ASYMPTOTIC BEHAVIOR IN DEGENERATE PARABOLIC FULLY NONLINEAR EQUATIONS AND ITS APPLICATION TO ELLIPTIC EIGENVALUE PROBLEMS

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Abstract. We study the asymptotic behavior of the nonlinear parabolic flows
$$u_t = F(D^2u^m)$$
when \( t \to \infty \) for \( m \geq 1 \), and the geometric properties for solutions of
the following elliptic nonlinear eigenvalue problems:
$$F(D^2\varphi) + \mu \varphi^p = 0, \quad \varphi > 0 \quad \text{in } \Omega$$
$$\varphi = 0 \quad \text{on } \partial \Omega$$
posed in a (strictly) convex and smooth domain \( \Omega \subset \mathbb{R}^n \) for \( 0 < p \leq 1 \), where \( F() \) is uniformly elliptic, positively homogeneous of order one and concave. We establish
that \( \log(\varphi) \) is concave in the case \( p = 1 \) and that the function \( \varphi^{\frac{1}{p-1}} \) is concave for
\( 0 < p < 1 \).

1. Introduction

In this paper, we consider the asymptotic behavior of \( u \) satisfying

$$\begin{cases}
  u_t(x,t) - F(D^2u^m(x,t)) = 0 & \text{in } Q = \Omega \times (0, +\infty), \\
  u(x,0) = 0 & \text{in } \Omega, \\
  u(x,t) = 0 & \text{on } \partial \Omega \times (0, +\infty), 
\end{cases}$$

(1.1)

and then we show a renormalized flow converges to \( \varphi(x) \) which satisfies the following nonlinear eigenvalue problem

$$\begin{cases}
  F(D^2\varphi) + \mu \varphi^p = 0 & \text{in } \Omega, \\
  \varphi > 0 & \text{in } \Omega, \\
  \varphi = 0 & \text{on } \partial \Omega 
\end{cases}$$

(1.2)

for some \( \mu > 0 \).

Such parabolic approach to nonlinear eigenvalue problem has been considered
at [LV2] for Laplace operator and extended to Fully nonlinear operator at [KSL]
with super-linear exponent (i.e. \( 1 < p < p_{\text{crit}} \) for some critical number \( p_{\text{crit}} > 1 \)).

In this paper, we consider linear and sublinear case (\( 0 < p \leq 1 \)) which have very
different behavior from the super-linear case. The super-linear nonlinear eigen
value problem can be described by the solutions of fast diffusion equations, where
the solution will extinct at the finite time. So the Harnack type estimate plays an
important role to analyze the asymptotic behavior near the finite extinct time. On
the other hand, the solution of sub-linear eigenvalue problem will be approximated
by the solutions of slow diffusion equation, where the parabolic solution exists for

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all time. This difference allows us to have different method based on barriers and then have sharper results than the super-linear case. When $F$ is Laplace operator, the asymptotic behavior of the solution in the degenerate or singular diffusion has been studied by many authors, Aronson, Berryman, Bonforte, Carrillo, Friedman, Galaktionov, Holland, Kamin, Kwong, Peletier, Toscani, Vazquez, et al. We refer [Va] for its details and references.

We also show that the geometric property can be preserved in the degenerate fully nonlinear flow under the concavity condition of the operator and hence such property also holds for the limit $\psi$. To study the concavity of a solution, the second difference of $u(x, t)$

$$C(x, y; u) = C(x, y) = 2(u(x) + u(y)) - u \left(\frac{x + y}{2}\right)$$

is considered. Lastly, we show the eventual concavity of parabolic flow which means that the parabolic solution itself has such geometric property in finite time.

This analysis gives us sharp description of the asymptotic profile of the parabolic flow and affirmative answer for the well-known question on the convexity of level sets of the solution when the domain is convex. We refer [LV2], [KsL], [GG] for the detailed history on the geometric issue.

1.1. Let $S^{n \times n}$ denote the set of $n \times n$ symmetric matrices and the norm of a matrix $\|M\|$, for $M \in S^{n \times n}$ is defined as the maximum absolute value among eigenvalues of $M$. For $0 < \lambda \leq \Lambda$ (called ellipticity constants), the Pucci’s extremal operators, that play a crucial role in the study of fully nonlinear elliptic equations, are defined as, for $M \in S^{n \times n}$,

$$M^{+}_{\lambda, \Lambda}(M) = \sup_{A \in A_{\lambda, \Lambda}} [\text{tr}(AM)],$$

$$M^{-}_{\lambda, \Lambda}(M) = \inf_{A \in A_{\lambda, \Lambda}} [\text{tr}(AM)],$$

where $A_{\lambda, \Lambda}$ consists of the symmetric matrices whose the eigenvalues lie in $[\lambda, \Lambda]$. We note that when $\lambda = \Lambda = 1$, the Pucci’s extremal operators $M^\pm$ simply coincide with the Laplace operator.

In this paper, we assume that the nonlinear operator $F : S^{n \times n} \to \mathbb{R}$ satisfies the following hypotheses unless it is specifically mentioned:

(F1) $F$ is a uniformly elliptic operator; for all $M, N \in S^{n \times n}$,

$$M^{-}(M - N) \leq F(M) - F(N) \leq M^{+}(M - N).$$

(F2) $F$ is positively homogeneous of order one; for all $t \geq 0$ and $M \in S^{n \times n}$

$$F(tM) = tF(M).$$

In addition, we assume that

(F3) $F$ is concave.

The concavity condition of $F$ will be required when we show geometric property of parabolic flows. The Pucci’s extremal operator $M^+$ is one of nontrivial examples of the operator satisfying (F1), (F2) and (F3). We may extend $F$ on $\mathbb{R}^{n \times n}$ by defining $F(A) := F \left(\frac{A + A^T}{2}\right)$ for a nonsymmetric matrix $A$. 


Throughout this paper, we assume that $\Omega$ is a bounded domain with a smooth boundary in $\mathbb{R}^n$.

We consider viscosity solutions of $(1.1), (1.2)$ which are proper notion of the weak solution for the fully nonlinear uniformly elliptic equation. A continuous function $u \in \Omega$ is said to be a viscosity subsolution (respectively, viscosity supersolution) of $F(D^2u(x)) = f(x)$ in $\Omega$ when the following condition holds: for any $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at $x_0$, we have
\[
F(D^2\phi(x_0)) \geq f(x_0)
\]
(respectively, if $u - \phi$ has a local minimum at $x_0$, we have $F(D^2\phi(x_0)) \leq f(x_0)$). We say that $u$ is a viscosity solution of $F(D^2u(x)) = f(x)$ in $\Omega$ when it is both a viscosity subsolution and supersolution. Viscosity solutions have been used to prove existence of solutions by Perron’s method via the comparison principle. We refer the details and regularity theory of the viscosity solutions to [CIL], [CC].

1.2. When the operator is fully nonlinear, there are several crucial issues to be discussed.

(i) Parabolic approach relies on the convergence of the parabolic flows, $u(x, t)$, to eigen functions, $\varphi(x)$, after normalizations, $(4.8), (5.16)$. For nonlinear parabolic flow of divergence type, some key steps for the analysis of asymptotic behavior of are based on the integration by parts, for example the existence of monotone integral quantities, [Va], which cannot be applicable to the fully nonlinear operator.

On the other hand, asymptotic Analysis in nondivergence form can be achieved in a couple of steps. First, it is crucial to find an exact decay rate of $u(x, t)$, which will give us the right normalization of $u(x, t)$ so that the normalized parabolic flows converge to eigen functions, $\varphi(x)$. In fact, the exact decay rate is related to the first eigen value when $p = 1$ and $m = 1$. We show the existence of the unique limit of normalized flow, $v(x, t)$, of $u(x, t)$, at Proposition $4.6$. When $0 < p < 1$ (or $1 < m = \frac{1}{p} < \infty$), we prove Aronson-Benilan type estimate, Lemma $5.2$ for degenerate fully nonlinear parabolic flows, which will give us almost monotonicity of $u(x, t)$. The uniqueness of the limit of normalized flows is proven at Proposition $5.3$.

(ii) Finally, we need to show that geometric properties of $u(x, t)$ will be preserved under the fully nonlinear parabolic flows, $(1.1)$. Geometric computation requires sophisticated computations to construct geometric quantities which satisfies maximum principle at Lemmas $4.9, 5.10$. The log-concavity of $u$ for $p = 1$, the square-root concavity of its pressure $u^{m-1}$ for $0 < p < 1$, turn out to be preserved geometric quantities. The difference of exponents comes from the difference in homogeneities of the operators, [Le].

1.3. This paper is organized into four parts as follows. At Section $2$, we summarize the known facts about fully nonlinear uniformly elliptic or parabolic partial differential equations. And at Section $3$, we show some known results about the fully nonlinear elliptic eigenvalue problem $(1.2)$ and the existence results of positive eigen-functions for fully nonlinear elliptic problem as well as solutions of the nonlinear diffusion equations in the range $0 < p \leq 1$. 

In Section 4, we deal with the fully nonlinear uniformly parabolic and elliptic equations and we discuss the log concavity of the first eigenfunction of nonlinear elliptic problem. First, Bernstein’s technique gives uniform estimates for normalized solutions \( v(x, t) = e^{\mu t} u(x, t) \) and then we use it to get the eigen-function as the limit of \( v(\cdot, t) \) at Proposition 4.6. On the other hand, we can choose initial data for this evolution having the desired geometric property, and then the evolution preserves the geometric property. Therefore the result for the elliptic problem will be obtained in the limit \( t \to \infty \).

Finally at Section 5, we show the long-time behavior of the parabolic flow for \( 0 < p < 1 \), Proposition 5.3. It is also proved that the pressure of the solution preserves square-root concavity under the parabolic flow and hence the concavity of eigenfunction is proved.

Notations. Let us make a summary of the notations and definitions that will be used.

- We denote by \( \nabla u \) or \( Du \) the spatial gradient of a function \( u(x, t) \), and by \( D^2 u \) the Hessian matrix. \( D_e f \) denotes the directional derivative in the direction \( e \in S^{n-1} \).
- The expressions \( D^2 u \geq 0 \), \( D^2 u \leq 0 \) are understood in the usual sense of quadratic forms.
- In order to avoid confusion between coordinates and partial derivatives, we will use the standard subindex notation to denote the former, while partial derivatives will be denoted in the form \( f_\alpha \) for \( \frac{\partial f}{\partial \alpha} \). In general, \( f_\alpha = \nabla_{e_\alpha} f \) for a unit direction \( e_\alpha \in S^{n-1} \) with a parameter \( \alpha \). And second partial derivatives will be denoted in the form \( f_{\alpha\beta} \) for \( \frac{\partial^2 f}{\partial \alpha \partial \beta} = e_\alpha^T D^2 f e_\beta \). If the computation is invariant under the rotation, we may assume that \( \alpha = 1, \cdots, n \) and that \( \{e_1, \cdots, e_n\} \) is an orthonormal basis. This notation is usual in some parts of the physics literature. But we will write \( f_\nu \) and \( f_\tau \) for the normal and tangential derivatives since no confusion is expected.
- h.o.t. means ‘higher order terms’.

2. Preliminaries

For the reader’s convenience, we are going to summarize basic facts and estimates for elliptic fully nonlinear equation \( F(D^2 u) = f(x) \) in a bounded domain \( \Omega \subset \mathbb{R}^n \), \([\text{CC}, \text{CIL}]\) and for parabolic fully nonlinear equations \( u_t = F(D^2 u) + f(x) \) in \( Q_T = \Omega \times (0, T] \), \([\text{CIL}, \text{L}, \text{W1}, \text{W2}]\), where \( F \) satisfies the condition (F1).

1. The existence and uniqueness of the viscosity solution, comparison principle between super- and sub-solutions, minimum principle and maximum principle in elliptic or parabolic Dirichlet problem, and their references can be found at \([\text{CIL}, \text{CC}, \text{W1}]\).

2. The strong maximum principle holds for \( f = 0 \) at elliptic equation, Proposition 4.9, \([\text{CC}]\) and the same argument with Corollary 3.21, \([\text{W1}]\), gives us the strong maximum principle for the parabolic equation. A version of strong maximum principle for fully nonlinear equation with nonhomogeneous operator has been proved at Lemma 5.3. The strong maximum principle for elliptic or parabolic equation says that whenever a subsolution \( u \) touches
a super-solution \( v \) from below at an interior point, \( u \equiv v \) on the domain \( \Omega \) or \( Q_T \), respectively.

(3) [**Local Regularity**] We refer the regularity theory for elliptic equation to \([GT, CC]\) and parabolic case to \([L, W1, W2]\). In \([CC]\), we can find Hölder continuity \((k = 0, 0 < \alpha < 1)\) at Proposition 4.10, Local \( C^{1,\alpha}\)-regularity \((k = 1, 0 < \alpha < 1)\) at Theorem 8.3, Local \( C^{1,1}\)-regularity \((k = 2, \alpha = 0)\) for convex or concave operator \( F \) at Proposition 9.3, local \( C^{2,\alpha}\)-regularity \((k = 2, 0 < \alpha < 1)\) for Hölder continuous \( f \) at Theorem 8.1, local \( C^\infty\)-regularity for smooth \( F \) and \( f \). When \( F \) and \( f \) is analytic, \( u \) will be analytic following Theorem 10 at Section 2.2, \([E]\).

(4) [**Global Regularity**] We also refer the global Schauder theory, \[ \|u\|_{C^{2,\alpha}([\Omega])} \leq C(\|u\|_{L^\infty([\Omega])} + ||f||_{C^{k,\alpha}([\Omega])}) \] to Theorem 9.5, \([CC]\). Therefore if \( \partial \Omega \) is \( C^{2,\alpha}\)-surface, then the viscosity solution will be classical. The similar results hold for parabolic equation, \([L]\).

(5) [**Harnack Inequality**] The Harnack inequality for a nonnegative elliptic solution is the following, Theorem 4.8, \([CC]\): for a nonnegative elliptic solution \( u \) in \( B_3 \), we have \[ \sup_{B_{1/3}} u \leq C(\inf_{B_{1/3}} u + ||f||_{L^n(B_2)}) \] for a uniform constant \( C > 0 \). Similar parabolic version can be found at \([W1]\).

### 3. Nonlinear eigenvalue problem

In this section we are going to study solutions to the fully nonlinear elliptic eigenvalue problem

\[
\begin{aligned}
F(D^2\phi(x)) &= -\mu\phi'(x) \quad \text{in } \Omega, \\
\phi &> 0 \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(NLEV)

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \) and \( F \) is a uniformly elliptic and positively homogeneous operator of order one defined on \( S^{n \times n} \). First, let us introduce the existence theorem of the positive eigen-function that was proven by Ishii and Yoshimura. The simplified proof can be found at \([A]\).

**Theorem 3.1.** \([IY]\) **Suppose that** \( F \) **satisfies (F1) and (F2) and that** \( \Omega \) **is a smooth bounded domain in** \( \mathbb{R}^n \). **Then there exist** \( \varphi \in C^{1,\alpha}(\overline{\Omega}) \), \( 0 < \alpha < 1 \) **and** \( \mu > 0 \) **such that** \( \varphi > 0 \) **in** \( \Omega \) **and** \( \varphi \) **satisfies**

\[
\begin{aligned}
-F(D^2\varphi(x)) &= \mu\varphi(x) \quad \text{in } \Omega, \\
\varphi(x) &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(EV)

Moreover, \( \mu \) is unique in the sense that if \( \rho \) is another eigen-value of \( F \) in \( \Omega \) associated with a nonnegative eigen-function, then \( \mu = \rho \); and is simple in the sense of that if \( \psi \) in \( C^0(\overline{\Omega}) \) is a solution of (EV) with \( \psi \) in place of \( \varphi \), then \( \psi = c\varphi \) for some \( c \in \mathbb{R} \).

Now let us state the Hopf’s Lemma that will be used frequently when we compare a solution with barrier.
Theorem 3.2 (Hopf’s Lemma). Suppose that $\Omega$ satisfies an interior sphere condition. Let $u \in C(\Omega)$ be a nonzero viscosity supersolution of

$$M(D^2 u) \leq 0 \quad \text{in } \Omega.$$  

Then for $x_0 \in \partial \Omega$ such that $u(x) > u(x_0)$ for all $x \in \Omega$, we have

$$\liminf_{x \in \Omega \rightarrow x_0} \frac{u(x) - u(x_0)}{|x - x_0|} > 0.$$  

Especially, if the outer normal derivatives of $u$ at $x_0$ exists, then

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where $\nu$ is the outer normal vector to $\partial \Omega$ at $x_0$.

