Extremal convex bodies for affine measures of symmetry

by

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Abstract. This paper is devoted to measures of symmetry based on the distance from the centroid to one of the centers of the John and the Löwner ellipsoid. The accuracy of the derived upper bounds for the relevant measures of symmetry is proven.

1. Introduction. By a convex body or a simply body we mean a compact convex subset of $\mathbb{R}^n$ with nonempty interior. We denote by $\mathcal{K}_n$ the set of all convex bodies in $\mathbb{R}^n$. Following [8] we call a map $p : \mathcal{K}_n \rightarrow \mathbb{R}^n$ an affine invariant point if it is continuous in the Hausdorff metric and if for any nonsingular affine map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ one has

$$p(T(K)) = T(p(K)).$$

In [3] B. Grünbaum introduced measures of symmetry as functions on $\mathcal{K}_n$ with values from 0 to 1 satisfying some additional conditions. In order to identify the convex bodies with a center of symmetry he required the measures of symmetry to take the value 1 exactly on this class of convex bodies.

M. Meyer, C. Schütt and E. M. Werner [7] defined another class of such measures. Given two affine invariant points $p_1, p_2$ such that $p_1(K), p_2(K) \in K$ for all $K \in \mathcal{K}_n$, put

$$d(p_1(K), p_2(K)) = \begin{cases} 0 & \text{if } p_1(K) = p_2(K) \\ \|p_1(K) - p_2(K)\| & \text{if } p_1(K) \neq p_2(K), \end{cases}$$

where $\ell$ is the line through $p_1(K)$ and $p_2(K)$. The corresponding measure of symmetry is defined on $\mathcal{K}_n$ by

$$K \mapsto \varphi_{p_1, p_2}(K) = 1 - d(p_1(K), p_2(K)).$$

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The quantity $d(p_1(K), p_2(K))$ is affine invariant, continuous in the Hausdorff metric and takes values from 0 to 1. The difference from Grünbaum’s definition is that such measures of symmetry are 1 on convex bodies with a unique fixed point under the group of symmetries.

The authors of [7] consider the case of the dual classic affine invariant points such as the centroid $g$ and the Santaló point $s$, and the center $j$ of the John ellipsoid and the center $l$ of the Löwner ellipsoid (for simplicity, below we call them the John point and the Löwner point, respectively). They provided the estimate

$$d(g(K), s(K)) \leq 1 - \frac{2}{n + 1}$$

(see [7], and also [10] for a small correction). The main result of [7] is that both $\max_{K \in \mathcal{K}_n} \varphi_{g,s}(K)$ and $\max_{K \in \mathcal{K}_n} \varphi_{j,l}(K)$ can be bounded from below by positive absolute constants.

Developing the idea of P. Kuchment [6], O. Mordhorst [10] has proved that in any dimension $n \geq 2$ there are some affine invariant points $p_1, p_2$ and $K \in \mathcal{K}_n$ such that $\phi_{p_1,p_2}(K) = 0$. Furthermore, in [10] the same upper bound for the John and Löwner points is provided by arguments similar to the ones in [7]. Modifying the construction of [7] Mordhorst proved the asymptotic sharpness of this bound.

This article is devoted to the dependence on $n$ of

$$\max_{K \in \mathcal{K}_n} d(p_1(K), p_2(K))$$

for pairs of some classical affine invariant points such as the centroid, and the John and Löwner points. Again by similar arguments to [7] and [10] one can show

(1.1) $d(g(K), j(K)) \leq 1 - \frac{2}{n + 1}$,

(1.2) $d(g(K), l(K)) \leq 1 - \frac{2}{n + 1}$

for $K \in \mathcal{K}_n$.

In Section 6 we give a construction of convex bodies $F_n, W_n \in \mathcal{K}_n$ for which estimates (1.1) and (1.2) have the asymptotically optimal order of magnitude:

$$d(g(F_n), j(F_n)) = 1 - \frac{1}{n} C^* + o\left(\frac{1}{n}\right)$$

and

$$d(g(W_n), l(W_n)) = 1 - \frac{1}{n} C^{**} + o\left(\frac{1}{n}\right)$$

where $C^* \approx 13$ and $C^{**} \approx 20$. 
In order to show the universality of the given approach we construct convex bodies $M_n \in \mathcal{K}_n$ such that
\[ d(j(M_n), l(M_n)) = 1 - \frac{8}{n} + o\left(\frac{1}{n}\right). \]

We prove that the following inequalities are sharp up to an error of order $1/n$:
\[ \frac{2}{n + 1} \leq \varphi_{g,j} \leq 1, \quad \frac{2}{n + 1} \leq \varphi_{g,l} \leq 1, \quad \frac{2}{n + 1} \leq \varphi_{j,l} \leq 1. \]

2. Definitions and auxiliary results. For any $x \in \mathbb{R}^n$ and for $1 \leq k \leq n$ we denote by $x^k$ the $k$th coordinate, the scalar product $\mathbb{R}^n$ by $\langle x, y \rangle = \sum_{k=1}^{n} x^k y^k$ and the corresponding euclidean norm by $\|x\| = \sqrt{\langle x, x \rangle}$. For convex bodies $K$ and $L$ in $\mathbb{R}^n$ containing the origin in their interior and some real number $c > 0$ we introduce the $(n+1)$-dimensional convex body by
\[ \text{conv}((K,0), (L,c)) = \{ \lambda (x,0) + (1-\lambda) (y,c) : x \in K, y \in L, 0 \leq \lambda \leq 1 \}. \]

Fix $t \in \mathbb{R}$, a unit vector $\xi \in S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$ and a convex body $K \in \mathcal{K}_n$. The $(n-1)$-dimensional section of $K$, orthogonal to the direction $\xi$ and passing through the point $t\xi$, is
\[ K(\xi,t) = \{ x \in K : \langle x, \xi \rangle = t \}. \]

We are interested in cross sections orthogonal to the direction $e_n = (0, \ldots, 0, 1)$. We denote them by $K(t) = K(e_n, t)$ and identify them with an $(n-1)$-dimensional convex body in $\mathbb{R}^{n-1}$.

We denote by $B_1^n = \{ x \in \mathbb{R}^n : \sum_{k=1}^{n} |x|^k \leq 1 \}$ the cross-polytope, by $B_2^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$ the euclidean ball and by $B_{\infty}^n = \{ x \in \mathbb{R}^n : \max_k |x|^k \leq 1 \}$ the cube. We write $\Delta_n$ for a fixed regular simplex with nonempty interior inscribed into $B_2^n$. The $n$-dimensional Lebesgue measure is $\text{vol}_n$. The boundary of a convex body $K$ is denoted by $\partial K$.

It is well-known (see e.g. [2, 16]) that for any convex body $K \in \mathcal{K}_n$ there exists a unique inscribed ellipsoid $J(K)$ of maximal volume and a unique circumscribed ellipsoid $L(K)$ of minimal volume. These ellipsoids are solutions of the corresponding extremal problems
\[ \text{vol}_n(J(K)) = \max_{\mathcal{E} \subset K} \text{vol}_n(\mathcal{E}), \]
\[ \text{vol}_n(L(K)) = \min_{K \subset \mathcal{E}} \text{vol}_n(\mathcal{E}), \]
where $\mathcal{E}$ is an ellipsoid. We say that a convex body is in John’s position if the ball $B_2^n$ is the John ellipsoid. The centers of the John and Löwner ellipsoids of $K$ are denoted by $j(K)$ and $l(K)$, respectively. They are affine invariant points.
We need the next lemma in order to compute $j$ and $l$ for preliminary convex bodies considered in Sections 4 and 5.

**Lemma 2.1 (M. Meyer, C. Schütt, E. M. Werner [7]).** If the John (respectively, Löwner) ellipsoid of a convex body $K$ is $\alpha B_n^2$ for some $\alpha > 0$, then, for every $s, t \geq 0$, the John (respectively, Löwner) ellipsoid of $sK + tB_n^2$ is $(s\alpha + t)B_n^2$.

