ON TRANSITIVE OPERATOR ALGEBRAS IN REAL BANACH SPACES

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Abstract. We consider weakly closed transitive algebras of operators containing non-zero compact operators in real Banach spaces (Lomonosov algebras). It is shown that they are naturally divided in three classes: the algebras of real, complex and quaternion classes. The properties and characterizations of algebras in each class as well as some useful examples are presented. It is shown that in separable real Hilbert spaces there is a continuum of pairwise non-similar Lomonosov algebras of complex type and of quaternion type.

1. Introduction

Almost fifty years ago Victor Lomonosov [7] proved that any algebra of operators on a complex Banach space $X$ that contains a non-zero compact operator either has a non-trivial closed invariant subspace or is dense in the algebra $\mathcal{B}(X)$ of all operators with respect to the strong operator topology (SOT) (or equivalently weak operator topology). Thus the only (SOT)-closed transitive (= having only trivial invariant subspaces) operator algebra containing a non-zero compact operator is $\mathcal{B}(X)$ itself. We call (SOT)-closed transitive algebras containing non-zero compact operators in a (real or complex) Banach space Lomonosov algebras. Then the Lomonosov result states that in any complex Banach space there is only one Lomonosov algebra. In this paper we establish that the structure of Lomonosov algebras in real spaces is much more complicated, pithy and intriguing.

The considerable interest in the theory of invariant subspaces in real Banach spaces that arose in recent times can be partially explained by its relations to infinite-dimensional extremal problems and representation theory (see for example the work of Atzmon [1] and references therein). Another reason is the fact that some interesting technical tools developed in this theory allowed to solve several classical problems which are still open for complex spaces. As a revealing example one can mention the theorem of Simoniec [12] on the existence of an invariant subspace for an operator with compact imaginary part (see also the earlier result of Lomonosov [6] and the work of Lomonosov and Shulman [8], where the commutative families of such operators and their analogues for Banach spaces were considered). The situation with the Lomonosov algebras is opposite: in real spaces their study is more difficult than in complex spaces.

Even if $X$ is finite-dimensional, the list of Lomonosov algebras is exhausted by the algebra $\mathcal{B}(X)$ only in the case of odd dim $X$. If dim $X = 2n$ then the algebra $M_n(\mathbb{C})$ of all complex $n \times n$ matrices is isomorphic to the subalgebra

$$\mathcal{A} = \left\{ \begin{pmatrix} T & -R \\ R & T \end{pmatrix} : T, R \in M_n(\mathbb{R}) \right\} \text{ of } \mathcal{B}(X) = M_{2n}(\mathbb{R}).$$

It is transitive, so that $\mathcal{A}$ is a Lomonosov algebra. Similarly, if dim $X = 4n$ then the algebra $M_n(\mathbb{H})$ of all quaternion $n \times n$ matrices is isomorphic to a transitive subalgebra of the algebra...
$B(\mathcal{X}) = M_{4n}(\mathbb{R})$. So this is another example of a Lomonosov algebra. Moreover, every finite-dimensional Lomonosov algebra is isomorphic either to $M_n(\mathbb{R})$, or to $M_n(\mathbb{C})$, or to $M_n(\mathbb{H})$, for some $n$.

It was natural to expect that in infinite dimensional spaces there exist a rich variety of Lomonosov algebras, but no attempt has been made to describe them or to give new meaningful examples up to now. Our aim is to fill in this gap: to analyze the general properties and present non-trivial classes of examples of Lomonosov algebras in infinite dimensional real Banach spaces.

In Section 2 we study not necessarily closed transitive algebras containing non-zero finite rank operators (L-algebras, for short) and obtain for them a general density type theorem. It states that if $\mathcal{A}$ is an L-algebra then, given a family $y_1, ..., y_n \in X$, $\varepsilon > 0$ and linearly independent set $W \subset X$ of cardinality $4n$, one can find $x_1, ..., x_n \in W$ and $T \in \mathcal{A}$ with $\|Tx_i - y_i\| < \varepsilon$. We show that L-algebras are naturally divided in three classes: the algebras of real, complex and quaternion types, and that for each class the density theorem can be given in a more informative form. A consequence of these results is the fact that the only Lomonosov algebra of real type is the algebra $B(\mathcal{X})$. We deduce also some other characterizations of algebras of complex and quaternion types. For example we show that an algebra $\mathcal{A}$ belongs to the quaternion type if and only if the rank of each finite-rank operator in $\mathcal{A}$ is a multiple of 4. This allows us to show that each L-algebra is contained in a maximal L-algebra of the same type.

In Section 3 we consider Lomonosov algebras of complex type. To each closed operator $S$ satisfying condition $S^{-1} = -S$, we relate a Lomonosov algebra $A_S$ of complex type. We study the properties of such algebras and show that they form a dominating class: Each Lomonosov algebra of complex type is contained in some algebra $A_S$. We prove a locality theorem for such algebras: Each algebra $A_S$ is the closure of its ideal of finite rank operators. Then, constructing a special system of algebras $A_S$ and using the technique of operator ranges, we prove that in a separable infinite-dimensional real Hilbert space there is a continuum of pairwise non-similar Lomonosov algebras of complex type.

In Section 4 similar results are obtained for the algebras of quaternion type. However, as the quaternion case is more complicated than the complex case, our approach differs. We introduce and study the notion of a closed representation of a finite group on a Banach space. Then, applying the results to the quaternion group $G_H$, we construct Lomonosov algebras $A_\pi \subset B(\mathcal{X})$ starting with any closed and regular representations $\pi$ of $G_H$ on $\mathcal{X}$. Using this construction we show that in a real separable, infinite-dimensional Hilbert space there is a continuum of pairwise non-similar Lomonosov algebras of quaternion type.

It should be underlined that our picture of the variety and the structure of Lomonosov algebras is still very far from completeness. It suffices to say that we have no examples of non-unital Lomonosov algebras and we do not even know if they exist (see the discussion of some open problems at the end of the paper). We hope that this subject will be further developed in subsequent studies.

2. Density theorems

In what follows, unless stated otherwise, by $\mathcal{X}$ we denote an infinite-dimensional real Banach space. Here we consider transitive algebras of operators on $\mathcal{X}$ that contain a non-zero finite rank operator; for brevity we call them L-algebras.
Any Lomonosov algebra is an $L$-algebra: the presence of non-zero finite-rank operators (moreover, projections) in Lomonosov algebras was proved in [9, Lemma 5.5]. Unfortunately the proof in [9] does not seem to be very transparent; anyway since we need somewhat less we may give a shorter and simpler proof.

**Lemma 2.1.** Let $X$ be a real Banach space and $T$ be a compact operator on $X$ that has a non-zero eigenvalue $\lambda$. Then the norm-closed algebra $A = A(T)$ generated by $T$ contains a non-zero finite rank operator.

**Proof.** It is well known that the statement holds for complex spaces: the Riesz projection $P$ corresponding to $\{\lambda\}$ is a limit of a sequence of polynomials of $T$. Let $Z$ be a complexification of $X$, i.e. $Z = X \oplus X$ and $X$ is included into $Z$ by identification of $x \in X$ with $x \oplus 0 \in Z$. We denote by $I$ the operator on $Z$ acting by the rule

$$I(x \oplus y) = (-y) \oplus x,$$

or in matrix form $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

One can consider $Z$ as a complex space setting $(\alpha + i\beta)z = (\alpha 1 + \beta I)z$. For each operator $K$ on $X$, the operator $K \oplus K$ on $Z$ is complex-linear. The operator $T \oplus T$ is compact and $\lambda$ is its eigenvalue. Let $p_n(t) = \sum_{k=1}^{N_n} (\alpha_{nk} + i\beta_{nk})t^k$ be a sequence of polynomials with complex coefficients such that $p_n(t)$ tends to a non-zero finite rank operator $W$. Then

$$p_n(T \oplus T) = \sum_{k=1}^{N_n} (\alpha_{nk} + i\beta_{nk}) \begin{pmatrix} T^k & 0 \\ 0 & T^k \end{pmatrix} = \sum_{k=1}^{N_n} \begin{pmatrix} \alpha_{nk}T^k & -\beta_{nk}T^k \\ \beta_{nk}T^k & \alpha_{nk}T^k \end{pmatrix} = \begin{pmatrix} q_n(T) & r_n(T) \\ -r_n(T) & q_n(T) \end{pmatrix},$$

where $q_n$ and $r_n$ are polynomials with real coefficients. It follows that

$$W = \begin{pmatrix} W_1 & -W_2 \\ W_2 & W_1 \end{pmatrix}$$

and $q_n(T) \to W_1$, $r_n(T) \to W_2$. Since at least one of operators $W_1, W_2$ is non-zero, $A(T)$ contains a non-zero finite rank operator. $\blacksquare$

The famous Lomonosov Lemma [7] establishes the presence of compact operators with non-zero spectra in transitive algebras containing compact operators; it is easy to see that its proof in [7] works for real spaces as well as for complex ones. Applying Lemma 2.1 we obtain the needed result:

**Corollary 2.2.** Every transitive norm-closed algebra of operators on a real Banach space containing a non-zero compact operator contains a non-zero finite rank operator.

So Lomonosov algebras can be equivalently defined as (SOT)-closed $L$-algebras.

Recall that an algebra $A$ of operators on a linear space $X$ over a field or, more generally, over a division ring is called strictly transitive, if for any $x, y \in X$ with $x \neq 0$, there is $T \in A$ with $Tx = y$. Furthermore $A$ is called strictly dense if for each finite linearly independent family $(x_1, \ldots, x_n) \subset X$ and any family $(y_1, \ldots, y_n) \subset X$, there is $T \in A$ with $Tx_i = y_i$, $i = 1, \ldots, n$.

Let $k \in N$. Let us say that $A$ is strictly $(1/k)$-dense, if for every linearly independent family $(x_1, \ldots, x_k) \subset X$ and any family $(y_1, \ldots, y_n) \subset X$, there are $j_1, \ldots, j_n \in \{1, \ldots, kn\}$ and $T \in A$ with $\{y_i : 1 \leq i \leq n\} \subset \{Tx_j : j = 1, \ldots, kn\}$. In other words $y_i = Tx_{j_i}$, $1 \leq i \leq n$. 


In the operator theory setting an algebra \( \mathcal{A} \) of bounded linear operators on a Banach space \( \mathcal{X} \) is called

transitive if for any \( \varepsilon > 0 \) and any \( x, y \in \mathcal{X} \) with \( x \neq 0 \), there is \( T \in \mathcal{A} \) with \( \|Tx - y\| < \varepsilon \) (it is easy to check that this definition is equivalent to one given above, namely the absence of non-trivial closed invariant subspaces).

dense if for any \( \varepsilon > 0 \), and for each finite linearly independent family \( (x_1, ..., x_n) \subset \mathcal{X} \) and any family \( (y_1, ..., y_n) \subset \mathcal{X} \), there is \( T \in \mathcal{A} \) with \( \|Tx_i - y_i\| < \varepsilon \), \( i = 1, ..., n \).

\((1/k)\)-dense, if for any \( \varepsilon > 0 \), every linearly independent family \( (x_1, ..., x_{kn}) \subset \mathcal{X} \) and any family \( (y_1, ..., y_n) \subset \mathcal{X} \), there are \( j_1, ..., j_n \in \{1, ..., kn\} \) and \( T \in \mathcal{A} \) with \( \|y_i - T x_{j_i}\| < \varepsilon \), \( 1 \leq i \leq n \).

Note that if \( \mathcal{A} \) is \((1/k)\)-dense then it is \((1/(k+1))\)-dense.

In this section our aim is to establish the following general version of density theorem for \( L \)-algebras.

**Theorem 2.3.** Every \( L \)-algebra of operators in a real Banach space is \((1/4)\)-dense.

This theorem will be deduced from more strong and detailed results. Before proving them we introduce some notation and establish several auxiliary results.

