SURJECTIVITY OF $p$-ADIC REGULATOR ON $K_2$ OF TATE CURVES

MASANORI ASAKURA

1. Introduction

Let $X$ be a nonsingular projective variety over a field $K$ of char($K$) $\neq p$. Then there are the $p$-adic regulator maps

$$c_{i,j} : K_i(X) \longrightarrow H^{2j-i}_{\text{ét}}(X, \mathbb{Z}_p(j)), \quad i, j \geq 0 \quad (1.1)$$

from Quillen’s $K$-groups to the étale cohomology groups with coefficients in the Tate twist $\mathbb{Z}_p(j)$ ([8], [25]). When $K$ is a local field, it is a long-standing problem whether the maps $K_i(X) \otimes \mathbb{Q}_p \rightarrow H^{2j-i}_{\text{ét}}(X, \mathbb{Q}_p(j))$ for $2j > i$ are surjective, in relation to the Beilinson conjectures (cf. [11] §3). The main result of this paper is to give an affirmative answer to this problem for $K_2$ of the Tate curves over certain $p$-adic fields:

**Theorem 1.1.** Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $E_K = K^* / q \mathbb{Z}$ be the Tate curve over $K$ where $q \in K^*$ is a non-zero element with its order $\text{ord}(q) > 0$. Suppose that $K \subset \mathbb{Q}_p(\zeta)$ for some root of unity $\zeta$. Then the $p$-adic regulator

$$K_2(E_K) \otimes \mathbb{Q}_p \longrightarrow H^2_{\text{ét}}(E_K, \mathbb{Q}_p(2)) \quad (1.2)$$

is surjective.

The surjectivity is also true in the integral sense. Namely due to Suslin’s exact sequence ([27] Cor.23.4), Theorem 1.1 implies that $H^0_{\text{Zar}}(E_K, \mathbb{K}_2) \otimes \mathbb{Z}_p \rightarrow H^2_{\text{ét}}(E_K, \mathbb{Z}_p(2))$ is surjective, and it induces an isomorphism $H^0_{\text{Zar}}(E_K, \mathbb{K}_2)/p^\nu \cong H^2_{\text{ét}}(E_K, \mathbb{Z}_p(2))/p^\nu$ for all $\nu \geq 1$ (Corollary 9.7).

We will prove Theorem 1.1 in the following steps. Recall that there is a standard way to obtain elements of $H^0_{\text{Zar}}(E_K, \mathbb{K}_2) \otimes \mathbb{Q}$ from torsion points of $E_K$ (e.g. [4] (5.1)). The proof of Theorem 1.1 is done by showing that such $K_2$-symbols span the étale cohomology group $H^2_{\text{ét}}(E_K, \mathbb{Q}_p(2))$. Due to the weight filtration on the étale cohomology, $H^2_{\text{ét}}(E_K, \mathbb{Q}_p(2))$ is divided into $H^2_{\text{ét}}(K, \mathbb{Q}_p(1))$ and $H^2_{\text{ét}}(K, \mathbb{Q}_p(2))$. The proof of the first part (Part I, §5) is to show that $K_2$-symbols from torsion points span $H^1_{\text{ét}}(K, \mathbb{Q}_p(1))$. To do this, we will give a quite explicit formula of the regulator maps. The technical results for it are given in §4. The second result (Part II, §7) is to show that some $K_2$-symbols from torsion points span the image of the natural map

$$\lim_{F^S} H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) \longrightarrow H^1_{\text{ét}}(K, \mathbb{Q}_p(2)) \quad (1.3)$$

1
where \( F \) runs over all subfields of \( K \) which are finite abelian extensions of \( \mathbb{Q} \) (i.e. \( F \subset K \cap \mathbb{Q}(\mu) \) for some root \( \mu \) of unity). To do this, we will relate some symbols in \( K_2(E_K) \) with indecomposable elements of \( K_3(K) \) and apply a theorem of Soulé (Theorem 8.1). §6 is the preliminary for it. Finally, we will show that the map (1.3) is surjective when \( K \subset \mathbb{Q}_p(\zeta) \) for some root of unity \( \zeta \) (Part III, §8). All over the proof (except Part III), \( p \)-adic theta function is a basic tool.

The surjectivity of (1.2) has an application to torsion of \( K_1(E_K) \) by Suslin’s exact sequence:

**Theorem 1.2** (Corollary 9.4). Let the notations and assumptions be as in Theorem 1.1. Then the torsion subgroup of \( K_1(E_K) \) is finite. More precisely, let \( \mu_n \) be the group of all roots of unity in \( K \) where \( n \) denotes its cardinality. Let \( (\cdot,\cdot)_n : K^*/n \times K^*/n \to \mu_n \) be the Hilbert symbol (cf. [16] §15). Then the torsion subgroup of \( K_1(E_K) \) is isomorphic to

\[
\mu_n \oplus \mu_n \oplus \mu_n/(q, K^*)_n. \tag{1.4}
\]

The decomposition in (1.4) corresponds to the decomposition \( K_1(E_K) = K^* \oplus K^* \oplus V(E_K) \).

There are previous works on the \( l \)-power torsion of \( K_1 \) for \( l \neq p \). T. Sato proved that the \( l \)-adic regulator \( K_2(E_K) \otimes \mathbb{Q}_l \to H^2_{\acute{e}t}(E_K, \mathbb{Q}_l(2)) \) is surjective and obtained the finiteness of the \( l \)-power torsion part of \( K_1(E_K) \) ([20]). When \( X \) is a nonsingular projective curve over a \( p \)-adic field which has a good reduction, the \( l \)-power torsion of \( K_1(X) \) is finite and described by the rational points of the jacobian of the special fiber (Colliot-Thélène and Raskind [2]). See §9.3 for details. However, very little has been known about the \( p \)-power torsion of \( K_1 \) or the surjectivity of \( p \)-adic regulators on \( K_2 \).

Our proof of Theorem 1.1 is comparable with T. Sato [20]. See §5.1 (in particular Remark 5.5) for his proof. However there is a big difference between \( l \)-adic and \( p \)-adic cases. It is based on the fact that \( \dim H^2_{\acute{e}t}(E_K, \mathbb{Q}_l(2)) = 1 \) for \( l \neq p \) whereas \( \dim H^2_{\acute{e}t}(E_K, \mathbb{Q}_p(2)) = 2[K : \mathbb{Q}_p] + 1 \) where \( [K : \mathbb{Q}_p] \) denotes the degree of \( K \) over \( \mathbb{Q}_p \). In the \( l \)-adic case we only need to construct one \( K_2 \)-symbol which has nontrivial boundary (see [21] for more calculation of the boundary). However it is not enough in the \( p \)-adic case. We need to calculate \( p \)-adic regulators of symbols with trivial boundary.

This paper is organized as follows. §2 is the preliminaries on algebraic \( K \)-theory. §3 is the summary of Tate curves and theta functions. §4 – §8 are devoted to prove Theorem 1.1. In §9, we give several corollaries of Theorem 1.1, including Theorem 1.2. In §10, we show that the \( l \)-adic regulator on \( K_2 \) of any open subscheme of Tate curves is surjective. §10 is independent of the previous sections.

**Acknowledgements.** This paper was written during my stay at the University of Chicago supported by JSPS Postdoctoral Fellowships for Research Abroad. I express sincere
gratitude for their hospitality, especially to professor Spencer Bloch. I also thank professor Shuji Saito for sending me [20], and giving many valuable suggestions.

2. Preliminaries on algebraic $K$-theory.

For an abelian group $M$, we denote by $M[n]$ (resp. $M/n$) the kernel (resp. cokernel) of multiplication by $n$.

2.1. Higher $K$-theory and regulator maps. Let $X$ be a separated quasi-projective scheme over a field $F$. Let $P(X)$ be the exact category of locally free sheaves, and $BQP(X)$ the simplicial set attached to $P(X)$ by Quillen ([18], [26]). The higher $K$-groups of $X$ are defined as the homotopy groups of $BQP(X)$:

$$K_i(X) = \pi_{i+1}BQP(X), \quad i \geq 0.$$  

We refer [26] for the general properties of higher $K$-theory such as, products, localization exact sequences, norm maps (also called transfer maps) etc.

Let $n$ be an integer which is prime to char($F$). Then there are the regulator maps

$$c_{i,j} : K_i(X) \to H^{2j-i}_{\text{ét}}(X, \mathbb{Z}/n(j)), \quad i, j \geq 0$$  

(2.1)

to the étale cohomology groups with coefficients in the Tate twist $\mathbb{Z}/n(j)$ ([8], [25]). They are compatible with products, pull-backs and norm maps. When $X = \text{Spec} F$ and $i = j = 1$, the regulator map is also known as the Galois symbol

$$F^* \to H^1_{\text{ét}}(F, \mathbb{Z}/n(1)), \quad f \mapsto [f]$$  

(2.2)

in which $[f]$ is defined as the cocycle

$$[f] : \text{Gal}(\overline{F}/F) \to \mathbb{Z}/n(1), \quad \sigma \mapsto \sigma(f^{1/n})/f^{1/n}.$$  

Of particular interest to us is the case that $i = j = 2$ and $X$ is a curve.

Lemma 2.1. Let $X$ be a curve over $F$. Put $X_{\overline{F}} = X \otimes_F \overline{F}$. Then the composition

$$K_2(X) \to H^2_{\text{ét}}(X, \mathbb{Z}/n(2)) \to H^0(F, H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/n(2)))$$  

(2.3)

is zero.

Proof. We may assume $n = p^\nu$ with $p \neq \text{char}(F)$. Since there is the isomorphism $H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/p^\nu(2)) \cong \bigoplus H^2_{\text{ét}}(X_i_{\overline{F}}, \mathbb{Z}/p^\nu(2))$ where $X_i_{\overline{F}}$ are the irreducible components of $X_{\overline{F}}$, we may assume that $X$ is irreducible. If $X$ is not complete, there is nothing to prove because of $H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/p^\nu(2)) = 0$. Assume that $X$ is complete. Assume further $H^0_{\text{ét}}(F, \mathbb{Z}_p(1)) = 0$. Then, the assertion follows from the fact that the composition (2.3) factors through

$$\lim_{\nu} H^0(F, H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/p^\nu(2))) = H^0(F, \mathbb{Z}_p(1)) = 0.$$
When $F$ is arbitrary, we choose an inductive limit $F = \lim_i F_i$ and a projective limit $X = \lim X_{F_i}$ where $F_i$ are finitely generated fields over the prime field and $X_{F_i}$ are curves over $F_i$. Since $H^0(F_i, \mathbb{Z}_p(1)) = 0$, we have

$$K_2(X) = \lim_i K_2(X_{F_i}) \to \lim_i H^2_{\text{ét}}(X_{F_i}, \mathbb{Z}/p^n(2))) = H^2_{\text{ét}}(X, \mathbb{Z}/p^n(2)))$$

is zero. \hfill \Box

By Lemma 2.1 and the Hochschild-Serre spectral sequence, the regulator map (2.1) gives rise to a map

$$\rho_X : K_2(X)/n \to H^1(F, H^1_{\text{ét}}(X, \mathbb{Z}/n(2))) \quad (2.4)$$

for a curve $X$.

2.2. $K$-cohomology. Let $\mathcal{K}_i$ be the Zariski sheaf on $X$ associated to the presheaf

$$U \mapsto K_i(U) \quad (U \subset X).$$

Assume that $X$ is a nonsingular variety over $F$. We denote by $X^i$ the set of points of height $i$. Then the Gersten conjecture (proved by Quillen) says the complex

$$0 \to K_i \to K_i(F(X)) \to \bigoplus_{x \in X^1} K_{i-1}(\kappa(x)) \to \cdots \to \bigoplus_{x \in X^{\text{dim}X}} K_{i-\text{dim}X}(\kappa(x)) \to 0$$

of Zariski sheaves is exact. The above complex gives the flasque resolution of the sheaf $\mathcal{K}_i$. Therefore we have the isomorphism

$$H^j_{\text{Zar}}(X, \mathcal{K}_i) \cong \frac{\ker\left(\bigoplus_{x \in X^j} K_{i-j}(\kappa(x)) \to \bigoplus_{x \in X^{j+1}} K_{i-j-1}(\kappa(x))\right)}{\text{Image}\left(\bigoplus_{x \in X^{j-1}} K_{i-j+1}(\kappa(x)) \to \bigoplus_{x \in X^j} K_{i-j}(\kappa(x))\right)} \quad (2.5)$$

In particular, when $X$ is a nonsingular curve over $F$, we have the exact sequence

$$0 \to H^0_{\text{Zar}}(X, \mathcal{K}_2) \to K^M_2(F(X)) \xrightarrow{\tau} \bigoplus_{x \in X^1} \kappa(x)^* \to H^1_{\text{Zar}}(X, \mathcal{K}_2) \to 0. \quad (2.6)$$

Here $K^M_2$ denotes the Milnor $K$-theory (which coincides with Quillen’s $K_2$ by Matsumoto’s theorem), and $\tau = \sum \tau_x$ is the sum of the tame symbol $\tau_x$ at $x \in X^1$:

$$\tau_x : K^M_2(F(X)) \to \kappa(x), \quad \{f, g\} \mapsto (-1)^{\text{ord}_x(f)\text{ord}_x(g)} \frac{f^{\text{ord}_x(g)}}{g^{\text{ord}_x(f)}}. \quad (2.7)$$

Hereafter, we always identify the $K$-cohomology $H^0_{\text{Zar}}(X, \mathcal{K}_2)$ (resp. $H^1_{\text{Zar}}(X, \mathcal{K}_2)$) with the kernel of $\tau$ (resp. cokernel of $\tau$) for a nonsingular curve $X$. Due to the localization exact sequence of $K$-theory, we see that there is a natural surjection $K_2(X) \to H^0_{\text{Zar}}(X, \mathcal{K}_2)$ and an exact sequence

$$0 \to H^1_{\text{Zar}}(X, \mathcal{K}_2) \to K_1(X) \to F(X)^* \xrightarrow{\text{ord}} \bigoplus_{x \in X^1} \mathbb{Z}. \quad (2.8)$$
Suppose further that \( X \) is a complete nonsingular curve. Then the norm maps \( N_{\kappa(x)/F} : \kappa(x)^* \to F^* \) induce the norm map \( H^1_{\text{Zar}}(X, \mathcal{K}_2) \to F^* \) on \( K \)-cohomology. We denote by \( V(X) \) the kernel of it:

\[
0 \to V(X) \to H^1_{\text{Zar}}(X, \mathcal{K}_2) \to F^*.
\]  

(2.9)

If \( X \) has a \( F \)-rational point the right map is surjective and we have a decomposition

\[
K_1(X) = F^* \oplus H^1_{\text{Zar}}(X, \mathcal{K}_2) = F^* \oplus F^* \oplus V(X).
\]

2.3. Suslin’s exact sequence. Let \( X \) be a nonsingular curve over \( F \). It follows from the Riemann-Roch theorem ([8]) that the regulator map \( c_{2,2} : K_2(X) \to H^2_{\text{ét}}(X, \mathbb{Z}/n(2)) \) induces a map \( H^0_{\text{Zar}}(X, \mathcal{K}_2) \to H^2_{\text{ét}}(X, \mathbb{Z}/n(2)) \). A. Suslin proved that there is the natural exact sequence ([27] Cor.23.4)

\[
0 \to H^0_{\text{Zar}}(X, \mathcal{K}_2)/n \to H^1_{\text{ét}}(X, \mathbb{Z}/n(2)) \to H^1_{\text{Zar}}(X, \mathcal{K}_2)[n] \to 0
\]

(2.10)

for \( \text{char}(F) \not| n \). (It is proved not only for curves but also for any nonsingular varieties. However, it is not used in this paper.) Suslin’s sequence (2.10) will be used for the proof of Theorem 1.2.

By Lemma 2.1 and the Hochschild-Serre spectral sequence, we have a map

\[
H^0_{\text{Zar}}(X, \mathcal{K}_2)/n \to H^1(\mathbb{F}, H^1_{\text{ét}}(X_{\mathbb{F}}, \mathbb{Z}/n(2))),
\]

(2.11)

which is compatible with (2.4) under the natural surjection \( K_2(X) \to H^0_{\text{Zar}}(X, \mathcal{K}_2) \).

Without confusing, we also write the map (2.11) by \( \rho_X \).

3. Tate curves and \( p \)-adic theta functions

We give a brief review on Tate curves and theta functions. No proofs are in this section. A good reference is Silverman’s book [23].

3.1. Let \( K \) be a finite extension of \( \mathbb{Q}_p \), and \( \text{ord}_K : K^* \to \mathbb{Z} \) the map of order such that \( \text{ord}_K(\pi_K) = 1 \) where \( \pi_K \) denotes a uniformizer of \( K \). Let \( q \in K^* \) satisfy \( \text{ord}_K(q) > 0 \). The Tate curve \( E_K \) with the period \( q \) is defined as the elliptic curve over \( K \) defined by the equation

\[
y^2 + x y = x^3 + a_4(q)x + a_6(q)
\]

(3.1)

where

\[
a_4(q) = -5 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad a_6(q) = -\sum_{n=1}^{\infty} \frac{(5n^3 + 7n^5)q^n}{1 - q^n}.
\]

(3.2)

This is a \( p \)-adic analogue of the complex torus \( \mathbb{C}^*/q^\mathbb{Z} \). As is so in the classical case, the discriminant \( \Delta \) of \( E_K \) is given by

\[
\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}
\]
and the \( j \)-invariant

\[
j(E_K) = \frac{1}{q} + 744 + 196884q + \cdots.
\]

The series

\[
X(u) = \sum_{n \in \mathbb{Z}} \frac{q^nu}{(1 - q^n u)^2} - 2\sum_{n \geq 1} \frac{nq^n}{1 - q^n}
\]

\[
Y(u) = \sum_{n \in \mathbb{Z}} \frac{(q^n u)^2}{(1 - q^n u)^3} + \sum_{n \geq 1} \frac{nq^n}{1 - q^n}
\]

converge for all \( u \in \overline{K} - q\mathbb{Z} \). They induce a bijective homomorphism

\[
\overline{K}^*/q\mathbb{Z} \xrightarrow{\sim} E_K(\overline{K}), \quad u \mapsto \begin{cases} (X(u), Y(u)) & \text{if } u \not\in q\mathbb{Z} \\ O & \text{if } u \in q\mathbb{Z} \end{cases}
\]  

(3.3)

where \( O \in E_K(K) \) denotes the infinity point. We often identify \( E_K(\overline{K}) \) with \( \overline{K}^*/q\mathbb{Z} \) by the isomorphism (3.3).

**Definition 3.1 (Theta function).**

\[
\theta(u) = \theta(u, q) \overset{\text{def}}{=} (1 - u) \prod_{n=1}^{\infty} (1 - q^n u)(1 - q^n u^{-1})
\]

\( \theta(u) \) converges for all \( u \in \overline{K}^* \) and satisfy

\[
\theta(qu) = \theta(u^{-1}) = -\frac{1}{u} \theta(u).
\]  

(3.4)

Using (3.4), we see that a function

\[
f(u) = c \prod_i \frac{\theta(\alpha_i u)}{\theta(\beta_i u)}
\]

is \( q \)-periodic if \( \prod_i \alpha_i / \beta_i = 1 \). Conversely, for any rational function \( f(u) \) on \( E_K(\overline{K}) := E_K \otimes \overline{K}, \) one can find \( c, \alpha_i, \beta_i \in \overline{K}^* \) such that \( f(u) \) is given as in (3.5). Thus we have a one-one correspondence

\[
\overline{K}(E_K)^* \xleftrightarrow{1:1} \left\{ c \prod_i \frac{\theta(\alpha_i u)}{\theta(\beta_i u)} \mid c, \alpha_i, \beta_i \in \overline{K}^* \text{ with } \prod_i \alpha_i / \beta_i = 1 \right\}.
\]  

(3.6)

We often identify the both sides of (3.6). Since the correspondence (3.6) is compatible with the action of the Galois group \( G_K \), a rational function \( f(u) \in \overline{K}(E_K)^* \) is contained in \( K(E_K) \) if and only if \( c \in K^* \) and

\[
\sum_i [\alpha_i] - [\beta_i] = \sum_i [\alpha_i^\sigma] - [\beta_i^\sigma]
\]  

(3.7)

as divisor on \( \mathbb{G}_m \) for all \( \sigma \in \text{Gal}(\overline{K}/K) \).
3.2. Definition of $K\langle u \rangle$. Let $R$ be the integer ring of $K$. We define a ring
\[
R\langle u \rangle \overset{\text{def}}{=} \lim_{\leftarrow n} R/\pi_K^n[[u]][u^{-1}]
\]
\[
= \{ \sum_{i=-\infty}^{+\infty} a_i u^i \in R[[u, u^{-1}]] : \text{ord}_K(a_i) \to +\infty \text{ as } i \to -\infty \}.
\]
This is a discrete valuation ring with a uniformizer $\pi_K$. We write by $K\langle u \rangle$ the quotient field of $R\langle u \rangle$:
\[
K\langle u \rangle = R\langle u \rangle / \pi_K^{-1} = R\langle u \rangle \otimes_R K.
\]
The field $K\langle u \rangle$ contains $K(u)$, but not $K((u))$.

