EXISTENCE OF APPROXIMATE HERMITIAN–EINSTEIN STRUCTURES ON SEMISTABLE PRINCIPAL BUNDLES

INDRANIL BISWAS, ADAM JACOB, AND MATTHIAS STEMMLER

Abstract. Let $E_G$ be a principal $G$–bundle over a compact connected Kähler manifold, where $G$ is a connected reductive linear algebraic group defined over $\mathbb{C}$. We show that $E_G$ is semistable if and only if it admits approximate Hermitian–Einstein structures.

1. Introduction

A holomorphic vector bundle $E$ over a compact connected Kähler manifold $(X, \omega)$ is said to admit approximate Hermitian–Einstein metrics if for every $\varepsilon > 0$, there is a Hermitian metric $h$ on $E$ such that

$$\sup_X \left| \sqrt{-1} \Lambda \omega F(h) - \lambda \cdot \text{Id}_E \right|_h < \varepsilon.$$ 

In [Ja11 Theorem 2] it was shown that a holomorphic vector bundle $E$ over a compact Kähler manifold $(X, \omega)$ is semistable if and only if it admits approximate Hermitian–Einstein metrics. This generalizes a result of Kobayashi [Ko87, p. 234, Theorem 10.13] for complex projective manifolds. It is an analogue, for semistable bundles, of the classical Hitchin–Kobayashi correspondence, which is given by the famous theorem of Donaldson, Uhlenbeck and Yau. This theorem relates polystable bundles to (exact) solutions of the Hermitian–Einstein equation, and was first proven for curves by Narasimhan and Seshadri [NS65], then for algebraic surfaces by Donaldson [Do85], and finally in arbitrary dimension by Uhlenbeck and Yau [UY86].

Our aim here is to generalize the above result of [Ja11] to principal $G$–bundles over $X$, where $G$ is a connected reductive linear algebraic group defined over $\mathbb{C}$. Fix a maximal compact subgroup $K \subset G$. A holomorphic principal $G$–bundle $E_G$ over $X$ is said to admit approximate Hermitian–Einstein structures if for every $\varepsilon > 0$, there exists a $C^\infty$ reduction of structure group $E_K \subset E_G$ to $K$ and an element $\lambda$ in the center of Lie($G$) such that

$$\sup_X \left| \Lambda \omega F(\nabla^{E_K}) - \lambda \right|_{h^{E_K}} < \varepsilon,$$

where $F(\nabla^{E_K})$ is the curvature form of the Chern connection of $E_K$, and $h^{E_K}$ is the Hermitian metric on $\text{ad}(E_G)$ induced by $E_K$ (this Hermitian metric $h^{E_K}$ is described in [2.4]).

We prove the following:

2000 Mathematics Subject Classification. 53C07, 32L05.

Key words and phrases. Approximate Hermitian-Einstein structure, semistable $G$-bundle, Kähler metric.
Theorem 1. A holomorphic principal $G$–bundle $E_G$ over $X$ is semistable if and only if it admits approximate Hermitian–Einstein structures.

2. Preliminaries

Let $(X, \omega)$ be a compact connected Kähler manifold of complex dimension $n$, and let $E$ be a holomorphic vector bundle over $X$.

Recall that the degree of a torsion–free coherent analytic sheaf $\mathcal{F}$ on $X$ is defined to be

$$\deg(\mathcal{F}) := \int_X c_1(\mathcal{F}) \wedge \omega^{n-1};$$

if $\text{rank}(\mathcal{F}) > 0$, the slope of $\mathcal{F}$ is

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\text{rank}(\mathcal{F})}.$$

Definition 2. A holomorphic vector bundle $E$ is called semistable if $\mu(\mathcal{F}) \leq \mu(E)$ for every nonzero coherent analytic subsheaf $\mathcal{F}$ of $E$.

Definition 3. A holomorphic vector bundle $E$ is said to admit approximate Hermitian–Einstein metrics if for every $\varepsilon > 0$, there exists a Hermitian metric $h$ on $E$ such that

$$\sup_X |\sqrt{-1} \Lambda_\omega F(h) - \lambda \cdot \text{Id}_E|_h < \varepsilon.$$

Here $\Lambda_\omega$ is the adjoint of the wedge product with $\omega$, $F(h)$ is the curvature form of the Chern connection for $h$, and $\lambda$ is given by

$$\lambda = \frac{2\pi \cdot \mu(E)}{(n-1)! \cdot \text{vol}(X)},$$

where $\text{vol}(X)$ denotes the volume of $X$ with respect to the Kähler form $\omega$.

