Abstract

For a graph $G$ of order $n$ and with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, the HL-index $R(G)$ is defined as $R(G) = \max \{ |\lambda_{\lfloor (n+1)/2 \rfloor}|, |\lambda_{\lceil (n+1)/2 \rceil}| \}$. We show that for every connected bipartite graph $G$ with maximum degree $\Delta \geq 3$, $R(G) \leq \sqrt{\Delta - 2}$ unless $G$ is the incidence graph of a projective plane of order $\Delta - 1$. We also present an approach through graph covering to construct infinite families of bipartite graphs with large HL-index.

Keywords: adjacency matrix, graph eigenvalues, median eigenvalues, covers.

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1 Introduction

Unless explicitly stated, we assume that all graphs in this paper are simple, i.e. multiple edges and loops are not allowed. The adjacency matrix of...
G, denoted by \( A(G) = (a_{uv})_{u,v \in V(G)} \), is a \((0,1)\)-matrix whose rows and columns are indexed by the vertices of \( G \) such that \( a_{uv} = 1 \) if and only if \( u \) is adjacent to \( v \). We use \( \deg_G(v) \) to denote the degree of vertex \( v \) in \( G \). The set of all neighbours of \( v \) is denoted by \( N_G(v) \) and we write \( N_G[v] = N_G(v) \cup \{v\} \). The smallest and largest degrees of \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively.

Let \( \lambda_1 \geq \cdots \geq \lambda_n \) be the eigenvalues of \( G \). Then \( \lambda_{\lfloor (n+1)/2 \rfloor} \) and \( \lambda_{\lceil (n+1)/2 \rceil} \) are called the median eigenvalue(s) of \( G \). These eigenvalues play an important role in mathematical chemistry since they are related to the HOMO-LUMO separation, see, e.g. [7] and [3, 4]. Following [9], we define the HL-index \( R(G) \) of the graph \( G \) as

\[
R(G) = \max \left\{ |\lambda_{\lfloor (n+1)/2 \rfloor}|, |\lambda_{\lceil (n+1)/2 \rceil}| \right\}.
\]

If \( G \) is a bipartite graph, then \( R(G) \) is equal to \( \lambda_{n/2} \) if \( n \) is even and 0, otherwise. In this paper, we show that for every connected bipartite graph \( G \) with maximum degree \( \Delta \), \( R(G) \leq \sqrt{\Delta - 2} \) unless \( G \) is the incidence graph of a projective plane of order \( \Delta - 1 \), in which case it is equal to \( \sqrt{\Delta - 1} \). This extends the result of one of the authors [13] who proved the same for subcubic graphs.

On the other hand, we present an approach through graph covering to construct infinite families of connected graphs with large HL-index. Graph coverings and analysis of their eigenvalues were instrumental in a recent breakthrough in spectral graph theory by Marcus, Spielman, and Srivastava who used graph coverings to construct infinite families of Ramanujan graphs of arbitrary degrees [11] (and for solving the Kadison-Singer Conjecture [12]). In our paper, we find another application of a different character. As opposed to double covers used in [11], we use \( k \)-fold covering graphs with cyclic permutation representation and show that the behavior of median eigenvalues can be controlled in certain instances. The main ingredient is a generalization of a result of Bilu and Linial [1] that eigenvalues of double covers over a graph \( G \) are the union of the eigenvalues of \( G \) and the eigenvalues of certain cover matrix \( A^- \) that is obtained from the adjacency matrix by replacing some of its entries by \(-1\). In our case, we use a family \( A^\lambda \) of such matrices, where instead of \(-1\) we use certain powers of a parameter \( \lambda \in [-1,1] \). This result seems to be of independent interest.
2 Bounds for bipartite graphs

In this section we obtain upper bounds on the HL-index of bipartite graphs in terms of maximum and minimum degrees of graphs. We consider regular graphs first.

**Theorem 1.** Let $G$ be a connected bipartite $k$-regular graph, where $k \geq 3$. If $R(G) > \sqrt{k-2}$, then $R(G) = \sqrt{k-1}$ and $G$ is the incidence graph of a projective plane of order $k - 1$.

