Research Article

A modification of the Prudnikov and Laguerre polynomials

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ABSTRACT

A two-parameter sequence of orthogonal polynomials \( \{P_n(x; \lambda, t)\}_{n \geq 0} \) with respect to the weight function \( x^\alpha e^{-\lambda x} \rho_\nu(x) \), \( \alpha > -1 \), \( \lambda, t \geq 0 \), \( \rho_\nu(x) = 2^{\nu/2} K_\nu(2\sqrt{x}) \), \( x > 0 \), \( \nu \geq 0 \), where \( K_\nu(z) \) is the modified Bessel function, is investigated. The case \( \lambda = 0 \) corresponds to the Prudnikov polynomials and \( t = 0 \) is related to the Laguerre polynomials. A special one-parameter case \( \{P_n(x; 1-t, t)\}_{n \geq 0}, \ t \in [0, 1] \) is analysed as well.

1. Introduction and preliminary results

Recently [1], the author gave an interpretation of the Prudnikov sequence of orthogonal polynomials with the weight function \( x^\alpha \rho_\nu(x) \), where \( \rho_\nu(x) = 2^{\nu/2} K_\nu(2\sqrt{x}) \), \( x > 0 \), \( \nu \geq 0 \), \( \alpha > -1 \) and \( K_\nu(z) \) is the modified Bessel function [2, Vol. II], in terms of the so-called composition orthogonality with respect to the measure \( x^\nu e^{-x} \) dx related to the Laguerre polynomials [3]. Our main goal here is to characterize a two-parameter sequence of orthogonal polynomials \( \{P_n(x; \lambda, t)\}_{n \geq 0} \) with respect to the weight function \( x^\alpha e^{-\lambda x} \rho_\nu(x) \), \( \alpha > -1 \), \( \lambda, t \geq 0 \), satisfying the following orthogonality conditions:

\[
\int_0^\infty P_m(x; \lambda, t)P_n(x; \lambda, t) e^{-\lambda x} \rho_\nu(x) x^\alpha \, dx = \delta_{n,m}, \tag{1.1}
\]

where \( \delta_{n,m}, \ n, m \in \mathbb{N}_0 \) is the Kronecker symbol. Up to a normalization factor, conditions (1.1) are equivalent to the following equalities, respectively:

\[
\int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_\nu(x) x^\alpha + m \, dx = 0, \quad m = 0, 1, \ldots, n - 1, \ n \in \mathbb{N}. \tag{1.2}
\]

The function \( \rho_\nu \) has the Mellin-Barnes integral representation

\[
\rho_\nu(x) = \frac{1}{2\pi i} \int_{y-i\infty}^{y+i\infty} \Gamma(v+s)\Gamma(s)x^{-s} \, ds, \quad x, y \in \mathbb{R}_+, \ v \in \mathbb{R}, \tag{1.3}
\]

where \( \Gamma(z) \) is Euler’s gamma-function [2, Vol. I]. Moreover, the Parseval equality for the Mellin transform [4] permits to write (1.5) as the Laplace integral representation for this
function. In fact, we obtain
\[ \rho_v(x) = \int_0^\infty y^{v-1} e^{-y-x/y} \, dy, \quad x > 0, \, v \in \mathbb{R}. \] (1.4)

Since \( \rho_v(0) = \Gamma(v), \) \( v > 0, \) the polynomials \( P_n(x; \lambda, 0), \) \( \lambda > 0 \) are associated with Laguerre polynomials, and, correspondingly, \( P_n(x; 0, t), \) \( t > 0, \) \( n \in \mathbb{N}_0 \) are related to the Prudnikov polynomials [1]. The asymptotic behaviour of the modified Bessel function at infinity and near the origin [2], Vol. II gives the corresponding values for the function \( \rho_v, \) \( v \in \mathbb{R}. \) Precisely, we have
\[ \rho_v(x) = O(x^{(v-|v|)/2}), \quad x \to 0, \, v \neq 0, \quad \rho_0(x) = O(\log x), \quad x \to 0, \] (1.5)
\[ \rho_v(x) = O(x^{v/2-1/4} e^{-2\sqrt{x}}), \quad x \to +\infty. \] (1.6)

The moments for the weight \( e^{-\lambda x} \rho_v(xt), \) \( \lambda^2 + t^2 \neq 0 \) can be calculated via integral (1.4). Indeed, we have
\[ I_n = \int_0^\infty e^{-\lambda x} \rho_v(xt)x^{n+\alpha} \, dx = \lambda^{n-\alpha-1} \int_0^\infty e^{-x} x^{n+\alpha} \int_0^\infty u^{v-1} e^{-u-xt/\lambda u} \, du \, dx \]
\[ = \Gamma(1 + n + \alpha) \int_0^\infty \frac{u^{v+n+\alpha} e^{-u}}{(\lambda u + t)^{n+\alpha+1}} \, du, \quad n \in \mathbb{N}_0, \] (1.7)
where the interchange of the order of integration is justified by Fubini’s theorem. The latter integral is expressed in terms of the Tricomi function \( \Psi(a, b; z) \) (cf. [2, Vol. I] and [5, Vol. I, Entry 2.3.6.9]). Hence, we get finally the values from (1.7)
\[ I_n = \lambda^{-v-\alpha-n-1} t^\nu \Gamma(n + v + \alpha + 1) \Gamma(n + \alpha + 1) \Psi \left( 1 + n + \alpha + v, 1 + v; \frac{t}{\lambda} \right). \] (1.8)

