Error Correction with Euclidean Qubits

Alexander Yu. Vlasov*

November 1999

Abstract

In classical case there is simplest method of error correction with using three equal bits instead of one. In the paper is shown, how the scheme fails for quantum error correction with complex vector spaces of usual quantum mechanics, but works in real and quaternionic cases. It is discussed also, how to implement the three qubits scheme with using encoding of quaternionic qubit by Majorana spinor. Necessary concepts and formulae from area of quantum error corrections are closely introduced and proved.

1 Introduction

A simple question, why quantum error correction code for one qubit needs for 5, or 7, or 9 ... qubits instead of 3 in classical case, is concerned with some rather deep topics. In the article is shown that difference between Hilbert and Euclidean spaces is also matter here, for example, three Euclidean qubits would be enough and it is discussed below.

Together with obvious example with real vector spaces, Euclidean case also related with quaternionic representation of qubits and quantum gates. Noncommutative algebra of quaternions is richer than complex numbers and can represent some nonstandard view on a qubit.

To show, that the view has some relation with physical reality the quaternionic qubit may be considered as some subsystem of four component Dirac spinoral wave function (end of Sec.5), but the relativistic example should not be considered as only way of interpretation of the modified qubit model.

*E-mail: Alexander.Vlasov@pobox.spbu.ru
The Sec. 2 devoted to simple case of error correction for real vector space. A closed introduction to topics of quantum error correction necessary for questions under consideration is given in Sec. 3. Quaternionic case is introduced in Sec. 4 and example with Majorana spinors is described in Sec. 5.

**Standard definitions**

The Pauli matrices are:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\(\mathbb{R}\) — real numbers, \(\mathbb{C}\) — complex numbers.

\(\mathbb{C}^n = \mathbb{C}^n - \{0\}\) — vector space without origin.

\(\mathbb{S}^n\) — unit sphere in \(\mathbb{R}^{n+1}\), i.e. \(\mathbb{S}^2\) — usual sphere and \(\mathbb{S}^1\) — circle.

\(\mathbb{C}P^n = \mathbb{C}^{n+1}/\mathbb{C}_*, \mathbb{R}P^n = \mathbb{R}^{n+1}/\mathbb{R}_*\) — complex and real projective spaces.

**Maps**: \(M \ni \rightarrow N\) — projection, \(N \ni \rightarrow M\) — injection.

\(A \cong B\) — isomorphism of groups, algebras (also equivalence of two topological spaces).

\(a \equiv b\) means: \(a \in A, b \in B, A \cong B, A \ni \rightarrow B, \iota(a) = b\)

\(a \approx b\) — equivalence relation for elements (used in definition of quotient spaces like \(SU(2)/U(1)\)).

All other necessary definitions (\(\mathbb{H}, \gamma_i, \text{etc.}\)) are given below in main text of the paper.

2 \(SO(2)\) (or \(U(1)\)) error correction

A trivial classical error correction method for one bit uses simplest 3 bit encoding: \(0 \rightarrow 000, 1 \rightarrow 111\). Let us consider, as toy model, qubits with real coefficients: \(a|0\rangle + b|1\rangle, (a, b \in \mathbb{R})\). The \(\mathbb{R}\)-qubit is described by some point on \(2\)-plane. Let us suggest now, that error — is rotation of the plane for one of the \(\mathbb{R}\)-qubits:

\[
|0\rangle \rightarrow \alpha |0\rangle + \beta |1\rangle; \quad |1\rangle \rightarrow -\beta |0\rangle + \alpha |1\rangle; \quad \alpha = \cos \theta; \quad \beta = \sin \theta \quad (1)
\]

