HÖLDER GRADIENT REGULARITY FOR THE INHOMOGENEOUS NORMALIZED $p(x)$-LAPLACE EQUATION

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Abstract. We prove the local gradient Hölder regularity of viscosity solutions to the inhomogeneous normalized $p(x)$-Laplace equation

$$-\Delta^N_{p(x)} u = f(x),$$

where $p$ is Lipschitz continuous, $\inf p > 1$, and $f$ is continuous and bounded.

1. Introduction

We study the inhomogeneous normalized $p(x)$-Laplace equation

$$-\Delta^N_{p(x)} u = f(x) \quad \text{in } B_1, \quad (1.1)$$

where

$$-\Delta^N_{p(x)} u := -\Delta u - \langle p(x) - 2, \frac{D^2 u Du, Du}{|Du|^2} \rangle$$

is the normalized $p(x)$-Laplacian, $p : B_1 \to \mathbb{R}$ is Lipschitz continuous, $1 < p_{\min} := \inf_{B_1} p \leq \sup_{B_1} p =: p_{\max}$ and $f \in C(B_1)$ is bounded. Our main result is that viscosity solutions to (1.1) are locally $C^{1,\alpha}$-regular.

Normalized equations have attracted a significant amount of interest during the last 15 years. Their study is partially motivated by their connection to game theory. Roughly speaking, the value function of certain stochastic tug-of-war games converges uniformly up to a subsequence to a viscosity solution of a normalized equation as the step-size of the game approaches zero [PS08, MPR10, MPR12, BG15, BR19]. In particular, a game with space-dependent probabilities leads to the normalized $p(x)$-Laplace equation [AHP17] and games with running pay-offs lead to inhomogeneous equations [Ruo16]. In addition to game theory, normalized equations have been studied for example in the context of image processing [Doe11, ETT15].

The variable $p(x)$ in (1.1) has an effect that may not be immediately obvious: If we formally multiply the equation by $|Du|^{p(x)-2}$ and rewrite it in a divergence form, then a logarithm term appears and we arrive at the expression

$$- \text{div}(|Du|^{p(x)-2} Du) + |Du|^{p(x)-2} \log(|Du|) Du \cdot Dp = |Du|^{p(x)-2} f(x). \quad (1.2)$$

For $f \equiv 0$, this is the so called strong $p(x)$-Laplace equation introduced by Adamowicz and Hästö [AH10, AH11] in connection with mappings of finite distortion. In the homogeneous case viscosity solutions to (1.1) actually coincide with weak solutions of (1.2) [Sil18], yielding the $C^{1,\alpha}$-regularity of viscosity solutions as a consequence of a result by Zhang and Zhou [ZZ12].

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In the present paper our objective is to prove $C^{1,\alpha}$-regularity of solutions to (1.1) directly using viscosity methods. The Hölder regularity of solutions already follows from existing general results, see [KS79, KS80, Caf89, CC95]. More recently, Imbert and Silvestre [IS12] proved the gradient Hölder regularity of solutions to the elliptic equation

$$|Du|^{\gamma} F(D^2u) = f,$$

where $\gamma > 0$ and Imbert, Jin and Silvestre [JS17, IJS19] obtained a similar result for the parabolic equation

$$\partial_t u = |Du|^{\gamma} \Delta^N_p u,$$

where $p > 1$, $\gamma > -1$. Furthermore, Attouchi and Parviainen [AP18] proved the $C^{1,\alpha}$-regularity of solutions to the inhomogeneous equation $\partial_t u - \Delta^N_p u = f(x,t)$. Our proof of Hölder gradient regularity for solutions of (1.1) is in particular inspired by the papers [JS17] and [AP18].

We point out that recently Fang and Zhang [FZ21a] proved the $C^{1,\alpha}$-regularity of solutions to the parabolic normalized $p(x,t)$-Laplace equation

$$\partial_t u = \Delta^N_{p(x,t)} u,$$

where $p \in C^{1}_{\text{loc}}$. The equation (1.3) naturally includes (1.1) if $f \equiv 0$. However, in this article we consider the inhomogeneous case and only suppose that $p$ is Lipschitz continuous. More precisely, we have the following theorem.

**Theorem 1.1.** Suppose that $p$ is Lipschitz continuous in $B_1$, $p_{\min} > 1$ and $f \in C(B_1)$ is bounded. Let $u$ be a viscosity solution to

$$-\Delta^N_{p(x)} u = f(x) \quad \text{in } B_1.$$

Then there is $\alpha(N, p_{\min}, p_{\max}, p_L) \in (0,1)$ such that

$$\|u\|_{C^{1,\alpha}(B_1/2)} \leq C(N, p_{\min}, p_{\max}, p_L, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}),$$

where $p_L$ is the Lipschitz constant of $p$.

The proof of Theorem 1.1 is based on suitable uniform $C^{1,\alpha}$-regularity estimates for solutions of the regularized equation

$$-\Delta v - (p_{\varepsilon}(x) - 2) \frac{\langle D^2v Dv, Dv \rangle}{|Dv|^2 + \varepsilon^2} = g(x),$$

where it is assumed that $g$ is continuous and $p_{\varepsilon}$ is smooth. In particular, we show estimates that are independent of $\varepsilon$ and only depend on $N$, $\sup p$, $\inf p$, $\|Dp_{\varepsilon}\|_{L^\infty}$ and $\|g\|_{L^\infty}$. To prove such estimates, we first derive estimates for the perturbed homogeneous equation

$$-\Delta v - (p_{\varepsilon}(x) - 2) \frac{\langle D^2v(Dv + q), Dv + q \rangle}{|Dv|^2 + \varepsilon^2} = 0,$$

where $q \in \mathbb{R}^N$. Roughly speaking, $C^{1,\alpha}$-estimates for solutions of (1.5) are based on “improvement of oscillation” which is obtained by differentiating the equation and observing that a function depending on the gradient of the solution is a supersolution to a linear equation. The uniform $C^{1,\alpha}$-estimates for solutions of (1.5) then yield uniform estimates for the inhomogeneous equation (1.4) by an adaption of the arguments in [SI12, AP18].

With the a priori regularity estimates at hand, the plan is to let $\varepsilon \to 0$ and show that the estimates pass on to solutions of (1.1). A problem is caused by the
fact that, to the best of our knowledge, uniqueness of solutions to (1.1) is an open problem for variable $p(x)$ and even for constant $p$ if $f$ is allowed to change signs. To deal with this, we fix a solution $u_0 \in C(\overline{B}_1)$ to (1.1) and consider the Dirichlet problem

$$-\Delta^N_{p(x)} u = f(x) - u_0(x) - u \quad \text{in } B_1$$

with boundary data $u = u_0$ on $\partial B_1$. For this equation the comparison principle holds and thus $u_0$ is the unique solution. We then consider the approximate problem

$$-\Delta u_\varepsilon - (p_\varepsilon(x) - 2) \frac{\langle D^2 u_\varepsilon, Du_\varepsilon \rangle}{|Du_\varepsilon|^2 + \varepsilon^2} = f_\varepsilon(x) - u_{0,\varepsilon}(x) - u_\varepsilon$$

with boundary data $u_\varepsilon = u_0$ on $\partial B_1$ and where $p_\varepsilon, f_\varepsilon, u_{0,\varepsilon} \in C^\infty(B_1)$ are such that $p \to p_\varepsilon$, $f \to f_\varepsilon$ and $u_{0,\varepsilon} \to u_0$ uniformly in $B_1$ and $\|Dp_\varepsilon\|_{L^\infty(B_1)} \leq \|Dp\|_{L^\infty(B_1)}$. As the equation (1.7) is uniformly elliptic quasilinear equation with smooth coefficients, the solution $u_\varepsilon$ exists in the classical sense by standard theory. Since $u_\varepsilon$ also solves (1.4) with $g(x) = f_\varepsilon(x) - u_{0,\varepsilon}(x) - u_\varepsilon(x)$, it satisfies the uniform $C^{1,\alpha}$-regularity estimate. We then let $\varepsilon \to 0$ and use stability and comparison principles to show that $u_0$ inherits the regularity estimate.

For other related results, see for example the works of Attouchi, Parviainen and Ruosteenoja [APR17] on the normalized $p$-Poisson problem $-\Delta^N_{p(x)} u = f$, Attouchi and Ruosteenoja [AR18] [AR20] [Att20] on the equation $-|Du|^r \Delta^N_{p(x)} u = f$ and its parabolic version, De Filippis [DF21] on the double phase problem $(|Du|^r + a(x)|Du|^s) F(D^2 u) = f(x)$ and Fang and Zhang [FZ21b] on the parabolic double phase equation $\partial_t u = (|Du|^r + a(x,t)|Du|^s) \Delta^N_{p(x)} u$. We also mention the paper by Bronzi, Pimentel, Rampasso and Teixeira [BPR17] where they consider fully nonlinear variable exponent equations of the type $|Du|^{p(x)} F(D^2 u) = 0$.

The paper is organized as follows: Section 2 is dedicated to preliminaries, Sections 3 and 4 contain $C^{1,\alpha}$-regularity estimates for equations (1.5) and (1.7), and Section 5 contains the proof of Theorem (1.1). Finally, the Appendix contains an uniform Lipschitz estimate for the equations studied in this paper and a comparison principle for equation (1.6).

2. Preliminaries

2.1. Notation. We denote by $B_R \subset \mathbb{R}^N$ an open ball of radius $R > 0$ that is centered at the origin in the $N$-dimensional Euclidean space, $N \geq 1$. The set of symmetric $N \times N$ matrices is denoted by $S^N$. For $X, Y \in S^N$, we write $X \preceq Y$ if $X - Y$ is negative semidefinite. We also denote the smallest eigenvalue of $X$ by $\lambda_{\min}(X)$ and the largest by $\lambda_{\max}(X)$ and set

$$\|X\| := \sup_{\xi \in \mathcal{B}_1} |X\xi| = \sup \{ |\lambda| : \lambda \text{ is an eigenvalue of } X \}.$$ 

We use the notation $C(a_1, \ldots, a_k)$ to denote a constant $C$ that may change from line to line but depends only on $a_1, \ldots, a_k$. For convenience we often use $C(\hat{p})$ to mean that the constant may depend on $p_{\min}$, $p_{\max}$ and the Lipschitz constant $p_L$ of $p$.

For $\alpha \in (0, 1)$, we denote by $C^\alpha(B_R)$ the set of all functions $u : B_R \to \mathbb{R}$ with finite Hölder norm

$$\|u\|_{C^\alpha(B_R)} := \|u\|_{L^\infty(B_R)} + [u]_{C^\alpha(B_R)}, \quad \text{where } [u]_{C^\alpha(B_R)} := \sup_{x, y \in B_R} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$
Similarly, we denote by $C^{1,\alpha}(B_R)$ the set of all functions for which the norm
\[ \|u\|_{C^{1,\alpha}(B_R)} := \|u\|_{C^0(B_R)} + \|Du\|_{C^0(\partial B_R)} \]
is finite.

2.2. Viscosity solutions. Viscosity solutions are defined using smooth test functions that touch the solution from above or below. If $u, \varphi : \mathbb{R}^N \to \mathbb{R}$ and $x \in \mathbb{R}^N$ are such that $\varphi(x) = u(x)$ and $\varphi(y) < u(y)$ for $y \neq x_0$, then we say that $\varphi$ touches $u$ from below at $x_0$.

