BUTTERFLY RESAMPLING: ASYMPTOTICS FOR PARTICLE FILTERS WITH CONSTRAINED INTERACTIONS

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We generalize the elementary mechanism of sampling with replacement \( N \) times from a weighted population of size \( N \), by introducing auxiliary variables and constraints on conditional independence characterised by modular congruence relations. Motivated by considerations of parallelism, a convergence study reveals how sparsity of the mechanism’s conditional independence graph is related to fluctuation properties of particle filters which use it for resampling, in some cases exhibiting exotic scaling behaviour. The proofs involve detailed combinatorial analysis of conditional independence graphs.

1. Introduction. Let \( \mathbb{X} \) and \( \mathbb{Y} \) be Polish state-spaces with Borel \( \sigma \)-algebras \( \mathcal{X} \) and \( \mathcal{Y} \). Let \( \pi_0 \) be a probability measure on \( \mathcal{X} \) and let \( f : \mathbb{X} \times \mathbb{X} \to [0,1] \) and \( g : \mathbb{X} \times \mathbb{Y} \to [0,1] \) be probability kernels. A hidden Markov model is a bi-variate process \( (X,Y) \) where the signal process \( X = (X_n)_{n \in \mathbb{N}} \) is a Markov chain with initial distribution \( \pi_0 \) and transition kernel \( f \), and the observations \( Y = (Y_n)_{n \in \mathbb{N}} \) are conditionally independent given \( X \), with the conditional distribution of \( Y_n \) given \( X \) being \( g(X_n, \cdot) \).

Suppose that for each \( x \in \mathbb{X} \), \( g(x, \cdot) \) admits a strictly positive density \( g(x,y) \) w.r.t. a \( \sigma \)-finite measure. Fix a \( \mathbb{Y} \)-valued sequence \( (y_n)_{n \in \mathbb{N}} \) and define the operators \( (\Phi_n)_{n \geq 1} \) acting on probability measures,

\[
\Phi_n(\mu)(A) := \frac{\int_{\mathbb{X}} g(x,y_{n-1}) f(x,A) \mu(dx)}{\int_{\mathbb{X}} g(x,y_{n-1}) \mu(dx)}, \quad A \in \mathcal{X}.
\]

Consider \( \pi_n := \Phi_n(\pi_{n-1}) \), \( n \geq 1 \). If one replaces \( (y_n)_{n \in \mathbb{N}} \) in (1) with the random variables \( (Y_n)_{n \in \mathbb{N}} \) then \( \pi_n \) is a version of the regular conditional distribution of \( X_n \) given \( Y_0, \ldots, Y_{n-1} \). Particle filters [10] approximate \( (\pi_n)_{n \in \mathbb{N}} \) by sampling \( (\zeta_i^{(n)})_{i=1}^N \) i.i.d. \( \sim \pi_0 \), and for \( n \geq 1 \),

\[
(\hat{\zeta}_{n-1}^i)_{i=1}^N \text{ i.i.d. } \sim \frac{\sum_j g(\zeta_{n-1}^j, y_{n-1}) \delta_{\hat{\zeta}_{n-1}^j}}{\sum_i g(\zeta_{n-1}^i, y_{n-1})}, \quad \zeta_n^i \sim f(\hat{\zeta}_{n-1}^i, \cdot), \quad i = 1, \ldots, N,
\]

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so in effect \((\zeta^i_n)_{i=1}^N \overset{\text{i.i.d.}}{\sim} \Phi_n(\pi^N_{n-1})\), where \(\pi^N_{n-1} := N^{-1}\sum_i \delta_{\zeta^i_{n-1}}\). This remarkably simple mechanism has found a huge number of applications. Under mild assumptions – it suffices that for each \(n\), \(g(x, y_n)\) is bounded in \(x\) – a law of large numbers and central limit theorem hold [5, 3, 12, 7]; for \(\mathbb{R}\)-valued, bounded functions \(\varphi\),

\[
\pi_n^N(\varphi) \xrightarrow{\text{a.s.}} \pi_n(\varphi), \quad \sqrt{N}(\pi_n^N(\varphi) - \pi_n(\varphi)) \xrightarrow{d} \mathcal{N}(0, \sigma^2_n(\varphi)),
\]

where for a measure \(\mu\), \(\mu(\varphi) := \int \varphi(x)\mu(dx)\). The asymptotic fluctuations of the particle approximation error are thus of order \(1/\sqrt{N}\), as they would be if \((\zeta^i_n)_{i=1}^N \overset{\text{i.i.d.}}{\sim} \pi_n\), and it can be shown that \(\sigma^2_n(\varphi)\) is never less than the asymptotic variance which would arise from such i.i.d. samples.

1.1. Conditional independence and convergence. The conditional independence and sampling with replacement, or resampling, in (2) leads to the \(\sqrt{N}\) scaling in (3). This dependence structure also influences how particle filters are implemented and resampling hinders their parallelization [15]. Our contribution is to lay rigorous foundations for the design of algorithms better suited to modern computing architectures. We provide insight into consequences for convergence of imposing constraints on the conditional independence structure of a particle filter as a proxy for its communication pattern – an important factor in efficiency of parallel and distributed algorithms [1]. As a taster: for some new algorithms we establish results of the general form

\[
s(N, r) \left(\pi_n^N(\varphi) - \pi_n(\varphi)\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2_n(\varphi, r)),
\]

where \(s(N, r)\) is some increasing function of \(N\) possibly other than \(\sqrt{N}\), and \(r\) is a parameter related to the sparsity of the algorithm’s conditional independence graph. We shall investigate the relationship between \(r, s(N, r)\) and \(\sigma^2_n(\varphi, r)\).

1.2. Outline. In Section 2 we introduce a new augmented resampling algorithm, which generalizes the i.i.d. sampling part of (2). We construct two instances of this algorithm, which we call butterfly resampling, since their conditional independence graphs have the butterfly pattern well known from the Cooley-Tukey fast Fourier transform, but which is also a standard network topology in parallel computing [18]. The butterfly structure stems from equivalence classes of conditionally i.i.d. samples in our algorithms, characterized by modular congruence relations, i.e. equivalence relations expressed
in terms of modular arithmetic. In turn this demands that we develop some non-standard tools for studying convergence.

- For the first butterfly algorithm, \( s(N, r) = \sqrt{N/\log_r N} \). This exotic scaling is the price to pay for the number of incoming edges per vertex in its conditional independence graph being \( r \) and the total number of edges being \( rN \log_r N \), versus respectively \( N \) and \( N^2 \) for a standard particle filter.

- To achieve a more even balance between fluctuations and interaction constraints, we devise a second butterfly algorithm for which \( s(N, r) = \sqrt{N} \), with an asymptotic variance upper bounded by \((2 - r^{-1})\sigma_n^2(\phi)\) where \( \sigma_n^2(\phi) \) is as in (3). For this algorithm some vertices have \( r \) incoming edges, no vertex has greater than \( N/r \) incoming edges and the total number of edges is \( rN + N^2/r \).

Proofs and supporting results are in Section 3 onwards, prefaced by a guide for the reader to aid navigation of our analysis. Two key ingredients that are not usually encountered in theoretical accounts of particle filters are:

- we establish error bounds for certain sub-populations of the particle system, subsequently put to use in establishing limit theorems,
- we conduct a detailed combinatorial analysis of conditional independence graphs, overcoming the biggest technical challenge in analysis of the second moment properties of butterfly sampling, which differ from those of standard particle filters.

The more technical results and most proofs are in the Supplement.

1.3. Notation and conventions. For all \( x, y \in \mathbb{R} \), such that \( y \neq 0 \), we define \( \lfloor x \rfloor := \max\{i \in \mathbb{Z} : i \leq x\} \), \( \lceil x \rceil := \min\{i \in \mathbb{Z} : i \geq x\} \) and \( x \mod y := x - y \lfloor x/y \rfloor \). For all \( n \in \mathbb{N} \), we write \([n] := \{1, \ldots, n\}\). Whenever a summation symbol \( \Sigma \) appears without the summation set made explicit, the summation set is taken to be \([N]\), for example we write \( \Sigma_i \) for \( \Sigma_{i=1}^{N} \). Also \( \sum_{(i_0, \ldots, i_k)} \) is short for \( \sum_{i_0} \cdots \sum_{i_k} \).

For a sequence \((M_k)_{k=1}^{m}\) of square matrices \( \prod_{k=1}^{m} M_k := M_1 \cdots M_m \). Also the shorthand notations \( M_{pq} := \prod_{k=p}^{q} M_k \), where \( p \leq q \), and \( M_{p,q} := \prod_{k=0}^{p-q} M_{p-k} \), where \( p \geq q \), will occasionally be used. The symbol \( \otimes \) denotes: Kronecker product for matrices, direct product for measures, and tensor product for functions. The interpretation will always be clear from the context. For \( n \in \mathbb{N} \), \( I_n \) denotes the \( n \times n \) identity matrix and \( 1_{1/n} \) denotes the \( n \times n \) matrix which has \( 1/n \) as every entry. The notation \( Id \) will be used for identity mappings in various contexts.
We denote by $\mathcal{M}(\mathcal{X})$, $\mathcal{P}(\mathcal{X})$ and $\mathcal{B}_b(\mathcal{X})$ respectively the collections of measures, probability measures and of $\mathbb{R}$-valued, measurable and bounded functions on $(\mathcal{X}, \mathcal{X})$. For $\mu \in \mathcal{M}(\mathcal{X})$, $\varphi \in \mathcal{B}_b(\mathcal{X})$, $A \in \mathcal{X}$ and an integral kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ we write $K(\varphi)(x) := \int K(x, dx') \varphi(x')$, $(\mu K)(A) := \int K(x, A) \mu(dx)$. For $\varphi \in \mathcal{B}_b(\mathcal{X})$, define $\|\varphi\|_\infty := \sup_{x \in \mathcal{X}} |\varphi(x)|$ and $\text{osc}(\varphi) := \sup_{x,y \in \mathcal{X}} |\varphi(x) - \varphi(y)|$. We assume an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all the random variables we encounter are defined. Convergence in probability under $\mathbb{P}$ is denoted by $\xrightarrow{\mathbb{P}}$. For random variables $X, Y, Z$ we write $X \perp \!\!\!\perp Y \mid Z$ to mean $X$ and $Y$ are conditionally independent given $Z$.

2. Algorithms and main results.

2.1. Basics of particle filtering. Since we consider a fixed observation sequence $(y_n)_{n \in \mathbb{T}}$, we shall write $g_n(x) := g(x, y_n)$. The following mild regularity condition is assumed to hold throughout this paper.

**Assumption 1.** For each $n \in \mathbb{N}$, $\sup_x g_n(x) < \infty$ and $g_n(x) > 0$, $\forall x$.

Algorithm 1 is a basic particle filter. There are a number of ways to perform the resample operation. The multinomial method is:

$$\tag{4} \underbrace{(\hat{\xi}_i^0)_{i \in [N]}}_{\text{i.i.d}} \sim \frac{\sum_{i} g_n(\hat{\xi}_i^0) \delta_{\hat{\xi}_i^0}}{\sum_{i} g_n(\hat{\xi}_i^0)},$$

and in that case Algorithm 1 is known as the Bootstrap Particle Filter (BPF).

**Algorithm 1** Particle filter

```plaintext
for $i = 1, \ldots, N$ do
    sample $\hat{\xi}_i^0 \sim \pi_0$
    set $(\hat{\xi}_i^0)_{i \in [N]} \leftarrow \text{RESAMPLE} \left((\hat{\xi}_i^0)_{i \in [N]}, g_0\right)$

for $n = 1, 2 \ldots$ do
    for $i = 1, \ldots, N$ do
        sample $\hat{\xi}_i^1 \sim f(\hat{\xi}_i^{n-1}, \cdot)$
        set $(\hat{\xi}_i^1)_{i \in [N]} \leftarrow \text{RESAMPLE} \left((\hat{\xi}_i^1)_{i \in [N]}, g_n\right)$
```

The following formulae are well defined and finite for $\varphi \in \mathcal{B}_b(\mathcal{X})$,

$$\sigma_n^2(\varphi) := \pi_0((\varphi - \pi_0(\varphi))^2),$$

$$\sigma_n^2(\varphi) := \hat{\sigma}_n^2 + \hat{\pi}_n^{-1}(f((\varphi - f(\varphi))^2)), \quad n \geq 1,$$

$$\hat{\sigma}_n^2(\varphi) := \hat{\pi}_n((\varphi - \hat{\pi}_n(\varphi))^2) + \hat{\pi}_n(g_n)^{-2} \sigma_n^2(g_n(\varphi - \hat{\pi}_n(\varphi))), \quad n \geq 0,$$
where \( \hat{\pi}_n(\varphi) = \frac{\pi_n(\varphi)}{\pi_n(g_n)} \). Considering the empirical measures \( \pi^N_n = N^{-1}\sum_i \delta_{\xi_i^n} \) and \( \hat{\pi}^N_n = N^{-1}\sum_i \delta_{\hat{\xi}_i^n} \), a direct application of e.g. the results of [3] (assuming for the convergence in distribution that the quantities in (5) are strictly positive) gives:

**Theorem 1.** For any \( n \geq 0 \) and \( \varphi \in B_b(X) \), the BPF has the properties that

\[
\pi^N_n(\varphi) - \pi_n(\varphi) \xrightarrow{a.s.} 0, \quad \sqrt{N} \left( \pi^N_n(\varphi) - \pi_n(\varphi) \right) \xrightarrow{d} N(0, \sigma^2_n(\varphi)),
\]

\[
\hat{\pi}^N_n(\varphi) - \hat{\pi}_n(\varphi) \xrightarrow{a.s.} 0, \quad \sqrt{N} \left( \hat{\pi}^N_n(\varphi) - \hat{\pi}_n(\varphi) \right) \xrightarrow{d} N(0, \hat{\sigma}^2_n(\varphi)).
\]

This result will serve as a point of reference against which to compare convergence properties of our new algorithms. Various refinements and extensions of Theorem 1 exist [5, 12, 7], but to emphasize the novel aspects of our comparisons we eschew some technical generalities, many of our results can be generalized to larger function classes and settings beyond HMM’s, and the structure of our algorithms can also be generalized without difficulty so as to incorporate other proposal and resampling schemes.

2.2. Considerations of parallelism and the motivation for our approach.

It is standard practice in computer science to reason about parallelism by introducing a graphical computation/communication model which captures some essence of a practical architecture [18, Ch. 7]. We adopt this philosophy. It is not the purpose of this paper to discuss implementation-specific details of programming etc.

Key to efficiency is an algorithm’s *communication pattern* – the structure via which computational elements exchange information [1]. The bottleneck in this regard for particle filters is the resampling operation, and its conditional independence graph, henceforth “graph” for brevity, provides a convenient and very simple model for its communication pattern if we associate
each vertex in the graph with a separate processing unit and each edge with a communication link. Figure 1 shows the graph for multinomial resampling (4); one can think of each \( \zeta_n^i \) and its weight \( g_n(\zeta_n^i) \) as being stored locally at the \( i \)th vertex in the top row, and the \( i \)th vertex in the bottom row being tasked with sampling \( \tilde{\zeta}_n^i \). To achieve full parallelism, one would need \( O(N^2) \) separate communication paths, ideally a separate physical connection corresponding to each edge in the graph. In practice, communication will be achieved through shared memory or a common data bus, inevitably leading to extensive memory traffic and delays as processors synchronize.

Our interest therefore turns to algorithms with more sparse graphs and – again as is standard in parallel computing [16, Ch. 3] – we can quantitatively summarize sparsity in terms of the total number of edges in the graph and the number of incoming edges per vertex, respectively \( N^2 \) and \( N \) for multinomial resampling. Our aim is to explore the mathematical connections between these quantities and convergence properties as per Theorem 1. Moreover, the graphs for the butterfly algorithms we devise match the structure of butterfly networks – well known communication topologies in parallel computing [18, Ch. 7].

2.3. Literature. There is a small but growing literature on theoretical analysis of particle algorithms with parallelism. The algorithms of [19] involve resampling at two hierarchical levels, and are presented with a study of asymptotic bias and variance. A recent preprint [20] gives a central limit theorem. Some authors of the present paper [22, 14] have studied the non-asymptotic stability properties of an “\( \alpha \)SMC” algorithm in which interaction between particles occurs adaptively, so as to keep the effective sample size above a given threshold. Despite some superficial similarities, the butterfly algorithms we devise are distinct from \( \alpha \)SMC in a number of ways, they do not involve any adaptation, their butterfly structure is entirely original and our study is focused on asymptotics. Some comments on stability are given in Section 2.7, Remark 1. Various issues of computational efficiency for standard algorithms are addressed by e.g. [17] and references therein.

2.4. Augmented resampling. We now introduce a new and general procedure called augmented resampling, which involves the following parameters:

- \( N \), the population size, as in Algorithm 1
- \( m \), a positive integer
- \( (A_k)_{k \in [m]} \), a sequence of non-negative matrices, each of size \( N \times N \)

The main idea is that we can use the matrices \( (A_k)_{k \in [m]} \) to impose constraints on conditional independence of the random variables \( \{\xi_k^i : i \in \end{equation}
Algorithm 2 Augmented resampling

$(\xi_{\text{out}})_i \in [N] = \text{resample } ((\xi_{\text{in}})_i)_i \in [N], g)$

for $i = 1, \ldots, N$ do
  $\xi_0^i \leftarrow \xi_{\text{in}}^i$
  $V_0^i \leftarrow g(\xi_0^i)$

for $k = 1, \ldots, m$ do
  for $i = 1, \ldots, N$ do
    set $V_k^i \leftarrow \sum_j A_{ij}^k V_{k-1}^j$
    sample $\xi_k^i \sim (V_k^i)^{-1} \sum_j A_{ij}^k V_{k-1}^j \delta_{\xi_{k-1}^j}$

for $i = 1, \ldots, N$ do
  $\xi_{\text{out}}^i \leftarrow \xi_{\text{in}}^i$

As a special case, consider $m = 1$ and let $A_1 = 1_{1/N}$. Algorithm 2 then delivers, by inspection,

$$\xi_{\text{out}}^i = \xi_1^i \sim \frac{1}{N} \sum_j V_0^j \delta_{\xi_0^j} = \frac{\sum_j g(\xi_{\text{in}}^j) \delta_{\xi_{\text{in}}^j}}{\sum_j g(\xi_{\text{in}}^j)}, \quad i \in [N],$$

thus augmented resampling generalizes the multinomial resampling scheme (4). With $m \geq 1$ it turns out that a fruitful approach is to consider certain $m$-fold factorizations of $1_{1/N}$ embodied by the following assumption.

**Assumption 2.** For all $k \in [m]$, $A_k$ is a doubly-stochastic matrix and $\prod_{k=1}^m A_k = 1_{1/N}$.

Under this assumption, we can establish some simple but fundamental lack-of-bias and moment properties of augmented resampling. The proof of the following proposition is in Section 3.3.

**Proposition 1.** Fix $N \geq 1$, and consider Algorithm 2 with $g \in \mathcal{B}(X)$ such that $g(x) > 0$ for all $x \in X$. Fix $m \geq 1$ and suppose that $(A_k)_{k \in [m]}$ satisfy Assumption 2. Then for any $\varphi \in \mathcal{B}(X)$,

$$\mathbb{E} \left[ \frac{1}{N} \sum_i \varphi(\xi_{\text{out}}^i) \left| (\xi_{\text{in}}^i)_i \in [N] \right. \right] = \frac{\sum_i g(\xi_{\text{in}}^i) \varphi(\xi_{\text{in}}^i)}{\sum_i g(\xi_{\text{in}}^i)},$$

and for any $p \geq 1$ there exists a finite constant $b_p$, depending only on $p$, such
that no matter what the distribution of \((\xi_{in}^i)_{i \in [N]}\) is,

\[
\mathbb{E} \left[ \left( \frac{1}{N} \sum_i g(\xi_{in}^i) \left( \frac{1}{N} \sum_i \varphi(\xi_{out}^i) - \frac{1}{N} \sum_i g(\xi_{in}^i) \varphi(\xi_{in}^i) \right) \right)^p \right] 
\leq b_p \left( \frac{m}{N} \right)^{\frac{p}{2}} \|g\|_\infty^p \text{osc} (\varphi)^p.
\]

(7)

It is of course implicit in the notation here that \(m\) and the matrices \((A_k)_{k \in [m]}\) may depend on \(N\). An immediate consequence of (7) is that if, for example, \(m\) is some non-decreasing function of \(N\), \((A_k)_{k \in [m]}\) satisfy Assumption 2 for every \(N\), and \(\sum_{N=1}^{\infty} (m/N)^{p/2} < \infty\) for some \(p \geq 1\), then

\[
\left( \frac{1}{N} \sum_i g(\xi_{in}^i) \left( \frac{1}{N} \sum_i \varphi(\xi_{out}^i) - \frac{1}{N} \sum_i g(\xi_{in}^i) \varphi(\xi_{in}^i) \right) \right) \xrightarrow{\text{a.s.}} 0,
\]

without requiring any convergence of \(N^{-1} \sum_i g(\xi_{in}^i)\) or \(N^{-1} \sum_i g(\xi_{in}^i) \varphi(\xi_{in}^i)\). However even if these quantities do converge, without further assumption there is no guarantee of a corresponding central limit theorem and more structure is needed to establish non-trivial limits for the moments in (7) when suitably rescaled. We next introduce parameterised families of the matrices \((A_k)_{k \in [m]}\) which give rise to this structure and which are pursuant to the aims described in Section 2.2.

2.5. Radix-\(r\) resampling algorithm. For each \(r \geq 2\) and \(m \geq 1\), consider the family of matrices

\[
\mathbb{A}_{\text{radix}}^{(r,m)} := (A_k)_{k \in [m]}, \quad A_k = I_{r^{m-k}} \otimes 1_{r^k} \otimes I_{r^{k-1}}, \quad k \in [m].
\]

We shall refer to Algorithm 2 applied with the matrices in (8) and \(N = r^m\) as the radix-\(r\) butterfly resampling algorithm. Examples of the matrices in (8) are shown in Figure 2.

The algebraic structure of (8) dictates the conditional independence structure of butterfly resampling. As a step towards illustrating this connection we now derive a modular congruence characterization of the non-zero matrix entries. For each \(k \in [m]\) and \(r \geq 2\) introduce the following congruence relation on \([N]\),

\[
i \sim_r^{(k,r)} j \iff \begin{cases} 
\left\lfloor \frac{i - 1}{r^k} \right\rfloor = \left\lfloor \frac{j - 1}{r^k} \right\rfloor, \\
\text{and}
\end{cases} \quad (i - 1) \mod r^{k-1} = (j - 1) \mod r^{k-1}.
\]
The matrices $A^{(2,3)}_{\text{radix}} = (A_1, A_2, A_3)$ for the radix-2 algorithm.

**Lemma 1.** The matrices in (8) satisfy Assumption 2. Moreover they are symmetric, have entries which are either $1/r$ or zero, and the non-zero entries are characterized by:

$$A_k^{ij} > 0 \iff i^{(k,r)} \sim j.$$

Since the matrices in (8) are a key and novel ingredient in our algorithms, we present the proof of the lemma before discussing its interpretation.

**Proof.** First we recall the mixed product property of Kronecker product, that is, for any matrices $A,B,C$ and $D$, such that the products $AC$ and $BD$ are defined, one has (see, e.g. [11])

$$A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Also we note that for any two square matrices $A$ of size $M$ and $B$ of size $N$, the Kronecker product has the element-wise formula:

$$A \otimes B)^{ij} = A_{\lfloor i/M \rfloor + 1 \lfloor j/N \rfloor + 1} B((i-1) \mod N+1,(j-1) \mod N)+1,$$

where $i,j \in [MN]$. From the element-wise formula we see immediately that $A \otimes B$ is symmetric if $A$ and $B$ are symmetric. Hence, by (8), $A_k$ is symmetric for all $k \in [m]$. By applying (9) twice to the definition of $A_k$, one has $A_k A_k = A_k$, i.e. $A_k$ is idempotent. By the associativity of the Kronecker product and two applications of the element-wise formula, we also have for the matrices in (8) the expression

$$A_k^{ij} = I_{\lfloor i/r \rfloor + 1 \lfloor j/r \rfloor + 1} (\lfloor i/r \rfloor \mod r)+1, (\lfloor j/r \rfloor \mod r)+1$$

$$\times I_{\lfloor (i-1) r^{k-1} \rfloor + 1 \lfloor (j-1) r^{k-1} \rfloor + 1}.$$
where we have also used the fact that \( \lfloor (i - 1)/r^{k-1} \rfloor/r = \lfloor (i - 1)/r^k \rfloor \) and \( \lfloor (j - 1)/r^{k-1} \rfloor/r = \lfloor (j - 1)/r^k \rfloor \). From this we see immediately that \( A^{ij}_k \in \{0,1/r\} \).

By the idempotence, symmetry and the facts that by (11), \( A^j_k = 1/r \) and \( A^{ij}_k \in \{0,1/r\} \) one has

\[
\frac{1}{r} = A^j_k = (A_k A_k)^j = (A^T_k A_k)^j = \sum_{j \in \{r_m\}} (A^{ij}_k)^2 = \frac{p}{r^2} \iff p = r,
\]

where \( p \) is the number of non-zero elements on the \( j \)th column of \( A_k \). Hence the double stochasticity of Assumption 2 follows by symmetry.

To prove the remaining part of Assumption 2, we assume that for some \( k > 1 \), \( \prod_{q=1}^{k-1} A_q = I_{r^{m-k+1}} \otimes 1_{1/r^{k-1}} \). By (8), this clearly holds for \( k = 2 \). Then by the associativity and the mixed product property (9)

\[
\prod_{q=1}^{k} A_q = (\prod_{q=1}^{k-1} A_q) A_k = (I_{r^{m-k+1}} \otimes 1_{1/r^{k-1}}) (I_{r^{m-k}} \otimes 1_{1/r} \otimes I_{r_{k-1}}) = (I_{r^{m-k}} \otimes 1_{1/r}) \otimes (1_{1/r^{k-1}} I_{r_{k-1}}) = I_{r^{m-k}} \otimes 1_{1/r^k},
\]

i.e. \( \prod_{q=1}^{k} A_q = I_{r^{m-k}} \otimes 1_{1/r^k} \) for all \( k \in \{m\} \), from which the remaining part of Assumption 2 follows by substituting \( k = m \).

Finally, the required equivalence then holds by (11), because \( 1_{1/r} \) has all entries strictly positive and \( I_{r^{m-k}} \) and \( I_{r_{k-1}} \) are identity matrices.

Using Lemma 1, we have by inspection of Algorithm 2 that for radix-\( r \) resampling with any \( i \in [N] \) and \( k \in [m] \),

\[
\xi^i_k \sim \frac{\sum_j A^j_k V^j_{k-1} \delta_{k}^j}{\sum_j A^j_k V^j_{k-1}} = \sum_{\{j:i \sim^{(k,r)} j\}} V^j_{k-1} \delta_{k}^j,
\]

and the following conditional independence holds:

\[
i \sim^{(k,r)} j \implies \xi^i_k \perp \xi^j_k \mid (\xi^u_{k-1}, V^u_{k-1}; u \sim^{(k,r)} i).
\]

These kind of considerations underly much of our convergence study. As illustrated in Figure 3 (a), for radix-\( r \) resampling, the parameter \( r \), which is equal to \( |\{j:i \sim (k,r) j\}| \) for all \( i \in [N], k \in [m] \), is the number of incoming
Fig 3. The conditional independence structure of (a) the radix-\(r\) algorithm with \(r = 2\), \(m = 3\) and \(N = 8\) and (b) the mixed radix-\(r\) algorithm with \(r = 2\), \(c = 4\) and \(N = 8\).

Radix-2 butterfly. \(N = 2^m\), \(m = 1, 2, 3, 4\).

\(N = 2\): \(N = 4\): \(N = 8\): \(N = 16\):

Mixed radix-2 butterfly. \(N = 2c\), \(c = 1, 2, 3, 4\).

\(N = 2\): \(N = 4\): \(N = 6\): \(N = 8\):

Fig 4. Growth of the conditional independence graphs for radix-2 and mixed radix-2 algorithms.

edges for the vertices corresponding to the random variables \(\{\xi_i^k; i \in [N], k \in [m]\}\). Recalling that here \(N = r^m\), the total number of edges in the graph is then \(rN \log_r N\).

