Probing Equilibrium and Dynamical Criticality Through Single-Site Observables

Ceren B. Da˘g,1,* Philipp Uhrich,2 and Jad C. Halimeh2

1Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA
2INO-CNR BEC Center and Department of Physics, University of Trento, Via Sommarive 14, I-38123 Trento, Italy

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Extracting critical behavior in the wake of quantum quenches has recently been at the forefront of theoretical and experimental investigations in condensed matter physics and quantum synthetic matter, with particular emphasis on experimental feasibility. Here, we investigate the potential of single-site observables in probing equilibrium phase transitions and dynamical criticality in short-range transverse-field Ising chains. For integrable and near-integrable models, our exact and mean-field-theory analyses reveal a truly out-of-equilibrium universal scaling exponent in the vicinity of the transition that is independent of the initial state and the location of the probe site so long as the latter is sufficiently close to the edge of the chain. Signature of a dynamical crossover survives when integrability is strongly broken. Our work provides a robust scheme for the experimental detection of quantum critical points and dynamical scaling laws in short-range interacting models using modern ultracold-atom setups.

Out-of-equilibrium quantum many-body dynamics has become a major research area [1] due to recent impressive progress in the control and precision achieved in quantum synthetic matter (QSM) [2–8]. Not only have concepts from equilibrium physics been extended to the out-of-equilibrium realm such as with dynamical phase transitions (DPTs) [9–13] and dynamical scaling laws [11, 14–19], but there have also been concerted efforts to probe equilibrium quantum critical points (QCPs) and universal scaling laws through quench dynamics [14, 16, 18, 20–26] or with infinite-temperature initial states [27–29]. Such techniques obviate the need for undertaking the usually difficult task of cooling the system into its ground state over a range of its microscopic parameters in order to construct its equilibrium phase diagram.

In a global quench, the ensuing dynamics of a quantum many-body system can yield signatures of dynamical criticality, such as DPTs. Several major concepts of DPTs have been proposed [9, 30] with some of them converging [31]. One such DPT is of the Landau paradigm, i.e., it is based on nonanalytic behavior in the long-time dynamics of the local order parameter. This indicates that, in principle, such nonanalytic behavior may be used to extract equilibrium criticality that manifests itself dynamically. Indeed, it has been shown that this is possible through, e.g., out-of-time-ordered correlators [23, 28, 32] and spin-spin correlations [15, 21, 25] in the wake of a quench. Given that quench protocols in modern QSM setups are relatively straightforward to implement compared to the preparation of a system in its ground state, it is worth further investigating experimentally feasible methods aimed at extracting equilibrium criticality through quench dynamics.

In this spirit, we show here that the dynamics of single-site observables close to the boundary of a short-range transverse-field Ising chain (TFIC) is a promising venue for the detection of a QCP. Whereas a single-site observables at an arbitrary site of a periodic chain decays exponentially for quenches starting in the ordered phase of the nearest-neighbor TFIC [33–35], a hard-boundary condition gives rise to a quasi-stationary regime [36]. This allows one to probe equilibrium phase transitions and dynamical criticality by utilizing the degrees of freedom close to the boundaries. Although the profile of the on-site order dynamics naturally differs for each single-site observable and depends on the initial state, we observe a universal behavior in the vicinity of the transition for sites sufficiently close to the chain boundaries independently of the initial state.

The same behavior is also observed in a near-integrable model under mean-field theory (MFT) analysis, which, together with the above, suggests a universal scaling exponent in the vicinity of the transition. This observation stems from the fact that the relaxation time to the quasi-stationary value diverges as we move towards the transition point, which is a consequence of critical slowing down. Single-site observables at different sites approach the quasi-stationary value in qualitatively the same way and independently of the initial state. Therefore, the information of the precise site location is effectively washed away in the vicinity of the transition. We determine the dynamical critical point (DCP) [25] in the near-integrable model, which we find to be close in value to the QCP. Furthermore, time-dependent density-matrix renormalization group (t-DMRG) calculations show a dynamical crossover when integrability is strongly broken, albeit data is inconclusive as to whether this is possibly a DPT due to numerical limitations on accessible evolution times.

We emphasize that we use the simplest possible probe, a single-site observable, in a chain with hard boundaries, which is experimentally more relevant than a periodic chain. Our work also forms a complementary approach to recent works on local probes in DPT [37] and other dynamical schemes for detecting equilibrium phase tran-
transverse-field strength, where $\sigma_h J$ we fix the latter to a value $h$ is the integrable nearest-neighbor TFIC. Let us consider the short-range TFIC with interaction boundaries [18, 27, 28].

The quasi-stationary temporal regime in chains with boundaries—. The short-range TFIC with interaction strength $\Delta$ is given by

$$H = -J \sum_{r=1}^{N-1} \sigma_r^x \sigma_{r+1}^x - \Delta \sum_{r=1}^{N-2} \sigma_r^z \sigma_{r+2}^z + h \sum_{r=1}^{N} \sigma_r^z, \tag{1}$$

where $\sigma_r^{x,z}$ are the Pauli spin matrices on site $r$, $h$ is the transverse-field strength, $N$ is the length of the chain, and we fix $J = 1$ as the energy scale. At $\Delta = 0$, this model is the integrable nearest-neighbor TFIC. Let us consider as initial state the ground state $|\psi_0\rangle$ of $H$ at initial value $h_i$ of the transverse-field strength, and then we quench the latter to a value $h$. Even though in the case of the TFIC under periodic-boundary conditions the single-site nonequilibrium response $C_r(t) = \langle \psi_0 | \sigma_r^z(t) | \psi_0 \rangle$ decays exponentially [18, 34], it has long been realized that open-boundary conditions stabilize a quasi-stationary regime in the integrable TFIC when $h_i < h \leq h_c$ [36] where $h_c$ is the QCP. Figure 1a shows the quasi-stationary regime of $C_{r=6}(t)$ in the integrable TFIC for various $h$ and systems sizes ranging between $N = 96 - 1440$ with $h_i = 0$ where the initial state is the fully $z$-up product state, $|\psi_0\rangle = |\uparrow \cdots \uparrow\rangle$ (see [38] for the methods). It is straightforward to demonstrate that this stationary regime is not thermal by simply observing the strong dependence of the stationary value on the initial state in Fig. 1a when $h_i = 0.1$. Since the stationary value carries the initial state information, and given the fact that this regime persists for all accessible times up to recurrences [36] at all sites $r \ll N/2$, degrees of freedom near the boundaries do not thermalize. The origin of this boundary effect is in fact independent of the integrability of TFIC, c.f. Fig. 1b for various nonintegrable cases up to some oscillations computed via $t$-DMRG, confirming the quasi-stationary nature of this temporal regime. Recurrences observed in some cases occur due to finite-size effects [38]. The stationary value again carries the initial state information. In a similar vein, this boundary effect is observed when interactions are power-law decaying long-range interactions [38–40]. Therefore, our work contributes another example of a strongly nonintegrable system exhibiting nonthermal behavior [41]. We note that this boundary effect is robust against changing the hard boundaries to smooth ones [38, 42], does not originate from strong zero modes [38, 43], and clearly is not confined to only the edge of the chain. Rather, the reason is simply the geometry of the open-boundary chain where the asymmetric location of a site $r \ll N/2$ causes destructive interference between two signals, one of which reflects back from the closest edge much earlier than the other moving towards the farthest edge [36]. This stabilizes a quasi-stationary nonthermal temporal regime regardless of the integrability of the system. In the rest of our work, we will utilize this quasi-stationary regime of single-site observables to probe equilibrium phase transitions and dynamical criticality in the short-range TFIC. Unless otherwise specified, we use polarized states as initial states and set hard boundaries in our discussion.

Quench dynamics in the integrable TFIC—. In equilibrium, the TFIC has two phases, i) the ferromagnetically ordered phase for $h < h_c$ and ii) the paramagnetic disordered phase for $h > h_c$. The QCP shifts to favor order upon introducing interactions ($\Delta \neq 0$). The local order parameter is the magnetization averaged over all sites, and when it is finite it indicates spontaneous symmetry breaking in the ground state.

