Tsallis statistics approach to the transverse momentum distributions in p-p collisions

Maciej Rybczyńskiab, Zbigniew Włodarczyka

aInstitute of Physics, Jan Kochanowski University, ul. Świętokrzyska 15, PL-25406 Kielce, Poland
bDepartment of Physics, University of Cergy-Pontoise, 95000 Cergy-Pontoise, France

1 Introduction

Transverse momentum (\(p_T\)) distributions of identified hadrons are the most common tools used to study the dynamics of high energy collisions. The p-p interactions are used as a baseline and are important to understand the particle production mechanisms. In the framework of Tsallis statistics, the momentum distribution is given by

\[
\frac{d^3N}{dp_T^3} = \frac{gV}{(2\pi)^3} \left[ 1 + (q-1) \frac{E - \mu}{T} \right]^{\frac{1}{1-q}} \exp \left( - \frac{E - \mu}{T} \right),
\]

where \(T\) and \(\mu\) are the temperature and the chemical potential, \(V\) is the volume and \(g\) is the degeneracy factor. In this form, Eq. (1) is usually supposed to represent a nonextensive generalization of the Boltzmann-Gibbs exponential distribution, with \(q\) being a new parameter, in addition to previous "temperature" \(T\). Such an approach is known as nonextensive statistics [2] in which the parameter \(q\) summarily describes all features causing a departure from the usual Boltzmann-Gibbs statistics. In particular it was shown in [5] that \(q - 1 = \text{Var}(T)/\langle T \rangle^2\) and directly describes intrinsic fluctuations of temperature (however, the Tsallis distribution also emerges from a number of other more dynamical mechanisms, for example see [6] for more details and references). This approach has been shown to be very successful in describing multiparticle production processes of a different kind (see [6, 7] for recent reviews). In terms of transverse momentum, transverse mass, \(m_T = \sqrt{m^2 + p_T^2}\), and rapidity \(y\), Eq. (1) becomes

\[
\frac{d^2N}{p_T dp_T dy} = \frac{gV m_T \cosh(y)}{(2\pi)^2} \times
\]

\[
\times \left[ 1 + (q-1) \frac{m_T \cosh(y) - \mu}{T} \right]^{\frac{1}{1-q}}.
\]

It has been shown repeatedly that the Tsallis distribution gives an excellent description of \(p_T\) spectra measured in p-p collisions at RHIC (\(\sqrt{s} = 62.4\) and 200 GeV) and LHC (\(\sqrt{s} = 0.9, 2.76\) and 7 TeV) energies [3, 8–11]. In particular changes in the transverse momentum distribution with energy (used data at energies 0.54, 0.9, 2.36 and 7 TeV) are studied using the Tsallis distribution (2) as a parametrization [12]. In this paper we extend this analysis to transverse momentum spectra obtained in p-p collisions at \(\sqrt{s} = 6.27, 7.4, 8.76, 12.32\) and 17.27 GeV by the NA61/SHINE collaboration [13] 1. In addition to possibility of study collis-

1Recently, the experimental results on inclusive spectra of negatively charged pions produced in inelastic p-p interactions at beam momenta 20, 31, 40, 80 and 158 GeV/c were presented [13]. The measurements were performed using the large acceptance NA61/SHINE hadron spectrometer at the CERN Super Proton Synchrotron. Numerical results corresponding to the two dimensional spectra in transverse momentum and rapidity corrected for experimental biases were given in Ref. [14].
2 Analysis of transverse momentum distributions

Transverse momentum spectra of negatively charged pions are fitted using Tsallis distribution given by Eq. (2) with \( g_{\pi^-} = 1 \) and \( \mu = 0 \). It is worth to be noted that the variable \( T \) and \( V \) are functions of \( \mu \) at fixed values of \( q \),

\[
T = T_0 + (q - 1) \mu, \\
V = V_0 [1 + (q - 1) \mu / T_0]^{q/(1-q)} = V_0 (T / T_0)^{q/(1-q)}
\]

and they can be calculated if the parameters \( T = T_0 \) and \( V = V_0 \) at \( \mu = 0 \) are known [12].

The Tsallis distribution describes the transverse momentum distributions of negatively charged pions in p-p collisions as obtained by the NA61/SHINE collaboration [13] in all rapidity intervals remarkably well as shown in Fig. 1. The values of nonextensivity parameters \( q \) needed to describe the transverse momentum distributions of negatively charged pions are shown in Fig. 2. The values of temperature parameter \( T \) for different energies and rapidity intervals are shown in Fig. 3. The temperature parameter \( T \) shows a clear rapidity dependence which we have parametrized as \( T \simeq 0.09 \cosh(y) \).

