Density Perturbations in Multifield Inflationary Models

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Abstract

We derive a closed-form, analytical expression for the spectrum of long-wavelength density perturbations in inflationary models with two (or more) inflaton degrees of freedom that is valid in the slow-roll approximation. We illustrate several classes of potentials for which this expression reduces to a simple, algebraic expression.
I. INTRODUCTION

Inflationary cosmology, in addition to resolving the cosmological flatness and horizon problems of the standard big bang model, predicts a remnant spectrum of density perturbations. The perturbations may have seeded large-scale structure formation and may have left an imprint on the cosmic microwave background (CMB) anisotropy. Therefore, considerable attention has been given to precise derivations of the perturbation spectrum when a single scalar (inflaton) field drives inflation and simultaneously this field is responsible for origin of initial inhomogeneities. Recently, we have shown how different methods of computing the perturbation are related and compared their accuracy. These results have been applied to obtain predictions of CMB anisotropy and large-scale structure.

In this paper, we derive the explicit, analytical solution for the perturbations in the models with two (or more) inflaton degrees of freedom in a case of general potential for the scalar fields. To obtain this expression, we must assume that the slow-roll approximation for the evolution of the scalar fields is valid. With two or more inflaton fields, the perturbation spectrum has additional contributions which do not occur in the single-field case; in particular, the single field spectrum consists purely of adiabatic fluctuations whereas the multi-field spectrum generically has an entropic contribution as well, as discussed for some specific cases previously. Previously, a formalism has been developed for cases in which the two fields are not coupled or have very simple couplings. However, in more complicated cases, only heuristic arguments based on single-field inflation have been applied. Yet, the single-field case is unusual because the evolving expectation value of the one field serves as a clock that determines when the universe exits inflation and returns to Friedmann-Robertson-Walker expansion. This was the basis, for example, of the time-delay formalism introduced by Guth and Pi for computing the perturbation spectrum for single-inflaton models. With two or more fields, fluctuations in one field can affect the evolution of the other field, and the complex conditions under which inflation ends cannot be expressed
in terms of one degree of freedom (e.g., some linear combination of fields). Consequently, the heuristic arguments are suspect. The issue has become more important in recent years because intriguing new models of inflation have been proposed which entail two or more inflaton degrees of freedom: e.g., extended and hyperextended inflation, hybrid inflation, and supernatural inflation.

For these reasons, it has become essential to have a formalism that applies to the multi-field case. Below we develop a procedure based on the natural generalization for the single-field case. We assume the slow-roll approximation in which the inflation kinetic energy is negligible compared to its potential energy during inflation. We first review the single-field case, then a simple case with two decoupled fields, and finally we solve the equations for perturbations in the general case. The central result is in Section IVb, Eqs. (46) through (49), the closed-form expressions for the perturbation spectrum. We present a series of potential forms for which the closed-form expressions reduce to simple algebraic expressions.

II. PERTURBATIONS IN SINGLE-FIELD INFLATION

We consider first the case of a single scalar field $\phi$ with potential $V(\phi)$. Assuming the spatial part of the energy-momentum tensor is diagonal, the metric in longitudinal gauge is:

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(1 - 2\Phi)\delta_{ik}dx^i dx^k,$$

where $a$ is the Robertson-Walker scale factor and $\Phi$ is the gravitational potential. The perturbed Klein-Gordon equation which describes the evolution of perturbations in $\phi$ is (in units with $\hbar = 4\pi G = c = 1$):

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a^2}\nabla^2\delta\phi + V''\delta\phi - 4\dot{\phi}\dot{\Phi} + 2V'\Phi = 0,$$

and the perturbed $0 - i$ Einstein equation is

$$\dot{\Phi} + H\Phi = \dot{\phi}_0\delta\phi,$$
where \( \phi = \phi_0 + \delta \phi \), dot means \( \partial / \partial t \), and prime is used for \( \partial / \partial \phi \). Throughout we use dimensionless units where \( 4\pi G = 1 \), where \( G \) is Newton’s constant.

