“Slimming” of power law tails by increasing market returns

D. Sornette\textsuperscript{1,2}
\textsuperscript{1} Institute of Geophysics and Planetary Physics
and Department of Earth and Space Science
University of California, Los Angeles, California 90095
\textsuperscript{2} Laboratoire de Physique de la Matière Condensée
CNRS UMR6622 and Université de Nice-Sophia Antipolis
B.P. 71, Parc Valrose, 06108 Nice Cedex 2, France
e-mail: sornette@moho.ess.ucla.edu

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Abstract

We introduce a simple generalization of rational bubble models which removes the fundamental problem discovered by [20] that the distribution of returns is a power law with exponent less than 1, in contradiction with empirical data. The idea is that the price fluctuations associated with bubbles must on average grow with the mean market return $r$. When $r$ is larger than the discount rate $r_\delta$, the distribution of returns of the observable price, sum of the bubble component and of the fundamental price, exhibits an intermediate tail with an exponent which can be larger than 1. This regime $r > r_\delta$ corresponds to a generalization of the rational bubble model in which the fundamental price is no more given by the discounted value of future dividends. We explain how this is possible. Our model predicts that, the higher is the market remuneration $r$ above the discount rate, the larger is the power law exponent and thus the thinner is the tail of the distribution of price returns.
1 The fundamental constraint on distribution of returns of rational bubbles

Since the publication of the original contributions on rational expectations (RE) bubbles by [3] and [4], a huge literature has emerged on theoretical refinements of the original concept and the empirical detectability of RE bubbles in financial data (see [7] and [1], for surveys of this literature).

[4] proposed a model with periodically collapsing bubbles in which the bubble component of the price follows an exponential explosive path (the price being multiplied by $a_t = \bar{a} > 1$) with probability $\pi$ and collapses to zero (the price being multiplied by $a_t = 0$) with probability $1 - \pi$. It is clear that, in this model, a bubble has an exponential distribution of lifetimes with a finite average lifetime $\pi / (1 - \pi)$. Bubbles are thus transient phenomena. In order to allow for the start of new bubble after the collapse, a stochastic zero mean normally distributed component $b_t$ is added to the systematic part of $B_t$.

Recently, [20] studied the implications of the bubble models for the unconditional distribution of prices, price changes and returns resulting from a more general discrete-time formulation of rational speculative bubbles:

$$B_{t+1} = a_t B_t + b_t,$$

(1)

where $a_t$ can take arbitrary values and are i.i.d. random variables drawn from some non-degenerate probability density function (pdf) $P_a(a)$. The model can also be generalized by considering non-normal realizations of $b_t$ with distribution $P_b(b)$ with $E[b_t] = 0$. In addition, the additive term is one way among many (see [26] for alternatives) to nucleate the bubble from a non-zero value. Indeed, most of the time, $a_t < 1$ and $B_t$ fluctuates with a scale set by $b_t$. When an amplification phase starts up, $B_t$ increases exponentially from this initial value and the additive contributions of succeeding $b_t$ becomes subdominant in the exponential growth. In (1), $B_t$ denotes the difference between the observed price and the fundamental price which defines the bubble component. The term “bubble” refers to the regimes when $B_t$ explodes exponentially under the action of successive multiplications by factor $a_t, a_{t+1}, ...$ with a majority of them larger than 1 but different, thus adding an additional stochastic component to the standard model of [4].

Denoting $E[.]$ the expectation operator, provided $E[\ln a] < 0$ (stationarity condition) and if there is a number $\mu$ such that $0 < E[|b|^\mu] < +\infty$, such that

$$E[|a|^\mu] = 1$$

(2)

and such that $E[|a|^\mu \ln |a|] < +\infty$, then the tail of the distribution of $B$ is asymptotically (for large $B$’s) a power law [14, 3]

$$P_B(B) \, dB \approx \frac{C}{|B|^{1+\mu}} \, dB,$$

(3)

with an exponent $\mu$ given by the real positive solution of (3). Rational expectations require in addition that the bubble component in asset prices obeys

$$\delta E[B_{t+1}] = B_t$$

(4)

where $\delta$ is the discount factor $< 1$. This implies

$$E[|a|] = 1/\delta > 1.$$  

(5)

Since the function $E[|a|^\mu]$ is upward convex, [20] showed that this automatically enforce $\mu < 1$. It is easy to show that the distribution of price differences has the same power law tail with the exponent...
$\mu < 1$ and the distribution of returns is dominated by the same power-law over an extended range of large returns [20]. Although power-law tails are a pervasive feature of empirical data, these characterizations are in strong disagreement with the usual empirical estimates which find $\mu$ in the range $3 - 5$ [34, 18, 24, 12, 10]. [20] concluded that exogenous rational bubbles are thus hardly reconcilable with some of the stylized facts of financial data at a very elementary level. This result has been extended to multi-dimensional bubbles involving an arbitrary number of coupled assets [21].

