On the Uniqueness of $L_\infty$ bootstrap: Quasi-isomorphisms are Seiberg-Witten Maps

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Abstract

In the context of the recently proposed $L_\infty$ bootstrap approach, the question arises whether the so constructed gauge theories are unique solutions of the $L_\infty$ relations. Physically it is expected that two gauge theories should be considered equivalent if they are related by a field redefinition described by a Seiberg-Witten map. To clarify the consequences in the $L_\infty$ framework, it is proven that Seiberg-Witten maps between physically equivalent gauge theories correspond to quasi-isomorphisms of the underlying $L_\infty$ algebras. The proof suggests an extension of the definition of a Seiberg-Witten map to the closure conditions of two gauge transformations and the dynamical equations of motion.
1 Introduction

The connection between gauge theories and $L_\infty$ algebras goes back to Zwiebach’s formulation of Closed String field theory (CSFT) \cite{1}. The idea was to consistently decompose the moduli space of compact Riemann surfaces without boundaries into closed string vertices and propagators. There each vertex corresponds to a function taking $n$ fields as input and giving back one output field. Using these functions, an action for bosonic CSFT was written down. In order to quantize the theory the action should satisfy the quantum BV-master action which, roughly speaking, is the case if the string $n$-products of the action define a loop $L_\infty$ algebra \cite{2}.

When only looking at tree level diagrams the action should satisfy the classical master equation which yields an $L_\infty$ algebra for the $n$-products. Using the geometric language of formal supermanifolds, in \cite{3} the connection between the classical master action of the BV-formalism and $L_\infty$ algebras was further analyzed, leading to the physics folklore that any consistent classical gauge theory should define an $L_\infty$ algebra. This picture was further elucidated in the more recent work by Hohm and Zwiebach \cite{4}, where it was explicitly shown how both the gauge transformation rules and the equations of motion of Chern-Simons and Yang-Mills theories are encoded in underlying $L_\infty$ algebras. New connections between $L_\infty$ algebras and extended symmetry algebras, called $W$-algebras, of two-dimensional conformal field theories were established \cite{5,6}.

Besides the $n$-product defining a vertex in the action, to actually write down an action one also needs an appropriate inner product. When trying to construct gauge theories in situations where the notion of an inner product and hence an action is a priori not clear it is still a good guiding principle to demand that the constructed theory should define an $L_\infty$ algebra. This point of view was applied in a recent paper \cite{7} where the authors propose an $L_\infty$ bootstrap program for the construction of non-commutative gauge theories with non-constant non-commutativity parameter. Starting with a free theory and a fixed gauge group the bootstrap idea proceeds by first making an ansatz for the lowest order products and then by solving the $L_\infty$ relations order by order. This in turn defines the higher order $L_\infty$ products. By this procedure one can derive both interaction terms in the resulting equations of motion and higher products describing the gauge transformations of the fields.

A natural question is if this procedure leads to a unique solution and how physically equivalent solutions are related. From one side, two $L_\infty$ algebras are known to be equivalent if they are quasi-isomorphic \cite{8}. From the other, following \cite{9}, gauge theories describing the same physics should be related via a Seiberg-Witten map, which ensures that gauge orbits of the respective theories get mapped onto each other. Seiberg-Witten maps have been studied in terms of the antibracket formalism for gauge theories (see \cite{10} and references therein), which is related to $L_\infty$ algebras. In this note we show that a Seiberg-Witten map
is the same as having a specific subset of quasi-isomorphism (QISO) relations of the $L_\infty$ algebras underlying the two gauge theories.

We first motivate this connection with a simple example from an abelian Chern-Simons theory. Then we formulate and prove a general theorem stating clearly the equivalence of the physical notion of a Seiberg-Witten map and the mathematical notion of an $L_\infty$ QISO. This means in particular that different solutions to the bootstrap program yield equivalent physics if they are related via an $L_\infty$ quasi-isomorphism. Let us emphasize that this does not show uniqueness of the bootstrap.\footnote{In other words, $L_\infty$ QISOs define equivalence classes of $L_\infty$ algebras. To illustrate this we remind the reader of a similar situation with the definition of the star product. Two star products $\star$ and $\star'$ are equivalent if they are related by a “gauge” transformation $f \star' g = D^{-1}(Df \star Dg)$, where $D$ is a QISO \cite{11}. In general, $\star$ and $\star'$ may have very different expressions and properties, e.g., $\star$ can be closed with respect to the integral $\int f \star g = \int f \cdot g$, while $\star'$ is not. However some features are conserved, if $\star$ is associative and represents the quantization of the Poisson bracket $\{f, g\}$, then $\star'$ is also associative and represents a different quantization of the same bracket. Thus star products related by QISOs represent different quantization prescriptions of the same classical system. This is similar to our statement that QISOs relate physically equivalent gauge theories via SW maps.}

In \cite{12} (see \cite{13} for a complete review) the question of consistent interactions for classical Yang-Mills gauge theory was addressed using local BRST-cohomology. This corresponds to an $L_\infty$ bootstrap starting with a free Yang-Mills theory. The results there depend crucially on the concrete gauge theory under consideration, hence it seems unlikely that there exist a general answer to the question of uniqueness. Instead, in this paper we take a more moderate step and investigate solely the question when two a priori different solutions to the bootstrap describe the same physics.

The paper is organized as follows. In section 2 we briefly recall the definition of $L_\infty$ quasi-isomorphisms and Seiberg-Witten maps. In section 3, we recall the bootstrap approach and present a simple example showing how possible redundancies can appear when solving the $L_\infty$ equations. We point out that these are related to Seiberg-Witten maps and to the existence of quasi-isomorphisms. Building on these examples we give a general proof of the equivalence of the two structures in section \footnote{In other words, $L_\infty$ QISOs define equivalence classes of $L_\infty$ algebras. To illustrate this we remind the reader of a similar situation with the definition of the star product. Two star products $\star$ and $\star'$ are equivalent if they are related by a “gauge” transformation $f \star' g = D^{-1}(Df \star Dg)$, where $D$ is a QISO \cite{11}. In general, $\star$ and $\star'$ may have very different expressions and properties, e.g., $\star$ can be closed with respect to the integral $\int f \star g = \int f \cdot g$, while $\star'$ is not. However some features are conserved, if $\star$ is associative and represents the quantization of the Poisson bracket $\{f, g\}$, then $\star'$ is also associative and represents a different quantization of the same bracket. Thus star products related by QISOs represent different quantization prescriptions of the same classical system. This is similar to our statement that QISOs relate physically equivalent gauge theories via SW maps.} We realize that from the QISO structure, one can learn some new aspects of the Seiberg-Witten map, namely how the closure conditions of the two related gauge theories are mapped and how the equations of motion are related.