**Proof.** Let $|V(G)| = 2n$. The adjacency matrix of $G$ can be written as

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where $B$ is a square matrix of order $n$. The matrix $E = BB^T - kI$ is a symmetric matrix of order $n$. Assuming that $R(G) > \sqrt{k-2}$, every eigenvalue $\lambda$ of $G$ satisfies $\lambda^2 > k - 2$ and hence all eigenvalues of $E$ are greater than $-2$. Hence, $E + 2I$ is a positive definite matrix. All diagonal entries of this matrix are equal to 2. Positive definiteness in turn implies that all off diagonal entries are 0 or 1. It follows that $E$ is the adjacency matrix of a graph $H$ with the least eigenvalue greater than $-2$. We see that $H$ is regular since $Ej = (BB^T - kI)j = (k^2 - k)j$. The connectedness of $G$ also yields that $H$ is connected. By Corollary 2.3.22 of [2], a connected regular graph with least eigenvalue greater than $-2$ is either a complete graph or an odd cycle. If $H$ is an odd cycle, then it is 2-regular and so from $k^2 - k = 2$, we have $k = 2$, a contradiction. Hence $H$ is a complete graph. It is easy to see that this implies that $G$ is the incidence graph of a projective plane of order $k - 1$. $\square$

For the next theorem, we need the following result [2, Theorem 2.3.20].

**Theorem 2 ([2]).** If $G$ is a connected graph with the least eigenvalue greater than $-2$, then one of the following holds:

(i) $G$ is the line graph of a multigraph $K$, where $K$ is obtained from a tree by adding one edge in parallel to a pendant edge;
(ii) $G$ is the line graph of a graph $K$, where $K$ is a tree or is obtained from a tree by adding one edge giving a nonbipartite unicyclic graph;
(iii) $G$ is one of the 573 exceptional graphs on at most 8 vertices.

We can now prove an analogue to Theorem 1 for non-regular graphs.
Theorem 3. Let $G$ be a connected bipartite nonregular graph with maximum degree $\Delta \geq 3$. Then $R(G) \leq \sqrt{\Delta - 2}$.

Proof. Let $d = \Delta - 1$. Suppose, for a contradiction, that $R(G) > \sqrt{d - 1}$. Let $\{U, W\}$ be the bipartition of $V(G)$. Then $U$ and $W$ have the same size, say $m$, since otherwise $R(G)$ would be zero. We proceed in the same way as in the proof of Theorem 1. The adjacency matrix of $G$ can be written in the form

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where the rows of $B$ are indexed by the elements of $U$ and the columns by $W$. The matrix $E = BB^T - (d - 1)I$ is a symmetric matrix of order $m$. Since $R(G) > \sqrt{d - 1}$, we have $\lambda_m(E) > 0$. Hence $E$ is a positive definite matrix whose diagonal entries are the integers $\deg_G(u) - (d - 1) \leq 2, u \in U$. Since $E$ is positive definite, these are all equal to 1 or 2 and hence the degrees of vertices in $U$ are either $\Delta$ or $\Delta - 1$. Moreover, this in turn implies that all off-diagonal entries of $E$ are either 0 or 1. Since the off-diagonal entries in $E$ count the number of walks of length 2 between vertices in $U$, the last conclusion in particular implies that $G$ has no 4-cycles. Let $D$ be the diagonal matrix whose diagonal is the same as the main diagonal of $E$. Let $H$ be the graph on $U$ with the adjacency matrix $A(H) = E - D$. Then the least eigenvalue of $H$ is greater than $-2$ and $A(H) + D$ is positive definite. The connectedness of $G$ yields that $H$ is connected.

Suppose that $v_1, v_2 \in U$ are distinct vertices of degree $d$ in $G$. Let $P$ be a shortest path in $H$ connecting $v_1$ to $v_2$. The vertices $v_1, v_2$ and the path $P$ can be chosen so that all internal vertices on $P$ are of degree $d + 1$ in $G$. Then $A(P) + \text{diag}(1, 2, \ldots, 2, 1)$, which is a principal submatrix of $A(H) + D$, has eigenvalue 0 with the eigenvector $(1, -1, 1, -1, \ldots)^T$, a contradiction. This shows that $U$ contains at most one vertex which has degree $d$ in $G$. Note that the same argument can be applied to $W$. As $G$ is not regular, we conclude that it has precisely two vertices of degree $d$, one in $U$ and one in $W$. Therefore, we may assume that $D = \text{diag}(2, \ldots, 2, 1)$.

