HILFER-KATUGAMPOLA FRACTIONAL DERIVATIVE

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Abstract. We propose a new fractional derivative, the Hilfer-Katugampola fractional derivative. Motivated by the Hilfer derivative this formulation interpolates the well-known fractional derivatives of Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, Liouville, Weyl, generalized and Caputo-type. As an application, we consider a nonlinear fractional differential equation with an initial condition using this new formulation. We show that this equation is equivalent to a Volterra integral equation and demonstrate the existence and uniqueness of solution to the nonlinear initial value problem.

Keywords: Generalized fractional integral; Hilfer-Katugampola fractional derivative; fractional differential equation; Volterra integral equation.
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1. Introduction

The number of operators of fractional integration and differentiation has been increasing during the last years [3, 4, 5, 8, 13, 14, 15, 17, 20, 21, 22]. In order to solve fractional differential equations, we mention the following works [1, 2, 6, 11, 18], where the authors propose and prove the equivalence between an initial value problem and the Volterra integral equation. Moreover, the existence and uniqueness theorems whose solution lies in a convenient space are demonstrated.

Motivated by some of these formulations and results, we introduce a new fractional derivative, which resembles the Hilfer and Hilfer-Hadamard fractional derivatives [9, 12]. This new fractional derivatives interpolates the Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, generalized and Caputo-type fractional derivatives, as well as the Weyl and Liouville fractional derivatives for particular cases of integration extremes. More details on this fractional derivatives mentioned above can be found in [5, 7, 9, 10, 14, 15, 17, 20].

The fact that there exist more than one definition for fractional derivatives makes choosing the convenient approach a crucial issue in solving a given problem. Thus, the Hilfer-Katugampola fractional derivative proposed in this paper is a generalization of the classical fractional derivatives, in order to overcome the problem of choosing the operator of fractional differentiation.
In order to obtain the analytical solution of some differential equations involving this new operator of fractional differentiation, we firstly prove the equivalence of a nonlinear differential equation, with adequate initial condition, to a Volterra integral equation. In the sequence we present a theorem on the existence and uniqueness of solution for such nonlinear differential equation and the space in which this solution exists.

The text is organized as follows: in section 2, we present results that will be used in the remaining sections. In section 3, we define our derivative of non-integer order, the Hilfer-Katugampola fractional derivative, together with some of its properties. In section 4, we discuss the equivalence between an initial value problem and a Volterra integral equation. In section 5, we present and prove the existence and uniqueness theorem for the initial value problem presented in the previous section. In section 6 we discuss, using the method of successive approximations, the analytical solution of some fractional differential equations involving this differentiation operator.

2. Preliminaries

Before we introduce Hilfer-Katugampola fractional derivatives, we present in this section some results which will be useful throughout the text, e.g. the Mittag-Leffler functions, the fixed point theorem and generalized fractional integrals and derivatives. Though Mittag-Leffler functions are well defined for complex parameters, in this paper we use only real parameters [17].

**Definition 1.** A two-parameter Mittag-Leffler function $E_{\alpha,\beta}(x)$, $\alpha, \beta, x \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$, is defined by

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}. \tag{2.1}$$

If $\beta = 1$, we have the one-parameter Mittag-Leffler function, given by the following series:

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}. \tag{2.2}$$

The generalized Mittag-Leffler function $E_{\alpha,l,m}(x)$, introduced by Kilbas and Saigo [16], is defined as follows:

**Definition 2.** Let $\alpha, l, m, x \in \mathbb{R}$ such that $\alpha > 0$, $m > 0$ and $\alpha(jm+l) \notin \mathbb{Z}^-$. The generalized Mittag-Leffler function is defined by

$$E_{\alpha,l,m}(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{2.3}$$
where
\[ c_0 = 1 \quad \text{and} \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma[\alpha(jm + l) + 1]}{\Gamma[\alpha(jm + l + 1) + 1]}, \quad k \in \mathbb{N}. \]

**Definition 3.** Let \( \Omega = [a, b] \) \((0 < a < b < \infty)\) be a finite interval on the half-axis \( \mathbb{R}^+ \) and the parameters \( \rho > 0 \) and \( 0 \leq \gamma < 1 \).

1. We denote by \( C[a, b] \) the space of continuous functions \( g \) on \( \Omega \) with the norm
\[ \|g\|_C = \max_{x \in \Omega} |g(x)|. \]

2. The weighted space \( C_{\gamma, \rho}[a, b] \) of functions \( g \) on \( (a, b) \) is defined by
\[ C_{\gamma, \rho}[a, b] = \left\{ g : (a, b] \to \mathbb{R} : \left( \frac{x^\rho - a^\rho}{\rho} \right)^\gamma g(x) \in C[a, b] \right\}, \]
where \( 0 \leq \gamma < 1 \) and with the norm
\[ \|g\|_{C_{\gamma, \rho}} = \left\| \left( \frac{x^\rho - a^\rho}{\rho} \right)^\gamma g(x) \right\|_C = \max_{x \in \Omega} \left( \frac{x^\rho - a^\rho}{\rho} \right)^\gamma |g(x)|, \]
where \( C_{0, \rho}[a, b] = C[a, b] \).

3. Let \( \delta_{\rho} = \left( t^{\rho-1} \frac{d}{dt} \right) \). For \( n \in \mathbb{N} \) we denote by \( C^n_{\delta_{\rho}, \gamma}[a, b] \) the Banach space of functions \( g \) which are continuously differentiable on \( [a, b] \), with operator \( \delta_{\rho} \), up to order \( (n-1) \) and which have the derivative \( \delta^n_{\rho} g \) of order \( n \) on \( (a, b] \) such that \( \delta^n_{\rho} g \in C_{\gamma, \rho}[a, b] \), that is,
\[ C^n_{\delta_{\rho}, \gamma}[a, b] = \left\{ g : [a, b] \to \mathbb{R} : \delta^k_{\rho} g \in C[a, b], k = 0, 1, \ldots, n-1, \delta^n_{\rho} g \in C_{\gamma, \rho}[a, b] \right\}, \]
where \( n \in \mathbb{N} \), with the norms
\[ \|g\|_{C^n_{\delta_{\rho}, \gamma}} = \sum_{k=0}^{n-1} \|\delta^k_{\rho} g\|_{C_{\gamma, \rho}} + \|\delta^n_{\rho} g\|_{C_{\gamma, \rho}}, \quad \|g\|_{C^n_{\delta_{\rho}}} = \sum_{k=0}^{n} \max_{x \in \Omega} |\delta^k_{\rho} g(x)|. \]
For \( n = 0 \), we have
\[ C^0_{\delta_{\rho}, \gamma}[a, b] = C_{\gamma, \rho}[a, b]. \]

