On the bi-Hamiltonian structure of
Bogoyavlensky system on so(4)

A V Vershilov
St.Petersburg State University, St.Petersburg, Russia
e–mail: alexander.vershilov@gmail.com

Abstract

We discuss bi-Hamiltonian structure for the Bogoyavlensky system on
so(4) with an additional integral of fourth order in momenta. An explicit
procedure to find the variables of separation and the separation relations
is considered in detail.

PACS: 45.10.Na, 45.40.Cc
MSC: 70H20; 70H06; 37K10

1 Introduction

We address the problem of the separation of variables for the Hamilton-Jacobi
equation within the theoretical scheme of bi-hamiltonian geometry. The main
aim is the construction of variables of separation for the given integrable system
without any additional information (Lax matrices, r-matrices, links with soliton
equations etc.)

The paper is organized as follows. In Section 2 we determine the Bogoy-
avlensky system on so(4). In Section 3, the necessary aspects of bi-hamiltonian
geometry are briefly reviewed. Then, we discuss a possible application of these
methods to calculation of the polynomial bi-hamiltonian structures for the given
Bogoyavlensky system. In Section 4, the problem of finding variables of separa-
tion and corresponding separation relations is treated and solved.

All the computations for the paper have been done by the computer algebra
system MAPLE. It allows us to solve overdetermined polynomial differential and
algebraic systems of equations. A major concern for any operations performed by
the program is the complexity of resulting expressions. So, we can say that this
note belongs mostly to so called “experimental” or ”computational” mathematical
physics.
2 The Bogoyavlensky system on so(4)

Let \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) and \( M = (M_1, M_2, M_3) \) be the two vectors of coordinates and momenta, respectively. We postulate the following Poisson bracket on this six-dimensional phase space

\[
\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = \varepsilon_{ijk} M_k. \tag{2.1}
\]

Here \( \varepsilon_{ijk} \) is the totally skew-symmetric and \( \varepsilon \) is a parameter. It is well-known that any linear Poisson bracket is defined by an appropriate Lie algebra. The cases \( \varepsilon = 0, \varepsilon^2 > 0 \) and \( \varepsilon^2 < 0 \) correspond to the Lie algebras \( e(3), so(4) \) and \( so(3,1) \).

The Poisson bracket (2.1) has the two Casimir functions

\[
C_1 = M_1 \gamma_1 + M_2 \gamma_2 + M_3 \gamma_3, \quad C_2 = \varepsilon^2 (M_1^2 + M_2^2 + M_3^2) + \gamma_1^2 + \gamma_2^2 + \gamma_3^2. \tag{2.2}
\]

Hence, for the Liouville integrability of the equations of motion only one additional integral functionally independent of the Hamiltonian and the Casimir functions is necessary.

The nontrivial class of quadratic homogeneous Hamiltonians of the form

\[
H = (M, A M) + (M, B \gamma) + (\gamma, C \gamma), \tag{2.3}
\]

where \( A, B \) and \( C \) are constant \( 3 \times 3 \)-matrices, has many important applications in the rigid body dynamics [3].

There are two classical integrable cases, one found by Chaplygin and one by Goryachev, where the additional integral of motion is of fourth degree. Namely, at \( \varepsilon = 0 \) and \( C_1 = 0 \) integrals of motion are in the involution with respect to the Poisson bracket (2.1)

\[
H_1 = M_1^2 + M_2^2 + 2M_3^2 + c_1 (\gamma_1^2 - \gamma_2^2) \quad c_1 \in \mathbb{R}, \tag{2.4}
\]

\[
H_2 = (M_1^2 - M_2^2 + c_1 \gamma_3^2)^2 + 4 M_1^2 M_2^2.
\]

It is so-called Chaplygin system on the sphere [4]. The Lax matrices and \( r \)-matrix formalism for this system have been obtained in [6, 8], the corresponding bi-hamiltonian geometry has been studied in [12, 15, 16].