Analog of the classical scheme for \(\mathbb{R}\)-qubits is code:

\[
|0\rangle \rightarrow |000\rangle, \quad |1\rangle \rightarrow |111\rangle; \quad a|0\rangle + b|1\rangle \rightarrow a|000\rangle + b|111\rangle \quad (2)
\]
then any 1-qubit error Eq. (1) can be corrected. It is enough to append some auxiliary *ancilla* qubits and to apply special transformation to “transfer” an error on the extra qubits. In the example under consideration we can add two ancilla qubits and use orthogonal transformation\(^\text{1}\):

\[
\begin{align*}
|000\rangle|00\rangle & \rightarrow |000\rangle|00\rangle \\
|100\rangle|00\rangle & \rightarrow |000\rangle|10\rangle \\
|010\rangle|00\rangle & \rightarrow |000\rangle|01\rangle \\
|001\rangle|00\rangle & \rightarrow |000\rangle|11\rangle \\
|111\rangle|00\rangle & \rightarrow |111\rangle|00\rangle \\
|011\rangle|00\rangle & \rightarrow |011\rangle|00\rangle \\
|101\rangle|00\rangle & \rightarrow |101\rangle|00\rangle \\
|110\rangle|00\rangle & \rightarrow |110\rangle|00\rangle
\end{align*}
\]

(3)

Let us consider error in first qubit as an example:

\[
a |000\rangle + b |111\rangle \leadsto a (\alpha_1 |000\rangle + \beta_1 |100\rangle) + b (-\beta_1 |011\rangle + \alpha_1 |111\rangle) \leadsto
\]

After appending ancilla |00\rangle:

\[
\leadsto a (\alpha_1 |000\rangle|00\rangle + \beta_1 |100\rangle|00\rangle) + b (-\beta_1 |011\rangle|00\rangle + \alpha_1 |111\rangle|00\rangle) \leadsto
\]

And after application of operator Eq. (3):

\[
\leadsto a (\alpha_1 |000\rangle|00\rangle + \beta_1 |100\rangle|10\rangle) + b (\beta_1 |111\rangle|10\rangle + \alpha_1 |111\rangle|00\rangle)
\]

\[
= (a |000\rangle + b |111\rangle) (\alpha_1 |00\rangle + \beta_1 |10\rangle)
\]

The scheme works also for usual (C-)qubit with complex coefficients \(a, b\) and only important condition is Eq. (1) with real \(\alpha, \beta\), i.e. \(SO(2)\) group of errors. Similar 3-qubit scheme was used for experimental phase error correction\([1]\). It is possible because \(U(1) \cong SO(2)\) and so phase, \(U(1)\), error can be considered as \(SO(2)\) error in other basis\([2]\).

### 3 Quantum (complex) error correction

The method above does not work for more general set of errors. Error matrix in Eq. (1) could be written as:

\[
\left(\begin{array}{cc}
\alpha & \beta \\
-\beta & \alpha
\end{array}\right); \quad \alpha, \beta \in \mathbb{R}; \quad \alpha^2 + \beta^2 = 1
\]

\(^1\)Only nontrivial transformations of 8 basis vectors (between 32) are shown.\(^2\)For paper [1] it is \(|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\) and \(|-\rangle = i (|0\rangle - |1\rangle)/\sqrt{2}\).
and general 1-qubit error can be expressed as element of $SU(2)$ group:

\[
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}; \quad \alpha, \beta \in \mathbb{C}; \quad |\alpha|^2 + |\beta|^2 = 1
\]  

(4)

If we use code like Eq. (2) with complex $a, b$, then we can correct error Eq. (4) if $\alpha$ and $\beta$ are real, but if, for example, any qubit suffers ‘$E_{\pi}$’ error with $\alpha = i$, $\beta = 0$; $|0\rangle \rightarrow i|0\rangle$, $|1\rangle \rightarrow -i|1\rangle$ then $a|000\rangle + b|111\rangle$ is transformed as $a i|000\rangle - b i|111\rangle$ and it could not be distinguished from case without error, but with initial coefficients $a' = ia$, $b' = -ib$. Because only pairs with same phase multiplier are physically equal, $(a, b) \simeq (\phi a, \phi b)$, the $(a', b')$ correspond to other state if $a \neq 0$ and $b \neq 0$.