**Lemma 1.** Let \( n \in \mathbb{N}_0 \) and \( \mu_1, \mu_2 \in \mathbb{R} \) such that
\[ 0 \leq \mu_1 \leq \mu_2 < 1. \]
Then,
\[ C^n_{\delta_{\rho}}[a, b] \longrightarrow C^n_{\delta_{\rho}, \mu_1}[a, b] \longrightarrow C^n_{\delta_{\rho}, \mu_2}[a, b], \]
with
∥f∥_{C^\mu_2,a,b} \leq K_{\delta_\rho} ∥f∥_{C^\mu_1,a,b},

where

\[ K_{\delta_\rho} = \min \left[ 1, \left( \frac{b^\rho - a^\rho}{\rho} \right)^{\mu_2 - \mu_1} \right], \quad \text{and} \quad a \neq 0. \]

In particular,

C[a, b] → C_{\mu_1,\rho}[a, b] → C_{\mu_2,\rho}[a, b],

with

\[ ∥f∥_{C_{\mu_2,\rho}[a, b]} \leq \left( \frac{b^\rho - a^\rho}{\rho} \right)^{\mu_2 - \mu_1} ∥f∥_{C_{\mu_1,\rho}[a, b]}, \quad a \neq 0. \]

**Lemma 2.** Let 0 ≤ γ < 1, a < c < b, g ∈ C_{γ,\rho}[a, c], g ∈ C[c, b] and g continuous at c. Then, g ∈ C_{γ,\rho}[a, b].

**Theorem 1.** [17] Let (U, d) be a nonempty complete metric space; let 0 ≤ ω < 1, and let T : U → U be the map such that, for every u, v ∈ U, the relation

\[ d(Tu, Tv) \leq ωd(u, v), \quad (0 ≤ ω < 1) \]

holds. Then the operator T has a unique fixed point u* ∈ U.

Furthermore, if T^k (k ∈ ℕ) is the sequence of operators defined by

\[ T^1 = T \quad \text{and} \quad T^k = TT^{k-1}, \quad (k ∈ ℕ \setminus \{1\}), \]

then, for any u_0 ∈ U, the sequence \{T^k u_0\}_{k=1}^\infty converges to the above fixed point u*.

The map T : U → U satisfying condition Eq.(2.7) is called a contractive map.

**Definition 4.** [14] Let α, c ∈ ℝ with α > 0 and ϕ ∈ X^p_c(a, b), where X^p_c(a, b) consists of those complex-valued Lebesgue measurable functions. The generalized fractional integrals, left- and right-sided, are defined, respectively, by

\[ (^\rho \mathcal{J}^\alpha_a ϕ)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{p-1} ϕ(t)}{(x^\rho - t^\rho)^{1-\alpha}} dt, \quad x > a \]

and

\[ (^\rho \mathcal{J}^\alpha_b ϕ)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{p-1} ϕ(t)}{(t^\rho - x^\rho)^{1-\alpha}} dt, \quad x < b \]

with ρ > 0.

Similarly, we define generalized fractional derivatives which correspond to generalized fractional integrals, Eq.(2.9) and Eq.(2.10).
Definition 5. Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha, \rho > 0$, $\alpha \notin \mathbb{N}$, and let $n = [\alpha] + 1$, where $[\alpha]$ is the integer part of $\alpha$. The generalized fractional derivatives, $(^\rho D_a^\alpha \varphi)(x)$ and $(^\rho D^\alpha_a \varphi)(x)$, left- and right-sided, are defined by

\[
(\,^\rho D_a^\alpha \varphi)(x) = \delta^\rho_n(\,^\rho \mathcal{J}_a^n-a^\alpha \varphi)(x)
\]

\[= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left( x^{1-\rho} d \right)^n \int_a^x \frac{t^{\rho-1} \varphi(t)}{(x^\rho - t^\rho)^{1-n+\alpha}} dt,
\]

and

\[
(\,^\rho D^\alpha_a \varphi)(x) = (-1)^n \delta^\rho_n(\,^\rho \mathcal{J}_a^n-a^\alpha \varphi)(x)
\]

\[= \frac{(-1)^n \rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left( x^{1-\rho} d \right)^n \int_x^b \frac{t^{\rho-1} \varphi(t)}{(x^\rho - t^\rho)^{1-n+\alpha}} dt,
\]

respectively, if the integrals exist and $\delta^\rho_n = \left( x^{1-\rho} d \right)^n$.

Theorem 2. Let $\alpha > 0$, $\beta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$ and $\rho, c \in \mathbb{R}$, $\rho \geq c$. Then, for $\varphi \in X^p_f(a, b)$ the semigroup property is valid, i.e.

\[(^\rho \mathcal{J}_a^\alpha \cdot ^\rho \mathcal{J}_a^\beta \varphi)(x) = (^\rho \mathcal{J}_a^{\alpha+\beta} \varphi)(x).
\]

Proof. See [14].

Lemma 3. Let $x > a$, $^\rho \mathcal{J}_a^\alpha$ and $^\rho D_a^\alpha$, as defined in Eq. (2.9) and Eq. (2.11), respectively. Then, for $\alpha \geq 0$ and $\xi > 0$, we have

\[
\left[ ^\rho \mathcal{J}_a^\alpha \left( \frac{x^\rho - a^\rho}{\rho} \right)^\xi \right](x) = \frac{\Gamma(\xi)}{\Gamma(\alpha + \xi)} \left( x^\rho - a^\rho \right)^{\alpha+\xi-1},
\]

\[
\left[ ^\rho D_a^\alpha \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha-1} \right](x) = 0, \quad 0 < \alpha < 1.
\]

Proof. See [2].

Lemma 4. For $\alpha > 0$, $^\rho \mathcal{J}_a^\alpha$ maps $C[a, b]$ into $C[a, b]$.

Lemma 5. Let $\alpha > 0$ and $0 \leq \gamma < 1$. Then, $^\rho \mathcal{J}_a^\alpha$ is bounded from $C_{\gamma,p}[a, b]$ into $C_{\gamma,p}[a, b]$.

Lemma 6. Let $\alpha > 0$ and $0 \leq \gamma < 1$. If $\gamma \leq \alpha$, then $^\rho \mathcal{J}_a^\alpha$ is bounded from $C_{\gamma,p}[a, b]$ into $C[a, b]$.

Lemma 7. Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \gamma < 1$ and $\varphi \in C_{\gamma,p}[a, b]$. If $\alpha > \gamma$, then $^\rho \mathcal{J}_a^\alpha \varphi$ is continuous on $[a, b]$ and

\[
(^\rho \mathcal{J}_a^\alpha \varphi)(a) = \lim_{x \to a^+} (^\rho \mathcal{J}_a^\alpha \varphi)(x) = 0.
\]
Proof. Since \( \varphi \in C_{\gamma, \varphi}[a, b] \), then \( \left( \frac{x^\rho - a^\rho}{\rho} \right)^\gamma \varphi(x) \) is continuous on \([a, b]\) and
\[
\left| \left( \frac{x^\rho - a^\rho}{\rho} \right)^\gamma \varphi(x) \right| \leq M, \quad x \in [a, b],
\]
for some positive constant \( M \). Consequently,
\[
|\left( \frac{\rho}{J^\alpha_a} \varphi \right)(x)| \leq M \frac{\Gamma(1 - \gamma)}{\Gamma(\alpha - \gamma + 1)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha - \gamma},
\]
and by Lemma 3, we can write
\[
(\rho J^\alpha_a + \rho D^\alpha_a)(x) = \varphi(x),
\]
for all \( x \in (a, b) \).

