THE UNIVERSAL K3 SURFACE OF GENUS 14 VIA CUBIC FOURFOLDS

GAVRIL FARKAS AND ALESSANDRO VERRA

Abstract. Using Hassett’s isomorphism between the Noether-Lefschetz moduli space \( C_{26} \) of special cubic fourfolds \( X \subseteq \mathbb{P}^5 \) of discriminant 26 and the moduli space \( F_{14} \) of polarized K3 surfaces of genus 14, we use the family of 3-nodal scrolls of degree seven in \( X \) to show that the universal K3 surface over \( F_{14} \) is rational.

1. Introduction

For a very general cubic fourfold \( X \subseteq \mathbb{P}^5 \), the lattice \( A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z}) \) of middle Hodge classes contains only classes of complete intersection surfaces, so \( A(X) = \langle h^2 \rangle \), where \( h \in \text{Pic}(X) \) is the hyperplane class (see \[V\]). Hassett, in his influential paper \[H1\], initiated the study of Noether-Lefschetz special cubic fourfolds. If \( C \) is the 20-dimensional coarse moduli space of smooth cubic fourfolds \( X \subseteq \mathbb{P}^5 \), let \( C_d \) be the locus of special cubic fourfolds \( X \) characterized by the existence of an embedding of a saturated rank 2 lattice

\[ L := \langle h^2, [S] \rangle \hookrightarrow A(X), \]

of discriminant \( \text{disc}(L) = d \), where \( S \subseteq X \) is an algebraic surface not homologous to a complete intersection. Hassett \[H1\] showed that \( C_d \subseteq C \) is an irreducible divisor, which is nonempty if and only if \( d > 6 \) and \( d \equiv 0, 2 \pmod{6} \). The study of the divisors \( C_d \) for small \( d \) has received considerable attention. For instance, \( C_8 \) consists of cubic fourfolds containing a plane, whereas \( C_{14} \) corresponds to cubic fourfolds containing a quintic del Pezzo surface, see \[H2\]. Relying on Fano’s work \[Fa\], recently Bolognesi and Russo \[BR\] have shown that all fourfolds \( [X] \in C_{14} \) are rational.

For every \( [X] \in C \), we denote by \( F(X) := \{ \ell \in G(1, 5) : \ell \subseteq X \} \) the Hilbert scheme of the lines contained in \( X \). It is well known \[BD\] that \( F(X) \) is a hyperkähler fourfold deformation equivalent to the Hilbert square of a K3 surface. For discriminant \( d = 2(n^2 + n + 1) \), where \( n \geq 2 \), it is shown in \[H1\] that \( F(X) \) is isomorphic to the Hilbert scheme \( S^{[2]} \) of a polarized K3 surface \((S, H)\) with \( H^2 = d \). If \( F_g \) denotes the moduli space of polarized K3 surfaces of genus \( g \), the previous assignment induces a rational map

\[ F_d \to F_g, \]

which is a birational isomorphism for \( d \equiv 2(\text{mod } 6) \) and a degree 2 cover for \( d \equiv 0(\text{mod } 6) \). This map, though non-explicit for it is defined at the level of moduli spaces of weight-2 Hodge structures, opens the way to the study of \( F_{n^2+n+2} \) via the concrete geometry of cubic fourfolds, without making a direct reference to K3 surfaces! The main result of this paper concerns the universal K3 surface \( F_{g,1} \to F_g \).

Theorem 1.1. The universal K3 surface \( F_{14,1} \) of genus 14 is rational.

Nuer \[Nu\] proved that \( C_{26} \) (and hence \( F_{14} \) as well) is unirational. His proof relies on the fact that a general fourfold \([X] \in C_{26}\) contains certain smooth rational surfaces, whose
shall prove that indeed, a general fourfold singularities, the double point formula implies that $R$.

We shall show in Section 3 that the secant variety $\text{Sec} R$ is unirational, see [Ve2] and also [Nu]. Recently, Lai [L] showed that $C_{42}$ is uniruled.

Mukai in a celebrated series of papers [M1], [M2], [M3], [M4], [M5] established structure theorems for polarized $K3$ surfaces of genus $g \leq 12$, as well as $g = 13, 16, 18, 20$. In particular, $F_g$ is unirational for those value of $g$. No structure theorem for the general $K3$ surface of genus $14$ is known. A quick inspection of Mukai’s methods shows that the universal $K3$ surface $F_{g,1}$ is unirational for $g \leq 11$ as well. On the other hand, Gritsenko, Hulek and Sankaran [GHS] have proved that $F_g$ is a variety of general type for $g > 62$, as well as for $g = 47, 51, 53, 55, 58, 59, 61$. In a similar vein, recently it has been established in [TVA] that $C_d$ is of general type for all $d$ sufficiently large. As pointed out in Remark 5.4, whenever $F_g$ is of general type, the Kodaira dimension of $F_{g,1}$ is equal to 19.

The proof of Theorem [1.1] relies on the connection between singular scrolls and special cubic fourfolds. We fix a general point $[X] \in C_{26}$ and denote by $S$ the associated $K3$ surface, such that $S^{[2]} \equiv F(X) \hookrightarrow G(1, 5)$. For each $p \in S$, we introduce the rational curve

$$\Delta_p := \{ \xi \in S^{[2]} : \{p\} = \text{supp}(\xi) \}.$$ 

Under the Plücker embedding $G(1, 5) \subseteq \mathbb{P}^{14}$, the degree of $\Delta_p \subseteq F(X)$ is equal to 7, which suggests that each point of $p \in S$ parametrizes a septic scroll $R = R_p \subseteq X$. Imposing the condition $\text{disc}(h^2, [R]) = 26$, one obtains $R^2 = 25$. Assuming $R$ has isolated non-normal nodal singularities, the double point formula implies that $R$ has precisely 3 non-normal nodes. We shall prove that indeed, a general fourfold $[X] \in C_{26}$ carries a 2-dimensional family of 3-nodal scrolls $R \subseteq X$ with $\deg(R) = 7$. Furthermore, this family of scrolls is parametrized by the $K3$ surface $S$ associated to $X$.

We now describe the moduli space of 3-nodal septic scrolls. We start with the Hirzebruch surface $F_1 := \text{Bl}_\sigma(\mathbb{P}^2)$, where $\sigma \in \mathbb{P}^2$, and denote by $\ell$ the class of a line and by $E$ the exceptional divisor. The smooth septic scroll $R' = S_{3,4} \subseteq \mathbb{P}^8$ is the image of the linear system

$$\phi_{[4\ell - 3E]} : F_1 \hookrightarrow \mathbb{P}^8.$$ 

We shall show in Section 3 that the secant variety $\text{Sec}(R') \subseteq \mathbb{P}^8$ is as expected 5-dimensional. Choose general points $a_1, a_2, a_3 \in \text{Sec}(R')$ and denote by $\Lambda := \langle a_1, a_2, a_3 \rangle \in G(2, 8)$ their linear span. The image $R \subseteq \mathbb{P}^5$ of the projection with center $\Lambda$

$$\pi_\Lambda : R' \rightarrow \mathbb{P}^5$$

is a 3-nodal septic scroll. Conversely, up to the action of $PGL(6)$ on the ambient projective space $\mathbb{P}^5$, each such scroll appears in this way. We denote by $\mathcal{S}_{\text{scr}}$ the moduli space of un-parametrized 3-nodal septic scrolls in $\mathbb{P}^5$, that is, the quotient of the corresponding Hilbert scheme under the action of $PGL(6)$. Then as showed in Proposition 3.6, the space $\mathcal{S}_{\text{scr}}$ turns out to be birationally isomorphic to the 9-dimensional unirational variety

$$\mathcal{S}_{\text{scr}} \cong \text{Sym}^3(\text{Sec}(R')) / \text{Aut}(R').$$

Fix a general 3-nodal septic scroll $R \subseteq \mathbb{P}^5$. A general $X \in \mathbb{P}(H^0(I_R/\mathcal{O}_{\mathbb{P}^5}(3))) = \mathbb{P}^{12}$ is a smooth cubic fourfold. Since $R$ has no further singularities apart from the three non-normal nodes, the double point formula implies that $[X] \in C_{26}$. One sets up the following incidence
correspondence between scrolls and cubic fourfolds of discriminant 26:

\[ \mathcal{X} := \left\{ (X, R) : R \subseteq X \right\} / PGL(6) \]

Thus \( \mathcal{X} \) is birational to a \( \mathbb{P}^{12} \)-bundle over the unirational variety \( \mathcal{H}_{\text{scr}} \). We then show that the fibre over a general cubic fourfold \( [X] \in \mathcal{C}_{26} \) of the projection \( \pi_1 \) is 2-dimensional and isomorphic to the K3 surface \( S \) appearing in the identification \( F(X) \cong S^{[2]} \). We summarize the discussion above.

**Theorem 1.2.** The universal K3 surface \( \mathcal{F}_{14,1} \) is birational to the \( \mathbb{P}^{12} \)-bundle \( \mathcal{X} \) over the moduli space \( \mathcal{H}_{\text{scr}} \) of 3-nodal septic scrolls \( R \subseteq \mathbb{P}^5 \). A general fourfold \( [X] \in \mathcal{C}_{26} \) contains a two-dimensional family of such scrolls \( R \subseteq X \subseteq \mathbb{P}^3 \). The space of such scrolls is isomorphic to the K3 surface associated to \( X \).

Theorem 1.2 allows us to elucidate the structure of \( \mathcal{F}_{14,1} \) even further and prove its rationality. We fix a 3-nodal septic scroll \( R \subseteq \mathbb{P}^5 \) as above and denote its nodes by \( p_1, p_2, p_3 \). The curve \( \Gamma_R \subseteq G(1,5) \) induced by the rulings of \( R \) is a smooth rational septic curve admitting biseant lines \( L_1, L_2 \) and \( L_3 \) in the Plücker embedding of \( G(1,5) \). Precisely, \( L_i \) parametrizes the lines passing through \( p_i \) and contained in the 2-plane \( P_i \) spanned by the two rulings of \( R \) that intersect at the node \( p_i \), for \( i = 1, 2, 3 \). Since \( \Gamma_R \) spans a 7-dimensional linear space in projective space \( \mathbb{P}^{14} \) containing \( G(1,5) \), using Mukai’s work [M6] on realizing canonical genus 8 curves as linear sections of the Grassmannian \( G(1,5) \), it follows that the intersection \( G(1,5) \cdot \langle \Gamma_R \rangle \) is a semi-stable curve of genus 8. We denote by \( Q \subseteq \langle \Gamma_R \rangle = \mathbb{P}^{7} \) the residual curve defined by the following equality:

(1) \[ G(1,5) \cdot \langle \Gamma_R \rangle = \Gamma_R + L_1 + L_2 + L_3 + Q. \]

We shall establish in Lemmas 4.1 and 4.2 that \( Q \) is a smooth rational quartic curve and \( Q \cdot L_i = 1 \) for \( i = 1, 2, 3 \), as well as \( Q \cdot \Gamma_R = 3 \). Therefore \( Q \) is the curve of rulings of a quartic scroll \( R_Q \subseteq \mathbb{P}^5 \), which contains three rulings \( \ell_1, \ell_2, \ell_3 \), such that that \( p_i \in \ell_i \) and \( \ell_i \in P_i \), for \( i = 1, 2, 3 \). In particular, \( R_Q \) contains the three nodes of the septic scroll \( R \). We can show furthermore that \( R_Q \) is smooth and isomorphic to \( F_0 \), see Theorem 4.10.

