TRAVELING WAVE SOLUTIONS IN A DIFFUSIVE PRODUCER-SCROUNGER MODEL

JUNHAO WEN AND PEIXUAN WENG
School of Mathematics, South China Normal University
Guangzhou, Guangdong 510631, China

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Abstract. This paper looks into the stability of equilibria, existence and non-existence of traveling wave solutions in a diffusive producer-scrounger model. We find that the existence and non-existence of traveling wave solutions are determined by a minimum wave speed \( c_m \) and a threshold value \( R_0 \). By constructing a suitable invariant convex set \( \Gamma \) and applying Schauder fixed point theorem, the existence for \( c > c_m, R_0 > 1 \) was established. Besides, a Lyapunov function is constructed subtly to explore the asymptotic behaviors of traveling wave solutions. The non-existences of traveling wave solutions for both \( c < c_m, R_0 > 1 \) and \( R_0 \leq 1, c > 0 \) were obtained by two-sides Laplace transform and reduction method to absurdity.

1. Introduction. Very recently, Cosner and Nevai [2] established a reaction-diffusion model to describe an ecological interaction of two species named Kleptoparasitism:

\[
\begin{align*}
\frac{\partial p}{\partial t} &= d_1 \Delta p + ( -b_1 - a_1 p ) p + m(x) \frac{dp}{s+d}, \quad x \in \Omega, \\
\frac{\partial s}{\partial t} &= d_2 \Delta s + ( -b_2 - a_2 s ) s + \theta m(x) \frac{ds}{s+d}, \quad x \in \Omega.
\end{align*}
\]

(1)

In general, Kleptoparasitism is one kind of interspecies reaction that the scrounger species steals food from another species named producer. Here \( \Omega \) is a bounded domain in \( \mathbb{R}^n (n \geq 1) \) with smooth boundary; \( p(x,t), s(x,t) \) are the density of producer and scrounger at location \( x \in \Omega \) and time \( t \geq 0 \), respectively; \( d_1, d_2 > 0 \) are diffusion rates and \( a_i, b_i > 0 \) (\( i = 1, 2 \)) denote the density-independent and density-dependent death rates. Furthermore, \( m(x) \) represents the per-capita rate of resource discovery at location \( x \) for the producer; \( d > 0 \) represents the producer's ability to avoid the resource to be stolen by scrounger, and \( \theta > 0 \) is the conversion rate.

For more pertinence, the model [1] assumes that the scroungers are incapable of discovering food resource themselves, i.e., they just can steal food from the resource discovered by producers. The per-capita rate of resource discovery \( m(x) \) and the birth rate of producers are in proportion. In particular, the independent of location in \( m(x) \) means that the birth rate is the same everywhere in the domain.

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Corresponding author: Peixuan Weng, Tel:0086-20-85213533.
Under no-flux boundary condition, Cosner and Nevai \cite{2} discussed the stability of steady-states for (1). In their results, “it is found that (i) both species can persist when the habitat has high productivity, (ii) neither species can persist when the habitat has low productivity, and (iii) slower dispersal of both the producer and scrounger is favored when the habitat has intermediate productivity”.

When the domain is unbounded, for example, let $\Omega = \mathbb{R}$, the movement region of the species will be unlimited. Besides the stability, another key problem for the model will be the asymptotic pattern of spatio-temporal propagation. In the present article, we are interested in the existence and non-existence of planar traveling wave solutions, and thus we assume that $m(x) \equiv m$ is a positive constant, for technical reason. Therefore, we shall consider the following model through this article

$$
\begin{align*}
\frac{\partial p}{\partial t} &= d_1 \frac{\partial^2 p}{\partial x^2} + (-b_1 - a_1 p) p + m \frac{\partial p}{\partial x}, \quad x \in \mathbb{R}, \\
\frac{\partial s}{\partial t} &= d_2 \frac{\partial^2 s}{\partial x^2} + (-b_2 - a_2 s) s + \theta m \frac{p s}{s + \theta}, \quad x \in \mathbb{R}.
\end{align*}
$$

Note that (2) is a mixed quasi-monotone system. Other familiar examples for mixed quasi-monotone systems are prey-predator models and epidemic models. The studies for traveling wave solutions of mixed quasi-monotone systems are developed during these thirty years. But it seems that the relative studies are very tricky indeed. In the 1980’s, Dunbar \cite{4, 5, 6} established the existence of traveling wave solutions for prey-predator models with Holling type-I and Holling type-II functional responses, using a shooting method based on Wazewski theorem, invariant manifold theorem and LaSalle theorem. His idea was used, developed and simplified in \cite{10, 11, 12, 13, 18} for prey-predator models with more complicated functional responses. Another important technique for the study of traveling wave solutions of mixed quasi-monotone systems is Schauder fixed point theorem (e.g., see \cite{1, 7, 9, 19, 21, 22, 25}). However, as emphasized in \cite{19}, the challenging and difficult task is to construct and verify a suitable invariant convex set $\Gamma$ for using Schauder fixed point theorem.

Ducrot & Magal \cite{3}, Li & Yang \cite{14}, Li, Li & Yang \cite{15}, Wang & Wu \cite{19}, Xu \cite{23}, Yang, Li, Li & Wang \cite{24} used some independent analytic techniques to construct an invariant convex set $\Gamma$ for some SIR epidemic models. Their discussions to a large extent depend heavily on the forms of models, and some techniques maybe very challenging and different subtly. Actually, the producer and scrounger system (1) is different with the SIR models, the producer and scrounger in (1) both have their own dynamical function $(-b_1 - a_1 p)p$ and $(-b_2 - a_2 s)s$, and the raise of producer (or scrounger) is embodied in the factor $\frac{mp}{s + \theta}$ (or $\frac{m\theta sp}{s^2 + \theta}$). To the best of our knowledge, the model (1) is very new, and there is no research done for the spatio-temporal propagation and traveling wave solution on it now.

The purpose of the current paper is to study the existence and non-existence of traveling wave solutions for model (1). Our idea is mainly motivated by the works of above mentioned studies, and our concrete technique and method depend on the specific form of (1). Here we will firstly show that there exists a threshold constant $R_0 > 0$, the function of which is similar to the basic reproducing number of SIR epidemic model. Assume that $R_0 > 1$, one could find a constant $c_m > 0$ such that it plays the function of minimal wave speed. In particular, we will construct four functions satisfying four inequalities and then a suitable invariant convex set $\Gamma$ with the aid of these four functions as upper and lower bounds. We then show that a traveling wave solution of model (1) exists in $\Gamma$ if $c > c_m$ by Schauder fixed point theorem. In order to discuss the asymptotic behaviors of traveling wave solutions, an
important Lyapunov function is constructed and its bounds are argued subtly. The non-existences of traveling wave solutions for both \( c < c_m, R_0 > 1 \) and \( R_0 \leq 1, c > 0 \) are discussed by reduction method to absurdity and two-sides Laplace transform.

The paper is organized as follow. In section 2, the existence and linear stability of constant equilibria are discussed in view of \( R_0 > 1 \) or \( R_0 < 1 \). In section 3, we construct an invariant convex set \( \Gamma \), and apply Schauder fixed point theorem to establish the existence of traveling waves. We then introduce a Lyapunov function in order to discuss the asymptotic behaviors of traveling wave solutions. Section 4 focuses on the non-existence of traveling wave solutions. A concluding discussion is given in Section 5, and an appendix for proofs of some lemmas is at the last.

2. Linear stability of constant equilibria. In this section, the existence and the linear stability of constant equilibria of (2) are discussed. We first give a proposition about the existence of constant equilibria.

**Proposition 1.** The following conclusions for the existence of constant equilibria of (2) hold.

1. The zero equilibrium \( E_0 = (0, 0) \) always exists.
2. A nonzero boundary equilibrium \( E_1 = \left( \frac{m-b_1}{a_1}, 0 \right) \) := \( (p_1, 0) \) exists such that \( p_1 > 0 \) if and only if \( m > b_1 \).
3. A unique positive equilibrium \( E^+ = (p^+, s^+) \) exists if and only if \( R_0 := \frac{\theta m (m-b_1)}{da_1 b_2} > 1 \). In addition, if \( E^+ \) exists, then \( E_1 \) also exists.

