Statistics of the occupation time
for a class of Gaussian Markov processes

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Abstract

We revisit the work of Dhar and Majumdar [Phys. Rev. E 59, 6413 (1999)] on the limiting distribution of the temporal mean $M_t = t^{-1} \int_0^t du \sign y_u$, for a Gaussian Markovian process $y_t$ depending on a parameter $\alpha$, which can be interpreted as Brownian motion in the scale of time $t' = t^{2\alpha}$. This quantity, for short the mean ‘magnetization’, is simply related to the occupation time of the process, that is the length of time spent on one side of the origin up to time $t$. Using the fact that the intervals between sign changes of the process form a renewal process in the time scale $t'$, we determine recursively the moments of the mean magnetization. We also find an integral equation for the distribution of $M_t$. This allows a local analysis of this distribution in the persistence region ($M_t \to \pm 1$), as well as its asymptotic analysis in the regime where $\alpha$ is large. We finally put the results thus found in perspective with those obtained by Dhar and Majumdar by another method, based on a formalism due to Kac.
1 Introduction

Consider the stochastic process $y_t$ defined by the Langevin equation

$$\frac{dy_t}{dt} = \sqrt{2\alpha} t^{\alpha-1/2} \eta_t, \quad (1.1)$$

where $\alpha$ is a positive parameter, and $\eta_t$ is a Gaussian white noise such that $\langle \eta_t \rangle = 0$ and $\langle \eta_{t_1} \eta_{t_2} \rangle = \delta(t_2 - t_1)$. In the new time scale

$$t' = t^{2\alpha},$$

this process satisfies the usual Langevin equation for one-dimensional Brownian motion,

$$\frac{dy'_{t'}}{dt'} = \zeta_{t'},$$

where $\zeta_{t'}$ is still a Gaussian white noise, with $\langle \zeta_{t'} \rangle = 0$ and $\langle \zeta_{t'_1} \zeta_{t'_2} \rangle = \delta(t'_2 - t'_1)$. The process defined by (1.1) is a simple example of subordinated Brownian motion \[1\]. As Brownian motion itself, it is Gaussian, Markovian, and non-stationary.

This process appears in various situations of physical interest. For instance, it is described in \[2\] as a Markovian approximation to fractional Brownian motion. It also appears in ref. \[3\] for the special case $\alpha = \frac{1}{4}$, as describing the time evolution of the total magnetization of a Glauber chain undergoing phase ordering.

Dhar and Majumdar \[4\] raised the question of computing the distribution of the occupation time of this process, that is the length of time spent by the process on one side of the origin up to time $t$,

$$T_t^\pm = \int_0^t du \frac{1 \pm \sigma_u}{2}, \quad (1.2)$$

where $\sigma_t = \text{sign } y_t$, or equivalently of

$$M_t = \frac{1}{t} \int_0^t du \sigma_u, \quad (1.3)$$

where $M_t$, the temporal mean of $\sigma_t$, is hereafter referred to as the ‘mean magnetization’ by analogy with physical situations where $\sigma_t$ represents a spin. The distribution of the occupation time bears information on the statistics of persistent events of the process beyond that contained in the persistence exponent \[5, 6, 7, 8, 9, 10, 11\]. This exponent governs the decay $\sim t^{-\theta}$ of the survival probability of the process, that is the probability that the process did not cross the origin up to time $t$. Actually, for the present case, the determination of $\theta$ is trite, as shown by a simple reasoning \[4\]: the probability for the Brownian process $y_{t'}$ not to change sign up to time $t'$ is known to decay as $(t')^{-1/2}$, hence for the original process as $t^{-\alpha}$. This shows that $\theta = \alpha$. 

When $\alpha = \frac{1}{2}$, the distribution of the fraction of time spent on one side of the origin by a random walker, or by Brownian motion, is given, in the long-time regime, by the arcsine law [12, 1]. In contrast, when $\alpha \neq \frac{1}{2}$, the explicit determination of this distribution, or equivalently of the distribution of $M_t$, seems very difficult. However, as shown in [4], in the long-time regime, the computation of the asymptotic moments $\langle M_t^k \rangle$ can be done recursively, using two different methods, yielding the same results. The first method relies on a formalism due to Kac [13], while the second one originates from ref. [5].

The method used in ref. [5] can be applied to any (smooth) process for which the intervals of time between sign changes are independent, when taken on a logarithmic scale, with finite (i.e., non zero) mean $\bar{\ell}$. It eventually leads to a recursive determination of the moments of $M_t$, as $t \to \infty$ (see equation (3.9) below).

Dhar and Majumdar make the observation that, since relations (3.9) are independent of $\bar{\ell}$, they can be applied to the determination of the moments of $M_t$ for the process (1.1). Comparing the resulting expressions of the moments thus obtained to those derived by their alternative method shows that this is indeed the case.

However it is not obvious to understand why relations (3.9) hold for the (non-smooth) process (1.1), since the assumptions made in order to derive them do not hold for such a process. In particular while, for the class of models with independent time intervals on a logarithmic scale, and finite (i.e., non zero) mean $\bar{\ell}$ (for which the method of ref. [5] has been devised), it is natural to work in a logarithmic scale of time, since the mean number of sign changes between 0 and $t$ scales as $\langle N_t \rangle \approx (\ln t) / \bar{\ell}$, this is not so in the present case, since $\bar{\ell}$ asymptotically vanishes, and the mean number of sign changes scales as $\langle N_t \rangle \approx 2 \pi^{-1/2} t^{\alpha-1/2}$ [11]. The validity of relations (3.9) for the process (1.1) therefore requires an explanation.

In the present work, we revisit and extend the study done in [4].

We first give a new derivation of the asymptotic expressions of the moments $\langle M_t^k \rangle$. We start from the same premises as in ref. [5], then follow another route –more adapted to the process under study– because of the difficulties encountered in applying step by step the method of ref. [5] to the present case (sections 2-5). We then identify the symmetry properties of the distributions of the random variables that appear in the computations, and derive a functional integral equation, the solution of which yields the distribution of $M_t$ (section 6). This approach is first checked on the case $\alpha = \frac{1}{2}$ (section 7). It is then successively applied to the study of the local behavior of this distribution in the persistence region, for general $\alpha$ (section 8), and to the large-$\alpha$ regime (section 9).

We finally discuss some aspects of ref. [4]. We explain why a formal application of the method of ref. [5] to the present case is only heuristic, and give a new interpretation of the results obtained in [4] with the method of Kac, under the light of the present work (section 10).
2 Observables of interest

Changes of sign of the process \( y_t \) (or zero crossings) occur at discrete instants of time \( t_1, t_2, \ldots, t_n, \ldots \), once the process is suitably regularized at short times. We assume that the process starts at the origin, so that \( t_0 = 0 \) is also a sign change. Let \( N_t \) be the number of sign changes which occurred between 0 and \( t \), i.e., \( N_t \) is the random variable for the largest \( n \) for which \( t_n \leq t \).

In the scale \( t' \), where the process is (regularized) Brownian motion, sign changes occur at the instants of time \( t_n' = (t_n)^{2\alpha} \), and \( N_t' \equiv N_t \) is the random variable for the largest \( n \) for which \( t_n' \leq t' \). The intervals of time between sign changes are denoted by \( \tau_n' = t_n' - t_{n-1}' \). These are independent, identically distributed random variables, with a probability density function \( \rho(\tau') \). For large values of \( \tau' \), \( \rho(\tau') \) decays proportionally to \( (\tau')^{-3/2} \). This behavior is independent of the regularizing procedure, while its prefactor just reflects the choice of time units. The density \( \rho(\tau') \) is therefore in the basin of attraction of a Lévy law of index \( \frac{1}{2} \). We choose units so that we have in Laplace space

\[
\mathcal{L}_{\tau'} \rho(\tau') = \hat{\rho}(s) = \left( e^{-s\tau'} \right) \approx 1 - \sqrt{s}.
\] (2.1)

The process formed by the independent intervals of time \( \tau_1', \tau_2', \ldots \), is known as a renewal process. In the original scale \( t \), the intervals of time \( \tau_n = t_n - t_{n-1} \) are not independent.

We denote by \( t_N \) the instant of the last change of sign of the process before time \( t \). This random variable depends implicitly on time \( t \) through \( N_t \). In the scale \( t' \), we have \( t_N' = (t_N)^{2\alpha} \).