As \( \alpha > \gamma \), the right-hand side of Eq. (2.13) goes to zero when \( x \to a^+ \). \( \square \)

Lemma 8. Let \( \alpha > 0, 0 \leq \gamma < 1 \) and \( \varphi \in C_{\gamma}[a, b] \). Then,
\[
(\rho D^\alpha_a + \rho J^\alpha_a \varphi)(x) = \varphi(x),
\]
for all \( x \in (a, b) \).

Proof. See [15]. \( \square \)

Lemma 9. Let \( 0 < \alpha < 1, 0 \leq \gamma < 1 \). If \( \varphi \in C_{\gamma}[a, b] \) and \( \rho J^{1 - \alpha}_a \varphi \in C_{\gamma}^1[a, b] \), then
\[
(\rho J^\alpha_a + \rho D^\alpha_a \varphi)(x) = \varphi(x) - \frac{(\rho J^{1 - \alpha}_a \varphi)(a)}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha - 1},
\]
for all \( x \in (a, b) \).

Proof. The proof uses integration by parts, with the choice \( u = (x^\rho - t^\rho)^{\alpha - 1} \) and \( dv = \frac{d}{dt} (\rho J^{1 - \alpha}_a \varphi)(t) dt \). \( \square \)

Lemma 10. Let \( \alpha > 0, 0 \leq \gamma < 1 \) and \( \varphi \in C_{\gamma}[a, b] \). Then,
\[
(\rho D^\alpha_a + \rho J^\alpha_a \varphi)(x) = \varphi(x),
\]
for all \( x \in (a, b) \).

Proof. See [15]. \( \square \)

Lemma 11. Let \( 0 < \alpha < 1, 0 \leq \gamma < 1 \). If \( \varphi \in C_{\gamma}[a, b] \) and \( \rho J^{1 - \alpha}_a \varphi \in C_{\gamma}^1[a, b] \), then
\[
(\rho J^\alpha_a + \rho D^\alpha_a \varphi)(x) = \varphi(x) - \frac{(\rho J^{1 - \alpha}_a \varphi)(a)}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha - 1},
\]
for all \( x \in (a, b) \).
3. Hilfer-Katugampola fractional derivative

In this section, our main result, we introduce the Hilfer-Katugampola fractional derivative and discuss other formulations for fractional derivatives.

**Definition 6.** Let order $\alpha$ and type $\beta$ satisfy $n - 1 < \alpha \leq n$ and $0 \leq \beta \leq 1$, with $n \in \mathbb{N}$. The fractional derivative (left-sided/right-sided), with respect to $x$, with $\rho > 0$ of a function $\varphi \in C_{1-\gamma,\rho}[a, b]$, is defined by

$$
(\rho \mathcal{D}_{a^+}^{\alpha,\beta} \varphi)(x) = \left( \pm \rho \mathcal{J}_{a^+}^{\beta(n-\alpha)} \left( t^{\rho-1} \frac{d}{dt} \right)^n \rho \mathcal{J}_{a^+}^{(1-\beta)(n-\alpha)} \varphi \right)(x)
$$

where $\mathcal{J}$ is the generalized fractional integral given in Definition 4. In this paper we consider the case $n = 1$ only, because the Hilfer derivative and the Hilfer-Hadamard derivative are discussed with $0 < \alpha < 1$.

Note that we present and discuss our new results involving the Hilfer-Katugampola fractional derivative using only the left-sided operator. An analogous procedure can be developed using the right-sided operator. The following property shows that it is possible to write operator $\rho \mathcal{D}_{a^+}^{\alpha,\beta}$ in terms of the operator given in Definition 5.

**Property 1.** The operator $\rho \mathcal{D}_{a^+}^{\alpha,\beta}$ can be written as

$$
\rho \mathcal{D}_{a^+}^{\alpha,\beta} = \rho \mathcal{J}_{a^+}^{\beta(1-\alpha)} \delta_{\rho} \mathcal{J}_{a^+}^{1-\gamma} = \rho \mathcal{J}_{a^+}^{\beta(1-\alpha)} \rho \mathcal{D}_{a^+}^{\gamma}, \quad \gamma = \alpha + \beta(1-\alpha).
$$

Proof. From definition of the generalized fractional integral, we have

$$
(\rho \mathcal{D}_{a^+}^{\alpha,\beta} \varphi)(x) = \rho \mathcal{J}_{a^+}^{\beta(1-\alpha)} \left( x^{1-\rho} \frac{d}{dx} \right) \left\{ \frac{\rho^{1-(1-\beta)(1-\alpha)}(1-\alpha)}{\Gamma((1-\beta)(1-\alpha))} \int_a^x t^{\rho-1} (x^\rho - t^\rho)^{1-(1-\beta)}(1-\alpha) \varphi(t)dt \right\}
$$

$$
= \left[ \rho \mathcal{J}_{a^+}^{\beta(1-\alpha)} \frac{\rho^{1+\alpha+\beta} - \alpha \beta}{\Gamma((1-\beta)(1-\alpha))} (1-\alpha) - 1 \right]
$$

$$
\times \int_a^x (x^\rho - t^\rho)^{1+\alpha+\beta - \alpha \beta} \varphi(t)dt \right] (x)
$$

$$
= (\rho \mathcal{J}_{a^+}^{\beta(1-\alpha)} \rho \mathcal{D}_{a^+}^{\gamma} \varphi)(x),
$$

where operator $\mathcal{D}$ is the generalized fractional derivative given in Definition 5.

Proof. The proof uses integration by parts, with the choice $u = (x^\rho - t^\rho)^{\alpha-1}$ and $dv = \frac{d}{dt}(\rho \mathcal{J}_{a^+}^{1-\alpha} \varphi)(t)dt$. $lacksquare$
Property 2. The fractional derivative $\rho D_{a+}^{\alpha,\beta}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho \to 1$) [9], Hilfer-Hadamard ($\rho \to 0^+$) [12], generalized ($\beta = 0$) [14], Caputo-type ($\beta = 1$) [20], Riemann-Liouville ($\beta = 0, \rho \to 1$) [17], Hadamard ($\beta = 0, \rho \to 0^+$) [17], Caputo ($\beta = 1, \rho \to 1$) [17], Caputo-Hadamard ($\beta = 1, \rho \to 0^+$) [7], Liouville ($\beta = 0, \rho \to 1, a = 0$) [17] and Weyl ($\beta = 0, \rho \to 1, a = -\infty$) [10]. This fact is illustrated in the diagram below.
Hilfer derivative $\rho \to 1$