**Definition 2.1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution to
\[ -\Delta^N_{p(x)} u \geq f(x, u) \quad \text{in } \Omega \]
if the following holds: Whenever $\varphi \in C^2(\Omega)$ touches $u$ from below at $x \in \Omega$ and $D\varphi(x) \neq 0$, we have
\[ -\Delta \varphi(x) - (p(x) - 2) \frac{\langle D^2 \varphi(x) D\varphi(x), D\varphi(x) \rangle}{|D\varphi(x)|^2} \geq f(x, u(x)) \]
and if $D\varphi(x) = 0$, then
\[ -\Delta \varphi(x) - (p(x) - 2) \langle D^2 \varphi(x) \eta, \eta \rangle \geq f(x, u(x)) \quad \text{for some } \eta \in B_1. \]

Analogously, a lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution if the above inequalities hold reversed whenever $\varphi$ touches $u$ from above. Finally, we say that $u$ is a viscosity solution if it is both viscosity sub- and supersolution.

**Remark.** The special treatment of the vanishing gradient in Definition 2.1 is needed because of the singularity of the equation. Definition 2.1 is essentially a relaxed version of the standard definition in [CIL92] which is based on the so called semicontinuous envelopes. In the standard definition one would require that if $\varphi$ touches a viscosity supersolution $u$ from below at $x$, then
\[
\begin{cases}
-\Delta^N_{p(x)} \varphi(x) \geq f(x, u(x)) & \text{if } D\varphi(x) \neq 0, \\
-\Delta \varphi(x) - (p(x) - 2) \lambda_{\min}(D^2 \varphi(x)) \geq f(x, u(x)) & \text{if } D\varphi(x) = 0 \text{ and } p(x) \geq 2, \\
-\Delta \varphi(x) - (p(x) - 2) \lambda_{\max}(D^2 \varphi(x)) \geq f(x, u(x)) & \text{if } D\varphi(x) = 0 \text{ and } p(x) < 2.
\end{cases}
\]
Clearly, if $u$ is a viscosity supersolution in this sense, then it is also a viscosity supersolution in the sense of Definition 2.1.

3. Hölder gradient estimates for the regularized homogeneous equation

In this section we prove $C^{1,\alpha}$-regularity estimates for solutions to the equation
\[ -\Delta u - (p(x) - 2) \frac{\langle D^2 u(Du + q), Du + q \rangle}{|Du + q|^2 + \varepsilon^2} = 0 \quad \text{in } B_1, \quad (3.1) \]
where $p : B_1 \to B_1$ is Lipschitz, $p_{\min} > 1$, $\varepsilon > 0$ and $q \in \mathbb{R}^N$. Our objective is to obtain estimates that are independent of $q$ and $\varepsilon$. Observe that (3.1) is a uniformly elliptic quasilinear equation with smooth coefficients. Viscosity solutions to (3.1) can be defined in the standard way and they are smooth if $p$ is smooth.

**Proposition 3.1.** Suppose that $p$ is smooth. Let $u$ be a viscosity solution to (3.1) in $B_1$. Then $u \in C^\infty(B_1)$.
HÖLDER GRADIENT REGULARITY

It follows from classical theory that the corresponding Dirichlet problem admits a smooth solution (see [GT01] Theorems 15.18 and 13.6 and the Schauder estimates [GT01] Theorem 6.17). The viscosity solution $u$ coincides with the smooth solution by a comparison principle [KK98, Theorem 3].

3.1. Improvement of oscillation. Our regularity estimates for solutions of (3.1) are based on improvement of oscillation. We first prove such a result for the linear equation

$$- \text{tr}(G(x)D^2u) = f \quad \text{in} \ B_1, \quad (3.2)$$

where $f \in C^1(B_1)$ is bounded, $G(x) \in S^N$ and there are constants $0 < \lambda < \Lambda < \infty$ such that the eigenvalues of $G(x)$ are in $[\lambda, \Lambda]$ for all $x \in B_1$. The result is based on the following rescaled version of the weak Harnack inequality found in [CC95, Theorem 4.8]. Such Harnack estimates for non-divergence form equations go back to at least Krylov and Safonov [KS79, KS80].

**Lemma 3.2** (Weak Harnack inequality). Let $u \geq 0$ be a continuous viscosity supersolution to (3.2) in $B_1$. Then there are positive constants $C(\lambda, \Lambda, N)$ and $q(\lambda, \Lambda, N)$ such that for any $\tau < \frac{1}{4\sqrt{N}}$ we have

$$\tau^{-\frac{q}{N}} \left( \int_{B_{\tau}} |u|^q \, dx \right)^{1/q} \leq C \left( \inf_{B_{2\tau}} u + \tau \left( \int_{B_{4\sqrt{N}\tau}} |f|^N \, dx \right)^{1/N} \right). \quad (3.3)$$

**Proof.** Suppose that $\tau < \frac{1}{4\sqrt{N}}$ and set $S := 8\tau$. Define the function $v : B_{\sqrt{N}/2} \rightarrow \mathbb{R}$ by

$$v(x) := u(Sx)$$

and set

$$\tilde{G}(x) := G(Sx) \quad \text{and} \quad \tilde{f}(x) := S^2 f(Sx).$$

Then, if $\varphi \in C^2$ touches $v$ from below at $x \in B_{\sqrt{N}/2}$, the function $\phi(x) := \varphi(x/S)$ touches $u$ from below at $Sx$. Therefore

$$-\text{tr}(G(Sx)D^2\phi(Sx)) \geq f(Sx).$$

Since $D^2\phi(Sx) = S^{-2}D^2\varphi(x)$, this implies that

$$-\text{tr}(G(Sx)D^2\varphi(x)) \geq S^2 f(Sx).$$

Thus $v$ is a viscosity supersolution to

$$-\text{tr}(\tilde{G}(x)D^2v) \geq \tilde{f}(x) \quad \text{in} \ B_{\sqrt{N}/2}.$$ 

We denote by $Q_R$ a cube with side-length $R/2$. Since $Q_1 \subset B_{\sqrt{N}/2}$, it follows from [CC95] Theorem 4.8 that there are $q(\lambda, \Lambda, N)$ and $C(\lambda, \Lambda, N)$ such that

$$\left( \int_{B_{1/8}} |v|^q \, dx \right)^{1/q} \leq \left( \int_{Q_{1/4}} |v|^q \, dx \right)^{1/q} \leq C \left( \inf_{Q_{1/2}} v + \left( \int_{Q_1} |\tilde{f}|^N \, dx \right)^{1/N} \right) \leq C \left( \inf_{B_{1/4}} v + \left( \int_{B_{\sqrt{N}/2}} |\tilde{f}|^N \, dx \right)^{1/N} \right).$$
By the change of variables formula we have
\[
\int_{B_{1/8}} |v|^q \, dx = \int_{B_{1/8}} |u(Sx)|^q \, dx = S^{-N} \int_{B_{2/8}} |u(x)|^q \, dx
\]
and
\[
\int_{B_{\sqrt{\pi}/2}} |\tilde{f}|^N \, dx = S^{2N} \int_{B_{\sqrt{\pi}/2}} |f(Sx)|^N \, dx = S^N \int_{B_{S\sqrt{\pi}/2}} |f(x)|^N \, dx.
\]
Recalling that \( S = 8\tau \), we get
\[
8^{-\frac{N}{q}} \frac{N}{q} \left( \int_{B_{2\tau}} |u(x)|^q \, dx \right)^{1/q} \leq C \left( \inf_{B_{2\tau}} u + 8\tau \left( \int_{B_{S\sqrt{\pi}/2}} |f(x)|^N \, dx \right)^{1/N} \right).
\]
Absorbing \( 8^{-\frac{N}{q}} \) into the constant, we obtain the claim. \( \square \)

**Lemma 3.3** (Improvement of oscillation for the linear equation). Let \( u \geq 0 \) be a continuous viscosity supersolution to (3.2) in \( B_1 \) and \( \mu, l > 0 \). Then there are positive constants \( \tau(\lambda, \Lambda, N, \mu, l, \|f\|_{L^\infty(B_1)}) \) and \( \theta(\lambda, \Lambda, N, \mu, l) \) such that if
\[
|\{ x \in B_\tau : u \geq l \}| > \mu |B_\tau|, \tag{3.4}
\]
then we have
\[
u \geq \theta \text{ in } B_\tau.
\]

**Proof.** By the weak Harnack inequality (Lemma 3.2) there exist constants \( C_1(\lambda, \Lambda, N) \) and \( q(\lambda, \Lambda, N) \) such that for any \( \tau < 1/(4\sqrt{N}) \), we have
\[
\inf_{B_{2\tau}} u \geq C_1 \tau^{-\frac{N}{q}} \left( \int_{B_{\tau}} |u|^q \, dx \right)^{1/q} - \tau \left( \int_{B_{4\sqrt{\pi}\tau}} |f|^N \, dx \right)^{1/N}. \tag{3.5}
\]
In particular, this holds for
\[
\tau := \min \left( \frac{1}{4\sqrt{N}}, \frac{\sqrt{C_1 |B_1|^{\frac{1}{q}} \mu^{\frac{1}{q}} l}}{2 \cdot 4\sqrt{N} (\|f\|_{L^\infty(B_1)} + 1)} \right).
\]
We continue the estimate (3.5) using the assumption (3.4) and obtain
\[
\inf_{B_{2\tau}} u \geq \inf_{B_{2\tau}} u \geq C_1 \tau^{-\frac{N}{q}} (|\{ x \in B_\tau : u \geq l \}| l^{\frac{q}{q-1}})^{1/q} - \tau \left( \int_{B_{4\sqrt{\pi}\tau}} |f|^N \, dx \right)^{1/N}
\]
\[
\geq C_1 \tau^{-\frac{N}{q}} \mu^{\frac{1}{q}} l^\frac{1}{q} \cdot \tau^{\frac{1}{q}} \cdot \|f\|_{L^\infty(B_1)}
\]
\[
= C_1 |B_1|^{\frac{1}{q}} \mu^{\frac{1}{q}} l^\frac{1}{q} \tau^{\frac{N}{q}} - 4\sqrt{N} \cdot |B_1|^{\frac{1}{q}} \|f\|_{L^\infty(B_1)} \tau^2
\]
\[
= C_1 |B_1|^{\frac{1}{q}} \mu^{\frac{1}{q}} l - 4\sqrt{N} \cdot |B_1|^{\frac{1}{q}} \|f\|_{L^\infty(B_1)} \tau^2.
\]
\[
\geq \frac{1}{2} C_1 |B_1|^{\frac{1}{q}} \mu^{\frac{1}{q}} l,
\]
where the last inequality follows from the choice of \( \tau \). \( \square \)
We are now ready to prove an improvement of oscillation for the gradient of a solution to (3.1). We first consider the following lemma, where the improvement is considered towards a fixed direction. We initially also restrict the range of $|q|$.

The idea is to differentiate the equation and observe that a suitable function of $Du$ is a supersolution to the linear equation (3.2). Lemma 3.3 is then applied to obtain information about $Du$.

**Lemma 3.4 (Improvement of oscillation to direction).** Suppose that $p$ is smooth. Let $u$ be a smooth solution to (3.1) in $B_1$ with $|Du| \leq 1$ and either $q = 0$ or $|q| > 2$. Then for every $0 < l < 1$ and $\mu > 0$ there exist positive constants $\tau(N,\hat{p},l,\mu) < 1$ and $\gamma(N,\hat{p},l,\mu) < 1$ such that

$$|\{x \in B_{\tau} : Du \cdot d \leq l\}| > \mu |B_{\tau}| \quad \text{implies} \quad Du \cdot d \leq \gamma \quad \text{in} \quad B_{\tau}$$

whenever $d \in \partial B_1$.

