A NOTE ON THE SAGNAC EFFECT IN GENERAL RELATIVITY AS A FINSLERIAN EFFECT

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Abstract. The geometry of the Sagnac effect in a stationary region of a spacetime is reviewed with the aim of emphasizing the role of asymmetry of a Finsler metric defined on a spacelike hypersurface associated to a stationary splitting and related to future-pointing null geodesics of the spacetime. We show also that an analogous asymmetry comes into play in the Sagnac effect for timelike geodesics.

1. Introduction

A light beam constrained to follow a closed path on a rotating system takes different times to reach a detector according to its travel direction. This is the Sagnac effect which allows a non-inertial observer to compute the angular speed of the rotating system and which is at the base of corrections adopted in synchronization of atomic clocks in GPS (see e.g. [1]).

Even though it has been the subject of a longstanding quarrel about the validity of special relativity (see, for example, [2]), P. Langevin [3] made clear that the Sagnac effect can be explained inside special relativity and, since the effect involves a non-inertial reference frame, he suggested its compatibility with general relativity too. Actually the Sagnac effect was considered in the context of general relativity only several years later by A. Ashtekar and A. Magnon [4], who introduced the notion of a Sagnac tube as a timelike surface $\mu$ in the spacetime $(M, g)$. Under the assumption that the Sagnac tube is foliated by the flow lines of a timelike Killing vector field in $\mu$, they computed the Sagnac shift $\Delta \tau$, i.e. the difference of the values of the proper time of an observer (whose world line coincides with one of the Killing field flow line as an unparameterized curve) at the arrival points of two future-pointing null curves winding round the tube. Although, by the point of view of an experimental apparatus, it is completely reasonable that the null curves considered are not necessarily lightlike geodesics of the spacetime $(M, g)$, it would be desirable to analyze the possibility to detect the effect with freely falling light rays. We point out in the next section that, in general, this is not guaranteed, even under the stronger assumption that the tube is inside a stationary region, because of the non-reversibility of the Finsler metric describing the lightlike geodesic flow in a stationary spacetime. We consider then a simple situation where two winding round future-pointing lightlike geodesics can be found, and we submit evidence for the relations of the effect with the topology of the spacetime. In particular a non-trivial topology of the spacetime can cause the effect also for a static local observer as it was already observed in [5]. Finally, we remark that the arguments above extend to freely falling massive particles.

Key words and phrases. Sagnac effect, Finsler metrics, stationary spacetime, lightlike, timelike, geodesic.

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2. Sagnac tube in a stationary region

Let \((M, g)\) be a spacetime and \(D \subset M\) be an open region which is invariant by the flow of a timelike Killing vector field \(K\). Let us assume for simplicity that \(D\) splits as \(S \times \mathbb{R}\), with \(S\) a spacelike hypersurface (with boundary), and the natural coordinate \(t\) associated to \(\mathbb{R}\) induces the vector field \(\partial_t = K\). Thus the metric \(g\) in \(D\) is given by

\[
g = -\Lambda dt^2 + \omega \otimes dt + dt \otimes \omega + g_0,
\]

where \(g_0\) is the Riemannian metric induced by \(g\) on \(S\), \(\omega\) is the one-form on \(S\) metrically equivalent to the vector field on \(S\) obtained as the pointwise \(g\)-orthogonal projection of \(K_x\), \(x \in S\), on \(T_xS\), and \(\Lambda = -g(K, K)|_S > 0\). As it is well-known, the mixed term \(\omega dt\) in the metric expression is at the base of the Sagnac effect. In fact, a lightlike vector \((\tau, v) \in TD\) has \(\tau\)-component, in dependence of \(v \in TS\), which is a root of the equation \(g((\tau, v), (\tau, v)) = 0\):

\[
\tau = \frac{\omega(v)}{\Lambda} \pm \left(\frac{g_0(v, v)}{\Lambda} + \frac{\omega^2(v)}{\Lambda^2}\right)^{1/2}.
\]

Notice that the positive root corresponds to a future-pointing lightlike vector while the negative one to a past-pointing one. Let us denote by \(h\) the Riemannian metric on \(S\) defined as

\[
h := \frac{g_0}{\Lambda} + \frac{\omega}{\Lambda} \otimes \frac{\omega}{\Lambda}.
\]

The expressions in (2) define two positive Lagrangians on \(TS\)

\[
F_{\pm}(v) := \pm \frac{\omega(v)}{\Lambda} + \left(\frac{g_0(v, v)}{\Lambda} + \frac{\omega^2(v)}{\Lambda^2}\right)^{1/2}
\]

which are two Finsler metrics of Randers type, called Fermat metrics in [6]\(^1\).