In particular, if $u = 0$ on $\partial \Omega$, $u > 0$ in $\Omega$ and $u \geq M > 0$ on $\Omega' \Subset \Omega$, then

$$\frac{\partial u}{\partial \nu}(x_0) < -c_0(M, \text{dist}(\Omega', \Omega)).$$

We refer to Lemma 3.4 at [GT] for uniformly elliptic linear equation and Lemma 2.6 at [L] and Appendix at [A] for uniformly parabolic fully nonlinear equation. The Hopf’s lemma for uniformly elliptic fully nonlinear equation follows by the comparison between super-solution and a barrier $R - \alpha - |x| - \alpha$ for large $\alpha > 0$ and a small $R > 0$ as Lemma 2.6, [L]. Hopf’s Lemma for the parabolic equation [L] holds in the following way:

$$\liminf_{x \rightarrow x_0, s \rightarrow t} \frac{u(x,s) - u(x_0,t)}{\sqrt{|x - x_0|^2 + (t - s)}} > 0$$

for any $x \in \Omega$ and $s \leq t$.

3.1. Case $0 < p < 1$. In this case, we consider the following equation

$$\begin{cases}
-F(D^2 f^m(x)) = \frac{1}{m-1} f(x) & \text{in } \Omega, \ m > 1, \\
 f = 0 & \text{on } \partial \Omega, \\
 f > 0 & \text{in } \Omega,
\end{cases}$$

which is the asymptotic profiles of the equation

$$\begin{cases}
H[u] = u_t(x,t) - F(D^2 u^m(x,t)) = 0 & \text{in } Q = \Omega \times (0,T], \\
u(x,0) = u(x), \\
u(x,t) = 0 & \text{on } \partial \Omega \times (0,T].
\end{cases}$$

We assume $u_0$ has nontrivial bounded gradient on $\partial \Omega$, i.e,

$$u_0^m \in C_b(\overline{\Omega}),$$

where

$$C_b(\overline{\Omega}) := \{ h \in C^0(\overline{\Omega}) | c_0 \text{dist}(x, \partial \Omega) \leq h(x) \leq C_0 \text{dist}(x, \partial \Omega) \text{ for } 0 < c_0 \leq C_0 < +\infty \}.$$  

If we set $\varphi = f^m$ and $p = \frac{1}{m-1}$, then $\varphi$ is the solution of [NLEV] with an eigenvalue $\frac{1}{m-1}$. For the sub-linear case, $0 < p < 1$, we have the following comparison principle and the existence and uniqueness result of nonlinear eigenfunction.
Lemma 3.3 (Comparison principle). Suppose \( F \) satisfies (F1), \( F(0) = 0 \) and that either (F2) or (F3). Let \( v \) and \( w \) be in \( C^2(\Omega) \cap C^1(\bar{\Omega}) \) such that \( v, w \geq 0 \). If \( F(D^2v) + \frac{1}{m-1}v^\frac{m}{m-1} \leq 0 \leq F(D^2w) + \frac{1}{m-1}w^\frac{m}{m-1} \) in \( \Omega \) and if \( v \geq w \) on \( \partial\Omega \), then \( v \geq w \) in \( \Omega \).

Proof. Suppose that \( v < w \) for some point in \( \Omega \). Since \( v \) satisfies

\[
M^*(D^2v) \leq F(D^2v) \leq -\frac{1}{m-1}v^\frac{m}{m-1} \leq 0,
\]

we have \( v > 0 \) in \( \Omega \) and \( |\nabla v| > 0 \) on \( \partial\Omega \) by the strong minimum principle and Hopf’s lemma. Let \( t^* = \inf\{t > 0 \mid v < tw \text{ for some point in } \Omega\} \). Then \( 0 < t^* < 1 \).

Set \( z = v - t^*w \), and then the nonnegative function \( z \) vanishes at some point in \( \bar{\Omega} \) and \( z \) satisfies

\[
M^*(D^2z) \leq F(D^2v) - F(D^2t^*w) \leq \frac{1}{m-1}(t^*w^\frac{m}{m-1} - v^\frac{m}{m-1}) \leq \frac{1}{m-1}((t^*w)^\frac{m}{m-1} - v^\frac{m}{m-1}) \leq 0.
\]

Assume that \( z \neq 0 \). From the strong minimum principle and Hopf’s lemma, we have \( z > 0 \) in \( \Omega \) and \( |\nabla z| > 0 \) on \( \partial\Omega \). Then we can choose \( \varepsilon > 0 \) such that \( z - \varepsilon v \geq 0 \) in \( \Omega \). It’s a contradiction to the definition of \( t^* \). Thus we get \( z \equiv 0 \) and \( v = t^*w \) in \( \bar{\Omega} \).

(i) First, let us assume that \( v \) is a strictly supersolution, i.e., \( F(D^2v) - \frac{1}{m-1}v^\frac{m}{m-1} < 0 \), we have

\[
0 > F(D^2v) + \frac{1}{m-1}v^\frac{m}{m-1} \geq t^*F(D^2w) + \frac{1}{m-1}(t^*w)^\frac{m}{m-1}
\]

\[
= t^* \left\{ F(D^2w) + \frac{m-1}{m-1}w^\frac{m}{m-1} \right\} \geq t^* \left\{ F(D^2w) + \frac{1}{m-1}w^\frac{m}{m-1} \right\} \geq 0,
\]

which is a contradiction.

(ii) Now, assume that \( v \) is a supersolution, i.e., \( F(D^2v) \leq -\frac{1}{m-1}v^\frac{m}{m-1} \). Then, we have that \( v > 0 \) in \( \Omega \) by the strong minimum principle and Hopf’s lemma.

Let \( v^\varepsilon := (1 + \varepsilon)v \) for \( \varepsilon > 0 \). Then \( v^\varepsilon \) satisfies

\[
F(D^2v^\varepsilon) + \frac{1}{m-1}(v^\varepsilon)^\frac{m}{m-1} \leq (1 + \varepsilon)F(D^2v) + \frac{1 + \varepsilon}{m-1}v^\frac{m}{m-1}
\]

\[
\leq \frac{1}{m-1}v^\frac{m}{m-1}[(1 + \varepsilon)^\frac{m}{m-1} - (1 + \varepsilon)] < 0,
\]

i.e., \( v^\varepsilon \) is a strictly supersolution. By (i), we get \( v^\varepsilon = (1 + \varepsilon)v \geq w \) in \( \Omega \). Letting \( \varepsilon \to 0 \), we have \( v \geq w \) in \( \Omega \).

\[\square\]

Theorem 3.4. Suppose \( F \) satisfies (F1) and (F2). The nonlinear eigenvalue problem has a unique positive viscosity solution \( \phi \in C^{0,1}(\bar{\Omega}) \cap C^1(\Omega) \), i.e.,

\[
\begin{align*}
\text{NLEV} & \quad \begin{cases} -F(D^2\phi(x)) = \frac{1}{m-1}\phi^p(x) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \\ \phi > 0 & \text{in } \Omega, \end{cases}
\end{align*}
\]

where \( p = \frac{1}{m} \). The eigen-function \( \phi \) satisfies \( \inf_{\partial\Omega} |\nabla \phi| \geq \delta_0 > 0 \). Moreover, if \( F \) is \( C^1 \), \( \phi \) is of \( C^\infty(\Omega) \).
Proof. (i) The uniqueness of the solution follows from Comparison Principle. It suffices to establish the existence of positive super and sub-solutions with zero boundary value in order to prove the existence of the solution. Let \( h \) be the solution of

\[
\begin{align*}
F(D^2h(x)) &= -1 & \text{in } \Omega, \\
h &= 0 & \text{on } \partial\Omega, \\
h &> 0 & \text{in } \Omega.
\end{align*}
\]

(3.5)

If we select \( t > 0 \) satisfying \( t^{1 - \frac{2}{m}}\|h\|_{L^\infty(\Omega)}^{-\frac{1}{m}} = \frac{1}{m - 1} \), then

\[
F(D^2(th)) = -t^{1 - \frac{2}{m}}h^{\frac{1}{m}}(th)^{\frac{1}{m}} \leq -\frac{1}{m - 1}(th)^{\frac{1}{m}}
\]

i.e., \( h^+ := th \) is a super-solution.

On the other hand, let \( \phi \) be the first eigen-function of \((\text{EV})\). Choose \( s > 0 \) so that

\[
\mu(s)\|\phi\|_{L^\infty(\Omega)} \leq \frac{1}{m - 1},
\]

then \( F(D^2(s\phi)) \geq -\frac{1}{m - 1}(s\phi)^{\frac{1}{m}} \). Thus \( h^- := s\phi \) is a sub-solution.

Thus the comparison principle Lemma 3.3 gives that \( h^- \leq h^+ \) and there is a viscosity solution \( \bar{\phi} \) such that \( h^- \leq \bar{\phi} \leq h^+ \) from \[CC\]. Since \( \phi^p \in L^\infty(\Omega) \), \( \phi \) is of \( C^{1,\alpha}(\Omega) \) from the regularity theory, \[CC\]. Since \( F(D^2\phi(x)) = -\frac{1}{m - 1}\phi^p(x) \leq 0 \), \( \phi \) satisfies \( \inf_{\partial\Omega}|\nabla\phi| \geq \delta_o > 0 \) from Hopf’s lemma.

(ii) Now we are going to show \( \phi \in C^{0,1}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega) \). First, we note that there are \( 0 < c_o \leq C_o < \infty \) such that \( c_o \operatorname{dist}(x, \partial\Omega) \leq h^- \leq \phi \leq h^+ \leq C_o \operatorname{dist}(x, \partial\Omega) \) from Hopf’s Lemma for \( h^- \) and \( C^{0,1}(\overline{\Omega}) \) - regularity of \( h^+ \), \[CC\].

Let \( \delta > 0 \) be a constant such that \( B_\delta(x) \subset \Omega \) for \( \operatorname{dist}(x, \partial\Omega) > \delta \). For \( x_o \in \Omega \) such that \( \operatorname{dist}(x_o, \partial\Omega) < \delta \), set \( \operatorname{dist}(x_o, \partial\Omega) = 2\varepsilon \). Now we scale the function \( \phi \),

\[
\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(x_o + \varepsilon x).
\]

Then \( 0 < c_o \leq \phi_\varepsilon(x) \leq 3C_o \) in \( B_1(0) \) and \( \phi_\varepsilon \) satisfies

\[
F(D^2\phi_\varepsilon) = -\varepsilon^{1+p}\phi_\varepsilon^p \in L^\infty(B_1(0)) \quad \text{uniformly.}
\]

From the regularity theory, \[CC\], we have

\[
|D\phi(x_o)| = |D\phi_\varepsilon(0)| \leq \tilde{C} \text{ for some uniform constant } \tilde{C} > 0,
\]

Therefore, we have \( |D\phi(x_o)| \leq \tilde{C} \) and we deduce that \( \phi \in C^{0,1}(\overline{\Omega}) \).

(iii) When \( F \) is \( C^1 \), the operator becomes a linear operator from the positive homogeneity of order one. Thus the result follows. \( \square \)

We state the following comparison principle of the solution, \( u \), of the parabolic flow \((3.4)\) for the case \( m > 1 \) and we consider the following equation:

\[
\begin{align*}
F(D^2v(x, t)) &= (\sigma^\frac{1}{2}\sigma)(x, t) & \text{in } Q_T = \Omega \times (0, T], \quad m > 1, \\
v(x, 0) &= v_o(x) = u_0^m \in C_b(\overline{\Omega}), \\
v(x, t) &= 0 & \text{on } x \in \partial\Omega.
\end{align*}
\]

(3.6)

The proof of Comparison principle for the case \( m > 1 \), is the same as the case \( m_o < m < 1 \), \[KsL\]. The similar argument as Lemma 3.3 gives us the following Lemma.
Lemma 3.5 (Comparison principle). Suppose $F$ satisfies (F1), $F(0) = 0$ and either (F2) or (F3). Let $u, w \in C^{2,1}(Q_T) \cap C^0(Q_T)$ such that $u, w > 0$ in $Q_T$. If $F(D^2v) - (v^\pm) \leq 0 \leq F(D^2w) - (w^\pm)$ in $Q_T$ and if $v \geq w$ on $\partial \Omega \times (0, T)$, then $v \geq w$ in $Q_T$.

**Theorem 3.6.** Suppose $F$ satisfies (F1), $F(0) = 0$ and either (F2) or (F3). Let $u_{m}^{n}$ be in $C_{b}(\overline{\Omega})$. When $m > 1$, there exists a unique solution of porous medium type (3.4). Moreover, the solution $u$ is positive in $\Omega \times (0, +\infty)$.

**Proof.** Let $f = \phi^\pm$ for $\phi$ in Theorem 3.4. First, we note that 0 and $f(x)(k + t)^{-\frac{m}{k}}$ (for any $k > 0$) are solutions of $u_t(x, t) = F(D^2u(x, t))$ in $Q_T$, with zero boundary.

We construct a supersolution using self-similar solutions. Let $\phi^+$ be an eigenfunction with the Pucci's operator $\mathcal{M}^+$ in Theorem 3.4. For a given $\varepsilon > 0$, we can choose $K > 0$ such that $0 < u_{m}^{n}(x) \leq \phi^+(x)K^{-\frac{m}{k}}$ since $\inf_{\partial \Omega} |\nabla \phi| > 0$. Then $\phi^+(x)(K + t)^{-\frac{m}{k}}$ is a supersolution of (3.4) with any $F$ as the operator since $\mathcal{M}^+ \leq F \leq \mathcal{M}^*$. Therefore there exists a unique solution $u$ of (3.4). Moreover, $u$ satisfies

$$0 \leq u(x, t) \leq \phi^+(x)(K + t)^{-\frac{m}{k}}$$

in $Q_T$ by the Comparison principle.

In addition, we are going to show that $u > 0$ in $Q_T$ if $u_0 \in C_b(\overline{\Omega})$. Let $\Omega'$ be any smooth compact subset of $\Omega$. From Theorem 3.4, there is a positive eigenfunction $\varphi_1$ corresponding to $\Omega'$ with the operator $\mathcal{M}^+$. Set $g(x) = \varphi_1^{1/m}$ and then $U_1 = g(x)(K + t)^{-\frac{m}{k}}$ solves (3.4) in $\Omega' \times (0, T]$ with the operator $\mathcal{M}^+$. Since $u_0(x) > 0$ on a compact set $\overline{\Omega}$ and $u_0(x)$ is continuous, there is a large $K > 0$ such that $U_1(x, 0) = g(x)K^{-\frac{m}{k}} \leq u_0(x)$ on $\overline{\Omega}$. From the comparison principle in $\Omega'$, we have $u(x, t) \geq U_1(x, t) > 0$ on $\Omega' \times (0, T]$. By taking $\Omega'$ arbitrary, we have $u > 0$ in $\Omega \times [0, +\infty)$.

**4. Uniformly fully nonlinear equation**

We consider the solutions $u(x, t)$ of the problem

$$\begin{align*}
H[u] &= u_t(x, t) - F(D^2u(x, t)) = 0 \quad \text{in } Q = \Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x) \in C^0(\overline{\Omega}), \\
u(x, t) &= 0 \quad \text{for } x \in \partial \Omega \times (0, +\infty),
\end{align*}$$

(4.7)

where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary.

4.1. **Asymptotic Behavior.** In this subsection, we are going to analyze the asymptotic behavior of the solution $u$ of (4.7). First, we will find the exact decay rate of $u$ comparing it with barriers constructed by using the principal eigen-value, $\mu$, and a positive eigen-function, $\varphi(x)$.

**Lemma 4.1.** Suppose $F$ satisfies (F1) and (F2). For any positive $u_0 \in C_b(\overline{\Omega})$, there are $0 < C_1 \leq C_2$ such that

$$C_1\varphi(x)e^{-\mu t} < u(x, t) < C_2\varphi(x)e^{-\mu t},$$

for $t > 0$. 

Proof. By Hopf’s lemma and $C^{0,1}$-regularity of $\varphi$, we have $0 < |\nabla \varphi| < +\infty$ on $\partial \Omega$. So we can choose $C_2 > C_1 > 0$ such that $C_1 \varphi(x) < u_o(x) < C_2 \varphi(x)$ in $\Omega$. Since $C\varphi(x)e^{-\mu t}$ is a solution of (4.7) for any constant $C > 0$, the comparison principle gives us the result. 

