Abstract

In a Robertson-Walker space-time a spinning particle model is investigated and we show that in a stationary case, there exists a class of new structures called $f$-symbols which can generate reducible Killing tensors and supersymmetry algebras.

Keywords: $f$-symbols, Killing-Yano tensors, hidden symmetries

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1 Introduction

The constants of motion of a scalar particle in a curved space are determined by the symmetries of the manifold, and are expressible in terms of the Killing vectors and tensors. A similar result hold for a spinning particle, with the modification that the constants of motion related to a Killing vector contain spin dependent parts and, there are Grassmann odd constants of motion, which do not have a counterpart in the scalar model. An illustration of the existence of extra conserved quantities is provided by Kerr-Newmann and Taub-NUT geometry. For the geodesic motion in the Taub-NUT space,
conserved vector analogous to the Runge-Lenz vector of the Kepler type problem is quadratic in 4-velocities, its components are Stackel-Killing tensors and they can be expressed as symmetrized products of Killing-Yano tensors \[2, 3, 4, 5, 6\]. The configuration space of spinning particles (spinning space) is an extension of an ordinary Riemannian manifold, parameterized by local coordinates \(\{x^\mu\}\), to a graded manifold parameterized by local coordinates \(\{x^\mu, \psi^\mu\}\), with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anti-commuting). In the spinning case the generalized Killing equations are more involved and new procedures have been conceived.

The aim of this paper is to show that the existence of the Killing-Yano tensors and its properties in a stationary case of Robertson-Walker space-times, may be understood in a systematic way as a particularly interesting example of a more general structures called \(f\)-symbols [9].

The plan of this paper is as follows. In Sec. 2 we give a short review of the formalism for pseudo-classical spinning point particles in an arbitrary background space-time, using anticommuting Grassmann variables to describe the spin degrees of freedom. In Sec. 3 we present the properties of \(f\)-symbols and the role played in generating new supersymmetries. In Sec. 4 we apply the results previously presented to show that in a stationary case of a Robertson-Walker space-time we obtain explicit solutions for \(f\)-symbols equations and construct corresponding Killing tensors. Conclusions are presented in Sec. 5.

## 2 Symmetries of Spinning Particle Model

The symmetries of spinning particle model can be divided into two classes [7, 8]: generic ones, which exists for any spinning particle model and non-generic ones, which depend on the specific background space considered. To the first class belong proper-time translations and supersymmetry, generated by the hamiltonian and supercharge: \(Q_0 = \Pi_\mu \psi^\mu\). To obtain all symmetries, including the non-generic ones, one has to find all functions \(J(x, \Pi, \psi)\) which commute with the Hamiltonian in the sense of Poisson-Dirac brackets: \(\{H, J\} = 0\). Expanding \(J\) in a power series in the covariant momentum

\[
J = \sum_{n=0}^{\infty} \frac{1}{n!} J^{(n)\mu_1...\mu_n}(x, \psi) \Pi_{\mu_1} ... \Pi_{\mu_n},
\]

(1)
then the components of $J$ satisfy

$$D_{(\mu_{n+1}} J^{(n)}_{\mu_1...\mu_n)} + \omega^a_{(\mu_{n+1}} b \psi^b \frac{\partial J^{(n)}_{\mu_1...\mu_n)}}{\partial \psi^a} = R_{\nu(\mu_{n+1}} J^{(n+1)}_{\mu_1...\mu_n) \nu}, \tag{2}$$

where the parentheses denote full symmetrization over the indices enclosed, $\omega^a_{\mu b}$ the spin connection and $R_{\mu\nu}$ given by

$$R_{\mu\nu} = \frac{i}{2} \psi^a \psi^b R_{ab\mu\nu}. \tag{3}$$

Eqs. (2) are the generalizations of the Killing equations to spinning space first obtained in [7]. Writing for $J$ the series expansion

$$J(x, \Pi, \psi) = \sum_{m,n=0}^{\infty} \frac{[m]}{m!n!} \psi^{a_1} ... \psi^{a_m} f^{(m,n)}_{a_1...a_m} (x) \Pi^{\mu_1} ... \Pi^{\mu_n}, \tag{4}$$

