Regularized Tyler’s Scatter Estimator: Existence, Uniqueness, and Algorithms

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Abstract—This paper considers the regularized Tyler’s scatter estimator for elliptical distributions, which has received considerable attention recently. Various types of shrinkage Tyler’s estimators have been proposed in the literature and proved work effectively in the “small n large p” scenario. Nevertheless, the existence and uniqueness properties of the estimators are not thoroughly studied, and in certain cases the algorithms may fail to converge. In this work, we provide a general result that analyzes the sufficient condition for the existence of a family of shrinkage Tyler’s estimators, which quantitatively shows that regularization indeed reduces the number of required samples for estimation and the convergence of the algorithms for the estimators. For two specific shrinkage Tyler’s estimators, we also proved that the condition is necessary and the estimator is unique. Finally, we show that the two estimators are actually equivalent. Numerical algorithms are also derived based on the majorization-minimization framework, under which the convergence is analyzed systematically.

Index Terms—Tyler’s scatter estimator, shrinkage estimator, existence, uniqueness, majorization-minimization.

I. INTRODUCTION

Covariance estimation has been a long existing problem in various signal processing related fields, including multiantenna communication systems, social networks, bioinformatics, as well as financial engineering. A well known and easy to implement estimator is the sample covariance matrix. Under the assumption of clean samples, the estimator is consistent by the Law of Large Numbers. However, the performance of the sample covariance matrix is vulnerable to data corrupted by noise and outliers, which is often the case in real-world applications.

As a remedy, robust estimators are proposed aimed at limiting the influence of erroneous observations so as to achieve better performance in non-Gaussian scenarios [1], [2]. Recently, Tyler’s scatter estimator [3] has received considerable attention both theoretically and practically in signal processing related fields, e.g., [4]–[8] to name a few, see [9] for a comprehensive overview. Tyler’s estimator estimates the normalized scatter matrix (equivalently the normalized covariance matrix if the covariance exists) assuming that the underlying distribution is elliptically symmetric. The estimator exists, where

\[ \alpha N > \text{dim}(S) \]

is the number of samples, \( N \) is the dimension of the samples and \( \alpha_0 \) controls the amount of penalty added to the cost function, \( P_N (S) \) stands for the proportion of samples contained in a proper subspace \( S \). In addition, we prove it is also a necessary condition, provided that \( \alpha_0 > 0 \). Although derived from different cost functions, and also with different estimation equation, we prove that the two shrinkage estimators are actually equivalent. Under the assumption that the underlying distribution is continuous, the condition simplifies to \( N > \frac{K}{1+\alpha_0} \). Comparing with the existence condition for
Tyler’s estimator, which is \( P_N (S) < \frac{\text{dim}(S)}{K} \), or \( N > K \) under continuity assumption, this result clearly demonstrates that regularization can relax the requirement on the number of samples, hence shows its capability of handling large dimension estimation problems. Algorithms for the shrinkage estimators based on majorization-minimization framework are provided, where the convergence can be analyzed systematically.

It is worth mentioning that in the work [14], where the same condition \( N > \frac{K}{1+\alpha_0} \) is also independently derived for the KL penalty based shrinkage estimator that shrinks the covariance matrix to identity in the complex field, assuming the samples are linearly independent. [14] refutes the additional trace normalization step in [13] by showing that the trace of the inverse of the estimator is equal to \( K \), and propose dropping the normalization step. Different from that approach, our work gives an interpretation of the estimator as the minimizer of a KL divergence penalized cost function. Starting from the cost function, we establish the existence condition with a different proof from [14]. In addition, we extend the result (in the real field), since the condition \( P_N (S) < \frac{1+\alpha_0}{1+\alpha_0} \) implies \( N > \frac{K}{1+\alpha_0} \) if the samples are linearly independent, and we consider a general positive definite shrinkage target matrix as in [13].

The paper is organized as follows: In Section II, we briefly review Tyler’s estimator for samples drawn from the elliptical family. In Section III, the two types of shrinkage estimators, i.e., one proposed in [13] and another derived based on KL divergence are considered, and a rigorous proof for the existence and uniqueness of the estimators is provided. Algorithms based on majorization-minimization are presented in Section IV. Numerical examples follow in Section V, and we conclude in Section VI.

**Notation**

\( \mathbb{R}^n \) stands for \( n \)-dimensional real-valued vector space, \( \| \cdot \|_2 \) stands for vector Frobenius norm, \( \mathbb{S}^K \) stands for symmetric positive semidefinite \( K \times K \) matrices, which is a closed cone in \( \mathbb{R}^K \times \mathbb{K} \), \( \mathbb{S}^K_{++} \) denotes symmetric positive definite \( K \times K \) matrices. \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) stand for the largest and smallest eigenvalue of a matrix \( \Sigma \) respectively. \( \det(\cdot) \) and \( \text{Tr}(\cdot) \) stand for matrix determinant and trace respectively. \( \| \cdot \|_F \) is the matrix Frobenius norm.

The boundary of the open set \( \mathbb{S}^K_{++} \) is conventionally defined as \( \mathbb{S}^K_{++} \setminus \mathbb{S}^K_{++} \), which contains all rank deficient matrices in \( \mathbb{S}^K_{++} \). With a slightly abuse of notation, we also include matrices with all eigenvalues \( \lambda \to +\infty \) into the boundary of \( \mathbb{S}^K_{++} \). Therefore a sequence of matrices \( \Sigma^k \) converges to the boundary of \( \mathbb{S}^K_{++} \) iff \( \lambda_{\text{max}}^k \to +\infty \) or \( \lambda_{\text{min}}^k \to 0 \). In the rest of the paper, we will use the statement “\( \Sigma \) converges” equivalently as “a sequence of matrices \( \Sigma^k \) converges” for notation simplicity.

**II. Robust Covariance Matrix estimation**

In this paper, we assume a number \( N \) of \( K \)-dimensional samples \( \{x_1, \ldots, x_N\} \) are drawn from an elliptical population distribution with probability density function (pdf) of the form

\[
 f(x) = \det(\Sigma_0)^{-\frac{1}{2}} g((x - \mu_0)^T \Sigma_0^{-1} (x - \mu_0)) \quad (1)
\]

with location and scatter parameter \((\mu_0, \Sigma_0)\) in \( \mathbb{R}^K \times \mathbb{S}^K_{++} \). The nonnegative function \( g(\cdot) \), which is called the density generator, determines the shape of the pdf. In most of the popularly used distributions, e.g., the Gaussian and the Student’s \( t \)-distribution, \( g(\cdot) \) is a decreasing function and determines the decay of the tails of the distribution. Given \( \mu_0 \), our problem of interest is to estimate the covariance matrix. We can always center the pdf by defining \( \tilde{x} = x - \mu_0 \), hence without loss of generality in the rest of the paper we assume \( \mu_0 = 0 \). We use the notation \( P_N \) and \( f(\cdot) \) for the empirical and the population distributions, respectively. It is known that the covariance matrix of elliptical distribution takes the form \( c_g \Sigma_0 \) with \( c_g \) being a constant that depends on \( g(\cdot) \) [11], hence it is unlikely to have a good covariance estimator without prior knowledge of \( g \). In this paper, instead of trying to find the parametric form of \( g \) and get an estimator of \( c_g \Sigma_0 \), we are interested in estimating the normalized covariance matrix \( \Sigma_0^{-\frac{1}{2}} \cdot \Sigma_{\tilde{x}} \).

The commonly used sample covariance matrix, which also happens to be the maximum likelihood estimator for the normal distribution, estimates \( c_g \Sigma_0 \) asymptotically, however it is sensitive to outliers. This motivates the research for estimators robust to outliers in the data and, in fact, many researchers in the statistics literature have addressed this problem by proposing various robust covariance estimators like M-estimators [15], S-estimators [16], MVE [17], and MCD [18] to name a few, see [1], [2] for a complete overview. For example, in [15], Maronna analyzed the properties of the M-estimators, which are given as the solution \( \Sigma \) to the equation

\[
 \Sigma = \frac{1}{N} \sum_{i=1}^{N} (x_i^T \Sigma^{-1} x_i) x_i x_i^T \quad (2)
\]

where the choice of function \( u(\cdot) \) determines a whole family of different estimators. Under some technical conditions on \( u(s) \) (i.e., \( u(s) \geq 0 \) for \( s > 0 \) and nonincreasing, and \( su(s) \) is strictly increasing), Maronna proved that there exists a unique \( \Sigma \) that solves (2), and gave an iterative algorithm to arrive at that solution. He also established its consistency and robustness. A number of well known estimators take the form (2) and in [15] Maronna gave two examples, with one being the maximum likelihood estimator for multivariate Student’s \( t \)-distribution, and the other being the Huber’s estimator [19]. Both of them are popular for handling heavy tails and outliers in the data.

For all the robust covariance estimators, there is a tradeoff between their efficiency, which measures the variance (estimation accuracy) of the estimator, and robustness, which quantifies the sensitivity of the estimator to outliers. As these two quantities are opposed in nature, a considerable effort has to be put in designing estimators that achieve the right balance between these two quantities. In [3], Tyler dealt with this problem by proposing an estimator that is distribution-free and the “most robust” estimator in mini-max sense. Tyler’s estimator of \( \Sigma \) is given as the solution of the following equation

\[
 \Sigma = \frac{K}{N} \sum_{i=1}^{N} \frac{x_i x_i^T}{x_i^T \Sigma x_i} \quad (3)
\]
where the results of [15] cannot be applied since $s_u(s) = K$ is not strictly increasing. Tyler established the conditions for the existence of a solution to the fixed-point equation (3), as well as the fact that the estimator is unique up to a positive scaling factor, in the sense that $\Sigma$ solves (3) if and only if $c\Sigma$ solves (3) for some positive scalar $c$. The estimator was shown to be strongly consistent and asymptotically normal with its asymptotic standard deviation independent of $c$.

