Conditional Ergodic Averages for Asymptotically Additive Potentials

Yun Zhao

Department of Mathematics, Soochow University, Suzhou 215006, Jiangsu, P.R.China

e-mail: zhaoyun@suda.edu.cn

Abstract. Using an asymptotically additive sequence of continuous functions as a restrictive condition, this paper studies the relations of several ergodic averages for asymptotically additive potentials. Basic properties of conditional maximum ergodic averages are studied. In particular, if the dynamical systems satisfy the specification property, the maximal growth rate of an asymptotically additive potential on the level set is equal to its conditional maximum ergodic averages and the maximal growth rates on the irregular set is its maximum ergodic averages. Finally, the applications for suspension flows are given in the end of the paper.

Key words and phrases Ergodic optimization; Multifractal analysis; Asymptotically additive potentials

1 Introduction

Throughout this paper $X$ is a compact metric space with metric $d$ and $T : X \rightarrow X$ is a continuous transformation. Such a tuple $(X, T)$ is called a topological dynamical systems (TDS for short). Let $C(X)$ denote the Banach space of continuous functions from $X$ to $\mathbb{R}$ with supremum norm $\| \cdot \|$. Let $\mathcal{M}_T$ and $\mathcal{E}_T$ denote the space of $T$-invariant Borel probability measures on $X$ and the set of all $T$-invariant ergodic Borel probability measures on $X$, respectively.

Given a continuous function $f : X \rightarrow \mathbb{R}$, the maximum ergodic average for $f$ is defined as

$$\beta(f) := \sup_{\mu \in \mathcal{M}_T} \int f d\mu.$$
Since $\mathcal{M}_T$ is weak$^*$ compact and the map $\mu \mapsto \int f d\mu$ is continuous, there always exists a measure which attains the supremum in the above formula, such a measure is called $f$–maximizing measure. Jenkinson [20] proved the following basic relations between different time averages of a continuous function $f$

$$\beta(f) = \sup_{x \in X} \limsup_{n \to \infty} \frac{1}{n} S_n f(x) = \lim_{n \to \infty} \frac{1}{n} \max_{x \in X} S_n f(x) = \sup_{x \in \text{Reg}(f, T)} \lim_{n \to \infty} \frac{1}{n} S_n f(x)$$

where $S_n f(x) := \sum_{i=0}^{n-1} f(T^i x)$ and $\text{Reg}(f, T)$ is the set of points $x \in X$ such that the limit of the sequence $\{\frac{1}{n} S_n f(x)\}_{n \geq 1}$ exists. Let

$$\mathcal{M}_{\text{max}}(f) := \left\{ \mu \in \mathcal{M}_T : \int f d\mu = \beta(f) \right\}$$

denote the set of all $f$–maximizing measures. The study of the variational problem of the functional $\beta(\cdot)$ and the set $\mathcal{M}_{\text{max}}(\cdot)$ has been termed ergodic optimization, and has attracted some recent research interest [6, 7, 8, 10, 11, 16, 20, 21, 22, 23]. Analogous problems for sub-additive potentials has been investigated for deterministic dynamical systems in [17, 30, 31] and for random dynamical systems in [13]. In [15], author studied the sub-growth rate of asymptotically sub-additive potentials and subordination principle for sub-additive potentials. One of the motivation of the present paper is to study the ergodic optimization for a particular sequence of asymptotically additive potentials (see precise definition in the next section and use AAP for short) which arises naturally in the study of the dimension theory in dynamical systems (see [19, 38] for examples of AAP).

The study of this paper is also motivated by the theory of multifractal analysis. The theory of multifractal analysis is a subfield of the dimension theory of dynamical systems, its main purpose is to study the complexity of the level sets or irregular sets of invariant local quantities obtained from a given dynamical system, e.g., the topological entropy or pressure on these sets. See the books [2, 26] for details about the theory of multifractal analysis. For a sequence of asymptotically additive continuous functions $\Phi = \{\varphi_n\}_{n \geq 1}$, the level sets induced by the asymptotically additive potential $\Phi$ are defined by

$$K_{\Phi}(\alpha) := \{ x \in X : \lim_{n \to \infty} \frac{1}{n} \varphi_n(x) = \alpha \}.$$ 

Since these level sets are pairwise disjoint for different real numbers $\alpha$, they induce the natural decomposition

$$X = \hat{X}_\Phi \cup \bigcup_{\alpha \in \mathbb{R}} K_{\Phi}(\alpha)$$

where $\hat{X}_\Phi := \{ x \in X : \lim_{n \to \infty} \frac{1}{n} \varphi_n(x) \text{ does not exist} \}$ is the irregular set for the AAP $\Phi = \{\varphi_n\}_{n \geq 1}$. 

Given an AAP $\mathcal{F} = \{f_n\}_{n \geq 1}$ and a $T$-invariant measure $\mu \in \mathcal{M}_T$, let

$$\mathcal{F}_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int f_n \, d\mu.$$  

Using an asymptotically additive potential $\Phi = \{\varphi_n\}_{n \geq 1}$ as a restrictive condition, this paper investigates the ergodic optimization of an AAP. Precisely, consider two asymptotically additive potentials $\mathcal{F} = \{f_n\}_{n \geq 1}$ and $\Phi = \{\varphi_n\}_{n \geq 1}$, for any real number $\alpha \in \mathbb{R}$ define

$$\Lambda_{\mathcal{F}|\Phi}(\alpha) := \sup \{ \mathcal{F}_*(\mu) : \mu \in M_T(\Phi, \alpha) \}$$

where $M_T(\Phi, \alpha) := \{\mu \in M_T : \Phi_*(\mu) = \alpha\}$. The quantity $\Lambda_{\mathcal{F}|\Phi}(\alpha)$ is called the conditional maximum ergodic average of the AAP $\mathcal{F} = \{f_n\}_{n \geq 1}$ (with respect to $\Phi$), and the following quantity

$$\beta(\mathcal{F}) := \sup \{ \mathcal{F}_*(\mu) : \mu \in M_T \}$$

is called the maximum ergodic average of the AAP $\mathcal{F} = \{f_n\}_{n \geq 1}$. As a direct consequence of the main result in [15], we can prove that

$$\beta(\mathcal{F}) = \limsup_{x \in X} \sup_{n \to \infty} \frac{1}{n} f_n(x) = \lim_{n \to \infty} \frac{1}{n} \max_{x \in X} f_n(x) = \sup_{x \in \text{Reg}(\mathcal{F}, T)} \lim_{n \to \infty} \frac{1}{n} f_n(x),$$

where $\text{Reg}(\mathcal{F}, T)$ is the set of points $x \in X$ for which the limit of the sequence $\{\frac{1}{n} f_n(x)\}_{n \geq 1}$ exists. Some basic properties of the function $\Lambda_{\mathcal{F}|\Phi}(\cdot)$ are studied, e.g., the continuity and monotonicity of the function $\Lambda_{\mathcal{F}|\Phi}(\cdot)$. Furthermore, if the TDS $(X, T)$ satisfies the specification property (see the exact definition in Section 2), we prove that the maximal growth rate of an AAP $\mathcal{F} = \{f_n\}_{n \geq 1}$ on the level sets is equal to the conditional maximum ergodic average of $\mathcal{F}$, that is,

$$\sup_{x \in K_{\Phi}(\alpha)} \limsup_{n \to \infty} \frac{1}{n} f_n(x) = \sup \{ \mathcal{F}_*(\mu) : \mu \in M_T(\Phi, \alpha) \},$$

and the maximum ergodic average of an AAP $\mathcal{F} = \{f_n\}_{n \geq 1}$ is exactly its maximum ergodic averages on the irregular sets, i.e.,

$$\beta(\mathcal{F}) = \sup \left\{ \mathcal{F}_*(\mu) : \mu \in \bigcup_{x \in X} \mathcal{V}(x) \right\}$$

where $\mathcal{V}(x)$ is the set of all the limit points of the empirical measures $\delta_{x,n} := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x}$ and $\delta_x$ is the dirac measure at the point $x$.

The remainder of this paper is organized as follows. The precise statements of the main results is given in Section 2. Section 3 provides some examples to illustrate our main results. In section 4, using the methods in the theory of multifractal analysis, we provides the proofs of all the statement is section 2. In section 5, we apply our main results to suspension flows.
2 Statements of the results

This section first provides some preliminaries and notations, and then gives the statements of the main results of this paper, the proofs are postponed to Section 4.

A sequence of continuous functions \( \mathcal{F} = \{ f_n \}_{n \geq 1} \subseteq C(X) \) is called an asymptotically additive potential on \( X \), if for each \( \xi > 0 \), there exists a continuous function \( f_\xi \in C(X) \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \| f_n - S_n f_\xi \| < \xi.
\]

(2.3)

This kind of potential was introduced by Feng and Huang [19].

Given an AAP \( \mathcal{F} = \{ f_n \}_{n \geq 1} \), Feng and Huang [19] proved the following result.

**Theorem 2.1.** Let \( \mu \) be a \( T \)-invariant measure. Then the following properties hold:

1. The limit \( \mathcal{F}_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int f_n \, d\mu \) exists and is finite. Furthermore, the limit \( \lambda_\mathcal{F}(x) := \lim_{n \to \infty} \frac{1}{n} f_n(x) \) exists for \( \mu \)-almost every \( x \in X \), and \( \int \lambda_\mathcal{F}(x) \, d\mu = \mathcal{F}_*(\mu) \).

