On Bisimulation in Absence of Restriction

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Abstract We revisit the standard bisimulation equalities in process models free of the restriction operator. As is well-known, in general the weak bisimilarity is coarser than the strong bisimilarity because it abstracts from internal actions. In absence of restriction, those internal actions become somewhat visible, so one might wonder if the weak bisimilarity is still 'weak'. We show that in both CCScore (i.e., Milner’s standard CCS without $\tau$-prefix, summation and relabelling) and its higher-order variant (named HOCCScore), the weak bisimilarity indeed remains weak, i.e., still strictly coarser than the strong bisimilarity, even without the restriction operator. These results can be extended to other first-order or higher-order process models. Essentially, this is due to the direct or indirect existence of the replication operation, which can keep a process retaining its state (i.e., capacity of interaction). By virtue of these observations, we examine a variant of the weak bisimilarity, called quasi-strong bisimilarity. This quasi-strong bisimilarity requires the matching of internal actions to be conducted in the strong manner, as for the strong bisimilarity, and the matching of visible actions to have no trailing internal actions. We exhibit that in CCScore without the restriction operator, the weak bisimilarity exactly collapses onto this quasi-strong bisimilarity, which is moreover shown to coincide with the branching bisimilarity. These results reveal that in absence of the restriction operation, some ingredient of the weak bisimilarity indeed turns into strong, particularly the matching of internal actions.

keywords: Strong Bisimulation, Weak Bisimulation, Restriction, Higher-order, First-order, Processes

2000 MSC: 68Q85

1 Introduction

Process models study the behaviour of concurrent systems, particularly their equivalence or degree of similarity [13, 14]. Bisimulation equality, called bisimilarity, is the most exploited notion of such equivalence [8, 12]. For two concurrent systems, a strong bisimulation requires each action, whether external (i.e., visible) or internal (i.e., invisible), of one system to be precisely matched by the other. In contrast, a weak bisimulation, as its name suggests, allows the bisimulation to be observational. Namely, an external action of one system can be matched by the same action of the other, possibly mingled with some internal actions. As is well-known, the weak bisimulation equality is usually coarser than the strong bisimulation equality, because the former can hide some computation inside the system (e.g., implementation). A typical way to achieve such hiding is by the restriction operator, sometimes called the localization operator, which literally has the effect of concealing a port/channel name from being discovered. For example, in the language of pure CCS [8], the following process $M$ has three concurrent components $P, Q, R$ connected by the operation of parallel composition ($\parallel$), and the components $P, Q$ share a restricted (or local) name $m$, represented by the restriction operation $(\langle m \rangle)$, that can be used to keep

*The author is supported by NSF of China (61872142, 62072299) and partially by the project ANR 12IS02001 PACE.
$R$ from knowing or sensing something happening between $P$ and $Q$, virtually forming a subsystem.

$$M \overset{\text{def}}{=} (m(P \mid Q) \mid R)$$

Restriction is a frequent and powerful operator in process models. Intensionally it can hide from outside world the critical information, and extensionally it facilitates internal or silent movement (possibly forced synchronization) so that the intrinsic implementation details are transparent to observers. From the viewpoint of computation, it provides a recourse to Turing completeness as well as interactional completeness in the measurement of computability [1, 14]. Concomitantly, some undecidability issues come along with the high computability, e.g., undecidability of bisimulations, making it sometimes succumb to efficient use in practice. So sometimes it is tempting to work without the restriction operator. Although this may bring about certain decrease in expressiveness, the resulting model can avoid being too powerful to be tractable, and in effect be advantageous for practical applications. Moreover, fortuitously some models without the restriction operator still turn out to be computational complete, e.g., higher-order processes [6, 14]. In another notable work, Hirschkoff et al. [5] study a sub-calculus of the pi-calculus without the restriction and the choice operations but featured with a special top-level replication. The focus of that work is to provide a new congruence result for that sub-calculus, by means of a syntactic characterisation of the (strong) bisimilarity. In this work, by contrast, we are interested in the relationship between the strong and weak bisimilarities in absence of the restriction operation.

It is natural to consider this: if the restriction operator is removed (so one loses the power of hiding), would weak bisimilarity still be coarser than strong bisimilarity? Or would the ‘gap’ between weak and strong bisimilarities become less pronounced? In this work, we look into this question and provide some answer. For our purpose, we take CCSCORE and its higher-order variant (named HOCCSCORE) as the test bed. CCSCORE (respectively HOCCSCORE) represents the standard CCS by Milner [8] (respectively plain CHOCS by Thomsen [15]) with the basic first-order (respectively higher-order) concurrency formalism, and without the $\tau$-prefix, summation and relabelling. In turn, CCS$^-$ (respectively HOCCS$^-$) denotes CCSCORE (respectively HOCCSCORE) free of the restriction operation. It is worth noting that CCS$^-$ and HOCCS$^-$ still admit the replication operation (in the latter, it is a derived operation), otherwise these models would be far less interesting. We choose CCSCORE and HOCCSCORE because they contain the interesting minimal part of the original models suitable (and non-trivial) for our study. Though useful, summation and relabelling are not essential for our work here. In particular, having the $\tau$-prefix would actually defeat the purpose of this work, since it can be deemed as an operation derivable from the restriction operation. To see this, think of the $\tau$-prefix $\tau . P$ as $(c)(c . P \mid \tau . 0)$ where $(c)Q, c . Q, \tau . Q$ denote the standard CCS restriction, input, and output respectively.

The main goal of this paper is to examine the bisimulation equalities in calculi CCS$^-$ and HOCCS$^-$.

We are interested in the relationship between the weak bisimilarity and the strong bisimilarity, particularly in absence of the restriction operation. Importantly, the lack of the restriction operator makes the interaction completely exposed to the environment, so that two weakly bisimilar processes may be forced to behave in the manner of strong bisimilarity, from certain perspective. Moreover, the situation can further vary when it comes to the style of interaction, say first-order synchronization (i.e., implicit name-passing) or higher-order process-passing.

**Contribution** We demonstrate that in CCS$^-$, the weak bisimilarity does somehow collapse onto a bisimulation equality called quasi-strong bisimilarity. In particular, this quasi-strong bisimilarity requests strong bisimulation on internal moves, and almost the same as the weak bisimilarity does for external actions. By ‘almost’, we mean that in the matching of an external (i.e., visible) action $\alpha$, one can make
a few internal (τ) actions before α but none after it, i.e., \( \equiv \xrightarrow{\alpha} \) in standard notation (also known as the ‘delay’ transition \([14]\)). The quasi-strong bisimilarity, as it appears, strengthens the weak bisimilarity, and moves toward the strong bisimilarity. Moreover, as a corollary, we show that the quasi-strong bisimilarity actually concides with the branching bisimilarity, which in turn implies the coincidence between the weak bisimilarity and the branching bisimilarity.

Specifically, we prove in detail that in CCS\(^-\), the weak bisimilarity can indeed be tightened, to be coincident with the quasi-strong bisimilarity. That being said, there appears to be still some distance from the strong bisimilarity. This is essentially attributed to the replication operator, which can introduce infinity that in turn generates some kind of state-preserving behaviour (if a process maintains all its interactional capability after doing some action, we refer to that action as state-preserving; otherwise it is state-changing). Consequently to some extent, replication plays a role of realizing immutability in a concurrent system even without the restriction operator.

In contrast, in HOCCS\(^-\) the weak bisimilarity appears resilient to the removal of the restriction and stays put anyhow. That is, the weak bisimilarity retains being strictly coarser than the strong bisimilarity, and what is more, we do not know how to strengthen it to the quasi-strong bisimilarity as for CCS\(^-\). This intrinsically develops from the complexity of process-passing, which enables one to encode recursion and yield richer behaviours subsequently. In that sense, erasing the restriction operator turns out to have little effect on the behavioural equivalences.

This work can potentially help to precisely identify the boundaries between different notions of bisimulations, while clarifying further the features of the considered process models and the properties of the processes that are classified (differently) by the different bisimulations. In the theoretical regard, the results of this work give evidence and lend confidence to related study of concurrent models concerning the relationship between the weak and strong bisimilarities in a setting free of the restriction operator. In regard to application, the results of this work can hopefully help in choosing a suitable bisimulation (for instance, a less demanding one with clauses easy to handle) when it comes to practical scenarios. Also the technical arguments for these results might be of independent interest.

**Organization** The remainder of the paper is organized as follows. Section 2 tackles the bisimilarity in HOCCS\(^-\). Section 3 deals with the CCS\(^-\) situation. In both sections, we first define the syntax and semantics of the corresponding calculus, and then present the main results with detailed discussion. Section 4 concludes the paper and points to some future work.

## 2 On the bisimulation equality in HOCCS\(^-\)

We first define HOCCS\(^-\), i.e., HOCCScore without the restriction operation, and then discuss the relationship between the strong and weak bisimulation equalities, that is, the bisimilarities.