**Proof.** To simplify notation, we set

$$A_{ij}(x,\eta) := \delta_{ij} + (p(x) - 2)\frac{(\eta_i + q_i)(\eta_j + q_j)}{|\eta + q|^2 + \epsilon^2}.$$

We also denote the functions $A_{ij} : x \mapsto A_{ij}(x,Du(x))$, $A_{ij,\eta} : x \mapsto (\partial_\eta A_{ij})(x,Du(x))$ and $A_{ij,\eta_k} : x \mapsto (\partial_{\eta_k} A_{ij})(x,Du(x))$. Then, since $u$ is a smooth solution to (3.1) in $B_1$, we have in Einstein’s summation convention

$$-A_{ij} u_{ij} = 0 \quad \text{pointwise in} \quad B_1.$$

Differentiating this yields

$$0 = (A_{ij} u_{ij})_k = A_{ij} u_{ijk} + (A_{ij})_k u_{ij} = A_{ij} u_{ijk} + A_{ij,\eta_m} u_{ij} u_{km} + A_{ij,\eta_k} u_{ij} \quad \text{for all} \quad k = 1, \ldots N. \quad (3.6)$$

Multiplying these identities by $d_k$ and summing over $k$, we obtain

$$0 = A_{ij} u_{ijk} d_k + A_{ij,\eta_m} u_{ij} u_{km} d_k + A_{ij,\eta_k} u_{ij} d_k = A_{ij}(Du \cdot d - l)_{ij} + A_{ij,\eta_m} u_{ij} (Du \cdot d - l)_m + A_{ij,\eta_k} u_{ij} d_k. \quad (3.7)$$

Moreover, multiplying (3.6) by $2u_k$ and summing over $k$, we obtain

$$0 = 2A_{ij} u_{ijk} u_k + 2A_{ij,\eta_m} u_{ij} u_{km} u_k + 2A_{ij,\eta_k} u_{ij} u_k = A_{ij}(u_k^2)_{ij} - 2A_{ij,\eta_k} u_{ij} u_k = A_{ij}(Du^2)_{ij} + A_{ij,\eta_m} u_{ij} (Du^2)_m + 2A_{ij,\eta_k} u_{ij} u_k = A_{ij}(Du^2)_{ij} + A_{ij,\eta_m} u_{ij} (Du^2)_m + 2A_{ij,\eta_k} u_{ij} u_k - 2A_{ij} u_{ij} u_k. \quad (3.8)$$

We will now split the proof into the cases $q = 0$ or $|q| > 2$, and proceed in two steps: First we check that a suitable function of $Du$ is a supersolution to the linear equation (3.3) and then apply Lemma 3.3 to obtain the claim.

**Case $q = 0$, Step 1:** We denote $\Omega_+ := \{x \in B_1 : h(x) > 0\}$, where

$$h := (Du \cdot d - l + \frac{l}{2} |Du|^2)^+.$$

If $|Du| \leq l/2$, we have

$$Du \cdot d - l + \frac{l}{2} |Du|^2 \leq -\frac{l}{2} + \frac{l^3}{8} < 0.$$
This implies that $|Du| > l/2$ in $\Omega_+$. Therefore, since $q = 0$, we have in $\Omega_+$

$$
|A_{ij,\eta_m}| = |p(x)| - 2\left| \frac{\delta_{ij}(u_j + q_j) + \delta_{jm}(u_i + q_i)}{|Du + q|^2 + \epsilon^2} - \frac{2(u_m + q_m)(u_i + q_i)(u_j + q_j)}{(|Du + q|^2 + \epsilon^2)^2} \right| 
\leq 8l^{-1} \|p - 2\|_{L^\infty(B_1)}.
$$

(3.9)

Summing up the equations (3.7) and (3.8) multiplied by $2^{-1}l$, we obtain in $\Omega_+$

$$
0 = A_{ij}(Du \cdot d - l)_{ij} + A_{ij,\eta_m}u_{ij}(Du \cdot d - l)_{m} + A_{ij,x_k}u_{ij}d_k
+ 2^{-1}l(A_{ij}(|Du|^2)_{ij} + A_{ij,\eta_m}u_{ij}(|Du|^2)_{m} + 2A_{ij,x_k}u_{ij}u_{k} - 2A_{ij}u_{kj}u_{ki})
= A_{ij}h_{ij} + A_{ij,\eta_m}u_{ij}h_{m} + A_{ij,x_k}u_{ij}d_k + lA_{ij,x_k}u_{ij}u_{k} - lA_{ij}u_{kj}u_{ki}
\leq A_{ij}h_{ij} + |A_{ij,\eta_m}| |h_{m}| + |A_{ij,x_k}| |u_{ij}||d_k + lu_k| - lA_{ij}u_{kj}u_{ki}.
$$

Since $|Du| \leq 1$, we have $|d_k + lu_k|^2 \leq 4$ and by uniform ellipticity $A_{ij}u_{kj}u_{ki} \geq \min(p_{\min} - 1, 1)|u_{ij}|^2$. Therefore, by applying Young’s inequality with $\epsilon > 0$, we obtain from the above display

$$
0 \leq A_{ij}h_{ij} + N^2 \epsilon^{-1}(|h_{m}|^2 + |d_k + lu_k|^2) + \epsilon(|A_{ij,\eta_m}|^2 + |A_{ij,x_k}|^2)|u_{ij}|^2 - lA_{ij}u_{kj}u_{ki}
\leq A_{ij}h_{ij} + N^2 \epsilon^{-1}(|Dh|^2 + 4) + \epsilon C(N, \hat{p})(l^2 + 1)|u_{ij}|^2 - l\min(p_{\min} - 1, 1)|u_{ij}|^2,
$$

where in the second estimate we used (3.9) and (3.10). By taking $\epsilon$ small enough, we obtain

$$
0 \leq A_{ij}h_{ij} + C_0(N, \hat{p})\frac{|Dh|^2}{l^3} + \frac{1}{l^3} \text{ in } \Omega_+,
$$

(3.11)

Next we define

$$
\overline{h} := \frac{1}{\nu}(1 - e^{\nu(h - H)}), \text{ where } H := 1 - \frac{l}{2} \text{ and } \nu := \frac{C_0}{l^3\min(p_{\min} - 1, 1)}.
$$

(3.12)

Then by (3.11) and uniform ellipticity we have in $\Omega_+

$$
-A_{ij}\overline{h}_{ij} = A_{ij}(h_{ij}e^{\nu(h - H)} + \nu h_{ij}e^{\nu(h - H)})
\geq e^{\nu(h - H)}(-C_0\frac{|Dh|^2}{l^3} - \frac{C_0}{l^3} + \nu \min(p_{\min} - 1, 1)|Dh|^2)
\geq -\frac{C_0}{l^3}.
$$

Since the minimum of two viscosity supersolutions is still a viscosity supersolution, it follows from the above estimate that $\overline{h}$ is a non-negative viscosity supersolution to

$$
-A_{ij}\overline{h}_{ij} \geq -\frac{C_0}{l^3} \text{ in } B_1.
$$

(3.13)

Case $q = 0$, Step 2: We set $l_0 := \frac{1}{\nu}(1 - e^{\nu(l - l)})$. Then, since $\overline{h}$ solves (3.13), by Lemma 3.3 there are positive constants $\tau(N, p, l, \mu)$ and $\theta(N, p, l, \mu)$ such that

$$
|\{x \in B_\tau : \overline{h} \geq l_0\}| > \mu |B_\tau| \text{ implies } \overline{h} \geq \theta \text{ in } B_\tau.
$$

If $Du \cdot d \leq l$, we have $\overline{h} \geq l_0$ and therefore

$$
|\{x \in B_\tau : \overline{h} \geq l_0\}| \geq |\{x \in B_\tau : Du \cdot d \leq l\}| > \mu |B_\tau|,
$$

where

$$
|\{x \in B_\tau : Du \cdot d \leq l\}| > \mu |B_\tau|.
$$

8  JARKKO SILTAKOSI
where the last inequality follows from the assumptions. Consequently, we obtain
\[ H \geq \theta \quad \text{in } B_{r}. \]

Since \( h - H \leq 0 \), by convexity we have \( H - h \geq \overline{h} \). This together with the above estimate yields
\[ 1 - 2^{-1}l - (Du \cdot d - l + 2^{-1}l|Du|^2) \geq \theta \quad \text{in } B_{r} \]
and so
\[ Du \cdot d + 2^{-1}l(Du \cdot d)^2 \leq Du \cdot d + 2^{-1}l|Du|^2 \leq 1 + 2^{-1}l - \theta \quad \text{in } B_{r}. \]

Using the quadratic formula, we thus obtain the desired estimate
\[ Du \cdot d \leq \frac{-1 + \sqrt{1 + 2l(1 + 2^{-1}l - \theta)}}{l} = \frac{-1 + \sqrt{(1 + l)^2 - 2l\theta}}{l} =: \gamma < 1 \quad \text{in } B_{r}. \]

**Case** \( |q| > 2 \): Computing like in (3.7) and (3.10), we obtain this time in \( B_{1} \)
\[ |A_{ij,\eta m}| \leq 4 \|p - 2\|_{L^\infty(B_{1})} \quad \text{and} \quad |A_{ij,x_k}| \leq p_{L} \]
Moreover, this time we set simply
\[ h := Du \cdot d - l + 2^{-1}l|Du|^2. \]

Summing up the identities (3.7) and (3.8) and using Young’s inequality similarly as in the case \( |q| = 0 \), we obtain in \( B_{1} \)
\[ 0 \leq A_{ij}h_{ij} + N^{2}\epsilon^{-1}(h_{m}^2 + d_{k} + lu_{k}^2) + \epsilon(|A_{ij,\eta m}|^2 + |A_{ij,x_k}|^2)|u_{ij}|^2 - lA_{ij}u_{kj}u_{ki} \leq A_{ij}h_{ij} + N^{2}\epsilon^{-1}(Dh^2 + 4) + \epsilon C(\hat{p})|u_{ij}|^2 - lC(\hat{p})|u_{ij}|^2. \]

By taking small enough \( \epsilon \), we obtain
\[ 0 \leq A_{ij}h_{ij} + C_{0}(N, \hat{p}) \frac{|Dh|^2 + 1}{l} \quad \text{in } B_{1}. \]

Next we define \( \overline{h} \) and \( H \) like in (3.12), but set instead \( \nu := C_{0}/(l \min(p_{\min} - 1, 1)). \)
The rest of the proof then proceeds in the same way as in the case \( q = 0 \). \( \Box \)

Next we inductively apply the previous lemma to prove the improvement of oscillation.

**Theorem 3.5** (Improvement of oscillation). *Suppose that \( p \) is smooth. Let \( u \) be a smooth solution to (3.7) in \( B_{r} \) with \( |Du| \leq 1 \) and either \( q = 0 \) or \( |q| > 2 \). Then for every \( 0 < l < 1 \) and \( \mu > 0 \) there exist positive constants \( \tau(N, \hat{p}, l, \mu) < 1 \) and \( \gamma(N, \hat{p}, l, \mu) < 1 \) such that if
\[ \{|x \in B_{r+1} : Du \cdot d \leq l\gamma^i\} \geq \mu |B_{r+1}| \quad \text{for all } d \in \partial B_{1}, \ i = 0, \ldots, k, \quad (3.14) \]
then
\[ |Du| \leq \gamma^{i+1} \quad \text{in } B_{r+1} \quad \text{for all } i = 0, \ldots, k. \quad (3.15) \]

*Proof.* Let \( k \geq 0 \) be an integer and suppose that (3.14) holds. We proceed by induction.

**Initial step:** Since (3.14) holds for \( i = 0 \), by Lemma 3.4 we have \( Du \cdot d \leq \gamma \) in \( B_{r} \) for all \( d \in \partial B_{1} \). This implies (3.15) for \( i = 0 \).