The quotient of the scaled Macdonald functions \( \rho_v, \rho_{v+1} \) is given by the Ismail integral [6]
\[ \frac{\rho_v(x)}{\rho_{v+1}(x)} = \frac{1}{\pi^2} \int_0^\infty \frac{y^{-1} \, dy}{(x+y)[J_v^2(2\sqrt{y}) + Y_{v+1}^2(2\sqrt{y})]}, \] (1.9)
where \( J_v(z), \) \( Y_v(z) \) are Bessel functions of the first and second kind, respectively [2]. Moreover, Entry 2.19.4.13 in [5], Vol. II represents the product \( x^n \rho_v(x) \) in terms of the associated Laguerre polynomials [3]. Precisely, it has
\[ \frac{(-1)^n x^n}{n!} \rho_v(x) = \int_0^\infty y^{v+n-1} e^{-y-x/y} L_n^v(y) \, dy, \quad n \in \mathbb{N}_0. \] (1.10)

Appealing to the Riemann-Liouville fractional integral [7]
\[ (I_x^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t - x)^{\alpha-1} f(t) \, dt, \quad \text{Re} \alpha > 0 \] (1.11)
and Entry 2.16.3.8 in [2, Vol. II], we get the formula
\[ \rho_v(x) = (I_x^\alpha \rho_0)(x). \] (1.12)
Besides, the index law for fractional integrals immediately implies
\[
\rho_{\nu+\mu}(x) = (I^\nu_{-\mu})(x) = (I^\mu_{-\nu})(x).
\] (1.13)
The corresponding definition of the fractional derivative presumes the relation \( D^\mu_{-\nu} = -D_{-\mu}^{1-\nu} \). Hence for the ordinary \( n \)th derivative of \( \rho_\nu \), we find
\[
D^n \rho_\nu(x) = (-1)^n \rho_{\nu-n}(x), \quad n \in \mathbb{N}_0.
\] (1.14)
Recalling the Mellin-Barnes integral (1.3) and reduction formula for the gamma-function, it is not difficult to derive the recurrence relation
\[
\rho_{\nu+1}(x) = v \rho_\nu(x) + x \rho_{\nu-1}(x), \quad v \in \mathbb{R}.
\] (1.15)
As it follows from the theory of orthogonal polynomials [3], a sequence \( \{p_n\}_{n \geq 0} \) satisfies the three term recurrence relation
\[
x p_n(x) = A_{n+1} p_{n+1}(x) + B_n p_n(x) + A_n p_{n-1}(x),
\] (1.16)
where \( p_{-1}(x) = 0 \), \( p_n(x) = a_n x^n + b_n x^{n-1} + \cdots \), and
\[
A_n = \frac{a_{n-1}}{a_n}, \quad B_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}.
\] (1.17)
As a consequence of (1.16) the Christoffel-Darboux formula takes place
\[
\sum_{k=0}^n \frac{p_k(x) p_k(y)}{x-y} = A_{n+1} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x-y}.
\] (1.18)
Denoting the weight function as \( \omega(x) = e^{-\lambda x} \rho_\nu(x)t^\alpha \), we establish a differential equation whose solution is \( \omega \).

**Lemma 1.1:** The weight function \( \omega \) satisfies the following second order differential equation
\[
x^2 \omega'' - (2(\alpha - \lambda x) + v - 1)x \omega' + ((\alpha - \lambda x)^2 + x(\lambda(1-v) - 1) + \alpha v) \omega = 0.
\] (1.19)

**Proof:** In fact, recalling (1.14), (1.15), we have
\[
[x \omega]' = \frac{d}{dx} [x^\alpha e^{-\lambda x} (\rho_{\nu+1}(x) - (v+1) \rho_{\nu+1}(xt))] = \frac{d}{dx} [x^\alpha e^{-\lambda x} \rho_{\nu+1}(xt)] = t(\alpha - \lambda x + v + 1) \omega - e^{-\lambda x} \rho_{\nu+1}(xt) x^\alpha.
\] (1.20)
Hence with one more differentiation, it gives
\[
[x \omega]' = t(1-\lambda) \omega + t(\alpha - \lambda x + v + 1) \omega' - \left( \frac{\alpha}{x} - \lambda \right) e^{-\lambda x} \rho_{\nu+1}(xt)x^\alpha.
\] (1.21)
Combining (1.20), (1.21), we find after simplification
\[
[x \omega]' = (2(\alpha - \lambda x) + v + 1) \omega' - \left( \frac{\alpha}{x} - \lambda \right) (\alpha - \lambda x + v + \lambda - 1) \omega.
\]
Therefore fulfilling the differentiation and multiplying by \( x \), we arrive at (1.19). □
2. Differential-difference properties

We begin this section with the following result.

**Theorem 2.1:** Let \( \alpha > 0, \lambda, t, \nu \geq 0, \lambda^2 + t^2 \neq 0, n \in \mathbb{N} \). Orthogonal polynomials \( P_n(x; \lambda, t) \) satisfy the integro-differential-difference equation

\[
x \frac{\partial}{\partial x} [P_n(x; \lambda, t)] = \frac{xtA_n}{\pi^2} \int_0^\infty \int_0^\infty [P_n(x; \lambda, t)P_{n-1}(y; \lambda, t) - P_n(y; \lambda, t)P_{n-1}(x; \lambda, t)] \times P_n(y; \lambda, t) e^{-\nu y} \rho_v(yt) y^\alpha \, \rho_v(ty) y^{\alpha-1} \, dy \, du,
\]

(2.1)

where \( A_n \) is defined in (1.17), \( P_n(x; \lambda, t) = a_n x^n + b_n x^{n-1} + \cdots + a_0 \) and, clearly, polynomial coefficients are functions of \( \lambda, t \).

