Assisted Learning and Imitation Privacy

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Abstract

Motivated by the emerging needs of decentralized learners with personalized learning objectives, we present an Assisted Learning framework where a service provider Bob assists a learner Alice with supervised learning tasks without transmitting Bob’s private algorithm or data. Bob assists Alice either by building a predictive model using Alice’s labels, or by improving Alice’s private learning through iterative communications where only relevant statistics are transmitted. The proposed learning framework is naturally suitable for distributed, personalized, and privacy-aware scenarios. For example, it is shown in some scenarios that two suboptimal learners could achieve much better performance through Assisted Learning. Moreover, motivated by privacy concerns in Assisted Learning, we present a new notion of privacy to quantify the privacy leakage at learning level instead of data level. This new privacy, named imitation privacy, is particularly suitable for a market of statistical learners each holding private learning algorithms as well as data.

I. INTRODUCTION

Rapid developments in communications, networking, robotics, genomics, novel materials, and powerful computation platforms are rapidly bringing data-generating people, processes and devices together. We are in an era where there exists an emerging number of statistical learners, each holding personalized data and domain-specific objective goals. Each learner often needs to integrate data from diverse sources or coordinate with other learners in order to facilitate personalized learning objectives. With a decentralized set of learners, an urgent related issue is to protect learners’ privacy with respect to data as well as sophisticated algorithms (or models). The interactions between multiple learners in privacy-aware scenarios motivate us to consider a learning framework where each learner can be assisted by others without transmitting sensitive information, and a related notion of privacy.

From the perspective of Machine-Learning-as-a-Service (MLaaS) [1], [2], existing assistance between learners is mainly in the following scenario. A service provider receives predictor-label pairs $(x, y)$ from data curators and then learn a private supervised model from such data. The service provider then provides prediction services for future data of the data curator or possibly other users, who send future predictor $\tilde{x}$ of a similar nature to inquire a prediction of the corresponding label $\tilde{y}$. Consider a set of learners who collect relevant features from a common population of interest, e.g., a group of patients, a cohort of mobile users, a basket of financial assets, etc. An excelling learner may provide services that does not transmit data but still convey information relevant to others’ learning objectives.

Suppose that there exist two learners Alice and Bob who collect various features from the same group of people. Alice wants to develop a new product, and she has the labels of interest from each person. Bob holds unique features which Alice does not, and those features may be relevant to Alice’s prediction goal. Bob intends to assist Alice, but Bob will not disclose data even if they are reasonably perturbed. Then, a natural way of Bob assisting Alice is to simply receive her labels, collate them with his own private data, and learn a supervised algorithm privately. Bob then provides prediction services for Alice who inquires with future data, for example in the form of an application programming interface (API). Moreover, suppose that Alice also has a private learning algorithm and private data features that can be (partially) collated to Bob’s. Is it possible to still benefit from the algorithm as well as data held by Bob? A classical approach is to for Alice to perform model selection from her own model and Bob’s private model (through Bob’s API), and then decide whether to use Bob’s service in the future. A related approach is to perform statistical model averaging over the two learners. However, neither approach will significantly outperform the better one of Alice and Bob [3], [4]. The above approaches, however, do not fully utilize all the available data which is a union of Alice’s and Bob’s. Is it possible for Alice to achieve the performance as if all the private information of Alice and Bob were centralized? This motivates us to propose the concept of Assisted Learning, where the main idea is to treat labels $y$ as public (to transmit) and predictors $x$ as private (not to transmit). And
TABLE I: Examples of Bob assisting Alice (none of whom will transmit personalized models or data). The * indicates an example where a statistician is received with all the cases but only a part of the predictors.

| Alice                  | Research group A | Pharmacy | Mobile device | Investor | Experimental scientist |
|------------------------|------------------|----------|---------------|----------|------------------------|
| Bob                    | Research group B | Hospital | Cloud service | Financial trader | Statistician         |
| Collating Index        | Data index       | Patient ID | User ID       | Time stamp | Case number*           |

we show that for a suitably chosen \( y \) at each iteration of communications, Alice may benefit from Bob as if she had Bob’s data. Some common scenarios of this nature are showed in Tab. I.

In correspondence to the privacy concerns in assisted learning, we also develop a new notion called imitation privacy. The proposed notion of privacy is motivated by a reasonable concern of Bob that his capability to generate algorithms during assisting other is to be imitated, when an adversary Alice keeps querying. Such an imitation, if accurate or near-accurate, could cause a considerable damage to Bob when his core competitive advantage is his black-box learning procedure that includes not only data but also sophisticated algorithms being deployed. This black-box level privacy is different from data level privacy since its focus is on protecting a learner’s capability to generate predictive models, instead of the data itself. In other words, Bob’s assisted learning service is the object to protect. For instance, in many domains such as financial trading and environmental prediction, data may be easily accessible by many learners but what really matters is an effective algorithm being deployed. We will show examples where imitation privacy and data privacy (in particular differential privacy [5]–[7]) do not imply each other.

Our main contributions in this work are three folds: (i) We introduce the notion of Assisted Learning that is naturally suitable for a variety of learning scenarios. (ii) Based on assisted learning, we develop some concrete protocols so that a service provider can assist others by improving their predictive performances. We show that the proposed learning protocol can be applied for a wide range of nonlinear and nonparametric learning tasks, where near-oracle performance can be achieved. Some preliminary results on the oracle performance are developed. (iii) We propose a concept of privacy that focuses on the protection of service-providing modules (or data-model pairs), and discuss it through concrete examples.

II. ASSISTED LEARNING

A. Notation

Throughout the paper, we let \( X \in \mathcal{X}^{n \times p} \) denote a general data matrix which consists of \( n \) items and \( p \) features, and \( y \in \mathcal{Y}^{n} \) be a vector of labels (or responses), where \( \mathcal{X}, \mathcal{Y} \subseteq \mathbb{R} \). Let \( x_i \) denotes the \( i \)th row of \( X \). A supervised function \( f \) approximates \( x_i \mapsto \mathbb{E}(y_i \mid x_i) \) for a pair of predictor (or feature) \( x_i \in \mathbb{X}^p \) and \( y_i \in \mathcal{Y} \). Let \( f(X) \) denote an \( \mathbb{R}^n \) vector whose \( i \)th element is \( f(x_i) \). We say two matrices or column vectors \( A, B \) are collated if rows of \( A \) and \( B \) are aligned with some common index. For example, the index can be date or time stamps for datasets of time series, or personal identification number for datasets of mobile users. Let \( \mathcal{N}(\mu, \sigma^2) \) denote the Gaussian distribution with mean \( \mu \) and standard deviation \( \sigma \).

B. Supervised Learning with Personalized Services

We first depict how we envision Assisted Learning through a concrete usage scenario. Alice is equipped with a set of labelled data \( (X_A, Y_A) \) and a supervised learning algorithm for the data. Some other researchers, say Bob, may be performing different learning tasks with distinct data \( (X_B, Y_B) \) and learning models, where \( X_A \) and \( X_B \) can be (partially) collated. Alice wishes to be assisted by Bob to facilitate her own learning, while maintaining both of their sensitive information. On the other hand, Alice would also be glad to assist others for potential rewards. A set of learning modules such as Alice constitute a statistical learning market where each module can either provide or receive assistance to facilitate personalized learning goals. Figures 1 illustrates Assisted Learning from a user’s perspective and a service provider’s perspective.

C. General Description of Assisted Learning

We first introduce our notions of algorithm and module in the context of supervised learning.
Definition 1 (Algorithm). A learning algorithm $A$ is a mapping from a dataset $X \in \mathbb{R}^{n \times p}$ and label vector $y \in \mathbb{R}^n$ to a prediction function $f_{A,X,y} : \mathbb{R}^p \rightarrow \mathbb{R}$.

An algorithm may represent linear regression, ensemble method, neural networks, or a set of models from which a suitable one is chosen using model selection techniques [4], [8]. For example, when the least squares method is used to learn the supervised relation between $X$ and $y$, then $f_{A,X,y}$ is a linear operator: $\hat{x} \mapsto \hat{x}^\top (X^\top X)^{-1} X^\top y$ for a predictor $\hat{x} \in \mathbb{R}^p$. The above $f_{A,X,y}$ is also called a hypothesis in some literature of classification.

Definition 2 (Module). A module $M = (A,X)$ is a pair of algorithm $A$ and observed dataset $X$. For a given label vector $y \in \mathbb{R}^n$, a module naturally induces a prediction function $f_{A,X,y}$. We simply write $f_{A,X,y}$ as $f_{M,y}$ whenever there is no ambiguity.

