LEFT-SYMMETRIC SUPERALGEBRAS ON SPECIAL LINEAR LIE SUPERALGEBRAS

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ABSTRACT. In this paper, we study the existence and classification problems of left-symmetric superalgebras on special linear Lie superalgebras \(\mathfrak{sl}(m|n)\) with \(m \neq n\). The main three results of this paper are: (i) a complete classification of the left-symmetric superalgebras on \(\mathfrak{sl}(2|1)\), (ii) \(\mathfrak{sl}(m|1)\) does not admit left-symmetric superalgebras for \(m \geq 3\), and (iii) \(\mathfrak{sl}(m+1|m)\) admits a left-symmetric superalgebra for every \(m \geq 1\). To prove these results we combine previous results on the existence and classification of left-symmetric algebras on the Lie algebras \(\mathfrak{gl}_m\) with a detailed analysis of small representations of the Lie superalgebras \(\mathfrak{sl}(m|1)\). We also conjecture that \(\mathfrak{sl}(m|n)\) admits left-symmetric superalgebras if and only if \(m = n + 1\).

1. INTRODUCTION

A superalgebra \( (\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1, \cdot) \) over a field \(k\) is called a left-symmetric superalgebra (or an LSSA for short) if the associator \( (x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) \) is supersymmetric in \(x\) and \(y\), i.e., \( (x, y, z) = (-1)^{|x||y|}(y, x, z) \); or, equivalently,

\[
(x \cdot y) \cdot z - x \cdot (y \cdot z) = (-1)^{|x||y|}((y \cdot x) \cdot z - y \cdot (x \cdot z)), \quad \forall x, y, z \in \mathcal{L}.
\]

It is clear that each associative superalgebra is an LSSA. The supercommutator \([x, y] = x \cdot y - (-1)^{|x||y|}y \cdot x\) defines a Lie superalgebra structure on \(\mathcal{L}\). The resulting Lie superalgebra \(\mathfrak{g}_\mathcal{L}\) is called the associated Lie superalgebra of \(\mathcal{L}\), and \(\mathcal{L}\) is called an LSSA on \(\mathfrak{g}_\mathcal{L}\). Note that if \(\mathcal{L}\) is an LSSA on a Lie superalgebra \(\mathfrak{g}\), then \(\mathcal{L}_0\) is a left-symmetric algebra on \(\mathfrak{g}_0\). Extending Segal’s remark, [13], we consider determining whether the set of LSSAs on a Lie superalgebra is non-empty and classifying all such LSSAs to be a fundamental task.

The ”non-super” version of this problem arises in the theory of affine structures on differentiable manifolds and Lie groups. Assume that \(G\) is a connected, simply connected Lie group with Lie algebra \(\mathfrak{g}\). It is known that endowing \(G\) with a left-invariant affine structure is equivalent to endowing \(\mathfrak{g}\) with a left-symmetric product. For more details, see for example [11, 12, 13]. Left-symmetric algebra structures on both finite-dimensional and infinite-dimensional Lie algebras have been studied extensively, see [1, 3, 16] and [5, 10, 14], respectively. Kong and Bai, [9], studied LSSAs on the Virasoro superalgebra.

In this paper we use repeatedly the following results. Medina, [11], demonstrated that finite-dimensional complex semisimple Lie algebras do not admit left-symmetric algebra structures.
Baues, [2], classified all left-symmetric algebras on $\mathfrak{gl}_n$ and proved that $\mathfrak{gl}_n$ is the only reductive Lie algebra with one-dimensional center and a simple semisimple ideal which admits left-symmetric algebras over an algebraically closed ground field (see also [4]).

Unlike the case of Lie algebras, there do exist finite-dimensional complex simple Lie superalgebras that admit LSSAs. The problem of classifying the LSSA-structures on Lie superalgebras is even more challenging. Indeed, Xu in [15] stated "it looks more challenging to classify LSSAs on all the well-known simple Lie superalgebras". Based on the classification of finite-dimensional simple Lie superalgebras over $\mathbb{C}$, the field of complex numbers, due to Kac, the even parts of classical Lie superalgebras except for $\mathfrak{sl}(m|n), m \neq n \geq 1$ and $\mathfrak{osp}(2,2n), n \geq 1$ are semisimple Lie algebras. Moreover, the even part of $\mathfrak{osp}(2,2n)$ is isomorphic to $\mathfrak{sp}_{2n} \oplus \mathbb{C}$. Using the results about left-symmetric algebras on Lie algebras, we conclude that $\mathfrak{sl}(m|n), m \neq n \geq 1$ are the only classical Lie superalgebras that may admit LSSAs.

The present paper is devoted to the existence and classification of LSSAs on the Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m > n$. We completely solve these problems in the case $n = 1$. Namely, we prove the following theorems.

**Theorem 1.1.** There do not exist LSSAs on simple Lie superalgebras $\mathfrak{sl}(m|1)$ for $m \geq 3$.

**Theorem 1.2.** Let $\mathcal{L}$ be an LSSA on simple Lie superalgebra $\mathfrak{sl}(2|1)$.

1. $\mathcal{L}$ corresponds to a bijective evaluation map associated with an appropriate $\mathfrak{sl}(2|1)$-module.
2. $\mathcal{L}$ is isomorphic to an LSSA in one of these three families:
   - $\mathcal{A}_k$, $k \in \mathbb{C} \setminus \{-1, -3\}$;
   - $\mathcal{B}_{k_1,k_2}$, $k_1, k_2 \in \mathbb{C} \setminus \{0\}, k_1 + k_2 \neq -2$;
   - $\mathcal{C}_k$, $k \in \mathbb{C} \setminus \{0, -1\}$.
3. $\mathcal{A}_k \cong \mathcal{A}_{-2-k}$, $\mathcal{B}_{k_1,k_2} \cong \mathcal{B}_{k_2,k_1} \cong \mathcal{B}_{-2-k_1,-2-k_2} \cong \mathcal{B}_{-2-k_2,-2-k_1}$, and $\mathcal{C}_k \cong \mathcal{C}_{-2-k}$. Moreover, these are the only isomorphisms among LSSAs in (2) above.

**Remark 1.3.** The families $\mathcal{A}_k, \mathcal{B}_{k_1,k_2}$, and $\mathcal{C}_k$ are constructed in Section 5.1.

To prove these results we start with Baues’ classification of left-symmetric algebras on $\mathfrak{gl}_n$ and then study how a left-symmetric algebra structure on $\mathfrak{g}_0$ can be extended to an LSSA-structure on $\mathfrak{g}$. The basic idea is that such an LSSA-structure on $\mathfrak{g}$ exists if and only if $\mathfrak{g}$ admits a bijective 1-cocycle corresponding to the respective representation of $\mathfrak{g}$.

Understanding the LSSAs on $\mathfrak{g} = \mathfrak{sl}(m|n)$ for $n > 1$ is more difficult because there is not a complete classification of the left-symmetric algebras on the even part $\mathfrak{g}_0 = \mathfrak{sl}_m \oplus \mathfrak{sl}_n \oplus \mathbb{C}$. Somewhat surprisingly, we prove the following result.

**Theorem 1.4.** There exists an LSSA on $\mathfrak{sl}(m+1|m)$ for every natural number $m$.

Baues [2, Proposition 5.1] proved that each left-symmetric algebra on $\mathfrak{gl}_n$ has a unique right identity. We conjecture that each LSSA on $\mathfrak{sl}(m|n)$ also has a unique right identity, which means that $\dim \mathfrak{sl}(m|n)_0 = \dim \mathfrak{sl}(m|n)_1$ by Proposition 2.1 below. Then we have $m = n + 1$. So we state the following conjecture:
Conjecture 1.5. There do not exist LSSAs on any Lie superalgebras \( \mathfrak{sl}(m|n) \) other than \( \mathfrak{sl}(m+1|m) \).

This paper is organized as follows. In Section 2, we discuss the relationship between LSSAs and bijective 1-cocycles on a given Lie superalgebra. We also study evaluation maps, which form a special class of 1-cocycles and are useful in proving isomorphisms of LSSAs on Lie superalgebras. In Section 3, we present some preliminaries on the special linear Lie superalgebras \( \mathfrak{sl}(m|n) \). We recall the construction of Kac modules and extensions between irreducible modules of \( \mathfrak{sl}(m|n) \). Section 4 investigates \( m^2|2m \)-dimensional \( \mathfrak{sl}(m|1) \)-modules for \( m \geq 3 \), whose even parts are isomorphic to the direct sum of \( m \) copies of the standard module or \( m \) copies of the dual module of the standard module as \( \mathfrak{sl}_m \)-modules. We prove that there are no bijective 1-cocycles of \( \mathfrak{sl}(m|1) \) for \( m \geq 3 \), proving Theorem 1.1. In Section 5, we show that bijective 1-cocycles and bijective evaluation maps associated with \( 4|4 \)-dimensional \( \mathfrak{sl}(2|1) \)-modules coincide and classify all the LSSAs on \( \mathfrak{sl}(2|1) \). Finally, in Section 6, we prove Theorem 1.4 by constructing a bijective evaluation map on each \( \mathfrak{sl}(m+1|m) \) for \( m \geq 1 \). In preparation for the proof of Theorem 1.4, we also establish some facts about representations of (possibly infinite-dimensional) Lie (super)algebras which may be of independent interest.

**Notation and conventions.** All vector spaces, algebras, superalgebras, etc. are over \( \mathbb{C} \). Elements of \( \mathbb{Z}_2 \) are denoted by \( 0 \) and \( 1 \). Homomorphisms (isomorphisms, automorphisms) of superalgebras are assumed to be homogeneous linear maps of degree zero. If \( W = W_0 \oplus W_1 \) is a \( \mathbb{Z}_2 \)-graded vector space and \( L : W \to W \) is a linear map, the supertrace \( \text{str}(L) \) of \( L \) is defined as \( \text{tr}(L_{00}) - \text{tr}(L_{11}) \), where \( L_{\gamma'\gamma''} := p_{\gamma''} \circ L \circ t_{\gamma'} \) is the natural map \( W_{\gamma'} \to W_{\gamma''} \) defined by \( L \), see also Section 3.1 for a coordinate definition of \( \text{str}(L) \). A module \( V \) of a superalgebra \( A = A_0 \oplus A_1 \) is always assumed to be \( \mathbb{Z}_2 \)-graded, that is \( V = V_0 \oplus V_1 \) and \( A_{\gamma'} V_{\gamma''} \subseteq V_{\gamma'+\gamma''} \) for \( \gamma', \gamma'' \in \mathbb{Z}_2 \). We use the terms \( \mathfrak{g} \)-module and representation of \( \mathfrak{g} \) interchangeably to mean a finite-dimensional representation of a Lie (super)algebra \( \mathfrak{g} \).

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2. **Left-symmetric superalgebras**

2.1. **Left regular representations.** Let \( \mathcal{L}, \cdot \) be an LSSA and \( (\mathfrak{g}_{\mathcal{L}}, [, ]) \) its associated Lie superalgebra. Then there are two product operations \( \cdot \) and \( [, ] \) on the underlying \( \mathbb{Z}_2 \)-graded vector space of \( \mathcal{L} \). For an element \( x \in \mathcal{L} \), the left multiplication operator \( \rho(x) : \mathcal{L} \to \mathcal{L} \) sends \( y \in \mathcal{L} \) to \( x \cdot y \), and the right multiplication operator \( \tau(x) : \mathcal{L} \to \mathcal{L} \) sends \( y \in \mathcal{L} \) to \( (-1)^{|x||y|} y \cdot x \). Define \( \rho : \mathfrak{g}_{\mathcal{L}} \to \mathfrak{gl}(\mathcal{L}), x \mapsto \rho(x) \). It is easy to check that \( \rho([x,y]) = [\rho(x), \rho(y)] \) for all \( x, y \in \mathfrak{g}_{\mathcal{L}} \), so
the map \( \rho \) gives a representation of Lie superalgebra \( \mathfrak{g}_L \), which is called the left regular representation of \( \mathfrak{g}_L \).

The following proposition relates LSSAs on Lie superalgebras to left and right identities.

**Proposition 2.1.** Let \( \mathfrak{g} \) be a Lie superalgebra of dimension \( p|q \) satisfying \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\). Suppose that there exists an LSSA on \( \mathfrak{g} \). The following statements hold.