Let us now consider degrees of vertices in $H$. Since $G$ has no 4-cycles, we have for every $u \in U$:

$$\deg_H(u) = \sum_{v \in E(G)} (\deg_G(v) - 1). \quad (1)$$

Since $U$ and $W$ each has precisely one vertex whose degree in $G$ is $d$, (1) implies the following: If $\deg_G(u) = d$, then $\deg_H(u) \in \{d^2, d^2 - 1\}$; if $\deg_G(u) = d + 1$, then $\deg_H(u) \in \{d^2 + d, d^2 + d - 1\}$ with the smaller
value only when \( u \) is adjacent to the vertex of degree \( d \) in \( W \). Thus, \( H \) has a unique vertex of degree \( d^2 \) or \( d^2 - 1 \), at most \( d \) vertices of degree \( d^2 + d - 1 \), and all other vertices are of degree \( d^2 + d \).

Let \( v \in U \) be the vertex with \( \deg_G(v) = d \). We claim that the neighbourhood of \( v \) in \( H \) is a complete graph. This is since

\[
\begin{bmatrix}
2 & 0 & 1 \\
0 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

is not positive definite and so it cannot be a principal submatrix of \( A(H) + D \). We also claim that \( N_H[v_1] \cap N_H[v_2] \subseteq N_H[v] \) for arbitrary distinct vertices \( v_1, v_2 \in N_H(v) \). If this were not the case, the matrix

\[
\begin{bmatrix}
2 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

which is not positive definite, would be a principal submatrix of \( A(H) + D \), a contradiction. Therefore, \( H \) has at least \( 1 + (d^2 - 1) + (d^2 - 1)((d^2 + d - 1) - (d^2 - 1)) = d^3 + d^2 - d \) or \( 1 + d^2 + d^2((d^2 + d - 1) - d^2) = d^3 + 1 \) vertices.

In any case, we have \( |V(H)| \geq 9 \) and so \( H \) is the case (i) or (ii) of Theorem 2.

Let \( H \) be the line graph of a multigraph \( K \) as in Theorem 2 where \( |V(K)| = m \) or \( m + 1 \) and \( |E(K)| = m \). Note that \( K \) is not a cycle, since then \( H \) would be regular in that case. So, from \( \sum_{v \in V(K)} \deg_K(v) = 2m \) and \( |V(K)| \geq m \), we have a vertex \( u \) such that \( \deg_K(u) = 1 \). Let \( u' \) be the unique neighbour of \( u \) in \( K \). The degree of the vertex in \( H \) corresponding to the edge \( uu' \) is at least \( d^2 - 1 \). Thus, \( \deg_K(u) + \deg_K(u') - 2 \geq d^2 - 1 \) and so \( \deg_K(u') \geq d^2 \). Let \( r \) be the number of vertices of degree one in \( K \). The sum of degrees in \( K \) is at least \( r + d^2 + 2(|V(K)| - r - 1) \leq 2m \) which gives \( r \geq d^2 - 2 \geq 2 \). Since \( H \) has only one vertex of degree \( d^2 - 1 \) or \( d^2 \), we may take \( u \) to be a vertex of degree 1 in \( K \) such that \( \deg_K(u) + \deg_K(u') - 2 \geq d^2 + d - 1 \) and now a similar argument as above gives \( r \geq d^2 + d - 2 > d + 1 \). Now again since \( H \) has at most \( d + 1 \) vertices of degree \( d^2 - 1 \), \( d^2 \) or \( d^2 + d - 1 \), we may take \( u \) such that \( \deg_K(u) + \deg_K(u') - 2 = d^2 + d \). It follows that \( H \) has the complete graph of order \( d^2 + d + 1 \) as a subgraph which in turn implies that \( H \) is in fact the complete graph of order \( d^2 + d + 1 \) since \( H \) is connected with maximum degree \( d^2 + d \). And this contradicts the fact that \( H \) is nonregular. \( \square \)
Theorems 1 and 3 can be combined into our main result.