Recall from Subsection II-A that the above $X$ is assumed to be in $\mathbb{R}^{n \times p}$, representing $n$ items and $p$ features. In the context of assisted learning, $M = (A,X)$ is treated as private and $y$ is public. If $y$ is from a benign user Alice, it represents a particular task of interest. The prediction function $f_{M,y} : \mathbb{R}^p \rightarrow Y$ is thus regarded as a particular model learned by $M$ (Bob), driven by $y$, in order to provide assistance. Typically $f_{M,y}$ is also treated as private.

Definition 3 (Assisted Learning System). An assisted learning system consists of a module $M$, a learning protocol, a prediction protocol, and the following two-stage procedure.

- In stage I (‘learning protocol’), module $M$ receives a user’s query of a label vector $y \in \mathcal{Y}^n$ that is collated with the rows of $X$; a prediction function $f_{M,y}$ is produced and privately stored; the fitted value $f_{M,y}(X) = [f_{M,y}(x_1), \ldots, f_{M,y}(x_n)]^\top$ is sent to the user.
- In stage II (‘prediction protocol’), module $M$ receives a query of future predictor $\hat{x}$; its corresponding prediction $\hat{y} = f_{M,y}(\hat{x})$ is calculated and returned to the user.

In the above Stage I, the fitted value, $f_{M,y}(X)$, returned from the service module (Bob) upon an inquiry of $y$, is supposed to inform the user module (Alice) of the training error so that Alice can take subsequent actions. Bob’s actual predictive performance is reflected in the Stage II. The querying user in Stage II may or may not be the same user as in stage I.

Related work. In many recent MLaaS interfaces in industry, e.g., Google AI Cloud and Microsoft Azure, a service provider Bob helps a user Alice to construct learning models based on the labeled data uploaded by Alice. Bob holds the trained model private and provides a query interface for Alice to perform future predictions. This procedure is similar to our Stage II, except that Alice’s data (including both labels and predictors) have to be held by Bob. In a decentralized scenario where the trustworthiness or learning capability of Bob is doubtful, Alice tends not to release private data. Assisted learning is more suitable in such scenarios.

A recent advancement in decentralized learning is Federated Learning [9]–[11], where the central server sends the current global model to a set of selected clients, and then each client updates the model parameter with local
data and returns the updates back to the central server. The main idea of federated learning is to learn from machine learning models based on data that are distributed among participants (e.g., mobile devices) to avoid direct sharing of data, while all participants share the same model. In contrast, assisted learning is designed to allow diverse learning goals and models, and each participant can be either a user or service provider without the need of a central coordinator.

Another related homomorphic encryption framework is the Secure Multi-party Computation [12], [13]. The main idea is that any function can be computed securely, equivalently, it address that no players can learn anything more than its prescribed output. Several works [14], [15] under this framework studied machine learning on vertically partitioned data [16]–[18], which share certain similar feature with assisted learning. Secure Multi-party Computation naturally relies on the external service provider. This is different from assisted learning where each module is both service provider and learner. In addition, assisted learning is perhaps more suitable for large-scale network and complex dataset.

D. A Specific Learning Scenario: Iterative Assistance

Consider a situation where the goal is to improve the learning quality of a given learner, say Alice, by allowing its module, \( \mathcal{M}_a = (A_a, X_A) \) to exchange statistics with other \( m \) modules \( \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m \). First, we note that for Alice to receive assistance from other modules, their data \( (D_1, D_2, \ldots, D_m) \) and \( D_0 = X_A \) should be aligned or partially aligned (as defined in section II-A). We also consider a general setting where module \( k \) has features \( \{X_i, i \in S_j\}, j = 0, 1, \ldots, m \), and the feature sets \( S_k \) are overlapping, partially overlapping, or non-overlapping. Without any privacy constraint, it is natural to consider the following oracle performance as the limit of learning. Such limit has been widely adopted in classical statistical learning theory (see e.g. [19] and the references therein). Let \( \ell \) denote some loss function (e.g. squared loss for regression).

**Definition 4 (Oracle Performance).** The oracle performance of module \( \mathcal{M}_a \) is defined by

\[
\min_{\hat{A}_j} \mathbb{E} \{ \ell(y^*, \hat{A}_j(x^*)) \}
\]

where \( \hat{A}_j \) is the trained \( A_j \) using all the pulled data \( \bigcup_{j=0}^m D_j \).

In other words, it is the optimal out-sample loss produced from the candidate methods and the pulled data of all modules (including Alice itself). The above quantity provides a theoretical limit or benchmark on what assisted learning can bring to Alice.

Suppose that Alice not only has a specific learning goal (labels), but also has private predictors and algorithm, how could Alice benefit from other modules/learners through the two stages of assisted learning? We address this by developing a specific user scenario of assisted learning. For Alice to receive assistance from other modules, their data should be at least partially collated. For brevity, we assume that the data of all the modules can be collated by developing a specific user scenario of assisted learning. For Alice to receive assistance from other modules, their data should be at least partially collated. For brevity, we assume that the data of all the modules can be collated (defined in Subsection II-A) using public indices. Procedure 1 outlines a realization of assisted learning between Alice and other modules. In the **training stage** (Stage I), at each round \( k \), Alice first sends a query to each module \( \mathcal{M}_j \) by transmitting its latest statistics \( e_{j,k} \); upon receipt of the query, if module \( k \) agrees, it treats \( e_{j,k} \) as labels and fit a model \( \hat{A}_{j,k} \) (based on the data aligned with such labels); module \( j \) then fits residual \( \tilde{e}_{j,k} \) and sends it back to module \( 0 \). Alice processes the collected responses \( \tilde{e}_{j,k}, \ldots, (j = 1, \ldots, m) \), and initializes the \( k+1 \) round of communications. After the above procedure stops at an appropriate stopping time \( k = K \), the training stage for Alice is suspended. In the **prediction stage** (Stage II), upon arrival of a new feature vector \( x^* \), user 0 queries the prediction results \( \hat{A}_{j,k}(x^*) (k = 1, 2, \ldots, K) \) from module \( j \), and combine them to form the final prediction \( \tilde{y}^* \).

In Procedure 2, we consider regression methods and two modules for brevity. The key idea is to allow each module to only transmit fitted residuals to the other module, iterating until the learning loss is reasonably small. In particular, in the **training stage** (Stage I), Alice fits the data using her algorithm \( A_a \) and sends the fitted residuals \( e_1 \) to Bob. Then Bob treats them as labels and fits \( e_1 \) using his algorithm \( \hat{A}_b \), and sends the fitted residuals \( \tilde{e}_1 \) back to Alice. Alice then initializes the second round of communication by treating \( \tilde{e}_1 \) as the new labels. Such a procedure repeats \( K \) times until the out-sample error (measured by, e.g., cross-validation) of Alice no longer decreases. In the **prediction stage** (Stage II), for any new predictor, Alice queries the corresponding prediction from Bob using his trained models from the first stage, and forms the final prediction by suitably aggregating all the predictors.
Our theoretical analysis and numerical experiments will be based on this simplified Procedure 2. For theoretically analysis, it is straightforward to extend this result to the general Procedure 1. Our numerical experiments show that this result holds for a wide range of models including nonlinear models such decision tree method and ensemble methods as well. In the experimental section, an interesting observation is also presented, which is a tradeoff between communication bandwidth, cost constraints, and computational overhead, Alice may only select a subset of \( m \) modules.

- Multi-Armed Bandit & Online Learning. In deploying a practical assisted learning protocol, due to communication bandwidth, cost constraints, and computational overhead, Alice may only select a subset of \( m \) modules. It could be helpful to cast the module selection as a multi-armed bandit problem or expert learning problem.
- Adversarial Learning. In a network of modules, some of them may be malicious and propagate misleading results to Alice under assistance. Since our ultimate goal in Assisted Learning is to reduce bias and variance, it is crucial to detect adversarial modules.
- Design of efficient feedback information. In our preliminary Procedure 2, the final prediction is aggregated from each module at each communication round. While this achieves the oracle performance in many cases (see Section IV), it is not clear whether such design is the most efficient.
- Optimal stopping criterion. When to stop is a key ingredient part in assisted learning. On one hand, more communications typically to bring more information exchange and better fitting to the data. On the other
Theorem 1. Suppose that Alice and Bob use linear regression models. Then for any label $y$, Alice will achieve the oracle performance for a sufficiently large number of communications $k$ in Procedure 2.