1. \( \text{str}(\rho(x)) = 0 \) and \( \text{str}(\tau(x)) = 0 \) for all \( x \in \mathfrak{g} \).
2. If there is an element \( e \in \mathfrak{g} \) such that \( \rho(e) = \text{id} \) or \( \tau(e) = \text{id} \), then \( p = q \).
3. If \( \mathfrak{g} \) is simple, then there is no element \( e \in \mathfrak{g} \) such that \( \rho(e) = \text{id} \).

**Proof.**

(1) Since for all \( x, y \in \mathfrak{g} \), \( \text{str}(\rho([x, y])) = \text{str}(\rho(x)\rho(y)) = 0 \) and \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\), we have \( \text{str}(\rho(x)) = 0 \) for all \( x \in \mathfrak{g} \). Similarly, \( \tau([x, y]) = \rho([x, y]) - \text{ad}_{[x, y]} = [\rho(x), \rho(y)] - [\text{ad}_x, \text{ad}_y] \) gives \( \text{str}(\tau(x)) = 0 \) for all \( x \in \mathfrak{g} \).

(2) By (1) one has \( 0 = \text{str}(\rho(e)) = \text{str}(\text{id}) = p - q \) or \( 0 = \text{str}(\tau(e)) = \text{str}(\text{id}) = p - q \), so (2) follows.

(3) Assume that there is an element \( e \in \mathfrak{g} \) such that \( \rho(e) = \text{id} \). Since \( \rho \) gives a representation of \( \mathfrak{g} \), we have that \( H = \text{Ker}(\rho) \) is an ideal of \( \mathfrak{g} \). By the assumption that \( \mathfrak{g} \) is simple and \( \rho \) is nonzero, then \( H = 0 \). On the other hand, \( \rho([e, x]) = [\rho(e), \rho(x)] = [\text{id}, \rho(x)] = 0 \) for all \( x \in \mathfrak{g} \) implies \([e, \mathfrak{g}] \subseteq H = 0 \), thus \( e \in \mathfrak{z}(\mathfrak{g}) \), the center of \( \mathfrak{g} \). Since \( \mathfrak{z}(\mathfrak{g}) = 0 \), we have \( e = 0 \) and \( \rho(e) = 0 \), which is a contradiction.

**2.2. LSSAs and 1-cocycles.** Given a Lie superalgebra \( \mathfrak{g} \) and a representation \( f : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \) of \( \mathfrak{g} \), an even linear map \( q : \mathfrak{g} \rightarrow V \) satisfying

\[
q([x, y]) = f(x)q(y) - (-1)^{|x||y|}f(y)q(x), \quad \forall x, y \in \mathfrak{g},
\]

is called an (even) 1-cocycle on \( \mathfrak{g} \) and denoted by the pair \((f, q)\). A 1-cocycle \((f, q)\) is called bijective if \( q \) is a bijection.

**Lemma 2.2.** Let \( \mathfrak{g} \) be a Lie superalgebra and \( f : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \) a representation of \( \mathfrak{g} \). If \((f, q)\) is a 1-cocycle on \( \mathfrak{g} \) such that \( q|_{\mathfrak{g}_0} = 0 \), then \( q|_{\mathfrak{g}_1} : \mathfrak{g}_1 \rightarrow V_1 \) is a homomorphism of \( \mathfrak{g}_0 \)-modules.

**Proof.** Since \( q|_{\mathfrak{g}_0} = 0 \), we have

\[
q([x, y]) = f(x)q(y) - f(y)q(x) = f(x)q(y), \quad \forall x \in \mathfrak{g}_0, y \in \mathfrak{g}_1.
\]

This means that \( q|_{\mathfrak{g}_1} \circ \text{ad}_x = f(x) \circ q|_{\mathfrak{g}_1} \) for all \( x \in \mathfrak{g}_0 \), as desired.

Given a Lie superalgebra \( \mathfrak{g} \), we denote by \( \mathcal{S} \) the set of all LSSAs on \( \mathfrak{g} \) and denote by \( \mathcal{O} \) the set of all bijective 1-cocycles on \( \mathfrak{g} \). Following Bai, [1], we note a close relation between \( \mathcal{S} \) and \( \mathcal{O} \). Suppose \( \mathcal{S} \) and \( \mathcal{O} \) are not empty. Given an element \( \mathcal{L} \in \mathcal{S} \), the left regular representation

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1 Since we will be interested in bijective 1-cocycles up to a quasi-equivalence, see below, we may fix the vector space \( V = \mathfrak{g} \) to avoid set-theoretic pitfalls that may arise when considering the collection of all 1-cocycles on \( \mathfrak{g} \).
\( \rho \) induced by \( \mathcal{L} \), together with the identity map, gives rise to a bijective 1-cocycle \( (\rho, \text{id}) \in \mathcal{O} \). Conversely, for each \( (f, q) \in \mathcal{O} \), the multiplication

\[
x \cdot y = q^{-1}(f(x)q(y)), \quad \forall x, y \in \mathfrak{g},
\]

gives rise to an LSSA \( (\mathcal{L}, \cdot) \in \mathcal{S} \). We denote the maps defined above by

\[
\Psi: \mathcal{S} \longrightarrow \mathcal{O} \quad \text{and} \quad \Phi: \mathcal{O} \longrightarrow \mathcal{S}.
\]

To classify LSSAs up to isomorphism, we introduce the notion of quasi-equivalence between 1-cocycles so that \( \Psi \) and \( \Phi \) above induce a bijection between isomorphic classes \( \mathcal{S}/ \cong \) in \( \mathcal{S} \) and quasi-equivalent classes \( \mathcal{O}/ \simeq \) in \( \mathcal{O} \).

**Definition 2.3.** Let \( \mathfrak{g} \) be a Lie superalgebra and \( f_i: \mathfrak{g} \longrightarrow \mathfrak{gl}(V_i), i = 1, 2 \), be two representations of \( \mathfrak{g} \). Two 1-cocycles \( (f_1, q_1) \) and \( (f_2, q_2) \) on \( \mathfrak{g} \) are called equivalent, denoted by \( (f_1, q_1) \cong (f_2, q_2) \), if there exists a linear isomorphism \( \varphi : V_2 \longrightarrow V_1 \) such that

\[
f_2(x) = \varphi^{-1} \circ f_1(x) \circ \varphi \quad \text{and} \quad q_2 = \varphi^{-1} \circ q_1, \quad \forall x \in \mathfrak{g}.
\]

We say that two 1-cocycles \( (f_1, q_1) \) and \( (f_2, q_2) \) on \( \mathfrak{g} \) are quasi-equivalent, denoted by \( (f_1, q_1) \simeq (f_2, q_2) \), if there exists an automorphism \( T \) of \( \mathfrak{g} \) such that \( f_1 \circ T, q_1 \circ T \) and \( f_2, q_2 \) are equivalent.

**Proposition 2.4.** The maps \( \Psi \) and \( \Phi \) induce a bijection between the set \( \mathcal{S}/ \cong \) of isomorphic classes of LSSAs on a Lie superalgebra \( \mathfrak{g} \) and the set \( \mathcal{O}/ \simeq \) of quasi-equivalent classes of bijective 1-cocycles on \( \mathfrak{g} \).

**Proof.** Suppose that \( T: \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \) is an isomorphism of two LSSAs \( (\mathcal{L}_1, \cdot_1) \) and \( (\mathcal{L}_2, \cdot_2) \) on \( \mathfrak{g} \). Then \( T \) is also an automorphism of the Lie superalgebra \( \mathfrak{g} \). We have

\[
\rho_{\mathcal{L}_2}(x)(y) = x \cdot_2 y = T^{-1}(T(x) \cdot_1 T(y)) = (T^{-1} \circ \rho_{\mathcal{L}_1} \circ T)(x) \circ T(y), \quad \forall x, y \in \mathfrak{g}.
\]

Together with the fact that \( \text{id} = T^{-1} \circ \text{id} \circ T \), we deduce that \( (\rho_{\mathcal{L}_1}, \text{id}) \) and \( (\rho_{\mathcal{L}_2}, \text{id}) \) are quasi-equivalent. Hence isomorphic LSSAs are mapped to quasi-equivalent bijective 1-cocycles.

Conversely, suppose that \( (f_1, q_1) \) and \( (f_2, q_2) \) are two quasi-equivalent bijective 1-cocycles on \( \mathfrak{g} \). Then there exists a linear isomorphism \( \varphi \) and an automorphism \( T \) of \( \mathfrak{g} \) such that \( f_2(x) = \varphi^{-1} \circ (f_1 \circ T)(x) \circ \varphi \) and \( q_2 = \varphi^{-1} \circ q_1 \circ T \) for all \( x \in \mathfrak{g} \). Let \( (\mathcal{L}_1, \cdot_1) \) and \( (\mathcal{L}_2, \cdot_2) \) be the corresponding LSSAs induced from \( (f_1, q_1) \) and \( (f_2, q_2) \) by Eq. (2.2), respectively. Then

\[
T(x \cdot_2 y) = T(q_2^{-1}(f_2(x)q_2(y))) = q_1^{-1}(f_1(T(x))q_1(T(y))) = T(x) \cdot_1 T(y)
\]

for all \( x, y \in \mathfrak{g} \). Hence quasi-equivalent bijective 1-cocycles are mapped to isomorphic LSSAs.

Note that \( \Phi \circ \Psi(\mathcal{L}) = \mathcal{L} \) for all \( \mathcal{L} \in \mathcal{S}/ \cong \). For all \( (f, q) \in \mathcal{O}/ \simeq \), we see that \( \Psi \circ \Phi((f, q)) = \Psi(\mathcal{L}) = (\rho_{\mathcal{L}}, \text{id}) \). Since \( \rho_{\mathcal{L}}(x)y = q^{-1}(f(x)q(y)) \) for all \( x, y \in \mathfrak{g} \) and \( \text{id} = q^{-1} \circ q \), we have \( (\rho_{\mathcal{L}}, \text{id}) \) is quasi-equivalent to \( (f, q) \). Therefore, \( \Phi \) and \( \Psi \) induce mutually inverse bijections between \( \mathcal{S}/ \cong \) and \( \mathcal{O}/ \simeq \). \( \square \)
2.3. **Evaluation maps.** Given a representation \( f : \mathfrak{g} \to \mathfrak{gl}(V) \) of a Lie superalgebra \( \mathfrak{g} \) and an element \( a \in V_0 \), the map \( \text{ev}_a : \mathfrak{g} \to V \) defined by \( \text{ev}_a(x) = f(x)a \) for all \( x \in \mathfrak{g} \) is called the **evaluation map** of \( \mathfrak{g} \) associated with \( f \) at the point \( a \). It is immediate that \( (f, \text{ev}_a) \) is a 1-cocycle on \( \mathfrak{g} \). Evaluation maps are very useful in establishing isomorphisms of LSSAs on \( \mathfrak{g} \); see Propositions 2.6 and 2.7 below.

**Lemma 2.5.** Let \( f : \mathfrak{g} \to \mathfrak{gl}(V) \) be a representation of a Lie superalgebra \( \mathfrak{g} \). If there exist \( a, b \in V_0 \) satisfying \( f(g)a = f(g)b = V \), then \( (f, \text{ev}_a) \) and \( (f, \text{ev}_b) \) are quasi-equivalent.

**Proof.** It is clear that \( f(g)a = f(g)b = V \). Let \( G_0 \) be the simply connected algebraic group with Lie algebra \( \mathfrak{g}_0 \) and let \( F : G_0 \to \text{GL}(V_0) \) be the representation of \( G_0 \) with \( dF = f|_{\mathfrak{g}_0} \). Then both \( F(G_0)a \) and \( F(G_0)b \) are open in \( V \) and hence \( F(G_0)a = F(G_0)b \). Choose \( t \in G_0 \) such that \( F(t)a = b \) and define \( T \in \text{Aut}(\mathfrak{g}) \) by \( T = \text{Ad}_t \) and let \( \phi : V \to V \) by \( \phi = F(t) \). Then \( f \circ T : \mathfrak{g} \to \mathfrak{gl}(V) \) sends every element \( x \in \mathfrak{g} \) to \( \phi \circ f(x) \circ \phi^{-1} \). Hence, \( f(x) = \phi^{-1} \circ (f \circ T)(x) \circ \phi \). Further, \( \text{ev}_b(T(x)) = \text{ev}_b(T(x)) = (f(T(x)))(b) = (\phi \circ f(x) \circ \phi^{-1})(b) = (\phi \circ f(x))(a) = \phi(\text{ev}_a(x)) \) for all \( x \in \mathfrak{g} \) and hence \( \text{ev}_a = \phi^{-1} \circ (\text{ev}_b \circ T) \), as desired. \( \square \)

**Proposition 2.6.** Let \( f : \mathfrak{g} \to \mathfrak{gl}(V) \) be a representation of a Lie superalgebra \( \mathfrak{g} \) with \( \dim_{\mathfrak{g}_\gamma} = \dim V_\gamma \) for \( \gamma \in \mathbb{Z}_2 \). If there exists an element \( a \in V_0 \) such that the evaluation map \( \text{ev}_a \) is bijective and bijective 1-cocycles and bijective evaluation maps associated with \( f \) coincide, then there exists a unique LSSA up to isomorphism on \( \mathfrak{g} \) associated with \( f \).

