SHEPHARD-TODD-CHEVALLEY THEOREM
FOR SKEW POLYNOMIAL RINGS

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ABSTRACT. We prove the following generalization of the classical Shephard-Todd-Chevalley Theorem. Let $G$ be a finite group of graded algebra automorphisms of a skew polynomial ring $A := k[p_{ij}[x_1, \cdots, x_n]]$. Then the fixed subring $A^G$ has finite global dimension if and only if $G$ is generated by quasi-reflections. In this case the fixed subring $A^G$ is isomorphic a skew polynomial ring with possibly different $p_{ij}$’s. A version of the theorem is proved also for abelian groups acting on general quantum polynomial rings.

0. Introduction

The classical Shephard-Todd-Chevalley Theorem states that if $G$ is a finite group acting faithfully on a finite dimensional $k$-vector space $V := \bigoplus_{i=1}^{n} kx_i$, then the fixed subring $k[x_1, \cdots, x_n]^G$ is isomorphic to $k[x_1, \cdots, x_n]$ if and only if $G$ is generated by pseudo-reflections of $V$. (In the rest of this paper a pseudo-reflection will be called simply a reflection). It is natural to ask if a version of the Shephard-Todd-Chevalley Theorem holds for regular algebras, which are algebras of finite global dimension that are noncommutative analogs of commutative polynomial rings. In [KKZ1] the authors began our investigation of conditions on a group $G$ of graded algebra automorphisms of a regular algebra $A$ so that $A^G$ is isomorphic to $A$, or that $A^G$ itself has finite global dimension and hence is regular.

The main goal of this paper is to provide evidence that a noncommutative version of Shephard-Todd-Chevalley Theorem should hold. Let $k$ be a base commutative field of characteristic zero, and let $n$ be a positive integer. Let $\{p_{ij} \mid 1 \leq i < j \leq n\}$ be a set of nonzero scalars in $k$. The skew polynomial ring $A := k[p_{ij}[x_1, \cdots, x_n]]$ is defined to be the $k$-algebra generated by $x_1, \cdots, x_n$ and subject to the relations

$$x_j x_i = p_{ij} x_i x_j$$

for all $1 \leq i < j \leq n$. The algebra $A$ is a connected $\mathbb{N}$-graded algebra with $\deg x_i = 1$ for all $i$. Let $\text{Aut}_{gr}(A)$ be the group of all graded algebra automorphisms of $A$. Our main result is the following theorem.

Theorem 0.1 (Theorem 4.5). Let $A = k[p_{ij}[x_1, \cdots, x_n]$ and $G$ a finite subgroup of $\text{Aut}_{gr}(A)$. Then the fixed subring $A^G$ has finite global dimension if and only if $G$ is generated by quasi-reflections.

When the fixed subring $A^G$ has finite global dimension, it need not be isomorphic to $A$ itself (as in the classical Shephard-Todd-Chevalley Theorem), but it is isomorphic to another skew polynomial ring $k[p_{ij}^*[x_1, \cdots, x_n]]$. The definition of a

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quasi-reflection is given in Definition 1.2. As we will see in Section 1 most quasi-
reflections are reflections in the classical sense, with only one interesting exception,
called mystic reflections [Definition 1.3(b)].

If \( p_{ij} \neq \pm 1 \) for all \( i < j \), then the quasi-reflections of \( A \) are easy to describe and
the proof of Theorem 0.1 is simple. The complication appears only when some of
\( p_{ij} \)’s are \( \pm 1 \). The idea in the proof is to group together certain quasi-reflections so
that we have a partition on the space \( \bigoplus_{i=1}^{n} kx_i \) under certain rules determined by
the parameter set \( \{p_{ij}\} \) and the group \( G \). Using this partition we can reduce the
question to each block, and within each block the algebra is either commutative or
connected by mystic reflections. We believe that this kind of partition should be
useful for proving a more general version of the Shephard-Todd-Chevalley Theorem
conjectured as follows.

**Conjecture 0.2.** Let \( B \) be a quantum polynomial ring [Definition 1.1(2)] and let \( G \) be a finite subgroup of \( \text{Aut}_{gr}(B) \). Then \( B^G \) has finite global dimension if and
only if \( G \) is generated by quasi-reflections.

Other evidence for the above conjecture is the following theorem, which settles
the case when \( G \) is abelian.

**Theorem 0.3** (Theorem 5.3). Let \( B \) be a quantum polynomial ring and let \( G \) be
a finite abelian subgroup of \( \text{Aut}_{gr}(B) \). Then \( B^G \) has finite global dimension if and
only if \( G \) is generated by quasi-reflections.

**Theorem 0.3** is a generalization of [KKZ1, Theorem 0.5]. The combination
of Theorem 0.1 with Theorem 0.3 covers only a small part of Conjecture 0.2. For a
different reason (see Proposition 0.5) we also verify the Shephard-Todd-Chevalley
Theorem for the quantum \( 2 \times 2 \) matrix algebra \( O_q(M_2) \), which has some non-abelian
reflection groups [Proposition 5.8].

The new quasi-reflections possessed by some regular algebras suggest extending
the notion of “reflection group” to include groups \( G \) with a representation by au-
tomorphisms of a regular algebra \( A \) giving a regular fixed subring \( A^G \). This leads
to a few natural questions:

**Question 0.4.** (a) Can every “reflection group” of a noncommutative regular
algebra be realized as a reflection group in the classical sense?
(b) Is there a version of the Shephard-Todd-Chevalley Theorem for finite quan-
tum group (i.e., finite dimensional Hopf algebra) actions?
(c) Using Hopf algebra actions, instead of group actions, can we obtain more
regular fixed subrings of a given quantum polynomial ring?

A secondary goal of this paper is to try to answer the above questions. J. Alev
asked a question similar to Question 0.4(c). In Example 6.2 we show that some
groups of mystic reflections are not reflection groups in the classical sense, which
answers Question 0.4(a) negatively. In Section 5, we prove the following result.

**Proposition 0.5.** [Propositions 5.7 and 5.8] Let \( C \) be the quantum \( 2 \times 2 \) matrix
algebra \( O_q(M_2) \). Suppose \( q \neq \pm 1 \).

(i) For every finite group \( G \subset \text{Aut}_{gr}(C) \), \( C^G \) is not isomorphic to a skew
polynomial ring (as defined before Theorem 0.1). For every non-trivial finite
group \( G \subset \text{Aut}_{gr}(C) \), \( C^G \) is not isomorphic to \( C \).
Suppose \( q \) is a root of 1. Then there is a finite dimensional Hopf algebra \( H \) with a Hopf action on \( C \) such that \( CH \) is isomorphic to a skew polynomial ring. As a consequence, \( CH \) has finite global dimension and is not isomorphic to \( CG \) for any finite group \( G \subset \text{Aut}_{gr}(C) \).

Hence by Proposition 0.5(ii) we do obtain more regular fixed subrings using Hopf algebra actions, which answers Question 0.4(c). As far as Question 0.4(b), Proposition 5.7 suggests that there should be some version of Shephard-Todd-Chevalley Theorem for Hopf algebra actions, however, we do not have an explicit conjectural statement. Some related work about Hopf algebra actions on Artin-Schelter Gorenstein algebras will be presented in [KKZ2].

The paper is organized as follows: Some preliminary material is reviewed in Section 1. Sections 2 and 3 contain some analysis of quasi-reflections of skew polynomial rings. Theorem 0.1 is proved in Section 4 and Theorem 0.3 is proved in Section 5. Section 6 contains some remarks about mystic reflection groups.

1. Definitions

Throughout \( k \) is a base field of characteristic zero. In [KKZ1] the authors assume that \( k \) is algebraically closed. For simplicity and our convenience we continue to assume that \( k \) is algebraically closed since we will use several results from [KKZ1], though all the main assertions in this paper hold without that assumption. The opposite ring of an algebra \( A \) is denoted by \( A^{op} \). Usually we are working with left \( A \)-modules, and a right \( A \)-module can be viewed as a left \( A^{op} \)-module.

An algebra \( A \) is called connected \( \mathbb{N} \)-graded or connected graded if

\[
A = k \oplus A_1 \oplus A_2 \oplus \cdots
\]

and \( A_i A_j \subset A_{i+j} \) for all \( i, j \in \mathbb{N} \). The Hilbert series of \( A \) is defined to be

\[
H_A(t) = \sum_{i \in \mathbb{Z}} (\dim A_i) t^i.
\]

The class of algebras considered in this paper are the graded algebras with finite global dimension. Sometimes we need the following more restrictive class of quantum polynomial rings.

**Definition 1.1.** Let \( A \) be a connected graded algebra.

1. We call \( A \) Artin-Schelter Gorenstein if the following conditions hold:
   a. \( A \) has graded injective dimension \( d < \infty \) on the left and on the right,
   b. \( \text{Ext}_A^i(k, A) = \text{Ext}_{A^{op}}^i(k, A) = 0 \) for all \( i \neq d \), and
   c. \( \text{Ext}_A^d(k, A) \cong \text{Ext}_{A^{op}}^d(k, A) \cong k(l) \) for some \( l \).

   If in addition,
   d. \( A \) has finite (graded) global dimension, and
   e. \( A \) has finite Gelfand-Kirillov dimension,

   then \( A \) is called Artin-Schelter regular (or regular for short) of dimension \( d \).

2. If \( A \) is a noetherian, regular graded domain of global dimension \( n \) and \( H_A(t) = (1-t)^{-n} \), then we call \( A \) a quantum polynomial ring of dimension \( n \).

Skew polynomial rings (with \( \deg x_i = 1 \)) are quantum polynomial rings. Next we recall some definitions of the noncommutative versions of reflections from [KKZ1].
Let $g \in \text{Aut}_\text{gr}(A)$, the graded algebra automorphisms of $A$, then the trace function of $g$ is defined to be

$$Tr_A(g, t) = \sum_{i=0}^{\infty} \text{tr}(g|_A)_it^i \in k[[t]],$$

where $\text{tr}(g|_A)_i$ is the trace of the linear map $g|_A_i$. Note that $Tr_A(g, 0) = 1$.

**Definition 1.2.** [KKZ1] Definition 2.2] Let $A$ be a regular graded algebra such that

$$H_A(t) = \frac{1}{(1-t)^nf(t)}$$

where $f(1) \neq 0$ (so that $n = \text{GKdim } A$). Let $g$ be a graded algebra automorphism of $A$. We say that $g$ is a quasi-reflection of $A$ if

$$Tr_A(g, t) = \frac{1}{(1-t)^{n-1}q(t)}$$

for $q(1) \neq 0$. If $A$ is a quantum polynomial ring, then $H_A(t) = (1-t)^{-n}$. In this case $g$ is a quasi-reflection if and only if

$$Tr_A(g, t) = \frac{1}{(1-t)^{n-1}(1-\lambda t)}$$

for some scalar $\lambda \neq 1$. (Note that we have chosen not to call the identity map a quasi-reflection).

Quasi-reflections of quantum polynomial rings are characterized in [KKZ1] Theorem 3.1].

**Definition 1.3.** Let $A$ be a quantum polynomial ring of global dimension $n$. If $g \in \text{Aut}_\text{gr}(A)$ is a quasi-reflection of finite order, by [KKZ1] Theorem 3.1] $g$ is described in one of the following two cases.

(a) There is a basis of $A_1$, say $\{y_1, \cdots, y_n\}$, such that $g(y_j) = y_j$ for all $j \leq n - 1$ and $g(y_n) = \lambda y_n$. Namely, $g|_{A_1}$ is a reflection. In this case $g$ is called a reflection of $A$, $y_n$ is called a non-invariant eigenvector of $g$, and $E_g := \oplus_{i<n}ky_i$ is called the invariant eigenspace of $g$.

(b) The order of $g$ is 4, and there is a basis of $A_1$, say $\{y_1, \cdots, y_n\}$, such that $g(y_j) = y_j$ for all $j \leq n - 2$, $g(y_{n-1}) = i y_{n-1}$, and $g(y_n) = -i y_n$ (where $i^2 = -1$). In this case $g$ is called a mystic reflection of $A$, $\{y_{n-1}, y_n\}$ is called a pair of non-invariant eigenvectors of $g$, and $E_g := \oplus_{i<n-1}ky_i$ is called the invariant eigenspace of $g$.

The homological determinant $\text{hdet}$ of $g$ is defined in [JoZ] Definition 2.3]. We refer to [JoZ] for some details. We use $\text{det}$ for the usual determinant of a $k$-linear map.

**Proposition 1.4.** Let $A$ be a quantum polynomial ring and let $g$ be a quasi-reflection of $A$.

(a) If $g$ is a reflection, then the non-invariant eigenvector $y_n$ is a normal element of $A$, and $\text{det } g|_{A_1} = \text{hdet } g = \lambda \neq 1$.

(b) Suppose $g$ is a mystic reflection with a pair of non-invariant eigenvectors $y_{n-1}$ and $y_n$. Then $E_g y_{n-1} = y_{n-1} E_g$, $E_g y_n = y_n E_g$, $\text{det } g|_{A_1} = 1$, and $\text{hdet } g = -1$.?
(c) If \( g \) is a mystic reflection, then \( y_{n-1}' = y_n^2 \) up to a scalar, and there are two elements \( y_{n-1}' = y_n' \) with \( ky_{n-1}' = ky_n + ky_n' \) and \( y_n'y_{n-1}' = -y_{n-1}y_n' \). Furthermore, \( g(y_{n-1}') = y_n' \) and \( g(y_n') = -y_{n-1}' \).

(d) Let \( G \) be generated by mystic reflections. Then \( G \) does not contain any reflections.

**Proof.** (a) The first assertion follows from the proof of [KKZ1, Lemma 5.2(a,d)]. The second assertion is clear from the definition.

(b) The first assertion follows from the proof of [KKZ1, Proposition 4.3].

(c) The first two assertions are [KKZ1, Proposition 4.3(c)]. For the last assertion we set \( y_{n-1}' = y_{n-1} + iy_n \) and \( y_n' = g(y_{n-1}') = iy_{n-1} + y_n \).

(d) By part (b), \( \det f|_{A_n} = 1 \) for all \( f \in G \). By part (a), such an \( f \) cannot be a reflection. \( \Box \)

**Proposition 1.5.** Let \( A \) be a noetherian regular algebra and \( G \) be a finite group of automorphisms of \( A \).