The occupation times \( T_t^+ \) and \( T_t^- \) (see equation (1.2)) are the lengths of time spent by the sign process \( \sigma_t \) respectively in the + and − states, up to time \( t \), hence

\[ t = T_t^+ + T_t^- . \]

They are simply related to the mean magnetization (1.3) by

\[ tM_t = T_t^+ - T_t^- = 2T_t^+ - t = t - 2T_t^- . \]

Assume that \( y_t > 0 \) at \( t = 0^+ \), i.e., \( \sigma_{t=0} = +1 \). Then

\[ tM_t = \begin{cases} (t - t_N) + (t_N - t_{N-1}) - \cdots & \text{if } N_t = 2k + 1 \text{ (i.e., } \sigma_t = -1 \text{),} \\ (t - t_N) - (t_N - t_{N-1}) + \cdots & \text{if } N_t = 2k \text{ (i.e., } \sigma_t = +1 \text{).} \end{cases} \]

The converse holds if \( \sigma_{t=0} = -1 \). Hence we have, with equal probabilities,

\[ M_t = \pm(1 - 2\xi_t) , \] (2.2)

where

\[ \xi_t = \frac{1}{t} (t_N - t_{N-1} + \cdots) \]

Hereafter we denote by a prime any temporal variable in this scale.
is the fraction of time spent in the state + if \( \sigma_t = -1 \), and conversely. The last formula can be rewritten as

\[
\xi_t = \frac{t_N}{t} X_N, \tag{2.3}
\]

where the \( X_N \) obey the recursion

\[
X_N = 1 - \frac{t_{N-1}}{t_N} X_{N-1}, \tag{2.4}
\]

with \( X_1 = 1 \). Both random variables \( X_N \) and \( t_{N-1}/t_N \) depend implicitly on time \( t \) through \( N_t \).

For instance, if \( \sigma_{t=0} = +1 \) and \( N_t = 4 \), then

\[
M_t = \frac{1}{t} \left( (t - t_4) - (t_4 - t_3) + (t_3 - t_2) - (t_2 - t_1) + t_1 \right) = 1 - 2\xi_t,
\]

where \( \xi_t = (t_4 - t_3 + t_2 - t_1)/t = t_4 X_4/t \), with

\[
X_4 = 1 - \frac{t_3}{t_4} \left( 1 - \frac{t_2}{t_3} \left( 1 - \frac{t_1}{t_2} \right) \right).
\]

3 Methods of solution

Equations (2.2), (2.3) and (2.4) contain in essence the solution to the problem posed, namely the determination of the limiting distribution of the mean magnetization \( M_t \) for \( t \to \infty \). Unfortunately, no explicit solution can be attained in general.

However, from (2.2)-(2.4), one can obtain recursively the moments of \( M_t \), in the long-time limit. This can be done either along the lines of ref. [5], as done in [4], or by the method of the present work. In this section, we explain the difficulty encountered when applying the method of ref. [5] to the process (1.1), in order to justify the more lengthy path we have adopted for the derivation of the moments. We shall come back to the comparison between the two methods in section 10.

3.1 General framework

Assume that, in the long-time regime, the dimensionless random variables \( t_N/t \), \( t_{N-1}/t_N \), \( X_N \), \( \xi_t \), and \( M_t \) possess a limiting joint distribution. Define

\[
H = \lim_{t \to \infty} \frac{t_N}{t}, \quad F = \lim_{t \to \infty} \frac{t_{N-1}}{t_N},
\]

\[
X = \lim_{t \to \infty} X_N, \quad \xi = \lim_{t \to \infty} \xi_t, \quad M = \lim_{t \to \infty} M_t.
\]
Then the equations to be solved are:

\begin{align}
X &= 1 - FX, \quad (3.1) \\
\xi &= HX, \quad (3.2) \\
M &= \pm (1 - 2\xi). \quad (3.3)
\end{align}

These equalities hold in distribution, and the random variables entering them are not independent a priori. Equation (3.1) is to be understood as the fixed-point equation corresponding to the recursion (2.4), while (3.2) and (3.3) respectively correspond to (2.3) and (2.2).

Assume that the distribution of the random variable $F$ is given, and that $F$ is independent of $X$. Even so, solving (3.1) is difficult in general [14, 15, 16, 17]. However, obtaining the moments of $X$ recursively is easier. If furthermore $H$ and $X$ are independent and the moments of $H$ are known, then (3.2) and (3.3) determine the moments of $M$.

### 3.2 The diffusion equation: a reminder

Such a situation precisely arises in the example treated in [5]: the process $y_t$ is the diffusing field at a fixed point of space, evolving from random initial conditions, and the so-called independent-interval approximation is used [18, 19]. In the long-time regime, the process is stationary in the logarithmic scale of time $T = \ln t$. As a consequence, the autocorrelation function of the sign process, $A(|\Delta T|) = \langle \sigma_T \sigma_{T+\Delta T} \rangle$, only depends on the difference of logarithmic times [18, 19].

Consider the intervals of time $\ell_N$ between successive sign changes of the process in the logarithmic scale of time, $\ell_N = T_N - T_{N-1}$, or

\[
e^{-\ell_N} = \frac{t_{N-1}}{t_N}.
\]

The independent-interval approximation consists in considering the intervals $\ell_N$ as independent, thus defining a renewal process. The distribution of the random variable $\ell_N$ can then be derived, in Laplace space, from the knowledge of the correlation function $A(|\Delta T|)$. This distribution is found to be independent of time, because the process is stationary in logarithmic time. Its average, $\langle \ell_N \rangle = \bar{\ell}$, is some time-independent positive number. Explicitly,

\[
\hat{f}_{\ell_N}(s) = \langle e^{-st_N} \rangle = \langle F_s \rangle = \frac{1 - \bar{\ell} \hat{g}(s)}{1 + \bar{\ell} \hat{g}(s)},
\]

with

\[
g(s) = \frac{s}{2} \left( 1 - s \hat{A}(s) \right),
\]

and

\[
\hat{f}_{\ell_N}(s) = \frac{1 - \bar{\ell} \hat{g}(s)}{1 + \bar{\ell} \hat{g}(s)}.
\]
where \( \hat{A}(s) \) is the Laplace transform of \( A(T) \). In particular, the moments
\[
 f_k = \langle F^k \rangle = \left\langle \left( \frac{t_{N-1}}{t_N} \right)^k \right\rangle = \left\langle e^{-k \ell_N} \right\rangle = \hat{f}_{\ell_N}(k)
\] (3.7)
are independent of time. Thus from (3.1) the moments of \( X \) are determined recursively in terms of the \( f_k \) (see (A.18)).

In the long-time regime, the distribution of the backward recurrence time of the process in the logarithmic scale, \( \lambda = T - T_N \), is also independent of time. This logarithmic recurrence time is related to the random variable \( H \) by
\[
e^{-\lambda} = \frac{t_N}{t} = H.
\]
Its distribution in Laplace space reads
\[
 \hat{f}_\lambda(s) = \left\langle e^{-s\lambda} \right\rangle = \langle H^s \rangle = \frac{2g(s)}{s\left(1 + \ell g(s)\right)}.
\] (3.8)

The random variables \( X \) and \( H \) (or \( \lambda \)) are independent. Hence (3.2) and (3.3) determine the moments of \( M \). A remarkable fact is that the moments thus obtained, which are functions of the \( f_k \) and of \( \bar{\ell} \), become independent of \( \bar{\ell} \) when the \( f_k \) are expressed in terms of the \( \hat{A}(k) \equiv \hat{A}_k \), using equations (3.5), (3.6) and (3.7). Thus
\[
 \langle M^2 \rangle = \hat{A}_1,
\]
\[
 \langle M^4 \rangle = 1 - \frac{\left(1 - 3\hat{A}_1 + 4\hat{A}_2\right)\left(1 - 3\hat{A}_3\right)}{1 - 2\hat{A}_2},
\] (3.9)
and so on.

More generally, the method used in [5] can be applied to any process for which the intervals of time between sign changes are independent, when taken on a logarithmic scale. It eventually leads to a recursive determination of the moments of \( M \), resulting in (3.9).

