Branching: the Essence of Constraint Solving

Antonio J. Fernández¹ and Pat Hill²

¹ Departamento de Lenguajes y Ciencias de la Computación, E.T.S.I.I., 29071 Teatinos, Málaga, Spain email:afdez@lcc.uma.es
² School of Computing, University of Leeds, Leeds, LS2 9JT, England email:hill@comp.leeds.ac.uk

Abstract This paper focuses on the branching process for solving any constraint satisfaction problem (CSP). A parametrised schema is proposed that (with suitable instantiations of the parameters) can solve CSP’s on both finite and infinite domains. The paper presents a formal specification of the schema and a statement of a number of interesting properties that, subject to certain conditions, are satisfied by any instances of the schema. It is also shown that the operational procedures of many constraint systems (including cooperative systems) satisfy these conditions. Moreover, the schema is also used to solve the same CSP in different ways by means of different instantiations of its parameters.

Keywords: constraint solving, filtering, branching.

1 Introduction

To solve a constraint satisfaction problem (CSP), we need to find an assignment of values to the variables such that all constraints are satisfied. A CSP can have many solutions; usually either any one or all of the solutions must be found. However, sometimes, because of the cost of finding all solutions, partial CSP’s are used where the aim is just to find the best solution within fixed resource bounds. An example of a partial CSP is a constraint optimisation problem (COP) that assigns a value to each solution and tries to find an optimal solution (with respect to these values) within a given time frame.

A common method for solving CSP’s is to apply filtering algorithms (also called arc consistency algorithms or propagation algorithms) that remove inconsistent values from the initial domain of the variables that cannot be part of any solution. The results are propagated through the whole constraint set and the process is repeated until a stable set is obtained. However, filtering algorithms are, often, incomplete in the sense that they are not adequate for solving a CSP and, as consequence, it is necessary to employ some additional strategy called constraint branching that divides the variable domains and then continues with the propagation on each branch independently.

* This work was partly supported by EPSRC grants GR/L19515 and GR/M05645 and by CICYT grant TIC98-0445-C03-03.
Constraint Solving algorithms have received intense study from many researchers, although the focus has been on developing new and more efficient methods to solve classical CSP’s [? ,?] and partial CSP’s [? ,?]. See [? ,? ,? ,?] for more information on constraint solving algorithms and [? ,?] for selected comparisons.

To our knowledge, despite the fact that it is well known that branching step is a crucial process in complete constraint solving, papers concerned with the general principles of constraint solving algorithms have mainly focused on the filtering step [? ,? ,? ].

In this paper, we propose a schema for constraint solving for both classical and partial CSP’s that includes a generic formulation of the branching process. (This schema may be viewed as a generalisation and extension of the interval lattice-based constraint-solving framework in [? ].) The schema can be used for most existing constraint domains (finite or continuous) and, as for the framework in [? ], is also applicable to multiple domains and cooperative systems. We will show that the operational procedures of many constraint systems (including cooperative systems) satisfy these conditions.

The paper is organised as follows. Section 2 shows the basic notions used in the paper and Section 3 describes the main functions involved in constraint solving with special attention to those involved in the branching step. In Section 4 a generic schema for classical constraint solving is developed and its main properties are declared. Then, Section 5 extends the original schema for partial constraint solving and more properties are declared. Section 6 shows several instances of the schema to solve both different CSP’s and different solvings for the same CSP. Section 7 contains concluding remarks. Proofs of the properties are found in the Appendixes.

2 Basic concepts

Let $D, D_1, \ldots, D_n$ be sets or domains. Then $\# D$ denotes the cardinality of $D$, $\wp(D)$ its power set and $D_{\prec}$ denote any totally ordered domain. $\bot_D$ and $\top_D$ denote respectively, if they exist, the bottom and top element of $D$ and fictitious bottom and top elements otherwise. Throughout the paper, $\Delta$ denotes a set of domains called computation domains.

**Definition 1.** (Constraint satisfaction problem) A Constraint satisfaction problem (CSP) is a tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{C} \rangle$ where

- $\mathcal{V} = \{ v_1, \ldots, v_n \}$ is a non-empty finite set of variables.
- $\mathcal{D} = \wp(D_1) \times \ldots \times \wp(D_n)$ where $D_i \in \Delta$.
- $\mathcal{C} \subseteq \wp(D_1, \ldots, D_n)$ is a set of constraints for $\mathcal{D}$.

If, as in the above definition, $\mathcal{D} = \wp(D_1) \times \ldots \times \wp(D_n)$, where $D_i \in \Delta$ for all $i \in \{ 1, \ldots, n \}$, then the set of all constraints for $\mathcal{D}$ is denoted as $\mathcal{C}_\mathcal{D}$ and the set $\{ D_i \mid 1 \leq i \leq n \}$ is denoted as $\Delta_\mathcal{D}$. 
Definition 2. (Constraint store) Let $S = (d_1, \ldots, d_n) \in D$. Then $S$ is called a constraint store for $(\mathcal{V}, D, C)$. $S$ is consistent if, for all $i \in \{1, \ldots, n\}$, $d_i \neq \emptyset$. $S$ is divisible if $S$ is consistent and for some $i \in \{1, \ldots, n\}$, $\#d_i > 1$. Let $S' = (d'_1, \ldots, d'_n)$ be another constraint store for $(\mathcal{V}, D, C)$. Then $S \preceq S'$ if and only if $d_i \subseteq d'_i$ for $1 \leq i \leq n$.

$S$ is a solution for $(\mathcal{V}, D, C)$ if $S = (\{s_1\}, \ldots, \{s_n\})$ and $(s_1, \ldots, s_n) \in c$, for all $c \in C$. $S'$ is a partial solution for $(\mathcal{V}, D, C)$ if there exists a solution $S''$ for $(\mathcal{V}, D, C)$ such that $S'' \preceq S'$. In this case we say that $S'$ covers $S''$.

The set of all solutions for $(\mathcal{V}, D, C)$ is denoted as $\text{Sol}((\mathcal{V}, D, C))$. Note that, if $S \in \text{Sol}((\mathcal{V}, D, C))$, then $S$ is consistent and not divisible. If $(d_1, \ldots, d_n) \in D$ and $i \in \{1, \ldots, n\}$, then $(d_1, \ldots, d_n)[d_i/d'] = (d_1, \ldots, d_{i-1}, d', d_{i+1}, \ldots, d_n)$.

Example 1. Let $D = \wp(\text{Bool}) \times \wp(\text{Bool}) \times \wp(\text{Bool})$. Let $c = x \lor (y \land z)$ be a constraint for $D$. Then $c = \{(0, 1, 1), (1, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0)\}$. Let

$$S_1 = (\{1\}, \{0\}, \{0\}), \quad S_2 = (\{0\}, \{1\}, \{0\}), \quad S_3 = (\{0\}, \{0\}, \emptyset), \quad S_4 = (\emptyset, \emptyset, \emptyset).$$

Then, $S_1$ is a solution but $S_2, S_3$ and $S_4$ are not. Note also that $S_1, S_2$ and $S_3$ are consistent and $S_4$ is inconsistent.