## 2 \ L_\infty\text{-QISO} and Seiberg-Witten maps

In this section we set the stage and present the relevant definitions and properties of $L_\infty$ algebras, their quasi-isomorphisms and Seiberg-Witten maps. We will start with some aspects of $L_\infty$ algebras.
2.1 Basics of $L_\infty$ algebras

For convenience, as signs are simpler, we will work with the following definition [14] of an $L_\infty$ algebra in the so-called $b$-picture [14]:

**Definition 1.** An $L_\infty$ algebra in the $b$-picture consists of a graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$ equipped with graded symmetric multilinear maps $b_n : V^\otimes n \to V$ of constant degree $|b_n| = -1$ satisfying

$$\sum_{\sigma \in \text{Unsh}(k+l=n)} \epsilon(\sigma;x) b_{1+l}(b_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}) = 0 \quad (2.1)$$

with $\epsilon(\sigma;x)$ denoting the Koszul sign defined via

$$x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)} = \epsilon(\sigma;x) x_1 \wedge \cdots \wedge x_n. \quad (2.2)$$

Here we used $x_i \wedge x_j = (-1)^{x_i x_j} x_j \wedge x_i$ where an $x_i$ in the exponent stands for the degree of the corresponding element. Moreover, the sum in (2.1) runs over the unshuffled permutations that satisfy

$$\sigma(1) < \ldots < \sigma(k), \quad \sigma(k+1) < \ldots < \sigma(n). \quad (2.3)$$

In contrast to the definition in [14] we used a homological grading on the graded vector space which merely results in having the maps being of degree $-1$. The first few defining equations explicitly read

$n=1$:

$$0 = b_1 (b_1(x)) \quad (2.4)$$

$n=2$:

$$0 = b_1 (b_2(x_1, x_2)) + b_2 (b_1(x_1), x_2) + (-1)^{x_1} b_2 (x_1, b_1(x_2)) \quad (2.5)$$

$n=3$:

$$0 = b_1 (b_3(x_1, x_2, x_3)) + b_3 (b_1(x_1), x_2, x_3) + (-1)^{x_1} b_3 (x_1, b_1(x_2), x_3)$$

$$+ (-1)^{x_1+x_2} b_3 (x_1, x_2, b_1(x_3)) + b_3 (b_2(x_1, x_2), x_3)$$

$$+ (-1)^{x_3(x_2+x_1)} b_2 (b_2(x_3, x_1), x_2) + (-1)^{x_1(x_2+x_3)} b_2 (b_2(x_2, x_3), x_1). \quad (2.6)$$

Like in the usual definition ($\ell$-picture), the first map is a differential. The second equation is like a Leibniz rule between the differential and the two-bracket, but now with slightly unusual signs. The same applies for the third equation, which still means that the two-bracket satisfies the Jacobi identity up to homotopy. The crucial difference to the $\ell$-picture is that the $b$-maps are graded symmetric and of
constant degree $-1$. Having this definition at hand we can now follow \[14\] and provide the notion of an $L_{\infty}$ morphism.$^2$

**Definition 2.** Let $(V, \{b_i\})$ and $(W, \{\tilde{b}_j\})$ be $L_{\infty}$ algebras.

1) An $L_{\infty}$ **morphism** $F : (V, \{b_i\}) \to (W, \{\tilde{b}_j\})$ consists of multilinear, graded symmetric maps $\{F_n\} : \bigotimes^n V \to W$ of degree $|F_n| = 0$ such that

$$
\sum_{\sigma \in \text{Unsh}(k+l=n)} \epsilon(\sigma; x) F_{1+l}\left(b_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}\right)
= \sum_{\sigma \in \text{Unsh}(k_1+\cdots+k_j=n)} \frac{\epsilon(\sigma; x)}{j!} \tilde{b}_j(F_{k_1} \otimes \cdots \otimes F_{k_j})(x_{\sigma(K)}).
$$

(2.7)

In the second line the index of the entry is a multindex $K = (k_1, \ldots, k_j)$ of length $n$. It keeps track of the partition of the $n$ entries $(x_1, \ldots, x_n)$ inserted into the $j$ maps $F_{k_i}$. The unshuffles $\sigma(K)$ consist of all permutations keeping the entries in the $j$-partitions ordered.

2) An $L_{\infty}$ **quasi-isomorphism** is an $L_{\infty}$ morphism $F$, whose lowest map $F_1 : V \to W$ induces an isomorphism on the homology of the chain complexes underlying the $L_{\infty}$ algebras.

Concretely, at lowest order the definition of an $L_{\infty}$ morphism gives the following equations

$n=1$: $$F_1(b_1(x)) = \tilde{b}_1(F_1(x))$$

(2.8)

$n=2$: $$F_1(b_2(x_1, x_2)) + F_2(b_1(x_1), x_2) + (-1)^{x_1} F_2(x_1, b_1(x_2))$$

$$= \tilde{b}_1(F_2(x_1, x_2)) + \frac{1}{2} \tilde{b}_2(F_1(x_1), F_1(x_2)) + \frac{1}{2} \tilde{b}_2(F_1(x_1), F_1(x_2)).$$

(2.9)

Since later we will choose $F_1 = \text{id}$, (i.e. we are considering automorphisms of $L_{\infty}$ algebras on a given chain complex), the second equation can be rewritten as

$$\tilde{b}_2(x_1, x_2) - b_2(x_1, x_2) =$$

$$F_2(b_1(x_1), x_2) + (-1)^{x_1} F_2(x_1, b_1(x_2)) - \tilde{b}_1(F_2(x_1, x_2)).$$

(2.10)

Thus for two $L_{\infty}$ algebras $(V, \{b_i\})$ and $(V, \{\tilde{b}_i\})$ with $b_1 = \tilde{b}_1$ to be quasi-isomorphic the product $\tilde{b}_2$ must be decomposable in the form \[2.10\]. It is not too

$^2$In \[8\] it was shown that the $\ell$-picture of an $L_{\infty}$ algebra is equivalent to having a nilpotent coderivation $Q$ of degree $-1$ on the coalgebra over the suspension of the graded vector space. A morphism between two $L_{\infty}$ algebras $(C(V), Q_V)$ and $(C(W), Q_W)$ is then a cohomomorphism $F : C(V) \to C(W)$ of the coalgebras satisfying $Q_W \circ F = F \circ Q_V$. Although this gives a closed expression of an $L_{\infty}$ morphism this expression is not very useful for actual calculations.
hard to see, that this procedure also works in higher degree equations. The right hand side of (2.7) always contains terms with \( j = n \), i.e. a summand of the form

\[
\sum_{\sigma \in S_n} \frac{\epsilon(\sigma;x)}{n!} \tilde{b}_n(F_1 \otimes \cdots \otimes F_1)(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) .
\]  

(2.11)

The sum runs over all permutations of \( \{1, \ldots, n\} \). As in the step from (2.9) to (2.10) we use \( F_1 = \text{id} \). Upon interchanging the entries, all terms are equal in this case and a single term \( \tilde{b}_n(x_1, \ldots, x_n) \) is left. This shows that the defining relation can always be solved for the highest appearing product \( \tilde{b}_n \). Bringing all other terms to the left produces a necessary and sufficient condition on the products \( \tilde{b}_n \) for being related to the \( b \)'s via a quasi-isomorphism

\[
\tilde{b}_n(x_1, \ldots, x_n) - b_n(x_1, \ldots, x_n) = \sum_{\sigma \in \text{Unsh}(k+l=n)_{k<n}} \epsilon(\sigma;x) F_{1+l}\left( b_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)} \right) \bigg|_{F_1=\text{id}} \]  

(2.12)

Since in the bootstrap approach one solves the \( L_\infty \) equations in an iterative way starting with the lowest products, the above relation can be applied for relating two possible solutions \( b \) and \( \tilde{b} \). We will provide a simple example for this procedure in section 3.2.

### 2.2 Seiberg-Witten maps

Next we recall the notion of a Seiberg-Witten (SW) map between gauge theories. In their seminal paper [9] Seiberg and Witten analyzed the behavior of open strings in backgrounds with non-zero but constant Kalb-Ramond \( B \)-field. It turns out that the \( B \)-dependence can be completely captured by making space-time non-commutative, i.e. by introducing the Moyal-Weyl star product between functions with \( \theta \sim B^{-1} \). It is then further argued that in the limit of large \( B \)-field one arrives at a description in terms of non-commutative Yang-Mills (YM) theory with the Moyal-Weyl star product. As this corresponds to a specific choice of regularization of the world sheet theory, Seiberg and Witten argued that there has to be a one to one correspondence between non-commutative and ordinary Yang-Mills theory.

