Noether’s Problem for Some Semidirect Products

Ming-chang Kang(1) and Jian Zhou(2)

(1)Department of Mathematics, National Taiwan University, Taipei
E-mail: kang@math.ntu.edu.tw

(2)School of Mathematical Sciences, Peking University, Beijing
E-mail: zhjn@math.pku.edu.cn

Abstract. Let $k$ be a field, $G$ be a finite group, $k(x(g) : g \in G)$ be the rational function field with the variables $x(g)$ where $g \in G$. The group $G$ acts on $k(x(g) : g \in G)$ by $k$-automorphisms where $h \cdot x(g) = x(hg)$ for all $h, g \in G$. Let $k(G)$ be the fixed field defined by $k(G) := k(x(g) : g \in G)^G = \{ f \in k(x(g) : g \in G) : h \cdot f = f \text{ for all } h \in G \}$. Noether’s problem asks whether the fixed field $k(G)$ is rational (= purely transcendental) over $k$. Let $m$ and $n$ be positive integers and assume that there is an integer $t$ such that $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ is of order $n$. Define a group $G_{m,n} := \langle \sigma, \tau : \sigma^m = \tau^n = 1, \tau^{-1}\sigma\tau = \sigma^t \rangle \simeq C_m \rtimes C_n$. We will find a sufficient condition to guarantee that $k(G)$ is rational over $k$. As a result, it is shown that, for any positive integer $n$, the set $S := \{ p : p \text{ is a prime number such that } \mathbb{C}(G_{p,n}) \text{ is rational over } \mathbb{C} \}$ is of positive Dirichlet density; in particular, $S$ is an infinite set.

§1. Introduction

Let $k$ be any field, $G$ be a finite group and $G \to GL(V)$ be a faithful linear representation of $G$ where $V$ is a finite-dimensional vector space over $k$. Then $G$ acts naturally on the function field $k(V)$ by $k$-automorphisms. Noether’s problem asks whether the fixed field $k(V)^G$ is rational (= purely transcendental) over $k$. In particular, when $V = V_{\text{reg}}$ is the regular representation, we will write $k(G) := k(V_{\text{reg}})^G$. Explicitly, $k(V_{\text{reg}}) := k(x(g) : g \in G)$ is the rational function field in the variables $x(g)$

2010 Mathematics Subject Classification. 14E08, 12F10, 13A50, 11R29.

Keywords and phrases. Noether’s problem, rationality problem, algebraic tori, class groups.
(where \( g \in G \)), \( G \) acts on \( k(V_{\text{reg}}) \) by \( k \)-automorphisms defined by \( h \cdot x(g) = x(hg) \) for any \( h, g \in G \), and \( k(G) = k(V_{\text{reg}})^G = \{ f \in k(V_{\text{reg}}) : h \cdot f = f \text{ for all } h \in G \} \). Note that Noether’s problem is a special case of the famous Lüroth problem.

When the group \( G \) is abelian and the field \( k \) contains enough roots of unity, the following theorem of Fischer guarantees that \( k(G) \) is rational.

**Theorem 1.1** (Fischer [Sw2, Theorem 6.1]) Let \( G \) be a finite abelian group of exponent \( e \), \( k \) be a field containing \( \zeta_e \), a primitive \( e \)-th root of unity. For any finite-dimensional representation \( G \rightarrow GL(V) \) over \( k \), the fixed field \( k(V)^G \) is rational over \( k \).

When \( G \) is abelian and \( k \) is any field (e.g. \( k = \mathbb{Q} \)), the rationality problem of \( k(G) \) was investigated by Swan, Endo and Miyata, Voskresenskii, Lenstra, etc.. Swan’s survey paper [Sw2] gives an excellent account of Noether’s problem for abelian groups.

Now we turn to Noether’s problem for non-abelian groups. We define the group \( G_{m,n} \) first.

**Definition 1.2** Let \( m \) and \( n \) be positive integers and assume that there is an integer \( t \) such that \( t \in \left( \mathbb{Z}/m\mathbb{Z} \right)^x \) is of order \( n \). Define a group \( G_{m,n} := \langle \sigma, \tau : \sigma^m = \tau^n = 1, \tau^{-1} \sigma \tau = \sigma^t \rangle \cong C_m \times C_n \).

We remark that the integer \( t \) always exists provided that \( m \) is an odd prime power and \( n \mid \phi(m) \). Also note that the group \( G_{m,n} \) depends on the choice of \( t \). However, for different choices of \( t \), the rationality criterion we are concerned about (e.g. Theorem [4.8]) is not affected, which may be justified by applying Lemma [3.3]. Thus we will not emphasize the dependence of \( G_{m,n} \) on the choice of \( t \).

We will study under what situation the fixed field \( \mathbb{C}(G_{m,n}) \) will be rational. The following theorem is quite useful, provided that the ring \( \mathbb{Z}[\zeta_n] \) is a UFD (unique factorization domain).

**Theorem 1.3** ([Ka2, Theorem 1.4]) Let \( k \) be a field and \( G \) be a finite group. Assume that (i) \( G \) contains an abelian normal subgroup \( H \) so that \( G/H \) is cyclic of order \( n \), (ii) \( \mathbb{Z}[\zeta_n] \) is a UFD where \( \zeta_n \) is a primitive \( n \)-th root of unity, and (iii) \( \zeta_e \in k \) where \( e := \text{lcm}\{\text{ord}(g) : g \in G\} \) is the exponent of \( G \). If \( G \rightarrow GL(V) \) is any finite-dimensional linear representation of \( G \) over \( k \), then \( k(V)^G \) is rational over \( k \).

It is unknown for a long time whether \( \mathbb{C}(G_{p,q}) \) is rational or not if \( p \) and \( q \) are distinct prime numbers and \( \mathbb{Z}[\zeta_q] \) is not a UFD. We note that the assumption that \( \mathbb{Z}[\zeta_n] \) is a UFD in Theorem [1.3] imposes a severe restriction to the integer \( n \), because of the theorem of Masley and Montgomery.

**Theorem 1.4** (Masley and Montgomery [MM]) \( \mathbb{Z}[\zeta_n] \) is a unique factorization domain if and if \( 1 \leq n \leq 22 \), or \( n = 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 40, 42, 44, 45, 48, 50, 54, 60, 66, 70, 84, 90 \).

A recent work of Chu and Huang [CH] found a sufficient condition for the rationality of \( \mathbb{C}(G_{m,q}) \) where \( q \) is a prime number.
Theorem 1.5 (Chu and Huang [CH Main Theorem]) Let $m$ and $q$ be positive integers where $q$ is a prime number and assume that there is an integer $t$ such that $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ is of order $q$. Define $m' = m/\gcd(m,t-1)$. Assume that there exist integers $a_0, a_1, \ldots, a_{q-2}, b$ such that $\gcd\{a_0, a_1, \ldots, a_{q-2}, b\} = 1$, $bn' = a_0 + a_1 t + \cdots + a_{q-2} t^{q-2}$ and $N_{q(\zeta_q)/q}(\alpha) = m'$ where $\alpha := a_0 + a_1 \zeta_q + \cdots + a_{q-2} \zeta_q^{q-2}$. If $k$ is a field with $\zeta_m, \zeta_q \in k$, then $k(G_{m,q})$ is rational over $k$.

In particular, for distinct prime numbers $p$ and $q$ with $q \mid \phi(p)$, if there is an element $\alpha \in \mathbb{Z}[\zeta_q]$ such that $N_{q(\zeta_q)/q}(\alpha) = p$ and $k$ is a field with $\zeta_p, \zeta_q \in k$, then $k(G_{p,q})$ is rational over $k$.

As an application, Chu and Huang show that $\mathcal{C}(G_{p,q})$ is rational when $(p,q) = (5801,29), (6263,31), (32783,37), (101107,41)$; for more examples, see Section 4 of [CH]. Note that $\mathbb{Z}[\zeta_q]$ is not a UFD when $q = 29, 31, 37$ or 41 by Theorem 1.4 thus these examples escape the application of Theorem 1.3.

We remark that the proof of Theorem 1.5 given in [CH] is rather computational and lengthy. Moreover, the assumptions, e.g. the number $m'$, look abrupt at first sight.

The purpose of this paper is to provide a conceptual approach to the rationality of $k(G_{m,n})$ different from the computational verification in [CH]. Moreover, a generalized form of Theorem 1.5 can be found in Theorem 4.7 and Theorem 4.8. A clarification of the number $m'$ is given in Lemma 4.6; we will show that the “purpose” of the complicated assumptions in Theorem 1.5 is just to ensure that the ideal $\langle \zeta_q - t, m' \rangle$ is a principal ideal of $\mathbb{Z}[\zeta_q]$ (see Step 3 in the proof of Theorem 4.7). The examples constructed by computer computing in [CH Section 4] turn out to be heralds of Theorem 1.9 which asserts that, for any positive integer $n$, there are infinitely many prime numbers $p$ such that the fixed field $\mathcal{C}(G_{p,n})$ is rational over $\mathbb{C}$.

In this article we will study the rationality problem of $k(G_{m,n})$ where $n$ is any positive integer; we don’t assume that $n$ is a prime number as in [CH]. Here is a sample of our results.

Theorem 1.6 Let $m$ and $n$ be positive integers such that $m$ is an odd integer. Assume that (i) there is an integer $t$ satisfying that $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ is of order $n$, and (ii) for any $e \mid n$, the ideal $\langle \zeta_e - t, m \rangle$ in $\mathbb{Z}[\zeta_e]$ is a principal ideal. If $k$ is a field with $\zeta_m, \zeta_n \in k$, then $k(G_{m,n})$ is rational over $k$.

For other results, see Theorem 4.3 and Theorem 4.8.

The main idea in the proofs of Theorem 4.6 and its variants is to apply the methods developed by Endo, Miyata, Lenstra etc. in solving Noether’s problem for abelian groups [EM1, EM2, Le]. These methods were reformulated by Colliot-Thélène and Sansuc [CTS, Section 1] (see Section 2 for a brief summary). Armed with these tools, we will embark on the investigation of the rationality problem of $k(G_{m,n})$ in Section 4.

In Section 4, the reader will find that the rationality of $\mathcal{C}(G_{m,n})$ is reduced to the rationality of $\mathcal{C}(M)^\pi$ where $\pi \simeq C_n$ and $M$ is a $\pi$-lattice (see Section 2 for the definition of a $\pi$-lattice and the multiplicative invariant field $\mathcal{C}(M)^\pi$). Such a rationality problem was studied by Saltman [Sa2], Beneish and Ramsey [BR]. One of the aims of Salmon
in [Sa2] is to find a group $\pi$ and a $\pi$-lattice $M$ such that $C(M)^\pi$ is not retract rational (and thus not stably rational); it is necessary that such a group $\pi$ is not cyclic by [Ka3]. On the other hand, Beneish and Ramsey considered a cyclic group $\pi$ and proposed a notion, the Property * for $\pi$ [BR, Definition 3.3]. Assuming the Property *, they were able to prove two significant results.

**Theorem 1.7** (Beneish and Ramsey [BR, Theorem 3.12 and Theorem 3.13]) Assume that the Property * for a cyclic group of order $n$ is valid.

(i) Let $\pi \simeq C_n$ and $M$ be any $\pi$-lattice. If $k$ is a field with $\zeta_n \in k$, then $k(M)^\pi$ is stably rational over $k$.

(ii) Let $G := A \rtimes C_n$ where $A$ is a finite abelian group of exponent $e$. If $k$ is a field with $\zeta_e, \zeta_n \in k$, then $k(G)$ is stably rational over $k$.

In Theorem 5.1 we will prove that the Property * for a cyclic group of order $n$ is equivalent to the assertion that $\mathbb{Z}[\zeta_n]$ is a UFD. As a result an alternative proof of Theorem 1.7 will be given in Section 5. In fact, similar results are valid, say, for the dihedral group, because such kind of theorems are consequences of Endo-Miyata’s Theorem [EM2, Theorem 3.3; EK, Theorem 1.4]; see Theorem 5.2 and Lemma 5.3.