The construction above can be reversed. Using the automorphism group of the scroll \( R_Q \subseteq \mathbb{P}^5 \), we fix three of its rulings \( \ell_1, \ell_2, \ell_3 \in G(1,5) \), as well as points \( p_i \in \ell_i \). We set

\[ P_i^\beta := \{ P_i \in G(2,5) : \ell_i \subseteq P_i \}, \]

for \( i = 1, 2, 3 \), then define a map

\[ \varpi : P_1^\beta \times P_2^\beta \times P_3^\beta / S_3 \rightarrow \mathcal{H}_{\text{scr}}, \]

by reversing the above construction and using the decomposition (1). Along with the fixed point \( p_i \), each 2-plane \( P_i \in P_i^\beta \) defines a line \( L_i \subseteq G(1,5) \) meeting the curve \( Q \) at the point \( \ell_i \). Precisely, \( L_i \) is the line of lines in \( P_i \) passing through the point \( p_i \). To the triple \( (P_1, P_2, P_3) \) we associate the scroll \( R \subseteq \mathbb{P}^5 \) whose associated curve of rulings \( \Gamma_R \) is defined by the formula (1).

The above discussion indicates that \( \varpi \) is dominant. In fact more can be proved:

**Theorem 1.3.** The moduli space of scrolls \( \mathcal{H}_{\text{scr}} \) is birational to \( P_1^\beta \times P_2^\beta \times P_3^\beta / S_3 \) and is thus rational.
Indeed, using the theorem on symmetric functions, see [Ma] or [GKZ] Theorem 2.8 for a recent reference, all symmetric products of projective spaces are known to be rational. It is now clear that Theorem 1.3 coupled with Theorem 1.2 implies that \( \mathcal{F}_{14,1} \) is a rational variety.

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2. K3 SURFACES AND CUBIC FOURFOLDS

We begin by setting some notation. Let \( U \subseteq |\mathcal{O}_{\mathbb{P}^3}(3)| \) be the locus of smooth cubic fourfolds and set

\[ \mathcal{C} := U / \text{PGL}(6) \]

to be the 20-dimensional moduli space of cubic fourfolds. For an integer \( d \equiv 0, 2 \pmod{6} \), as pointed out in the Introduction, \( \mathcal{C}_d \) denotes the irreducible divisor of \( \mathcal{C} \) consisting of special cubic fourfolds of discriminant \( d \). As usual, \( \mathcal{F}_g \) is the irreducible 19-dimensional moduli space of smooth polarized K3 surfaces \((S, H)\) of genus \( g \), that is, with \( H^2 = 2g - 2 \). We denote by \( u : \mathcal{F}_{g,1} \to \mathcal{F}_g \) the universal K3 surface of genus \( g \) in the sense of stacks. Each fibre \( u^{-1}([S, H]) \) is identified with the K3 surface \( S \).

Using the Hodge-theoretic similarity between K3 surfaces of genus \( g = n^2 + n + 1 \) and special cubic fourfolds of degree \( 2g - 2 \), Hassett [H1] constructed a morphism of moduli spaces

\[ \varphi : \mathcal{F}_{n^2+n+2} \to \mathcal{C}_{2(n^2+n+1)} \]

which is birational for \( n \equiv 0, 2 \pmod{3} \), and of degree 2 for \( n \equiv 1 \pmod{3} \) respectively. In particular, for \( n = 3 \) there is a birational isomorphism of spaces of weight 2 Hodge structures

\[ \varphi : \mathcal{F}_{14} \xrightarrow{\sim} \mathcal{C}_{26} \]

that will be of use throughout the paper. At the moment, there is no geometric construction of the polarized K3 surface \( \varphi^{-1}([X]) \) associated to a general fourfold \( [X] \in \mathcal{C}_{26} \).

We recall basic facts on Hilbert squares of K3 surfaces and refer to [HT1] for a general reference on these matters. Let \((S, H)\) be a polarized K3 surface with \( \text{Pic}(S) = \mathbb{Z} \cdot H \) and \( H^2 = 2g - 2 \). We denote by \( S^{[2]} \) the Hilbert scheme of length two 0-dimensional subschemes on \( S \). Then \( H^2(S^{[2]}, \mathbb{Z}) \) is endowed with the Beauville-Bogomolov quadratic form \( q \). We denote by \( \Delta \subseteq S^{[2]} \) the diagonal divisor consisting of zero-dimensional subschemes supported only at a single point and by \( \delta := \frac{\Delta}{2} \in H^2(S^{[2]}, \mathbb{Z}) \) the reduced diagonal class. One has \( q(\delta, \delta) = -2 \).

Note the canonical identification

\[ \Delta = \mathcal{P}(T_S) = \cup\{\Delta_p : p \in S\}, \]

where \( \Delta_p \) is the rational curve consisting of those 0-dimensional subschemes \( \xi \in \Delta \) such that \( \text{supp}(\xi) = \{p\} \). We set \( \delta_p := [\Delta_p] \in H_2(S^{[2]}, \mathbb{Z}) \).

For a curve \( C \in |H| \) in the polarization class, we introduce the divisor

\[ f_C := \{\xi \in S^{[2]} : \text{supp}(\xi) \cap C \neq \emptyset\} \]

and set \( f := [f_C] \in H^2(S^{[2]}, \mathbb{Z}) \). If \( p \in S \) is a general point, we also define the curve

\[ F_p := \{\xi = p + x \in S^{[2]} : x \in C\} \]

and set \( f_p := [F_p] \in H_2(S^{[2]}, \mathbb{Z}) \). The Beauville-Bogomolov form can be extended to a quadratic form on \( H_2(S^{[2]}, \mathbb{Z}) \), by setting \( q(\alpha, \alpha) := q(w_\alpha, w_\alpha) \), with \( w_\alpha \in H^2(S^{[2]}, \mathbb{Z}) \) being the
curves $R$ correspond to quintic scroll, but the class characterized by the property $\alpha \cdot u = q(w_\alpha, u)$, for every $u \in H^2(S^{[2]}, \mathbb{Z})$. Here $\alpha \cdot u$ denotes the usual intersection product.

One has the following decompositions, orthogonal with respect to $q$, both for the Picard group and for the group $N_1(S^{[2]}, \mathbb{Z})$ of 1-cycles modulo numerical equivalence:

$$\text{Pic}(S^{[2]}) \cong \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \delta$$

and

$$N_1(S^{[2]}, \mathbb{Z}) \cong \mathbb{Z} \cdot f_p \oplus \mathbb{Z} \cdot \delta_p.$$

We record, the more or less obvious relations:

$$f \cdot f_p = 2g - 2, \quad \delta \cdot \delta_p = -1, \quad f \cdot \delta_p = 0 \text{ and } \delta \cdot f_p = 0. \tag{2}$$

Assume now that $X \subseteq \mathbb{P}^5$ is a general special cubic fourfold of discriminant 26 and let $[S, H] = \varphi^{-1}([X]) \in \mathcal{F}_{14}$ be the associated polarized $K3$ surface such that

$$S^{[2]} = F(X) \subseteq G(1, 5) \hookrightarrow \mathbb{P}^{14}. \tag{3}$$

Following [BD], let $\gamma_S := [\mathcal{O}_{S^{[2]}}(1)]$ be the hyperplane class of $G(1, 5)$ restricted to the Hilbert square under the identification (3). Since $q(\gamma_S, \gamma_S) = 6$, using (2), it quickly follows that

$$\gamma_S = 2f - 7\delta \in H^2(S^{[2]}, \mathbb{Z}).$$

**Proposition 2.1.** Suppose $[S, H] \in \mathcal{F}_{26}$ is a general element and let $R \subseteq S^{[2]}$ be an effective 1-cycle such that $R \cdot \gamma_S = 7$. Then $R$ is one of the rational irreducible curves $\Delta_p$, for $p \in S$. In particular, $R$ is smooth.

**Proof.** Assume that $R$ is an effective 1-cycle and write $[R] = af_p - b\delta_p \in N_1(S^{[2]}, \mathbb{Z})$. Since $7 = R \cdot \gamma_S = 52a - 7b$, hence we can write $a = 7a_1$, with $a_1 \in \mathbb{Z}$, and then $b = 52a_1 - 1$. Using [BM] Proposition 12.6, we have $q(R, R) \geq -\frac{5}{2}$. We obtain $39a_1^2 - 26a_1 - 1 \leq 0$, and the only integer solution of this inequality is $a_1 = 0$, therefore $[R] = \delta_p$. Since $[R] \cdot \delta = -1$, it follows that $R \subseteq \Delta$. We claim that $R$ lies in one of the fibres of the $\mathbb{P}^1$-bundle $\pi: \Delta = \pi(T_S) \to S$, which implies that $R = \Delta_p$, for some $p \in S$. Indeed, otherwise $\pi(R) \equiv mH$, for some $m > 0$. Accordingly, we write

$$mH^2 = R \cdot \pi^{-1}(H) = R \cdot f = \delta_p \cdot f = 0,$$

which is a contradiction. \hfill \Box

**Remark 2.2.** Unlike degree 26, for other values of $d$, a general $[X] \in \mathcal{C}_d$ may contain several types of scrolls. For instance when $d = 14$ and $\gamma_S = 2f - 5\delta$, the curves $\Delta_p$ with $p \in S$ correspond to quintic scroll, but $X$ also contains quartic scrolls corresponding to rational curves $R \subseteq F(X)$ with $[R] = 3f_p - 16\delta_p$. Note that $q(R, R) = -2$.