Let \( \mathcal{F}(\mathcal{X}) \) be the ideal of all finite-rank operators on \( \mathcal{X} \). For an \( L \)-algebra \( \mathcal{A} \) on \( \mathcal{X} \), we set

\[
\mathcal{A}^F = \mathcal{A} \cap \mathcal{F}(\mathcal{X}) \quad \text{and} \quad \mathcal{X}^F = \mathcal{A}^F \mathcal{X} := \text{lin}\{Tx : T \in \mathcal{A}^F, x \in \mathcal{X}\}.
\]

Since \( \mathcal{A}^F \) is an ideal of \( \mathcal{A} \), the linear subspace \( \mathcal{X}^F \) is invariant for \( \mathcal{A} \) and therefore dense in \( \mathcal{X} \).

**Lemma 2.4.** The restriction of \( \mathcal{A}^F \) to \( \mathcal{X}^F \) is a strictly transitive algebra of operators on \( \mathcal{X}^F \).

**Proof.** Let \( 0 \neq x_0 \in \mathcal{X}^F \) and \( \mathcal{Y} = \mathcal{A}^F x_0 \). Then \( \mathcal{Y} \) is \( \mathcal{A} \)-invariant and therefore dense in \( \mathcal{X} \). It follows that for each \( T \in \mathcal{A}^F \), \( T \mathcal{Y} \) is dense in \( T \mathcal{X} \). Since \( \dim T \mathcal{X} < \infty \), \( T \mathcal{Y} = T \mathcal{X} \). Therefore \( T \mathcal{X} \subset \mathcal{Y} \). Since \( \mathcal{X}^F = \text{lin}(\cup_{T \in \mathcal{A}^F} T \mathcal{X}) \) we see that \( \mathcal{X}^F \subset \mathcal{Y} \). The converse inclusion follows from the definition of \( \mathcal{X}^F \). We proved that \( \mathcal{Y} = \mathcal{X}^F \), so \( \mathcal{A}^F \) is strictly transitive. \( \blacksquare \)

Let \( \mathcal{D} \) be the algebra of all linear operators on \( \mathcal{X}^F \) commuting with \( \mathcal{A}^F \). Since \( \mathcal{A}^F \) is strictly transitive each non-zero operator in \( \mathcal{D} \) is invertible (its kernel and range are invariant for \( \mathcal{A}^F \)). Thus \( \mathcal{D} \) is a division algebra. By Jacobson’s Density Theorem (see [2, Section 5] for several convenient formulations), we obtain the following result.

**Corollary 2.5.** \( \mathcal{A}^F \) is strictly dense on \( \mathcal{X}^F \), which is considered as a linear space over \( \mathcal{D} \).

**Lemma 2.6.** The \( \mathbb{R} \)-algebra \( \mathcal{D} \) is finite-dimensional.

**Proof.** The range \( \mathcal{M} = T \mathcal{X}^F \) of a non-zero operator \( T \in \mathcal{A}^F \) is invariant for \( \mathcal{D} \). The map \( S \mapsto S|_{\mathcal{M}} \) from \( \mathcal{D} \) to the algebra \( \mathcal{L}(\mathcal{M}) \) of all linear operators on \( \mathcal{M} \) is injective. Indeed if \( S|_{\mathcal{M}} = 0 \), then \( \ker S = 0 \), but \( \ker S \) must be trivial being an invariant subspace for \( \mathcal{A}^F \). So \( \dim(\mathcal{D}) \leq \dim(\mathcal{L}(\mathcal{M})) < \infty \). \( \blacksquare \)

Applying the famous Frobenius’s Theorem on finite-dimensional division algebras over \( \mathbb{R} \), we conclude that only 3 cases are possible:

a) \( \dim(\mathcal{D}) = 1 \), \( \mathcal{D} \) is isomorphic to \( \mathbb{R} \); in this case we say that \( \mathcal{A} \) is an algebra of real type,
b) \( \dim(\mathbb{D}) = 2 \), \( \mathbb{D} \) is isomorphic to \( \mathbb{C} \); \( \mathcal{A} \) is an algebra of complex type,
c) \( \dim(\mathbb{D}) = 4 \), \( \mathbb{D} \) is isomorphic to the algebra of quaternions, \( \mathbb{H} \); \( \mathcal{A} \) is an algebra of quaternion type.

**Lemma 2.7.** Each operator \( T \in \mathcal{A} \) is \( \mathbb{D} \)-linear on \( \mathcal{X}^F \), i.e., \( T \) commutes with each \( \Lambda \in \mathbb{D} \) on \( \mathcal{X}^F \).

**Proof.** Let \( x \in \mathcal{X}^F \). Since \( \mathcal{A}^F \) is strictly transitive on \( \mathcal{X}^F \), there is \( K \in \mathcal{A}^F \) with \( Kx = x \). So for each \( \Lambda \in \mathbb{D} \), \( T\Lambda x = T\Lambda Kx = \Lambda(TK)x = \Lambda T x \). Thus \( T \) commutes with \( \mathbb{D} \) on \( \mathcal{X}^F \). \( \blacksquare \)

**Lemma 2.8.** For every linearly independent set \( \{u_1, ..., u_m\} \subset \mathcal{X} \), there is \( K \in \mathcal{A}^F \) such that the set \( \{Ku_1, ..., Ku_m\} \) is linearly independent.

**Proof.** For \( m = 1 \) this is evident. Using induction, assume that for \( m \)-element sets the statement is true. Aiming at a contradiction, suppose that for a system \( \{u_1, ..., u_{m+1}\} \) the statement fails. Then for each \( K \in \mathcal{A}^F \) satisfying the condition

\[
(2.1) \quad \text{the set } \{Ku_1, Ku_2, ..., Ku_m\} \text{ is linearly independent,}
\]

there are numbers \( t_i(K) \in \mathbb{R} \), \( 1 \leq i \leq m \), such that

\[
(2.2) \quad Ku_{m+1} = \sum_{i=1}^{m} t_i(K)Ku_i.
\]

Fix \( K_0 \in \mathcal{A}^F \) satisfying \( 2.1 \) and set \( \lambda_i = t_i(K_0) \), \( 1 \leq i \leq m \). Let us denote \( \text{lin}(u_1, ..., u_n) \) by \( \mathcal{L} \) and let \( \mathcal{E} \) be the set of all operators \( T \in \mathcal{A}^F \) satisfying the conditions

\[
(2.3) \quad \dim T\mathcal{L} = m \text{ and } T\mathcal{L} \cap K_0\mathcal{L} = 0.
\]

To see that \( \mathcal{E} \) is non-void, let \( \mathcal{M} = K_0\mathcal{L} \) and \( \widetilde{\mathcal{M}} \) be the \( \mathbb{D} \)-linear span of \( \mathcal{M} \). This is a finite-dimensional \( \mathbb{D} \)-linear subspace of \( \mathcal{X}^F \) and (since \( \dim(\mathcal{X}^F) = \infty \)) there is a \( \mathbb{D} \)-linear subspace \( \mathcal{N} \subset \mathcal{X}^F \) with \( \widetilde{\mathcal{M}} \cap \mathcal{N} = 0 \) and \( \dim(\mathcal{N}) = \dim(\mathcal{M}) \). By Corollary 2.5, \( \mathcal{A}^F \) as a \( \mathbb{D} \)-algebra is strictly dense in \( \mathcal{X}^F \). Hence there is \( R \in \mathcal{A}^F \) with \( R\widetilde{\mathcal{M}} = \mathcal{N} \). Clearly the operator \( RK_0 \) belongs to \( \mathcal{E} \).

We claim that \( t_i(T) = \lambda_i, 1 \leq i \leq m \), for all \( T \in \mathcal{E} \).

The standard argument shows that \( \mathcal{E} \) is open in \( \mathcal{A}^F \): if \( T \in \mathcal{E} \) then \( T + K \in \mathcal{E} \) for all \( K \in A \) with sufficiently small norm. So fix \( T \in \mathcal{E} \) and, for some sufficiently small \( \alpha \), let \( \mu_i = t_i(T + \alpha K^F) \). Then

\[
\sum_{i=1}^{m} \lambda_i \alpha K_0 u_i + \sum_{i=1}^{m} t_i(T) Tu_i = \alpha K_0 u_{m+1} + Tu_{m+1}
\]

\[
= (\alpha K_0 + T) u_{m+1} = \sum_{i=1}^{m} \mu_i(\alpha K_0 + T) u_i = \sum_{i=1}^{m} \mu_i \alpha K_0 u_i + \sum_{i=1}^{m} \mu_i Tu_i.
\]

whence

\[
\sum_{i=1}^{m} (\lambda_i - \mu_i) \alpha K_0 u_i = \sum_{i=1}^{m} (\mu_i - t_i(T)) Tu_i.
\]

It follows from \( 2.3 \) that both parts of the above equality are equal to zero, whence

\[
\lambda_i = \mu_i = t_i(T) \quad \text{for } i = 1, ..., m.
\]
So we proved that \( Tu_{i+1} = \sum_{i=1}^{m} \lambda_i Tu_i \), for all \( T \in \mathcal{E} \). In other words

\[
  u_{i+1} - \sum_{i=1}^{m} \lambda_i u_i \in \bigcap_{T \in \mathcal{E}} \ker T.
\]

Since \( \mathcal{E} \) is open in \( \mathcal{A}^F \), \( \bigcap_{T \in \mathcal{E}} \ker T = \ker \mathcal{A}^F = \{0\} \). Therefore \( u_{m+1} = \sum_{i=1}^{m} \lambda_i u_i \), a contradiction. ■

**Corollary 2.9.** If \( \mathcal{A}^F \) is strictly \((1/k)\)-dense on \( \mathcal{X}^F \) then \( \mathcal{A} \) is \((1/k)\)-dense.

**Proof.** Let \( \varepsilon > 0 \), a linearly independent set \( \{x_1, ..., x_{kn}\} \subset \mathcal{X} \) and a set \( \{y_1, ..., y_n\} \subset \mathcal{X} \) be given. Since \( \mathcal{X}^F \) is dense in \( \mathcal{X} \), there is a set \( \{z_1, ..., z_n\} \subset \mathcal{X}^F \) with \( \|z_i - y_i\| < \varepsilon \), \( 1 \leq i \leq n \).

Using Lemma 2.8 we find \( K \in \mathcal{A}^F \) such that the set \( \{Kx_1, ..., Kx_{kn}\} \) is linearly independent. Since \( Kx_i \in \mathcal{X}^F \), it follows from the assumption of the corollary, that there are \( P \in \mathcal{A}^F \) and numbers \( j_1, ..., j_n \) with \( PKx_{j_i} = z_i \), so it remains to set \( T = PK \). ■

Now we can prove our "detailed" density theorem.

**Theorem 2.10.** An \( L \)-algebra of real type is dense, an \( L \)-algebra of complex type is \((1/2)\)-dense, an \( L \)-algebra of quaternion type is \((1/4)\)-dense.

**Proof.** By Corollary 2.9 it suffices to show that, for a real type algebra \( \mathcal{A} \), the algebra \( \mathcal{A}^F \) is strictly dense on \( \mathcal{X}^F \); for a complex type algebra \( \mathcal{A} \), the algebra \( \mathcal{A}^F \) is strictly \((1/2)\)-dense on \( \mathcal{X}^F \), and for a quaternion type algebra \( \mathcal{A} \), the algebra \( \mathcal{A}^F \) is strictly \((1/4)\)-dense on \( \mathcal{X}^F \).

The needed statement for real type algebras follows from the Jacobson Density Theorem because in this case \( \mathbb{D} = \mathbb{R} \).