Here, we provide a simple and natural extension of model (1) that removes this constraint $\mu < 1$ and thus the discrepancy with empirical analysis. The key to understanding intuitively how the result of [20] derives is to realize that the rational expectations condition (4) means that the bubble price grows locally exponentially with an instantaneous growth rate equal to the discount rate

$$r_\delta = -\ln \delta > 0.$$ (6)

Since the stochastic auto-regressive equation (1) describes a stationary process when the condition $E[\ln a] < 0$ is satisfied, the only possibility to reconcile the non-stationary exponential growth with the stationary regression is that the distribution has no mean, which indeed occurs only for $\mu < 1$.

In practice for any finite time intervals, the absence of a mean implies that averages grow with the size of the time window. This remark points to the remedy that we now present.

## 2 Generalization and breakdown of the constraint

### 2.1 Rational expectation bubble model

For the following discussion, it is useful to recall that pricing of an asset under rational expectations theory is based on the two following hypothesis: the rationality of the agents and the “no-free lunch” condition.

Under the rational expectation condition, the best estimation of the price $p_{t+1}$ of an asset at time $t+1$ viewed from time $t$ is given by the expectation of $p_{t+1}$ conditioned upon the knowledge of the filtration $\{F_t\}$ (i.e. sum of all available information accumulated) up to time $t$: $E[p_{t+1}|F_t]$.

The “no-free lunch” condition imposes that the expected returns of every assets are all equal under a given probability measure $Q$ equivalent to the historical probability measure $P$. In particular, the expected return of each asset is equal to the return $r_\delta$ of the risk-free asset (which is assumed to exist), and thus the probability measure $Q$ is named the risk neutral probability measure.

Puting together these two conditions, one is led to the following valuation formula for the price $p_t$:

$$p_t = \delta \cdot E_Q[p_{t+1}|F_t] + d_t \quad \forall\{p_t\}_{t \geq 0},$$ (7)

where $d_t$ is an exogeneous “dividend”, and $\delta = (1 + r_\delta)^{-1}$ is the discount factor. The first term in the r.h.s. quantifies the usual fact that something tomorrow is less valuable than today by a factor called the discount factor. Intuitively, the second term, the dividend, is added to express the fact that the expected price tomorrow has to be decreased by the dividend since the value before giving the dividend incorporates it in the pricing.

The “forward” solution of (7) is well-known to be the fundamental price

$$p_t^f = \sum_{i=0}^{+\infty} \delta^i \cdot E_Q[d_{t+i}|F_t].$$ (8)
It is straightforward to check by replacement that the sum of the forward solution (8) and of an arbitrary component \( B_t \)
\[ p_t = p_f^t + B_t , \]
where \( B_t \) has to obey the single condition of being an arbitrary martingale:
\[ B_t = \delta \cdot \mathbb{E}_Q[B_{t+1} | \mathcal{F}_t] \]
is also the a solution of (7). In fact, it can be shown [11] that (8) is the general solution of (7).

Here, it is important to note that, in the framework of the Blanchard and Watson model, the speculative bubbles appear as a natural consequence of the valuation formula (7), i.e., of the no free-lunch condition and the rationnality of the agents. Thus, the concept of bubbles is not an addition to the theory, as sometimes believed, but is entirely embedded in it.

### 2.2 Model and predictions

Consider again \( B_t \) as the price difference between observed price and the fundamental price. We propose the following extension of the model initially defined by (1) which consists in introducing an exponentially growing additive term \( b_t \rightarrow e^{rt} b_t \) with \( r > 0 \) such that the dynamics of \( B_t \) is
\[ B_{t+1} = a_t B_t + e^{rt} b_t . \]
The justification for modifying the additive term into \( e^{rt} b_t \) is the following. As we have seen, the additive term represents the background of “normal” fluctuations around the fundamental price. As we recalled above in the definition of the standard RE model, the additive term ensures that bubbles can nucleate spontaneously. If the economy and the market exhibit a long-term growth rate \( r \), their “normal” fluctuations have also to grow with the same growth rate in order to remain stationary in relative value. The factor \( e^{rt} \) in the additive term \( e^{rt} b_t \) thus reflect nothing but the average exponential growth of the underlying economy. In this framework, for the model to make sense, we need to impose an additional condition \( \mathbb{E}[\ln a] < r \) that the average growth rate \( \mathbb{E}[\ln a] \) of \( B_t \) is smaller than the growth rate \( r \) of the background economy. This condition ensures that most of time \( B_t \) is a “normal” fluctuation with stationary average relative deviations from the fundamental price. Only intermittent exponential amplifications will create transient bubbles that will be significantly above this (exponentially growing) background.