**Lemma 4.2.** Suppose that $F$ satisfies (F1) and (F2). For any nonnegative and nonzero $u_o \in C^0(\overline{\Omega})$, there is $t_0 > 0$ such that

$$C_1 \varphi(x) < u(x, t_0) < C_2 \varphi(x),$$

for some $0 < C_1 < C_2$ and then for $t \geq t_0$

$$C_1 \varphi(x)e^{-\mu t} < u(x, t) < C_2 \varphi(x)e^{-\mu t}.$$  

Proof. We are going to construct a subsolution of (4.7) which expands in time. Define $g(x, t) = \frac{r}{\alpha} \exp\left(-\frac{\alpha^2}{t^2}\right)$, where $\alpha = \frac{1}{4\lambda}$, $\beta = \frac{2\alpha}{\lambda}$ and $r = |x|$. We can easily see that at the point $(r, 0, \cdots, 0)$,

- $\partial_{ij} g = 0$ if $i \neq j$,
- $\partial_{11} g = \frac{2\alpha}{t^2}(2\alpha r^2 - t)$,
- and $\partial_{ii} g = -\frac{2\alpha}{t}$ if $i > 1$.

Then we check for $r^2 < \frac{1}{4\alpha}$

$$\mathcal{M}^-(D^2 g) - g_t = \lambda \partial_{11} g + (n-1)\Lambda \partial_{22} g - g_t = \frac{g}{t^2}[t(\beta - 2\alpha \Lambda n) + \alpha r^2(4\Lambda \alpha - 1)] \geq 0$$

and for $r^2 \geq \frac{1}{4\alpha}$

$$\mathcal{M}^-(D^2 g) - g_t = \frac{g}{t^2}[t(\beta - 2\alpha (n-1)\Lambda) + \alpha r^2(4\Lambda \alpha - 1)] \geq 0.$$  

Thus $g$ is a subsolution of $F(D^2u) - u_t = 0$. For positive constants $\tau_o, c_o$ and $\delta_o$, we define

$$h(x, t) := \max\left\{c_o, \frac{1}{(t + \tau_o)^\beta} \exp\left(-\alpha \frac{|x - x_o|^2}{t + \tau_o}\right) - \delta_o, 0\right\}$$

and then $h$ is also a subsolution as long as $\text{supp}\, h(\cdot, t) \subset \Omega$.

Since $u_o \not\equiv 0$, there exists a point $x_o \in \Omega$ such that $u(x_o) := m_1 > 0$. We choose $\rho > 0$ and $\eta > 0$ small so that $B_\rho(x_o) \subset \subset \Omega$, $u_o(x) \geq \frac{m_o}{m} = m_o$ in $B_\rho(x_o)$, and $2\rho < \text{dist}(x_o, \partial \Omega)$ and that $0 < \eta \leq \rho$, and $B_{2\rho}(y) \subset \Omega$ for $y \in \{x \in \Omega | \text{dist}(x, \partial \Omega) \geq 2\eta\} \equiv \Omega_{2\eta}$. By taking $\tau_o, c_o$ and $\delta_o$ such that

$$\eta^2 = 4\Lambda n \tau_o(2\Lambda n \tau_o), \quad \frac{c_o}{\tau_o^\beta} \exp\left(-\alpha \frac{\eta^2}{\tau_o}\right) = \delta_o \quad \text{and} \quad \frac{c_o}{\tau_o^\beta} - \delta_o = m_0,$$

then the support of $h(x, t)$ is increasing for $0 < t \leq \frac{1}{e} \left(\frac{c_o}{\alpha \beta}\right)^{1/\beta} - \tau_o$ with $h(x, 0) < u_o(x)$. In fact, at time $t_0 = \frac{1}{e} \left(\frac{c_o}{\alpha \beta}\right)^{1/\beta} - \tau_o = \frac{e - 1}{4\Lambda \alpha \eta^2}$, the support of $h(x, t)$ becomes the ball with radius $\sqrt{\frac{e}{2}} \eta$ centered at $x_o$. Comparison principle implies $h(x, t) \leq u(x, t)$ in $\Omega \times (0, t_0]$ and hence $u(x, t_0) > 0$ in $B_{\sqrt{\frac{e}{2}} \eta}(x_o)$ at $t_0 = \frac{e - 1}{4\Lambda \alpha \eta^2} > 0$. 


For any point $y \in \partial \Omega$, we have a chain of uniform number of balls with radius $\sqrt{\frac{1}{2}} \eta$ from $x$ to $y$ and each ball will be filled by the above subsolution $h$ starting at the previous ball. Since all of argument can be carried out at finite step only depending on the initial data and the domain, there is a time $t_1$ such that $u(t, t_1) > 0$ in $\Omega$ and $|Vu(y, t_1)| > 0$ for $y \in \partial \Omega$. Thus, there is $C_1 > 0$ such that $C_1 v(x)e^{-\mu t} < u(x, t_1)$ in $\Omega$. Since $u$ is $C^{1,\beta}(\Omega \times [t_1, t_1 + 1])$, there is $C_2 > 0$ such that $u(x, t) < C_2 v(x)e^{-\mu t}$ in $\Omega \times [t_1, \infty)$. Therefore, the result follows.

To refine the asymptotic behavior, let us introduce the normalized function

$$v(x, t) = e^{\mu t} u(x, t).$$

Then, $v(x, t)$ satisfies $v_t = F(D^2 v) + \mu v$ if the operator $F$ satisfies the condition (F2) and we deduce the following Corollary from Lemma 4.2.

**Corollary 4.3.** Under the same assumption of Lemma 4.2, $v(x, t) = e^{\mu t} u(x, t)$ has the following estimate:

$$\|v(x, t)\|_{L^\infty(\Omega \times [t_0, \infty))} \leq C\|v(x, t_0)\|_{L^\infty(\Omega)},$$

where $t_0 > 0$ is in Lemma 4.2.

Before studying fine asymptotic behavior of parabolic solutions, let us summarize the regularity theory of uniformly parabolic equation.

**Theorem 4.4 (Global Regularity for $m = 1$).** Suppose that the domain $\Omega$ is bounded and smooth and $F$ satisfies (F1).

(i) Let $u$ be a solution of (3.7) and let $Q = \Omega \times (\delta_0, T)$ for any $T > \delta_0 > 0$.
   (a) $u$ is of $C^{1,\beta}(\overline{Q})$ for some $0 < \beta < 1$.
   (b) If $F$ is concave, $u$ is of $C^{1,1}(\overline{Q})$.
   (c) If $u \in C^{1,\overline{\alpha}}(\overline{Q})$ and if $F$ is concave or convex, $u$ is of $C^{2,\overline{\alpha}}(Q)$ for some $0 < \beta < 1$.
   (d) If $u \in C^{2,\beta}(\overline{Q})$ and $F \in C^{\alpha}$, $u$ is of $C^\alpha(\overline{Q})$.

(ii) Let $v(x, t)$ be a bounded solution of $v_t = F(D^2 v) + \mu v$. (a),(b),(c) and (d) for $v$ also hold.

We refer the regularity theory to [GT, CC, L, W1, W2]. We note that in this parabolic setting, $C^\alpha$ means that $C^\alpha$ in $x$ and $C^{\alpha/2}$ in $t$.

Let us prove the interior $C^{1,1}_x$ estimate for reader’s convenience through Bernstein’s computation.

**Lemma 4.5.** Suppose that $F$ satisfies (F1), (F3) and $F(0) = 0$ and $F$ is of $C^2$. Then, a bounded solution $v \in C^4$ of $v_t = F(D^2 v) + \mu v$, $\mu \in \mathbb{R}$ satisfies

$$\|v(x, t)\|_{C^{1,\alpha}(\overline{Q}_\frac{1}{4})} \leq C\|v\|_{L^\infty(Q_1)}$$

and

$$\|D^2 v\|_{L^\infty(Q_\frac{1}{4})} + \|v_t\|_{L^\infty(Q_\frac{1}{4})} \leq C\|v\|_{L^\infty(Q_1)}$$

where $Q_K := B_R(0) \times (-R^2, 0)$. Moreover if $F$ is smooth,

$$\|v(x, t)\|_{C^{1,\alpha}(Q_{\frac{1}{4}})} \leq C\|v(x, t)\|_{L^\infty(Q_1)}$$

for $k = 1, 2, \ldots$. 

Proof. (i) Let \( M := \|v(x, t)\|_{L^\infty(Q_t)} \) and let \( \psi \in C^\infty(\overline{Q_t}) \) be a parabolic cutoff function such that \( 0 \leq \psi \leq 1 \) in \( Q_t \), \( \psi = 1 \) in \( Q_{t/2} \), \( \psi = 0 \) on \( \partial P Q_t \) and \( |\psi| + |\nabla \psi| + |D^2 \psi| < c = c(\psi) \).

For large \( \delta > 0 \) (to be chosen later), define

\[
h(x, t) = \delta(M - v)^2 + \psi^2 |\nabla v|^2 + M^2 \frac{8\delta|\mu|}{\lambda} x_i^2.
\]

Now, we define the uniformly elliptic operator

\[
L[w] := F_{ij}(D^2 v)D_{ij} w,
\]

and the uniformly parabolic operator \( H[w] := L[w] - w_t \) and we have that \( H[v] \leq -\mu v, H[v_{\infty}] = -\mu v_{\infty}, \) and \( H[v_{\infty}] \geq -\mu v_{\infty} \) from the concavity of \( F \) and \( F(0) = 0 \) using the function \( a(s) = F((1 - s)D^2 v) + (1 - s)(\mu v - vt) \) as in the chapter 9 at \([CC]\). Using Bernstein’s technique, we get

\[
H[h] = Lh - h_t \geq 2\delta\lambda|\nabla v|^2 + 2\delta(M - v)(-F_{ij}D_{ij}v + v_t) + 2|\nabla v|^2 \lambda|\nabla \psi|^2
- 2\psi|\nabla v|^2 DF||D^2 \psi| + 8\psi F_{ij}D_{ij}vD_{ij}v + 2\psi^2 \lambda|D^2 v|^2
+ 2\psi^2 D_{ij}(F_{ij}D_{ij}v - v_t) - 2\psi|\psi_t||\nabla v|^2 + M^2 \frac{8\delta|\mu|}{\lambda} F_{11}
\geq 2\delta\lambda|\nabla v|^2 + 2\delta(M - v)\mu v + 8M^2 \delta|\mu| + 2|\nabla v|^2 \lambda|\nabla \psi|^2
- 2\psi|\nabla v|^2 DF||D^2 \psi| + 8\psi F_{ij}D_{ij}vD_{ij}v + 2\psi^2 \lambda|D^2 v|^2 + 2\psi^2 |\nabla v|^2
- 2\psi |\nabla v|^2 \geq 0
\]

for large \( \delta = \delta(c(\psi), \Lambda, \lambda, n) > 0 \).

Since

\[
h \leq \delta M^2 + M^2 \frac{8\delta|\mu|}{\lambda} \leq CM^2 \quad \text{on} \quad \partial \Omega,
\]

we obtain that \( \sup_{Q_t} h \leq CM^2 \) from the maximum principle and hence

\[
||\nabla v(x, t)||_{L^\infty(Q_{t/2})} \leq C||v(x, t)||_{L^\infty(Q_t)}.
\]

(ii) \( ||D^2 v||_{L^\infty(Q_{t/2})} \leq C||v||_{L^\infty(Q_t)} \) comes from applying the maximum principle on

\[
g = \delta(v_t)^2 + \psi^2(v_{\infty})^2 + \delta CM^2|\mu|x_i^2
\]

for any direction \( e \in S^{n-1} \), as Proposition 9.3, \([CC]\). \( \square \)

Now, we are going to show normalized parabolic flow \( v(x, t) = e^{\alpha t}u(x, t) \) has the unique limit as \( t \to \infty \) and use the approach presented at [AI] to obtain the uniqueness of the limit.

**Proposition 4.6.** Suppose \( F \) satisfies (F1) and (F2). Let \( \varphi(x) \) be an eigenfunction of \([EV]\) and let \( v(x, t) = e^{\alpha t}u(x, t) \) where \( u \) solves \([LA]\) with nonnegative initial data. Then, there exists a unique constant \( \gamma^* > 0 \) depending on initial data such that

\[
||v(x, t) - \gamma^* \varphi(x)||_{C^0(\overline{Q})} \to 0 \quad \text{as} \quad t \to \infty.
\]

**Proof.** Let us recall that \( v \) is bounded and

\[
\sup_{s \geq 1} ||v(s^\alpha + s)||_{C^0(\partial Q_{(0, +\infty)})} < +\infty \quad \text{for} \quad \alpha > 0,
\]

and that

\[
||\nabla v||_{L^\infty(Q_{t/2})} \leq C||v||_{L^\infty(Q_t)}
\]

holds. Using the maximum principle, we have

\[
h = \delta(M - v)^2 + \psi^2 |\nabla v|^2 + M^2 \frac{8\delta|\mu|}{\lambda} x_i^2
\]

for large \( \delta > 0 \). Since

\[
\delta \leq \delta M^2 + M^2 \frac{8\delta|\mu|}{\lambda} \leq CM^2 \quad \text{on} \quad \partial \Omega,
\]

we obtain that \( \sup_{Q_t} h \leq CM^2 \) from the maximum principle and hence

\[
||\nabla v(x, t)||_{L^\infty(Q_{t/2})} \leq C||v(x, t)||_{L^\infty(Q_t)}.
\]

By the maximum principle, we have

\[
||D^2 v||_{L^\infty(Q_{t/2})} \leq C||v||_{L^\infty(Q_t)}
\]

for any direction \( e \in S^{n-1} \), as Proposition 9.3, [CC]. \( \square \)
which can be proved by the Weak Harnack inequalities, [W1]. Then for any sequence \( \{s_n\} \), there are a subsequence \( \{s_{n_k}\} \) and a function \( w(x, t) \) such that

\[
v(x, t + s_{n_k}) \to w(x, t) \quad \text{locally in } \Omega \times [0, +\infty) \quad \text{as} \quad n_k \to \infty
\]

and \( w \) satisfies \( F(D^2w) + \mu w - w_1 = 0 \) in \( \Omega \times (0, \infty) \). Now let \( A \) be the set of all sequential limits of \( \{v(\cdot, \cdot + s)\}_{s \geq 0} \) and let

\[
\gamma' = \inf \{\gamma > 0 : \exists w \in A \text{ such that } w \leq \gamma \phi \text{ in } \Omega \times (0, \infty)\}.
\]

We note that \( 0 < \gamma' < \infty \) from Lemma 4.2. We are going to prove that \( A = \{\gamma' \phi\} \).

First, we show that \( w \leq \gamma' \phi \) for any \( w \in A \). Fix \( \epsilon > 0 \). There exists \( w \in A \) such that \( w \leq (\gamma' + \epsilon) \phi \) by the definition of \( \gamma' \). Then we have a sequence of functions, \( \{v_n := v(\cdot, \cdot + s_n)\} \), converging to \( w \) as \( s_n \to \infty \), i.e., for a fixed \( T > 0 \), there is \( N > 0 \) such that \( |v_n(x, T) - w(x, T)| < \epsilon \) for all \( n > N \). Maximum principle for \( e^{-\mu t}(w_n - w) \) gives us that \( |v_n(x, t) - w(x, t)| < \epsilon \) for \( \Omega \times (T, \infty) \). From the Regularity Theory, we have

\[
||\nabla v_n(\cdot, T + 1) - w(\cdot, T + 1)||_{L^\infty(\Omega)} \leq C \epsilon
\]

and hence we deduce

\[
|v_n(\cdot, T + 1) - w(\cdot, T + 1)| \leq C \epsilon \phi
\]

for a uniform constant \( C > 0 \) depending on \( \Omega \) and \( \phi \), i.e.,

\[
v(x, T + 1 + s_n) \leq (\gamma' + C \epsilon) \phi(x) \quad \text{for large } s_n > 0.
\]

Comparison principle implies that

\[
v(x, t) = e^{\mu t} u(x, t) \leq (\gamma' + C \epsilon) \phi(x) \quad \text{for } t \geq T + 1 + s_n.
\]

and also

\[
w \leq (\gamma' + C \epsilon) \phi \quad \text{for all } w \in A.
\]

Since \( \epsilon \) is arbitrary and \( C \) is uniform, \( w \leq \gamma' \phi \quad \text{for all } w \in A \).

Second, we are going to show \( A \) has only one element. Assume that \( w \neq \gamma' \phi \) for some \( w \in A \). Then it is obvious that \( w(\cdot, 0) \leq \gamma' \phi \) because \( u_1(x, t) := e^{-\mu t} w(x, t) \) and \( u_2(x, t) := \gamma' \phi(x) e^{-\mu t} \) solve the same equation,

\[
F(D^2u) - u_1 = 0 \quad \text{in } \Omega \times (0, \infty).
\]

Maximum principle and Hopf’s Lemma imply that \( u_2(x, 1) - u_1(x, 1) > 0 \) in \( \Omega \) and \( u_2(x, 1) - u_1(x, 1) \geq \delta \phi(x) \) for all \( x \in \Omega \) for some \( \delta > 0 \), i.e., \( w(x, 1) \leq (\gamma' - \delta) \phi(x) \) in \( \Omega \). Therefore, we have that \( e^{\mu t} u_2(x, t + 1) = w(x, t + 1) \leq (\gamma' - \delta) \phi(x) \) in \( \Omega \times (0, \infty) \) from the comparison principle. Now, setting \( t_n := s_n + 1 \) we get

\[
v(x, t + t_n) \to w(x, t + 1) \quad \text{in } \Omega \times [0, +\infty) \quad \text{as} \quad n \to \infty,
\]

which is a contradiction to the definition of \( \gamma' \). Therefore we conclude that \( A = \{\gamma' \phi\} \) and the result follows.