In particular, when \( \mu \in \mathcal{E}_T \), \( \lambda_\mathcal{F}(x) = \mathcal{F}_*(\mu) \) for \( \mu \)-a.e. \( x \in X \);

2. The map \( \mathcal{F}_* : \mathcal{M}_T \to \mathbb{R} \) is continuous;

3. Let \( \mu = \int_{\mathcal{E}_T} m \, d\tau(m) \) be the ergodic decomposition of \( \mu \in \mathcal{M}_T \) (here \( \tau \) is a probability measure on the space \( \mathcal{M}_T \) such that \( \tau(\mathcal{E}_T) = 1 \)), then \( \mathcal{F}_*(\mu) = \int_{\mathcal{E}_T} \mathcal{F}_*(m) \, d\tau(m) \).

Now we recall the definition of the specification property. Roughly speaking, a TDS \((X,T)\) has specification property if one can always find a real orbit to interpolate between different pieces of orbits, up to a pre-assigned error.

**Definition 2.1.** A TDS \((X,T)\) satisfies the specification property if for each \( \epsilon > 0 \), there exists an integer \( m = m(\epsilon) \) such that for any collection \( \{ I_j := [a_j, b_j] \subset \mathbb{N} : j = 1, 2, \ldots, k \} \) of finite intervals with \( a_{j+1} - b_j \geq m(\epsilon) \) for \( j = 1, 2, \ldots, k - 1 \) and any \( x_1, x_2, \ldots, x_k \in X \), there exists a point \( x \in X \) such that

\[
d(T^{p+a_j}(x), T^p(x_j)) < \epsilon
\]

(2.4)

for all \( p = 0, 1, \ldots, b_j - a_j \) and every \( j = 1, 2, \ldots, k \).

This definition is weaker than the original Bowen’s definition of specification. In Bowen’s definition, it is required that the shadowing point \( x \) is a periodic point of period \( p \geq b_k - a_1 + m(\epsilon) \). There are well-know examples of dynamical systems has the specification property. The most basic example is the shift map on the full symbolic space; another example is the topologically mixing shift map on the subshift space of
finite type [18]. Blokh [5] proved that a topologically mixing map of the interval has Bowen’s specification, a recent proof can be found in [12].

Given an AAP $\Phi = \{\varphi_n\}_{n \geq 1}$ as the restrictive condition, let

$$\eta(\Phi) := \min\{\Phi_*(\mu) : \mu \in M_T\}$$

be the minimum ergodic average of $\Phi$. By (2) of Theorem 2.1 we know that the set $M_T(\Phi, \alpha) = \{\mu \in M_T : \Phi_*(\mu) = \alpha\}$ is non-empty, convex and compact for each $\alpha \in [\eta(\Phi), \beta(\Phi)]$, so the supremum can be replaced by maximum in the definition of $\Lambda_{F|\Phi}(\alpha)$ in (1.1).

The first theorem says that the function $\alpha \mapsto \Lambda_{F|\Phi}(\alpha)$ is continuous on the interval $[\eta(\Phi), \beta(\Phi)]$.

**Theorem A.** Let $F = \{f_n\}_{n \geq 1}$ and $\Phi = \{\varphi_n\}_{n \geq 1}$ be two asymptotically additive potentials on $X$. Then the map $\alpha \mapsto \Lambda_{F|\Phi}(\alpha)$ is continuous on the interval $[\eta(\Phi), \beta(\Phi)]$.

Since the map $F_* : M_T \to \mathbb{R}$ is continuous and $M_T$ is weak* compact, by ergodic decomposition theorem there exists a $T$-invariant ergodic measure $\mu^* \in E_T$ such that $\beta(F) = F_*(\mu^*)$. Assume that $\Phi_*(\mu^*) = \alpha^*$. Note that $\mu^*$ may be not unique and $\beta(F) = \Lambda_{F|\Phi}(\alpha^*)$. The following theorem studies the monotonicity of the function $\alpha \mapsto \Lambda_{F|\Phi}(\alpha)$.

**Theorem B.** Let $F = \{f_n\}_{n \geq 1}$ and $\Phi = \{\varphi_n\}_{n \geq 1}$ be two asymptotically additive potentials on $X$. Then the conditional maximum ergodic average $\Lambda_{F|\Phi}(\cdot)$ is monotone increasing on $[\eta(\Phi), \alpha^*]$ and monotone decreasing on $[\alpha^*, \beta(\Phi)]$.

**Remark 1.** Let $M_{\max}(F) := \{\mu \in M_T | F_*(\mu) = \beta(F)\}$ be the set of maximizing measures for the AAP $F = \{f_n\}_{n \geq 1}$. Since the map $F_* : M_T \to \mathbb{R}$ is continuous, the set $M_{\max}(F)$ is a non-empty, convex and compact subset of $M_T$. Let

$$\alpha_1 := \min_{\mu \in M_{\max}(F)} \Phi_*(\mu) \quad \text{and} \quad \alpha_2 := \max_{\mu \in M_{\max}(F)} \Phi_*(\mu).$$

By Theorem 2 we know that $\Lambda_{F|\Phi}(\alpha) \equiv \beta(F)$ for each $\alpha \in [\alpha_1, \alpha_2]$.

In the following, we consider the conditional maximum ergodic average of an AAP $F = \{f_n\}_{n \geq 1}$ at the extreme point $\alpha = \beta(\Phi)$ or $\eta(\Phi)$.

**Proposition 2.1.** Let $F = \{f_n\}_{n \geq 1}$ and $\Phi = \{\varphi_n\}_{n \geq 1}$ be two asymptotically additive potentials on $X$. Then

$$\Lambda_{F|\Phi}(\alpha) = \sup\{F_*(\mu) : \mu \in E_T \text{ and } \Phi_*(\mu) = \alpha\}$$

when $\alpha = \beta(\Phi)$ or $\eta(\Phi)$. 
For any two asymptotically additive potentials $F = \{f_n\}_{n \geq 1}$ and $\Phi = \{\varphi_n\}_{n \geq 1}$, the following theorem considers the time averages of $F$ on the level sets of $\Phi$.

**Theorem C.** Let $(X, T)$ be a TDS satisfying the specification property, and $F = \{f_n\}_{n \geq 1}$ and $\Phi = \{\varphi_n\}_{n \geq 1}$ two asymptotically additive potentials on $X$. Then

$$
\Lambda_{F,\Phi}(\alpha) = \sup_{x \in K_{\Phi}(\alpha)} \limsup_{n \to \infty} \frac{1}{n} f_n(x) = \sup \left\{ \mathcal{F}_s(\mu) : \mu \in \bigcup_{x \in K_{\Phi}(\alpha)} \mathcal{V}(x) \right\}
$$

for each $\alpha \in [\eta(\Phi), \beta(\Phi)]$.

On the irregular set of $\Phi$, we have the following theorem.

**Theorem D.** Let $(X, T)$ be a TDS satisfying the specification property, and $F = \{f_n\}_{n \geq 1}$ an AAP on $X$. Assume that $\Phi = \{\varphi_n\}_{n \geq 1}$ is an AAP on $X$ satisfying $\eta(\Phi) < \beta(\Phi)$, then we have

$$
\beta(F) = \sup_{x \in \hat{X}_\Phi} \limsup_{n \to \infty} \frac{1}{n} f_n(x) = \sup \left\{ \mathcal{F}_s(\mu) : \mu \in \bigcup_{x \in \hat{X}_\Phi} \mathcal{V}(x) \right\}.
$$

**Remark 2.** (1) It is easy to see that if $\eta(\Phi) = \beta(\Phi) := \lambda$, then the sequence $\frac{1}{n}\varphi_n(x)$ converges uniformly to the constant $\lambda$. Therefore, in this case there is no irregular points for the AAP $\Phi = \{\varphi_n\}_{n \geq 1}$; (2) Both Theorem C and Theorem D use the assumption that the TDS $(X, T)$ has specification property. Under this assumption we have $[\eta(\Phi), \beta(\Phi)] = \{\alpha \in \mathbb{R} : K_{\Phi}(\alpha) \neq \emptyset\}$ (see [35] for a proof), this result is essentially contained in [32].

To apply the main results to suspension flows, we consider the following more general cases of ratios of sequence of AAPs. Precisely, let $\Phi = \{\varphi_n\}_{n \geq 1}$ and $\Psi = \{\psi_n\}_{n \geq 1}$ be two asymptotically additive potentials on $X$. We always assume that

\begin{equation}
\frac{1}{n}\psi_n(x) \geq \sigma \quad \forall n \in \mathbb{N}, \forall x \in X
\end{equation}

for some constant $\sigma > 0$. For each $\alpha \in \mathbb{R}$, let

$$
E_{\Phi,\Psi}(\alpha) := \{x \in X : \lim_{n \to \infty} \frac{\varphi_n(x)}{\psi_n(x)} = \alpha\}.
$$

The ergodic optimization on this level set has a similar property as Theorem C.

**Theorem E.** Let $(X, T)$ be a TDS satisfying the specification property, $F = \{f_n\}_{n \geq 1}$ and $\Phi = \{\varphi_n\}_{n \geq 1}$ are AAPs on $X$. Assume that $G = \{g_n\}_{n \geq 1}$ and $\Psi = \{\psi_n\}_{n \geq 1}$ are two AAPs satisfying (2.5). Then, for any real number $\alpha$ with $E_{\Phi,\Psi}(\alpha) \neq \emptyset$

$$
\sup_{x \in E_{\Phi,\Psi}(\alpha)} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} = \sup \left\{ \frac{\mathcal{F}_s(\mu)}{\mathcal{G}_s(\mu)} : \mu \in \mathcal{M}_T \text{ and } \Phi_s(\mu)/\Psi_s(\mu) = \alpha \right\} \sup \left\{ \frac{\mathcal{G}_s(\mu)}{\mathcal{G}_s(\mu)} : \mu \in \bigcup_{x \in E_{\Phi,\Psi}(\alpha)} \mathcal{V}(x) \right\}.
$$
Remark 3. In the proof of Theorem E, we only use the fact that the generic set \( G_\mu \) is non-empty for each \( \mu \in \mathcal{M}_T \), where \( G_\mu = \{ x \in X : \delta_{x,n} \to \mu (n \to \infty) \} \). This property is always called saturated property of a TDS \((X, T)\), and is satisfied when a TDS \((X, T)\) satisfying specification property. This observation means that Theorem E remains true for a broader class of systems. For example, Pfister and Sullivan [27] consider a weak specification property which is called the \( g \)-almost product property in that paper. In [27], they proved that a TDS \((X, T)\) satisfies the saturated property when it has \( g \)-almost product property. Therefore, Theorem E remains true if a TDS \((X, T)\) has \( g \)-almost product property.