(a) Let \( R \) be the subgroup of \( G \) generated by quasi-reflections. Then \( R \) is a normal subgroup of \( G \), and if both \( A^R \) and \( A^G \) have finite global dimension, then \( R = G \).

(b) Suppose, for every finite \( H \subset \text{Aut}_gr(A) \) generated by quasi-reflections, \( A^H \) has finite global dimension. Then \( A^G \) has finite global dimension if and only if \( G \) is generated by quasi-reflections.

**Proof.** (a) By [KKZ1, Lemma 1.10(c)], if \( A^G \) has finite global dimension, then it is regular.

Let \( n = \text{GKdim} \ A \). Since the property of \( h \) being a quasi-reflection is determined by \( Tr(h,t) \) and \( Tr(ghg^{-1},t) = Tr(h,t) \) for any \( g \in G \), then \( R \) is a normal subgroup of \( G \). Thus the quotient group \( G/R \) acts on the fixed subring \( A^R \), and the fixed subring \( (A^R)^{G/R} \) equals \( A^G \). Let \( g \in G \setminus R \). Then by [JiZ, Lemma 5.2]

\[
Tr_{A^R}(g|_{A_R}, t) = \frac{1}{|R|} \sum_{h \in R} Tr_A(hg, t).
\]

Note that no \( hg \) can be a quasi-reflection, for otherwise we would have \( g \in R \). Hence \( Tr_A(hg, t) \) must have the form

\[
Tr_A(hg, t) = \frac{1}{(1 - t)^m p(t)}
\]

where \( m \leq n - 2 \) and \( p(1) \neq 0 \). This means that \( (1 - t)^{n-2}Tr_{A^R}(g|_{A_R}, t) \) is analytic at \( t = 1 \) and \( g|_{A_R} \) cannot be a quasi-reflection. Since \( (A^R)^{G/R} \) is regular, \( G/R \) must contain a quasi-reflection by [KKZ1, Theorem 2.4]. Hence \( G/R \) is trivial and \( G = R \).

(b) This is immediate from part (a). \( \Box \)

It follows from this proposition that to prove a result like Theorem 0.1 we need only to show one direction, namely, to show that if \( G \) is generated by quasi-reflections then \( A^G \) has finite global dimension.

In the rest of the section we review two elementary lemmas.
Lemma 1.6. Let $A(i)$, for $i = 1, \cdots, d$, be regular algebras (or connected graded algebras having finite global dimension). Then the tensor product $A(1) \otimes_k \cdots \otimes_k A(d)$ is a regular algebra (or an algebra having finite global dimension).

Proof. By induction we may assume $d = 2$. Then we use the proof of [YZ, Proposition 4.5(a)]. □

The algebra $A(1) \otimes_k \cdots \otimes_k A(d)$ in the above lemma is connected $\mathbb{N}^d$-graded. Let $B$ be any $\mathbb{N}^d$-graded algebra. Let $\phi := \{\phi_s \mid s = 1, \cdots, d\}$ be a sequence of commuting $\mathbb{N}^d$-graded algebra automorphisms of $B$. Then we can define a twisted algebra, as in [Z], $B^\phi$ as follows: $B^\phi = B$ as a $\mathbb{N}^d$-graded vector space, and the new multiplication $*$ of $B^\phi$ is determined by

$$a \ast b = a\phi^{i|a|}(b)$$

where $\phi^{i|a|} = \phi_1^{i_1} \cdots \phi_d^{i_d}$ if the degree of $a$ is $|a| = (a_1, \cdots, a_d)$. We refer to [Z] for some basic properties of twisted algebras. The following lemma is elementary.

Lemma 1.7. Let $B$ be a connected $\mathbb{N}^d$-graded algebra that is regular as a connected graded algebra. Let $\phi := \{\phi_s \mid s = 1, \cdots, d\}$ be a sequence of commuting $\mathbb{N}^d$-graded algebra automorphisms of $B$.

(a) The algebra $B^\phi$ is a connected $\mathbb{N}^d$-graded algebra that is regular as a connected graded algebra.

(b) Suppose $B$ is a skew polynomial ring with generators $\{x_i\}$. ($B$ may not be generated in degree 1.) If each $x_i$ is $\mathbb{N}^d$-homogeneous and if $\phi_s$ maps $x_i$ to $q_{si}x_i$ for all $s$ and $i$, where $q_{si}$ are some nonzero scalars in $k$, then $B^\phi$ is isomorphic to a skew polynomial ring.

2. Block decomposition, elementary transformations and quasi-reflections of skew polynomial rings

In Sections 2, 3 and 4 we fix an integer $n \geq 2$ and a set of nonzero scalars $\{p_{ij} \mid 1 \leq i < j \leq n\}$. Let $A$ be the skew polynomial ring $k_{p_{ij}}[x_1, \cdots, x_n]$ that is generated by $\{x_1, \cdots, x_n\}$ and subject to the relations $x_jx_i = p_{ij}x_ix_j$ for all $1 \leq i < j \leq n$. We will try to use other letters for a general algebra. For $i \geq j$ set

$$p_{ij} = \begin{cases} 1 & i = j \\ p_{ji}^{-1} & i > j. \end{cases}$$

Let $[n]$ be the set $\{1, 2, \cdots, n\}$ and let $p$ be the set $\{p_{ij} \mid i, j \in [n]\}$. Following (2.0.1) we have $x_jx_i = p_{ij}x_ix_j$ for all $i, j \in [n]$. Let $A_1 = \oplus_{i=1}^n kx_i$ which is called the generating space of $A$. Let $\{x'_1, \cdots, x'_n\}$ be another basis of $A_1$. We call $\{x'_1, \cdots, x'_n\}$ a p-basis if $x'_jx'_i = p_{ij}x'_ix'_j$ holds in $A$ for all $i, j \in [n]$.

Definition 2.1. Fix a set $p$ satisfying (2.0.1) and fix an $i \in [n]$.

(a) We define the block containing $i$ to be

$$B(i) = \{i' \in [n] \mid p_{ij} = p_{i'j} \quad \forall j \in [n]\}.$$ 

We say that $i$ and $j$ are in the same block if $B(i) = B(j)$.

(b) We use the blocks to define an equivalence relation on $[n]$, and then $[n]$ is a disjoint union of blocks

$$[n] = \bigcup_{i \in W} B(i).$$
Lemma 2.2. Let \( a, b, c, d \) These are straightforward.

**Proof.**

\( \bigoplus \) rings. If there is a permutation \( \sigma \) then \( i \neq j \) for all \( i, j \) in the block decomposition is a partition of \( [n] \) and it is uniquely determined by \( p \). By abusing the language, we also call the subspace \( \bigoplus_{s \in B(i)} kx_s \) a block.

In the lemma below we will rescale using a square root. In this and in other times that we do this rescaling, we choose a particular square root \( q_{ij} = \sqrt{p_{ij}} \) for \( i < j \), and then take \( q_{ji} = 1/q_{ij} \), using the square root chosen for \( q_{ij} \).

**Lemma 2.2.** Let \( W = [m] \) and let \( \{ B_w \mid w \in [m] \} \) be the set of blocks. Let \( P_w = \bigoplus_{i \in B_w} kx_i \) for each \( w \).

(a) If \( i, j \in B_w \), then \( p_{ij} = 1 \). This means that the subalgebra generated by \( x_i \) for all \( i \in B_w \) is commutative. As a consequence, if \( p_{ij} \neq 1 \) for all \( i < j \), then \( W = [n] \) and \( B(i) = \{ i \} \) for all \( i \in [n] \).

(b) The algebra \( A \) is an \( \mathbb{N}^m \)-graded algebra with degree \( x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) where 1 is in the \( w \)th position if \( i \in B_w \).

(c) For every \( w \in W \), pick any \( k \)-linear basis \( \{ x'_i \mid i \in B_w \} \) for \( P_w \), then there is an \( \mathbb{N}^m \)-graded algebra automorphism of \( A \) determined by the map \( \theta : x_i \to x'_i \) for all \( i \in [n] \).

(d) For each \( w \in W \), pick \( j \in B_w \). Define the automorphism \( \phi_w : x_i \to q_{ij} x_i \) for all \( i \). (Recall that \( q_{ij} = \sqrt{p_{ij}} \).) Then \( \phi = \{ \phi_w \} \) is a sequence of commuting \( \mathbb{N}^m \)-graded algebra automorphisms of \( A \).

(e) The graded twist \( A^\phi \) is the commutative polynomial ring \( k[x_1, \ldots, x_n] \).

**Proof.** (a,b,c,d) These are straightforward.

(e) The new multiplication of the graded twist \( A^\phi \) is determined by

\[
x_s * x_t = x_s \phi_w(x_t) = x_s q_{st} x_t
\]

for \( s \in B_w \). Then

\[
x_t * x_s = x_t q_{ts} x_s = p_{st} x_s x_t q_{ts} = q_{st} x_s x_t = x_s * x_t.
\]

Therefore \( A^\phi \) is commutative. \( \square \)

We call the algebra automorphism in Lemma 2.2(c) an elementary transformation. There is a class of obvious isomorphisms between two skew polynomial rings. If there is a permutation \( \sigma \in S_n \) such that \( p_{ij} = p_{\sigma(i)\sigma(j)} \) for all \( i, j \in [n] \), then \( k_{p_{\sigma}}[x_1, \ldots, x_n] \) is isomorphic to \( k_{p_{ij}}[x_1, \ldots, x_n] \) by sending \( x_i \to x_{\sigma(i)} \) for all \( i \in [n] \). Such an isomorphism is called a permutation.

Next we define some standard quasi-reflections of skew polynomial rings.

**Definition 2.3.** As usual we fix a set of scalars \( p \) and let \( A \) be the skew polynomial ring \( k_p[x_1, \ldots, x_n] \). Let \( \lambda \) be a nonzero scalar.

(a) Let \( s \in [n] \) and suppose that \( \lambda \) is not 1. Let \( \theta_{s, \lambda} \) be the automorphism of \( A \) determined by

\[
\theta_{s, \lambda}(x_i) = \begin{cases} x_i & i \neq s \\ \lambda x_s & i = s. \end{cases}
\]

This map is called a standard reflection of \( A \).
Let $s, t \in [n]$. Suppose that $p_{st} = -1$ and that $p_{sj} = p_{ij}$ for all $j \in [n] \setminus \{s, t\}$. Let $\tau_{s, t, \lambda}$ be the automorphism of $A$ determined by

$$
\tau_{s, t, \lambda}(x_i) = \begin{cases} 
  x_i & i \neq s, t \\
  \lambda x_i & i = s \\
  -\lambda^{-1} x_s & i = t.
\end{cases}
$$

This map is called a standard mystic reflection of $A$.

(c) If $g$ is an elementary transformation, then $g \theta_{s, \lambda}^{-1}$ is called an elementary reflection.

(d) If $g$ is an elementary transformation, then $g \tau_{s, t, \lambda}^{-1}$ is called an elementary mystic reflection.

The following easy lemma justifies the names given above.

**Lemma 2.4.** Using the notation above:

(a) Each $\theta_{s, \lambda}$ is an algebra automorphism of $A$ that is a reflection of $A$ in the sense of Definition 1.3(a).

(b) Each $\tau_{s, t, \lambda}$ is an algebra automorphism of $A$ that is a mystic reflection of $A$ in the sense of Definition 1.3(b).

(c) If $g$ is in $\text{Aut}_{gr}(A)$ and $\theta$ is a reflection, then $g \theta^{-1}$ is a reflection in the sense of Definition 1.3(a).

(d) If $g$ is in $\text{Aut}_{gr}(A)$ and $\tau$ is a mystic reflection, then $g \tau^{-1}$ is a mystic reflection in the sense of Definition 1.3(b).

In the rest of this section we study the reflections of $A$. For any $f \in A_1$, we write $f = \sum_i a_i x_i$. If $f \neq 0$, we define $I_f = \{i \mid a_i \neq 0\}$. We also write $f = \sum_{i \in I_f} a_i x_i$.

**Lemma 2.5.** Let $f$ and $g$ be nonzero elements in $A_1$.

(a) Suppose $f$ is a normal element of $A$. Then $p_{ij} = p_{i'j}$ for all $i, i' \in I_f$ and for all $j \in [n]$, or equivalently, $I_f \subset B_w$ for some $w \in W$. As a consequence, $p_{ij} = 1$ for all $i, i' \in I_f$.

(b) Suppose $fg = qf$ for some $1 \neq q \in k^\times := k \setminus \{0\}$. Then $p_{ij} = q$ for all $j \in I_f$ and $i \in I_g$. As a consequence, $I_f \cap I_g = \emptyset$.

(c) Suppose $fg = gf$. Then $p_{ij} = 1$ for all $j \in I_f$ and $i \in I_g$ and $\{i, j\} \not\subseteq I_f \cap I_g$.

(d) Suppose that $p_{ij} \neq 1$ for all $i < j$. Then every normal element in $A$ is of the form $c x_i$ for some $c \in k$ and for some $i \in [n]$.

(e) Suppose that $p_{ij} \neq 1$ for all $i < j$. Let $\phi$ be an automorphism of $A$. Then there is a permutation $\sigma \in S_n$ and $c_s \in k^\times$ for all $s \in [n]$ such that

$$
\phi(x_s) = c_s x_{\sigma(s)}
$$

for all $s \in [n]$.

Proof. (d) is a consequence of (a), and (e) is a consequence of (d). So we prove (a,b,c) next.

All three statements are easy to check when $|I_f| = 1$, or $p_{ij} = 1$ for all $i$ and $j$, or $n \leq 2$.

We now assume that (i) $n \geq 3$, (ii) $|I_f| \geq 2$, and (iii) $p_{ij} \neq 1$ for some $i, j$. Let $A' = A/(x_{i_0})$ for a choice of $i_0 \in [n]$, and let $f'$ and $g'$ be the images of $f$ and $g$ in $A'$, respectively.
(a) Since $|I_f| \geq 2$, $f'$ is nonzero for any $i_0$. Since $f$ is normal, we have

$$x_{i_0} f = f(\sum_i b_i x_i)$$

which implies

$$0 = f'(\sum_{i \neq i_0} b_i x_i)$$

in $A'$. Hence $b_i = 0$ for all $i \neq i_0$ and

$$x_{i_0} f = b_{i_0} f x_{i_0}.$$ 

Therefore $p_{i_0} = p_{i_0}' = b_{i_0}$ for all $i, i' \in I_f$. Since $i_0$ is arbitrary, part (a) is proved.