Generally, quantum error correction suggests also entanglement of qubit with environment [2].

The extension of error correction method discussed in beginning of Sec.2 is following. Let the $|w_l\rangle$ is codeword for $|l\rangle$, $E_p$ is some error operator and $|0_A\rangle$, $|A_p\rangle$ are initial and final states of ancillas. Then unitary error correction operator $U_{ec}$ acts as:

\[
U_{ec}((E_p|w_l\rangle)|0_A\rangle) = |w_l\rangle|A_p\rangle
\]  

(5)

To show, that the operator $U_{ec}$ corrects a linear combination of errors, let us consider a state $|W\rangle = \sum_l c_l|w_l\rangle$ and error operator $E = \sum_p e_p E_p$:

\[
U_{ec}((E|W\rangle)|0_A\rangle) = \sum_{p,l} c_l e_p (U_{ec}E_p|w_l\rangle)|0_A\rangle) = \sum_{p,l} c_l e_p|w_l\rangle|A_p\rangle = \sum_l c_l|w_l\rangle \sum_p e_p|A_p\rangle
\]  

(6)

The same expression Eq. (6) is valid for entanglement with environment if to consider instead of complex numbers $e_p$ operators $e_p$ those act on environment term in product $|Env\rangle \otimes |W\rangle$, i.e. $E = \sum_p e_p \otimes E_p$.

To write instead of Eq. (3) two standard (see [3, 4, 5]) conditions, let us note, the unitary operator $U_{ec}$ does not change scalar products of any two vectors, i.e.:

\[
\langle w_{l_1}|E_{p_1}^\dagger E_{p_2}|w_{l_2}\rangle = \langle w_{l_1} | w_{l_2} \rangle \langle A_{p_1} | A_{p_2} \rangle
\]  

(7)

4
where expression includes also case with $E$ is no-error, i.e. identity operator and $E_p^\dagger$ is Hermitian conjugation, $(E_p w, u) = \langle w, E_p^\dagger u \rangle$. Two different cases: $l_1 \neq l_2$ and $l_1 = l_2$ produce two sets of equations:

$$\langle w_{l_1} | E_{p_1}^\dagger E_{p_2} | w_{l_2} \rangle = 0 \quad (l_1 \neq l_2) \tag{8}$$
$$\langle w_m | E_{p_1}^\dagger E_{p_2} | w_m \rangle = \langle w_n | E_{p_1}^\dagger E_{p_2} | w_n \rangle \tag{9}$$

With using Eqs. (8, 9) it is possible to show, why non-entangled code like Eq. (2) does not work with $SU(2)$ error. For one qubit phase error $(E_\pi)$ discussed earlier $\langle 0 | E_\pi | 0 \rangle = i$, $\langle 1 | E_\pi | 1 \rangle = -i$ does not compatible with Eq. (1) and the wrong multiplier breaks simple ‘product’ code Eq. (2).

On the other hand, the same Eq. (1) shows that for effective qubit error like Eq. (1) we have $\langle l | E_p | l \rangle = 0$ and code would work. It should be mentioned, the discussed effectiveness condition is sufficient, but not necessary for error correction code.

## 4 Quaternionic qubits

### 4.1 Preliminaries

The *quaternions*, $\mathbb{H} \equiv \mathbb{C}_4$ (real) algebra with basis $i, j, k$, and $1$ (unit), $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $ji = -kj = i$, $ki = -ik = j$. It is algebra with multiplicative norm, like complex numbers, i.e. for $q = q_01 + q_1i + q_2j + q_3k$ norm is Euclidean length of $q$: $|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$ and $|q| |h| = |q\ h|$.