Other examples of affine invariant points are the centroid $g(K)$ and the Santaló point $s(K)$ of $K \in K_n$ [8, 9] given by

$$g(K) = \frac{1}{\text{vol}_n(K)} \int_K x \, dx, \quad \text{vol}_n(K^{s(K)}) = \min_{x \in K} \text{vol}_n(K^x),$$

where $K^x$ is the polar of $K$ with respect to the point $x$,

$$K^x = \{ y \in \mathbb{R}^n : \langle z - x, y \rangle \leq 1 \text{ for all } z \in K \}.$$  

From Minkowski’s theorem (see e.g. [14, §4.2] or [4, Chapter 6, §1]) it follows that for all $K, L \in K_n$ and $\lambda, \mu \geq 0$ the volume of the body $\lambda K + \mu L$ is a polynomial of degree $n$ in $\lambda$ and $\mu$, i.e.

$$\text{vol}_n(\lambda K + \mu L) = \sum_{k=0}^n C_n^k \lambda^{n-k} \mu^k V_{n-k,k}(K, L).$$

The coefficients $V_{n-k,k}(K, L)$ are called *mixed volumes*. They are homogeneous of degree $n - k$ and $k$ in $K$ and $L$ respectively.

In order to calculate centroids of the preliminary convex bodies considered in Section 4 we need the following lemma.

**Lemma 2.2 (M. Meyer, C. Schütt, E. M. Werner [7]).** Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$, $c > 0$ and $P_n = \text{conv}((K, 0), (L, c)) \subset \mathbb{R}^{n+1}$. Then the $(n + 1)$th coordinate $g^{n+1}(P_n)$ of the centroid $g(P_n)$ of $P_n$ satisfies

$$g^{n+1}(P_n) = \frac{c}{n + 2} \frac{\sum_{k=0}^n (k + 1)V_{n-k,k}(K, L)}{\sum_{k=0}^n V_{n-k,k}(K, L)}.$$  

Also we need the mixed volumes $V_{n-k,k}(B_1^n, B_\infty^n)$ and $V_{n-k,k}(B_2^n, B_\infty^n)$. They are easily calculated by the next lemma.

**Lemma 2.3 (A. Pajor [11, Theorem 1.10], [12, Theorem 6]).** Let $K$ be a convex body in $\mathbb{R}^n$. Then, for $0 \leq k \leq n$,

$$V_{n-k,k}(K, B_\infty^n) = \frac{2^k}{(n)} \left( \sum_{I \subset \{1, \ldots, n\} \atop |I| = n-k} \text{vol}_{n-k}(P^I K) \right),$$

where for a subset $I \subset \{1, \ldots, n\}$ with $|I| = n - k$ we denote by $P^I$ the projection from $\mathbb{R}^n$ into the $(n - k)$-dimensional subspace

$$\mathbb{R}^I = \{ x \in \mathbb{R}^n : x^i = 0 \text{ for } i \notin I \}.$$
Note that \( P_I B_n^1 = B_I^1 \cong B_1^{n-k} \) and \( P_I B_n^2 = B_I^2 \cong B_2^{n-k} \) for all subsets \( I \subset \{1, \ldots, n\} \) with \( |I| = n - k \). Therefore
\[
V_{n-k,k}(B_n^1, B_\infty^1) = 2^k \text{vol}_{n-k}(B_1^{n-k}), \quad V_{n-k,k}(B_n^2, B_\infty^2) = 2^k \text{vol}_{n-k}(B_2^{n-k}).
\]

By homogeneity of the mixed volumes \( V_{n-k,k} \) we have
\[
V_{n-k,k}(\lambda B_n^1, \mu B_\infty^1) = (2\mu)^k \lambda^{n-k} \text{vol}_{n-k}(B_1^{n-k}), \quad V_{n-k,k}(\lambda B_n^2, \mu B_\infty^2) = (2\mu)^k \lambda^{n-k} \text{vol}_{n-k}(B_2^{n-k})
\]
for any \( \lambda, \mu > 0 \).

We shall use the notation \( a \asymp b \) when there exist \( c_1, c_2 > 0 \) such that
\[
c_1 a \leq b \leq c_2 a.
\]

It is well-known that
\[
\text{vol}_n(B_n^1) = \frac{2^n}{\Gamma(1 + n)} = \frac{2^n}{n!} \asymp \left(\frac{2e}{n}\right)^n \frac{1}{\sqrt{n}},
\]
\[
\text{vol}_n(B_n^2) = \frac{\pi^{n/2}}{\Gamma(1 + n/2)} \asymp \left(\frac{2e\pi}{n}\right)^{n/2} \frac{1}{n^{(n+1)/2}}, \quad \text{where } n \geq 1.
\]

3. Affine invariant points of cartesian products of convex bodies.

In order to prove Lemma 3.2 we need John’s theorem (see e.g. [1]).

**Theorem 3.1 (F. John [5]).** Let \( K \subset \mathbb{R}^n \) be a convex body. The ball \( B_n^2 \) is the John ellipsoid if and only if \( B_n^2 \subset K \) and (for some \( k \)) there are vectors \( \{u_i\}_{i=1}^k \subset \partial K \cap \partial B_n^2 \) and positive numbers \( \{c_i\}_{i=1}^k \) which satisfy
\[
\sum_{i=1}^k c_i u_i = 0 \quad \text{and} \quad \sum_{i=1}^k c_i \langle x, u_i \rangle^2 = \|x\|^2 \quad \text{for all } x \in \mathbb{R}^n.
\]

The ball \( B_n^2 \) is the Löwner ellipsoid if and only if \( K \subset B_n^2 \) and (for some \( s \)) there are vectors \( \{v_i\}_{i=1}^s \subset \partial K \cap \partial B_n^2 \) and positive numbers \( \{b_i\}_{i=1}^s \) which satisfy the conditions
\[
\sum_{i=1}^s b_i v_i = 0 \quad \text{and} \quad \sum_{i=1}^s b_i \langle x, v_i \rangle^2 = \|x\|^2 \quad \text{for all } x \in \mathbb{R}^n.
\]

It is a well-known consequence of John’s theorem (see e.g. [1]) that the positive numbers \( b_i \) and \( c_i \) satisfy
\[
\sum_{i=1}^s b_i = \sum_{i=1}^k c_i = n.
\]

**Lemma 3.2.** Let \( D \subset \mathbb{R}^n \) and \( K \subset \mathbb{R}^m \) be convex bodies. Then the centroid, and the John, Löwner and Santaló points of the convex body
\(D \times K \subset \mathbb{R}^n \times \mathbb{R}^m\) satisfy the identities
\[
g(D \times K) = (g(D), g(K)), \quad j(D \times K) = (j(D), j(K)),
\]
\[
l(D \times K) = (l(D), l(K)), \quad s(D \times K) = (s(D), s(K)).
\]

