Representations of quivers and mixed graphs*

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Abstract

This is a survey article for Handbook of Linear Algebra, 2nd ed., Chapman & Hall/CRC, 2014. An informal introduction to representations of quivers and finite dimensional algebras from a linear algebraist’s point of view is given. The notion of quiver representations is extended to representations of mixed graphs, which permits one to study systems of linear mappings and bilinear or sesquilinear forms. The problem of classifying such systems is reduced to the problem of classifying systems of linear mappings.

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Introduction

In Sections 1–3, we give an informal introduction to quivers from a linear algebraist’s point of view. Exact definitions, results, and their proofs can be found in surveys [7, 30] and monographs [1, 2, 8, 16, 20, 27, 28].

After Gabriel’s article [15], in which the notions of a quiver and its representations were introduced, it became clear that a whole range of problems about systems of linear mappings can be formulated and studied in a uniform way. Quivers arise naturally in many areas of mathematics (representation

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theory, algebraic and differential geometry, number theory, Kac-Moody algebras, quantum groups, geometric invariant theory) and physics (string theory, supersymmetry, black holes, particle physics). Each finite dimensional algebra can be given by a quiver with relations, and representations of the algebra can be identified with representations of this quiver; that is, with finite systems of linear mappings satisfying some relations. Thus, the modern theory of representations of finite dimensional algebras can be considered as a branch of linear algebra.

In Sections 4 and 5, we extend the notion of quiver representations to representations of mixed graphs, which permits one to study systems of linear mappings and bilinear or sesquilinear forms. We reduce the problem of classifying such systems to the problem of classifying systems of linear mappings.

1 Systems of linear mappings as representations of quivers

DEFINITIONS. A quiver $Q$ is a directed graph where multiple loops and multiple arrows between two vertices are allowed. We suppose that the vertices of $Q$ are $1,\ldots,t$ and denote by $\alpha : i \rightarrow j$ an arrow $\alpha$ from a vertex $i$ to a vertex $j$.

A representation $A$ of $Q$ over a field $F$ is given by assigning to each vertex $i$ a finite dimensional vector space $A_i$ over $F$ and to each arrow $\alpha : i \rightarrow j$ a linear mapping $A_\alpha : A_i \rightarrow A_j$.

The dimension of $A$ is the vector $z = (\dim A_1, \ldots, \dim A_t)$.

A morphism $\varphi : A \rightarrow B$ between representations $A$ and $B$ of $Q$ is a family of linear mappings

$$\varphi_1 : A_1 \rightarrow B_1, \ldots, \varphi_t : A_t \rightarrow B_t$$

such that the diagram

$$\begin{array}{ccc}
A_i & \xrightarrow{A_\alpha} & A_j \\
\downarrow{\varphi_i} & & \downarrow{\varphi_j} \\
B_i & \xrightarrow{B_\alpha} & B_j
\end{array}$$
is commutative (i.e., $\varphi_j A_\alpha = B_\alpha \varphi_i$) for each arrow $\alpha : i \rightarrow j$.

An isomorphism $\varphi : A \rightarrow B$ is a morphism $\varphi : A \rightarrow B$ in which all $\varphi_i$ are bijections.

The direct sum $A \oplus B$ of representations $A$ and $B$ of $Q$ is the representation of $Q$ defined by

$$(A \oplus B)_i := A_i \oplus B_i, \quad (A \oplus B)_\alpha := A_\alpha \oplus B_\alpha$$

for all vertices $i$ and arrows $\alpha$. (The direct sum of linear mappings $A : U \rightarrow V$ and $A' : U' \rightarrow V'$ is the linear mapping $A \oplus A' : U \oplus U' \rightarrow V \oplus V'$ defined by $A \oplus A' : u + u' \mapsto Au + A'u'$ for all $u \in U$ and $u' \in U'$.)

A representation of nonzero dimension is indecomposable if it is not isomorphic to a direct sum of representations of smaller dimensions.

**FACT.** The Krull–Schmidt theorem [20, Corollary 2.4.2]: Each representation of a quiver is isomorphic to a direct sum of indecomposable representations. This direct sum is uniquely determined, up to permutation and isomorphisms of direct summands; that is, if

$$A_1 \oplus \cdots \oplus A_r \simeq B_1 \oplus \cdots \oplus B_s,$$

in which all $A_i$ and $B_j$ are indecomposable representations, then $r = s$ and all $A_i \simeq B_i$ after a suitable renumbering of $A_1, \ldots, A_r$.

**EXAMPLES.**

1. Each representation

$$
\begin{align*}
A_\alpha & \quad A_1 \quad A_2 \quad A_\gamma \\
A_\beta & \quad A_1 \quad A_3 \\
A_\gamma & \quad A_3 \quad A_\zeta
\end{align*}
$$

of the quiver

$$
\begin{align*}
\alpha & \quad 1 \quad 2 \quad 3 \\
\beta & \quad 2 \quad \zeta \\
\gamma & \quad \delta \quad \zeta
\end{align*}
$$

over a field $\mathbb{F}$ is a system of vector spaces $A_1, A_2, A_3$ over $\mathbb{F}$ and linear mappings $A_\alpha : A_1 \rightarrow A_1$, $A_\beta : A_1 \rightarrow A_2$, \ldots
2. Consider the problems of classifying representations of the quivers

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\circ & & \circ \\
1 & \rightarrow & 1 \\
\end{array}
\]

- Each matrix \( A \in \mathbb{F}^{m \times n} \) defines the representation \( \mathbb{F}^n \xrightarrow{A} \mathbb{F}^m \) of the quiver \( 1 \rightarrow 2 \) by assigning to its arrow the linear mapping \( x \mapsto Ax \) with \( x \in \mathbb{F}^n \). Thus, the problem of classifying representations of the quiver \( 1 \rightarrow 2 \) is the canonical form problem for matrices under equivalence transformations \( A \mapsto R^{-1}AS \) with nonsingular \( R \) and \( S \). Its canonical matrices are \( I \oplus 0 \), and so each representation is isomorphic to a direct sum, uniquely determined up to permutations of summands, of representations of the form

\[
\begin{array}{ccc}
\mathbb{F} & \xrightarrow{I} & \mathbb{F}, \\
0 & \xrightarrow{01} & \mathbb{F}, \\
\mathbb{F} & \xrightarrow{001} & 0
\end{array}
\]  

(1)

(it is agreed that \( \mathbb{F}^0 = 0 \) and there exist exactly one matrix \( 0_{n0} \) of size \( n \times 0 \) and exactly one matrix \( 0_{0n} \) of size \( 0 \times n \) for every nonnegative integer \( n \); they are the matrices of linear mappings \( 0 \rightarrow \mathbb{F}^n \) and \( \mathbb{F}^n \rightarrow 0 \)).