We now recall the correspondence between scrolls and rational curves in Grassmannians. Following for instance [Dol] 10.4, we define a rational scroll to be the image $R \subseteq \mathbb{P}^n$ of a $\mathbb{P}^1$-bundle $\pi: R' = \pi(\mathcal{E}) \to \mathbb{P}^1$ under a map $\phi: R' \to \mathbb{P}^n$ given by a linear subsystem of $|\mathcal{O}_{\mathbb{P}^1}(1)|$, thus sending the fibres of $\pi$ to lines in $\mathbb{P}^n$. Let $f_R: \mathbb{P}^1 \to G(1, n)$ be the map

$$f_R(t) := [\phi(\pi^{-1}(t))]$$

and denote by $\Gamma_R$ its image. Conversely, start with a non-degenerate map $f: \mathbb{P}^1 \to G(1, n)$, then consider the pull-back under $f$ of the projectivization of tautological rank 2 vector over $G(1, n)$, that is,

$$\Xi := \left\{ (t, x) : t \in \mathbb{P}^1, x \in L_{f(t)} \right\} \subseteq \mathbb{P}^1 \times \mathbb{P}^n. \tag{4}$$
Here $L_{\varphi(t)} \subseteq \mathbb{P}^n$ denotes the line whose moduli point in $G(1, n)$ is precisely $\varphi(t)$.

The projection $\pi_2 : \Xi \rightarrow \mathbb{P}^n$ is a finite map and its image is a scroll $R \subseteq \mathbb{P}^n$ of degree
\[
\deg(G_R) = \deg f^* \left( \mathcal{O}_{G(1, n)}(1) \right).
\]

Throughout the paper, we interpret scrolls in terms of their associated curves of rulings. It will be useful to determine, using this language, when a scroll is smooth.

**Proposition 2.3.** Let $R \subseteq \mathbb{P}^n$ be a scroll which is not a cone and such that $\Gamma_R$ is a smooth rational curve in $G(1, n)$ which is not contained in a plane. Then there is a bijective correspondence between singularities of $R$ and bisecant lines to $\Gamma_R$ lying on $G(1, n)$. In particular, if $\Gamma_R$ admits no bisecant lines contained in $G(1, n)$, then $R$ is smooth.

**Proof.** We consider the projection $\pi_2 : \Xi \rightarrow R$ defined by $\ell$. Then $\Xi$ is a smooth variety and the assumptions made on $R$ imply that $\pi_2$ is a finite map. If a point $x \in R$ corresponds to a singularity, then one of the two following possibilities occur: (i) the fibre $\pi_2^{-1}(x)$ consists of more than point, or (ii) the differential of $\pi_2$ at a point of $(t, x) \in \pi_2^{-1}(x)$ is not an isomorphism.

In case (i), we choose distinct points $t_1, t_2 \in \pi_1(\pi_2^{-1}(x))$. Denoting by $\ell_1 := f_R(t_1)$ and $\ell_2 := f_R(t_2)$ the rulings of $\Xi$ corresponding to these points, we observe that $x \in \ell_1 \cap \ell_2$. The set $L$ of lines in the 2-plane $\langle \ell_1, \ell_2 \rangle$ passing through $x$ is a line in $G(1, n)$ such that $\Gamma_R \cap L \supseteq \{ \ell_1, \ell_2 \}$, that is, $\Gamma_R$ possesses a secant line lying inside $G(1, n)$ in its Plücker embedding. Note that $L$ is a genuine secant line in the sense that it meets the curve $\Gamma_R$ in two distinct points $\ell_1$ and $\ell_2$. All lines lying inside $G(1, n)$ in its Plücker embedding correspond to pencils of lines in a 2-plane passing through a point in $\mathbb{P}^n$. Thus conversely, when such a line meets $\Gamma_R$ in two distinct points, these will correspond to two incident rulings of $R$. In particular $R$ is singular at their point of intersection.

To deal with case (ii), we carry out a local calculation. Assume $(t_0, x) \in \Xi$ is a point at which the differential of $\pi_2$ is not an isomorphism. We set $\ell_0 := f_R(t_0)$ and denote by
\[
p_{ij}(t) = a_i(t)b_j(t) - a_j(t)b_i(t), \quad \text{where } 0 \leq i < j \leq n
\]
the Plücker coordinates of the curve $\Gamma_R$ in a neighborhood of $\ell_0$, where $a(t) = (a_0(t), \ldots, a_n(t))$ and $b(t) = (b_0(t), \ldots, b_n(t))$.

In local coordinates, the map $\pi_2$ is given by $\mathbb{P}^1 \times \mathbb{C} \ni ([\lambda, \mu], t) \mapsto [(\lambda a_i(t) + \mu b_i(t))] =: x$. By direct calculation, the condition that $(d\pi_2)_{(t_0, x)}$ is not an isomorphism is equivalent to
\[
b'(t_0) \wedge a(t_0) = 0 \in \mathbb{C}^{n+1}.
\]

Setting $a_i := a_i(t_0)$, $b_i := b_i(t_0)$, $a'_i := a'_i(t_0)$ and $b'_i := b_i(t_0)$, we then observe that the Plücker coordinates of a point on the tangent line $T_{\ell_0}(\Gamma_R) \subseteq \mathbb{P}^{(n+1)/2}$ are given by
\[
a_i b_j - a_j b_i + \mu(a'_i b_j + a'_j b_i - a_j b'_i) = b_j(a_i + \mu a'_i) - b_i(a_j + \mu a'_j),
\]
for some scalar $\mu$. It follows that the tangent line to $\Gamma_R$ at $\ell_0$ is contained in $G(1, n)$. The argument being reversible, we finish the proof. \hfill \Box

The scrolls $R \subseteq \mathbb{P}^n$ we consider most of the time have at worst non-normal nodal singularities $x \in R$, corresponding to the case $|\varphi^{-1}(x)| = 2$. The tangent cone of $R$ at $x$ is isomorphic to the union of two 2-planes in $\mathbb{P}^4$ meeting in one point. According to Proposition 2.3 to each
such singularity corresponds a line in the Plücker embedding of $G(1, n)$ meeting $\Gamma_R$ in two distinct points.

Suppose now that $R \subseteq X \subseteq P^5$ is a rational scroll with isolated nodal singularities contained in a cubic fourfold. Using the double point formula [Ful] 9.3 applied to the map $\phi : R' \to X$, we find the number of singularities of $R = \phi(R')$:

$$D(\phi) = R^2 - 6h^2 - K_R^2 - 3h \cdot K_R + 2\chi_{\text{top}}(R). \quad (5)$$

When $[X] \in C_{26}$, assuming that $A(X) = \langle h^2, [R] \rangle$, where $h^2 \cdot [R] = \deg(R) = 7$, necessarily $R^2 = 25$. From formula (5), we compute $D(\phi) = 3$, that is, if $R$ has only (isolated) improper nodes, then it is 3-nodal.

Before stating our next result, we recall that $\mathcal{M}_0(F(X), 7)$ denotes the space of stable maps $f : C \to F(X)$, from a nodal curve $C$ of genus zero such that $\deg(f^*(\mathcal{O}_{F(X)}(1))) = 7$. We denote by $\mathcal{M}_0(F(X), 7)$ the open sublocus consisting of maps with source $P^1$ and denote by $\mathcal{M}_7(X)$ the closure of $\mathcal{M}_0(F(X), 7)$ inside $\mathcal{M}_0(F(X), 7)$.

**Corollary 2.4.** Let $[X] \in C_{26}$ a general special fourfold of discriminant 26 and $[S, H] \in F_{26}$ its associated K3 surface. Then there is an isomorphism $S \cong \mathcal{M}_7(X)$.

**Proof.** Using the identification $S^{[2]} \cong F(X)$, we define the map $j : S \to \mathcal{M}_7(X)$, by setting $j(p) := \Delta_p \subseteq F(X)$. All points in the image of $j$ consist of embedded smooth rational curves $P^1 \cong \Delta_p$ and we identify $\Delta_p$ with the corresponding map $P^1 \to F(X)$. In a neighborhood of this map, the moduli space $\mathcal{M}_0(F(X), 7)$ is locally isomorphic to the Hilbert scheme of septic rational curves on $F(X)$.

The tangent space of $\mathcal{M}_7(X)$ at the point $[\Delta_p]$ is canonically isomorphic to $H^0(N_{\Delta_p/F(X)}/\Delta_p)$. Using the following exact sequence on $\Delta_p \cong P^1$

$$0 \to N_{\Delta_p/\Delta} \to N_{[\Delta_p]/F(X)} \to \mathcal{O}_{\Delta_p}(\Delta) \to 0,$$

since $N_{\Delta_p/\Delta} = \mathcal{O}_{\Delta_p}^{\oplus 2}$ and $\mathcal{O}_{\Delta_p}(\Delta) = \mathcal{O}_{\Delta_p}(-1)$, we compute $N_{[\Delta_p]/F(X)} = \mathcal{O}_{\Delta_p}^{\oplus 2} \oplus \mathcal{O}_{\Delta_p}(-1)$. It follows that $H^1([\Delta_p], N_{[\Delta_p]/F(X)}) = 0$, hence the obstruction space for deformations vanishes and

$$\dim T_{[\Delta_p]}(\mathcal{M}_0(F(X), 7)) = h^0(\Delta_p, N_{\Delta_p/F(X)}) = 2.$$

We conclude that $[\Delta_p]$ is a smooth point of expected dimension of $\mathcal{M}_7(X)$, for every $p \in S$.