3.3 The case of the process (1.1)
The situation for the process (1.1) is more difficult because, in the limit \( t \to \infty \), \( t_{N-1}/t_N \to F = 1 \). Hence (3.1) no longer determines \( X \), and furthermore \( \langle \ell_N \rangle \to 0 \). Now, for the class of models with independent time intervals \( \ell_N \) on a logarithmic scale, and finite (i.e., non-zero) \( \langle \ell_N \rangle = \bar{\ell} \), for which the method of ref. [5], sketched above, has been devised, the mean number of sign changes between 0 and \( t \) scales as \( \langle N_t \rangle \approx (\ln t)/\bar{\ell} \). So, in contrast, in the present case there is no obvious reason to work
with this logarithmic scale of time, since $\langle \ell_N \rangle$ asymptotically vanishes, and the mean number of sign changes scales as $\langle N_t \rangle \approx 2 \pi^{-1/2} t^{\alpha}$\).

On the other hand, if time is kept finite, then the time intervals $\ell_N$ are not independent and the process (1.1) is not stationary, precluding again the application of the method of ref. [3].

A way out is to formally apply this method to the process (1.1), without paying attention to the difficulties mentioned above, and taking advantage of the fact that the moments $\langle M^k \rangle$, given by (3.9), are independent on $\bar{\ell}$, and therefore (hopefully) insensitive to the fact that $\langle \ell_N \rangle \to 0$. This approach, which is the one followed by Dhar and Majumdar [4], is however only heuristic, as further discussed in section 10.

Our approach relies instead on the fact that the time intervals $\tau'_n$ between two sign changes of the process (1.1) form a renewal process (the $\tau'_n$ are independent, identically distributed random variables with density $\rho(\tau')$, given by (2.1) for large $\tau'$). This is a fundamental property of the process (1.1), and in particular of Brownian motion if $\alpha = \frac{1}{2}$.

This property allows us to determine the limiting distribution $f_H$ of $H$ when $t \to \infty$, and the moments $f_{k,t}$ of the random variable $t_{N-1}/t_N$, which are now time dependent. We also find the explicit time-dependent expression of $\langle \ell_N \rangle$. Using the original equation (2.4), instead of (3.1), and equations (3.2) and (3.3), we eventually recover the expressions (3.9) of the $\langle M^k \rangle$, thus extending their range of applicability.

We then establish an integral equation for $f_X$, and study its consequences.

## 4 Distribution of $t_N/t$

Using the independence of the $\tau'_1, \tau'_2, \ldots$, we first determine the distribution of the random variable $t_N'/t$, from which we then deduce that of $t_N$. The method used below is borrowed from ref. [11], where a thorough study of the statistics of the occupation time of renewal processes can be found.

We denote by $f_{t_N',N}$ the joint probability density of the random variables $t_N'$ and $N_t$. It reads

$$f_{t_N',N}(t'; y, n) = \frac{d}{dy} \mathcal{P}(t_N' < y, N_{t'} = n) = \langle \delta(y - t'_N) I(t'_n < t' < t'_{n+1}) \rangle,$$

where $I(t'_n < t' < t'_{n+1}) = 1$ if the event inside the parenthesis occurs, and 0 if not. The brackets denote the average over $\tau'_1, \tau'_2, \ldots$ Summing over $n$ gives the distribution of $t_N'$,

$$f_{t_N'}(t'; y) = \sum_{n=0}^{\infty} f_{t_N',N}(t'; y, n) = \langle \delta(y - t'_N) \rangle.$$

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In Laplace space, where $s$ is conjugate to $t'$ and $u$ to $y$,
\[
\mathcal{L}_{t',y} f_{t'N}(t'; y, n) = \hat{f}_{t'N}(s; u, n) = \left\langle e^{-ut_n} \int_{t'_n}^{t'_{n+1}} dt' e^{-st'} \right\rangle
\]
\[
= \left\langle e^{-ut_n} e^{-st'_n} \frac{1 - e^{-s\tau'_{n+1}}}{s} \right\rangle
\]
\[
= \hat{\rho}(s + u)^{n} \frac{1 - \hat{\rho}(s)}{s} \quad (n \geq 0).
\]
(4.1)

Note that setting $u = 0$ in (4.1) gives the distribution of $N_{t'}$. We finally obtain
\[
\mathcal{L}_{t',y} f_{t'N}(t'; y) = \mathcal{L}_{t'} \left\langle e^{-ut'_N} \right\rangle = \hat{f}_{t'N}(s; u)
\]
\[
= \sum_{n=0}^{\infty} \hat{f}'_{t'N}(s; u, n) = \frac{1}{1 - \hat{\rho}(s + u)} \frac{1 - \hat{\rho}(s)}{s}.
\]

In the long-time regime, i.e., for $s$ and $u$ simultaneously small, we get the scaling form
\[
\hat{f}_{t'N}(s; u) \approx \frac{1}{\sqrt{s(s + u)}},
\]
which yields
\[
f_{t'N}(t'; y) \approx \frac{1}{\sqrt{y(t' - y)}}.
\]

As a consequence, the random variable $H' = \lim_{t' \to \infty} t'^{-1} t'_N$ possesses the limiting distribution
\[
f_{H'}(x) = \frac{1}{\pi \sqrt{x(1 - x)}},
\]
(4.2)

which is the arcsine law on $[0, 1]$.

Using the equality $t_{N'/t} = (t'_{N'/t'})^{1/2\alpha}$, this last result yields immediately the distribution of
\[
H = \lim_{t \to \infty} t_{N'/t} = (H')^{1/2\alpha},
\]
which reads
\[
f_H(x) = \frac{2\alpha x^{\alpha-1}}{\pi \sqrt{1 - x^{2\alpha}}} = \frac{2\alpha}{\pi x \sqrt{x-2\alpha - 1}}.
\]
(4.3)

This is the main result of this section. Let us define
\[
h(s, \alpha) = \langle H^s \rangle = \frac{1}{\pi} B \left( \frac{s}{2\alpha} + \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{s}{2\alpha} + \frac{1}{2} \right),
\]
(4.4)
where \( B(a, b) = \Gamma(a)\Gamma(b) / \Gamma(a + b) \) is the beta function. For integer values of \( s \), (4.4) gives the moments of \( f_H \), denoted by

\[
\begin{align*}
\hat{h}_k^{(s)} &= B(k + \frac{1}{2}, k) = \frac{(2k)!}{2^{2k}(k!)^2}.
\end{align*}
\]

In the particular case \( \alpha = \frac{1}{2} \), corresponding to Brownian motion, the distribution of \( H \equiv H' \) is the arcsine law (4.2), with moments

\[
\begin{align*}
\hat{h}(1/2, k) &= \frac{1}{\pi B(k + 1/2, 1/2)} = \frac{(2k)!}{2^{2k}(k!)^2},
\end{align*}
\]

(4.5)

### 5 Determination of the moments

In order to obtain recursion relations for the moments of the random variable \( X \), we proceed in two steps. We first compute the moments of \( t_{N-1}/t_N \), from which we deduce those of \( t_{N-1}/t_N \). The recursion relations for the \( \langle X^k \rangle \) then emerge from (2.4). Equations (3.2) and (3.3) finally determine the moments of \( M \).

#### 5.1 Moments of \( t_{N-1}/t_N \)

We first determine the probability density function of the joint variables \( t'_{N-1} \) and \( t'_N \). In Laplace space, we have

\[
\hat{f}_{t'_{N-1},t'_N}(s; u, v, n) = \hat{f}_{t'_{N-1},t'_N}(s; u, v) = \sum_{n=0}^{\infty} \hat{f}_{t'_{N-1},t'_N}(s; u, v, n)
\]

Summing over \( n \) gives

\[
\begin{align*}
\mathcal{L}_{t'} \langle e^{-ut'_{N-1}}e^{-vt'} \rangle &= \hat{f}_{t'_{N-1},t'_N}(s; u, v) = \sum_{n=0}^{\infty} \hat{f}_{t'_{N-1},t'_N}(s; u, v, n) \\
&= \frac{1 - \hat{\rho}(s)}{s} \left( 1 + \frac{\hat{\rho}(s + v)}{1 - \hat{\rho}(s + u + v)} \right),
\end{align*}
\]

(5.1)

so that in particular \( \hat{f}_{t'_{N-1},t'_N}(s; u = 0, v = 0) = 1/s \).

