CHANGE POINT DETECTION IN RANDOM COEFFICIENT AUTOREGRESSIVE MODELS

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Abstract. We propose a family of CUSUM-based statistics to detect the presence of change-points in the deterministic part of the autoregressive parameter in a Random Coefficient AutoRegressive (RCA) sequence. In order to ensure the ability to detect breaks at sample endpoints, we thoroughly study weighted CUSUM statistics, analysing the asymptotics for virtually all possible weighing schemes, including the standardised CUSUM process (for which we derive a Darling-Erdős theorem) and even heavier weights (studying the so-called Rényi statistics). Our results are valid irrespective of whether the sequence is stationary or not, and indeed prior knowledge of stationarity or lack thereof is not required from a practical point of view. From a technical point of view, our results require the development of strong approximations which, in the nonstationary case, are entirely new. Similarly, we allow for heteroskedasticity of unknown form in both the error term and in the stochastic part of the autoregressive coefficient, proposing a family of test statistics which are robust to heteroskedasticity; again, our tests can be readily applied, with no prior knowledge as to the presence or type of heteroskedasticity. Simulations show that our procedures work well in finite samples, under all cases considered (stationarity versus nonstationarity, homoskedasticity versus various forms of heteroskedasticity). We complement our theory with applications to financial, economic and epidemiological time series.
1. Introduction

In this paper we study the stability of the autoregressive parameter of an RCA(1) sequence:

\[ y_i = \begin{cases} 
(\beta_0 + \epsilon_{i,1}) y_{i-1} + \epsilon_{i,2}, & \text{if } 1 \leq i \leq k^*, \\
(\beta_A + \epsilon_{i,1}) y_{i-1} + \epsilon_{i,2}, & \text{if } k^* + 1 \leq i \leq N,
\end{cases} \]

where \( y_0 \) denotes an initial value. We test for the null hypothesis of no change versus the alternative of at most one change (AMOC) i.e.

\[ H_0 : k^* > N, \]
\[ H_A : 1 < k^* < N \text{ and } \beta_0 \neq \beta_A. \]

The RCA model was firstly studied by Anděl (1976) and Nicholls and Quinn (2012). It belongs in the wider class of nonlinear models for time series (see Fan and Yao, 2008), which have been proposed “as a reaction against the supremacy of linear ones - a situation inherited from strong, though often implicit, Gaussian assumptions” (Akharif and Hallin, 2003). Arguably, (1.1) is very flexible, allowing for the autoregressive “root” \( \beta_0 + \epsilon_{i,1} \) to vary over time, and thus for the possibility of having stationary and nonstationary regimes. This may be a more appropriate model than a linear specification (see Lieberman, 2012; Leybourne et al., 1996). Giraitis et al. (2014) argue that a time-varying parameter model like (1.1) can be viewed as a competitor for a model with an abrupt break in the autoregressive root. Furthermore, equation (1.1) also allows for the possibility of (conditional) heteroskedasticity in \( y_i \); Tsay (1987) shows that the widely popular ARCH model by Engle (1982) can be cast into (1.1), which therefore can be viewed as a second-order equivalent. Finally, a major advantage of (1.1) compared to standard autoregressive models is that estimators of \( \beta_0 \) are always asymptotically normal, irrespective of whether \( y_i \) is stationary or nonstationary, thus avoiding the risk of over-differencing (see Leybourne et al., 1996).
Given such generality and flexibility, (1.1) has been used in many applied sciences, including biology (Stenseth et al., 1998), medicine (Fryz, 2017), and physics (Ślęzak et al., 2019). The RCA model has also been applied successfully in the analysis of economic and financial data, and we refer to the recent contribution by Regis et al. (2021) for a comprehensive review.

The inferential theory for (1.1) has been studied extensively. Schick (1996), Koul and Schick (1996) and Janečková and Prášková (2004) study Weighted Least Squares (WLS) estimation of $\beta_0$; Berkes et al. (2009) and Aue and Horváth (2011) study Quasi Maximum Likelihood estimation, and Hill et al. (2016) develop an Empirical Likelihood estimator. Several tests have also been developed, including tests for stationarity (see e.g. Zhao and Wang, 2012; and Trapani, 2021) and for the randomness of the autoregressive coefficient (Akharif and Hallin, 2003; Nagakura, 2009; and Horváth and Trapani, 2019).

In contrast, changepoint detection is still underexplored in the RCA framework. To the best of our knowledge, the only exceptions are Lee (1998), Lee et al. (2003) and Aue (2004); in these papers, a CUSUM test is proposed, but only for the stationary case and based on the unweighted CUSUM process. The latter is well-known to suffer from low power, being in particular less able to detect changepoints occurring at the beginning/end of the sample. As a solution, the literature has proposed weighted versions of the CUSUM process on the interval $[0,1]$, where more emphasis is given to observations at the sample endpoints (see Csörgő and Horváth, 1997). Weighing functions are typically of the form $[t(1-t)]^\kappa$ with $0 \leq \kappa < \infty$, for $t \in [0,1]$, with more weight placed on observations at the endpoints as $\kappa$ increases. In particular, the case $\kappa = \frac{1}{2}$ corresponds to the standardised CUSUM process also proposed by Andrews (1993), whereas the more heavily weighted case $\kappa > \frac{1}{2}$ corresponds to a family of test statistics known as “Rényi statistics” (see Horváth et al., 2020b). When $\kappa > 0$, the asymptotics becomes more complicated, since the weighted statistics diverge at the endpoints $t = 0$ and $t = 1$, and one can no longer rely on weak convergence to derive the limiting distributions. In order to overcome this issue, Andrews (1993) proposes trimming the interval on which the weighted CUSUM process is studied; however, this has the undesirable consequence that tests are unable to detect breaks when these occurs e.g. at the end of the sample.
In this paper, we bridge all the gaps mentioned above by proposing a family of weighted, untrimmed CUSUM statistics. Our paper makes the following four contributions.

First, we study virtually all possible weighing schemes, deriving the asymptotics for all \( 0 \leq \kappa < \infty \). From a practical viewpoint, this entails that our test statistics are designed to detect breaks even when these are very close to the sample endpoints. Second, all our results hold irrespective of whether \( y_t \) is stationary or not; this robustness arises from using the WLS estimator, and from the well-known fact that the RCA model does not suffer from the "knife edge effect" which characterizes linear models (Lumsdaine, 1996). From a practical point of view, this entails that the tests can be applied with no modifications required, and no prior knowledge of the stationarity of \( y_t \) or lack thereof. This feature is particularly desirable e.g. in the context of detecting the beginning (or end) of bubbles (see Harvey et al., 2016): with our set-up, it is possible to detect changes from stationary to nonstationary/explosive behaviour (as e.g. in Horváth et al., 2020; and Horváth et al., 2021) which characterize the emergence of a bubble, but it is also possible - again with no modifications required - to detect changes from explosive to non-explosive behaviour, as would be the case at the end of a bubble. Being able to accommodate both cases is a distinctive advantage of the RCA set-up: whilst tests for changes towards an explosive behaviour have been developed in the literature (see, inter alia, Phillips et al., 2011; Phillips et al., 2015; and the review by Homm and Breitung, 2012), tests to detect changes from an explosive behaviour are more rare, possibly due to the more complicated asymptotics in this case. Third, we allow for heteroskedasticity in both \( \epsilon_{i,1} \) and \( \epsilon_{i,2} \), which is usually not considered in the RCA context; interestingly, for the case \( \kappa \geq \frac{1}{2} \), we recover the same, nuisance free distribution as in the homoskedastic case (in particular, when \( \kappa = \frac{1}{2} \), we obtain a "classical" Darling-Erdős limit theorem). Hence, our modified test statistics can be used from the outset, with no prior knowledge required as to whether \( \epsilon_{i,1} \), or \( \epsilon_{i,2} \), or both, is heteroskedastic. Fourth, our asymptotics is based on strong approximations for the partial sums of an RCA sequence, which are valid irrespective of the stationarity or lack thereof of \( y_t \);
the strong approximation for the nonstationary case is entirely new.

The remainder of the paper is organised as follows. We present our test statistics in Section 2 and study their asymptotics in the homoskedastic case, as a benchmark, in Section 3. The heteroskedastic case is studied in Section 4. In Section 5 we report a simulation exercise; applications to real data are in Section 6. Section 7 concludes. Extensions, technical lemmas and all proofs are relegated to the Supplement.

NOTATION. We use the following notation: “\( \mathcal{D} \)” for weak convergence; “\( \mathcal{P} \)” for convergence in probability; “a.s.” for “almost surely”; “\( \mathbb{D} \)” for equality in distribution; \( \lfloor \cdot \rfloor \) is the integer value function. Positive, finite constants are denoted as \( c_0, c_1, \ldots \) and their value may change from line to line. Other notation is introduced further in the paper.

2. The test statistics

Our approach is based on comparing the estimates of \( \beta_0 \) before and after each point in time \( k \), by dividing the data into two subsets at \( k \) and estimating the autoregressive parameter in both subsamples. As mentioned above, we use WLS, with weights \( 1 + y_{i-1}^2 \). This has the advantages of (i) avoiding restrictions on the moments of the observations, and (ii) ensuring standard normal asymptotics irrespective of whether \( y_i \) is stationary or not. The WLS estimators are

\[
\hat{\beta}_{k,1} = \left( \sum_{i=2}^{k} \frac{y_{i-1}^2}{1 + y_{i-1}^2} \right)^{-1} \left( \sum_{i=2}^{k} \frac{y_i y_{i-1}}{1 + y_{i-1}^2} \right), \quad 2 \leq k \leq N,
\]

and

\[
\hat{\beta}_{k,2} = \left( \sum_{i=k+1}^{N} \frac{y_{i-1}^2}{1 + y_{i-1}^2} \right)^{-1} \left( \sum_{i=k+1}^{N} \frac{y_i y_{i-1}}{1 + y_{i-1}^2} \right), \quad 1 \leq k \leq N - 1.
\]
Our test statistics will be functionals of the process

\[ Q_N(t) = \begin{cases} 
0, & \text{if } 0 \leq t < 2/(N + 1), \\
N^{1/2}(t(1 - t))(\hat{\beta}_{(N+1)t,1} - \hat{\beta}_{(N+1)t,2}), & \text{if } 2/(N + 1) \leq t < 1 - 2/(N + 1), \\
0, & \text{if } 1 - 2/(N + 1) < t \leq 1.
\]

A “natural” choice to detect the presence of a possible change is to use the sup-norm of (2.3), viz. \( \sup_{0 < t < 1} |Q_N(t)| \), but, as mentioned above, this choice may have low power in detecting changes which occur early or late in the sample. In order to enhance the power at sample endpoints, one can use weight functions:

\[ \sup_{0 < t < 1} \frac{|Q_N(t)|}{w(t)}. \]

**Assumption 2.1.** It holds that: (i) \( \inf_{\delta \leq t \leq 1 - \delta} w(t) > 0 \) for all \( 0 < \delta < 1/2 \); (ii) \( w(t) \) is non decreasing in a neighborhood of 0; (iii) \( w(t) \) is non increasing in a neighborhood of 1.

The functions \( w(t) \) satisfying Assumption 2.1 belong in a very wide class; a possible example is \( w(t) = (t(1 - t))^\kappa \) with \( \kappa > 0 \). The existence of the limit of (2.4) can be determined based on the finiteness of the integral functional (see Csörgő and Horváth, 1993)

\[ I(w, c) = \int_0^1 \frac{1}{t(1 - t)} \exp \left( -\frac{cw^2(t)}{t(1 - t)} \right) dt. \]

As we show below, (2.5) entails that \( w(t) = (t(1 - t))^\kappa \) with \( 0 < \kappa < \frac{1}{2} \) can be employed in this context.

In order to further enhance the power of our testing procedures, functions which place more weight at the sample endpoints can also be used, i.e.

\[ \sup_{0 < t < 1} \frac{|Q_N(t)|}{(t(1 - t))^\kappa}, \]

with \( \kappa \geq \frac{1}{2} \). As mentioned above, when \( \kappa = \frac{1}{2} \), the corresponding limit theorems will be of the Darling-Erdős type (Darling and Erdős, 1956); when \( \kappa > \frac{1}{2} \), the test statistics defined in (2.6) are known as “Rényi statistics” (Horváth et al., 2020b).
3. Testing for changepoint under homoskedasticity

We begin by assuming that the errors \( \{ \varepsilon_{i,1}, \varepsilon_{i,2}, -\infty < i < \infty \} \) have constant variance.

**Assumption 3.1.** It holds that: (i) \( \{ \varepsilon_{i,1}, -\infty < i < \infty \} \) and \( \{ \varepsilon_{i,2}, -\infty < i < \infty \} \) are independent sequences; (ii) \( \{ \varepsilon_{i,1}, -\infty < i < \infty \} \) are independent and identically distributed random variables with \( E \varepsilon_{i,1} = 0, 0 < E \varepsilon_{i,1}^2 < \infty \) and \( E|\varepsilon_{i,1}|^4 < \infty \); (iii) \( \{ \varepsilon_{i,2}, -\infty < i < \infty \} \) are independent and identically distributed random variables with \( E \varepsilon_{i,2} = 0, 0 < E \varepsilon_{i,2}^2 = \sigma_2^2 < \infty \) and \( E|\varepsilon_{i,2}|^4 < \infty \).

In (1.1), the stationarity or lack thereof of \( y_i \) is determined by the value of \( E \ln |\beta_0 + \varepsilon_{0,1}| \) (see Aue et al., 2006). In particular, if \(-\infty \leq E \ln |\beta_0 + \varepsilon_{0,1}| < 0\), then \( y_i \) converges exponentially fast to a strictly stationary solution for all initial values \( y_0 \). Conversely, if \( E \ln |\beta_0 + \varepsilon_{0,1}| \geq 0\), then \( y_i \) is nonstationary - specifically, \( |y_i| \) diverges exponentially fast a.s. when \( E \ln |\beta_0 + \varepsilon_{0,1}| > 0\), whereas it diverges in probability, but at a rate slower than exponential, in the boundary case \( E \ln |\beta_0 + \varepsilon_{0,1}| = 0 \) (see Horváth and Trapani, 2016).

We show that the asymptotic variance of the limiting process depends on whether \( y_i \) is stationary or not: we therefore study the two cases (stationarity versus lack thereof) separately. We show that the variance of the weak limit of \( Q_N(t) \) is

\[
\eta^2 = \begin{cases} 
E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \sigma_1^2 + E \left( \frac{\bar{y}_0}{1 + \bar{y}_0^2} \right)^2 \sigma_2^2 & \text{if } -\infty \leq E \ln |\beta_0 + \varepsilon_{0,1}| < 0, \\
\sigma_1^2 & \text{if } E \ln |\beta_0 + \varepsilon_{0,1}| \geq 0.
\end{cases}
\]

We require the following notation. In order to study the case \( \kappa = \frac{1}{2} \), we define

\[
a(x) = (2 \ln x)^{1/2} \quad \text{and} \quad b(x) = 2 \ln x + \frac{1}{2} \ln \ln x - \frac{1}{2} \ln \pi.
\]
Also, in order to study the case $\kappa > \frac{1}{2}$, let

\begin{equation}
(3.3) \quad r_N = \min(r_1(N), r_2(N)), \quad \lim_{N \to \infty} \frac{r_N}{r_1(N)} = \gamma_1 \quad \text{and} \quad \lim_{N \to \infty} \frac{r_N}{r_2(N)} = \gamma_2.
\end{equation}

\begin{equation}
(3.4) \quad a(\kappa) = \sup_{1 \leq t < \infty} \frac{|W(t)|}{t^\kappa}.
\end{equation}

**Assumption 3.2.** It holds that $r_1(N) \to \infty$, $r_1(N)/N \to 0$, and $r_2(N) \to \infty$, $r_2(N)/N \to 0$.

We start with the stationary case $-\infty \leq E \ln |\beta_0 + \epsilon_{0,1}| < 0$. In this case, the solution of (1.1) under the null hypothesis is close to $\overline{y}_i$, the unique anticipative stationary solution of

\begin{equation}
(3.5) \quad \overline{y}_i = (\beta_0 + \epsilon_{i,1})\overline{y}_{i-1} + \epsilon_{i,2}, \quad -\infty < i < \infty.
\end{equation}

We need the following (technical) assumption, to rule out the degenerate case that, under stationarity, the denominator of $\eta^2$ defined in (3.1) is zero with probability 1.

**Assumption 3.3.** It holds that $P\{\overline{y}_0 = 0\} < 1$.

**Theorem 3.1.** We assume that $H_0$ of (1.2), Assumptions 2.1, 3.1 and 3.3 hold, and $-\infty \leq E \ln |\beta_0 + \epsilon_{0,1}| < 0$.

(i) If $I(w, c) < \infty$ for some $c > 0$, then it holds that

$$
\sup_{0 < t < 1} \frac{|Q_N(t)|}{w(t)} \xrightarrow{D} \eta \sup_{0 < t < 1} \frac{|B(t)|}{w(t)},
$$

where $\{B(t), 0 \leq t \leq 1\}$ is a standard Brownian bridge and $\eta$ is defined in (3.1).

(ii) For all $x$, it holds that

$$
\lim_{N \to \infty} P\left\{ a(\ln N) \frac{1}{\eta} \max_{1 < k \leq N} \left( \frac{k(N - k)}{N} \right)^{1/2} \left| \widehat{\beta}_{k,1} - \hat{\beta}_{k,2} \right| \leq x + b(\ln N) \right\} = \exp(-2e^{-x}).
$$

(iii) If Assumption 3.2 is satisfied, then it holds that

$$
\left( \frac{r_N}{N} \right)^{-1/2} \frac{1}{\eta} \frac{1}{r_1(N)/N \leq \epsilon_{1-r_2(N)/N}} \sup_{t (1-t)^{\kappa}} \frac{|Q_N(t)|}{(t(1-t))^{\gamma}} \xrightarrow{D} \max \left( \gamma_1^{-1/2} a_1(\kappa), \gamma_2^{-1/2} a_2(\kappa) \right),
$$

where $a_1(\kappa)$ and $a_2(\kappa)$ are defined in (3.4).
for all $\kappa > 1/2$, where and $r_N$, $\gamma_1$ and $\gamma_2$ are defined in (3.3), and $a_1(\kappa)$ and $a_2(\kappa)$ are independent copies of $a(\kappa)$ defined in (3.4).

We now turn to the nonstationary case. We need an additional technical condition:

**Assumption 3.4.** It holds that $\epsilon_{0,2}$ has a bounded density.

**Theorem 3.2.** We assume that $H_0$ of (1.2), Assumptions 2.1, 3.1, 3.4 hold, and $0 \leq E \ln |\beta_0 + \epsilon_{0,1}| < \infty$.

(i) If $I(w, c) < \infty$ for some $c > 0$, then it holds that

$$\sup_{0 < t < 1} \frac{|Q_N(t)|}{w(t)} \overset{p}{\to} \eta \sup_{0 < t < 1} \frac{|B(t)|}{w(t)},$$

where $\{B(t), 0 \leq t \leq 1\}$ is a standard Brownian bridge and $\eta$ is defined in (3.1).

(ii) For all $x$

$$\lim_{N \to \infty} P \left\{ a(N) \frac{1}{\eta} \max_{1 < k < N} \left( \frac{k(N - k)}{N} \right)^{1/2} |\hat{\beta}_{k,1} - \hat{\beta}_{k,2}| \leq x + b(N) \right\} = \exp(-2e^{-x}),$$

where $a(x)$ and $b(x)$ are defined in (3.2).

(iii) If Assumption 3.2 is satisfied, then it holds that

$$\left( \frac{r_N}{N} \right)^{\kappa-1/2} \frac{1}{\eta} \frac{1}{r_1(N)/N \leq t \leq 1-r_2(N)/N} \frac{|Q_N(t)|}{t(1-t)^{1/2}} \overset{p}{\to} \max \left( \gamma_1^{\kappa-1/2} a_1(\kappa), \gamma_2^{\kappa-1/2} a_2(\kappa) \right),$$

for all $\kappa > 1/2$, where and $r_N$, $\gamma_1$ and $\gamma_2$ are defined in (3.3), and $a_1(\kappa)$ and $a_2(\kappa)$ are independent copies of $a(\kappa)$ defined in (3.4).