(b) By symmetry, we may also assume that $|I_g| \geq 2$. Let $i_0 = n$. Write $f = f' + ax_n$ and $g = g' + bx_n$. Then $f'g = gg'f'$ in $A'$. If $a = b = 0$, then the assertion follows trivially from $f'g' = gg'f'$ and induction. Now assume that $a \neq 0$, and so we can assume that $a = 1$ without loss of generality. Note that the equations $fg = ggf$ and $f'g' = gg'f'$ imply that

$$x_n g' + b f' x_n + bx_n^2 = q(g' x_n + bx_n f' + bx_n^2).$$

Since $q \neq 1$, then $b = 0$ and the assertion follows from that fact $x_n g' = gg' x_n$.

(c) For any $i \in I_f$ and $j \in I_f$ such that $\{i, j\} \not\subset I_f \cap I_g$. Since $n \geq 3$, then there is an $i_0$ that is neither $i$ nor $j$. Then we have $f'g = g'f'$ in $A' = A/(i_0)$. The assertion follows by induction. □

The following proposition is one of the main results of this section.

**Proposition 2.6.** Let $\theta$ be a reflection of $A$. Then $\theta$ is an elementary reflection, namely, up to an elementary transformation, $\theta = \theta_{i, \lambda}$ for some $i \in [n]$ and for some $1 \neq \lambda \in k^\times$.

**Proof.** Let $v$ be a non-invariant eigenvector of $\theta$. Then $\theta(v) = \lambda v$ for some nonzero scalar $\lambda \neq 1$, and $\theta$ becomes the identity on $A/(v)$.

By Lemma 2.5(a) $I_v \subset B_w$ for some $w \in W$. We claim that

(i) $\theta(x_s) = x_s$ for all $s \notin B_w$.

(ii) $\theta(x_s) \in \sum_{i \in B_w} k x_i$ for all $s \in B_w$.

Since $\theta$ becomes the identity on $A/(v)$, for every $i$, $\theta(x_i) = x_i + c_i v$ for some $c_i \in k$.

If $s \notin B_w$, then $\theta(x_s) = x_s + c_s v$ for $s \notin B_w$, establishing condition (ii). For any $s \notin B_w$, we claim that $\theta(x_s) = x_s$. If not, then $\theta(x_s) = x_s + cv$ for some $c \neq 0$. Then $I_{\theta(x_s)} = \{s\} \cup I_v$. Since $x_s$ is normal, so is $\theta(x_s)$. By Lemma 2.5(a), $I_{\theta(x_s)} \subset B_w$, and hence $s \in B_w$, a contradiction. Therefore condition (i) holds. Let $g$ be an elementary transformation such that $g(x_i) = v$ for some $i \in B_w$ and $g(x_s)$ is fixed by $\theta$ for all $s \in B_w \setminus \{i\}$. Then $g^{-1} \theta g = \theta_{i, \lambda}$.

□

A partition $D = \{D_w \mid w \in W\}$ of $[n]$ (that is, $[n]$ is a disjoint union $\bigcup_{w \in W} D_w$) is called a p-partition if, for any two distinct $i, i' \in D_w$, $p_{ij} = p_{i'j}$ for all $j \in [n] \setminus D_w$. The block decomposition is a p-partition. For a given partition $D$, let $\text{Aut}_{\text{gr}}(A)$ be the subgroup of $\text{Aut}(A)$ consisting of the automorphisms $g$ satisfying

\[
\begin{cases}
  g(x_i) = x_i & \text{if } i \notin D_w, \\
  g(x_i) = \sum_{s \in D_w} k x_s & \text{if } i \in D_w.
\end{cases}
\]
Lemma 2.7. Fix a p-partition $D = \{D_w \mid w \in W\}$. Let $G$ be a finite group of graded algebra automorphisms of $A$. Let $G_w = G \cap \text{Aut}_{\text{gr}}(A)$. In parts (d,e) suppose that $G = \prod_{w \in W} G_w$.

(a) For any $w$, pick any $j \in D_w$ and define

$$\phi_w : x_i \mapsto \begin{cases} q_{ij}x_i & i \not\in D_w \\ x_i & i \in D_w. \end{cases}$$

Then $\phi_w$ commutes with automorphisms in $\text{Aut}_{\text{gr}}(A)$ for all $w'$.

(b) Let $A_w$ be the subalgebra generated by $\sum_{i \in D_w} kx_i$. Let $\phi = \{\phi_w\}$ as in part (a). We can use this set of automorphisms to twist algebra by $A^\phi$. Then $A^\phi$ is isomorphic to $\bigotimes_{w \in W} A_w$, and hence $A$ is a graded twist of the tensor product $\bigotimes_{w \in W} A_w$.

(c) If each $D_w$ is a subset of some $B_w$, then $A^\phi$ is a commutative polynomial ring.

(d) The group $G$ acts naturally on the twisted algebra $A^\phi$, and $(A^\phi)^G$ is isomorphic to a twisted algebra $(A^G)^\phi$, where $\phi$ is a set of graded automorphisms of $A^G$ induced from $A$.

(e) The fixed ring $A^G$ has finite global dimension (respectively, is regular) if and only if $(A^G)^\phi$ has finite global dimension (respectively, is regular) if and only if each part $A_{w}^{G_{w}}$ has finite global dimension (respectively, is regular).

Proof. (a) Straightforward. Recall that $q_{ij} = \sqrt{p_{ij}}$.

(b) Use the proof of Lemma 2.2(c)

(c) This follows from part (b) and the fact that each $A_w$ is a commutative polynomial ring.

(d) Since any $\phi_w$ commutes with elements in $G$, $G$ acts on $A^\phi$ naturally. The rest is easy to check.

(e) The first assertion is Lemma 1.7(a). By part (d) we may assume $\phi$ is trivial, and hence $A$ is a tensor product of $A_w$’s and $A^G$ is a tensor product of $A_{w}^{G_{w}}$. Then assertion follows from Lemma 1.6. □

Lemma 2.8. Fix a p-partition $D = \{D_w \mid w \in W\}$ and let $\phi_w$ be as in Lemma 2.7(a). Suppose $G = \prod_{w \in W} G_w$. Then an element $\eta$ in $G$ is a reflection (respectively, mystic reflection) of $A$ if and only if it is a reflection (respectively, mystic reflection) of $A^\phi$.

Proof. By Lemma 2.7(a) and by the hypothesis that $G = \prod_{w \in W} G_w$, $\phi_w$ commutes with every element in $G$. Therefore every element in $G$ is also a graded algebra automorphism of $A^\phi$ when $A^\phi$ is identified with $A$ as graded vector spaces. This also implies that $\text{Tr}_{A}(\eta, t) = \text{Tr}_{A^{\phi}}(\eta, t)$ for all $\eta \in G$. Therefore $\eta$ is a quasi-reflection of $A$ if and only if it is a quasi-reflection of $A^\phi$. Since $\det \eta|_{A_i} = \det \eta|_{A^{\phi}_i}$, $\eta$ is a mystic reflection of $A$ if and only if it is a mystic reflection of $A^\phi$ by Proposition 1.4(b). □

3. Circles induced by finite group actions

In this section we will study further the structure of $A$ when $A$ admits a mystic reflection. As in the last section we fix a parameter set $p$. Our starting point is the following proposition.

Proposition 3.1. Let $\tau$ be a mystic reflection of $A$. 


(a) Up to a sequence of elementary transformations, $\tau$ is a standard mystic reflection $\tau_{i,j,1}$ for some $i,j \in [n]$.

(b) There is a pair $(i,j)$ such that $p_{ij} = -1$.

(c) Let $i,j \in [n]$ be the integers described in part (a). Then $B(i) = \{i\}$ and $B(j) = \{j\}$. As a consequence, $\tau = \tau_{i,j,\lambda}$ (without conjugating elementary transformations).

**Proof.** By Proposition 1.4(b,c), there is a basis $\{y_i \mid i \in [n]\}$ such that

$$
\tau(y_i) = \begin{cases} 
  y_i & i \leq n - 2 \\
  y_n & i = n - 1 \\
  -y_{n-1} & i = n
\end{cases}
$$

and $y_n y_{n-1} = -y_{n-1} y_n$. Since $y_n y_{n-1} = -y_{n-1} y_n$, Lemma 2.5(b) implies that $I_{y_n} \cap I_{y_{n-1}} = \emptyset$, $p_{ij} = -1$ for all $i \in I_{y_n}$, and $j \in I_{y_{n-1}}$. Thus we have proved part (b).

We claim that part (c) follows from part (a). By part (a) and induction on the number of elementary transformations we may assume that $\tau = g\tau_{i,j,1}g^{-1}$ where $g$ is an elementary transformation. Suppose $|B(i)| = \alpha > 1$ and then write $B(i) = \{i_1 = i, i_2, \ldots, i_\alpha\}$. We have $g(x_{i_s}) = x'_{i_s} \in g\Theta_k x_{i_s}$ for all $s = 1, \ldots, \alpha$.

Similarly, we write $B(j) = \{j_1 = j, j_2, \ldots, j_\beta\}$ and $g(x_{j_s}) = x'_{j_s} \in g\Theta_k x_{j_s}$ for all $s = 1, \ldots, \beta$. Since $\tau = g\tau_{i,j,1}g^{-1}$, we have $\tau(x_{i_s}) = x_{i_s}'$, $\tau(x_{j_s}) = -x_{j_s}'$, and $\tau(x_{i_1}) = x'_{i_1}$ for all $s \neq i, j$. By the last paragraph, $p_{i,j} = -1$ for some $i, j \in B(i)$ and $i,j \in B(j)$.

By the definition of the blocks, we have $p_{i,j} = -1$ for all $i,j \in B(i)$ and $i,j \in B(j)$. Since $x'_{i} = x'_{i_s}$ commutes with $x'_{j}$, (they are in the same block), $x_{i} = x'_{i_s}$ commutes with $x'_{i_s}$ after applying $\tau$. This contradicts the fact that $p_{i,j} = -1$. Therefore $B(i) = \{i\}$. By symmetry, $B(j) = \{j\}$. Up to a scalar, we may assume that $x'_{i} = x_{i}$ and $x'_{j} = x_{j}$. Hence $\tau(x_{i}) = x_{i}$, $\tau(x_{j}) = -x_{j}$, and $\tau(x_{i}) = x_{j}$ for all $s \neq i, j$. For $x_{s}$ outside of the blocks $B(i)$ and $B(j)$, we may assume $g$ is the identity. Therefore $\tau = \tau_{i,j,1}$ up to a scalar change on $x_{i}$.

It remains to prove (a). We assume (3.1.1) and, without loss of generality, we may assume that, up to elementary transformations, $|I_{y_{n-1}}|$ is minimal.

Case 1: $|I_{y_{n-1}}| = 1$, or, equivalently, up to a scalar, $y_{n-1} = x_t$ for some $t \in [n]$.

Up to a permutation we may assume that $y_{n-1} = x_{n-1}$. Then $y_{n-1}$ is normal and so is $y_n = \tau(y_{n-1})$. Lemma 2.5(a) says that $I_{y_n}$ is a subset of a block $B_w$ for some $w$ (denote this block as $B(I_{y_n})$). Using an elementary transformation and a permutation if necessary, we may assume that $y_n = x_n$. Since $\tau$ becomes the identity on $A/(x_{n-1}, x_n)$, we have

$$
\tau(x_i) = x_i + a_i x_{n-1} + b_i x_n
$$

for all $i \in [n-2]$. Recall that $p_{n-1} = -1$. Using Lemma 2.5(a) we see the following:

(i) If $i \in B(n)$, then $a_i = 0$ because $n - 1 \notin B(n)$. Now the fact that $\tau^2(x_i) = x_i + b_i x_n - b_i x_{n-1}$ is normal implies that $b_i = 0$.

(ii) If $i \in B(n-1)$, then $b_i = 0$. A similar argument to that above shows that $a_i = 0$.

(iii) If $i \notin B(n) \cup B(n-1)$, then $a_i = b_i = 0$.

Therefore $\tau = \tau_{i,j,1}$ up to an elementary transformation and a permutation.

Case 2: $|I_{y_n}| = 1$. This is similar to Case 1.
Case 3: $|I_{y_{n-1}}| > 1$ and $|I_{y_n}| > 1$. Since $\tau$ is the identity on $A_1/(ky_{n-1} + ky_n)$, we have
\[
\tau(x_i) = x_i + a_1y_{n-1} + b_1y_n
\]
for all $i \in [n]$. Since $x_i$ is normal, so is $\tau(x_i)$. Let $i \not\in I_{y_n} \cup I_{y_{n-1}}$. If $a_i \neq 0$, then $B(I_{\tau(x_i)}) \supset \{i\} \cup I_{y_{n-1}}$, by Lemma 2.5(a). This implies that $I_{y_{n-1}}$ is a single element after an elementary transformation, a contradiction. Therefore $a_i = 0$. Similarly, $b_i = 0$ for all $i \not\in I_{y_n} \cup I_{y_{n-1}}$.

For the final argument we consider two cases.

First we consider the case when $|I_{y_n}| = |I_{y_{n-1}}| = 2$. Without loss of generality, we may assume that $y_{n-1} = x_1 + x_2$ and $y_n = x_{n-1} + x_n$. So we have $\tau(x_i) = x_i$ for all $2 < i < n - 1$. So we have
\[
\tau(x_1) = x_1 + a_1(x_1 + x_2) + b_1(x_{n-1} + x_n), \quad \text{and}
\tau(x_2) = x_2 + a_2(x_1 + x_2) + b_2(x_{n-1} + x_n).
\]
Since $\tau(x_1 + x_2) = \tau(y_{n-1}) = y_n = x_{n-1} + x_n$, either $b_1$ or $b_2$ is nonzero. Suppose $b_1 \neq 0$. Then $\{n-1, n\} \in B(I_{\tau(x_1)}).$ Therefore up to an elementary transformation, $|I_{y_n}| = 1.$ But this was done in Case 1.

Second we assume that $|I_{y_{n-1}}| \geq 3$. Write $\tau(x_i) = x_i + a_iy_{n-1} + b_iy_n$.

