On a class of linearisable Abel equations

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Abstract

Applying symmetry reduction to a class of SL(2, R)-invariant third-order ODEs, we obtain Abel equations whose general solution can be parametrised by hypergeometric functions. Particular case of this construction provides a general parametric solution to the Kudashev equation, an ODE arising in the asymptotic analysis of a simultaneous solution to the KdV equation and the stationary part of its higher non-autonomous symmetry.

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1 Introduction

The first-kind and the second-kind Abel equations,
\[ u' = f_3(x)u^3 + f_2(x)u^2 + f_1(x)u + f_0(x) \]
and
\[ (G_1(x)U + G_0(x))U' = F_3(x)U^3 + F_2(x)U^2 + F_1(x)U + F_0(x), \]
which are related by a point transformation \( u = 1/(G_1U + G_0) \), have been of interest since the work of Abel on elliptic functions where Abel equations appeared in the form
\[ (\eta + h_0(x))\eta' = H_2(x)\eta^2 + H_1(x)\eta + H_0(x). \]

Although Abel equations can be seen as a slight generalization of well-understood Riccati equations, there is no general approach to solving them. By and large, there has been no real progress on this issue since the classical works of Abel, Liouville and Appell at the end of the 19th century [1, 3, 14]. In fact, the classical textbooks on solutions of ODEs (Kamke [13], Polyanin and Zaitsev [19]) contain about a hundred of integrable Abel equations, however, most of them are equivalent to the 11 canonical forms [4, Appendix A] under the transformations
\[ x \to \phi(x), \quad u \to \psi(x)u + \eta(x); \]

note that under these transformations every Abel equation of the first kind can be reduced to the normal form \( u' = u^3 + f(x) \). We also note that Chiellini integrability condition, a discovery of 1930s and a working horse of some modern progress [16, 17], is merely a means of checking whether an Abel equation at hand is equivalent to a separable Abel equation.

In the present paper we obtain a new two-parameter family of integrable Abel equations as a reduction of SL(2, \( \mathbb{R} \))-invariant third-order ODEs (examples of solvable Abel equations obtained as symmetry reductions of some second-order ODEs can be found in [18]). Although the found equations possess first integrals which are clear generalizations of that of known integrable Abel equations, these integrals are quite cumbersome, and we present more palatable parametric solutions in terms of hypergeometric functions. As an application, we provide a general parametric solution to the Kudashev equation,
\[ \frac{dR}{dz} = \frac{486R^4 - 171R^2 + 9zR + 5}{9(54R^3 - 9R + z)(2R + 3z)}, \] (1.1)

which takes a second-kind Abel form when written in terms of \( z(R) \). Equation (1.1) arises in the following context (see [11, 21] and references therein): consider the KdV equation constrained by the stationary part of its higher-order non-autonomous symmetry,
\[ u_t + uu_x + u_{xxx} = 0, \quad u_{xxxx} + \frac{5}{3} uu_{xx} + \frac{5}{6} u_x^2 + \frac{5}{18} (x - tu + u^3) = 0. \] (1.2)

Under suitable (Gurevich-Pitaevskii) boundary conditions, these equations possess a common solution with the asymptotic expansion
\[ u(t, x) = \sqrt{t} \left( v_0(z, \phi) + t^{-\frac{7}{4}} v_1(z, \phi) + t^{-\frac{7}{2}} v_2(z, \phi) + \ldots \right). \] (1.3)
where $z = xt^{-\frac{3}{2}}$ and $\phi = t^2 f(z) + S(z)$ are the slow and fast variables, respectively (the functions $v_0, v_1, v_2, \ldots$ are assumed $2\pi$-periodic in the fast variable $\phi$). Introducing $R(z) = \frac{1}{4} f_{\phi} - \frac{3}{2} z$, one can show that $R$ satisfies ODE (1.1). This ODE was first derived by Vadim Kudashev in the late 1990s, but was never published during his lifetime. It has first appeared in [11], see also [21] where a peculiar hypergeometric integral was provided, thus confirming its integrability. We also refer to [8, 7] for the universality property of system (1.2) and its rigorous asymptotic theory.

We show that equation (1.1) possesses the general parametric solution

$$R = \epsilon \sqrt{\frac{1 - \omega}{3(6\omega + 1)}}, \quad z = -\frac{\epsilon}{6} \sqrt{\frac{1 - \omega}{3(6\omega + 1)}} \frac{2(1 - \omega)(576\omega^2 - 333\omega + 2)\psi + 245}{(\omega - 1)^2(6\omega + 1)} \quad (1.4)$$

(here and below $\epsilon = \pm 1$) where $\omega$ and $\psi$ are the following functions of the parameter $s$:

$$\omega = \frac{144s(1 - s)w^2_s}{144s(1 - s)w^2_s - 35w^2}, \quad \psi = \frac{(144s(1 - s)w^2_s - 35w^2)^3}{10080s(1 - s)w^2,w^2_s(144s(1 - s)w^2_s + 24swsw_s + 35w^2)};$$

here $w(s)$ is the general solution to the hypergeometric differential equation,

$$s(1 - s)w_{ss} + \left(\frac{1}{2} - \frac{5}{6}s\right)w_s + \frac{35}{144}w = 0, \quad (1.5)$$

corresponding to the parameter values $(\alpha, \beta, \gamma) = (\frac{5}{12}, -\frac{7}{12}, \frac{1}{2})$. Equation (1.1) also possesses a special algebraic solution given by the same formula (1.4) where $\omega$ and $\psi$ are rational functions of the parameter $\sigma$:

$$\omega = \frac{(7\sigma - 5)^2}{12(7\sigma^2 + 5)}, \quad \psi = \frac{6(7\sigma^2 + 5)^3}{35\sigma(\sigma + 1)^2(7\sigma - 5)^2}; \quad (1.6)$$

it can be represented implicitly as

$$20(1 - 3R^2)^3 - 27(z + 14R^3 - 4R)^2 = 0. \quad (1.7)$$

The structure of the paper is as follows. In Section 2 we carry out symmetry reduction of a general $\text{SL}(2, \mathbb{R})$-invariant third-order ODE to a first-order ODE and identify a class of Abel equations among such reductions. Using the fact that one can construct a parametric solution of the third-order equations, we give its analogue for the identified Abel equations. In Section 3 we exemplify our method with some known $\text{SL}(2, \mathbb{R})$-invariant third-order ODEs, in particular those satisfied by modular forms, and elaborate on the Kudashev equation. The phase portrait of the Kudashev equation is discussed in Section 4. Following [11], in Section 5 we present the leading term $v_0$ of the asymptotic expansion (1.3). In Section 6 we generalise the linearisability result of Section 2. Finally, Section 7 is left for conclusions.
2 Crux of the method