Suppose that \( \mathbb{D} \) is isomorphic to \( \mathbb{C} \), so we may consider \( \mathcal{X}^F \) as a \( \mathbb{C} \)-module. Let \( x_1, ..., x_{2n} \) be an \( \mathbb{R} \)-linearly independent family in \( \mathcal{X}^F \), and let \( \mathcal{M} = \text{lin}_\mathbb{C}(x_1, ..., x_{2n}) \) the \( \mathbb{C} \)-linear hull of \( x_1, ..., x_{2n} \). If \( \dim_\mathbb{C} \mathcal{M} < n \) then \( \dim_\mathbb{R} \mathcal{M} < 2n \), a contradiction with our conditions. Thus \( \dim_\mathbb{C} \mathcal{M} \geq n \). Hence \( \mathcal{M} \) contains a \( \mathbb{C} \)-linearly independent subfamily \( x_{i_1}, ..., x_{i_n} \). Therefore, by Corollary 2.5, for each \( y_1, ..., y_n \in \mathcal{X}^F \), there is \( T \in \mathcal{A}^F \) with \( Tx_{i_k} = y_k, k = 1, ..., n \).

For algebras of quaternion type the argument is similar. ■

Since any dense and any \((1/2)\)-dense algebra is \((1/4)\)-dense, Theorem 2.3 immediately follows from Theorem 2.10.

Below we will prove a kind of converse statement. Let us begin with a lemma which will be repeatedly used below.

**Theorem 2.11.** For each \( L \)-algebra \( \mathcal{A} \subset \mathcal{B}(\mathcal{X}) \), all operators in \( \mathbb{D} \) are closable on \( \mathcal{X} \).

**Proof.** Let \( S \in \mathbb{D} \), \( x_n \in \mathcal{X}^F \), \( x_n \to 0 \) and \( Sx_n \to y \in \mathcal{X} \). For every \( T \in \mathcal{A}^F \), \( STx_n = TSx_n \to Ty \). On the other hand the subspace \( T\mathcal{X}^F \) is invariant for \( S \) and finite-dimensional so \( S \) is bounded on \( T\mathcal{X}^F \). Since \( Tx_n \to 0 \), we get that \( STx_n \to 0 \). Therefore \( Ty = 0 \), for each \( T \in \mathcal{A}^F \). Since \( \cap_{T \in \mathcal{A}^F} \ker T = 0 \) (it is a closed \( \mathcal{A} \)-invariant subspace), \( y = 0 \). Thus \( S \) is closable. ■

**Theorem 2.12.** (i) An \( L \)-algebra \( \mathcal{A} \) is dense if and only if it is of real type.

(ii) An \( L \)-algebra \( \mathcal{A} \) is \((1/2)\)-dense but not dense if and only if it is of complex type.

(iii) An \( L \)-algebra \( \mathcal{A} \) is \((1/4)\)-dense but neither dense, nor \((1/2)\)-dense if and only if it is of quaternion type.
**Proof.** In all cases the "only if" part needs to be proved.

(i) Suppose that a dense \( L \)-algebra \( A \) is not of real type. Then there is an operator \( S \in D \) which is is not a scalar multiple of \( 1_{\mathcal{H}^F} \). Therefore there is \( x \in \mathcal{H}^F \) such that \( x \) and \( Sx \) are linearly independent. Choose \( 0 \neq y \in \mathcal{H} \). Since \( A \) is dense then there is a sequence \( T_n \in A \) with \( T_n x \to 0 \), \( T_n Sx \to y \). So \( ST_n x \to y \) which is impossible since \( S \) is closable by Theorem 2.11.

(ii) Using (i) we have only to show that an \((1/2)-dense\) \( L \)-algebra cannot be of quaternion type. Suppose that \( A \) is of quaternion type, so \( \dim D = 4 \). Choose \( 0 \neq x_0 \in \mathcal{H}^F \). Then the space \( K = D x_0 \) is 4-dimensional. Choose \( 0 \neq y \in \mathcal{H} \). Since \( A \) is \((1/2)-dense\) there are \( x_1, x_2 \in K \) and a sequence \( T_n \in A \) with \( T_n x_1 \to 0 \), \( T_n x_2 \to y \). By definition there are \( S_1, S_2 \in D \) such that \( x_i = S_i x_0 \). Therefore \( x_2 = S x_1 \) where \( S = S_2 S_1^{-1} \). Then arguing as above we get \( y = \lim T_n x_2 = \lim T_n S x_1 = \lim S T_n x_1 \). Since \( S \) is closable, we get \( y = 0 \), a contradiction.

(iii) Follows directly from Theorem 2.10. □

**Corollary 2.13.** An \( L \)-algebra \( A \) belongs to real, complex or quaternion type if and only if its \( (SOT) \)-closure \( \overline{A} \) belongs to the same type.

**Proof.** By Theorem 2.12 it suffices to show that for \( k = 1, 2, 4 \), an algebra \( A \) is \((1/k)-dense\) if and only if \( \overline{A} \) (as well as any algebra that lies between them) is \((1/k)-dense\). Clearly if \( A \) is \((1/k)-dense\) then each set containing \( A \) is \((1/k)-dense\).

Conversely suppose that \( \overline{A} \) (or some intermediate algebra) is \((1/k)-dense\). By definition, for any \( \varepsilon > 0 \), for every linearly independent family \( (x_1, ..., x_{kn}) \subset \mathcal{H} \) and any family \( (y_1, ..., y_n) \subset \mathcal{H} \), there are \( j_1, ..., j_n \in \{1, ..., kn\} \) and \( T \in \overline{A} \) with \( \|y_i - Tx_{j_i}\| < \varepsilon/2, 1 \leq i \leq n \).

Now by definition of \((SOT)\), there is \( K \in A \) such that \( \|K x_{j_i} - T x_{j_i}\| < \varepsilon/2, 1 \leq i \leq n \).

Then \( \|y_i - K x_{j_i}\| < \varepsilon \), for \( 1 \leq i \leq n \). So \( A \) is \((1/k)-dense\). □

Now we will characterize \( L \)-algebras of all types in terms of the ranks of its elements. Recall that the \( \text{rank}, \text{rank}(T) \), of an operator \( T \) acting on a linear space \( \mathcal{M} \) is defined as the dimension of its range \( TM \).

**Proposition 2.14.** (i) If \( A \) is an \( L \)-algebra of complex type then \( \text{rank}(T) \) is even, for any \( T \in A^F \).

(ii) If \( A \) is an \( L \)-algebra of quaternion type then \( \text{rank}(T) \) is divisible by four, for any \( T \in A^F \).

**Proof.** (i) If \( A \) is an \( L \)-algebra of complex type then \( \mathcal{H}^F \) has the structure of complex space and all operators in \( A^F \) are \( \mathbb{C} \)-linear. In particular the range of an operator \( T \in A^F \) is a complex subspace of \( \mathcal{H}^F \) and its dimension equals its complex dimension multiplied by 2.

(ii) Similarly, the dimension of an \( \mathbb{H} \)-linear subspace equals its quaternion dimension multiplied by 4. □

For any \( L \)-algebra \( A \), we set

\[ r(A) = \min \{ \text{rank}(T) : T \in A^F \} \]

**Theorem 2.15.** An \( L \)-algebra \( A \) is of real, complex or quaternion type if and only if \( r(A) = 1, 2 \) or 4, respectively.
Proof. Let $T_0$ be an operator of minimal rank in $A$. We denote by $Z$ the subspace $T_0\mathcal{X}^F$ of $\mathcal{X}^F$ and let $z_1, \ldots, z_m$ be a basis of $Z$ as a linear space over $\mathbb{D}$. Our aim is to show that $m = 1$.

We claim that the algebra $B = T_0A|_{Z} = \{T_0K|_{Z} : K \in \mathcal{A}^F\}$ coincides with the algebra of all $\mathbb{D}$-linear operators on $Z$.

Indeed, let $V$ be an arbitrary $\mathbb{D}$-linear operator on $Z$ and $w_i = Vz_i$, $1 \leq i \leq m$. Since $Z = T_0\mathcal{X}^F$ there are vectors $x_1, \ldots, x_m \in \mathcal{X}^F$ such that $w_i = T_0x_i$, for all $i$. By Corollary 2.5 there is $K \in \mathcal{A}$ with $Kz_i = x_i$, whence the operator $Q = T_0K|_{Z} \in B$ satisfies the condition $Qz_i = Vz_i$, $1 \leq i \leq m$, and therefore coincides with $V$.

It follows that $B$ contains an operator $T_1 = T_0K_1|_{Z}$ whose range as a $\mathbb{D}$-linear space has dimension 1. Clearly the operator $T_0K_1T_0 \in A$ has the same range as $T_1$. Thus, by the choice of $T_0$, the $\mathbb{D}$-rank of $T_0$ equals 1. So the $\mathbb{R}$-rank of $T_0$ is 1, 2 or 4, if $A$ is of real, complex or quaternion type respectively.

Corollary 2.16. Each $L$-algebra is contained in a maximal $L$-algebra of the same type.

Proof. By Zorn’s lemma it suffices to show that the union of a linearly ordered family of algebras of given type belongs to this type. Let $A = \cup_{\alpha \in \Lambda} A_{\alpha}$, where the family $(A_{\alpha})_{\alpha \in \Lambda}$ is linearly ordered (or up-directed). If all $A_{\alpha}$ have the same type, then by Theorem 2.15 they have the same minimal rank: $r(A_{\alpha}) = k$, for all $\alpha$. It follows that $r(A) = k$, because each operator in $A$ belongs to some $A_{\alpha}$. Again applying Theorem 2.15 we conclude that type of $A$ is the same as the type of $A_{\alpha}$. ■

Remark 2.17. It follows from Corollary 2.13 that maximal algebras of a given type are Lomonosov algebras.

3. LOMONOSOV ALGEBRAS OF COMPLEX TYPE

It follows from Theorem 2.10 that Lomonosov algebras of real type are dense and therefore can be described trivially:

$$A = B(\mathcal{X})$$

because the density of an algebra $A$ is equivalent to the condition that the (SOT)-closure of $A$ in $B(\mathcal{X})$ coincides with $B(\mathcal{X})$.

Passing on to algebras of complex type, we begin with discussion of some examples.

We call by a partial complex structure on $\mathcal{X}$ (PCS, for brevity) an operator $S$ defined on a dense subspace $\mathcal{D}(S) \subset \mathcal{X}$ and satisfying the condition

$$(3.1) \quad S^{-1} = -S,$$

which implies the equality $S\mathcal{D}(S) = \mathcal{D}(S)$.

One of the ways to construct a PCS is to choose a sequence of linearly independent 2-dimensional subspaces $\mathcal{X}_n$ and operators $S_n$ on $\mathcal{X}_n$ satisfying the condition $S_n^2 + 1\mathcal{X}_n = 0$ (they have matrices of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in appropriate bases. Then we set $\mathcal{D} = \text{lim}(\cup_n \mathcal{X}_n)$ and define $S$ on $\mathcal{D}$ as the direct sum of $S_n$. The properties of such operators depend on the choice of subspaces $\mathcal{X}_n$ and the bases in $\mathcal{X}_n$. For example, if $\mathcal{X}$ is a Hilbert space one can take mutually orthogonal $\mathcal{X}_n$ and orthonormal
bases in each $X_n$, then $S$ will be bounded. If the bases in $X_n$ is such that $\|S_n\| \to \infty$ then $S$ will be closable but not bounded.

It is important that the closure of a closable PCS is a PCS, as the following lemma shows.

**Lemma 3.1.** Let $X$ be a Banach space, $X_0$ its dense subspace, and let $S: X_0 \to X_0$ be a closable bijective operator satisfying the condition $S^{-1} = -S$. Then the closure $\overline{S}$ of $S$ is a closed bijective operator satisfying the condition $(\overline{S})^{-1} = -\overline{S}$.