Finally we remark that, besides the sufficient condition for the rationality of $C(G_{m,n})$, the retract rationality, one of the necessary conditions, is already known before. For a field extension $L/k$ (where $k$ is an infinite field), the notion that $L$ is retract rational over $k$ is introduced by Saltman [Sa1]. It is known that “rational” $\Rightarrow$ “stably rational” $\Rightarrow$ “retract rational” $\Rightarrow$ “unirational”. The reader may consult Theorem 1.6, Theorem 1.7 and Theorem 1.8 of [Ka3] for a quick review of the retract rationality of $C(A \rtimes C_n)$ where $A$ is a finite abelian group.

**Standing notations.** Throughout this article, we consider only finite groups. The following notations are adopted:

- $C_n$: the cyclic group of order $n$,
- $\pi$: a finite group,
- $\mathbb{Z}\pi$: the integral group ring of $\pi$,
- $\Phi_n(X)$: the $n$-th cyclotomic polynomial,
- $\phi(n)$: the value of the Euler $\phi$-function at $n$,
- $\left(\mathbb{Z}/m\mathbb{Z}\right)^\times$: the group of units of the ring $\mathbb{Z}/m\mathbb{Z}$,
- $\text{ord}_p(n)$: the exponent of $p$ in $n$, i.e. if $\text{ord}_p(n) = e$, then $p^e \mid n$ but $p^{e+1} \nmid n$,
- $\zeta_n$: a primitive $n$-th root of unity.

When we say that $\zeta_n \in k$ ($k$ is a field), it is understood that either $\text{char} \, k = 0$ or $\text{char} \, k > 0$ with $\text{char} \, k \nmid n$. Recall that a field extension $L/k$ is rational if $L$ is purely transcendental over $k$, i.e. $L \simeq k(x_1, \ldots, x_n)$ over $k$ where $k(x_1, \ldots, x_n)$ is the rational function field of $n$ variables over $k$. A field extension $L/k$ is stably rational if $L(y_1, \ldots, y_m)$ is rational over $k$ where $y_1, \ldots, y_m$ are some elements algebraically independent over $L$. If $\pi$ is a group and $\tau \in \pi$, then $\langle \tau \rangle$ denotes the subgroup generated by $\tau$; similarly, if $R$ is a commutative ring and $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$, then $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ denotes the ideal generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. 

4
Let \( m \) and \( n \) be positive integers. For the sake of simplicity, we will simply say that \( t \in (\mathbb{Z}/m\mathbb{Z})^\times \) is of order \( n \), when we mean that \( t \in \mathbb{Z} \), \( \gcd\{t, m\} = 1 \) and the subgroup \( \langle \bar{t} \rangle \cong C_n \) where \( \bar{t} \) is the residue class of \( t \) in \((\mathbb{Z}/m\mathbb{Z})^\times\).

Finally we remind the reader that the fixed field \( k(G) \) is defined at the beginning of this section.

Acknowledgments. The proof of Theorem 4.9 was suggested by Prof. Ching-Li Chai (Univ. of Pennsylvania), whose help is highly appreciated.

\section{Preliminaries}

Let \( \pi \) be a finite group. We recall the definition of \( \pi \)-lattices.

**Definition 2.1** Let \( \pi \) be a finite group. A finitely generated \( \mathbb{Z}[\pi] \)-module \( M \) is called a \( \pi \)-lattice if \( M \) is a free abelian group when it is regarded as an abelian group.

If \( M \) is a \( \pi \)-lattice and \( L \) is a field with \( \pi \)-action, we will associate a rational function field over \( L \) with \( \pi \)-action as follows. Suppose that \( M = \bigoplus_{1 \leq i \leq m} \mathbb{Z} \cdot u_i \). Define \( L(M) = L(x_1, \ldots, x_m) \), a rational function field of \( m \) variables over \( L \). For any \( \sigma \in \pi \), if \( \sigma \cdot u_i = \sum_{1 \leq j \leq m} a_{ij} u_j \) in \( M \) (where \( a_{ij} \in \mathbb{Z} \)), we define \( \sigma \cdot x_i = \prod_{1 \leq j \leq m} x_j^{a_{ij}} \) in \( L(M) \) and, for any \( \alpha \in \mathbb{L} \), define \( \sigma \cdot \alpha \) by the prescribed \( \pi \)-action on \( L \).

Note that, if \( \pi \) acts faithfully on \( L \) and \( K = L^\pi \) (i.e. \( \pi \cong \text{Gal}(L/K) \)), then the fixed field \( L(M)^\pi = \{ f \in L(M) : \sigma \cdot f = f \text{ for all } \sigma \in \pi \} \) is the function field of an algebraic torus defined over \( K \), split by \( L \) and with character lattice \( M \) (see [Vo; Sw2, Section 12; Sa2]).

On the other hand, if \( \pi \) acts trivially on \( L \) (i.e. \( \sigma(\alpha) = \alpha \) for all \( \sigma \in \pi \), for all \( \alpha \in \mathbb{L} \)), the action of \( \pi \) on \( L(M) \) is called a purely monomial action in some literature. When we write \( k(M)^\pi \) without emphasizing the action of \( \pi \) on \( k \), it is understood that \( \pi \) acts trivially on \( k \), i.e. the situation of purely monomial actions.

**Definition 2.2** Let \( \pi \) be a finite group and \( M \) be a \( \pi \)-lattice. \( M \) is called a permutation lattice if \( M \) has a \( \mathbb{Z} \)-basis permuted by \( \pi \). A \( \pi \)-lattice \( M \) is called an invertible lattice if it is a direct summand of some permutation lattice. A \( \pi \)-lattice \( M \) is called a flabby lattice if \( H^{-1}(\pi', M) = 0 \) for all subgroup \( \pi' \) of \( \pi \); it is called a coflabby lattice if \( H^1(\pi', M) = 0 \) for all subgroups \( \pi' \) of \( \pi \). For the basic properties of \( \pi \)-lattices, see [CTS]; Sw2.

**Definition 2.3** Let \( \pi \) be a finite group. Denote by \( \mathcal{L}_\pi \) (resp. \( \mathcal{F}_\pi \)) the class of all the \( \pi \)-lattices (resp. all the flabby \( \pi \)-lattices). We introduce a similarity relation on \( \mathcal{L}_\pi \) and \( \mathcal{F}_\pi \): two lattices \( M_1 \) and \( M_2 \) are similar, denoted by \( M_1 \sim M_2 \), if \( M_1 \oplus Q_1 \cong M_2 \oplus Q_2 \) for some permutation \( \pi \)-lattices \( Q_1 \) and \( Q_2 \). Let \( \mathcal{L}_\pi /\sim \) and \( \mathcal{F}_\pi /\sim \) be the sets of similarity classes of \( \mathcal{L}_\pi \) and \( \mathcal{F}_\pi \) respectively; we define \( F_\pi = \mathcal{F}_\pi /\sim \). For each \( \pi \)-lattice \( M \), denote by \([M]\) the similarity class containing \( M \).
We define an addition on \( \mathcal{L}_\pi/\sim \) and \( F_\pi \) as follows: \([M_1] + [M_2] := [M_1 \oplus M_2]\) for any \( \pi \)-lattices \( M_1 \) and \( M_2 \). In this way, \( \mathcal{L}_\pi/\sim \) becomes an abelian monoid and \( F_\pi \) is a submonoid of \( \mathcal{L}_\pi/\sim \). Note that \([M] = 0\) in \( F_\pi \) if and only if \( M \) is stably permutation, i.e. \( M \oplus Q \) is isomorphic to a permutation \( \pi \)-lattice where \( Q \) is some permutation \( \pi \)-lattice. See [Sw2] for details.

**Definition 2.4** Let \( \pi \) be a finite group, \( M \) be a \( \pi \)-lattice. The \( M \) have a flabby resolution, i.e. there is an exact sequence of \( \pi \)-lattices: \( 0 \to M \to Q \to E \to 0 \) where \( Q \) is a permutation lattice and \( E \) is a flabby lattice [EM2, Lemma 1.1; CTS; Sw2].

Although the above flabby resolution is not unique, the class \([E] \in F_\pi\) is uniquely determined by \( M \). Thus we define the flabby class of \( M \), denoted as \([M]_{fl} \), by \([M]_{fl} \in F_\pi\) (see [Sw2]). Sometimes we say that \([M]_{fl}\) is permutation or invertible if the class \([E] \) contains a permutation lattice or an invertible lattice.

**Theorem 2.5** Let \( L/K \) be a finite Galois extension with \( \pi = \text{Gal}(L/K) \), and let \( M \) be a \( \pi \)-lattice. The group \( \pi \) acts on the field \( L(M) \) as in Definition 2.1.

1. ([EM1, Theorem 1.6; Vo; Le, Theorem 1.7; CTS]) The fixed field \( L(M)^\pi \) is stably rational over \( K \) if and only if \([M]_{fl} = 0\) in \( F_\pi \).

2. ([Sa1, Theorem 3.14]) Assume that \( K \) is an infinite field. Then the fixed field \( L(M)^\pi \) is retract rational over \( K \) if and only if \([M]_{fl}\) is invertible.

Finally we recall a variant of the No-Name Lemma.

**Theorem 2.6** ([HK, Theorem 1]) Let \( L \) be a field and \( G \) be a finite group acting on \( L(x_1, \ldots, x_m) \), the rational function field of \( m \) variables over \( L \). Assume that

(i) for any \( \sigma \in G \), \( \sigma(L) \subset L \),

(ii) the restriction of the action of \( G \) to \( L \) is faithful, and

(iii) for any \( \sigma \in G \),

\[
\begin{pmatrix}
  \sigma(x_1) \\
  \vdots \\
  \sigma(x_m)
\end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\
  \vdots \\
  x_m \end{pmatrix} + B(\sigma)
\]

where \( A(\sigma) \in GL_m(L) \) and \( B(\sigma) \) is an \( m \times 1 \) matrix over \( L \).

Then \( L(x_1, \ldots, x_m) = L(z_1, \ldots, z_m) \) for some elements \( z_1, \ldots, z_m \in L(x_1, \ldots, x_m) \) such that \( \sigma(z_i) = z_i \) for all \( \sigma \in G \) for all \( 1 \leq i \leq m \). Consequently, \( L(x_1, \ldots, x_m)^G = L^G(z_1, \ldots, z_m) \).
§3. Some ideals of $\mathbb{Z}[\zeta_n]$

Let $\mathbb{Z}[\zeta_n]$ be the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_n)$.

**Lemma 3.1** (1) ([Ar Lemma 1; Le, Lemma 3.8]) Let $p \geq 3$ be a prime number. For simplicity, we write $\text{ord}(m)$ for $\text{ord}_p(m)$. Suppose that $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ is of order $n$, then

$$\text{ord}(\Phi_n(t)) = \text{ord}(t^n - 1) \geq 1;$$

$$\text{ord}(\Phi_{p^e}(t)) = 1 \text{ if } d \geq 1;$$

$$\text{ord}(\Phi_p(t)) = 0 \text{ if } e \in \mathbb{N} \text{ and } e \text{ is not of the form } p^d n \text{ (where } d \geq 0);$$

$$\text{ord}(t^l - 1) = 0 \text{ if } l \in \mathbb{N} \text{ and } n \nmid l;$$

$$\text{ord}(t^l - 1) = 1 \text{ if } t \in \mathbb{N} \text{ and } n \nmid l.$$ 

(2) ([EM1 page 15]) Let $p \geq 3$ and $\text{ord}(m)$ be the same as in (1). Let $d \geq 1$, $n \mid \phi(p^d)$ and $n = p^d n_0$ where $0 \leq d_0 \leq d - 1$ and $p \nmid n_0$. Let $t \in (\mathbb{Z}/p^d\mathbb{Z})^\times$ be of order $n$. Then

$$\text{ord}(\Phi_{n_0}(t)) \geq d \text{ if } d_0 = 0, \text{ i.e. } p \nmid n;$$

$$\text{ord}(\Phi_{n_0}(t)) = d - d_0 \text{ if } d_0 \geq 1;$$

$$\text{ord}(\Phi_{p^{e_0}n_0}(t)) = 1 \text{ if } d_0 \geq 1 \text{ and } 1 \leq d' \leq d_0,$$

$$\text{ord}(\Phi_{p^{e}n_0}(t)) = 0 \text{ if } 0 \leq d' \leq d_0, n' \mid n_0 \text{ and } n' < n_0.$$

(3) ([Ar Lemma 1]) $\text{ord}_2(\Phi_{2^d}(t)) = 1 \text{ if } d \geq 2 \text{ and } t \text{ is an odd integer.}$

**Lemma 3.2** Let $m_1$, $m_2$ and $n$ be positive integer such that $\gcd\{m_1, m_2\} = 1$. If $J$ is an ideal of $\mathbb{Z}[\zeta_n]$, then $\langle J, m_1 m_2 \rangle = \langle J, m_1 \rangle \cdot \langle J, m_2 \rangle = \langle J, m_1 \rangle \cap \langle J, m_2 \rangle$.