**Proof.** The proofs of (1) and (2) are easy and omitted. The positive equilibrium exists if and only if the following algebraic system has a positive solution:

\[
\begin{align*}
-b_1 - a_1 p + m \frac{d}{s + d} &= 0, \\
-b_2 - a_2 s + \theta m \frac{p}{s + d} &= 0.
\end{align*}
\] (3)

Note that, the first curve in (3) decreases for \( s > 0 \), while the second curve of (3) increases for \( s > 0 \) and comes to infinity as \( s \to +\infty \). Then the two curves have an intersection point inside the first quadrant if and only if \( \frac{m-b_1}{a_1} > \frac{b_2 d}{\theta m} \) \( (R_0 > 1) \), see Figure 1. Furthermore, we know that the intersection point \( E^+ \) in the first quadrant is unique. \( \square \)

![Figure 1. Constant equilibria of model (2)](image-url)
In the following, we always assume \( m > b_1 \), by a biological significance. This implies that (2) has at least two nonnegative equilibria. Now we begin to discuss the linear stability of the constant equilibria of (2). For any constant equilibrium \((p_*, s_*)\), the linearized system of (2) at \((p_*, s_*)\) is

\[
\begin{align*}
\frac{dp}{dt} &= d_1 \frac{\partial^2 p}{\partial x^2} - b_1 p - 2a_1 p s + m \frac{dp}{s + d}, \\
\frac{ds}{dt} &= d_2 \frac{\partial^2 s}{\partial x^2} - b_2 s - 2a_2 s^2 + \theta m \frac{s}{s + d} + \theta m \frac{dp}{s + d}.
\end{align*}
\]

(4)

Substituting \((p(x, t), s(x, t))^T = e^{\lambda x + i\sigma x}(c_1, c_2)^T\) into (4) leads to the characteristic equation:

\[
\begin{vmatrix}
\lambda + d_1 \sigma^2 + b_1 + 2a_1 p_* - m \frac{d}{s_* + d} & m \frac{dp}{s_* + d} \\
-\theta m \frac{s_*}{s_* + d} & \lambda + d_2 \sigma^2 + b_2 + 2a_2 s_* - \theta m \frac{dp}{s_* + d}
\end{vmatrix} = 0,
\]

(5)

where \( \lambda \) is a complex number, \( \sigma \) is a real number, \( i \) is the imaging unit, and \((c_1, c_2)^T \) is a constant vector.

Replacing \((p_*, s_*)\) by \(E_0, E_1, E_+\) in (5), respectively, and applying the method of eigenvalue analysis, we can obtain the following proposition easily. The proof is simple and omitted here.

**Proposition 2.** Assume \( m > b_1 \). The following conclusions for the linear stability of constant equilibria of (2) hold.

1. The zero equilibrium \( E_0 = (0, 0) \) is linearly unstable.
2. If \( R_0 < 1 \), then the nonzero boundary equilibrium \( E_1 = (p_1, 0) \) is linearly asymptotically stable.
3. If \( R_0 > 1 \), then the nonzero boundary equilibrium \( E_1 \) is linearly unstable, and the positive equilibrium \( E_+ = (p_+, s_+) \) is linearly asymptotically stable.

3. **Existence of traveling wave solution.** A traveling wave solution of (2) is a solution with the form \((p(x, t), s(x, t)) = (P(x + ct), S(x + ct))\), where \( c \) is a constant called wave speed. Denote \( \xi = x + ct \), then \((P(\xi), S(\xi))\) satisfies the so-called wave profile system

\[
\begin{align*}
cP'(\xi) &= d_1 P''(\xi) + (-b_1 - a_1 P(\xi)) P(\xi) + m \frac{dP(\xi)}{s(\xi) + d}, \\
\frac{cS'(\xi)}{s(\xi) + d} &= d_2 S''(\xi) + (-b_2 - a_2 S(\xi)) S(\xi) + \theta m \frac{P(\xi)S(\xi)}{S(\xi) + d}.
\end{align*}
\]

(6)

In this section, we shall focus on the existence of solutions for (6) with asymptotic boundary conditions

\[
(P, S)(-\infty) = E_1, \quad (P, S)(+\infty) = E_+.
\]

(7)

The linearized equation of the second equation in (6) at the nonzero boundary equilibrium \( E_1 \) is

\[
d_2 S''(\xi) - \frac{\theta m p_1}{d} - b_2 \right) S(\xi) = 0,
\]

(8)

and then the characteristic equation of (5) is \( \Delta(\lambda, c) = 0 \), where

\[
\Delta(\lambda, c) = d_2 \lambda^2 - c \lambda + \frac{\theta m p_1}{d} - b_2.
\]

(9)

Note that if \( R_0 > 1 \), then \( \frac{\theta m p_1}{d} - b_2 = b_2 (R_0 - 1) > 0 \). Therefore, one can obtain the following properties about the characteristic equation \( \Delta(\lambda, c) = 0 \) without proof.
Lemma 3.1. Assume $R_0 > 1$, then there exist only one pair of positive numbers $\lambda_m = \frac{c}{2d_2}$ and $c_m = 2\sqrt{d_2b(R_0 - 1)}$ such that
\[
\frac{\partial \Delta}{\partial \lambda}(\lambda_m, c_m) = 0, \quad \Delta(\lambda_m, c_m) = 0.
\]
In addition,
1. For any fixed $c > c_m$, the characteristic equation $\{\}$ exists two different positive roots $\lambda_2(c) > \lambda_1(c) > 0$ such that $\Delta(\lambda, c) > 0$ for any $\lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), +\infty)$, and $\Delta(\lambda, c) < 0$ for any $\lambda \in (\lambda_1(c), \lambda_2(c))$.
2. For any $0 < c < c_m$, $\Delta(\lambda, c) > 0$ for all $\lambda \in (0, +\infty)$, and $\lim_{\lambda \to \infty} \Delta(\lambda, c) = \infty$.

In what follows in this section, we always assume $R_0 > 1$, $c > c_m$, and denote $\lambda_i = \lambda_i(c)$ ($i = 1, 2$) for a given $c > c_m$ without causing confusion.

3.1. An invariant convex set $\Gamma$. In order to apply Schauder fixed point theorem to obtain a solution of $\{\}$, we have to construct a nonempty, closed and convex set $\Gamma$. The following four lemmas give four functions, which are used as the upper and lower bounds of $\Gamma$. In order for complete understanding of the construction of $\Gamma$, we will put the argument details into the Appendix.

Lemma 3.2. For any $S(\xi) \geq 0$, the constant function $P(\xi) \equiv p_1 = \frac{m-b_1}{a_1}$ satisfies
\[
cP'(\xi) \geq d_1P''(\xi) + (-b_1 - a_1P(\xi)) P(\xi) + m \frac{dP(\xi)}{S(\xi) + d}.
\]

Lemma 3.3. For $\lambda_1 := \lambda_1(c)$ and $P(\xi) \equiv p_1$ defined in Lemma 3.1 and Lemma 3.2 respectively, the function $S(\xi) = \min\{e^{\lambda_1\xi}, s_1\}$ satisfies
\[
cS'(\xi) \geq d_2S''(\xi) + (-b_2 - a_2S(\xi)) S(\xi) + \theta m \frac{P(\xi)S(\xi)}{S(\xi) + d}
\]
for $\xi \neq \xi_1 := \frac{1}{\lambda_1} \ln s_1$, where $s_1 = \frac{-d_2 + \sqrt{d_2^2 + 4d_2\theta m p_1}}{2d_2}$ is the positive root of a quadratic equation $a_2s_1^2 + (b_2 + a_2)s_1 + b_2d - \theta mp_1 = 0$.

Lemma 3.4. For $S(\xi)$ defined in Lemma 3.3, $P(\xi) = \max\{p_1 - \sigma e^{\alpha_\xi}, 0\}$ satisfies
\[
cP'(\xi) \leq d_1P''(\xi) + (-b_1 - a_1P(\xi)) P(\xi) + m \frac{dP(\xi)}{S(\xi) + d}
\]
for $\xi \neq \xi_2 := \frac{1}{\alpha} \ln \frac{p_1}{\sigma} < 0$, where $\alpha \in (0, \min\{\lambda_1, \frac{c}{s_2}\})$, and $\sigma > p_1$ is a sufficiently large number.

Lemma 3.5. For $P(\xi)$ defined in Lemma 3.4, $S(\xi) = \max\{e^{\lambda_1\xi} - qe^{(\lambda_1 + \eta)\xi}, 0\}$ satisfies
\[
cS'(\xi) \leq d_2S''(\xi) + (-b_2 - a_2S(\xi)) S(\xi) + \theta m \frac{P(\xi)S(\xi)}{S(\xi) + d}
\]
for $\xi \neq \xi_3 := \frac{1}{\eta} \ln \frac{1}{q} < 0$, where $\eta \in (0, \min\{\lambda_1, \alpha, \lambda_2 - \lambda_1\})$, and $q > 1$ is a sufficiently large number.

We could define a set $\Gamma$ as follows:
\[
\Gamma = \{(P, S) \in C(\mathbb{R}, \mathbb{R}^2) | P(\xi) \leq P(\xi) \leq P(\xi), \ S(\xi) \leq S(\xi) \leq S(\xi)\}.
\]
It is obvious that $\Gamma$ is a nonempty, closed and convex set in $C(\mathbb{R}, \mathbb{R}^2)$ and bounded with the maximum norm.
3.2. Traveling wave solutions. Define an operator \( Q = (Q_1, Q_2) : \Gamma \to C(\mathbb{R}, \mathbb{R}^2) \) by

\[
\begin{cases}
Q_1(P, S)(\xi) = \beta_1 P(\xi) + (-b_1 - b_1 P(\xi)) P(\xi) + m \frac{dP(\xi)}{S(\xi) + d}, \\
Q_2(P, S)(\xi) = \beta_2 S(\xi) + (-b_2 - a_2 S(\xi)) S(\xi) + \theta m \frac{P(\xi) S(\xi)}{S(\xi) + d},
\end{cases}
\]

where \( \beta_i > 0 (i = 1, 2) \) are large enough such that for any \( \xi \in \mathbb{R} \),

\[
\begin{cases}
Q_1(P_1, S_1)(\xi) \leq Q_1(P_2, S_1)(\xi), & Q_1(P_1, S_1)(\xi) \geq Q_1(P_1, S_2)(\xi), \\
Q_2(P_1, S_1)(\xi) \leq Q_2(P_2, S_1)(\xi), & Q_2(P_1, S_1)(\xi) \leq Q_2(P_2, S_2)(\xi),
\end{cases}
\]

hold for any \( (P_1, S_1), (P_2, S_2) \in \Gamma \) satisfying \( (P_1, S_1) \leq (P_2, S_2) \). The equation \( (11) \) can be rewritten as

\[
\begin{align*}
&Q_1(P, S)(\xi) = \alpha_1 P(\xi) + \beta_1 P(\xi) + \gamma_1 S(\xi) + \delta_1 S(\xi) + \eta_1 P(\xi) S(\xi) + \theta_1 \frac{P(\xi) S(\xi)}{S(\xi) + d}, \\
&Q_2(P, S)(\xi) = \alpha_2 S(\xi) + \beta_2 S(\xi) + \gamma_2 P(\xi) + \delta_2 P(\xi) + \eta_2 P(\xi) S(\xi) + \theta_2 \frac{P(\xi) S(\xi)}{S(\xi) + d},
\end{align*}
\]

