Perfect Cuboid and Congruent Number Equation Solutions

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Abstract

A perfect cuboid (PC) is a rectangular parallelepiped with rational sides $a$, $b$, $c$ whose face diagonals $d_{ab}$, $d_{bc}$, $d_{ac}$ and space (body) diagonal $d_s$ are rationals. The existence or otherwise of PC is a problem known since at least the time of Leonhard Euler. This research establishes equivalent conditions of PC by nontrivial rational solutions $(X, Y)$ and $(Z, W)$ of congruent number equation

$$y^2 = x^3 - N^2x,$$

where product $XZ$ is a square. By using such pair of solutions five parametrizations of nearly-perfect cuboid (NPC) (only one face diagonal is irrational) and five equivalent conditions for PC were found. Each parametrization gives all possible NPC. For example, by using one of them – invariant parametrization for sides and diagonals of NPC are obtained:

$$a = 2XZN, \quad b = |YW|, \quad c = |X - Z|\sqrt{XZ}N,$$
$$d_{bc} = |XZ - N^2|\sqrt{XZ}, \quad d_{ac} = |X + Z|\sqrt{XZ}N,$$
$$d_s = (XZ + N^2)\sqrt{XZ};$$

and condition of the existence of PC is the rationality of
Because each parametrization is complete, inverse problem is discussed. For given NPC is found corresponding congruent number equation (i.e. congruent number) and its solutions.

**Keywords.** Perfect cuboid, congruent number equation, nearly-perfect cuboid, congruent curve, rational cuboid, rational parametrization, complete parametrization.

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### 1 Introduction

Perfect cuboid (PC) problem is equivalent to the system of Diophantine equations:

\[
\begin{align*}
    a^2 + b^2 &= d_{ab}^2, \\
    b^2 + c^2 &= d_{bc}^2, \\
    a^2 + c^2 &= d_{ac}^2, \\
    a^2 + b^2 + c^2 &= d_s^2; 
\end{align*}
\]

integer (rational) solution existence. If we remove the integer (rational) condition for space diagonal, then we get Euler cuboid (EC). If among the seven:

\[ a, \ b, \ c, \ d_{ab}, \ d_{bc}, \ d_{ac}, \ d_s \]

only one face diagonal or side is irrational, then this is called nearly-perfect cuboid (NPC). Numerous EC and NPC have been found by the help of computers [1–7]. As for the search for a PC, the computer programs of
many researchers in many countries have been unsuccessful. Among the recent research with the help of computers it was proved that there exists no PC, the smallest side of which is less than

$$2, 1 \cdot 10^{10}.$$  

Eventually, having recorded such large number, we come to a hypothesis that there exists no PC, but for now, this hypothesis has not been proved right yet.

2 Perfect Cuboid First Equation

Rewrite the system (I) of PC as follows

$$a^2 + d_{bc}^2 = d_s^2,$$

$$b^2 + d_{ac}^2 = d_s^2,$$

$$c^2 + d_{ab}^2 = d_s^2.$$  

Divide each equation by $d_s^2$:

