Kazakov–Migdal Model with Logarithmic Potential and the Double Penner Matrix Model

Lori Paniak and Nathan Weiss

Department of Physics, University of British Columbia,
Vancouver, British Columbia, Canada V6T 1Z1

Abstract

The Kazakov–Migdal (KM) Model is a U(N) Lattice Gauge Theory with a Scalar Field in the adjoint representation but with no kinetic term for the Gauge Field. This model is formally soluble in the limit $N \to \infty$ though explicit solutions are available for a very limited number of scalar potentials. A “Double Penner” Model in which the potential has two logarithmic singularities provides an example of a explicitly soluble model. We begin by reviewing the formal solution to this Double Penner KM Model. We pay special attention to the relationship of this model to an ordinary (one) matrix model whose potential has two logarithmic singularities (the Double Penner Model). We present a detailed analysis of the large N behavior of this Double Penner Model. We describe the various one cut and two cut solutions and we discuss cases in which “eigenvalue condensation” occurs at the singular points of the potential. We then describe the consequences of our study for the KM Model described above. We present the phase diagram of the model and describe its critical regions.
I. INTRODUCTION

Several years ago Kazakov and Migdal [1] proposed a model which they hoped would provide a description of Quantum Chromodynamics (QCD) in the limit in which the number \( N \) of colors is large. Their model is a Lattice Field Theory defined on a \( D \) dimensional hypercubic lattice with sites labeled as \( x \) and links labeled by pairs of nearest neighbor sites \( \langle x, y \rangle \). The fields in the model are a scalar field \( \phi(x) \) which, for every site \( x \), is an \( N \times N \) Hermetian matrix and a Gauge field \( U(x, y) \) which, for every link \( \langle x, y \rangle \) of the lattice, is an \( N \times N \) Unitary matrix. The Action for a given configuration \( \phi(x) \) and \( U(x, y) \) in their model is given by

\[
S_{\text{KM}} = N \sum_x \text{Tr} V[\phi(x)] - N \sum_{\langle x, y \rangle} \text{Tr} \left( \phi(x)U(x, y)\phi(y)U^\dagger(x, y) \right) \tag{1}
\]

where \( V(\phi) \) is a potential function for the scalars which is, at this stage, arbitrary. The factors of \( N \) are included to assure a smooth large \( N \) limit for the model. The model is defined by the Partition Function \( Z_{\text{KM}} \) given by the following Functional Integral:

\[
Z_{\text{KM}} = \int \prod_x d\phi(x) \prod_{\langle x, y \rangle} [dU(x, y)] \exp \left[ -S_{\text{KM}} \right] \tag{2}
\]

Here \( d\phi(x) \) is the Hermetian integration measure over the matrix \( \phi(x) \) and \( [dU(x, y)] \) is the invariant Haar measure for integration of the matrix \( U(x, y) \) over the unitary group \( U(N) \).

This model is invariant under the gauge transformations

\[
\phi(x) \rightarrow \omega(x)\phi(x)\omega^\dagger(x); \quad U(xy) \rightarrow \omega(x)U(xy)\omega^\dagger(y) \tag{3}
\]

where \( \omega(x) \) is an arbitrary \( U(N) \) valued function of \( x \). The second term in Eq. (1) is the usual gauge invariant kinetic term for a scalar field in the adjoint representation of a gauge group. In fact this model (which is called the Kazakov–Migdal Model or the KM Model) is simply a model of an adjoint scalar coupled to a Gauge Field except that the usual kinetic term for the Gauge Field is omitted. It is the absence of this term that makes the model soluble in the large \( N \) limit. We begin by reviewing how this works.
The first step, which can be done for finite $N$, is to explicitly integrate over all the unitary matrices $U(x,y)$ in Eq. (2). This can be done since there is no coupling between the various $U(x,y)$. The result is a Functional Integral over the fields $\phi(x)$ which. As a result of the Gauge Invariance (3) the integral depends only on the eigenvalues $\phi_i(x)$ ($i = 1..N$) of the matrices $\phi(x)$. In fact the integral can be done explicitly using the “Itzykson–Zuber” formula \[2\]. (For details see Ref. \[1\].) The result is:

$$Z_{KM} \propto \int \prod_{x,i} d\phi_i(x) \Delta^2[\phi(x)] \exp \left( -N \sum_x \text{Tr}V[\phi(x)] \right) \prod_{<x,y>} \frac{\det_{ij} e^{N\phi_i(x)\phi_j(y)}}{\Delta[\phi(x)]\Delta[\phi(y)]}$$

(4)

where $d\phi_i(x)$ is the ordinary integration measure over the real numbers $\phi_i(x)$ and $\Delta[\phi] = \det_{ij} (\phi_i - \phi_j)$ is the Vandermonde determinant for $\phi$.

In this form of the Functional Integral it is well known \[3\] how to go to the limit of large $N$. When $N \to \infty$ the partition function (4) is dominated by the stationary points of the action (defined as minus the logarithm of the integrand). In such circumstances the stationary points ($\Phi_i(x)$) are called the Master Fields for the Theory. The value of the integral when $N \to \infty$ is equal to the integrand evaluated when $\phi_i(x)$ is equal to the Master Field $\Phi_i(x)$.

When $N$ is infinite it is customary and convenient to describe the Master Field $\Phi_i$ will will usually be independent of $x$ by a density of eigenvalues. The idea is to order the eigenvalues so that $\Phi_i$ is monotonically increasing and then to define the density of eigenvalues

$$\rho(\lambda) = \frac{1}{N} \left( \frac{d\Phi_i}{di} \right)^{-1} (\Phi_i = \lambda)$$

(5)

so that

$$\int_{-\infty}^{\infty} \rho(\lambda)d\lambda = 1$$

(6)

$N\rho(\lambda)d\lambda$ is equal to the number of eigenvalues in a range $d\lambda$ about $\lambda$.

The simplest KM Model in which the potential $V(\phi) = m^2\phi^2/2$ is quadratic was first solved by Gross \[4\]. There is a wealth of literature both on the solutions to the saddle point equations and on the relationship of the KM Model with QCD. A selection of references are Refs. \[5\] - \[18\].
It was originally thought that this model would have a second order phase transition and some evidence was given in the case of a quadratic potential that this occurs when $m^2 = 2D$. It was then argued that the critical behavior of this model should be represented by QCD, the only known nontrivial four dimensional field theory with non-Abelian gauge symmetry. Unfortunately the solution of Gross \cite{4} showed that the Gaussian model had no critical behavior. This problem, combined with a further problem of an additional local $Z_N$ symmetry which implies the vanishing of Wilson Loops \cite{19} and a better understanding of why the Gaussian Model fails to induce QCD has led to a consensus that the KM Model does not induce QCD.

Nonetheless the model \cite{2} is interesting in its own right both as a Gauge Theory which is soluble in large $N$ and, as we shall review below, as an interesting example of a Matrix Model. Fortunately there has recently \cite{20}, \cite{21} been some progress in finding an explicit solution to a non-Gaussian Kazakov-Migdal Model with a logarithmic potential of a very specific form. This solution was found by relating the density of eigenvalues for the KM Model to that of the an ordinary (one) Matrix Model whose potential has two logarithmic singularities. Although much is known about Matrix Models for polynomial potentials and for potentials with a single logarithmic singularity (the Penner Model) little is known about the model with two logarithmic singularities (which might be called a “Double Penner” Model).

The purpose of this paper is twofold. First of all to study ordinary Matrix Models with a “Double Penner” Potential (which has two logarithmic singularities) and then to relate the solution of this problem to the solution of the Kazakov Migdal Model with a specific class of logarithmic potentials.

The results of this paper are applicable in a much wider context than just the KM Model. If, for example, we choose the dimension $D = 1/2$ we recover the solution of the ordinary “Two Matrix Model” \cite{21}. The techniques discussed in this paper and in Ref. \cite{20} are applicable to ordinary Matrix Models whose wide range of applicability can be seen in Refs. \cite{22} and \cite{23}.
The plan of the paper is as follows. In Section II we review the method of [20] of finding the eigenvalue distributions for both Kazakov–Migdal Models and ordinary Matrix Models. We then show in general how this allows for explicit solutions in the case of a ‘Double Penner” potential and how the density of eigenvalues for the KM Model is related to that of an ordinary matrix model. In Section III we describe in detail the variety of solutions to the ordinary Matrix Model with a Double Penner potential. We discuss the various one cut and two cut solutions and the cases in which there is “condensation of eigenvalues” at the singularities. In Section IV we apply the results of Section III to the Double Penner KM Model. We present a phase diagram for the model and review its critical behavior. In Section V we summarize our results and conclusions.