From the Regularity theory and the approximation lemma 4.6 we obtain the following corollary.

**Corollary 4.7.** Suppose that \( F \) satisfies (F1), (F2) and \( F \) is concave. Let \( \phi(x) \) be an eigenfunction of \( \{F\} \) and let \( v(x, t) = e^{\mu t} u(x, t) \) where \( u \) solves (4.7) with nonnegative initial data. Then we have

\[
||v(x, t) - \gamma' \phi(x)||_{C^1(\overline{\Omega})} \to 0 \quad \text{for some } \gamma' > 0
\]

for \( k = 1, 2 \).
4.2. **Log-concavity.** In this subsection, we are going to study a geometric property of solutions of (4.7) and (EV) provided \( \Omega \) is convex. First, let us approximate the operator as follows.

**Lemma 4.8.** Let \( F \) satisfy (F1), (F2) and (F3). Then there are smooth \( F_\varepsilon \) converging to \( F \) uniformly in \( \text{Lip}(\mathbb{S}^{n-1}) \) satisfying (F1), (F3) and

\[
|DF_\varepsilon(z) \cdot z - F_\varepsilon(z)| \leq \sqrt{n} \Lambda \varepsilon.
\]

**Proof.** Let \( \psi \in C^\infty_c(\mathbb{R}^{n \times n}) \) be a standard mollifier with \( \int \psi(z)dz = 1 \) and let \( \psi_\varepsilon(z) = \frac{1}{\varepsilon^n} \psi(\frac{z}{\varepsilon}) \). Let us define \( F_\varepsilon \) by \( F \ast \psi_\varepsilon \). We note that \( F_\varepsilon \) is smooth, uniformly elliptic and concave and satisfies

\[
|F(z) - F_\varepsilon(z)| \leq \sqrt{n} \Lambda \varepsilon
\]
since \( F \) is uniformly elliptic.

Now we are going to show that for all \( z \),

\[
|DF_\varepsilon(z) \cdot z - F_\varepsilon(z)| \leq \sqrt{n} \Lambda \varepsilon.
\]

Since \( F \) is Lipschitz continuous with a Lipschitz constant \( \sqrt{n} \Lambda \), \( F \) is differentiable almost everywhere from Rademacher’s Theorem. Moreover, we get \( \|DF\|_\infty \leq \sqrt{n} \Lambda \) and

\[
DF(z) \cdot z = F(z) \quad \text{a.e. } z \in \mathbb{R}^{n \times n}
\]

using the fact that

\[
\frac{F((1 + t)z) - F(z)}{t} = F(z) \quad \text{for all } z \text{ and for } t > 0 \text{ from (F2).}
\]

Then we have

\[
DF_\varepsilon(z) \cdot z - F_\varepsilon(z) = \int (DF(y) \cdot z - F(y)) \psi_\varepsilon(z - y) dy
\]

\[
= \int DF(y) \cdot (z - y) \psi_\varepsilon(z - y) dy
\]

and therefore we deduce \( |DF_\varepsilon(z) \cdot z - F_\varepsilon(z)| \leq \sqrt{n} \Lambda \varepsilon. \)

**Lemma 4.9.** Let \( F \) satisfy (F1), (F2), and (F3) and let \( \Omega \) be strictly convex. Assume that \( u_0 \in C^0(\overline{\Omega}) \) be a positive initial data in \( \Omega \). If \( \log(u_0) \) is concave, then the solution \( u(x,t) \) of (4.7) is log-concave in the spatial variable for all \( 0 < t < \infty \), i.e.,

\[
D^2_x \log(u(x,t)) \leq 0 \quad \text{for } (x,t) \in \Omega \times (0, \infty).
\]

**Proof.** (i) Let us assume that \( u_0 \) is smooth in \( \overline{\Omega} \) and that \( D^2 \log u_0(x) \leq -cl \) in \( \Omega \) for some \( c > 0 \) and approximate \( F \) by \( F_\varepsilon \) from Lemma 4.8. We also approximate \( u_0 \) by \( u_{\varepsilon,0} \) for small \( \varepsilon > 0 \) such that

\[
D^2 \log u_{\varepsilon,0} \leq 0 \quad \text{in } \Omega, \quad F_\varepsilon(D^2 u_{\varepsilon,0}) = 0 \quad \text{on } \partial \Omega.
\]

Then there is the positive smooth solution \( u_\varepsilon \) of (4.7) with an operator \( F_\varepsilon(\cdot) - F_\varepsilon(0) \) and an initial data \( u_{\varepsilon,0} \). Let us put \( g(x,t) = \log u_\varepsilon(x,t) \), which is finite and smooth for \( x \in \Omega \) and takes the value \( g = -\infty \) on \( \partial \Omega \times (0, \infty) \). It also satisfies the equation

\[
\partial_t g = e^{-\varepsilon} F_\varepsilon \left( e^\varepsilon \left( D^2 g + \nabla g \nabla g^T \right) \right) - e^{-\varepsilon} F_\varepsilon(0) \quad \text{in } \Omega \times (0, +\infty).
\]

First, let us consider a domain \( \Omega \times (0, T) \) for \( T > 0 \). To estimate the maximum of its second derivatives, for small \( \delta > 0 \), consider the function \( Z \) defined as

\[
Z(t) = \sup _{x \in \Omega} \sup _{|\eta| = 1} g \delta \eta (y,t) + \psi(t),
\]
where \( c_\beta \in S^{n-1} \) and \( \psi(t) := -\delta \tan(2K \sqrt{\delta} t) \). The constant \( K > 0 \) independent of \( \epsilon > 0 \) and \( \delta > 0 \) will be chosen later. Now, let us assume there exists \( t_0 \in \left[ 0, \min\left( \frac{\pi}{4K \sqrt{\delta}}, T \right) \right] \) such that
\[
Z(t) = \sup_{y \in \Omega} \sup_{|y| = 1} g_{\beta\beta}(y, t) + \psi(t) = 0 \quad \text{at } t = t_0.
\]
We may assume that
\[
Z(t_0) = g_{\alpha\alpha}(x_\alpha, t_0) + \psi(t_0) = 0
\]
for some direction \( e_\alpha \) and \( x_\alpha \in \overline{\Omega} \). Then \( e_\alpha \) is an eigen-direction of the symmetric matrix \( D^2 g(x_\alpha, t_0) \) which means that, using orthonormal coordinates in which \( e_\alpha \) is taken as one of the coordinate axes, \( g_{\alpha\beta} \) is zero at \((x_\alpha, t_0)\) for \( \beta \neq \alpha \). We note that \( Z(0) < 0 \) from the assumption.

Then, we claim that
\[
g_{\alpha\alpha}(x, t_0) = \frac{u_\epsilon u_{\epsilon,\alpha\alpha} - u_\epsilon^2}{u_\epsilon^2} \to -\infty \quad \text{as } x \in \Omega \to \partial \Omega.
\]
This holds when \( e_\alpha \) is not a tangential direction, since \( \partial \Omega \) is smooth, \(|D^2 u_\epsilon|\) is bounded and \(|V u_\epsilon| > 0\) on \( \partial \Omega \) by Hopf’s lemma. For a tangential direction \( e_\alpha \), we take a coordinate system such that \( x_n = 0 \) and that the tangent plane is \( x_n = 0 \). Let the boundary be given locally by the equation \( x_n = f(x') \), and \( x' = (x_1, \ldots, x_{n-1}) \).

We introduce the change of variables
\[
y_i = x_i \quad (i = 1, \ldots, n - 1), \quad y_n = x_n - f(x'), \quad v(y, t) = u_\epsilon(x, t).
\]
Then along tangent directions \( e_\alpha \) we have
\[
 u_{\epsilon,\alpha\alpha}(x, t) = v_{\alpha\alpha}(y, t) - 2v_{\alpha\alpha}(y, t) f_{\alpha\alpha}(x') + v_{\alpha\alpha}(y, t) (f_{\alpha\alpha}(x'))^2 - v_{\alpha\alpha}(y) f_{\alpha\alpha}(x').
\]
Using the fact that \( \partial_j v(0, t) = 0 \) from the boundary condition and \( f_j(0) = 0 \) for \( j = 1, \ldots, n-1 \), we obtain
\[
u_{\epsilon,\alpha\alpha}(0, t_0) = -v_{\alpha\alpha}(0) f_{\alpha\alpha}(0) < 0,
\]
for a tangential vector \( e_\alpha \). We note that \( f_{\alpha\alpha}(0) > 0 \) since \( \Omega \) is strictly convex. Thus \( g_{\alpha\alpha}(x, t_0) \) tends to \( -\infty \) as \( x \in \Omega \) goes to \( \partial \Omega \). And from the uniform global \( C^2 \) estimate of \( u_\epsilon \), there is a small \( \eta > 0 \) independent of \( \epsilon, \delta \) such that \( g_{\alpha\alpha}(x, t) < -10 \) for \( x \in \Omega \setminus \Omega_{(\epsilon, \eta)} \times (0, T) \), where \( \Omega_{(\epsilon, \eta)} = \{ x \in \Omega : d(x, \partial \Omega) > \eta \} \). So we deduce that the maximum of \( Z \) can only be achieved at an interior point \( x_\alpha \in \Omega_{(\epsilon, \eta)} \).

Next, we look at the evolution equation of \( g_{\alpha\alpha}(x, t) \), which is given by the equation below
\[
g_{\alpha\alpha,t} = F_{ij} \cdot (D_{ij}g_{\alpha\alpha} + D_{ij}g_{\alpha\alpha}D_{ij}g + D_{ij}D_{ij}g_{\alpha\alpha} + 2D_{ij}g_{\alpha\alpha}D_{ij}g_{\alpha\alpha})
+ (g_{\alpha\alpha}^2 - g_{\alpha\alpha}) \{ e^{-8F} \left( e^F \left( D_{ij}g + D_{ij}D_{ij}g \right) \right) - F_{ij} \cdot \left( D_{ij}g + D_{ij}D_{ij}g \right) \}
+ e^{-8F} F_{ijkl} : \left( e^F \left( D_{ij}g + D_{ij}D_{ij}g \right) \right) \left( e^F \left( D_{ij}g + D_{ij}D_{ij}g \right) \right) \cdot \left( e^F \left( D_{ij}g + D_{ij}D_{ij}g \right) \right) \cdot \left( e^F \left( D_{ij}g + D_{ij}D_{ij}g \right) \right)
- (g_{\alpha\alpha}^2 - g_{\alpha\alpha}) e^{-8F} F_{ij}(0)
\]
where \( F_{ij} = F_{\epsilon,ij} \left( e^F \left( D_{ij}g + D_{ij}D_{ij}g \right) \right) \). Since \( F_{ij} \) satisfies (F1), concavity and (4.9), it follows that
\[
g_{\alpha\alpha,t} \leq F_{ij} \cdot (D_{ij}g_{\alpha\alpha} + D_{ij}g_{\alpha\alpha}D_{ij}g + D_{ij}D_{ij}g_{\alpha\alpha} + 2D_{ij}g_{\alpha\alpha}D_{ij}g_{\alpha\alpha}) + 2\sqrt{n} \Lambda \epsilon e^{-8F} [g_{\alpha\alpha}^2 - g_{\alpha\alpha}].
\]
At the point of maximum \((0, t_0)\), we see that
\[
g_{aa} = -\psi \geq 0, \quad \nabla_s g_{aa} = 0, \quad D_s^2 g_{aa} \leq 0
\]
as well as \(g_{\alpha\beta} = 0\) for \(\beta \neq \alpha\). Thus we get at the point of maximum \((0, t_0)\),
\[
g_{aa,t} \leq F_{ij} \cdot (D_{ij}g_{aa} + D_i g_{aa} D_j g + D_i D_j g_{aa} + 2D_i g_a D_j g_a) + 2 \sqrt{n} \Lambda \varepsilon^{-\delta} |g_{aa}^2 - g_{aa}|^\frac{1}{2}
\leq 2F_{aa} \frac{\alpha^2}{\alpha^2} + 2 \sqrt{n} \Lambda \varepsilon^{-\delta} (g_{aa}^2 + g_{aa})
\leq 2\Lambda g_{aa}^2 + \varepsilon 2 \sqrt{n} \Lambda \frac{|u_k|}{ue}.
\]

On the other hand, when the supremum of \(Z(t) - \psi(t) = \sup_{y \in \Omega} \sup_{|z_j| = 1} g_{\beta\beta}(y, t)\) is achieved at a point \(x(t) \in \Omega\) with a unit vector \(e_{\beta(t)}\) at each time \(t\), we check that \(g_{\beta\beta}(t, \beta(0)) = 0\) and \(\nabla_s g_{\beta\beta}(t, \beta(0)) = 0\) at the point \((x(t), t)\). Therefore, we have at the maximum point \((0, t_0)\),
\[
0 \leq Z'(t_0) = g_{aa,t} + \psi_t
\leq \psi_t + 2\Lambda \psi^2 + \varepsilon 2 \sqrt{n} \Lambda \frac{|u_k|}{ue} \leq \psi_t + K(\psi^2 + \varepsilon),
\]
when we select a uniform number \(K > 0\) bigger that \(C(\Lambda, \eta) \left(1 + \max_{\Omega_{(-\eta)} \times (0, T)} \frac{|D^2 u_k|}{ue}\right)\).
Now, it is easy to check that
\[
\psi_t + K(\psi^2 + \varepsilon) < \frac{2K(-\delta^3/2 + \delta^2)}{\cos(2K \sqrt{\delta} t)} < 0
\]
for \(0 < \varepsilon < \delta\) and for \(2K \sqrt{\delta} t < \frac{\pi}{8}\), which implies a contradiction. Therefore, we obtain
\[
\sup_{y \in \Omega} \sup_{|z_j| = 1} \partial_{aa} \log(u_k)(y, t) < -\psi(t) = \delta \tan(2K \sqrt{\delta} t) \leq \delta
\]
for \(0 < t < \min\left(\frac{\pi}{8K \sqrt{\delta}}, T\right)\) and for \(0 < \varepsilon < \delta\) from the uniform interior \(C^{2,\beta}\)-estimates of \(u_k\) in \(\Omega_{(-\eta)} \times (0, T)\). Letting \(\delta \to 0\) we conclude that
\[
\partial_{aa} \log(u) \leq 0 \quad \text{in} \quad \Omega \times (0, T).
\]
Therefore \(u(x, t)\) is log-concave with respect to \(x \in \Omega \times (0, \infty)\) since \(T\) is arbitrary.

(ii) The proof in the general case uses a density argument which is more or less standard. Briefly, if \(u_k\) is not smooth and strictly log-concave, we perform a mollification to obtain an approximating sequence \(u_{o,j}\) of smooth and log-concave functions. To make \(u_{o,j}\) strictly log-concave we may put for instance,
\[
\tilde{u}_{o,j}(x) = u_{o,j}(x) \exp(-c_j |x|^2)
\]
for some \(c_j > 0, c_j \to 0\) as \(j \to \infty\). From (i), we get the result for \(\tilde{u}_{o,j}\), the solution of the problem with data \(\tilde{u}_{o,j}\). Uniform Hölder regularity let us take a subsequence \(\tilde{u}_j\) converging uniformly to \(u\) in each compact subset and then uniform convergence on each compact subset will preserve the sign in the second difference in the limit.

\textbf{Corollary 4.10.} \textit{Let \(F\) satisfy (F1),(F2) and (F3) and let \(\Omega\) be convex. If \(u_o\) is log-concave, so is the viscosity solution \(u(x, t)\).}
Remark 4.11. We note that any concave function in a convex domain \( \Omega \) is log-concave. On the other hand, it is well-known that the distance function \( \text{dist}(x, \partial \Omega) \) is concave for a convex domain, so the lemma is not void.

Remark 4.12.
\begin{enumerate}
\item Let \( \sigma_k(D^2 u) = \sum_{i < c_k} \lambda_i \cdots \lambda_k \) for the eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \) of \( D^2 u \).
\item If a differentiable operator \( F \) satisfies the conditions (F2) and (F3), \( F(D^2 u) = \sigma_k(D^2 u)^t \) satisfies the conditions (F2) and (F3).
\end{enumerate}

Corollary 4.13 (Log-concavity). Let \( F \) satisfy (F1),(F2) and (F3) and let \( \Omega \) be convex. Then, the stationary profile \( \varphi(x) \) is log-concave, i.e., \( D^2 \log(\varphi(x)) \leq 0 \).

