A generalization of the concept of $\mathcal{PT}$ symmetry

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Abstract

We show that the $\mathcal{PT}$ symmetric Hamiltonians (and their generalizations $H = H^\dagger$ defined in the text) may be all assigned the projected (so called Feshbach or effective) nonlinear Hamiltonians which are “locally” Hermitian. This implies that many (if not all) of the bound-state energies may be real in a broad domain of Hermiticity-violating interactions. A complexification of a superintegrable $D = 2$ example is conjectured as an illustration.

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1 Introduction

Evolution of bound states in quantum mechanics is mediated (or generated) by their Hamiltonian, $|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$. In the models with Hermitian $H = H^\dagger$ the availability of solutions of the time-independent Schrödinger equation

$$H|\psi_n\rangle = E_n|\psi_n\rangle, \quad n = 0, 1, \ldots \quad (1)$$

simplifies this rule since all the eigenvalues $E_n$ remain real and the time-dependence of the separate eigenstates becomes elementary,

$$|\psi_n(t)\rangle = e^{-iE_n t}|\psi_n(0)\rangle. \quad (2)$$

A puzzling situation is encountered when the real energies $E_n$ are derived from a non-Hermitian Hamiltonian $H \neq H^\dagger$. Recently, Bender et al [1] conjectured that many models of such a type exist and are characterized by a “weaker analogue” of Hermiticity, $H = H^\dagger$. For the sake of definiteness they restricted their attention to a small class of the complex one-dimensional models and performed a number of numerical and semi-classical calculations showing that the spectra $\{E_n\}$ of their non-Hermitian $H = H^\dagger = p^2 + x^{2N}(ix)^\epsilon$ with $\epsilon > 0$ are real, discrete and bounded below. Generalizing this example they conjectured the definition

$$H^\dagger = \mathcal{PT} H \mathcal{PT} \quad (3)$$

of the required “weaker Hermiticity”. The operator $\mathcal{P}$ is defined as changing the parity, $\mathcal{P} x = -x$, while $\mathcal{T}$ mimics the time reversal, $\mathcal{T} i = -i$. Our present remark is inspired by the comparatively narrow range of the existing applications of the definition (3) which are mainly single-particle or one-dimensional (cf. [2]-[8]).

In a preparatory step we shall clarify an algebraic background of the apparently unmotivated assumption (3) (section 2). In the main body of this paper (sections 3 and 4) we shall propose a more general definition of the weakened Hermiticity $H = H^\dagger$. An applicability of the scheme is illustrated on a complexification of an elementary two-dimensional superintegrable example of ref. [9].

Our definition extends the $\mathcal{PT}$ symmetric class of non-Hermitian Hamiltonians supporting the real spectra and generating the oscillatory, unitary-like time-dependence (2) of bound states in quantum mechanics. In section 5 we shall argue
that the generalized scheme (with the parity replaced by a more general operator $Q$) remains mathematically selfconsistent. It admits many physical interpretations of the operator $Q$ which plays the role of an indefinite metric in our Hilbert (or rather Krein or Pontrjagin) space of admissible wavefunctions \[3, 4, 10\].

\[2\] Explanation

The $\mathcal{PT}$ symmetric Hamiltonians $H = \mathcal{PT} H \mathcal{PT}$ (with $\mathcal{P}^2 = 1$ and $\mathcal{T}^2 = 1$) which possess a real-energy solution $|\psi_n\rangle$ of eq. \((\text{I})\) resemble their Hermitian analogues in several aspects. They may be split in the real and imaginary part, $H = S + iA$ and, since $\mathcal{P} H \mathcal{P} = \mathcal{T} H \mathcal{T}$ by assumption, we have $\mathcal{P} S \mathcal{P} = S$, $\mathcal{P} A \mathcal{P} = -A$. Each wavefunction $|\psi_n\rangle$ may be complemented by another eigenstate $\mathcal{PT} |\psi_n\rangle$ at the same (real) energy $E_n$. In the generic non-degenerate case this means that among the two linearly dependent superposition solutions $\pm |\psi_n\rangle + \mathcal{PT} |\psi_n\rangle = |\psi^{[\pm]}_n\rangle$ of eq. \((\text{II})\), we are free to pick up one with the even or odd $\mathcal{PT} -$parity. In what follows we shall assume that the latter generalized parity has been fixed as positive which means that we can write

$$|\psi_n\rangle = |\sigma_n\rangle + i |\tau_n\rangle, \quad \mathcal{P} |\sigma_n\rangle = |\sigma_n\rangle, \quad \mathcal{P} |\tau_n\rangle = -|\tau_n\rangle.$$  