Definition 3. (Stacks) Let $P = (S_1, \ldots, S_\ell) \in \wp(D)$. Then $P$ is a stack for $(\mathcal{V}, D, C)$.

Let $P' = (S'_1, \ldots, S'_\ell')$ be another stack for $(\mathcal{V}, D, C)$. Then $P \preceq P'$ if and only if for all $S_i \in P$ ($1 \leq i \leq \ell$), there exists $S'_j \in P'$ ($1 \leq j \leq \ell'$) such that $S_i \preceq S'_j$. In this case we say that $P'$ covers $P$.

3 The Branching Process

This section describes the main functions used in the branching process.

First we define a filtering function which removes inconsistent values from the domains of a constraint store.

Definition 4. (Filtering function) $\text{filtering}_D : \wp(C_D) \times D \rightarrow D$ is a called a filtering function for $D$ if, for all $S \in D$,

(a) $\text{filtering}_D(C, S) \preceq S$;
(b) $\forall R \in \text{Sol}((\mathcal{V}, D, C)) : R \preceq S \implies R \preceq \text{filtering}_D(C, S)$.
(c) If $\text{filtering}_D(C, S)$ is consistent and not divisible then $\text{filtering}_D(C, S)$ is a solution for $(\mathcal{V}, D, C)$.

Condition (a) ensures that the filtering never gains values, condition (b) guarantees that no solution covered by a constraint store is lost in the filtering process and condition (c) guarantees the correctness of the filtering function.

Variable ordering is an important step in constraint branching. We define a selecting function which provides a schematic heuristic for variable ordering.
Definition 5. (Selecting function) Let $S = (d_1, \ldots, d_n) \in \mathcal{D}$. Then

$\text{choose} :: \{S \in \mathcal{D} \mid S \text{ is divisible}\} \to \{\varphi(D) \mid D \in \Delta_\mathcal{D}\}$

is called a selecting function for $\mathcal{D}$ if $\text{choose}(S) = d_j$ where $1 \leq j \leq n$ and $\#d_j > 1$.

Example 2. Here is a naive strategy to select the left-most divisible domain.

Precondition: $\{S = (d_1, \ldots, d_n) \in \mathcal{D} \text{ is divisible}\}$

$\text{choose}_{\text{naive}}(S) = d$

Postcondition: $\exists j \in \{1, \ldots, n\} . d = d_j, \ #d_j > 1 \text{ and } \forall i \in \{1, \ldots, j-1\} : \#d_i = 1$.

In the process of branching, some computation domain has to be partitioned, in two or more parts, in order to introduce a choice point. We define a splitting function which provides a heuristic for value ordering.

Definition 6. (Splitting function) Let $D \in \Delta$ and $k > 1$. Then

$\text{split}_D :: \varphi(D) \to \varphi(D) \times \cdots \times \varphi(D) \text{ k times}$

is called a splitting function for $D$ if, for all $d \in \varphi(D)$, $\#d > 1$, this function is defined $\text{split}_D(d) = (d_1, \ldots, d_k)$ such that the following properties hold:

Completeness: $d_1 \cup \ldots \cup d_k = d$.

Contractance: $d_i \subset d, \forall i \in \{1, \ldots, k\}$.

To guarantee termination, even on continuous domains, an extension of the concept of precision map shown in [7] is applied here.

Definition 7. (Precision map) Let $\mathcal{RI} = (\mathbb{R}^+, \text{Integer})$ where $\mathbb{R}^+$ is the domain of non-negative reals. Then precision $\mathcal{D}$ is a precision map for $D \in \Delta$, if precision $\mathcal{D}$ is a strict monotonic function from $\varphi(D)$ to $\mathcal{RI}$.

Let $S = (d_1, \ldots, d_n)$ be a constraint store for $\langle V, \mathcal{D}, C \rangle$ and, for each $D \in \Delta_\mathcal{D}$, precision $\mathcal{D}$ is defined for $D$. Then, a precision map for $D = (D_1, \ldots, D_n)$ is defined as

$\text{precision}(S) = \sum_{1 \leq i \leq n} \text{precision}_{D_i}(d_i),$

where the sum in $\mathcal{RI}$ is defined as $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$.

The monotonicity of the precision is a direct consequence of the definition\(^1\).

\(^1\) $\mathcal{RI}$ is continuous and infinite so that it is supposed that we can define a map from $D$ to $\mathcal{RI}$, even if $D$ is infinite. $\mathcal{RI}$ was chosen since it is valid for the interval domain as it is shown in Section 6, but any domain totally ordered supporting the operator $-$ may also be adequate (see Line 3 in Figure 1).
Proposition 1. Let \( S, S' \) be two constraint stores for \( \langle V, D, C \rangle \). If \( S \preceq_s S' \) then \( \text{precision}(S) <_{\mathbb{R}^I} \text{precision}(S') \).

The precision map also means a novel way to normalise the selecting functions when the constraint system supports multiple domains.

Example 3. The well known first fail principle chooses the variable constrained with the smallest domain. For multiple domain constraint systems to emulate the first fail principle, we define \( \text{choose}/1 \) so that it selects the domain with the smallest precision\(^2\). We denote this procedure by \( \text{choose}_{ff} \).

\[
\begin{align*}
\text{Precondition} : & \{S = (d_1, \ldots, d_n) \in D \text{ is divisible}\} \\
\text{choose}_{ff}(S) &= d \\
\text{Postcondition} : & \{\exists j \in \{1, \ldots, n\}. d = d_j, \#d_j > 1 \text{ and} \\
& \forall i \in \{1, \ldots, n\}\setminus\{j\} : \#d_i > 1 \implies \text{precision}_{D_i}(d_i) \leq_{\mathbb{R}^I} \text{precision}_{D_j}(d_j)\}.
\end{align*}
\]

4 Branching in Constraint Solving

Figure 1 shows a generic schema for solving any CSP \( \langle V, D, C \rangle \). This schema requires the following parameters: \( C \), the set of constraints to solve, a constraint store \( S \) for \( \langle V, D, C \rangle \), a bound \( p \in \mathbb{R}^I \) and a non-negative real bound \( \varepsilon \). There are a number of values and subsidiary procedures that are assumed to be defined externally to the main branch procedure:

- a filtering function \( \text{filtering}_{D}/2 \) for \( D \);
- a selecting function \( \text{choose}/1 \) for \( D \);
- a splitting function \( \text{split}_{D} \) for each domain \( D \in \Delta_D \);
- a precision map for \( D \) (therefore it is assumed that there is defined one precision map for each \( D \in \Delta_D \));
- a stack \( P \in \wp(D) \) for \( \langle V, D, C \rangle \).

It is assumed that all the external procedures have an implementation that terminates for all possible values.