Obviously, the linear parts of \( (11) \) leads to two algebraic equations \( d_1 r^2 - cr - \beta_i = 0 \) \((i = 1, 2)\), each of which has two roots \( r_{11} < 0 < r_{12} \), where

\[
r_{11} = \frac{c - \sqrt{c^2 + 4d_1 \beta_i}}{2d_1}, \quad r_{12} = \frac{c + \sqrt{c^2 + 4d_1 \beta_i}}{2d_1}.
\]

Define another operator \( F = (F_1, F_2) : \Gamma \to C(\mathbb{R}, \mathbb{R}^2) \) by

\[
\begin{cases}
F_1(P, S)(\xi) = \frac{1}{\alpha_1 r_{12} - r_{11}} \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} + \int_{\xi}^{\infty} e^{r_{12}(\xi - \eta)} \right) Q_1(P, S)(\eta) d\eta, \\
F_2(P, S)(\xi) = \frac{1}{\alpha_2 r_{22} - r_{21}} \left( \int_{-\infty}^{\xi} e^{r_{21}(\xi - \eta)} + \int_{\xi}^{\infty} e^{r_{22}(\xi - \eta)} \right) Q_2(P, S)(\eta) d\eta.
\end{cases}
\]

It is obvious that a fixed point of \( F \) is a solution of \( (10) \), and vice versa. Now we shall argue that \( F \) defined on \( \Gamma \) satisfies all the conditions in Schauder fixed point theorem. Since some proofs are standard, and we will ask the readers to refer \[9\][22] for some details.

Lemma 3.6. The operator \( F = (F_1, F_2) \) maps \( \Gamma \) into \( \Gamma \).

Proof. For \( \xi \neq \xi_2 \) (see \( \xi_2 \) in Lemma [3.4]), we have

\[
F_1(P, S)(\xi) \geq \frac{1}{\alpha_1 (r_{12} - r_{11})} \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} + \int_{\xi}^{\infty} e^{r_{12}(\xi - \eta)} \right) \left( \beta_1 P(\eta) + c P'(\eta) - d_1 P''(\eta) \right) d\eta.
\]

If \( \xi > \xi_2 \), we have

\[
F_1(P, S)(\xi) \geq \frac{1}{\alpha_1 r_{12} - r_{11}} \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} + \int_{\xi}^{\xi_2} e^{r_{11}(\xi - \eta)} + \int_{\xi_2}^{\infty} e^{r_{12}(\xi - \eta)} \right) \left( \beta_1 P(\eta) + c P'(\eta) - d_1 P''(\eta) \right) d\eta.
\]

Now, integrating by part, we have

\[
\begin{align*}
&\int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} (\beta_1 P(\eta) + c P'(\eta) - d_1 P''(\eta)) d\eta \\
&= \beta_1 \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} P(\eta) d\eta + c \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} dP(\eta) - d_1 \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} dP'(\eta) \\
&= \beta_1 \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} P(\eta) d\eta + c \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} dP(\eta) - d_1 \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} dP'(\eta) \\
&= \beta_1 \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} P(\eta) d\eta + c \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} dP(\eta) \\
&= -d_1 e^{r_{11}(\xi - \xi_2)} P'(\xi - 0) - d_1 r_{11} \int_{-\infty}^{\xi_2} e^{r_{11}(\xi - \eta)} P'(\eta) d\eta \\
&= -d_1 e^{r_{11}(\xi - \xi_2)} P'(\xi - 0) - d_1 r_{11} \int_{-\infty}^{\xi_2} e^{r_{11}(\xi - \eta)} P'(\eta) d\eta
\end{align*}
\]
\[
\int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} (\beta_1 P(\eta) + cP'(\eta)) d\eta
\]

\[
= \beta_1 \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} P(\eta) d\eta + c \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} dP(\eta) - d_1 \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} dP'(\eta)
\]

\[
= \beta_1 \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} P(\eta) d\eta + cP(\xi) + cr_{11} \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} P(\eta) d\eta
\]

\[
- d_1 \left( P'(\xi) - e^{r_{11}(\xi-\xi_2)} P'(\xi_2 + 0) \right) - d_1 r_{11} \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} P'(\eta) d\eta
\]

\[
= \beta_1 \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} P(\eta) d\eta + cP(\xi) + cr_{11} \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} P(\eta) d\eta
\]

\[
- d_1 \left( P'(\xi) - e^{r_{11}(\xi-\xi_2)} P'(\xi_2 + 0) \right) - d_1 r_{11} \left( P(\xi) + r_{11} \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} P(\eta) d\eta \right)
\]

\[
= (c - d_1 r_{11}) \left( P(\xi) - d_1 P'(\xi) + d_1 e^{r_{11}(\xi-\xi_2)} P'(\xi_2 + 0) \right)
\]

and

\[
\int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} (\beta_1 P(\eta) + cP'(\eta) - d_1 P''(\eta)) d\eta
\]

\[
= \beta_1 \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} P(\eta) d\eta + c \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} dP(\eta) - d_1 \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} dP'(\eta)
\]

\[
= \beta_1 \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} P(\eta) d\eta + cP(\xi) + cr_{12} \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} P(\eta) d\eta
\]

\[
+ d_1 P'(\xi) - d_1 r_{12} \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} P'(\eta) d\eta
\]

\[
= \beta_1 \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} P(\eta) d\eta + cP(\xi) + cr_{12} \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} P(\eta) d\eta
\]

\[
+ d_1 P'(\xi) - d_1 r_{12} \left( -P(\xi) + r_{12} \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} P(\eta) d\eta \right)
\]

\[
= (c + d_1 r_{12}) P(\xi) + d_1 P'(\xi).
\]

Therefore, we have from (12)-(15) that

\[
F_1(P, S)(\xi) \geq \frac{1}{d_1 (r_{12} - r_{11})} \left( \int_{-\infty}^{\xi_2} e^{r_{11}(\xi-\eta)} + \int_{\xi_2}^{\xi} e^{r_{11}(\xi-\eta)} + \int_{\xi}^{\xi} e^{r_{12}(\xi-\eta)} \right)
\]

\[
\times \left( \beta_1 P(\eta) + cP'(\eta) - d_1 P''(\eta) \right) d\eta
\]

\[
\geq P(\xi) + \frac{1}{r_{12} - r_{11}} e^{r_{11}(\xi-\xi_2)} \left( P'(\xi_2 + 0) - P'(\xi_2 - 0) \right) \geq P(\xi).
\]

Similarly, if \( \xi < \xi_2 \), we could also derive that

\[
F_1(P, S)(\xi) \geq P(\xi) + \frac{1}{r_{12} - r_{11}} e^{r_{12}(\xi-\xi_2)} \left( P'(\xi_2 + 0) - P'(\xi_2 - 0) \right) \geq P(\xi).
\]

Summarizing the above results, we know that \( F_1(P, S)(\xi) \geq P(\xi) \) for any \( \xi \neq \xi_2 \).

For \( \xi \in \mathbb{R} \), we have from Lemma 3.2 that

\[
F_1(P, S)(\xi) = \frac{1}{d_1 (r_{12} - r_{11})} \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)} + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)} \right) Q_1(P, S)(\eta) d\eta
\]

\[
\leq \frac{\beta_1 p_1}{d_1 (r_{12} - r_{11})} \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)} + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)} \right) d\eta
\]

\[
= \frac{\beta_1 p_1}{d_1 (r_{12} - r_{11})} \cdot \left( \frac{1}{r_{12}} - \frac{1}{r_{11}} \right) = p_1 = \bar{P}(\xi).
\]

By similar arguments, we can get

\[
\bar{S}(\xi) \leq F_2(P, S)(\xi) \quad \text{for} \ \xi \neq \xi_3,
\]

\[
\bar{S}(P, \bar{S})(\xi) \leq \bar{S}(\xi) \quad \text{for} \ \xi \neq \xi_1,
\]

where \( \xi_1, \xi_3 \) are defined in Lemma 3.3 and 3.5 respectively.
From the continuity of the functions, for any $\xi \in \mathbb{R}$, there hold
\[
P(\xi) \leq F_1(P, S)(\xi), \quad F_1(P, S)(\xi) \leq P(\xi),
\]
\[
S(\xi) \leq F_2(P, S)(\xi), \quad F_2(P, S)(\xi) \leq S(\xi).
\]
Now, for any $(P, S) \in \Gamma$, we have
\[
P(\xi) \leq F_1(P, S)(\xi) \leq F_1(P, S)(\xi) \leq P(\xi),
\]
\[
S(\xi) \leq F_2(P, S)(\xi) \leq F_2(P, S)(\xi) \leq S(\xi),
\]
which means $F = (F_1, F_2)$ maps $\Gamma$ into $\Gamma$. The proof is complete.