In any (e.g., harmonic-oscillator) basis \{ $|n^{(\alpha)}\rangle$ \} numbered by the integers $n = 0, 1, \ldots$ and by the even and odd parity $\alpha = \pm 1$ we can expand

$$|\sigma_n\rangle = \sum_{k=0}^{\infty} |n^{(+)}\rangle s_k, \quad |\tau_n\rangle = \sum_{n=0}^{\infty} |n^{(-)}\rangle t_k$$

and re-write our Schrödinger equation \((\text{I})\) in the linear algebraic form

$$\sum_{k=0}^{\infty} \langle m^{(+)} | S | k^{(+)}\rangle s_k - \sum_{j=0}^{\infty} \langle m^{(+)} | A | j^{(-)}\rangle t_j = E \ s_m,$$

$$\sum_{k=0}^{\infty} \langle m^{(-)} | S | k^{(-)}\rangle t_k + \sum_{j=0}^{\infty} \langle m^{(-)} | A | j^{(+)}\rangle s_j = E \ t_m$$

with $m = 0, 1, \ldots$. Switching to a compactified notation

$$\sum_{k=0}^{\infty} S_{mk}^{(+)} s_k - \sum_{j=0}^{\infty} A_{mj} t_j = E \ s_m,$$
\[ \sum_{k=0}^{\infty} S_{mk}^{(-)} t_k + \sum_{j=0}^{\infty} A_{mj}^s s_j = E t_m \]

and to its further matrix (non-Hermitian) abbreviation
\[ \begin{pmatrix} F - EI & -A \\ A^\dagger & G - EI \end{pmatrix} \begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} = 0, \tag{4} \]

we eliminate \( \vec{t} = -(G - EI)^{-1} A^\dagger \vec{s} \) and get the reduced Schrödinger equation with the so called Feshbach or effective energy-dependent Hamiltonian [1,2],
\[ H_{eff}(E) \vec{s} = E \vec{s}, \quad H_{eff}(E) = F + A (G - EI)^{-1} A^\dagger. \tag{5} \]

We can formulate our first important conclusion: The reality of the spectrum of many \( \mathcal{PT} \) symmetric Hamiltonians is the consequence of an elementary observation that their Feshbach’s effective Hamiltonian \( H_{eff}(E) \) in eq. (5) can be approximated by its energy-independent forms \( H_{eff}(\varrho) \). All of these effective Hamiltonians are Hermitian and possess the real spectra \( \{ E_n(\varrho) \} \). The exact energy levels are obtained from them via the nonlinear selfconsistency condition
\[ \varrho = \varrho_n = E_n(\varrho). \tag{6} \]

For comparison, it is extremely useful to imagine that our equation (4) is formally related to the problem where one replaces the upper right submatrix \(-A\) by the block \(+A\) with an opposite sign. The new Hamiltonian becomes Hermitian (and real and symmetric) and we derive its current effective \( H_{Herm.-eff}(E) \) which differs from its \( \mathcal{PT} \) symmetric predecessor in eq. (5) by the artificial sign-change,
\[ H_{Herm.-eff}(E) = F - A (G - EI)^{-1} A^\dagger. \]

Besides the Hermitian case, equation (5) may have solely real solutions in a broad domain of the coupling strengths in the \( \mathcal{PT} \) symmetric regime. An elementary illustration of such an expectation is provided by the two-by-two matrix with the four real matrix elements,
\[ \begin{pmatrix} f - E & -a \\ a & g - E \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = 0. \]
This mimics our Schrödinger eq. (4) and the dimension of its effective Hamiltonian is one. The exact spectrum

\[ E = E_{1,2} = \frac{1}{2} \left( f + g \pm \sqrt{(f-g)^2 - 4a^2} \right) \]

proves all real and non-degenerate if and only if \( 2|a| < |f-g| \). This is to be compared with the parallel Hermitian illustration the spectrum of which is always real,

\[ E_{\text{Herm.-eff}} = \frac{1}{2} \left( f + g \pm \sqrt{(f-g)^2 + 4a^2} \right). \]

We may summarize: In contrast to the Hermitian case the complete reality of the spectrum of non-Hermitian models is not robust and can be violated by a change of the magnitude of matrix elements. The \( \mathcal{PT} \) symmetry offers a firm guidance of our understanding of the stability of the spectrum in terms of the fairly transparent selfconsistency condition (6). The frequent occurrence of the \( \mathcal{PT} \) symmetric models with real spectra has been constructively confirmed by the numerous examples which are solvable non-numerically, without any recourse to their matrix representation (cf. refs. [5]).