Theorems 3.1 and 3.2 stipulate that the limiting distributions of the weighted CUSUM statistics are the same irrespective of whether $y_i$ is stationary, explosive or at the boundary: the impact of nonstationarity is only on $\eta^2$. Hence, it is important to find an estimator for $\eta^2$ which is consistent for all cases. Let

$$\hat{a}_{N,1} = \frac{1}{N-1} \sum_{i=2}^{N} \frac{(y_i - \hat{\beta}_{N,1} y_{i-1})^2 y_{i-1}^2}{(1 + y_{i-1}^2)^2} \quad \text{and} \quad \hat{a}_{N,2} = \frac{1}{N-1} \sum_{i=2}^{N} \frac{y_{i-1}^2}{1 + y_{i-1}^2}$$
We use the following estimator for $\eta^2$

\begin{equation}
\hat{\eta}^2_N = \frac{\hat{a}_{N,1}}{\hat{a}_{N,2}^2}.
\end{equation}

**Corollary 3.1.** The results of Theorems 3.1–3.2 remain true if $\eta$ is replaced with $\hat{\eta}_N$.

Corollary 3.1 states that the feasible versions of our test statistics, based on $\hat{\eta}_N$, have the same distribution as the infeasible ones, based on $\eta$. Practically, this means that the test statistics developed above can be implemented with no prior knowledge as to whether $y_i$ is stationary or not.

4. Change point detection with heteroskedastic errors

In the previous section we assumed, as is typical in the RCA literature, that the innovations $\{\epsilon_{i,1}, \epsilon_{i,2}, 1 \leq i \leq N\}$ are homoskedastic, which may be an undesirable restriction. The literature on the changepoint problem has recently considered this issue, but contributions are still relatively rare: exceptions include Xu (2015), Górecki et al. (2018) Bardsley et al. (2017) and Horváth et al. (2020a) (see also Xu and Phillips, 2008, for adaptive estimation in autoregressive models). Heteroskedasticity is particularly interesting and challenging in the RCA case: if the distribution of $\epsilon_{i,1}$ is allowed to change, the observations might change from stationarity to non stationarity even if $\beta_0$ does not undergo any change; however, inference on the RCA model will still be asymptotically normal in light of the properties of the WLS estimator discussed above.

In this section, we extend all the results above allowing for heteroskedasticity in both $\epsilon_{i,1}$ and $\epsilon_{i,2}$. Our results are valid also in the baseline case of homoskedasticity, and do not require any explicit knowledge of the form of heteroskedasticity.

Changes in the distribution of $\{\epsilon_{i,1}, \epsilon_{i,2}, 1 \leq i \leq N\}$ at times $1 < m_1 < \ldots < m_M < N$ are allowed through the following assumption.

**Assumption 4.1.** It holds that $m_\ell = \lfloor N \tau_\ell \rfloor$, for $1 \leq \ell \leq M$, with $0 < \tau_1 < \tau_2 < \ldots < \tau_M < 1$.

Henceforth, we will use the notation: $m_0 = 0, m_{M+1} = N, \tau_0 = 0$ and $\tau_{M+1} = 1$. 
For each subsequence \( \{y_i, m_{\ell-1} < i \leq m_{\ell}\}, 1 \leq \ell \leq M + 1 \), the condition for stationarity can be satisfied; in this case, the elements of this subsequence can be approximated with stationary variables \( \{\bar{y}_{\ell,j}, -\infty < j < \infty\} \) defined by the recursion

\[
\bar{y}_{\ell,j} = (\beta_0 + \epsilon_{\ell,j,1})\bar{y}_{\ell,j-1} + \epsilon_{\ell,j,2}, \quad -\infty < j < \infty,
\]

where \( \epsilon_{\ell,j,1} = \epsilon_{j,1}, m_{\ell-1} < j \leq m_{\ell} \), and \( \epsilon_{\ell,j,1}, -\infty < j < \infty, j \not\in (m_{\ell-1}, m_{\ell-1} + 1, \ldots, m_{\ell}] \) are independent and identically distributed copies of \( \epsilon_{m_{\ell},1} \). The random variables \( \epsilon_{\ell,j,2} \) are defined in the same way.

To allow for changes in the distributions of the errors, we replace Assumptions 3.1-3.4 with

**Assumption 4.2.** It holds that: (i) \( \{\epsilon_{i,1}, -\infty < i < \infty\} \) and \( \{\epsilon_{i,2}, -\infty < i < \infty\} \) are independent sequences of independent random variables; (ii) \( P\{y_0 = 0\} < 1 \); (iii) for each \( 1 \leq \ell \leq M + 1 \), \( \{\epsilon_{i,1}, m_{\ell-1} < i \leq m_{\ell}\} \) are identically distributed with \( E\epsilon_{m_{\ell},1} = 0, E\epsilon_{m_{\ell},1}^2 = \sigma_{m_{\ell},1}^2 \) and \( E|\epsilon_{m_{\ell},1}|^4 < \infty \); (iv) for each \( 1 \leq \ell \leq M + 1 \), \( \{\epsilon_{i,2}, m_{\ell-1} < i \leq m_{\ell}\} \) are identically distributed with \( E\epsilon_{m_{\ell},2} = 0, E\epsilon_{m_{\ell},2}^2 = \sigma_{m_{\ell},2}^2 \) and \( E|\epsilon_{m_{\ell},2}|^4 < \infty \); (v) if \( E \ln |\beta_0 + \epsilon_{m_{\ell},1}| < 0 \), then \( \epsilon_{m_{\ell},2} \) has a bounded density, \( 1 \leq \ell \leq M + 1 \).

By Assumption 4.2, the WLS estimator may have different variances in the various regimes. In order to study the limit theory, consider the following notation:

\[
\eta_{\ell}^2 = \begin{cases} 
E \left( \frac{\bar{y}_{\ell,0}^2}{1 + \bar{y}_{\ell,0}^2} \right)^2 \sigma_{\ell,1}^2 + E \left( \frac{\bar{y}_{\ell,0}^2}{1 + \bar{y}_{\ell,0}^2} \right)^2 \sigma_{\ell,2}^2, \\
\sigma_{\ell,1}^2, \quad \text{if } -\infty \leq E \ln |\beta_0 + \epsilon_{m_{\ell},1}| < 0, \\
\end{cases}
\]

and

\[
a_{\ell} = \begin{cases} 
E \left( \frac{\bar{y}_{\ell,0}^2}{1 + \bar{y}_{\ell,0}^2} \right), \quad \text{if } -\infty \leq E \ln |\beta_0 + \epsilon_{m_{\ell},1}| < 0, \\
1, \quad \text{if } E \ln |\beta_0 + \epsilon_{m_{\ell},1}| \geq 0,
\end{cases}
\]
1 ≤ ℓ ≤ M + 1. Also, let

\[ \bar{\eta}(t) = \sum_{j=1}^{\ell-1} \eta_j^2 a_j^2 (\tau_j - \tau_{j-1}) + \frac{\eta_{\ell}^2}{a_{\ell}^2} (t - \tau_{\ell-1}), \quad \tau_{\ell-1} < t \leq \tau_{\ell}, 1 \leq \ell \leq M + 1, \]

and define the zero mean Gaussian process \( \{ \Gamma(t), 0 \leq t \leq 1 \} \), with \( E[\Gamma(t) \Gamma(s)] = \bar{\eta}(\min(t, s)) \).

We begin by investigating how the limits in Theorems 3.1 and 3.2 behave under heteroskedasticity.

**Theorem 4.1.** We assume that \( H_0 \) of (1.2), and Assumptions 2.1, 4.1 and 4.2 hold.

(i) If \( I(w, c) < \infty \) for some \( c > 0 \), then it holds that

\[ \sup_{0 < t < 1} \frac{|Q_N(t)|}{w(t)} \overset{D}{\to} \sup_{0 < t < 1} \frac{|\Gamma(t) - t\Gamma(1)|}{w(t)}. \]

(ii) For all \( x \), it holds that

\[ \lim_{N \to \infty} P \left\{ a(\ln N) \sup_{0 < t < 1} \frac{|Q_N(t)|}{\eta_0^{1/2}(t, t)} \leq x + b(\ln N) \right\} = \exp(-2e^{-x}). \]

(iii) If Assumption 3.2 is also satisfied, then it holds that, for all \( \kappa > 1/2 \)

\[ \left( \frac{r_N}{N} \right)^{\kappa-1/2} \sup_{r_1(N) / N \leq t \leq r_2(N) / N} \frac{(t(1-t))^{-\kappa+1/2}}{\eta_0^{1/2}(t, t)} |Q_N(t)| \overset{D}{\to} \max \left( \gamma_1^{\kappa-1/2} a_1(\kappa), \gamma_2^{\kappa-1/2} a_2(\kappa) \right). \]

Theorem 4.1 is only of theoretical interest, but we point out that heteroskedasticity impacts only on part (i). In that case, the limiting distribution of the weighted \( Q_N(t) \) is given by a Gaussian process with covariance kernel

\[ \eta_0(t, s) = E[(\Gamma(t) - t\Gamma(1)) (\Gamma(s) - s\Gamma(1))] = \bar{\eta}(\min(t, s)) - t\bar{\eta}(s) - s\bar{\eta}(t) + st\bar{\eta}(1). \]

Parts (ii)-(iii) of the theorem are the same as in the case of homoskedasticity. Distribution is driven only by the observations which are as close to sample endpoints as \( o(N) \). On these intervals, (4.3) ensures that the asymptotic variance \( \eta_0(t, t) \) is proportional to \( t(1-t) \).
Finally, note that, in light of the definitions of $\eta_0(t, t)$ and $\eta^2$ and $a_\ell$, heteroskedasticity in $\epsilon_{i,2}$ does not play a role in the nonstationary case.

4.1. Feasible tests under heteroskedasticity. By Theorem [4.1], the implementation of tests based on $Q_N(t)$ requires an estimate of $\eta_0(t, t)$. However, this is fraught with difficulties, since it requires knowledge of the different regime dates, $m_\ell$. Thus, we consider a modification of $Q_N(t)$ to reflect the possible changes in the variances of the errors.

Let

\begin{equation}
\hat{c}_{N,1}(t) = \frac{1}{N} \sum_{i=2}^{[(N+1)t]} \frac{y^2_{i-1}}{1+y^2_{i-1}} \quad \text{and} \quad \hat{c}_{N,2}(t) = \frac{1}{N} \sum_{i=\{(N+1)t\}+1}^{N} \frac{y^2_{i-1}}{1+y^2_{i-1}}, \quad 0 \leq t \leq 1;
\end{equation}

clearly, $\hat{c}_{N,2}(t) = \hat{c}_{N,1}(1) - \hat{c}_{N,1}(t)$. We then define the modified test statistic

$$\overline{Q}_N(t) = \begin{cases} 
0, & \text{if } 0 < t < 2/(N+1), \\
N^{1/2}\hat{c}_{N,1}(t)\hat{c}_{N,2}(t) \left( \hat{\beta}_{\{(N+1)t\},1} - \hat{\beta}_{\{(N+1)t\},2} \right), & \text{if } 2/(N+1) \leq t < 1 - 2/(N+1), \\
0, & \text{if } 1 - 2/(N+1) \leq t < 1.
\end{cases}$$

Under the null of no change, the same arguments as in the proof of Corollary 3.1 guarantee that $\hat{c}_{N,1}(t)$ and $\hat{c}_{N,2}(t)$ converge to the functions

\begin{equation}
c_1(t) = \sum_{j=1}^{\ell-1} (\tau_\ell - \tau_{\ell-1})a_\ell + (t - \tau_{\ell-1})a_\ell, \quad \tau_{\ell-1} < t \leq \tau_\ell, 1 \leq \ell \leq M + 1,
\end{equation}

and $c_2(t) = c_1(1) - c_1(t)$, for $0 \leq t \leq 1$, where $\tau_\ell$ is defined in Assumption 4.1 and $a_\ell, 1 \leq \ell \leq M + 1$ is defined in (1.2). In order to present our main results, we define the zero mean Gaussian process

\begin{equation}
\Theta(t) = c_2(t)\Delta(t) - c_1(t)(\Delta(1) - \Delta(t)), \quad 0 \leq t \leq 1;
\end{equation}

$\{\Delta(t), 0 \leq t \leq 1\}$ is also a zero mean Gaussian process with $E\Delta(t)\Delta(s) = b(\min(t, s))$, where

\begin{equation}
b(t) = \sum_{j=1}^{\ell-1} (\tau_j - \tau_{j-1})\eta^2_j + (t - \tau_{\ell-1})\eta^2_\ell, \quad \tau_{\ell-1} < t \leq \tau_\ell, 1 \leq \ell \leq M + 1.
\end{equation}
Let $g(t,s) = E(Θ(t)Θ(s))$; elementary calculations yield

$$
(4.9) \quad g(t,s) = c_1(1)c_1(1)b(\min(t,s)) - c_1(1)c_1(t)b(s) - c_1(1)c_1(s)b(t) + c_1(t)c_1(s)b(1).
$$

**Theorem 4.2.** We assume that $H_0$ of (1.2), and Assumptions 2.1, 4.1 and 4.2 hold.

(i) If $I(w,c) < \infty$ for some $c > 0$, then

$$
\sup_{0<t<1} \frac{|Q_N(t)|}{w(t)} \xrightarrow{\mathcal{D}} \sup_{0<t<1} \frac{|Θ(t)|}{w(t)},
$$

where $\{Θ(t), 0 \leq t \leq 1\}$ is the Gaussian process defined in (1.7).

(ii) For all $x$

$$
\lim_{N \to \infty} P\left\{a(\ln N) \sup_{0<t<1} \frac{|Q_N(t)|}{g^{1/2}(t,t)} \leq x + b(\ln N)\right\} = \exp(-2e^{-x}),
$$

where $a(x)$, $b(x)$ are defined in (3.3) and $g(t,s)$ is given in (4.9).

(iii) If Assumption 3.2 also holds and $κ > 1/2$, then

$$
\left(\frac{r_N}{N}\right)^{κ-1/2} \sup_{t_1<t_2} \frac{(t(1-t))^{-κ+1/2}}{g^{1/2}(t,t)} |Q_N(t)| \xrightarrow{\mathcal{D}} \max(γ_1^{κ-1/2}a_1(κ), γ_2^{κ-1/2}a_2(κ)),
$$

where $t_1 = r_N/N$, $t_2 = 1 - t_1$, and $r_N$, $γ_1$, $γ_2$ are defined in (3.3), $a_1(κ)$ and $a_2(κ)$ are independent copies of $a(κ)$ defined in (3.4), and $g(t,s)$ is given in (4.9).

Some comments on the practical implementation of the results in Theorem 4.2 are in order.

Parts (ii) and (iii) require an estimate of $g(t,t)$; to this end, we use $\hat{c}_{N,1}(t)$ defined in (4.5) instead of $c_1(t)$, and we estimate $b(t,s)$ as

$$
\hat{b}_N(t) = \frac{1}{N} \sum_{i=2}^{[Nt]} \left(\frac{y_i - \hat{β}_N y_{i-1} y_{i-1}}{1 + y_{i-1}^2}\right)^2, \quad 0 \leq t \leq 1.
$$

Then we can define

$$
(4.10) \quad \hat{g}(t,s) = \hat{c}_{N,1}^2(1)\hat{b}_N(\min(t,s)) - \hat{c}_{N,1}(1)\left(\hat{c}_{N,1}(t)\hat{b}_N(s) + \hat{c}_{N,1}(s)\hat{b}_N(t)\right) + \hat{c}_{N,1}(t)\hat{c}_{N,1}(s)\hat{b}_N(1).
$$
The implementation of part (i) of Theorem 4.2 is more complicated, since the presence of nuisance parameters is not relegated to a multiplicative function. We reject the null hypothesis in (1.2) if

$$\sup_{0 < t < 1} \frac{|Q_N(t)|}{w(t)} \geq c(\alpha),$$

with $c(\alpha)$ defined as $P\left\{ \sup_{0 < t < 1} \frac{|\Theta(t)|}{w(t)} \geq c(\alpha) \right\} = \alpha$. Computing the covariance functions, one can verify that

$$\{ \Delta(t), 0 \leq t \leq 1 \} \overset{D}{=} \{ W(b(t)), 0 \leq t \leq 1 \},$$

where $\{ W(x), 0 \leq x < \infty \}$ is a Wiener process. In order to approximate the critical values, one can simulate independent Wiener processes $W_i(x), 1 \leq i \leq L$, and compute the empirical distribution function

$$F_{N,L}(x) = \frac{1}{L} \sum_{i=1}^{L} I \left\{ \sup_{0 < t < 1} \frac{|\hat{\Theta}_i(t)|}{w(t)} \leq x \right\},$$

where $\hat{\Theta}_i(t) = \hat{\Theta}_{N,2}(t)W_i(\hat{b}_N(t)) - \hat{\Theta}_{N,1}(t)(W_i(\hat{b}_N(1)) - W_i(\hat{b}_N(t))).$

Let $c_{N,L}(\alpha)$ be defined as $c_{N,L}(\alpha) = \inf\{ x : F_{N,L}(x) \geq 1 - \alpha \}$.

**Corollary 4.1.** We assume that $H_0$ of (1.2) holds. Under the same assumptions as Theorem 4.2(i), it holds that

$$\lim_{\min(N,L) \to \infty} P\left\{ \sup_{0 < t < 1} \frac{|Q_N(t)|}{w(t)} \geq c_{N,L}(\alpha) \right\} = \alpha.$$

The results of Theorem 4.2(ii)-(iii) remain true if $g(t,s)$ is replaced with $\tilde{g}(t,s)$.

**4.2. Consistency versus alternatives.** We study the consistency of our tests versus the AMOC alternative

$$y_i = \begin{cases} (\beta_0 + \epsilon_{i,1}) y_{i-1} + \epsilon_{i,2}, & \text{if } 1 \leq i \leq k^* \\ (\beta_A + \epsilon_{i,1}) y_{i-1} + \epsilon_{i,2}, & \text{if } k^* + 1 \leq i \leq N \end{cases}.$$

Let $\Delta_N = |\beta_0 - \beta_A|$, and define $t^*$ as $\lfloor Nt^* \rfloor = k^*$.

---

1In Section A.2 in the Supplement, we also discuss the power of our tests versus the alternative of multiple breaks.
Theorem 4.3. We assume that $H_0$ of (1.3) holds. Under the same assumptions as Theorem 4.2, if, as $N \to \infty$

\begin{equation}
N^{1/2} \Delta_N \left( \frac{k^* \left( \frac{N - k^*}{N} \right)}{w(t^*)} \right) \to \infty,
\end{equation}

then it holds that, as $\min (N, L) \to \infty$

\begin{equation}
\sup_{0 < t < 1} \frac{\left| Q_N(t) \right|}{w(t)} \to \infty.
\end{equation}

Further, if, as $N \to \infty$

\begin{equation}
\frac{N^{1/2}}{(\ln \ln N)^{1/2}} \Delta_N \left( \frac{k^* \left( \frac{N - k^*}{N} \right)}{w(t^*)} \right)^{1/2} \to \infty,
\end{equation}

then it holds that

\begin{equation}
a(\ln N) \sup_{0 < t < 1} \frac{1}{\hat{g}^{1/2}(t, t)} \left| Q_N(t) \right| - b(\ln N) \to \infty.
\end{equation}

Finally, if

\begin{equation}
\left( \frac{r_N}{N} \right)^{\kappa - 1/2} N^{1/2} \Delta_N \left( \frac{k^* \left( \frac{N - k^*}{N} \right)}{w(t^*)} \right)^{1-\kappa} \to \infty,
\end{equation}

then it holds that, for all $\kappa > \frac{1}{2}$

\begin{equation}
\left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{t_1 < t < t_2} \frac{(t(1 - t))^{-\kappa + 1/2}}{\hat{g}^{1/2}(t, t)} \left| Q_N(t) \right| \to \infty.
\end{equation}

The theorem ensures that, as long as (4.12), (4.14) and (4.16) hold, our tests reject the null with probability (asymptotically) 1. Conditions (4.12), (4.14) and (4.16) essentially state that breaks will be detected as long as they are “not too small”, and “not too close” to the endpoints of the sample.

Consider (4.12). This condition can be understood by considering two cases. First, when $\frac{k^*}{N} \to c \in (0, 1)$, it is required that $N^{1/2} \Delta_N \to \infty$: this entails that $\beta_A$ may depend on the sample size $N$, so that even small changes in the regression parameter are allowed. When $\Delta_N > 0,$
(4.12) holds as long as \( k^* N^{\frac{1}{2(1-\kappa)}} \to \infty \): tests based on weight functions \( w(t) = (t - (1 - t))^\kappa \) can detect breaks almost as close to the sample endpoints as \( O\left( N^{\frac{1}{2(1-\kappa)}} \right) \).