### 2.1 Calculus HOCCS\(^-\)

A HOCCS\(^-\) process is given by the following grammar. We denote names by lowercase letters, processes by uppercase letters, and process variables by \( X, Y, Z \).

\[
P, P' ::= 0 \mid X \mid a(X) . P \mid \overline{a}P'.P \mid P \mid P'
\]

The operators have their standard meaning: input prefix \( a(X) . P \), output prefix \( \overline{a}P'.P \), and parallel composition \( P \mid P' \). We stipulate parallel composition to have the least precedence.
A process variable $X$ occurring in $P$ is bound by input-prefix $a(X).P$ and free otherwise. We use $\text{fpv}(-)$, $\text{bpv}(-)$, $\text{pvt}(-)$ respectively to denote free process variables, bound process variables and process variables in a set of processes. Additionally, we use $n(-)$ to denote the names in a set of processes. A name or process variable is fresh if it does not appear in the processes under consideration. Closed processes are those having no free variables and considered by default in discussion. As usual, we use $a.0$ as a shortcut for $a(X).0$, and $\pi.0$ for $\pi\{X\}0$; moreover, the trailing 0 is often omitted. Sometimes for clarity, we may write $\pi[A]$ for higher-order output. A tilde $\sim$ represents a tuple. A higher-order substitution $P[A/X]$ replaces free occurrences of variable $X$ with $A$ and can be extended to tuples in the expected entry-wise way.

A context $C$, or $C\cdot$ to emphasize the hole in it, is a process with some subprocess replaced by the hole $\cdot$, and $C[A]$ is the process obtained by substituting $A$ for the hole. We denote by $E[X]$ the process expression $E$ (possibly) with free occurrence of the variables $X$, and $E[A]$ stands for $E\{A/X\}$. Essentially, $E[X]$ can be treated as a multi-hole context $[14]$, and sometimes we also write $E[\cdot]$ in the discussion.

The semantics of HOCCS$^-$ (on closed processes) is as below. The symmetric rules are omitted.

\[
\begin{align*}
\frac{}{a(X).P \xrightarrow{a(A)} P[A/X]} & \quad \frac{}{E[A].P \xrightarrow{E[A]} P} \\
\frac{P \xrightarrow{\lambda} P'}{P | Q \xrightarrow{\lambda} P' | Q} & \quad \frac{P \xrightarrow{a(A)} P' \quad Q \xrightarrow{E[A]} Q'}{P | Q \xrightarrow{\sim} P' | Q'}
\end{align*}
\]

We denote by $\alpha, \lambda$ the actions: internal move ($\tau$), input $(a(A))$, output $(\pi A)$. Operations $\text{fpv}(-)$, $\text{bpv}(-)$, $\text{pvt}(-)$, $n(-)$ can be similarly defined on actions. As usual, $\Rightarrow$ is the reflexive transitive closure of internal actions, and $\xrightarrow{\lambda}$ is $\Rightarrow \lambda \Rightarrow$. Also $\xrightarrow{\tau}$ is $\Rightarrow \tau \Rightarrow$ otherwise. We use $\xrightarrow{\tau^k}$ to mean $k$ consecutive $\tau$’s. For a binary relation $\Re$, we use $P \Re Q$ as a shortcut for $(P, Q) \in \Re$. Sometimes we write $P \xrightarrow{\tau^k} Q$ to mean that there exists $P''$ such that $P \xrightarrow{\tau^k} P''$ and $P'' \Re Q$.

We denote by $\equiv$ the standard structural congruence $[9][14]$. It is the smallest equivalence relation satisfying $\alpha$-convertibility over (bound) process variables, the monoid laws and commutative laws for parallel composition. That is,

\[
\begin{align*}
a(X).P & \equiv a(Y).P\{Y/X\} \quad (Y \text{ fresh}) & P|0 & \equiv P \\
| & & | \\
P|\langle Q|R \equiv \langle P|Q | R | P
\end{align*}
\]

As is well-known, replication can be derived in HOCCS$^-$ $[7][13][15]$. That is, we can define

\[
\begin{align*}
!P & \overset{\text{def}}{=} \tau Q_{c,p} | Q_{c,p} & Q_{c,p} & \overset{\text{def}}{=} c(X).(\tau X | X | P) \quad (c \text{ fresh})
\end{align*}
\]

Sometimes the name $c$ used to achieve such a replication is referred to as a replicator name. Further, we can also define the so-called guarded replication for a prefix $\phi$ (input or output).

\[
\begin{align*}
!\tau \phi.P & \overset{\text{def}}{=} \tau Q_{c,\phi,p} | Q_{c,\phi,p} \\
Q_{c,\phi,p} & \overset{\text{def}}{=} c(X).(\phi.(\tau X | X | P))
\end{align*}
\]
Context bisimulation

Throughout, we are relying on the following standard notion of context bisimulation \[10,11\].

**Definition 1** (Context bisimulation). A symmetric relation \( \mathcal{R} \) on (closed) HOCCS\(^-\) processes is a (weak) context bisimulation (respectively strong context bisimulation), if \( P \mathcal{R} Q \) implies the following properties:

1. if \( P \xrightarrow{\alpha} P' \) in which \( \alpha \) is \( a(A) \) or \( \tau \), then \( Q \xrightarrow{\alpha} Q' \) (respectively \( Q \xrightarrow{\alpha} Q' \)) for some \( Q' \) and \( P' \mathcal{R} Q' \).
2. if \( P \xrightarrow{\pi A} P' \), then \( Q \xrightarrow{\pi B} Q' \) (respectively \( Q \xrightarrow{\pi B} Q' \)) for some \( B \) and \( Q' \), and for every \( E[X] \) it holds that

\[
E[A] \mid P' \mathcal{R} E[B] \mid Q'
\]

The (weak) context bisimilarity (respectively strong context bisimilarity), denoted by \( \approx \) (respectively \( \sim \)), is the largest context bisimulation (respectively strong context bisimulation).

It is well-known that both \( \approx \) and \( \sim \) are congruences \[10,11,14,15\]. Relation \( \approx \) (similar for \( \sim \)) can be extended to open processes in the usual way: suppose \( X = \text{fpv}(P,P') \), then \( P \approx P' \) if and only if \( P\{A/X\} \approx P'\{A/X\} \) for all closed \( A \).

It should be clear that by the definitions, the following implications are true (see \[10,14\]).

**Lemma 2.** It holds that \( \equiv \subseteq \sim \subseteq \approx \).

### 2.2 The weak context bisimilarity is still weaker than strong bisimilarity in HOCCS\(^-\)

The following lemma states that the weak context bisimilarity is strictly coarser than the strong context bisimilarity.

**Lemma 3.** On (closed) HOCCS\(^-\) process we have \( \sim \subseteq \approx \).

**Proof.** To see that in HOCCS\(^-\), the weak bisimilarity is strictly coarser than the strong bisimilarity, i.e., \( \approx \subseteq \sim \), we examine the ‘replication’, which is reproduced below for convenience.

\[
!P \overset{\text{def}}{=} \overline{\tau}Q_{.,P} \mid Q_{.,P}, \quad Q_{.,P} \overset{\text{def}}{=} c(X).\overline{\tau}X \mid X \mid P
\]

We claim that

1. \( !P \approx !P \mid P \); but
2. \( !P \not\approx !P \mid P \)

For part (1) of the claim, it should be clear that every action by \( !P \) can be matched by \( !P \mid P \) because the former is a component of the latter. Moreover, an action by \( !P \mid P \), say \( !P \mid P \xrightarrow{\lambda} T \) where \( \lambda \) can be \( \tau \) or visible, can be matched by \( !P \xrightarrow{\overline{\tau}} !P \mid P \xrightarrow{\lambda} T \).

For part (2) of the claim, suppose that \( P \) can make an action \( P \xrightarrow{\lambda'} P' \) other than those over \( c \), e.g., \( P \overset{\text{def}}{=} \overline{\tau}0.0 \). Then \( !P \mid P \xrightarrow{\overline{\tau}} !P \mid P' \) cannot be matched by \( !P \) because \( !P \) can only fire immediate visible actions over \( c \).

As demonstrated by Lemma\[3\], that the second inclusion of Lemma\[2\] is strict is essentially attributed to the fact that the replication that can be derived in HOCCS\(^-\). Indeed, the capacity of HOCCS\(^-\) to encode recursive behaviour somewhat compensates the loss in expressiveness incurred by the absence of the restriction operator. We remark that actually the first inclusion of Lemma\[2\] is strict as well. That is, \( \equiv \not\subseteq \sim \), where the *difference* between \( \equiv \) and \( \sim \) results from some distributive law \[4\]; see \[6\] for more details.
3 On the bisimulation equality in CCS$^-$

In this section, we first define CCS$^-$, i.e., CCScore without the restriction operation. Then we discuss the relationship between the strong and weak bisimilarities. In particular, we show that the strong bisimilarity is still strictly finer than the weak bisimilarity. However, unlike the situation for HOCCS$^-$, the distance between the strong bisimilarity and the weak bisimilarity can be shortened. This is evidenced by the so-called quasi-strong bisimilarity, which requests more than the weak bisimilarity and moves closer to the strong bisimilarity, but still turns out to be coincident with the weak bisimilarity.