**Proof.** The identity for the centroid follows from Fubini’s theorem.
We prove the statement for the John point of \(D \times K\).
Suppose that the bodies \(D\) and \(K\) are in John’s position. We consider for \(D\) and \(K\) the corresponding vectors \(\{u_i\}_{i=1}^k \subset \partial D \cap \partial B_2^n\) and \(\{w_j\}_{j=1}^s \subset \partial K \cap \partial B_2^m\) and positive numbers \(\{c_i\}_{i=1}^k\) and \(\{d_j\}_{j=1}^s\) satisfying the conditions of Theorem 3.1. Then it follows from the definition that the sets of vectors \(\{(u_i, 0)\}_{i=1}^k \cup \{(0, w_j)\}_{j=1}^s \subset \partial(D \times K)\) and the positive numbers \(\{c_i\}_{i=1}^k \cup \{d_j\}_{j=1}^s\) satisfy all the conditions of Theorem 3.1 for \(D \times K\). Therefore, \(B_2^{n+m}\) is the John ellipsoid. Thus
\[
j(D \times K) = (0, 0) = (j(D), j(K)).
\]
The general case of the statement follows from affine invariance of the John ellipsoid.
Let us turn to the proof of the statement for the Löwner point of \(D \times K\).
By the same argument as in the statement for the John point it is sufficient to consider the case where the Löwner ellipsoids of \(D\) and \(K\) are the balls \(B_2^n\) and \(B_2^m\) respectively. Then it follows from Theorem 3.1 that there exist vectors \(\{v_i\}_{i=1}^k \subset \partial D \cap \partial B_2^n\) and \(\{z_j\}_{j=1}^s \subset \partial K \cap \partial B_2^m\) and positive numbers \(\{a_i\}_{i=1}^k\) and \(\{b_j\}_{j=1}^s\) for \(D\) and \(K\) respectively. Consider the sets of vectors \(\{(v_i, z_j)\}_{i=1}^k,\{a_i\}_{i=1}^k,\{b_j\}_{j=1}^s\subset \partial D \times \partial K \subset \partial(D \times K)\) and the positive numbers \(\lambda a_i b_j\) \(\{a_i\}_{i=1}^k,\{b_j\}_{j=1}^s\) for \(i, j = 1, \ldots, k, s\), where \(\lambda\) is a positive number which we determine below. So we have
\[
(3.1) \quad \sum_{i=1}^k \sum_{j=1}^s \lambda a_i b_j (v_i, z_j) = \lambda \left( \sum_{j=1}^s b_j \sum_{i=1}^k a_i v_i, \sum_{i=1}^k a_i \sum_{j=1}^s b_j z_j \right) = 0
\]
and
\[
(3.2) \quad \sum_{i=1}^k \sum_{j=1}^s \lambda a_i b_j ((x, y), (v_i, z_j))^2
\]
\[
= \lambda \sum_{j=1}^s b_j \|x\|^2 + \lambda \sum_{i=1}^k a_i \|y\|^2 = \lambda (m \|x\|^2 + n \|y\|^2)
\]
for all \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\).
Define the \((n + m)\)-dimensional ellipsoid
\[
\mathcal{E} = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \lambda \|x\|^2 m + \lambda \|y\|^2 n \leq 1 \},
\]
where $\lambda$ is chosen so that $\partial B_2^n \times \partial B_2^m \subset \partial E$. In other words,

$$\lambda := \frac{1}{m + n}.$$ 

We note that $E$ is the affine image of $B_2^{n+m}$ of the diagonal matrix $T$ defined by

$$T(x, y) = \left( \frac{x}{\sqrt{\lambda m}}, \frac{y}{\sqrt{\lambda n}} \right).$$

From identities (3.1) and (3.2) and Theorem 3.1 it follows that the ball $B_2^{n+m}$ is the Löwner ellipsoid of $T(D \times K)$. Since the Löwner ellipsoid is affine invariant, $E = T^{-1}B_2^{n+m}$ is the Löwner ellipsoid for $D \times K$. Thus, we have

$$l(D \times K) = (0, 0) = (l(D), l(K)).$$

In order to prove the last statement about the Santaló point we need the next well-known fact:

**FACT 1** (see e.g. [13] or [14, Chapter 10 §5]). The interior point $x$ of the convex body $K$ is the Santaló point if and only if $0$ is the centroid of $(K - x)^\circ$.

**FACT 2.** Let $D \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^m$ be convex bodies containing the origin in their interior. Then the polar of their cartesian product, $(D \times K)^\circ$, is

$$\text{conv}((D^\circ, 0), (0, K^\circ)) = \{\lambda(x, 0) + (1 - \lambda)(0, y) : x \in D^\circ, y \in K^\circ\}.$$

It is sufficient to consider the case $s(D) = 0$ and $s(K) = 0$. By Fact 1, we need to prove that from the conditions for centroids $g(D^\circ) = 0$ and $g(K^\circ) = 0$ it follows that $g(\text{conv}((D^\circ, 0), (0, K^\circ))) = 0$. Using Fubini’s theorem we have

$$\text{vol}_{n+m}(\text{conv}((D^\circ, 0), (0, K^\circ))) \cdot g(\text{conv}((D^\circ, 0), (0, K^\circ)))$$

$$= \left( \int_{\text{conv}((D^\circ, 0), (0, K^\circ))} z \, dz \, dw, \int_{\text{conv}((D^\circ, 0), (0, K^\circ))} w \, dz \, dw \right)$$

$$= \left( \int_{K^\circ} \int_{D^\circ} z \, dz \, dw, \int_{D^\circ} \int_{(1-\|w\|_{K^\circ})^{D^\circ}} w \, dz \, dw \right) = (0, 0).$$

4. Constructions of convex bodies for $\varphi_{g,j}$ and $\varphi_{g,l}$. In this section we shall consider preliminary convex bodies $F_n^{(1)}$ and $F_n^{(2)}$ (respectively, $W_n^{(1)}$ and $W_n^{(2)}$) in $\mathbb{R}^{n+1}$ for extremal convex bodies for the measure $\varphi_{g,j}$ (respectively, $\varphi_{g,l}$).

The centroid of $F_n^{(1)}$ (respectively, $W_n^{(1)}$) is “separated” from its boundary with increasing $n$, whereas its John point (respectively, Löwner point) is
“close” to the boundary. The centroid and the John point (respectively, the Löwner point) of \( F_n^{(2)} \) (respectively, \( W_n^{(2)} \)) will have the opposite properties.

The convex bodies above are defined as follows:

\[
F_n^{(1)} = \text{conv}((B_\infty^n, 0), (B_2^n, 1)), \\
F_n^{(2)} = \text{conv}\left(\bigg(\sqrt{\frac{n}{2e\pi}}B_2^n, 0\bigg), \bigg(\frac{1}{2}B_\infty^n, 1\bigg)\right), \\
W_n^{(1)} = \text{conv}\left(\bigg(B_2^n, 0\bigg), \bigg(\frac{1}{\sqrt{n}}B_\infty^n, 1\bigg)\right), \\
W_n^{(2)} = \text{conv}\left(\bigg(\frac{n}{e}B_1^n, 0\bigg), (B_\infty^n, 1)\right).
\]

We note that each of the convex bodies above is invariant with respect to rotations around the axis with the direction vector \( e_{n+1} = (0, \ldots, 0, 1) \). Hence the centroids and the Löwner and John points of \( F_n^{(1)}, F_n^{(2)}, W_n^{(1)} \) and \( W_n^{(2)} \) have the forms

\[
(4.1) \quad g(F_n^{(i)}) = g^{n+1}(F_n^{(i)})e_{n+1} \quad \text{and} \quad j(F_n^{(i)}) = j^{n+1}(F_n^{(i)})e_{n+1}, \\
g(W_n^{(i)}) = g^{n+1}(W_n^{(i)})e_{n+1} \quad \text{and} \quad l(W_n^{(i)}) = l^{n+1}(W_n^{(i)})e_{n+1}.
\]

where \( g^{n+1}, j^{n+1} \) and \( l^{n+1} \) are the corresponding \((n+1)\)th coordinates of \( g, j \) and \( l \) for \( F_n^{(i)} \) and \( W_n^{(i)} \) for \( i = 1, 2 \).

Now we are ready to prove the next four lemmas.

**Lemma 4.1.** Consider \( F_n^{(1)} = \text{conv}((B_\infty^n, 0), (B_2^n, 1)) \). Then

\[
j(F_n^{(1)}) = \frac{1}{2}e_{n+1} \quad \text{and} \quad g(F_n^{(1)}) = g^{n+1}(F_n^{(1)})e_{n+1},
\]

where

\[
g^{n+1}(F_n^{(1)}) = \frac{1}{n}\left(1 + \frac{\pi}{2} + \frac{e^{-\pi/4}}{\text{erf}(\sqrt{\pi}/2) + 1}\right) + o(1/n).
\]

Here \( \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt \) is the error function (see e.g. [15, §16.2]).