Tyler’s fixed-point equation (3) can be alternatively interpreted as follows. Consider the normalized samples defined as $s = \frac{x}{||x||_2^2}$, it is known that the probability distribution of $s$ takes the form.

$$f(s) = \frac{\Gamma \left( \frac{K}{2} \right)}{2\pi^{K/2}} \det(\Sigma)^{-\frac{K}{2}} (s^T \Sigma^{-1} s)^{-K/2}.$$ (4)

Given $N$ samples from the normalized distribution $\{s_i\}$, the maximum likelihood estimator of $\Sigma$ can be obtained by minimizing the negative log-likelihood function.

$$L(\Sigma) = \sum_{i=1}^N \frac{K}{2} \log (s_i^T \Sigma^{-1} s_i) + \frac{N}{2} \log \det(\Sigma).$$ (5)

which is equivalent to minimizing

$$L_{\text{Tyler}}(\Sigma) = \sum_{i=1}^N \frac{K}{2} \log (x_i^T \Sigma^{-1} x_i) + \frac{N}{2} \log \det(\Sigma).$$ (6)

If a minimum $\Sigma > 0$ of the function $L_{\text{Tyler}}(\Sigma)$ exists, it needs to satisfy the stationary equation given in (5), which was originally derived by Tyler in [3]. In [3], [21], the authors provided the condition for existence of a nonsingular solution to (3) based on the following reasoning. Notice that $\Sigma$ must be nonsingular, and the function $L_{\text{Tyler}}(\Sigma)$ is unbounded above on the boundary of positive definite matrices, implies the existence of a minimum. Based on these observations, Kent and Tyler established the existence conditions by showing $L_{\text{Tyler}}(\Sigma) \rightarrow +\infty$ on the boundary. Specifically, under the condition that: (i) no $x_i$ lies on the origin, and (ii) for any proper subspace $S \subseteq \mathbb{R}^K$, $P_N(S) < \frac{\dim(S)}{K}$, where $P_N(S) \equiv \frac{\sum_{i=1}^N 1_{x_i \in S}}{N}$ stands for the proportion of samples in $S$, then a nonsingular minimum of the problem (6) exists, which is equivalent to equation (3) having a solution. In words, the above mentioned conditions require the number of samples to be sufficiently large, and the samples should be spread out in the whole space.

To arrive at the estimator satisfying (3), Tyler proposed the following iterative algorithm:

$$\Sigma_{t+1} = \frac{K}{N} \sum_{i=1}^N \frac{x_i x_i^T}{x_i^T \Sigma_t^{-1} x_i},$$
$$\Sigma_{t+1} = \frac{\Sigma_{t+1}}{\text{Tr}(\Sigma_{t+1})}.$$ (7)

that converges to the unique (up to a positive scaling factor) solution of (3).

The robust property of Tyler’s estimator can be understood intuitively as follows: by normalizing the samples, i.e., $s = \frac{x}{||x||_2^2}$, the magnitude of an outlier is more unlikely to make the estimator break down. In other words, the estimator is not sensitive to the magnitude of samples, only their direction can affect the performance.

III. REGULARIZED COVARIANCE MATRIX ESTIMATION

The regularity conditions for the existence of Tyler’s estimator leads to a condition on the number of samples that $N \geq K + 1$ [21], [23]. In some practical applications the number of samples is not sufficient, in those cases Tyler’s iteration (7) may not converge. In these scenarios, a most sensible approach is to shrink Tyler’s estimator to some known a priori estimate of $\Sigma$. In the literature of robust estimators, there exists two different shrinkage based approaches.

In the first approach, the authors in [10], [12] proposed the following estimator:

$$\Sigma_{t+1} = \frac{1}{1 + \alpha_0 N} \sum_{i=1}^N \frac{x_i x_i^T}{x_i^T \Sigma_t^{-1} x_i} + \frac{\alpha_0}{1 + \alpha_0} I,$$
$$\Sigma_{t+1} = \frac{\Sigma_{t+1}}{\text{Tr}(\Sigma_{t+1})}.$$ (8)

which is a slightly modified version of the original Tyler’s iteration in (7), with the modification being including an identity matrix in the first step of the iteration that aims at shrinking the estimator towards the identity matrix. This resembles the idea of regularizing an estimator via diagonal loading [11], [24]. In [12], Chen et al. proved the uniqueness of the estimator obtained by the iteration (8) based on concave Perron-Frobenius theory, and gave a method to choose the regularization weight $\alpha_0$. Although this estimator is widely used and performs well in practice, it is still considered to be heuristic as it does not have an interpretation based on minimizing a cost function.

As a second approach, in [13], the author took a different route and derived a new shrinkage-based Tyler’s estimator that has a clear interpretation based on minimizing the penalized negative log-likelihood function

$$L_{\text{Wiesel}}(\Sigma) = \frac{2}{N} L_{\text{Tyler}}(\Sigma) + \alpha_0 h_{\text{target}}(\Sigma)$$ (9)

where $h_{\text{target}}(\Sigma) = K \log (\text{Tr}(\Sigma^{-1} T)) + \log \det(\Sigma)$ is a function with minimum at the desired target matrix $T$, hence it will shrink the solution of (9) towards the target. By showing the cost function $L_{\text{Wiesel}}(\Sigma)$ is geodesic convex, the author proved that any local minimum over the set of positive definite matrices is a global minimum [13]. He then derived an iterative algorithm based on majorization-minimization that monotonically decreases the cost function at each iteration:

$$\Sigma_{t+1} = \frac{1}{1 + \alpha_0 N} \sum_{i=1}^N \frac{x_i x_i^T}{x_i^T \Sigma_t^{-1} x_i} + \frac{\alpha_0}{1 + \alpha_0} \frac{K T}{\text{Tr}(\Sigma_t^{-1} T)}.$$ (10)

Even though the author in [13] showed that the cost function is convex in geodesic space, the existence and uniqueness of the global minimizer remains unknown. Moreover, it is mentioned in [13] that for some values of $\alpha_0$ the cost function becomes unbounded below and the iterations do not converge.

In this section, we address the following points: (i) we give the missing interpretation based on minimizing a cost function for the estimator in (3), and we also prove its existence and uniqueness; (ii) we prove the iteration in (10) with an additional trace normalization step converges to a unique point and also establish the conditions on the regularization parameter $\alpha_0$ to ensure the existence of the solution. For both
cases, the cost function takes the form of penalized negative
log-likelihood function with different penalizing functions.
Our methodology for the proofs hinges on techniques used
by Tyler in [21, 23].

We start with a proof of existence for a minimizer of a gen-
eral penalized negative log-likelihood function in the follow-
ing theorem, the proof of existence of the two aforementioned
cases $L_{\text{Tyler}}(\Sigma)$ and $L_{\text{Wiesel}}(\Sigma)$ are just special cases of the
general result.

The idea of proving the existence is to establish the regular-
arity conditions under which the cost function takes value $+\infty$
on the boundary of the set $\mathbb{S}_{++}^d$, a minimum then exists by the
continuity of the cost function. The main result is established
in Theorem 3 and the following lemma is needed.

**Lemma 1.** For any continuous function $f(\cdot)$ defined on the
set $\mathbb{S}_{++}^d$, there exists a $\Sigma > 0$ such that $f(\Sigma) \leq f(\Sigma)$ $\forall \Sigma > 0$
if $f(\Sigma) \rightarrow +\infty$ on the boundary of the set $\mathbb{S}_{++}^d$.

**Definition 2.** For any continuous function $f(s)$ defined on $s > 0$, define the quanti-
ties

\[ a_f = \sup \left\{ a | a^{s/2} \exp(-f(s)) \rightarrow 0 \text{ as } s \rightarrow +\infty \right\} \quad (11) \]

and

\[ a_f' = \inf \left\{ a | a^{s/2} \exp(-f(s)) \rightarrow 0 \text{ as } s \rightarrow 0 \right\} \quad (12) \]

In this paper we are particularly interested in the functions
$f(s) = c \log s$ and $f(s) = cs$ with some positive scalar $c < +\infty$. For $f(s) = c \log s$, $a_f = a_f' = 2c$, and for $f(s) = cs$, $a_f = +\infty$, $a_f' = 0$. We restrict our attention to the case $a_f \geq 0$.

Consider the penalized cost function takes the general form

\[ \tilde{L}(\Sigma) = L^\rho(\Sigma) + \varrho(\Sigma) \quad (13) \]

where $\varrho(\cdot)$ is a continuous function, and the penalty term

\[ \varrho(\Sigma) = a_D \log \det(\Sigma) + \sum_{i=1}^L \alpha_i h_i \left( \text{Tr} \left( \mathbf{A}_i^T \Sigma^{-1} \mathbf{A}_i \right) \right) \quad (14) \]

where $\text{Tr} \left( \mathbf{A}_i^T \Sigma^{-1} \mathbf{A}_i \right)$ measures the difference between $\Sigma$ and the positive semidefinite matrix $\mathbf{A}_i \mathbf{A}_i^T$. $h_i(\cdot)$ is, in gen-
eral, an increasing function that increases the penalty as $\Sigma$
deviates from $\mathbf{A}_i \mathbf{A}_i^T$, which is considered to be the prior target
that we wish to shrink $\Sigma$ to.

We first give an intuitive argument on the condition that
ensures the existence of the estimator. Since the estimator $\hat{\Sigma}$
is defined as the minimizer to the penalized loss function, it
exists if $\hat{L}(\Sigma) \rightarrow +\infty$ on the boundary of $\mathbb{S}_{++}^d$ by Lemma
1 and clearly $\hat{\Sigma}$ is nonsingular. We infer $\Sigma$ by the samples
$\{x_i\}$, if the samples are concentrated on some subspace,
naturally we “guess” the distribution is degenerate, i.e., $\Sigma$
is singular. Therefore, the samples are required to be suffi-
ciently spread out in the whole space so that the inference leads
to a nonsingular $\hat{\Sigma}$. Under the case when we have a prior
information that $\Sigma$ should be close to the matrix $\mathbf{A}_i \mathbf{A}_i^T$, to
ensure $\Sigma$ being nonsingular we need to distribute more $x_i$’s in
the null space of $\mathbf{A}_i \mathbf{A}_i^T$ and hence less in the range of $\mathbf{A}_i \mathbf{A}_i^T$
. To formalize this intuition, we give the following theorem.