- The problem of classifying representations of the quiver \( 1 \circ \) is the canonical form problem for an \( m \times m \) matrix \( A \) over a field \( \mathbb{F} \) under similarity transformations \( S^{-1}AS \) with nonsingular \( S \in \mathbb{F}^{m \times m} \). Its canonical matrix is a direct sum of companion matrices

\[
C_n(q) = \begin{bmatrix}
0 & 0 & -c_n \\
1 & \ddots & \ddots \\
& \ddots & 0 & -c_2 \\
0 & 1 & \ddots & \ddots \\
0 & \ddots & \ddots & 0 & -c_1
\end{bmatrix}
\]  

(2)

whose characteristic polynomials

\[ q(x) = x^n + c_1x^{n-1} + \cdots + c_n \]

are powers of irreducible polynomials. This canonical matrix is called the elementary divisors rational canonical form of \( A \), or the Frobenius canonical form of \( A \). Thus, each representation of \( 1 \circ \) is isomorphic to a direct sum, uniquely determined up to permutation of summands, of representations of the form \( \mathbb{F}^n \circ C_n(q) \). If \( \mathbb{F} \) is an algebraically closed field, then a Jordan block \( J_n(\lambda) \) can be taken instead of \( C_n(q) \).
• The problem of classifying representations of the quiver $1 \rightarrow 2$ is the canonical form problem for pairs $(A, B)$ of matrices of the same size under equivalence transformations $(R^{-1}AS, R^{-1}AS)$ with nonsingular $R$ and $S$.

By Kronecker’s theorem on pencils of matrices, each representation of $1 \rightarrow 2$ is isomorphic to a direct sum, uniquely determined up to permutations of summands, of representations of the form

$$
\mathbb{F}^n \xrightarrow{I_n} \mathbb{F}^n, \quad \mathbb{F}^n \xrightarrow{J_n(0)} I_n \mathbb{F}^n, \quad \mathbb{F}^n \xrightarrow{L_n} R_n^{-1} \mathbb{F}^{n-1}, \quad \mathbb{F}^{n-1} \xrightarrow{L_n^T} R_n \mathbb{F}^n,
$$

in which $n = 1, 2, \ldots$.

$L_1 = R_1 = 0_{01}$, and $C_n(q)$ is a block $(2)$, which can be replaced by a Jordan block if $\mathbb{F}$ is algebraically closed.

• The problem of classifying representations of the quiver $1 \rightarrow 2$ is the canonical form problem for pairs $(A, B)$ of $p \times q$ and $q \times p$ matrices under contragredient equivalence transformations $(R^{-1}AS, S^{-1}AR)$ with nonsingular $R$ and $S$.

Dobrovol'skaya and Ponomarev [9] (see also [21]) proved that each representation of $1 \rightarrow 2$ is isomorphic to a direct sum, determined uniquely up to permutation of summands, of representations of the form

$$
\mathbb{F}^n \xrightarrow{I_n} \mathbb{F}^n, \quad \mathbb{F}^n \xrightarrow{J_n(0)} I_n \mathbb{F}^n, \quad \mathbb{F}^n \xrightarrow{L_n} R_n^{-1} \mathbb{F}^{n-1}, \quad \mathbb{F}^{n-1} \xrightarrow{L_n^T} R_n \mathbb{F}^n,
$$

in which $n = 1, 2, \ldots$, the matrices $L_n$ and $R_n$ are defined in (4), and $C_n(q)$ is a block $(2)$, which can be replaced by a Jordan block if $\mathbb{F}$ is algebraically closed.
2 Tame and wild quivers

The problem of classifying pairs of \( n \times n \) matrices up to similarity transformations
\[
(A, B) \mapsto (S^{-1}AS, S^{-1}BS)
\]
with nonsingular \( S \) (i.e., representations of the quiver \( \mathcal{G}_1 \)) plays a special role in the theory of quiver representations: it contains the problem of classifying representations of each quiver.

DEFINITIONS. A quiver is of **finite type** if it has only finitely many nonisomorphic indecomposable representations. A quiver is of **wild type** if the problem of classifying its representations contains the problem of classifying matrix pairs up to similarity, otherwise the quiver is of **tame type** (see formal definitions in [16, Section 14.10]).

The **Tits quadratic form** \( q_Q : \mathbb{Z}^t \to \mathbb{Z} \) of a quiver \( Q \) with vertices \( 1, \ldots, t \) is the form
\[
q_Q(x_1, \ldots, x_t) := x_1^2 + \cdots + x_t^2 - \sum_{i \to j} x_i x_j
\]
in which the sum is taken over all arrows of the quiver.

FACTS. 1. The problem of classifying pairs of commuting nilpotent matrices up to similarity contains the problem of classifying arbitrary matrix pairs up to similarity (see [18] and Example 1).

2. The problem of classifying matrix pairs up to similarity contains the problem of classifying representations of any quiver (see [18, 5] and Example 2).

3. **Gabriel’s theorem** [15]: Let \( Q \) be a connected quiver with \( t \) vertices.
   - \( Q \) is of finite type if and only if the Tits form \( q_Q \) (considered as a form over \( \mathbb{R} \)) is positive definite, if and only if \( Q \) can be obtained by directing edges in one of the Dynkin diagrams
   \[
   A_t \qquad D_t
   \]
   \[
   E_6 \qquad E_7 \qquad E_8
   \]
   (6)
Let $Q$ be of finite type and let $z = (z_1, \ldots, z_t)$ be an integer vector with nonnegative components. There exists an indecomposable representation of dimension $z$ if and only if $q_Q(z) = 1$; this representation is determined by $z$ uniquely up to isomorphism. (Representations of quivers of finite type were classified by Gabriel [15]; see also [20, Theorem 2.6.1].)

4. The Donovan–Freislich–Nazarova theorem [10, 26]: Let $Q$ be a connected quiver with $t$ vertices.

- $Q$ is of tame type if and only if the Tits form $q_Q$ is positive semidefinite, if and only if $Q$ can be obtained by directing edges in one of the Dynkin diagrams (6) or the extended Dynkin diagrams

\[
\begin{align*}
\tilde{A}_{t-1} & \quad \tilde{D}_{t-1} \\
\tilde{E}_6 & \quad \tilde{E}_7 \\
\tilde{E}_8 &
\end{align*}
\]

(the index plus one is the number of vertices).