$$\left( \frac{a}{d_s} \right)^2 + \left( \frac{d_{bc}}{d_s} \right)^2 = 1,$$

$$\left( \frac{b}{d_s} \right)^2 + \left( \frac{d_{ac}}{d_s} \right)^2 = 1,$$

$$\left( \frac{c}{d_s} \right)^2 + \left( \frac{d_{ab}}{d_s} \right)^2 = 1.$$
Use rational parametrization formulae for unit circle
\[ x^2 + y^2 = 1, \]
its positive rational solutions are:
\[ x = \left| \frac{1 - t^2}{1 + t^2} \right|, \quad y = \left| \frac{2t}{1 + t^2} \right|, \]
where \( t \) is arbitrary nontrivial rational number
\[ t \in \mathbb{Q} \setminus \{0; \pm 1\}. \]
Take \( \alpha_1, \beta_1, \gamma_1 \) for parametrization variables, then we obtain a system:
\[
\begin{align*}
\frac{a}{d_s} &= \left| \frac{2\alpha_1}{1 + \alpha_1^2} \right|, & \frac{d_{bc}}{d_s} &= \left| \frac{1 - \alpha_1^2}{1 + \alpha_1^2} \right|, \\
\frac{b}{d_s} &= \left| \frac{1 - \beta_1^2}{1 + \beta_1^2} \right|, & \frac{d_{ac}}{d_s} &= \left| \frac{2\beta_1}{1 + \beta_1^2} \right|, & \tag{2} \\
\frac{c}{d_s} &= \left| \frac{2\gamma_1}{1 + \gamma_1^2} \right|, & \frac{d_{ab}}{d_s} &= \left| \frac{1 - \gamma_1^2}{1 + \gamma_1^2} \right|. 
\end{align*}
\]
Insert these expressions into the system (1):
\[
\begin{align*}
\left( \frac{2\alpha_1}{1 + \alpha_1^2} \right)^2 + \left( \frac{1 - \beta_1^2}{1 + \beta_1^2} \right)^2 &= \left( \frac{1 - \gamma_1^2}{1 + \gamma_1^2} \right)^2, \\
\left( \frac{1 - \beta_1^2}{1 + \beta_1^2} \right)^2 + \left( \frac{2\gamma_1}{1 + \gamma_1^2} \right)^2 &= \left( \frac{1 - \alpha_1^2}{1 + \alpha_1^2} \right)^2, \\
\left( \frac{2\alpha_1}{1 + \alpha_1^2} \right)^2 + \left( \frac{2\gamma_1}{1 + \gamma_1^2} \right)^2 &= \left( \frac{2\beta_1}{1 + \beta_1^2} \right)^2, \\
\left( \frac{2\alpha_1}{1 + \alpha_1^2} \right)^2 + \left( \frac{2\gamma_1}{1 + \gamma_1^2} \right)^2 &= \left( \frac{2\beta_1}{1 + \beta_1^2} \right)^2.
\end{align*}
\]
\[
\left(\frac{2\alpha_1}{1 + \alpha_1^2}\right)^2 + \left(\frac{1 - \beta_1^2}{1 + \beta_1^2}\right)^2 + \left(\frac{2\gamma_1}{1 + \gamma_1^2}\right)^2 = 1.
\]

By using elementary properties, each equation of a system is reduced to third equation. So,

**Theorem 1.** The existence of PC is equivalent to the existence of non-trivial rational solution of equation:

\[
\left(\frac{2\alpha_1}{1 + \alpha_1^2}\right)^2 + \left(\frac{2\gamma_1}{1 + \gamma_1^2}\right)^2 = \left(\frac{2\beta_1}{1 + \beta_1^2}\right)^2.
\]

### 3 Perfect Cuboid Second Equation

Rewrite the system (1) of PC as follows

\[
\begin{align*}
\frac{d_s^2}{a^2} - \frac{d_{bc}^2}{a^2} &= a^2, \\
\frac{d_{ac}^2}{a^2} - c^2 &= a^2, \\
\frac{d_{ab}^2}{a^2} - b^2 &= a^2.
\end{align*}
\]

Divide each equation by \(a^2\):

\[
\begin{align*}
\left(\frac{d_s}{a}\right)^2 - \left(\frac{d_{bc}}{a}\right)^2 &= 1, \\
\left(\frac{d_{ac}}{a}\right)^2 - \left(\frac{c}{a}\right)^2 &= 1, \\
\left(\frac{d_{ab}}{a}\right)^2 - \left(\frac{b}{a}\right)^2 &= 1.
\end{align*}
\]
Use rational parametrization formulae for unit hyperbola
\[ x^2 - y^2 = 1, \]
its positive rational solutions are:
\[ x = \left| \frac{1 + t^2}{1 - t^2} \right|, \quad y = \left| \frac{2t}{1 - t^2} \right|, \]
where \( t \) is arbitrary nontrivial rational number.
Take \( \alpha_2, \beta_2 \) and \( \gamma_2 \) for parametrization variables, then we obtain the system:
\[
\begin{align*}
\frac{d_s}{a} &= \left| \frac{1 + \alpha_2^2}{1 - \alpha_2^2} \right|, \quad \frac{d_{bc}}{a} = \left| \frac{2\alpha_2}{1 - \alpha_2^2} \right|, \\
\frac{d_{ac}}{a} &= \left| \frac{1 + \beta_2^2}{1 - \beta_2^2} \right|, \quad \frac{c}{a} = \left| \frac{2\beta_2}{1 - \beta_2^2} \right|, \\
\frac{d_{ab}}{a} &= \left| \frac{1 + \gamma_2^2}{1 - \gamma_2^2} \right|, \quad \frac{b}{a} = \left| \frac{2\gamma_2}{1 - \gamma_2^2} \right|. 
\end{align*}
\]
(3)
Insert these expressions into system (1):
\[
\begin{align*}
1 + \left( \frac{2 \gamma_2}{1 - \gamma_2^2} \right)^2 &= \left( \frac{1 + \gamma_2^2}{1 - \gamma_2^2} \right)^2, \\
\left( \frac{2 \gamma_2}{1 - \gamma_2^2} \right)^2 + \left( \frac{2 \beta_2}{1 - \beta_2^2} \right)^2 &= \left( \frac{2 \alpha_2}{1 - \alpha_2^2} \right)^2, \\
1 + \left( \frac{2 \beta_2}{1 - \beta_2^2} \right)^2 &= \left( \frac{1 + \beta_2^2}{1 - \beta_2^2} \right)^2, \\
1 + \left( \frac{2 \gamma_2}{1 - \gamma_2^2} \right)^2 + \left( \frac{2 \beta_2}{1 - \beta_2^2} \right)^2 &= \left( \frac{1 + \alpha_2^2}{1 - \alpha_2^2} \right)^2.
\end{align*}
\]
The first and the third equations are identities, whereas the second and fourth are equivalent. So,