Let \( \Phi = \{ \varphi_n \}_{n \geq 1} \) and \( \Psi = \{ \psi_n \}_{n \geq 1} \) be two asymptotically additive potentials given as above, the irregular set associated with \( \Phi \) and \( \Psi \) is defined as

\[
\hat{X}_{\Phi, \Psi} := \{ x \in X : \lim_{n \to \infty} \frac{\varphi_n(x)}{\psi_n(x)} \text{ does not exist} \}.
\]

Theorem F. Let \((X, T)\) be a TDS satisfying the specification property, \( \mathcal{F} = \{ f_n \}_{n \geq 1} \) and \( \Phi = \{ \varphi_n \}_{n \geq 1} \) are AAPs on \( X \). Assume that \( \mathcal{G} = \{ g_n \}_{n \geq 1} \) and \( \Psi = \{ \psi_n \}_{n \geq 1} \) are two AAPs satisfying (2.5), and \( \inf_{\mu \in \mathcal{M}_T} \frac{\Phi_*(\mu)}{\Psi_*(\mu)} < \sup_{\mu \in \mathcal{M}_T} \frac{\Phi_*(\mu)}{\Psi_*(\mu)} \), then we have

\[
\sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{E}_T \right\} = \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \right\} = \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \bigcup_{x \in \hat{X}_{\Phi, \Psi}} \mathcal{V}(x) \right\} = \sup_{x \in \hat{X}_{\Phi, \Psi}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)}.
\]

If the TDS \((X, T)\) satisfying the \( g \)-almost product property (see [27] for the definition), then the proof of Theorem F generalizes unproblematically to this setting and thus Theorem F holds for continuous maps with the \( g \)-almost product property.

3 Examples

This section provides examples to illustrate the main results in Section 2.

We first provide a system that does satisfy weak specification property but does not satisfy the specification property, however, the main results are still valid for this kind of systems.

Example 1. Consider the piecewise expanding maps of the interval \([0, 1)\) given by \( T_\beta(x) = \beta x (\text{mod } 1) \), where \( \beta > 1 \). This family is known as beta transformations and it was introduced by Rényi in [28]. From [12], we know that for all but countable
many values of $\beta$ the transformation $T_\beta$ does not satisfy the specification property. However, it follows from [27, 36] that every $\beta$-transformation satisfies the $g$-almost product property. By Remark 3 Theorem [1, 4] are true for $\beta$-transformation for every $\beta > 1$.

The following particular asymptotically additive potentials were introduced by Barreira [4] and Mummert [24] independently to study the theory of thermodynamic formalism for a broader class of potentials.

**Example 2.** Let $\mathcal{F} = \{f_n\}_{n \geq 1}$ be an almost additive potential, i.e., there exists $C > 0$ such that $f_n + f_m \circ T^n - C \leq f_{n+m} \leq f_n + f_m \circ T^n + C$ for any $n, m \in \mathbb{N}$. This kind of potential was introduced by Barreira [4] and Mummert [27] independently. They independently investigated the theory of thermodynamic formalism for almost additive potentials, including the existence and uniqueness of equilibrium state and Gibbs state, variational principle for almost additive topological pressure.

It is not hard to see that an almost additive potential is asymptotically additive, see [19, Proposition A.5] or [38, Proposition 2.1] for proofs. Hence, Theorem [1, 4] are true for any asymptotically additive potentials.

The last example deals with families of potentials responsible for computing the largest and smallest Lyapunov exponents are asymptotically additive.

**Example 3.** Let $M$ be a $d$-dimensional smooth manifold and $J$ a compact expanding invariant set for a $C^1$ map $f$. Let $\mathcal{E}(J, f)$ denote the set of all $f$-invariant ergodic measures supported on $J$. We say that $J$ is an average conformal repeller if all Lyapunov exponents of each ergodic measure $\mu \in \mathcal{E}(f | J)$ are equal and positive. In particular, it follows from [1, Theorem 4.2] that

$$\lim_{n \to \infty} \frac{1}{n} \left( \log \|Df^n(x)\| - \log \|Df^n(x)^{-1}\|^{-1} \right) = \lim_{n \to \infty} \frac{1}{n} \log \frac{\|Df^n(x)\|}{\|Df^n(x)^{-1}\|^{-1}} = 0 \quad (3.6)$$

uniformly on $J$. Let $\Psi_1 = \{\log \|Df^n(x)\|\}_{n \geq 1}$ and $\Psi_2 = \{\log \|Df^n(x)^{-1}\|^{-1}\}_{n \geq 1}$, it is not hard to check that these two potentials are asymptotically additive since they can be approximated by the additive potentials $\left\{\frac{1}{d} \log | \det(Df^n(x))| \right\}_{n \geq 1}$. Note that the potential $\Psi_3 = \left\{\log \frac{\|Df^n(x)\|}{\|Df^n(x)^{-1}\|^{-1}}\right\}_{n \geq 1}$ is also asymptotically additive.

Assume further that $J$ is topological mixing, then $(J, f)$ satisfies the specification property. The following facts hold for the expanding system $(J, f)$:

(i) By (3.6) there is no irregular point for the AAP $\Psi_3$;

(ii) Given any AAP $\Phi = \{\varphi_n\}_{n \geq 1}$ satisfying $\eta(\Phi) < \beta(\Phi)$, the following properties hold:
(1) by Theorem 3 and (3.6), for each \( \alpha \in [\eta(\Phi), \beta(\Phi)] \) we have
\[
\Lambda_{\Psi_1|\Phi}(\alpha) = \sup_{x \in K_{\Phi}(\alpha)} \limsup_{n \to \infty} \frac{1}{n} \log ||Df^n(x)||
\]
\[
= \sup \left\{ \Psi_1^*(\mu) : \mu \in \bigcup_{x \in K_{\Phi}(\alpha)} V(x) \right\} = \Lambda_{\Psi_2|\Phi}(\alpha);
\]

(2) by (1.2), Theorem D and (3.6) we have
\[
\sup \left\{ \Psi_1^*(\mu) : \mu \in \bigcup_{x \in \hat{X}_{\Phi}} V(x) \right\} = \sup \left\{ \Psi_2^*(\mu) : \mu \in \bigcup_{x \in \hat{X}_{\Phi}} V(x) \right\}
\]
\[
= \sup \limsup_{n \to \infty} \frac{1}{n} \log ||Df^n(x)||
\]
\[
= \sup \limsup_{n \to \infty} \frac{1}{n} \log ||Df^n(x)||;
\]

(iii) Let \( \Psi_3 \) be the restrictive condition, by (3.6), Theorems C and D, for any asymptotically additive potential \( \mathcal{G} = \{g_n\}_{n \geq 1} \) we have
\[
\Lambda_{\mathcal{G}|\Psi_3}(0) = \sup_{x \in K_{\Psi_3}(0)} \limsup_{n \to \infty} \frac{1}{n} g_n(x) = \sup \left\{ \mathcal{G}_*(\mu) : \mu \in \bigcup_{x \in K_{\Psi_3}(0)} V(x) \right\} = \beta(\mathcal{G}).
\]

4 Proofs

This section provides the proofs of the theorems stated in Section 2.

Proof of Theorem A. Assume that \( \eta(\Phi) = \Phi_*(\mu_1) \) and \( \beta(\Phi) = \Phi_*(\mu_2) \) for some invariant measures \( \mu_1, \mu_2 \in \mathcal{M}_T \). Let \( \{\alpha_n\}_{n \geq 1} \subseteq [\eta(\Phi), \beta(\Phi)] \) so that \( \alpha_n \to \alpha \) as \( n \to \infty \), to prove the continuity of the map \( \Lambda_{\mathcal{F}|\Phi}(\cdot) \) it suffices to prove that
\[
\lim_{n \to \infty} \Lambda_{\mathcal{F}|\Phi}(\alpha_n) = \Lambda_{\mathcal{F}|\Phi}(\alpha).
\]

For each \( n \geq 1 \), since \( \mathcal{M}_T(\Phi, \alpha_n) \) is non-empty and compact, there exists an invariant measure \( \mu_n \in \mathcal{M}_T(\Phi, \alpha_n) \) such that \( \Lambda_{\mathcal{F}|\Phi}(\alpha_n) = \mathcal{F}_*(\mu_n) \). Choose a subsequence of integers \( \{n_j\}_{j \geq 1} \) such that
\[
\limsup_{n \to \infty} \Lambda_{\mathcal{F}|\Phi}(\alpha_n) = \lim_{j \to \infty} \Lambda_{\mathcal{F}|\Phi}(\alpha_{n_j})
\]
and \( \mu_{n_j} \to \mu \) (\( j \to \infty \)) for some \( \mu \in \mathcal{M}_T \). The strategy for the proof is to approximate the asymptotically additive potential by appropriate Birkhoff sums associated with some continuous function. Indeed, fix a small number \( \xi > 0 \) and \( \varphi_\xi \) is a continuous function given by (2.3) approximating \( \Phi \). We have
\[
\int \varphi_\xi \, d\mu = \lim_{j \to \infty} \int \varphi_\xi \, d\mu_{n_j}.
\]
Therefore, for all sufficiently large \( j \) the following holds:

\[
\left| \Phi_*(\mu) - \alpha \right| \leq \left| \Phi_*(\mu) - \int \varphi \, d\mu \right| + \left| \int \varphi \, d\mu - \int \varphi \, d\mu_{n_j} \right| + \left| \int \varphi \, d\mu_{n_j} - \alpha \right|
\]
\[
\leq \xi + \xi + \left| \int \varphi \, d\mu_{n_j} - \alpha \right|
\]

for all sufficiently large \( j \). The above two inequalities yield that

\[
\left| \Phi_*(\mu) - \alpha \right| \leq 4\xi.
\]