Since $I_{y_{n-1}} \cap I_{y_n} = \emptyset$, $I_{\tau(x_i)} \supset I_{y_{n-1}} \setminus \{i\}$ if $a_i \neq 0$. This means that we can strictly reduce the cardinality of $I_{y_{n-1}}$ by an elementary transformation. This contradicts the minimality of $|I_{y_{n-1}}|$. Therefore $a_i = 0$ for all $i$. Thus $\tau$ is an identity on $A_1/(ky_n)$, which yields a contradiction. By symmetry, it is impossible to have $|I_{y_n}| \geq 3$. This completes the proof. \(\Box\)

**Definition 3.2.** Fix a finite subgroup $G$ of $\text{Aut}_{gr}(A)$.

(a) Any single element subset of $[n]$ is called a **trivial circle**.

(b) Let $C$ be a subset of $[n]$ consisting of at least two integers. We call $C$ a **circle** if, for each pair of distinct integers $(i, j) \subset C$, there is a sequence of mystic reflections $\{r_{i, j, \lambda_s} \in G \mid s = 0, \cdots, l\}$ for some $\lambda_s \in k^\times$ such that $i = i_0, i_{s+1} = j_s$ for all $s$, and $j_1 = j$. By Proposition 3.1(c), every non-trivial circle is a union of some single-element blocks.

(c) We call $C$ a **maximal circle** if $C$ is a non-trivial circle and it is not properly contained in another circle.

(d) A **circle decomposition** is a disjoint union of maximal circles and trivial circles
\[
[n] = \bigcup_{u \in U} C_u.
\]

(e) A **block-circle decomposition** is a disjoint union
\[
[n] = \bigcup_{u \in U_1} C_u \cup \bigcup_{w \in W_1} B_w
\]
where each $C_u$ for $u \in U_1$ is a maximal circle and where each $B_w$ for $w \in W_1$ is a block in $[n]$. This decomposition can be formed by first having a block decomposition and then joining single-element blocks together to possible maximal circles.
For simplicity, we write block-circle decomposition as

\[ [n] = \bigcup_{v \in V} D_v \]

where \( V = U_1 \cup W_1 \) and \( D_v \) is either \( B_v \) or \( C_v \).

The block-circle decomposition always exists and is uniquely determined by the parameters \((p,G)\). As we noted before, the block decomposition is a \( p \)-partition. Next we show that the block-circle decomposition is a \( p \)-partition.

**Proposition 3.3.** Let \( G \) be a finite group of \( \text{Aut}_{\text{gr}}(A) \) and let \([n] = \bigcup_{v \in V} D_v\) be the block-circle decomposition. Let \( A_v \) be the subalgebra generated by \( \sum_{i \in D_v} kx_i \).

1. Let \( G_v = G \cap \text{Aut}_{\text{gr},v}(A) \). Assume that \( G \) is generated by quasi-reflections. Then \( G = \bigcap_{v \in V} G_v \) and each \( G_v \) is generated by quasi-reflections.
2. The block-circle decomposition is a \( p \)-partition, namely, for any \( i \neq i' \) in the same \( D_v \) and \( j \notin D_v \), \( p_{ij} = p_{i'j} \).

**Proof.** (a) Note that the block decomposition is a sub-partition of the circle-block decomposition by Proposition 2.6. Every reflection is an elementary transformation, hence it belongs to \( \text{Aut}_{\text{gr},w}(A) \cap G = G_w \) for some \( w \). By Proposition 3.1, every mystic reflection is in \( G_v \) for some \( v \in U_1 \). Therefore the assertion follows.

(b) We need to show this only for \( v \in U_1 \). By induction we may assume that there is a mystic reflection of the form \( \tau_{i,i',1} \). Since \( \tau_{i,i',1}(x_j) = x_j \) for all \( j \notin D_v \), \( \tau_{i,i',1}(x_i) = x_{i'} \) implies that \( p_{ij} = p_{i'j} \).

The next corollary follows from Proposition 3.3, Lemma 2.7 and Lemma 4.1 in the next section.

**Corollary 3.4.** Let \( G \) be a finite group of \( \text{Aut}_{\text{gr}}(A) \) and let \([n] = \bigcup_{v \in V} D_v\) be the block-circle decomposition. Let \( A_v \) be the subalgebra generated by \( \sum_{i \in D_v} kx_i \).

1. For any \( v \) and any \( j \in D_v \), there is an algebra automorphism determined by

\[ \phi_v : x_i \mapsto \begin{cases} 
q_{ij}x_i & i \notin D_v \\
ix_i & i \in D_v.
\end{cases} \]

Further \( \phi_v \) commutes with automorphisms in \( \text{Aut}_{\text{gr},v}(A) \) for all \( v' \).
2. Let \( \phi = \{ \phi_v \mid v \in V \} \). We can use this set of automorphisms to twist \( A \); denote the resulting algebra by \( A^\phi \). Then \( A^\phi \) is isomorphic to \( \bigotimes_{v \in V} A_v \), and hence \( A \) is a graded twist of \( \bigotimes_{v \in V} A_v \).
3. The fixed ring \( A^G \) has finite global dimension (respectively, is regular) if and only if \( (A^G)^\phi \) has finite global dimension (respectively, is regular) if and only if each part \( A_v^G \) has finite global dimension (respectively, is regular).
4. If \( D_v \) is a block, then \( A_v \) is a commutative polynomial ring.
5. If \( D_v \) is a circle, then \( A_v \) is isomorphic to \( k_{-1}[x_1, \ldots, x_m] \) where \( m = |D_v| \).
6. If \( G \) is generated by quasi-reflections of \( A \), then each \( G_v \), when restricted to \( A_v \), is generated by quasi-reflections of \( A_v \).

**Proof.** By Proposition 3.3, the block-circle decomposition is a \( p \)-partition. See Lemma 2.7 for parts (a,b,c,d).

Part (e) will be proved in Lemma 4.1 in the next section.

(f) By parts (b,c) we may assume that \( A \) is the tensor product \( \bigotimes_{v \in V} A_v \). Then the trace formula implies that \( g \in G_v \) is a quasi-reflection if and only if \( g|A_v \) is a
quasi-reflection. The assertion follows because $G_v$ is generated by quasi-reflections of $A$ by the proof of Proposition 3.3(a).

By Corollary 3.3(c,f) we need to prove the Shephard-Todd-Chevalley Theorem only for each block/circle $A_0$. If $D_v$ is a block, then the classical Shephard-Todd-Chevalley Theorem applies. The next section deals with the circle case.

4. One circle case and the proof of the Main Theorem

In most of this section we let $G$ be a finite subgroup of $\text{Aut}_G(A)$ that contains mystic reflections such that $[n]$ is a circle. Assume that $n \geq 2$.

Lemma 4.1. Suppose $[n]$ is a circle for the group $G$, which is generated by quasi-reflections of $A$. Up to scalar change of a basis for $A_1$, $\tau_{i,j,1} \in G$ for all $i \neq j$. As a consequence, $p_{ij} = -1$ for all $i < j$.

Proof. We use induction on $m$ to show that $\tau_{i,j,1} \in G$ for all $1 \leq i, j \leq m$. Nothing is to be proved for $m = 1$, so we begin with $m = 2$. By Definition 3.2(b), there is a sequence of $\{\tau_{i,s,\lambda}, \in G \mid s = 0, \cdots, t\}$ such that $1 = i_0, i_{s+1} = j_s, j_1 = 2$. Choose $t$ to be minimal; then we claim that $t = 0$. If $t > 0$, we have $\tau_{i_1,j_1,\lambda_1} = \tau_{i_1,j_1,-\lambda_1}$, and

$$\tau_{i_1,j_1,\lambda_0,\lambda_1} = \tau_{i_1,j_1,\lambda_1} \tau_{i_1,i_0,\lambda_0} \tau_{i_0,j_1,\lambda_1}^{-1}$$

Then we can make a shorter sequence, a contradiction. Therefore $t = 0$ and $\tau_{i,2,\lambda} \in G$ for some $\lambda \in k^\times$. Replacing $x_2$ with $\lambda x_2$, we have $\tau_{1,2,1} \in G$, and this completes the proof for $m = 2$. Suppose now $m > 2$ and the assertion holds for $m - 1$. By a similar argument, we have $\tau_{m-1,m,\lambda} \in G$. Letting the new $x_m$ be $\lambda x_m$, we have $\tau_{m-1,m,1} \in G$. For any $i < m - 1$, we have

$$\tau_{i,m,1} = \tau_{m-1,m,1} \tau_{i,m-1,1} \tau_{m-1,m,1}^{-1}$$

By the induction hypothesis, $\tau_{i,j,1} \in G$ for all $1 \leq i, j \leq m$. The assertion follows now by induction on $m$.

We now assume that $p_{ij} = -1$ for all $i < j$; fix a basis $\{x_i \mid i \in [n]\}$ so that $\tau_{i,j,1} \in G$ for all $i \neq j$. Let

$$\Theta_i = \{\lambda \in k^\times \mid \theta_{i,\lambda} \in G\},$$

$$T_{i,j} = \{\lambda \in k^\times \mid \tau_{i,j,\lambda} \in G\},$$

and

$$S_{i,j} = \{\lambda \in k^\times \mid s_{i,j,\lambda} \in G\},$$

where $s_{i,j,\lambda}$ is the automorphism determined by

$$s_{i,j,\lambda}(x_s) = \begin{cases} x_s & s \neq i, j \\ \lambda x_i & s = i \\ \lambda^{-1} x_j & s = j. \end{cases}$$

Lemma 4.2. Suppose $[n]$ is a circle for a group $G$ that is generated by quasi-reflections of $A$.

(a) $\Theta_i = \Theta_j$ for all $i, j$, and $\Theta_i \cong \mathbb{Z}/(\alpha)$ for some integer $\alpha$.

(b) $S_{i,j} = S_{i',j'}$ for all $(i, j)$ and $(i', j')$ and $S_{i,j} \cong \mathbb{Z}/(\beta)$ for some even integer $\beta$.

(c) $T_{i,j} = T_{i',j'}$ for all $(i, j)$ and $(i', j')$, and $T_{i,j} = S_{i,j} \cong \mathbb{Z}/(\beta)$ for some even integer $\beta$. 
(d) $\alpha$ divides $\beta$.

Proof. First note that $\Theta_i, T_{i,j}$, and $S_{i,j}$ are all subgroups of $k^\times$.

(a) The first assertion follows from

$$\theta_{i,j} = \tau_{i,j,1} \theta_{i,\lambda} \tau_{i,j,1}^{-1}.$$  

The second follows from the fact that any finite subgroup of $k^\times$ is cyclic, whence it is of the form $\{\lambda | \lambda^\alpha = 1\}$ for some $\alpha$.

(b) Since $S_{i,j} = S_{j,i}$ for $i \neq j$, we may assume that $i = i'$ without loss of generality. In this case $i, j, j'$ are all different. The first assertion follows from the fact that

$$s_{i,j',\lambda} = \tau_{j',j,1} s_{i,j,\lambda} \tau_{j',j,1}^{-1}.$$  

Similar to the proof of (a), $S_{i,j} \cong \mathbb{Z}/(\beta)$. The reason that $\beta$ is even is that $s_{i,j,-1} = \tau_{i,j,1}^2$ is in $G$.

(c) This is true because $\tau_{i,j,1}^{-1} \tau_{i,j,\lambda} = s_{i,j,\lambda}$.

(d) This is true because $s_{i,j,\lambda} = \theta_{i,\lambda} \theta_{i,j,\lambda}^{-1}$. $\square$

Suppose $\alpha | \beta$ and $2 | \beta$. Let $M(n, \alpha, \beta)$ be the subgroup of $\text{Aut}_{gr}(A)$ generated by $\{\theta_{i,\lambda} | \lambda^\alpha = 1\} \cup \{\tau_{i,j,\lambda} | \lambda^\beta = 1\}$. Lemma 4.2, Proposition 2.6, and Proposition 3.1(c) imply the following fact. Recall that in this section (except for the last theorem) $A = k_{-1}[x_1, \ldots, x_n]$.

Lemma 4.3. If $G$ is a finite subgroup of $\text{Aut}_{gr}(A)$ generated by quasi-reflections and if $[n]$ is circle, then $G$ is $M(n, \alpha, \beta)$ for some $\alpha$ and $\beta$.

Let $G = M(n, \alpha, \beta)$. Let $B$ be the subalgebra of $A$ generated by $x_1^\alpha, \ldots, x_n^\alpha$. If $\alpha$ is even then $B$ is the commutative polynomial ring. If $\alpha$ is odd, then $B \cong A$ under the map that sends $x_i \to x_i^\alpha$. Let $z_i = x_i^\alpha$ for all $i$. To cover both cases of $\alpha$, we write $B = k_{\pm 1}[z_1, \ldots, z_n]$. Let $R$ be the subgroup generated by all $\theta_{i,\lambda}$. Then $B = A^R$. Let $G_B$ be the induced group of $G$ acting on $B$. Note the similarity between the invariants described below and those obtained when the symmetric group $S_n$ acts on $k[x_1, \ldots, x_n]$.

Proposition 4.4. Retain the notation described above.

(a) $A^G = B^{G_B}$ as subalgebras of $A$.

(b) If $\alpha$ is even, then $G_B$ is a classical reflection group of the commutative polynomial ring $B$. As a consequence, $B^{G_B}$ is regular.

(c) If $\alpha$ is odd, then $G_B$ contains no reflections of $B$. In this case $G_B$ is generated by mystic reflections of $B$.

(d) Suppose $\alpha$ is odd. Let $2t$ be the $\beta$ for the group $G_B$. Then $B^{G_B}$ is the commutative polynomial ring $D := k[(z_1, \ldots, z_n), \sum_i z_i^{2t}, \sum_i z_i^{4t}, \ldots, \sum_i z_i^{(n-1)2t}]$. As a consequence, $B^{G_B}$ is regular.

By (b, d), $A^G$ is always isomorphic to a commutative polynomial ring.

Proof. (a) Let $R$ be the subgroup generated by the reflections $\theta_{i,\lambda}$. Then $R$ is a normal subgroup of $G$, and it is easy to see that $A^R = B$. Since $R$ is normal, $G$ induces a subgroup $G_B$ of $\text{Aut}_{gr}(B)$ that is generated by the induced maps of $\tau_{i,j,\lambda}$, denoted by $\tau'_{i,j,\lambda}$. Clearly, $A^G = B^{G_B}$.

(b) If $\alpha$ is even, all of the $\tau'_{i,j,\lambda}$ are reflections of $B$. Hence $B^{G_B}$ is regular.
(c) If $\alpha$ is odd, then all of the $\tau^i_{j,\lambda}$ are equal to $\tau_{i,j,\lambda}$ for the skew polynomial ring $B = k - [z_1, \ldots, z_n]$, and so these are mystic reflections. By Proposition 4.3(d), $G_B$ does not contain any reflection.