**Theorem 4.** Let $G$ be a connected bipartite graph with maximum degree $\Delta \geq 3$. Then $R(G) \leq \sqrt{\Delta - 2}$ unless $G$ is the incidence graph of a projective plane of order $\Delta - 1$, in which case $R(G) = \sqrt{\Delta - 1}$.

There are connected graphs that are not incidence graphs of a projective planes and attain the bound $\sqrt{\Delta - 2}$ of Theorem 4. The incidence graph of a biplane ($(v, k, 2)$ symmetric design) has degree $k$ and HL-index $\sqrt{k - 2}$. Only 17 biplanes are known and the question of existence of infinitely many biplanes is an old open problem in design theory [8]. An infinite family of cubic graphs with HL-index equal to 1 is constructed in [6].

Finally, we give an upper bound for the HL-index of bipartite graphs in term of the minimum degree.

**Theorem 5.** Let $G$ be a bipartite graph with minimum degree $\delta$. Then $R(G) \leq \sqrt{\delta}$. Moreover, if $\delta \geq 2$ or $\delta = 1$ and $G$ has a component with minimum degree 1 that is not isomorphic to $K_2$, then $R(G) < \sqrt{\delta}$.

**Proof.** We may assume that $G$ is connected and of even order $n = 2m$. Let $v$ be a vertex of degree $\delta$ and $H = G - v$. Since $G$ is connected, $\lambda_1(G) > \lambda_1(H)$. By interlacing, $\lambda_i(G) \geq \lambda_i(H)$ for $1 < i \leq m$. We also have $\lambda_m(H) = 0$ since $H$ is bipartite and has an odd number of vertices.

The sum of the squares of the eigenvalues of $G$ is the trace of $A^2$, which is equal to $2|E(G)|$. By considering only half of the eigenvalues and using the fact that eigenvalues of a bipartite graph are symmetric about zero, we have:

$$
\lambda_m^2(G) = |E(G)| - \sum_{i=1}^{m-1} \lambda_i^2(G) \\
= \delta + \sum_{i=1}^{m-1} \lambda_i^2(H) - \sum_{i=1}^{m-1} \lambda_i^2(G) \\
\leq \delta.
$$

If $m \geq 2$, then for $i = 1$, we have $\lambda_1^2(H) - \lambda_1^2(G) < 0$, so the last inequality is strict. This proves the assertion of the theorem.

3 Covering graphs and their eigenvalues

If $\hat{G}$ is a covering graph of $G$, then all eigenvalues of $G$ are included in the spectrum of $\hat{G}$. The essence of this section is to show how to control the newly arising eigenvalues in the covering graph.
We will denote the eigenvalues of a symmetric $n \times n$ matrix $M$ by $
abla_1(M) \geq \cdots \geq \nabla_n(M)$. Also, if $x$ is an eigenvector of $M$, then we denote the corresponding eigenvalue by $\nabla_x(M)$. For a positive integer $t$, let $I_t$ and $O_t$ denote the $t \times t$ identity matrix and the $t \times t$ all-zero matrix, respectively. A permutation matrix $C$ of size $t$ and order $m$ is a $t \times t$ $(0,1)$-matrix that has exactly one entry $1$ in each row and each column and $m$ is the smallest positive integer such that $C^m = I_t$.

Let us replace each edge of a multigraph $G$ by two oppositely oriented directed edges joining the same pair of vertices and let $\overrightarrow{E}(G)$ denote the resulting set of directed edges. We denote by $(e, u, v) \in \overrightarrow{E}(G)$ the directed edge in $\overrightarrow{E}(G)$ corresponding to an edge $e = uv$ that is oriented from $u$ to $v$. Let $S_t$ denote the symmetric group of all permutations of size $t$. We shall consider a representation of $S_t$ as the set of all permutation matrices of size $t$. A function $\phi : \overrightarrow{E}(G) \to S_t$ is a permutation voltage assignment for $G$ if $\phi(e, u, v) = \phi(e, v, u)^{-1}$ for every $e \in E(G)$. A $t$-lift of $G$ associated to $\phi$ and denoted by $G(\phi)$, is a multigraph with the adjacency matrix obtained from the adjacency matrix of $G$ by replacing any $(u, v)$-entry of $A(G)$ by the $t \times t$ matrix $\sum_{(e, u, v) \in \overrightarrow{E}(G)} \phi(e, u, v)$. Note that if $G$ is bipartite, then so is $G(\phi)$. We say that $G(\phi)$ is an Abelian lift if all matrices in the image of $\phi$ commute with each other.