The first moment of the random variable \( t'_{N-1}/t'_N \) is obtained by considering

\[
\begin{align*}
\mathcal{L}_{t'} \langle t'_{N-1} \rangle &= \int_0^\infty \frac{dv}{v} \left( -\frac{d}{dv} \right)_{u=0} \mathcal{L}_{t'} \langle e^{-ut'_{N-1}}e^{-vt'} \rangle \\
&= \frac{\hat{\rho}(s)}{s} + \frac{1 - \hat{\rho}(s)}{s} \ln(1 - \hat{\rho}(s)) \\
&\approx \frac{1}{s} + \ln s \frac{1}{2\sqrt{s}}.
\end{align*}
\]

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which leads to
\[ \langle \frac{t'_{N-1}}{t'_N} \rangle \approx 1 - \frac{\ln t'}{2\sqrt{\pi t'}}, \] (5.2)

omitting the finite parts of the logarithms.

This computation generalizes to higher-order moments, using the asymptotic form (2.1) in (5.1). We have
\[
\mathcal{L}_\nu \left( \left( \frac{t'_{N-1}}{t'_N} \right)^k \right) = \left( \int_0^\infty dv \right)^k \left( -\frac{d}{du} \right)^k
\mathcal{L}_\nu \left( e^{-u't'_{N-1}e^{-vt'_N}} \right)
\approx \frac{1}{s} + kh_k^{(1/2)} \ln s \sqrt{s},
\]
which leads to
\[ \left( \frac{t'_{N-1}}{t'_N} \right)^k \approx 1 - kh_k^{(1/2)} \frac{\ln t'}{\sqrt{\pi t'}}, \] (5.3)

where \( h_k^{(1/2)} \) is given by equation (4.5). In particular, since \( h_1^{(1/2)} = \frac{1}{2} \), (5.2) is recovered.

The result (5.3) can be extended to non-integer values of \( k \). We have thus (see (3.4))
\[ \langle \ell'_N \rangle = -\langle \ln \frac{t'_{N-1}}{t'_N} \rangle = \lim_{k \to 0} \left( \frac{1 - \left( \frac{t'_{N-1}}{t'_N} \right)^k}{k} \right) \approx \frac{\ln t'}{\sqrt{\pi t'}}, \]
as \( \lim_{k \to 0} h_k^{(1/2)} = 1 \). Equation (5.3) can thus be rewritten as
\[ \left( \frac{t'_{N-1}}{t'_N} \right)^k \approx 1 - kh_k^{(1/2)} \langle \ell'_N \rangle. \] (5.4)

As announced above, when \( t \to \infty \), the random variable \( t'_{N-1}/t'_N \) converges to 1, in distribution.

The moments \( f_{k,t} \) of \( t_{N-1}/t_N \) are obtained from (5.4) as
\[ f_{k,t} = \left( \frac{t_{N-1}}{t_N} \right)^k = \left( \frac{t'_{N-1}}{t'_N} \right)^{k/2\alpha} \approx 1 - \frac{k}{2\alpha} h_k^{(\alpha)} \langle \ell'_N \rangle, \]
because \( h(k/2\alpha, 1/2) = h(k, \alpha) \equiv h_k^{(\alpha)} \). On the other hand,
\[ \bar{\ell}_t = \langle \ell_N \rangle = -\langle \ln \frac{t_{N-1}}{t_N} \rangle = \frac{1}{2\alpha} \langle \ell'_N \rangle \approx \frac{\ln t}{\sqrt{\pi t^\alpha}}, \]
hence finally
\[ f_{k,t} \approx 1 - k h_k^{(\alpha)} \bar{\ell}_t. \] (5.5)
5.2 Moments of $X$

From the recursion relation (2.4), we have

$$\langle X_N^k \rangle = \left\langle 1 - \frac{t_{N-1}}{t_N} X_{N-1} \right\rangle^k.$$  (5.6)

In the long-time regime, there is an asymptotic decoupling of the variables $X_{N-1}$ and $t_{N-1}/t_N$, so that it is legitimate to take $X_N \to X$, while keeping the leading time dependence of $f_{k,t}$, given by (5.5). This procedure can be justified along the lines of ref. [11]. Consider first the simple situation $\alpha = \frac{1}{2}$. The difference between unity and $t_{N-1}/t_N$, which gives rise to the result (5.5), is proportional to the interval $\tau_N = t_N - t_{N-1}$. This quantity has been shown in [11] to be, asymptotically for large $t$, independent of $t_{N-1}$, and distributed according to the a priori law $\rho(\tau)$. A similar decoupling asymptotically takes place for generic $\alpha$.

Denoting the moments $\langle X_N^k \rangle$ by $x_k$, we obtain

$$x_k = B(f_{k,t} x_k),$$  (5.7)

with $x_0 = f_{0,t} = 1$, and where we have introduced the notation $B$ for the linear binomial operator

$$B(x_k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j x_j.$$  (5.8)

As shown in the Appendix, (5.7) implies the following recursion relations, according to the parity of $k$,

$$x_k(1 + f_{k,t}) = B(x_k(1 + f_{k,t})) \quad (k \text{ odd}),$$  (5.9)

$$x_k(1 - f_{k,t}) = -B(x_k(1 - f_{k,t})) \quad (k \text{ even}).$$  (5.10)

Using the expression (5.5) of $f_{k,t}$, we obtain, in the limit $t \to \infty$, where $\bar{\ell}_t \to 0$,

$$x_k = B(x_k) \quad (k \text{ odd}),$$  (5.11)

$$k h^{(\alpha)}_k x_k = -B(k h^{(\alpha)}_k x_k) \quad (k \text{ even}).$$  (5.12)

These relations, which can be rewritten as

$$x_k = \frac{1}{2} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \quad (k \text{ odd}),$$  (5.13)

$$x_k = -\frac{1}{2k h^{(\alpha)}_k} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j j h^{(\alpha)}_j x_j \quad (k \text{ even}),$$  (5.14)

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determine the stationary values of the $x_k$ recursively. We thus obtain

\[
\begin{align*}
x_1 &= \frac{1}{2}, & x_2 &= \frac{h_1^{(a)}}{4h_2^{(a)}}, & x_3 &= -\frac{1}{4} + \frac{3h_1^{(a)}}{8h_2^{(a)}}, \\
x_4 &= -\frac{h_1^{(a)} + 3h_3^{(a)}}{8h_4^{(a)}} + \frac{9h_1^{(a)}h_3^{(a)}}{16h_2^{(a)}h_4^{(a)}}, & \ldots
\end{align*}
\]

5.3 Moments of $M$

For reasons similar to those exposed below equation (5.6), the random variables $H$ and $X$ (defined in the limit $t \to \infty$) are independent. Thus, by (3.2) we have

\[
\langle \xi^k \rangle = h_k^{(a)} x_k, \tag{5.15}
\]

which, together with equation (3.1), leads to a determination of the even moments of the mean magnetization $M$ in terms of the $x_k$:

\[
\langle M^k \rangle = \langle (1 - 2\xi)^k \rangle = B(2^k h_k^{(a)} x_k) \quad (k \text{ even}). \tag{5.16}
\]

We thus finally obtain

\[
\begin{align*}
\langle M^2 \rangle &= 1 - h_1^{(a)}, \\
\langle M^4 \rangle &= 1 + 2h_3^{(a)} - \frac{3h_1^{(a)}h_3^{(a)}}{h_2^{(a)}}, \\
\langle M^6 \rangle &= 1 - 5h_1^{(a)} - 10h_3^{(a)} - 16h_5^{(a)} + \frac{h_1^{(a)}(15h_3^{(a)} + 20h_5^{(a)})}{h_2^{(a)}} \\
+ \frac{(10h_1^{(a)} + 30h_3^{(a)})h_5^{(a)}}{h_4^{(a)}} - \frac{45h_1^{(a)}h_3^{(a)}h_5^{(a)}}{h_2^{(a)}h_4^{(a)}}, \tag{5.17}
\end{align*}
\]

and so on.

For instance, if $\alpha = \frac{1}{2}$, corresponding to Brownian motion, the successive even moments of $M$ are equal to $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \ldots$, i.e.,

\[
\langle M^{2j} \rangle = \frac{(2j)!}{2^{2j}(j!)^2} = h_j^{(1/2)},
\]

which are the even moments of the arcsine law on $[-1, 1]$ (see (7.2) below).
6 An integral equation for the determination of $f_M$

The recursion relation (5.11) expresses a symmetry property of the distribution $f_X$:

$$f_X(x) = f_X(1 - x)$$  \hspace{1cm} (6.1)

(see Appendix). This is also obvious from (3.1), since formally $F = 1$ in the present case.