Hilfer-Katugampola derivative $\beta = 0$

$\rho \to 1$

Katugampola derivative $\beta = 1$

$\rho \to 0^+$

Riemann-Liouville $a = 0$

Liouville

Hadamard $a = -\infty$

Weyl

Caputo-type derivative $\rho \to 1$

$\rho \to 0^+$

Caputo $\rho \to 1$

Caputo-Hadamard $\rho \to 0^+$

Hilfer-Hadamard derivative
**Property 3.** We consider the following parameters $\alpha, \beta, \gamma, \mu$ satisfying
$$
\gamma = \alpha + \beta(1 - \alpha), \quad 0 < \alpha, \beta, \gamma < 1, \quad 0 \leq \mu < 1.
$$
Thus, we define the spaces
$$
C^{\alpha, \beta}_{1-\gamma, \mu}[a, b] = \{ \varphi \in C_{1-\gamma, \mu}[a, b], \rho D_{a^+}^{\alpha, \beta} \varphi \in C_{\mu, \rho}[a, b] \},
$$
and
$$
C^{\gamma}_{1-\gamma, \rho}[a, b] = \{ \varphi \in C_{1-\gamma, \rho}[a, b], \rho D_{a^+}^{\gamma} \varphi \in C_{1-\gamma, \rho}[a, b] \},
$$
where $C_{\mu, \rho}[a, b]$ and $C_{1-\gamma, \rho}[a, b]$ are weighted spaces of continuous functions on $(a, b)$ defined by item (2) in Definition 3. Since $\rho D_{a^+}^{\alpha, \beta} \varphi = \rho J^{(1-\alpha)}_{a^+} \rho D_{a^+}^{\gamma} \varphi$, it follows from Lemma 5 that
$$
C^{\gamma}_{1-\gamma}[a, b] \subset C^{\alpha, \beta}_{1-\gamma}[a, b].
$$

**Lemma 12.** Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(1 - \alpha)$. If $\varphi \in C^{\gamma}_{1-\gamma}[a, b]$, then
$$
(3.1) \quad \rho J^{\gamma}_{a^+} \rho D_{a^+}^{\gamma} \varphi = \rho J^{\gamma}_{a^+} \rho D_{a^+}^{\gamma} \varphi
$$
and
$$
(3.2) \quad \rho D_{a^+}^{\gamma} \rho J^{\alpha}_{a^+} \varphi = \sigma D_{a^+}^{\beta(1-\alpha)} \varphi.
$$

**Proof.** We first prove Eq. (3.1). Using Theorem 2 and Property 1 we can write
$$
\rho J^{\gamma}_{a^+} \rho D_{a^+}^{\gamma} \varphi = \rho J^{\gamma}_{a^+} \rho J^{\gamma}_{a^+} \rho D_{a^+}^{\gamma} \varphi = \rho J^{\gamma}_{a^+} \rho J^{\gamma}_{a^+} \rho D_{a^+}^{\gamma} \varphi = \rho J^{\gamma}_{a^+} \rho D_{a^+}^{\gamma} \varphi.
$$
To prove Eq. (3.2), we use Definition 6 and Theorem 2 to get
$$
\rho D_{a^+}^{\gamma} \rho J^{\alpha}_{a^+} \varphi = \delta \rho J^{\gamma}_{a^+} \rho J^{\alpha}_{a^+} \varphi = \delta \rho J^{\gamma}_{a^+} \rho J^{\alpha}_{a^+} \varphi = \delta \rho J^{\gamma}_{a^+} \rho J^{\alpha}_{a^+} \varphi
$$
$$
= \delta \rho J^{\gamma}_{a^+} \rho J^{\alpha}_{a^+} \varphi = \rho D_{a^+}^{\beta(1-\alpha)} \varphi.
$$

**Lemma 13.** Let $\varphi \in L^1(a, b)$. If $\rho D_{a^+}^{\beta(1-\alpha)} \varphi$ exists on $L^1(a, b)$, then
$$
\rho D_{a^+}^{\alpha, \beta} \rho J^{\alpha}_{a^+} \varphi = \rho J^{\beta(1-\alpha)}_{a^+} \rho D_{a^+}^{\beta(1-\alpha)} \varphi.
$$

**Proof.** From Lemma 3, Definition 5 and Definition 6 we obtain
$$
\rho D_{a^+}^{\alpha, \beta} \rho J^{\alpha}_{a^+} \varphi = \rho J^{\beta(1-\alpha)}_{a^+} \rho D_{a^+}^{\alpha, \beta} \rho J^{\alpha}_{a^+} \varphi = \rho J^{\beta(1-\alpha)}_{a^+} \delta \rho J^{\gamma}_{a^+} \rho J^{\alpha}_{a^+} \varphi
$$
$$
= \rho J^{\beta(1-\alpha)}_{a^+} \delta \rho J^{\gamma}_{a^+} \rho J^{\alpha}_{a^+} \varphi = \rho J^{\beta(1-\alpha)}_{a^+} \rho D_{a^+}^{\beta(1-\alpha)} \varphi.
$$

**Lemma 14.** Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(1 - \alpha)$. If $\varphi \in C_{1-\gamma}[a, b]$ and $\rho J^{\gamma}_{a^+} \rho J^{\alpha}_{a^+} \varphi$ exists on $(a, b)$ and
$$
(3.3) \quad \rho D_{a^+}^{\alpha, \beta} \rho J^{\alpha}_{a^+} \varphi = \varphi, \quad x \in (a, b),
$$
Using Lemma 11, Lemma 3 and Lemma 13 we obtain
\[(^{\rho}D_{a+}^{\alpha,\beta} J_{a+}^{\alpha}) (x) = (^{\rho}J_{a+}^{\beta(1-\alpha)} D_{a+}^{\beta (1-\alpha)}) (x) \]
\[= \varphi(x) - \frac{(^{\rho}J_{a+}^{\beta(1-\alpha)} \varphi)(a)}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\beta(1-\alpha)-1} \]
\[= \varphi(x), \quad x \in (a, b]. \]

\[\square\]

4. Equivalence between the generalized Cauchy problem and the Volterra integral equation

We consider the following nonlinear fractional differential equation
\[(4.1) \quad (^{\rho}D_{a+}^{\alpha,\beta}) (x) = f(x, \varphi(x)), \quad x > a > 0 \]
where \(0 < \alpha < 1, 0 \leq \beta \leq 1, \rho > 0\), with the initial condition
\[(4.2) \quad (^{\rho}J_{a+}^{1-\gamma}) (a) = c, \quad \text{with} \quad \gamma = \alpha + \beta(1-\alpha), \quad c \in \mathbb{R}. \]

The following theorem yields the equivalence between the problem Eq.(4.1)-Eq.(4.2) and the Volterra integral equation, given by
\[(4.3) \quad \varphi(x) = \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha-1} t^{\rho-1} f(t, \varphi(t)) dt. \]

**Theorem 3.** Let \(\gamma = \alpha + \beta(1-\alpha)\), where \(0 < \alpha < 1\) and \(0 \leq \beta \leq 1\). If \(f : (a, b] \times \mathbb{R} \to \mathbb{R}\) is a function such that \(f(\cdot, \varphi(\cdot)) \in C_{1-\gamma}[a, b]\) for any \(\varphi \in C_{1-\gamma}[a, b]\), then \(\varphi\) satisfies Eq.(4.1)-Eq.(4.2) if and only if it satisfies Eq.(4.3).