Proof. Take the distance function as an initial data of parabolic flow (4.7). Then Corollary 4.10 yields that for \( x, y \in \Omega \),
\[ 2(\log u(x, t) + \log u(y, t)) - \log u \left( \frac{x + y}{2}, t \right) \leq 0. \]
From the asymptotic result, Proposition 4.6 we have the uniform convergence
\[ \|e^{it}u(x, t) - \gamma^* \varphi(x)\|_{C^0(\Omega)} \to 0 \text{ as } t \to \infty \]
and hence the result follows.

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and hence the result follows.

For a differentiable operator, the foregoing is a classical result, [LY2] for a domain which is smooth and strictly convex.

Lemma 4.14 (Strict log-concavity). Suppose that \( F \) satisfies (F1), (F2) and (F3) and is differentiable and that \( \Omega \) is smooth and strictly convex. Then the positive eigenfunction \( \varphi \) of (EV) is strictly log concave, i.e., there exists a constant \( c_1 > 0 \) such that
\[ D^2(\log \varphi) \leq -c_1 I. \]

Theorem 4.15 (Eventual log-concavity). We assume the same hypothesis as Lemma 4.14. Let \( u_0 \) be a nonnegative initial function. Then, the solution \( u(x, t) \) of (4.7) is strictly log-concave in the spatial variable for large \( t > 0 \), i.e., for every \( \epsilon > 0 \) there is \( t_0 = t_0(u_0, \epsilon) \) such that
\[ D^2(\log u(x, t)) \leq -(c_1 - \epsilon) I \quad \text{for all } t \geq t_0, \]
where \( c_1 > 0 \) is the constant of Lemma 4.14.

5. Degenerate Parabolic Fully Nonlinear Equation

In this section, we consider the solution \( u(x, t) \) of the fully nonlinear degenerate parabolic equation
\begin{equation}
\begin{cases}
u(x, t) = F(D^2 u^m(x, t)) & \text{in } Q_T = \Omega \times (0, T), \ m > 1, \\
u(x, 0) = u_0(x), & \\
u(x, t) = 0 & \text{on } x \in \partial \Omega,
\end{cases}
\end{equation}
where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with a smooth boundary. We assume that \( u_0 \) belongs to
\[ C_b(\overline{\Omega}) := \{ h \in C^0(\overline{\Omega}) | c_0 \text{ dist}(x, \partial \Omega) \leq h(x) \leq C_0 \text{ dist}(x, \partial \Omega) \text{ for some } 0 < c_0 \leq C_0 < +\infty \}. \]
We define \( w := u^m \), then \( w \) satisfies
\[
\begin{aligned}
\alpha m - \frac{1}{2} F(D^2 w) - w_t &= 0 & \text{in } Q_T = \Omega \times (0, T), \quad m > 1, \\
 w(x, 0) &= w_0(x) = u_0^m(x) \in C_{\alpha}(\overline{\Omega}), \\
 w(x, t) &= 0 & \text{on } x \in \partial \Omega.
\end{aligned}
\]  
(5.11)

We also introduce the pressure in the form \( v = \frac{m}{m - 1} u^{m-1} \). If \( F \) satisfies (F2), the pressure \( v \) solves
\[
\begin{aligned}
 v_t &= F((m - 1)vD^2 v + DvDV^T) & \text{in } Q_T = \Omega \times (0, T), \quad m > 1, \\
 v(x, 0) &= v_0(x) = \frac{m}{m - 1} u_0^{m-1}, \\
 v(x, t) &= 0 & \text{on } x \in \partial \Omega.
\end{aligned}
\]  
(5.12)

Before studying asymptotic behaviors of degenerate parabolic flows, let us state the regularity of the solution.

**Proposition 5.1 (Regularity for \( m > 1 \)).** Let \( F \) satisfy (F1), (F2) and let \( u \) be the solution of (5.10).

1. If \( u_0 \) is nonzero and nonnegative, then
   - (i) there is a time \( t_o = t_o(u_0, \Omega) > 0 \) such that
     \[ u(x, t) > 0 \quad \text{in } \Omega \times (t_o, \infty) \]
     for a uniform constant \( t_o = t_o(\lambda, \Lambda, u_o) > 0 \).
   - (ii) \( 0 \leq u(x, t) \leq C_o t^{-\frac{1}{\alpha}} \text{dist}(x, \partial \Omega)^{\frac{1}{\alpha}} \) \text{in } \Omega \times (0, \infty).

2. If \( u_0 \) is an initial data in \( C_{\alpha}(\overline{\Omega}) \), then
   - (i) we have
     \[
     C_o(t + \tau_1)^{-\frac{1}{\alpha}} \text{dist}(x, \partial \Omega)^{\frac{1}{\alpha}} \leq u(x, t) \leq C_o(t + \tau_2)^{-\frac{1}{\alpha}} \text{dist}(x, \partial \Omega)^{\frac{1}{\alpha}} \quad \text{in } \Omega \times (0, \infty)
     \]
     for some constant \( \tau_1, \tau_2 \) depending on \( u_0 \). Moreover, for \( Q_T = \Omega \times [s, T] \), \((0 < s < T)\),
     - (a) \( u \) is of \( C^{1,\beta}(Q_T) \) for some \( 0 < \beta < 1 \),
     - (b) \( u \) is of \( C^{1,1}(Q_T) \) if \( F \) is concave or convex,
     - (c) \( u \) is of \( C^{2,\beta}(Q_T) \) for some \( 0 < \beta \) if \( F \) is concave or convex and \( u \in C^{1,1} \),
     - (d) \( u \) is of \( C^\infty(Q_T) \) if \( F \) is \( C^\infty \) and \( u \in C^2 \).

   - (ii) \( u \) is of \( C^{1,\beta}_o(\overline{\Omega} \times [s, T]) \cap C^{1,\beta}(\Omega \times [s, T]) \) for some \( 0 < \beta < 1 \).

**Proof.** (1) For \( c > 0 \), let

\[
V(x, t) = t^{-a} \left( c - k \frac{|x|^2}{t^b} \right),
\]

where \( a = \frac{m(m-1)\lambda}{2(1+n(m-1)\lambda)}, \quad \beta = \frac{2\lambda}{2(1+n(m-1)\lambda)}, \quad \text{and} \quad k = \frac{1}{2(1+n(m-1)\lambda)}. \) Then we can check
\[
F((m - 1)VD^2 V + DVDV^T) - V_t \geq M^+((m - 1)VD^2 V + M^+(DVDV^T) - V_t = 0 \quad \text{in } \{V > 0\}
\]
and hence \( V \) is a sub-solution of (5.12) as long as \( \text{supp}(V) \subset \overline{\Omega} \).

We define \( \overline{U}(x, t) = \left( \frac{m-1}{m} V(x, t) \right)^{\frac{1}{m-1}} = \left( \frac{m-1}{m} \right)^{\frac{1}{m-1}} t^{-a/(m-1)} \left( c - k \frac{|x|^2}{t^b} \right)^{\frac{1}{m-1}} \), and hence \( \overline{U} \) is a sub-solution of (5.10) in \( \text{supp}(\overline{U}) \) as long as \( \text{supp}(\overline{U}) \subset \overline{\Omega} \). We note that the
support of $\bar{U}$ is compact and expands in time. So the previous argument in Lemma 4.2 gives the result that $u$ is positive for large time $t$.

(ii) To get the upper bound, we are going to show that

$$u(x,t) \leq f(x) t^{-\frac{1}{m}} \quad \text{in} \quad \Omega \times (0, \infty),$$

where $f$ is the solution of (3.3). Define $u_{\varepsilon,t} := (u_0 - \varepsilon)_{+} = \max(u_0 - \varepsilon, 0)$ for $\varepsilon > 0$ and let $u_t$ be the solution of (5.10) with initial data $u_{\varepsilon,t}$. We choose $\tau_\varepsilon > 0$ converging to 0 as $\varepsilon \to 0$ such that $u_{\varepsilon,t}(x) \leq f(x)(\tau_\varepsilon)^{-\frac{1}{m}}$. Comparison principle yields that

$$u_t(x,t) \leq f(x)(\tau_\varepsilon + t)^{-\frac{1}{m}} \leq f(x)t^{-\frac{1}{m}}$$

in $\Omega \times (0, \infty)$ since $f(x)(\tau + t)^{-\frac{1}{m}}$ is a similarity solution for any $\tau > 0$.

From the comparison principle, $u_t$ is nondecreasing as $\varepsilon$ decreases and

$$u_{\varepsilon,t} \leq u_t \leq u \leq \max_{\Omega} u_0$$

if $\varepsilon < \varepsilon_0$ for any $\varepsilon_0 > 0$. Then for each compact subset $K$ of $\Omega \times (0, \infty)$, $w_t := u_{\varepsilon,t}^m$ satisfies a uniformly parabolic equation, $w_t^{1+\frac{1}{m}} f(D^2 w_t) - w_t = 0$, and uniform parabolic estimates tell us that $w_t \to \check{w}$ as $\varepsilon \to 0$ in $K$ for some locally Hölder continuous function $\check{w}$, which is the solution of (5.11). Therefore, we obtain

$$u(x,t) \leq f(x) t^{-\frac{1}{m}} \quad \text{in} \quad \Omega \times (0, \infty)$$

and hence $0 \leq u(x,t) \leq C \langle t, \varpi \rangle^\frac{1}{m} \quad \text{dist} (x, \partial \Omega)^\frac{1}{m} \quad \text{in} \quad \Omega \times (0, \infty)$ since $\inf_{\partial \Omega} |V_s f|^m > 0$.

(2) (i) We choose $\tau_1 > 0, \tau_2 > 0$ such that

$$f \cdot (\tau_1 + t)^{-\frac{1}{m}} \leq u(x,t) \leq f \cdot (\tau_2 + t)^{-\frac{1}{m}}$$

because $u_{\varepsilon,t} \in C_0(\bar{\Omega})$. Since $f(x)(\tau_1 + t)^{-\frac{1}{m}}$ is a solution of (5.10), the comparison principle implies

$$f \cdot (\tau_1 + t)^{-\frac{1}{m}} \leq u(x,t) \leq f \cdot (\tau_2 + t)^{-\frac{1}{m}}.$$

Thus the first result comes from the gradient estimate of the positive eigenfunction on the boundary. On the other hand, for each compact subsets $K \Subset \bar{K} \subseteq \Omega$, there exist $0 < c_0 \leq C_0 < +\infty$ such that

$$0 < c_0 \leq w(x,s) = u_{\varepsilon,t}^m(x,s) \leq C_0 < +\infty \quad \text{in} \quad \bar{K} \times [s/2, T],$$

which means that the operator $w^{1+\frac{1}{m}} F(\cdot)$ becomes uniformly elliptic in $\bar{K} \times [s/2, T]$. So the estimates follow from Theorem 4.1.

(ii) We use the fact (i) and scaling property to prove the Hölder regularity on the boundary. In fact, since we have a linear growth of $w = u^m$ away from the boundary: let $\delta_0 > 0$ be a constant such that $B_{\delta_0}(x) \subset \Omega$ for $\text{dist} (x, \partial \Omega) > \delta_0$. For $x_0 \in \Omega$ such that $\text{dist} (x_0, \partial \Omega) < \delta_0$, we set $\text{dist} (x_0, \partial \Omega) = 2\sigma$. According to (i), it follows that

$$c_0 \sigma < w(x_0, t) = u^m(x_0, t) < C_0 \sigma, \quad \text{for} \quad t \in [s/2, T],$$

where $|x - x_0| = \text{dist} (x, \partial \Omega) = 2\sigma < \delta_0$.

Now we scale $w$ linearly with the distance $\sigma$ to the boundary so the scaled function $\check{w}$ has a value of order one. Then $\check{w}$ will satisfy a uniformly parabolic equation and have a uniform gradient estimate. Define $\check{w}(\check{x}, \check{t}) = w^{1+\frac{1}{m}}(\check{x}, \check{t}) := \frac{1}{\sigma} w(x_0 + \sigma \check{x}, \sigma^{1+1/m} \check{t})$. From scaling property, $\check{w}$ satisfies $\check{w}^{1+\frac{1}{m}} F(D^2 \check{w}) - \check{w}_t = 0$ for an elliptic
operator $\tilde{T}(\cdot) = \sigma F \left( \frac{\cdot}{t} \right)$ with the same ellipticity constants $\lambda, \Lambda$ and this transform sends $\{ x \in \Omega : \| x - x_0 \| = \alpha \}$ to $\{ \tilde{x} : \| \tilde{x} \| = 1 \}$. Thus (5.13) implies that

$$c_\alpha < \tilde{w}(\tilde{x}, \tilde{t}) < C_\alpha \quad \text{for } (\tilde{x}, \tilde{t}) \in B_1(0) \times [\sigma^{-1/\varepsilon} s/2, \sigma^{-1/m} T],$$

and then we have

$$|\nabla w(x_0, t)| = |\nabla \tilde{w}(0, \tilde{t})| < C \quad \text{for } t \in [s, T]$$

from uniform gradient estimates for uniformly parabolic equations. (We refer to [L], [W].)

On the region $K := \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \frac{\delta}{2} \}$, $u$ is positive so we have $u \geq c_\alpha$ in $K \times [s/2, T]$ for some constant $c_\alpha > 0$. Then the operator is uniformly parabolic in $K \times [s/2, T]$ and hence we also have

$$|\nabla w(x, t)| < C \| w \|_{L^\infty(K \times [s/2, T])}$$

for $\text{dist}(x, \partial \Omega) \geq \frac{\delta}{2}$ and $t \in [s, T]$ from the regularity theory of the uniformly parabolic equations. Therefore, $w = u^m$ is of $C^0,1(\Omega \times [s, T])$. \hfill \Box

5.1. Asymptotic Behavior. First, we are going to show Aronson-Bénilan inequality for the degenerate fully nonlinear equation with $m > 1$, which tells us almost monotonicity of parabolic flows as $t \to \infty$.

**Lemma 5.2** (Aronson-Benilan inequality). Suppose that $F$ satisfies (F1), (F3) and $F(0) = 0$. Let $u$ be the solution of (5.10) with initial data $u_0^m \in C_b(\overline{\Omega})$ and let $v = u^{m-1}$. Then we have for large $C = C(m) > 0$,

$$u_t \geq -C \frac{u}{t} \quad \text{and} \quad v_t \geq -C \frac{v}{t} \quad \text{for } t > 0.$$  

**Proof.** (i) First, let us also assume that $F$ is of $C^{1}$. Let $w := u^m$ and $w$ solves (5.11). Let $\delta > 0$ and $\varepsilon > 0$ and let $C$ be a positive constant bigger than $\frac{1}{m-1}$. We can select $-\delta < \tau_{\varepsilon, \delta} < 0$ so that $w_t + C \frac{w + \varepsilon}{t + \tau_{\varepsilon, \delta}} > 0$ at $t = \delta$ because $w_t = w = 0$ on $\partial \Omega \times (0, \infty)$.