The recursion relation (5.12), which can be rewritten as

$$k \left\langle \xi^k \right\rangle = -B(k \left\langle \xi^k \right\rangle) \quad (k \text{ even}),$$  \hspace{1cm} (6.2)

expresses a symmetry property of the distribution $f_\xi$, as we now show. First, it is easy to prove that

$$B(k \left\langle \xi^k \right\rangle) = -k \left\langle (1 - \xi)^{k-1} \right\rangle.$$  \hspace{1cm} (6.2)

Therefore (6.2) yields

$$\left\langle \xi^k \right\rangle = \left\langle (1 - \xi)^{k-1} \right\rangle \quad (k \text{ even}),$$

which is equivalent to the following symmetry property

$$\xi f_\xi(\xi) = (1 - \xi)f_\xi(1 - \xi),$$  \hspace{1cm} (6.3)

or

$$\phi(\xi) = \phi(1 - \xi),$$  \hspace{1cm} (6.4)

introducing the function

$$\phi(\xi) = \xi f_\xi(\xi).$$  \hspace{1cm} (6.5)

On the other hand, as a consequence of (3.2) and of the independence of $H$ and $X$, the distribution $f_\xi$ is equal to the convolution of $f_H$, given by (4.3), and of $f_X$:

$$f_\xi(\xi) = \int_\xi^1 \frac{dx}{x} f_X(x) f_H \left( \frac{\xi}{x} \right) = \frac{2\alpha}{\pi} \xi^{\alpha-1} \int_\xi^1 dx \frac{f_X(x)}{\sqrt{x^{2\alpha} - \xi^{2\alpha}}},$$  \hspace{1cm} (6.6)

hence

$$\phi(\xi) = \frac{2\alpha}{\pi} \xi^{\alpha} \int_\xi^1 dx \frac{f_X(x)}{\sqrt{x^{2\alpha} - \xi^{2\alpha}}}. $$  \hspace{1cm} (6.7)

In summary, two conditions determine the distribution $f_X(x)$: it obeys the symmetry property (6.1), and the function $\phi(\xi)$, given by (6.7), obeys the symmetry property (6.4).

Once the probability density $f_X$ is known, $f_\xi$ is given by (6.6). Finally, (3.3), (6.5), and (6.4) imply

$$f_M(m) = \frac{1}{1 - m^2} \phi \left( \frac{1 \pm m}{2} \right).$$  \hspace{1cm} (6.8)

We explore the consequences of this general setup in the next three sections.
7 The case $\alpha = \frac{1}{2}$

This situation corresponds to Brownian motion. It is easy to check that the uniform distribution on $[0, 1]$,

$$f_X(x) = 1,$$  \hspace{1cm} (7.1)

solves the problem. Indeed, equations (6.6) and (6.7) yield

$$\phi(\xi) = \frac{2}{\pi} \sqrt{\xi(1 - \xi)}, \quad f_\xi(\xi) = \frac{2}{\pi} \sqrt{\frac{1 - \xi}{\xi}},$$

which satisfy (6.3) and (6.4). Finally, by (6.8), the limiting distribution of $M_t$ is obtained:

$$f_M(m) = \frac{1}{\pi \sqrt{1 - m^2}},$$  \hspace{1cm} (7.2)

which is the arcsine law on $[-1, 1]$.

All these results can be derived by more direct means, using the fact that in the present case the time intervals $\tau_1, \tau_2, \ldots$ between sign changes define a renewal process [11].

8 Local analysis in the persistence region

The persistence region is defined by the condition $M \to \pm 1$, i.e., $\xi \to 0$ or $\xi \to 1$.

Considering (6.6) for $\xi \to 0$ yields at once

$$f_\xi(\xi) \approx \frac{2\alpha}{\pi} \left< X^{-\alpha} \right> \xi^{\alpha - 1},$$  \hspace{1cm} (8.1)

provided the average $\left< X^{-\alpha} \right>$ is convergent (see the comment below equation (9.12)). As a consequence, using (6.5) and (6.8), we obtain

$$f_M(m) \approx C (1 - m^2)^{\alpha - 1},$$  \hspace{1cm} (8.2)

with

$$C = \frac{2^{1 - 2\alpha} \alpha}{\pi} \left< X^{-\alpha} \right>.$$  \hspace{1cm} (8.3)

The behavior of the distribution $f_X(x)$ as $x \to 0$ can be determined as well. Assuming $f_X(x) \approx Ax^\gamma$ ($x \to 0$), and using (6.1), we obtain

$$\phi(\xi) \approx A \sqrt{\frac{2\alpha}{\pi}} \int_0^\xi dx \frac{\tilde{x}^\gamma}{\sqrt{\xi - \tilde{x}}} = A \sqrt{\frac{2\alpha}{\pi} \frac{\Gamma(\gamma + 1)}{\Gamma\left(\gamma + \frac{3}{2}\right)}} \xi^{\gamma + 1/2},$$
with $\tilde{\xi} = 1 - \xi$, $\tilde{x} = 1 - x$. An identification with (8.1), using again (6.3) and (6.3), yields the values of $\gamma$ and $A$, hence

$$f_X(x) \approx \sqrt{\frac{2\alpha}{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \langle X^{-\alpha} \rangle x^{\alpha - 1/2}. \quad (8.4)$$

Let us compare the singular behavior (8.2) of $f_M$ in the persistence region with the beta law on $[-1, 1]$ of same index:

$$f_M^\beta(m) = C^\beta (1 - m^2)^{\alpha - 1}, \quad (8.5)$$

where

$$C^\beta = \frac{\Gamma \left( \alpha + \frac{1}{2} \right)}{\sqrt{\pi \Gamma(\alpha)}}. \quad (8.6)$$

A measure of the difference between the two distributions is provided by the enhancement factor

$$E = \frac{C}{C^\beta}. \quad (8.7)$$

For $\alpha = \frac{1}{2}$, the distribution $f_M(m)$ is the arcsine law (7.2), which is a beta law. Equation (7.1) yields $\langle X^{-1/2} \rangle = 2$, so that $C = C^\beta = 1/\pi$, and $E = 1$. The estimate (8.4) also agrees with (7.1).

For $\alpha \neq \frac{1}{2}$, the distribution $f_M(m)$ is no longer a beta law, so that the enhancement factor $E$ is non-trivial.

## 9 Asymptotic analysis for large values of $\alpha$

For large values of $\alpha$, the distributions $f_X(x)$, $f_\xi(\xi)$, and $f_M(m)$ are expected to share, at least qualitatively, some resemblance with the beta law (8.5). This observation suggests to set

$$f_X(x) \sim_{\alpha \gg 1} \exp \left( - \alpha S(x) \right), \quad (9.1)$$

with

$$S(x) = S(1 - x), \quad (9.2)$$

as a consequence of (6.1). The function $S(x)$ is expected to be regular, and positive, with a minimum at $S(\frac{1}{2}) = 0$, just as its counterpart

$$S^\beta(x) = - \ln(4x(1 - x)), \quad (9.3)$$

associated with the beta law (8.5).
With these hypotheses, \( \phi(\xi) \), given by (6.7), can be estimated as follows. Setting \( x = \xi + \varepsilon \) with \( \varepsilon \ll 1 \), we have \( f_X(x) \approx e^{-\alpha S(\xi) - \alpha \varepsilon S'(\xi)} \) and \( x^{2\alpha} - \xi^{2\alpha} \approx \xi^{2\alpha}(e^{2\alpha\varepsilon/\xi} - 1) \).

The change of integration variable from \( x \) to \( z = 2\alpha\varepsilon/\xi \) yields

\[
\phi(\xi) \approx P(\xi) f_X(\xi),
\]

with

\[
P(\xi) = \frac{\xi}{\pi} \int_0^\infty dz \frac{e^{-\xi S'(\xi)z/2}}{\sqrt{e^z - 1}}.
\]

Setting \( y = e^{-z} \), and returning to the variable \( x \), we finally obtain

\[
P(x) = \frac{x}{\sqrt{\pi}} \frac{\Gamma \left( \frac{1}{2} + \frac{x}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{xS'(x)}{2} \right)}
\]

provided the arguments of both Gamma functions are positive (this will indeed be the case). Note that (9.5) does not involve the parameter \( \alpha \) any more.