Bogoyavlensky found pull-back of the Chaplygin system to the so(4) algebra

\[
H_1 = (1 - \varepsilon^2 a_1) M_1^2 + (1 - \varepsilon^2 a_2) M_2^2 + (2 - \varepsilon^2 a_2 - \varepsilon^2 a_1) M_3^2 + (a_1 - a_2) (\gamma_1^2 - \gamma_2^2), \tag{2.5}
\]

\[
H_2 = \left( (1 - \varepsilon^2 a_2) M_1^2 - (1 - \varepsilon^2 a_1) M_2^2 - (a_2 - a_1) \gamma_3^2 \right)^2 + 4 (1 - \varepsilon^2 a_2) (1 - \varepsilon^2 a_1) M_1^2 M_2^2.
\]

and proved that equations of motion can be integrated by means of elliptic functions [1].
Remark 1 In fact we can found two integrable at $C_1 = 0$ systems on $so(4)$ in the Bogoyavlensky book \cite{1}. The Hamilton function for the first system

$$
\tilde{H}_1 = b_1 M_1^2 + b_2 M_2^2 + b_3 M_3^2 + \frac{2b_2 - b_3}{\varepsilon^2} \gamma_1^2 + \frac{2b_1 - b_3}{\varepsilon^2} \gamma_2^2 + \frac{b_2 - b_3 + b_1}{\varepsilon^2} \gamma_3^2
$$

(2.6)

coincides with the Hamiltonian $H_1$ (2.5) up to Casimir function

$$
\tilde{H}_1 = H_1 + \frac{b_1 + b_2 - b_3}{\varepsilon^2} C_2 ,
$$

if we put

$$
b_1 - b_3 = \varepsilon^2 a_2 - 1, \quad b_2 - b_3 = \varepsilon^2 a_1 - 1 .
$$

The second system with the Hamilton function

$$
\tilde{H}_1 = c_1 M_1^2 + c_2 M_2^2 + c_3 M_3^2 + \frac{1}{2\varepsilon^2} \left( (c_2 + c_3) \gamma_1^2 + (c_1 + c_3) \gamma_2^2 + (c_1 + c_2) \gamma_3^2 \right)
$$

(2.7)

has the following additional integrals of motion

$$
K_i = \left( (c_i - c_k) \gamma_j^2 + (c_j - c_i) \gamma_k^2 + (c_j - c_k) \gamma_i^2 \right)^2 + 4(c_i - c_j)(c_i - c_k) \gamma_j^2 \gamma_k^2 ,
$$

where $\{ijk\}$ is one of the possible cyclic permutations of subscripts $\{123\}$. It is easy to prove that

$$
\alpha K_1 + \beta K_2 + \gamma K_3 = 0, \quad \text{iff} \quad \alpha + \beta + \gamma = 0 .
$$

According to \cite{9}, Hamiltonians (2.6) and (2.7) are related by the Poisson map

$$
\begin{align*}
M_1 &\to \frac{\gamma_1}{\varepsilon}, \quad \gamma_1 \to \varepsilon M_1, \quad M_2 \to \frac{\gamma_2}{\varepsilon}, \quad \gamma_2 \to \varepsilon M_2, \quad M_3 \to M_3, \quad \gamma_3 \to \gamma_3 ,
\end{align*}
$$

and change of parameters

$$
c_1 = (2b_2 - b_3), \quad c_2 = (2b_1 - b_3), \quad c_3 = b_3 .
$$

3 The bi-hamiltonian structure

In order to get variables of separation according to general usage of bi-hamiltonian geometry firstly we have to calculate the bi-hamiltonian structure for the given integrable system with integrals of motion $H_{1,2}$ (2.5) on the Poisson manifold $so(4)$ with the kinematic Poisson bivector $P$ and the Casimir functions $C_{1,2}$ (2.2):