Since the theta function $\theta(u)$ is contained in $R\langle u \rangle$, the correspondence (3.6) defines an inclusion $K(E_K) \hookrightarrow K\langle u \rangle$. Thus we have a dominant morphism
\[
\text{Spec} K\langle u \rangle \longrightarrow E_K.
\]

3.3. Semistable reduction of Tate curves. Let $C/R$ be a minimal proper regular model of $E_K/K$ over the integer ring $R$. Put $n = \text{ord}_K(q)$. By a result of Kodaira and Néron, the special fiber $Y = Y_1 + \cdots + Y_n$ is type $I_n$, namely, if $n \geq 2$, $Y_i$ are nonsingular rational curves which are arranged in the shape of a $n$-gon, and if $n = 1$, $Y = Y_1$ is an irreducible rational curve with a node.

Let $E/R$ be the Néron model of $E_K/K$. It is the largest subscheme of $C/R$ which is smooth ([23] Theorem 6.1). The group law on $E_K$ extends to make $E/R$ into a commutative group scheme over $R$. The special fiber $E_0$ of $E/R$ is a commutative group scheme over the residue field $k$ which consists of $n$-copies of $\mathbb{G}_{m,k}$. More precisely, we have an isomorphism $E_0 \cong \mathbb{G}_{m,k} \times \mathbb{Z}/n$ as group schemes. By the Néron mapping property, we have $\mathcal{E}(R) = E_K(K)$. Therefore we have a homomorphism $E_K(K) = K^*/q^n \rightarrow E_0(k) = k^* \times \mathbb{Z}/n$. It is explicitly given by $aq^{i/n} \mapsto (a \mod \pi_K, i \mod n)$.

For an integer $r \geq 1$, we put $R_r := R/\pi_K^{r+1}$ and $\mathcal{E}_r := \mathcal{E} \otimes_R R_r$. Then $\mathcal{E}_r/R_r$ is a group scheme. Let $\mathcal{E}_r^0$ be the identity component of $\mathcal{E}_r$. As we have seen in the above, $\mathcal{E}_0^0 \cong \mathbb{G}_{m,k}$. Due to the rigidity of algebraic tori ([SGA3] exp.IX §3), there is an isomorphism $\mathcal{E}_r^0 \cong \mathbb{G}_{m,R_r}$ of group schemes over $R_r$. The embedding
\[
h : \mathbb{G}_{m,R_r} \longrightarrow \mathcal{E}_r^0 \subset \mathcal{E} \subset C
\]
is (locally) defined by
\[
h^*(x) = X(u) \mod \pi_K^{r+1}, \quad h^*(y) = Y(u) \mod \pi_K^{r+1}.
\]
Taking the inductive limit, we have a homomorphism
\[
\mathbb{G}_m^{\text{for}} := \lim_{\leftarrow r} \mathbb{G}_{m,R_r} \longrightarrow \mathcal{E}^{\text{for}} := \lim_{\leftarrow r} \mathcal{E}_r^{\text{for}}
\]
of formal schemes. (Note that it is not algebraizable.) Composing with \( E^\text{for} \to C \), we get a morphism \( \mathbb{G}_m^\text{for} \otimes_R K \to C \otimes_R K = E_K \). It gives a homomorphism
\[
K(E_K) \to \left( \lim_{\to} R/\pi_K^r[u, u^{-1}] \right) \otimes_R K \to \left( \lim_{\to} R/\pi_K^r[[u]][u^{-1}] \right) \otimes_R K = K\langle u \rangle
\]
of fields. This gives another definition of (3.8).

4. The weight exact sequence

4.1. Weight exact sequence. The algebraic fundamental group \( \pi_1(E_K) \) of the Tate curve \( E_K := E_K \otimes \overline{K} \) is isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) since the characteristic of \( K \) is zero. Let us give its generators explicitly. For a Galois covering \( f : X \to E_K \), we denote by \( \text{Aut}(f) \) the group of \( \overline{K} \)-automorphisms \( T : E_K \to E_K \) such that \( fT = f \). Let \( \nu_n : E_{K^n} \to E_K \) be the Galois covering given by the multiplication \( x \mapsto x^n \). Then \( \text{Aut}(\nu_n) \) is isomorphic to \( \mathbb{Z}/n \times \mathbb{Z}/n \). The generators are translations
\[
T_{\zeta_n} : \mathbb{K}^* / q^\mathbb{Z} \to \mathbb{K}^* / q^\mathbb{Z}, \quad x \mapsto x \zeta_n,
\]
\[
T_{q^n} : \mathbb{K}^* / q^\mathbb{Z} \to \mathbb{K}^* / q^\mathbb{Z}, \quad x \mapsto x q^n
\]
where \( \zeta_n \) is a primitive \( n \)-th root of unity. \( \pi_1(E_K) \) is isomorphic to \( \lim_{\to} \text{Aut}(\nu_n) \), and its (topological) generators are \( T_{\zeta_{\infty}} = \lim_{\to} T_{\zeta_n} \) and \( T_{q_{\infty}} = \lim_{\to} T_{q^n} \). There is the fibration exact sequence
\[
1 \to \pi_1(E_K) \to \pi_1(E_K) \to G_K \to 1
\]
where \( G_K \) denotes the absolute Galois group of \( K \). The sequence (4.3) is split by the map coming from a \( K \)-rational point \( \text{Spec} K \to E_K \). Thus \( \pi_1(E_K) \) is isomorphic to the semidirect product \( \pi_1(E_K) \cdot G_K \) with
\[
\sigma T_{\zeta_{\infty}} \sigma^{-1} = T_{\zeta_n}, \quad \sigma T_{q_{\infty}} \sigma^{-1} = T_{q^n}
\]
for \( \sigma \in G_K \). Denote by \( \langle T_{\zeta_{\infty}} \rangle \subset \pi_1(E_K) \) the closed subgroup generated by \( T_{\zeta_{\infty}} \), and \( \pi \subset \pi_1(E_K) \) the closed subgroup generated by \( \langle T_{\zeta_{\infty}} \rangle \) and \( G_K \):
\[
1 \longrightarrow \pi_1(E_K) \longrightarrow \pi_1(E_K) \longrightarrow G_K \longrightarrow 1
\]

Due to (4.4), \( \pi \) is isomorphic to the semidirect product \( \langle T_{\zeta_{\infty}} \rangle \cdot G_K \) with the relation \( \sigma T_{\zeta_{\infty}} \sigma^{-1} = T_{\zeta_n} \). Therefore the natural map
\[
H^1_{\text{ét}}(E_K, \mathbb{Z}/n(j + 1)) = \text{Hom}(\pi_1(E_K), \mathbb{Z}/n(j + 1)) \longrightarrow \text{Hom}(\langle T_{\zeta_{\infty}} \rangle, \mathbb{Z}/n(j + 1))
\]
is compatible with respect to $G_K$-action, and the target is isomorphic to $\mathbb{Z}/n(j)$ as $G_K$-module. Similarly we can see that the map

$$\text{Hom}(\pi_1(E_\mathcal{R})/\langle T_{\zeta_\infty} \rangle, \mathbb{Z}/n(j + 1)) \rightarrow \text{Hom}(\pi_1(E_\mathcal{R}), \mathbb{Z}/n(j + 1))$$

is compatible with $G_K$-action, and the source is isomorphic to $\mathbb{Z}/n(j+1)$ as $G_K$-module. As a result, we have an exact sequence of $G_K$-modules:

$$0 \rightarrow \mathbb{Z}/n(j + 1) \rightarrow H^1_{\text{et}}(E_\mathcal{R}, \mathbb{Z}/n(j + 1)) \rightarrow \mathbb{Z}/n(j) \rightarrow 0. \quad (4.5)$$

This is called the weight exact sequence.

**Lemma 4.1.** Let $E_{m, K} = K*/q^m\mathbb{Z}$. Denote by $\psi_m : E_K \rightarrow E_{m, K}$ and $\phi_m : E_{m, K} \rightarrow E_K$ the homomorphism given by $x \mapsto x^m$ and the natural surjection respectively. Then the pull-backs $\psi_m^*$ and $\phi_m^*$ induce the following commutative diagrams:

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z}/n(j) \\
\downarrow & & \downarrow \psi_m \\
0 & \rightarrow & H^1_{\text{et}}(E_\mathcal{R}, \mathbb{Z}/n(j)) \\
\downarrow \text{mult. by } m & & \downarrow \\
0 & \rightarrow & \mathbb{Z}/n(j - 1) \\
\end{array}
\]

Proof. The map $\psi_m : \pi_1(E_K) \rightarrow \pi_1(E_{m, K})$ is given as follows

$$\psi_m T_{\zeta_\infty} = \lim_i \psi_m T_{\zeta_i} = \lim_i T_{\zeta_i}^m = T_{\zeta_\infty}^m,$$

$$\psi_m T_{q^m} = \lim_i \psi_m T_{q^m i} \equiv \lim_i T_{(q^m) i} \equiv (q^m)_{\infty} \mod \langle T_{\zeta_\infty} \rangle.$$ 

Thus the commutative diagram for $\psi_m$ follows. The diagram for $\phi_m$ follows in a similar way. \hfill \Box

### 4.2. Definition of $\tau^\text{\text{et}}_{\infty}$

From the weight exact sequence (4.5), we have

$$H^0_{\text{zar}}(E_K, \mathcal{K}_2)/n \rho \downarrow$$

$$H^1(K, \mathbb{Z}/n(2)) \rightarrow a H^1(K, H^1_{\text{et}}(E_\mathcal{R}, \mathbb{Z}/n(2))) \rightarrow b H^1(K, \mathbb{Z}/n(1)) = K^*/n.$$ 

Here $\rho = \rho_{E_K}$ is as in (2.11). We define $\tau_{\infty}$ as the composition of $b$ and $\rho$:

$$\tau_{\infty}^\text{\text{et}} \overset{\text{def}}{=} b \cdot \rho : H^0_{\text{zar}}(E_K, \mathcal{K}_2)/n \rightarrow K^*/n.$$ 

By the construction, the maps $\rho$ and $\tau_{\infty}^\text{\text{et}}$ are compatible with the pull-back and the norm map for any finite extension $L/K$. 
4.3. Put $K\langle u \rangle_K = K\langle u \rangle \otimes_K \bar{K}$. Consider a map
\[
(K\langle u \rangle_K)^* / n \to \mathbb{Z} / n, \quad f \mapsto \text{Res} \frac{df}{f}
\] (4.6)
where Res denotes the residue map at $u = 0$, namely if we express $\omega = \sum_{n \in \mathbb{Z}} a_n u^n du$ in the unique way then $\text{Res}(\omega) = a_{-1}$. The map (4.6) is clearly a homomorphism of $G_K$-module. On the other hand the morphism (3.8) induces
\[
H^1_{\text{ét}}(E, \mathbb{Z}/n(1)) \to H^1_{\text{ét}}(K\langle u \rangle_K, \mathbb{Z}/n(1)) = (K\langle u \rangle_K)^* / n.
\] (4.7)

Lemma 4.2. The diagram
\[
\begin{array}{ccc}
H^1_{\text{ét}}(E, \mathbb{Z}/n(1)) & \longrightarrow & \mathbb{Z} / n \\
\downarrow & & \downarrow \\
(K\langle u \rangle_K)^* / n & \longrightarrow & \mathbb{Z} / n
\end{array}
\] (4.8)
is commutative. Here the maps are as in (4.5), (4.6) and (4.7).

Proof. Fix $q^{1/n} \in \bar{K}$ and a primitive $n$-th root of unity $\zeta_n$. We put
\[
f_1(u) := \frac{\theta(q^{1/n}u)^n}{\theta(u)^n - \theta(q^{1/n}u)} = -u \left( \frac{\theta(q^{1/n}u)}{\theta(u)} \right)^n
\]
\[
f_2(u) := \left( \frac{\theta(\zeta_n u)}{\theta(u)} \right)^n.
\]
The divisors of $f_1$ and $f_2$ are $n([q^{-1/n}] - [1])$ and $n([\zeta_n^{-1}] - [1])$ respectively where $[\alpha]$ denotes the divisor of a closed point $\alpha \in \bar{K}^* / q^\mathbb{Z}$. Therefore, each $f_i$ defines the cohomology class $[f_i] \in H^1_{\text{ét}}(E, \mathbb{Z}/n(1))$. We claim that $[f_1]$ and $[f_2]$ span the cohomology group $H^1_{\text{ét}}(E, \mathbb{Z}/n(1))$. Recall that the cohomology class $[f_i]$ is defined as
\[
T_{\zeta_n} \mapsto T_{\zeta_n} (\phi^*_n f_1^n) / \phi^*_n f_1^n, \quad T_{q^{1/n}} \mapsto T_{q^{1/n}} (\phi^*_n f_2^n) / \phi^*_n f_2^n
\]
under the isomorphism $H^1_{\text{ét}}(E, \mathbb{Z}/n(1)) \cong \text{Hom}(\pi_1(E), \mathbb{Z}/n(1))$. Note
\[
\phi^*_n f_1^n = (-1)^{1/n} v \frac{\theta(q^{1/n}u^n)}{\theta(u^n)}, \quad \phi^*_n f_2^n = \frac{\theta(\zeta_n u^n)}{\theta(u^n)}
\]
and $T_{\zeta_n}$ and $T_{q^{1/n}}$ are given by $v \mapsto \zeta_n v$ and $v \mapsto q^{1/n} v$ respectively. Therefore we see
\[
[f_1] : T_{\zeta_n} \mapsto \zeta_n, \quad T_{q^{1/n}} \mapsto 0, \quad [f_2] : T_{\zeta_n} \mapsto 0, \quad T_{q^{1/n}} \mapsto \zeta_n^{-1}.
\]
This shows that $[f_1]$ and $[f_2]$ span $H^1_{\text{ét}}(E, \mathbb{Z}/n(1)) = \text{Hom}(\pi_1(E), \mathbb{Z}/n(1))$.

To show the commutativity of the diagram (4.8), it suffices to show that
\[
\text{Res} \frac{df_1}{f_1} = 1, \quad \text{Res} \frac{df_2}{f_2} = 0 \quad \text{mod} \ n \mathbb{Z}.
\]
Each of them is straightforward.
Put \((K_2(K\langle u \rangle)/n)'/\) by
\[
0 \rightarrow (K_2(K\langle u \rangle)/n)'/ \rightarrow K_2(K\langle u \rangle)/n \rightarrow H^1_{\text{et}}(K\langle u \rangle, \mathbb{Z}/n(2)).
\]
Then we have a commutative diagram
\[
\begin{array}{ccc}
K_2(E_K)/n & \xrightarrow{\phi} & H^1(K, H^1_{\text{et}}(E_K, \mathbb{Z}/n(2))) \\
\downarrow & & \downarrow \\
(K_2(K\langle u \rangle)/n)' & \xrightarrow{\psi} & K^*/n
\end{array}
\]
where the commutativity of the right square is due to Lemma 4.2. We denote by \(\hat{\tau}_{\infty}\) the composition of the below arrows:
\[
\hat{\tau}_{\infty} : (K_2(K\langle u \rangle)/n)' \rightarrow K^*/n.
\]

**Lemma 4.3.** Let \(K\langle u \rangle_{\overline{K}} \rightarrow \mathbb{Z}/n\) be as in (4.6). Then the composition \(K\langle u \rangle_{\overline{K}}^* \rightarrow K\langle u \rangle_{\overline{K}}^*/n \rightarrow \mathbb{Z}/n\) is given by
\[
f \mapsto \sum_{\alpha} \text{Res}_\alpha \frac{df}{f}
\]
where \(\alpha\) runs over all \(\alpha \in \overline{K}\) such that \(\text{ord}_K(\alpha) > 0\) (including \(\alpha = 0\)).

**Proof.** This is straightforward because
\[
(u - \alpha)^{-1} = \begin{cases} u^{-1} \sum_{n=0}^{\infty} (\alpha u^{-1})^n & \text{ord}_K(\alpha) > 0 \\ -\alpha^{-1} \sum_{n=0}^{\infty} (\alpha^{-1} u)^n & \text{ord}_K(\alpha) \leq 0 \end{cases}
\]
in \(K\langle u \rangle_{\overline{K}}\). \qed

It is well-known that the composition
\[
K_2(K\langle u \rangle)/n \rightarrow H^1(K, H^1_{\text{et}}(K\langle u \rangle, \mathbb{Z}/n(2))) \xrightarrow{\text{Res}_2} H^1(K, \mathbb{Z}/n(1)) = K^*/n
\]
coincides with the tame symbol \(\tau_\alpha\) at \(u = \alpha\). Therefore the following map
\[
K_2(K\langle u \rangle)/n \rightarrow (K_2(K\langle u \rangle)/n)' \xrightarrow{\hat{\tau}_{\infty}} K^*/n.
\]
is given by
\[
\xi \mapsto \sum_{\alpha} \tau_\alpha(\xi)
\]
where \(\alpha\) runs over all \(\alpha \in \overline{K}\) such that \(\text{ord}_K(\alpha) > 0\) (including \(\alpha = 0\)).

Summarizing the above results, we have the following:
Theorem 4.4. The diagram

\[
\begin{array}{ccc}
K_2(E_K)/n & \xrightarrow{\tau_\infty^\text{et}} & K^*/n \\
\downarrow & & \downarrow \\
(K_2(K(u))/n)' & \xrightarrow{\tau_\infty^\text{et}} & K^*/n \\
\uparrow & & \uparrow \\
K_2(K(u))/n & \sum_{\alpha} \tau_\alpha & K^*/n
\end{array}
\]

is commutative where \( \alpha \) runs over all \( \alpha \in \overline{K} \) such that \( \text{ord}_K(\alpha) > 0 \) (including \( \alpha = 0 \)).

This theorem enables us to calculate the map \( \tau_\infty^\text{et} \) explicitly (cf. proof of Proposition 5.2).

