A LINEAR WEGNER ESTIMATE FOR ALLOY TYPE
SCHRÖDINGER OPERATORS ON METRIC GRAPHS

MARIO HELM¹ AND IVAN VESELIĆ¹,²

Abstract. We study spectra of alloy-type random Schrödinger operators on
metric graphs. For finite edge subsets of general graphs we prove a Wegner
estimate which is linear in the volume (i.e. the number of edges) and the length
of the considered energy interval. The single site potential of the alloy-type
model needs to have fixed sign, but the considered metric graph does not need
to have a periodic structure. The second result we obtain is an exhaustion
construction of the integrated density of states for ergodic random Schrödinger
operators on metric graphs with a \( \mathbb{Z}^d \)-structure. For certain models the two
above results together imply the Lipschitz continuity of the integrated density
of states.

1. Introduction

In the present paper we study spectral properties of random Schrödinger opera-
tors on a metric graph. More precisely, we consider a countable metric graph and
a random Hamiltonian with a so-called alloy type potential. Under suitable as-
sumptions we are able to prove a Wegner estimate for the restriction of the random
Hamiltonian to a finite part of the graph. Our Wegner estimate is optimal in the
sense that it is linear in the energy and the volume.

In the case that the integrated density of states (IDS) of the random Schrödinger
operator exists, the Wegner estimate implies that the IDS is Lipschitz continuous.
For certain random operators on \( \mathbb{Z}^d \)-metric graphs we establish the existence of the
IDS via an exhaustion procedure. More generally, the existence of an selfaverag-
ing IDS is ensured by certain amenability assumptions on the metric graph and
ergodicity assumptions on the random Hamiltonian, see Remark 4.

The literature on Wegner estimates for Schrödinger operators in the continuum,
i.e. on \( \mathbb{R}^d \), is quite extensive. We refer to the textbook accounts [CFKS87, CL90,
PF92, Sto01, Ves04, KM] and the references therein. For the application to spectral
localization in random media, a quite weak form of Wegner estimate is sufficient.
However, upper bounds which are linear both in the volume and the energy interval
length are of independent interest, since they imply the regularity of the IDS. To
obtain such a Wegner estimate, the proofs which are presented in the literature need
some assumptions, beyond those necessary for a weak form of Wegner estimate. In
particular, mostly the covering condition

$$\sum_{k \in \mathbb{Z}^d} u(x - k) \geq \text{const.} > 0 \quad \text{for all } x \in \mathbb{R}^d$$

was assumed. Here \( u \) is a compactly supported, non-negative single site potential.
Papers which are devoted to the question how this assumption can be removed or at
least weakened include [Klo95, Kir96, Ves96, CHN01, KV02, Gie, CHK03]. In the
recent [CHK] a linear Wegner estimate is proposed without the abovementioned
covering condition. However, the deterministic part of the Schrödinger operator
needs to be translation invariant. The reason for this assumption is that it implies a

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uniform, translation invariant kind of unique continuation principle for the solutions of the Schrödinger equation, cf. Section 4 in [CHK03].

In our setting, on metric graphs, the situation is somewhat simpler, since in certain aspects the Schrödinger operators behave as in the one dimensional case. In particular, on the (one dimensional) edges one has a stronger (and simpler) unique continuation principle at disposal. We use techniques developed for Wegner estimates for operators on $L^2(\mathbb{R})$ in [KV02]. (See also [Ves96, Gie, CHK03] for similar results for one dimensional random Schrödinger operators.)

This allows us to prove linear Wegner estimates for random operators on very general metric graphs. In particular, we do not need any periodicity condition on the graph. Our assumptions are general enough to include, for instance, operators on the metric graph associated to $\mathbb{Z}^d$ or, more generally, the Cayley graph of a finitely generated discrete group, or a Penrose tiling graph.

Note that for quantum graphs there exists a certain dichotomy concerning the unique continuation principle. While along a single edge one has a quite strong version of this property, globally on the graph it does not hold. More precisely, a Laplacian (even without potential) on a metric graph with “free boundary conditions” at the vertices may exhibit compactly supported eigenfunctions. For ergodic Hamiltonians on discrete graphs it is well known (see for instance [DLM+03, KLS03, Ves05, Ves06, LV]) that compactly supported eigenfunctions are related to discontinuities of the IDS. Thus, for a complete understanding of the continuity properties of the IDS a systematic analysis of compactly supported eigenfunctions is necessary. See also [GLV] for related results and discussion.

The structure of the paper is as follows: in the next section we formulate our results, Section 3 contains the bulk of the proof of the Wegner estimate, while arguments relating to boundary condition perturbations are deferred to Section 4, and those to the unique continuation principle to Section 5. The last section contains a proof of the construction of the IDS via an exhaustion procedure.

2. Model and Results

First we define the geometric structure of the metric graphs and the operators acting on the associated $L^2$-Hilbert space.

Definition 1. Let $V$ and $E$ be countable sets and $\mathcal{G}$ a map

$$\mathcal{G}: E \to V \times V \times [0, \infty), \quad e \mapsto (\iota(e), \tau(e), l_e).$$

We call the triple $G = (V, E, \mathcal{G})$ a metric graph, elements of $V = V(G)$ vertices, elements of $E = E(G)$ edges, $\iota(e)$ the initial vertex of $e$, $\tau(e)$ the terminal vertex of $e$ and $l_e$ the length of $e$. Both $\iota(e)$ and $\tau(e)$ are called endvertices of $e$, or incident to $e$. The two endvertices of an edge are allowed to coincide. The number of edges incident to the vertex $v$ is called the degree of $v$. We assume that the degree is finite for all vertices. The elements of $V$ which have degree equal to one we call boundary vertices and denote their set by $V^\partial$. The elements of $V^i := V \setminus V^\partial$ are called interior vertices.

We will consider the graph $G = (V, E, \mathcal{G})$ as a topological space, more precisely as an one dimensional CW-complex, which we will again denote by $G$. The 0-skeleton of $G$ is $V$ and the collection of its one dimensional cells is given by $E$. Each one-dimensional cell $e \in E$ is attached either to one or to two zero dimensional cells $v \in V$, namely $\iota(e)$ and $\tau(e)$. This defines the topological structure of $G$.