As a visual preface to our convergence results, Figure 4 shows the sequence of graphs corresponding to \(A_{\text{radix}}^{(2,m)}\) for \(m = 1, 2, 3, 4\). The bound of Proposition 1 with \(m = \log_r N\) for radix-\(r\) resampling and \(p = 1\) is

\[
 b_1 \sqrt{\frac{\log_r N}{N}} \|g\|_\infty \text{osc} (\varphi)
\]

It turns out that \(\sqrt{\log_r N/N}\) is, asymptotically, the exact scale of the stochastic error for the particle filter when radix-\(r\) resampling is used. However,
this is far from trivial to prove due to the intricacies of the butterfly dependence structure and, in particular, the fact that there are several equivalence classes of conditionally-i.i.d. samples as per (12)-(13), rather than a single such equivalence class for multinomial resampling (4). For \( r \geq 2, \varphi \in \mathcal{B}_b(\mathbb{X}) \) and \( n \geq 1 \) define

\[
\begin{align*}
\sigma^2_{R,0}(\varphi, r) &:= \pi_0((\varphi - \pi_0(\varphi))^2), \\
\sigma^2_{R,n}(\varphi, r) &:= \hat{\sigma}^2_{R,n-1}(f(\varphi), r), \\
\hat{\sigma}^2_{R,0}(\varphi, r) &:= (1 - r^{-1})\hat{\pi}_0((\varphi - \hat{\pi}_0(\varphi))^2), \\
\hat{\sigma}^2_{R,n}(\varphi, r) &:= (1 - r^{-1})\hat{\pi}_n((\varphi - \hat{\pi}_n(\varphi))^2) \\
&\quad + \pi_n(g_n)^{-2}\sigma^2_{R,n}(g_n(\varphi - \hat{\pi}_n(\varphi)), r).
\end{align*}
\]

(14)

Assuming that the above quantities are all strictly positive, we have:

**Theorem 2.** For any \( \varphi \in \mathcal{B}_b(\mathbb{X}) \) and \( r \geq 2 \), the particle filter with radix-\( r \) butterfly resampling has the properties that

\[
\begin{align*}
\pi^N_0(\varphi) - \pi_0(\varphi) &\xrightarrow{a.s.} 0, \\
\sqrt{N}(\pi^N_0(\varphi) - \pi_0(\varphi)) &\xrightarrow{d} \mathcal{N}(0, \sigma^2_{R,0}(\varphi, r)), \\
\hat{\pi}^N_0(\varphi) - \hat{\pi}_0(\varphi) &\xrightarrow{a.s.} 0, \\
\sqrt{N}\log N(\hat{\pi}^N_0(\varphi) - \hat{\pi}_0(\varphi)) &\xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2_{R,0}(\varphi, r)),
\end{align*}
\]

(15)

and for any \( n \geq 1, \)

\[
\begin{align*}
\pi^N_n(\varphi) - \pi_n(\varphi) &\xrightarrow{a.s.} 0, \\
\sqrt{N}\log N(\pi^N_n(\varphi) - \pi_n(\varphi)) &\xrightarrow{d} \mathcal{N}(0, \sigma^2_{R,n}(\varphi, r)), \\
\hat{\pi}^N_n(\varphi) - \hat{\pi}_n(\varphi) &\xrightarrow{a.s.} 0, \\
\sqrt{N}\log N(\hat{\pi}^N_n(\varphi) - \hat{\pi}_n(\varphi)) &\xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2_{R,n}(\varphi, r)),
\end{align*}
\]

(16)

where in (15)-(16) the convergence is as \( N \to \infty \) along the sequence of integer population sizes \( (r^m; m = 1, 2, \ldots) \) for which the radix-\( r \) butterfly resampling algorithm is defined.

**Remark 1.** Under various conditions on the HMM and observation sequence, [3, 21] have proved uniform bounds of the form \( \sup_n \sigma^2_n(\varphi) < \infty \) and [6, 9, 8] have shown that the sequence \( (\sigma^2_n(\varphi))_{n \geq 0} \), regarded as a function of random observations, is tight. In the present setting, it is easily checked that \( \sigma^2_{R,n}(\varphi, r) \leq \sigma^2_n(\varphi) \) and \( \hat{\sigma}^2_{R,n}(\varphi, r) \leq \hat{\sigma}^2_n(\varphi) \), allowing immediate transfer of the aforementioned results to the particle filter with radix-\( r \) resampling.
One interpretation of Theorem 2 is that constraining interaction so that the degree of any vertex in graph does not grow with $N$ leads to slower convergence than the BPF. This leads us to consider our second butterfly resampling scheme.

2.6. Mixed radix-$r$ resampling algorithm. For each $r \geq 2$ and $c \geq 1$

consider the pair of matrices,

\[(17) \quad A_{\text{mixed}}^{(r,c)} = (A_1, A_2), \quad A_k = I_{r^{2-k}} \otimes 1_{1/(r^{k-1}c^2-k)} \otimes I_{c^{k-1}}, \quad k \in \{1, 2\}.\]

We shall refer to Algorithm 2 applied with the matrices in (17), $m = 2$ and $N = rc$ as the mixed radix-$r$ butterfly resampling algorithm. For each $k \in \{1, 2\}$ and $r \geq 2$ introduce the following congruence relation on $[N]$:

\[(18) \quad i \overset{(k,r)}{\sim} j \iff \begin{cases} 
\left\lfloor \frac{i - 1}{r^{k-1}c} \right\rfloor = \left\lfloor \frac{j - 1}{r^{k-1}c} \right\rfloor, \\
(i - 1) \mod c^{k-1} = (j - 1) \mod c^{k-1}.
\end{cases}\]

**Lemma 2.** The matrices in (17) satisfy Assumption 2. Moreover they are symmetric, for $k \in \{1, 2\}$, $A_k$ has entries which are either $1/(r^{k-1}c^2-k)$ or zero, and the non-zero entries are characterized by:

\[A_{ij}^k > 0 \iff i \overset{(k,r)}{\sim} j.\]

**Proof.** The symmetry follows from (10) and the fact that $A_1$ and $A_2$ are defined as Kronecker products of symmetric matrices. Also the idempotence of $A_1$ and $A_2$ as well as

\[(I_r \otimes 1_{1/c})(1_{1/r} \otimes I_c) = I_r 1_{1/r} \otimes 1_{1/c} I_c = 1_{1/r} \otimes 1_{1/c} = 1_{1/(rc)},\]

follow from the mixed product property (9), proving the product part of Assumption 2. Similarly as in the proof of Lemma 1, we have by two applications of the element-wise formula (10)

\[A_{ij}^k = I_{r^{2-k}} \otimes 1_{1/c} + 1_{1/(r^{k-1}c^2-k)} \otimes I_{c^{k-1}}, \quad k \in \{1, 2\},\]

where we have also used the fact that $\left\lfloor (i - 1)/r^{k-1}c \right\rfloor, (j - 1)/r^{k-1}c$, and $\left\lfloor (j - 1)/r^{k-1}c^2-k \right\rfloor = [(i - 1)/r^{k-1}c]$. 


From this it is clear that $A_k^{ij} \in \{0,1/(r^{k-1}c^{2-k})\}$, and $A_k^i = 1/(r^{k-1}c^{2-k})$.

The double stochasticity then follows from these facts similarly as in the proof of Lemma 1 by the symmetry and idempotence. Finally, by the positivity of all elements of $1/(r^{k-1}c^{2-k})$, the required equivalence follows. 

The formulae (12)-(13) hold for the the mixed radix-$r$ algorithm, with the congruence relation (18). Figures 3 (b) and 4 show the graphs, the latter for the case $r = 2$ and $c = 1,2,3,4$. For the mixed radix $r$-algorithm, note that the number of rows $m+1 = 3$ is fixed, $r$ is equal to the degree of the vertices in the bottom row, and $c$ is equal to the number of incoming edges for the vertices in the middle row.

It turns out that the mixed radix $r$-algorithm has the same rate of convergence as the BPF. For all $r \geq 2$ and $\varphi \in \mathcal{B}_h(\mathcal{X})$ define

\[
\begin{align*}
\sigma_{M,0}^2(\varphi, r) &:= \pi_0((\varphi - \pi_0(\varphi))^2), \\
\sigma_{M,n}^2(\varphi, r) &:= \sigma_{M,n-1}^2(f(\varphi), r) + \hat{\pi}_{n-1}(f((\varphi - f(\varphi))^2)), \quad n \geq 1, \\
\hat{\sigma}_{M,n}^2(\varphi, r) &:= (2 - r^{-1}) \hat{\pi}_n((\varphi - \hat{\pi}_n(\varphi))^2) \\
&\quad + \pi_n(g_n)^{-2} \sigma_{M,n}^2(g_n(\varphi - \hat{\pi}_n(\varphi)), r), \quad n \geq 0.
\end{align*}
\]

Assuming the quantities in (19) are strictly positive, we have:

**Theorem 3.** For any $\varphi \in \mathcal{B}_h(\mathcal{X})$ and $r \geq 2$, the particle filter with mixed radix-$r$ butterfly resampling has the properties that for any $n \geq 0$,

\[
\begin{align*}
\pi_n^N(\varphi) - \pi_n(\varphi) &\xrightarrow{a.s.} 0, \quad \sqrt{N}(\pi_n^N(\varphi) - \pi_n(\varphi)) \xrightarrow{d} \mathcal{N}(0, \sigma_{M,n}^2(\varphi, r)), \\
\hat{\pi}_n^N(\varphi) - \hat{\pi}_n(\varphi) &\xrightarrow{a.s.} 0, \quad \sqrt{N}(\hat{\pi}_n^N(\varphi) - \hat{\pi}_n(\varphi)) \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}_{M,n}^2(\varphi, r)),
\end{align*}
\]

where the convergence is as $N \to \infty$ along the sequence of integer population sizes $(rc; c = 1,2,\ldots)$ for which the mixed radix-$r$ butterfly scheme is defined.

A simple induction shows that for any $n \geq 0$, $\sigma_n^2(\varphi) \leq \sigma_{M,n}^2(\varphi, r) \leq (2 - r^{-1}) \sigma_n^2(\varphi)$, and the same inequalities hold with $\sigma_{M,n}^2(\varphi, r), \sigma_n^2(\varphi)$ replaced by $\hat{\sigma}_{M,n}^2(\varphi, r), \hat{\sigma}_n^2(\varphi)$. Thus the stability properties of Remark 1 also apply to the particle filter with mixed radix-$r$ resampling.

**2.7. Discussion.** A summary of the edge characteristics for the graphs of the algorithms we have considered is as follows (excluding vertices $(\xi_{ih})_{i \in [N]}$).
Incoming edges per vertex | Total edges
---|---
Multinomial | $N$ | $N^2$
Radix-$r$ butterfly | $r$ | $rN \log_r N$
Mixed radix-$r$ butterfly | $r$ or $N/r$ | $rN + N^2/r$

With this as a backdrop, let us compare and contrast Theorems 1-3. The behaviour of $\pi_0^N(\varphi)$ is of course common to all three results. Theorem 2 shows the unusual scaling of the radix-$r$ algorithm; the higher the value of $r$ the faster the convergence, but for any finite $r$, the convergence is slower than that of the BPF. This phenomenon and the factor of $(1-r^{-1})$ present in $\tilde{\sigma}_{R,0}^2(\varphi, r)$ and $\tilde{\sigma}_{R,n}^2(\varphi, r)$ have underlying connections to the facts displayed in the table above, namely that the number of incoming edges per node for the radix-$r$ butterfly is fixed to $r$ and in particular is non-increasing in $N$, a characteristic not shared with the BPF, for which the number of incoming edges is $N$.

Note the term $\hat{\pi}_{n-1}(f((\varphi - f(\varphi))^2))$ is present in the functional $\sigma_n^2(\varphi)$ in (5) but absent from $\sigma_{R,n}^2(\varphi, r)$ in (14); the explanation is that for radix-$r$ resampling, the error associated with resampling is of order $\sqrt{\log_r N/N}$, whereas the error associated with sampling $\hat{c}_n^i \sim f(\hat{c}_{n-1}^i, \cdot)$ for each $i \in [N]$ is of order $\sqrt{1/N}$, and therefore makes no contribution to the asymptotic variance (although it will contribute to the non-asymptotic variance in general). On the other hand Theorem 3 shows that the mixed radix-$r$ algorithm has the same scaling as the BPF, and the term $\hat{\pi}_{n-1}(f((\varphi - f(\varphi))^2))$ does appear in $\sigma_{M,n}^2(\varphi, r)$. The difference is the factor of $(2 - 1/r)$ in $\sigma_{M,n}^2(\varphi, r)$, which has underlying connections to the facts that for the mixed radix-$r$ algorithm, $m = 2$ is a constant, and some vertices have $r$ incoming edges.

Let us close with some remarks about generality. One can derive as many instances of augmented resampling as one can factorizations of $1_{1/N}$ into non-negative matrices, there are many alternatives to the two butterfly algorithms we have studied. Also, in practice, one could easily combine butterfly sampling with other techniques such as stratified and adaptive resampling leading to variance reductions. Lastly, we note that the butterfly resampling schemes could be applied as part of many other algorithms and statistical procedures, not just particle filters.

3. Analysis part I - augmented resampling and preparatory results.

3.1. A guide for the reader. The remainder of the paper is structured so that the main results and ideas are given in Sections 3-5, which we recom-
mend the reader browse first to get a sense for our strategy, before getting into the details of the proofs and more technical results in the Supplement. After some preliminaries in Section 3.2, the cornerstone of our analysis is a novel block-wise martingale difference decomposition result, Proposition 2 of Section 3.3, which allows us to quantify the errors associated with certain sub-populations of the particle system, and we later put it to use in establishing the CLT’s.

Theorem 4 in Section 3.4 is a conditional CLT for triangular martingale arrays proved by [7], which we shall apply, while Section 3.5 describes how we map the martingales of Proposition 2 in the cases of the two butterfly resampling schemes onto the triangular array format. Propositions 3 and 4 provide novel tools to quantify second moment properties of augmented resampling, with a view to verifying the conditions of Theorem 4.

Statements and main proof steps of LLN’s and CLT’s for single applications of butterfly resampling, Theorems 5−8, are then given in Section 4. These rely on a number of novel but highly technical results given in the Supplement, in turn utilizing Propositions 2−4. An outline of proofs for Theorems 2 and 3, the LLN’s and CLT’s for particle filters, is given in Section 5, with the details in the Supplement.

3.2. Probability law of the augmented resampling algorithm. We begin building the theory with a more explicit probabilistic description of a single instance of Algorithm 2. Consider ξ := (ξi)i∈[N] and (ξk)k∈[m], where ξk := (ξ1k, . . . , ξNk) and each ξi and each ξk are X-valued random elements. By convention, set ξ0 := ξin, ξ0 := ξi and ξout := ξm, ξout := ξm. Unless otherwise explicitly stated, the parameters N, m ≥ 1 are assumed fixed and we write A(N,m) := (Ak)k∈[m] for the sequence of matrices parameterizing the augmented resampling algorithm. Moreover, the following regularity condition, prototypical of Assumption 1, is imposed from henceforth on the function g passed to Algorithm 2.

Assumption 3. The function g belongs to Bb(X) and is strictly positive.

Define for i ∈ [N] and k ∈ [m],

\[ V_0^i := g(ξ_0^i), \quad V_k^i := \sum_j A_k^j V_{k-1}^j. \]

The following facts about the \( V_k^i \)’s shall be used repeatedly.

Lemma 3. Fix N, m ≥ 1. For any i ∈ [N] and 0 ≤ k ≤ m,
(i) $V^i_k$ is measurable w.r.t. $\sigma(\xi_{in})$.

(ii) $V^i_k \leq \|g\|_\infty$.

If, in addition, $\mathbb{A}^{(N,m)}$ satisfies Assumption 2, then $V^i_m = N^{-1} \sum_j g(\xi^i_{in})$ for all $i \in [N]$.

**Proof.** From (21) we have $V^i_0 = g(\xi^i_{in})$ and a simple induction shows that for $k \in [m]$,

$$V^i_k = \sum_{(i_0, \ldots, i_{k-1})} g(\xi^i_{in}) \prod_{q=1}^{k} A_{i_q}^{i_{q-1}}.$$  

It is then clear that $V^i_k$ is measurable w.r.t. $\sigma(\xi_{in})$. Since each $A_k$ is a row-stochastic matrix, the bound $V^i_k \leq \|g\|_\infty$ holds. Applying (22) in the case $k = m$ and using the assumption $\prod_{k=1}^{m} A_k = 1_{1/N}$ we find

$$V^i_m = \sum_{(i_0, \ldots, i_{m-1})} g(\xi^i_{in}) \prod_{q=1}^{m} A_{i_q}^{i_{q-1}} = \sum_{i_0} g(\xi^i_{in}) \left( \prod_{q=1}^{m} A_q \right)^{i_{m-1}} = \frac{1}{N} \sum_{i_0} g(\xi^i_{in}).$$

Algorithm 2 corresponds to the following distributional prescription. For each $k \in [m]$ the random elements $(\xi^i_k)_{i \in [N]}$ are conditionally independent given $(\xi_0, \ldots, \xi_{k-1})$, a property which will be frequently referred to as one step conditional independence. Moreover, for each $i \in [N]$ and $S \in \mathcal{X}$,

$$\mathbb{P}(\xi^i_k \in S \mid \xi_0, \ldots, \xi_{k-1}) = \frac{1}{V_k} \sum_j A_k^{ij} V^j_{k-1} 1_{S(\xi^j_{k-1})}.$$  

Since $V^i_{k-1}$ is measurable w.r.t. $\sigma(\xi_0)$, we notice from (23) that in fact

$$\mathbb{P}(\xi^i_k \in S \mid \xi_0, \ldots, \xi_{k-1}) = \mathbb{P}(\xi^i_k \in S \mid \xi_0, (\xi^j_{k-1}; j \in [N], A_k^{ij} > 0)).$$

We have also an explicit expression for the conditional marginal distribution of $\xi^i_k$, given $(\xi_0, \ldots, \xi_q)$ where $0 \leq q \leq k-1$, according to the following result for which the proof is given in Section A of the Supplement.

**Lemma 4.** Fix $N, m \geq 1$. If $\mathbb{A}^{(N,m)}$ satisfies Assumption 2, then for all $i \in [N]$, $k \in [m]$ and $S \in \mathcal{X}$

$$\mathbb{P}(\xi^i_m \in S \mid \xi_0, \ldots, \xi_{m-k}) = \frac{1}{V_m} \sum_j \left( \prod_{q=0}^{k-1} A_{m-q} \right)^{ij} V^j_{m-k} 1_{S(\xi^j_{m-k})}.$$
3.3. Block-wise martingale decomposition. Given $N \geq 1$ and a partition $\mathcal{I}$ of $[N]$, $\mathcal{I} = \{\mathcal{I}_u \subset [N] : u \in [I]\}$, let $\mathcal{J}(\mathcal{I})$ be the set of all functions $J : [I] \to [N]$ such that for each $u \in [I]$, $J(u)$ is some member of $\mathcal{I}_u$.

This section addresses martingale decomposition of error terms of the form

\begin{equation}
\left( \frac{1}{N} \sum_i g(\xi_{in}) \right) \left( \frac{1}{|I|} \sum_{i=1}^{[I]} \varphi(\xi_{out}^{(i)}) \right) - \frac{1}{N} \sum_i g(\xi_{in}) \varphi(\xi_{in}).
\end{equation}

Note that in the special case $I = \{\{u\}; u \in [N]\}$, we have $|J(I)| = 1$, the unique member of $J(I)$ is $J = \text{Id}$ and (24) reduces to the quantity in (7).

We shall use the generality of (24) beyond this special case to help prove our CLT’s. Loosely speaking, we shall be concerned with partitions $I$ such that for any $(i, j) \in \mathcal{I}_u \times \mathcal{I}_v$ and some $d \in [m]$, (25)

\begin{align*}
u = v & \implies P(\xi_{out}^i \in \cdot | \xi_0, \ldots, \xi_{m-d}) = P(\xi_{out}^j \in \cdot | \xi_0, \ldots, \xi_{m-d}), \\
u \neq v & \implies \xi_{out}^i \perp \perp \xi_{out}^j | \xi_0, \ldots, \xi_{m-d}.
\end{align*}

Whether or not (25) holds obviously depends on the choice of matrices $\mathcal{A}_{(N,m)}$, a matter which we shall formalize in Assumption 4 below.

Let us now proceed with the precise details. We shall make multiple uses of the objects which we define next and this flexibility is accommodated by our notation, which is a little intricate, but provides just what we need.

For $m \geq 1$, define the index mappings $p_N : [Nm] \to [N]$ and $s_N : [Nm] \to [m]$ for each $\rho \in [Nm]$ as

\begin{align*}
p_N(\rho) := ((\rho - 1) \text{ mod } N) + 1, \\
s_N(\rho) := \left\lfloor \frac{\rho}{N} \right\rfloor.
\end{align*}

Now for given $d \in [m]$, a partition $\mathcal{I}$ of $[N]$ and $J \in \mathcal{J}(\mathcal{I})$, we define the $\sigma$-algebras $(\mathcal{F}_{\rho}^{(N,m)})_{0 \leq \rho \leq (m-d)N + |I|}$ as

\begin{equation}
\mathcal{F}_{\rho}^{(N,m)} = \begin{cases} 
\sigma(\xi_{in}), & \rho = 0, \\
\mathcal{F}_{\rho-1}^{(N,m)} \vee \sigma(\xi_{s\rho N(\rho)}), & 0 < \rho \leq N^*, \\
\mathcal{F}_{\rho-1}^{(N,m)} \vee \sigma(\xi_{mP\rho N(\rho)}), & \rho > N^*,
\end{cases}
\end{equation}

where $N^* := (m - d)N$.

For $\varphi \in \mathcal{B}_b(\mathbb{X})$, let

\begin{equation}
\mathcal{F}_N(x) := \varphi(x) - \frac{\sum_i g(\xi_{bb}) \varphi(\xi_{bb})}{\sum_i g(\xi_{bb})},
\end{equation}
and by writing $\varphi_{N,q}^i = \varphi_N(\xi_q^i)$ for brevity, for all $i \in [N]$ and $0 \leq q \leq m$, define the sequence $(X_{\varphi}^{(N,m)})_{\varphi \in [(m-d)N + |I|]}$,

$$X_{\varphi}^{(N,m)} := \begin{cases} \frac{S_{N,m,d} V_i}{N} \left( \varphi_{N,q} - \frac{1}{V_q} \sum_j A_q^i V_q^j \varphi_{N,q-1}^j \right), & \varphi \leq N^*, \\ \frac{S_{N,m,d} V_i}{|I|} \left( \varphi_{N,m} - \frac{1}{V_m} \sum_j \left( \prod_{p=0}^{d-1} A_{m-p}^j \right) V_m^{j-d} \varphi_{N,m-d}^j \right), & \varphi > N^*, \end{cases}$$

where $q = s_N(\varphi), i = p_N(\varphi)$ for all $0 < \varphi \leq N^*$ and $i = J(p_N(\varphi))$ for all $N^* < \varphi \leq N^* + |I|$. The scaling factor $S_{N,m,d}$ is

$$S_{N,m,d} := \left( \frac{m-d}{N} + \frac{1}{|I|} \right)^{-1/2}.$$ 

We stress that $F_{\varphi}^{(N,m)}$ depends on $d, J; X_{\varphi}^{(N,m)}$ depends on $d, |I|, J, \varphi$; and $S_{N,m,d}$ depends on $|I|$; but these dependencies are suppressed from the notation.

The following assumption, which we shall invoke in Proposition 2, demands some specific relationships between the matrices $A_{(N,m)}$, the partition $I$ and the parameter $d$.

Assumption 4. For given $N, m \geq 1, d \in [m], A_{(N,m)}$, and $I = \{I_u \subset [N] : u \in [|I|]\}$, the sequence of matrices $A_{(N,m)}$ satisfies Assumption 2 and the triple $(A_{(N,m)}, I, d)$ has the following properties:

(i) $I$ is a partition of $[N]$ such that for all $u \in [|I|], |I_u| = N/|I| \geq d$.
(ii) For all $u \in [|I|], j_1, j_2 \in I_u$ and $i \in [N],

$$\left( \prod_{q=0}^{d-1} A_{m-q}^i \right)^{j_1 i} = \left( \prod_{q=0}^{d-1} A_{m-q}^i \right)^{j_2 i}.

(iii) For all $u, v \in [|I|]$ such that $u \neq v$, and $(i, j) \in I_u \times I_v, \xi_{out}^i \perp \xi_{out}^j | \xi_0, \ldots, \xi_{m-d}$.

Remark 2. The condition (i) means that $I$ partitions $[N]$ into sets of equal sizes. By Lemma 4, (ii) ensures that the random variables $\xi_{out}^i$ and $\xi_{out}^j$, where $i$ and $j$ belong to the same element of the partition $I$, have conditionally identical distributions given $\xi_0, \ldots, \xi_{m-d}$. Together with (iii) this formalizes (25).
Remark 3. Assumption 4 reduces to exactly Assumption 2 in the case that $d = 1$ and $\mathcal{I} = \{\{u\}; u \in [N]\}$. To see this, note that then: $|\mathcal{I}| = N$, so (i) is satisfied; $\mathcal{I}_u = \{u\}$, so (ii) is satisfied; and (iii) is satisfied due to the one step conditional independence property of augmented resampling, stated above (23).

We can now present the martingale decomposition. The proof is given Section A of the Supplement.

Proposition 2. If for some $N, m \geq 1$ and $d \in [m]$, $(A^{(N,m)}(\mathcal{I},d)$ satisfies Assumption 4, then for all $\varphi \in \mathcal{B}_b(\mathcal{X})$, $J \in \mathcal{J}(\mathcal{I})$ and $\varphi \in [(m - d)N + |\mathcal{I}|]$, the following hold:

(i) $X^{(N,m)}_\varphi$ is measurable w.r.t. $F^{(N,m)}_\varphi$,

(ii) $E \left[ X^{(N,m)}_\varphi | F^{(N,m)}_{\varphi-1} \right] = 0$,

(iii) $X^{(N,m)}_\varphi$ is bounded by

\[
|X^{(N,m)}_\varphi| \leq \begin{cases} 
S_{N,m,d}N^{-1} \|g\|_\infty \text{osc} (\varphi), & \varphi \leq (m - d)N, \\
S_{N,m,d}|\mathcal{I}|^{-1} \|g\|_\infty \text{osc} (\varphi), & \varphi > (m - d)N,
\end{cases}
\]

(iv) and we have the decomposition

\[
\frac{1}{S_{N,m,d}} \sum_{\varphi=1}^{(m-d)N + |\mathcal{I}|} X^{(N,m)}_\varphi = \frac{1}{|\mathcal{I}|} \sum_{i=1}^{|\mathcal{I}|} V_m^{J(i_m)} \mathcal{P}_N (\xi^{J(i_m)}_m)
\]

\[
= \left( \frac{1}{N} \sum_i g(\xi^i_{in}) \right) \left( \frac{1}{|\mathcal{I}|} \sum_{i=1}^{|\mathcal{I}|} \varphi(\xi^{J(i)}) \right) - \frac{1}{N} \sum_i g(\xi^i_{in}) \varphi(\xi^i_{in}).
\]

We can now prove Proposition 1.

Proof of Proposition 1. Let us choose $d = 1$, $\mathcal{I} = \{\{u\}; u \in [N]\}$ and $J = Id$. In this case, $|\mathcal{I}| = N$, $(m - d)N + |\mathcal{I}| = Nm$, $S_{N,m,d} = \sqrt{N/m}$. Assumption 4 is satisfied for any $(A_k)_{k\in[m]}$ satisfying Assumption 2 – see Remark 3. Therefore we can apply Proposition 2. The lack-of-bias property (6) follows immediately from Proposition 2(ii), (32) and the tower property of conditional expectation. For the moment bound (7), we apply
the Burkholder-Davis-Gundy inequality and (30) to obtain
\[
\mathbb{E} \left[ \left| \sum_{\varrho \in [Nm]} X^{(N,m)}_{\varrho} \right|^p \right] \leq b_p \mathbb{E} \left[ \left( \sum_{\varrho \in [Nm]} \left( X^{(N,m)}_{\varrho} \right)^2 \right)^{p/2} \right] \leq b_p \|g\|_\infty \text{osc}(\varphi)^p.
\]

\[\square\]

**Warning:** Throughout the remainder of Sections 3-5, whenever the sequences \( (\mathcal{F}_{\varrho}^{(N,m)})_{0 \leq \varrho \leq (m-d)N + |I|} \) and \( (X^{(N,m)}_{\varrho})_{\varrho \in [(m-d)N + |I|]} \) appear, they are taken to be as in (26) and (28) with specifically \( d = 1, I = \{\{u\} : u \in [N]\} \) and \( J = Id. \)

### 3.4. Conditional CLT for martingale array

In light of Proposition 2, for each \( N \) and \( m, \) \( (X^{(N,m)}_{\varrho})_{\varrho \in [Nm]} \) is clearly a martingale difference sequence w.r.t. \( (\mathcal{F}_{\varrho}^{(N,m)})_{0 \leq \varrho \leq Nm}. \) Our strategy is to study its behaviour using the following result, which is a special case of [7, Theorem A.3].