For $\mu \in (0, \min\{-r_{11}, r_{12}, -r_{21}, r_{22}\})$, define
\[
B_\mu(\mathbb{R}, \mathbb{R}^2) = \{(P, S) \in C(\mathbb{R}, \mathbb{R}^2) \mid \sup_{\xi \in \mathbb{R}} |(P, S)(\xi)| e^{-\mu |\xi|} < \infty\}.
\]
and
\[
|(P, S)|_\mu = \sup_{\xi \in \mathbb{R}} |(P, S)(\xi)| e^{-\mu |\xi|}.
\]
Then $(B_\mu(\mathbb{R}, \mathbb{R}^2), | \cdot |_\mu)$ is a Banach space.

**Lemma 3.7.** The operator $F = (F_1, F_2)$ is continuous with respect to the norm $| \cdot |_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^2)$.

**Lemma 3.8.** The operator $F = (F_1, F_2)$ is compact with respect to the norm $| \cdot |_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^2)$.

In view of Lemma 3.2 and applying Schauder fixed point theorem, we obtain the following existence theorem about traveling wave solution $(P(x + ct), S(x + ct))$ for (2).

**Theorem 3.9.** Assume $R_0 > 1$ and $c > c_\mu$, then system (2) admits a traveling wave solution $(P(x + ct), S(x + ct)) \in \Gamma$.

### 3.3. Asymptotic behaviors of traveling wave solutions
In this subsection, we shall discuss the asymptotic behaviors of traveling wave solutions of system (2).

In the following, we always assume that $(P, S) \in \Gamma$ is a traveling wave solution of system (2) obtained in Theorem 3.9.

**Proposition 3.** Assume $R_0 > 1$ and $c > c_\mu$, then the traveling wave solution $(P(x + ct), S(x + ct)) \in \Gamma$ of system (2) satisfies

1. $0 < P(\xi) \leq p_1$ and $0 < S(\xi) \leq s_1$ for any $\xi \in \mathbb{R}$;
2. $P(-\infty) = p_1$, $S(-\infty) = 0$, $\lim_{\xi \to -\infty} S(\xi) e^{-\lambda_1 \xi} = 1$;
3. $P'(\infty) = 0$, $S'(-\infty) = 0$;

**Proof.** (1) Since $(P, S) \in \Gamma$ is a fixed point of $F = (F_1, F_2)$, $(P(\xi), S(\xi)) \leq (p_1, s_1)$ for $\xi \in \mathbb{R}$. Further we have
\[
P(\xi) = F_1(P, S)(\xi) \geq F_1(P, S)(\xi)
\]
\[
= \frac{1}{d_1(r_{12} - r_{11})} \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)} + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)} \right) Q_1(P, S)(\eta) d\eta
\]
\[
\geq \left( \frac{\beta_1 - b_1 - a_1 p_1 + \frac{md}{s_1 + d}}{d_1(r_{12} - r_{11})} \right) \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)} + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)} \right) P(\eta) d\eta > 0,
\]
Similarly,
\[ S(\xi) = F_2(P, S)(\xi) \geq F_2(P, \bar{S})(\xi) \]
\[ \geq \frac{(\beta_2 - b_2 - a_2s_1)}{d_1(r_{12} - r_{11})} \left( \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} + \int_{\xi}^{\infty} e^{r_{12}(\xi - \eta)} \right) S(\eta) d\eta > 0. \]
Thus, for any \( \xi \in \mathbb{R} \)
\[ 0 < P(\xi) \leq \bar{P}(\xi) \leq p_1, \quad 0 < S(\xi) \leq \bar{S}(\xi) \leq s_1. \]
(2) Note that for any \( \xi \in \mathbb{R} \),
\[ p_1 - \sigma e^{p_1 \xi} \leq P(\xi) < p_1, \quad e^{\lambda_2 \xi - q e^{(\lambda_1 + \eta)\xi}} \leq S(\xi) \leq e^{\lambda_1 \xi}, \]
and it follows
\[ P(-\infty) = p_1, \quad S(-\infty) = 0, \quad \lim_{\xi \to -\infty} S(\xi)e^{-\lambda_1 \xi} = 1. \]
(3) For any \( \xi \in \mathbb{R} \), we have
\[ P'(\xi) = \frac{1}{d_1(r_{12} - r_{11})} \left( r_{11} \int_{-\infty}^{\xi} e^{r_{11}(\xi - \eta)} + r_{12} \int_{\xi}^{\infty} e^{r_{12}(\xi - \eta)} \right) Q_1(P, S)(\eta) d\eta. \]
By applying L’Hospital’s rule, we get
\[ d_1(r_{12} - r_{11})P'(-\infty) = \lim_{\xi \to -\infty} \left( r_{11} \int_{-\infty}^{\xi} e^{-r_{11}\eta}Q_1(P, S)(\eta) d\eta + r_{12} \int_{\xi}^{\infty} e^{-r_{12}\eta}Q_1(P, S)(\eta) d\eta \right) \]
\[ = \lim_{\xi \to -\infty} \left( r_{11} \frac{e^{-r_{11}\xi}Q_1(P, S)(\xi)}{-r_{11}e^{-r_{11}\xi}} + r_{12} \frac{e^{-r_{12}\xi}Q_1(P, S)(\xi)}{-r_{12}e^{-r_{12}\xi}} \right) = 0. \]
That is \( P'(-\infty) = 0 \). Similarly, we have \( S'(-\infty) = 0 \).

Now, we begin to discuss the asymptotic behaviour of traveling wave solutions at \(+\infty\). Let \( f(x) := x - 1 - \ln(x(x > 0)) \) and \( q := \frac{s_+ d}{b_2 s_+ + d} > 0 \). Consider a Lyapunov function (see [10] for a similar function) as follows:
\[ V(P, S)(\xi) = cf \left( \frac{P(\xi)}{p_+} \right) + d_1 P'(\xi) \left( \frac{1}{P(\xi)} - \frac{1}{p_+} \right) + q \left[ cf \left( \frac{S(\xi)}{s_+} \right) + d_2 S'(\xi) \left( \frac{1}{S(\xi)} - \frac{1}{s_+} \right) \right]. \]
In order to estimate the lower bound of \( V(P, S)(\xi) \) on \( \mathbb{R} \), we need to introduce the following lemma.

**Lemma 3.10.** Assume \( R_0 > 1 \) and \( c > c_m \), then for the traveling wave solution \( (P(x + ct), S(x + ct)) \in \Gamma \) of system [3], there exists four positive numbers \( M_1, M_2, M_3, M_4 \) such that
\[ -M_1 P(\xi) < P'(\xi) < M_3 P(\xi), \quad -M_2 S(\xi) < S'(\xi) < M_4 S(\xi) \]
for any \( \xi \geq 0 \).
Proof. (1) For a sufficiently large number $M_1 > 0$ satisfying $P'(0) + M_1 P(0) > 0$ and $cM_1 - m > 0$, define $W_1(\xi) = P'(\xi) + M_1 P(\xi)$. We claim that $W_1(\xi) > 0$ for any $\xi \geq 0$. For contradiction, assume that $\Theta_1 = \{\eta > 0 | W_1(\eta) = 0\}$ is not an empty set and $\eta_1 = \min \Theta_1$. Then $W_1(\xi) > 0$ for $\xi \in [0, \eta_1)$ and $W_1(\eta_1) = 0$, $W'_1(\eta_1) \leq 0$. If $W'_1(\eta_1) = 0$, i.e. $P''(\eta_1) = -M_1 P'(\eta_1) = M_1^2 P(\eta_1) > 0$, from the first equation of (6), we obtain
\[
0 = d_1 P''(\eta_1) - cP'(\eta_1) + (-b_1 - a_1 P(\eta_1)) P(\eta_1) + m \frac{dP(\eta_1)}{S(\eta_1)} + d
\]
\[> cM_1 P(\eta_1) + (-b_1 - a_1 P(\eta_1)) P(\eta_1) = (cM_1 - m) P(\eta_1) > 0.
\]
It is a contradiction. If $W'_1(\eta_1) < 0$, from the first equation of (6), we also have
\[
d_1 P''(\xi) = cP'(\xi) + (b_1 + a_1 P(\xi)) P(\xi) - m \frac{dP(\xi)}{S(\xi)} + d
\]
\[< -cM_1 P(\xi) + (b_1 + a_1 P(\xi)) P(\xi)
\]
\[= -(cM_1 - m) P(\xi) < 0, \quad \xi > \eta_1.
\]
That is $P'(\xi)$ is decreasing in $(\eta_1, \infty)$, and thus $P'(\xi) < P'(\eta_1) < -M_1 P(\eta_1)$ for $\xi \in (\eta_1, \infty)$. It contradicts to the boundedness of $P$.