We identify each edge $e$ with the open interval $(0, l_e)$, where the point 0 corresponds to the vertex $\iota(e)$ and $l_e$ to $\tau(e)$. Assume that there exist constants $0 < l_- < l_e < l_+ < \infty$ such that for all $e \in E$

$$0 < l_- \leq l_e \leq l_+ < \infty.$$
The identification of edges by intervals allows us to define in a natural way the length of a path between two points in the topological space $G$. Taking the infimum over the lengths of paths connecting two given points in $G$, one obtains a distance function $d : G \times G \to [0, \infty)$. Since we assumed that each vertex of $G$ has bounded degree, the map $d$ is indeed a metric, cf. for instance Section 2.2 in [Sch06]. Thus we have turned $G$ into a metric space $(G, d)$.

For a finite subset $\Lambda \subset E$ we define the subgraph $G_{\Lambda}$ by deleting all edges $e \in E \setminus \Lambda$ and the arising isolated vertices. We denote the set of vertices of $G_{\Lambda}$ by $V_{\Lambda}$, the set of vertices $v \in V_{\Lambda}$ of degree one (in $G_{\Lambda}$) by $V_{\Lambda}^{1}$, and its complement $V_{\Lambda} \setminus V_{\Lambda}^{1}$ by $V_{\Lambda}^{0}$. Again, elements of $V_{\Lambda}^{0}$ are called boundary vertices of $G_{\Lambda}$ and elements of $V_{\Lambda}^{1}$ interior vertices of $G_{\Lambda}$. Similarly as above we may consider $G_{\Lambda}$ as a sub-CW-complex of $G$ with an induced topology and metric.

For any $\Lambda \subset E$ the Hilbert spaces $L_{2}(G_{\Lambda})$ have a natural direct sum representation $L_{2}(G_{\Lambda}) = \oplus_{e \in E}L_{2}(0, l_{e})$. In particular for $\Lambda = E$ we have $L_{2}(G) = \oplus_{e \in E}L_{2}(0, l_{e})$, and for $\tilde{\Lambda} \subset \Lambda \subset E$ we have $L_{2}(G_{\tilde{\Lambda}}) = \oplus_{e \in \Lambda}L_{2}(0, l_{e}) \subset L_{2}(G_{\Lambda}) = \oplus_{e \in \Lambda}L_{2}(0, l_{e})$.

For a function $\phi : G \to \mathbb{C}$ and an edge $e \in E$ we denote by $\phi_{e} := \phi|_{e}$ its restriction to $e$ (which is identified with $(0, l_{e})$). We denote by $C(G)$ the space of continuous, complex-valued functions on the metric space $(G, d)$. Similarly, $C(G_{\Lambda})$ denotes the space of continuous, complex-valued functions on the metric sub-space $(G_{\Lambda}, d)$. For each $v \in V$, any edge $e$ incident to $v$, and function $f \in W^{2,2}(e) \subset C^{1}(e) \cong C^{1}(0, l_{e})$ we define the derivatives

(1) \[ \partial_{e}f(v) := \partial_{e}f(0) := \lim_{\epsilon \searrow 0} \frac{f(\epsilon) - f(0)}{\epsilon} \quad \text{if } v = \iota(e) \]

and

(2) \[ \partial_{e}f(v) := \partial_{e}f(l_{e}) := \lim_{\epsilon \searrow 0} \frac{f(l_{e}) - f(l_{e} - \epsilon)}{\epsilon} \quad \text{if } v = \tau(e). \]

Note that, since $f|_{e} \in W^{2,2}(e)$, by the Sobolev imbedding theorem the function $f$ is not only continuously differentiable on the open segment $(0, l_{e})$, but also that its derivative has well defined limits at both boundaries $0$ and $l_{e}$.

For any $\Lambda \subset E$ it will be convenient to use the following Sobolev space

\[ W^{2,2}(\Lambda) := \oplus_{e \in \Lambda}W^{2,2}(e) \subset C^{1}(\Lambda) := \oplus_{e \in \Lambda}C^{1}(e) \]

with the norm $\|\phi\|^{2}_{W^{2,2}(\Lambda)} := \sum_{e \in \Lambda}\|\phi_{e}\|^{2}_{W^{2,2}(0, l_{e})}$.

Note that this space is defined on the edge set only and does not see the graph structure of $G$. The operators which we consider will be defined on the space

\[ \mathcal{D}(\Delta_{\Lambda}) := \{f \in W^{2,2}(\Lambda) \cap C(G_{\Lambda}) \mid \forall v \in V_{\Lambda}^{1} : \sum_{e \in E, \iota(e) = v} \partial_{e}f(v) = \sum_{e \in E, \tau(e) = v} \partial_{e}f(v), \forall v \in V_{\Lambda}^{0} : f(v) = 0\}. \]

For each $\Lambda \subset E$ we define a linear operator

\[ -\Delta_{\Lambda} : \mathcal{D}(\Delta_{\Lambda}) \to L^{2}(\Lambda) \]

by the rule

\[ (-\Delta_{\Lambda}f)(x) := -\frac{\partial^{2}f_{e}(x)}{\partial x^{2}} \]

if $x \in G$ is contained in the edge $e$. This way the function $-\Delta_{\Lambda}f$ is defined on the set $E \subset G$, whose complement $V = G \setminus E$ in the metric space $G$ has Hausdorff measure zero.
The operator $-\Delta$ is self-adjoint on the domain $\mathcal{D}(\Delta)$, see for instance [Kuc04, KS99] or Section 3.3 in [Sch06]. At the boundary vertices it has clearly Dirichlet boundary conditions, while the type of boundary conditions it has at the interior vertices is called “free boundary conditions” by some authors and “Kirchhoff boundary conditions” by others.

Next we describe the potential energy part of the Hamiltonian. Since it is random, we need an appropriate probability space.

Let $\omega_- < \omega_+$ be real numbers, $\Omega$ a probability space, and $\mathbb{P}$ a probability measure on $\Omega$. Denote by $\mathbb{E}$ the mathematical expectation on $\Omega$ with respect to $\mathbb{P}$. Let $W: \Omega \times G \to [\omega_-, \omega_+]$ be a stochastic process which is jointly measurable in the variables $\omega \in \Omega$ and $x \in G$. For a fixed $\omega \in \Omega$ we denote by $W(\omega): L_2(G) \to L_2(G)$ the multiplication operator $W(\omega)f(x) = (W(\omega,x)f)(x), x \in G$.

To derive our Theorems 2 and 3 below we will need more specific hypotheses on $\mathbb{P}$ and $W(\omega)$. These are formulated in what follows. The first assumption describes random potentials of alloy type for which we are able to prove a Wegner estimate.

