ON THE RELATIVIZED ALON SECOND EIGENVALUE CONJECTURE VI: SHARP BOUNDS FOR RAMANUJAN BASE GRAPHS

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ABSTRACT. This is the sixth in a series of articles devoted to showing that a typical covering map of large degree to a fixed, regular graph has its new adjacency eigenvalues within the bound conjectured by Alon for random regular graphs.

In this article we show that if the fixed graph is regular Ramanujan, then the algebraic power of the model of random covering graphs is $+\infty$. This implies a number of interesting results, such as (1) one obtains the upper and lower bounds—matching to within a multiplicative constant—for the probability that a random covering map has some new adjacency eigenvalue outside the Alon bound, and (2) with probability smaller than any negative power of the degree of the covering map, some new eigenvalue fails to be within the Alon bound without the covering map containing one of finitely many “tangles” as a subgraph (and this tangle containment event has low probability).

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1. Introduction

This paper is the sixth in a series of six articles whose main results are to prove a relativized version of Alon’s Second Eigenvalue Conjecture, conjectured in [Fri03], in the case where the base graph is regular.

The relativized Alon conjecture for regular base graphs was proven in Article V (i.e., the fifth article in this series). In this article we give a sharper version of the relativized Alon conjecture that holds for all of our basic models of a random covering map of degree $n$ to a fixed base graph, $B$, provided that $B$ is $d$-regular and Ramanujan. Roughly speaking, for a fixed such $B$, and for a random covering map $G \to B$ of degree $n$, for $n$ large we determine—to within a constant factor independent of $n$—the probability that this map fails to be a relative expander, in the sense that its new spectral radius is larger than the bound conjectured by Alon for random $d$-regular graphs; this probability is proportional to a negative power of $n$ which we call the tangle power of the model.

Curiously, in [Fri08] such upper and lower bounds were established for random $d$-regular graphs formed from $d/2$ permutations (for $d$ even) for all $d$ except those that are one more than a perfect odd square (e.g., 10, 26, 50, ...). However, the upper and lower bounds for these exceptional $d$ differed by a factor of $n$ in [Fri08], and the results in this article close this bound (since such random graphs are included in our basic models, where the base graph, $B$, is a bouquet of whole-loops and easily seen to be Ramanujan).

In Article V we proved that the probability that a random covering graph has a new eigenvalue outside the Alon bound is bounded above proportional to $n^{-\tau_1}$ and below proportional to $n^{-\tau_2}$, where

$$\tau_1 = \min(\tau_{\text{tang}}, \tau_{\text{alg}}), \quad \tau_2 = \min(\tau_{\text{tang}}, \tau_{\text{alg}} + 1),$$
where \( \tau_{\text{tang}} \) is a positive integer and \( \tau_{\text{alg}} \) is either a positive integer or \( +\infty \) (both depending on the base graph, \( B \), and the model of random covering map). However, the integer \( \tau_{\text{alg}} \) appears to be very difficult to compute directly: there is—in principle—a finite algorithm to determine if \( \tau_{\text{alg}} \) is larger than any given integer, but (1) we know of no finite algorithm to check that \( \tau_{\text{alg}} = +\infty \), and (2) when \( \tau_{\text{alg}} \) is larger than 1 or 2 the direct computation of \( \tau_{\text{alg}} \) seems quite laborious. On the other hand, the integer \( \tau_{\text{tang}} \) has a simple meaning and is much easier to compute in practice.

In this article we show that for all of our basic models of random covering maps of a \( d \)-regular Ramanujan graph, \( B \), \( \tau_{\text{alg}} = +\infty \); the method of this proof goes back to [Fri91], which uses the fact that \( \tau_{\text{alg}} \)—at least when \( B \) is Ramanujan—is the order of the first coefficient of an asymptotic expansion involving traces that grows as an exponential function with base \((d-1)^{1/2}\). So rather than compute these asymptotic expansions directly, we use the existence of these coefficients and apply other facts about random graphs—namely Alon’s notion of magnification—and standard counting arguments to infer that the growth rates of these asymptotic expansion coefficients are strictly less than \((d-1)^{1/2}\). As a consequence, we prove that \( \tau_{\text{alg}} = +\infty \) (without directly computing asymptotic expansion coefficients); hence to determine \( \tau_1 \) and \( \tau_2 \) above we need compute only \( \tau_{\text{tang}} \).

Once we formally define \( \tau_{\text{alg}} \), it becomes clear that \( \tau_{\text{alg}} = +\infty \) implies something quite strong for a \( d \)-regular \( B \): namely, the probability of having a new eigenvalue outside the Alon bound—namely, larger than \( 2(d-1)^{1/2} + \epsilon \) in absolute value for any fixed \( \epsilon > 0 \)—can be made smaller than any positive power of \( n \), provided that we discard the covering maps that contain certain tangles (which are graph theoretically local events that occur with probability proportional to \( n^{-\tau_{\text{tang}}} \)).

Beyond our theorems in this article, we conjecture that for our “basic models” of covering maps to a fixed graph \( B \) (regular or not), \( \tau_{\text{alg}} = +\infty \) (\( \tau_{\text{alg}} \) and \( \tau_{\text{tang}} \) are defined for any \( B \), regular or not).

The rest of this article is organized as follows. In Section 2, we review the definitions we will need in this article; for more details, see Article I in this series. In Section 3 we state the main theorems in this article, and quote the results we will need from Article V. In Section 4 we review Alon’s notion of magnification and introduce a variant of this notion, pseudo-magnification, that will be useful to us. In Section 5 we prove that our basic models of random covering maps to a base graph \( B \) are pseudo-magnifying in the case where \( B \) has no half-loops; this proof is computationally simpler than the general case, although it illustrates all the main ideas. In Section 6 we prove pseudo-magnification for our basic models over general \( B \). In Section 7 we use the pseudo-magnification results to prove our main theorem, that \( \tau_{\text{alg}} = +\infty \) if \( B \) is regular and Ramanujan. In this case the probability of a cover having new adjacency eigenvalues of absolute value outside the Alon bound is controlled by \( \tau_{\text{tang}} \); we devote Section 8 to proving estimates on \( \tau_{\text{tang}} \) for our basic models when \( B \) is \( d \)-regular.

2. Review of the Main Definitions

We refer the reader to Article I for the definitions used in this article, the motivation of such definitions, and an appendix there that lists all the definitions and notation. In this section we briefly review these definitions and notation.
2.1. Basic Notation and Conventions. We use $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{N}$ to denote, respectively, the real numbers, the complex numbers, the integers, and positive integers or natural numbers; we use $\mathbb{Z}_{\geq 0}$ ($\mathbb{R}_{>0}$, etc.) to denote the set of non-negative integers (of positive real numbers, etc.). We denote $\{1, \ldots, n\}$ by $[n]$.

If $A$ is a set, we use $\mathbb{N}^A$ to denote the set of maps $A \to \mathbb{N}$; we will refer to its elements as vectors, denoted in bold face letters, e.g., $k \in \mathbb{N}^A$ or $k: A \to \mathbb{N}$; we denote its component in the regular face equivalents, i.e., for $a \in A$, we use $k(a) \in \mathbb{N}$ to denote the $a$-component of $k$. As usual, $\mathbb{N}^n$ denotes $\mathbb{N}^{[1, \ldots, n]}$. We use similar conventions for $\mathbb{N}$ replaced by $\mathbb{R}, \mathbb{C}$, etc.

If $A$ is a set, then $\#A$ denotes the cardinality of $A$. We often denote a set with all capital letters, and its cardinality in lower case letters; for example, when we define $\text{SNBC}(G, k)$, we will write $\text{snbc}(G, k)$ for $\# \text{SNBC}(G, k)$.

If $A' \subset A$ are sets, then $\mathbb{1}_{A'}: A \to \{0, 1\}$ (with $A$ understood) denotes the characteristic function of $A'$, i.e., $\mathbb{1}_{A'}(a)$ is 1 if $a \in A'$ and otherwise is 0; we also write $\mathbb{1}_{A'}$ (with $A$ understood) to mean $\mathbb{1}_{A' \cap A}$ when $A'$ is not necessarily a subset of $A$.

All probability spaces are finite; hence a probability space is a pair $\mathcal{P} = (\Omega, P)$ where $\Omega$ is a finite set and $P: \Omega \to \mathbb{R}_{>0}$ with $\sum_{\omega \in \Omega} P(\omega) = 1$; hence an event is any subset of $\Omega$. We emphasize that $\omega \in \Omega$ implies that $P(\omega) > 0$ with strict inequality; we refer to the elements of $\Omega$ as the atoms of the probability space. We use $\mathcal{P}$ and $\Omega$ interchangeably when $P$ is understood and confusion is unlikely.

A complex-valued random variable on $\mathcal{P}$ or $\Omega$ is a function $f: \Omega \to \mathbb{C}$, and similarly for real-, integer-, and natural-valued random variable; we denote its $\mathcal{P}$-expected value by

$$\mathbb{E}_{\omega \in \Omega}[f(\omega)] = \sum_{\omega \in \Omega} f(\omega)P(\omega).$$

If $\Omega' \subset \Omega$ we denote the probability of $\Omega'$ by

$$\text{Prob}_\mathcal{P}[\Omega'] = \sum_{\omega \in \Omega'} P(\omega') = \mathbb{E}_{\omega \in \Omega}[\mathbb{1}_{\Omega'}(\omega)].$$

At times we write $\text{Prob}_\mathcal{P}[\Omega']$ where $\Omega'$ is not a subset of $\Omega$, by which we mean $\text{Prob}_\mathcal{P}[\Omega' \cap \Omega]$.

2.2. Graphs, Our Basic Models, Walks. A directed graph, or simply a digraph, is a tuple $G = (V_G, E_G^{\text{dir}}, h_G, t_G)$ consisting of sets $V_G$ and $E_G^{\text{dir}}$ (of vertices and directed edges) and maps $h_G, t_G$ (heads and tails) $E_G^{\text{dir}} \to V_G$. Therefore our digraphs can have multiple edges and self-loops (i.e., $e \in E_G^{\text{dir}}$ with $h_G(e) = t_G(e)$). A graph is a tuple $G = (V_G, E_G^{\text{dir}}, h_G, t_G, \iota_G)$ where $(V_G, E_G^{\text{dir}}, h_G, t_G)$ is a digraph and $\iota_G: E_G^{\text{dir}} \to E_G^{\text{dir}}$ is an involution with $t_G\iota_G = h_G$: the edge set of $G$, denoted $E_G$, is the set of orbits of $\iota_G$, which (notation aside) can be identified with $E_G^{\text{dir}}/\iota_G$, the set of equivalence classes of $E_G^{\text{dir}}$ modulo $\iota_G$; if $\{e\} \in E_G$ is a singleton, then necessarily $e$ is a self-loop with $t_G e = e$, and we call $e$ a half-loop; other elements of $E_G$ are sets $\{e, \iota_G e\}$ of size two, i.e., with $e \neq \iota_G e$, and for such $e$ we say that $e$ (or, at times, $\{e, \iota_G e\}$) is a whole-loop if $h_G e = t_G e$ (otherwise $e$ has distinct endpoints).

Hence these definitions allow our graphs to have multiple edges and two types of self-loops—whole-loops and half-loops—as in [Fri93, Fri08]. The indegree and outdegree of a vertex in a digraph is the number of edges whose tail, respectively whose head, is the vertex; the degree of a vertex in a graph is its indegree (which
it is closed, non-backtracking, and a morphism whose head is $e$.

A morphism $\pi: G \to H$ of directed graphs is a pair $\pi = (\pi_V, \pi_E)$ where $\pi_V: V_G \to V_H$ and $\pi_E: E^\text{dir}_G \to E^\text{dir}_H$ are maps that intertwine the heads maps and the tails maps of $G, H$ in the evident fashion; such a morphism is covering (respectively, étale, elsewhere called an immersion) if for each $v \in V_G$, $\pi_E$ maps those directed edges whose head is $v$ bijectively (respectively, injectively) to those whose head is $\pi_V(v)$, and the same with tail replacing head. If $G, H$ are graphs, then a morphism $\pi: G \to H$ is a morphism of underlying directed graphs where $\pi_E^{\text{dir}} = i_H \pi_E$; $\pi$ is called covering or étale if it is so as a morphism of underlying directed graphs. We use the words morphism and map interchangeably.

A walk in a graph or digraph, $G$, is an alternating sequence $w = (v_0, e_1, \ldots, e_k, v_k)$ of vertices and directed edges with $t_G e_i = v_{i-1}$ and $h_G e_i = v_i$ for $i \in [k]$; $w$ is closed if $v_0 = v_k$; if $G$ is a graph, $w$ is non-backtracking, or simply NB, if $t_G e_i \neq e_{i+1}$ for $i \in [k-1]$; strictly non-backtracking closed, or simply SNBC, if it is closed, non-backtracking, and $t_G e_k \neq e_1$. The visited subgraph of a walk, $w$, in a graph $G$, denoted $\text{VisSub}_G(w)$ or simply $\text{VisSub}(w)$, is the smallest subgraph of $G$ containing all the vertices and directed edges of $w$; $\text{VisSub}_G(w)$ generally depends on $G$, i.e., $\text{VisSub}_G(w)$ cannot be inferred from the sequence $v_0, e_1, \ldots, e_k, v_k$ alone without knowing $\iota_G$.

The adjacency matrix, $A_G$, of a graph or digraph, $G$, is defined as usual (its $(v_1, v_2)$-entry is the number of directed edges from $v_1$ to $v_2$); if $G$ is a graph on $n$ vertices, then $A_G$ is symmetric and we order its eigenvalues (counted with multiplicities) and denote them

$$\lambda_1(G) \geq \cdots \geq \lambda_n(G).$$

If $G$ is a graph, its Hashimoto matrix (also called the non-backtracking matrix), $H_G$, is the adjacency matrix of the oriented line graph of $G$, $\text{Line}(G)$, whose vertices are $E^\text{dir}_G$ and whose directed edges are the subset of $E^\text{dir}_G \times E^\text{dir}_G$ consisting of pairs $(e_1, e_2)$ such that $e_1, e_2$ form the directed edges of a non-backtracking walk (of length two) in $G$ (the tail of $(e_1, e_2)$ is $e_1$, and its head $e_2$); therefore $H_G$ is the square matrix indexed on $E^\text{dir}_G$, whose $(e_1, e_2)$ entry is $1$ or $0$ according to, respectively, whether or not $e_1, e_2$ form a non-backtracking walk (i.e., $t_G e_1 = t_G e_2$ and $h_G e_1 \neq e_2$).

We use $\mu_1(G)$ to denote the Perron-Frobenius eigenvalue of $H_G$, and use $\mu_i(G)$ with $1 < i \leq \#E^\text{dir}_G$ to denote the other eigenvalues of $H_G$ (which are generally complex-valued) in any order.