Proof: In the appendix. It is straightforward to extend this result to the general assisted learning protocol as outlined in Procedure 1. The above result is applicable to linear models and additive regression models [20] on a linear basis, e.g., spline, wavelet, or polynomial basis. Its proof is included in the supplementary material. The proof actually implies that the prediction loss decays exponentially with the number of communications. The result also indicates that if the true data generating model is consistent nonparametric algorithms [21], [22] are used. Moreover, suppose that $\mathbb{E}(y \mid x)$ cannot be written as $f_a(x_a) + f_b(x_b)$ but the interactive terms (such as $x_a \cdot x_b$ if both are scalars) involve categorical variables or continuous variables that can be well-approximated by quantizers. The Assisted Learning procedure could be modified so that Alice sends stratified dataset to Bob which involves only additive regression functions. An illustrating example is $\mathbb{E}(y \mid x) = \beta_a x_a + \beta_b x_b + \beta_{a,1} x_{a,1} x_{b,1}$ where $x_{a,1} \in \{0, 1\}$, and Alice sends data $\{x_a : x_{a,1} = 0\}$ and $\{x_a : x_{a,1} = 1\}$ separately to Bob. In Section IV, we will show by experiments that the oracle performance can be well-approximated in general with a sufficiently number of communications.

E. Another Learning Scenario: Learning with Neural Networks in Assisted Learning

The past decades have witnessed the explosion of neural network techniques, and the extraordinary performances of deep neural network have made it the standard tool for several machine learning tasks. Therefore, in this section, we give the solution for neural networks in the context of assisted learning. The general setting will be the same as described in Section II-D. For simplicity, we consider the learning protocol of Alice and Bob with a two-layer neural network. Their data are fully aligned and have non-overlapping features. The full batch data is used to train the neural network, i.e., one iteration equals one epoch, and back propagation algorithm is used to update the weights in the nets. Let $w_{a,k}$ and $w_{b,k}$ be weights from input to hidden layer at the $k$th iteration for Alice and Bob respectively. Denote all the rest weights in the neural network at the $k$th iteration by $\tilde{w}_k$.

The assisted learning protocol is depicted in Fig. 2 and is summarized in Procedure 3. Intuitively speaking, the weights for neural nets in assisted learning will be update in a coordinate-wise fashion, i.e., each module will and can only update those weights corresponding to her/his own data. This is because the data can never be shared between different modules in assisted learning for privacy concern. In the training stage (Stage I), at the $k$th iteration, Alice first calculates $w_{a,k}^T X_A$, then inquires Bob’s $w_{b,k}^T X_B$, and jointly uses the information to feed the neural network. If $k$ is even, then Alice will update her current weight $w_{a,k}$ to $w_{a,k+1}$ and $\tilde{w}_k$ to $\tilde{w}_{k+1}$, while Bob will fix his weight for the next iteration, i.e. $w_{b,k+1} = w_{b,k}$. If $k$ is odd, Alice will fix her weight for next iteration, i.e. $w_{a,k+1} = w_{a,k}$ and updates $\tilde{w}_k$ to $\tilde{w}_{k+1}$. Then she sends $\tilde{w}_k$ and other related information to Bob. Bob will use the information to update his current weight $\tilde{w}_k$ to $\tilde{w}_{k+1}$. Such a procedure repeats $K$ times until the out-sample error (measured by, e.g., cross-validation) of Alice no longer decreases. In the prediction stage (Stage II), for any new predictor, Alice queries the corresponding $w_{b,K} x_i^T (i \in S_o)$ from Bob, and uses the trained neural network to get the assisted learning prediction. In Section IV-A3, we experimentally verify that the oracle score can achieved via this learning procedure on various kinds of datasets. Also, note that such kind of learning protocol can be easily applied to deep neural network with multiple modules/learners.

III. PRIVACY AT MODULE LEVEL: ‘IMITATION GAME’

A. Imitation Privacy

Concerns centering data privacy have led to more stringent regulations on the use of data in machine learning [23]. It has also raised various research interests in designing machine learning architectures that facilitate privacy as well as accuracy. A popular data-level privacy is the differential privacy [5]–[7] and its many variations including,
Fig. 2: A simple illustration of updating scheme for Alice (a) and Bob (b) with two-layer neural network (back propagation algorithm is used) in assisted learning. Similar strategy can be directly applied to deep neural network with multiple participants/modules. Red solid line represents the weight being updated at current iteration. Black dashed line means that this weight is fixed at current iteration.

**Procedure 3** Assisted learning of Module ‘Alice’ (‘a’) using Module ‘Bob’ (‘b’) for neural network

**Input:** Module Alice, its initial label $y \in \mathbb{R}^n$, initial weight $w_{a,1}$ and $\tilde{w}_1$ for the neural network, assisting module Bob, (optional) new predictors $\{x_i^*, i \in S\}$  

**Initialisation:** round $k = 1$  

1: repeat  
2: Alice calculates $w_{a,k}^T X_A$ and receives Bob’s $w_{b,k}^T X_B$ to train the neural network  
3: if $k$ is odd then  
4: Alice updates $w_{a,k}, \tilde{w}_k$ by using back-propagation to get $w_{a,k+1}, \tilde{w}_{k+1}$  
5: Bob sets $w_{b,k+1} \leftarrow w_{b,k}$  
6: else (k is even)  
7: Alice sets $w_{a,k+1} \leftarrow w_{a,k}$ and updates $\tilde{w}_k$ by using back-propagation to get $\tilde{w}_{k+1}$  
8: Bob updates $w_{b,k}$ by using back-propagation to get $w_{b,k+1}$  
9: end if  
10: Alice initializes the $k+1$ round  
11: until Stop criterion satisfied  
12: On arrival of a new data $\{x_i^*, i \in S\}$, Alice queries $w_{b,K} x_i^*$, ($x_i^*, i \in S_B$) from Bob  
13: Alice also calculates the $w_{a,K} x_i^*$ ($x_i^*, i \in S_A$) and feed them to the neural network to get the final prediction $\tilde{y}^*$.  

**Output:** The Assisted Learning prediction $\tilde{y}^*$

local differential privacy [24], [25], concentrated differential privacy [26], Rényi differential privacy [27], and information-theoretic differential privacy [28], [29].

While data privacy is important in assisted learning, we also need a notion of privacy to protect the service provider is worried about potential leakage of his capability to generate algorithms during assisting others’ learning. A competitor Alice may attempt to imitate the learning service provided by Bob, through consecutively querying or other side information. Such an imitation, if successful, will cause an undesirable leakage of Bob’s black-box learning capacity. In the rest of this section, we let $M = (A_b, X_B)$ denote the module of Bob and $f_M, y$ the learned model from label $y$. Let $\mathcal{I}$ denote any side information available to Alice, and $f_{\mathcal{I}, y}$ the learned model of Alice using $\mathcal{I}$ and $y$. Mathematically, $\mathcal{I}$ could be treated as filtrations associated with an appropriate probability space.
Definition 5 (Imitation Privacy). The imitation privacy for an imitation \( \mathcal{I} \) and module \( M \) is
\[
\rho_{\mathcal{I},M} = \mathbb{E}_{y \sim p_Y} \frac{\mathbb{E}_{\tilde{x} \sim p_X} |f_{\mathcal{I},y}(\tilde{x}) - f_{M,y}(\tilde{x})|^2}{\mathbb{E}_{\tilde{x} \sim p_X} |f_{M,y}(\tilde{x})|^2},
\]
where \( p_Y \) and \( p_X \) denote the distribution of query label \( y \) and (unobserved) future data \( \tilde{x} \), respectively.

In the above definition, the denominator is to remove the unit of \( y \). The definition is for regression scenarios, and it extensions for classification are left as future work. Without loss of generality, we assume \( \mathbb{E}_{\tilde{x} \sim p_X} |f_{M,y}(\tilde{x})|^2 = 1 \) in the rest of the paper.