In [Ja11], the following was proved:

Theorem 4 ([Ja11, Theorem 2]). A holomorphic vector bundle $E$ over $X$ is semistable if and only if it admits approximate Hermitian–Einstein metrics.

Now let $G$ be a connected reductive linear algebraic group defined over $\mathbb{C}$, and let $E_G$ be a holomorphic principal $G$–bundle over $X$.

Definition 5. $E_G$ is called semistable if for every triple $(P, U, \sigma)$, where

- $P \subset G$ is a maximal proper parabolic subgroup,
- $U \subset X$ is a dense open subset such that the complement $X \setminus U$ is a complex analytic subset of $X$ of codimension at least 2, and
- $\sigma : U \rightarrow E_G/P$ is a holomorphic reduction of structure group, over $U$, of $E_G$ to the subgroup $P$, satisfying the condition that the pullback $\sigma^* T_{rel}$, which is a holomorphic vector bundle over $U$, extends to $X$ as a coherent analytic sheaf (here $T_{rel}$ is the relative tangent bundle over $E_G/P$ for the natural projection $E_G/P \rightarrow X$),
the inequality
\[ \deg(\sigma^*T_{\text{rel}}) \geq 0 \]
holds. The degree of \( \sigma^*T_{\text{rel}} \) is
\[ \deg(\sigma^*T_{\text{rel}}) := \int_X c_1(\iota_*\sigma^*T_{\text{rel}}) \wedge \omega^{n-1}, \]
where \( \iota : U \to X \) is the inclusion map.

We will now define approximate Hermitian–Einstein structures on \( E_G \). Let
\[ 0 \to \text{ad}(E_G) \to \text{At}(E_G) \xrightarrow{q} TX \to 0 \]
be the Atiyah exact sequence for \( E_G \) (see [At57] for the construction of (2.1)). Recall that a complex connection on \( E_G \) is a \( C^\infty \) splitting \( D : TX \to \text{At}(E_G) \) of this exact sequence, meaning \( q \circ D = \text{Id}_{TX} \). Note that (2.1) is a short exact sequence of sheaves of Lie algebras. The curvature form of a connection \( D \),
\[ F(D) \in H^0(X, \Lambda^{1,1}T^*X \otimes \text{ad}(E_G)), \]
measures the obstruction of the homomorphism \( D \) to be Lie algebra structure preserving; see [At57] for the details.

Fix a maximal compact subgroup
\[ K \subset G. \]
A Hermitian structure on \( E_G \) is a smooth reduction of structure group \( E_K \) of \( E_G \) to \( K \). Given a Hermitian structure \( E_K \) on \( E_G \), there is a unique complex connection on \( E_G \) which is induced by a connection on \( E_K \). This connection on \( E_G \) is called the Chern connection of the Hermitian structure \( E_K \), and it will be denoted by \( \nabla^{E_K} \). The connection on \( E_K \) inducing the Chern connection on \( E_G \) will also be called the Chern connection. Let
\[ (2.2) \quad F(\nabla^{E_K}) \in H^0(X, \Lambda^{1,1}T^*X \otimes \text{ad}(E_G)) \]
be the curvature of \( \nabla^{E_K} \). Note that \( F(\nabla^{E_K}) \) lies in the image of \( \Lambda^2(T^R X)^* \otimes \text{ad}(E_K) \), where \( T^R X \) is the real tangent bundle.

Let \( g \) be the Lie algebra of \( G \). Consider the adjoint representation
\[ (2.3) \quad \rho : G \to GL(g). \]
Fix a maximal compact subgroup \( \tilde{K} \subset GL(g) \) containing \( \rho(K) \). Let
\[ E_{GL(g)} = E \times^G GL(g) \to X \]
be the principal \( GL(g) \)-bundle obtained by extending the structure group of \( E_G \) to \( GL(g) \) using the homomorphism \( \rho \) in (2.3). We note that the vector bundle associated to the principal \( GL(g) \)-bundle \( E_{GL(g)} \) for the standard action of \( GL(g) \) on \( g \) is identified with the adjoint bundle \( \text{ad}(E_G) \).

Given a Hermitian structure \( E_K \subset E_G \), we obtain a reduction of structure group
\[ (2.4) \quad E_K(\tilde{K}) = E_K \times^K \tilde{K} \subset E_{GL(g)} \]
of \( E_{GL(g)} \) to \( \tilde{K} \). This reduction corresponds to a Hermitian metric on the adjoint vector bundle \( \text{ad}(E_G) \).
Let \( \mathfrak{z} \) be the center of the Lie algebra \( \mathfrak{g} \). Since the adjoint action of \( G \) on \( \mathfrak{z} \) is trivial, an element \( \lambda \in \mathfrak{z} \) defines a smooth section of \( \text{ad}(E_G) \), which will also be denoted by \( \lambda \).