**Proof.** Let $x \in D(\overline{S})$ and $y = \overline{S}x$. Then there is a sequence $x_n \in X_0$ with $x_n \to x$, $Sx_n \to y$. Since $x_n = -S(Sx_n)$, we see that $y \in D(\overline{S})$ (so $\overline{S}(D(\overline{S})) \subset D(\overline{S})$) and $x = -\overline{S}y$ (so $D(\overline{S}) \subset \overline{S}(D(\overline{S}))$). Thus $\overline{S}(D(\overline{S})) = D(\overline{S})$ and $\overline{S}^2x = -x$, for all $x \in D(\overline{S})$. Hence $(\overline{S})^{-1} = -\overline{S}$ on $D(\overline{S})$. \hfill \blacksquare

Recall that for any operator $W$ defined on a linear subspace $D(W)$ of a Banach space $X$, its graph $G_W = \{(x, Wx) : x \in D(W)\}$ is a linear subspace of the Banach space $X \oplus X$. We identify the space $X^* \oplus X^*$ with the dual space of $X \oplus X$, setting

$$(f_1 + f_2)(x_1 + x_2) = f_1(x_2) - f_2(x_1).$$

If $D(W)$ is dense in $X$ then the annihilator $G_W^\perp$ of $G_W$ in $X^* \oplus X^*$ does not contain pairs of the form $(0, f)$, $f \neq 0$, and therefore there is an operator $V$ defined on a subspace $D_V \subset X^*$ such that $G_W^\perp = \{(f, V^*f) : f \in D_V\}$. The operator $V$ is called the adjoint of $W$ and denoted by $W^*$. The domain $D(W^*)$ of $W^*$ can be described as follows

$$(3.2) \quad D(W^*) = \{f \in X^* : \text{there is } C > 0 \text{ such that } |f(Wx)| \leq C\|x\| \text{ for all } x \in D(W)\}.$$ 

If $W$ is closable then $D_{W^*}$ is weak*-dense in $X^*$. If, moreover, $W$ is closed then $G_W$ is closed and therefore (see for example [11, Theorem 4.4.6]), $(G_W^\perp)^\perp \cap (X \oplus X) = G_W$. In other words,

$$(3.3) \quad \text{if } x, y \in X \text{ and } f(x) = (W^*f)(y), \text{ for all } f \in D(W^*), \text{ then } y \in D(W), \ x = Wy.$$ 

In particular, if $S$ is a closable PCS, then the adjoint operator $S^*$ is defined on a weak*-dense subspace $D(S^*) \subset X^*$. If $f \in D(S^*)$ and $g = S^*f$ then $g \in D(S^*)$ and $S^*g = -f$. Indeed,

$$Sx \in D(S) \text{ by } (3.1), \text{ and } g(Sx) = S^*f(Sx) = f(SSx) = -f(x), \text{ for } x \in D(S),$$

whence $|g(Sx)| \leq \|f\| \|x\|$ and the claim follows from (3.2).

For any $v \in D(S)$ and $f \in D(S^*)$, we set

$$(3.4) \quad T_{v,f} = v \otimes f - Sv \otimes S^*f.$$ 

It is obvious that on $D(S)$ the operator $T_{v,f}$ acts by the rule

$$T_{v,f}x = f(x)v - f(Sx)Sv.$$

Clearly, $T_{v,f}$ is a rank 2 operator that maps $X$ to $D$. Let us show that it commutes with $S$ on $D(S)$. Indeed,

$$T_{v,f}Sx = f(Sx)v - f(S^2x)Sv = -f(Sx)S^2v + f(Sx)v = S(-f(Sx)Sv + f(x)v) = ST_{v,f}x,$$

for $x \in D(S)$.

For any PCS $S$, we set

$$\mathcal{A}_S = \{T \in \mathcal{B}(X) : TS \subset ST\}.$$
Theorem 3.2. If a partial complex structure $S$ on a real Banach space $\mathcal{X}$ is closed, then $\mathcal{A}_S$ is a Lomonosov algebra of complex type on $\mathcal{X}$.

Proof. Clearly, $\mathcal{A}_S$ is an algebra. To see that $\mathcal{A}_S$ is (SOT)-closed let $T_n \rightarrow T \in \mathcal{B}(\mathcal{X})$. Then, for each $x \in \mathcal{D}(S)$, $T_n x \rightarrow Tx$ and $ST_n x = T_n S x \rightarrow TS x$. Since $S$ is closed, $Tx \in \mathcal{D}(S)$ and $ST x = TS x$. This means that $TS \subset ST$, $T \in \mathcal{A}_S$.

Let $x_0 \in \mathcal{X}$, $y_0 \in \mathcal{X}$ and $\varepsilon > 0$. Our aim is to find an operator $T \in \mathcal{A}_S$ with $\|Tx_0 - y_0\| < \varepsilon$. In fact, we will find such $T$ in $\mathcal{A}_F^\mathcal{X} := \mathcal{A}_S \cap \mathcal{F}(\mathcal{X})$.

Let us show that there is $f_0 \in \mathcal{D}(S^*)$ with $f_0(x_0) = 1$ and $(S^* f_0)(x_0) = 0$. Indeed, it suffices to find $f \in \mathcal{D}(S^*)$ with $(S^* f)(x_0) = 0$ and $f(x_0) \neq 0$. If this is impossible then considering functionals $F_1(f) = (S^* f)(x_0)$, $F_2(f) = f(x_0)$ on $\mathcal{D}(S^*)$ we have that $\ker F_1 \subset \ker F_2$. Therefore there is $\alpha \in \mathbb{R}$ with $F_2 = \alpha F_1$. Thus $f(x_0) = \alpha (S^* f)(x_0)$, for all $f \in \mathcal{D}(S^*)$. By (3.3), $\alpha x_0 \in \mathcal{D}(S)$ and $x_0 = S(\alpha x_0)$. However, $S$ has no eigenvalues (since $S^2 \subset -1 \mathcal{X}$). This contradiction proves our claim.

Now we choose $u \in \mathcal{D}(S)$ with $\|y_0 - u\| < \varepsilon$ and set $T = T_{u,f_0}$. Then it follows from (3.4) and the definition of $f_0$ that

$$Tx_0 = f_0(x_0)u - (S^* f_0)(x_0)Su = u.$$ 

Thus $\|Tx_0 - y_0\| = \|u - y_0\| < \varepsilon$. This means that the algebra $\mathcal{A}_S$ is transitive. Since $T$ commutes with $S$ and the rank of $T$ is 2, $T \in \mathcal{A}_F^\mathcal{X}$ whence $\mathcal{A}_S$ is an $L$-algebra. Being (SOT)-closed, $\mathcal{A}_S$ is a Lomonosov algebra.

Let us show that the space $\mathcal{X}^F := \mathcal{A}_F^\mathcal{X} := \text{lin}(\cup_{T \in \mathcal{A}_F^\mathcal{X}} T \mathcal{X})$ coincides with $\mathcal{D}(S)$. Indeed, if $T \in \mathcal{A}_F^\mathcal{X}$ then $TD(S) \subset \mathcal{D}(S)$ (this is true for all operators in $\mathcal{A}_S$). Since $\mathcal{D}(S)$ is dense in $\mathcal{X}$, we have that $T \mathcal{X} \subset \overline{T \mathcal{D}(S)}$. But $T \mathcal{D}(S)$ is closed because $\text{dim} T \mathcal{D}(S) < \infty$. Therefore $T \mathcal{X} \subset T \mathcal{D}(S) \subset \mathcal{D}(S)$. It follows that $\mathcal{X}^F \subset \mathcal{D}(S)$. On the other hand, we have shown above that for arbitrary $u \in \mathcal{D}(S)$, there is such $f \in \mathcal{D}(S^*)$ that the range of the operator $T_{u,f} \in \mathcal{A}_F^\mathcal{X}$ contains $u$. Therefore $\mathcal{D}(S) \subset \mathcal{X}^F$.

It follows that $S \in \mathbb{D}$, so $\mathcal{A}_S$ is not of real type. Since the rank of $T_{u,f}$ equals 2, $\mathcal{A}_S$ is not of quaternion type by Proposition 3.3. Thus $\mathcal{A}_S$ is of complex type.

It will be convenient to formulate separately a statement which was obtained during the proof of Theorem 3.2.

Proposition 3.3. If $S$ is a closed complex partial structure then the subspace $\mathcal{X}^F := \mathcal{A}_F^\mathcal{X}$ coincides with $\mathcal{D}(S)$ and $S \in \mathbb{D}$.

Our next aim is to show that the algebras $\mathcal{A}_S$ majorize all $L$-algebras of complex type.

Theorem 3.4. Every $L$-algebra $\mathcal{A}$ of complex type is contained in the algebra $\mathcal{A}_S$, for some closed partial complex structure $S$.

Proof. By our assumption, the commutant $\mathbb{D}$ of $\mathcal{A}^F$ on $\mathcal{X}^F$ is isomorphic to $\mathbb{C}$. This means that $\mathbb{D} = \text{lin}\{1_{\mathcal{X}^F}, W\}$ where $W$ is an operator on $\mathcal{X}^F$ commuting with all $K \in \mathcal{A}^F$ and satisfying the condition $W^{-1} = -W$.

By Theorem 2.11 $W$ is closable; let $S$ be its closure. By Lemma 3.1 $S$ is a PCS. We will consider the Lomonosov algebra $\mathcal{A}_S$.

To show that $\mathcal{A} \subset \mathcal{A}_S$, we have to prove that $TS \subset ST$, for each $T \in \mathcal{A}$. Firstly, let $x \in \mathcal{X}^F$. Then $Tx \in \mathcal{X}^F$ and $STx = TSx$ since $S|_{\mathcal{X}^F} = W$ and $T$ commutes with $W$ by Lemma 2.7.
Let now \( x \in \mathcal{D}(S) \) and \( y = Sx \). As \( S \) is the closure of \( W \), there are \( x_n \in \mathcal{X}^F \) with \( x_n \to x \) and \( Sx_n = Wx_n \to y \). Therefore \( Tx_n \to Tx \) and, by above, \( STx_n = TSx_n \to Ty \). It follows that \( Tx \in \mathcal{D}(S) \) and \( STx = Ty = TSx \). Thus \( TS \subset ST \) and we are done. \( \Box \)

An operator algebra \( \mathcal{A} \) is called local (see [4]) if it coincides with the (SOT)-closure of \( \mathcal{A} \cap \mathcal{K}(\mathcal{X}) \), where \( \mathcal{K}(\mathcal{X}) \) is the ideal of all compact operators. Now we will show that each algebra \( \mathcal{A}_S \) has a property which can be considered as a reinforced version of locality.

**Theorem 3.5.** If \( \mathcal{A} = \mathcal{A}_S \), where \( S \) is a closed partial complex structure, then \( \mathcal{A}^F \) is (SOT)-dense in \( \mathcal{A} \).

**Proof.** Suffices to show that for every finite-dimensional subspace \( \mathcal{L} \subset \mathcal{X} \), and every \( T \in \mathcal{A} \), the restriction \( T|_{\mathcal{L}} \) is close to \( \mathcal{A}^F|_{\mathcal{L}} \). This means that for any basis \( \{e_i : i = 1, \ldots, \dim \mathcal{L}\} \) of \( \mathcal{L} \) and any \( \varepsilon > 0 \), one can find \( V \in \mathcal{A}^F \) with \( \|Ve_i - Te_i\| < \varepsilon \).