The dynamical order parameter is set as the time-averaged nonequilibrium response, $C_r(t) =$
Hence, the behavior of the singularity at the same plot in logarithmic scale that exhibits a clear... with increasing system size. This observation suggests... different functional form of \( r \) in the thermodynamic limit, we would observe... for the value of the quasi-stationary regime at... \( \delta \)... in the dynamically ordered phase varies for different... \( v \)... can be estimated based on the maximal quasi-particle... center of the open-boundary chain [38]. For quenches... yielding... that reflects the underlying ground-state phase... \( \gamma \sim 1 \). This power-law decay is independent of the choice of ultraviolet temporal... as long as \( h_i < h_c \) holds [38]. Therefore, we demonstrate a DPT for different sites... reflecting the underlying ground-state phase transition. The functional form of the dynamic order in the dynamically ordered phase varies for different \( r \). Nevertheless, we observe that the site information is effectively washed away in the vicinity of the transition.

Figure 3a shows how the time-average \( \bar{C}_r(t) \) scales with the reduced control parameter \( h_n = (h_c - h)/h_c \) for different \( r \ll N/2 \) for quenches to the vicinity of \( h_c = 1 \). As we move closer to the phase transition, all sites exhibit the same scaling behavior \( \bar{C}_r(h) - \bar{C}_r(h_c) \propto h_n^\beta \) where \( \beta \sim 4/3 \) is obtained through fit functions [38]. This is a main result of our work, indicating a possibly truly out-of-equilibrium critical exponent, since it is not found in equilibrium. A comparison with the analytic expression of the edge magnetization in the quasi-stationary regime \( C_{r=1}^{qs}(h) \) [36] (black-solid line in Fig. 3a), reveals that \( C_{r=1}^{qs}(h) \) does not describe well the time-average of the data in the close vicinity of the transition. The origin of this discrepancy lies in the fact that the analytic expression is strictly valid when \( t \to \infty \), whereas the relaxation time diverges in the vicinity of the transition. This means that in practice the quasi-stationary regime cannot be reached in the close vicinity of the transition, neither numerically nor in experiments. A similar effect has been recently observed in the periodic TFIC [18], where the analytic expressions are derived in the \( t \to \infty \) limit [34]. We do not study the region \( h_n < 10^{-4} \), because the nonequilibrium response does not show true equilibration to a quasi-stationary state over accessible evolution times due to the diverging relaxation time [38]. Figure 3b displays how nonequilibrium response functions of different sites collapse onto each other when rescaled according to the coefficients of the fit function appearing in Fig. 3a, e.g., \( C_r'(h_n) = a_r h_n^\beta \) and \( C_r'(h_c) = a_r h_c^\beta \) and hence the scaling factor reads \( C_r'(h_n)/C_r'(h_c) = a_r/a_r' \), i.e., \( C_r'(t) = C_r'(t) a_r/a_r' \). Even though the frequencies differ, all nonequilibrium responses converge to a single envelope function when rescaled, displaying a universal behavior in the relaxation to the quasi-stationary value. We emphasize that this region, where scaling at all sites \( r \ll N/2 \) yields the critical exponent \( \beta \), still describes the critical relaxation regime to the quasi-stationary value, although it lasts for very long times, i.e., the time-averages in Fig. 3b (black-solid) would not perfectly match with the value in the quasi-stationary regime (green-dotted).

This observation can also be confirmed by the scaling change in the edge magnetization from the analytic prediction of the quasi-stationary value \( C_{r=1}^{qs}(h_n) = 2 h_n - h_n^2 \) [38]. For quenches in the vicinity of the transition, the...
latter scales as $C_{\tau=1}(h_n) \propto h_n$. However, our numerical results show that a critical relaxation regime preceding the quasi-stationary state should persist indefinitely as $N \to \infty$ due to critical slowing down, leading to $C_{\tau=1}(h_n) \propto h_n^{1/3}$. Changing the initial state does not alter the numerically obtained scaling exponents of the magnetization per site in the vicinity of the transition [38]. As a consequence, we demonstrate the presence of a slowed down critical relaxation regime in the vicinity of the QCP which precedes the quasi-stationary regime and results in a dynamical critical exponent of $\sim 4/3$.

**Nonintegrable TFIC**. We apply MFT analysis for $\Delta = -0.1$ [38] which is a near-integrable TFIC. This model has a QCP at $h_c \sim 1.16$ [25]. Figure 4a shows the dynamical phase diagram for different $r = 1 - 12$, which looks similar to the case of the integrable TFIC ($\Delta = 0$) except for a shift in the QCP to favor order, $h_c > 1$, as expected. We notice that the quasi-stationary value of the edge magnetization, which our numerical results access for quenches far away from the vicinity of the transition, can be fitted well with a functional form that is reminiscent of that of the integrable TFIC, $C_{\tau=1}(h) = \alpha(h_{dc} - h)^\nu$ for $h \leq h_{dc}$ and zero otherwise, where we denote the DCP as $h_{dc}$. We focus on $C_{\tau=6}(h)$ in Fig. 4b where we observe a singularity at $h_{dc} \sim 1.144 \pm 0.001 < h_c$, whose magnetization decreases with increasing system size [38]. Meanwhile, the same (opposite) behavior is observed in the dynamically disordered (ordered) phase, suggesting a phase transition at $h_{dc}$. In Fig. 4c we plot the nonequilibrium responses around the DCP for $N = 1440$, which shows a qualitative change in the nonequilibrium response across the DCP. For example, up until $h = 1.144$ which is denoted by the gray-dashed line in Fig. 4c, there is evidence of equilibrium, whereas starting at $h = 1.145$, which is denoted by the pink-dotted line, the response starts to develop a low-frequency oscillatory feature seen in the downward trend of its dynamics (see [38] for data on $h > 1.145$). This feature is captured in Fig. 4b as a singularity. Based on this method, we further refine the DCP to be at $h_{dc} = 1.1437 \pm 0.0001$, which is slightly smaller than the corresponding QCP at $h_c \sim 1.16$. Although it is possible that the DCP we obtain is actually the QCP and this small numerical difference is a mere artifact of MFT, it is worth noting that MFT usually predicts a larger critical point than the physical one since it neglects fluctuations, whereas here $h_{dc} < h_c$. With $h_{dc}$ substituted into the fit function of the edge magnetization, we obtain $\alpha = 0.78$ and $\nu = 1.89$. It is important to note that although the MFT treatment gives rise to a quasi-stationary regime for long intervals of time as seen in Fig. 4c for quenches away from the vicinity of the DCP, this is not conclusive evidence for infinitely long-lived nonthermal behavior in a nonintegrable model. Indeed, MFT is not expected to adequately capture thermalization as it may neglect fluctuations that are essential for the latter.

Next we study the close vicinity of the transition in the MFT data. Figure 4d reveals an out-of-equilibrium critical exponent $\beta \sim 4/3$ based on the fit functions in the relaxation regime for different single-site observables $r = 3 - 12$ [38], in congruence with the conclusions of the integrable case. Once again, this shows how the analytic prediction for the quasi-stationary steady state does not capture this exponent since $C_{\tau=1}(h_n) \propto h_n$ as $h_n \to 0$ [38]. As in the integrable TFIC, a diverging relaxation time for quenches in the vicinity of the DCP gives rise to a very long-lived critical relaxation regime where the quasi-
stationary steady state is accessible neither numerically nor in experiment.

Finally, we consider the strongly nonintegrable TFIC with \( \Delta = -1 \), whose QCP lies at \( h_c \sim 2.46 \) [18]. At such large values of \( \Delta \), MFT is inadequate, and we therefore employ \( t \)-DMRG. The latter is numerically exact, but within a given fidelity threshold the accessible evolution times are limited and far shorter than those achievable for the (near-)integrable model. Figure 5a shows the local order profile for \( r = 3 \) and \( N = 48 \) spins, which exhibit behavior similar to that of our previous results. Figure 5b focuses on \( r = 6 \) and exhibits the increasing trend of order with increasing system size in the dynamically ordered phase. We apply a temporal cutoff of \( t_l = N/3 \) for both to calculate \( C_r(h) \). Although we cannot precisely determine a DCP due to lack of data for larger times (a constraint due to increasing bond dimension with evolution time in \( t \)-DMRG), our data still suggests a dynamical crossover. Intricate details of this crossover, e.g., whether it is actually a DPT if longer evolution times are available, are not possible to discern within the limited evolution times we can achieve in \( t \)-DMRG.

