Topological frustration and structural balance in strongly correlated itinerant electron systems: an extension of Nagaoka’s theorem

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We prove that Nagaoka’s theorem, that the large-$U$ Hubbard model with exactly one hole is ferromagnetic, holds for any balanced Hamiltonian. Simply put, if a positive bond encodes friendship and a negative bond encodes enmity, then balance implies when that enemy of one’s enemy is one’s friend. We argue that, in itinerant electron systems, a balanced Hamiltonian, rather than bipartite lattice, defines an unfrustrated system. The proof is valid for multi-orbital models with arbitrary two-orbital interactions provided that no exchange interactions are antiferromagnetic: a class of models including the Kanomori Hamiltonian.

I. INTRODUCTION

Geometrical frustration plays a central role in the understanding of quantum magnetism [1]. The distinction between bipartite and geometrically frustrated lattices is fundamental for spin-models of localized electrons. But, we will argue below, in itinerant electron systems the bipartite/geometrically frustrated distinction is not relevant. Rather a topological definition of frustration is required. We will argue below that structural balance, introduced in an anthropological context, where one is often concerned with networks containing both beneficial and antagonistic relationships, [2], provides an appropriate measure of topological frustration.

Previous attempts to classify the frustration of itinerant electrons have focused on the reduced bandwidth in frustrated systems [3][4]. Therefore, these measures miss the fundamental role that the sign of the hopping integral plays in frustrated itinerant electron systems. Balance considers this.

To ground the above claims we study one of the few exact results of strongly correlated itinerant electrons: Nagaoka’s theorem. This theorem concerns the proper results of strongly correlated itinerant electrons: considers this.

\[ H = \hat{H}_t + \hat{H}_U + \hat{H}_V + \hat{H}_J + \hat{H}_P + \hat{H}_X \]

where

\[ \hat{H}_t = -\sum_{i\mu\sigma} t_{i\mu\sigma} \hat{c}_{i\mu\sigma}^\dagger \hat{c}_{i\mu\sigma}, \]

\[ \hat{H}_U = U \sum_{i\mu} \hat{n}_{i\mu\uparrow} \hat{n}_{i\mu\downarrow}, \]

\[ \hat{H}_V = \hat{V}(\{\hat{n}_{\mu}\}), \]

\[ \hat{H}_J = \sum_{(i,\mu)\neq(j,\nu)} \rho_{\sigma} J_{ij\mu\nu} \hat{S}_{i\mu} \cdot \hat{S}_{j\nu} + \frac{1}{4} \hat{n}_{i\mu} \hat{n}_{i\nu}, \]

\[ \hat{H}_P = \sum_{(i,\mu)\neq(j,\nu)} P_{ij\mu\nu} \hat{c}_{i\mu\sigma}^\dagger \hat{c}_{i\mu\sigma} \hat{c}_{j\nu\sigma}^\dagger \hat{c}_{j\nu\sigma}, \]

\[ H_x = \sum_{(i,\mu)\neq(j,\nu)} \sum_{\sigma} X_{ij\mu\nu} \hat{c}_{i\mu\sigma}^\dagger \hat{c}_{j\nu\sigma}(\hat{n}_{i\mu\sigma} + \hat{n}_{j\nu\sigma} - 1), \]

\[ \hat{c}_{i\mu\sigma} \] creates (annihilates) an electron with spin $\sigma$ on the $\mu$th Wannier orbital centered on site $i$, $\sigma \neq \sigma$, $\hat{n}_{i\mu\sigma} = \hat{c}_{i\mu\sigma}^\dagger \hat{c}_{i\mu\sigma}$, $\hat{n}_{i\mu\uparrow} + \hat{n}_{i\mu\downarrow} = \frac{1}{2} \sum_{\alpha\beta} \hat{c}_{i\mu\alpha}^\dagger \hat{c}_{i\mu\beta}^\dagger \hat{c}_{i\mu\beta} \hat{c}_{i\mu\alpha}$, and $\sigma$ is the vector of Pauli matrices. Here $t_{ij\mu\nu}$ is the amplitude for hopping between the $\mu$th orbital on site $i$ and the $\nu$th orbital on site $j$ [5]. $U$ is the effective Coulomb interaction between two electrons on the same orbital, $\hat{V}(\{\hat{n}_{\mu}\})$ is any potential that depends only on the orbital occupation numbers – obviously this includes arbitrary pairwise direct Coulomb interaction: $\sum_{i\mu\nu} V_{i\mu\nu} \hat{n}_{i\mu\uparrow} \hat{n}_{i\nu\downarrow}$; but in fact the proofs below hold for arbitrary terms of this form, including multi-site interactions, the $J_{ij\mu\nu}$ are the exchange couplings, the $P_{ij\mu\nu}$ are hopping amplitudes, and the $X_{ij\mu\nu}$ are correlated hopping amplitudes. Note that no assumption about the intra-site hopping has been made, in particular $t_{i\mu\nu}$ may be non-zero.