The arbitrariness of \( \xi \) implies that \( \mu \in \mathcal{M}_T(\Phi, \alpha) \). Therefore

\[
\limsup_{n \to \infty} \Lambda_{\mathcal{F}\mid \Phi}(\alpha_n) = \lim_{j \to \infty} \Lambda_{\mathcal{F}\mid \Phi}(\alpha_n) = \lim_{j \to \infty} \mathcal{F}_*(\mu_n) = \mathcal{F}_*(\mu) \leq \Lambda_{\mathcal{F}\mid \Phi}(\alpha). \tag{4.7}
\]

On the other hand, choose \( \mu \in \mathcal{M}_T(\Phi, \alpha) \) such that \( \mathcal{F}_*(\mu) = \Lambda_{\mathcal{F}\mid \Phi}(\alpha) \). For each \( n \geq 1 \), if \( \eta(\Phi) \leq \alpha_n \leq \alpha \), take a \( T \)-invariant measure \( \mu_p \) of the form \( p\mu_1 + (1 - p)\mu \) such that \( \Phi_*(\mu_p) = \alpha_n \). If \( \alpha \leq \alpha_n \leq \beta(\Phi) \), consider a \( T \)-invariant measure \( \mu_p \) of the form \( p\mu_2 + (1 - p)\mu \) such that \( \Phi_*(\mu_p) = \alpha_n \). Note that \( p \to 0 \) as \( n \to \infty \), and

\[
\mathcal{F}_*(\mu_p) = p\mathcal{F}_*(\mu_1) + (1 - p)\mathcal{F}_*(\mu) \text{ or } p\mathcal{F}_*(\mu_2) + (1 - p)\mathcal{F}_*(\mu).
\]

Hence,

\[
\liminf_{n \to \infty} \Lambda_{\mathcal{F}\mid \Phi}(\alpha_n) \geq \lim_{p \to 0} \mathcal{F}_*(\mu_p) = \mathcal{F}_*(\mu) = \Lambda_{\mathcal{F}\mid \Phi}(\alpha). \tag{4.8}
\]

Combining inequalities (4.7) and (4.8), the desired result immediately follows. \( \square \)

**Remark 4.** Note that in the proof of the inequalities (4.7) and (4.8), it is enough to require that the map \( \mu \mapsto \mathcal{F}_*(\mu) \) is upper-semi-continuous. Therefore, Theorem 4.4 is also true for asymptotically sub-additive potentials. A sequence \( \mathcal{F} = \{f_n\}_{n \geq 1} \subseteq C(X) \) is called an asymptotically sub-additive potential on \( X \), if for each \( \xi > 0 \) there exists a sub-additive potential \( \Phi^\xi = \{\varphi^\xi_n\}_{n \geq 1} \subseteq C(X) \), i.e. \( \varphi^\xi_{n+m} \leq \varphi^\xi_n + \varphi^\xi_m \circ T^m \) for any \( x \in X \) and any \( n, m \in \mathbb{N} \), such that

\[
\limsup_{n \to \infty} \frac{1}{n} \|f_n - \varphi^\xi_n\| < \xi.
\]

In this case, it was proved by Feng and Huang (cf. [19]) that the map \( \mu \mapsto \mathcal{F}_*(\mu) \) is upper-semi-continuous.
Proof of Theorem \[B\] We first show that $\Lambda_{\mathcal{F}|\Phi}(\alpha)$ is monotone increasing on $[\eta(\Phi), \alpha^*]$. Let $\alpha_1, \alpha_2 \in [\eta(\Phi), \alpha^*]$ and $\alpha_1 < \alpha_2$, choose a $T$–invariant measure $\nu \in \mathcal{M}_T(\Phi, \alpha_1)$ so that $\mathcal{F}_*(\nu) = \Lambda_{\mathcal{F}|\Phi}(\alpha_1)$. For $t \in [0, 1]$, put $\mu_t = (1 - t)\nu + t\mu^*$, then we have

$$\mathcal{F}_*(\mu_t) = (1 - t)\Lambda_{\mathcal{F}|\Phi}(\alpha_1) + t\Lambda_{\mathcal{F}|\Phi}(\alpha^*).$$

Since $\Lambda_{\mathcal{F}|\Phi}(\alpha^*) = \beta(\mathcal{F})$, we have

$$\frac{d}{dt}\mathcal{F}_*(\mu_t) = \Lambda_{\mathcal{F}|\Phi}(\alpha^*) - \Lambda_{\mathcal{F}|\Phi}(\alpha_1) \geq 0.$$

Therefore, the map $\mathcal{F}_*(\mu_t)$ is monotone increasing w.r.t. $t$ on the interval $[0, 1]$. Since $\alpha_1 < \alpha_2 \leq \alpha^*$, there exists $t_0 \in [0, 1]$ such that

$$\Phi_*(\mu_{t_0}) = (1 - t_0)\alpha_1 + t_0\alpha^* = \alpha_2.$$

Hence,

$$\Lambda_{\mathcal{F}|\Phi}(\alpha_1) = \mathcal{F}_*(\mu_0) \leq \mathcal{F}_*(\mu_{t_0}) \leq \Lambda_{\mathcal{F}|\Phi}(\alpha_2)$$

which shows that $\Lambda_{\mathcal{F}|\Phi}(\alpha)$ is monotone increasing on $[\eta(\Phi), \alpha^*].$

Next we prove that $\Lambda_{\mathcal{F}|\Phi}(\alpha)$ is monotone decreasing on $[\alpha^*, \beta(\Phi)]$. The methods used here are similar as the above arguments. Let $\alpha_3, \alpha_4 \in [\alpha^*, \beta(\Phi)]$ and $\alpha_3 < \alpha_4$, choose a $T$–invariant measure $m \in \mathcal{M}_T(\Phi, \alpha_4)$ so that $\mathcal{F}_*(m) = \Lambda_{\mathcal{F}|\Phi}(\alpha_4)$. For $t \in [0, 1]$, put $\nu_t = (1 - t)\mu^* + tm$, then we have

$$\mathcal{F}_*(\nu_t) = (1 - t)\Lambda_{\mathcal{F}|\Phi}(\alpha^*) + t\Lambda_{\mathcal{F}|\Phi}(\alpha_4).$$

Since

$$\frac{d}{dt}\mathcal{F}_*(\nu_t) = \Lambda_{\mathcal{F}|\Phi}(\alpha_4) - \Lambda_{\mathcal{F}|\Phi}(\alpha^*) \leq 0,$$

the map $\mathcal{F}_*(\nu_t)$ is monotone decreasing w.r.t. $t$ on the interval $[0, 1]$. And since $\alpha^* \leq \alpha_3 < \alpha_4$, there exists $t_1 \in [0, 1]$ such that

$$\Phi_*(\nu_{t_1}) = (1 - t_1)\alpha^* + t_1\alpha_4 = \alpha_3.$$

Hence,

$$\Lambda_{\mathcal{F}|\Phi}(\alpha_4) = \mathcal{F}_*(\nu_t) \leq \mathcal{F}_*(\nu_{t_1}) \leq \Lambda_{\mathcal{F}|\Phi}(\alpha_3).$$

This shows that $\Lambda_{\mathcal{F}|\Phi}(\alpha)$ is monotone decreasing on $[\alpha^*, \beta(\Phi)]$. \(\square\)

The same reason as presented in Remark \[4\] Theorem \[B\] is also true for asymptotically sub-additive potentials.
Proof of Proposition 2.1. This proof only deals with the case \( \alpha = \beta(\Phi) \), since the other case of \( \alpha = \eta(\Phi) \) can be proven in a similar fashion. Since \( \mathcal{M}_T(\Phi, \beta(\Phi)) \) is a compact subset of \( \mathcal{M}_T \), there exists a \( T \)-invariant measure \( \mu \in \mathcal{M}_T(\Phi, \beta(\Phi)) \) such that \( F_*(\mu) = \Lambda_{F|\Phi}(\beta(\Phi)) \). Let \( \mu = \int_{\mathcal{E}_T} m \, d\tau(m) \) be the ergodic decomposition of \( \mu \), then there exists a full \( \tau \)-measure set \( \Omega \subset \mathcal{E}_T \) such that \( F_*(m) = \beta(\Phi) \) for each \( m \in \Omega \). Using (3) of Theorem 2.1, we have

\[
\Lambda_{F|\Phi}(\beta(\Phi)) = F_*(\mu) = \int_{\mathcal{E}_T} F_*(m) \, d\tau(m) = \int_{\Omega} F_*(m) \, d\tau(m).
\]

Notice that \( F_*(m) \leq \Lambda_{F|\Phi}(\beta(\Phi)) \) for each \( T \)-invariant ergodic measure \( m \in \Omega \), therefore there must exist some \( T \)-invariant ergodic measure \( m \in \Omega \) such that \( \Lambda_{F|\Phi}(\beta(\Phi)) = F_*(m) \). This completes the proof. \( \square \)

Proof of Theorem C. We consider the asymptotically additive potentials \( g_n(x) \equiv n \) and \( \psi_n(x) \equiv n \) for each \( x \in X \) and \( n \in \mathbb{N} \), then Theorem C is a direct consequence of Theorem E. \( \square \)

Proof of Theorem D. In Theorem E we consider the particular asymptotically additive potentials \( g_n(x) \equiv n \) and \( \psi_n(x) \equiv n \) for each \( x \in X \) and \( n \in \mathbb{N} \), then Theorem D is a direct consequence of Theorem E. \( \square \)

Proof of Theorem E. We divide the proof into two parts.