Recall that non-commutative Yang Mills theory on the Moyal-Weyl plane with a star product

\[
(f \star g)(x) := e^{i\theta_{ij}\frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j}} f(y)g(z) \bigg|_{y=z=x}
\]  

(2.13)

formally looks like usual YM theory where all point-wise products are replaced by star products. This means that for a non-commutative gauge field \( \hat{A} \) the
infinitesimal gauge variation takes the form
\[ \delta_\lambda \hat{A}_j = \partial_j \hat{\lambda} + i \hat{\lambda} \star \hat{A}_j - i \hat{A}_j \star \hat{\lambda} \]  
(2.14)
while the field strength is defined as
\[ \hat{F}_{jk} = \partial_j \hat{A}_k - \partial_k \hat{A}_j - i \hat{A}_j \star \hat{A}_k + i \hat{A}_k \star \hat{A}_j . \]  
(2.15)
As pointed out in [9], in order to relate non-commutative Yang-Mills theory to ordinary Yang-Mills it suffices that the gauge orbits of the respective theories are mapped onto each other. This ensures that the degrees of freedom on both sides are the same\(^3\). The relation between the non-commutative and ordinary YM theory turned out to be of the following type.

**Definition 3.** Two gauge theories with data \((\lambda, A)\) and \((\hat{\lambda}, \hat{A})\) are related via a Seiberg-Witten map, if there exist two maps
\[ \hat{\lambda} = \hat{\lambda}(\lambda, A) , \quad \hat{A} = \hat{A}(A) \]  
(2.16)
so that (at linear order in \(\lambda\)) gauge orbits are mapped onto gauge orbits
\[ \hat{A}(A + \delta_\lambda A) = \hat{A}(A) + \hat{\delta}_{\hat{\lambda}(A)} \hat{A}(A) . \]  
(2.17)
For the concrete case of non-commutative YM theory, this map has been worked out order by order in \(\theta^{ij}\) [9]. As we will see later when comparing \(L_\infty\) QISO to SW maps, this definition can be extended by further requirements. To see how, let us consider the closure of two gauge transformations
\[ [\delta_{\lambda_1}, \delta_{\lambda_2}] A = \delta_{[\lambda_1, \lambda_2]} A ; \quad [\delta_{\hat{\lambda}_1}, \delta_{\hat{\lambda}_2}] \hat{A} = \delta_{[\hat{\lambda}_1, \hat{\lambda}_2]} \hat{A} \]  
(2.18)
on both sides of the the original SW duality. The naive guess for the mapping between the two closure conditions would be
\[ \hat{A} \left( A + \delta_{[\lambda_1, \lambda_2]} A \right) = \hat{A}(A) + \hat{\delta}_{[\hat{\lambda}_1, \hat{\lambda}_2]} \hat{A}(A) . \]  
(2.19)
However, by inspection of the \(U(1)\) case this cannot be true. The left-hand side of the equation is \(\hat{A}(A)\) as \(U(1)\) is abelian, which is not true for the non-commutative \(U(1)\) on the right hand side. Hence the second term on the right side is non-vanishing and the equation cannot hold. However, there is something we can say about the closure relation from (2.17)
\[ \hat{A} \left( A + \delta_{[\lambda_1, \lambda_2]} A \right) = \hat{A}(A) + \hat{\delta}_{\lambda([\lambda_1, \lambda_2],\lambda) \dot{A}(A)} . \]  
(2.20)
\(^3\)Note that this does not imply that the gauge groups are equivalent, as can be inferred by looking at a non-commutative \(U(1)\) YM theory, which is non-abelian.
Thus the question is what \( \hat{\lambda}(\lambda_1, \lambda_2, A) \) really is. We can use the explicit form of the original SW map up to first order in the non-commutativity parameter \( \theta^i{}_j \), (c.f. [9] (3.5)) to derive an equation to first order. An elementary but tedious computation shows that the following relation holds

\[
\hat{\lambda}(\lambda_1, \lambda_2, A)\bigg|_{O(\theta)} = [\hat{\lambda}_1, \hat{\lambda}_2] + \hat{\lambda}_1(\lambda_2, \delta_{\lambda_1} A) - \hat{\lambda}_2(\lambda_1, \delta_{\lambda_2} A)\bigg|_{O(\theta)}.
\] (2.21)

Holding up to first order in \( \theta^i{}_j \), it is a reasonable extension to require (2.21) to hold to all orders. Therefore, we conjecture that the gauge closures should map as

\[
\hat{A}(A + \delta_{\lambda_1, \lambda_2} A) = \hat{A}(A) + \delta_{\lambda_1, \lambda_2} \hat{A}(A) + \delta_{\lambda, \lambda_2} \hat{A}(A) - \delta_{\lambda, \lambda_1} \hat{A}(A).
\] (2.22)

In the main section [4] by exploiting the intriguing relation of the SW map to an \( L_\infty \) QISO we collect further evidence that this is the right formula. For gauge transformations that close only on-shell there will be an extra term in (2.22). Moreover, we will find how SW maps should be extended to the equations of motion of the two equivalent theories.

### 3 Redundancies in the \( L_\infty \) bootstrap

We begin with recalling how a classical gauge theory with irreducible gauge freedom can be described in terms of an \( L_\infty \) algebra. By irreducible gauge freedom we mean that the gauge parameters of the field theory do not have gauge redundancies themselves. This very much builds on the dictionary established in [4].

#### 3.1 Basics of the \( L_\infty \) bootstrap approach

In order to define an \( L_\infty \) algebra in the sense of definition (1) we need a graded vector space. We take \( X = X_1 \oplus X_0 \oplus X_{-1} \) and all others trivial. The assignment is as follows. The vector space \( X_1 \) contains the gauge parameters, \( X_0 \) is the space of fields and \( X_{-1} \) contains the equations of motion of the gauge theory. A standard gauge transformation is then of the form

\[
\delta_\lambda A = b_1(\lambda) + b_2(\lambda, A) + \frac{1}{2} b_3(\lambda, A, A) + \cdots = \sum_{n=0}^\infty \frac{1}{n!} b_{n+1}(\lambda, A^n).
\] (3.1)

The equations of motion can be written as

\[
\mathcal{F} = b_1(A) + \frac{1}{2} b_2(A, A) + \frac{1}{3!} b_3(A, A, A) + \cdots = \sum_{n=1}^\infty \frac{1}{n!} b_n(A^n).
\] (3.2)
Then, the equation of motion transforms under gauge transformations as follows

\[
\delta_{\lambda} F = b_2(\lambda, F) + b_3(\lambda, F, A) + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} b_{n+2}(\lambda, F, A^n) .
\] (3.3)

Using the L\(_\infty\) equations one can show that the gauge commutator yields \[4, 15\]

\[
[\delta_{\lambda_1}, \delta_{\lambda_2}] A \sim \delta_{C(\lambda_1, \lambda_2, A)} A + \delta^T_{C(\lambda_1, \lambda_2, F, A)} A
\] (3.4)

with

\[
C(\lambda_1, \lambda_2, A) = b_2(\lambda_1, \lambda_2) + b_3(\lambda_1, \lambda_2, A) + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} b_{n+2}(\lambda_1, \lambda_2, A^n) \] (3.5)

and where the second term on the right hand side of (3.4) vanishes on-shell. It can be expanded as

\[
C(\lambda_1, \lambda_2, F, A) = b_3(\lambda_1, \lambda_2, F) + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} b_{n+3}(\lambda_1, \lambda_2, F, A^n) .
\] (3.6)

We come back to this term in section [4] but for now we drop it and assume that the gauge structure defines a closed algebra. Comparing (3.1) to (2.14), for the non-commutative YM theory one can read off

\[
b_1(\hat{\lambda}) = \partial \hat{\lambda}, \quad b_2(\hat{\lambda}, \hat{A}) = i \hat{\lambda} \star \hat{A}_j - i \hat{A}_j \star \hat{\lambda}, \quad b_n(\hat{\lambda}, \hat{A}^n) = 0, \quad n \geq 2 .
\] (3.7)

Instead of reading-off the L\(_\infty\) maps from a given gauge structure and equations of motion and checking the L\(_\infty\) equations one could proceed in the other direction. This the L\(_\infty\) bootstrap approach followed in [7]. Let us explain briefly how this works, for more details we refer to [7].