Quaternionic conjugation is introduced as $\bar{q} \equiv q_01 - q_1i - q_2j - q_3k$, with properties: $q\bar{q} = |q|^2$, $\bar{q}\bar{u} = \bar{u}\bar{q}$, $q^{-1} = \bar{q}/|q|^2$. Quaternions may be used for representation of 3D rotations; if $v$ is pure imaginary quaternion, i.e. $v_0 = 0$ (or $\bar{v} = -v$) any rotation of vector $v$ can be represented as:

$$v' = q v q^{-1}; \quad \text{or simply } v' = q v q. \quad |q| = 1 \tag{10}$$

Qubit $a|0\rangle + b|1\rangle$ ($a = a_x + i a_y$, $b = b_x + i b_y$) can be considered as element of 2D complex or 4D real vector space and can be expressed as quaternion $q = a_01 + a_yi + b_j + b_yk$ or simply $q = a + bj$. Here ‘usual’ complex $i$ is equivalent with *left* multiplication on $i$, for example: $\exp(\varphi i)q = e^{i\varphi}a + e^{i\varphi}b$.

---

3 *I.e* different $E_p$ map same vector in orthogonal linear spaces
With the notation, physically equivalent states could be described as:

$$q \simeq e^{i\varphi} q$$  \hfill (11)

Where $\varphi$ is a real number and $|q| = 1$. The action of $SU(2)$ group in Eq. (4) is expressed via right multiplication. To show it, let us consider $u = c + dj$, $\tilde{u} = c - dj$ and so:

$$qu = (a + bj)(c + di) = ac - bd + (ad + bc)j$$
$$q\tilde{u} = (a + bj)(c - di) = ac + bd + (-ad + bc)j$$

or

$$q\tilde{u} \equiv \begin{pmatrix} \overline{c} & -\overline{d} \\ -d & \overline{c} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad (|u| = |c|^2 + |d|^2 = 1) \hfill (12)$$

It is equivalent to Eq. (4) with $\alpha = \overline{c}$, $\beta = d$.

Action of usual Pauli matrices in the notation is expressed as:

$$\sigma_x(q) = i q i, \sigma_y(q) = i q j, \sigma_z(q) = i q k$$

and arbitrary complex matrix can be expressed via:

$$M(q) = qu + iqw$$  \hfill (13)

It corresponds to isomorphism of algebra $\mathbb{C} \otimes \mathbb{H}$ with algebra of all $2 \times 2$ complex matrices.

### 4.2 Qubit and Hopf fibration

Here is only introduced quaternionic notation for usual ($\mathbb{C}^2$) qubit. A quaternionic qubit will be discussed later. Now let us use the notation for some simplification of description of usual qubit.

The qubit is example of mathematical object known as Hopf fibration:

$$SU(2) \xrightarrow{U(1)} S^2 \quad \text{or} \quad S^3 \xrightarrow{\mathbb{S}^1} S^2$$

Here normalized 2D complex vectors correspond to quaternions with unit length $|q| = 1$ and the subspace is isomorphic with $S^3$ (sphere in 4D) or with $SU(2)$ (see Eq. (12)). The quotient of the space on equivalence relation Eq. (11) is $SU(2)/U(1) \cong S^2$ i.e. sphere in 3D.
A qubit $|q\rangle$ maps to a sphere by projections $S^3 \mapsto S^2$ or $\mathbb{C}^2 \mapsto S^2$:

$$q \mapsto v = \bar{q} i q, \ |q| = 1; \ (\text{or } q \mapsto v = q^{-1} i q, \ |q| \neq 0) \quad (14)$$

with properties:

$$e^{\varphi i} q \mapsto v' = e^{\varphi i} q i e^{\varphi i} q = \bar{q} e^{-\varphi i} i e^{\varphi i} q = \bar{q} i q = v \quad (15)$$

$$q u \mapsto v' = \bar{q} i q u = u ^\dagger q i q u = uvu \quad (16)$$

The Eq. (15) shows that map Eq. (14) does not depend on phase and meets Eq. (11). The Eqs. (10, 16) show that unitary operation Eq. (12) corresponds to rotation of sphere $S^2$.

