Quantum footprints of Liouville integrable systems

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Abstract

We discuss the problem of recovering geometric objects from the spectrum of a quantum integrable system. In the case of one degree of freedom, precise results exist. In the general case, we report on the recent notion of good labellings of asymptotic lattices.

Keywords: Liouville integrable system, quantization, inverse spectral problem, Morse Hamiltonian, semiclassical analysis, asymptotic lattice, good labelling

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1 Inverse problems

In this paper, we use for convenience the vocabulary of classical and quantum mechanics, but one should keep in mind that inverse problems can be stated in a more universal way. Our general question is: “What footprints does a classical system leave on its quantum counterparts, and are they sufficient to recognize the classical system that produced these footprints?”

Imagine you’re walking in a snowy landscape, trying to take a good photograph of a wild animal. A silhouette appears in the distance, you have no idea what beast it can be; you seize your camera, look at the small screen..., and the ghost has just disappeared. Just as if it knew you wanted to capture it. The same goes for quantum particles, they want to delocalize when they are observed. Yet we know they live there. On the other hand, the footprints on the snow, they are real, and stable. You can take your time and study them, until, maybe, by clever induction, you find out what kind of animal was standing there.

What we have just described is an inverse problem: from the observation of a signal emitted by some device, can we recognize the device that has emitted the signal? If we hear the sound produced by various instruments playing the same note C, can we tell the instrument without looking? This question

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can easily be turned into a mathematical problem, up to some simplifications, and was, for instance, popularized by the “Can one hear the shape of a drum” paper by Kac [18]. The “sound” is the superposition of all possible “frequencies” of a drum, i.e. the spectrum of the Laplace operator on a Euclidean domain \( \Omega \subset \mathbb{R}^n \), and the question is whether the shape of \( \Omega \) (which is the equivalence class of \( \Omega \) under the action of the orthogonal group \( O(n) \) and translations) can be determined by the spectrum of the Dirichlet Laplacian. Natural variants of this question exist. One can consider Riemannian metrics on a compact manifold \( X \): can one recover the metric from the (discrete) spectrum of the corresponding Laplace-Beltrami operator? Or, back to quantum mechanics, we may consider a Schrödinger operator \(-\hbar^2 \Delta + V\) on \( \mathbb{R}^n \), and ask whether its spectrum determines the electric potential \( V \). Both questions gave rise to an important literature, see for instance the references cited in [35, 7] and [13, 14], and have applications in non-quantum situations, for instance in seismology, see [6, 9].

In this paper we consider the so-called “symplectic case”. Given an arbitrary “semiclassical operator” \( \hat{H}_\hbar \) depending on a small parameter \( \hbar > 0 \) (see the next section) and its symbol \( H \), which is a smooth function on a symplectic manifold \( (M,\omega) \), can you recover the triple \( (M,\omega,H) \) from the \( \hbar \)-family of spectra of \( \hat{H}_\hbar \), where \( \hbar \) varies in a set accumulating at zero? This natural question can be found for instance in [17]. Actually, Kac’s problem, in disguise, is such a symplectic inverse problem; in this case indeed, the semiclassical Laplacian is simply \( \hbar^2 \Delta \), and the Hamiltonian \( H \) is the metric on the cotangent bundle \( M = \mathbb{T}^* X \) induced by \( g \).

2 Quantization and Semiclassical analysis

Semiclassical analysis is a general framework for obtaining a “geometric limit” from PDEs with highly oscillating solutions; the name “semiclassical” comes from the model situation where classical mechanics can be seen as a (singular) limit of quantum mechanics, as Planck’s constant \( \hbar \) tends to zero. Our conventions in this text are the following:

By Classical observables, or classical Hamiltonians, we mean smooth functions \( H \in C^\infty(M) \) on a symplectic manifold \( (M,\omega) \), the classical phase space. For instance, \( M = \mathbb{R}^{2n} \) with the canonical symplectic structure \( \omega = d\xi \wedge dx \). Each function \( H \) defines an evolution equation, the flow of the associated Hamiltonian vector field \( X_H \) defined by \( \iota_{X_H} \omega = -dH \).