It will be convenient below to differentiate between four versions of this model: (a) the multiorbital model – Eq. (1); (b) the single orbital model – one orbital per site (henceforth we drop the orbital subscripts when discussing single orbital models) (c) the extended Hubbard model – the single orbital model with $J_{ij} = P_{ij} = X_{ij} = 0$ for all $i, j, \mu, \nu$; and (d) the Hubbard model – the extended Hubbard model with $\hat{V}(\{\hat{n}_{\mu}\}) = 0$.

Note that the hole doped sector of each of these models ($N < L$, where $N$ is the number of electrons and $L$ is the number of orbitals on the entire lattice) is equivalent to the electron sector of that model ($N > L$) with the signs of all $t_{ij\mu\nu}$ reversed. A particle-hole transformation maps between the Hamiltonians, up to constants, even in the absence of particle-hole symmetry. Henceforth we will only discuss the hole doped problem; however it is implicit throughout that all results hold for the electron doped problem if the signs of all $t_{ij\mu\nu}$ reversed.

Nagaoka showed [6] that in the Nagaoka limit ($U \to \infty$ and $N = L - 1$) the ground state of the Hubbard model on certain lattices is a fully polarized ferromagnet – i.e., the magnetization, $M = N/2$. Nagaoka showed that for nearest neighbor hopping only ($t_{ij} = t$ for nearest neighbors, $t_{ij} = 0$ otherwise) and $t < 0$ [7] this result...
holds for simple cubic, body centered cubic, face centered cubic and hexagonal close packed lattices.

However, for \( t > 0 \) Nagaoka found that the theorem holds on the simple cubic and body centered cubic lattices, but not for the face centered cubic or hexagonal close packed lattices. The former lattices are bipartite, while the latter are not. On a bipartite lattice the latter are not. On a bipartite lattice the latter are not. On a bipartite lattice the gauge transformation \( \hat{c}_\sigma \rightarrow -\hat{c}_\sigma \) on one sublattice only changes the sign of all hopping integrals.

In 1989 Tasaki gave a more general proof of Nagaoka’s theorem [7]. Specifically, Tasaki proved that Nagaoka’s theorem holds for the extended Hubbard model on all lattices where \( t_{ij} \leq 0 \) for all \( i,j \). Thus, one is moved to ask which other Hamiltonians with some or all \( t_{ij} > 0 \) have ferromagnetic ground states? In particular, for which geometrically frustrated (non-bipartite) Hamiltonians can one prove Nagaoka’s theorem? This is particularly important as for a simple covalent bond one expects that \( t_{ij} > 0 \) [8].

In 1996 Kollar, Strack and Vollhardt [9] extend Nagaoka’s theorem in a different direction – discussing the role of other two-body interactions. Among other things they showed that Nagaoka’s theorem holds for the infinite-\( U \) single orbital model on periodic lattices if the hopping and all interactions are constrained to be between nearest neighbors only, exchange is ferromagnetic (or zero), and either \( t < 0 \) or the lattice is bipartite. It is therefore natural to ask what other (e.g., longer range) two-orbital interactions admit that a proof of Nagaoka’s theorem?

Furthermore, given that ferromagnetism is observed in many materials where multiple orbitals are relevant to the low-energy physics it is natural to ask whether multiple orbital models exhibit Nagaoka-like ferromagnetism. Below, we give partial answers to the above questions by proving the following:

**Theorem 1:** Consider the multiorbital model [1] with \( U \) infinite, \( V \) and \( \{P_{ij\mu\nu}\} \) arbitrary, \( J_{ij\mu\nu} \leq 0 \) for all \( i,j,\mu,\nu \) and \( N = L - 1 \). If the signed graph \( \mathcal{S} \) defined by the set of renormalized hopping integrals \( \{t^{*}_{ij\mu\nu}\} \), where \( t^{*}_{ij\mu\nu} \equiv t_{ij\mu\nu} + X_{ij\mu\nu} \), is balanced (defined below) then among the ground states there exist at least \( L \) states with \( S = S_{\text{max}} \equiv N/2 \).