Conclusions—. We have studied the quasi-stationary regime and the critical relaxation to it at sites close to the edge in short-range TFICs with open-boundary conditions. We have shown that single-site observables are able to extract quantum critical points at or near integrability. The corresponding dynamical phase transition is present independently of the measurement location and initial state. In the vicinity of the transition, a universal scaling behavior with a truly out-of-equilibrium exponent \( \beta \sim 4/3 \) emerges in the very long-lived critical relaxation regime to the quasi-stationary steady state.

Our setup is experimentally convenient, because (i) hard-boundary chains are a more natural setup than their periodic counterparts in experimental realizations, (ii) single-site observables are readily accessible in modern quantum simulators [3]. Further, since we have shown that the physics near the edge of the chain is independent of the initial state, so long as \( h_i < h_c \), one can prepare the most convenient initial state in a lab, e.g., a polarized state. Most theoretical works have focused on periodic chains naturally to utilize the translational symmetry, which removes site-dependency of the dynamical order parameter within the ordered phase [25, 44, 45]. In this sense, our work complements the literature via explicitly demonstrating the potential of single-site observables in open-boundary chains. One can reproduce the periodic chain results in the middle of an open-boundary chain [18], however most of the chain would actually diverge from this behavior due to the boundary effects exemplified above. Our setup does not require a precise choice of location \( r \ll N/2 \), yet one can probe a universal dynamical criticality in the vicinity of the transition.

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* cbdag@umich.edu

[1] J. Eisert, M. Friesdorf, and C. Gogolin, Nature Physics 11, 124–130 (2015).
[2] M. Greiner, O. Mandel, T. W. Hänsch, and I. Bloch, Nature 419, 51–54 (2002).
[3] W. S. Bakr, J. I. Gillen, A. Peng, S. Fölling, and M. Greiner, Nature (London) 462, 74 (2009), arXiv:0908.0174 [cond-mat.quant-gas].
[4] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauß, T. Fukuhara, C. Gross, I. Bloch, C. Kollath, and S. Kuhr, Nature 481, 484–487 (2012).
[5] R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. Röspringi, and M. Greiner, Nature 528, 77 (2015).
[6] A. M. Kaufman, M. E. Tai, A. Lukin, M. Röspringi, R. Schittko, P. M. Preiss, and M. Greiner, Science 353, 794–800 (2016).
[7] J. Zhang, G. Pagano, P. W. Hess, A. Kyriakidis, P. Becker, H. Kaplan, A. V. Goshkov, Z.-X. Gong, and C. Monroe, Nature 551, 601–604 (2017).
[8] M. Gättnner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, Nature Physics 13, 781–786 (2017).
[9] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Journal of Physics B: Atomic, Molecular and Optical Physics 51, 112001 (2018).
[10] M. Heyl, A. Polkovnikov, and S. Kehrein, Phys. Rev. Lett. 110, 135704 (2013).
[11] N. Tsuji, M. Eckstein, and P. Werner, Phys. Rev. Lett. 110, 136404 (2013).
[12] P. Jurcevic, H. Shen, P. Hauke, C. Maier, T. Brydges, C. Hempel, B. P. Lanyon, M. Heyl, R. Blatt, and C. F. Roos, Phys. Rev. Lett. 119, 080501 (2017).
[13] N. Fläschner, D. Vogel, M. Tarnowski, B. S. Rem, D.-S. Lühmann, M. Heyl, J. C. Budich, L. Mathey, K. Sengstock, and C. Weitenberg, Nature Physics 14, 265 (2018).
[14] B. Sciolla and G. Biroli, Phys. Rev. B 88, 201110 (2013).
[15] E. Nicklas, M. Karl, M. Höffler, A. Johnson, W. Mussel, H. Strobel, J. Tomković, T. Gasenzer, and M. K. Oberthaler, Phys. Rev. Lett. 115, 245301 (2015).
[16] A. Ciocchetta, M. Tavora, A. Gambassi, and A. Mitra, Phys. Rev. B 91, 220302 (2015).
[17] V. Gurarie, Phys. Rev. A 100, 031601 (2019).
[18] C. B. Da˘ g and K. Sun, arXiv e-prints, arXiv:2004.12287 (2020), arXiv:2004.12287 [quant-ph].
[19] D. Trapin, J. C. Halimeh, and M. Heyl, (2020), arXiv:2005.06481 [cond-mat.stat-mech].
[20] C. Kollath, A. M. Läuchli, and E. Altman, Phys. Rev. Lett. 98, 180601 (2007).
[21] M. Karl, H. Cakir, J. C. Halimeh, M. K. Oberthaler,
M. Kastner, and T. Gasenzer, Phys. Rev. E 96, 022110 (2017).

[22] C. B. Dağ, S.-T. Wang, and L.-M. Duan, Phys. Rev. A 97, 023603 (2018).

[23] M. Heyl, F. Pollmann, and B. Dóra, Phys. Rev. Lett. 121, 016801 (2018).

[24] H.-X. Yang, T. Tian, Y.-B. Yang, L.-Y. Qiu, H.-Y. Liang, A.-J. Chu, C. B. Dağ, Y. Xu, Y. Liu, and L.-M. Duan, Phys. Rev. A 100, 013622 (2019).

[25] P. Titum, J. T. Iosue, J. R. Garrison, A. V. Gorshkov, and Z.-X. Gong, Phys. Rev. Lett. 123, 115701 (2019).

[26] P. Uhrich, N. Defenu, R. Jafari, and J. C. Halimeh, Phys. Rev. B 101, 245148 (2020).

[27] F. J. Gómez-Ruiz, J. J. Mendoza-Arenas, F. J. Rodríguez, C. Tejedor, and L. Quiroga, Phys. Rev. B 97, 235134 (2018).

[28] C. B. Dağ, L.-M. Duan, and K. Sun, Phys. Rev. B 101, 104415 (2020).

[29] Z.-H. Sun, J.-Q. Cai, Q.-C. Tang, Y. Hu, and H. Fan, Annalen der Physik 532, 1900270 (2020).

[30] M. Heyl, Reports on Progress in Physics 81, 054001 (2018).

[31] J. Lang, B. Frank, and J. C. Halimeh, Phys. Rev. B 97, 174401 (2018).

[32] C. B. Dağ, K. Sun, and L.-M. Duan, Phys. Rev. Lett. 123, 140602 (2019).

[33] P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006).

[34] P. Calabrese, F. H. L. Essler, and M. Fagotti, Journal of Statistical Mechanics: Theory and Experiment 2012, P07016 (2012).

[35] P. Calabrese, F. H. L. Essler, and M. Fagotti, Phys. Rev. Lett. 106, 227203 (2011).

[36] F. Iglói and H. Rieger, Phys. Rev. Lett. 106, 035701 (2011).

[37] J. C. Halimeh, D. Trapin, M. V. Damme, and M. Heyl, (2020), arXiv:2010.07307 [cond-mat.quant-gas].

[38] See Supplemental Material at [link provided by publisher].

[39] Z.-X. Gong and L.-M. Duan, New Journal of Physics 15, 113051 (2013).

[40] B. Neyenhuis, J. Zhang, P. W. Hess, J. Smith, A. C. Lee, P. Richerme, Z.-X. Gong, A. V. Gorshkov, and C. Monroe, Science Advances 3 (2017), 10.1126/sciadv.1700672.

[41] C. Gogolin, M. P. Müller, and J. Eisert, Phys. Rev. Lett. 106, 040401 (2011).

[42] M. Vekić and S. R. White, Phys. Rev. Lett. 71, 4283 (1993).

[43] J. Kemp, N. Y. Yao, C. R. Laumann, and P. Fendley, Journal of Statistical Mechanics: Theory and Experiment 2017, 063105 (2017).