**Proof.** (\(\Rightarrow\)) Let \(\varphi \in C_{1-\gamma}[a, b]\) be a solution of the problem Eq.(4.1)-Eq.(4.2). We prove that \(\varphi\) is also a solution of Eq.(4.3). From the definition of \(C_{1-\gamma}[a, b]\), Lemma 5 and using Definition 3, we have
\[^{\rho}J_{a+}^{1-\gamma} \varphi \in C[a, b]\]
and
\[^{\rho}D_{a+}^{\gamma} \varphi = {\delta}_{\rho}^{\rho} J_{a+}^{1-\gamma} \varphi \in C_{1-\gamma}[a, b].\]

By Definition 3 it follows that
\[^{\rho}J_{a+}^{1-\gamma} \varphi \in C_{1-\gamma}[a, b].\]

Using Lemma 11 with \(\alpha = \gamma\), and Eq.(4.2), we can write
\[(4.4) \quad (^{\rho}J_{a+}^{\gamma} J_{a+}^{\gamma} \varphi)(x) = \varphi(x) - \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma-1}, \]
where \(x \in (a, b]\). By hypothesis, \(^{\rho}D_{a+}^{\gamma} \varphi \in C_{1-\gamma}[a, b]\), using Lemma 12 with \(\alpha = \gamma\) and Eq.(4.1), we have
\[(4.5) \quad (^{\rho}J_{a+}^{\gamma} D_{a+}^{\gamma} \varphi)(x) = (^{\rho}J_{a+}^{\alpha} J_{a+}^{\alpha} \varphi)(x) \]
\[= (^{\rho}J_{a+}^{\alpha} f(t, \varphi(t)))(x).\]
Comparing Eq. (4.4) and Eq. (4.5), we see that
\[
\phi(x) = \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma-1} + (\rho^\alpha_+ f(t, \phi(t))(x),
\]
with \( x \in (a, b] \), that is, \( \phi(x) \) satisfies Eq. (4.3).

\((\Leftarrow)\) Let \( \varphi \in C_{1-\gamma}[a, b] \) satisfying Eq. (4.3). We prove that \( \varphi \) also satisfies the problem Eq. (4.1)-Eq. (4.2). Apply operator \( \rho^\gamma_{a+} \) on both sides of Eq. (4.6). Then, from Lemma 3, Lemma 12 and Definition 6 we obtain
\[
(\rho^\gamma_{a+} \varphi)(x) = (\rho^\beta_{a+}(1-\alpha) f(t, \phi(t))(x).
\]
By hypothesis, \( \rho^\gamma_{a+} \varphi \in C_{1-\gamma}[a, b] \); then, Eq. (4.7) implies that
\[
(\rho^\gamma_{a+} \varphi)(x) = (\rho^\beta_{a+}(1-\alpha) \varphi) \in C_{1-\gamma}[a, b]
\]
As \( f(\cdot, \varphi(\cdot)) \in C_{1-\gamma}[a, b] \) and from Lemma 5 follows
\[
\rho^\beta_{a+}(1-\alpha) f \in C_{1-\gamma}[a, b].
\]
From Eq. (4.8), Eq. (4.9) and Definition 3, we obtain
\[
\rho^\beta_{a+}(1-\alpha) \varphi \in C_{1-\gamma}[a, b].
\]
Applying operator \( \rho^\beta_{a+}(1-\alpha) \) on both sides of Eq. (4.8) and using Lemma 11, Lemma 3 and Theorem 2 to get
\[
(\rho^\beta_{a+}(1-\alpha) \rho^\gamma_{a+} \varphi)(x) = f(x, \varphi(x)) + \frac{(\rho^\beta_{a+}(1-\alpha) f(t, \phi(t))(a)}{\Gamma(\beta(1-\alpha))} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\beta(1-\alpha)-1}
= (\rho^\beta_{a+}(1-\alpha) \rho^\gamma_{a+} \varphi)(x) = f(x, \varphi(x),
\]
that is, Eq. (4.1) holds. Next, we prove that if \( \varphi \in C_{1-\gamma}[a, b] \) satisfies Eq. (4.3), it also satisfies Eq. (4.2). To this end, we multiply both sides of Eq. (4.6) by \( \rho^\beta_{a+}(1-\alpha) \) and use Lemma 3 and Theorem 2 to get
\[
(\rho^\beta_{a+}(1-\alpha) f(\cdot, \phi(t))(x).
\]
Finally, taking \( x \to a \) in Eq. (4.10), Eq. (4.2) follows.

5. Existence and uniqueness of solution for the Cauchy problem

In this section we prove the existence and uniqueness of the solution for the problem Eq. (4.1)-Eq. (4.2) in the space \( C^{\alpha,\beta}_{1-\gamma}[a, b] \) defined in Property 3 under the hypotheses of Theorem 3 and the Lipschitz condition on \( f(\cdot, \varphi) \) [17 p. 136] with respect to the second variable, that is, that \( f(\cdot, \varphi) \) is bounded in a region \( G \subset \mathbb{R} \) such that
\[(5.1) \quad \| f(x, \varphi_1) - f(x, \varphi_1) \|_{C_{1-\gamma,\rho}[a,b]} \leq A \| \varphi_1 - \varphi_2 \|_{C_{1-\gamma,\rho}[a,b]},\]

for all \( x \in (a, b) \), and for all \( \varphi_1, \varphi_2 \in G \), where \( A > 0 \) is constant.

**Theorem 4.** Let \( 0 < \alpha < 1 \), \( 0 \leq \beta \leq 1 \) and \( \gamma = \alpha + \beta(1 - \alpha) \). Let \( f : (a, b) \times \mathbb{R} \to \mathbb{R} \) be a function such that \( f(\cdot, \varphi(\cdot)) \in C_{\mu,\rho}[a,b] \) for any \( \varphi \in C_{\mu,\rho}[a,b] \) with \( 1 - \gamma \leq \mu < 1 - \beta(1 - \alpha) \) and satisfying the Lipschitz condition Eq.(5.1) with respect to the second variable. Then, there exists a unique solution \( \varphi \) for the problem Eq.(4.1)-Eq.(4.2) in the space \( C_{1-\gamma,\mu}^{\alpha,\beta}[a,b] \).