$$
P = \begin{pmatrix}
0 & \varepsilon^2 M_3 & -\varepsilon^2 M_2 & 0 & \gamma_3 & -\gamma_2 \\
-\varepsilon^2 M_3 & 0 & \varepsilon^2 M_1 & -\gamma_3 & 0 & \gamma_1 \\
\varepsilon^2 M_2 & -\varepsilon^2 M_1 & 0 & \gamma_2 & -\gamma_1 & 0 \\
0 & \gamma_3 & -\gamma_2 & 0 & M_3 & -M_2 \\
-\gamma_3 & 0 & \gamma_1 & -M_3 & 0 & M_1 \\
\gamma_2 & -\gamma_1 & 0 & M_2 & -M_1 & 0
\end{pmatrix}, \quad PdC_{1,2} = 0 .
$$

(3.1)
Following to [11, 13, 14, 18, 16] we are looking for solution $P'$ of the equations

$$\{H_1, H_2\}' = \langle P'dH_1, dH_2 \rangle = 0, \quad [P, P'] = [P', P'] = 0,$$

(3.2)

where $[., .]$ means the Schouten bracket.

Obviously enough, in their full generality equations (3.2) are too difficult to be solved because it has infinitely many solutions [10, 14]. In order to get some particular solutions we will use the additional assumption

$$P'dC_{1,2} = 0,$$

(3.3)

and polynomial in momenta $M$ ansätze for the components $P'_{ij}$ of the desired Poisson bivector $P' = \sum P_{ij} \partial_i \wedge \partial_j$.

Substituting polynomial ansätze into the equations (3.2-3.3) and demanding that all the coefficients at powers of $M$ vanish one gets the over determined system of algebro-differential equations on functions of $x$ which can be easily solved in the modern computer algebra systems. The computation was performed using the computer algebra system MAPLE and, in contrast with the Chaplygin case [16], these calculations were performed non-automatically with essential manual interaction. In this way we get a lot of real and complex solutions, which will be classified and studied at a future date.

In this note we will not consider a complete classification and restrict ourselves by discussion of one example only. In order to describe this solution we introduce some special notations. It is easy to see that at $C_1 = 0$ we can rewrite kinematic bivector (3.1) in the following form

$$P = \begin{pmatrix} x^2 \partial_2 \Lambda & \Lambda \\ -\Lambda^\top & \partial_2 \Lambda \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}$$

(3.4)

where antisymmetric matrix $\partial_\gamma \Lambda$ is defined by

$$(\partial_\gamma \Lambda)_{ij} = \sum_{k=1}^3 \frac{1}{\gamma_k} \left( \frac{\partial A_{jk}}{\partial \gamma_i} \gamma_i M_k - \frac{\partial A_{ik}}{\partial \gamma_j} \gamma_j M_j \right)$$

(3.5)

In the similar notations at $C_1 = 0$ the second bivector is equal to

$$P' = \alpha \begin{pmatrix} x^2 \partial_2 \Lambda & \Lambda \\ -\Lambda^\top & 0 \end{pmatrix} + \begin{pmatrix} \partial_M \Pi & \Pi \\ -\Pi^\top & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Lambda' \\ -\Lambda'^\top & -\partial_2 \Lambda' \end{pmatrix},$$

(3.6)

where

$$\alpha = \frac{2x^2(\beta_1 M_1^2 + \beta_2 M_2^2)}{(\beta_1 - \beta_2) \gamma_3^2}, \quad \beta_1 = x^2 a_1 - 1, \quad \beta_2 = x^2 a_2 - 1.$$