We impose again \( \mathbb{E}[b_t] = 0 \) and \( b_t \) is a white noise process. The rational expectations (4) leads, as before [20], to
\[ \mathbb{E}[a] = \delta^{-1} . \]
We introduce the reduced price variable \( A_t \) such that
\[ B_t = e^{rt} A_t . \]
Equation (11) then reads
\[ A_{t+1} = a_t e^{-r} A_t + e^{-r} b_t , \]
which is of the standard form (1) with stationary coefficients and obeys the usual conditions. Note that the rational expectations condition (12) translates into \( \mathbb{E} [a_t e^{-r}] = \delta^{-1} \) which can now be smaller than 1 (see below).
Note that the condition $E[\ln a] < r$ ensures that $E[\ln(ae^{-r})] < 0$ which is now the stationarity condition for the process $A_t$ defined by (14). The conditions $0 < E[|e^{rt}b_t|^\mu] < +\infty$ (which is the same condition $0 < E[|b_t|^\mu] < +\infty$ as before) and the solution of
\[ E[|ae^{-rt}|^\mu] = 1 \]

together with the constraint $E[|ae^{-rt}|^\mu \ln |ae^{-r}|] < +\infty$ (which the same as $E[|a|^\mu \ln |a|] < +\infty$) leads to an asymptotic power law distribution for the reduced price variable $A_t$ of the form $P_A(A) \approx C_A/|A|^{1+\mu}$, where $\mu$ is the real positive solution of (15). Note that the condition $E[\ln(ae^{-r})] < 0$ which is $E[\ln(a)] < r$ now allows for positive average growth rate of the product $a_1a_{t-1}a_{t-2}...a_2a_1a_0$. 

[24] have demonstrated rigorously that, if $A_t$ is distributed with a power law distribution with exponent $\mu$, then its increments $A_t - A_{t-1}$ are also distributed according to a power law with the same exponent $\mu$. Here, we are interested in the distributions of the returns of the price which we now discuss.

### 2.3 Distribution of returns

In order for the model to be meaningful with respect to returns, we need to recognize that the observable market price is the sum of the bubble component $B_t$ and of a “fundamental” price $p_t^f$

\[ p_t = p_t^f + B_t. \]

Then,
\[ p_{t+1} = p_{t+1}^f + a_tB_t + e^{rt}b_t = a_t p_t + (e^r - a_t)p_t^f + e^{rt}b_t, \]

where we have assumed that the economy and the market exhibit a long-term growth rate $r$, i.e., $p_{t+1} = e^{rt}p_t$, leading to
\[ p_t^f = p_0e^{rt}. \]

Expression (17) allows us to make concrete the meaning of the additive term $e^{rt}b_t$ previously introduced in (14). Indeed, we pointed out that the additive term represents the background of “normal” fluctuations around the fundamental price (18). Their “normal” fluctuations have also to grow with the same growth rate in order to remain stationary in relative value. This is shown by replacing $p_t^f$ in (14) by $p_0e^{rt}$ given in (18), which leads to
\[ p_t = e^{rt}(p_0 + A_t). \]

In addition, replacing $p_t^f$ in (17) again by $p_0e^{rt}$ leads to
\[ p_{t+1} = a_t p_t + e^{rt}[p_0(e^r - a_t) + b_t]. \]

The expression (20) has the same form as (14) with a different additive term $[p_0(e^r - a_t) + b_t]$ replacing $b_t$. The structure of this new additive term makes clear the origin of the factor $e^{rt}$ introduced in (14): as we said, it reflects nothing but the average exponential growth of the underlying economy. The contributions $e^{rt}b_t$ are then nothing but the fluctuations around this average growth.