Turning to (4.14), when \( k^* \to c > 0 \), the test is powerful as long as \( \left( \frac{N}{\ln \ln N} \right)^{1/2} \Delta N \to \infty \): again small changes are allowed for, but these are now “less small” by a \( O(\ln \ln N) \) factor.

Conversely, when \( \Delta N > 0 \), (4.12) holds as long as \( k^* (\ln \ln N)^{-1/2} \to \infty \): breaks that are as close as \( O\left( \sqrt{\ln \ln N} \right) \) periods to the sample endpoints can be detected. This effect is reinforced in the case of Rényi statistics, where, on account of (4.16), the only requirement is that \( k^* > r_N \).

5. Simulations

We provide some Monte Carlo evidence on the performance of the test statistics proposed in Section 4.1.

Data are generated using (1.1). In all experiments, we use \( \beta_0 \in \{0.5, 0.75, 1, 1.05\} \) to consider both the cases of stationary and nonstationary \( y_i \). We have experimented also with different values of \( \beta_0 \), but results are essentially the same. Under the alternative, we consider both a mid-sample and an end-of-sample break

\[
y_i = (\beta_0 + \Delta I (i \geq 0.5N) + \epsilon_{i,1}) y_{i-1} + \epsilon_{i,2},
\]

\[
y_i = (\beta_0 + \Delta I (i \geq 0.9N) + \epsilon_{i,1}) y_{i-1} + \epsilon_{i,2}.
\]

The shocks \( \epsilon_{i,1} \) and \( \epsilon_{i,2} \) are simulated as independent of one another and \( i.i.d. \) with distributions \( N(0, \sigma_1^2) \) and \( N(0, \sigma_2^2) \) respectively. We report results for \( \sigma_1^2 = 0.01 \) and \( \sigma_2^2 = 0.5 \) - the value of \( \sigma_1^2 \) is based on “typical” values as found e.g. in the empirical applications in Horváth and Trapani (2019). We note however that, in unreported simulations using different values of \( \sigma_1^2 \) and \( \sigma_2^2 \), the main results do not change, except for the (expected) fact that tests have better properties (in terms of size and power) for smaller values of \( \sigma_2^2 \). Similarly, the test performs better (with empirical rejection frequencies closer to their nominal value) when \( \sigma_1^2 \) is larger, and tends to be undersized for smaller values of \( \sigma_1^2 \). Both effects (of \( \sigma_1^2 \) and \( \sigma_2^2 \)) vanish as \( N \) increases. When

\footnote{In Section A.1 in the Supplement, we also evaluate, as a benchmark, the performance of our tests under homoskedasticity, based on the theory in Section 3.}
allowing for heteroskedasticity, we generate $\epsilon_{i,1}$ and $\epsilon_{i,2}$ as $i.i.d. N(0, \sigma_1^2)$ and $i.i.d. N(0, \sigma_2^2)$ for $1 \leq i \leq N/2$, and $i.i.d. N(0, 1.5\sigma_1^2)$ and $i.i.d. N(0, 1.5\sigma_2^2)$ for $N/2 + 1 \leq i \leq N$.

Finally, we generate $N + 1, 000$ values of $y_{i}$ from (1.1) - with $y_0 = 0$ - and discard the first 1,000 values. All our routines are based on 2,000 replications, and we use critical values corresponding to a nominal level equal to $5\%$ - hence, empirical rejection frequencies under the null have a 95% confidence interval $[0.04, 0.06]$.

We consider four different cases: (i) homoskedasticity in both $\epsilon_{i,1}$ and $\epsilon_{i,2}$; (ii) homoskedasticity in $\epsilon_{i,1}$ and heteroskedasticity in $\epsilon_{i,2}$; (iii) homoskedasticity in $\epsilon_{i,2}$ and heteroskedasticity in $\epsilon_{i,1}$; and, finally, (iv) heteroskedasticity in both $\epsilon_{i,1}$ and $\epsilon_{i,2}$.

### Table 5.1. Empirical rejection frequencies under the null - part 1

| $\beta$ | N   | 0.5 | 0.75 | 1   | 1.05 |
|---------|-----|-----|------|-----|------|
| $\kappa$ | 200 | 400 | 800  | 1600| 200  | 400  | 800  | 1600| 200  | 400  | 800  | 1600|
| 0       | 0.056 | 0.066 | 0.048 | 0.060 | 0.058 | 0.049 | 0.054 | 0.058 | 0.058 | 0.059 | 0.056 | 0.060 | 0.057 | 0.057 | 0.063 | 0.059 |
| 0.25    | 0.048 | 0.053 | 0.046 | 0.053 | 0.046 | 0.050 | 0.050 | 0.058 | 0.052 | 0.060 | 0.058 | 0.056 | 0.056 | 0.057 | 0.056 | 0.062 | 0.060 |
| 0.45    | 0.033 | 0.034 | 0.036 | 0.038 | 0.035 | 0.027 | 0.031 | 0.040 | 0.033 | 0.042 | 0.038 | 0.036 | 0.029 | 0.034 | 0.038 | 0.041 | 0.041 |
| 0.5     | 0.033 | 0.026 | 0.033 | 0.050 | 0.035 | 0.030 | 0.034 | 0.035 | 0.037 | 0.038 | 0.042 | 0.041 | 0.040 | 0.039 | 0.042 | 0.045 | 0.045 |
| 0.51    | 0.015 | 0.020 | 0.019 | 0.037 | 0.016 | 0.019 | 0.028 | 0.038 | 0.016 | 0.017 | 0.025 | 0.038 | 0.015 | 0.025 | 0.028 | 0.036 | 0.036 |
| 0.75    | 0.034 | 0.047 | 0.042 | 0.040 | 0.042 | 0.043 | 0.047 | 0.047 | 0.039 | 0.043 | 0.041 | 0.051 | 0.035 | 0.042 | 0.045 | 0.034 | 0.045 |
| 0.85    | 0.034 | 0.050 | 0.045 | 0.046 | 0.047 | 0.050 | 0.050 | 0.050 | 0.043 | 0.049 | 0.049 | 0.053 | 0.043 | 0.060 | 0.048 | 0.040 | 0.040 |
| 1       | 0.038 | 0.050 | 0.047 | 0.045 | 0.050 | 0.048 | 0.049 | 0.043 | 0.043 | 0.051 | 0.046 | 0.052 | 0.043 | 0.061 | 0.046 | 0.042 | 0.042 |

| $\beta$ | N   | 0.5 | 0.75 | 1   | 1.05 |
|---------|-----|-----|------|-----|------|
| $\kappa$ | 200 | 400 | 800  | 1600| 200  | 400  | 800  | 1600| 200  | 400  | 800  | 1600|
| 0       | 0.055 | 0.058 | 0.051 | 0.053 | 0.054 | 0.050 | 0.055 | 0.057 | 0.058 | 0.052 | 0.052 | 0.058 | 0.058 | 0.054 | 0.058 | 0.058 | 0.060 |
| 0.25    | 0.054 | 0.053 | 0.053 | 0.054 | 0.057 | 0.043 | 0.053 | 0.055 | 0.052 | 0.056 | 0.054 | 0.054 | 0.050 | 0.059 | 0.058 | 0.060 | 0.060 |
| 0.45    | 0.033 | 0.033 | 0.033 | 0.043 | 0.034 | 0.029 | 0.033 | 0.037 | 0.030 | 0.038 | 0.040 | 0.034 | 0.027 | 0.034 | 0.034 | 0.043 | 0.043 |
| 0.5     | 0.036 | 0.026 | 0.043 | 0.044 | 0.038 | 0.030 | 0.036 | 0.040 | 0.039 | 0.037 | 0.040 | 0.040 | 0.037 | 0.040 | 0.040 | 0.048 | 0.048 |
| 0.51    | 0.018 | 0.019 | 0.025 | 0.041 | 0.019 | 0.023 | 0.032 | 0.034 | 0.029 | 0.018 | 0.027 | 0.040 | 0.017 | 0.029 | 0.026 | 0.045 | 0.045 |
| 0.75    | 0.031 | 0.045 | 0.043 | 0.043 | 0.045 | 0.046 | 0.052 | 0.050 | 0.046 | 0.043 | 0.045 | 0.054 | 0.039 | 0.050 | 0.040 | 0.042 | 0.042 |
| 0.85    | 0.036 | 0.050 | 0.049 | 0.050 | 0.050 | 0.047 | 0.053 | 0.048 | 0.046 | 0.047 | 0.049 | 0.051 | 0.044 | 0.055 | 0.042 | 0.046 | 0.046 |
| 1       | 0.037 | 0.055 | 0.048 | 0.051 | 0.051 | 0.050 | 0.051 | 0.047 | 0.045 | 0.050 | 0.046 | 0.051 | 0.044 | 0.053 | 0.045 | 0.047 | 0.047 |

Empirical rejection frequencies under the null of no change - the first panel refers to homoskedastic innovations; in the second panel we consider heteroskedasticity in $\epsilon_{i,2}$ only.

3When using (1.11), we use $L = 200$. Results are however not particularly sensitive to this specification.
the Rényi statistic with κ

95%

not even in small samples - conversely, there are some cases of (severe) under-rejection in small cases where asymptotic critical values for Rényi statistics, and the method described in Section 4.1 for the

κ

0 0 0 0

1 0

0.1 0.015 0.017 0.021 0.008 0.014 0.023 0.020 0.009 0.013 0.021 0.021 0.013 0.018 0.020 0.025

0.75 0.035 0.040 0.041 0.043 0.031 0.050 0.043 0.045 0.035 0.046 0.047 0.043 0.036 0.044 0.040 0.040

0.85 0.040 0.046 0.043 0.043 0.035 0.055 0.046 0.048 0.040 0.058 0.053 0.046 0.043 0.050 0.042 0.041

1 0.044 0.045 0.045 0.044 0.038 0.058 0.045 0.047 0.042 0.063 0.051 0.047 0.047 0.054 0.045 0.041

Empirical rejection frequencies under the null of no change - in the first panel we consider heteroskedasticity in ε_{i,1} only; in the second panel we consider heteroskedasticity in both ε_{i,1} and ε_{i,2} only.

Table 5.2. Empirical rejection frequencies under the null - part 2

| β   | 0.5 | 0.75 | 1   | 1.05 |
|-----|-----|------|-----|------|
| N   |     |      |     |      |
| κ  | 0   | 0.046 0.061 0.045 0.057 0.065 0.045 0.056 0.052 0.066 0.063 0.056 0.058 0.058 0.054 0.062 0.059 |
| 0.25 | 0.044 0.055 0.050 0.056 0.062 0.047 0.056 0.054 0.059 0.058 0.054 0.060 0.051 0.050 0.062 0.059 |
| 0.45 | 0.023 0.031 0.029 0.036 0.035 0.030 0.040 0.034 0.036 0.033 0.041 0.043 0.027 0.036 0.038 0.045 |
| 0.5  | 0.027 0.033 0.026 0.040 0.040 0.026 0.042 0.040 0.040 0.029 0.043 0.039 0.036 0.035 0.039 0.050 |
| 0.51 | 0.015 0.006 0.017 0.021 0.008 0.014 0.023 0.020 0.009 0.013 0.021 0.021 0.013 0.018 0.020 0.025 |
| 0.75 | 0.035 0.040 0.041 0.043 0.031 0.050 0.043 0.045 0.035 0.046 0.047 0.043 0.036 0.044 0.040 0.040 |
| 0.85 | 0.040 0.046 0.043 0.043 0.035 0.055 0.046 0.048 0.040 0.058 0.053 0.046 0.043 0.050 0.042 0.041 |
| 1   | 0.044 0.045 0.045 0.044 0.038 0.058 0.045 0.047 0.042 0.063 0.051 0.047 0.047 0.054 0.045 0.041 |

Empirical rejection frequencies under the null are in Tables 5.1 and 5.2. We have used asymptotic critical values for Rényi statistics, and the method described in Section 4.1 for the cases where κ < 0.5. When using the Darling-Erdős statistic (κ = 0.5), asymptotic critical values yield hugely undersized tests; we have therefore used the critical values in Table I in Gombay and Horváth (1996). From Tables 5.1 and 5.2 all tests work very well in all cases considered, possibly being slightly worse in the fully homoskedastic case. Tests never over-reject, not even in small samples - conversely, there are some cases of (severe) under-rejection in small samples, especially when κ is around 0.5. As N increases, however, this vanishes and the empirical rejection frequencies all lie within their 95% confidence interval. The only exception is the Rényi statistic with κ = 0.51, which is severely undersized even in large samples.
The empirical power of the tests is reported in Figures C.1-C.4 where we only consider a sample size of $N = 400$ to save space. The figures illustrate the robustness of the approach proposed in Section 4.1 showing, essentially, the same pattern: tests work well in all cases considered, with the power increasing monotonically in $\Delta$. Test statistics with lower $\kappa$ exhibit more power versus alternatives with “small” values of $\Delta$: in this case, the power is monotonically decreasing in $\kappa$, with virtually no exceptions. Figures C.1-C.2 compared with Figures C.3-C.4 show an interesting feature: the power of the test is virtually unaffected by the presence or absence in heteroskedasticity in $\epsilon_{i,2}$; conversely, heteroskedasticity in $\epsilon_{i,1}$ does have an impact. This is particularly apparent in the cases where $\beta_0 \geq 1$, which could be explained by noting that, in the nonstationary case the asymptotics of the WLS estimator is driven only by $\epsilon_{i,1}$. This can be read in conjunction with the fact that, as shown in Figures C.1-C.2, the value of $\beta_0$ has virtually no impact on the power of our tests when $\epsilon_{i,1}$ is constant.

We also consider the case of end-of-sample breaks (5.2). Results are in Figures C.5-C.8. The results show, essentially, the same pattern as above: all test statistics have monotonic power in $\Delta$, and whilst heteroskedasticity in $\epsilon_{i,2}$ does not affect the whole picture, heteroskedasticity in $\epsilon_{i,1}$ gives very different results, with its presence increasing power especially for $\beta_0 \geq 1$. However, the impact of $\kappa$ here is, as expected, completely reversed: the power versus breaks that occur at the end of the sample increases monotonically, ceteris paribus, with $\kappa$. This makes a difference particularly in the case of medium-sized changes - e.g., when $\Delta = 0.35$, increases in power from $\kappa = 0$ to $\kappa = 1$ are in the region of $10 - 15\%$.

Finally, in Figures C.9-C.10 we report a small scale exercise where we evaluate the empirical rejection frequencies when $\beta_0$ is close to unity. We only consider heteroskedasticity in $\epsilon_{i,2}$: results for other cases are available upon request and, in general, no major differences are noted compared to the other results. These “boundary” cases should be helpful to shed more light on the performance of our procedure when detecting changes from stationarity to nonstationarity (when $\beta_0 < 1$ and changes are positive), and vice versa (when $\beta_0 > 1$ and changes are negative). The main message of Figures C.9-C.10 is that our tests work very well in these boundary cases.
In particular, the tests are very effective in detecting changes from stationarity to explosive behaviour, and vice versa. The power is especially high when $\beta_0 > 1$ - i.e. when the RCA process changes from an explosive to a stationary behaviour. This suggests a possible, effective test to detect e.g. the collapse of a bubble in financial econometrics applications.

6. Empirical applications

We illustrate our approach through three applications to real data. In Sections 6.1-6.2, we use economic and financial time series; in Section 6.3 we use Covid-19 data.

6.1. Application I: changes in the persistence of US CPI. In his landmark paper, Engle (1982) applies an ARCH(1) specification to monthly inflation data, showing that these exhibit conditional heteroskedasticity. Inspired by this, and by the fact that the RCA model is a second-order equivalent to the ARCH model, in this section we test for the presence of changepoints in the dynamics of US CPI over the last century. There is an increasing literature on testing for changes in the persistence of inflation: in particular, Benati and Kapetanios (2003) carry out a systematic study, applying tests for structural breaks to an AR($p$) model for inflation for several countries. Their analysis shows that not only the average level of inflation (as is well-documented), but also its serial correlation, may be subject to numerous changes.

We use monthly CPI data taken from the FRED dataset over a period spanning from January 1913 until January 2021, with $N = 1297$. We use monthly inflation rates, calculated as the month-on-month log differences of the series. Given that the series is quite long, we expect to see more than one break; hence, we use binary segmentation (as suggested in Vostrikova, 1981), reporting the point in time at which the relevant test statistic is maximised as the breakdate estimate.

Results in Table XXX differ across tests only marginally, and suggest the presence of several changepoints in the autoregressive coefficient - see also Figure C.11 in the Supplement. Some of the estimated breakdates have a clear economic interpretation. In chronological order, the first break is found (only by Rényi statistics) around January 1918, which should reflect not only the war effort, but also the increasingly more comprehensive data collection from the Bureau of
Table 6.1. Changepoint detection with US CPI data.

| Breakdate | Notes                                                                 |
|-----------|----------------------------------------------------------------------|
| Jan 1918  | Date found with $\kappa = 0.85$, 1 (found at Apr 1918 with other $\kappa > 0.51$; not found with $\kappa = 0.51$). Found at 10% level with $\kappa = 0.25$ and 0.5 |
| Sep 1921  | Date found with $\kappa = 0.85$, 1 (found Jan 1922 with $\kappa > 0.65$; not found with other values of $\kappa$) |
| May 1929  | Break found at 10% nominal level only, with $\kappa < 0.5$ only |
| Jul 1957  | Same date found by all Rényi-type stats (also found at Jan 1957 with $\kappa = 0.5$). Break not found with $\kappa < 0.5$ |
| Nov 1966  | Same date found by all tests - except $\kappa = 0.5$ (found at Feb 1966), and $\kappa = 0.25$ (at 10% level) |
| Jun 1982  | Same date found by all tests (found at Dec 1981 with $\kappa = 0.5$) - Rényi-type stats at 5%, nominal level, all others at 10% |
| Nov 1989  | Same date found by all Rényi-type stats (found at Mar 1989 with $\kappa = 1$). Break not found with $\kappa \leq 0.5$ |

We use the monthly log differences of CPI. Detected changepoints, and their estimated date, are presented in chronological order; as mentioned in the text, breakdates have been estimated as the earliest detection date across different values of $\kappa$. Whilst details are available upon request, we note that breaks were detected with this order (from the first to be detected to the last one): November 1966; July 1957 and January 1918; June 1982 and September 1921; March 1989 and May 1929.

Labor Statistics; evidence in favour of the changepoint increases as $\kappa$ increases, as the theory would suggest given that this is an early change (occurring circa at 5% of the sample). Indeed, when $\kappa \leq 0.5$, the break is found at 10%, but not at 5% level. Similar considerations, on the detection ability and timing corresponding to different values of $\kappa$, can be made for the break found in 1921; in this case, a possible cause is the impact of the severe recession at the beginning of the decade, as well as the very rapid deflation which had occurred in 1920. Conversely, there is limited evidence for a break in the autoregressive parameter of inflation around the Great Depression - looking at Figure C.11 this may be explained as a shift which occurred only in the mean as opposed to the persistence. The break occurring in 1957 may be viewed as related to inflation reemerging, albeit modestly, in the spring of 1956 after a long period of price stability, with the All-Items CPI increasing by 3.6 percent from April 1956 to April 1957 (comparing with the previous period, the All-Items CPI had risen by 0.2 percent annualized rate from July 1952 to April 1956). The changepoint in 1966 (which is the first one to be found, by all tests) can be explained by noting that food prices had started accelerating early at the end of 1965; and, by October 1966, the change in the All-Items CPI reached its highest since 1957. The change in 1981 is documented also in other studies (Eo, 2016), and it corresponds to the beginning of an aggressive FED policy to rein in inflation after the 1970s. Finally, the evidence for the break in
1989 (which is not picked up by all tests) is less clear, but it may be the outcome of the cooling off in FED policy towards the end of the 1980s.

6.2. Application II: monthly IBM returns. We use IBM monthly returns, over a period spanning January 1962 till March 2021 (corresponding to $T = 710$). The data used in this application has also been employed, in the context of testing for changepoints, by Yau and Zhao (2016), who find evidence of two breaks: one around June 1987 (with confidence interval between June 1986 and June 1988), and another around October 2002 (with confidence interval between April 2001 and April 2004). As in the previous section, we have applied the tests using binary segmentation, using all the test statistics developed above. None of them rejected the null of no break at 5% level; when applying the tests at 10% level, though, several breaks were found.