**Proof.** By (4.1) it is sufficient to prove the statement for the corresponding \((n+1)\)th coordinates.

We first prove the statement for \( j^{n+1}(F_n^{(1)}) \). Let \( J \) be the John ellipsoid of \( F_n^{(1)} \). Since the John ellipsoid is unique, the ellipsoid \( J \) satisfies

\[
J = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : \frac{\|x\|^2}{a^2} + \frac{(t - j^{n+1}(F_n^{(1)}))^2}{b^2} \leq 1 \right\},
\]

where \( a \) and \( b \) are positive numbers depending on \( n \). For \( t \in [0; 1] \) the section \( J(t) = \{ x \in \mathbb{R}^n : (x, t) \in J \} \) is the \( n \)-dimensional euclidean ball. Since the John ellipsoid for \( B_\infty^n \) is equal to \( B_2^n \), we have the inclusions \( J(0) \subset B_2^n \) and
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\( J(1) \subset B_2^n \). Consequently,

\[ J \subset \text{conv}((B_2^n, 0), (B_2^n, 1)) \subset F_n^{(1)} = \text{conv}((B_\infty^n, 0), (B_2^n, 1)). \]

From maximality of the volume of the John ellipsoid \( J \) it follows that \( J \) is also the John ellipsoid for the cylinder \( \text{conv}((B_2^n, 0), (B_2^n, 1)) \). Since \( \text{conv}((B_2^n, 0), (B_2^n, 1)) \) is centrally symmetric with respect to \((0, \ldots, 0, 1/2)\) this cylinder has only a single affine invariant point. Consequently, this point coincides with the center of the ellipsoid \( J \). Hence \( j_n^{n+1}(F_n^{(1)}) = 1/2 \).

Now we prove the asymptotic formula for \( g_n^{n+1}(F_n^{(1)}) \). We apply Lemma 2.2 to \( F_n^{(1)} \). From (2.2) and (2.4) we get

\[
\begin{align*}
g_n^{n+1}(F_n^{(1)}) &= \frac{1}{n+2} \sum_{k=0}^{n} \frac{(k+1)V_{n-k,k}(B_\infty^n, B_2^n)}{\sum_{k=0}^{n} V_{n-k,k}(B_\infty^n, B_2^n)} \\
&= \frac{1}{n+2} \sum_{k=0}^{n} \frac{(k+1)2^{-k} \text{vol}_k(B_2^n)}{\sum_{k=0}^{n} 2^{-k} \text{vol}_k(B_2^n)} \\
&= \frac{1}{n+2} + \frac{1}{n+2} \sum_{k=0}^{n} \frac{k \left( \frac{\sqrt{\pi}}{2} \right)^k}{\Gamma(k/2+1)} \\
&= \frac{1}{n+2} + \frac{1}{n+2} \sum_{k=0}^{n} \left( \frac{\sqrt{\pi}}{2} \right)^k \frac{1}{\Gamma(k/2+1)}.
\end{align*}
\]

Denote by \( f \) the function

\[ f(x) = \sum_{k=0}^{\infty} \left( \frac{x}{2} \right)^k \frac{1}{\Gamma(k/2+1)}. \]

By the equality

\[ \lim_{n \to \infty} \sum_{k=0}^{n} \frac{k \left( \frac{\sqrt{\pi}}{2} \right)^k}{\Gamma(k/2+1)} \frac{1}{\Gamma(k/2+1)} = \frac{\sqrt{\pi} f'(\sqrt{\pi})}{f(\sqrt{\pi})}, \]

it is sufficient to find an explicit formula for \( f \). We shall seek \( f \) as the solution of a differential equation with initial condition \( f(0) = 1 \). From the identity

\[ \sum_{k=0}^{n} \frac{k \left( \frac{x}{2} \right)^k}{\Gamma(k/2+1)} = x \sum_{k=1}^{n} \frac{x^{k-1}}{\Gamma(k/2)} = x \left( \frac{1}{\sqrt{\pi}} + \frac{1}{2} \left( \sum_{k=0}^{n-2} \left( \frac{x}{2} \right)^k \frac{1}{\Gamma(k/2+1)} \right) \right), \]
we get the differential equation
\[ xf'(x) = x \left( \frac{1}{\sqrt{\pi}} + \frac{x}{2} f(x) \right). \]
Solving this equation, we have
\[ f(x) = \left( \frac{1}{\sqrt{\pi}} \int_0^x e^{-t^2/4} \, dt + 1 \right) e^{x^2/4} = \left( \text{erf} \left( \frac{x}{2} \right) + 1 \right) e^{x^2/4}. \]
Hence,
\[ \lim_{n \to \infty} \sum_{k=0}^n \frac{k \left( \frac{\sqrt{\pi}}{2} \right)^k}{\Gamma(k/2 + 1)} \frac{1}{f'(\sqrt{\pi})} = \sqrt{\pi} f'(\sqrt{\pi}) = \frac{\pi}{2} + \frac{e^{-\pi/4}}{\text{erf}(\sqrt{\pi}/2) + 1}. \]

**Lemma 4.2.** Consider \( F_{n}^{(2)} = \text{conv} \left( (\sqrt{\frac{n}{2e\pi}} B_2^n, 0), \left( \frac{1}{2} B_\infty^n, 1 \right) \right) \). Then
\[ j(F_{n}^{(2)}) = j^{n+1}(F_{n}^{(2)}) e_{n+1} \quad \text{and} \quad g(F_{n}^{(2)}) = g^{n+1}(F_{n}^{(2)}) e_{n+1}, \]
where \( j^{n+1}(F_{n}^{(2)}) = \frac{1}{n} + o\left(\frac{1}{n}\right) \) and \( g^{n+1}(F_{n}^{(2)}) = 1 - \frac{1}{c} - \frac{1}{n}(1 - \frac{2}{c}) + o\left(\frac{1}{n}\right) \).

**Proof.** Let \( J \) be the John ellipsoid of \( F_{n}^{(2)} \). From its uniqueness it follows that \( J \) has the form
\[ \mathcal{E}_{a,b,c} = \left\{ (x, t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} : \frac{\|x\|^2}{a^2} + \frac{(t-c)^2}{b^2} \leq 1 \right\} \]
for some \( a, b > 0 \) and \( c \in [0; 1] \).
Denote \( \sqrt{\frac{n}{2e\pi}} \) by \( r_n \). Note that the John ellipsoids of \( r_n B_1^n \) and \( \frac{1}{2} B_\infty^n \) are equal to \( r_n B_2^n \) and \( \frac{1}{2} B_2^n \) respectively. For any \( t \in [0; 1] \) it follows from Lemma 2.1 that \( J(t) \subset r_n(1-t) B_2^n + \frac{1}{2} t B_2^n = (r_n-t(r_n-1/2)) B_2^n \). Therefore, the parameters \( a, b \) and \( c \) satisfy
\[ a^2 \left( 1 - \frac{(t-c)^2}{b^2} \right) \leq \left( r_n - t \left( r_n - \frac{1}{2} \right) \right)^2 \]
for any \( t \in [0; 1] \). Since the John ellipsoid has the maximal volume among all ellipsoids contained in the convex body, this inequality turns into equality for some \( t \in [0; 1] \). This condition is equivalent to
\[ a^2 = \left( r_n - c \left( r_n - \frac{1}{2} \right) \right)^2 - \left( r_n - \frac{1}{2} \right)^2 b^2. \]
Consider the function \( f \) of the square of the volume
\[ f(b, c) = \text{vol}_{n+1}(\mathcal{E}_{a,b,c}) = b^2 a^{2n} = b^2 ((r_n + c(r_n - 1/2))^2 - (r_n - 1/2)^2 b^2)^n. \]
The John ellipsoid \( J \) maximizes this function. Since \( f(b, c) \) is decreasing in \( c \) on \([0; 1]\) if \( b \leq \min\{c, 1 - c\} \), the maximum is attained in \( b = c \). Thus
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\[ j^{n+1}(W_n^{(2)}) \] maximizes the function \( f(c, c) = c^2(r_n^2 - 2r_n(r_n - 1/2)c)^n \) where \( c \in [0; 1] \). Hence we have

\[ j^{n+1}(F_n^{(2)}) = \frac{r_n}{(r_n - 1/2)(n + 2)} = \frac{\sqrt{\frac{n}{2e\pi}}}{\left(\sqrt{\frac{n}{2e\pi}} - \frac{1}{2}\right)(n + 2)} = \frac{1}{n + o\left(\frac{1}{n}\right)}. \]

The statement about the centroid follows from the same computations as in [7, Appendix A]. These arguments are given here for the sake of completeness.