3 Generalization

A core of our preceding explanation (why the \( \mathcal{PT} \) symmetric Hamiltonians (3) can have real energies) lies in the demonstration of the Hermiticity of their effective form \( H_{\text{eff}}(\varrho) \) (so that all the auxiliary \( E_n(\varrho) \) are real). The Hermitian and non-Hermitian alternative mechanisms of breaking the parity mean that we start from a doublet of independent even-parity and odd-parity Hamiltonians \( F \) and \( G \) and couple them in Schrödinger equation

\[
\begin{pmatrix}
F - EI \\
A^\dagger
\end{pmatrix}
\begin{pmatrix}
\alpha A \\
G - EI
\end{pmatrix}
\begin{pmatrix}
\vec{s} \\
\vec{t}
\end{pmatrix} = 0
\]

where either \( \alpha = 1 \) (Hermitian case) or \( \alpha = -1 \) (\( \mathcal{PT} \) symmetric case). The partitioning need not necessarily be related to the usual parity of the basis. Our argument has been entirely general. For example, in the Feshbach’s re-interpretation of eq. (7), the upper partition of the size \( \text{dim } F \) represents the more relevant part of the Hilbert
space determined by the so called “model space” projector. The lower Feshbach’s partition is usually treated in less detail. Thus, after we truncate the basis in the Hilbert space (say, in a variational setting), we can have very different partitions, with $m = \dim F \neq n = \dim G$. This is a comparatively easy generalization of the $\mathcal{PT}$ symmetry but does not seem to exhaust all the possibilities.

### 3.1 Partitioning three by three

The next step of our analysis is based on the triple partitioning of the basis, finite or infinite. Let us assume that $\alpha, \beta, \gamma = \pm 1$ and postulate a parallel between the $\mathcal{PT}$ symmetry and Hermiticity in the triply partitioned equation

$$
\begin{pmatrix}
F - EI & \alpha A & \beta B \\
A^\dagger & G - EI & \gamma C \\
B^\dagger & C^\dagger & Z - EI
\end{pmatrix}
\cdot
\begin{pmatrix}
\vec{r} \\
\vec{s} \\
\vec{t}
\end{pmatrix}
= 0.
$$

The elimination of $\vec{t} = -(Z - EI)^{-1} \left( B^\dagger \vec{r} + C^\dagger \vec{s} \right)$ gives us the two by two effective Schrödinger equation

$$
\begin{pmatrix}
F - EI & \alpha A \\
A^\dagger & G - EI
\end{pmatrix}
- \begin{pmatrix}
\beta B \\
\gamma C
\end{pmatrix}
\cdot
\begin{pmatrix}
(Z - \varrho I)^{-1} \left( B^\dagger C^\dagger \right) \\
\vec{r} \\
\vec{s}
\end{pmatrix}
= 0 \quad (8)
$$

where $\varrho \equiv E$. We intend to guarantee that the effective Hamiltonians $H_{eff}(\varrho)$ remain Hermitian. At any $\varrho = constant$ the reality of all the energy roots $E_n(\varrho)$ of the linearized eq. $(8)$ near $\varrho$ will be guaranteed by this Hermiticity. It is true for the diagonal blocks in $H_{eff}(\varrho)$. In order to satisfy also the “off-diagonal” condition,

$$
\alpha' A' = \alpha A - \beta B C^\dagger,
\quad (A')^\dagger = A^\dagger - \gamma C B^\dagger
$$

we choose $\alpha' = \alpha$ and arrive at the constraint

$$
\alpha \beta = \gamma \quad (9)
$$

and menu

| $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|
| + | + | + |
| + | - | - |
| - | + | - |
| - | - | + |

5
which lists all the available possibilities. The first line represents the Hermitian choice.

