Perturbative calculation of the clover term for Wilson fermions in any representation of the gauge group SU($N$)

S. Musberg, G. Münster, S. Piemonte

Universität Münster, Institut für Theoretische Physik
Wilhelm-Klemm-Str. 9, D-48149 Münster, Germany

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We calculate the Sheikholeslami-Wohlert coefficient of the O($a$) improvement term for Wilson fermions in any representation of the gauge group SU($N$) perturbatively at the one-loop level. The result applies to QCD with adjoint quarks and to $\mathcal{N} = 1$ supersymmetric Yang-Mills theory on the lattice.

In recent years gauge theories with fermions in representations of the gauge group different from the fundamental representation have gained interest in the context of technicolor and orbifold models. A special case is that of fermions in the adjoint representation. Non-perturbative properties of such models have been investigated by means of numerical simulations on a lattice, see [1] for a recent review. Another physically relevant model with fermions in a non-fundamental representation of the gauge group is $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, which contains Majorana fermions in the adjoint representation. Recent numerical studies of this model are presented in [2, 3, 4].

The numerical simulations of such models are commonly done with Wilson fermions, which are afflicted with discretisation errors of $O(a)$, where $a$ is the lattice spacing. These cutoff effects are not negligible in the currently used parameter ranges. Therefore it is important to employ Symanzik-improved actions [5, 6] in order to reduce the cutoff effects to $O(a^2)$. The $O(a)$ improvement of Wilson fermions can be achieved by adding the so-called clover term

$$\mathcal{L}_1 = -a c_{sw} \frac{1}{4} \bar{\psi} \sigma_{\mu\nu} \hat{F}_{\mu\nu} \psi$$  (1)
to the lattice Lagrangian \( \mathcal{L} \), where one defines \( \sigma_{\mu\nu} = (1/2i)[\gamma_{\mu}, \gamma_{\nu}] \) in the Euclidean domain, and \( \tilde{F}_{\mu\nu} \) is the clover version of the gauge field strength. The Sheikholeslami-Wohlert coefficient \( c_{\text{SW}} \) depends on the gauge group representation of the fermions and on the bare gauge coupling \( g \). Its knowledge is crucial for the implementation of \( \mathcal{O}(a) \) improvement. The coefficient \( c_{\text{SW}} \) has a perturbative expansion

\[
c_{\text{SW}} = c^{(0)}_{\text{SW}} + c^{(1)}_{\text{SW}} g^2 + \mathcal{O}(g^4).
\]

For Wilson fermions in the fundamental representation of gauge group \( SU(N) \) the tree level coefficient \( c^{(0)}_{\text{SW}} \) and the one-loop coefficient \( c^{(1)}_{\text{SW}} \) have been calculated by Wohlert [8], and confirmed in different settings in [9, 10, 11]. Non-perturbative determinations of \( c_{\text{SW}} \) for fundamental Wilson fermions have been done in [12, 13] within numerical simulations of lattice QCD.

For Wilson fermions in the adjoint representation of \( SU(2) \) numerical results for \( c_{\text{SW}} \) have been obtained in [14]. The corresponding perturbative calculation had, however, not yet been done. In this article we present the perturbative result for \( c_{\text{SW}} \) in the one-loop approximation. The gauge action considered is the common plaquette action. Our calculation follows [11], where the fermion-gluon vertex has been considered in lattice perturbation theory with the clover-improved Wilson action. The coefficient \( c_{\text{SW}} \) is then chosen such that \( \mathcal{O}(a) \) terms vanish. Intermediate infrared divergences are regulated by a gluon mass, which is set to zero at the end.

Let \( R \) be an irreducible representation of \( SU(N) \) with dimension \( d_R \). Dirac fermions in the representation \( R \) are in some basis described through their components \( \psi_j(x) \), \( j = 1, \ldots, d_R \), which are Dirac spinors. The hermitean generators of \( SU(N) \) in representation \( R \) are denoted \( T^a_R \) with \( a \) running from 1 to \( N^2 - 1 \). They are \( d_R \times d_R \) matrices, and their commutators obey the Lie algebra

\[
[T^a_R, T^b_R] = i f^{abc} T^c_R.
\]

with structure constants \( f^{abc} \). The generators in the fundamental representation with dimension \( d_F = N \) are just denoted \( T^a \). They are normalized according to

\[
\text{tr} \left( T^a T^b \right) = \frac{1}{2} \delta^{ab}.
\]

The adjoint representation has dimension \( d_A = N^2 - 1 \) and its generators are given by

\[
(T^a_A)^{bc} = -i f^{abc}.
\]

In the continuum the gauge covariant derivative of the fermion field \( \psi(x) = (\psi^j(x)) \) is given by

\[
D_\mu \psi = \partial_\mu \psi + ig A^a_\mu T^a_R \psi
\]

with the gauge field

\[
A_\mu(x) = A^a_\mu(x) T^a.
\]

The gauge field strength \( F_{\mu\nu}(x) = F^a_{\mu\nu}(x) T^a \) is

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].
\]
On the lattice the Wilson action for Dirac fermions in representation $R$ is written as

$$S = \sum_{x,y} a^4 \bar{\psi}_x (D_x)_{x,y} \psi_y,$$

(9)

where the Wilson-Dirac operator

$$(D_x)_{x,j,\alpha;k,\beta} = \delta_{xy} \delta_{jk} \delta_{\alpha\beta} - \kappa \sum_{\mu=1}^4 \left[ (r - \gamma_{\mu})_{\alpha\beta} (V_{R,\mu}(x))^{jk} \delta_{x+\mu,y} + (r + \gamma_{\mu})_{\alpha\beta} (V_{R,\mu}^\dagger(x - \mu))^{jk} \delta_{x-\mu,y} \right]$$

(10)

contains gauge links $V_{R,\mu}(x)$ in the representation $R$. In case of the adjoint representation they are related to the usual fundamental gauge links by

$$(V_{A,\mu}(x))^{ab} = 2 \text{tr}[U_{\mu}(x)T^a U_{\mu}(x)T^b].$$

(11)

In the following, the Wilson parameter is set to $r = 1$ – which is the usual choice.