Furthermore, $j$ is injective, because for distinct points $p, q \in S$, since $\Delta_p \cap \Delta_q = \emptyset$, the associated scrolls $R_p$ and $R_q$ share no rulings. We finally observe that $j$ is an immersion. Indeed, for each $p \in S$, we have the identification $\Delta_p = P\left(T_p(S) \oplus T_p(S)/T_p(S)\right)$, the quotient being given by the diagonal embedding. Thus the differential $dj(p)$ is essentially the identity map, via the identification $P(T_S) \cong \bigcup_{p \in S} P(N_{\Delta_p/\Delta})$. Since according to Proposition 2.1 we have that $\mathcal{M}_0(F(X), 7) \subseteq \text{Im}(j)$, we can conclude the proof. \qed

### 3. Nodal Septic Scrolls and Cubic Fourfolds

In this section we study in more detail the moduli space $\mathcal{M}_{3\text{scr}}$ of 3-nodal septic scrolls that will be used to parametrize the universal $K3$ surface of degree 26. We fix once and for all the smooth septic scroll

$$R' := S_{3, 4} \hookrightarrow P^8,$$
given as the image of the map $\phi_{|4\ell-3E|}$ on the Hirzebruch surface $F_1 = Bl_3(P^2)$. We denote by $h : R' \to P^1$ the map induced by the linear system $|\ell - E|$. The fibres of $h$ are pairwise disjoint lines in $P^8$. Equivalently, we consider the vector bundle on $P^1$

$$G = O_{P^1}(3) \oplus O_{P^1}(4)$$

and then $R' \cong P(G)$. One has the canonical identification between space of sections:

$$H^0(R', O_{R'}(1)) \cong H^0(P(G), O_{P(G)}(1)) \cong H^0(P^1, G).$$

Later, when computing the dimension of the parameter space of 3-nodal septic scrolls, we shall make use of the basic fact

$$\dim \text{Aut}(R') = \dim \text{Aut}(F_1) = 6.$$

Every smooth septic scroll in $P^8$ is obtained from $R'$ by applying a linear transformation of $P^8$. In particular, the Hilbert scheme of septic scrolls in $P^8$ has dimension equal to

$$\dim PGL(9) - \dim \text{Aut}(R') = 80 - 6 = 74.$$

Using coordinates in $P^8$, if $P^3_{y_0,...,y_3} \subseteq P^8$ is the linear span of the twisted cubic $E$ corresponding to the exceptional divisor on $F_1$ and $P^3_{x_0,...,x_4} \subseteq P^8$ is the linear span of a rational quartic curve linearly equivalent to $\ell$, then the ideal of $R'$ in $P^8$ is given by the following determinantal condition, see for instance [Ha] Lecture 9:

$$\rk \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 \\ x_1 & x_2 & x_3 & x_4 & y_1 & y_2 & y_3 \end{pmatrix} \leq 1.$$

The secant variety $\text{Sec}(R') \subseteq P^8$ is also determinantal, with equations given by the $3 \times 3$ minors of the following $1$-generic matrix:

$$\rk \begin{pmatrix} x_0 & x_1 & x_2 & y_0 & y_1 \\ x_1 & x_2 & x_3 & y_1 & y_2 \\ x_2 & x_3 & x_4 & y_2 & y_3 \end{pmatrix} \leq 2.$$ 

It follows from [CC] Lemma 3.1 that, as expected, $\text{Sec}(R')$ is 5-dimensional. Furthermore, applying e.g. [Ei] Corollary 3.3, it follows that the singular locus of $\text{Sec}(R')$ coincides with the scroll $R'$.

**Lemma 3.1.** Let $a_1, a_2, a_3 \in \text{Sec}(R')$ be general points and set $\Lambda := \langle a_1, a_2, a_3 \rangle \in G(2, 8)$. The image $R$ of the projection $\pi : R' \to P^5$ with center $\Lambda$ has three non-normal nodes corresponding to the three bisecant lines passing through $a_1, a_2$ and $a_3$ and no further singularities.

**Proof.** The chosen points $a_1, a_2, a_3$ can be assumed to lie in $\text{Sec}(R') - (R' \cup \text{Tan}(R'))$. Since $\dim \text{Sec}(R') = 5$, by using the Trisecant lemma, see for instance [CC] Proposition 2.6, it follows that the scheme-theoretic intersection of $\text{Sec}(R')$ with $\Lambda$ consists only of the points $a_1, a_2, a_3$. In particular, $\Lambda \cap R' = \emptyset$, hence the projection $\pi = \pi_\Lambda : R' \to R$ is a regular morphism. Furthermore, each point $a_i$ lies on a unique bisecant line $\langle x_i, y_i \rangle$, where $x_i$ and $y_i$ are distinct points of $R'$, for $i = 1, 2, 3$.

Suppose now that for $x, y \in R'$, one has $\pi(x) = \pi(y)$. This happens if and only if $\langle x, y \rangle \cap \Lambda \neq \emptyset$, hence $\emptyset \neq \langle x, y \rangle \cap \Lambda \subseteq \langle a_1, a_2, a_3 \rangle$ and then necessarily $\{x, y\} = \{x_i, y_i\}$, for $i \in \{1, 2, 3\}$. Since $\Lambda \cap \text{Tan}(R') = \emptyset$, it follows that the differential of $\pi$ is everywhere injective. To summarize, the only singularities of $R$ are the three non-normal nodes $\pi(x_i) = \pi(y_i)$, for $i = 1, 2, 3$. 

\[\square\]
We now fix a general projection $\pi = \pi_\Lambda : R' \to \mathbb{P}^5$ as in Lemma 3.1. We denote by $p_i$ the three singularities of the image scroll $R$. The map $\pi_\Lambda$ is defined by the 6-dimensional subspace $V := H^0(\mathbb{P}^5, \mathcal{I}_\Lambda/\mathcal{I}(p_1)(1))$ of $H^0(\mathbb{P}^1, \mathcal{G})$. To give $\Lambda$ amounts to specifying $V \subseteq H^0(\mathbb{P}^1, \mathcal{G})$. Since $\Lambda \cap R' = \emptyset$, it follows that the evaluation map $ev_V : V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{G}$ is surjective. Hence $ev_V$ defines a morphism

$$f : \mathbb{P}^1 \to G(1,5).$$

This map is induced by the ruling of the image scroll $R$, that is, $f_R = f$ is the map given by $f_R(t) := [\pi(h^{-1}(t))]$, for $t \in \mathbb{P}^1$. Set $\Gamma_R := \text{Im}(f_R)$.

**Proposition 3.2.** For a general choice of the 3-secant plane $\Lambda$ to Sec$(R')$, the following hold:

(i) $\dim \langle p_1, p_2, p_3 \rangle = 2$.

(ii) $\langle p_1, p_2, p_3 \rangle \cap R = \{p_1, p_2, p_3\}$.

**Proof.** It suffices to consider a codimension 2 general linear section $Z \subseteq R' \subseteq \mathbb{P}^8$. Then $Z$ is a smooth 0-dimensional scheme supported at seven distinct points $x_1, y_1, x_2, y_2, x_3, y_3$ and $z$, spanning a 6-dimensional linear space in $\mathbb{P}^8$. In particular, $z$ does not lie in the 5-plane spanned by the points $\{x_i, y_i\}_{i=1}^3$ and no line intersecting the lines $\langle x_1, y_1 \rangle$, $\langle x_2, y_2 \rangle$, $\langle x_3, y_3 \rangle$ exists. Pick general points $a_i \in \langle x_i, y_i \rangle$, for $i = 1, 2, 3$. Then the projection $\pi_\Lambda$ defined by the plane $\Lambda = \langle a_1, a_2, a_3 \rangle$ satisfies both conditions (i) and (ii). \qed

For a projection $\pi_\Lambda$ satisfying the assumptions of Lemma 3.1 the map $f_R : \mathbb{P}^1 \to G(1,5)$ is an embedding, for $\Lambda$ intersects no ruling of $R'$. We record the conclusion of Proposition 2.3 for a scroll $R$ as above:

**Proposition 3.3.** The rational curve $\Gamma_R \subseteq G(1,5)$ admits three secant lines that lie in $G(1,5)$. Conversely, a rational septic curve $\Gamma \subseteq G(1,5)$ having three secant lines lying in $G(1,5)$ is the curve of rulings of a 3-nodal septic scroll in $\mathbb{P}^5$.

We establish a couple of properties concerning the linear system of cubic fourfolds containing a 3-nodal septic scroll:

**Proposition 3.4.** The following statements hold for a general 3-nodal septic scroll $R \subset \mathbb{P}^5$:

(i) $\dim |\mathcal{I}_{\mathcal{R}/\mathbb{P}^5}(3)| = 12$ and (ii) $H^1(\mathbb{P}^5, \mathcal{I}_{\mathcal{R}/\mathbb{P}^5}(3)) = 0$.

**Proof.** Recall that $R$ is the image of a projection $\pi = \pi_\Lambda : R' \to R$ with center $\Lambda$, and denote by $p_1, p_2, p_3 \in R$ the three (non-normal) singularities of $R$ and by $\{x_i, y_i\} = \pi^{-1}(p_i)$, for $i = 1, 2, 3$. By Proposition 3.2, the points $p_1, p_2$ and $p_3$ are in general linear position in $\mathbb{P}^5$ and thus impose independent conditions on cubic hypersurfaces, that is, $H^1(\mathbb{P}^5, \mathcal{I}_{\mathcal{Sing}(R)/\mathbb{P}^5}(3)) = 0$.

By passing to cohomology in the short exact sequence

$$0 \to \mathcal{I}_{\mathcal{R}/\mathbb{P}^5}(3) \to \mathcal{I}_{\mathcal{Sing}(R)/\mathbb{P}^5}(3) \to \mathcal{I}_{\mathcal{Sing}(R)/\mathcal{R}}(3) \to 0,$$

we write the following exact sequence:

$$0 \to H^0(\mathcal{I}_{\mathcal{R}/\mathbb{P}^5}(3)) \to H^0(\mathcal{I}_{\mathcal{Sing}(R)/\mathbb{P}^5}(3)) \to H^0(\mathcal{I}_{\mathcal{Sing}(R)/\mathcal{R}}(3)) \to H^1(\mathcal{I}_{\mathcal{R}/\mathbb{P}^5}(3)) \to 0.$$

Clearly $h^0(\mathbb{P}^5, \mathcal{I}_{\mathcal{Sing}(R)/\mathbb{P}^5}(3)) = \binom{8}{3} - 3 = 53$. Furthermore, we have the following identification of linear systems:

$$\pi^*(|\mathcal{I}_{\mathcal{Sing}(R)/\mathcal{R}}(3)|) = |\mathcal{I}_{\{x_1, y_1, x_2, y_2, x_3, y_3\}/R'}(12\ell - 9E)|.$$
The scroll \( [R] \in \mathfrak{H}_{\text{scr}} \) is obtained as a general projection like in Lemma \[\text{3.1}\]. In particular, the points \( \{x_i, y_i\}_{i=1}^3 \subseteq R' \) are general as well, hence impose independent conditions on the linear system \( |12\ell - 9E| \) on \( R' \). Using the identification \[\text{6}\], we compute:

\[
h^0(R, \mathcal{I}_{\text{Sing}(R)/R}(3)) = h^0(R', \mathcal{O}_{R'}(12\ell - 9E)) - 6 = h^0(P^2, \mathcal{O}_{P^2}(12)) - \binom{10}{2} - 6 = 40.
\]