Bilu and Linial [1] found an expression for the spectrum of 2-lifts. They proved that the spectrum of a 2-lift $G(\phi)$ consists of the spectrum of $G$ together with the spectrum of the matrix $A^-$ which is obtained from the adjacency matrix of $G$ by replacing each $(u, v)$-entry by $-1$ whenever the voltage $\phi(e, u, v)$ is not the identity. Note that 2-lifts are always Abelian since the permutation matrices of size 2 commute with each other. Below we extend the result of [1] to arbitrary Abelian $t$-lifts. Since permutation matrices are diagonalizable and any commuting family of diagonalizable $t \times t$ matrices has a common basis of eigenvectors, we observe that any commuting set of permutation matrices of the same size has a common basis of eigenvectors.

In the proofs we will use the following result, see [10] Theorem 1.

**Theorem 6** ([10]). Let $t$ and $n$ be positive integers and for $i, j \in \{1, \ldots, n\}$, let $B_{ij}$ be $t \times t$ matrices over a commutative ring that commute pairwise. Then

$$
\det \begin{bmatrix}
B_{11} & \cdots & B_{1n} \\
\vdots & \ddots & \vdots \\
B_{n1} & \cdots & B_{nn}
\end{bmatrix} = \det \left( \sum_{\sigma \in S_n} \text{sign}(\sigma) B_{1\sigma(1)} \cdots B_{n\sigma(n)} \right).
$$

1
where $S_n$ is the set of all permutations of $\{1, \ldots, n\}$.

**Theorem 7.** Let $G$ be a multigraph and $\phi$ be a permutation voltage assignment for an Abelian $t$-lift $G(\phi)$ of $G$. Let $B$ be a common basis of eigenvectors of the permutation matrices in the image of $\phi$. For every $x \in B$, let $A_x$ be the matrix obtained from the adjacency matrix of $G$ by replacing any $(u, v)$-entry of $A(G)$ by $\sum_{(e, u, v) \in \overrightarrow{E}(G)} \lambda_x(\phi(e, u, v))$. Then the spectrum of $G(\phi)$ is the multiset union of the spectra of the matrices $A_x$ ($x \in B$).

**Proof.** The adjacency matrix of a $t$-lift can be written in the block form, with the blocks being indexed by $V(G)$, where the $(u, v)$-block $D_{uv}$ is equal to the permutation matrix $\phi(e, u, v)$ if $u, v$ are joined by a single edge $e$, or to $\sum_{(e, u, v) \in \overrightarrow{E}(G)} \phi(e, u, v)$ if there are multiple edges, or $0$, if $u, v$ are not adjacent in $G$. Thus, assuming $V(G) = \{1, \ldots, n\}$, we can write

$$
\lambda I - A(G(\phi)) = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix},
$$

where the diagonal blocks are $\lambda I$, while the off-diagonal blocks are $B_{uv} = -D_{uv}$. All block matrices $B_{uv}$ commute with each other and all their products and sums also commute and have $B$ as a common basis of eigenvectors. By Theorem 6 we have

$$
\det(\lambda I - A(G(\phi))) = \det\left(\sum_{\sigma \in S_n} \text{sign}(\sigma) B_{1\sigma(1)} \cdots B_{n\sigma(n)}\right)
$$

$$
= \prod_{x \in B} \lambda_x \left(\sum_{\sigma \in S_n} \text{sign}(\sigma) B_{1\sigma(1)} \cdots B_{n\sigma(n)}\right)
$$

$$
= \prod_{x \in B} \left(\sum_{\sigma \in S_n} \text{sign}(\sigma) \lambda_x(1\sigma(1)) \cdots \lambda_x(n\sigma(n))\right)
$$