The map Eq. (14) also can be considered as stereographic projection of complex projective plane $\mathbb{C}P \cong \mathbb{R}^2 + \{\infty\}$ to sphere $S^2$, if qubit $a|0\rangle + b|1\rangle$ is represented as element $a/b$ of $\mathbb{C}P$.

The description of qubit as Hopf fibration $S^3/S^1 \to S^2$ here devoted to following problem. We have two manifolds: $S^3$ as space of normalized wave vectors $|\psi\rangle = 1$ and $S^2$ as physical space of states produced by phase equivalence relation like Eq. (11). The relation describes points of some big circle on the sphere $S^3$ and the circle maps to one point on sphere $S^2$. The sphere $S^2$ forms base of Hopf fibration, the big circle on $S^3$ ‘over’ a point of base is fiber and whole $S^3$ is total space.

The problem is: the physical space of states like Bloch sphere for spin system corresponds to base and so is described by quite nonlinear way. We introduce linearity of states in space $\mathbb{C}^2$ that related with physical states by surjections: $\mathbb{C}^2 \mapsto S^3 \mapsto S^2$. The Hopf fibration let us manage with the last projection $S^3 \mapsto S^2$, or $S^3/S^1 \cong S^2$ as with a standard mathematical object.

The Hopf fibration is simplest example of nontrivial fiber bundle i.e. the total space $S^3$ does not equivalent to direct product of base and fiber $S^2 \times S^1$. Physically it is related with following problem — we may not consider normalized wave vector for qubit $|\psi\rangle = a|0\rangle + b|1\rangle$, $|a|^2 + |b|^2 = 1$ simply as some pair $(\phi, s)$ with $\phi \in S^1$ – phase and $s \in S^2$ – phase-independent description of qubit state, for example point on Bloch sphere.

The other property of nontrivial bundle is absence of continuous inverse map from points of base ($S^2$) to fiber (big circle $S^1$) over given point, i.e. we cannot consider space of physically different states $S^2$ as some continuous subset of space $S^3$ of complex 2-vectors with unit norm.

---

4 "Nonlinear" means, the space $S^2$ does not accept some additive structure.
The property of qubit as Hopf fibration often makes rigorous mathematical consideration of different constructions with qubit rather difficult.

### 4.3 H-qubits and $SU(2)$ error correction

The *quaternionic qubit* ($\mathbb{H}$-qubit) is introduced here as 1D quaternionic space, i.e., 4D real space with *Euclidean norm* and action of group $SU(2)$ via right quaternionic multiplication.

$\mathbb{H}$-qubit can be considered as a physical system with state space isomorphic to $S^3$ rather than $S^2 \cong S^3/S^1$. The idea could be regarded as some allusion with *quaternionic quantum mechanics*, but further in Sec.5 will be described an application of the model to usual quantum mechanics by embedding quaternions as 4D real subspace in 4D complex space of Dirac spinors.

Let us denote basis of the space $\mathbb{H}$ as:

\[
1 \rightarrow |0\rangle, \ i \rightarrow |0\rangle, \ j \rightarrow |1\rangle, \ \ell \rightarrow |1\rangle
\]

or

\[
1 \rightarrow |00\rangle, \ i \rightarrow |01\rangle, \ j \rightarrow |10\rangle, \ \ell \rightarrow |11\rangle
\]

The two notations emphasize that $\mathbb{H}$-qubit extends 1-qubit system, but can be included in 2-qubit space:

\[
\mathbb{C}P \hookrightarrow S^3 \hookrightarrow \mathbb{C}P^3 \\
\mathbb{C}^2 \cong \mathbb{H} \subset \mathbb{C}^4
\]

The $n$-$\mathbb{H}$-qubit space is introduced as $4^n$ real space of tensor product:

\[
\mathbb{H}^{\otimes n} = \mathbb{H} \otimes \mathbb{R} \mathbb{H} \otimes \mathbb{R} \cdots \otimes \mathbb{R} \mathbb{H}
\]