5. Proof of Theorem 1.1: Part I

There is the Hochschild-Serre spectral sequence

\[
E_2^{ij} = H^i_{\text{ét}}(K, H^j(E_K, \mathbb{Q}_p(2))) \Rightarrow H^{i+j}_{\text{ét}}(E_K, \mathbb{Q}_p(2)).
\]

It degenerates at \( E_2 \)-terms. Since the cohomological dimension of \( K \) is 2 ([22] II 4.3), we have \( E_2^{ij} = 0 \) for \( i \geq 3 \). Moreover, since \([K : \mathbb{Q}_p] < \infty\),

\[
E_2^{02} = H^0_{\text{ét}}(K, H^2(E_K, \mathbb{Q}_p(2))) = H^0_{\text{ét}}(K, \mathbb{Q}_p(1)) = 0.
\]

Due to the duality theorem for the Galois cohomology of local fields (loc.cit. II. 5.2, Theorem. 2), we have

\[
H^2_{\text{ét}}(K, \mathbb{Z}/p^\nu(2)) \cong \text{Hom}(H^0_{\text{ét}}(K, \mathbb{Z}/p^\nu(-1)), \mathbb{Q}/\mathbb{Z}) \cong H^0_{\text{ét}}(K, \mathbb{Z}/p^\nu(1)),
\]

and hence \( E_2^{00} = H^2_{\text{ét}}(E_K, \mathbb{Q}_p(2)) = 0 \). Therefore we have an isomorphism

\[
H^2_{\text{ét}}(E_K, \mathbb{Q}_p(2)) \cong H^1_{\text{ét}}(K, H^1(E_K, \mathbb{Q}_p(2))).
\]

(5.1)

Thus in order to prove Theorem 1.1 it suffices to prove that the cardinality of the cokernel of the map

\[
\rho : H^0_{Zar}(E_K, \mathfrak{K}_2)/p^\nu \rightarrow H^1_{\text{ét}}(K, H^1(E_K, \mathbb{Z}/p^\nu(2)))
\]

has an upper bound which does not depend on \( \nu \). Due to the weight exact sequence (4.5), we have an exact sequence

\[
H^1_{\text{ét}}(E_K, \mathbb{Z}/p^\nu(2)) \xrightarrow{a} H^1_{\text{ét}}(K, H^1(E_K, \mathbb{Z}/p^\nu(2))) \xrightarrow{b} H^1_{\text{ét}}(K, \mathbb{Z}/p^\nu(1)) = K^*/p^\nu.
\]

Note that both of the kernel of \( a \) and the cokernel of \( b \) are finite whose orders are at most \( K^*[p^\infty] \). Then, we first show the following.
(Part I): The cardinality of the cokernel of the map
\[ \tau_\infty^{\text{ét}} : H^0_{\text{Zar}}(E_K, K_2)/p^{\nu} \longrightarrow K^*/p^{\nu} \]
has an upper bound which does not depend on \( \nu \).

Second we put
\[ H_{K_2} := \text{Image}(\ker \tau_\infty^{\text{ét}} \rightarrow H^1_{\text{ét}}(K, \mathbb{Z}/p^{\nu}(2))/\ker a) \quad (5.3) \]
and
\[ H_{ab} := \text{Image}(\lim_{F \rightarrow K} H^1_{\text{ét}}(F, \mathbb{Z}/p^{\nu}(2)) \rightarrow H^1_{\text{ét}}(K, \mathbb{Z}/p^{\nu}(2))/\ker a), \quad (5.4) \]
where \( F \) runs over all subfields of \( K \) which are finite abelian extensions of \( \mathbb{Q} \).

(Part II): \( H_{K_2} \supset mH_{ab} \) for some \( m \neq 0 \) which does not depend on \( \nu \).

Final step is to show that the index of \( H_{ab} \) has an upper bound which does not depend on \( \nu \), or equivalently

(Part III): The map (1.3) is surjective if \( K \subset \mathbb{Q}(\zeta) \) for some root of unity \( \zeta \).

Remark 5.1. We do not need any assumption on \( K \) for the proofs of (Part I) and (Part II). Therefore, we have the surjectivity of the \( p \)-adic regulator (1.2) only if \( K \) satisfies that (1.3) is surjective.

5.1. Proof of Part I: Step 1. Let \( \text{ord}_K : K^* \rightarrow \mathbb{Z} \) be the map of order such that \( \text{ord}_K(\pi_K) = 1 \) for a uniformizer \( \pi_K \in K \). We first show that the map
\[ \text{ord}_K \cdot \tau_\infty^{\text{ét}} : H^0_{\text{Zar}}(E_K, K_2)/p^{\nu} \longrightarrow \mathbb{Z}/p^{\nu} \quad (5.5) \]
is surjective. More precisely, let
\[ o_K : H^0_{\text{Zar}}(E_K, K_2) \longrightarrow \lim_{n \rightarrow \mathbb{Z}/n} H^0_{\text{Zar}}(E_K, K_2)/n \longrightarrow \lim_{n \rightarrow \mathbb{Z}} = \hat{\mathbb{Z}} \quad (5.6) \]
be the composition. Then we construct a symbol \( \xi \in H^0_{\text{Zar}}(E_K, K_2) \) (which comes from torsion points of \( E_K \)) such that \( o_K(\xi) \) is a nonzero integer.

Let \( L/K \) be a finite extension such that there is a uniformizer \( \pi_0 \) of \( L \) satisfying \( q = \pi_0^r \) for some \( r \geq 3 \). Let \( 0 < a < b < r \) be integers. We consider the following rational functions
\[ f(u) := \frac{\theta(\pi_0^a u)^r}{\theta(u)^{r-a} \theta(q u)^a} = (-u)^a \left( \frac{\theta(\pi_0^a u)}{\theta(u)} \right)^r \]
and
\[ g(u) := \frac{\theta(\pi_0^b u)^r}{\theta(u)^{r-b} \theta(q u)^b} = (-u)^b \left( \frac{\theta(\pi_0^b u)}{\theta(u)} \right)^r \]
on $E_L := E_K \otimes_K L$. It is easy to see that the symbol
\[
\xi_L := \left\{ \frac{f(u)}{f(\pi_0^u)}, \frac{g(u)}{g(\pi_0^u)} \right\}
\]  
(5.7)
is contained in the $K$-cohomology group $H^0_{\text{Zar}}(E_L, K^2)$.

**Proposition 5.2.** Put
\[
S(\alpha) = \prod_{k=1}^{\infty} \left( \frac{1 - \alpha q^k}{1 - \alpha q^k} \right)^k \quad (\alpha \in \overline{K}^* - q\mathbb{Z}).
\]  
(5.8)

Then
\[
\tilde{\tau}^\text{et}_\infty(\xi_L) = (-1)^{(a-r)b} \pi_0^{a(b-a)(b-r)} \left( \frac{\theta(\pi_0^b)}{\theta(\pi_0^{b-a})^{b-a} \theta(\pi_0^a)} \right)^r \left( \frac{S(\pi_0^b)}{S(\pi_0^{b-a}) S(\pi_0^a)} \right)^{r^2} \in L^*/n.
\]

**Proof.** We denote by $\hat{\xi}_L \in K_2(L\langle u \rangle)/n$ the image of the symbol $\xi_L$. Due to Theorem 4.4, we have $\tilde{\tau}^\text{et}_\infty(\xi_L) = \hat{\tau}^\text{et}_\infty(\hat{\xi}_L)$. Note
\[
\prod_{k>N} (1 - \pi_0^a q^k u) (1 - \pi_0^{-a} q^k u^{-1}) \in (K\langle u \rangle)^n
\]
for sufficiently large $N \gg \nu$. Therefore we see
\[
\hat{\xi}_L \equiv \left\{ \frac{(-u)^a}{f(\pi_0^b)} \left( \frac{\theta(\pi_0^a)}{\theta(\pi_0^a)} \right)^r, \frac{(-u)^b}{g(\pi_0^a)} \left( \frac{\theta(\pi_0^b)}{\theta(\pi_0^a)} \right)^r \right\} \mod nK_2(L\langle u \rangle)
\]  
(5.9)
where we put
\[
\theta_N(u) := (1 - u) \prod_{k=1}^{N} (1 - q^k u) (1 - q^k u^{-1}).
\]
The right hand side of (5.9) comes from $K_2(K\langle u \rangle)$, so that we can calculate $\hat{\tau}^\text{et}_\infty(\hat{\xi}_L)$ by the tame symbol (Theorem 4.4). The following are straightforward:
\[
\hat{\tau}^\text{et}_\infty \{ u, c \} = c^{-1}
\]  
(5.10)
\[
\hat{\tau}^\text{et}_\infty \{ \theta_N(\pi_0^i u), c \} = 1
\]  
(5.11)
\[
\hat{\tau}^\text{et}_\infty \{ \theta_N(\pi_0^i u), u \} = 1
\]  
(5.12)
\[
\hat{\tau}^\text{et}_\infty \{ \theta_N(\pi_0^i u), \theta_N(\pi_0^j u) \} = S(\pi_0^{i-j})
\]  
(5.13)
for all $0 \leq i, j < r$. Using the aboves, we have
\[
\hat{\tau}^\text{et}_\infty \{ f, g \} = (-1)^{ab} \left( \frac{S(\pi_0^b)}{S(\pi_0^{b-a}) S(\pi_0^a)} \right)^{r^2},
\]
and
\[
\hat{\tau}^\text{ét}_\infty(\xi_L) = \hat{\tau}^\text{ét}_\infty(\xi_L) = \frac{f(\pi_0^{-b}) - b}{g(\pi_0^{-a}) - a} \cdot \hat{\tau}^\text{ét}_\infty\{f, g\}
\]
\[
= (-1)^{a(r-b)} \pi_0^{a(b-a)(b-r)} \left( \frac{\theta(\pi_0^b)}{\theta(\pi_0^b)} \right)^r \left( \frac{S(\pi_0^b)}{S(\pi_0^b)} \right)^{r^2}.
\]

**Corollary 5.3.** Let \( N_{L/K} : H^0_{\text{Zar}}(E_L, \mathcal{O}_L) \rightarrow H^0_{\text{Zar}}(E_K, \mathcal{O}_K) \) be the norm map. Put \( \xi = N_{L/K}(\xi_L) \). Then \( o_K(\xi) \) is a nonzero integer. In particular the cokernel of the map (5.5) is finite.

**Proof.** \( \tau^\text{ét}_\infty(\xi_L) \) can be written as
\[
\tau^\text{ét}_\infty(\xi_L) = (-1)^{a(r-b)} \pi_0^{a(b-a)(b-r)} \left( \frac{M_b}{M_{b-a}M_a} \right)^r \in L^*/n
\]

where
\[
M_i := \theta(\pi_0^i)^r S(\pi_0^i)^r = \prod_{n=1}^\infty \frac{(1 - \pi_0^{nr-i})^{nr-i}}{(1 - \pi_0^{nr+i})^{nr+i}}.
\]

Therefore we have \( o_L(\xi_L) = a(b-a)(b-r) \). Denote by \( f \) the degree of the residue field of \( L \) over the residue field of \( K \). Since the diagram
\[
H^0_{\text{Zar}}(E_L, \mathcal{O}_L) \xrightarrow{o_L} \hat{\mathbb{Z}}
\]
\[
\downarrow N_{L/K}
\]
\[
H^0_{\text{Zar}}(E_K, \mathcal{O}_K) \xrightarrow{o_K} \hat{\mathbb{Z}}
\]

is commutative we have \( o_K(\xi) = fa(b-a)(b-r) \). This is a nonzero integer. \( \square \)

**Corollary 5.4** (T. Sato [20]). The l-adic regulator \( H^0_{\text{Zar}}(E_K, \mathcal{O}_K) \otimes \mathbb{Q}_l \rightarrow H^2_{\text{ét}}(E_K, \mathbb{Q}_l(2)) \) is surjective for \( l \neq p \).

**Proof.** In the same way as the p-adic case, we can show that there is an exact sequence
\[
0 \rightarrow H^1_{\text{ét}}(K, \mathbb{Q}_l(2)) \rightarrow H^2_{\text{ét}}(E_K, \mathbb{Q}_l(2)) \rightarrow H^1_{\text{ét}}(K, \mathbb{Q}_l(1)) \rightarrow 0.
\]

Since \( l \neq p \), it follows from the Euler-Poincaré characteristic (cf. [22] II 5.7.) that we have \( H^1_{\text{ét}}(K, \mathbb{Q}_l(2)) = 0 \) and \( H^1_{\text{ét}}(K, \mathbb{Q}_l(1)) \cong \mathbb{Q}_l \). The composition
\[
H^0_{\text{Zar}}(E_K, \mathcal{O}_K) \rightarrow H^2_{\text{ét}}(E_K, \mathbb{Q}_l(2)) \rightarrow H^1_{\text{ét}}(K, \mathbb{Q}_l(1)) \rightarrow \mathbb{Q}_l
\]

coinsides with the composition of \( o_K \) and the natural map \( \hat{\mathbb{Z}} \rightarrow \mathbb{Q}_l \). Therefore the surjectivity follows from Corollary 5.3. \( \square \)
Remark 5.5 (T.Sato’s thesis). The proof of Corollary 5.4 is different from his one in [20]. His proof is done in the following way.

Let $E/R$ be the Néron model of $E_L/L$, and $G_{m,k} \subset \mathcal{E}_k$ the identity component of the special fiber. Let

$$\partial = \partial_2 \partial_1 : K_2(E_L) \xrightarrow{\partial_1} K_1(G_{m,k}) \xrightarrow{\partial_2} K_0(k) = \mathbb{Z}$$

be the composition where $\partial_i$ is the boundary map coming from the localization sequence of $K$-theory. (It seems $\partial = o_L$ under the inclusion $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$.) In his thesis, T.Sato constructed the symbol $\xi_L$ and showed $\partial(\xi_L) = a(b - a)(b - r)$ (cf. [21] for more calculation of the boundary). This implies

$$\text{corank } H^0(E_K, K_2) \otimes Q_l / \mathbb{Z} \geq 1,$$

and thus he obtained the surjectivity of the $l$-adic regulator $K_2(E_K) \otimes Q_l \to H^2_{\text{ét}}(E_K, Q_l(2))$ and the finiteness of $l$-power torsion part $K_1(E_K)[l^\infty]$ for $l \neq p$ by using Suslin’s exact sequence (cf. Proposition 9.1 below).

The map $\partial$ is enough to study the $l$-adic regulator on $K_2$. However, it is not enough to study the $p$-adic regulator. In fact, our map $\tau_{\infty}^{\text{ét}}$ plays an essential role in the next step.

5.2. Proof of Part I : Step 2. Put $U_K := (1 + \pi_K R) / p^{\nu} \subset K^*/p^{\nu}$ and

$$U_K^{\tau_{\infty}^{\text{ét}}} := U_K \cap \tau_{\infty}^{\text{ét}}(H^0(E_K, K_2))$$

Our next step is to show that the cardinality of $U_K / U_K^{\tau_{\infty}^{\text{ét}}}$ has an upper bound which does not depend on $\nu$. Step 1 and Step 2 immediately imply (Part I).

It follows from the norm map that we may replace $K$ with any finite extension $L$ of $K$. Thus we may assume that there is a uniformizer $\pi_0 \in K^*$ such that $\pi_0^r = q$. The proof is done in the following steps:

(Step 2-1) Let $i m \geq 1$ be any integers and $\zeta_m$ any $m$-th root of unity. Suppose $\zeta_m \in K^*$. Then we have $(1 - \zeta_m \pi_0^i)^{mi} \in U_K^{\tau_{\infty}^{\text{ét}}}$.

(Step 2-2) Let $V_K \subset U_K$ be the subgroup generated by all $(1 - \zeta_m \pi_0^i)^{mi}$ where $i \geq 1$ and $\zeta_m \in K^*$ are roots of unity with $\zeta_m^m = 1$. (By Step 2-1, we have $V_K \subset U_K^{\tau_{\infty}^{\text{ét}}} \subset U_K$.) Then the cardinality of $U_K / V_K$ has an upper bound which does not depend on $\nu$.

To prove the above steps, we use the following lemmas.

Lemma 5.6. Let $m_1, m_2 \geq 1$ be integers, and $\zeta_i \in K^*$ $m_i$-th roots of unity with $\zeta_1 \neq \zeta_2$. Put

$$f(u) := \left( \frac{\theta(\zeta_1^{-1} u)}{\theta(u)} \right)^{m_1}, \quad g(u) := \left( \frac{\theta(\zeta_2^{-1} u)}{\theta(u)} \right)^{m_2}.$$

Then

$$\tau_{\infty}^{\text{ét}} \left\{ \begin{array}{c} f(u) \\ g(u) \end{array} \right\} = \left( \frac{S(\zeta_1^{-1} \zeta_2)}{S(\zeta_1^{-1}) S(\zeta_2)} \right)^{m_1 m_2}.$$

Here \( S(\alpha) \) is as in (5.8).

**Lemma 5.7.** Let \( m \geq 1 \) and \( 1 \leq b < a \) be integers, and \( \zeta \in K^* \) a root of unity with \( \zeta^m = 1 \). Suppose that there is a \( q_0 \in K^* \) such that \( q_0^b = q \). Put

\[
  f(u) := \frac{\theta(q_0^b u)^a}{\theta(u)^a - \theta(q^{-b} u)}, \quad g(u) := \left( \frac{\theta(\zeta^{-1} u)}{\theta(u)} \right)^m.
\]

Then

\[
  \tau_{\text{ét}} \left\{ \frac{f(u) \cdot g(u)}{f(\zeta) \cdot g(q_0^b)} \right\} = \left( \frac{S(q_0^{-b} \zeta)}{S(\zeta) S(q_0^b)} \right)^{ma} \left( \frac{\theta(q_0^b)}{\theta(q_0^b \zeta^{-1})} \right)^{mb}
  = \left( S(\zeta)^{-a} \left( \frac{1 - q_0^{-b}}{1 - q_0^{-b} \zeta^{-b}} \right)^b \prod_{k=1}^{\infty} \left( \frac{1 - q_0^{-b} q^{k}}{1 - q_0^{-b} q^{k}} \right) \right)^{ak+b} \left( \frac{1 - q_0^{-b} q^{k}}{1 - q_0^{-b} q^{k}} \right)^{ak-b}.
\]

The proofs of Lemmas 5.6 and 5.7 are similar to the one of Proposition 5.2.

5.3. Proof of Step 2-1.

**Lemma 5.8.** Let \( \zeta \in K^* \) be any root of unity with \( \zeta^m = 1 \). Then \( S(\zeta)^m \in U_{K}^{\text{ét}} \).

**Proof.** Let \( \mu \in \overline{K}^* \) be a primitive \( N \)-th root of unity with \( (N, pm) = 1 \). Put \( L = K(\mu) \) and \( E_L = E_K \otimes L \). By Lemma 5.6, we have

\[
  \left( \frac{S(\zeta \mu)}{S(\zeta) S(\mu)} \right)^{mN} \in U_{L}^{\text{ét}} \subset L^*/(p^*).
\]

Since \( p \nmid N \), we have

\[
  \left( \frac{S(\zeta \mu)}{S(\zeta) S(\mu)} \right)^{m} \in U_{L}^{\text{ét}} \subset L^*/(p^*).
\]

Applying the norm map for \( L/K \), we have

\[
  \left( \prod_{i=0}^{N-1} \frac{S(\zeta \mu^i)}{S(\zeta) S(\mu^i)} \right)^{m} = \left( S(\zeta)^{-N} \prod_{i=0}^{N-1} \frac{S(\zeta \mu^i)}{S(\mu^i)} \right)^{m} \in U_{K}^{\text{ét}} \subset K^*/(p^*).
\]

Choosing a sufficiently large \( N \gg 1 \) with \( (N, pm) = 1 \), we have

\[
  \prod_{i=0}^{N-1} S(\zeta \mu^i) = \prod_{k=1}^{\infty} \left( \frac{1 - \zeta^N q^{Nk}}{1 - \zeta^{-N} q^{Nk}} \right)^k \equiv 1 \mod (K^*)^{p^*}
\]

and \( \prod_{i=0}^{N-1} S(\mu^i) \equiv 1 \). Thus we have

\[
  S(\zeta)^{-N m} \in U_{K}^{\text{ét}}.
\]

Since \( p \nmid N \), we have \( S(\zeta)^m \in U_{K}^{\text{ét}} \). \( \square \)

**Lemma 5.9.** \( (1 - \pi_0^i)^i \in U_{K}^{\text{ét}} \) for all \( i \geq 1 \).
Proof. Let $m$, $a \geq 1$ be integers. Let $\zeta \in \overline{K}^\ast$ be a primitive $m$-th root of unity. Take $q_0 \in \overline{K}$ such that $q_0^a = \pi_0$ (and hence $q_0^{ar} = q$). We put $L_1 = K(\zeta) \subset L_2 = K(q_0, \zeta)$ and $E_{L_i} = E_K \otimes L_i$. Due to Lemmas 5.7 and 5.8, we have
\[
\left(\frac{1 - q_0^b}{1 - \zeta^{-1}q_0^b}\right)^b \prod_{k=1}^{\infty} \left(\frac{1 - q_0^b q^{ak}}{1 - \zeta^{-1}q_0^b q^{ak}}\right)^{ark+b} \left(\frac{1 - \zeta q_0^b q^{ak}}{1 - q_0^b q^{ak}}\right)^{ark-b}\bigg|_{m} \in U_{L_2}^{\text{et}}
\] (5.18)
for any $1 \leq b < ar$. Suppose that $a$ is large enough and $(a, bm) = 1$. Taking the norm map for $L_2/L_1$, we have
\[
\left(\frac{1 - \pi_0^b}{1 - \zeta^{-a}\pi_0^b}\right)^{mb} \prod_{k=1}^{\infty} \left(\frac{1 - \pi_0^b \pi^{ak}}{1 - \zeta^{-a}\pi_0^b \pi^{ak}}\right)^{ark+b} \left(\frac{1 - \zeta \pi_0^b \pi^{ak}}{1 - \pi_0^b \pi^{ak}}\right)^{ark-b}\bigg|_{m} \in U_{L_1}^{\text{et}}.
\] (5.19)
Since $a \gg 1$, we have
\[
\left(\frac{1 - \pi_0^b}{1 - \zeta^{-a}\pi_0^b}\right)^{mb} \in U_{L_1}^{\text{et}}.
\] (5.20)
Suppose further $m \gg 1$ and $p \not| m$. Then by taking the norm map for $L_1/K$, we have
\[
\left(\frac{1 - \pi_0^b}{1 - \pi_0^{mb}}\right)^{mb} \equiv (1 - \pi_0^b)^{m^2b} \in U_{K}^{\text{et}}.
\] (5.21)
Since $p \not| m$, we have $(1 - \pi_0^b)^{mb} \in U_{K}^{\text{et}}$.