**Assumption 1.** Let $c_+ \geq c_- > 0$, $s > 0$ and $c_0$ be real numbers. For each edge $e \in E$ let $\mu_e$ be a probability measure which is absolutely continuous with respect to Lebesgue measure. More precisely, for $e \in E$ let $g_e \in L_\infty[\omega_-, \omega_+]$, $\|g_e\|_\infty \leq c_0$, and $d\mu_e(t) = g_e(t) dt$ on the interval $[\omega_-, \omega_+]$. Let the probability space $\Omega$ be given by the cartesian product $x \in E [\omega_-, \omega_+]$, and $\mathbb{P}$ by the product measure $\otimes_{e \in E} \mu_e$.

For each $e \in E$ let $u_e \in L_\infty(0, l_e)$ be a single site potential satisfying

$$c_- \chi_{\{S_e\}} \leq u_e \leq c_+ \chi_{[0,l_e]},$$

where $S_e \subset [0, l_e]$ is an interval of length $|S_e| \geq s$. We imbed $L_\infty(0, l_e) \cong L_\infty(\mathbb{E})$ in $L_\infty(G)$ and thus consider $u_e$ as an element of the latter space. Let the random potential $W(\omega)$ have the form

$$W(\omega): L_2(G) \to L_2(G), \quad W(\omega) = \sum_{e \in E} \omega_e u_e.$$

For quantum graphs with a $\mathbb{Z}^\nu$-structure we will establish the existence of the integrated density of states in Theorem 3. The following hypothesis formulates precisely what type of $\mathbb{Z}^\nu$-structure we need for this result.

**Assumption 2.** The vertex set $V$ consists of the points $\mathbb{Z}^\nu \subset \mathbb{R}^\nu$. The set of edges $E$ consists of segments $e = [x,y]$ parallel to the coordinate axes in $\mathbb{R}^\nu$ where $x, y \in V$ have Euclidean distance equal to one. The union $V \cup E \subset \mathbb{R}^\nu$ inherits the metric structure of $\mathbb{R}^\nu$. This structure coincides with the one defined earlier and we denote the corresponding metric space again by $(G, d)$.

Assume that for each $x \in \mathbb{Z}^\nu$ there is a measure preserving map $\tau_x: \Omega \to \Omega$ such that $\{\tau_x\}_{x \in \mathbb{Z}^\nu}$ forms an additive group which acts ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$.

Assume that the random potential transforms in the following way under translations: $W(\tau_x \omega, y) = W(\omega, y - x)$ for all $x \in \mathbb{Z}^\nu$, $y \in G$ and $\omega \in \Omega$. Thus the Schrödinger operator is equivariant in the sense that

$$U_x H(\omega) U_x^* = H(\tau_x \omega) \quad \text{for all } x \in \mathbb{Z}^\nu \text{ and } \omega \in \Omega$$

where $U_x f(y) = f(y - x), y \in G$ denotes the unitary translation operator by $x \in \mathbb{Z}^\nu$.

The restriction of the operator $W(\omega)$ to $L_2(\Lambda)$ will be denoted by $W_\Lambda(\omega)$ or, if the set $\Lambda$ is clear from the context, simply by $W(\omega)$ again. Note that the norm of the operator $W_\Lambda(\omega)$ is uniformly bounded in $\omega \in \Omega$ and $\Lambda \subset E$. Thus the operator $H_\Lambda(\omega) := -\Delta + W_\Lambda(\omega)$ is self-adjoint on $\mathcal{D}(\Delta)$ for any $\omega \in \Omega$ and $\Lambda \subset E$. If the set $\Lambda \subset E$ contains a single element $e$ we write $-\Delta_e(\omega)$ for $-\Delta_\Lambda(\omega)$ and $H_e(\omega)$ for $H_\Lambda(\omega)$.

A metric graph $G$ together with a Schrödinger $H$ operator which is defined on it we call a quantum graph.
Our main result is the following:

**Theorem 2.** If Assumption 1 holds, then there exists for any $\lambda \in \mathbb{R}$ a constant $C$ such that for all $\epsilon \in [0, 1]$,

$$
\mathbb{E}\{\text{tr} \chi_{(-\epsilon, \lambda+\epsilon)}(H_\Lambda(\omega))\} \leq C \cdot \epsilon \cdot \sharp \Lambda.
$$

If in the terminology of Assumption 1 we have $S_\epsilon = [0, l_\epsilon]$ for all $\epsilon \in E$, then the constant appearing in Theorem 2 can be chosen uniformly in the energy parameter $\lambda \in \mathbb{R}$.

Now we turn to the situation where the quantum graph has a $\mathbb{Z}^d$-structure as formulated in Assumption 2, and describe how the integrated density of states can be defined by an exhaustion procedure.

For any $l \in \mathbb{N}$ denote by $\Lambda_l$ the set of edges which are contained in

$$\{x \in \mathbb{R}^d \mid x_i \in (0, l) \text{ for all } i = 1, \ldots, d\},$$

and abbreviate $V_{\Lambda_l}$ by $V_l$, and $G_{\Lambda_l}$ by $G_l$. Denote by $\partial G_l$ the boundary of $G_l$ as a subset of the metric space $G$, i.e. the vertices in $V_l$ which have degree one. Similarly abbreviate $\Delta_l := \Delta_{\Lambda_l}$ and $H_l(\omega) := H_{\Lambda_l}(\omega)$. These are restrictions of the operators $\Delta$ and $H(\omega)$ to a finite cube with sidelength $l$ and with Dirichlet boundary conditions on the boundary of the cube. The finite cube Schrödinger operator $H_l(\omega)$ is again self-adjoint, lower bounded and has purely discrete spectrum. Let us enumerate the eigenvalues of $H_l(\omega)$ in ascending order $\lambda_1(H_l(\omega)) < \lambda_2(H_l(\omega)) \leq \lambda_3(H_l(\omega)) \leq \ldots$ and counting multiplicities.

Thus for each $\lambda \in \mathbb{R}$ and $l \in \mathbb{N}$, the counting function

$$F_\omega^l(\lambda) := \sharp\{n \in \mathbb{N} \mid \lambda_n(H_l(\omega)) \leq \lambda\}$$

is monotone increasing and right-continuous, i.e. a distribution function, which is associated to a pure point measure. Denote by $N_\omega^l(\lambda) := \frac{1}{l} F_\omega^l(\lambda)$ the volume-scaled version of $F_\omega^l(\lambda)$.

**Theorem 3.** Let Assumption 2 hold, then there exists a distribution function $N : \mathbb{R} \rightarrow \mathbb{R}$ and a subset $\Omega' \subset \Omega$ of measure one such that for all $\omega \in \Omega$ and for all $\lambda \in \mathbb{R}$ where $N$ is continuous the convergence

$$
\lim_{l \rightarrow \infty} N_\omega^l(\lambda) = N(\lambda)
$$

holds.