*Proof.* Since $m_1$ and $m_2$ are relatively prime, the ideal $\langle J, m_1 \rangle$ and $\langle J, m_2 \rangle$ are co-maximal. Thus $\langle J, m_1 \rangle \cdot \langle J, m_2 \rangle = \langle J, m_1 \rangle \cap \langle J, m_2 \rangle$.

It is clear that $\langle J, m_1 m_2 \rangle \subset \langle J, m_1 \rangle \cap \langle J, m_2 \rangle \subset \langle J, m_1 \rangle \cdot \langle J, m_2 \rangle$.

Done. □

Recall that, if $J$ is an ideal of $\mathbb{Z}[\zeta_n]$, then the (absolute) norm of $J$, denoted by $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(J)$, is the index of $J$ in $\mathbb{Z}[\zeta_n]$; in other words, $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(J) = |\mathbb{Z}[\zeta_n]/J|$ (see, for examples, [IR] pages 203–204). Consequently, if $a \in \mathbb{Z}[\zeta_n] \setminus \{0\}$, then $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(a) = |\text{Norm}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(a)|$.

**Lemma 3.3** Let $m = p^d$ where $p \geq 3$ is a prime number and $d \geq 1$. Let $n$ be a positive integer with $n \mid \phi(m)$. Write $n = p^{d_0} n_0$ where $d_0 \geq 0$ and $p \nmid n_0$. Suppose that $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ is of order $n$.

(1) If $e \mid n$ and $e = p^d n'$ with $0 \leq d' \leq d_0$ and $p \nmid n'$, then

$$\langle \zeta_e - t, p \rangle = \begin{cases} a \text{ proper ideal of } \mathbb{Z}[\zeta_e], \text{ if } n' = n_0; \\ \mathbb{Z}[\zeta_e], \text{ otherwise.} \end{cases}$$
Moreover, for $1 \leq d'' \leq d$, $\langle \zeta_e - t, p^{d''} \rangle = \langle \zeta_e - t, p \rangle^{d''}$.

(2) Every prime ideal of $\mathbb{Z}[\zeta_n]$ lying over $p$ is of the form $\langle \zeta_n - t', p \rangle$ where $t'$ is an integer and $t' \in (\mathbb{Z}/m\mathbb{Z})^\times$ is of order $n$. All of these prime ideals are conjugate in $\mathbb{Z}[\zeta_n]$. In fact, if $e \mid n$ and $1 \leq d'' \leq d$, the ideals $\langle \zeta_e - t, p^{d''} \rangle$ and $\langle \zeta_e - t', p^{d''} \rangle$ are conjugate in $\mathbb{Z}[\zeta_e]$.

(3) The following assertions are equivalent: The ideal $\langle \zeta_n - t, p \rangle$ is a principal ideal $\iff$ There is an element $\alpha \in \mathbb{Z}[\zeta_n]$ such that $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\alpha) = \pm p \iff$ For any $e \mid n$, the ideal $\langle \zeta_e - t, p \rangle$ is a principal ideal.

Proof. (1) If $e \mid n$ with $e = p^d n'$ where $n' = n_0$, we will show that $\langle \zeta_e - t, p \rangle \subseteq \mathbb{Z}[\zeta_e]$. Otherwise, there are $x, y \in \mathbb{Z}[\zeta_e]$ such that $x(\zeta_e - t) + py = 1$. For any $g \in \text{Gal}(\mathbb{Q}(\zeta_e)/\mathbb{Q})$, we have $g(x) \cdot (g(\zeta_e) - t) + pg(y) = 1$. Hence $1 = \prod g(x)(g(\zeta_e) - t) + pg(y) = \alpha \cdot \Phi_e(t) + p \cdot \beta$ for some $\alpha, \beta \in \mathbb{Z}[\zeta_e]$. By Lemma 3.1 (2), we have $p \mid \Phi_e(t)$. Thus $p \mid \alpha \Phi_e(t) + p\beta$ in $\mathbb{Z}[\zeta_e]$. A contradiction.

Now suppose that $e \mid n$ with $n' < n_0$. By Lemma 3.1 (2) again, we find that $p \nmid \Phi_e(t)$. Thus we may find integers $a$ and $b$ with $a \Phi_e(t) + pb = 1$. Since $\Phi_e(t) = \prod_i (t - \zeta_i)$ where $i$ runs over integers in $(\mathbb{Z}/\mathbb{Z})^\times$, it follows that $\langle \zeta_e - t, p \rangle = \mathbb{Z}[\zeta_e]$.

We will prove that $\langle \zeta_e - t, p^{d''} \rangle = \langle \zeta_e - t, p \rangle^{d''}$ if $\langle \zeta_e - t, p \rangle = \mathbb{Z}[\zeta_e]$, then $\alpha(\zeta_e - t) + p\beta = 1$ for some $\alpha, \beta \in \mathbb{Z}[\zeta_e]$. Hence $1 = (\alpha(\zeta_e - t) + p\beta)^{d''} = \gamma(\zeta_e - t) + p^{d''}\beta^{d''} \in \langle \zeta_e - t, p^{d''} \rangle$ for some $\gamma \in \mathbb{Z}[\zeta_e]$. It remains to consider the case when $\langle \zeta_e - t, p \rangle$ is a proper ideal of $\mathbb{Z}[\zeta_e]$. The ideals $\langle \zeta_e - t, p^{d''} \rangle$ and $\langle \zeta_e - t, p^{d''} \rangle$ are of the same norm $p^{d''}$. Since $\langle \zeta_e - t, p^{d''} \rangle \subseteq \langle \zeta_e - t, p^{d''} \rangle$, thus they are equal.

(2) By (1), the ideal $\langle \zeta_n - t, p \rangle$ is of norm $p$; thus it is a prime ideal lying over $p$. By [IR] page 182, Proposition 3.3 all the prime ideals of $\mathbb{Z}[\zeta_n]$ lying over $p$ are conjugate to each other. Thus they are of the form $\langle \zeta_0^n - t, p \rangle$ where $u$ is an integer, $1 \leq u \leq n - 1$ and $\gcd\{n, u\} = 1$. For each $u$, choose an integer $v$ with $1 \leq v \leq n - 1$ and $uv \equiv 1$ (mod $n$). We will show that $\langle \zeta_0^n - t, p \rangle = \langle \zeta_n - t^v, p \rangle$. Since $\zeta_n - t^v = \zeta_0^n(t^v - t^v)$ is divisible by $\zeta_0^n - t$, we find that $\langle \zeta_0^n - t, p \rangle \supset \langle \zeta_n - t^v, p \rangle$. These two ideals are of the same norm $p$ by (1) (note that $\mathbb{Z}[\zeta_n] = \mathbb{Z}[\zeta_0^n]$). Thus they are equal. We conclude that all the prime ideals over $p$ are conjugate and are of the required form.

We will show that $\langle \zeta_e - t, p^{d''} \rangle$ and $\langle \zeta_e - t', p^{d''} \rangle$ are conjugate. If $\langle \zeta_e - t, p \rangle = \mathbb{Z}[\zeta_e]$, then $\langle \zeta_e - t, p^{d''} \rangle = \mathbb{Z}[\zeta_e]$ because it depends only on the factorizations of $n$ and $e$ by (1). Using (1) again, we find that $\langle \zeta_e - t, p^{d''} \rangle = \mathbb{Z}[\zeta_e] = \langle \zeta_e - t', p^{d''} \rangle$. Now assume that both $\langle \zeta_e - t, p \rangle$ and $\langle \zeta_e - t', p \rangle$ are proper ideal of $\mathbb{Z}[\zeta_e]$. Then they are prime ideals lying over $p$. Hence they are conjugate in $\mathbb{Z}[\zeta_e]$. By (1) we find that $\langle \zeta_e - t, p^{d''} \rangle$ and $\langle \zeta_e - t', p^{d''} \rangle$ are also conjugate.

(3) If $\alpha \in \mathbb{Z}[\zeta_n]$ is of norm $\pm p$, then $|\mathbb{Z}[\zeta_n]/\langle \alpha \rangle| = p$. Thus $\langle \alpha \rangle$ is a prime ideal and $p \in \langle \alpha \rangle$. Hence $\langle \alpha \rangle$ is a prime ideal over $p$. It follows that $\langle \alpha \rangle = \langle \zeta_n - t, p \rangle$ for some $t' \in (\mathbb{Z}/m\mathbb{Z})^\times$ of order $n$. Since $\langle \zeta_n - t, p \rangle$ and $\langle \zeta_n - t', p \rangle$ are conjugate by (2), we find that $\langle \zeta_n - t, p \rangle$ is a principal ideal.

Now assume that $\langle \zeta_n - t, p \rangle$ is a principal ideal. For any $e \mid n$, consider $\langle \zeta_e - t, p \rangle$. If $\langle \zeta_e - t, p \rangle = \mathbb{Z}[\zeta_e]$, there is nothing to prove. Now assume that $\langle \zeta_e - t, p \rangle \not\subseteq \mathbb{Z}[\zeta_e]$. Then $\langle \zeta_e - t, p \rangle$ is a prime ideal lying over $p$. Since there is an element $\alpha \in \mathbb{Z}[\zeta_n]$ with
Let \( N_{\mathbb{Q}(\zeta_e)/\mathbb{Q}}(\alpha) = \pm p \), define \( \beta = N_{\mathbb{Q}(\zeta_e)/\mathbb{Q}}(\alpha) \). Then \( N_{\mathbb{Q}(\zeta_e)/\mathbb{Q}}(\beta) = \pm p \). It follows that \( \langle \beta \rangle \) is a prime ideal of \( \mathbb{Z}[\zeta_e] \) over \( p \). But \( \langle \zeta_e - t, p \rangle \) is also a prime ideal over \( p \). Thus \( \langle \zeta_e - t, p \rangle \) is conjugate to \( \langle \beta \rangle \). Hence \( \langle \zeta_e - t, p \rangle \) is a principal ideal of \( \mathbb{Z}[\zeta_e] \).

§4. The main results

Recall that, for positive integers \( m, n \) and the integer \( t \in (\mathbb{Z}/m\mathbb{Z})^\times \) of order \( n \), the group \( G_{m,n} \) is defined in Definition 1.2. In this section, we will find sufficient conditions to guarantee that \( \mathcal{C}(G_{m,n}) \) is rational over \( \mathbb{C} \) under various situations for \( m \) and \( n \).

**Theorem 4.1** ([EM1, Theorem 1.11; Vo; Le, Theorem 2.6]) Let \( K/k \) be a finite Galois extension with \( \pi = \text{Gal}(K/k) \). Let \( M \) be a \( \pi \)-lattice.

Assume furthermore that \( \pi = \langle \tau \rangle \cong C_n \) and \( M \) is a projective \( \mathbb{Z}\pi \)-module. Then the following three statements are equivalent:

(i) the field \( K(M)^\pi \) is stably rational over \( k \);

(ii) the field \( K(M)^\pi \) is rational over \( k \);

(iii) for any \( e \mid n \), the module \( M/\Phi_e(\tau)M \) is a free \( \mathbb{Z}[\zeta_e] \)-module. (Note that \( M/\Phi_e(\tau)M \) is a module over \( \mathbb{Z}\pi/\Phi_e(\tau) \cong \mathbb{Z}[\zeta_e] \).)