Lemma 5.10. $(1 - \zeta \pi_0^i)^{mi} \in U_{K}^{\text{et}}$ for all $i \geq 1$ and all roots of unity $\zeta \in K^\ast$ such that $\zeta^m = 1$.

Proof. In the proof of Lemma 5.9, the same argument works until (5.9). Thus we have
\[
\left(\frac{1 - \pi_0^b}{1 - \zeta^{-a}\pi_0^b}\right)^{mb} \in U_{K}^{\text{et}}.
\] (5.22)
for all $b \geq 1$. By Lemma 5.9, we have $(1 - \zeta \pi_0^b)^{mb} \in U_{K}^{\text{et}}$.

Lemma 5.10 completes the proof of Step 2-1.

5.4. Proof of Step 2-2. Let $U_{K}^i$ be the subgroup of $K^\ast/p^i$ generated by $1 + \pi_K^i R$ and we put $V_{K}^i := V_K \cap U_{K}^i$. By definition $U_{K}^1 = U_K$, and $U_{K}^i = 0$ for $i \gg \nu$. Let $e$ be the ramified index of $K/Q_p$ (i.e. $\pi_K R = pR$). Then we show that the map
\[
V_{K}^i/V_{K}^{i+1} \longrightarrow U_{K}^i/U_{K}^{i+1}
\] (5.23)
is surjective for all $i \geq e^2 + 2e$. This implies $U_{K}^i = V_{K}^i$ for $i \geq e^2 + 2e$, and hence we obtain an upper bound of $U_{K}/V_{K}$ which does not depend on $\nu$. 

Write \( i = ek + l \) with \( e + 1 \leq l \leq 2e \). Since \( i \geq e^2 + 2e \), we have \( k \geq e \). Let \( a \in R^* \) be any invertible element. Since \( i/e > k + 1/(p-1) \), there is an invertible element \( a' \in R^* \) such that \( 1 + a\pi^i_0 = (1 + a'\pi^i_0)^p^k \). It follows from \( l \leq 2e \leq p^e < p^{k+1} \) that \( \text{ord}_p(l) \leq k \) and therefore we have \( 1 + a\pi^i_0 \in V^i_K \cdot (U^{l+1}_K)^p^k \). On the other hand, since \( l + 1 \geq e \), we have \( \text{ord}_K((p^k\pi^s_0)^{s^l+s}) = e(k - \text{ord}_p(s)) + sl + s \geq i + 1 \) for all \( 1 \leq s \leq p^k \). This shows \((U^{l+1}_K)^p^k \subset U^{i+1}_K \). Thus we have \( 1 + a\pi^i_0 \in V^i_K \cdot U^{i+1}_K \).

6. Nodal rational curves and the Bloch groups

Before going to (Part II), we study \( K_2 \) of nodal rational curves. To do this, we need \( K \)-theory and regulator not only for schemes but also for simplicial schemes. We work in A. Huber’s theory [10] which suffices for our purpose.

6.1. Higher \( K \)-theory of simplicial schemes. Let \( F \) be a field of characteristic zero. We work over the category \((\text{Sch}/F)\) of separated schemes of finite type over \( \text{Spec} F \). Let \( \Delta \) be the category of finite sets \( \{0, \cdots, n\} \) with ordering \( \leq \). A simplicial scheme is a functor from \( \Delta^{op} \) to \((\text{Sch}/F)\). We write \( X_n = X\bullet(\{0, \cdots, n\}) \) for a simplicial scheme \( X\bullet \). A scheme \( X \) is canonically considered as the simplicial scheme such that \( X_n = X \) for \( n \geq 0 \).

The \( K \)-groups \( K(X\bullet) \) of a simplicial scheme \( X\bullet \) are defined. They are functorial and agree with the usual \( K \)-theory if \( X\bullet \) is a scheme. We refer [7] or [10] for the details. Rather than going into the general theory, we pick up the results which we will use later.

**Theorem 6.1** ([10] Proposition 18.1.2). Let \( X\bullet \) be a simplicial scheme. Then there is a natural spectral sequence

\[
E_1^{pq} = \begin{cases} 
0 & p < 0 \\
K_q(X_p) \cap \ker s^0 \cap \cdots \ker s^{p-1} & \text{others} 
\end{cases} \Rightarrow K_{q-p}(X\bullet) \tag{6.1}
\]

where \( s^i : K_q(X_p) \to K_q(X_{p-1}) \) are the degeneracy maps.

A simplicial scheme \( X\bullet \) is called split if

\[
N(X_n) = X_n - \bigcup_s(X_{n-1})
\]

is an open and closed subscheme of \( X_n \). Here \( s : X_{n-1} \to X_n \) runs over all degeneracy maps. We mostly work over split simplicial schemes with finite combinatorial dimension, namely simplicial schemes which are split and such that \( N(X_n) \) is empty for large \( n \). If \( X\bullet \) is a split simplicial scheme with finite combinatorial dimension, then the spectral sequence (6.1) converges and \( E_1^{pq} = K_q(N(X_p)) \).
Theorem 6.2 (loc.cit. 18). There is the regulator map
\[ c_{i,j} : K_i(X_n) \longrightarrow H^{2j-i}_\text{ét}(X_n, \mathbb{Z}/p^\nu(j)), \quad i, j \geq 0. \]
If \( F = \mathbb{C} \) and each \( X_n \) is nonsingular, then we also have the regulator map
\[ c_{i,j}^D : K_i(X_n) \longrightarrow H^{2j-i}_D(X_n, \mathbb{Z}(j)), \quad i, j \geq 0 \]
to the Deligne-Beilinson cohomology. They are functorial and agree with the usual regulator maps (2.1) when \( X_n \) is a scheme.

6.2. Nodal rational curves. Let \( C \) be an irreducible rational curve over \( F \) with one node. It is obtained by attaching 0 to \( \infty \) of \( \mathbb{P}^1 \). We denote by * the node of \( C \). Let \( f : \mathbb{P}^1 \rightarrow C \) be the normalization such that \( f^{-1}(*) = \{0, \infty\} \). We have the simplicial scheme \( C_* \) from \( C \) in the usual way, namely, \( C_0 = \mathbb{P}^1 \), \( C_1 = \mathbb{P}^1 \amalg \{,*\} \), \( \ldots \), and \( d_i : C_1 \rightarrow C_0 \) is defined as the identity on \( \mathbb{P}^1 \) and on \( \{,*\} \), \( d_0 = i_0 \) the inclusion into 0 and \( d_{\infty} = i_{\infty} \) the inclusion into \( \infty \), etc. The natural map \( C_* \rightarrow C \) is a proper hypercovering ([3] 5.3.5 V). Since \( C_* \) is a split simplicial scheme with finite combinatorial dimension, we have an exact sequence

\[ \cdots \longrightarrow K_{i+1}(F) \longrightarrow K_i(C_*) \longrightarrow K_i(\mathbb{P}^1) \xrightarrow{id \otimes (i_0^*-i_{\infty}^*)} K_i(F) \longrightarrow \cdots \]  

from (6.1). The composition of the natural maps \( K_i(C) \rightarrow K_i(C_*) \) and \( K_i(C_*) \rightarrow K_i(\mathbb{P}^1) \) is equal to the pull-back \( f^* \). Moreover, we claim \( i_0^*-i_{\infty}^* = 0 \). In fact, \( K_i(\mathbb{P}^1) \) is isomorphic to \( K_i(F) \otimes K_0(\mathbb{P}^1) \cong K_i(F)^{\oplus 2} \). In the commutative diagram

\[
\begin{array}{ccc}
K_i(F) \otimes K_0(\mathbb{P}^1) & \xrightarrow{id \otimes (i_0^*-i_{\infty}^*)} & K_i(F) \otimes K_0(F) \\
\cong & & \cong \\
K_i(\mathbb{P}^1) & \xrightarrow{i_0^*-i_{\infty}^*} & K_i(F)
\end{array}
\]

the above map is clearly zero. Thus we have \( i_0^*-i_{\infty}^* = 0 \). Now the exact sequence (6.2) becomes

\[ 0 \longrightarrow K_{i+1}(F) \longrightarrow K_i(C_*) \longrightarrow K_i(\mathbb{P}^1) \longrightarrow 0. \]  

Put \( K_i(C)_0 := \ker(f^* : K_i(C) \rightarrow K_i(\mathbb{P}^1)) \). From (6.3), we have a natural map
\[ \delta : K_i(C)_0 \longrightarrow K_{i+1}(F). \]  

The similar argument also works on étale cohomology, and they are compatible under the regulator maps. Therefore we have a commutative diagram

\[
\begin{array}{ccc}
K_i(C)_0 & \xrightarrow{\delta} & K_{i+1}(F) \\
c_{i,j} & & c_{i+1,j} \\
H^{2j-i}_\text{ét}(C, \mathbb{Z}/p^\nu(j)) & \xrightarrow{\delta_{\text{ét}}} & H^{2j-i-1}_\text{ét}(F, \mathbb{Z}/p^\nu(j))
\end{array}
\]
where we put $H^{2j-i}_{\text{ét}}(C, \mathbb{Z}/p'(j))_0 := \ker(f^* : H^{2j-i}_{\text{ét}}(C, \mathbb{Z}/p'(j)) \to H^{2j-i}_{\text{ét}}(\mathbb{P}^1, \mathbb{Z}/p'(j)))$.

Of particular interest to us is the case $i = j = 2$. Write $C_\mathcal{P} = C \otimes_F \mathbb{F}$. We can easily see $H^2_{\text{ét}}(C, \mathbb{Z}/p'(2))_0 = H^1(F, H^1_{\text{ét}}(C_\mathcal{P}, \mathbb{Z}/p'(2)))$ and $\delta_{\text{ét}}$ is the map defined from the natural isomorphism $H^1_{\text{ét}}(C_\mathcal{P}, \mathbb{Z}/p'(2)) \cong \mathbb{Z}/p'(2)$ up to sign. As a result, we obtain

**Proposition 6.3.** The diagram

$$
\begin{array}{ccc}
K_2(C)_0 & \xrightarrow{\delta} & K_3(F) \\
\rho_C \downarrow & & \downarrow \kappa_{3,2} \\
H^1_{\text{ét}}(F, H^1_{\text{ét}}(C_\mathcal{P}, \mathbb{Z}/p'(2))) & \cong & H^1_{\text{ét}}(F, \mathbb{Z}/p'(2))
\end{array}
$$

(6.6)

is commutative up to sign. Here $\rho_C$ is given in (2.4), and the isomorphism below is induced from the natural isomorphism $H^1_{\text{ét}}(C_\mathcal{P}, \mathbb{Z}/p'(2))) \cong \mathbb{Z}/p'(2)$ up to sign.

**Remark 6.4.** To remove the sign ambiguity, we need a careful looking at the relation between the map $K_3(F) \to K_2(C_*)$ and the isomorphism $H^1_{\text{ét}}(C_\mathcal{P}, \mathbb{Z}/p'(2))) \cong \mathbb{Z}/p'(2)$. Since it is nothing important for our purpose, we omit it.

### 6.3. Local ring of the Nodal curve.

Let $\mathcal{O}_{0=\infty}$ be the local ring of $C$ at $\ast$. More explicitly, it is given as follows:

$$
\mathcal{O}_{0=\infty} = \{ f(t) \in F(t) | f(0) = f(\infty) \neq \infty \}
$$

$$
= \left\{ \alpha + \beta \frac{t^n + a_1 t^{n-1} + \cdots + a_n}{t^n + b_1 t^{n-1} + \cdots + b_n} | \alpha, \beta, a_i, b_j \in F, a_n = b_n \neq 0 \right\}.
$$

Quillen’s localization theorem ([9], [26] Thm.(9-1)) yields the exact sequence

$$
0 \longrightarrow K_2(C)_Q \longrightarrow K_2(\mathcal{O}_{0=\infty})_Q \xrightarrow{\tau} \bigoplus_{x \in C-\{\ast\}} \kappa(x)_Q^\ast
$$

(6.7)

where $\tau$ is the tame symbol (2.7). Moreover, by a theorem of van der Kallen [13], $K_2(\mathcal{O}_{0=\infty})$ is isomorphic to Milnor’s $K_2^M(\mathcal{O}_{0=\infty})$. Thus we can think of $K_2(C)_Q$ being a subgroup of $K_2^M(\mathcal{O}_{0=\infty})_Q$.

Let $\mathcal{O}_{0,\infty}$ be the semi-local ring of $\mathbb{P}^1$ at 0 and $\infty$. Let $\mathcal{O}_*$ be the simplicial scheme associated to $\text{Spec} \mathcal{O}_{0=\infty}$. Similarly to (6.2), we have

$$
K_i^M(\mathcal{O}_{0=\infty}) \downarrow
$$

$$
\cdots \longrightarrow K_{i+1}(F) \longrightarrow K_i(\mathcal{O}_*) \longrightarrow K_i(\mathcal{O}_{0,\infty}) \xrightarrow{i_0^* - i_\infty^*} K_i(F) \longrightarrow \cdots
$$

(6.8)

where $i_0 : \{\ast\} \to \text{Spec} \mathcal{O}_{0,\infty}$ and $i_\infty : \{\ast\} \to \text{Spec} \mathcal{O}_{0,\infty}$ are the inclusions into 0 and $\infty$ respectively.
Lemma 6.5. Define the indecomposable $K_3$-group $K_3^{\text{ind}}(F)$ as the cokernel of the natural map $K_3^M(F) \to K_3(F)$. Then the cokernel of $i_0^* - i_{\infty}^* : K_3(\mathcal{O}_{0,\infty}) \to K_3(F)$ is isomorphic to $K_3^{\text{ind}}(F)$.

Proof. By Quillen’s localization theorem, we have
\[
K_3(\mathbb{P}^1 - \{1\}) \to K_3(\mathcal{O}_{0,\infty}) \to \bigoplus_{x \neq 0,1,\infty} K_2(\kappa(x)).
\]

Note that $K_3(\mathbb{P}^1 - \{1\}) = K_3(F)$. The composition of the maps $K_3^M(\mathcal{O}_{0,\infty}) \to K_3(\mathcal{O}_{0,\infty}) \to \bigoplus_{x \neq 0,1,\infty} K_2(\kappa(x))$ is the same symbol. A direct calculation yields that it is surjective. This shows that the map $K_3^M(\mathcal{O}_{0,\infty}) \to K_3(\mathcal{O}_{0,\infty})/K_3(F)$ is surjective. Therefore the cokernel of $i_0^* - i_{\infty}^* : K_3^M(\mathcal{O}_{0,\infty}) \to K_3(F)$ is equal to the cokernel of $i_0^* - i_{\infty}^* : K_3^M(\mathcal{O}_{0,\infty}) \to K_3(\mathcal{O}_{0,\infty}) \to K_3(F)$ is the image of $K_3^M(F)$. This completes the proof. □

We put $K_3^M(\mathcal{O}_{0,\infty}) = \ker(f^* : K_2^M(\mathcal{O}_{0,\infty}) \to K_2(\mathcal{O}_{0,\infty}))$. By Lemma 6.5 and (6.8), we have a map
\[
K_2^M(\mathcal{O}_{0,\infty}) \to K_3^{\text{ind}}(F).
\]

It is clearly compatible with (6.4):
\[
\begin{array}{ccc}
K_2^M(\mathcal{O}_{0,\infty}) & \longrightarrow & K_3^{\text{ind}}(F) \\
\uparrow & & \uparrow \\
K_2(C)_0 & \longrightarrow & K_3(F).
\end{array}
\]

Note that the natural map $K_2(C)_0 \otimes \mathbb{Q} \to K_2^M(\mathcal{O}_{0,\infty})_0 \otimes \mathbb{Q}$ is bijective due to (6.7) and the injectivity of $K_2(\mathbb{P}^1)_\mathbb{Q} \to K_2(\mathcal{O}_{0,\infty})_\mathbb{Q}$.

6.4. Bloch groups. Let $D(F)$ be the free abelian group with basis $[x] (x \in F^* - \{1\})$, and $P(F)$ the quotient group of $D(F)$ by the subgroup generated by the following
\[
[x] - [y] + [y/x] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \quad (x \neq y \in F^* - \{1\}).
\]

The relation (6.11) is called the scissors congruence relations. Then one can easily derive the following basic relations in $P(F) \otimes \mathbb{Q}$ (cf. [5] §5):
\[
[x] + [x^{-1}] = 0 \quad (x \in F^*),
\]
\[
[x] + [1 - x] = 0 \quad (x \in F^* - \{1\}).
\]

If $F$ contains a primitive $m$-th root $\zeta$ of unity, then
\[
m \sum_{i=1}^{m} [\zeta^i x] = [x^n].
\]
A homomorphism
\[ \lambda : P(F) \rightarrow F^* \wedge F^*, \quad [x] \mapsto x \wedge (1 - x). \] (6.15)
is well-defined. The kernel of \( \lambda \) is called the B\textit{loch group} which we denote by \( B(F) \):
\[ 0 \rightarrow B(F) \rightarrow P(F) \xrightarrow{\lambda} F^* \wedge F^* \rightarrow K_2^M(F) \rightarrow 0. \]

Using some ideas of Bloch, Suslin proved the following remarkable theorem.

\textbf{Theorem 6.6} (Suslin [28]). \( K_3^{\text{ind}}(F)_Q \cong B(F)_Q \).

See also related works by Dupont and Sah [5]. Hereafter we identify \( K_3^{\text{ind}}(F)_Q \) with \( B(F)_Q \) by the above theorem.

\section*{6.5. Explicit Description of \( \delta \).}
Passing to the projective limit and tensoring with \( Q \), we have from (6.6)
\[ K_2(C)_{0,Q} \xrightarrow{\rho_C} K_3(F)_Q \xrightarrow{\delta} K_3^M(O_{0=\infty})_{0,Q} \]

(6.16)

Note \( K_2(C)_{0,Q} \cong K_2^M(O_{0=\infty})_{0,Q} \). As is well-known, the regulator map \( c_{3,2} \) factors through \( K_3^{\text{ind}}(F) \). Moreover, \( K_3^{\text{ind}}(F) \) is isomorphic to the Bloch group \( B(F)_Q \) by Theorem 6.6. We thus have a diagram
\[ K_2^M(O_{0=\infty})_{0,Q} \xrightarrow{\delta} B(F)_Q \]

(6.17)

which is commutative up to sign.

We want to describe the map \( \delta \) explicitly. Unfortunately, it is done only when \( F \subset \overline{Q} \), because we use Borel’s theorem in the proof.