To find the nondecaying solution for the long-wavelength inhomogeneities (for which the spatial derivatives in the equations of motion can be neglected) in slowroll approximation, the terms proportional to \( \dot{\Phi} \) or depending on second derivatives in \( \delta \phi \) can be dropped, resulting in the simplified equations

\[
3H\ddot{\phi} + V''\delta \phi + 2V'\Phi = 0 \quad (4)
\]

\[
H\Phi = \dot{\phi}_0 \delta \phi. \quad (5)
\]

The (unperturbed) background equations are:

\[
3H\dot{\phi}_0 = -V' \\
H^2 = \frac{2}{3}V \\
\dot{H} = -\dot{\phi}_0^2 \quad (6)
\]

where the slow-roll approximation has been assumed in dropping \( \ddot{\phi}_0 \) in the first equation and \( \dot{\phi}_0^2 \) in the second.

If we introduce a new variable, \( x \equiv \delta \phi / V' \), the perturbative equations reduce to:

\[
3H\dot{x} = -2\Phi \\
\Phi = \frac{\dot{V}}{H} x. \quad (7)
\]

Substituting the second expression into the first and using the background equation to express \( H \) in terms of \( V \), we obtain

\[
\dot{x} = -\frac{\dot{V}}{V} x, \quad (8)
\]

whose solution is

\[
x = \frac{C}{V}, \quad (9)
\]
where \( C \) is an integration constant. Therefore, using the definition of \( x \) and Eq. (3), we obtain

\[
\delta \phi = V' x = C \frac{V'}{V} = -2C \frac{\dot{\phi}_0}{H}.
\]

To fix the integration constant, we use the quantum de Sitter fluctuation result that \( \delta \phi_k \sim H \) evaluated at horizon-crossing, \( k = aH \). Solving for \( C \) above, we obtain

\[
C \approx \left( -H \frac{\delta \phi}{2\phi} \right)_{k=aH}.
\]

According to Eq. (7), \( \Phi = \frac{\dot{V}}{3V} \). Since our solution is \( x = C/V \), we obtain

\[
\Phi = C \frac{\dot{V}}{HV} = 2C \frac{\dot{H}}{H^2}.
\]

The last expression is obtained by noting \( H^2 = \frac{2}{3}V \) and \( \dot{H}/H = 2\dot{V}/V \) where, applying the slow-roll approximation, we have ignored the inflaton kinetic energy contribution to total energy density during inflation. At the end of inflation, \( \dot{H} = \mathcal{O}(1)H^2 \), which means that

\[
\Phi_f \approx \mathcal{O}(1)C \approx \mathcal{O}(1) \left( H \frac{\delta \phi}{\phi} \right)_{k=aH},
\]

the standard, lowest-order result. The fluctuations described in this relation are adiabatic. For the cosmic microwave background (CMB) anisotropy in large angular scales, the gravitational potential sets the CMB temperature fluctuations \( \delta T/T \approx \Phi/3 \).

### III. TWO DECOUPLED FIELDS

Next, consider the case of two decoupled fields \( \phi_1 \) and \( \phi_2 \) with potential \( V(\phi_1, \phi_2) = V_1(\phi_1) + V_2(\phi_2) \). This case has been considered previously. In this case, the Klein-Gordon equations for the fields in the slow-roll, long-wavelength approximation takes the form:

\[
3H\dot{\delta \phi}_1 + V''_1\delta \phi_1 + 2V'_1\Phi = 0
\]

\[
3H\dot{\delta \phi}_2 + V''_2\delta \phi_2 + 2V'_2\Phi = 0
\]
and

\[ H \Phi = \dot{\phi}_1 \delta \phi_1 + \dot{\phi}_2 \delta \phi_2, \quad (15) \]

where \( V'_1 \equiv \partial V_1 / \partial \phi_1 \) and \( V'_2 \equiv \partial V_2 / \partial \phi_2 \).