**Proof.** According to Theorem 3, we just have to prove that there exists a unique solution for the Volterra integral equation, Eq.(4.3). This equation can be written as

\[ \varphi(x) = T\varphi(x), \]

where

\[(5.2) \quad T\varphi(x) = \varphi_0(x) + \left[ \rho \mathcal{J}_\alpha^\alpha f(t, \varphi(t)) \right](x),\]

with

\[(5.3) \quad \varphi_0(x) = \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma-1}.\]

Thus, we divide the interval \((a, b)\) into subintervals on which operator \( T \) is a contraction; we then use Banach fixed point theorem, Theorem 1. Note that \( \varphi \in C_{1-\gamma,\rho}[a,x_1] \), where \( a = x_0 < x_1 < \ldots < x_M = b \) and \( C_{1-\gamma,\rho}[a,x_1] \) is a complete metric space with metric

\[ d(\varphi_1, \varphi_2) = \| \varphi_1 - \varphi_2 \|_{C_{1-\gamma,\rho}[a,x_1]} = \max_{x \in [a,x_1]} \left| \left( \frac{x^\rho - a^\rho}{\rho} \right)^{1-\gamma} [\varphi_1 - \varphi_2] \right|. \]

Choose \( x_1 \in (a, b) \) such that the inequality

\[(5.4) \quad w_1 = \frac{A \Gamma(\gamma)}{\Gamma(\alpha + \gamma)} \left( \frac{x_1^\rho - a^\rho}{\rho} \right)^\alpha < 1,\]

where \( A > 0 \) is a constant, holds, as in Eq.(5.1). Thus, \( \varphi_0 \in C_{1-\gamma,\rho}[a,x_1] \) and from Lemma 5 we have \( T\varphi \in C_{1-\gamma,\rho}[a,x_1] \) and \( T \) maps \( C_{1-\gamma,\rho}[a,x_1] \) into \( C_{1-\gamma,\rho}[a,x_1] \). Therefore, from
Eq. (5.1), Eq. (5.2), Lemma 5] and for any \( \varphi_1, \varphi_2 \in C_{1-\gamma,\rho}[a, x_1] \), we can write
\[
\| T \varphi_1 - T \varphi_2 \|_{C_{1-\gamma,\rho}[a, x_1]} = \| \rho J_a^\alpha f(t, \varphi_1(t)) - \rho J_a^\alpha f(t, \varphi_2(t)) \|_{C_{1-\gamma,\rho}[a, x_1]} \\
= \| \rho J_a^\alpha [f(t, \varphi_1(t)) - f(t, \varphi_2(t))] \|_{C_{1-\gamma,\rho}[a, x_1]} \\
\leq \left( \frac{x^\rho - a^\rho}{\rho} \right)^\alpha \frac{\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} \times \| f(t, \varphi_1(t)) - f(t, \varphi_2(t)) \|_{C_{1-\gamma,\rho}[a, x_1]} \\
\leq A \left( \frac{x^\rho - a^\rho}{\rho} \right)^\alpha \frac{\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} \| \varphi_1(t) - \varphi_2(t) \|_{C_{1-\gamma,\rho}[a, x_1]} \\
\leq w_1 \| \varphi_1(t) - \varphi_2(t) \|_{C_{1-\gamma,\rho}[a, x_1]}.
\]

By hypothesis Eq. (5.4) we can use the Banach fixed point to get a unique solution \( \varphi^* \in C_{1-\gamma,\rho}[a, x_1] \) for Eq. (4.3) on the interval \((a, x_1)\). This solution \( \varphi^* \) is obtained as a limit of a convergent sequence \( T^k \varphi_0^* \):
\[
\lim_{k \to \infty} \| T^k \varphi_0^* - \varphi^* \|_{C_{1-\gamma,\rho}[a, x_1]} = 0,
\]
where \( \varphi_0^* \) is any function in \( C_{1-\gamma,\rho}[a, x_1] \) and
\[
(T^k \varphi_0^*)(x) = (TT^{k-1} \varphi_0^*)(x) = \varphi_0(x) + [\rho J_a^\alpha f(t, (T^{k-1} \varphi_0^*)(t))](x), \quad k \in \mathbb{N}.
\]
We take \( \varphi_0^*(x) = \varphi_0(x) \) with \( \varphi_0(x) \) defined by Eq. (5.3). Denoting
\[
\varphi_k(x) = (T^k \varphi_0^*)(x), \quad k \in \mathbb{N},
\]
then Eq. (5.6) admits the form
\[
\varphi_k(x) = \varphi_0(x) + [\rho J_a^\alpha f(t, \varphi_{k-1}(t))](x), \quad k \in \mathbb{N}.
\]
On the other hand, Eq. (5.5) can be rewritten as
\[
\lim_{k \to \infty} \| \varphi_k - \varphi^* \|_{C_{1-\gamma,\rho}[a, x_1]} = 0.
\]
We consider the interval \([x_1, x_2]\), where \( x_2 = x_1 + h_1, \ h_1 > 0 \) and \( x_2 < b \), then by Eq. (4.3), we can write
\[
\varphi(x) = \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma - 1} \times \int_a^{x_1} t^{\rho - 1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha - 1} f(t, \varphi(t)) dt \\
+ \frac{1}{\Gamma(\alpha)} \int_a^{x} t^{\rho - 1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha - 1} f(t, \varphi(t)) dt \\
= \varphi_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^{x} t^{\rho - 1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha - 1} f(t, \varphi(t)) dt,
\]
where \( \varphi_{01}(x) \), defined by

\[
\varphi_{01}(x) = \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^x t^{\rho-1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha-1} f(t, \varphi(t)) dt,
\]

is the known function and \( \varphi_{01}(x) \in C_{1-\gamma,\rho}[x_1, x_2] \). Using the same arguments as above, we conclude that there exists a unique solution \( \varphi^* \in C_{1-\gamma,\rho}[x_1, x_2] \) for Eq.\((1.3)\) on the interval \([x_1, x_2] \). The next interval to be considered is \([x_2, x_3] \), where \( x_3 = x_2 + h_2 \), \( h_2 > 0 \) and \( x_3 < b \). Repeating this process, we conclude that there exists a unique solution \( \varphi^* \in C_{1-\gamma,\rho}[a, b] \) for Eq.\((1.3)\) on the interval \([a, b] \). We must show that such unique solution \( \varphi^* \in C_{1-\gamma,\rho}[a, b] \) is also in \( C_{1-\gamma,\mu}^{\alpha,\beta}[a, b] \). Thus, we need show that \( (D_{a+}^\alpha \varphi^*) \in C_{\mu,\rho}[a, b] \). We emphasize that \( \varphi^* \) is the limit of the sequence \( \varphi_k \), where \( \varphi_k = T_k \varphi_0^* \in C_{1-\gamma,\rho}[a, b] \), that is,

\[
(5.7) \quad \lim_{k \to \infty} \| \varphi_k - \varphi^* \|_{C_{1-\gamma,\rho}[a, b]} = 0,
\]

for an adequate choice of \( \varphi_0^*(x) \) on each subinterval \([a, x_1], \ldots, [x_{M-1}, b] \). If \( \varphi_0(x) \neq 0 \), then we can admit \( \varphi_0(x) = \varphi(x) \) and once \( \mu \geq 1 - \gamma \), from Lipschitz condition, Eq.\((5.1)\), and by Lemma \ref{lemma1} we can write

\[
|D_{a+}^\alpha \varphi_k - D_{a+}^\alpha \varphi^*|_{C_{\mu,\rho}[a, b]} = |f(x, \varphi_k) - f(x, \varphi^*)|_{C_{\mu,\rho}[a, b]} \leq A \left( \frac{b^\rho - a^\rho}{\rho} \right)^{\mu-1+\gamma} \| \varphi_k - \varphi^* \|_{C_{1-\gamma,\rho}[a, b]}.
\]