$$
= \prod_{x \in B} \det(\lambda I - A_x).
$$

This equality gives the conclusion of the theorem. \qed

Let us keep the notation of Theorem 7 and its proof. There are two things to be observed. Since every permutation matrix $D_{uv}$ satisfies $D_{uv}^{-1} = D_{uv}^T$, we have that the matrices $A_x$ ($x \in B$) are Hermitian. The $(u, v)$-entry of $A_x$ is the eigenvalue of $D_{uv}$ corresponding to the eigenvector $x$. Thus the
characteristic polynomial \( \varphi(A_x, \lambda) = \det(\lambda I - A_x) \) is a polynomial in \( \lambda \), whose coefficients are polynomials in the \( x \)-eigenvalues of the permutation matrices in the image of \( \phi \).

4 Median eigenvalues of covering graphs

In this section we present an approach through graph covering to construct infinite families of graphs with large HL-index. The main tool is Theorem 7 which will be invoked in a very special situation.

Let us select a set \( F \subseteq \vec{E}(G) \) of oriented edges such that whenever \( (e, u, v) \in F \), the opposite edge \( (e, v, u) \) is not in \( F \). For every positive integer \( t \), let \( C_t \) be a cyclic permutation of size and order \( t \). Now, let us consider an infinite family of Abelian lifts \( \phi_1, \phi_2, \phi_3, \ldots \) such that \( \phi_t \) is an Abelian \( t \)-lift over the graph \( G \), whose voltages are given by the following rule:

\[
\phi_t(e, u, v) = \begin{cases} 
C_t, & (e, u, v) \in F; \\
C_t^{-1}, & (e, v, u) \in F; \\
I, & \text{otherwise.}
\end{cases}
\] (2)

In this way, we obtain an infinite family of graphs \( G(\phi_t) \). By Theorem 7 we can express the characteristic polynomial of \( A(G(\phi_t)) \) as a product of the characteristic polynomials of matrices \( A_x \). For each \( x \in B \), the characteristic polynomial of \( A_x \) depends only on \( \lambda \) and on the eigenvalue \( \alpha = \lambda_x(C_t) \) and on \( \lambda_x(C_t^{-1}) = \alpha^{-1} = \bar{\alpha} \). The dependence on \( \alpha \) can be expressed in terms of the real parameter \( \nu = \alpha + \bar{\alpha} \). For cyclic permutations, every such \( \nu \) is an eigenvalue of the \( t \)-cycle, which is of the form \( \nu = 2 \cos(2\pi j/t) \) for some \( j \in \{0, 1, \ldots, t-1\} \). Thus, there is a polynomial \( \Phi(\lambda, \nu) \) such that

\[
\det(\lambda I - A(G(\phi_t))) = \prod_{0 \leq j < t} \Phi(\lambda, 2 \cos(2\pi j/t)).
\] (3)

Note that \( \Phi(\lambda, \nu) \) is independent of \( t \) and only depends on the underlying graph \( G \) and the values for \( \nu \) lie in the interval \([-2, 2]\) for every \( t \). All eigenvalues of \( G(\phi_t) \) correspond to the zero-set of the polynomial \( \Phi(\lambda, \nu) \) with \( \nu \in [-2, 2] \). When \( t \) gets large, the appropriate values of \( \nu \) become dense in the interval \([-2, 2]\). This shows that if \( G \) is bipartite, then \( R(G(\phi_t)) \) converges to some value when \( t \) goes to infinity. This is better seen in a special case which is given in the following theorem.

**Theorem 8.** Let \( G \) be a bipartite graph and let \( E_0 \) be a set of edges all incident with some fixed vertex \( v_0 \). Let \( F \subseteq \vec{E}(G) \) be the set of directed edges
\{(e, u, v_0) \mid uv_0 \in E_0\}. For each positive integer \(t\), fix a cyclic permutation matrix \(C_t\) of size and order \(t\) and define a permutation voltage assignment \(\phi_t\) by (2). Then
\[
R(G(\phi_{2t})) = R(G(\phi_2))
\]
for every \(t \geq 1\), whereas the values \(R(G(\phi_{2t+1}))\) are non-increasing as a function of \(t\) and
\[
\lim_{t \to \infty} R(G(\phi_{2t+1})) = R(G(\phi_2)).
\]