By Quantum observables, or quantum Hamiltonians, we mean a self-adjoint operator \( \hat{H} \) on a Hilbert space \( \mathcal{H} \), and the Hilbert space itself must be the quantization of a classical phase space \( M \). Each quantum Hamiltonian gives rise to the evolution governed by the Schrödinger equation

\[
\frac{\hbar}{i} \partial_t \psi = \hat{H} \psi,
\]
so, formally, $\psi(t) = e^{i\hat{H}t}\psi(0)$. Stationary states are solutions of the form $\psi(t) = e^{i\lambda t}\hat{H}\ u$, where $u \in \mathcal{H}$ and $\hat{H}u = \lambda u$. It is a fascinating subject to understand the relationships between the classical Hamiltonian flow and the quantum Schrödinger evolution.

Two rigorous quantization schemes allow us to realise the above picture: when $M$ is a cotangent bundle, $M = T^*X$, one can use pseudo-differential quantization, see for instance [22]. When $M$ is a prequantizable Kähler manifold, one can use Berezin-Toeplitz quantization, see for instance [20] (Berezin-Toeplitz quantization was later extended to general symplectic manifolds, see [3]). Berezin-Toeplitz and pseudo-differential quantization are similar in many respects, and both benefit from the power and flexibility of microlocal analysis à la Maslov, Hörmander, etc..

3 1D Hamiltonians : the Morse case

Let $(M, \omega)$ be a 2-dimensional symplectic manifold. Let $H : M \to \mathbb{R}$ a proper Morse function. Following the usual Morse approach we will be interested in the (singular) foliation of $M$ by level sets of $H$. An important object, the Reeb graph $\mathcal{G}$, is the set of leaves, i.e. connected components of level sets of $H$; in a neighbourhood of a regular level set, $\mathcal{G}$ is a smooth, one-dimensional manifold. The smooth parts are the edges of the graph. Each critical point of $H$ contributes to a graph vertex, its degree is one for elliptic singularities (the vertex is then a leaf), and three for hyperbolic singularities.

The Reeb graph turns out to be essential in the description of the spectrum of a 1D quantum Hamiltonian. Let $\hat{H} := \text{Op}_h(H_h) = H + hH_1 + h^2H_2 + \cdots$ be the quantization of a symbol $H_h := H + hH_1 + h^2H_2 + \cdots$ on $M$. Let $I \subset \mathbb{R}$ be a closed, bounded interval. Since $H$ is proper, $H^{-1}(I)$ is compact. In the case of pseudo-differential quantization, we assume that $H_h$ belongs to a symbol class and is elliptic at infinity, see for instance [22]. Then the spectrum of $\hat{H}$ in any compact subset of $\text{int} \ I$ is discrete. In this case, the inverse spectral theory is well understood, and summarized by the following statement.

**Theorem 3.1 ([34, 19])** Let $M_I := H^{-1}(\text{int} \ I)$. Suppose that $H_{|M_I}$ is a simple Morse function. Assume that the graphs of the periods of all trajectories of the Hamiltonian flow defined by $H_{|M_I}$, as functions of the energy, intersect generically.

Then the knowledge of the spectrum $\sigma(\hat{H}) \cap I + O(h^2)$ determines $(M_I, \omega, H)$. The proof of this theorem, like many of its kind, is divided in two steps. The first one is to recover the Reeb graph $\mathcal{G}$ of $(M_I, H_I)$ from the spectrum. The second step is to prove that $\mathcal{G}$, decorated with appropriate numerical invariants that we can also recover from the spectrum, completely determines the classical system $(M_I, \omega, H)$. The last step was proven by Dufour-Toulet-Molino [11]. The first step was established in the pseudo-differential case.
in [34], and in the case of Berezin-Toeplitz operators by Le Floch [19]. It involved microlocal analysis in the (time/energy) phase space to be able to separate the various connected components of $G$ contributing to the same region of the spectrum.