**Induction step:** Suppose that \( 0 < i \leq k \) and that (3.15) holds for \( i - 1 \). We define
\[ v(x) := \tau^{-i}\gamma^{-i}u(\tau^ix). \]
Then \( v \) solves
\[
-\Delta v - (p(\tau^i x) - 2) \frac{\langle D^2 v(Dv + \gamma^{-i} q), Dv + \gamma^i q \rangle}{|Dv + \gamma^{-i} q|^2 + (\gamma^{-i} \varepsilon)^2} = 0 \quad \text{in } B_1.
\]
Moreover, by induction hypothesis \(|Dv(x)| = \gamma^{-i} |Du(\tau^i x)| \leq \gamma^{-i} \gamma^i = 1 \) in \( B_1 \).
Therefore by Lemma 3.4 we have that
\[
\{ x \in B_t : Du \cdot d \leq \mu \} > \mu |B_t| \quad \text{implies } Du \cdot d \leq \gamma \quad \text{in } B_t
\]
whenever \( d \in \partial B_1 \). Since
\[
\{ x \in B_t : Du \cdot d \leq \mu \} \iff \{ x \in B_{t+1} : Du \cdot d \leq \mu \} > \mu |B_{t+1}|
\]
we have by (3.14) and (3.16) that \( Du \cdot d \leq \gamma \) in \( B_t \). This implies that \( Du \cdot d \leq \gamma^{i+1} \) in \( B_{t+1} \). Since \( d \in \partial B_1 \) was arbitrary, we obtain (3.15) for \( i \).

3.2. Hölder gradient estimates. In this section we apply the improvement of oscillation to prove \( C^{1, \alpha} \)-estimates for solutions to (3.1). We need the following regularity result by Savin [Sav07].

**Lemma 3.6.** Suppose that \( p \) is smooth. Let \( u \) be a smooth solution to (3.1) in \( B_1 \) with \( |Du| \leq 1 \) and either \( q = 0 \) or \(|q| > 2 \). Then for any \( \beta > 0 \) there exist positive constants \( \eta(N, \hat{p}, \beta) \) and \( C(N, \hat{p}, \beta) \) such that if
\[
|u - L| \leq \eta \quad \text{in } B_1
\]
for some affine function \( L \) satisfying \( 1/2 \leq |DL| \leq 1 \), then we have
\[
|Du(x) - Du(0)| \leq C \left| \frac{x}{4} \right|^{\beta} \quad \text{for all } x \in B_{1/2}.
\]

**Proof.** Set \( v := u - L. \) Then \( v \) solves
\[
-\Delta v - \frac{(p(x) - 2) \langle D^2 u(Du + q + DL), Du + q + DL \rangle}{|Du + q + DL|^2 + \varepsilon^2} = 0 \quad \text{in } B_1. \quad (3.17)
\]
Observe that by the assumption \( 1/2 \leq |DL| \leq 1 \) we have \(|Du + q + DL| \geq 1/4 \) if \(|Du| \leq 1/4 \). It therefore follows from [Sav07, Theorem 1.3] (see also [Wan13]) that \( \|v\|_{C^{2, \beta}(B_{1/2})} \leq C \) which implies the claim. \( \square \)

We also use the following simple consequence of Morrey’s inequality.

**Lemma 3.7.** Let \( u : B_1 \to \mathbb{R} \) be a smooth function with \( |Du| \leq 1 \). For any \( \theta > 0 \) there are constants \( \varepsilon_1(N, \theta), \varepsilon_0(N, \theta) < 1 \) such that if the condition
\[
\{ x \in B_1 : |Du - d| > \varepsilon_0 \} \leq \varepsilon_1
\]
is satisfied for some \( d \in S^{N-1} \), then there is a \( a \in \mathbb{R} \) such that
\[
|u(x) - a - d \cdot x| \leq \theta \quad \text{for all } x \in B_{1/2}.
\]

**Proof.** By Morrey’s inequality (see for example [EG15, Theorem 4.10])
\[
\text{osc}_{x \in B_{1/2}} (u(x) - d \cdot x) = \sup_{x, y \in B_{1/2}} |u(x) - d \cdot x - u(y) + d \cdot y|
\]
\[
\leq C(N) \left( \int_{B_1} |Du - d|^{2N} \, dx \right)^{1/N}
\]
\[
\leq C(N)(\varepsilon_1^{2N} + \varepsilon_0).
\]
Therefore, denoting \( a := \inf_{x \in B_{1/2}} (u(x) - d \cdot x) \), we have for any \( x \in B_{1/2} \)
\[
|u(x) - a - d \cdot x| \leq \text{osc}_{B_{1/2}} (u(x) - d \cdot x) \leq C(N)(\varepsilon \frac{1}{N} + \varepsilon_0) \leq \theta,
\]
where the last inequality follows by taking small enough \( \varepsilon_0 \) and \( \varepsilon_1 \).

We are now ready to prove a Hölder estimate for the gradient of solutions to (3.1). We first restrict the range of \(|q|\).

**Lemma 3.8.** Suppose that \( p \) is smooth. Let \( u \) be a smooth solution to (3.1) in \( B_1 \) with \(|Du| \leq 1\) and either \( q = 0 \) or \(|q| > 2\). Then there exists a constant \( \alpha(N, \hat{p}) \in (0, 1) \) such that
\[
\|Du\|_{C^\alpha(B_{1/2})} \leq C(N, \hat{p}).
\]

**Proof.** For \( \beta = 1/2 \), let \( \eta > 0 \) be as in Lemma 3.6. For \( \theta = \eta/2 \), let \( \varepsilon_0, \varepsilon_1 \) be as in Lemma 3.7. Set
\[
l := 1 - \frac{\varepsilon_0^2}{2} \quad \text{and} \quad \mu := \frac{\varepsilon_1}{|B_1|}.
\]
For these \( l \) and \( \mu \), let \( \tau, \gamma \in (0, 1) \) be as in Theorem 3.5. Let \( k \geq 0 \) be the minimum integer such that the condition (3.14) does not hold.

**Case** \( k = \infty \): Theorem 3.5 implies that
\[
|Du| \leq \gamma^{i+1} \quad \text{in} \quad B_{\gamma^{i+1}} \quad \text{for all} \quad i \geq 0.
\]
Let \( x \in B_1 \setminus \{0\} \). Then \( \tau^{i+1} \leq |x| \leq \tau^i \) for some \( i \geq 0 \) and therefore
\[
i \leq \frac{\log |x|}{\log \tau} \leq i + 1.
\]
We obtain
\[
|Du(x)| \leq \gamma^i = \frac{1}{\gamma} \gamma^{i+1} \leq \frac{1}{\gamma} \frac{\log |x|}{\log \gamma} = \frac{1}{\gamma} \frac{\log |x|}{\log \gamma} \frac{\log \gamma}{\log \tau} =: C |x|^\alpha,
\]
where \( C = 1/\gamma \) and \( \alpha = \log \gamma/\log \tau \).

**Case** \( k < \infty \): There is \( d \in \partial B_1 \) such that
\[
\left| \left\{ x \in B_{\gamma^{k+1}} : Du \cdot d \leq l \gamma^k \right\} \right| \leq \mu |B_{\gamma^{k+1}}|.
\]
We set
\[
v(x) := \tau^{-k} \gamma^{-k} u(\tau^{k+1} x).
\]
Then \( v \) solves
\[
-\Delta v - (p(\tau^{k+1} x) - 2)\frac{\langle D^2 v(\tau^{k+1} x) + \gamma^{-k} q, Dv + \gamma^{-k} q \rangle}{|Dv + \gamma^{-k} q|^2 + \gamma^{-2k} \varepsilon^2} = 0 \quad \text{in} \quad B_1
\]
and by (3.19) we have
\[
\left| \left\{ x \in B_1 : Du \cdot d \leq l \right\} \right| = \left| \left\{ x \in B_1 : Du(\tau^{k+1} x) \cdot d \leq l \gamma^k \right\} \right|
\leq \left| \left\{ x \in B_{\gamma^{k+1}} : Du(x) \cdot d \leq l \gamma^k \right\} \right|
\leq \tau^{-N(k+1)} \mu |B_{\gamma^{k+1}}| = \mu |B_1| = \varepsilon_1.
\]
Since either \( k = 0 \) or (3.14) holds for \( k - 1 \), it follows from Theorem 3.5 that \(|Du| \leq \gamma^k \) in \( B_{\gamma^k} \). Thus
\[
|Du(x)| = \gamma^{-k} |Du(\tau^{k+1} x)| \leq 1 \quad \text{in} \quad B_1.
\]
For vectors $\xi, d \in B_1$, it is easy to verify the following fact
$$|\xi - d| > \varepsilon_0 \implies \xi \cdot d \leq 1 - \varepsilon_0^2/2 = l.$$ 

Therefore, in view of (3.20) and (3.21), we obtain
$$|\{x \in B_1 : |Dv - d| > \varepsilon_0\}| \leq \varepsilon_1.$$ 

Thus by Lemma 3.7 there is $a \in \mathbb{R}$ such that
$$|v(x) - a - d \cdot x| \leq \theta = \eta/2 \quad \text{for all } x \in B_{1/2}.$$ 

Consequently, by applying Lemma 3.6 on the function $2v(2^{-1}x)$, we find a positive constant $C(N, \hat{p})$ and $e \in \partial B_1$ such that
$$|Dv(x) - e| \leq C |x| \quad \text{in } B_{1/4}.$$ 

Since $|Dv| \leq 1$, we have also
$$|Dv(x) - e| \leq C |x| \quad \text{in } B_1.$$ 

Recalling the definition of $v$ and taking $\alpha' \in (0, 1)$ so small that $\gamma/\tau^{\alpha'} < 1$ we obtain
$$|Du(x) - \gamma^k e| \leq C\gamma^k \tau^{-k-1} |x| \leq \frac{C}{\tau^{\alpha'}} \left( \frac{\gamma}{\tau^{\alpha'}} \right)^k |x|^{\alpha'} \leq C |x|^{\alpha'} \quad \text{in } B_{\tau^{k+1}}, \quad (3.22)$$

where we absorbed $\tau^{\alpha'}$ into the constant. On the other hand, we have
$$|Du| \leq \gamma^{i+1} \quad \text{in } B_{\tau^{i+1}} \quad \text{for all } i = 0, \ldots, k - 1$$

so that, if $\tau^{i+2} \leq |x| \leq \tau^{i+1}$ for some $i \in \{0, \ldots, k - 1\}$, it holds that
$$|Du(x) - \gamma^k e| \leq 2\gamma^{i+1} |x|^{\alpha'} \leq \frac{2}{\tau^{\alpha'}} \left( \frac{\gamma}{\tau^{\alpha'}} \right)^{i+1} |x|^{\alpha'} \leq C |x|^{\alpha'}.$$ 

Combining this with (3.22) we obtain
$$|Du(x) - \gamma^k e| \leq C |x|^{\alpha'} \quad \text{in } B_{\tau}, \quad (3.23)$$

The claim now follows from (3.18) and (3.23) by standard translation arguments. 

**Theorem 3.9.** Let $u$ be a bounded viscosity solution to (3.1) in $B_1$ with $q \in \mathbb{R}^N$. Then
$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(N, \hat{p}, \|u\|_{L^\infty(B_1)}) \quad (3.24)$$

for some $\alpha(N, \hat{p}) \in (0, 1)$.