where $f^{(n,m)}$ is completely symmetric in the $n$ upper indices $\{\mu_k\}$ and completely anti-symmetric in the $m$ lower indices $\{a_i\}$ we obtains the component equation

$$n f^{(m+1,n-1)}_{a_0a_1...a_m} (\mu_1...\mu_{n-1} e^{\mu_n}) a_0 = m D_{[a_1} f^{(m-1,n)}_{a_2...a_m]} \mu_1...\mu_n, \tag{5}$$

where $D_a = e^{\mu_a} D_\mu$ are ordinary covariant derivatives, and indices in parentheses are to be symmetrized completely, whilst those in square brackets are to be anti-symmetrized, all with unit weight. Note in particular for $m = 0$, $f^{(1,n)}_{a} (\mu_1...\mu_n e^{\mu_{n+1}} a_0) = 0$. In a certain sense these equations represent a square root of the generalized Killing equations, although they only provide sufficient, not necessary conditions for obtaining solutions. Having found $\Theta$ we can then reconstruct the corresponding $J$. Eqs. (5) partly solve $f^{(m+1,n-1)}$ in terms of $f^{(m-1,n)}$ and only that part of $f^{(m+1,n-1)}$ is solved which is symmetrized in one flat index and all $(n - 1)$ curved indices. On the other hand eqs. (5) do not automatically imply that $f^{(m+1,n-1)}$ is completely anti-symmetric in the first $(m + 1)$ indices. Imposing that condition on eqs. (5) one finds a new set of equations which are precisely the generalized Killing equations for that part of $f^{(m+1,n-1)}$ which was not given in terms of $f^{(m-1,n)}$, and which should still be solved for. This is the part of $f^{(m+1,n-1)}$ which is anti-symmetrized in one curved index and all $(m + 1)$ flat indices. Hence eqs. (5) clearly have advantages over the generalized Killing equations (2).
order to find the constant of motion corresponding to a Killing tensor of rank $n$,

$$D_{(\mu_{n+1}} J^{(n)}_{\mu_{1}...\mu_{n})} = 0, \quad (6)$$

with $D_\mu$ given by

$$D_\mu = \partial_\mu + \Gamma^\lambda_{\mu\nu} \Pi_\lambda \frac{\partial}{\partial \Pi_\nu} + \omega_{\mu}^a \psi^b \frac{\partial}{\partial \psi^a}, \quad (7)$$

one has to solve the hierarchy of equations (2) for $(J^{(n-1)}_{\mu_{1}...\mu_{n-1}},...,J^{(0)}_{\mu_{1}...\mu_{n}})$ and add the terms, as in expression (1). Having a solution $f_{(m,n)(\mu_{1}...\mu_{n})}^{\alpha}$ of the equation $f_{(m,n)(\mu_{1}...\mu_{n})}^{\alpha} = 0$, then we generate at least part of the components $f_{(m+2,n-\alpha)(\mu_{1}...\mu_{n-\alpha})}^{\alpha}$ for $\alpha = 1,...,n$ by mere differentiation.

3 New Supersymmetries and $f$-symbols

The constants of motion generate infinitesimal transformations of the coordinates leaving the equations of motion invariant: $\delta x^\mu = \delta \alpha \{ x^\mu, J \}$, $\delta \psi^a = \delta \alpha \{ \psi^a, J \}$, with $\delta \alpha$ the infinitesimal parameter of the transformation. The theory might admit other (non-generic) supersymmetries [9] of the type $\delta x^\mu = -i \epsilon f^\mu_a \psi^a$ with corresponding supercharges

$$Q_f = -i \epsilon f^\mu_a \psi^a + \frac{i}{3!} c_{abc}(x) \psi^a \psi^b \psi^c, \quad (8)$$

provided the tensors $f^\mu_a$ and $c_{abc}$ satisfy the differential constraints

$$D_\mu f^\mu_a + D_\nu f^\nu_a = 0, \quad (9)$$

$$D_\mu c_{abc} = -(R_{\mu\nu ab} f^\nu_c + R_{\mu\nu bc} f^\nu_a + R_{\mu\nu ca} f^\nu_b). \quad (10)$$

