}S_{+} \) of the system \([68, 69]\). The behaviour of free energy \( F_{+} \) vs temperature \( T_{+} \) for different values of pressure \( P \) and GB coupling \( \alpha \) is depicted in the Fig. 8. As shown in the Fig. 8, Gibbs Free develops swallow tail structure when the pressure is below than the critical pressure \( P_{c} \), which infers the first order phase transition. When \( P = P_{c} \), the shallow tail disappear corresponding to the critical point, and when the thermodynamic pressure is larger than the critical pressure \( P_{c} \) no phase transition will occur. Using the temperature \( T_{+} \) and specific volume \( v = 2r_{+} \) \([70]\), we can obtain the following equation of state from Eq. \((24)\)  

\[
P_{+} = \frac{3(2\alpha + r_{+}^{2})T_{+}}{2r_{+}^{2}(5r_{+} - k)} + \frac{3(r_{+}^{2}(r_{+} - k) + \alpha(k + r_{+}))}{8\pi r_{+}^{6}(5r_{+} - k)}.
\]

\[
\tag{31}
\]
FIG. 8: Plot of free energy $F_+$ vs temperature $T_+$ for different values of pressure $P$. The value of critical pressure is $P_c = 0.000197371$ for $\alpha = 0.20$ and $P_c = 0.0000711748$ for $\alpha = 0.40$.

FIG. 9: Plot of pressure $P_+$ vs radii $r_+$ for different values of temperature $T < T_c$, $T = T_c$ and $T > T_c$. The critical temperature is $T_c = 0.0332604$ for $\alpha = 0.20$ and $T_c = 0.026007$ for $\alpha = 0.40$.

Now to calculate the critical values, one can use the inflection point properties [69]

$$
\left( \frac{\partial P}{\partial r_+} \right)_T = 0, \quad \left( \frac{\partial^2 P}{\partial r_+^2} \right)_T = 0.
$$

We solved the Eq. (32) for critical horizon radius $r_c$ and critical temperature $T_c$, and used Eq. (31) to obtain the critical pressure $P_c$. We summarized the numerical values of critical parameters in Table I for various values of NED charge parameter $k$. In order to elaborate the effect of NED parameter $k$ and GB coupling $\alpha$ we plot the pressure $P_+$ with horizon radius $r_+$ for various isotherms in Fig. 9. From the Table I, one can see that for fixed $\alpha$ and increasing $k$, the critical radius $r_c$ increases, whereas critical pressure $P_c$ and critical temperature $T_c$ decrease. In addition the global parameter also decreases with increasing the NED charge $k$. 

\[ \text{FIG. 8: Plot of free energy } F_+ \text{ vs temperature } T_+ \text{ for different values of pressure } P. \text{ The value of critical pressure is } P_c = 0.000197371 \text{ for } \alpha = 0.20 \text{ and } P_c = 0.0000711748 \text{ for } \alpha = 0.40. \] 

\[ \text{FIG. 9: Plot of pressure } P_+ \text{ vs radii } r_+ \text{ for different values of temperature } T < T_c, T = T_c \text{ and } T > T_c. \text{ The critical temperature is } T_c = 0.0332604 \text{ for } \alpha = 0.20 \text{ and } T_c = 0.026007 \text{ for } \alpha = 0.40. \] 

\[ \text{Now to calculate the critical values, one can use the inflection point properties [69]} \]

\[ \left( \frac{\partial P}{\partial r_+} \right)_T = 0, \quad \left( \frac{\partial^2 P}{\partial r_+^2} \right)_T = 0. \quad (32) \]

\[ \text{We solved the Eq. (32) for critical horizon radius } r_c \text{ and critical temperature } T_c, \text{ and used Eq. (31) to obtain the critical pressure } P_c. \text{ We summarized the numerical values of critical parameters in Table I for various values of NED charge parameter } k. \text{ In order to elaborate the effect of NED parameter } k \text{ and GB coupling } \alpha \text{ we plot the pressure } P_+ \text{ with horizon radius } r_+ \text{ for various isotherms in Fig. 9. From the Table I, one can see that for fixed } \alpha \text{ and increasing } k, \text{ the critical radius } r_c \text{ increases, whereas critical pressure } P_c \text{ and critical temperature } T_c \text{ decrease. In addition the global parameter also decreases with increasing the NED charge } k. \]
TABLE I: The numerical value of critical exponents (radius $r_c$, temperature $T_c$, pressure $P_c$) and GB Coupling $\alpha = 0.2$ and $\alpha = 0.4$.