**Proof.** Suppose first that $p$ is smooth. Let $\nu_0(N, \hat{p}, \|u\|_{L^\infty(B_1)})$ and $C_0(N, \hat{p}, \|u\|_{L^\infty(B_1)})$ be as in the Lipschitz estimate (Theorem A.2 in the Appendix) and set
$$M := 2 \max(\nu_0, C_0).$$

If $|q| > M$, then by Theorem A.1 we have
$$|Du| \leq C_0 \quad \text{in } B_{1/2}.$$ 

We set $\tilde{u}(x) := 2u(x/2)/C_0$. Then $|D\tilde{u}| \leq 1$ in $B_1$ and $\tilde{u}$ solves
$$-\Delta \tilde{u} - (p(x/2) - 2)D^2\tilde{u}(D\tilde{u} + q/C_0), D\tilde{u} + q/C_0) = 0 \quad \text{in } B_1,$$

where $q/C_0 > 2$. Thus by Theorem 3.8 we have
$$\|D\tilde{u}\|_{C^{1,\alpha}(B_{1/2})} \leq C(N, \hat{p}),$$

and taking
which implies (3.24) by standard translation arguments.

If \(|q| \leq M\), we define

\[ w := u - q \cdot x. \]

Then by Theorem [A.2] we have

\[ |Dw| \leq C(N, \hat{p}, \|w\|_{L^\infty(B_1)}) =: C'(N, \hat{p}, \|u\|_{L^\infty(B_1)}) \quad \text{in } B_{1/2}. \]

We set \( \tilde{w}(x) := 2w(x/2)/C' \). Then \(|D\tilde{w}| \leq 1\) and so by Theorem 3.6 we have

\[ \|D\tilde{w}\|_{C^0(B_{1/2})} \leq C(N, \hat{p}), \]

which again implies (3.24).

Suppose then that \( p \) is merely Lipschitz continuous. Take a sequence \( p_j \in C^\infty(B_1) \) such that \( p_j \to p \) uniformly in \( B_1 \) and \( \|Dp_j\|_{L^\infty(B_1)} \leq \|Dp\|_{L^\infty(B_1)} \). For \( r < 1 \), let \( u_j \) be a solution to the Dirichlet problem

\[
\begin{cases} 
-\Delta u_j - (p_j(x) - 2)\frac{2u_j(Du_j + q, Du_j + q)}{|Du_j + q|^2 + \epsilon^2} = 0 & \text{in } B_r, \\
\quad u_j = u & \text{on } B_r.
\end{cases}
\]

As observed in Proposition 3.1, the solution exists and we have \( u_j \in C^\infty(B_r) \). By comparison principle \( \|u_j\|_{L^\infty(B_r)} \leq \|u\|_{L^\infty(B_1)} \). Then by the first part of the proof we have the estimate

\[ \|u_j\|_{C^{1,\beta}(B_{r/2})} \leq C(N, \hat{p}, \|u\|_{L^\infty(B_1)}). \]

By [CC95, Theorem 4.14] the functions \( u_j \) are equicontinuous in \( B_1 \) and so by the Ascoli-Arzelà theorem we have \( u_j \to v \) uniformly in \( B_1 \) up to a subsequence. Moreover, by the stability principle \( v \) is a solution to (3.1) in \( B_r \) and thus by comparison principle [KK07, Theorem 2.6] we have \( v \equiv u \). By extracting a further subsequence, we may ensure that also \( Du_j \to Du \) uniformly in \( B_{r/2} \) and so the estimate \( \|Du\|_{C^{1,\beta}(B_{r/2})} \leq C(N, \hat{p}, \|u\|_{L^\infty(B_1)}) \) follows. \( \square \)

4. Hölder Gradient Estimates for the Regularized Inhomogeneous Equation

In this section we consider the inhomogeneous equation

\[
-\Delta u - (p(x) - 2)\frac{2u(Du + q, Du + q)}{|Du|^2 + \epsilon^2} = f(x) \quad \text{in } B_1, 
\]

where \( p : B_1 \to \mathbb{R} \) is Lipschitz continuous, \( p_{\text{min}} > 1, \epsilon > 0, q \in \mathbb{R}^N \) and \( f \in C(B_1) \) is bounded. We apply the \( C^{1,\alpha} \)-estimates obtained in Theorem 3.9 to prove regularity estimates for solutions of (4.1) with \( q = 0 \). Our arguments are similar to those in [AP18, Section 3], see also [IS12]. The idea is to use the well known characterization of \( C^{1,\alpha} \)-regularity via affine approximates. The following lemma plays a key role: It states that if \( f \) is small, then a solution to (4.1) can be approximated by an affine function. This combined with scaling properties of the equation essentially yields the desired affine functions.

\[ \text{Lemma 4.1.} \quad \text{There exist constants } c(N, \hat{p}), \tau(N, \hat{p}) \in (0, 1) \text{ such that the following holds: If } \|f\|_{L^\infty(B_1)} \leq c \text{ and } w \text{ is a viscosity solution to (4.1) in } B_1 \text{ with } q \in \mathbb{R}^N, \]

\[ w(0) = 0 \text{ and } \text{osc}_{B_1} w \leq 1, \text{ then there exists } q' \in \mathbb{R}^N \text{ such that} \]

\[ \text{osc}_{B_r}(w(x) - q' \cdot x) \leq \frac{1}{2}\tau. \]

\[ \text{Lemma 4.1.} \quad \text{There exist constants } c(N, \hat{p}), \tau(N, \hat{p}) \in (0, 1) \text{ such that the following holds: If } \|f\|_{L^\infty(B_1)} \leq c \text{ and } w \text{ is a viscosity solution to (4.1) in } B_1 \text{ with } q \in \mathbb{R}^N, \]

\[ w(0) = 0 \text{ and } \text{osc}_{B_1} w \leq 1, \text{ then there exists } q' \in \mathbb{R}^N \text{ such that} \]

\[ \text{osc}_{B_r}(w(x) - q' \cdot x) \leq \frac{1}{2}\tau. \]
Moreover, we have $|q'| \leq C(N, \hat{p})$.

Proof. Suppose on the contrary that the claim does not hold. Then, for a fixed \( \tau(N, \hat{p}) \) that we will specify later, there exists a sequence of Lipschitz continuous functions \( p_j : B_1 \to \mathbb{R} \) such that

\[
p_{\min} \leq \inf_{B_1} p_j \leq \sup_{B_1} p_j \leq p_{\max} \quad \text{and} \quad (p_j)_L \leq p_L,
\]

functions \( f_j \in C(B_1) \) such that \( f_j \to 0 \) uniformly in \( B_1 \), vectors \( q_j \in \mathbb{R}^N \) and viscosity solutions \( w_j \) to

\[-\Delta w_j - (p_j(x) - 2) \frac{\langle D^2 w_j (Dw_j + q_j), Dw_j + q_j \rangle}{|Dw_j + q_j|^2 + \varepsilon^2} = f_j(x) \quad \text{in} \quad B_1\]

such that \( w_j(0) = 0 \), \( \text{osc}_{B_1} w_j \leq 1 \) and

\[\text{osc}_{B_1}(w_j(x) - q' \cdot x) > \frac{\tau}{2} \quad \text{for all} \quad q' \in \mathbb{R}^N. \quad (4.2)\]

By [CC95, Proposition 4.10], the functions \( w_j \) are uniformly Hölder continuous in \( B_r \) for any \( r \in (0,1) \). Therefore by the Ascoli-Arzela theorem, we may extract a subsequence such that \( w_j \to w_\infty \) and \( p_j \to p_\infty \) uniformly in \( B_r \) for any \( r \in (0,1) \). Moreover, \( p_\infty \) is \( p_L \)-Lipschitz continuous and \( p_{\min} \leq p_\infty \leq p_{\max} \). It then follows from (4.2) that

\[\text{osc}_{B_1}(w_\infty(x) - q' \cdot x) > \frac{\tau}{2} \quad \text{for all} \quad q' \in \mathbb{R}^N. \quad (4.3)\]

We have two cases: either \( q_j \) is bounded or unbounded.

Case \( q_j \) is bounded: In this case \( q_j \to q_\infty \in \mathbb{R}^N \) up to a subsequence. It follows from the stability principle that \( w_\infty \) is a viscosity solution to

\[-\Delta w_\infty - (p_\infty(x) - 2) \frac{\langle D^2 w_\infty (Dw_\infty + q_\infty), Dw_\infty + q_\infty \rangle}{|Dw_\infty + q_\infty|^2 + \varepsilon^2} = 0 \quad \text{in} \quad B_1. \quad (4.4)\]

Hence by Theorem 3.5 we have \( \|Dw_\infty\|_{C^1(B_{1/2})} \leq C(N, \hat{p}) \leq C(N, \hat{p}) \) for some \( \beta_1(N, \hat{p}) \). The mean value theorem then implies the existence of \( q' \in \mathbb{R}^N \) such that

\[\text{osc}_{B_r}(u - q' \cdot x) \leq C_1(N, \hat{p}) r^{1+\beta_1} \quad \text{for all} \quad r \leq \frac{1}{2}. \]

Case \( q_j \) is unbounded: In this case we take a subsequence such that \( |q_j| \to \infty \) and the sequence \( d_j := d_j/|d_j| \) converges to \( d_\infty \in \partial B_1 \). Then \( w_j \) is a viscosity solution to

\[-\Delta w_j - (p_\infty(x) - 2) \frac{\langle D^2 w_j(|q_j|^{-1}Dw_j + d_j), |q_j|^{-1}Dw_j + d_j \rangle}{|||q_j|^{-1}Dw_j + d_j|^2 + |q_j|^{-2} \varepsilon^2} = f_j(x) \quad \text{in} \quad B_1.\]

It follows from the stability principle that \( w_\infty \) is a viscosity solution to

\[-\Delta w_j - (p_\infty(x) - 2) \langle D^2 w_\infty d_\infty, d_\infty \rangle = 0 \quad \text{in} \quad B_1. \]

By [CC95, Theorem 8.3] there exist positive constants \( \beta_2(N, \hat{p}), C_2(N, \hat{p}), r_2(N, \hat{p}) \) and a vector \( q' \in \mathbb{R}^N \) such that

\[\text{osc}_{B_r}(w_\infty - q' \cdot x) \leq C_2 r^{1+\beta_2} \quad \text{for all} \quad r \leq r_2. \]
We set $C_0 := \max(C_1, C_2)$ and $\beta_0 := \min(\beta_1, \beta_2)$. Then by the two different cases there always exists a vector $q' \in \mathbb{R}^N$ such that

$$\text{osc}_{B_r}(w_\infty - q' \cdot x) \leq C_0 r^{1+\beta_0} \quad \text{for all } r \leq \min\left(\frac{1}{2}, r_2\right).$$

We take $\tau$ so small that $C_0 \tau^{\beta_0} \leq \frac{1}{4}$ and $\tau \leq \min\left(\frac{1}{2}, r_2\right)$. Then, by substituting $r = \tau$ in the above display, we obtain

$$\text{osc}_{B_r}(w_\infty - q' \cdot x) \leq C_0 \tau^{\beta_0} \tau \leq \frac{1}{4} \tau,$$

which contradicts (4.3).