**Theorem 2.** The existence of PC is equivalent to the existence of non-trivial rational solution of equation:

\[
\left( \frac{2\gamma_2}{1 - \gamma_2^2} \right)^2 + \left( \frac{2\beta_2}{1 - \beta_2^2} \right)^2 = \left( \frac{2\alpha_2}{1 - \alpha_2^2} \right)^2.
\]

4 **Perfect Cuboid Third Equation**

Discuss second type rational parametrization for unit hyperbola

\[x^2 - y^2 = 1,
\]

its positive rational solutions are:

\[x = \left| \frac{1 + t^2}{2t} \right|, \quad y = \left| \frac{1 - t^2}{2t} \right|,
\]

where \(t\) is arbitrary nontrivial rational number.

Take \(\alpha_3, \beta_3\) and \(\gamma_3\), for parametrization variables, then we obtain the system:

\[
\begin{align*}
\frac{d_s}{a} &= \left| \frac{1 + \alpha_3^2}{2\alpha_3} \right|, & \frac{d_{bc}}{a} &= \left| \frac{1 - \alpha_3^2}{2\alpha_3} \right|, \\
\frac{d_{ac}}{a} &= \left| \frac{1 + \beta_3^2}{2\beta_3} \right|, & c &= \left| \frac{1 - \beta_3^2}{2\beta_3} \right|, \\
\frac{d_{ab}}{a} &= \left| \frac{1 + \gamma_3^2}{2\gamma_3} \right|, & b &= \left| \frac{1 - \gamma_3^2}{2\gamma_3} \right|.
\end{align*}
\]
Insert these expressions into the system (i):

\[
1 + \left(\frac{1 - \gamma_3^2}{2\gamma_3}\right)^2 = \left(\frac{1 + \gamma_3^2}{2\gamma_3}\right)^2,
\]
\[
\left(\frac{1 - \gamma_3^2}{2\gamma_3}\right)^2 + \left(\frac{1 - \beta_3^2}{2\beta_3}\right)^2 = \left(\frac{1 - \alpha_3^2}{2\alpha_3}\right)^2,
\]
\[
1 + \left(\frac{1 - \beta_3^2}{2\beta_3}\right)^2 = \left(\frac{1 + \beta_3^2}{2\beta_3}\right)^2,
\]
\[
1 + \left(\frac{1 - \gamma_3^2}{2\gamma_3}\right)^2 + \left(\frac{1 - \beta_3^2}{2\beta_3}\right)^2 = \left(\frac{1 + \alpha_3^2}{2\alpha_3}\right)^2.
\]

The first and the third equations are identities, whereas the second and fourth are equivalent. So,

**Theorem 3.** The existence of PC is equivalent to the existence of non-trivial rational solution of the equation:

\[
\left(\frac{1 - \gamma_3^2}{2\gamma_3}\right)^2 + \left(\frac{1 - \beta_3^2}{2\beta_3}\right)^2 = \left(\frac{1 - \alpha_3^2}{2\alpha_3}\right)^2.
\]

Theorem 3 and Theorem 2 equations are birationally equivalent over \(\mathbb{Q} \setminus \{0; \pm 1\}\).