**Theorem 3.** For cost function

\[ \tilde{L}(\Sigma) = \frac{N}{d} \log \det(\Sigma) + \sum_{i=1}^N \rho \left( x_i^T \Sigma^{-1} x_i \right) \]

\[ + \left( \alpha \log \det(\Sigma) + \sum_{i=1}^L \alpha_i h_i \left( \text{Tr} \left( \mathbf{A}_i^T \Sigma^{-1} \mathbf{A}_i \right) \right) \right) \quad (15) \]

defined on positive definite matrices $\Sigma > 0$ with $\rho(\cdot)$ and $h(\cdot)$
being continuous functions, define $a_D$ and $a_f$ for $\rho$, $a_1$ and $a_1$
for $\alpha_i h_i$’s according to (17) and (12), then $\tilde{L}(\Sigma) \rightarrow +\infty$ on
the boundary of the set $\mathbb{S}_{++}^d$ if the following conditions are
satisfied:

(i) no $x_i$ lies on the origin;

(ii) for any proper subspace $S$

\[ P_N(S) < \min \left\{ 1 - \frac{(N+2\alpha)(K - \dim(S)) - \sum_{i=1}^L a_i}{a_D N}, \right. \]

\[ \left. \frac{(N+2\alpha)(K - \dim(S)) - \sum_{i=1}^L a_i}{a_D N} \right\} \]

where sets $\omega$ and $\upsilon$ are defined as $\omega = \{ l | A_i \subset S \}$, $\upsilon = \{ l | A_i \not\subset S \}$.

(iii) $\left( \frac{a_D}{d} - a_1 \right) K + \frac{a_1}{d} N + \frac{1}{2} \sum a_1 < 0$ and $\frac{a_D}{d} N - \left( \frac{a_D}{d} + a_1 \right) K + \frac{1}{2} \sum a_1 > 0$.

**Proof:** See Appendix A.

**Remark 4.** Condition (i) avoids the scenario when $x_i^T \Sigma^{-1} x_i$
takes value 0 and $\rho(\cdot)$ is undefined at $s = 0$, for example
$\rho(\cdot) = \log(\cdot)$ for the log-likelihood function. The first part
in condition (ii), $P_N(S) < 1 - \frac{(N+2\alpha)(K - \dim(S)) - \sum_{i=1}^L a_i}{a_D N}$,
enures $\tilde{L}(\Sigma) \rightarrow +\infty$ under the case that some but not all eigenvalues $\lambda_j$ of $\Sigma$
tend to zero, and the second part in condition (ii), $P_N(S) < \frac{(N+2\alpha)(K - \dim(S)) - \sum_{i=1}^L a_i}{a_D N}$,
enures $\tilde{L}(\Sigma) \rightarrow +\infty$ under the case that some but not all eigenvalues $\lambda_j$ of $\Sigma$
tend to positive infinity. Together they force $\tilde{L}(\Sigma) \rightarrow +\infty$ when $\frac{a_D}{d}$ tends to 0. The first part of condition
(iii) ensures $\tilde{L}(\Sigma) \rightarrow +\infty$ when all $\lambda \rightarrow +\infty$ and the second part
ensures $\tilde{L}(\Sigma) \rightarrow +\infty$ when all $\lambda \rightarrow 0$.

**Corollary 5.** Assuming the population distribution $f(\cdot)$ is
continuous, and the matrices $A_i$ are full rank, condition (ii)
in Theorem 3 simplifies to:

\[ \left\{ \begin{array}{l}
\sum a_i - (N + 2\alpha)(K - d) > a_D (d - N) \\
\alpha > \frac{a_D - N}{2}
\end{array} \right\} \quad , \forall 1 \leq d \leq K - 1.
\]

**Proof:** The conclusion follows easily from the following
two facts: given that the population distribution $f(\cdot)$ is
continuous, and no $x_i$ lies on the origin, any $1 \leq d < K$ sample
points define a proper subspace $S$ with $\dim(S) = d$ with
probability one; and since $A_i$’s are full rank, the set $\omega = 0$.

Under the regularity conditions provided in Theorem 3
Lemma 1 implies a minimizer of $\hat{L}(\Sigma)$ exists and is
positive definite, therefore it needs to satisfy the condition
$\frac{\partial \hat{L}(\Sigma)}{\partial \Sigma} = 0$.

We then show how Theorem 3 works for Tyler’s esti-
imator defined as the nonsingular minimizer of [6]. Notice
that the loss function $L_{\text{Tyler}}(\Sigma)$ is scale-invariant, we have
\( L_{\text{Tyler}} (c \Sigma_0) = L_{\text{Tyler}} (\Sigma_0) = \text{constant for any positive definite } \Sigma_0 \). This implies that there are cases when \( \Sigma \) goes to the boundary of \( S_{++} \), and \( L_{\text{Tyler}} (\Sigma) \) will not go to positive infinity. Due to this reason, condition (iii) is violated in Theorem 3. To handle the scaling issue, we introduce a trace constraint \( \text{Tr}(\Sigma) = 1 \).

For the Tyler’s problem of minimizing (6), we seek for the condition that ensures \( L_{\text{Tyler}} (\Sigma) \to +\infty \) when \( \Sigma \) goes to the boundary of the set \( \{ \Sigma | \Sigma > 0, \text{Tr}(\Sigma) = 1 \} \). The condition implies that there is a unique minimizer \( \Sigma^* \) that minimizes \( L_{\text{Tyler}} (\Sigma) \) over the set \( \{ \Sigma | \Sigma > 0, \text{Tr}(\Sigma) = 1 \} \), and by it is equivalent to the existence of a unique (up to a positive scaling factor) minimizer \( \Sigma^* \) that minimizes \( L_{\text{Tyler}} (\Sigma) \) over the set \( S_{++} \) since \( L_{\text{Tyler}} (\Sigma) \) is scale-invariant.

The constraint \( \text{Tr}(\Sigma) = 1 \) excludes the case that any of \( \lambda_j \to +\infty \) and the case all \( \lambda_j \to 0 \), hence we only need to let \( L_{\text{Tyler}} (\Sigma) \to +\infty \) under the case that some but not all \( \lambda_j \to 0 \), which corresponds to the condition \( P_N (S) < 1 - \frac{a_{\text{opt}}}{N} (K - \text{dim}(S)) \) in Theorem 3. For Tyler’s cost function \( L_{\text{Tyler}} (\Sigma) \), we have \( \rho (s) = \frac{K}{2} \log s \) and \( \alpha = 0 \), \( \alpha_0 = \alpha_p = K \), therefore Theorem 3 leads to the condition on the samples: \( P_N (S) < \frac{\text{dim}(S)}{K} \), or \( N \geq K + 1 \) if the population distribution \( f (\cdot) \) is continuous, which reduces to the condition given in (17).

### A. Regularization via Wiesel’s penalty

In (13), Wiesel proposed a regularization penalty \( h (\Sigma) \) that results in a shrinkage estimator. Specifically, the penalty terms that encourage shrinkage towards an identity matrix and more generally towards an arbitrary prior matrix \( T \) are defined as follows:

- \( h_{\text{identity}} (\Sigma) = K \log (\text{Tr}(\Sigma^{-1})) + \log \det (\Sigma) \)
- \( h_{\text{target}} (\Sigma) = K \log (\text{Tr}(\Sigma^{-1}T)) + \log \det (\Sigma) \).

As can be seen the penalty terms are scale-invariant. Wiesel justified the choice of the above mentioned penalty functions by showing that the minimizer of the penalty functions would be some scaled multiple of \( I \) (or \( T \)). Thus adding this penalty terms to the Tyler’s cost function would yield estimators that are shrink towards \( I \) (or \( T \)). In the rest of this subsection we consider the general case \( h_{\text{target}} \) only, where the penalty term shrinks \( \Sigma \) to scalar multiples of \( T \), and we make the assumption that \( T \) is positive definite, which is reasonable since \( \Sigma \) must be a positive definite matrix. The cost function is restated below for convenience

\[
L_{\text{Wiesel}} (\Sigma) = \log \det (\Sigma) + K \sum_{i=1}^{N} \log (x_i^T \Sigma^{-1} x_i) + \alpha_0 (K \log (\text{Tr}(\Sigma^{-1}T)) + \log \det (\Sigma)).
\]

Minimizing \( L_{\text{Wiesel}} (\Sigma) \) gives the fixed-point condition

\[
\Sigma = \frac{1}{1 + \alpha_0} \frac{K \sum_{i=1}^{N} x_i x_i^T}{\text{Tr}(\Sigma^{-1}T)}. \tag{18}
\]

Recall that in the absence of regularization (i.e., \( \alpha_0 = 0 \)), a solution to the fixed-point equation exists under the condition \( P_N (S) < \frac{\text{dim}(S)}{N} \). With the regularization, however, it is not clear. We start giving a result for the uniqueness and then come back to the existence.

**Theorem 6.** If (17) has a solution, then it is unique up to a positive scaling factor.

**Proof:** It’s easy to see if \( \Sigma \) solves (13), \( c \Sigma \) is also a solution for \( c > 0 \). Without loss of generality assume \( \Sigma = I \) is a solution, otherwise define \( \hat{x}_i = \Sigma^{-\frac{1}{2}} x_i \), and \( T = \Sigma^{-\frac{1}{2}} T \Sigma^{-\frac{1}{2}} \), and that there exists another solution \( \Sigma_1 \). Denote the eigenvalues of \( \Sigma_1 \) as \( \lambda_1 \geq \cdots \geq \lambda_K \) with at least one strictly inequality, then under the condition that \( T \) is positive definite

\[
\Sigma_1 = \frac{1}{1 + \alpha_0} \frac{K \sum_{i=1}^{N} x_i x_i^T}{\text{Tr}(\Sigma^{-1}T)} + \alpha_0 (K \log (\text{Tr}(\Sigma^{-1}T)) + \log \det (\Sigma)).
\]

where the inequality follows from the fact that \( \text{Tr}(S \Sigma_1^{-1}) > \text{Tr}(\lambda_1^{-1} S) \) for any positive definite matrix \( S \) and the last equality follows from the assumption that \( I \) is a solution to (13). We have the contradiction \( \lambda_1 < \lambda_1 \), hence all the eigenvalues of \( \Sigma_1 \) should be equal, i.e., \( \Sigma_1 = \Sigma \).