Set \( \mathcal{L}_0 = \mathcal{L} \cap \mathcal{X}^F \). If \( \mathcal{L}_0 \) is not \( \mathbb{C} \)-linear (= \( S \)-invariant), then we denote by \( \mathcal{L}_0' \) its \( \mathbb{C} \)-linear span and replace \( \mathcal{L} \) by \( \mathcal{L} + \mathcal{L}_0' \). So we may assume that \( \mathcal{L}_0 \) is \( \mathbb{C} \)-linear. Denote by \( C \) the norm of \( S|_{\mathcal{L}_0} \). Since \( S \) is antiinvolutive, \( C \geq 1 \). Let \( u_1, \ldots, u_s \) be a \( \mathbb{C} \)-basis in \( \mathcal{L}_0 \), and let \( w_i = Su_i, 1 \leq i \leq s \). Let \( \mathcal{M} \) be a complement of \( \mathcal{L}_0 \) in \( \mathcal{L} \), and \( u_{s+1}, \ldots, u_N \) be a basis in \( \mathcal{M} \).

Then \( \{u_1, \ldots, u_N, w_1, \ldots, w_s\} \) is a basis in \( \mathcal{L} \).

Our main step will be the proof of the following

Claim: there is an operator \( K \in \mathcal{A}^F \) such that the family \( Ku_1, \ldots, Ku_N \) is \( \mathbb{C} \)-linearly independent.

Indeed, if this is established then one can finish the proof as follows.

For all \( i = 1, \ldots, N \) choose \( q_i \in \mathcal{X}^F \) with \( \|Tu_i - q_i\| < \varepsilon/C \). By strict \( \mathbb{C} \)-density of \( \mathcal{A}^F \) on \( \mathcal{X}^F \) (see Corollary 2.5), there is \( R \in \mathcal{A}^F \) with \( RKu_i = q_i, i = 1, \ldots, N \). So, setting \( V = RK \), we get all we need:

\[
\|Vu_i - Tu_i\| = \|q_i - p_i\| < \varepsilon/C \leq \varepsilon
\]

and, using the fact that \( \mathcal{A} \) commutes with \( S \) on \( \mathcal{X}^F \) (see Lemma 2.7),

\[
\|Vw_i - Tw_i\| = \|VSu_i - TSu_i\| = \|S(V - T)u_i\| \leq C\varepsilon/C = \varepsilon.
\]

We will prove the above Claim by induction. More precisely we will show by induction that for each \( n \leq N \), there is an operator \( K \in \mathcal{A}^F \) such that the family \( Ku_1, \ldots, Ku_n \) is \( \mathbb{C} \)-linearly independent.

For \( n = 1 \), the Claim is evident. Assume that it holds for \( n = m < N \) and does not hold for \( n = m + 1 \). So for every \( K \in \mathcal{A}^F \),

\[
Ku_{m+1} = \sum_{k=1}^{m} t_k(K) Ku_k
\]

as in the proof of Lemma 2.8 with the only difference that coefficients \( t_k(K) \) now belong to \( \mathbb{C} \) (recall that the multiplication by a complex number \( a + ib \) acts in \( \mathcal{X}^F \) as the operator \( a1_{\mathcal{X}^F} + bS \)). Repeating the arguments in the proof of Lemma 2.8, we again come to the conclusion that the numbers \( t_k(K) \) do not depend on \( K \):

\[
t_k(K) = a_k1_{\mathcal{X}^F} + b_kS.
\]
Thus
\[
Ku_{m+1} = \sum_{k=1}^{m} (a_k + b_kS)Ku_k, \text{ for all } K \in A^F.
\]

Let now \( 0 \neq v \in X^F = D(S), \) \( f \in D(S^*) \) and \( K = T_{v,f}, \) the operator defined in (3.4). Denoting \( S^*f \) by \( g, \) one can write
\[
Kx = f(x)v - g(x)Sv, \text{ for all } x \in X.
\]
So we get from (3.5) that
\[
f(u_{m+1})v - g(u_{m+1})Sv = \sum_{k=1}^{m} (a_k1 + b_kS)(f(u_k)v - g(u_k)Sv),
\]
whence
\[
(f(u_{m+1}) - \sum_{k=1}^{m} (a_kf(u_k) + b_kg(u_k)))v = (g(u_{m+1}) - \sum_{k=1}^{m} (a_kg(u_k) - b_kf(u_k)))Sv.
\]
Since \( v \) and \( Sv \) are not proportional, both parts are zero. Setting \( x_0 = \sum_{k=1}^{m} a_ku_k, \) \( y_0 = \sum_{k=1}^{m} b_ku_k, \) we rewrite this in the form:
\[
f(u_{m+1}) = f(x_0) + g(y_0) \text{ and } g(u_{m+1}) = g(x_0) - f(y_0).
\]
Writing the second equality as \( g(u_{m+1} - x_0) = -f(y_0) \) and setting \( z = x_0 - u_{m+1}, \) we have \((S^*f)(z) = f(y_0)\) for all \( f \in D(S^*). \) By (3.3), \( z \in D(S) \) and \( y_0 = Sz = S(x_0 - u_{m+1}). \)

Since \( D(S) = X^F, \) by Proposition 3.3 we see that \( u_{m+1} - x_0 \in X^F. \) On the other hand, \( x_0 \in \text{lin}(u_1, ..., u_m) \subset L \) and \( u_{m+1} \in L, \) so we obtain that \( u_{m+1} - x_0 \in X^F \cap L = L_0 \) whence \( u_{m+1} \in L_0 + \text{lin}(u_1, ..., u_m). \) But this is possible only if \( m + 1 \leq s. \)

Indeed, let \( q : L \rightarrow L/L_0 \) be the quotient map. Then \( q(u_j) = 0, \) for \( j \leq s. \) So if \( m + 1 > s \) then \( q(u_{m+1}) \) is a linear combination of \( (u_j)_{s<j<s+1}. \) Since \( M \cap L_0 = \{0\} \) the map \( q \) is injective on \( M. \) Therefore \( u_{m+1} \) is a linear combination of \( (u_j)_{s<j<s+1}, \) a contradiction.

Thus \( m + 1 < s, \) so we have that \( y_0 \in D(S). \) Hence the validity of the equality \( f(u_{m+1}) = f(x_0) + g(y_0) \) for all \( f \in D(S^*), \) implies that
\[
u_{m+1} = x_0 + Sy_0 = \sum_{k=1}^{m} (a_ku_k + b_kSu_k) = \sum_{k=1}^{m} (a_k + ib_k)u_k,
\]
which contradicts the \( \mathbb{C} \)-linear independence of the family \( (u_1, ..., u_s). \) The obtained contradiction proves the Claim. \[\blacksquare\]

It is not clear if closed PCSs exist in all real Banach spaces. If \( X = H, \) an infinite-dimensional real Hilbert space, then one can easily construct a bounded PCS — it suffices to present \( H \) as the orthogonal direct sum of two copies of a space \( H_0: \) \( H = H_0 \oplus H_0 \) and set \( S_I(x \oplus y) = (-y) \oplus x. \) In fact we turn \( H \) into a complex Hilbert space \( H_c \) if we denote the action of \( S_I \) by \( i. \)

Now one can introduce a large family of closed PCSs as follows. Choose in the obtained complex Hilbert space \( H_c \) two closed \( \mathbb{C} \)-linear subspaces \( M, N \) forming a "generic pair":
\[
M \cap N = 0, \quad \overline{M + N} = H_c.
\]
Setting $D = M + N$ we define an operator $S_{M,N}$ on $D$ by the rule

$$S_{M,N}(x + y) = ix - iy, \quad \text{for } x \in M, y \in N;$$

it is easy to check that $S_{M,N}$ is a closed CPS in $H_c$. Indeed, if $x_n + y_n \to z$ and $ix_n - iy_n \to w$ where $x_n \in M, y_n \in N$, then

$$p := (z - iw)/2 = \lim x_n \in M \quad \text{and} \quad q := (z + iw)/2 = \lim y_n \in N.$$ 

So $z = p + q \in D, w = ip - iq = S_{M,N}(p + q) = S_{M,N}z$ and $S_{M,N}$ is closed. The equality $S_{M,N}^{-1} = -S_{M,N}$ is evident.

Using this construction we will show that at least in Hilbert spaces there are many Lomonosov algebras of complex type.

**Corollary 3.6.** *In a separable real infinite-dimensional Hilbert space $H$ there is a continuum of pairwise non-similar Lomonosov algebras of complex type.*

**Proof.** We will say that two linear subspaces $Y_1, Y_2$ of Banach spaces $X_1, X_2$ are isomorphic, if there is a bounded invertible operator $T : X_1 \to X_2$ with $TY_1 = Y_2$. Let us show that if partial complex structures $S_1, S_2$ are such that algebras $A_{S_1}$ and $A_{S_2}$ are similar then their domains $D(S_1), D(S_2)$ are isomorphic.

Indeed if $T A_{S_1} T^{-1} = A_{S_2}$, for some bounded invertible operator $T$, then

$$T(A_{S_1} \cap \mathcal{F}(X_1)) T^{-1} = A_{S_2} \cap \mathcal{F}(X_2).$$

Thus

$$T(A_{S_1} \cap \mathcal{F}(X_1)) = A_{S_2}^{\mathcal{F}} X_2.$$

By Proposition 3.3 $TD(S_1) = D(S_2)$.

Now it remains to find in a real Hilbert space $H$ a continuum of closed PCS’s whose domains are pairwise non-isomorphic.

Using the complex structure in $H$ defined by the operator $S_1$, we may assume that $H$ is a complex space. The construction described before the corollary, relates any generic pair $M, N$ of closed subspaces in $H$ to the PCS $S = S_{M,N}$ with $D(S) = M + N$. So we will look for non-isomorphic subspaces of this form.

Recall that a linear subspace $L$ of $H$ is called an *operator range* if there is a Hilbert space $H'$ and a bounded operator $W : H' \to H$ with $WH' = L$. It is easy to see that any domain of a closed operator is an operator range.

Furthermore it is known (see [5] Theorems 5.1, 5.11, [5] Theorem 2.6) that any dense operator range $L \subseteq H$ that contains an infinite-dimensional closed subspace of $H$ can be presented in the form $M + N$, for some generic pair $M, N$.

To any sequence $(H_k)_{k=0}^\infty$ of pairwise orthogonal subspaces of $H$ such that $H = \oplus_{k=0}^\infty H_k$ there corresponds a dense linear subspace

$$(3.6) \quad L_{(H_k)} = \left\{ \sum_{k=0}^\infty x_k : x_k \in H_k, \sum_{k=0}^\infty 2^k \|x_k\| < \infty \right\}.$$ 

It is easy to show that $L_{(H_k)}$ is an operator range: $L_{(H_k)} = T H$, where $T = \sum_{k=0}^\infty 2^{-k} P_{H_k}$.

It was proved in [5] Theorem 3.3 that if $(K_k)_{k=0}^\infty$ is another sequence of pairwise orthogonal subspaces, then the subspaces $L_{(H_k)}$ and $L_{(K_k)}$ are isomorphic if and only if the dimensions of $H_k$ and $K_k$ satisfy the following condition:
there is \( p \in \mathbb{N} \) such that \( \sum_{k=n}^{m} \dim H_k \leq \sum_{k=n-p}^{m+p} \dim K_k \), for any pair \((n, m)\) with \( n < m \),
and dually (it is assumed that \( H_k = K_k = 0 \) if \( k < 0 \)).