One initially starts with a free theory, i.e. an assignment of \(b_1(\lambda), b_1(A)\) and a fixed gauge group represented through its algebra captured in \(b_2(\lambda_1, \lambda_2)\) and the other products left open a priori. Trying to solve subsequently the L\(_\infty\) equation will fix higher products and define a consistent theory by the arguments above. Due to the grading of the elements a specific L\(_\infty\) equation has to be solved only on particular inputs. Note that every L\(_\infty\) relation in the \(b\)-picture will lower the degree by two. Take for example the equation with two inputs (2.3). It is only non-trivial on insertions \(\lambda_1, \lambda_2 \in X_1\) of total degree 2 and \(\lambda \in X_1, A \in X_0\) of total degree 1. The first case yields

\[
0 = b_1(b_2(\lambda_1, \lambda_2)) + b_2(b_1(\lambda_1), \lambda_2) - b_2(\lambda_1, b_1(\lambda_2)) .
\] (3.8)

The first term is fixed by our assignments, but we are free to choose \(b_2(\lambda, A)\) such that the above holds with \(A = b_1(\lambda)\). Hence we are defining a new product
corresponding to the transformation law of the field. In the same fashion the insertion of \( \lambda, A \) will then fix a product \( b_2(A, B) \) with \( A, B \in X_0 \).

The question we are concerned with in this paper is, if the \( L_\infty \) relations uniquely fix the products or whether there are other choices \( \tilde{b}_2(\lambda, A) \) and \( \tilde{b}_2(A, B) \) solving them. Note that this is in general a tough question and will very much depend on the specific theory we are looking at. Hence we do not expect an answer to this question solely from the \( L_\infty \) perspective. In this paper we rather want to address the situation when there are different solutions to the equations, and discuss how to decide if they lead to equivalent field theories.

From what we already said, we expect that two solutions related by a SW map should be considered physically equivalent. It would be nice if mathematically this corresponded to the existence of a corresponding \( L_\infty \) QISO. Before we formulate a general theorem relating the physical and the mathematical approach, let us discuss a simple example.

### 3.2 Redundancies in the abelian Chern-Simons theory

Let us consider the \( L_\infty \) algebra for a free 3D Chern-Simons theory [4] with gauge group \( U(1) \). The vector space is \( X = X_1 \oplus X_0 \oplus X_{-1} \) and the only non-trivial maps are

\[
\begin{align*}
  b_1(\lambda)_a &= \partial_a \lambda \\
  b_1(A)^a &= \epsilon^{ac}_{\phantom{ac}b} \partial_b A_c \\
  b_2(\lambda_1, \lambda_2) &= 0.
\end{align*}
\]  

(3.9)

Choosing all other maps to vanish satisfies the \( L_\infty \) algebra so the bootstrap approach is rather trivial. The gauge transformations, gauge algebra and field equations are very simple

\[
\begin{align*}
  \delta_\lambda A &= \partial_a \lambda, \quad [\delta_{\lambda_1}, \delta_{\lambda_2}] = 0, \quad F = \epsilon^{bc}_{\phantom{bc}d} \partial_d A_c.
\end{align*}
\]  

(3.10)

Revisiting the defining equation (2.5) for inputs \( \lambda_1, \lambda_2 \in X_1 \), the condition on \( b_2(\lambda, A) \) is not to vanish identically but rather

\[
0 = b_2(b_1(\lambda_1), \lambda_2) - b_2(\lambda_1, b_1(\lambda_2)).
\]  

(3.11)

This admits other solutions, for instance

\[
\tilde{b}_2(\hat{\lambda}, \hat{A})_a = -v^i (\hat{A}_i \partial_a \hat{\lambda} + \hat{A}_a \partial_i \hat{\lambda})
\]  

(3.12)

with some constant vector \( v^i \). Comparing with (3.1) this map plays a role in the gauge transformation of the fields \( \hat{A} \in X_0 \). The \( L_\infty \) structure forces us to introduce further maps to satisfy all defining equations, essentially bootstrapping a theory with deformed gauge transformations. Up to order \( \tilde{b}_3 \), the maps are given
by
\begin{align*}
b_1(\hat{\lambda})_a &= \partial_a \hat{\lambda} \\
b_1(\hat{A})_a &= \epsilon^{bc}_a \partial_b \hat{A}_c \\
\tilde{b}_2(\hat{A}, \hat{\lambda})_a &= v^i (\hat{A}_i \partial_a \hat{\lambda} + \hat{A}_a \partial_i \hat{\lambda}) \\
\tilde{b}_2(\hat{E}, \hat{\lambda})_a &= v^i \hat{E}_a \partial_i \hat{\lambda} \\
\tilde{b}_2(\hat{A}, \hat{B})_a &= v^i \epsilon^{bc}_a (\hat{A}_c \partial_b \hat{B}_i + \partial_b \hat{A}_i \hat{B}_c) \\
\tilde{b}_3(\hat{A}, \hat{B}, \hat{\lambda})_a &= -v^i v^j \hat{A}_i \hat{B}_j \partial_a \hat{\lambda} + \frac{1}{2} v^i v^j \hat{A}_i \hat{A}_j \partial_a \hat{\lambda} + \ldots \quad (3.13)
\end{align*}

The deformed gauge transformation can be read-off as
\begin{equation}
\hat{\delta}_\hat{\lambda} \hat{A} = \partial_a \hat{\lambda} - v^i \hat{A}_i \partial_a \hat{\lambda} - v^i \hat{A}_a \partial_i \hat{\lambda} - \frac{1}{2} v^i v^j \hat{A}_i \hat{A}_j \partial_a \hat{\lambda} + \ldots \quad (3.14)
\end{equation}

and the deformed field equation are
\begin{equation}
\hat{\mathcal{F}} = \epsilon^{bc}_a \left( \partial_b \hat{A}_c + v^i \partial_b \hat{A}_i \hat{A}_c + 2v^i v^j \partial_b \hat{A}_i \hat{A}_j \hat{A}_c + \ldots \right) \quad (3.15)
\end{equation}

while the closure remains trivial. As the ellipsis indicate the equations are only exact up to terms coming from maps higher than those given above. In the following we will suppress the ellipsis for better readability. Now let us analyze whether these two bootstrapped solutions are related via a SW map and an L_∞ QISO, respectively.