[44] J. C. Halimeh, V. Zauner-Stauber, I. P. McCulloch, I. de Vega, U. Schollwöck, and M. Kastner, Phys. Rev. B 95, 024302 (2017).

[45] J. C. Halimeh, M. Van Damme, V. Zauner-Stauber, and L. Vanderstraeten, Phys. Rev. Research 2, 033111 (2020).
Supplemental Material: Probing Equilibrium and Dynamical Criticality Through Single-Site Observables

In this supplementary we provide additional evidence for the observations stated in the main text as well as details of the methods utilized in this paper.

METHODS

Mapping to noninteracting fermions and cluster theorem

We map the integrable TFIC to a noninteracting fermionic model in 1D via the transformation \[ S1, \]

\[
\sigma^z_r = -\prod_{s\neq r} (1 - 2c^\dagger_s c_s) (c_r + c_r^\dagger),
\]

(\textbf{S1})

\[
\sigma^x_r = 1 - 2c^\dagger_r c_r,
\]

\[
\sigma^y_r = -i \prod_{s\neq r} (1 - 2c^\dagger_s c_s) (c_r - c_r^\dagger),
\]

to obtain the noninteracting Hamiltonian

\[
H = -J \sum_r \left( c^\dagger_r c_{r+1} + c^\dagger_{r+1} c_r + c^\dagger_r c_{r+1} c_{r+1} + c_{r+1} c_r - 2hc^\dagger_r c_r \right).
\]

(\textbf{S2})

One can immediately see that calculating the dynamical evolution of a bulk spin \( \langle \sigma^z_r(t) \rangle \) in the noninteracting picture brings a string of operators and is not really tractable. Hence we instead calculate equal-time two-point correlators and invoke the cluster theorem \[ S2, \]

\[
\langle \sigma^z_r(t) \sigma^z_{N-r+1}(t) \rangle \sim \langle \sigma^z_r(t) \rangle \langle \sigma^z_{N-r+1}(t) \rangle,
\]

(\textbf{S3})

where \( r \ll N/2 \) is close to the left boundary. The cluster theorem holds in the lightcone, meaning up until the time \( t_l \) when the two sites \( r \) and \( N - r + 1 \) become correlated with one another due to operator spreading. This time can be estimated based on the maximum quasiparticle velocities \( v_q \), as \( t_l = \Delta x / (2v_q) \) where \( \Delta x = N - 2r + 1 \) is the distance between two spins that are equidistant from the symmetry center of an open-boundary chain in Eq. (S3), which is the middle of the chain. For such symmetrically placed sites, the non-equilibrium response is the same and one can thus write

\[
\langle \sigma^z_r(t) \rangle = \sqrt{\langle \sigma^z_r(t) \sigma^z_{N-r+1}(t) \rangle} \equiv \langle C_r(t) \rangle.
\]

(\textbf{S4})

Therefore, we can extract the dynamical evolution of a bulk spin at site \( r \) from the equal-time two-point correlators of sites \( r \) and \( N - r + 1 \). The latter can be written in terms of auxiliary operators \( \phi^+_{r}(t) = c^\dagger_r \pm c_r \) as

\[
\langle \sigma^z_r(t) \sigma^z_{N-r+1}(t) \rangle = \left\langle \phi^+_{r}(t) \left( \prod_{s=r+1}^{N-r} \phi^+_{s}(t) \phi^-_{s}(t) \right) \phi^+_{N-r+1}(t) \right\rangle.
\]

(\textbf{S5})

This is, in fact, the expectation value of the so-called string order parameter (SOP) \[ S3 \]. It can be calculated by invoking Wick’s theorem, which allows one to re-express the above expectation value as a sum over products of elementary contractions, which in turn is the Pfaffian of an appropriately constructed antisymmetric matrix \( T(t) \) i.e. \( (C_r(t))^2 = \text{Pf}(T(t)) \) \[ S1, S2 \]. Although this is in general a complex number, it follows from (S3) that for \( t < t_l \) we actually have \( (C_r(t))^2 \in \mathbb{R} \) so that we may compute \( C_r(t) = |\text{Pf}(T(t))|^{1/2} = |\sqrt{\det(T(t))}|^{1/2} \). This is advantageous since it is numerically more efficient to calculate determinants as compared to Pfaffians. To construct the matrix \( T(t) \) we only need to calculate all possible elementary two-point contractions \( \langle \phi^p_{a}(t) \phi^q_{b}(t) \rangle \) where \( p, q = \pm \) and \( r \leq a \leq b \leq N - r + 1 \). Additionally, we need to incorporate the mechanism of a sudden quench in this picture. For this we follow the procedure outlined in Ref. [S4], and we briefly review it here.

Since we quench from a polarized state, we have \( h_i = 0 \), where \( h_i \) is the transverse field of the initial Hamiltonian
of the quench procedure. Therefore, we first solve the initial Hamiltonian $H_i$, 

$$H_i = \sum_k E_k^i \alpha_k^\dagger \alpha_k,$$

(S6)

where $E_k$ and $\alpha_k$ are the single particle eigenenergies and eigenstates, respectively. The solution reads in general terms,

$$\begin{pmatrix} \alpha \\ \alpha^\dagger \end{pmatrix} = \begin{pmatrix} G_i & F_i \\ F_i & G_i \end{pmatrix} \begin{pmatrix} c_i \\ c_i^\dagger \end{pmatrix},$$

(S7)

where $c_i = (c_1, c_2, \ldots, c_N)^T$ and similarly for the creation operator $c_i^\dagger$. Note that one can work in this Bogoliubov-de Gennes (BdG) basis with its size doubled [S5], however here we work with the block matrices $G$ and $F$ [S6] which is computationally more efficient. By solving the eigensystem of

$$[(A_i - B_i)(A_i + B_i)] |\Phi_k^i \rangle = (E_k^i)^2 |\Phi_k^i \rangle,$$

(S8)

we obtain the eigenenergies $E_k^i$ and eigenvectors $|\Phi_k^i \rangle$. Here $A_i$ and $B_i$ are the nearest neighbor hopping and the pairing terms in the Hamiltonian, respectively, so that the Hamiltonian could be written as,

$$H_i = \begin{pmatrix} A_i & B_i \\ B_i^\dagger & -A_i \end{pmatrix},$$

(S9)

in the $(c \ c^\dagger)$ basis. Then we use the eigensystem $(E_k^i \ |\Phi_k^i \rangle)$ to find

$$|\Psi_k^i \rangle = \frac{1}{E_k^i} \langle \Phi_k^i | (A_i - B_i) \rangle^T.$$

(S10)

Now we can calculate the $G_i$ and $F_i$ in terms of $|\Phi_k^i \rangle$ and $|\Psi_k^i \rangle$. Defining

$$\Phi_i = [|\Phi_1^i \rangle \ |\Phi_2^i \rangle \cdots |\Phi_N^i \rangle],$$

$$\Psi_i = [|\Psi_1^i \rangle \ |\Psi_2^i \rangle \cdots |\Psi_N^i \rangle],$$

the block matrices follow

$$G_i = \frac{1}{2} \left( \Phi_i^T + \Psi_i^T \right), \quad F_i = \frac{1}{2} \left( \Phi_i^T - \Psi_i^T \right).$$

(S11)

A similar procedure follows for the final Hamiltonian $H_f$ with transverse field $h$ with

$$\begin{pmatrix} \beta \\ \beta^\dagger \end{pmatrix} = \begin{pmatrix} G_f & F_f \\ F_f & G_f \end{pmatrix} \begin{pmatrix} c_f \\ c_f^\dagger \end{pmatrix},$$