Our starting point is third-order ordinary differential equations \( F(z, g, g', g'', g''') = 0 \) for \( g(z) \) (here prime denotes differentiation by \( z \)) that possess \( SL(2, \mathbb{R}) \)-symmetry of the form

\[
\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \tilde{g} = (\gamma z + \delta)^2 g + \gamma (\gamma z + \delta).
\] (2.8)

The corresponding Lie invariance algebra is \( \mathfrak{g} = \langle \partial_z, z\partial_z - g\partial_g, z^2\partial_z - (2zg + 1)\partial_g \rangle \). It turns out that the presence of symmetry (2.8) implies linearisability of the equations under study. The following statement is, essentially, contained in [5]:

**Theorem 1.** A general third-order equation \( F(z, g, g', g'', g''') = 0 \) possessing \( SL(2, \mathbb{R}) \)-symmetry (2.8) can be represented in the form \( F(I_2, I_3) = 0 \) where \( I_2 \) and \( I_3 \) are the basic differential invariants of the order two and three, respectively:

\[
I_2 = \frac{(g'' - 6gg' + 4g^3)^2}{(g' - g^2)^3}, \quad I_3 = \frac{g''' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4}{(g' - g^2)^2}.
\]

The general solution of any such equation can be represented parametrically as

\[
z = \frac{\tilde{w}}{w}, \quad g = \frac{ww'}{W}
\] (2.9)

where \( w(s) \) and \( \tilde{w}(s) \) are two linearly independent solutions of a second-order linear equation \( w_{ss} + pw_s + qw = 0 \) and \( W = \tilde{w}_s w - w_s \tilde{w} \) is the Wronskian of \( w \) and \( \tilde{w} \) (the coefficients \( p(s) \) and \( q(s) \) depend on the equation \( F = 0 \) and can be efficiently reconstructed, see the proof).

**Proof:**

Consider a linear equation \( w_{ss} + pw_s + qw = 0 \), take its two linearly independent solutions \( w(s) \), \( \tilde{w}(s) \) and introduce parametric relations (2.9). Using \( ds/dz = w^2/W \), \( W_s = -pW \) and the chain rule we obtain

\[
g' - g^2 = -q \frac{w^4}{W^2},
\]

\[
g'' - 6gg' + 4g^3 = -(q_s + 2pq) \frac{w^6}{W^3},
\]

\[
g''' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4 = -(q_{ss} + 2p_sq + 5q_sp + 6p^2q) \frac{w^8}{W^4};
\]

recall that prime denotes differentiation by \( z \). Thus, one arrives at the relations

\[
I_2 = \frac{(g'' - 6gg' + 4g^3)^2}{(g' - g^2)^3} = -\frac{(q_s + 2pq)^2}{q^3},
\]

\[
I_3 = \frac{g''' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4}{(g' - g^2)^2} = -\frac{q_{ss} + 2p_sq + 5q_sp + 6p^2q}{q^2}.
\]

4
To solve the equation \( F(I_2, I_3) = 0 \), it is therefore sufficient to find coefficients \( p(s), q(s) \) such that

\[
F \left( -\frac{(q_s + 2pq)^2}{q^3}, -\frac{q_{ss} + 2p_s q + 5p q + 6p^2 q^3}{q^2} \right) = 0. \tag{2.10}
\]

This finishes the proof. \( \square \)

**Remark.** Parametric formula \((2.9)\) can be generalised as

\[
z = \frac{\bar{w}}{w}, \quad g = \frac{ww_s + rw^2}{W}
\]

where, as in Theorem 1, \( w(s) \) and \( \bar{w}(s) \) are two linearly independent solutions of a second-order linear equation \( w_{ss} + pw_s + qw = 0 \) and \( W = \bar{w}_s w - w_s \bar{w} \) is their Wronskian. Here the coefficients \( p(s), q(s) \) and \( r(s) \) depend on the equation \( F \) and can be efficiently reconstructed, see Section \( 6 \). Introducing an extra function \( r(s) \) allows more flexibility in the construction.

As a next step, we reduce third-order equation for \( g(z) \) to a first-order equation by carrying out the symmetry reduction with respect to the two-dimensional subalgebra \( \langle \partial_z, z\partial_z - g\partial_g \rangle \) of \( g \).

In the new independent variable \( \omega \) and the new dependent variable \( \psi \),

\[
\omega = \frac{g^2}{g'}, \quad \psi = \frac{(g')^3}{g^2(2(g')^2 - gg'')}, \tag{2.11}
\]

the invariants \( I_2 \) and \( I_3 \) take the form

\[
\hat{I}_2 = \frac{(2(\omega - 1)(2\omega - 1)\omega \psi - 1)^2}{(1 - \omega)^3 \omega^3 \psi^2}, \quad \hat{I}_3 = \frac{\omega \psi - 6\omega^2(\omega - 1)(2\omega - 1)^2 \psi^3 + (12\omega - 7)\omega \psi^2 + 3\psi}{(\omega - 1)^2 \omega^3 \psi^3},
\]

so that the reduced first-order equation for \( \psi(\omega) \) can be represented as \( F(\hat{I}_2, \hat{I}_3) = 0 \). By construction, this equation will also be linearisable. Using the expressions for \( g, g', g'' \) in terms of the linear equation \( w_{ss} + pw_s + qw = 0 \), one can rewrite parametric formulae \((2.11)\) as follows:

\[
\omega = \frac{w_s^2}{w_s^2 - qw^2}, \quad \psi = \frac{(w_s^2 - qw^2)^3}{w^2w_s^2(2qw_s^2 + (qs + 2pq)ww_s + 2q^2w^2)}, \tag{2.12}
\]

here \( w(s) \) is an arbitrary solution of the linear equation.