**Proof:** Since \( \partial^2 [P_n(x; \lambda, t)] \) is a polynomial of degree \( n-1 \), we write it in the form

\[
\frac{\partial}{\partial x} [P_n(x; \lambda, t)] = \sum_{k=0}^{n-1} c_{n,k} P_k(x; \lambda, t),
\]

(2.2)

where, owing to the orthogonality,

\[
c_{n,k} = \int_0^\infty \frac{\partial}{\partial x} [P_n(x; \lambda, t)] P_k(x; \lambda, t) e^{-\lambda x} \rho_v(xt) x^\alpha \, dx.
\]

(2.3)

Then, integrating by parts with the use of the orthogonality (1.1) and relation (1.14), we eliminate integrated terms by virtue of the asymptotic formulas (1.5), (1.6) to obtain

\[
c_{n,k} = t \int_0^\infty P_n(x; \lambda, t)P_k(x; \lambda, t) e^{-\lambda x} \rho_v((xt)^\alpha x^\alpha - 1) \, dx
\]

(2.4)

In the meantime, we easily observe from the orthogonality that

\[
\int_0^\infty P_n(y; \lambda, t) \sum_{k=0}^{n-1} P_k(y; \lambda, t) P_k(x; \lambda, t) e^{-\nu y} \rho_v(yt) y^\alpha \frac{\rho_v(xt)}{\rho_v((xt)^\alpha x^\alpha - 1)} \, dy = 0,
\]

\[
\int_0^\infty P_n(y; \lambda, t) \sum_{k=0}^{n-1} P_k(y; \lambda, t) P_k(x; \lambda, t) e^{-\nu y} \rho_v(yt) y^{\alpha-1} \, dy = 0.
\]
Therefore from (1.2), (1.9), (2.2), (2.4) and Christoffel-Darboux formula (1.18) we derive

\[
\frac{\partial}{\partial x} [P_n(x; \lambda, t)]
\]

\[
= -\alpha \sum_{k=0}^{n-1} P_k(x; \lambda, t) \int_0^\infty P_n(y; \lambda, t) P_k(y; \lambda, t) e^{-\lambda y} \rho_v(yt) y^\alpha \left[ \frac{1}{y} - \frac{1}{x} \right] dy
\]

\[
+ t \sum_{k=0}^{n-1} P_k^t(x, t) \int_0^\infty P_n(y; \lambda, t) P_k(y; \lambda, t) e^{-\lambda y} \rho_v(yt) y^\alpha \left[ \frac{\rho_{v-1}(yt)}{\rho_v(yt)} - \frac{\rho_{v-1}(xt)}{\rho_v(xt)} \right] dy
\]

\[
= -\frac{\alpha A_n}{x} \int_0^\infty P_n(y; \lambda, t) [P_n(x; \lambda, t) P_{n-1}(y; \lambda, t) - P_n(y; \lambda, t) P_{n-1}(x; \lambda, t)] e^{-\lambda y} \rho_v(yt) y^{\alpha-1} dy
\]

\[
+ tA_n \int_0^\infty P_n(y; \lambda, t) [P_n(x; \lambda, t) P_{n-1}(y; \lambda, t) - P_n(y; \lambda, t) P_{n-1}(x; \lambda, t)]
\]

\[
\times e^{-\lambda y} \rho_v(yt) y^\alpha \left[ \frac{\rho_{v-1}(yt)}{\rho_v(yt)} - \frac{\rho_{v-1}(xt)}{\rho_v(xt)} \right] dy
\]

\[
= \frac{tA_n}{\pi^2} \int_0^\infty \int_0^\infty [P_n(x; \lambda, t) P_{n-1}(y; \lambda, t) - P_n(y; \lambda, t) P_{n-1}(x; \lambda, t)]
\]

\[
\times \frac{P_n(y; \lambda, t) e^{-\lambda y} \rho_v(yt) y^\alpha}{u(y+u)(x+u)[J_\nu^2(2\sqrt{u}) + Y_\nu^2(2\sqrt{u})]} \, du \, dy
\]

\[
- \frac{\alpha A_n}{x} [a_{n-1,0} P_n(x; \lambda, t) - a_{n,0} P_{n-1}(x; \lambda, t)] \int_0^\infty P_n(y; \lambda, t) e^{-\lambda y} \rho_v(yt) y^{\alpha-1} dy.
\]

This completes the proof of Theorem 2.1. \qed

Now, assuming that polynomial coefficients are continuously differentiable functions of \( \lambda, t \), we differentiate through equalities (1.2) with respect to \( \lambda \) and \( t \) under the integral sign and integrate by parts with the use of the orthogonality and equalities (1.14), (1.15). We note that the differentiation under the integral sign in (1.2) can be motivated by the absolute and uniform convergence on the subset of \( \mathbb{R}^2 M = \{(\lambda, t) \in \mathbb{R}^2_+ | \lambda \geq \lambda_0 > 0, t \geq t_0 > 0\} \). Hence, we obtain from (1.2) three types of orthogonality conditions

\[
\int_0^\infty \frac{\partial}{\partial \lambda} [P_n(x; \lambda, t)] e^{-\lambda x} \rho_v(xt)x^{\alpha+m} \, dx
\]

\[
- \int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_v(xt)x^{\alpha+m+1} \, dx = 0, \quad m = 0, 1, \ldots, n - 1, \quad (2.5)
\]

\[
\int_0^\infty \frac{\partial}{\partial t} [P_n(x; \lambda, t)] e^{-\lambda x} \rho_v(xt)x^{\alpha+m} \, dx
\]