Let us consider 3-$\mathbb{H}$-qubit error correction code \{ |000\>, |111\> \}. The 1-$\mathbb{H}$-qubit $SU(2)$ errors act via right multiplication like in Eq. (12). The examples of error in first $\mathbb{H}$-qubit are shown in next tables with two different notations:

| code-word | $\times i$ | $\times j$ | $\times \ell$ |
|-----------|-----------|-----------|-----------|
| |000\>   | |100\>   | |000\>   | |100\>   |
| |111\>   | |011\>   | |111\>   | |011\>   |

(17)
or

\[
\begin{array}{c|c|c|c}
\text{codeword} & \times i & \times j & \times k \\
\hline
|00000\rangle & |10000\rangle & |01000\rangle & |11000\rangle \\
|10101\rangle & -|00100\rangle & -|11000\rangle & |01101\rangle \\
\end{array}
\] (18)

The SU(2) errors acts effectively on one qubit, general error correction conditions Eqs. (8, 9) are satisfied (in Euclidean norm) for the 3-\(n\)-qubit code and after appending few ancilla \(n\)-qubits any such errors can be corrected via orthogonal error correction operator.

5 Example with Dirac, Majorana spinors

The quaternionic qubit can be considered as real subspace of 4D complex vector space. Here it is described as real subspace of 4-components Dirac spinor.

Dirac equation in system of unit \(\hbar = 1, c = 1\) is [6]:

\[
(i\gamma_0 \frac{\partial}{\partial t} + i\gamma_1 \frac{\partial}{\partial x} + i\gamma_2 \frac{\partial}{\partial y} + i\gamma_3 \frac{\partial}{\partial z} - m)\Psi = 0 \tag{19}
\]

Here \(\Psi\) is 4-components complex function and \(\gamma_i\) are \(4 \times 4\) complex Dirac matrices are expressed via \(2 \times 2\) Pauli matrices as:

\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \\
\gamma_1 = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} ; \\
\gamma_2 = \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix} ; \\
\gamma_3 = \begin{pmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{pmatrix} 
\] (20)

(here 0 and 1 are \(2 \times 2\) matrices).

It is also useful to introduce \(4 \times 4\) \(\gamma_5\) matrix:

\[
\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} 
\] (21)

The five gamma matrices have following algebraic properties:

\[
\gamma_i\gamma_j = -\gamma_j\gamma_i \, (i \neq j); \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1; \quad \gamma_0^2 = \gamma_5^2 = 1 \tag{22}
\]
The Eq. (22) do not depend on basis. If we choose other representation with \( \Psi' = U \Psi \) for some unitary operator \( U \), in the new basis \( \gamma \) matrices may have other numerical form instead of Eqs. (20, 21):

\[
\gamma'_i = U \gamma_i U^{-1} = U \gamma_i U^\dagger
\]

but relations Eq. (22) do not change.

Now it is necessary to find transformations of 4D spinor \( \Psi \) those correspond to \( SU(2) \) errors of nonrelativistic 2D Pauli spinor.

As a good candidate here is considered transformations:

\[
e_0 I + e_1 \gamma_2 \gamma_3 + e_2 \gamma_3 \gamma_1 + e_3 \gamma_1 \gamma_2
\]

where \( e_k \) are real numbers, \( I \) is a 4 × 4 matrix unit and in usual spinor notation Eq. (20) other three matrices are represented as:

\[
\gamma_2 \gamma_3 = i \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}; \quad \gamma_3 \gamma_1 = i \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}; \quad \gamma_1 \gamma_2 = i \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}
\]

The representation meets with correspondence principle; 4-spinor can be expressed as two 2-spinors: \( \Psi = (\xi \eta) \). In so called standard representation of Dirac equation are used other two 2-vectors \( \varphi = (\xi + \eta)/\sqrt{2} \) and \( \chi = (\xi - \eta)/\sqrt{2} \). Then for rest particle \( \chi = 0 \), i.e. in nonrelativistic limit: \( v \ll c, \chi \approx 0 \) can be omitted and we can work with one Pauli spinor \( \varphi \).