If $B, G$ are both digraphs, we say that $G$ is a coordinatized graph over $B$ of degree $n$ if

$$V_G = V_B \times [n], \quad E^\text{dir}_G = E^\text{dir}_B \times [n], \quad t_G(e, i) = (t_B e, i), \quad h_G(e, i) = (h_B e, \sigma(e) i)$$

for some map $\sigma: E^\text{dir}_B \to \mathcal{S}_n$, where $\mathcal{S}_n$ is the group of permutations on $[n]$; we call $\sigma$ (which is uniquely determined by (1)) the permutation assignment associated to $G$. [Any such $G$ comes with a map $G \to B$ given by “projection to the first component of the pair,” and this map is a covering map of degree $n$.] If $B, G$ are graphs, we say that a graph $G$ is a coordinatized graph over $B$ of degree $n$ if (1) holds and also

$$\iota_G(e, i) = (\iota_B e, \sigma(e) i),$$

equals its outdegree) in the underlying digraph; therefore a whole-loop about a vertex contributes 2 to its degree, whereas a half-loop contributes 1.
which implies that

\[(e, i) = \iota_G \sigma_G(e, i) = (e, \sigma(i_B e) \sigma(e) i) \quad \forall e \in E_B^{\text{dir}}, \quad i \in [n],\]

and hence \(\sigma(i_B e) = (\sigma(e))^{-1}\); we use \(\text{Coord}_n(B)\) to denote the set of all coordinatized covers of a graph, \(B\), of degree \(n\).

The \textit{order} of a graph, \(G\), is \(\text{ord}(G) \triangleq (\#E_G) - (\#V_G)\). Note that a half-loop and a whole-loop each contribute 1 to \(\#E_G\) and to the order of \(G\). The \textit{Euler characteristic} of a graph, \(G\), is \(\chi(G) \triangleq (\#V_G) - (\#E_G)/2\). Hence \(\text{ord}(G) \geq -\chi(G)\), with equality iff \(G\) has no half-loops.

If \(w\) is a walk in any \(G \in \text{Coord}_n(B)\), then one easily sees that \(\text{VisSub}_G(w)\) can be inferred from \(B\) and \(w\) alone.

If \(B\) is a graph without half-loops, then the \textit{permutation model over} \(B\) refers to the probability spaces \(\{C_n(B)\}_{n \in \mathbb{N}}\) where the atoms of \(C_n(B)\) are coordinatized coverings of degree \(n\) over \(B\) chosen with the uniform distribution. More generally, a \textit{model} over a graph, \(B\), is a collection of probability spaces, \(\{C_n(B)\}_{n \in \mathbb{N}}\), defined for \(n \in \mathbb{N}\) where \(\mathbb{N} \subseteq \mathbb{N}\) is an infinite subset, and where the atoms of each \(C_n(B)\) are elements of \(\text{Coord}_n(B)\). There are a number of models related to the permutation model, which are generalizations of the models of [Fri08], that we call our \textit{basic models} and are defined in Article I; let us give a rough description.

All of our \textit{basic models} are \textit{edge independent}, meaning that for any orientation \(E_B^{\text{or}} \subseteq E_B^{\text{dir}}\), the values of the permutation assignment, \(\sigma\), on \(E_B^{\text{or}}\) are independent of one another (of course, \(\iota_G \sigma_G\) = \((\sigma(e))^{-1}\), so \(\sigma\) is determined by its values on any orientation \(E_B^{\text{or}}\)); for edge independent models, it suffices to specify the (\(S_n\)-valued) random variable \(\sigma(e)\) for each \(e \in E_B^{\text{dir}}\) or \(E_B^{\text{or}}\). The permutation model can be alternatively described as the edge independent model that assigns a uniformly chosen permutation to each \(e \in E_B^{\text{dir}}\) (which requires \(B\) to have no half-loops); the \textit{full cycle} (or simply \textit{cyclic}) model is the same, except that if \(e\) is a whole-loop then \(\sigma(e)\) is chosen uniformly among all permutations whose cyclic structure consists of a single \(n\)-cycle. If \(B\) has half-loops, then we restrict \(C_n(B)\) either to \(n\) even or \(n\) odd and for each half-loop \(e \in E_B^{\text{dir}}\) we choose \(\sigma(e)\) as follows: if \(n\) is even we choose \(\sigma(e)\) uniformly among all perfect matchings, i.e., involutions (maps equal to their inverse) with no fixed points; if \(n\) is odd then we choose \(\sigma(e)\) uniformly among all \textit{nearly perfect matchings}, meaning involutions with one fixed point. We combine terms when \(B\) has half-loops: for example, the term \textit{full cycle-involution} (or simply \textit{cyclic-involution}) \textit{model of odd degree over} \(B\) refers to the model where the degree, \(n\), is odd, where \(\sigma(e)\) follows the full cycle rule when \(e\) is not a half-loop, and where \(\sigma(e)\) is a near perfect matching when \(e\) is a half-loop; similarly for the \textit{full cycle-involution} (or simply \textit{cyclic-involution}) \textit{model of even degree} and the \textit{permutation-involution model of even degree or of odd degree}.

If \(B\) is a graph, then a model, \(\{C_n(B)\}_{n \in \mathbb{N}}\), over \(B\) may well have \(N \neq \mathbb{N}\) (e.g., our basic models above when \(B\) has half-loops); in this case many formulas involving the variable \(n\) are only defined for \(n \in N\). For brevity, we often do not explicitly write \(n \in N\) in such formulas; for example we usually write

\[
\lim_{n \to \infty} \quad \text{to abbreviate} \quad \lim_{n \in N, \ n \to \infty}.
\]

Also we often write simply \(C_n(B)\) or \(\{C_n(B)\}\) for \(\{C_n(B)\}_{n \in N}\) if confusion is unlikely to occur.
A graph is *pruned* if all its vertices are of degree at least two (this differs from the more standard definition of *pruned* meaning that there are no leaves). If \( w \) is any SNBC walk in a graph, \( G \), then we easily see that \( \text{VisSub}_G(w) \) is necessarily pruned; i.e., any of its vertices must be incident upon a whole-loop or two distinct edges [note that a walk of length \( k = 1 \) about a half-loop, \((v_0, e_1, v_1)\), by definition, is not SNBC since \( e_G e_1 e_k = e_1 \)]. It easily follows that \( \text{VisSub}_G(w) \) is contained in the graph obtained from \( G \) by repeatedly “pruning any leaves” (i.e., discarding any vertex of degree one and its incident edge) from \( G \). Since our trace methods only concern (Hashimoto matrices and) SNBC walks, it suffices to work with models \( C_n(B) \) where \( B \) is pruned. It is not hard to see that if \( B \) is pruned and connected, then \( \text{ord}(B) = 0 \) iff \( B \) is a cycle, and \( \mu_1(B) > 1 \) iff \( \chi(B) < 0 \); this is formally proven in Article III (Lemma 6.4). Our theorems are not usually interesting unless \( \mu_1(B) > \mu_1^{1/2}(B) \), so we tend to restrict our main theorems to the case \( \mu_1(B) > 1 \) or, equivalently, \( \chi(B) < 0 \); some of our techniques work without these restrictions.

2.3. Asymptotic Expansions. A function \( f : \mathbb{N} \to C \) is a *polyexponential* if it is a sum of functions \( p(k)\mu^k \), where \( p \) is a polynomial and \( \mu \in \mathbb{C} \), with the convention that for \( \mu = 0 \) we understand \( p(k)\mu^k \) to mean any function that vanishes for sufficiently large \( k \); we refer to the \( \mu \) needed to express \( f \) as the *exponents* or *bases* of \( f \). A function \( f : \mathbb{N} \to C \) is of *growth* \( \rho \) for a \( \rho \in \mathbb{R} \) if \( |f(k)| = o(1)(\rho + \epsilon)^k \) for any \( \epsilon > 0 \). A function \( f : \mathbb{N} \to C \) is \((B, \nu)\)-bounded if it is the sum of a function of growth \( \nu \) plus a polyexponential function whose bases are bounded by \( \mu_1(B) \) (the Perron-Frobenius eigenvalue of \( H_B \)); the *larger bases* of \( f \) (with respect to \( \nu \)) are those bases of the polyexponential function that are larger in absolute value than \( \nu \). Moreover, such an \( f \) is called \((B, \nu)\)-Ramanujan if its larger bases are all eigenvalues of \( H_B \).

We say that a function \( f = f(k, n) \) taking some subset of \( \mathbb{N}^2 \) to \( C \) has a \((B, \nu)\)-bounded expansion of order \( r \) if for some constant \( C \) we have

\[
f(k, n) = c_0(k) + \cdots + c_{r-1}(k) + O(1)c_r(k)/n^r,
\]

whenever \( f(k, n) \) is defined and \( 1 \leq k \leq n^{1/2}/C \), where for \( 0 \leq i \leq r - 1 \), the \( c_i(k) \) are \((B, \nu)\)-bounded and \( c_r(k) \) is of growth \( \mu_1(B) \). Furthermore, such an expansion is called \((B, \nu)\)-Ramanujan if for \( 0 \leq i \leq r - 1 \), the \( c_i(k) \) are \((B, \nu)\)-Ramanujan.

Typically our functions \( f(k, n) \) as in (4) are defined for all \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \) for an infinite set \( N \subseteq \mathbb{N} \) representing the possible degrees of our random covering maps in the model \( \{C_n(B)\}_{n \in N} \) at hand.

2.4. Tangles. A \((\geq \nu)\)-tangle is any connected graph, \( \psi \), with \( \mu_1(\psi) \geq \nu \), where \( \mu_1(\psi) \) denotes the Perron-Frobenius eigenvalue of \( H_B \); a \((\geq \nu, < r)\)-tangle is any \((\geq \nu)\)-tangle of order less than \( r \); similarly for \((> \nu)\)-tangles, i.e., \( \psi \) satisfying the weak inequality \( \mu_1(\psi) > \nu \), and for \((> \nu, r)\)-tangles. We use TangleFree\((\geq \nu, < r)\) to denote those graphs that don’t contain a subgraph that is \((\geq \nu, < r)\)-tangle, and HasTangles\((\geq \nu, < r)\) for those that do; we never use \((> \nu)\)-tangles in defining TangleFree and HasTangles, for the technical reason (see Article III or Lemma 9.2

\[\text{This convention is used because then for any fixed matrix, } M, \text{ any entry of } M^k, \text{ as a function of } k, \text{ is a polyexponential function of } k; \text{ more specifically, the } \mu = 0 \text{ convention is due to the fact that a Jordan block of eigenvalue 0 is nilpotent.}\]
of [Fri08]) that for \( \nu > 1 \) and any \( r \in \mathbb{N} \) that there are only finitely many \((\geq \nu, < r)\)-tangles, up to isomorphism, that are minimal with respect to inclusion\(^2\).

2.5. **B-Graphs, Ordered Graphs, and Strongly Algebraic Models.** An ordered graph, \( G^\leq \), is a graph, \( G \), endowed with an ordering, meaning an orientation (i.e., \( tG \)-orbit representatives), \( E_{G}^\text{dir} \subset E_{G}^\text{dir} \), and total orderings of \( V_G \) and \( E_G \); a walk, \( w = (v_0, \ldots, e_k, v_k) \) in a graph endows \( \text{VisSub}(w) \) with a first-encountered ordering: namely, \( v \leq v' \) if the first occurrence of \( v \) comes before that of \( v' \) in the sequence \( v_0, v_1, \ldots, v_k \), similarly for \( e \leq e' \), and we orient each edge in the order in which it is first traversed (some edges may be traversed in only one direction). We use \( \text{VisSub}^\leq(w) \) to refer to \( \text{VisSub}(w) \) with this ordering.

A morphism \( G^\leq \to H^\leq \) of ordered graphs is a morphism \( G \to H \) that respects the ordering in the evident fashion. We are mostly interested in isomorphisms of ordered graphs; we easily see that any isomorphism \( G^\leq \to G^\leq \) must be the identity morphism; it follows that if \( G^\leq \) and \( H^\leq \) are isomorphic, then there is a unique isomorphism \( G^\leq \to H^\leq \).

If \( B \) is a graph, then a \( B \)-graph, \( G_B \), is a graph \( G \) endowed with a map \( G \to B \) (its \( B \)-graph structure). A morphism \( G_B \to H_B \) of \( B \)-graphs is a morphism \( G \to H \) that respects the \( B \)-structures in the evident sense. An ordered \( B \)-graph, \( G^\leq_B \), is a graph endowed with both an ordering and a \( B \)-graph structure; a morphism of ordered \( B \)-graphs is a morphism of the underlying graphs that respects both the ordering and \( B \)-graph structures. If \( w \) is a walk in a \( B \)-graph, \( G_B \), we use \( \text{VisSub}_B(w) \) to denote \( \text{VisSub}(w) \) with the \( B \)-graph structure it inherits from \( G \) in the evident sense; we use \( \text{VisSub}^\leq_B(w) \) to denote \( \text{VisSub}_B(w) \) with its first-encountered ordering.

At times we drop the superscript \( \leq \) and the subscript \( _B \); for example, we write \( G \in \text{Coord}_n(B) \) instead of \( G_B \in \text{Coord}_n(B) \) (despite the fact that we constantly utilize the \( B \)-graph structure on elements of \( \text{Coord}_n(B) \)).

A \( B \)-graph \( G_B \) is covering or étale if its structure map \( G \to B \) is.

If \( \pi: S \to B \) is a \( B \)-graph, we use \( a = a_{S_B} \) to denote the vector \( E_{B}^\text{dir} \to \mathbb{Z}_{\geq 0} \) given by \( a_{S_B}(e) = \# \pi^{-1}(e) \); since \( a_{S_B}(v_Be) = a_{S_B}(e) \) for all \( e \in E_{B}^\text{dir} \), we sometimes view \( a \) as a function \( E_B \to \mathbb{Z}_{\geq 0} \), i.e., as the function taking \( \{e, v_Be\} \) to \( a_{S_B}(e) = a_{S_B}(v_Be) \). We similarly define \( b_{S_B}: V_B \to \mathbb{Z}_{\geq 0} \) by setting \( b_{S_B}(v) = \# \pi^{-1}(v) \). If \( w \) is a walk in a \( B \)-graph, we set \( a_w \) to be \( a_{S_B} \) where \( S_B = \text{VisSub}_B(w) \), and similarly for \( b_w \). We refer to \( a, b \) (in either context) as \( B \)-fibre counting functions.

If \( S_B^\leq \) is an ordered \( B \)-graph and \( G_B^\leq \) is a \( B \)-graph, we use \( [S_B^\leq] \cap G_B \) to denote the set of ordered graphs \( G_B' \) such that \( G_B' \subset G_B^\leq \) and \( G_B' \simeq S_B^\leq \) (as ordered-\( B \)-graphs); this set is naturally identified with the set of injective morphisms \( S_B^\leq \to G_B \), and the cardinality of these sets is independent of the ordering on \( S_B^\leq \).

A \( B \)-graph, \( S_B \), or an ordered \( B \)-graph, \( S_B^\leq \), occurs in a model \( \{C_n(B)\}_{n \in \mathbb{N}} \) if for all sufficiently large \( n \in \mathbb{N} \), \( S_B^\leq \) is isomorphic to a \( B \)-subgraph of some element of \( \text{Coord}_n(B) \); similarly a graph, \( S \), occurs in \( \{C_n(B)\}_{n \in \mathbb{N}} \) if it can be endowed with a \( B \)-graph structure, \( S_B \), that occurs in \( \{C_n(B)\}_{n \in \mathbb{N}} \).