**Seiberg-Witten map**

Inspection reveals that indeed there exists a Seiberg-Witten map relating the two solutions. The field redefinitions are
\begin{align*}
\hat{A}_a(A) &= A_a - v^i A_i A_a + \frac{1}{2} v^i v^j A_i A_j A_a \\
\hat{\lambda}(\lambda, A) &= \lambda.
\end{align*}

(3.16)

The Seiberg-Witten condition (2.17) can be easily verified by direct computation. Since the gauge algebra is trivial in both cases and the gauge parameters map directly to each other, the deformed gauge algebra maps to the original gauge algebra. One can also show that the deformed field equation map to a product of original one and a function of the gauge field,
\begin{equation}
\hat{\mathcal{F}}(\mathcal{F}, A) = \mathcal{F}(A) \left( 1 - v^i A_i + \frac{1}{2} v^i v^j A_i A_j \right). \quad (3.17)
\end{equation}
\textbf{\(L_\infty\) QISO}

Continuing the discussion from the end of section 2.1 let us now investigate whether we can also identify an \(L_\infty\) QISO. First recall that we have \(\tilde{b}_1 = b_1\) and \(b_n \geq 2 = 0\). Then the first quasi-isomorphism relation (2.8) is the fact that the free theories are the same, \(F_1(b_1(x)) = \tilde{b}_1(F_1(x)) = b_1(x)\) with \(F_1 = \text{id}\) and for \(x \in \{\lambda, A\}\). The quasi-isomorphism relation (2.10) for \(\tilde{b}_2\) is

\[
\tilde{b}_2(x_1, x_2) = F_2(b_1(x_1), x_2) + (-1)^{x_1}F_2(x_1, b_1(x_2)) - b_1(F_2(x_1, x_2)).
\]

Evaluating this on the possible combinations of entries \((\lambda_1, \lambda_2), (\lambda, A), (\lambda, E), \) and \((A, B)\) allows us to read off the next quasi-isomorphism maps.

\[
\begin{align*}
F_2(E, A) &= -v^i E_a A_i & \in X_{-1} \\
F_2(A, B) &= -v^i (A_i B_a + A_a B_i) & \in X_0 \\
F_2(E, \lambda) &= F_2(A, \lambda) = 0.
\end{align*}
\]

The quasi-isomorphism relation for \(\tilde{b}_3\) simplifies drastically thanks to the simple form of the original \(L_\infty\) algebra. Separated into knowns on the left hand side and unknowns on the right, it reads

\[
\begin{align*}
\tilde{b}_3(x_1, x_2, x_3) + \tilde{b}_2(x_1, F_2(x_2, x_3)) \\
+ \tilde{b}_2(F_2(x_1, x_2), x_3) + (-1)^{x_2 x_3} \tilde{b}_2(F_2(x_1, x_3), x_2) \\
= F_3(b_1(x_1), x_2, x_3) + (-1)^{x_1}F_3(x_1, b_1(x_2), x_3) \\
+ (-1)^{2x_1 + x_2} F_3(x_1, x_2, b_1(x_3)) - b_1(F_3(x_1, x_2, x_3)).
\end{align*}
\]

Evaluating these on the list of inputs \((E_1, E_2, \lambda), (E, A, B), (E, A, \lambda), (A, B, C), (A, B, \lambda), (E, \lambda_1, \lambda_2), \) and \((A, \lambda_1, \lambda_2)\) we can read off the non-vanishing QISO maps \(F_3\):

\[
\begin{align*}
F_3(E, A, B) &= v^i v^j E_a A_i B_j & \in X_{-1} \\
F_3(A, B, C) &= v^i v^j (A_i B_j C_a + A_j B_a C_i + A_a B_i C_j) & \in X_0.
\end{align*}
\]

Let us take special note of the quasi-isomorphism acting on identical gauge fields

\[
\begin{align*}
F_2(A, A)_a &= -2v^i A_i A_a \\
F_3(A, A, A)_a &= 3v^i v^j A_i A_j A_a.
\end{align*}
\]

Noting also that except for \(F_1 = \text{id}\) the quasi-isomorphism acts on gauge parameters trivially, we see a nice connection to the Seiberg-Witten map:

\[
\begin{align*}
\hat{A}_a(A) &= F_1(A) + \frac{1}{2} F_2(A, A) + \frac{1}{6} F_3(A, A, A) + \ldots \\
\hat{\lambda}(\lambda, A) &= F_1(\lambda).
\end{align*}
\]
Moreover, the new field equation (3.17) can be expressed in terms of the QISO as
\[ \hat{F}(F, A) = F_1(F) + F_2(F, A) + \frac{1}{2}F_3(F, A, A) + \ldots \] (3.24)
In addition, the form of the maps (3.19) and (3.21) suggests the general solution
\[ F_n(A^1, \ldots, A^n)_a = (-1)^{n-1}v^i_1 \ldots v^n_{i-1}\left(A^1_{i_1} \ldots A^{n-1}_{i_{n-1}} A^n_a + \text{cycl}\right) \]
\[ F_{n+1}(E, A^1, \ldots, A^n)_a = (-1)^{n}v^i_1 \ldots v^n_{i-1}E_a A^1_{i_1} \ldots A^n_{i_n} \] (3.25)
with all other maps being zero. By inspection of equation (2.12) one can see that starting with a given \( L_\infty \) algebra \((V, \{b_i\})\) an arbitrary set of graded symmetric, multilinear maps \( F_n : V \to V, n \geq 2 \) of degree 0 together with \( F_1 = \text{id} \) defines a new, quasi-isomorphic \( L_\infty \) algebra \((V, \{\hat{b}_i\})\). Therefore we can take the maps (3.25) to complete the solution (3.13) to all orders. The constructed theory is then related to the usual Chern-Simons theory via the Seiberg-Witten maps
\[ \hat{A}_a(A) = A_a \exp(-v^i A_i) \]
\[ \hat{\lambda}(\lambda, A) = \lambda \]
\[ \hat{F}(F, A) = F(A) \exp(-v^i A_i) \] (3.26)
Thus, analogously to the fact that the gauge transformations and the field equations are encoded in the higher products \( b_n \) of an \( L_\infty \) algebra, also the SW-like field redefinitions seem to be encoded in the \( F_n \) maps of an \( L_\infty \) QISO. In the following section we will make this connection more precise.

4 SW maps are \( L_\infty \) QISOs

Based on the example in the last section we analyze the general relation between Seiberg-Witten maps and \( L_\infty \) quasi-isomorphisms. Assume we are given a quasi-isomorphism \( \{F_n\} : V \to W \) between two gauge theories \((V = V_1 \oplus V_0 \oplus V_{-1}, \{b_i\}), (W = W_1 \oplus W_0 \oplus W_{-1}, \{\hat{b}_i\})\). Recall that the maps of a quasi-isomorphism are of degree 0. Thus an element \( \hat{\lambda} \in W_1 \) can be of the schematic form
\[ \hat{\lambda} \sim \sum_n F_{n+1}(\lambda, A^n) + \sum_k F_{k+3}(\lambda, \mu, E, A^k) + \sum_k F_{k+5}(\lambda, \mu_1, \mu_2, E_1, E_2, A^k) + \ldots \] (4.1)
The first sum is what is expected from the point of view of a Seiberg-Witten map. All other terms contain at least two gauge parameters \( \lambda, \mu \). As we are working on the level of infinitesimal gauge transformations those terms are suppressed and
we don’t include them in the expansion. Similarly, for a gauge field \( \hat{A} \in W_0 \) there is the expansion

\[
\hat{A} \sim \sum_n F_n(A^n) + \sum_k F_{k+2}(\mu, E, A^k) + \sum_k F_{k+4}(\mu_1, \mu_2, E_1, E_2, A^n) + \ldots. \tag{4.2}
\]

Again, the first term is expected. Terms in the third sum and all higher terms are at least of second order in the gauge parameters and therefore suppressed. From a physics point of view the second sum does not make much sense either, as the field of the hatted theory would depend on the gauge parameter \( \mu \). It is therefore reasonable to ignore those, too and take only the first sum. Next, an element \( \hat{E} \in W_{-1} \) can have the expansion

\[
\hat{E} \sim \sum_n F_{n+1}(E, A^n) + \sum_k F_{k+3}(\mu, E, A^k) + \ldots. \tag{4.3}
\]

Elements of degree \(-1\) in the \( L_\infty \) algebra are related to the equations of motion. A possibly non-vanishing term in the second sum would mean that the field equations of the hatted theory depend on a random gauge parameter \( \mu \). Thus it is sensible to drop those.