This novel additive term brings in a new feature which turns out to be essential to describe realistic return time series: it has a non-fluctuating component which ensures that $p_t$ grows on the average exponentially with time. For positive bubbles $A_t > 0$ or for large fundamental price $p_0 > |A_t|$, a return $\ln(p_{t+1}/p_t)$ can not be made artificially large by the approach of $p_t$ arbitrarily close to 0. In contrast, the pure bubble component $B_t$ can cross 0 an arbitrary number of times, as
shown in figure 1. Close to these crossings, the returns \([B_{t+1} - B_t]/B_t\) of the bubble component are very large since the denominator is close to 0. Since such approach of \(B_t\) to 0 occurs with a uniform probability in the vicinity of 0, the distribution of \(1/B_t\) is determined by the rule of change of probability density under a change of variable and is found to be a power law with exponent equal to 1: this is one of the possible mechanisms known to generate power law distributions known as the “power law change of variable close to the origin” \([5]\) (see Chap. 14). In the context of price returns, this is an artifact as the observable price contains the additional contribution of the fundamental price and therefore cannot go close to zero.

The observable return is

\[
R_t = \frac{p_{t+1} - p_t}{p_t} = \frac{p^f_{t+1} - p^f_t + B_{t+1} - B_t}{p^f_t + B_t} = \chi_t \left( \frac{p^f_{t+1} - p^f_t}{p^f_t} + \frac{B_{t+1} - B_t}{p^f_t} \right) = \chi_t \left( r + \frac{A_{t+1} - A_t}{p_0} \right),
\]

where

\[
\chi_t = \frac{p^f_t}{p^f_t + B_t} = \frac{1}{1 + (A_t/p_0)}.
\]

In order to derive the last equality in the right-hand-side of (21), we have used the definition of the return of the fundamental price (neglecting the small second order difference between \(e^r - 1\) and \(r\)) and we have combined (13) and (18). Expression (21) shows that the distribution of returns \(R_t\) of the observable prices is the same as that of the product of the random variable \(\chi_t\) by \(r + (A_{t+1} - A_t)/p_0\). Now, the tail of the distribution of \(r + (A_{t+1} - A_t)/p_0\) is the same as the tail of the distribution of \(A_{t+1} - A_t\), which is a power law with exponent \(\mu\) solution of (13), as shown rigorously by [20].

It remains to show that the product of this variable \(r + (A_{t+1} - A_t)/p_0\) by \(\chi\) has the same tail behavior as \(r + (A_{t+1} - A_t)/p_0\) itself. If \(r + (A_{t+1} - A_t)/p_0\) and \(\chi\) were independent, this would follow from results in [5] who demonstrates that for two independent random variables \(\phi\) and \(\chi\) with \(\text{Proba}(|\phi| > x) \approx cx^{-k}\) and \(E[\chi^\epsilon + \epsilon] < \infty\) for some \(\epsilon > 0\), the random product \(\phi \chi\) obeys \(\text{Proba}(\phi \chi > x) \approx E[\chi]^\epsilon x^{-k}\).

\(r + (A_{t+1} - A_t)/p_0\) and \(\chi\) are not independent as both contain a contribution from the same term \(A_t\). However, when \(A_t < p_0\), \(\chi\) is close to 1 and the previous result should hold. The impact of \(A_t\) in \(\chi\) becomes important when \(A_t\) becomes comparable to \(p_0\).

It then convenient to rewrite (21) using (22) as

\[
R_t = \frac{r}{1 + (A_t/p_0)} + \frac{A_{t+1} - A_t}{p_0 + A_t} = \frac{r}{1 + (A_t/p_0)} + \frac{(a_t e^{-r} - 1)A_t + e^{-r} b_t}{p_0 + A_t}.
\]

We can thus distinguish two regimes:

- for not too large values of the reduced bubble term \(A_t\), specifically for \(A_t < p_0\), the denominator \(p_0 + A_t\) changes more slowly than the numerator of the second term, so that the distribution of returns will be dominated by the variations of this numerator \((a_t e^{-r} - 1)A_t + e^{-r} b_t\) and, hence, will follow approximately the same power-law as for \(A_t\), according to the results of [5].

- For large bubbles, \(A_t\) of the order of or greater than \(p_0\), the situation changes, however: from (23), we see that when the reduced bubble term \(A_t\) increases without bound, the first term \(r/(1 + (A_t/p_0))\) goes to 0 while the second term becomes asymptotically \(a_t e^{-r} - 1\). This leads to the existence of an absolute upper bound for the absolute value of the returns given by

\[
\max|R_t| = \max\{|\min(a_t e^{-r} - 1)|, |\max(a_t e^{-r} - 1)|\}.
\]

(24)
To summarize, we expect that the distribution of returns will therefore follow a power-law with the same exponent $\mu$ as for $A_t$, but with a finite cut-off given in equation (24).