The bound $|q'| \leq C(N, \hat{p})$ follows by observing that (4.5) together with the assumption $\text{osc}_{B_1} w \leq 1$ yields $|q'| \leq C$. Thus the contradiction is still there even if (4.3) is weakened by requiring additionally that $|q'| \leq C$. \hfill \Box

**Lemma 4.2.** Let $\tau(N, \hat{p})$ and $\epsilon(N, \hat{p})$ be as in Lemma 4.1. If $\|f\|_{L^\infty(B_1)} \leq \epsilon$ and $u$ is a viscosity solution to (4.1) in $B_1$ with $q = 0$, $u(0) = 0$ and $\text{osc}_{B_1} u \leq 1$, then there exists $\alpha \in (0, 1)$ and $q_\infty \in \mathbb{R}^N$ such that

$$\sup_{B_{\tau k}} |u(x) - q_\infty \cdot x| \leq C(N, \hat{p}) \tau^{k(1+\alpha)} \quad \text{for all } k \in \mathbb{N}.$$

**Proof.**

**Step 1:** We show that there exists a sequence $(q_k)_{k=0}^\infty \subset \mathbb{R}^N$ such that

$$\text{osc}_{B_{\tau k}}(u(x) - q_k \cdot x) \leq \tau^{k(1+\alpha)}.$$ (4.6)

When $k = 0$, this estimate holds by setting $q_0 = 0$ since $u(0) = 0$ and $\text{osc}_{B_1} u \leq 1$. Next we take $\alpha \in (0, 1)$ such that $\tau^\alpha > \frac{1}{2}$. We assume that $k \geq 0$ and that we have already constructed $q_k$ for which (4.6) holds. We set

$$w_k(x) := \tau^{-k(1+\alpha)}(u(\tau^k x) - q_k \cdot (\tau^k x))$$

and

$$f_k(x) := \tau^{k(1-\alpha)} f(\tau^k x).$$

Then by induction assumption $\text{osc}_{B_1}(w_k) \leq 1$ and $w_k$ is a viscosity solution to

$$-\Delta w_k - \frac{(p(\tau^k x) - 2) \left(D^2 w_k(Dw_k + \tau^{-k\alpha} q_k), Dw_k + \tau^{-k\alpha} q_k\right)}{|Dw_k + \tau^{-k\alpha} q_k|^2 + (\tau^{-k\alpha} \epsilon)^2} = f_k(x) \quad \text{in } B_1.$$

By Lemma 4.1 there exists $q'_k \in \mathbb{R}^N$ with $|q'_k| \leq C(N, \hat{p})$ such that

$$\text{osc}_{B_{\tau k}}(w_k(x) - q'_k \cdot x) \leq \frac{1}{2} \tau.$$

Using the definition of $w_k$, scaling to $B_{\tau k+1}$ and dividing by $\tau^{-k(\alpha+1)}$, we obtain from the above

$$\text{osc}_{B_{\tau k+1}}(u(x) - (q_k + \tau^{k\alpha} q'_k) \cdot x) \leq \frac{1}{2} \tau^{1+k(1+\alpha)} \leq \tau^{(k+1)(1+\alpha)}.$$

Denoting $q_{k+1} := q_k + \tau^{k\alpha} q'_k$, the above estimate is condition (4.6) for $k+1$ and the induction step is complete.

**Step 2:** Observe that whenever $m > k$, we have

$$|q_m - q_k| \leq \sum_{i=k}^{m-1} \tau^{i\alpha} |q'_i| \leq C(N, \hat{p}) \sum_{i=k}^{m-1} \tau^{i\alpha}.$$
Therefore $q_k$ is a Cauchy sequence and converges to some $q_\infty \in \mathbb{R}^N$. Thus
\[
\sup_{x \in B_{r_k}} (q_k \cdot x - q_\infty \cdot x) \leq \tau^k |q_k - q_\infty| \leq \tau^k \sum_{i=k}^{\infty} \tau^{i\alpha} q_i' \leq C(N, \hat{p}) \tau^{k(1+\alpha)}.
\]
This with (4.6) implies that
\[
\sup_{x \in B_{r_k}} |u(x) - q_\infty \cdot x| \leq C(N, \hat{p}) \tau^{k(1+\alpha)}.
\]
\[\Box\]

**Theorem 4.3.** Suppose that $u$ is a viscosity solution to (4.7) in $B_1$ with $q = 0$ and $\text{osc}_{B_1} \leq 1$. Then there are constants $\alpha(N, \hat{p})$ and $C(N, \hat{p}, \|f\|_{L^\infty(B_1)})$ such that
\[
\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C.
\]

**Proof.** Let $\epsilon(N, \hat{p})$ and $\tau(N, \hat{p})$ be as in Lemma 4.2. Set
\[
v(x) := \kappa u(x/4)
\]
where $\kappa := \epsilon(1 + \|f\|_{L^\infty(B_1)})^{-1}$. For $x_0 \in B_1$, set
\[
w(x) := v(x + x_0) - v(x_0).
\]
Then $\text{osc}_{B_1} w \leq 1$, $w(0) = 0$ and $w$ is a viscosity solution to
\[
-\Delta w - \frac{(p(x/4 + x_0/4) - 2) \langle D^2 w Dw, Dw \rangle}{|Dw|^2 + \epsilon^2 k^2/4^2} = g(x) \quad \text{in } B_1,
\]
where $g(x) := \kappa f(x/4 + x_0/4)/4^2$. Now $\|g\|_{L^\infty(B_1)} \leq \epsilon$ so by Lemma 4.2 there exists $q_\infty(x_0) \in \mathbb{R}^N$ such that
\[
\sup_{x \in B_{r_k}} |w(x) - q_\infty(x_0) \cdot x| \leq C(N, \hat{p}) \tau^{1+\alpha} \quad \text{for all } k \in \mathbb{N}.
\]
Thus we have shown that for any $x_0 \in B_1$ there exists a vector $q_\infty(x_0)$ such that
\[
\sup_{x \in B_r(x_0)} |v(x) - v(x_0) - q_\infty(x_0) \cdot (x - x_0)| \leq C(N, \hat{p}) r^{1+\alpha} \quad \text{for all } r \in (0, 1].
\]
This together with a standard argument (see for example AP18, Lemma A.1) implies that $[Dv]_{C^{1,0}(B_1)} \leq C(N, \hat{p})$ and so by definition of $v$, also $[Du]_{C^{1,0}(B_1)} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)})$. The conclusion of the theorem then follows by a standard translation argument. \[\Box\]

5. PROOF OF THE MAIN THEOREM

In this section we finish the proof our main theorem.

**Proof of Theorem 1.1** We may assume that $u \in C(\overline{B_1})$. By Comparison Principle (Lemma B.2 in the Appendix) $u$ is the unique viscosity solution to
\[
\begin{cases}
-\Delta v - \frac{(p(x) - 2) \langle D^2 w Dw, Dw \rangle}{|Dw|^2} = f(x) + u - v & \text{in } B_1, \\
v = u & \text{on } \partial B_1.
\end{cases}
\]
(5.1)

By GT01 Theorem 15.18 there exists a classical solution $u_\varepsilon$ to the approximate problem
\[
\begin{cases}
-\Delta u_\varepsilon - \frac{(p(x) - 2) \langle D^2 w_\varepsilon Dw_\varepsilon, Dw_\varepsilon \rangle}{|Dw_\varepsilon|^2 + \epsilon^2} = f_\varepsilon(x) + u - u_\varepsilon & \text{in } B_1, \\
u_\varepsilon = u & \text{on } \partial B_1,
\end{cases}
\]
where \( p_\varepsilon, f_\varepsilon, u_\varepsilon \in C^\infty(B_1) \) are such that \( p_\varepsilon \to p, f_\varepsilon \to f \) and \( u_\varepsilon \to u_0 \) uniformly in \( B_1 \) as \( \varepsilon \to 0 \) and \( \|Dp_\varepsilon\|_{L^\infty(B_1)} \leq \|Dp\|_{L^\infty(B_1)} \). The maximum principle implies that \( \|u_\varepsilon\|_{L^\infty(B_1)} \leq 2\|f\|_{L^\infty(B_1)} + 2\|u\|_{L^\infty(B_1)} \). By [CC95, Proposition 4.14] the solutions \( u_\varepsilon \) are equicontinuous in \( \overline{B}_1 \) (their modulus of continuity depends only on \( N, p, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)} \) and modulus of continuity of \( u \)). Therefore by the Ascoli-Arzela theorem we have \( u_\varepsilon \to v \in C(\overline{B}_1) \) uniformly in \( \overline{B}_1 \) up to a subsequence. By the stability principle, \( v \) is a viscosity solution to \([5.1]\) and thus by uniqueness \( v \equiv u \).

By Corollary 4.3 we have \( \alpha(N, \hat{p}) \) such that
\[
\|Du_\varepsilon\|_{C^\alpha(B_{1/2})} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)})
\]
and by the Lipschitz estimate A.2 also
\[
\|Du_\varepsilon\|_{L^\infty(B_{1/2})} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}).
\]

Therefore by the Ascoli-Arzela theorem there exists a subsequence such that \( Du_\varepsilon \to \eta \) uniformly in \( B_{1/2} \), where the function \( \eta : B_{1/2} \to \mathbb{R}^N \) satisfies
\[
\|\eta\|_{C^\alpha(B_{1/2})} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}).
\]

Using the mean value theorem and the estimate (5.2), we deduce for all \( x, y \in B_{1/2} \)
\[
|u(y) - u(x) - (y - x) \cdot \eta(x)|
\leq |u_\varepsilon(x) - u_\varepsilon(y) - (y - x) \cdot Du_\varepsilon(x)|
+ |u(y) - u_\varepsilon(y) - u(x) + u_\varepsilon(x)| + |x - y| |\eta(x) - Du_\varepsilon(x)|
\leq C(N, \hat{p}, \|u\|_{L^\infty(B_1)}) |x - y|^{1+\alpha} + o(\varepsilon)/\varepsilon.
\]

Letting \( \varepsilon \to 0 \), this implies that \( Du(x) = \eta(x) \) for all \( x \in B_{1/2} \).

\[ \square \]

**APPENDIX A. Lipschitz estimate**

In this section we apply the method of Ishii and Lions [IL90] to prove a Lipschitz estimate for solutions to the inhomogeneous normalized \( p(x) \)-Laplace equation and its regularized or perturbed versions. We need the following vector inequality.

**Lemma A.1.** Let \( a, b \in \mathbb{R}^N \setminus \{0\} \) with \( a \not= b \) and \( \varepsilon \geq 0 \). Then
\[ \left| \frac{a}{\sqrt{|a|^2 + \varepsilon^2}} - \frac{b}{\sqrt{|b|^2 + \varepsilon^2}} \right| \leq \frac{2}{\max(|a|, |b|)} |a - b|. \]

**Proof.** We may suppose that \( |a| = \max(|a|, |b|) \). Let \( s_1 := \sqrt{|a|^2 + \varepsilon^2} \) and \( s_2 := \sqrt{|b|^2 + \varepsilon^2} \). Then
\[ \left| \frac{a}{s_1} - \frac{b}{s_2} \right| = \frac{1}{s_1} \left| a - b + \frac{b}{s_2} (s_2 - s_1) \right| \leq \frac{1}{s_1} (|a - b| + \frac{|b|}{s_2} |s_2 - s_1|)
\leq \frac{1}{|a|} (|a - b| + |s_2 - s_1|). \]
Moreover
\[
|s_2 - s_1| = \left| \sqrt{|a|^2 + \varepsilon^2} - \sqrt{|b|^2 + \varepsilon^2} \right| = \frac{|a|^2 - |b|^2|}{\sqrt{|a|^2 + \varepsilon^2 + \sqrt{|b|^2 + \varepsilon^2}}}
\geq \frac{|a| - |b|}{|a| + |b|} \leq |a - b|.
\]