Omitting again pre-exponential factors, (9.4) implies

\[
f_M(m) \approx \alpha \gg 1 \exp \left( -\alpha S \left( \frac{1 \pm m}{2} \right) \right).
\]

In the regime of large \( \alpha \), the three distributions of interest are therefore given by a single function \( S(x) \). The problem then amounts to finding \( S(x) \), with the symmetry property (9.2), and such that the corresponding function \( P(x) \), given by (9.5), obey

\[
P(x) = P(1 - x),
\]

as a consequence of (6.4). The function \( S(x) \) is entirely determined by the above conditions. This property is more evident in the present regime than in the general case of section 6, because (9.3) is explicit, while (6.7) is an integral relationship.

Let us first investigate the behavior of \( S(x) \) for \( x \to \frac{1}{2} \), i.e., \( m \to 0 \), corresponding to the center of the distributions. Inserting the expansion

\[
S(x) = c_2 \left( x - \frac{1}{2} \right)^2 + c_4 \left( x - \frac{1}{2} \right)^4 + \cdots
\]

in (9.3), (9.7), and expanding the Gamma functions accordingly, we obtain

\[
c_2 = \frac{2}{\ln 2}, \quad c_4 = \frac{4}{3 \ln 2} + \frac{\pi^2}{3(\ln 2)^3} - \frac{\zeta(3)}{(\ln 2)^4}, \quad \ldots
\]

and

\[
P(x) = \frac{1}{2} + \left( \frac{\pi^2}{12(\ln 2)^2} - 3 \right) \left( x - \frac{1}{2} \right)^2 + \cdots
\]
To leading order, keeping only the quadratic term in $S(x)$, we find that the bulk of the distributions is asymptotically given by narrow Gaussians for $\alpha$ large, namely

$$f_X(x) \sim_{\alpha \gg 1} f_\xi(x) \sim_{\alpha \gg 1} \exp \left( -\frac{2\alpha}{\ln 2} \left( x - \frac{1}{2} \right)^2 \right), \quad f_M(m) \sim_{\alpha \gg 1} \exp \left( -\frac{\alpha}{2 \ln 2} m^2 \right).$$

The latter result is in agreement with the expressions (5.17) of the moments of $M$, which behave as $\langle M^2 \rangle \approx (\ln 2)/\alpha$, $\langle M^4 \rangle \approx 3(\ln 2)^2/\alpha^2$, and so on, for $\alpha \gg 1$.

It is also worthwhile noticing that the beta law (8.5), (9.3) also becomes a narrow Gaussian for $\alpha$ large. We have $S^{\text{beta}}(x) \approx 4(x - \frac{1}{2})^2$ and $f_M^{\text{beta}}(m) \approx e^{-\alpha m^2}$, so that the beta law misses a finite factor $2 \ln 2 \approx 1.3862$ in the variance of the mean magnetization.

The expression (9.8) of the subleading amplitude $c_4$, involving Riemann’s zeta function, shows however that the function $S(x)$ is altogether non-trivial.

![Figure 1](image_url)

**Figure 1:** Plot of the function $S(x)$ characterizing the limiting distributions of the variables $X$ and $\xi$ and of the mean magnetization $M$ in the large-$\alpha$ region, against $x$ (full line), compared with the function $S^{\text{beta}}(x)$ associated with the beta law (dashed line).

Let us now turn to the behavior of $S(x)$ deep in the tails of the distributions, i.e., for $x \to 0$ or 1, or $m \to \pm 1$, corresponding to the persistence region. The general result (8.2) shows that $S(x) \approx -\ln x$ has a logarithmic divergence as $x \to 0$. As a consequence, the Gamma function in the numerator of expression (9.3) for $P(x)$ becomes singular, as its argument goes to zero. Furthermore, in the same expression for $P(1 - x)$, the arguments of both Gamma functions tend to infinity. A careful
treatment of (9.5) yields the more complete expansions as \( x \to 0 \)

\[
S(x) = - \ln x + S_0 + 2 \sqrt{\frac{2x}{\pi}} + \cdots, \quad P(x) = \sqrt{\frac{2x}{\pi}} + \cdots, \quad (9.9)
\]

while the constant \( S_0 \) cannot be predicted by this local analysis. The square-root behavior of \( P(x) \) and its prefactor agree with the general results (8.1), (8.4).

We have numerically determined the solution of (9.5), (9.7) over the whole range \( 0 < x < \frac{1}{2} \), obtaining thus accurate values of \( S(x) \). This approach yields in particular \( S_0 \approx -2.0410 \).

Figure 1 shows a plot of the function \( S(x) \) thus obtained, compared with \( S_{\text{beta}}(x) \).

As the amplitude \( C_{\text{beta}} \) of the beta law (8.3) remains of order unity, within exponential accuracy, the result (9.9) for \( S(x) \) implies that the amplitude \( C \) of the power law (8.2) in the persistence region, and the enhancement factor \( E \) defined in (8.7), blow up exponentially, as

\[
C \sim \alpha \gg 1 \quad E \sim \alpha \gg 1 \exp(G\alpha), \quad (9.10)
\]

with

\[
G = \lim_{x \to 0} (S_{\text{beta}}(x) - S(x)) = -S_0 - 2 \ln 2 \approx 0.6547. \quad (9.11)
\]

In order to test the relevance of this large-\( \alpha \) approach, we have evaluated numerically \( E \) for various values of the parameter \( \alpha \), and compared the results with the exponential law (9.10) predicted for large \( \alpha \). The computation of \( E \) can be done in (at least) two different ways.

The first method consists in directly evaluating the limit

\[
E = \lim_{n \to \infty} \frac{\langle M^{2n} \rangle}{\langle M^{2n} \rangle_{\text{beta}}},
\]

The moments of the beta law (8.7) read

\[
\langle M^{2n} \rangle_{\text{beta}} = \frac{\Gamma \left( \alpha + \frac{1}{2} \right) \Gamma \left( n + \frac{1}{2} \right)}{\sqrt{\pi} \Gamma \left( n + \alpha + \frac{1}{2} \right)},
\]

while the true moments \( \langle M^{2n} \rangle \) are determined from (5.13), (5.14), and (5.16), up to some maximal order, typically \( n_{\text{max}} = 100 \) to 150, beyond which numerical accuracy rapidly deteriorates, because the computation of \( \langle M^{2n} \rangle \) involves alternating sums.

The second method consists in combining (8.3) and (8.8), getting thus

\[
E = \frac{2^{1-2\alpha}}{\sqrt{\pi}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})} \langle X^{-\alpha} \rangle,
\]

\[
19
\]
and in evaluating $\langle X^{-\alpha} \rangle$ as

$$\langle X^{-\alpha} \rangle = \langle (1 - X)^{-\alpha} \rangle = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha)}{n!} x_n.$$  \hspace{1cm} (9.12)

The behavior (8.4) implies that the $x_n$ decay as $n^{-\alpha-1/2}$, so that the term of order $n$ in the above sum decays as $n^{-3/2}$. Hence this sum is convergent, and truncating it at some order $n_{\text{max}}$ brings a correction proportional to $n_{\text{max}}^{-1/2}$. The $x_n$ are again determined from (5.13) and (5.14). A linear extrapolation in $n_{\text{max}}^{-1/2}$ of the results of both schemes turns out to yield consistent results. We have for instance $E \approx 1.443$ for $\alpha = 1$.

![Figure 2: Logarithmic plot of the enhancement factor $E$ in the persistence region, against $\alpha$, evaluated numerically as described in the text (symbols). The straight line has the theoretical slope $G$ of (9.11).](image)

Figure 2 shows our numerical results for the enhancement factor $E$, for values of $\alpha$ up to 3. The comparison with the exponential law (9.10) is convincing, in spite of the moderate values of $\alpha$ used.

10 Revisiting the work of Dhar and Majumdar

10.1 Using the method of section 3.2

Let us first show that the expressions (5.17) for the even moments of $M$ are the same as those obtained by using (3.9), with the expression of $\hat{A}(s)$ appropriate to the
process (1.1), as done in [4].

The autocorrelation $A(|\Delta T|) = \langle \sigma_T \sigma_{T+\Delta T} \rangle$ of the sign process $\sigma_t$ in the logarithmic scale of time $T = \ln t$ reads $A(T) = (2/\pi) \arcsin(e^{-aT})$ [4], with Laplace transform

$$\hat{A}(s) = \frac{1}{s} \left[ 1 - \frac{1}{\pi} B \left( \frac{s}{2\alpha}, \frac{1}{2}, \frac{1}{2} \right) \right].$$

(10.1)

We notice that $\hat{A}(s)$ is related to $h(s,\alpha)$, defined in (4.4), by

$$h(s,\alpha) = 1 - s\hat{A}(s),$$

(10.2)

where the right side is equal to $2g(s)/s$ by (3.6). Using this identity, it is easy to check that the moments (3.9) obtained by the method of section 3.2, with $\hat{A}_k$ given by (10.1) for $s = k$ integer, are identical to the moments (5.17) obtained by the method of the present work.