More recently, a new interpretation of this result has been proposed by several mathematicians, in particular Leonid Polterovich and the author. Suppose you add a generic non-selfadjoint perturbation to the quantum operator $\hat{H}$. Then, the connected components of $G$, instead of leading to overlapping parts of the spectrum — and hence potentially difficult to tell apart — should instead give rise to different complex branches of the spectrum of the non-selfadjoint operator. Thanks to the recent result by Rouby [30] explaining the non-selfadjoint version of Bohr-Sommerfeld quantization conditions, we believe that this conjectural interpretation should produce new rigorous results.

**Theorem 3.2 (Rouby [30])**  Let $P_\epsilon$ be an analytic pseudodifferential operator on $\mathbb{R}$ or $S^1$ of the form $P_\epsilon = \hat{H} + i\epsilon Q$, where $\hat{H}$ is selfadjoint with discrete spectrum, and $Q$ is $\hat{H}$-bounded.

Then, near any regular value of the symbol $H$, with connected fibers, the spectrum of $P_\epsilon$ is given by $\{g(\hbar m;\epsilon); m \in \mathbb{Z}\}$, where $g : \mathbb{C} \to \mathbb{C}$ is holomorphic and

$$g \sim g_0 + \hbar g_1 + \hbar^2 g_2 + \cdots$$

Moreover, $g_0$ is the inverse of the action variable, and

$$g_0 \sim H + i\epsilon(q) + \mathcal{O}(\epsilon^2).$$

Rouby’s theorem is technically quite involved, because one has to take advantage of analyticity to fight non-selfadjoint instability (pseudo-spectral effects), and usual $C^\infty$ microlocal analysis is not strong enough for this. No analogue of this result for Berezin-Toeplitz quantization exists yet. However, very recent advances on the analyticity of the Bergman projection give some hope, see [31, 10, 4].

### 4 Integrable systems

In view of Rouby’s theorem, one can notice that a particular case where analyticity is not required occurs when $P_\epsilon$ is normal, i.e. the non-selfadjoint perturbation $Q$ commutes with the selfadjoint part $\hat{H}$. More generally, a number of results exist in the presence of a completely integrable quantum system, by which we mean the data of $n$ pairwise commuting selfadjoint operators $P_1, \ldots, P_n$, when the phase space $M$ is $2n$-dimensional. In fact, even
for operators that are not quantum integrable but still have a completely integrable classical limit, quite precise results can be obtained, for both direct and inverse problems; see [16, 15], and references therein.

This notion of quantum integrability parallels the usual Liouville integrability of classical Hamiltonians, where we dispose of $n$ independent Poisson-commuting functions $f_1, \ldots, f_n$ on $M$. Note that, near a regular level set of the joint map $F := (f_1, \ldots, f_n) : M \to \mathbb{R}^n$, one has action-angle coordinates, due to the celebrated Liouville-Mineur-Arnold theorem, but Liouville integrability is more general: it allows for singularities where the action-angle theorem cannot apply.

The natural multi-dimensional generalization of the Reeb graph is the leaf space of the “moment map” $F$, which is equipped with a natural integral affine structure (see for instance [32]). The quantum analogue of this singular integral affine manifold is the joint spectrum of the commuting operators $P_1, \ldots, P_n$. Hence, we are naturally lead to the following inverse problem: given an $\hbar$-family of joint spectra, can one recover the triple $(M, \omega, F)$?

A first approach to this question is to restrict oneself to Hamiltonian systems with many compact symmetries; namely the toric and semitoric cases. See [29] for a description of a general program of study, and conjectures.

## 5 Asymptotic lattices

Having in mind the general inverse problem for quantum integrable systems, another angle of attack is to consider the regular part of the integral affine structure, and exploit the lattice structure of the joint spectrum, which was already established by Colin de Verdière [5]. This leads to the notion of asymptotic lattices, whose systematic study was initiated recently in [8]. Although the initial goal of that paper was to recover from the quantum spectrum a specific classical invariant, the rotation number, we believe that the general setup should help understanding all invariants related to the integral affine structure. In particular, we hope that it will allow to finally obtain a complete result on the inverse theory of semitoric systems.