(d) Let $C = k[[z_1, \ldots, z_n]^{2t}, \sum_{i=1}^{2t} z_i, \ldots, \sum_{i}^{(2n-2)t}]$. It is clear that $C$ is a subalgebra of $\mathcal{D}$. For any homogeneous element $w \in B$, we define

$$\Gamma(w) := \frac{1}{|G_B|} \sum_{g \in G_B} g(w).$$

It suffices to show that $\Gamma(w) \in D$ for all $w \in B$. (The definition of $D$ is given in Proposition 4.3(d).) Since $\Gamma$ is linear, we may take $w$ to be a monomial. The following analysis uses induction on the degree of $w$.

Case 1: Suppose $(z_1 \cdots z_n)w$. Then $w = (z_1 \cdots z_n)w'$ for some $w'$ with its degree less than the degree of $w$, and $\Gamma(w) = (z_1 \cdots z_n)\Gamma(w')$. The assertion follows by induction.

Case 2: Suppose $(z_1 \cdots z_n) \not{|} w$. If $\Gamma(w) = 0$ we are done. Otherwise $\Gamma(w) = \sum_{(d)} c(d) z_1^{d_1} \cdots z_n^{d_n} \neq 0$ where $(d) = (d_1, \ldots, d_n)$. By the definition of $\Gamma(w)$ and the hypothesis that $(z_1 \cdots z_n) \not{|} w$, we see that for each $c(d) \neq 0$ there is an $i$ such that $d_i = 0$. If, for each $c(d) \neq 0$, $2t | d_i$ for all $i$, then $\Gamma(w)$ is in $k[z_1^{2t}, \ldots, z_n^{2t}]$, which is the ring of symmetric polynomials in $\{z_1^{2t}, \ldots, z_n^{2t}\}$. By the Newton identities (see e.g. [CLO, p. 317]) this is $k[[z_1^{2t}, \ldots, z_n^{2t}]]$, and we are done in this case. Otherwise there is $c(d) \neq 0$ such that $2t \not{|} d_i$ for some $i$. Without loss of generality, we may assume that:

(a) $c(d) \neq 0$ for some $(d) = (d_1, \ldots, d_{n-1}, 0)$, (this means that we pick $d_n = 0$),

(b) $2t \not{|} d_i$ for some $i$, and

(c) $2t | d_j$ for all $j > i$.

Note that every $\tau_{i,j,\lambda}$ maps a monomial in $\Gamma(w)$ to another monomial in $\Gamma(w)$ (possibly the same). In particular, $\tau_{i,n,\lambda}$ maps $c(d) z_1^{d_1} \cdots z_i^{d_i} \cdots z_n^{d_{n-1}}$ to

$$c(d) z_1^{d_1} \cdots (\lambda z_n)^{d_i} \cdots z_n^{d_{n-1}} = \lambda^{d_i} c(d) z_1^{d_1} \cdots z_i^{d_i} \cdots z_n^{d_{n-1}}.$$ 

Since $\tau_{i,n,\lambda}(\Gamma(w)) = \Gamma(w) = \tau_{i,n,\lambda}(\Gamma(w))$, the element $\lambda^{d_i} c(d) z_1^{d_1} \cdots z_i^{d_i} \cdots z_n^{d_{n-1}}$ is independent of $\lambda$. This implies that $\lambda^{d_i} = 1$ or $2t | d_i$, a contradiction. This completes the proof. \hfill $\square$

Now we are ready to prove our main result, Theorem 0.1.

**Theorem 4.5.** Let $A$ be a skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_n]$ and $G$ a finite subgroup of $\text{Aut}_{gr}(A)$. Then the fixed subring $A^G$ has finite global dimension if and only if $G$ is generated by quasi-reflections. In this case the fixed subring $A^G$ is isomorphic to a skew polynomial ring with possibly different $\{p_{ij}\}$.

**Proof.** By Proposition 1.5 we need to show only one direction, namely, that if $G$ is generated by quasi-reflections then $A^G$ has finite global dimension. It suffices to show that $A^G$ is isomorphic to a skew polynomial ring.

By Corollary 3.3(c,f) and Lemma 1.7(b), we need to show the claim for $A_v$, where $D_v$ is either a block or a circle in the block-circle decomposition of $[n]$. If $D_v$ is a block, then $A_v$ is a commutative polynomial ring, and the claim follows from
the classical Shephard-Todd-Chevalley Theorem. If $D_n$ is a circle, the claim follows from Lemma 4.3 and Proposition 4.4.

**Corollary 4.6.** Let $A = k_{p_1}[x_1, \ldots, x_n]$ and let $G$ be a finite subgroup of $\text{Aut}_{gr}(A)$. If $G$ is generated by quasi-reflections, then, as an abstract group, $G$ is a product of classical reflection groups and copies of some $M(n, \alpha, \beta)$.

**Proof.** Using the block-circle decomposition $[n] = \bigcup_{v \in V} D_v$ [Definition 3.2(e)], $G$ is a product $\prod_{v \in V} G_v$ and each $G_v$ is generated by quasi-reflections [Proposition 3.3]. If $D_v$ is a block, then $G_v$ is isomorphic to a classical reflection group [Corollary 3.4(d)]. If $D_v$ is a circle, then $G_v$ is isomorphic to $M(n, \alpha, \beta)$ by Lemma 4.3. The assertion follows.

**Remark 4.7.** We will see that there are “mystic reflection groups” $M(n, \alpha, \beta)$ that cannot be realized as classical reflection groups [Example 6.2 and Example 6.3].

The following conjecture, which is suggested by Corollary 4.6, is related to Conjecture 4.8.

**Conjecture 4.8.** Let $B$ be a quantum polynomial ring. Suppose $G$ is a finite subgroup of $\text{Aut}_{gr}(B)$ such that $B^G$ has finite global dimension. Then $G$ is a product of classical reflection groups and copies of $M(n, \alpha, \beta)$.

## 5. Toward a Quantum Shephard-Todd-Chevalley Theorem

In the first half of this section we prove Theorem 0.3, namely, a version of Shephard-Todd-Chevalley Theorem for general quantum polynomial rings when the group is abelian. The proof of Theorem 0.3 is quite different from the proof of Theorem 0.1.

**Lemma 5.1.** Let $B$ be a quantum polynomial ring and let $G$ be a finite abelian subgroup of $\text{Aut}_{gr}(B)$. Suppose that $\{g_1, g_2, \ldots, g_r\}$ is a minimal generating set for $G$ where each $g_i$ is a quasi-reflection. If $i \neq j$ and $y \in B_1$ is a common eigenvector for $g_i$ and $g_j$ with $g_i(y) = \lambda_i y$ and $g_j(y) = \lambda_j y$, then either $\lambda_i = 1$ or $\lambda_j = 1$.

**Proof.** Since $G$ is abelian, we can find a $k$-linear basis $\{y_1, y_2, \ldots, y_n\}$ of $B_1$ such that the action of each $g_i$ on $B_1$ is diagonal with respect to this basis. Suppose that neither $\lambda_i$ nor $\lambda_j$ is equal to 1. For simplicity assume that $g_i = g_1$ and that $g_j = g_2$. Suppose to the contrary that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$. We will obtain a contradiction in each of the following four cases.

Case 1: Suppose that $g_1$ and $g_2$ are both reflections. Since the dimensions of the corresponding non-invariant eigenspaces are each 1 [Definition 3.3(a)], there is no loss of generality in assuming that $y = y_n$. By Proposition 1.4(a) $y_n$ is a normal element of $B$. In this case $\lambda_1$ and $\lambda_2$ are $m_1$th and $m_2$th roots of unity, respectively. Let $\lambda$ be the generator of the subgroup $\langle \lambda_1, \lambda_2 \rangle$ of $k^\times$. Then $\lambda$ is an $m$th root of unity for $m = \text{lcm}(m_1, m_2)$, and $\lambda = \lambda_1^{r_1} \lambda_2^{r_2}$ for integers $r_1$ and $r_2$. Let $g = g_1^{r_1} g_2^{r_2}$. Then $g$ is an automorphism of $B$ with $g(y_n) = \lambda y_n$ and $g(y_s) = y_s$ for $s \neq n$. Thus $g$ induces the identity automorphism on the factor algebra $B = B/(y_n)$ so that

$$\text{Tr}_B(g, t) = \frac{1}{(1-t)^{n-1}}.$$ 

But it is also the case that $\text{Tr}_B(g, t) = (1 - \lambda t) \text{Tr}_B(g, t)$. Hence

$$\text{Tr}_B(g, t) = \frac{1}{(1-t)^{n-1}(1-\lambda t)},$$ 

and
and $g$ is a quasi-reflection. Since both $g_1$ and $g_2$ are powers of $g$, we have contradicted the minimality of $\{g_1, g_2, \ldots, g_r\}$ as a generating set.

Case 2: Suppose that $g_1$ is a reflection and $g_2$ is a mystic reflection. Then as in Case 1 we may assume that $y = y_n$, and, since $g_1$ is a reflection, that $y_n$ is a normal element of $B$. By Definition 1.3(b) we have that $g_2(y_n) = \pm iy_n$; without loss of generality, say $g_2(y_n) = iy_n$. There then must be another eigenvector, which we may take as $y_{n-1}$, such that $g_2(y_{n-1}) = -iy_{n-1}$. Furthermore, we assume that the subalgebra generated by $y_{n-1}$ and $y_n$ is isomorphic to $C := k\langle y_{n-1}, y_n \rangle / (y_n^2 - y_{n-1}^2)$ [Proposition 1.4(c)]. Since $y_n$ is a normal element of $B$ we have that

$$y_n y_{n-1} = (a_{n-1} y_{n-1} + a_n y_n + \sum_{i=1}^{n-2} a_i y_i) y_n.$$  

Applying $g_2$ we have

$$y_n y_{n-1} = (a_{n-1} y_{n-1} - a_n y_n + i \sum_{i=1}^{n-2} a_i y_i) y_n.$$  

Subtracting gives

$$(2a_n y_n + (1 - i) \sum_{i=1}^{n-2} a_i y_i) y_n = 0 	ext{ and } 2a_n y_n + (1 - i) \sum_{i=1}^{n-2} a_i y_i = 0,$$

since $B$ is a domain. From linear independence it follows that $y_n y_{n-1} = a_{n-1} y_{n-1} y_n$, and $y_n$ is a normal element of $C$, which is a contradiction because $y_n$ is not normal in $C$.

Case 3: Suppose that both $g_1$ and $g_2$ are mystic reflections that share only one of the non-invariant eigenvectors $y_n$. Without loss of generality we may assume that $g_1(y_n) = g_2(y_n) = -iy_n$ (by replacing $g_s$ by $g_s^{-1}$ if necessary). Again by Definition 1.3(b) we may take another non-invariant eigenvector of $g_1$ as $y_{n-1}$ such that $g_1(y_{n-1}) = iy_{n-1}$ and the subalgebra generated by $y_{n-1}$ and $y_n$ is isomorphic to $k\langle y_{n-1}, y_n \rangle / (y_n^2 - y_{n-1}^2)$. Since $g_1$ and $g_2$ share only the one non-invariant eigenvector $y_n$, $g_2(y_{n-1}) = y_{n-1}$. In this case

$$y_{n-1}^2 = g_2(y_{n-1}) = g_2(y_n^2) = (-iy_n)^2 = -y_n^2 = -y_{n-1}^2,$$

which is a contradiction.

Case 4: Suppose that both $g_1$ and $g_2$ are mystic reflections that share the pair of non-invariant eigenvectors that we can take to be $y_{n-1}$ and $y_n$. By Definition 1.3(b) we may assume that $g_1(y_{n-1}) = iy_{n-1}$ and $g_1(y_n) = -iy_n$. Then either $g_2(y_{n-1}) = iy_{n-1}$ and $g_2(y_n) = -iy_n$, in which case $g_2 = g_1$, or $g_2(y_{n-1}) = -iy_{n-1}$ and $g_2(y_n) = iy_n$, in which case $g_2 = g_1^3$. In either case we have contradicted the minimality of the set of generators.

Let $o(g)$ denote the order of $g$.

**Proposition 5.2.** Let $B$ be a quantum polynomial ring and let $G$ be a finite abelian subgroup of $\text{Aut}_e(B)$ generated by a minimal set of generators

$$S = \{g_1, g_2, \ldots, g_p, h_1, h_2, \ldots, h_q\}$$

where each $g_i$ is a reflection and each $h_j$ is a mystic reflection.
(a) There is a $k$-linear basis
\[ \{x_1, x_2, \ldots, x_p, y_1, z_1, y_2, z_2, \ldots, y_q, z_q, w_1, \ldots, w_m \} \]
of $B_1$ such that $g_i(x_i) = \lambda_i x_i$ for $i = 1, 2, \ldots, p$ and $g_i(x) = x$ if $x$ is any of the other elements of the basis, and $h_j(y_j) = iy_j, h_j(z_j) = -iz_j$ for $j = 1, 2, \ldots, q$ and $h_j(x) = x$ if $x$ is any other element of the basis.

(b) The group $G$ decomposes as
\[ G = \langle g_1 \rangle \oplus \cdots \oplus \langle g_p \rangle \oplus \langle h_1 \rangle \oplus \cdots \oplus \langle h_q \rangle. \]

(c) The algebra $B$ is expressed as a free module over $B^G$ by
\[ B = \bigoplus x_1^{r_1} x_2^{r_2} \cdots x_p^{r_p} y_1^{s_1} y_2^{s_2} \cdots y_q^{s_q} z_q^{t_q} B^G \]
where $(s_j, t_j) \in \{(0,0), (1,0), (0,1), (2,0)\}$ for $j = 1, 2, \ldots, q$ and $0 \leq r_i < o(g_i)$ for $i = 1, 2, \ldots, p$.

(d) The fixed subring $B^G$ is regular.

Proof. Since $G$ is abelian, there is a basis for $B_1$ for which the action of each element of $G$ on $B_1$ is diagonal. Then, by Lemma [5.1], the elements of $S$ do not share any non-invariant eigenvectors, and part (a) follows.