**Remark 4.** Under certain additional assumptions it is possible to prove that the normalized finite volume eigenvalue counting functions converge uniformly in the energy parameter to the IDS, see [GLV]. For metric graphs whose isometry group does not exhibit a $\mathbb{Z}^d$-structure, but is merely amenable, it should be still possible to define the IDS of an ergodic Hamiltonian by an exhaustion, proceeding analogously as in the paper [LPV04].

3. **Proof of the Wegner estimate**

For the purposes of the proof it will be necessary to differentiate the spectral projection with respect to the energy parameter, which motivates the introduction of the following smooth 'switch function'.

Let $\rho$ be a smooth, non-decreasing function such that on $(-\infty, -\epsilon]$ it is identically equal to $-1$, on $[\epsilon, \infty)$ it is identically equal to zero and $\|\rho'\|_{\infty} \leq 1/\epsilon$. Then

$$
\chi_{(-\epsilon, \lambda+\epsilon)}(x) \leq \rho(x - \lambda + 2\epsilon) - \rho(x - \lambda - 2\epsilon) = \int_{-2\epsilon}^{2\epsilon} \rho'(x - \lambda + t) \, dt
$$
Thus by the spectral theorem

$$\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\Lambda(\omega)) \leq \int_{-2\epsilon}^{2\epsilon} dt \rho'(H_\Lambda(\omega) - \lambda + t)$$

in the sense of quadratic forms. Since $B_\epsilon(\lambda) = (\lambda-\epsilon,\lambda+\epsilon)$ is bounded and $\sigma(H_\Lambda(\omega))$ discrete, the above operators are trace class and we may estimate:

$$\text{tr} \left[ \chi_{B_\epsilon(\lambda)}(H_\Lambda(\omega)) \right] \leq \text{tr} \left[ \int_{-2\epsilon}^{2\epsilon} \rho'(H_\Lambda(\omega) - \lambda + t) dt \right] = \sum_{n \in \mathbb{N}} \int_{-2\epsilon}^{2\epsilon} \rho'(\lambda^*_n(\omega) - \lambda + t) dt$$

where $\lambda^*_n(\omega)$ denotes the eigenvalues of $H_\Lambda(\omega)$ enumerated in non-decreasing order and counting multiplicities. Only a finite number of terms in the sum are non-zero.

**Proof of Theorem 2.** Let $\rho$ be as above. Similarly as in [Kir96], p. 509, we estimate

$$\mathbb{E}\{\text{tr} \chi_{[\lambda-\epsilon,\lambda+\epsilon]}(H_\Lambda(\omega))\} \leq \int \sum_{n \in \mathbb{N}} \int_{-2\epsilon}^{2\epsilon} \rho'(\lambda^*_n(\omega) - \lambda + t) dt \prod_{\tilde{\epsilon} \in \Lambda} d\mu_{\tilde{\epsilon}}.$$  

To bound the right hand side we use Corollary 9 in Section 5 and the monotone convergence theorem to obtain the estimate

$$C_1 \sum_{n \in \mathbb{N}} \int_{-2\epsilon}^{2\epsilon} \int_{\Lambda} \sum_{\epsilon \in \Lambda} \frac{\partial \rho(\lambda^*_n(\omega) - \lambda + t)}{\partial \omega_\epsilon} dt \prod_{\tilde{\epsilon} \in \Lambda} d\mu_{\tilde{\epsilon}}(\omega_\tilde{\epsilon})$$

$$\leq C_1 \sum_{\epsilon \in \Lambda} \int_{\Lambda} \prod_{\tilde{\epsilon} \in \Lambda} d\mu_{\tilde{\epsilon}}(\mu_\epsilon) \sum_{n \in \mathbb{N}} \int_{-2\epsilon}^{2\epsilon} \int_{\omega_-}^{\omega_+} \frac{\partial \rho(\lambda^*_n(\omega) - \lambda + t)}{\partial \omega_\epsilon} d\mu_\epsilon(\omega_\epsilon) dt.$$  

Here we used the abbreviation $A := \Lambda \setminus \{\epsilon\}$. Denote by $H_\Lambda(e,\omega_+)$ and $H_\Lambda(e,\omega_-)$ the operator $H_\Lambda$, where the random variable $\omega_\epsilon$ is set to its maximum respectively its minimum value. The $n$-th eigenvalues of this operators are abbreviated by $\lambda^*_n(e,\omega_+)$ and $\lambda^*_n(e,\omega_-)$. Using monotonicity, the sum over $n$ in (6) can be estimated by

$$\|g_\epsilon\|_\infty \sum_{n \in \mathbb{N}} \int_{-2\epsilon}^{2\epsilon} \{ \rho(\lambda^*_n(e,\omega_+) - \lambda + t) - \rho(\lambda^*_n(e,\omega_-) - \lambda + t) \} dt.$$  

We introduce now the operators $H_\Lambda^*(e,\omega)$, $* \in \{D,N\}$, that coincide with $H_\Lambda(\omega)$ up to additional Dirichlet, respectively Neumann b.c. at the vertices $\iota(e)$ and $\tau(e)$. Their eigenvalues are denoted by $\lambda^*_n(*)$. By Lemma 7 on Dirichlet-Neumann bracketing in Section 4 we have

$$\rho(\lambda^*_n(e,\omega_+) - \lambda + t) - \rho(\lambda^*_n(e,\omega_-) - \lambda + t) \leq \rho(\lambda^*_nD(e,\omega_+) - \lambda + t) - \rho(\lambda^*_nN(e,\omega_-) - \lambda + t).$$

Because of the decoupling of the edge $e$, the latter operators can be written as the direct sum $H_\Lambda^*(e,\omega) = H_\Lambda^{*D}(e,\omega) \oplus H_\Lambda^{*N}(e,\omega)$ of operators acting on $L_2(0,l_e)$ and $L_2(G_{\Lambda \setminus \epsilon})$, respectively. For the eigenvalues of these operators we use the notation $\lambda^*_n(*)$ and $\lambda^*_n(*)$. Hence the sum over the terms in (8) can be separated in
the corresponding parts:

\begin{align}
\sum_{n \in \mathbb{N}} & \rho(\lambda_n^{\Lambda,D}(e, \omega_+) - \lambda + t) - \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t) \\
& = \sum_{n \in \mathbb{N}} \rho(\lambda_n^{\Lambda,D}(e, \omega_+) - \lambda + t) - \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t) \\
& + \sum_{n \in \mathbb{N}} \rho(\lambda_n^{\Lambda,D}(e, \omega_+) - \lambda + t) - \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t).
\end{align}