Interpretations. For a given \( M \), smaller \( \rho_{\mathcal{I},M} \) means a closer imitation and less privacy. The minimal value \( \rho_{\mathcal{I},M} = 0 \) is achieved at \( f_{\mathcal{I},y} = f_{M,y} \) almost everywhere for every \( y \in \mathcal{Y} \), meaning that Alice performs as well as Bob and there is ‘0’ privacy for Bob. This can be clearly achieved when, for example, Alice holds both data and algorithm of Bob. The privacy value is typically greater than zero when Alice only holds side information such as a part of \( X_B \), a transformation of \( X_B \), or some other data that we will demonstrate in the sequel. On the other hand, the value of \( \rho_{\mathcal{I},M} \) is typically no larger than 1, since a trivial imitation \( f_{\mathcal{I},y}(x) = 0 \) for all \( x \) leads to \( \rho_{\mathcal{I},M} = 1 \). As a result, it is expected that \( \rho_{\mathcal{I},M} \in [0, 1] \) and it is paramount to keep a large \( \rho_{\mathcal{I},M} \) for the benefit of Bob.

In the definition of \( \rho_{\mathcal{I},M} \), the closeness of \( f_{\mathcal{I},y} \) and \( f_{M,y} \) is evaluated on unobserved data (through \( \mathbb{E} \)). To enable easier computation, the privacy may be approximated by the training data \( X = \{x_1, \ldots, x_n\} \) if \( x_i \)'s are assumed to be i.i.d. generated. In other words, \( \mathbb{E} \) may be replaced with \( \mathbb{E}_n \), where \( \mathbb{E}_n f(X) = n^{-1} \sum_{i=1}^n f(x_i) \) for any measurable function \( f \).

The notion of imitation privacy may be extended in the following way. For two constants \( \varepsilon, \delta \in [0, 1] \), the module \( M \) is said to be \((\varepsilon, \delta)\)-private with respect to \( I \) if with probability at most \( \delta \), \( \rho_{\mathcal{I},M} \leq \varepsilon \). The module \( M \) is said to be \((\varepsilon, \delta)\)-private with respect to a class of imitations \( \mathcal{I} \), if \( \inf_{\mathcal{I} \in \mathcal{I}} \rho_{\mathcal{I},M} \leq \varepsilon \) with probability at most \( \delta \). The probability is due to possible randomizations of \( \mathcal{I} \) or \( M \).

Example 1 (Algorithm Leakage). Consider a scenario where Bob’s algorithm (Definition 1) is not available to Alice, but Bob’s full data \( X_B \) and the fitted response \( f_{M,y}(X_B) \) are available to Alice. Suppose that there is a small fraction of data that is mismatched or overly noisy, and Bob uses a robust learning algorithm to circumvent those outliers in \( X_B \) and learn an accurate model. Suppose that Alice holds a rudimentary algorithm that is sensitive to outliers. As a consequence of observing \( f_{M,y}(X_B) \), Alice would be able to identify outliers as those with significant gaps between \( f_{M,y}(X_B) \) and \( y \). In this case, Bob’s learning capability (in handling outliers) is implicitly leaked even if Bob’s algorithm is not transmitted.

Example 2 (Data Leakage). Consider a scenario where Bob’s data \( X_B \) is not available to Alice, but his learning algorithm and fitted response \( f_{M,y}(X_B) \) are available to Alice. A learning algorithm will demonstrate unique information regarding the dataset, e.g. column space revealed by linear regression and data structure implicitly shown from decision Tree. In some cases, Alice will be able to reverse-engineer some key statistics of Bob’s data or even precise values of data, e.g. in Example 3 and 4 of the next section.
B. Imitation Privacy in Assisted Learning

In the context of an assisted learning system, how do we interpret and measure the privacy for the service module Bob? Suppose that a user module Alice has no prior information before contacting Bob in the Assisted Learning. The only way to gain information from Bob (or “hack the system”) is through queries at either Stage I or Stage II (in Definition 3). Such information is quantified below.

Definition 6 (Query Set). A query set $Q$ from an Assisted Learning system consists of ordered quadruplets $\{(y_\ell, \hat{y}_\ell, \bar{X}_\ell, \mu), \ell = 1, \ldots, k\}$, where

1) $y_\ell$ is the $\ell$th query sent to the system in Stage I,
2) $\hat{y}_\ell$ is the fitted value returned by the system in Stage I,
3) $\bar{X}_\ell \in \mathbb{R}^{n_x \times p}$ consists of $n_\ell$ queries (by row) sent to the system during in II,
4) $\mu$ consists of predictions returned by the system in Stage II (that corresponds to $\bar{X}_\ell$).

The above $k$ is the number of Stage I queries, and $n_1, \ldots, n_k$ are the numbers of Stage II queries.

There are two components of a query set. The first component is concerned with the queries at Stage II that aim to hack $f_{M,y}(\cdot)$ for a particular $y$. The second component is to query Stage I in order to hack the internal functionality of $M$ itself. The positive integer $k$ in our context is interpreted as the communication complexity between modules. We will show by examples that a joint query to both Stage I and II is necessary to successfully imitate Bob.

If Alice is not a benign user, the above query set enables her to possess useful knowledge from the service module Bob. Simply speaking, Alice aims to provide assistance to other users whatever Bob could provide, as if Alice had the algorithm and data that Bob privately holds. This is formulated by the following system in parallel to a regular assisted learning system.

Definition 7 (Imitation System). An imitation $I$ is a pair of query set $Q$ and hacking algorithm $H$ that maps $Q$ and any label vector $y$ to a prediction function $f_{I,y} : \mathbb{R}^p \rightarrow \mathbb{R}$ (which has the similar functionality as $f_{M,y}$).

An imitation system $S'$ consists of an imitation $I$ from an assisted learning system $S$, a learning protocol, a prediction protocol, and the two-stage procedure introduced in Definition 3, except that the prediction function, written as $f_{I,y}$, is produced from $I$ instead of $M$.

An illustration of the above concepts are included in Fig. 3. We now give some examples to concretely demonstrate the idea of Imitation Privacy. Technical details of these examples are included in the supplement.

Example 3 (Linear regression imitation privacy). Suppose that in an imitation, the number of Stage I queries is $k_1$, and the numbers of Stage II queries are $n_1 = \cdots = n_k = k_2$. Suppose that Bob employs a linear regression model and Alice knows about it. By querying $k_1 \geq \min\{p, n - p\}$ random label vectors $y_\ell (\ell = 1, \ldots, k_1)$ from Stage I, Alice can obtain the column space $\text{span}(X_B)$ of Bob’s data $X_B$ with probability one. Additionally, if Alice also knows about the true covariance matrix of Bob’s features, Alice is able to develop an imitation system with $\rho_{I,M} = o(n^{-1})$, where $n$ is the data size (or the number of rows in $X_B$). Note that Bob’s data are never transmitted.

Solution. Suppose that in an imitation, the number of Stage I queries is $k_1$. Suppose that Bob employs a linear regression model and Alice knows that Bob is using a linear regression. By querying $k_1 \geq \min\{p, n - p\}$ random label vectors $y_\ell (\ell = 1, \ldots, k_1)$ from Stage I, Alice can obtain the column space $\text{span}(X_B)$ of Bob’s data $X_B$ with probability one. This is because in each fitting process, Bob will project the random query $y_i$ onto $\text{span}(X_B)$, i.e. $y_i \mapsto P_{X_B}y_i$. Hence, with $k_1 = p$ fitted values in the form of $P_{X_B}y_i, i = 1, 2, \ldots, k_1$, Alice is able to uniquely identify the column space of $X_B$ with probability 1. On the other hand, with $n - p$ queries, Alice will identify the orthogonal space of $\text{span}(X_B)$, which further implies $\text{span}(X_B)$.

However, Alice cannot obtain the Bob’s data $X_B$ exactly without further side-information. One such side information is the true covariance matrix of Bob. Alice is able to develop an imitation system with $\rho_{I,M} = o_p(1)$ as Bob’s data size $n \rightarrow \infty$. To see this, suppose without loss of generality that the underlying covariance of $X_B$ is an identity matrix. Let Alice arbitrarily pick up a matrix $\tilde{X} \in \mathbb{R}^{n_x \times p}$ whose column space is $\text{span}(X_B)$. We only need to find such a $Q$ that $\tilde{X} = X_BQ$. For each label $Y_t, t = 1, 2, \ldots, p$ sent in Stage I, Alice can calculate the empirical covariance between $Y$ and $X_B$, say $K_t \in \mathbb{R}^{p \times 1}$. By the law of large numbers,
$K_t \to_p \text{cov}(Q X_B, Y_t) = Q \text{cov}(X_B, Y_t)$ as $n \to \infty$. If each query $Y_t$ is in the form: $Y_t = X_B \beta_t + \eta_t$, with some fixed $\beta_t \in \mathbb{R}^{p \times 1}$, then Alice could solve $Q$ by letting

$$[K_1 \ K_2 \ldots \ K_p] = Q [\beta_1 \ \beta_2 \ldots \ \beta_p]$$

As long as $\beta$ is linearly independent, Alice obtains a unique $\hat{Q}_n$ that converges in probability to the true $Q$. Consequently, Alice would use $\hat{X} \hat{Q}_n^{-1}$ as if it were $X_B$ to provide assistance, with an $o_p(1)$ imitation privacy.