The remainder of the proof will be by induction on $p + q$. We will consider the generators of $G$ ordered with the reflections listed first, followed by the mystic reflections. If $G = \langle g_1 \rangle$, then $B = \bigoplus_{k=1}^{o(g_1)} x_k B^G$ by the proofs of [KKZ1] Lemma 5.1 for $o(g_1) > 2$ and [KKZ1] Lemma 5.2 for $o(g_1) = 2$. If $G = \langle h_1 \rangle$, then $B = B^G \oplus y_1 B^G \oplus y_1^2 B^G \oplus z_1 B^G$ by the proof of [KKZ1] Proposition 4.3.

Now suppose that the result holds whenever $|S| < p + q$. First suppose that $S = \{g_1, g_2, \ldots, g_p\}$ and let $H = \langle g_1, g_2, \ldots, g_{p-1} \rangle$; that is, consider the case where $q = 0$. By induction we may assume that
\[ H = \langle g_1 \rangle \oplus \langle g_2 \rangle \oplus \cdots \oplus \langle g_{p-1} \rangle \]
and that
\[ B = \bigoplus x_1^{r_1} x_2^{r_2} \cdots x_{p-1}^{r_{p-1}} B^H. \]

Since $G$ is abelian, $g_p(B^H) = B^H$. By part (a) we have that $h(x_p) = x_p$ for all $h \in H$, and thus $x_p \in B^H$. Since $g_p(x_p) = \lambda_p x_p$ where $\lambda_p$ is an $o(g_p)$th root of unity, $g_p^k(x_p) \neq x_p$ for $1 \leq k < o(g_p)$. Consequently, $g_p^k \notin H$ for $1 \leq k < o(g_p)$, and $G = \langle g_1 \rangle \oplus \cdots \oplus \langle g_p \rangle$. Clearly $B^G = B^H \cap B^G$. As above
\[ B = \bigoplus_{k=1}^{o(g_p)} x_p^k B^G. \]

Thus if $a \in B^H$, then $a = \sum_{k=1}^{o(g_p)} x_p^k a_k$ for unique $a_k$'s in $B^G$. Let $h \in H$. Then
\[ h(a) = a = \sum_{k=1}^{o(g_p)} h(x_p^k) h(a_k) = \sum_{k=1}^{o(g_p)} x_p^k h(a_k). \]

Since $G$ is abelian, we have that $h(a_k) \in B^G$, and from the uniqueness of the sum, that $h(a_k) = a_k$. Consequently each $a_k \in B^G \cap B^H = B^G$, and $B^H = \bigoplus_{k=1}^{o(g_p)} x_p^k B^G$. It follows that
\[ B = \bigoplus x_1^{r_1} x_2^{r_2} \cdots x_{p-1}^{r_{p-1}} B^H = \bigoplus x_1^{r_1} x_2^{r_2} \cdots x_{p-1}^{r_{p-1}} x_p^{r_p} B^G. \]

Now suppose that $S = \{g_1, g_2, \ldots, g_p, h_1, h_2, \ldots, h_q\}$ with $q \geq 1$, and let
\[ H = \langle g_1, g_2, \ldots, g_p, h_1, h_2, \ldots, h_{q-1} \rangle. \]
Then by induction we have that
\[ H = \langle g_1 \rangle \oplus \cdots \oplus \langle g_p \rangle \oplus \langle h_1 \rangle \oplus \cdots \oplus \langle h_{q-1} \rangle, \]
and \( B \) is a free module over \( B^H \) written as
\[ B = \bigoplus x_1^{r_1} x_2^{r_2} \cdots x_p^{r_p} y_1^{s_1} z_1^{t_1} y_2^{s_2} z_2^{t_2} \cdots y_{q-1}^{s_{q-1}} z_{q-1}^{t_{q-1}} B^H. \]
By part (a) we have that \( y_q \in B^H \). The order of \( h_q \) is 4, and since \( h_q(y_q) = iy_q, h_q^2(y_q) = -y_q, \) and \( h_q^3(y_q) = -iy_q, \) we have that no nonidentity power of \( h_q \) is in \( H \). Thus \( \langle h_q \rangle \cap H = \{ e \} \) and
\[ G = \langle g_1 \rangle \oplus \cdots \oplus \langle g_p \rangle \oplus \langle h_1 \rangle \oplus \cdots \oplus \langle h_q \rangle. \]
As noted above \( B \) as a free module over \( B^{h_q} \) decomposes as
\[ B = B^{h_q} \oplus y_q B^{h_q} \oplus y_q^2 B^{h_q} \oplus z_q B^{h_q}. \]
Using that \( B^G = B^{h_q} \cap B^H, \) and the inductive decomposition of \( B \) as a free module over \( B^H \) yields that
\[ B = \bigoplus x_1^{r_1} x_2^{r_2} \cdots x_p^{r_p} y_1^{s_1} z_1^{t_1} y_2^{s_2} z_2^{t_2} \cdots y_{q-1}^{s_{q-1}} z_{q-1}^{t_{q-1}} B^G. \]
Thus parts (b,c) follow by induction.
(d) This follows from [KKZ1, Lemma 1.10].

**Theorem 5.3.** Let \( B \) be a quantum polynomial ring and let \( G \) be a finite abelian subgroup of Aut\(_{gr}(B)\). Then \( B^G \) is regular if and only if \( G \) is generated by quasi-reflections.

**Proof.** One implication is Proposition 5.2(d) and the other implication is Proposition 1.5(a). □

By Proposition 5.2(b) Conjecture 1.8 holds in this case.

In the rest of this section we give an example that indicates that we might need to go to the world of Hopf algebras for the most general version of the Shephard-Todd-Chevalley Theorem.

**Example 5.4.** Let \( C \) be the quantum \( 2 \times 2 \)-matrix algebra \( O_q(M_2) \) [BC, Definition I.1.7] that is generated by \( x_{11}, x_{12}, x_{21}, x_{22} \) subject to the relations
\[ x_{12} x_{11} = q x_{11} x_{12} \]
\[ x_{21} x_{11} = q x_{11} x_{21} \]
\[ x_{22} x_{12} = q x_{12} x_{22} \]
\[ x_{22} x_{21} = q x_{21} x_{22} \]
\[ x_{21} x_{12} = x_{12} x_{21} \]
\[ x_{22} x_{11} = x_{11} x_{22} + (q^{-1} - q) x_{12} x_{21}. \]
This algebra is in fact a bialgebra, with coproduct \( \Delta \) and counit \( \epsilon \) given by:
\[ \Delta(x_{ij}) = \sum_{s=1}^2 x_{is} \otimes x_{sj} \]
\[ \epsilon(x_{ij}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]
for all \( i, j \in [2] \).
Lemma 5.5. Let $C = O_q(M_2)$ be defined as in Example 5.4.

(a) The algebra $C$ is an iterated Ore extension $k[x_{11}] [x_{12}; \tau_2][x_{21}; \tau_3][x_{22}; \delta_4]$, and it is a connected graded algebra with deg $x_{ij} = 1$.

(b) It is a quantum polynomial ring of dimension 4.

(c) It is an Auslander regular, Cohen-Macaulay, noetherian domain.

(d) Suppose $q \neq \pm 1$. Then every normal element in degree 1 is of the form $c_1 x_{12} + c_2 x_{21}$ for $c_i \in k$. As a consequence, $C$ is not isomorphic to a skew polynomial ring.

Proof. (a) [BG Example I.1.16].

(b,c) These follows from part (a) and [BG] Lemma I.15.4.

(d) This can be checked directly using the relations listed above. \qed

In the next lemma and the following proposition let $q$ be a primitive $m$th root of unity where $m \geq 3$. Let

$$n = \begin{cases} m & \text{if } m \text{ is odd} \\ \frac{m}{2} & \text{if } m \text{ is even}. \end{cases}$$

Let $H$ be a factor bialgebra of $C$ modulo $x_{11}^n - 1, x_{12}^n, x_{21}^n, x_{22}^n - 1$. Let $C(n)$ be the subspace of $C$ spanned by $x_{11}^b x_{12}^c x_{21}^d x_{22}^e$ for all $0 \leq a, b, c, d \leq n - 1$.

Lemma 5.6. Let $H = C/I$ where $I$ is the ideal generated by $x_{11}^n - 1, x_{12}^n, x_{21}^n, x_{22}^n - 1$.

(a) Then $H$ is a finite dimensional Hopf algebra with the coalgebra structure determined by (6.4.1).

(b) The canonical map $\phi : C \rightarrow H$ induces an isomorphism of vector spaces $C(n) \cong H$.

(c) Let $y_{ij}$ be the image of $x_{ij}$ in $H$. Then $H$ has a $k$-basis

$$\{ y_{11}^a y_{12}^b y_{21}^c y_{22}^d \mid 0 \leq a, b, c, d \leq n - 1 \}.$$

Proof. (a) First we prove that $H$ is a bialgebra. It suffices to show that $I$ is a bialgebra ideal of $C$. Note that

$$\Delta(x_{ij}) = x_{i1} \otimes x_{1j} + x_{i2} \otimes x_{2j},$$

and

$$(x_{i2} \otimes x_{2j})(x_{i1} \otimes x_{1j}) = q^2(x_{i1} \otimes x_{1j})(x_{i2} \otimes x_{2j}).$$

Since $q^2$ is a primitive $n$th root of unity,

$$\Delta(x_{ij}^n) = x_{i1}^n \otimes x_{1j}^n + x_{i2}^n \otimes x_{2j}^n.$$

Hence

$$(5.6.1) \quad \Delta(x_{11}^n - 1) = (x_{11}^n - 1) \otimes x_{11}^n + 1 \otimes (x_{11}^n - 1) + x_{12}^n \otimes x_{21}^n,$$

$$\Delta(x_{12}^n) = x_{11}^n \otimes x_{12}^n + x_{12}^n \otimes x_{22}^n,$$

$$\Delta(x_{21}^n) = x_{21}^n \otimes x_{11}^n + x_{22}^n \otimes x_{21}^n,$$

$$\Delta(x_{22}^n - 1) = (x_{22}^n - 1) \otimes x_{22}^n + 1 \otimes (x_{22}^n - 1) + x_{21}^n \otimes x_{12}^n.$$

Therefore $I$ is a bialgebra ideal and $H$ is a bialgebra. It remains to show that $H$ is a Hopf algebra. Let $B$ be the localization $C[D_q^{-1}]$, where $D_q$ is the quantum determinant $x_{11} x_{22} - q^{-1} x_{12} x_{21}$ (which is a central element in $C$). Then $B$ is a Hopf algebra denoted by $O_q(GL_2)$ [BG Definition I.1.10]. Since $D_q^n$ is equal to 1 in $H$, $H$ is a factor bialgebra of $B$ modulo $J = (x_{11}^n - 1, x_{12}^n, x_{21}^n, x_{22}^n - 1)$. Using
the definition of the antipode [BG] Definition I.1.10(7), it is easy to see that \( J \) is a Hopf ideal of \( B \). Thus \( H \) is a Hopf algebra.

(b,c) Clear.

We define a right \( H \)-comodule structure on \( C \) by

\[
\rho(x_{ij}) = \sum_{s=1}^{2} x_{is} \otimes y_{sj}
\]

for all \( i, j \in [2] \).

By the next proposition, we may think of \( H \) as a quantum reflection group of \( C \). By the definition of \( n \), \( q^{n^2} \) is either 1 or -1.

**Proposition 5.7.** The algebra \( C \) defined above is a right \( H \)-comodule algebra, and the coinvariant subring \( C^{coH} \) is the skew polynomial ring \( k_{\pm 1}[x_{11}^n, x_{12}^n, x_{21}^n, x_{22}^n] \) subject to the relations

\[
\begin{align*}
x_{12}^nx_{11}^n &= q^{n^2} x_{11}^nx_{12}^n \\
x_{21}^nx_{11}^n &= q^{n^2} x_{11}^nx_{21}^n \\
x_{22}^nx_{12}^n &= q^{n^2} x_{12}^nx_{22}^n \\
x_{22}^nx_{21}^n &= q^{n^2} x_{21}^nx_{22}^n \\
x_{21}^nx_{12}^n &= x_{12}^nx_{21}^n \\
x_{22}^nx_{11}^n &= x_{11}^nx_{22}^n.
\end{align*}
\]

**Proof.** Since \( C \) is a bialgebra, it is a right \( C \)-comodule algebra. Since \( H \) is a factor bialgebra of \( C \), \( C \) is also a right \( H \)-comodule algebra.

Let \( B \) be the subring of \( C \) generated by \( x_{11}^n, x_{12}^n, x_{21}^n, x_{22}^n \). Then \( B \) is isomorphic to \( O_{q^{n^2}}(M_2) \), which is a skew polynomial ring \( k_{\pm 1}[x_{11}^n, x_{12}^n, x_{21}^n, x_{22}^n] \) described in the statement. Using (5.6.1) we see that

\[
\begin{align*}
\rho(x_{11}^n - 1) &= (x_{11}^n - 1) \otimes 1, \\
\rho(x_{12}^n) &= x_{12}^n \otimes 1, \\
\rho(x_{21}^n) &= x_{21}^n \otimes 1, \\
\rho(x_{22}^n - 1) &= (x_{22}^n - 1) \otimes 1.
\end{align*}
\]

This implies that \( B \subset C^{coH} \). It remains to show that \( C^{coH} \subset B \). Let \( f \) be a nonzero element in \( C^{coH} \). Write \( f = \sum h_i g_i \), where \( h_i \) are linear combinations of monomials of the form \( x_{11}^a x_{12}^b x_{21}^c x_{22}^d \) for \( 0 \leq a, b, c, d < n \), and where \( \{g_i \in B\} \) are linearly independent over \( k \). Then

\[
\sum_i h_i g_i \otimes 1 = f \otimes 1 = \rho(f) = \sum_i \rho(h_i) \rho(g_i) = \sum_i \rho(h_i)(g_i \otimes 1).
\]

Hence \( \sum_i (h_i \otimes 1 - \rho(h_i))(g_i \otimes 1) = 0 \), where \( g_i \otimes 1 \in B \otimes k \) and \( h_i \otimes 1 - \rho(h_i) \in C(n) \otimes H \). Recall that \( C(n) = \sum_{0 \leq a, b, c, d < n} \sum_{x_{11}^a x_{12}^b x_{21}^c x_{22}^d \otimes \Psi_{a,b,c,d}} \sum_{0 \leq a, b, c, d < n} \sum_{x_{11}^a x_{12}^b x_{21}^c x_{22}^d \otimes \Psi_{a,b,c,d}} \). Then \( \sum_{0 \leq a, b, c, d < n} x_{11}^a x_{12}^b x_{21}^c x_{22}^d \otimes \Psi_{a,b,c,d} = 0 \) implies that \( \Psi_{a,b,c,d} = 0 \) for all \( a, b, c, d, i \) because \( \{x_{11}^a x_{12}^b x_{21}^c x_{22}^d g_i \} \) are linearly independent over \( k \). This means that \( \rho(h_i) = h_i \otimes 1 \) or \( h_i \in C^{coH} \) for all \( i \). Since \( H \) is a factor bialgebra of \( C \), \( \rho(h_i) = h_i \otimes 1 \) implies that \( \Delta_H(\phi(h_i)) = \phi(h_i) \otimes 1 \) where \( \phi : C \rightarrow H \) is the canonical map. Hence \( \phi(h_i) \in k \). By Lemma (6.6.1),
\( \phi : C(n) \rightarrow H \) is an isomorphism of vector spaces which sends \( x_{11}^{a}x_{12}^{b}x_{21}^{c}x_{22}^{d} \) to \( y_{11}^{a}y_{12}^{b}y_{21}^{c}y_{22}^{d} \). By Lemma [5.6(c)], \( h_{i} \in k \). Therefore \( C^{\circ}H = B \). \( \square \)

The next proposition is a Shephard-Todd-Chevalley Theorem for the algebra \( C \). It also shows that \( C \) is a rigid graded algebra, i.e. \( C^{G} \not\cong C \) for all finite groups \( G \) of graded automorphisms of \( C \). Combined with Proposition [5.7] it shows that the “quantum reflection group” \( H \) of \( C \) provides a regular ring of invariants that is different from the fixed subrings under any group action. This suggests that Hopf actions may provide the proper context for a generalized Shephard-Todd-Chevalley Theorem.