One now obtains the following algebra of Poisson-Dirac brackets of the conserved charges $Q_i$:

$$\{ Q_i, Q_j \} = -2i Z_{ij}, \quad (11)$$

with

$$Z_{ij} = \frac{1}{2} K_{ij}^{\mu\nu} \Pi_\mu \Pi_\nu + I_{ij}^{\mu} \Pi_\mu + G_{ij}, \quad (12)$$

where

$$K_{ij}^{\mu\nu} = \frac{1}{2} \left( f^\mu_{ia} f^\nu_{ja} + f^\nu_{ia} f^\mu_{ja} \right), \quad (13)$$
showing that \( K_{ij \mu \nu} \) is a symmetric Killing tensor of 2nd rank: \( D(\lambda \ K_{ij \ (\mu \nu)}) = 0 \), whilst \( I_{ij}^\mu \) is the corresponding Killing vector and \( G_{ij} \) the corresponding Killing scalar.

In order to study the properties of the new supersymmetries, it is convenient to introduce the 2nd rank tensor \( f_{\mu \nu} = f_{\mu a} e_a^\nu \), which will be referred to as the \( f \)-symbol \[9]. The defining relation (9) implies

\[ D_\nu f_{\lambda \mu} + D_\lambda f_{\nu \mu} = 0. \]

(14)

It follows that the \( f \)-symbol is divergence-less on its first index \( D_\nu f_{\nu \mu} = 0 \), and by contracting of eq.(14) one finds \( D_\nu f_{\mu \nu} = -\partial_\mu f_{\nu \nu} \). Hence the divergence on the second index vanishes if and only if the trace of the \( f \)-symbol is constant:

\[ D_\nu f_{\nu \mu} = 0 \Leftrightarrow f_{\mu} = \text{const.} \]

(15)

If the trace is constant, it may be subtracted from the \( f \)-symbol without spoiling condition (14). It follows, that in this case one may without loss of generality always take the constant equal to zero and hence \( f \) to be traceless. The symmetric part of the \( i \)th \( f \)-symbol is the tensor

\[ S_{\mu \nu} \equiv K_{i0 \mu \nu} = \frac{1}{2} (f_{\mu \nu} + f_{\nu \mu}) , \]

(16)

which satisfies the generalized Killing equation \( D_{(\mu} S_{\nu \lambda)} = 0 \). We can also construct the anti-symmetric part

\[ B_{\mu \nu} = -B_{\nu \mu} = \frac{1}{2} (f_{\mu \nu} - f_{\nu \mu}) . \]

(17)

which obeys the condition \( D_\nu B_{\lambda \mu} + D_\lambda B_{\nu \mu} = D_\mu S_{\nu \lambda} \). It follows, that if the symmetric part does not vanish and is not covariantly constant, then the anti-symmetric part \( B_{\mu \nu} \) by itself is not a solution of eq.(14). But by the same token the anti-symmetric part of \( f \) can not vanish either, hence \( f \) can be completely symmetric only if it is covariantly constant.

Anti-symmetric \( f \)-symbols, Killing-Yano tensors \( f_{\mu \nu} \) found by Penrose and Floyd \[15\], and their corresponding Killing-tensors have been studied extensively in refs.\[1\] \[10\] \[11\] \[12\] \[13\] in the related context of finding solutions of the Dirac-equation in non-trivial curved space-time.
4 $f$-symbols in Robertson-Walker space-times

We consider space-time to be $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ represents the time direction and $\Sigma$ is a homogeneous and isotropic three-manifold (maximally symmetric space), with metric of the form

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}(u)du^i du^j.$$  \hspace{1cm} (18)

Here $t$ is the timelike coordinate, and $(u^1, u^2, u^3)$ are the coordinates on $\Sigma$; $\gamma_{ij}$ is the maximally symmetric metric on $\Sigma$. The function $a(t)$ is known as the scale factor and the coordinates used here, in which the metric is free of cross terms $dt \, du^i$ and the spacelike components are proportional to a single function of $t$, are known as comoving coordinates.