Let $B \subset \mathbb{R}^n$ be a simply connected bounded open set. Let $\mathcal{L}_h \subset B$ be a discrete subset of $B$ depending on the small parameter $h \in \mathcal{I}$, where $\mathcal{I} \subset \mathbb{R}_+^*$ is a set of positive real numbers admitting 0 as an accumulation point. Here is a slightly imprecise definition of asymptotic lattices (we don’t delve into multiplicity issues and the details of the $(\hbar^\infty)$ topology).

**Definition 5.1 (Asymptotic lattice [33, 8])** We say that $(\mathcal{L}_h, \mathcal{I}, B)$ is an asymptotic lattice if

$$\mathcal{L}_h = G_h (h\mathbb{Z}^n \cap U) + \mathcal{O}(\hbar^\infty)$$

with

$$G_h = G_0 + \hbar G_1 + \hbar^2 G_2 + \cdots$$
in the $C^\infty(U)$ topology, where $G_0 : U \to \mathbb{R}^n$ is a diffeomorphism on its image.

The definition is motivated by the following older result.

**Theorem 5.2 ([5, 1, 2])** Let $P := (P_1, \ldots, P_n)$ be a Quantum integrable system. Let $c \in \mathbb{R}^n$ be a regular value of the classical moment map $F$ with connected fiber. Then the joint spectrum of $P$ near $c$ is an asymptotic lattice.

### 6 Good labellings

In order to recover the integral affine structure from the spectrum, one needs to recover the map $G_0$ in the previous theorem. In order to do this, we claim that it is enough to find a “good labelling” of all joint eigenvalues. By this we mean, to assign to each joint eigenvalue $\lambda$ a $n$-uple of integers $(k_1, \ldots, k_n)$ such that

$\lambda = G_\hbar(\hbar k_1, \ldots, \hbar k_n) + \mathcal{O}(\hbar^\infty)$.

In [8], we investigated the case of two degrees of freedom, $n = 2$. To our surprise, this question turned out to be more intricate (and more interesting) than what we initially thought.

On the other hand, the process of finding a good labelling is elementary, and can be described algorithmically, which is important for the following reason. The informal question “can one hear the shape of a drum” has two possible interpretations. The minimal one is to prove injectivity of the map sending a classical system to its quantum spectrum. In this case, the classical system is determined by the quantum spectrum in a weak sense: two different classical systems cannot give rise to the same quantum spectrum. A stronger result would be to obtain the classical system that produced the spectrum in a constructive way. Writing an algorithm contributes to the latter.

The algorithm will be constructed in two steps. In the first one, the value $\hbar$ is fixed, and the algorithm returns a candidate labelling $\lambda \mapsto (k_1, k_2)$. However, this candidate does not have the required continuity property in the variable $\hbar$. Hence we perform a second step where we consider now a full sequence of values $\hbar_i \to 0$, and we correct the discontinuity by an inductive algorithm in the variable $i \in \mathbb{N}$. We don’t know whether a direct approach, in one step, would be possible. When constructing a good labelling, another difficulty comes from the choice of a valid “origin” for the lattice. For this purpose, the set of values of $\hbar$ must be “dense enough” when accumulating at zero. Values of the form $\hbar = \frac{1}{k}$ for $k \in \mathbb{N} \setminus \{0\}$ do not fulfill this requirement, which is an issue for Berezin-Toeplitz quantization. However, in many applications, the choice of the lattice origin is irrelevant. Introducing the notion of “linear labelling” as a good labelling “modulo its origin”, we have the following result.

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Theorem 6.1 There exists an explicit algorithm such that the following holds. Let \((\mathcal{L}_h, \mathcal{I}, B)\) be an asymptotic lattice, where \(B \subset \mathbb{R}^2\). Let \(h_j \in \mathcal{I}, \ j \geq 1\), be a decreasing sequence tending to 0. Then, from this data, the algorithm produces a linear labelling of the asymptotic lattice \((\mathcal{L}_h, \mathcal{I}', B)\), where \(\mathcal{I}' = \{h_j, j \in \mathbb{N}^*\}\).