**Proof.** We denote by \( \mathcal{L} \) the LSSA on \( \mathfrak{g} \) given by \( (f, \text{ev}_a) \). Suppose there exists another LSSA \( \mathcal{L}' \) on \( \mathfrak{g} \) given by the bijective evaluation map \( (f, \text{ev}_b) \) with \( b \in V_0 \). Then \( \text{ev}_a(\mathfrak{g}) = \text{ev}_b(\mathfrak{g}) = V \), that is, \( f(\mathfrak{g})a = f(\mathfrak{g})b = V \). Lemma 2.5 implies that \( (f, \text{ev}_a) \) and \( (f, \text{ev}_b) \) are quasi-equivalent. It follows from Proposition 2.4 that \( \mathcal{L} \) is isomorphic to \( \mathcal{L}' \). \( \square \)

**Proposition 2.7.** Let \( f_i : \mathfrak{g} \to \mathfrak{gl}(V_i), i = 1, 2, \) be two quasi-equivalent representations of \( \mathfrak{g} \), i.e., there exist \( T \in \text{Aut}(\mathfrak{g}) \) and an isomorphism \( \phi : V_2 \to V_1 \) such that \( f_2(x) = \phi^{-1} \circ (f_1 \circ T)(x) \circ \phi \) for all \( x \in \mathfrak{g} \). Assume further that \( \dim(V_1)_\gamma = \dim_{\mathfrak{g}_\gamma} \) for \( i = 1, 2, \gamma \in \mathbb{Z}_2 \) and that bijective 1-cocycles and bijective evaluation maps associated with \( f_i \) coincide for each \( i = 1, 2 \). Then we have

1. if there exists a bijective evaluation map associated with one of them, then there exists a bijective evaluation map associated with the other one;
2. LSSAs associated with \( f_1 \) and \( f_2 \) are isomorphic.

**Proof.** (1) Suppose there exists an element \( b \in (V_2)_0 \) such that the evaluation map \( \text{ev}_b \) associated with \( f_2 \) is bijective. Then \( \text{ev}_b(\mathfrak{g}) = f_2(\mathfrak{g})b = V_2 \), and hence \( V_2 = (\phi^{-1} \circ f_1(T(\mathfrak{g})) \circ \phi)b = (\phi^{-1} \circ f_1(\mathfrak{g}) \circ \phi)b \). Let \( a := \phi(b) \in (V_1)_0 \). Then \( \text{ev}_a(\mathfrak{g}) = f_1(\mathfrak{g})a = \phi(V_2) = V_1 \), i.e., \( \text{ev}_a \) is surjective. Note that \( \dim V_1 = \dim \mathfrak{g} \). Then \( \text{ev}_a \) is injective and hence there exists a bijective evaluation map \( \text{ev}_a \) associated with \( f_1 \).
(2) It follows from Proposition 2.6 that there exists a unique LSSA associated with each \( f_i, i = 1, 2 \). We denote by \( \mathcal{L} \) and \( \mathcal{L}' \) the LSSAs on \( \mathfrak{g} \) given by \( (f_1, ev_a) \) and \( (f_2, ev_b) \), respectively. Since \( ev_b(x) = f_2(x)b = (\varphi^{-1} \circ f_1(T(x)))a = (\varphi^{-1} \circ ev_a \circ T)(x) \), we see that \( (f_1, ev_a) \) and \( (f_2, ev_b) \) are quasi-equivalent. By Proposition 2.4, we conclude that \( \mathcal{L} \) and \( \mathcal{L}' \) are isomorphic. \( \Box \)

3. REPRESENTATIONS OF \( \mathfrak{sl}(m|n) \)

We present some preliminaries and calculations on the representations of the Lie superalgebras \( \mathfrak{sl}(m|n) \). For more details, see [7, 8].

3.1. Definitions. Let \( \mathfrak{gl}(m|n) \) be the space of \( (m+n) \times (m+n) \) matrices. We write an element \( X \in \mathfrak{gl}(m|n) \) in a block-diagonal form \( X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \), where \( X_1, X_2, X_3, X_4 \) are matrices of sizes \((m \times m), (m \times n), (n \times m), \) and \((n \times n)\) respectively. Setting

\[
\mathfrak{gl}(m|n)_0 = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{gl}(m|n)_1 = \left\{ \begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix} \right\}
\]

endows \( \mathfrak{gl}(m|n) \) with a \( \mathbb{Z}_2 \)-grading. The corresponding supercommutator defined by

\[
[X, Y] = XY - (-1)^{\gamma_x\gamma_y}YX
\]

where \( X \in \mathfrak{gl}(m|n)_{\gamma_x} \) and \( Y \in \mathfrak{gl}(m|n)_{\gamma_y} \) turns \( \mathfrak{gl}(m|n) \) into a Lie superalgebra.

Furthermore \( \mathfrak{gl}(m|n) \) admits a \( \mathbb{Z} \)-grading

\[
\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{-1} \oplus \mathfrak{gl}(m|n)_0 \oplus \mathfrak{gl}(m|n)_1
\]

defined by

\[
\mathfrak{gl}(m|n)_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ X_3 & 0 \end{pmatrix} \right\}, \quad \mathfrak{gl}(m|n)_0 = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} \right\}, \quad \mathfrak{gl}(m|n)_1 = \left\{ \begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix} \right\}.
\]

The two gradings are compatible, i.e.,

\[
\mathfrak{gl}(m|n)_0 = \mathfrak{gl}(m|n)_0 \quad \text{and} \quad \mathfrak{gl}(m|n)_1 = \mathfrak{gl}(m|n)_{-1} \oplus \mathfrak{gl}(m|n)_1.
\]

The supertrace of \( X \in \mathfrak{gl}(m|n) \) is defined as \( \text{str}(X) = \text{tr}(X_1) - \text{tr}(X_4) \). The special linear Lie superalgebra \( \mathfrak{sl}(m|n) \) is the subalgebra of \( \mathfrak{gl}(m|n) \) of traceless matrices:

\[
\mathfrak{sl}(m|n) = \{ X \in \mathfrak{gl}(m|n) | \text{str}(X) = 0 \}.
\]

Clearly, \( \mathfrak{sl}(m|n)_0 = \mathfrak{sl}_m \oplus \mathfrak{sl}_n \oplus \mathbb{C} \) is a reductive Lie algebra. Since \( \mathfrak{sl}(m|n) \) is isomorphic to \( \mathfrak{sl}(m|m) \), we always assume that \( m \geq n \geq 1 \). If \( m \neq n \), then \( \mathfrak{sl}(m|n) \) is a simple Lie superalgebra. On the other hand, \( \mathfrak{sl}(m|m) \) has a one-dimensional centre \( \mathbb{C} \mathfrak{l}_{2m} \) and the Lie superalgebra \( \mathfrak{psl}(m|m) := \mathfrak{sl}(m|m)/\mathbb{C} \mathfrak{l}_{2m} \) is simple; its even part \( \mathfrak{psl}(m|m)_0 \cong \mathfrak{sl}_m \oplus \mathfrak{sl}_m \) is semisimple.

For the rest of the paper \( \mathfrak{g} \) will be the Lie superalgebra \( \mathfrak{gl}(m|n) \) or \( \mathfrak{sl}(m|n) \). If \( \theta \) is an automorphism of \( \mathfrak{g} \) and \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \) is a representation of \( \mathfrak{g} \), the \( \theta \)-twist \( V^\theta \) of \( V \) is the module corresponding to the representation \( \rho \circ \theta \). If \( \theta \) is an inner automorphism, then \( V^\theta \cong V \). The supertranspose of \( X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathfrak{g} \) is the matrix \( X^t := \begin{pmatrix} X_1^t & X_3^t \\ -X_2^t & X_4^t \end{pmatrix} \). The supertranspose \( st \) is
an antiautomorphism of \( g \) while \(-st\) is an automorphism which is not an inner automorphism. If \( V \) is an irreducible \( g \)-module then \( V^{-st} \cong V^* \). However, \( V^{-st} \not\cong V^* \) in general.

### 3.2. Roots, positive roots

Denote the subalgebra of \( g \) of diagonal matrices by \( \mathfrak{h} \). As usual, we denote by \( E_{ij}, 1 \leq i, j \leq m+n \) the elementary matrix, i.e., the matrix with 1 in position \((i, j)\) and zeroes elsewhere. If \( g = gl(m|n) \), let \( \{ \varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n \} \) be the basis of \( \mathfrak{h}^* \) dual to the basis \( \{ E_{11}, \ldots, E_{mm}, E_{m+1,m+1}, \ldots, E_{nn} \} \) of \( \mathfrak{h} \). For \( g = sl(m|n) \), slightly abusing notation, we denote the restriction of \( \varepsilon_i \) and \( \delta_j \) to \( \mathfrak{h} \) by \( \varepsilon_i \) and \( \delta_j \) as well. Note that \( \{ \varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n \} \) span \( \mathfrak{h}^* \) and satisfy the relation

\[
\varepsilon_1 + \ldots + \varepsilon_m = \delta_1 + \ldots + \delta_n.
\]

The Lie superalgebra \( g \) admits a root decomposition

\[
g = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} g^{\alpha}),
\]

where, for any \( \alpha \in \mathfrak{h}^* \),

\[
g^{\alpha} = \{ X \in g \mid [h, X] = \alpha(h)X \text{ for every } h \in \mathfrak{h} \}
\]

and \( \Delta = \{ \alpha \in \mathfrak{h}^* \setminus \{0\} \mid g^\alpha \neq 0 \} \).

The elements of \( \Delta \) are called roots of \( g \). Furthermore, the decomposition \( g = g_0 \oplus g_1 \) induces the decomposition of \( \Delta \) as

\[
\Delta = \Delta_0 \cup \Delta_1, \quad \text{where} \quad \Delta_\gamma = \{ \alpha \in \Delta \mid g^{\alpha} \subseteq g_\gamma \}, \gamma \in \mathbb{Z}_2.
\]

Explicitly,

\[
\Delta_0 = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq m \} \cup \{ \delta_i - \delta_j \mid 1 \leq i \neq j \leq n \},
\]

and

\[
\Delta_1 = \{ \pm(\varepsilon_i - \delta_j) \mid 1 \leq i \leq m, 1 \leq j \leq n \}.
\]

All root spaces are one-dimensional and spanned by elementary matrices. Namely, \( g^{\varepsilon_i - \varepsilon_j}, g^{\delta_i - \delta_j}, g^{\varepsilon_i - \delta_j}, g^{\delta_i - \varepsilon_j} \) are spanned by \( E_{ij}, E_{m+i,m+j}, E_{i,m+j}, E_{m+i,j} \) respectively.

Let \( \Delta_{-1} \) and \( \Delta_1 \) denote the roots of \( g_{-1} \) and \( g_1 \) respectively and let

\[
\Delta_0 = \Delta_0^+ \cup \Delta_0^-
\]

be the triangular decomposition of \( \Delta_0 \) defined by \( \Delta_0^+ = \{ \pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq m \} \). Fixing the set \( \Delta^+ = \Delta_0^+ \cup \Delta_1 \) of positive roots of \( g \), we denote the corresponding Borel subalgebra of \( g \) by \( \mathfrak{b} \):

\[
\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta^+} g^{\alpha}).
\]

Note that \( \mathfrak{b}_0 = \mathfrak{b} \cap g_0 \) is a Borel subalgebra of \( g_0 \) with roots \( \Delta_0^+ \). The \( \mathbb{Z} \)-grading of \( g \) defines the parabolic subalgebra \( \mathfrak{p} = g_0 \oplus g_1 \) with roots \( \Delta_0 \cup \Delta_1^+ \).
3.3. **Representations.** Given a (finite-dimensional) \( g_0 \)-module \( L \), setting \( g_1 \cdot L = 0 \), we turn it into a \( p \)-module and define the corresponding parabolically induced module \( K(L) \) by

\[
K(L) := \text{Ind}_0^L \cong \wedge(g_{-1}) \otimes_C L.
\]

Here \( \wedge(g_{-1}) \) denotes the exterior algebra of the vector space \( g_{-1} \) and the isomorphism is an isomorphism of \( g_{-1} \)-modules.