\textbf{Proposition 6.7.} Suppose \( F \subset \overline{Q} \). Put \( [a,b] := [a^{-1}b] - [a^{-1}] - [b] \in P(F) \). Let
\[ \xi = \sum \left\{ c_i \prod_{i} \frac{1 - a_i^{-1}t}{1 - b_i^{-1}t}, c'_j \prod_{j} \frac{1 - c_j^{-1}t}{1 - d_j^{-1}t} \right\} \in K_2^M(O_{0=\infty}) \]
be a symbol with \( \prod a_i/b_i = \prod c_j/d_j = 1 \). Assume \( \xi \in K_2^M(O_{0=\infty})_0 \). Then we have
\[ \delta(\xi) = \pm \sum \sum [a_i, c_j] - [a_i, d_j] - [b_i, c_j] + [b_i, d_j] \in B(F)_Q. \]

\textbf{Remark 6.8.} I believe that the above formula holds without the assumption “\( F \subset \overline{Q} \)”.

However we use Proposition 6.7 only for the following special case (see §7.3 Step 4).
Corollary 6.9. Let $F$ be an arbitrary field of characteristic zero. Suppose that there are distinct roots of unity $\zeta_1, \zeta_2 \in F$ such that $\zeta_1^{m_1} = \zeta_2^{m_2} = 1$. Let

$$\eta_0 := \left\{ \left( \frac{1-\zeta_1^{-1}t}{1-t} \right)^{m_1}, \left( \frac{1-\zeta_1^{-1}\zeta_2}{1-\zeta_2} \right)^{m_1}, \left( \frac{1-\zeta_2^{-1}t}{1-t} \right)^{m_2}, \left( \frac{1-\zeta_2^{-1}\zeta_1}{1-\zeta_1} \right)^{m_2} \right\}$$

be a symbol in $K_2^M(\mathcal{O}_{0=\infty})$. Then $\eta_0$ is contained in $K_2^M(\mathcal{O}_{0=\infty})_0$, and

$$\delta(\eta_0) = \pm m_1 m_2 ([\zeta_1^{-1}\zeta_2^{-1}] - [\zeta_1] - [\zeta_2^{-1}]) \in B(F)_Q.$$

Proof. Since everything are defined over $Q(\zeta_1, \zeta_2)$, we may assume $F = Q(\zeta_1, \zeta_2)$. Thus we can apply Proposition 6.7 if we show $\eta_0 \in K_2^M(\mathcal{O}_{0=\infty})_0$.

Letting $\eta'_0 = f^*\eta_0 \in K_2(\mathcal{O}_{0,\infty})$, we want to show $\eta'_0 = 0$. Recall the localization exact sequence

$$K_2(F) = K_2(\mathbb{P}^1 - \{1\}) \rightarrow K_2(\mathcal{O}_{0,\infty}) \rightarrow \bigoplus_{x \not\in \{0,1,\infty\}} \kappa(x)^*.$$

The composition of the maps $K_2^M(\mathcal{O}_{0=\infty}) \rightarrow K_2(\mathcal{O}_{0,\infty}) \rightarrow \bigoplus_{x \not\in \{0,1,\infty\}} \kappa(x)^*$ is the tame symbol, and a direct calculation yields the tame image of $\eta_0$ is zero. Therefore $\eta'_0$ is in the image of $K_2(F)$. We have

$$\eta'_0 = \eta_0'|_{t=0} = \left\{ \left( \frac{1-\zeta_1^{-1}\zeta_2}{1-\zeta_2} \right)^{-m_1}, \left( \frac{1-\zeta_2^{-1}\zeta_1}{1-\zeta_1} \right)^{-m_2} \right\}$$

in $K_2(\mathcal{O}_{0,\infty})$. We can see that the right hand side of (6.18) is zero in the following way.

$$\text{R.H.S of (6.18)} = m_1 m_2 \left\{ \frac{1-\zeta_1^{-1}\zeta_2}{1-\zeta_2}, \frac{1-\zeta_2^{-1}\zeta_1}{1-\zeta_1} \right\}
= m_1 m_2 \left\{ \frac{\zeta_1 - \zeta_2}{1-\zeta_2}, \frac{\zeta_2 - \zeta_1}{1-\zeta_1} \right\}
= m_1 m_2 \left\{ 1-x, 1-x^{-1} \right\} \quad (x := \frac{1-\zeta_1}{1-\zeta_2})
= 0.$$

\[ \square \]

6.6. Proof of Proposition 6.7. We prove the assertion by using the complex regulators. For a complex place $\sigma : F \hookrightarrow \mathbb{C}$, we denote by $c_{\sigma}$ the composition of $K_3^{\text{ind}}(F) \rightarrow K_3^{\text{ind}}(\mathbb{C})$ and the complex regulator $c_{\sigma}^D : K_3^{\text{ind}}(\mathbb{C}) \rightarrow \mathbb{R}$. Note $K_3^{\text{ind}}(F)_Q = K_3(F)_Q$ for any number field $F$. Borel’s theorem asserts the isomorphism

$$K_3^{\text{ind}}(F) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\text{\sim}} \mathbb{R}^{r_2}, \quad x \mapsto (\cdot, \cdot, c_{\sigma}(x), \cdot, \cdot).$$
Put $C_\sigma = C \otimes_{F, \sigma} C$, $O_{0=\infty, \sigma} = O_{0=\infty} \otimes_{F, \sigma} C$ etc. We denote by $\rho'_\sigma$ the composition of $K_2^M(O_{0=\infty})_0 \to K_2^M(O_{0=\infty, \sigma})_0$ and the complex regulator map $K_2^M(O_{0=\infty, \sigma})_0 \to \text{Ext}_{\text{MHS}}(R, H^1(C_\sigma, R(2)))$. Similarly to (6.17), we have a diagram

$$
\begin{array}{ccc}
K_2^M(O_{0=\infty})_0 & \xrightarrow{\delta} & B(F) \\
\rho'_\sigma \downarrow & & \downarrow c_\sigma \\
\text{Ext}_{\text{MHS}}(R, H^1(C_\sigma, R(2))) & \xrightarrow{i} & R
\end{array}
$$

(6.19)

which is commutative up to sign. Here the isomorphism $i$ is induced from the isomorphism $H^1(C_\sigma, Z(2)) \cong Z(2)$. Due to Borel’s theorem, it suffices to show

$$i\rho'_\sigma(\xi) = \pm c_\sigma \delta(\xi) = \pm \sum_{i,j} c_\sigma[a_i, c_j] - c_\sigma[a_i, d_j] - c_\sigma[b_i, c_j] + c_\sigma[b_i, d_j] \in R$$

(6.20)

for all complex places $\sigma$.

The map $c_\sigma$ in (6.19) is given by the Bloch-Wigner function $D_2 ([1])$:

$$c_\sigma[x] = D_2(\sigma(x)), \quad x \in F^* - \{1\}.$$  

(6.21)

Here $D_2$ is defined in the following way.

$$D_2(x) = \arg(1 - x) \log |x| - \text{Im} \int_0^x \log(1 - t) \frac{dt}{t}.$$  

This is a single valued function on $C - \{0, 1\}$. On the other hand, the map $\rho'_\sigma$ in (6.19) is given in the following way. Let $\Sigma\{f, g\}$ be a symbol which is contained in $K_2^M(O_{0=\infty})_0$. We denote by $f^\sigma$ the image of $f$ in $K_2^M(O_{0=\infty, \sigma})_0$. Choose a path $\gamma \subset P^1(C)$ from 0 to $\infty$ which does not meet either poles or zeros of $f^\sigma$ and $g^\sigma$, and such that its homotopy class $[\gamma]$ is a generator of $\pi_1(C_\sigma, *)$. Then $\rho_\sigma$ is given by

$$i\rho_\sigma(\Sigma\{f, g\}) = \sum \int_\gamma \log |f^\sigma| \text{darg}(g^\sigma) - \log |g^\sigma| \text{darg}(f^\sigma).$$  

(6.22)

One can easily check that $\rho_\sigma$ does not depend on the choice of $\gamma$.

Now a direct calculation using (6.21) and (6.22) yields (6.20). Left to the reader for the details.

7. Proof of Theorem 1.1 : Part II

In this section, we prove

\textbf{(Part II):} $H_{K_2} \supset mH_{ab}$ for some $m \neq 0$ which does not depend on $\nu$ (See (5.3) and (5.4) for the notations.)
7.1. Proof of Part II : Step 1. We consider the Tate curve $E_{n,K} = K^*/q^n\mathbb{Z}$ with the period $q^n$ for an integer $n \geq 1$. Recall the diagram (cf. §4.2):\[ H^0_{\text{Zar}}(E_{n,K}, \mathcal{K}_2)/p^\nu \xrightarrow{\rho} H^0_{\text{Zar}}(E_{n,K}, \mathcal{K}_2)/p^\nu \]
\[ H^1_{\text{ét}}(K, \mathbb{Z}/p^\nu(2)) \xrightarrow{a} H^1_{\text{ét}}(K, H^1_{\text{ét}}(E_{n,K}, \mathbb{Z}/p^\nu(2))) \xrightarrow{b} K^*/p^\nu. \] (7.1)

By (Part I), the cardinality of the cokernel of $\tau^\infty_{\text{ét}}$ has an upper bound which does not depend on $\nu$. The kernel of $a$ is dominated by $H^0(K, \mathbb{Z}/p^\nu(1))$ whose order is at most $N := \sharp K^*[p^\infty]$. Let $m_i \geq 1$ ($i = 1, 2$) be integers, and $\zeta_i \in K^*$ $m_i$-th roots of unity with $\zeta_1 \neq \zeta_2$. Let
\[
\begin{align*}
f(v) &:= \left(\frac{\theta(\zeta_1^{-1}v, q^n)}{\theta(v, q^n)}\right)^{m_1}, \quad g(v) := \left(\frac{\theta(\zeta_2^{-1}v, q^n)}{\theta(v, q^n)}\right)^{m_2},
\end{align*}
\]
be rational functions on $E_{n,K}$, where $\theta(v, q^n)$ is the theta function with the period $q^n$. Then we consider a symbol
\[
\eta := \left\{ \frac{f(v)}{f(\zeta_2)}, \frac{g(v)}{g(\zeta_1)} \right\} \in H^0_{\text{Zar}}(E_{n,K}, \mathcal{K}_2)/p^\nu.
\]
By Lemma 5.6, we have $\tau^\infty_{\text{ét}}(\eta) = 1$ when $\text{ord}_p q^n > \nu + 1/(p - 1)$. Thus we get a class
\[
\rho(\eta) \in H^1_{\text{ét}}(K, \mathbb{Z}/p^\nu(2))
\]
such that $a(\rho(\eta)) = \rho(\eta)$.

On the other hand, let $C := \mathbb{P}^1_K/0 \sim \infty$ the nodal curve over $K$ which is obtained by attaching the two points $0$ and $\infty$ (cf. §6.2). We put
\[
\eta_0 := \left\{ \left(\frac{1 - \zeta_1^{-1}t}{1 - t}\right)^{m_1}, \left(\frac{1 - \zeta_1^{-1}\zeta_2}{1 - \zeta_2}\right)^{-m_1}, \left(\frac{1 - \zeta_2^{-1}t}{1 - t}\right)^{m_2}, \left(\frac{1 - \zeta_2^{-1}\zeta_1}{1 - \zeta_1}\right)^{-m_2} \right\}
\]
a symbol in $H^0_{\text{Zar}}(C, \mathcal{K}_2)/p^\nu$. Let
\[
\rho_C : H^0_{\text{Zar}}(C, \mathcal{K}_2)/p^\nu \rightarrow H^1_{\text{ét}}(K, H^1_{\text{ét}}(C_K, \mathbb{Z}/p^\nu(2))) \cong H^1_{\text{ét}}(K, \mathbb{Z}/p^\nu(2))
\]
be the regulator as in (2.11). Thus we get a class
\[
\rho_C(\eta_0) \in H^1_{\text{ét}}(K, \mathbb{Z}/p^\nu(2)).
\]

**Theorem 7.1.** Let $N$ be the cardinality of $K^*[p^\infty]$. Suppose $\text{ord}_p q^n \geq 2\nu + 3$ if $p \geq 3$ and $\text{ord}_p q^n \geq 2\nu + 5$ if $p = 2$. Then we have
\[
N \cdot \rho(\eta) = \pm N \cdot \rho_C(\eta_0) \in H^1_{\text{ét}}(K, \mathbb{Z}/p^\nu(2)).
\]

Note that $N \cdot \rho(\eta)$ does not depend on the choice of $\rho(\eta)$.

**Remark 7.2.** The above equality seems true only if $\text{ord}_p q^n > \nu + 1/(p - 1)$. 
7.2. Proof of Theorem 7.1. With an indeterminant \( s \), we put
\[
\mathcal{O}_i := R[[q^i s]] \subset R[[s]] \quad (i \geq 0).
\]
Since \( \mathcal{O}_i \) is isomorphic to \( R[[t]] \) as ring, it is a complete local ring whose maximal ideal is \( (\pi_K^i, q^i s) \). Moreover it is a unique factorization domain (i.e. any ideal of height 1 is a principal ideal).

Let \( A_i := \mathcal{O}_i[q^{-1}, s^{-1}] \subset R[[s]][q^{-1}, s^{-1}] \). Note \( \mathcal{O}_i[q^{-1}, s^{-1}] = \mathcal{O}_i[\pi_K^{-1}, (q^i s)^{-1}] \) and hence \( A_i \cong R[[s]][\pi_K^{-1}, s^{-1}] \). Let \( \mathcal{E}_i \) be the Tate curve over \( A_i \) with the period \( q^i s \)
\[
\pi : \mathcal{E}_i \rightarrow \text{Spec} A_i.
\]
Since \( \pi \) is a projective and smooth morphism, the regulator map
\[
H^0_{\text{Zar}}(\mathcal{E}_i, \mathcal{K}_2)/p^r \rightarrow H^1_{\text{ét}}(\mathcal{E}_i, \mathbb{Z}/p^r(2))
\]
gives rise to a map
\[
\rho_i : H^0_{\text{Zar}}(\mathcal{E}_i, \mathcal{K}_2)/p^r \rightarrow H^1_{\text{ét}}(A_i, R^1\pi_*\mathbb{Z}/p^r(2)).
\]
Moreover, we have an exact sequence
\[
0 \rightarrow \mathbb{Z}/p^r(2) \rightarrow R^1\pi_*\mathbb{Z}/p^r(2) \rightarrow \mathbb{Z}/p^r(1) \rightarrow 0 \quad (7.3)
\]
of étale sheaves on \( \text{Spec} A_i \) similarly to (4.5). Therefore we have
\[
H^1_{\text{ét}}(A_i, \mathbb{Z}/p^r(2)) \xrightarrow{\alpha_i} H^1_{\text{ét}}(A_i, R^1\pi_*\mathbb{Z}/p^r(2)) \xrightarrow{\beta_i} H^1_{\text{ét}}(A_i, \mathbb{Z}/p^r(1)). \quad (7.4)
\]
The kernel of \( \alpha_i \) is dominated by \( H^0_{\text{ét}}(A_i, \mathbb{Z}/p^r(1)) = H^0(K, \mathbb{Z}/p^r(1)) \), which is finite of order \( \leq N \).

**Lemma 7.3.** \( H^1_{\text{ét}}(A_i, \mathbb{Z}/p^r(1)) \cong A_i^*/p^r \cong (\mathcal{O}_i^* \times \pi_K^r \times s^r)/p^r \).

**Proof.** Since \( R[[s]] \) is a unique factorization domain, so is \( A_i \cong R[[s]][\pi_K^{-1}, s^{-1}] \). Then the assertion follows from Hilbert 90. \( \square \)

Let
\[
f_\mathcal{E}(v) := \left( \frac{\theta(\zeta_{1}^{-1} v, q^i s)}{\theta(v, q^i s)} \right)^{m_1}, \quad g_\mathcal{E}(v) := \left( \frac{\theta(\zeta_{2}^{-1} v, q^i s)}{\theta(v, q^i s)} \right)^{m_2}
\]
be rational functions on \( \mathcal{E}_i \). Consider the symbol
\[
\eta_\mathcal{E} := \left\{ \frac{f_\mathcal{E}(v)}{f_\mathcal{E}(\zeta_1)}, \frac{g_\mathcal{E}(v)}{g_\mathcal{E}(\zeta_1)} \right\} \in H^0(\mathcal{E}_i, \mathcal{K}_2)/p^r.
\]
For an integer \( m \geq 1 \) we put by \( s_m : A_i \rightarrow K \) the \( R \)-ring homomorphism given by \( s \mapsto q^m \). Then we have
\[
\mathcal{E}_i \otimes_{s_m} K = E_{i+m,K} = K^*/(q^{i+m})^\mathbb{Z},
\]
\[
s_m^* f_\mathcal{E}(v) = \left( \frac{\theta(\zeta_{1}^{-1} v, q^{i+m})}{\theta(v, q^{i+m})} \right)^{m_1}, \quad s_m^* g_\mathcal{E}(v) = \left( \frac{\theta(\zeta_{2}^{-1} v, q^{i+m})}{\theta(v, q^{i+m})} \right)^{m_2}.
\]
Lemma 7.4. Put

\[ S_{\mathcal{E}_i}(\alpha) \overset{\text{def}}{=} \prod_{k=1}^{\infty} \left( 1 - \frac{\alpha(q^i)^k}{1 - \alpha^{-1}(q^i)^k} \right)^k \quad (\alpha \in \mathcal{O}_i^\ast). \]

Then we have

\[ (\beta_i\rho_i)(\eta\varepsilon) = \left( \frac{S_{\mathcal{E}_i}(\zeta_1^{-1}\zeta_2)}{S_{\mathcal{E}_i}(\zeta_1^{-1})S_{\mathcal{E}_i}(\zeta_2)} \right)^{m_1m_2} \in A_i^\ast/p^\nu. \] \quad (7.5)

Proof. Put

\[ S_l(\alpha) \overset{\text{def}}{=} \prod_{k=1}^{\infty} \left( 1 - \frac{\alpha(q^l)^k}{1 - \alpha^{-1}(q^l)^k} \right)^k, \quad l \geq 1. \]

By Lemma 5.6 we have

\[ s_m ((\beta_i\rho_i)(\eta\varepsilon)) = \tau^{\text{et}} \left\{ \frac{s_m f\varepsilon(v)}{s_m f\varepsilon(\zeta_2)}, \frac{s_m g\varepsilon(\zeta_1)}{s_m g\varepsilon(\zeta_2)} \right\} \]

\[ = (\frac{S_{i+m}(\zeta_1^{-1}\zeta_2)}{S_{i+m}(\zeta_1^{-1})S_{i+m}(\zeta_2)})^{m_1m_2} \]

\[ = s_m \left( \frac{S_{\mathcal{E}_i}(\zeta_1^{-1}\zeta_2)}{S_{\mathcal{E}_i}(\zeta_1^{-1})S_{\mathcal{E}_i}(\zeta_2)} \right)^{m_1m_2} \in K^\ast/p^\nu \]

for any \( \nu \geq 1 \) and \( m \geq 1 \). This implies (7.5) because of the injectivity of

\[ \prod_{m \geq 1} s_m : \lim_{\nu} A_i^\ast/p^\nu \longrightarrow \prod_{m \geq 1} \lim_{\nu} K^\ast/p^\nu. \]

Let \( i = n - 1 \). Since \( \text{ord}_p q^n \geq 2\nu + 3 \) if \( p \geq 3 \) and \( \text{ord}_p q^n \geq 2\nu + 5 \) if \( p = 2 \), we can choose an integer \( n_0 \) such that \( 0 < n_0 < n - 1 \) and

\[ n_0 \cdot \text{ord}_p q > \nu + 1/(p - 1), \quad (n - 1 - n_0) \cdot \text{ord}_p q > \nu + 1/(p - 1). \]

Due to Lemma 7.4, the class \((\beta_{n-1}\rho_{n-1})(\eta\varepsilon)\) goes to zero in \( A_{n_0}^\ast/p^\nu \) via the natural inclusion \( A_{n-1} \to A_{n_0} \). Therefore, by the exact sequence (7.4), we have a class

\[ \rho_{n-1}(\eta\varepsilon) \in H^{1}_{et}(A_{n_0}, \mathbb{Z}/p^\nu(2)) \]
such that \( \alpha_{n_0}(\rho_{n_1}(\eta_E)) = \rho_{n-1}(\eta_E) \). Next we go on to another inclusion \( A_{n_0} \to A_0 \).
Write \( A_{i,K} \):= \( A_i \otimes_K K \). There is a commutative diagram

\[
\begin{array}{cccc}
0 & \to & H^1_{\text{ét}}(K, \mathbb{Z}/p^n(2)) & \to H^1_{\text{ét}}(A_0, \mathbb{Z}/p^n(2)) \\
\downarrow & & \downarrow r_1 & \\
H^1_{\text{ét}}(A_{n_0}, \mathbb{Z}/p^n(2)) & \to & H^1_{\text{ét}}(A_{n_0,0}, \mathbb{Z}/p^n(2)) & \to H^1_{\text{ét}}(A_{n_0}, \mathbb{Z}/p^n(2)) \\
\end{array}
\]

(7.6)

with exact columns.