Equivalency is given by rational transformation

\[
\gamma_2 = \frac{1 - \gamma_3}{1 + \gamma_3}, \quad \beta_2 = \frac{1 - \beta_3}{1 + \beta_3}, \quad \alpha_2 = \frac{1 - \alpha_3}{1 + \alpha_3}.
\]
5 Congruent Number Equation Solutions Properties

By using solutions of congruent number equation (congruent curve)

\[ C_N : \quad y^2 = x^3 - N^2x \]

it is possible to construct rational right triangles with area \( N \) and 3-term arithmetical progressions of squares with common difference \( N \) \[9, 10\]. In both cases there is one-to-one correspondence between above mentioned sets and only points of congruent curve which are obtained by drawn tangent line through some points.

The usage of solutions of congruent number equation is not limited to the mentioned two cases. This research found the third usage.

By two solutions of congruent number equation every NPC is constructed and PC existence equivalency condition is found. It is impossible to choose arbitrary pair of solutions, they must satisfy the following condition – the product of solutions must be a square.

First of all prove that arbitrary congruent number equation has such infinitely many solutions.

Denote the addition operation of rational points of \( C_N \) elliptic curve (congruent curve is a case of elliptic curve) by \( \oplus \) symbol, then the point which are obtained by drawn tangent line through \( P(X, Y) \) is

\[ P \oplus P = 2P, \]

then the first coordinate of \( 2P \) is

\[ [2P]_x = \left( \frac{X^2 + N^2}{2Y} \right)^2. \]

Meaning that by drawing tangent line all first coordinates for followings

\[ 2P \oplus 2P = 4P, \quad 3P \oplus 3P = 6P, \quad 4P \oplus 4P = 8P, \ldots \]
are squares.

By drawing a secant line through $P$ and $2P$ points we obtain

$$P \oplus 2P = 3P,$$

the first coordinates of which satisfy:

$$[P]_x \cdot [2P]_x \cdot [3P]_x = d^2,$$

where $d$ is a $y$-intercept of secant line

$$d = \frac{X \cdot [2P]_y - Y \cdot [2P]_x}{X - [2P]_x}.$$

Meaning that the product $[P]_x \cdot [3P]_x$ is a square.

By the consequent use of the reasoning we obtain

**Property 1.** For arbitrary point $P$ of congruent curve $C_N$, the product

$$[kP]_x \cdot [mP]_x$$

is a square, if $k$ and $m$ numbers have the same parity.

So, congruent number equation has infinitely many rational pair solutions, the product of which is a square.

By using the solutions $X$ and $Z$ of congruent curves we can find all nontrivial rational solutions for special type of Kummer’s surface $[11]$.

**Property 2.** All nontrivial rational solutions of the equation

$$\eta^2 = \xi \zeta (\xi^2 - 1)(\zeta^2 - 1),$$
are obtained by formulae

\[(\xi, \zeta, \eta) = \left( \frac{X}{N}, \frac{Z}{N}, \frac{YW}{N^3} \right),\]

where \((X, Y)\) and \((Z, W)\) are arbitrary nontrivial different rational solutions of arbitrary \(C_N\) congruent number equation.

Nontrivial solutions of congruent equation are the solutions with \(y \neq 0\) and Kummer's surface equation nontrivial solutions are the solutions with \(\eta \neq 0, \xi \neq \zeta\).

The points obtained by drawing a secant line through point \(P(X, Y)\) of congruent curve \(C_N\) and trivial \((0, 0)\) and \((N, 0)\) points are called first and second reflected points

\[(X, Y) \rightarrow \left( -\frac{N^2}{X}, -\frac{N^2Y}{X^2} \right),\]

\[(X, Y) \rightarrow \left( \frac{N(X + N)}{X - N}, \frac{2N^2Y}{(X - N)^2} \right).\]

By drawing a secant line through the third trivial point \((-N, 0)\), we obtain point

\[\left( \frac{N(N - X)}{X + N}, \frac{2N^2Y}{(X + N)^2} \right),\]

which is the result of the composition of the first and second reflected transformations.

**Property 3.** If \(X\) and \(Z\) are two solutions of congruent number equation the product of which is a square, then the product of the first reflected solutions and second reflected solutions
\[( - \frac{N^2}{X}) \cdot ( - \frac{N^2}{Z}) \quad \text{and} \quad \frac{N(X + N)}{X - N} \cdot \frac{N(Z + N)}{Z - N}\]

are also squares.