We define a symmetric bilinear form on \( \mathfrak{h}^* \) by \( (e_i, e_j) = 1, (\delta_j, \delta_j) = -1 \) and setting all other pairings between elements \( e_1, \ldots, e_m, \delta_1, \ldots, \delta_n \) to be equal to zero. A weight \( \lambda \in \mathfrak{h}^* \) is said to be integral if \( (\lambda, \beta) \in \mathbb{Z} \) for all roots \( \beta \in \Delta_0 \), and dominant if \( 2(\lambda, \beta) \geq 0 \) for all \( \beta \in \Delta_0^+ \).

We define a symmetric bilinear form on \( \mathfrak{h}^* \) by \( (e_i, e_j) = 1, (\delta_j, \delta_j) = -1 \) and setting all other pairings between elements \( e_1, \ldots, e_m, \delta_1, \ldots, \delta_n \) to be equal to zero. A weight \( \lambda \in \mathfrak{h}^* \) is said to be integral if \( (\lambda, \beta) \in \mathbb{Z} \) for all roots \( \beta \in \Delta_0 \), and dominant if \( 2(\lambda, \beta) \geq 0 \) for all \( \beta \in \Delta_0^+ \).

We denote by \( X^+ \) the set of dominant integral weights in \( \Delta_0 \). It parametrizes the isomorphism classes of irreducible finite-dimensional \( g_0 \)-modules. For a given \( \lambda \in X^+ \), the corresponding \( g_0 \)-module is denoted by \( L(\lambda) \). The Kac module \( K(\lambda) \) is simply \( K(L(\lambda)) \). It admits a unique proper maximal submodule \( I(\lambda) \) and, respectively, a unique irreducible quotient \( V(\lambda) = K(\lambda)/I(\lambda) \).

Every irreducible \( g \)-module is isomorphic to \( V(\lambda) \) or \( \Pi V(\lambda) \) for a unique \( \lambda \in X^+ \), where \( \Pi \) is the parity change functor. Note that, whenever using the notations \( K(\lambda) \) and \( V(\lambda) \), we assume that the highest weight space is even.

The weight \( \lambda \in X^+ \) is called typical if \( (\lambda + \rho, \alpha) \neq 0 \) for all \( \alpha \in \Delta_1^+ \), where

\[
\rho = \rho_0 - \rho_1,
\]

with

\[
\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha \quad \text{and} \quad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha.
\]

If \( \lambda \) is not typical, it is called atypical. The modules \( K(\lambda) \) and \( V(\lambda) \) are called typical (respectively, atypical) if the corresponding weight \( \lambda \) is typical (respectively, atypical). Note that \( K(\lambda) \) is irreducible if and only if it is typical; otherwise \( K(\lambda) \) is indecomposable but reducible.

The degree of atypicality of \( \lambda \) (and of the respective modules \( K(\lambda) \) and \( V(\lambda) \)) is defined as the number of distinct elements \( \alpha \in \Delta_1^+ \) for which \( (\lambda + \rho, \alpha) = 0 \). If there exists one and only one such \( \alpha \in \Delta_1^+ \), \( \lambda \) is called singly atypical. Note that any dominant integral weight of \( \mathfrak{sl}(m|1), m \geq 2 \) is either typical or singly atypical ([6, Lemma 3.2.1]). Germony studied singly atypical representations in [6] and we will rely on the results therein.

If \( \lambda \) is a singly atypical weight, then \( K(\lambda) \) contains a unique proper submodule which is irreducible. Let \( T^- \lambda \) denote the highest weight of the unique proper submodule of \( K(\lambda) \). In other words, \( K(\lambda) \) is a non-split extension of \( V(\lambda) \) by \( V(T^- \lambda) \) or by \( \Pi V(T^- \lambda) \), depending on the parity of the highest weight space of the proper submodule of \( K(\lambda) \). The operator \( T^- \) has an inverse denoted by \( T^+ \). The extensions among simple \( g \)-modules is described in [6, Proposition 6.1.2]:

**Proposition 3.1.** Let \( \lambda, \mu \in X^+ \) be dominant integral weights.
(1) If $\lambda$ is typical, then

$$\text{Ext}^1(V(\lambda), V(\mu)) = \begin{cases} C & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

(2) If $\lambda$ is singly atypical, then

$$\dim \text{Ext}^1(V(\lambda), V(\mu)) = \begin{cases} 1 & \text{if } \mu \in \{T^+\lambda, T^-\lambda\}, \\ 0 & \text{otherwise.} \end{cases}$$

As mentioned above, $K(\lambda)$ is a non-split extension of $V(\lambda)$ by $V(T^-\lambda)$ or by $\Pi V(T^-\lambda)$. To describe a non-split extension of $V(\lambda)$ by $V(T^+\lambda)$ or by $\Pi V(T^+\lambda)$, we introduce the following notation. Given $\mu \in X^+$, the opposite Kac module $K'(\mu)$ is the module

$$K'(\mu) := \text{Ind}_{0}^{g} L(\mu)$$

and the weight $\mu'$ is defined as the unique element of $X^+$ for which $V(\mu)$ is a quotient of $K'(\mu')$. The module $K'(\lambda')$ is a non-split extension of $V(\lambda)$ by $V(T^+\lambda)$ or by $\Pi V(T^+\lambda)$. Thus $K(\lambda)$ and $K'(\lambda')$ provide examples for Proposition 3.1 (2).

Next we provide an example for Proposition 3.1 (1). Let $g = \mathfrak{sl}(m|n)$ and denote by $\mathbb{C}^{(2)}$ the two-dimensional $g_{00}$-module on which every element acts trivially, except that a fixed nonzero central element $z$ acts via a nilpotent matrix of order 2 and set

$$K(\lambda)^{(2)} = K(L(\lambda) \otimes \mathbb{C}^{(2)}) = \text{Ind}_{0}^{g} (L(\lambda) \otimes \mathbb{C}^{(2)}).$$

It is clear that $K(\lambda)^{(2)}$ is a non-split extension of $K(\lambda)$ by itself. If $\lambda$ is singly atypical, the structure of $K(\lambda)^{(2)}$ is described in [6, Lemma 6.1.1]:

**Lemma 3.2.** Let $\lambda$ be a singly atypical dominant integral weight. The module $K(\lambda)^{(2)}$ is uniserial with composition factors (listed from top to socle) $V(\lambda), V(T^-\lambda), V(\lambda)$, and $V(T^-\lambda)$ or $V(\lambda), \Pi V(T^-\lambda), V(\lambda)$, and $\Pi V(T^-\lambda)$.

### 3.4 Irreducible representations of $\mathfrak{sl}(2|1)$

Let $\mathfrak{g} = \mathfrak{sl}(2|1)$. Fix the elements $h = E_{11} - E_{22}$ and $z = E_{11} + E_{22} + 2E_{33}$ which form a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $V(i,k)$ for $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{C}$ denote the irreducible $g$-module with highest weight $\lambda$ defined by $\lambda(h) = i$ and $\lambda(z) = k$. The corresponding Kac module and irreducible module are denoted respectively by $K(i,k)$ and $V(i,k)$. Denote by $S_i$ the irreducible $\mathfrak{sl}_2$-module of dimension $i + 1$; by convention, $S_{-1} = 0$. The following proposition describes the modules $V(i,k)$ and the extensions among them.

**Proposition 3.3.** Let $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{C}$.

(1) $(i,k)$ is typical if and only if $k \notin \{i, -i - 2\}$.

(2) If $k \notin \{i, -i - 2\}$, $V(i,k) = K(i,k)$ is of dimension $2(i+1)(2i+1)$ and, as an $\mathfrak{sl}_2$-module,

$$V(i,k)_0 \cong S_i \oplus S_{i+1}, \quad V(i,k)_1 \cong S_{i-1} \oplus S_{i+1}.$$
Moreover, $z$ acts on $V(i,k)$ as multiplication by $k+1$ and the decomposition above can be chosen so that $z$ acts on one of the copies of $V(i,k)$ as multiplication by $k$ and on the other one as multiplication by $k+2$.

(3) The module $V(i,i)$ is of dimension $i+1|1$ and, as an $\mathfrak{sl}_2$-module,

$$V(i,i)_0 \cong S_i, \quad V(i,i)_1 \cong S_{i-1}.$$ 

Moreover, $z$ acts on $V(i,i)_0$ as multiplication by $k$ and on $V(i,i)_1$ as multiplication by $k+1$.

The module $V(i, -i - 2)$ is of dimension $i + 1|1$ and, as an $\mathfrak{sl}_2$-module,

$$V(i,i)_0 \cong S_i, \quad V(i,i)_1 \cong S_{i+1}.$$ 

Moreover, $z$ acts on $V(i,i)_0$ as multiplication by $k$ and on $V(i,i)_1$ as multiplication by $k+1$.

(4) The operator $T^-$ acts on atypical weights as

$$T^-(i,k) = \begin{cases} (i+1, i+1) & \text{if } k = i \\ (i-1, -i-1) & \text{if } i > 0, k = -i - 2 \\ (0,0) & \text{if } i = 0, k = -2. \end{cases}$$

Proof. Since $\rho = -\epsilon_2 + \delta$, for $\lambda = (i,k)$, we have

$$(\lambda + \rho, \epsilon_2 - \delta) = \frac{k-i}{2} \quad \text{and} \quad (\lambda + \rho, \epsilon_1 - \delta) = \frac{k+i}{2} + 1,$$

proving (1). The remaining statements follow from an easy calculation using the explicit $\mathfrak{gl}_2$-structure of the module $K(i,k)$. We leave these to the reader.

It will be convenient to index the atypical weights of $\mathfrak{sl}(2|1)$ by $\mathbb{Z}$. Namely, set

$$\lambda_i := \begin{cases} (i,i) & \text{if } i \geq 0 \\ (-i-1, i-1) & \text{if } i < 0. \end{cases}$$

In this notation $T^- (\lambda_i) = \lambda_{i+1}$ for any $i \in \mathbb{Z}$. Setting $V_i := V(\lambda_i)$, we conclude that there are non-trivial extensions between $V_i$ and $V_j$ if and only if $|i-j| = 1$. Taking into account parity, the non-trivial extensions between $V_{i-1}$ and $V_i$ require that these modules are taken in different parities except for $i = 0$ when the parities have to be the same.

We complete the discussion of irreducible $\mathfrak{sl}(2|1)$-modules by describing their twists by the automorphism $-st$.

**Proposition 3.4.** Let $V$ be an irreducible $\mathfrak{sl}(2|1)$-module. Then

$$V^{-st} \cong \begin{cases} K(i, -k-2) & \text{if } V = K(i,k) \text{ is typical} \\ \Pi V_{-i} & \text{if } V = V_i, i \neq 0 \\ V_0 & \text{if } V = V_0. \end{cases}$$

Proof. If $w$ is the lowest weight vector of $V$ and the weight of $w$ is $\mu$, then $w$ is the highest weight vector of $V^{-st}$ of weight $-\mu$. An explicit calculation which determines the lowest weights of the irreducible representations of $\mathfrak{sl}(2|1)$ completes the proof. We omit this calculation here. □
3.5. **Small irreducible modules of \( \mathfrak{sl}(m|1) \) for \( m \geq 3 \).** Let \( \mathbb{C}^m \) be the standard \( m \)-dimensional module of \( \mathfrak{sl}_m \) and \( \mathbb{C}^{m|n} \) the standard \( m|n \)-dimensional module \( \mathfrak{sl}(m|n) \). As \( \mathfrak{sl}_m \)-modules, we have \( \mathfrak{sl}(m|1)_{-1} \cong (\mathbb{C}^m)^* \) and \( \mathfrak{sl}(m|1)_1 \cong \mathbb{C}^m \), where \( (\mathbb{C}^m)^* \) is the dual module of the standard module \( \mathbb{C}^m \) of \( \mathfrak{sl}_m \). In this section, we denote by \( \text{tr} \) the 1-dimensional trivial module of \( \mathfrak{sl}_m \) and by \( \tilde{\text{tr}} \) – the \( 0|1 \)-dimensional trivial module of \( \mathfrak{sl}(m|1) \).

**Proposition 3.5.** Let \( V \) be an irreducible \( \mathfrak{sl}(m|1) \)-module with \( m \geq 3 \).

1. If \( V \) is purely odd, i.e., \( V_0 = 0 \) (respectively, purely even), then \( V \) is the \( 0|1 \)-dimensional (respectively, \( 1|0 \)-dimensional) trivial module.