Therefore \( h^0(P^5, \mathcal{I}_{R/P^5}(3)) = 13 \) if and only if \( h^1(P^5, \mathcal{I}_{R/P^5}(3)) = 0 \). This last statement can be proved via a simple Macaulay calculation by choosing the points \( a_1, a_2, a_3 \) randomly in the variety \( \text{Sec}(R') \) whose equations have been given explicitly. \[\square\]

**Remark 3.5.** It is possible to realize the rational curve \( \Gamma_R \) inside the linear system \( |\mathcal{O}_R(1)| \) as follows. Recall that we have denoted by \( \phi : F_1 \hookrightarrow P^5 \) the embedding whose image is the smooth scroll \( R' \). In \( |4\ell - 3E| \cong P^5 \), we consider the space of reducible hyperplane sections:

\[
\left\{ A' + L' : A' \in |3\ell - 2E|, L' \in |\ell - E| \right\}.
\]

Note that \( L' \) is a ruling of \( R' \), whereas \( A' \subseteq P^5 \) is a sextic with \( \langle A' \rangle = P^6 \) and with \( L' \cdot A' = 1 \). In the linear system \( |3\ell - 2E| \) there exists a unique sextic \( A'_0 \) such that \( \Lambda \subseteq \langle A'_0 \rangle \subseteq P^5 \). The curve \( A'_0 \) corresponds to the unique curve in the linear system

\[
\mathcal{I}(x_1, y_1, x_2, y_2, x_3, y_3) / R' (3\ell - 2E)
\]

on \( R' \). Indeed, \( x_i, y_i \in A'_0 \), therefore \( a_i \in \langle x_i, y_i \rangle \subseteq \langle A'_0 \rangle \), for \( i = 1, 2, 3 \). It then follows that \( \Lambda = \langle a_1, a_2, a_3 \rangle \subseteq \langle A'_0 \rangle \). The projection \( A_0 := \pi(A'_0) \subseteq P^5 \) is a sextic curve on \( R \) passing through the nodes \( p_1, p_2, p_3 \). One identifies \( \Gamma_R \) with \( A_0 \) via the map \( L \mapsto L \cdot A_0 \).

We denote by \( \mathcal{H}_{\text{scr}} \) the Hilbert scheme of 3-nodal septic scrolls in \( R \subseteq P^5 \) and set

\[
\mathfrak{H}_{\text{scr}} := \mathcal{H}_{\text{scr}} / \text{PGL}(6).
\]

We regard \( \mathfrak{H}_{\text{scr}} \) as the coarse moduli space of 3-nodal septic scrolls.

**Proposition 3.6.** The parameter space \( \mathfrak{H}_{\text{scr}} \) is birationally isomorphic to \( \text{Sym}^3(\text{Sec}(R')) / \text{Aut}(R') \). In particular, \( \mathfrak{H}_{\text{scr}} \) is a unirational 9-dimensional variety.

**Proof.** We identify \( \text{Aut}(R') \) with the group consisting of linear automorphisms \( \sigma \in \text{PGL}(9) \) such that \( \sigma(R') = R' \). Every \( \sigma \in \text{Aut}(R') \) clearly invariates \( \text{Sec}(R') \). Since \( \text{Sing}(\text{Sec}(R')) = R' \), conversely, every automorphism \( \sigma \in \text{PGL}(9) \) invariating \( \text{Sec}(R') \) belongs actually to \( \text{Aut}(R') \). One has a birational action of \( \text{Aut}(R') \) on \( \text{Sym}^3(\text{Sec}(R')) \) given by

\[
\sigma(a_1, a_2, a_3) := \langle \sigma(a_1), \sigma(a_2), \sigma(a_3) \rangle,
\]

for \( \sigma \in \text{Aut}(R') \) and \( a_1, a_2, a_3 \in \text{Sec}(R') \). We can now define a birational morphism

\[
\vartheta : \text{Sym}^3(\text{Sec}(R')) / \text{Aut}(R') \dashrightarrow \mathfrak{H}_{\text{scr}}, \quad \text{by setting}
\]

\[
\Lambda := \langle a_1, a_2, a_3 \rangle \mapsto \pi_\Lambda(R') \mod \text{PGL}(6),
\]

where \( \pi_\Lambda : P^9 \dashrightarrow P^5 \) is a projection of center \( \Lambda \). The assignment is clearly \( \text{Aut}(R') \)-invariant, hence \( \vartheta \) is well-defined. Applying Lemma \[\text{3.1}\], we obtain that \( \vartheta \) is a birational isomorphism.

The secant variety \( \text{Sec}(R') \) being a \( P^1 \)-bundle over the rational variety \( \text{Sym}^2(R') \) is unirational. This implies that \( \text{Sym}^3(\text{Sec}(R')) \) and thus \( \mathfrak{H}_{\text{scr}} \) are unirational as well. \[\square\]
THE UNIVERSAL K3 SURFACE OF GENUS 14 VIA CUBIC FOURFOLDS

Over the Hilbert scheme $\mathcal{H}_{\text{scr}}$ we consider the universal family of scrolls:

$$\mathcal{H}_{\text{scr}} \leftarrow p_1 \quad \mathcal{V}_{\text{scr}} \rightarrow p_2 \rightarrow \mathbb{P}^5.$$  

We introduce the incidence correspondence between cubic fourfolds of discriminant 26 and nodal septic scrolls in $\mathbb{P}^5$:

$$|O_{\mathbb{P}^5}(3)| \leftarrow \mathcal{X} := \mathbb{P}\left(\langle p_1 \rangle \ast (\mathcal{I}_{\mathcal{V}_{\text{scr}}/\mathcal{H}_{\text{scr}} \times \mathbb{P}^5 \otimes p_2^*O_{\mathbb{P}^5}(3))\right) \rightarrow \mathcal{H}_{\text{scr}}$$

The group $PGL(6)$ acts on the entire diagram. By quotienting out this action, if we set $\mathcal{X} := \mathcal{X}/PGL(6)$, we obtain two projections:

$$\mathcal{C}_{26} \leftarrow \pi_1 \quad \mathcal{X} \rightarrow \pi_2 \rightarrow \mathcal{H}_{\text{scr}}$$

The 21-dimensional variety $\mathcal{X}$ being a $\mathbb{P}^{12}$-bundle over the unirational variety $\mathcal{H}_{\text{scr}}$ is unirational as well. A general scroll $[R] \in \mathcal{H}_{\text{scr}}$ has precisely 3 non-normal nodes. Checking that a general cubic fourfold $X \supset R$ is smooth, reduces to a standard Macaulay calculation. Applying (5), we obtain that the lattice $A(X)$ contains a 2-dimensional lattice $\langle h^2, [R] \rangle$ of discriminant 26, therefore the map $\pi_1$ is well-defined. Proposition 2.1 implies $\dim \pi_1^{-1}([X]) \leq 2$, for all $[X] \in \mathcal{C}_{26}$, hence $\mathcal{X}$ dominates $\mathcal{C}_{26}$. In fact one can prove something more precise and establish in the process Theorem 1.2.

**Theorem 3.7.** The incidence correspondence $\mathcal{X}$ is birational to the universal K3 surface $\mathcal{F}_{14,1}$.

**Proof.** We define a map $\theta : \mathcal{X} \rightarrow \mathcal{F}_{14,1}$ as follows. We start with a pair $[X, R] \in \mathcal{X}$ and denote by $f_R : \mathbb{P}^1 \rightarrow F(X)$ the rational curve of rulings described in Proposition 3.3. Denoting by $[S, H] := \phi^{-1}([X]) \in \mathcal{F}_{14}$ the polarized K3 surface provided by the identification (5), applying Proposition 2.1 there exists a uniquely determined point $p \in S$ such that $\Delta_p = \Gamma_R$.

The map $\theta$ is clearly generically injective. Since both $\mathcal{X}$ and $\mathcal{F}_{14,1}$ are irreducible varieties of the same dimension 21, it follows that $\theta$ is birational. In particular, in the isomorphism $S \cong \mathcal{M}_7(X)$ constructed in Corollary 2.4 the general point on both sides corresponds to a septic scroll $R \subset X$ which is 3-nodal and has no further singularities. \hfill $\square$

4. **The rationality of $\mathcal{F}_{14,1}$**

In this section, using in an essential way the characterization given in Proposition 3.3 of the rational curves $\Gamma_R$ of rulings of 3-nodal scrolls $R \subset \mathbb{P}^5$, we show that the universal K3 surface of genus 14 is rational.

We begin by recalling the structure of the moduli space of curves of genus 8. Consider the Grassmannian $G(1, 5) \subset \mathbb{P}^{14}$ in its Plücker embedding. Denote by

$$\mathcal{M}_8 := G \left(7, \mathbb{P} \left(\bigwedge^2 \mathbb{C}^6\right)\right)/PGL(6)$$

the space of codimension 7 linear sections of $G(1, 5)$. Mukai [M6] has shown that the map

$$\mathcal{M}_8 \rightarrow \overline{\mathcal{M}}_8,$$

sending a general 7-plane $[P(V) \subset \mathbb{P}^5] \in \mathcal{M}_8$ to the intersection $[G(1, 5) \cdot P(V)] \in \overline{\mathcal{M}}_8$ viewed as a canonical curve of genus 8, is a birational isomorphism. For more details on how to extend Mukai’s isomorphism over parts of the boundary of $\overline{\mathcal{M}}_8$, see also [PV2].

Recall that we introduced in Section 3 the smooth septic scroll $R' \cong F_1 \subset \mathbb{P}^8$, then considered a singular scroll $R \subset \mathbb{P}^5$, defined as the image of a linear projection $\pi_A : R' \rightarrow \mathbb{P}^5$.  


whose center is a general plane $\Lambda \subset P^8$, which is 3-secant to $\text{Sec}(R')$. We denote by $p_1, p_2, p_3$ the three nodes of $R$ and $\{x_i, y_i\} = \pi^{-1}(p_i)$. As explained in the Introduction, $P_i \subset P^9$ denotes the 2-plane spanned by the rulings of $R$ passing through $p_i$, for $i = 1, 2, 3$. The line

$$L_i \subset G(1, 5) \subset P^{14}$$

parametrizes the lines in the plane $P_i$ passing through the point $p_i$. If $\Gamma = \Gamma_R \subset G(1, 5)$ is the curve of rulings associated to $R$ introduced in Proposition 3.3, then $L_i$ meets $\Gamma$ in two distinct points. We keep this notation throughout this section.