It is, however, not possible to identify the intermediate results of both methods, as can be seen by comparing respectively equation (5.3) to equations (3.7) and (3.5), and equation (4.4) to equation (3.8). This demonstrates the formal character of the application of the method of section 3.2 to the process (1.1). (See also the discussion in section 3.3.)

10.2 Comments on the results obtained using Kac’s formalism

A first comment is that the recursion relations for the coefficients $c_k$ appearing in equations (14) of ref. [4] can be easily recognized to be identical to the recursion relations (5.13), (5.14) for the $x_k$, by noting the correspondences

$$c_k = \frac{2^k}{k! D_{-k/\alpha}(0)} x_k,$$

$$\frac{D_{-k/\alpha+1}(0)}{D_{-k/\alpha}(0)} = \frac{\sqrt{2\pi}}{\alpha} \frac{k h_k^{(\alpha)}}{2}.$$  

(10.3)

The second comment concerns the continuity conditions expressed in equations (13) of ref. [4]. Using (10.3), these conditions yield, with the notations of the present work,

$$\langle e^{a(2X-1)} \rangle = \langle e^{a(1-2X)} \rangle,$$

$$\langle \xi e^{a(2\xi-1)} \rangle = \langle \xi e^{a(1-2\xi)} \rangle.$$  

These equations hold for $a$ arbitrary, hence they are equivalent to (6.1) and (6.3), respectively.
11 Summary and discussion

In this paper we have revisited and extended the work of Dhar and Majumdar [4]. Besides providing a new recursive determination of the moments of the mean magnetization $M$, the present study leads to a functional integral equation for the distribution of the latter quantity. This framework allows a local analysis of this distribution, and of other relevant quantities, in the persistence region ($M_t \to \pm 1$), as well as a detailed investigation of the regime where $\alpha$ is large.

The present work casts new light on the status of the expressions (3.9) for the moments of $M$. The method recalled in section 3.2, which leads to these relations, can be applied to any smooth process for which the intervals of time $\ell_N$ between sign changes are independent, on a logarithmic scale. For this class of processes $\langle \ell_N \rangle = \bar{\ell}$ is finite (i.e., non-zero), and the mean number of sign changes between 0 and $t$ scales as $\langle N_t \rangle \approx (\ln t)/\bar{\ell}$.

Relations (3.9) are also verified for the class of processes considered in this work. This was observed in [4] (by comparing the expressions thus found to those obtained by another method, based on a formalism due to Kac), and justified by the absence of $\bar{\ell}$ in equations (3.9). Yet, as discussed in section 3.3, in the present case there is no obvious reason to work with a logarithmic scale of time, since $\langle \ell_N \rangle$ asymptotically vanishes, and the mean number of sign changes scales as $\langle N_t \rangle \approx 2\pi^{-1/2}t^{\alpha-1/2}$ [11]. (See also the discussion in section 10.1.)

Most of the effort of the present work was to provide a new derivation of (3.9) for the class of processes (1.1). Our approach relies on the fact that the time intervals $\tau'_n$ between two sign changes of the process (1.1) form a renewal process (the $\tau'_n$ are independent, identically distributed random variables). The derivation proceeds in two steps. First, relations (5.17) for the $\langle M^k \rangle$ are obtained; then, using (10.2), equations (5.17) yield (3.9). This extends the range of applicability of relations (3.9). Note that for diffusion (in the independent-interval approximation) (see section 3.2), relations (3.9) hold but neither (5.17) nor (10.2) do.

We conclude by a few additional comments.

In passing, let us mention another equivalent formulation of (10.2), namely that the two-time autocorrelation of the sign process reads, with $t < t'$,

$$C(t,t') = \int_0^{t/t'} dx f_H(x).$$  \hfill (11.1)

Another situation where (3.9), (5.17), and (10.2) or (11.1) hold is for the renewal processes considered in [11] (provided $\theta < 1$), which are yet another deformation of Brownian motion.

Note that the first relation of (3.9), $\langle M^2 \rangle = \hat{A}_1$, holds whenever the two-time autocorrelation function is a scaling function of the ratio of the two times [3], while the first relation of (5.17), $\langle M^2 \rangle = 1 - \langle H \rangle$, does not in general. For instance, for the
random acceleration problem, using results of ref. [20], we find $\langle M^2 \rangle = 3\sqrt{3}/\pi - 1 \approx 0.653986$ and $1 - \langle H \rangle \approx 0.791335$.

This work also underlines the importance of the random variables $X$ and $H$. The distribution of the latter is known exactly in the present case. This quantity, which is a natural one to consider for Brownian motion [1], and more generally for renewal processes [11], appears also in the context of phase ordering [21].

As mentioned in the introduction, the process (1.1) has been proposed [2] as a Markovian approximation to fractional Brownian motion. Let us compare the expressions of $\langle M^2 \rangle$ for these two processes. For the present model we have (see (5.17))

$$\langle M^2 \rangle = 1 - \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{1}{2\alpha} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2\alpha} + 1 \right)}, \quad (11.2)$$

while for fractional Brownian motion, with Hölder index $0 < h < 1$, we have

$$\langle M^2 \rangle = \frac{2}{\pi} \int_0^1 dx \arcsin \frac{x^{2h} + 1 - (1 - x)^{2h}}{2x^h}. \quad (11.3)$$

The correspondence between the two processes is made by identifying their persistence exponents: $\theta = \alpha = 1 - h$. For $\theta = 1/2$, we have $\langle M^2 \rangle = 1/2$ in both cases. For $\theta = 1$, (11.2) yields $\langle M^2 \rangle = 1 - 2/\pi \approx 0.363380$, while (11.3) yields $\langle M^2 \rangle = 1/3$. For $\theta \to 0$, we have $\langle M^2 \rangle = 1 - c\sqrt{\theta}$, with (11.2) yielding $c = \sqrt{2/\pi} \approx 0.797885$, and (11.3) yielding $c \approx 0.812233$. The distributions of the mean magnetization for the two processes are therefore expected to be rather similar (for $0 < \theta < 1$).

Finally, let us comment on the changes in behaviour induced by letting the persistence exponent $\alpha$ vary, and compare the present process to other ones in this respect. The distribution of $M$ shows a change in shape as $\alpha$ increases, the most probable value of the mean magnetization shifting from the edges to the center [4]. More precisely, as shown in section 8, as long as $\alpha < 1$, $f_M(m)$ diverges at $m \to \pm 1$, while for $\alpha > 1$ it vanishes at these points (see equation (8.2)). However for any arbitrary value of $\alpha$ the magnetization $M$ remains distributed.

This behaviour is actually generic, whenever the two-time autocorrelation function of the process is asymptotically a function of the ratio of the two time variables $\tau$. In particular, this is so for diffusion. In the independent-interval approximation the persistence exponent $\theta(D) \approx 0.1454\sqrt{D}$ increases without bounds when the dimension of space $D$ is large [13, 19]. As originally noted in [3], as long as $\theta < 1$ the density $f_M(m)$ diverges at the edges, while it vanishes there if $\theta > 1$. This was also emphasized in [4], on the basis of scaling arguments, and recently confirmed by direct numerical computations [22].

In contrast, there are other processes for which the change in behaviour at $\theta = 1$ is more radical. For fractional Brownian motion, $\theta = 1$ appears as a maximum
persistence exponent. For the renewal processes considered in ref. [11], the mean magnetization possesses a non-trivial asymptotic distribution only if $\theta < 1$. 

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A Properties of the binomial operator $B$

The aim of this Appendix is to prove the following property, used in section 5.2. Assume that the sequence $x_k$ satisfies

$$x_k = B(f_k x_k), \quad (A.1)$$

with $x_0 = f_0 = 1$, and where $B(x_k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j x_j$ (see (5.8)). Then

$$x_k(1 + f_k) = B(x_k(1 + f_k)) \quad (k \text{ odd}), \quad (A.2)$$

$$x_k(1 - f_k) = -B(x_k(1 - f_k)) \quad (k \text{ even}), \quad (A.3)$$

which are respectively equations (5.9) and (5.10) in the text.