3.1 Calculus CCS$^-$

The syntax of CCS$^-$ is given as follows. We use capital letters to stand for processes.

$$P, Q := 0 \mid a.P \mid P \mid Q \mid P$$

The operational semantics is also standard and presented below (we skip the symmetric rules).

$$a.P \xrightarrow{a} P$$

$$\overline{\alpha}.P \xrightarrow{\overline{\alpha}} P$$

$$P \xrightarrow{\alpha} P' \quad Q \xrightarrow{\alpha} Q'$$

$$\overline{\alpha}.P \xrightarrow{\overline{\alpha}} P' \quad \overline{\alpha}.Q \xrightarrow{\overline{\alpha}} Q'$$

$$P \mid Q \xrightarrow{\overline{\alpha}} P' \mid Q'$$

$$\overline{\alpha}.P \mid Q \xrightarrow{\overline{\alpha}} P' \mid Q'$$

$$P \gamma \xrightarrow{P} P'$$

$$P \mid Q \gamma \xrightarrow{P} P' \mid Q'$$

$$\overline{\gamma}.P \xrightarrow{\overline{\gamma}} P'$$

There are three kinds of actions (ranged over by $\alpha, \beta, \gamma$): input ($a$), output ($\overline{a}$), and internal ($\tau$). The $\tau$ action is often referred to as silent or invisible, and the others as visible. We write $\overline{\gamma}$ for the complement of $\gamma$, i.e., $\overline{\gamma}$ is $\overline{a}$ if $\gamma$ is $a$ and $a$ if $\gamma$ is $\overline{a}$. Sometimes we will omit the trailing 0 in a process, e.g., $a, \overline{a}$ are shortcuts for $a.0, \overline{a}.0$ respectively. Like HOCCS$^-$, a context $C[\cdot]$ is a process with some subprocess replaced by a hole $[\cdot]$. And $C[Q]$ means substituting the hole in $C$ with process $Q$. We reuse $\equiv$ to stand for the standard structural congruence for CCS$^-$ \cite{8,14}, like in HOCCS$^-$ except that the rule for $\alpha$-convertibility disappears. A name is said to be fresh if it does not appear in the current processes. Other conventions in HOCCS$^-$ are carried over here (e.g., $n(\cdot)$) and we will use them in need without further notice; this shall not cause confusion in contexts. A process is divergent if it can fire an infinite sequence of $\tau$ actions, e.g., $!(\overline{\alpha}).a$ and $\overline{\alpha}!a$. Said otherwise, if a process is not divergent, then it can only engage finitely many internal actions.

Bisimulation

We now define the bisimulations, strong and weak. It should be noted that the bisimulations here explicitly consider divergence property \cite{12,14}. We say that a relation $R$ is divergence-sensitive, if for every $P, Q$, $P$ diverges if and only if $Q$ does. We impose divergence sensitivity in the bisimulation as it makes sense to distinguish between divergent and non-divergent processes, particularly from the standpoint of programming languages. Specifically, divergence sensitivity requires a pair of processes to simultaneously diverge or not unconditionally.
Definition 4. A symmetric binary relation $\mathcal{R}$ on CCS processes is a strong (respectively weak) bisimulation if it is divergence-sensitive and whenever $P \mathcal{R} Q$, it holds that

- if $P \xrightarrow{\alpha} P'$, then $Q \xrightarrow{\alpha} Q'$ (respectively $Q \xrightarrow{\widehat{\alpha}} Q'$) and $P' \mathcal{R} Q'$;

Two processes $P$ and $Q$ are strongly (respectively weakly) bisimilar, notation $P \sim_{\text{ccs}} Q$ (respectively $P \approx_{\text{ccs}} Q$), if there exists some strong (respectively weak) bisimulation $\mathcal{R}$ such that $P \mathcal{R} Q$.

We call $\approx_{\text{ccs}}$ and $\sim_{\text{ccs}}$ the weak bisimilarity and strong bisimilarity respectively. As is well-known, they are equivalence relations and congruences. In the discussion of the bisimilarities, we may use the up-to techniques to build bisimulations, e.g., bisimulation up-to context. These are well-established proof method for process models; see [14] for a comprehensive introduction. For the sake of convenience, we give the definition of the (weak) bisimulation up-to context.

Definition 5. A symmetric binary relation $\mathcal{R}$ on CCS processes is a (weak) bisimulation up-to context if it is divergence-sensitive and whenever $P \mathcal{R} Q$, it holds that

- if $P \xrightarrow{\alpha} P'$, then $Q \xrightarrow{\widehat{\alpha}} Q'$ and there exist some context $C$, $P_1$ and $Q_1$ such that $P' \equiv C[P_1]$, $Q' \equiv C[Q_1]$, and $P_1 \mathcal{R} Q_1$.

As the following lemma states, if a relation is a bisimulation up-to context, then it is subsumed by the weak bisimilarity. The proof of this lemma amounts to showing that the weak bisimilarity is contextual (i.e., preserved by contexts), and divergence sensitivity would not raise obstacle because it does not involve explicit action matching; see [14] for a reference of proof and more discussion.

Lemma 6. If $\mathcal{R}$ is a bisimulation up-to context, then it holds that $\mathcal{R} \subseteq \approx_{\text{ccs}}$.

In terms of the bisimilarity, actions can be divided into two classes: state-changing and state-preserving. We say that an action $\alpha$ as occurring in $P \xrightarrow{\alpha} P'$ is state-changing if $P \not\approx_{\text{ccs}} P'$; otherwise it is state-preserving. We distinguish between these two kinds of actions because they are significant for the incoming arguments.

It should be clear that the following implications are true [8, 14].

Lemma 7. It holds that $\equiv \subseteq \sim_{\text{ccs}} \subseteq \approx_{\text{ccs}}$.

Both of the implications of Lemma 7 are immediate from the definitions. For the first implication of the lemma to be strict, we notice that $!a.0$ and $!a.0 | !a.0$ are strongly bisimilar but not structurally congruent.

Now we define a bisimulation that stands in between the strong and weak bisimilarities. As will be shown, it coincides with the weak bisimilarity and is slightly weaker than the strong bisimilarity (thus so is the weak bisimilarity).

Definition 8. A symmetric binary relation $\mathcal{R}$ on CCS processes is a quasi-strong bisimulation if it is divergence-sensitive, and whenever $P \mathcal{R} Q$, the following properties hold.

- if $P \xrightarrow{\alpha} P'$ and $\alpha$ is not $\tau$, then $Q \xrightarrow{\alpha} Q'$ and $P' \mathcal{R} Q'$;

Two processes $P$ and $Q$ are quasi-strongly bisimilar, notation $P \sim_{\text{ccs}}^q Q$, if there exists some quasi-strong bisimulation $\mathcal{R}$ such that $P \mathcal{R} Q$.

For a quasi-strong bisimulation, since internal actions are bisimulated in a strong manner, divergence sensitivity somehow becomes a derived condition, and we include it in the definition for convenience. By definition, it is straightforward to see that $\sim_{\text{ccs}} \subseteq \sim_{\text{ccs}}^q \subseteq \approx_{\text{ccs}}$. 
3.2 The relationship between the weak and strong bisimilarities in CCS−

We first remark that the approach for Lemma 3 does not apply to the case of CCS−, because the semantics ascertain that !P ∼ccs− !P | P (actually this holds in any CCS-like calculus, unless the replication is defined differently). This observation lends confidence to the coincidence between ≈ccs− and ∼ccs−. But this turns out to be wrong. We will elaborate this in the current section.

In CCS−, the weak bisimilarity is still strictly coarser than the strong bisimilarity. The intrinsic reason is that we still have the replication operator, from which infinite behaviour arises. This renders false the matching of visible actions in a strong manner (the best we can have is as Proposition 22 illustrates).

To see a counterexample, take
\[ P_1 \overset{\text{def}}{=} !c.d | \tau | d \]
\[ P_2 \overset{\text{def}}{=} !c.d | \tau | !c \]

Obviously \( P_1 \not\sim_{ccs} P_2 \). However \( P_1 \approx_{ccs} P_2 \), because the action
\[ P_1 \overset{d}{\rightarrow} !c.d | \tau | 0 \]

can be simulated by
\[ P_2 \overset{\tau}{\rightarrow} !c.d | d | 0 | \tau | !c \overset{d}{\rightarrow} !c.d | 0 | 0 | \tau | !c \]

Henceforth, every \( d \) produced by \( P_1 \) in simulating the subprocess \( !c \) in \( P_2 \) can be simulated in a similar way. So we have the following proper inclusion.