Due to the results of the previous section, our strategy is now to describe the family $U \subset \text{Hom}(P^1, G(1, 5))$ of smooth rational septic curves $\Gamma_R \subset G(1, 5)$ carrying three bisecant lines contained in $G(1, 5)$. From Proposition 3.3 it follows that $U$ is birational to the Hilbert scheme $H_{\text{sc}}$ of 3-nodal septic scrolls in $P^9$. Then we show that the quotient $U/PGL(6)$ is rational. Since $U/PGL(6)$ is birational to $H_{\text{sc}}$ and, as proven in Theorem 1.2, the universal $K3$ surface of genus 14 is a $P^{12}$-bundle over $H_{\text{sc}}$, its rationality will follow.

The nodal curve $\Gamma + L_1 + L_2 + L_3 \subset \langle \Gamma \rangle \cdot G(1, 5)$ has arithmetic genus 3. It follows from Mukai’s work [M1] that the intersection $\langle \Gamma \rangle \cdot G(1, 5)$ is a canonical curve of genus 8, provided (i) it is proper and reduced and (ii) $\text{dim} \langle \Gamma \rangle = 7$. Using the surjectivity of the period map for polarized $K3$ surfaces of genus 8, we shall show that both assumptions (i) and (ii) are satisfied. Granting both (i) and (ii) for the moment, we consider the canonically embedded curve in $\langle \Gamma \rangle = P^7$, pictured also below:

$$C := \langle \Gamma \rangle \cdot G(1, 5) = Q + \Gamma + L_1 + L_2 + L_3.$$

Bertini’s Theorem implies that a general 8-dimensional space $\langle \Gamma \rangle \subset P^8 \subset P^{14}$ cuts out on $G(1, 5)$ a smooth 2-dimensional linear section $T$, see also [Ve1], Propositions 3.2 and 3.3. By the adjunction formula, $T \hookrightarrow P^8$ is a smooth $K3$ surface (of genus 8) polarized by $O_T(C)$. We now describe the Picard lattice of $T$:

**Lemma 4.1.** One has the following intersection products on $T$:

$$Q^2 = -2, \ Q \cdot \Gamma = 3, \ Q \cdot L_i = 1, \ \Gamma \cdot L_i = 2, \ L_i \cdot L_j = -2\delta_{ij}, \ \text{for} \ i, j = 1, 2, 3.$$
Proof. The generality assumptions ensure that $L_i$ and $L_j$ are disjoint lines, for $i \neq j$. Else, if $L_i \cap L_j \neq \emptyset$, then $\langle p_i, p_j \rangle \subseteq P_i \cap P_j \subseteq \mathbf{P}^5$. It follows that the four rulings of $R'$ passing through the points $x_i, y_i, x_j, y_j$ respectively, span a 6-dimensional space in $\mathbf{P}^8$, which is impossible for

$$h^0\left( R', O_{R'}(1)(-4(\ell - E)) \right) = h^0(R', O_{R'}(E)) = 1,$$

where recall that $\ell, E \in \text{Pic}(R')$ denote the line class and the exceptional divisor respectively. This implies that there exists a unique hyperplane in $\mathbf{P}^8$ containing the four rulings, therefore they must span a 7-dimensional linear space.

Since $L_i^2 = -2$, by intersecting (7) with $L_i$, we obtain $Q \cdot L_i = 1$. Furthermore $7 = \Gamma \cdot C$ and since $\Gamma^2 = -2$, we obtain $\Gamma \cdot Q = 3$. Finally, $C \cdot Q = \deg(Q) = 4$, therefore $Q^2 + \Gamma \cdot Q + 3 = 4$, implying $Q^2 = -2$ and thus finishing the proof. □

In particular $Q \subseteq \langle T \rangle = \mathbf{P}^8$ is a reduced, connected quartic curve of arithmetic genus zero. Since $C - Q \equiv \Gamma + L_1 + L_2 + L_3$, we obtain $h^0(T, O_T(C - Q)) = 4$. The next lemma summarizes the situation.

Lemma 4.2. The span $\langle Q \rangle$ is 4-dimensional and $Q$ is a connected nodal quartic curve with $p_a(Q) = 0$.

In fact, we shall construct a $K3$ surface $T$, such that the curve $Q$ described in Lemma 4.2 is actually smooth.

To establish the validity of the assumptions (i) and (ii) and thus the existence of the special $K3$ surface $T$, we use Hodge theory. We consider the following sublattice

$$\mathbb{L} := \mathbb{Z} \cdot [Q] \oplus \mathbb{Z} \cdot [\Gamma] \oplus \mathbb{Z} \cdot [L_1] \oplus \mathbb{Z} \cdot [L_2] \oplus \mathbb{Z} \cdot [L_3]$$

(8)

generated by the $(-2)$ classes corresponding to $Q, \Gamma, L_1, L_2$ and $L_3$ respectively, and with intersection pairing as given in Lemma 4.1. We invoke the surjectivity of the period map for $K3$ surfaces. The rank 5 lattice $\mathbb{L}$ is even and has signature $(1, 4)$. Applying [Mo] Corollary 2.9, there exists a smooth $K3$ surface $T$, such that $\text{Pic}(T) \cong \mathbb{L}$. We define the following class on $T$

$$C := \Gamma + Q + L_1 + L_2 + L_3.$$

The genus zero curves $\Gamma, Q, L_1, L_2, L_3 \subseteq T$ cannot have multiple components, for that would make $\text{Pic}(T)$ larger than $\mathbb{L}$, therefore they are all smooth, rational curves on $T$.

Lemma 4.3. The linear system $|O_T(C)|$ is very ample.

Proof. We use Reider’s Theorem [R], which, in the case of $K3$ surfaces, had been proven before in [SD]. It suffices to show that there exists no curve $E$ on $T$ with $E^2 = 0$ and $E \cdot C \in \{1, 2\}$, nor a curve $F$ on $T$ with $F^2 = -2$ and $F \cdot C = 0$. We prove the first statement, the second follows similarly. Assuming there is such a curve $E$, we express it as an integral combination $E \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$ of the generators of $\text{Pic}(T)$. If $C \cdot E = 1$, we obtain

$$-15x^2 - 12xy - 5y^2 + 2x + y = z_1^2 + z_2^2 + z_3^2.\]$$

By comparing the signs of the two sides, one concludes that this equation has no integral solutions. The case $C \cdot E = 2$ is similar. Finally, if $F \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$ is a $(-2)$-curve with $C \cdot F = 0$, we obtain

$$-15x^2 - 12xy - 5y^2 + 1 = z_1^2 + z_2^2 + z_3^2,$$

which implies $x = y = 0$ and, say $z_2 = z_3 = 0$ and then $z_1 = 1$. Thus $F = L_1$, but $C \cdot L_1 = 1$, hence this case does not appear. We conclude that $C$ is very ample. □
We show that the K3 surface $T$ constructed in Lemma 4.3 is a linear section of $\mathbf{G}(1,5)$. In particular, Mukai’s results [M6] will apply for its hyperplane section $C$.

**Proposition 4.4.** The K3 surface $T$ carries a globally generated rank two vector bundle $T$ with $\det(T) = \mathcal{O}_T(C)$, providing an embedding $T \hookrightarrow \mathbf{G}(1,5)$ such that

$$\langle T \rangle \cdot \mathbf{G}(1,5) = S.$$

**Proof.** We use [M7] and need to show that the polarized K3 surface $(T, \mathcal{O}_T(C))$ is Brill-Noether general, that is, for all pairs of line bundles $M, N$ on $T$ such that $M \otimes N = \mathcal{O}_T(C)$, one has $h^0(T, M) \cdot h^0(T, N) < h^0(T, C)$. Under these circumstances, it is shown in loc.cit. that $T$ carries a rigid, globally generated, stable rank 2 vector bundle $E$ with $h^0(T, E) = 6$ and $\det(E) = \mathcal{O}_T(C)$, inducing a map $\varphi_E : T \rightarrow \mathbf{G}(1,5)$. Reasoning along the lines of [M7] Theorem 3.10, the K3 surface $T$ is then a linear section of $\mathbf{G}(1,5)$ in its Plücker embedding, that is, $T = \mathbf{G}(1,5) \cdot \langle T \rangle$.

To establish the Brill-Noether generality of $(T, \mathcal{O}_T(C))$, we use for instance [GLT] Lemma 2.8. It suffices to show that in the lattice $L$ there exists no vector $D$ such that $D^2 = 2$ and $D \cdot C \in \{7, 6\}$, nor is there a vector $D$ with $D^2 = 0$ and $D \cdot C \leq 4$.

We treat in detail only the first case, the remaining ones being similar. We write

$$D \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3.$$

The conditions $D^2 = 2$ and $D \cdot C = 7$ translate into the equalities $z_1 + z_2 + z_3 + 7x + 4y = 7$ and $-15x^2 - 5y^2 - 12xy + 14x + 7y + 1 = z_1^2 + z_2^2 + z_3^2 \geq 0$. It is elementary to see that there are no integral solutions.

Using Proposition 4.4 we conclude that the intersection $\langle \Gamma \rangle$ corresponding to a general curve $\Gamma_R \in \mathcal{U}$ corresponds to a semistable canonical curve of genus 8.

It will be useful to have a criterion for determining when the curve $\Gamma$ spans a space of maximal possible dimension in the Plücker space $\mathbb{P}^{14} \supseteq \mathbf{G}(1,5)$. To that end, recall that the Plücker embedding of the dual Grassmannian $\mathbf{G}(1,5)^\vee = \mathbf{G}(3,5) \hookrightarrow (\mathbb{P}^{14})^\vee$ assigns to a point $p \in \mathbf{G}(1,5)^\vee$ corresponding to a 3-plane $P_p^3 \subseteq \mathbb{P}^5$ the Schubert cycle

$$\sigma_p := \{ \ell \in \mathbf{G}(1,5) : \ell \cap P_p^3 \neq \emptyset \}.$$

Note that $\dim \langle \Gamma \rangle + 1 = \text{codim}(\Gamma)^\perp$. Setting

$$W^1(\Gamma) := \mathbf{G}(3,5) \cap \langle \Gamma \rangle^\perp = \{ p \in \mathbf{G}(3,5) : \Gamma \subseteq \sigma_p \},$$

for dimension reasons, the next lemma follows immediately:

**Lemma 4.5.** Assume $W^1(\Gamma)$ is finite. Then $\dim \langle \Gamma \rangle = 7$.

Keeping the previous notation, let $f_R : \mathbb{P}^1 \rightarrow \mathbf{G}(1,5)$ be a sufficiently general element of $\mathcal{U}$ and set again $\Gamma = \Gamma_R$. Then under the assumption $R' = S_{3,4}$, we can prove that:

**Theorem 4.6.** The set $W^1(\Gamma)$ is finite. In particular $\dim \langle \Gamma \rangle = 7$ and $\Gamma$ is a rational normal septic curve.