Consider the illustrative case where the multiplicative factors $a_t$ are distributed according to a log-normal distribution such that $\mathbb{E}[\ln a] = \ln a_0$ (where $a_0$ is thus the most probable value taken by $a_t$) and of variance $\sigma^2$. Then,

$$\mathbb{E}[|ae^{-r}|^\mu] = \exp \left[ -r\mu + \mu \ln a_0 + \frac{\mu^2 \sigma^2}{2} \right].$$

Equating (25) to 1 to get $\mu$ according to equation (15) gives

$$\mu = \frac{2 \frac{r - \ln a_0}{\sigma^2}}{\frac{r - \ln a_0}{r_\delta - \ln a_0} + 1}.$$  

We have used the notation (6) for the discount rate defined in terms of the discount factor. The second equality in (26) results from (12) using $\mathbb{E}[a] = a_0 e^{\sigma^2/2}$.

First, we retrieve the result (20) that $\mu < 1$ for the initial RE model (1) for which $r = 0$ and $\ln a_0 < 0$. However, as soon as $r > r_\delta = -\ln \delta$, we get

$$\mu > 1,$$

and $\mu$ can take arbitrary values. Technically, this results fundamentally from the structure of (11) in which the additive noise grows exponentially to mimick the growth of the bubble which alleviates the bound $\mu < 1$. Note that $r$ does not need to be large for the result (27) to hold. Take for instance an annualized discount rate $r_\delta^y = 2\%$, an annualized return $r^y = 4\%$ and $a_0 = 1.01$. Expression (26) predicts $\mu = 3$, which is compatible with empirical data.

### 2.4 Implications of the condition $r > r_\delta$

When the price fluctuations associated with bubbles grow on average with the mean market return $r$, we find that the exponent of the power law tail of the returns is no more bounded by 1 as soon as $r$ is larger than the discount rate $r_\delta$ and can take essentially arbitrary values. As can be seen from equation (8), this condition $r > r_\delta$ corresponds to the unsolved regime in fundamental valuation theory where the forward valuation solution (8) loses its meaning, as discussed recently in [28]. Indeed, changing $d_{t+i}$ into $d_{t+i} e^{ry}$ to account for the growth of dividend associated with the growth of the fundamental price, the sum in (8) then behaves as $\sum_{i=0}^{+\infty} [\delta e^r]^i$ which diverges for $\delta e^r \geq 1$, i.e., $r > r_\delta$ (neglecting the difference between $e^r$ and $1 + r$).

Two attitudes can be taken with respect to the condition $r > r_\delta$, entailing two different modeling strategies.

#### 2.4.1 The “hard-line” attitude

The “hard-line” attitude (summarized from private exchanges with T. Lux) can be described as follows. The assumption $r > r_\delta$ corresponds to abandon the background of the rational bubble theory which is the rational valuation formula which fails. Once this starting point is abandoned, there is no theory of rational bubbles anymore as the transition from (8) to (9) with (10) is no more defined. The question of whether the results of (20) are saved or changed then becomes irrelevant. In other words, giving up the fundamental valuation formula, the argument goes, would mean that we give up “rationality” (in the very limited and special sense of the rational valuation
formula and similar theories in economics). We would then find ourselves within the realm of not-fully-rational behavior which implies that we have already given up also the possibility of having “rational speculative bubbles”. This does not preclude any kind of near-rational bubbles like those in the agents-based models (see for instance [19, 23, 13, 17, 31]) but the strictly rational bubbles cease to exist. To sum up the hard-line attitude, the theory of RE bubbles is of no interest anymore when one gives up the hypothesis of rational valuation because there is then readily available a universe of possible alternative bubble theories.

2.4.2 The “intrinsic RE bubble” model

The other attitude advocated here and elsewhere [14, 15, 32, 33] decouples the rational valuation formula from the RE bubble model, which we term the “intrinsic RE bubble” model. It is true that, in the context of rational pricing based on the flow of dividend with infinite time horizon, an exponential growth of the price is associated with the same exponential growth of the dividend. This leads to an inconsistency as the valuation formula leads to a diverging price if the growth rate is larger than the discount rate. As discussed in [28], the price is of course not infinite, only the rational valuation formula is of no use to determine it. The price has to be defined by processes other than the dividend valuation formula.