Below is the complete description of the first step. If \((k_1, k_2)\) is a label for a point \(\lambda \in \mathcal{L}_h\), we shall denote this point \(\lambda = \lambda_{k_1,k_2}\). The complete algorithm is as follows [8], and pictured in Figure 1.

1. Choose an open subset \(B_0 \Subset B\), and fix \(c \in B_0\).
2. Choose a closest point to \(c\). Label it as \((0, 0)\).
3. Choose a closest point to \(\lambda_{0,0}\) (in the set \(\mathcal{L}_h \setminus \{\lambda_{0,0}\}\)). Label it as \((1, 0)\).
4. Choose a closest point to \(2\lambda_{1,0} - \lambda_{0,0} = \lambda_{1,0} + (\lambda_{1,0} - \lambda_{0,0})\) and label it as \((2, 0)\).

Continuing in this fashion (if \(k_1-1,0\) is chosen, take \(\lambda_{k_1,0}\) to be a closest point to \(\lambda_{k_1-1,0} + (\lambda_{k_1-1,0} - \lambda_{k_1-2,0})\), label points \(\lambda_{k_1,0}\), \(k_1 > 0\), until the next point lies outside of \(B_0\).

Label \(\lambda_{k_1,0}\) for negative \(k_1\) in the same way, starting from the closest point to \(2\lambda_{0,0} - \lambda_{1,0}\).
5. Choose a closest point to \(\lambda_{0,0}\) not already labeled and label it as \((0, 1)\).
6. Label a closest point to \(\lambda_{0,1} + (\lambda_{1,0} - \lambda_{0,0})\) as \((1, 1)\).
7. Use the points $\lambda_{0,1}, \lambda_{1,1}$ to repeat the process in step 4, labelling as many points $\lambda_{k,1}$ as possible (if $\lambda_{k-1,1}$ is chosen, take $\lambda_{k,1}$ to be a closest point to $\lambda_{k-1,1} + (\lambda_{k-1,1} - \lambda_{k-2,1})$).

8. Label a closest point to $2\lambda_{0,1} - \lambda_{0,0}$ as $\lambda_{0,2}$. Repeat steps 6-7 to label all points $\lambda_{k,2}$.

9. Continuing as above, label all points $\lambda_{k,1,k_2}; k_2 > 0$ which lie in the given neighborhood.

10. Label a closest point to $2\lambda_{0,0} - \lambda_{0,1}$ as $(0, -1)$.

11. Repeat steps 6, 7, 8, 9 with negative $k_2$ indices.

12. Finally, if the determinant of the vectors $(\lambda_{1,0} - \lambda_{0,0}, \lambda_{0,1} - \lambda_{0,0})$ is negative, switch the labelling $\lambda_{k,1,k_2} \mapsto \lambda_{-k,1,k_2}$ (in order to make it oriented).

This algorithm should be useful in several inverse spectral problems. For instance, in [8], Theorem 6.1 was used to prove that the classical rotation number of any Liouville torus can be recovered from the joint spectrum. From a quite different perspective, it could also be interesting to investigate the proximity of our approach with topological data analysis and manifold learning.

7 Prospects

The detection of good labellings should allow the complete reconstruction of the integral affine structure, at least for its regular part. The next step would be to globalize the notion of asymptotic lattice, and include singularities, to obtain quantized integral affine structures with singularities. For instance, the singular limit of the rotation number, as explained in [12], should be a feature of asymptotic lattices with focus-focus singularities.

As far as inverse spectral theory is concerned, we certainly hope to use the notion of asymptotic lattice to advance towards the Spectral semitoric conjecture [28, 21]: can you detect the five symplectic invariants of a semitoric system on the joint spectrum (or asymptotic lattice)? In an ongoing work with Le Floch, which initially focussed on the reconstruction of the twisting index invariant, we finally expect to obtain not only the injectivity of the “semiclassical joint spectrum map” for simple semitoric systems, but also a full reconstruction procedure. A more general result should include multi-pinched tori (see [27, 24, 23]), and systems with non-proper circle moment map [25, 26].
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