Lemma 9. In CCS−, \( \sim_{ccs} \subseteq \approx_{ccs} \).

In the remainder of this section, we prove the follow-up theorem (Theorem 10), which says that the weak bisimilarity can somehow be approximated by a partially strong bisimilarity, i.e., the quasi-strong bisimilarity. It reveals that in CCS−, there is truly some stronger characterization of the weak bisimilarity, or in other words, the gap between the strong and weak bisimilarities is diminished to some (arguably) noticeable extent. In a broader sense, this exhibits that the weak bisimilarity can indeed be strengthened in a process model without the restriction operator, at least in the first-order paradigm.

Theorem 10. Assume \( P \) and \( Q \) are CCS− processes. Then \( P \approx_{ccs} Q \) implies \( P \sim_{ccs}^q Q \).

This theorem immediately leads to the corollary below.

Corollary 11. In CCS−, it holds that \( \approx_{ccs} = \sim_{ccs}^q \).

To establish the implication claimed by Theorem 10 we exploit the structure of CCS− processes, and go through multiple analyses toward our goal, i.e., the weak bisimilarity is indeed a quasi-strong bisimulation.

The following lemma claims that every \( \tau \) action changes the interactional capability, i.e., the ‘state’ of a process, except for those with infinite actions. We recall that a process \( P \) diverges if it has an infinite sequence of \( \tau \) actions, i.e., \( P \overset{\tau}{\rightarrow} \ldots \overset{\tau}{\rightarrow} \ldots \), and we say that a process \( P \) has an infinite number of visible action \( \alpha \) (\( \alpha \) is not \( \tau \)) if \( P \Rightarrow \overset{\alpha}{\rightarrow} \overset{\alpha}{\rightarrow} \ldots \Rightarrow \overset{\alpha}{\rightarrow} \ldots \), among which a noticeable special case is \( P \overset{\alpha}{\rightarrow} \overset{\alpha}{\rightarrow} \ldots \overset{\alpha}{\rightarrow} \ldots \).

Lemma 12. If \( P \) is not divergent and \( P \overset{\tau}{\rightarrow} P' \), then \( P \not\sim_{ccs} P' \).
Proof. First, we note that the only possibility of having infinite actions is through replication, since the replication operator is the mere way of producing infinity, by a simple induction. We have the following observations that lead to the result of this lemma.

1. If $P$ is not divergent, then for any $a$, $P$ cannot have both an infinite number of $\tau$ actions and an infinite number of $a$ actions (notice that only $\tau$ actions occur between each two neighbouring visible actions). Assume for a contradiction that $P$ has both an infinite number of $\tau$ actions and an infinite number of $a$ actions. Then there are two possibilities.

   (a) These two threads, i.e., infinite numbers of $\tau$ actions and $a$ actions respectively, are in parallel composition. That is, $P \equiv P_1 \parallel P_2$ in which $P_1$ is capable of an infinite number of $\tau$ actions and $P_2$ is capable of an infinite number of $a$ actions;

   (b) The two thread are intertwined, i.e., $P \Rightarrow \gamma_1 \Rightarrow \gamma_2 \Rightarrow \gamma_3 \Rightarrow \ldots$ in which $\gamma_i$ is either $\tau$ or $a$.

   We claim that both case (a) and case (b) would lead to the divergence of $P$, a contradiction. Case (a) is obvious. For case (b), to yield the infinite visible action sequence (separated by $\tau$ actions only), all the actions $\gamma_i$ must be consumable through interactions, i.e., each $\gamma_i$ should go away by taking part in some interaction. This immediately results in divergence.

2. Due to 1, if $P$ is not divergent and can make a $\tau$ that comes from an interaction, say over $a$, between its parallel components, then $P \xrightarrow{\tau} P_1$ and also $P \xrightarrow{a}$ but not both of these two actions incur infinity of the same action. This, in turn, means that $P$ either has a finite (non-zero) number of $\tau$ actions or a finite number of $a$ actions. Suppose it is the latter, and the former is similar. Then we conclude that $P$ cannot be bisimilar to $P'$ because $P'$ is short of one action $a$ for such a name $a$. This suffices to obtain the result because once that finite visible action is fired it never comes back.

We have a straightforward corollary of Lemma 12, since $\sim_{ccs}$ is included in $\approx_{ccs}$.

Corollary 13. If $P$ is not divergent and $P \xrightarrow{\tau} P'$, then $P \not\approx_{ccs} P'$.

Another corollary also follows straight away.

Corollary 14. If $P \xrightarrow{\tau} P'$ and $P \sim_{ccs} P'$ (or $P \approx_{ccs} P'$), then $P$ is divergent.

The following lemma describes that $\tau$ actions resulting from a replication in a process do not change the state of that process.

Lemma 15. Whenever $!P \xrightarrow{=} P'$, it holds that $!P \approx_{ccs} P'$.

Proof. We first look at a particular case, and the general one can be proven in essentially the same way.

1. In particular, we first show the result for the case when the length of the $\tau$ action by $!P$ is 1, i.e., $!P \xrightarrow{\tau} P'$.

   From the semantics, the $\tau$ action by $!P$ must be one of the following cases.

   (a) $!P \xrightarrow{\tau} P_1 \mid !P$ and $P_1 \equiv P_1 \mid !P$ due to $P \xrightarrow{\tau} P_1$.

   We show that the following relation $R$ is a weak bisimulation up-to context (Definition 5; see also [14]), so that $!P \approx_{ccs} P_1'$.

   $$R \equiv \{(P, P_1')\} \cup \approx_{ccs}$$

   Assume $!P R P_1'$. We have two simulation scenarios.
i. Suppose \( P' \xrightarrow{\alpha} P'' \). Then !\( P \) simulates by \( !P \xrightarrow{\tau} P' \xrightarrow{\alpha} P'' \), and \( P'_1 \not\approx P'' \) because \( P'_1 \approx_{ccs} P'' \). In the spirit of “up-to context”, one simply sets the context to be \([\cdot]\).

ii. Conversely, suppose \( !P \xrightarrow{\alpha} P_2 \) !\( P \) \( \equiv P'_2 \) (when \( \alpha = \tau \), we may assume \( P_1 = P_2 \) because otherwise the simulation is trivial). Then \( P'_1 \) simulates by

\[
\begin{align*}
P'_1 & \equiv P_1 | !P \xrightarrow{\alpha} P_2 | !P \\
& \equiv P_2 | P_1 | !P \\
& \equiv P_2 | P'_1
\end{align*}
\]

The simulation now continues by taking advantage of the “up-to context”. By setting \( C \overset{\text{def}}{=} P_2 | [\cdot] \), we have

\[
P'_2 \equiv P_2 | !P \equiv C[!P]
\]

in which \( !P \not\approx P'_1 \). So we are done.

(b) \( !P \xrightarrow{\tau} P_1 | [\cdot] \) !\( P \) \( \equiv P'_2 \) due to \( P \xrightarrow{\tau} P_1 \) and \( P \xrightarrow{\tau} P_2 \).

We show that the following relation \( \mathcal{R}' \) is a weak bisimulation up-to context, which implies that \( !P \approx_{ccs} P'' \).

\[
\mathcal{R}' \overset{\text{def}}{=} \{(!P, P'_1)\} \cup \approx_{ccs}
\]

Assume \( !P \not\approx P'' \) \( \mathcal{R}' \). We have two simulation scenarios.

i. Suppose \( P'_1 \not\approx P'' \). Then \( !P \) simulates by \( !P \xrightarrow{\tau} P'_1 \not\approx P'' \), and \( P'' \not\approx \mathcal{R}' P'' \) follows due to \( P'' \approx_{ccs} P'' \). One can set the context to be \([\cdot] \) to comply with the requirement of “up-to context”.

ii. Conversely, suppose \( !P \overset{\tau}{\not\approx} P_3 \) !\( P \) \( \overset{\text{def}}{=} P'_3 \) (we may assume \( P_3 \) is not \( P_1 \) \( P_2 \) when \( \alpha = \tau \) because otherwise the simulation is trivial). Then \( P'_1 \) simulates by

\[
\begin{align*}
P'_1 & \equiv P_1 | P_2 | !P \overset{\tau}{\not\approx} P_1 | P_2 | P_3 | !P \\
& \equiv P_3 | P_1 | P_2 | !P \\
& \equiv P_3 | P'_1
\end{align*}
\]

Taking advantage of the “up-to context” and setting \( C' \overset{\text{def}}{=} P_3 | [\cdot] \), we have

\[
P'_3 \equiv P_3 | !P \equiv C'[!P]
\]

in which \( !P \not\approx P'' \). So we are done.