**Theorem A.2** (Lipschitz estimate). Suppose that \( p : B_1 \to \mathbb{R} \) is Lipschitz continuous, \( p_{\min} > 1 \) and that \( f \in C(B_1) \) is bounded. Let \( u \) be a viscosity solution to
\[
-\Delta u - (p(x) - 2)\frac{\langle D^2 u(Du + q) \rangle}{|Du + q|^2 + \varepsilon^2} = f(x) \quad \text{in } B_1,
\]
where \( \varepsilon \geq 0 \) and \( q \in \mathbb{R}^N \). Then there are constants \( C_0(N, \hat{\rho}, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}) \) and \( \nu_0(N, \hat{\rho}) \) such that if \( |q| > \nu_0 \) or \( |q| = 0 \), then we have
\[
|u(x) - u(y)| \leq C_0|x - y| \quad \text{for all } x, y \in B_{1/2}.
\]

**Proof.** We let \( r(N, \hat{\rho}) \in (0, 1/2) \) denote a small constant that will be specified later. Let \( x_0, y_0 \in B_{r/2} \) and define the function
\[
\Psi(x, y) := u(x) - u(y) - L\varphi(|x - y|) - \frac{M}{2} |x - x_0|^2 - \frac{M}{2} |y - y_0|^2,
\]
where \( \varphi : [0, 2] \to \mathbb{R} \) is given by
\[
\varphi(s) := s - s^\gamma \kappa_0, \quad \kappa_0 := \frac{1}{\gamma 2^{\gamma + 1}},
\]
and the constants \( L(N, \hat{\rho}, \|u\|_{L^\infty(B_1)}), M(N, \hat{\rho}, \|u\|_{L^\infty(B_1)}) > 0 \) and \( \gamma(N, \hat{\rho}) \in (1, 2) \) are also specified later. Our objective is to show that for a suitable choice of these constants, the function \( \Psi \) is non-positive in \( \overline{B_r \times B_r} \). By the definition of \( \varphi \), this yields \( u(x_0) - u(y_0) \leq L|x_0 - y_0| \) which implies that \( u \) is \( L \)-Lipschitz in \( B_r \). The claim of the theorem then follows by standard translation arguments.

Suppose on contrary that \( \Psi \) has a positive maximum at some point \((\hat{x}, \hat{y}) \in \overline{B_r \times B_r} \). Then \( \hat{x} \neq \hat{y} \) since otherwise the maximum would be non-positive. We have
\[
0 < u(\hat{x}) - u(\hat{y}) - L\varphi(|\hat{x} - \hat{y}|) - \frac{M}{2} |\hat{x} - x_0|^2 - \frac{M}{2} |\hat{y} - y_0|^2
\leq |u(\hat{x}) - u(\hat{y})| - \frac{M}{2} |\hat{x} - x_0|^2. \tag{A.1}
\]
Therefore, by taking
\[
M := \frac{8 \text{osc}_{B_1} u}{r^2}, \tag{A.2}
\]
we get
\[
|\hat{x} - x_0| \leq \sqrt{\frac{2}{M} |u(\hat{x}) - u(\hat{y})|} \leq r/2,
\]
and similarly \( |\hat{y} - y_0| \leq r/2 \). Since \( x_0, y_0 \in B_{r/2} \), this implies that \( \hat{x}, \hat{y} \in B_r \).

By [CC95, Proposition 4.10] there exist constants \( C'(N, \hat{\rho}, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}) \) and \( \beta(N, \hat{\rho}) \in (0, 1) \) such that
\[
|u(x) - u(y)| \leq C'|x - y|^\beta \quad \text{for all } x, y \in B_r. \tag{A.3}
\]
Moreover, choosing $\phi$ where

$$M |\hat{x} - x_0| \leq C_0 |\hat{x} - \hat{y}|^{\beta/2},$$

$$M |\hat{y} - y_0| \leq C_0 |\hat{x} - \hat{y}|^{\beta/2}. \quad (A.4)$$

Since $\hat{x} \neq \hat{y}$, the function $(x, y) \mapsto \varphi(|x - y|)$ is $C^2$ in a neighborhood of $(\hat{x}, \hat{y})$ and we may invoke the Theorem of sums [CIL92, Theorem 3.2]. For any $\mu > 0$ there exist matrices $X, Y \in S^N$ such that

$$(D_x(L \varphi(|x - y|))(\hat{x}, \hat{y}), X) \in \mathcal{T}^{2+} \left( u - \frac{M}{2} |x - x_0|^2 \right)(\hat{x}),$$

$$(-D_y(L \varphi(|x - y|))(\hat{x}, \hat{y}), Y) \in \mathcal{T}^{2-} \left( u + \frac{M}{2} |y - y_0|^2 \right)(\hat{y}),$$

which by denoting $z := \hat{x} - \hat{y}$ and

$$a := L \varphi(|z|) \frac{z}{|z|} + M(\hat{x} - x_0),$$

$$b := L \varphi(|z|) \frac{z}{|z|} - M(\hat{y} - y_0),$$

can be written as

$$(a, X + MI) \in \mathcal{T}^{2+} u(\hat{x}), \quad (b, Y - MI) \in \mathcal{T}^{2-} u(\hat{y}). \quad (A.5)$$

By assuming that $L$ is large enough depending on $C_0$, we have by (A.4) and the fact $\varphi' \in \left[ \frac{2}{3}, 1 \right]$

$$|a|, |b| \leq L |\varphi'(|\hat{x} - \hat{y}|)| + C_0 |\hat{x} - \hat{y}|^{\beta/2} \leq 2L, \quad (A.6)$$

$$|a|, |b| \geq L |\varphi'(|\hat{x} - \hat{y}|)| - C_0 |\hat{x} - \hat{y}|^{\beta/2} \geq \frac{1}{2} L. \quad (A.7)$$

Moreover, we have

$$-(\mu + 2 \|B\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \frac{2}{\mu} \begin{pmatrix} B^2 & -B^2 \\ -B^2 & B^2 \end{pmatrix}, \quad (A.8)$$

where

$$B = L \varphi''(|z|) \frac{z}{|z|} \otimes \frac{z}{|z|} + L \varphi'(|z|) \left( I - \frac{z}{|z|} \otimes \frac{z}{|z|} \right),$$

$$B^2 = BB = L^2(\varphi''(|z|))^2 \frac{z}{|z|} \otimes \frac{z}{|z|} + \frac{L^2(\varphi''(|z|))^2}{|z|^2} \left( I - \frac{z}{|z|} \otimes \frac{z}{|z|} \right).$$

Using that $\varphi''(|z|) < 0 < \varphi'(|z|)$ and $|\varphi''(|z|)| \leq \varphi'(|z|)/|z|$, we deduce that

$$\|B\| \leq \frac{L \varphi'(|z|)}{|z|} \quad \text{and} \quad \|B^2\| \leq \frac{L^2(\varphi'(|z|))^2}{|z|^2}. \quad (A.9)$$

Moreover, choosing

$$\mu := 4L \left( |\varphi''(|z|)| + \frac{|\varphi'(|z|)|}{|z|} \right),$$

and using that $\varphi''(|z|) < 0$, we have

$$\left( B \frac{z}{|z|}, \frac{z}{|z|} \right) + \frac{2}{\mu} \left( B^2 \frac{z}{|z|}, \frac{z}{|z|} \right) = L \varphi''(|z|) + \frac{2}{\mu} L^2 |\varphi''(|z|)| \leq \frac{L}{2} \varphi''(|z|). \quad (A.10)$$
We set \( \eta_1 := a + q \) and \( \eta_2 := b + q \). By (A.6) and (A.7) there is a constant \( \nu_0(L) \) such that if \( |q| = 0 \) or \( |q| > \nu_0 \), then

\[
|\eta_1|, |\eta_2| \geq \frac{L}{2}.
\]  

We denote \( A(x, \eta) := I + (p(x) - 2)\eta \otimes \eta \) and \( \eta := \frac{\eta}{\sqrt{|\eta|^2 + \varepsilon^2}} \). Since \( u \) is a viscosity solution, we obtain from (A.5)

\[
0 \leq \text{tr}(A(\hat{x}, \eta_1)(X + MI)) - \text{tr}(A(\hat{y}, \eta_2)(Y - MI)) + f(\hat{x}) - f(\hat{y})
\]

\[
= \text{tr}(A(\hat{y}, \eta_2)(X - Y)) + \text{tr}((A(\hat{x}, \eta_2) - A(\hat{y}, \eta_2))X)
\]

\[
+ \text{tr}(A(\hat{x}, \eta_1) - A(\hat{x}, \eta_2))X) + M\text{tr}(A(\hat{x}, \eta_1) + A(\hat{y}, \eta_2))
\]

\[
+ f(\hat{x}) - f(\hat{y})
\]

\[
=: T_1 + T_2 + T_3 + T_4 + T_5.
\]  

We will now proceed to estimate these terms. The plan is to obtain a contradiction by absorbing the other terms into \( T_1 \) which is negative by concavity of \( \varphi \).

**Estimate of \( T_1 \):** Multiplying (A.8) by the vector \( (\frac{\hat{z}}{\|\hat{z}\|}, -\frac{\hat{z}}{\|\hat{z}\|}) \) and using (A.10), we obtain an estimate for the smallest eigenvalue of \( X - Y \)

\[
\lambda_{\min}(X - Y) \leq \left\langle (X - Y)\frac{\hat{z}}{\|\hat{z}\|}, \frac{\hat{z}}{\|\hat{z}\|} \right\rangle
\]

\[
\leq 4\left\langle B\frac{\hat{z}}{\|\hat{z}\|}, \frac{\hat{z}}{\|\hat{z}\|} \right\rangle + \frac{8}{\mu}\left\langle B^2\frac{\hat{z}}{\|\hat{z}\|}, \frac{\hat{z}}{\|\hat{z}\|} \right\rangle \leq 2L\varphi''(|\hat{z}|).
\]

The eigenvalues of \( A(\hat{y}, \eta_2) \) are between \( \min(1, p_{\min} - 1) \) and \( \max(1, p_{\max} - 1) \). Therefore by (The75)

\[
T_1 = \text{tr}(A(\hat{y}, \eta_2)(X - Y)) \leq \sum_i \lambda_i(A(\hat{y}, \eta_2))\lambda_i(X - Y)
\]

\[
\leq \min(1, p_{\min} - 1)\lambda_{\min}(X - Y)
\]

\[
\leq C(\dot{p})L\varphi''(|\hat{z}|).
\]

**Estimate of \( T_2 \):** We have

\[
T_2 = \text{tr}((A(\hat{x}, \eta_2) - A(\hat{y}, \eta_2))X) \leq |p(\hat{x}) - p(\hat{y})| |X| \eta_2| \leq C(\dot{p}) \|X\|,
\]

where by (A.8) and (A.9)

\[
\|X\| \leq \|B\| + \frac{2}{\mu} \|B\|^2 \leq \frac{L\varphi'(|\hat{z}|)}{\|\hat{z}\|} + \frac{2L^2(\varphi'(|\hat{z}|))^2}{4L(\|\varphi''(|\hat{z}|)| + \frac{\varphi''(\|\hat{z}\|)}{\|\hat{z}\|})^2 |\hat{z}|^2}
\]

\[
\leq \frac{2L\varphi'(|\hat{z}|)}{\|\hat{z}\|}.
\]  