Matrix $\Pi$ is equal to

$$\Pi = \frac{2x^2}{(\beta_1 - \beta_2) \gamma_3^2} \begin{pmatrix} \beta_1 M_1 (M_2 \gamma_2 - \gamma_2 M_3) & -\beta_1 (\gamma_2 M_1 M_3 + \gamma_3 M_2^2) & \beta_1 \gamma_2 (M_1^2 + M_2^2) \\ \beta_2 (\gamma_1 M_1 M_3 + \gamma_3 M_2^2) & -\beta_2 M_2 (M_1 \gamma_3 - \gamma_1 M_3) & -\beta_2 \gamma_1 (M_1^2 + M_2^2) \\ \beta_2 \gamma_3 M_2 M_3 & -\beta_1 \gamma_3 M_1 M_3 & (\beta_1 - \beta_2) \gamma_3 M_1 M_2 \end{pmatrix}.$$
and antisymmetric matrix $\partial_M \Pi$ reads as
\[
\partial_M \Pi = \varkappa^2 \begin{pmatrix}
0 & M_3 & M_2 \\
-M_3 & 0 & M_1 \\
-M_2 & -M_1 & 0
\end{pmatrix} + \frac{2\varkappa^2 M_3}{(\beta_1 - \beta_2)\gamma_3} \begin{pmatrix}
0 & \beta_2 \gamma_3 & -\beta_1 \gamma_2 \\
-\beta_2 \gamma_3 & 0 & \beta_2 \gamma_1 \\
\beta_1 \gamma_2 & -\beta_2 \gamma_1 & 0
\end{pmatrix}.
\]

Instead of antisymmetric matrix $\Lambda (3.4)$ in the second bivector we have symmetric matrix
\[
\Lambda' = \begin{pmatrix}
0 & -\gamma_3 & \gamma_2 \\
-\gamma_3 & 0 & \gamma_1 \\
\gamma_2 & \gamma_1 & -2\gamma_1 \gamma_2
\end{pmatrix},
\]
whereas definition of $\partial_\gamma \Lambda'$ is completely similar to (3.5)
\[
(\partial_\gamma \Lambda')_{ij} = \sum_{k=1}^3 \frac{1}{\gamma_k} \left( \frac{\partial \Lambda'_{jk}}{\partial \gamma_i} \gamma_i M_i - \frac{\partial \Lambda'_{ik}}{\partial \gamma_j} \gamma_j M_j \right).
\]

At $\varkappa \to 0$ one get bi-hamiltonian structure for the Chaplygin system, which has been obtained in [16]. We believe that bivector $P'$ has some algebro-geometric justification, similar to compatible bivectors on $so(n)$ from [2].

**Remark 2** Usually the second Poisson bivector $P'$ is the Lie derivative of $P$ along some polynomial Liouville vector field $X$
\[
P' = \mathcal{L}_X(P),
\]
see [11, 13, 18, 16]. For the Bogoyavlensky system we could not find such Liouville vector field. So, we can not say that bivector $P'$ (3.6) is the 2-coboundary associated with the Liouville vector field $X$ in the Poisson-Lichnerowicz cohomology defined by $P$.

To sum up, using applicable ansätze for the Liouville vector field $X$ we get a real relatively simple quadratic bivector (3.6) and some more complicated complex bivectors. Modern computer software allows to do it on a personal computer wasting only few seconds. The application of this Poisson bivector will be given in the next section.

## 4 Variables of separation and separation relations

The second step in the bi-hamiltonian method of separation of variables is calculation of canonical variables of separation $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and separation relations of the form
\[
\phi_i(q_i, p_i, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \quad \text{with } \det \left[ \frac{\partial \phi_i}{\partial H_j} \right] \neq 0.
\]
The reason for this definition is that the stationary Hamilton-Jacobi equations for the Hamiltonians $H_i$ can be collectively solved by the additively separated complete integral

$$ W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} W_i(q_i; \alpha_1, \ldots, \alpha_n), \quad (4.2) $$

where $W_i$ are found by quadratures as solutions of ordinary differential equations.