Before stating Theorem 4.2, we explain some terminology. We denote by \( \mathcal{C}(\mathbb{Z}[\zeta_e]) \) the ideal class group of \( \mathbb{Z}[\zeta_e] \), i.e. the quotient group of the group of non-zero fractional ideals of \( \mathbb{Z}[\zeta_e] \) by the subgroup of principal ideals. Thus, if \( M \) is a finitely generated torsion-free \( \mathbb{Z}[\zeta_e] \)-module and \( M \cong I_1 \oplus I_2 \oplus \cdots \oplus I_l \) where each \( I_j \) is a non-zero ideal of \( \mathbb{Z}[\zeta_e] \), then the class of \( M \) in \( \mathcal{C}(\mathbb{Z}[\zeta_e]) \), denoted by \( [M] \), is defined as \( [M] = [I_1 \cdot I_2 \cdot \cdots \cdot I_l] \in \mathcal{C}(\mathbb{Z}[\zeta_e]) \). If \( M \) is a finitely generated \( \mathbb{Z}[\zeta_e] \)-module, \( (M)_0 \) denotes \( M/t(M) \) where \( t(M) \) is the torsion submodule of \( M \).

**Theorem 4.2** ([EM2, page 86; Sw3, Theorem 2.10; EK, Theorem 1.4]) Let \( \pi = \langle \tau \rangle \cong C_n \), \( F_\pi \) be the flabby class monoid of Definition 2.3. Then \( F_\pi \) is a finite group and the group homomorphism \( c : F_\pi \to \bigoplus_{e \mid n} \mathcal{C}(\mathbb{Z}[\zeta_e]) \) defined by \( c([M]) = (\ldots, [(M/\Phi_e(\tau)M)_0], \ldots) \) is an isomorphism where \( M \) is a flabby \( \pi \)-lattice.

**Theorem 4.3** Let \( m \) and \( n \) be positive integers. Assume that (i) \( \text{gcd}\{m, n\} = 1 \), there is an integer \( t \) such that \( t \in (\mathbb{Z}/m\mathbb{Z})^\times \) is of order \( n \), and (ii) for any \( e \mid n \), the ideal \( \langle \zeta_e - t, m \rangle \) in \( \mathbb{Z}[\zeta_e] \) is a principal ideal. If \( k \) is a field with \( \zeta_m, \zeta_n \in k \), then \( k(G_{m,n}) \) is rational over \( k \).

**Proof.** The situation \( m = 1 \) or \( n = 1 \) is trivial. Thus we may assume that \( m, n \geq 2 \). For simplicity, write \( G = G_{m,n} \) and \( G = \langle \sigma, \tau : \sigma^m = \tau^n = 1, \tau^{-1}\sigma \tau = \sigma^t \rangle \).
Step 1. Let $V := V_{\text{reg}} = \bigoplus_{g \in G} k \cdot u_g$ be the regular representation space of $G$ such that $h \cdot u_g = u_{hg}$ for any $g, h \in G$. Let \{x(g) : g \in G\} be the dual basis of \{u_g : g \in G\}. Then the induced action of $G$ acts on the dual space $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by $h \cdot x(g) = x(hg)$ for any $g, h \in G$. Since $k[V]$ is the symmetric algebra of $V^*$, we find that $k(G) = k(V)^G = k(x(g) : g \in G)^G$.

Define $X = \sum_{0 \le i \le m-1} \zeta^{-i}_m x(\sigma^i) \in V^*$. Then $\sigma \cdot X = \zeta_m X$.

Define $y_j = \tau^j \cdot X \in V^*$ for $0 \le j \le n - 1$.

It is easy to verify that

$$\sigma : y_j \mapsto \zeta^j_m y_j \text{ for } 0 \le j \le n - 1,$$

$$\tau : y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{n-1} \mapsto y_0.$$

By Theorem 2.6, $k(G) = k(y_j : 0 \le j \le n - 1)^G(z_1, \ldots, z_{mn-n})$ where $g \cdot z_i = z_i$ for all $1 \le i \le mn-n$ and for all $g \in G$.

Now $k(y_j : 0 \le j \le n - 1)^{(\sigma)} = k(y_0^m, y_j/y_{j-1}^m : 1 \le j \le n - 1)$.

Let $\pi = \langle \tau \rangle$ and $\langle y_j : 0 \le j \le n - 1 \rangle$ be the multiplicative subgroup of $k(y_j : 0 \le j \le n - 1)\backslash\{0\}$. As a $\pi$-lattice, $\langle y_j : 0 \le j \le n - 1 \rangle \cong \mathbb{Z}\pi$. Define a $\pi$-sublattice $M$ of $\mathbb{Z}\pi$ by

$$(4.1) \quad M = \langle \tau - t, m \rangle \subset \mathbb{Z}\pi$$

Under the isomorphism $\langle y_j : 0 \le j \le n - 1 \rangle \cong \mathbb{Z}\pi$, $M$ corresponds to $\langle y_0^m, y_j/y_{j-1}^m : 1 \le j \le n - 1 \rangle$. In other words, $k(y_0^m, y_j/y_{j-1}^m : 1 \le j \le n - 1) \cong k(M)$ and $k(G) = k(y_j : 0 \le j \le n - 1)^G(z_1, \ldots, z_{mn-n}) \cong k(M)^{(\pi)}(z_1, \ldots, z_{mn-n})$.

The $\pi$-lattice $M$ is called the Masuda’s ideal in [EM1, page 14]. We will show that it is a projective ideal of $\mathbb{Z}\pi$ in Step 4, i.e. a left ideal of of $\mathbb{Z}\pi$ which is also a $\mathbb{Z}\pi$-projective module.

Step 2. Let $M' = \bigoplus_{0 \le j \le n-1} \mathbb{Z} \cdot v_j$ be the $\pi$-lattice defined by $\tau : v_0 \mapsto v_1 \mapsto \cdots \mapsto v_{n-1} \mapsto v_0$, i.e. $M' \cong \mathbb{Z}\pi$. Consider $k(M \oplus M')^{(\sigma)}$.

Since $\pi = \langle \tau \rangle$ acts faithfully on $k(M)$ and $k(M \oplus M') = k(M)(v_j : 0 \le j \le n - 1)$, it follows that $k(M \oplus M')^{(\sigma)} = k(M)^{(\sigma)}(w_1, \ldots, w_n)$ where $\tau \cdot w_j = w_j$ for all $1 \le j \le n$ by Theorem 2.6. It follows that $k(G) \cong k(M \oplus M')^{(\sigma)}(z'_1, z'_2, \ldots, z'_{mn-2n})$.

We will show that $k(M \oplus M')^{(\sigma)}$ is rational over $k$. Once this is finished, we find that $k(G)$ is rational over $k$.

Step 3. Regard $k(M \oplus M')^{(\sigma)} = k(M')(M)^{(\sigma)}$. Write $K := k(M')$, $k_0 = k(M')^{(\sigma)}$. Then $K/k_0$ is a finite Galois extension with $\text{Gal}(K/k_0) = \pi$. Moreover, $k_0 = k(M)^{(\sigma)}$ is rational over $k$ by Theorem 3.1 since $\zeta_n \in k$ and $\pi = \langle \tau \rangle$ is abelian. If we show that $K(M)^{(\sigma)} = k(M')(M)^{(\tau)}$ is rational over $k_0$, then $k(M \oplus M')^{(\tau)}$ is rational over $k$.

Step 4. From the definition of $M$, i.e. Formula 4.1, it is clear that $[\mathbb{Z}\pi : M] = m$. Since $\gcd(m, n) = 1$, it follows that $M$ is a projective $\mathbb{Z}\pi$-module by [Sw1] Proposition 7.1; Ka2, Theorem 3.9]. Hence we may apply Theorem 4.1 to $M$. 

10
We remark that this is the only situation in which the assumption $\gcd\{m, n\} = 1$ is used.

Note that, since $M$ is $\mathbb{Z}\pi$-projective, it follows that $M/\Phi_e(\tau)M = \mathbb{Z}\pi/\Phi_e(\tau) \otimes M$ is torsion-free, i.e. $M/\Phi_e(\tau)M = (M/\Phi_e(\tau)M)_0$ in the notation of Theorem 4.2.

Since $M$ is a sublattice of $\mathbb{Z}\pi$ with $\mathbb{Z}\pi/M$ torsion, we may use [LC, Proposition 2.2] to evaluate $M/\Phi_e(\tau)M$. We find that, $M/\Phi_e(\tau)M$ is the image of $M = \langle \tau - t, m \rangle$ in $\mathbb{Z}\pi/\Phi_e(\tau) \cdot \mathbb{Z}\pi \simeq \mathbb{Z}[\zeta_e]$. We find that $M/\Phi_e(\tau)M \simeq \langle \zeta_e - t, m \rangle$.

By assumption, for all $e | n$, $\langle \zeta_e - t, m \rangle$ is a principal ideal, i.e. the class of $\langle \zeta_e - t, m \rangle$ in $C(\mathbb{Z}[\zeta_e])$ is the zero class. By Theorem 4.2, we obtain that $c([M]) = 0$ and thus $[M] = 0$ in $F_\pi$.

In conclusion, $M$ is a projective ideal of $\mathbb{Z}\pi$ and $[M] = 0$ in $F_\pi$ (equivalently, there is permutation $\pi$-lattices $Q_1$ and $Q_2$ such that $M \oplus Q_1 \simeq Q_2$).

Step 5. Since $M$ is $\mathbb{Z}\pi$-projective, there is a projective $\mathbb{Z}\pi$-module $P$ such that $M \oplus P \simeq \mathbb{Z}\pi^l$ for some integer $l$. It follows that $0 \rightarrow M \rightarrow \mathbb{Z}\pi^l \rightarrow P \rightarrow 0$ is a flabby resolution. Thus $[M]^l = [P]$.

From $M \oplus P \simeq \mathbb{Z}\pi^l$ and $M \oplus Q_1 \simeq Q_2$, we get $Q_2 \oplus P \simeq Q_1 \oplus M \oplus P \simeq Q_1 \oplus \mathbb{Z}\pi^l$. Hence $[M]^l = [P] = 0$ in $F_\pi$.

By Theorem 2.5, $K(M)^{(\tau)}$ is stably rational over $k_0$ (remember that the definitions of $K$ and $k_0$ in Step 3). Apply Theorem 4.1. We obtain that $K(M)^{(\tau)}$ is rational over $k_0$ because $M$ is $\mathbb{Z}\pi$-projective.

Remark. In the above theorem, the assumption $\zeta_m, \zeta_n \in k$ may be replaced by $\zeta_m \in k$ and $k(C_n)$ is rational over $k$, because we may apply Theorem 1.1.

Lemma 4.4 Let $\pi$ be a cyclic group of order $n$ and $M$ be a $\pi$-lattice satisfying that $[M]^l = 0$. If $k$ is a field with $\zeta_n \in k$, then $k(M)^\pi$ is stably rational over $k$.

Proof. Step 1. Define $\pi' = \{ \lambda \in \pi : \lambda \text{ acts trivially on } M \}$. Define $\pi'' = \pi/\pi'$. Then $M$ is a faithful $\pi''$-lattice. As a $\pi$-lattice, $[M]^l = 0$. It follows that, as a $\pi''$-lattice, we also have $[M]^l = 0$ by [CTS, page 180, Lemma 2].

Step 2. We use the ideas of Step 2 and Step 3 in the proof of Theorem 4.3.