Theorem 1. (Properties of the branch/4 schema) Let \( S \) be the top element in \( D \) (i.e., \( S = (D_1, \ldots, D_n) \)), \( \varepsilon \in \mathbb{R}^+ \) and \( p = \top_{\mathbb{R}^I} \). Then, the following properties are guaranteed:

1. Termination: if \( \varepsilon > 0.0 \) then \( \text{branch}(C, S, p, \varepsilon) \) terminates;
2. Completeness: if \( \varepsilon = 0.0 \) and the execution of \( \text{branch}(C, S, p, \varepsilon) \) terminates, then the final state for the stack \( P \) contains all the solutions for \( \langle V, D, C \rangle \);
3. Approximate completeness: if \( \varepsilon > 0.0 \) and \( R \) is a solution for \( \langle V, D, C \rangle \), then an execution of \( \text{branch}(C, S, p, \varepsilon) \) will result in \( P \) containing either \( R \) or a partial solution \( R' \) that covers \( R \).

\(^2\) It is straightforward to include more conditions e.g., if \( d_i, d_k, d_j \) have the same (minimum) precision, the most left domain can be chosen i.e., \( d_{\text{minimum}(i,k,j)} \).
procedure branch(C, S, p, ε)
begin
S ← filtering_D(C, S); % (1)
if S is consistent then
if (S is not divisible or p < ⊤ℜI and p − precision(S) ≤ (ε, 0)) then
    push(P, S); %% Add S to top of P % (4)
else
    dj ← choose(S); % (6)
    (d_j1, . . . , d_jk) ← split_Dj(d_j), where d_j ⊆ D_j; % (7)
    branch(C, S[d_j/d_j1], precision(S), ε) % (8)
    . . . . . . . . . . . . % Choice Points
branch(C, S[d_j/d_jk], precision(S), ε);
endif;
endif;
end.

Figure 1. branch/4: A Generic Schema for Constraint Solving

4. Correctness: if ε = 0.0, the stack P is initially empty and the execution of branch(C, S, p, ε) terminates with R in the final state of P, then R is a solution for ⟨V, D, C⟩.

5. Approximate correctness or control on the result precision: If P_0.0, P_ε₁, and P_ε₂ are stacks resulting from any terminating execution of branch(C, S, p, ε) (where initially P is empty) when ε has the values 0.0, ε₁ and ε₂, respectively, 0.0 < ε₁ < ε₂ and P_0.0 is not empty, then P_0.0 ≼_p P_ε₁ ≼_p P_ε₂.
(In other words, the set of (possibly partial) solutions in the final state of the stack is dependent on the value of ε in the sense that lower ε is, closer to the real set of solutions is).

Observe that the bound ε guarantees termination and allows to control the precision of the results.

5 Solving optimisation problems

The schema in Figure 1 can be adapted to solve COPs by means of three new subsidiary functions.

Definition 8. (Subsidiary functions and values) Let D_< be a totally ordered domain\(^3\). Then we define

- a cost function, fcost :: D → D_<;

\(^3\) Normally D_< would be ℜ.
– an ordering relation, $\diamond ::= D_< \times D_< \in \{>, <, =\}$;
– a bound, $\delta \in D_<$.

Then the extended schema, \textit{branch}_+/4, is obtained from the schema \textit{branch}/4 by replacing Line 4 in Figure 1 with:

$$\text{if } fcost(S) \diamond \delta \text{ then } \delta \leftarrow fcost(S); \text{ push}(P, S) \text{ endif;} \quad (4^*)$$

\textbf{Theorem 2.} (Properties of the branch+/4 schema) Let $S$ be the top element in $D$ (i.e., $S = (D_1, \ldots, D_n)$), $\varepsilon \in \mathbb{R}^+$ and $p = \top_{\mathbb{R}_I}$. Then, the following properties hold:

1. Termination: if $\varepsilon > 0.0$, then the execution of \textit{branch}_+(C, S, p, \varepsilon) terminates;
2. If $fcost$ is a constant function with value $\delta$ and $\diamond$ is $=$, then all properties shown in Theorem 1 hold for the execution of \textit{branch}_+(C, S, p, \varepsilon).
3. Soundness on optimisation: if $\varepsilon = 0.0$, $\diamond$ is $>$ (resp. $<$), $\delta = \bot_{D_<}$ (resp. $\top_{D_<}$), the stack $P$ is initially empty and the execution of \textit{branch}_+(C, S, p, \varepsilon) terminates with $P$ non-empty, then the top element of $P$ is the first solution found that maximises (resp. minimises) the cost function.

Unfortunately, if $\varepsilon$ is higher than 0.0, we cannot guarantee that the top of the stack contains a solution or even a partial solution for the optimisation problem. However, by imposing a monotonicity condition on the cost function $fcost/1$, we can compare solutions.

\textbf{Theorem 3.} (More properties on optimisation) Suppose that, for $i \in \{1, 2\}$, $P_{\varepsilon_i}$ is a stack resulting from the execution of \textit{branch}_+(C, S, p, \varepsilon_i) where $\varepsilon_i \in \mathbb{R}^+$. Suppose also that $\text{top}(P)$ returns the top element of a non-empty stack $P$. Then, if $\varepsilon_1 < \varepsilon_2$ the following property hold.

Approximate soundness: If for $i \in \{1, 2\}$, $P_{\varepsilon_i}$ is not empty, and $\text{top}(P_{\varepsilon_2})$ is a solution or covers a solution for $(V, D, C)$, then, if $fcost/1$ is monotone and $\diamond$ is $<$ (i.e., a minimisation problem),

$$fcost(\text{top}(P_{\varepsilon_1})) \leq_{D_<} fcost(\text{top}(P_{\varepsilon_2})), \quad \text{and, if } fcost/1 \text{ is anti-monotone and } \diamond \text{ is } > (i.e., a maximisation problem),

$$fcost(\text{top}(P_{\varepsilon_1})) \geq_{D_<} fcost(\text{top}(P_{\varepsilon_2})).$$

Therefore, by using a(n) (anti-)monotone cost function, the lower $\varepsilon$ is, the better the (probable) solution is. Moreover, decreasing $\varepsilon$ is a means to discard approximate solutions. For instance, in a minimisation problem, if $fcost/1$ monotone, then, by the approximate soundness property it is deduced that $\text{top}(P_{\varepsilon_2})$ cannot be a solution or cover a solution.
6 Examples

To illustrate the schemas $\text{branch}/4$ and $\text{branch}_+/4$ presented in the previous two sections, several instances of $\text{branch}/4$ are given for some well-known domains of computation. In addition, we explain how the choice of instantiation of the additional global functions and parameters in the definition of $\text{branch}_+/4$ can determine the method of solution for the CSP.