According to [5, 13, 18], separated coordinates $q_j$ are the eigenvalues of the control matrix $F$ defined by

$$ P' dH = P(F dH). $$

Its eigenvalues coincide with the Darboux-Nijenhuis coordinates (eigenvalues of the recursion operator) on the corresponding symplectic leaves. Using control matrix $F$ we can avoid the procedure of restriction of the bivectors $P$ and $P'$ on symplectic leaves, that is a necessary intermediate calculation for the construction of the recursion operator [5].

In our case for the Poisson bivector $P'$ (3.6) control matrix $F$ reads as

$$ F = \frac{1}{2v} \begin{pmatrix} 2u & 1 \\ H_2 & 2u \end{pmatrix}, \quad (4.3) $$

where

$$ u = -(\beta_1 M_1^2 + \beta_2 M_2^2), \quad v = -\kappa^{-2}(\beta_1 - \beta_2)p_3^2. $$

The eigenvalues of this matrix $F$ are the required variables of separation $q_{1,2}$

$$ q_1 = \frac{u + \sqrt{H_2}}{v}, \quad q_2 = \frac{u - \sqrt{H_2}}{v}. \quad (4.4) $$

These variables has been introduced in [1] without any explanations and reasonable arguments. We reproduce this result in framework of the generic method based on direct solution of the equations (3.2). According to [5], eigenvectors of the control matrix $F$ form the Stäckel matrix $S$

$$ F = S \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} S^{-1} $$

whose entries $S_{ij}$ depend only on a pair $(q_i, p_i)$ of the canonical variables of separation. In our case matrix $S$ is equal to

$$ S = \begin{pmatrix} 1 & 1 \\ 2\sqrt{H_2} & -2\sqrt{H_2} \end{pmatrix}. $$

It means that we have non-Stäckel integrable system with non-affine in $H_1$ or $H_2$ separated relations (4.1), similar to the generalized Chaplygin system [16] and the Kowalevski top [17].
From the definitions of separation coordinates $q_{i,2}$ (4.4) and $H_2$ (2.5) we immediately obtain

$$M_1 = \sqrt{\frac{(1 - q_1)(1 - q_2)}{2\beta_1(q_1 - q_2)}} H_2^{1/4}, \quad M_2 = \sqrt{\frac{(1 + q_1)(1 + q_2)}{2\beta_2(q_2 - q_1)}} H_2^{1/4},$$

and

$$\gamma_3 = \frac{2\chi H_2^{1/4}}{\sqrt{2(\beta_1 - \beta_2)(q_2 - q_1)}}.$$

Such as $C_1 = 0$ and

$$\dot{q}_k = \{H, q_k\} = -\frac{4\beta_1 p_2 M_1(1 + q_k)}{\gamma_3} - \frac{4\beta_2 \gamma_1 M_2(1 - q_k)}{\gamma_3},$$

we have

$$\gamma_1 = \frac{\chi}{4(q_1 - q_2)\sqrt{\beta_2(\beta_1 - \beta_2)}} \frac{(1 + q_2)\dot{q}_1 - (1 + q_1)\dot{q}_2}{\sqrt{(1 + q_1)(1 + q_2)}},$$

$$\gamma_2 = \frac{\chi}{4(q_1 - q_2)\sqrt{\beta_1(\beta_2 - \beta_1)}} \frac{(1 - q_2)\dot{q}_1 - (1 - q_1)\dot{q}_2}{\sqrt{(1 - q_1)(1 - q_2)}},$$

$$M_3 = \frac{\dot{q}_1(1 - q_2^2) + \dot{q}_2(1 - q_1^2)}{4(q_1 - q_2)\sqrt{-\beta_1\beta_2(1 - q_1^2)(1 - q_2^2)}}.$$