II. METHOD OF SOLUTION

A. Ordinary Matrix Model

Consider first an ordinary Hermitian Matrix Model whose only variable is a single $N \times N$ Hermetian Matrix $\phi$. The model is defined by by the partition function

$$Z = \int d\phi \ \exp \left[ -N \text{tr} V(\phi) \right]$$

where $d\phi$ is the Hermetian integration measure. If (as is always assumed) $\text{Tr} V(\phi)$ is invariant under $U(N)$ transformations of $\phi$ then the integrand in Eq. (3) depends only on the eigenvalues $\phi_i$ of $\phi$ and the partition function can be written as

$$Z \propto \int \prod_i d\phi_i \ \Delta^2(\phi) \ \exp \left[ -N \text{tr} V(\phi) \right]$$

As discussed in the Introduction, in the limit $N \to \infty$ this integral is dominated by a matrix $\Phi$ whose eigenvalues $\mu$ are distributed according to some distribution $\rho(\mu)$.

There are many methods of studying and solving such Matrix Models. The method we discuss here is most suitable for generalization to the KM Model [II]. The density of eigenvalues can be found by defining the quantity
where $\lambda$ is an arbitrary complex number. Once $E_\lambda$ has been computed the density of eigenvalues can easily be determined since at large $N$

$$E_\lambda = \frac{1}{N} \text{Tr} \frac{1}{\lambda - \phi} = \frac{1}{N} \text{Tr} \frac{1}{\lambda - \Phi} = \int_{-\infty}^{\infty} d\eta \rho(\eta) \frac{1}{\lambda - \eta}$$

(10)

So that

$$2\pi i \rho(\eta) = E_{\lambda - i\epsilon} - E_{\lambda + i\epsilon}$$

(11)

which is nonzero only along the branch cuts of $E_\lambda$. Thus by finding the branch cuts of $E_\lambda$ and computing the discontinuities across these cuts we can compute the density of eigenvalues $\rho(\lambda)$.

Following Ref. [20] we begin with the equation

$$\int d\phi \frac{d}{d\phi_{ij}} \left\{ \left( \frac{1}{\lambda - \phi} \right)_{ij} \exp[-N\text{tr}V(\phi)] \right\} = 0$$

(12)

This is now written in terms of an integral over the matrices $\phi$ rather than in terms of their eigenvalues. Eq. (12) leads to the following equation for $E_\lambda$

$$E^2_\lambda - \left< \frac{1}{N} \text{Tr} \frac{1}{\lambda - \phi} V'(\phi) \right> = 0$$

(13)

In the simplest case in which $V(\phi) = m^2 \phi^2/2$ this equation leads immediately to a quadratic equation for $E_\lambda$

$$E_\lambda = \frac{1}{2} \left( m^2 \lambda - \sqrt{m^4 \lambda^2 - 4m^2} \right)$$

(14)

and thus to the well known semicircle distribution of the eigenvalues of $\phi$

$$\rho(\lambda) = \frac{m}{\pi} \sqrt{1 - \frac{m^2 \lambda^2}{4}}$$

(15)

which has support on the interval $(-2/m, 2/m)$. 
B. Kazakov–Migdal Model

The situation for the Kazakov–Migdal Model is significantly more complicated mostly due to the presence of the gauge field. One option for solving this model is to begin with the Functional Integral (2) and to define the two quantities $E_\lambda$ and $G_{\lambda\mu}$ as follows: Let $a$ and $b$ be two adjacent sites on the lattice and let $U_{ab}$ be the gauge field on the link joining them then define the average $<Q>$ of any quantity $Q$ as

$$<Q> = \frac{1}{Z_{KM}} \int \prod_x d\phi(x) \prod_{<x,y>} [dU(x,y)] \; Q \; \exp[-S_{KM}]$$

(16)

We then define

$$E_\lambda = \langle \frac{1}{N} \text{Tr} \frac{1}{\lambda - \phi(a)} \rangle$$

(17)

which is expected to be independent of the chosen site $a$ (in the limit of large volume) and

$$G_{\lambda\mu} = \langle \frac{1}{N} \text{tr} \left( \frac{1}{\lambda - \phi(a)} U_{ab} \frac{1}{\mu - \phi(b)} U_{ab}^{-1} \right) \rangle$$

(18)

which is expected to be independent of the chosen link $(a,b)$. It is useful to note that asymptotically, for large $\lambda$,

$$E_\lambda \sim \frac{1}{\lambda} + \sum_{n=1} \frac{\langle \phi^n \rangle}{\lambda^{n+1}}$$

(19)

$$G_{\lambda\mu} \sim \frac{E_\mu}{\lambda} + ...$$

(20)

The next step is to write two equations analogous to Eq. 12; one for $E_\lambda$ and one for $G_{\lambda\mu}$

$$\int \prod_x d\phi(x) \prod_{<x,y>} [dU(x,y)] \frac{d}{d\phi(a)_{ij}} \left\{ \left( \frac{1}{\lambda - \phi(a)} \right)_{ij} \exp[-S_{KM}(\phi,U)] \right\} = 0$$

(21)

and

$$\int \mathcal{D}\phi \mathcal{D}U \frac{d}{d\phi(a)_{ij}} \left\{ \left( \frac{1}{\lambda - \phi(a)} U_{ab} \frac{1}{\mu - \phi(b)} U_{ab}^{-1} \right)_{ij} \exp[-S_{KM}(\phi,U)] \right\} = 0$$

(22)

where $S(\phi, U)$ is the Kazakov–Migdal Action 11.
Recall that in the limit of infinite $N$ the functional integral is dominated by a single, translationally invariant Master Field $\Phi$. Due to the Gauge Invariance of the Action we can choose $\Phi$ to be diagonal with eigenvalues $\Phi_i$ ($i = 1..N$) without any loss of generality.

Before proceeding it is useful to define a quantity $\Lambda(\phi)$ as follows: Consider first the quantity

$$C_{ij}(\Phi) = \frac{1}{Z_{KM}} \int [dU] |U_{ij}|^2 \exp \left( -S_{KM}(U, \Phi) \right)$$

Then define

$$\Lambda_i(\Phi) = C_{ij}(\Phi)\Phi_j$$

With this definition in hand one can derive the following equations (see Ref. [20] for details) corresponding to Eqs. 21 and 22 respectively:

$$E_\lambda^2 - \left\langle \frac{1}{N} \text{Tr} \frac{V'(\Phi) - 2D\Lambda(\Phi)}{\lambda - \Phi} \right\rangle = 0$$

$$(E_\lambda + \mu)G_{\lambda\mu} - E_\lambda - \left\langle \frac{1}{N} \text{Tr} \frac{V'(\Phi) - (2D - 1)\Lambda(\Phi)}{\lambda - \Phi} U_{xy} \frac{1}{\mu - \Phi} U^{-1}_{xy} \right\rangle = 0$$

Note $\Lambda(\phi)$ is defined so that $\langle \cdot \cdot \cdot U_{xy}\phi(y)U^{-1}_{xy} \rangle$ is replaced by $\langle \cdot \cdot \cdot \Lambda(\Phi) \rangle$ in the large $N$ limit provided the full expression is gauge invariant and that $U_{xy}$ does not appear in the expression again. At this stage we must make an important comment about notation. In Equation (26) the only average remaining is that over the Gauge field $s$ which is done in the background field $\Phi$. After doing this average all that remains is to perform the trace which can be thought of as an average over the distribution of eigenvalues of $\Phi$. It is thus common to write, for example, $V'(\phi)$ or $\Lambda(\phi)$ instead of $V'_i(\Phi)$ or $\Lambda_i(\Phi)$ where we identify the label $i$ with the eigenvalue $\phi$ so that $\Phi_i = \phi$. In the case when the potential is the trace of a function of the matrix scalar field, $V'(\phi)$ will have the same form as the derivative of the potential with respect to the matrix valued field.

To proceed further we need to make a choice for the potential $V(\phi)$. The solution to the above equations is simplest in the case of a Gaussian potential which is discussed in detail.
in Ref. [20]. In this case it turns out that $\Lambda(\phi)$ is simply proportional to $\phi$. As a result one obtains a quadratic equation for $E_\lambda$ whose solution (see [20] and [4]) yields a semicircle distribution of eigenvalues where the edge of the distribution $2/M(m)$ is a calculable function of the parameter $m$ in the potential.