- Let $Q$ be of tame type and let $z = (z_1, \ldots, z_t)$ be an integer vector with nonnegative components. There exists an indecomposable representation of dimension $z$ if and only if $q_Q(z) = 0$ or 1. (Representations of quivers of tame type were classified independently in [10] and [26].)

5. Kac’s theorem [24, 25]: Let $\mathbb{F}$ be an algebraically closed field. The set of dimensions of indecomposable representations of a quiver $Q$ with $t$ vertices over $\mathbb{F}$ coincides with the positive root system $\Delta_+ (Q)$ defined in [24]. The following holds for $z \in \Delta_+ (Q)$:

(a) $q_Q(z) \leq 1$.

(b) If $q_Q(z) = 1$, then all representations of dimension $z$ are isomorphic.
(c) If $q_Q(z) \leq 0$, then there are infinitely many nonisomorphic representations of dimension $z$ and the number of parameters of the set of indecomposable representations of dimension $z$ is

$$1 - q_Q(z) = \sum_{i \to j} z_i z_j - (z_1^2 + \cdots + z_t^2 - 1)$$ (see Example 3).

6. Belitskii’s algorithm [3, 4]: Let $\mathbb{F}$ be an algebraically closed field. Belitskii constructed an algorithm that transforms each pair $(A, B)$ of $n \times n$ matrices over $\mathbb{F}$ to a pair $(A_{can}, B_{can})$ that is similar to $(A, B)$ and is such that

$$(A, B) \text{ is similar to } (C, D) \iff (A_{can}, B_{can}) = (C_{can}, D_{can}).$$

The pair $(A_{can}, B_{can})$ is called Belitskii’s canonical form of $(A, B)$ under similarity. We can define Belitskii’s canonical pairs as those matrix pairs that are not changed by Belitskii’s algorithm, but we cannot expect to obtain an explicit description of them. Friedland [13] gave an alternative approach to the problem of classifying matrix pairs up to similarity.

7. The tame and wild theorem [33]: Belitskii’s algorithm was extended to a wide class of matrix problems that includes the problems of classifying representations of quivers and representations of finite dimensional algebras. For each matrix problem from this class over an algebraically closed field $\mathbb{F}$, denote by $\text{Bel}_{mn}$ the set of $m \times n$ indecomposable Belitskii canonical matrices and consider $\text{Bel}_{mn}$ as a subset in the affine space of $m \times n$ matrices $\mathbb{F}^{m \times n}$. Then

- either $\text{Bel}_{mn}$ consists of a finite number of points and straight lines for every $m \times n$ (then the matrix problem is of tame type),
- or $\text{Bel}_{mn}$ contains a 2-dimensional plane for a certain $m \times n$ (then the matrix problem is of wild type).

This statement is a geometric form of Drozd’s tame and wild theorem [11].
EXAMPLES. 1. Two pairs \((A, B)\) and \((A', B')\) of \(n \times n\) matrices are similar if and only if the pairs
\[
\begin{bmatrix}
0 & I_n & 0 & 0 \\
0 & 0 & I_n & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & A & 0 & I_n \\
0 & 0 & A & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & B & 0
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
0 & I_n & 0 & 0 \\
0 & 0 & I_n & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & A' & 0 & I_n \\
0 & 0 & A' & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & B' & 0
\end{bmatrix}
\]
of commuting nilpotent \(4n \times 4n\) matrices are similar. Thus, a solution of the problem of classifying pairs of commuting nilpotent matrices up to similarity would imply a solution of the problem of classifying pairs of arbitrary matrices up to similarity.

2. Two representations

\[
\begin{array}{cccc}
A & \mathbb{F}^p & B & \mathbb{F}^q \\
C & D & E & F
\end{array}
\quad\text{and}\quad
\begin{array}{cccc}
A' & \mathbb{F}'^p & B' & \mathbb{F}'^q \\
C' & D' & E' & F'
\end{array}
\]
are isomorphic over a field \(\mathbb{F}\) with at least 4 distinct elements \(\alpha, \beta, \gamma, \delta\) if and only if the pairs
\[
\begin{bmatrix}
\alpha I_p & 0 & 0 & 0 \\
0 & \beta I_q & 0 & 0 \\
0 & 0 & \gamma I_r & 0 \\
0 & 0 & 0 & \delta I_r
\end{bmatrix}
\begin{bmatrix}
A & 0 & 0 & 0 \\
B & 0 & 0 & 0 \\
C & 0 & 0 & 0 \\
D & E & I_r & F
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
\alpha I_{p'} & 0 & 0 & 0 \\
0 & \beta I_{q'} & 0 & 0 \\
0 & 0 & \gamma I_{r'} & 0 \\
0 & 0 & 0 & \delta I_{r'}
\end{bmatrix}
\begin{bmatrix}
A' & 0 & 0 & 0 \\
B' & 0 & 0 & 0 \\
C' & 0 & 0 & 0 \\
D' & E' & I_{r'} & F'
\end{bmatrix}
\]
are similar. This example can be extended to representations of any quiver over any field as in [12, Section 5].
3. The statement about the number of parameters in Fact 5(c) is intuitively clear: Let \( A \) be a representation of dimension \( z \). In some bases of \( A_1, \ldots, A_t \), let \( M_\alpha \) be the matrix of \( A_\alpha \) for an arrow \( \alpha : i \to j \). Let \( S_1, \ldots, S_t \) be the change of basis matrices. Then the \( \sum_{i \to j} z_iz_j \) entries of \( M_\alpha \)'s are reduced by \( z_1^2 + \cdots + z_t^2 \) entries of \( S_i \)'s. But really only \( z_1^2 + \cdots + z_t^2 - 1 \) independent parameters are used since multiplying all \( S_i \) by the same nonzero scalar does not change the transformation \( M_\alpha \mapsto S_j^{-1}M_\alpha S_i \) for all arrows \( \alpha : i \to j \).

### 3 Quivers of finite dimensional algebras

All representations of a finite dimensional algebra can be identified with all representations of some quiver with relations.

**DEFINITIONS.** A relation in a quiver \( Q \) over a field \( \mathbb{F} \) is a formal expression of the form

\[
\sum_{i=1}^{m} c_i \alpha_{i_1} \cdots \alpha_{i_2} A_{i_1} = 0, \quad 0 \neq c_i \in \mathbb{F},
\]

(7)

in which all

\[
u \xrightarrow{\alpha_{i_1}} u_{i_2} \xrightarrow{\alpha_{i_2}} \cdots \xrightarrow{\alpha_{i_p}} u_{i_p} \xrightarrow{\alpha_{i_1}} \nu,
\]

\( i = 1, \ldots, m, \)

are directed paths in \( Q \) with the same start vertex \( u \) and the same end vertex \( v \) (it is possible that \( u = v \)).