**Example 4 (Decision tree imitation privacy).** Suppose that Bob uses a decision tree with width 2 and depth at least $p$, with $p$ being the number of Bob’s features. Then with $k_1 \geq n$ and $k_2 = \infty$, there exists an imitation such that $\rho_{I,M} = 0$ with high probability.

**Solution.** Suppose that Bob uses a decision tree with width 2 and depth at least $p$, with $p$ being the number of Bob’s features. Then with $k_1 \geq n$ and $k_2 = \infty$, there exists an imitation such that $\rho_{I,M} = o_p(1)$ as $n \to \infty$. In fact, in the $i$th Stage I query, Alice sends label $y_i$ such that its $i$th entry is sufficiently large and all other entries are zero ($i = 1, \ldots, n$) to identify the structure of Bob’s data. From the infinite Stage II queries corresponding to the $i$th Stage I query, Alice is able to reconstruct the tree built by Bob, which puts a mass at the finest neighborhood of $x_i$. Finally, Alice is able to reconstruct Bob’s data up to a precision that goes to 0 as $n$ grows. Therefore, with $k_1 \geq n$ and $k_2 = \infty$, by using the strategy described above, Alice can create an imitation system such that $\rho_{I,M} = o_p(1)$.

In fact, there exist multiple ways to obtain the data structure of Bob. Next, we demonstrate another way that does not even need Stage II queries. Consider a one-dimensional case where Bob’s data is $X_B^n = [7, 1, 10, 5, 18, 9]$, and a decision tree with width 2 is employed. We use $x_j$ to denote the $j$th entry in $X_B^n$. Alice can obtain the structure of Bob’s data by sending queries $e_i$, for $i = 1, 2, \ldots, 6$ to Bob in Stage I only, where $e_i$ is the standard basis for $\mathbb{R}^6$, and observing the fitted values $o_i$ for $i = 1, 2, \ldots, 6$.

| Input    | $\Delta$ | Fitted Value |
|----------|----------|--------------|
| $e_1 = [1,0,0,0,0,0]$ | $o_1 = [\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0]$ |
| $e_2 = [0,1,0,0,0,0]$ | $o_2 = [0, 0, 0, 0, 0, 0]$ |
| $e_3 = [0,0,1,0,0,0]$ | $o_3 = [0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0]$ |
| $e_4 = [0,0,0,1,0,0]$ | $o_4 = [0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0]$ |
| $e_5 = [0,0,0,0,1,0]$ | $o_5 = [0, 0, 0, 0, 0, 0]$ |
| $e_6 = [0,0,0,0,0,1]$ | $o_6 = [0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0]$ |

From $o_2$ and $o_5$, Alice knows that $x_2, x_5$ are beginning/ending points, and without loss of generality, we assume $x_2 < x_5$. From $o_4$ and $o_1$, Alice knows $x_4$ must lie between $x_1$ and $x_2$. Similarly, the order of $x_6, x_3, x_5$ can be inferred from $o_3, o_6$. Therefore, Alice has successfully recovered the structure of Bob’s data, i.e., $[x_2, x_4, x_1, x_6, x_3, x_5] = [1, 5, 7, 9, 10, 18]$. Note that without the information from Stage II, Alice can never know the exact value or even ranges of Bob’s data. In fact, it is straightforward to apply such kind of strategy on complex dataset.

**Example 5 (Restricted outcome imitation privacy).** Suppose that $y$ is generated from $y = f(x) + \eta$, where $\eta \sim \mathcal{N}(0, \sigma^2)$ and $f$ is randomly generated from a compact space $\mathcal{F}$ with a suitable probability measure $F$ and metric $L_2(P_X)$. The distribution of $y$ conditional on Bob’s data $X, p_Y|X$, is Gaussian with mean $\int_{\mathcal{F}} f(X) dF$ and variance $\sigma^2$. Then there exists an imitation $\mathcal{I}$ with $k_1 = \exp\{H_{d,\epsilon}(\mathcal{F})\}$ and $k_2 = \infty$, such that $\rho_{I,M} \leq \epsilon$ with high probability, where $H_{d,\epsilon}(\mathcal{F})$ denotes the Kolmogorov $\epsilon$-entropy of $\mathcal{F}$.

**Solution.** Suppose that $y$ is generated from $y = f(x) + \eta$, where $\eta \sim \mathcal{N}(0, \sigma^2)$ and $f$ is randomly generated from a compact space $\mathcal{F}$ with a suitable probability measure $F$ and metric $L_2(P_X)$. The marginal distribution of $y$ conditional on Bob’s data $X, p_Y|X$, is thus Gaussian with mean $\int_{\mathcal{F}} f(X) dF$ and variance $\sigma^2$. Then there exists an imitation $\mathcal{I}$ with $k_1 = \exp\{H_{d,\epsilon}(\mathcal{F})\}$ and $k_2 = \infty$, such that $\rho_{I,M} \leq \epsilon$ with high probability, where $H_{d,\epsilon}(\mathcal{F})$ denotes the Kolmogorov $\epsilon$-entropy of $\mathcal{F}$, namely the logarithm of the smallest number of $\epsilon$-covers of $\mathcal{F}$. In fact, Alice may be able to develop the following imitation system.

Let $f_1, \ldots, f_k$, with $k = \exp\{H_{d,\epsilon}(\mathcal{F})\}$ denote the $\epsilon$-quantizations of the function space $\mathcal{F}$. Suppose that $k_1$ queries are constructed in such a way that $y_j$ is generated from $f_j$, namely $y_j = f_j(x) + \eta_j$, $j = 1, \ldots, k_1$. For any future query sent to Alice, say $y_* = f_*(x) + \eta_*$, Alice can search from the dictionary of $y_j$, $j = 1, \ldots, k_1$, and
find the $j$ that minimizes $\|y_j - y_*\|$. Since $n^{-1}\|y_j - y_*\|^2 = 2\sigma^2 + \|f_j - f_*\|_{L_2(P_x)} + o_p(1)$ as $n \to \infty$ (assuming independent noises), Alice would obtain such $j$ that $f_j$ is $\varepsilon$-away from $f_*$. Consequently, when Alice uses the Stage II queries corresponding to $y_j$ to assist others, she obtains an imitation privacy of $\rho_{T,M} \leq \varepsilon$ with high probability (for large $n$).

**Related work.** A related concept is Optimal Experimental Design or Active Learning [30], [31], where the focus is to study economical collection of labeled data to train a learning algorithm with comparable accuracy. Model Extraction [32], [33], the process of reconstructing machine learning models through prediction APIs, is similar to the hacking process in Stage II. In model extraction, the goal is to extract specific model trained on given label. While in assisted learning, we aim to acquire the ability of mimicking the functionality of module, which shall hold for arbitrary label.

Knowledge distillation [34], [35] utilizes information from complex model (teacher network) to train a smaller one (student network) with comparable accuracy. The student network will be trained on ‘soft target’, which is teacher network’s output. This is similar to the idea that ‘key statistics’ transmitted so as to improving learning performance in assisted learning.

Another closely related concept is differential privacy. The main goal of differential privacy is to secure the privacy of data. In the context of imitation privacy, the focus is the to secure the privacy of both the data and learning model. Below we give two examples to demonstrate that differential privacy and imitation privacy do not imply each other.

**Example 6 (Ensured differential privacy and breached imitation privacy).** Suppose that Bob’s data $X_B$ contains $n$ i.i.d. observations of a random variable supported on $[-b, b]$. Bob can apply Laplacian mechanism to his data $X_B$ to get $\alpha$-locally deferentially private data $\tilde{X}_B$, and then releases $\tilde{X}_B$ to Alice.