6.1 Some instances

In the following, $\text{branch}_X$ denotes an instance of the schema $\text{branch}/4$ for solving the CSP $\langle V, D, C \rangle$ where $X \subseteq \Delta_D$. We assume that

$$\Delta = \{ \text{Bool, Integer, } \mathbb{R}, \text{Set Integer} \} \cup \{ \text{Interv}(D) \mid D \text{ is a lattice} \}.$$  

where $\text{Interv}(D)$ denotes the set $\{ (d_1, d_2) \mid d_1, d_2 \in D, d_1 \leq d_2 \}$.

To identify $\text{branch}_D$, we indicate a possible definition for both the splitting function and the precision map for each $D \in \Delta_D$ and assume that both a selecting function (e.g., $\text{choose}_D$ as defined in Example 3) and a filtering function for $D$ have been already defined. We also indicate the initial value of $S \in D$, so that the execution of $\text{branch}_D(C, S, p, \varepsilon)$ allows to solve the CSP where $\varepsilon \in \mathbb{R}^+$. 

The finite domain (FD) Constraint solving in a FD of sparse elements is solved by an instance $\text{branch}_{FD}$ as defined below where $\text{split}_{FD}$ is defined as a naive enumeration strategy in which values are chosen from left to right. For example, consider a finite domain of integers $\mathbb{Z}$, Booleans $\mathbb{B}$ or finite sets of integers $\mathbb{S}$.

$$FD \in \{ \text{Integer, Bool, Set Integer} \},$$

$$\text{branch}_{FD} \left\{ \begin{array}{l}
S = (FD, \ldots, FD); \\
n \times \text{ precision}_{FD}(d) = (\#d, 0); \\
\text{split}_{FD}(\{a_1, a_2, a_3, \ldots, a_k\}) = (\{a_1\}, \{a_2, a_3, \ldots, a_k\}) \end{array} \right\}.$$ 

Finite closed intervals Many existing FD constraint systems solve constraints defined in the domain of closed intervals $[a, b]$ where $a, b \in FD$ and denoted here by $a..b$. Usually $a, b$ are either integers $\mathbb{Z}$, Booleans $\mathbb{B}$ or finite sets of integers $\mathbb{S}$. Here are two instances of our schema that solve CSP’s on these domains:

$$FD \in \{ \text{Integer, Bool} \},$$

$$\text{branch}_{\text{Interv}(FD)} \left\{ \begin{array}{l}
S = (\bot_{FD}, \top_{FD}, \ldots, \bot_{FD}, \top_{FD}); \\
n \times \text{ precision}_{\text{Interv}(FD)}(a..b) = (b - a, 0); \\
\text{split}_{\text{Interv}(Integer)}(a..b) = (a..a, a + 1..b) \end{array} \right\}.$$ 

4 The Boolean domain is considered as the integer subset $\{0, 1\}$. 


\( FD = \text{Set Int}, \)

\[
\begin{aligned}
\text{branch}_{\text{Interv}(FD)} & \quad \left\{ \begin{array}{l}
S = (\emptyset..\text{Integer}, \ldots, \emptyset..\text{Integer}) \\
n \text{times}
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
\text{precision}_{\text{Interv}(FD)}(a..b) &= (\#b - \#a, 0) \\
\text{split}_{\text{Interv}(FD)}(a..b) &= (a..b \{c\}, a \cup \{c\}..b) \text{ where } c \in b\setminus a.
\end{aligned}
\]

**Lattice (interval) domain** In [?], we have described a generic filtering algorithm that propagates interval constraints on any domain \( L \) with lattice structure subject to the condition that a function \( \circ_L : L \times L \to \mathbb{R} \) is defined that is strictly monotonic on its first argument and strictly anti-monotonic on its second argument. Below we provide an instance to solve any CSP defined on \( \text{Interv}(L) \):

\[
\begin{aligned}
\text{branch}_{\text{Interv}(L)} & \quad \left\{ \begin{array}{l}
L \text{ is a lattice and } S = ([\perp_L, \top_L], \ldots, [\perp_L, \top_L]) \\
n \text{times}
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
\text{precision}_{\text{Interv}(L)}(r) &= \left\{ \begin{array}{ll}
(b \circ_L a, 2) & \text{if } r = [a, b] \\
(b \circ_L a, 1) & \text{if } r = (a, b) \\
(b \circ_L a, 1) & \text{if } r = [a, b] \\
(b \circ_L a, 0) & \text{if } r = (a, b)
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
\text{split}_{\text{Interv}(L)}(\{a, b\}) &= (\{a, c\}, (c, b}) \text{ where } a \preceq_L c \prec_L b.
\end{aligned}
\]

\( \{a, b\} \) denotes any interval in \( L \). With this instance we have a constraint solving mechanism for solving (interval) constraints defined on any domain with lattice structure. Thus it is a good complement to the filtering algorithm in [?]. Note also that if \( L = \mathbb{R} \) and \( \circ_L \) is \( - \), we obtain the instance \( \text{branch}_{\text{Interv}(\mathbb{R})} \) (also, if \( c = \frac{b-a}{2.0} \) we have a usual strategy of real interval division at the mid point).

**A cooperative domain** The schema also supports cooperative instances that solve CSP’s defined on multiple domains. This is done by mixing together several instances of the schema \( \text{branch}/4 \). As an example, consider \( \text{branch}_{\text{BNR}} \) as defined below where \( \text{split}_{\text{Interv}(D)} \) and \( \text{precision}_{\text{Interv}(D)} \) are defined as in previous examples for \( D \in \{\text{Bool}, \text{Integer}, \mathbb{R}\} \):

\[
\begin{aligned}
\text{branch}_{\text{BNR}} & \quad \left\{ \begin{array}{l}
\Delta = \{\text{Interv}(D) \mid D \in \{\text{Bool}, \text{Integer}, \mathbb{R}\}\} \\
S = (\emptyset..D_1, \ldots, \emptyset..D_n), \{D_1, \ldots, D_n\} \subseteq \{\text{Bool}, \text{Integer}, \mathbb{R}\}.
\end{array} \right.
\end{aligned}
\]

This instance simulates the well known \( \text{splitsolve} \) method of CLP(BNR) [?].

The generic schema is also valid for solving non-linear constraints provided the filtering function \( \text{filtering}_{D}/2 \) solves constraints in non-linear form.
6.2 Different ways to solve the instances of a CSP

Here we show that, for any instance, the schema $branch_+4$ also allows a CSP to be solved in many different ways, depending on the values for $fcost$, $\delta$ and $\varnothing$.

For instance, a successful result for a classical CSP can either be all possible solutions or a set of partial solutions that cover the actual solutions. As stated in Theorem 2(2), to solve classical CSP’s, $fcost$ should be defined as the constant function $\delta \in \mathbb{R}$ and the parameter $\varnothing$ should have the value $\neq$. In Table 1 this case is given in the first row.