The (unperturbed) background equations are:

\[
\begin{align*}
3H \dot{\phi}_1 &= -V'_1 \\
3H \dot{\phi}_2 &= -V'_2 \\
H^2 &= \frac{2}{3} (V_1 + V_2) \\
\dot{H} &= -(\dot{\phi}_1^2 + \dot{\phi}_2^2)
\end{align*}
\]

\( (16) \)

Similar to the single-field case, we introduce two new variables \( x_1 \) and \( x_2 \) via

\[
\delta \phi_1 = V'_1 x_1 \quad \text{and} \quad \delta \phi_2 = V'_2 x_2.
\]

\( (17) \)

Then, the system of perturbed equations, Eq. (14-15), can be rewritten as

\[
\begin{align*}
3H \dot{x}_1 &= -2\Phi \\
3H \dot{x}_2 &= -2\Phi
\end{align*}
\]

\( (18) \hspace{1cm} (19) \)

\[
\Phi = \frac{1}{H} (\dot{V}_1 x_1 + \dot{V}_2 x_2).
\]

\( (20) \)

Subtracting Eq. (19) from Eq. (18), we obtain \( \dot{x}_1 - \dot{x}_2 = 0 \), or

\[
x_2 = x_1 + D,
\]

\( (21) \)

where \( D \) is an integration constant. Substituting this into Eq (20), one gets that

\[
\Phi = \frac{1}{H} [(\dot{V}_1 + \dot{V}_2) x_1 + \dot{D} V_2].
\]

\( (22) \)

Using this expression to replace the right-hand-side of Eq. (18), we then obtain the closed form equation for \( x_1 \):
\[ \dot{x}_1 = \frac{\dot{V}_1 + \dot{V}_2}{V_1 + V_2} x_1 - \frac{D\dot{V}_2}{V_1 + V_2} \]  
(23)

(where the background equation expressing \( H^2 \) in terms of \( V_1 \) and \( V_2 \) has been used). This equation may be rewritten as

\[ \frac{\partial}{\partial t}(x_1(V_1 + V_2) + DV_2) = 0, \]  
(24)

and then can be integrated to obtain

\[ x_1 = \frac{C - DV_2}{V_1 + V_2}, \]  
(25)

where \( C \) is the integration constant. The corresponding result for \( x_2 \) is

\[ x_2 = \frac{C + DV_1}{V_1 + V_2}. \]  
(26)

The integration constants \( C \) and \( D \) should be fixed by assuming that \( \delta \phi_{1,2} \sim H \) at horizon crossing during inflation, \( k = aH \). Using the relation between \( x_{1,2} \) and \( \delta \phi_{1,2} \), we obtain

\[ D = \left[ \frac{1}{3H} \left( \frac{\delta \phi_1}{\dot{\phi}_1} - \frac{\delta \phi_2}{\dot{\phi}_2} \right) \right]_{k = aH} \]

\[ C = -\frac{1}{2} \left[ H \left( \frac{V_1}{V_1 + V_2} \frac{\delta \phi_1}{\dot{\phi}_1} + \frac{V_2}{V_1 + V_2} \frac{\delta \phi_2}{\dot{\phi}_2} \right) \right]_{k = aH} \]  
(27)

Also, substituting Eq. (23) into the expression for \( \Phi \), Eq. (22), we derive

\[ \Phi = 2C \frac{\dot{H}}{H^2} + D \frac{1}{H} \frac{V_1 \dot{V}_2 - \dot{V}_1 V_2}{V_1 + V_2} \]  
(28)

The first term above is the same as the total contribution in the single field case (see Eq. (12)); as in the single-field case, it can be interpreted as the adiabatic contribution to the fluctuation spectrum. The second term is a new contribution due to entropic fluctuations which arises whenever there are two or more inflaton fields. The entropic contribution arises because two or more components of the cosmic fluid are undergoing different fluctuations and evolution. Hence, the entropic contribution is a characteristic feature of multicomponent inflation models.\[ \square \]
IV. GENERAL CASE

We now generalize our method to models with two scalar fields in which the kinetic energy terms have non-linear sigma-model form and the potential $V(\phi, \psi)$ is an arbitrary function of $\phi, \psi$:

$$S = \int \left[ \frac{1}{2} f_1(\phi, \psi) \phi_{,\alpha} \phi^{,\alpha} + \frac{1}{2} f_2(\phi, \psi) \psi_{,\alpha} \psi^{,\alpha} - V(\phi, \psi) \right] \sqrt{-g} d^4x \quad (29)$$

Kinetic terms of this type occur, for example, in supergravity models with non-trivial Kahler potential.