| $k$ | $r_c$ | $P_c$ | $T_c$ | $P_{c/v}/T_c$ | $r_c$ | $P_c$ | $T_c$ | $P_{c/v}/T_c$ |
|-----|-------|-------|-------|---------------|-------|-------|-------|---------------|
| 0.1 | 1.46092 | 0.000821072 | 0.0452288 | 0.053022 | 1.99844 | 0.00201186 | 0.0721711 | 0.1114181 |
| 0.2 | 1.62635 | 0.000299952 | 0.0373109 | 0.026149 | 2.16228 | 0.000009706 | 0.0283228 | 0.0148199 |
| 0.3 | 1.79649 | 0.000197371 | 0.0332604 | 0.0213212 | 2.32881 | 0.000007117 | 0.0239742 | 0.0110315 |
| 0.4 | 1.97222 | 0.000133079 | 0.0298641 | 0.017577 | 2.4988 | 0.000005292 | 0.0269926 | 0.0146365 |
| 0.5 | 2.15382 | 0.000091715 | 0.0260071 | 0.0127468 | 2.67269 | 0.00003984 | 0.0221814 | 0.0096007 |

IV. CONCLUSION

Lately there has been a surge of interest in regularization of EGB gravity, the limit $D \to 4$ of the $D$-dimensional solutions of EGB gravity. Interestingly, the exact static spherically symmetric solutions in the various proposed $D \to 4$ gravities coincide, and incidentally some other theories also admit the same exact solution. We have obtained an exact $4D$ static spherically symmetric black hole solution to the $4D$ EGB gravity coupled to the NED which encompasses the black hole solutions of Glavan and Lin [20] when NED is switched off ($k = 0$) and asymptotically ($r \gg k$) mimics the charged black holes of Fernandes [26]. The NED charged $4D$ EGB-AdS black hole metric is characterized by horizons which could be at most two, describing different objects including an extremal black hole with degenerate horizons and non-extremal black holes with two distinct horizons. The thermodynamic quantities associated with NED charged $4D$ EGB-AdS black hole have been analysed as a function of $r_+$, $k$ and $\alpha$. With regard to the thermodynamics properties, we noticed some significant NED effects and corrections to the previously obtained $4D$ EGB black holes were discovered. The Hawking temperature, as in the asymptotically flat case, does not diverge as the event horizon shrinks down; instead, it has a local minimum before taking a maximum value for a critical radius and then drops down to zero at degenerate horizons, which happens for larger values of the radius $r_+$ due to NED (cf. Fig. 3) and also increases with parameter $\alpha$. The entropy of a $4D$ EGB-AdS black hole is exactly the same as in the asymptotically flat case. The entropy of a black hole in GR obeys the area law, but not for neutral $4D$ EGB-AdS black holes where it has a logarithmic correction term whereas the NED charged $4D$ EGB-AdS black hole is more complicated. The heat capacity diverges at critical horizon radii $r_{c+}$ and $r_{c-}$, which depends on the NED parameter $k$, and incidentally local extrema of the Hawking temperature also occur at these radii. The phase transition is detectable by the divergence of the heat capacity ($C_+$) at critical radii (changes with NED parameter $k$), such that the black hole is stable in the region $r_+ < r_+ < r_{c+}$ and $r_+ > r_{c+}$ with positive heat capacity ($C_+ > 0$), on the other hand the heat capacity is negative ($C_+ < 0$), when $r_{c+} < r_+ < r_{b+}$ and $r_+ < r_0$, indicating the instability of black holes.

We find that the NED has a profound influence on the properties of black holes which may have several astrophysical consequences, for example, on wormholes and accretion onto black holes. Some of the results presented here are generalizations of previous discussions on $4D$ EGB [20, 26] and GR black holes [12], to a more general setting. The possibility of a further generalization of these results to Lovelock gravity [45] is an interesting problem for the future.

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