On the stability of Forward in Time and Centred in Space (FTCS) scheme for scalar hyperbolic equation∗

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Abstract

It is well known that Forward Time and Centred in Space (FTCS) scheme for scalar Hyperbolic Conservation Law (HCL) is unconditionally unstable. The main contribution of this work to show that FTCS is conditionally stable for HCL. A new approach is used to give bounds on the initial data profile by transforming FTCS into two point convex combination scheme. Numerical results are given in support of the claim.

keyword Numerical oscillations; Von-Neumann stability; smoothness parameter; finite difference schemes; hyperbolic equations.

1 Introduction

We consider for the simplicity, the linear transport problem,

\[
\frac{\partial}{\partial t} u(x, t) + a \frac{\partial}{\partial x} u(x, t) = 0, a \in \mathbb{R} \tag{1}
\]

\[
u_0(x) = f(x) \tag{2}
\]

where \( u = u(x, t) \) is a scalar field transported by flow of constant velocity \( a \). The exact solution of linear transport equation is given by

\[
u(x, t) = v_0(x \pm at). \tag{3}
\]

Seemingly simple, the linear transport equation \([1]\) has played a crucial role in the development of the numerical methods for general hyperbolic conservation laws. It is the closed form solution \([3]\) of \([1]\) which pave the way to analyze any new numerical scheme devised for more complex non-linear hyperbolic problems for its convergence and stability. Considering the solution \([3]\), it is natural to seek a numerical scheme which is stable in the sense that it does not induce numerical oscillations. Therefore, the notion of stability of a numerical schemes evolves around the induced spurious numerical oscillations. In order to elaborate it, we consider a uniform grid with the spatial width \( h \), time step \( k \) and denote the discrete mesh point \((x_j, t_n)\) by \( x_j = jh, j \in \mathbb{Z} \) and \( t_n = nk, k \in \mathbb{N} \).

In the seminal work Courant-Friedrichs and Levy shown that for the convergence a difference scheme must contains the physical domain of dependence of partial differential equation \([1]\). In other words, they gave a necessary condition on ratio of the spatial and time discretization step for the convergence of the difference scheme known as CFL number. Since then, the notion of

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CFL condition has been an indispensable tool for defining the stability of numerical schemes. It is known that the linear Von-Neumann stability analysis of a numerical scheme for \( C = |a|\lambda, \lambda = \frac{k}{h} \). Moreover, almost all notion of non-linear stability such as upwind range condition, monotone stability [10, 2, 15], positivity preserving [6, 12], total variation stability [4] heavily relies on CFL number to devise any new stable scheme e.g., TVD schemes in [3, 7, 14], essentially non-oscillatory (ENO) schemes [5, 16], weighted essentially non-oscillatory schemes [18]. Apart from the notion of stability, in a recent paper CFL number is exploited for improved approximation by the flux limiters based scheme in [8]. In this work, we are interested in the stability of the following FTCS scheme. It is obtained by the discretization of (1) by replacing the time derivative with a forward difference, and the space derivative with a centred difference formula i.e.,

\[
 u^n_{j+1} = u^n_j - \frac{a \lambda}{2h} (u^n_{j+1} - u^n_{j-1}).
\]  

Above three point centred FTCS scheme (4) seems to be a correct and natural choice as the spatial discretization in FTCS does not violets the physical domain of dependence of (1) given in [1]. It is also interesting to note that the FTCS (4) and the centered Lax-Wendroff (LxW) scheme [11] shares the same spatial stencil of grid points. Note that for the CFL number \( C \leq 1 \) the three point centred LxW scheme is linearly stable [17]. Contrary to the expectation, the solution obtained by FTCS scheme (4) is diverging and and induced oscillations grow exponentially no matter how small the time step is compared to the space step. The classical Von-Neumann stability analysis also shows that FTCS (4) is unconditionally unstable. Moreover, FTCS does not satisfies any of the notion of non-linear stability mentioned above see [9].

2 Non-oscillatory condition on FTCS

The above, unconditional unstability FTCS scheme (4) is surprising as it can be observed that for smooth initial data, such as sinusoidal wave, the induced oscillations by FTCS does not grow immediately. Moreover, the magnitude and occurrence of induced oscillations can be controlled by choosing small CFL number \( C = \lambda |a| \), see Figure 2. On the other hand when applied on discontinuous data, FTCS introduces strong oscillations immediately see Figure 3(a). These observations have been the motivation for the present study on the dependence of induced oscillations by FTCS scheme on data type and CFL number. In the following result, a classification on data type is given in terms of ratio of consecutive gradients which is defined as

\[
 \theta^n_j = \begin{cases} 
 \frac{\Delta_- u^n_j}{\Delta_+ u^n_j} & \text{if } a \geq 0, \\
 \frac{\Delta_+ u^n_j}{\Delta_- u^n_j} & \text{if } a < 0,
\end{cases}
\]

\( \Delta \pm u^n_j = \pm u_{j+1} - u_j \). We assume that the time and space step ratio satisfies the CFL condition \( a \lambda \leq 1 \) and denote the sign of \( x \) by \( \text{sgn}(x) \).