Before establishing the existence condition, we give an example when the solution to (18) does not exist for illustration.

**Example 7.** Consider the case when all \( x_i \)'s are aligned in one direction. Eigendecompose \( \Sigma = U \Lambda U^T \) and choose \( u_1 \) to be aligned with the \( x_i \)'s, let \( \lambda_1 \to +\infty \) while others \( 0 < c \leq \lambda < +\infty \). Ignoring the constant terms, the boundedness of \( L_{\text{Wiesel}} (\Sigma) \) is equivalent to the boundedness of \( (1 + \alpha_0 - K) \log \lambda_1 \), hence it is unbounded below if \( \alpha_0 < K - 1 \).

The example shows that \( L_{\text{Wiesel}} (\Sigma) \) can be unbounded below implying that (13) has no solution if the data are too concentrated and \( \alpha_0 \) is small. The following theorems gives the exact tradeoff between data dispersion and the choice of \( \alpha_0 \).

**Theorem 8.** A unique solution to (18) exists (up to a positive scaling factor) if the following conditions are satisfied:

(i) no \( x_i \) lies on the origin; 
(ii) for any proper subspace \( S \subseteq \mathbb{R}^K \), \( P_N (S) < \frac{(1 + 2\alpha_0) \text{dim}(S)}{K} \), 
and they are the global minima of the loss function (17).

**Proof:** We start by rewriting the function including a the scaling factor \( \frac{N}{2} \) w.r.t. (17) for convenience:

\[
L_{\text{Wiesel}} (\Sigma) = \frac{N}{2} \log \det (\Sigma) + \frac{K}{2} \sum_{i=1}^{N} \log (x_i^T \Sigma^{-1} x_i) + \frac{\alpha_0}{2} (K \log (\text{Tr}(\Sigma^{-1}T)) + \log \det (\Sigma)).
\]

Invoke Theorem 3 with \( \rho (s) = \frac{K}{2} \log (s) \), \( h_1 (s) = K \log (s) \), \( \alpha = 0 \), \( \alpha_1 = 0 \), and \( A_1 = T^2 \), hence \( \alpha_p = \alpha_p = K \) and \( \alpha_1 = 2 \alpha K \). By the same reasoning as for the Tyler’s loss function, the condition \( P_N (S) < \frac{1}{(1 + 2\alpha_0) (K - \text{dim}(S)) - \sum_{i=1}^{\text{dim}(S)} \alpha_i} \), which is \( P_N (S) < \frac{(1 + 2\alpha_0) \text{dim}(S)}{K} \) since \( T \) is full rank, ensures
the existence of a unique solution to (18) under the constraint \( \Sigma > 0 \) and \( \text{Tr} (\Sigma) = 1 \). Hence a unique (up to a positive scaling factor) solution to (18) exists on the set of \( S^K_{++} \) by the scale-invariant property of \( L_{\text{Wiesel}} \) (\( \Sigma \)).

To make the existence condition checkable, we use Corollary 5 Theorem 8 then simplifies to \( \alpha_0 > \frac{N}{\lambda} - 1 \) or, equivalently \( N > \frac{\lambda}{\alpha_0 + \lambda} \), from which we can see that compared to the condition without regularization shrinkage allows less number of samples, and the minimum number depends on \( \alpha_0 \).

At last, we show that the condition \( P_N (S) < \frac{\dim(S)}{K} \) is also necessary in the following proposition.

**Proposition 9.** If (18) admits a solution on \( S^K_{++} \), then for any proper subspace \( S \subseteq \mathbb{R}^K \), \( P_N (S) < \frac{\dim(S)}{K} \), provided that \( T \) is positive definite and \( \alpha_0 > 0 \).

**Proof:** For a proper subspace \( S \), define \( P \) as the orthogonal projection matrix associated to \( S \), i.e., \( P x = x, \forall x \in S \). Assume the solution is \( I \). Multiplying both sides of equation (18) by matrix \( I - P \) and taking the trace we have

\[
K - \dim(S) = \frac{1}{1 + \alpha_0} \frac{K}{N} \sum_{i=1}^{N} x_i^T (I - P) x_i + \frac{\alpha_0 K}{1 + \alpha_0} \text{Tr} (T - TP) \quad \text{Tr} (T)
\]

If \( x_i \in S \), then \( x_i^T (I - P) x_i = 0 \), and \( x_i^T (I - P) x_i \leq x_i^T x_i \) if \( x_i \notin S \). Moreover, \( \text{Tr} (TP) > 0 \) since \( T \) is positive definite. This therefore implies

\[
K - \dim(S) < \frac{1}{1 + \alpha_0} \frac{K}{N} (N - NP_N (S)) + \frac{\alpha_0 K}{1 + \alpha_0} \text{Tr} (T) - \text{Tr} (T)
\]

Rearranging the terms yields

\[
P_N (S) < \frac{(1 + \alpha_0) \dim(S)}{K}.
\]

**B. Regularization via Kullback-Leibler Divergence Penalty**

An ideal penalty term should increase as \( \Sigma \) deviates from the prior target \( T \). Wiesel’s penalty function discussed in the last subsection satisfies this property and, in this subsection, we propose another penalty that has this property. The penalty that we choose is the KL divergence between \( N_T (\Sigma) \) and \( N_T (0, T) \), i.e., two zero-mean Gaussians with covariance matrices \( \Sigma \) and \( T \), respectively. The formula for the KL divergence is as follows (25), (26)

\[
D_{KL} (N_T || N_{\Sigma}) = \frac{1}{2} \left( \text{Tr} (\Sigma^{-1} T) - K - \log \left( \frac{\det(T)}{\det(\Sigma)} \right) \right)
\]

Ignoring the constant terms results in the following loss function:

\[
L_{KL} (\Sigma) = \log \det (\Sigma) + \frac{K}{2} \sum_{i=1}^{N} \log (x_i^T \Sigma^{-1} x_i) + \frac{\alpha_0}{2} \left( \text{Tr} (\Sigma^{-1} T) + \log \det (\Sigma) \right)
\]

with the following fixed-point condition:

\[
\Sigma = \frac{1}{1 + \alpha_0} \frac{K}{N} \sum_{i=1}^{N} x_i x_i^T + \frac{\alpha_0}{1 + \alpha_0} T.
\]

Unlike the penalty function discussed in the last subsection, KL divergence penalty encourages shrinkage towards \( T \) without scaling ambiguity. This can be easily seen, as the minimizer for the KL divergence penalty is just \( T \). Notice that (21) is similar to the diagonal loading in (8), but without the heuristic normalizing step.

**Theorem 10.** If (21) has a solution, then it is unique.

**Proof:** Without loss of generality, we assume \( \Sigma = I \) solves (21). Assume there is another matrix \( \Sigma_1 \) that solves (21), and denote the largest eigenvalue of \( \Sigma_1 \) as \( \lambda_1 \) and suppose \( \lambda_1 > 1 \). We then have the following contradiction:

\[
\Sigma_1 = \frac{1}{1 + \alpha_0} \frac{K}{N} \sum_{i=1}^{N} x_i x_i^T + \frac{\alpha_0}{1 + \alpha_0} T
\]

which gives contradiction \( \lambda_1 < \lambda_1 \), hence \( \lambda_1 \leq 1 \). Similarly, suppose the smallest eigenvalue of \( \Sigma_1 \) satisfies \( \lambda_K < 1 \). We then have

\[
\Sigma_1 = \frac{1}{1 + \alpha_0} \frac{K}{N} \sum_{i=1}^{N} x_i x_i^T + \frac{\alpha_0}{1 + \alpha_0} T + \frac{\alpha_0 \lambda_K}{1 + \alpha_0} T = \lambda_K I
\]

which is a contradiction and hence \( \lambda_K \geq 1 \), from which \( \Sigma_1 \) is I follows.

**Theorem 11.** A unique solution to (21) exists, if

(i) no \( x_i \) lies on the origin;

(ii) \( P_N (S) < \frac{(1 + \alpha_0) \dim(S)}{K} \)

and it is the global minimum of loss function (20).

**Proof:** Equivalently, we can define

\[
L^{KL} (\Sigma) = \frac{N}{2} \log \det (\Sigma) + \frac{K}{2} \sum_{i=1}^{N} \log (x_i^T \Sigma^{-1} x_i) + \frac{\alpha_0}{2} \left( \text{Tr} (\Sigma^{-1} T) + \log \det (\Sigma) \right)
\]

Invoke Theorem 3 with \( \rho (s) = \frac{K}{2} \log (s) \), \( h_1 (s) = s, \alpha = \frac{\alpha_1}{\alpha_0} N + \lambda_1 = T^{-2} \), hence \( \alpha_0 \rho = a_0 = K, \alpha_1 = +\infty, \alpha_1 = 0 \). Since \( T \) is full rank and \( a_1 = +\infty \), condition (ii) reduces to \( P_N (S) < \frac{(1 + \alpha_0) \dim(S)}{K} \). Condition (iii) is satisfied, hence an interior minimum exists. Furthermore, it is the unique minimum, hence it is global.

**Remark 12.** The only difference between the regularized estimator discussed in this subsection and the heuristic estimator in (8) is the extra normalizing step in (8). With the trace normalization, (12) proved that the iteration implied by (8) converges to a unique solution without any assumption of the data. However, the iteration implied by (21), which is based on minimizing a negative log-likelihood function penalized via the KL divergence function, requires some regularity conditions to be satisfied (cf. Theorem 11). According to Corollary 5 the condition simplifies to \( \alpha_0 > \frac{N}{\lambda} - 1 \) if the population distribution is continuous.

**Proposition 13.** If (21) admits a solution on \( S^K_{++} \), then for any proper subspace \( S \subseteq \mathbb{R}^K \), \( P_N (S) < \frac{(1 + \alpha_0) \dim(S)}{K} \), provided that \( T \) is positive definite and \( \alpha_0 > 0 \).
Proof: Multiply both sides of equation (21) by $T^{-\frac{1}{2}}$ and define $\Sigma = T^{-\frac{1}{2}} \Sigma T^{-\frac{1}{2}}$, $\hat{x}_i = T^{-\frac{1}{2}} x_i$ yields
\[
\tilde{\Sigma} = \frac{1}{1 + \alpha_0} \frac{K}{N} \sum_{i=1}^{N} \hat{x}_i \hat{x}_i^T + \frac{\alpha_0}{1 + \alpha_0} I. \quad (23)
\]
The rest of the proof follows the same token as Proposition 9.