(S12)

and corresponding $\Phi_f$ and $\Psi_f$. Based on the pairs of block matrices, we calculate the transfer matrices,

$$T_1 = G_f G_i^T + F_f F_i^T,$$

$$T_2 = G_f F_i^T + F_f G_i^T.$$

Now we want to calculate the Pfaffian matrix elements, $\alpha \langle \psi_0 | \phi_b^\dagger \phi_b^\dagger \psi_0 \rangle$ where Greek subscripts imply in which basis we have the states and the operators. Since we would like to make use of $\alpha \langle \psi_0 | = 0$, we use the above transfer matrices to write $[\phi_b^\dagger \phi_b^\dagger]_{\alpha}$ in the $\alpha$ basis as

$$[\phi_b^\dagger \phi_b^\dagger]_{\alpha} = \left[ (c_b^\dagger(t) \pm c_b(t)) \right]_{\alpha}, = \left[ (G_f^T \pm F_f^T) (e^{i\xi t} T_1 \pm e^{-i\xi t} T_2) \alpha^\dagger \right]_b \psi_0 \rangle_{\alpha},$$

where $\xi$ is a diagonal matrix with eigenenergies of the final Hamiltonian as the entries, $\xi = \text{diag} \{E_1^f, E_2^f, \ldots, E_N^f\}$. Based on this formulation, we construct matrices $M_\alpha(t)$ in an explicit form,

$$M_+ (t) = \Phi_f \left( e^{-i\xi t} T_1 + e^{i\xi t} T_2 \right),$$

$$M_- (t) = \left( T_1^T e^{i\xi t} - T_2^T e^{-i\xi t} \right) \Psi_f^T,$$

(S13)
Figure S1. Benchmarking mean field theory (MFT) analysis. All subfigures compare the results of MFT, t-DMRG and exact diagonalization (ED) results (see individual legends for system size information) for $C_{r=3}(t)$. The external fields are (a) $h = 0.5$, (b) $h = 1.1$ and (c) $h = 1.2$. In all, the MFT nonequilibrium response matches well with the nonequilibrium responses of the exact methods.

To utilize in the following contractions,

$$
\langle \phi^+_a(t) \phi^+_b(t) \rangle = [M_+(t)M_+(t)^\dagger]_{ab},
$$

$$
\langle \phi^-_a(t) \phi^-_b(t) \rangle = -[M_-^+(t)M_-^+(t)^\dagger]_{ab},
$$

$$
\langle \phi^+_a(t) \phi^-_b(t) \rangle = [M_+(t)M_-^+(t)^\dagger]_{ab},
$$

$$
\langle \phi^-_a(t) \phi^+_b(t) \rangle = -[M_-^+(t)M_+^+(t)^\dagger]_{ab}.
$$

(S14)

Now we can construct the Pfaffian matrix $T(t)$ at time $t$ with the matrix elements $T_{ks}(t) = \langle \phi^k_+(t) \phi^s_+(t) \rangle$ where $1 \leq k < s \leq 2\Delta x$, $p = +(-)$ for $k$ even(odd) and $q = +(-)$ for $s$ even(odd). The relation between parameters $a, b$ and $k, s$ reads $a = r + \lfloor k/2 \rfloor$ and $b = r + \lfloor s/2 \rfloor$, because $r \leq a \leq b \leq N - r + 1$. Having constructed $T(t)$, one can then extract $\langle C_r(t) \rangle = |\sqrt{\det{T(t)}}|^{1/2}$, as discussed below (S5).

**Mean-field theory (MFT) Analysis**

The next nearest-neighbor (NNN) term in the TFIC Hamiltonian reads in the fermionic picture as,

$$
\Delta \sum_r \left( c^+_r - c^+_r \right) \left( 1 - 2c^+_r c^-_{r+2} \right) \left( c^+_{r+2} + c^-_{r+2} \right),
$$

$$
= -\Delta \sum_r \left( c^+_r - c^+_r \right) \left( 1 - 2c^+_r c^-_{r+1} c^+_{r+2} + c^-_{r+2} \right),
$$

$$
= -\Delta \sum_r \phi^-_r \phi^+_r \phi^-_{r+1} \phi^+_r \phi^-_{r+2},
$$

(S15)

where $\Delta < 0$ and $\phi^\pm_r$ stand for the auxiliary fermions of type-I or -II.

In Hartree-Fock expansion, we assume $|\Delta| \ll |J|$ where $J$ is the nearest-neighbor coupling strength, and write Eq. S15 as

$$
\Delta \sum_r \left[ \langle \phi^-_r \phi^+_r \rangle_{t \rightarrow \infty} \phi^-_{r+1} \phi^+_{r+2} + \phi^-_r \phi^+_r \langle \phi^+_r \phi^-_{r+1} \rangle_{t \rightarrow \infty} - \phi^-_r \langle \phi^+_r \phi^-_{r+1} \rangle_{t \rightarrow \infty} \phi^-_{r+2} + \phi^-_r \langle \phi^+_r \phi^-_{r+2} \rangle_{t \rightarrow \infty} \phi^+_{r+1} \phi^-_{r+1} \right].
$$

(S16)

Here the $\langle \cdot \rangle_{t \rightarrow \infty}$ means that we calculate the free fermion problem and obtain the correlators in the infinite-time limit (instead of ground state which would be for the static problem). In our numerics we treat the largest time point allowed by the cluster theorem as the asymptotically infinite time limit. The quench MFT formalism was previously applied to two-point correlators in a periodic chain [S7]. Note that for an open-boundary chain, one needs to carefully
take the edges of the chain into account based on Eq. (S16). Using the above expansion, we obtain an effective mean field Hamiltonian which has slightly stronger nearest-neighbor (NN) coupling compared to the free problem, as well as new NNN couplings. Further, the effective chemical potential slightly decreases, which is reasonable when we think about how the critical point shifts to favor order, e.g. for $\Delta = -0.1$, $h_c \sim 1.16$ [S7].

Based on these equations, we can calculate a quench phase diagram for the interacting problem in the mean field picture as shown in the main text. Fig. S1 shows comparisons between MFT, t-DMRG and exact diagonalization (ED) results at different $h = 0.5$, $h = 1.1$ and $h = 1.2$, where the QCP is $h_c \sim 1.16$. We observe that the MFT analysis can even capture the correct frequency of the oscillations early times and the general trend of the nonequilibrium response successfully. However, MFT does not completely match with the exact methods, which is expected since MFT analysis is an approximate method that averages out the interactions.

### t-DMRG calculations

We utilize the ITensor environment [S8] to construct our matrix product states (MPS) and Trotter decomposition for the time evolution of the MPS. We set the maximum bond dimension as 100 of the resulting compressed MPS and the initial truncation error cutoff for the compression of the MPS as $\epsilon \sim 10^{-5}$. The truncation error cutoff is adaptive: As the maximum bond dimension is reached for the resulting MPS, the error cutoff increases systematically up until a hard error threshold of $\epsilon \sim 10^{-5}$ to be able to access longer times. Setting a maximum allowed bond dimension thus introduces an error which grows with time. Consequently, we are confined to early times for which the above interval of the error thresholds is satisfied.

### THE QUASI-STATIONARY REGIME IN LONG-RANGE INTERACTING TFIC

The quasi-stationary temporal regime also emerges in the long-range hard-boundary TFIC with power-law decaying interactions. This boundary effect on the long-range TFIC has been previously noticed in the context of prethermalization [S9]. Here we provide numerical evidence for this boundary effect. All data presented in this section was obtained from TEBD (time-evolving block decimation).

The Hamiltonian for the long-range TFIC reads,

$$ H = -\sum_{r,r'} J(r,r') \sigma^z_r \sigma^z_{r'} + h \sum_r \sigma^x_r; $$

where $J(r,r') = J/|r - r'|^\alpha$. In the limit where $\alpha = 0$, the model becomes integrable with all-to-all interactions, e.g. LMG model; whereas in the limit of $\alpha \to \infty$ the model reduces to short-range NN TFIC. When $\alpha = 10$, which is effectively a short-range TFIC with power-law decaying interactions, we reproduce the results in the main text for the TFIC with nearest-neighbor couplings and $J = 1$. Fig. S2a shows that a spin close to the boundary develops a quasi-stationary regime whereas the spin in the middle of the chain does not. These nonequilibrium responses are compared to the total magnetization (green-dotted line), whose behavior in the long-time limit is not conclusive based on the data. When $\alpha$ decreases to $\alpha = 4$ and $\alpha = 3$, the quasi-stationary regime still survives for $C_{r=a}(t)$, up to some oscillations. We note that the nonintegrable short-range TFIC also develops such oscillations in the quasi-stationary regime, as demonstrated in the main text. Ref. [S10] found a decaying nonequilibrium response for the spin in the middle of an open-boundary chain in the nonintegrable short-range TFIC with next-nearest neighbor interactions. Similarly, it also seems that the spin in the middle of the chain tends to decay in our results for the long-range TFIC with power-law decaying interactions. When $\alpha = 2$, the model becomes truly long-ranged and we do not observe a quasi-stationary regime in any of the spins. In fact the spins close to the boundary and in the middle of the chain behave quite similarly. This result points to the importance of locality in the Hamiltonian to observe the quasi-stationary temporal regime in spins close to the boundary, confirming the role of chain geometry rather than integrability.