\[
- \int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_{v+1}(xt)x^{\alpha+m} \, dx = 0, \quad m = 0, 1, \ldots, n - 1, \quad (2.6)
\]

\[
\lambda \int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_v(xt)x^{\alpha+m+1} \, dx + \int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_{v+1}(xt)x^{\alpha+m} \, dx
\]

\[
- \int_0^\infty \frac{\partial}{\partial x} [P_n(x; \lambda, t)] e^{-\lambda x} \rho_v(xt)x^{\alpha+m+1} \, dx = 0, \quad m = 0, 1, \ldots, n - 1. \quad (2.7)
\]
Then as a direct consequence of (2.6), (2.7), we find the equalities

\[
\int_0^\infty \left( t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} \right) [P_n(x; \lambda, t)] e^{-\lambda x} \phi_v(xt) x^{\alpha+m} \, dx \\
+ \lambda \int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \phi_v(xt) x^{\alpha+m+1} \, dx = 0, \quad m = 0, 1, \ldots, n - 1. \tag{2.8}
\]

Furthermore, (1.16), (2.5), (2.8) and the differentiation through (1.1) for \( n = m \) by \( \lambda \) and by \( t \) yield

\[
\int_0^\infty \left[ \frac{\partial}{\partial \lambda} [P_n(x; \lambda, t)] - A_n P_{n-1}(x; \lambda, t) \right] e^{-\lambda x} \phi_v(xt) x^{\alpha+m} \, dx = 0, \quad m = 0, 1, \ldots, n - 1, \tag{2.9}
\]

\[
\int_0^\infty \left[ \left( \frac{t}{\partial t} - x \frac{\partial}{\partial x} \right) [P_n(x; \lambda, t)] + \lambda A_n P_{n-1}(x; \lambda, t) \right] e^{-\lambda x} \phi_v(xt) x^{\alpha+m} \, dx = 0, \quad m = 0, 1, \ldots, n - 1, \tag{2.10}
\]

\[
2 \int_0^\infty P_n(x; \lambda, t) \frac{\partial}{\partial t} [P_n(x; \lambda, t)] e^{-\lambda x} \phi_v(xt) x^{\alpha} \, dx \\
- \int_0^\infty [P_n(x; \lambda, t)]^2 e^{-\lambda x} \phi_v(xt) x^{\alpha+1} \, dx = 0, \tag{2.11}
\]

\[
2 \int_0^\infty P_n(x; \lambda, t) \frac{\partial}{\partial t} [P_n(x; \lambda, t)] e^{-\lambda x} \phi_v(xt) x^{\alpha} \, dx \\
- \int_0^\infty [P_n(x; \lambda, t)]^2 e^{-\lambda x} \phi_{v-1}(xt) x^{\alpha+1} \, dx = 0. \tag{2.12}
\]

As a result, we are ready to prove the following theorems.

**Theorem 2.2:** Let \( \alpha > -1, \lambda, t, \nu \geq 0, \lambda^2 + t^2 \neq 0, n \in \mathbb{N} \). Orthogonal polynomials \( P_n(x; \lambda, t) \) satisfy the first order ordinary and partial differential-difference equations

\[
\frac{\partial}{\partial \lambda} [P_n(x; \lambda, t)] - \frac{1}{a_n} \frac{\partial a_n}{\partial \lambda} P_n(x; \lambda, t) - A_n P_{n-1}(x; \lambda, t) = 0, \tag{2.13}
\]

\[
\left( \frac{t}{\partial t} - x \frac{\partial}{\partial x} \right) [P_n(x; \lambda, t)] - \left[ \frac{t}{a_n} \frac{\partial a_n}{\partial t} - n \right] P_n(x; \lambda, t) + \lambda A_n P_{n-1}(x; \lambda, t) = 0. \tag{2.14}
\]

**Proof:** Indeed, equations (2.13), (2.14) are direct consequences of the uniqueness and orthogonality conditions (2.9), (2.10), respectively. In fact, polynomials within the brackets under the integral sign are of degree \( n \) and up to constants are equal to \( P_n \). These constants, in turn, can be found, equating coefficients in front of \( x^n \). ■

Equating coefficients in front of \( x^{n-1} \) in (2.13), (2.14) to zero and dividing the obtained equalities by \( a_n \), we find after simplification
Corollary 2.1: The following differential recurrence relations for coefficients \(a_n, b_n, A_n, B_n\) hold

\[
\frac{\partial}{\partial \lambda} \left( \frac{b_n}{a_n} \right) = A_n^2,
\]

\[
\frac{\partial}{\partial t} \left( \frac{tb_n}{a_n} \right) = -\lambda A_n^2,
\]

\[
\left( \frac{\lambda}{\partial \lambda} + \frac{t}{\partial t} \right) B_n + B_n = 0,
\]

\[
\left( \frac{\lambda}{\partial \lambda} - \frac{t}{\partial t} \right) \left( \frac{b_n}{a_n} \right) - \frac{b_n}{a_n} = 2\lambda A_n^2,
\]

\[
\left( \frac{\lambda}{\partial \lambda} - \frac{t}{\partial t} \right) B_n - B_n = 2\lambda [A_n^2 - A_{n+1}^2].
\]

We note that relations (2.17)-(2.19) follow immediately from (2.15), (2.16) via (1.17). Moreover, as we will see, identities (2.11), (2.12) generate