**Definition 6.** A holomorphic principal \( G \)-bundle \( E_G \) over \( X \) is said to admit approximate Hermitian-Einstein structures if for every \( \varepsilon > 0 \), there exists a Hermitian structure \( E_K \subset E_G \) and an element \( \lambda \in \mathfrak{z} \), such that

\[
\sup_X \| \Lambda_{\omega} F(\nabla^{E_K}) - \lambda \|_{h_{E_K}} < \varepsilon,
\]

where \( F(\nabla^{E_K}) \) is the curvature form of the Chern connection of \( E_K \) (see (2.2)), and \( h_{E_K} \) is the Hermitian metric on \( \text{ad}(E_G) \) induced by \( E_K \) (see (2.4)).

### 3. Proof of Theorem 1

We will first show that it is enough to prove the theorem under the assumption that \( G \) is semisimple.

Let \( Z_0(G) \) be the connected component of the center of \( G \) which contains the identity element. The normal subgroup \([G, G] \subset G\) is semisimple because \( G \) is reductive. We have a natural surjective homomorphism

\[
G \longrightarrow (G/Z_0(G)) \times (G/[G, G])
\]

whose kernel is a finite group contained in the center of \( G \). In particular, the induced homomorphism of Lie algebras is an isomorphism.

Let \( \rho : A \longrightarrow B \) be a homomorphism of Lie groups such that the induced homomorphism of Lie algebras

\[
d\rho : \text{Lie}(A) \longrightarrow \text{Lie}(B)
\]

is an isomorphism, and \( \text{kernel}(\rho) \) is contained in the center of \( A \). Let \( E_A \) be a principal \( A \)-bundle, and let \( E_B := E_A \times^\rho B \) be the principal \( B \)-bundle obtained by extending the structure group of \( E_A \) to \( B \) using \( \rho \). The isomorphism of Lie algebras \( d\rho \) produces an isomorphism

\[
\tilde{\rho} : \text{ad}(E_A) \longrightarrow \text{ad}(E_B)
\]

between the adjoint bundles. There is a natural bijective correspondence between the connections on \( E_A \) and the connections on \( E_B \). To construct this bijection, first note that \( \rho \) induces a map

\[
\tilde{\rho} : E_A \longrightarrow E_B
\]

that sends \( z \in E_A \) to the element of \( E_B \) given by \((z, e)\) (recall that \( E_B \) is a quotient of \( E_A \times B \)). This map \( \tilde{\rho} \) intertwines the actions of \( A \), with \( A \) acting on \( E_B \) through \( \rho \). Since \( \text{kernel}(\rho) \) is a finite group contained in the center of \( A \), any \( A \)-invariant vector field on \( E_A|_U \), where \( U \) is some open subset of the base manifold, produces a \( B \)-invariant vector field on \( E_B|_U \). This way we get an isomorphism of \( \text{At}(E_A) \) with \( \text{At}(E_B) \). This identification of \( \text{At}(E_A) \) with \( \text{At}(E_B) \) produces a bijection between the connections on \( E_A \) and the connections on \( E_B \). The curvature of a connection on \( E_B \) is given by the curvature of the corresponding connection on \( E_A \) using the isomorphism \( \tilde{\rho} \).
Therefore, to prove the theorem, it suffices to prove it for $G/Z_0(G)$ and $G/[G,G]$ separately. Since $G/[G,G]$ is a product of copies of $\mathbb{C}^*$, in this case the theorem follows immediately from Theorem 4. Since $G/Z_0(G)$ is semisimple, it is enough to prove the theorem under the assumption that $G$ is semisimple.