Let \( (\ell_n)_{n \geq 1} \) be a sequence of positive integer constants. Let \( (U_{n,\varrho})_{\varrho \in [\ell_n]} \) be a triangular array of random variables and let \( ( \mathcal{G}_{n,\varrho} )_{0 \leq \varrho \leq \ell_n} \) be a triangular array of sub-\( \sigma \)-algebras of the \( \sigma \)-algebra \( \mathcal{F} \) of the underlying probability space, such that for each \( n \) and \( \varrho \in [\ell_n], U_{n,\varrho} \) is \( \mathcal{G}_{n,\varrho} \)-measurable and \( \mathcal{G}_{n,\varrho-1} \subseteq \mathcal{G}_{n,\varrho}. \)

**Theorem 4.** Assume that \( \mathbb{E} \left[ U_{n,\varrho}^2 \mathbb{I}_{\mathcal{G}_{n,\varrho-1}} \right] < \infty \) for any \( n \) and \( \varrho \in [\ell_n], \) and

\[
\mathbb{E} \left[ U_{n,\varrho} \mathbb{I}_{\mathcal{G}_{n,\varrho-1}} \right] = 0,
\]

\[
\sum_{\varrho \in [\ell_n]} \mathbb{E} \left[ U_{n,\varrho}^2 \mathbb{I}_{\{U_{n,\varrho} \geq \epsilon\}} \mathbb{I}_{\mathcal{G}_{n,\varrho-1}} \right] \xrightarrow{P} 0, \quad \text{for any } \epsilon > 0,
\]

\[
\sum_{\varrho \in [\ell_n]} \mathbb{E} \left[ U_{n,\varrho}^2 \mathbb{I}_{\mathcal{G}_{n,\varrho-1}} \right] \xrightarrow{P} \sigma^2, \quad \text{for some } \sigma^2 > 0.
\]

Then, for any real \( u, \)

\[
\mathbb{E} \left[ \exp \left( iu \sum_{\varrho \in [\ell_n]} U_{n,\varrho} \right) \mathbb{I}_{\mathcal{G}_{n,0}} \right] \xrightarrow{P} \exp \left( -\frac{u^2}{2} \sigma^2 \right).
\]
3.5. Triangular martingale array representation of butterfly resampling algorithms. In order to apply Theorem 4 we need to map the martingales of Section 3.3 onto the format of Theorem 4. This is done in a different way for each of the two butterfly resampling algorithms.

For the radix-$r$ algorithm, we have a fixed positive integer $r \geq 2$ and $N = r^m$ with $m \geq 1$. For the variables in Theorem 4 we take $n = m$, $\ell_n = Nm = r^m m$, and $U_{n,\phi} = X_{\phi}^{(r^m, m)}$ for all $\phi \in [mr^m]$ and $\mathcal{G}_{n,\phi} = \mathcal{F}_{\phi}^{(r^m, m)}$ for $0 \leq \phi \leq mr^m$. In simple terms, the $m$th row of the array involves the random variables in an instance of the butterfly resampling scheme with population size $N = r^m$.

For the mixed radix-$r$ algorithm, we have a fixed positive integer $r \geq 2$ and the population size $N$ is taken to be an integer multiple of $r$, i.e. $N = rc$ where $c \geq 1$. $m = 2$ is a constant. For the variables in Theorem 4 we take $n = c$, $\ell_n = 2N = 2rc$, $U_{n,\phi} = X_{\phi}^{(rc, 2)}$ for all $\phi \in [2rc]$, and $\mathcal{G}_{n,\phi} = \mathcal{F}_{\phi}^{(rc, 2)}$ for $0 \leq \phi \leq 2rc$.

For each of the butterfly algorithms, it is then easily checked that: $\mathcal{G}_{n,\phi-1} \subseteq \mathcal{G}_{n,\phi}$, using (26); $U_{n,\phi}$ is $\mathcal{G}_{n,\phi}$-measurable, using Proposition 2; and finally $\mathbb{E} [U_{n,\phi}^2 | \mathcal{G}_{n,\phi-1}] < \infty$ using (30).

Our aim is to verify the remaining conditions of Theorem 4, the most challenging is (35), and our next step is to develop some tools which help.

3.6. Conditional variance and collision analysis. We shall use the following proposition to establish the connection between the conditional second moment of the martingale of Proposition 2 and the conditional independence structure of the augmented resampling algorithm through the matrices $(A_k)_{k \in [m]}$. The proof of the proposition is given in Section B of Supplement, and is partly inspired by [2].

**Proposition 3.** For any $N \geq 2$, $m \geq 1$, $\varphi \in \mathcal{B}_b(\mathbb{X})$ and for any sequence of row stochastic matrices $(A_k)_{k \in [m]}$

\begin{equation}
\frac{m}{N} \mathbb{E}\left[\left( \sum_{\phi \in [Nm]} X_{\phi}^{(N, m)} \right)^2 \bigg| \mathcal{F}_{0}^{(N, m)} \right] = \sum_{(i_0, j_0, \ldots, i_m, j_m)} \left( \frac{1}{N^2} \prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}} A_{k+1}^{j_{k+1}} \right) g(\xi_0^{i_0}) g(\xi_0^{j_0}) C_{i_1, m, j_1, m} (\Phi)(\xi_0^{i_0}, \xi_0^{j_0})
\end{equation}

where $\Phi = \mathbb{F}_N^{\mathbb{R}^2}$, $C_{i_1, m, j_1, m} := C_{[i_1 = j_1]} \cdots C_{[i_m = j_m]}$, and $C_0$ and $C_1$ act on functions $\mathcal{B}_b(\mathbb{X}^2) \to \mathcal{B}_b(\mathbb{X}^2)$ to the right as $C_0 := \text{Id}$ and $(C_1 \Phi)(x, x') := \Phi(x, x)$. 

When operating on the function $\varphi^\otimes 2$, the composite operator $C_{i_1,m,j_1,m}$ satisfies

$$C_{i_1,m,j_1,m}(\varphi^\otimes 2)(x,x') = \begin{cases} \varphi(x)\varphi(x), & \text{if } i_k = j_k \text{ for some } k \in [m], \\ \varphi(x)\varphi(x'), & \text{otherwise}. \end{cases}$$

To determine which of the cases in (37) is true, is equivalent to asking whether the sequences $(i_0, \ldots, i_m)$ and $(j_0, \ldots, j_m)$ have a common element $i_k = j_k$ for some $k \in [m]$, i.e. if these sequences collide. Consequently, formulating more tractable expressions for the r.h.s. of (36) boils down to finding the sets of pairs $(i_0, \ldots, i_m)$, $(j_0, \ldots, j_m)$ for which the term $\prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}i_{k}} A_{k+1}^{j_{k+1}j_{k}}$ is non-zero, and identifying their collisions. We term this collision analysis. In order to state a resulting expression for the r.h.s. of (36), we need to introduce the following notations.

For all $i \in [N]$ and $k \in [m],$

(38) $P_{\mathcal{A}} := \{(j_0, \ldots, j_m) \in [N]^{m+1} : \prod_{k=0}^{m-1} A_{k+1}^{j_{k+1}j_{k}} \neq 0\},$

(39) $P_{\mathcal{A}}^{(i)} := \{(j_0, \ldots, j_m) \in P_{\mathcal{A}} : j_0 = i\},$

(40) $A_{\mathcal{A}}^{(k,i)} := \{j \in [N] : A_{k}^{ij} \neq 0\},$

(41) $\overline{A}_{\mathcal{A}}^{(k,i)} := \{j \in [N] : \left(\prod_{q=0}^{k-1} A_q^{ij}\right)^{-1} > 0\},$

(42) $R_{\mathcal{A}}^{(k,i)} := \overline{A}_{\mathcal{A}}^{(k,i)} \setminus A_{\mathcal{A}}^{(k-1,i)},$ where $\overline{A}_{\mathcal{A}}^{(0,i)} := \{i\}.$

To interpret these sets, consider a directed graph $G_{\mathcal{A}} := (V_{\mathcal{A}}, E_{\mathcal{A}})$ with vertices and edges defined by

(43) $V_{\mathcal{A}} := \{\xi_k^i : 0 \leq k \leq m, i \in [N]\},$

(44) $E_{\mathcal{A}} := \{(\xi_{k-1}^i, \xi_k^j) : A_{k}^{ij} \neq 0, k \in [m], i, j \in [N]\},$

respectively. Suppose that the graph is arranged in the form of an array where $\xi_k^i$ is the vertex on the $k$th row and $i$th column, as shown in Figure 5. In this case, $P_{\mathcal{A}}$ denotes the set of all paths in the graph starting from the top row and ending at the bottom row, and $P_{\mathcal{A}}^{(i)}$ is this set restricted to those paths starting from $\xi_0^i$. Sets $A_{\mathcal{A}}^{(k,i)}$ determine the column indices of the parents of $\xi_k^i$ and sets $\overline{A}_{\mathcal{A}}^{(k,i)}$ determine the column indices of those vertices on the first row from which there exists a path to the vertex $\xi_k^i$. An illustration of these definitions is given in Figure 5.

The following assumption shall be invoked in Proposition 4. It serves to impose some structure which is common to the matrices which define
radix-$r$ and the mixed radix-$r$ butterfly resampling algorithms. For fixed $N, m \geq 1$ and for any sequence $\mathbb{A} = (A_k)_{k=1}^m$, we write $\mathbb{A}_{p,q} := (A_k)_{k=p}^q$ where $0 < p \leq q \leq m$.

**Assumption 5.** For given $N, m \geq 1$, the matrices $\mathbb{A} = \mathbb{A}^{(N,m)}$ satisfy Assumption 2, and, in addition, one has for all $p, q \in [m]$ and $i, j \in [N]$

(i) **Symmetry:** $A_{ij} = A_{ji}$,
(ii) **Commutativity:** $A_q A_p = A_p A_q$,
(iii) **Idempotence:** $A_p A_p = A_p$,
(iv) **Equal number of non-zero elements:** $|\mathbb{A}_{p,j}^{(p,j)}| = |\mathbb{A}_{p,j}^{(p,j)}|$, 
(v) **For all $i_p, j_p \in [N]$ and $(i_p, \ldots, i_q), (j_p, \ldots, j_q) \in \mathcal{P}_{i_p} \times \mathcal{P}_{j_p}$, $\mathcal{P}_{i_p} \times \mathcal{P}_{j_p}$, where $0 \leq p < q \leq m$ and $(i_p, i_q) = (j_p, j_q)$, one has $(i_p, \ldots, i_q) = (j_p, \ldots, j_q)$.

**Remark 4.** The conditions (i)–(iii) are standard matrix properties. Condition (iv) states that each row in each element of $\mathbb{A}$ has the same number of non-zero elements. Property (v) states that given any two vertices of the graph $\mathcal{G}_\mathbb{A}$ with column indices $i_p$ and $i_q$, there exists at most one directed path between those vertices. This condition is closely related to the existence of unique paths between any vertices in an undirected tree graph (see, e.g. [13]).

We are then ready to state the second main result on the conditional
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second moment, whose proof is given in Section B of the Supplement.

**Proposition 4.** Fix $N, m \geq 1$ and $\varphi \in \mathcal{B}_b(\mathbb{X})$. If $A = A^{(N,m)}$ satisfies Assumption 5, then

$$
\frac{m}{N} \mathbb{E} \left[ \left( \sum_{\varrho \in [Nm]} X^{(N,m)}_\varrho \right)^2 \right]_0^{(N,m)} = \frac{1}{N^2} \sum_i g^2(\xi^i_0) \mathbb{P}_N(\xi^i_0) + \frac{1}{N^4} \sum_i \sum_{j \neq i} g(\xi^i_0) \mathbb{P}_N(\xi^j_0) g(\xi^j_0) D^{(i,j)}_A
$$

where $D^{(i,j)}_A$ and $P^{(i,j)}_A$ are given by

$$
D^{(i,j)}_A = \sum_{k=1}^m \left| \mathcal{L}_A(k, u_0) \right|^2 \mathbb{1} \left( j \in \mathcal{R}_A^{(k,i)} \right),
$$

$$
P^{(i,j)}_A = \sum_{k=1}^m \left( N^2 - \left| \mathcal{L}_A(k, u_0) \right|^2 \left| \mathcal{A}_A^{(k,i)} \right| \right) \mathbb{1} \left( j \in \mathcal{R}_A^{(k,i)} \right),
$$

where $u_0 \in [N]$, and for all $0 \leq k < m$ and $i \in [N]$, $\mathcal{L}_A(k, i) := P^{(i)}_{A_{k+1,m}}$ and $\mathcal{L}_A(m, i) := \{i\}$.

It is now apparent that in order to study the asymptotic behaviour of the conditional second moment, one needs to study the quantities $D^{(i,j)}_A$ and $P^{(i,j)}_A$. This involves detailed combinatorial analysis, specific to each of the two butterfly resampling schemes.

**4. Analysis part II - LLN and CLT for butterfly resampling algorithms.** The next step towards proving the LLN and CLT for particle filters deploying the butterfly resampling, is to prove the corresponding results for a single application of butterfly resampling.

**4.1. Radix-\(r\) algorithm.** Throughout Section 4.1, $r \geq 2$ is a fixed integer and for each $m \geq 1$ we assume $A^{(r^m,m)} = A^{(r,m)}$ as defined in (8).

**Theorem 5.** No matter what the distribution of the input random variables $(\xi^i_m)_{i \in [r^m]}$ is, for any $\varphi \in \mathcal{B}_b(\mathbb{X})$,

$$
\sqrt{\frac{m}{r^m}} \sum_{\varrho \in [r^m]} X^{(r^m,m)}_\varrho \overset{\text{a.s.}}{\longrightarrow} 0.
$$
PROOF. By Lemma 1, $A_{\text{radix}}^{(r,m)}$ satisfies Assumption 2 and hence we can apply Proposition 1 to give, for any $p \geq 1$,

$$
\mathbb{E} \left[ \left( \frac{1}{r^m} \sum_{\varphi \in [r^m]} X_{\varphi}^{(r,m)} \right)^p \right] \leq b_p \left( \frac{m}{r^m} \right)^{p/2} \|g\|_{\infty, \text{osc}}^p \|\varphi\|^p,
$$

and the claim then follows from the Borel-Cantelli lemma. \hfill \Box

We note that the following result has as a hypothesis a bound on errors associated with certain subsets of the input random variables $(\xi_{in}^i)_{i \in [N]}$, which is unusual compared to similar results for multinomial resampling, e.g. [3].

**Theorem 6.** If for all $\varphi \in \mathcal{B}_b(\mathcal{X})$ there exists $b(\varphi) \in \mathbb{R}$ such that for some $\mu \in \mathcal{P}(\mathcal{X})$, and for all $m \geq 1$, $d \in [m]$ and $q \in [q^{d-1}]$

(45)

$$
\mathbb{E} \left[ \left( \frac{1}{r^{m-d+1}} \sum_{i \in [r^{m-d+1}]} \varphi(\xi_{in}^i) - \mu(\varphi) \right)^{2m} \right] \leq b(\varphi) \sqrt{\frac{m - d}{r^m} + \frac{1}{r^{m-d+1}}},
$$

where $J(i) := i + (q-1)r^{m-d+1}$ for all $i \in [r^{m-d+1}]$, then for any $\varphi \in \mathcal{B}_b(\mathcal{X})$ and any $u \in \mathbb{R}$,

$$
\mathbb{E} \left[ \exp \left( iu \sum_{\varphi \in [r^m]} X_{\varphi}^{(r,m)} \right) \right] \overset{\mathcal{L}}{\longrightarrow} \exp \left( -(u^2/2)\sigma^2(\varphi) \right),
$$

where

$$
\sigma^2(\varphi) = (1 - r^{-1})\mu \left( g \left( \varphi - \frac{\mu(g,\varphi)}{\mu(g)} \right) \right)^2 \mu(g).
$$

**Proof.** In order to apply Theorem 4, by the discussion in Section 3.5, we need to verify conditions (33)-(35). Condition (33) holds immediately by Proposition 2(ii). To check (34), we have by (30) that

$$
\sum_{\varphi \in [r^m]} \mathbb{E} \left[ \left( X_{\varphi}^{(r,m)} \right)^2 \mathbb{I} \left\{ \left| X_{\varphi}^{(r,m)} \right| \geq \epsilon \right\} \right] \mathcal{F}_{\varphi}^{(r,m)}
$$

$$
\leq \|g\|_{\infty, \text{osc}}^2 \mathbb{I} \left( \frac{\|g\|_{\infty, \text{osc}}(\varphi)}{\sqrt{m^m}} \geq \epsilon \right) \overset{m \to \infty}{\longrightarrow} 0.
$$

It remains to verify (35), i.e.,

(46)

$$
\sum_{\varphi \in [r^m]} \mathbb{E} \left[ \left( X_{\varphi}^{(r,m)} \right)^2 \mathcal{F}_{\varphi}^{(r,m)} \right] \overset{p}{\longrightarrow} \sigma^2(\varphi), \text{ for some } \sigma^2(\varphi) > 0.
$$
To do this, we first use Proposition 2(ii) and the tower property of conditional expectations to obtain the decomposition:

$$\sum_{\varrho \in [r^m m]} \mathbb{E} \left[ \left( \sum_{\varrho \in [r^m m]} X^{(r^m, m)}_{\varrho} \right)^2 \bigg| \mathcal{F}_{m, \varrho-1} \right]$$

(47) $$= \mathbb{E} \left[ \left( \sum_{\varrho \in [r^m m]} X^{(r^m, m)}_{\varrho} \right)^2 \bigg| \mathcal{F}_0 \right] + \sum_{\varrho \in [r^m m]} Z^{(r^m, m)}_{\varrho},$$

where

$$Z^{(r^m, m)}_{\varrho} := \mathbb{E} \left[ \left( X^{(r^m, m)}_{\varrho} \right)^2 \bigg| \mathcal{F}_{m, \varrho-1} \right] - \mathbb{E} \left[ \left( X^{(r^m, m)}_{\varrho} \right)^2 \bigg| \mathcal{F}_0 \right].$$

By Proposition 6 in Section C.1 of the Supplement, \( A^{(r, m)} \) satisfies Assumption 5 and hence Propositions 3-4 together with the hypothesis (45) can be used to establish Proposition 7 in Section C.2 of the Supplement, from which it follows that

$$\mathbb{E} \left[ \left( \sum_{\varrho \in [r^m m]} X^{(r^m, m)}_{\varrho} \right)^2 \bigg| \mathcal{F}_0 \right] \overset{P}{\underset{m \to \infty}{\longrightarrow}} \sigma^2(\varphi).$$

Proposition 8, in Section C.3 of the Supplement, shows that \( Z^{(r^m, m)}_{\varrho} \) converges in probability to 0. This establishes (46) and the proof of the theorem is complete.

4.2. Mixed radix-\( r \) algorithm. For the mixed radix-\( r \) algorithm, we fix, throughout Section 4.2, \( m = 2 \) and \( r \geq 2 \), and for all \( c \geq 1 \) we assume \( A^{(r, c)}_{\varrho} = A^{(r, c)}_{\text{mixed}} \) as defined in (17). Analogous to Theorem 5, we have by Proposition 1:

**Theorem 7.** No matter what the distribution of the input random variables \( (\xi_{i\varrho})_{i \in [rc]} \) is, for any \( \varphi \in \mathcal{B}_b(X) \),

$$\sqrt{\frac{2}{rc}} \sum_{\varrho \in [2rc]} X^{(r, c, 2)}_{\varrho} \overset{\text{a.s.}}{\underset{c \to \infty}{\longrightarrow}} 0.$$

**Proof.** Similar to the proof of Theorem 5.

Similarly as in the case of the radix-\( r \) algorithm, a hypothesis on the errors associated with certain sub-populations of the input random variables plays a role in the CLT for the mixed radix-\( r \) algorithm.
Theorem 8. If for all $\varphi \in \mathcal{B}_b(X)$ there exists $b(\varphi) \in \mathbb{R}$ such that for some $\mu \in \mathcal{P}(X)$, and for all $c \geq 1$, $d \in \{1, 2\}$ and $q \in [r^{d-1}]$.

\begin{equation}
E \left[ \left\| \frac{r^{d-1}}{rc} \sum_{i \in [cr^{2-d}]} \varphi(\xi_{0}^{J(i)}) - \mu(\varphi) \right\|^2 \right]^\frac{1}{2} \leq b(\varphi) \sqrt{\frac{2 - d}{rc} + \frac{r^{d-1}}{rc}},
\end{equation}

where $J(i) = i + (q - 1)c \in [cr^{2-d}]$ for all $i \in [cr^{2-d}]$, then for any $\varphi \in \mathcal{B}_b(X)$ and any $u \in \mathbb{R}$,

\begin{equation}
E \left[ \exp \left( iu \sum_{\rho \in [2rc]} X_{\rho}^{(rc,2)} \right) \left| J_{0}^{(rc,2)} \right\|^P \right] \xrightarrow{c \to \infty} \exp \left( -(u^2/2)\sigma^2(\varphi) \right),
\end{equation}

where

\[ \sigma^2(\varphi) = \left( 1 - \frac{1}{2r} \right) \mu \left( g(\varphi - \frac{\mu(g\varphi)}{\mu(g)})^2 \right) \mu(g). \]

Proof. The proof is similar to that of Theorem 6, with the exceptions that we use Proposition 9 in Section D.1 of the Supplement instead of Proposition 6, Proposition 10 of Section D.2 in the Supplement instead of Proposition 7, and Proposition 11 of Section D.3 in the Supplement instead of Proposition 8. Also, the hypothesis (49) as well as Propositions 3 and 4 are needed in the proof of Proposition 10.

5. Analysis part III - particle filters. Finally, we address the proofs of the two main results of the paper, Theorems 2 and 3. To extend the results of Section 4 to the particle filter we need to ensure that the hypotheses of Theorems 6 and 8 are valid and that their validity is preserved throughout the filtering sequence. The next result, when applied with appropriate $\mathbb{A}^{(N,m)}$, $\mathcal{I}$, $d$ and $J$, allows us to do this, by quantifying the errors associated with certain sub-populations of the particle system.

Proposition 5. Fix $N, m \geq 1$, $d \in [m]$ and a partition $\mathcal{I}$, and let $(\zeta_n^i, \zeta_n^i)_{n \geq 0, i \in [m]}$ be the random variables associated with the augmented re-sampling particle filter deploying matrices $\mathbb{A}^{(N,m)}$. If the triple $(\mathbb{A}^{(N,m)}, \mathcal{I}, d)$ satisfies Assumption 4, then for all $n \geq 0$, $p > 1$, $\varphi \in \mathcal{B}_b(X)$ there exist $b_n(\varphi, p), b_n(\varphi, p) \in \mathbb{R}$,

\begin{equation}
E \left[ \left\| \frac{1}{|\mathcal{I}|} \sum_{i \in [|\mathcal{I}|]} \varphi(\zeta_n^{J(i)}) - \pi_n(\varphi) \right\|^p \right] \leq b_n(\varphi, p) \sqrt{\frac{m - d}{N} + \frac{1}{|\mathcal{I}|}},
\end{equation}

where $J(i) = i + (q - 1)c \in [cr^{2-d}]$ for all $i \in [cr^{2-d}]$. 

$\square$
and

$$E \left[ \frac{1}{|I|} \sum_{i \in |I|} \varphi(\hat{\zeta}_n^{J(i)}) - \hat{\pi}_n(\varphi) \right]^{\frac{1}{p}} \leq \hat{b}_n(\varphi, p) \sqrt{\frac{m - d}{N} + \frac{1}{|I|}}.$$ 

The strategy of the proof is an induction, showing that if the first bound holds for the entire population given as input to the augmented resampling algorithm, then the second bound holds for certain blocks in the output of the resampling, including the entire population, and moreover that this bound is preserved in the mutation step of the particle filter. The proof is given in Section E of the Supplement.

The steps required to complete the proofs of Theorems 2 and 3, in outline, follow those of [3]. An inductive argument is used to show that the LLN and CLT are preserved at each time step. Although some of the scaling in the CLT’s is unusual, the proof techniques are standard and so the proofs are given in Section E of the Supplement.

SUPPLEMENTARY MATERIAL

Supplement: “Butterfly resampling: asymptotics for particle filters with constrained interactions” ().
SUPPLEMENTARY MATERIAL FOR “BUTTERFLY RESAMPLING: ASYMPTOTICS FOR PARTICLE FILTERS WITH CONSTRAINED INTERACTIONS”

By Kari Heine, Nick Whiteley, A. Taylan Cemgil and Hakan Güldaş

A. Proofs for Sections 3.2 and 3.3.

Proof of Lemma 4. By Assumption 2 and the definition of $\xi^i_k$ in Algorithm 2 we can assume that for all $k \in [m]$ and $i \in [N]$

$$\xi^i_k = \xi^I_{k-1}, \text{ where } I_k \sim \frac{1}{V_k} \sum_j A^j_k V^j_{k-1} \delta_j,$$

and the $I^i_k$ are independent given $\xi_0$. Let $\ell_0 = i$. By the law of total probability, conditional independence of $I^i_k$ and the one step conditional independence, for all $0 \leq k \leq m - 1$,

$$\mathbb{P}(\xi^i_m \in S \mid \xi_0, \ldots, \xi_{m-k-1}) = \sum_{(\ell_1, \ldots, \ell_{k-1})} \mathbb{E} \left[ \prod_{q=0}^{k-1} \mathbb{I}(I^q_{m-q} = \ell_{q+1}) \mathbb{I}(\xi^I_{m-k} \in S) \mid \xi_0, \ldots, \xi_{m-k-1} \right]$$

(S.1) yielding

$$\sum_{(\ell_1, \ldots, \ell_{k-1})} \prod_{q=0}^{k-1} \mathbb{P}(I^q_{m-q} = \ell_{q+1} \mid \xi_0) = (V^q_{m-q})^{-1} A^q_{m-q} V^{q+1}_{m-q-1} \mathbb{P}(I^q_{m-q} = \ell_{q+1} \mid \xi_0).$$

By (S.1), $\mathbb{P}(I^q_{m-q} = \ell_{q+1} \mid \xi_0) = (V^q_{m-q})^{-1} A^q_{m-q} V^{q+1}_{m-q-1}$ yielding

$$\sum_{(\ell_1, \ldots, \ell_{k-1})} \prod_{q=0}^{k-1} \mathbb{P}(I^q_{m-q} = \ell_{q+1} \mid \xi_0) = \frac{V^k_{m-k}}{V^0} \sum_{(\ell_1, \ldots, \ell_{k-1})} \left( \prod_{q=0}^{k-1} A^q_{m-q} \right) \ell_0 \ell_k.$$

(S.3)

Because by (S.1)

$$\mathbb{P}(\xi^I_{m-k} \in S \mid \xi_0, \ldots, \xi_{m-k-1}) = (V^k_{m-k})^{-1} \sum_j A^j_{m-k} V^j_{m-k-1} \mathbb{I}(\xi^j_{m-k-1} \in S),$$
by substituting (S.3) into (S.2) we have
\[
\mathbb{P}(\xi_{m}^{\ell_{0}} \in S \mid \xi_{0}, \ldots, \xi_{m-k-1}) = \frac{1}{V_{\ell_{0}}^{j_{0}}} \sum_{j} V_{m-k-1}^{j} \mathbb{I}(\xi_{m-k-1}^{j} \in S) \sum_{\ell_{k}} \left( \prod_{q=0}^{k-1} A_{m-q} \right) A_{\ell_{k}j}.
\]

from which the claim follows by recalling that \(\ell_{0} = i\).