To demonstrate that the quasi-stationary temporal regime is not a finite-size effect, we show in Fig. S3 the nonequilibrium responses of $C_{r=a}(t)$, a spin close to the boundary, for different system sizes and different $\alpha$. For a given $\alpha$, one can determine the time at which finite-size effects kick in by observing when the data for different system sizes $N = 32 - 64$ no longer overlap. Note that, for $\alpha = 10$ and $\alpha = 4$, the quasi-stationary regime develops before the finite-size effects appear.
Figure S2. Nonequilibrium responses of $\sigma_{N/2}^z$ (spin in the middle of the chain), $\sigma_6^z$ (spin close to the boundary) and the total magnetization, when (a) $\alpha = 10$, (b) $\alpha = 4$, (c) $\alpha = 3$ and (d) $\alpha = 2$.

Figure S3. The nonequilibrium responses of $C_{r=6(t)}$ for different $\alpha$ and different system sizes (see legend).
NUMERICAL EVIDENCE ON QUASI-STATIONARY REGIME NOT BEING RELATED TO STRONG-ZERO MODES

In this section, we plot the coherence times of edge magnetization (Fig. S4c), as well as the magnetization of a bulk spin $\sigma^z_3$ (Fig. S4a) with respect to time for different interaction strengths $\Delta$. An important evidence of strong zero modes is the presence of resonances, which would result in a non-monotonous trend of the steady-state value with respect to $\Delta$ [S11]. The absence of such a behavior can be seen in Figs. S4a and S4c. Additionally, we provide a 2D plot of the long-time steady state value of $C_{r=3}(t)$ with respect to the external field strength $h$ and the interaction strength $\Delta$. The behavior is monotonous everywhere between $0 < h < 1$ and $0 < \Delta < 2$, which allows us to exclude the physics of strong zero modes as a possible explanation of the quasi-stationary temporal regime observed in the open-boundary chains.

We also compare the spatial profiles of the single-site magnetization at a fixed time — determined according to the breakdown of the cluster theorem $t_1$ — with that of a Majorana edge mode $\gamma^1 = \phi^+_1$ (see Secs. and for the details of the temporal cutoffs in the study). For $h$ in the ordered phase and $r \ll N/2$, where $N = 480$ is set for concreteness, this time corresponds to the quasi-stationary regime, and hence the single-site order parameter value is compared with the probability of the edge mode being found at $r$. As is evident from Fig. S5, the spatial profiles for the magnetization decay exponentially in space, which is expected [S12]. However, we observe that the decay rate
Figure S6. The single-site dynamical phase diagrams with an ultraviolet temporal cutoff of $t^* = 10$ for (a) $C_{r=3}(t)$, (b) $C_{r=9}(t)$ and (c) $C_{r=12}(t)$. (d) The system size scaling of the single-site magnetization at the critical point $h_c$ with cutoff $t^* = 10$. Downward trend can be seen with $N^{-\gamma}$ where $\gamma \sim 1$.

of an edge mode and of magnetization at the same $h$ differ by at least two orders of magnitude (red-diamonds vs. blue-circles, respectively). Their spatial profiles coincide perfectly at $r = 1$, which is the boundary of the chain, and start to differ as $r$ increases. This is further evidence that the leakage of a zero mode into the bulk of the chain, alone, cannot explain the presence of a quasi-stationary temporal regime of bulk single-site observables. Fig. S5 also compares the spatial profiles of magnetization at different $h$, showing them to decay faster for external fields close to the DCP.

### TEMPORAL CUTOFFS

There are two relevant temporal cutoffs in our results: i) ultraviolet (short-time, short-distance) cutoff and ii) infrared (long-time, long-distance) cutoff [S10]. We set the infrared cutoff as being a parametric cutoff due to the application of the cluster theorem (see Sec. ), and we test here whether our results depend on the choice of ultraviolet cutoff.

The results in the main text are produced with a fixed ultraviolet (UV) cutoff of $t^* = 10$ for all $h$. However, none of our results depend on the choice of ultraviolet cutoff: Figs. S6, Figs. S7 and S8 all show the same qualitative behavior for single-site dynamical phase diagrams for various choices of ultraviolet cutoff. Figs. S6 complement the $t^* = 10$ data of the main text by showing the single-site phase diagrams of the observables at $r = 3, 9, 12$. Figs. S7 exhibit another fixed temporal cutoff of $t^* = 20$, whereas Figs. S8 demonstrate the results of a parametric temporal cutoff for all studied sites. This parametric UV cut-off is determined as follows: We roughly estimate the onset of the quasi-stationary regime as the time required for the quasiparticles to reflect back from the closest edge to the observation site. Therefore, the estimate can be mathematically stated as, $t^* = 2\alpha \Delta x/v_q$ where the distance $\Delta x = r - 1$ is the distance between the observation site, $r = 3, 6, 9, 12$ and the closest edge site, $r' = 1$, in our case. The parameter $\alpha$ is a tuning parameter, as our analytical formula is only an estimate. In fact we find that $\alpha = 2$ presents phase diagrams qualitatively the same with others for all $r$ that we studied.

Additionally, in all cases the single-site magnetization at the critical point $h_{dc}$ scales with similar exponents in a
Figure S7. The single-site dynamical phase diagrams with an ultraviolet temporal cutoff of $t^* = 20$ for (a) $C_{r=3}(t)$, (b) $C_{r=6}(t)$, (c) $C_{r=9}(t)$ and (d) $C_{r=12}(t)$. The behavior is qualitatively the same with the results of $t^* = 10$. (e) The system size scaling of the single-site magnetization at the critical point $h_c$ with cutoff $t^* = 20$. Downward trend can be seen with $N^{-\gamma}$ where $\gamma \sim 1$.

| obs. | cutoff | $\beta$ | cutoff | $\beta$ | cutoff | $\beta$ | cutoff | $\beta$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\sigma_i^2$ | $t^* = 10$ | 1.2972 $\pm$ 0.0018 | | | | | | |
| $\sigma_i^3$ | $t^* = 20$ | 1.2925 $\pm$ 0.0016 | | | | | | |
| $\sigma_i^6$ | | 1.2948 $\pm$ 0.0016 | parm, $\alpha = 1$ | 1.3011 $\pm$ 0.0016 | | | | |
| $\sigma_i^9$ | | 1.3106 $\pm$ 0.0038 | | | | | | |
| $\sigma_i^{12}$ | | 1.3478 $\pm$ 0.0069 | | | | | | |
| | | 1.3456 $\pm$ 0.0057 | | | | | | |

Table I. Fit parameters for the universal scaling law with different ultraviolet temporal cutoffs.

power-law fashion, $N^{-\gamma}$ where $\gamma \sim 1$. These fits can be seen in Figs. S6d, S7e and S8e. Although the choice of cutoff slightly affects this exponent, it does not change the fact that there is a decreasing trend of the magnetization at $h_{dc}$ with system size. This is a numerical evidence for the presence of a DCP.

Finally, we test the presence of the universal scaling in the vicinity of the transition when we change the ultraviolet cutoff. Table I provides all fit parameters for the exponent $\beta$ in the integrable TFIC performed with different temporal cutoffs, either fixed or parametric, with the latter denoted as $parm$ in the table. All exponents are very similar and around $\beta \sim 4/3$.