A representation \( A \) of \( Q \) satisfies the relation (7) if

\[
\sum_{i=1}^{m} c_i A_{\alpha_{i_1}} \cdots A_{\alpha_{i_2}} A_{i_1} = 0.
\]

(8)

If \( u = v \), then (7) may have a summand \( c_i \varepsilon_u \), in which \( \varepsilon_u \) is the path without arrows. This “lazy” path \( \varepsilon_u \) (to stand in place) is replaced in (8) by the identity operator on \( A_u \).

By a quiver with relations \((Q, L)\) we mean a quiver \( Q \) with a finite set \( L \) of relations in \( Q \). Its set of representations consists of all representations of \( Q \) that satisfy all relations from \( L \).
The **path algebra** $\mathbb{F}Q$ of a quiver $Q$ is a finite dimensional algebra over a field $\mathbb{F}$ whose elements are formal linear combinations

$$\sum_{i=1}^{m} c_i \alpha_{i_1} \cdots \alpha_{i_2} \alpha_{i_1},$$

in which $c_i \in \mathbb{F}$ and $\alpha_{i_1} \cdots \alpha_{i_2} \alpha_{i_1}$ are directed paths (they may be lazy paths and may have distinct start vertices and distinct end vertices). Their multiplication is determined by the distributive law and the rule:

$$(\beta_q \cdots \beta_1)(\alpha_p \cdots \alpha_1) =
\begin{cases}
\beta_q \cdots \beta_1 \alpha_p \cdots \alpha_1 & \text{if the end vertex of } \alpha_p \text{ is the start vertex of } \beta_1, \\
0 & \text{otherwise}.
\end{cases}$$

The multiplicative identity of the algebra $\mathbb{F}Q$ is the sum $\varepsilon_1 + \cdots + \varepsilon_t$ of all lazy paths taken over all vertices. If $(Q, L)$ is a quiver with relations, then its **path algebra** $\mathbb{F}(Q, L)$ is determined modulo these relations; that is, $\mathbb{F}(Q, L) := \mathbb{F}Q / \mathcal{L}$ in which $\mathcal{L}$ is the two-sided ideal of $\mathbb{F}Q$ generated by the left-hand sides of relations from $L$.

A **representation** of a finite dimensional algebra $\Lambda$ over $\mathbb{F}$ is a homomorphism $\varphi : \Lambda \to \text{End } V$ to the algebra $\text{End } V$ of linear operators on a vector space $V$ over $\mathbb{F}$.

An algebra over an algebraically closed field $\mathbb{F}$ is called a **basic algebra** if for some positive integer $m$ it is isomorphic to an algebra $\Lambda$ of upper triangular $m \times m$ matrices over $\mathbb{F}$ that satisfies the condition:

$$\begin{bmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & \ddots & \vdots \\
0 & \cdots & a_{mm}
\end{bmatrix} \in \Lambda \implies \begin{bmatrix}
a_{11} & 0 \\
\vdots & \ddots \\
0 & \cdots & a_{mm}
\end{bmatrix} \in \Lambda.$$

**FACTS.** 1. Each finite dimensional algebra $\Lambda$ over a field $\mathbb{F}$ is isomorphic to the path algebra $\mathbb{F}(Q, L)$ of a quiver with relations $(Q, L)$, which is constructed in \(\text{[I Chapter II]}\). We give a simplified construction of $(Q, L)$ in the following algorithm.
Algorithm 1: From an algebra $\Lambda$ to a quiver $(Q, L)$.

1. Decompose the unit of $\Lambda$ into a sum of orthogonal idempotents:
   \[ 1 = e_1 + \cdots + e_t, \quad e_i e_j = 0 \text{ if } i \neq j, \quad e_i^2 = e_i \neq 0. \]  
   (9)

2. Choose a set $a_1, \ldots, a_n$ of elements of $\Lambda$ such that $e_1, \ldots, e_t, a_1, \ldots, a_n$ generate $\Lambda$ and each $a_i$ is equal to $e_{q(i)}a_i e_{p(i)}$ for some $p(i)$ and $q(i)$ (such a set exists since if $b_1, b_2, \ldots$ generate $\Lambda$, then all $e_i b_j e_k$ also generate $\Lambda$).

3. Denote by $Q$ the quiver with vertices $1, \ldots, t$ and $n$ arrows $\alpha_i : p(i) \rightarrow q(i)$.

4. Denote by $\pi : FQ \rightarrow \Lambda$ the epimorphism of algebras such that $\pi(e_1) = e_1, \ldots, \pi(e_t) = e_t, \pi(a_1) = a_1, \ldots, \pi(a_n) = a_n$.

5. Construct a set $L$ of relations in $Q$ by choosing a finite subset of $\bigcup e_j \text{Ker}(\pi)$ that generates $\text{Ker}(\pi)$, expressing its elements through $e_1, \ldots, e_t, a_1, \ldots, a_n$, and equating them to zero.

Then $F(Q, L) \simeq FQ / \text{Ker}(\pi) \simeq \Lambda$.

2. In the following algorithm, we construct a canonical correspondence $\varphi \mapsto R$ between representations of $\Lambda$ and representations of the quiver $(Q, L)$ constructed by Algorithm 1 such that $\varphi$ and $\varphi'$ are isomorphic if and only if the corresponding representations $R$ and $R'$ are isomorphic.

Algorithm 2: From a representation $\varphi : \Lambda \rightarrow \text{End} V$ of $\Lambda$ to a representation $R$ of $(Q, L)$.

1. Since (9) holds for $\tau_i := \varphi(e_i) : V \rightarrow V$ instead of $e_i$, we have $V = \tau_1 V \oplus \cdots \oplus \tau_t V$. Put $R_i := \tau_i V$ for every vertex $i = 1, \ldots, t$.

2. For each $a_i$ from Step 2 of Algorithm 1, define $\rho_i := \varphi(a_i) : V \rightarrow V$. Since $\rho_i = \tau_{q(i)} \rho_i \tau_{p(i)}$, we have $\rho_i(\tau_{p(i)} V) \subseteq \tau_{q(i)} V$ and $\rho_i(\tau_k V) = 0$ if $k \neq p(i)$, and so each $\rho_i$ is fully determined by its restriction $\rho_i |_{\tau_{p(i)} V} : \tau_{p(i)} V \rightarrow \tau_{q(i)} V$. Put $R_{\alpha_i} := \rho_i |_{\tau_{p(i)} V}$ for every arrow $\alpha_i : p(i) \rightarrow q(i)$.
3. If the field $\mathbb{F}$ is algebraically closed, then it suffices to study representations of basic algebras since for each finite dimensional algebra over $\mathbb{F}$ there exists a basic algebra over $\mathbb{F}$ such that the categories of representations of these algebras are equivalent; see [11, Corollary I.6.10].