In what follows, we will consider a special two-parameter class of SL(2, \( \mathbb{R} \))-invariant third-order equations for \( g(z) \) with a linear function \( F \), namely, \( I_3 + c_1I_2 + c_2 = 0 \). In explicit form,

\[
(g' - g^2)(g'' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4) + c_1(g'' - 6gg' + 4g^3)^2 + c_2(g' - g^2)^3 = 0. \tag{2.13}
\]

The corresponding first-order equation \( A_{c_1,c_2} \) for \( \psi(\omega) \) is \( \hat{I}_3 + c_1\hat{I}_2 + c_2 = 0 \), explicitly,

\[
\psi - \omega(\omega - 1)(4c_1(2\omega - 1)^2 - c_2(2\omega - 1) + 6(2\omega - 1)^2)\psi^3
+ (4c_1(2\omega - 1) + 12\omega - 7)\psi^2 + \frac{3\omega - c_1 - 3}{\omega(\omega - 1)}\psi = 0, \tag{2.14}
\]
which is an Abel equation of the first kind depending on two parameters $c_1, c_2$. Its general solution can be represented in parametric form \( (2.12) \) where \( w(s) \) is the general solution of a second-order linear equation \( w_{ss} + pw_s + qw = 0 \) whose coefficients \( p(s) \) and \( q(s) \) can be recovered from the corresponding relation \( (2.10) \):

\[
q(q_{ss} + 2p_s q + 5q_s p + 6p^2 q) + c_1 (q_s + 2pq)^2 - c_2 q^3 = 0.
\]  

(2.15)

Note that we have a single constraint for the two unknown coefficients \( p(s) \) and \( q(s) \): this allows some flexibility in selecting a linear equation with desired analytic properties. Remarkably, in the case of \( (2.15) \), one can choose the linear equation to be hypergeometric (for regular values of \( c_1 \) and \( c_2 \): \( c_1 \neq -3/2, \ c_2 \neq 0 \)):

\[
s(1 - s)w_{ss} + (\gamma - (1 + \alpha + \beta)s)w_s - \alpha \beta w = 0.
\]  

(2.16)

Indeed, substituting the corresponding coefficients \( p(s) = \frac{\gamma - (1 + \alpha + \beta)s}{s(1-s)} \), \( q(s) = -\frac{\alpha \beta}{s(1-s)} \) into \( (2.15) \) one obtains the following relations among hypergeometric parameters \( \alpha, \beta, \gamma \) and the parameters \( c_1, c_2 \) of the Abel equation \( A_{c_1,c_2} \):

\[
(4c_1 + 6)\gamma^2 - (4c_1 + 7)\gamma + c_1 + 2 = 0,
\]

\[
(4c_1 + 6)(\alpha + \beta)^2 - c_2 \alpha \beta = 0,
\]

\[
c_2 \alpha \beta - (8c_1 + 12)(\alpha + \beta)\gamma + (4c_1 + 5)(\alpha + \beta) + 2\gamma - 1 = 0.
\]

These relations can be explicitly solved for \( \gamma, \ \alpha + \beta \) and \( \alpha \beta \), leading to the four cases:

\[
\gamma = \frac{1}{2}, \ \ \alpha + \beta = 0, \ \ \alpha \beta = 0;
\]

\[
\gamma = \frac{1}{2}, \ \ \alpha + \beta = \frac{1}{4c_1 + 6}, \ \ \alpha \beta = \frac{1}{c_2(4c_1 + 6)};
\]

\[
\gamma = \frac{c_1 + 2}{2c_1 + 3}, \ \ \alpha + \beta = \frac{1}{2c_1 + 3}, \ \ \alpha \beta = \frac{2}{c_2(2c_1 + 3)};
\]

\[
\gamma = \frac{c_1 + 2}{2c_1 + 3}, \ \ \alpha + \beta = \frac{1}{4c_1 + 6}, \ \ \alpha \beta = \frac{1}{c_2(4c_1 + 6)}.
\]

Thus, there can be several different hypergeometric equations linearising the same Abel equation. Note that the first case can be disregarded since it leads to the inconsistent condition \( \omega = 1 \) in the formula \( (2.12) \). Furthermore, hypergeometric equations in the second and the fourth cases are equivalent under the transformation \( s \to 1 - s, \ w \to w \). In what follows, we will not distinguish between collections \( (\alpha, \beta, \gamma) \) and \( (\beta, \alpha, \gamma) \) since they correspond to the same hypergeometric equation.

It is important to note that besides the general solutions expressed via hypergeometric functions, the Abel equations \( A_{c_1,c_2} \) possess special algebraic solutions given by parametric formulae \( (2.12) \) where \( w \) satisfies a linear equation \( w_{ss} + pw_s + qw = 0 \) with constant coefficients \( p \).
and \( q \). The substitution into (2.15) gives a single relation among the parameters, \((6 + 4c_1)p^2 = c_2q\), where without any loss of generality one can set \( p = 1 \). Thus, the required linear equation is

\[
\frac{6 + 4c_1}{c_2} w = 0.
\]

(2.17)

3 Examples

In this section we discuss four examples of integrable Abel equations \( A_{c_1,c_2} \) given by (2.14) that correspond to different choices of constants \( c_1, c_2 \). The first three of them originate from the theory of modular forms, and the last example is related to the Kudashev equation.