We estimate first the sum in (10). The difference $H^{c,D}_\Lambda - H^{c,N}_\Lambda$ is a perturbation of rank 2 in resolvent sense, see for instance [Sim95]. By Lemma 6 in Section 4 the first term in (10) obeys the bound

$$\rho(\lambda_n^{\Lambda,D}(e, \omega_+) - \lambda + t) \leq \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t).$$

By a telescoping argument we obtain the estimate

$$\sum_{n \in \mathbb{N}} \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t) - \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t) \leq 2$$

where we used that the total variation of $\rho$ equals one. Thus we are left with estimating

\begin{align}
\sum_{n \in \mathbb{N}} & \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t) - \rho(\lambda_n^{\Lambda,N}(e, \omega_-) - \lambda + t) \\
& = \text{tr} \left[ \rho(H^{c,N}_\Lambda(e, \omega_+)) - \rho(H^{c,N}_\Lambda(e, \omega_-)) \right].
\end{align}

Note that by definition of $\rho$ we have $\rho(x - \lambda + t) \leq \chi_{(\lambda - 3\epsilon, \infty)}(x)$ and $-\rho(x - \lambda + t) \leq \chi_{(\lambda + 3\epsilon, \infty)}(x)$ for all $t \in [-2\epsilon, 2\epsilon]$. Thus the trace in (12) is bounded by

\begin{align}
\text{tr} \left[ \chi_{(\lambda - 3\epsilon, \infty)}(H^{c,N}_\Lambda(e, \omega_+)) - \chi_{(\lambda + 3\epsilon, \infty)}(H^{c,N}_\Lambda(e, \omega_-)) \right] & \\
& \leq \text{tr} \left[ \chi_{(\lambda - 3\epsilon, \infty)}(H^{c,N}_\Lambda(e, \omega_-) + (\omega_+ - \omega_-)\|u_e\|_\infty) - \chi_{(\lambda + 3\epsilon, \infty)}(H^{c,N}_\Lambda(e, \omega_-)) \right] \\
& = 2 + \text{tr} \left( \chi_{(\lambda - 3\epsilon, \omega_- - \omega_+)}(\|u_e\|_\infty, \lambda + 3\epsilon)(H^{c,N}_\Lambda(e, \omega_-)) \right) \\
& \leq C_2.
\end{align}

Here $C_2$ is a constant which is independent of $\Lambda$ and depends only on the considered energy interval. In fact, a careful look reveals that $C_2$ depends only on the length $(6\epsilon + (\omega_+ - \omega_-)\|u_e\|_\infty)$ of the energy interval, but not on $\Lambda \in \mathbb{R}$. See for instance Section 3 in [GLV].

Next we want to estimate the sum in (11). Let $\tilde{d} := \deg(t(e)) + \deg(\tau(e)) - 2$. Similarly as above, one sees that the difference $H^{c,D}_\Lambda(e, \omega_+) - H^{c,N}_\Lambda(e, \omega_+)$ is a perturbation of rank $\tilde{d}$ in resolvent sense. Consequently, the first term in (11) obeys the bound

$$\rho(\lambda_n^{\Lambda,D}(e, \omega_+) - \lambda + t) \leq \rho(\lambda_n^{\Lambda,N}(e, \omega_+) - \lambda + t).$$

A telescoping argument bounds the whole sum in (11) by $\tilde{d}$ times the total variation of $\rho$, i.e. by $\tilde{d}$.

Hence the following upper bound on (6) completes the proof:

$$C_1 \sum_{e \in \Lambda} \|u_e\|_\infty \int_{-2\epsilon}^{2\epsilon} (\tilde{d} + C_2) \, dt \leq 4C_1(c_2 + \tilde{d}) \epsilon \cdot \xi \Lambda.$$
4. Changing boundary conditions

In this section we show how to control the shifting of eigenvalues when one modifies boundary conditions. This is used in the proof of Theorem 2 above.

Lemma 6 shows that changing the boundary conditions from the Dirichlet operator $H^{e,D}_\Lambda(e,\omega_-)$ to the Neumann one $H^{e,N}_\Lambda(e,\omega_-)$ shifts the eigenvalue index at most by the rank of the perturbation, i.e. by 2. The proof of this fact is based on the min-max formula for eigenvalues, see e.g. [RS78], Theorem XIII.1.

Lemma 7 is a monotonicity statement about boundary conditions known as Dirichlet-Neumann-bracketing. By quadratic form considerations one sees that the passage from Neumann to Dirichlet boundary condition shifts the eigenvalues up.

First we proof an auxiliary lemma for bounded operators.

**Lemma 5.** Let $S,T = S+V$ be bounded, selfadjoint operators on a separable Hilbert space $\mathcal{H}$ with $\dim (\ker V) = d < \infty$. Assume that the spectra of $S$ and $T$ below $\inf \sigma_{ess}(S)$, respectively $\inf \sigma_{ess}(T)$, consist of an infinite, discrete set of eigenvalues.

Then for the $m$-th eigenvalues of $S$ and $T$, counted in ascending order including multiplicities, we have $\lambda_m(S) \leq \lambda_{m+d}(T)$ ($\forall m \in \mathbb{N}$).

**Proof.** Let $\tilde{L} := (\ker V)^\perp$. By the min-max principle one has

$$\lambda_m(T) = \max_{\dim L = m-1} \min_{\|\varphi\| = 1, \varphi \perp L} (\varphi | T \varphi) \leq \max_{\dim L = m-1} \min_{\|\varphi\| = 1, \varphi \perp L+\tilde{L}} [(\varphi | S \varphi) + (\varphi | V \varphi)]_{=0} \leq \max_{\dim L = m+d-1} \min_{\|\varphi\| = 1, \varphi \perp L} (\varphi | S \varphi) = \lambda_{m+d}(S) \quad \blacksquare$$

The lemma above can of course not be applied to the Hamiltonians under consideration because they are unbounded operators. But there are some related operators for which the statement is true - one can simply compare suitable resolvents and rearrange the eigenvalues by the spectral theorem.

**Lemma 6.** For the operators $H^{e,D}_\Lambda(e,\omega_-)$ and $H^{e,N}_\Lambda(e,\omega_-)$ we have

$$\lambda_m(H^{e,D}_\Lambda(e,\omega_-)) \leq \lambda_{m+2}(H^{e,N}_\Lambda(e,\omega_-)). \quad (13)$$

**Proof.** We shift the spectrum of both operators by addition of a suitable constant and work in the following with the two arising strictly positive operators $H_1$ (Dirichlet case) and $H_2$ (Neumann case).