3.2 Partitioning four by four

Let us now preserve the latter rule (3), add the three new variables $\mu, \nu, \rho = \pm 1$ and postulate

$$
\begin{pmatrix}
F - EI & \alpha A & \beta B & \mu U \\
A^\dagger & G - EI & \gamma C & \nu V \\
B^\dagger & C^\dagger & Z - EI & \rho W \\
U^\dagger & V^\dagger & W^\dagger & K - EI
\end{pmatrix}
\begin{pmatrix}
\vec{p} \\
\vec{r} \\
\vec{s} \\
\vec{t}
\end{pmatrix}
= 0.
$$

The insertion of $\vec{r} = -(K - EI)^{-1} (U^\dagger \vec{p} + V^\dagger \vec{r} + W^\dagger \vec{s})$ reduces the Schrödinger equation to a three by three partitioned problem

$$
\begin{pmatrix}
F - EI & \alpha A & \beta B \\
A^\dagger & G - EI & \gamma C \\
B^\dagger & C^\dagger & Z - EI
\end{pmatrix}
\begin{pmatrix}
\vec{p} \\
\vec{r} \\
\vec{s}
\end{pmatrix}
- \begin{pmatrix}
\mu U \\
\nu V \\
\rho W
\end{pmatrix}
(K - \varrho I)^{-1}
\begin{pmatrix}
U^\dagger & V^\dagger & W^\dagger
\end{pmatrix}
\begin{pmatrix}
\vec{p} \\
\vec{r} \\
\vec{s}
\end{pmatrix}
= 0, \quad \varrho \equiv E.
$$

In order that the effective Hamiltonians $H_{\text{eff}}(\varrho)$ preserve the three by three symmetry $H = H^\dagger$, we satisfy the elementary "diagonal" conditions by fixing $\alpha' = \alpha$, $\beta' = \beta$ and $\gamma' = \gamma$. The three "off-diagonal" constraints

$$
\alpha \beta = \gamma, \quad \alpha \mu = \nu, \quad \beta \mu = \rho, \quad \gamma \nu = \rho.
$$
lead to the following eight solutions,

\[
\begin{array}{cccccccc}
\alpha & \beta & \gamma & \mu & \nu & \rho \\
+ & + & + & + & + & + \\
+ & + & + & - & - & - \\
+ & - & - & + & + & - \\
+ & - & - & - & + & + \\
- & + & - & + & - & + \\
- & + & - & - & - & - \\
- & - & + & - & + & + \\
- & - & + & - & - & - \\
\end{array}
\]  

(11)

The first line is the Hermitian choice and the fourth line reproduces the result of the two by two partitioning. The non-square two by two partitioning gives the lines number two and seven. The remaining four items offer the genuine three by three structures. Two contain the three minuses and the other two the four ones.

### 3.3 Example

The partitioning of the bases appears in the majority of their (e.g., variational) applications. For illustration, let us consider a two-dimensional Hamiltonian

\[ H = -\partial_x^2 - \partial_y^2 + x^2 + y^2 + \frac{g}{x^2} + \frac{g}{y^2} \]

which is superintegrable [9]. As a consequence, its two-dimensional time-independent Schrödinger equation allows the separation of variables not only in the cartesian system \((x, y) \in \mathbb{R}^2\) but also in polar coordinates where it degenerates to the Pöschl-Teller problem in the quadruple-well potential,

\[
\left( -\frac{d^2}{d\varphi^2} + \frac{g}{\cos^2 2\varphi} \right) \psi(\varphi) = k^2 \psi(\varphi), \quad \varphi \in (-\pi, \pi).
\]

In the standard Hermitian setting the latter equation decays in the four independent and exactly solvable eigenvalue problems with \(2\varphi \in (k\pi, k\pi + \pi)\) and \(k = -2, -1, 0, 1\), respectively. The symmetry

\[ [H, \mathcal{R}] = 0 \]
of the Hamiltonian with respect to the shift $\mathcal{R} : \varphi \to \varphi + \pi/2$ resembles the parity once we put $\mathcal{P} = \mathcal{R}^2$.

Let us now consider a breaking of the Hermiticity of the Hamiltonian. By a suitable complex deformation of the coordinate line, $\varphi = \varphi(t) = t + i \varepsilon(t), t \in (-\pi, \pi)$ we can avoid the barriers (i.e., poles of the potential which lie at the integer multiples of $\pi/2$) so that a tunneling takes place. Still, the energies need not become complex after such a regularization of the potential (cf. the three recent solvable examples in [8]), provided that we restrict our attention to the complexifications which preserve the commutativity

$$[H, \mathcal{RT}] = 0. \quad (12)$$

As long as we have $\mathcal{P}^2 = \mathcal{R}^4 = 1$, our model is not $\mathcal{PT}$ symmetric. Having a sample bound-state solution $|\psi\rangle$ and the new symmetry $(12)$ we infer that the state $\mathcal{R}|\psi\rangle$ lies within the same Hilbert space and satisfies Schrödinger differential equation at the identical energy. Superpositions