Example 1: \( c_1 = 0, c_2 = 24 \). In this case equation (2.13) is the Chazy equation,

\[
g''' - 12gg'' + 18g^2 = 0,
\]

which is satisfied by the weight 2 Eisenstein series \( E_2(z) \) associated with the full modular group \( SL(2, \mathbb{Z}) \). Setting in this equation \( g = \frac{E'}{2\Delta} \) we obtain a fourth-order ODE for the modular discriminant \( \Delta \), see e.g. [20, 15]. The corresponding Abel equation \( A_{0,24} \) is

\[
\psi + wss + ws + \frac{6 + 4c_1}{c_2} w = 0.
\]

(3.18)

Its general solution can be represented in parametric form (2.12) where \( w \) satisfies hypergeometric equation (2.16) with any of the following parameter values \((\alpha, \beta, \gamma)\): \((1, \frac{1}{12}, \frac{1}{12}), (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}), (\frac{1}{12}, \frac{1}{12}, \frac{3}{4})\). Equation (3.18) also possesses an algebraic solution given by parametric formula (2.12) where \( w \) satisfies the linear equation (2.17), \( wss + ws + \frac{1}{4} w = 0 \). Taking \( w(s) = ae^{-\frac{s}{2}} + bse^{-\frac{1}{2}s} \), where without any loss of generality one can set \( a = 0, b = 1 \), gives

\[
\omega = -\frac{(s - 2)^2}{4(s - 1)}, \quad \psi = -\frac{8(s - 1)^3}{s^2(s - 2)^2},
\]

or in the explicit form,

\[
\psi(\omega) = \frac{(2\omega - 2\Omega - 1)^3}{2(\Omega - \omega + 1)^2(\Omega - \omega)^2} \quad \text{where} \quad \Omega = \pm \sqrt{\omega(\omega - 1)}.
\]

Example 2: \( c_1 = -1, c_2 = 9 \). In this case equation (2.13) takes the form

\[
g'''(g' - g^2) = (g'')^2 - 4g^3g'' - 3(g')^3 + 9g^2(g')^2 - 3g^4g' + g^6.
\]

(3.19)

It has appeared in the classification of integrable Euler–Lagrange equations; setting \( g = \frac{F'}{T} \) one obtains a fourth-order ODE for \( f \) satisfied by the Eisenstein series \( E_{1,3}(z) \) [9]. The corresponding Abel equation \( A_{-1,9} \) is

\[
\psi + \omega(\omega - 1)(\omega - 2)(\omega + 1)\psi^3 + (4\omega - 3)\psi^2 + \frac{3\omega - 2}{\omega(\omega - 1)} \psi = 0.
\]

(3.20)
Its general solution can be represented in parametric form \((2.12)\) where \(w\) satisfies hypergeometric equation \((2.16)\) with any of the following parameter values \((\alpha, \beta, \gamma)\): \((\frac{1}{3}, \frac{1}{6}, \frac{1}{2})\), \((\frac{1}{3}, \frac{2}{3}, 1)\), \((\frac{1}{3}, \frac{1}{6}, 1)\). Equation \((3.20)\) also possesses an algebraic solution given by parametric formula \((2.12)\) where \(w\) satisfies the linear equation \((2.17)\), \(w_{ss} + w_{s} + \frac{2}{3}w = 0\). Taking \(w(s) = ae^{-\frac{s}{2}} + be^{-\frac{s}{2}}\), where without any loss of generality one can set \(a = 1\), \(b = 1\), gives
\[
\omega = \frac{(2\sigma + 1)^2}{2\sigma^2 - 1}, \quad \psi = -\frac{(2\sigma^2 - 1)^3}{4\sigma(2\sigma + 1)^2(\sigma + 1)^2} \quad \text{where} \quad \sigma = e^{-s/3},
\]
or in the explicit form,
\[
\psi(\omega) = -\frac{2(2\Omega + 3\omega - 2)^3}{(\Omega + \omega)^2(\Omega + 2\omega - 2)^2(\Omega + 2)(\omega - 2)} \quad \text{where} \quad \Omega = \pm \sqrt{2\omega(\omega - 1)}.
\]
Finally, this equation possesses the discrete symmetry \(\tilde{\omega} = 1 - \omega\), \(\tilde{\psi} = -\psi\).

**Example 3:** \(c_1 = -1, \ c_2 = 8\). In this case equation \((2.13)\) takes the form
\[
g''''(g' - g^2) = (g'')^2 - 4g^3g'' - 2(g')^3 + 6g^2(g')^2.
\]
Up to a scaling factor, it has appeared in [2] as the equation satisfied by the Eisenstein series \(E_2(z)\) of the level two congruence subgroup \(\Gamma_0(2)\) of the modular group. The corresponding Abel equation \(A_{-1,8}\) is
\[
\psi_\omega - 2\omega(\omega - 1)\psi^3 + (4\omega - 3)\psi^2 + \frac{3\omega - 2}{\omega(\omega - 1)} \psi = 0.
\]
Its general solution can be represented in parametric form \((2.12)\) where \(w\) satisfies hypergeometric equation \((2.16)\) with any of the following parameter values \((\alpha, \beta, \gamma)\): \((\frac{1}{4}, \frac{1}{4}, \frac{1}{2})\), \((\frac{1}{2}, \frac{1}{2}, 1)\), \((\frac{1}{4}, \frac{1}{4}, 1)\). Equation \((3.21)\) also possesses an algebraic solution, the same as in Example 1, since the corresponding linear equations \((2.17)\) are identical.

**Example 4:** \(c_1 = -3, \ c_2 = 24/35\). In this case equation \((2.13)\) takes the form
\[
(g' - g^2)(g'''' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4) = 3(g'' - 6gg' + 4g^3)^2 + \frac{24}{35}(g' - g^2)^3 = 0.
\]
We were not able to uncover its ‘modular’ origin. The corresponding Abel equation \(A_{-3,24/35}\) is
\[
\psi_\omega + \frac{6}{35}\omega(\omega - 1)(12\omega - 5)(12\omega - 7)\psi^3 - (12\omega - 5)\psi^2 + \frac{3}{\omega - 1} \psi = 0.
\]
Its general solution can be represented in parametric form \((2.12)\) where \(w\) satisfies hypergeometric equation \((2.16)\) with any of the following parameter values \((\alpha, \beta, \gamma)\): \((\frac{1}{12}, -\frac{7}{12}, \frac{1}{2})\), \((\frac{5}{6}, -\frac{7}{6}, \frac{1}{3})\), \((\frac{5}{12}, -\frac{7}{12}, \frac{1}{3})\). This equation also possesses an algebraic solution given by parametric formula \((2.12)\) where \(w\) satisfies the linear equation \((2.17)\), \(w_{ss} + w_{s} - \frac{20}{3}w = 0\). Taking \(w(s) = ae^{-\frac{s}{2}} + be^{-\frac{s}{2}}\), where without any loss of generality one can set \(a = 1\), \(b = 1\), gives
\[
\omega = \frac{(7\sigma - 5)^2}{12(7\sigma^2 + 5)}, \quad \psi = \frac{6(7\sigma^2 + 5)^3}{35\sigma(\sigma + 1)^2(7\sigma - 5)^2} \quad \text{where} \quad \sigma := e^{-6s},
\]
or in the explicit form,
\[ \psi(\omega) = -\frac{1225}{2} \frac{(2\Omega + 2\omega - 7)^3}{(\Omega - 5\omega)^2(\Omega + 7\omega - 7)^2(12\Omega - 35)(12\omega - 7)} \]

where \( \Omega = \pm \sqrt{35\omega(1 - \omega)} \).