**Proof.** By Theorem 7, there is a polynomial \(\Phi(\lambda, \nu)\) such that (2) holds and every eigenvalue of any \(G(\phi_t)\) lies among the values \(\lambda\) for which there is a \(\nu \in [-2, 2]\) such that \(\Phi(\lambda, \nu) = 0\). It is easy to see that our choice of \(F\) implies that \(\Phi(\lambda, \nu)\) is linear in \(\nu\), so it can be expressed in the form
\[
\Phi(\lambda, \nu) = p(\lambda) - \nu q(\lambda).
\]
If \(R(G(\phi_2))\) is zero, then \(R(G(\phi_{2t})) = 0\) for every \(t \geq 1\), since by Theorem 7 the spectrum of \(G(\phi_2)\) is contained in the spectrum of \(G(\phi_{2t})\). Hence, we may assume that \(R(G(\phi_2)) \neq 0\). Note that \(R(G) \geq R(G(\phi_2))\), so we also have \(R(G) \neq 0\).

Let us first assume that \(q(0) \neq 0\). Let \(\Phi(0, \nu_0) = 0\). Then \(\nu_0 = p(0)/q(0)\). We have \(\Phi(0, 2) = p(0) - 2q(0)\) and \(\Phi(0, -2) = p(0) + 2q(0)\) which results in
\[
\nu_0 = \frac{p(0)}{q(0)} = \frac{2(\Phi(0, -2) + \Phi(0, 2))}{\Phi(0, -2) - \Phi(0, 2)}.
\]
On the other hand, Eq.(3) gives that \(\Phi(0, 2) = \det(-A(G))\) and \(\Phi(0, -2) = \det(-A(G(\phi_2)))/\det(-A(G))\). Since the eigenvalues of bipartite graphs \(G\) and \(G(\phi_2)\) are symmetric about zero, this implies that the above determinants, and thus also \(\Phi(0, 2)\) and \(\Phi(0, -2)\), have the same sign. It follows that \(|\nu_0| > 2\). Since \(\Phi(\lambda, \nu)\) is linear in \(\nu\), for each \(\lambda\) there exists at most one value \(\nu\) such that \(\Phi(\lambda, \nu) = 0\) (and there is exactly one if \(q(\lambda) \neq 0\)). Therefore, the continuity of \(\Phi(\lambda, \nu)\) and its linearity in \(\nu\) show that the eigenvalue \(R(G(\phi_t))\) is either a root of \(\Phi(\lambda, 2)\) or a root of \(\Phi(\lambda, -2)\) (if \(t\) is even) or a root of \(\Phi(\lambda, 2 \cos(\pi(t-1)/t))\) (if \(t\) is odd). This is independent of \(t\) when \(t\) is even and is already among the eigenvalues of \(G(\phi_2)\). For odd values of \(t\), this shows the behavior as claimed in the theorem.

Suppose next that \(q(0) = 0\). Then \(p(0) \neq 0\), since otherwise we have \(\Phi(0, 2) = 0\) and so \(R(G) = 0\), a contradiction. This shows that if \(\Phi(\lambda_0, \nu_0) = 0\) and \(\lambda_0\) goes to zero, then \(\nu_0\) goes to infinity. Again the continuity of \(\Phi(\lambda, \nu)\) and its linearity in \(\nu\) show that \(R(G(\phi_t))\) is either a root of \(\Phi(\lambda, 2)\) or a root of \(\Phi(\lambda, -2)\) (if \(t\) is even) or a root of \(\Phi(\lambda, 2 \cos(\pi(t-1)/t))\) (if \(t\) is odd). We now complete the proof in the same was as above. \(\square\)
Theorem 9. For any integer \( k \) for which \( k - 1 \) is a prime power, there exist infinitely many connected bipartite \( k \)-regular graphs \( G \) with \( \sqrt{k-2} - 1 < R(G) < \sqrt{k-1} - 1 \).