In this paper we shall discuss the more complicated case of a “Penner” potential. Previous work on this subject can be found in Ref. [21]. The case which can be solved involves choosing a potential for which

$$V'(\phi) - (2D - 1)\Lambda(\phi) = \frac{q}{\phi - \xi} + B$$

which has a pole at $\phi = \xi$ with residue $q$ and an asymptotic value of $B$ at $\phi \to \infty$. The idea is to use this form of $V' - (2D - 1)\Lambda(\phi)$ and to solve for the potential $V$ which leads to this function.

Using Eqs. (25) and (26) (see [20] for more details) we can derive the following equation which relates $G_{\lambda\mu}$ to $E_\lambda$ and $G_{\xi\mu}$ (which thus involves $G$ at the singularity of $V$):

$$G_{\lambda\mu} = (\lambda - \xi)E_\lambda - qG_{\xi\mu}$$

This can be written in a more useful form by using the asymptotic condition for $G_{\lambda\mu}$. If we expand $G_{\lambda\nu}$ in Eq. (28) in large $\lambda$ using Eq. (20) we find that

$$qG_{\xi\mu} = 1 + (B - \mu)E_\mu$$

Substituting this back into Eq. (28) we relate $G_{\lambda\nu}$ to $E_\lambda$ and $E_\nu$:

$$G_{\lambda\mu} = (\lambda - \xi)E_\lambda + (\mu - B)E_\mu - \frac{1}{(\lambda - \xi)(E_\lambda + \mu - B) - q}$$

The next step is to notice from Eq. (28) that $G_{\lambda\nu}$ is symmetric under interchange of $\lambda$ and $\nu$ so that $G_{\lambda\nu} = G_{\nu\lambda}$. Applying this condition to Eq. (30) one finds, after some algebra, the following quadratic equation for $E_\lambda$:

$$(\lambda - B)\lambda - \xi) E_\lambda^2 + [(\lambda - B)(\lambda - \xi)(\xi - B) + q(B - \xi) - (\lambda - \xi)] E_\lambda + \lambda(B - \xi) = \text{constant}$$

(31)
where the constant is independent of $\lambda$. It is possible to determine this constant in terms of the mean value of $\phi$: $\bar{\phi} = \int \rho(\phi) \phi d\phi$ using the asymptotic condition (19). The equation for $E_\lambda$ then becomes:

$$(\lambda - B)(\lambda - \xi) E_\lambda^2 + [(\lambda - B)(\lambda - \xi)(\xi - B) + q(B - \xi) - (\lambda - \xi)] E_\lambda$$

$$+ \left[ (\lambda + \bar{\phi})(B - \xi) + (\xi^2 - B^2) \right] = 0$$  \hspace{1cm} (32)

In principle it is necessary to determine the value of $\bar{\phi}$ self-consistently by extracting the density of eigenvalues $\rho$, which will depend on $\bar{\phi}$, from Eq. (32) and then demanding that $\int d\phi \rho(\phi) \phi = \bar{\phi}$. We shall see however that in many cases a properly normalized solution will exist for range of values of $\bar{\phi}$. The physical reason for this is that due to the singularities of the potential the eigenvalues (in large $N$) can be arbitrarily distributed among two minima on different sides of a singularity. This point will be discussed further later in this paper.

It is now possible to extract the functions $\Lambda(\phi)$ and the potential $V(\phi)$ by comparing the equation (32) with Equation (25). Notice that if $V' - 2DLambda$ would have only simple poles we would also obtain (from Eq. (25) a quadratic equation for $E_\lambda$. By comparing these two equations one finds (see [20] for details)

$$V'(\lambda) - 2D\Lambda(\lambda) = \frac{1 - q}{\lambda - B} + \frac{q}{\lambda - \xi} + (B - \xi)$$  \hspace{1cm} (33)

Using the ansatz (27) we find

$$\Lambda(\lambda) = \frac{q - 1}{\lambda - B} + \xi$$  \hspace{1cm} (34)

which leads finally to the expression for $V'$

$$V'(\lambda) = \frac{q}{\lambda - \xi} + \frac{(2D - 1)(q - 1)}{\lambda - B} + (2D - 1)\xi + B$$  \hspace{1cm} (35)

which results from our ansatz (27). This now yields the potential $V$ for which we have found a solution.

The preceding results thus establish an interesting relationship between the KM Model with two logarithmic singularities as in Eq. (33) and a ordinary one matrix model with the potential $W(\phi)$ whose derivative is of the general form
\[ W'(\phi) = \frac{r_1}{\phi - \eta_1} + \frac{r_2}{\phi - \eta_2} + C \]  

(36)

with

\[ r_1 + r_2 = 1; \quad \eta_2 - \eta_1 = C \]  

(37)

where \( r_1 \) is identified with \( q \), \( \eta_1 \) with \( \xi \) and \( \eta_2 \) with \( B \). A Matrix Model with the potential \( W(\phi) \) will have the same function \( E(\lambda) \) and thus, in particular, the same distribution of eigenvalues as the KM Model. This One Matrix Model can be discussed for arbitrary values of \( r_1, r_2 \) and \( C \) (i.e. without requiring the conditions (37)) and it has many interesting aspects. For example it generalizes the Penner Model which contains a single logarithmic singularity to a model with two singularities and, as will be seen later, it admits a multi-phase solution space with non-trivial critical behavior. In Section III we thus study the double penner One Matrix Model whose results will later be applied to the KM Model.

### C. Interpretation of Solutions

Before proceeding it is useful to recall that the infinite \( N \) behavior of both the ordinary Matrix Model and the K–M Model can be described by the solution to an analogue mechanical problem. In the case of the ordinary Matrix Model the partition function Eq. (7) is written in terms of the eigenvalues \( \phi_i \) of the matrix \( \phi \)

\[
Z = \int \prod_{i=1}^{N} d\phi_i \quad \Delta^2(\phi) \exp \left[ -N \sum_i V(\phi_i) \right]
\]

(38)

In the large \( N \) limit in which the solution is given by the classical extremum of the action we see that we have an analogue mechanical problem of \( N \) particles which are constrained to lie on a line at locations \( \phi_1 \cdots \phi_N \). Each particle is subjected to an overall potential \( V(\phi_i) \) and to a logarithmically repulsive two–body potential \( \log (\phi_i - \phi_j)^2 \).

The solution to the KM Model corresponds to another, more complicated, analogue mechanical model. If we look at the partition function in Eq. (38) and recall that in the
infinite $N$ limit the $\phi(x)$ are independent of $x$ and that on a square lattice each site has $D$ independent nearest neighbors (where $D$ is the dimensionality of the spacetime) then for infinite $N$ Eq. (4) is equivalent to solving the following one matrix model

$$Z = \int \prod_{i=1}^{N} d\phi_i \Delta^2(\phi) \exp \left[ -N \sum_i V(\phi_i) - \left( \frac{\det_{ij} e^{N\phi_i\phi_j}}{\Delta^2(\phi)} \right)^{D} \right] \right) \int \prod_{i=1}^{N} d\phi_i \exp \left[ -N \sum_i \left( V(\phi_i) - D\phi_i^2 \right) - (D-1) \sum_{i<j} \log(\phi_i - \phi_j)^2 + D \log \det_{ij} e^{N(\phi_i - \phi_j)^2/2} \right]$$

This is once more a problem of $N$ particles on a line at locations $\phi_i$. Their central potential is now $V(\phi) - D\phi^2$ and their interaction is no longer a two–body interaction since it involves a determinant. We can, however, use the fact that the integral over the Gauge Group in Eq. (2) which lead to Eq. (4) is nonsingular when any two eigenvalues approach each other. Using this we see that the effective interaction is logarithmically repulsive at short distances (due to the extra factor of $\Delta^2$) and attractive at long distances if $D > 1$.

The analogue mechanical problem presented in this section is a very useful tool for visualizing and checking the solution we obtain using the mathematical machinery of Matrix Models.