A model \( \{C_n(B)\}_{n \in \mathbb{N}} \) of coverings of \( B \) is strongly algebraic if

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2 By contrast, there are infinitely many minimal \((\geq \nu, < r)\)-tangles for some values of \( \nu > 1 \) and \( r \): indeed, consider any connected pruned graph \( \psi \), and set \( r = \text{ord}(\psi) + 2, \nu = \mu_1(\psi) \). Then if we fix two vertices in \( \psi \) and let \( \psi_s \) be the graph that is \( \psi \) with an additional edge of length \( s \) between these two vertices, then \( \psi_s \) is an \((\geq \nu, < r)\)-tangle. However, if \( \psi' \) is \( \psi \) with any single edge deleted, and \( \psi'_s \) is \( \psi_s \) with this edge deleted, then one can show that \( \mu_1(\psi'_s) < \nu \) for \( s \) sufficiently large. It follows that for \( s \) sufficiently large, \( \psi_s \) are minimal \((\geq \nu, < r)\)-tangles.
(1) for each \( r \in \mathbb{N} \) there is a function, \( g = g(k) \), of growth \( \mu_1(B) \) such that if \( k \leq n/4 \) we have

\[
\mathbb{E}_{G \in \mathcal{C}_n(B)}[\text{snbc}_r^c(G, k)] \leq g(k)/n^r
\]

where \( \text{snbc}_r^c(G, k) \) is the number of SNBC walks of length \( k \) in \( G \) whose visited subgraph is of order at least \( r \);

(2) for any \( r \) there exists a function \( g \) of growth 1 and real \( C > 0 \) such that the following holds: for any ordered \( B \)-graph, \( S_{\mathbb{B}} \), that is pruned and of order less than \( r \),

(a) if \( S_{\mathbb{B}} \) occurs in \( \mathcal{C}_n(B) \), then for \( 1 \leq \#E^{\text{dir}}_{S_{\mathbb{B}}} \leq n^{1/2}/C \),

\[
\mathbb{E}_{G \in \mathcal{C}_n(B)}\left[\#\left(\left[S_{\mathbb{B}}^c\right] \cap G\right)\right] = c_0 + \cdots + c_{r-1}/n^{r-1} + O(1)g(#E_{S}/n^r)
\]

where the \( O(1) \) term is bounded in absolute value by \( C \) (and therefore independent of \( n \) and \( S_{\mathbb{B}} \)), and where \( c_i = c_i(S_{\mathbb{B}}) \in \mathbb{R} \) such that \( c_i \) is 0 if \( i < \ord(S) \) and \( c_i > 0 \) for \( i = \ord(S) \); and

(b) if \( S_{\mathbb{B}} \) does not occur in \( \mathcal{C}_n(B) \), then for any \( n \) with \( #E_{S_{\mathbb{B}}}^{\text{dir}} \leq n^{1/2}/C \),

\[
\mathbb{E}_{G \in \mathcal{C}_n(B)}\left[\#\left(\left[S_{\mathbb{B}}^c\right] \cap G\right)\right] = 0
\]

(or, equivalently, no graph in \( \mathcal{C}_n(B) \) has a \( B \)-subgraph isomorphic to \( S_{\mathbb{B}}^c \));

(3) \( c_0 = c_0(S_{\mathbb{B}}) \) equals 1 if \( S \) is a cycle (i.e., \( \ord(S) = 0 \) and \( S \) is connected) that occurs in \( \mathcal{C}_n(B) \);

(4) \( S_{\mathbb{B}} \) occurs in \( \mathcal{C}_n(B) \) iff \( S_{\mathbb{B}} \) is an étale \( B \)-graph and \( S \) has no half-loops; and

(5) there exist polynomials \( p_i = p_i(a, b) \) such that \( p_0 = 1 \) (i.e., identically 1), and for every étale \( B \)-graph, \( S_{\mathbb{B}}^c \), we have that

\[
c_{\ord(S)+1}(S_{\mathbb{B}}) = p_i(a_{S_{\mathbb{B}}}, b_{S_{\mathbb{B}}})
\]

Notice that condition (3), regarding \( S \) that are cycles, is implied by conditions (4) and (5); we leave in condition (3) since this makes the definition of algebraic (below) simpler. Notice that (6) and (8) are the main reasons that we work with ordered \( B \)-graphs: indeed, the coefficients depend only on the \( B \)-fibre counting function \( a, b \), which depend on the structure of \( S_{\mathbb{B}}^c \) as a \( B \)-graph; this is not true if we don’t work with ordered graphs: i.e., (6) fails to hold if we replace \( [S_{\mathbb{B}}^c] \) with \( [S_{\mathbb{B}}] \) (when \( S_{\mathbb{B}} \) has nontrivial automorphisms), where \( [S_{\mathbb{B}}] \cap G \) refers to the number of \( B \)-subgraphs of \( G \) isomorphic to \( S_{\mathbb{B}} \); the reason is that

\[
\#\left[S_{\mathbb{B}}^c\right] \cap G_{\mathbb{B}} = \left(\#\text{Aut}(S_{\mathbb{B}})\right)\left(\#\left[S_{\mathbb{B}}\right] \cap G_{\mathbb{B}}\right)
\]

where \( \text{Aut}(S_{\mathbb{B}}) \) is the group of automorphisms of \( S_{\mathbb{B}} \), and it is \( [S_{\mathbb{B}}^c] \cap G_{\mathbb{B}} \) rather than \( [S_{\mathbb{B}}] \cap G_{\mathbb{B}} \) that turns out to have the “better” properties; see Section 6 of Article I for examples. Ordered graphs are convenient to use for a number of other reasons.

2.6. Homotopy Type. The homotopy type of a walk and of an ordered subgraph are defined by suppressing its “uninteresting” vertices of degree two; examples are given in Section 6 of Article I. Here is how we make this precise.

A bead in a graph is a vertex of degree two that is not incident upon a self-loop. Let \( S \) be a graph and \( V' \subset V_S \) be a proper bead subset of \( V_S \), meaning that \( V' \) consists only of beads of \( V \), and that no connected component of \( S \) has all its vertices in \( V' \) (this can only happen for connected components of \( S \) that
are cycles); we define the bead suppression $S/V'$ to be the following graph: (1) its vertex set $V_{S/V'}$ is $V'' = V_S \setminus V'$, (2) its directed edges, $E_{S/V'}^{\text{dir}}$, consist of the $V'$-beaded paths, i.e., non-backtracking walks in $S$ between elements of $V''$ whose intermediate vertices lie in $V'$, (3) $t_{S/V'}$ and $h_{S/V'}$ give the first and last vertex of the beaded path, and (4) $e_{S/V'}$ takes a beaded path to its reverse walk (i.e., takes $(v_0, e_1, \ldots, v_k)$ to $(v_k, t e_k, \ldots, t e_1, v_0)$). One can recover $S$ from the suppression $S/V'$ for pedantic reasons, since we have defined its directed edges to be beaded paths of $S$. If $S^\subseteq = \text{VisSub}^\subseteq(w)$ where $w$ is a non-backtracking walk, then the ordering of $S$ can be inferred by the naturally corresponding order on $S/V'$, and we use $S^\subseteq/V'$ to denote $S/V'$ with this ordering.

Let $w$ be a non-backtracking walk in a graph, and $S^\subseteq = \text{VisSub}^\subseteq(w)$ its visited subgraph; the reduction of $w$ is the ordered graph, $R^\subseteq$, denoted $S^\subseteq/V'$, whose underlyingly graph is $S/V'$ where $V'$ is the set of beads of $S$ except the first and last vertices of $w$ (if one or both are beads), and whose ordering is naturally arises from that on $S^\subseteq$; the edge lengths of $w$ is the function $E_{S/V'} \to \mathbb{N}$ taking an edge of $S/V'$ to the length of the beaded path it represents in $S$; we say that $w$ is of homotopy type $T^\subseteq$ for any ordered graph $T^\subseteq$ that is isomorphic to $S^\subseteq/V'$; in this case the lengths of $S^\subseteq/V'$ naturally give lengths $E_T \to \mathbb{N}$ by the unique isomorphism from $T^\subseteq$ to $S^\subseteq/V'$. If $S^\subseteq$ is the visited subgraph of a non-backtracking walk, we define the reduction, homotopy type, and edge-lengths of $S^\subseteq$ to be that of the walk, since these notions depend only on $S^\subseteq$ and not the particular walk.

If $T$ is a graph and $k : E_T \to \mathbb{N}$ a function, then we use VLG$(T, k)$ (for variable-length graph) to denote any graph obtained from $T$ by gluing in a path of length $k(e)$ for each $e \in E_T$. If $S^\subseteq$ is of homotopy type $T^\subseteq$ and $k : E_T \to \mathbb{N}$ its edge lengths, then VLG$(T, k)$ is isomorphic to $S$ (as a graph). Hence the construction of variable-length graphs is a sort of inverse to bead suppression. If $T^\subseteq$ is an ordering on $T$ that arises as the first encountered ordering of a non-backtracking walk on $T$ (whose visited subgraph is all of $T$), then this ordering gives rise to a natural ordering on VLG$(T, k)$ that we denote VLG$^\subseteq(T^\subseteq, k)$. Again, this ordering on the variable-length graph is a sort of inverse to bead suppression on ordered graphs.

2.7. B-graphs and Wordings. If $w_B = (v_0, e_1, \ldots, e_k, v_k)$ with $k \geq 1$ is a walk in a graph $B$, then we can identify $w_B$ with the string $e_1, e_2, \ldots, e_k$ over the alphabet $E_B^{\text{dir}}$. For technical reasons, the definitions below of a $B$-wording and the induced wording, are given as strings over $E_B^{\text{dir}}$ rather than the full alternating string of vertices and directed edges. The reason is that doing this gives the correct notion of the eigenvalues of an algebraic model (defined below).

Let $w$ be a non-backtracking walk in a B-graph, whose reduction is $S^\subseteq/V'$, and let $S_B^\subseteq = \text{VisSub}^\subseteq_B$. Then the wording induced by $w$ on $S^\subseteq/V'$ is the map $W$ from $E_{S/B}^{\text{dir}}$ to strings in $E_B^{\text{dir}}$ of positive length, taking a directed edge $e \in E_{S/B}^{\text{dir}}$ to the string of $E_B^{\text{dir}}$ edges in the non-backtracking walk in $B$ that lies under the walk in $S$ that it represents. Abstractly, we say that a $B$-wording of a graph $T$ is a map $W$ from $E_T^{\text{dir}}$ to words over the alphabet $E_B^{\text{dir}}$ that represent (the directed edges of) non-backtracking walks in $B$ such that (1) $W(t e)$ is the reverse word (corresponding to the reverse walk) in $B$ of $W(e)$, (2) if $e \in E_B^{\text{dir}}$ is a half-loop, then $W(e)$ is of length one whose single letter is a half-loop, and (3) the tail of the first directed edge in $W(e)$ (corresponding to the first vertex in the associated
walk in \( B \) depends only on \( t_T e \); the *edge-lengths* of \( W \) is the function \( E_T \rightarrow \mathbb{N} \) taking \( e \) to the length of \( W(e) \). [Hence the wording induced by \( w \) above is, indeed, a \( B \)-wording.]

Given a graph, \( T \), and a \( B \)-wording \( W \), there is a \( B \)-graph, unique up to isomorphism, whose underlying graph is \( \text{VLG}(T, k) \) where \( k \) is the edge-lengths of \( W \), and where the \( B \)-graph structure maps the non-backtracking walk in \( \text{VLG}(T, k) \) corresponding to an \( e \in E_T^{\text{dir}} \) to the non-backtracking walk in \( B \) given by \( W(e) \). We denote any such \( B \)-graph by \( \text{VLG}(T, W) \); again this is a sort of inverse to starting with a non-backtracking walk and producing the wording it induces on its visited subgraph.

Notice that if \( S_n^{B} = \text{VLG}(T^\leq, W) \) for a \( B \)-wording, \( W \), then the \( B \)-fibre counting functions \( a_{S_n} \) and \( b_{S_n} \) can be inferred from \( W \), and we may therefore write \( a_W \) and \( b_W \).

### 2.8. Algebraic Models

By a \( B \)-type we mean a pair \( T^{\text{type}} = (T, \mathcal{R}) \) consisting of a graph, \( T \), and a map from \( E_T^{\text{dir}} \) to the set of regular languages over the alphabet \( E_T^{\text{dir}} \) (in the sense of regular language theory) such that (1) all words in \( \mathcal{R}(e) \) are positive length strings corresponding to non-backtracking walks in \( B \), (2) if for \( e \in E_T^{\text{dir}} \) we have \( w = e_1 \ldots e_k \in \mathcal{R}(e) \), then \( w^R \) is a word in \( \mathcal{R}(\tau e) \), and (3) if \( W : E_T^{\text{dir}} \rightarrow (E_T^{\text{dir}})^* \) is a regular mapping in \( \mathcal{R}(\tau e) \) satisfies \( W(e) \in \mathcal{R}(e) \) and \( W(\tau e) = W(e)^R \) for all \( e \in E_T^{\text{dir}} \), then \( W \) is a \( B \)-wording. A \( B \)-wording \( W \) of \( T \) is of type \( T^{\text{type}} \) if \( W(e) \in \mathcal{R}(e) \) for each \( e \in E_T^{\text{dir}} \).

Let \( C_n(B) \) be a model that satisfies (1)–(3) of the definition of strongly algebraic. If \( T \) is a subset of \( B \)-graphs, we say that the model is algebraic restricted to \( T \) if either all \( S_n \in T \) occur in \( C_n(B) \) or they all do not, and if so there are polynomials \( p_0, p_1, \ldots \) such that \( c_i(S_n) = p_i(S_n) \) for any \( S_n \in T \). We say that \( C_n(B) \) is algebraic if

1. setting \( h(k) \) to be the number of \( B \)-graph isomorphism classes of \( \eta \)-type \( B \)-graphs \( S_n \) such that \( S \) is a cycle of length \( k \) and \( S \) does not occur in \( C_n(B) \), we have that \( h \) is a function of growth \( (d - 1)^{1/2} \); and
2. for any pruned, ordered graph, \( T^\leq \), there is a finite number of \( B \)-types, \( T_j^{\text{type}} = (T^\leq, R_j) \), \( j = 1, \ldots, s \), such that (1) any \( B \)-wording, \( W \), of \( T \) belongs to exactly one \( R_j \), and (2) \( C_n(B) \) is algebraic when restricted to \( T_j^{\text{type}} \).

[In Article I we show that if instead each \( B \)-wording belong to at least one \( B \)-type \( T_j^{\text{type}} \), then one can choose a another set of \( B \)-types that satisfy (2) and where each \( B \)-wording belongs to a unique \( B \)-type; however, the uniqueness is ultimately needed in our proofs, so we use uniqueness in our definition of algebraic.]

We remark that one can say that a walk, \( w \), in a \( B \)-graph, or an ordered \( B \)-graphs, \( S_n^{B} \), is of homotopy type \( T^\leq \), but when \( T \) has non-trivial automorphism one cannot say that is of \( B \)-type \((T, \mathcal{R})\) unless—for example—one orders \( T \) and speaks of an ordered \( B \)-type, \((T^\leq, \mathcal{R})\). [This will be of concern only in Article II.]