A. Derivation

Variation of this action with respect to $\phi, \psi$ leads to the equations

$$\left( f_1 \phi^{,\alpha} \right)_{,\alpha} - \frac{1}{2} f_1 \phi_{,\alpha} \phi^{,\alpha} - \frac{1}{2} f_2 \psi_{,\alpha} \psi^{,\alpha} + V_{,\phi} = 0 \quad (30)$$

$$\left( f_2 \psi^{,\alpha} \right)_{,\alpha} - \frac{1}{2} f_1 \phi_{,\alpha} \phi^{,\alpha} - \frac{1}{2} f_2 \psi_{,\alpha} \psi^{,\alpha} + V_{,\psi} = 0 \quad (31)$$

We consider a Friedmann-Robertson-Walker Universe with small perturbations

$$ds^2 = (1 + 2\Phi) dt^2 - a^2 (1 - 2\Phi) \delta_{ik} dx^i dx^k \quad (32)$$

and decompose fields into homogeneous and inhomogeneous components

$$\phi = \phi_0(t) + \delta\phi(x,t), \psi = \psi_0 + \delta\psi \quad (33)$$

The equation for the background then take the form

$$\left( \ddot{\phi}_0 + 3H \dot{\phi}_0 \right) f_1 + \dot{f}_1 \dot{\phi}_0 - \frac{1}{2} f_{1,\phi} \phi_0^2 - \frac{1}{2} f_{2,\phi} \psi_0^2 + V_{,\phi} = 0 \quad (34)$$

Here all functions depend only on the background variables. The equation for $\psi$ can be obtained from the above one if we make the following substitutions: $f_1 \leftrightarrow f_2; \phi \leftrightarrow \psi$.  

The first linearized equation for perturbations is

\[ f_1 (\delta \ddot{\phi} + 3H \delta \dot{\phi} - \frac{1}{a^2} \nabla^2 \delta \phi - 4 \dot{\Phi} \dot{\phi}_0) + 2\Phi V,_{\phi} + f_{1,\phi} \dot{\phi}_0 \delta \dot{\phi} - f_{2,\phi} \dot{\psi}_0 \delta \dot{\psi} + f_{1,\psi} \left( \phi_0 \delta \dot{\psi} + \dot{\psi}_0 \delta \dot{\phi} \right) \]

\[ \left[ f_{1,\phi} \left( \dot{\phi}_0 + 3H \dot{\phi}_0 \right) + \frac{1}{2} f_{1,\phi \phi} \dot{\phi}_0^2 - \frac{1}{2} f_{2,\phi \phi} \dot{\psi}_0^2 + f_{1,\phi \psi} \dot{\phi}_0 \dot{\psi}_0 + V,_{\phi} \right] \delta \phi + \]

\[ \left[ f_{1,\psi} \left( \dot{\phi}_0 + 3H \dot{\phi}_0 \right) + \frac{1}{2} f_{1,\psi \psi} \dot{\phi}_0^2 - \frac{1}{2} f_{2,\phi \psi} \dot{\psi}_0^2 + f_{1,\phi \psi} \dot{\phi}_0 \dot{\psi}_0 + V,_{\psi} \right] \delta \psi = 0 \]  (35)

The second equation is obtained by the above mentioned substitution. Because we have three unknown functions, we need a third relation. It is convenient to choose the 0-i Einstein equation:

\[ \dot{\Phi} + H \Phi = f_1 \dot{\phi}_0 \delta \phi + f_2 \dot{\psi}_0 \delta \psi \]  (36)