### III. THE DOUBLE PENNER MODEL

#### A. Formal Solution

In this section we investigate the large $N$ solutions of the (non–KM) One Matrix Model with the potential (36), for all possible values of the parameters $r_1, r_2, \eta_1, \eta_2$ and $C$. The basic equation is Eq. (13) and leads to the quadratic equation

$$E^2 - W'(\lambda)E + \frac{r_1 E_{\eta_1}}{\lambda - \eta_1} + \frac{r_2 E_{\eta_2}}{\lambda - \eta_2} = 0 \quad (40)$$

with the $E_{\eta_i}$ determined by the asymptotic expansion of $E_{\lambda}$ as

$$r_1 E_{\eta_1} + r_2 E_{\eta_2} = C$$
\begin{equation}
    r_1 E_{\eta_1} \eta_1 + r_2 E_{\eta_2} \eta_2 = C\bar{\phi} + r_1 + r_2 - 1 \tag{41}
\end{equation}

which allows the \( E_{\eta_i} \) to be determined explicitly in terms of the mean value \( \bar{\phi} \) of \( \phi \).

\begin{equation}
    \begin{pmatrix}
        r_1 E_{\eta_1} \\
        r_2 E_{\eta_2}
    \end{pmatrix}
    = \frac{1}{\eta_2 - \eta_1}
    \begin{pmatrix}
        (\eta_2 - \bar{\phi})C + 1 - (r_1 + r_2) \\
        (\bar{\phi} - \eta_1)C + (r_1 + r_2) - 1
    \end{pmatrix} \tag{42}
\end{equation}

The solution to Eq. (40) is simply

\begin{equation}
    E_\lambda = \frac{1}{2} \left( W'(\lambda) - \sqrt{(W'(\lambda))^2 - 4 \sum_{i=1}^{2} \frac{r_i E_{\eta_i}}{\lambda - \eta_i}} \right) \tag{43}
\end{equation}

where

\begin{equation}
    W'(\lambda) = \sum_{i=1}^{2} \frac{r_i}{\lambda - \eta_i} + C \tag{44}
\end{equation}

is given by Eq. (36). Notice that we have chosen to write the solution to the quadratic with a “−” rather than with a “±” and to discuss the two solutions in terms of the possible branches of the square root.

The correct choice of the branch of the square root in Eq. (43) is very subtle, even in the case of the ordinary Penner Model (see [23]). First of all we must choose the branch of the square root so that \( E_\lambda \) satisfies the asymptotic condition (19): \( E_\lambda \sim 1/\lambda \) as \( \lambda \to \infty \). This will be satisfied provided the square root is chosen to have no branch cuts going out to infinity and that the square root approaches \(+C\) at infinity. There are however two other conditions which must also be satisfied. Notice from Eq. (12) than \( E_{\eta_i} \) is some finite number so that \( E_\lambda \) is not singular at the singularities of the potential. We shall see that this condition is not automatically satisfied by the solution (13). Furthermore we would like the “Master Field” \( \Phi \) which is encoded by \( E_\lambda \) to be a Hermetian Matrix with real eigenvalues. This requires the branch cuts to be on the real axis with a purely imaginary discontinuity so that the density of eigenvalues is real and positive. We shall see below that it is often impossible to satisfy all these conditions simultaneously though it may be possible to relax these conditions somewhat and still maintain an interesting solution. In fact we shall see that despite the fact that in the ordinary Double Penner model there are both one–cut and
two–cut solutions in the KM case in which the conditions \((37)\) must be satisfied, only one–cut solutions will be possible (i.e. the second cut will correspond to two degenerate branch points leading to a zero density of eigenvalues).

Before proceeding to analyze the branch cuts it is useful to write Eq. \((43)\) in an alternate form. Note that the location of the branch points are found by finding the zeros of the function under the square root in Eq. \((43)\). This is a quartic equation for \(\lambda\). Let \(\xi_1, \xi_2, \xi_3, \xi_4\) be the solutions to this quartic equation. The requirement of a real, positive definite, density of eigenvalues will require these roots to be real though there may be some degeneracy among them. We thus expect, in general, both one–cut and two–cut solutions to the Double Penner Matrix Model. Let us assume, without loss of generality, that \(\xi_1 \leq \xi_2 \leq \xi_3 \leq \xi_4\) Eq. \((43)\) for \(E_\lambda\) can now be written

\[
E_\lambda = \frac{1}{2} \left( W'(\lambda) - \frac{C}{(\lambda - \eta_1)(\lambda - \eta_2)} \sqrt{(\lambda - \xi_1)(\lambda - \xi_2)(\lambda - \xi_3)(\lambda - \xi_4)} \right) \quad (45)
\]

As discussed above, the asymptotic condition on \(E_\lambda\) requires the branch cuts of the square root to be chosen so that the square root approaches \(\lambda^2\) at infinity and that there are no branch cuts which go out to infinity.

Implementation of the further conditions described above will be discussed in Sec. III C. Before doing so we should point out that the density of eigenvalues which is extracted from Eq. \((13)\) will always be normalized and have the correct value of \(\bar{\phi}\) provided only that \(E_\lambda\) has no singularities and that there are no cuts going out to infinity. To see this recall that \(E_\lambda \sim 1/\lambda\) as \(\lambda \to \infty\). Thus

\[
\frac{1}{2\pi i} \oint_C E_\lambda \, d\lambda = 1 \quad (46)
\]

where \(C\) is a circle at infinity. If \(E_\lambda\) has no additional singularities then this integral can be written as an integral over the discontinuities across the cuts of \(E_\lambda\) which is just the total normalization \(\int \rho(\phi) d\phi\) which is thus equal to 1. The mean value of \(\phi\) can similarly be calculated. From the asymptotic expansion \((19)\) of \(E_\lambda\) we see that

\[
\frac{1}{2\pi i} \oint_C \lambda E_\lambda \, d\lambda = \bar{\phi} \quad (47)
\]
which is now valid for the $E_\lambda$ in Eq. (45). Recall however that the parameters $\xi_i$ in (45) depended on $\bar{\phi}$. The result (47) is valid, however, independently of this value of $\bar{\phi}$. Again, if $E_\lambda$ has no additional singularities the contour can be deformed to a contour surrounding the cuts so that $\int \phi \rho(\phi) d\phi = \bar{\phi}$ is this same value of $\bar{\phi}$ which, as we recall, was extracted from the same asymptotic expansion of $E_\lambda$. (If the cuts are chosen so that $E_\lambda$ is singular at either one or both of the $\eta_i$ then both the normalization and the mean value of $\phi$ will get an extra contribution from the singularity. It is not difficult to show, for example, that the normalization will get a contribution $r_i$ from a singularity at $\eta_i$.

B. Review of Single Penner Case

Before discussing the various possibilities in the Double Penner Case let us review briefly how the cuts work in the ordinary Single Penner Model. In this case the derivative of the potential is of the form

$$W'_p(\phi) = \frac{r}{\phi - \eta} + Q$$

and the quadratic equation for $E_\lambda$ resulting from Eq. (13) has the solution:

$$E_\lambda = \frac{1}{2} \left\{ \left( \frac{r}{\lambda - \eta} + Q \right) - \sqrt{\left( \frac{r}{\lambda - \eta} + Q \right)^2 - 4 \frac{Q}{\lambda - \eta}} \right\}$$

with

$$rE_{\eta} = Q$$

The location $\xi_\pm$ of the branch points occurs when the square root vanishes i.e.

$$\xi_\pm = \eta + \frac{2 - r}{Q} \pm \frac{2}{Q} \sqrt{1 - r}$$

When $r > 1$ there are no real solutions. This is the case when the potential $W_p(\phi)$ has an attractive logarithmic singularity at $\eta$ with a strength greater than 1. When $r < 0$ the potential has a repulsive singularity which results in a local minimum of $W_p$ which occurs
at $\phi > \eta$ if $Q > 0$ and at $\phi < \eta$ if $Q < 0$. In this case it is easy to check that both $\xi_\pm$ are on the same side of the singularity as this local minimum. In the case $0 < r < 1$ the potential has an attractive singularity and $W_\rho$ has only a local maximum. In this case both branch points $\xi_\pm$ lie side of the singularity opposite to this local maximum.