Now for any \( t > 1 \), we denote by \( \mathcal{L}(t) \) a subspace of the form \( \mathcal{L}(H_n) \), where \( \dim H_k = [k^t] := \max\{n \in \mathbb{N} : n \leq k^t\} \), for all \( k > 0 \), while \( \dim H_0 = \infty \). The last condition guarantees that \( \mathcal{L}(t) \) contains an infinite-dimensional closed subspace and therefore is a sum of two subspaces forming a generic pair. Let us check that \( \mathcal{L}(t) \) and \( \mathcal{L}(r) \) are not isomorphic, if \( t \neq r \). Indeed, if \( t > r \) and (3.7) holds, for some \( p \) and all \( m, n \), then choosing \( n = p + 1 \) we obtain that

\[
\sum_{k=p+1}^{m} [k^t] \leq \sum_{k=1}^{m+p} [k^r], \text{ for all } m.
\]

However, since \( a - 1 < [a] \leq a \), the left hand side of (3.8), when \( m \to \infty \), is asymptotically equivalent to

\[
\sum_{k=p+1}^{m} k^t \sim \int_0^m x^t \, dx = \frac{m^{t+1}}{t+1},
\]

while the right hand side, similarly, is asymptotically equivalent to

\[
\frac{(m + p)^{r+1}}{r+1} \sim \frac{m^{r+1}}{r+1}.
\]

So the inequality (3.7) contradicts the condition \( t > r \). \( \blacksquare \)

4. LOMONOSOV ALGEBRAS OF QUATERNION TYPE

If \( \mathcal{A} \) is an \( L \)-algebra of quaternion type then, by definition, there is an isomorphism \( \pi_{\mathcal{A}} \) of the algebra \( \mathbb{H} \) onto \( \mathbb{D} \), the commutant of \( \mathcal{A}^F|_{\mathcal{X}^F} \). It can be considered as a representation of \( \mathbb{H} \) on the space \( \mathcal{X}^F \). If \( G_{\mathbb{H}} \subset \mathbb{H} \) is the quaternion group

\[
G_{\mathbb{H}} = \{ \pm 1, \pm i, \pm j, \pm k : (-1)^2 = 1, \ i^2 = j^2 = k^2 = -1, \ ijk = -1 \}
\]

then the restriction of \( \pi_{\mathcal{A}} \) to \( G_{\mathbb{H}} \) will also be denoted by \( \pi_{\mathcal{A}} \).

It will be convenient to begin the study of this representation in a more general context.

Let \( G \) be a finite group with the unit element \( e_G \), and let \( \pi \) be a representation of \( G \) by linear operators on a dense linear subspace \( \mathcal{E} \) of a Banach space \( \mathcal{X} \). The subspace \( \mathcal{E} \) is called the domain of \( \pi \). We will write \( \mathcal{E}_{\pi} \) instead of \( \mathcal{E} \) when it will be necessary to underline that \( \mathcal{E} \) is the domain of \( \pi \).

We say that \( \pi \) is closed if, whenever

\[
(4.1) \quad \pi(g)x_n \to w(g) \in \mathcal{X} \text{ for all } g \in G \text{ and some sequence } (x_n)_{n=1}^{\infty} \text{ in } \mathcal{E}_{\pi},
\]

then \( w(g) = \pi(g)x \), for some \( x \in \mathcal{E}_{\pi} \) and all \( g \in G \).

A representation \( \pi \) is called closable if, whenever a function \( w : G \to \mathcal{X} \) satisfies (4.1), the condition \( w(e_G) = 0 \) implies \( w(g) = 0 \), for all \( g \in G \).

Let \( C(G, \mathcal{X}) \) be the Banach space of all \( \mathcal{X} \)-valued functions on \( G \) with the norm \( \|F\| = \sup_{g \in G} \|F(g)\| \). We denote by \( \Gamma(\pi) \) the subset of the space \( C(G, \mathcal{X}) \) that consists of all functions \( F(g) = \pi(g)x \) where \( x \in \mathcal{E} \).
Lemma 4.1. The following conditions are equivalent:

(i) $\pi$ is closed;
(ii) $\mathcal{E}$ is complete with respect to the norm $\|x\|_G = \sum_{g \in G} \|\pi(g)x\|$;
(iii) $\Gamma(\pi)$ is closed in $C(G, \mathcal{X})$.

**Proof.** (ii) $\iff$ (iii) follows from the fact that the map $x \mapsto F(g) = \pi(g)x$ is a topological isomorphism between $(\mathcal{E}, \| \cdot \|_G)$ and $\Gamma(\pi)$.

(i) $\Rightarrow$ (ii). Let $x_n$ be a Cauchy sequence in $(\mathcal{E}, \| \cdot \|_G)$. Since $\|\pi(g)x\| \leq \|x\|_G$, the sequence $(\pi(g)x_n)$, for any $g \in G$, is a Cauchy sequence in $\mathcal{X}$. Hence, for each $g \in G$ there is $w(g) \in \mathcal{X}$ with $\pi(g)x_n \to w(g)$. By (i), there is $x \in \mathcal{E}$ with $w(g) = \pi(g)x$. Therefore $\|x_n - x\|_G \to 0$.

(iii) $\Rightarrow$ (i). If $\pi(g)y_n \to w(g)$ for all $g$, then the function $F(g) = w(g)$ belongs to the closure of $\Gamma(\pi)$. Since $\Gamma(\pi)$ is closed, $F(g)$ belongs to $\Gamma(\pi)$. So there is $y \in \mathcal{E}$ with $w(g) = \pi(g)y$.

Let $\overline{\Gamma}$ be the closure of $\Gamma(\pi)$ in $C(G, \mathcal{X})$. It is easy to see that $\Gamma(\pi)$ and $\overline{\Gamma}$ are linear subspaces of $C(G, \mathcal{X})$ and

$$ F \in \overline{\Gamma} \text{ if there are } x_n \in \mathcal{E}_\pi \text{ such that } F(g) = \lim_{n \to \infty} \pi(g)x_n \text{ for all } g \in G. $$

Setting $\mathcal{Z} = \{ F(e_G) : F \in \overline{\Gamma} \}$, we see that $\mathcal{Z}$ is a linear subspace of $\mathcal{X}$ containing $\mathcal{E}_\pi$.

**Lemma 4.2.** If $\pi$ is closable then $\overline{\Gamma} = \Gamma(\rho)$ where $\rho$ is a closed representation of $G$ on $\mathcal{Z}$.

**Proof.** Firstly, we claim that, for each $g \in G$, there is a linear operator $\rho(g) : \mathcal{Z} \to \mathcal{X}$ with

$$ F(g) = \rho(g)F(e_G) \text{ for each function } F \in \overline{\Gamma}. $$

Indeed, let $F \in \overline{\Gamma}$. By (4.2), there is a sequence $(x_n)$ in $\mathcal{E}_\pi$ with $F(g) = \lim_n \pi(g)x_n$. So the assumption of closability of $\pi$ implies that if $F(e_G) = 0$ then $F(g) = 0$, for all $g \in G$. Since the maps $F \mapsto F(e_G) \in \mathcal{Z}$ and $F \mapsto F(g)$ are linear, this shows that $F(g)$ depends on $F(e_G)$ linearly and our claim is proved.

Now we have to show that $\rho(g)\mathcal{Z} \subset \mathcal{Z}$ and $\rho(gh) = \rho(g)\rho(h)$, for all $g, h \in G$.

Let $x \in \mathcal{Z}$. Then there is $F \in \overline{\Gamma}$ with $F(e_G) = x$. So, by (4.2), there are $x_n \in \mathcal{E}_\pi$ such that

$$ \pi(g)x_n \to F(g) = \rho(g)F(e_G) = \rho(g)x \text{ for all } g \in G. $$

Let $t_n = \pi(h)x_n$ for some $h \in G$. Set $F_h(g) = F(gh)$. Then $t_n \in \mathcal{E}_\pi$ and, by (4.4),

$$ t_n \to \rho(h)x \text{ and } \pi(g)t_n = \pi(gh)x_n \to \rho(gh)x = F(gh) = F_h(g). $$

It follows from (4.2) and (4.5) that $F_h \in \overline{\Gamma}$. Hence $\rho(h)x = F(h) = F_h(e_G) \in \mathcal{Z}$ and

$$ \rho(gh)x = F(gh) = F_h(g) = \rho(g)F_h(e_G) = \rho(g)\rho(h)x. $$

Since also $\rho(e_G)x = x$, we have that $g \mapsto \rho(g)$ is a representation of $G$ on $\mathcal{Z}$.

Since $\Gamma(\rho) = \overline{\Gamma(\pi)}$, the representation $\rho$ is closed.

The representation $\rho$ defined in the proof of Lemma 4.2 is called the **closure of $\pi$** and denoted by $\overline{\pi}$.

For a representation $\pi$ of $G$ on $\mathcal{E}$, we set

$$ \mathcal{E}^* = \{ f \in \mathcal{X}^* : |f(\pi(g)x)| \leq C\|x\|, \text{ for some } C > 0 \text{ and all } x \in \mathcal{E}, g \in G \}. $$


Lemma 4.3. (i) Every regular representation is weak*-dense in $\mathcal{X}^*$.  

(ii) The closure of a regular representation is regular.

Proof. Let $\pi$ be a regular representation of a group $G$. 

(i) If $(x_n)_{n=1}^\infty \in \mathcal{E}_\pi$, $\lim \pi(g)x_n = w(g)$ and $w(e_G) = 0$, then $\|x_n\| \to 0$ and therefore $|f(\pi(g)x_n)| \leq C\|x_n\| \to 0$, for any $g \in G$ and any $f \in \mathcal{E}^*$. It follows that $f(w(g)) = \lim f(\pi(g)x_n) = 0$. Hence $w(g) = 0$, since by our assumptions $\mathcal{E}^*$ is weak*-dense in $\mathcal{X}^*$. We proved (i).

(ii) Let $\mathcal{E}_{\pi^*}$ be the domain of $\pi$. To prove (ii) it suffices to show that $\mathcal{E}^* \subseteq \mathcal{E}_{\pi^*}$, that is, $|f(\pi(g)x)| \leq C\|x\|$, for all $g \in G$, $f \in \mathcal{E}^*$ and $x \in \mathcal{E}_{\pi^*}$.

Let $x \in \mathcal{E}_{\pi^*}$. By (4.4), there are $x_n \in \mathcal{E}_\pi$ with $x_n \to x$ and $\pi(g)x_n \to \pi^*(g)x$. Therefore $f(\pi(g)x) = \lim f(\pi(g)x_n)$, so that $|f(\pi^*(g)x)| \leq \lim |f(\pi(g)x_n)| \leq \lim sup C\|x_n\| = C\|x\|$. $\blacksquare$

For each $g \in G$ and each $f \in \mathcal{E}^*$, the map $x \mapsto f(\pi(g^{-1})x)$ is a bounded linear functional on $\mathcal{E}_\pi$. The extension of this functional to $\mathcal{X}$ by continuity will be denoted by $\pi^*(g)f$. It is easy to check that in this way we define a representation of $G$ on $\mathcal{E}^*$; we will denote this representation by $\pi^*$.

For every operator $K$ acting on $\mathcal{E}_\pi$, its "mean" $M_G(K)$ is defined by the formula

$$M_G(K) = \sum_{g \in G} \pi(g)K\pi(g^{-1}).$$  \hspace{1cm} (4.7)

It is easy to check that $M_G(K)$ commutes with all operators $\pi(g)$.

In particular, for $y \in \mathcal{E}_\pi$ and $f \in \mathcal{E}^*$, we set

$$T_{y,f} = M_G(y \otimes f) = \sum_{g \in G} \pi(g)(y \otimes f)\pi(g^{-1}) = \sum_{g \in G} \pi(g)y \otimes \pi^*(g)f,$$ \hspace{1cm} (4.8)

where as usual $y \otimes f$ is the rank one operator on $\mathcal{E}$ acting by the rule

$$(y \otimes f)(x) = f(x)y.$$  

Since the operators $\pi(g)y \otimes \pi^*(g)f$ are in fact defined and bounded on $\mathcal{X}$ we may (and will) consider $T_{y,f}$ as an operator on $\mathcal{X}$. Clearly $\text{rank}(T_{y,f}) \leq |G|$ so

$$T_{y,f} \in \mathcal{F}(\mathcal{X}).$$

If $\pi$ is a representation of a group $G$ on a dense subspace $\mathcal{E}$ of a real Banach space $\mathcal{X}$ then we denote by $\mathcal{A}_\pi$ the set of all operators $T \in B(\mathcal{X})$ that preserve $\mathcal{E}$ and commute with all operators $\pi(g)$ on $\mathcal{E}$:

$$\mathcal{A}_\pi = \{T \in B(\mathcal{X}) : T\mathcal{E} \subseteq \mathcal{E} \text{ and } T\pi(g)|_{\mathcal{E}} = \pi(g)T|_{\mathcal{E}} \text{ for } g \in G\}.$$  

Proposition 4.4. If a representation $\pi$ is closed then $\mathcal{A}_\pi$ is a (SOT)-closed algebra of operators on $\mathcal{X}$.