INDEPENDENCY OF THE RESULTS FROM THE INITIAL STATES

In this section, we change the initial state to the ground state of an initial Hamiltonian with $h_i = 0.1$, and test whether any of our results depend on the initial state. Figs. S9 show the single-site dynamical phase diagrams computed with this initial state. We do not observe a change in the qualitative behavior. The single-site magnetization
Figure S8. The single-site dynamical phase diagrams with an ultraviolet temporal cutoff of $t^* = 2\Delta x/v_q$ where $\Delta x = r - 1$ with $r$ being the single-site observable location and $\alpha$ is a tuning parameter for (a) $C_{r=3}(t)$, (b) $C_{r=6}(t)$, (c) $C_{r=9}(t)$ and (d) $C_{r=12}(t)$. The behavior is qualitatively the same with the results of other cutoffs. In all subfigures $\alpha = 2$. (e) The system size scaling of the single-site magnetization at the critical point $h_c$ with cutoff $t^* = 2\Delta x/v_q$ and $\alpha = 2$. Downward trend can be seen with $N^{-\gamma}$ where $\gamma \sim 1$. $t$ reflect in the plot is the same with $t^*$. 

### Table II. Fit parameters for the universal scaling law with a different initial state.

| Observable | $\beta$ |
|------------|---------|
| $\sigma_1^z$ | 1.3092 ± 0.0059 |
| $\sigma_6^z$ | 1.3257 ± 0.0079 |
| $\sigma_{12}^2$ | 1.3184 ± 0.0067 |

at the critical point still decreases with increasing system size.

We also test whether the scaling in the vicinity of the transition changes in Fig. S10. The fit parameters for some observables in the vicinity of the transition is given in Table II all of which demonstrates a dynamical critical exponent of $\beta \sim 4/3$. Based on this observation, we demonstrate in Fig. S11 the collapse of the nonequilibrium responses of the system at $r = 6$ for different initial states $h_i = 0$ and $h_i = 0.1$. This collapse is achieved by utilizing the nonuniversal fit parameters in the scaling functions: $C_{r,h_i}(h_n) = a_{r,h_i}h_n^\beta$ and $C_{r',h_i'}(h_n) = a_{r',h_i'}h_n^\beta$ and hence the scaling factor reads $\tilde{C}_{r,h_i}(h_n)/\tilde{C}_{r',h_i'}(h_n) = a_{r,h_i}/a_{r',h_i'}$, i.e., $\tilde{C}_r(t) = C_r(t)a_{r,h_i}/a_{r',h_i'}$ in Fig. S11.

**Analytical expression for the edge magnetization in the vicinity of the transition**

In this subsection we show that the quasi-stationary value of the edge magnetization does not change its dynamical scaling in the vicinity of the transition. We remind the reader that this scaling is different from the dynamical universal scaling that we numerically observed in the vicinity of the transition. This is because of the divergent relaxation times which means that the times accessible to computation and experiment are still within the relaxation regime when

...
Figure S9. The single-site dynamical phase diagrams with an initial state as the ground state of an initial Hamiltonian with $h_i = 0.1$ and an ultraviolet temporal cutoff of $t^* = 10$ for (a) $C_{r=3}(t)$, (b) $C_{r=6}(t)$, (c) $C_{r=9}(t)$ and (d) $C_{r=12}(t)$. The behavior is qualitatively the same as the results of $h_i = 0$. (e) The system size scaling of the single-site magnetization at the critical point $h_c$ with cutoff $t^* = 10$. Downward trend can be seen with $N^{-\gamma}$ where $\gamma \sim 1$.

Figure S10. The scaling behavior in the vicinity of the transition for $r = 1$, $r = 6$ and $r = 12$ in descending order when the initial state is the ground state of $h_i = 0.1$. All scaling exponents are $\beta \sim 4/3$.

we are in the close vicinity of the transition. The analytical expression for the quasi-stationary value of the edge magnetization in the dynamically-ordered phase is,

$$C^{qs}_{r}(h, h_i) = \frac{(1 - h^2)(1 - h_i)^{1/2}}{1 - hh_i},$$

(S18)
Figure S11. \( C_{r=0}(t) \) for quenches from \( h_i = 0 \) (blue-solid) or \( h_i = 0.1 \) (dashed-red) to \( h = 0.999 \) where the latter is rescaled to collapse on the other according to the fitted power-laws discussed in the main text. Rescaled quantity is denoted by tilde.

For \( h, h_i < 1 \). Let us rewrite it in terms of the reduced control parameter, \( h_n \),

\[
C_{r=1}^{qs}(h_n, h_i) = \frac{(2 - h_n)h_n(1 - h_i)^{1/2}}{1 + (h_n - 1)h_i}. \tag{S19}
\]

In the vicinity of the transition, \( h_n \to 0 \), we can expand this expression and find up to the third order in \( h_n \) and \( h_i \)

\[
C_{r=1}^{qs}(h_n \to 0, h_i) = a_1(h_i)h_n + a_2(h_i)h_n^2 + a_3(h_i)h_n^3 + \cdots, \tag{S20}
\]

\[
a_1(h_i) = 2 + h_i + \frac{3}{4}h_i^2 + \frac{5}{8}h_i^3 + \cdots, \tag{S21}
\]

\[
a_2(h_i) = -1 - \frac{5}{2}h_i - \frac{27}{8}h_i^2 - \frac{65}{16}h_i^3 + \cdots, \tag{S22}
\]

\[
a_3(h_i) = h_i + \frac{7}{2}h_i^2 + \frac{55}{8}h_i^3 + \cdots. \tag{S23}
\]

Therefore, one can see that \( C_{r=1}^{qs}(h_n) \propto h_n \) in the vicinity of the transition, \( h_n \to 0 \) regardless of the choice of initial state. The initial state only changes the coefficient in front of \( h_n \), which is known to be nonuniversal. Based on the observation that the edge magnetization also exhibits \( \beta \sim 4/3 \) in its relaxation regime to the quasi-stationary value, it is possible that the rest of the single-site observables close enough to the boundary will also experience a similar scaling \( \sim 1 \) in their quasi-stationary regime. This is yet to be discovered, most likely analytically.

We note that the form of the fit function for the quasi-stationary regime of the edge magnetization is the same as Eq. (S18), and hence the scaling in the vicinity of the transition is the same as well. Let us show this briefly,

\[
C_{r=1}^{qs}(h_n) = \alpha(h_{dc}^\beta - (h_{dc} - h_n)^\beta), \tag{S24}
\]

\[
\lim_{h_n \to 0} C_{r=1}^{qs}(h_n) = \alpha\beta h_{dc}^{\beta - 1}h_n - \frac{1}{2}\alpha(\beta - 1)\beta h_{dc}^{\beta - 2}h_n^2 + \cdots. \tag{S25}
\]

We note that for a different initial state the general trend will remain the same, but that the coefficients \( \alpha \) and \( \beta \) might change. In such a case, the scaling in the vicinity of the transition should remain the same as well.

**NONEQUILIBRIUM RESPONSE IN THE CLOSE VICINITY OF THE TRANSITION**

In this section we plot the nonequilibrium response in the close vicinity of the transition, to demonstrate that the dynamics slow down critically so that the onset of a quasi-stationary regime diverges as we move closer to the critical point. This naturally implies that the dynamical critical exponents that we probe in the vicinity of the transition are of nonequilibrium type, instead of equilibrium which could have been so if we were probing the quasi-stationary regime in the close vicinity. Fig. S12a and S12b show the nonequilibrium responses of edge magnetization and bulk
Figure S12. The nonequilibrium response of integrable TFIC in the vicinity of the transition (see legend) for (a) the edge magnetization and (b) bulk $r = 6$ magnetization at system size $N = 1440$. The solid-black lines are the time-average with a fixed UV temporal cutoff of $t^* = 10$.

Figure S13. The nonequilibrium response of integrable TFIC (a) at DCP $h = 1$ and (b) in the dynamically-disordered phase $h = 1.1$ for $r = 6$ magnetization at various system sizes (see legends).