Along the lines of [11] we have calculated the necessary bare vertices for lattice perturbation theory with clover-improved adjoint Wilson fermions. Using these vertices the proper fermion-antifermion-gluon vertex is calculated in the one-loop approximation. As $c_{SW}$ is independent of the fermion mass, we perform the calculation with massless fermions. For on-shell fermions with momenta $p$ and $p'$, the vertex is of the form

$$\Lambda(p,p')^{jk} = g \left( i \gamma_{\mu} A + \frac{1}{2} (p+p')_{\mu} \left( B - c_{SW} \right) + \mathcal{O}(p^2,p'^2,p \cdot p') + \mathcal{O}(a^2) \right) (T_R)^{kj}. $$

(12)

For the coefficients the calculation at tree level yields

$$A = 1 + \mathcal{O}(g^2)$$

(13)

and

$$B = 1 + \mathcal{O}(g^2).$$

(14)

Therefore we find at tree level that $\mathcal{O}(a)$ terms in the on-shell vertex are cancelled if

$$c_{SW}^{(0)} = 1$$

(15)

as is the case for fermions in the fundamental representation.

On the one-loop level there are six Feynman diagrams contributing to the vertex, displayed in [11]. The infrared divergent part turns out to be proportional to

$$\left( c_{SW}^{(0)} - 1 \right) \ln(\lambda^2 a^2),$$

(16)

where the gluon mass $\lambda$ has been introduced as an infrared regulator. Therefore the above choice of $c_{SW}^{(0)} = 1$ guarantees the cancellation of infrared divergences.
For the calculation of the one-loop vertex we make use of the following relations for the generators:

\[ T_R^a T_R^a = C_R 1, \]  
\[ f^{abc} T_R^b T_R^c = \frac{i}{2} N T_R^a, \]  
\[ T_R^b T_R^a T_R^b = \left( C_R - \frac{N}{2} \right) T_R^a, \]

where double indices are summed, and \( C_R \) is the quadratic Casimir invariant. The Casimir invariant can be computed in terms of the Dynkin labels or the highest weight vector of the representation \( R \) by means of the Racah formula, see e.g. [15, 16]. For the fundamental and for the adjoint representation we have

\[ C_F = \frac{N^2 - 1}{2N}, \quad C_A = N. \]  

The result for the \( \mathcal{O}(a) \) contribution to the vertex is

\[ B = 1 + g^2 (0.16764(3) C_R + 0.01503(3) N) + \mathcal{O}(g^4), \]  

where the decimal number results from numerical loop integrations. So, requiring the vanishing of \( \mathcal{O}(a) \) contributions leads to the result for the one-loop coefficient

\[ c_{SW}^{(1)} = 0.16764(3) C_R + 0.01503(3) N. \]  

For the case of the adjoint representation of gauge group SU(2), the estimate from Monte Carlo simulations is represented in [14] by the interpolation

\[ c_{SW}^{(MC)} = 1 + 0.032653 g^2 - 0.002844 g^4. \]  

Expanding the fraction yields

\[ c_{SW}^{(MC)} = 1 + 0.346806 g^2 + \mathcal{O}(g^4). \]  

Comparing with our perturbative result

\[ c_{SW} = 1 + 0.36533(4) g^2 + \mathcal{O}(g^4), \]  

the coefficients \( c_{SW}^{(1)} \) differ by about 5%.

In \( \mathcal{N} = 1 \) supersymmetric Yang-Mills theory the gluinos \( \lambda(x) = \lambda^a(x) T^a \) are Majorana fermions in the adjoint representation of the gauge group, obeying \( \bar{\lambda} = \lambda^T C \). In this case the lattice action for the fermions is

\[ S = \frac{1}{2} \sum_{x,y} a^4 \bar{\lambda}_x (D_w)_{x,y} \lambda_y. \]
As $\lambda$ and $\bar{\lambda}$ are not independent, the fermions are represented by a real Grassmann algebra and the fermionic functional integral is given by

\[ \int D\lambda e^{-S}. \]  \hfill (27)

The Majorana nature of the fermions implies certain differences in perturbation theory. Wick contractions between $\lambda$ and $\bar{\lambda}$ also contribute, leading in general to additional Feynman diagrams and different symmetry factors compared to Dirac fermions \[17\] \[18\]. In the case of the six diagrams contributing to the gluino-gluon vertex, the bare vertices have an additional factor 1/2 from Eq. (26), which however is cancelled by a factor 2 arising from the modified symmetry factor of the diagrams. Therefore, at the end, the result for the improvement coefficient $c_{SW}$ up to one-loop is the same as for Dirac fermions.

With our calculation we have obtained the improvement coefficient $c_{SW}$ to one-loop order for supersymmetric Yang-Mills theory and models with Dirac fermions in the any representation of SU$(N)$. The result is in good agreement with the numerical investigations.

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