**Theorem 2:** Consider the multiorbital model [1] with \( U \) infinite, \( V \) and \( \{P_{ij\mu\nu}\} \) arbitrary, \( J_{ij\mu\nu} \leq 0 \) for all \( i,j,\mu,\nu \) and \( N = L - 1 \). If the signed graph \( \mathcal{S} \) defined by the set of renormalized hopping integrals \( \{t^{*}_{ij\mu\nu}\} \) is balanced and \( \hat{\mathcal{H}} + \hat{\mathcal{H}}_X \) satisfies the connectivity condition (defined below) then the ground state has \( S = S_{\text{max}} \equiv N/2 \) and is unique up to the trivial \( N \)-fold degeneracy.

The remainder of the paper is laid out as follows. Having introduced balanced and unbalanced lattices in section [I] we prove that balance is a sufficient condition for the proof of Nagaoka’s theorem in the Nagaoka limit (section [II]). All of the lattices for which Nagaoka, Tasaki, or Kollar et al. proved Nagaoka’s theorem previously are balanced. Finally, in section [IV] discuss the implications of balance for the orbital part of the ground state wave-function, clarifying why balance favours ferromagnetism.

**II. BALANCE**

The sign of the hopping between a pair of orbitals, \( t^{*}_{ij\mu\nu} \), is not gauge invariant: the transformation \( \hat{c}_{\mu\nu} \rightarrow e^{-i\theta_{\mu\nu}}\hat{c}_{\mu\nu} \) takes \( t^{*}_{ij\mu\nu} \rightarrow t^{*}_{ij\mu\nu} e^{i\theta_{\mu\nu}} \). Nevertheless, gauge invariant information is contained in the signs of the set \( \{t^{*}_{ij\mu\nu}\} \) associated with a particular Hamiltonian. This is a topological problem – the magnitudes of the \( t^{*}_{ij\mu\nu} \) are unimportant; only their signs matter. Thus, rather than considering every \( \{t^{*}_{ij\mu\nu}\} \) separately, it is sufficient to instead study a related ‘signed graph’. We define this signed graph as follows: We introduce a vertex of the graph corresponding to each orbital in the physical Hamiltonian, \( \{\mathbf{K}\} = \{(i,\mu)\} \) (throughout this paper we use Latin characters for sites in the Hamiltonian, Greek for orbitals, and Hebrew for vertices in the signed graph). Furthermore, we introduce a set of edges defined by \( \{\tau_{(i,\mu),(j,\nu)}\} \), where \( \tau_{(i,\mu),(j,\nu)} \equiv -\text{sgn}(t^{*}_{ij\mu\nu}) \) if and only if \( t^{*}_{ij\mu\nu} \neq 0 \). We now ask whether there exists a series of gauge transformations that make all \( \tau_{(i,\mu),(j,\nu)} = 1 \)? If so, the gauge transformation makes all \( t^{*}_{ij\mu\nu} \leq 0 \).

A walk on a signed graph is defined as a sequence of vertices such that consecutive vertices in the sequence are connected by an edge, e.g., \( \mathbf{K} \rightarrow \mathbf{Z} \rightarrow \mathbf{I} \rightarrow \cdots \rightarrow \mathbf{W} \rightarrow \mathbf{N} \). A walk in which all the sites that are visited are distinct (i.e., a self avoiding walk) is called a path. A path that visits at least three sites and is closed (e.g., \( \mathbf{K} \rightarrow \mathbf{N} \) in the example above) is called a cycle. The sign of a path or cycle on a signed graph is defined as the product of signs \( \tau_{(i,\mu),(j,\nu)} \cdots \tau_{(i,\mu),(j,\nu)} \) of the edges forming the path/cycle. Thus every positive cycle has an even number (including zero) of negative edges. A signed graph is balanced if all cycles in the corresponding signed graph are positive, cf. Fig. [I]. We will call a Hamiltonian balanced if it maps onto a balanced signed graph.