As well, as shown in Theorem 2(3), a CSP is solved as a COP by instantiating $\varnothing$ as either $>$ (for maximisation problems) or $<$ (for minimisation problems). The value $\delta$ should be instantiated to the initial cost value from which an optimal solution must be found. Traditionally, the range of the cost function (i.e., $D_<$) is the domain $\mathbb{R}$. Rows 2 and 3 in Table 1 show how to initialise both $\delta$ and $\varnothing$ in these two cases.

| CSP Type           | $fcost$         | $D_<$ | $\varnothing$ | $\delta$         |
|-------------------|----------------|------|---------------|------------------|
| Classical CSP     | constant       | $\mathbb{R}$ | $\neq$       | $fcost(S)$       |
| Typical Minimisation COP | any cost function       | $\mathbb{R}$ | $<$       | $\top \mathbb{R}$ |
| Typical Maximisation COP | any cost function       | $\mathbb{R}$ | $>$       | $\bot \mathbb{R}$ |
| Max-Min COP       | any cost function       | $\mathbb{R} \times \mathbb{R}$ | $<$       | $(\top \mathbb{R}, \bot \mathbb{R})$ |

Table 1. CSP type depends on parameters instantiation

In contrast to typical COP’s that maintain either a lower bound or an upper bound, our schema also permits a mix of the maximization and minimization criteria (or even to give priority to some criteria over others). This is the case when $D_<$ is a compound domain. Then the ordering in $D_<$ determines how the COP will be solved.

Example 4. Let $\langle V, \mathcal{D}, \mathcal{C} \rangle$ be a COP, $D_<$ the domain $\mathbb{R}^2$ with ordering $(a, b) < (c, d) \iff a < c \wedge b > d$, $fcost(S) = (fcost_1(S), fcost_2(S))$ a cost function on $\mathbb{R}^2$ for any $S \in \mathcal{D}$ where $fcost_1, fcost_2 : \mathcal{D} \to \mathbb{R}$. Then, if $\delta$ and $\varnothing$ are as shown in Row 4 of Table 1, this COP is solved by minimising $fcost_1$ and maximising $fcost_2$. However, if $<$ is defined lexicographically, this COP is solved by giving priority to the minimisation of $fcost_1$ over the minimisation of $fcost_2$ e.g. suppose $S_1$, $S_2$ and $S_3$ are solutions with costs $(1.0, 5.0)$, $(3.0, 1.0)$ and $(1.0, 8.0)$, respectively. Then with the first ordering the optimal solution is $S_3$ whereas with the lexicographic ordering $S_1$ is the optimal solution.

7 Concluding remarks

This paper analyses the branching process in constraint solving. We have provided a generic schema for solving CSP’s on finite or continuous domains as well on multiple domains. We have proved key properties such as correctness and
completeness. We have shown how termination may be guaranteed by means of a precision map. We have also shown, by means of an example, how, for systems supporting multiple domains, the precision map can be used to normalise the heuristic for variable ordering.

By using a schematic formulation for the branching process, we have indicated which properties of main procedures involved in branching are responsible for the key properties of constraint solving. For optimisation problems, we have also shown by means of examples that, in some cases, the methods for solving CSP’s depend on the ordering of the range of the cost functions.

By combining a filtering function satisfying our conditions with an appropriate instance of our schema, we obtain an operational semantics for a constraint programming domain (for example: FD, sets of integers, Booleans, multiple domains, ...,etc) and systems designed for constraint solving such as clp(FD) [?], clp(B) [?], DecLic [?], clp(B/FD) [?], CLIP [?], Conjunto [?] or CLP(BNR) [?].

Further work is needed to consider how to construct an efficient implementation. Moreover, it would be useful to examine how the efficiency of a COP solver in our schema could be improved by adding constraints $f_{cost}(S) \circ \delta$ to the original set of constraints for solving $C$, so that exhaustive search is replaced by a forward checking mechanism.

Appendix: Proofs

A path $q \in (\text{Natural} \setminus \{0\})^*$ is any finite sequence of (non-zero) natural numbers. The empty path is denoted by $\varepsilon$, whereas $q . i$ denotes the path obtained by concatenating the sequence formed by the natural number $i \neq 0$ with the sequence of the path $q$. The length of the sequence $q$ is called the length of the path $q$.

Given a tree, we label the nodes by the paths to the nodes. The root node is labelled $\varepsilon$. If a node with label $q$ has $k > 0$ children, then they are labelled, from left to right, $q . 1, \ldots, q . k$.

**Definition 9.** (Search tree). Let $S$ be a constraint store for $\langle V, D, C \rangle$, $\varepsilon \in \mathbb{R}$ and $p \in \mathbb{R} ^ I$. The search tree for branch($C, S, p, \varepsilon$) is a tree that has $S$ at the root node and, as children, has the search trees for the recursive executions of branch/4 as consequence of reaching Line 8 of Figure 1.

Given a search tree for branch($C, S, p, \varepsilon$), we say that $S_\varepsilon = S$ is the constraint store and $p_\varepsilon = p$ the precision at the root node $\varepsilon$. Let $S_q$ be the constraint store and $p_q$ the precision at a node $q$. If $q$ has $k > 0$ children $q . 1, \ldots, q . k$, then $S_q$ is consistent and, if $S^f_q = \text{filtering}_D(C, S_q)$, then $S^f_q$ is divisible so that choose($S^f_q$) = $d_j$ and, for some $k > 0$, split$_D(d_j) = (d_{j1}, \ldots, d_{jk})$. Then we say that $S_{q . i} = S^f_q[d_j/d_{ji}]$ is the constraint store and $p_{q . i} = \text{precision}(S^f_q)$ the precision at node $q . i$, for $i \in \{1, \ldots, k\}$.

---

5 For example, in COP problems, the top of $P$ can be removed after Line 4 of the schema so that $P$ contains only the optimal solution found so far and memory can be saved.
Lemma 1. Let choose/1 be a selecting function for $D$, split $D/1$ a splitting function for $D \in \Delta D$, $S = (d_1, \ldots, d_n)$ a consistent and divisible constraint store for $(\forall, D, C)$, $d_j = \text{choose}(S)$, $d_j \subseteq D$ and $(d_{j1}, \ldots, d_{jk}) = \text{split}_D(d_j)$. Then

(a) $\forall i \in \{1, \ldots, k\} : S[d_j/d_{ji}] \prec_s S$.

(b) Also, if $S'$ is a solution for $(\forall, D, C)$ and $S' \prec_s S$, then

$$\exists i \in \{1, \ldots, k\} : S' \preceq_s S[d_j/d_{ji}].$$

Proof. We prove the cases separately.

Case (a). By Definition 5, $\#d_j > 1$ and, by the contractance property shown in Definition 6, for all $i \in \{1, \ldots, k\} d_{ji} \subset d_j$. Therefore, by Definition 2, for all $i \in \{1, \ldots, k\} S[d_j/d_{ji}] \prec_s S$.