Define $M' = \mathbb{Z}\pi''$. Consider the fixed field $k(M \oplus M')^{\pi} (= k(M \oplus M')^{\pi''})$. Since $M$ is a faithful $\pi''$-lattice, we may apply Theorem 2.6. We get $k(M \oplus M')^{\pi} = k(M)^{\pi}(w_1, \ldots, w_e)$ where $e = |\pi''|$. On the other hand, since $[M]^l = 0$ as a $\pi''$-lattice, we may apply Theorem 2.5. It follows that $k(M \oplus M')^{\pi}$ is stably rational over $k(M')^{\pi''}$. Since $\zeta_n \in k$, clearly $\zeta_e \in k$ (remember that $e = |\pi''|$). Hence $k(M')^{\pi''}$ is rational over $k$ by Theorem 1.1. In conclusion, $k(M \oplus M')^{\pi}$ is stably rational over $k$. Thus we find that $k(M)^\pi$ is also stably rational over $k$.

Remark. In the above lemma, the condition $[M]^l = 0$ is a mild restriction. In fact, it may happen that $M$ may be a projective $\mathbb{Z}\pi$-module which is not stably free and is not a permutation lattice, while $[M] = 0$ (equivalently, $[M]^l = 0$). Here is such an example taken from [EMI] page 18, line -17.
Let \( \pi = \langle \tau \rangle \simeq C_{12} \), \( t \) be an integer such that \( t \in (\mathbb{Z}/13\mathbb{Z})^\times \) is of order 12, i.e. a primitive root of \( (\mathbb{Z}/13\mathbb{Z})^\times \). Then \( \pi = \langle \tau - t, 13 \rangle \subset \mathbb{Z}[\pi] \) is a projective \( \mathbb{Z}\pi \)-module by [EM1 Proposition 2.6]. \( M \) is a projective ideal of \( \mathbb{Z}\pi \).

Clearly \( M \) is not a free module, i.e. \( M \) is not a principal ideal of \( \mathbb{Z}\pi \). Obviously it is not a permutation \( \pi \)-lattice. We claim that \( M \) is not a stably free \( \mathbb{Z}\pi \)-module. Otherwise, we have \( M \oplus \mathbb{Z}\pi^{(s)} \simeq \mathbb{Z}\pi^{(s+1)} \) for some positive integer \( s \). Taking the determinant of both sides, we find that \( M \) is a free module, which is a contradiction.

However, the class \([M] \in F_{\pi}\) is the zero class because \( c([M]) = 0 \) by Theorem 4.2 (by Theorem 1.3 \( \mathbb{Z}[\zeta_e] \) is a UFD for all divisors \( e \) of 12). It follows that \( M \) is stably permutation.

**Lemma 4.5** Let \( m, n \) be positive integers. Assume that (i) \( m = p^d \) \((p \geq 3 \) is a prime number, \( d \geq 1)\), \( t \) is an integer such that \( t \in (\mathbb{Z}/m\mathbb{Z})^\times \) is of order \( n \), and (ii) for any \( e \mid n \), the ideal \( \langle \zeta_e - t, m \rangle \) in \( \mathbb{Z}[\zeta_e] \) is a principal ideal. If \( k \) is a field with \( \zeta_m, \zeta_n \in k \), then \( k(G_{m,n}) \) is rational over \( k \).

**Proof.** Write \( G_{m,n} = \langle \sigma, \tau : \sigma^m = \tau^n = 1, \tau^{-1}\sigma\tau = \sigma^t \rangle \).

All the assumptions and the conclusion in this lemma are the same as those in Theorem 4.3 except that we don’t assume that \( \gcd\{m, n\} = 1 \) and replace it by \( m = p^d \).

The proof of Theorem 4.3 remains valid till Step 3. We will also show that the ideal \( M = \langle \tau - t, m \rangle \subset \mathbb{Z}\pi \) is a projective \( \mathbb{Z}\pi \)-module.

The fact that \( M \) is \( \mathbb{Z}\pi \)-projective follows from [EM1 Proposition 2.6]. In fact, it is the Masuda’s ideal mentioned in Step 1 of the proof of Theorem 4.3. The ideal \( M \) is denoted by \( M_V \) (and also by \( I_k(p^i) \)) in [EM1 page 14–15].

For the convenience of the reader, we explain briefly the main idea of the proof [EM1 Proposition 2.6]. Using our notation, consider the module \( \mathbb{Z}\pi/M \). By a direct verification, we show that \( \mathbb{Z}\pi/M \) is cohomologically trivial [RIi Theorem 4.12] (the verification is straightforward). Hence \( M \) is also cohomologically trivial. Then we may apply [RIi Theorem 4.11] to conclude that \( M \) is projective.

Once we know that \( M \) is \( \mathbb{Z}\pi \)-projective, the remaining proof of Theorem 4.3 works as before. 

**Proof of Theorem 4.3.**

Let \( G := G_{m,n} = \langle \sigma, \tau : \sigma^m = \tau^n = 1, \tau^{-1}\sigma\tau = \sigma^t \rangle \).

Write \( m = \prod_{1 \leq i \leq r} p_i^{d_i} \) where \( p_1, \ldots, p_r \) are distinct prime number and \( d_i \geq 1 \).

Define \( m_i := m/p_i^{d_i} \) for \( 1 \leq i \leq r \); and define \( \sigma_i = \sigma^{m_i}, H_i = \langle \sigma_j : j \neq i \rangle \). Then \( \langle \sigma \rangle = \langle \sigma_1 \rangle \times H_1, \) and \( \langle \sigma \rangle = \langle \sigma_1, \ldots, \sigma_r \rangle \).

Let \( V^* = \bigoplus_{g \in G} k \cdot x(g) \) be the same as in Step 1 of the proof of Theorem 4.3. For \( 1 \leq j \leq r \), define

\[
X_j = \sum_{g \in H_j} \sum_{0 \leq l \leq p_j^{d_j} - 1} \zeta_{p_j^{d_j}}^{-l} x(g \sigma_j^l) \in V^*, \quad y_{t}^{(j)} = t^l X_j \quad \text{for } 0 \leq l \leq n - 1.
\]
It is routine to check that
\[
\sigma_i : X_i \mapsto \zeta_{p_i^d_i} X_i, \quad X_j \mapsto X_j \quad \text{if } j \neq i,
\]
\[
y_l^{(i)} \mapsto \zeta_{p_i^d_i}^{l}, \quad y_l^{(i)} \mapsto y_l^{(i)} \quad \text{if } j \neq i
\]
\[
\tau : y_0^{(i)} \mapsto y_1^{(i)} \mapsto \cdots \mapsto y_{n-1}^{(i)} \mapsto y_0^{(i)} \quad \text{for } 1 \leq i \leq r.
\]

By Theorem \ref{th:2.6} \( k(G) \) is rational over \( k(y_l^{(i)} : 1 \leq i \leq r, 0 \leq l \leq n-1)^G \). Moreover, \( k(y_l^{(i)} : 1 \leq i \leq r, 0 \leq l \leq n-1)^c = k((y_0^{(i)})^{p_i^d_i}, y_l^{(i)}/(y_{l-1}^{(i)})^t : 1 \leq i \leq r, 1 \leq l \leq n-1) \).

Write \( \pi = \langle \tau \rangle \). Define a \( \pi \)-lattice \( M_i \) of \( \mathbb{Z} \pi \) by
\[
M_i = \langle \tau - t, p_i^{d_i} \rangle \quad \text{for } 1 \leq i \leq r.
\]

Since each \( p_i \) is odd, it follows that \( M_i \) is \( \mathbb{Z} \pi \)-projective by \cite{EMI} Proposition 2.6 (see the proof of Lemma \ref{lem:4.5}). Note that \( k(y_l^{(i)} : 1 \leq i \leq r, 0 \leq l \leq n-1)^G = k(M_1 \oplus \cdots \oplus M_r)^{(\tau)} \).

For any \( e \mid n \), \( \langle \zeta_e - t, m \rangle = \prod_{1 \leq i \leq r} \langle \zeta_e - t, p_i^{d_i} \rangle = \prod_{1 \leq i \leq r} [M_i/\Phi_e(\tau)M_i] \) is the \( e \)-th component of \( c(M) \) where \( M = \bigoplus_{1 \leq i \leq r} M_i \) and \( c : F_\pi \to \bigoplus_{e\mid n} C(\mathbb{Z}[\zeta_e]) \) is the isomorphism defined in Theorem \ref{th:4.2}. By assumption \( \langle \zeta_e - t, m \rangle \) is a principal ideal. Hence \( c(M) = 0 \) and thus \( [M] = 0 \) in \( F_\pi \) by Theorem \ref{th:4.2}. The remaining part of the proof is the same as in Theorem \ref{th:4.3}. Done.

In Theorem \ref{th:4.7} we will give a generalization of Theorem \ref{th:4.5} using the method of Theorem \ref{th:4.3}.

Recall the assumptions in Theorem \ref{th:4.5} \( q \) is a prime number, \( t \in (\mathbb{Z}/m\mathbb{Z})^\times \) is of order \( q \), \( G_{m,n} = \langle \sigma, \tau : \sigma^m = \tau^q = 1, \tau^{-1}\sigma = \sigma^t \rangle \), and \( m' = m/\gcd\{m, t-1\} \).

Write \( m = q^{d_0} \prod_{1 \leq i \leq s} p_i^{d_i} \cdot \prod_{1 \leq j \leq s'} q_j^{e_j} \) where \( d_0 = \ord_q(m) \geq 0 \), \( d_i, e_j \geq 1 \), \( q_i, q_j \) are distinct prime numbers such that \( p_i \mid t-1 \) for \( 1 \leq i \leq s \), and \( q_j \mid t-1 \) for \( 1 \leq j \leq s' \).

Define \( m'' = \prod_{1 \leq i \leq s} p_i^{d_i} \). Define \( m_1 \) and \( m_2 \) by the formula \( m = m_1 m_2 \) and \( m_2 \) is defined by
\[
m_2 = \begin{cases} 
q^{d_0} m'' & \text{if } q \mid m', \\
m'' & \text{if } q \nmid m'.
\end{cases}
\]

\begin{lemma}
Let \( m', m'', m_1 \) and \( m_2 \) be defined as above. Then
\[
m' = \begin{cases} 
qm'' & \text{if } 1 \leq \ord_q(t-1) < \ord_q(m), \\
m'' & \text{otherwise}.
\end{cases}
\]

In fact, \( q \mid m' \) if and only if \( \ord_q(t-1) = d_0 - 1 \geq 1 \) with \( \ord_q(m_2) = d_0 \).
\end{lemma}
Proof. If \( q \nmid m \), then \( q \nmid m' \). Thus we will consider the situation \( q \mid m \) in the sequel. We will show that, if \( 1 \leq \text{ord}_q(t - 1) < \text{ord}_q(m) \), then \( \text{ord}_q(t - 1) = \text{ord}_q(m) - 1 \).

Since \( q \mid m \), it follows that \( t^q \equiv 1 \pmod{q^{d_0}} \) where \( d_0 = \exp_q(m) \). From \( t \equiv t^q \pmod{q} \), we get \( q \mid t - 1 \).

If \( \text{ord}_q(t - 1) \geq \text{ord}_q(m) \), then \( q \nmid m' \) and \( q \nmid m_2 \). It remains to consider the situation \( 1 \leq \text{ord}_q(t - 1) < \text{ord}_q(m) \).

Write \( t = 1 + aq^{d_0} \) and \( m = bq^{d_0} \) where \( 1 \leq d' < d_0 \) and \( q \nmid ab \). From \( t^q \equiv 1 \pmod{q^{d_0}} \), we find that \( t^q - 1 = (1 + aq^{d_0})^q - 1 = aq^{d_0 + 1} + cq^{2d_0 + 2} \) for some integer \( c \). Thus \( d' + 1 = d_0 \) as we expected. Clearly \( \text{ord}_q(m_2) = 1 \) and \( \text{ord}_q(m_2') = d_0 \).

Suppose that \( q_j = q \) is a prime divisor of \( m \) with \( q_j \mid t - 1 \). Write \( t = 1 + aq_j^{d''} \), \( m = bq_j^{d''} \) where \( d'', e_j \geq 1 \) and \( q_j \nmid ab \). We claim that \( d'' \geq e_j \). Otherwise, \( d'' \leq e_j - 1 \). Then \( q_j^{d''} \) will not divide \( t^q - 1 = (1 + aq_j^{d''})^q - 1 = aqq_j^{d''} + cq_j^{2d''} \) where \( c \) is some integer. We conclude that \( d'' \geq e_j \) and \( q_j \mid m' \).