Proposition 14. The solution to fixed point equation (21) solves (18) and, conversely, any solution of (18) solves (21) with a proper scaling factor.

Proof: If $\alpha_0$ is zero, the statement is trivial. We consider the case $\alpha_0 \neq 0$. Following the argument of previous proposition we arrive at equation (23). It has been shown in [14] that the unique solution $\Sigma$ to (23) satisfies $\text{Tr} (\Sigma^{-1}) = K$ given $\alpha_0 > 0$, hence $\text{Tr} (\Sigma^{-1} T) = K$. Substitute it into equation (18) yields exactly equation (21) with solution $\Sigma$, which indicates $\Sigma$ solves (18). The second part of the proposition follows from the fact that Wiesel’s fixed-point equation (18) has a unique solution up to a positive scaling factor.

IV. ALGORITHMS

Before going the specific algorithms, we briefly introduce the concepts of majorization-minimization [27], [28]. Consider the following optimization problem
\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \mathcal{X} \quad (24)
\end{align*}
\]
where $f(\cdot)$ is assumed to be a continuous function, not necessarily convex, and $\mathcal{X}$ is a closed convex set.

At a given point $x_i$, the majorization-minimization algorithm finds a surrogate function $g(x|x_i)$ that satisfies the following properties:
\[
\begin{align*}
&f(x_i) = g(x_i|x_i) \\
&f(x) \leq g(x|x_i) \quad \forall x \in \mathcal{X} \\
&f'(x_i; d) = g'(x_i; d|x_i) \quad \forall x_i + d \in \mathcal{X}
\end{align*}
\]
with $f'(x; d)$ stands for directional derivative. The surrogate function $g(x|x_i)$ is assumed to be continuous in $x$ and $x_i$.

The majorization-minimization algorithm updates $x$ as
\[
x_{i+1} = \arg \min_{x \in \mathcal{X}} g(x|x_i).
\]

It is proved that every limit point of the sequence $\{x_i\}$ converges to a stationary point of problem (24), and under the assumption that the level set $\{x : f(x) \leq f(x_0)\}$ is compact, the distance between $\{x_i\}$ and the set of stationary points reduces to zero in the limit [28].

In the rest of this section, for any continuous differentiable function $f(y)$, we define $f(y) = +\infty$ when $\lim_{x \to y} f(x) = +\infty$.

\[\text{Algorithm 1 Wiesel’s shrinkage estimator}\]

1) Initialize $\Sigma_0$ as an arbitrary positive definite matrix.
2) Do iteration
\[
\begin{align*}
\Sigma_{t+1} &= \frac{1}{1 + \alpha_0} \frac{K}{N} \sum_{i=1}^{N} x_i x_i^T + \frac{\alpha_0}{1 + \alpha_0} K T \\
\Sigma_{t+1} &= \frac{\Sigma_{t+1}}{\text{Tr}(\Sigma_{t+1})}
\end{align*}
\]
until convergence.

A. Regularization via Wiesel’s Penalty

In [13], Wiesel derived Tyler’s iteration [7] but without the trace normalization step, from the majorization-minimization perspective, with surrogate function $g(\Sigma | \Sigma_t)$ for (6) defined as
\[
g(\Sigma | \Sigma_t) = \frac{N}{2} \log \det (\Sigma) + \sum_{i=1}^{N} \frac{K}{2} x_i^T \Sigma_t^{-1} x_i + \text{const.} \quad (26)
\]

A positive definite stationary point of $g(\Sigma | \Sigma_t)$ satisfies the first equation of (7). By the same technique, to solve the problem
\[
\begin{align*}
\text{minimize} & \quad \log \det (\Sigma) + \sum_{i=1}^{N} \frac{K}{2} x_i^T \Sigma_t^{-1} x_i + \alpha_0 (K \log (\text{Tr}(\Sigma_t^{-1} T)) + \log \det (\Sigma)) \\
\text{subject to} & \quad \Sigma \succeq 0
\end{align*}
\]

Wiesel derived the iteration (10) by majorizing (17) with function
\[
(1 + \alpha_0) \log \det (\Sigma) + \frac{K}{N} \sum_{i=1}^{N} x_i^T \Sigma_t^{-1} x_i + \frac{\alpha_0 K}{\text{Tr}(\Sigma_t^{-1} T)} \text{Tr}(\Sigma_t^{-1} T) + \text{const.} \quad (28)
\]

It is worth pointing out that if we do the change of variable $\psi = \Sigma^{-1}$ in $L_{\text{Wiesel}}(\Sigma)$ and linearize the term $\log (x_i^T \Sigma^{-1} x_i)$, this also leads to the same iteration (10).

In the rest of this subsection, we prove the convergence of the iteration (10) proposed by Wiesel, but with an additional trace normalization step, i.e., our modified iteration takes the form:
\[
\begin{align*}
\Sigma_{t+1} &= \frac{1}{1 + \alpha_0} \frac{K}{N} \sum_{i=1}^{N} x_i x_i^T + \frac{\alpha_0}{1 + \alpha_0} K T \\
\Sigma_{t+1} &= \frac{\Sigma_{t+1}}{\text{Tr}(\Sigma_{t+1})}
\end{align*}
\]

Denote the set $S = \{\Sigma | \text{Tr}(\Sigma) = 1, \Sigma \succ 0\}$.

Lemma 15. The set $\lambda^0 = \{\Sigma | L_{\text{Wiesel}} (\Sigma) \leq L_{\text{Wiesel}} (\Sigma_0)\} \cap S$ is a compact set.

Proof: $\text{Tr}(\Sigma) = 1$ implies the set $\lambda^0$ is bounded. The set is closed follows easily from the fact that $L_{\text{Wiesel}}(\Sigma) \to +\infty$ when $\Sigma$ tends to be singular.

Lemma 16. The $\tilde{\Sigma}_{t+1}$ given in (29) is the unique minimizer to surrogate function (28).

Proof: For surrogate function (28), its value goes to positive infinity when $\frac{\lambda_{\text{min}}(\Sigma)}{\lambda_{\text{max}}(\Sigma)} \to +\infty$, since it majorizes
$L_{Wiesel}(\Sigma)$ and $L_{Wiesel}(\Sigma) \rightarrow +\infty$ in this case. Now consider the case when \( \lambda_{\min} = O(1) \). Define \( \Sigma = \Sigma_{K,K} \), then function \((28)\) can be rewritten as

\[
(1 + \alpha_0) \log \det (\Sigma) + K \log \lambda_{\min}
\]

\[
+ \frac{K}{N} \sum_{i=1}^{N} \lambda_{\min}^{-1} x_i^T \Sigma^{-1} x_i + \frac{\alpha_0 K}{\text{Tr} (\Sigma^{-1} T)} \text{Tr} \left( \lambda_{\min}^{-1} \Sigma^{-1} T \right) + \text{const}
\]

The terms $\log \det (\Sigma)$, $x_i^T \Sigma^{-1} x_i$, and $\text{Tr} (\Sigma^{-1} T)$ are all constants bounded away from both $0$ and $+\infty$. It is easy to see that when $\lambda_{\min} \rightarrow 0$ or $\lambda_{\min} \rightarrow +\infty$, \((30)\) goes to $+\infty$. Therefore we conclude that the value of Wiesel’s surrogate function \((28)\) goes to $+\infty$ when $\Sigma$ approaches the boundary of $S_{K,K}^+$. The fact that $\Sigma_{t+1}$ given in \((29)\) is the unique solution to the stationary equation implies that it is the unique minimizer of \((29)\) on the set $S_{K,K}^+$.

**Proposition 17.** The sequence \( \{\Sigma_t\} \) generated by Algorithm 7 converges to the global minimizer of problem \((29)\).

**Proof:** It is proved in Theorem 8 that under the conditions provided in Theorem 8 the minimizer $\Sigma$ for problem

\[
\text{minimize } \log \det (\Sigma) + \frac{K}{N} \sum_{i=1}^{N} \lambda_{\min}^{-1} x_i^T \Sigma^{-1} x_i \\
+ \alpha_0 \left( K \log \left( \text{Tr} (\Sigma^{-1} T) \right) + \log \det (\Sigma) \right)
\]

subject to $\text{Tr} (\Sigma) = 1$ exists and is unique, furthermore, it solves problem \((31)\). It is also proved that the objective function $L_{Wiesel}(\Sigma) \rightarrow +\infty$ on the boundary of the set $S$. We now show that the sequence \( \{\Sigma_t\} \) converges to unique minimizer of \((31)\).

Denote the surrogate function in general as $g(\Sigma|\Sigma_t)$, by Lemma 16 we therefore have the following inequality

\[
L_{Wiesel}(\Sigma_t) = g(\Sigma_t|\Sigma_t) \geq g(\Sigma_{t+1}|\Sigma_t)
\]

\[
\geq L_{Wiesel}(\Sigma_{t+1}) = L_{Wiesel}(\Sigma_{t+1}),
\]

which means \( \{L_{Wiesel}(\Sigma_t)\} \) is a non-increasing sequence.

Assume that there exists converging subsequence $\Sigma_{t_j} \rightarrow \Sigma_\infty$, then

\[
g(\Sigma_t|\Sigma_{t_j}) \geq g(\Sigma_{t_j+1}|\Sigma_{t_j}) \geq L_{Wiesel}(\Sigma_{t_j+1})
\]

\[
= L_{Wiesel}(\Sigma_{t_j+1}) \geq L_{Wiesel}(\Sigma_{t_{j+1}}) = g(\Sigma_{t_{j+1}}|\Sigma_{t_{j+1}}),
\]

$\forall \Sigma > 0$.