Notice that these two metrics are non-reversible, i.e. \(F_{\pm}(v) \neq F_{\pm}(-v)\), and induce then two different asymmetric distances on \(S\) that can be used by the observer \(K/\sqrt{\Lambda}\) to determine, at least locally, both the time and the radar distance (see, e.g. [8]) on \(D\). In fact, each geodesic \(x:\ [a, b] \to S\) of \(F_+\) (resp. \(F_-\)) parameterized with \(\frac{g_0(x, x)}{\Lambda} + \omega^2(x) = \text{const.}\) lifts, up to translation by the flow of the Killing field \(K\), to a unique future-pointing (resp. past-pointing) lightlike geodesic \(\gamma_{\pm}\) of \((D, g)\) given by \(\gamma_{\pm}(s) = \left(\int_a^s F_+(\dot{x}(u))du, x(s)\right)\) (resp. \(\gamma_{-} = \left(-\int_a^s F_-(\dot{x}(u))du, x(s)\right)\)) (see [6, Th. 4.1]).

Remark 1. The Lagrangians \(F_{\pm}\) make sense whenever \(h\) is Riemannian without necessarily assuming that \(g_0\) is a Riemannian metric. Of course, the signature of \(g_0\) is related to the causal character of the hypersurface \(S\) considered in the stationary splitting \(S \times \mathbb{R}\). Notice however that, if we assume that \(S\) is timelike or lightlike, then there exists a vector \(v \in TS\) such that \(\frac{1}{\Lambda}g_0(v, v) + \frac{1}{\Lambda}\omega^2(v) = h(v, v) \leq \frac{1}{\Lambda^2}\omega^2(v)\). This implies that the norm of \(\omega/\Lambda\) w.r.t. \(h\) is not strictly less than 1. As a consequence, the Lagrangians \(F_{\pm}\) are not Finsler in the classical sense (i.e. they are not positive, and they do not have strongly convex indicatrices).

Nevertheless, the critical curves of their action functionals (they satisfy the Euler-Lagrange equation (4) below) can still be lifted to, respectively, future-pointing and past-pointing null curves of the spacetime and, since \(F_{\pm}(v) \neq F_{\pm}(-v)\), they are in general not invariant by orientation reversing reparametrizations. Thus, also in this more general situation, the arguments below related to the Sagnac effect hold unchanged.

\(^1\)We point out that in other references, as e.g. [7], the name Fermat metric has been attributed to the Riemannian metric \(h\).
Let us denote by $\ell_{F_\pm}$ the length functional associated to the Fermat metrics $F_\pm$, i.e. $\ell_{F_\pm}(\sigma) := \int_a^b F_\pm(\dot{\sigma}) ds$, where $\sigma : [a, b] \to S$, $\sigma = \sigma(s)$, is a curve on $S$. Let us assume that there exists a (non-constant) geodesic loop or a periodic geodesic $x : [a, b] \to S$ of $F_+$ which remains a geodesic also when parameterized in the opposite direction. Then the Sagnac shift for the observer $K/\sqrt{\Lambda}$ along its world line passing through $x(\bar{s})$, $\bar{s} \in [a, b],^2$ is given by

$$|\Delta \tau| = \frac{1}{2} \int_0^b A(x(s)) \left| \ell_{F_+}(x) - \ell_{F_-}(x) \right| = \frac{2}{\sqrt{\Lambda(x(\bar{s}))}} \left| \int A / \Lambda \right|$$

that, by Stokes’ theorem, coincides with the value of the surface integral

$$\frac{2}{\sqrt{\Lambda(x(\bar{s}))}} \left| \int_A d(\omega / \Lambda) \right|$$

if there exists a surface $A \subset S$ spanning $x$ (see also [4, 9]).

The above reasoning is based on the fact that $x$ remains a geodesic of $F$ when it is parameterized in the opposite direction. In general, the non-reversibility of the Fermat metric implies that this might be not the case.\(^3\) Indeed, if there exists a loop or a periodic geodesic $x : [a, b] \to S$ for the Fermat metric $F_+$, we can consider the Sagnac tube defined by $x$ and the flow lines of $K$ passing through the points in the image of $x$ and then the future-pointing lightlike geodesic $\gamma_+$ which connects, on this tube, the events $(0, x(a))$ and $(\int_0^b F_+(\dot{x}) ds, x(a))$. Now, in general, the opposite curve of $x$ will be not a pregeodesic of $F_+$ and then there is no other distinct future-pointing lightlike geodesic in the Sagnac tube. It is not difficult to construct examples where this can happen. It is enough to consider a Randers space $S$ with a non-reversible geodesic and then to associate to it a standard stationary spacetime $\mathbb{R} \times S$ by stationary-to-Randers correspondence [10].