From Lemma 2.2 it follows that

\[ g^{n+1}(F_n^{(2)}) = \frac{1}{n + 2} \sum_{k=0}^{n} \frac{(k + 1) V_{n-k,k}(\sqrt{\frac{n}{2e\pi}} B_2^n, \frac{1}{2} B_\infty^n)}{\sum_{k=0}^{n} V_{n-k,k}(\sqrt{\frac{n}{2e\pi}} B_2^n, \frac{1}{2} B_\infty^n)} \]

\[ = \frac{1}{n + 2} \sum_{k=0}^{n} \frac{(k + 1)}{\left(\frac{n}{2e}\right)^{k/2} \Gamma(1 + \frac{n-k}{2})} \]

\[ = \frac{n + 1}{n + 2} - \frac{1}{n + 2} \sum_{k=0}^{n} \frac{\left(\frac{n}{2e}\right)^{k/2}}{\Gamma(1 + k/2)} \]

For every \( x \geq 0 \),

\[ \sum_{k=0}^{n} \frac{k^x}{\Gamma(1 + k/2)} = \sum_{k=1}^{n} \frac{k^x}{(k/2) \Gamma(k/2)} = 2x \sum_{k=1}^{n} \frac{x^{k-1}}{\Gamma(k/2)} \]

\[ = 2x \left(\frac{1}{\Gamma(1/2)} + \sum_{k=2}^{n} \frac{x^{k-1}}{\Gamma(k/2)}\right) = 2x \left(\frac{1}{\Gamma(1/2)} + x \sum_{k=0}^{n-2} \frac{x^k}{\Gamma(1 + k/2)}\right). \]

Substituting \( x = \sqrt{\frac{n}{2e}} \) and denoting

\[ A_n = \sum_{k=0}^{n} \frac{k^x}{\Gamma(1 + k/2)} \quad \text{and} \quad B_n = \sum_{k=0}^{n} \frac{\left(\frac{n}{2e}\right)^{k/2}}{\Gamma(1 + k/2)} \]

it follows that

\[ A_n = 2\sqrt{\frac{n}{2e}} \left(\frac{1}{\Gamma(1/2)} + \sqrt{\frac{n}{2e}} \sum_{k=0}^{n-2} \frac{\left(\frac{n}{2e}\right)^{k/2}}{\Gamma(1 + k/2)}\right) \]

\[ = 2\sqrt{\frac{n}{2e}} \left(\frac{1}{\Gamma(1/2)} + \sqrt{\frac{n}{2e}} \left(B_n - \frac{\left(\frac{n}{2e}\right)^{n-1}}{\Gamma(1 + n-1/2)} - \frac{\left(\frac{n}{2e}\right)^{n \frac{2}{2}}}{\Gamma(1 + n \frac{2}{2})}\right)\right). \]
Consequently,
\[
g^{n+1}(F_n^{(2)}) = \frac{n+1}{n+2} - \frac{1}{n+2} \cdot \frac{A_n}{B_n} = \frac{n+1}{n+2} \left( 1 - \frac{\sqrt{\frac{n}{2e}}}{(1+\frac{n-1}{2}) \Gamma(1+\frac{n}{2})} + \frac{(\frac{n}{2e})^{\frac{n-1}{2}}}{(1+\frac{n-1}{2}) \Gamma(1+\frac{n}{2})} \right).
\]

Taking into account that \( B_n \geq \left( \frac{n}{2e} \right)^2 / \Gamma(3) \) for \( n \geq 4 \), we get
\[
g^{n+1}(F_n^{(2)}) = 1 - \frac{1}{e} - \frac{1}{n} \left( 1 - \frac{2}{e} \right) + o\left( \frac{1}{n} \right). \]

**Lemma 4.3.** Let \( W_n^{(1)} = \text{conv}((B_n^n, 0), \left( \frac{1}{\sqrt{n}} B_n^\infty, 1 \right)) \). Then
\[
l(W_n^{(1)}) = \frac{1}{2} e_{n+1} \quad \text{and} \quad g(W_n^{(1)}) = g^{n+1}(W_n^{(1)}) e_{n+1},
\]
where
\[
g^{n+1}(W_n^{(1)}) = \frac{1}{n} \left( \frac{1}{1 - \sqrt{2/\pi}} + o\left( \frac{1}{n} \right) \right).
\]

**Proof.** We show that \( l^{n+1}(W_n^{(1)}) = 1/2 \). Denote by \( L \) the Löwner ellipsoid of \( W_n^{(1)} \). Since the Löwner ellipsoid is unique, \( L \) has the form
\[
L = \left\{ (x, t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} : \frac{\|x\|^2}{a^2} + \frac{(t - l^{n+1}(W_n^{(1)}))^2}{b^2} \leq 1 \right\},
\]
where \( a \) and \( b \) are some positive numbers depending on \( n \). For \( t \in [0; 1] \) the section \( L(t) = \{ x \in \mathbb{R}^n : (x, t) \in L \} \) is the \( n \)-dimensional euclidean ball containing the Löwner ellipsoid of the section \( W_n^{(1)}(t) \). Since the Löwner ellipsoid for \( \frac{1}{\sqrt{n}} B_n^\infty \) is \( B_n^2 \), we have the inclusions \( B_n^2 \subset L(0) \) and \( B_n^2 \subset L(1) \). Consequently,
\[
W_n^{(1)} = \text{conv}\left( (B_n^n, 0), \left( \frac{1}{\sqrt{n}} B_n^\infty, 1 \right) \right) \subset \text{conv}\left( (B_n^n, 0), (B_n^n, 1) \right) \subset L.
\]

From minimality of the Löwner ellipsoid \( L \) it follows that \( L \) is also the Löwner ellipsoid for the cylinder \( \text{conv}( (B_n^n, 0), (B_n^n, 1) ) \). Since this cylinder is symmetric about \( (0, \ldots, 0, 1/2) \), this body has only a single affine invariant point. Therefore, this point is also the center of the ellipsoid \( L \). Hence \( l^{n+1}(W_n^{(1)}) = 1/2 \).