Equation (3.22) is related to the Kudashev equation (1.1) by the point transformation (1.4)

4 Phase portrait of the Kudashev equation

Recall that solutions \( R(z) \) of the Kudashev equation (1.1) are given by parametric formula (1.4) where for the general solution we choose the general solution of the associated hypergeometric equation (1.5),

\[ w(s) = a \, \text{}_2F_1 \left( \frac{5}{12}, -\frac{7}{12}; \frac{1}{2}; s \right) + b\sqrt{s} \, \text{}_2F_1 \left( -\frac{1}{12}, \frac{11}{12}; \frac{3}{2}; s \right) \]

with arbitrary constants \( a \) and \( b \) (one of them is not essential and therefore can be set to be equal to one without loss of generality). Depicted below is the phase portrait for the Kudashev equation. Its apparent symmetry reflects the symmetry \( z \mapsto -z, \, R \mapsto -R \) of the equation (1.1).

Figure 1: The phase portrait of the Kudashev equation

There are six equilibrium points, where both the numerator and the denominator of the right-hand side of the equation (1.1) vanish, (left to right),

\[ P_1 = \left( -\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9} \right), \quad P_2 = \left( -\frac{1}{3\sqrt{2}}, -\sqrt{2} \right), \quad P_3 = \left( -\frac{1}{5} \sqrt{\frac{5}{2}}, \frac{2}{27} \sqrt{\frac{5}{2}} \right), \]

1We slightly abuse notation here: \( z \) in (1.4) is the independent variable of the Kudashev equation (1.1), and it has nothing to do with the independent variable of the \( \text{SL}(2, \mathbb{R}) \)-invariant equation for \( g(z) \) at the beginning of the example. Both the variables are denoted \( z \) in the literature and we wanted to keep the notation.
\[ P_4 = \left( \frac{1}{5} \sqrt{\frac{5}{2}}, -\frac{2}{27} \sqrt{\frac{5}{2}} \right), \quad P_5 = \left( \frac{1}{3\sqrt{2}}, \sqrt{2} \right), \quad P_6 = \left( \frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9} \right). \]

The apparent separatrix, shown in black in Figure 1, is nothing else but the algebraic solution of the Kudashev equation. It has the implicit form (1.7), and the explicit form can be written as

\[ z(R) = \frac{2\epsilon}{9} (3R^2 - 1) \sqrt{15(1 - 3R^2)} - 2R(7R^2 - 2). \]

Another separatrix of interest, which passes through the equilibrium points \( P_4 \) and \( P_5 \), is not parametrised by algebraic solutions, but by the solution (4.23) with the values of the constants

\[ a = \frac{\sqrt{\pi}}{\Gamma(11/12)\Gamma(19/12)}, \quad b = \pm \frac{2i\sqrt{\pi}}{\Gamma(5/12)\Gamma(13/12)}. \]

Note that this is one of the Kummer solutions of (1.5), \( w(s) = e^{\frac{5\pi}{12}i} s^{-5/12} {}_2F_1 \left( \frac{5}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{s} \right) \). This separatrix has a symmetric counterpart passing through the equilibrium points \( P_2 \) and \( P_3 \).

It can also be noted that integral curves, lying outside the algebraic separatrix and having the endpoints \( (P_4, P_6) \) and the endpoints \( (P_5, P_6) \) (resp., \( (P_1, P_2) \) and \( (P_1, P_3) \)), are also separated by a separatrix, but we were unable to find its parametrisation.

5 Leading term of the asymptotic solution

Recall that the fast variable \( \phi \) in the asymptotic expansion (1.3) depends on the function \( f(z) \) that solves the first-order ODE \( R(z) = \frac{7f}{4z} - \frac{3}{2} z \), where \( R(z) \) is the solution of the Kudashev equation (1.1). With the help of the parametric formula of the solution of this equation, we can find the parametric representation of \( f(z) \). For the general solution of the Kudashev equation, the function \( f \) takes the form

\[ f = \frac{c |s|^{5/4} |s - 1|^{5/6} |w|^{5/2}}{|144s(s - 1)w^2 + 5w^2|^{7/4}}, \quad (5.24) \]

and for its algebraic solution it has a simpler form, \( f = \frac{c |7\sigma - 5|^{5/2}}{(9\sigma^2 - 10\sigma + 5)^{7/4}} \).

Below we follow [11] to show that the knowledge of the coefficients \( f(z) \) and \( R(z) \) leads to an explicit formula for the leading term \( v_0(z, \phi) \) of the asymptotic expansion (1.3). We set \( v_0 \equiv v \) to simplify the notation.