One usually applies Algorithms 1 and 2 to a basic algebra $\Lambda$ over $\mathbb{F}$, chooses $a_1, \ldots, a_n$ among its nilpotent elements, and takes the numbers $t$ and $n$ to be maximal and minimal, respectively.

**EXAMPLES.**

1. The path algebra $\mathbb{F}(Q, L)$ of the quiver with relation

$$
\begin{array}{ccc}
1 & \overset{\alpha}{\rightarrow} & 2 \\
\downarrow & & \downarrow \\
3 & \overset{\beta}{\rightarrow} & 4 \\
\downarrow & & \downarrow \\
\delta & \gamma \\
\end{array}
$$

has the basis

$\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha, \beta, \gamma, \delta, \beta\alpha$

over $\mathbb{F}$. The product of $\varepsilon_3 - \delta + \beta\alpha$ and $\varepsilon_1 + 2\gamma$ in $\mathbb{F}(Q, L)$ is

$$
\varepsilon_3\varepsilon_1 - \delta\varepsilon_1 + \beta\alpha\varepsilon_1 + 2\varepsilon_3\gamma - 2\delta\gamma + 2\beta\alpha\gamma = \beta\alpha + 2\gamma - 2\delta\gamma = -\beta\alpha + 2\gamma.
$$

Each representation $A$ of $(Q, L)$ defines a representation of $\mathbb{F}(Q, L)$ by operators on the space $A_1 \oplus A_2 \oplus A_3 \oplus A_4$.

2. The problem of classifying representations of the quiver with relations

$$
\begin{array}{ccc}
\alpha & \bigcap & 1 \\
\downarrow & & \downarrow \\
1 & \bigcap & \beta \\
\downarrow & & \downarrow \\
\delta & \gamma \\
\end{array}
$$

is the problem of classifying pairs of mutually annihilating linear operators, which was solved in [17] (see also [6]). Its path algebra is an infinite dimensional algebra whose elements are finite linear combinations of $\varepsilon_1, \alpha, \alpha^2, \alpha^3, \ldots, \beta, \beta^2, \beta^3, \ldots$ over $\mathbb{F}$.

3. Let us apply Algorithm 1 to the basic algebra

$$
\Lambda := \left\{ \begin{bmatrix} u & x & z \\ 0 & u & y \\ 0 & 0 & v \end{bmatrix} : u, v, x, y, z \in \mathbb{C} \right\}.
$$

Decompose its unit into a sum of orthogonal idempotents: $I_3 = e_1 + e_2$, in which

$$
e_1 := \text{diag}(1, 1, 0), \quad e_2 := \text{diag}(0, 0, 1).
$$
Write
\[ a_1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = e_1 a_1 e_1, \quad a_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = e_1 a_2 e_2. \]

The elements \( e_1, e_2, a_1, a_2 \) generate \( \Lambda \). We obtain the quiver with relations
\[ \alpha_1 \circlearrowleft 1 \overset{\alpha_2}{\rightarrow} 2 \quad \alpha_1^2 = 0 \quad (10) \]
whose path algebra is isomorphic to \( \Lambda \). Each representation of \( \Lambda \) is obtained from a representation of (10) and vice versa.

4 Systems of linear mappings and forms as representations of mixed graphs

By analogy with quiver representations, systems of linear mappings and forms can be considered as representations of mixed graphs, in which forms are assigned to undirected edges.

**Definitions.** Let \( F \) be a field with a fixed involution \( a \mapsto \overline{a} \); i.e., a bijection \( F \to F \) (which can be the identity) satisfying
\[ \overline{a + b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{a} \overline{b}, \quad \overline{\overline{a}} = a. \]

A mixed graph \( G \) is a graph in which loops and multiple edges are allowed and that may contain both directed and undirected edges; we suppose that the vertices are \( 1, \ldots, t \).

A representation \( \mathcal{A} \) of \( G \) over \( F \) is given by assigning to each vertex \( i \) a finite dimensional vector space \( A_i \) over \( F \), to each directed edge \( \alpha : i \rightarrow j \) a linear mapping \( A_\alpha : A_i \rightarrow A_j \), and to each undirected edge \( \lambda : i \overset{\sim}{\rightarrow} j \) (\( i \leq j \)) a sesquilinear form \( A_\lambda : A_j \times A_i \rightarrow F \); this form is bilinear if the fixed involution on \( F \) is the identity. We suppose that \( A_\lambda \) is linear on \( A_i \) and semilinear on \( A_j \).

An isomorphism \( \varphi : \mathcal{A} \cong \mathcal{B} \) between representations \( \mathcal{A} \) and \( \mathcal{B} \) of \( G \) is a family of linear bijections
\[ \varphi_1 : A_1 \rightarrow B_1, \ldots, \varphi_t : A_t \rightarrow B_t. \]
such that
\[ \varphi_j A_\alpha = B_\alpha \varphi_i \quad \text{for each directed edge } \alpha : i \to j \]
and
\[ A_\lambda(y, x) = B_\lambda(\varphi_j y, \varphi_i x) \quad \text{for each undirected edge } \lambda : i \to j. \]

The notions of the \textit{dimension} of a representation, \textit{direct sum}, and \textit{finite, tame, and wild types} are defined for mixed graphs in the same way as for quivers. The \textbf{Tits form} is defined as in (5), but the sum is also taken over all undirected edges \( i \to j \) (\( i \leq j \)).

\textbf{FACTS.} 1. The Krull–Schmidt, Gabriel, and Donovan–Freislich–Nazarova theorems (the Fact in Section 1 and Facts 3 and 4 in Section 2) remain true if we replace the word “quiver” by “mixed graph”; see [32].

2. A \textit{generalization of Kac’s theorem} (from Fact 5 of Section 2): Let \( \mathbb{F} \) be an algebraically closed field of characteristic not 2. The set of dimensions of indecomposable representations of a mixed graph \( G \) over \( \mathbb{F} \) coincides with the positive root system \( \Delta_+(G) \) (its definition in [24] does not depend on the orientation of edges).