We derive an asymptotic formula for \( g^{n+1}(W_n^{(1)}) \). From Lemma 2.2 it follows that
Now we derive asymptotic formulas for the sums

\begin{align*}
\sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{\frac{n-k}{2}} \frac{1}{\Gamma\left(1 + \frac{n-k}{2}\right)}
\end{align*}

and

\begin{align*}
\sum_{k=0}^{n} k \left( \frac{\pi n}{4} \right)^{\frac{n-k}{2}} \frac{1}{\Gamma\left(1 + \frac{n-k}{2}\right)}.
\end{align*}

Each term of both sums is less than

\begin{align*}
\left( \frac{\pi n}{4} \right)^{n/2} \frac{1}{\Gamma\left(1 + n/2\right)} \sim \frac{1}{\sqrt{\pi n}} \left( \frac{\pi e}{2} \right)^{n/2}.
\end{align*}

The sum \([4.2]\) can be written in the form

\begin{align*}
\sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{\frac{n-k}{2}} \frac{1}{\Gamma\left(1 + \frac{n-k}{2}\right)}
\end{align*}

\begin{align*}
= \sum_{0 \leq k < n^{1/4}} \left( \frac{\pi n}{4} \right)^{\frac{n-k}{2}} \frac{1}{\Gamma\left(1 + \frac{n-k}{2}\right)} + \sum_{n^{1/4} \leq k \leq n} \left( \frac{\pi n}{4} \right)^{\frac{n-k}{2}} \frac{1}{\Gamma\left(1 + \frac{n-k}{2}\right)}.
\end{align*}

The second sum is of order \(o(n^{-3/2}(\pi e/2)^{n/2})\) since

\begin{align*}
\sum_{n^{1/4} \leq k \leq n} \left( \frac{\pi n}{4} \right)^{\frac{n-k}{2}} \frac{1}{\Gamma\left(1 + \frac{n-k}{2}\right)} \leq n \left( \frac{\pi n}{4} \right)^{\frac{n-n^{1/4}}{2}} \frac{1}{\Gamma\left(1 + \frac{n-n^{1/4}}{2}\right)}.
\end{align*}
\[ \leq Cn \left( \frac{\pi n}{4} \right)^{n-1/4} e^{n^{-1/2}} \left( \frac{n-n^{1/4}}{2} \right)^{-n^{-1/4}+1} \]

\[ \leq Cn^{-3/2} \left( \frac{\pi e}{2} \right)^{n/2} n^2 \frac{2}{\pi} n^{1/4}/2. \]

In order to compute the asymptotic behaviour of \( \sum_{0 \leq k \leq n^{1/4}} \ldots \) we use Stirling’s formula for the gamma function [15, §12.33]: if \( x > 0 \), then

\[ \Gamma(x + 1) = x^xe^{-x} \sqrt{2\pi x} e^{\theta(x)}, \]

where \( 0 < \theta(x) < \frac{1}{12x} \). Assume that \( x \) depends on \( n \) so that \( n-n^{1/4} < x < n \). We get \( 0 < \theta(x) < \frac{1}{12(n-n^{1/4})} \) and hence

\[ \Gamma(x + 1) = x^xe^{-x} \sqrt{2\pi x} (1 + O(1/n)) \]

as \( n \to +\infty \). Substituting \( x = \frac{n-k}{2} \) for \( 0 \leq k \leq n^{1/4} \) we obtain

\[ \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} = \left( \frac{\pi n}{4} \right)^{n-k} \left( \frac{n-k}{2} \right)^{-n-k} e^{\frac{n-k}{2}} \frac{1 + o(1)}{\sqrt{\pi(n-k)}} \]

\[ = \frac{1}{\sqrt{\pi n}} \left( \frac{\pi e}{2} \right)^{n/2} \left( \frac{2}{\pi} \right)^{k/2} (1 + o(1)). \]

Consequently,

\[ \sum_{k=0}^{n^{1/4}} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} = \frac{1}{\sqrt{\pi n}} \left( \frac{\pi e}{2} \right)^{n/2} \sum_{k=0}^{n^{1/4}} \left( \frac{2}{\pi} \right)^{k/2} (1 + o(1)) \]

\[ = \frac{1}{\sqrt{\pi n}} \left( \frac{\pi e}{2} \right)^{n/2} \left( \frac{1}{1 - \sqrt{2/\pi}} + o(1) \right). \]

The asymptotic formula for (4.3) is computed in a similar way:

\[ \sum_{k=0}^{n^{1/4}} k \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} = \frac{1}{\sqrt{\pi n}} \left( \frac{\pi e}{2} \right)^{n/2} \sum_{k=0}^{n^{1/4}} k \left( \frac{2}{\pi} \right)^{k/2} (1 + o(1)) \]

\[ = \frac{1}{\sqrt{\pi n}} \left( \frac{\pi e}{2} \right)^{n/2} \left( \frac{\sqrt{2/\pi}}{(1 - \sqrt{2/\pi})^2} + o(1) \right). \]

Hence we have

\[ \sum_{k=0}^{n} \frac{n}{k} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ \sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ = \frac{1}{n+2} + \frac{1}{n+2} \sum_{k=0}^{n} \frac{n}{k} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ + \frac{1}{n+2} \sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ \sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ \sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ \sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ \sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]

\[ \sum_{k=0}^{n} \left( \frac{\pi n}{4} \right)^{n-k} \frac{1}{\Gamma(1+\frac{n-k}{2})} \]
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\[
= \left( \frac{1}{n} + o\left( \frac{1}{n} \right) \right) \left( 1 + \frac{\sqrt{2/\pi}}{1 - \sqrt{2/\pi}} + o(1) \right) = \frac{1}{n} \frac{1}{1 - \sqrt{2/\pi}} + o\left( \frac{1}{n} \right). \]

**Lemma 4.4.** Consider \( W_n(2) = \text{conv}\left( (\frac{n}{c} B_1^n, 0), (B_\infty^n, 1) \right) \). Then

\[
l(W_n(2)) = l^{n+1}(W_n(2)) e_{n+1} \quad \text{and} \quad g(W_n(2)) = g^{n+1}(W_n(2)) e_{n+1},
\]

where \( l^{n+1}(W_n(2)) = \frac{1}{n} + o\left( \frac{1}{n} \right) \) and \( g^{n+1}(W_n(2)) = 1 - \frac{1}{e} - \frac{1}{n} (1 - \frac{2}{e}) + o\left( \frac{1}{n} \right) \).

**Proof.** Let \( L \) be the Löwner ellipsoid of \( W_n(2) \). From uniqueness of the Löwner ellipsoid it follows that \( L \) has the form

\[
\mathcal{E}_{a,b,c} = \left\{ (x,t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} : \|x\|^2 / a^2 + (t - c)^2 / b^2 \leq 1 \right\}
\]

for some \( a > 0, b > 0 \) and \( c \in [0; 1] \).

Denote \( n/e \) by \( r_n \). From the definition of the Löwner ellipsoid the inclusion \( W_n(2)(t) = tB_\infty^n + (1 - t)r_nB_1^n \subset L(t) \) follows for any \( t \in [0; 1] \). On the other hand, for any ellipsoid \( \mathcal{E}_{a,b,c} \) from convexity and inclusions \( B_\infty^n \subset L(0) \) and \( B_1^n \subset L(1) \) we get \( W_n(2)(t) = tB_\infty^n + (1 - t)r_nB_1^n \subset \mathcal{E}_{a,b,c}(t) \) for any \( t \in [0; 1] \). Note that the Löwner ellipsoids for \( r_nB_1^n \) and \( B_\infty^n \) are \( r_nB_2^n \) and \( \sqrt{n}B_2^n \). The sections \( L(t) = \{ x \in \mathbb{R}^n : (x,t) \in L \} \) of the ellipsoid \( L \) are \( n \)-dimensional euclidean balls up to some scaling. Minimality of the Löwner ellipsoid implies \( L(0) = r_nB_2^n; L(1) = \sqrt{n}B_2^n \) and the following conditions for the parameters \( a, b \) and \( c \):

\[
r_n^2 = a^2 \left( 1 - \frac{c^2}{b^2} \right) \quad \text{and} \quad n = a^2 \left( 1 - \frac{(1 - c)^2}{b^2} \right).
\]

Hence

\[
b^2 = \frac{r_n^2(1 - c)^2 - nc^2}{r_n^2 - n} \quad \text{and} \quad a^2 = \frac{r_n^2(1 - c)^2 - nc^2}{1 - 2c}.
\]

From the positivity of the left-hand sides the restriction \( c \in (0; 1/2) \) follows.