**Lemma 7.5.** \((r_2 \iota_1)(N \cdot \rho_{n-1}(\eta_E)) = 0\). In other wards, \( \iota_1(N \cdot \rho_{n-1}(\eta_E)) \) is in the image of \( H^1_{\text{ét}}(K, \mathbb{Z}/p^n(2)) \).

**Proof.** For a finite extension \( L/K \), we denote by \( R_L \) the integer ring of \( L \) and by \( \pi_L \) a uniformizer of \( L \). Then we have

\[
H^1_{\text{ét}}(A_{i,K}, \mathbb{Z}/p^n(2)) = \lim_{L/K \text{ finite}} H^1_{\text{ét}}(A_i \otimes_K L, \mathbb{Z}/p^n(1)) \otimes \mathbb{Z}/p^n(1)
\]

and

\[
R_L^{((q^i s)[q^{-1}, s^{-1}])^*} = R_L^{((q^i s)[q^{-1}, s^{-1}])^*} \times \pi_L^Z \times s^Z
\]

Therefore we have

\[
H^1_{\text{ét}}(A_{i,K}, \mathbb{Z}/p^n(2)) = \lim_{L/K \text{ finite}} \left((1 + q^i s R_L^{((q^i s)[q^{-1}, s^{-1}])}) \times \pi_L^Z \times s^Z\right) /p^n \otimes \mathbb{Z}/p^n(1).
\]

Since \( n_0 > n + 1/(p - 1) \), the component \((1 + q^{n_0} s R_L^{((q^{n_0} s)[q^{-1}, s^{-1}])})/p^n \) goes to zero via \( \iota_2 \):

\[
(1 + q^{n_0} s R_L^{((q^{n_0} s)[q^{-1}, s^{-1}])}) /p^n \xrightarrow{0} (1 + s R_L[s])/p^n.
\]
Therefore, to prove the lemma, it suffices to show that \( \iota_1(\rho_{n-1}(\eta \cdot \tilde{\eta})) \) goes to a \( N \)-torsion element via the composition of the following:

\[
H^1_{\text{ét}}(A_0, \mathbb{Z}/p'(2)) \xrightarrow{r_2} H^1_{\text{ét}}(A_0, \mathbb{Z}/p'(2)) \xrightarrow{\rho} H^1_{\text{ét}}(\mathbb{Z}/p'(2)) = \mathbb{Z}/p' \otimes \mathbb{Z}/p'(1).
\]

However, the image of \( H^1_{\text{ét}}(A_0, \mathbb{Z}/p'(2)) \) must be contained in \( G_K \)-invariant part of \( s^2 \otimes \mathbb{Z}/p' \). Since it is \( N \)-torsion, the assertion follows.

We put by \( \hat{\eta} \) the element of \( H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)) \) such that \( r_1(\hat{\eta}) = \iota_1(N \cdot \rho_{n-1}(\eta \cdot \tilde{\eta})) \).

**Lemma 7.6.** \( \hat{\eta} = N \cdot \rho(\eta) \) in \( H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)) \).

**Proof.** It is enough to show \( r_1(\hat{\eta}) = \iota_1(N \cdot \rho_{n-1}(\eta \cdot \tilde{\eta})) = r_1(N \cdot \rho(\eta)) \) in \( H^1_{\text{ét}}(A_0, \mathbb{Z}/p'(2)) \).

Since they are in the image of \( H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)) \), we may specialize them via the map \( A_0 \to K \) given by \( s \mapsto q \):

\[
\begin{array}{ccc}
H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)) & \xrightarrow{r_1} & H^1_{\text{ét}}(A_0, \mathbb{Z}/p'(2)) \\
& \xrightarrow{x \mapsto x|_{s=q}} & H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)).
\end{array}
\]

We want to show that \( \iota_1(N \cdot \rho_{n-1}(\eta \cdot \tilde{\eta}))|_{s=q} = r_1(N \cdot \rho(\eta))|_{s=q} = N \cdot \rho(\eta) \). Note \( E_{n-1}|_{s=q} = E_{n, K} = K^* / q^n \mathbb{Z} \), \( f \cdot (v)|_{s=q} = f(v) \) and \( g \cdot (v)|_{s=q} = g(v) \) and therefore \( \eta \cdot \tilde{\eta}|_{s=q} = \eta \). This shows that the specialization \( \iota_1(N \cdot \rho_{n-1}(\eta \cdot \tilde{\eta}))|_{s=q} \) coincides with \( \rho(\eta) \) in the cohomology \( H^1_{\text{ét}}(K, E_{n, K}^* / q^n \mathbb{Z}, \mathbb{Z}/p'(2)) \). Hence we have \( \iota_1(N \cdot \rho_{n-1}(\eta \cdot \tilde{\eta}))|_{s=q} \) coincides with \( \rho(\eta) \) modulo \( \ker a \). Since \( \# \ker a \leq N \) the lemma follows.

Next, we take another specialization of \( \iota_1(N \cdot \rho_{n-1}(\eta \cdot \tilde{\eta})) \) via the natural inclusion \( \mathcal{O}_0[(qs)^{-1}] \hookrightarrow M := K((s)) \):

\[
\begin{array}{ccc}
H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)) & \xrightarrow{r_1} & H^1_{\text{ét}}(A_0, \mathbb{Z}/p'(2)) \\
& \xrightarrow{x \mapsto x|_{M}} & H^1_{\text{ét}}(M, \mathbb{Z}/p'(2)).
\end{array}
\]

Let \( E_{n-1, M} = M^* / (q^{n-1} s)^2 \) be the Tate curve defined over the field \( M \), \( \eta_{E, M} \in H^0_{\text{Zar}}(E_{n-1, M}, \mathbb{Z}/p') \) the restriction of \( \eta \), and \( \rho_{n-1, M} : H^0_{\text{Zar}}(E_{n-1, M}, \mathbb{Z}/p') \to H^1_{\text{ét}}(M, H^1_{\text{ét}}(E_{n-1, M}, \mathbb{Z}/p'(2))) \) the regulator map. We have the classes

\[
\rho_{n-1, M}(\eta_{E, M}) \in H^1_{\text{ét}}(M, H^1_{\text{ét}}(E_{n-1, M}, \mathbb{Z}/p'(2)))
\]
\[
\rho_{n-1,M}(\eta_{E,M}) \in H^1_{\text{ét}}(M, \mathbb{Z}/p'\mathbb{Z}(2))
\]
as before. Then we have

\[
r_1(\hat{\eta})|_M = \iota_1(N \cdot \rho_{n-1}(\eta_{E}))|_M = N \cdot \rho_{n-1,M}(\eta_{E,M}) \in H^1_{\text{ét}}(M, \mathbb{Z}/p'\mathbb{Z}(2)).
\]

(7.7)

We can think of the right hand side as an element of \(H^1_{\text{ét}}(K, \mathbb{Z}/p'\mathbb{Z}(2))\):

\[
\hat{\eta} = N \cdot \rho_{n-1,M}(\eta_{E,M}) \in H^1_{\text{ét}}(K, \mathbb{Z}/p'\mathbb{Z}(2)).
\]

(7.8)

We calculate the right hand side of (7.8).

**Lemma 7.7.** \(N \cdot \rho_{n-1,M}(\eta_{E,M}) = \pm N \cdot \rho_C(\eta_0) \in H^1_{\text{ét}}(K, \mathbb{Z}/p'\mathbb{Z}(2))\).

**Proof.** Take a semistable model \(X\) of \(E_{n-1,M}\) over \(K[[s]]\), and let \(Y \subset X\) be the special fiber:

\[
\begin{array}{c}
Y \\
\downarrow
\end{array} \quad \begin{array}{c}
X \\
\downarrow
\pi_X
\end{array} \quad \begin{array}{c}
E_{n-1,M} \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{Spec} K \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Spec} K[[s]] \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Spec} M.
\end{array}
\]

We can easily see that the symbol \(\eta_{E,M}\) comes from a symbol in the \(K\)-cohomology \(H^0_{\text{Zar}}(X, K_2)/p'\mathbb{Z}\) of \(X\), which we denote by \(\eta_X\). Note that there is a commutative diagram

\[
\begin{array}{c}
H^0_{\text{Zar}}(X, K_2)/p'\mathbb{Z} \\
\downarrow
\end{array} \quad \begin{array}{c}
H^1_{\text{ét}}(K[[s]], R^1\pi_*\mathbb{Z}/p'\mathbb{Z}(2)) \\
\downarrow
\end{array} \quad \begin{array}{c}
H^1_{\text{ét}}(E_{n-1,M}, \mathbb{Z}/p'\mathbb{Z}(2)) \\
\downarrow
\end{array}
\]

Moreover, by the proper base change theorem, we have

\[
H^1_{\text{ét}}(K[[s]], R^1\pi_*\mathbb{Z}/p'\mathbb{Z}(2)) \xrightarrow{\sim} H^1_{\text{ét}}(K, (R^1\pi_*\mathbb{Z}/p'\mathbb{Z}(2))|_{s=0})
\]

\[
\cong H^1_{\text{ét}}(K, H^1_{\text{ét}}(Y\overline{Y}, \mathbb{Z}/p'\mathbb{Z}(2)))
\]

\[
\cong H^1_{\text{ét}}(K, \mathbb{Z}/p'\mathbb{Z}(2))
\]

\[
\cong H^1_{\text{ét}}(M, \mathbb{Z}/p'\mathbb{Z}(2))
\]

where the 3rd isomorphism is due to the fact that \(Y\) is a chain of rational curves. Therefore \(\eta_X\) defines a class \(\eta'_{X} \in H^1_{\text{ét}}(K, \mathbb{Z}/p'\mathbb{Z}(2))\), and it coincides with the class \(\rho_{n-1,M}(\eta_{E,M})\) up to \(N\)-torsion:

\[
N \cdot \eta_X = \pm N \cdot \rho_{n-1,M}(\eta_{E,M}) \in H^1_{\text{ét}}(K, \mathbb{Z}/p'\mathbb{Z}(2)).
\]

(7.9)
On the other hand, there is also a commutative diagram

\[
\begin{array}{ccc}
H^0_{\text{Zar}}(Y, \mathcal{K}_2)/p' & \overset{\rho_Y}{\longrightarrow} & H^1_{\text{ét}}(K, H^1_{\text{ét}}(Y \otimes_K \overline{K}, \mathbb{Z}/p')(2)) \\
\uparrow & & \uparrow \\
H^0_{\text{Zar}}(X, \mathcal{K}_2)/p' & \longrightarrow & H^1_{\text{ét}}(K[[s]], R^1\pi_*(\mathbb{Z}/p')(2)).
\end{array}
\]

Here \(\rho_Y\) is as in (2.11). Thus we have

\[\eta'_X = \pm \rho_Y(\eta_X|_Y) \in H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)).\]

By the definition,

\[\eta_X|_Y = \left\{ \left( \frac{1 - \zeta_1^{-1}v}{1 - v} \right)^{m_1} \left( \frac{1 - \zeta_1^{-1} \zeta_2}{1 - \zeta_2} \right)^{-m_1} \left( \frac{1 - \zeta_2^{-1}v}{1 - v} \right)^{m_2} \left( \frac{1 - \zeta_2^{-1} \zeta_1}{1 - \zeta_1} \right)^{-m_2} \right\}.\]

This means the following (cf. §3.3). Let \(Y_0 \subset Y\) be the identity component. Let \(0, \infty \in Y_0\) be the singular points of \(Y\) which are contained in \(Y_0\). By attaching 0 with \(\infty\), we have a nodal curve \(C' := Y_0/0 \sim \infty\). There is the morphism \(\text{col} : Y \to C'\) which collapses the other components. Viewing \(\eta_0\) as a symbol of \(C'\), (7.11) means

\[\eta_X|_Y = \text{col}^* \eta_0 \in H^0_{\text{Zar}}(Y, \mathcal{K}_2)/p'.\]

(7.9), (7.10) and (7.12) yield

\[N \cdot \rho_{n-1,M}(\eta_E,M) = \pm N \cdot \eta'_X = \pm N \cdot \rho_Y(\eta_X|_Y) = \pm N \cdot \rho_Y(\text{col}^*\eta_0) = \pm N \cdot \rho_C(\eta_0)\]

in \(H^1_{\text{ét}}(K, \mathbb{Z}/p'(2))\). This completes the proof. \(\square\)

Now Theorem 7.1 is straightforward from (7.8) and Lemmas 7.6 and 7.7.

### 7.3. Proof of Part II : Step 2

We finish the proof of (Part II).

Let \(\psi_n : E_K \to E_{n,K}\) be the surjective homomorphism given by \(x \mapsto x^n\). The map \(\rho\) in (7.1) and the map (5.2) are compatible under the pull-back \(\psi_n^*\). Therefore by Lemma 4.1, we have

\[\rho(\eta) \in a(H_{K_2}) \subset H^1_{\text{ét}}(K, H^1_{\text{ét}}(E, \mathbb{Z}/p')(2))).\]

By Theorem 7.1, we have

\[N \cdot a \rho_C(\eta_0) = \pm N \cdot \rho(\eta) \in a(H_{K_2}).\]

Let \(\hat{\rho}_C\) be the composition

\[\hat{\rho}_C : H^0_{\text{Zar}}(C, \mathcal{K}_2) \longrightarrow \lim_{\psi_n^*} H^0_{\text{Zar}}(C, \mathcal{K}_2)/p' \longrightarrow H^1_{\text{ét}}(K, H^1_{\text{ét}}(C, \mathbb{Z}/p')(2))) \cong H^1_{\text{ét}}(K, \mathbb{Z}/p'(2)).\]

Then \(\rho_C(\eta_0) = \hat{\rho}_C(\eta_0) \mod p'\), and hence we have from (7.14) that

\[\text{Image of } N \cdot \rho_C(\eta_0) \in H_{K_2}.\]
On the other hand, by Corollary 6.9 we have
\[ \hat{\rho}_C(\eta_0) = \pm m_1m_2c_{3,2}([\zeta_1^{-1}\zeta_2] - [\zeta_1^{-1}] - [\zeta_2]) \in H^1_{\text{et}}(K, \mathbb{Q}_p(2)) \] (7.16)
where \( c_{3,2} \) is the regulator map
\[ c_{3,2} : B(K)_\mathbb{Q} \cong K^\text{ind}_3(K)_\mathbb{Q} \rightarrow H^1_{\text{et}}(K, \mathbb{Q}_p(2)). \]

Put by \( H_{K_2}' \) (resp. \( H_{ab}' \)) the image of \( H_{K_2} \) (resp. \( H_{ab} \)) by the following map
\[ H^1_{\text{et}}(K, \mathbb{Z}/p^\nu(2))/\ker a \rightarrow H^1_{\text{et}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))/\ker a. \]
Since the kernel of \( H^1_{\text{et}}(K, \mathbb{Z}/p^\nu(2)) \rightarrow H^1_{\text{et}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) \) is dominant by a finite group \( H_{\text{et}}^0(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) \), to say that \( H_{K_2}' \supset mH_{ab}' \) for some \( m \neq 0 \) is equivalent to say that \( H_{K_2}' \supset m'H_{ab}' \) for some \( m' \neq 0 \) which does not depend on \( \nu \). Noting the commutative diagram
\[
\begin{array}{ccc}
H^1_{\text{et}}(K, \mathbb{Z}_p(2)) & \xrightarrow{\text{mult. by } p^{-\nu}} & H^1_{\text{et}}(K, \mathbb{Q}_p(2)) \\
\downarrow & & \downarrow \\
H^1_{\text{et}}(K, \mathbb{Z}/p^\nu(2)) & \longrightarrow & H^1_{\text{et}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))
\end{array}
\]
we have from (7.15) and (7.16) that
\[ \text{Image of } \left( \frac{Nm_1m_2}{p^\nu}c_{3,2}([\zeta_1^{-1}\zeta_2] - [\zeta_1^{-1}] - [\zeta_2]) \right) \in H_{K_2}'. \] (7.17)

Assume \( m = m_1 = m_2 \). Due to (6.14) we have
\[ \sum_{\zeta_1}[\zeta_1^{-1}\zeta_2] - [\zeta_1^{-1}] - [\zeta_2] = -m[\zeta_2] \]
in the Bloch group \( B(K)_\mathbb{Q} \) where \( \zeta_1 \) runs over all \( m \)-th roots of unity such that \( \zeta_1 \neq \zeta_2 \).
Therefore we have from (7.17) that
\[ \text{Image of } \left( \frac{Nm^3}{p^\nu}c_{3,2}[\zeta] \right) \in H_{K_2}' \] (7.18)
for any \( \zeta \in K^* \) such that \( \zeta^m = 1 \).

Let \( F \subset K \) be any finite abelian extension over \( \mathbb{Q} \). Then \( F \) is contained in a cyclotomic field \( \mathbb{Q}(\mu) \). Since \( H^1_{\text{et}}(F, \mathbb{Q}_p(2)) \) is spanned by \( c_{3,2}c_{3,2}^\nu[\mu] \) (cf. Theorem 8.1 below), so is \( H^1_{\text{et}}(F, \mathbb{Q}_p(2)) \). Therefore, by (7.18) and the norm argument there exists an integer \( m' \neq 0 \) which does not depend on \( \nu \) (but does on \( F \)) such that
\[ \text{Image of } H^1_{\text{et}}(F, \mathbb{Z}_p(2)) \otimes \frac{m'}{p^\nu}\mathbb{Z}/\mathbb{Z} \subset H_{K_2}'. \] (7.19)
This means \( H_{K_2}' \supset m'H_{ab}' \), and hence \( H_{K_2} \supset mH_{ab} \) for some \( m \neq 0 \). This completes the proof of (Part II).
8. Proof of Theorem 1.1: Part III

In this section, we prove

(Part III): The map (1.3) is surjective if \( K \subset \mathbb{Q}_p(\zeta) \) for some root of unity \( \zeta \).

To do this, the following results are crucial.

**Theorem 8.1.**

1. ([24] Thm.1) Let \( F \) be a number field. Then the regulator map
   \[ c_{3,2} : B(F) \otimes \mathbb{Q}_p \cong K_{3}^{\text{ind}}(F) \otimes \mathbb{Q}_p \rightarrow H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) \]
   is bijective. The dimension of both sides is \( r_2 \), where \( r_2 \) denotes the number of complex places of \( F \).

2. Let \( \zeta \) be a primitive \( n \)-th root of unity. Then the basis of the Bloch group
   \[ B(\mathbb{Q}(\zeta))_\mathbb{Q} \]
   is given by \( \{ [\zeta^i] ; 1 \leq i < n/2, (i, n) = 1 \} \). (cf. [1] Thm.7.2.4, [17].)

3. \( \dim H^1_{\text{ét}}(K, \mathbb{Q}_p(2)) = [K : \mathbb{Q}_p] \).

4. If \( l \neq p \), then \( H^1_{\text{ét}}(K, \mathbb{Q}_l(2)) = 0 \).

Note that (3) and (4) follow from the Euler-Poincare characteristic ([22] II 5.7).

Due to (1) and (2) one can see that the map (1.3) is surjective if and only if any \( x \in H^1_{\text{ét}}(K, \mathbb{Q}_p(2)) \) can be written as a linear combination of \( c_{3,2}(\zeta)^i \)’s.