(2) For a sufficiently large number $M_2 > 0$ satisfying $S'(0) + M_2 S(0) > 0$ and $cM_2 - b_2 - a_2 s_1 > 0$, define $W_2(\xi) = S'(\xi) + M_2 S(\xi)$. We claim that $W_2(\xi) > 0$ for any $\xi \geq 0$. The argument is similar to (1), and we omit it.

(3) For a sufficiently large number $M_3$ satisfying $P'(0) - M_3 P(0) < 0$ and $d_1 M_3^2 - cM_3 - m > 0$, define $W_3(\xi) = P'(\xi) - M_3 P(\xi)$. We claim that $W_3(\xi) < 0$ for any $\xi \geq 0$. For contradiction, assume that $\Theta_3 = \{\eta > 0 | W_3(\eta) = 0\}$ is not an empty set and $\eta_3 = \min \Theta_3$. Then $W'_3(\eta_3) \geq 0$, that is $P''(\eta_3) \geq M_3 P'(\eta_3) = M_3^2 P(\eta_3) > 0$. From the first equation of (6), we have
\[
0 = d_1 P''(\eta_3) - cP'(\eta_3) + (-b_1 - a_1 P(\eta_3)) P(\eta_3) + m \frac{dP(\eta_3)}{S(\eta_3)} + d
\]
\[> d_1 M_3^2 P(\eta_3) - cM_1 P(\eta_3) + (-b_1 - a_1 P(\eta_3)) P(\eta_3) = (d_1 M_3^2 - cM_3 - m) P(\eta_3) > 0.
\]
It is a contradiction.

(4) For a sufficiently large number $M_4$ satisfying $S'(0) - M_4 S(0) < 0$ and $d_2 M_4^2 - cM_2 - b_2 - a_2 s_1 > 0$, define $W_4(\xi) = S'(\xi) - M_4 S(\xi)$. We claim that $W_4(\xi) < 0$ for any $\xi \geq 0$. The argument is similar to (3), and we omit it.

Proposition 4. Assume $R_0 > 1$ and $c > c_m$, then the traveling wave solution $(P(x + ct), S(x + ct)) \in \Gamma$ of system (2) satisfies $P(\infty) = p_+$, $S(\infty) = s_+$.

Proof. Noting that $(p_+, s_+)$ and $(p_1, s_1)$ satisfy
\[
a_2 s_1^2 + (b_2 + a_2 d) s_1 + b_2 d - \theta m p_+ = 0 \quad \text{(see (3))},
\]
\[a_2 s_1^2 + (b_2 + a_2 d) s_1 + b_2 d - \theta m p_1 = 0 \quad \text{(see Lemma 3.3)},
\]
respectively, and $0 < p_+ < p_1$, we obtain $0 < s_+ < s_1$. Furthermore, we get from (1) in Proposition 3 that $0 < P(\xi) \leq p_1$, $0 < S(\xi) \leq s_1$ on $\xi \in \mathbb{R}$, and thus $\frac{1}{S(0)} - \frac{1}{p_+}$ and $\frac{1}{S(0)} - \frac{1}{s_+}$ could change sign on $\xi \in \mathbb{R}$. Anyway, by using Lemma 3.10, one could show that $V(P, S)(\xi)$ is bounded below on $\mathbb{R}^+$. Without loss of generality, assume $V(P, S)(\xi) \geq M_0$, where $M_0$ is a constant.
Let $\mathcal{V}(\xi) := V(P, S)(\xi)$. Calculating the derivative along the wave profile system \eqref{eq:TW_solution}, we obtain

\[
\frac{d\mathcal{V}(\xi)}{d\xi} = (cP'(\xi) - d_1P''(\xi)) \left( \frac{1}{p_+} - \frac{1}{P(\xi)} \right) - \left( \frac{P'(\xi)}{P(\xi)} \right)^2
+ q \left[ (cS'(\xi) - d_2S''(\xi)) \left( \frac{1}{s_+} - \frac{1}{S(\xi)} \right) - \left( \frac{S'(\xi)}{S(\xi)} \right)^2 \right]
\]

\[
= (-b_1 - a_1P(\xi) + m \frac{d}{S(\xi) + d}) (P(\xi) - p_+) - \frac{a_1}{p_+} (P(\xi) - p_+)^2
+ q \left[ (-b_2 - a_2S(\xi) + \theta m \frac{P(\xi)}{S(\xi) + d}) (S(\xi) - s_+) - \frac{a_2q}{s_+} (S(\xi) - s_+)^2 \right]
\]

\[
= \frac{1}{p_+} (-b_1 - a_1p_+ + m \frac{d}{S(\xi) + d}) (P(\xi) - p_+)
+ \frac{q}{s_+} (-b_2 - a_2s_+ + \theta m \frac{P(\xi)}{S(\xi) + d}) (S(\xi) - s_+)
\]

\[
= T(\xi) - \frac{a_1}{p_+} (P(\xi) - p_+)^2 - \frac{a_2q}{s_+} (S(\xi) - s_+)^2 - \left( \frac{P'(\xi)}{P(\xi)} \right)^2 - q \left( \frac{S'(\xi)}{S(\xi)} \right)^2,
\]

where

\[
T(\xi) = \frac{1}{p_+} (-b_1 - a_1p_+ + m \frac{d}{S(\xi) + d}) (P(\xi) - p_+)
+ \frac{q}{s_+} (-b_2 - a_2s_+ + \theta m \frac{P(\xi)}{S(\xi) + d}) (S(\xi) - s_+)
\]

\[
= \frac{1}{p_+} (-m \frac{d}{s_+ + d} + m \frac{d}{S(\xi) + d}) (P(\xi) - p_+)
+ \frac{q}{s_+} (-\theta m \frac{p_+}{s_+ + d} + \theta m \frac{P(\xi)}{S(\xi) + d}) (S(\xi) - s_+)
\]

\[
= \frac{md}{p_+ (s_+ + d)(S(\xi) + d)} (P(\xi) - p_+)
+ \frac{q\theta m}{s_+} \left( \frac{P(\xi)(s_+ + d) - p_+(S(\xi) + d)}{(S(\xi) + d)(s_+ + d)} \right) (S(\xi) - s_+)
\]

\[
= \left( \frac{q\theta m s_+ + d}{s_+} - \frac{md}{p_+} \right) \frac{(S(\xi) - s_+) (P(\xi) - p_+)}{(s_+ + d)(S(\xi) + d)}
+ \frac{q\theta m}{s_+} \left( \frac{(P(\xi) - p_+)s_+ + p_+(s_+ - S(\xi))}{(S(\xi) + d)(s_+ + d)} \right) (S(\xi) - s_+)
\]

\[
= \left( \frac{q\theta m (s_+ + d)}{s_+} - \frac{md}{p_+} \right) \frac{(S(\xi) - s_+) (P(\xi) - p_+)}{(s_+ + d)(S(\xi) + d)}
- \frac{q\theta m p_+}{s_+} \frac{(S(\xi) - s_+)^2}{(S(\xi) + d)(s_+ + d)}
\]

\[
= \frac{D}{\xi} \left( \frac{V(\xi)}{\xi} \right)
\]
trivial and bounded positive traveling wave solution

\[ \text{Proof.} \]
Assume the equation (2) has a non-trivial and bounded positive traveling wave solution.

\[ \text{Theorem 4.2.} \]

Suppose \( f \) is non-decreasing. Since \( f \) is non-decreasing, we see from the definition of \( V \) that \( \lim\limits_{n \to \infty} V_n(\xi) = V(\xi + \xi_n) \leq V(\xi) \).

Furthermore, there exist a number \( \nu \) such that \( \lim\limits_{n \to \infty} V_n(\xi) = \nu \). According to the Lebesgue dominated convergence theorem, we obtain
\[ V(\infty) = \lim\limits_{n \to \infty} V_n(\xi) = \nu. \]

Thus \( \lim\limits_{n \to \infty} \frac{dV_n(\xi)}{d\xi} = 0 \), and \( P(\infty) = p_+ \), \( S(\infty) = s_+ \), \( P'(\infty) = 0 \) and \( S'(\infty) = 0 \). This implies the traveling wave solution \((P(\xi), S(\xi))\) satisfies the asymptotic boundary conditions.

4. Nonexistence of traveling waves. In this section, the nonexistence of traveling waves is discussed. We first give a lemma called Babálat Lemma (see [8, Lemma 1.2.5]) for the usage of convenience.

\[ \text{Lemma 4.1.} \]
Suppose \( f(\xi) \) is defined on \( [\xi, \infty) \) and \( \lim\limits_{\xi \to +\infty} f(\xi) = \alpha \) with \( |\alpha| < \infty \), and \( f'(\xi) \) is uniformly continuous on \( [\xi, \infty) \). Then \( \lim_{\xi \to +\infty} f'(\xi) = 0 \).

\[ \text{Theorem 4.2.} \]
Assume \( R_0 > 1 \) and \( 0 < c < c_m \), then equation (2) has no non-trivial and bounded positive traveling wave solution \((P(x + ct), S(x + ct))\) satisfies (7).

\[ \text{Proof.} \]
Assume the equation (2) has a non-trivial and bounded positive traveling wave solution \((P(x + ct), S(x + ct))\) satisfies \((P, S)(-\infty) = (p_1, 0), (P, S)(\infty) = (p_+, s_+) \). We have from (2) that \( P''(\xi) \) and \( S''(\xi) \) are uniformly bounded, which leads to \( P'(-\infty) = 0 \) and \( S'(-\infty) = 0 \), according to Babálat Lemma.