Let $D_0 := D(H_1) \cap D(H_2)$. Then one has

$$H_1^{-1} - H_2^{-1} \mid_{D_0} = H_1^{-1}(H_2 - H_1)H_2^{-1} \mid_{H_2D_0}. \quad (14)$$

By definition, $H_1 \mid_{D_0} - H_2 \mid_{D_0} = 0$, such that by (14) and continuity we get

$$H_1^{-1} - H_2^{-1} \mid_{H_2D_0} = 0,$$

i.e. $\ker (H_1^{-1} - H_2^{-1}) \perp \subset H_2D_0$.

We want to apply Lemma 5 to the bounded operators $-H_1^{-1}$ and $-H_2^{-1}$. So we have to show that $\dim \ker (H_1^{-1} - H_2^{-1}) \perp \leq 2$, for what in turn $\dim H_2D_0 \perp \leq 2$ is sufficient. For this purpose we show next that $D_0$ and $D(H_2)$ differ by a 2-dimensional subspace $L$.

Let $\phi_1, \phi_2 \in C^\infty(0, l_c)$ be such that $\phi_1 \equiv 1$ in a neighborhood of 0 and $\phi_1 \equiv 0$ in a neighborhood of $l_c$, and similarly let $\phi_2 \equiv 0$ in a neighborhood of 0 and $\phi_2 \equiv 1$ in a neighborhood of $l_c$. Thus $\phi_1, \phi_2$ are linearly independent vectors and moreover elements of $D(H_2) \setminus D_0$. 
Let \( \psi \) an arbitrary element of \( D(H_2) \), \( c_1 := \psi(0) \) and \( c_2 := \psi(l_e) \). Then
\[
(\psi - c_1 \phi_1 - c_2 \phi_2)'(0) \quad \text{and} \quad (\psi - c_1 \phi_1 - c_2 \phi_2)'(l_e) = 0
\]
as well as
\[
(\psi - c_1 \phi_1 - c_2 \phi_2)(0) = 0 \quad \text{and} \quad (\psi - c_1 \phi_1 - c_2 \phi_2)(l_e) = 0.
\]
Hence \psi - c_1 \phi_1 - c_2 \phi_2 \in D_0 \) what implies that \( \dim D(H_2) \setminus D_0 = 2 \).

Now, \( H_2 : D(H_2) \to \mathcal{H} \) is one to one, and we get
\[
\mathcal{H} = H_2 D(H_2)
\]
\[
= H_2(D_0 + L)
\]
\[
= H_2 D_0 + H_2 L.
\]
Hence \( \dim H_2 D_0^\perp \leq \dim H_2 L \leq 2 \).

So Lemma 5 is applicable to \( -H_1^{-1} \) and \( -H_2^{-1} \), and inequality (13) follows by the spectral theorem. \( \square \)

It can be seen easily that the operators \( H^{e,D}_\Lambda(e,\omega) \) and \( H^{e,N}_\Lambda(e,\omega) \) in the proof of Theorem 2 can be treated in the same way.

**Lemma 7.** Let \( \omega \in \Omega, \Lambda \subset E \) finite, and \( e \in \Lambda \) be arbitrary. Consider the operators \( H_\Lambda(e,\omega) \), \( H^{e,D}_\Lambda(e,\omega) \) and \( H^{e,N}_\Lambda(e,\omega) \) and their eigenvalues \( \lambda^{e,D}_n(\Lambda,\omega) \), \( \lambda^{e,N}_n(\Lambda,\omega) \) (defined in Section 6). Then the following inequalities hold for all \( n \in \mathbb{N} \)
\[
\lambda^{e,N}_n(\Lambda,\omega) \leq \lambda^{e,D}_n(\Lambda,\omega) \leq \lambda^{e,D}_n(\Lambda,\omega).
\]

**Proof.** Denote by \( h_\Lambda(e,\omega) \), \( h^{D}_\Lambda(e,\omega) \) and \( h^{N}_\Lambda(e,\omega) \) the quadratic forms associated to the operators \( H_\Lambda(e,\omega) \), \( H^{e,D}_\Lambda(e,\omega) \) and \( H^{e,N}_\Lambda(e,\omega) \). Then by Section 4.2 in [Sch06] the quadratic form domains obey \( h^{N}_\Lambda(e,\omega) \supset h_\Lambda(e,\omega) \supset h^{D}_\Lambda(e,\omega) \).

Thus the statement of the lemma follows immediately if one applies the quadratic form version of the min-max formula for eigenvalues, see e.g. [RS78], Theorem XIII.2. \( \square \)

## 5. Single site potentials of small support

In this section we prove a uniform lower bound on the sum of derivatives of eigenvalues which is formulated in

**Lemma 8.**
\[
\sum_{e \in \Lambda} \frac{\partial \lambda^e_n(\omega)}{\partial \omega_e} \geq C_1(I) > 0
\]
for all eigenvalues \( \lambda^e_n \) of \( H_\Lambda(\omega) \) inside a bounded energy interval \( I \). The bound \( C(I) \) does not depend on the set of edges \( \Lambda \subset E \) and on the eigenvalue index \( n \in \mathbb{N} \).

We infer immediately:

**Corollary 9.** Let \( \rho : \mathbb{R} \to [0,1] \) be a smooth, monotone function with \( \rho = -1 \) on \( (-\infty,-\epsilon] \) and \( \rho = 0 \) on \( [\epsilon,\infty) \). Then, for the \( n+1 \)th eigenvalue \( \lambda^e_n \) of \( H_\Lambda(\omega) \) we have
\[
\rho'(\lambda^e_n(\omega) - \lambda + t) \leq C_1(I) \sum_{e \in \Lambda} \frac{\partial \rho(\lambda^e_n(\omega) - \lambda + t)}{\partial \omega_e}.
\]

**Proof.** By the chain rule
\[
\sum_{e \in \Lambda} \frac{\partial \rho(\lambda^e_n(\omega) - \lambda + \theta)}{\partial \omega_e} = \rho'(\lambda^e_n(\omega) - \lambda + \theta) \sum_{e \in \Lambda} \frac{\partial \lambda^e_n(\omega)}{\partial \omega_e},
\]
(16) implies the estimate:
\[ \rho'(\lambda_n^A(\omega) - \lambda + \theta) \leq C_1(I)^{-1} \sum_{e \in \Lambda} \frac{\partial \rho(\lambda_n^A(\omega) - \lambda + \theta)}{\partial \omega_e}. \]

To infer the lower bound (16), set \( S = \bigcup_{e \in \Lambda} S_e \) and apply the Hellman-Feynman theorem, i.e. first order perturbation theory. For a normalized eigenfunction \( \psi_n \) corresponding to \( \lambda_n^A(\omega) \) we have:
\[
\sum_{e \in \Lambda} \frac{\partial \lambda_n^A(\omega)}{\partial \omega_e} = \sum_{e \in \Lambda} (\psi_n | u_e \psi_n) \geq \int_S |\psi_n|^2.
\]
If the integral on the right hand side would extend over the whole of \( \Lambda \), it would be equal to 1 due to the normalization of \( \psi_n \). A priori the integral over \( S \) could be much smaller, but the following Lemma shows that we can control the ratio of the two integrals.