Note that these general considerations are consistent with the example in the last section where the only non-trivial quasi-isomorphism maps were \( F_2(A, B), F_2(A, E), F_3(A, B, C) \) and \( F_3(A, B, E) \). In addition, the discussion in the example went beyond that of a mere Seiberg-Witten map since we included the field equations. After these considerations we are now ready to formulate a theorem that clearly relates a SW map and an \( L_\infty \) QISO. The following theorem is the main result of this paper:

**Theorem 1.** Let \((V = V_1 \oplus V_0 \oplus V_{-1}, \{ b_i \}), (W = W_1 \oplus W_0 \oplus W_{-1}, \{ \tilde{b}_i \})\) be two \( L_\infty \) algebras underlying two classical gauge theories. Then

A) There exists a Seiberg-Witten map \( \hat{\lambda} = \hat{\lambda}(\lambda, A) \) and \( \hat{A} = \hat{A}(A) \), satisfying

\[
\hat{A}(A + \delta_\lambda A) = \hat{A}(A) + \hat{\delta}_\lambda(\lambda, A) \hat{A}(A) \tag{4.4}
\]

and the closure mapping

\[
\hat{A}(A + \delta_C(\lambda_1, \lambda_2, A) A + \delta_T^C(\lambda_1, \lambda_2, F) A) = \hat{A}(A) + \hat{\delta}_C(\lambda_1, \lambda_2, \hat{A}(A)) + \hat{\delta}_T^C(\lambda_1, \lambda_2, F) \hat{A}(A), \tag{4.5}
\]

between the gauge theories if and only if there exist graded symmetric maps \( \{ F_n \} : V \to W \) of degree 0 with

\[
F_{n+k+l}(\lambda_1, \ldots, \lambda_k, E_1, \ldots, E_l, A^n) = 0, \quad \text{for all } k, l \in \{1, 2\}, \ n \geq 0 \tag{4.6}
\]

satisfying the \( L_\infty \) quasi-isomorphism relations for inputs \((\lambda, A^n), (\lambda_1, \lambda_2, A^n), (\lambda_1, \lambda_2, E, A^n)\) for all \( n \geq 0 \).
The Seiberg-Witten map of $A$ maps the dynamics of the field theories according to

\[ \hat{F} = \hat{F}(F, A), \]
\[ \hat{F}(F + \delta_\lambda F, A + \delta_\lambda A) = \hat{F}(F, A) + \hat{\delta}_\lambda(A, F) \]

if and only if the graded symmetric maps of $A$ satisfy the $L_\infty$ quasi-isomorphism relations on inputs $(A^n, (\lambda, E, A^n))$ for all $n \geq 0$.

In the remainder of this section we prove this theorem.

**Proof.** We start with the proof of $A$). Assume that there exist graded symmetric maps $\{F_n\} : V \to W$ of degree 0 s.th. (4.6) holds and the maps satisfy the $L_\infty$ quasi-isomorphism relations on the inputs stated in $A$). We then define the corresponding Seiberg-Witten maps as

\[ \hat{A}(A) = \sum_{n=1}^\infty \frac{1}{n!} F_n(A^n), \quad \hat{\lambda}(\lambda, A) = \sum_{k=0}^\infty \frac{1}{k!} F_{k+1}(\lambda, A^k). \]

Recall from section 3 the form of the gauge variations (3.1) in terms of $L_\infty$ brackets

\[ \delta_\lambda A = \sum_{n=0}^\infty \frac{1}{n!} b_{n+1}(\lambda, A^n), \quad \hat{\delta}_\lambda \hat{A} = \sum_{n=0}^\infty \frac{1}{n!} \hat{b}_n(\hat{\lambda}, \hat{A}^n). \]

Using the defining relations of an $L_\infty$ quasi-isomorphism we prove a first lemma that says that gauge orbits are mapped to gauge orbits.

**Lemma 1.**

\[ \hat{A}(A + \delta_\lambda A) = \hat{A}(A) + \hat{\delta}_\lambda \hat{A}(A) \]

**Proof.** We start with the left hand side of (4.10) for which we can calculate

\[ \hat{A}(A + \delta_\lambda A) = \hat{A} \left( A + \sum_{k=0}^\infty \frac{1}{k!} b_{k+1}(\lambda, A^k) \right) \]
\[ = \sum_{n=1}^\infty \frac{1}{n!} F_n \left( A + \sum_{k=0}^\infty \frac{1}{k!} b_{k+1}(\lambda, A^k), \ldots, A + \sum_{k=0}^\infty \frac{1}{k!} b_{k+1}(\lambda, A^k) \right) \]
\[ = \sum_{n=1}^\infty \frac{1}{n!} \left[ F_n(A^n) + n F_n \left( A^{n-1}, \sum_{k=0}^\infty \frac{1}{k!} b_{k+1}(\lambda, A^k) \right) + O(\lambda^2) \right] \]
\[ = \hat{A}(A) + \sum_{n=0}^\infty \frac{1}{n!} \left[ F_{n+1} \left( \sum_{k=0}^\infty \frac{1}{k!} b_{k+1}(\lambda, A^k), A^n \right) \right] + O(\lambda^2) \]
\[ = \hat{A}(A) + \sum_{m=0}^\infty \frac{1}{m!} \sum_{k+m=1} \frac{m!}{n! k!} F_{n+1} \left( b_{k+1}(\lambda, A^k), A^n \right). \]
The first two equalities are just writing the variation in terms of \( L_\infty \) brackets and using the definition of \( \hat{A}(A) \) in terms of the \( F_n \)’s. In the next equality the linearity of the map \( F_n \) is used and we truncated to linear order in the infinitesimal gauge parameter \( \lambda \). Next we changed the index of the outer sum and switched the order of the entries, which doesn’t cause extra minus signs as all elements are of degree 0. In the last step we just rewrote the sums in a more convenient form and inserted \( \frac{m!}{m!} \).

To proceed, we recall the left hand side of the defining equation for an \( L_\infty \) quasi-isomorphism (2.7):

\[
\sum_{\sigma \in \text{Unsh}(k+n=m+1)} \epsilon(\sigma; x) F_{1+n}(b_{k+1}(x_{\sigma(1)}, \ldots, x_{\sigma(k+1)}), x_{\sigma(k+2)}, \ldots, x_{\sigma(m+1)}) \cdot (4.12)
\]

Choosing the input \( x_1 = \lambda, x_2 = \cdots = x_{m+1} = A \) and taking into account that interchanging any two elements never causes a minus sign this expression becomes

\[
\sum_{k+n=m} \frac{m!}{k!n!} F_{1+n}(b_{k+1}(\lambda, A^k), A^n) + \frac{m!}{(k+1)!(n-1)!} F_{1+n}(b_{k+1}(A^{k+1}), \lambda, A^{n-1}) \cdot \quad (4.13)
\]

In the first summand the prefactor \( \frac{m!}{k!n!} \) is the number of unshuffles for the \( m \) gauge fields \( A \) into partitions of length \( k \) and \( n \). The same holds for the second summand. Upon bringing the second term in (4.13) to the right hand side, the left hand side of the defining relation for an \( L_\infty \) quasi-isomorphism appears in (4.11). We first investigate what happens to the second term in (4.13):

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k+n=m} \frac{m!}{(k+1)!(n-1)!} F_{n+1}(b_{k+1}(A^{k+1}), \lambda, A^{n-1})
\]

\[
= \sum_{m=0}^{\infty} \sum_{k+n=m+1} \frac{1}{k!(n-1)!} F_{n+1}(b_k(A^k), \lambda, A^{n-1})
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(n-1)!} F_{n+1}\left(\sum_{k=1}^{\infty} \frac{1}{k!} b_k(A^k), \lambda, A^{n-1}\right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(n-1)!} F_{n+1}(\mathcal{F}, \lambda, A^{n-1}) = 0 . \quad (4.14)
\]

In the last step the equation of motion is abbreviated by \( \mathcal{F} \). Thus, we realize that in general there will appear more terms on the right hand side of the SW-condition (4.10). However, these terms are proportional to the equation of motion and are of the type appearing in (4.6) so that they actually vanish.