We define the eigenvalues of a regular language, \( R \), to be the minimal set \( \mu_1, \ldots, \mu_m \) such that for any \( k \geq 1 \), the number of words of length \( k \) in the language is given as

\[
\sum_{i=1}^{m} p_i(k) \mu_i^k
\]
for some polynomials $p_i = p_i(k)$, with the convention that if $\mu_i = 0$ then $p_i(k)\mu_i^k$ refers to any function that vanishes for $k$ sufficiently large (the reason for this is that a Jordan block of eigenvalue 0 is a nilpotent matrix). Similarly, we define the eigenvalues of a $B$-type $T^{\text{type}} = (T, \mathcal{R})$ as the union of all the eigenvalues of the $\mathcal{R}(e)$. Similarly a set of eigenvalues of a graph, $T$ (respectively, an algebraic model, $\mathcal{C}_n(B)$) is any set containing the eigenvalues containing the eigenvalues of some choice of $B$-types used in the definition of algebraic for $T$-wordings (respectively, for $T$-wordings for all $T$).

[In Article V we prove that all of our basic models are algebraic; some of our basic models, such as the permutation-involution model and the cyclic models, are not strongly algebraic.]

We remark that a homotopy type, $T^\le$, of a non-backtracking walk, can only have beads as its first or last vertices; however, in the definition of algebraic we require a condition on all pruned graphs, $T$, which includes $T$ that may have many beads and may not be connected; this is needed when we define homotopy types of pairs in Article II.

2.9. SNBC Counting Functions. If $T^\le$ is an ordered graph and $k : E_T \to \mathbb{N}$, we use $\text{SNBC}(T^\le, k; G, k)$ to denote the set of SNBC walks in $G$ of length $k$ and of homotopy type $T^\le$ and edge lengths $k$. We similarly define $\text{SNBC}(T^\le, \ge \xi; G, k) \overset{\text{def}}{=} \bigcup_{k \ge \xi} \text{SNBC}(T^\le, k; G, k)$ where $k \ge \xi$ means that $k(e) \ge \xi(e)$ for all $e \in E_T$. We denote the cardinality of these sets by replacing SNBC with snbc; we call $\text{snbc}(T^\le, \ge \xi; G, k)$ the set of $\xi$-certified traces of homotopy type $T^\le$ of length $k$ in $G$; in Article III we will refer to certain $\xi$ as certificates.

3. The Main Theorems in this Article

In this section we formally state the main theorems in this article. We first review some definitions and results of Article V.

3.1. Results from Article V. If $B$ is a graph, $\|A_B\|_2$ denotes the $L^2$ norm of the adjacency operator on a universal cover, $B$, of $B$; it is well-known that if $B$ is $d$-regular, then $\|A_B\|_2 = 2\sqrt{d-1}$ [MW89]. If $\pi : G \to B$ is a covering map graphs, and $\epsilon > 0$, the $\epsilon$-non-Alon multiplicity of $G$ relative to $B$ is $\text{NonAlon}_B(G; \epsilon) \overset{\text{def}}{=} \#\{\lambda \in \text{Spec}_B^{\text{new}}(A_G) \mid |\lambda| > \|A_B\|_2 + \epsilon\}$, where the above $\lambda$ are counted with their multiplicity in $\text{Spec}_B^{\text{new}}(A_G)$.

In Article V the Relativized Alon Conjecture was proven when $B$ is $d$-regular. The statement regards any algebraic model $\{\mathcal{C}_n(B)\}_{n \in \mathbb{N}}$ an algebraic model over a $d$-regular graph $B$; it says that for $\epsilon > 0$ there is a constant $C = C(\epsilon)$ for which $\text{Prob}_{G \in \mathcal{C}_n(B)}[\text{NonAlon}_B(G; \epsilon) > 0] \le C(\epsilon)/n$.

The point of this article is to give matching upper and lower bounds for this probability when $B$ is, furthermore, a Ramanujan graph in the following sense.

Definition 3.1. We say that a $d$-regular graph $B$ is Ramanujan if all eigenvalues of $A_B$ lie in

$$\{d, -d\} \cup \left[-2\sqrt{d-1}, 2\sqrt{d-1}\right].$$
We now give the more precise form of the Relativized Alon Conjecture proven in Article V.

**Definition 3.2.** Let \( \{ C_n(B) \}_{n \in \mathbb{N}} \) be a model over a graph, \( B \). By the tangle power of \( \{ C_n(B) \} \), denoted \( \tau_{\text{tang}} \), we mean the smallest order, \( \text{ord}(S) \), of any graph, \( S \), that occurs in \( \{ C_n(B) \} \) and satisfies \( \mu_1(S) > \mu_1^{1/2}(B) \).

In this article we prove some results regarding \( \tau_{\text{tang}} \); for example, the results of Section 6.3 of [Fri08] show that for any algebraic model over a \( d \)-regular graph, \( B \),

\[
\tau_{\text{tang}} \geq m = m(d)
\]

where

\[
m(d) = \left\lfloor \left( (d - 1)^{1/2} - 1 \right)/2 \right\rfloor + 1
\]

(and for any \( d \geq 3 \) there is a \( d \)-regular \( B \) where equality holds).

The most difficult theorem in this series of articles, to which most of Articles II-V are devoted, is the following result.

**Theorem 3.3.** Let \( C_n(B) \) be an algebraic model over a \( d \)-regular graph \( B \). For any \( \nu \) with \( (d - 1)^{1/2} < \nu < d - 1 \), let \( \epsilon' > 0 \) be given by

\[
2(d - 1)^{1/2} + \epsilon' = \nu + \frac{d - 1}{\nu}.
\]

Then

1. there is an integer \( \tau = \tau_{\text{alg}}(\nu, r) \geq 1 \) such that for any sufficiently small \( \epsilon > 0 \) there are constants \( C = C(\epsilon), C' > 0 \) such that for sufficiently large \( n \) we have

\[
n^{-\tau} C' \leq \mathbb{E}_{G \in C_n(B)}[\mathbb{I}_{\text{TangleFree}(\geq \nu, < r)}(G) \text{NonAlon}_d(G; \epsilon' + \epsilon)] \leq n^{-\tau} C(\epsilon),
\]

or

2. for all \( j \in \mathbb{N} \) and \( \epsilon > 0 \) we have

\[
\mathbb{E}_{G \in C_n(B)}[\mathbb{I}_{\text{TangleFree}(\geq \nu, < r)}(G) \text{NonAlon}_d(G; \epsilon' + \epsilon)] \leq O(n^{-j})
\]

in which case we use the notation \( \tau_{\text{alg}}(\nu, r) = +\infty \).

Moreover, if \( \tau = \tau_{\text{alg}}(\nu, r) \) is finite, then for some eigenvalue, \( \ell \in \mathbb{R} \), of the model with \( |\ell| > \nu \), there is a real \( C_\ell > 0 \) such that for sufficiently small \( \theta > 0 \)

\[
\lim_{n \to \infty} \mathbb{E}_{G \in C_n(B)}[\#(\text{Spec}_{B}^{\text{new}}(H_G) \cap B_{n^{-\theta}}(\ell)) \mathbb{I}_{\text{TangleFree}(\geq \nu, < r)}(G)] = C_\ell n^{-\tau} + o(n^{-\tau}).
\]

Notice if \( \nu_1 \leq \nu_2 \) and \( r_1 \geq r_2 \) then

\[
\mathbb{I}_{\text{TangleFree}(\geq \nu_2, < r_2)}(G) \leq \mathbb{I}_{\text{TangleFree}(\geq \nu_1, < r_1)}(G),
\]

for the simple reason that \( \mathbb{I}_{\text{TangleFree}(\geq \nu_2, < r_2)}(G) = 1 \) implies that \( G \) has no \((\geq \nu_2, < r_2)\)-tangles, and hence no \((\geq \nu_1, < r_1)\)-tangles; then (9) and (10) imply that

\[
\tau_{\text{alg}}(\nu_1, r_1) \leq \tau_{\text{alg}}(\nu_2, r_2).
\]

**Definition 3.4.** Let \( \{ C_n(B) \}_{n \in \mathbb{N}} \) be an algebraic model over a \( d \)-regular graph \( B \). For each \( r \in \mathbb{N} \) and \( \nu \) with \( (d - 1)^{1/2} < \nu < d - 1 \), let \( \tau(\nu, r) \) be as in Theorem 3.3.

We define the algebraic power of the model \( C_n(B) \) to be

\[
\tau_{\text{alg}} = \max_{\nu > (d - 1)^{1/2}, r} \tau(\nu, r) = \limsup_{r \to \infty, \nu \to (d - 1)^{1/2}} \tau(\nu, r)
\]
where $\nu$ tends to $(d-1)^{1/2}$ from above (and we allow $\tau_{\text{alg}} = +\infty$ when this maximum is unbounded or if $\tau(\nu, r) = \infty$ for some $r$ and $\nu > (d-1)^{1/2}$).

Of course, according to Theorem 3.3, $\tau(\nu, r) \geq 1$ for all $r$ and all relevant $\nu$, and hence $\tau_{\text{alg}} \geq 1$.

Here is the more precise form of the Relativized Alon Conjecture proven in Article V.

**Theorem 3.5.** Let $B$ be a $d$-regular graph, and let $C_n(B)$ be an algebraic model of tangle power $\tau_{\text{tang}}$ and algebraic power $\tau_{\text{alg}}$ (both of which are at least 1). Let

$$\tau_1 = \min(\tau_{\text{tang}}, \tau_{\text{alg}}), \quad \tau_2 = \min(\tau_{\text{tang}}, \tau_{\text{alg}} + 1).$$

Then $\tau_2 \geq \tau_1 \geq 1$, and for $\epsilon > 0$ sufficiently small there are $C, C'$ such that for sufficiently large $n$ we have

$$C' n^{-\tau_2} \leq \operatorname{Prob}_{G \in C_n(B)}[\text{NonAlon}_d(G; \epsilon) > 0] \leq C n^{-\tau_1}.$$  \hfill (13)

The last result we need from Article V regards a set of eigenvalues for our basic models.

**Lemma 3.6.** Let $B$ be a connected, pruned graph with $\mu_1(B) > 1$ (equivalently $\chi(B) < 0$). All our basic models are algebraic, and a set of eigenvalues for each model consist of possibly 1 and some subset of the eigenvalues $\mu_i(B)$ of the Hashimoto matrix $H_B$.

[The Ihara determinantal formula (see Articles I or V) easily implies that all $B$'s in the above lemma have at least one $H_B$ eigenvalue equal to either $\pm 1$; hence the possible addition of 1 to the set of eigenvalues in the lemma is not particularly significant.]

3.2. **Main Result of This Article.** In principle we can compute $\tau_{\text{alg}}$, using the methods of Articles II-V, which involve analyzing the main term of certain asymptotic expansions involving certified traces. However this computation is difficult to carry out. We will borrow the method of [Fri91] that uses the existence of these asymptotic expansion and an indirect method to draw conclusions about the main terms we need.

**Theorem 3.7.** Let $d \geq 3$ be an integer, and let $\{C_n(B)\}_{n \in \mathbb{N}}$ be one of our basic models over $d$-regular Ramanujan graph, $B$. Then $\tau_{\text{alg}} = +\infty$.

The idea behind the proof is to show that (11) cannot hold for any fixed value of $\tau$ with $r \to \infty$ if $\ell = d - 1$, due to the fact that a new eigenvalue of $H_G$ near $d - 1$ implies that $G$ has a “nearly disconnected component,” a notion which is made precise by Alon’s notion of magnification (one could also use an analog of “Cheeger’s” inequality for graphs, e.g., [Dod84, SJ89]). To prove this one needs to prove a (fairly weak) magnification result for most graphs in the model $C_n(B)$. This result holds for all of our basic models.

If $B$ is $d$-regular Ramanujan and connected, then the larger $H_B$ eigenvalues of all our basic models are either $d - 1$ or $\pm (d - 1)$ (the latter iff $B$ is bipartite), and we easily see that if (11) for some $\ell$ then it must hold for $\ell = d - 1$. Since this is impossible, we must have $\tau_{\text{alg}} = +\infty$. 
3.3. Results on $\tau_{\text{tang}}$. Whenever $\tau_{\text{alg}} = +\infty$, or merely $\tau_{\text{alg}} \geq \tau_{\text{tang}} + 1$, Theorem 3.5 determines matching upper and lower bounds on

$$\text{Prob}_{G \in C_n(B)}[\text{NonAlon}_d(G; \epsilon) > 0]$$

for any fixed $\epsilon > 0$ sufficiently small, both bounds being proportional to $n^{-\tau_{\text{tang}}}$. It therefore becomes interesting to compute $\tau_{\text{tang}}$ or to give bounds on it.

In Section 8 we shall give such bounds on $\tau_{\text{tang}}$. Let us state the main bounds.

If we fix $d \geq 3$, then the lower bound we give on $\tau_{\text{tang}}$ for any $d$-regular $B$ is

$$\tau_{\text{tang}} \geq \left\lfloor \frac{(d-1)^{1/2} + 1}{2} \right\rfloor$$

where $\lfloor \cdot \rfloor$ denotes the floor function, i.e., the largest integer lower bound; this bound is tight when $B$ is a bouquet of $d/2$ whole-loops (so that $d$ is even) and $C_n(B)$ is the permutation model. Furthermore, in models over $B$ in which whole-loops don’t occur, we have

$$\tau_{\text{tang}} \geq \left\lfloor (d-1)^{1/2} \right\rfloor;$$

this bound is tight in our basic model whenever $B$ is a bouquet of $d$ half-loops, and is also tight, except for possibly $d = 4$, for the full cycle model of $d/2$ whole-loops (hence $d$ is even).

As noted in [Fri08], this implies that the full cycle model has a much lower probability of having non-Alon new eigenvalues than does the permutation model, at least when $B$ is a bouquet of sufficiently many whole-loops.

We also prove that for fixed $d$, as the girth of $B$ tends to infinity, then so does $\tau_{\text{tang}}$. Hence the lower bounds quoted above can be very far from tight. Our proof, however, does not give an explicit relationship between the girth and $\tau_{\text{tang}}$.

4. Magnifiers and Tangles

In this section we describe some technical results we will prove about the relative magnification of random graphs in our basic models. One could alternatively use a graph theoretic analog [Dod84, SJ89, JS89] of “Cheeger’s” inequality [Che70]; in this article we will use magnification.

4.1. Magnifiers. We review the results of Alon on magnifiers.

**Definition 4.1.** Let $G$ be a graph, and $U \subset V_G$. We define the neighbourhood of $U$, denoted $\Gamma_G(U)$, to be the subset of $V_G$ consisting of those vertices joined by an edge of $G$ to a vertex of $U$. If $\gamma > 0$ is a real number, we say that a graph, $G$, is a $\gamma$-magnifier if for all $U \subset V_G$ of size at most $\frac{\# V_G}{2}$ we have

$$\#(\Gamma_G(U) \setminus U) \geq \gamma(\# U);$$

moreover, we say that $G$ is a $\gamma$-spreader if for all such $U$ we have

$$\#(\Gamma_H(U)) \geq (1 + \gamma)(\# U).$$

The notion of a magnifier was introduced in [Alo86], where Alon proved the following theorem.