In this section, we also plot the nonequilibrium response at the DCP which coincides with the QCP (Fig. S13a), as well as in the dynamically-disordered phase at $h = 1.1$ (Fig. S13b). One notices the power-law decay of the envelope of the nonequilibrium response, which suggests that there is no quasi-stationary regime appearing in the accessible magnetization at $r = 6$, respectively. (i) The first observation is that the nonequilibrium responses in the vicinity of the transition exhibit a very similar trend for both cases up to different time-average values (solid-black lines). We already presented the collapse of nonequilibrium responses of different sites in the main text. (ii) As we move closer to the critical point, we observe that the onset of the quasi-stationary regime diverges: This effect starts to be visible from $h = 0.99$. Therefore, the time-average of the signal no longer matches perfectly with the magnetization value of the quasi-stationary regime, which is beyond accessible simulation times in these figures. We note that it is also around this value of $h$ where we observe the onset of universal scaling appearing with $\beta \sim 4/3$ (see for instance Fig. 3a of the main text). We do not study the dynamical scaling closer than $h_n = 10^{-4}$ (purple), as it is not completely clear whether the dynamics relax to a quasi-stationary regime past this point due to constrained simulation time for a system size of $N = 1440$. Nevertheless, the allowed parameter regime still demonstrates a robust exponent of $\beta \sim 4/3$. In the future, one can extend the simulation time by increasing the system size further and thus test the dynamical critical exponent closer to the critical point.
Figure S14. Mean-field nonequilibrium responses of a near-integrable model with $\Delta = -0.1$ past the DCP (see legend) at a system size of $N = 1440$.

| observable | $\beta$  |
|------------|----------|
| $\sigma^z_1$ | $1.3378 \pm 0.0043$ |
| $\sigma^z_3$ | $1.3568 \pm 0.0039$ |
| $\sigma^z_6$ | $1.3537 \pm 0.0038$ |
| $\sigma^z_9$ | $1.3647 \pm 0.0023$ |
| $\sigma^z_{12}$ | $1.3618 \pm 0.0132$ |

Table III. Fit parameters for the scaling law of the near-integrable model in the vicinity of the transition.

Details on the Near-Integrable Model and the Associated Fit Functions

We plot the nonequilibrium responses of our near-integrable model with $\Delta = -0.1$ for $h$ past the DCP in Fig. S14. This plot should be compared with Fig. 4c of the main text. One can observe the emergence of long wavelength oscillations as we pass the dynamical critical point, which is helpful in determining the DCP. This is because, such behavior of long wavelength oscillations is a characteristic of the disordered dynamical phase. One can also notice that the envelope of the nonequilibrium response in the dynamically-disordered phase decays in a power-law fashion, suggesting the absence of a quasi-stationary regime. Similar behavior is observed in the integrable TFIC (see Sec. ).

Table III presents the fitting parameters for the dynamical scaling law in the vicinity of the transition for the near-integrable model for different sites $r \ll N/2$. For all sites we find a similar exponent of $\beta \sim 4/3$, which matches that found in the integrable model. We also study the system size scaling at the DCP with different ultraviolet cutoffs $t^*$, all of which gives similar exponents $C_r(h_c) \propto N^{-\gamma}$ where $\gamma \sim 0.7$. Table IV summarizes the fit parameters for some temporal cutoffs $t^*$.

| cutoff  | $\sigma^z_1, \gamma$          | $\sigma^z_3, \gamma$          | $\sigma^z_6, \gamma$          |
|---------|-------------------------------|-------------------------------|-------------------------------|
| $t^* = 10$ | $0.6842 \pm 0.0437$ | $0.6973 \pm 0.052$ | $0.7003 \pm 0.0505$ |
| $t^* = 15$ | $0.6957 \pm 0.0449$ | $0.6865 \pm 0.0492$ | $0.6710 \pm 0.0448$ |

Table IV. Fit parameters for the system-size scaling of single-site observable at $r = 1, 3, 6$ at the DCP.
Figure S15. Different boundary conditions are compared for the integrable TFIC at $h = 0.5$ and system size $N = 24$. One can obtain a quasi-stationary regime with smooth boundary conditions too.

Figure S16. The dynamic order profile of the nonintegrable TFIC with $\Delta = -1$ in logarithmic scale.

When applying the cluster theorem to the MFT data of this weakly-interacting nonintegrable TFIC, one needs to estimate the lightcone (correlation) velocity of the model. While for the integrable TFIC this velocity is analytically known, this is not true when we introduce nonintegrability to the model. Here we approximate a quasiparticle velocity based on the analytical prediction of the integrable TFIC: $v_q = 2h$ for $h \leq h_c$ and $v_q = 2h_c$ for $h > h_c$. Since this is only an approximation, we sometimes exceed the time when the cluster theorem breaks down. This is because distant sites of the chain become correlated with one another (see Sec. ). This time can be observed with a recurrence attempt in the figures both in the main text and the supplementary, which is also a sign of finite-size effects.

**ROBUSTNESS OF THE QUASI-STATIONARY REGIME**

In this section, we show that the quasi-stationary temporal regime emerges not only when we introduce hard boundaries [S13], but also smooth boundaries. A smooth boundary condition can be applied by smoothly turning off the Hamiltonian parameters towards the edges of the chain [S14]. Fig. S15 shows the single-site nonequilibrium responses of the integrable TFIC with hard boundaries (red-diamonds), smooth boundaries (green-solid) and periodic boundary condition (blue-dotted). As shown before [S10], the middle of a hard-boundary chain (yellow triangles) acts like an arbitrary site in a periodic chain.
We thus conclude that the quasi-stationary regime is robust against altering the boundary conditions, so long as they remain open. This provides further evidence that the stationary regime is a result of the geometry of the chain rather than of the zero modes.

**T-DMRG RESULTS IN DETAIL**

Finally, in Fig. S16 we show the data of Fig. 5b of the main text in logarithmic scale to demonstrate why this data is insufficient to claim the presence of a dynamical phase transition (DPT) in the nonintegrable TFIC with strong integrability breaking. We remind the reader that the interaction strength is set to $\Delta = -1$ and the results are obtained via t-DMRG (see Sec. ). Although the dynamic order tends to increase with increasing system size in the dynamically-ordered regime and hence demonstrates a persistent dynamic order, it is not clear where the transition really happens. The DCP is expected to happen either before or at the QCP, which is at $h_c \sim 2.46J$ for this model. Longer simulations times, with sufficiently high precision, are needed in order to decide whether this crossover is actually a DPT.

* cbdag@umich.edu

[S1] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, 2001).
[S2] P. Calabrese, F. H. L. Essler, and M. Fagotti, *Journal of Statistical Mechanics: Theory and Experiment* 2012, P07016 (2012).
[S3] J. C. Budich and M. Heyl, *Phys. Rev. B* 93, 085416 (2016).
[S4] P. Uhrich, N. Defenu, R. Jafari, and J. C. Halimeh, *Phys. Rev. B* 101, 245148 (2020).
[S5] C. B. Dağ, L.-M. Duan, and K. Sun, *Phys. Rev. B* 101, 104415 (2020).
[S6] E. Lieb, T. Schultz, and D. Mattis, *Annals of Physics* 16, 407 (1961).
[S7] P. Titum, J. T. Isosue, J. R. Garrison, A. V. Gorshkov, and Z.-X. Gong, *Phys. Rev. Lett.* 123, 115701 (2019).
[S8] iTensor Library (version 2.0.11) http://itensor.org.
[S9] B. Neyenhuis, J. Zhang, P. W. Hess, J. Smith, A. C. Lee, P. Richerme, Z.-X. Gong, A. V. Gorshkov, and C. Monroe, *Science Advances* 3 (2017), 10.1126/sciadv.1700672.
[S10] C. B. Dağ and K. Sun, arXiv e-prints, arXiv:2004.12287 (2020), arXiv:2004.12287 [quant-ph].
[S11] J. Kemp, N. Y. Yao, C. R. Laumann, and P. Fendley, *Journal of Statistical Mechanics: Theory and Experiment* 2017, 063105 (2017).
[S12] P. Calabrese and J. Cardy, *Phys. Rev. Lett.* 96, 136801 (2006).
[S13] F. Iglói and H. Rieger, *Phys. Rev. Lett.* 106, 035701 (2011).
[S14] M. Vekić and S. R. White, *Phys. Rev. Lett.* 71, 4283 (1993).