The following theorem is slightly different from Theorem 4.5. In fact, we don’t require that \( \gcd\{a_0, a_1, \ldots, a_{q-2}, b\} = 1 \) and the positivity condition of the norm in Theorem 4.5 is waived.

**Theorem 4.7** Let \( m \) and \( q \) be positive integers where \( q \) is a prime number and assume that there is an integer \( t \) such that \( t \in (\mathbb{Z}/m\mathbb{Z})^\times \) is of order \( q \). Define \( m' = m/\gcd\{m, t - 1\} \). Assume that there exist integers \( a_0, a_1, \ldots, a_{q-2}, b \) such that \( bm' = a_0 + a_1t + \cdots + a_{q-2}t^{q-2} \) and \( N_{Q(\zeta_q)/Q}(\alpha) = \pm m' \) where \( \alpha := a_0 + a_1\zeta_q + \cdots + a_{q-2}\zeta_q^{q-2} \). If \( k \) is a field with \( \zeta_m, \zeta_q \in k \), then \( k(G_{m,n}) \) is rational over \( k \).

**Proof.** Write \( G := G_{m,n} = \langle \sigma, \tau : \sigma^m = \tau^q = 1, \tau^{-1}\sigma\tau = \sigma^t \rangle \).

Step 1. If \( q = 2 \), we may apply Theorem 4.3. Thus we assume that \( q \geq 3 \) from now on. The notations \( m', m'', m_1, m_2 \) and \( p_i (1 \leq i \leq s) \), \( q_j (1 \leq j \leq s') \) in Lemma 4.6 remain in force throughout the proof.

Note that \( m'' \) and \( m_2 \) are odd integers. Otherwise, \( p_i = 2 \) for some \( 1 \leq i \leq s \). By definition, \( t \in (\mathbb{Z}/p_i^{d_i}\mathbb{Z})^\times \) is of order \( q > 1 \). If \( p_i = 2 \), then \( (\mathbb{Z}/p_i^{d_i}\mathbb{Z})^\times \) is a group of order \( 2^{d_i-1} \). Since \( q \) is odd, we get a contradiction.

Step 2. By Lemma 4.6 and the definitions of \( m_1 \) and \( m_2 \) in Equation (4.3), it is routine to verify that \( m_1 \mid t - 1 \) no matter whether \( q \mid m' \) or not.

Define \( \lambda = \sigma^{m_2}, \rho = \sigma^{m_1} \). Then \( \tau^{-1}\lambda\tau = \lambda \) because \( m_1 \mid t - 1 \).

Define \( H := \langle \lambda \rangle \simeq C_{m_1} \) and \( G_0 := \langle \rho, \tau \rangle \simeq G_{m_2,q} \).

Thus \( G = H \times G_0 \). If both \( k(H) \) and \( k(G_0) \) are rational, then \( k(G) \) is also rational by [KP] Theorem 1.3. Note that \( k(H) \) is rational by Theorem 4.1. It remains to show that \( k(G_0) \) is rational over \( k \).

To show that \( k(G_0) \) is rational over \( k \), we will apply Theorem 4.6 (remember that both \( m_2 \) and \( q \) are odd by Step 1). Thus the goal is to show that \( \langle \zeta_q - t, m_2 \rangle \) is a principal ideal of \( \mathbb{Z}[\zeta_q] \).

Step 3. From the assumption of Theorem 4.5 there exist integers \( a_0, a_1, \ldots, a_{q-2}, b \) such that \( bm' = a_0 + a_1t + \cdots + a_{q-2}t^{q-2} \) and \( N_{Q(\zeta_q)/Q}(\alpha) = \pm m' \) where \( \alpha := a_0 + \cdots + a_{q-2}t^{q-2} \). Then \( \tau^{-1}\lambda\tau = \lambda \) because \( m_1 \mid t - 1 \).
$a_1 \zeta_q + \cdots + a_q - 2 \zeta_q^{q-2}$. It follows that $\alpha - bm' = (\zeta_q - t) \cdot \beta$ for some $\beta \in \mathbb{Z}[\zeta_q]$, i.e. $\alpha \in (\zeta_q - t, m')$.

We will show that $N_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\langle \zeta_q - t, m' \rangle) = m'$.

If $q \nmid m'$, then $m' = m'' = \prod p^d$ and $\langle \zeta_q - t, m' \rangle = \prod \langle \zeta_q - t, p_i^d \rangle$ by Lemma 3.2. Since $\langle \zeta_q - t, p_i^d \rangle \subset \langle \zeta_q - t, p_i \rangle$ and $t \in (\mathbb{Z}/p_i^d \mathbb{Z})^\times$ is of order $q$, it follows that $\langle \zeta_q - t, p_i \rangle \subset \mathbb{Z}[\zeta_q]$ is of index $p_i$ by Lemma 3.3. Hence $N_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\langle \zeta_q - t, p_i \rangle) = p_i$ and $N_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\langle \zeta_q - t, p_i^d \rangle) = p_i^d$. Thus $N_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\langle \zeta_q - t, m' \rangle) = m'$.

For the other situation, if $q \mid m'$, then $m'' = qm''$. We have $\langle \zeta_q - t, m'' \rangle = \langle \zeta_q - t, q \rangle \cdot \langle \zeta_q - t, m'' \rangle$. Note that $t - 1 = aq^{d_0} - 1$ where $d_0 - 1 \geq 1$ and $q \nmid a$ by Lemma 1.6. It follows that $\zeta_q - t = \zeta_q - 1 - (t - 1) = (\zeta_q - 1) \cdot (1 + (\zeta_q - 1) \cdot \beta)$ for some $\beta \in \mathbb{Z}[\zeta_q]$. Thus $\langle \zeta_q - t, q \rangle = \langle \zeta_q - 1 \rangle$ is of norm $q$. We can show that $N_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\langle \zeta_q - t, m'' \rangle) = m''$ as the previous situation. Hence we find $N_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\langle \zeta_q - t, m' \rangle) = m'$.

Since $\langle \alpha \rangle \subset \langle \zeta_q - t, m' \rangle$ and both ideals have the same norm $m'$, it follows that $\langle \alpha \rangle = \langle \zeta_q - t, m' \rangle$, i.e. the ideal $\langle \zeta_q - t, m' \rangle$ is a principal ideal.

Step 4. We will show that the ideals $\langle \zeta_q - t, m'' \rangle$ and $\langle \zeta_q - t, m_2 \rangle$ are principal ideals. This will finish the proof by Step 2.

If $q \nmid m'$, then $m' = m'' = m_2$. Thus $\langle \zeta_q - t, m_2 \rangle = \langle \zeta_q - t, m'' \rangle = \langle \zeta_q - t, m' \rangle$ is a principal ideal by Step 3.

It remains to consider the case $q \mid m'$, i.e. $m' = qm''$ and $m_2 = q^{d_0}m''$.

From $\langle \alpha \rangle = \langle \zeta_q - t, m' \rangle = \langle \zeta_q - t, q \rangle \langle \zeta_q - t, m'' \rangle = \langle \zeta_q - 1 \rangle \cdot \langle \zeta_q - t, m'' \rangle$, we get $\alpha = (\zeta_q - 1) \cdot \beta$ for some $\beta \in \mathbb{Z}[\zeta_q]$. Hence $\langle \zeta_q - t, m'' \rangle = \langle \beta \rangle$ is a principal ideal.

Now $\langle \zeta_q - t, m_2 \rangle = \langle \zeta_q - t, q^{d_0} \rangle\cdot\langle \zeta_q - t, m'' \rangle = \langle \zeta_q - t, m_2 \rangle = \langle \zeta_q - t, m'' \rangle$ is a principal ideal because so is the ideal $\langle \zeta_q - t, m'' \rangle$.

\textbf{Theorem 4.8} Let $p$ and $n$ be positive integers where $p$ is a prime number and assume that $t$ is an integer such that $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ is of order $n$. Assume that there is some element $\alpha \in \mathbb{Z}[\zeta_n]$ satisfying that $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\alpha) = \pm p$. If $k$ is a field with $\zeta_p$, $\zeta_n \in k$, then $k(G_{p,n})$ is rational over $k$.

\textbf{Proof.} The ideals $\langle \alpha \rangle$ and $\langle \zeta_n - t, p \rangle$ are of norm $p$. Both of them are prime ideals of $\mathbb{Z}[\zeta_n]$ lying over $p$. Thus they are conjugate to each other. In particular, the ideal $\langle \zeta_n - t, p \rangle$ is a principal ideal. For any $e \mid n$, if $e < n$, then $\langle \zeta_e - t, p \rangle = \mathbb{Z}[\zeta_e]$ by Lemma 3.3. Now apply Theorem 4.3.

\textbf{Theorem 4.9} Let $n$ be a positive integer. Define $S := \{ p \in \mathbb{N} : p$ is a prime number and $p$ splits completely into the product of principal prime ideals of $\mathbb{Z}[\zeta_n] \}$, and define $S_0 = \{ p \in \mathbb{N} : p$ is a prime number such that $C(G_{p,n})$ is rational over $\mathbb{C} \}$. Then the Dirichlet densities of $S$ and $S_0$ are positive; in particular, $S_0$ is an infinite set. Consequently, there are infinitely many prime numbers $p$ satisfying that $C(G_{p,n})$ is rational over $\mathbb{C}$.

\textbf{Proof.} Step 1. If $p \in S$, then $p$ splits completely in $\mathbb{Z}[\zeta_n]$. Thus $n \mid \phi(p)$ by considering the factorization of $\Phi_n(X) \pmod{p}$; alternatively, apply [Wee page 103]. It
follows that there is an integer $t$ such that $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ is of order $n$. Hence we may define the group $G_{p,n}$ as in Definition 1.2.

Since $p$ splits completely into the product of principal prime ideals of $\mathbb{Z}[\zeta_n]$, there is some element $\alpha \in \mathbb{Z}[\zeta_n]$ such that $\langle \alpha \rangle$ is a prime ideal lying over $p$ with $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\alpha) = \pm p$ by Lemma 3.3. It follows that $\mathbb{C}(G_{p,n})$ is rational by Theorem 4.8. In summary, if $p$ is a prime number and $p \in S$, then $\mathbb{C}(G_{p,n})$ is rational, i.e. $S \subset S_0$.

It remains to show that the Dirichlet density of $S$ is positive. We denote by $d(S)$ the Dirichlet density of $S$; the reader is referred to [Ne, page 130] for its definition.

Step 2. Suppose that $p \in S$ and $P$ is a non-zero principal prime ideal of $\mathbb{Z}[\zeta_n]$ lying over $p$, then $P$ is of degree one, i.e. natural map $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}[\zeta_n]/P$ is an isomorphism. Define $T_1 = \{P : P$ is a non-zero principal prime ideal of degree one in $\mathbb{Z}[\zeta_n]\}$, and $T = \{P : P$ is a non-zero principal prime ideal of $\mathbb{Z}[\zeta_n]\}$. We will show that $d(T_1) = d(T) = h_n$ where $h_n$ is the class number of $\mathbb{Q}(\zeta_n)$.

Step 3. Let $L$ be the Hilbert class field of $\mathbb{Q}(\zeta_n)$; note that $\text{Gal}(L/\mathbb{Q}(\zeta_n)) \simeq C(\mathbb{Z}[\zeta_n])$ is the ideal class group of $\mathbb{Z}[\zeta_n]$. If $P$ is a non-zero prime ideal of $\mathbb{Z}[\zeta_n]$, denote by $(P, L/\mathbb{Q}(\zeta_n))$ the Artin symbol of $P$, if $P$ is unramified in $L$ (see [Ne, page 105]). If $P$ is a non-zero prime ideals of $\mathbb{Z}[\zeta_n]$, then $P \in T$ if and only if $P$ splits completely in $L$ by [Ne] page 107, Corollary 8.5] (alternatively, this fact is one of the conditions in the definition of the Hilbert class field). On the other hand, $P$ splits completely in $L$ is equivalent to $(P, L/\mathbb{Q}(\zeta_n)) = 1$. It follows from the Tchebotarev density theorem that $d(T) = 1/h_n$ [Ne page 132, Theorem 6.4].