We compute \( l^{n+1}(W_n(2)) \) as the minimum point of the square of the volume

\[
f(c) = \text{vol}^2_{n+1}(\mathcal{E}_{a,b,c}) = b^2 a^{2n} \text{vol}^2_{n+1}(B_2^{n+1}) = \frac{r_n^2(1 - c)^2 - nc^2}{r_n^2 - n} \left( \frac{r_n^2(1 - c)^2 - nc^2}{1 - 2c} \right)^n \text{vol}^2_{n+1}(B_2^{n+1}),
\]

where \( c \in (0; 1/2) \).

The roots of the equation \( f'(c) = 0 \) are

\[
\beta^+_n = \left( n + 3 \right)r_n^2 - (n + 1)n \pm \sqrt{\left( (n + 3)r_n^2 - (n + 1)n \right)^2 - 4(n + 2)(r_n^2 - n)r_n^2} / 2(n + 2)(r_n^2 - n).
\]
Since \( r_n = n/e \), the root \( \beta_n^+ \) does not belong to the interval \((0; \frac{1}{2})\) for \( n \) large enough. Consequently,
\[
l^{n+1}(W_n^{(2)}) = \beta_n^- = \frac{1 + \frac{3}{n} - \frac{n}{r_n} + o\left(\frac{1}{n}\right) - \sqrt{1 + \frac{2}{n} - \frac{2n}{r_n} + o\left(\frac{1}{n}\right)}}{2\left(1 + \frac{2}{n}\right)\left(1 - \frac{n}{r_n}\right)} = \frac{1}{n} + o\left(\frac{1}{n}\right).
\]

Let us turn to the proof of the statement for centroids. Applying Lemma 2.2 to \( W_n^{(2)} \) and using (2.1) together with (2.3) we get
\[
g^{n+1}(W_n^{(2)}) = \frac{1}{n+2} \sum_{k=0}^{n} (k+1)2^k \left(\frac{n}{e}\right)^{n-k} \frac{\text{vol}_{n-k}(B_1^{n-k})}{\text{vol}_{n-k}(B_1^{n-k})} = \frac{1}{n+2} \sum_{k=0}^{n} \frac{n^{n-k}(k+1)}{e^{n-k}(n-k)!} = \frac{1}{n+2} \sum_{k=0}^{n} \frac{n^k(n-k+1)}{e^k k!} \sum_{k=0}^{n} \frac{n^k}{e^k k!}.
\]

Therefore, it suffices to prove the asymptotic formula
\[
(4.4) \quad \frac{1}{n+2} \sum_{k=0}^{n} \frac{k n^k}{e^k k!} = \frac{1}{e - n e} + o\left(\frac{1}{n}\right).
\]

Note that for any \( x \in \mathbb{R} \),
\[
\sum_{k=0}^{n} \frac{k x^k}{k!} = \sum_{k=1}^{n} \frac{x^{k-1}}{(k-1)!} = x \sum_{k=1}^{n} \frac{x^{k-1}}{(k-1)!} = x \sum_{k=0}^{n-1} \frac{x^k}{k!}.
\]

Substituting \( x = n/e \) and dividing both parts by \( \sum_{k=0}^{n} \frac{n^k}{e^k k!} \), we have
\[
\frac{\sum_{k=0}^{n} \frac{k n^k}{e^k k!}}{\sum_{k=0}^{n} \frac{n^k}{e^k k!}} = \frac{n}{e} \left(1 - \frac{n^1}{\sum_{k=0}^{n} \frac{n^k}{e^k k!}}\right) = \frac{n}{e} \left(1 + o\left(\frac{1}{n}\right)\right).
\]
The last equality holds since $\frac{n^n}{en^n!}$ is less than 1 by Stirling’s formula and $\sum_{k=0}^{n} \frac{n^k}{e^k k!}$ is greater than $\frac{n^2}{2e^2}$. This yields (4.4). \hfill \box

5. Constructions of convex bodies for $\varphi_{j,l}$. As in the previous section, we provide convex bodies $M_n^{(1)}$ and $M_n^{(2)}$ in $\mathbb{R}^{n+1}$ with the following properties: the John point of $M_n^{(1)}$ is “close” to the boundary, whereas the Löwner point is “separated” from the boundary; for $M_n^{(2)}$ the corresponding points have the opposite properties.

The construction of $M_n^{(1)}$ is taken from [7].

**Lemma 5.1** (M. Meyer, C. Schütt, E. M. Werner [7]). For $M_n^{(1)} = \text{conv}((B_2^n, 0), (\Delta_n, 1))$

we have $l(M_n^{(1)}) = \frac{1}{2} e_{n+1}$ and $j(M_n^{(1)}) = j^{n+1}(M_n^{(1)}) e_{n+1}$, where $j^{n+1}(M_n^{(1)}) = 1/n + o(1/n)$.

By the same argument as in [7] the second convex body $M_n^{(2)}$ can be constructed in the following way.

**Lemma 5.2.** For $M_n^{(2)} = \text{conv}((\Delta_n, 0), (\frac{1}{n} B_2^n, 1))$ we have $j(M_n^{(2)}) = \frac{1}{2} e_{n+1}$ and $l(M_n^{(2)}) = l^{n+1}(M_n^{(2)}) e_{n+1}$, where $l^{n+1}(M_n^{(2)}) = 1/n + o(1/n)$.

**Proof.** First, we prove the statement for the John point of $M_n^{(2)}$. Denote the John ellipsoid of $M_n^{(2)}$ by $J$.

We note that $M_n^{(2)}$ is invariant with respect to rotations around the axis directed along $e_{n+1} = (0, \ldots, 0, 1)$. Since the John ellipsoid is unique, $J$ has the form

$$E_{a,b,c} = \left\{ (x, t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} : \frac{\|x\|^2}{a^2} + \frac{(t-c)^2}{b^2} \leq 1 \right\}$$

for some $a > 0$, $b > 0$ and $c \in [0, 1]$.

By Lemma 2.1 the John ellipsoid for $M_n^{(2)}(t) = (1-t)\Delta_n + t\frac{1}{n} B_2^n$ equals $\frac{1}{n} B_2^n$ for $t \in [0, 1]$. From (5.1) it follows that $J(t) \subset \frac{1}{n} B_2^n$. Thus, we have the necessary condition for the parameters $a$, $b$, $c$:

$$a \sqrt{1 - \frac{(t-c)^2}{b^2}} \leq \frac{1}{n}$$

valid for every $t \in [0; 1]$. Since the John ellipsoid has the maximal volume, there exists some $t \in [0; 1]$ for which (5.2) turns into equality. It is easy to check that equality is attained at $t = c$ and hence $a = 1/n$. Since the
John ellipsoid is inscribed into $M_n^{(2)}$ it follows that $b \leq \min(c, 1 - c)$. From $\text{vol}_{n+1}(\mathcal{E}_{1/n,b,c}) = b \frac{1}{n^2} \text{vol}_{n+1}(B_2^{n+1})$, we find that the volume of $\mathcal{E}_{1/n,b,c}$ is maximized at $b = c = 1/2$.

On the other hand, 

$$\mathcal{E}_{1/n,1/2,1/2} \subset \left( \frac{1}{n} B_2^n \right) \times [0; 1] \subset M_n^{(2)}.$$ 

Consequently, by uniqueness of the John ellipsoid, we have $J = \mathcal{E}_{1/n,1/2,1/2}$ and $J^{n+1}(M_n^{(2)}) = 1/2$.

We turn to the proof of the statement about the Löwner point of $M_n^{(2)}$. By the same argument as above the Löwner ellipsoid has the form (5.1).