Case (b). By Definition 5, $\#d_j > 1$ and, by the completeness property of the splitting functions shown in Definition 6,

$$\forall s \in d_j. \exists i \in \{1, \ldots, k\} : s \in d_{ji} \quad (1)$$

Suppose that $S' = (d'_{1}, \ldots, d'_{n})$. By Definition 2, for all $i \in \{1, \ldots, n\} d'_i = \{s'_i\}$ and also, as $S' \prec_s S$, $s'_j \in d_j$. As consequence, by (1), $\exists i \in \{1, \ldots, k\} : s'_j \in d_{ji}$ so that $d'_j \subseteq d_{ji}$. Therefore, by Definition 2, $S' \preceq_s S[d_j/d_{ji}]$. $\blacksquare$

Theorem 1 on page 5.

Proof. (Property (1). Termination) In the following, we show that the search tree for $\text{branch}(C, S, p, \varepsilon)$ is finite so that the procedure effectively terminates.

Let $S_e = S$ and $p_e = p$. If the search tree for $\text{branch}(C, S, p_e, \varepsilon)$ has only one node then the procedure terminates. Otherwise, the root node $\epsilon$ has $k$ children with constraint stores $S_i$ where $i \in \{1, \ldots, k\}$ and $S_i = S'[d_j/d_{ji}]$. By Lemma 1(a) and Definition 4, for all $i \in \{1, \ldots, k\}$, $S_i \prec_s S_e$ and, by Proposition 1, $\text{precision}(S_i) \prec_{\mathbb{R}^I} \text{precision}(S_e)$. Then, $\text{precision}(S_i) \prec_{\mathbb{R}^I} \top_{\mathbb{R}^I}$. Suppose now that $\text{precision}(S_i) = (\top_{\mathbb{R}^I}, n)$ for some $n \in \text{Integer}$. Then the test in Line 2 $p_i - \text{precision}(S_i) \leq (\varepsilon, 0)$ holds and the node containing $S_i$ has no children. Otherwise, for all $i \in \{1, \ldots, k\}$,

$$p_i - \text{precision}(S_i) \geq_{\mathbb{R}^I} (\varepsilon, 0) \quad (2)$$

and there exists some constant $\ell \in \mathbb{R}$ such that

$$\text{precision}(S_i) \prec_{\mathbb{R}^I} (\ell \times \varepsilon, 0).$$

We show by induction on the length $j \geq 1$ of a path $q$ in the search tree that

$$\text{precision}(S_i) - \text{precision}(S'_j) \geq_{\mathbb{R}^I} ((j - 1) \times \varepsilon, 0).$$

It follows that $j \leq \ell$ and that, all paths have length $\leq \ell + 1$ (since the second condition in Line 3 in Figure 1 holds) and thus there are no infinite branches.
The base case when \( j = 1 \) follows from (2). Suppose next that \( j > 1 \) and that the hypothesis holds for a path \( q \) of length \( j - 1 \). Let \( q . i_q \) be a child of \( q \) of length \( j \). Then, by the condition in Line 3 of the if sentence,

\[
p_{q.i_q} - \text{precision}(S_{q.i_q}^f) > \mathbb{R}_I (\epsilon, 0),
\]

However, by the inductive hypothesis,

\[
\text{precision}(S_i) - \text{precision}(S_{q.i_q}^f) \geq_{\mathbb{R}_I} ((j - 2) \times \epsilon, 0)
\]

so that, as \( \text{precision}(S_q^f) \) is \( p_{q.i_q} \),

\[
\text{precision}(S_i) - \text{precision}(S_{q.i_q}^f) \geq_{\mathbb{R}_I} (\epsilon, 0) + ((j - 2) \times \epsilon, 0) = ((j - 1) \times \epsilon, 0).
\]

\[\Box\]

\textbf{Proof.} (Property (2). Completeness) Let \( R \) be a solution for \( (\mathcal{V}, \mathcal{D}, C) \). Then, \( R \preceq_s S_i \) and, by Definition 4, \( R \preceq_s S_i^f \). If \( R = S_i^f \), then \( S_i^f \) is consistent and not divisible by Definition 2, tests in Lines 2-3 hold and \( R \) is pushed in the stack \( P \). Otherwise, \( R \preceq S_i^f \) and, by Definition 2, \( S_i^f \) is consistent and divisible. As \( p_r = \top_{\mathbb{R}_I} \), then condition in Line 3 does not hold and node \( \epsilon \) has \( k \) children. By Lemma 1(a) and Definition 4, for any \( q \) of length \( m \geq 1 \), \( S_{q.i_q}^f \preceq_s S_{q.i_q}^f \) and by Proposition 1, \( \text{precision}(S_i^f) - \text{precision}(S_{q.i_q}^f) > (0, 0) \) that means that the condition \( p_{q.i_q} - \text{precision}(S_{q.i_q}^f) \leq (\epsilon, 0) \) in Line 3 never holds. It follows that all the branches in the tree terminate either with an inconsistent store (because test in Line 2 does not hold) or with a consistent and not divisible store (as result of holding tests in Lines 2 and 3). Now, we show by induction on the length \( j \geq 1 \) of a path \( q \) in the search tree that

\[
R \preceq S_q^f \implies \exists i_q \in \{1, \ldots, k\} : R \preceq_s S_{q.i_q}^f.
\]

As, by hypothesis, the procedure terminates then the search tree is finite and it follows that there exists some path \( p = q . q' \) such that \( R = S_p^f \). Then, by Definition 2 \( S_p^f \) is consistent and not divisible and tests in Lines 2 and 3 hold so that \( R \) is put in the stack \( P \). In the base case, when \( j = 1 \), \( S_i = S_i^f [d_j/d_{ji}] \) \((i \in \{1, \ldots, k\})\) and by Lemma 1(b) and Definition 4 \( \exists i \in \{1, \ldots, k\} : R \preceq_s S_i^f \).

Suppose next that \( j > 1 \) and that the hypothesis holds for a path \( q \) of length \( j - 1 \) so that \( R \preceq_s S_q^f \). If \( R \preceq S_q^f \) then, by Definition 2, \( S_q^f \) is divisible so that the node \( S_q^f \) has \( k \) children by Lemma 1(b) and Definition 4 \( \exists i_q \in \{1, \ldots, k\} : R \preceq_s S_{q.i_q}^f \).

\[\Box\]

\textbf{Proof.} (Property (3). Approximate completeness) Let \( R \) be a solution for \( (\mathcal{V}, \mathcal{D}, C) \) so that \( R \preceq_s S_e \) and, by Definition 4, \( R \preceq_s S_i^f \). Since \( \epsilon > 0 \), and as shown in Theorem 1(1), all paths have length \( \leq \ell + 1 \). Therefore, as shown in completeness proof, by following (3), there must exists some path \( q \) with no children and length \( j \geq 1 \) such that \( R \preceq_s S_q^f \). If \( R = S_q^f \) then \( R \) is put in the stack since, by Definition 2, \( R \) is consistent and not divisible and thus tests in Lines 2 and 3 hold. Otherwise, as shown in termination proof, the node \( S_q^f \) has no more children
since the test $p_q - \text{precision}(S_q^f) \leq \varepsilon I (\varepsilon, 0)$ holds and $S_q^f$ is put in the stack. As $R \preceq_s S_q^f$, by Definition 2, $S_q^f$ is either a solution for $(\mathcal{V}, \mathcal{D}, \mathcal{C})$ or a partial solution that covers $R$. ■