**Proof.** If $p \in W^1(\Gamma)$, then $P_p^3$ contains an integral curve intersecting each line of $R$. Its strict transform by $\pi_A : R' \rightarrow R$ is an integral section $A$ of the ruled surface $R'$. Set $d := \deg(A)$, hence $A \equiv (d - 3)\ell - (d - 4)E \in \text{Pic}(F_1)$. Clearly $\langle A \rangle \subseteq \pi_A^{-1}(P_p^3)$, implying $\dim \langle A \rangle \leq 6$. 

Let $I_A := |H - A|$ be the linear system of hyperplanes in $\mathbb{P}^8$ containing the curve $A \subseteq R'$. By direct calculation, we find $\dim(I_A) = \dim |H - A| = 7 - d \geq 1$ and $\dim |A| = 2d - 6$. It follows that $3 \leq d \leq 6$. Recalling that $V = H^0(\mathbb{P}^8, I_A/\mathbb{P}^8(1))$, the condition

$$\dim(\mathbb{P}V \cap I_A) \geq 1$$

is equivalent to the condition that the curve $q_5(A)$ be contained in a 3-space $\mathbb{P}^3_p$. For $3 \leq d \leq 6$ let $G(7 - d, |H|)$ denote the Grassmannian of $(7 - d)$-subspaces of $|H| \cong \mathbb{P}^8$ and introduce the $(2d - 6)$-dimensional variety

$$S_d := \left\{ I_{A'} \in G(7 - d, |H|) : A' \in |(d - 3)\ell - (d - 4)E| \right\}.$$

For an integer $k \geq 1$, we consider the Schubert cycle

$$\sigma^k : = \{ I \in G(7 - d, |H|) : \dim(\mathbb{P}V \cap I) \geq k \}.$$

The cycle $\sigma^k \cdot S_d$ is finite for $k = 1$ and empty for $k \geq 2$, provided the intersection is proper. By Kleiman’s transversality of a general translate this is true for a general translate of $\sigma^k V$ in $G(d - 7, |H|)$, that is, for a general choice of $\Lambda$ (or equivalently, of $V$). Hence $W^1(\Gamma)$ is finite.

**Remark 4.7.** The theorem above fails for rational septic scrolls in $\mathbb{P}^8$ containing sections of degree $d \leq 2$, that is, for the scrolls $S_{a,7-a}$, where $a \neq 3$.

We turn to the smooth residual rational curve $Q \subseteq G(1,5)$ defined by $\emptyset$. Let $R_Q \subseteq \mathbb{P}^5$ be the quartic scroll whose rulings are parametrized by the curve $Q$.

**Lemma 4.8.** $R_Q$ is a non-degenerate smooth rational normal scroll in $\mathbb{P}^5$.

**Proof.** First, observe that $R_Q$ cannot be a cone. Let us assume $R_Q$ is a cone of vertex $v \in \mathbb{P}^5$. Then $\langle Q \rangle \cong \mathbb{P}^4 \subseteq G(1,5)$ parametrizes the lines passing through $v$. This is a contradiction because $\langle Q \rangle \subseteq \langle \Gamma \rangle \cdot G(1,5) = C$. Now assume that $R_Q$ is contained in a hyperplane $H \subseteq \mathbb{P}^5$. Then $Q$ is contained in the Grassmannian $G_H := G(1, H) \subseteq G(1,5)$ of lines of $H$. Since $K_{G_H} = O_{G_H}(-5)$, we observe that, by adjunction, the curvilinear sections of $G_H$ are curves of arithmetic genus 1. Because of this fact and since $\deg(G_H) = 5$, it follows that

$$\langle Q \rangle \cdot G_H = Q + L \subseteq C,$$

where $L$ is a bisecant line to $Q$. But the only line components in $C$ are $L_1, L_2, L_3$ and none of them is bisecant to $Q$. Via Proposition 2.3, the same argument shows that the scroll $R_Q$ has no incident rulings, therefore $R_Q$ is smooth.

**Lemma 4.9.** The scroll $R_Q$ contains no other lines except the ruling parametrized by $Q$.

**Proof.** Assume $R_Q$ contains a line $\ell_0$ not parametrized by a point of $Q$. We prove that this implies that $W^1(\Gamma)$ is not finite, thus contradicting Theorem 4.6. Consider the family $G$ of codimension 1 Schubert cycles $\sigma_p$ defined by a 3-space $\mathbb{P}^3_p \supseteq \ell_0$. Note that $G \cong G(1,3)$. We have $G \subseteq \langle Q \rangle \perp$. Since $\langle Q \rangle \subseteq \langle \Gamma \rangle$, we also have $\langle \Gamma \rangle \perp \subseteq \langle Q \rangle \perp$. Counting dimensions it follows $\dim(G \cap \langle \Gamma \rangle \perp) \geq 1$, which implies that $W^1(\Gamma)$ is not finite.

There are two types of smooth quartic scrolls in $\mathbb{P}^5$, namely $S_{1,3} = \mathbb{P}(O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(3))$ and $S_{2,2} = \mathbb{P}(O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(2))$. The latter case is characterized by the property that every line contained in the scroll is a ruling. Lemma 4.9 implies the following:
Theorem 4.10. Let $\Gamma \subseteq G(1,5)$ be a smooth septic rational curve corresponding to a general element of $U$ and $Q \subseteq G(1,5)$ the residual quartic curve. Then $R_Q$ is isomorphic to $S_{2,2}$.

To summarize, to a general rational curve $\Gamma = \Gamma_R \in U$, we associated the quartic scroll $R_Q$, equipped with three rulings $\ell_1, \ell_2, \ell_3$ corresponding to the points $L_i \cdot Q \subseteq G(1,5)$, for $i = 1, 2, 3$. Each ruling $\ell_i$ passes through the node $p_i$ of the scroll $R$ and is contained in the 2-plane $P_i$ whose existence is established in Proposition 3.3.

To prove the rationality of $S_{\text{scr}}$ and thus that $F_{14,1}$, we reverse this construction. We denote by $\mathcal{V}$ the variety classifying elements $(R_Q, p_1, p_2, p_3)$, where $R_Q \subseteq P^5$ is a smooth quartic scroll isomorphic to $S_{2,2}$ and $p_i \in R_Q$ for $i = 1, 2, 3$.

Lemma 4.11. The $\text{PGL}(6)$-stabilizer of a general point $(R_Q, p_1, p_2, p_3) \in \mathcal{V}$ is trivial. In particular, $\text{PGL}(6)$ acts transitively on $\mathcal{V}$.

Proof. The automorphism group of $S_{2,2} \cong F_0$ is the semidirect product of $\text{PGL}(2) \times \text{PGL}(2)$ with $\mathbb{Z}/2\mathbb{Z}$. The last factor corresponds to the automorphism $u \in \text{Aut}(F_0)$ permuting the two factors. In particular, $\text{Aut}(S_{2,2})$ is 6-dimensional. This implies that the space $\mathcal{V}$ has dimension

$$\dim \text{PGL}(6) - \dim \text{Aut}(S_{2,2}) + 3 \dim(R_Q) = 35 = \dim \text{PGL}(6).$$

Choose general points $p_i = (a_i, b_i) \in F_0 \cong S_{2,2}$, with $a_i \neq b_i$, for $i = 1, 2, 3$. Up to the action of $u \in \text{Aut}(F_0)$, the stabilizer $\text{Stab}_{\text{PGL}(6)}(R_Q, p_1, p_2, p_3)$ corresponds to pairs of automorphism $(\sigma_1, \sigma_2) \in \text{PGL}(2) \times \text{PGL}(2)$, such that $\sigma_1(a_i) = a_i$ and $\sigma_2(b_i) = b_i$. Thus $\sigma_1 = \sigma_2 = 1$. The points $p_i$ not lying on the diagonal of $F_0$, the automorphism $u$ does not fix any of them, thus the stabilizer in question is trivial. Since $\mathcal{V}$ and $\text{PGL}(6)$ have the same dimension, this also implies the transitivity of the $\text{PGL}(6)$-action on $\mathcal{V}$, as claimed. \qed

We can thus start by fixing once and for all the quartic scroll $R_Q$. Precisely, we embed the surface $F_0 := P^1 \times P^1$ in $P^5$ via the linear system $|O_{F_0}(1,2)|$ and denote by

$$R_0 \subseteq P^5$$

the image quartic scroll. The rulings on $R_0$ are the elements of the linear system $|O_{F_0}(0,1)|$. Let $Q_0 \subseteq G(1,5)$ be the curve of rulings of $R_0$. We then fix three points in $F_0$, for instance

$$\alpha_1 := ([1 : 0], [0 : 1]), \quad \alpha_2 := ([0 : 1], [1 : 0]) \quad \text{and} \quad \alpha_3 := ([1 : 1], [-1 : -1]),$$

which we identify with their images in $R_0$. As explained in Lemma 4.11, the stabilizer subgroup $G$ of $\text{PGL}(6)$ fixing both $R_0$ as well as the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is isomorphic to the subgroup of $\text{PGL}(2) \times \text{PGL}(2)$ fixing the set $\{\alpha_1, \alpha_2, \alpha_3\}$. Therefore $G = S_3$.

For $i = 1, 2, 3$, we denote by $\ell_i$ the ruling of $R_0$ passing through the point $\alpha_i$. Then, let $P^3_i$ be the projective space consisting of 2-planes $\Pi_i \subseteq P^5$ containing the line $\ell_i$. Giving a plane $\Pi_i$ is equivalent to specifying a line $L_i \subseteq G(1,5)$ in the Plücker embedding of the Grassmannian. Note that $L_i$ meets $Q_0$ transversally at precisely one point, namely $\ell_i \in G(1,5)$.