**Estimate of \( T_3 \):** From Lemma (A.1) and the estimate (A.11) it follows that

\[
|\eta_1 - \eta_2| \leq \frac{2|\eta_1 - \eta_2|}{\max(|\eta_1|, |\eta_2|)} \leq \frac{4}{L} |\eta_1 - \eta_2| = \frac{4}{L} |a - b|
\]

\[
\leq \frac{4}{L} (M|\hat{x} - x_0| + M|\hat{y} - y_0|) \leq \frac{8C_0}{L} |\hat{z}|^{\beta/2},
\]  

where in the last inequality we used (A.4). Observe that

\[
|\eta_1 \otimes \eta_1 - \eta_2 \otimes \eta_2| = ||(\eta_1 - \eta_2) \otimes \eta_1 - \eta_2 \otimes (\eta_2 - \eta_1)|| \leq (|\eta_1| + |\eta_2|) |\eta_1 - \eta_2|.
\]
Using the last two displays, we obtain by [The75] and (A.13)
\[ T_3 = \text{tr}(A(\dot{x}, \vec{\eta}_1) - A(\dot{x}, \vec{\eta}_2))X) \leq N \|A(x, \vec{\eta}_1) - A(x, \vec{\eta}_2)\| \|X\| \]
\[ \leq N |p(x_1) - 2||\vec{\eta}_1| + |\vec{\eta}_2|)|\vec{\eta}_1 - \vec{\eta}_2| \|X\| \]
\[ \leq \frac{C(N, \hat{p})C_0}{L} |z|^\beta/2 \|X\| \]
\[ \leq C(N, \hat{p}, \|u\|_{L^\infty} \|f\|_{L^\infty}) \sqrt{M} \varphi'(|z|) |z|^\beta/2 - 1. \]

**Estimate of $T_4$ and $T_5$:** By Lipschitz continuity of $p$ we have
\[ T_4 = M \text{tr}(A(\dot{x}, \vec{\eta}_1) + A(\dot{y}, \vec{\eta}_2)) \leq 2MC(N, \hat{p}). \]
We have also
\[ T_5 = f(\dot{x}) - f(\dot{y}) \leq 2 \|f\|_{L^\infty(B_1)}. \]
Combining the estimates, we deduce the existence of positive constants $C_1(N, \hat{p})$ and $C_2(N, \hat{p}, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)})$ such that
\[ 0 \leq C_1 L \varphi''(|z|) + C_2 (L \varphi'(|z|) + \sqrt{M} \varphi'(|z|) |z|^\beta/2 - 1 + M + 1) \]
\[ \leq C_1 L \varphi''(|z|) + C_2 (L + \sqrt{M} |z|^\beta/2 - 1 + M + 1) \]
(A.15)
where we used that $\varphi'(|z|) \in [\frac{\beta}{2}, 1]$. We take $\gamma := \frac{\beta}{2} + 1$ so that we have
\[ \varphi''(|z|) = \frac{1 - \gamma}{2\gamma + 1} |z|^{\gamma - 2} = \frac{-\beta}{2\gamma + 3} |z|^\frac{\beta}{2 - 1} =: -C_3 |z|^\frac{\beta}{2 - 1}. \]
We apply this to (A.15) and obtain
\[ 0 \leq (C_2 \sqrt{M} - C_1 C_3 L) |z|^\frac{\beta}{2 - 1} + C_2 (L + M + 1) \]
(A.16)
We fix $r := \frac{1}{2} \left( \frac{6C_2}{C_1 C_3} \right)^{\frac{1}{\beta - 1}}$. By (A.2) this will also fix $M = (N, \hat{p}, \|u\|_{L^\infty(B_1)})$. We take $L$ so large that
\[ L > \max \left( \frac{2C_2 \sqrt{M}}{C_1 C_3}, M + 1 \right). \]
Then by (A.16) we have
\[ 0 < -\frac{1}{2} C_1 C_3 L |z|^\frac{\beta}{2 - 1} + 2C_2 L \leq L(-\frac{1}{2} C_1 C_3 (2r)^\frac{\beta}{2 - 1} + 2C_2) \]
\[ = -LC_2 \leq 0, \]
which is a contradiction. \qed

**Appendix B. Stability and Comparison Principles**

**Lemma B.1.** Suppose that $p \in C(B_1)$, $p_{\min} > 1$ and that $f : B_1 \times \mathbb{R} \to \mathbb{R}$ is continuous. Let $u_\varepsilon$ be a viscosity solution to
\[ -\Delta u_\varepsilon - (p_\varepsilon(x) - 2) \frac{\langle D^2 u_\varepsilon Du_\varepsilon, Du_\varepsilon \rangle}{|Du_\varepsilon|^2 + \varepsilon^2} = f_\varepsilon(x, u(x)) \quad \text{in} \ B_1 \]
and assume that $u_\varepsilon \to u \in C(B_1)$, $p_\varepsilon \to p$ and $f_\varepsilon \to f$ locally uniformly as $\varepsilon \to 0$. Then $u$ is a viscosity solution to
\[ -\Delta u - (p(x) - 2) \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} = f(x, u(x)) \quad \text{in} \ B_1. \]
Proof. It is enough to consider supersolutions. Suppose that \( \varphi \in C^2 \) touches \( u \) below at \( x \). Since \( u_\varepsilon \to u \) locally uniformly, there exists a sequence \( x_\varepsilon \to x \) such that \( u_\varepsilon - \varphi \) has a local minimum at \( x_\varepsilon \). We denote \( \eta_\varepsilon := D\varphi(x_\varepsilon)/\sqrt{|D\varphi(x_\varepsilon)|^2 + \varepsilon^2} \). Then \( \eta_\varepsilon \to \eta \in \overline{B}_1 \) up to a subsequence. Therefore we have

\[
0 \leq -\Delta \varphi(x_\varepsilon) - (p_\varepsilon(x_\varepsilon) - 2) \langle D^2 \varphi(x_\varepsilon) \eta_\varepsilon, \eta_\varepsilon \rangle - f_\varepsilon(x_\varepsilon, u_\varepsilon(x_\varepsilon))
- \Delta \varphi(x) - (p(x) - 2) \langle D^2 \varphi(x) \eta, \eta \rangle - f(x, u(x)),
\]

which is what is required in Definition 2.1 in the case \( D\varphi(x) = 0 \). If \( D\varphi(x) \neq 0 \), then \( D\varphi(x_\varepsilon) \neq 0 \) when \( \varepsilon \) is small and thus \( \eta = D\varphi(x)/|D\varphi(x)| \). Therefore (B.1) again implies the desired inequality. \( \square \)

Lemma B.2. Suppose that \( p : B_1 \to \mathbb{R} \) is Lipschitz continuous, \( p_{\min} > 1 \) and that \( f \in C(B_1) \) is bounded. Assume that \( u \in C(\overline{B}_1) \) is a viscosity subsolution to \( -\Delta_{N}^pu \leq f - u \) in \( B_1 \) and that \( v \in C(\overline{B}_1) \) is a viscosity supersolution to \( -\Delta_{N}^pv \geq f - v \) in \( B_1 \). Then

\[
u \leq v \quad \text{on } \partial B_1
\]

implies

\[
u \leq v \quad \text{in } B_1.
\]

Proof. Step 1: Assume on the contrary that the maximum of \( u - v \) in \( \overline{B}_1 \) is positive. For \( x, y \in \overline{B}_1 \), set

\[
\Psi_j(x, y) := u(x) - v(y) - \varphi_j(x, y),
\]

where \( \varphi_j(x, y) := \frac{j}{4} |x - y|^4 \). Let \( (x_j, y_j) \) be a global maximum point of \( \Psi_j \) in \( \overline{B}_1 \times \overline{B}_1 \). Then

\[
u(x_j) - v(y_j) - \frac{j}{4} |x_j - y_j|^4 \geq u(0) - v(0)
\]

so that

\[
\frac{j}{4} |x_j - y_j|^4 \leq 2 \|u\|_{L^\infty(B_1)} + 2 \|v\|_{L^\infty(B_1)} < \infty.
\]

By compactness and the assumption \( u \leq v \) on \( \partial B_1 \) there exists a subsequence such that \( x_j, y_j \to \hat{x} \in B_1 \) and \( u(\hat{x}) - v(\hat{x}) > 0 \). Finally, since \( (x_j, y_j) \) is a maximum point of \( \Psi_j \), we have

\[
u(x_j) - v(y_j) \leq u(x_j) - v(y_j) - \frac{j}{4} |x_j - y_j|^4,
\]

and hence by continuity

\[
\frac{j}{4} |x_j - y_j|^4 \leq v(x_j) - v(y_j) \to 0
\]

as \( j \to \infty \).

Step 2: If \( x_j = y_j \), then \( D^2\varphi_j(x_j, y_j) = D^2\varphi_j(x_j, y_j) = 0 \). Therefore, since the function \( x \mapsto u(x) - \varphi_j(x, y_j) \) reaches its maximum at \( x_j \) and \( y \mapsto v(y) - (-\varphi_j(x_j, y)) \) reaches its minimum at \( y_j \), we obtain from the definition of viscosity sub- and supersolutions that

\[
0 \leq f(x_j) - u(x_j) \quad \text{and} \quad 0 \geq f(y_j) - v(y_j).
\]

That is \( 0 \leq f(x_j) - f(y_j) + v(y_j) - u(x_j) \), which leads to a contradiction since \( x_j, y_j \to \hat{x} \) and \( v(\hat{x}) - u(\hat{x}) < 0 \). We conclude that \( x_j \neq y_j \) for all large \( j \). Next we
apply the Theorem of sums \cite[Theorem 3.2]{CIL92} to obtain matrices $X, Y \in S^N$

\begin{align*}
(D_x \varphi(x, y), X) \in T^{2,+} u(x), \quad (-D_y \varphi(x, y), Y) \in T^{2,-} v(y)
\end{align*}

and

\begin{align*}
\begin{pmatrix}
X & 0 \\
0 & -Y
\end{pmatrix} \leq D^2 \varphi(x, y) + \frac{1}{j} (D^2(x, y))^2,
\end{align*}

(B.3)

where

\begin{align*}
D^2(x, y) = \begin{pmatrix} M & -M \\
-M & M
\end{pmatrix}
\end{align*}

with $M = j (2(x - y) \otimes (x - y) + |x - y|^2 I)$. Multiplying the matrix inequality (B.3) by the $\mathbb{R}^{2N}$ vector $(\xi_1, \xi_2)$ yields

\begin{align*}
\langle X \xi_1, \xi_1 \rangle - \langle Y \xi_2, \xi_2 \rangle &\leq \langle (M + 2j^{-1}M^2)(\xi_1 - \xi_2), \xi_1 - \xi_2 \rangle \\
&\leq (\|M\| + 2j^{-1}\|M\|^2) |\xi_1 - \xi_2|^2.
\end{align*}

Observe also that $\eta := D_x \varphi(x, y) = -D_y (x, y) = j |x - y|^2 (x - y) \neq 0$ for all large $j$. Since $u$ is a subsolution and $v$ is a supersolution, we thus obtain

\begin{align*}
f(y) - f(x) + u(x) - v(y) &\leq \text{tr}(X - Y) + (p(x) - 2) \left\langle X \frac{\eta}{|\eta|}, \frac{\eta}{|\eta|} \right\rangle - (p(y) - 2) \left\langle Y \frac{\eta}{|\eta|}, \frac{\eta}{|\eta|} \right\rangle \\
&\leq (p(x) - 1) \left\langle X \frac{\eta}{|\eta|}, \frac{\eta}{|\eta|} \right\rangle - (p(y) - 1) \left\langle Y \frac{\eta}{|\eta|}, \frac{\eta}{|\eta|} \right\rangle \\
&\leq (\|M\| + 2j^{-1}\|M\|^2) \left| \sqrt{p(x)} - 1 - \sqrt{p(y)} - 1 \right|^2 \\
&\leq C j |x - y|^2 \left( \frac{|p(x) - p(y)|^2}{\sqrt{p(x)} - 1 + \sqrt{p(y)} - 1} \right)^2 \\
&\leq C(j) j |x - y|^4.
\end{align*}

This leads to a contradiction since the left-hand side tends to $u(\hat{x}) - v(\hat{y}) > 0$ and the right-hand side tends to zero by (B.2).

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