- From analogies with statistical physics and with the theory of bifurcations and their normal forms, in which similar situations occurs, I proposed in [28] a scenario according to which this regime may be associated with a spontaneous symmetry breaking phase corresponding to a spontaneous valuation in absence of dividends.

- Another approach is to realize that the fundamental valuation formula is a statement of equilibrium. It is thus natural to interpret the regime \( r > r_d \) where the fundamental valuation formula fails as intrinsically dynamical: the fundamental price becomes determined by dynamical processes other than just the flow of dividends.

- It is possible to develop models of general equilibrium [2] in which one can treat the growth of the economy reflected in the stock market as the accumulation of composite capital, plus a change of available technology that affects both the old and new capital. Capital’s market value grows as people add new units of capital, or as existing units grow in value as they become more effective. If \( K_t \) is the composite capital or market value at time \( t \), \( r_t \) is the return and \( c_t \) is consumption, then \( K_{t+1} = r_t K_t - c_t \). In the long run, consumption seems to grow at the same rate as wealth, income and other measures of available resources. Thus \( c_t \) is roughly proportional to \( K_t \), i.e., \( c_t = \gamma_t K_t \) with \( \gamma_t < r_t \) as people in general consume less that the expected return on composite capital, leading to expected growth. In this framework [3], dividends are seen as depending on current and past prices rather than prices as depending on expected future dividends as in the standard valuation formula. It thus is perfectly possible to model a market growing at an average rate \( r \) equal to the average of \( r_t - \gamma_t \) which can be larger than the risk-free rate or discounting rate or inflation rate. This does not invalidate the assumption of rational expectations, which simply states that people make economic decisions in a way that tends to take into account all available information bearing significantly on the future consequences of their decisions. We can still form rational expectations from an intelligent appraisal of circumstances, though the process behind such circumstances may be hard to discern. An economic system doing the most efficient possible job of reading the information being reflected in price signals will still experience some
irreducible business cycle swings and bubbles. This results from the fact that the economic process contains inherent mechanisms that convert random shocks on prices into a more persistent, short-term misreading of changing profit opportunities, in other words the rational interpretation of noise can stimulate a cumulative swing in output that will continue until the misreading is realized and retrenchment sets in. Random shocks to prices and markets are always with us and rational bubbles are thus naturally occurring market phases. Burke [6] has also established, by dropping continuity and weakening utility representation, that commodity prices and consumptions can approach approximate equilibrium to within any practical tolerance after dropping the standard general-equilibrium assumption that preference orders discount future consumption faster than the economy grows.

• Another fix is based on the understanding that the divergence of the sum in (8) stems from the fact that the investor assumes an infinite time horizon; in practice, we would discount future cash flows only up to a maximum time, say ten or thirty years at most. As a consequence, the rational valuation formula has to be truncated to a finite number of terms and provides a finite price.

Therefore, the apparent contradiction is resolved by abandoning the pricing based on dividends discounted over an infinite time horizon and replacing it by an exogeneous dynamical process. In other words, we assume here that the fundamental price is growing exponentially, without reference to dividends. We thus propose not to “throw away the baby with the bath”, i.e., to decouple the fundamental pricing mechanism from the RE bubbles. Indeed, investors use various methods for estimating the fundamental price and the rational valuation formula is only one among an arsenal of techniques that, in real life situations, are usually combined.

The point of view advocated here is that it is reasonable to keep the RE bubble model [10], independently of what determines the specific underlying fundamental price. This is actually the point of view initially introduced by [3] and further developed in [14, 15, 33], where the bubble price may follow essentially arbitrary trajectories as long as there is a jump process (crash) with a hazard rate ensuring the no-arbitrage condition.

In a way, the choice between the “hard-line” attitude and the “intrinsic RE bubble” model is almost a matter of philosophy with respect to the meaning of models. Ultimately, a model is useful if it helps thinking about the problem and if it allows one to formulate new questions. It seems interesting to use the modelling strategy in which the RE condition is relaxed partially by abandoning only the rational valuation formula. Following this viewpoint, we have the following logical sequence:

1. RE bubbles give an exponent $\mu > 1$ for the distribution of returns only when the fundamental price is growing faster than the discount rate;

2. when the fundamental price is growing faster than the discount rate, the rational valuation formula breaks down;

3. empirical studies show that $\mu > 1$ in real financial time series;

4. as a consequence, since the distribution of returns are usually constructed over rather long time histories, it is sufficient that the fundamental valuation departs from the dividend formula at some times to explain item 3.