Substituting (1.3) into the first equation (1.2), at the leading order \( t^{5/4} \) one obtains

\[ Q^2 v_{\phi\phi\phi} + v_{\phi\phi} + Rv_{\phi} = 0. \]

Similarly, substituting (1.3) into the second equation (1.2), at the leading order \( t^{3/2} \) one obtains

\[ Q^4 v_{\phi\phi\phi\phi} + \frac{5}{6} Q^2 \left( 2v v_{\phi\phi} + v_{\phi}^2 \right) + \frac{5}{18} (z - v + v^3) = 0; \]
here the coefficients $Q(z) = f_z$ and $R(z) = \frac{7}{9} f - \frac{3}{2} z$ are functions of $z$ only. These two equations for $v$ are equivalent to a single first-order equation,

$$Q^2 v_\phi^2 + \frac{1}{3} v^3 + R v^2 + \left(6 R^2 - \frac{5}{3}\right) v + 5 R - 18 R^3 - \frac{5}{3} z = 0.$$  \hspace{1cm} (5.25)

We look for a solution of (5.25) in the form

$$v = A \text{dn}^2 \left(\frac{B}{Q} \phi, k\right) - C - R$$  \hspace{1cm} (5.26)

where $\text{dn}(r, k)$ is the Jacobi elliptic function and the coefficients $A, B, C, k$ are functions of the slow variable $z$. Recall that $y = \text{dn}(r, k)$ satisfies the equation $y_r^2 = (y^2 - 1)(1 - k^2 - y^2)$, which implies $Y_r^2 = 4Y(Y - 1)(1 - k^2 - Y)$ for $Y = \text{dn}^2(r, k)$. Substituting ansatz (5.26) into (5.25) we obtain four relations for the coefficients:

$$A - 12 B^2 = 0,$$  \hspace{1cm} (5.27a)

$$4(2 - k^2) B^2 - C = 0,$$  \hspace{1cm} (5.27b)

$$12(2 k^2 - 1) A B^2 + 3 C^2 + 15 R^2 - 5 = 0,$$  \hspace{1cm} (5.27c)

$$C^3 + (15 R^2 - 5)C + 70 R^3 - 20 R + 5 z = 0.$$  \hspace{1cm} (5.27d)

One can solve the equations (5.27a) and (5.27b) for $A$ and $B$:

$$A = \frac{3 C}{2 - k^2}; \hspace{1cm} B^2 = \frac{C}{4(2 - k^2)};$$

here $C$ and $k$ can be recovered from (5.27c) and (5.27d):

$$9(k^2 - 1) C^2 + (3 C^2 + 15 R^2 - 5)(k^2 - 2)^2 = 0, \hspace{1cm} C^3 + (15 R^2 - 5)C + 70 R^3 - 20 R + 5 z = 0.$$  

The further analysis splits into two different cases depending on whether $R(z)$ is a generic or the algebraic solution of the Kudashev equation.

### 5.1 Generic solution of the Kudashev equation

If $R(z)$ is the general solution (1.4) of the Kudashev equation, then the equation (5.27d) has three distinct roots, which for $s \leq 0$ are real-valued and take the forms $C_1 = \mathcal{C}(-4, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}})$, $C_2 = \mathcal{C}(4, 1, 1)$, $C_3 = \mathcal{C}(-4, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}})$, where

$$\mathcal{C}(\alpha, \beta, \gamma) = -\frac{\alpha e^{\sqrt{15 w}(s - 1)^{1/3}}}{\sqrt{144 s(s - 1) w^2 + 5 w^2}} \beta^{(\sqrt{s} + 1)^{1/3}} + \gamma(\sqrt{s} - 1)^{1/3}.$$  

After introducing $\zeta = (\sqrt{s} + 1)^{1/3}$ and $\theta = (\sqrt{s} - 1)^{1/3}$, we can write the corresponding values of $k^2$ as

$$k_1^2 = \frac{e^{\frac{\pi i}{3}} \theta + e^{\frac{2\pi i}{3}} \zeta}{\theta - \zeta}, \hspace{1cm} k_2^2 = \frac{\theta + e^{\frac{\pi i}{3}} \zeta}{e^{\frac{\pi i}{3}} \theta + \zeta},$$

$$k_3^2 = \frac{e^{\frac{2\pi i}{3}} \theta + e^{\frac{\pi i}{3}} \zeta}{\zeta - \theta},$$

and

$$k_1^2 = \frac{\theta - e^{\frac{2\pi i}{3}} \zeta}{\zeta + e^{\frac{\pi i}{3}} \theta}, \hspace{1cm} k_2^2 = \frac{\theta - e^{\frac{2\pi i}{3}} \zeta}{\zeta - e^{\frac{2\pi i}{3}} \theta}. $$

11
In the Gurevich–Pitaevskii problem it is required that the function \(v\) is periodic \([11]\), which translates into the condition

\[
\frac{B}{Q} = \frac{K(k)}{\pi},
\]

(5.28)

where \(K\) is the complete elliptic integral of the first kind, and the constant \(\pi\) appears to make a period of the function \(v\) to be equal \(2\pi\). Although the equation (5.28) contains two unknowns, the constant \(c\) appearing in the formula (5.24) for \(f\) and the essential constant determining a solution (4.23) of the hypergeometric equation (1.5), they are both determined therefrom. The constant \(c\) regulates a period of the function \(v\), and therefore the period of the function \(v\) may be taken arbitrary. On the other hand, the latter constant is uniquely determined from the equation (5.28). Another way to look at the rôle of the latter constant is as follows. It assures the function \(B/Q\) satisfy a hypergeometric differential equation as a function of \(k^2\) (since \(K(k) = \frac{\pi}{2} 2F_1(1/2, 1/2, 1, k^2)\)).

With the help of numerical computations one can show that the condition (5.28) is satisfied for \(C = C_3\) with either of the values of \(k^2\) can be chosen, \(a = 1\) and \(b = -0.92i\), and \(c = 2664.54\). Note that the value of \(b\) is close to \(2i \frac{(11/12)(19/12)}{(5/12)(13/12)}\), and this the solution of the hypergeometric equation (1.5) that ensures the condition (5.28) is the one that determines the hypergeometric separatrix in Section 4. It was shown in \([11]\) that the phase shift for the Gurevich–Pitaevskii solution is equal to \(\pi\). The numerical simulation of the obtained developing bore is depicted in Figure 2 below.

Figure 2: Development of undular bore over time

Here \(s \in [-100, 0]\), \(k^2 = \frac{e^{2\pi i} + e^{\pi i}}{\zeta - \theta}, w(s) = 2F_1 \left(\frac{5}{12}, -\frac{7}{12}; \frac{1}{2}, s\right) - 0.9233i\sqrt{s} 2F_1 \left(-\frac{1}{12}, \frac{11}{12}; \frac{3}{2}, s\right)\).