Proof of Proposition 2. For brevity, let us write \(J_{i} := J(i)\) for all \(i \in [|I|]\). Then, by defining
\[
A := \frac{1}{|I|} \sum_{i_{m} \in [|I|]} V_{m}^{J_{im}} \varphi_{N}(\xi_{m}^{J_{im}}) - \frac{1}{N} \sum_{i_{m-d}} V_{m-d}^{J_{im-d}} \varphi_{N}(\xi_{m-d}^{J_{im-d}}),
\]
\[
B := \sum_{q \in [m-d]} \left( \frac{1}{N} \sum_{i_{q}} V_{q}^{J_{iq}} \varphi_{N}(\xi_{q}^{J_{iq}}) - \frac{1}{N} \sum_{i_{q-1}} V_{q-1}^{J_{iq-1}} \varphi_{N}(\xi_{q-1}^{J_{iq-1}}) \right),
\]
we have the telescoping decomposition
\[
(S.4) \quad \frac{1}{|I|} \sum_{i_{m} \in [|I|]} V_{m}^{J_{im}} \varphi_{N}(\xi_{m}^{J_{im}}) = A + B.
\]

By Assumption 2, the matrices \((A_{k})_{k \in [m]}\) are doubly stochastic and by (i) and (ii) of Assumption 4, we have by writing \(\varphi_{m} := \varphi_{N}(\xi_{m})\) for brevity
\[
A = \frac{1}{|I|} \sum_{i_{m} = 1}^{[|I|]} V_{m}^{J_{im}} \varphi_{m}^{J_{im}} \varphi_{N}(\xi_{m}^{J_{im}}) - \frac{1}{N} \sum_{j} \sum_{i_{m-d}} \left( \prod_{q=0}^{d-1} A_{m-q} \right) V_{m-d}^{J_{im-d}} \varphi_{m-d}^{J_{im-d}} \varphi_{N}(\xi_{m-d}^{J_{im-d}})
\]
\[
= \frac{1}{|I|} \sum_{i_{m} = 1}^{[|I|]} V_{m}^{J_{im}} \varphi_{m}^{J_{im}} \varphi_{N}(\xi_{m}^{J_{im}})
\]
\[
- \frac{1}{N} \sum_{j=1}^{[|I|]} N \sum_{i_{m-d}} \left( \prod_{q=0}^{d-1} A_{m-q} \right) V_{m-d}^{J_{im-d}} \varphi_{m-d}^{J_{im-d}} \varphi_{N}(\xi_{m-d}^{J_{im-d}})
\]
\[
(S.5) \quad = \frac{1}{|I|} \sum_{i_{m} = 1}^{[|I|]} V_{m}^{J_{im}} \varphi_{m}^{J_{im}} \varphi_{N}(\xi_{m}^{J_{im}})
\]
\[
\times \left( \varphi_{m}^{J_{im}} - \frac{1}{V_{m}^{J_{im}}} \sum_{i_{m-d}} \left( \prod_{q=0}^{d-1} A_{m-q} \right) V_{m-d}^{J_{im-d}} \varphi_{m-d}^{J_{im-d}} \right).
\]
Similarly for \( B \) we have

\[
B = \sum_{q \in [m - d]} \left( \frac{1}{N} \sum_{i_q} V_{i_q}^q \mathcal{N}(\xi_i^q) - \frac{1}{N} \sum_j \sum_{i_{q-1}} A_q^{i_q-1} V_{i_{q-1}}^{q-1} \mathcal{N}(\xi_{i_{q-1}}^{q-1}) \right)
\]

(S.6) \[= \sum_{q \in [m - d]} \sum_{i_q} \frac{V_{i_q}^q}{N} \left( \mathcal{N}(\xi_i^q) - \frac{1}{V_{i_{q-1}}^{q-1}} \sum_{i_{q-1}} A_q^{i_q-1} V_{i_{q-1}}^{q-1} \mathcal{N}(\xi_{i_{q-1}}^{q-1}) \right),
\]

and by combining (S.4), (S.5) and (S.6) we have established (31). Using (21) and Lemma 3, we can establish (32):

\[
\left( \frac{1}{N} \sum_i g(\xi_{in}) \right) \left( \frac{1}{|Z|} \sum_{i \in [|Z|]} \varphi(\xi_{out}^i) \right) - \frac{1}{N} \sum_i g(\xi_{in}) \varphi(\xi_{in}^i)
\]

\[= \frac{1}{|Z|} \sum_{i \in [|Z|]} V_{m}^i \varphi(\xi_{in}^i) - \frac{1}{N} \sum_i V_0^i \varphi(\xi_0^i)
\]

\[= \frac{1}{|Z|} \sum_{i \in [|Z|]} V_{m}^i \left( \varphi(\xi_{in}^i) - \frac{1}{N} \sum_j V_0^j \varphi(\xi_0^j) \right)
\]

\[= \frac{1}{|Z|} \sum_{i \in [|Z|]} V_{m}^i \mathcal{N}(\xi_{out}^i).
\]

Using (27), (26), (28) and Lemma 3(i) we find that (i) holds. For \( q \leq (m - d)N \), (ii) follows from (23) and the one step conditional independence. For \( (m - d)N < q \leq (m - d)N + |Z| \), (ii) follows from Lemma 4 and Assumption 4(iii). Finally (30) holds by Lemma 3(ii).

\[\square\]

**B. Proofs for Section 3.6.**

**Proof of Proposition 3.** We will fix \( N \geq 2 \) and \( m \geq 1 \). For all \( 0 \leq p, q \leq m \) we define \( i_{p,q} := (i_p, \ldots, i_q) \in [N]^{q-p+1} \), \( j_{p,q} := (j_p, \ldots, j_q) \in [N]^{q-p+1} \) and for all \( i_{p,q}, j_{p,q} \in [N]^{q-p+1} \) we define

\[
\mathcal{C}_{i_{p,q},j_{p,q}} := \mathcal{C}_{[i_p=j_p]} \cdots \mathcal{C}_{[i_q=j_q]},
\]

where \( \mathcal{C}_0 \) and \( \mathcal{C}_1 \) are as defined in the statement of the proposition. For all \( k \in [m] \) and \( i \in [N] \) we define also the measure

\[
\Gamma_{k,i} := \sum_j A_k^{ij} \psi_{k-1}^j \delta_{\xi_{k-1}^j}.
\]
To proceed, we will in fact prove a more general result:

\[
E \left[ \left( \frac{1}{N} \sum_i V_{im}^i \delta_{\xi_{im}} \right) \otimes \left( \frac{1}{N} \sum_i V_{im}^i \delta_{\xi_{im}} \right) (\Phi) \right]_{N(m)}^{(N,m)} = \sum_{(i_0,j_0,\ldots,i_m,j_m)} \left( \frac{1}{N^2^m} \prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}+j_{k+1}} \right) g(\xi_0) g(\xi_0) C_{i_1:m,j_1:m} (\Phi) (\xi_0, \xi_0),
\]

where \( \Phi \in B_b(\mathbb{R}^2) \). The claim then follows by noting that when \( \Phi = \phi \otimes \phi \), then by (31)

\[
\left( \sum_{g \in [Nm]} X_{N,m}^{(N,m)} \right)^2 = \left( \frac{1}{N} \sum_i V_{im} \phi (\xi_{im}) \right)^2 = \sum_{(i_0,j_0,\ldots,i_m,j_m)} \left( \frac{1}{N^2^m} \prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}+j_{k+1}} \right) g(\xi_0),
\]

(S.7)

We first derive an expression for \( E \left[ \left( V_{ik}^i \delta_{\xi_{ik}} \right) \otimes \left( V_{jk}^j \delta_{\xi_{jk}} \right) (\Phi) \right]_{N,m}^{(N,m)} \), where \( k \in [m] \) and \( \Phi \in B_b(\mathbb{R}^2) \). In the case \( i_k = j_k \), and by writing \( \Phi(x) = \Phi(x,x) \)

\[
E \left[ \left( V_{ik}^i \delta_{\xi_{ik}} \right) \otimes \left( V_{jk}^j \delta_{\xi_{jk}} \right) (\Phi) \right]_{N,m}^{(N,m)} \xi_0, \ldots, \xi_{k-1} = \left( V_{ik}^i \right)^2 \frac{\Gamma_{k,i_k}(\Phi)}{\Gamma_{k,i_k}(1)} \Gamma_{k,i_k}(1) = (\Gamma_{k,i_k} \otimes (C_1(\Phi))),
\]

and in the case \( i_k \neq j_k \),

\[
E \left[ \left( V_{ik}^i \delta_{\xi_{ik}} \right) \otimes \left( V_{jk}^j \delta_{\xi_{jk}} \right) (\Phi) \right]_{N,m}^{(N,m)} \xi_0, \ldots, \xi_{k-1} = \left( V_{ik}^i \right)^2 \frac{\Gamma_{k,i_k}(\Phi)}{\Gamma_{k,i_k}(1)} \Gamma_{k,j_k}(1) = (\Gamma_{k,i_k} \otimes (C_1(\Phi))),
\]

so in any case,

\[
E \left[ \left( V_{ik}^i \delta_{\xi_{ik}} \right) \otimes \left( V_{jk}^j \delta_{\xi_{jk}} \right) (\Phi) \right]_{N,m}^{(N,m)} \xi_0, \ldots, \xi_{k-1} = (\Gamma_{k,i} \otimes (C_1[i_k=j_k] \Phi)),
\]

(S.8)
The proof now proceeds by a backward induction. Our first application of (S.8) is with \( k = m \) to initialize this induction, with the identity:

\[
E \left[ \left( \frac{1}{N} \sum_{i_m} V_{m}^{i_m} \delta_{\xi_{m}^{i_m}} \right) \otimes \left( \frac{1}{N} \sum_{j_m} V_{m}^{j_m} \delta_{\xi_{m}^{j_m}} \right) \Phi \right] | \xi_0, \ldots, \xi_{m-1} = \frac{1}{N^2} \sum_{(i_{m}, j_{m})} (\Gamma_{m, i_{m}} \otimes \Gamma_{m, j_{m}}) (C_{[i_{m} = j_{m}]} \Phi).
\]

The inductive hypothesis is that at rank \( k \), with \( 1 \leq k \leq m \), the following holds:

\[
E \left[ \left( \frac{1}{N} \sum_{i_m} V_{m}^{i_m} \delta_{\xi_{m}^{i_m}} \right) \otimes \left( \frac{1}{N} \sum_{j_m} V_{m}^{j_m} \delta_{\xi_{m}^{j_m}} \right) \Phi \right] | \xi_0, \ldots, \xi_{k-1} = \frac{1}{N^2} \sum_{(i_{k}, j_{k}, \ldots, i_{m}, j_{m})} \left( \prod_{q=k}^{m-1} A_{q+1}^{i_{q+1}, i_{q}} A_{q+1}^{j_{q+1}, j_{q}} \right)
\times \left( V_{k}^{i_k} \delta_{\xi_k^{i_k}} \otimes V_{k}^{j_k} \delta_{\xi_k^{j_k}} \right) \left( C_{i_{k+1} = j_{k+1}} (\Phi) \right).
\]

By (S.8) we have

\[
E \left[ \left( V_{k}^{i_k} \delta_{\xi_k^{i_k}} \otimes V_{k}^{j_k} \delta_{\xi_k^{j_k}} \right) \left( C_{i_{k+1} = j_{k+1}} (\Phi) \right) \right] | \xi_0, \ldots, \xi_{k-1} = (\Gamma_{i_{k+1}, j_{k+1}}) (C_{i_{k+1} = j_{k+1}} (\Phi)),
\]

and therefore at rank \( k - 1 \), applying the tower property of conditional expectation gives

\[
E \left[ \left( \frac{1}{N} \sum_{i_m} V_{m}^{i_m} \delta_{\xi_{m}^{i_m}} \right) \otimes \left( \frac{1}{N} \sum_{j_m} V_{m}^{j_m} \delta_{\xi_{m}^{j_m}} \right) \Phi \right] | \xi_0, \ldots, \xi_{k-1} = \frac{1}{N^2} \sum_{(i_{k-1}, j_{k-1}, \ldots, i_{m}, j_{m})} \left( \prod_{q=k-1}^{m-1} A_{q+1}^{i_{q+1}, i_{q}} A_{q+1}^{j_{q+1}, j_{q}} \right)
\times \left( V_{k-1}^{i_{k-1}} \delta_{\xi_{k-1}^{i_{k-1}}} \otimes V_{k-1}^{j_{k-1}} \delta_{\xi_{k-1}^{j_{k-1}}} \right) \left( C_{i_{k+1} = j_{k+1}} (\Phi) \right).
\]
That is, the hypothesis then also holds at rank $k - 1$. Thus the induction is complete, and so we can conclude that for $k = 1$,
\[
E \left[ \left( \frac{1}{N} \sum_{i_m} V_{i_m}^j \delta_{\xi_{i_m}^j} \right) \otimes \left( \frac{1}{N} \sum_{m} V_{m,j}^j \delta_{\xi_{m,j}^j} \right) (\Phi) \mid \xi_0 \right]
\]
\[
= \frac{1}{N^2} \sum_{(i_0,j_0,\ldots,i_m,j_m)} \left( \prod_{q=0}^{m-1} A_{q+1}^{i_q+1} \right) \left( V_{0}^{i_0} \delta_{\xi_{i_0}^0} \otimes V_{0}^{j_0} \delta_{\xi_{j_0}^0} \right) (C_{1,m,j_1,m} (\Phi))
\]
\[
= \sum_{(i_0,j_0,\ldots,i_m,j_m)} \left( \frac{1}{N^2} \prod_{q=0}^{m-1} A_{q+1}^{i_q+1} \right) \left( g(\xi_{i_0}^0) g(\xi_{j_0}^0) (C_{1,m,j_1,m} (\Phi)) \right),
\]
as required. 

The proof of Proposition 4 consists of several technical results which we state and prove first while the actual proof of Proposition 4 is postponed to the end of this section. First we establish some key implications of Assumption 5 that will be found useful throughout the remainder of the work.

**Lemma 5.** If $A = A^{(N,m)}$ satisfies Assumption 5 for some $N, m \geq 1$, then for all $((i_0, \ldots, i_m), (j_0, \ldots, j_m)) \in P^m_k$, $i, j \in [N]$, and $k \in [m]$

\( (i) \) $i \in \mathcal{A}_{\Lambda}^{(k,i)}$ if and only if $j \in \mathcal{A}_{\Lambda}^{(k,j)}$, 
\( (ii) \) If $j \in \mathcal{A}_{\Lambda}^{(k,i)}$, then $\mathcal{A}_{\Lambda}^{(k,i)} = \mathcal{A}_{\Lambda}^{(k,j)}$, if $j \notin \mathcal{A}_{\Lambda}^{(k,i)}$, then $\mathcal{A}_{\Lambda}^{(k,i)} \cap \mathcal{A}_{\Lambda}^{(k,j)} = \emptyset$, 
\( (iii) \) $A_{ij}^k > 0$, 
\( (iv) \) If $q \leq k$ and $j \in \mathcal{A}_{\Lambda}^{(k,i)}$, then $\mathcal{A}_{\Lambda}^{(q,j)} \subset \mathcal{A}_{\Lambda}^{(k,i)}$, 
\( (v) \) If $j_0 \in \mathcal{A}_{\Lambda}^{(k,i)}$, then either $i_k = j_k$ or for all $q \geq k$, $i_q \neq j_q$, 
\( (vi) \) $\bigcup_{k=1}^m R_{\Lambda}^{(k,i)} = [N] \setminus \{i\}$, and for all $k, k' \in [m]$ such that $k \neq k'$, 
$R_{\Lambda}^{(k,i)} \cap R_{\Lambda}^{(k',i)} = \emptyset$. 

**Proof.** (i) follows from (41) and parts (i) and (ii) of Assumption 5.

To check (ii), suppose that $j \in \mathcal{A}_{\Lambda}^{(k,i)}$, and there exists $u \in [N]$ such that $u \in \mathcal{A}_{\Lambda}^{(k,i)}$. Then, by parts (ii), (iii), (i) of Assumption 5 and Assumption 2,
$A_{k:1}^{j_u} = (A_{k:1} A_{k:1})^{j_u} = \sum_{\ell} A_{k:1}^{j_{u_{\ell}}} A_{k:1}^{j_u} \geq A_{k:1}^{j_{u_{\ell}}} A_{k:1}^{j_u} > 0$ wherever the last inequality
holds by assumption. Thus \( u \in \mathcal{A}^{(k,j)}_h \) proving \( \mathcal{A}^{(k,i)}_h \subset \mathcal{A}^{(k,j)}_h \). The converse inclusion follows from the symmetry of the arguments. For the case \( j \notin \mathcal{A}^{(k,i)}_h \) we assume there exists \( u \in \mathcal{A}^{(k,i)}_h \cap \mathcal{A}^{(k,j)}_h \), from which it follows that 

\[
0 < A^{u}_{k:1}A^{u}_{k:1} \leq \sum_{\ell} A^{u}_{k:1} A^{\ell}_{k:1} = (A_{k:1} A_{k:1})^{ij} = A^{ij}_{k:1} \iff j \in \mathcal{A}^{(k,i)}_h \]

constituting a contradiction, which completes the proof of (ii).

To prove (iii) we have by Assumption 2 and parts (iii) and (i) of Assumption 5, \( A^{ii}_k = (A_k A_k)^{ii} = \sum_{\ell} A^{\ell}_{k:1} A^{\ell}_{k:1} > 0 \).

To prove (iv), suppose that for some \( q \leq k \) and \( j \in \mathcal{A}^{(k,i)}_h \) we take \( u \in \mathcal{A}^{(q,j)}_h \). By (iii), there exists \((i_q, \ldots, i_k) \in \mathcal{P}_{h+1:k}\) such that \( i_q = i_k = j \). Since \( u \in \mathcal{A}^{(q,j)}_h \) there exists \((i_0, \ldots, i_k) \in \mathcal{P}_{h+1:k}\) such that \( i_0 = u \) and \( i_k = j \) implying, by (ii), that \( u \in \mathcal{A}^{(k,j)}_h = \mathcal{A}^{(k,j)}_h \), hence proving (iv).

To prove (v), we observe that by (ii), \( j_0 \in \mathcal{A}^{(k,i_0)}_h \) implies \( \mathcal{A}^{(k,i_0)}_h = \mathcal{A}^{(k,j_0)}_h \). Because \( j_0, \ldots, j_m \in \mathcal{P}_{h,k}\), we have \( j_0 \in \mathcal{A}^{(k,j_0)}_h \) and by (i), \( j_k \in \mathcal{A}^{(k,j_0)}_h = \mathcal{A}^{(k,j_0)}_h \). Hence, by (i), \( i_0 \in \mathcal{A}^{(k,j_k)}_h \) and there exists \((i', j') \in \mathcal{P}_{h,k}\), where \( i' = (i'_0, \ldots, i'_m) \) and \( j' = (j'_0, \ldots, j'_m) \), such that \( i'_0 = j'_0 = i_0, i'_k = i_k, j'_k = j_k \). Now suppose that \( i_k \neq j_k \) and there exists \( q \geq k \) such that \( i_q = j_q \). By the existence of \((i', j')\), we can construct paths \( i'' = (i''_0, \ldots, i''_k, i''_{k+1}, \ldots, i''_m) \) and \( j'' = (j''_0, \ldots, j''_k, j''_{k+1}, \ldots, j''_m) \) for which we have \( i''_0 = j''_0, i''_k \neq j''_k \) and \( i''_q = j''_q \) contradicting Assumption 5(v) which completes the proof of (v).

To prove (vi), we observe that by (iv) and (iii), \( \mathcal{A}^{(k-1,j)}_h \subset \mathcal{A}^{(k,j)}_h \). Moreover, by definition \( \mathcal{A}^{(0,i)}_h = \{i\} \) and by Assumption 2, \( \mathcal{A}^{(m,i)}_h = [N] \). Therefore it is a matter of elementary set operations to check that 

\[
\bigcup_{k=1}^m \mathcal{R}^{(k,i)}_h = \bigcup_{k=1}^m \mathcal{A}^{(k,i)}_h \setminus \mathcal{A}^{(k-1,i)}_h = \mathcal{A}^{(m,i)}_h \setminus \{i\} = [N] \setminus \{i\}.
\]

Empty intersections follow straightforwardly by definition (42) and the fact that \( \mathcal{A}^{(k-1,i)}_h \subset \mathcal{A}^{(k,i)}_h \).

We start proving Proposition 4 by writing \( i_0:m = (i_0, \ldots, i_m) \) and \( j_0:m = (j_0, \ldots, j_m) \) for brevity. Then, for any \( N, m \geq 1 \), the set \( \mathcal{P}^2_h \), where \( h = h(N,m) \), can be decomposed in three disjoint sets

\[
D^1_h(m) := \{(i_0:m, j_0:m) \in \mathcal{P}^2_h : i_0 = j_0\},
\]

\[
D^2_h(m) := \bigcup_{k=1}^m \{(i_0:m, j_0:m) \in \mathcal{P}^2_h : i_0 \neq j_0, i_k = j_k\},
\]

\[
D^3_h(m) := \bigcap_{k=1}^m \{(i_0:m, j_0:m) \in \mathcal{P}^2_h : i_0 \neq j_0, i_k \neq j_k\}.
\]

Clearly, the sets \( D^1_h(m), D^2_h(m) \) and \( D^3_h(m) \) form a partition of \( \mathcal{P}^2_h \).
LEMMA 6.  Fix $N, m \geq 1$ and $\mathbb{A} = \mathbb{A}^{(N,m)}$. The sets $D^2_\mathbb{A}(m)$ and $D^3_\mathbb{A}(m)$ admit the decompositions:

\begin{align*}
(S.10) \quad D^2_\mathbb{A}(m) &= \bigcup_{i \in [N]} \bigcup_{j \in [N]} \bigcup_{k \in [m]} \bigcup_{u \in [N]} D_\mathbb{A}(k, u, i, j). \\
(S.11) \quad D^3_\mathbb{A}(m) &= \bigcup_{i \in [N]} \bigcup_{j \in [N]} \left( (\mathcal{P}^{(i)}_\mathbb{A} \times \mathcal{P}^{(j)}_\mathbb{A}) \setminus \bigcup_{k \in [m]} \bigcup_{u \in [N]} D_\mathbb{A}(k, u, i, j) \right),
\end{align*}

where for all $k \in [m]$ and $i, j, u \in [N]$

$$D_\mathbb{A}(k, u, i, j) := \bigcap_{q=0}^{k-1} \left\{ ((i'_0, \ldots , i'_m), (j'_0, \ldots , j'_m)) \in \mathcal{P}^{(i)}_\mathbb{A} \times \mathcal{P}^{(j)}_\mathbb{A} : i'_k = j'_k = u, \ i'_q \neq j'_q \right\},$$

and for all $(k, u, i, j) \neq (k', u', i', j')$, $D_\mathbb{A}(k, u, i, j) \cap D_\mathbb{A}(k', u', i', j') = \emptyset$.

PROOF. First we observe that for the sets in (S.10) the inclusion $\supset$ is trivial by definition. Then take a pair $(i_0, \ldots , i_m)$ and $(j_0, \ldots , j_m)$ belonging to $D^2_\mathbb{A}(m)$. Then there exists $p = \min(q \in [m] : i_q = j_q)$, and thus the pair also belongs to $D_\mathbb{A}(p, i_p, i_0, j_0)$ and therefore also the inclusion $\subset$ holds, establishing (S.10). By elementary set theory it follows by (S.9) that $D^3_\mathbb{A}(m) = (\bigcup_{i \in [N]} \bigcup_{j \in [N]} (\mathcal{P}^{(i)}_\mathbb{A} \times \mathcal{P}^{(j)}_\mathbb{A})) \setminus D^2_\mathbb{A}(m)$ and since for all $k \in [m]$ and $u \in [N]$, $D_\mathbb{A}(k, u, i, j) \subset \mathcal{P}^{(i)}_\mathbb{A} \times \mathcal{P}^{(j)}_\mathbb{A}$, (S.11) can be checked by elementary set theory.

To prove that the sets $D_\mathbb{A}(k, u, i, j)$ are disjoint, assume that

\begin{align*}
(S.12) \quad ((i_0, \ldots , i_m), (j_0, \ldots , j_m)) \in D_\mathbb{A}(k, u, i, j) \cap D_\mathbb{A}(k', u', i', j').
\end{align*}

If $i \neq i'$ and (S.12) was true, then $i = i_0 = i' \neq i$, and similarly for $j$ and $j'$. In the case $k \neq k'$, since we are not assuming anything about the values of $u$, $u'$, $i$, $i'$, $j$, and $j'$, we can assume without loss of generality that $k < k'$. Now if (S.12) was true, then $i_k = j_k$ and $i_k \neq j_k$, which is a contradiction. Finally it suffices to consider the case $k = k'$ and $u \neq u'$. If (S.12) was true, then one must have $u = i_k = j_k = u' \neq u$ which is a contradiction completing the proof.

\[\square\]

The cardinality of a set can be evaluated by constructing a bijection between the set in question and some other set with known cardinality. For this purpose, we have the following result. Note that throughout the
remainder of this document, for given \( N, m \geq 1 \), \( \Lambda = \Lambda^{(N, m)} \), \( 0 \leq k \leq m \) and \( u \in [N] \), we let \( \mathcal{L}_\Lambda(k, u) \) be as defined in the statement of Proposition 4.

**Lemma 7.** Fix \( N, m \geq 1 \) and \( \Lambda = \Lambda^{(N, m)} \). For all \( i, j, u \in [N] \), such that \( i \neq j \) and \( k \in [m] \), define

\[
\mathcal{U}_\Lambda(k, u, i, j) := \bigcap_{q=0}^{k-1} \left\{ ((i_0', \ldots, i_k'), (j_0', \ldots, j_k')) \in \mathcal{P}_\Lambda^{(i)} \times \mathcal{P}_\Lambda^{(j)} : i'_k = j'_k = u, \ i'_q \neq j'_q \right\},
\]

and let the mapping

\[
\kappa : D_\Lambda(k, u, i, j) \to \mathcal{U}_\Lambda(k, u, i, j) \times \mathcal{L}_\Lambda(k, u) \times \mathcal{L}_\Lambda(k, u),
\]

be defined as

\[
\kappa : (i_{0:m}, j_{0:m}) \mapsto (((i_0, \ldots, i_k), (j_0, \ldots, j_k)), (i_k, \ldots, i_m), (j_k, \ldots, j_m)),
\]

where \( i_{0:m} := (i_0, \ldots, i_m) \), \( j_{0:m} := (j_0, \ldots, j_m) \). Then \( \kappa \) is a bijection.

**Proof.** By the definitions of \( \mathcal{L}_\Lambda \), \( D_\Lambda \) and \( \mathcal{U}_\Lambda \), for any \((i_{0:m}, j_{0:m}) \in D_\Lambda(k, u, i, j)\), where \( i, j, u \in [N] \) such that \( i \neq j \) and \( k \in [m] \)

\[
(((i_0, \ldots, i_k), (j_0, \ldots, j_k)), (i_k, \ldots, i_m), (j_k, \ldots, j_m))
\]

\[
\in \mathcal{U}_\Lambda(k, u, i, j) \times \mathcal{L}_\Lambda(k, u) \times \mathcal{L}_\Lambda(k, u).
\]

If \((i_{0:m}, j_{0:m}) \neq (i'_{0:m}, j'_{0:m}) \in D_\Lambda(k, u, i, j)\) then \( \kappa(i_{0:m}, j_{0:m}) \neq \kappa(i'_{0:m}, j'_{0:m}) \), from which we conclude that \( \kappa \) is an injection. To see that \( \kappa \) is a surjection, take any

\[
(((i_0, \ldots, i_k), (j_0, \ldots, j_k)), (i_k', \ldots, i_m'), (j_k', \ldots, j_m'))
\]

\[
\in \mathcal{U}_\Lambda(k, u, i, j) \times \mathcal{L}_\Lambda(k, u) \times \mathcal{L}_\Lambda(k, u).
\]

Then \( i_0 = i, j_0 = j \), and by the definitions of \( \mathcal{L}_\Lambda \) and \( \mathcal{U}_\Lambda \), \( i_k = j_k = i'_k = j'_k = u \), for all \( 0 \leq p < k \), \( i_p \neq j_p \), and \((i_0, \ldots, i_k, i'_{k+1}, \ldots, i'_m) \in \mathcal{P}_\Lambda^{(i)}, (j_0, \ldots, j_k, j'_{k+1}, \ldots, j'_m) \in \mathcal{P}_\Lambda^{(j)} \). From these observations we conclude by the definition of \( D_\Lambda(k, u, i, j) \) that

\[
(((i_0, \ldots, i_k, i'_{k+1}, \ldots, i'_m), (j_0, \ldots, j_k, j'_{k+1}, \ldots, j'_m)) \in D_\Lambda(k, u, i, j),
\]

and hence \( \kappa \) is a surjection. \( \square \)
By using the bijectivity result, Lemma 7, we can find an expression for the cardinalities of the sets $D_{\mathcal{A}}(k, u, i, j)$ in terms of the cardinalities of the sets $\mathcal{L}_{\mathcal{A}}(k, u)$ as defined in the statement of Proposition 4. This is established by the following result.

**Lemma 8.** If $\mathcal{A} = \mathcal{A}^{(N, m)}$ satisfies Assumption 5 for some $N, m \geq 1$, then for all $k \in [m]$ and $i, j, u \in [N]$ such that $i \neq j$

$$|D_{\mathcal{A}}(k, u, i, j)| = |\mathcal{L}_{\mathcal{A}}(k, u)|^2 \mathbb{I}(u \in \mathcal{A}_{\mathcal{A}}^{(k, i)}) \mathbb{I}(j \in \mathcal{R}_{\mathcal{A}}^{(k, i)}).$$

**Proof.** First we prove the part $D_{\mathcal{A}}(k, u, i, j) = \emptyset$ if $(u, j) \notin \mathcal{A}_{\mathcal{A}}^{(k, i)} \times \mathcal{R}_{\mathcal{A}}^{(k, i)}$. If $u \notin \mathcal{A}_{\mathcal{A}}^{(k, i)}$, then by Lemma 5(i) $i \notin \mathcal{A}_{\mathcal{A}}^{(k, u)}$ and hence $D_{\mathcal{A}}(k, u, i, j) = \emptyset$. Next, if $j \in \mathcal{A}_{\mathcal{A}}^{(k-1, i)}$, then by Lemma 5(v), for all $(i'_0, \ldots, i'_m) \in \mathcal{P}_{\mathcal{A}}^{(i)}$, $(j'_0, \ldots, j'_m) \in \mathcal{P}_{\mathcal{A}}^{(j)}$, either $j'_{k-1} = j'_{k-1}$ or $i'_k \neq j'_{k}$ and hence $D_{\mathcal{A}}(k, u, i, j) = \emptyset$. For $j \notin \mathcal{A}_{\mathcal{A}}^{(k, i)}$, suppose that $D_{\mathcal{A}}(k, u, i, j) \neq \emptyset$. In this case, $i, j \in \mathcal{A}_{\mathcal{A}}^{(k, u)}$ and by Lemma 5(i) $u \in \mathcal{A}_{\mathcal{A}}^{(k, i)} \cap \mathcal{A}_{\mathcal{A}}^{(k, j)}$ and hence by Lemma 5(ii) $j \in \mathcal{A}_{\mathcal{A}}^{(k, i)}$, which concludes the proof for $(u, j) \notin \mathcal{A}_{\mathcal{A}}^{(k, i)} \times \mathcal{R}_{\mathcal{A}}^{(k, i)}$.