Proof. Clearly, $\mathcal{A}_\pi$ is an algebra. If $T_n \xrightarrow{\text{SOT}} T$, for some $T_n \in \mathcal{A}_\pi$ and $T \in B(\mathcal{X})$, then $\pi(g)T_n x = T_n \pi(g)x \to T\pi(g)x$ for any $x \in \mathcal{E}, g \in G$.

Since $\pi$ is closed on $\mathcal{E}$ and $T_n x \to Tx$, $T\pi(g)x = \pi(g)y$ for some $y \in \mathcal{E}$. In particular, $Tx = y \in \mathcal{E}$, so that $T$ preserves $\mathcal{E}$, and $T(\pi(g)x) = \pi(g)Tx$. Thus $T$ commutes with $\pi(g)$ on $\mathcal{E}$. It follows that $T \in \mathcal{A}_\pi$ and $\mathcal{A}_\pi$ is SOT-closed. \hfill $\blacksquare$
In what follows we consider only the quaternion group \( G = G_\mathbb{H} \) and those representations of \( G_\mathbb{H} \) on a linear subspace \( \mathcal{E} \subset \mathcal{X} \) that satisfy the condition

\[
\pi(1) = -\pi(-1) = 1_E.
\]

Any such representation extends to a representation of the algebra \( \mathbb{H} \): for \( q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H} \), one sets

\[
\pi(q) = \alpha 1_\mathcal{E} + \beta \pi(i) + \gamma \pi(j) + \delta \pi(k).
\]

One can obtain examples of closed representations of \( G_\mathbb{H} \) as follows. Let \( \mathcal{X} = \ell^2 \) with the standard basis \( \{e_n\}_{n \in \mathbb{N}} \) and let \( \mathcal{X}_m = \text{span}\{e_{4m-3}, e_{4m-2}, e_{4m-1}, e_{4m}\}, m = 1, 2, \ldots \) We identify \( \mathcal{X} \) with the orthogonal direct sum of all \( \mathcal{X}_m \). For each \( m \), we choose a linear bijection \( \phi_m: \mathbb{H} \to \mathcal{X}_m \) and define a representation \( \pi_m \) of \( \mathbb{H} \) on \( \mathcal{X}_m \) by

\[
\pi_m(q)(x) = \phi_m(q \phi_m^{-1}(x)) \quad \text{for} \quad q \in \mathbb{H} \quad \text{and} \quad x \in \mathcal{X}_m.
\]

Let \( s_m = \max_{g \in G_\mathbb{H}} \| \pi_m(g) \| \), and let \( \mathcal{E} \) be the set of all elements \( x = \bigoplus_{m=1}^\infty x_m \in \mathcal{X} \) for which \( \sum_m s_m^2 \| x_m \|^2 < \infty \). The representation \( \pi \) of \( G_\mathbb{H} \) on \( \mathcal{E} \) is defined as the direct sum of representations \( \pi_m \).

**Theorem 4.5.** If a representation \( \pi \) of the group \( G = G_\mathbb{H} \) on a dense subspace \( \mathcal{E} \subset \mathcal{X} \) is closed and regular, then \( \mathcal{A}_\pi \) is a Lomonosov algebra of quaternion type on \( \mathcal{X} \).

**Proof.** By Proposition 4.4, \( \mathcal{A}_\pi \) is a (SOT)-closed algebra. Note that all operators \( T_{y,f} \) defined by the formula (4.8) for any pair \((y,f)\), where \( y \in \mathcal{E}, f \in \mathcal{E}^\ast \), belong to \( \mathcal{A}_\pi \), since they preserve \( \mathcal{E} \) (moreover, they map \( \mathcal{X} \) to \( \mathcal{E} \)) and commute with \( \pi(G) \) on \( \mathcal{E} \). Let \( 0 \neq y \in \mathcal{E} \) and \( 0 \neq x \in \mathcal{X} \); we are going to find \( f \in \mathcal{E}^\ast \) such that \( T_{y,f}x = y \).

We claim that there exists \( f \in \mathcal{E}^\ast \) satisfying conditions

\[
(4.9) \quad f(x) = \frac{1}{2} \quad \text{and} \quad (\pi^\ast(i)f)(x)(\pi^\ast(j)f)(x)(\pi^\ast(k)f)(x) = 0.
\]

To prove the claim, we consider the linear maps \( f \mapsto f(x) \) and \( v: f \mapsto v(f) \) on \( \mathcal{E}^\ast \), where

\[
v(f) = ((\pi^\ast(i)f)(x),(\pi^\ast(j)f)(x),(\pi^\ast(k)f)(x)) \in \mathbb{R}^3.
\]

It suffices to show that there is \( f \in \mathcal{E}^\ast \) with \( v(f) = 0 \) while \( f(x) \neq 0 \). If, by contradiction, \( v(f) = 0 \) implies \( f(x) = 0 \), then the map \( v(f) \mapsto f(x) \), \( f \in \mathcal{E}^\ast \), is well defined and linear on a subspace \( \nu(\mathcal{E}^\ast) \subseteq \mathbb{R}^3 \). Extending it to a linear functional on \( \mathbb{R}^3 \) we get that there exist \( \alpha, \beta, \gamma \in \mathbb{R} \) with

\[
f(x) = \alpha(\pi^\ast(i)f)(x) + \beta(\pi^\ast(j)f)(x) + \gamma(\pi^\ast(k)f)(x), \quad \text{for all} \quad f \in \mathcal{E}^\ast.
\]

This can be written in the form

\[
(\pi^\ast(q)f)(x) = 0, \quad \text{for all} \quad f \in \mathcal{E}^\ast, \quad \text{where} \quad q = 1 - \alpha i - \beta j - \gamma k.
\]

In other words, \( \pi^\ast(q)(\mathcal{E}^\ast) \subset \mathcal{E}_0^\ast \subset \mathcal{E}_0^\ast \) where \( \mathcal{E}_0^\ast = \{ f \in \mathcal{E}^\ast : f(x) = 0 \} \). Since \( q \) is invertible in \( \mathbb{H} \), the operator \( \pi^\ast(q) \) is invertible. So we get the equality \( \mathcal{E}^\ast = \mathcal{E}_0^\ast \). This contradicts the assumption that \( \pi \) is regular, because the subspace \( \mathcal{E}_0^\ast \) is not weak\(^\ast\)-dense in \( \mathcal{X}^\ast \).

Let now \( f \in \mathcal{E}^\ast \) satisfy (4.9). Then, by (4.8),

\[
T_{y,f} = \sum_{g \in G}(\pi(g)^{-1}y \otimes \pi(g)^\ast f)x = \sum_{g \in G}(\pi(g)^\ast f)(x)\pi(g)^{-1}y
\]

\[
= (\pi^\ast(1)f)(x)\pi(1)y + (\pi^\ast(-1)f)(x)\pi(-1)y = 2f(x)y = y.
\]

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Thus we proved that, for any $0 \neq x \in \mathcal{X}$, the space $A_\pi x$ contains $\mathcal{E}$. Since $\overline{\mathcal{E}} = \mathcal{X}$, $A_\pi$ is transitive on $\mathcal{X}$.

Taking in account that all operators $T_{y,f}$ are of finite rank, we conclude that $A_\pi$ is an $L$-algebra. Since $A_\pi$ is (SOT)-closed, it is a Lomonosov algebra. As $\dim(\mathbb{H}) = 4$ (the representation $\pi$ of $\mathbb{H}$ is injective because $\mathbb{H}$ has no non-trivial ideals) and the commutant $D$ of $A_\pi$ contains $\pi(\mathbb{H})$, we conclude that $D = \pi(\mathbb{H})$ and $A_\pi$ is a Lomonosov algebra of quaternion type.

**Corollary 4.6.** In assumptions of Theorem 4.5 the space $\mathcal{X}^F := A_\pi^F \mathcal{X}$ coincides with $\mathcal{E}$.

**Proof.** Let $x \in \mathcal{E}$. By Theorem 4.5 there is $f \in \mathcal{E}^*$ satisfying condition (4.9). It follows that $T_{x,f}x = x$, where $T_{x,f} \in A_\pi^F$ is the operator defined in (4.8) with $y = x$. Therefore $x \in \mathcal{X}^F$ whence $\mathcal{E} \subset \mathcal{X}^F$. On the other hand, if $T \in A_\pi^F$ then $T \mathcal{X} = T \mathcal{E} \subset T \mathcal{E}$. Since $T \mathcal{E}$ is finite-dimensional, $T \mathcal{E} = T \overline{\mathcal{E}} \subset \mathcal{E}$. Therefore $\mathcal{X}^F \subset \mathcal{E}$.

Now we return to the representations $\pi_A$ of $G_\mathbb{H}$ related to arbitrary $L$-algebras $A$ of quaternion type (see the discussion at the beginning of Section 4). Recall that operators $\pi_A(g)$ act on the space $\mathcal{X}^F = A_\pi^F \mathcal{X}$, so $\mathcal{E} = \mathcal{X}^F$.

**Lemma 4.7.** Every representation $\pi_A$ is regular.

**Proof.** For any functional $f \in \mathcal{X}^*$ and operator $T \in A_\pi^F$, we denote by $f_T$ the functional on $\mathcal{X}$ defined by the equality $f_T(x) = f(Tx)$. Then, for each $x \in \mathcal{X}^F$,

$$|f_T(\pi_A(g)x)| = |f_T(T_{\pi_A}(g)x)| = |f(\pi_A(g)Tx)| \leq \|f\|\|\pi_A(g)\|_T\|T\|\|x\|,$$

where $\|\pi_A(g)\|_T$ is the norm of the restriction of $\pi_A(g)$ to the finite-dimensional subspace $T\mathcal{X}$. Since $G_\mathbb{H}$ is finite, $\sup_{g \in G_\mathbb{H}}\|\pi_A(g)\|_T < \infty$ whence

$$|f_T(\pi_A(x))| < C\|x\|, \text{ for all } g \in G_\mathbb{H} \text{ AND } x \in \mathcal{X}^F.$$ Hence all functionals $f_T$ belong to $(\mathcal{X}^F)^*$ (see (4.6)). So to see that the subspace $(\mathcal{X}^F)^*$ is weak*-dense in $\mathcal{X}^*$ it suffices to show that the intersection of kernels of all functionals $f_T$ is $\{0\}$. If $x \in \mathcal{X}$ is such that $f_T(x) = 0$, for all $f \in \mathcal{X}^*$ and all $T \in A_\pi^F$, then $Tx \in \cap_{f \in \mathcal{X}^*} \ker f = \{0\}$, for all $T \in A_\pi^F$. Hence $x \in \ker A_\pi^F = \{0\}$.

**Theorem 4.8.** Every $L$-algebra $A$ of quaternion type is contained in the Lomonosov algebra $A_\pi$ of quaternion type, where $\pi$ is some closed regular representation of $G_\mathbb{H}$.

**Proof.** It follows from Lemma 4.7 that the representation $\pi_A$ of $G_\mathbb{H}$ on $\mathcal{X}^F$ is regular. Hence, by Lemma 4.3 it is closable and its closure $\pi = \overline{\pi_A}$ is a regular closed representation. So by Theorem 4.5 the algebra $A_\pi$ is a Lomonosov algebra of quaternion type.