Part I: For any real number \( \alpha \in \mathbb{R} \) such that \( E_{\Phi, \Psi}(\alpha) \neq \emptyset \), we will show that

\[
\sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \text{ and } \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \right\} = \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \bigcup_{x \in E_{\Phi, \Psi}(\alpha)} V(x) \right\}.
\]

Choose a \( T \)-invariant measure \( \mu \in \mathcal{M}_T \) such that \( \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \). Since the TDS \( (X, T) \) satisfies the specification property, there exists some point \( x_0 \in G_\mu \), i.e., \( \delta_{x_0,n} \to \mu \) as \( n \to \infty \). This means that \( V(x_0) = \{\mu\} \). Fix a small number \( \xi > 0 \), let \( \varphi_\xi \) and \( \psi_\xi \) be continuous functions given by (2.3) approximating \( \Phi \) and \( \Psi \) respectively. Note that

\[
\lim_{n \to \infty} \frac{1}{n} S_n \varphi_\xi(x_0) = \int \varphi_\xi \, d\mu \text{ and } \lim_{n \to \infty} \frac{1}{n} S_n \psi_\xi(x_0) = \int \psi_\xi \, d\mu.
\]

Therefore, for all sufficiently large \( n \) we have that

\[
\frac{\varphi_n(x_0)}{\psi_n(x_0)} \leq \frac{S_n \varphi_\xi(x_0) + \xi}{S_n \psi_\xi(x_0) - \xi} \leq \frac{\varphi_\xi \, d\mu + 2\xi}{\psi_\xi \, d\mu - 2\xi} \leq \frac{\Phi_*(\mu) + 3\xi}{\Psi_*(\mu) - 3\xi}.
\]

Similarly, for all sufficiently large \( n \) we obtain

\[
\frac{\varphi_n(x_0)}{\psi_n(x_0)} \geq \frac{\Phi_*(\mu) - 3\xi}{\Psi_*(\mu) + 3\xi}.
\]

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The above two inequalities yield that
\[
\lim_{n \to \infty} \frac{\varphi_n(x_0)}{\psi_n(x_0)} = \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha.
\]

Hence, we know that \( x_0 \in E_{\Phi, \Psi}(\alpha) \). In consequence we have that \( \mu \in \bigcup_{x \in E_{\Phi, \Psi}(\alpha)} \mathcal{V}(x) \).

This yields that
\[
\sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \text{ and } \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \right\} \leq \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \bigcup_{x \in E_{\Phi, \Psi}(\alpha)} \mathcal{V}(x) \right\}.
\]

Conversely, given \( \mu \in \mathcal{V}(x) \) for some \( x \in E_{\Phi, \Psi}(\alpha) \), we may assume that there exists a subsequence of positive integers \( \{n_k\}_{k \geq 1} \) such that \( \delta_{x,n_k} \to \mu \) as \( k \to \infty \). Clearly, \( \mu \) is a \( T \)-invariant measure. Fix a small number \( \xi > 0 \), let \( \varphi_\xi \) and \( \psi_\xi \) be continuous functions given by (2.3) approximating \( \Phi \) and \( \Psi \) respectively. Then we have
\[
\int \varphi_\xi \, d\mu = \lim_{k \to \infty} \int \varphi_\xi \, d\delta_{x,n_k} = \lim_{k \to \infty} \frac{1}{n_k} S_{n_k} \varphi_\xi(x)
\]
and
\[
\int \psi_\xi \, d\mu = \lim_{k \to \infty} \int \psi_\xi \, d\delta_{x,n_k} = \lim_{k \to \infty} \frac{1}{n_k} S_{n_k} \psi_\xi(x).
\]

Hence, for all sufficiently large \( k \) we have
\[
\frac{\Phi_*(\mu)}{\Psi_*(\mu)} \leq \frac{\int \varphi_\xi \, d\mu + \xi}{\int \psi_\xi \, d\mu - \xi} \leq \frac{\frac{1}{n_k} S_{n_k} \varphi_\xi(x) + 2\xi}{\frac{1}{n_k} S_{n_k} \psi_\xi(x) - 2\xi} \leq \frac{\varphi_{n_k}(x) + 3\xi}{\psi_{n_k}(x) - 3\xi} \leq \alpha + 5\xi,
\]
the last inequality holds since \( x \in E_{\Phi, \Psi}(\alpha) \). Similarly, we can obtain that
\[
\frac{\Phi_*(\mu)}{\Psi_*(\mu)} \geq \alpha - 5\xi.
\]

The arbitrariness of \( \xi \) implies that
\[
\frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha.
\]

This yields that
\[
\sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \text{ and } \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \right\} \geq \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \bigcup_{x \in E_{\Phi, \Psi}(\alpha)} \mathcal{V}(x) \right\}.
\]

Combining the above arguments, the desired result immediately follows.
**Part II:** In this part, we will show that
\[
\sup_{x \in E_{\Phi, \Psi}(\alpha)} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} = \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \text{ and } \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \right\}.
\]

The strategy of the proof is to approximate the asymptotically additive potentials by appropriate Birkhoff sums associated with some continuous function.

Given a small number \(\xi > 0\), since \(\mathcal{F} = \{f_n\}_{n=1}^\infty\) is an AAP, there exists a continuous function \(f_\xi \in C(X)\) such that
\[
\frac{1}{n} S_n f_\xi(x) - \xi \leq \frac{1}{n} f_n(x) \leq \frac{1}{n} S_n f_\xi(x) + \xi, \quad \forall x \in X
\]
for all sufficiently large \(n\). This yields that
\[
F_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int f_n(x) \, d\mu = \lim_{\xi \to 0} \frac{1}{n} \int f_\xi(x) \, d\mu
\]
for each \(T\)-invariant measure \(\mu \in \mathcal{M}_T\). Similarly, for the AAP \(\mathcal{G} = \{g_n\}_{n \geq 1}\), there exists a continuous function \(g_\xi\) such that the corresponding properties hold.

Fix a number \(\alpha \in \mathbb{R}\) such that \(E_{\Phi, \Psi}(\alpha) \neq \emptyset\). Take \(x \in E_{\Phi, \Psi}(\alpha)\) and choose a subsequence of positive integers \(\{n_j\}_{j \geq 1}\) such that the following properties hold:

(i) \(\lim_{j \to \infty} \frac{f_{n_j}(x)}{g_{n_j}(x)} = \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)}\);

(ii) \(\delta_{x,n_j} \to \mu\) for some \(\mu \in \mathcal{M}_T\) as \(j \to \infty\).

Take the same \(\xi > 0\) as above, and \(\varphi_\xi\) and \(\psi_\xi\) are continuous functions given by (2.3) approximating \(\Phi\) and \(\Psi\) respectively. Note that
\[
\int \varphi_\xi \, d\mu = \lim_{j \to \infty} \int \varphi_\xi \, d\delta_{x,n_j} = \lim_{j \to \infty} \frac{1}{n_j} S_{n_j} \varphi_\xi(x)
\]
and
\[
\int \psi_\xi \, d\mu = \lim_{j \to \infty} \int \psi_\xi \, d\delta_{x,n_j} = \lim_{j \to \infty} \frac{1}{n_j} S_{n_j} \psi_\xi(x).
\]
Therefore, for all sufficiently large \(j\) we have
\[
\frac{\Phi_*(\mu)}{\Psi_*(\mu)} \leq \frac{\int \varphi_\xi \, d\mu + \xi}{\int \psi_\xi \, d\mu - \xi} \leq \frac{\frac{1}{n_j} S_{n_j} \varphi_\xi(x) + 2\xi}{\frac{1}{n_j} S_{n_j} \psi_\xi(x) - 2\xi} \leq \frac{\varphi_{n_j}(x) + 3\xi}{\psi_{n_j}(x) - 3\xi} \leq \alpha + 5\xi,
\]
where the last inequality holds since \(x \in E_{\Phi, \Psi}(\alpha)\). Similarly, we can prove that
\[
\frac{\Phi_*(\mu)}{\Psi_*(\mu)} \geq \alpha - 5\xi.
\]
By the arbitrariness of $\xi$, we have
\[
\frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha.
\]

On the other hand, note that
\[
\frac{1}{n_j} \| f_{n_j}(x) - S_{n_j} f_\xi \| < \xi \quad \text{and} \quad \frac{1}{n_j} \| g_{n_j}(x) - S_{n_j} g_\xi \| < \xi
\]
for all sufficiently large $j$. Using properties (i) and (ii), we have that
\[
\limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} = \lim_{j \to \infty} \frac{f_{n_j}(x)}{g_{n_j}(x)} \\
\leq \lim_{j \to \infty} \frac{S_{n_j} f_\xi(x) + n_j \xi}{S_{n_j} g_\xi(x) - n_j \xi} \\
= \lim_{j \to \infty} \frac{\int f_\xi \, d\delta_{x,n_j} + \xi}{\int g_\xi \, d\delta_{x,n_j} - \xi} \\
= \frac{\int f_\xi \, d\mu + \xi}{\int g_\xi \, d\mu - \xi}.
\]