So the matrices Eq. (24) are only appropriate, because they do not break condition \( \chi = (\xi - \eta)/\sqrt{2} = 0 \).

The usual Dirac equation Eq. (19) cannot be considered as equation for 4D-real vector \( \Psi \). To make the equation real it is necessary to find representation with all matrices \( i\gamma_k \), \( k = 0, \ldots, 3 \) are real by some transformation like Eq. (23). Such form of Dirac equation is called by name of Italian physicist E. Majorana after his work at 1937.

There are many different real representations related via Eq. (23) with orthogonal matrices \( U \in SO(4) \subset SU(4) \) and here is used most convenient for particular purpose. An unitary transformation ‘swaps’ \( \gamma_3 \leftrightarrow i\gamma_0, \gamma_1 \leftrightarrow i\gamma_5 \) (the ‘\( \gamma_2 \)’ term is used to change signs of all matrices except \( \gamma_2 \)):

\[
U_M = \gamma_2 (\gamma_5 + i\gamma_1)(\gamma_0 + i\gamma_3)/2 = \frac{i+1}{2} \begin{pmatrix} 0 & 1 & i & 0 \\ -i & 0 & 0 & 1 \\ -1 & 0 & 0 & i \\ 0 & -i & 0 & 0 \end{pmatrix}
\]

\[\text{They introduce Dirac algebra as some abstract object — 4D Clifford algebra}\]
\[ \gamma'_0 = -i\gamma_3; \quad \gamma'_1 = i\gamma_5; \quad \gamma'_2 = \gamma_2; \quad \gamma'_3 = i\gamma_0; \quad \text{(and} \quad \gamma'_5 = -i\gamma_1) \]  

(26)

In the new basis expressions for errors Eq. (24) include only matrices with all 16 elements are real:

\[ \gamma'_2 \gamma'_3 = \begin{pmatrix} -i\sigma_y & 0 \\ 0 & i\sigma_y \end{pmatrix}; \quad \gamma'_3 \gamma'_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad \gamma'_1 \gamma'_2 = \begin{pmatrix} 0 & i\sigma_y \\ i\sigma_y & 0 \end{pmatrix} \]  

(27)

or

\[ \gamma'_2 \gamma'_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \gamma'_3 \gamma'_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & +1 & 0 & 0 \end{pmatrix}; \quad \gamma'_1 \gamma'_2 = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]  

(28)

those correspond to right multiplication on quaternion units represented as 4 × 4 real matrices, i.e. for \( q = q_0 + q_1i + q_2j + q_3k \):

\[ \begin{align*}
q \cdot i &= q_0 + q_3i - q_2k \equiv -E^M_1|\Psi_q\rangle \\
q \cdot j &= q_2 - q_3i + q_0j + q_1k \equiv -E^M_2|\Psi_q\rangle \\
q \cdot k &= q_3 + q_2i - q_1j + q_0k \equiv -E^M_3|\Psi_q\rangle
\end{align*} \]  

(29)

where \( E^M_1 = -\gamma'_2\gamma'_3, \quad E^M_2 = -\gamma'_3\gamma'_1, \quad E^M_3 = E^M_1 E^M_2 = \gamma'_1\gamma'_2 \). Finally:

\[ (e_0 + e_1E^M_1 + e_2E^M_2 + e_3E^M_3)|\Psi_q\rangle \equiv q (e_0 - e_1i - e_2j - e_3k) \equiv q \tilde{e} \]  

(30)

Because the existence of the ‘true neutral’ Majorana particles are not proved yet, it is useful to consider relation of the formulae with usual complex Dirac equation.

In the case, after transformation Eq. (24) of 4D complex wave function \( \Psi \) in Dirac equation to new basis, the function:

\[ |\Psi'\rangle = U_M|\Psi\rangle \]

may again be complex, but real and imaginary parts of the vector are transformed separately by errors like Eq. (30). If the function \( \Psi' \) prepared as pure real, it will be real after any such error, i.e. real subspace of the complex vector space is invariant in respect to action of group of errors Eq. (30).