A.1 Basic properties

In order to prove (A.2) and (A.3) we need the following auxiliary properties.

First, $B$ is its own inverse:

$$B = B^{-1}. \quad (A.4)$$

A combinatorial proof of this result can be found in ref. [23]. An alternative proof is obtained by noting that the action of $B$ on exponential sequences $x_k = y^k$ reads

$$B(y^k) = (1 - y)^k. \quad (A.5)$$

This relation is invariant in the change of $y$ to $1 - y$, hence (A.4) follows.

Then, we have the properties

$$x_k = B(x_k) \quad \text{for } k \text{ even} \quad \text{implies} \quad x_k = B(x_k) \quad \text{for all } k, \quad (A.6)$$

$$x_k = B(x_k) \quad \text{for } k \text{ odd} \quad \text{implies} \quad x_k = B(x_k) \quad \text{for all } k, \quad (A.7)$$

$$x_k = -B(x_k) \quad \text{for } k \text{ even} \quad \text{implies} \quad x_k = -B(x_k) \quad \text{for all } k, \quad (A.8)$$

$$x_k = -B(x_k) \quad \text{for } k \text{ odd} \quad \text{implies} \quad x_k = -B(x_k) \quad \text{for all } k. \quad (A.9)$$

Before giving the proofs, let us explicit the meaning of (A.6)-A.9.

Let us take the example of (A.6). By hypothesis, the sequence $x_k$ satisfies the condition $x_k = B(x_k)$ for $k$ even, with $x_0$ arbitrary, which is equivalent to saying that

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j = 0 \quad (k \text{ even}). \quad (A.10)$$

This recursion determines $x_k$ for $k$ odd in terms of the $x_\ell$ with $\ell = 0, \ldots, k - 1$ even:

$$x_1 = \frac{1}{2} x_0, \quad x_3 = \frac{3}{2} x_2 - \frac{1}{4} x_0, \quad x_5 = \frac{5}{2} x_4 - \frac{5}{2} x_2 + \frac{1}{2} x_0, \quad \ldots$$
The property (A.6) states that $x_k = B(x_k)$ for $k$ odd, or equivalently,

$$2x_k = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \quad (k \text{ odd}),$$

which provides an infinite number of consistency relations amongst the $x_k$ satisfying (A.10).

Similarly, taking the example of (A.8), by hypothesis we have $x_k = -B(x_k)$ for $k$ even, with $x_0 = 0$, which is equivalent to

$$2x_k = -\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \quad (k \text{ even}). \quad (A.11)$$

This recursion determines $x_k$ for $k$ even in terms of the $x_{\ell}$ with $\ell = 1, \ldots, k - 1$ odd:

$$x_2 = x_1, \quad x_4 = 2x_3 - x_1, \quad x_6 = 3x_5 - 5x_3 + 3x_1, \quad \ldots$$

The property (A.8) states that $x_k = -B(x_k)$ for $k$ odd, or equivalently,

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j = 0 \quad (k \text{ odd}),$$

which provides an infinite number of consistency relations amongst the $x_k$ satisfying (A.11).

We now prove the properties (A.6-A.9). In order to do so, let us define, for a given sequence $x_k$, the Laurent series

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k}, \quad G(z) = \sum_{k=0}^{\infty} B(x_k) z^{-k}.$$ 

We assume that these series are convergent for $|z|$ larger than some radius $R$. This happens e.g. if the $x_k$ are bounded.

The functions $F(z)$ and $G(z)$ are related to each other by

$$F(z) = \frac{z}{z-1} G(1 - z), \quad (A.12)$$

$$G(z) = \frac{z}{z-1} F(1 - z), \quad (A.13)$$

as we now show. We have

$$x_k = \oint \frac{dy}{2\pi i y} y^k F(y),$$

hence, using (A.5),

$$B(x_k) = \oint \frac{dy}{2\pi i y} (1-y)^k F(y),$$

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so that

\[ G(z) = \oint \frac{dy}{2\pi iy} \frac{z}{y + z - 1} F(y). \]

This integral is equal to the contribution of the pole at \( y = 1 - z \), yielding (A.13), from which (A.12) follows. The symmetric form of the formulas (A.12), (A.13) is due to the property (A.4).

**Proof of (A.6) and (A.7)**

The hypothesis in (A.6) implies \( F(z) + F(-z) = G(z) + G(-z) \), i.e.,

\[ \Phi(z) = -\Phi(-z), \]  

(A.14)

with, using (A.13),

\[ \Phi(z) = F(z) - G(z) = F(z) - \frac{z}{z - 1} F(1 - z). \]

Therefore \((z - 1)\Phi(z) + z\Phi(1 - z) = 0\), which can be rewritten, using (A.14), as

\[ \frac{\Phi(z)}{z} = \frac{\Phi(z - 1)}{z - 1}. \]

The function \( \Phi(z)/z \) is thus periodic, with unit period, and decaying at infinity, as we have \( \Phi(z)/z \approx (x_0 - 2x_1)/z^2 \) a priori. We conclude that \( \Phi(z) = 0 \) identically, that is \( F(z) = G(z) \), implying the property (A.6).

For the case where the \( x_k = \langle X^k \rangle \) are the moments of a random variable \( X \), with density \( f_X \) on [0, 1], an alternative proof of (A.6) is as follows. The hypothesis in (A.6) expresses the property

\[ \langle X^k \rangle = \langle (1 - X)^k \rangle \quad (k \text{ even}). \]

As both random variables \( X \) and \( 1 - X \) are positive, this last condition is sufficient to imply \( f_X(x) = f_X(1 - x) \), hence \( \langle X^k \rangle = \langle (1 - X)^k \rangle \) for all \( k \), which proves (A.6).

The proof of the second property, (A.7), is very similar. The hypothesis in (A.7) implies

\[ \frac{\Phi(z)}{z} = -\frac{\Phi(z - 1)}{z - 1}. \]

The function \( \Phi(z)/z \) is therefore periodic, with period two, and decaying at infinity, hence \( \Phi(z) = 0 \) identically.

**Proof of (A.8) and (A.9)**

The hypothesis in (A.8) implies \( F(z) + F(-z) = -(G(z) + G(-z)) \), i.e.,

\[ \Psi(z) = -\Psi(-z), \]  

(A.15)
with, using (A.13),

\[ \Psi(z) = F(z) + G(z) = F(z) + \frac{z}{z-1} F(1-z). \]

Therefore \((z - 1)\Psi(z) - z\Psi(1 - z) = 0\), which can be rewritten, using (A.13), as

\[ \frac{\Psi(z)}{z} = -\frac{\Psi(z-1)}{z-1}. \]

The function \(\Psi(z)/z\) is thus again periodic, with period two, and decaying at infinity, hence identically zero. The proof of the fourth property, (A.9), is very similar.

### A.2 Proofs of equations (A.2) and (A.3)

Equation (A.1) implies the relations

\[ x_{k}(1 + f_{k}) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j f_j x_j \quad (k \text{ odd}), \quad (A.16) \]
\[ x_{k}(1 - f_{k}) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j f_j x_j \quad (k \text{ even}), \quad (A.17) \]

which determines the \(x_k\) recursively. We have thus

\[ x_1 = \frac{1}{1+f_1}, \quad x_2 = \frac{1-f_1}{(1+f_1)(1-f_2)}, \quad \ldots \quad (A.18) \]

Since the operator \(\mathcal{B}\) is its own inverse, (A.1) is equivalent to \(f_k x_k = \mathcal{B}(x_k)\), which itself implies

\[ x_{k}(1 + f_{k}) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \quad (k \text{ odd}), \quad (A.19) \]
\[ x_{k}(f_{k} - 1) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \quad (k \text{ even}). \quad (A.20) \]

Comparing (A.16) and (A.19) shows that

\[ \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j (1-f_j) = 0 \quad (k \text{ odd}), \]

hence, using the property (A.9),

\[ x_{k}(1 - f_{k}) = -\mathcal{B}(x_k(1 - f_k)) \quad (k \text{ even}), \]

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which is equation (A.3). Similarly, comparing (A.17) and (A.20) shows that
\[
\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j (1 + f_j) = 0 \quad (k \text{ even}),
\]
or, using the property (A.6),
\[
x_k (1 + f_k) = B(x_k (1 + f_k)) \quad (k \text{ odd}),
\]
which is equation (A.2).
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