2. Now we show the lemma in its general case, i.e., \( !P \leftrightarrow P' \). We remark that by a routine (transition) induction, it can be shown that \( P' \) must be of the form \( P'' | !P \) for some \( P'' \) (up-to \( \equiv \)). We can obtain \( !P \approx_{ccs} P' \) by showing the following relation \( \mathcal{R}_1 \) to be a weak bisimulation up-to context.

\[
\mathcal{R}_1 \overset{\text{def}}{=} \{(!P, P'')\} \cup \approx_{ccs}
\]

To achieve this and complete the proof, we make the arguments in the way exactly as case 1 above in showing \( \mathcal{R} \) or \( \mathcal{R}' \) to be a weak bisimulation up-to context. The only difference is that there may be more than one \( \tau \) in the weak transitions by \( !P \), but the form of the derived process \( P' \) does not change at all and the arguments stay put.
Remark It is noteworthy that Lemma 15 is also true in the standard CCS (with restriction), as the proof does not rely on the absence of the restriction operation. However, if \( P \) has some visible actions, then its state is not preserved any longer. That is, for \( P' \) such that \( P \overset{\tau}{\Rightarrow} a_1 \overset{\tau}{\Rightarrow} a_2 \overset{\tau}{\Rightarrow} \cdots \overset{\tau}{\Rightarrow} a_i \overset{\tau}{\Rightarrow} P' \) in which \( a_i \) (\( i = 1, \ldots, k \)) is visible (i.e., not \( \tau \)), it does not necessarily hold that \( P \overset{\text{ccs}}{=} P' \). The reason is that the action by \( \! P \) may reveal some action \( \! P \) cannot match without first doing another visible action. It is not hard to contrive a counterexample, say \( P' \) can do some action that \( \! P \) cannot do. For instance, let \( P \overset{\text{def}}{=} ! a.c \). Then \( P \overset{\alpha_i}{\Rightarrow} c \mid \! P \overset{\tau}{=} P' \), and \( P' \overset{\tau}{\Rightarrow} \) whereas \( \! P \) cannot.

We continue to examine the state-preserving/state-changing \( \tau \) actions in the coming up lemmas. Again by state-changing, we mean that if \( P \overset{\alpha}{\Rightarrow} P' \), then \( P \not\overset{\text{ccs}}{=} P' \) (more often than not, \( \alpha \) here is \( \tau \), though it appears to still make sense if it is visible); otherwise it is state-preserving.

Lemma 16. Assume \( P \) is a CCS\(^-\) process. Then there exists \( k \geq 0 \) and \( P' \) such that \( P \overset{\tau}{\Rightarrow} P' \) and \( P' \overset{\text{ccs}}{=} P'' \) for any \( P'' \) such that \( P'' \overset{\tau}{=} P'' \).

Proof. Before going ahead, we provide some observation. If \( P \) is not divergent, then the result obviously hold, because we can consume all the finite \( \tau \) actions \( P \) can fire to reach a state that can make no more \( \tau \). That state satisfies the claim of the lemma (in a void way). Now assume \( P \) is divergent. Intuitively, the number \( k \) is referring to those \( \tau \)'s that are not introduced by the replication. Consuming these \( \tau \) actions leads \( P \) to a state ready for starting the divergent path. By “the divergent path”, we mean that the process can make the same \( \tau \) over and again, without changing the states in absence of the restriction operation.

Now we make the formal proof by induction on the structure of \( P \).

- The cases when \( P = 0 \), \( a.P \), or \( \overline{a}.P \) are trivial.
- \( P \) is \( P_1 \mid P_2 \). There are several subcases.
  1. \( P_1 \) is divergent while \( P_2 \) is not. (For example, \( P_1 \overset{\text{def}}{=} ! a \mid \overline{a} \overline{a} \), \( P_2 \overset{\text{def}}{=} c \). In here and what follows, we may use examples simply for more of an illustration, and they are not meant to be part of the proof anyhow.)
     In this case, the claim of the lemma holds for \( P_1 \) by induction hypothesis. That is, there exists \( k_1 \geq 0 \) and \( P_1' \) such that \( P_1 \overset{\tau}{\Rightarrow} P_1' \), and \( P_1' \overset{\text{ccs}}{=} P_1'' \) for any \( P_1'' \) such that \( P_1'' \overset{\tau}{=} P_1'' \). Now since \( P_2 \) is not divergent, we can expire all the possible \( \tau \) actions concerning \( P_2 \), including those by \( P_2 \) alone or from finite interactions between \( P_2 \) and \( P_1 \). By “finite interactions” we mean that the interactions are composed by (i.e., result from) visible actions that cannot be repeated for infinitely many times. In addition, notice that by means of interaction, we only eliminate (i.e., consume) those finite \( \tau \) actions from inside \( P_2 \) or between \( P_1 \) and \( P_2 \). After this \( P_2 \) may still have infinite visible actions and have infinite \( \tau \)'s with \( P_1 \), but this does not matter because they do not change the state of \( P \), as there is an infinite repository of these actions (so the capacity of \( P \) remains unchanged). Suppose this expiration operation constitutes \( k_1' \) (\( k_1' \geq 0 \)) \( \tau \) actions. Then we know that \( P \overset{\tau}{\Rightarrow} P_1 + k_1' P_2' \), and \( P'' \overset{\text{ccs}}{=} P_1'' \) for any \( P_1'' \) such that \( P_1'' \overset{\tau}{=} P_1'' \).
  2. \( P_2 \) is divergent while \( P_1 \) is not. Similar to the previous case.
  3. Both \( P_1 \) and \( P_2 \) are divergent. (For example, \( P_1 \overset{\text{def}}{=} ! a \mid \overline{a} \overline{a} \), \( P_2 \overset{\text{def}}{=} ! b \mid \overline{b} \).)
     By induction hypothesis, we know that there exists \( k_i \geq 0 \) (\( i = 1, 2 \)) and \( P_i' \) such that \( P_i \overset{\tau}{\Rightarrow} P_i' \), and \( P_i' \overset{\text{ccs}}{=} P_i'' \) for any \( P_i'' \) such that \( P_i'' \overset{\tau}{=} P_i'' \). Similar to case 1, we can remove those finite,
say $k_3 \geq 0$, $\tau$ actions between $P_1$ and $P_2$. Then we know that $P \xrightarrow{\tau} P' \equiv_{ccs} P''$, and $P' \xrightarrow{\tau} P''$ for any $P''$ such that $P' \equiv_{ccs} P''$.

4. Neither of $P_1$ and $P_2$ is divergent, but $P$ is divergent. We break into two subcases.

(a) We first separate a subcase that $P_1$ is indirectly divergent (the case for $P_2$ is similar).

(For example, $P_1 \equiv \tau.(!a \mid \tau P)$, $P_2 \equiv c$)

That is, $P_1$ becomes divergent after having some finite interaction with $P_2$. Suppose after some finite interaction with $P_2$, $P_1$ and $P_2$ become $P'_1$ and $P'_2$ respectively, where $P'_1$ is divergent. Then $P \equiv_{ccs} P'_1 \mid P'_2$, and the result holds by proceeding as in case 1.

(b) Excluding the indirectly divergent cases as in (a), we may suppose $P \equiv P_1 \mid P_2 \xrightarrow{\tau} k P'_1 \mid P'_2$ in which $k \geq 0$ and neither $P'_1$ nor $P'_2$ can do $\tau$ action.

(For example, $P_1 \equiv !a \mid \tau$, $P_2 \equiv c \mid \tau P$)

That is, we can deplete all the finite interactions in between $P_1$ and $P_2$ as in case 1. Now since $P$ is divergent, we have two observations: first, there must be some interaction between $P'_1$ and $P'_2$; second, the visible actions comprising the interaction, and thus the interaction itself, can recur forever. Thus suppose, in particular, that $P'_1 \equiv_{ccs} P'_2$ and these actions run infinitely. Then $P'_1 \mid P'_2$ is exactly the $P'$ as needed, because it holds that $P'_1 \mid P'_2 \xrightarrow{\tau} k P''_1 \mid P''_2 \equiv_{ccs} P'_1 \mid P'_2$ for any $k' \geq 0$, due to the infinite repository of the same actions.

- $P$ is $P_1$. Suppose $P_1$ can do some $\tau$ (the result trivially holds otherwise). Then taking $k$ as zero would be sufficient, by Lemma[15] That is, all the incoming $\tau$ actions do not change the state of $P_1$.

Now the proof is completed. □

By virtue of Lemma[16] the number of such $\tau$ actions as state-changing is finite in a sense. That is to say, a CCS$^-$ process can only make finite state-changing $\tau$ actions in a sequence, without any intertwining visible actions.

**Corollary 17.** Suppose $P$ can make a (possibly infinite) sequence of $\tau$ actions, then only a finite initial segment of it is state-changing. This $\tau$ sequence can be written as $P \xrightarrow{\tau} P_1 \xrightarrow{\tau} P_2 \cdots \xrightarrow{\tau} P_k \xrightarrow{\tau} \cdots$, where $P$ and $P_1$ through $P_k$ are state-changing and those afterwards (if any) are state-preserving.