8.1. Proof of Part III. We may assume \( K = \mathbb{Q}_p(\zeta_m) \) where \( m \geq 1 \) is an integer and \( \zeta_m \) is a primitive \( m \)-th roots of unity. In fact, if we show the surjectivity of (1.3) for \( K = \mathbb{Q}_p(\zeta_m) \), then we have it for any \( K \subset \mathbb{Q}_p(\zeta_m) \) by using the norm map.

We use the following result:

**Lemma 8.2.** Let \( F \) be a number field, and \( P \) the set of all finite places of \( F \). For \( v \in P \), we denote by \( F_v \) the completion of \( F \) by \( v \). Then the natural map

\[ H^1_{\text{ét}}(F, \mathbb{Q}_p(j)) \rightarrow \prod_{v \in P} H^1_{\text{ét}}(F_v, \mathbb{Q}_p(j)) \]

is injective for \( j \neq 0 \).

**Proof.** See [12] Theorem 3 a). \( \square \)

Let \( l \) be a prime number such that \( l \equiv -1 \mod 4 \), \( (l, m) = 1 \) and \( p \) is complete split in \( \mathbb{Q}(\sqrt{-l}) \) (equivalently, \( l \equiv -1 \mod 8 \) if \( p = 2 \) and \( \left( \frac{-l}{p} \right) = 1 \) if \( p \geq 3 \)). Put
\[ F = \mathbb{Q}(\zeta_m, \sqrt{-l}) \]. There are two finite places \( v_1 \) and \( v_2 \) of \( \mathbb{Q}(\sqrt{-l}) \) lying over \( p \). Denote by \( e, f \) and \( g \) the customary meaning for \( F/\mathbb{Q}(\sqrt{-l}) \). Then there are \( g \)-finite places \( p_i \) (resp. \( p'_i \)) \( 1 \leq i \leq g \), of \( \mathbb{Q}(\zeta_m, \sqrt{-l}) \) lying over \( v_1 \) (resp. \( v_2 \)). The completions \( F_{p_i} \) and \( F_{p'_i} \) are isomorphic to \( K = \mathbb{Q}_p(\zeta_m) \). We have \( [F : \mathbb{Q}] = 2efg = 2\varphi(m) \) and \( [\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p] = ef \).

Due to Lemma 8.2 and Theorem 8.1 (4) the natural map

\[ H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) \rightarrow \prod_{i=1}^{g} H^1_{\text{ét}}(F_{p_i}, \mathbb{Q}_p(2)) \oplus H^1_{\text{ét}}(F_{p'_i}, \mathbb{Q}_p(2)) \] (8.1)
is injective. By Theorem 8.1 (1) and (3) we have \( \dim H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) = efg \) and \( \dim H^1_{\text{ét}}(F_p, \mathbb{Q}_p(2)) = \dim H^1_{\text{ét}}(F'_p, \mathbb{Q}_p(2)) = ef \). Let

\[
f_1 : H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) \longrightarrow \prod_{i=1}^{\theta} H^1_{\text{ét}}(F_p, \mathbb{Q}_p(2)), \quad f_2 : H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) \longrightarrow \prod_{i=1}^{\theta} H^1_{\text{ét}}(F'_p, \mathbb{Q}_p(2))
\]

be the natural ones. We show that \( f_1 \) and \( f_2 \) are bijective. Let \( \sigma : F \to F' \) be the automorphism such that \( \sigma \sqrt{-l} = \sqrt{-l} \) and \( \sigma(\zeta_m) = \zeta_m^{-1} \). It extends to an isomorphism \( \bar{\sigma} : \Pi_i F'_p \to \Pi_i F_p \) such that the diagram

\[
\begin{array}{ccc}
H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) & \xrightarrow{f_1} & \prod_i H^1_{\text{ét}}(F_p, \mathbb{Q}_p(2)) \\
\downarrow \sigma^* & & \downarrow \bar{\sigma}^* \\
H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) & \xrightarrow{f_2} & \prod_i H^1_{\text{ét}}(F'_p, \mathbb{Q}_p(2))
\end{array}
\]

(8.2)

is commutative. On the other hand, let \( \tau : F \to F' \) be the automorphism such that \( \tau \sqrt{-l} = \sqrt{-l} \) and \( \tau(\zeta_m) = \zeta_m^{-1} \). Since \( \tau \) does not change \( \nu_i \), it extends to an isomorphism \( \bar{\tau} : \Pi_i F_p \to \Pi_i F'_p \) which makes a commutative diagram

\[
\begin{array}{ccc}
H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) & \xrightarrow{f_1} & \prod_i H^1_{\text{ét}}(F_p, \mathbb{Q}_p(2)) \\
\downarrow \tau^* & & \downarrow \bar{\tau}^* \\
H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) & \xrightarrow{f_2} & \prod_i H^1_{\text{ét}}(F'_p, \mathbb{Q}_p(2)).
\end{array}
\]

(8.3)

We see the action of \( \sigma^* \) and \( \tau^* \) on \( H^1_{\text{ét}}(F, \mathbb{Q}_p(2)) \) explicitly. To do this, it is enough to see it on the Bloch group \( B(F)_{\mathbb{Q}} \) by Theorem 8.1 (1). Let \( \zeta_l \) be a primitive \( l \)-th root of unity. Since \( l \equiv -1 \mod 4 \) we have \( \mathbb{Q}(\sqrt{-l}) \subset \mathbb{Q}(\zeta_l) \). Embedding \( B(F)_{\mathbb{Q}} \mapsto B(\mathbb{Q}(\zeta_m, \zeta_l))_{\mathbb{Q}} \), we can see that \( B(F)_{\mathbb{Q}} \) is generated by

\[
\begin{align*}
\beta_1(\zeta_m^k) &:= \sum_{i=1}^{l-1} [\zeta_m^i \zeta_m^k], \\
\beta_2(\zeta_m^k) &:= \sum_{i=1}^{l-1} \left( \frac{i}{l} \right) [\zeta_m^i \zeta_m^k], \\
\quad (1 \leq k < m, \ (k, m) = 1)
\end{align*}
\]

with relations

\[
\beta_1(\zeta_m^k) = -\beta_1(\zeta_m^{-k}), \quad \beta_2(\zeta_m^k) = \beta_2(\zeta_m^{-k}).
\]

Here the second equality is due to the fact that \( \left( \frac{-1}{l} \right) = -1 \). Letting \( r \in (\mathbb{Z}/l)^* \) be a generator, we see

\[
\sigma^* \beta_1(\zeta_m^k) = \sum_{i=1}^{l-1} [\zeta_m^i r \zeta_m^k] = \beta_1(\zeta_m^k),
\]

(8.4)

\[
\sigma^* \beta_2(\zeta_m^k) = \sum_{i=1}^{l-1} \left( \frac{i}{l} \right) [\zeta_m^i r \zeta_m^k] = -\sum_{i=1}^{l-1} \left( \frac{i}{l} \right) [\zeta_m^i \zeta_m^k] = -\beta_2(\zeta_m^k),
\]

(8.5)
and
\[ \tau^* \beta_1(\zeta_m^k) = \beta_1(\zeta_m^{-k}) = -\beta_1(\zeta_m^k), \quad \tau^* \beta_2(\zeta_m^k) = \beta_2(\zeta_m^{-k}) = \beta_2(\zeta_m^k). \] (8.6)

Now we show that \( f_i \) are bijective. Since the dimensions of the target and source are same, it is enough to show that \( f_i \) are injective. Suppose that \( \sum a_k \beta_1(\zeta_m^k) + b_k \beta_2(\zeta_m^k) \) is in the kernel of \( f_1 \). Due to the diagram (8.3) and (8.6), both of \( \sum a_k \beta_1(\zeta_m^k) \) and \( \sum b_k \beta_2(\zeta_m^k) \) are contained in the kernel of \( f_1 \). Then, by the diagram (8.2), both of \( \sigma^* \left( \sum a_k \beta_1(\zeta_m^k) \right) = \sum a_k \beta_1(\zeta_m^k) \) and \( \sigma^* \left( \sum b_k \beta_2(\zeta_m^k) \right) = -\sum b_k \beta_2(\zeta_m^k) \) are contained in the kernel of \( f_2 \). This shows \( \ker f_1 = \ker f_2 \). On the other hand, since (8.1) is injective, we have \( \ker f_1 \cap \ker f_2 = 0 \). Thus we have \( \ker f_i = 0 \) for each \( i \).

This completes the proof of (Part III) and hence Theorem 1.1.

Remark 8.3. I don’t think that the map (1.3) is always surjective. However, I know of no examples where it is not surjective.

9. Applications of Theorem 1.1

For an abelian group \( M \) we denote by \( M_{\text{tor}} \) and \( M_{\text{div}} \) the torsion subgroup and the maximal divisible subgroup of \( M \) respectively:

\[ M_{\text{tor}} := \bigcup_{m \geq 1} M[m], \quad M_{\text{div}} := \bigcap_{m \geq 1} mM. \]

9.1. Consequence of Suslin’s exact sequence. Let \( F \) be any field, and \( X \) a nonsingular curve over \( F \). Let \( p \) be a prime number such that \( p \neq \text{char}(F) \). Passing to the inductive limit over \( p^\nu \), we have from (2.10)

\[ 0 \to H^0_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p / \mathbb{Z}_p \to H^2_{\text{ét}}(X, \mathbb{Q}_p / \mathbb{Z}_p(2)) \to H^1_{\text{Zar}}(X, \mathcal{K}_2)[p^\infty] \to 0. \] (9.1)

The following is an easy consequence of Suslin’s exact sequence.

Proposition 9.1. Assume that for any finite \( p \)-torsion \( G_F \)-module \( M \), \( H^2_{\text{ét}}(F, M) \) is finite for all \( i \geq 0 \). Then the following are equivalent.

1. The map \( H^0_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p / \mathbb{Z}_p \to H^2_{\text{ét}}(X, \mathbb{Q}_p / \mathbb{Z}_p(2)) \) is surjective.
2. The map \( H^0_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p \to H^2_{\text{ét}}(X, \mathbb{Q}_p(2)) \) is surjective.
3. The map \( H^0_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p / \mathbb{Z}_p \to H^2_{\text{ét}}(X, \mathbb{Q}_p / \mathbb{Z}_p(2)) \otimes \mathbb{Q}_p / \mathbb{Z}_p \) is surjective (and hence bijective by (9.1)).
4. The corank of \( H^0_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p / \mathbb{Z}_p \) is greater than or equal to the corank of \( H^2_{\text{ét}}(X, \mathbb{Q}_p / \mathbb{Z}_p(2)) \).
5. \( H^2_{\text{Zar}}(X, \mathcal{K}_2)[p^\infty] \cong H^2_{\text{ét}}(X, \mathbb{Z}_p(2))[p^\infty] \).
6. \( H^2_{\text{Zar}}(X, \mathcal{K}_2)[p^\infty] \) is finite.
7. \( H^0_{\text{Zar}}(X, \mathcal{K}_2)[p^\nu] \cong H^2_{\text{ét}}(X, \mathbb{Z}_p(2))[p^\nu] \) for all \( \nu \geq 1 \).
8. \( H^1_{\text{Zar}}(X, \mathcal{K}_2)[p^\nu] \cong H^2_{\text{ét}}(X, \mathbb{Z}_p(2))[p^\nu] \) for all \( \nu \geq 1 \).
Proof. By the Hochschild-Serre spectral sequence, the assumption on $F$ implies that $H^i_{\text{ét}}(X, \mathbb{Z}/p^n(j))$ is finite for all $i$, $j$ and $\nu$, and hence we have that $H^i_{\text{ét}}(X, \mathbb{Z}_p(j))$ is a finitely generated $\mathbb{Z}_p$-module and there are exact sequences

$$0 \rightarrow H^i_{\text{ét}}(X, \mathbb{Z}_p(j))/p^n \rightarrow H^i_{\text{ét}}(X, \mathbb{Z}/p^n(j)) \rightarrow H^{i+1}_{\text{ét}}(X, \mathbb{Z}_p(j))[p^n] \rightarrow 0 \quad (9.2)$$

for all $i$, $j$ and $\nu$ (see for example [15] p.165, Lemma 1.11).

Since $H^2_{\text{ét}}(X, \mathbb{Z}_p(2))$ is a finitely generated $\mathbb{Z}_p$-module, (2) is equivalent to (3). Due to the exact sequences (9.1) and (9.2), we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & H^2_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H^2_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \rightarrow & H^2_{\text{Zar}}(X, \mathcal{K}_2)[p^\infty] & \rightarrow & 0 \\
\downarrow & & \downarrow= & & \downarrow & & \\
0 & \rightarrow & H^2_{\text{ét}}(X, \mathbb{Z}_p(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H^2_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \rightarrow & H^3_{\text{ét}}(X, \mathbb{Z}_p(2))[p^\infty] & \rightarrow & 0.
\end{array}
$$

Therefore, (3) is equivalent to (5). Since the map $H^2_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^2_{\text{ét}}(X, \mathbb{Z}_p(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is always injective, (3) is equivalent to (4). Thus we have completed

$$(2) \iff (3) \iff (4) \iff (5).$$

(5)$\implies$(6). This follows from the fact that $H^3_{\text{ét}}(X, \mathbb{Z}_p(2))$ is a finitely generated $\mathbb{Z}_p$-module.

(6)$\implies$(4). This follows from (9.1).

(3)$\implies$(7) and (8). Due to Suslin’s exact sequences (2.10) and (9.1), we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & H^2_{\text{ét}}(X, \mathbb{Z}_p(2))/p^n & \rightarrow & H^2_{\text{ét}}(X, \mathbb{Z}/p^n(2)) & \rightarrow & H^3_{\text{ét}}(X, \mathbb{Z}_p(2))[p^n] & \rightarrow & 0 \\
& \uparrow a_1 & & \uparrow= & & \uparrow a_2 & & \\
0 & \rightarrow & H^2_{\text{Zar}}(X, \mathcal{K}_2)/p^n & \rightarrow & H^2_{\text{ét}}(X, \mathbb{Z}/p^n(2)) & \rightarrow & H^2_{\text{Zar}}(X, \mathcal{K}_2)[p^n] & \rightarrow & 0 \\
& & \downarrow \partial & & \downarrow b & & \\
0 & \rightarrow & H^2_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H^2_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \rightarrow & H^2_{\text{Zar}}(X, \mathcal{K}_2)[p^\infty] & \rightarrow & 0.
\end{array}
$$

We show that $a_1$ is surjective. It implies that $a_1$ and $a_2$ are bijective. To do this, it is enough to see that $H^2_{\text{ét}}(X, \mathbb{Z}_p(2))/p^n$ goes to zero by the map $\partial$. Since $b$ is injective, it is enough to see that it goes to zero by $b\partial$. However since $H^2_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p = H^2_{\text{ét}}(X, \mathbb{Z}_p(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p$, it is clear.

(7)$\implies$(1). Nakayama’s lemma.

(1)$\implies$(2). Clear.

(7)$\implies$(3). Clear.

(8)$\implies$(5). Clear. \qed
Proposition 9.2. If the equivalent conditions in Proposition 9.1 are satisfied, then we have
\[ \varprojlim_{\nu} H^1_{\text{Zar}}(X, \mathcal{K}_2)[p^\nu] = 0. \]
In particular, any $p$-divisible subgroup $D \subset H^1_{\text{Zar}}(X, \mathcal{K}_2)$ (i.e. $D = pD$) is uniquely $p$-divisible.

Proof. Since (2.10) is an exact sequence of finite groups, the exactness is preserved after taking the projective limit:
\[ 0 \longrightarrow \varprojlim_{\nu} H^0_{\text{Zar}}(X, \mathcal{K}_2)/p^\nu \longrightarrow H^2_{\text{et}}(X, \mathbf{Z}_p(2)) \longrightarrow \varprojlim_{\nu} H^1_{\text{Zar}}(X, \mathcal{K}_2)[p^\nu] \longrightarrow 0. \]
The vanishing of the last term follows from (1). If $D \subset H^1_{\text{Zar}}(X, \mathcal{K}_2)$ is a $p$-divisible subgroup, then we have $\varprojlim_{\nu} D[p^\nu] = 0$ and therefore $D$ has no $p$-torsion. \hfill \Box

9.2. Applications to $V(E_K)$. Let us go back to the Tate curve $E_K = K^*/q\mathbf{Z}$. Suppose that $K \subset \mathbf{Q}_p(\zeta)$ for some root of unity $\zeta$. Then by Theorem 1.1 and Proposition 9.1 we have
\[ H^1_{\text{Zar}}(E_K, \mathcal{K}_2)[p^\nu] \cong H^2_{\text{et}}(E_K, \mathbf{Z}_p(2))[p^\nu], \quad V(E_K)[p^\nu] \cong H^2_{\text{et}}(K, H^1_{\text{et}}(E_K, \mathbf{Z}_p(2)))[p^\nu] \]
for all $\nu \geq 1$. Here we note
\[ H^1_{\text{et}}(K, H^1_{\text{et}}(E_K, \mathbf{Z}_p(r))) \overset{\text{def}}{=} \varprojlim_{\nu} H^1_{\text{et}}(K, H^1_{\text{et}}(E_K, \mathbf{Z}_p)[p^\nu(r)]). \]
Due to T.Sato [20] (cf. Corollary 5.4), the above is also true for the $l$-torsion parts. Thus we have
\[ V(E_K)_{\text{tor}} = \bigoplus_{l} V(E_K)[l^\infty] \cong \bigoplus_{l} H^2_{\text{et}}(K, H^1_{\text{et}}(E_K, \mathbf{Z}_l(2)))[l^\infty]. \quad (9.3) \]

Lemma 9.3. Let $\nu_0 \geq 0$ be the largest integer such that a primitive $l^{\nu_0}$-th root of unity is contained in $K^*$. Then there is the natural isomorphism
\[ H^2_{\text{et}}(K, H^1_{\text{et}}(E_K, \mathbf{Z}_l(2)))[l^\infty] \cong K^M_2(K)/(l^{\nu} K^M_2(K) + \{q, K^*\}) \]
for $\nu \geq \nu_0$.

Proof. By the weight exact sequence (4.5), we have an exact sequence
\[ H^1_{\text{et}}(K, \mathbf{Z}_l(1)) \overset{\delta}{\longrightarrow} H^2_{\text{et}}(K, \mathbf{Z}_l(2)) \longrightarrow H^2_{\text{et}}(K, H^1_{\text{et}}(E_K, \mathbf{Z}_l(2))) \longrightarrow H^2_{\text{et}}(K, \mathbf{Z}_l(1)). \]
Recall the isomorphism
\[ K^M_2(K)/l^{\nu} \overset{\cong}{\longrightarrow} \varprojlim_{i} K^M_2(K)/l^i \overset{\cong}{\longrightarrow} H^2_{\text{et}}(K, \mathbf{Z}_l(2)) \]
for $\nu \geq \nu_0$. Under this isomorphism and $H^1_{\text{et}}(K, \mathbf{Z}/l^i(1)) \cong K^*/l^i$, the map $\delta$ is given by $x \mapsto \{x, q\}$. Therefore the cokernel of $\delta$ is isomorphic to $K^M_2(K)/(l^{\nu} K^M_2(K) + \{q, K^*\})$. On the other hand, $H^2_{\text{et}}(K, \mathbf{Z}_l(1))$ is isomorphic to $\mathbf{Z}_l$ which has no torsion. Thus we have the assertion. \hfill \Box
By Lemma 9.3 and the isomorphism (9.3), we have

**Corollary 9.4.** Let the notations and assumption be as in Theorem 1.1. Write by \( \mu_n \) the group of all roots of unity in \( K \) where \( n \) denotes the cardinality. Then there is the natural isomorphism

\[
V(E_K)_{\text{tor}} \cong K_2^M(K)/(nK_2^M(K) + \{q, K^*\}) \xrightarrow{\cong} \mu_n/(q, K^*)_n
\]

(9.4)

where the last isomorphism is the map induced from the Hilbert symbol \((\cdot, \cdot)_n : K^*/n \times K^*/n \to \mu_n \) (cf. [6] IX (4.3)). In particular, \( V(E_K)_{\text{tor}} \) and hence \( K_1(E_K)_{\text{tor}} \) are finite.