The fundamental theorem of signed graphs [10] states that for a signed graph, \( \mathcal{S} \), the following three conditions are equivalent:
FIG. 2. Three geometrically frustrated but balanced periodic lattices. (a) The square lattice with nearest neighbour hopping \( t_1 > 0 \) and next nearest neighbour \( t_2 < 0 \). (b) The honeycomb lattice with \( t_1 > 0 \) and \( t_2 < 0 \). (c) The anisotropic triangular lattice, relevant to organic superconductors and magnets [14], with \( t_1 < 0 \) and \( t_2 > 0 \). In each case two sublattices are marked: all hopping integrals are made positive by the transformation \( \hat{c}_{i\sigma} \rightarrow \hat{c}_{i\sigma} \) on one sublattice only.

1. \( \mathcal{S} \) is balanced, i.e., all cycles within \( \mathcal{S} \) are positive.

2. For any pair of vertices \( \mathcal{S} \) and \( \mathcal{S}' \) in \( \mathcal{S} \) all paths joining \( \mathcal{S} \) and \( \mathcal{S}' \) have the same sign.

3. There exists a partition of \( \mathcal{S} \) into two subsets, \( \mathcal{S}_a \) and \( \mathcal{S}_b \), (one of which may be empty) such that \( \tau_{\mathcal{S}_a} \geq 0 \) for all \( \mathcal{S} \) and \( \mathcal{S}' \) within the same subset, but \( \tau_{\mathcal{S}_a} = \tau_{\mathcal{S}_b} \leq 0 \) for \( \mathcal{S} \in \mathcal{S}_a \) and \( \mathcal{S}' \in \mathcal{S}_b \).

Thus, for example, bipartite lattices with only nearest neighbor hopping (and the same sign of hopping between all neighbors) are balanced. Some simple examples of geometrically frustrated but balanced lattices are shown in Fig. 2.

### III. BALANCE IS SUFFICIENT FOR NAGAOKA

In the Nagaoka limit all states with finite energy can be written as a superposition of ‘single hole states’ of the form

\[
| i, \mu, \tau \rangle = (-1)^{\rho(i, \mu)} \prod_{(j, \nu) \neq (i, \mu)} \hat{c}_{j\nu}^\dagger \rho(i, \mu) \rho(i, \mu) | 0 \rangle
\]

(3)

where \( \tau = \{ \sigma_j \}_{j \neq i} \) is a binary vector describing the spins of all of the electrons, \( | 0 \rangle \) is the vacuum state defined by \( \hat{c}_{i\sigma} | 0 \rangle = 0 \) for all \( i, \mu, \sigma \), and \( \rho(i, \mu) \in [0, L - 1] \) is an arbitrary ordering of the orbitals. \( \rho(i, \mu) \) need have no correlation to the physical pattern of the lattice, but is required to enforce the correct antisymmetrization of the hole states – the operators in the product are to be ordered by \( \rho(i, \mu) \) with lower values to the left.

We describe two states as ‘directly connected’ if \( \langle i, \mu, \tau | \hat{H}_t + \hat{H}_X | j, \nu, \tau' \rangle \neq 0 \). We will denote direct connection by \( (i, \mu, \tau) \leftrightarrow (j, \nu, \tau') \). For directly connected states

\[
\langle i, \mu, \tau | \hat{H}_t + \hat{H}_X | j, \nu, \tau' \rangle = t'^{ij}_{\mu\nu}. \quad (4)
\]

A Hamiltonian is said to satisfy the connectivity condition if there exists an integer \( n \) for which \( \langle i, \mu, \tau | (\hat{H}_t + \hat{H}_X)^n | j, \nu, \tau' \rangle \neq 0 \) for every pair of states with the same \( S^z \). Notably, one dimensional single orbital models with only nearest neighbor hopping are not connected in this sense [6, 7].

### A. Proof of theorem 1

Compare the arbitrary superposition of states of single hole states

\[
| A_h \rangle = \sum_{ij\mu\tau} \alpha_{ij\mu\tau} | i, \mu, \tau \rangle,
\]

(5)

with a fully spin polarized superposition

\[
| \Phi_h \rangle = \sum_{ij\mu} \phi_{ij\mu} | i, \mu, \uparrow \rangle,
\]