Henceforth, we will assume that $G$ is semisimple. This implies that the center $\mathfrak{z}$ of its Lie algebra $\mathfrak{g}$ is trivial, and thus the constant $\lambda$ in Definition 6 is zero. The Killing form on $\mathfrak{g}$, being $G$–invariant, produces a holomorphic bilinear form on the fibers of $\text{ad}(E_G)$. Since the Killing form is nondegenerate (as $G$ is semisimple), this bilinear form on the fibers of $\text{ad}(E_G)$ is nondegenerate. Hence we get a trivialization

$$\det(\text{ad}(E_G)) := \bigwedge^{\text{top}} \text{ad}(E_G) \sim \mathcal{O}_X.$$ 

Therefore, $\deg(\text{ad}(E_G)) = \deg(\det(\text{ad}(E_G))) = 0$, or equivalently

$$\mu(\text{ad}(E_G)) = 0.$$ 

First assume that the principal bundle $E_G$ admits approximate Hermitian–Einstein structures. Given $\varepsilon > 0$, we thus obtain a Hermitian structure $E_K \subset E_G$ satisfying the condition that

$$\sup_X |\Lambda_\omega F(\nabla^{E_K})|_{h_{E_K}} < \varepsilon.$$ 

The Hermitian structure $E_K$ on $E_G$ induces a Hermitian metric $h_{E_K}$ on $\text{ad}(E_G)$ (see (2.4)). The Chern connection on $\text{ad}(E_G)$ for $h_{E_K}$ coincides with the connection $\nabla^{\text{ad}}$ on $\text{ad}(E_G)$ induced by $\nabla^{E_K}$. The curvature forms of $\nabla^{E_K}$ and $\nabla^{\text{ad}}$ are related by

$$F(\nabla^{\text{ad}}) = \text{ad}(F(\nabla^{E_K})), \quad \text{where}$$

$$\text{ad} : \text{ad}(E_G) \longrightarrow \text{End}(\text{ad}(E_G)) = \text{ad}(E_G) \otimes \text{ad}(E_G)^*$$

is the homomorphism of vector bundles induced by the homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}^*$ given by the adjoint action of $G$ on $\mathfrak{g}$.

Since the Hermitian metric $h_{E_K}$ on $\text{ad}(E_G)$ is induced by a Hermitian structure on $E_G$, there is a real number $c_0 > 0$ such that $\frac{1}{c_0} \cdot \text{ad}$ is an isometry, where $\text{ad}$ is the homomorphism in (3.5). Now from (3.4) it follows that

$$\left|\sqrt{-1} \Lambda_\omega F(\nabla^{\text{ad}})\right|_{h_{E_K}}^2 = \text{tr}(\Lambda_\omega F(\nabla^{\text{ad}}) \circ (\Lambda_\omega F(\nabla^{\text{ad}}))^*)$$

$$= \text{tr}(\text{ad}(\Lambda_\omega F(\nabla^{E_K})) \circ (\text{ad}(\Lambda_\omega F(\nabla^{E_K})))^*)$$

$$= c_0^2 \cdot h_{E_K}(\Lambda_\omega F(\nabla^{E_K}), \Lambda_\omega F(\nabla^{E_K}))$$

$$= c_0^2 \cdot |\Lambda_\omega F(\nabla^{E_K})|_{h_{E_K}}^2,$$

where “*” denotes the adjoint with respect to $h_{E_K}$. From this and (3.3) we conclude that

$$\sup_X \left|\sqrt{-1} \Lambda_\omega F(\nabla^{\text{ad}})\right|_{h_{E_K}} < c_0^2 \cdot \varepsilon.$$ 

Therefore, $\text{ad}(E_G)$ admits approximate Hermitian–Einstein metrics, and is semistable by Theorem 4. A holomorphic principal $G$–bundle $F_G$ over $X$ is semistable if and only if
its adjoint vector bundle $\text{ad}(F_G)$ is semistable \cite[Proposition 2.10]{AB01}. Therefore, a principal $G$–bundle admitting approximate Hermitian–Einstein structures is semistable.

For the converse direction, assume that $E_G$ is semistable. As we have stated, this is equivalent to the vector bundle $\text{ad}(E_G)$ being semistable. Let $\mathcal{H}(\text{ad}(E_G))$ be the space of all $C^\infty$ Hermitian metrics $h$ on $\text{ad}(E_G)$ satisfying the following condition: the isomorphism in \eqref{eq:3.1} takes the Hermitian metric on $\det(\text{ad}(E_G))$ induced by $h$ to the constant Hermitian metric on $\mathcal{O}_X$ given by the absolute value. For any initial $h \in \mathcal{H}(\text{ad}(E_G))$, we can evolve the metric by the following parabolic equation, which we call the Donaldson heat flow:

$$h^{-1}\partial_t h = -\sqrt{-1}\Lambda_\omega F(h) - \lambda \cdot \text{Id}_{\text{ad}(E_G)}.$$  