This provides a parsimonious model for the empirical distribution of returns.
2.5 Numerical simulations

We now present numerical simulations that illustrate these results. We do not pretend to capture reality accurately but show nevertheless that the stylized facts of empirical data are recovered by this simple model. In order to make a precise comparison with empirical data, one should specify the possible shapes of the distributions $P_a(t)$ and $P_b(t)$ and one should also add the fundamental price to the bubble price $B_t$ since only the sum is observable. Tests of the model would thus be a joined test of the relevance of the RE model together with an assumption on the dynamics of the fundamental price.

Figure 1 shows a typical trajectory of the price $B_t$ of a bubble generated by equation (11) with $r = 0.33\%$. This choice is such that one time step can be approximately interpreted as one month, since the compounded return over 12 time steps gives a realistic yearly return of $4\% (1.0033^{12} = 1.04)$, as can be seen in figure 2.

The multiplicative factors $a_t$ are generated from the formula

$$a_t = a_0 \exp[\sigma \eta_t], \quad \text{where} \quad a_0 = 1.001 \quad \text{and} \quad \sigma = 0.0374,$$

and the $\eta_t$’s are centered Gaussian random variables with unit variance. With this parameterization, $a_t$ are log-normal random variables of variance $\sigma^2$ with $E[\ln a] = \ln a_0 = 0.001$ and $E[a] = a_0 e^{\sigma^2/2} = 1/\delta = 1.0017$. Therefore, $\delta = 0.9983$ and $r_\delta$ defined by (6) is $r_\delta = 0.17\%$. With the values $r = 0.33\%, r_\delta = 0.17\%$ and $\ln a_0 = 0.1\%$, expression (26) predicts an exponent $\mu = 3.3$.

The additive term $b_t$ is taken uniformly distributed in the interval $[-0.05; +0.05]$. Note that $b_t$ sets the scale of $B(t)$. The bubble price $B_t$ stays on average at 0, as seen from (11) which predicts $E[B_t] = 0$. However, like a correlated random walk with fat tails, it wanders around, with an appearance quite reminiscent of active and volatile markets.

As discussed in the previous section, in order to make a meaningful comparison with empirical data, it is necessary to add a fundamental component to the bubble to form an observable price. To remain as simple as possible, we follow the specification (18) with (18), with $p_0 = 1$. Figure 2 shows the time dynamics of $B_t + e^{rt}$, where $B_t$ is the same as shown in figure 1 and we have assumed that the fundamental price follows a deterministic growth at the “annualized” rate of $4\%$ (corresponding to the monthly rate $r = 0.33\%$) according to (18). The trajectory of the Dow Jones Industrial Average (DJIA) extrapolated back from 1790 till sept. 2000 is also shown as the thin line. The Dow Jones index was constructed by The Foundation for the Study of Cycles (http://www.cycles.org/cycles.htm). It is striking to observe how the simple RE bubble model added to a simple exponentially growing fundamental price can capture the large scale variability of the DJIA.

Figure 3 shows a double logarithmic scale representation of the complementary cumulative distribution of the “monthly” returns $R_t$ defined in (27), constructed from the synthetic time series shown in figure 2. As predicted, the distribution is well-described by the asymptotic power law with an exponent in agreement with the prediction $\mu \approx 3.3$ given by the equations (15) and (26) and shown as the straight line. Note that the expected cut-off at the maximum return is not observed as the returns have not yet explored the large values of the order of 1 in the finite-time series.

For comparison, figure 4 shows the distribution of positive and negative returns of the Dow Jones Industrial Average price over the 20th century. The tails are very well quantified by power laws with exponents respectively equal to $\mu_+ = 2.9 \pm 0.3$ and $\mu_- = 2.4 \pm 0.3$. The error bars are estimated from the theory of maximum likelihood applied to the Hill estimator which predict $\delta \mu/\mu = 1/\sqrt{N}$, where $N$ is the number of data in the power law tail. Using $N \approx 100$, we find $\delta \mu \approx 0.3$. The data does not reject the hypothesis that $\mu_+ = \mu_- \approx 2.7 \pm 0.3$. These values are compatible
with those previously reported in the literature on smaller time scales \[54, 18, 24, 12, 10\]. The comparison between figures 3 and 4 suggests that the RE bubble model \([11]\) goes a long way into capturing the structure of real market prices.