(6)

where \( \tau = \uparrow \) indicates that all the electrons are up and \( \phi_{ij\mu} = \sqrt{\sum_\tau |\alpha_{ij\mu\tau}|^2} \). We have

\[
\langle A_h | \hat{H}_V | A_h \rangle = \sum_{ij\mu\tau} |\alpha_{ij\mu\tau}|^2 \langle i, \mu, \tau | \hat{H}_V | i, \mu, \tau \rangle
\]

\[
= \sum_{ij\mu\tau} |\alpha_{ij\mu\tau}|^2 \langle i, \mu, \uparrow | \hat{H}_V | i, \mu, \uparrow \rangle
\]

\[
= \sum_{ij\mu} |\phi_{ij\mu}|^2 \langle i, \mu, \uparrow | \hat{H}_V | i, \mu, \uparrow \rangle
\]

\[
= \langle \Phi_h | \hat{H}_V | \Phi_h \rangle. \quad (7)
\]

That is, because \( U \) is infinite and \( V(\{ \hat{n}_i^\uparrow + \hat{n}_i^\downarrow \}) \) only depends on the site occupation numbers, \( \langle \hat{H}_V \rangle \) is independent of the spin degrees of freedom for single hole states.

Now note that

\[
\langle i, \mu, \uparrow | \hat{S}_{i\mu' \nu} \cdot \hat{S}_{j\mu' \nu'} + \frac{1}{4} \hat{n}_i^\mu \hat{n}_j^\nu \rangle | j, \nu, \uparrow \rangle
\]

\[
\geq \langle i, \mu, \tau | \hat{S}_{i\mu' \nu} \cdot \hat{S}_{j\mu' \nu'} + \frac{1}{4} \hat{n}_i^\mu \hat{n}_j^\nu \rangle | j, \nu, \tau \rangle
\]

\[
\geq 0 \quad (8)
\]

for all \( i, j, \mu, \nu, \tau \). Thus, if \( J_{ij\mu\nu} \leq 0 \) for all \( i, j, \mu, \nu \) then

\[
0 \geq \langle j, \nu, \tau | \hat{H}_J | i, \mu, \tau \rangle \equiv K_{ij\mu\nu}^{\mu\nu} \geq K_{ij\mu\nu}^{\mu\nu} \quad (9)
\]

for all \( i, j, \mu, \nu, \tau \). Therefore,

\[
\langle A_h | \hat{H}_J | A_h \rangle = \sum_{ij\mu\nu\tau} K_{ij\mu\nu}^{\mu\nu} \alpha_{ij\mu\tau} \alpha_{j\nu\tau}^*
\]

\[
\geq \sum_{ij\mu\nu\tau} K_{ij\mu\nu}^{\mu\nu} \sum_{\tau} \alpha_{ij\mu\tau} \alpha_{j\nu\tau}^*
\]

\[
\geq \sum_{ij\mu\nu} K_{ij\mu\nu}^{\mu\nu} \phi_{ij\mu} \phi_{ij\nu}^* \langle \Phi_h | \hat{H}_J | \Phi_h \rangle. \quad (10)
\]

where the second inequality follows from the Cauchy-Schwartz inequality.
Because double occupancy is forbidden in the infinite-
U limit \((i, \mu, \tau)|\mathcal{H}_V|j, \nu, \upsilon\rangle = 0\).

Finally, we specialize to the case of a balanced lattice. Property 3 of the fundamental theorem of signed
graphs implies that we can construct a gauge transformation that maps the Hamiltonian onto one where all
\(t_{ij}^* \leq 0\). An explicit example of such a gauge transformation is
\[
\hat{c}_{\mu\sigma} \rightarrow -\hat{c}_{\mu\sigma} \text{ for all } (i, \mu) \in S_a,
\hat{c}_{\mu\sigma} \rightarrow \hat{c}_{\mu\sigma} \text{ for all } (i, \mu) \in S_b.
\]
Furthermore,
\[
\langle A_h | \mathcal{H}_t + \mathcal{H}_X | A_h \rangle = t_{ij}^* \varphi_{ij\mu\nu} \sum_{\tau \upsilon} \alpha_{\mu\tau} \alpha_{\nu\upsilon}
\geq t_{ij}^* \varphi_{ij\mu\nu} \alpha_{\mu\tau} \alpha_{\nu\upsilon} = \langle \Phi_h | \mathcal{H}_t + \mathcal{H}_X | \Phi_h \rangle,
\]
where we have again made use of the Cauchy-Schwartz
inequality and \(\sum_{\tau \upsilon} \alpha_{\mu\tau} \alpha_{\nu\upsilon}\) indicates that the sum is restricted
to run only over \(\tau\) and \(\upsilon\) such that \((i, \mu, \tau) \leftrightarrow (j, \nu, \upsilon)\), as the
overlap integral vanished otherwise. Thus, there are
no states with energy lower than \(\langle \Phi_h \rangle\). Theorem 1 follows
immediately from the SO(3) symmetry of the model. \(\square\)