By Eq.\((5.7)\) and Eq.\((5.8)\), we obtain

\[
\lim_{k \to \infty} \| D_{a+}^\alpha \varphi_k - D_{a+}^\alpha \varphi^* \|_{C_{\mu,\rho}[a, b]} = 0.
\]

From this last expression, we have \( (D_{a+}^\alpha \varphi^*) \in C_{\mu,\rho}[a, b] \) if \( (D_{a+}^\alpha \varphi_k) \in C_{\mu,\rho}[a, b] \), \( k = 1, 2, \ldots \). Since \( (D_{a+}^\alpha \varphi_k)(x) = f(x, \varphi_{k-1}(x)) \), then by the previous argument, we obtain that \( f(\cdot, \varphi^*(\cdot)) \in C_{\mu,\rho}[a, b] \) for any \( \varphi^* \in C_{\mu,\rho}[a, b] \). Consequently, \( \varphi^* \in C_{1-\gamma,\mu}^{\alpha,\beta}[a, b] \). \hfill \square

6. Cauchy-type problems for fractional differential equations

In this section, we present explicit solutions to fractional differential equations involving the Hilfer-Katugampola differential operator \( (D_{a+}^\alpha \varphi)(x) \) of order \( 0 < \alpha < 1 \) and type \( 0 \leq \beta \leq 1 \) in the space \( C_{1-\gamma,\rho}^{\alpha,\beta}[a, b] \) defined in Property \ref{property3}

We consider the following Cauchy problem

\[
(6.1) \quad (D_{a+}^\alpha \varphi)(x) - \lambda \varphi(x) = f(x), \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1,
\]

\[
(6.2) \quad (J_{a+}^{1-\gamma})(c) = c, \quad \gamma = \alpha + \beta(1 - \alpha),
\]

where \( c, \lambda \in \mathbb{R} \). We suppose that \( f(x) \in C_{\mu,\rho}[a, b] \) with \( 0 \leq \mu < 1 \) and \( \rho > 0 \). Then, by Theorem \ref{theorem3}, the problem Eq.\((6.1)\)-Eq.\((6.2)\) is equivalent to solve the following integral
Using Eq. (2.9), Eq. (6.4) and Lemma 3, we have the following expression for $\varphi$:

\begin{align}
\varphi(x) &= \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha)} \int_a^x t^{\rho-1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha-1} \varphi(t) dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^x t^{\rho-1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha-1} f(t) dt.
\end{align}

(6.3)

In order to solve Eq. (6.3), we use the method of successive approximations, that is,

\begin{align}
\varphi_0(x) &= \frac{c}{\Gamma(\gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma-1}, \\
\varphi_k(x) &= \varphi_0(x) + \frac{\lambda}{\Gamma(\alpha)} \int_a^x t^{\rho-1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha-1} \varphi_{k-1}(t) dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^x t^{\rho-1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha-1} f(t) dt, \quad (k \in \mathbb{N}).
\end{align}

(6.4)

Using Eq. (2.9), Eq. (6.4) and Lemma 3, we have the following expression for $\varphi_1(x)$:

\begin{align}
\varphi_1(x) &= \varphi_0(x) + (\alpha J^\alpha_a \varphi_0)(x) + (\alpha J^\alpha_a f)(x) \\
&= c \sum_{j=1}^2 \frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha j + \beta(1-\alpha)-1} + (\alpha J^\alpha_a f)(x).
\end{align}

(6.5)

Similarly, using Eq. (6.4), Eq. (6.5), Eq. (6.6) and Property 2, we get an expression for $\varphi_2(x)$, as follows:

\begin{align}
\varphi_2(x) &= \varphi_0(x) + (\alpha J^\alpha_a \varphi_1)(x) + (\alpha J^\alpha_a f)(x) \\
&= c \sum_{j=1}^3 \frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha j + \beta(1-\alpha)-1} \\
&+ \lambda (\alpha J^\alpha_a \alpha J^\alpha_a f)(x) + (\alpha J^\alpha_a f)(x) \\
&= c \sum_{j=1}^3 \frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha j + \beta(1-\alpha)-1} \\
&+ \int_a^x \sum_{j=1}^2 \frac{\lambda^{j-1}}{\Gamma(\alpha j)} t^{\rho-1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha j-1} f(t) dt.
\end{align}

(6.6)

Continuing this process, the expression for $\varphi_k(x)$ is given by

\begin{align}
\varphi_k(x) &= c \sum_{j=1}^{k+1} \frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha j + \beta(1-\alpha)-1} \\
&+ \int_a^x \sum_{j=1}^k \frac{\lambda^{j-1}}{\Gamma(\alpha j)} t^{\rho-1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha j-1} f(t) dt.
\end{align}
Taking the limit $k \to \infty$, we obtain the expression for $\varphi(x)$, that is,

$$\varphi(x) = c \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j + \beta (1-\alpha))} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha j + \beta (1-\alpha) - 1}$$

$$+ \int_a^x \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j)} t^{\rho - 1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha j - 1} f(t) dt.$$

Changing the summation index in this last expression, $j \to j + 1$, we have

$$\varphi(x) = c \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j + \gamma)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha j + \beta (1-\alpha) - 1}$$

$$+ \int_a^x \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j + \alpha)} t^{\rho - 1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha j + \alpha - 1} f(t) dt.$$

Using Definition 1, we can rewrite the solution in terms of two-parameters Mittag-Leffler functions,

$$\varphi(x) = c \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\gamma - 1} E_{\alpha,\gamma} \left[ \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^\alpha \right]$$

$$+ \int_a^x t^{\rho - 1} \left( \frac{x^\rho - t^\rho}{\rho} \right)^{\alpha - 1} E_{\alpha,\alpha} \left[ \lambda \left( \frac{x^\rho - t^\rho}{\rho} \right)^\alpha \right] f(t) dt.$$

The function $f(x, \varphi) = \lambda \varphi(x) + f(x)$ satisfies the Lipschitz condition, Eq.(5.1), for any $x_1, x_2 \in (a, b]$ and any $y \in G$, where $G$ is an open set on $\mathbb{R}$. If $\mu \geq 1 - \gamma$, then by Theorem 4, the problem Eq.(6.1)-Eq.(6.2) has a unique solution given by Eq.(6.7) in the space $C_{\alpha,\beta}^{\gamma,\mu} [a, b]$. Note that the problem Eq.(6.1)-Eq.(6.2), whose solution is given by Eq.(6.7), includes the following particular cases:

- If $\rho \to 1$ and $\beta = 0$, then $\gamma = \alpha$ and we have a problem involving the Riemann-Liouville fractional derivative; its solution can be found in [17, p.224].
- For $\rho \to 1$ and $\beta = 1$ our derivative becomes the Caputo fractional derivative; the solution can be found in [17, p.231].
- Considering $\rho \to 0^+$ and $\beta = 0$, we have $\gamma = \alpha$ we have a Cauchy problem formulated with the Hadamard fractional derivative; the solution can be found in [17, p.235].
- Other particular cases arise when we vary the parameters as described in Property 2. Some of them are presented below.