Letting $\xi \to 0$, by (4.10) we have
\[
\limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \leq \frac{\mathcal{F}_*(\mu)}{\mathcal{G}_*(\mu)}.
\]

This implies that
\[
\sup_{x \in E_{\Phi, \Psi}(\alpha)} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \leq \sup \left\{ \frac{\mathcal{F}_*(\mu)}{\mathcal{G}_*(\mu)} : \mu \in \mathcal{M}_T \text{ and } \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \right\}.
\]

To prove the reverse inequality, note that the set of $T$-invariant measure $\mu$ such that $\frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha$ is a compact subset of $\mathcal{M}_T$. Choose such a measure $\mu$ so that
\[
\frac{\mathcal{F}_*(\mu)}{\mathcal{G}_*(\mu)} = \sup \left\{ \frac{\mathcal{F}_*(\mu)}{\mathcal{G}_*(\mu)} : \mu \in \mathcal{M}_T \text{ and } \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \right\}.
\]

Fix a small number $\xi > 0$, and $\varphi_\xi$ and $\psi_\xi$ are continuous functions given by (2.3) approximating $\Phi$ and $\Psi$ respectively. By the specification property of the TDS $(X,T)$, there exists $x_0 \in G_\mu$, i.e., $\delta_{x_0,n} \to \mu$ as $n \to \infty$. Hence,
\[
\lim_{n \to \infty} \frac{1}{n} S_n \varphi_\xi(x_0) = \lim_{n \to \infty} \int \varphi_\xi \, d\delta_{x_0,n} = \int \varphi_\xi \, d\mu
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} S_n \psi_\xi(x_0) = \lim_{n \to \infty} \int \psi_\xi \, d\delta_{x_0,n} = \int \psi_\xi \, d\mu.
\]
Therefore, for all sufficiently large $n$ we have that
\[
\frac{\varphi_n(x_0)}{\psi_n(x_0)} \leq \frac{S_n\varphi(x_0) + \xi}{S_n\psi(x_0) - \xi} \leq \frac{\int \varphi \, d\mu + 2\xi}{\int \psi \, d\mu - 2\xi} \leq \frac{\Phi_*(\mu) + 3\xi}{\Psi_*(\mu) - 3\xi}.
\]
Similarly, for all sufficiently large $n$ we can prove that
\[
\frac{\varphi_n(x_0)}{\psi_n(x_0)} \geq \frac{\Phi_*(\mu) - 3\xi}{\Psi_*(\mu) + 3\xi}.
\]
Hence,
\[
\lim_{n \to \infty} \frac{\varphi_n(x_0)}{\psi_n(x_0)} = \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha.
\]
This means that $x_0 \in E_{\Phi,\Psi}(\alpha)$. On the other hand, by (4.9) we have that
\[
\limsup_{n \to \infty} \frac{f_n(x_0)}{g_n(x_0)} \geq \limsup_{n \to \infty} \frac{S_n f_\xi(x_0) - n\xi}{S_n g_\xi(x_0) + n\xi} = \limsup_{n \to \infty} \frac{\int f_\xi \, d\delta_{x_0,n} - \xi}{\int g_\xi \, d\delta_{x_0,n} + \xi} = \frac{\int f_\xi \, d\mu - \xi}{\int g_\xi \, d\mu + \xi}.
\]
Letting $\xi \to 0$, by (4.10) we have
\[
\limsup_{n \to \infty} \frac{f_n(x_0)}{g_n(x_0)} \geq \frac{F_*(\mu)}{G_*(\mu)}.
\]
It follows that
\[
\sup_{x \in E_{\Phi,\Psi}(\alpha)} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \geq \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \text{ and } \frac{\Phi_*(\mu)}{\Psi_*(\mu)} = \alpha \right\}.
\]
This finishes the proof of Theorem E. \hfill \Box

Next we turn to prove Theorem F, some of the methods was used by Thompson in [34] for the multifractal analysis of Birkhoff averages of a continuous function, in that paper he proved that the irregular set is either empty or carries full topological pressure when the system has specification property. His ideas is originally due to the work of Takens and Verbitskiy [32, 33] and Chen et al. [14].

Proof of Theorem F. We first prove that
\[
\sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{E}_T \right\} = \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \right\}.
\]  (4.11)

Since $\mathcal{E}_T \subset \mathcal{M}_T$, it is clear that
\[
\sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{E}_T \right\} \leq \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \right\}.
\]
To prove the reverse inequality, since the map $\mu \mapsto \frac{F_*(\mu)}{G_*(\mu)}$ is continuous on the space $\mathcal{M}_T$ we can choose a $T$-invariant measure $\mu \in \mathcal{M}_T$ such that
\[
\frac{F_*(\mu)}{G_*(\mu)} = \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \right\}.
\]
By (3) of Theorem 2.1, using the ergodic decomposition theorem (see [37]) we have that
\[
F_*(\mu) = \int_{E_T} F_*(m) \, d\tau(m) \quad \text{and} \quad G_*(\mu) = \int_{E_T} G_*(m) \, d\tau(m),
\]
where $\tau$ is a probability measure on the space $\mathcal{M}_T$ such that $\tau(E_T) = 1$. Let $\lambda := \frac{F_*(\mu)}{G_*(\mu)}$, we have
\[
0 = \int_{E_T} F_*(m) - \lambda G_*(m) \, d\tau(m) = \int_{E_T} G_*(m) \left( \frac{F_*(m)}{G_*(m)} - \lambda \right) \, d\tau(m).
\]
Since $G_*(m) \geq \sigma > 0$ and $\frac{F_*(m)}{G_*(m)} \leq \lambda$ for each $m \in E_T$, for $\tau$-a.e. $m \in E_T$ we have
\[
\lambda = \frac{F_*(m)}{G_*(m)}.
\]
Thus formula (4.11) immediately follows.

Next, we prove the second equality of the theorem. It is clear that
\[
\sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{M}_T \right\} \geq \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \bigcup_{x \in \hat{X}_{\Phi,\Psi}} V(x) \right\},
\]
since every measure $\mu \in \bigcup_{x \in \hat{X}_{\Phi,\Psi}} V(x)$ is $T$-invariant.

To prove the reverse inequality, by formula (4.11) it suffices to prove that $E_T \subset \bigcup_{x \in \hat{X}_{\Phi,\Psi}} V(x)$.

Given $\mu_1 \in E_T$, there must exists some $T$-invariant ergodic measure $\mu_2$ satisfying
\[
\frac{\Phi_*(\mu_1)}{\Psi_*(\mu_1)} \neq \frac{\Phi_*(\mu_2)}{\Psi_*(\mu_2)},
\]
since $\inf_{\mu \in \mathcal{M}_T} \frac{\Phi_*(\mu)}{\Psi_*(\mu)} \leq \sup_{\mu \in \mathcal{M}_T} \frac{\Phi_*(\mu)}{\Psi_*(\mu)}$. For $i = 1, 2$, choose a point $x_i$ satisfies
\[
\lim_{n \to \infty} \frac{\varphi_n(x_i)}{\psi_n(x_i)} = \frac{\Phi_*(\mu_i)}{\Psi_*(\mu_i)}.
\]
Let $m_k := m(\epsilon/2^k)$ be as in the definition of specification and $N_k$ be a sequence of integers chosen to grow to infinity sufficiently rapidly that $N_{k+1} > \exp \sum_{i=1}^k (N_i + m_i)$. 

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We define a sequence of points \( \{z_i\}_{i \geq 1} \subset X \) inductively using the specification property. For \( x, y \in X \) and \( n \in \mathbb{N} \), define a new metric on \( X \) as follows

\[
d_n(x, y) := \max \{ d(T^i x, T^i y) : i = 0, 1, \ldots, n - 1 \}.
\]

And the dynamical ball centered at \( x \) of radius \( r \) and length \( n \) is denoted by \( B_n(x, r) := \{ y \in X : d_n(x, y) < r \} \). Let \( t_1 = N_1, t_k = t_{k-1} + m_k + N_k \) for \( k \geq 2 \) and \( \rho(k) := (k + 1)(\text{mod} \ 2) + 1 \). Let \( z_1 = x_1 \). Let \( z_2 \) satisfy

\[
d_{N_1}(z_2, z_1) < \frac{\varepsilon}{4} \text{ and } d_{N_2}(T^{N_1 + m_2} z_2, x_2) < \frac{\varepsilon}{4}.
\]

For \( k > 2 \), let \( z_k \) satisfy

\[
d_{t_{k-1}}(z_k, z_{k-1}) < \frac{\varepsilon}{2^{k-1}} \text{ and } d_{N_k}(T^{t_{k-1} + m_k} z_k, x_{\rho(k)}) < \frac{\varepsilon}{2^k}.
\]

Note that if \( q \in \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1}) \), then

\[
d_{t_{k-1}}(q, z_{k-1}) \leq d_{t_{k-1}}(q, z_k) + d_{t_{k-1}}(z_k, z_{k-1}) \leq \frac{\varepsilon}{2^{k-1}} + \frac{\varepsilon}{2^k} < \frac{\varepsilon}{2^{k-2}},
\]

and thus \( \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1}) \subset \overline{B}^*_{t_{k-1}}(z_{k-1}, \varepsilon/2^{k-2}) \). Hence, we can define a point by

\[
p := \bigcap_{k \geq 1} \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1}).
\]

For each continuous function \( \phi \in C(X) \), since the orbit of \( p \) alternates between approximating increasingly long orbit segments of \( x_1 \) and \( x_2 \), we can show that

\[
\frac{1}{t_k} S_{t_k} \phi(p) \to \int \phi \, d\mu_{\rho(k)} \ (k \to \infty).
\]