Indeed,

\[
\frac{N(X + N)}{X - N} \cdot \frac{N(Z + N)}{Z - N} = \frac{N^2Y^2W^2}{(X - N)^2(Z - N)^2XZ}.
\]

6 First and its Reflected Parametrizations of NPC. PC Conditions

From the Theorem 1

\[
\left( \frac{\gamma_1}{1 + \gamma_1^2} \right)^2 = \frac{\alpha_1^2\left( 1 - \left( \frac{\alpha_1}{\beta_1} \right)^2 \right)(1 - (\alpha_1\beta_1)^2)}{\left( \frac{\alpha_1}{\beta_1} \right)^2(1 + \alpha_1^2)^2(1 + \beta_1^2)^2}.
\]

Denoted the numerator by \( \eta_1^2 \)

\[
\eta_1^2 = \left( \frac{\alpha_1}{\beta_1} \right)(\alpha_1\beta_1)\left( 1 - \left( \frac{\alpha_1}{\beta_1} \right)^2 \right)(1 - (\alpha_1\beta_1)^2),
\]

and using Property 2:

\[
\alpha_1\beta_1 = \frac{X_1}{N_1}, \quad \frac{\alpha_1}{\beta_1} = \frac{Z_1}{N_1}.
\]
So

\[ \alpha_1 = \frac{\sqrt{X_1Z_1}}{N_1}, \quad \beta_1 = \frac{\sqrt{X_1}}{Z_1}, \quad \eta_1 = \frac{Y_1W_1}{N_1^3}, \]

where \((X_1, Y_1)\) and \((Z_1, W_1)\) are arbitrary nontrivial different solutions of arbitrary \(C_{N_1}\) congruent number equation. By using Property 1 the solutions \(X_1\) and \(Z_1\), the product (ratio) of which is a square, are infinitely many. For these solutions the equation (5) is:

\[ \gamma_1 + \gamma_1^2 = \frac{Y_1W_1}{(X_1Z_1 + N_1^2)(X_1 + Z_1)}. \]

If in the last expression \(\gamma_1\) is rational, we obtain PC.

Insert the given expressions for variables \(\alpha_1\), \(\beta_1\) and \(\gamma_1\) in system [2]. We obtain the first parametrization formulae for sides and diagonals.

**Theorem 4.** Complete parametrization of NPC is given by formulae:

\[
\begin{align*}
a &= \frac{2N_1\sqrt{X_1Z_1}}{X_1Z_1 + N_1^2} \cdot d_s, \\
b &= \frac{|X_1 - Z_1|}{X_1 + Z_1} \cdot d_s, \\
c &= \frac{2Y_1W_1}{(X_1Z_1 + N_1^2)(X_1 + Z_1)} \cdot d_s, \\
d_{ac} &= \frac{2\sqrt{X_1Z_1}}{|X_1 + Z_1|} \cdot d_s, \\
d_{bc} &= \frac{|X_1Z_1 - N_1^2|}{X_1Z_1 + N_1^2} \cdot d_s,
\end{align*}
\]
where \((X_1, Y_1)\) and \((Z_1, W_1)\) are arbitrary nontrivial different rational solutions of arbitrary \(C_{N_1}\) congruent number equation, the product \(X_1Z_1\) is a square.

Condition of the existence of PC is the rationality of

\[
d_{ab} = \sqrt{1 - \left( \frac{2Y_1 W_1}{(X_1 Z_1 + N_1^2)(X_1 + Z_1)} \right)^2} \cdot d_s,
\]

Based on Property 3 solutions \((X_1, Y_1)\) and \((Z_1, W_1)\) can be replaced by the first and second reflected points. Obtained formulae do not change by the first reflected transformation, while they are changed by second reflected transformation. As a result we obtain reflected parametrization.

**Corollary 1.** Complete parametrization of NPC is given by formulae:

\[
a' = \frac{|Y_1 W_1|}{(X_1 Z_1 + N_1^2)\sqrt{X_1 Z_1}} \cdot d_s',
\]

\[
b' = \frac{|N_1(Z_1 - X_1)|}{X_1 Z_1 - N_1^2} \cdot d_s',
\]

\[
c' = \frac{2N_1 Y_1 W_1}{X_1^2 Z_1^2 - N_1^4} \cdot d_s',
\]

\[
d_{ac}' = \frac{Y_1 W_1}{(X_1 Z_1 - N_1^2)\sqrt{X_1 Z_1}} \cdot d_s',
\]

\[
d_{bc}' = \frac{|N_1(X_1 + Z_1)|}{X_1 Z_1 + N_1^2} \cdot d_s',
\]

where \((X_1, Y_1)\) and \((Z_1, W_1)\) are arbitrary nontrivial different rational solutions of arbitrary \(C_{N_1}\) congruent number equation, the product \(X_1Z_1\) is a square.