In such a case equation with pure imaginary \( \gamma_k \) Eq. (26) is considered as an equivalent form of Dirac equation in other basis with complex-valued:

\[ |\Psi'\rangle = |\Psi'_1\rangle + i|\Psi'_2\rangle \]
with two ‘independent’ quaternionic parts \( \Psi'_1 \) and \( \Psi'_2 \). It can be also written as:

\[
\mathbb{S}^1 \times \mathbb{S}^3 \subset \mathbb{C} \otimes \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H}
\]

where \( \mathbb{S}^3 \) and \( \mathbb{H} \) are spaces related with \( \mathbb{R} \)-qubits and \( \mathbb{S}^1 \) is phase. The complexification Eq. (31) of \( \mathbb{H} \)-qubit is represented as simple direct product (i.e. trivial bundle) in comparison with nontrivial Hopf fibration of usual \( \mathbb{C}^2 \)-qubit.

6 Conclusion

As not very formal answer to the question, why 3 is enough for classical case, but is not enough in quantum one, may be used suggestion that 2D space \( \mathbb{S}^2 \) of one qubit is ‘too small’ in comparison with 3D space of all possible errors. In classical case we have only one possible error — flip of a bit. In case with \( \mathbb{R} \)-qubit it is 1D space (\( \mathbb{S}^1 \)) with 1D space of errors and in \( \mathbb{H} \) case it is 3D space (\( \mathbb{S}^3 \)) with 3D space of errors.

The idea is also related with initial Shor’s 9-qubit code \( \mathbb{H} \), because the code can be considered as two-steps process: first, we preencode qubit to 3 qubits:

\[
\begin{align*}
|0\rangle & \rightarrow |B_3^+\rangle = (|000\rangle + |111\rangle)/\sqrt{2}, \\
|1\rangle & \rightarrow |B_3^-\rangle = (|000\rangle - |111\rangle)/\sqrt{2}
\end{align*}
\]

The code \( B_3^\pm \) belongs to 8D space, has 7 different kinds of errors and second step is repeating:

\[
|0\rangle \rightarrow |B_3^+B_3^+B_3^+\rangle, \quad |1\rangle \rightarrow |B_3^-B_3^-B_3^-\rangle
\]

Then the last example with Dirac spinors can be considered as an analogue of Shor idea, but with preencoding due to additional physical degrees of freedom of relativistic particle in some subspace invariant with respect to 3D group of ‘slack’, nonrelativistic errors. Here is only question, do the preencoding and error correction operator \( U_{ec} \) physically possible — they are unitary, but not all unitary operation with relativistic particles would be performed under realistic conditions.

\(^6\)Here 7 = 3 \times 3 − 2 (all three phase errors with different qubits act in the same way)
References

[1] D. G. Cory, W. Mass, M. Price, R. Laflamme, W. H. Zurek, T. F. Havel and S. S. Somaroo, “Experimental Quantum Error Correction,” Phys. Rev. Lett. 81 (1998), 2152–2155, quant-ph/9802018.

[2] P. W. Shor, “Scheme for reducing decoherence in quantum computer memory,” Phys. Rev. A 52 (1995), R2493–2496.

[3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, “Mixed State Entanglement and Quantum Error Correction,” Phys. Rev. A 54, (1996) 3824–3851, quant-ph/9604024

[4] A. R. Calderbank, E. M. Rains, P. W. Shor, N. J. A. Sloane, “Quantum Error Correction and Orthogonal Geometry,” Phys. Rev. Lett. 78 (1997), 405–408, quant-ph/9605005.

[5] E. Knill, R. Laflamme, “A Theory of Quantum Error-Correcting Codes,” quant-ph/9604034.

[6] Landau and Lifschitz, Course of Theoretical Physics Vol. IV (Quantum Electrodynamics), Moscow, Nauka, 1988