There is the exact sequence

\[
0 \rightarrow H^3_{\text{zar}}(E_K, \mathcal{O}_K)/m \rightarrow H^3_{\text{et}}(E_K, \mathbb{Z}/m(2)) \rightarrow H^3_{\text{et}}(K(E_K), \mathbb{Z}/m(2))
\]

for all \( m \geq 1 \) ([14] §18). From this we have an isomorphism

\[
V(E_K)/m \cong K_2^M(K)/(mK_2^M(K) + \{q, K^*\}).
\]

(9.5)

**Corollary 9.5.** Let the notations and assumption be as in Theorem 1.1. Then we have a decomposition

\[
V(E_K) = V(E_K)_{\text{tor}} \oplus V(E_K)_{\text{div}},
\]

(9.6)

and \( V(E_K)_{\text{div}} \) is uniquely divisible.

**Proof.** Due to (9.4) and (9.5) we have \( V(E_K)_{\text{tor}}/m \xrightarrow{\cong} V(E_K)/m \) for all \( m \geq 1 \). Therefore \( V(E_K)/V(E_K)_{\text{tor}} \) is uniquely divisible. Since \( V(E_K)_{\text{tor}} \) is finite whose order is divided by \( n \), we have that \( nV(E_K) = V(E_K)_{\text{div}} \) and hence it maps onto \( V(E_K)/V(E_K)_{\text{tor}} \). Moreover \( V(E_K)_{\text{div}} \cap V(E_K)_{\text{tor}} = 0 \) because of the finiteness of \( V(E_K)_{\text{tor}} \). Thus we have \( V(E_K)_{\text{div}} \xrightarrow{\cong} V(E_K)/V(E_K)_{\text{tor}} \). \( \square \)

**Remark 9.6.** It is known that \( \dim_{\mathbb{Q}} V(E_K)_{\text{div}} = +\infty \) ([29] Appendix).

9.3. **Earlier works on** \( V(X) \). Several people studied \( V(X) \) (mainly from the viewpoint of the class field theory) and obtained related results to §9.2. Here are some:

1. (T.Sato [20]). If \( X \) is an elliptic curve over a \( p \)-adic field with bad reduction, \( V(X)_{\text{div}} \) is uniquely \( l \)-divisible for \( l \neq p \).
2. (S.Saito [19]). Let \( X \) be a nonsingular projective curve over a \( p \)-adic field. Then \( V(X)/V(X)_{\text{div}} \) is finite.
3. (Colliot-Thélène and Raskind [2]). If \( X \) is a nonsingular projective curve over a \( p \)-adic field with good reduction, \( V(X) \) is a direct sum of a uniquely divisible group and \( V(X)_{\text{tor}} \). Moreover, \( V(X)[l^{\infty}] \cong J(k)[l^{\infty}] \) for any \( l \neq p \) where \( k \) is the residue field of \( K \) and \( J/k \) is the Jacobian variety of the special fiber.

(1) is a consequence of the surjectivity of the \( l \)-adic regulator on \( K_2(E_K) \) (Proposition 9.2). Suppose that \( X \) has a good reduction. Then it follows from the Euler-Poincare characteristic ([22] II 5.7) that we have \( H^3_{\text{et}}(X, \mathbb{Q}_l(2)) = 0 \). Therefore \( V(X)[l^{\infty}] \) is finite and \( V(X)_{\text{div}} \) is uniquely \( l \)-divisible for \( l \neq p \) (Propositions 9.1, 9.2). However I do not
know any previous results about finiteness of $V(X)[p^\infty]$. (Note that (2) and (3) do not imply anything about finiteness of $p$-power torsion.) When $X$ has a bad reduction and the genus of $X$ is greater than 1, the question of the finiteness remains open even for the $l$-power torsion part.

9.4. Other Corollaries.

**Corollary 9.7.** Let the notations and assumption be as in Theorem 1.1. Then the $p$-adic regulator $H^0_{\text{Zar}}(E_K, \mathcal{K}_2) \otimes \mathbb{Z}_p \to H^2_{\text{\acute{e}t}}(E_K, \mathbb{Z}_p(2))$ is surjective, and it induces an isomorphism

$$H^0_{\text{Zar}}(E_K, \mathcal{K}_2)/p^\nu \xrightarrow{\cong} H^2_{\text{\acute{e}t}}(E_K, \mathbb{Z}_p(2))/p^\nu$$

for all $\nu \geq 1$.

**Proof.** Straightforward from Theorem 1.1 and Proposition 9.1. □

**Corollary 9.8.** Let the notations and assumption be as in Theorem 1.1. Denote by $n_0$ the cardinality of the subgroup $(q, K^*) \subset \mu_n$. Let $m \geq 1$ be an integer. Put by $e_m$ the order of the kernel of a map

$$\mathbb{Z}/(n_0, m) \xrightarrow{k \mapsto k \cdot n_0^{-1}} \mathbb{Z}/(n, m). \quad (9.7)$$

Then the cohomology of the sequence

$$H^0_{\text{Zar}}(E_K, \mathcal{K}_2)/m \xrightarrow{\tau^\ell} K^*/m \xrightarrow{x \mapsto (x, q)} K_2(K)/m \quad (9.8)$$

at the middle term is a cyclic group of order $e_m$. In particular, (9.8) is exact if and only if (9.7) is injective.

**Proof.** Since we have the exact sequence

$$0 \to H^0_{\text{Zar}}(E_K, \mathcal{K}_2)/m \to H^1_{\text{\acute{e}t}}(K, H^1(E_K, \mathbb{Z}/m(2))) \xrightarrow{b} H^2_{\text{\acute{e}t}}(K, H^1(E_K, \hat{\mathbb{Z}}(2)))[m] \to 0,$$

the cohomology of (9.8) is isomorphic to the cokernel of the map

$$H^0_{\text{Zar}}(E_K, \mathcal{K}_2)/m \to \text{Coker } a.$$

By Proposition 9.1 and Theorem 1.1, we have an exact sequence

Moreover there is a commutative diagram

$$\begin{array}{ccc}
H^1_{\text{\acute{e}t}}(K, H^1(E_K, \mathbb{Z}/m(2))) & \xrightarrow{a} & H^2_{\text{\acute{e}t}}(K, H^1(E_K, \hat{\mathbb{Z}}(2)))[m] \\
\uparrow a & & \uparrow a' \\
H^1_{\text{\acute{e}t}}(K, \mathbb{Z}/m(2)) & \xrightarrow{b'} & H^2_{\text{\acute{e}t}}(K, \hat{\mathbb{Z}}(2))[m]
\end{array}$$
where $b'$ is surjective. Therefore the cohomology of (9.8) is isomorphic to the cokernel of $a'$. As we have seen in the proof of Lemma 9.3, we have

$$H^2_{\text{ét}}(K, H^1(E_K, \hat{\mathbb{Z}}(2)))_{\text{tor}} \cong \mu_n/(q, K^*)_n \cong \mathbb{Z}/mn_0^{-1}, \quad H^2_{\text{ét}}(K, \hat{\mathbb{Z}}(2))_{\text{tor}} \cong \mu_n \cong \mathbb{Z}/n,$$

and the map $a'$ can be identified with the natural map

$$\mathbb{Z}/n[m] \rightarrow \mathbb{Z}/nn_0^{-1}[m].$$

Its cokernel is isomorphic to the kernel of (9.7).

\[\square\]

10. Surjectivity of $l$-adic regulator on $K_2$ of open Tate curves

**Theorem 10.1.** Let $l \neq p$ be a prime number. Let $U_K \subset E_K$ be an arbitrary Zariski open set (no assumption on $K$). Then the $l$-adic regulator

$$K_2(U_K) \otimes \mathbb{Q}_l \longrightarrow H^2_{\text{ét}}(U_K, \mathbb{Q}_l(2))$$

(10.1)

is surjective.

This is a generalization of the main result of T.Sato’s thesis [20] which proved the above in case $U_K = E_K$. When $l = p$, the question of the surjectivity for $U_K \neq E_K$ remains open.

**Proof.** Using the norm map, we can replace $K$ with an arbitrary finite extension $L$ over $K$. Thus we may assume that $E_K$ is defined by an equation

$$y^2 = x^3 + x^2 + c$$

(10.2)

with $\text{ord}_K(c) = n \geq 1$, and $D_K := E_K - U_K = P_1 + \cdots + P_s$ with each $P_i \in E_K(K)$. In the same way as the proof of (5.1), we have

$$H^2_{\text{ét}}(U_K, \mathbb{Q}_l(2)) \cong H^1_{\text{ét}}(K, H^1(U_K, \mathbb{Q}_l(2))).$$

(10.3)

It follows from the exact sequence (cf. §4)

$$0 \rightarrow \mathbb{Q}_l(2) \rightarrow H^1_{\text{ét}}(U_K, \mathbb{Q}_l(2)) \rightarrow \mathbb{Q}_l(1) \oplus \bigoplus_{i=2}^s \mathbb{Q}_l(1)([P_i] - [P_1]) \rightarrow 0$$

(10.4)

and Theorem 8.1 (4) that we have

$$H^1_{\text{ét}}(K, H^1(U_K, \mathbb{Q}_l(2))) \cong H^1_{\text{ét}}(K, \mathbb{Q}_l(1)) \oplus \bigoplus_{i=2}^s H^1_{\text{ét}}(K, \mathbb{Q}_l(1)([P_i] - [P_1]))$$

$$\cong \mathbb{Q}_l \oplus \bigoplus_{i=2}^s \mathbb{Q}_l([P_i] - [P_1]).$$

Thus we can rewrite the $l$-adic regulator (10.1) in the following form:

$$K_2(U_K) \otimes \mathbb{Q}_l \longrightarrow \mathbb{Q}_l \oplus \bigoplus_{i=2}^s \mathbb{Q}_l([P_i] - [P_1]).$$

(10.5)
Since \( K_2(E_K) \otimes \mathbb{Q}_l \) is onto the first component (Corollary 5.4), it is enough to show that the composition

\[
K_2(U_K) \otimes \mathbb{Q}_l \longrightarrow \mathbb{Q}_l \oplus \bigoplus_{i=2}^{s} \mathbb{Q}_l([P_i] - [P_1]) \overset{\text{pr}}{\longrightarrow} \bigoplus_{i=2}^{s} \mathbb{Q}_l([P_i] - [P_1])
\]

is surjective. To do this, we note that the map (10.6) is obtained from the tame symbols. More precisely let \( \tau_{P_i} \) be the tame symbol at \( P_i \) and \( \text{ord}_{K^*} : K^* \to \mathbb{Z} \) the valuation such that \( \text{ord}_{K}(\pi_K) = 1 \). Then (10.6) is obtained by tensoring

\[
\sum_{i=2}^{s} \text{ord}_{K} \cdot \tau_{P_i} : H^0_{\text{Zar}}(U_K, \mathcal{K}_2) \longrightarrow \bigoplus_{i=2}^{s} \mathbb{Z}([P_i] - [P_1])
\]

with \( \mathbb{Q}_l \). Therefore it suffices to show that the cokernel of (10.7) is finite.

Obviously we can reduce it to the case \( s = 2 \). Moreover, by using the translation \( x \mapsto x - a \), we may assume \( P_1 = O \). Put \( P = P_2 \). Write \( P = \alpha \zeta^{r/n} (\alpha \in \mathbb{R}^*) \) under the identification \( K^*/\mathbb{Z} = E_K(K) \). Put \( Q = \zeta^{r/n} \) with \( \zeta^m = 1 \). Then the cokernel of the tame symbol \( \tau_Q : H^0_{\text{Zar}}(E_K - \{Q, O\}, \mathcal{K}_2) \to K^* \) is finite. In fact, putting \( f(u) = \theta(\zeta^{-1} q^{r/n} u)^m/\theta(u)^{n-1} \theta(q^{-r/n} u) \), the symbol \( \{a, f(u)\} \) goes to \( a^n \zeta^m \). Therefore, in order to show that the cokernel of (10.7) is finite in case \( D_K = P + O \), it suffices to show it in case \( D_K = P + Q + O \). It is also reduced to the case \( D_K = P + Q \).

By the translation, it is reduced to the case \( D_K = P' + O \) where \( P' = \alpha \zeta^{-1} \in \mathbb{R}^* \). By choosing a suitable \( \zeta \), we can assume \( \alpha \zeta^{-1} \equiv 1 \mod \pi_K \). Moreover, replacing \( K \) with \( K(\sqrt{\alpha \zeta^{-1}}) \), we may assume that there is \( \beta \in \mathbb{R}^* \) such that \( \alpha \zeta^{-1} = \beta^2 \). By using the translation, we can reduce the case \( D_K = P'' + (-P'') \) with \( P'' = \beta \) and \( -P'' = \beta^{-1} \). Summarizing the above, we have reduced the proof to the following claim:

**Claim 10.2.** Suppose \( P = \alpha \in \mathbb{R}^* \) and \( \alpha \equiv \pm 1 \mod \pi_K \). Then the cokernel of \( \text{ord}_K \cdot \tau_P : H^0_{\text{Zar}}(E_K - \{P, (-P)\}, \mathcal{K}_2) \to \mathbb{Z} \) is finite.

Let us go back to the equation (10.2). Let \( P = (a, b) \) be the coordinate expression by \( (x, y) \) with \( a, b \in \mathbb{R} \). Note \( (-P) = (a, -b) \). We consider a \( K_2 \)-symbol

\[
\xi := \left\{ \begin{array}{ll}
y - \sqrt{a + 1} x, & \text{if } c = 0, \\
y + \sqrt{a + 1} x, & \text{if } c = 0.
\end{array} \right.
\]

We have

\[
\tau_Q(\xi) = \begin{cases} 
(b - \sqrt{a + 1} a)/(b + \sqrt{a + 1} a) & Q = P \\
(b + \sqrt{a + 1} a)/(b - \sqrt{a + 1} a) & Q = (-P) \\
1 & \text{otherwise.}
\end{cases}
\]

This shows \( \xi \in H^0_{\text{Zar}}(E_K - \{P, (-P)\}, \mathcal{K}_2) \). The assumption \( \alpha \in \mathbb{R}^* \) and \( \alpha \equiv \pm 1 \mod \pi_K \) implies \( a \equiv \pm 1 \mod \pi_K \). Suppose \( p = 2 \). Since \( (b + \sqrt{a + 1} a) - (b - \sqrt{a + 1} a) = \)
\[2\sqrt{a + 1}a \text{ is a unit, either } (b - \sqrt{a + 1}a) \text{ or } (b + \sqrt{a + 1}a) \text{ is a unit in } R. \text{ On the other hand, the order of } (b - \sqrt{a + 1}a)(b + \sqrt{a + 1}a) = c \text{ is } n. \text{ We have}
\]
\[
\text{ord}_K \cdot \tau_P(\xi) = \text{ord}_K \left( \frac{b - \sqrt{a + 1}a}{b + \sqrt{a + 1}a} \right) = \pm n.
\]
This means that the cokernel of \(\text{ord}_K \cdot \tau_P\) is finite in case \(p \neq 2\). Suppose \(p = 2\). If \(n/2 > \text{ord}_K(2)\), one can show that the order of \((b - \sqrt{a + 1}a)/(b + \sqrt{a + 1}a)\) is not zero by the same argument as above. For a small \(n\), we take a finite covering \(E_{m,K} = K^* / q^nZ \to E_K\) given by \(x \mapsto x^m\) with \(m \gg 1\). Since we have obtained the surjectivity of the \(l\)-adic regulator for any Zariski open set of \(E_{m,K}\), we can obtain it for \(E_K\) by using the transfer map for \(E_{m,K} \to E_K\). This completes the proof for \(p = 2\) and all \(n \geq 1\).

**Corollary 10.3.** The \(l\)-power torsion \(K_1(U_K)[l^\infty]\) is finite.

**Proof.** This follows from Theorem 10.1 and Proposition 9.1. \qed

**References**

[1] S. Bloch: Higher Regulators, Algebraic K-theory, and Zeta functions of Elliptic curves. CRM monograph Series, 11, AMS, 2000.

[2] J.-L. Colliot-Thélène and W. Raskind: \(K_2\)-cohomology and the second Chow group. Math. Amn. 270 (1985), 165–199.

[3] P. Deligne: Théorie de Hodge III. Publ.Math.IHES. 44 (1974) 5-78.

[4] C. Deninger and K. Wingberg: On the Beilinson conjectures for elliptic curves with complex multiplication. In Beilinson’s Conjectures on Special Values of L-Functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math. Vol.4, pp.249–272, 1987.

[5] J. Dupont and C.-H. Sah: Scissors congruences II. J. Pure and Appl. Algebra 25 (1982) no.2, 159–195.

[6] I.B. Fesenko and S.V. Vostokov: Local fields and Their extensions. Trans. of Math. Monographs, Vol. 121. (2002), AMS.

[7] E. Friedlander: Etale Homotopy of Simplicial Schemes. Ann. of Math. Studies 104, Princeton, 1982.

[8] H. Gillet: Riemann-Roch theorem for higher Algebraic K-theory. Adv. Math. 40 (1981), 203–289.

[9] D. Grayson: Higher algebraic K-theory. II (after D. Quillen). pp. 217–240. Lecture Notes in Math. Vol. 551, Springer, 1976.

[10] A. Huber: Mixed Motives and their Realization in Derived Category. Lecture Notes in Math. Vol. 1604, Springer, 1995.

[11] U. Jannsen: Deligne homology, Hodge-D-conjecture, and motives. In Beilinson’s conjectures on special values of L-functions, 305–372, Perspect. Math., 4, 1988.

[12] On the \(l\)-adic cohomology of varieties over number fields and its Galois cohomology. In Galois groups over \(Q\) (Berkeley, 1987), 315–360. Springer, 1989.

[13] van der Kallen: Generators and relations in algebraic K-theory. Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 305–310, Acad. Sci. Fennica, Helsinki, 1980.
[14] A.S. Merkur’ev and A.A. Suslin: $K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism. Math. USSR Izv. 21 No.2 (1983), 307–340.
[15] J. Milne: Étale cohomology. Princeton, 1980.
[16] J. Milnor: Introduction to Algebraic $K$-theory. Ann. of Math Studies 72, Princeton 1970.
[17] J. Neukirch: The Beilinson Conjecture for Algebraic Number Fields. In Beilinson’s Conjectures on Special Values of $L$-Functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math. Vol.4, 193–247, 1988.
[18] D. Quillen: Higher algebraic $K$-theory. I. pp. 85–147. Lecture Notes in Math. Vol. 341, Springer, Berlin 1973.
[19] S. Saito: Class field theory for curves over local fields. J. Number Theory, Vol. 21, No.1 (1985) 44–80.
[20] T. Sato: On $K$-cohomology of elliptic curves over local fields: (Japanese). Tokyo University Master thesis. Unpublished.
[21] N. Schappacher and A. J. Scholl: The boundary of the Eisenstein symbol. Math. Ann. 290 (1991), 419–430.
[22] J.-P. Serre: Cohomologie galoisienne. Lecture Notes in Math. 5, 1964. Springer.
[23] J. Silverman: Advanced Topics in the Arithmetic of Elliptic curves. GTM 151, Springer, 1994.
[24] C. Soulé: On higher $p$-adic regulators. Algebraic $K$-theory, Evanston 1980, pp.372–401, LNM 854, Springer 1981.
[25] ______: $K$-théorie des anneaux d’entiers de corps de nombres et cohomologie étale. Invent. Math. 55 (1979) pp. 251–295.
[26] V. Srinivas: Algebraic $K$-theory. Progress in Math. Vol 90 (1991), Birkhäuser.
[27] A. Suslin: Algebraic $K$-theory and the norm residue homomorphism. J. Soviet Math. 30 (1985), 2556–2611.
[28] ______: Algebraic $K$-theory of fields. Proceedings of the International Congress of Mathematicians, Berkeley, 1986 pp. 222-243.
[29] T. Szamuely: Sur la théorie des corps de classes pour les variétés sur les corps $p$-adiques. (With appendix by Colliot-Thélène) J. Reine Angew. Math. 525 (2000), 183–212.

Graduate School of Mathematics, Kyushu University, Hakozaki Higashi-ku Hukuoka 812-8581, JAPAN

E-mail address : asakura@math.kyushu-u.ac.jp