The figure on the left is for \(t = 3\), on the right is for \(t = 6\).
5.2 Algebraic solution of the Kudashev equation

Algebraic solution \( R(z) \) of the Kudashev equation can be obtained by substituting (1.6) into (1.4), which results in

\[
R = \frac{\epsilon \sqrt{10}(\sigma + 1)}{6\sqrt{9\sigma^2 - 10\sigma + 5}}, \quad z = -\epsilon \sqrt{10}(\sigma - 1)(\sigma^2 - 10\sigma + 5) \frac{9\sigma^2 - 10\sigma + 5}{(9\sigma^2 - 10\sigma + 5)^{3/2}}.
\]

(5.29)

Note that the implicit equation (1.7) defining the algebraic solution is equivalent to the vanishing of the discriminant of the cubic equation (5.27d). This means that the equation (5.27d) has a multiple root. Indeed, its roots are \( C_1 = -\frac{\epsilon \sqrt{10}(7\sigma - 5)}{6\sqrt{9\sigma^2 - 10\sigma + 5}} \) (of multiplicity two) and \( C_2 = \frac{\epsilon \sqrt{10}(7\sigma - 5)}{3\sqrt{9\sigma^2 - 10\sigma + 5}} \). The corresponding values of \( \kappa^2 \) are 1 and 0, respectively. In what follows we consider the solitonic limit, \( \kappa^2 = 1 \). In this case ansatz (5.26) degenerates into

\[
v = A \text{sech}^2 \left( \frac{B}{Q} \phi \right) - C - R
\]

(5.30)

where the parameter values are as follows: \( A = 3C, \quad B = \frac{1}{2} \sqrt{C}, \quad C = \frac{\sqrt{5/3} - 5R^2}{5} \). This solution can be interpreted as the asymptotic form of the leading soliton in the developing undular bore. Here the determination of the phase shift \( S(z) \) requires analysis of higher-order terms in the asymptotic expansion. As an example we take \( S(z) = 0 \) in Figure 3 below. We refer to [12, 22, 6] for the general asymptotic theory of evolution of soliton parameters and the phase shift problem.

Figure 3: Development of a soliton solution over time

The left and the right parts of this soliton solution are parametrised differently. One part corresponds to \( x = z\sqrt{t} - 3/2, \quad \epsilon = -1, \quad \sigma \in [2, 1000] \), and another one corresponds to \( x = -z\sqrt{t} - 3/2, \quad \epsilon = 1, \quad \sigma \in [-1000, 2] \). The phase shift \( S(z) \) is taken to be 0.
6 Generalisation of the linearisability result

Here we provide the following generalisation of Theorem 1.

**Theorem 2.** A general third-order equation \( F(z, g, g', g'', g''') = 0 \) possessing \( SL(2, \mathbb{R}) \)-symmetry (2.8) can be represented in the form \( F(I_2, I_3) = 0 \) where \( I_2 \) and \( I_3 \) are the basic differential invariants of the order two and three, respectively:

\[
I_2 = \frac{(g'' - 6gg' + 4g^3)^2}{(g' - g^2)^3}, \quad I_3 = \frac{g''' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4}{(g' - g^2)^2}.
\]

The general solution of any such equation can be represented parametrically as

\[
z = \frac{\tilde{w}}{w}, \quad g = \frac{ww_s + rw^2}{W} \quad (6.31)
\]

where \( w(s) \) and \( \tilde{w}(s) \) are two linearly independent solutions of a second-order linear equation \( w_{ss} + pw_s + qw = 0 \) and \( W = \tilde{w}_s w - w_s \tilde{w} \) is the Wronskian of \( w \) and \( \tilde{w} \). Here the coefficients \( p(s), q(s) \) and \( r(s) \) depend on the equation \( F = 0 \) and can be efficiently reconstructed, see the proof below.

**Proof:**

We consider a linear equation \( w_{ss} + pw_s + qw = 0 \), take two linearly independent solutions \( w(s), \tilde{w}(s) \) and introduce parametric relations (6.31). Using \( ds/dz = w^2/W, \ W_s = -pW \) and the chain rule we obtain

\[
g' - g^2 = -\tilde{q}\frac{w^4}{W^2},
\]

\[
g'' - 6gg' + 4g^3 = -(\tilde{q}_s + 2\tilde{p}\tilde{q})\frac{w^6}{W^3},
\]

\[
g''' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4 = -(\tilde{q}_{ss} + 2\tilde{p}_s\tilde{q} + 5\tilde{p}\tilde{q}_s + 6\tilde{p}^2\tilde{q})\frac{w^8}{W^4};
\]

where \( \tilde{p} = p - 2r, \tilde{q} = q - pr + r^2 - r_s \). Thus, one arrives at the relations

\[
I_2 = \frac{(g'' - 6gg' + 4g^3)^2}{(g' - g^2)^3} = -\frac{(\tilde{q}_s + 2\tilde{p}\tilde{q})^2}{\tilde{q}^3},
\]

\[
I_3 = \frac{g''' - 12gg'' - 6(g')^2 + 48g^2g' - 24g^4}{(g' - g^2)^2} = -\frac{\tilde{q}_{ss} + 2\tilde{p}_s\tilde{q} + 5\tilde{p}\tilde{q}_s + 6\tilde{p}^2\tilde{q}}{\tilde{q}^2}.
\]

To solve the equation \( F(I_2, I_3) = 0 \), it is therefore sufficient to find coefficients \( p(s), q(s) \) and \( r(s) \) such that

\[
F \left( -\frac{(\tilde{q}_s + 2\tilde{p}\tilde{q})^2}{\tilde{q}^3}, -\frac{\tilde{q}_{ss} + 2\tilde{p}_s\tilde{q} + 5\tilde{p}\tilde{q}_s + 6\tilde{p}^2\tilde{q}}{\tilde{q}^2} \right) = 0. \quad (6.32)
\]

This finishes the proof. \( \Box \)
Having an extra function $r(s)$ allows one some more freedom in choosing the desired linear equation for $w$. For example, the general solution of the Kudashev equation can be parametrised by the associated Legendre functions $P_{1/2}^{2/3}(s)$, $Q_{1/2}^{2/3}(s)$ if we choose $r(s) = s^2$. In this case, it is given by the parametric formulae (1.4) where

$$
\omega = \frac{4(3(s^2 - 1)w_s + 2sw)^2}{\Lambda(36, 48, 51)} , \quad \psi = \frac{\Lambda(36, 48, 51)^3}{280(s^2 - 1)w^2(3(s^2 - 1)w_s + 2sw)^2\Lambda(-36, -36, 27)}
$$

$\Lambda(\alpha, \beta, \gamma) = \alpha(s^2 - 1)^2w_s^2 + \beta s(s^2 - 1)ww_s + (\gamma s^2 - 35)w^2$ and $w$ is the general solution to the associated Legendre equation $(1 - s^2)w_{ss} - 2sw_s + \left(\frac{3}{4} - \frac{4}{3(1-s^2)}\right)w = 0$. Another advantage of the parametrisation (6.31) over the parametrisation (2.9) is that the former is invariant under the transformations $s \rightarrow T(s), w \rightarrow S(s)w$, unlike the latter.