**Table 6.2. Changepoint detection with IBM data.**

| Changepoint 1 | Changepoint 2 | Changepoint 3 |
|---------------|---------------|---------------|
| Date          | Sep 1973      | Nov 1987      | Oct 1999      |
| Notes         | Rényi stats (10% nominal level) | Rényi stats (10% nominal level) | Weighted stats (10% nominal level) |

We use the logs of the original data, with no further transformations. For each estimated changepoint, we indicate which statistic has detected it, and at which (nominal) level. When more than one statistic finds a break, the estimated date is computed as the majority vote across statistics, using the point in time at which the relevant statistic finds a break as estimator. Whilst details are available upon request, we note that breaks were detected with this order (from the first to be detected to the last one): break in 1973; break in 1987; break in 1999.

We found three changepoints in the whole series (see Table 6.2). The first one, whose date corresponds to the well-known 1973-74 market crash (due to the collapse of the Bretton-Woods system, and compounded by the oil shock), is relatively close to the beginning of the sample, and indeed it has been identified by the Rényi statistics (the other tests do not identify such break). The second changepoint can also be related to a specific event, i.e. the Black Monday (the break is found in November 1987, i.e. one month later the actual event). Finally, Rényi statistics do not find the third changepoint, which occurs mid-sample, confirming the idea that mid-sample breaks are better detected using milder weight functions (indeed, not even the Darling-Erdős test finds evidence of such a break); the break is found before the collapse of the dot-com bubble.
(traditionally dated around March 2000), reflecting the trouble brewing in the months leading to the event.

6.3. **Application III: Covid-19 UK hospitalisation data.** In this section, we consider UK data on Covid-19 - in particular, we use data on hospitalisations rather than cases, as the latter may be less reliable due to the change in number of tests administered. Shatatland and Shatatland (2008) *inter alia* advocate using a low-order autoregression as an approximation of the popular SIR model, especially as a methodology for the early detection of outbreaks. In this context, the autoregressive root is of crucial importance since, as the authors put it, if “the parameter is greater than one, we have an explosive case (an outbreak of epidemic)”. It is therefore important to check whether the observations change from an explosive to a stationary regime (meaning that the epidemic is slowing down), or vice versa whether the change occurs from a stationary to an explosive regime (i.e., the epidemic undergoes a surge, or “wave”). In this respect, the empirical exercise in this section should be read in conjunction with Figures C.9-C.10.

We use (logs of) UK daily data, for the four UK nations, and for the various regions of England, again using binary segmentation to detect multiple breaks. We only report results obtained using Rényi statistics (with $\kappa = 0.51, 0.55, 0.65, 0.75, 0.85$ and $1$); the other tests give very similar results, available upon request. As far as breakdates are concerned, we pick the ones corresponding to the “majority vote” across $\kappa$, although discrepancies are, when present, in the region of few days ($2 - 5$ at most).

The results in Table 6.3 suggest that, with the exception of Wales, there were multiple breaks in all series considered; we note that Wales is an outlier as regards hospital admissions, because these are counted in a different way than the rest of the UK.

Some breaks occur closely to the sample endpoints, highlighting the importance of using Rényi statistics. Also, all changepoints indicate a transition of the autoregressive coefficient $\beta_0$ around

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4 The data are available from https://ourworldindata.org/enerator/uk-daily-covid-admissions?tab=chart&stackMode=absolute&time=2020-03-29..latest&region=World
5 Figure 6.1 contains the same information, albeit limited to the four UK nations only to save space
6 Specifically, Wales reports also suspected Covid-19 cases, whereas all the other nations only report confirmed cases; see https://www.cebm.net/covid-19/the-flaw-in-the-reporting-of-welsh-data-on-covid-hospital-admissions/
Table 6.3. Changepoint analysis for Covid-19 daily hospitalisation UK data.

| Region            | Start Date | Sample size | Changepoint 1 | Changepoint 2 | Changepoint 3 | Changepoint 4 | Changepoint 5 |
|-------------------|------------|-------------|---------------|---------------|---------------|---------------|---------------|
| East of England   | 19/3/20    | 317         | 12 Apr        | 25 Aug        | 08 Aug        | 08 Jan        | 08 Jan        |
|                   |            |             | [0.013;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| London            | 19/3/20    | 317         | 10 Apr        | 05 Aug        | 05 Apr        | 05 Aug        | 07 Jan        |
|                   |            |             | [0.014;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| Midlands          | 19/3/20    | 317         | 07 Apr        | 12 Aug        | 19 Aug        | 14 Nov        | 13 Jan        |
|                   |            |             | [0.013;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| North East        | 19/3/20    | 317         | 09 Apr        | 08 Aug        | 08 Aug        | 14 Nov        | 12 Dec        |
|                   |            |             | [0.023;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| North West        | 19/3/20    | 317         | 09 Apr        | 18 Aug        | 18 Aug        | 29 Oct        | 13 Dec        |
|                   |            |             | [0.013;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| South East        | 19/3/20    | 317         | 12 Apr        | 01 Sep        | 01 Sep        | 13 Nov        | 07 Jan        |
|                   |            |             | [0.013;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| South West        | 19/3/20    | 317         | 12 Apr        | 01 Sep        | 01 Sep        | 13 Nov        | 07 Jan        |
|                   |            |             | [0.013;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| England           | 19/3/20    | 317         | 10 Apr        | 26 Aug        | 29 Oct        | 29 Oct        | 12 Jan        |
|                   |            |             | [0.009;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| Northern Ireland  | 02/3/20    | 325         | 04 Apr        | 12 Aug        | 12 Aug        | 12 Aug        | 12 Jan        |
|                   |            |             | [0.009;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| Scotland          | 01/3/20    | 326         | 04 Apr        | 04 Apr        | 04 Apr        | 04 Apr        | 12 Jan        |
|                   |            |             | [0.052;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |
| Wales             | 23/3/20    | 313         | 10 Apr        | 21 Aug        | 21 Aug        | 21 Aug        | 21 Jan        |
|                   |            |             | [0.009;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] | [0.001;0.006] |

All series end at 30 January 2021. We use the logs of the original data (plus one, given that, in some days, hospitalisations are equal to zero): no further transformations are used.

All changepoints have been detected by all Rényi-type tests - no discrepancies were noted. Detected changepoints, and their estimated date, are presented in chronological order; breakdates have been estimated as the points in time where the majority of tests identifies a changepoint. Whilst details are available upon request, we note that breaks were detected with this order (from the first to be detected to the last one): breaks in August; breaks in April and January; breaks in October-November; breaks in December.

For each changepoint, we report in square brackets, for reference, the left and right WLS estimates of $\beta_0$.

unity. Differences between pre- and post-break values of $\beta_0$ are small, but sufficient to trigger, or quench, an outbreak - on account of the Monte Carlo evidence contained in Figures C.9, C.10 we would not expect spurious detection of breaks when these are absent.

Considering first regions of England, all of these experience a break in early April as a consequence of the first national lockdown, which started on March 23rd, 2020, but was preceded by growing concerns, and closures in the education and hospitality sectors, the week before. Similarly, all series have a subsequent change (with $\beta_0$ exceeding unity after the breaks) in late August - one exception is London, where the change occurred in early August. These breaks indicate the beginning of the “second wave” in the UK, which has been ascribed (also) to an increase in travelling during the holiday season and which was officially acknowledged by the PM on September 18th, 2020. The breaks in autumn, where present, can be explained as the effect of the local and national lockdowns which were implemented at the end of October, and of the easing of restrictions in early December. Finally, all series have a change towards stationarity
around mid-January, which again can be explained as the effect of the national lockdown announced on January 4th, 2021, and of the growing concerns about a third wave voiced before and during the Christmas holidays.

The same picture applies to England as a whole. Conversely, the other UK nations experienced slightly different patterns, likely as a consequence of different policies implemented by local governments. With the exception of Wales, which seems to have only one break (but note the caveat about Welsh data mentioned above), Scotland and Northern Ireland are essentially aligned with the results for England in terms of the effects of the first lockdown, the summer holiday, and the third lockdown.

### 7. Discussion and conclusions

In this paper, we study changepoint detection in the deterministic part of the autoregression coefficient of a Random Coefficient AutoRegressive model. We use the CUSUM process based

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**Figure 6.1. Daily Covid-19 hospitalisations for the four UK nations**

![Graph showing hospitalisations for four UK nations](image-url)
on comparing the left and right WLS estimators. In order to be able to detect changepoints close to the sample endpoints, we study weighted statistics, where more weight is placed at the sample endpoints. We consider a very wide class of weighing functions, studying: (i) weighing schemes based on the functions $w(t)$, which drift to zero, at sample endpoints, more slowly than $(t (1 - t))^{1/2}$; (ii) standardised statistics, with weighting $(t (1 - t))^{1/2}$; and (iii) Rényi statistics, where heavier weights are used. The last class of statistics is still not fully studied (with the notable exception of Horváth et al., 2020b), and looks extremely promising in the detection of early or late breaks.

From a practical point of view, our tests can be applied in the presence of heteroskedasticity (requiring no knowledge as to the actual presence, or the form, thereof), and simulations show that our procedures work very well in practice - indeed, they work even better than procedures based on asymptotic critical values in the baseline case of homoskedasticity. Technically, all our results are based on a (strong) approximation of the weighted maximum of partial sums. We have developed these both in the stationary and in the nonstationary case: in the latter case (nonstationary data), our approximations are entirely novel, and yield the same results as in the stationary case. This, too, has important practical implications: our tests can be applied with no prior knowledge as to the stationarity or lack thereof of the data. This robustness reinforces the case made by Aue and Horváth (2011) for RCA models, where the authors advocate the use of these models as an alternative to the AR(1) model, which does not possess the same property and may require differencing, with the well-known problems attached to this transformation (Leybourne et al., 1996). Hence, our procedures lend themselves to several interesting applications and extensions. As a leading example, our theory could be used as the building block to develop procedures for the sequential detection of bubbles starting or collapsing. This, and other extensions, are under investigation by the authors.

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We discuss the computation of asymptotic critical values, and report some of them (Section A.1). Further, in Section A.2 we study the power versus more general alternatives that the AMOC one considered in Section 4.2.

A.1. Computation of critical values and further simulations under homoskedasticity.

The asymptotic critical values for the homoskedastic case are in Table A.1.

| \( \kappa \) | 5%  | 10% |
|--------|-----|-----|
| 0.0    | 1.3700 | 1.2238 |
| 0.25   | 2.0142 | 1.8106 |
| 0.45   | 3.0320 | 2.8988 |
| 0.50   | 3.7052 | 3.2507 |
| 0.55   | 2.9414 | 2.7992 |
| 0.60   | 2.9946 | 2.7050 |
| 0.75   | 2.5896 | 2.3860 |
| 0.85   | 2.5745 | 2.2996 |
| 1.0    | 2.4948 | 2.2365 |

Critical values at 5% and 10% nominal levels, for the limiting distributions in Theorems 3.1 and 4.2.

When \( \kappa < \frac{1}{2} \), we have simulated the critical values using the algorithm proposed in [Franke et al. (2020)]; our results differ marginally from the values reported in the original paper, but unreported experiments show that our critical values yield less undersizement than the original ones, at least in small samples.

In the case \( \kappa > \frac{1}{2} \), we know that critical values are the same for both the homoskedastic and the heteroskedastic case. In all experiments (and in the computation of critical values), we use symmetric trimming - i.e., \( r_1(N) = r_2(N) \); in this case, it is easy to see that

\[
P \left[ \max (a_1(\kappa), a_2(\kappa)) \leq c_\alpha \right] = (P [a(\kappa) \leq c_\alpha])^2,
\]

and our critical values are based on Table 1 in [Horváth et al. (2004)].
The most critical case is the case $\kappa = \frac{1}{2}$. For a given nominal level $\alpha$, asymptotic critical values are given by $c_\alpha = -\ln(-0.5\ln(1-\alpha))$, and Theorems 3.1-4.2 state that the limiting distribution of the max-type statistics is the same in both the homoskedastic and the heteroskedastic cases. Interestingly (and contrary to the heteroskedastic case), our simulations show that in the homoskedastic case, asymptotic critical values work well, with no under-rejection and good power.

To complement Section 5, we also report, as a benchmark, some evidence on the size of our tests under homoskedasticity, using the theory in Section 3. Empirical rejection frequencies under the null are reported in Table A.2.

| $\beta$ | $0.5$ | $0.75$ | $1$ | $1.05$ |
|---------|------|------|----|------|
| $N$     | 200  | 400  | 800| 1600 | 200  | 400  | 800| 1600 | 200  | 400  | 800| 1600 |
| $\kappa$ | 0    | 0.048| 0.044| 0.040| 0.037| 0.044| 0.053| 0.037| 0.039| 0.048| 0.052| 0.054| 0.045| 0.057| 0.045| 0.038| 0.037 |
|         | 0.25 | 0.060| 0.052| 0.045| 0.043| 0.061| 0.062| 0.046| 0.048| 0.069| 0.069| 0.058| 0.049| 0.070| 0.059| 0.042| 0.045 |
|         | 0.45 | 0.073| 0.059| 0.034| 0.030| 0.062| 0.071| 0.054| 0.039| 0.086| 0.090| 0.079| 0.067| 0.079| 0.081| 0.062| 0.060 |
|         | 0.5  | 0.060| 0.044| 0.023| 0.026| 0.047| 0.056| 0.053| 0.030| 0.062| 0.077| 0.057| 0.061| 0.058| 0.065| 0.051| 0.047 |
|         | 0.51 | 0.018| 0.019| 0.020| 0.028| 0.029| 0.026| 0.054| 0.034| 0.034| 0.028| 0.037| 0.042| 0.028| 0.041| 0.037| 0.037 |
|         | 0.75 | 0.045| 0.037| 0.038| 0.039| 0.058| 0.049| 0.051| 0.051| 0.070| 0.060| 0.050| 0.060| 0.067| 0.072| 0.058| 0.053 |
|         | 0.85 | 0.051| 0.035| 0.043| 0.041| 0.066| 0.052| 0.055| 0.051| 0.076| 0.061| 0.057| 0.052| 0.074| 0.076| 0.059| 0.057 |
|         | 1    | 0.053| 0.036| 0.044| 0.042| 0.066| 0.051| 0.053| 0.050| 0.077| 0.061| 0.053| 0.053| 0.075| 0.076| 0.059| 0.055 |

Table A.2. Empirical rejection frequencies, homoskedastic case using asymptotic critical values

| $\beta$ | $0.5$ | $0.75$ | $1$ | $1.05$ |
|---------|------|------|----|------|
| $N$     | 200  | 400  | 800| 1600 | 200  | 400  | 800| 1600 |
| $\kappa$ | 0    | 0.048| 0.044| 0.040| 0.037| 0.044| 0.053| 0.037| 0.039| 0.048| 0.052| 0.054| 0.045| 0.057| 0.045| 0.038| 0.037 |
|         | 0.25 | 0.060| 0.052| 0.045| 0.043| 0.061| 0.062| 0.046| 0.048| 0.069| 0.069| 0.058| 0.049| 0.070| 0.059| 0.042| 0.045 |
|         | 0.45 | 0.073| 0.059| 0.034| 0.030| 0.062| 0.071| 0.054| 0.039| 0.086| 0.090| 0.079| 0.067| 0.079| 0.081| 0.062| 0.060 |
|         | 0.5  | 0.060| 0.044| 0.023| 0.026| 0.047| 0.056| 0.053| 0.030| 0.062| 0.077| 0.057| 0.061| 0.058| 0.065| 0.051| 0.047 |
|         | 0.51 | 0.018| 0.019| 0.020| 0.028| 0.029| 0.026| 0.054| 0.034| 0.034| 0.028| 0.037| 0.042| 0.028| 0.041| 0.037| 0.037 |
|         | 0.75 | 0.045| 0.037| 0.038| 0.039| 0.058| 0.049| 0.051| 0.051| 0.070| 0.060| 0.050| 0.060| 0.067| 0.072| 0.058| 0.053 |
|         | 0.85 | 0.051| 0.035| 0.043| 0.041| 0.066| 0.052| 0.055| 0.051| 0.076| 0.061| 0.057| 0.052| 0.074| 0.076| 0.059| 0.057 |
|         | 1    | 0.053| 0.036| 0.044| 0.042| 0.066| 0.051| 0.053| 0.050| 0.077| 0.061| 0.053| 0.053| 0.075| 0.076| 0.059| 0.055 |

The table contains the empirical rejection frequencies under the null of no changepoint for different sample sizes and different values of $\kappa$, in the homoskedastic case.

Asymptotic critical values have been used, based on the limiting distributions in Theorems 3.1 (see Table A.1).

Broadly speaking, all test statistics have the correct size for large samples; as mentioned above, this also includes the Darling-Erdős statistic, based on asymptotic critical values, despite the notoriously slow convergence to the extreme value distribution. When $N = 200$ (i.e., in small samples), tests are, occasionally, mildly oversized. This also happens when $\kappa = 0.45$ (and

\[\text{We do not discuss the power for brevity, also on account of the fact that, in practice, one would apply the tests discussed in Section 4.1. Results are anyway in line with the rest of the simulations, and available upon request.}\]
\( \kappa > 0.5 \) with \( \beta_0 = 1.05 \); conversely, the test is grossly undersized almost under any circumstances when \( \kappa = 0.51 \), although this seems to improve as \( N \) increases. We note that, with the few exceptions mentioned above, the value of \( \beta_0 \) does not affect the empirical rejection frequencies in any obvious way (the case \( \beta_0 = 1 \) is marginally worse than the other ones for small \( N \), but this vanishes as \( N \) increases).

A.2. Consistency versus multiple breaks. As a complement to the results in Section 4.2, we briefly discuss the power of our tests against the alternative of \( R \) changes:

\[
y_i = \begin{cases} 
(\beta_1 + \epsilon_{i,1})y_{i-1} + \epsilon_{i,2}, & \text{if } 1 \leq k \leq k_1, \\
(\beta_2 + \epsilon_{i,1})y_{i-1} + \epsilon_{i,2}, & \text{if } k_1 < k \leq k_2, \\
& \vdots \\
(\beta_{R+1} + \epsilon_{i,1})y_{i-1} + \epsilon_{i,2}, & \text{if } k_R < k \leq k_{R+1},
\end{cases}
\]

with \( k_0 = 0 \) and \( k_{R+1} = N \). For the sake of simplicity we require

**Assumption A.1.** It holds that \( k_\ell = \lfloor N \tau_\ell \rfloor \) where \( 0 < \tau_1 < \tau_2 < \ldots < \tau_R < 1 \), with \( \tau_0 = 0 \) and \( \tau_{R+1} = 1 \).

Extending the theory developed above, it can be shown that, for all \( 1 \leq \ell \leq R \) and \( 1 < i < N \) whereby \( -\infty \leq E \ln |\epsilon_\ell + \epsilon_{i,1}| < \infty \), the following limits exist:

\[
(A.1) \quad \frac{1}{N} \sum_{i=k_{\ell-1}+1}^{k_\ell} \frac{y_{i-1}^2}{1 + y_{i-1}^2} \xrightarrow{p} \alpha_\ell > 0, \quad 1 \leq \ell \leq R + 1.
\]

Elementary arguments give that \( (A.1) \) implies

\[
\tilde{\beta}_{k_\ell,1} \xrightarrow{p} \beta_{\ell,1} = \sum_{i=1}^{\ell} \beta_i h_i \left( \sum_{i=1}^{\ell} h_i \right)^{-1}, \quad 1 \leq \ell \leq R + 1
\]
and
\[ \hat{\beta}_{k,2} \xrightarrow{p} \beta_{\ell,2} = \sum_{i=\ell+1}^{R+1} \beta_i h_i \left( \sum_{i=\ell+1}^{R+1} h_i \right)^{-1}, \quad 1 \leq \ell \leq R + 1. \]

Let
\[ \Delta_N(R) = \max_{1 \leq \ell \leq R} |\beta_{\ell,1} - \beta_{\ell,2}|. \]

**Corollary A.1.** We assume that \( H_0 \) of (1.2) holds. Under the same Assumptions as Theorem 4.3 and Assumption A.1, if, as \( N \to \infty \)
\[ N^{1/2} \Delta_N(R) \to \infty, \]
then it holds that
\[ \sup_{0 < t < 1} \frac{|Q_N(t)|}{w(t)} \xrightarrow{p} \infty. \]

If
\[ \left( \frac{N}{\ln \ln N} \right)^{1/2} \Delta_N(R) \to \infty, \]
then
\[ a(\ln N) \sup_{0 < t < 1} \frac{|Q_N(t)|}{\tilde{g}^{1/2}(t, t)} - b(\ln N) \xrightarrow{p} \infty. \]

If
\[ \left( \frac{r_N}{N} \right)^{k-1/2} N^{1/2} \Delta_N(R) \left( \frac{k^*}{N} \left( \frac{N - k^*}{N} \right) \right)^{1-\kappa} \to \infty, \]
then
\[ \left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{t_1 < t < t_2} \frac{(t(1-t))^{-\kappa + 1/2}}{\tilde{g}^{1/2}(t, t)} \left| Q_N(t) \right| \xrightarrow{p} \infty. \]

Corollary A.1 states that our test has power even in the presence of many breaks, and that the consistency (or lack thereof) depends on the largest break only, irrespective of the magnitude and number of all the other ones. The proof follows from the same arguments as that of Theorem 4.3.
Appendix B. Technical lemmas

We begin by stating some preliminary facts.