We denote the Löwner ellipsoid of $M_n^{(2)}$ by $L$. From the definition of $L$ it follows $M_n^{(2)}(t) = (1 - t)\Delta_n + t\frac{1}{n} B_2^n \subset L(t)$ for $t \in [0; 1]$. Furthermore, if the ellipsoid $\mathcal{E}_{a,b,c}$ satisfies $\Delta_n \subset \mathcal{E}_{a,b,c}(0)$ and $\frac{1}{n} B_2^n \subset \mathcal{E}_{a,b,c}(1)$, then for every $t \in [0; 1]$ we have $M_n^{(2)}(t) = (1 - t)\Delta_n + t\frac{1}{n} B_2^n \subset \mathcal{E}_{a,b,c}(t)$. Thus, the minimality of the volume of the Löwner ellipsoid implies the necessary conditions

$$\frac{1}{n^2} = a^2 \left( 1 - \frac{(1 - c)^2}{b^2} \right) \quad \text{and} \quad 1 = a^2 \left( 1 - \frac{c^2}{b^2} \right).$$

Therefore,

$$b^2 = \frac{\left(1 - \frac{1}{n^2}\right)c^2 - 2c + 1}{1 - \frac{1}{n^2}} \quad \text{and} \quad a^2 = \frac{\left(1 - \frac{1}{n^2}\right)c^2 - 2c + 1}{1 - 2c}.$$

Since the left-hand side is positive, we have $c < 1/2$.

We shall find $l_{n+1}(M_n^{(2)})$ as the solution of the minimization problem for the square volume

$$f(c) = \text{vol}^2_{n+1}(\mathcal{E}_{a,b,c}) = b^2 a^{2n} \text{vol}^2_{n+1}(B_2^{n+1})$$

$$= \frac{(1 - \frac{1}{n^2})c^2 - 2c + 1}{1 - \frac{1}{n^2}} \left( \frac{(1 - \frac{1}{n^2})c^2 - 2c + 1}{1 - 2c} \right)^n \text{vol}^2_{n+1}(B_2^{n+1})$$

where $c \in (0; 1/2)$.

The roots of $f'(c) = 0$ are

$$\alpha_n^\pm = \frac{n^3 + 3n^2 - n - 1 \pm \sqrt{(n^3 + 3n^2 - n - 1)^2 - 4n^2(n^2 - 1)(n + 2)}}{2(n^2 - 1)(n + 2)}.$$ 

For large enough $n$ we have $\alpha_n^+ \not\in (0; 1/2)$. From uniqueness of the Löwner ellipsoid it follows $l_{n+1}(M_n^{(2)}) = \alpha_n^-$. Thus,

$$l_{n+1}(M_n^{(2)}) = \alpha_n^- = \frac{1 + \frac{3}{n} + o\left(\frac{1}{n}\right) - \sqrt{1 + \frac{2}{n} + o\left(\frac{1}{n}\right)}}{2\left(1 - \frac{1}{n^2}\right)\left(1 + \frac{2}{n}\right)} = \frac{1}{n} + o\left(\frac{1}{n}\right). \quad \blacksquare$$
6. Asymptotical sharpness of the upper bounds for measures of symmetry. In this section a method is provided for constructing the extremal convex bodies for the measures of symmetry under study.

We need the geometrical remark below, following from direct computations.

**Lemma 6.1.** For two different interior points \((x^1, y^1), (x^2, y^2) \in [0; 1] \times [0; 1]\) denote by \(s\) the line passing through them. Then
\[
\frac{||(x^1, y^1) - (x^2, y^2)||}{\text{vol}_1(s \cap [0; 1] \times [0; 1])} = \frac{|x^2 - x^1| |y^2 - y^1|}{|x^1y^2 - x^2y^1|}.
\]

We shall illustrate the approach on the measure of symmetry \(\varphi_{j,l}\).

**Theorem 6.2.** There exist convex bodies \(M_n \in \mathcal{K}_n, n \in \mathbb{N}\), such that
\[
d(j(M_n), l(M_n)) = 1 - \frac{8}{n} + o\left(\frac{1}{n}\right).
\]

**Proof.** First, we consider the case of even dimensions. Then \(n = 2k\) for some \(k \in \mathbb{N}\). We introduce the following \(2k\)-dimensional convex bodies
\[
M_{2k} = M^{(1)}_{k-1} \times M^{(2)}_{k-1},
\]
where \(M^{(1)}_{k-1}\) and \(M^{(2)}_{k-1}\) are defined as in Lemmas 5.1 and 5.2. Applying Lemma 3.2 for \(M_{2k}\) and using Lemmas 5.1 and 5.2 we have the following representations for the John and Löwner points:
\[
j(M_{2k}) = (j(M^{(1)}_{k-1}), j(M^{(2)}_{k-1})) = \left(0, \ldots, 0, j^k, 0, \ldots, 0, \frac{1}{2}\right),
\]
\[
l(M_{2k}) = (l(M^{(1)}_{k-1}), l(M^{(2)}_{k-1})) = \left(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0, l^{2k}\right),
\]
where \(j^k = \frac{1}{k} + o\left(\frac{1}{k}\right)\) and \(l^{2k} = \frac{1}{k} + o\left(\frac{1}{k}\right)\).

Denote by \(\ell\) the line passing through \(j(M_{2k})\) and \(l(M_{2k})\). The representations for \(j(M_{2k})\) and \(l(M_{2k})\) imply that \(\ell\) is contained in the 2-dimensional subspace
\[
\mathcal{L} = \{x \in \mathbb{R}^{2k} : x^i = 0 \text{ for } i \not\in \{k, 2k\}\}.
\]

From the definition of \(M_{2k}\) one can deduce
\[
M_{2k} \cap \mathcal{L} = \{x \in \mathbb{R}^{2k} : x^i = 0 \text{ for } i \not\in \{k, 2k\} \text{ and } 0 \leq x^k, x^{2k} \leq 1\}.
\]

Thus, applying Lemma 6.1 to the points \((j^k, 1/2)\) and \((1/2, l^{2k})\), for large
enough $n$ we obtain

$$d(j(M_n), l(M_n)) = d(j(M_{2k}), l(M_{2k})) = \frac{\|j(M_{2k}) - l(M_{2k})\|}{\text{vol}_1(\ell \cap (M_{2k} \cap L))}$$

$$= \frac{|j - \frac{1}{2}|}{\frac{1}{4} - l^2 j^k} = \frac{(1 - \frac{2}{k} + o(\frac{1}{k})) (1 - \frac{2}{k} + o(\frac{1}{k}))}{1 + o(\frac{1}{k})} = 1 - \frac{4}{k} + o\left(\frac{1}{k}\right) = 1 - \frac{8}{n} + o\left(\frac{1}{n}\right).$$

In the case of odd dimension the proof is similar. For $n = 2k + 1$ it is sufficient to consider $M_{2k+1} = M_{k-1} \times M_k$. ■

Using the same argument for the convex bodies

$$F_{2k} = F_{k-1}^{(1)} \times F_{k-1}^{(2)}, \quad F_{2k+1} = F_{k-1}^{(1)} \times F_k^{(2)}$$

and

$$W_{2k} = W_{k-1}^{(1)} \times W_{k-1}^{(2)}, \quad W_{2k+1} = W_{k-1}^{(1)} \times W_k^{(2)}$$

and taking into account Lemmas 4.1–4.4 we provide extremal convex bodies for the measures of symmetry $\varphi_{g,l}$ and $\varphi_{g,j}$, i.e.

$$d(g(F_n), j(F_n)) = 1 - \frac{1}{n} \left(4 + \frac{2e}{e - 1} \left(1 + \frac{\pi}{2} + \frac{e^{-\pi/4}}{\text{erf}(\sqrt{\pi}/2) + 1}\right)\right) + o\left(\frac{1}{n}\right)$$

and

$$d(g(W_n), l(W_n)) = 1 - \frac{1}{n} \left(4 + \frac{2e}{(e - 1)(1 - \sqrt{2/\pi})}\right) + o\left(\frac{1}{n}\right),$$

where $4 + \frac{2e}{e - 1} (1 + \frac{\pi}{2} + \frac{e^{-\pi/4}}{\text{erf}(\sqrt{\pi}/2) + 1}) \approx 13$ and $4 + \frac{2e}{(e - 1)(1 - \sqrt{2/\pi})} \approx 20$.

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