**Proof.** This is a consequence of what we have proven in Lemma[16] One simply chooses the minimal $k$ as designated by that lemma. □

**Remark.** In general, it is not true that a process can only have finite state-changing $\tau$ actions (not necessarily from the start and with-in a sequence). For example, the process $!a.(b \mid \tau P.c)$ can have infinite state-changing $\tau$ actions. However, the $\tau$ actions in a row without any in-between visible actions can only have finite state-changing (consecutive) $\tau$ actions as stated in the foregoing corollary.

Furthermore, a $\tau$ action that changes the state of a process, which may not come from a replication, should be simulated also by a state-changing $\tau$. We capture these ideas in the following two lemmas.

**Lemma 18.** Assume $P \equiv_{ccs} Q$. Then in the $\tau$ sequences that $P$ and $Q$ can make respectively (as described in Corollary[17]), the numbers of state-changing $\tau$ actions are equal.
Proof. By Corollary 17 we can assume the $\tau$ sequences by $P$ and $Q$ are respectively

$$P \xrightarrow{\tau} P_1 \xrightarrow{\tau} P_2 \cdots \xrightarrow{\tau} P_k \xrightarrow{\tau} \cdots$$

$$Q \xrightarrow{\tau} Q_1 \xrightarrow{\tau} Q_2 \cdots \xrightarrow{\tau} Q_k \xrightarrow{\tau} \cdots$$

where $P$ (respectively $Q$) and $P_1$ (respectively $Q_1$) through $P_k$ (respectively $Q_k$) are state-changing and those afterwards (if any) are state-preserving. We need to show $k = k'$. Assume to the contrary that $k \neq k'$, say $k < k'$ (the case $k > k'$ is similar).

We then show that this would break the bisimulation between $P$ and $Q$. We observe that every $\tau$ action is composed of two complementary visible actions (since there is no restriction operation, every interaction is based on a visible name), so one can make these two visible actions in a sequential manner to reach the same state. Say $P \xrightarrow{\tau} P'$ through interaction over $\alpha$, then from the operational semantics we must have $P \xrightarrow{\alpha} \pi P'$, which we call a pair-action. Moreover, because the $\tau$ considered here is state-changing, these complementary visible actions must not be repeated forever (otherwise they would lead to repeated $\tau$ actions, making it state-preserving). So $P$ can reach $P_k$ in finite such pair-actions. Since $P \approx_{ccs} Q$, it must be the case that $Q$ consumes exactly the same number of pair actions (of visible actions), and this corresponds to $k$ state-changing $\tau$ actions. That is, $Q$ must evolve to $Q_k$ in order to keep pace with $P$ so that the bisimulation can be maintained. Yet now as assumed, $Q$ can make more state-changing $\tau$ actions, i.e., more pair-actions correspondingly. These (non-repeating) pair actions cannot be bisimulated by $P$ since $P$ at that stage only has state-preserving $\tau$ actions (and consequently repeating pair actions). So we have a contradiction. □

Lemma 18 helps to develop the following intuition: in order to simulate a state-changing $\tau$, one must do at least one $\tau$ so as to change its state; moreover, it cannot do more than one state-changing $\tau$ because it has only finite such $\tau$ actions in the (bisimulating) sequence of internal actions, even though it can do a couple of state-preserving $\tau$ actions; making such a state-changing $\tau$ is sufficient for the bisimulation.

Hence we have the follow-up lemma on how to match internal actions that are state-changing.

**Lemma 19.** Suppose $P \approx_{ccs} Q$. If $P \xrightarrow{\tau} P'$ in which the $\tau$ is state-changing, then $Q \xrightarrow{\tau} Q' \approx_{ccs} P'$ in which the $\tau$ is state-changing as well.

Proof. Suppose $P \xrightarrow{\tau} P'$ in which the $\tau$ is state-changing, since $P \approx_{ccs} Q$, we know that $Q \approx_{ccs} Q_1$ for some $Q_1$ and $P' \approx_{ccs} Q_1$. Because the $\tau$ by $P$ is state-changing, it must originate from two complementary visible actions, say $\alpha$ and $\beta$, that cannot be simulated by $P'$ anyhow. Thus $Q$ must not do nothing, so we can rewrite the simulation by $Q$ as $Q \xrightarrow{\tau} Q_2 \approx_{ccs} Q_1 \approx_{ccs} P'$. We claim that in this simulation, the $\tau$ in $Q \xrightarrow{\tau} Q_2$ is state-changing, and the $\tau$ actions in $Q_2 \approx_{ccs} Q_1$ are state-preserving. Then the result of the lemma follows.

To see why $Q$ is forced to simulate with exactly one state-changing $\tau$, we note that $Q$ can only make finite state-changing $\tau$ action sequence (by Corollary 17) whose length is the same as that of $P'$s (by Lemma 18). If the simulation consumes any more or fewer state-changing $\tau$ actions in the sequence, $Q$ will eventually lose the pace with $P$ and fail to bisimulate it. Thus $P$ and $Q$ are obliged to engage a step-wise state-changing bisimulation, as they only hold a finite number of such actions.

Therefore, it must be the case that the $\tau$ in $Q \xrightarrow{\tau} Q_2$ is state-changing, and the $\tau$ actions from $Q_2$ to $Q_1$ are state-preserving, i.e., $Q_2 \approx_{ccs} Q_1 \approx_{ccs} P'$. To recap, now we have $Q \xrightarrow{\tau} Q_2 \approx_{ccs} P'$, and $Q_2$ is the $Q'$ we seek in the statement of the lemma. □

A corollary out of Lemma 18 and Lemma 19 is that a state-preserving $\tau$ must be simulated by state-preserving one(s), since it cannot be bisimulated by a state-changing $\tau$ action.
Corollary 20. Suppose $P \approx_{ccs} Q$. If $P \overset{\tau}{\rightarrow} P'$ in which the $\tau$ is state-preserving, then $Q \overset{\tau}{\rightarrow} Q' \approx_{ccs} P'$ in which the $\tau$ is state-preserving as well.

The upcoming lemma somehow generalizes the results in the foregoing lemmas.

Lemma 21. Assume $P$ and $Q$ are $CCS$ processes, and $k$ is as described in Lemma 16 for $P$. If $P \approx_{ccs} Q$, then

1. $Q \overset{k}{\rightarrow} Q'$, and $Q' \approx_{ccs} Q''$ for any $Q''$ such that $Q' \Rightarrow Q''$;
2. there is a minimal $k$ satisfying both (1) for $Q$ and the property as described in Lemma 16 for $P$; i.e., $P$ and $Q$ make the same number of state-changing $\tau$ action sequence before (synchronously) entering (state-preserving) divergence (if any).

Proof. It should be clear that (2) follows from (1) and Lemma 16.

For (1), one has to make sure that $Q$ has the same behavioural pattern. To see this, we remember that the bisimulation here requires synergistic divergence, so $Q$ is divergent too whenever $P$ is. We also notice that $Q$ has the property due to Lemma 16, i.e., there exists $k' \geq 0$ and $Q'$ such that $Q \overset{\tau}{\rightarrow} Q'$, and $Q' \approx_{ccs} Q''$ for any $Q''$ such that $Q' \Rightarrow Q''$.

For the sake of convenience and w.l.o.g., we assume that $k$ and $k'$ are both the minimal such integers respectively (the result for ‘not minimal’ follows from this case). We need to show that $k = k'$. Intuitively, if $k < k'$ or $k > k'$, then we can derive that $P$ and $Q$ are not bisimilar. Assume to the contrary that $k < k'$ (the case $k > k'$ is similar). This means somehow $Q$ becomes state-preserving, say divergent, later than $P$. In light of Corollary 13 starting from $P \approx_{ccs} Q$ and before entering the state-preserving stage (diverging if any), both $P$ and $Q$ change their states stepwise in each ‘bisimulating’ action. Specifically, if $k < k'$, consider the following two sequences.

$$P \equiv P_0 \overset{\tau}{\rightarrow} P_1 \overset{\tau}{\rightarrow} P_2 \cdots \overset{\tau}{\rightarrow} P_k \overset{\tau}{\rightarrow} P_{k+1}$$

$$Q \equiv Q_0 \overset{\tau}{\rightarrow} Q_1 \overset{\tau}{\rightarrow} Q_2 \cdots \overset{\tau}{\rightarrow} Q_k \overset{\tau}{\rightarrow} Q_{k+1} \overset{\tau}{\rightarrow} \cdots \overset{\tau}{\rightarrow} Q_{k'}$$

We have

1. $P_i \not\approx_{ccs} P_{i+1}$ ($i = 0, \ldots, k - 1$). Before jumping into state-preserving stage (i.e., divergence), every $\tau$ changes the state of the process initiating from $P$ (recall that we choose the minimal $k$)
2. $P_k \approx_{ccs} P_{k+1}$. Once entering the phase of state-preserving, no $\tau$ actions changes the state.
3. $Q_i \not\approx_{ccs} Q_{i+1}$ ($i = 0, \ldots, k' - 1$). For a reason similar to (1).
4. $P_i \approx_{ccs} Q_i$ ($i = 0, \ldots, k$). This, can be ensured by Lemma 19 and Lemma 18 since the $\tau$ actions (until $P_k$ and $Q_k$) change the states.