Letting $j \rightarrow +\infty$ results in

\[
g(\Sigma|\Sigma_\infty) \geq g(\Sigma_\infty|\Sigma_\infty) \forall \Sigma > 0,
\]

which implies that the directional derivative $L_{Wiesel}'(\Sigma_\infty; \Delta) \geq 0, \forall \Sigma_\infty + \Delta > 0$. The limit $\Sigma_\infty$ is nonsingular if $\Sigma_\infty$ is singular $L_{Wiesel}(\Sigma_\infty) = +\infty$, but $L_{Wiesel}(\Sigma_\infty) \leq L_{Wiesel}(\Sigma_0) < +\infty$ given that $\Sigma_0 > 0$, which is a contradiction. Since $\Sigma_{t_j} \rightarrow +\infty$ and the function is continuously differentiable, we have $\frac{\partial L_{Wiesel}(\Sigma)}{\partial \Sigma} |_{\Sigma_\infty} = 0$.

Since $\text{Tr} (\Sigma_\infty) = 1$, $\Sigma_\infty = \Sigma$.

The set $\mathcal{X}^0 = \{\Sigma|L(\Sigma) \leq L(\Sigma_0)\}$ is a compact set, and $\{\Sigma_t\}$ lies in this set, hence $\{\Sigma_t\}$ converges to $\Sigma$.

**Algorithm 2 KL divergence penalized shrinkage estimator**

1. Initialize $\Sigma_0$ as an arbitrary positive definite matrix.
2. Do iteration

\[
\Sigma_{t+1} = \frac{1}{1+\alpha_0} \frac{K}{N} \sum_{i=1}^{N} \frac{x_i x_i^T}{\Sigma_t^{-1} x_i^T} + \frac{\alpha_0}{1+\alpha_0} T
\]

until convergence.

**B. Regularization via Kullback-Leibler Penalty**

Following the same approach, for the KL divergence penalty problem:

\[
\text{minimize } \log \det (\Sigma) + \frac{K}{N} \sum_{i=1}^{N} \log (x_i^T \Sigma^{-1} x_i) + \alpha_0 \left( \text{Tr} (\Sigma^{-1} T) + \log \det (\Sigma) \right)
\]

subject to $\Sigma > 0$

We can majorize $L_{KL}(\Sigma)$ at $\Sigma_t$ by function

\[
(1 + \alpha_0) \log \det (\Sigma) + \frac{K}{N} \sum_{i=1}^{N} \frac{x_i^T \Sigma^{-1} x_i}{\Sigma_t^{-1} x_i} + \alpha_0 \left( \text{Tr} (\Sigma^{-1} T) + \log \det (\Sigma) \right)
\]

(33)

the stationary condition leads to the iteration

\[
\Sigma_{t+1} = \frac{1}{1+\alpha_0} \frac{K}{N} \sum_{i=1}^{N} \frac{x_i x_i^T}{\Sigma_t^{-1} x_i^T} + \frac{\alpha_0}{1+\alpha_0} T.
\]

Algorithm 2 summarizes the procedure for KL shrinkage estimator.

**Proposition 18.** The sequence \( \{\Sigma_t\} \) generated by Algorithm 2 converges to the global minimizer of problem \((32)\).

**Proof:** We verify the assumptions required for the convergence of algorithm \((28)\), namely \((25)\) and the compactness of initial level set $\mathcal{X}^0 = \{\Sigma|L_{KL}(\Sigma) < L_{KL}(\Sigma_0), \Sigma > 0\}$.

The first condition in \((25)\) is satisfied by construction. To verify the second condition, we see that the gradient of the surrogate function $g(\Sigma|\Sigma_t)$ has a unique zero. Since $g(\Sigma|\Sigma_t)$ is a global upperbound for $L_{KL}(\Sigma)$, $g(\Sigma|\Sigma_t) \rightarrow +\infty$ as $\Sigma$ goes to the boundary of $S_{+}^K$. By the continuity of $g(\Sigma|\Sigma_t)$, a minimizer $\Sigma^* > 0$ exists and has to satisfy $\frac{\partial g}{\partial \Sigma} = 0$. Therefore the unique zero has to be the global minimum, i.e., $\Sigma_{t+1} = \arg \min_{\Sigma > 0} g(\Sigma|\Sigma_t)$. The last condition is satisfied since $L_{KL}(\Sigma)$ is continuously differentiable on $S_{+}^K$.

It is proved in Theorems 10 and 11 that on set $S_{+}^K$, $L_{KL}(\Sigma)$ has a unique stationary point and it is the global minimum. Furthermore, the conditions in Theorem 11 ensures $L_{KL}(\Sigma) \rightarrow +\infty$ when $\Sigma$ goes to the boundary of $S_{+}^K$. The initial set $\mathcal{X}^0 = \{\Sigma|L_{KL}(\Sigma) < L_{KL}(\Sigma_0), \Sigma > 0\}$ is compact follows easily.

Therefore the sequence $\{\Sigma_t\}$ converges to the set of stationary points, hence the global minimum of problem \((32)\).

**C. Estimation with Structure Constraints**

In this subsection, we briefly discuss the covariance estimation problem with structure constraints. In general, the
uniqueness of the estimator cannot be guaranteed. However, algorithms can still be derived based on majorization-minimization when the constraint set $C$ is convex. In this case, we can majorize the objective functions $L^{\text{Wiesel}}(\Sigma)$ and $L^{\text{KL}}(\Sigma)$ by
\[
g^{\text{Wiesel}}(\Sigma|\Sigma_t) = (1 + \alpha_0) \text{Tr}(\Sigma_t^{-1}\Sigma) + \frac{K}{N} \sum_{i=1}^{N} \frac{x_i^T \Sigma_{t}^{-1} x_i}{\text{Tr}(\Sigma_{t}^{-1} T)} + \frac{\alpha_0 K}{\text{Tr}(\Sigma_{t}^{-1} T)}
\]
and
\[
g^{\text{KL}}(\Sigma|\Sigma_t) = (1 + \alpha_0) \text{Tr}(\Sigma_t^{-1}\Sigma) + \frac{K}{N} \sum_{i=1}^{N} \frac{x_i^T \Sigma_{t}^{-1} x_i}{\text{Tr}(\Sigma_{t}^{-1} T)} + \alpha_0 \text{Tr}(\Sigma_{t}^{-1} T)
\]
respectively, ignoring the constant term. Without any additional constraint, setting the gradient of $g(\cdot)$ to zero yields update
\[
\Sigma_{t+1} = \left( \Sigma_{t}^{-1} M_t \right)^{1/2} \left( \Sigma_{t}^{-1} M_t \right)^{1/2}
\]
where
\[
M_t^{\text{Wiesel}} = \frac{1}{1 + \alpha_0 N} \sum_{i=1}^{N} x_i x_i^T + \frac{\alpha_0}{1 + \alpha_0 N} K T
\]
and
\[
M_t^{\text{KL}} = \frac{1}{1 + \alpha_0 N} \sum_{i=1}^{N} x_i x_i^T + \frac{\alpha_0}{1 + \alpha_0 N} T.
\]
Notice that $M_t$ is exactly the update we derived by only majorizing the log $	ext{Tr}(\cdot)$ terms in the previous subsection, and $\Sigma_{t+1}$ is the geometric mean between matrices $\Sigma_t$ and $M_t$. Intuitively, $\Sigma_{t+1}$ can be viewed as a smoothed update of $\Sigma_t$.

However, when constrained, a closed-form solution for $\Sigma_{t+1}$ cannot be obtained in general. The surrogate function $g(\cdot)$ is convex since $\text{Tr}(\Sigma_{t}^{-1} \Sigma)$ is linear and $\text{Tr}(\Sigma_{t}^{-1} T)$ is convex, $\Sigma_{t+1} = \arg \min_{\Sigma \in C} g(\Sigma|\Sigma_t)$ can be found numerically if $C$ is convex. We consider two such examples.

1) Covariance Matrix with Toeplitz Structure: Toeplitz structure arises frequently in various signal processing related fields. For example, in time series analysis, the autocovariance matrix of a stationary process is Toeplitz. Imposing the Toeplitz structure on $\Sigma$ we need to solve
\[
\begin{align*}
\text{minimize} & \quad g(\Sigma|\Sigma_t) \\
\text{subject to} & \quad \Sigma \succeq 0 \\
& \quad \Sigma_{ij} = \Sigma_{i+1,j+1} \quad \forall i, j = 1, \ldots, K - 1
\end{align*}
\]
for each iteration. The additional constraint is linear.

2) Linear Additive Structure: Suppose $\Sigma$ can be decomposed as $\Sigma = S + \text{diag}(\sigma_1, \ldots, \sigma_K)$, where $S \succeq 0$ is signal covariance and $\text{diag}(\sigma_1, \ldots, \sigma_K)$ with $\sigma_i \in \mathcal{I}_i$ is noise covariance restricted to some interval. Then, at each iteration we solve
\[
\begin{align*}
\text{minimize} & \quad g(\Sigma|\Sigma_t) \\
\text{subject to} & \quad \Sigma \succeq 0 \\
& \quad S \succeq 0 \\
& \quad \Sigma = S + \text{diag}(\sigma_1, \ldots, \sigma_K) \\
& \quad \sigma_i \in \mathcal{I}_i.
\end{align*}
\]
The additional constraint is convex.

D. Parameter Tuning

A crucial issue in regularized covariance estimator is to choose the penalty parameter $\alpha_0$. We have shown that if the population distribution is continuous, for both Wiesel's penalty and KL divergence penalty, we require $\alpha_0 > \frac{K}{N} - 1$ to guarantee the existence of the regularized estimator.

There is a rich literature discussing the rules of parameter tuning developed for specific estimators. A standard way is to select $\alpha_0$ by cross-validation, method based on random matrix theory has also been investigated in a recent paper [30].

V. Numerical Results

In all of the simulations, the estimator performance is evaluated according to the criteria in [13], namely, the normalized mean-square error
\[
\text{NMSE} = \frac{E \left( \left\| \hat{\Sigma} - \Sigma^{\text{true}} \right\|_F^2 \right)}{\left\| \Sigma^{\text{true}} \right\|_F^2}
\]
where all matrices $\Sigma$ are all normalized by their trace. The expected value is approximated by 100 times Monte-Carlo simulations.