2. Assume that \( 1 < \dim V \leq m^2/2m \), i.e., \( V \) is non-trivial and \( \dim V_0 \leq m^2 \) and \( \dim V_1 \leq 2m \). Furthermore, assume that, as an \( \mathfrak{sl}_m \)-module, \( V_0 \) is isomorphic to a direct sum of copies of \( \mathbb{C}^m \). Then \( V \) is isomorphic to one of the following modules:
   
   (a) \( \mathbb{C}^{m|1} \);
   (b) \( \Pi \wedge^2 (\mathbb{C}^{m|1}) \) for \( m = 3, 4 \);
   (c) \( \Pi S^2(\mathbb{C}^3|1) \) for \( m = 3 \).

**Proof.** (1) If \( V \) is purely even or purely odd, then \( \mathfrak{sl}(m|1) \cdot V = 0 \). Since \( \mathfrak{sl}(m|1)_{-1} \) generates \( \mathfrak{sl}(m|1) \), we conclude that \( \mathfrak{sl}(m|1) \cdot V = 0 \). The irreducibility of \( V \) implies \( \dim V = 0|1 \) or \( \dim V = 1|0 \).

(2) We assume that \( V \) is induced from an irreducible \( \mathfrak{gl}_m \)-module \( L(\lambda) \) with the highest weight \( \lambda \), then, up to parity, \( V \) is isomorphic to the Kac module \( K(\lambda) \) or a quotient \( K(\lambda)/I(\lambda) \) of \( K(\lambda) \). Since \( L(\lambda) \) is an irreducible \( \mathfrak{gl}_m \)-module, it is irreducible as an \( \mathfrak{sl}_m \)-module. Our proof will be separated into two cases: (I) \( L(\lambda) \subseteq V_0 \) and (II) \( L(\lambda) \subseteq V_1 \).

Case (I). Since, as an \( \mathfrak{sl}_m \)-module, \( V_0 \) is isomorphic to a direct sum of copies of \( \mathbb{C}^m \), we have \( L(\lambda) \cong \mathbb{C}^m \). Thus \( \lambda = \epsilon_1 + \mu \delta \), where \( \delta = \delta_1 \), cf. Section 3.2. We analyze the module structure of Kac module \( K(\lambda) = \wedge^\cdot (\mathfrak{sl}(m|1)_{-1}) \otimes \mathbb{C}^m \). The \( \mathbb{Z} \)-grading on \( \wedge (\mathfrak{sl}(m|1)_{-1}) \) induces a grading

\[
K(\lambda) = \oplus_{i=0}^m K_i.
\]

The component \( K_1 \) is contained in the odd part of \( K(\lambda) \) and, as an \( \mathfrak{sl}_m \)-module, it is isomorphic to the direct sum of the trivial module and the adjoint module of dimension \( m^2 - 1 \). Since \( m^2 - 1 \geq 2m \) for \( m \geq 3 \), the adjoint \( \mathfrak{sl}_m \)-module must be contained in the unique submodule \( I(\lambda) \) of \( K(\lambda) \). Consequently, \( \mu = 0 \) and \( \lambda = \epsilon_1 \), proving that \( V(\lambda) \cong \mathbb{C}^{m|1} \) with \( V_0 \cong \mathbb{C}^m \) and \( V_1 \cong \text{tr} \) as \( \mathfrak{sl}_m \)-modules.

Case (II). Since \( L(\lambda) \subseteq V_1 \), the inequalities

\[
\dim L(\lambda) \leq \dim V_1 \leq 2m
\]

imply that \( L(\lambda) \), as an \( \mathfrak{sl}_m \)-module, is isomorphic to one of the following:

\[
\text{tr}, \mathbb{C}^m, (\mathbb{C}^m)^* \quad \text{for } m \geq 6;
\text{tr}, \mathbb{C}^m, (\mathbb{C}^m)^*, \wedge^2 (\mathbb{C}^m), \wedge^2 ((\mathbb{C}^m)^*) \quad \text{for } m = 4, 5;
\text{tr}, \mathbb{C}^3, (\mathbb{C}^3)^*, S^2(\mathbb{C}^3), S^2((\mathbb{C}^3)^*) \quad \text{for } m = 3.
\]
Here $\wedge^2(W)$ and $S^2(W)$ denote respectively the second exterior and symmetric powers of $W$. Below we consider each of these cases for $L(\lambda)$.

(i) If $L(\lambda) \equiv \text{tr}$, then $\lambda = \mu \delta$. Then, as an $\mathfrak{sl}_m$-module, $K_1 \cong (\mathbb{C}^m)^\ast$. However $K_1$ is contained in the even part of $K(\lambda)$ which must be a sum of copies of $\mathbb{C}^m$. Since $\mathbb{C}^m \not\cong (\mathbb{C}^m)^\ast$, we conclude that $K_1 \subset I(\lambda)$ which leads to $\lambda = 0$. This contradicts the assumption that $V$ is non-trivial.

(ii) If $L(\lambda) \cong \mathbb{C}^m$, then $\lambda = \varepsilon_1 + \mu \delta$. As in Case (i), $K_1$ is the direct sum of the trivial and the adjoint modules of $\mathfrak{sl}_m$. These two modules are not isomorphic to $\mathbb{C}^m$. Thus $K_1 \subset I(\lambda)$ and $V(\lambda) = K(\lambda)/I(\lambda) = K_0$. In particular, $V$ is purely odd and, by (1), $V$ is trivial.

(iii) If $L(\lambda) \cong (\mathbb{C}^m)^\ast$, then $\lambda = -\varepsilon_m + \mu \delta$. Then, as an $\mathfrak{sl}_m$-module, $K_1$ is the direct sum of $S^2((\mathbb{C}^m)^\ast)$ and $\wedge^2((\mathbb{C}^m)^\ast)$, neither of which is isomorphic to $\mathbb{C}^m$ if $m \geq 4$. Arguing as in (ii) above, we conclude that, for $m \geq 4$, $V$ is purely odd and thus trivial. When $m = 3$, we observe that $\wedge^2((\mathbb{C}^3)^\ast) \cong \mathbb{C}^3$ but $S^2((\mathbb{C}^m)^\ast) \not\cong \mathbb{C}^3$. Thus $S^2((\mathbb{C}^m)^\ast) \subset I(\lambda)$, $\lambda = -\varepsilon_3 + \delta = \varepsilon_1 + \varepsilon_2$, and $V \cong \Pi \wedge^2(\mathbb{C}^3[1])$.

(iv) If $m = 5$ and $L(\lambda) \cong \wedge^2(\mathbb{C}^5)$, then $\lambda = \varepsilon_1 + \varepsilon_2 + \mu \delta$. Then, as an $\mathfrak{sl}_5$-module, $K_1$ is the direct sum of $\mathbb{C}^5$ and the 45-dimensional module with highest weight $\varepsilon_1 + \varepsilon_2 - \varepsilon_5$. The assumption $K_1 \subset I(\lambda)$ leads to a contradiction as $V$ would be purely odd. Alternatively, $K_1 \cap I(\lambda)$ equals the 45-dimensional module above. Then $\lambda = \varepsilon_1 + \varepsilon_2$ and $V = \Pi \wedge^2(\mathbb{C}^5[1])$. However, $\dim(\Pi \wedge^2(\mathbb{C}^5[1]))_1 = 11 > 10$, contradicting the assumption on $V$.

(v) If $m = 5$ and $L(\lambda) \cong \wedge^2((\mathbb{C}^5)^\ast)$, then $\lambda = -\varepsilon_4 - \varepsilon_5 + \mu \delta$. Then, as an $\mathfrak{sl}_5$-module, $K_1$ is the direct sum of $\wedge^2(\mathbb{C}^5)$ and the 40-dimensional module with highest weight $-\varepsilon_4 - 2\varepsilon_5$. The assumption on $V$ implies that $K_1 \subset I(\lambda)$ which leads to a contradiction as $V$ would be purely odd.

(vi) If $m = 4$ and $L(\lambda) \cong \wedge^2(\mathbb{C}^4) \cong \wedge^2((\mathbb{C}^4)^\ast)$, then $\lambda = \varepsilon_1 + \varepsilon_2 + \mu \delta$. Then, as an $\mathfrak{sl}_4$-module, $K_1$ is the direct sum of $\mathbb{C}^4$ and the 20-dimensional module with highest weight $\varepsilon_1 + \varepsilon_2 - \varepsilon_4$. As in (iv) above, we conclude that $K_1 \cap I(\lambda)$ equals the 20-dimensional module above. Then $\lambda = \varepsilon_1 + \varepsilon_2$ and $V = \Pi \wedge^2(\mathbb{C}^4[1])$.

(vii) If $m = 3$ and $L(\lambda) \cong S^2(\mathbb{C}^3)$, then $\lambda = 2\varepsilon_1 + \mu \delta$. Then, as an $\mathfrak{sl}_3$-module, $K_1$ is the direct sum of $\mathbb{C}^3$ and the 15-dimensional module with highest weight $2\varepsilon_1 - \varepsilon_3$. As in (vi) above, we conclude that $V = \Pi S^2(\mathbb{C}^3[1])$.

(viii) If $m = 3$ and $L(\lambda) \cong S^2((\mathbb{C}^3)^\ast)$, then $\lambda = -2\varepsilon_3 + \mu \delta$. Then, as an $\mathfrak{sl}_3$-module, $K_1$ is the direct sum of the adjoint module and $S^3((\mathbb{C}^3)^\ast)$. Arguing as in (v) above, we reach a contradiction. \hfill $\square$

**Remark 3.6.** Let $V$ be an irreducible $\mathfrak{sl}(m|1)$-module for $m \geq 3$. Assume that the dimension of $V$ is less than or equal to $m^2|2m$ and $V_0$, as an $\mathfrak{sl}_m$-module, is isomorphic to a direct sum of copies of $(\mathbb{C}^m)^\ast$. Then $V^\ast$ satisfies the assumptions of Proposition 3.5 and hence $V$ is isomorphic to a module dual to one of the modules listed in Proposition 3.5.

We complete the discussion of $\mathfrak{sl}(m|1)$-modules by recording some information about the modules $\wedge^2(\mathbb{C}^m[1])$ and $S^2(\mathbb{C}^m[1])$. The proof is trivial and we omit it here.
Proposition 3.7. We have

(1) \( \dim \wedge^2(C^m|1) = \frac{m^2 - m + 2}{2}m \) and \( \dim S^2(C^m|1) = \frac{m^2 + m}{2}m \);

(2) \( (\wedge^2(C^m|1))_0 \cong \wedge^2(C^m) \oplus \tr, \ (\wedge^2(C^m|1))_1 \cong \bar{C}^m \)

and

\( (S^2(C^m|1))_0 \cong S^2(C^m), \ (S^2(C^m|1))_1 \cong \bar{C}^m \) as \( \sln_m \)-modules.

4. LSSAs on \( \sln |1 \)

The purpose of this section is to give a proof of Theorem 1.1, that is, we want to prove that there are no LSSAs on \( \sln |1 \) for \( m \geq 3 \). By Lemma 2.4, it suffices to show that the set \( \mathcal{O} \) of bijective 1-cocycles of \( \sln |1 \) is empty.

Throughout this section we assume \( m \geq 3 \). Let \( P_m := mC^m|1 \oplus \tr \) be the \( \sln |1 \)-module which is the direct sum of \( m \) copies of \( C^m|1 \) and \( m \) copies of \( \tr \) and let \( P_m^* \) be module dual to \( P_m \).

Proposition 4.1. Let \( W \) be an \( \sln |1 \)-module of dimension \( m^2 |2m \) such that, as an \( \sln_m \)-module, \( W_0 \) is isomorphic to the direct sum of \( m \) copies of \( C^m \) or \( m \) copies of \( (C^m)^* \).

1. If \( m \geq 4 \), then \( W \) is isomorphic to either \( P_m \) or \( P_m^* \).

2. If \( m = 3 \), then \( W \) is isomorphic to one of \( P_3, P_3^*, Q_3, \) or \( Q_3^* \), where

\( Q_3 = 2C^3|1 \oplus \Pi \wedge^2(C^3|1), \) see Proposition 3.5, and \( Q_3^* \) is the module dual to \( Q_3 \).

Proof. We will prove the proposition in the case when \( W_0 \) is isomorphic to the direct sum of \( m \) copies of \( C^m \). The case when \( W_0 \) is isomorphic to the direct sum of \( m \) copies of \( (C^m)^* \) then follows by duality.