Theorem 2.3: Let \(\alpha > -1, t > 0, \lambda, \nu \geq 0, n \in \mathbb{N}_0\). Then it has the equalities

\[
B_n = \frac{2}{a_n} \frac{\partial a_n}{\partial \lambda},
\]

\[
2t \frac{\partial a_n}{\partial t} = 2n + \alpha + 1 - \lambda B_n,
\]

\[
\left( \frac{\lambda}{\partial \lambda} + \frac{t}{\partial t} \right) a_n = \left( n + \frac{\alpha + 1}{2} \right) a_n,
\]

\[
\left( \frac{\lambda}{\partial \lambda} + \frac{t}{\partial t} \right) A_n + A_n = 0.
\]

Proof: Identity (2.20) follows from (2.11), orthogonality (1.1), the three term recurrence relation (1.16) and, as a consequence, from the integral

\[
\int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_\nu(xt) x^{\alpha + n} dx = \frac{1}{a_n}.
\]

In order to prove (2.21), we have from (2.12) via (1.14), (1.15), (2.24) and integration by parts

\[
\frac{2}{a_n} \frac{\partial a_n}{\partial t} - \frac{\alpha + 1}{t} + \lambda B_n
\]

\[
- \frac{2}{t} \int_0^\infty P_n(x; \lambda, t) \frac{\partial}{\partial x} [P_n(x; \lambda, t)] e^{-\lambda x} \rho_\nu(xt) x^{\alpha + 1} dx = 0.
\]

Using again (2.24), it is easily seen that the latter integral is equal to \(n\). Hence we end up with (2.21), and equalities (2.22), (2.23) are immediate consequences of (2.20), (2.21).

Further, differentiating through by \(\lambda\) and by \(t\) the three term recurrence relation (1.16) and employing (2.17), (2.23), we arrive at the following result.
Corollary 2.2: Let $\alpha > -1$, $\lambda, t, \nu \geq 0$, $\lambda^2 + t^2 \neq 0$, $n \in \mathbb{N}_0$, $m \in \mathbb{Z}$. Orthogonal polynomials $P_n(x; \lambda, t)$ obey the partial differential-difference recurrence relation of the form

$$
\left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + m\mathcal{E} \right) xP_n(x; \lambda, t)
= A_{n+1} \left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + (m - 1)\mathcal{E} \right) P_{n+1}(x; \lambda, t)
+ B_n \left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + (m - 1)\mathcal{E} \right) P_n(x; \lambda, t)
+ A_n \left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + (m - 1)\mathcal{E} \right) P_{n-1}(x; \lambda, t),
$$

(2.26)

where $\mathcal{E}$ is the identity operator.

The differential-difference recurrence relation (2.26) can be extended for the integer power of the operator $\lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + m\mathcal{E}$, $m \in \mathbb{Z}$. Namely, employing (2.17), (2.23) and the method of mathematical induction, it is not difficult to establish the following differential-difference recurrence relation

$$
\left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + m\mathcal{E} \right)^r xP_n(x; \lambda, t)
= A_{n+1} \left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + (m - 1)\mathcal{E} \right)^r P_{n+1}(x; \lambda, t)
+ B_n \left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + (m - 1)\mathcal{E} \right)^r P_n(x; \lambda, t)
+ A_n \left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} + (m - 1)\mathcal{E} \right)^r P_{n-1}(x; \lambda, t),
$$

(2.27)

where $r \in \mathbb{N}_0$.

Now, recalling Theorem 2.2, we multiply (2.13) by $\lambda$ and sum with (2.14) to get

Corollary 2.3: Let $\alpha > -1$, $\lambda, t, v > 0$, $n \in \mathbb{N}_0$. Orthogonal polynomials $P_n(x; \lambda, t)$ satisfy the partial differential equation of the first order

$$
\left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} - \alpha \right) P_n(x; \lambda, t) - x \frac{\partial}{\partial x} [P_n(x; \lambda, t)] - \frac{\alpha + 1}{2} P_n(x; \lambda, t) = 0.
$$

(2.28)

Moreover, analogously to (2.27) for the operator powers it has the relation

$$
\left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} - \frac{\alpha + 1}{2} \right)^m P_n(x; \lambda, t) = \left( x \frac{\partial}{\partial x} \right)^m P_n(x; \lambda, t), \quad m \in \mathbb{N}_0.
$$

(2.29)
Equating coefficient in front of $x^{n-1}$ to zero, we find a companion of the equality (2.22)
\[
\left( \lambda \frac{\partial}{\partial \lambda} + t \frac{\partial}{\partial t} \right) b_n = \left( n + \frac{\alpha - 1}{2} \right) b_n.
\] (2.30)

Further use of the three term recurrence relation (1.16) and the orthogonality (1.1) generates more values of the integrals similar to (2.24) and identities between polynomial coefficients. In fact, we have

**Lemma 2.1:** Writing the polynomial $P_n$ in the form
\[
P_n(x; \lambda, t) = a_n x^n + b_n x^{n-1} + d_n x^{n-2} + \text{lower degrees},
\]
the following values of integrals take place
\[
\int_0^\infty \left[ P_n(x; \lambda, t) \right]^2 e^{-\lambda x} \rho_\nu(x^\alpha+2) \, dx = A_{n+1}^2 + B_n^2 + A_n^2,
\] (2.31)
\[
\int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_\nu(x^\alpha+n+1) \, dx = -\frac{b_{n+1}}{a_{n+1}a_n},
\] (2.32)
\[
\int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_\nu(x^\alpha+n+2) \, dx = \frac{b_{n+2}b_{n+1}}{a_{n+2}a_{n+1}a_n} - \frac{d_{n+2}}{a_{n+2}a_n},
\] (2.33)
\[
\int_0^\infty \left[ P_n(x; \lambda, t) \right]^2 e^{-\lambda x} \rho_\nu(x^\alpha+3) \, dx = A_{n+1}^2 [B_{n+1} + 2B_n]
\text{ } + A_n^2 B_{n-1} + [2A_n^2 + B_n^2]B_n,
\] (2.34)

and the identity
\[
\frac{d_n}{a_n} - \frac{d_{n+2}}{a_{n+2}} - \frac{b_{n+1}}{a_{n+1}} [B_n + B_{n+1}] = A_{n+1}^2 + B_n^2 + A_n^2.
\] (2.35)