For example, consider the standard stationary spacetime in $\mathbb{R} \times D(0, 2)$, where $D(0, 2) \subset \mathbb{R}^2$ is the open disk centered in $0$ and having radius $2$, with the metric

$$g = -dt^2 + \omega \otimes dt + dt \otimes \omega + h - \omega \otimes \omega,$$

where $h$ is the Euclidean metric in $\mathbb{R}^2$ and $\omega$ is the one-form whose components are given by $\frac{1}{2} (y, -x)$, $(x, y) \in D(0, 2)$. It is immediate to see that the Fermat metric $F_+$ associated to this spacetime is the Randers metric $F_+(v) = \sqrt{h(v, v) + \omega(v)}$. It can be shown that $F_+$ has closed geodesics given by all the circles of radius $1$ inside $D(0, 2)$ but only if traversed anticlockwise [11].

On the other hand, it is possible that a closed geodesic of a Randers metric remains a geodesic also when parameterized in the opposite direction. This is for example the case of a Katok metric on the sphere $S^n$, i.e. a Randers metric obtained via Zermelo navigation by considering as a wind a Killing vector field on the sphere [12, §4]. The orbits of the Killing vector field corresponding to a great circle on $S^n$ are geodesics of the Katok metric and when parameterized in the direction the rotation have Finslerian length equal to $\frac{4\pi a}{1 - a^2}$ while in the opposite direction the length is $\frac{2\pi}{1 - a^2}$, where $0 < a < 1$ is a parameter associated to the rotation considered. Thus, from Eq. (3), the Sagnac effect, in the standard stationary spacetime associated to these data as above, is equal to

$$\Delta \tau = \frac{2\pi}{1 - a} - \frac{2\pi}{1 + a} = \frac{4\pi a}{1 - a^2};$$

independently of the world line of the observer $\partial_t$ considered.

\(^2\)Notice that if $x$ is a reversible geodesic loop based at $x(a) = x(b)$ and $\bar{s} \in (a, b)$ then the two future-pointing lightlike curves defined by $x$ are piecewise lightlike geodesics.

\(^3\)Of course, if $x = x(s)$ is a geodesic of $F_\pm$ then $\bar{x}(s) = x(a + b - s)$ is a geodesic of $F_\mp$. 

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3. Sagnac effect in a static spacetime with a stationary splitting

A case when any geodesic $x$ of $F_\pm$ remains a geodesic also when parameterized in the opposite direction is exactly when the one-form $\omega/\Lambda$ is closed. Indeed, by using the Levi-Civita connection $\nabla$ of $h$ we obtain the Euler-Lagrange equation of the length functionals associated to $F_\pm$ as

$$\nabla_x \dot{x} = \pm \sqrt{h(x, \dot{x})} \Omega(x),$$

(see, e.g. [13, Eq. (5)]) where $\Omega : TS \to TS$ is the endomorphism $h$-metrically equivalent to $d(\omega/\Lambda)$, when $x$ is parameterized with $h(\dot{x}, \dot{x}) = \text{const}$. Therefore, if $d(\omega/\Lambda) = 0$ then $\Omega = 0$ and (4) reduces to $\nabla_x \dot{x} = 0$ (i.e., in this case, $x$ is a geodesic of $F_\pm$ if and only if $x$ is a geodesic of $h$ and, moreover, if $x$ is a geodesic of $F_\pm$ then the opposite curve $\dot{x}$ remains a geodesic of $F_\pm$).

Thus, if $S$ is not simply connected, the equation $d(\omega/\Lambda) = 0$ does not imply that $\omega/\Lambda$ is exact; then the integral in (3) is not in general equal to 0.

Let us finally observe that if $\omega/\Lambda$ is closed then $(D, g)$ is static with respect to the observer field $K/\sqrt{\mathbf{A}}$ (see [14, Def. 12.35]). In fact, let $(\ell, \bar{p}) \in D$ and consider a neighborhood $U \subset S$ of $\bar{p}$ such that a local primitive $f$ of $\omega/\Lambda$ is defined in $U$. By adding a constant to $f$ we can assume that $f(\bar{p}) = \ell$. The graph $G$ of $f$ is then an integral manifold of the orthogonal distribution defined by $K$ passing through $(\ell, \bar{p})$. Indeed, any vector $\zeta \in T\mathcal{G}$ is given by $(df(\xi), \xi)$ with $\xi \in TU$ and then $g(K, \zeta) = \omega(\xi) - df(\xi) = 0$.