Since the maximally symmetric metrics obey $^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$, where $k$ is some constant, the Ricci tensor is $^{(3)}R_{jl} = 2k\gamma_{jl}$ and the metric on $\Sigma$ can be put in the form

$$d\sigma^2 = \gamma_{ij}du^i du^j = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2)$$ \hspace{1cm} (19)

we obtain the following metric on space-time:

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right].$$ \hspace{1cm} (20)

with $k = -1, 0, 1$, known as Robertson-Walker metric.

We apply the results obtained previously to show that in a stationary case $a(t) = \text{const}$ of this metric we obtain solutions of (14) which are not just Killing - Yano tensors of order two, also investigated in [16, 17, 18]. In the stationary case and $k \neq 0$ we have seven Killing vector fields:

$$\zeta^{(1)} = -r\sqrt{1 - kr^2} \sin(\theta) \frac{\partial}{\partial \theta} + \frac{1}{1 - kr^2} \cos(\theta) \frac{\partial}{\partial r}$$

$$\zeta^{(2)} = \cos(\theta) \cos(\phi) \frac{\partial}{\partial \theta} - \frac{1}{\sqrt{1 - kr^2}} \sin(\theta) \cos(\phi) \frac{\partial}{\partial r}$$

$$\zeta^{(3)} = r\sqrt{1 - kr^2} \sin(\phi) \frac{\partial}{\partial \theta} + \frac{1}{\sqrt{1 - kr^2}} \sin(\theta) \sin(\phi) \frac{\partial}{\partial r}$$

$$+ \, r\sqrt{1 - kr^2} \sin(\theta) \cos(\phi) \frac{\partial}{\partial \phi}$$

$$\zeta^{(4)} = r^2 \cos(\phi) \frac{\partial}{\partial \theta} - r^2 \sin(\theta) \cos(\phi) \sin(\phi) \frac{\partial}{\partial \phi}$$ \hspace{1cm} (21)
\[
\vec{\xi}^{(5)} = r^2 \sin(\phi) \frac{\partial}{\partial \theta} + r^2 \sin(\theta) \cos(\theta) \frac{\partial}{\partial \phi}
\]
\[
\vec{\xi}^{(6)} = r^2 \sin^2(\theta) \frac{\partial}{\partial \phi}
\]
\[
\vec{\xi}^{(7)} = \frac{\partial}{\partial t}
\]

We obtain from (14) the following three independent solutions
\[
f^{(1)}_{tt} = 1 \quad \text{(22)}
\]
\[
f^{(2)}_{rt} = \frac{\cos(\theta)}{\sqrt{1 - kr^2}}, \quad f^{(2)}_{\theta t} = -r \sqrt{1 - kr^2} \sin(\theta) \quad \text{(23)}
\]
\[
f^{(3)}_{\varphi t} = r^2 \sin^2(\theta) \quad \text{(24)}
\]

and other four antisymmetric solutions (Killing-Yano tensors) \cite{17,18},
\[
Y^{(1)}_{r\theta} = \frac{r \cos \varphi}{\sqrt{1 - kr^2}}, \quad Y^{(1)}_{r\varphi} = -\frac{r \sin \theta \cos \theta \sin \varphi}{\sqrt{1 - kr^2}}, \quad Y^{(1)}_{\theta \varphi} = r^2 \sqrt{1 - kr^2} \sin^2 \theta \sin \varphi \quad \text{(25)}
\]
\[
Y^{(2)}_{r\theta} = \frac{r \sin \varphi}{\sqrt{1 - kr^2}}, \quad Y^{(2)}_{r\varphi} = \frac{r \sin \theta \cos \theta \cos \varphi}{\sqrt{1 - kr^2}}, \quad Y^{(2)}_{\theta \varphi} = -r^2 \sqrt{1 - kr^2} \sin^2 \theta \cos \varphi \quad \text{(25)}
\]
\[
Y^{(3)}_{r\varphi} = \frac{r \sin^2 \theta}{\sqrt{1 - kr^2}}, \quad Y^{(3)}_{\theta \varphi} = r^2 \sqrt{1 - kr^2} \sin \theta \cos \theta \quad \text{(25)}
\]
\[
Y^{(4)}_{\theta \varphi} = r^3 \sin \theta. \quad \text{(25)}
\]