Here $F(h)$ is the curvature of the Chern connection on $\text{ad}(E_G)$ for $h$, and

$$\lambda = \frac{2\pi \cdot \mu(\text{ad}(E_G))}{(n-1)! \cdot \text{vol}(X)}.$$  

Since $\mu(\text{ad}(E_G)) = 0$ (see \eqref{eq:3.2}), for us the Donaldson heat flow is given by the simpler expression

\begin{equation}  
(3.6) \quad h^{-1}\partial_t h = -\sqrt{-1}\Lambda_\omega F(h). 
\end{equation}

As shown in \eqref{eq:2.4}, a Hermitian structure on $E_G$ produces a Hermitian metric on $\text{ad}(E_G)$. Such a Hermitian metric on $\text{ad}(E_G)$ satisfies the condition that the isomorphism in \eqref{eq:3.1} takes the induced Hermitian metric on $\det(\text{ad}(E_G))$ to the constant Hermitian metric on $\mathcal{O}_X$ given by the absolute value. In other words, it lies in $\mathcal{H}(\text{ad}(E_G))$. Let

$$\mathcal{H}(E_G) \subset \mathcal{H}(\text{ad}(E_G))$$

be the subspace corresponding to the Hermitian structures on $E_G$.

**Lemma 7.** The Donaldson heat flow on $\mathcal{H}(\text{ad}(E_G))$ preserves $\mathcal{H}(E_G)$.

**Proof.** Let $E_K \subset E_G$ be a $C^\infty$ reduction of structure group to $K$. The element of $\mathcal{H}(\text{ad}(E_G))$ given by the Hermitian structure $E_K$ will be denoted by $h$. The Chern connection $\nabla(h)$ on $\text{ad}(E_G)$ for $h$ is given by the Chern connection $\nabla^{E_K}$ on $E_K$. In particular, the curvature of $\nabla(h)$ coincides with that of $\nabla^{E_K}$. Therefore, the curvature $F(h)$ in \eqref{eq:3.6}, which is a priori a real two–form with values in $\text{End}(\text{ad}(E_G))$, is actually a real two–form with values in $\text{ad}(E_K)$ (the adjoint bundle of $E_K$). Consequently, $\Lambda_\omega F(h)$ is a $C^\infty$ section of $\text{ad}(E_K)$ (the operator $\Lambda_\omega$ takes real two–forms to real valued functions). This implies that the Donaldson heat flow on $\mathcal{H}(\text{ad}(E_G))$ preserves $\mathcal{H}(E_G)$. \hfill \Box

Fix a Hermitian metric $h_0 \in \mathcal{H}(\text{ad}(E_G))$, and consider the Donaldson heat flow with initial metric $h_0$. The space $\mathcal{H}(\text{ad}(E_G))$ was defined so that $h_0$ satisfies the normalization

$$c_1(\text{ad}(E_G), h_0) = \frac{\sqrt{-1}}{2\pi} \text{tr}(F(h_0)) = 0.$$  

This guarantees that

$$\det(h_0^{-1}h) = 1$$

along the flow. From this and the semistability of $\text{ad}(E_G)$, it follows that approximate Hermitian–Einstein metrics on $\text{ad}(E_G)$ are realized along the flow for sufficiently large
time (see the proof of Theorem 4 in [Ja11] for details). Consequently, taking \( h_0 \) to be an element of \( \mathcal{H}(E_G) \), from Lemma 7 we conclude that \( E_G \) admits approximate Hermitian–Einstein structures. This completes the proof of Theorem 1.

References

[AB01] B. Anchouche and I. Biswas: Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold, Am. Jour. Math. 123, 207–228 (2001).

[At57] M. F. Atiyah: Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85, 181–207 (1957).

[Do85] S. K. Donaldson: Anti self–dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. 3, 1–26 (1985).

[Ja11] A. Jacob: Existence of approximate Hermitian–Einstein structures on semi–stable bundles, arXiv:1012.1888v2 [math.DG].

[Ko87] S. Kobayashi: Differential geometry of complex vector bundles, Princeton University Press, Princeton, NJ, Iwanami Shoten Publishers, Tokyo, 1987.

[NS65] M. S. Narasimhan and C. S. Seshadri: Stable and unitary bundles on a compact Riemann surface, Math. Ann. 85, 540–564 (1965).

[UY86] K. Uhlenbeck and S. T. Yau: On the existence of Hermitian–Yang–Mills connections in stable vector bundles, Comm. Pure and Appl. Math. 39, 257–293 (1986).

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Department of Mathematics, Columbia University, New York, NY 10027, USA

E-mail address: ajacob@math.columbia.edu

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: stemmler@math.tifr.res.in