Now these observations lead to a contradiction, i.e., we have $P_k \approx_{ccs} P_{k+1} \approx_{ccs} Q_{k+1}$ since $P_k$ must simulate $Q_k$’s $\tau$, but then $Q_k \approx_{ccs} Q_{k+1}$, which is contradictory. This scenario is depicted as in Fig. 1 in which the place where the contradiction emerges is highlighted with the bold font. That is, the transitive path $Q_k \approx_{ccs} P_k \approx_{ccs} P_{k+1} \approx_{ccs} Q_{k+1}$ yields a contradiction with the edge connecting $Q_k$ and $Q_{k+1}$, i.e., $Q_k \not\approx_{ccs} Q_{k+1}$.

Next comes a crucial property for analyzing the matching of $\tau$ actions in the sense of the weak bisimulation equality.
Proof. By the premise, \( Q \approx_{ccs} P \). Consequently, we have

\[
P \approx_{ccs} Q.
\]

In this simulation, by Lemmas 18, 19 and Corollary 20, the \( \tau \) bisimilarities (see Lemma 9).

*)Figure 1: Figure for the proof of Lemma 21.*

**Proposition 22.** Assume \( P \approx_{ccs} Q \) and \( P \xrightarrow{\tau} P' \), then \( Q \xrightarrow{\tau} \approx_{ccs} P' \).

**Proof.** By the assumption, we know that \( P \xrightarrow{\tau} P' \) implies \( Q \Rightarrow \approx_{ccs} P' \). We need to more precisely analyze the matching weak transition by \( Q \) (in absence of the restriction operation), with the help of the lemmas thus far and the accompanying corollaries.

The analysis separates the cases whether \( P \) is divergent or not. We first tackle the case when \( P \) is not divergent. By Lemma 12, we know that \( P \not\approx_{ccs} P' \), i.e., that \( \tau \) is state-changing. Then By Lemma 19, we have \( Q \xrightarrow{\tau} Q' \approx_{ccs} P' \) in which \( \tau \) is state-changing.

Now consider the case when \( P \) is divergent. Since \( P \approx_{ccs} Q \), we know that \( Q \) diverges too. Assume \( k \) is as decided by Lemma 21. That is, \( P \) and \( Q \) make the same number of state-changing \( \tau \) action sequence before (simultaneously) entering divergence (in which \( \tau \) is state-preserving). There are two subcases.

1. If \( k \) equals 0, then the \( \tau \) in \( P \xrightarrow{\tau} P' \) must be state-preserving. By Corollary 20, \( Q \xrightarrow{\tau} Q' \approx_{ccs} P' \) in which \( \tau \) is state-preserving.

2. If \( k \) is not 0, then the \( \tau \) in \( P \xrightarrow{\tau} P' \) can be state-changing or state-preserving (i.e., from the divergence after \( k \)). In the former, the result follows as above in the case \( P \) is not divergent. In the latter, we conclude as in (1).

Having done analyzing the \( \tau \) action in a bisimulation, we finally deal with the situation for visible actions.

**Proposition 23.** Assume \( P \approx_{ccs} Q \) and \( P \xrightarrow{\alpha} P' \) in which \( \alpha \) is not \( \tau \), then \( Q \Rightarrow Q_1 \xrightarrow{\alpha} Q_2 \approx_{ccs} P' \) for some \( Q_1, Q_2 \), in which \( Q \approx_{ccs} Q_1 \) and the \( \tau \) actions in \( Q \Rightarrow Q_1 \) are state-preserving.

**Proof.** By the premise, \( Q \) must simulate by \( Q \Rightarrow Q_1 \xrightarrow{\alpha} Q_2 \Rightarrow Q' \approx_{ccs} P' \) for some \( Q_1, Q_2 \), and \( Q' \). In this simulation, by Lemmas 18, 19 and Corollary 20, the \( \tau \) sequence between \( Q \) and \( Q_1 \), and the \( \tau \) sequence between \( Q_2 \) and \( Q' \), contain no state-changing \( \tau \) actions. If assumed otherwise, \( Q' \) would be unable to match \( P' \) due to being short of enough state-changing \( \tau \) actions as compared with those of \( P' \). So the internal action sequences in the bisimulation must be state-preserving. Consequently, we have \( Q \Rightarrow Q_2 \approx_{ccs} P' \), as needed.

In general, the result stated in Proposition 23 turns out to be the best we can do to strengthen the simulation of a visible action, as opposed to the counterexample that distinguishes the strong and weak bisimilarities (see Lemma 9).
Nonetheless, Proposition 23 can be refined further if one simply focuses on non-divergent processes. In that case, we end up with the strong bisimilarity, as Corollary 24 reveals. With this corollary in position, it makes sense to speculate that, if the replication operator were to be eliminated from $\text{CCS}^-$ (though this makes the calculus less interesting), then the weak bisimilarity would flat onto the strong bisimilarity.

**Corollary 24.** Assume $P$ is not divergent. If $P \approx_{ccs^-} Q$ and $P \xrightarrow{\alpha} P'$ where $\alpha$ is not $\tau$, then $Q \xrightarrow{\alpha} \approx_{ccs^-} P'$.

**Proof.** Since $P$ is not divergent, neither is $Q$. By Proposition 23, we know that $Q \Rightarrow Q_1 \xrightarrow{\alpha} Q_2 \approx_{ccs^-} P'$, where $Q \Rightarrow Q_1$ has only state-preserving $\tau$ actions. However, $Q$ is not divergent. This means that the $\tau$ actions in $Q \Rightarrow Q_1$ can only be state-changing. This is a contradiction, which leads to the only possibility that $Q \Rightarrow Q_1$ contains zero $\tau$ actions, i.e., $Q_1$ is $Q$. Thus we are done.

Now, Proposition 22 and Proposition 23 amount to Theorem 10, the main result.

**Proof of Theorem 10.** We define

$$\mathcal{R} \overset{\text{def}}{=} \{(P, Q) \mid P \approx_{ccs^-} Q\}.$$ 

We show that $\mathcal{R}$ is a quasi-strong bisimulation. Assume $P \mathcal{R} Q$. We have two cases.

- $P \xrightarrow{\alpha} P'$ in which $\alpha$ is not $\tau$. By Proposition 23, $Q \Rightarrow Q_1 \xrightarrow{\alpha} Q_2 \approx_{ccs^-} P'$, so we have $P' \mathcal{R} Q'$.
- $P \xrightarrow{\tau} P'$. By Proposition 22, $Q \xrightarrow{\tau} Q' \approx_{ccs^-} P'$. So we have $P' \mathcal{R} Q'$.

**3.3 Further results and discussion**

We argue that Corollary 11 also holds for the branching bisimilarity (in place of the weak bisimilarity). This, in turn, would lead to the coincidence between the weak bisimilarity and the branching bisimilarity. It is an intriguing and practicable work that we now solidify.

First of all, it is helpful to exploit further Theorem 10, particularly the relationship with the branching bisimilarity [2, 3], a well-known equivalence relation on processes that preserves the branching structure of processes. The definition of branching bisimulation is as follows.

**Definition 25.** A symmetric binary relation $\mathcal{R}$ on $\text{CCS}^-$ processes is a branching bisimulation if it is divergence-sensitive, and whenever $P \mathcal{R} Q$, the following properties hold.

- if $P \xrightarrow{\alpha} P'$, then either
  - $\alpha$ is $\tau$ and $P' \mathcal{R} Q$; Or
  - $Q \Rightarrow Q'' \xrightarrow{\alpha} Q'$, $P \mathcal{R} Q''$ and $P' \mathcal{R} Q'$.

Two processes $P$ and $Q$ are branching bisimilar, notation $P \approx_{ccs^-}^{br} Q$, if there exists some branching bisimulation $\mathcal{R}$ such that $P \mathcal{R} Q$.

We call $\approx_{ccs^-}^{br}$ the branching bisimilarity. To be consistent with the current setting, we impose divergence-sensitiveness on the branching bisimulation. It is not hard to see that the branching bisimilarity implies the weak bisimilarity, i.e., $\approx_{ccs^-}^{br} \subseteq \approx_{ccs^-}$.

Here comes an important observation of the proof of Theorem 10. It uses Proposition 23 which states that the internal actions in $Q \Rightarrow \xrightarrow{\alpha} Q' \approx_{ccs^-} P'$ in the first clause of the proof of Theorem 10 are actually state-preserving. Following this observation, we can strengthen the definition of quasi-strong bisimulation without changing any distinguishing power.
Definition 26. A symmetric binary relation $R$ on CCS$^-$ processes is a quasi-strong branching bisimulation if it is divergence-sensitive, and whenever $P \mathrel{R} Q$, the following properties hold.