$$|\psi^{[k,l,m,n]}\rangle = (1 + i\mathcal{R})^k(1 - i\mathcal{R})^l(1 + \mathcal{R})^m(1 - \mathcal{R})^n|\psi\rangle$$

such that

$$\mathcal{R}|\psi^{[0,1,1,1]}\rangle = i|\psi^{[0,1,1,1]}\rangle, \quad \mathcal{R}|\psi^{[1,0,1,1]}\rangle = -i|\psi^{[1,0,1,1]}\rangle,$$

$$\mathcal{R}|\psi^{[1,1,0,1]}\rangle = -|\psi^{[1,1,0,1]}\rangle, \quad \mathcal{R}|\psi^{[1,1,1,0]}\rangle = +|\psi^{[1,1,1,0]}\rangle.$$ 

can be expanded in a basis $|n^{[k,l,m,n]}\rangle$ with $n = 0, 1, \ldots$ and with the superscript which marks the symmetry. Due to the Schur’s lemma, the Hermitian Hamiltonian matrix becomes block-diagonal and is partitioned accordingly.

Paralleling the two-by-two partitioning of $\mathcal{PT}$ symmetric Hamiltonians, we now have a freedom of adding interactions compatible with the four by four partitioning specified by the four different $\mathcal{R}$–parities. Such complexifications should obey any one of the conjugations $H = H^\dagger$ as listed in eq. $(11)$. In the light of what has been said before, we may expect a priori that at least a finite number of energies $E_n$ remains real for a number of non-Hermitian interaction terms.
4 Recurrences and re-orderings of the basis

One could construct the further conditions \( H = H^\dagger \) based on the partitioning \( N \) by \( N \) with \( N = 5 \) etc. The construction is recurrent in \( N \). A key to its efficient simplification exists and lies in a modification of the projection technique. One has to re-order the basis states and check how this changes the schemes of the type (11). The result is unexpected since all the complicated multiply partitioned solutions prove reducible to the single two by two structure of eq. (7) with the non-equal partitioning dimensions in general. The “generic”, \( (m+n) \times (m+n) \)–dimensional Schrödinger operator reads

\[
\begin{pmatrix}
G - EI & -C \\
C^\dagger & Z - EI
\end{pmatrix}
\]

with, by assumption, \( \dim G = m \), \( \dim Z = n \) and \( G = G^\dagger \), \( Z = Z^\dagger \). The inverse matrix exhibits the same structure,

\[
\left( \begin{array}{cc}
G - EI & -C \\
C^\dagger & Z - EI
\end{array} \right)^{-1} = \left( \begin{array}{cc}
G'(E) & -C'(E) \\
(C'^\dagger)(E) & Z'(E)
\end{array} \right).
\]

In the mathematical induction step, the dimension increases by one. With a new one-dimensional partition added on the top,

\[
\begin{pmatrix}
F - EI & \alpha A & \beta B \\
\alpha A^\dagger & G - EI & -C \\
\beta B^\dagger & C^\dagger & Z - EI
\end{pmatrix}
\]

the effective secular equation is one-dimensional,

\[
F - \left[ \begin{pmatrix}
\alpha A & \beta B
\end{pmatrix}
\begin{pmatrix}
G'(E) & -C'(E) \\
(C'^\dagger)(E) & Z'(E)
\end{pmatrix}
\right] \left( \begin{pmatrix}
A^\dagger \\
B^\dagger
\end{pmatrix} \right) = E.
\]

The effective Hamiltonian remains real if and only if

\[
\alpha = -\beta.
\]

This is the only condition required. We can very easily permute the basis and re-derive all the three by three solutions (11) as well as all the four by four schemes.
in eq. \((\text{[11]}\)\) etc. The latter case with \(m + n = 4\) is the first one which gives either the square-shaped \(A\) with \(m = n = 2\) or the oblong blocks \(A\) with dimensions \(3 = \max(m, n) > \min(m, n) = 1\).

