Geometric modeling of the parallel approach method in some transport problems

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Abstract. This article presents a quasidiscrete geometric model based on the problem of simple pursuit on a plane by the method of parallel approximation. Apollonius and the characteristic lines associated with it in this geometric model for a predetermined trajectory are in the optimal path of the pursuer. The geometric model is also used in solving some transport problems: visualization, construction of calculation grids, etc. Modeling was performed in the computer mathematics system MathCAD. Based on the simulation results, an animation clip was created, where you can view the movement and transformation of the Apollonius circle and associated characteristic points and lines.

1. Introduction

Apollonius circles are used in solving problems of simple pursuit on a plane using the parallel approach strategy. A simple motion of a point is called motion when the traveled distance \( S \) is a linear function of time: \( S(t) = v \cdot t \), where \( v = \text{Const} \) is the modulus of the speed of the point. The Apollonius circle is the locus of the points of the plane when \( |KP|/|KT| = \text{Const} \) (Figure 1).

![Figure 1. Apollonius Circum.](image-url)
In relation to the pursuit, the circle of Apollonius contains the following meaning. If the pursuer and the goal at some point in time have $P$ and $T$ positions on the plane and the speed values are equal to modulo $V_P$ and $V_T$ respectively, then the geometric set of points $K$, as the place of possible meetings of the pursuer $P$ with the goal $T$, is a circle of radius $|QK|$ with center $Q$ [1], [2].

The direction of the pursuer’s speeds and the goals are interconnected. That is, the direction of the speed of the goal dictates the direction of the speed of the pursuer, or vice versa: the direction of the pursuer determines the direction of the goal in order to ensure the meeting at points belonging to the circle of Apollonius.

The purpose of this article is to formalize and algorithmize the method of the parallel approximation of the pursuer and the goal.

2. The calculation of the parameters of the Apollonius circle

The fact that the set of points $K$ is a circle was known to ancient mathematicians, but we will give calculations of the calculation of the circle and its center.

Let’s introduce an orthonormal coordinate system $(e_1, e_2)$ with center $T$ (Figure 1), vector $e_1$ and vector $PT$ are co-directional. Let’s say that $K = \begin{pmatrix} x \\ y \end{pmatrix}$ and $P = \begin{pmatrix} -a \\ 0 \end{pmatrix}$, where $a = |PT|$. Then $|TK| = \sqrt{x^2 + y^2}$, and $|PK| = |K - P| = \sqrt{(x + a)^2 + y^2}$. The condition that the pursuer and the goal come to the point $K$ simultaneously implies following: $\frac{|PK|}{V_T} = \frac{|TK|}{V_T}$. Which means that $V_T \cdot \sqrt{(x + a)^2 + y^2} = V_P \cdot \sqrt{x^2 + y^2}$. After the squaring and removing the brackets we obtain the following equation:

$$\left(x - \frac{V_T^2}{V_P^2 - V_T^2} \cdot a\right)^2 + y^2 = \left(V_P \cdot \frac{V_T}{V_P^2 - V_T^2} \cdot a\right)^2.$$

The obtained equation in system $(e_1, e_2)$ with center $T$ circumscribes the circle with radius $R$ and center $Q$:

$$Q = \begin{pmatrix} \frac{V_P^2}{V_P^2 - V_T^2} \cdot a \\ 0 \end{pmatrix}, R = \frac{V_P \cdot V_T}{V_P^2 - V_T^2} \cdot a, a = |PT|.$$

Let us note the characteristic point called Apollonius point:

$$A = \begin{pmatrix} \frac{V_P^2}{V_P^2 - V_T^2} \cdot a + \frac{V_P \cdot V_T}{V_P^2 - V_T^2} \cdot a \\ 0 \end{pmatrix}.$$

3. The modeling of the iterative process of the aim of pursuit

The fact that the pursuer’s strategy in the pursuit problem using the parallel approach method is optimal in terms of minimizing the aim capture time was proved in the works of L. Petrosyan [1-8].
Figure 2. Iterative scheme.

We assume that for the iterative process the initial data is known as $P_0$ and $T_0$. The pursuer’s speeds and the goals are constant and equal modulo $V_p$ and $V_T$ respectively. The aim trajectory in the model is predefined, so we can calculate the array of points $\{T_i\}$, where the distance between the points $T_i$ and $T_{i+1}$ equals:

$$|T_{i+1} - T_i| = V_T \cdot \Delta T, \quad \Delta T - \text{sampling period over time}.$$ 

The iterative scheme for calculating the coordinates of the pursuer, the coordinates of the centers of the Apollonius circles, the radii of the circles of Apollonius and characteristic points is presented in Figure 2.