**Proof:** Identity (2.31) is a direct consequence of (1.16), (1.1). In the same manner one gets (2.32), (2.33), involving the polynomial $P_{n+1}$, $P_{n+2}$, respectively. Then a combination with (2.31) and (1.17) yields (2.35). Finally, the value of the integral (2.34) can be obtained, writing it via (1.16), namely,
\[
\int_0^\infty \left[ P_n(x; \lambda, t) \right]^2 e^{-\lambda x} \rho_\nu(x^\alpha+3) \, dx
\]
\[
= \int_0^\infty [A_{n+1}P_{n+1}(x; \lambda, t) + B_n P_n(x; \lambda, t) + A_n P_{n-1}(x; \lambda, t)]^2 e^{-\lambda x} \rho_\nu(x^\alpha+1) \, dx,
\]
and appealing to (1.1), (1.16).

On the other hand, we observe from (2.12), (2.21)
\[
\int_0^\infty \left[ P_n(x; \lambda, t) \right]^2 e^{-\lambda x} \rho_{\nu+1}(xt)x^\alpha \, dx = 2n + \alpha + \nu + 1 - \lambda B_n.
\] (2.36)

Then, we employ (1.14), (1.15), (2.31), (2.32) and integration by parts to get
\[
\int_0^\infty \left[ P_n(x; \lambda, t) \right]^2 e^{-\lambda x} \rho_{\nu+1}(xt)x^{\alpha+1} \, dx = vB_n + t \int_0^\infty \left[ P_n(x; \lambda, t) \right]^2 e^{-\lambda x} \rho_{\nu-1}(xt)x^{\alpha+2} \, dx
\]
\[ (\alpha + 2 + \nu)B_n - \lambda [A_{n+1}^2 + B_n^2 + A_n^2] + 2 \int_0^\infty P_n(x; \lambda, t) \frac{\partial}{\partial x} [P_n(x; \lambda, t)] e^{-\lambda x} \rho_{v+1}(xt)x^{\alpha+2} \, dx \]

\[ = (\alpha + 2 + \nu)B_n - \lambda [A_{n+1}^2 + B_n^2 + A_n^2] + 2(n - 1) \frac{b_n}{a_n} - 2n \frac{b_{n+1}}{a_{n+1}}, \]

i.e. we derive the identity

\[ \int_0^\infty [P_n(x; \lambda, t)]^2 e^{-\lambda x} \rho_{v+1}(xt)x^{\alpha+1} \, dx = (\alpha + 2(n + 1) + \nu)B_n - \lambda [A_{n+1}^2 + B_n^2 + A_n^2] - 2 \frac{b_n}{a_n}. \] (2.37)

Now, returning to equalities (2.8), we observe that for \( \lambda \neq 0 \) the second integral is zero when \( m = 0, 1, \ldots, n - 2 \). Consequently, if \( t = 0 \), we end up with a modification of Laguerre polynomials \( \tilde{L}_n^\alpha \), getting the known property

\[ \frac{d}{dx} [P_n(x; \lambda, 0)] = \tilde{L}_{n-1}^\alpha (x; \lambda), \] (2.38)

and can be calculated explicitly, invoking properties of Laguerre polynomials. Otherwise, in the case \( t \neq 0 \), we write (2.8) in the form

\[ \int_0^\infty \left( t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} \right) [P_n^\alpha(x; \lambda, t)] e^{-\lambda x} \rho_{v}(xt)x^{\alpha+m} \, dx = 0, \quad m = 0, 1, \ldots, n - 2, \] (2.39)

\[ \int_0^\infty \left( t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} \right) [P_n^\alpha(x; \lambda, t)] e^{-\lambda x} \rho_{v}(xt)x^{\alpha+n-1} \, dx \neq 0. \] (2.40)

This means that the sequence \( \{ (t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x})[P_n(x; \lambda, t)] \}_{n \geq 0} \) is quasi-orthogonal with respect to the weight \( e^{-\lambda x} \rho_{v}(xt)x^{\alpha} \). The latter property is confirmed, in particular, by virtue of the recurrence relation (2.14) as a combination of the above quasi-orthogonal polynomials of degree \( n \) in terms of their orthogonal counterparts \( P_n, P_{n-1} \).

### 3. Integral-difference equations. Composition orthogonality

In this section we will establish the integral-difference equations for the sequence \( \{P_n\}_{n \geq 0} \) and characterize it in terms of the composition orthogonality for the operator polynomials \( P_n(\theta) \), where \( \theta = xDx \), \( D = \frac{d}{dx} \) and the usual product of polynomials \( P_n(x)P_m(x) \) is substituted by the composition of operators \( P_n(\theta)P_m(\theta) \). This notion was introduced in [1] and as we will see below, the orthogonality relations (1.2) can be rewritten in the composition sense by virtue of the key property for the operator \( \theta \), namely, its nonnegative integer power can be represented in the form

\[ \theta^m = x^m D^m x^m, \quad m \in \mathbb{N}_0. \] (3.1)