3 Concluding remarks

In the numerical example given in the previous section, the stationary process \(A_t\) with zero mean and finite variance may in principle eventually reach the value \(-p_0\) with very low but not zero probability, at which the observable price vanishes. Such large negative bubbles will thus produce artifacts in the distribution of returns which, as we demonstrated for the returns of \(B_t\), are controlled by the excursions of \(p_t\) close to 0. The general formulation of the RE bubbles allows in principle for both positive and negative bubbles with the sign of each new bubble depending on that of the additive stochastic term in its inception period. However, it is well known that there are conceptual problems with negative bubbles \([8]\). Therefore, we deliberately confined ourselves to bubbles of amplitudes smaller than the fundamental price. Numerical tests using only positive bubbles, obtained by imposing a strictly positive additive term \(b_t\), confirm the results of figures 2 and 3. We thus emphasize that none of our results hinges on this slight modification of the model and both modeling strategies are equally viable: consider only the bubbles that do not become too negative so that the total observable price does not become too small or construct only positive bubbles. This conclusion has also been checked extensively by \([24]\).

While investors enjoy getting a larger return \(r\), this comes usually at the price of increasing risks. Here, the situation is different because a higher market return leads to a thinner tail of the return distribution since the exponent \(\mu\), for instance given by \([26]\), increases with \(r\). Hence, increasing the average market return \(r\) decreases the extreme risks. In this context, it is interesting to bring into focus the long-standing paradox that the Dow Jones average index has been argued to exhibit an anomalously large return, averaging 6% per year over the 1889-1978 period \([22]\), which cannot be explained by any reasonable risk aversion coefficient. What our study shows is that the excess remuneration \(r - r_{\delta}\) has an unexpected “good” consequence in decreasing drastically the large risks of the market by increasing the exponent of the asymptotic power law distribution. Following recent ideas in the theory of complex systems (see for instance \([30]\) and references therein), we argue that the market has self-organized such that the excess remuneration has reached a level which compensate for the huge risks associated with the intermittent bubbles created by investors. We conjecture that the presently observed value of the exponent \(\mu\) in the range 3 – 5 \([54, 18, 24, 12, 11]\) and the anomalous returns of the market over long period of times are the tools that the system has developed to tame the fat tails that bubbles tend to create. Extending the present model to derive a dynamics on \(r\) would allow us to understand how this organization may proceed.

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Figure 1: Bubble price $B_t$ generated by equation (11) with $r = 0.33\%$. The multiplicative factors $a_t$ are generated from the formula (28). $a_t$ are thus log-normal random variables with $E[\ln a] = \ln a_0 = 0.1\%$ and $E[a] = 1.0017$. The additive term $b_t$ is uniformly distributed in the interval $[-0.05; +0.05]$. With these parameters, one time step can be interpreted as approximately one month. The time interval shown covers thus approximately 80 model-years.
Figure 2: The thick line shows the time dynamics of $B_t + p_t^f$ (bubble + fundamental price), where we have assumed that the fundamental price $p_t^f = e^{rt}$ follows a deterministic growth at the “annualized” rate of 4% corresponding to the value $r = 0.33\%$. The trajectory of the Dow Jones Industrial Average (DJIA) extrapolated back from 1790 till 2000 is also shown as the thin line.
Figure 3: Double logarithmic scale representation of the complementary cumulative distribution of the “monthly” returns $R_t$ defined in (21) of the synthetic total price shown in figures 2. The continuous (resp. dashed) line corresponds to the positive (resp. negative) returns. The distribution is well-described by an asymptotic power law with an exponent in agreement with the prediction $\mu \approx 3.3$ given by the equations (15) and (26) and shown as the straight line. The small differences between the predicted slope and the numerically generated ones are within the error bar of $\pm 0.3$ obtained from a standard maximum likelihood Hill estimation.
Figure 4: Double logarithmic scale representation of the complementary cumulative distribution of the monthly returns $\ln p_{t+1}/p_t$ of Dow Jones Industrial Average price $p_t$ from Jan. 1900 till Sept. 2000. The +’s (resp empty circles) correspond to positive (resp. negative) returns. The two dashes straight lines give the best fits to a power law for the largest 100 positive and negative monthly returns. For positive (resp. negative) returns, we find an exponent $\mu_+ = 2.9 \pm 0.3$ (resp. $\mu_- = 2.4 \pm 0.3$). This is in agreement with previous empirical works [34, 18, 24, 12, 10].