In this Single Penner case we can, as discussed above, try to implement the three conditions: $E_\lambda \sim 1/\lambda$ as $\lambda \to \infty$; $E_\lambda$ nonsingular at $\eta$; and $\rho(\phi)$ real and positive with support on the real axis. The third condition implies that the branch points and branch cuts must lie on the real axis. This guarantees that $\rho$ is real but not necessarily that it is positive. To analyze the other conditions we write, as in Eq. (45)

$$E_\lambda = \frac{1}{2} \left\{ \frac{r}{\lambda - \eta} + Q - \frac{Q \sqrt{(\lambda - \xi_+)(\lambda - \xi_-)}}{\lambda - \eta} \right\}$$

(52)

The asymptotic condition on $E_\lambda$ requires that there be no branch cuts at infinity and that the square root be positive for $\lambda > \xi_\pm$. This, together with the positivity requirement of $\rho$ implies that the branch cut joins $\xi_\pm$ in a straight line along the real axis. Furthermore the positivity of $\rho$ requires that $\xi_\pm > \eta$ if $Q > 0$ and $\xi_\pm < \eta$ if $Q < 0$. This is precisely the same result as Eq. (51) so that in the single penner case we are guaranteed that the eigenvalue density will be positive if the branch cut joins the $\xi_\pm$ in a straight line along the real axis. From Eq. (49) we can determine the behavior of the square root in Eq. (52) at the singularity up to a possible sign.

$$E_\lambda \to \frac{1}{2} \frac{1}{\lambda - \eta} \left\{ r \mp \frac{Q}{|Q|} \frac{|r|}{r} \right\} \quad \text{as} \quad \lambda \to \eta$$

(53)

where the minus sign is used if the square root in (52) is positive which occurs if $\eta > \xi_\pm$ and the plus sign is used if it is negative which occurs if $\eta < \xi_\pm$. Thus we establish the following condition for $E_\eta$ to be nonsingular: If $\eta > \xi_\pm$ then the sign of $Q$ and $r$ must be the same whereas if $\eta < \xi_\pm$ the $Q$ and $r$ must have opposite sign. Thus, for the cancellation of singularities, we must have

$$(\eta - \xi_\pm)Qr > 0$$

(54)
In the Single Penner Model the above condition is satisfied if and only if \( r < 0 \). This is evident from Eq. (51) which implies that \( \xi_+ > \eta \) if \( Q > 0 \) whereas \( \xi_- < \eta \) if \( Q < 0 \). Thus both cases require \( r < 0 \) for the regularity of \( E_\lambda \) at the singularity. This is physically reasonable since this is precisely the case in which the potential has a local minimum.

The conventional way to analyze the case \( 1 > r > 0 \) has been to relax the positivity condition on the density of eigenvalues (see [23] for example) and to allow the branch cut to go around the singularity so that the branch cuts goes from \( \xi_- \) to \( \xi_+ \) by first going around \( \eta \). In the case \( 1 > r > 0 \) this will lead to a distribution of eigenvalues with complex support but \( E_\lambda \) will have both the correct asymptotics and it will be nonsingular at \( \eta \). One of the reasons this choice of cut is interesting is that it behaves nearly the same as a distribution of eigenvalues with a delta function singularity (of strength \( r \)) at \( \eta \) and and the remainder of the eigenvalues along the real axis between the \( \xi_\pm \). This can be seen by noting that if we compute the average of any analytic function \( f(\lambda) \)

\[
I = \int_{\text{cut}} d\lambda \rho(\lambda)f(\lambda) \tag{55}
\]

then this integral will be independent of the precise path which the cut takes around the singularity. In fact by taking the cut from \( \xi_- \) just below the real axis then around the singularity and back just above the real axis to \( \xi_+ \) one can easily check that

\[
I = rf(\eta) + \int_{\xi_-}^{\xi_+} d\lambda \hat{\rho}(\lambda)f(\lambda) \tag{56}
\]

with \( \hat{\rho} \) normalized to \( 1 - r \). This looks as if we could write

\[
\rho(\lambda) = r\delta(\lambda - \eta) + \hat{\rho}(\lambda) \tag{57}
\]

with \( \hat{\rho} \) having support on \( (\xi_-, \xi_+) \). This is often called “condensation of eigenvalues” since some fraction of the eigenvalues “condense” at the singularity. Unfortunately this interpretation is not quite correct since if we were to evaluate the average of a nonanalytic function \( g(\lambda) \) (for example if we were to evaluate the Free Energy which contains a logarithmic cut precisely in the region of interest) then the result would be dependent on the precise path of
the cut. In fact, for a general path, the Free Energy is not even real though some authors
have chosen the contour in such a way that the Free Energy is real.

C. Structure of Cuts in the Double Penner Case

In the previous subsection we saw that even in the ordinary Penner model proper one–cut
solutions which satisfy all our requirements do not necessarily exist and there are cases when
either no solutions exist or when only very unusual solutions with “eigenvalue condensation”
are present. We now continue with the discussion in Subsection (IIIA) of the Double Penner
Case where we shall find a similar situation. Let us focus attention on Eqs. (43) and (45).
As in the Single Penner case the requirement that $E_\lambda$ have no cut singularities at infinity and
that $\rho(\lambda)$ be real and positive requires all the branch cuts to be on the real line. Recalling
that $\xi_1 < \xi_2 < \xi_3 < \xi_4$ we must then have one cut from $\xi_1$ to $\xi_2$ and the other from $\xi_3$ to $\xi_4$.
It is also clear from Eqs. (43) and (45) that if $E_\lambda$ is to be regular at the singularities they
cannot be on a branch cut. Thus each singularity is either below $\xi_1$, between $\xi_2$ and $\xi_3$ or
above $\xi_4$. Finally recall that the asymptotic condition on $E_\lambda$ requires that the square root
in Eq. (45) must be positive for $\lambda > \xi_4$. This in turn implies that it is negative between the
branch cuts and positive for $\lambda < \xi_1$.

We are now ready to study the possible singularity of $E_\lambda$ near the $\eta_i$. Near the singularity
at $\eta_1$

$$E_\lambda \to \frac{1}{2} \frac{1}{\lambda - \eta_1} \left\{ r_1 \pm \frac{C}{|C|} \frac{|r_1|}{r_1} \frac{|\eta_1 - \eta_2|}{|\eta_1 - \eta_2|} r_1 \right\} \quad \text{as } \lambda \to \eta_1$$

(Near the singularity at $\eta_2$ simply replace $1 \leftrightarrow 2$ everywhere in Eq. (58).) The correct
sign in the above equation depends on the sign of the square root near the singularity. The
minus sign is to be used when the square root is positive i.e if $\eta_1 > \xi_4$ or $\eta_1 < \xi_3$ (in this
case we shall say that the singularity is “outside” the cuts) and the plus sign must be used
when it is negative i.e. if $\xi_2 < \eta_1 < \xi_3$ (in which case we shall say that the singularity is
“inside” (i.e. between) the cuts). If the minus sign is used the singularity cancels provided
\((\eta_1 - \eta_2)Cr_1 > 0\) whereas if the plus sign is used the cancellation occurs if and only if \((\eta_1 - \eta_2)Cr_1 < 0\). Let us assume without loss of generality that \(\eta_2 > \eta_1\). (It is obvious that this can be done for the general double Penner case but for the KM Model the requirement that \(C > 0\) makes this not so obvious. There are however symmetries which relate the case \(\eta > B\) to the case \(\eta < B\) so that the assumption \(\eta_2 > \eta_1\) is completely general.)

Our conclusion is then that:

\[
\begin{align*}
Cr_1 &> 0 & \eta_1 \text{ inside } (\xi_2 < \eta_1 < \xi_3) & (\sqrt{-}) \\
Cr_1 &< 0 & \eta_1 \text{ outside } (\eta_1 < \xi_1 \text{ or } \xi_4 < \eta_1) & (\sqrt{+}) \\
Cr_2 &> 0 & \eta_2 \text{ outside } (\eta_2 < \xi_1 \text{ or } \xi_4 < \eta_2) & (\sqrt{+}) \\
Cr_2 &< 0 & \eta_2 \text{ inside } (\xi_2 < \eta_2 < \xi_3) & (\sqrt{-})
\end{align*}
\]

(59)

(If \(\eta_1 > \eta_2\) this would of course be reversed.) We shall consider the case \(C > 0\) throughout.

Although the above conditions guarantee that \(E_\lambda\) is nonsingular it does not guarantee that \(\rho\) is positive. The best way to see this is to look at Eq. (45). It is clear that if there are two cuts with no singularity separating them or with both singularities separating them, then one will have a positive \(\rho\) and the other a negative \(\rho\). If we define the regions I, II and III as the regions \(\lambda < \eta_1\), \(\eta_1 < \lambda < \eta_2\) and \(\lambda > \eta_2\) respectively then one cut must be in region II and the other must be in region III. There is however a loophole. If one of the cuts is degenerate (i.e. \(\xi_1 = \xi_2\) or \(\xi_3 = \xi_4\)) then \(\rho\) is zero along the degenerate cut and it does not matter if it is in the wrong region. We can thus have single cut solutions as follows: Either the cut \((\xi_1, \xi_2)\) is in region II or the cut \((\xi_3, \xi_4)\) is in either region I or III with the other cut being degenerate.