Substituting these expressions into the Hamiltonian $H_1$ and Cazimir $C_2$ and solving the resulting equations with respect to $\dot{q}_{1,2}$ one gets a pair of the Abel-Jacobi equations

$$\frac{\chi\dot{q}_1}{\sqrt{4(q_1^2 - 1)(\lambda_1 q_1 + \mu_1)}} = 1, \quad \frac{\chi\dot{q}_2}{\sqrt{4(q_2^2 - 1)(\lambda_2 q_2 + \mu_2)}} = 1. \quad (4.5)$$

where

$$\lambda_{1,2} = (\beta_1 - \beta_2)\left(C_2(\beta_1 + \beta_2) + \chi^2(H_1 \pm \sqrt{H_2})\right),$$

$$\mu_{1,2} = (\beta_1 - \beta_2)^2 C_2 + \chi^2(\beta_1 + \beta_2)(H_1 \pm \sqrt{H_2}).$$

Let us note that (4.5) are degenerate Abel-Jacobi equations, i.e. each of them depends on a unique variable $q_1$ or $q_2$ only, and a two-dimensional Abel torus splits into one-dimensional tori.

According to [9], the remaining separation variables $p_{1,2}$ are equal to

$$p_k = \frac{\dot{q}_k}{4(1 - q_k^2)(\beta_1(1 + q_k) + \beta_2(1 - q_k))}, \quad \{q_i, p_k\} = \delta_{ik}, \quad i, k = 1, 2.$$  

They satisfy to the following separated relations which directly follow from the Abel-Jacobi equations (4.5).

$$\Phi_{1,2}(q, p) = 4\chi^2(q^2 - 1)\left(\beta_1(1 + q) + \beta_2(1 - q)\right)^2 p^2 - \lambda_{1,2} q - \mu_{1,2} = 0, \quad (4.6)$$
Here \( q = q_{1,2} \) and \( p = p_{1,2} \). At \( \varkappa \to 0 \) these equations coincide with the separated relations for the Chaplygin system, see [16].

The third part of the Jacobi method consists of the construction of new integrable systems starting with known variables of separation and some other separated relations. Namely, if we substitute our variables of separation \( q_{1,2} \) and \( p_{1,2} \) into the following deformation of (4.6)

\[
\Phi^{(d)}(q, p) = \Phi_1 \Phi_2 - d_1 q - d_2 = 0, \quad d_1, d_2 \in \mathbb{R},
\]

and solve the resulting equations with respect to integrals of motion \( H_{1,2} \), then we get rational generalization of the initial polynomial Hamilton function

\[
H_1^{(d)} = H_1 - \frac{\kappa^2}{16\beta_1^2\beta_2^2(\beta_1 - \beta_2)(\varkappa^2M_1^2 + \varkappa^2M_2^2 + \gamma_3^2)^2}(\beta_1 \varkappa^2(d_1 - d_2)(\beta_1 - \beta_2)M_1^2
- \beta_2 \varkappa^2(d_1 + d_2)(\beta_1 - \beta_2)M_2^2 + (\beta_1^2(d_1 - d_2) + \beta_2^2(d_1 + d_2))\gamma_3^2).
\]

If \( d_1 = d_2(\beta_1 - \beta_2)/(\beta_1 + \beta_2) \) this Hamiltonian looks like

\[
H_1^{(d)} = H_1 + \frac{2d_2\kappa^2\beta_1\beta_2(\beta_1 - \beta_2)}{(\beta_1 + \beta_2)(\varkappa^2M_1^2 + \varkappa^2M_2^2 + \gamma_3^2)}.
\]

At \( \varkappa \to 0 \) we obtain the Hamilton function for the generalized Chaplygin system studied in [16]. The main problem of this part of the Jacobi method is how to get the Hamiltonian to be interesting to physics.

5 Conclusion

Starting with the integrals of motion for the Bogoyavlensky system on \( so(4) \) we found polynomial in momenta Poisson bivector \( P' \), which are compatible with the canonical Poisson bivector \( P \) on zero-level of the Casimir function \( C_1 \). Then in framework of the bi-hamiltonian geometry we reproduce known separation variables and separated relations. Some rational generalization of the Bogoyavlensky system is considered.