The first two simulations aims at illustrating the existence conditions for both Wiesel’s shrinkage estimator and KL shrinkage estimator. We choose $N = 8$ and $K = 10$ with the samples drawn a Gaussian distribution $N(0, \Sigma_0)$, where $\Sigma_0$ is a randomly generated positive definite covariance matrix. The shrinkage target $T$ is also an arbitrary positive definite matrix. According to the result in Section III, $\alpha_0 > \frac{K}{N} - 1$, i.e., $\alpha_0 > 0.25$, is the necessary and sufficient condition for the existence of a positive definite estimator. We simulate two scenarios with $\alpha_0 = 0.24$ and 0.26. Fig. 1 plots $\text{NMSE}$ with the condition number, namely $\alpha^{\text{min}}(\Sigma)$, as a function of the number of iterations in log-scale for Wiesel’s shrinkage estimator and with $\alpha_0 = 0.24$ (left) and $\alpha_0 = 0.26$ (right) respectively. Fig. 1 shows that for Wiesel’s shrinkage estimator, when $\alpha_0 = 0.24$ $\Sigma$ diverges, and when $\alpha_0 = 0.26$ $\Sigma$ converges to a nonsingular limit. Fig. 2 shows similar situation happens for KL shrinkage estimator.

For the rest of the simulations, the shrinkage parameter $\alpha_0$ is selected by grid search. That is, we define $\rho = \frac{1}{1 + \alpha_0}$ and enumerate $\rho$ uniformly on interval $(0, 1)$, and select the $\rho$ (equivalently $\alpha_0$) that gives the smallest error.

Fig. 3 demonstrates the performance of shrinkage Tyler’s estimator in the sample deficient case. The tuning parameter is selected to be the one that yields the smallest NMSE for each estimator as proposed in [13]. We choose the example
\[
\Sigma(\beta)_{ij} = |i-j|^{\beta}
\]
with $K = 30$. In this simulation, the underlying distribution is chosen to be a Student’s $t$-distribution with parameters $\mu_0 = 0$, $\Sigma_0 = \Sigma(0, 8)$, and $\nu = 3$, and the shrinkage target is set to be an identity matrix. The number of samples $N$ starts from 11 to 61. The curve corresponding to Tyler’s estimator.
of them exist, shrinkage estimators exist even when the existence conditions are not satisfied with $\alpha_0 = 0.24$, and (b) when the existence conditions are satisfied with $\alpha_0 = 0.26$.

Figure 1: Algorithm convergence of Wiesel’s shrinkage estimator: (a) when the existence conditions are not satisfied with $\alpha_0 = 0.24$, and (b) when the existence conditions are satisfied with $\alpha_0 = 0.26$.

Figure 2: Algorithm convergence of KL shrinkage estimator: (a) when the existence conditions are not satisfied with $\alpha_0 = 0.24$, and (b) when the existence conditions are satisfied with $\alpha_0 = 0.26$.

starts at $N = 31$ since the condition for Tyler’s estimator to exist is $N > K$, i.e., $N > 30$ in this case. The figure illustrates that both Tyler’s estimator and shrinkage Tyler’s estimator outperform the sample covariance matrix when all of them exist, shrinkage estimators exist even when $N \leq K$ and, moreover, achieve the best performance in all cases.

Fig. 4 and 5 compare the performance of different shrinkage Tyler’s estimators following roughly one of the simulation set-up in [13] for a fair comparison. The samples are drawn from a Student’s $t$-distribution with parameters $\mu_0 = 0$, $\Sigma_0 = \Sigma(0.8)$ and $\nu = 3$. The number of samples $N$ varies from 20 to 100. Fig. 4 shows the estimation error when setting $T = I$ and Fig. 5 shows that when setting $T = \Sigma(0.7)$, the searching step size of $\rho$ is set to be 0.01. The result indicates that estimation accuracy is increased due to shrinkage when the number of sample is not enough. Wiesel’s shrinkage estimator and KL shrinkage estimator yield the same NMSE. Interestingly, Chen’s shrinkage estimator gives roughly the same NMSE, although with a different shrinkage parameter $\alpha_0$. Chen’s and KL shrinkage estimator thus find their advantage in practice since an easier way of choosing $\alpha_0$ rather than cross-validation has been investigated in the literature [12], [30], a detailed comparison of them from random matrix theory perspective has also been provided in [30].

In both of the simulations, we include Tyler’s estimator with a Toeplitz structure constraint as introduced in the previous section. The figures show that the structure constrained estimator achieves relatively better performance than all other estimators both when shrinking to $I$ and shrinking to $T$. Although structure constraint can be imposed on shrinkage estimators to achieve potentially even smaller estimation error, we leave out this simulation due to the heavy computational cost introduced both by a lack of a closed-form solution per iteration (a SDP need to be solved numerically) and grid searching for the best regularization parameter. The problem of accelerating the algorithm and investigating the effect of imposing structure constraint on shrinkage estimator are left for future work.

Finally, the performance of Tyler’s estimator is tested on a real financial data set. We choose daily close prices $p_t$ from Jan 1, 2008 to July 31, 2011, 720 days in total, of $K = 45$ stocks from the Hang Seng Index provided by Yahoo Finance. The samples are constructed as $r_t = \log p_t - \log p_{t-1}$, i.e., the daily log-returns. The process $r_t$ is assumed to be stationary. The vector $r_t$ is constructed by stacking the log-returns of all $K$ stocks. $r_t$ that is close to 0 (all elements are less than $10^{-6}$) is discarded. We compare the performance of different covariance estimators in the minimum variance portfolio set

Figure 3: Illustration of the benefit of shrinkage estimators with $K = 30$ and shrinkage target matrix $I$.

Figure 4: Illustration of the benefit of shrinkage estimators with $K = 10$ and shrinkage target matrix $I$.
up, that is, we allocate the portfolio weights to minimize the overall risk. The problem can be formulated formally as

\[
\begin{align*}
\text{minimize} & \quad w^T \Sigma w \\
\text{subject to} & \quad 1^T w = 1 \\
\end{align*}
\]

(35)

with \( \Sigma \) being the covariance matrix of \( r_t \). Clearly the scaling of \( \Sigma \) does not affect the solution to this problem.

The simulation takes the following procedure. For non-shrinkage estimators, at day \( t \), we take the \( r_i \)'s with \( i \in [t - N^{\text{train}} - N^{\text{val}}, t - 1] \) as samples to estimate the normalized covariance matrix \( \Sigma \). For a particular shrinkage estimator, the target matrix is set to be \( I \) and the tuning parameter \( \rho \) is chosen as follows: for each value of \( \rho \in \{0.01, 0.02, \ldots, 1\} \), we calculate the shrinkage estimator \( \Sigma^\rho \) with samples \( r_i \), \( i \in [t - N^{\text{train}} - N^{\text{val}}, t - 1] \) and the corresponding \( w^\rho \) by solving (35). We then take the \( r_i \)'s with \( i \in [t - N^{\text{val}}, t - 1] \) as validation data and evaluate the variance of portfolio series \( \{(w^\rho)^T r_i\} \) in this period, the best \( \rho^* \) is chosen to be the one that yields the smallest variance. Finally the shrinkage estimator is obtained using samples \( r_i \) with \( i \in [t - N^{\text{train}} - N^{\text{val}}, t - 1] \) and tuning parameter \( \rho^* \). With the allocation strategy \( w \) for each of the estimators as the solution to (35), we construct portfolio for the next \( N^{\text{test}} \) days and collect the returns. The procedure is repeated every \( N^{\text{test}} \) days till the end and the variance of the portfolio constructed based on different estimators is calculated.

In the simulation, we choose \( N^{\text{val}} = N^{\text{test}} = 10 \) and vary \( N^{\text{train}} \) from 70 to 100. Fig. 6 compares the variance (risk) of portfolio constructed based on different estimators, with one additional baseline portfolio constructed by equal investment in each asset. From the figure we can see shrinkage estimators achieves relatively better performance than the nonshrinkage ones.

**VI. Conclusion**

In this work, we have given a rigorous proof for the existence and uniqueness of the regularized Tyler’s estimator proposed in [13], and justified the heuristic diagonal loading shrinkage estimator in [10] by KL divergence. Under the condition that samples are reasonably spread out, i.e., \( P_N(\Sigma) < (1 + a_0) \text{dim}(\Sigma) \), or \( N > \frac{K}{1 + a_0} \) if the underlying distribution is continuous, the estimators have been shown to exist and unique (up to a positive scaling factor for Wiesel’s estimator). Algorithms based on the majorization-minimization framework have also been provided with guaranteed convergence. Finally we have discussed structure constrained estimation and have shown in the simulation that imposing such constraint helps improving estimation accuracy.
\( x_i \) lies on the origin, we have \( \sum_{j=1}^{m} P_N(D_j) = P_N(S_m) \) and \( \sum_{j=m}^{m} P_N(D_j) = 1 - P_N(S_{m-1}) \).

Partition the samples \( x_i \) according to \( D_j \), \( j = 0 \) is excluded hereafter, define function
\[
G_j = \left\{ \begin{array}{ll}
\lambda_j \frac{1}{\alpha - \epsilon} \prod_{x_i \in D_j} g_p \left( x_i^T \Sigma^{-1} x_i \right) & \text{if } \exists x_i \in D_j \\
\frac{1}{\alpha - \epsilon} & \text{if } \forall x_i \in D_j
\end{array} \right.
\]
and we have \( G(\Sigma) = \prod_{j=1}^{K} G_j(\Sigma) \prod g_l \left( \sum_{j=1}^{K} \lambda_j^{-1} \left\| \hat{a}_{lj} \right\|^2 \right) \).

For the \( \lambda_j \)'s, denote \( \lambda_j = [\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{jm}] \). For each \( a_j \), there exists some \( D_j \) such that \( a_j \in D_j \), since the \( D_j \)'s partition the whole space. Define \( q_l \) to be the maximum index of \( D_j \) that the \( \lambda_j \)'s belongs to. Therefore we have \( \| \hat{a}_{lj} \| \neq 0 \) and \( \| \hat{a}_{lj} \| = 0 \) for \( j > q_l \).