A special case is the case when $f(x) = 0$; we then have the following problem

$$\left( \rho D_{ax}^{\alpha,\beta} \varphi \right)(x) - \lambda \varphi(x) = 0, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1,$$

$$\left( \rho J_{ax}^{1-\gamma} \right)(a) = c, \quad \gamma = \alpha + \beta (1-\alpha),$$
with \( \lambda \in \mathbb{R} \) and \( a < x \leq b \). The solution is given by

(6.10) \[
\varphi(x) = c \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\gamma-1} E_{\alpha, \gamma} \left[ \lambda \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha} \right].
\]

Also, consider the following Cauchy problem:

(6.11) \[
\left( \rho \mathcal{D}_{a^+}^{\alpha, \beta} \varphi \right)(x) - \lambda \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\xi} \varphi(x) = 0, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1,
\]

(6.12) \[
\left( \rho \mathcal{J}_{a^+}^{1-\alpha} \right)(a) = c, \quad c \in \mathbb{R}, \quad \rho > 0,
\]

with \( \lambda, \xi \in \mathbb{R}, \ a < x \leq b \) and \( \xi > -\alpha \). We suppose \( \left[ \lambda \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\xi} \right] \in C_{1-\alpha, \rho}[a, b] \).

Then, by Theorem 3, the problem Eq.(6.11)-Eq.(6.12) is equivalent to the following integral equation:

(6.13) \[
\varphi(x) = \frac{c}{\Gamma(\alpha)} \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha-1} + \frac{\lambda}{\Gamma(\alpha)} \int_{a}^{x} t^{\rho-1} \left( \frac{x^{\rho} - t^{\rho}}{\rho} \right)^{\alpha-1} \left( \frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\xi} \varphi(t) dt.
\]

We apply the method of successive approximations to solve the integral equation Eq.(6.13), that is, we consider

(6.14) \[
\varphi_0(x) = \frac{c}{\Gamma(\alpha)} \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha-1}
\]

and

(6.15) \[
\varphi_k(x) = \varphi_0(x) + \frac{\lambda}{\Gamma(\alpha)} \int_{a}^{x} t^{\rho-1} \left( \frac{x^{\rho} - t^{\rho}}{\rho} \right)^{\alpha-1} \left( \frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\xi} \varphi_{k-1}(t) dt.
\]

For \( k = 1 \) and using Lemma 3 we have

(6.16) \[
\varphi_1(x) = \varphi_0(x) + \lambda \left( \rho \mathcal{J}_{a^+}^{\alpha} \left( \frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\xi} \varphi_0 \right)(x) = \frac{c}{\Gamma(\alpha)} \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha-1} + \frac{c\lambda}{\Gamma(\alpha) \Gamma(2\alpha + \xi)} \left( \frac{x^{\rho} - a^{\rho}}{\rho} \right)^{2\alpha+\xi-1}.
\]
For \( k = 2 \) and using again Lemma 3, we can write
\[
\varphi_2(x) = \varphi_0(x) + \lambda \left( t^\rho - a^\rho \right)^\xi \varphi_1(x)
\]
\[
= \varphi_0(x) + \frac{c\lambda}{\Gamma(\alpha)} \left( t^\rho - a^\rho \right)^{\alpha+\xi-1}(x)
\]
\[
+ \frac{c\lambda^2}{\Gamma(\alpha)\Gamma(2\alpha+\xi)} \left( t^\rho - a^\rho \right)^{2\alpha+2\xi-1}(x)
\]
\[
= \frac{c}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha-1} \left\{ 1 + c_1 \left[ \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha+\xi} \right] \right\}
\]
\[
+ c_2 \left[ \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha+\xi} \right]^2,
\]
where

\[
c_1 = \frac{\Gamma(\alpha + \xi)}{\Gamma(2\alpha + \xi)} \quad \text{and} \quad c_2 = \frac{\Gamma(\alpha + \xi)\Gamma(2\alpha + 2\xi)}{\Gamma(2\alpha + \xi)\Gamma(3\alpha + 2\xi)}.
\]

Continuing this process, we obtain the expression for \( \varphi_k(x) \), given by

\[
\varphi_k(x) = \frac{c}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha-1} \left\{ 1 + \sum_{j=1}^{k} c_j \left[ \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha+\xi} \right] \right\},
\]

where

\[
c_j = \prod_{r=1}^{j} \frac{\Gamma[r(\alpha + \xi)]}{\Gamma[r(\alpha + \xi) + \alpha]}, \quad j \in \mathbb{N}.
\]

Using Definition 2 we can write the solution, Eq. (6.18), in terms of a generalized Mittag-Leffler function:

\[
\varphi_k(x) = \frac{c}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha-1} \times E_{\alpha,1+\xi/\alpha,1+(\xi-1)/\alpha} \left[ \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha+\xi} \right].
\]

If \( \xi \geq 0 \), then \( f(x, \varphi) = \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^\xi \varphi(x) \) satisfies the Lipschitz condition, Eq. (5.1), for any \( x_1, x_2 \in (a, b) \) and for all \( \varphi_1, \varphi_2 \in G \), where \( G \) is an open set on \( \mathbb{R} \). If \( \mu \geq 1 - \gamma \), then by Theorem 4, there exists a unique solution to the problem Eq. (6.11)-Eq. (6.12), given by Eq. (6.20), in space \( C_{1-\gamma,\mu}^{\alpha,\beta}[a, b] \). Note that the problem Eq. (6.11)-Eq. (6.12), whose solution is given by Eq. (6.20), admits the following particular cases:
• For $\rho \to 1$ and $\beta = 0$, we have formulation for this problem, as well as it solution considering the Riemann-Liouville fractional derivative which can be found in [17, p.227].
• Consider $\rho \to 1$ and $\beta = 1$, we have formulation of the problem and it solution, considering the Caputo fractional derivative, which can be found in [17, p.233].
• For $\rho \to 0^+$ and $\beta = 0$, we have formulation for this Cauchy problem and its solution considering the Hadamard fractional derivative that can be found in [17, p.237].
• According to the parameters presented in Property 2 it is also possible to obtain other particular cases.

We have presented a new fractional derivative, the Hilfer-Katugampola fractional derivative. This formulation admits as particular cases the well-known fractional derivatives of Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, generalized, Caputo-type, Weyl and Liouville.

The equivalence between a nonlinear initial value problem and a Volterra integral equation was proved. We also discussed the existence and uniqueness of the solution for this initial value problem. Finally, we obtained the analytical solutions, using the method of successive approximations, to some fractional differential equations; some particular cases were recovered.

A possible continuation of this paper consists in defining a fractional integral whose kernel contains a Mittag-Leffler function and, using such definition, solving an initial value problem similar to the one discussed in this paper. This will be published in a future work [19].

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