Next we prove that $D_{\mathcal{A}}(k, u, i, j) \neq \emptyset$, if $(u, j) \in \mathcal{A}_{\mathcal{A}}^{(k, i)} \times \mathcal{R}_{\mathcal{A}}^{(k, i)}$. Take $(u, j) \in \mathcal{A}_{\mathcal{A}}^{(k, i)} \times \mathcal{R}_{\mathcal{A}}^{(k, i)}$. Because $u \in \mathcal{A}_{\mathcal{A}}^{(k, i)}$, then by Lemma 5(ii), $\mathcal{A}_{\mathcal{A}}^{(k, i)} = \mathcal{A}_{\mathcal{A}}^{(k, u)}$, and because $j \in \mathcal{A}_{\mathcal{A}}^{(k, i)} = \mathcal{A}_{\mathcal{A}}^{(k, u)}$, then by Lemma 5(i), $i, j \in \mathcal{A}_{\mathcal{A}}^{(k, u)}$ from which we conclude that there exists $(i'_0, \ldots, i'_m) \in \mathcal{P}_{\mathcal{A}}^{(i)}$ and $(j'_0, \ldots, j'_m) \in \mathcal{P}_{\mathcal{A}}^{(j)}$ such that $i'_k = j'_k = u$ and $i'_0 = i$ and $j'_0 = j$. Suppose then that $i'_{k-1} = j'_{k-1}$. This would imply that $i, j \in \mathcal{A}_{\mathcal{A}}^{(k-1, i'_{k-1})}$ and, by Lemma 5(ii), $\mathcal{A}_{\mathcal{A}}^{(k-1, i)} = \mathcal{A}_{\mathcal{A}}^{(k-1, j)} = \mathcal{A}_{\mathcal{A}}^{(k-1, i'_{k-1})}$, and hence $j \in \mathcal{A}_{\mathcal{A}}^{(k-1, i)}$ which is a contradiction implying that $i'_{k-1} \neq j'_{k-1}$. By Assumption 5(v) we then deduce that $i'_q \neq j'_q$ for all $q < k$ and hence $((i'_0, \ldots, i'_m), (j'_0, \ldots, j'_m)) \in D_{\mathcal{A}}(k, u, i, j)$ which can therefore not be empty.

Finally, by Lemma 7, for nonempty $D_{\mathcal{A}}(k, u, i, j)$ we have

$$|D_{\mathcal{A}}(k, u, i, j)| = |U_{\mathcal{A}}(k, u, i, j)| |\mathcal{L}_{\mathcal{A}}(k, u)| |\mathcal{L}_{\mathcal{A}}(k, u)|,$$

and by Assumption 5(v) we have $|U_{\mathcal{A}}(k, u, i, j)| = 1$ which concludes the proof.

By Lemma 8 we observe that in order to have explicit expressions for the cardinalities of $D_{\mathcal{A}}(k, u, i, j)$, it suffices to have expressions for the cardinalities of the sets $\mathcal{L}_{\mathcal{A}}(k, u)$. In order to evaluate these cardinalities, we follow the principle mentioned earlier of constructing appropriate bijections to sets with known cardinalities, according to the following result.
LEMMA 9. Suppose that $\mathbb{A} = \mathbb{A}^{(N, m)}$ satisfies Assumption 5 for some $N, m \geq 1$. For all $i \in [N]$ and $k \in [m]$, let $\tau_k := [A_{\mathbb{A}}^{(k,i)}]$ and let $\phi_i^k : A_{\mathbb{A}}^{(k,i)} \rightarrow [r_k]$ be arbitrary bijections. Then for any $u \in [N]$ and $k \in [m]$, the mapping $\gamma : \mathcal{L}_\mathbb{A}(k,u) \rightarrow \{u\} \times [r_{k+1}] \times \cdots \times [r_m]$, defined as $\gamma : (i_0, \ldots, i_{m-k}) \mapsto (c_0, \ldots, c_{m-k})$, where $c_0 = i_0$ and for all $0 \leq p < m-k$, $c_{p+1} = \phi_{p+k+1}^{i_{p+1}}(i_{p+1})$, is a bijection.

PROOF. From the definition of $\mathcal{L}_\mathbb{A}$ and Assumption 5(i), it follows that for given $(i_0, \ldots, i_{m-k}) \in \mathcal{L}_\mathbb{A}(k,u)$, one has $i_{p+1} \in A_{\mathbb{A}}^{(p+k+1,i_p)}$ for all $0 \leq p < m-k$. It then follows that $c_{p+1} = \phi_{p+k+1}^{i_{p+1}}(i_{p+1}) \in [r_{p+k+1}]$ for all $0 \leq p < m-k$ and thus $\gamma(i_0, \ldots, i_{m-k}) \in \{u\} \times [r_{k+1}] \times \cdots \times [r_m]$.

For $((i_0, \ldots, i_{m-k}), (i_0', \ldots, i_{m-k}') \in \mathcal{L}_\mathbb{A}(k,u)^2$ such that $(i_0, \ldots, i_{m-k}) \neq (i_0', \ldots, i_{m-k}')$, one can take $q = \max(p \in \{0, \ldots, m-k\} : i_p = i_p')$ for which $i_{q+1} \neq i_{q+1}'$. By the bijectivity of $\phi_{q+k+1}^{i_q}$, one has $\phi_{q+k+1}^{i_q}(i_{q+1}) \neq \phi_{q+k+1}^{i_q'}(i_{q+1}')$. From this it follows that $\gamma(i_0, \ldots, i_{m-k}) \neq \gamma(i_0', \ldots, i_{m-k}')$ proving that $\gamma$ is injection.

For given $0 \leq p < m-k$, $c \in [r_{p+k+1}]$ and $i \in [N]$, one has $(\phi_{p+k+1}^{i})^{-1}(c) \in A_{\mathbb{A}}^{(p+k+1,i)}$ and hence if for any given $(c_0, \ldots, c_{m-k}) \in \{u\} \times [r_{k+1}] \times \cdots \times [r_m]$, $(i_0, \ldots, i_{m-k})$ is defined recursively as $i_0 = c_0$ and $i_{p+1} = (\phi_{p+k+1}^{i_p})^{-1}(c_{p+1})$ for all $0 \leq p < m-k$, then $(i_0, \ldots, i_{m-k}) \in \mathcal{L}_\mathbb{A}(k,n)$ and $\gamma(i_0, \ldots, i_{m-k}) = (c_0, \ldots, c_{m-k})$, which completes the proof. □

REMARK 5. By Lemma 9, the primary implication of Assumption 5(iv) becomes clear. Effectively it implies that that the cardinalities of $\mathcal{L}_\mathbb{A}(k,u)$ are independent of $u$ and, as a corollary of Lemma 9, we have for any $u \in [N]$ and $(i_{k+1}, \ldots, i_m) \in [N]^{m-k}$,

$$|\mathcal{L}_\mathbb{A}(k,u)| = \prod_{q=k+1}^{m} |A_{\mathbb{A}}^{(q,i_q)}|,$$

which is simple to evaluate given the explicit definition of $A_{\mathbb{A}}$.

We have now all the ingredients to prove Proposition 4.

PROOF OF PROPPOSITION 4. Throughout the proof we will use the notations $i_{p,q} := (i_p, \ldots, i_q) \in [N]^{q-p+1}$ and $j_{p,q} := (j_p, \ldots, j_q) \in [N]^{q-p+1}$, for
all \(0 \leq p \leq q \leq m\). First note that by Proposition 3

\[(S.13)\]

\[
\frac{m}{N} E \left[ \left( \sum_{g \in \mathcal{N}m} X_g^{(N,m)} \right)^2 \right] \mathcal{F}_0^{(N,m)} = \sum_{(i_0,j_0,\ldots,i_m,j_m)} \frac{1}{N^2} \left( \prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}i_k} A_{k+1}^{j_{k+1}j_k} \right) g(\xi_{i_0}^0)g(\xi_{j_0}^0) C_{i_1, m+j_1, m} (\mathcal{F}_N^2) (\xi_{i_0}^0, \xi_{j_0}^0).
\]

By Assumption 5(v), there exists at most one sequence \((i'_0, \ldots, i'_m)\) in \(\mathcal{P}_k\) for which \((i_0, i_m) = (i'_0, i'_m)\). Therefore, by Assumption 5(i) and Assumption 2, we have

\[
\prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}i_k} A_{k+1}^{j_{k+1}j_k} = \sum_{(i_1, \ldots, i_m-1)} \prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}i_k} \left( \prod_{k=0}^{m-1} A_{k+1}^{i_mi_0} \right) = \frac{1}{N},
\]

and hence \(\prod_{k=0}^{m-1} A_{k+1}^{i_{k+1}i_k} A_{k+1}^{j_{k+1}j_k} = N^{-2} \mathbb{I}((i_0, \ldots, i_m), (j_0, \ldots, j_m)) \in \mathcal{P}_k^2\), and from (S.13) we then have

\[(S.14)\]

\[
\frac{m}{N} E \left[ \left( \sum_{g \in \mathcal{N}m} X_g^{(N,m)} \right)^2 \right] \mathcal{F}_0^{(N,m)} = \frac{1}{N^4} \sum_{(i,j) \in \mathcal{P}_k^2} g(\xi_{i_0}^0)g(\xi_{j_0}^0) C_{i_1, m+j_1, m} (\mathcal{F}_N^2) (\xi_{i_0}^0, \xi_{j_0}^0).
\]

By Lemma 6, Assumption 5(iv) and Lemma 8 (see also Remark 5),

\[(S.15)\]

\[
\left| \bigcup_{k \in [m]} \bigcup_{u \in [N]} D_k(k, u, i_0, j_0) \right| = \sum_{k=1}^{m} \sum_{u=1}^{N} \left| D_k(k, u, i_0, j_0) \right| = D_{i_0, j_0}^{(i_0, j_0)}.
\]

By Assumption 2, for all \(u \in [N]\), \(\mathcal{A}_k^{(m,u)} = [N]\), and therefore for all \(i \in [N]\), there exists a sequence \((i_0, \ldots, i_m) \in \mathcal{P}_k^{(i)}\) such that \(i_m = u\). On the other hand, by Assumption 5(v) there exist at most one such sequence from which we conclude that \(|\mathcal{L}_k(0, i)| = N\). Therefore, by the definition of \(\mathcal{L}_k\),

\[(S.16)\]

\[
|\mathcal{P}_k^{(i)} \times \mathcal{P}_k^{(j)}| = |\mathcal{L}_k(0, i)| |\mathcal{L}_k(0, j)| = N^2.
\]

Using the fact that by Lemma 5(vi) for all \(i_0 \neq j_0\), \(\sum_{k=1}^{m} \mathbb{I}(j_0 \in \mathcal{R}_k^{(k, i_0)}) = \)
1, we then have by (S.16), Lemma 6, Lemma 8 and (S.15) that

\[
\left| (P_{\Delta}^{(i_0)} \times P_{\Delta}^{(j_0)}) \setminus \bigcup_{k \in [m]} \bigcup_{u \in [N]} D_{\Delta}(k, u, i_0, j_0) \right|
\]

\[
= N^2 \sum_{k=1}^{m} \mathbb{I}(j_0 \in R_{\Delta}^{(k,i_0)}) - \sum_{k=1}^{m} \sum_{u=1}^{N} |D_{\Delta}(k, u, i_0, j_0)|
\]

(S.17) \[
= P_{\Delta}^{(i_0,j_0)}.
\]

By (37) and (S.9)

\[
g(\xi_0^{i_0})g(\xi_0^{j_0})\left( C_{i_0:m,j_0:m}(\varphi_N^{(i_0,j_0)}) \right) (\xi_0^{i_0}, \xi_0^{j_0})
\]

\[
= \begin{cases} 
  g^2(\xi_0^{i_0})\varphi_N^{(i_0)}(\xi_0^{i_0}), & (i_0,m,j_0,m) \in D_{\Delta}^1(m), \\
  g(\xi_0^{i_0})\varphi_N^{(i_0,j_0)}(\xi_0^{i_0})g(\xi_0^{j_0}), & (i_0,m,j_0,m) \in D_{\Delta}^2(m), \\
  g(\xi_0^{i_0})\varphi_N^{(i_0)}(\xi_0^{i_0})\varphi_N(\xi_0^{j_0}), & (i_0,m,j_0,m) \in D_{\Delta}^3(m).
\end{cases}
\]

For the set \(D_{\Delta}^1(m)\) we have the disjoint decomposition

\[
D_{\Delta}^1(m) = \bigcup_{u=1}^{N} P_{\Delta}^{(u)} \times P_{\Delta}^{(u)}.
\]

By Lemma 6, (S.16), (S.15) and (S.17) we have

\[
\sum_{(i_0,m,j_0,m) \in D_{\Delta}^1(m)} g^2(\xi_0^{i_0})\varphi_N^{(i_0)}(\xi_0^{i_0}) = N^2 \sum_{i_0} g^2(\xi_0^{i_0})\varphi_N^{2}(\xi_0^{i_0}),
\]

\[
\sum_{(i_0,m,j_0,m) \in D_{\Delta}^2(m)} g(\xi_0^{i_0})\varphi_N^{(i_0,j_0)}(\xi_0^{i_0})g(\xi_0^{j_0}) = \sum_{i_0} \sum_{j_0 \neq i_0} g(\xi_0^{i_0})\varphi_N^{(i_0,j_0)}(\xi_0^{i_0})g(\xi_0^{j_0})D_{\Delta}^{(i_0,j_0)}
\]

and

\[
\sum_{(i_0,m,j_0,m) \in D_{\Delta}^3(m)} g(\xi_0^{i_0})\varphi_N^{(i_0)}(\xi_0^{i_0})g(\xi_0^{j_0})\varphi_N(\xi_0^{j_0})
\]

\[
= \sum_{i_0} \sum_{j_0 \neq i_0} g(\xi_0^{i_0})\varphi_N^{(i_0,j_0)}(\xi_0^{i_0})g(\xi_0^{j_0})\varphi_N(\xi_0^{j_0})D_{\Delta}^{(i_0,j_0)}.
\]

The proof is completed by substituting the last three equations into (S.14).

\[\square\]

C. Proofs for Section 4.1. In this section, we essentially focus on establishing the condition (35) of Theorem 4 for the radix-\(r\) algorithm. Because of the lengthy analysis, this task is divided into the three subsequent
sections. In Section C.1 we establish that the specific choice of matrices \( \mathcal{A}(r^m,m) = \mathcal{A}_{\text{radix}}^{(r,m)} \) associated with the radix-\( r \) algorithm enables us to construct partitions, call them \( \mathcal{T}_{\text{radix}}^{(r,m,d)} \), such that for any given \( d \in [m] \), the triple \( (\mathcal{A}_{\text{radix}}^{(r,m)}, \mathcal{T}_{\text{radix}}^{(r,m,d)}, d) \) satisfies all the required conditions, namely Assumptions 4 and 5, that we need to establish (35). The task then becomes two fold due to the structure of the proof of Theorem 6 where the sum in (35) is decomposed into two parts. For the first part, in Section C.2, the limit is shown to be exactly as desired and in Section C.3 the remainder part of the decomposition is shown to vanish by further analysis of the conditional independence structure of the radix-\( r \) algorithm.

C.1. Conditional independence structure of the radix-\( r \) algorithm.

**Proposition 6.** The matrices \( \mathcal{A}_{\text{radix}}^{(r,m)} \) satisfy Assumption 5 for all \( r \geq 2 \) and \( m \geq 1 \). Moreover, define for all \( r \geq 2 \), \( m \geq 1 \) and \( d \in [m] \)

\[
\begin{align*}
\mathcal{I}_{\text{radix}}^{(r,m,d)} & := \{ \mathcal{T}_u^{(r,m,d)} : u \in [r^{m-d+1}] \}, \\
\mathcal{T}_u^{(r,m,d)} & := \{ u + (q - 1)r^{m-d+1} : q \in [r^{d-1}] \}, \text{ } u \in [r^{m-d+1}].
\end{align*}
\]  

(S.18)

Then the triple \( (\mathcal{A}_{\text{radix}}^{(r,m)}, \mathcal{I}_{\text{radix}}^{(r,m,d)}, d) \) satisfies Assumption 4 for all \( r \geq 2 \), \( m \geq 1 \) and \( d \in [m] \).

The proof is divided into several technical lemmata that we will prove first. The proof of Proposition 6 itself is postponed to the end of this section.

**Lemma 10.** Fix \( m \geq 1 \), \( r \geq 2 \) and \( \mathcal{A} = \mathcal{A}_{\text{radix}}^{(r,m)} \). Then for all \( k \in [m] \)

(i) \( \prod_{q=0}^{k-1} A_{m-q} = 1_{1/r^k} \otimes I_{r^{m-k}} \),

(ii) \( \prod_{q=1}^{k} A_q = I_{r^{m-k}} \otimes 1_{1/r^k} \).

**Proof.** For both cases, the proof is by induction and the case \( k = 1 \) is obvious by (8). To check (i), assume then that \( \prod_{q=0}^{k-2} A_{m-q} = 1_{1/r^{k-1}} \otimes I_{r^{m-(k-1)}} \) for some \( k > 1 \). Then by (8) and the associativity and the mixed product property (9) we have

\[
\begin{align*}
\prod_{q=0}^{k-1} A_{m-q} & = (\prod_{q=0}^{k-2} A_{m-q}) A_{m-k+1} \\
& = (1_{1/r^{k-1}} \otimes I_{r^{m-k+1}})(I_{r^{k-1}} \otimes 1_{1/r} \otimes I_{r^{m-k}}) \\
& = (1_{1/r^{k-1}} I_{r^{k-1}}) \otimes (I_{r^{m-k+1}} (1_{1/r} \otimes I_{r^{m-k}})) \\
& = 1_{1/r^{k-1}} \otimes 1_{1/r} \otimes I_{r^{m-k}} \\
& = 1_{1/r^k} \otimes I_{r^{m-k}},
\end{align*}
\]
concluding the proof of (i). The part (ii) follows from the proof of Lemma 1.

We introduce the following additional set notation for all \( N, m \geq 1, k \in [m] \) and \( i \in [N] \)

\[
\tilde{\mathcal{A}}^{(k,i)}_k := \{ j \in [N] : \left( \prod_{q=0}^{k-1} A_{m-q} \right)^{ij} \neq 0 \}.
\]

By (40) and (41), these sets admit the following special cases for all \( i \in [N] \)

\[
(\text{S.19}) \quad \tilde{\mathcal{A}}^{(1,i)}_k = \mathcal{A}^{(m,i)}_k, \quad \tilde{\mathcal{A}}^{(m,i)}_k = \mathcal{A}^{(m,i)}_k.
\]

**Lemma 11.** Fix \( m \geq 1, r \geq 2 \) and \( \mathcal{A} = \mathcal{A}_{\text{radix}}^{(r,m)} \). Then for all \( k \in [m] \) and \( i \in [r^m] \)

\[
(\text{S.20}) \quad \mathcal{A}^{(k,i)}_{\mathcal{A}} = \left\{ (i - 1) \mod r^{k-1} + (q - 1)r^{k-1} + r^k \left\lfloor \frac{i - 1}{r^k} \right\rfloor + 1 : q \in [r] \right\},
\]

and

\[
(\text{S.21}) \quad \tilde{\mathcal{A}}^{(k,i)}_{\mathcal{A}} = \left\{ (i - 1) \mod r^{m-k} + (q - 1)r^{m-k} + 1 : q \in [r^k] \right\},
\]

\[
(\text{S.22}) \quad \mathcal{A}^{(k,i)}_{\mathcal{A}} = \left\{ r^k \left\lfloor \frac{i - 1}{r^k} \right\rfloor + q : q \in [r^k] \right\}.
\]

Moreover, if \( u_1, u_2 \in \mathcal{A}^{(k,i)}_{\mathcal{A}} \) and \( u_1 \neq u_2 \), then \( \mathcal{A}^{(k-1,u_1)}_{\mathcal{A}} \cap \mathcal{A}^{(k-1,u_2)}_{\mathcal{A}} = \emptyset \).

**Proof.** We start with the element-wise definition of the Kronecker product. For any \( N_1 \times N_2 \) matrix \( A \), \( M_1 \times M_2 \) matrix \( B \) and \( 0 \leq \alpha < N_1 M_1 \) and \( 0 \leq \beta < N_2 M_2 \), we have

\[
(\text{S.23}) \quad (A \otimes B)^{\alpha+1,\beta+1} = A^{\left\lfloor \frac{\alpha}{N_1} \right\rfloor + 1, \left\lfloor \frac{\beta}{N_2} \right\rfloor + 1} B^{(\alpha \mod M_1)+1,(\beta \mod M_2)+1}.
\]

By the definition in (8), the associativity of the Kronecker product, and two applications of (S.23), we have for all \( 0 \leq \alpha < r^m \) and \( 0 \leq \beta < r^m \)

\[
(\text{S.24}) \quad \mathcal{A}^{\alpha+1,\beta+1} = \left\lfloor \frac{\alpha}{r^{m-k}} \right\rfloor + 1, \left\lfloor \frac{\beta}{r^k} \right\rfloor + 1 \left( \left\lfloor \frac{\alpha}{r^{m-k}} \right\rfloor \mod r \right) + 1, \left( \left\lfloor \frac{\beta}{r^k} \right\rfloor \mod r \right) + 1 \times \left( \frac{\alpha \mod r^{k-1}}{r^{k-1}} + 1, \frac{\beta \mod r^{k-1}}{r^{k-1}} + 1 \right)
\]
where also the facts that \(\lfloor \alpha/r^k \rfloor - 1 \) and \(\lfloor \beta/r^k \rfloor - 1 \) have been used. From this, by considering only the diagonal elements of the identity matrices, we have readily

\[
A_{k,\alpha+1} = \left\{ i \in [r^m] : \left\lfloor \frac{\alpha}{r^k} \right\rfloor = \left\lfloor \frac{i-1}{r^k} \right\rfloor, (\alpha \mod r^{-1}) = ((i-1) \mod r^{-1}) \right\}.
\]

To prove the ‘⊃’ part of the equation (S.20), suppose that

(S.24) \( \beta = (\alpha \mod r^{-1}) + (q-1)r^{-1} + r^{-1} \lfloor \alpha/r^k \rfloor, \quad q \in [r] \).

It is then simple to check by substituting the \( \beta \) specified by (S.24) that

\[
\left\lfloor \frac{\beta}{r^k} \right\rfloor = \left\lfloor \frac{\alpha}{r^k} \right\rfloor \quad \text{and} \quad (\beta \mod r^{-1}) = (\alpha \mod r^{-1}).
\]

To prove the converse inclusion, suppose that \(\lfloor \alpha/r^k \rfloor = \lfloor \beta/r^k \rfloor \) and \((\alpha \mod r^{-1}) = (\beta \mod r^{-1})\). Then one can check that

\[
\beta = (\alpha \mod r^{-1}) + r^{-1} \left\lfloor \frac{\alpha}{r^k} \right\rfloor + r^{-1} \left( \left\lfloor \frac{\beta}{r^k} \right\rfloor \mod r \right),
\]

and since \((\lfloor \beta/r^{-1} \rfloor \mod r) + 1 \in [r]\), the claim follows.

To prove (S.21) we have by (S.23) and Lemma 10 for all \(0 \leq \alpha, \beta < r^m\)

\[
\left( \prod_{p=0}^{k-1} A_{m-p} \right)^{\alpha+1,\beta+1} = \left( \frac{\alpha}{r^k} \right)^{\lfloor \alpha/r^k \rfloor + 1} \left( \frac{\beta}{r^k} \right)^{\lfloor \beta/r^k \rfloor + 1} \times (I_{r^m-k})^{(\alpha \mod r^m-k)+1, (\beta \mod r^m-k)+1},
\]

from which we have readily that

\[
\tilde{A}_{k,\alpha+1} = \left\{ i \in [r^m] : (\alpha \mod r^{m-k}) = ((i-1) \mod r^{m-k}) \right\}.
\]

Take \(i \in \tilde{A}_{k,\alpha+1}^\alpha\), for which \(i = (\alpha \mod r^{m-k}) + (i-1)/r^{m-k} \) \(r^{m-k} + 1\) and since \(i \in [r^m]\), we have \((i-1)/r^{m-k} + 1 \in [r^k]\) and therefore ‘⊂’ holds for (S.21). To prove the converse inclusion, suppose that \(i = (\alpha \mod r^{m-k}) + (q-1)r^{m-k} + 1\), where \(q \in [r]\). Then, by the substitution of this particular choice of \(i\) one can check that \(((i-1) \mod r^{m-k}) = (\alpha \mod r^{m-k})\). The equation (S.22) follows analogously by (S.23) and Lemma 10(ii).
To check the empty intersection, by (S.20) and the assumption that \( u_1 \neq u_2 \), we have for \( \ell \in \{1, 2\} \)

\[
\text{(S.25)} \quad u_\ell = ((i - 1) \mod r^{k-1}) + q_\ell r^{k-1} + r^k \left\lfloor \frac{i - 1}{r^k} \right\rfloor + 1,
\]

where \( 0 \leq q_1, q_2 < r \) and \( q_1 \neq q_2 \). Without loss of generality, we can assume \( q_1 < q_2 \) and by (S.22) it suffices to show that

\[
r^{k-1} \left\lfloor \frac{u_2 - 1}{r^{k-1}} \right\rfloor + 1 - r^{k-1} \left\lfloor \frac{u_1 - 1}{r^{k-1}} + 1 \right\rfloor > 0,
\]

which follows from elementary calculations using (S.25). \( \square \)

**Lemma 12.** Fix \( m \geq 1, r \geq 2, d \in [m], A = A_{\text{radix}}^{(r,m)}, \) and let \( \mathcal{I}_u^{(r,m,d)} \) be as in (S.18) for all \( u \in [r^{m-d+1}] \).

(i) If \( u_1, u_2 \in [r^{m-d+1}], u_1 \neq u_2, d > 1 \) and \( (i,j) \in \mathcal{I}_u^{(r,m,d)} \times \mathcal{I}_u^{(r,m,d)} \), then

\[
\overline{A}_{u}^{(d-1,i)} \cap \overline{A}_{u}^{(d-1,j)} = \emptyset.
\]

(ii) If \( u \in [r^{m-d+1}] \) and \( i, j \in \mathcal{I}_u^{(r,m,d)} \), then \( \overline{A}_{u}^{(d,i)} = \overline{A}_{u}^{(d,j)} \).

**Proof.** To prove (i) we have by (S.18) \( i = u_1 + (q_1 - 1)r^{m-d+1} \) and \( j = u_2 + (q_2 - 1)r^{m-d+1} \) for some \( q_1, q_2 \in [r^{d-1}] \), from which it follows that \((i - 1) \mod r^{m-d+1}) = u_1 - 1 \) and \((j - 1) \mod r^{m-d+1}) = u_2 - 1 \). Now, suppose that \( \overline{A}_{u}^{(d-1,i)} \cap \overline{A}_{u}^{(d-1,j)} \neq \emptyset \). Then, by (S.21) one must have

\[
q_1' - q_2' = \frac{1}{r^{m-d+1}} \left( ((i - 1) \mod r^{m-d+1}) - ((j - 1) \mod r^{m-d+1}) \right) = \frac{1}{r^{m-d+1}} (u_1 - u_2),
\]

for some \( q_1', q_2' \in [r^{d-1}] \). Since \( u_1, u_2 \in [r^{m-d+1}] \) and \( u_1 \neq u_2 \), \( (u_1 - u_2)r^{-m+d-1} \in (-1, 1) \setminus \{0\} \) while \( q_1' - q_2' \in \mathbb{Z} \), which is a contradiction proving (i).