The domain $\mathcal{E}_\pi$ of $\pi$ clearly contains the subspace $\mathcal{E}_{\pi_A} = \mathcal{X}^F$ of $\mathcal{X}$. For $z \in \mathcal{E}_\pi$, there are $y_n \in \mathcal{X}^F$ such that $\pi_A(g)y_n \to \pi(g)z$, for all $g \in G$. Hence, for each $T \in A$, the sequence $t_n = Ty_n$ satisfies conditions

$$t_n \to Tz \text{ and } \pi_A(g)t_n = \pi_A(g)Ty_n = T\pi_A(g)y_n \to T\pi(g)z.$$ It follows that $Tz \in \mathcal{E}_\pi$ and $T\pi(g)z = \pi(g)Tz$.

Thus all operators in $A$ preserve the subspace $\mathcal{E}_\pi$ and commute with operators $\pi(g)$ on $\mathcal{E}_\pi$. By the definition of $A_\pi$, this means that $A \subset A_\pi$. ■
We will need a special construction of closed regular representations of the group $G_\mathbb{H}$, resembling the construction of Lomonosov algebras of complex type considered in the previous section.

Let $\mathcal{H}_q$ be an infinite-dimensional separable quaternion Hilbert space. The multiplication by $i,j,k$ defines a representation of $G_\mathbb{H}$ in $\mathcal{H}_q$ which we denote by $\tau$. Let $\mathcal{M}, \mathcal{N}$ be a generic pair of quaternion-linear subspaces in $\mathcal{H}_q$:
\[
\mathcal{M} \cap \mathcal{N} = \{0\} \quad \text{and} \quad \overline{\mathcal{M} + \mathcal{N}} = \mathcal{H}_q.
\]

On the space $\mathcal{E} = \mathcal{M} + \mathcal{N}$ we define a representation $\pi$ of $G_\mathbb{H}$ by setting
\[
\pi(g)(x + y) = \tau(g)x + \tau(\alpha(g))y, \quad \text{for} \ x \in \mathcal{M}, y \in \mathcal{N},
\]
where $\alpha$ is an automorphism of $G_\mathbb{H}$ such that $\alpha(i) = -i$.

**Lemma 4.9.** The representation $\pi$ is closed and regular.

**Proof.** Let $\pi(g)(x_n + y_n) \rightarrow z(g)$ for all $g \in G_\mathbb{H}$. Taking subsequently $g = 1$ and $g = i$, we obtain that
\[
x_n + y_n \rightarrow z(1), \quad \text{and} \quad ix_n - iy_n \rightarrow z(i),\]
whence
\[
x_n \rightarrow x := \frac{1}{2}(z(1) - iz(i)) \in \mathcal{M} \quad \text{and} \quad y_n \rightarrow y := \frac{1}{2}(z(1) + iz(i)) \in \mathcal{N}.
\]
It follows that, for all $g \in G_\mathbb{H}$,
\[
\pi(g)(x_n + y_n) = \tau(g)x_n + \tau(\alpha(g))y_n \rightarrow \tau(g)x + \tau(\alpha(g))y = \pi(g)(x + y).
\]

Thus $z(g) = \pi(g)(x + y)$, so that the representation $\pi$ is closed.

Furthermore, let $f \in \mathcal{N}^\perp$, the annihilator of $\mathcal{N}$ in $\mathcal{X}^*$. Since $\mathcal{N}$ is $\tau$-invariant, we get that, for each $x \in \mathcal{M}, y \in \mathcal{N}$ and $g \in G$,
\[
f(\pi(g)(x + y)) = f(\tau(g)x + \tau(\alpha(g))y) = f(\tau(g)x) = f(\tau(g)x + \tau(g)y),
\]
whence $|f(\pi(g)(x + y))| \leq ||f|| ||x + y||$. Thus $f \in \mathcal{E}^*$, so that $\mathcal{N}^\perp \subset \mathcal{E}^*$. Similarly $\mathcal{M}^\perp \subset \mathcal{E}^*$. Since $(\mathcal{M}^\perp + \mathcal{N}^\perp)_\perp = \mathcal{M} \cap \mathcal{N} = \{0\}$, we obtain that $\mathcal{E}^*$ is weak*-dense in $\mathcal{X}^*$. Hence the representation $\pi$ is regular. 

We denote the representation $\pi$ defined in (4.10) by $\pi_{\mathcal{M},\mathcal{N}}$; the corresponding Lomonosov algebra $\mathcal{A}_\pi$ is denoted by $\mathcal{A}_{\mathcal{M},\mathcal{N}}$.

**Corollary 4.10.** In an infinite-dimensional, separable real Hilbert space there is a continuum of pairwise non-similar Lomonosov algebras of quaternion type.

**Proof.** An infinite-dimensional real Hilbert space $\mathcal{H}$ can be realized as a tensor product of a four-dimensional real space $l_2^4$ and a real Hilbert space $\mathcal{H}_0$. Identifying in a natural way $l_2^4$ with $\mathbb{H}$, one can turn $\mathcal{H} = \mathbb{H} \otimes \mathcal{H}_0$ into a quaternion Hilbert space by setting
\[
q_1(q_2 \otimes x) = q_1q_2 \otimes x,
\]
for $q_1, q_2 \in x \in \mathcal{H}_0$. Let $\mathcal{M}_0, \mathcal{N}_0$ be a generic pair of subspaces in $\mathcal{H}_0$. Then the quaternion linear subspaces $\mathcal{M} = \mathbb{H} \otimes \mathcal{M}_0, \mathcal{N} = \mathbb{H} \otimes \mathcal{N}_0$ form a generic pair in $\mathcal{H}$.

We have to show that there is a continuum of pairwise non-similar algebras of the form $\mathcal{A}_{\mathcal{M},\mathcal{N}}$. Arguing as in the proof of Corollary 3.6 and using Corollary 4.6 instead of Corollary...
we see that it suffices to show that there is a continuum of non-isomorphic subspaces \( \mathcal{E} = \mathcal{M} + \mathcal{N} \) of the form \( \mathcal{M} = \mathbb{H} \otimes \mathcal{M}_0, \mathcal{N} = \mathbb{H} \otimes \mathcal{N}_0 \), for some generic pairs \((\mathcal{M}_0, \mathcal{N}_0)\).

We will choose subspaces \( \mathcal{E} \) in the form \( \mathcal{L}_{(H_k)} \) (see (3.6)). Namely, for each \( t > 1 \), we take \( H_k(t) = \mathbb{H} \otimes H_k^0(t) \), where \( \dim H_k^0(t) = [k^t] \) for \( k > 0 \), and \( \dim H_0^0 = \infty \). Then \( \dim_{\mathbb{R}} H_k(t) = 4[k^t] \) and it follows from (3.7) that the subspaces \( \mathcal{L}_{(H_k(t))} \) are pairwise non-isomorphic. On the other hand, since the subspaces \( \mathcal{L}_{(H_0^0(t))} \) are operator ranges containing infinite-dimensional closed subspaces, they can be presented in the form \( \mathcal{M}_0(t) + \mathcal{N}_0(t) \), for generic pairs \( \mathcal{M}_0(t), \mathcal{N}_0(t) \). Therefore

\[
\mathcal{L}_{(H_k(t))} = \mathcal{M}(t) + \mathcal{N}(t) = \mathbb{H} \otimes \mathcal{M}_0(t) + \mathbb{H} \otimes \mathcal{N}_0(t)
\]

and we are done. \( \blacksquare \)

5. Questions and Commentaries

1. Is it true that \( \mathcal{A} = \overline{\mathcal{A}^F} \), for every Lomonosov algebra \( \mathcal{A} \)? (As always the bar over a set of operators denotes the closure in SOT).

2. Is it true that each Lomonosov algebra contains the identity operator?

**Proposition 5.1.** The questions 1 and 2 are equivalent.

**Proof.** Firstly, we show that if the answer to Question 2 is positive then the answer to Question 1 is positive. Indeed, under this assumption, for any Lomonosov algebra \( \mathcal{A} \), the closure \( \overline{\mathcal{A}^F} \) of \( \mathcal{A}^F \) is unital. Since \( \overline{\mathcal{A}^F} \) is an ideal of \( \mathcal{A} \), it coincides with \( \mathcal{A} \).

Conversely, assume that the answer to Question 1 is positive. If a Lomonosov algebra \( \mathcal{A} \) is not unital, then \( \mathcal{B} = \mathcal{A} + \mathbb{R}1 \) is a Lomonosov algebra, so that \( \mathcal{B} = \overline{\mathcal{B}^F} \) by our assumption. Therefore \( \mathcal{B}^F \) is not contained in \( \mathcal{A} \), so there is a finite rank operator \( K \in \mathcal{B}^F \setminus \mathcal{A} \). Clearly, \( K = \lambda 1 + T \) where \( T \in \mathcal{A} \) and \( 0 \neq \lambda \in \mathbb{R} \). Thus \( T = -\lambda 1 + K \in \mathcal{A} \) whence the algebra \( \mathcal{C} \subset \mathcal{A} \) generated by \( T \) contains an operator \( T_1 \) of the form \( 1 + R \) where \( R \in \mathcal{F}(\mathcal{X}) \). Since \( \mathcal{C} \) is commutative and finite-dimensional, idempotents lift from any quotient of \( \mathcal{C} \) (much more general results can be found e.g. in [10]). This means that if \( \mathcal{J} \) is an ideal of \( \mathcal{C} \) and \( W^2 - W \in \mathcal{J} \), for some \( W \in \mathcal{C} \), then there is \( P \in \mathcal{C} \) such that \( P^2 = P \) and \( P - W \in \mathcal{J} \). Applying this to \( \mathcal{J} = \mathcal{C} \cap \mathcal{F}(\mathcal{X}) \) and \( W = T_1 \), we get that there is a projection \( P \in \mathcal{A} \) such that \( P - 1 - R \in \mathcal{F}(\mathcal{X}) \). Thus \( \mathcal{A} \) contains a projection \( P \) such that the projection \( Q = 1 - P \) is of finite rank.

For \( Z \in \mathcal{A} \), the operators \( QZQ = (1 - P)Z(1 - P) = Z - PZ - ZP + PZP \) belong to \( \mathcal{A} \). So \( \mathcal{U} = \{ T \in \mathcal{A}; QTQ = T \} = \{ QZQ; Z \in \mathcal{A} \} \subset \mathcal{A} \) is an algebra and \( \mathcal{U}|_Y \) is an operator algebra on the space \( Y = Q\mathcal{X} \), \( \dim Y < \infty \). It is transitive, since if \( 0 \neq y \in Y \) then \( Uy = QA_y = Q\overline{A_y} = Q\mathcal{X} = Y \). As every such algebra coincides with the algebra of all \( \mathbb{D} \)-linear operators (where \( \mathbb{D} = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \)), \( \mathcal{U}|_Y \) is unital. Let \( R \in \mathcal{U} \) be such that \( R|_Y = 1_Y \). Then \( QRQ = R \) and \( RQ = Q \). So \( R = Q(RQ) = Q^2 = Q \). Thus \( Q \in \mathcal{A} \). Therefore \( 1 = P + Q \in \mathcal{A} \), a contradiction. So \( \mathcal{A} \) is unital. \( \blacksquare \)

3. Let \( \mathcal{A} \) be a Lomonosov algebra. Is it true that all operators in \( \mathbb{D} \) are closed?

4. For which operators \( S \) (respectively, representations \( \pi \)) the algebra \( \mathcal{A}_S \) (respectively, \( \mathcal{A}_\pi \)) is a maximal Lomonosov algebra of complex (respectively, quaternion) type?
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