If \( 0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = W \) is a composition series of \( W \), then each \( V^i := W^i/W^{i-1} \) is an irreducible \( \sln |1 \)-module for \( 1 \leq i \leq k \). If \( \dim V^i = a_i | b_i \), then \( \dim W = m^2 | m \) implies that \( \sum_{i=1}^k a_i = m^2 \) and \( \sum_{i=1}^k b_i = 2m \). Hence \( \dim V^i \leq m^2 | 2m \) for \( 1 \leq i \leq k \) and \( V^i \) is isomorphic to \( \tr \) or one of the modules from Proposition 3.5. Combining Propositions 3.5 and 3.7, we also get \( \dim(V^i)_0 \leq m, \dim(V^i)_1 \geq 1, \) and \( k \geq m \).

As an \( \sln_m \)-module, \( W_0 \cong \oplus_{i=1}^k (V^i)_0 \), implying that each \( (V^i)_0 \) itself is isomorphic to a (possibly empty) direct sum of \( C^m \) for \( 1 \leq i \leq k \). Proposition 3.5 implies that each \( V^i \) is isomorphic to one of the following modules

\( C^m|1 \) or \( \tr \) for \( m \geq 5 \);
\( C^4|1, \tr, \) or \( \Pi \wedge^2(C^4|1) \) for \( m = 4 \);
\( C^3|1, \tr, \Pi \wedge^2(C^3|1), \) or \( \Pi S^2(C^3|1) \) for \( m = 3 \).

First we note that, for \( m = 4 \), \( V^i \) cannot be isomorphic to \( \Pi \wedge^2(C^4|1) \). If, to the contrary, \( V^i \cong \Pi \wedge^2(C^4|1) \) for some \( i \), then

\( \dim W_i = \dim(V^i)_1 + \sum_{j \neq i} \dim(V^j)_1 \geq 7 + (k - 1) \geq 7 + 3 > 8, \)
which contradicts the assumption on $W$. A similar argument shows that, for $m = 3$, $V^i$ cannot be isomorphic to $\Pi S^2(C^{3|1})$. This proves that each $V^i$ is isomorphic to one of the modules $C^{3|1}, \tilde{\tau}$, or $\Pi \Lambda^2(C^{3|1})$.

Counting dimensions we conclude that either

(i) $k = 2m$ and $V^i \cong C^{m|1}$ for $m$ values of $i$ and $V^i \cong \tilde{\tau}$ for the remaining $m$ values of $i$

or

(ii) $m = k = 3$ and $V^i \cong \Pi \Lambda^2(C^{3|1})$ for one value of $i$ and $V^i \cong C^{5|1}$ for two values of $i$.

However, by Proposition 3.1, there are no non-trivial extensions between $C^{m|1}$ and $\tilde{\tau}$ and there are no non-trivial extensions between $\Pi \Lambda^2(C^{3|1})$ and $C^{3|1}$. Thus $W$ is isomorphic to $P_m$ or $Q_3$.

\textbf{Lemma 4.2.} All bijective 1-cocycles of $\mathfrak{gl}_m$ are bijective evaluation maps for $m > 1$.

\textbf{Proof.} Let $q : \mathfrak{gl}_m \to V$ be a bijective 1-cocycle of $\mathfrak{gl}_m$ associated with the representation $f : \mathfrak{gl}_m \to \mathfrak{gl}(V)$. Then $q$ induces a left-symmetric algebra structure on $\mathfrak{gl}_m$, see [1, Theorem 2.1]. The results of Bauers, [2], imply that $q$ is a bijective evaluation map. Namely, it follows from [2, Section 2.2] that there exists an étale affine representations of $\mathfrak{gl}_m$ with base point $0 \in V$ and evaluation map $ev_0 = q$. Furthermore, all étale affine representations of $\mathfrak{gl}_m$ are linear, see [2, Propositions 5.1 and 2.2] and hence all bijective 1-cocycles of $\mathfrak{gl}_m$ are bijective evaluation maps.

\textbf{Lemma 4.3.} If $q$ is a bijective 1-cocycle of $\mathfrak{sl}(m\mid 1)$ associated with one of the modules $P_m, P_m^*$, for $m \geq 3$, $Q_3$, or $Q_3^*$, then $q$ is a bijective evaluation map.

\textbf{Proof.} Let $q$ be a bijective 1-cocycle of $\mathfrak{sl}(m\mid 1)$ associated with the module $V$. Since the even part of $\mathfrak{sl}(m\mid 1)$ is $\mathfrak{gl}_m$, we see that the restriction $q|_{\mathfrak{gl}_m}$ is a bijective 1-cocycle of $\mathfrak{gl}_m$. Then $q|_{\mathfrak{gl}_m}$ is a bijective evaluation map by Lemma 4.2. Let $q|_{\mathfrak{gl}_m} = ev_a : \mathfrak{gl}_m \to V_0$ for some point $a \in V_0$ and extend $ev_a$ to the evaluation map $\tilde{ev}_a : \mathfrak{sl}(m\mid 1) \to V$ at the same point $a$ by setting $\tilde{ev}_a(y) = y \cdot a$ for all $y \in \mathfrak{sl}(m\mid 1)$. Define $p := q - \tilde{ev}_a$. Then $p$ is a 1-cocycle of $\mathfrak{sl}(m\mid 1)$ associated with $V$ and $p|_{\mathfrak{gl}_m} = 0$. It follows from Lemma 2.2 that $p|_{\mathfrak{sl}(m\mid 1)} : \mathfrak{sl}(m\mid 1) \to V_1$ is a homomorphism of $\mathfrak{gl}_m$-modules and hence also of $\mathfrak{sl}_m$-modules.

If $V \in \{P_m, P_m^*\}$, then as $\mathfrak{sl}_m$-modules, $\mathfrak{sl}(m\mid 1) \cong C^{m} \oplus (C^{m})^*$ while $V_1$ is isomorphic to the direct sum of $2m$ copies of the trivial module. Schur’s lemma implies that $p|_{\mathfrak{sl}(m\mid 1)} = 0$ and hence $q = \tilde{ev}_a$ is an evaluation map of $\mathfrak{sl}(m\mid 1)$.

If $m = 3$ and $V = Q_3$, then as $\mathfrak{sl}_3$-modules, $\mathfrak{sl}(3\mid 1) \cong C^{3} \oplus (C^{3})^*$ while $V_1$ is isomorphic to the direct sum of $(C^{3})^*$ and 3 copies of the trivial module. Moreover, the central element $E_{11} + E_{22} + E_{33} + 3E_{44}$ acts on the copy of $(C^{3})^*$ in $\mathfrak{sl}(3\mid 1)$ as multiplication by zero, while it acts on the copy of $(C^{3})^*$ in $(Q_3)_{1}$ as multiplication by 2. Applying Schur’s lemma as above completes the argument in this case. The case when $m = 3$ and $V = (Q_3)^*$ is dealt with in a similar manner.

\[\square\]
Proposition 4.4. There are no bijective evaluation maps of $\mathfrak{sl}(m|1)$ associated with the modules $P_m, P^*_m$ for $m \geq 3$, $Q_3$, and $Q^*_3$.

Proof. First we show that there are no bijective evaluation maps associated with $P_m$ for $m \geq 3$. Since $P_m = m \mathbb{C}^{m|1} \oplus \tilde{m}$, any point $a$ of $(P_m)_0$ is annihilated by the odd positive root vector $E_{1,m+1}$, i.e., $\text{ev}_a(E_{1,m+1}) = E_{1,m+1} \cdot a = 0$. Thus, $\text{ev}_a$ associated with the module $P$ is not bijective for any point $a \in (P_m)_0$. Similarly, the odd negative root vector $E_{m+1,1}$ annihilates any point of $(P^*_m)_0$ and hence there are not bijective evaluation maps associated with the module $P^*_m$.

Assume now that $m = 3$, $V = Q_3$, and $a \in V_0$. A direct calculation shows that

$$\dim \text{span}\{\text{ev}_a(E_{14}), \text{ev}_a(E_{24}), \text{ev}_a(E_{34})\} \leq 2 < 3 = \dim \text{span}\{E_{14}, E_{24}, E_{34}\},$$

proving that $\text{ev}_a$ is not a bijective evaluation map. The case when $V = Q^*_3$ is dealt with in a similar way. \hfill \Box

Proof of Theorem 1.1. Let $m \geq 3$. Assume to the contrary that $\mathcal{L}$ is an LSSA on $\mathfrak{sl}(m|1)$. Let $V$ be the $m^2|2m$-dimensional $\mathfrak{sl}(m|1)$-module given by $\mathcal{L}$. Then there exists a bijective 1-cocycle of $\mathfrak{sl}(m|1)$ associated with $V$, and $V_0$ induces an LSA $\mathcal{L}_0$ on $\mathfrak{gl}_m$. It follows from [2, Theorem 4.5] that $V_0$, as an $\mathfrak{sl}_m$-module, is isomorphic to $m \mathbb{C}^m$ or $(m \mathbb{C}^m)^*$. Hence, by Proposition 4.1, $V$ is isomorphic to one of $P_m, P^*_m, Q_3$, or $Q^*_3$. Proposition 4.4 completes the proof. \hfill \Box

5. Proof of Theorem 1.2

Throughout this section $\mathfrak{g} = \mathfrak{sl}(2|1)$ and we use the notation introduced in Section 3 some of which we recall for convenience. Let $h = E_{11} - E_{22}$ and $z = E_{11} + E_{22} + 2E_{33}$. A dominant integral weight $\lambda$ of $\mathfrak{g}$ is of the form $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{C}$, where $i = \lambda(h)$ and $k = \lambda(z)$; the corresponding irreducible highest weight module and Kac module are denoted respectively by $V(i,k)$ and $K(i,k)$. The weight $(i,k)$ is atypical if and only if $k = i$ or $k = -i - 2$. We index the atypical irreducible $\mathfrak{g}$-modules by $\mathbb{Z}$: $V_i := V(i,i)$ for $i \geq 0$ and $V_i := V(-i - 1, i - 1)$ for $i < 0$. Finally, $S_j$ denotes the $j + 1$-dimensional irreducible $\mathfrak{sl}_2$-module.

The proof of Theorem 1.2 is carried out in the rest of this section. Namely, in Section 5.1 we construct the LSSAs $A_k, B_{k_1,k_2},$ and $C_k$. In Section 5.2 we describe the 4/4-dimensional modules which may be associated with LSSAs and prove that every LSSA on $\mathfrak{sl}(2|1)$ is isomorphic to an LSSA among $A_k, B_{k_1,k_2},$ and $C_k$. Finally, in Section 5.3 we establish the isomorphisms among $A_k, B_{k_1,k_2},$ and $C_k$.

5. The LSSAs $A_k, B_{k_1,k_2},$ and $C_k$. In this section we define the LSSAs $A_k, B_{k_1,k_2},$ and $C_k$ by providing a $\mathfrak{g}$-module $M$ along with a vector $a \in M_0$ for which the evaluation map $\text{ev}_a$ is bijective.

5.1. $A_k$ for $k \in \mathbb{C} \setminus \{-1, -3\}$. Let $M = K(1,k)$. Then, as an $\mathfrak{sl}_2$-module, $M_0 \cong S_1 \oplus S_1$. Moreover, as a $\mathfrak{gl}_2$-module, $M_0 = M_1 \oplus M_2$, where $z$ acts on $M_1$ and $M_2$ as multiplication by $k$ and $k + 2$ respectively. Denoting the highest weight vector of $M$ by $v_0$, we note that

$$\{v_0, v_1 := E_{21}v_0, w_0 := E_{31}E_{32}v_0, w_1 := E_{21}E_{31}E_{32}v_0\}$$
is a basis of $M_0$. Similarly,

$$\{E_{32}v_0, E_{21}E_{32}v_0, E_{21}^2E_{32}v_0, E_{31}v_0\}$$

is a basis of $M_1$.

Consider $a := v_0 + w_1$. The action of $g$ on $a$ is as follows:

$$\begin{align*}
h \cdot a &= v_0 - w_1 \\
z \cdot a &= kv_0 + (k + 2)w_1 \\
E_{12} \cdot a &= w_0 \\
E_{21} \cdot a &= v_1 \\
E_{13} \cdot a &= \frac{k+1}{2}E_{21}E_{32}v_0 + \frac{k-1}{2}E_{31}v_0 \\
E_{23} \cdot a &= \frac{k+3}{4}E_{21}^2E_{32}v_0 \\
E_{31} \cdot a &= E_{31}v_0 \\
E_{32} \cdot a &= E_{32}v_0.
\end{align*}$$

(5.1)

It is immediate that, for $k \neq -1, -3$, the vectors in the right hand side of (5.1) are linearly independent and hence $ev_a$ is bijective. Thus, for $k \neq -1, -3$, the pair $(M, a)$ defines an LSSA on $\mathfrak{sl}(2|1)$ which we denote by $A_k$.