\textbf{EXAMPLE.} Each representation
\[
\begin{array}{c}
A_\lambda \xrightarrow{A_1} A_\alpha \\
A_\mu \xrightarrow{A_2} A_\beta \xrightarrow{A_3} A_\gamma
\end{array}
\]
of
\[
\begin{array}{c}
\mu \xrightarrow{2} \xrightarrow{1} \xrightarrow{3} \xrightarrow{\gamma}
\end{array}
\]
is a system of vector spaces \( A_1, A_2, A_3 \) over \( \mathbb{F} \), linear mappings \( A_\alpha, A_\beta, A_\gamma \), and sesquilinear forms
\[ A_\lambda : A_2 \times A_1 \to \mathbb{F}, \quad A_\mu : A_2 \times A_2 \to \mathbb{F}, \quad A_\nu : A_3 \times A_2 \to \mathbb{F} \]
(these forms are bilinear if the fixed involution on \( \mathbb{F} \) is the identity).
5 Generalization of the law of inertia to representations of mixed graphs

The problem of classifying systems of forms and linear mappings over $\mathbb{C}$ and $\mathbb{R}$ is reduced to the problem of classifying systems of linear mappings.

DEFINITIONS. Let $V$ be a finite dimensional vector space over a field $F$ with a fixed involution (which can be the identity). By the *dual space* of $V$, we mean the space $V^\ast$ of all mappings $\varphi : V \to F$ that are semilinear, i.e.,

$$\varphi(au + bv) = \bar{a}\varphi(u) + \bar{b}\varphi(v)$$

for all $u, v \in V$ and $a, b \in F$.

For each linear mapping $A : U \to V$, we define the *adjoint mapping*

$$A^\ast : V^\ast \to U^\ast$$

by putting

$$A^\ast \varphi := \varphi A \quad \text{for all } \varphi \in V^\ast.$$

For each mixed graph $G$, we denote by $G$ the quiver that is obtained from $G$ by replacing

- each vertex $i$ of $G$ by the vertices $i$ and $i^\ast$,
- each arrow $\alpha : i \to j$ by the arrows

$$\alpha : i \to j, \quad \alpha^\ast : j^\ast \to i^\ast,$$

- each undirected edge $\lambda : i \to j$ ($i \leq j$) by the arrows

$$\lambda : i \to j^\ast, \quad \lambda^\ast : j \to i^\ast.$$

We consider $G$ as a quiver with involution on the set of vertices and on the set of arrows.

For each representation $\mathcal{A}$ over $F$ of a mixed graph $G$, we denote by $\mathcal{A}$ the representation of $G$ that is obtained from $\mathcal{A}$ by replacing (see Example 3)

- each vector space $A_i$ by the mutually *dual spaces $A_i$ and $A_i^\ast$,
- each linear mapping $A_{\alpha} : A_i \to A_j$ by the mutually *adjoint mappings

$$A_{\alpha} : A_i \to A_j, \quad A_{\alpha}^\ast : A_j^\ast \to A_i^\ast.$$
each sesquilinear form $A_\lambda : A_j \times A_i \to F$ by the mutually *adjoint mappings

$$A_\lambda : u \in A_i \mapsto A_\lambda(?, u) \in A_j^*, \quad A_\lambda^* : v \in A_j \mapsto A_\lambda(v, ?) \in A_i^*.$$

For each representation $\mathcal{M}$ of $\mathcal{G}$, we define the adjoint representation $\mathcal{M}^\circ$ of $\mathcal{G}$ that is formed by the vector spaces $M_v^\circ := M_v^*$ for all vertices $v$ of $\mathcal{G}$ and the linear mappings $M_\tau^\circ := M_\tau^*$ for all arrows $\tau$ of $\mathcal{G}$ (see Example 1).

A representation $\mathcal{M}$ of $\mathcal{G}$ is selfadjoint if $\mathcal{M}^\circ = \mathcal{M}$.

A mixed graph with relations $(G, L)$ is a mixed graph $G$ with a finite set $L$ of relations in $\mathcal{G}$. By representations of $(G, L)$ we mean those representations $A$ of $G$ for which $A$ satisfies $L$.

For each relation

$$\sum_{i=1}^m c_i \tau_{i_p} \cdots \tau_{i_2} \tau_{i_1} = 0 \quad \text{in } \mathcal{G} \quad \text{(see (7))},$$

we define the adjoint relation

$$\sum_{i=1}^m \bar{c}_i \tau_{i_1}^* \tau_{i_2}^* \cdots \tau_{i_p}^* = 0 \quad \text{in } \mathcal{G}.$$

For each set $L$ of relations in $\mathcal{G}$, we denote by $L^*$ the set of relations that are adjoint to the relations from $L$.

For each representation $A$ of $G$, we denote by $A^-$ the representation of $G$ that is obtained from $A$ by replacing all its forms $A_\lambda$ by $-A_\lambda$.

FACT. In the following algorithm, the problem of classifying representations of a mixed graph with relations $(G, L)$ over $\mathbb{C}$ and $\mathbb{R}$ is reduced to the problem of classifying representations of the quiver with relations $(\mathcal{G}, L \cup L^*)$. The algorithm is a special case of the method [32] (see also [29, 23, 31, 35]) for reducing the problem of classifying representations of a mixed graph $(G, L)$ over a field or skew field $F$ of characteristics not 2 to the problem of classifying representations of the quiver $(\mathcal{G}, L \cup L^*)$ over $F$ and the problem of classifying Hermitian and symmetric forms over fields and skew fields that are finite extensions of the center of $F$. 
Algorithm 3: Classification of representations of a mixed graph with relations \((G, L)\).

1. Construct a set \(\text{ind}(G, L \cup L^*)\) of indecomposable representations of \((G, L \cup L^*)\) such that every indecomposable representation of \((G, L \cup L^*)\) is isomorphic to exactly one representation from \(\text{ind}(G, L \cup L^*)\).

2. Improve \(\text{ind}(G, L \cup L^*)\) such that
   - if \(M \in \text{ind}(G, L \cup L^*)\) is isomorphic to a selfadjoint representation, then \(M\) is selfadjoint,
   - if \(M \in \text{ind}(G, L \cup L^*)\) is not isomorphic to \(M^\circ\), then \(M^\circ \in \text{ind}(G, L \cup L^*)\).

Then every representation of \((G, L)\) over \(F\) is isomorphic to a direct sum, uniquely determined up to permutation of summands, of representations of the types

\[
\begin{align*}
A, B & \quad \text{if } F = \mathbb{C} \text{ with the identity involution,} \\
A, B, B^- & \quad \text{if } F = \mathbb{C} \text{ with complex conjugation,} \\
A, B, \text{ and also } B^- & \quad \text{if } F = \mathbb{R},
\end{align*}
\]

in which \(A = M \oplus M^\circ\) for each unordered pair \(\{M, M^\circ\}\) such that \(M^\circ \neq M \in \text{ind}(G, L \cup L^*)\) and \(B \in \text{ind}(G, L \cup L^*)\).