The above equations can be simplified and integrated explicitly for the longwave perturbations in slowroll approximation. In this approximation the equations for the background simplifies to

\[ 3H f_1 \dot{\phi}_0 = -V,_{\phi} \]

\[ 3H f_2 \dot{\psi}_0 = -V,_{\psi} \]

\[ H^2 = \frac{2}{3} V \]  (37)

and the equations for perturbations take the form

\[ 3H \delta \ddot{\phi} + \left( \frac{V,_{\phi}}{f_1} \right)_{,\phi} \delta \phi + \left( \frac{V,_{\psi}}{f_1} \right)_{,\psi} \delta \psi + 2\Phi \frac{V,_{\phi}}{f_1} = 0 \]

\[ 3H \delta \ddot{\psi} + \left( \frac{V,_{\psi}}{f_2} \right)_{,\phi} \delta \phi + \left( \frac{V,_{\phi}}{f_2} \right)_{,\psi} \delta \psi + 2\Phi \frac{V,_{\psi}}{f_2} = 0 \]

\[ H \Phi = f_1 \dot{\phi}_0 \delta \phi + f_2 \dot{\psi}_0 \delta \psi \]  (38)
Let us define the variables $x$ and $y$ as
\[ \delta \phi = \frac{V_{\phi}}{f_1} x \quad \text{and} \quad \delta \psi = \frac{V_{\psi}}{f_2} y. \tag{39} \]
In terms of these variables, the above equations-of-motion for the perturbations become
\[ 3H \dot{x} + \frac{(V_{\phi}/f_1)_{,\psi}(V_{\psi}/f_2)}{(V_{\phi}/f_1)} (y - x) + 2\Phi = 0 \]
\[ 3H \dot{y} + \frac{(V_{\psi}/f_2)_{,\phi}(V_{\phi}/f_1)}{(V_{\psi}/f_2)} (x - y) + 2\Phi = 0 \tag{40} \]
These equations are very similar to Eqs. (18) and (19) for the decoupled case, except that there is here the middle term which vanishes if $V = V_1(\phi) + V_2(\psi)$. Nevertheless, we can obtain a closed-form solution. Subtracting the two equations, we obtain
\[ 3H (\dot{y} - \dot{x}) = \left( \frac{(V_{\phi}/f_1)_{,\psi}(V_{\psi}/f_2)}{(V_{\phi}/f_1)} + \frac{(V_{\psi}/f_2)_{,\phi}(V_{\phi}/f_1)}{(V_{\psi}/f_2)} \right) (y - x) \tag{41} \]
which can be integrated
\[ y - x = \gamma \exp \left\{ \int \left[ \frac{(V_{\phi}/f_1)_{,\psi}(V_{\psi}/f_2)}{(V_{\phi}/f_1)} + \frac{(V_{\psi}/f_2)_{,\phi}(V_{\phi}/f_1)}{(V_{\psi}/f_2)} \right] \frac{dt}{3H} \right\}, \tag{42} \]
where $\gamma$ is a constant of integration. Using the $0 - i$ Einstein equation, $\Phi$ can be expressed in terms of $x$
\[ \Phi = \frac{1}{H} \left( f_1 \dot{\phi} \delta \phi + f_2 \dot{\psi} \delta \psi \right) = \frac{1}{H} \left( V_{\phi} \dot{x} + V_{\psi} \dot{y} \right) = \frac{1}{H} \left( \dot{V} x + V_{\psi} \dot{\psi} (y - x) \right) \tag{43} \]
Eqs. (40) and (42) can then be used to find an integral expression for $x$:
\[ x = -\frac{\gamma}{V} \int \left[ \frac{H (V_{\phi}/f_1)_{,\psi}(V_{\psi}/f_2)}{2 (V_{\phi}/f_1)} + V_{\psi} \dot{\psi} \right] \frac{F}{V} dt \tag{44} \]
where
\[ F = V \exp \left\{ \int \left[ \frac{(V_{\phi}/f_1)_{,\psi}(V_{\psi}/f_2)}{(V_{\phi}/f_1)} + \frac{(V_{\psi}/f_2)_{,\phi}(V_{\phi}/f_1)}{(V_{\psi}/f_2)} \right] \frac{dt}{3H} \right\} \tag{45} \]
Similar expressions can be obtained for $y$. Using the definitions of $x$ and $y$ (Eq. (39)) and the equations for the background (37), the expressions can be simplified to the final, compact, closed forms given in the next section.
B. A General, Closed-form Expression