If $H_0$ holds, then

$$
\hat{\beta}_{k,1} - \hat{\beta}_{k,2} = \left( \sum_{i=2}^{k} \frac{y_{i-1}^2}{1 + y_{i-1}^2} \right)^{-1} \left( \sum_{i=2}^{k} \frac{y_{i-1}^2 \epsilon_{i,1} + y_{i-1} \epsilon_{i,2}}{1 + y_{i-1}^2} \right) - \left( \sum_{i=k+1}^{N} \frac{y_{i-1}^2}{1 + y_{i-1}^2} \right)^{-1} \left( \sum_{i=k+1}^{N} \frac{y_{i-1}^2 \epsilon_{i,1} + y_{i-1} \epsilon_{i,2}}{1 + y_{i-1}^2} \right).
$$

Under the null hypothesis the recursion is

$$
y_i = \rho_i y_{i-1} + \epsilon_{i,2}, \quad 1 \leq i < \infty,
$$

where $\rho_i = \beta_0 + \epsilon_{i,1}$. We can solve the recursion in (B.2) explicitly

$$
y_i = \sum_{\ell=1}^{i} \epsilon_{\ell,2} \prod_{j=\ell+1}^{i} \rho_j + y_0 \prod_{j=1}^{i} \rho_j = \sum_{\ell=0}^{i-1} \epsilon_{i-\ell,2} \prod_{j=1}^{\ell} \rho_{i-j+1} + y_0 \prod_{j=1}^{i} \rho_j.
$$

If it holds that

$$
-\infty \leq E \ln |\rho_0| = -\kappa, \quad \text{where } \kappa > 0,
$$

then the unique anticipative solution of (3.5) is (see Aue et al., 2006)

$$
\bar{y}_i = \sum_{\ell=0}^{\infty} \epsilon_{i-\ell,2} \prod_{j=1}^{\ell} \rho_{i-j+1}.
$$

We note that

$$
\bar{y}_i = \sum_{\ell=0}^{i-1} \epsilon_{i-\ell,2} \prod_{j=1}^{\ell} \rho_{i-j+1} + y_0 \prod_{j=1}^{i} \rho_j.
$$

We are now ready to state our technical lemmas. The first one states that we can replace $y_i$ with $\bar{y}_i$ in the sums in (B.1), and the difference will be small.
Lemma B.1. If $H_0$ of (1.2) and Assumption 3.1 are satisfied, and (B.3) holds, then

\begin{equation}
\sum_{i=2}^\infty \left| \frac{y_i^2 - y_{i-1}^2}{1 + y_i^2} - \frac{y_{i-1}^2}{1 + y_{i-1}^2} \right| < \infty \quad \text{a.s.,}
\end{equation}

\begin{equation}
\sum_{i=2}^\infty \left| \frac{y_i - y_{i-1}}{1 + y_i^2} - \frac{\overline{y}_{i-1}}{1 + \overline{y}_{i-1}^2} \right| < \infty \quad \text{a.s.,}
\end{equation}

and

\begin{equation}
\sum_{i=2}^\infty \left| \frac{y_{i-1}^2 - y_{i-1}^2}{1 + y_{i-1}^2} - \frac{\overline{y}_{i-1}^2}{1 + \overline{y}_{i-1}^2} \right| < \infty \quad \text{a.s.}
\end{equation}

Proof. Using (B.3), $E(\ln |\rho_j| + \overline{\kappa}/2) = -\overline{\kappa}/2 < 0$; hence, by Lemma 2 in Aue et al. (2006), there are $\nu_1 > 0$ and $c_1 < 1$ such that $E|e^{\overline{\kappa}/2}\rho_0|^{\nu_1} = c_1 < 1$. Lemma 2 in Aue et al. (2006) also entails that

\begin{equation}
E|\overline{y}_0|^{\nu_2} < \infty \quad \text{and} \quad E|\overline{y}_0|^{\nu_2} \leq c_2
\end{equation}

with some $\nu_2 > 0$. Hence

\begin{equation}
\prod_{j=1}^i |\rho_j| = \exp \left( \sum_{j=1}^i \ln |\rho_j| \right) = e^{-\overline{\kappa}/2} \exp \left( \sum_{j=1}^i (\ln |\rho_j| + \overline{\kappa}/2) \right)
\end{equation}

and

\begin{equation}
E \left( \exp \left( \sum_{j=1}^i (\ln |\rho_j| + \overline{\kappa}/2) \right) \right)^{\nu_1} \leq c_2 \quad \text{for all } 1 \leq i < \infty.
\end{equation}

Given that

\[ \left| \frac{y_i^2 - \overline{y}_i^2}{1 + y_i^2} - \frac{\overline{y}_{i-1}^2}{1 + \overline{y}_{i-1}^2} \right| \leq 2|y_i - \overline{y}_i||y_i + \overline{y}_i|, \]

we have

\[ \sum_{i=2}^\infty |\varepsilon_i| \left| \frac{y_i^2 - \overline{y}_{i-1}^2}{1 + y_i^2} - \frac{\overline{y}_{i-1}^2}{1 + \overline{y}_{i-1}^2} \right| \leq 2 \sum_{i=2}^\infty |\varepsilon_i||y_i - \overline{y}_{i-1}| |y_{i-1} - \overline{y}_{i-1}|. \]
We can assume that $0 < \nu_3 = \min(\nu_1, \nu_2)/3 < 1$ and we conclude

$$
E \left( \sum_{i=2}^{\infty} |\epsilon_i| \left| \frac{y_i^2}{1 + y_{i-1}^2} - \frac{\bar{y}_{i-1}^2}{1 + \bar{y}_{i-1}^2} \right| \right)^{\nu_3} \leq 2^{\nu_3} \sum_{i=2}^{\infty} E \left( |\epsilon_i| |y_{i-1} + \bar{y}_{i-1}| |y_{i-1} - \bar{y}_{i-1}| \right)^{\nu_3} \leq 2^{\nu_3} \sum_{i=2}^{\infty} \left( E |\epsilon_i| |y_{i-1} + \bar{y}_{i-1}| |y_{i-1} - \bar{y}_{i-1}| \right)^{1/3} \leq c_3 \sum_{i=2}^{\infty} e^{-3\nu_3 r/2},
$$

completing the proof of (B.6). The same arguments give (B.7) and (B.8). □

Lemma B.1 entails that we only need to work with the stationary solution. Equation (B.4) entails that there is a function $g(\cdot) : R^{\infty \times \infty} \to R$ such that

(B.12) \[ \bar{y}_i = g(\epsilon_i, \epsilon_{i-1}, \ldots), \quad \epsilon_i = (\epsilon_{i,1}, \epsilon_{i,2})'. \]

The representation in (B.12) means that $\{y_i, -\infty < i < \infty\}$ is a Bernoulli shift. Let $\{\epsilon_i^*, -\infty < i < \infty\}$ be independent copies of $\epsilon_0$ with $\epsilon_i^* \overset{D}{=} \epsilon_0$, independent of $\{\epsilon_i, -\infty < i < \infty\}$ and define the construction

(B.13) \[ y_{i,\ell} = g(\epsilon_i, \epsilon_{i-1}, \ldots, \epsilon_{i-\ell-1}, \epsilon_{i-\ell}^*, \epsilon_{i-\ell-1}^*, \ldots), \quad \ell \geq 1. \]

**Lemma B.2.** If $H_0$ of (1.2) and Assumption 3.1 are satisfied, and (B.3) holds, then

(B.14) \[ E \left| \frac{\epsilon_{i+1,1} \bar{y}_i^2}{1 + \bar{y}_i^2} - \frac{\epsilon_{i+1,1} \bar{y}_{i,\ell}^2}{1 + \bar{y}_{i,\ell}^2} \right|^4 \leq c\ell^{-\alpha}, \]

(B.15) \[ E \left| \frac{\epsilon_{i+1,2} \bar{y}_i}{1 + \bar{y}_i^2} - \frac{\epsilon_{i+1,2} \bar{y}_{i,\ell}}{1 + \bar{y}_{i,\ell}^2} \right|^4 \leq c\ell^{-\alpha}, \]

and

(B.16) \[ E \left| \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - \frac{\bar{y}_{i,\ell}^2}{1 + \bar{y}_{i,\ell}^2} \right|^4 \leq c\ell^{-\alpha}, \]

for all $\alpha > 0$ with some $c > 0$. 


Proof. We begin by showing (B.16). Using (B.5), we have \( \bar{y}_0 = \ell + u_\ell \), and \( y_0,\ell = \ell + u^*_\ell \), where

\[
\bar{y}_\ell = \sum_{j=0}^{\ell-1} \epsilon_{-j,2} \prod_{k=1}^{j} \rho_{-k+1},
\]

\[
u_\ell = \prod_{j=0}^{\ell-1} \rho_{-j},
\]

\[
u^*_\ell = \prod_{j=0}^{\ell-1} \rho^*_{-j},
\]

with

\[
\bar{y}^*_\ell = g(\epsilon^*_{\ell-1}, \epsilon^*_{\ell-2}, \cdots) \quad \text{and} \quad \rho^*_j = \beta_0 + \epsilon^*_{-j,1}.
\]

Consider now the set \( U_\ell = \{ u_\ell : |u_\ell| \leq \ell - \beta, u^*_\ell : |u^*_\ell| \leq \ell - \beta \} \), with \( \beta > 0 \). It holds that

\[
E \left| \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - \frac{y^2_{i,\ell}}{1 + y^2_{i,\ell}} \right|^4 \leq E \left( \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - \frac{y^2_{i,\ell}}{1 + y^2_{i,\ell}} \right)^4 (u_\ell, u^*_\ell) \in U_\ell \right) P ((u_\ell, u^*_\ell) \in U_\ell)

+ E \left( \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - \frac{y^2_{i,\ell}}{1 + y^2_{i,\ell}} \right)^4 (u_\ell, u^*_\ell) \notin U_\ell \right) P ((u_\ell, u^*_\ell) \notin U_\ell)

\leq E \left( \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - \frac{y^2_{i,\ell}}{1 + y^2_{i,\ell}} \right)^4 (u_\ell, u^*_\ell) \in U_\ell \right) + P ((u_\ell, u^*_\ell) \notin U_\ell) = I + II.

Consider II first. Using (B.9), (B.10) and (B.11) we get via Markov’s inequality,

\[
P\{|u_\ell| > \ell - \beta\} \leq c_1 \ell^{\beta \nu_1} \leq c_1 e^{-\ell \nu_2}.
\]

with some constants \( \nu_1 > 0, \nu_2 > 0, \nu_3 > 0 \) and \( c_1, c_2 = c_2(\beta) \), and similarly it can be shown that

\[
P\{|u^*_\ell| > \ell - \beta\} \leq c_2 e^{-\ell \nu_3}.
\]
Combining (B.17) and (B.18), it follows that

\[(B.19) \quad II \leq c_2 e^{-\ell \omega_3}.
\]

Turning to \(I\), elementary calculations give

\[
\left| \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} - \frac{y_{0,\ell}^2}{1 + y_{0,\ell}^2} \right| \leq \frac{4|z_\ell||u_\ell|}{1 + \bar{y}_0^2} + \frac{2(u_\ell^2 + (u_\ell^*)^2)}{1 + \bar{y}_0^2}.
\]

In (B.13), we can assume \(\ell\) is large enough that \(\ell^{-\beta} < 1/8\) and therefore

\[
\frac{|z_\ell|}{1 + (z_\ell + u_\ell)^2} \leq c_3.
\]

Therefore, on the set \(U_\ell\), we have

\[(B.20) \quad \left| \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - \frac{y_{i,\ell}^2}{1 + y_{i,\ell}^2} \right| \leq c_4 \ell^{-\beta}.
\]

Putting together (B.17), (B.18) and (B.20), (B.16) follows immediately. Also, noting that \(\epsilon_1\) and \((\bar{y}_0, y_{0,\ell})\) are independent, (B.14) also obtains (note that the equation does not depend on \(i\)). Similar arguments give (B.15). \(\square\)

**Appendix C. Proofs**

**Proof of Theorem 3.1.** Let

\[
a_0 = E \frac{\bar{y}_0^2}{1 + \bar{y}_0^2}.
\]

Combining equation (B.16) with the approximations in Aue et al. (2014) and Berkes et al. (2014), we can define a sequence of Wiener processes \(\{W_{N,1}(x), x \geq 0\}\) such that

\[(C.1) \quad \sup_{1 \leq k \leq N} \frac{1}{k^{\xi_1}} \sum_{i=1}^{k} \left( \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - a_0 \right) - c_1 W_{N,1}(k) = O_P(1),
\]
with some $c_1 > 0$ and $\zeta_1 < 1/2$. It follows from (C.1) and the Law of the Iterated Logarithm (LIL henceforth) that

\[
\max_{1 \leq k \leq N} \frac{1}{k^{\zeta_2}} \left| \sum_{i=1}^{k} \left( \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - a_0 \right) \right| = O_P(1),
\]

for all $\zeta_2 < 1/2$. Therefore, using Taylor’s expansion, we obtain from (C.2) that

\[
\max_{1 \leq k \leq N} \frac{1}{k^{\zeta_2}} \left| \left( \frac{1}{k} \sum_{i=1}^{k} \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} \right)^{-1} \right| = O_P(1).
\]

Since the proofs of the approximations in Aue et al. (2014) and Berkes et al. (2014) are based on the blocking technique, we can define Wiener processes $\{W_{N,2}(x), x \geq 0\}$, independent of $W_{N,1}(x)$, such that

\[
\sup_{1 \leq k < N} \frac{1}{(N - k)^{\zeta_1}} \left| \sum_{i=k+1}^{N} \left( \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} - a_0 \right) - c_1 W_{N,2}(N - k) \right| = O_P(1)
\]

which implies

\[
\max_{1 \leq k < N} \frac{1}{(N - k)^{\zeta_2}} \left| \left( \frac{1}{N - k} \sum_{i=k+1}^{N} \frac{\bar{y}_i^2}{1 + \bar{y}_i^2} \right)^{-1} \right| = O_P(1).
\]

Using the decomposability of the Bernoulli shifts in (B.15) and (B.16) and the approximations of Aue et al. (2014) and Berkes et al. (2014), we can define two independent Wiener processes $\{W_{N,3}(x), 0 \leq x \leq N/2\}$ and $\{W_{N,4}(x), 0 \leq x \leq N/2\}$ such that

\[
\max_{1 \leq x \leq N/2} \frac{1}{x^{\zeta_3}} \left| \sum_{i=2}^{x/2} \frac{\bar{y}_{i-1}^2 \epsilon_{i,1} + \bar{y}_{i-1}^2 \epsilon_{i,2}}{1 + \bar{y}_{i-1}^2} - a_0 \eta W_{N,3}(x) \right| = O_P(1),
\]

and

\[
\max_{N/2 \leq x \leq N-1} \frac{1}{(N - x)^{\zeta_3}} \left| \sum_{i=[x]+1}^{N} \frac{\bar{y}_{i-1}^2 \epsilon_{i,1} + \bar{y}_{i-1}^2 \epsilon_{i,2}}{1 + \bar{y}_{i-1}^2} - a_0 \eta W_{N,4}(N - x) \right| = O_P(1),
\]
with some \( \zeta_3 < 1/2 \). Putting together we get

\[
\max_{1 \leq k \leq N} \frac{1}{k^\zeta_4} \left| \left( \frac{1}{k} \sum_{i=2}^{k} \frac{\overline{y}_{i-1}^2}{1 + \overline{y}_{i-1}^2} \right)^{-1} - \frac{1}{a_0} \right| \frac{1}{a_0} \left| \sum_{i=2}^{k} \frac{\overline{y}_{i-1}^2 \epsilon_{i,1} + \overline{y}_{i-1} \epsilon_{i,2}}{1 + \overline{y}_{i-1}^2} \right| = O_P(1),
\]

and

\[
\max_{1 \leq k \leq N-1} \frac{1}{(N-k)^\zeta_4} \left| \left( \frac{1}{N-k} \sum_{i=k+1}^{N} \frac{\overline{y}_{i-1}^2}{1 + \overline{y}_{i-1}^2} \right)^{-1} - \frac{1}{a_0} \right| \frac{1}{a_0} \left| \sum_{i=k+1}^{N} \frac{\overline{y}_{i-1}^2 \epsilon_{i,1} + \overline{y}_{i-1} \epsilon_{i,2}}{1 + \overline{y}_{i-1}^2} \right| = O_P(1),
\]

for all \( \zeta_4 > 0 \). It follows from (B.1), (C.4) and (C.5)

\[
(C.6) \quad \max_{2 \leq x \leq N/2} \frac{x(N-x)}{N} \left( \hat{\beta}_{[x],1} - \hat{\beta}_{[x],2} \right)
\]

\[
- \eta \left( W_{N,3}(x) - \frac{x}{N} [W_{N,4}(N/2) + W_{N,3}(N/2)] \right) = O_P(1),
\]

and

\[
(C.7) \quad \max_{N/2 \leq x \leq N-1} \frac{1}{(N-x)^\zeta} \left| \left( \frac{1}{N-x} \sum_{i=x+1}^{N} \frac{\overline{y}_{i-1}^2}{1 + \overline{y}_{i-1}^2} \right)^{-1} - \frac{1}{a_0} \right| \frac{1}{a_0} \left| \sum_{i=x+1}^{N} \frac{\overline{y}_{i-1}^2 \epsilon_{i,1} + \overline{y}_{i-1} \epsilon_{i,2}}{1 + \overline{y}_{i-1}^2} \right| = O_P(1),
\]

with some \( \zeta_5 < 1/2 \). The computation of the covariance function shows that

\[
B_N(t) = \begin{cases} 
N^{-1/2} \left( W_{N,3}(Nt) - t [W_{N,4}(N/2) + W_{N,3}(N/2)] \right), & 0 \leq t \leq 1/2, \\
N^{-1/2} \left( -W_{N,4}(N(1-t)) + (1-t) [W_{N,4}(N/2) + W_{N,3}(N/2)] \right), & 1/2 \leq t \leq 1,
\end{cases}
\]

is a Brownian bridge.

Let \( 0 < \delta < 1/2 \). Recalling Assumption 2.1(i), it follows from (C.6) and (C.7) that

\[
\sup_{\delta \leq t \leq 1-\delta} \frac{|Q_N(t) - \eta B_N(t)|}{w(t)} \leq c_0 \sup_{\delta \leq t \leq 1-\delta} |Q_N(t) - \eta B_N(t)| = O_P \left( N^{-1/2+\zeta_5} \right) = o_P(1).
\]

It follows from (C.6) that

\[
\sup_{2/(N+1) \leq t \leq \delta} \frac{|Q_N(t) - \eta B_N(t)|}{w(t)} \leq \sup_{0 < t \leq \delta} \frac{t^{1/2}}{w(t)} \sup_{2/(N+1) \leq t \leq \delta} \frac{|Q_N(t) - \eta B_N(t)|}{t^{1/2}}
\]
Using again Csörgő et al. (1986) we obtain for all $N$
and (C.7) implies

$$
\sup_{1-\delta \leq t \leq 1-2/(N+1)} \frac{|Q_N(t) - B_N(t)|}{w(t)} = O_P(1) \sup_{1-\delta \leq t < 1} \frac{(1-t)^{1/2}}{w(t)}.
$$

Csörgő et al. (1986) proved that, if $I(w, c) < \infty$ for some $c > 0$, then

(C.8)

$$
\lim_{t \to 0^+} t^{1/2} \frac{1}{w(t)} = 0 \quad \text{and} \quad \lim_{t \to 1^-} \frac{(1-t)^{1/2}}{w(t)} = 0.
$$

Hence for every $x > 0$ we have

$$
\lim_{\delta \to 0^+} \limsup_{N \to \infty} P \left\{ \sup_{2/(N+1) \leq t \leq \delta} \frac{|Q_N(t) - B_N(t)|}{w(t)} \geq x \right\} = 0,
$$

and

$$
\lim_{\delta \to 0^+} \limsup_{N \to \infty} P \left\{ \sup_{1-\delta \leq t \leq 1-2/(N+1)} \frac{|Q_N(t) - B_N(t)|}{w(t)} \geq x \right\} = 0.
$$

Using again Csörgő et al. (1986) we obtain for all $N$

$$
\sup_{\delta \leq t \leq 1-\delta} \frac{|B_N(t)|}{w(t)} \overset{p}{\to} \sup_{\delta \leq t \leq 1-\delta} \frac{|B(t)|}{w(t)} \overset{a.s.}{\to} \sup_{0 < t < 1} \frac{|B(t)|}{w(t)},
$$
as $\delta \to 0$, where $\{B(t), 0 \leq t \leq 1\}$ is a Brownian bridge. The first part of Theorem 3.1 now follows from putting everything together.