Thus \( \delta_{p, t_{2k+1}} \to \mu_1, \delta_{p, t_{2k}} \to \mu_2 \) as \( k \to \infty \) and, so \( \mu_i \in \mathcal{V}(p) \) for \( i = 1, 2 \). To complete the proof of the second equality of this theorem, it is left to show that

\[
p \in \hat{X}_{\phi, \psi}.
\]

Indeed, fix a small number \( \xi > 0 \), by the definition of asymptotically additive potentials there exist continuous functions \( \varphi_\xi \) and \( \psi_\xi \) approximating \( \Phi \) and \( \Psi \) respectively. Applying \((4.12)\) for continuous functions \( \varphi_\xi \) and \( \psi_\xi \), for all sufficiently large \( k \) we have

\[
\frac{\varphi_{t_{2k}}(p)}{\psi_{t_{2k}}(p)} \leq \frac{S_{t_{2k}} \varphi_\xi(p) + t_{2k} \xi}{S_{t_{2k}} \psi_\xi(p) - t_{2k} \xi} = \frac{\int \varphi_\xi \, d\delta_{p, t_{2k}} + \xi}{\int \psi_\xi \, d\delta_{p, t_{2k}} - \xi} \leq \frac{\int \varphi_\xi \, d\mu_2 + 2 \xi}{\int \psi_\xi \, d\mu_2 - 2 \xi} \leq \frac{\Phi_\xi(\mu_2) + 3 \xi}{\Phi_\xi(\mu_2) - 3 \xi}.
\]
Similarly, we can prove that
\[
\frac{\varphi_{t_{2k}}(p)}{\psi_{t_{2k}}(p)} \geq \frac{\Phi_*(\mu_2) - 3\xi}{\Psi_*(\mu_2) + 3\xi}.
\]
Hence,
\[
\lim_{k \to \infty} \frac{\varphi_{t_{2k}}(p)}{\psi_{t_{2k}}(p)} = \frac{\Phi_*(\mu_2)}{\Psi_*(\mu_2)}.
\]
By the same arguments, we can prove that
\[
\lim_{k \to \infty} \frac{\varphi_{t_{2k+1}}(p)}{\psi_{t_{2k+1}}(p)} = \frac{\Phi_*(\mu_1)}{\Psi_*(\mu_1)}.
\]
This implies that \( p \in \hat{X}_{\Phi, \Psi} \). Hence,
\[
\mathcal{E}_T \subset \bigcup_{x \in \hat{X}_{\Phi, \Psi}} \mathcal{V}(x).
\]
This completes the proof of the second equality of Theorem F.

To prove the last equality, by the Proposition 4.1 below and (4.11) we have
\[
\sup_{x \in \hat{X}_{\Phi, \Psi}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \leq \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{E}_T \right\}.
\]
Next we prove the reverse inequality. Pick a \( T \)-invariant ergodic measure \( \tilde{\mu}_1 \) such that
\[
\frac{F_*(\tilde{\mu}_1)}{G_*(\tilde{\mu}_1)} = \sup \left\{ \frac{F_*(\mu)}{G_*(\mu)} : \mu \in \mathcal{E}_T \right\}.
\]
Then we choose another \( T \)-invariant ergodic measure \( \tilde{\mu}_2 \) so that
\[
\frac{\Phi_*(\tilde{\mu}_1)}{\Psi_*(\tilde{\mu}_1)} \neq \frac{\Phi_*(\tilde{\mu}_2)}{\Psi_*(\tilde{\mu}_2)}.
\]
Repeating the arguments in the proof of the second equality, we can find a point \( \tilde{p} \in \hat{X}_{\Phi, \Psi} \) such that \( \delta_{\tilde{p}, t_{2k+1}} \to \tilde{\mu}_1 \) and \( \delta_{\tilde{p}, t_{2k}} \to \tilde{\mu}_2 \) as \( k \to \infty \). Fix a small number \( \xi > 0 \), by the definition of asymptotically additive potentials there exist continuous functions \( f_\xi \) and \( g_\xi \) approximating \( F \) and \( G \) respectively. Applying (4.12) for continuous functions \( f_\xi \) and \( g_\xi \), for all sufficiently large \( k \) we have
\[
\frac{f_{t_{2k+1}}(\tilde{p})}{g_{t_{2k+1}}(\tilde{p})} \leq \frac{S_{t_{2k+1}} f_\xi(\tilde{p}) + t_{2k+1}\xi}{S_{t_{2k+1}} g_\xi(\tilde{p}) - t_{2k+1}\xi} = \frac{\int f_\xi \, d\delta_{\tilde{p}, t_{2k+1}} + \xi}{\int g_\xi \, d\delta_{\tilde{p}, t_{2k+1}} - \xi} \leq \frac{\int f_\xi \, d\tilde{\mu}_1 + 2\xi}{\int g_\xi \, d\tilde{\mu}_1 - 2\xi} \leq \frac{F_*(\tilde{\mu}_1) + 3\xi}{G_*(\tilde{\mu}_1) - 3\xi}.
\]
Similarly, we can prove that
\[
\frac{f_{t_{2k+1}}(\tilde{p})}{g_{t_{2k+1}}(\tilde{p})} \geq \frac{F_*(\tilde{\mu}_1) - 3\xi}{G_*(\tilde{\mu}_1) + 3\xi}.
\]
Hence,
\[ \lim_{k \to \infty} \frac{f_{2k+1}(\bar{p})}{g_{2k+1}(\bar{p})} = \frac{F_s(\bar{\mu}_1)}{G_s(\bar{\mu}_1)}. \]

This implies that
\[ \sup_{x \in \mathcal{X}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \geq \sup \left\{ \frac{F_s(\mu)}{G_s(\mu)} : \mu \in \mathcal{M}_T \right\}. \]

The proof of this theorem is completed. \(\square\)

**Proposition 4.1.** Let \( \mathcal{F} = \{f_n\}_{n \geq 1} \) and \( \mathcal{G} = \{g_n\}_{n \geq 1} \) be two AAPs. Assume that \( \mathcal{G} \) satisfies (2.5), then
\[ \sup_{x \in \mathcal{X}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} = \sup_{x \in \mathcal{R}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} = \limsup_{n \to \infty} \max_{x \in \mathcal{X}} \frac{f_n(x)}{g_n(x)} = \sup \left\{ \frac{F_s(\mu)}{G_s(\mu)} : \mu \in \mathcal{M}_T \right\}, \]
where \( \mathcal{R} = \{ x \in \mathcal{X} : \lim_{n \to \infty} \frac{f_n(x)}{g_n(x)} \text{ exists} \} \).

**Proof.** It is clear that
\[ \sup_{x \in \mathcal{R}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \leq \sup_{x \in \mathcal{X}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \leq \limsup_{n \to \infty} \max_{x \in \mathcal{X}} \frac{f_n(x)}{g_n(x)}. \]

Given a \( T \)-invariant ergodic measure \( \mu \), by Theorem 2.1 there exist a point \( x \) such that
\[ \lim_{n \to \infty} \frac{f_n(x)}{g_n(x)} = \frac{F_s(\mu)}{G_s(\mu)}. \]

This together with (4.11) imply that
\[ \sup_{x \in \mathcal{R}} \limsup_{n \to \infty} \frac{f_n(x)}{g_n(x)} \geq \sup \left\{ \frac{F_s(\mu)}{G_s(\mu)} : \mu \in \mathcal{M}_T \right\}. \]

On the other hand, for each \( n \) choose a point \( x_n \in \mathcal{X} \) such that
\[ \frac{f_n(x_n)}{g_n(x_n)} = \max_{x \in \mathcal{X}} \frac{f_n(x)}{g_n(x)}. \]

Put \( \mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x_n} \), and let \( \mu \) be a weak* limit point of \( \{\mu_n\}_{n \geq 1} \). Without loss of generality, assume that \( \mu_n \to \mu \) as \( n \to \infty \). Fix a small number \( \xi > 0 \), let \( f_\xi \) and \( g_\xi \) be the continuous functions in the definition of asymptotically additive potentials that approximating \( \mathcal{F} \) and \( \mathcal{G} \) respectively. Hence, for all sufficiently large \( n \) we have
\[ \frac{f_n(x_n)}{g_n(x_n)} \leq \frac{1}{n} S_n f_\xi(x_n) + \xi = \int g_\xi d\mu_n + \xi \]
and this yields that
\[ \limsup_{n \to \infty} \max_{x \in \mathcal{X}} \frac{f_n(x)}{g_n(x)} \leq \lim_{n \to \infty} \int g_\xi d\mu_n - \xi \leq \int g_\xi d\mu + \xi. \]

Letting \( \xi \to 0 \) yields that
\[ \limsup_{n \to \infty} \max_{x \in \mathcal{X}} \frac{f_n(x)}{g_n(x)} \leq \frac{F_s(\mu)}{G_s(\mu)}. \]

This yields the desired result. \(\square\)
5 Application to suspension flows

This section provides applications of our main results to suspension flows. Let \( T : X \to X \) be a homeomorphism of a compact metric space \( X \) with metric \( d \), and \( \tau : X \to (0, \infty) \) is a continuous roof function. The suspension space is defined as follows:

\[
X_\tau := \{(x, s) \in X \times \mathbb{R} : 0 \leq s \leq \tau(x)\},
\]

where \((x, \tau(x))\) is identified with \((T(x), 0)\) for all \( x \). There is a natural topology on \( X_\tau \) which makes \( X_\tau \) a compact topological space. This topology is induced by a distance introduced by Bowen and Walters in [9] (see the appendix in [3] for details).