To prove (ii), we observe that if \( i \in \mathcal{I}_u^{(r,m,d)} \), then by (S.18) \( i = u + (q - 1)r^{m-d+1} \) where \( q \in [r^{d-1}] \) and thus

\[
((i - 1) \mod r^{m-d}) = u - 1 + (q - 1)r^{m-d+1} - \left\lfloor \frac{u - 1}{r^{m-d}} + (q - 1)r \right\rfloor r^{m-d} = u - 1 - \left\lfloor \frac{u - 1}{r^{m-d}} \right\rfloor r^{m-d},
\]

Since, the same can be repeated for \( j \in \mathcal{I}_u^{(r,m,d)} \), we have \((i - 1) \mod r^{m-d}) = ((j - 1) \mod r^{m-d}) \) which, by (S.21) is sufficient for (ii) to hold. \( \square \)
LEMMA 13. Fix $N, m \geq 1$, $\mathbb{A} = \mathbb{A}^{(N,m)}$, $\mathcal{G} \subset \mathcal{F}$ and $k, k' \in [m]$. If $\mathbb{A}$ satisfies Assumption 2 and for some $i, j \in [N]$

(S.26) \[
\mathcal{A}^{(k,i)}_\mathbb{A} \cap \mathcal{A}^{(k',j)}_\mathbb{A} = \emptyset,
\]
\[
\mathcal{A}^{(k,i)}_\mathbb{A} \times \mathcal{A}^{(k',j)}_\mathbb{A} \subset \{(u, v) \in [N]^2 : \xi_{k-1}^u \perp \perp \xi_{k'-1}^v \mid \mathcal{G} \},
\]
then $(i, j) \in \{(u, v) \in [N]^2 : \xi_k^u \perp \perp \xi_{k'}^v \mid \mathcal{G} \}$.

PROOF. By Assumption 2 we can use (S.1), (S.26) and the law of total probability, for all $i, j \in [N]$ and $S^i, S^j \in \mathcal{X}$

\[
\mathbb{P}(\xi_k^i \in S^i, \xi_k^j \in S^j \mid \mathcal{G})
= \sum_{\ell_i \in \mathcal{A}^{(k,i)}_\mathbb{A}} \sum_{\ell_j \in \mathcal{A}^{(k',j)}_\mathbb{A}} \mathbb{P}(I_k^i = \ell_i, \xi_{k-1}^i \in S^i, I_k^j = \ell_j, \xi_{k-1}^j \in S^j \mid \mathcal{G})
= \sum_{\ell_i \in \mathcal{A}^{(k,i)}_\mathbb{A}} \mathbb{P}(I_k^i = \ell_i, \xi_{k-1}^i \in S^i \mid \mathcal{G}) \sum_{\ell_j \in \mathcal{A}^{(k',j)}_\mathbb{A}} \mathbb{P}(I_k^j = \ell_j, \xi_{k-1}^j \in S^j \mid \mathcal{G})
= \mathbb{P}(\xi_k^i \in S^i \mid \mathcal{G}) \mathbb{P}(\xi_k^j \in S^j \mid \mathcal{G}),
\]
concluding the proof.

To prove (iii) of Assumption 4 we need the following conditional independence result.

LEMMA 14. Fix $N, m \geq 1$ and $\mathbb{A} = \mathbb{A}^{(N,m)}$. If $\mathbb{A}$ satisfies Assumption 5 and $\tilde{\mathcal{A}}^{(d-1,i)}_\mathbb{A} \cap \tilde{\mathcal{A}}^{(d-1,j)}_\mathbb{A} = \emptyset$ for some $i, j \in [N]$ and $1 < d \leq m$, then $\xi_{m} \perp \perp \xi_{m} \mid \xi_0, \ldots, \xi_{m-1}$.

PROOF. In the case $d = 2$, by (S.19), $\tilde{\mathcal{A}}^{(d-1,i)}_\mathbb{A} \cap \tilde{\mathcal{A}}^{(d-1,j)}_\mathbb{A} = \emptyset$ and because by the one step conditional independence we also have

\[
\mathcal{A}^{(m,i)}_\mathbb{A} \times \mathcal{A}^{(m,j)}_\mathbb{A} \subset \{(u, v) \in [N]^2 : \xi_{m-1}^u \perp \perp \xi_{m-1}^v \mid \xi_0, \ldots, \xi_{m-2} \},
\]
the claim holds by Lemma 13 for $d = 2$.

In order to prove the claim for $2 < d \leq m$, we first show that if for any $k' \in [m-2]$ and $\mathcal{G} \subset \mathcal{F}$, one has

(S.27) \[
\tilde{\mathcal{A}}^{(k+1,i)}_\mathbb{A} \cap \tilde{\mathcal{A}}^{(k+1,j)}_\mathbb{A} = \emptyset,
\]
\[
\tilde{\mathcal{A}}^{(k+1,i)}_\mathbb{A} \times \tilde{\mathcal{A}}^{(k+1,j)}_\mathbb{A} \subset \{(u, v) \in [N]^2 : \xi_{m-k-1}^u \perp \perp \xi_{m-k-1}^v \mid \mathcal{G} \},
\]
where \( k = k' \), then (S.27) is also true for \( k = k' - 1 \). To do this, we first observe that by (S.27) and Assumption 5(ii)

\[
\sum_{\ell} A_{m-k: m}^{\ell} A_{m-k: m}^{\ell} = \sum_{(u,v) \in \mathcal{A}_{\hat{m}}^{(k,i)} \times \mathcal{A}_{\hat{m}}^{(k,j)}} A_{m-k+1: m}^{iu} A_{m-k+1: m}^{jv} + \sum_{u} A_{m-k+1: m}^{iu} A_{m-k+1: m}^{iu} A_{m-k}^{u} A_{m-k}^{u} = 0.
\]

(S.28)

In the second sum of the decomposition, by Assumption 2, \( \sum_{\ell} A_{m-k}^{u}\ell A_{m-k}^{u}\ell > 0 \), and from this we conclude that \( \sum_{\ell} \left( \prod_{q=0}^{k-1} A_{m-q} \right)^{i\ell} \left( \prod_{q=0}^{k-1} A_{m-q} \right)^{j\ell} = 0 \), which is equivalent to \( \mathcal{A}_{\hat{m}}^{(k,i)} \cap \mathcal{A}_{\hat{m}}^{(k,j)} = \emptyset \), proving the first part of (S.27) for \( k = k' - 1 \). To prove the second part, we show that for all \((p, q) \in \mathcal{A}_{\hat{m}}^{(k,i)} \times \mathcal{A}_{\hat{m}}^{(k,j)}\)

(S.29)

\[
\mathcal{A}_{\hat{m}}^{(m-k,p)} \cap \mathcal{A}_{\hat{m}}^{(m-k,q)} = \emptyset, \\
\mathcal{A}_{\hat{m}}^{(m-k,p)} \times \mathcal{A}_{\hat{m}}^{(m-k,q)} \subset \{(u, v) \in [N]^2 : \xi_{m-k-1} \perp \xi_{m-k-1} | \mathcal{G}\}.
\]

To see this, we observe that in the first sum of the decomposition (S.28), \( A_{m-k+1: m}^{iu} > 0 \) and \( A_{m-k+1: m}^{jv} > 0 \), and hence by the non-negativity of the matrices \((A_{\hat{m}})^{k\in[m]}\), one must also have \( \sum_{\ell} A_{m-k}^{u}\ell A_{m-k}^{u}\ell = 0 \) which is equivalent to \( \mathcal{A}_{\hat{m}}^{(m-k,u)} \cap \mathcal{A}_{\hat{m}}^{(m-k,v)} = \emptyset \). This establishes the first part of (S.29). To prove the second part of (S.29), one can check that by definitions \( \mathcal{A}_{\hat{m}}^{(d-1,i)} = \bigcup_{\ell \in \mathcal{A}_{\hat{m}}^{(k,i)}} \mathcal{A}_{\hat{m}}^{(m-k,\ell)} \) and hence by (S.27), also the conditional independence in (S.29) holds for all \((p, q) \in \mathcal{A}_{\hat{m}}^{(k,i)} \times \mathcal{A}_{\hat{m}}^{(k,j)}\).

Finally the conditional independence in (S.27) for \( k = k' - 1 \) follows by (S.29) and Lemma 13.

By assumption, \( \mathcal{A}_{\hat{m}}^{(d-1,i)} \cap \mathcal{A}_{\hat{m}}^{(d-1,j)} = \emptyset \) and by the one step conditional independence (S.27) holds for \( k = d - 2 \) and \( \mathcal{G} = \sigma(\xi_0, \ldots, \xi_{m-d}) \). By (S.19), \( \mathcal{A}_{\hat{m}}^{(1,i)} = \mathcal{A}_{\hat{m}}^{(m,i)} \) and \( \mathcal{A}_{\hat{m}}^{(1,j)} = \mathcal{A}_{\hat{m}}^{(m,j)} \). Thus by the backward induction enabled by (S.27) we have,

\[
\mathcal{A}_{\hat{m}}^{(m,i)} \cap \mathcal{A}_{\hat{m}}^{(m,j)} = \emptyset, \\
\mathcal{A}_{\hat{m}}^{(m,i)} \times \mathcal{A}_{\hat{m}}^{(m,j)} \subset \{(u, v) \in [N]^2 : \xi_{m-1} \perp \xi_{m-1} | \xi_0, \ldots, \xi_{m-d}\},
\]

from which the claim then follows by Lemma 13. \( \square \)

**Proof of Proposition 6.** First we prove that \( \mathcal{A} = \mathcal{A}_{\text{radix}}^{(r,m)} \) satisfies Assumption 5. Assumption 2 and parts (i), (iii) of Assumption 5 follow from
the proof of Lemma 1. Assumption 5(ii) can be checked by using the mixed product property (9). Assumption 5(iv) follows from (S.20).

To prove the only non-trivial condition Assumption 5(v), we prove that if there are \((i_0, \ldots, i_m)\) and \((j_0, \ldots, j_m)\) in \(\mathcal{P}_m\) such that for some \(p \in [m]\), one has \(i_p = j_p\) and \(i_{p-1} \neq j_{p-1}\) then \(i_q \neq j_q\) for all \(q < p\). To do this, suppose that \(i_q = j_q\) for some \(q < p \in [m]\). From the definition (41) it follows that for any \(k \in [m]\), \(\mathcal{A}_{k}^{(k-1, i_{k-1})} \subset \mathcal{A}_{k}^{(k, i_k)}\). Therefore \(\mathcal{A}_{k}^{(p-1, i_{p-1})} \supset \mathcal{A}_{k}^{(q, j_q)} \subset \mathcal{A}_{k}^{(p-1, j_{p-1})}\), which is a contradiction with \(\mathcal{A}_{k}^{(p-1, i_{p-1})} \cap \mathcal{A}_{k}^{(p-1, j_{p-1})} = \emptyset\), which we know by Lemma 11 since \(i_{p-1}, j_{p-1} \in \mathcal{A}_{k}^{(0, p)}\) and \(i_{p-1} \neq j_{p-1}\).

It remains to prove that \((\mathcal{A}, \mathcal{I}^{(r, m, d)}_{\text{radix}}, d)\) satisfies Assumption 4. Assumption 4(i) follows from (S.18), and Assumption 4(ii) follows from Lemma 12(ii) since \((\prod_{k=0}^{d-1} A_{m-k})^{ij} \in \{0, r-d\}\). To verify Assumption 4(iii), we observe first that for \(d = 1\), the claim follows trivially by the one step conditional independence. For \(1 < d \leq m\) we observe that by Lemma 12(i), if \((i, j) \in \mathcal{I}^{(r, m, d)}_{u_1} \times \mathcal{I}^{(r, m, d)}_{u_2}\) where \((u_1, u_2) \in [r^{m-d+1}]^2\) such that \(u_1 \neq u_2\), then \(\mathcal{A}_{k}^{(d-1, i)} \cap \mathcal{A}_{k}^{(d-1, j)} = \emptyset\), and the claim thus follows from Lemma 14.

C.2. Convergence of the conditional variance. The main result of this section is the following proposition whose proof is postponed to the end of this section.

**Proposition 7.** Under the hypotheses of Theorem 6,

\[
\mathbb{E} \left[ \left( \sum_{\varphi = 1}^{r^{m}} X_\varphi^{(r, m, m)} \right) \mid \mathcal{F}_0 \right] \xrightarrow{p_{m \to \infty}} \left( 1 - \frac{1}{r} \right) \mu \left( g \left( \varphi - \frac{\mu(g \varphi)}{\mu(g)} \right) \right)^2 \mu(g).
\]

In order to prove Proposition 7 we need the following auxiliary result which is the main application of the block-wise absolute second moment bound hypothesis in Theorem 6.

**Lemma 15.** Under the hypotheses of Theorem 6, for all \(\varphi, \varphi' \in B_0(\mathcal{X})\)

\[
e_m(\varphi, \varphi') := \frac{1}{m} \sum_{k=1}^{m} \frac{1}{r^m} \sum_{i} \frac{1}{\mathcal{R}_{\hat{A}(m)}^{(k, i)}} \sum_{j \in \mathcal{R}_{\hat{A}(m)}^{(k, i)}} \varphi(\xi_0^k)\varphi'(\xi_0^l) - \mu(\varphi)\mu(\varphi') \xrightarrow{p_{m \to \infty}} 0,
\]

where \(\hat{A}(m) = \mathcal{A}_{\text{radix}}^{(r, m)}\).
PROOF. By defining

\[ A_m := \frac{1}{m} \sum_{k=1}^{m} \frac{1}{r^m} \sum_{i} \varphi(\xi_0^i) \frac{1}{R_{A(m)}} \sum_{j \in R_{A(m)}} \left( \varphi'(\xi_0^j) - \mu(\varphi') \right), \]

\[ B_m := \left( \frac{1}{r^m} \sum_{i} \varphi(\xi_0^i) \right) \mu(\varphi') - \mu(\varphi) \mu(\varphi'), \]

we have the decomposition \( |e_m(\varphi, \varphi')| = |A_m + B_m| \leq |A_m| + |B_m| \). From the hypotheses of Theorem 6 it follows that if we set \( d = 1 \) and \( q = 1 \) in (45), then for all \( m \geq 1 \)

\[ \mathbb{E} \left[ \left| \frac{1}{r^m} \sum_{j} \varphi(\xi_0^j) - \mu(\varphi) \right|^2 \right] \leq b(\varphi) \sqrt{\frac{m}{r^m}}, \]

implying that \( |B_m| \) converges to zero in probability as \( m \to \infty \). To prove the same for \( |A_m| \) we apply triangle inequality, Cauchy-Schwartz inequality and Jensen’s inequality, yielding

\[ \mathbb{E} \left[ |A_m| \right] \]

\[ \leq \frac{1}{m} \sum_{k=1}^{m} \mathbb{E} \left[ \left| \frac{1}{r^m} \sum_{i} \frac{1}{R_{A(m)}} \sum_{j \in R_{A(m)}} \left( \varphi'(\xi_0^j) - \mu(\varphi') \right) \right|^2 \right] \]

\[ \leq \frac{\|\varphi\|_{\infty}}{m} \sum_{k=1}^{m} \mathbb{E} \left[ \left| \frac{1}{r^m} \sum_{i} \left( \frac{1}{R_{A(m)}} \sum_{j \in R_{A(m)}} \left( \varphi'(\xi_0^j) - \mu(\varphi') \right) \right) \right|^2 \right] \]

\[ \leq \frac{\|\varphi\|_{\infty}}{m} \sum_{k=1}^{m} \mathbb{E} \left[ \left| \frac{1}{R_{A(m)}} \sum_{j \in R_{A(m)}} \left( \varphi'(\xi_0^j) - \mu(\varphi') \right) \right|^2 \right]. \]

By reversing the summation order in the last sum, we need to consider the sets \( R_{A(m)}^{(m-k+1,i)} \). Using (S.22) one can check that

\[ R_{A(m)}^{(m-k+1,i)} = \{ j + (q(p) - 1)r^{m-k} : p \in [r] \setminus \{p^*\}, j \in [r^{m-k}] \} \]

where

\[ p^* := \left( \left\lfloor \frac{\left( i - 1 \right)}{r^{m-k}} \right\rfloor \bmod r \right) + 1, \quad q(p) := r \left\lfloor \frac{\left( i - 1 \right)}{r^{m-k+1}} \right\rfloor + p \]
from which we readily have

\[(S.33) \quad |R_{A(m)}^{(m-k+1,i)}| = (r-1)r^{m-k}.\]

Note that \(p^*\) and \(q(p)\) both depend on \(m, k\) and \(i\) but in the following we will consider these quantities for fixed \(m, k\) and \(i\) only.

Because \(|R_{A(m)}(1,i)| = r - 1\) we have

\[(S.34) \quad \frac{1}{m} \sum_{i} \mathbb{E} \left[ \left( \frac{1}{|R_{A(m)}^{(1,i)}|} \sum_{j \in R_{A(m)}^{(1,i)}} \left( \varphi'(\xi_0^j) - \mu(\varphi') \right) \right)^2 \right] \leq 2 \frac{\|\varphi'\|_\infty}{m},\]

for the term \(k = m\) in the sum in \((S.31)\).

For all \(m \geq 1, k \in [m-1], i \in [r^m]\) and \(p \in [r]\), we have \(k + 1 \in [m]\) and \(q(p) \in [\mu^{(k+1)-1}]\). Hence by \((S.32), (S.33)\), Minkowski’s inequality and \((45)\) that

\[
\mathbb{E} \left[ \left( \frac{1}{|R_{A(m)}^{(m-k+1,i)}|} \sum_{j \in R_{A(m)}^{(m-k+1,i)}} \left( \varphi'(\xi_0^j) - \mu(\varphi') \right) \right)^2 \right]^{1/2} \leq \frac{1}{r-1} \sum_{p \in [r] \setminus \{p^*\}} \mathbb{E} \left[ \left( \frac{1}{r^{m-(k+1)+1}} \sum_{j \in [r^{m-(k+1)+1}]} \left( \varphi'(\xi_0^{J(j)}) - \mu(\varphi) \right) \right)^2 \right]^{1/2} \leq b(\varphi') \sqrt{\frac{m-k-1}{r^m} + \frac{1}{r^{m-k}}},
\]

where \(J(j) = j + (q(p) - 1)r^{m-(k+1)-1}\). By substituting this and \((S.34)\) into \((S.31)\) we have for all \(m \geq 1\)

\[
\mathbb{E} \left[ |A_m| \right] \leq 2 \frac{\|\varphi\|_\infty^2}{m} + b(\varphi') \|\varphi\|_\infty \frac{1}{m} \sum_{k=1}^{m-1} \sqrt{\frac{m-k-1}{r^m} + \frac{1}{r^{m-k}}},
\]

and by Cauchy-Schwartz inequality we have

\[
\left( \frac{1}{m} \sum_{k=1}^{m-1} \sqrt{\frac{m-k-1}{r^m} + \frac{1}{r^{m-k}}} \right)^2 \leq \frac{1}{m} \sum_{k=1}^{m-1} \left( \frac{m-k-1}{r^m} + \frac{1}{r^{m-k}} \right) \leq \frac{(m-1)(m-2)}{mr^m} + \frac{1}{m} \left( \frac{1}{1-r^{-m}} - 1 \right) \xrightarrow{m \to \infty} 0,
\]
implying that \( \mathbb{E}[|A_m|] \) converges to zero as \( m \to \infty \) which concludes the proof.

Proof of Proposition 7. Because \( r \geq 2 \) is assumed fixed, let us write \( \mathcal{K}(m) = \mathcal{K}_\text{radix}(m) \). First we observe that by (S.22), (S.20) and Lemma 9 we have for all \( k \in [m] \), \( i, u_0 \in [r^m] \)

\[
|A_{\mathcal{K}(m)}^{(k,i)}| = r^k, \quad |A_{\mathcal{K}(m)}^{(k,i)}| = r, \quad |\mathcal{L}_{\mathcal{K}(m)}(k, u_0)| = r^{m-k},
\]

and by Proposition 6 we can apply Proposition 4 and by substitution we have

\[
\begin{align*}
\mathbb{E} \left[ \left( \sum_{m=1}^{m} X_{g}^{(r^m, m)} \right)^2 \mathcal{F}_{0}^{(r^m, m)} \right] &= \frac{m}{m} \frac{1}{r^{2m}} \sum_{i} g^2(\xi_0^i) \mathcal{F}_{r^m(\xi_0^i)} \mu(g(\xi_0^i))
\end{align*}
\]

\[
+ \frac{r}{m} \frac{1}{r^{2m}} \sum_{k=1}^{m} \sum_{i} \sum_{j \neq i} g(\xi_0^i) \mathcal{F}_{r^m(\xi_0^i)} g(\xi_0^j) \mathbb{I}(j \in \mathcal{R}_{\mathcal{K}(m)}^{(k,i)}) r^{-k}
\]

\[
+ \frac{r}{m} \frac{1}{r^{2m}} \sum_{k=1}^{m} \sum_{i} \sum_{j \neq i} g(\xi_0^i) \mathcal{F}_{r^m(\xi_0^i)} g(\xi_0^j) \mathbb{I}(j \in \mathcal{R}_{\mathcal{K}(m)}^{(k,i)}) (1 \!-\! r^{-k}).
\]

The three nested sums on the r.h.s. will each be considered separately. For the first sum, we have by (S.30) and the continuous mapping theorem

\[
\frac{r}{m} \frac{1}{r^{2m}} \sum_{i} g^2(\xi_0^i) \mathcal{F}_{r^m(\xi_0^i)} \mu(\xi_0^i) \xrightarrow{\mathbb{P} \ m \to \infty} 0.
\]

For the second sum we see by normalizing the nested sums and by using (S.33) that

\[
\frac{r}{m} \frac{1}{r^{2m}} \sum_{k=1}^{m} \sum_{i} \sum_{j \neq i} g(\xi_0^i) \mathcal{F}_{r^m(\xi_0^i)} g(\xi_0^j) \mathbb{I}(j \in \mathcal{R}_{\mathcal{K}(m)}^{(k,i)}) r^{-k}
\]

\[
= \left(1 - \frac{1}{r}\right) \frac{1}{m} \frac{1}{r^{m}} \sum_{k=1}^{m} \frac{1}{|\mathcal{R}_{\mathcal{K}(m)}^{(k,i)}|} \sum_{j \in \mathcal{R}_{\mathcal{K}(m)}^{(k,i)}} g(\xi_0^i) \mathcal{F}_{r^m(\xi_0^i)} g(\xi_0^j)
\]

\[
\xrightarrow{\mathbb{P} \ m \to \infty} \left(1 - \frac{1}{r}\right) \mu(g^2) \mu(g),
\]

where the convergence follows from Lemma 15 and several applications of (S.30) and continuous mapping theorem.
For the third sum we define

\[ A_m := \frac{r^m}{m} \sum_{k=1}^{m} \sum_{i \neq j} g(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) g(\xi_0^j) \mathcal{P}_{r^m} (\xi_0^j) I \left( j \in \mathcal{R}^{(k,i)}_{k(m)} \right), \]

\[ B_m := \frac{r^m}{m} \sum_{k=1}^{m} \sum_{i \neq j} g(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) g(\xi_0^j) \mathcal{P}_{r^m} (\xi_0^j) I \left( j \in \mathcal{R}^{(k,i)}_{k(m)} \right) r^{-k}, \]

and show that

\[ \frac{r^m}{m} \sum_{k=1}^{m} \sum_{i \neq j} g(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) g(\xi_0^j) \mathcal{P}_{r^m} (\xi_0^j) I \left( j \in \mathcal{R}^{(k,i)}_{k(m)} \right) (1 - r^{-k}) \]

(S.38) \[ = A_m + B_m \xrightarrow{p \rightarrow \infty} 0. \]

To do this, for \( A_m \) we use Lemma 5(vi) by which

\[ A_m = \frac{r^m}{m} \sum_{k=1}^{m} \sum_{i \neq j} g(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) g(\xi_0^j) \mathcal{P}_{r^m} (\xi_0^j) \]

\[ = \frac{r^m}{m} \sum_{k=1}^{m} \left( \sum_{i} g(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) \right) \left( \sum_{j} g(\xi_0^j) \mathcal{P}_{r^m} (\xi_0^j) - g(\xi_0^j) \mathcal{P}_{r^m} (\xi_0^j) \right) \]

(S.39) \[ = \frac{r^m}{m} \left( \frac{1}{r^m} \sum_{i} g(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) \right)^2 - \frac{1}{m} \sum_{i} g^2(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i). \]

Because for the first term in (S.39) we have \( r^{-m} \sum_{i} g(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) = 0 \) and for the second term we have \( r^{-m} \sum_{i} g^2(\xi_0^i) \mathcal{P}_{r^m} (\xi_0^i) \leq \|g\|_\infty^2 \text{osc} (\varphi)^2 \). Hence we see that \( |A_m| \) converges to zero in probability as \( m \to \infty \). For \( B_m \) we have similarly as for (S.37) that

\[ B_m \xrightarrow{p \rightarrow \infty} \left( 1 - \frac{1}{r} \right) \mu(g\varphi) \mu(g\varphi) = 0. \]

The proof is completed by combining (S.36), (S.37), (S.38) and (S.35). \( \square \)

C.3. Approximation of the conditional variance and independence analysis. The main result of this section is the following proposition, which is the last remaining part in completing the proof of Theorem 6.

**Proposition 8.** Under the hypotheses of Theorem 6,

\[ \sum_{\theta=1}^{r^m m} \left( \mathbb{E} \left[ \left( X^{(r^m m)}_{\theta} \right)^2 | \mathcal{F}^{(r^m m)}_{\theta-1} \right] - \mathbb{E} \left[ \left( X^{(r^m m)}_{\theta} \right)^2 | \mathcal{F}^{(r^m m)}_{0} \right] \right) \xrightarrow{p \rightarrow \infty} 0. \]
Proof. We take $Z_{\varrho}^{(r^m,m)}$ to be as defined in (48) of the proof of Theorem 6. By Markov’s inequality, for any $\epsilon > 0$

$$
\Pr \left( \left| \sum_{\varrho \in [r^m m]} Z_{\varrho}^{(r^m,m)} \right| \geq \epsilon \right) \leq \frac{1}{\epsilon^2} \mathbb{E} \left[ \left( \sum_{\varrho \in [r^m m]} Z_{\varrho}^{(r^m,m)} \right)^2 \right]
$$

(S.40)

$$
= \frac{1}{\epsilon^2} \sum_{\varrho=1}^{mr^m} \mathbb{E} \left[ \left( Z_{\varrho}^{(r^m,m)} \right)^2 \right] + \frac{1}{\epsilon^2} \sum_{\varrho=1}^{mr^m} \sum_{\varrho' \neq \varrho} \mathbb{E} \left[ Z_{\varrho}^{(r^m,m)} Z_{\varrho'}^{(r^m,m)} \right].
$$

By Proposition 2, $|X_{\varrho}^{(r^m,m)}| \leq (r^m m)^{-1/2} \|g\|_\infty \text{osc}(\varphi)$. Therefore for any $\varrho, \varrho' \in [r^m m]$,

(S.41)

$$
\left| Z_{\varrho}^{(r^m,m)} Z_{\varrho'}^{(r^m,m)} \right| \leq 4(r^m m)^{-2} \|g\|_\infty^4 \text{osc}(\varphi)^4.
$$

Therefore the first term on the r.h.s. of (S.40) converges to zero as $m \to \infty$. It remains to establish the convergence of the second term. This is not equally straightforward as the number of cross terms is of order $(r^m m)^2$ and therefore the reasoning applied to the first term does not work without additional delicacy. The key step, which we shall take next, is to establish that a suitably large proportion of the terms $\mathbb{E} \left[ Z_{\varrho}^{(r^m,m)} Z_{\varrho'}^{(r^m,m)} \right]$ are in fact zero.

To proceed, we observe that if $\varrho, \varrho' \in [r^m m]$ are such that $Z_{\varrho}^{(r^m,m)}$ and $Z_{\varrho'}^{(r^m,m)}$ are conditionally independent given $\mathcal{F}_0^{(r^m,m)}$, then by the tower property and Proposition 2(ii), $\mathbb{E} \left[ Z_{\varrho}^{(r^m,m)} Z_{\varrho'}^{(r^m,m)} \right] = 0$.

There are altogether $m^2 r^{2m} - mr^m$ pairs $(Z_{\varrho}^{(r^m,m)}, Z_{\varrho'}^{(r^m,m)})$ with $\varrho \neq \varrho'$, and by Lemma 17, there are at most

(S.42)

$$
a_m = m^2 r^{2m} - mr^m - r(r - 1) \sum_{i=0}^{m-2} (i + 1)^2 r^{m+i}
$$

pairs which are not conditionally independent given $\mathcal{F}_0^{(r^m,m)}$. Therefore in order to establish that the second term on the r.h.s. of (S.40) converges to zero as $m \to \infty$, it is enough to once again apply (S.41), and check that

(S.43)

$$
\lim_{m \to \infty} \frac{a_m}{m^2 r^{2m}} = 0.
$$

By shifting the summation index, reversing the summation order and ex-
Fig 1. All the subsets $V_A(k, w)$, where $A = A^{(2,4)}_{\text{radix}}$, $k \in [4]$ and $w \in [r^{4-k}]$ depicted by rectangles, and the set $V_A(2, 3)$ is highlighted by the rectangle with thick border.

Expanding the square expression, we have

$$r \sum_{i=0}^{m-2} (i + 1)^2 r^{m+i} = r^{2m} \sum_{i=1}^{m-1} (m-i)^2 r^{-i}$$

\[(S.44)\]

where each of the three sums converges to a finite value as $m \to \infty$. By elementary calculations one can then check that $(S.43)$ follows by combining $(S.42)$ and $(S.44)$.