Thus, each system of linear mappings and bilinear forms over \(\mathbb{C}\) or \(\mathbb{R}\) and each system of linear mappings and sesquilinear forms over \(\mathbb{C}\) are decomposed into direct sums of indecomposables uniquely, up to isomorphisms of summands. This is the Krull–Schmidt theorem for representations of mixed graphs; see Fact 1 in Section 4.

**EXAMPLES.** 1. The problems of classifying representations over a field \(F\) of the mixed graphs with relations

- \(1 \leq \lambda\)
- \(\lambda \leq 1 \leq \mu\) \(\lambda = \varepsilon \lambda^*, \quad \mu = \delta \mu^*\)
- \(\alpha \leq 1 \leq \lambda\) \(\lambda = \varepsilon \lambda^*\) is nonsingular, \(\alpha^* \lambda = \lambda \alpha\)
- \(\alpha \leq 1 \leq \lambda\) \(\lambda = \varepsilon \lambda^* = \alpha^* \lambda \alpha\) is nonsingular

(in which \(\varepsilon, \delta \in \{-1, 1\}\)) are the problems of classifying
• sesquilinear (bilinear if the involution on $F$ is the identity) forms,
• pairs of $\varepsilon$, $\delta$-Hermitian (symmetric, or skew-symmetric) forms,
• triples $(V, H, A)$, in which $V$ is a vector space, $H$ is a nonsingular $\varepsilon$-Hermitian (symmetric, or skew-symmetric) form on $V$, and $A$ is a linear operator on $V$ that is $H$-selfadjoint, i.e.,
$$H(Ax, y) = H(x, Ay) \quad \text{for all } x, y \in V.$$
• triples $(V, H, A)$, in which $V$ is a vector space, $H$ is a nonsingular $\varepsilon$-Hermitian (symmetric, or skew-symmetric) form on $V$, and $A$ is a linear operator on $V$ that is $H$-unitary, i.e.,
$$H(Ax, Ay) = H(x, y) \quad \text{for all } x, y \in V.
$$
Canonical matrices for these problems are given in [32] over any field $F$ of characteristic not 2 up to classification of Hermitian and symmetric forms over finite extensions of $F$; they are based on the elementary divisors rational canonical form (see Example 2 in Section 1). Simpler canonical matrices over $\mathbb{C}$ and $\mathbb{R}$ that are based on the Jordan canonical form are given in [14, 19, 22, 23, 35, 36].

2. The problem of classifying representations of a mixed graph $(G, L)$ is hopeless if the quiver $(G, L \cup L^*)$ is of wild type. For example, the problem of classifying triples of Hermitian forms and the problem of classifying normal operators on a complex space with scalar product given by a nonsingular Hermitian form are hopeless.

3. If
$$A: \begin{array}{c}
A_1 \\
\downarrow \alpha
\end{array} \begin{array}{c}
A_\lambda \\
\downarrow \lambda
\end{array} \begin{array}{c}
A_\mu \\
\downarrow \mu
\end{array} \begin{array}{c}
A_2
\end{array}
$$
is a representation of the mixed graph
$$G: \begin{array}{c}
1 \\
\downarrow 1
\end{array} \begin{array}{c}
\alpha \\
\downarrow \lambda
\end{array} \begin{array}{c}
\lambda^* \\
\downarrow \mu^*
\end{array} \begin{array}{c}
\mu
\end{array}$$
then
$$A: \begin{array}{c}
A_1 \\
\downarrow \alpha
\end{array} \begin{array}{c}
A_\lambda \\
\downarrow \lambda
\end{array} \begin{array}{c}
A_\mu \\
\downarrow \mu
\end{array} \begin{array}{c}
A_2
\end{array} \begin{array}{c}
A_1^* \\
\downarrow \alpha^*
\end{array} \begin{array}{c}
A_\lambda^* \\
\downarrow \lambda^*
\end{array} \begin{array}{c}
A_\mu^* \\
\downarrow \mu^*
\end{array} \begin{array}{c}
A_2^*
\end{array}$$
and
$$G: \begin{array}{c}
1 \\
\downarrow 1
\end{array} \begin{array}{c}
\alpha \\
\downarrow \lambda
\end{array} \begin{array}{c}
\lambda^* \\
\downarrow \mu^*
\end{array} \begin{array}{c}
\mu
\end{array}$$

(11)

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4. For each representation $\mathcal{M}$ of the quiver $G$ in (11), the adjoint representation $\mathcal{M}^\circ$ is constructed as follows:

\begin{equation}
\mathcal{M} : \begin{array}{c}
U_1 \\
\downarrow A_1
\end{array} \begin{array}{c}
B_1 \\
\downarrow C_1 \\
V_1
\end{array} \begin{array}{c}
U_2 \\
\downarrow A_2
\end{array} \begin{array}{c}
B_2 \\
\downarrow C_2 \\
V_2
\end{array} \begin{array}{c}
M \\
\downarrow \circ
\end{array} \begin{array}{c}
U_1 \\
\downarrow A_1
\end{array} \begin{array}{c}
B_1 \\
\downarrow C_1 \\
V_1
\end{array} \begin{array}{c}
U_2 \\
\downarrow A_2
\end{array} \begin{array}{c}
B_2 \\
\downarrow C_2 \\
V_2
\end{array} \begin{array}{c}
M^\circ \\
\downarrow \circ
\end{array} \begin{array}{c}
U_1 \\
\downarrow A_1
\end{array} \begin{array}{c}
B_1 \\
\downarrow C_1 \\
V_1
\end{array} \begin{array}{c}
U_2 \\
\downarrow A_2
\end{array} \begin{array}{c}
B_2 \\
\downarrow C_2 \\
V_2
\end{array}
\end{equation}

5. Applying Algorithm 3 to Hermitian or symmetric forms gives the law of inertia. Indeed, these forms are representations of the mixed graph with relations $(G, L) : 1 = \lambda = \lambda^*$. Its quiver

\begin{equation}
(G, L \cup L^*) : \begin{array}{c}
1 \\
\downarrow \lambda \\
\uparrow \lambda^*
\end{array}
\end{equation}

By (1), $\text{ind}(G, L \cup L^*)$ consists of 3 representations:

\begin{equation}
\mathcal{M} : \begin{array}{c}
0 \\
\downarrow 0
\end{array} \begin{array}{c}
F
\end{array} , \quad \mathcal{M}^\circ : \begin{array}{c}
0 \\
\downarrow 0
\end{array} \begin{array}{c}
F
\end{array} , \quad \mathcal{N} = \mathcal{N}^\circ : \begin{array}{c}
1 \\
\downarrow 1
\end{array} \begin{array}{c}
F
\end{array}
\end{equation}

Since $\mathcal{M} \oplus \mathcal{M}^\circ = A$ for $A : \mathbb{C}[0]$ and $\mathcal{N} = B$ for $B : \mathbb{C}[1]$, Algorithm 3 ensures that each representation of $1 = \lambda = \lambda^*$ is isomorphic to a direct sum, uniquely determined up to permutation of summands, of representations of the form:

- $\mathbb{C}[0], \mathbb{C}[1]$ if $F = \mathbb{C}$ with the identity involution,
- $\mathbb{C}[0], \mathbb{C}[1], \mathbb{C}[-1]$ if $F = \mathbb{C}$ with complex conjugation or $\mathbb{R}$.