However, the above mechanism typically does not admit a non-vanishing imitation privacy (Definition 5, for any label distribution $p_Y$). For example, suppose that Bob uses a linear regression model, then Alice can create the following imitation system with a vanishing imitation privacy. For any queried $y$, Alice calculates $\tilde{\beta}_a = (\tilde{X}_B^T\tilde{X}_B - \tau^2 I)^{-1}\tilde{X}_B^Ty = (\tilde{X}_B^T\tilde{X}_B - \tau^2 I/n)^{-1}(\tilde{X}_B^Ty/n)$, and uses $f_{M,y} : x \mapsto \tilde{\beta}_a^Tx$ for prediction, where $\tau^2 = 8b^2/\alpha^2$ is the variance of the Laplacian noise that could be estimated from Stage II if not known to Alice. By the law of large numbers, the above $\tilde{\beta}_a$ converges in probability to the same limit as Bob’s estimator $\hat{\beta}_b = (X_B^T X_B)^{-1}X_B^Ty$. This implies a vanishing imitation privacy as the data size $n$ becomes large.

**Example 7 (Ensured imitation privacy and breached differential privacy).** Suppose that a module Bob is equipped with a linear regression algorithm. Suppose that one predictor/feature is released to the public, then his dataset will not be differentially private at any privacy level. However, such direct release of partial data will only decrease the imitation privacy by a small amount. Alice still can not estimate the functionality of Bob with arbitrary accuracy.

**IV. Experimental Study**

We provide numerical demonstrations of the proposed method in Section II-D, II-E. For synthetic data, blue line represents the mean of training errors from 20 replicates, and the red line stands for the mean of test errors from 20 replicates. The oracle performance (denoted by black dashed line) is the testing error obtained by the model trained on the pulled data. In each replicate, we use a training dataset with size 500, and a testing data with size 10000. We chose a testing size much larger than training size in order to produce a fair comparison of out-sample predictive performance (see [8], [19] for a detailed discussion). For real data, the red line indicates mean of the testing error on a testing dataset which is 30% of whole data, resampled 20 times. In addition, the oracle performance (denoted by black dashed line) is the testing error obtained by the model that is previously trained on the pulled data. Also, the shaded regions describe the corresponding -1/+1 standard errors. Besides, for all nonlinear learning algorithms, tuning-parameters are finely tuned.

**A. Synthetic Data**

1) Synthetic Data with linear underlying pattern: We first consider the case where the true data generating function is linear. Let $x_j = [x_{j1}, x_{j2}, \ldots, x_{j6}]$, where $x_{ji} \overset{IID}{\sim} \mathcal{N}(0, 1)$. The data generating model is $y = X\beta + \varepsilon$, where
where \( \beta \sim \mathcal{N}(0, I_6) \), and \( \varepsilon \sim \mathcal{N}(0, 1) \). Module A holds data \( X_A = [x_1, x_2, x_3] \) and Module B holds data \( X_B = [x_4, x_5, x_6] \). The experiments are independently replicated 20 times. Each time a training size of 500 and a testing size of 100000 are used.

Figures below show the prediction performances of module A with 3 different learning algorithms. In Fig. 4a, with linear regression model, we observe that the error terms converge in two iterations and exactly match the oracle score, which verifies the statement of ‘convergence in one round with high probability for independent data’ in Theorem 1. Fig. 4b is the prediction performance of using decision tree (regression). The error term decreases to the oracle score with 25 rounds of communication. Gradient boosting algorithm is used in Fig. 4c. Note that the testing error first decreases to the oracle score and then begin to increase. Interestingly, this phenomenon strikingly resembles the classical tradeoff between overfitting and underfitting due to model complexity. In our case the communication complexity is the counterpart of model complexity.

![Graphs showing prediction performances of module A](image)

Fig. 4: Prediction performances for Module A (as measured by RMSE) on synthetic linear data with multiple learning algorithms. Each line is the mean of 20 replicates and the shaded regions describe the corresponding +1/-1 standard errors. In (a), with linear regression model being employed, the testing error converges in two iterations and exactly matches the oracle score. In (b), with decision tree (regression) being used, the testing error decreases to the oracle score with 25 rounds of communication. Gradient boosting algorithm is used in (c). Note that the testing error first decreases to the oracle score and then begin to increase. (Overfitting with respect to the communication complexity.)

2) Synthetic Data with non-linear underlying pattern: We next test on the data generated by Friedman1 benchmark [36], [37], i.e.,

\[
 f(x) = 10 \sin(\pi x_1 x_2) + 20(x_3 - \frac{1}{2})^2 + 10x_4 + 5x_5 + \varepsilon, \tag{2}
\]

where \( \varepsilon \sim \mathcal{N}(0, 1) \) and \( x_i \sim \text{Unif}(0, 1) \) for \( i = 1, 2, ..., 5 \). Module A has the first 2 features \( X_A = [x_1, x_2] \) and Module B has the rest 3 features \( X_B = [x_3, x_4, x_5] \). Fig. 5a, 5b, below show the prediction performances of module A with linear regression and additive model with spline basis. We observe that both methods attain their corresponding oracle scores, and the error terms converge very fast, which is expected from Theorem 1. Fig. 5c, 5d, 5e are the prediction performances of module A with decision tree (regression), random forest, and gradient boosting. All the error terms achieve their corresponding oracles, and similarly, we observe the overfitting issue with respect to the communication complexity.

In fact, from the data generating function eq. (2), we notice that the interaction effect is only between covariates \( x_1 \) and \( x_2 \). The effects contributed by other features are independent. Therefore, it is reasonable for module A with features \( [x_1, x_2, x_3] \) to achieve the oracle score in assisted learning, since the interaction term can be well learned by the machine learning model. Next, we consider the case where module A holds features \( [x_1, x_3, x_4] \) and module B holds features \( [x_2, x_5] \). Fig. 5f demonstrates the prediction performances of module A using gradient boosting. There is a gap between the oracle score and the assisted learning result. Such gap is due to the fact that interaction terms \( x_1 \) and \( x_2 \) are now fitted separately. However, compared to the result without using assisted learning assisted learning, it does bring significant improvement to module A. Indeed, this gap can be closed by neural network protocol in assisted learning, and we shall numerically address this issue in the upcoming section.
3) Assisted Learning for Neural Network: In addition, we use two-layer neural network, with 4 nodes in the hidden layer and Relu function, to illustrate the proposed method. The training size is 500 and the testing size is 10000 for each replicate. Full batch is used to train the network each time. We test on the same two datasets used in the previous section. Module A holds features \([x_1, x_3, x_4]\) and module B will hold the rest features corresponding to each dataset. We follow the learning protocol described in Section II-E. Fig. 6a, 6b show the prediction performances of module A on linear data and Friedman 1 respectively. In both cases, assisted learning eventually approaches the performance of vanilla neural network (oracle). Note that the features held by A are not complete for interaction term, i.e. \(x_1\) and \(x_2\) are separate for the Friedman1 data. Yet, the oracle performance is achieved by assisted learning and therefore we solve the problem raised in the previous section.

B. Real Data Study

We demonstrate our approach using the Superconductor Data [38] that consists of 21263 entries and 81 features. The learning task is to use chemical characteristics to predict the superconducting critical temperatures. The features is partitioned into two sets, 40 features held by module A and the other 41 features held by B. We consider two settings where one module uses gradient boosting and the other one uses linear regression.

The results as depicted in Fig. 7a, 7b show that both modules can approximately achieve their corresponding oracle performances. The prediction performance for gradient boosting is much better than Linear Regression. In
(a) Prediction performances of module A with two-layer neural network on linear data.

(b) Prediction performances of module A with two-layer neural network on Friedman 1.

Fig. 6: Prediction performances (as measured by RMSE) of module A with two-layer neural work on two different datasets. Oracle means the result obtained from vanilla two-layer neural network. Each line is the mean of 100 replicates and the shaded regions describe +1/-1 standard errors.

Fig. 7: Prediction performances for Module A (as measured by RMSE) on Superconductor Data with two learning algorithms. Each line is the mean of 20 replicates and the shaded regions describe +1/-1 standard errors.

terms of convergence rate, gradient boosting converges faster than linear regression. In Fig. 7b, we observe the ‘over-fitting’ issue with respect to communication complexity.

C. The interactions between Federated Learning and Assisted Learning

In this section, we demonstrate the interactions between federated learning and assisted learning. We will first see how these two frameworks benefit from each other. Suppose that the whole feature space of interests is \([x_1, x_2, ..., x_p] \in \mathbb{R}^p\) and Alice’s data has the first three features \([x_1, x_2, x_3]\) and \(n_1\) entries. Consider the following 2 types of learners: 1. C-type learners, who hold the whole features \([x_1, x_2, ..., x_p]\) but with limited number of data entries; 2. B-type learners, who holds partial features, e.g. \([x_3, x_5, ..., x_6]\), and the data entries can be aligned (fully or partially) with Alice’s data. C-types learners are the typical participants in federated learning and B-type learners
are commonly seen in assisted learning. Now, Alice wants to learn a supervised relationship with her data, what is a good strategy for her in the presence of other learners?