We introduce a rational map

$$\varphi : P^3_1 \times P^3_2 \times P^3_3 / S_3 \longrightarrow \mathcal{S}_{\text{scr}}$$

defined as follows. To a triple of planes $(\Pi_1, \Pi_2, \Pi_3)$, we attach the lines $L_1, L_2, L_3 \subseteq G(1,5)$. Since $Q_0 \subseteq G(1,5)$ is a smooth rational quartic curve, in the Plücker embedding we have that $\langle Q_0 \rangle \cong P^4$. Attaching one general 1-secant line to $Q_0$ increases the dimension of the linear span of the union by one, therefore by attaching three general 1-secant lines, we have

$$\langle Q_0 + L_1 + L_2 + L_3 \rangle \cong P^7 \subseteq P^{14}.$$
We write 
\[(Q_0 + L_1 + L_2 + L_3) \cdot G(1, 5) = Q_0 + L_1 + L_2 + L_3 + \Gamma,\]
where \(\Gamma\) is a degree 7 curve. Applying Lemma 4.11 it follows that \(\Gamma\) is a rational curve and \(\Gamma \cdot L_i = 2\), for \(i = 1, 2, 3\). We denote by \(\ell_i\) and \(\ell_i'\) the intersection points \(L_i \cdot \Gamma\). From Proposition 3.3 it follows that the scroll \(R := R_{\ell_i}\) induced by \(\Gamma\) is 3-nodal, with nodes given by the intersection \(\ell_i \cap \ell_i'\) taken in the 2-plane \(\Pi_i\). We set 
\[\kappa(\Pi_1 + \Pi_2 + \Pi_3) := [R].\]

We conclude the proof of the rationality of the Hilbert scheme of 3-nodal scrolls in \(\mathbb{P}^5\):

**Proof of Theorem 1.3** We first observe that \(\kappa\) is well-defined. To that end, we choose the polarized \(K3\) surface \((T, \mathcal{O}_T(C))\) constructed in Propositions 4.3 and 4.4 and we keep the notation used there. Applying Theorem 4.10 the residual quartic rational curve \(Q \subseteq G(1, 5)\) parametrizes the rulings of a quartic scroll \(R_Q \subseteq \mathbb{P}^5\), which is isomorphic to \(S_{2,2}\). Applying Lemma 4.11 there exists a unique automorphisms \(\sigma \in PGL(6)\) such that \(\sigma(R_Q) = R_0\) and \(\sigma(p_i) = a_i\), for \(i = 1, 2, 3\). Set \(\sigma(P_i) = \Pi_i \in \mathbb{P}^3\) and then \(\kappa(\Pi_1 + \Pi_2 + \Pi_3) = [R_\Gamma]\).

To finish the proof it suffices to observe that \(\kappa\) is generically injective. A general septic curve \(\Gamma \in \mathcal{U}\) corresponding to a 3-nodal septic scroll \([\mathcal{R}_\Gamma]\) in \(\mathcal{S}_{3,\text{nec}}\) has precisely 3 bisecant lines lying in \(G(1,5)\). Giving \(\Gamma\) determines its linear span \(\langle \Gamma \rangle\), hence the set \(\{L_1, L_2, L_3\}\) as well. \(\square\)

5. **The Unirationality of the Universal \(K3\) Surface of Genus at Most 12**

We denote by \(\mathcal{F}_{g,n}\) the universal \(n\)-pointed \(K3\) surface of genus \(g\). Thus \(\mathcal{F}_{g,n}\) is an irreducible variety of dimension \(19 + 2n\). Similarly, one can consider the universal Hilbert scheme of \(0\)-dimensional cycles of length \(n\), that is, \(\nu^{[n]} : \mathcal{F}_{g}^{[n]} \to \mathcal{F}_g\). We also introduce the notation \(\mathcal{C}_{g,n} := M_{g,n}/\mathbb{S}_n\) for the degree \(n\) universal symmetric product over \(M_g\), where the symmetric group \(\mathbb{S}_n\) acts by permuting the marked points.

The aim of this short last section is to point out how Mukai’s results determine the birational type of \(\mathcal{F}_{g,n}\) and that of \(\mathcal{F}_{g}^{[n]}\) for small \(g\), and thus put our Theorem 1.1 better into context:

**Theorem 5.1.** The following results on the Kodaira dimension of \(\mathcal{F}_{g,n}\) hold:

(i) \(\mathcal{F}_{g,g+1}\) is unirational for \(g \leq 10\).

(ii) \(\mathcal{F}_{11,11}\) is unirational. The Kodaira dimension of both \(\mathcal{F}_{11,11}\) and \(\mathcal{F}_{11}^{[11]}\) equals 19.

**Proof.** For \(g \leq 5\), the general \(K3\) surface of genus \(g\) is a complete intersection in a projective space and the result follows easily. For details, see the table after Theorem 1.10 in [MZ].

For \(6 \leq g \leq 10\), Mukai [M1] has constructed a rational homogeneous variety \(V_g \subseteq \mathbb{P}^{N_g}\), where \(N_g = g + \dim(V_g) - 2\), such that the general \(K3\) surface of genus \(g\) is obtained as a general linear section \(S = V_g \cap \Lambda_g\), where \(\Lambda_g \subseteq \mathbb{P}^{N_g}\) is a \(g\)-dimensional plane, with the polarization being the one induced by \(\mathcal{O}_{\mathbb{P}^{N_g}}(1)\). Moreover, one has the following birational isomorphism, see [M1] Corollary 0.3:

\[
\mathcal{F}_g \cong \frac{G(g,N_g)}{\text{Aut}(V_g)}.
\]

These results imply the existence of a dominant map \(\chi_g : V_g^{g+1} \to \mathcal{F}_{g,g+1}\) given by

\[
\chi(x_1, \ldots, x_{g+1}) := [V_g \cap \langle x_1, \ldots, x_{g+1}\rangle, x_1, \ldots, x_{g+1}].
\]

This proves that \(\mathcal{F}_{g,g+1}\) (and hence \(\mathcal{F}_{g,n}\) for \(n \leq g + 1\)) is unirational in this range.
For $g = 11$, we use [M8], where it is shown that a general curve $[C] \in \mathcal{M}_{11}$ lies on a unique K3 surface $C \subseteq S$ as a hyperplane section, with $\text{Pic}(S) = \mathbb{Z} \cdot C$. This implies the existence of a rational map $\chi_n : \mathcal{M}_{11,n} \to \mathcal{F}_{11,n}$ defined by

$$\chi_n([C, x_1, \ldots, x_n]) := [S, x_1, \ldots, x_n].$$

The map $\chi_n$ is dominant for $n \leq 11$ and a birational isomorphism for $n = 11$. Indeed, in this last case, given an embedded K3 surface $S \frac{\langle H \rangle}{\langle H \rangle} \mathbb{P}^{11}$ and general points $x_1, \ldots, x_{11} \in S$, the hyperplane $\langle x_1, \ldots, x_{11} \rangle \cong \mathbb{P}^{10}$ cuts out a canonical genus 11 curve $C$ on $S$, which comes equipped with the marked points $x_1, \ldots, x_{11}$. By quotienting the action of the symmetric group $S_{11}$, the map $\chi_{11, 11}$ induces a birational isomorphism between the universal symmetric product $\mathcal{C}_{11, 11}$ and $\mathcal{F}_{11, 11}$. Now we use [FVII] Theorem 0.5. Both varieties $\mathcal{M}_{11, 11}$ and $\mathcal{C}_{11, 11}$ have Kodaira dimension 19, hence we conclude.

We now pass on to the universal K3 surface $\mathcal{F}_{11, 1}$. To that end we define a rational map

$$\vartheta : \mathcal{M}_{10, 2} \dashrightarrow \mathcal{F}_{11, 1},$$

associating to a 2-pointed curve $[C, p_1, p_2] \in \mathcal{M}_{10, 2}$, the unique K3 surface $S$ of genus 11 containing the curve $[X := C/p_1 \sim p_2]$ obtained from $C$ by identifying $p_1$ and $p_2$. To show that $\vartheta$ is well-defined, that is, Mukai’s construction [M8] can be also carried out for the 1-nodal curve $[X] \in \mathcal{M}_{11}$, we use [CLM] Proposition 4.4. Observe that the K3 surface $S$ has a distinguished point corresponding to the image of the singularity of $X$. The map $\vartheta$ is clearly dominant, for in each linear system on a K3 surface, the 1-nodal curves fill-up a divisor. The unirationality of $\mathcal{F}_{11, 1}$ now follows from that of $\mathcal{M}_{10, 2}$, which can be established in a variety of ways, see for instance [BCF] Theorem B.

\[\square\]

Remark 5.2. It is claimed incorrectly in [L] Table 3, that $\mathcal{M}_{11, n}$ is unirational for $n \leq 10$. The argument sketched in loc.cit. only establishes the uniruledness of $\mathcal{M}_{11, n}$ when $n \leq 10$, precisely using the map $\chi_n : \mathcal{M}_{11, n} \to \mathcal{F}_{11, n}$, which is birationally a $\mathbb{P}^{11-n}$-bundle. But this argument alone offers no indications concerning the birational nature of the base variety $\mathcal{F}_{11, n}$. One can establish partial results on the birational nature of $\mathcal{F}_{11, n}$, for $n \leq 10$. For instance, it is shown in [Ve1] that the universal product $\mathcal{C}_{11, 6}$ is unirational, which implies that $\mathcal{F}_{11, 6}$ is unirational as well.

Remark 5.3. Mukai [M4] gives an explicit orbit space realization over a projective space for the universal K3 surface $\mathcal{F}_{13, 1}$. The unirationality of $\mathcal{F}_{13, 1}$ thus follows. Presumably, a similar argument works for genus 12, when $\mathcal{F}_{12}$ is known to be birational to a $\mathbb{P}^{13}$-bundle over the rational moduli space $\mathcal{M} \mathcal{F}_{22}$ of Fano 3-folds $V_2 \subseteq \mathbb{P}^{13}$, see again [M1].

Remark 5.4. Since $u : \mathcal{F}_{g, 1} \to \mathcal{F}_g$ is a morphism fibred in Calabi-Yau varieties, by Iitaka’s easy addition formula $\kappa(\mathcal{F}_{g, 1}) \leq \dim(\mathcal{F}_g) = 19$, in particular, $\mathcal{F}_{g, 1}$ is never of general type. Furthermore, by [K], we also write $\kappa(\mathcal{F}_{g, 1}) \geq \kappa(\mathcal{F}_g)$. In particular, when $\mathcal{F}_g$ is of general type, then $\kappa(\mathcal{F}_{g, 1}) = 19$.

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Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6
10099 Berlin, Germany
E-mail address: farkas@math.hu-berlin.de

Università Roma Tre, Dipartimento di Matematica, Largo San Leonardo Murialdo
1-00146 Roma, Italy
E-mail address:verra@mat.uniroma3.it