Proof. (Property (4). Correctness) Let $R$ be an element in the final state of $P$. As shown in completeness proof, if $\varepsilon = 0.0$ the test $p_q - \text{precision}(S_q^f) \leq (\varepsilon, 0)$ never holds, for all path $q$ of length $m \geq 1$ (also, Line 3 is never satisfied when $q = \epsilon$). Therefore, $R$ is in $P$ because there exists a path $q$ where $S_q^f = R$ is consistent and not divisible so that tests in Lines 2 and 3 hold. Moreover, by Definition 4, $S_q^f$ is a solution for $(\mathcal{V}, \mathcal{D}, \mathcal{C})$. ■

Proof. (Property (5). Approximate correctness or control on the result precision) By Theorem 1(4), if $R \in P_{0.0}$ then $R$ is a solution for $(\mathcal{V}, \mathcal{D}, \mathcal{C})$ and, by Theorem 1(2), if $R$ is a solution for $(\mathcal{V}, \mathcal{D}, \mathcal{C})$ then $R \in P_{0.0}$. Also, by Theorem 1(3), $\exists S_{e_1} \in P_{e_1}$ and $\exists S_{e_2} \in P_{e_2}$ such that $R \preceq_s S_{e_1}$ and $R \preceq_s S_{e_2}$. Thus, by Definition 3, $P_{0.0} \subseteq P_{e_1}$ and $P_{0.0} \subseteq P_{e_2}$ (also $P_{e_1}$ and $P_{e_2}$ are not empty).

Now we prove that $P_{e_1} \subseteq P_{e_2}$. Suppose that $S_{e_2} \in P_{e_2}$. Then, exists a path $q$ of length $m \geq 0$ such that $S_q^f = S_{e_2}$ and $S_{e_2}$ was pushed in the stack because tests in Lines 2 and 3 hold so that $S_{e_2}$ is consistent and (a) also not divisible (i.e., it is a solution), or (b) the test $p_q - \text{precision}(S_q^f) \leq (\varepsilon_2, 0)$ holds.

Suppose (a). If $q = \epsilon$ then it is obvious that $S_{e_2}$ is also pushed in $P_{e_1}$. Otherwise, for all path $q_1$ with length $\geq 1$ and where $q = q_1 \cdot q_2$ the test $p_q - \text{precision}(S_q^f) > (\varepsilon, 0)$ holds for $\varepsilon = \varepsilon_2$ and thus also holds for $\varepsilon = \varepsilon_1$. Therefore, $S_{e_2}$ is also pushed in $P_{e_1}$. Now suppose (b). Then, if $p_q - \text{precision}(S_q^f) \leq (\varepsilon_1, 0)$ then $S_{e_2}$ is also pushed in $P_{e_1}$, otherwise since $S_q^f$ covers a solution and as shown in proof of Theorem 1(3) there is be some path $q' = q \cdot p$ such that the test $p_q - \text{precision}(S_q^f) \leq (\varepsilon_1, 0)$ holds and thus $S_q^f$ is pushed in $P_{e_1}$. By induction on the length of $p$ and by (3), it is straightforward to prove that $S_q^f \preceq_s S_q^f$. Thus, by Definition 3, $P_{e_1} \subseteq P_{e_2}$. ■

**Theorem 2 on page 7.**

Proof. (Property (1). Termination) This proof is as that of Theorem 1(1). ■

Proof. (Property (2)) Observe that if $f_{cost}(S) = \delta$ for all $S \in \mathcal{D}$, then test in Line 4* of the extended schema always holds. It is straightforward to prove, in this case, that the schemas $\text{branch}/4$ and $\text{branch}_+ /4$ are equivalent so that all properties for the schema $\text{branch}/4$ are also held in the schema $\text{branch}_+ /4$. ■

Proof. (Property (3). Soundness on optimisation) We prove the case when $\circ$ and $\delta$ are, respectively, $>$ and $\bot_{\mathcal{D}_s}$. The respective case is proved analogously. As shown in proof of Theorem 1(2), for $\varepsilon = 0.0$, if $R$ is a solution for $(\mathcal{V}, \mathcal{D}, \mathcal{C})$ then there exists some path $q$ of length $j \geq 0$, such that $R = S_q^f$ and the tests in Lines 2–3 hold by Definition 2. Thus, Line 4* is reached for each solution $R \in \text{Sol}(\mathcal{V}, \mathcal{D}, \mathcal{C})$, and as consequence, the top of $P$ will contain the first solution found that maximises $f_{cost}/1$. ■

**Theorem 3 on page 7.**
Proof. (Property: Approximate soundness) We show that during the execution of $\text{branch}_+(C, S, p, \varepsilon_1)$, Line 4* is reached for some $S_q^f \preceq_s \text{top}(P_{\varepsilon_2})$ (where $q$ is a path of length $m_1 \geq 0$). As consequence, $f\text{cost}(S_q^f) \preceq_{D, <} f\text{cost}(\text{top}(P_{\varepsilon_2}))$.

It follows that either $S_q^f$ is in the top of $P_{\varepsilon_1}$ or $S_q^f$ is not in the top of $P_{\varepsilon_1}$ since $f\text{cost}(\text{top}(P_{\varepsilon_1})) \preceq_{D, <} f\text{cost}(S_q^f)$ so that effectively $f\text{cost}(\text{top}(P_{\varepsilon_1})) \preceq_{D, <} f\text{cost}(\text{top}(P_{\varepsilon_2}))$.

Observe that $\text{top}(P_{\varepsilon_2})$ is in $P_{\varepsilon_2}$ because there exist some path $q'$ with length $m_2 \geq 0$ such that $S_q^f = \text{top}(P_{\varepsilon_2})$ and tests in Lines 2-4* hold. Then, as shown in proof of Theorem 1(1), for all path $q_1$ such that $q' = q_1 . q_2$

\[ p_{q_1} - \text{precision}(S_{q_1}^f) > (\varepsilon_2, 0). \]

As $\varepsilon_1 < \varepsilon_2$, this also holds in the execution of $\text{branch}_+(C, S, p, \varepsilon_1)$. Therefore, if $S_q^f$ is in $P_{\varepsilon_2}$ because it is consistent and not divisible, then in the execution of $\text{branch}_+(C, S, S', \varepsilon_1)$ Line 4* is also reached with $S_q^f$. Otherwise, $S_q^f$ is in $P_{\varepsilon_2}$ because $p_{q'} - \text{precision}(S_{q'}^f) \leq (\varepsilon_2, 0)$. Then, as $\varepsilon_1 < \varepsilon_2$, and as shown in proof of Theorem 1(5), there exists some path $q''$ with length $r \geq m_2$ such that

\[ p_{q''} - \text{precision}(S_{q''}^f) \leq (\varepsilon_1, 0) \text{ and } S_{q''}^f \preceq_s S_q^f. \]

so that again Line 4* is reached for $S_q^f$. 