We now turn to proving part (ii) of the theorem. Let $c(N) = (\ln N)^4$. First we observe that according to (C.6)

$$
\begin{aligned}
\max_{2 \leq k \leq N/2} \frac{N^{1/2}}{(k(N-k))^{1/2}} \left| \frac{k(N-k)}{N} \left( \hat{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) - \eta \left( W_{N,3}(k) - \frac{k}{N} \left[ W_{N,4}(N/2) + W_{N,3}(N/2) \right] \right) \right| \\
\leq 2 \max_{2 \leq k \leq N/2} k^{\zeta-1/2} \left| \frac{k(N-k)}{N} \left( \hat{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) - \eta \left( W_{N,3}(k) - \frac{k}{N} \left[ W_{N,4}(N/2) + W_{N,3}(N/2) \right] \right) \right| \\
= O_P(1),
\end{aligned}
$$

and by (C.7)

\[
\max_{N/2 \leq k \leq N-1} \frac{N^{1/2}}{(k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \tilde{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) - \eta \left( -W_{N,4}(N - k) + \frac{N - k}{N} [W_{N,4}(N/2) + W_{N,3}(N/2)] \right) \right|
= O_P(1).
\]

Hence the LIL for the Wiener process implies

\[
\frac{1}{(2 \ln \ln N)^{1/2}} \max_{2 \leq k \leq N/2} \frac{N^{1/2}}{(k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \tilde{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) \right| \rightarrow \eta,
\]

\[
\frac{1}{(2 \ln \ln N)^{1/2}} \max_{N/2 \leq k \leq N-1} \frac{N^{1/2}}{(k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \tilde{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) \right| \rightarrow \eta,
\]

\[
\max_{2 \leq k \leq c(N)} \frac{N^{1/2}}{(k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \tilde{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) \right| = O_P((\ln \ln N)^{1/2}),
\]

and

\[
\max_{N-c(N) \leq k \leq N-1} \frac{N^{1/2}}{(k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \tilde{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) \right| = O_P((\ln \ln N)^{1/2}).
\]

Putting all these results together, we conclude that

(C.9) \[
\lim_{N \to \infty} \frac{1}{N} \max_{2 \leq k \leq N-1} \frac{N^{1/2}}{(k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \tilde{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) \right| = 1.
\]

Further, the approximations in (C.6) and (C.7) yield

(C.10) \[
\max_{c(N) \leq k \leq N/2} \frac{N^{1/2}}{(k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \tilde{\beta}_{k,1} - \tilde{\beta}_{k,2} \right) - \eta \left( W_{N,3}(k) - \frac{k}{N} [W_{N,4}(N/2) + W_{N,3}(N/2)] \right) \right|
= O_P\left( \left( \ln N \right)^{4(\kappa_4^{-1/2})} \right) = o_P(1),
\]

and

\[
\max_{N/2 \leq k \leq 1 - c(N)} \frac{N^{1/2}}{k(N - k))^{1/2}} \left| \frac{k(N - k)}{N} \left( \hat{\beta}_{k,1} - \hat{\beta}_{k,2} \right) - \eta \left( -W_{N,a}(N - k) + \frac{N - k}{N} \left[ W_{N,a}(N/2) + W_{N,b}(N/2) \right] \right) \right| = O_P \left( (\ln N)^{4(\zeta_5 - 1/2)} \right) = o_P (1),
\]

since \( \zeta_5 < 1/2 \). Finally, Theorem A.4.2 in Csörgő and Horváth (1997) states that

\[
\lim_{N \to \infty} P \left\{ a (\ln N) \max_{c(N) \leq k \leq N - c(N)} \left( \frac{k}{N} \left( 1 - \frac{k}{N} \right) \right)^{-1/2} |B(k/N)| \leq x + b (\ln N) \right\} = \exp(-2e^{-x}),
\]

for all \( x \), which implies the second part of the theorem. Finally, the proof of part (iii) follows automatically from the approximations to Wiener processes in (C.6), (C.7) and (C.24) and (C.25) - see Horváth et al. (2020b).

Proof of Theorem 3.2. We begin by showing that the approximations in (C.6) and (C.7) hold in the non stationary case too. These approximations imply immediately the limit results in the present theorem, repeating exactly the same passages as in the proof of Theorem 3.1. We begin by noting that Lemma A.4 of Horváth and Trapani (2016) implies that there are two constants, \( 0 < \delta < 1 \) and \( c_1 > 0 \), such that

\[
P\left\{ |y_i| \leq i^\delta \right\} \leq c_1 i^{-\delta}.
\]

Equation (C.12), in turn, implies that

\[
E \left( \frac{1}{1 + y_i^2} \right) = E \left( \frac{1}{1 + y_i^2} I\{ |y_i| \leq i^\delta \} \right) + E \left( \frac{1}{1 + y_i^2} I\{ |y_i| \geq i^\delta \} \right) \leq P\{ |y_i| \leq i^\delta \} + (1 + i^{2\delta})^{-1} \leq c_2 i^{-\delta},
\]

with some constant \( c_2 \). Using (C.13) and Markov’s inequality, we have for all \( x > 0 \) and \( \zeta_1 > 1 - \delta \)

\[
P \left\{ \max_{M \leq k < \infty} \frac{1}{k^{\zeta_1}} \sum_{i=1}^{k} \frac{1}{1 + y_i^2} > x \right\} \leq P \left\{ \max_{\ln M \leq \ell < \infty} \max_{e^\ell \leq k < e^{\ell + 1}} \frac{1}{k^{\zeta_1}} \sum_{i=1}^{k} \frac{1}{1 + y_i^2} > x \right\}.
\]
\[
\leq \sum_{\ell=\ln M}^{\infty} P \left\{ \max_{e^\ell \leq k < e^{\ell+1}} \frac{1}{k^{\zeta_1}} \sum_{i=1}^{k} \frac{1}{1+y_i^2} > x \right\}
\]
\[
\leq \sum_{\ell=\ln M}^{\infty} P \left\{ \max_{e^\ell \leq k < e^{\ell+1}} \frac{1}{k^{\zeta_1}} \sum_{i=1}^{k} \frac{1}{1+y_i^2} > x e^{\zeta_1 \ell} \right\}
\]
\[
= \sum_{\ell=\ln M}^{\infty} P \left\{ \frac{\exp(\ell+1)}{1+y_i^2} > x e^{\zeta_1 \ell} \right\}
\]
\[
\leq \frac{1}{x} \sum_{\ell=\ln M}^{\infty} e^{-\zeta_1 \ell} \sum_{i=1}^{\exp(\ell+1)} E \frac{1}{1+y_i^2}
\]
\[
\leq \frac{c_2}{x} \sum_{\ell=\ln M}^{\infty} e^{-\zeta_1 \ell} \sum_{i=1}^{\exp(\ell+1)} i^{-\delta}
\]
\[
\leq \frac{c_3}{x} M^{-(\zeta_1 - (1-\delta))}.
\]

Hence there is \(\zeta_2 < 1\) such that for all \(x > 0\)

(C.14) \[
\lim_{M \to \infty} P \left\{ \max_{M \leq k < \infty} \frac{1}{k^{\zeta_2}} \sum_{i=1}^{k} \frac{1}{1+y_i^2} > x \right\} = 0.
\]

Similar arguments give

(C.15) \[
\lim_{M \to \infty} \limsup_{N \to \infty} P \left\{ \max_{1 \leq k \leq N-M} \frac{1}{(N-k)^{\zeta_2}} \sum_{i=k+1}^{N} \frac{1}{1+y_i^2} > x \right\} = 0.
\]

We obtain immediately from (C.14) and (C.15)

(C.16) \[
\max_{1 \leq k \leq N} k^{1-\zeta_2} \left| \left( \frac{1}{k} \sum_{i=2}^{k} \frac{y_i^2}{1+y_i^2} \right)^{-1} - 1 \right| = O_P(1),
\]

and

(C.17) \[
\max_{1 \leq k \leq N-1} (N-k)^{1-\zeta_2} \left| \left( \frac{1}{N-k} \sum_{i=k+1}^{N} \frac{y_i^2}{1+y_i^2} \right)^{-1} - 1 \right| = O_P(1).
\]
Further, using Assumption 3.1 we get via the Cauchy–Schwartz inequality

\[ E\left(\sum_{\ell=1}^{j} \frac{\epsilon_{\ell,1}}{1 + y_{\ell-1}^2}\right)^4 \]

\[ = \sum_{\ell=1}^{j} E\epsilon_{\ell,1}^4 E\left(\frac{1}{1 + y_{\ell-1}^2}\right)^4 + \sum_{i \leq \ell \leq j} E\left[\frac{\epsilon_{\ell,1}}{1 + y_{\ell-1}^2}\right]^2 \left[\frac{\epsilon_{\ell',1}}{1 + y_{\ell'-1}^2}\right]^2 \]

\[ \leq \sum_{\ell=1}^{j} E\epsilon_{0,1}^4 E\left(\frac{1}{1 + y_{\ell-1}^2}\right)^4 + \sum_{i \leq \ell \leq j} E\epsilon_{0,1}^4 \left[ E\left(\frac{1}{1 + y_{\ell-1}^2}\right)^4 \right]^{1/2} \left[ E\left(\frac{1}{1 + y_{\ell'-1}^2}\right)^4 \right]^{1/2} \]

\[ \leq E\epsilon_{0,1}^4 \left[ \sum_{\ell=1}^{j} c_{\ell-1}^{1/2} \right]^2 + \left[ \sum_{\ell=1}^{j} c_{\ell-1} \right]^2 \]

\[ \leq 2E\epsilon_{0,1}^4 \left( \sum_{\ell=1}^{j} c_{\ell-1} \right)^2, \]

where

\[ c_{\ell} = E\left(\frac{1}{(1 + y_{\ell}^2)^4}\right). \]

Note that we can assume - without loss of generality - that \( c_{\ell} \leq 1 \) since, along the lines of the proof of (C.13), \( c_{\ell} = O(\ell^{-\delta}) \) as \( \ell \to \infty \). Theorem 3.1 of Móricz et al. (1982) implies

\[ E\left(\max_{2 \leq k \leq 1} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1} y_{i-1}^2}{1 + y_{i-1}^2} - \sum_{i=2}^{k} \epsilon_{i,1} \right| \right)^4 \leq c_4 \left( \sum_{\ell=1}^{j} c_{\ell-1} \right)^2. \]

Hence by Markov’s inequality for all \( x > 0 \)

\[ P\left\{ \max_{2 \leq k \leq N} \frac{1}{k^{\gamma_3}} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1} y_{i-1}^2}{1 + y_{i-1}^2} - \sum_{i=2}^{k} \epsilon_{i,1} \right| > x \right\} \leq P\left\{ \max_{0 \leq \ell \leq \ln N} \max_{\ell' \leq k \leq \ell' + 1} \frac{1}{k^{\gamma_3}} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1}}{1 + y_{i-1}^2} \right| > x \right\} \]

\[ \leq \sum_{\ell=0}^{\ln N} P\left\{ \max_{\ell \leq k \leq \ell' + 1} \frac{1}{k^{\gamma_3}} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1}}{1 + y_{i-1}^2} \right| > x \right\} \]

\[ \leq \sum_{\ell=0}^{\ln N} P\left\{ \max_{\ell \leq k \leq \ell' + 1} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1}}{1 + y_{i-1}^2} \right| > x e^{\delta_3 \ell} \right\} \]

\[ \leq \frac{1}{x^4} \sum_{\ell=0}^{\ln N} e^{-4\delta_3 \ell} E\max_{1 \leq k \leq \ell' + 1} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1}}{1 + y_{i-1}^2} \right|^4. \]
\[ \leq \frac{C_5}{x^4} \sum_{\ell=0}^{\ln N} e^{-4\zeta_3 \ell} e^{2\ell(1-\delta)} \]
\[ \leq \frac{C_6}{x^4}, \]

with the choice of \( \zeta_3 > (1 - \delta) / 2 \). Hence, there exists a \( \zeta_4 < 1/2 \) such that

\[ (C.18) \quad \max_{2 \leq k \leq N} \frac{1}{k^{\zeta_4}} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1} y_{i-1}^2}{1 + y_{i-1}^2} - \sum_{i=2}^{k} \epsilon_{i,1} \right| = O_P(1); \]

similar arguments give

\[ (C.19) \quad \max_{2 \leq k < N} \frac{1}{(N - k)^{\zeta_4}} \left| \sum_{i=k+1}^{N} \frac{\epsilon_{i,1} y_{i-1}^2}{1 + y_{i-1}^2} - \sum_{i=2}^{k} \epsilon_{i,1} \right| = O_P(1). \]

Following the proof of (C.18) and (C.19) one also can verify that

\[ (C.20) \quad \max_{2 \leq k \leq N} \frac{1}{k^{\zeta_5}} \left| \sum_{i=2}^{k} \frac{\epsilon_{i,1} y_{i-1}}{1 + y_{i-1}^2} \right| = O_P(1), \]

and

\[ (C.21) \quad \max_{1 \leq k < N} \frac{1}{(N - k)^{\zeta_5}} \left| \sum_{i=k+1}^{N} \frac{\epsilon_{i,1} y_{i-1}}{1 + y_{i-1}^2} \right| = O_P(1), \]

with some \( \zeta_5 < 1/2 \).

Combining (C.16), (C.17), (C.18), (C.19), (C.20) and (C.21), it is easy to see that it is possible to find a \( \zeta_6 < 1/2 \) such that

\[ (C.22) \quad \max_{1 \leq k \leq N/2} \frac{1}{k^{\zeta_6}} \left| k(\frac{N-k}{N}) \left( \beta_{k,1} - \beta_{k,2} \right) - \left( \sum_{i=1}^{k} \epsilon_{i,1} - \frac{k}{N} \sum_{i=1}^{N} \epsilon_{i,1} \right) \right| = O_P(1), \]

and

\[ (C.23) \quad \max_{N/2 \leq k \leq N-1} \frac{1}{(N-k)^{\zeta_6}} \left| \frac{k(\frac{N-k}{N})}{N} \left( \beta_{k,1} - \beta_{k,2} \right) - \left( -\sum_{i=k+1}^{N} \epsilon_{i,1} + \frac{N-k}{N} \sum_{i=1}^{N} \epsilon_{i,1} \right) \right| = O_P(1). \]

Finally, by the Komlós–Major–Tusnády approximation (see Komlós et al., 1975, and Komlós et al., 1976), we can define two independent Wiener processes \( \{W_{N,1}(x), 0 \leq x \leq N/2\} \) and \( \{W_{N,2}(x), 0 \leq x \leq N-1\} \).
\( x \leq N/2 \} \) such that

\[
(C.24) \quad \max_{1 \leq k \leq N/2} \frac{1}{k^{1/4}} \left| \sum_{i=1}^{k} \epsilon_{i,1} - \sigma_1 W_{N,1}(k) \right| = O_P(1),
\]

and

\[
(C.25) \quad \max_{1 \leq k \leq N/2} \frac{1}{(N-k)^{1/4}} \left| \sum_{i=k+1}^{N} \epsilon_{i,1} - \sigma_1 W_{N,2}(N-k) \right| = O_P(1).
\]

Putting together \((C.22), (C.23), (C.24)\) and \((C.25)\), it finally follows that, for some \( \zeta_7 < 1/2 \)

\[
(C.26) \quad \max_{1 \leq k \leq N/2} \frac{1}{k^{1/7}} \left| \frac{k(N-k)}{N} (\hat{\beta}_{k,1} - \hat{\beta}_{k,2}) - \sigma_1 \left( W_{N,1}(k) - \frac{k}{N} (W_{N,2}(N/2) + W_{N,1}(N/2)) \right) \right| = O_P(1),
\]

and

\[
(C.27) \quad \max_{N/2 \leq k \leq N} \frac{1}{k^{1/7}} \left| \frac{k(N-k)}{N} (\hat{\beta}_{k,1} - \hat{\beta}_{k,2}) - \sigma_1 \left( -W_{N,2}(k) + \frac{N-k}{N} (W_{N,2}(N/2) + W_{N,1}(N/2)) \right) \right| = O_P(1).
\]

The desired results now follow from repeating exactly the same passages as in the proof of Theorem 3.1

\[\square\]

**Proof of Corollary 3.1.** We note that

\[
(C.28) \quad (y_i - \hat{\beta}_{N,1} y_{i-1})^2 = \left( \left( (\beta_0 - \hat{\beta}_{N,1}) y_{i-1} + \epsilon_{i,1} \right) y_{i-1} + \epsilon_{i,2} \right)^2
\]

\[
= \epsilon_{i,1}^2 y_{i-1}^2 + \epsilon_{i,2}^2 + \left( \beta_0 - \hat{\beta}_{N,1} \right)^2 y_{i-1}^2 + 2 \left( \beta_0 - \hat{\beta}_{N,1} \right) y_{i-1}^2 \epsilon_{i,1} + \epsilon_{i,2} \frac{y_{i-1}^2 \epsilon_{i,1} \epsilon_{i,2}}{2},
\]
Hence, we have

\[ \hat{a}_{N,1} = \frac{1}{N - 1} \sum_{i=2}^{N} \frac{\epsilon_{i,1}^2 y_{i-1}^4}{(1 + y_{i-1}^2)^2} + \frac{1}{N - 1} \sum_{i=2}^{N} \frac{\epsilon_{i,2}^2 y_{i-1}^2}{(1 + y_{i-1}^2)^2} + \left( \beta_0 - \hat{\beta}_{N,1} \right)^2 \frac{1}{N - 1} \sum_{i=2}^{N} \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} 
+ 2 \frac{1}{N - 1} \left( \beta_0 - \hat{\beta}_{N,1} \right) \sum_{i=2}^{N} \frac{\epsilon_{i,1} y_{i-1}^2}{(1 + y_{i-1}^2)^2} + 2 \frac{1}{N - 1} \left( \beta_0 - \hat{\beta}_{N,1} \right) \sum_{i=2}^{N} \frac{\epsilon_{i,2} y_{i-1}^2}{(1 + y_{i-1}^2)^2} 
+ 2 \frac{1}{N - 1} \sum_{i=2}^{N} \frac{y_{i-1} \epsilon_{i,1} \epsilon_{i,2}}{(1 + y_{i-1}^2)^2}. \]

The proofs of Theorems 3.1 and 3.2 show that, irrespective of \( y_i \) being stationary or not,

\[ |\hat{\beta}_N - \beta_0| = O_P(N^{-1/2}), \]

\[ \frac{1}{N} \sum_{i=2}^{N} \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} = O_P(1), \quad \frac{1}{N} \sum_{i=2}^{N} \frac{y_{i-1}^4 \epsilon_{i,1}}{(1 + y_{i-1}^2)^2} = O_P(N^{-1/2}), \]

\[ \frac{1}{N} \sum_{i=2}^{N} \frac{y_{i-1}^3 \epsilon_{i-2}}{(1 + y_{i-1}^2)^2} = O_P(N^{-1/2}), \quad \frac{1}{N} \sum_{i=2}^{N} \frac{y_{i-1}^3 \epsilon_{i,1} \epsilon_{i-2}}{(1 + y_{i-1}^2)^2} = O_P(N^{-1/2}). \]

Therefore, it holds that

\[ (C.29) \quad \hat{a}_{N,1} = \frac{1}{N - 1} \sum_{i=2}^{N} \frac{\epsilon_{i,1}^2 y_{i-1}^4}{(1 + y_{i-1}^2)^2} + \frac{1}{N - 1} \sum_{i=2}^{N} \frac{\epsilon_{i,2}^2 y_{i-1}^2}{(1 + y_{i-1}^2)^2} + O_P(N^{-1/2}). \]