First, with Alice’s own data only, the learned model $\hat{f}_A$ will certainly be biased. So it is beneficial for Alice to acquire type-B learners’ information to modify the biased model, which can be achieved via assisted learning framework. If Alice manages to obtain a ‘correct’ model $\tilde{f}_A$, how can she further improve her predictive performance? A possible way is for Alice to communicate with C-type learners by averaging their models, i.e. $\tilde{f}_A \leftarrow w_A\tilde{f}_A + \sum_i w_{C,i}\tilde{f}_{C,i}$, to gain a better prediction performance. The above process can be implemented in the framework of federated learning.

In detail, for experimental study, there are in total 12 features $X = [x_1, x_2, \ldots, x_{12}] \in \mathbb{R}^{12}$ with $x_i \sim \mathcal{N}(0, 1)$ for $i = 1, 2, \ldots, 12$, and the data is generated from $y = X\beta + \varepsilon$, with $\varepsilon \sim \mathcal{N}(0, 1)$ and $\beta \sim \mathcal{N}(0, I_{12})$. Alice holds $[x_1, x_2, x_3]$ and has 100 data entries, and each B-type learner will hold 3 randomly selected features. All learners will employ linear regression as machine learning models. Alice follows Procedure 1 to communicate with B-type learners, and will communicate each of them only once ($K = 1$ in Procedure 1). Each C-type learner will have the whole 12 features and 25 data entries. Alice will follow the Fedavg algorithm and serve as the central service in federated learning to communicate with C-type learners. There are 4 lines in Fig. 8a, with each single line being the mean of 100 replicates. For convenience, we denote $C_1$ to be the learning performance with 2 C-type Learners (red line in the figure (a)), and similarly for $C_2, C_3, C_4$.

In Fig. 8a, we see that as the number of B-type learners increases, the out-sample prediction performances (on 10000 data points, as measured by RMSE) significantly reduce and eventually go to 0. This shows the significant benefit brought by assisted learning. To see the goodness of federated learning, we calculate the following ratios: $C_4/C_1$, $C_4/C_2$, $C_4/C_3$, and $C_4/C_1$, to use them as the statistical efficiency measurement (High = 1). In Fig. 8b, we observe that when B-type learners increase from 10 to 20, the ratios $C_4/C_1$, $C_4/C_2$ and $C_4/C_3$ increase due to the reduction of bias term in mean square error (MSE). After the number of B-type learners exceeding 25, the bias term almost goes to 0, and we see the benefit on efficiency brought by federated learning.

The above experiments show that these two frameworks can help each other to achieve a better learning performance. Next, we will demonstrate a unique advantage of assisted learning. As we discussed before, the choice of machine learning model is arbitrary in assisted learning. Consider the scenario where there are only B-type learners and a proportion of B-type learners have very noisy data. If they are all restricted to use the same model, e.g., ordinary linear regression, then the out-sample prediction performance can be very bad. However, if they follow the assisted learning protocol, i.e. learners with noisy data opt to robust learning technique, then the out-sample prediction performances will be very promising.

For the experimental study, we have in total 20 B-type learners, and learner $B_j$ for some $j = 1, 2, \ldots, [20\rho]$ have really noisy data. Here $\rho$ is the proportion of clients whose data contain outliers and larger noise. For vanilla method, all the learners will use the linear regression model. For assisted learning, those B-type learners with noisy data will use robust regression technique, and those normal B-type learners will use linear regression. Fig. 8c demonstrates the prediction performances of Alice for two proposed methods with different proportions of noisy B-type learners. With vanilla method, the RMSE significantly goes up as the portion of noisy B-type learners increases. For assisted learning, the RMSE remains around 1 in all cases.

V. Conclusion

The interactions between multiple learners in privacy-aware scenarios pose new challenges that cannot be well addressed by classical statistical learning with a single learning objective and algorithmic procedure. In this work, we propose the notion of Assisted Learning, where the key idea is to treat predictors as private and labels as public, and learners hold private predictors as well as private algorithms provide learning services for other learners. On the other hand, most of the existing literature on privacy focuses on protecting users’ data, there is also a growing demand for protecting the learners who manage data. A new notion of privacy, imitation privacy, is proposed to quantify the privacy leakage of a learner who provides assistance. This privacy enables a unified measurement for both data as well as private model being used.
Fig. 8: The interactions between Federated Learning and Assisted Learning. In (a), each line corresponds to a set of C-type learners. In (b), the relative efficiency is calculated as the ratio of results in (a) to ‘400 C-type Learners’ respectively. For instance, the red dotted line is the ratio of ‘400 C-type Learners’ to ‘2 C-type Learners’, and the yellow crossed line is the ratio of ‘400 C-type Learners’ to ‘400 C-type Learners’.

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VI. APPENDIX

We will use the following lemma. Let $M, N$ be two closed subspaces in a Hilbert Space $\mathcal{H}$. Their Friedrichs angle is defined to be the number $0 \leq \theta_F \leq \frac{\pi}{2}$ such that

$$\cos \theta_F = \sup_{x \in M^\perp, \|x\| = 1} \|y\| = 1 y^T x.$$  

(3)

Lemma 1. [39] Let $M_1, M_2, M_3, \ldots, M_k$ be closed subspaces in $\mathcal{H}$ with intersection $M = \bigcap_{i=1}^k M_i$. For $j = 1, 2, 3, \ldots, k$, we denote $\theta_F^j$ to be the Friedrichs angle between $M_j$ and $\bigcap_{i=j+1}^k M_i$, then for any $x \in \mathcal{H}$ and any integer $n \geq 1$, we have

$$\| (P_{k} \ldots P_{1})^n x - P_M x \| \leq c^n \| x - P_M x \|$$

(4)

where $c = \left(1 - \prod_{j=1}^{k-1} \sin^2 \theta_F^j \right)^{1/2}$.

Proof of Theorem 1. We prove for the ordinary linear regression. The same technique can be extended to general additive models. For any design matrix $X \in \mathbb{R}^{n \times p}$, we define the projection matrix $P_X = X(X^TX)^{-1}X^T$ and its orthogonal $P_X^\perp = I_n - P_X$. We let $X = [X_A, X_B]$, with $X_A \in \mathbb{R}^{n \times p_1}$, $X_B \in \mathbb{R}^{n \times p_2}$ ($p_1 + p_2 = p$), and $y \in \mathbb{R}^n$ be the corresponding labels. For simplicity, we use $A, B$ to denote span$(X_A), \text{span}(X_B)$ respectively. Also, we denote $\| \cdot \|$ to be the Euclidean norm and $\| \cdot \|_2$ to be the matrix operator norm.

Denote $e_{\text{orac}}$ to be the residual obtained from the linear regression of $y$ on $X$, i.e., $e_{\text{orac}} = y - \hat{y} = y - (X_A \hat{\beta}_a + X_B \hat{\beta}_b)$, where $[\hat{\beta}_a, \hat{\beta}_b]$ is the oracle least square estimator from all the data. Suppose that Alice holds data $X_A$ and the label $y$, and Bob only has data $X_B$. Let $e_i$ denote the residual at $i$th iteration and $e_0 = y$. Since they both use linear regression models, the residual $e_k$ at $k$th iteration is:

$$e_k = (P_B^\perp P_A^\perp)^k e_0,$$

and we also have the following identity:

$$e_{\text{orac}} = P_{A \cup B}^\perp e_0 = P_{A^\perp \cap B^\perp} e_0.$$

By Lemma 1, for any integer $k \geq 1$, we have

$$\| e_k - e_{\text{orac}} \| = \left\| \left( P_B^\perp P_A^\perp \right)^k e_0 - P_{A^\perp \cap B^\perp} e_0 \right\| \leq c^k \| e_0 - P_{A^\perp \cap B^\perp} e_0 \|$$

$$= c^k \| e_0 - e_{\text{orac}} \| = (1 - \sin^2 \theta_F)^{k/2} \| e_0 - e_{\text{orac}} \|$$

$$= (\cos \theta_F)^k \| e_0 - e_{\text{orac}} \|$$

(5)

where $\cos \theta_F$ is the Friedrichs angle between $A^\perp$ and $B^\perp$. Since $\cos \theta_F = \cos \theta_F^1$ [40], and $\cos \theta_F < 1$ (since $X$ has a full column rank), the error term will converge exponentially.