Define $Z(t) := \inf_{x \in \Omega} \left( w_t + C \frac{w + \varepsilon}{t + \tau_{\varepsilon, \delta}} \right)$. We note that $Z(\delta) > 0$. From the concavity of $F$ and $F(0) = 0$, the function $w$ satisfies

$$m w^{1-1/m} F_{ij} (D^2 w) D_{ij} w - w_t \leq 0.$$  

Let $t_0 \in (\delta, \infty)$ be the first time such that $Z(t_0) = 0$ and then we have that $w_t(x_0, t_0) < 0$, and that $w_t^2 = C^2 \frac{(w + \varepsilon)^2}{(t + \tau_{\varepsilon, \delta})^2} > 0$ at the minimum point $(x_0, t_0) \in \Omega \times \{ t_0 \}$. Indeed, the minimum point $x_0$ is interior in $\Omega$ because $\left( w_t + C \frac{w + \varepsilon}{t + \tau_{\varepsilon, \delta}} \right) > 0$ on $\partial \Omega$. At the
minimum point, we have

\[
Z_l = \left( w_l + C \frac{w + \varepsilon}{\tau + \tau_{\varepsilon, l}} \right)
\]

\[
= \left( 1 - \frac{1}{m} \right) mw^{-1/m} F(D^2 w) w_l + mw^{-1/m} F_{ij}(D^2 w) D_{ij} w_l + C \frac{w_l}{\tau + \tau_{\varepsilon, l}} - C \frac{w + \varepsilon}{(\tau + \tau_{\varepsilon, l})^2}
\]

\[
= \left( 1 - \frac{1}{m} \right) \frac{(w_l)^2}{w} + mw^{-1/m} F_{ij} D_{ij} w_l + C \frac{w_l}{\tau + \tau_{\varepsilon, l}} - C \frac{w + \varepsilon}{(\tau + \tau_{\varepsilon, l})^2}
\]

\[
\geq \left( 1 - \frac{1}{m} \right) \frac{(w_l)^2}{w} - C \frac{w_l}{\tau + \tau_{\varepsilon, l}} + C \frac{w_l}{\tau + \tau_{\varepsilon, l}} - C \frac{w + \varepsilon}{(\tau + \tau_{\varepsilon, l})^2}
\]

\[
\geq \left( 1 - \frac{1}{m} \right) C^2 \frac{(w + \varepsilon)^2}{w(\tau + \tau_{\varepsilon, l})^2} - C \frac{w + \varepsilon}{(\tau + \tau_{\varepsilon, l})^2} \geq C \frac{w + \varepsilon}{(\tau + \tau_{\varepsilon, l})^2} \left( \frac{m-1}{m} C - 1 \right) > 0,
\]

which is a contradiction. Therefore we have \( w_l > -C \frac{w + \varepsilon}{\tau + \tau_{\varepsilon, l}} \geq -C \frac{w + \varepsilon}{\tau - \delta} \) for \( t > \delta \). Since \( \varepsilon, \delta > 0 \) are arbitrary, we deduce that \( tw_l + Cw \geq 0 \) for \( \Omega \times (0, \infty) \) and hence \( u_l \geq -C_1 \frac{w}{t} \) for \( t > 0 \).

(ii) In general, let us approximate \( F(\cdot) \) by smooth \( F_{\varepsilon}(\cdot) \). Let \( u^\varepsilon \) be the solution of \( \ref{5.10} \) with the operator \( F_{\varepsilon} \) and with the same initial data and let \( u^\varepsilon \) be the solution of \( \ref{5.10} \) with the operator \( \mathcal{M}^\varepsilon \). Let us define \( w^\varepsilon := (u^\varepsilon)^m \). From Comparison principle, it follows that

\[ 0 < u^- \leq u^\varepsilon \leq \|u_0\|_{L^\infty(\Omega)} \quad \text{in} \quad \Omega \times (0, \infty), \]

which implies that \( w^\varepsilon \) solves the uniformly parabolic equation in each compact subset of \( \Omega \times (0, \infty) \). Then, \( w^\varepsilon \) and \( w^\varepsilon \) converge uniformly to \( w \) and \( w_i \), respectively, in each compact subset of \( \Omega \times (0, \infty) \) from the regularity theory. Therefore we conclude that \( w_l \geq -C \frac{w}{t} \) for large \( C = C(m) > 0 \) and hence \( \ref{5.14} \) holds by direct calculations.

**Proposition 5.3 (Approximation).** Suppose that \( F \) satisfies (F1) and (F2). Let \( u \) be the solution of \( \ref{5.10} \) with initial data \( u_0^\varepsilon \in C_0(\Omega) \). Set \( U(x, t) := \frac{f(x)}{(1 + t)^{m-1}} \), where \( f \) solves

\[
\begin{cases}
-F(D^2 f^m(x)) &= \frac{1}{m-1} f(x) \quad \text{in} \quad \Omega, \quad m > 1, \\
f &= 0 \quad \text{on} \quad \partial \Omega, \\
f > 0 \quad \text{in} \quad \Omega.
\end{cases}
\]

(5.15)

Then, we have

\[ \lim_{t \to \infty} t^{\frac{m}{m-1}} |u(x, t) - U(x, t)| \to 0 \quad \text{uniformly in} \quad \Omega, \]

and there exists \( t_0 > 0 \) such that \( u^m \) is \( C^1 \) up to the boundary and \( 0 < c_0 < t^{\frac{m}{m-1}} |V(\Omega)| \) for \( x \in \partial \Omega \), where \( c_0 \) and \( C_0 \) depend on \( u_0 \) and \( \Omega \).

**Proof.** (i) In the proof of (2) at Proposition \( \ref{5.1} \) we have

\[ f \cdot (\tau_1 + t)^{-\frac{m}{m-1}} \leq u(\cdot, t) \leq f \cdot (\tau_2 + t)^{-\frac{m}{m-1}} \]
since \( u^m \in C_b(\Omega) \). Then, we obtain
\[
\left| t^\frac{1}{m} (u - U) \right| \leq f \cdot \left( \frac{t^\frac{1}{m}}{(\tau_2 + t)^{\frac{1}{m}}} - \frac{t^\frac{1}{m}}{(\tau_1 + t)^{\frac{1}{m}}} \right) \to 0 \quad \text{uniformly as } t \to \infty.
\]

(ii) From (i), \( w = u^m \) satisfies
\[
\phi \cdot (\tau_1 + t)^{-\frac{1}{m}} \leq w \leq \phi \cdot (\tau_2 + t)^{-\frac{1}{m}} \quad \text{in } \Omega \times [0, \infty),
\]
where \( \phi = f^m \) is the solution of (NLEV). From Hopf’s Lemma for \( \phi \), we have
\[
\frac{c_1}{(1 + t)^{\frac{1}{m}}} \inf_{\Omega} \phi \leq w(x, t) \leq \frac{c_2}{(1 + t)^{\frac{1}{m}}} \sup_{\partial \Omega} \phi \quad \text{in } \Omega \times [0, \infty)
\]
and
\[
\frac{c_1}{(1 + t)^{\frac{1}{m}}} \inf_{\partial \Omega} \phi \leq w(x, t) \leq \frac{c_2}{(1 + t)^{\frac{1}{m}}} \sup_{\partial \Omega} \phi \quad \text{in } \Omega \times [1, \infty).
\]
We follow a similar argument as (2),(ii) at Proposition 5.1 and use scaling property for porous medium equation to estimate
\[
|\nabla w(x, t)| \leq \frac{C_o}{t^{\frac{1}{m}}} \quad \text{for } (x, t) \in \Omega \times [1, \infty)
\]
for some \( 0 < C_o < +\infty \). Moreover, we have that
\[
\frac{c_0}{(1 + t)^{\frac{1}{m}}} \leq |\nabla w(x, t)| \leq \frac{C_o}{t^{\frac{1}{m}}} \quad \text{for } (x, t) \in \partial \Omega \times [1, \infty)
\]
for some \( 0 < c_0 < C_o < +\infty \), which means that \( 0 < c_0 < t^\frac{1}{m} |\nabla u^m(x, t)| < C_o \) for \( x \in \partial \Omega \) and for large \( t > t_o \).

\textbf{Remark 5.4.} If we set \( z(x, t) := t^{\frac{1}{m}} u(x, t) \), the renormalized function, the estimate in Lemma 5.2 holds to \( z \). In fact, we have
\[
z_t = t^{\frac{1}{m}} \left( u_t + \frac{1}{m-1} u \right) \geq \left( -C + \frac{1}{m-1} \right) \frac{z}{t}.
\]

\textbf{Corollary 5.5.}
Under the same condition of Proposition 5.3,
\[
z(x, t) = t^{\frac{1}{m}} u(x, t) \to f(x) \quad \text{uniformly as } t \to +\infty.
\]
And if \( F \) is concave, \( z(x, t) \) converges to \( f(x) \) in \( C^0_b(\Omega) \cap C^{0,\alpha}_k(\Omega) \) for \( k = 1, 2 \).

\textbf{Proof.} The first part (i) at Proposition 5.3 directly gives the convergence of \( z(x, t) \) to \( f(x) \) as \( t \to \infty \) uniformly in \( \Omega \). So we will see the second estimate. For each compact subsets \( K \Subset K' \) of \( \Omega \), uniform convergence implies
\[
\frac{1}{2} \inf_{K'} f \leq z(x, t) \leq 2 \sup_{K'} f \quad \text{in } K' \times [T, \infty)
\]
for large \( T > 1 \). For \( w := u^m \), we have
\[
\frac{1}{2} \inf_{K'} f^m t^{\frac{1}{m}} \leq w(x, t) \leq 2 \sup_{K'} f^m t^{\frac{1}{m}} \quad \text{in } K' \times [T, \infty).
\]
Let \( t_o > 2T \). Then there exist uniform constants \( C_1, C_2 \) with respect to time such that
\[
C_1 t_{t_o}^{\frac{1}{m}} \leq w(x, t) \leq C_2 t_{t_o}^{\frac{1}{m}} \quad \text{on } K' \times \left[ \frac{t_o}{2}, t_o \right].
\]
We define \( \tilde{w}(x,t) = t^{\frac{m}{2-m}} w(x,t,t) \) in \( K' \times \left[ \frac{1}{2}, 1 \right] \) and we have
\[
C_1 \leq \tilde{w}(x,t) \leq C_2 \quad \text{on} \quad K' \times \left[ \frac{1}{2}, 1 \right].
\]

Then \( \tilde{w} \) satisfies uniformly parabolic equation, \( m^{1-1/m} F(D^2 \tilde{w}) = \tilde{w}_t \) in \( K' \times (1/2, 1) \) using scaling property. Thus we get
\[
\|\tilde{w}(\cdot, 1)\|_{L^\infty(K')} \leq C \|\tilde{w}\|_{L^\infty(K' \times [1/2, 1])} = C \|t^{\frac{m}{2-m}} w\|_{L^\infty(K' \times [t/2, 1])} \leq C \|f^m\|_{L^\infty(\Omega)}
\]
from the concavity of \( F \), which means that for any \( t_0 > 2T \),
\[
\|\tilde{w}(\cdot, t_0)\|_{L^\infty(K')} = \|\tilde{w}(\cdot, 1)\|_{L^\infty(K')} \leq C \|f^m\|_{L^\infty(\Omega)}.
\]
Therefore uniform convergence of \( z^m \) to \( f^m \) and uniform \( C^{2,\alpha}_k \) estimates will give that \( z^m \) converges to \( f^m \) in \( C^{2,\alpha}_k \) - norm. \( \square \)

5.2. Square-root concavity of the pressure. Let \( v = u^{m-1} \) be the pressure and let \( v = w^2 \). We are going to prove the concavity of \( w \) in spatial variables for \( m > 1 \).

The fact that \( w \) is a suitable function to perform geometrical investigations was demonstrated by Daskalopoulos, Hamilton and Lee at [DHL]. We remark that the following computation is also valid for the fast diffusion, \( m_{1,f} < m < 1 \).

First, let us approximate the equation: for \( 0 < \eta < 1 \),
\[
\begin{cases}
u_{\eta,t} = F(D^2 u_{\eta}^m) \quad \text{in} \quad \Omega \times (0, \infty) \\
u_{\eta} = \eta \quad \text{on} \quad \partial \Omega \times (0, \infty) \\
u_{\eta,0} \geq \eta \quad \text{in} \quad \Omega,
\end{cases}
\]
where we assume \( \eta + \frac{1}{2} u_{\eta} \leq u_{\eta,0} \leq \eta + 2 u_{\eta} \). Let \( g_{\eta} = u_{\eta}^m \). Then \( g_{\eta} \) satisfies the following equations:
\[
\begin{cases}
m g_{\eta}^{1-1/m} F(D^2 g_{\eta}) = g_{\eta,t} \quad \text{in} \quad \Omega \times (0, \infty) \\
g_{\eta} = \eta^m \quad \text{on} \quad \partial \Omega \times (0, \infty) \\
g_{\eta,0} > \eta^m \quad \text{on} \quad \Omega,
\end{cases}
\]
which is uniformly parabolic for a fixed \( \eta > 0 \) since \( g_{\eta} \geq \eta^m \) from the Comparison principle. We also assume that \( g_{\eta,0} \in C^\infty(\Omega) \) and \( \eta^m + \frac{1}{2} g_{\eta} \leq g_{\eta,0} = g_{\eta}(\cdot, 0) \leq 2 g_{\eta} + \eta^m \) in \( \Omega \). Then we have the following uniform estimate with respect to \( \eta \) so it suffices to show the concavity of \( w_{\eta} \).

Lemma 5.6. Let \( F \) satisfy (F1) and \( F(0) = 0 \) and let \( g_0 \in C_0(\overline{\Omega}) \). For each \( t > s > 0 \), there are uniform constants \( 0 < c_0(t), c_1, c_0(t,s) < \infty \) independent of \( \eta > 0 \) such that
\[
0 < c_0(t) < |\nabla_x g_{\eta} | < c_1 \quad \text{on} \quad \partial \Omega \times (0, t]
\]
and
\[
|\nabla_x g_{\eta} | < c_0(t, s) \quad \text{on} \quad \overline{\Omega} \times [s, t].
\]

Proof. We establish a subsolution and a supersolution of (5.17). Let \( \varphi^- \) be the positive eigen-function with respect to the eigenvalue \( \mu^- > 0 \) for the Pucci's operator \( \mathcal{M}^- \) from Theorem 3.1 that is, \( \varphi^- > 0 \) solves
\[
\begin{cases}
-\mathcal{M}^-(D^2 \varphi^-)(x)) = \mu^- \varphi^-(x) \quad \text{in} \quad \Omega, \\
\varphi(x)^- = 0 \quad \text{on} \quad \partial \Omega.
\end{cases}
\]
We may assume that $g_{\eta,0} \geq \varphi^- + \eta^m$ by multiplying a positive constant since $g_0 \in C_0(\bar{\Omega})$ and since $M^\ast$ is positively homogeneous of degree one. Define $K := \mu m(1 + ||\varphi^-||_\infty)^\gamma > 0$ for $\gamma := 1 - \frac{1}{m} > 0$ and $h(x, t) := \eta^m + \varphi^- e^{-Kt}$. Then we have

$$mh''F(D^2h) - h_t \geq mh''M^\ast(D^2h) - h_t$$

$$= mh''e^{-Kt} \left\{ - \frac{K\varphi^-}{m(\eta + \varphi^- e^{-Kt})^\gamma} \right\} \geq 0,$$

$h = \eta^m$ on $\partial\Omega$ and $h(\cdot, 0) \geq \varphi^- + \eta^m$. Thus the Comparison principle gives that $g_\eta \geq h = \eta^m + \varphi^- e^{-Kt}$, where $K$ depends on the initial data $g_0$. So it follows that

$$|\nabla g_\eta(\cdot, t)| \geq c_0 e^{-Kt} > 0 \quad \text{on} \quad \partial\Omega.$$

On the other hand, let $\varphi^+$ be the positive eigenfunction of

$$( Ev ) \quad \begin{cases} - M^\ast(D^2\varphi^+(x)) = \varphi^+ + h(x), & \text{in} \quad \Omega, \\ \varphi^+(x) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

From Theorem 3.1. Multiplying a positive constant, we assume that $g_{\eta,0} \leq \varphi^+ + \eta^m$ and $\varphi^+$ is the eigen-function with an eigen-value $\mu > 0$. If we define $h := \varphi^+ + \eta^m$, then $h$ satisfies $mh''F(D^2h) - h_t \leq mh''M^\ast(D^2h) - h_t \leq mh''(-\mu_0 \varphi^{1/m} < 0$ in $\Omega \times (0, \infty)$. From Comparison principle, we obtain

$$g_\eta \leq \varphi^+ + \eta^m,$$

which means that

$$|\nabla g_\eta| \leq C_0 = C_0(\varphi^+) \quad \text{on} \quad \partial\Omega \times (0, \infty).$$

uniformly in $\eta > 0$. A similar argument as in (2), (ii) at Proposition 5.1 gives

$$|\nabla g_\eta| \leq C_0 \quad \text{in} \quad \Omega \times (s, t).$$

\[ \square \]

**Lemma 5.7.** Suppose that $F$ satisfies (F1), (F2), (F3). Let $u$ be the solution of (5.10) with $F(D^2u_\eta) \leq 0$ in $\Omega$ and let $u_0$ be the solution of (5.12) with the initial data $u_{\eta,0}$ satisfying $F(D^2u_{\eta,0}) \leq 0$. Then $u$ and $u_\eta$ are nonincreasing in time.