**Lemma 10.** Let \( I \) be a bounded interval and \( S_e \subset [0,l_e] \) a non-degenerate interval. There exists a constant \( C_1(I) > 0 \) such that
\[
\int_{S_e} |\psi|^2 \geq C_1(I) \int_0^{l_e} |\psi|^2
\]
for all \( \Lambda \subset E \) finite, all \( e \in \Lambda \) and for any eigenfunction \( \psi \) corresponding to an eigenvalue \( \lambda \in I \) of \( H_\Lambda(\omega) \).

Thus \( \int_S |\psi|^2 \geq C_1(I) \int_{G_\Lambda} |\psi|^2 \) with the same constant as in (17). Hence Lemma 10 implies directly Lemma 8.

**Proof.** For \( y \in \mathbb{R} \) denote by \( S_e + y := \{ x \in (0,l_e) \mid x - y \in S_e \} \) the translations of the set \( S_e \) along the edge \( e \). The derivative of the function
\[
\phi(y) := \int_{S_e+y} |\psi(x)|^2 \, dx = \int_{S_e} |\psi(x-y)|^2 \, dx
\]
satisfies
\[
\left| \frac{\partial}{\partial y} \phi(y) \right| = \left| \int_{S_e} \left[ \frac{\partial}{\partial y} \psi(x-y) \right] \psi(x-y) \, dx + \int_{S_e} \psi(x-y) \frac{\partial}{\partial y} \psi(x-y) \, dx \right| \leq 2 ||\psi||_{L^2(S_e+y)} ||\psi'||_{L^2(S_e+y)}.
\]
Sobolev norm estimates (e.g. Theorems 7.25 and 7.27 in [GT83]) imply
\[
||\psi'||_{L^2(S_e+y)} \leq C_3 ||\psi||_{L^2(S_e+y)} + ||\psi''||_{L^2(S_e+y)}.
\]
By the eigenvalue equation we have
\[
\left| \frac{\partial}{\partial y} \phi(y) \right| \leq C_4 ||\psi||_{L^2(S_e+y)}^2 = C_4 \phi(y), \quad C_4 = C_4(||W - \lambda||_\infty).
\]
Gronwall’s Lemma implies \( \phi(y) \leq \exp(C_4|y|) \phi(0) \) and thus
\[
\int_0^{l_e} |\psi|^2 \leq e^{C_4 l_e} \frac{l_e}{|S_e|} \int_{S_e} |\psi|^2.
\]
\( \square \)
6. Proof of Theorem 3

The proof of Theorem 3 on the existence of the integrated density of states follows the arguments of [KMS82]. There the convergence (5) of the IDS was established for all rational energies $\lambda \in \mathbb{Q}$, whereas we establish the same fact for all $\lambda$ which are continuity points of the IDS. The main step consists in proving that a superadditive ergodic theorem of the paper [AK81] is applicable.

In the whole of this section we assume that the quantum graph has a $\mathbb{Z}^\nu$-structure as described in Assumption 2.

Let us first describe the type of superadditive processes considered in [AK81]. Denote by $T$ the semigroup of measurable transformations given by $\tau_x$, $x \in \mathbb{N}_0^\nu$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and by $\mathcal{I}$ the class of sets of the form

$$\{ x \in \mathbb{R}^\nu \mid a_i \leq x_i < b_i, \text{ for all } i = 1, \ldots, d \}$$

where $a, b \in \mathbb{N}_0^\nu$. For $x \in \mathbb{N}_0^\nu$ and $Q \in \mathcal{I}$ denote by $Q + x := \{ y \in \mathbb{R}^\nu \mid y - x \in Q \}$ the translation of the set $Q$ by $x$.

**Definition 11.** A function $F : \mathcal{I} \to L^1(\Omega, \mathbb{P})$ which satisfies

(i) $F_Q \circ \tau_x = F_{Q+x}$ for all $Q \in \mathcal{I}$ and $x \in \mathbb{N}_0^\nu$,

(ii) if $Q_1, \ldots, Q_n \in \mathcal{I}$ are disjoint sets and if their union $Q = \cup_{i=1}^n Q_i$ is again an element of $\mathcal{I}$, then

$$F_Q \geq \sum_{i=1}^n F_{Q_i},$$

(iii) $\gamma(F) := \sup_{Q \in \mathcal{I}, |Q| > 0} \frac{1}{|Q|} \mathbb{E}\{F_Q\} < \infty$

is called a (discrete) superadditive process.

We state Theorem 2.4 from [AK81]:

**Theorem 12.** If $F$ is a discrete superadditive process and $Q_l := \{ x \mid 0 \leq x_i < l \text{ for all } i = 1, \ldots, d \}$, then

$$\lim_{l \to \infty} \frac{F_{Q_l}(\omega)}{l^\nu}$$

exists for almost all $\omega \in \Omega$.

In fact, in the case that the semigroup $T$ acts ergodically on the probability space $(\Omega, \mathbb{P})$ one can identify the limit, see the remark on page 59 in [AK81]:

$$\lim_{l \to \infty} l^{-d} F_{Q_l}(\omega) = \gamma(F) \quad \text{almost surely.}$$

In Section 6.2 of [Kre85] the above statements are extended to the case that it is not the semigroup $T$ which acts ergodically on $\Omega$, but rather the full group $\tau_x$, $x \in \mathbb{Z}^\nu$.

To apply the superadditive ergodic theorem we consider for arbitrary, fixed $\lambda \in \mathbb{R}$ the process given by the eigenvalue counting functions of the Schrödinger operator with Dirichlet boundary conditions. For $Q \in \mathcal{I}$ denote by $\Lambda$ the set of edges $e \in E \subset \mathbb{R}^\nu$ which are contained in the interior of $Q$ and set

$$F_Q := F_Q(\lambda, \omega) := \sharp\{ n \mid \lambda_n(H_{\Lambda}(\omega)) < \lambda \}, \quad Q \in \mathcal{I}.$$  

The Dirichlet Schrödinger operator $H_{\Lambda}(\omega)$ has been defined in Section 2. We will show that $F_Q$, $Q \in \mathcal{I}$, is a superadditive process.