The following closed-form expressions are a compact, general representation of the perturbations with wavenumber $k$ in two-field models in the slow-roll approximation, the central result of this paper. The perturbed variables $\delta \phi$, $\delta \psi$, and $\Phi$ are $k$-dependent, but the subscript $k$ has been suppressed for simplicity.

$$\delta \phi(t) = \gamma \left( \ln \frac{(\ln V)_\phi}{f_1} \right) Fd\psi \left( \ln \frac{(\ln V)_\phi}{f_1} \right) + \alpha \left( \ln \frac{(\ln V)_\phi}{f_1} \right) \quad (46)$$

$$\delta \psi(t) = -\gamma \left( \ln \frac{(\ln V)_\psi}{f_2} \right) \int_{t_0}^t \left( \ln \frac{(\ln V)_\psi}{f_2} \right) Fd\phi \left( \ln \frac{(\ln V)_\psi}{f_2} \right) + \beta \left( \ln \frac{(\ln V)_\psi}{f_2} \right) \quad (47)$$

where

$$F(t) = \exp \left\{ -\int_{t_0}^t \left[ \left( \ln \frac{(\ln V)_\psi}{f_2} \right) d\phi + \left( \ln \frac{(\ln V)_\phi}{f_1} \right) d\psi \right] \right\} \quad (48)$$

(In re-expressing $F$, an overall constant has been removed and absorbed into the definition of $\gamma$.) Here $t_0$ is an arbitrary moment of time; we take $t_0$ to be the moment of horizon crossing $t_k$ when $k = aH$ for the given mode $k$. The limits of integration indicate that the integration variable is to be evaluated at the time $t_0$ and $t$.

There are three integrations constants, $\alpha$, $\beta$ and $\gamma$. Evaluating these expressions at horizon-crossing and using the fact that $\delta \phi \sim \delta \psi \sim H$ at horizon-crossing, the integration constants $\alpha$ and $\beta$ can be determined in terms of the inflaton potential and derivatives during inflation. The third integration constant, $\gamma$, must be taken in such a way as to satisfy Eq. (42), from where it follows that $\beta = \alpha + \gamma$. (One must make use of the definitions of $x$ and $y$ in Eq. (39) and the fact that, at horizon-crossing, the integral on the right-hand-side of Eq. (42) is $1/V_{k=aH}$. ) The value of $\gamma$ corresponds physically to the amplitude of the entropy perturbations. The gravitational potential is expressed in terms of $\delta \phi$ and $\delta \psi$ as

$$\Phi = -\frac{1}{2} \left[ (\ln V)_{,\phi} \delta \phi + (\ln V)_{,\psi} \delta \psi \right] \quad (49)$$
The above formulae can also be rewritten in other forms which can be useful for approximate evaluation of the integrals. For instance, in the standard case where the kinetic energy terms are canonical \((f_1 = f_2 = 1)\), the expressions above reduce to:

\[
\delta \phi = -\gamma \left( \ln V \right)_{,\phi} \int_{t_0}^{t} \frac{V_{,\phi}^2}{V_{,\psi}^2 + V_{,\phi}^2} dF + \alpha \left( \ln V \right)_{,\phi} \\
\delta \psi = \gamma \left( \ln V \right)_{,\phi} \int_{t_0}^{t} \frac{V_{,\phi}^2}{V_{,\psi}^2 + V_{,\phi}^2} dF + \beta \left( \ln V \right)_{,\psi}
\]

\[50\]

V. APPLICATIONS

In a surprisingly wide range of cases, the closed-form integral expressions above can be simplified. Our first two examples are simple cases which have been studied previously in the literature. We use these to show how our general expressions are to be used and to test that they reproduce known results. We then apply the method to more general and more realistic models.