**Theorem 2.1.** In the solution data region where \( \theta^n_j \in \mathcal{S}_{FTCS} = (\infty, -1] \cup \left[ \frac{\text{sgn}(a)}{2 - \text{sgn}(a)} \lambda, \infty \right) \),

the FTCS scheme (4) is stable and does not introduce numerical oscillations provided CFL number \( a \lambda \leq 1 \).

**Proof.** Rewrite FTCS (4) in the form

\[
 u^n_{j+1} = u^n_j - \frac{a \lambda}{2} (\Delta_+ u^n_j + \Delta_- u^n_j).
\]  

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• Let \( a > 0 \): FTCS can be written as,
\[
    u_j^{n+1} = u_j^n - \frac{a \lambda}{2} \left( \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} + 1 \right) \Delta^- u_j^n.
\]
which can be rewritten in upwind stencil form as,
\[
    u_j^{n+1} = \alpha^+ u_j^n + \beta^+ u_{j-1}^n.
\]
where the coefficients \( \alpha^+ = 1 - \frac{a \lambda}{2} \left( \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} + 1 \right) \) and \( \beta^+ = \frac{a \lambda}{2} \left( \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} + 1 \right) \). Note that \( \alpha + \beta = 1 \), thus to ensure non-oscillatory stable approximation such that \( u_j^{n+1} \leq u_j^n \leq u_j^{n+1} \) by (8), it is sufficient that
\[
    \alpha^+ = 1 - \frac{a \lambda}{2} \left( \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} + 1 \right) \geq 0 \quad \text{and} \quad \beta^+ = \frac{a \lambda}{2} \left( \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} + 1 \right) \geq 0.
\]
Inequalities (9) satisfies if,
\[
    \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} \leq \frac{2 - a \lambda}{a \lambda} \text{ and } \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} \geq -1.
\]
Which on inversion yield non-oscillatory condition for FTCS scheme (4) in case of \( a > 0 \),
\[
    \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \leq -1 \text{ OR } \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \geq \frac{a \lambda}{2 - a \lambda}.
\]

• Let \( a < 0 \), then (6) can be written in upwind stencil form as,
\[
    u_j^{n+1} = \alpha^- u_j^n + \beta^- u_{j+1}^n.
\]
where \( \alpha^- = \left( 1 + \frac{a \lambda}{2} \left( \frac{1}{\Delta^- u_j^n} + \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \right) \right) \) and \( \beta^- = -\frac{a \lambda}{2} \left( 1 + \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \right) \). Similar to above, (11) ensures for a non-oscillatory approximation provided
\[
    \left( \frac{a \lambda}{2} \left( 1 + \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \right) \right) \geq -1 \text{ and } -\frac{a \lambda}{2} \left( \frac{1}{\Delta^- u_j^n} + \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \right) \geq 0.
\]
Note that \( a \lambda < 0 \), therefore inequalities in (12) satisfy if
\[
    \left( 1 + \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \right) \leq \frac{-2}{a \lambda} \text{ and } 1 + \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \geq 0
\]
\[
    \text{or } \left( \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \right) \leq \frac{-(2 + a \lambda)}{a \lambda} \text{ and } \Delta^- u_j^n \geq \Delta^+ u_j^n \geq -1.
\]
Since \( \frac{-(2 + a \lambda)}{a \lambda} > 0 \), thus on inversion compound inequality satisfies if,
\[
    \frac{\Delta^+ u_j^n}{\Delta^- u_j^n} \leq -1 \text{ OR } \frac{\Delta^- u_j^n}{\Delta^+ u_j^n} \geq \frac{a \lambda}{(2 + a \lambda)}.
\]

The non-oscillatory stable region \( S_{FTCS} \) given by inequalities (10) is given for \( a > 0 \) in Figure (1) which clearly shows the effect of CFL number on the stability of FTCS.
2.1 Non-oscillatory bound for non-linear scalar problem

Consider the non-linear scalar hyperbolic conservation laws
\[ u_t + g(u(x,t))_x = 0, \quad u(x,0) = u_0(x). \] (14)

The characteristic speed \( g'(u) \) associated with (14) at interface \( x_{j+\frac{1}{2}} \) of the cell \([x_j, x_{j+1}]\) can be approximated by
\[ \alpha_{j+\frac{1}{2}} = \begin{cases} \frac{\Delta+g^n_j}{g'_j} & \text{if } \Delta+u^n_j \neq 0, \\ \text{else}. \end{cases} \] (15)

For (14), FTCS can be written as
\[ u^{n+1}_j = u^n_j - \frac{k}{2h} \left( g^n_{j+1} - g^n_{j-1} \right). \] (16)

where \( g^n_j = g(u^n_j) \). Which using (15) can be written as
\[ u^{n+1}_j = u^n_j - \frac{k}{2h} \left( a_{j+\frac{1}{2}}u^n_{j+1} - a_{j-\frac{1}{2}}u^n_{j-1} \right). \] (17)

In the non-sonic region, following result give bounds for non-oscillatory approximation of non-linear scalar problem (14) by FTCS.