**Proof.** According to Lemma 5.6, it suffices to show that $g_{\eta,t} \leq 0$ for any $\eta > 0$. Let us fix $\eta > 0$ and approximate the operator $F$ by smooth operators $\tilde{F}(\cdot) := F_\epsilon(\cdot) - F_\epsilon(0)$ in Lemma 4.8. Let $g_{\epsilon,\eta}$ be the solution of (5.18) with the same initial data $g_{\eta,0}$. For simplicity, we denote $g_{\epsilon,\eta}$ and $F_\epsilon$ by $g$ and $F$, where the equation (5.18) is uniformly parabolic in $\Omega \times (0, T)$ for a fixed $\eta$. Now define

$$h := g_t - \delta t - \delta$$

for small $\delta > 0$. Then $h$ is negative on the parabolic boundary. Indeed, at $t = 0$ we have $h = mg^{1-1/m} F(D^2g) - \delta \leq mg^{1-1/m} \sqrt{n} \Lambda \epsilon - \delta < 0$ for small $0 < \epsilon << \delta$ and $h < 0$ on $\partial\Omega \times (0, T)$. Assume that there is $t_0 \in (0, T]$ such that $h$ vanishes at some point $x_0 \in \Omega$ for the first time. Then at the maximum point $(x_0, t_0)$, we have

$$0 \geq mg^{1-1/m} F_{ij}(D^2g) h_{ij} - h_t = \left( 1 - \frac{1}{m} \right) \frac{\delta^2}{\eta^m} + \delta$$

$$\geq - \left( 1 - \frac{1}{m} \right) \frac{\partial^2(t_0 + 1)^2}{\eta^m} + \delta \geq - \left( 1 - \frac{1}{m} \right) \frac{\partial^2(T + 1)^2}{\eta^m} + \delta,$$
which is a contradiction if we select $\delta$ and $\varepsilon$ small enough. Thus for a given $\eta, T > 0$, there is $\delta(\eta, T), \varepsilon(\eta, T) > 0$ such that if $0 < \delta < \delta(\eta, T)$ and $0 < \varepsilon < \varepsilon(\eta, T)$, then
\[
g_{\varepsilon,\eta,t} < \delta t + \delta \quad \text{and} \quad g_{\varepsilon,\eta,t} \leq g_{\varepsilon,\eta} \quad \text{in} \quad \Omega \times (0, T).
\]
Letting $\varepsilon > 0$ and $\delta > 0$ go to 0, we have $g_{\eta,t} \leq 0$ in $\Omega \times (0, T]$ from the uniform Lipschitz estimates of $g_{\varepsilon,\eta}$ for a given $\eta > 0$. This completes the proof.

**Lemma 5.8.** Suppose that $F$ satisfies (F1), (F2), (F3) and that $\Omega$ is strictly convex. Let $u$ and $u_\eta$ be the solutions in Lemma 5.7. Then for each $T > 0$, there is $\eta(T) > 0$ such that for $0 < \eta < \eta(T)$, we have
\[
\delta \leq \frac{m - 1}{2m \varepsilon^2} (g_{\eta} g_{\eta,\eta} - \frac{m + 1}{2m} g_{\eta}^2) \leq \text{sign} (1 - m) \frac{c_\varepsilon}{\eta m} \eta \leq \frac{m - 1}{2m \varepsilon^2} (g_{\eta} g_{\eta,\eta} - \frac{m + 1}{2m} g_{\eta}^2) \leq \text{sign} (1 - m) \frac{c_\varepsilon}{\eta m}
\]
on $(x, t) \in \partial \Omega \times (0, T]$ for any direction $e_\alpha$, where $c_\varepsilon > 0$ is independent of $\eta > 0$.

**Proof.** (a) Let us fix $\eta > 0$. First, let us approximate the operator $F$ by $F_\varepsilon(\cdot) - F_\varepsilon(0)$ as in Lemma 4.8 and consider the approximated equation:
\[
\begin{cases}
u_{\varepsilon,t} = F_\varepsilon(D^2 u_\varepsilon) - F_\varepsilon(0) & \text{in} \quad \Omega \times (0, T), \\
u_\varepsilon = \eta & \text{on} \quad \partial \Omega \times (0, T), \\
u_{\eta,\varepsilon} > \eta & \text{on} \quad \Omega.
\end{cases}
\]
Let $g = u_\varepsilon$ and $g_{\varepsilon} = u_\varepsilon$ then $g_{\varepsilon}$ satisfies
\[
mc g_{\varepsilon}(F_\varepsilon(D^2 g_{\varepsilon}) - F_\varepsilon(0)) = g_{\varepsilon,t} \quad \text{in} \quad \Omega \times (0, T), \quad \gamma := 1 - 1/m > 0.
\]
We will denote $g_{\varepsilon}, F_\varepsilon$ by $g, F$, respectively, for the simplicity.

Let us fix a boundary point $(x_0, t_0) \in \partial \Omega \times (0, T)$. We denote $x_0$ by origin. Now we introduce the coordinate system such that $x_0 = 0$ and that the tangent plane is $x_n = 0$ at the origin. When $\tau = e_\tau, (\iota = 1, \ldots, n - 1)$ is a tangential direction at $x_0 = 0$, $g_\tau = 0$ and $g_\tau = g_{\varepsilon,\gamma,\tau}$ at the origin where $e_\tau$ is the outer normal vector to $\partial \Omega$ and $\gamma_\tau$ is the curvature of $\partial \Omega$ in the direction $\tau$.

(i.) According to boundary estimates at Lemma 5.6 and the strict convexity of $\partial \Omega$, we have $0 < c(T) < -g_{\tau,\tau} < C$ for any tangential vector $e_\tau$ and hence
\[
g_{\tau,\tau}(0, t) - \frac{m + 1}{2m} g_{\tau,\tau}^2(0, t) \leq -c_\varepsilon \eta^m \quad \text{on} \quad \partial \Omega \times (0, T)
\]
for some $c_\varepsilon(t) > 0$. We also have $|g_{\varepsilon,\eta}| \leq C$ on $\partial \Omega, \quad (1 \leq i, j \leq n - 1)$ for some constant $C$ depending on $\partial \Omega$ and $C_\varepsilon$ which is a uniform bound for gradients on the boundary).

(ii.) Near the origin, $\partial \Omega$ is represented by $x_n = \gamma(x') = \frac{1}{2} B_{ij} x_i x_j + O(|x'|^3)$. The estimate $c_\varepsilon < \gamma_{\tau,\tau} < C_\varepsilon$ says that the eigenvalues of $B_{ij}$ is in $[c_\varepsilon, C_\varepsilon]$. After a change of coordinate of $\mathbb{R}^{n-1}$, the boundary becomes $x_n = \gamma(x') = \frac{1}{2} |x'|^2 + O(|x'|^3)$ and the operator $F$ will be transformed to a new operator $\tilde{F}$ with new elliptic coefficients $\tilde{\lambda} = \tilde{\lambda}(\lambda, \Gamma_{\varepsilon,0}, C_\varepsilon)$ and $\tilde{\lambda} = \tilde{\lambda}(\lambda, \Gamma_{\varepsilon,0}, C_\varepsilon)$ that are uniformly bounded and positive. So $\partial \Omega$ is close to a unit ball with an error $O(|x'|^3)$ near the origin. For simplicity we are going to assume that $\Omega = B_1(e_\alpha)$. The general domain can be considered with a simple modification as $CNS$. 


(iii.) We claim that $|g_{e,e_i}(0, t_o)| \leq C$ for $1 \leq i \leq n - 1$. For positive constants $A, B$ and $D$, let us define in $\Omega \times (t_o/2, t_o)$

$$w_0(x, t) = \left\{ \partial_{\mathbf{n}} g + A \sum_{l=1}^{n-1} g_{l}^2 + D\mathbf{x}_n^2 \right\} T(t) = \left\{ (1 - x_n)g_t + x_l g_n + A \sum_{l=1}^{n-1} g_{l}^2 + D\mathbf{x}_n^2 \right\} T(t)$$

where $\partial_{\mathbf{n}} g := (1 - x_n)g_t + x_l g_n$ is a directional derivative and coincides with a tangential derivative on $\partial B_1$ and $T(t) := e^{M(t - 1/2)} - 1$. Let $v := C(A + D)x_n S(t)$ for $S(t) := K(t - t_o/2) \geq 0$. The constant $M, K > 0$ will be chosen so that $S(t) \geq T(t)$ in $(t_o/2, t_o)$.

Since $g = \eta^m$ on $\partial B_1$

$$\frac{1}{2} \|g_t\|^2 = \frac{1}{2}[(1 - x_n)g_t + x_l g_n + x_n g_l - x_l g_n]^2 \leq [(1 - x_n)g_l + x_l g_n]^2 + [x_n g_l - x_l g_n]^2$$

$$\leq [(1 - x_n)g_l + x_l g_n]^2 + C|x|^2$$

(we recall $|\nabla g| < C$ on $B_1 \times (t_o/2, t_o)$),

we see that for all $x \in \partial B_1$

$$-C(A + D)|x|^2 T(t) \leq w_0 \leq C(A + D)|x|^2 T(t).$$

Since $|x|^2 = 2x_n$ for any $x \in \partial B_1$, we obtain that

$$-C(A + D)x_n T(t) \leq w_0 \leq C(A + D)x_n T(t) \text{ for } x \in \partial_{\mathbf{p}}(B_1 \times (t_o/2, t_o))$$

and then we have

$$-v \leq w_0 \leq v \text{ for any } x \in \partial_{\mathbf{p}}(B_1 \times (t_o/2, t_o)).$$

Now, let us consider a linearized operator

$$H[u] = mg^\nu F_{ij}(D^2 g)D_{ij}u - u_t.$$  

If $H[w_+] \geq H[v]$ and $H[w_-] \leq H[-v]$ for some constants $A, D, M$ and $K$, then the comparison principle gives

$$-v \leq w_- \leq w_+ \leq v \text{ in } B_1 \times (t_o/2, t_o).$$

Therefore, we deduce that, for $1 \leq k \leq n - 1$,

$$|g_{kn}(0, t_o)| = |w_+(0, t_o)| \leq C(A + D)S(t_o).$$

So, it remains to show that $H[w_+] \geq H(v)$ and $H[w_-] \leq H(-v)$ if $A$ (uniform with respect to $\eta, \varepsilon$) and $D$ are chosen large enough. Using $mg^\nu F(D^2 g) - g_t = 0$ and the ellipticity of $F$, it follows that

$$|mg^\nu g_{mn}| \leq C \left( |mg^\nu|^2 \sum_{(i,j)\neq(n,n)} |g_{ij}|^2 + |g_t|^2 \right) \text{ in } \Omega,$$

$$mg^\nu \sum_{(i,j)\neq(n,n)} |g_{ij}|^2 \geq C mg^\nu g_{mn}^2 - \frac{C|g_t|^2}{g^\nu} \text{ in } \Omega.$$
Using the above inequalities, we have

\[ H[w_\ast] = -T(t) \frac{\partial g}{\partial t} \left( 1 - x_n \right) g_k + x_k g_n + 2A \sum_{i=1}^{n-1} g_i^2 \right) \\
+ T(t) 2mg^\gamma \left\{ - \sum_{i=1}^{n} F_m g_i + \sum_{i=1}^{n} F_i g_i + A \sum_{i=1}^{n-1} g_i^2 + DX_n^2 \right\} \\
- T'(t) \left\{ 1 - x_n \right\} g_k + x_k g_n + A \sum_{i=1}^{n-1} g_i^2 + DX_n^2 \right\} \\
\geq -T(t) \frac{\partial g}{\partial t} \left( 1 - x_n \right) g_k + x_k g_n + 2A \sum_{i=1}^{n-1} g_i^2 \right) + T(t)D \lambda \eta^m \\
+ 2T(t)mg^\gamma \left\{ -C \sum_{i=1}^{n} |g_i|^2 + \frac{\lambda A}{C} \sum_{i=1}^{n} g_i^2 + A \lambda / 2 \right\} - T(t)C \frac{|g|^2}{g^2} g^{2-\gamma} \\
- T'(t) \left\{ 1 - x_n \right\} g_k + x_k g_n + A \sum_{i=1}^{n-1} g_i^2 + DX_n^2 \right\} \\
\geq 2T(t)mg^\gamma \left\{ -C \sum_{i=1}^{n} |g_i|^2 + \frac{\lambda A}{C} \sum_{i=1}^{n} g_i^2 + A \lambda / 2 \right\} \\
+ CT(t) \left(D \eta^m - C\right) - CT'(t) \left(D - C\right). \]

We note that \( g \), \( \frac{\partial g}{\partial t} \) and \( |g| \) are uniformly bounded with respect to \( \eta \) and small \( \varepsilon (\eta) \) in \( B_1 \times (t_0/2, t_0) \) according to Aronson-Benilan inequality at Lemma 5.2 and Lemma 5.6. Thus if \( A > C/\sqrt{\lambda}, \ D > 2A \) and \( D \min(\eta^m, 1) \) is big enough, we get

\[ H[w_\ast] \geq C_1 D( C_2 \eta^m T - T'). \]

Setting \( M = C_2 \eta^m = C_2 \eta^m -1 \) and \( K = \frac{1}{\varepsilon} \left( C_2 \eta^m -1 \right) \), we have

\[ H[w_\ast] \geq 0 \geq H[v] = -C(A + D)x_n S'(t). \]

Similarly, we have \( H[w_\ast] \leq H[-v] \) in \( B_1 \times (t_0/2, t_0) \). Therefore, we have proved that

\[ |g_{in}(0, t_0)| = |(w_\ast)_n(0, t_0)| \leq \frac{1}{\eta^m} \left(C_2 \eta^{m-1} T - 1\right) \leq 2C_2 T, \]

for \( 1 \leq k \leq n-1 \) and for small \( 0 < \eta < \eta(T) \), where \( C_2 \) and \( \eta(T) \) are uniform with respect to \( \eta, \varepsilon \).

(iv) Lastly, since \( g_{in}^2(0, t_0) \leq C \sum_{i,j \neq (n,n)} |g_{ij}|^2 \) from (5.22), we have

\[ |g_{in}(0, t_0)| \leq C(T) \quad \text{and} \quad |D^2 g(0, t_0)| \leq C(T), \]

where \( C(T) \) is independent of \( \eta > 0 \) and \( \varepsilon > 0 \). Therefore, we have that for any unit vector \( e_\beta := \beta_1 e_1 + \beta_2 e_2, \)

\[ g g_{\beta \beta}(0, t_0) - \frac{m + 1}{2m} g_{\beta}(0, t_0) \leq g \left( c_0 \beta_1^2 - C(T) \left( \beta_1^2 + 2\beta_1 \beta_2 \right) \right) - \beta_2 \delta_0 \]

\[ \leq -\frac{c_0}{2} \eta^m \beta_1^2 + \left( \eta^m C(T) \left( \frac{1}{2c_0} + 1 \right) - \delta_0 \right) \beta_2^2 \leq -\frac{c_0}{2} \eta^m \]

\[ \text{for} \ \beta_1, \beta_2 \in \mathbb{R}. \]
for a small \( \eta > 0 \), using Young’s inequality and gradient estimate at Lemma 5.6.

(b) For the general operator instead of smooth operators, the result follows from the uniform \( C^{2,p} \) estimates since \( g_\varepsilon \) satisfies the uniformly parabolic equation with ellipticity constants related to \( \eta > 0 \), (5.21).

\[ \tag{5.21} \]

**Remark 5.9.**

(i) The boundary estimate (5.19) holds if \( |D^2 g_\varepsilon| \) is uniformly bounded in \( \overline{\Omega} \times (0,T] \) with respect to \( \varepsilon > 0 \). In Lemma 5.8, we have proved the estimate for the solutions with initial condition that \( F(D^2 u_\varepsilon) \leq 0 \).

(ii) To prove the estimate (5.19) up to the boundary, we need to prove a weighted \( C^{2,p} \) estimate of \( u_\varepsilon = g_\eta^{1/m} \) up to the boundary, which will be studied in the future work. When \( F(D^2 u) = \Delta u \), Schauder theory has been proved in [KL].

**Lemma 5.10.** Let \( F \) satisfy (F1), (F2) and (F3) and let \( \Omega \) be a strictly convex bounded domain. Let \( u \) be the solution of (5.10) with an initial data \( u^m_\varepsilon \in C^0(\Omega) \) and let \( u_\varepsilon \) be an approximated solution of (5.18). Assume that the boundary estimate of Lemma 5.8 holds for approximated solutions \( u_\varepsilon \).

If \( \sqrt{u_\varepsilon} = u_\varepsilon^{1/2} \) is concave, then the pressure \( v(x,t) = u^{m-1}(x,t) \) of (5.12) is square root-concave in the spatial variables, i.e., \( D^2 \sqrt{v(x,t)} \leq 0 \) in \( \Omega \times (0,\infty) \).

**Proof.** (i) First, we fix \( T > 0 \). We may assume that \( u_{\eta,0} \in C^\infty(\Omega) \) satisfies \( \eta^m + \frac{1}{2} u_{\eta}^m \leq u_{\eta,0}^m \leq \eta^m + 2 u_{\eta}^m \), \( F(D^2 u_{\eta,0}^m) \leq 0 \) and \( D^2 \sqrt{u_{\eta,0}^{m-1}} \leq 0 \) in \( \Omega \) and also assume that there is small \( \eta(T) \) such that for \( 0 < \eta < \eta(T) \), the boundary estimate (5.19) is true from the assumption.