Before stating the next result, it is worth recalling the graph theoretical interpretation of the conditional independence structure of the augmented resampling algorithm defined in $(43)$ and $(44)$ in Section 3.6. The following result establishes the conditional independence of specific subsets of vertices of the graph $G_A$, where $A = A^{(r,m)}_{\text{radix}}$. For all $r \geq 2$, $m \geq 1$, $k \in [m]$ and $w \in [r^{m-k}]$ these subsets are defined as

\[(S.45)\] $V_A(k, w) := \{\xi_i^q : 0 \leq q \leq k, (w-1)r^k < i \leq wr^k\}$,

where $A = A^{(r,m)}_{\text{radix}}$. See Figure 1 for an illustrations of these sets.

**Lemma 16.** Fix $m \geq 1$, $r \geq 2$, $A = A^{(r,m)}_{\text{radix}}$, $k \in [m]$, $w_1, w_2 \in [r^{m-k}]$, such that $w_1 \neq w_2$ and $u_1, u_2 \in [r^m]$ and $0 \leq q_1, q_2 \leq m$ such that $(\xi_{u_1}^{q_1}, \xi_{u_2}^{q_2}) \in V_A(k, w_1) \times V_A(k, w_2)$. Then $\xi_{q_1}^{u_1} \perp \perp \xi_{q_2}^{u_2} | \xi_0$. 

By definition, the sets \( V_i \) define, for any \( 1 < k \leq q \) and \( r \) where for all \( j \) established that \( A \) to the graph \( \{ \xi \} \subset V \) to denote the graph theoretical parents of the elements of \( V \). Moreover we define, for \( i \in \{ 1, 2 \} \), \( V_i := \{ \xi^i_q : A^i_{p+1/q} \neq 0, \ 0 \leq p < q \} \cup \{ \xi^i_q \} \). In simple terms, \( V_i^1 \) and \( V_i^2 \) are the ancestor sets of \( \xi^i_q \) and \( \xi^u_q \), respectively.

First we show that \( V_i^1 \cap V_i^2 = \emptyset \). To do this, assume that there exists \( \xi^u_q \subset V_i^1 \cap V_i^2 \), where \( q \leq \min(q_1, q_2) \) and \( u^* \in [r^m] \). By Lemma 5(iii) there exists \( (i_0, \ldots, i_q) \in \mathcal{P}_{A_{1q}} \) and \( (j_0, \ldots, j_{q^2}) \in \mathcal{P}_{A_{1q^2}} \) such that \( i_0 = j_0 = u^* \), \( q_1 = u_1 \) and \( q_2 = u_2 \). Hence \( u^* \in \mathcal{A}_{A_i} \cap \mathcal{A}_{A_j} \). On the other hand, by Lemma 11, \( \mathcal{A}_{A_i} \cap \mathcal{A}_{A_j} = \{ (w_1 - 1)k^k, k^k, \ldots, w_{q^2}k^k \} \) and \( \mathcal{A}_{A_i} \cap \mathcal{A}_{A_j} = \{ (w_1 - 1)k^k, k^k, \ldots, w_{q^2}k^k \} \) and since \( q_1, q_2 \leq k \), by Lemma 5(iv) \( \mathcal{A}_{A_i} \cap \mathcal{A}_{A_j} = \emptyset \) and thus \( \mathcal{A}_{A_i} \cap \mathcal{A}_{A_j} = \emptyset \), which is a contradiction proving that \( V_i^1 \cap V_i^2 = \emptyset \).

The conditional distribution of \( (\xi_k)_{0 \leq k \leq m} \) given \( \xi_{in} \) factorizes according to the graph \( G_A \) with conditional densities

\[
\phi_{\xi_i}^{\xi_i}(x_{\xi_i}, \gamma_{\xi_i}) = \begin{cases} 
\frac{1}{V_i} \sum_{j \in A_{A_i}} A^{ij}_{A_i} V_{A_{ij}}^j \delta_{x_i}(x_k^j), & (k, i) \in [m] \times [N] \\
\delta_{x_i}(x_k^j), & k = 0, \ i \in [N].
\end{cases}
\]

By definition, the sets \( V_i^1 \) and \( V_i^2 \) are ancestral (see, e.g. [13]) and having established that \( V_i^1 \cap V_i^2 = \emptyset \), we can apply [13, Corollary 3.23] to yield the claimed conditional independence.

**Lemma 17.** Fix \( m > 1 \), \( r \geq 2 \) and \( A = A_{\text{radix}} \). Then for all \( 1 < k \leq m \), \( w \in [r^m-k] \)

\[
|Q_A(k, w)| \geq r(r - 1) \sum_{i=0}^{k-2} (i + 1)^2 r^{i+k+1},
\]

where for all \( r \geq 2 \), \( m \geq 1 \), \( k \in [m] \) and \( w \in [r^m-k] \)

\[
Q_A(k, w) := \{ (\xi_{k_1}^i, \xi_{k_2}^i) \in V_A(k, w)^2 : k_1, k_2 \in [m], \xi_{k_1}^i \perp \xi_{k_2}^i \mid F_0(r_{m^{-1}}) \}
\]

**Proof.** The proof is by induction over \( k > 1 \). First we observe that for any \( 1 < k \leq m \) and \( w \in [r^m-k] \), there are \( r \) subsets \( V_A(k-1, w_{k-1}) \subset V_A(k, w) \), where \( wr - (r - 1) \leq w_{k-1} \leq wr \).
By Lemma 16, if \((\xi, \xi') \in \mathcal{V}_k(1, w_1) \times \mathcal{V}_k(1, w'_1)\) where \(wr - (r - 1) \leq w_1, w'_1 \leq wr\) and \(w_1 \neq w'_1\), then \((\xi, \xi') \in \mathcal{Q}_k(2, w)\). By (S.45) one can check that \(|\mathcal{V}_k(k, w) \setminus \{\xi_0^i : i \in [r^m]\}| = kr^k\) and hence

\[
|\mathcal{V}_k(1, w_1) \setminus \{\xi_0^i : i \in [r^m]\}| = |\mathcal{V}_k(1, w'_1) \setminus \{\xi_0^i : i \in [r^m]\}| = r.
\]

Therefore the first element of the pair \((\xi, \xi')\) can be chosen among the \(r\) elements of \(r\) sets and the second element from the \(r\) elements of the remaining \(r - 1\) sets implying that, when \(k = 2\), for all \(w \in [r^{m-2}]\)

\[
|\mathcal{Q}_k(2, w)| \geq r(r-1)r^2 + \sum_{i=wr-(r-1)}^{wr} |\mathcal{Q}_k(1, i)| \geq r(r-1)r^2,
\]

where the second inequality follows from the simplifying observation that for all \(w \in [r^{m-2}]\) and \(wr - (r - 1) \leq i \leq wr\), one trivially has \(|\mathcal{Q}_k(1, i)| \geq 0\). This completes the proof for \(k = 2\).

Let us then assume that the claim holds for some \(2 \leq k < m\). Therefore each of the \(r\) subsets \(\mathcal{V}_k(k, w_k)\) of \(\mathcal{V}_k(k+1, w)\), where \(w \in [r^{m-(k+1)}]\) and \(wr - (r - 1) \leq w_k \leq wr\), admits at least

\[
(S.46) \quad a_1 = r(r-1) \sum_{i=0}^{k-2} (i+1)^2 r^{k+i},
\]

pairs of vertices that are conditionally independent given \(\mathcal{F}_0^{(r^m,m)}\). By applying Lemma 16 again, similarly as above, and by observing that there are

\[
a_2 = r(kr^k)(r-1)(kr^k)
\]

pairs of vertices \((\xi, \xi') \in \mathcal{V}_k(k, w_k) \times \mathcal{V}_k(k, w'_k)\) where \(wr - (r - 1) \leq w_k, w'_k \leq wr\) and \(w_k \neq w'_k\). From this together with (S.46) we conclude that

\[
|\mathcal{Q}_k(k+1, w)| \geq r(kr^k)(r-1)(kr^k) + r \cdot r(r-1) \sum_{i=0}^{k-2} (i+1)^2 r^{k+i}
\]

\[
= r(r-1) \sum_{i=0}^{(k+1)-2} (i+1)^2 r^{k+1+i},
\]

completing the proof.

D. Proofs for Section 4.2. In this section we undertake the task of establishing the condition (35) of Theorem 4 for the mixed radix-\(r\) algorithm. Because the proof of Theorem 8 is similar to that of Theorem 6, also the structure of this section is analogous to Section C.
D.1. Conditional independence structure of the mixed radix-$r$ algorithm.

**Proposition 9.** The matrices $A_{\text{mixed}}^{(r,c)}$ satisfy Assumption 5 for all $r \geq 2$ and $c \geq 1$. Moreover, define for all $r \geq 2$, $c \geq 1$ and $d \in \{1, 2\}$

\begin{align*}
I_{\text{mixed}}^{(r,c,d)} &:= \left\{ I_{u}^{(r,c,d)} : u \in [cr^2 - d] \right\}, \\
I_{u}^{(r,c,d)} &:= \left\{ u + (q - 1)cr^2 - d : q \in [r^d - 1] \right\}, \quad u \in [cr^2 - d].
\end{align*}

Then the triple $(A_{\text{mixed}}^{(r,c)}, I_{\text{mixed}}^{(r,c,d)}, d)$ satisfies Assumption 4 for all $r \geq 2$, $c \geq 1$ and $d \in \{1, 2\}$.

Before the proof of Proposition 9, we state the following technical result establishing explicit expression for the sets needed in the collision analysis in the case of the mixed radix-$r$ algorithm.

**Lemma 18.** Fix $r \geq 2$, $c \geq 1$ and $A = A_{\text{mixed}}^{(r,c)}$. For all $i \in [rc]$

\begin{align*}
A_{\tilde{A}}^{(1,i)} &= 1 + \left\lfloor \frac{i - 1}{c} \right\rfloor + q : q \in [c], \\
A_{\tilde{A}}^{(2,i)} &= \left\{ ((i - 1) \mod c) + (q - 1)c + 1 : q \in [r] \right\},
\end{align*}

and $\overline{A}_{\tilde{A}}^{(1,i)} = A_{\tilde{A}}^{(1,i)}$, $\overline{A}_{\tilde{A}}^{(2,i)} = [rc]$.

**Proof.** By the element-wise definition (S.23) of the Kronecker product and (17) it follows similarly as in the proof of Lemma 11 that $A_{\tilde{A}}^{(1,\alpha+1)} = \{ j \in [rc] : [\alpha/c] = [(j - 1)/c] \}$. From this (S.48) follows by elementary calculation. Equation (S.49) can be verified exactly as in the proof of (S.21) in Lemma 11. The identity $\overline{A}_{\tilde{A}}^{(1,i)} = A_{\tilde{A}}^{(1,i)}$ follows immediately by definition and finally the claim $\overline{A}_{\tilde{A}}^{(2,i)} = [rc]$ holds because $A_1A_2 = 1_{1/rc}$.

**Proof of Proposition 9.** To prove that $\tilde{A} = A_{\text{mixed}}^{(r,c)}$ satisfies Assumption 5 we observe, as in the proof of Proposition 6, that the only non-trivial property is Assumption 5(v), which follows similarly as in the proof of Proposition 6 by using Lemma 18.

Assumption 4(i) can be checked with elementary calculation using (17) and (S.47). Assumption 4(ii) is verified simply by noting that for $d = 1$, $u \in [rc]$, $|I_{u}^{(r,c,d)}| = 1$, and for $d = 2$ one has $\prod_{k=0}^{d-1} A_{m-k} = 1_{1/rc}$.

To check Assumption 4(iii) we first note that for $d = 1$ the claim follows from the one step conditional independence. For $d = 2$, one can check using
(S.49) of Lemma 18 analogously to the proof of Lemma 12(i), that for all \((i, j) \in T_{u_1}^{(rc, 2)} \times T_{u_2}^{(rc, 2)}\) where \((u_1, u_2) \in [c]^2\) such that \(u_1 \neq u_2\), one has \(A_{h}^{(2, i)} \cap A_{h}^{(2, j)} = \emptyset\). By the one step conditional independence, we have \(A_{h}^{(2, i)} \times A_{h}^{(2, j)} \subset \{(u, v) \in [rc]^2 : \xi^u_1 \perp \xi^v_1 | \xi_0\}\) and hence Assumption 4(iii) follows from Lemma 13.

D.2. **Convergence of the conditional variance.** The main result of this section is the following proposition.

**Proposition 10.** Under the hypotheses of Theorem 8

\[
\mathbb{E} \left[ \left( \sum_{g=1}^{2rc} X_{g}^{(rc, 2)} \right)^2 \bigg| X_0^{(rc, 2)} \right] \xrightarrow{P_{c \to \infty}} \left( 1 - \frac{1}{2r} \right) \mu \left( g \left( \varphi - \frac{\mu(g, \varphi)}{\mu(\varphi)} \right)^2 \right) \mu(g).
\]

We have the following result, which serves a purpose analogous to Lemma 15 in the case of radix-\(r\) algorithm, although it is somewhat different by nature.

**Lemma 19.** Under the hypotheses of Theorem 8, if for all \(i \in [rc], q_i \in [r^{k-1}]\), for some \(k \in \{1, 2\}\), then for all \(\varphi, \varphi' \in \mathcal{B}_b(X)\)

\[
\frac{1}{rc} \sum_{i} \frac{r^{k-1}}{rc} \sum_{j \in [rc^{2-k}]} \varphi(\xi_i^j)\varphi'(\xi_i^{J(j)}) - \mu(\varphi)\mu(\varphi') \xrightarrow{P_{c \to \infty}} 0.
\]

where \(J(j) = j + (q_i - 1)cr^{2-k} \forall j \in [rc^{2-k}]\).

**Proof.** By defining

\[
A_c := \frac{1}{rc} \sum_{i} \varphi(\xi_i^j) \frac{r^{k-1}}{rc} \sum_{j \in [rc^{2-k}]} (\varphi' (\xi_i^{J(j)}) - \mu(\varphi'))),
\]

\[
B_c := \frac{1}{rc} \sum_{i} \varphi(\xi_i^j)\mu(\varphi') - \mu(\varphi)\mu(\varphi'),
\]

we have by the triangle inequality

\[
\left| \frac{1}{rc} \sum_{i} \frac{r^{k-1}}{rc} \sum_{j \in [rc^{2-k}]} \varphi(\xi_i^j)\varphi' (\xi_i^{J(j)}) - \mu(\varphi)\mu(\varphi') \right| = |A_c + B_c| \leq |A_c| + |B_c|.
\]
By the hypotheses of Theorem 8, by setting \( d = 1 \) and \( q = 1 \) in (49) yields for all \( c \geq 1 \)

\[
E \left[ \left( \frac{1}{rc} \sum_{i} \varphi(\xi^i) - \mu(\varphi) \right)^2 \right] \leq b(\varphi) \sqrt{\frac{2}{rc}},
\]

(S.50)

from which we deduce that \( |B_c| \) converges to zero in probability as \( c \to \infty \). It remains to show the same for \( |A_c| \). By Jensen’s inequality, Cauchy-Schwartz inequality and (49) we have

\[
E[|A_c|] \leq \sqrt{1 + \sum_{i} \mathbb{E} \left[ \left( \frac{r^{k-1}}{rc} \sum_{j \in [cr^2-k]} (\varphi'(\xi^j) - \mu(\varphi'))^2 \right) \right]}
\]

\[
\leq \|\varphi\|_\infty \sqrt{1 + \sum_{i} \mathbb{E} \left[ \left( \frac{r^{k-1}}{rc} \sum_{j \in [cr^2-k]} (\varphi'(\xi^j) - \mu(\varphi'))^2 \right) \right]}
\]

\[
\leq \|\varphi\|_\infty \frac{\sqrt{2-k} r^{k-1}}{rc} + \frac{\sqrt{2-k} r^{k-1}}{rc} \frac{\mathbb{P}}{c \to \infty} \to 0,
\]

completing the proof.

\[\square\]

**Proof of Proposition 10.** Because \( r \geq 2 \) is assumed fixed, let us write \( \Lambda(c) = \Lambda^{(r,c)}_{mixed}. \) By Proposition 9 we can apply Lemma 9 and on the other hand we can also use Lemma 18 yielding for all \( i, u_0 \in [rc] \)

\[
|\mathcal{A}_{\Lambda(c)}^{(k,i)}| = cr^{k-1}, \quad |\mathcal{A}_{\Lambda(c)}^{(k,i)}| = c^2 k r^{k-1}, \quad |\mathcal{L}_{\Lambda(c)}(k, u_0)| = r^{2-k},
\]

and therefore by substitution in Proposition 4

\[
E \left[ \left( \sum_{g=1}^{2rc} X^{(rc,2)}_g \right)^2 \right] = \mathcal{F}_0^{(rc,2)} \frac{1}{2 (rc)^2} \sum_{i} g^2(\xi^i) \overline{g^2(\xi^i)}
\]

\[
+ \frac{rc}{2 (rc)^3} \sum_{k=1}^{m} \sum_{i} \sum_{j \neq i} g(\xi^i) \overline{g(\xi^j)} \mathbb{E}_{\Lambda(c)}(j \in \mathcal{R}_{\Lambda(c)}^{(k,i)}) r^{2-k}
\]

\[
+ \frac{rc}{2 (rc)^3} \sum_{k=1}^{m} \sum_{i} \sum_{j \neq i} g(\xi^i) \overline{g(\xi^j)} \mathbb{E}_{\Lambda(c)}(j \in \mathcal{R}_{\Lambda(c)}^{(k,i)}) (rc - r^{2-k}).
\]
To obtain the limit of (S.52) we first observe that by (S.50), (27), and the continuous mapping theorem
\[(S.53) \quad \lim_{c \to \infty} \frac{1}{(rc)^2} \sum_i g^2(\xi_0^i) \overline{\varphi}_{rc}(\xi_0^i) \xrightarrow{p} \frac{1}{2} \mu(g^2 \overline{\varphi}^2)\].

For the second sum in (S.52) we define
\[
A_c := \frac{1}{rc} \sum_i g(\xi_0^i) \overline{\varphi}_{rc}(\xi_0^i) \frac{r}{rc} \sum_j g(\xi_0^j) \|I_j \in \mathcal{R}_{(1;i)}(c)\|
\]
and
\[
B_c := \frac{1}{rc} \sum_i g(\xi_0^i) \overline{\varphi}_{rc}(\xi_0^i) \frac{1}{rc} \sum_j g(\xi_0^j) \|I_j \in \mathcal{R}_{(2;i)}(c)\|
\]
in which case the second sum is equal to \((A_c + B_c)/2\). By Lemma 18 we have
\[
\mathcal{R}_{(1;i)}(c) = \{j + (q_i - 1)c : j \in [c] \} \setminus \{i\}
\]
where \(q_i := \lfloor (i - 1)/c \rfloor + 1\). Since \(q_i \in [r]\) we can use Lemma 19 with \(k = 2\), the continuous mapping theorem, and the fact that
\[
\left| \frac{1}{rc} \sum_i g^2(\xi_0^i) \overline{\varphi}_{rc}(\xi_0^i) \right| \leq \|g\|^2_{\infty} \text{osc} (\varphi)^2,
\]
and we have
\[
A_c = \frac{1}{rc} \sum_i g(\xi_0^i) \overline{\varphi}_{rc}(\xi_0^i) \left( \frac{r}{rc} \sum_{j \in [c]} g(\xi_0^{(j)}) \frac{r}{rc} - \frac{r}{rc} g(\xi_0^i) \right)
\]
\[
= \frac{1}{rc} \sum_i g(\xi_0^i) \overline{\varphi}_{rc}(\xi_0^i) \frac{r}{rc} \sum_{j \in [c]} g(\xi_0^{(j)}) - \frac{r}{rc} \frac{1}{rc} \sum_i g^2(\xi_0^i) \overline{\varphi}_{rc}(\xi_0^i)
\]
\[(S.54) \quad \lim_{c \to \infty} \frac{\mu(g^2 \overline{\varphi}^2)}{\mu(g)}\]
where \(J(j) = j + (q_i - 1)c\) for all \(j \in [c]\). Using Lemma 18, it can be checked that
\[
\mathcal{R}_{(2;i)}(c) = \{j + (q_i - 1)c : q \in [r] \setminus \{q_i\}, j \in [c]\}
\]
Then, by Lemma 19 with $k = 2$, (S.50), and the continuous mapping theorem

$$B_c = \frac{1}{rc} \sum_i g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) \frac{1}{rc} \sum_{q \in [r]} \sum_{j \in [c]} g(\xi_{jq}^j)$$

$$= \left( \frac{1}{rc} \sum_i g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) \right) \left( \frac{1}{rc} \sum_j g(\xi_j^0) \right)$$

$$- \frac{1}{r rc} \sum_i g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) \frac{1}{c} \sum_{j \in [c]} g(\xi_{jq}^j)$$

(S.55) \[ \xrightarrow{c \to \infty} \left( 1 - \frac{1}{r} \right) \mu(g \mathcal{P}^2) \mu(g), \]

where $J_q(j) = j + (q - 1)c$ for all $j \in [c]$ and $q \in [r]$. By combining (S.54) and (S.55) we have

$$\frac{rc}{2} \sum_{k=1}^m \sum_i \sum_{j \neq i} g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) g(\xi_j^0) \mathcal{P}_{rc}(\xi_j^0) \mathbb{I}(j \in \mathcal{R}_{k(c)}^{(k,i)}) r^{2-k}$$

(S.56) \[ \xrightarrow{c \to \infty} \frac{1}{2} (A_c + B_c) \xrightarrow{p \to \infty} \left( 1 - \frac{1}{2r} \right) \mu(g \mathcal{P}^2) \mu(g). \]

To conclude the proof we define

$$A'_c := \frac{1}{rc} \sum_{k=1}^m \sum_i \sum_j g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) g(\xi_j^0) \mathcal{P}_{rc}(\xi_j^0) \mathbb{I}(j \in \mathcal{R}_{k(c)}^{(k,i)})$$

$$B'_c := \frac{1}{(rc)^2} \sum_{k=1}^m \sum_i \sum_j g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) g(\xi_j^0) \mathcal{P}_{rc}(\xi_j^0) \mathbb{I}(j \in \mathcal{R}_{k(c)}^{(k,i)}) r^{2-k},$$

in which case the third sum in (S.52) equals $(A'_c + B'_c)/2$. By Lemma 5(vi), and the fact that $(rc)^{-1} \sum_i g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) = 0$, for $A'_c$ we have

$$A'_c = \frac{1}{rc} \sum_i \sum_{j \neq i} g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) g(\xi_j^0) \mathcal{P}_{rc}(\xi_j^0)$$

$$= rc \left( \frac{1}{rc} \sum_i g(\xi_i^0) \mathcal{P}_{rc}(\xi_i^0) \right)^2 - \frac{1}{rc} \sum_i g^2(\xi_i^0) \mathcal{P}^2_{rc}(\xi_i^0)$$

(S.57) \[ \xrightarrow{c \to \infty} -\mu(g \mathcal{P}^2), \]
where the second equality follows similarly as in (S.39) and the convergence follows from (S.50) together with the continuous mapping theorem. By arguments identical to those used in proving (S.56) we see that \( B'_n \) converges in probability to \((2 - r^{-1}) \mu(g^2) = 0\) and combining this with (S.57) gives

\[
\frac{rc}{(rc)^3} \sum_{k=1}^m \sum_i \sum_{j \neq i} g(\xi_0^i)u_{rc}(\xi_0^i) g(\xi_0^j)u_{rc}(\xi_0^j) \mathbb{I} \left( j \in \mathcal{R}^{(k,i)}_{A(c)} \right) (rc - r^{2-k})
\]

\[
= \frac{1}{2} (A'_c - B'_c) \xrightarrow{c \to \infty} - \frac{1}{2} \mu(g^2)\phi^2).
\]

The proof is completed by combining this limit and the limits in (S.53) and (S.56) with (S.52).

D.3. **Approximation of the conditional variance and independence analysis.** The main result of this section is the following proposition, which is the last remaining part in completing the proof of Theorem 8.

**Proposition 11.** Under the hypotheses of Theorem 8,

\[
\sum_{q \in \{2rc\}} \left( \mathbb{E} \left[ \left( X^{(rc,2)}_q \right)^2 \right] \bigg| \mathcal{F}^{(rc,2)}_{q-1} \right) - \mathbb{E} \left[ \left( X^{(rc,2)}_q \right)^2 \bigg| \mathcal{F}^{(rc,2)}_0 \right] \right) \xrightarrow{c \to \infty} 0.
\]

**Proof of Proposition 11.** Recall the definition of \( Z^{(rc,2)}_q \) in (48). By Markov’s inequality we have the same decomposition (S.40) as in the case of the radix-\( r \) algorithm. By (30), \( |X^{(rc,2)}_q| \leq (2rc)^{-1/2} \|g\| \text{osc } (\phi) \), hence for any \( q, q' \in \{2rc\}, \)

\[
Z^{(rc,2)}_q Z^{(rc,2)}_{q'} \leq \frac{1}{(rc)^2} \|g\|_\infty^4 \text{osc } (\phi)^4,
\]

and the first term on the r.h.s. of (S.40) converges to zero as \( c \to \infty \). It remains to establish the convergence of the second term in a manner similar to that in the proof of Proposition 8.

There are altogether \( 2rc(2rc - 1) \) pairs \( (Z^{(rc,2)}_q, Z^{(rc,2)}_{q'}) \) with \( q \neq q' \), and thus by Lemma 20, there are at most

\[
a_c = 2rc(2rc - 1) - rc(rc - 1) - 3rc(rc - r) = 3r^2c - rc
\]

pairs which are not conditionally independent given \( \mathcal{F}^{(rc,2)}_0 \). Therefore it is enough to apply (S.58), and check that \( \lim_{c \to 0} a_c/(rc)^2 = 0 \), which is trivial. \( \Box \)
Lemma 20. Fix \( r \geq 2, c \geq 1 \) and \( \mathcal{A} = A_{\text{mixed}}^{(r,c)} \). Then

\[
|Q_\mathcal{A}| \geq rc(rc - 1) + 3rc(rc - r),
\]

where \( Q_\mathcal{A} := \{(\xi_{k_1}^{i_1}, \xi_{k_2}^{i_2}) : k_1, k_2 \in \{1, 2\}, i_1, i_2 \in [rc], \xi_{k_1}^{i_1} \perp \perp \xi_{k_2}^{i_2} \mid F_0^{(rc,2)}\} \).

Proof. By the one step conditional independence

\[
A := \{(\xi_{k_1}^{i_1}, \xi_{k_2}^{i_2}) : k_1 = k_2 = 1, i_1, i_2 \in [rc], i_1 \neq i_2\} \subset Q_\mathcal{A},
\]

and readily \(|A| = rc(rc - 1)|. For the set

\[
B := \{(\xi_{k_1}^{i_1}, \xi_{k_2}^{i_2}) : i, j \in [rc], i \notin A^{(2,j)}_\mathcal{A}\},
\]

we also have \( B \subset Q_\mathcal{A} \), since by the one step conditional independence, for all \( i, j \in [rc] \) such that \( i \notin A^{(2,j)}_\mathcal{A} \) and for all \( S_1, S_2 \in \mathcal{X} \) we have by (S.1)

\[
P(\xi_1^i \in S_1, \xi_2^j \in S_2 \mid \xi_0) = \sum_{\ell \in A^{(2,j)}_\mathcal{A}} P(\xi_1^i \in S_1 \mid \xi_0) P(I_2^j = \ell, \xi_1^i \in S_2 \mid \xi_0)
= P(\xi_1^i \in S_1 \mid \xi_0) P(\xi_2^j \in S_2 \mid \xi_0).
\]

Also, because by (S.51), \(|A^{(2,j)}_\mathcal{A}| = r\), one has \(|B| = rc(rc - r)|. Similarly we have

\[
C := \{(\xi_1^i, \xi_1^j) : i, j \in [rc], j \notin A^{(2,i)}_\mathcal{A}\} \subset Q_\mathcal{A},
\]

and \(|C| = rc(rc - r)|. Moreover, by Lemma 13 and the one step conditional independence, we also have

\[
D := \{(\xi_1^i, \xi_2^j) : A^{(2,i)}_\mathcal{A} \cap A^{(2,j)}_\mathcal{A} = \emptyset\} \subset Q_\mathcal{A},
\]

and by Proposition 9 one can check similarly as in the proof of Lemma 5(ii) that \( A^{(2,i)}_\mathcal{A} \cap A^{(2,j)}_\mathcal{A} = \emptyset \) if and only if \( j \notin A^{(2,i)}_\mathcal{A} \), and hence \(|D| = rc(rc - r)|. Finally the claim follows by observing that \( A \cap B \cap C \cap D = \emptyset \), and hence

\[
|Q_\mathcal{A}| = |A| + |B| + |C| + |D|.
\]

E. Proofs for Section 5. Our next objective is to prove Proposition 5. The first step is a generalization of (7).
Lemma 21. For all \( \varphi \in \mathcal{B}_b(X) \) and \( p > 1 \) there exists \( b_p \in \mathbb{R} \), depending only on \( p \), such that if \((\tilde{A}^{(N,m)}, I, d)\) satisfies Assumption 4 for some \( N, m \geq 1 \) and \( d \in [m] \), then for all \( J \in \mathcal{J} (I) \)

\[
\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i \in I} g(\xi_{i_{in}}) \right) \left( \frac{1}{|I|} \sum_{i=1}^{|I|} \varphi(\xi_{J(i)}^{J(i)}) \right) - \frac{1}{N} \sum_{i} g(\xi_{i_{in}}) \varphi(\xi_{i_{in}}) \right]^{\frac{1}{p}} \leq \sqrt{\frac{m - d}{N} + \frac{1}{N_{m,d}} b_p \|g\|_{\infty} \text{osc} (\varphi)}.
\]

Proof. Follows from Proposition 2 similarly as in the proof of Proposition 1.