3 Energy dependence of parameters

The energy dependence of the various parameters is displayed in Figs. 4, 5 and 6. For comparison with higher energy data [12] which are for mid-rapidity \( y = 0 \), we show parameters as evaluated for rapidity interval \( 0 < y < 0.2 \). All analysed parameters show a clear but weak energy dependence which we have parametrized as

\[
q(s) = 1.027 \left( \sqrt{s} \right)^{0.01326} \\
T(s) = 0.1014 \left( \sqrt{s} \right)^{-0.03262} \\
R(s) = \left( \frac{3 V(s)}{4 \pi} \right)^{1/3} = 2.31 \left( \sqrt{s} \right)^{0.09}
\]

The value of \( R \) is not necessarily related to the size of the system as deduced from a HBT analysis [15, 16] but serves...
Fig. 4 (Color online) Energy dependence of the parameter $q$ appearing in the Tsallis distribution. Open points are from ATLAS, ALICE and UA1 Collaborations data (taken from Ref. [12]). Solid points are from NA61/SHINE Collaboration data [13]. Data are fitted by Eq. (5).

Fig. 5 (Color online) Energy dependence of the temperature parameter $T$ appearing in the Tsallis distribution. Open points are from ATLAS, ALICE and UA1 Collaborations data (taken from Ref. [12]). Solid points are from NA61/SHINE Collaboration data [13]. Data are fitted by Eq. (6).

Fig. 6 (Color online) Energy dependence of the radius $R$ appearing in the volume factor, $V = 4/3\pi R^3$. Open points are from ATLAS, ALICE and UA1 Collaborations data (taken from Ref. [12]). Solid points are from NA61/SHINE Collaboration data [13]. Data are fitted by Eq. (7).

Fig. 7 (Color online) $dN/dy$ of charged particles produced in the central rapidity region as a function of center-of mass energy in $p-\bar{p}$ and $p-\bar{p}$ collisions. Energy dependence given by Eq. (9) is compared with inelastic measurements from NA61/SHINE [13] ($p-\bar{p}$), NAL Bubble Chamber ($p-\bar{p}$), ISR ($p-\bar{p}$), UA5 ($p-\bar{p}$), PHOBOS ($p-\bar{p}$) and ALICE ($p-\bar{p}$) experiments taken from compilation [17].

For evaluated above energy dependence of parameters $q(s)$, $T(s)$ and $R(s)$ given by Eqs. (5-7) we have

\[
d\frac{N}{dy}\bigg|_{y=0} \simeq 0.1 + 0.56 (\sqrt{s})^{0.24}.
\]

Energy dependence of $dN/dy$ in the central rapidity region in comparison with inelastic measurements is shown in Fig. 7.

We can treat the size of the system, $R$, more seriously. The radius given by Eq. (7) is calculated for $\mu = 0$. For other values of chemical potential, the size is smaller (cf. Eqs. (3) and (4)). Comparing $R(s)$ with experimental data deduced from HBT analysis we can see that $R_{HBT} \simeq R/\kappa$, where $\kappa = 3.5$. In Fig. 8 we displayed $R(s)/\kappa$ in comparison with data obtained from HBT analysis [18].

Following this observation we assume

\[
V_{\mu=0} = V_{\mu} \cdot \kappa^3
\]

and from Eqs. (3) and (4) we have

\[
\mu = \frac{T_{\mu=0}}{q-1} \left( \kappa^{q-1}/q - 1 \right)
\]
and using parametrizations (5) and (6) we have energy dependence of chemical potential in the form

$$\mu(s) \simeq 0.39 \left(\frac{s}{1 \text{GeV}}\right)^{-0.022}. \quad (12)$$

4 Different parametrizations

Almost fifty years ago Hagedorn develop a statistical description of momentum spectra observed in multiparticle production processes [19]. Hagedorn’s approach predicts an exponential decay of momentum distribution

$$E \frac{d^3N}{dp^3} \simeq C \exp\left(-\frac{p_T}{T}\right) \quad (13)$$

for transverse momenta, whereas in experiments one observes non-exponential behavior for large transverse momenta. Subsequently, Hagedorn proposed the "QCD inspired" empirical formula describing the data of the invariant momentum distribution of hadrons as a function of $p_T$ over a wide range [20]:

$$E \frac{d^3N}{dp^3} = C \left(1 + \frac{p_T}{p_0}\right)^{-n} \rightarrow \begin{cases} \exp(-np_T/p_0) & \text{for } p_T \rightarrow 0 \\ (p_T/p_0)^{-n} & \text{for } p_T \rightarrow \infty \end{cases} \quad (14)$$

with $C$, $p_0$, and $n$ being fit parameters. This becomes pure exponential for small $p_T$ and pure power law for large $p_T$. For $n = q/(q-1)$ and $p_0 = T/(q-1)$, the Hagedorn formula (14) coincides with Tsallis distribution [2],

$$E \frac{d^3N}{dp^3} = C \left[1 - (1 - q) \frac{p_T}{T}\right]^{\frac{q}{1-q}}. \quad (15)$$