Define \( I(\xi) = \int_{-\infty}^{\xi} S(\eta) d\eta \) (\( \xi \in \mathbb{R} \)). Note \( S(\xi) \geq 0 \) for \( \xi \in \mathbb{R} \), thus \( I(\xi) \geq 0 \) is non-decreasing. Since
\[ -b_2 - a_2 S(\xi) + \theta m \frac{P(\xi)}{S(\xi) + d} \to -b_2 + \frac{\theta m p_1}{d} > 0 \quad (\xi \to -\infty), \]
there exists $\hat{\xi} < 0$, such that for any $\xi \leq \hat{\xi}$,
\[
c S'(\xi) = d_2 S''(\xi) + \left(-b_2 - a_2 S(\xi)\right)S(\xi) + \frac{\theta m}{S(\xi) + d} P(\xi) S(\xi)
\]
\[
> d_2 S''(\xi) + \frac{1}{2} \left(\frac{\theta m p_1}{d} - b_2\right) S(\xi).
\]
Integrating two sides of the last inequality from $-\infty$ to $\xi$ ($\xi \leq \hat{\xi}$), we obtain
\[
\frac{1}{2} \left(\frac{\theta m p_1}{d} - b_2\right) I(\xi) < c S(\xi) - d_2 S'(\xi).
\]
Again integrating two sides of (16) from $-\infty$ to $\xi$ ($\xi \leq \hat{\xi}$), we further have
\[
0 < \frac{1}{2} \left(\frac{\theta m p_1}{d} - b_2\right) \int_{-\infty}^{\xi} I(\eta)d\eta < c I(\xi) - d_2 S(\xi) < c I(\xi).
\]
On the other hand, for any $\hat{\xi} > 0$, we have
\[
\int_{-\infty}^{\xi} I(\eta)d\eta = \int_0^{+\infty} I(\xi - \eta)d\eta \geq \int_0^{\hat{\xi}} I(\xi - \eta)d\eta = \hat{\xi} I(\xi - \hat{\xi}).
\]
This together with (17) leads to
\[
\frac{1}{2} \left(\frac{\theta m p_1}{d} - b_2\right) \hat{\xi} I(\xi - \hat{\xi}) \leq c I(\xi).
\]
Let $\hat{\xi}$ be large enough such that $\mu_0 := \frac{1}{\hat{\xi}} \ln \left(\frac{\theta m p_1}{d} - b_2\right) > 0$. Denote $Q(\xi) := I(\xi)e^{-\mu_0 \xi}$, then from (18), we have $Q(\xi - \hat{\xi}) < Q(\xi)$ for any $\xi \leq \hat{\xi}$, which implies the existence of a constant $M > 0$ such that
\[
I(\xi) \leq Me^{\mu_0 \xi} \text{ for any } \xi \leq \hat{\xi}.
\]
Furthermore, for $0 < \mu \leq \mu_0$, (16) and (17) imply
\[
S(\xi) \leq \frac{c}{d_2} I(\xi) \leq \frac{c}{d_2} Me^{\mu_0 \xi} \leq \frac{c}{d_2} Me^{\mu \xi}, \quad S'(\xi) \leq \frac{c}{d_2} S(\xi) \leq \frac{c^2}{d_2^2} Me^{\mu_0 \xi} \leq \frac{c^2}{d_2^2} Me^{\mu \xi}
\]
for $\xi \leq \hat{\xi}$. Since $S(\xi)$ is bounded in $\mathbb{R}$, we can have
\[
\sup_{\xi \in \mathbb{R}} \{ S(\xi)e^{-\mu \xi} \} < \infty, \quad \sup_{\xi \in \mathbb{R}} \{ S'(\xi)e^{-\mu \xi} \} < \infty, \quad \sup_{\xi \in \mathbb{R}} \{ S''(\xi)e^{-\mu \xi} \} < \infty \quad (0 < \mu \leq \mu_0).
\]
We declare that there exists $\mu_0' > 0$ such that
\[
\sup_{\xi \in \mathbb{R}} \left\{ e^{-\mu \xi} \left( a_2 S(\xi) + \frac{\theta m p_1}{d} \right) - \frac{\theta m P(\xi)}{S(\xi) + d} \right\} < \infty \quad (0 < \mu \leq \mu_0').
\]
In fact, integrating the first equation of (6) from $-\infty$ to $\xi$, we obtain
\[
c (P(\xi) - p_1) = d_1 P(\xi) - \int_{-\infty}^{\xi} \left( b_1 + a_1 P(\eta) - \frac{md}{S(\eta) + d} \right) P(\eta)d\eta.
\]
Note
\[ J(\xi) := \int_{-\infty}^{\xi} \left( b_1 + a_1 P(\eta) - \frac{md}{S(\eta) + d} \right) P(\eta) d\eta \]
\[ \leq \int_{-\infty}^{\xi} \left( b_1 + a_1 P(\eta) - \frac{md}{S(\eta) + d} \right) P(\eta) d\eta \]
\[ = \int_{-\infty}^{\xi} \left( m - \frac{md}{S(\eta) + d} \right) P(\eta) d\eta \]
\[ \leq \frac{mp_1}{d} \int_{-\infty}^{\xi} S(\eta) d\eta. \]

Since \( \sup_{\xi \in \mathbb{R}} \{ S(\xi) e^{-\mu_0 \xi} \} < \infty \), there exist a constant \( G > 0 \) such that \( S(\xi) \leq G e^{\mu_0 \xi} \) for \( \xi \in \mathbb{R} \). Thus, \( J(\xi) \leq \frac{mp_1 G e^{\mu_0 \xi}}{d} \) for \( \xi \in \mathbb{R} \). Let \( R(\xi) = p_1 - P(\xi) \), then \( (22) \) can be rewritten as \( d_1 R'(\xi) - cR(\xi) = -J(\xi) \). Multiplying the two sides of \( d_1 R'(\xi) - cR(\xi) = -J(\xi) \) by \( e^{-\mu_0 \xi} \) and integrating it from \( \xi \) to zero, this yields
\[ R(\xi) = R(0) e^{-\mu_0 \xi} + \frac{e^{\frac{\mu_0}{d}}}{d_1} \int_{\xi}^{0} J(\eta) e^{-\mu_0 \eta} d\eta. \]

Due to \( J(\xi) \leq \frac{mp_1 G e^{\mu_0 \xi}}{d} \), we can choose \( \mu_0 = \min\{\mu_0, \frac{\mu_0}{d}\} > 0 \) such that \( \sup_{\xi \in \mathbb{R}} \{ R(\xi) e^{-\mu_0 \xi} \} < \infty \). Furthermore,
\[ \sup_{\xi \in \mathbb{R}} \left\{ e^{-\mu_0 \xi} \left( a_2 S(\xi) + \frac{\theta m p_1}{d} - \frac{\theta m P(\xi)}{S(\xi) + d} \right) \right\} \]
\[ = \sup_{\xi \in \mathbb{R}} \left\{ e^{-\mu_0 \xi} \left( a_2 S(\xi) + \frac{\theta m d R(\xi)}{d S(\xi) + d} + \frac{\theta m p_1}{d} P(\xi) \right) \right\} \]
\[ \leq \sup_{\xi \in \mathbb{R}} \left\{ e^{-\mu_0 \xi} \left( a_2 S(\xi) + \frac{\theta m d}{d^2} (d R(\xi) + p_1 S(\xi)) \right) \right\}, \]
which leads to \( (21) \).

For \( \lambda \in \mathbb{C} \) with \( 0 < \Re \lambda \leq \mu_0 \), define a so-called two-side Laplace transform of \( S(\cdot) \) by
\[ \ell(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} S(\xi) d\xi. \]

Note
\[ \int_{-\infty}^{+\infty} e^{-\lambda \xi} S'(\xi) d\xi = \int_{-\infty}^{+\infty} e^{-\lambda \xi} dS(\xi) = - \int_{-\infty}^{+\infty} S(\xi) d e^{-\lambda \xi} = \lambda \ell(\lambda), \]
\[ \int_{-\infty}^{+\infty} e^{-\lambda \xi} S''(\xi) d\xi = \int_{-\infty}^{+\infty} e^{-\lambda \xi} dS'(\xi) = - \int_{-\infty}^{+\infty} S'(\xi) d e^{-\lambda \xi} = \lambda^2 \ell(\lambda), \]
which yields
\[ \Delta(\lambda, c) \ell(\lambda) = \left( d_2 \lambda^2 - c\lambda + \left( \frac{\theta m p_1}{d} - b_2 \right) \right) \ell(\lambda) \]
\[ = \int_{-\infty}^{+\infty} e^{-\lambda \xi} \left( d_2 S''(\xi) - c S'(\xi) + \left( \frac{\theta m p_1}{d} - b_2 \right) S(\xi) \right) d\xi \]
\[ = \int_{-\infty}^{+\infty} e^{-\lambda \xi} \left( a_2 S(\xi) + \frac{\theta m p_1}{d S(\xi) + d} - \frac{\theta m P(\xi)}{S(\xi) + d} \right) S(\xi) d\xi \]
for any \( \lambda \in \mathbb{C} \) with \( 0 < \Re \lambda < \mu_0 \), where \( \Delta(\lambda, c) \) is defined in \( (9) \). In view of the property of Laplace transform, either \( \ell(\lambda) \) can be analytically continued for any \( \lambda \in \mathbb{C} \) with \( \Re \lambda > 0 \), or there exists a positive number \( \mu_* \), a singular point of \( \ell(\lambda) \), such that \( \ell(\lambda) \) is analytic for any \( \lambda \in \mathbb{C} \) with \( 0 < \Re \lambda < \mu_* \).
If there exists a singular point $\mu_+ < \infty$ such that $\ell(\lambda)$ is analytic just for $0 < \text{Re}\lambda < \mu_+$. Then it must be $\mu_+ > \mu_0$. Choosing $\bar{\mu} > \mu_+$ and $n \in \mathbb{N}$ such that $\bar{\mu} - \frac{n-1}{n} \mu^* < \mu_0$, we have