In the above arguments, we showed that $e_k$ will converge to $e_{\text{orac}}$ as $k$ become large. Next we explicitly show the the aggregated coefficients obtained by Alice ad Bob will asymptotically approach the oracle least square estimators defined above. As a result, Alice will attain near-oracle performance from the assistance of Bob.

Proposition 1. Let $\hat{\beta}_a^k, \hat{\beta}_b^k$ be the coefficients obtained at the $k$th round of communication for Alice and Bob respectively, and $\hat{\beta}_a, \hat{\beta}_b$ be the oracle coefficients (defined as above). Then we have:

$$\lim_{k \to \infty} \sum_{i=1}^k \hat{\beta}_a^i = \hat{\beta}_a$$

(6)

$$\lim_{k \to \infty} \sum_{i=1}^k \hat{\beta}_b^i = \hat{\beta}_b$$

Proof of Proposition 1. We proof for the case of Alice, i.e. $\lim_{k \to \infty} \sum_{i=1}^k \hat{\beta}_a^i = \hat{\beta}_a$. The similar technique can be used to prove Bob’s case. From the procedure of assisted learning, the $k$th coefficient for Alice
is \((X_A^T X_A)^{-1} X_A^T (P_B^+ P_A^+)^k y\), and we know \(\hat{\beta}_a = (X_A^T P_B^+ X_A)^{-1} X_A^T P_B^+ y\) by some calculations. Then it suffices to show
\[
(X_A^T P_B^+ X_A)^{-1} X_A^T P_B^+ = (X_A^T X_A)^{-1} X_A^T \left( \sum_{k=0}^{\infty} (P_B^+ P_A^+)^k \right).
\]
(7)

By Gelfand’s formula, we have
\[
\rho(P_B^+ P_A^+) \leq \|P_B^+ P_A^+\|_2,
\]
(8)
where \(\rho(\cdot)\) is the spectral radius (the largest absolute value of eigenvalues). From Spectral Theorem, we know that for any square matrix \(A\), \(A\) is normal if and only if the operator norm equals the spectral radii. Therefore, we consider the following two cases.

Case 1: If \(P_B^+ P_A^+\) is normal, then
\[
P_B^+ P_A^+ P_B^+ = P_A^+ P_B^+ P_A^+
\]
(9)
We just need to show that
\[
X_A^T P_B^+ = X_A^T P_B^+ X_A (X_A^T X_A)^{-1} X_A^T \left( \sum_{k=0}^{\infty} (P_B^+ P_A^+)^k \right)
\]
(10)
Plugging (9) into the right hand side of (10), we have
\[
X_A^T P_B^+ X_A (X_A^T X_A)^{-1} X_A^T \left( \sum_{k=0}^{\infty} (P_B^+ P_A^+)^k \right) = X_A^T P_B^+ P_A + X_A^T P_B^+ P_A^+ P_B^+ = X_A^T P_B^+ P_A + X_A^T P_B^+ (I_n - P_A^+) P_B^+ = X_A^T P_B^+ P_A + X_A^T P_B^+ P_A^+ P_B^+. \]
(11)

Since \(X_A^T P_B^+ P_A^+ P_B^+ = X_A^T P_B^+ P_B^+ P_A^+ = (P_A^+ X_A)^T P_B^+ P_A^+ = 0\), then Eq. (11) reduces to
\[
X_A^T P_B^+ P_A + X_A^T P_B^+ P_A^+ = X_A^T P_B^+ (P_A + P_A^+) = X_A^T P_B.
\]

Therefore, Eq. (10) is correct and Case 1 holds.

Case 2: If \(P_B^+ P_A^+\) is not normal, then the equality in Eq. (8) will not hold. By a simple fact that \(\|P_B^+ P_A^+\|_2 \leq 1\), we have \(\rho(P_B^+ P_A^+) < 1\). By the property of Neumann Series, the following holds:
\[
\sum_{t=0}^{\infty} (P_B^+ P_A^+)^t = (I_n - P_B^+ P_A^+)^{-1},
\]
and \((I_n - P_B^+ P_A^+)^{-1}\) exists.

We just need to show
\[
(X_A^T X_A)^{-1} X_A^T (I_n - P_B^+ P_A^+)^{-1} = (X_A^T P_B^+ X_A)^{-1} X_A^T P_B^+
\]
(12)
by multiplying \((I_n - P_B^+ P_A^+)\) on both sides of (12), it reduces to
\[
(X_A^T X_A)^{-1} X_A^T = (X_A^T P_B^+ X_A)^{-1} X_A^T P_B^+ - (X_A^T P_B^+ X_A)^{-1} X_A^T P_B^+ P_A.
\]
Since \(X_A\) is with full column rank, then \(X_A X_A^T\) is invertible. Multiplying it on both sides, we have
\[
(X_A^T X_A)^{-1} X_A^T X_A X_A^T = (X_A^T P_B^+ X_A)^{-1} X_A^T P_B^+ X_A X_A^T - (X_A^T P_B^+ X_A)^{-1} X_A^T P_B^+ P_A X_A X_A^T,
\]
and it remains to show \(X_A^T = X_A^T\), which is obviously true. Hence, Case 2 holds and we conclude the proof of Proposition 1. In conclusion, if Alice and Bob use linear regression models, then for a sufficiently large number of communications \(k\), the oracle performance will be achieved and the error will decay exponentially.

In fact, the Theorem 1 above concerns a finite-sample result when the data size \(n\) remains fixed. The following result extends Theorem 1 to a probabilistic setting with random observations and varying \(n\). Suppose that the data generating model is \(y = \beta_a^T x_a + \beta_b^T x_a + \varepsilon\), where \(\varepsilon\) has zero mean and \(\sigma^2\) variance, \(x \in [x_a, x_b] \in \mathbb{R}^p\) follows from a subGaussian distribution with zero mean and correlation matrix \(R\), and \(x, \varepsilon\) are independent. Suppose that \(n\) independent observations \((y_i, x_{a,i})\) are available to Alice, and \((x_{b,i})\) are available to Bob, \(i = 1, \ldots, n\). Let
$X = [x_1, \ldots, x_n]^\top$ denotes the design matrix centralizing all the data. Recall that $S_a, S_b$ denote the variable indices of Alice and Bob, respectively.

**Corollary 1.** Assume that the smallest eigenvalue of $X^\top X/n$ is almost surely lower bounded by a positive constant. Also assume that $x$ is sub-Gaussian with a fixed covariance matrix, and $\mathbb{E} y^2 < \infty$. Then the final predictor of Alice $\tilde{y}_n^*$ satisfies $\mathbb{E}(\tilde{y}_n^* - y)^2 \rightarrow \sigma^2$ as $n \rightarrow \infty$, meaning that it is a consist predictor.

**Proof of Corollary 1.** Let $e_{n,k}$ and $e_{n,\text{orac}}$ denote the residual of Alice at step $k$ of stage I, and the oracle residual by pulling all the data, respectively, where the subscript $n$ highlights their dependence on the data size. Suppose there are $k$ communications in Stage I. In stage II, suppose that the aggregated linear prediction function of Alice forms has a coefficient vector $\tilde{\beta}_{n,k}$; also suppose the oracle least square estimate by pulling the data is $\hat{\beta}_n$. It suffices to prove that $\tilde{\beta}_{n,k} - \hat{\beta}_n \rightarrow 0$ in probability as $n \rightarrow \infty$. By the subGaussian assumption, the Friedrichs angle between $X_A$ and $X_B$, $\cos \theta_F$, is bounded away from 1 with probability at most $c_1 p^2 e^{-c_2 nt^2}$ for some constants $c_1, c_2$. Using Theorem 1 and the assumption on the smallest eigenvalue, there exists a constant $c$ that

$$||\tilde{\beta}_{n,k} - \hat{\beta}_n||^2 \leq cn^{-1}||X\tilde{\beta}_{n,k} - X\hat{\beta}_n||^2 = cn^{-1}||e_{n,k} - e_{n,\text{orac}}||^2$$

which goes to zero in probability.

In the above corollary, it is possible that $p \rightarrow \infty$ and $k/p \rightarrow 0$ as $n \rightarrow \infty$, maintaining a high privacy for Bob since only a small fraction of column space is available to Alice. Next, we elaborate on Example 3, 4 and 5 in the main part of the paper.