1. Example 1: \(f_1 = f_2 = 1, \quad V = V_1(\phi) + V_2(\psi)\)

We have already presented a derivation for the most trivial case where the two inflaton fields are decoupled, \(V = V_1(\phi) + V_2(\psi)\) (see Section III). Here we show that the answer can be reproduced by our more general formulae. We detail a few steps to aid the reader in becoming familiar with applying our general formulae.

If \(V = V_1(\phi) + V_2(\psi)\), then the expression for \(F(t)\) in Eq. \((18)\) can be simplified. The first integrand in the exponent can be reduced to \((-\left( \ln V \right)_{,\phi} d\phi\) and the second integrand to \(-\left( \ln V \right)_{,\psi} d\psi\). The two can be combined into a total differential, \(-d(\ln V)\); as a result, \(F\) reduces to \(F(t) = V/V_{k=H_0}\), where \(V_{k=H_0}\) is the value of potential taken at the moment of horizon crossing \(t_0 = t_k\). In the first term of Eq. \((16)\), the integral reduces to

\[
-\int_{t_k}^{t} \left( V_2/V_{k=H_0} \right) d\psi = (V_2/V)_{k=H_0} - (V_2/V_{k=H_0})
\]
\[ \delta \phi = - \frac{\gamma}{V_{k=H_a}} \frac{V_1 V_2'}{V} + \left( \alpha + \gamma \left( \frac{V_2}{V} \right)_{k=H_a} \right) \frac{V_1'}{V}, \]  

and

\[ \delta \psi = \frac{\gamma}{V_{k=H_a}} \frac{V_2' V_1}{V} + \left( \beta - \gamma \left( \frac{V_1}{V} \right)_{k=H_a} \right) \frac{V_2'}{V} \]

which agrees with Eqs. (17) and (25) in our earlier derivation. Comparing these formulae with Eqs. (17) and (25) we see that the integration constants \(C\) and \(D\) in Eq. (25) are linear combinations of \(\gamma, \alpha\) and \(\beta\):

\[ C = \alpha + \gamma \left( \frac{V_2}{V} \right)_{k=H_a} = \beta - \gamma \left( \frac{V_1}{V} \right)_{k=H_a}, \]

\[ D = \frac{\gamma}{V_{k=H_a}}. \]

This identification between the three integration constants and the two coefficients, \(C\) and \(D\), requires use of the constraint that \(\alpha\) and \(\beta\) must satisfy, \(\beta - \alpha = \gamma\); see remarks above Eq. (49) in Section IVB. Also note that the integration constants are functions of the wavenumber \(k\) which can be related to \(\delta \phi\) and \(\delta \psi\) at the moment of horizon crossing \(k = H_a\) using the relations (46), (47) and taking in these formulae \(t = t_k\):

\[ \alpha = - \frac{1}{2} \left( H \frac{\delta \phi}{\dot{\phi}} \right)_{k=H_a}, \quad \beta = - \frac{1}{2} \left( H \frac{\delta \psi}{\dot{\psi}} \right)_{k=H_a} \]

\[ \gamma = \frac{1}{2} \left[ H \left( \frac{\delta \phi}{\dot{\phi}} - \frac{\delta \psi}{\dot{\psi}} \right) \right]_{k=H_a} \]

Note that the above expressions for coefficients are valid in general case.