**Theorem 4.2** (Alon, [Alo86]). If $G$ is $d$-regular and a $\gamma$-magnifier, then for all $i > 1$ we have

$$\lambda_i(G) \leq d - \frac{\gamma^2}{4 + 2\gamma^2}.$$
The notion of a spreader appears in [Fri91, Fri08] but in this article we will only use the notion of magnification; the point is that it is easier to prove that most random \(d\)-regular graphs on \(n\) vertices are \(\gamma\)-spreaders, since spreading is a less subtle feature than magnification. However, in this article (unlike [Fri08]) a graph \(G \in C_n(B)\) will never be a spreader if \(B\) is a connected, bipartite graph (since the subset of all vertices lying over one side of a bipartition \(B\) has all its neighbours in the other side).

4.2. Pseudo-Magnification. In article we will study the following variant of magnification.

**Definition 4.3.** For real \(\gamma > 0\) and \(R \in \mathbb{N}\), we say that a graph, \(G\) is an \((R, \gamma)\)-pseudo-magnifier if for each \(U \subset V_G\) with \(R \leq \#U \leq (\#V_G)/2\) we have

\[
\#(\Gamma_G(U) \setminus U) \geq \gamma(\#U).
\]

Our interest in this definition is evident in the following definition and easy lemma.

**Definition 4.4.** Let \(C_n(B)\) be a model over a connected graph, \(B\). We say that the model is pseudo-magnifying if for every \(i \in \mathbb{N}\) there is a \(\gamma > 0\) and \(R \in \mathbb{N}\) for which

\[
\Pr_{G \in C_n(B)}[G \text{ is not an } (R, \gamma)\text{-pseudo-magnifier}] = O(n^{-i}).
\]

**Lemma 4.5.** Let \(C_n(B)\) be a pseudo-magnifying, algebraic model over a connected graph, \(B\). Then for any \(i\) and \(\theta > 0\), there is an \(r \in \mathbb{N}\) such that for \(\ell = \mu_1(B)\) and any \(\nu \leq \mu_1(B)\) we have

\[
\mathbb{E}_{G \in C_n(B)}[\text{TangleFree}(\geq \nu, < r)](G) \left(\#(\text{Spec}_{new}^B(H_G) \cap B_{n-\rho}(\ell))\right) = O(n^{-i}).
\]

This will give us a way to prove that \(\tau_{\text{alg}} = +\infty\) in our basic models when \(B\) is Ramanujan.

4.3. Results on Pseudo-Magnification. Here is the main result we need.

**Lemma 4.6.** All of our basic models over a connected, pruned graph, \(B\), with \(\chi(B) > 0\) are pseudo-magnifying.

We remark that the proof we give can be modified to work without the condition that \(B\) be pruned, but the assumption of being pruned simplifies the proof (and in our applications, \(B\) will be \(d\)-regular for \(d \geq 3\), so \(B\) is necessarily pruned).

We prove this with a standard type of counting argument. The case where \(B\) has no half-loops is a bit simpler and illustrates all the main ideas; hence we first prove Lemma 4.6 in this case.

5. Pseudo-Magnification in Base Graphs Without Half-Loops

The point of this section is to prove that when \(B\) has no half-loops, then our (two) basic models over \(B\) are pseudo-magnifying.

**Lemma 5.1.** Let \(B\) be a connected, pruned graph without half-loops and with \(\chi(B) < 0\). Then the permutation and full-cycle models over \(B\) are pseudo-magnifying.
We will address the case where \( B \) has half-loops in the next section; however, the case where \( B \) has no half-loops makes the estimates simpler, and yet gives all the main ideas we will need for the general case. Hence we prove this special case first.

5.1. **The Counting Argument.** We will prove Lemma 5.1 by a counting argument. Let us give basic definitions we need.

Our counting argument works as follows: if \( G \in \text{Coord}_n(B) \), then
\[
V_G = V_B \times [n].
\]
If such a \( G \) is not an \((R, \gamma)\)-pseudo-magnifier, then by definition it follows that there are sets
\[
U \subset U' \subset V_G = V_B \times [n]
\]
whose sizes satisfy
\[
R \leq \#U \leq \#V_G, \quad #(U' \setminus U) = \lceil \gamma(#U) \rceil - 1
\]
where \( \lceil \cdot \rceil \) denotes the ceiling function (the smallest integer upper bound), such that
\[
\Gamma_G(U) \subset U'.
\]
Our counting argument is the simple one: the probability that \( G \in \mathcal{C}_n(B) \) is not a pseudo-magnifier is bounded by
\[
\sum_{U, U'} \text{Prob}_{G \in \mathcal{C}_n(B)}[\Gamma_G(U) \subset U']
\]
where we sum over each pair \( U, U' \) satisfying (14) and (15); we will show that for each \( i \) there are \( R, \nu \) such that the above sum is bounded by \( O(n^{-i}) \).

Now we build up the tools we need. We begin by setting
\[
p(U, U') \overset{\text{def}}{=} \text{Prob}_{G \in \mathcal{C}_n(B)}[\Gamma_G(U) \subset U'];
\]
we now study \( p(U, U') \).

5.2. **Almost Equal Fibre Sizes.** First we prove out that \( p(U, U') = 0 \) unless \( U \subset V_B \times [n] \) has nearly equal “fibre sizes.” Let us make this precise.

**Definition 5.2.** Let \( B \) be a connected graph, and let \( U \subset V_B \times [n] \) for some \( n \in \mathbb{N} \). By the \( V_B \)-fibres (or simply fibres) of \( U \) we mean the family of subsets of \([n]\), \( \{U_v\}_{v \in V_B} \), indexed on \( v \in V_B \), defined by
\[
U_v \overset{\text{def}}{=} \{i \in [n] \mid (v, i) \in U\} \subset [n].
\]

**Lemma 5.3.** Let \( B \) be a connected graph. Then for any \( \epsilon \in (0, 1) \) there is a \( \nu_1(\epsilon) > 0 \) for which the following is true: for \( n \in \mathbb{N} \) sufficiently large (depending only on \( B, \epsilon \)), let \( U \subset V_B \times [n] \) satisfy
\[
\min_{v \in V_B} \#U_B < (1 - \epsilon) \max_{v \in V_B} \#U_B.
\]
Then for any \( G \in \text{Coord}_n(B) \),
\[
\#(\Gamma_G(U) \setminus U) \geq \nu_1(#U).
\]
Proof. Let \( m = \#V_B \), and let \( \epsilon' > 0 \) be such that
\[
(1 - \epsilon')^{m-1} = 1 - \epsilon.
\]
Let \( \nu_{\min}, \nu_{\max} \) be respective vertices where \( \#U_v \) (in the above definition) takes its minimum and maximum values. Since \( B \) is connected there is a path from \( \nu_{\min} \) to \( \nu_{\max} \) consisting of vertices
\[
\nu_{\max} = v_1, v_2, \ldots, v_k = \nu_{\min}
\]
with \( k \leq m - 1 \). If (18) holds, then
\[
\#U_{\min} < (1 - \epsilon')^{m-1}(\#U_{\max}) \leq (1 - \epsilon')^k(\#U_{\max})
\]
and therefore for some \( i \in [k-1] \) we must have
\[
(1 - \epsilon')(\#U_{v_{i+1}}) < \#U_v;
\]
consider the smallest value of \( i \) for which the above holds. Then we have
\[
\#U_v \geq (1 - \epsilon')(\#U_{v_{i+1}}) \geq \cdots \geq (1 - \epsilon')^{i-1}(\#U_{v_1}) \geq (1 - \epsilon)(\#U_{v_1}).
\]
Since there is an edge from \( v_i \) to \( v_{i+1} \), we have that \( \Gamma(U) \) has a fibre of size at least \( \#U_{v_i} \) over \( v_i+1 \), and hence
\[
(\Gamma_G(U) \setminus U) \geq (\#U_{v_i}) - (\#U_{v_{i+1}}) \geq \epsilon'(\#U_{v_i}) \geq \epsilon'(1 - \epsilon)(\#U_{v_1}).
\]
Since \( \#U \leq m(\#U_{\max}) = m(\#U_{\min}) \), applying this to the rightmost term above yields
\[
(\Gamma_G(U) \setminus U) \geq \epsilon'(1 - \epsilon)(1/m)(\#U).
\]
Hence the lemma holds, i.e., (18) holds, with
\[
\nu_1(\epsilon) = \epsilon'(1 - \epsilon)/m > 0.
\]

5.3. The Probability Bound. Next we give a simple bound for \( p(U,U') \) in (17).

Lemma 5.4. Let \( n \in \mathbb{N} \), and let \( W, W' \subset [n] \) be subsets with \( \#W \leq \#W' \). If \( \sigma \in S_n \) is a random permutation, then
\[
\text{Prob}_\sigma[\sigma(W) \subset W'] = \frac{(\#W')^{\#W}}{\binom{n}{\#W}},
\]
and if \( \sigma \) is a random full-cycle then the above probability is at most \( n \) times the above right-hand-side.

Proof. The formula for \( \sigma \) a random permutation is immediate. Each random full-cycle occurs with probability \( 1/(n - 1)! \), which is exactly \( n \) times its probability of occurring as a random permutation; this implies the statement about the full-cycle case.

Corollary 5.5. Let \( B \) be a graph, and \( C_n(B) \) be one of our basic models. Let \( n \in \mathbb{N} \), and let \( U, U' \subset V_B \times [n] \). For each \( v \in V_B \) let \( s_v = \#U_v \) and \( s'_v = \#U'_v \). If \( e \in E_B \) is not a half-loop, and \( \sigma \) is the permutation assignment \( E_B^{\text{dir}} \to S_n \) associated to a \( G \in C_n(B) \), then the probability that
\[
\sigma(e)U_e \subset U'_e \quad \text{and} \quad \sigma(e^{-1})U_{e^c} \subset U'_{e^c}
\]
is 0 if $s'_{he} < s_{te}$ or $s''_{te} < s_{he}$, and is otherwise less than $p_1p_2$, where
\[ p_1 = \sqrt{\binom{n(s_{he})}{n(s_{te})}}, \quad p_2 = \sqrt{\binom{n(s''_{te})}{n(s_{he})}}. \]

Proof. The statement about when the probability is zero is clear. Otherwise, if $p_1 \leq p_2$ then we have $\sigma(e)U_{te} \subset U'_{he}$ occurs with probability at most $p_1^2 \leq p_1p_2$; similarly if $p_1 > p_2$ and for $\sigma(e^{-1})U_{he} \subset U'_{te}$. \qed

5.4. A Binomial Coefficient Estimate. In this section we introduce some useful formulas regarding binomial coefficients, and prove a lemma that will be useful to us in the counting argument we give. This lemma, as is typical in our counting arguments, is straightforward but involves some calculation.

First, Stirling’s formula shows that for all integers $0 \leq b \leq a$ we have that
\[ C_12^{aH_2(b/a)}a^{-1/2} \leq \binom{a}{b} \leq C_22^{aH_2(b/a)} \]
for some absolute constants $C_1, C_2$, where
\[ H_2(\mu) \overset{\text{def}}{=} -\mu \log_2 \mu - (1 - \mu) \log_2(1 - \mu). \]
It follows that for $0 \leq b \leq a$ and $\alpha \geq 1$ we have
\[ \log_2 \binom{a}{b} = aH_2(b/a) + O(\log_2 a) \]
where the $O(\log_2 a)$ is bounded by an absolute constant (i.e., independent of $a, b$) times $\log_2 a$.

We will also use the formula for the second derivative of $H_2(x)$
\[ H_2''(x) = \frac{-\log_2 e}{x(1-x)}, \quad \forall x \in (0, 1). \]

**Lemma 5.6.** For any $C > 0$ and $j \in \mathbb{N}$, for any sufficiently small $\theta > 0$ the following holds: there are natural numbers $S_0 = S_0(\theta)$ and $n_0 = n_0(\theta)$ such that for $n \geq n_0$ and any non-negative integers $s', s$ with $S_0 \leq s \leq n(1/2 + \theta)$, and $s' \leq \theta s$ we have
\[ \binom{n}{s'} \leq n^{-j} \binom{n}{s}^{1/C}. \]

Proof. Taking logs and dividing by $n$ it suffices to show that
\[ H_2(s'/n) \leq -j\frac{\log_2 n}{n} + (1/C)H_2(s/n) \]
where $j'$ is $j$ plus constants to absorb the $O(\log_2 n)$ terms in (20) with $a = n$. Hence it suffices to prove that all sufficiently small $\theta > 0$, there are $S_0, n_0$ such that for all $x \in [S_0/n, 1/2 + \theta]$ we have
\[ g(x) \geq j\frac{\log_2 n}{n}, \quad \text{where } g(x) \overset{\text{def}}{=} (1/C)H_2(x) - H_2(\theta x). \]

We shall do so by first showing that for fixed $C > 0$, for sufficiently small $\theta > 0$ we have that
\[ g''(x) \leq 0 \quad \forall x \in (0, 1). \]
It follows that to establish (24) it suffices to check this at $x = S_0/n$ and $x = 1/2 + \theta$. 

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Using (21) we have
\[ g''(x) \log_e 2 = \frac{-1}{Cx(1-x)} + \frac{\theta^2}{\theta x(1-\theta x)} = \frac{-1}{Cx(1-x)} + \frac{\theta}{x(1-\theta x)}. \]

Note that
\[ \frac{-1}{Cx(1-x)} \leq \frac{-1}{Cx}, \]
and for \( x \in (0, 1) \) and \( \theta \leq 1/2 \) we have \( 1 - \theta x \geq 1/2 \) and hence
\[ \frac{\theta}{x(1-\theta x)} \leq \frac{2\theta}{x}. \]

It follows that for \( x \in (0, 1) \) and \( \theta \in (0, 1/2) \) we have
\[ g''(x) \log_e 2 \leq (2\theta - 1/C) \frac{1}{x} \leq 0 \]
provided that \( 2\theta \leq 1/C \). This establishes (25) for \( \theta > 0 \) with \( \theta \leq 1/(2C) \) and \( \theta \leq 1/2 \).

So consider only those \( \theta > 0 \) with \( \theta \leq \min(1/(2C), 1/4) \).

Then \( g''(x) \leq 0 \) for all \( x \in (0, 1) \); it follows that to prove (24) for all \( x \in [S_0/n, 1/2 + \theta] \), it remains to show that
\[ g(S_0/n), g(1/2 + \theta) \geq j' \frac{\log_2 n}{n} \]
for some some fixed \( S_0 \) and \( n \) sufficiently large.

Since \( \theta \leq 1/4 \), and since \( H_2 \) is monotone increasing on \((0, 1/2)\) and monotone decreasing on \((1/2, 1)\), we have
\[ g(1/2 + \theta) = (1/C)H_2(1/2 + \theta) - H_2(\theta(1/2 + \theta)) \]
\[ \geq (1/C)H_2(3/4) - H_2(\theta(3/4)) \]
(26)
which is strictly positive for sufficiently small \( \theta > 0 \) (since \( H_2(x) \to 0 \) as \( x \to 0 \), and \( H_2(3/4) > 0 \)). For such a \( \theta \) we have
\[ g(1/2 + \theta) > 0 \]
and is therefore greater than \( j' \log_2 n/n \) for \( n \) sufficiently large.