\footnote{This is slightly cheating, since there is a term with \( n = 0 \) and we set \((-1)^!=1.\)}

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Employing now the quasi-isomorphism equation (4.11) for $x_1 = \lambda$, $x_2 = \cdots = x_{n+1} = A$, one can express (4.11) as

$$
\hat{A}(A) + \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\sigma \in \text{Unsh}(k_1 + \cdots + k_j = m+1)} \epsilon(x; \sigma) \tilde{b}_j \left( F_{k_1} \otimes \cdots \otimes F_{k_j} \right) (x_{\sigma(K)}). \tag{4.15}
$$

We still stated the equation in an abstract form to highlight the point where (2.7) is used. In order to unwrap the expression we note that

$$
\left| \text{Unsh}(k_1 + \cdots + k_j = n) \right| = \binom{n}{k_1, \ldots, k_j} \tag{4.16}
$$

meaning that there are $(\binom{n}{k_1, \ldots, k_j})$ possibilities to order a set of size $n$ into $j$ partitions of length $k_i$, $i = \{1, \ldots, j\}$ preserving the order in each partition. Using this we get for the second term in (4.15)

$$
\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1 + \cdots + k_j = m+1} \frac{1}{j!} \left[ \tilde{b}_j \left( (F_{k_1}(\lambda, A^{k_1-1}), F_{k_2}(A^{k_2-1}), \ldots, F_{k_j}(A^{k_j-1})) \right) \left( (k_1, k_2, \ldots, k_j) \right) 
+ \tilde{b}_j \left( F_{k_1}(A^{k_1}), F_{k_2}(\lambda, A^{k_2-1}), \ldots, F_{k_j}(A^{k_j}) \right) \left( (k_1, \ldots, k_{j-1}) \right) 
+ \tilde{b}_j \left( F_{k_1}(A^{k_1}), \ldots, F_{k_j}(\lambda, A^{k_j}) \right) \left( m \right) \right]. \tag{4.17}
$$

Of course the terms in the squared bracket are all the same. Using the definition of the multinomial-coefficient this gives

$$
\sum_{m=0}^{\infty} \sum_{k_1 + \cdots + k_j = m+1} \frac{1}{(j-1)!} \frac{1}{(k_1-1)!} \cdots \frac{1}{(k_j-1)!} \tilde{b}_j \left( F_{k_1}(\lambda, A^{k_1-1}), F_{k_2}(A^{k_2-1}), \ldots, F_{k_j}(A^{k_j-1}) \right) 
= \sum_{m=0}^{\infty} \sum_{k_0 + \cdots + k_j = m} \frac{1}{j!} \frac{1}{k_0!} \tilde{b}_{j+1} \left( F_{k_0+1}(\lambda, A^{k_1}), F_{k_1}(A^{k_1}), \ldots, F_{k_j}(A^{k_j}) \right) 
= \sum_{j=0}^{\infty} \sum_{k_0 + \cdots + k_j = m} \frac{1}{j!} \tilde{b}_{j+1} \left( (F_{k_0+1}(\lambda, A^{k_0}), F_{k_1}(A^{k_1}), \ldots, F_{k+j}(A^{k_j})) \right) 
$$

(4.18)

where we changed the summation indices $j$ and $k_1$ in the first step and rewrote the summation in a more convenient way in the second.

All this was for the left hand side of (4.10). Next we compute the second
term on the right hand of (4.10)

\[
\hat{\delta}_\lambda \hat{A}(A) = \sum_{j=0}^{\infty} \frac{1}{j!} \tilde{b}_{j+1} \left( \hat{\lambda}(\lambda, A), \hat{A}(A)^j \right)
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{j!} \tilde{b}_{j+1} \left( \sum_{n=0}^{\infty} \frac{1}{n!} F_{n+1}(\lambda, A^n), \left( \sum_{k=1}^{\infty} \frac{1}{k!} F_k(A^k) \right)^j \right)
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_j \geq 1} \frac{1}{n! k_1! \ldots k_j!} \tilde{b}_{j+1} \left( F_{n+1}(\lambda, A^n), F_{k_1}(A^{k_1}), \ldots, F_{k_j}(A^{k_j}) \right).
\]

But this is exactly (4.18) which is equivalent to the second term in (4.15). This proves Lemma 1.

Next we want to derive the closure statement. Note that we can be more general than in section 3 and consider a gauge algebra which closes only on shell, i.e.

\[
[\delta_{\lambda_1}, \delta_{\lambda_2}] A = \delta_{C(\lambda_1, \lambda_2, A)} A + \delta^T_{C(\lambda_1, \lambda_2, \mathcal{F})} A.
\] (4.20)

If we ignore this term for the moment we can prove the guess of section 3.

**Lemma 2.**

\[
\hat{A}(A + \delta_{C(\lambda_1, \lambda_2, A)} A) = \hat{A}(A) + \hat{\delta}_{\hat{C}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{A})} + \hat{\lambda}_{\lambda_1, \delta_{\lambda_2} A - \hat{\lambda}(\lambda_1, \delta_{\lambda_2} A)} \hat{A}(A)
\] (4.21)

**Proof.** Upon using lemma 1 this amounts to

\[
\hat{\lambda}(C(\lambda_1, \lambda_2, A), A) = \hat{C}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{A}) + \hat{\lambda}(\lambda_2, \delta_{\lambda_1} A) - \hat{\lambda}(\lambda_1, \delta_{\lambda_2} A).
\] (4.22)

This equality is readily proven by going through the analog steps as in the proof of Lemma 1.

Now we check what happens when we include the term $\delta^T_{C(\lambda_1, \lambda_2, \mathcal{F})} A$ in the closure condition. Since the gauge algebras in the original Seiberg-Witten map closed off-shell, there is no obvious guess. We will utilize the quasi-isomorphism to derive a transformation rule. We start with

\[
\hat{A}(A + \delta_{C(\lambda_1, \lambda_2, A)} A + \delta^T_{C(\lambda_1, \lambda_2, \mathcal{F})} A)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n!} F_n \left( (A + \delta_{C(\lambda_1, \lambda_2, A)} A + \delta^T_{C(\lambda_1, \lambda_2, \mathcal{F})} A)^n \right)
\]

\[
= \sum_{n=1}^{\infty} \left[ \frac{1}{n!} F_n (A^n) + \frac{1}{(n-1)!} F_n \left( \delta_{C(\lambda_1, \lambda_2, A)} A, A^{n-1} \right) \right] + o(\lambda^3),
\] (4.23)
where the first two terms are the ones we already computed for (4.21). The third term is new. Inserting (3.6), using the defining equation for an $L_\infty$ morphism and conditions (4.6) this term is computed to be

\[ \sum_{n=1}^{\infty} \frac{1}{(n-1)!} F_n \left( \delta^T_{C(\lambda_1,\lambda_2,\mathcal{F})} A, A^{n-1} \right) = \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{b}_n (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\mathcal{F}}, \hat{A}^n) = \delta^T_{C(\lambda_1,\lambda_2,\mathcal{F})} \hat{A} \] (4.24)

Thus we get the transformation rule

\[ \hat{A}(A + \delta_{C(\lambda_1,\lambda_2,A)} A + \delta^T_{C(\lambda_1,\lambda_2,\mathcal{F})} A) = \hat{A}(A) + \hat{\delta}_{C(\lambda_1,\lambda_2,A)} (A) + \hat{\delta}^T_{C(\lambda_1,\lambda_2,\mathcal{F})} \hat{A}(A) . \] (4.25)

This proves one direction of the Theorem 1A). For the other direction note that we can start with a Seiberg-Witten map $\hat{A}(A)$ and $\hat{\lambda}(\lambda, A)$ of the form (4.8) satisfying the relations (2.17) and (4.25). Going all the steps in the computations backwards then reveals that the $F_n$ with the same gauge field inserted satisfy the defining equation of an $L_\infty$ quasi-isomorphism on the stated inputs with the trivial assertions (4.6). Using the graded symmetry and polarization identities (see [4]) this is enough to get the equations on general inputs of gauge fields. This finally proves Theorem 1A).