In summary, we have

\begin{equation}
\frac{1}{h_n} = \lim_{s \to 1^+} \frac{\sum_{P \in T} \frac{1}{N(P)^s}}{\sum_{P} \frac{1}{N(P)^s}}
\end{equation}

where $N(P)$ is the abbreviation of $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(P)$.

Step 4. Note that $\sum_{P} \frac{1}{N(P)^s} \sim \log \frac{1}{s-1}$ (see [Ne page 130]).

Also note that $\sum_{P \in T} \frac{1}{N(P)^s} \sim \sum_{\deg P \leq 2} \frac{1}{N(P)^s}$ because $\sum_{\deg P \geq 2} \frac{1}{N(P)^s}$ is an analytic function at $s = 1$ (see [Ne page 130]). We find that

\begin{equation}
\frac{1}{h_n} = \lim_{s \to 1^+} \frac{\sum_{P \in T_1} \frac{1}{N(P)^s}}{\log \frac{1}{s-1}}.
\end{equation}

Step 5. Define a function $\Psi : T_1 \to S$ by $\Psi(P) = P \cap \mathbb{Z}$ (here we identify a prime number $p$ with the prime ideal $p\mathbb{Z}$ in $\mathbb{Z}$). Note that $\Psi$ is well-defined. Also note that $\Psi^{-1}(p)$ is a set of $\phi(n)$ elements for any $p \in S$. Thus

$$\sum_{P \in T_1} \frac{1}{N(P)^s} = \phi(n) \cdot \sum_{p \in S} \frac{1}{p^s}.$$
We have also \( \sum_p \frac{1}{p^r} \sim \log \frac{1}{s-1} \) as in Step 4. From Formula (4.5), we obtain

\[
d(S) = \lim_{s \to 1^+} \frac{\sum_{p\leq S} \frac{1}{p^r}}{\sum_{p} \frac{1}{p^r}} = \lim_{s \to 1^+} \frac{1}{\phi(n)} \sum_{P \in \mathcal{T}_1(N(P))} \frac{1}{\phi(n)} = \frac{1}{\phi(n) \cdot h_n}.
\]

\[\square\]

**Example 4.10** We will supply examples of groups \( G \cong A \times C_n \) (where \( A \) is an abelian group) such that \( C(G) \) is rational while we cannot apply Theorem 1.3 to assert the rationality (because \( \mathbb{Z}[\zeta_n] \) is not a UFD).

Let \( d_1, \ldots, d_s, d \) be integers with \( 1 \leq d \leq \min\{d_1 - 1, d_2 - 1, \ldots, d_s - 1\} \). Define \( n = 3^d, m_i = 3^{d_i} \) for \( 1 \leq i \leq s \). For each \( 1 \leq i \leq s \), find an integer \( t_i \) such that \( t_i \in (\mathbb{Z}/m_i\mathbb{Z})^\times \) is of order \( n \). Define \( G := \langle \sigma_1, \ldots, \sigma_s, \tau : \sigma_i^{m_i} = \tau^n = 1, \sigma_i \sigma_j = \sigma_j \sigma_i, \tau^{-1} \sigma_i \tau = \sigma_i^{t_i} \rangle \) for \( 1 \leq i \leq s \). Note that \( G \) is a 3-group. If \( \zeta_{3^n} \in k \) (where \( e \geq d_i \) for all \( 1 \leq i \leq s \)), we claim that \( k(G) \) is rational over \( k \).

Note that, if \( d \geq 4 \), then \( \mathbb{Z}[\zeta_n] \) is not a UFD by Theorem 1.4. Thus we cannot apply Theorem 1.3 to show that \( k(G) \) is rational if \( d \geq 4 \).

On the other hand, the method in Lemma 4.3 and in the proof of Theorem 1.6 may be adapted to show that \( k(G) \) is rational. Write \( \pi = \langle \tau \rangle \) and define \( \pi \)-lattices \( M_i \) as in Formula (4.2) by

\[
M_i = \langle \tau - t_i, m_i \rangle \subset \mathbb{Z}[\tau].
\]

As before, it is easy to see that each \( M_i \) is a \( \mathbb{Z}[\pi] \)-projective module and \( k(G) \) is rational over \( k(M_1 \oplus \cdots \oplus M_s)^{(\tau)} \). By the same proof as [EM1, Corollary 3.3], \( [M_i]^{fl} = 0 \) for all \( i \) (rigorously speaking, \( [M_i]^{fl} = 0 \) because of [EM1, Proposition 3.2] and its proof). Thus \( [M_1 \oplus M_2 \oplus \cdots \oplus M_s]^{fl} = 0 \) and the same arguments in the proof of Theorem 1.6 work as well. Done.

The same method can be applied to 2-groups. Let \( d_1, \ldots, d_s, d \) be integers with \( 1 \leq d \leq \min\{d_1 - 2, d_2 - 2, \ldots, d_s - 2\} \). Define \( n = 2^d, m_i = 2^{d_i} \). Find an integer \( t_i \) such that \( t_i \in (\mathbb{Z}/m_i\mathbb{Z})^\times \) is of order \( n \) (this is possible because \( d \leq d_i - 2 \)). Define \( G := \langle \sigma_1, \ldots, \sigma_s, \tau : \sigma_i^{m_i} = \tau^n = 1, \sigma_i \sigma_j = \sigma_j \sigma_i, \tau^{-1} \sigma_i \tau = \sigma_i^{t_i} \rangle \) for \( 1 \leq i \leq s \). If \( \zeta_{2^n} \in k \) (where \( e \geq d_i \) for all \( 1 \leq i \leq s \)), we claim that \( k(G) \) is rational over \( k \).

Note that, if \( d \geq 6 \), then \( \mathbb{Z}[\zeta_n] \) is not a UFD by Theorem 1.4. The proof of the rationality is similar to the above situation of 3-groups, but some modification is necessary.

If \( d = 1 \), we may apply Theorem 1.3. Thus we assume that \( d \geq 2 \). Since \( \langle t_i \rangle \cong C_n \) is a cyclic group of order \( \geq 4 \), it follows that \( -1 \notin \langle t_i \rangle \subset (\mathbb{Z}/m_i\mathbb{Z})^\times \). By [EM1, Proposition 2.6], the \( \pi \)-lattice \( \langle \tau - t_i, m_i \rangle \subset \mathbb{Z}[\tau] \) is projective where \( \pi = \langle \tau \rangle \). Note that \( t - 1 \) is an even integer. If \( e \mid n \) and \( e \geq 2 \), then \( \langle \zeta_{e - t_i}, m_i \rangle = \langle \zeta_{e - 1} \rangle \). Thus \( [M_1 \oplus M_2 \oplus \cdots \oplus M_s]^{fl} = 0 \). Done.

We remark that, if \( G \) is a metacyclic \( p \)-group (a \( p \)-group which is an extension of a cyclic group by another cyclic group), then \( C(G) \) is always rational by [Kal].

**Example 4.11** Let \( p \geq 3 \) be a prime number, \( a_1, a_2, \ldots, a_s \) be positive integers with \( p \nmid a_i a_2 \cdots a_s \). Define \( m_i = a_i p^{d_i}, t_i = 1 + a_i p^{d_i - 1} \) where \( d_i \geq 2 \) for \( 1 \leq i \leq s \). Define
\[ G = \langle \sigma_1, \ldots, \sigma_s, \tau : \sigma_i^{m_i} = \tau^p = 1, \sigma_i\sigma_j = \sigma_j\sigma_i, \tau^{-1}\sigma_i\tau = \sigma_i^{t_i} \text{ for } 1 \leq i \leq s \rangle. \] If \( \zeta_\ell \in k \) (where \( e = \text{lcm}\{m_i : 1 \leq i \leq s\} \)), we claim that \( k(G) \) is rational.

Note that the case when \( s = 1 \) is proved in [CH Corollary 18].

Define \( \lambda_i = \sigma_i^{p^d_i}, \rho_i = \sigma_i^{q_i}, H_1 = \langle \lambda_i : 1 \leq i \leq s \rangle, H_2 = \langle \rho_i, \tau : 1 \leq i \leq s \rangle \). Then \( G = H_1 \times H_2 \). Note that \( k(H_1) \) is rational by Theorem 4.9. We will prove that \( k(H_2) \) is rational. Then we may apply [KP Theorem 1.3] to conclude that \( k(G) \) is rational.

Now we consider \( k(H_2) \). Define \( \pi = \langle \tau \rangle \) and \( \pi \)-lattices \( M_i \) defined by \( \lambda_i = \langle \tau - t_i, p^d_i \rangle \) as before. Note that \( \zeta_p - t_i = \zeta_p - 1 - (t_i - 1) = (\zeta_p - 1)(1 + \alpha(\zeta_p - 1)^{p - 2}) \) for some \( \alpha \in \mathbb{Z}[\zeta_p] \). Hence \( \langle \zeta_p - t_i, p^d_i \rangle = \langle \zeta_p - 1 \rangle \) is a principal ideal. Thus \( [M_1 \oplus \cdots \oplus M_s]^{\pi} = 0 \). Done.

**Example 4.12** We will give an application of Theorem 4.9

Let \( k \) be an algebraic number field. Define \( P_0 = \{p : p \text{ is a prime number, } p \leq 43\} \cup \{61, 67, 71\} \) and \( P_k = \{p : p \text{ ramifies in } k\} \). It is known that, if \( p \) is a prime number, then (1) \( \mathbb{Q}(\zeta_p) \) is rational over \( \mathbb{Q} \) if and only if \( p \in P_0 \) [P], and (2) if \( p \notin (P_0 \cup P_k) \), then \( k(C_p) \) is not stably rational over \( k \) [Ka4]. If \( p \in P_0 \), then \( k(C_p) \) is rational over \( k \) because \( \mathbb{Q}(C_p) \) is rational over \( \mathbb{Q} \). However, it is not clear whether \( k(C_p) \) is rational over \( k \) if \( p \in P_k \).

We will construct infinitely many pairs \((p, k)\) where \( p \) is a prime number, \( k \) is an algebraic number field such that \( p \) ramifies in \( k \) and \( k(C_p) \) is rational over \( k \).

Given a positive integer \( n \), let \( S \) be the set defined in Theorem 4.9. For each \( p \in S \), \( n \mid \phi(p) \) (see Step 1 in the proof of Theorem 4.9). Choose \( k \) to be the subfield of \( \mathbb{Q}(\zeta_p) \) such that \( [\mathbb{Q}(\zeta_p) : k] = n \). Note that the field \( k \) depends on the choice of \( p \in S \). Also note that \( p \) ramifies in \( \mathbb{Q}(\zeta_p) \) (and also in \( k \)). Thus \( \mathbb{Q}(\zeta_p) \) and \( \mathbb{Q}(\zeta_n) \) are linearly disjoint over \( \mathbb{Q} \), because \( p \) splits completely in \( \mathbb{Q}(\zeta_n) \).

Since \( k(\zeta_p) = \mathbb{Q}(\zeta_p) \) is of degree \( n \) over \( k \) and there is some element \( \alpha \in \mathbb{Z}[\zeta_n] \) with \( N_{\mathbb{Q}(\zeta_n) / \mathbb{Q}}(\alpha) = \pm p \) (see Step 1 in the proof of Theorem 4.9), we find that \( k(C_p) \) is rational over \( k \) by [Le page 321, Corollary 7.1].

### §5. The multiplicative invariant fields

Let \( \pi \) be the cyclic group of order \( n \). Beneish and Ramsey [BR] introduced a notion the Property * for \( \pi \) in [BR Definition 3.3] and proved Theorem 5.7.