We now proceed to discuss the various cases individually.

**D. The Case \(r_1 < 0\) and \(r_2 < 0\)**

The simplest situation occurs when both \(r_1 < 0\) and \(r_2 < 0\). This corresponds to a potential \(W(\phi)\) which has two repulsive logarithmic singularities and two local minima. One
minimum is between the two singularities and the other is above \( \eta_2 \) (since we have assumed that \( C > 0 \)). (We call this the “two up” or “2u” potential.) In this case the condition \([59]\) for a nonsingular \( E_\lambda \) is that the branch cuts lie in the same region as the minima of the potential namely in regions II and II. This is what we expect since the eigenvalues of a Matrix Model are expected to be distributed about the minima of the potential as discussed in Sec. \( \text{II C} \).

The only remaining question is whether the branch cuts which are a solution to the quartic equation resulting from Eq. \([13]\) do in fact lie in the correct place. It is possible to show that they always do which also guarantees us that the density of eigenvalues is everywhere positive. (Recall from the previous section that when positivity of a double cut solution requires the cuts to be in regions II and III when \( C > 0 \).)

The solution to the quartic equation leading to the branch points \( \xi_1, \xi_2, \xi_3, \xi_4 \) of \( E_\lambda \) in Eq. \([13]\) is in general quite complicated. The simplest way to see the types of solutions which are possible is by a graphical method. Our goal is to solve the equation

\[
\left( \frac{r_1}{\lambda - \eta_1} + \frac{r_2}{\lambda - \eta_2} + C \right)^2 - 4 \left( \frac{r_1 E_1}{\lambda - \eta_1} + \frac{C - r_1 E_1}{\lambda - \eta_2} \right) = 0
\]  

with \( r_1 E_1 \) and \( r_2 E_2 \) given in Eq. \([13]\) in terms of \( \bar{\phi} \). Without loss of generality we may choose \( \eta_1 = 0 \) and call \( \eta_2 = \eta \). Let us also call \( r_1 E_1 = \delta \) so that \( r_2 E_2 = C - \delta \). Eq. \([60]\) is thus equivalent to:

\[
[r_1 (\lambda - \eta_2) + r_2 (\lambda - \eta_1) + C (\lambda - \eta_1) (\lambda - \eta_2)]^2 = 4((C\lambda - \delta\eta) ((\lambda - \eta_1) (\lambda - \eta_2))
\]  

The left hand side (LHS) has zeros precisely at the extrema of the potential \( W(\lambda) \) which we call \( \lambda_1 \) and \( \lambda_2 \). In our present case \((r_1 < 0, r_2 < 0)\) we have \( 0 < \lambda_1 < \eta < \lambda_2 \). We now sketch both the LHS and the RHS of Eq. \([61]\) for various values of \( \delta \) (which, we recall, is related linearly to \( \bar{\phi} \)). The LHS is a quartic with the two degenerate roots \( \lambda_1 \) and \( \lambda_2 \) whereas the RHS is a cubic which goes like \( 4C\lambda^3 \) as \( \lambda \to +\infty \) and with roots at \( \eta_1, \eta_2 \) and \( \delta\eta/C \). We now notice the following:

For \( \delta < 0 \) there is always a pair of roots in region III \((\lambda > \eta)\) but there are never any roots in region II. Depending on the values of the various parameters there may be a pair
of roots in region I. These would however lead to a negative density of eigenvalues and, correspondingly, the \( \rho(\lambda) \) for \( \lambda > 0 \) would have \( \int_{\lambda>0} \rho(\lambda)d\lambda > 1 \). In case there are no roots in region I the two additional roots are complex and, consequently, the density of eigenvalues for \( \lambda > 0 \) will not be normalized to 1.

When \( \delta \) is slightly positive, the roots in region III persist but there are no roots in region I nor in region II. At some critical \( \delta_{c1} \) which lies in the interval \((0, \lambda_1)\) a pair of roots begins to appear in region II. This pair of roots persists for all \( \delta > \delta_{c1} \). At some critical \( \delta_{c2} > \lambda_2 \) the pair of roots in region III disappears.

Thus for all values of \( \tilde{\phi} \) for which \( \delta_{c1} < \delta < \delta_{c2} \), \( E_\lambda \) has two branch cuts in regions II and III for which the density of eigenvalues is positive, normalized to 1 and for which \( \int \phi \rho(\phi)d\phi = \tilde{\phi} \) as required. For the two special cases \( \delta = \delta_{c1} \) and \( \delta = \delta_{c2} \) two of the branch points become degenerate and the above two-cut solution reduces to a one-cut solution. Notice also that for all the above cases \( \rho(\lambda) \sim |\lambda - \xi|^{\frac{1}{2}} \) near any branch point \( \xi \).

The “physical” reason for the existence of this large class of classical solutions can be seen by referring to the mechanical analogue problem discussed in Sec. II C. Since there is an infinite barrier separating the regions II and III we expect that a solution will exist with any number \( n_1 \) of particles in region II and \( N - n_1 \) particles in region III. The degenerate cases \( \delta = \delta_{c1} \) and \( \delta = \delta_{c2} \) correspond to the cases when all \( N \) particles are either in region I or in region II.

E. The Case \( r_1 > 0 \) and \( r_2 > 0 \)

The next case we consider is the case in which both \( r_1 \) and \( r_2 \) are positive. (This case could correspond to a KM Model provided the conditions (37) were satisfied.) The next case we consider is the case in which both \( r_1 \) and \( r_2 \) are positive. (This case could correspond to a KM Model provided the conditions in Eq. (37) were satisfied.) In this case the potential has no minimum but it has two maxima at points \( \lambda_1 \) and \( \lambda_2 \) in regions I and II respectively. We expect no real normalizable solutions in this case though there may
be cases in which “eigenvalue condensation” occurs as was discussed in Sec. III B for the Single Penner case.

Using the conditions given in Eq. (59) (and recalling that $C > 0$) we see that in order to avoid singularities of $E_{\lambda}$ at $\eta_1$ and $\eta_2$ we must have one cut in region I and the other in region II (i.e. in the same regions as the extrema of $W(\lambda)$). This, unfortunately, leads to a negative density of eigenvalues in both regions I and II. We thus conclude that there are no normalizable one-cut or two-cut solutions in this case.

In order to examine the possibility of “eigenvalue condensation” we use the graphical approach discussed in Sec. III D. As in this previous section we assume without loss of generality that $\eta_1 = 0$ and we call $\eta_2 = \eta$ and we conclude as follows: For many values of the parameters there will not be four real branch points. In case four real branch points exist there are two possibilities. Either the two branch cuts are in regions II and III or in regions I and III. Let us begin by discussing the case when the cuts are in regions II and III. In this case $\rho(\lambda)$ is positive along both cuts but $E_{\lambda}$ is singular at both $\eta_1$ and $\eta_2$ unless we deform both branch cuts in regions II and III to surround the singularities $\eta_1$ and $\eta_2$ respectively. Recall from the single Penner case that for the purposes of computing averages of analytic functions we can choose a contour which circles around each singularity but otherwise goes along the real axis between ($\xi_1$ and $\xi_2$) and between ($\xi_3$ and $\xi_4$). The singularities at $\eta_1$ and $\eta_2$ contribute an amount $r_1$ and $r_2$ respectively to the normalization. We thus expect that such solutions should exist only if $r_1 + r_2 < 1$.

Finally we examine the possible case in which the cuts lie in regions I and III. In this case $\rho(\lambda)$ is positive in region III but negative in region I. Thus, in order to have a sensible solution, the two branch points in region I must be degenerate ($\xi_1 = \xi_2$) so that the negative $\rho$ in region I is of no concern. We will however have a singularity of $E_{\lambda}$ at $\eta_2$ unless the cut in region III surrounds the singularity $\eta_2$. In this case the singularity will contribute $r_2$ to the normalization so we expect such solutions only if $r_2 < 1$. 