$$\Delta(\bar{\mu},c)\ell(\bar{\mu}) = \int_{-\infty}^{+\infty} e^{-\bar{\mu} \xi} \left( a_2 S(\xi) + \frac{\theta m p_1}{d} - \frac{\theta m P(\xi)}{S(\xi) + d} \right) S(\xi) d\xi$$

$$\leq \int_{-\infty}^{+\infty} e^{-\mu \xi} e^{-\left(\bar{\mu} - \frac{n-1}{n} \mu^*\right) \xi} \left( a_2 S(\xi) + \frac{\theta m p_1}{d} - \frac{\theta m P(\xi)}{S(\xi) + d} \right) S(\xi) d\xi$$

$$\leq \ell(\frac{\mu_+}{n}) \sup_{\xi \in \mathbb{R}} \left\{ e^{-\left(\bar{\mu} - \frac{n-1}{n} \mu^*\right) \xi} \left( a_2 S(\xi) + \frac{\theta m p_1}{d} - \frac{\theta m P(\xi)}{S(\xi) + d} \right) \right\}.$$ 

Now we have from [21] that $\ell(\bar{\mu})$ is actually analytic with $\bar{\mu} > \mu_+$. It is a contradiction. Therefore $\ell(\lambda)$ can be analytically continued for any $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > 0$.

In fact, [24] can be rewritten as

$$\int_{-\infty}^{+\infty} e^{-\mu \xi} \left( S(\xi) \Delta(\mu, c) - \left( a_2 S(\xi) + \frac{\theta m p_1}{d} - \frac{\theta m P(\xi)}{S(\xi) + d} \right) S(\xi) \right) d\xi = 0,$$

and in view of Lemma 3.1, for $c \in (0, c_m)$, one has $\lim_{\mu \to +\infty} \Delta(\mu, c) = +\infty$. It is a contradiction again. The proof is complete. ☐

**Theorem 4.3.** Assume $R_0 \leq 1$ and $c > 0$, then equation (2) has no non-trivial and bounded positive traveling wave solution $(P(x + ct), S(x + ct))$ satisfies (7).

**Proof.** Assume that the model (2) has a non-trivial and bounded positive traveling wave solution $(P(x + ct), S(x + ct))$ satisfies $(P, S)(-\infty) = (p_1, 0), (P, S)(\infty) = (p_+, s_+)$. Integrating the second equation of (6) from $-\infty$ to $+\infty$, we obtain

$$cs_+ = c \int_{-\infty}^{+\infty} S'(\xi) d\xi = \int_{-\infty}^{+\infty} \left( d_2 S''(\xi) + (-b_2 - a_2 S(\xi)) S(\xi) + \theta m p_1 P(\xi) S(\xi) + d_2 S(\xi) + d_1 S(\xi) \right) d\xi$$

$$= \int_{-\infty}^{+\infty} \left( -b_2 + \theta m p_1 P(\xi) S(\xi) + d_2 S(\xi) \right) S(\xi) d\xi - a_2 \int_{-\infty}^{+\infty} S''(\xi) d\xi$$

$$< \int_{-\infty}^{+\infty} \left( -\theta m p_1 + \theta m p_1 P(\xi) S(\xi) + d_2 S(\xi) \right) S(\xi) d\xi < 0.$$ 

It is a contradiction. The proof is complete. ☐

5. **Concluding discussions.** This article established the existence of traveling wave solutions connecting the nonzero boundary equilibrium $E_1$ to the positive equilibrium $E_+$ for the diffusive producer-scrounger model [2]. It is showed that the existence of traveling wave is determined by a threshold value $R_0$ and a wave speed threshold $c_m$. One may call that $c_m > 0$ is the minimal wave speed although we cannot show the existence of traveling wave solution for $c = c_m$ at the moment. In detail, when $R_0 > 1$ and $c > c_m$, the traveling waves do exist. But in the case $R_0 > 1$ and $c < c_m$ or the case $R_0 \leq 1$, the traveling waves do not exist. However the dynamics in the case $c = c_m$ remains an open problem. It seems that $R_0$ plays a role like the basic reproducing number in SIR epidemic models.

In biology, the existence of traveling wave solutions connecting $E_1$ to $E_+$ implies a spatio-temporal co-existence state between the two species: producer and scrounger, while the scrounger species invade the habitat of producer species. In this sense, the producer species’ high productivity and their weak ability to avoid food to be
stolen, as well as the slow diffusion of the scrounger species are more suitable for their co-existence.

**Appendix.** In this appendix, we shall give all proofs of Lemma 3.2-3.5.

**Proof of Lemma 3.2.** \( \overline{P}(\xi) = p_1, \overline{P}'(\xi) = \overline{P}''(\xi) = 0, \) and then
\[
d_1 \overline{P}''(\xi) - c \overline{P}'(\xi) + (-b_1 - a_1 \overline{P}(\xi)) \overline{P}(\xi) + m \frac{d \overline{P}(\xi)}{\overline{S}(\xi) + d}
= (-b_1 - a_1 p_1) p_1 + m \frac{d p_1}{\overline{S}(\xi) + d} \leq (-b_1 - a_1 p_1) p_1 + mp_1 = 0.
\]

**Proof of Lemma 3.3.** If \( \xi > \xi_1 \) and \( \overline{S}(\xi) = s_1 \), then \( \overline{S}'(\xi) = \overline{S}''(\xi) = 0 \), and
\[
d_2 \overline{S}''(\xi) - c \overline{S}'(\xi) + (-b_2 - a_2 \overline{S}(\xi)) \overline{S}(\xi) + \theta m \frac{\overline{P}(\xi) \overline{S}(\xi)}{\overline{S}(\xi) + d}
= \frac{s_1}{s_1 + d} \left[-(b_2 + a_2 s_1)(s_1 + d) + \theta mp_1\right]
= - \frac{s_1}{s_1 + d} \left[a_2 s_1^2 + (b_2 + a_2 d)s_1 + b_2 d - \theta mp_1\right] = 0.
\]
If \( \xi < \xi_1 \) and \( \overline{S}(\xi) = e^{\lambda_1 \xi}, \overline{S}'(\xi) = \lambda_1 e^{\lambda_1 \xi}, \overline{S}''(\xi) = \lambda_2^2 e^{\lambda_1 \xi} \), and then
\[
d_2 \overline{S}''(\xi) - c \overline{S}'(\xi) + (-b_2 - a_2 \overline{S}(\xi)) \overline{S}(\xi) + \theta m \frac{\overline{P}(\xi) \overline{S}(\xi)}{\overline{S}(\xi) + d}
\leq d_2 \overline{S}''(\xi) - c \overline{S}'(\xi) - b_2 \overline{S}(\xi) + \theta m \frac{\overline{P}(\xi) \overline{S}(\xi)}{d} = e^{\lambda_1} \Delta(\lambda_1, c) = 0.
\]

**Proof of Lemma 3.4.** If \( \xi > \xi_2 \) and \( \overline{P}(\xi) = 0 \), the inequality holds. If \( \xi < \xi_2 \), then \( \overline{P}(\xi) = p_1 - \sigma e^{\alpha_\xi}, \overline{P}'(\xi) = -\alpha \sigma e^{\alpha_\xi}, \overline{P}''(\xi) = -\alpha^2 \sigma e^{\alpha_\xi} \). Note that for any \( \xi \in \mathbb{R} \), we have \( \overline{S}(\xi) \leq e^{\lambda_1 \xi} \). Therefore,
\[
d_1 \overline{P}''(\xi) - c \overline{P}'(\xi) + (-b_1 - a_1 \overline{P}(\xi)) \overline{P}(\xi) + m \frac{d \overline{P}(\xi)}{\overline{S}(\xi) + d}
\geq -d_1 \alpha^2 \sigma e^{\alpha_\xi} + c \alpha \sigma e^{\alpha_\xi} + (p_1 - \sigma e^{\alpha_\xi}) \left[-b_1 - a_1 (p_1 - \sigma e^{\alpha_\xi}) + m \frac{\sigma e^{\alpha_\xi}}{e^{\lambda_1 \xi} + d}\right]
= -d_1 \alpha^2 \sigma e^{\alpha_\xi} + c \alpha \sigma e^{\alpha_\xi} + (p_1 - \sigma e^{\alpha_\xi}) \left[-m + a_1 \sigma e^{\alpha_\xi} + m \frac{\sigma e^{\alpha_\xi}}{e^{\lambda_1 \xi} + d}\right]
= -d_1 \alpha^2 \sigma e^{\alpha_\xi} + c \alpha \sigma e^{\alpha_\xi} + (p_1 - \sigma e^{\alpha_\xi}) \left[a_1 \sigma e^{\alpha_\xi} - m \frac{e^{\lambda_1 \xi}}{e^{\lambda_1 \xi} + d}\right].
\]
Since \( 0 < p_1 - \sigma e^{\alpha_\xi} < p_1 \), we have \( 0 < \sigma e^{\alpha_\xi} < p_1 \). Let \( \tilde{p} = \sigma e^{\alpha_\xi} \), and
\[
f(\tilde{p}) := (p_1 - \sigma e^{\alpha_\xi})(a_1 \sigma e^{\alpha_\xi} - m \frac{e^{\lambda_1 \xi}}{e^{\lambda_1 \xi} + d})
= (p_1 - \tilde{p})(a_1 \tilde{p} - m \frac{e^{\lambda_1 \xi}}{e^{\lambda_1 \xi} + d}).
\]
f(\tilde{p}) is a quadratic function with the variable \( \tilde{p} \in (0, p_1) \). Note that \( f(\tilde{p}) \) has two positive zero point \( p_1 \) and \( \frac{m}{a_1 \lambda_1 \xi + a} \). If \( p_1 \leq \frac{m}{a_1 \lambda_1 \xi + a} \), then \( f(\tilde{p}) \) increases in \( (0, p_1) \).
Proof of Lemma 3.5. That is, if \( \sigma > p \) is sufficiently large, we have