Here is our interpretation of the Property * for \( \pi \).

**Theorem 5.1** Let \( \pi \) be a cyclic group of order \( n \). Then the following are equivalent:

(i) the Property * for \( \pi \);
(ii) \( F_\pi = 0 \);
(iii) \( \mathbb{Z}[\zeta_n] \) is a UFD.

**Proof.** (iii) \( \Rightarrow \) (i) by [BR Corollary 3.9].

18
(i) ⇒ (ii). By [EK, Lemma 2.2] we have an isomorphism $T(\pi) \to F_\pi$ where $T(\pi)$ is defined in [EM2, page 86; EK, Definition 1.3]. In order to show that $F_\pi = 0$, it is necessary and sufficient to show that $T(\pi) = 0$. By [EK, Theorem 1.4], we have an isomorphism $C(\mathbb{Z}[\pi])/C^O(\mathbb{Z}[\pi]) \to T(\pi)$ where $C(\mathbb{Z}[\pi])$ is the subgroup of $K_0(\mathbb{Z}[\pi])$, the Grothendieck group of the category of finitely generated projective $\mathbb{Z}[\pi]$-modules, defined by

$$C(\mathbb{Z}[\pi]) := \{[A] - [\mathbb{Z}[\pi]] \in K_0(\mathbb{Z}[\pi]) : A \text{ is a projective ideal over } \mathbb{Z}[\pi] \} \quad \text{(see EK, Definition 2.11)}.$$ 

The map $\varphi$ is induced by the map $\psi : C(\mathbb{Z}[\pi]) \to T(\pi)$ defined by sending $[A] - [\mathbb{Z}[\pi]] \in C(\mathbb{Z}[\pi])$ to $[A] \in T(\pi)$; note that $C^O(\mathbb{Z}[\pi]) := \{[A] - [\mathbb{Z}[\pi]] \in C(\mathbb{Z}[\pi]) : [A] = 0 \text{ in } T(\pi)\}$ is the kernel of $\psi$ (see EK, Definition 2.12). By [BR, Theorem 3.11] every projective ideal $A$ is stably permutation; thus $C(\mathbb{Z}[\pi])/C^O(\mathbb{Z}[\pi]) = 0$ and so $T(\pi) = 0$.

(ii) ⇒ (iii) by Theorem 1.2. Done.

We indicate a direct proof of (iii) ⇒ (ii). Suppose that $\mathbb{Z}[\zeta_n]$ is a UFD. For any $e | n$, $\mathbb{Z}[\zeta_n]$ is also a UFD by [Wa, page 39, Proposition 4.11]. Hence $\bigoplus_{e|n} C(\mathbb{Z}[\zeta]) = 0$. By Theorem 1.2 again we find $F_\pi = 0$.

**Proof of Theorem 1.7**

(i) By Theorem 5.1, $F_\pi = 0$. Thus $[M]^f_\pi = 0$ for any $\pi$-lattice $M$. Apply Lemma 4.4.

(ii) By Theorem 5.1, $\mathbb{Z}[\zeta_n]$ is a UFD. Apply Theorem 1.3. Note that we prove that $k(G)$ is not only stably rational, but also rational. ■

**Theorem 5.2** Let $\pi \simeq D_n$ the dihedral group of order $2n$ where $n$ is an odd integer. Then $F_\pi = 0$ if and only if $\mathbb{Z}[\zeta_n + \zeta_n^{-1}]$ is a UFD.

**Proof.** As in the proof of Theorem 5.1 let $T(\pi)$ be defined in [EM2, page 86; EK, Definition 1.3]. Then we have $F_\pi \simeq T(\pi) \simeq C(\Omega_{\mathbb{Z}[\pi]})$ where $\Omega_{\mathbb{Z}[\pi]}$ is a maximal $\mathbb{Z}$-order in $\mathbb{Q}[\pi]$ containing $\mathbb{Z}[\pi]$ and $C(\Omega_{\mathbb{Z}[\pi]})$ is the class group of $\Omega_{\mathbb{Z}[\pi]}$ [EM2, Theorem 3.3; EK, Theorem 1.4]. It can be shown that $C(\Omega_{\mathbb{Z}[\pi]}) \simeq \bigoplus_{d|n} C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}])$ (see, for examples, EK, Theorem 6.3). If $F_\pi = 0$, then $C(\mathbb{Z}[\zeta_n + \zeta_n^{-1}]) = 0$.

On the other hand, if $C(\mathbb{Z}[\zeta_n + \zeta_n^{-1}]) = 0$, then $C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]) = 0$ for all $d | n$ by [Wa, page 39, Proposition 4.11]. Thus $F_\pi = 0$. ■

**Lemma 5.3** Let $\pi \simeq D_n$ the dihedral group of order $2n$ where $n$ is an odd integer. Assume that $\mathbb{Z}[\zeta_n + \zeta_n^{-1}]$ is a UFD.

(i) If $M$ is any $\pi$-lattice and $k$ is a field with $\zeta_n \in k$, then $k(M)^\pi$ is stably rational over $k$.

(ii) Suppose that $G := A \rtimes D_n$ where $A$ is a finite abelian group of exponent $e$ and $n$ is an odd integer. If $k$ is a field with $\zeta_e, \zeta_n \in k$, then $k(G)$ is stably rational over $k$.

**Proof.** By Theorem 5.2, $F_\pi = 0$.

(i) Since $F_\pi = 0$, we find that $[M]^f_\pi = 0$. The idea of showing that $k(M)^\pi$ is stably rational is almost the same as that of Lemma 4.4. It remains to prove that $k(\pi''')$ is rational where $\pi'''$ is some quotient group of $\pi$.

19
Since $n$ is odd, the quotients groups of $\pi \cong D_n$ are isomorphic to $D_m$ ($m$ divides $n$), $C_2$, or the trivial group.

Suppose that $\pi''$ is a dihedral group. Write $\pi'' = \langle \sigma_0, \tau : \sigma_0^m = \tau^2 = 1, \tau^{-1}\sigma_0\tau = \sigma_0^{-1} \rangle$ where $m$ is some integer dividing $n$. Define $\pi_0 := \langle \tau \rangle \cong C_2$. If $\text{char } k \neq 2$, apply Theorem 1.3. Here is a proof for the general case.

Since $\zeta_n \in k$, we may use the same arguments as in Step 1 of the proof of Theorem 4.3. We find that $k(\pi'')$ is rational over $k(\pi_0)$ where $\pi_0 = \langle \tau \rangle \cong C_2$. If $\text{char } k \neq 2$, apply Theorem 1.3. It follows that $k(\pi_0) \cong k(C_2) = k(x,y)$ where $\tau : x \mapsto y \mapsto x$. Define $X = y/x$. Then $k(x,y) \cong k(X,x)$ is rational over $k(X)$ by Theorem 2.6. Note that $k(X) \cong k(X/(1 + X^2))$ is rational. Done.

If $\pi'' \cong C_2$, the rationality of $k(C_2)$ has been proved in the above paragraph.

(ii) By the same method as in the proof of Theorem 1.6 (given in Section 4), it can be shown that $k(G)$ is rational over $k(M)^{D_n}$ where $M$ is some $D_n$-lattice. Then apply the result of Part (i).

Remark. Let $\pi = D_n$, the dihedral group of order $2n$. It is important that we assume that $n$ is odd in Theorem 5.2 and Lemma 5.3. If $4 \mid n$, by [Ba], there exist $\pi$-lattices $M$ such that the unramified Brauer groups $\text{C}(M)^{\pi}$ are not zero; thus $\text{C}(M)^{\pi}$ are not retract rational and hence are not stably rational. For a concrete construction of such lattices, see [HKY, Theorem 6.2]. If $2 \mid n$ and $4 \nmid n$, by [Ba] again, the unramified Brauer group of $\text{C}(M)^{\pi}$ is zero for any $\pi$-lattice $M$, but it is unknown whether $\text{C}(M)^{\pi}$ is always stably rational.
References

[Ar] E. Artin, *The orders of the linear groups*, Comm. Pure Appl. Math. 8 (1955), 355-366.

[Ba] J. Barge, *Cohomologie des groupes et corps d’invariants multiplicatifs*, Math. Ann. 283 (1989), 519-528.

[BR] E. Beneish and N. Ramsey, *Lattices over cyclic groups and Noether settings*, J. Alg. 452 (2016), 212–226.

[CH] H. Chu and S. Huang, *Noether’s problem on semidirect product groups*, to appear in “J. Alg.”.

[CTS] J.-L. Colliot-Thélène and J.-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) 10 (1977), 175–229.

[EK] S. Endo and M. Kang, *Function fields of algebraic tori revisited*, arXiv: 1406.0949v3, to appear in “Asian J. Math.”.

[EM1] S. Endo and T. Miyata, *Invariants of finite abelian groups*, Journal Math. Soc. Japan 25 (1973), 7–26.

[EM2] S. Endo and T. Miyata, *On a classification of the function fields of algebraic tori*, Nagoya Math. J. 56 (1975), 85–104.

[HK] M. Hajja and M. Kang, *Some actions of symmetric groups*, J. Algebra 177 (1995), 511–535.

[HKY] A. Hoshi, M. Kang and A. Yamasaki, *Multiplicative invariant fields of dimension ≤ 6*, arXiv: 1609.04142.

[IR] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, Springer GTM vol. 84, Springer-Verlog 1990, New York.

[Ka1] M. Kang, *Noether’s problem for metacyclic p-groups*, Advances in Math. 203 (2006), 554–567.

[Ka2] M. Kang, *Rationality problem for some meta-abelian groups*, J. Alg. 322 (2009), 1214–1219.

[Ka3] M. Kang, *Bogomolov multipliers and retract rationality for semidirect products*, J. Alg. 397 (2014), 407–425.

[Ka4] M. Kang, *Noether’s problem for cyclic groups of prime order*, arXiv: 1606.04611.
[KP] M. Kang and B. Plans, *Reduction theorems for Noether’s problem*, Proc. Amer. Math. Soc. 137 (2009), 1867–1874.

[Le] H. W. Lenstra Jr., *Rational functions invariant under a finite abelian group*, Invent. Math. 25 (1974), 299–325.

[MM] J. M. Masley and H. L. Montgomery, *Cyclotomic fields with unique factorization*, J. Reien Angew. Math. 286/287 (1976), 248–256.

[Ne] J. Neukirch, *Class field theory*, Springer-Verlag, 1986, Berlin.

[Pl] B. Plans, *On Noether’s rationality problem for cyclic groups over ℚ*, arXiv: 1605.09228.

[Re] I. Reiner, *Integral representations of cyclic groups of prime order*, Proc. Amer. Math. Soc. 8 (1957), 142–146.

[Ri] D. S. Rim, *Modules over finite groups*, Ann. Math. 69 (1959), 700–712.

[Sa1] D. J. Saltman, *Retract rational fields and cyclic Galois extensions*, Israel J. Math. 47 (1984), 165–215.

[Sa2] D. J. Saltman, *Multiplicative invariant fields*, J. Alg. 106 (1987), 221–238.

[Sw1] R. G. Swan, *Induced representations and projective modules*, Ann. Math. 71 (1960), 552–578.

[Sw2] R. G. Swan, *Noether’s problem in Galois theory*, in “Emmy Noether in Bryn Mawr”, edited by B. Srinivasan and J. Sally, Springer-Verlag, Berlin, 1983, pp. 21–40.

[Sw3] R. G. Swan, *The flabby class group of a finite cyclic group*, in “Proceedings of the 4th International Congress of Chinese Mathematicians vol. 1”, edited by L. Ji, K. Liu, L. Yang and S.-T. Yau, Higher Education Press and International Press, Beijing and Somerville, MA, 2008, pp. 241–251.

[Vo] V. E. Voskresenskii, *Algebraic groups and their birational invariants*, Amer. Math. Soc. Transl. of Math. Monographs, vol. 179, Providence, 1998.

[Wa] L. C. Washington, *Introduction to cyclotomic fields*, Second edition, Springer GTM vol. 83, Springer-Verlag 1997, New York.