By taking the symmetric parts \cite{16} for solutions \(f^{(1)}, f^{(2)}\) and \(f^{(3)}\), we obtain the following three Killing tensors:
\[
K^{(1)}_{tt} = 1 \quad \text{(26)}
\]
\[
K^{(2)}_{rt} = \frac{1}{2} \frac{\cos(\theta)}{\sqrt{1 - kr^2}}, \quad K^{(1)}_{\theta \varphi} = -\frac{1}{2} r \sqrt{1 - kr^2} \sin(\theta) \quad \text{(27)}
\]
\[
K^{(3)}_{t\varphi} = \frac{1}{2} r^2 \sin^2(\theta) \quad \text{(28)}
\]

From \cite{13,22,23,24} we obtain six more Killing tensors
\[
K^{(4)}_{tt} = 1 \quad \text{(29)}
\]
\[
K^{(5)}_{rt} = \frac{1}{2} \frac{\cos(\theta)}{\sqrt{1 - kr^2}}, \quad K^{(5)}_{t\theta} = -\frac{1}{2} r \sqrt{1 - kr^2} \sin(\theta) \quad \text{(30)}
\]
We observe that the Killing tensors obtained from \( f^{(1)} \), \( f^{(2)} \) and \( f^{(3)} \) are reducible

\[
K^{(1)}_{\mu\nu} = \xi^{(7)}_\mu \xi^{(7)}_\nu
\]
\[
K^{(2)}_{\mu\nu} = \frac{1}{2} \left( \xi^{(1)}_\mu \xi^{(7)}_\nu + \xi^{(1)}_\nu \xi^{(7)}_\mu \right)
\]
\[
K^{(3)}_{\mu\nu} = \frac{1}{2} \left( \xi^{(6)}_\mu \xi^{(7)}_\nu + \xi^{(6)}_\nu \xi^{(7)}_\mu \right)
\]
\[
K^{(4)}_{\mu\nu} = \xi^{(7)}_\mu \xi^{(7)}_\nu
\]
\[
K^{(5)}_{\mu\nu} = \frac{1}{2} \left( \xi^{(1)}_\mu \xi^{(7)}_\nu + \xi^{(1)}_\nu \xi^{(7)}_\mu \right)
\]
\[
K^{(6)}_{\mu\nu} = \frac{1}{2} \left( \xi^{(6)}_\mu \xi^{(7)}_\nu + \xi^{(6)}_\nu \xi^{(7)}_\mu \right)
\]
\[
K^{(7)}_{\mu\nu} = \xi^{(1)}_\mu \xi^{(1)}_\nu
\]
\[
K^{(8)}_{\mu\nu} = \xi^{(6)}_\mu \xi^{(6)}_\nu
\]
\[
K^{(9)}_{\mu\nu} = \frac{1}{2} \left( \xi^{(1)}_\mu \xi^{(6)}_\nu + \xi^{(1)}_\nu \xi^{(6)}_\mu \right)
\]

as well as Killing tensors \([17, 18]\) constructed from Killing-Yano tensors \([25]\), where reducible means that it can be written as a linear combination of the metric and symmetrized products of Killing vectors, i.e.

\[
K_{\mu\nu} = a_0 g_{\mu\nu} + \sum_{I=1}^{N} \sum_{J=1}^{N} a_{IJ} \xi^{(I)}_\mu \xi^{(J)}_\nu
\]

where \( \xi^{(I)} \) for \( I = 1 \ldots N \) are the Killing vectors admitted by the manifold and \( a_0 \) and \( a_{IJ} \) for \( 1 \leq I \leq J \leq N \) are constants, the quadratic constant of motion associated with a reducible Killing tensor simply being a linear combination of existing first integrals.
The case investigated does not correspond to any of the special cases \cite{19} as admitting non-reducible Killing tensors of order two that exist in spherically symmetric static space-times.

5 Conclusions

In this paper a spinning particle model was investigated and was shown, that in a stationary case of Robertson-Walker space-time, there exists a class of new structures called \textit{f-symbols} which can generate reducible Killing tensors and supersymmetry algebras. As to our knowledge a curved space-time possessing such structures have not been exemplified until now. Further implications of \textit{f-symbols} in the context of Dirac-type operators \cite{10,11,12,13,14} and geometric duality \cite{20,21,22,23} are under investigations \cite{24}.

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