The suspension flow \( \Theta = (\theta_t)_t \) on \( X_\tau \) is defined by \( \theta_t(x, s) = (x, s + t) \). For a continuous function \( \Phi : X_\tau \to \mathbb{R} \), we associate the function \( \varphi : X \to \mathbb{R} \) by

\[
\varphi(x) = \int_0^{\tau(x)} \Phi(x, t) \, dt. \tag{5.13}
\]

Since the roof function \( \tau \) is continuous, so is the function \( \varphi \). If \( x \in X \) and \( s \in [0, \tau(x)] \), we have (see [31, Lemma 5.3])

\[
\lim \inf_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\theta_t(x, s)) \, dt = \lim \inf_{n \to \infty} \frac{S_n \varphi(x)}{S_n \tau(x)}. \tag{5.14}
\]

The above formula remains true if we replace \( \lim \inf \) by \( \lim \sup \).

For any \( T \)-invariant measure \( \mu \in \mathcal{M}_T \), we define the measure \( \mu_\tau \) on the suspension space \( X_\tau \) by

\[
\int_{X_\tau} \Phi \, d\mu_\tau = \frac{\int_X \varphi \, d\mu}{\int_X \tau \, d\mu}
\]

for all continuous function \( \Phi \in C(X_\tau) \), where \( \varphi \) is defined as (5.13). It is well-known that the measure \( \mu_\tau \) is \( \Theta \)-invariant, i.e., \( \mu(\theta_t^{-1}A) = \mu(A) \) for all \( t \geq 0 \) and measurable sets \( A \). The map \( \mathcal{L} : \mathcal{M}_T \to \mathcal{M}_\Theta \) given by \( \mu \mapsto \mu_\tau \) is a bijection, where \( \mathcal{M}_\Theta \) is the space of all \( \Theta \)-invariant measures on the suspension space \( X_\tau \).

For any real number \( \alpha \), we consider the level set associated to a continuous \( \Phi : X_\tau \to \mathbb{R} \) as follows:

\[
K(\Phi, \alpha) := \{(x, s) \in X_\tau : \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\theta_t(x, s)) \, dt = \alpha\}.
\]

**Theorem 5.1.** Let \( T : X \to X \) be a homeomorphism on a compact metric space \((X, d)\) with the specification property, \( \tau : X \to (0, \infty) \) a continuous roof function, and \((X_\tau, \Theta)\) the corresponding suspension flow over \( X \). Let \( \Phi, H \in C(X_\tau) \), for each \( \alpha \) with \( K(\Phi, \alpha) \neq \emptyset \) we have

\[
\sup_{(x,s) \in K(\Phi, \alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T H(\theta_t(x, s)) \, dt = \sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in \mathcal{M}_\Psi, \int \Phi \, d\mu_\tau = \alpha \right\}
\]

\[
= \sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in \bigcup_{(x,s) \in K(\Phi, \alpha)} \mathcal{V}((x,s)) \right\}
\]

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where \( \mathcal{V}((x, s)) \) is the set of limit point of the sequence \( \delta_{(x, s), T} := \frac{1}{T} \int_0^T \delta_{\theta_t(x, s)} \, dt. \)

**Proof.** Let \( \varphi, h \in C(X) \) be the continuous function on \( X \) associated to \( \Phi \) and \( H \) respectively, that is

\[
\varphi(x) = \int_0^{\tau(x)} \Phi(x, t) \, dt \quad \text{and} \quad h(x) = \int_0^{\tau(x)} H(x, t) \, dt.
\]

For a real number \( \alpha \) with \( K(\Phi, \alpha) \neq \emptyset \), let

\[
\Gamma(\alpha) := \left\{ x \in X : \lim_{n \to \infty} \frac{S_n \varphi(x)}{S_n \tau(x)} = \alpha \right\}.
\]

Using (5.14), \( x \in \Gamma(\alpha) \) if and only if \( (x, s) \in K(\Phi, \alpha) \) for all \( 0 \leq s \leq \tau(x) \). Hence,

\[
\sup_{(x, s) \in K(\Phi, \alpha)} \limsup_{T \to \infty} \frac{1}{T} \int_0^T H(\theta_t(x, s)) \, dt = \sup_{x \in \Gamma(\alpha)} \limsup_{n \to \infty} \frac{S_n h(x)}{S_n \tau(x)}.
\]

In Theorem 1, consider particular asymptotically additive potentials \( g_n(x) = \psi_n(x) = S_n \tau(x), f_n(x) = S_n h(x) \) and \( \varphi_n(x) = S_n \varphi(x) \) for each \( x \in X \) and \( n \in \mathbb{N} \). Then we have

\[
\sup_{x \in \Gamma(\alpha)} \limsup_{n \to \infty} \frac{S_n h(x)}{S_n \tau(x)} = \sup \left\{ \int h \, d\mu : \mu \in \mathcal{M}_T \text{ and } \int \frac{\varphi}{\tau} \, d\mu = \alpha \right\}
\]

\[
= \sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in \mathcal{M}_T \text{ and } \int \Phi \, d\mu_\tau = \alpha \right\},
\]

the above last equality holds since

\[
\int H \, d\mu_\tau = \int h \, d\mu \quad \int \Phi \, d\mu_\tau = \int \varphi \, d\mu
\]

for some \( \mu \in \mathcal{M}_T \) and the map \( \mu \mapsto \mu_\tau \) is a bijection. To finish the proof of the theorem, by Theorem 2 it suffices to prove that

\[
\sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in \bigcup_{(x, s) \in K(\Phi, \alpha)} \mathcal{V}((x, s)) \right\} = \sup \left\{ \int h \, d\mu : \mu \in \bigcup_{x \in \Gamma(\alpha)} \mathcal{V}(x) \right\}.
\]

Indeed, for each \( \mu \in \mathcal{V}(x) \) for some \( x \in \Gamma(\alpha) \), we know that \( (x, s) \in K(\Phi, \alpha) \) for any \( 0 \leq s < \tau(x) \). Furthermore, by (5.14) we know that the corresponding measure \( \mu_\tau \) is a limit point of the sequence \( \delta_{(x, s), T} \). Conversely, for each \( \mu_\tau \in \bigcup_{(x, s) \in K(\Phi, \alpha)} \mathcal{V}((x, s)) \), by a standard argument we can show that the corresponding measure \( \mu \) is a limit point of the sequence \( \delta_{x, n} \). This observation yields (5.15).

For a continuous function \( \Phi : X_\tau \to \mathbb{R} \) on the suspension space, we consider its irregular set as follows:

\[
\hat{X}_\Phi := \left\{ (x, s) \in X_\tau : \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\theta_t(x, s)) \, dt \text{ does not exist} \right\}.
\]
Theorem 5.2. Let $T : X \to X$ be a homeomorphism on a compact metric space $(X, d)$ with the specification property, $\tau : X \to (0, \infty)$ a continuous roof function, and $(X, \Theta)$ the corresponding suspension flow over $X$. Let $\Phi, H \in C(X, \Theta)$, if

$$\inf_{\mu_\tau \in M_\Theta} \int \Phi \, d\mu_\tau < \sup_{\mu_\tau \in M_\Theta} \int \Phi \, d\mu_\tau$$

then we have

$$\sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in M_\Theta \right\} = \sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in \bigcup_{(x,s) \in \hat{X}_\Phi} \mathcal{V}((x,s)) \right\}$$

$$= \sup \limsup_{(x,s) \in \hat{X}_\Phi} \frac{1}{T} \int_0^T H(\theta_t(x,s)) \, dt.$$

Proof. First note that

$$\sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in M_\Theta \right\} = \sup \left\{ \int \frac{h \, d\mu}{\tau \, d\mu} : \mu \in M_T \right\}.$$

Let $\hat{X}_{\varphi,\tau} = \left\{ x \in X : \lim_{n \to \infty} \frac{S_n \varphi(x)}{S_n \tau(x)} \text{ does not exist} \right\}$, by (5.14) we know that $x \in \hat{X}_{\varphi,\tau}$ if and only if $(x,s) \in \hat{X}_\Phi$ for all $0 \leq s \leq \tau(x)$. Using similar arguments as the proof of Theorem 5.1 we have that

$$\sup \left\{ \int H \, d\mu_\tau : \mu_\tau \in \bigcup_{(x,s) \in \hat{X}_\Phi} \mathcal{V}((x,s)) \right\} = \sup \left\{ \int \frac{h \, d\mu}{\tau \, d\mu} : \mu \in \bigcup_{x \in \hat{X}_{\varphi,\tau}} \mathcal{V}(x) \right\}.$$

On the other hand, note that

$$\sup \limsup_{(x,s) \in \hat{X}_\Phi} \frac{1}{T} \int_0^T H(\theta_t(x,s)) \, dt = \sup_{x \in \hat{X}_{\varphi,\tau}} \lim_{n \to \infty} \frac{S_n h(x)}{S_n \tau(x)}$$

and $\inf_{\mu_\tau \in M_\Theta} \int \Phi \, d\mu_\tau < \sup_{\mu_\tau \in M_\Theta} \int \Phi \, d\mu_\tau$ is equivalent to $\inf_{\mu \in M_T} \int \frac{\varphi \, d\mu}{\tau \, d\mu} < \sup_{\mu \in M_T} \int \frac{\varphi \, d\mu}{\tau \, d\mu}$. Consider $G = \Psi = \{S_n \tau\}_{n \geq 1}$, $\Phi = \{S_n \varphi\}_{n \geq 1}$ and $F = \{S_n h\}_{n \geq 1}$ in Theorem 14 the desired result immediately follows.

Acknowledgements. This work was finally completed when I am supported by CSC to visit Pennsylvania State University in the academic year 2013-2014. I would like to take this chance to thank Professor Yakov Pesin for the warm hospitality. This work is partially supported by NSFC (11371271).

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