Proof. Let \( G \) be the incidence graph of a projective plane of order \( q = k - 1 \). Note that \( G \) is bipartite and \( k \)-regular. It is well-known (see, e.g., [5]) that \( G \) has eigenvalues \( \pm k \) and \( \pm \sqrt{q} \). Thus, \( R(G) = \sqrt{q} \). Let \( e_0 \) be any edge of \( G \) and \( E_0 = \{e_0\} \). For each positive integer \( t \), define the permutation assignment \( \phi_t \) as in Theorem 8.

The adjacency matrix of \( G(\phi_2) \) can be written as \( A(G(\phi_2)) = A(G) \otimes I_2 + B \), where \( B \) is a matrix with only \( \pm 2 \) as nonzero eigenvalues. Let \( r \) be the number of vertices of \( G(\phi_2) \). By the Courant-Weyl inequalities \( \lambda_i + \lambda_j - r (A + B) \geq \lambda_i(A) + \lambda_j(B) \), we have \( \lambda_{r/2-1}(A(G(\phi_2))) \geq \lambda_{r/2}(A(G) \otimes I_2) + \lambda_{r-1}(B) \) which gives \( \lambda_{r/2-1}(G(\phi_2)) \geq \sqrt{q} \). In fact, since \( G(\phi_2) \) has \( \sqrt{q} \) as an eigenvalue with big multiplicity, one observes that \( \lambda_{r/2-1}(G(\phi_2)) = \sqrt{q} \) and so \( R(G) = \lambda_{r/2}(G(\phi_2)) \leq \sqrt{q} \).

Let \( e_0 = \{v_0, v_1\} \). Let us consider the partition \( \{v_0, v_1\} \cup W \cup W' \) of \( V(G) \), where \( W \) is the set of all vertices adjacent to \( v_0 \) or \( v_1 \) and \( W' \) is the set of vertices nonadjacent to \( v_0, v_1 \). Define the vector \( x \) on \( V(G) \) as

\[
x(w) = \begin{cases} 1 & w = v_0 \text{ or } w = v_1, \\ a & w \in W, \\ b & w \in W'. \\
\end{cases}
\]

We now consider the vector \( y = (x, -x) \) on \( V(G(\phi_2)) \). Since the girth of \( G \) is six, it is easy to see that \( y \) is an eigenvector of \( G(\phi_2) \) with the corresponding eigenvalue \( \lambda \) if and only if \( \lambda = qa - 1, a \lambda = qb + 1, \) and \( b \lambda = qb + a \). Solving these equations in term of \( \lambda \) gives \( \lambda^3 + (1 - q)\lambda^2 - 3q\lambda + q^2 - q = 0 \). The value of the expression on the left side of this equation is \( 2 \) and \( \sqrt{q} - q \) for \( \lambda = \sqrt{q - 1} - 1 \) and \( \lambda = \sqrt{q} - 1 \), respectively. Therefore, there is a root between \( \sqrt{q - 1} - 1 \) and \( \sqrt{q} - 1 \). This implies that \( \sqrt{q - 1} - 1 < R(G(\phi_2)) < \sqrt{q} - 1 \). Finally, since \( R(G) = \sqrt{q} \), Theorem 8 implies that \( \sqrt{q - 1} - 1 < R(G(\phi_2)) = R(G(\phi_2)) < \sqrt{q} - 1 \) for all \( t \).

The proof of Theorem 9 can be used to obtain a slightly better bound on \( R(G) \). Let \( t = \sqrt{k-1} \). The value of \( \lambda^3 + (1 - q)\lambda^2 - 3q\lambda + q^2 - q \) is

\[
(\sqrt{h^2 - h^3})t^5 + (h^3 - 2h^2)t^4 + (4h^2 - h)t^3 + (3h^2 - h^2)t^2 - 2ht - 1)
\]

for \( \lambda = t - 1 - (ht)^{-1} \). Note that (4) is positive for \( h = 2 \) and any \( t \), whereas it is negative for any \( h > 2 \) if \( t \) is large enough. Therefore we find that \( \sqrt{k-1} - 1 - \frac{1}{2\sqrt{k-1}} < R(G) < \sqrt{k-1} - 1 - \frac{1}{(2+\epsilon)\sqrt{k-1}} \) for every \( \epsilon > 0 \) and any large \( k \).
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