22
F. The Case $r_1 > 0$ and $r_2 < 0$

In the case $r_1 > 0$ $r_2 < 0$ the potential has a minimum in region III and a maximum in region I. The condition \[^{[59]}\] for the existence of a nonsingular solution is that the cuts lie in regions I and III. In this case the density of eigenvalues $\rho(\lambda)$ will be positive in region III but negative in region I. It thus follows that the branch points in region I must coincide ($\xi_1 = \xi_2$). It is easy to see using the graphical method described in the previous sections that, independent of the parameters of the potential (provided $r_1 > 0$ $r_2 < 0$) such a solution to our quartic equation always exists. This leads in every instance to a normalizable distribution of eigenvalues in region III for which the eigenvalues are distributed about the minimum of the potential.

We now consider the more general possibility in which we allow “eigenvalue condensation” by relaxing the condition \[^{[59]}\]. Depending on the values of the parameters in the potential, there may be no real set of branch points or else the branch cuts will be in regions II and III. This leads to a positive density of eigenvalues along both cuts but $E_\lambda$ will be singular at $\eta_1$ unless the branch cut in region II surrounds $\eta_1$. Since this singularity will contribute $r_1$ to the normalization, we only expect this solution to occur if $r_1 < 1$.

G. The Case $r_1 < 0$ and $r_2 > 0$

Finally we turn to the case $r_1 < 0$ $r_2 > 0$. In this case there are four possibilities:

\[(a)\] $4|r_1|C\eta > (r_1 + r_2 - C\eta)^2$ \hspace{1cm} (62)

In this case $W(\lambda)$ has no extrema at all.

In the remaining cases $4|r_1|C\eta < (r_1 + r_2 - C\eta)^2$.

\[(b)\] $r_1 + r_2 > C\eta$ \hspace{1cm} (63)

$W(\lambda)$ has first a maximum then a minimum in region I.

\[(c)\] $-C\eta < r_1 + r_2 < C\eta$ \hspace{1cm} (64)
$W(\lambda)$ has first a minimum then a maximum in region II.

\begin{equation}
(d) \quad r_1 + r_2 < -C\eta
\end{equation}

$W(\lambda)$ has first a maximum then a minimum in region III.

In the case (a) we expect no nonsingular solution. Whereas in the cases (b),(c) and (d) we guess that if the minimum is not too shallow we should get a normalizable solution distributed about this minimum. The condition (65) for nonsingular solutions in this case allows the branch cuts to be either both in region I or both in region II or both in region III.

We begin by analyzing case (a) in which $W(\lambda)$ has no extrema. In this case, the graphical method implies that the four real solutions to the quartic equations, if they exist, may be either in regions I and III or in regions II and III. It can further be shown that they cannot be in regions I and III. Thus they can only be in regions II and III. We thus conclude, as expected, that in this case we do not have a nonsingular solution. However in case there are two pairs of roots in regions II and III, the resulting branch cuts lead to a real eigenvalue distribution for which $E_\lambda$ will be singular and $\eta_2$ unless the cut in region III is deformed to circle around $\eta_2$. This will lead to “eigenvalue condensation” at $\eta_2$ which can only occur if $r_2 < 1$.

Next we look at case (b) for which $W(\lambda)$ has first a maximum then a minimum in region I. In this case the quartic equation has four roots in region I (for suitable values of $\delta$) provided only that $(r_1 + r_2 - C\eta)^2 + 4r_1 C\eta$ is not too small (i.e. the minimum of $W$ is not too shallow). The cut connecting $\xi_1$ and $\xi_2$ must be degenerate since it would lead to a negative $\rho(\lambda)$. We thus have a normalizable, nonsingular one-cut solution which can be seen (using, for example, the graphical method) to lie around the minimum of $W$. A precisely analogous situation occurs for cases (c) and (d). We have a normalizable, nonsingular distribution of eigenvalues near the minimum of the potential provided the minimum is not too shallow.
IV. THE KAZAKOV–MIGDAL MODEL

We saw in the previous section that the ordinary “Two–Pole” Penner Model has a rich variety of one–cut, two–cut and singular solutions. In this section we apply these results to the KM Model. According to the results of Sec. II B the solution to the KM Penner Model with the potential $V(\lambda)$ given in Eq. (35) is related to an ordinary (non KMM) “Two–Pole” Penner Model with $W(\lambda)$ given in Eq. (36) provided (37)

$$r_1 + r_2 = 1; \quad \eta_2 - \eta_1 = C$$  \hspace{1cm} (66)

(Recall that $r_1 = q$, $\eta_1 = \xi$ and $\eta_2 = B$.) For simplicity we shall call $r_1 = r$ which is, of course, also equal to $q$. The results of Sec. II can now be applied directly to the KM Model.

The most obvious conclusion we can draw immediately is that the case $r_1 < 0$ $r_2 < 0$ is not applicable to the KM Model. This was the only case in the previous section which admitted a nonsingular two–cut solution. We thus conclude that in the KM Double–Penner Model we have at best only one–cut solutions.

Consider first the case $r > 1$. In this case $r = r_1 > 1 > 0$ and $r_2 = 1 - r < 0$. This case was discussed in Sec. III F. For all such values of $r$ and for all $C = \eta = \eta_2 - \eta_1$ we have a normalized one–cut solution in region III. In fact the degenerate branch points $\xi_1 = \xi_2 < 0$ and $\eta_2 < \xi_3 < \lambda_2 < \xi_4$ where $\lambda_2$ is the location of the minimum of $W(\lambda)$. The alternate solution in which there is condensation of eigenvalues at the pole $\eta_1$ cannot occur in the KM case since $r = r_1 > 1$.

It is instructive to examine the KM Potential in this case and then to compare our results with the physical expectation described in Sec. II C. From Sec. II C we recall that we are interested not simply in the potential $V$ which appears in the KM action but in $V(\lambda) - D\lambda^2$ (see Eq. (39)). Choosing, without loss of generality, $\xi = 0$ and calling $B = \eta$ we have

$$V'(\lambda) - 2D\lambda = \frac{r}{\lambda} + \frac{(2D - 1)(r - 1)}{\lambda - \eta} + \eta - 2D\lambda$$  \hspace{1cm} (67)

$$V(\lambda) - D\lambda^2 = r\log(\lambda) + (2D - 1)(r - 1)\log(\lambda - \eta) + \eta\lambda - D\lambda^2$$  \hspace{1cm} (68)
Noting that
\[ \eta \lambda - D \lambda^2 = - D \left( \lambda - \frac{\eta}{2D} \right)^2 + \text{constant} \] (69)
we see that, unlike \( W(\lambda) \) which has a maximum in region III, \( V(\lambda) - D \lambda^2 \) has a maximum in this region. This is an example of the situation discussed in Sec. II C in which the eigenvalue distribution has its support at the maximum of a potential. The eigenvalues can “straddle” the maximum due to the long range attraction of the eigenvalues which is a result of the integral over the Gauge Fields.

The next case we consider is when \( r < 0 \). In this case \( r_1 = r < 0 \) and \( r_2 = 1 - r > 1 \). This situation was discussed in Sec. III G. First note that there can be no eigenvalue condensation in this case since \( r_2 > 1 \). Furthermore Eq. (62) implies that if \( |r| > \frac{1}{4} \left( \eta - 1 \right)^2 \) (70) there are no real normalizable solutions. In fact even if \( |r| \) is somewhat less than this limit, the minimum of the potential \( W(\lambda) \) is too shallow to admit normalized solutions. When \( |r| \) is sufficiently small we can look for solutions by first noting the location of the extrema of \( W(\lambda) \) using Eqs. (63, 64, 65). If \( \eta < 1 \) the requirements of case (b) of Sec. III G is satisfied so \( W(\lambda) \) has a minimum in region I. It can be shown that in this case there are never four real roots in region I. (One way to see that this has to be the case is by noting that if such a solution did exist then \( \bar{\phi} \) would have to be in region I. This would imply that \( \delta = \eta - \bar{\phi} > \eta \). The graphical method of the previous section then shows that there can be no roots in region I.)