To prove Proposition 5, we first establish a bound for the mean of order \( p \) for the initialization of the filter. We then proceed to establish similar bounds inductively for the subsequent resampling and mutation steps. This strategy is embodied in the following three lemmata.

Lemma 22 (Initialization). Fix \( N \geq 1 \). For all \( \varphi \in \mathcal{B}_b(X) \) and \( p > 1 \), there exists \( b_0(p) \in \mathbb{R} \), depending only on \( p \), such that

\[
\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i} \varphi(\xi^i_{0}) - \pi_0(\varphi) \right)^{\frac{1}{p}} \right] \leq b_0(p) \sqrt{\frac{1}{N} \text{osc} (\varphi)}.
\]

Proof. Because \( \{\xi^i_{0}\}_{i \in [N]} \overset{\text{i.i.d.}}{\sim} \pi_0 \) the claim follows straightforwardly by Burkholder’s inequality.

Lemma 23 (Resampling). Let \( n \geq 0 \) and \( p > 1 \) be fixed. If the triple \((\tilde{A}^{(N,m)}, I, d)\), satisfies Assumption 4 for some \( N, m \geq 1 \) and \( d \in [m] \) and for all \( \varphi \in \mathcal{B}_b(X) \) there exists \( b_n(\varphi, p) \in \mathbb{R} \) such that

\[
(\text{S.}59) \quad \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i} \varphi(\xi^i_{n}) - \pi_n(\varphi) \right)^{\frac{1}{p}} \right] \leq b_n(\varphi, p) \sqrt{\frac{m}{N}},
\]

then for all \( \varphi \in \mathcal{B}_b(X) \) there exists \( \hat{b}_n(\varphi, p) \in \mathbb{R} \) such that for all \( J \in \mathcal{J} (I) \)

\[
\mathbb{E} \left[ \left( \frac{1}{|I|} \sum_{i=1}^{|I|} \varphi(\xi_{J(i)}^{J(i)}) - \hat{\pi}_n(\varphi) \right)^{\frac{1}{p}} \right] \leq \hat{b}_n(\varphi, p) \sqrt{\frac{m - d}{N} + \frac{1}{N_{m,d}}}.
\]
Proof. For brevity of notations, let us write $g_n^i := g_n(\zeta_n^i)$ and $\varphi_n^i := \varphi(\zeta_n^i)$. Define

$$\overline{\varphi}_N(x) := \varphi(x) - \frac{\sum_i g_n^i \varphi_n^i}{\sum_i g_n^i},$$

and

$$A := \frac{1}{\pi_n(g_n)} \left( \frac{1}{N} \sum_i g_n^i \left( \frac{1}{|\mathcal{I}|} \sum_{i=1}^{|\mathcal{I}|} \varphi(\hat{\zeta}_n^{J(i)}) \right) - \frac{1}{N} \sum_i g_n^i \varphi_n^i \right),$$

$$B := \sum_i g_n^i \varphi_n^i - \frac{\pi_n(g_n \varphi)}{\pi_n(g_n)}$$

$$C := \frac{1}{\pi_n(g_n)} \left( \frac{1}{|\mathcal{I}|} \sum_{i=1}^{|\mathcal{I}|} \overline{\varphi}_N(\hat{\zeta}_n^{J(i)}) \right) \left( \pi_n(g_n) - \frac{1}{N} \sum_i g_n^i \right),$$

for which we have the decomposition

$$(S.60) \quad \frac{1}{|\mathcal{I}|} \sum_{i=1}^{|\mathcal{I}|} \varphi(\hat{\zeta}_n^{J(i)}) - \frac{\pi_n(g_n \varphi)}{\pi_n(g_n)} = A + B + C.$$

By Lemma 21

$$(S.61) \quad \mathbb{E} \left[ |A|^p \right]^{\frac{1}{p}} \leq \left( \frac{m - d}{N} + \frac{1}{N_m d} \right)^{\frac{1}{2}} \frac{1}{\pi_n(g_n)} b_n \|g_n\|_{\infty} \text{osc}(\varphi).$$

For $B$ we then have, similarly as e.g. in [4, proof of Lemma 4], by Minkowski’s inequality and (S.59)

$$\mathbb{E} \left[ |B|^p \right]^{\frac{1}{p}} \leq \frac{\|\varphi\|_{\infty}}{\pi_n(g_n)} \mathbb{E} \left[ \left| \frac{1}{N} \sum_i g_n^i \varphi_n^i - \frac{1}{N} \sum_i g_n^i \varphi_n^i \right|^p \right]^{\frac{1}{p}}$$

$$+ \frac{1}{\pi_n(g_n)} \mathbb{E} \left[ \left| \frac{1}{N} \sum_i g_n^i \varphi_n^i - \pi_n(g_n \varphi) \right|^p \right]^{\frac{1}{p}}$$

$$(S.62) \quad \leq \frac{1}{\pi_n(g_n)} \left( \|\varphi\|_{\infty} b_n(g_n, p) + b_n(g_n \varphi, p) \right) \left( \frac{m - d}{N} + \frac{1}{|\mathcal{I}|} \right),$$

where we have also used the fact that by Assumption 4(i) $N/|\mathcal{I}| \geq d$ and
hence \( m/N \leq (m - d)/N + 1/|T| \). For \( C \) we have
\[
E[|C|^p]^\frac{1}{p} \leq \frac{\text{osc}(\varphi)}{\pi_n(g_n)} E \left[ \left( \frac{1}{N} \sum_i g_n^i \right)^p \right] ^{\frac{1}{p}} \leq \frac{\text{osc}(\varphi)}{\pi_n(g_n)} b_n(g_n,p) \sqrt{\frac{m - d}{N} + \frac{1}{|T|}}. \tag{S.63}
\]
Thus by combining (S.60), (S.61), (S.62) and (S.63) the claim follows by Minkowski’s inequality.

**Lemma 24 (Mutation).** Fix \( N, m \geq 1, n \geq 1, p > 1, d \in [m] \) and \( J \in \mathcal{J}(\mathcal{I}) \), where \( \mathcal{I} \) is a partition of \([N]\). If for all \( \varphi \in \mathcal{B}b(\mathbb{X}) \) there exists \( \hat{b}_n(\varphi, p) \in \mathbb{R} \), such that
\[
E \left[ \left| \frac{1}{|T|} \sum_{i=1}^{\left|\mathcal{I}\right|} \varphi(\zeta_{n-1}^J(i)) - \pi_n(\varphi) \right|^p \right] ^{\frac{1}{p}} \leq \hat{b}_n(\varphi, p) \sqrt{\frac{m - d}{N} + \frac{1}{|T|}},
\]
then for all \( \varphi \in \mathcal{B}b(\mathbb{X}) \) there exists \( b_n(\varphi, p) \in \mathbb{R} \) such that
\[
E \left[ \left| \frac{1}{|T|} \sum_{i=1}^{\left|\mathcal{I}\right|} \varphi(\zeta_{n}^J(i)) - \pi_n(\varphi) \right|^p \right] ^{\frac{1}{p}} \leq b_n(\varphi, p) \sqrt{\frac{m - d}{N} + \frac{1}{|T|}}.
\]

**Proof.** By defining
\[
A := \frac{1}{|T|} \sum_{i=1}^{\left|\mathcal{I}\right|} \varphi(\zeta_{n-1}^J(i)) - \pi_n(\varphi),
\]
\[
B := \frac{1}{|T|} \sum_{i=1}^{\left|\mathcal{I}\right|} f(\varphi)(\zeta_{n-1}^J(i)) - \hat{\pi}_{n-1}(f(\varphi)),
\]
we have the decomposition
\[
\frac{1}{|T|} \sum_{i=1}^{\left|\mathcal{I}\right|} \varphi(\zeta_{n}^J(i)) - \pi_n(\varphi) = A + B, \tag{S.65}
\]
With the sequence \( X_j := 1/|T| \sum_{i=1}^{\left|\mathcal{I}\right|} \varphi(\zeta_{n}^J(i)) - f(\varphi)(\zeta_{n-1}^J(i)) \) and \( \sigma \)-algebras \( \mathcal{A}_j := \sigma(\zeta_{n-1}^J, \zeta_{n-1}^{J(1)}, \ldots, \zeta_{n-1}^{J(j)}) \), the sequence \( (X_j, \mathcal{A}_j)_{j \in \left|\mathbb{I}\right|} \) is a martingale and by Burkholder’s inequality
\[
E[|A|^p]^{\frac{1}{p}} \leq b_p \text{osc} (\varphi) \sqrt{\frac{1}{|T|}} \leq b_p \text{osc} (\varphi) \sqrt{\frac{m - d}{N} + \frac{1}{|T|}}.
\]
For $B$ we have by (S.64)
\[
\mathbb{E} \left[ |B|^p \right]^{1/p} \leq \hat{b}_n(f(\varphi), p) \sqrt{\frac{m - d}{N} + \frac{1}{|T|}},
\]
and the claim follows from (S.65) by Minkowski’s inequality.

The proofs of Theorems 2 and 3 are composed of a number of lemmata. We start with the initialization of the particle filter, which is common to both Theorems. Results specific to each of the two butterfly resampling schemes then follow in Sections E.1 and E.2

**Lemma 25.** For all $\varphi \in \mathcal{B}_b(X)$,
\[
\begin{align*}
\text{(S.66)} \quad & \frac{1}{N} \sum_{i} \varphi(\zeta_{0}^i) - \pi_0(\varphi) \xrightarrow{a.s.} 0, \\
\text{(S.67)} \quad & \sqrt{N} \left( \frac{1}{N} \sum_{i} \varphi(\zeta_{0}^i) - \pi_0(\varphi) \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2_0(\varphi)).
\end{align*}
\]

**Proof.** Because $\{\zeta_{0}^i\}_{i \in [N]} \overset{\text{i.i.d.}}{\sim} \pi_0$, the claim follows straightforwardly from the strong law of large numbers and central limit theorem for i.i.d. random variables.

**E.1. Particle filter deploying the radix-$r$ algorithm.** For the following three Lemmata, we will assume $r \geq 2$ fixed and that for all $m \geq 1$, $(\zeta_{n}^i, \hat{\zeta}_{n}^i)_{n \geq 0, i \in [r^m]}$ are the random variables associated with the augmented resampling particle filter deploying matrices $\mathbb{A}^{(r,m)}_{\text{radix}}$.

**Lemma 26 (Resampling at time $n = 0$).** If for all $\varphi \in \mathcal{B}_b(X)$,
\[
\begin{align*}
\text{(S.68)} \quad & \frac{1}{r^m} \sum_{i} \varphi(\zeta_{0}^i) - \pi_0(\varphi) \xrightarrow{a.s.} 0, \\
\text{(S.69)} \quad & \sqrt{r^m} \left( \frac{1}{r^m} \sum_{i} \varphi(\zeta_{0}^i) - \pi_0(\varphi) \right) \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2_0(\varphi, r)),
\end{align*}
\]
then for all $\varphi \in \mathcal{B}_b(X)$,
\[
\begin{align*}
\text{(S.70)} \quad & \frac{1}{r^m} \sum_{i} \varphi(\hat{\zeta}_{0}^i) - \hat{\pi}_0(\varphi) \xrightarrow{a.s.} 0, \\
\text{(S.71)} \quad & \sqrt{r^m} \left( \frac{1}{r^m} \sum_{i} \varphi(\hat{\zeta}_{0}^i) - \hat{\pi}_0(\varphi) \right) \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2_{R,0}(\varphi, r)).
\end{align*}
\]
where $\hat{\sigma}^2_{R,0}(\varphi, r)$ is as defined in (14).

Proof. With

\[ \varphi_r(x) := \varphi(x) - \frac{\sum_i g_0(\zeta^i_0)\varphi(\zeta^i_0)}{\sum_i g_0(\zeta^i_0)}, \]

and the shorthand notations:

\[ A_m := \frac{1}{\pi_0(g_0)} \left( \frac{1}{r^m} \sum_i g_0(\zeta^i_0) \right) \left( \frac{1}{r^m} \sum_i \varphi(\zeta^i_0) \right) - \frac{1}{r^m} \sum_i g_0(\zeta^i_0) \varphi(\zeta^i_0), \]

\[ B_m := \sum_i g_0(\zeta^i_0) \varphi(\zeta^i_0) - \frac{\pi_0(g_0 \varphi)}{\pi_0(g_0)}, \]

\[ C_m := \frac{1}{\pi_0(g_0)} \left( \frac{1}{r^m} \sum_i \varphi_r(\zeta^i_0) \right) \left( \pi_0(g_0) - \frac{1}{r^m} \sum_i g_0(\zeta^i_0) \right) \]

we have

\[ \frac{1}{r^m} \sum_i \varphi(\zeta^i_0) - \hat{\pi}_0(\varphi) = A_m + B_m + C_m, \]

because of the fact that $\hat{\pi}_0(\varphi) = \pi_0(g_0 \varphi)/\pi_0(g_0)$. For the law of large numbers, (S.70), we shall check that the terms $A_m, B_m, C_m$, each converge to zero as $m \to \infty$, $\mathbb{P}$-almost surely. For $A_m$, note that the random variables $(\zeta^i_0)_{i \in [r^m]}$ are input to the resampling scheme, and $(\hat{\zeta}^i_0)_{i \in [r^m]}$ are the corresponding output, so the desired convergence follows from the identity (32) in Proposition 2 and Theorem 5. For $B_m$ the desired convergence follows from (S.68). For $C_m$, it follows from Theorem 5 and (S.68) that

\[ \frac{1}{r^m} \sum_i \varphi_r(\hat{\zeta}^i_0) \xrightarrow{a.s.} 0, \]

and the desired convergence then holds since

\[ |C_m| \leq \pi_0(g_0)^{-1} |r^{-m} \sum_i \varphi_r(\hat{\zeta}^i_0)|2 \|g_0\|\infty. \]

For the CLT, (S.71), first apply (S.69) to establish

\[ \sqrt{r^m} \left( \frac{1}{r^m} \sum_i g_0(\zeta^i_0) - \pi_0(g_0) \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2_{R,0}(g_0, r)), \]

and combining this fact with (S.74) and Slutsky’s theorem, we find that $(r^m/m)^{1/2}C_m$ converges to zero in probability.
Noting that

\( B_m = \frac{\sum g_0(\zeta^i_0)(\varphi(\zeta^i_0) - \hat{\pi}_0(\varphi))}{\sum g_0(\zeta^i_0)} \),

we have by (S.68), (S.69) and Slutsky’s theorem that \((r^m)^{1/2}B_m\) converges in distribution as \(m \to \infty\) to a Gaussian random variable, so \((r^m/m)^{1/2}B_m\) converges in probability to zero.

So, by another application of Slutsky’s theorem, in order to complete the proof, it suffices to show

\[(S.76) \sqrt{r^m/m} A_m \stackrel{d}{\to} \mathcal{N}(0, \hat{\sigma}^2_{R,0}(\varphi, r)).\]

By Propositions 6 and 5, we can apply Theorem 6 to the test function \(\varphi(\cdot)/\pi_0(g_0)\), yielding

\[\mathbb{E} \left[ \exp(iu(r^m/m)^{1/2}A_m) \right|_{\zeta_0} \stackrel{P}{\to} \exp(-(u^2/2)\hat{\sigma}^2_{R,0}(\varphi, r)),\]

and since the modulus of the complex exponential is no greater than 1, this convergence in fact holds in the \(L_1\) sense, and hence, by Levy’s continuity theorem, (S.76) holds.

**Lemma 27** (Mutation at time \(n \geq 1\)). *Fix \(n \geq 1\). If for all \(\varphi \in \mathcal{B}_b(X)\),

\[(S.77) \frac{1}{r^m} \sum_i \varphi(\zeta^i_{n-1}) - \hat{\pi}_{n-1}(\varphi) \xrightarrow{a.s.} 0,\]

\[(S.78) \sqrt{r^m/m} \left( \frac{1}{r^m} \sum_i \varphi(\zeta^i_{n-1}) - \hat{\pi}_{n-1}(\varphi) \right) \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2_{R,n-1}(\varphi, r)),\]

then for all \(\varphi \in \mathcal{B}_b(X)\),

\[(S.79) \frac{1}{r^m} \sum_i \varphi(\zeta^i_n) - \pi_n(\varphi) \xrightarrow{a.s.} 0,\]

\[(S.80) \sqrt{r^m/m} \left( \frac{1}{r^m} \sum_i \varphi(\zeta^i_n) - \pi_n(\varphi) \right) \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2_{R,n}(\varphi, r)).\]

where \(\hat{\sigma}^2_{R,n}(\varphi, r)\) and \(\hat{\sigma}^2_{R,n-1}(\varphi, r)\) are as defined in (14).

**Proof.** With

\[A_m := \frac{1}{r^m} \sum_i \varphi(\zeta^i_n) - f(\varphi)(\zeta^i_{n-1}), \ B_m := \frac{1}{r^m} \sum_i f(\varphi)(\zeta^i_{n-1}) - \hat{\pi}_{n-1}(f(\varphi)),\]
we have
\[
\frac{1}{r_m} \sum_i \varphi(\zeta^i_n) - \pi_n(\varphi) = A_m + B_m.
\]

With \(X_j := (r_m)^{-1/2} \sum_{i=1}^j \varphi(\zeta^i_n) - f(\varphi)(\hat{\zeta}^i_{n-1})\) and \(A_j := \sigma(\hat{\zeta}^i_{n-1}, \zeta^1_n, \ldots, \zeta^n_j)\), \((X_j, A_j)_{j \in \mathbb{N}}\) is a martingale, and by application of Burkholder’s inequality, Markov’s inequality, the fact that \(\varphi \in \mathcal{B}_b(\mathbb{X})\), and Borel-Cantelli, we find that \(A_m\) converges to zero as \(m \to \infty\), \(\mathbb{P}\)-almost surely, and \((r_m/m)^{1/2} A_m\) does too. \(B_m\) converges to zero almost surely by (S.77), and \((r_m/m)^{1/2} B_m\) converges to a \(\mathcal{N}(0, \hat{\sigma}^2_{R,n-1}(f(\varphi), r))\) by (S.78).

**Lemma 28 (Resampling at time \(n \geq 1\)).** Fix \(n \geq 1\). If for all \(\varphi \in \mathcal{B}_b(\mathbb{X})\),

\[
\begin{align*}
\text{(S.81)} & \quad \frac{1}{r_m} \sum_i \varphi(\zeta^i_n) - \pi_n(\varphi) \xrightarrow{a.s. \ m \to \infty} 0, \\
\text{(S.82)} & \quad \sqrt{\frac{r_m}{m}} \left( \frac{1}{r_m} \sum_i \varphi(\zeta^i_n) - \pi_n(\varphi) \right) \xrightarrow{d \ m \to \infty} \mathcal{N}(0, \sigma^2_{R,n}(\varphi, r)), \\
\text{(S.83)} & \quad \frac{1}{r_m} \sum_i \varphi(\zeta^i_n) - \hat{\pi}_n(\varphi) \xrightarrow{a.s. \ m \to \infty} 0, \\
\text{(S.84)} & \quad \sqrt{\frac{r_m}{m}} \left( \frac{1}{r_m} \sum_i \varphi(\zeta^i_n) - \hat{\pi}_n(\varphi) \right) \xrightarrow{d \ m \to \infty} \mathcal{N}(0, \hat{\sigma}^2_{R,n}(\varphi, r)).
\end{align*}
\]

then for all \(\varphi \in \mathcal{B}_b(\mathbb{X})\),

\[
\begin{align*}
\text{(S.83)} & \quad \frac{1}{r_m} \sum_i \varphi(\tilde{\zeta}^i_n) - \hat{\pi}_n(\varphi) \xrightarrow{a.s. \ m \to \infty} 0, \\
\text{(S.84)} & \quad \sqrt{\frac{r_m}{m}} \left( \frac{1}{r_m} \sum_i \varphi(\tilde{\zeta}^i_n) - \hat{\pi}_n(\varphi) \right) \xrightarrow{d \ m \to \infty} \mathcal{N}(0, \hat{\sigma}^2_{R,n}(\varphi, r)).
\end{align*}
\]

where \(\sigma^2_{R,n}(\varphi, r)\) and \(\hat{\sigma}^2_{R,n}(\varphi, r)\) are as defined in (14).

**Proof.** By defining \(\overline{\varphi}_{r,m}, A_m, B_m, C_m\) as in (S.72) and (S.73) but by replacing 0 with \(n\) we have

\[
\frac{1}{r_m} \sum_i \varphi(\zeta^i_n) - \hat{\pi}_n(\varphi) = A_m + B_m + C_m.
\]

For the law of large numbers, (S.83), very similar arguments to those in the proof of Lemma 26 establish that \(A_m, B_m, C_m\), each converge to zero as \(m \to \infty\), \(\mathbb{P}\)-almost surely.

The proof of the CLT (S.84), also uses arguments similar to those in the proof of Lemma 26, the main difference being that due to the statistically different nature of the input \((\zeta^i_n)_{i \in \mathbb{N}}\), the term \(B_m\) does not vanish. From
(S.81), (S.82) and (S.83) it follows that \( (r^m/m)^{1/2} C_m \) converges to zero in probability. So in order to complete the proof, it suffices to show

\[
\text{(S.85)} \quad \sqrt{\frac{r^m}{m}} A_m + \sqrt{\frac{r^m}{m}} B_m \xrightarrow{d_{m \to \infty}} N(0, \sigma_{R,n}^2(\varphi, r)).
\]

By Propositions 6 and 5, we can apply Theorem 6 to the test function \( \varphi(\cdot)/\pi_n(g_n) \),

\[
\mathbb{E} \left[ \exp(iu(r^m/m)^{1/2} A_m) \left| \zeta_n \right. \right] \xrightarrow{m \to \infty} \exp(-u^2/2)\sigma^2),
\]

where \( \sigma^2 = (1 - r^{-1})\hat{\pi}_n((\varphi - \hat{\pi}_n(\varphi))^2) \).

For \( B_m \) we have an expression analogous to (S.75) from which we see by by (S.81), (S.82) and Slutsky’s theorem that \( (r^m/m)^{1/2} B_m \) converges in distribution as \( m \to \infty \) to a Gaussian random variable, call it \( Z \), with mean zero and variance \( \pi_n(g_n)^{-2} \sigma_{R,n}^2(g_n(\varphi - \hat{\pi}_n(\varphi)), r) \). Then by the continuous mapping theorem, \( \exp(iu(r^m/m)^{1/2} B_m) \) converges in distribution to \( \exp(iuZ) \), and by yet another application of Slutsky’s theorem,

\[
\mathbb{E} \left[ \exp(iu(r^m/m)^{1/2} B_m) \left| \zeta_n \right. \right] \exp(iu(r^m/m)^{1/2} B_m) \xrightarrow{m \to \infty} \exp(-u^2/2)\sigma^2) \exp(iuZ),
\]

from which (S.85) follows. \( \square \)

From the Lemmata 26, 27 and 28, together with Lemma 25, Theorem 2 follows.

E.2. Particle filter deploying the mixed radix-r algorithm. For the following two Lemmata, we will assume \( r \geq 2 \) fixed and that for all \( c \geq 1 \), \((\zeta_n^c, \hat{\zeta}_n^c)_{n \geq 0, i \in [rc]} \) are the random variables associated with the augmented resampling particle filter deploying matrices \( A_{\text{mixed}}^{(r,c)}(r^m/m)^{1/2} C_m \) converges to zero in probability. So in order to complete the proof, it suffices to show

\[
\text{(S.86)} \quad \frac{1}{rc} \sum_i \varphi(\zeta_n^c) - \pi_n(\varphi) \xrightarrow{a.s.} c \to \infty 0,
\]

\[
\text{(S.87)} \quad \sqrt{rc} \left( \frac{1}{rc} \sum_i \varphi(\zeta_n^c) - \pi_n(\varphi) \right) \xrightarrow{d_{c \to \infty}} N(0, \sigma_{M,n}^2(\varphi, r)).
\]
then for all \( \varphi \in \mathcal{B}_b(X) \),

(S.88) \[
\frac{1}{rc} \sum_i \varphi(\hat{\zeta}_n^i) - \hat{\pi}_n(\varphi) \xrightarrow{a.s. \; c \to \infty} 0,
\]

(S.89) \[
\sqrt{rc} \left( \frac{1}{rc} \sum_i \varphi(\hat{\zeta}_n^i) - \hat{\pi}_n(\varphi) \right) \xrightarrow{d \; c \to \infty} \mathcal{N}(0, \hat{\sigma}_M^2, n(\varphi, r)).
\]

where \( \sigma^2_{M,n}(\varphi, r) \) and \( \hat{\sigma}_M^2(\varphi, r) \) are as defined in (19).

**Proof of Lemma 29.** By defining \( \varphi, A_c, B_c \) and \( C_c \) as in (S.72) and (S.73) but by replacing 0 with \( n \) and \( r^m \) with \( rc \) we have

\[
\frac{1}{rc} \sum_i \varphi(\hat{\zeta}_n^i) - \hat{\pi}_n(\varphi) = A_c + B_c + C_c.
\]

The law of large numbers follows from (32) of Proposition 2, Theorem 7 and (S.86) analogously to the proof of Lemma 26 so the details are omitted.

To prove (S.89) it suffices to show that

(S.90) \[
\sqrt{rc} A_c + \sqrt{rc} B_c + \sqrt{rc} C_c \xrightarrow{d \; c \to \infty} \mathcal{N}(0, \hat{\sigma}_M^2, n(\varphi, r)).
\]

For \( \sqrt{rc} B_c \) and \( \sqrt{rc} C_c \) we proceed similar to the proofs of Lemma 26 and Lemma 28. For \( \sqrt{rc} A_c \) we observe that by Proposition 9 and Proposition 5 we can apply Theorem 8 to the test function \( \sqrt{2} \varphi(\cdot)/\pi_n(g_n) \), yielding by (32) of Proposition 2

\[
\mathbb{E} \left[ \exp(iu\sqrt{rc} A_c) \right] \xrightarrow{p \; c \to \infty} \exp(-u^2/2\sigma^2),
\]

where \( \sigma^2 = (2 - r^{-1}) \hat{\pi}_n(\varphi - \hat{\pi}_n(\varphi))^2 \). We then proceed analogously to the proof of Lemma 28 to establish (S.90) completing the proof.

**Lemma 30 (Mutation at time \( n \geq 1 \)).** Fix \( n \geq 1 \). If for all \( \varphi \in \mathcal{B}_b(X) \),

(S.91) \[
\frac{1}{rc} \sum_i \varphi(\hat{\zeta}_{n-1}^i) - \hat{\pi}_{n-1}(\varphi) \xrightarrow{a.s. \; c \to \infty} 0,
\]

(S.92) \[
\sqrt{rc} \left( \frac{1}{rc} \sum_i \varphi(\hat{\zeta}_{n-1}^i) - \hat{\pi}_{n-1}(\varphi) \right) \xrightarrow{d \; c \to \infty} \mathcal{N}(0, \hat{\sigma}_{M,n-1}^2(\varphi, r)),
\]

...
then for all $\varphi \in \mathcal{B}$.\(\mathcal{B}(\mathbb{X})\),

\[\frac{1}{rc} \sum_{i} \varphi(\zeta_{n}^{i}) - \pi_{n}(\varphi) \xrightarrow{\text{a.s.}} c \to \infty 0,\]  

(S.93)

\[\sqrt{rc} \left( \frac{1}{rc} \sum_{i} \varphi(\zeta_{n}^{i}) - \pi_{n}(\varphi) \right) \xrightarrow{\text{d}} c \to \infty N(0, \sigma_{M,n}^{2}(\varphi, r)).\]  

(S.94)

where $\hat{\sigma}_{M,n-1}^{2}(\varphi, r)$ and $\sigma_{M,n}^{2}(\varphi, r)$ are as defined in (19).

**Proof of Lemma 30.** The proof of (S.93) is analogous to that in the proof of Lemma 27, and (S.94) follows from same arguments as [3, Lemma A.1].

From Lemmata 29 and 30, together with Lemma 25, Theorem 3 follows. \[\square\]

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