**Theorem 2.2.** Away from the sonic point i.e., where \( a_{j-\frac{1}{2}} \times a_{j+\frac{1}{2}} > 0 \), the FTCS scheme (17) is stable and does not introduce numerical oscillations in the solution region
\[ \theta^n_i \in \mathcal{S}_{\text{FTCS}} = \begin{cases} \left( \infty, \frac{\alpha_{j+12}}{\alpha_{j-12}} \right) \cup \left[ \frac{\lambda\alpha_{j+12}}{2-\lambda\alpha_{j-12}}, \infty \right] & \text{if } g'(u) > 0, \\ \left( \infty, -\frac{\alpha_{j-12}}{\alpha_{j+12}} \right) \cup \left[ -\frac{\lambda\alpha_{j-12}}{2+\lambda\alpha_{j+12}}, \infty \right] & \text{if } g'(u) < 0. \end{cases} \] (18)

provided \( \lambda \max_u |g'(u)| \leq 1 \).

**Proof.** Analogous to the proof of Theorem 2.1. \( \square \)

3 Numerical Results

3.1 Linear Case

Consider linear transport equation (1) along with the following initial conditions
\[ u_0(x) = \sin(\pi x), \quad x \in [-1 : 1] \] (19)
\[ u_0(x) = \begin{cases} 1 & \text{if } |x| \leq 1/3, \\ 0 & \text{else}, \quad x \in [-1 : 1] \end{cases} \] (20)
\[ u_0(x) = \begin{cases} \exp\left(\frac{1}{1-x^2}\right) & \text{if } |x| \leq 1, \\ 0 & \text{else} \end{cases}, \quad x \in [-2 : 4] \] (21)

In Figure 2 numerical results corresponding to initial condition (19) are given to show the effect of CFL number on induced oscillations by FTCS. It can be clearly seen that numerical oscillations disappear for small CFL number. This supports the result in Theorem 2.1 as \( C \to 0 \) reduce the region of oscillations of \( \theta \) to (-1,0) as shown in Figure 1.

In order to show data region which cause induced oscillations by FTCS, the following hybrid approach using first order non-oscillatory upwind scheme is used to show,
• **FTCS does not introduce oscillations for data region** $\mathcal{S}_{FTCS}$: Use upwind scheme if $\theta_j^n \notin (-1, \frac{2-C}{C})$ otherwise FTCS is used. Results obtained by this approach (using legend 'FTCSUP') in Figure 3(b) and Figure 4(a) show that oscillations does not arise is shown as FTCSUP.

• **FTCS introduces oscillations only for data region** $\mathbb{R} \setminus \mathcal{S}_{FTCS}$: If $\theta_j^n \in (-1, \frac{2-C}{C})$ use FTCS, otherwise first order upwind scheme is used. Results obtained by this approach in Figure 3(c) and Figure 4(b) (using legend 'FTUPCS') show occurrence of spurious oscillations by FTUPCS.

### 3.2 Nonlinear scalar

Consider the non-linear Burgers equations

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad (22)$$

with initial conditions

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0.5 \\ 0 & \text{if } x > 0.5, \; x \in [0, 1]. \setminus 1 \end{cases} \quad (23)$$

$$u_0(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0, 2] \setminus 1. \end{cases} \quad (24)$$

In Figure 5 and 6 numerical results are given corresponding to initial condition (23) and (24) respectively. Similar to the linear case, again results by FTCS and FTUPCS are oscillatory whereas results by FTCSUP does not show any numerical oscillations. These results confirm that FTCS is conditionally stable and introduce the oscillations only in the solution data region where $\theta \in \mathcal{S}_{FTCS} \setminus \mathbb{R}$

### 4 Conclusion

Non-oscillatory stability bounds on initial data profile are given and numerically tested for unconditionally unstable (in Von-Neumann sense) FTCS scheme. An extension of this preliminary study for other existing schemes for non-linear hyperbolic problem is being carried out for a separate work.
Figure 1: Plot $\theta$ v/s $a\lambda$, $a = 1$: $a\lambda \to 0$ stabilizes the FTCS as it corresponds to reduced oscillatory shaded region.

Figure 2: Effect of CFL on induced oscillations in the solution corresponding to (19), at $T = 4$ using $N = 80$ grid points (a) $C = 0.05$ (b) $C = 0.25$ and (c) $C = 0.5$.

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Figure 3: Solution corresponding to (20), with $N = 80$ grid point at $T = 0.1$ and $C = 0.1$, (a) FTCS does not introduce oscillations for top and bottom of left and right discontinuity respectively (b) No oscillations by FTCSUP, FTCS is applied only in its stability region and (c) Induced oscillations FTUPCS, FTCS is applied only in its unstability region.

Figure 4: Solution corresponding to (21), at $T = 1.0$ and $C = 0.6$ (a) No oscillations by FTCSUP (b) Induced spurious oscillations FTUPCS, FTCS is applied only in its unstability region.

Figure 5: Solution of Burgers equation corresponding to IC (23) using $CFL = 0.9, N = 80$, after 6 time steps at $T = 0.05$
Figure 6: Solution of Burgers equation corresponding to IC (23) using $CFL = 0.8, N = 40$, after 3 time steps.

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