If \( \eta > 1 \) we satisfy the requirements of case (c) resulting in a density of eigenvalues whose support is in region II. There is however a critical value \( \eta_{cr} > 1 \) of \( \eta \) with the property that for \( \eta < \eta_{cr} \) there is never a normalized distribution of eigenvalues (i.e. one never has two cuts in region II one of which is degenerate). For \( \eta > \eta_{cr} \) a proper solution exists provided \( |r| < |r_c(\eta)| < (\eta - 1/\eta)^2/4 \). The curve \( r_c(\eta) \) can be computed with the aid of Eq. (65). Using the fact that \( E_\lambda \) is nonsingular at both \( \lambda = 0 \) and at \( \lambda = \eta \) and using the fact that \( E_\lambda \sim 1/\lambda \) as \( \lambda \to \infty \) one derives the following equation:
\[ r^2 = \xi_1 \xi_2 \xi_3 \xi_4 \]
\[ (1 - r)^2 = (\eta - \xi_1)(\eta - \xi_2)(\eta - \xi_3)(\eta - \xi_4) \]
\[ 1 = \eta^{2\frac{\xi_1+\xi_2+\xi_3+\xi_4}{\xi_1+\xi_2+\xi_3+\xi_4}} - \eta \]

(71)

The critical line will occur when three of the \( \xi_i \) are degenerate. This leaves three equations with only two \( \xi \)'s unknown and allows us to determine \( r \) in terms of \( \eta \) with the result that
\[ r_c(\eta) = \frac{2 + 3\eta^{3/2} - \eta^2}{4} \]

(72)

When \( r < r_c(\eta) \) the KM potential \( V(\lambda) - D\lambda^2 \) has a minimum in the support of \( \rho \). Notice that there is no case when \( r < 0 \) for which the minimum of \( W \) is in region III.

Finally we consider the case \( 0 < r < 1 \). In this case \( 1 > r = r_1 > 0 \) and \( 1 > 1 - r = r_2 > 0 \) which is discussed in Sec. III E. There are no nonsingular solutions in this case. Unlike the previous examples this is a case in which the ordinary Penner model can have “eigenvalue condensation” since both \( r_1 \) and \( r_2 \) are \( < 1 \). In fact we found cases in which “condensation” occurred just at \( \lambda = \eta \) and cases in which is occurred at both \( \lambda = 0 \) and \( \lambda = \eta \). Unfortunately the KM Potential \( V - D\lambda^2 \) has a maximum at \( \lambda = \eta \) and thus eigenvalue condensation cannot occur at this point. The only possible explanation is that the procedure which lead from the Ordinary Penner Model to the KM Penner model does not work when the eigenvalue distribution is singular. This is not surprising, especially in light of the problems with interpreting these as real distributions. We thus conclude that for the case \( 0 < r < 1 \) there are no solutions at all.

In summary we see that we never have eigenvalue condensation in the KM Penner Model and that nonsingular normalized solutions exist for all \( \eta \) when \( r > 1 \) and for \( |r| < |r_c(\eta)| < (\eta - 1/\eta)^2/4 \) when \( r < 0 \) and \( \eta > 1 \). The phase diagram for this model is shown in Figure 1.

There are several interesting critical lines. Along the line \( \eta = 0 \) the potential \( W'(\lambda) = 1/\lambda \). This is an single–pole Penner potential with a critical value of the coupling \( (r = 1) \) but without a linear term. The line \( r = 1 \) (\( \eta > 0 \)) yields a potential \( W'(\lambda) = 1/\lambda + \eta \). This is again a critical single–pole Penner potential but this time with a linear term. The line
$r = 0$ is also a critical Penner potential though centered at $\lambda = \eta$ and with a linear term. Finally there is the critical line $r = r_c(\eta)$ given by Eq. (71) at which the potential in region II become sufficiently deep to admit normalized solutions.

The behavior of physical quantities such as the susceptibility

$$\chi = -\frac{d^2}{dr^2} F(r)$$

where $F$ is the Free Energy near the various critical points or lines is a subject of much interest in Matrix Models. In the case of the KM Penner Model this was studied in great detail by Makeenko [21] who computed the susceptibility as well as the various critical exponents of the model.
V. DISCUSSION

In this paper we have studied in detail the large $N$ solutions to a Kazakov–Migdal Model with two logarithmic singularities

$$V(\phi) = r \log \phi + (2D - 1)(r - 1) \log \phi - \eta + \eta$$  \hspace{1cm} (74)

and the solutions to the related problem of an ordinary Matrix Model with the Double Penner Potential

$$W(\phi) = r_1 \log \phi + r_2 \log \phi - \eta + C$$  \hspace{1cm} (75)

The ordinary Double Penner Model has a rich phase structure which includes regions in parameter space in which there are one–cut solutions and two–cut solutions, regions in which there are no solutions and regions in which the solutions are singular with “eigenvalue condensation” at the poles. The KM Penner Model on the other hand has either one–cut solutions or no solutions at all. Its phase diagram is shown in Fig. 1.

The method employed in this paper, which was first used for the KM Model in Ref. [20], emphasizes the power of the Path Integral in solving this difficult mathematical problem which otherwise would have involved finding the extrema of Actions involving the logarithm of the Itzykson–Zuber determinant in Eq. (4). The techniques of Ref. [20] reduce the problem to solving an ordinary quartic equation.

ACKNOWLEDGEMENTS

We wish to thank Richard Szabo and Gordon Semenoff for their advice and help. This work was supported in part by the Natural Science and Research Council of Canada. Their support is greatly acknowledged. Nathan Weiss wishes to thank the Department of Particle Physics at the Weizmann Institute, where he is presently on leave, for their support.
REFERENCES

[1] V.A. Kazakov and A.A. Migdal, Nucl. Phys. B397 214 (1993)

[2] Harish-Chandra, Amer. J. Math. 79 87 (1957); C.Itzykson and J.B. Zuber, J. Math. Phys. 21 411 (1980); M.L. Mehta, Comm. Math. Phys. 79 327 (1981)

[3] E. Witten, The 1/N Expansion in Atomic and Particle Physics, in ”Recent Developments in Gauge Theories ” edited by t’Hooft et al; Plenum Press New York (1980)

[4] D.J. Gross, Phys. Lett. B293 181 (1992)

[5] A.A. Migdal, Mod. Phys. Lett. A8 359 (1993); A.A. Migdal, Mod Phys. Lett. A8 153 (1993)

[6] I.I.Kogan, A.Morozov, G.W.Semenoff and N.Weiss, Nucl. Phys. B395 547 (1993)

[7] A.Gocksch and Y.Shen, Phys. Rev. Lett. 69 2747 (1992)

[8] S.B.Khokhlachev and Yu.M.Makeenko, Phys. Lett. B297 345 (1992)

[9] M.Caselle, A.D.’Adda and S.Panzeri, Phys. Lett. B302 80 (1993)

[10] A.A.Migdal, Mod. Phys. Lett. A8 139 (1993)

[11] Yu. M. Makeenko, Mod. Phys. Lett. A8 209 (1993)

[12] , S. Khokhlachev and Yu. Makeenko, Mod. Phys. Lett. A7 3653 (1992)

[13] I.I. Kogan, A.Morozov, G.W. Semenoff and N. Weiss, Int. J. Mod. Phys. A8 1411 (1993)

[14] A.A.Migdal, Mod. Phys. Lett. A8 245 (1993)

[15] S.L. Shatashvili, Commun. Math. Phys. 154 421 (1993)

[16] A. Yu. Morozov, Mod. Phys. Lett. A7 3503 (1992)

[17] M. I. Dobroliubov, A. Morzov, G.W. Semenoff, N. Weiss Int. J. Mod. Phys. A9 5033 (1994)
[18] V. A. Kazakov, Zh. Eksp. Teor. Fiz. 85, 1887 (1983) [Sov. Phys. JETP] 58 1096 (1983).

[19] I.I.Kogan, G.W.Semenoff and N.Weiss, Phys. Rev. Lett. 69 3435 (1992)

[20] M.I. Dobroliubov, Yu. Makeenko, G.W. Semenoff, Mod. Phys. Lett. A8 2387 (1993)

[21] Yu. Makeenko, Phys. Lett. B314 197 (1993)

Yu. Makeenko, “Critical Scaling and Continuum Limits in the D> 1 Kazakov–Migdal Model”, HEP-TH-9408029

[22] Mehta, M. L. Random Matrices 2nd ed; Academic Press Inc. (1991)

[23] S. Chauduri, H. Dykstra, H., J. Lykken, Mod. Phys. Lett. A6 1665 (1991)

J. Ambjorn, C. F. Kristjansen, Yu. Makeenko, Phys. Rev. D50 5193 (1994)

[24] Lori Paniak, M.Sc. Thesis, University of British Columbia (1994)

FIGURE CAPTIONS

Figure 1: Phase Diagram for the Kazakov–Migdal Penner Model
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9501037v1
Figure 1