**Proposition 5.8.** Let \( C \) be as defined above. Suppose \( q \neq \pm 1 \).

(a) \( C \) has no mystic reflections.

(b) All quasi-reflections of \( C \) are reflections of the form

\[
g_{b}(x_{11}) = x_{11}, \quad g_{b}(x_{12}) = bx_{21}, \quad g_{b}(x_{21}) = b^{-1}x_{12}, \quad g_{b}(x_{22}) = x_{22}
\]

for \( b \in k^{\times} \).

(c) The finite groups \( G \subset \text{Aut}_{gr}(C) \) that are generated by quasi-reflections of \( C \) are the dihedral groups and the cyclic group of order 2.

(d) For all finite groups \( G \subset \text{Aut}_{gr}(C) \) that are generated by quasi-reflections of \( C \), the fixed subring \( C^{G} \) is regular and is generated as an algebra by 3 elements. Hence \( C^{G} \not\cong C \) for any non-trivial finite group \( G \subset \text{Aut}_{gr}(C) \).

(e) Conjectures [12] and [4.8] hold for \( C \).

(f) Assume that \( q \) is a root of unity. Then for any finite group \( G \subset \text{Aut}_{gr}(C) \), \( C^{G} \not\cong C^{\circ}H \) for the Hopf algebra \( H \) in Proposition [5.7].

**Proof.** (a) If \( \sigma \) is a mystic reflection, then there are \( z_{1}, z_{2} \in C_{1} \) such that \( z_{1}z_{2} = -z_{2}z_{1} \). But it is easy to check that this is impossible when \( q \neq \pm 1 \).

(b) By Lemma [5.5(d)] every normal element of degree 1 is in \( kx_{12} + kx_{21} \).

Let \( g \) be a quasi-reflection that is a reflection. Then there is a normal element \( y \in C_{1} \) with \( g(y) = \lambda y \in kx_{12} + kx_{21} \) and a basis \( \{ y_{1}, y_{2}, y_{3}, y_{4} \} \) of \( C_{1} \) with \( y_{1} = y \) and \( g(y_{i}) = y_{i} \) for \( i \neq 1 \). Then \( g \) induces a graded automorphism that is the identity map on the factor algebra \( C/\langle y \rangle \), and so

\[
g(x_{12}) = x_{12} + \alpha y, \quad g(x_{21}) = x_{21} + \beta y, \quad g(x_{11}) = x_{11} + \gamma y, \quad g(x_{22}) = x_{22} + \delta y
\]

for scalars \( \alpha, \beta, \gamma, \delta \in k \). Since \( g(x_{12}x_{11}) = qg(x_{11}x_{12}) \) we get either \( \gamma = 0 \) or \( \alpha y = -x_{12} \). Suppose that \( \gamma \neq 0 \). Then \( \alpha \neq 0 \). In a similar way, we have that \( g(x_{21}x_{11}) = qg(x_{11}x_{21}) \) yields \( \gamma = 0 \) or \( \beta y = -x_{21} \). If \( \gamma \neq 0 \) then \( \beta \neq 0 \), and \( y = -x_{12}/\alpha = -x_{21}/\beta \) is a contradiction. Hence \( \gamma = 0 \). By symmetry \( \delta = 0 \), and whence \( x_{11} \) and \( x_{22} \) are fixed by \( g \). The last quadratic relation implies that \( g(x_{12}x_{21}) = x_{12}x_{21} \). To show that \( g \) is of the form indicated, write \( y = ax_{12} + bx_{21} \) and compute \( g(x_{12}x_{21}) = x_{12}x_{21} = (x_{12} + \alpha y)(x_{21} + \beta y) \). Equating terms gives the result.

(c) If \( G \) is generated by one reflection, then \( G \) is cyclic of order 2. Suppose \( G \) has at least two generators \( g_{a} \) and \( g_{b} \). Then \( g_{a}g_{b} = g_{ab}^{-1} \), where \( g_{c} \) is the “diagonal map” determined by

\[
d_{c}(x_{11}) = x_{11}, \quad d_{c}(x_{12}) = cx_{12}, \quad d_{c}(x_{21}) = c^{-1}x_{21}, \quad d_{c}(x_{22}) = x_{22}.
\]

Let \( N = \{ d_{c} \in G \} \); then \( N \) is a normal subgroup of \( G \) with \( |G : N| = 2 \). The group \( N \) is isomorphic to a finite subgroup of \( k^{\times} \), so it is cyclic, generated by an \( m \)th root
of unity \( \lambda \). Furthermore, \( g_1 \) and \( N \) generate \( G \), and \( G \) is isomorphic to the dihedral group \( D_m \) of order \( 2m \).

(d,e,f) It is easily checked that the fixed subring under the dihedral group \( D_m \) represented as above is generated as an algebra by \( x_{11}, x_{22}, x_{12}x_{21}, \) and \( x_{12}^{-1} + b^{-1}x_{21} \). When \( G = (g_b) \), it is generated by \( x_{11}, x_{22}, x_{12}x_{21}, \) and \( x_{12} + bx_{21} \). Since

\[
x_{12}x_{21} = (q^{-1} - q)^{-1}(x_{22}x_{11} - x_{11}x_{22})
\]

the fixed subring \( C^G \) is generated by \( 3 \) elements in both cases. It can be checked that it is an iterated Ore-extension, so is regular. As both \( C \) and \( C^{\co H} \) require four algebra generators, \( C^G \) is not isomorphic to either \( C \) or \( C^{\co H} \).

By Proposition \ref{prop}(b) \( C^G \) is regular (or has finite global dimension) if and only if \( G \) is generated by quasi-reflections, so Conjecture \ref{conj} holds. Conjecture \ref{conj} holds trivially since \( D_{2m} \) is a reflection group in the classical sense. Thus we have proven part (e).

Finally if \( C^G \) is isomorphic to either \( C \) or \( C^{\co H} \) for some finite group \( G \subset \Aut(C) \), then \( C^G \) is regular. By part (e) \( G \) is either trivial or generated by quasi-reflections. But \( C^G \neq C \) or \( C^{\co H} \) if \( G \) is generated by quasi-reflections, as proved in the first paragraph of the proof of (d,e). Hence \( G \) is trivial. In particular, \( C \) is rigid.

Proposition \ref{prop} follows from Propositions \ref{prop} and \ref{prop} immediately. Note that a Hopf algebra \( H \) coaction on \( C \) is equivalent to a Hopf algebra \( H^{\ast} \) action on \( C \). Neither the Hopf algebra \( H \) in Lemma \ref{lem} nor its dual \( H^{\ast} \), is semisimple.

6. The groups \( M(n, \alpha, \beta) \)

Classical (pseudo-)reflection groups are well-understood in many respects. As generalizations of reflection groups, the groups \( M(n, \alpha, \beta) \), generated by mystic reflections, merit further investigation. We begin their study here.

In this section we give some examples of groups \( M(n, \alpha, \beta) \) which answer Question \ref{question}(a). For simplicity let \( k = \mathbb{C} \). In the first example, we give some details about groups \( M(n, 1, \beta) \) generated by mystic reflections. These groups may be compared to the classical reflection groups classified by Shephard and Todd \cite{ShephardTodd}. In the classical case there are three infinite families of reflection groups (i.e. groups with a representation generated by reflections of \( k[x_1, \cdots, x_n] \) for some \( n \)):

\[
\begin{align*}
&\text{the cyclic groups,} \\
&\text{the symmetric groups, and} \\
&\text{the groups } G(m, p, n) \quad (\text{which include the dihedral groups});
\end{align*}
\]

there are also 34 exceptional groups. The group \( G(m, p, n) \) (for positive integers \( m, p, n \), where \( p \) divides \( m \), so \( m = pq \)) is a semidirect product of the set of \( n \times n \) diagonal matrices \( A(m, p, n) \) described below, and \( S_n \) represented as permutation matrices. The group \( A(m, p, n) \) is

\[
A(m, p, n) = \left\{ \begin{bmatrix} \omega_1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \omega_n \end{bmatrix} : \omega_1^m = 1 \text{ and } (\omega_1 \cdots \omega_n)^{m/p} = 1 \right\}.
\]
Then $G(m, p, n)$ is a subgroup of weighted $n \times n$ permutation matrices, and $G(m, p, n)$ contains a copy of the $n \times n$ permutation matrices. The order of $G(m, p, n)$ is $m^m n! / p$ (see e.g. [K] pp. 161-2 and p.166) and [ShT]). The group $G(m, m, 2)$ is isomorphic to the dihedral group with $2m$ elements, and for arbitrary $p$ the groups $G(m, p, 2)$ are realized as symmetry groups of certain complex polytopes.

**Example 6.1.** Let $A$ be the skew-polynomial ring $\mathbb{C}_{-1}[x_1, x_2, x_3]$ with $p_{ij} = -1$ for all $i \neq j$. Let $G$ be the group generated by the two mystic reflections $\tau_{1,2,1}$ and $\tau_{2,3,1}$; then $G = M(3, 1, 2)$. We first note that $G$ is the rotation group of the cube, viewing a cube centered at the origin, with the three coordinate axes through the center of the faces. Then the rotation group of the cube is isomorphic to the symmetric group $S_4$ and is generated by the two matrices:

$$g_1 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

that act on $A$ as the mystic reflections $g_1 = \tau_{1,2,1}$ and $g_2 = \tau_{2,3,1}$, respectively. By Proposition 4.4(d) the ring of invariants $A^G$ is the commutative polynomial ring

$$\mathbb{C}[x_1^2 + x_2^2 + x_3^2, x_1 x_2 x_3, x_1^2 + x_2^2 + x_3^2].$$

Hence $S_4$ is a “reflection group” of $A$ in the sense that it has a representation that is generated by quasi-reflections of $A$. However, as in Proposition 4.4(c), this representation of $S_4$ contains no non-identity reflections of the commutative polynomial ring $\mathbb{C}[x_1, x_2, x_3]$ under the usual action. It is interesting that the generators of $A^G$ in the case of quantum polynomials are nicer than those for the commutative polynomial ring $\mathbb{C}[x_1, x_2, x_3]^G$, see [CLO] p. 337, Problem 12]. Although the representation of $S_4$ generated by the mystic reflections above is not a “reflection group” in the commutative sense (the group of matrices above is not generated by classical reflections) the group $S_4$ has another representation (the representation as permutation matrices) that is a “reflection group” in the classical sense.

Our next examples show that there exist infinite families of finite groups $G$ having a representation that is generated by quasi-reflections of a regular algebra, yet the abstract group $G$ has no representation generated by classical reflections. Though not classical reflection groups, these groups are “reflection groups” in the sense that they have representations that act on a regular algebra producing a regular fixed ring (the fixed ring is even a commutative polynomial ring).

**Example 6.2.** We claim that the mystic reflection groups $M(2, 1, 2m)$, for $m \gg 0$, are not isomorphic to classical reflection groups as abstract groups.

Let $A$ be the quantum plane $\mathbb{C}_{-1}[x, y]$ and let $G$ be the group generated by the mystic reflections $g_1 = \tau_{2,1,1}$ and $g_2 = \tau_{2,1,\lambda}$ where $\lambda$ is a primitive $2m$th root of unity. Then

$$g_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{bmatrix}. $$

The group $G$ is also generated by $g_1$ and

$$g_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix},$$

so it contains a copy of the $n \times n$ permutation matrices.
and the fixed ring is $A^G = \mathbb{C}[x_1x_2, x_1^{2m} + x_2^{2m}]$. Then taking $a = g_3$ and $b = g_1$, it is not difficult (e.g. \text{[N] Exercise 16, p. 25-26}) to see that $G = M(2, 1, 2m)$ is isomorphic to the “dicyclic group” $Q_{4m}$ generated by $a$ and $b$ with relations:

$$a^{2m} = 1, \quad b^{-1}ab = a^{-1}, \quad b^2 = a^m.$$  

When $m = 1$ then $Q_4$ is the cyclic group of order 4, and when $m = 2$ then $Q_8$ is the quaternion group of order 8. It is easy to see that all elements of $G$ can be written in the form $a^i$ or $a^ib$, that $|G| = 4m$, and that $Q_{4m}$ has a unique element $a^m$ of order 2. When $m = 2^{n-2}$ then the dicyclic groups are called “generalized quaternion groups” $Q_{2^n}$, generated by elements $a$ and $b$ with relations:

$$a^{2^{n-1}} = 1, \quad b^{-1}ab = a^{-1}, \quad b^2 = a^{2^{n-2}}$$

for $n \geq 3$ (e.g. \text{[N] Exercise 15, p. 25}).