For the proof of Theorem 1B) we have to perform very similar computations. Recall that the equation of motion is expanded as

\[ \mathcal{F} = \sum_{n=1}^{\infty} \frac{1}{n!} b_n (A^n) . \] (4.26)

Given the maps $\{F_n\} : V \to W$ of $A$, which we assume to satisfy additionally the $L_\infty$ QISO relations on inputs $(A^n)$, $(\lambda, E, A^n)$, we define for the equation of motion in the hatted theory

\[ \hat{\mathcal{F}} = \sum_{n=0}^{\infty} \frac{1}{n!} F_{n+1} (\mathcal{F}, A^n) . \] (4.27)

In order for this to be consistent we have to show

**Lemma 3.**

\[ \sum_{n=0}^{\infty} \frac{1}{n!} F_{n+1} (\mathcal{F}, A^n) = \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{b}_n (\hat{A}^n) \] (4.28)

with $\hat{A}(A)$ given by (4.8).
Proof. Inserting \((4.26)\) in the left hand side of \((4.28)\) and using the defining relation for an \(L_\infty\) morphism this is can be verified by a similar computation as in the proof of Lemma 1.

**Lemma 4.**

\[ \hat{F}(\mathcal{F} + \delta \lambda \mathcal{F}, A + \delta \lambda A) = \hat{F}(\mathcal{F}, A) + \hat{\delta}_{\lambda_A} \hat{F}(\mathcal{F}, A). \] (4.29)

**Proof.** We start by computing the left hand side of \((4.29)\).

\[
\begin{align*}
\hat{F}(\mathcal{F} + \delta \lambda \mathcal{F}, A + \delta \lambda A) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ F_{n+1}(\mathcal{F}, A^n) + F_{n+2}(\mathcal{F}, \delta \lambda A, A^n) + F_{n+1}(\delta \lambda \mathcal{F}, A^n) \right] + \mathcal{O}(\lambda^2) \\
&= \hat{F}(\mathcal{F}, A) + \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n+k=m} \frac{m!}{n!k!} \left[ F_{n+2}(b_{k+1}(\lambda, A^k), \mathcal{F}, A^n) \\
&\quad + F_{n+1}(b_{k+2}(\lambda, \mathcal{F}, A^k), A^n) \right].
\end{align*}
\] (4.30)

In the second equality we used \((4.27)\) and inserted the definition of the gauge transformations for the field and the equations of motion. Next we use \((2.7)\) in the second summand which upon using \((4.6)\) and some combinatorics is equivalent to

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1 + \ldots + k_{j+2} = m} \frac{1}{j!k_1! \ldots k_{j+2}!} \tilde{b}_{j+2}\left(F_{k_1+1}(\lambda, A^{k_1}), F_{k_2+1}(\mathcal{F}, A^{k_2}), \right.
\]

\[
\left. F_{k_3}(A^{k_3}), \ldots, F_{k_{j+2}}(A^{k_{j+2}}) \right)
\]

\[= \sum_{j=0}^{\infty} \frac{1}{j!} \tilde{b}_{j+2}\left(\hat{\lambda}(\lambda, A), \hat{F}(\mathcal{F}, A), \hat{A}(A)^j\right) \]

\[= \hat{\delta}_{\lambda_A} \hat{F}(\mathcal{F}, A). \] (4.31)

This finally shows \((4.29)\). □

For the other direction of Theorem 1B) we note that we can make the ansatz \((4.27)\) and just go all the steps of the above computations backwards. This gives the defining relations for an \(L_\infty\) quasi-isomorphism on \((A^n)\) and \((\lambda, E, A^n)\). □

Note that it seems that we are only using the conditions for an \(L_\infty\) morphism. But the Seiberg-Witten map should of course be invertible, which implies quasi-isomorphism on the \(L_\infty\) side.
5 Conclusion

Motivated by redundancies in defining the $L_\infty$ structure in the bootstrap approach to gauge theories, we showed how mathematical equivalence of the solutions implies also physical equivalence. This is done by showing that a quasi-isomorphism between $L_\infty$ algebras of two gauge field theories is equivalent to the existence of a Seiberg-Witten map between the two. This ensures that there are the same degrees of freedom in both field theories. Note that when considering only the gauge $L_\infty$ algebras of the theories, i.e. setting $X_{-1} = 0$, the conditions (4.6) are trivially satisfied and we get a complete quasi-isomorphism between the gauge $L_\infty$ algebras from a Seiberg-Witten map.

In addition we derived a condition for the closure of the gauge algebra in terms of a Seiberg-Witten map. This was motivated by the original example of Seiberg-Witten discussed in [9], but we think that the equivalence in terms of $L_\infty$ algebras establishes (4.25) as the correct formula. Furthermore we argued that the existence of a quasi-isomorphism of the full theory implies that the equations of motion in both theories get mapped onto each other.

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References

[1] B. Zwiebach, “Closed string field theory: Quantum action and the B-V master equation,” Nucl. Phys. B390 (1993) 33–152, hep-th/9206084.

[2] M. Markl, “Loop homotopy algebras in closed string field theory,” Commun. Math. Phys. 221 (2001) 367–384, hep-th/9711045.

[3] M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich, “The Geometry of the master equation and topological quantum field theory,” Int. J. Mod. Phys. A12 (1997) 1405–1429, hep-th/9502010.

[4] O. Hohm and B. Zwiebach, “$L_\infty$ Algebras and Field Theory,” Fortsch. Phys. 65 (2017), no. 3-4, 1700014, 1701.08824.

[5] R. Blumenhagen, M. Fuchs, and M. Traube, “On the Structure of Quantum $L_\infty$ algebras,” JHEP 10 (2017) 163, 1706.09034.

[6] R. Blumenhagen, M. Fuchs, and M. Traube, “$W$ algebras are $L_\infty$ algebras,” JHEP 07 (2017) 060, 1705.00736.

[7] R. Blumenhagen, I. Brunner, V. Kupriyanov, and D. Lst, “Bootstrapping non-commutative gauge theories from $L_\infty$ algebras,” JHEP 05 (2018) 097, 1803.00732.

[8] T. Lada and M. Markl, “Strongly homotopy lie algebras,” Communications in Algebra 23 (1995), no. 6, 2147–2161.

[9] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 09 (1999) 032, hep-th/9908142.

[10] G. Barnich, F. Brandt, and M. Grigoriev, “Local BRST cohomology and Seiberg-Witten maps in noncommutative Yang-Mills theory,” Nucl. Phys. B677 (2004) 503–534, hep-th/0308092.

[11] M. Kontsevich, “Deformation quantization of Poisson manifolds. 1.,” Lett. Math. Phys. 66 (2003) 157–216, q-alg/9709040.

[12] G. Barnich and M. Henneaux, “Consistent couplings between fields with a gauge freedom and deformations of the master equation,” Phys. Lett. B311 (1993) 123–129, hep-th/9304057.

[13] G. Barnich, F. Brandt, and M. Henneaux, “Local BRST cohomology in gauge theories,” Phys. Rept. 338 (2000) 439–569, hep-th/0002245.

[14] H. Kajiura and J. Stasheff, “Homotopy algebras inspired by classical open-closed string field theory,” Commun. Math. Phys. 263 (2006) 553–581, math/0410291.
[15] R. Fulp, T. Lada, and J. Stasheff, “sh-Lie algebras induced by gauge transformations,” Commun. Math. Phys. 231 (2002) 25–43.