Indeed, basing on differential-difference equations (2.13), (2.14), we have
Theorem 3.1: Let \( \alpha > -1, \nu > 0, \lambda, t \geq 0, \lambda^2 + t^2 \neq 0, n \in \mathbb{N} \). Orthogonal polynomials \( P_n(x; \lambda, t) \) satisfy the following integral-difference equations

\[
P_n(x; \lambda, t) = \int_0^\lambda \exp \left( \frac{1}{2} \int_0^\lambda B_n(y, t) \, dy \right) A_n(\xi, t) P_{n-1}(x; \xi, t) \, d\xi
\]

\[
+ \exp \left( \frac{1}{2} \int_0^\lambda B_n(y, t) \, dy \right) P_n(x; 0, t),
\]

(3.2)

\[
P_n(x; \lambda, t) = \int_0^t \left[ x \frac{\partial}{\partial x} [P_n(x; \lambda, y)] - nP_n(x; \lambda, y) - \lambda A_n(\lambda, y) P_{n-1}(x; \lambda, y) \right] \frac{dy}{ya_n(\lambda, y)}
\]

\[
+ (-1)^n \lambda^{-n-\alpha} \Gamma(n + \alpha + 1) \Gamma(\nu)^{1/2} a_n(\lambda, t) \tilde{L}_n^\alpha(x; \lambda).
\]

(3.3)

**Proof:** The first equation (3.2) is obtained, solving the non-homogeneous ordinary differential-difference equation (2.13) in terms of \( \lambda \) with respect to \( P_n \), and using identity (2.20). Concerning the second equation, we will resolve (2.14) under the initial condition (cf. (2.38)) \( P_n(x; \lambda, 0) = \tilde{L}_n^\alpha(x; \lambda) \). Therefore it is straightforward to write from (2.14) via integration the equality

\[
P_n(x; \lambda, t) = \int_0^t \left[ x \frac{\partial}{\partial x} [P_n(x; \lambda, y)] - nP_n(x; \lambda, y) - \lambda A_n(\lambda, y) P_{n-1}(x; \lambda, y) \right] \frac{dy}{ya_n(\lambda, y)}
\]

\[
\times \frac{a_n(\lambda, t)}{a_n(\lambda, 0)} \tilde{L}_n^\alpha(x; \lambda),
\]

(3.4)

where the integral converges via (2.14) and since \( a_n \neq 0 \). Our goal now is to find the value \( a_n(\lambda, 0) \). At the same time the three term recurrence relation (1.16) for the modified Laguerre polynomials can be written explicitly, because due to (2.21), (2.23) and the orthogonality for the normalized Laguerre polynomials we find

\[
B_n(\lambda, 0) = \frac{1}{\lambda} [2n + \alpha + 1], \quad \lambda > 0,
\]

(3.5)

\[
A_n(\lambda, 0) = \frac{A_n(1, 0)}{\lambda} = - \frac{(n(n + \alpha))^{1/2}}{\lambda}, \quad \lambda > 0.
\]

(3.6)

Hence, taking into account (1.1) with \( t = 0, (1.17) \) and (3.6), we obtain

\[
\prod_{k=1}^n A_k(\lambda, 0) = \frac{a_0(\lambda, 0)}{a_n(\lambda, 0)} = \frac{(-1)^n}{\lambda^n} (n(1 + \alpha)_n)^{1/2}, \quad \lambda > 0,
\]

(3.7)

\[
a_0(\lambda, 0) = \frac{\lambda^{(1+\alpha)/2}}{(\Gamma(\nu) \Gamma(1+\alpha))^{1/2}}, \quad \nu, \lambda > 0,
\]

(3.8)

where \((z)_n\) is the Pochhammer symbol. Thus it yields the formula

\[
a_n(\lambda, 0) = \frac{(-1)^n \lambda^{n+(1+\alpha)/2}}{(n! \Gamma(n + \alpha + 1) \Gamma(\nu))^{1/2}}.
\]

(3.9)

Substituting this value in (3.4), we end up with (3.3), completing the proof of Theorem 3.1.
Concerning the composition orthogonality of the sequence \( \{P_n\}_{n \geq 0} \), we establish the following result.

**Theorem 3.2:** Let \( \alpha > -1, \lambda, t, \nu \geq 0, \lambda^2 + t^2 \neq 0 \). The sequence \( \{P_n\}_{n \geq 0} \) is compositionally orthogonal in the sense of Laguerre, i.e. the corresponding orthogonality conditions have the form

\[
\int_0^\infty y^{\nu} e^{-y} P_n \left( \frac{\theta}{t}; \lambda, t \right) \theta^m \left\{ \frac{\Gamma(1+\alpha)y^\alpha}{(\lambda y + t)^{\alpha+1}} \right\} \, dy = 0, \quad m = 0, 1, \ldots, n - 1, \ n \in \mathbb{N},
\]

(3.10)

where \( \theta = yDy \).