6. (See details in [22, 32].) Applying Algorithm 3 to sesquilinear or bilinear forms over $\mathbb{C}$ gives their canonical forms from [22]. Indeed, these forms are representations of the graph $G : 1 = \lambda$. Each representation $\mathcal{M} : \begin{array}{c}
U \\
\downarrow B
\end{array} \begin{array}{c}
V
\end{array}$ of the quiver $G : 1 = \lambda$ defines the representations

\begin{equation}
\mathcal{M}^\circ : V^* \begin{array}{c}
B^*
\end{array} \begin{array}{c}
U^*
\end{array} , \quad \mathcal{M} \oplus \mathcal{M}^\circ : \begin{array}{c}
U \oplus V
\end{array} \begin{array}{c}
\begin{bmatrix}
0 & B^* \\
A & 0
\end{bmatrix}
\end{array} \begin{array}{c}
U \oplus V
\end{array} \begin{array}{c}
\begin{bmatrix}
0 & A^* \\
B & 0
\end{bmatrix}
\end{array} \begin{array}{c}
U \oplus V
\end{array}
\end{equation}

of $G$.
(we interchanged the summands in $V \oplus U^*$ of $\mathcal{M} \oplus \mathcal{M}^*$ to make it selfadjoint); $\mathcal{M} \oplus \mathcal{M}^*$ corresponds to the representation

$$\mathcal{M}^* : U \oplus V^* \cong \begin{bmatrix} 0 & B^* \\ A & 0 \end{bmatrix}$$

of $G$.

By (3), there is a set $\text{ind}(G)$ consisting of the representations

$$\mathcal{M}_n(\lambda) : \mathbb{C}^n \xrightarrow{J_n(\lambda)} \mathbb{C}^n \quad (\lambda \neq 0)$$

and pairs of mutually adjoint representations

$$\mathbb{C}^n \xrightarrow{J_n(0)} \mathbb{C}^n \quad \text{and} \quad \mathbb{C}^n \xrightarrow{I_n} \mathbb{C}^n; \quad \mathbb{C}^n \xrightarrow{L_n} \mathbb{C}^{n-1} \quad \text{and} \quad \mathbb{C}^{n-1} \xrightarrow{R_n^T} \mathbb{C}^n \quad (12)$$

(For each matrix $M$, $M^*$ is $M^*$ if the fixed involution on $\mathbb{C}$ is complex conjugation, or $M^T$ if the involution is the identity.) $\mathcal{M}_n(\lambda)$ is isomorphic to $\mathcal{M}_n(\mu)^*$ if and only if $J_n(\mu)$ is similar to $J_n(\lambda)^{-1}$. $\mathcal{M}_n(\lambda)$ is isomorphic to a selfadjoint representation $\mathbb{C}^n \xrightarrow{A} \mathbb{C}^n$ if and only if $A^{-1}A$ is similar to $J_n(\lambda)$. If such $A$ exists, we fix one and denote it by $\sqrt{J_n(\lambda)}$.

By Fact 1, each representation of $G$ over $\mathbb{C}$ is isomorphic to a direct sum of representations of the following forms:

- $\mathcal{M}_n(\lambda)^* : \mathbb{C}^n \simeq \begin{bmatrix} 0 & I_n \\ J_n(\lambda) & 0 \end{bmatrix}$ if does not exist $\sqrt{J_n(\lambda)}$, in which $J_n(\lambda)$ is determined up to replacement by $J_n(\mu)$ that is similar to $J_n(\lambda)^{-1}$;

- $\mathbb{C}^n \xrightarrow{\varepsilon \sqrt{J_n(\lambda)}}$, in which $\varepsilon = \pm 1$ if the involution on $\mathbb{C}$ is complex conjugation and $\varepsilon = 1$ if the involution is the identity;

- $\mathbb{C}^m \simeq J_m(0)$, which is isomorphic to $\mathbb{C}^n \simeq \begin{bmatrix} 0 & I_n \\ J_n(0) & 0 \end{bmatrix}$ if $m = 2n$ or $\mathbb{C}^n \simeq \begin{bmatrix} 0 & R_n^T \\ L_n & 0 \end{bmatrix}$ if $m = 2n - 1$ (these forms are obtained from (12)).

The matrix $\sqrt{J_n(\lambda)}$ exists if and only if $|\lambda| = 1$ when the involution on $\mathbb{C}$ is complex conjugation, and $\lambda = (-1)^{n+1}$ when the involution on $\mathbb{C}$ is the identity. Respectively, one can take

$$\sqrt{J_n(\lambda)} = \sqrt{\lambda} \Delta_n, \quad \sqrt{J_n((-1)^{n+1})} = \Gamma_n,$$
where
\[
\Gamma_n := \begin{bmatrix}
0 & 1 \\
1 & 1 \\
-1 & -1 \\
1 & 1
\end{bmatrix}
\quad \text{and} \quad
\Delta_n := \begin{bmatrix}
0 & 1 \\
1 & i \\
1 & 0
\end{bmatrix}
\quad \text{(n-by-n)}.
\]

We obtain the following canonical forms of a square complex matrix \( A \):

- \( A \) is congruent to a direct sum of matrices of the form
  \[
  \begin{bmatrix}
  0 & I_n \\
  J_n(\lambda) & 0
  \end{bmatrix}, \quad \mu \Delta_n, \quad J_n(0),
  \]
  in which \(|\lambda| > 1\) and \(|\mu| = 1\);

- \( A \) is congruent to a direct sum of matrices of the form
  \[
  \begin{bmatrix}
  0 & I_n \\
  J_n(\lambda) & 0
  \end{bmatrix}, \quad \Gamma_n, \quad J_n(0),
  \]
  in which \(0 \neq \lambda \neq (-1)^{n+1}\) and \(\lambda\) is determined up to replacement by \(\lambda^{-1}\).

These direct sums are uniquely determined by \( A \), up to permutation of summands.

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