As each of the 34 exceptional complex reflection groups has order divisible by 4, it is possible that a finite number of $M(2, 1, 2m) \cong Q_{4m}$ could be isomorphic to an exceptional reflection group; however, by examining the list of the 34 exceptional groups and considering their orders, one can easily determine that e.g. $Q_8, Q_{12}, Q_{16}, Q_{20}, Q_{28}$, or any of the generalized quaternion groups cannot be isomorphic to an exceptional reflection group.

We claim that except when $m = 1$ (when $G \cong \mathbb{Z}/4\mathbb{Z}$), the groups $Q_{4m}$ are not isomorphic to any of the groups in the three infinite families of groups in the Shephard-Todd table of complex reflection groups (6.0.1). These non-abelian groups $Q_{4m}$ each have a unique element of order 2, so are not isomorphic to cyclic, symmetric or dihedral groups. Hence, once we show that $Q_{4m}$ is never isomorphic to a group of the form $G(m, p, n)$, there are an infinite number of new “reflection groups” arising from reflections of $\mathbb{C}_{-1}[x_1, x_2]$.

Finally, we show that the groups $Q_{4m^*}$ (we changed $m$ to a different integer $m^*$) are not isomorphic to a group $G(m, p, n)$. Since the dicyclic groups each have a unique element of order 2, then if a dicyclic group $Q_{4m^*}$ is isomorphic to the reflection group $G(m, p, n)$, it follows that $n = 2$ because it contains a subgroup isomorphic to $S_n$. The groups $G(m, p, 2)$ are generated by the transposition

$$t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the matrices

$$\begin{bmatrix} \omega^{a_1} & 0 \\ 0 & \omega^{a_2} \end{bmatrix}$$

with $a_1 + a_2 \equiv 0 \pmod{p}$, where $\omega$ is a primitive $m$th root of unity \text{[N] p. 147}. In order that $|G(m, p, 2)| = (m^2)2!/p = 4m^* = |Q_{4m^*}|$, $m$ must be even, and hence the element

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \omega^{m/2} & 0 \\ 0 & \omega^{m/2} \end{bmatrix},$$

as well as the transposition $t$, are two elements of $G$ with order 2, contradicting the uniqueness of the elements of order 2 in $Q_{4m^*}$. Hence it follows that the groups $G = M(2, 1, 2m)$ are not isomorphic to the groups $G(m, p, n)$, and so can be classical reflection groups only if they happen to be one of the 34 exceptional complex classical reflection groups in the Shephard-Todd table.

Therefore we proved the assertion by taking $m^* \gg 0$. 

In Example 6.1 we showed that $M(3,1,2)$ is isomorphic to the group of rotations of the cube. Next we examine the family of groups $G = M(n,1,2)$ that are generated by mystic reflections of $\mathbb{C}_{-1}[x_1, \cdots, x_n]$. It is easy to check that $G$ has some properties of the classical reflection group $G' = G(2,2,n)$; both groups have order $2^n-1$, and both contain the diagonal matrices $D = A(2,2,n)$, described above, as a normal subgroup. In the rest of this section we show that when $n$ is even the groups $G$ and $G'$ are not isomorphic, but when $n$ is odd they are isomorphic. Hence $M(n,1,2)$ for $n$ even provides a second family of new “reflection groups” generated by mystic reflections.

Example 6.3. The group $G = M(n,1,2)$ is generated by the mystic reflections $\tau_{i,j,1}$ for all $i \neq j$. Both $G$ and $G'$ are subgroups of the group $B$ of all signed $(\pm 1)$ permutation $n \times n$ matrices. The order of $B$ is $2^n n!$, and both $G$ and $G'$ have order $2^n-1$. The group $G$ is the kernel of the determinant map from $B \to \{\pm 1\}$. The diagonal matrices in $G$ are precisely the diagonal matrices $D = A(2,2,n)$, described above, as a subgroup of $G'(2,2,n)$; these matrices contain an even number of entries of $-1$ and are the diagonal matrices in $B$ with determinant $1$. The subgroup $D$ is normal in $B$ (and hence in both $G$ and $G'$) and $G/D$ is isomorphic to $S_n$ by reducing $-1$ entries to $1$ entries; it is clear from its definition that $G'/D$ is isomorphic to $S_n$. Hence, it is not surprising that some effort is required in distinguishing the groups $G$ and $G'$ when they are not isomorphic.

Case 1: $n$ is odd. Consider the elements $\sigma_i$, $i = 1, \ldots, n-1$ chosen so that the coset $D\sigma_i$ represents the transposition $(i, i+1)$ in the factor group $G/D$; since $n$ is odd we can choose $\sigma_i$ to be of the form

$$
\sigma_i = \begin{bmatrix}
-1 & & & \\
 & \ddots & & \\
 & & -1 & \\
 & & & 1 \\
& & & & -1 \\
& & & & & \ddots \\
& & & & & & -1
\end{bmatrix}
$$

as the determinant of $\sigma_i$ is $1$. One can check that the elements $\sigma_i$ of $G$ satisfy the relations of $S_n$:

$$
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \ldots, n-1,
$$

$$
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2, \ \text{and}
$$

$$
\sigma_i^2 = 1 \quad \text{for } i = 1, \ldots, n-1.
$$

Hence $G$ contains a copy of $S_n$ that acts on $D$ as it does in $G'$, so $G \cong G'$.

Case 2: $n$ is even. We will show that the groups $G$ and $G'$ differ in the distribution of elements of order $2^t$ for some $t$. First we prove the following lemmas:
Lemma 6.4. Let $M$ be a $t \times t$ weighted $t$-cycle matrix of the form

$$M = \begin{bmatrix}
0 & \cdots & 0 & \pm 1 \\
\pm 1 & 0 & \cdots & 0 \\
0 & \pm 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \pm 1 & 0
\end{bmatrix}.$$  

Then the order of $M$ is $t$ when the number of $-1$’s in $M$ is an even number, and it is $2t$ when the number of $-1$’s in $M$ is an odd number.

Proof. Thinking of $M$ as a function makes this easy to check. \qed

Lemma 6.5. Let $g$ be an element of $G'$ with $Dg$ an even permutation of $G'/D \cong S_n$. Then $g \in G$.

Proof. The coset $Dg \in G'/D$ contains a permutation of determinant 1, and hence all elements of $Dg$ have determinant 1, and so are in $G$. \qed

We next compare the orders of $g$ and $Dg$ when $g$ has order a power of 2. Note that it is possible for the order of $Dg$ to be half the order of $g$; e.g. if

$$g = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}$$

then $g$ has determinant 1 so is in $G$ and $|g| = 4$, but the order of $Dg$ is 2 because changing the sign of the first row and the third row gives the element

$$Dg = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}$$

in the coset $Dg$ that clearly has order 2; hence it is clear that the coset $Dg$ has order 2. The next lemma shows that the order of $Dg$ cannot decrease from $|g|$ by more than one power of 2.

Lemma 6.6. If the order of $g \in G$ is $2^\ell+1$, then the order of the coset $Dg$ in $G/D$ is either $2^\ell$ or $2^{\ell+1}$.

Proof. The order of the coset $Dg$ in $G/D \cong S_n$ must be $\leq 2^{\ell+1}$, so it can be represented by a block permutation matrix for a product of disjoint $2^{t_i}$-cycles for $t_i \leq \ell + 1$. Hence $g$ is obtained from the block permutation matrix representing the product of these $2^{t_i}$-cycles for $t_i \leq 2^{\ell+1}$ by an even number of sign changes. It follows from Lemma 6.4 that if all the cycles in $Dg$ are of length $\leq 2^{\ell-1}$, then $g$ cannot have order $2^{\ell+1}$. \qed

Lemma 6.7. Let $Dg$ be an element of $G/D \cong S_n$ of order $2^k$ for $k \geq \ell$ where $n = 2^\ell m$ for $m$ odd, and let $\sigma$ be the corresponding permutation in $S_n$. Let $a$ in $G'$ be a representative of the coset corresponding to $\sigma$ in $G'/D \cong S_n$. Then the number of elements in $Dg$ of order $2^{k+1}$ in $G$ is $\geq$ the number of elements in $Da$ of order $2^{k+1}$ in $G'$. 
Proof. Since $D$ is normal in $B$, and $G$ and $G'$ are normal in $B$, conjugation by any permutation matrix preserves the number of elements of each order in $Dg$ and $Da$. Hence we may assume that elements of $G$ and $G'$ have block form corresponding to disjoint weighted cycles of the form in Lemma 6.4. If the permutation $\sigma$ is even then the sets $Dg$ and $Da$ are the same set, and the result is obvious. Hence assume that $\sigma$ is an odd permutation of order $2^k$ and that $g$ has order $2^{k+1}$.

First we consider the case when $k = \ell$ and $\sigma$ is a product of $m$ disjoint $2^\ell$-cycles. Since changing an even number of signs in the representation of $g$ gives and element that remains in the coset $Dg$, we may assume that the first block in $g$ is a $2^\ell \times 2^\ell$ matrix of the form

$$C^- = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

the other blocks are the $2^\ell \times 2^\ell$ permutation matrix

$$C^+ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

and that $a$ is the matrix with $m$ blocks of the permutation matrix $C^+$. Every element in $Dg$ is of order $2^{k+1}$, since any even number of changes of signs to $g$ will leave an odd number of sign changes in at least one block. Since $a$ has order $2^\ell$, then $Da$ has fewer elements of order $2^{k+1}$.

Next consider the remaining case: where $\sigma$ is an odd permutation containing at least one $2^k$-cycle and at least one $2^t$-cycle for $0 \leq t < k$. Without loss of generality we may assume that $g$ is represented by a matrix with top block a $2^k \times 2^k$ matrix of the form $C^-$, and the final block is of strictly smaller size. Hence $a$ is represented by a matrix with a top block a $2^k \times 2^k$ matrix of the form $C^+$, and with smaller size matrices below. If $g$ has $s$ blocks of order $2^k$ let $d = (d_1, \cdots, d_s, d_{s+1})$ represent a diagonal matrix where the $d_1, \cdots, d_s$ are each $2^k$-tuples on the diagonal of $d \in D$. Elements of $G$ of order $2^{k+1}$ in $Dg$ are of the form $da$, where $d \in D$ has at least one of the $d_1, \cdots, d_s$ with an odd number of $-1$'s. To establish the result, we produce an injective map from the elements of $G'$ of order $2^{k+1}$ in $Da$ to elements of $G$ of order $2^{k+1}$ in $Dg$. If at least one of $d_2, \cdots, d_s$ has an odd number of $-1$'s, then associate $dg$ to $da$. If none of $d_2, \cdots, d_s$ has an odd number of $-1$'s then $d'_1$ has an odd number of $-1$'s so we associate $d'dg$ to $da$, where $d'$ is the diagonal matrices with exactly two $-1$'s (one in the first entry, and one in the last entry). \qed

Now we use the lemmas to show that the groups $G$ and $G'$ are not isomorphic when $n$ is even. We will show that $G$ has more elements than $G'$ of order $2^t$ for some $t$. The largest possible order of elements that are a power of 2 both in $G$ and in $G'$ is $2^{r+1}$, where $2^r$ is the largest power of 2 that is $\leq n$. This happens only when $Dg$ has order $2^r$ in $G/D \cong S_n$. By Lemma 6.7 the number of such elements in $G$ is $\geq$ the number of such elements in $G'$. If $G$ has more such elements than $G'$, we are done, so assume that both $G$ and $G'$ have the same number of elements of largest
possible order $2^{r+1}$; hence these cosets also contain the same number of elements of order $2^r$. Now consider all elements of order $2^r$ in $G$ and $G'$; by Lemma 6.6 these elements of $G$ are in the cosets $Dg$, where either $Dg$ has order $2^{r-1}$ or $2^r$ in $G/D$, but we are assuming the number of elements of order $2^r$ in $G/D$ and $G'/D$ is the same, so we need consider only elements of $G/D$ of order $2^{r-1}$. Again by Lemma 6.7 the number of elements of $G$ of order $2^r$ is at least the number of elements from $G'$ of order $2^r$. Again if these numbers are different, we are done, and hence without loss of generality we may assume that $G$ and $G'$ have the same number of elements of order $2^t$ for $t \geq \ell + 2$, where $n = 2^\ell m$ for $m$ odd. Now consider elements of order $2^{r+1}$. These elements arise only from $Dg$ with $Dg$ of order $2^\ell$ or $2^{\ell+1}$. Again we may assume that number of elements of order $2^{\ell+1}$ arising from cosets in $G/D$ of order $2^{\ell+1}$ is the same for $G$ and $G'$ because the number of elements of order $2^{\ell+2}$ in these cosets is the same. Finally consider cosets of order $2^\ell$. For each of these cosets the number of elements of order $2^{\ell+1}$ in $G$ is at least the number of elements of order $2^{\ell+1}$ in $G'$. But we have the element of $G$ that has $m$ blocks of size $2^\ell$, for $m$ odd, so this coset is represented by an element $g$ that has top block a $2^\ell \times 2^\ell$ block of the form $C^-$ and all other blocks $2^\ell \times 2^\ell$ blocks $C^+$. All elements in this coset have order $2^{\ell+1}$, while the corresponding coset of $D$ in $G'$ has the element with all $2^\ell \times 2^\ell$ blocks of the form $C^+$, so this coset of $g'$ has at least one element has order $2^{\ell}$. Hence $G$ and $G'$ have different numbers of elements of order $2^{\ell+1}$, and these groups are not isomorphic, establishing the claim that all $M(n,1,2)$ for $n$ even are not isomorphic to a classical reflection group.

Forgetting about the underlying quantum polynomial ring $A$, we can define a mystic reflection of a $k$-vector space.

**Definition 6.8.** Let $V$ be a finite dimensional vector space over $k$. A linear map $g : V \to V$ is called a **mystic reflection** if there is a basis of $V$, say $\{y_1, \ldots, y_n\}$, such that $g(y_j) = y_j$ for all $j \leq n - 2$ and $g(y_{n-1}) = iy_{n-1}$ and $g(y_n) = -iy_n$.

Theorem 6.4 says that every reflection group of a skew polynomial ring is generated by reflections and mystic reflections of the vector space $A_1$. Further such a reflection group is a product of classical reflection groups of $A_1$ and copies of $M(n, \alpha, \beta)$'s. We finish the paper by the following question.

**Question 6.9.** Let $G$ be a finite subgroup of $GL(V)$. If $G$ is generated by reflections in the classical sense and mystic reflections in the sense of Definition 6.8 then is $G$ isomorphic to a product of classical reflection groups of $V$ and copies of $M(n, \alpha, \beta)$'s?

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