So fix any \( \theta > 0 \) with \( \theta \leq 1/2, \theta \leq 1/(2C) \), and for which (26) is positive. For any fixed constant \( K \) we have
\[ H_2(K/n) = (K \log_2 n + O(1))/n \]
for large \( n \), and hence for fixed \( S_0 \) we have
\[ g(S_0/n) = (1/C)H_2(S_0/n) - H_2(\theta S_0/n) = ((1/C)S_0 - \theta S_0)(\log_2 n + O(1))/n \]
for \( n \) large. Hence for any \( S_0 \) with
\[ S_0((1/C) - \theta) > j' \]
we have
\[ g(S_0/n) \geq j' \frac{\log_2 n}{n} \]
for sufficiently large \( n \). Since \( \theta < 1/(2C) \), we have \( 1/C - \theta \) is positive; and hence the above inequality holds for sufficiently large \( n \) provided that
\[ S_0 > j'/(1/C - \theta). \]
It follows that for $\theta$ and $S_0$ as above, we have (24) when $x$ is either endpoint of $[S_0/n, 1/2 + \theta]$, and hence it holds for the entire interval. □

5.5. Some Notation and Our Counting Lemma. In this subsection we will introduce some helpful notation and give a lemma that summarizes the counting argument we shall use; the lemma is based on a simple union bound.

If $U \subset V_B \times [n]$ for some $n$, we use $\text{Sizes}(U)$ to denote the function $V_B \to \mathbb{Z}_{\geq 0}$ given by $\text{Sizes}(U)(v) = \#U_v$.

For any $s: V_B \to \mathbb{Z}_{\geq 0}$ we use the following notation:

\begin{equation}
(27) \quad s_{\min} = \min_{v \in V_B} s_v, \quad s_{\max} = \max_{v \in V_B} s_v, \quad \varpi = (\#V_B)^{-1} \sum_{v \in V_B} s_v;
\end{equation}

and similarly for any $s': V_B \to \mathbb{Z}_{\geq 0}$ (i.e., for $s'_{\min}, s'_{\max}, \varpi'$).

Here is a simple consequence of the union bound and Lemma 5.3. In this lemma we use $v \sim u$ to denote the fact that $v,u$ are adjacent vertices in $V_B$.

**Lemma 5.7.** Let $B$ be a graph (with or without half-loops), and $C_n(B)$ any model over $B$ (algebraic or not). Set $m = \#V_B$. Say that for any $i \in \mathbb{N}$ there are $R, \nu, \epsilon > 0$ such that the following holds: for any $s, s'$ from $V_B \to \{0, 1, \ldots, n\}$ such that

\begin{align*}
&\text{(28)} \quad s_v \leq s'_u \quad \text{whenever } v = u \text{ or } v \sim u, \\
&\text{(29)} \quad s' \cdot 1 \leq (1 + \nu)s \cdot 1, \quad s'_{\max} - s_{\min} \leq n/2, \\
&\text{(30)} \quad R \leq s \cdot 1 \leq nm/2, \quad s_{\min} \geq (1 - \epsilon)s_{\max}, \\
\end{align*}

we have

\begin{equation}
(31) \quad \max_{U, U'} \left( \text{Prob}_{G \in C_n(B)}[\Gamma_G(U) \subset U'] \right) \prod_{v \in V_B} \left( \binom{n}{s_n} \binom{n}{s'_n - s_n} \right) = O(n^{-i})
\end{equation}

where the above max is over all $U, U'$ such that

\begin{equation}
(32) \quad \text{Sizes}(U) = s, \quad \text{Sizes}(U') = s', \quad U \subset U'.
\end{equation}

Then $C_n(B)$ is pseudo-magnifying. Similarly provided that (31) is replaced with the bound

\begin{equation}
(33) \quad \left( \frac{n}{s'_{\max} - s_{\min}} \right)^{\#V_B} \max_{U, U'} \left( \text{Prob}_{G \in C_n(B)}[\Gamma_G(U) \subset U'] \right) \prod_{v \in V_B} \left( \binom{n}{s_n} \right) = O(n^{-i}).
\end{equation}

**Proof.** Given an integer $i' \in \mathbb{N}$, let us find $R, \gamma > 0$ such that

\begin{equation}
(34) \quad \text{Prob}_{G \in C_n(B)}[G \text{ is not a } (R, \gamma)-\text{pseudomagnifier}] = O(n^{-i'}).
\end{equation}

First, let $R, \nu, \epsilon > 0$ be such that (31) holds for $i = i' + 2m$ whenever $s, s'$ satisfy (28), (29), (30). Second, let $\nu', \epsilon' > 0$ satisfy

\begin{equation}
(35) \quad \nu' m + 1/(1 - \epsilon') - (1 - \epsilon') \leq 1.
\end{equation}

Let us show that (34) holds with

\begin{equation}
(36) \quad \gamma = \min(\nu, \nu', \nu_1(\epsilon), \nu_1(\epsilon')).
\end{equation}
The union bound implies that the probability that \( G \in \mathcal{C}_n(B) \) is not an \((R, \gamma)\)-pseudomagnifier is at most

\[
\sum_{R \leq \#U \leq \frac{nm}{2}} p_1(n, U, \gamma),
\]

where

\[
p_1(n, U, \gamma) \overset{\text{def}}{=} \text{Prob}_{G \in \mathcal{C}_n(B)}[\#(\Gamma_G(U) \setminus U) \leq \gamma(\#U)].
\]

In view of (36),

\[
p_1(n, U, \gamma) \leq p_1(n, U, \nu_1(\epsilon)),
\]

and Lemma 5.3 implies that

\[
p_1(n, U, \nu_1(\epsilon)) = 0
\]

whenever \( \text{Sizes}(U) = s \) and

\[
s_{\min} < (1 - \epsilon)s_{\max}.
\]

Hence in the union bound (37) we may restrict the sum to those \( s \) with

\[
s_{\min} \geq (1 - \epsilon)s_{\max}.
\]

Since the number of possible \( s \) is (crudely) bounded by \((n + 1)^m\), to establish (37), it suffices to show that for all \( s \) satisfying (30) we have

\[
\sum_{U, \text{Sizes}(U) = s} p_1(n, U, \gamma) \leq O(n^{-i - m}).
\]

Next note that for any \( U \) and \( G \in \mathcal{C}_n(B) \) for which

\[
\#(\Gamma_G(U) \setminus U) \leq \gamma(\#U),
\]

the set \( U' = \Gamma_G(U) \cup U \) satisfies

\[
\Gamma_G(U) \subset U';
\]

moreover setting \( s' = \text{Sizes}(U') \) then \( s' \) must satisfy

\[
s \leq s', \quad s' \cdot 1 \leq (1 + \gamma)s \cdot 1 \leq (1 + \nu)s \cdot 1,
\]

and if \( u \sim v \) then \( \Gamma_G(U) \subset U' \) implies that \( s_v \leq s'_v \) (or else the \( v \)-fibre over \( U \) could not “fit into” the \( u \)-fibre over \( U' \) under \( G \)-adjacency).

Once we fix \( s, s' \), the number of \( U \subset U' \) with those respective sizes is exactly

\[
\prod_{v \in V_B} \binom{n}{s_v} \binom{n - s_v}{s'_v - s_v},
\]

which is bounded from above by

\[
\prod_{v \in V_B} \binom{n}{s_v} \binom{n}{s'_v - s_v}.
\]

Since there are (crudely, again) at most \((n + 1)^m\) possible values for \( s' \), we therefore have that to prove (38), it suffices to show that for all \( s, s' \) as above we have

\[
\max_{U, U'} \left( \text{Prob}_{G \in \mathcal{C}_n(B)}[\Gamma_G(U) \subset U'] \right) \prod_{v \in V_B} \binom{n}{s_v} \binom{n}{s'_v - s_v} = O(n^{-i - 2m}) = O(n^{-i}).
\]

where the max is over all \( U, U' \) such that (32) holds.
In view of (36), we see that in (39) it suffices to sum over $s', s$ that additionally satisfy
\[ s' \cdot 1 \leq (1 + \nu')s \cdot 1, \quad s_{\min} \geq (1 - \epsilon')s_{\max}. \]
However, for such $s', s$ we claim that
\[ s'_{\max} - s_{\min} \leq n/2; \]
indeed,
\[ s'_{\max} - s_{\min} \leq (s'_{\max} - s_{\max}) + (s_{\max} - s_{\min}) \leq \gamma' m \bar{\pi} + (\bar{\pi} (1 - \epsilon') - \pi (1 - \epsilon')) \]
and using $\bar{\pi} \leq n/2$ we conclude that
\[ s'_{\max} - s_{\min} \leq n/2 (\gamma' m + 1/(1 - \epsilon') - (1 - \epsilon')) \leq n/2. \]
Hence we may also limit (39) to those $s, s'$ for which
\[ s'_{\max} - s_{\min} \leq n/2, \]
and hence in (31) we may restrict our consideration to $s, s'$ satisfying (28)–(30).

For the statement regarding (33), notice that since our restrictions on $s, s'$ include $s'_{\max} - s_{\min} \leq n/2$, for any $v$ we have
\[ \left( \frac{n}{s'_v - s_v} \right) \leq \left( \frac{n}{s'_{\max} - s_{\min}} \right), \]
and hence for all relevant $s, s'$ the bound (33) implies (31). \qed

5.6. Proof of Lemma 5.1.

Proof of Lemma 5.1. According to Lemma 5.7, it suffices to show that for each $i \in \mathbb{N}$ there exist $R, \nu, \epsilon > 0$ such that (31) holds for all $s, s'$ satisfying (28)–(30). So fix an $i \in \mathbb{N}$, and let us seek such $R, \gamma, \epsilon$.

According to Corollary 5.5,
\[ \text{Prob}_{G \in C_n(B)} \left[ \Gamma_G(U) \subset U' \right] \leq \prod_{v \in V_B} \prod_{u \sim v} \left[ \text{deg}_B(v)/2 \right] \left( \frac{n}{s_v} \right), \]
where $u \sim v$ is shorthand for multiplication over all $e$ with $te = v$ of $u = he$ (hence for multiple edges the factor of $u$ in the product occurs multiple times). Since
\[ \prod_{u \sim v} \left[ \frac{n}{s_v} \right] \leq \left( \frac{n}{s_v} \right)^{\text{deg}_B(v)/2} \leq \left( \frac{n}{\text{max}_v} \right)^{\text{deg}_B(v)/2} \left( \frac{n}{s_v} \right), \]
we have
\[ \text{Prob}_{G \in C_n(B)} \left[ \Gamma_G(U) \subset U' \right] \leq \prod_{v \in V_B} \left( \frac{n}{\text{max}_v} \right)^{\text{deg}_B(v)/2}. \]
Hence the left-hand-side of (33), namely
\[ \left( \frac{n}{s'_{\max} - s_{\min}} \right)^{\#V_B} \max_{U, U'} \left( \text{Prob}_{G \in C_n(B)} \left[ \Gamma_G(U) \subset U' \right] \right) \prod_{v \in V_B} \left( \frac{n}{s_v} \right), \]
is bounded above by

\[(42)\quad nC_1 \left( s'_{\text{max}} - s_{\text{min}} \right)^{C_2} \prod_{v \in V_B} \left( \frac{n}{s_v} \right)^{(2 - \deg_B(v))/2}\]

for constants \(C_1, C_2 > 0\). Since \(B\) is pruned, \(2 - \deg_B(v) \leq 0\) for all \(v \in V_B\), and since

\[\chi(B) = \sum_{v \in B} (2 - \deg_B(v))/2\]

is negative, \(2 - \deg_B(v) \leq -1\) for at least one \(v\). Hence to establish (33), it suffices to show that for any \(i\) there are \(R, \nu, \epsilon > 0\) such that for any \(v \in V_B\) we have

\[(43)\quad nC_1 \left( s'_{\text{max}} - s_{\text{min}} \right)^{C_2} \left( \frac{n}{s_v} \right)^{-1/2} = O(n^{-i})\]

provided that \(s', s\) satisfy (28)–(30).

(Since \(B\) has no half-loops, \(\chi(B)\) is an integer, so we could replace the \(-1/2\) exponent in (43) by \(-1\); in the next section, when \(B\) may have half-loops, we cannot do so, and the \(-1/2\) exponent will be sufficient for us.)

So let us apply Lemma 5.6 with \(C = 2C_2\) and any \(j \geq (i + C_1)/C_2\); fix a \(\theta > 0\) sufficiently small so that there exist \(S_0, n_0\) for which

\[(44)\quad \left( \frac{n}{r} \right) \leq n^{-(i+C_1)/C_2} \left( \frac{n}{s} \right)^{1/(2C_2)}\]

provided that

\[(45)\quad n \geq n_0, \quad S_0 \leq s \leq n(1/2 + \theta), \quad r \leq \theta s;\]

for such \(n, r, s\) we therefore have

\[\left( \frac{n}{s} \right)^{-1/2} \left( \frac{n}{r} \right)^{C_2} n^{C_1} \leq n^{-i}.\]

It follows that (43) holds provided that for all \(v\) we have

\[S_0 \leq s_v \leq n(1/2 + \theta), \quad s'_{\text{max}} - s_{\text{min}} \leq \theta s_v\]

or, in other words,

\[(46)\quad S_0 \leq s_{\text{min}}, \quad s_{\text{max}} \leq n(1/2 + \theta), \quad s'_{\text{max}} - s_{\text{min}} \leq \theta s_{\text{min}}.\]

Now we specify \(R, \nu, \epsilon\): take \(\nu = \theta/(2m)\), choose any \(\epsilon > 0\) sufficiently small so that

\[(\theta/2 + \epsilon)/(1 - \epsilon) \leq \theta\]

and

\[\frac{1/2}{1 - \epsilon} \leq 1/2 + \theta,\]

and finally any \(R \in \mathbb{N}\) with

\[R \geq S_0 m/(1 - \epsilon).\]

If \(s, s'\) satisfy (28)–(30) with these values of \(R, \nu, \epsilon\), let us verify the three conditions in (46) hold:

(1) \(S_0 \leq s_{\text{min}}\):

\[R \leq s \cdot 1 \leq m s_{\text{max}} \leq m s_{\text{min}}/(1 - \epsilon),\]

so \(s_{\text{min}} \geq R(1 - \epsilon)/m \geq S_0.\]
Proof. Consider a $W$ in $W$ that the subset, $W$ in Lemma 6.1. Let $s$ least $s$ element of $W$ subset, where $p$ element of $W$ is an odd, then $n$ is a random perfect matching for $n$, therefore (33) in Lemma 5.7. This lemma then implies that $C_n(B)$ is pseudo-magnifying.

6. Pseudo-Magnification in Graphs With Half-Loops

Let us now describe the ingredients needed to prove pseudo-magnification when our base graphs may have half-loops. The main point is that we have to include probability estimates involving random involutions.