### 7 Conclusion

Here are a few final comments.

- In the present paper we described a class $A_{c_1,c_2}$ of linearisable Abel equations and it would be interesting to know how many of these integrable equations are new. In [4], most of the known integrable Abel equations were categorised into 11 equivalence classes, with canonical representatives and their first integrals being provided therein. For regular values $c_1$ and $c_2$, the equation $A_{c_1,c_2}$ possesses a first integral of the form

$$
I = \Psi^{-1} \frac{\hat{\Psi} F_1\left(-\frac{1}{2a}, \frac{\sqrt{b(b-8a)-b}}{2ab}; \frac{a-1}{a}; \Psi\right)}{\Psi F_1\left(\frac{1}{2a}, \frac{\sqrt{b(b-8a)+b}}{2ab}; \frac{a+1}{a}; \Psi\right)} + \Psi^+ F_1\left(\frac{1}{2a} + 1, \frac{\sqrt{b(b-8a)+b}}{2ab}; \frac{a+1}{a} + 1; \Psi\right)
$$

where $\Psi$’s are (at most) rational functions of $\psi$ with $\omega$-dependent coefficients, $a = 2c_1 + 3$, $b = c_2$. Recall that the Kudashev equation (1.1) is known to possess a similar first integral, see [21]. Note that for both of the second hypergeometric functions in the numerator and the denominator, the first three parameters are greater by one than their counterparts on the left, and recall that it is the feature of the derivative of the hypergeometric function. Four of the canonical integrable Abel equations have similar first integrals with hypergeometric functions being replaced by other special functions with at most one parameter, which suggests that these equations can be special cases of the equation $A_{c_1,c_2}$. For example, the equation $A_{-\frac{3}{2},\frac{2}{3}}$ is related to the classical Abel equation $AD_\alpha, x^2y_x + xy^3 + (x^2 + \alpha)y^2 = 0$, via the point transformation $\omega = x^2/(x^2 + \alpha)$, $\psi = -(x^2 + \alpha)^3y/(2ax^3)$.

- An interesting observation is that although the first integral above contains four hypergeometric functions with various forms of the parameters $(\alpha, \beta, \gamma)$, in the special case
Unfortunately, it is computationally difficult to relate known integrable Abel equations to
The algebra
An interesting class of exactly solvable first-order ODEs (with nonlinear dependence on
tions
Aside from similarity of first integrals of known integrable Abel equations and the equa-
tions even stronger.
Moreover, these solutions often involve special functions, which makes a connection with
second-order linear ODEs even stronger.

- Unfortunately, it is computationally difficult to relate known integrable Abel equations to
equations $A_{c_1,c_2}$. In order to check whether two integrable Abel equations are equivalent,
the authors of [4] compared their invariants. Recall that the class of all Abel equations
$u' = f_3(x)u^3 + f_2(x)u^2 + f_1(x)u + f_0(x)$ is invariant under transformations $x \rightarrow \phi(x),
u \rightarrow \psi(x)u + \eta(x)$. It was shown in [3,14] that $I_1 = s_5^3/s_5^3$ and $I_2 = s_5s_7/s_5^4$ are invariants
of this action. Here $s_3$, $s_5$ and $s_7$ are relative invariants defined recursively as
\[ s_3 = f_0f_3^2 + \frac{1}{3} \left( \frac{2}{9} f_2^3 - f_1 f_2 f_3 + f_5 \frac{d}{dx} f_2 - f_2 \frac{d}{dx} f_3 \right), \]
\[ s_{2m+1} = f_3 \frac{d}{dx} s_{2m-1} - (2m - 1) s_{2m-1} \left( \frac{d}{dx} f_3 + f_1 f_3 - \frac{1}{3} f_2^2 \right). \]
Although comparing invariants for two specific Abel equations is highly efficient, for the
generic equation $A_{c_1,c_2}$ these invariants are quite lengthy and computation amounts to
calculating the resultant of two polynomials of high degrees. On the other hand, for some
equations $A_{c_1,c_2}$ invariants may be short. For instance, the equation $A_{-3/2,0}$ has constant
invariants, which implies that it is related to a separable Abel equation.

- An interesting class of exactly solvable first-order ODEs (with nonlinear dependence on
the derivative) whose singular solutions can be parametrised by hypergeometric functions
has appeared in [10] in the context of ring waves in stratified fluids (the so-called di-
rectional adjustment equations). In this connection, one should mention that algebraic
separatrix solutions of the equations $A_{c_1,c_2}$ constructed in our paper can be viewed as
singular solutions.

- The algebra $g = \langle \partial_z, z \partial_x - g \partial_y, z^2 \partial_x - (2zg + 1) \partial_y \rangle$ is one of four inequivalent realisations
of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, but it is the only one that leads to Abel equations as symmetry
reductions of $\mathfrak{sl}_2(\mathbb{R})$-invariant third-order ODEs. The other three realisations lead to
Riccati equations [5]. At the same time, Abel equations are not the only equations that
arise in this way. In a case when $g$-invariant equation $F(I_2, I_3) = 0$ is not of the form $I_3 + c_1 I_2 + c_2 = 0$, its symmetry reduction with respect to the algebra $\langle \partial_z, z \partial_z - g \partial_{\bar{z}} \rangle$ is not an Abel equation, but its solutions can still be expressed in terms of solutions of a second-order linear ODE (2.10), albeit its coefficients may be hard to find explicitly.

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