Taking into account the above relations (52) we see that these expressions for the constants of integration are in agreement with formulae (27). We refer to this as Example 1. Other examples where the expressions for the solutions can be significantly simplified are:
2. Example 2: \( f_1 = f_2 = 1, \ V = V_1(\phi)V_2(\psi) \)

The integrands in the expression for \( F(t) \) in Eq. (48) and in the expressions for \( \delta \phi \) and \( \psi \) are precisely zero. Hence, \( F(t) = 1 \) and

\[
\delta \phi = \alpha \frac{V_{1,\phi}}{V_1}, \quad \delta \psi = \beta \frac{V_{2,\psi}}{V_2}.
\]

(54)

Eq. (49) then reduces to a closed-form expression for \( \Phi \),

\[
\Phi = -\frac{1}{2} \left( \alpha \frac{V_{1,\phi}^2}{V_1^2} + \beta \frac{V_{2,\psi}^2}{V_2^2} \right),
\]

(55)

in agreement with results obtained previously for this kind of potential.

3. Example 3: \( f_1 = f_1(\psi), \ f_2 = f_2(\phi), \ V = V_1(\phi)V_2(\psi) \)

The solution is

\[
\delta \phi = -\frac{\gamma}{(f_1 f_2)_{k=H}} \left( V_{1,\phi} \int_{t_k}^{t} f_2 df_1 + \alpha \frac{V_{1,\phi}}{V_1 f_1} \right)
\]

\[
\delta \psi = -\frac{\gamma}{(f_1 f_2)_{k=H}} \left( V_{2,\psi} \int_{t_k}^{t} f_1 df_2 + \beta \frac{V_{2,\psi}}{V_2 f_2} \right)
\]

where \( \alpha, \beta \) and \( \gamma \) are given by the formulae (53). In particular, when \( f_1 = \exp(-\psi/\psi_1), \ f_2 = 1, \ V_2(\psi) = \exp(-\psi/\psi_2) \) we obtain

\[
\delta \phi = \frac{V_{1,\phi}}{V_1} (\beta \exp(\psi/\psi_1) - \gamma \exp(\psi_{k=H}/\psi_1))
\]

\[
\delta \psi = -\beta/\psi_2
\]

(56)

in agreement with the recent result obtained by Starobinskii & Yokoyama.
4. Example 4: \( f_1 = f_2 = 1, \quad V = \Lambda + V_1 (\phi) V_2 (\psi) \)

In this case we have

\[
\delta \phi = -\Lambda \gamma \left( 1 - \frac{\Lambda}{V_{k=H_\Lambda}} \right) (\ln V)_{,\phi} \int_{t_k}^{t} \frac{1}{V_1} d \left( \frac{1}{V_2} \right) + \alpha (\ln V)_{,\phi}
\]

\[
\delta \psi = \Lambda \gamma \left( 1 - \frac{\Lambda}{V_{k=H_\Lambda}} \right) (\ln V)_{,\psi} \int_{t_k}^{t} \frac{1}{V_2} d \left( \frac{1}{V_1} \right) + \beta (\ln V)_{,\psi}
\] (57)

where as before the coefficients \( \alpha, \beta \) and \( \gamma \) can be expressed in terms of \( \delta \phi \) and \( \delta \psi \) at the moment of horizon crossing via (53).

5. Example 5: \( f_1 = f_2 = 1, \quad V = \Lambda + \alpha V_1 (\phi) + \beta V_2 (\psi) + V_1 (\phi) V_2 (\psi) \)

This case can be reduced to the previous one by the redefinition of the potentials \( V_1 \) and \( V_2 \) (shifting them by constant terms).

In summary, the central results of this paper are in Section IVb, explicit analytical solutions of equations for the energy density perturbations in the general case of two scalar fields with an arbitrary potential. To apply these solutions to a particular inflationary model, one should find first the analytical solutions which describe the behavior of the background and then use the obtained solutions in the formulae we have provided. A significant feature compared to the single inflaton field case is the existence of two constants of integration, each set by conditions at horizon-crossing during inflation. The two coefficients can be chosen so as to correspond to adiabatic and entropic perturbations. The latter is absent in the single-field case, but here is shown to be a general feature. In a future publication, we shall discuss the observational consequences of these results; namely, how observations can be used to distinguish single- from multi-field inflationary models.

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