We study the leading term in (C.29), showing that, when \( E \ln |\beta_0 + \epsilon_{0,1}| < 0 \)

\[ (C.30) \quad \frac{1}{N - 1} \sum_{i=2}^{N} \frac{\epsilon_{i,1}^2 y_{i-1}^4 + y_{i-1}^2 \epsilon_{i,2}^2}{(1 + y_{i-1}^2)^2} = E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \sigma_1^2 + E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \sigma_2^2 + O_P(N^{-\zeta}), \]

for some \( \zeta > 0 \). We begin by showing

\[ (C.31) \quad \frac{1}{N - 1} \sum_{i=2}^{N} \frac{\epsilon_{i,1}^2 y_{i-1}^4}{(1 + y_{i-1}^2)^2} = E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \sigma_1^2 + O_P(N^{-\zeta}). \]

Note first that

\[ \frac{1}{N - 1} \sum_{i=2}^{N} \left| \frac{\epsilon_{i,1}^2 y_{i-1}^4}{(1 + y_{i-1}^2)^2} - \frac{\epsilon_{i,1}^2 y_{i-1}^4}{(1 + \bar{y}_{i-1}^2)^2} \right| \]
\[
\frac{1}{N-1} \sum_{i=2}^{N} E \left| \epsilon_{i,1}^2 \right| \left( E \left| y_{i-1} - \bar{y}_{i-1} \right|^2 \right)^{1/2} \left( E \left| y_{i-1} + \bar{y}_{i-1} \right|^2 \right)^{1/2} = O \left( \frac{1}{N} \right),
\]
which follows from the definition of \( \bar{y}_i \) and Assumption 3.1(ii). Recall that Aue et al. (2006) show that \( \theta_1 = E|\beta_0 + \epsilon_{0,1}|^k < 1 \), for some \( 0 < k < 1 \), and consider the construction

\[
\begin{align*}
\hat{y}_k(N) &= \sum_{j=i-\lfloor \theta \ln N \rfloor}^{i-1} \epsilon_{j,2} \prod_{z=j}^{i-1} (\beta_0 + \epsilon_{z,1}) + \sum_{j=-\infty}^{i-\lfloor \theta \ln N \rfloor-1} \epsilon_{j,2}^* \prod_{z=j}^{i-1} (\beta_0 + \epsilon_{z,1}),
\end{align*}
\]
where \( \epsilon_{j,1}^* \) and \( \epsilon_{j,2}^* \) are completely independent, and independent of \( \epsilon_{j,1} \) and \( \epsilon_{j,2} \), with \( \epsilon_{j,1}^* \overset{D}{=} \epsilon_{j,1}, \epsilon_{j,2}^* \overset{D}{=} \epsilon_{j,2}, \) and \( \theta = 2/(1 - \theta_1) \). Following the proof of Lemma A.3 in Horváth and Trapani (2016), it follows that

\[
E \left| \frac{\hat{y}_{i-1}^4(N)}{(1 + \hat{y}_{i-1}^2(N))^2} - \frac{\hat{y}_{i-1}^4(N)}{(1 + \hat{y}_{i-1}^2(N))^2} \right| \leq c_0 \theta_1^{[\theta \ln N]/(1+k)},
\]
so that it is easy to see that

\[
\frac{1}{N-1} \sum_{i=2}^{N} E \left| \frac{\epsilon_{i,1}^2 \bar{y}_{i-1}^4}{(1 + \bar{y}_{i-1}^2(N))^2} - \frac{\epsilon_{i,1}^2 \hat{y}_{i-1}^4(N)}{(1 + \hat{y}_{i-1}^2(N))^2} \right| = o \left( \frac{1}{N} \right).
\]
Finally, consider

\[
\begin{align*}
E \left( \frac{1}{N-1} \sum_{i=2}^{N} \epsilon_{i,1}^2 \left( \frac{\hat{y}_{i-1}^4(N)}{(1 + \hat{y}_{i-1}^2(N))^2} - E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \right) \right)^2 &= \frac{1}{(N-1)^2} \sum_{i=2}^{N} \sum_{j=2}^{N} E \left[ \epsilon_{i,1}^2 \epsilon_{j,1}^2 \left( \frac{\hat{y}_{i-1}^4(N)}{(1 + \hat{y}_{i-1}^2(N))^2} - E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \right) \left( \frac{\hat{y}_{j-1}^4(N)}{(1 + \hat{y}_{j-1}^2(N))^2} - E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \right) \right] \\
&\leq \frac{1}{(N-1)^2} \sum_{i=2}^{N} \sum_{j=2}^{N} E \left[ \epsilon_{i,1}^2 \left( \frac{\hat{y}_{i-1}^4(N)}{(1 + \hat{y}_{i-1}^2(N))^2} - E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \right) \epsilon_{j,1}^2 \left( \frac{\hat{y}_{j-1}^4(N)}{(1 + \hat{y}_{j-1}^2(N))^2} - E \left( \frac{\bar{y}_0^2}{1 + \bar{y}_0^2} \right)^2 \right) \right] \\
&\times I \left( |i-j| \leq 2\theta \lfloor \ln N \rfloor \right) \\
&= O \left( \frac{\ln N}{N} \right).
\end{align*}
\]
having used the fact that $\epsilon_{i,1}^2 \tilde{y}_{i-1}^4 (N)$ and $\epsilon_{j,1}^2 \tilde{y}_{j-1}^4 (N)$ are independent for $|i - j| > 2\theta [\ln N]$.

Putting all together, it follows that
\[
\frac{1}{N - 1} \sum_{i=2}^{N} \epsilon_{i,1}^2 \left( \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} - E \left( \frac{\tilde{y}_0^2}{1 + \tilde{y}_0^2} \right)^2 \right) = O_P(1),
\]

for some $\zeta > 0$, whence (C.31) follows immediately. Using exactly the same logic, it is also possible to show that
\[
(C.32) \quad \frac{1}{N - 1} \sum_{i=2}^{N} \epsilon_{i,2}^2 y_{i-1}^2 \left( \frac{y_{i-1}^2}{(1 + y_{i-1}^2)^2} \right) = E \left( \frac{\tilde{y}_0}{1 + \tilde{y}_0} \right)^2 \sigma_2^2 + O_P(1),
\]

which, with (C.31), finally implies (C.30). The same logic also yields
\[
\hat{a}_{N,2} = E \left( \frac{\tilde{y}_0^2}{1 + \tilde{y}_0^2} \right) + O_P(1),
\]

whence we have finally shown that $\hat{\eta}_N = \eta + O_P(1)$ when $y_i$ is stationary.

It remains to show that the corollary also holds in the nonstationary case $E \ln |\beta_0 + \epsilon_{0,1}| \geq 0$.

Note
\[
\frac{1}{N - 1} \sum_{i=2}^{N} \epsilon_{i,1}^2 y_{i-1}^4 \left( \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} \right) = \frac{1}{N - 1} \sigma_1^2 \sum_{i=2}^{N} y_{i-1}^4 \left( \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} \right) + \frac{1}{N - 1} \sum_{i=2}^{N} \left( \epsilon_{i,1}^2 - \sigma_1^2 \right) y_{i-1}^4 \left( \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} \right);
\]

following the proof of (B.16), it is easy to see that
\[
\frac{1}{N - 1} \sigma_1^2 \sum_{i=2}^{N} y_{i-1}^4 \left( \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} \right) = \sigma_1 + O_P \left( \frac{1}{N} \right),
\]

and
\[
E \left( \frac{1}{N - 1} \sum_{i=2}^{N} \left( \epsilon_{i,1}^2 - \sigma_1^2 \right) y_{i-1}^4 \left( \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} \right) \right)^2 = \frac{1}{(N - 1)^2} \sum_{i=2}^{N} \left( E \left( \epsilon_{i,1}^2 - \sigma_1^2 \right)^4 \right) \left( E \left( \frac{y_{i-1}^4}{(1 + y_{i-1}^2)^2} \right)^2 \right) \left( \frac{1}{N} \right),
\]

having used the fact that $\epsilon_{i,1}^2 \tilde{y}_{i-1}^4 (N)$ and $\epsilon_{i,1}^2 \tilde{y}_{i-1}^4 (N)$ are independent for $|i - j| > 2\theta [\ln N]$.
so that
\[
\frac{1}{N-1} \sum_{i=2}^{N} \frac{\epsilon_{i,1}^2 y_{i-1}^4}{(1 + y_{i-1}^2)^2} = \sigma_1^2 + O_p(N^{-1}).
\]
The same logic yields
\[
\frac{1}{N-1} \sum_{i=2}^{N} \frac{\epsilon_{i,2}^2 y_{i-1}^2}{(1 + y_{i-1}^2)^2} = O_p \left( \frac{1}{N} \right).
\]
Hence
\[
\hat{a}_{N,1} = \sigma_1^2 + O_p(N^{-1}),
\]
and, by the same token
\[
\hat{a}_{N,2} = 1 + O_p(N^{-1}),
\]
so that, in this case, it holds that \( \hat{\eta}_N = \eta + O_p(N^{-1}) \).

Proof of Theorem 4.1. We assume, without loss of generality, that \( M \geq 1 \) - the \( M = 0 \) case is already covered in Theorem 3.1.

We begin by showing two preliminary sets of results: (i) that the approximations developed in the proofs of Theorems 3.1 and 3.2 are valid on each segment \((m_{\ell-1}, m_\ell], 1 \leq \ell \leq M + 1\), with only the variances of the approximating Gaussian processes depending on \( \ell \); and (ii) that the approximating process on a segment is independent of the approximating processes on the other segments.

Let
\[
\tilde{z}_i = \begin{cases} 
\frac{\tilde{y}_{\ell,i-1}^2 \epsilon_{i,1} + \bar{y}_{\ell,i-1} \epsilon_{i,2}}{1 + \bar{y}_{\ell,i-1}^2}, & \text{if} \quad -\infty \leq E \ln |\beta_0 + \epsilon_{m_{\ell-1}}| < 0, \\
\epsilon_{i,1}, & \text{if} \quad E \ln |\beta_0 + \epsilon_{m_{\ell-1}}| \geq 0,
\end{cases}
\]

if \( m_{\ell-1} < i \leq m_\ell, 1 \leq \ell \leq M + 1 \). We define the sums
\[
S_\ell(j) = \sum_{i=m_{\ell-1}+1}^{m_\ell+j} \tilde{z}_i, \quad \text{if} \quad 1 \leq j \leq m_\ell - m_{\ell-1}, 1 \leq \ell \leq M + 1.
\]
Following the proofs of Lemma 3.1 (in the case $-\infty \leq \ln |\beta_0 + \epsilon_{m,1}| < 0$), and of Theorem 3.2 (in the case $0 \leq \ln |\beta_0 + \epsilon_{m,1}| < \infty$), it can be shown that

$$\sum_{i=m_{\ell-1}+1}^{m_\ell+j} \left| \frac{y_{i-1}^2 \epsilon_{i,1} + y_{i-1} \epsilon_{i,2}}{1 + y_{i-1}^2} - S_\ell(j) \right| = O_P(1), \quad \text{if} \quad 1 \leq j \leq m_\ell - m_{\ell-1}, \quad 1 \leq \ell \leq M + 1.$$ 

This entails that we can replace the partial sums of $(y_{i-1}^2 \epsilon_{i,1} + y_{i-1} \epsilon_{i,2})/(1 + y_{i-1}^2)$ with the partial sums of the $z_i$'s.

Using the approximations in Aue et al. (2014) or Berkes et al. (2014), we can define independent Wiener processes \( \{W_{N,\ell,1}(x), 0 \leq x \leq (m_\ell - m_{\ell-1})/2\} \), \( \{W_{N,\ell,2}(x), 0 \leq x \leq (m_\ell - m_{\ell-1})/2\} \), \( 1 \leq \ell \leq M + 1 \) such that

$$\max_{1 \leq k \leq (m_\ell - m_{\ell-1})/2} \frac{1}{k} |S_\ell(k) - \eta_\ell W_{N,\ell,1}(k)| = O_P(1),$$

and

$$\max_{(m_\ell - m_{\ell-1})/2 < k < m_\ell - m_{\ell-1}} \frac{1}{(m_\ell - k)^{\zeta}} |S_\ell(m_\ell) - S_\ell(k) - \eta_\ell W_{N,\ell,2}(k)| = O_P(1),$$

with some \( \zeta < 1/2 \) for all \( 1 \leq \ell \leq M + 1 \). Recall that the results in Aue et al. (2014) and Berkes et al. (2014) are based on blocking arguments: thus, \( W_{N,\ell,1}(k) \) and \( W_{N,\ell,2}(k) \) are - as well as independent of each other - independent across \( \ell \).

Let

$$W_{N,\ell}(x) = \begin{cases} 
W_{N,\ell,1}(x), & \text{if} \quad 0 \leq x \leq (m_\ell - m_{\ell-1})/2, \\
W_{N,\ell,1}((m_\ell - m_{\ell-1})/2) + W_{N,\ell,2}((m_\ell - m_{\ell-1})/2) - W_{N,\ell,2}((m_\ell - m_{\ell-1}) - x), & \text{if} \quad (m_\ell - m_{\ell-1})/2 \leq x \leq m_\ell - m_{\ell-1}, 
\end{cases}$$

\( 1 \leq \ell \leq M + 1 \). Now we define

$$\Delta_N(k) = \sum_{j=1}^{\ell-1} \frac{\eta_j}{a_j} W_{N,j}(m_j - m_{j-1}) + \frac{\eta_\ell}{a_\ell} W_{N,\ell}(x - m_{\ell-1}), \quad m_{\ell-1} < x \leq m_\ell, \quad 1 \leq \ell \leq M + 1.$$
Thus we obtain the following approximations

\[
\max_{1 \leq k \leq N/2} \frac{1}{k^\zeta} \left| \frac{k(N-k)}{N} (\hat{\beta}_{k,1} - \hat{\beta}_{k,2}) - \left( \Gamma_N(k) - \frac{k}{N} \Gamma_N(N) \right) \right| = O_P(1),
\]

and

\[
\max_{N/2 \leq k \leq N-1} \frac{1}{(N-k)^\zeta} \left| \frac{k(N-k)}{N} (\hat{\beta}_{k,1} - \hat{\beta}_{k,2}) - \left( -\left( \Gamma_N(N) - \Gamma_N(k) \right) + \frac{N-k}{N} \Gamma_N(N) \right) \right| = O_P(1),
\]

for some \(0 < \zeta < 1/2\). We further note that

\[
\{N^{-1/2} \Gamma_N(Nt), 0 \leq t \leq 1\} \overset{D}{=} \{\Gamma(t), 0 \leq t \leq 1\},
\]

and that \(\eta_0(t, t)\) is proportional to \(t(1-t)\), if \(0 < t \leq \tau_1\) or \(\tau_M < t < 1\).

We now prove part (i) of the theorem; the proof is based on the same arguments used in the proof of Theorem 3.1. Let \(0 < \delta < \min \{\tau_1, \tau_M\}\); then

\[
\sup_{1/(N+1) < t < \delta} \left| Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{\lfloor Nt \rfloor}{N} \right) \left( N^{-1/2} \Gamma_N(N) \right) \right| w(t)
\]

\[
\leq c_0 N^{-1/2+\zeta} \sup_{N\delta \leq k \leq N/2} \frac{1}{k^\zeta} \left| \frac{k(N-k)}{N} (\hat{\beta}_{k,1} - \hat{\beta}_{k,2}) - \left( \Gamma_N(k) - \frac{k}{N} \Gamma_N(N) \right) \right|
\]

\[
+ c_0 N^{-1/2+\zeta} \sup_{N/2 \leq k \leq N(1-\delta)} \frac{1}{(N-k)^\zeta} \left| \frac{k(N-k)}{N} (\hat{\beta}_{k,1} - \hat{\beta}_{k,2}) - \left( \Gamma_N(k) - \frac{k}{N} \Gamma_N(N) \right) \right|
\]

\[= o_P(1), \]

having used Assumption 2.1 in the second passage, and (C.33)–(C.34) in the last one. Also note

\[
\sup_{1/(N+1) < t < \delta} \frac{1}{t(1-t)^{1/2}} \left| Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{\lfloor Nt \rfloor}{N} \right) \left( N^{-1/2} \Gamma_N(N) \right) \right| \left( t(1-t) \right)^{1/2} w(t)
\]

\[= \sup_{1/(N+1) < t < \delta} \frac{1}{t(1-t)^{1/2}} \left| Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{\lfloor Nt \rfloor}{N} \right) \left( N^{-1/2} \Gamma_N(N) \right) \right| \left( t(1-t) \right)^{1/2} w(t) \]
\[
\leq \sqrt{2N^{-1/2+\varepsilon}} \frac{\delta^{1/2}}{w(\delta)} \sup_{1 \leq k \leq N\delta} \frac{k(N-k)}{N} \left( \frac{1}{k^2} \left( \hat{\beta}_{k,1} - \hat{\beta}_{k,2} \right) - \left( \Gamma_N(k) - \frac{k}{N} \Gamma_N(N) \right) \right) \sup_{1/(N+1) < t < \delta} t^{\varepsilon-1/2} \\
\leq c_0 \frac{\delta^{1/2}}{w(\delta)} O_P(1),
\]

having used (C.33). Using (C.8), it follows that

\[
\lim_{\delta \to 0^+} \lim_{N \to \infty} \sup_{1/(N+1) < t < \delta} \frac{w(t)}{w(\delta)} \left| Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{Nt}{N} N^{-1/2} \Gamma_N(N) \right) \right| > x = 0,
\]

for all \( x > 0 \). Similar arguments yield

\[
\lim_{\delta \to 0^+} \lim_{N \to \infty} \sup_{1/(N+1) < t < \delta} \frac{w(t)}{w(\delta)} \left| Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{Nt}{N} N^{-1/2} \Gamma_N(N) \right) \right| > x = 0,
\]

for all \( x > 0 \). Putting all together, it holds that

\[
\sup_{1/(N+1) \leq t \leq 1/(N+1)} \frac{\left| Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{Nt}{N} N^{-1/2} \Gamma_N(N) \right) \right|}{w(t)} = o_P(1),
\]

which proves part (i) of the theorem.