\[
\inf_{\tilde{p} \in (0, p_1)} f(\tilde{p}) = f(0) = -m \frac{p_1 e^{\lambda_1 \xi}}{\lambda_1 \xi + d}.
\]

That is,

\[
(p_1 - \sigma e^{\alpha \xi}) \left( a_1 \sigma e^{\alpha \xi} - m \frac{e^{\lambda_1 \xi}}{e^{\lambda_1 \xi} + d} \right) \geq -m p_1 e^{\lambda_1 \xi}.
\]

Therefore,

\[
d_1 P''(\xi) - c P'(\xi) + \left( -b_1 - a_1 P(\xi) \right) P(\xi) + m \frac{dP(\xi)}{S(\xi) + d} \geq \sigma (-d\alpha^2 + \alpha) - \frac{mp_1 e^{(\lambda_1 - \alpha) \xi}}{d}.
\]

For \( \sigma > p_1, \alpha \in (0, \min\{\lambda_1, \xi\}) \), we have \( \xi < \xi_2 < 0, e^{(\lambda_1 - \alpha) \xi} < 1 \) and \( -d\alpha^2 + \alpha > 0 \). It means that when \( \sigma \) is sufficiently large, we have \( \sigma (-d\alpha^2 + \alpha) - \frac{mp_1 e^{(\lambda_1 - \alpha) \xi}}{d} > 0 \). The proof is complete. \( \square \)

Proof of Lemma 3.6. Note that if \( \eta \in (0, \min\{\lambda_1, \alpha, \lambda_2 - \lambda_1\}) \), we have \( \Delta(\lambda_1 + \eta, c) < 0 \). Now, we discuss it by three cases.

1. When \( S(\xi) = 0 \), the inequality holds.
2. If \( S(\xi) = e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \) and \( P(\xi) = 0 \), that is \( \xi_2 < \xi < \xi_1, S'(\xi) = \lambda_1 e^{\lambda_1 \xi} - q(\lambda_1 + \eta) e^{(\lambda_1 + \eta) \xi}, S''(\xi) = \lambda_1^2 e^{\lambda_1 \xi} - q(\lambda_1 + \eta)^2 e^{(\lambda_1 + \eta) \xi} \), and \( 0 < S(\xi) \leq e^{\lambda_1 \xi} \), then

\[
d_2 S''(\xi) - c S'(\xi) + (-b_2 - a_2 S(\xi)) S(\xi) + \theta m \frac{P(\xi) S(\xi)}{S(\xi) + d} =
\]

\[
= d_2 \left( \lambda_1^2 e^{\lambda_1 \xi} - q(\lambda_1 + \eta)^2 e^{(\lambda_1 + \eta) \xi} \right) - c \left( \lambda_1 e^{\lambda_1 \xi} - q(\lambda_1 + \eta) e^{(\lambda_1 + \eta) \xi} \right)
\]

\[
+ \left( -b_2 - a_2 \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right) \right) \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)
\]

\[
= e^{\lambda_1 \xi} \Delta(\lambda_1 + \eta, c) - q e^{(\lambda_1 + \eta) \xi} \Delta(\lambda_1 + \eta, c) - a_2 \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)^2
\]

\[
- \frac{\theta m (m - b_1)}{da_1} \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)
\]

\[
\geq - q e^{(\lambda_1 + \eta) \xi} \Delta(\lambda_1 + \eta, c) - a_2 e^{2\lambda_1 \xi} - \frac{\theta m p_1}{d} e^{\lambda_1 \xi}
\]

\[
= e^{(\lambda_1 + \eta) \xi} \left( -q \Delta(\lambda_1 + \eta, c) - a_2 e^{(\lambda_1 - \eta) \xi} - \frac{\theta m p_1}{d} e^{-\eta \xi} \right).
\]

For \( \xi_2 < \xi < \xi_3, a_2 e^{(\lambda_1 - \eta) \xi} + \frac{\theta m (m - b_1)}{da_1} e^{-\eta \xi} \) is bounded. Let \( q \) be sufficiently large, we have

\[
-q \Delta(\lambda_1 + \eta, c) - a_2 e^{(\lambda_1 - \eta) \xi} - \frac{\theta m p_1}{d} e^{-\eta \xi} \geq 0 \quad \text{as} \quad \Delta(\lambda_1 + \eta, c) < 0.
\]
(3) If $S(\xi) = e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi}$ and $P(\xi) = p_1 - \sigma e^{\alpha \xi}$, it means $\xi < \min\{\xi_2, \xi_3\} < 0$, $0 < S(\xi) \leq e^{\lambda_1 \xi}$, and then

$$d_2 S''(\xi) - c S'(\xi) + (-b_2 - a_2 S(\xi)) S(\xi) + \theta m \frac{P(\xi) S(\xi)}{S(\xi) + d}$$

$$= e^{\lambda_1 \xi} \Delta(\lambda_1, c) - q e^{(\lambda_1 + \eta) \xi} \Delta(\lambda_1 + \eta, c) - a_2 \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)^2$$

$$- \frac{\theta m (m - b_1)}{da_1} \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right) + \theta m \left( p_1 - \sigma e^{\alpha \xi} \right) \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)$$

$$= - q e^{(\lambda_1 + \eta) \xi} \Delta(\lambda_1 + \eta, c) - a_2 \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)^2$$

$$- \frac{\theta m p_1}{d} \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right) + \theta m \left( p_1 - \sigma e^{\alpha \xi} \right) \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)$$

$$= - q e^{(\lambda_1 + \eta) \xi} \Delta(\lambda_1 + \eta, c) - a_2 \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)^2 - \frac{\theta m \sigma e^{\alpha \xi} \left( e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} \right)}{e^{\lambda_1 \xi} - q e^{(\lambda_1 + \eta) \xi} + d}$$

$$\geq - q e^{(\lambda_1 + \eta) \xi} \Delta(\lambda_1 + \eta, c) - a_2 e^{2\lambda_1 \xi} - \frac{\theta m \sigma e^{\alpha \xi} e^{\lambda_1 \xi}}{e^{\lambda_1 \xi} + d} - \frac{\theta m p_1}{d} e^{2\lambda_1 \xi}$$

$$\geq - q e^{(\lambda_1 + \eta) \xi} \Delta(\lambda_1 + \eta, c) - a_2 e^{2\lambda_1 \xi} - \frac{\theta m \sigma e^{(\alpha + \lambda_1) \xi}}{e^{(\alpha + \lambda_1) \xi} - \frac{\theta m p_1}{d^2} e^{2\lambda_1 \xi}}$$

$$= e^{(\lambda_1 + \eta) \xi} \left( -q \Delta(\lambda_1 + \eta, c) - a_2 e^{(\lambda_1 - \eta) \xi} - \frac{\theta m \sigma}{d} e^{(\alpha - \eta) \xi} - \frac{\theta m p_1}{d^2} e^{(\lambda_1 - \eta) \xi} \right).$$

For $\eta \in (0, \min\{\lambda_1, \alpha, \lambda_2 - \lambda_1\})$ and $\xi < \min\{\xi_2, \xi_3\}$, we have

$$-a_2 e^{(\lambda_1 - \eta) \xi} - \frac{\theta m \sigma}{d} e^{(\alpha - \eta) \xi} - \frac{\theta m p_1}{d^2} e^{(\lambda_1 - \eta) \xi} > -a_2 - \frac{\theta m \sigma}{d} - \frac{\theta m p_1}{d^2}.$$

Therefore, if $q$ is sufficiently large, we have

$$-q \Delta(\lambda_1 + \eta, c) - a_2 e^{(\lambda_1 - \eta) \xi} - \frac{\theta m \sigma}{d} e^{(\alpha - \eta) \xi} - \frac{\theta m p_1}{d^2} e^{(\lambda_1 - \eta) \xi} \geq 0.$$

The proof is complete. \hfill \Box

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E-mail address: wen_jh@sznu.edu.cn
E-mail address: wengpx@scnu.edu.cn