- if $P \xrightarrow{\alpha} P'$ and $\alpha$ is not $\tau$, then $Q \mathrel{=} Q'' \xrightarrow{\alpha} Q'$, $P \mathrel{R} Q''$ and $P' \mathrel{R} Q'$.
- if $P \xrightarrow{\tau} P'$, then $Q \xrightarrow{\tau} Q'$ and $P \mathrel{R} Q'$.

Two processes $P$ and $Q$ are quasi-strongly branching bisimilar, notation $P \sim_{qc}^{bs} Q$, if there exists some quasi-strong branching bisimulation $R$ such that $P \mathrel{R} Q$.

Through the same proof routine as that of Theorem 10, we can infer that $\sim_{qc}^{bs}$ also coincides with the weak bisimilarity.

Lemma 27. In CCS$^-$, it holds that $\approx_{ccs} = \sim_{qc}^{bs}$.

Now examining the difference between the quasi-strongly branching bisimulation and the branching bisimulation, it is straightforward to see that both of the clauses of the quasi-strongly branching bisimulation implies that of the branching bisimulation. Hence the following lemma.

Lemma 28. In CCS$^-$, it holds that $\sim_{qc}^{bs} \subseteq \approx_{br}^{bs}$.

Proof. To show that $\sim_{qc}^{bs}$ is a branching bisimulation, we focus on the $\tau$-clause, since it is the only distinct part. In particular, the second clause of the quasi-strongly branching bisimulation implies that of the branching bisimulation because we can rewrite $Q \xrightarrow{\tau} Q'$ as $Q \mathrel{=} Q'' \xrightarrow{\tau} Q'$.

The lemmas above lead to the follow-up corollary.

Corollary 29. In CCS$^-$, it holds that $\approx_{ccs} = \sim_{br}^{bs}$.

Proof. By Lemma 28 we have $\sim_{qc}^{bs} \subseteq \approx_{br}^{bs} \subseteq \approx_{ccs}$. Then the equality follows by Lemma 27.

To conclude, all the discussion so far boils down to the next theorem.

Theorem 30. In CCS$^-$, it holds that $\approx_{ccs} = \approx_{qc}^{bs} = \sim_{qc}^{bs} = \sim_{br}^{bs} = \approx_{br}^{bs}$.

We make some more remarks before ending this section. As mentioned, in spirit of Corollary and Proposition if CCS$^-$ is further deprived of the replication, we believe that the weak bisimilarity would fall onto the strong bisimilarity. This is virtually not hard to verify, by means of going through all the analysis above but ignoring those parts concerning the replication. We do not extend the discussion into details here, as a calculus with neither restriction nor replication appears not very interesting. Nevertheless, it still sheds light on the essential gap between the weak and strong bisimulation equalities. It might be intriguing to see if one can find a subcalculus of CCS in which the weak and strong similarities coincide, e.g., with a very special form of replication. Also to this point, the analysis and result on quasi-strong bisimilarity may offer some potential tool for analyzing finite-state processes in a broader field.

Conceivably, we can coin the quasi-strong bisimilarity in the higher-order model, in roughly the same vein as that of CCS$^-$. It is worthwhile to investigate whether the same coincidence exists between the quasi-strong bisimilarity and the weak bisimilarity. Due to the difference in communication machinery, the technical approach might be strikingly different.
4 Conclusion

This paper has been focusing on the relationship between the strong and the weak bisimilarities, in process models from which the restriction operator is removed. We have presented a few observations about such relationship in both a first-order model (CCS−) and a higher-order one (HOCCS−). Basically, it is shown invariant in both models that the weak bisimilarity remains strictly weaker than the strong bisimilarity, even without the capacity of hiding information. Essentially, this is a consequence of the replication operation, though the situation is a bit different in CCS− and HOCCS−, because the replication is primitive in the former but derivable in the latter. Anyhow, what we have shown illustrates that the replication operation somehow can also ‘hide’ information, but in a sharply different way, i.e., it offers plenty of the same processes (and actions). Though slightly beyond expectation, we can still succeed in reducing the distance between the strong and weak bisimilarities in CCS−. That is, we show that in CCS−, the strong bisimilarity can be approached by the so-called quasi-strong bisimilarity. Formally, this bisimilarity intensifies the weak bisimilarity in two respects: requiring strong bisimulation for silent actions and relinquishing trailing silent actions in matching a visible action. As it appears, the quasi-strong bisimilarity tightens up the weak bisimilarity, toward the strong bisimilarity. To this end, a key result is that the quasi-strong bisimilarity coincides with the weak bisimilarity. This coincidence conveys that the weak bisimilarity can be reinforced by demanding more than its original requirement about simulation, and thus becomes quasi-strong, while maintaining its original discriminating power. Moreover, it reveals that the absence of the restriction operator does not cause zero effect, and we can hopefully take advantage of this to make the weak bisimilarity more tractable. We have also discussed the relationship between the quasi-strong bisimilarity and the branching bisimilarity. As a significant spinoff, we show that the branching bisimilarity is coincident with the weak bisimilarity.

There are some questions worthy of further investigation. A first one is to prove or disprove the conjecture, aforementioned in Section 2 that in HOCCS− the strong context bisimilarity collapses onto the structural congruence. A second one is to seek in HOCCS− a counterpart of the quasi-strong bisimilarity like Definition 8 for CCS− (potentially with novel idea), and attempt to prove its agreement with the weak context bisimilarity. Intuitively, we believe that the weak bisimilarity can be tightened toward the strong bisimilarity in HOCCS−. This quest would become even more interesting, especially when taking into account the outcome of the conjecture on the coincidence between the strong context bisimilarity and the structural congruence. One more direction is to exploit more (process) models without the restriction operator, e.g., value-passing models or (higher-order) ambient models, for properties that stem from the absence of the operator.

References

[1] Y. Fu (2015): Theory of interaction. Theoretical Computer Science 611, pp. 1–49, doi [10.1016/j.tcs.2015.07.043].

[2] R.J. van Glabbeek & W.P. Weijland (1989): Branching time and abstraction in bisimulation semantics (extended abstract). In: Proceedings of the 11th IFIP World Computer Congress, Information Processing ’89, pp. 613–618, doi [10.1.1.85.625].

[3] Rob J. van Glabbeek & W. P. Weijland (1996): Branching time and abstraction in bisimulation semantics. Journal of the ACM 43(3), pp. 555–600, doi [10.1145/233551.233556].

[4] D. Hirschkoff & D. Pous (2008): A Distribution Law for CCS and a New Congruence Result for the pi-calculus. Logical Methods in Computer Science LMCS-4(2:4).
[5] D. Hirschkoff & D. Pous (2010): On Bisimilarity and Substitution in Presence of Replication. In: Proceedings of the 37th International Colloquium on Automata, Languages, and Programming (ICALP 2010), LNCS 6199, pp. 454–465, doi:10.1007/978-3-642-14162-1_38

[6] I. Lanese, J. A. Pérez, D. Sangiorgi & A. Schmitt (2011): On the Expressiveness and Decidability of Higher-Order Process Calculi. Information and Computation 209(2), pp. 198–226, doi:10.1016/j.ic.2010.10.001

[7] I. Lanese, J.A. Pérez, D. Sangiorgi & A. Schmitt (2008): On the Expressiveness and Decidability of Higher-Order Process Calculi. In: Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science (LICS 2008), IEEE Computer Society, pp. 145–155, doi:10.1109/LICS.2008.8. Journal version in [6].

[8] R. Milner (1989): Communication and Concurrency. Prentice Hall.

[9] R. Milner, J. Parrow & D. Walker (1992): A Calculus of Mobile Processes (Parts I and II). Information and Computation 100(1), pp. 1–77, doi:10.1016/0890-5401(92)90008-4, 10.1016/0890-5401(92)90009-5.

[10] D. Sangiorgi (1992): Expressing Mobility in Process Algebras: First-order and Higher-order Paradigms. Phd thesis, University of Edinburgh.

[11] D. Sangiorgi (1996): Bisimulation for Higher-order Process Calculi. Information and Computation 131(2), pp. 141–178, doi:10.1006/inco.1996.0096.

[12] D. Sangiorgi (2011): Introduction to Bisimulation and Coinduction. Cambridge University Press.

[13] D. Sangiorgi (2012): Concurrency theory: timed automata, testing, program synthesis. Distributed Computing 25(1), pp. 3–4, doi:10.1007/s00446-011-0156-2.

[14] D. Sangiorgi & D. Walker (2001): The Pi-calculus: a Theory of Mobile Processes. Cambridge University Press.

[15] B. Thomsen (1993): Plain CHOCS, a Second Generation Calculus for Higher-Order Processes. Acta Informatica 30(1), pp. 1–59, doi:10.1007/BF01200262.