If we show the square root - concavity of \( v_{\eta} \), the pressure of \( u_\varepsilon \) in \( \Omega \times (0,T] \), then the concavity for \( v = u^{m-1} \) follows from uniform convergence. Indeed, the uniform Lipschitz estimates of \( u_{\eta}^m(x,t) \) will give us uniform convergence of \( u_{\eta} \) to \( u \) in each compact subset of \( \Omega \times (0,T] \), from Lemma 5.6 and the limit \( u \) also satisfies

\[ u^{m-1}(x,t) + u^{m-1}(y,t) - 2 u^{m-1}(\frac{x + y}{2},t) \leq 0. \]

(ii) Now, let us fix \( T \) and \( \eta \) for \( 0 < \eta < \eta(T) \). It remains to show that \( u_\varepsilon^{1/2} \) is concave in \( \Omega \times (0,T] \) for small \( \eta > 0 \). Let us approximate \( F \) by a smooth \( F_\varepsilon \) from Lemma 4.8 and let \( u_{\varepsilon,\eta} \) be the solution of the approximated equation

\[ \begin{cases} u_t = F_\varepsilon(D^2 u^m) - F_\varepsilon(0) \quad \text{in} \quad \Omega \times (0,T), \\ u = \eta \quad \text{on} \quad \partial \Omega \times (0,T), \\ u(\cdot,0) = u_{\eta,0} > \eta \quad \text{on} \quad \Omega. \end{cases} \]

For simplicity, we denote \( u_{\varepsilon,\eta} u_{\varepsilon,\eta} \) by \( u, g = u^m \). The function \( g \) solves

\[ m g^{\gamma'}(F_\varepsilon(D^2 g) - F_\varepsilon(0)) = g \quad \text{in} \quad \Omega \times (0,T), \quad (\gamma = 1 - 1/m > 0), \]

which is uniformly parabolic for a given \( \eta > 0 \).

The geometric quantity \( w := \sqrt{v} = u^{1/2} \) satisfies

\[ w_t = \frac{m - 1}{2} w^{1/2} F_\varepsilon \left( \frac{2m}{m - 1} w^{1/2} \left( w^2 D^2 w + \frac{m + 1}{m - 1} w Dw Dw^{1/2} \right) \right) \quad \text{by} \quad \frac{m - 1}{2} w^{1/2} F_\varepsilon(0). \]
After the change of the time \( t \mapsto mt \), the equation will be simplified into

\[
(5.25) \quad w_{t} = \frac{m - 1}{2m} w^{\frac{m}{m - 1}} F\left( \frac{2m}{m - 1} w^{\frac{m}{m - 1}} \left( w^{2} D^2 w + rwDwDw' \right) \right) - \frac{m - 1}{2m} w^{\frac{m}{m - 1}} F_{t}(0)
\]

with \( r = \frac{m}{m - 1} \). By taking differentiation, we have

\[
\begin{align*}
w_{\alpha\beta t} &= \frac{m - 1}{2m} w^{\frac{m}{m - 1}} F_{\alpha\beta t} \cdot \left( \frac{2m}{m - 1} w^{\frac{m}{m - 1}} \left( w^{2} D_{\alpha\beta} w + \frac{m + 1}{m - 1} wD_{\alpha} wD_{\beta} w \right) \right) \\
&\quad + F_{ij} \cdot \left( 2w_{\alpha} w_{\beta} D_{ij} w + 2w_{\alpha} w_{\beta} D_{ij} w + 2w_{\alpha} D_{i} w_{\beta} + 2w_{\alpha} D_{i} w_{\beta} + w^{2} D_{ij} w_{\alpha\beta} \right) \\
&\quad + 2rwD_{\alpha} D_{ij} w_{\beta} + 2rwD_{\alpha} D_{ij} w_{\beta} \\
&\quad + \frac{m - 1}{2m} \frac{3}{m - 1} w^{3} - 2 w_{\alpha} w_{\beta} F_{ij} \left( w^{2} D_{ij} w + rwDwDw' \right) \\
&\quad - \frac{m - 1}{2m} \frac{3}{m - 1} \frac{1}{2} w^{2} w_{\alpha} w_{\beta} F_{ij} \left( \frac{2m}{m - 1} w^{2} D^{2} w + rwDwDw' \right) \\
&\quad + \frac{m - 1}{2m} \frac{3}{m - 1} \frac{1}{2} w^{2} w_{\alpha} w_{\beta} F_{ij} \left( w^{2} D_{ij} w + rwDwDw' \right) \\
&\quad - \frac{m - 1}{2m} \frac{3}{m - 1} \frac{1}{2} w^{2} w_{\alpha} w_{\beta} F_{ij} \left( \frac{2m}{m - 1} w^{2} D^{2} w + rwDwDw' \right) F_{t}(0) ,
\end{align*}
\]

for \( F_{ij} = F_{ij} \left( \frac{2m}{m - 1} w^{\frac{m}{m - 1}} \left( w^{2} D^2 w + rwDwDw' \right) \right) \).

In order to show the concavity of \( w \), consider

\[
\sup_{y \in \Omega} \sup_{|\xi| = 1} w_{\beta\beta}(y, t) + \psi(t),
\]

where \( \psi \in S^{n-1} \) and a negative function \( \psi(t) \) with \( \psi(0) < 0 \) will be chosen later. Let us assume that

\[
\sup_{y \in \Omega} \sup_{|\xi| = 1} w_{\beta\beta}(y, t) + \psi(t) = 0 \quad \text{at} \quad t = t_{0},
\]

for the first time. From the assumption that the pressure is initially square-root concave, the quantity \( \sup_{y \in \Omega} \sup_{|\xi| = 1} \tilde{g}_{\beta\beta}(y, t) + \psi(t) \) is negative at \( t = 0 \).

Now, we assume that the supremum

\[
\sup_{y \in \Omega} \sup_{|\xi| = 1} w_{\beta\beta}(x, t_{0}) = w_{\beta\beta}(x_{0}, t_{0}) = -\psi(t_{0})(> 0)
\]

is achieved at \( (x_{0}, t_{0}) \in \overline{\Omega} \times (0, T) \) with a unit vector \( e_{\beta}^{\star} \) and assume that \( x_{0} = 0 \) without losing of generality. Then, the assumption on the boundary that \( w_{\beta\beta} \leq 0 \) yields that \( (0, t_{0}) \) should be an interior point. We introduce an orthonormal coordinates in which \( e_{\beta}^{\star} \) is taken as one of the coordinate axes and we assume that

\[
w_{\beta\beta}(0, t_{0}) = 0 \quad \text{if} \quad \beta \neq \alpha.
\]

In order to create extra terms, we perturb second derivatives of \( w \) and we use the function

\[
Z(x, t) = w_{\alpha\beta}(x, t) \xi^{\alpha}(x) \xi^{\beta}(x)
\]
where $\mathcal{E}(x) = \delta_0, + c_0 \alpha + \frac{1}{2} c_0 \gamma x^2$. We are going to choose $c_0$ so that $-4\omega^2 c_0 + 4\omega \omega_0 = 0$ at the maximum point $(x(t), t(t))$ and then the function $Z$ will help the third derivatives cancel out, which appear in the porous medium equation after differentiations. We note that at the maximum point $(0, t_0)$, we have

$$w_{\alpha \beta} = 0 \text{ if } \beta \neq \alpha, \quad D^2 \xi Z = 0, \quad \text{and } \nabla \xi Z = 0$$

since

$$D^2 w(x(t), t) < -\psi(t) I \quad \text{for } 0 < t < t_0, \quad Z(x(t), t) = \xi^T D^2 w \xi, \quad Z(0, t_0) = w_{\alpha \alpha}(0, t_0) = -\psi(t_0).$$

A simple computation gives us, at $(x(t), t) = (0, t_0)$,

$$Z_i = w_{\alpha \beta} \xi^\alpha \xi^\beta + 2w_{\alpha \beta} c_\alpha \xi^\alpha \xi^\beta$$

$$Z_{ij} = w_{\alpha \beta \gamma} \xi^\alpha \xi^\beta + 4w_{\alpha \beta \gamma} c_\alpha \xi^\alpha \xi^\beta + 2w_{\alpha \beta \gamma} c_\alpha \xi^\alpha \xi^\beta + 2w_{\alpha \beta \gamma} c_\alpha \xi^\alpha \xi^\beta.$$

and hence we have at $(0, t_0)$,

$$Z_t = w_{\alpha \beta} \eta^\alpha \eta^\beta$$

$$\leq w^2 F_{ij} \cdot Z_{ij} + F_{i j} w_{\alpha \beta} \eta^\alpha \eta^\beta (-4w^2 c_\alpha + 4\omega \omega_\alpha) + 2F_{i j} w_{i j} (-w^2 c_\alpha + w^2)$$

$$+(2wF_{i j} D_{j i} + r F_{i j} D_{i j} w) w_{\alpha \beta} + 4r F_{i j} D_{j i} w w_{\alpha \beta} - 2w^2 F_{\alpha \beta} c_{\alpha \beta} w_{\alpha \beta}$$

$$+ 2w^2 F_{\alpha \beta} Z_{\alpha \beta} + 2r F_{\alpha \beta} Z_{\alpha \beta} Z + 2w^2 F_{\alpha \beta} c_{\alpha \beta} Z$$

$$\leq w^2 F_{\alpha \beta} Z_{\alpha \beta} + 2w^2 F_{\alpha \beta} c_{\alpha \beta} Z + 2w^2 F_{\alpha \beta} c_{\alpha \beta} Z$$

$$\leq 2w F_{\alpha \beta} Z_{\alpha \beta} + 2w^2 F_{\alpha \beta} c_{\alpha \beta} Z + 2w^2 F_{\alpha \beta} c_{\alpha \beta} Z$$

Now let us define

$$Y(x, t) := Z(x, t) + \psi(t).$$

We notice that $Y(x, 0) < 0$ for any $x \in \Omega$ since we know

$$\sup_{x \in \Omega} \sup_{|s| = 1} w_{\alpha \beta}(x, 0) < 0 \quad \text{and } \quad \psi(0) < 0$$

and $\partial_t Y(0, t_0) \geq 0$ since we have

$$D^2 w(0, t) < -\psi(t) I \quad \text{for } 0 < t < t_0, \quad Z(0, t_0) = w_{\alpha \alpha}(0, t_0) = -\psi(t_0).$$

Thus we obtain at the maximum point $(0, t_0)$,

$$0 \leq \partial_t Y(0, t) = Z_t + \psi_t \leq \psi_t + K(\psi^2 - \psi + \epsilon),$$

where $K = c(n, m, \Lambda) \sup_{\Omega \times [0, t]} \left( w + w^{-\frac{m}{2m}} \right)$, $(1 + w^{-\frac{m}{2m}})|\nabla w|^2 + w$. We note that $K$ is independent of $\epsilon$ (and $\delta$) since $\nabla \eta \xi \in \mathbb{R}$ is uniformly bounded and $\eta > 0$ is given.

If we set $\psi(t) := -\epsilon - e^{-\frac{1}{2}} \epsilon^{\frac{1}{2}} \tan(K \sqrt{\epsilon} t)$, a simple calculation yields

$$\psi_t + K(\psi^2 - \psi + \epsilon) < 0.$$
for $0 < \varepsilon << \delta$ and $K \sqrt{\delta t} < \frac{\pi}{2}$, which implies a contradiction if $t_0 < \frac{\pi}{2K \sqrt{\delta}}$. Therefore, we deduce that
\[
\sup_{y \in \Omega} \sup_{|\eta| = 1} w_{\beta}(y, t) < -\psi(t) \quad \text{for} \quad 0 < t < \frac{\pi}{2K \sqrt{\delta}} \quad \text{and for small} \quad \varepsilon << \delta,
\]
and hence
\[
\partial_{\beta} w = \frac{w_{\beta}}{\partial_{\beta} u_{\eta, \beta}} < \varepsilon + \varepsilon^{-1/\kappa} \frac{1}{\sqrt{\delta}} \quad \text{for any} \quad \varepsilon > 0.
\]

Therefore, it follows that
\[
\sup_{y \in \Omega} \sup_{|\eta| = 1} w_{\beta}(y, t) < -\psi(t) \quad \text{for} \quad 0 < t < \frac{\pi}{8K \sqrt{\delta}} \quad \text{and for small} \quad \varepsilon << \delta.
\]

Letting $\delta \to 0$, we conclude that
\[
\partial_{\beta} u_{\eta, \beta} \leq 0 \quad \text{in} \quad \Omega \times (0, T) \quad \text{for any} \quad \varepsilon > 0,
\]
that implies
\[
\sup_{(x, t) \in \Omega} u_{\eta, \beta}(x, t) \leq \sup_{t \in (0, T)} u_{\eta, \beta}(x, t) \leq \frac{\pi}{2}\sqrt{\delta t}.
\]

\[\square\]

**Corollary 5.11.** Let us assume that $F$ satisfies (F1), (F2) and (F3) and $\Omega$ is convex. If $\sqrt{\phi}$ is concave, so is the viscosity solution $v$ of (5.10) with an initial data $\phi_1 \in \mathcal{F}(\Omega)$. First, we may assume that $\phi$ is concave in $x$ variables for large $t$ is concave, so is the viscosity solution $v$ of (5.10) with an initial data $\phi_1 \in \mathcal{F}(\Omega)$. Then, the pressure $v$ of solutions to (3.3),(5.10) which follow from Lemma 5.8 and 5.10 imply
\[
\sup_{(x, t) \in \Omega} \sup_{|\eta| = 1} w_{\beta}(y, t) < -\psi(t) \quad \text{for} \quad 0 < t < \frac{\pi}{2K \sqrt{\delta}} \quad \text{and for small} \quad \varepsilon << \delta,
\]
and hence
\[
\partial_{\beta} w = \frac{w_{\beta}}{\partial_{\beta} u_{\eta, \beta}} < \varepsilon + \varepsilon^{-1/\kappa} \frac{1}{\sqrt{\delta}} \quad \text{for any} \quad \varepsilon > 0.
\]

Now we state the strict concavity of solutions to (3.3),(5.10) which follow from Lemma 5.8 and 5.10 imply
\[
D^2 \sqrt{\phi} \leq \frac{\pi}{2}\sqrt{\delta t} \quad \text{in} \quad \Omega \times (0, T).
\]

The uniform convergence at Proposition 5.3 Corollary 5.5 that is,
\[
tu^{m-1}(x, t) \to f(x) \quad \text{uniformly in} \quad \Omega \quad \text{as} \quad t \to +\infty,
\]
will preserve the concavity of $f^{m-1}$. Therefore, it follows that
\[
f^{m-1}(x) + f^{m-1}(y) - \frac{1}{2} f^{m-1}(\frac{x + y}{2}) \leq 0 \quad \text{for} \quad x, y \in \Omega.
\]

Now we state the strict concavity of solutions to (3.3),(5.10) which follow from Lemma 5.8 and 5.10 imply
\[
D^2 \sqrt{\phi} \leq \frac{\pi}{2}\sqrt{\delta t} \quad \text{in} \quad \Omega \times (0, T).
\]

**Lemma 5.13 (Strict Square-root Concavity).** Suppose that $F$ satisfies (F1), (F2) and (F3) and is differentiable. If $\Omega$ is smooth and strictly convex, $f^{m-1}$ is strictly concave: there exists a constant $c_1 > 0$ such that
\[
D^2 \sqrt{\phi} \leq -c_1 I.
\]

**Theorem 5.14 (Eventual square root-concavity).** We assume the same hypothesis as Lemma 5.13. Let $u_0 \in C^\infty(\Omega)$. Then, the pressure $v(x, t) = u^{m-1}(x, t)$ is strictly square root -concave in $x$ variables for large $t > 0$. More precisely, for any $\varepsilon > 0$, there is $t_0 = t_0(u_0, \varepsilon)$ such that
\[
D^2 \sqrt{v} \leq -(c_1 - \varepsilon) I.
\]
for \( t \geq t_0 \) and \( x \in \Omega = \{ x \in \Omega | d(x, \partial \Omega) > \epsilon \} \), where \( c_1 > 0 \) is the constant in Lemma 5.13.

**Remark 5.15.**

(i) \( F(\cdot) \) in Theorem 5.14 is basically Laplacian after a simple transformation if \( F(\cdot) \) is differentiable and satisfies (F1), (F2).

(ii) Condition (F2) is required to have the convergence of \( tv(x, t) \) to \( f^{m-1}(x) \) as \( t \to +\infty \) and the concavity of \( F \) is required when we consider a concavity of solutions.

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