**Proof:** Recalling integral representation (1.10) of the function \( \rho_\nu \) in terms of Laguerre polynomials, we substitute its right-hand side in (1.2) and interchange the order of integration by Fubini’s theorem via the absolute convergence of the iterated integral. Hence we write

\[
\int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_\nu(xt) x^{\alpha+m} \, dx = (-1)^m m! \int_0^\infty y^{\nu+m-1} e^{-y} L_\nu^\nu(y) \times \int_0^\infty P_n(x; \lambda, t) e^{-x(\lambda + t/y)} x^{\alpha} \, dx \, dy.
\]

(3.11)

Then, invoking the Rodrigues formula form Laguerre polynomials, we observe (see (3.1))

\[
m! y^{\nu+m} e^{-y} L_\nu^\nu(y) = \theta^m \{ y^\nu e^{-y} \}.
\]

(3.12)

Moreover, the differentiation by \( y \) under the integral sign via the absolute and uniform convergence allows to deduce the following operator formula

\[
\frac{1}{y} \int_0^\infty P_n(x; \lambda, t) e^{-x(\lambda + t/y)} x^{\alpha} \, dx = P_n^\nu \left( \frac{\theta}{t}; \lambda, t \right) \left\{ \frac{1}{y} \int_0^\infty e^{-x(\lambda + t/y)} x^{\alpha} \, dx \right\}
\]

\[
= P_n^\nu \left( \frac{\theta}{t}; \lambda, t \right) \left\{ \frac{\Gamma(1+\alpha)y^\alpha}{(\lambda y + t)^{\alpha+1}} \right\}.
\]

(3.13)

Plugging (3.12), (3.13) in the right-hand side of (3.11), we integrate by parts, eliminating the integrated terms. Thus we derive

\[
\int_0^\infty P_n(x; \lambda, t) e^{-\lambda x} \rho_\nu(xt) x^{\alpha+m} \, dx
\]

\[
= (-1)^m \int_0^\infty \theta^m \{ y^\nu e^{-y} \} P_n^\nu \left( \frac{\theta}{t}; \lambda, t \right) \left\{ \frac{\Gamma(1+\alpha)y^\alpha}{(\lambda y + t)^{\alpha+1}} \right\} \, dy
\]

\[
= \int_0^\infty y^\nu e^{-y} P_n^\nu \left( \frac{\theta}{t}; \lambda, t \right) \theta^m \left\{ \frac{\Gamma(1+\alpha)y^\alpha}{(\lambda y + t)^{\alpha+1}} \right\} \, dy.
\]

This implies (3.10) and completes the proof.■
4. One-parameter case $\lambda = 1 - t$

Let us consider the sequence of orthogonal polynomials $\{P_n(x; 1 - t, t)\}_{n \geq 0}$ with respect to the weight $e^{-(1-t)x} \rho_v(xt)$, $0 \leq t \leq 1$. The respective orthogonality conditions (1.2) will take the form

$$ \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_v(xt)x^{\alpha+m} \, dx = 0, \quad m = 0, 1, \ldots, n - 1, \quad n \in \mathbb{N}. \tag{4.1} $$

Then, making a differentiation with respect to parameter, we deduce from (4.1) similar to (2.5), (2.6)

$$ \int_0^\infty \frac{d}{dt} [P_n(x; 1 - t, t)] e^{-(1-t)x} \rho_v(xt)x^{\alpha+m} \, dx $$

$$ + \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_v(xt)x^{\alpha+m+1} \, dx $$

$$ - \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_{v-1}(xt)x^{\alpha+m+1} \, dx = 0, \quad m = 0, 1, \ldots, n - 1. \tag{4.2} $$

Now, employing again (1.15) and orthogonality (4.1), we find from the latter equalities (4.2)

$$ \int_0^\infty t \frac{d}{dt} [P_n(x; 1 - t, t)] e^{-(1-t)x} \rho_v(xt)x^{\alpha+m} \, dx $$

$$ + t \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_v(xt)x^{\alpha+m+1} \, dx $$

$$ - \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_{v+1}(xt)x^{\alpha+m} \, dx = 0, \quad m = 0, 1, \ldots, n - 1. \tag{4.3} $$

On the other hand, integrating by parts in (4.1) and using (1.14), (1.15) and the orthogonality, it gives (cf. (2.7))

$$ (1 - t) \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_v(xt)x^{\alpha+m+1} \, dx $$

$$ + \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_{v+1}(xt)x^{\alpha+m} \, dx $$

$$ - \int_0^\infty \frac{\partial}{\partial x} [P_n(x; 1 - t, t)] e^{-(1-t)x} \rho_v(xt)x^{\alpha+m+1} \, dx = 0, \quad m = 0, 1, \ldots, n - 1. \tag{4.4} $$

Eliminating the integral with $\rho_{v+1}$ via (4.3), we get

$$ \int_0^\infty \left( t \frac{d}{dt} - x \frac{\partial}{\partial x} \right) [P_n(x; 1 - t, t)] e^{-(1-t)x} \rho_v(xt)x^{\alpha+m} \, dx $$

$$ + \int_0^\infty P_n(x; 1 - t, t) e^{-(1-t)x} \rho_v(xt)x^{\alpha+m+1} \, dx = 0, \quad m = 0, 1, \ldots, n - 1. \tag{4.5} $$
This yields immediately an analogue of the differential-difference equation (2.14)

\[
\left( t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} \right) [P_n(x; 1 - t, t)] - \left[ \frac{t}{a_n} \frac{\partial a_n}{\partial t} - n \right] P_n(x; 1 - t, t) \\
+ A_n P_{n-1}(x; 1 - t, t) = 0.
\] (4.6)

Equating coefficients in front of \(x^{n-1}\) in (4.6), we derive the Toda-type equations (cf. (2.18), (2.19))

\[
\frac{\partial}{\partial t} \left( \frac{t b_n}{a_n} \right) = -A_n^2, \quad \text{(4.7)}
\]

\[
\frac{\partial}{\partial t} (t B_n) = [A_{n+1}^2 - A_n^2]. \quad \text{(4.8)}
\]

Finally, as in (2.39), (2.40) we conclude from (4.5) that the sequence \(\{(t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x}) [P_n(x; 1 - t, t)]\}_{n \geq 0}\) is quasi-orthogonal with respect to the weight \(e^{-(1-t)x} \rho_\nu(xt) x^\alpha\).

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