B. Proof of theorem 2

The Perron–Frobenius theorem \([11]\) states (among
other things) that if all the elements of an irreducible
real square matrix are non-negative then it has a unique
largest real eigenvalue and that the corresponding eigen-
vector has strictly positive components. A Hermitian
matrix is reducible if and only if it can be block diag-
onalized by a permutation matrix. Let use write the
Hamiltonian \((\ref{11})\) in the form
\[
\mathcal{H} = \sum_{m=(1-N)/2}^{(N-1)/2} \mathcal{H}_m,
\]
where \(m\) labels the \(z\)-component of the total spin of
the system. Each of the \(N\) matrices \(M_m = -\mathcal{H}_m\) is irre-
ducible provided the Hamiltonian satisfies the connectiv-
ity condition. Furthermore, we have seen above that all
of the matrix elements of \(\mathcal{H}_m \leq 0\). Therefore each of the
\(M_m\) satisfy the Perron–Frobenius theorem.

The SO(3) symmetry of the Hamiltonian means that
\(|\Phi_h\rangle\) must be \(N\)-fold degenerate. As no states have lower
energy than \(|\Phi_h\rangle\) this means that the lowest energy states
of the \(S_z\)-sectors are necessarily degenerate and that, up
to this required \(N\)-fold degeneracy, \(|\Phi_h\rangle\) is unique. \(\square\)

IV. FRUSTRATION AND THE ORBITAL PART
OF THE GROUND STATE WAVEFUNCTION

For the Hubbard model the explicit wavefunction can
be straightforwardly constructed. Of course one could
simply take Eq. \((\ref{9})\) as a variational wavefunction and
minimize all of the \(\phi_i\). However, a more elegant approach
is to introduce an ancillary model of non-interacting spin-
less fermions on the same lattice:
\[
\mathcal{H}_a = -\sum_{ij\mu\nu} t_{ij} \hat{c}_{\mu\nu}^\dagger \hat{c}_{\nu\mu},
\]
and then make a particle-hole transformation \(\hat{h}_i^\dagger = \hat{c}_i\).
As this is a single particle Hamiltonian the ground state
wavefunction of the Hubbard model from the ancillary
non-interaction model. Often, directly minimizing Eq.
\((\ref{15})\) is not the most efficient approach, for example, if the
lattice is periodic a Fourier transformation leads directly
to the solution.

If all \(t_{ij} > 0\) the ground state must have \(\phi_i = \psi_i > 0\)
for all \(i\). That is, the wavefunction is bonding between
all sites. In this sense, the ground state wavefunction is
unfrustrated. Note that, in a periodic system, it always
possible to construct a wavefunction that is strictly
positive at high symmetry points with wavevectors, \(\mathbf{k}\),
satisfying \(2k = \mathbf{G}\), where \(\mathbf{G}\) is a reciprocal lattice vector.
The set of all such high symmetry points always includes
the \(\Gamma\)-point (origin of the unit cell).

Returning to the multiorbital model, the Perron–Frobenius
theorem guarantees the existence of a gauge for which all \(\phi_{\mu}\) are strictly positive. Thus again the
ground state wavefunction is unfrustrated.

In this context it is interesting to note the recent discovery
that on some topologically frustrated lattices antiferromagnetic states with magnetization near the classical
limit occur in the Nagaoka limit \([15–17]\). Again here
relasing the frustration in the orbital part of the wave-
function appears to play a crucial role \([16]\).

V. CONCLUSIONS

We have shown that structural balance and the absence
of antiferromagnetic exchange are sufficient to prove that
the ground state of infinite-\(U\) multiorbital model, Eq.
\((\ref{1})\), with arbitrary pairwise interactions in ferromagnetic.

Structural balance implies the absence of topological
frustration – therefore, for itinerant electrons, balance is
the natural definition of an unfrustrated lattice. While
bipartite lattices (with no hopping within the sublattices) are always balanced, many non-bipartite lattices are also balanced, see Fig. 2 for some examples. An interesting question, beyond the scope of this paper, is the role of structural balance to other problems involving frustration and itinerant electrons.

Balance is important because it allows for an unfrustrated orbital part of the ground state wavefunction. This is consistent with the general insight that Nagaoka’s theorem arises from the minimization of the hole’s kinetic energy.

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