5.1.2. $B_{k_1,k_2}$ for $k_1, k_2 \in \mathbb{C} \setminus \{0\}$, $k_1 + k_2 \neq -2$. In this case we set $M := \Pi K(0, k_1) \oplus \Pi K(0, k_2)$, where, as usual $\Pi$ stands for the change-of-parity functor. As a $\mathfrak{gl}_2$-module, $M_0 = M_1 \oplus M_2$, where $\dim M_1 = \dim M_2 = 2$ and $z$ acts on $M_1$ and $M_2$ as multiplication by $k_1 + 1$ and $k_2 + 1$ respectively. As above, let $v_0$ be the highest weight vector of $M_1$ and $w_1$ be the lowest weight vector of $M_2$. Exactly as in the case of $A_k$, one checks that, for $a = v_0 + w_1$, the map $ev_a$ is bijective as long as $k_1 + k_2 \neq -2$ and $k_1, k_2 \neq 0$. We leave completing the details to the reader. The resulting LSSA is denoted by $B_{k_1,k_2}$.

5.1.3. $C_k$ for $k \in \mathbb{C} \setminus \{0, -1\}$. In this case we set $M := \Pi K(0, k)^{(2)}$. As a $\mathfrak{gl}_2$-module, $M_0$ is a non-trivial extension of $M_1$ by $M_2$, where $M_1 \cong M_2$, $\dim M_1 = \dim M_2 = 2$ and $z$ acts on $M_1$ and $M_2$ as multiplication by $k + 1$. Note that $z$ acts on the (two-dimensional) highest weight space of $M_0$ by the $2 \times 2$-matrix

$$\begin{pmatrix} k+1 & 0 \\ 1 & k+1 \end{pmatrix}.$$ 

Let $v_0$ be a preimage in $M_0$ of the highest weight vector of $M_1$ and $w_1$ be the lowest weight vector of $M_2 \subset M_0$. Exactly as in the case of $A_k$, one checks that, for $a = v_0 + w_1$, the map $ev_a$ is bijective as long as and $k \neq 0, -1$. We leave completing the details to the reader. The resulting LSSA is denoted by $C_k$.

5.2. $g$-modules associated with LSSAs. Let $L$ be an LSSA on $g = \mathfrak{sl}(2|1)$ and let $M$ denote the corresponding $g$-module. Since $L_0$ is a left-symmetric algebra with corresponding $\mathfrak{gl}_2$-module $M_0$, Baue’s classification theorem implies that, as an $\mathfrak{sl}_2$-module $M_0 \cong S_3$ or $M_0 \cong S_1 \oplus S_1$. This fact, along with $\dim M = 4|4$, imply that the composition factors of $M$ are among the following modules (cf. Section 3.4):

$$K(1, k), k \neq 1, -3, \quad \Pi K(0, k), k \neq 0, -2, \quad \Pi V_{-3}, \quad V_{-2}, \quad \Pi V_{-1}, \quad \Pi V_0, \quad V_{1}, \quad \Pi V_2, \quad V_{3}.$$ 

More precisely, to obtain a module of dimension $4|4$, we need to combine composition factors from one of the following sets:
(1) \(\{K(1,k)\}, k \neq 1, -3\);
(2) \(\{\Pi V_{-3}, \Pi V_0\} \text{ or } \{V_3, \Pi V_0\}\);
(3) \(\{V_{-2}, \Pi V_{-1}\}, \{V_{-2}, V_1\}, \{\Pi V_2, \Pi V_{-1}\}, \text{ or } \{\Pi V_2, V_1\}\);
(4) \(\{\Pi K(0,k_1), \Pi K(0,k_2)\}, k_1, k_2 \notin \{0, -2\}\);
(5) \(\{\Pi K(0,k), \Pi V_{-1}, \Pi V_0\} \text{ or } \{\Pi K(0,k), V_1, \Pi V_0\}\);
(6) \(\{\Pi V_{-1}, \Pi V_{-1}, \Pi V_0, \Pi V_0\}, \{\Pi V_{-1}, V_1, \Pi V_0, \Pi V_0\}, \text{ or } \{V_1, V_1, \Pi V_0, \Pi V_0\}\).

First we prove Theorem 1.2 (1):

**Proposition 5.1.** Let \(\mathcal{L}\) be an LSSA with corresponding \(g\)-module \(M\) and 1-cocycle \(q\). Then \(q\) is an evaluation map.

**Proof.** The list of possible composition factors of \(M\) above shows that every composition factor of \(M\) considered as an \(\mathfrak{sl}_2\)-module is isomorphic to \(S_0\) or \(S_2\). Noting that \(\mathfrak{sl}(2|1)\), considered as an \(\mathfrak{sl}_2\)-module, is isomorphic to \(S_1 \oplus S_1\), an argument as in Lemma 4.3 proves that \(q\) is an evaluation map.

Proposition 5.1 and Lemma 2.5 imply immediately:

**Corollary 5.2.** If \(\mathcal{L}_1\) and \(\mathcal{L}_2\) are two LSSAs on \(g\) corresponding to the same \(g\)-module \(M\), then \(\mathcal{L}_1 \cong \mathcal{L}_2\).

**Remark 5.3.** Recall from Section 3.1 that \(M^{-st}\) is isomorphic to the twist of the \(g\)-module \(M\) by the outer automorphism \(-st\) of \(g\). If \((M, q)\) is the pair of a \(g\)-module and a bijective 1-cocycle corresponding to an LSSA \(\mathcal{L}\), then the LSSA \(\mathcal{L}^{st}\) corresponding to the pair \((M^{-st}, q \circ (-st))\) is isomorphic to \(\mathcal{L}\). In particular, to list all LSSAs it suffices to determine which \(g\)-modules \(M\) (up to a twist by \(-st\)) admit bijective evaluation maps.

**Lemma 5.4.** Let \(M\) be a \(g\)-module whose composition factors are in the list above. Assume \(M\) satisfies one of the conditions:

1. Both \(\Pi V_{-1}\) and \(V_1\) are composition factors of \(M\);
2. \(\Pi V_0\) is a quotient of \(M\);
3. \(\Pi V_{-1}\) or \(V_1\) is a submodule of \(M\).

Then there is no bijective evaluation map associated with \(M\).

**Proof.** (1) If both \(\Pi V_{-1}\) and \(V_1\) are composition factors of \(M\), then, as a \(\mathfrak{gl}_2\)-module, \(M_0 = M_1 \oplus M_2\), where both \(M_1\) and \(M_2\) are 2-dimensional irreducible \(\mathfrak{sl}_2\)-modules and \(z\) acts on \(M_1\) as multiplication by 1 and on \(M_2\) by \(-1\). Hence, \(M_2 \cong M_1^*\), which implies that the \(\mathfrak{gl}_2\)-module \(M_0\) does not admit a bijective evaluation map.

(2) Assume \(M'\) is a submodule of \(M\) such that \(M/M' \cong \Pi V_0\). Then the image of \(\text{ev}_a\) is contained in \(M'\) and hence \(\text{ev}_a\) is not bijective.

(3) Assume that \(V_1\) is a submodule of \(M\) and consider the list of possible composition factors of \(M\) above. If the composition factors of \(M\) are \(\{V_{-2}, V_1\}\), then, for any \(a \in M_0\), \(\text{ev}_a(E_{13}) = \ldots\)
ev_a(E_{23}) = 0 and thus ev_a is not bijective. If these are \{ΠV_2, V_1\}, then, for any a ∈ M_0, ev_a(E_{31}) and ev_a(E_{32}) are linearly dependent and thus ev_a is not bijective. If these are \{ΠK(0,k), V_1, ΠV_0\} or \{V_1, V_1, ΠV_0, ΠV_0\}, then ΠV_0 is a quotient of M and we refer to (2). Finally, if these are \{ΠV_{-1}, V_1, ΠV_0, ΠV_0\}, then we refer to (1). The case when ΠV_{-1} is a submodule of M is dealt with in a similar way.

We are now ready to prove Theorem 1.2 (2).

**Proposition 5.5.** Let \( \mathcal{L} \) be an LSSA. Then \( \mathcal{L} \) is isomorphic to one of \( \mathcal{A}_k, k ∈ \mathbb{C} \setminus \{-1, -3\}, \mathcal{B}_{k_1,k_2}, k_1 + k_2 ≠ -2, k_1, k_2 ∈ \mathbb{C} \setminus \{0\}, \) or \( \mathcal{C}_k, k ∈ \mathbb{C} \setminus \{-1, 0\} \).

**Proof.** Let \( \mathcal{L} \) be an LSSA with corresponding \( g \)-module \( M \). In view of Corollary 5.2 and Remark 5.3, it suffices to show that \( M \) or \( M_{-1} \) is isomorphic to a module corresponding to one of the LSSAs \( \mathcal{A}_k, k ∈ \mathbb{C} \setminus \{-1, -3\}, \mathcal{B}_{k_1,k_2}, k_1 + k_2 ≠ -2, k_1, k_2 ∈ \mathbb{C} \setminus \{0\}, \) and \( \mathcal{C}_k, k ∈ \mathbb{C} \setminus \{-1, 0\} \). We consider the six cases listed above for the composition factors of \( M \).

1. \( M = K(1,k) \) for \( k ≠ -1, -3 \) is irreducible. Then, for \( k ≠ -1 \), \( M \) gives rise to the LSSA \( \mathcal{A}_k \). Consider \( M = K(1,-1) \). In this case, as a \( gl_2 \)-module, \( M_0 = M_1 ⊕ M_2 \), where \( M_2 \cong M_1 \). This means that \( M_0 \) does not admit a bijective evaluation map and hence \( M = K(1,-1) \) does not give rise to an LSSA.

2. Since there are no extensions between \( ΠV_0 \) and \( V_3 \) or \( ΠV_{-3} \), in this case \( ΠV_0 \) is a quotient of \( M \) and, by Lemma 5.4, \( M \) does not give rise to an LSSA.

3. Since there are no extensions between \( V_1 \) and \( V_{-3} \) or between \( ΠV_{-1} \) and \( ΠV_2 \), Lemma 5.4 excludes these cases. For the other two pairs, again Lemma 5.4, leaves only two possible modules: a non-trivial extension of \( V_1 \) by \( ΠV_2 \) and a non-trivial extension of \( ΠV_{-1} \) by \( V_{-2} \). In the former case \( M \cong K(1,1) \) and hence corresponds to the LSSA \( \mathcal{A}_1 \). In the latter case \( M_{-1} \cong K(1,1) \) and, by Remark 5.3, it corresponds to an LSSA isomorphic to \( \mathcal{A}_1 \).

4. (4) In this case we need to consider two cases: when \( M \) is completely reducible and when \( M \) is indecomposable.

Assume first that \( M = ΠK(0,k_1) ⊕ ΠK(0,k_2) \), where \( k_1, k_2 ∈ \mathbb{C} \setminus \{0, -2\} \). If \( k_1 + k_2 ≠ -2 \), \( M \) corresponds to the LSSA \( \mathcal{B}_{k_1,k_2} \). If \( k_1 + k_2 = -2 \), then \( M_0 \) is a self-dual \( gl_2 \)-module and hence it does not admit a bijective evaluation map. In particular, for \( k_1 + k_2 = -2 \), \( M = ΠK(0,k_1) ⊕ ΠK(0,k_2) \) does not give rise to an LSSA.

Next assume that \( M \) is indecomposable. Then \( k_1 = k_2 : k ≠ 0, -2 \) and \( M \cong ΠK(0,k)^{(2)} \). Thus, for \( k ≠ -1 \), \( M \) corresponds to the LSSA \( \mathcal{C}_k \). For \( k = -1 \), the \( gl_2 \)-module \( M_0 \) is self-dual; hence it does not admit a bijective evaluation map and nor does \( M \).