Appendix: A simple example

Here we show a simple example in the domain $\text{Interv}(\text{Integer})$, illustrating the flexibility of the schema to solve a CSP in different ways. Let $\langle V, D, C \rangle$ be a CSP where $V = \{x_1, x_2, x_3\}$, $\Delta_D = \{\text{Interv}(\text{Integer})\}$ and $C$ is the constraint set

$$\{x_1 + x_2 + x_3 \leq 1, \ x_1 \leq 1, \ x_2 \leq 1, \ x_3 \leq 1, \ x_1 \geq 0, \ x_2 \geq 0, \ x_2 \geq 0\}.$$ 

Consider also the following cost functions\(^6\) defined on different ranges:

- $f_{\text{cost}}_1(x_1, x_2, x_3) = 1.0$. Range: $\mathbb{R}$
- $f_{\text{cost}}_2(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Range: $\mathbb{R}$
- $f_{\text{cost}}_3(x_1, x_2, x_3) = (f_{\text{cost}}_2(x_1, x_2, x_3), x_1 + x_3)$. Range: $\mathbb{R}^2$
- $f_{\text{cost}}_4(x_1, x_2, x_3) = (f_{\text{cost}}_2(x_1, x_2, x_3), x_2 + x_3)$. Range: $\mathbb{R}^2$

Consider now the instance $\text{branch}_{\text{Interv}}(FD)$, as defined in Section 6.1, where $FD = \text{Integer}$ and assume that $\text{choose}_{\text{naive}}$ is as defined in Example 2, $p = \top_{\text{MT}}$, $\varepsilon = 0.0$ and initially the global stack $P$ is empty. Suppose that as filtering algorithm we define a simple consistency check on the consistency of constraint stores in such a way that $\text{filtering}_P(C, S)$ returns $S$ if $S$ is consistent and the inconsistent store $(\emptyset, \emptyset, \emptyset)$ otherwise. Now, assume that $\text{branch}_4(C, S, p, \varepsilon)$ is executed with different values for $\delta$, $\diamond$ and $f_{\text{cost}}/1$. Since the domain is finite, termination is guaranteed even if $\varepsilon = 0.0$. Each row in Table 2 corresponds with a different execution of the extended schema where

- Column 1 indicates the way in which the CSP is solved,
- Column 2 shows the value to which $\delta$ is initialised,
- Column 3 the cost function used in the current instance,
- Column 4 the initialisation of $\diamond$,
- Column 5 indicates where is, in the global stack, the solution(s) and
- Column 6 references the figure that shows the final state of the stack $P^7$.

By simplicity, suppose that during each execution of the extended schema, branches are solved by classical backtracking following a classical depth first strategy. Then, the CSP is solved in different ways. For instance, to solve the problem as a classical CSP (see Row 1 in Table 2), $f_{\text{cost}}$ is a constant function with value $\delta$ (where $\delta$ is 1.0) and $\diamond = \cdot$. Then, all possible solutions for the problem are pushed in the stack (see Figure 2(a)).

Also, Rows 2-3 in Table 2 show how to solve this CSP by maximising and minimising the function $f_{\text{cost}}_2$ respectively. The optimal solution is that on the top of the stack (see Figures 2(b) and 2(c)). On their turn, Rows 4-7 indicate

\[^6\] The sum is defined to return the mid point of the sum of operand intervals, e.g., the sum of two intervals $a..b$ and $c..d$ is $(a+c+b+d)/2$ that is exactly the mid point of the interval $a+c..b+d$.

\[^7\] To the right of each element in $P$ we write its cost.
Table 2. Different solvings of the CSP

| CSP Type                  | $\delta$ | cost function $\phi$ | Solution | Figure |
|--------------------------|----------|-----------------------|----------|--------|
| Classical CSP            | 1.0      | $f_{cost_1}$          | Each element in the stack | 2(a)   |
| Maximisation COP         | $\bot \mathbb{R}$ | $f_{cost_2}$ > | Stack Top | 2(b)   |
| Minimisation COP         | $\top \mathbb{R}$ | $f_{cost_2}$ < | Stack Top | 2(c)   |
| Max-Min COP (i)          | $(\bot \mathbb{R}, \top \mathbb{R})$ | $f_{cost_3}$ <1 | Stack Top | 2(d)   |
| Max-Min COP (ii)         | $(\bot \mathbb{R}, \top \mathbb{R})$ | $f_{cost_4}$ <1 | Stack Top | 2(e)   |
| Max-Min COP (iii)        | $(\bot \mathbb{R}, \top \mathbb{R})$ | $f_{cost_3}$ <2 | Stack Top | 2(f)   |
| Max-Min COP (iv)         | $(\bot \mathbb{R}, \top \mathbb{R})$ | $f_{cost_4}$ <2 | Stack Top | 2(g)   |

Table 3. Evaluation of the solutions to the problems

| Solution (S) | $f_{cost_1}(S)$ | $f_{cost_2}(S)$ | $f_{cost_3}(S)$ | $f_{cost_4}(S)$ |
|--------------|-----------------|-----------------|-----------------|-----------------|
| (1,0,0)      | 1.0             | 1.0             | (1.0,1.0)       | (1.0,0.0)       |
| (0,1,0)      | 1.0             | 1.0             | (1.0,0.0)       | (1.0,1.0)       |
| (0,0,1)      | 1.0             | 1.0             | (1.0,1.0)       | (1.0,1.0)       |
| (0,0,0)      | 1.0             | 0.0             | (0.0,0.0)       | (0.0,0.0)       |

how to mix optimisation criteria to solve the CSP. For instance, assume the following two orderings on $\mathbb{R}^2$:

$$(a, b) \leq_1 (c, d) \iff a \geq c\text{ and } b \leq d;$$

$$(a, b) \leq_2 (c, d) \iff a > c\text{ or } a = c\text{ and } b \leq d.$$  

Then, row 4 corresponds to the problem of maximising $x_1 + x_2 + x_3$ and minimising $x_1 + x_3$ whereas row 5 corresponds to the problem of maximising $x_1 + x_2 + x_3$ and minimising $x_2 + x_3$. Also, row 6 corresponds to the problem of firstly maximising $x_1 + x_2 + x_3$, and if this cannot be more optimised then minimise $x_1 + x_3$ (this is consequence of the ordering $<_2$) whereas row 7 does the same but minimising $x_2 + x_3$. Figure 2 shows the final state of the global stack for each of these cases (also Table 3 shows the evaluation of each solution to the CSP by the different cost functions

Figure 2. The final state of the global stack $P$ in the different solvings of the CSP