The proof of part (ii) is based again on the approximations in (C.33) and (C.34). First we show that

\[
(C.35) \quad \lim_{N \to \infty} P \left\{ \sup_{0 < t < 1} \frac{|Q_N(t)|}{\eta_0^{1/2}(t, t)} = \max \left( \sup_{t_1 \leq t \leq t_2} \frac{|Q_N(t)|}{\eta_0^{1/2}(t, t)}, \sup_{1-t_2 \leq t \leq 1-t_1} \frac{|Q_N(t)|}{\eta_0^{1/2}(t, t)} \right) \right\} = 1,
\]

where \( t_1 = (\ln N)^4/N \) and \( t_2 = 1/\ln N \). Indeed, recalling that \( \eta_0(t, t) = c_0 t (1-t) \) for \( t \leq \tau_1 \) and \( t \geq \tau_M \)

\[
(C.36) \quad \sup_{0 < t < 1/(N+1)} \frac{|Q_N(t)|}{\eta_0^{1/2}(t, t)} = N^{-1/2} |\hat{\beta}_N| \sup_{0 < t < 1/(N+1)} \eta_0^{-1/2}(t, t) \leq c_0 |\hat{\beta}_N| = O_P(1).
\]

Also, it holds that

\[
N^{-1/2} \sup_{1/(N+1) \leq t < t_1} \frac{k(N-k)}{N} \left( \hat{\beta}_{k,1} - \hat{\beta}_{k,2} \right) - \left( W_{N,1,1}(Nt) - \frac{Nt}{N} \Gamma_N(N) \right) \left( t(1-t) \right)^{1/2}
\]
\[ \leq N^{-1/2 + \zeta} \sup_{1/(N+1) \leq t < t_1} \left| \frac{k(N-k)}{N} \left( \hat{\beta}_{k,1} - \hat{\beta}_{k,2} \right) - \left( W_{N,1,1}(Nt) - \frac{|Nt|}{N} \Gamma_N(N) \right) \right| (Nt)^\zeta \sup_{1/(N+1) \leq t < t_1} t^{\zeta-1/2} = O_P(1) , \]

and

\[ N^{-1/2} \sup_{1/(N+1) \leq t < t_1} \frac{|Nt|}{N} \Gamma_N(N) \leq c_0 N^{-1/2} |\Gamma_N(N)| \sup_{1/(N+1) \leq t < t_1} t^{1/2} = O_P(1) t^{1/2} = O_P \left( N^{-1/2}(\ln N)^2 \right) . \]

Hence

\[ \sup_{1/(N+1) \leq t < t_1} \frac{|Q_N(t) - N^{-1/2} W_{N,1,1}(Nt)|}{(t (1 - t))^{1/2}} = O_P(1) . \]

Also

\[ \sup_{1/(N+1) \leq t < t_1} \frac{|N^{-1/2} W_{N,1,1}(Nt)|}{\eta_0^{1/2}(t,t)} \leq \sup_{1/(N+1) \leq t < t_1} \frac{|N^{-1/2} W_{N,1,1}(Nt)|}{(t (1 - t))^{1/2}} \sup_{1/(N+1) \leq t < t_1} \frac{(t (1 - t))^{1/2}}{\eta_0^{1/2}(t,t)} = O_P \left( (\ln \ln \ln N)^{1/2} \right) O(1) , \]

having used the Darling-Erdős theorem (see Theorem A.4.2 in Csörgő and Horváth, 1997). The equations above entail that

\[ (C.37) \quad \sup_{1/(N+1) \leq t < t_1} \frac{|Q_N(t)|}{\eta_0^{1/2}(t,t)} = O_P \left( (\ln \ln \ln N)^{1/2} \right) . \]

Finally, note that, by similar passages as above

\[ \sup_{t_2 \leq t < 1/2} \frac{|Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{|Nt|}{N} N^{-1/2} \Gamma_N(N) \right)|}{\eta_0^{1/2}(t,t)} \leq \sup_{t_2 \leq t < 1/2} \frac{|t \Gamma(1)|}{\eta_0^{1/2}(t,t)} + \sup_{t_2 \leq t \leq t_1} \frac{|Q_N(t) - N^{-1/2} W_{N,1,1}(Nt)|}{\eta_0^{1/2}(t,t)} + \sup_{\tau_1 \leq t \leq \tau_2} \frac{|Q_N(t) - N^{-1/2} W_{N,2,1}(Nt)|}{\eta_0^{1/2}(t,t)} + \ldots = O_P(1) , \]
with \( W_{2,1}(t) = N^{-1/2}W_{N,2,1}(Nt) \), etc... Since

\[
\sup_{t_2 \leq t < 1/2} \frac{|N^{-1/2}W_{N,1,1}(Nt)|}{\eta_0^{1/2}(t,t)} \leq \sup_{t_2 \leq t < 1/2} \frac{|N^{-1/2}W_{N,1,1}(Nt)|}{(t(1-t))^{1/2}} \sup_{t_2 \leq t < 1/2} \frac{(t(1-t))^{1/2}}{\eta_0^{1/2}(t,t)} = O_P \left( (\ln \ln N)^{1/2} \right),
\]

\[
\sup_{\tau_1 < t \leq \tau_2} \frac{|N^{-1/2}W_{N,2,1}(Nt)|}{\eta_0^{1/2}(t,t)} \leq \sup_{\tau_1 < t \leq \tau_2} \frac{|N^{-1/2}W_{N,2,1}(Nt)|}{(t(1-t))^{1/2}} \sup_{\tau_1 < t \leq \tau_2} \frac{(t(1-t))^{1/2}}{\eta_0^{1/2}(t,t)} = O_P(1),
\]

and the same for all \( W_{N,t,1}(Nt), 1 < \ell \), we conclude that

\[
(C.38) \quad \sup_{t_2 \leq t < 1/2} \frac{|Q_N(t)|}{\eta_0^{1/2}(t,t)} = O_P \left( (\ln \ln N)^{1/2} \right).
\]

The same results can be shown, with the same logic, over the intervals \( 1 - t_1 < t < 1 \) and \( 1/2 < t < 1 - t_2 \). Next we note that

\[
(C.39) \quad \sup_{t_1 \leq t \leq t_2} \left| \frac{\eta_0(t,t)}{t} - \frac{\eta_1^2}{a_1^2} \right| = O(1/\ln N),
\]

and

\[
(C.40) \quad \sup_{1 - t_1 \leq t \leq 1 - t_2} \left| \frac{\eta_0(t,t)}{1 - t} - \frac{\eta_{1,t+1}^2}{a_{1,t+1}^2} \right| = O(1/\ln N).
\]

It thus follows from (C.33) that

\[
\sup_{t_1 \leq t \leq t_2} \left| Q_N(t) - N^{-1/2}W_{N,1,1}(Nt) \right| \leq c_0 N^{-1/2+\zeta} \sup_{t_1 \leq t \leq t_2} \left| \frac{k(N-k)N}{\bar{\kappa}_{k,1} - \bar{\kappa}_{k,2}} - \frac{W_{N,1,1}(Nt) - \frac{Nt}{N} \Gamma_N(N)}{N(t)^{\zeta}} \right| \sup_{t_1 \leq t \leq t_2} t^{\zeta-1/2}
\]

\[
+ c_0 N^{-1/2} \left| \Gamma_N(N) \right| \sup_{t_1 \leq t \leq t_2} t^{1/2}
\]

\[
= O_P \left( (\ln N)^{4(\zeta-1/2)} \right).
\]

The same result can be shown, with exactly the same logic, in the interval \( 1 - t_2 < t < 1 - t_1 \), viz.

\[
\sup_{1 - t_2 \leq t \leq 1 - t_1} \left| Q_N(t) - N^{-1/2}W_{N,M,1,1}(Nt) \right| = O_P \left( (\ln N)^{4(\zeta-1/2)} \right).
\]
Further, using (C.39)

\[
\sup_{t_1 \leq t \leq t_2} \left| \frac{N^{-1/2}W_{N,1,1}(Nt)}{\eta_0^{1/2}(t,t)} - \left( \frac{\eta_1}{a_1^2} \right)^{-1/2} \frac{N^{-1/2}W_{N,1,1}(Nt)}{(t(1-t))^{1/2}} \right| \\
\leq c_0 \sup_{t_1 \leq t \leq t_2} \left( \frac{\eta_2}{a_1^2} \right)^{-1/2} \left| \frac{N^{-1/2}W_{N,1,1}(Nt)}{(t(1-t))^{1/2}} \right| \sup_{t_1 \leq t \leq t_2} \left| \frac{1/2}{\eta_0^{1/2}(t,t)} - \left( \frac{\eta_1}{a_1^2} \right)^{1/2} \right| \\
= O_P \left( \frac{\sqrt{2\ln\ln N}}{\ln N} \right),
\]

and, by the Darling-Erdős theorem

\[
\frac{1}{\sqrt{2\ln\ln N}} \sup_{t_1 \leq t \leq t_2} \left| \frac{N^{-1/2}W_{N,1,1}(Nt)}{(t(1-t))^{1/2}} \right| \xrightarrow{P} 1.
\]

This proves (C.35). Since \( \{W_{N,1,1}(x), Nt_1 \leq x \leq Nt_2\} \) and \( \{W_{N,M+1,2}(x), N(1-t_2) \leq x \leq N(1-t_1)\} \) are independent Wiener processes, Theorem A.4.2 in Csörgő and Horváth (1997) implies

\[
\lim_{N \to \infty} \mathbb{P} \left\{ a(\ln N) \max_{(\ln N)^4 \leq x \leq \ln N} x^{-1/2} |W_{N,1,1}(x)|, \right. \\
\left. \sup_{N-N/\ln N \leq x \leq N-(\ln N)^4} (N-x)^{-1/2} |W_{N,M+1,2}(N-x)| \right\} \leq x + b(\ln N)
\]

\[
= \lim_{N \to \infty} \mathbb{P} \left\{ a(\ln N) \max_{(\ln N)^4 \leq x \leq \ln N} x^{-1/2} |W_{N,1,1}(x)| \leq x + b(\ln N) \right\} \\
\times \lim_{N \to \infty} \mathbb{P} \left\{ \sup_{(\ln N)^4 \leq x \leq \ln N} \frac{x^{-1/2} |W_{N,M+1,2}(x)|}{x^{-1/2} |W_{N,M+1,2}(x)|} \leq x + b(\ln N) \right\}
\]

\[
= [\exp(-e^{-x})]^2.
\]

The proof of Theorem 4.1(ii) is now complete.

Part (iii) follows by repeating the proof of part (ii) with minor modifications, and therefore we only report some passages. Let \( r_{1,N} = r_{2,N} = r_N \) for simplicity; it is easy to see that

(C.41) \[
\sup_{r_N/N \leq t \leq 1-r_N/N} \left| \frac{t(1-t)^{1/2}}{\eta_0^{1/2}(t,t)} \right| = O(1),
\]
We have
\[
(C.42) \quad \sup_{r_N N \leq t \leq 1 - r_N N} \frac{(t(1 - t))^{1/2}}{(t(1 - t))^\kappa} = O \left( \left( \frac{r_N N}{N} \right)^{1/2 - \kappa} \right).
\]

Note now that
\[
\begin{align*}
\leq r_N N^{\kappa - 1/2} \sup_{r_N N \leq t \leq 1 - r_N N} \frac{Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{|N|}{N} N^{-1/2} \Gamma_N(N) \right)}{(t(1 - t))^{\kappa}} \\
\leq N^{-1/2 + \zeta} \left( \frac{r_N N}{N} \right)^{\kappa - 1/2} \sup_{r_N N \leq k \leq 1 - r_N N} \frac{k(N-k)(\beta_{k,1} - \hat{\beta}_{k,2}) - (\Gamma_N(k) - \frac{k}{N} \Gamma_N(N))}{(N t)^\zeta} \\
\times \sup_{r_N N \leq t \leq 1 - r_N N} \frac{(t(1 - t))^\zeta}{(t(1 - t))^\kappa} \\
\leq N^{\zeta - 1/2} \left( \frac{r_N N}{N} \right)^{\kappa - 1/2} O_P(1) \left( \frac{r_N N}{N} \right)^{-\kappa} = O_P \left( \frac{r_N N}{N} \right) = o_P(1).
\end{align*}
\]

Using \((C.41)\), this also entails that
\[
(C.43) \quad \left( \frac{r_N N}{N} \right)^{\kappa - 1/2} \sup_{r_N N \leq t \leq 1 - r_N N} \frac{(t(1 - t))^{1/2 - \kappa}}{\eta_0^{1/2}(t, t)} \left| Q_N(t) - \left( N^{-1/2} \Gamma_N(Nt) - \frac{|N|}{N} N^{-1/2} \Gamma_N(N) \right) \right| = o_P(1).
\]

We have
\[
\begin{align*}
\left( \frac{r_N N}{N} \right)^{\kappa - 1/2} \sup_{(r_N N)^{1/2} \leq t \leq 1/2} \frac{N^{-1/2} \Gamma_N(Nt) - \frac{|N|}{N} N^{-1/2} \Gamma_N(N)}{(t(1 - t))^{\kappa}} \\
\leq 2^\kappa \left( \frac{r_N N}{N} \right)^{\kappa - 1/2} \sup_{(r_N N)^{1/2} \leq t \leq 1/2} \frac{N^{-1/2} \Gamma_N(Nt) - \frac{|N|}{N} N^{-1/2} \Gamma_N(N)}{t^{1/2}} t^{1/2 - \kappa} \\
\leq 2^\kappa \left( \frac{r_N N}{N} \right)^{\kappa - 1/2} \left( \left( \frac{r_N N}{N} \right)^{1/2 - \kappa} \sup_{(r_N N)^{1/2} \leq t \leq 1/2} \frac{N^{-1/2} \Gamma_N(Nt) - \frac{|N|}{N} N^{-1/2} \Gamma_N(N)}{t^{1/2}} \right) = o_P(1).
\end{align*}
\]

By using the same logic as in the proof of \((C.38)\), it can be shown that
\[
\sup_{(r_N N)^{1/2} \leq t \leq 1/2} \frac{N^{-1/2} \Gamma_N(Nt) - \frac{|N|}{N} N^{-1/2} \Gamma_N(N)}{t^{1/2}} = O_P \left( \sqrt{ \ln \frac{r_N N}{N} } \right),
\]
so that ultimately
\begin{equation}
\left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{(r_N/N)^{1/2} \leq t \leq 1/2} \frac{\left| N^{-1/2} \Gamma_N(Nt) - \frac{[Nt]}{N} N^{-1/2} \Gamma_N(N) \right|}{(t (1 - t))^\kappa} = o_P(1).
\end{equation}

By the same token, it follows that
\begin{equation}
\left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{1/2 \leq t \leq 1 - (r_N/N)^{1/2}} \frac{\left| N^{-1/2} \Gamma_N(Nt) - \frac{[Nt]}{N} N^{-1/2} \Gamma_N(N) \right|}{(t (1 - t))^\kappa} = o_P(1).
\end{equation}

Further, note that
\begin{align*}
\left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{r_N/N \leq t < (r_N/N)^{1/2}} \frac{\left| \frac{[Nt]}{N} N^{-1/2} \Gamma_N(N) \right|}{(t (1 - t))^\kappa} \leq N^{-1/2} \Gamma_N(N) \left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{r_N/N \leq t < (r_N/N)^{1/2}} t^{1 - \kappa} = O_P \left( \left( \frac{r_N}{N} \right)^{\frac{1}{2} \min(1, \kappa)} \right) = o_P(1),
\end{align*}
and the same holds on the interval $1 - (r_N/N)^{1/2} \leq t < 1 - r_N/N$. Thus, we only need to focus on finding the limiting distributions of
\begin{align*}
\left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{r_N/N \leq t < (r_N/N)^{1/2}} \frac{\left| N^{-1/2} \Gamma_N(Nt) \right|}{(t (1 - t))^\kappa},
\left( \frac{r_N}{N} \right)^{\kappa - 1/2} \sup_{1 - (r_N/N)^{1/2} \leq t < 1 - r_N/N} \frac{\left| N^{-1/2} \Gamma_N(Nt) \right|}{(t (1 - t))^\kappa}.
\end{align*}

On the intervals $r_N/N \leq t < (r_N/N)^{1/2}$ and $1 - (r_N/N)^{1/2} \leq t < 1 - r_N/N$, however, $\Gamma_N(Nt)$ is given by the Wiener processes $W_{N,1,1}(Nt)$ and $W_{N,M+1,1}(Nt)$; the desired result now follows from Horváth et al. (2020b).

**Proof of Theorem 4.2.** Repeating the arguments used in the proofs of Theorems 3.1 and 3.2 on the intervals $(m_{\ell-1}, m_{\ell}]$, one can show that
\begin{equation}
\sup_{0 < t < 1} \left| \tilde{c}_{N,1}(t) - \psi_1(t) \right| = O_P \left( (\ln N)^{-\zeta} \right),
\end{equation}
with some $\zeta > 0$, and

\[(C.47)\quad \sup_{0<t<1} |\hat{b}_N(t) - b(t)| = O_P\left((\ln N)^{-\zeta}\right),\]

with some $\zeta_1 > 0$. Thus it holds that

\[(C.48)\quad \sup_{0<t,s<1} |\hat{g}_N(t, s) - g(t, s)| = o_P(1),\]

and we can use the results in the proof of Theorem 4.1 to establish Theorem 4.2.

\[\square\quad \square\]

**Proof of Theorem 4.3.** The proof follows from standard arguments (see e.g. Csörgő and Horváth, 1997), and therefore we only report the most important passages. We begin by considering (4.13), and show that, under (4.12)

\[N^{1/2} \left( k^* \left( \frac{N - k^*}{N} \right) \right) \left| \hat{\beta}_{k^*,1} - \hat{\beta}_{k^*,2} \right| \xrightarrow{P} \infty.\]

Under our assumptions, the same arguments as above yield

\[(C.49)\quad \hat{\beta}_{k^*,1} \xrightarrow{P} \beta_0,\]

\[(C.50)\quad \hat{\beta}_{k^*,2} \xrightarrow{P} \beta_A.\]

Hence (4.13) follows immediately. As far as (4.17) is concerned, we begin by noting that, under the conditions of Theorem 4.2 one can show following the same passages as in the proof of that theorem that, if $|\beta_0 - \beta_A|$ is bounded, as $N \to \infty$, then there are functions $c_1^*(t)$ and $b^*(t)$ such that

\[(C.51)\quad \sup_{0<t<1} \left| \hat{c}_{N,1}(t) - c_1^*(t) \right| = o_P(1),\]

\[(C.52)\quad \sup_{0<t<1} \left| \hat{b}_N(t) - b^*(t) \right| = o_P(1).\]
(C.51) and (C.52) entail that there exists a function \( g^* (t, s) \)

(C.53) \[
\sup_{0 < t, s < 1} |\hat{g}_N (t, s) - g^* (t, s)| = o_p(1),
\]

which entails that (4.13) follows as long as

\[
N^{1/2} \left( \frac{k^*}{N} \left( \frac{N - k^*}{N} \right) \right) \frac{r_N^{\kappa - 1/2}}{g^{1/2}(t^*, t^*)} \left( \frac{k^*}{N} \left( \frac{N - k^*}{N} \right) \right)^{-\kappa + 1/2} |\hat{\beta}_{k^*,1} - \bar{\beta}_{k^*,2}| \overset{p}{\to} \infty
\]

having let \([Nt^*] = k^*\). But this follows immediately from (C.49) and (C.50), noting that, by definition \( g(t^*, t^*) = O \left( \left( \frac{k^*}{N} \left( \frac{N - k^*}{N} \right) \right)^{1/2} \right) \). Finally, (4.15) follows from the same logic.
Figure C.1. Empirical rejection frequencies under alternatives - homoskedasticity and mid-sample break

(a). $\beta_0 = 0.5$

(b). $\beta_0 = 0.75$

(c). $\beta_0 = 1$

(d). $\beta_0 = 1.05$
Figure C.2. Empirical rejection frequencies under alternatives - heteroskedasticity in \( \epsilon_{i,2} \) and mid-sample break

(a). \( \beta_0 = 0.5 \)  
(b). \( \beta_0 = 0.75 \)  
(c). \( \beta_0 = 1 \)  
(d). \( \beta_0 = 1.05 \)

Figure C.3. Empirical rejection frequencies under alternatives - heteroskedasticity in \( \epsilon_{i,1} \) and mid-sample break

(a). \( \beta_0 = 0.5 \)  
(b). \( \beta_0 = 0.75 \)  
(c). \( \beta_0 = 1 \)  
(d). \( \beta_0 = 1.05 \)
Figure C.4. Empirical rejection frequencies under alternatives - heteroskedasticity in $\epsilon_{i,1}$ and $\epsilon_{i,2}$ and mid-sample break

(a). $\beta_0 = 0.5$
(b). $\beta_0 = 0.75$
(c). $\beta_0 = 1$
(d). $\beta_0 = 1.05$

Figure C.5. Empirical rejection frequencies under alternatives - homoskedasticity and end-of-sample break

(a). $\beta_0 = 0.5$
(b). $\beta_0 = 0.75$
(c). $\beta_0 = 1$
(d). $\beta_0 = 1.05$
Figure C.6. Empirical rejection frequencies under alternatives - heteroskedasticity in $\epsilon_{i,2}$ and end-of-sample break

(a). $\beta_0 = 0.5$

(b). $\beta_0 = 0.75$

(c). $\beta_0 = 1$

(d). $\beta_0 = 1.05$

Figure C.7. Empirical rejection frequencies under alternatives - heteroskedasticity in $\epsilon_{i,1}$ and end-of-sample break

(a). $\beta_0 = 0.5$

(b). $\beta_0 = 0.75$

(c). $\beta_0 = 1$

(d). $\beta_0 = 1.05$
Figure C.8. Empirical rejection frequencies under alternatives - heteroskedasticity in $\epsilon_{i,1}$ and $\epsilon_{i,2}$ and end-of-sample break

(a). $\beta_0 = 0.5$

(b). $\beta_0 = 0.75$

(c). $\beta_0 = 1$

(d). $\beta_0 = 1.05$

Figure C.9. Empirical rejection frequencies - heteroskedasticity in $\epsilon_{i,2}$ and mid-sample break

(a). $\beta_0 = 0.98$

(b). $\beta_0 = 0.99$

(c). $\beta_0 = 1.01$

(d). $\beta_0 = 1.02$
Figure C.10. Empirical rejection frequencies - heteroskedasticity in $\epsilon_{i,t}$ and end-of-sample break

(a). $\beta_0 = 0.98$

(b). $\beta_0 = 0.99$

(c). $\beta_0 = 1.01$

(d). $\beta_0 = 1.02$

Figure C.11. Monthly log differences of US CPI