The basic conceptual difference between (14) and (15) is in the underlying physical picture. In (14) the low-$p_T$ region is controlled by soft physics represented by some unknown unperturbative theory or model, and the high-$p_T$ region is governed by hard physics represented by perturbative QCD. In (15), the nonextensive formula works in the whole range of $p_T$ and it is not derived from some particular theory. It is only a generalization of the regular statistical mechanics and just offers the kind of universal unifying principle, namely the existence of some kind of equilibrium affecting all scales of $p_T$, which is described by two parameters, $T$ and $q$. The temperature $T$ characterizes its mean properties and the parameter $q$, known as the nonextensivity parameter, expresses action of the potentially non-trivial long range effects believed to be caused by fluctuations [5] (but also by some correlations or long memory effects [2]). It is worth to be noted that the invariant momentum distribution in the form (cf. Eq.(1))

$$E \frac{d^3N}{dp^3} = \frac{gV}{(2\pi)^3} \left[1 + (q - 1) \frac{E}{T}\right]^{\frac{1}{q}} \quad (16)$$

result in Eq. (2) without pre-factor $m_T \cosh(y)$ in the right hand side of the equation. For the non-relativistic energies ($E = p^2/(2m)$), Eq. (16) corresponds to Tsallis distribution

$$E \frac{d^3N}{dp^3} = \frac{gV}{(2\pi)^3} \left[1 + (q - 1) \frac{p^2}{2mT}\right]^{\frac{q}{1-q}} \quad (17)$$

originated from multiplicative noise [21, 22].

Exponential function Eq. (13) described data only in the limited range of transverse momentum, $0.15 < p_T < 0.6$ [13]. As shown in Fig. 1, the Tsallis distribution given by Eq. (2) describes all $p_T$ range remarkably well.

All Tsallis-like distributions lead to a power law tail

$$\frac{d^2N}{p_T dp_T dy} \propto p_T^{-n} \quad (18)$$

of the distribution for sufficiently large transverse momenta. The difference between them can be seen in low $p_T$ region, where

$$\frac{d^2N}{p_T dp_T dy} \propto \begin{cases} \alpha - \beta p_T + \gamma p_T^2 & \text{for Eqs. (13), (14), (15)} \\ \alpha - \gamma p_T^2 & \text{for Eqs. (1), (16), (17)} \end{cases} \quad (19)$$

$^2$The Langevin equation $dp/dt + \gamma(t)p = \xi(t)$ where both $\gamma(t)$ and $\xi(t)$ denote stochastic processes (traditional multiplicative noise and additive noise, respectively) leads to a power-law tail of the distribution for sufficiently large momenta. As shown in [21] in the case of Cov($\gamma, \xi$) = 0 and $E(\xi) = 0$ (i.e., for, respectively, no correlation between noises and no drift term due to the additive noise) the solution is given by the non-normalized Tsallis distribution for the variable $p^2$. 
Fig. 9 (Color online) Panel (a) - transverse momentum distribution of negatively charged pions produced in p − p collisions at √s = 17.27 GeV in the rapidity interval 0 < y < 0.2 [13] fitted by different parametrizations (with normalization at high p_T region). Panel (b) - ratio fit/data for the results presented in panel (a).

Parameters α, β and γ are positive valued functions of q and T (in case of Eq. (1), T < m is required for γ > 0).

In low p_T region, Tsallis-like distributions with variable p_T^2 differs from the one expressed in variable p_T. Comparison of different parametrizations is shown in Fig. 9.

5 Discussion and conclusions

In conclusion, the Tsallis distribution, Eq. (2) leads to an excellent description of data on transverse momentum. By comparing results from NA61/SHINE [13] to the results obtained at higher energies [12] it has been possible to extract energy dependence of the parameters q, T and R. A consistent picture emerges from comparison of fits using the Tsallis distribution in wide range of energies.

Different parametrizations lead not only to different quality of fits but also to different values of parameters. In Ref. [13] experimental data are fitted by exponential distribution (13) in limited range of transverse momenta (0.15 < p_T < 0.6 GeV/c) evaluating temperature parameters seemingly larger than our estimate based on parametrization (2). Such difference in values of temperature parameters is fully understandable.

For distributions with the same mean transverse momentum, ⟨p_T⟩, the parameter T exp evaluated from Eq. (13) is connected with parameter T evaluated from Eq. (2) by the relation

\[ T_{\text{exp}} \simeq a + b \cdot T, \]

(20)

where, numerically, \( a = 0.31 - 0.65 q + 0.354 q^2 \) and \( b = 27.35 - 55 q + 29.07 q^2 \). Moreover, it is remarkable to notice that parametrization (1) proposed by Cleymans [3, 4] is for momentum distribution, dN/dp^3 while the other Tsallis-like parametrizations (14)-(17) are for invariant distribution EdN/dp^3.

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