6.1. An Involution Probability Bound. Here is the alternate of Lemma 5.4 that we will use; if $n, t \in \mathbb{N}$ with $t$ even and $t \leq n$, we use the “odd binomial coefficient notation”:

$$\binom{n}{t}_{\text{odd}} \overset{\text{def}}{=} \frac{(n-1)(n-3)\ldots(n-2t+1)}{(2t-1)(2t-3)\ldots1}.$$ 

Lemma 6.1. Let $n \in \mathbb{N}$, and let $W \subset W' \subset [n]$ be subsets with $\#W \leq \#W'$. Let $s''$ be the largest non-negative even integer with $s'' \leq 2(\#W) - (\#W') - 1$. If $\sigma \in S_n$ is a random perfect matching for $n$ even, and a random near perfect matching for $n$ odd, then

$$\text{Prob}_{\sigma}[\sigma(W) \subset W'] \leq \binom{\#W}{s''}_{\text{odd}}.$$ 

Proof. Consider a $\sigma$ for which $\sigma(W) \subset W'$. Note that $\sigma$ matches every element in $W$ with some element of $W''$, except for possibly one element of $W$ (when $n$ is odd); since at most $(\#W') - (\#W)$ elements of $W$ can be matched with elements in $W' \setminus W$, and at most one element of $W$ can be matched with itself, it follows that the subset, $W'' = W''(\sigma) \subset W$, of elements that $\sigma$ matches in $W$ is of size at least $s''$. Since there are $\binom{\#W}{s''}$ possible values of $W'' = W''(\sigma)$, the union bound implies that

$$\text{Prob}_{\sigma}[\sigma(W) \subset W'] \leq \binom{\#W}{s''} p(n, s''),$$

where $p(n, s'')$ is the probability that a random involution $\sigma \in S_n$ matches a fixed subset, $W''$, of size $s''$ in pairs; this probability equals the probability that a fixed element of $W''$ is matched with another element of $W''$, times the probability that a fixed remaining element is paired with another remaining element, etc. Hence

$$p(n, s'') = \frac{s'' - 1}{n - 1} \frac{s'' - 3}{n - 3} \cdots \frac{1}{n - s'' + 3} = \frac{1}{\binom{n}{s''}_{\text{odd}}}$$

□
6.2. Odd Binomial Coefficient Estimates. It is be simpler for us to express odd binomial coefficients in terms of almost equal expressions involving binomial coefficients.

**Lemma 6.2.** Let \( n, t \in \mathbb{N} \) with \( t \) even. Then
\[
\frac{n-t}{n} \binom{n}{t} \leq \left( \binom{n}{t}_{\text{odd}} \right)^2 \leq t \binom{n}{t}.
\]

**Proof.** Comparing factor by factor, we have
\[
(t-1)! \leq \left( (t-1)(t-3) \ldots 1 \right)^2 \leq t!
\]
and
\[
(n-1)(n-2) \ldots (n-t) \leq \left( (n-1)(n-3) \ldots (n-2t+1) \right)^2 \leq n(n-1) \ldots (n-t+1).
\]
Dividing the second two inequalities by the first two yields
\[
\frac{(n-1)(n-2) \ldots (n-t)}{t!} \leq \left( \binom{n}{t}_{\text{odd}} \right)^2 \leq \frac{n(n-1) \ldots (n-t+1)}{(t-1)!}
\]
which is equivalent to the upper and lower bounds in the lemma. \( \square \)

6.3. Additional Binomial Coefficient Estimates. We will also use some easy binomial coefficient estimates. First, for any \( 0 \leq r' \leq r \leq n \) we have
\[
\binom{n}{r} = \binom{n}{r'} \binom{n-r'}{r-r'} \leq \binom{n}{r'} \binom{n}{r-r'}
\]
and hence
\[
\left( \binom{n}{r'} \right)^{-1/2} \leq \left( \binom{n}{r} \right)^{-1/2} \left( \binom{n}{r-r'} \right)^{1/2}
\]
We will need the trivial estimate that for \( r \geq 0 \) and any \( n \) (including \( n \leq r+1 \))
\[
\binom{n}{r+1} \leq \binom{n}{r} n, \quad \binom{n}{r+2} \leq \binom{n}{r} n^2.
\]

6.4. Proof of Lemma 4.6.

**Proof of Lemma 4.6.** According to Lemma 5.7, it suffices to show that for each \( i \in \mathbb{N} \) there exist \( R, \nu, \epsilon > 0 \) such that (31) holds for all \( s, s' \) satisfying (28)–(30). So fix an \( i \in \mathbb{N} \), and let us prove that such \( R, \nu, \epsilon \) exist; we shall reduce this proof to part of the proof given in Subsection 5.6 (the case where \( B \) has no half-loops).

For each \( v \in V_B \), let half(\( v \)) be the number of half-loops in \( V \) about \( v \). Let \( n \in \mathbb{N} \), and let \( U \subset U' \subset V_B \times [n] \); let \( s, s' \) denote the fibre sizes of \( U, U' \) respectively, and assume that they satisfy (28)–(30) for some \( R, \nu, \epsilon > 0 \) that we will later specify. Let us add the assumptions that
\[
s'_{\max} - s_{\min} \leq s_{\min}/3.
\]
From these assumptions on \( s, s' \), we have that for all \( v \in V_B \)
\[
2s_v - s'_v \geq 2s_{\min} - s'_{\max} \geq (2 - 4/3)s_{\min} \geq 0.
\]
For each \( v \in V_B \), set \( s''_v \) to be the largest even integer less than \( 2s_v - s'_v \); hence
\[
2s_v - s'_v - 2 \leq s''_v \leq 2s_v - s'_v - 1.
\]
Next, let us verify that for sufficiently large $n$ we have
\[
\frac{s_{\text{max}}'}{s_{\text{min}}} - s_{\text{min}} \leq (n/2) - 2, \quad s_{\text{max}} \leq 2n/3;
\]
the first inequality follows from
\[
s_{\text{max}}' - s_{\text{min}} \leq s_{\text{min}}/3 \leq \pi/3 \leq n/6
\]
which is at most $n/2 - 2$ for $n \geq 6$; the second inequality follows from
\[
s_{\text{max}} \leq s_{\text{max}}' \leq (4/3)s_{\text{min}} \leq (4/3)(n/2) = 2n/3.
\]

According to Lemmas 6.1 and 5.4,
\[
\text{Prob}_{G \in \mathcal{B}}[\Gamma_G(U) \subset U'] \leq \prod_{v \in V_B} Q(v)
\]
where
\[
Q(v) = \left(\frac{s_v'}{s_v''} \right)^{\text{half}(v)} \prod_{u \sim v} \sqrt{\frac{n(s_v''/n)}{n(s_v/\nu)}},
\]
where $u \sim v$ is the product over all edges $e$ that are not half-loops, and whose tail is $u$ and whose head is $v$. Let us show that for each $v \in V_B$ we have
\[
Q(v) \leq n^{K_1} \left(\frac{n}{s_{\text{max}}'} - s_{\text{min}}\right)^{K_2} \left(\frac{\text{deg}_B(v) - \text{half}(v)}{2}\right)
\]
where $K_1 = K_1(v), K_2 = K_2(v)$ are constants depending on $v$; once we do this we will obtain the same estimate as in (42)—with different constants $C_1, C_2$—and then finish the proof as it is finished below (42) (for the case there, where $B$ has no half-loops).

Since the number of such $e$ is $\text{deg}_B(v) - \text{half}(v)$, (41) and the equation above it imply that
\[
Q(v) \leq \left(\frac{s_v'}{s_v''} \right)^{\text{half}(v)} \left(\frac{n}{s_{\text{max}}'} - s_{\text{min}}\right)^{\text{deg}_B(v) - \text{half}(v)/2}
\]
To estimate the new term raised to the power $\text{half}(v)$ (whenever $\text{half}(v) > 0$), we note that
\[
\frac{s_v'}{s_v''} - \frac{s_v''}{s_v'} \leq \frac{s_v'}{s_v} - 2 - s_v \leq s_{\text{max}}' - s_{\text{min}} + 2 \leq n/2
\]
(using (52)), and therefore
\[
\left(\frac{s_v'}{s_v''} \right)^{\text{half}(v)} \leq \left(\frac{n}{s_{\text{max}}'} - s_{\text{min}}\right)^{\text{deg}_B(v) - \text{half}(v)/2}
\]
(whenever $\text{half}(v) > 0$). Also, in view of Lemma 6.2
\[
\left(\frac{n}{s_v''} \right)^{1/2} \geq \left(\frac{n-s_v''}{n} \right)^{1/2} \frac{n}{s_v''}.
\]
Hence
\[
\left(\frac{n}{s_v''} \right)^{1/2} \leq \left(\frac{n}{s_{\text{max}}'} - s_{\text{min}}\right) n \left(\frac{n-s_v''}{n} \right)^{1/2} \frac{n}{s_v''}.
\]
We also have
\[
s_v'' \leq 2s_v' - s_v \leq s_v
\]
Also, setting \( r' = s''_v \) and \( r = s_v \) in (47) we have
\[
\frac{n - s''_v}{n} \leq \frac{n - s_v}{n}.
\]

Also, setting \( r' = s''_v \) and \( r = s_v \) in (47) we have
\[
\left( \frac{n}{s''_v} \right)^{-1/2} \leq \left( \frac{n}{s_v} \right)^{-1/2} \left( \frac{n}{s_v - s''_v} \right)^{1/2};
\]

using (55) this implies
\[
\left( \frac{n}{s''_v} \right)^{-1/2} \leq \left( \frac{n}{s_v} \right)^{-1/2} \left( \frac{n}{s_{\text{max}} - s_{\text{min}} + 2} \right)^{1/2} \leq \left( \frac{n}{s_v} \right)^{-1/2} \left( \frac{n}{s_{\text{max}} - s_{\text{min}}} \right)^{1/2} n.
\]

Applying this inequality and (57) to (56) we get
\[
\frac{(s_v)}{(s''_v)} \leq \frac{(n)}{(n - s_v)} \left( \frac{n}{s_{\text{max}} - s_{\text{min}}} \right)^{3/2} \left( \frac{n}{s_v} \right)^{-1/2}.
\]

This establishes (54). It follows from (53) that
\[
\text{Prob}_{G \in C_n(B)}[\Gamma_G(U) \subset U'] \leq n K'_1 \left( \frac{n}{s'_{\text{max}} - s_{\text{min}}} \right) K'_2 \prod_{v \in V_B} \left( \frac{n}{s_v} \right)^{2 - \text{deg}_B(v)} / 2,
\]
where \( K'_1, K'_2 \) are the sum over the \( K_1(v), K_2(v) \). Hence for some \( v \in V_B \) we have
\[
\text{Prob}_{G \in C_n(B)}[\Gamma_G(U) \subset U'] \leq n K'_1 \left( \frac{n}{s'_{\text{max}} - s_{\text{min}}} \right) K'_2 \left( \frac{n}{s_v} \right)^{-1/2}
\]
(see the discussion between (42) and (43)). It follows that the left-hand-side of (33) is bounded by (42) for some \( C_1, C_2 \) (involving \( K'_1, K'_2 \)).

Now we mimic the rest of the proof of Lemma 5.1 following (43): to establish (33) it suffices to prove that for any \( i \) there are \( R, \nu, \epsilon > 0 \) that guarantee (43). We use Lemma 5.7 to find \( \theta > 0 \), \( S_0, n_0 \) such that (44) holds provided that (45) holds; since Lemma 5.7 implies that there are \( S_0, n_0 \) for arbitrarily small \( \theta > 0 \), we may insist that \( \theta \leq 1/3 \). Then we take \( R, \nu, \epsilon > 0 \) as given just below (46), and the same proof shows that for any \( s, s' \) that satisfy (28)–(30), then (46) holds; since we took \( \theta \leq 1/3 \), the last inequality in (46) implies that (49) holds. It follows that for any \( i \) there are \( R, \nu, \epsilon > 0 \) such that (42) holds, and therefore (33) holds. Therefore Lemma 5.7 implies that \( C_n(B) \) is pseudo-magnifying. \( \square \)

7. Proof of Theorem 3.7

In this section we prove Lemma 4.5, and then easily deduce Theorem 3.7.

Our proof really shows that if \( B \) is \( d \)-regular and \( C_n(B) \) is pseudo-magnifying, then \( \tau_{\text{alg}} \) can only be finite if for some eigenvalue \( \ell \) of the model, with \( |\ell| > (d-1)^{3/2} \), but \( \ell \neq d-1 \), we have that (11) holds for fixed \( \tau \) and arbitrarily small \( \nu > (d-1)^{3/2} \) and arbitrarily large \( r \).

Proof of Lemma 4.5. For any \( i \), the are \( R, r, \gamma \) such that the \( G \in C_n(B) \) probability that \( G \) is not an \((R, \gamma)\)-pseudo-magnifier is at most \( O(n^{-i}) \). However, for any \( G \in \text{Coord}_n(B) \) and \( R \), if \( U \subset V_G \) has \( \#U \leq R \), then either:

1. \( \Gamma_G(U) \setminus U = \emptyset \), in which case \( U \) is disconnected from the other vertices in \( V_G \), or
(2) \( \Gamma_G(U) \setminus U \) in nonempty, in which case
\[
\#(\Gamma_G(U) \setminus U) \geq 1 \geq (1/R)(\#U).
\]
Hence setting
\[
\gamma' = \min(\gamma, 1/R),
\]
if \( G \) is not \( \gamma' \)-magnifier, then \( G \) has a connected component, \( U' \), of size at most \( R \), which makes \( U' \) a \( \geq (d - 1) \)-tangle whose order is
\[
\text{ord}(U') = \text{ord}(B) \frac{\#V_{U'}}{\#V_B} \leq \text{ord}(B)R/(\#V_B).
\]
Hence taking \( r \in \mathbb{N} \) with \( r \geq 1 + \text{ord}(B)R/(\#V_B) \), the probability that \( G \) is \( (d - 1, < r) \)-tangle free and is not an \( \gamma' \)-magnifier is at most \( O(n^{-i}) \).

According to Alon’s theorem, if \( G \in \text{Coord}_n(B) \) is a \( \gamma' \)-magnifier, then all eigenvalues of \( A_G \) except the largest one are bounded away from \( d \); it follows that \( \text{Spec}_{B}^\text{new}(H_G) \) is bounded away from \( d - 1 \), and hence
\[
\text{Spec}_{B}^\text{new}(H_G) \cap B_{n^{-\phi}}(d - 1)
\]
is empty for \( n \) sufficiently large. Since the number of new eigenvalues of \( H_G \) is \( O(n) \), it follows that for the above value of \( r \) we have
\[
\mathbb{E}_{G \in \text{Coord}_n(B)} \left[ t_{\text{TangleFree}(\geq d-1, < r)}(G) \left( \#(\text{Spec}_{B}^\text{new}(H_G) \cap B_{n^{-\phi}}(\ell)) \right) \right] = O(n^{-i+1}).
\]
Since \( i \) is arbitrary, we conclude Lemma 4.5. \( \square \)

**Proof of Theorem 3.7.** If \( \tau_{\text{alg}} \) is finite, then for some \( \tau \in \mathbb{N} \) and \( \nu > (d - 1)^{1/2} \) sufficiently small and \( r \in \mathbb{N} \) sufficiently large we have
\[
\mathbb{E}_{G \in \text{Coord}_n(B)} \left[ t_{\text{TangleFree}(\geq \nu, < r)}(G) \left( \#(\text{Spec}_{B}^\text{new}(H_G) \cap B_{n^{-\phi}}(\ell)) \right) \right] = C \ell n^{-\tau} + o(n^{-\tau}).
\]
for some \( C > 0 \) (\( C \ell \) may depend on \( \nu \) and \( r \)), where \( \ell \) an eigenvalue of the model with \( |\ell| > (d - 1)^{1/2} \).