This example may be useful for creating a general theory, which takes the constructive answers on the main open questions:

- how to get the Poisson bivectors \( P' \) on \( so(n) \) compatible with \( P \);
- how to describe all the natural Hamilton functions associated with a given \( P' \).

Now we have some particular answers obtained by direct tedious computations only [2, 11, 13].

The author wish to thank A.V. Tsiganov for formulation of the problem and stimulating discussions.
References

[1] O. I. Bogoyavlensky, *Inverting Solitons. Nonlinear Integrable Equations*, M.: Nauka. 1991.

[2] A. V. Bolsinov, A. V. Borisov, *Compatible Poisson brackets on Lie algebras*, Matem. Notes, v. 72, p. 10-30, 2002.

[3] A.V. Borisov, I.S. Mamaev, *Rigid Body Dynamics. Hamiltonian Methods, Integrability, Chaos*, Moscow-Izhevsk, RCD, 2005.

[4] S.A. Chapligin, *A new partial solution of the problem of motion of a rigid body in a liquid*, Trudy otdel. Fiz. Nauk Obsh. Liub. Est. 11, p.7-10, 1903.

[5] G. Falqui, M. Pedroni, *Separation of variables for bi-Hamiltonian systems*, Math. Phys. Anal. Geom., 6, p.139-179, 2003.

[6] V.B. Kuznetsov, A.V. Tsiganov, *A special case of Neumann’s system and the Kowalewski-Chaplygin-Goryachev top*, J. Phys. A., 22, p.L73-79, 1989.

[7] F. Magri, *Eight lectures on Integrable Systems*. In: Integrability of Nonlinear Systems (Y. Kosmann-Schwarzbach et al. eds.), Lecture Notes in Physics 495, Springer Verlag, Berlin-Heidelberg, 1997, pp. 256–296.

[8] A.V. Tsiganov, *On the Kowalevski-Goryachev-Chaplygin gyrostat*, J. Phys. A, Math. Gen. 35, No.26, L309-L318, 2002.

[9] A.V. Tsiganov, *Integrable systems in the separation of variables method*, Moscow-Izhevsk, RCD, 2005.

[10] A.V. Tsiganov, *On the two different bi-Hamiltonian structures for the Toda lattice*, Journal of Physics A: Math. Theor. 40, pp. 6395-6406, 2007.

[11] A.V. Tsiganov, *Separation of variables for a pair of integrable systems on so*\(^*\)(4)*, Doklady Math., 76, p.839-842, 2007.

[12] A. V. Tsiganov, *A family of the Poisson brackets compatible with the Sklyanin bracket*, J. Phys. A: Math. Theor. v.40, pp.4803-4816, 2007.

[13] A.V. Tsiganov, *On bi-hamiltonian structure of some integrable systems on so*\(^*\)(4)*, J. Nonlinear Math. Phys., 15, p.171-185, 2008.

[14] A.V. Tsiganov, *On bi-hamiltonian geometry of the Lagrange top*, J. Phys. A: Math. Theor., 41, 315212 (12pp), 2008.

[15] A.V. Tsiganov, *The Poisson bracket compatible with the classical reflection equation algebra*, Regular and Chaotic Dynamics, 13, 191-203, 2008.

[16] A.V. Tsiganov, *On the generalized Chaplygin system*, Journal of Mathematical Sciences, v.168, n.8, p.901-911, 2010.
[17] A.V. Tsiganov, *New variables of separation for particular case of the Kowalevski top*, accepted to Reg. Chaot. Dynamics, Preprint: arXiv:1001.4599v1, 2010.

[18] A.V. Vershilov, A.V. Tsiganov, *On bi-Hamiltonian geometry of some integrable systems on the sphere with cubic integral of motion*, J. Phys. A: Math. Theor. **42**, 105203 (12pp), 2009.