We analyze the behavior of \( G(\Sigma) \) at the boundary of its feasible set \( S^{K+1}_k \), by Lemma 1 we only need to ensure \( G(\Sigma) \rightarrow 0 \), then there exists \( \tilde{L}(\Sigma) \leq \tilde{L}(\Sigma) \forall \Sigma > 0 \), and \( \Sigma > 0 \).

Consider the general case that some of the \( \lambda_j \)'s go to zero, some remain bounded away from both 0 and positive infinity, and the rests tend to positive infinity. Formally, define two integers \( r \) and \( s \) that \( 1 \leq r \leq s \leq K \), such that \( \lambda_j \rightarrow +\infty \) for \( j \in [r, s] \), \( \lambda_j \) is bounded for \( j \in (r, s) \) and \( \lambda_j \rightarrow 0 \) for \( j \in (s, K] \). Denote some arbitrary small positive quantity by \( \epsilon \).

First we analyze the terms \( G_j \) with \( \lambda_j \rightarrow 0 \). Consider the samples \( x_i \in D_h \) for some \( h \in (s, K] \), then \( x_i^T \Sigma^{-1} x_i = \sum_{j=1}^{h} \lambda_j^{-1} \left\| u_j^T u_j \right\|^2 \geq \lambda_h^{-1} \left\| u_h^T u_h \right\|^2 \), which is \( +\infty > (x_i^T \Sigma^{-1} x_i) \lambda_h > 0 \). Since \( \lambda_h \rightarrow 0 \), we have \( x_i^T \Sigma^{-1} x_i \rightarrow +\infty \). By definition
\[
\lim_{\lambda_h \rightarrow 0} g_p \left( x_i^T \Sigma^{-1} x_i \right) \left( x_i^T \Sigma^{-1} x_i \right)^{(\alpha_p - \epsilon)/2} = \lim_{\lambda_h \rightarrow 0} \left\{ g_p \left( x_i^T \Sigma^{-1} x_i \right) \lambda_h^{(\alpha_p - \epsilon)/2} \right\} = 0
\]
which implies \( \lim_{\lambda_h \rightarrow 0} g_p \left( x_i^T \Sigma^{-1} x_i \right) \lambda_h^{(\alpha_p - \epsilon)/2} = 0 \), i.e., \( g_p \left( x_i^T \Sigma^{-1} x_i \right) = o \left( \lambda_h^{\alpha_p - \frac{\epsilon}{2}} \right) \). Therefore, if \( x_i \in D_j \),
\[
g_p \left( x_i^T \Sigma^{-1} x_i \right) = o \left( \lambda_h^{\alpha_p - \frac{\epsilon}{2}} \right)
\]
For each \( G_j \) we have
\[
G_j = o \left( \lambda_j^{\alpha_p - \frac{\epsilon}{2}} \prod_{\lambda_j} \sum_{j=1}^{m} P_N(D_j) - \lambda_j - \epsilon \right) \forall j \geq 1.
\]
In the second step, we analyze the terms \( G_j \) with \( \lambda_j \rightarrow +\infty \). Consider the samples \( x_i \in D_h \) for some \( h \in [1, r] \), we have shown that \( 0 < (x_i^T \Sigma^{-1} x_i) \lambda_h < +\infty \). Since \( \lambda_h \rightarrow +\infty \), \( (x_i^T \Sigma^{-1} x_i) \rightarrow 0 \). Given that \( \alpha_p > -\infty \),
\[
g_p \left( x_i^T \Sigma^{-1} x_i \right) = o \left( \varphi_h^{\alpha_p - \epsilon} \right)
\]
Therefore for each \( G_j \) we have
\[
G_j = o \left( \varphi_j^{\alpha_p - \epsilon} \prod_{\lambda_j} \sum_{j=1}^{m} P_N(D_j) - \lambda_j - \epsilon \right) \forall j \leq r.
\]
For the \( G_j \) with \( \lambda_j \) being some constant, it is easy to see that \( g_p \left( x_i^T \Sigma^{-1} x_i \right) = O(1) \), which does not affect the order of \( G(\Sigma) \).

Now we have characterized the \( g_j \)'s, we move to the \( g_l \)'s. Since \( \| \hat{a}_{lj} \| \neq 0 \) and \( \| \hat{a}_{lj} \| = 0 \) for \( j \geq q_l \), by the same reasoning above, \( g_l = o \left( \varphi_{q_l}^{-\epsilon} \right) \) if \( q_l \leq r \) and \( g_l = o \left( \lambda_{q_l}^{\alpha_p - \epsilon} \right) \) if \( q_l \geq r + 1 \). Therefore
\[
G(\Sigma) = \prod_{j=1}^{K} G_j(\Sigma) \prod g_l \left( \sum_{j=1}^{K} \lambda_j^{-1} \left\| \hat{a}_{lj} \right\|^2 \right) = \prod_{j=1}^{K} o \left( \varphi_j^{\epsilon} N \prod_{j=1}^{K} P_N(D_j) - \lambda_j - \epsilon \right).
\]
with \( \prod_{l(1 \leq l \leq r)} \) defined to be 1 if the set \( \{ l \mid q_l \geq s + 1 \} \) is empty, and the same for \( \prod_{l(1 \leq l \leq r)} \).

We make the following assumption:
\[
\left( \frac{N}{2} + \alpha + \epsilon \right) m - \frac{C}{2} N \sum_{j=1}^{m} P_N(D_j) - \sum_{q_l \leq m} \frac{C}{2} \geq 0, \forall 1 \leq m \leq r
\]
\[
\frac{C}{2} \cdot N \sum_{j=1}^{K} P_N(D_j) \left( \frac{N}{2} + \alpha + \epsilon \right) - \sum_{q_l \geq m} \frac{C}{2} \geq 0, \forall K \geq m \geq s + 1
\]
by the order \( \lambda_1 \geq \cdots \geq \lambda_K \) hence \( \varphi_1 \leq \cdots \leq \varphi_K \), and base on the fact that
\[
o \left( \lambda_1^{\alpha_p} \right) o \left( \lambda_2^{\alpha_p} \right) = o \left( \lambda_1^{\alpha_p + \alpha_p} \right) \text{ if } \alpha_p \geq 0
\]
\[
o \left( \varphi_1^{\alpha_p} \right) o \left( \varphi_2^{\alpha_p} \right) = o \left( \varphi_2^{\alpha_p} \right) \text{ if } \alpha_p \geq 0
\]
the order of \( G(\Sigma) \) is
\[
G(\Sigma) = o \left( \frac{\varphi_r^{\alpha_p} \cdot N \sum_{j=1}^{K} P_N(D_j) - \left( \frac{N}{2} + \alpha + \epsilon \right) (K-s-1)}{\sum_{q_l \geq m} \frac{C}{2}} \right).
\]
and it goes to zero.

Now we simplify the assumption [36]. Since \( \sum_{j=1}^{m} P_N(D_j) = P_N(S_m) \) and \( \sum_{j=m}^{m} P_N(D_j) = 1-P_N(S_{m-1}) \), and \( r, s \) can take any value that satisfies \( 1 \leq r \leq s \leq K \leq 1 \), we end up with the following condition:
\[
\left( \frac{N}{2} + \alpha + \epsilon \right) d - \frac{\alpha_p^{\epsilon}}{\lambda} \sum_{d=1}^{K} P_N(D_d) - \sum_{q_l \leq d} \frac{C}{2} \geq 0
\]
\[
\frac{C}{2} \cdot N (1 - P_N(S_d)) - \left( \frac{N}{2} + \alpha + \epsilon \right) (K-d) + \sum_{q_l \geq d+1} \frac{C}{2} \geq 0
\]
for all \( 1 \leq d \leq K-1 \).

Define sets \( \omega = \{ l \mid q_l < d \} \) and \( \nu = \{ l \mid q_l > d \} \), consider when \( l \in \omega \), which means \( q_l \leq d \), by the definition of \( q_l \), is equivalent to range \( (A_l) \subseteq S_d \), similarly for \( l \in \nu \), which means \( q_l > d \), is equivalent to range \( (A_l) \nsubseteq S_d \).

The condition should be valid for any \( U \) and \( 1 \leq d \leq K-1 \), tidy up the expression and let \( \epsilon \rightarrow 0 \) results in: for any proper
subspace $S$

$$P_N(S) < \min \left\{ 1 - \frac{(N+2\alpha)(K - \dim(S)) - \sum_{i=1}^{\rho(N)} \alpha_l}{\alpha_p N} \right\},$$

where sets $\omega$ and $\nu$ are defined as $\omega = \{|l|\text{range}(A_l) \subseteq S\}$, $\nu = \{|l|\text{range}(A_l) \not\subseteq S\}$.

For the case $r = 0, 1 \leq s < K$, which means no $\lambda \to +\infty$ and some, not all $\lambda \to 0$ and some, not all $\lambda \to +\infty$, gives condition

$$P_N(S) < 1 - \frac{(N+2\alpha)(K - \dim(S)) - \sum_{i=1}^{\rho(N)} \alpha_l}{\alpha_p N}$$

and for $s = K, 1 \leq r < K$, which means no $\lambda \to 0$ and some, not all $\lambda \to +\infty$, gives condition

$$P_N(S) < \frac{(N+2\alpha)\dim(S) - \sum_{i=1}^{\rho(N)} \alpha_l}{\alpha_p N}$$

Notice that the above two conditions are included in the first one.

And finally under the scenario that all $\lambda \to +\infty$, it’s easy to see $G(\Sigma) = o \left( \frac{1}{N} \sum_{i=1}^{\rho(N)} (\alpha_l + \epsilon) \right)$ goes to zero if $(-\frac{N^2}{\alpha} - \alpha)K + \frac{a}{2} N + \frac{1}{2} \sum_{\rho(N)} \alpha_l < 0$, and under the case that all $\lambda \to 0$, $G(\Sigma) = o \left( \frac{1}{N} \sum_{i=1}^{\rho(N)} (\alpha_l - \epsilon) \right)$ goes to zero if $\frac{a}{2}N - \frac{1}{2} \sum_{\rho(N)} \alpha_l > 0$.

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