First consider the case where \( B \) is \( d \)-regular Ramanujan and not bipartite. Then \( \ell = d - 1 \) is the only eigenvalue of the model with \( |\ell| > (d - 1)^{1/2} \). In view of Lemma 4.5, for any \( i \in \mathbb{N} \), the left-hand-side of (58) is \( O(n^{-i}) \) for \( \nu = d - 1 \) and some \( r \in \mathbb{N} \), which contradicts (58) for these values of \( \nu, r \), and hence for any smaller \( \nu \) and any larger \( r \). Hence \( \tau_{\text{alg}} = +\infty \).

Next consider the case where \( B \) is connected, \( d \)-regular, Ramanujan, and bipartite. Then for any \( G \in \text{Coord}_n(B) \), \( G \) is also bipartite, and hence has the same multiplicity of the eigenvalue \( d - 1 \) in \( H_G \) as it does that of \( -(d - 1) \). Since \( H_G \) has one old eigenvalue of \( d - 1 \) and one of \( -(d - 1) \) (since \( B \) is bipartite and connected), it follows that the left-hand-side of (58) is the same for \( \ell = d - 1 \) and \( \ell = -(d - 1) \). Since (58) cannot hold for \( \ell = d - 1 \) with \( \nu \) arbitrarily close to \( (d - 1)^{1/2} \) and \( r \) arbitrarily large—by the argument in the previous paragraph—it also cannot hold for \( \ell = -(d - 1) \). Hence \( \tau_{\text{alg}} = +\infty \) also in this case. \( \square \)

**8. Bounds on \( \tau_{\text{tang}} \)**

If \( B \) is \( d \)-regular and Ramanujan, we now know that Theorem 3.5 holds with
\[
\tau_1 = \tau_2 = \tau_{\text{tang}},
\]
and therefore for fixed $\epsilon > 0$ sufficiently small we have matching upper and lower bounds proportional to $n^{−\tau_{\text{tang}}}$ for

$$\text{Prob}_{G \in \mathcal{C}_n(B)}[\text{NonAlon}_d(G; \epsilon) > 0].$$

It therefore becomes interesting to have bounds on $\tau_{\text{tang}}$. Let us just give those bounds that follow from [Fri08].

8.1. **Lower Bounds on $\tau_{\text{tang}}$ for any Model.** Chapter 6 of [Fri08] computes $\tau_{\text{fund}}$ of various models, which is the smallest order of a $\geq (d-1)^{1/2}$-tangle. By definition, $\tau_{\text{tang}}$ is the smallest order of a $\geq (d-1)^{1/2}$-tangle, so the same techniques apply.

**Theorem 8.1.** Let $B$ be a graph and let $m = m(B)$ be the smallest integer with

$$2m − 1 > \mu_{1/2}(B).$$

Then for any algebraic model, $\mathcal{C}_n(B)$, $\tau_{\text{tang}} \geq m − 1$, and equality holds if the bouquet of $m$ whole-loops occurs in $\mathcal{C}_n(B)$. In particular, if $B$ is $d$-regular then $(\mu_1(B) = d−1$ and) $m(B)$ depends only on $d$ and is given by

$$(59) \quad m = m(d) = \left\lceil \left(\frac{(d−1)^{1/2} + 1}{2}\right) \right\rceil + 1,$$

and

$$(60) \quad \tau_{\text{tang}} \geq \left\lceil \left(\frac{(d−1)^{1/2} + 1}{2}\right) \right\rceil;$$

furthermore, equality holds in the permutation model if $B$ has a vertex incident upon at least $m$ whole-loops.

**Proof.** Lemma 6.7 of [Fri08] shows that if $u, v$ are distinct vertices in a graph, $\psi$, joined by an edge, $e$, then the graph, $\psi'$ obtained by identifying $u$ and $v$ and discarding $e$ has the same order as $\psi$ and satisfies $\mu_1(\psi') \geq \mu_1(\psi)$. By repeatedly performing this operation on a connected graph, $\psi$, we get a graph $\psi'$ with one vertex, of the same order as $\psi$ and with $\mu_1(\psi') \geq \mu_1(\psi)$. It follows that if $\psi$ is a $(\geq \nu, < r)$-tangle, then so is $\psi'$. If $\psi'$ is $d'$-regular, then since $\psi'$ has one vertex we have $\mu_1(\psi') = d' − 1$ (by the Ihara Determinantal formula\(^3\)). If $\psi'$ has $m$ whole-loops and $m'$ half-loops, then $\text{ord}(\psi') = m + m' − 1$ and $d' = 2m + m'$. It follows that if $m = m(B)$ is the smallest integer for which

$$2m − 1 > \mu_{1/2}(B),$$

then $\tau_{\text{tang}} \geq 2m − 1$, and that equality holds if the graph with $m$ whole-loops occurs in $\mathcal{C}_n(B)$. Then (59) follows, and therefore (60) as well. \(\square\)

8.2. **Lower Bounds on $\tau_{\text{tang}}$ for Models Where Whole-Loops Do Not Occur.** Similarly we can use the methods of Chapter 6 of [Fri08] to determine $\tau_{\text{tang}}$ in cases where no graph with whole-loops occurs in $\mathcal{C}_n(B)$.

**Theorem 8.2.** Let $\mathcal{C}_n(B)$ be an algebraic model over a graph $B$ such that no graph with one or more whole-loops occurs in $\mathcal{C}_n(B)$. Then if $m' = m'(B)$ is the smallest integer with

$$m' − 1 > \mu_{1/2}(B),$$

\(^3\) One can also see $\mu_1(\psi') = d' − 1$ by noting that any non-backtracking walk can be augmented by one step that continues the walk to be non-backtracking in $d' − 1$ ways, and can be made to be SNBC in $d' − 2$ ways, which shows that $d'(d' − 1)^{k−2}(d' − 2) \leq \text{SNBC}(\psi', k) \leq d'(d' − 1)^{k−1}$.\(\)
then $\tau_{\text{tang}} \geq m' - 2$, and equality holds if some vertex of $B$ is incident upon $m' + 1$ self-loops (which may be any combination of whole-loops and half-loops). In particular, for the full cycle-involution model, of even or of odd degree, $C_n(B)$, we have $m' = m'(B)$ depends only on $d$ and is given by

$$m'(d) = \left\lfloor (d - 1)^{1/2} \right\rfloor + 2,$$

and

$$\tau_{\text{tang}} \geq \left\lfloor (d - 1)^{1/2} \right\rfloor.$$

**Proof.** Lemma 6.9 of [Fri08] shows that if $\psi$ is a graph with two vertices, $u, v$, of distance exactly two, so that they are both adjacent to some vertex $w$, then the graph, $\psi'$, obtained by identifying $u, v$ and discarding one edge from $w$ to $u$ (or to $v$), has $\mu_1(\psi') \geq \mu_1(\psi)$ (and, of course, $\text{ord}(\psi') = \text{ord}(\psi)$). Note that this operation doesn’t create any new self-loops. Repeated application of this process yields a graph, $\psi''$, with the same number of whole-loops and half-loops as in $\psi$, such that $\psi, \psi''$ have the same order and $\mu_1(\psi'') \geq \mu_1(\psi)$. If $\psi''$ contains a pair of vertices $u, v$ that are joined by only one edge, then the operation of Lemma 6.7 identifying $u$ and $v$ and discarding the edge between them (see the proof of Theorem 8.1 above) produces a graph with one fewer vertices, the same order, the same number of whole-loops and half-loops, and no smaller a $\mu_1$. Repeated application of this process yields a graph $\psi'''$ with $\text{ord}(\psi''') = \text{ord}(\psi)$, the same number of whole-loops and of half-loops as in $\psi$, and $\mu_1(\psi''') \geq \mu_1(\psi)$.

Now say that $\psi$ is a $(\geq \nu, < r)$-tangle that occurs in $C_n(B)$. Then $\psi$ has no whole-loops, and so the process above yields a $\psi'''$ that is another such tangle and has no whole-loops. Next we note:

1. If $\psi'''$ has one vertex, then $\psi'''$ is a bouquet of $d'$ half-loops, and

$$\mu_1(\psi''') = d' - 1 = \text{ord}(\psi''');$$

2. If $\psi'''$ has two vertices and $m$ edges, then

$$\mu_1(\psi''') = m - 1 = \text{ord}(\psi''') + 1$$

and $\mu_1(\psi''')$ is at most one less than maximum degree of a vertex, which is at most $m - 1$; furthermore this is attained for the graph with two vertices joined by $m$ edges (i.e., without half-loops);

3. If $\psi'''$ has $n \geq 3$ vertices, then $\mu_1(\psi''')$ is at most one less than the maximum degree of a vertex, which is at most

$$\#E_{\psi'''} - \left(\frac{n - 1}{2}\right)^2 - 1$$

since each pair of vertices of $\psi''$ is joined by at least two edges and there are no whole-loops so each edge incident upon a vertex can contribute at most 1 to its degree; hence

$$\mu_1(\psi''') \leq \#E_{\psi'''} - (n - 1)(n - 2) - 1$$

and since $\#E_{\psi'''} = \text{ord}(\psi''') - \#V_{\psi'''} = \text{ord}(\psi''') - n$, we have

$$\mu_1(\psi''') \leq \text{ord}(\psi''') + n - (n - 1)(n - 2) - 1 = \text{ord}(\psi''') + 1 - (n - 2)^2.$$ 

It follows that $\mu(\psi''') < \text{ord}(\psi''') + 1$ if $n \geq 3$. 

[ Fri08 ]
It follows that if \( m' = m'(B) \) is the smallest integer with
\[
m' - 1 > \mu_1(B),
\]
then
\[
\tau_{\text{tang}} = m' - 2,
\]
with equality if the graph with two vertices and \( m' \) edges occurs in \( C_n(B) \). \( \square \)

8.3. The One Vertex Case. The theorems proven so far (all drawn from [Fri08]) are sufficient to determine \( \tau_{\text{tang}} \) in our basic models for graphs for a bouquet of \( d/2 \) whole-loops (for \( d \geq 4 \) and even) and a bouquet of \( d \) half-loops (for \( d \geq 3 \), except for \( d \) small (since \( m(d), m'(d) \) are of order \( d^{1/2} \)). In fact, we easily check that these theorems determine \( \tau_{\text{tang}} \) in all cases except the full cycle model over a bouquet of 2 or 3 whole-loops (since whole-loops cannot occur in the full cycle model, and a graph with two vertices joined by, respectively, 3 or 4 edges does not occur in the model).

In [Fri08] these cases are dealt with by producing graphs that occur in these models that match the lower bounds. For example, the proof of Theorem 6.10 shows that for 3 whole-loops, the graph \( \psi \) with three vertices, \( \{v_1, v_2, v_3\} \), where \( v_1, v_2 \) are joined by three edges and \( v_2, v_3 \) by two edges has order 2 and \( \mu_1 \geq \sqrt{3} > \sqrt{5} \); since \( \psi \) occurs in this model, we still conclude the bound in Theorem 8.1. However, the example in the proof of Theorem 6.10 for the full cycle model over a bouquet of 2 whole-loops is not sufficient here (since this example has \( \mu_1(\psi) = \sqrt{3} \), while \( \tau_{\text{tang}} \)—as opposed to \( \tau_{\text{fund}} \) in [Fri08]—looks for \( \psi \) with \( \mu(\psi_1) > \sqrt{3} \), the inequality required to be strict.\(^4\)

Note that Theorems 8.1 and 8.2 aren’t strictly sufficient to determine \( \tau_{\text{tang}} \) for an arbitrary bouquet of whole-loops and half-loops. However, the method of the proof of Theorem 8.2 shows gives a method of bounding the number of vertices, \( n \), in a \((> \mu_1^{1/2}(B))\)-tangle, so for a fixed \( B \) this would become a finite procedure.

8.4. The Case of \( B \) with Sufficiently Large Girth. In this section we note that for fixed \( d \), \( \tau_{\text{tang}}(B) \) for a \( d \)-regular graph, \( B \), is bounded below by a function of the girth of \( B \) that tends to infinity as the girth tends to infinity. It follows that for fixed \( d \), one can find \( d \)-regular graphs where \( \tau_{\text{tang}}(B) \) is arbitrarily large.

**Theorem 8.3.** For a fixed \( r \in \mathbb{N} \) and a real \( \nu > 1 \), there is a \( g \in \mathbb{N} \) such that any \((\geq \nu, < r)\)-tangle has girth at most \( g \).

**Proof.** According to Article III or Lemma 9.2 of [Fri08], there are a finite number of \((\geq \nu, < r)\)-tangles \( \psi_1, \ldots, \psi_k \) such that any \((\geq \nu, < r)\)-tangle contains a subgraph isomorphic to \( \psi_i \) for some \( i \). Since \( \nu \geq 1 \), each \( \psi_i \) contains a cycle of some length \( L_i \). Hence any \((\geq \nu, < r)\)-tangle has girth at most \( \max_i L_i \). \( \square \)

Of course, the above proof does not give an explicit estimate of \( g \).

**Corollary 8.4.** For a fixed \( r, d \in \mathbb{N} \), there is a \( g \) such that if \( B \) is \( d \)-regular and of girth greater than \( g \), then \( \tau_{\text{tang}} \geq r \).

---

\(^4\) Certainly \( \tau_{\text{tang}} \leq 2 \) for the bouquet of 2 whole-loops under the cyclic model: consider the graph \( \psi \) with four vertices \( \{v_1, v_2, v_3, v_4\} \) where \( v_i \) and \( v_{i+1} \) are joined by two edges for \( i = 1, 2, 3 \); we claim that \( \mu_1(\psi) > \sqrt{3} \) (roughly since each time we are in a middle vertex, i.e., \( v_2, v_3 \), we have three possible choices, and we will encounter side vertices less than half the time in a typical non-backtracking walk). Since \( \text{ord}(\psi) = 2 \) and \( \psi \) occurs in this model. We believe that there is no \( > \sqrt{3} \) tangle of order 1 that occurs in this model, but there are a few cases to check.
Proof. If $\psi$ occurs in any model over $B$, then $\psi$ admits an étale map to $B$ and hence if $\psi$ has an SNBC closed walk of length $k$, then so does $B$; it follows that the girth of $\psi$ is at least that of $B$. Hence if $g$ is as in the above theorem with $\nu = (d - 1)^{1/2}$ and $r$ fixed, then the order of any graph, $\psi$, occurring in any model of $B$ where $B$ is $d$-regular (and therefore $\mu_1(B) = d - 1$) and of girth greater than $g$ has $\text{ord}(\psi) \geq r$. □

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