5. If the composition factors of \( M \) are \( \{ΠK(0,k), ΠV_{-1}, ΠV_0\} \) with \( k ≠ 0, -2 \), Lemma 5.4 implies that \( M = M_1 ⊕ M_2 \), where \( M_1 \cong ΠK(0,k) \) and \( M_2 \cong ΠK(0,-2) \). Thus \( M \) gives rise to \( \mathcal{B}_{k,-2} \). If the composition factors of \( M \) are \( \{ΠK(0,k), V_1, ΠV_0\} \) with \( k ≠ 0, -2 \), then the composition factors of \( M_{-1} \) are \( \{ΠK(0,-k-2), ΠV_{-1}, ΠV_0\} \) and, as above, \( M_{-1} \) (and hence \( M \)) corresponds to an LSSA isomorphic to \( \mathcal{B}_{-k-2,-2} \).
(6) Assume that the composition factors of $M$ are $\{\Pi V_{-1}, \Pi V_{-1}, \Pi V_0, \Pi V_0\}$. Lemma 5.4 implies that $M$ is isomorphic to $\Pi K(0, -2) \oplus \Pi K(0, -2)$ or to $\Pi K(0, -2)^{(2)}$. The former module gives rise to $B_{-2, -2}$, while that latter gives rise to $C_{-2}$. If the composition factors of $M$ are $\{V_1, V_1, \Pi V_0, \Pi V_0\}$, then $M^{-st}$ is isomorphic to $\Pi K(0, -2) \oplus \Pi K(0, -2)$ or to $\Pi K(0, -2)^{(2)}$ and hence $M$ gives rise to an LSSA isomorphic to $B_{-2, -2}$ or $C_{-2}$. Finally, Lemma 5.4 shows that a module with composition factors $\{\Pi V_{-1}, V_1, \Pi V_0, \Pi V_0\}$ does not give rise to an LSSA. □

5.3. Isomorphisms. We complete this section with the proof of Theorem 1.2 (3).

**Proposition 5.6.** $A_k \cong A_{-2-k}$, $B_{k_1,k_2} \cong B_{k_2,k_1} \cong B_{-2-k_1,-2-k_2} \cong B_{-2-k_2,-2-k_1}$, and $C_k \cong C_{-2-k}$.

Moreover, these are the only isomorphisms among $A_k, B_{k_1,k_2}$, and $C_k$.

**Proof.** Let $\mathcal{L}$ and $\mathcal{L}'$ be two LSSAs with corresponding modules $M$ and $M'$ respectively. Remark 5.3 implies that $\mathcal{L}' \cong \mathcal{L}$ if and only if $M' \cong M$ or $M' \cong M^{-st}$. Applying Proposition 3.4 to the list of $g$-modules corresponding to the LSSAs $A_k, B_{k_1,k_2}, C_k$ completes the proof. □

6. An Example of LSSAs on $\mathfrak{sl}(m+1|m)$

Throughout this section $m$ is a fixed positive integer. Before we prove Theorem 1.4 by providing an $\mathfrak{sl}(m+1|m)$-module $U$ which admits a bijective evaluation map, we recall some facts about the exterior square of a super vector space.

Let $W = W_0 \oplus W_1$ be a $\mathbb{Z}_2$-graded vector space. Then the exterior square of $W$ is by definition

$$\wedge^2 W := (W \otimes W)/\text{span}\{u \otimes v + (-1)^{|u||v|}v \otimes u\}.$$ 

As non-graded vector spaces, $(\wedge^2 W)_0 = \wedge^2 W_0 \oplus S^2 W_1$ and $(\wedge^2 W)_1 = W_0 \otimes W_1$, where $S^2 W_1$ is the usual symmetric square of $W_1$. In particular,

$$\dim \wedge^2 W = \left(\frac{\dim W_0}{2}\right) + \left(\frac{\dim W_1 + 1}{2}\right) \dim W_0 \dim W_1.$$ 

If $u, v \in W$, we denote the image of $u \otimes v$ in $\wedge^2 W$ under the natural projection $W \otimes W \to \wedge^2 W$ by $uv$. If $\{e_1, e_2, \ldots, e_p\}$ is a basis of $W_0$ and $\{\xi_1, \xi_2, \ldots, \xi_q\}$ is a basis of $W_1$, then $\{e_i e_j, \xi_i \xi_j | 1 \leq i < j \leq p, 1 \leq s \leq t \leq q\}$ is a basis of $(\wedge^2 W)_0$ and $\{e_i \xi_j | 1 \leq i \leq p, 1 \leq s \leq q\}$ is a basis of $(\wedge^2 W)_1$.

**Proof of Theorem 1.4.** Let $W := C^{m+1|m}$ be the standard module of $\mathfrak{sl}(m+1|m)$. Set

$$U := \Pi(\wedge^2 W) \oplus \Pi(\wedge^2 W),$$

i.e., $U$ is the direct sum of two copies of the exterior square of $W$ with the parity reversed.

Let $\{e_1, e_2, \ldots, e_{m+1}\}$ and $\{\xi_1, \xi_2, \ldots, \xi_m\}$ denote the standard bases of $W_0 = C^{0|m+1}$ and $W_1 = C^{m+1|m}$ respectively. Given an element $v \in \Pi(\wedge^2 W)$, we denote by $v'$ and $v''$ the elements $(v, 0)$ and $(0, v)$ in $U$. In this notation

$$\{e_i' \xi_j' | 1 \leq i \leq m+1, 1 \leq s \leq m\}$$

and

$$\{e_i' e_j', e_i'' e_j'' | 1 \leq i < j \leq m+1, 1 \leq s \leq t \leq m\}$$
are bases of $U_0$ and $U_1$ respectively.

Consider the element

$$a := \sum_{i=1}^{m} (e'_{i+1} \xi_i' + e''_{i} \xi''_i) \in U_0.$$ 

We show below that $\text{ev}_a : \text{sl}(m+1|m) \to U$ is a bijective evaluation map, thus proving Theorem 1.4. Since $\dim \text{sl}(m+1|m) = \dim U$, to prove that $\text{ev}_a$ is bijective, it suffices to prove it is injective, i.e., that $\ker \text{ev}_a = 0$.

Assume $X = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{sl}(m+1|m)$, where $A, B, C$, and $D$ are matrices of sizes $(m+1) \times (m+1)$, $(m+1) \times m$, $m \times (m+1)$, and $m \times m$ respectively. Let $A = (a_{i,j}), B = (b_{i,s}), C = (c_{s,i}),$ and $D = (d_{s,t})$ with $1 \leq i, j \leq m+1$ and $1 \leq s, t \leq m$. We calculate

$$\text{ev}_a(X) = \sum_{i=1}^{m+1} \sum_{s=1}^{m} a_{i,s} e'_{i} \xi_i' + \sum_{i=1}^{m} \sum_{s=1}^{m} d_{s,i} e'_{i+1} \xi_i'. $$

The expression above shows that $X \in \ker \text{ev}_a$ if and only if

$$a_{1,s+1} = 0 \quad a_{i+1,s+1} + d_{s,i} = 0 \quad \text{for} \quad 1 \leq i, s \leq m,$$

$$a_{m+1,s} = 0 \quad a_{i,s} + d_{s,i} = 0 \quad \text{for} \quad 1 \leq i, s \leq m,$$

$$b_{1,j} = 0 \quad b_{i+1,j} - b_{j+1,i} = 0 \quad \text{for} \quad 1 \leq i \neq j \leq m,$$

$$b_{m+1,j} = 0 \quad b_{i,j} - b_{j,i} = 0 \quad \text{for} \quad 1 \leq i \neq j \leq m,$$

$$c_{s,s+1} = 0 \quad c_{s,t+1} + c_{t,s+1} = 0 \quad \text{for} \quad 1 \leq s \neq t \leq m,$$

$$c_{s,s} = 0 \quad c_{s,t} + c_{t,s} = 0 \quad \text{for} \quad 1 \leq s \neq t \leq m.$$ 

An easy and somewhat tedious calculation shows that the solutions of the system above are the matrices $A = cI_{m+1}, D = -cI_m, B = 0, C = 0$, where $c$ is a scalar. Since $\text{str}X = (2m+1)c$ and $X \in \text{sl}(m+1|m)$, we conclude that $c = 0$, i.e., $X = 0$. This proves that $\text{ev}_a$ is injective and completes the proof of the theorem. \hfill $\square$

**APPENDIX**

Set

$$x_1 := E_{12}, \quad x_2 := E_{21}, \quad x_3 := E_{11} - E_{22}, \quad x_4 := E_{11} + E_{22} + 2E_{33},$$

$$y_1 := E_{31}, \quad y_2 := E_{32}, \quad y_3 := E_{13}, \quad y_4 := E_{23}.$$ 

Below we provide the multiplication tables for $A_k, B_{k1,k2}$, and $C_k$. 

### Table 1. The LSSAs $\mathcal{A}_k$, $k \neq -1, -3$

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) |
|---|---|---|---|
| \( 0 \) | \( \frac{1}{2(4k+1)} (-kx_3 + x_4) \) | \(-x_1\) | \( (k+2)x_1 \) |
| \( \frac{1}{x_1} \) | \( x_1 \) | \(-x_2\) | \( kx_2 \) |
| \( x_1 \) | \( kx_1 \) | \( kx_2 \) | \( kx_2 \) |
| \( y_1 \) | \( y_1 \) | \( y_1 \) | \( y_1 \) |
| \( y_2 \) | \( \frac{2}{k+1} (y_1 - 2y_3) \) | \( y_2 \) | \( y_2 \) |
| \( y_3 \) | \( 0 \) | \(-y_3\) | \( (k+2) y_3 \) |
| \( y_4 \) | \( \frac{(k+3)(k-1)}{4(k+1)} y_1 + \frac{2}{k+1} y_3 \) | \( 0 \) | \( (k+2) y_4 \) |

### Table 2. The LSSAs $\mathcal{B}_{k_1,k_2}$, $k_1 + k_2 \neq -2$

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) |
|---|---|---|---|
| \( 0 \) | \( \frac{1}{k_1+k_2+2} ((k_2+1)x_3 + x_4) \) | \(-x_1\) | \( (k_2+1)x_1 \) |
| \( \frac{1}{k_1+k_2+2} \) | \( (k_2+1)x_1 \) | \( x_2 \) | \( (k_1+1)x_2 \) |
| \( x_3 \) | \( x_3 \) | \(-x_2\) | \( (k_2+1)x_2 \) |
| \( x_3 \) | \( x_3 \) | \( (k_2+1)x_2 \) | \( (k_1+1)x_2 \) |
| \( y_1 \) | \( y_2 \) | \( y_2 \) | \( y_2 \) |
| \( y_3 \) | \( y_3 \) | \( y_3 \) | \( y_3 \) |
| \( y_4 \) | \( y_4 \) | \( y_4 \) | \( y_4 \) |
LEFT-SYMMETRIC SUPERALGEBRAS ON \( \mathfrak{sl}(m|n) \)

### Table 3. The LSSAs \( \mathcal{C}_k, k \neq -1 \)

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) |
|---|---|---|---|
| 0 | \( \frac{1}{2(k+1)}(x_4 - x_1) - \frac{k}{2}x_3 \) | \( x_2 \) | \( \frac{1}{2(k+1)}(x_4 - x_1) + \frac{2}{x_3} \) |
| \( \frac{1}{2(k+1)}(x_4 - x_1) - \frac{k}{2}x_3 \) | 0 | \( -x_1 \) | \( \frac{1}{x_1}(x_4 - x_1) + (k+1)x_2 - \frac{k}{2}x_3 \) |
| \( (k+1)x_1 \) | 0 | \( -x_1 \) | \( kx_3 \) |
| \( x_2 \) | \( x_2 \) | \( x_2 \) | \( kx_3 \) |
| \( y_1 \) | 0 | \( -x_2 \) | \( \frac{k}{4(k+1)}(x_4 - x_1) - \frac{k}{2}x_3 \) |
| \( y_2 \) | 0 | \( -y_2 \) | \( \frac{k}{4(k+1)}(x_4 - x_1) - \frac{k}{2}x_3 \) |
| \( y_3 \) | 0 | \( -y_3 \) | \( \frac{k}{4(k+1)}(x_4 - x_1) - \frac{k}{2}x_3 \) |
| \( y_4 \) | 0 | \( -y_4 \) | \( \frac{k}{4(k+1)}(x_4 - x_1) - \frac{k}{2}x_3 \) |

Note that the tables above include the LSSAs \( B_{k,0} \) with \( k \neq -2 \) (including \( B_{0,0} \)) and \( C_0 \). These correspond to the \( \mathfrak{sl}(2|1) \)-modules \( (\Pi K(0, -2 - k) \oplus \Pi K(0, -2))^{st} \) and \( (\Pi K(0, -2)(2))^{st} \) respectively. In particular, they are isomorphic to \( B_{-2-k,-2} \) and \( C_{-2} \) respectively.

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