We may conclude that the recurrent picture is consistent. At any partitioning \(N\) by \(N\) the number \(K\) of the sub-partitions with the minus sign (\(\alpha = -1, \ldots\)) is not arbitrary. Our construction admits one and two minus signs in the respective two by two and three by three partitioned matrices. At the higher \(N > 3\) our choice becomes non-unique and we can opt for the non-equivalent generalized “weakly non-Hermitian” structures of the Hamiltonian numbered by \(K = m \cdot n = 1 \cdot (N - 1)\) or \(2 \cdot (N - 2)\) etc. In each case a re-ordering of the basis states transforms the Hamiltonian into the canonical two by two structure

\[
H = H^\dagger = \begin{pmatrix} F & -A \\ A^\dagger & G \end{pmatrix}, \quad m = \dim F, \quad n = \dim G. \tag{14}
\]

with the negative sign attached to the \(m \times n\) matrix elements in the upper right submatrix of the Hamiltonian in Schrödinger eq. \((\text{[7]}\)\).

Once we increase the number of partitions \(\mathcal{M} = m + n\) of the Hamiltonian matrix by one, the necessary and sufficient condition \((\text{[13]}\)\) simply adds \(m\) or \(n\) blocks of minuses in the upper line of the new partitioned matrix. In the former case we can replace the upper-partition dimension \(m\) by \(m + 1\) after we permute the basis re-shuffling its topmost item to the bottom. In the latter case the two by two partitioning is unchanged and we replace \(n\) by \(n + 1\).

## 5 Summary

Generically, the models with non-Hermitian Hamiltonians \(H \neq H^\dagger\) possess the complex eigenvalues and make the evolution non-unitary. This is the reason why their so called \(\mathcal{P}\mathcal{T}\) symmetric special cases can be considered as an appealing alternative to their Hermitian predecessors. We have seen that there exists a formal connection between the Hermitian and \(\mathcal{P}\mathcal{T}\) symmetric form of \(H\). It is based on the similarity between their non-linear (so called effective) reductions constructed by the Feshbach’s projection method \(\text{[1]}\).
We described in detail this intimate relationship (i.e., sign-difference) between the respective Hamiltonians as well as effective Hamiltonians (the latter proved Hermitian in both these cases). We emphasized that even in the non-Hermitian, PT symmetric case the Schrödinger equation generated many (if not all) energies for a “very broad” range of the Hermiticity-violating components of the interaction.

The latter observation inspired an immediate generalization of the PT symmetry to a more general property (14). It is characterized by the partitioning dimensions \( m, n \) and degenerates to the Hermiticity at \( m = 0 \) or \( n = 0 \) and to the PT symmetry at \( m = n \neq 0 \). In all the non-Hermitian cases with \( m > 0 \) and \( n > 0 \) the left eigenvectors are different from the right ones. The validity of the equation

\[
\begin{pmatrix}
G - EI & -C \\
C^\dagger & Z -EI
\end{pmatrix}
\begin{pmatrix}
\vec{g} \\
\vec{z}
\end{pmatrix} = 0
\]

implies that

\[
\left[ (\vec{g}, -\vec{z}) \begin{pmatrix}
G - EI & -C \\
C^\dagger & Z -EI
\end{pmatrix} \right] = 0.
\]

The related “natural” normalization remains indefinite in its sign,

\[
\sum_{j=0}^{m} (g_j)^2 - \sum_{k=0}^{m} (z_k)^2 = \pm 1.
\]

(15)

This can be interpreted as a result of an overlap between the right eigenvector \( |\psi\rangle \) and its new conjugate \( \langle \langle \psi | = \langle \psi | Q \). The “metric” \( Q \) is a unit matrix with the last \( m \) diagonal elements replaced by \(-1\). In the PT symmetric special case where \( m = n \) this operator coincides with the parity \( P \).

The innovation of the bra vector leads to the modified inner product. It exhibits the (pseudo-)orthogonality feature

\[
\langle \langle \psi_j | \psi_k \rangle = \langle \psi_j | Q | \psi_k \rangle = \pm \delta_{jk}
\]

(16)

when computed between the two different eigenstates of \( H \). The alternative inner product has been used in many \( m = n \) studies of the perturbations of Hermitian Hamiltonians (cf. ref. [12] and, especially, Corollary II.7.6 there) as well as in the early stages of development of the elementary PT symmetric models (cf. [8] and eq. (14) there, or the text after eq. (6) in ref. [5]).
In the present context, the use of the $m \neq n$ “metric” $Q$ leads to a natural generalization of the concept of the Hilbert space [10]. The self-overlaps ([13] or ([16]) at $j = k$] can be understood as a pseudo-norm in a space where the time-evolution is pseudo-unitary [3]. This opens new perspectives and questions (e.g., about the possible physical interpretation of the wavefunctions) shared by our present generalized formalism with its increasingly popular $\mathcal{PT}$ symmetric predecessor [4].

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