The coordinates of the pursuer in the step $i$ of iterations will look as follows:

$$P_i = P_{i-1} + V_p \cdot \Delta T \cdot \frac{K_{i-1} - P_{i-1}}{|K_{i-1} - P_{i-1}|}.$$ 

Apollonius circle radius:

$$R_i = \frac{V_T^2}{V_T^2 - V_p^2} \cdot |T_i - P_i|.$$ 

The center of the circle of Apollonius is calculated as follows:

$$Q_i = T_i + \frac{V_T^2}{V_T^2 - V_p^2} \cdot (T_i - P_i).$$ 

The coordinates of the "$K_i$" point is a product of solving a system of equations for a continuous $t$ parameter:

$$\begin{cases} 
(K_i - Q_i)^2 = R_i^2 \\
K_i = T_i + V_T \cdot \frac{T_{i+1} - T_i}{|T_{i+1} - T_i|} \cdot t.
\end{cases}$$
The above resolved system as to parameter $t$, the system described above represents the roots of the quadratic equation, the calculation of which we will not give in this article because of cumbersome expressions. In a test program written on the basis of the article, the solution of the quadratic equation is written as a separate procedure – a function. The text of the test program can be found here [18].

The fact that the segment $[P_1, T_1]$ will remain parallel to the segment $[P_0, T_0]$, is beyond doubt. Let us look to the first segment $[P_1, T_1]$. Coordinates of points $P_1$ and $T_1$ equal (Пандусюк 3):

$$P_1 = P_0 + V_p \cdot \frac{P_0 K_0}{|P_0 K_0|} \cdot \Delta T$$

$$T_1 = T_0 + V_T \cdot \frac{T_0 K_0}{|T_0 K_0|} \cdot \Delta T$$

Based on the fact that the pursuer and the aim must come to the point $K$ on the Apollonius circle at the same time, we may conclude that:

$$V_p \cdot \frac{P_0 K_0}{|P_0 K_0|} \cdot \Delta T = V_T \cdot \frac{T_0 K_0}{|T_0 K_0|} \cdot \Delta T = \varepsilon.$$  

Then:

$$P_1 T_1 = T_1 - P_1 = (T_0 - P_0) + \varepsilon \cdot T_0 K_0 - \varepsilon \cdot P_0 K_0 = (1 - \varepsilon) \cdot (T_0 - P_0).$$

In other words, vector $P_1 T_1$ is co-directional with vector $P_0 T_0$ and perpendicular to normal vector $N$ (Пандусюк 3).

**4. The results of the modeling the pursuit problem by the parallel approach method**

We have developed a test program, a screenshot of the results of which is shown in Figure 4. Figure 4 shows that the segments $[P_i, T_i]$ form a one-parameter sequence of parallel $[P_0, T_0]$ lines.
Figure 4 also shows that the points $P_i, T_i, Q_i, A_i$ belong to one line. A moving triangle $P_i, Q_i, K_i$ is shown which converges to the meeting point of the pursuer and the aim.

Looking at Figure 4, namely, the one-parameter set of Apollonius circles that converges to the meeting point, a misleading impression may arise: the whole set of circles touches the pursuer and the aim at the meeting point. In the next model with a different aim path, we show that this is not true.

Figure 4 is supplemented by a link to an animated image [22], where you can see how the location of the pursuer, the aim and the points on the Apollonius circle changes over time.

Now then, the situation presented in Figure 4 can be interpreted as a simulation of an aim’s interception by a pursuer.

The situation presented in Figure 5 can be interpreted as a simulation of the process of the aim running away from the pursuer [23].
Figure 5. Simulation of the aim running away from a pursuer.

Figure 5 also shows a moving triangle $P_i, Q_i, K_i$, points $P_i, T_i, Q_i, A_i$ and Apollonius circles. Significantly, the behavior of the point $K_i$, it is similar to the behavior of the point of return with type 2. Here, the convergence of Apollonius circles to the meeting point of the pursuer and the aim is observed, but the set of circles is not tangent at one point [23].

5. Conclusion

The purpose of this article was to show the motion of all characteristic lines and points in the implementation of the parallel approach method in simple pursuit problems on the plane.

The movement of the Apollonius circle and its convergence to the point of meeting of the pursuer and the aim is shown. Also, it is shown how the segment converges connecting the pursuer and the aim, being parallel to itself.

In geometric modeling using the parallel approach method, it is not necessary to calculate the parameters of the Apollonius circle. It will be enough to construct a one-parameter set of parallel lines connecting the pursuer with the aim and the circle with a radius equal to the pursuer’s step to find the point of the next position of the pursuer.

The texts of the programs are posted on the author’s resource [18]. Links to the animated image made according to the results of the programs are available on the resources [22], [23].

When writing the article, the results obtained in the following sources [9-17], [19-21] are taken into account.
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