Spacetimes satisfying the above assumptions can be found inside the class of stationary cylindrical vacuum spacetimes (see, e.g., [15, §6.1] and the discussion in [5] about their existence as physical meaningful solutions of the Einstein field equations).

Remark 2. It makes sense to consider the more general case where the Killing field $K$ becomes null in some embedded submanifold of $D$ (a Killing horizon). In this case it is still possible to introduce a positive homogeneous Lagrangian on $S$ related to future-pointing lightlike vectors in $(D, g)$ (see Randers-Kropina metrics introduced in [16]). Anyway at the Killing horizon, all future-pointing lightlike vectors (except for the vectors collinear to $K$) can point outside $D$ making impossible, without further assumptions, to find a future-pointing null curve with endpoints at a flow line of $K$ and that remains in $D$ (apart from the same flow line). In particular, the existence of a closed geodesic for a Randers-Kropina metric on a compact manifold is open, in general, even in the case when the Killing field is null everywhere (see [16, 17]). On the other hand, if $K$ is timelike everywhere, several results about existence and multiplicity of lightlike future-pointing geodesics that project on geometrically distinct closed geodesics of the Fermat metric $F_\pm$ are available when $S$ is a compact manifold (see [18]). Such closed geodesics might be sources of the Sagnac effect with freely falling light rays.

4. Finslerian description of the Sagnac effect for massive particles

It has been observed that the Sagnac shift is universal in the sense that it does not depend on the physical nature of the two interfering beams: light rays, electromagnetic waves, neutron beams, etc. (see [19] and the references therein). This universal property holds in the setting considered above when considering freely falling particles. Indeed, future-pointing timelike geodesics $\gamma$ parameterized with proper time in $D$ can be obtained from the geodesics of the Fermat metric $F_\pm$ associated to the one-dimensional higher standard stationary spacetime $D \times \mathbb{R}$ with the product metric $\tilde{g} := g \oplus ds^2$, $g$ as in (1) and $s$ the natural coordinate on the added factor $\mathbb{R}$. Since $\partial_s$ is a Killing vector field for the metric $\tilde{g}$, geodesics $\gamma$ in
$\mathcal{D} \times \mathbb{R}$ have to satisfy the conservation law $\tilde{g}(\dot{\zeta}, \partial_s) = \text{const.}$, which implies that the $s$-component of a geodesic is an affine function. Moreover, the projection $\gamma$ on $\mathcal{D}$ of $\zeta$ is a geodesic for $(\mathcal{D}, \tilde{g})$. In particular lightlike geodesics $\zeta = (\gamma, u)$ for the metric $\tilde{g}$ satisfy the equation $g(\dot{\gamma}, \dot{\gamma}) = -\dot{u}^2 = \text{const}$. Thus, a timelike geodesic $\gamma$ in $(\mathcal{D}, g)$ parameterized with proper time is the projection of a lightlike geodesic in $(\mathcal{D} \times \mathbb{R}, \tilde{g})$ whose $s$-component $u$ has constant derivative equal to $\pm 1$. The Fermat metrics $\tilde{F}_\pm$ associated to $\mathcal{D} \times \mathbb{R}$ are now the two Randers metrics on $S \times \mathbb{R}$ defined by

$$\tilde{F}_\pm(v, \nu) = \pm \omega(v)/\Lambda + \left(\tilde{h}(v, \nu, (v, \nu))\right)^{1/2},$$

for all $(v, \nu) \in TS \times \mathbb{R}$, where $\tilde{h}$ is the Riemannian metric on $S \times \mathbb{R}$ defined as

$$\tilde{h} := \frac{g_0 \otimes ds^2}{\Lambda} + \frac{\omega \otimes \omega}{\Lambda},$$

(see [6, §4.3]). Thus, if $\gamma_i : [a, b] \to \mathcal{D}$, $i = 1, 2$ are two timelike geodesics of $(\mathcal{D}, g)$, and which project onto the same loop or closed curve $x$ in $\mathcal{S}$ traversed in opposite directions, then the Sagnac shift, measured by the observer $\partial_t/\sqrt{\Lambda}$ along its world line passing through $x(s), \ s \in [a, b]$, is given by

$$\Delta \tau = \frac{1}{\sqrt{\Lambda(x(s))}} \left| \ell_{\tilde{F}_+}(x, u) - \ell_{\tilde{F}_-}(x, u) \right| = \frac{2}{\sqrt{\Lambda(x(s))}} \int_x \frac{\omega}{\Lambda}$$

as in (3).

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