**Lemma 13.** For any fixed energy value $\lambda$, the process $F_Q$ is superadditive.
Proof. We have to check that the properties (i) – (iii) in Definition 11 hold.

Since \( \Lambda = \text{int } Q \cap E \) we have also \( \Lambda + x = (\text{int } Q + x) \cap E \) for all \( x \in \mathbb{Z}^n \).

The equivariance property (4) of the random operators carries over to the spectral projections and thus to the eigenvalue counting functions. Hence property (i) holds.

Property (ii) can be seen using Lemma 7 on Dirichlet-Neumann bracketing in Section 4. For \( Q \) and \( Q_1, \ldots, Q_n \in \mathcal{I} \) as in (ii), the set \( \Lambda \) contains \( \bigcup_{i=1}^n \Lambda_i \) and finitely many edges \( e_1, \ldots, e_N \) which lie in \( \text{int } Q \setminus \bigcup_{i=1}^n \text{int } Q_i \). By introducing Dirichlet boundary conditions at finitely many vertices one obtains from the operator \( H_\Lambda(\omega) \) the direct sum

\[
\bigoplus_{i=1}^n H_{\Lambda_i}(\omega) \oplus \bigoplus_{j=1}^N H_{e_j}(\omega).
\]

(20)

Hence the eigenvalue counting function of \( H_\Lambda(\omega) \) is an upper bound of the one of the direct sum operator (20). Obviously the counting function of \( \bigoplus_{i=1}^n H_{\Lambda_i}(\omega) \) can be estimated from above by the one of the operator (20). Now property (ii) follows.

Denote the eigenvalue counting function of the negative Dirichlet Laplacian on \( e = (0, 1) \) by

\[
n_0(\lambda) := \# \{ n \in \mathbb{N} \mid \lambda_n(\Delta_e) \leq \lambda \},
\]

Obviously the counting function of

\[
-\bigoplus_{e \in \Lambda} \Delta_e
\]

equals \( |\Lambda| \cdot n_0(\lambda) \). Since the operators \( -\Delta_\Lambda \) and (21) differ by Dirichlet boundary conditions at \( 2|\Lambda| \), or even less, vertices, the counting function of \( -\Delta_\Lambda \) is bounded by

\[
|\Lambda| \cdot n_0(\lambda) + 4d \cdot |\Lambda|.
\]

For this conclusion we use an argument analogous to Lemma 6 in Section 4.

Since the random potentials we are considering are uniformly bounded by a constant, say \( K \), we have

\[
F_Q(\lambda, \omega) \leq |\Lambda| \cdot n_0(\lambda + K) + 4d \cdot |\Lambda| \quad \text{for all } \lambda \in \mathbb{R}, Q \in \mathcal{I}.
\]

The number of edges in the set \( \Lambda \) which is associated to a box \( Q \in \mathcal{I} \) is linearly bounded by the volume of \( Q \). Hence (iii) is proven.

\( \Box \)

Now we can complete the

Proof of Theorem 3. For a fixed \( \lambda \in \mathbb{R} \) one applies the ergodic theorem of [AK81] to the superadditive process \( F_Q(\lambda, \omega), Q \in \mathcal{I} \). Let us denote the corresponding \( \gamma(F) \) by \( \gamma(\lambda) \). By definition \( F_Q(\lambda, \omega) \leq F_Q(\lambda, \omega) \) for all \( \lambda \leq \lambda \in \mathbb{R} \) and all \( \omega \in \Omega, Q \in \mathcal{I} \), thus \( \lambda \mapsto \gamma(\lambda) \) is a non-decreasing function. In particular, it has at most a countable set of discontinuity points. Denote the complement of this set by \( \mathcal{C} \) and choose a dense countable set \( S \subset \mathcal{C} \). Hence \( \gamma \) is continuous at each \( \lambda \in S \).

Since in our case the transformation group is ergodic, for each \( \lambda \) there is a set \( \Omega_\lambda \) of measure one on which the convergence \( \lim_{t \to \infty} t^{-d} F_Q(\omega) = \gamma(\lambda) \) holds.

Since \( S \) is countable, \( \Omega' = \cap_{\lambda \in S} \Omega_\lambda \) still has full measure and the convergence statement of the superadditive theorem holds for all \( \lambda \in S \) and \( \omega \in \Omega' \). Now define the distribution function \( N(\lambda) := \lim_{S \ni \lambda} N(\lambda) \). Thus on the set \( \mathcal{C} \) the functions \( \gamma \) and \( N \) coincide.
The monotonicity of \( \lambda \mapsto F_{Q_l}(\lambda, \omega) \) and the continuity of \( N \) on the set \( \mathcal{C} \) imply the convergence \((5)\). To see this, choose a sequence \( \lambda_n \in S, \lambda_n \geq \lambda, \lim_{n \to \infty} \lambda_n = \lambda \). Then we have
\[
l^{-d}F_{Q_l}(\lambda, \omega) - N(\lambda) \leq l^{-d}F_{Q_l}(\lambda_n, \omega) - N(\lambda_n) + N(\lambda_n) - N(\lambda).
\]
For arbitrary \( \omega \in \Omega' \) and \( \epsilon > 0 \) we choose first \( n \) sufficiently large such that \( N(\lambda_n) - N(\lambda) \leq \epsilon/2 \) and then \( l \) sufficiently large such that \( l^{-d}F_{Q_l}(\lambda_n, \omega) - N(\lambda_n) \leq \epsilon/2 \). Thus one sees that
\[
\limsup_{l \to \infty} l^{-d}F_{Q_l}(\lambda, \omega) \leq N(\lambda).
\]
Similarly one can choose a sequence \( \lambda_n \in S, \lambda_n \leq \lambda, \lim_{n \to \infty} \lambda_n = \lambda \) and then show that \( \liminf_{l \to \infty} l^{-d}F_{Q_l}(\lambda, \omega) \geq N(\lambda) \). Thus the theorem is proven. \( \square \)

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(1) Fakultät für Mathematik, 09107 TU-Chemnitz, Germany
E-mail address: Mario.Helm@mathematik.tu-chemnitz.de

(2) Emmy-Noether-Programme of the Deutsche Forschungsgemeinschaft
URL: www.tu-chemnitz.de/mathematik/schroedinger/members.php