Condition of the existence of PC is the rationality of
\[ d_{ab}' = \sqrt{1 - \left( \frac{2N_1Y_1W_1}{X_1^2Z_1^2 - N_1^4} \right)^2 \cdot d_s'} . \]

7 Second and its Reflected Parametrizations of NPC. PC Conditions

From the Theorem 2

\[
\left( \frac{\gamma_2}{1 - \gamma_2^2} \right)^2 = \frac{\beta_2^2 \left( 1 - \left( \frac{\beta_2}{\alpha_2} \right)^2 \right) \left( 1 - (\alpha_2 \beta_2)^2 \right)}{\frac{\beta_2^2}{\alpha_2^2} \left( 1 - \alpha_2^2 \right)^2 \left( 1 - \beta_2^2 \right)^2} .
\]

(6)

Denote the numerator by \( \eta_2^2 \)

\[
\eta_2^2 = \left( \frac{\beta_2}{\alpha_2} \right) (\beta_2 \alpha_2) \left( 1 - \left( \frac{\beta_2}{\alpha_2} \right)^2 \right) \left( 1 - (\alpha_2 \beta_2)^2 \right) ,
\]

and using Property 2

\[
\frac{\beta_2}{\alpha_2} = \frac{X_2}{N_2} , \quad \beta_2 \alpha_2 = \frac{Z_2}{N_2}
\]

where \((X_2, Y_2)\) and \((Z_2, W_2)\) are arbitrary nontrivial different solutions of arbitrary \( \mathcal{C}_{N_2} \) congruent number equation. So,

\[
\alpha_2 = \sqrt{\frac{Z_2}{X_2}} , \quad \beta_2 = \frac{\sqrt{X_2Z_2}}{N_2} , \quad \eta_2 = \frac{Y_2W_2}{N_2^3} .
\]
Based on Property 1, solutions the product (ratio) of which is a square are infinitely many. For such solutions the equation (6) is
\[
\frac{\gamma_2}{1 - \gamma_2^2} = \frac{Y_2W_2}{(X_2 - Z_2)(N_2^2 - X_2Z_2)}.
\]
If \(\gamma_2\) is rational, then we obtain PC.
Insert the obtained parametrization variables \(\alpha_2, \beta_2\) and \(\gamma_2\) into system \((3)\) we obtain third parametrization formulae for sides and diagonals.

**Theorem 5.** Complete parametrization of NPC is given by formulae:

\[
b = \left| \frac{2Y_2W_2}{(X_2 - Z_2)(N_2^2 - X_2Z_2)} \right| \cdot a,
\]
\[
c = \frac{2N_2\sqrt{X_2Z_2}}{|N_2^2 - X_2Z_2|} \cdot a,
\]
\[
d_{bc} = \frac{2\sqrt{X_2Z_2}}{|X_2 - Z_2|} \cdot a,
\]
\[
d_{ac} = \frac{N_2^2 + X_2Z_2}{|N_2^2 - X_2Z_2|} \cdot a,
\]
\[
d_s = \frac{X_2 + Z_2}{|X_2 - Z_2|} \cdot a,
\]

where \((X_2, Y_2)\) and \((Z_2, W_2)\) are arbitrary nontrivial different rational solutions of arbitrary \(C_{N_2}\) congruent number equation, the product \(X_2Z_2\) is a square.

Condition of the existence of PC is the rationality of
\[
d_{ab} = \sqrt{1 + \left( \frac{2Y_2W_2}{(X_2 - Z_2)(N_2^2 - X_2Z_2)} \right)^2} \cdot a.
\]
Obtained parametrization is invariable under the first reflected transformation but the second reflected transformation gives new parametrization.

**Corollary 2.** Complete parametrization of NPC is given by formulae:

\[
\begin{align*}
    b' &= \left| \frac{2Y_2W_2}{N_2(Z_2^2 - X_2^2)} \right| \cdot a', \\
    c' &= \left| \frac{Y_2W_2}{N_2(X_2 + Z_2)\sqrt{X_2Z_2}} \right| \cdot a', \\
    d_{bc}' &= \left| \frac{Y_2W_2}{N_2(Z_2 - X_2)\sqrt{X_2Z_2}} \right| \cdot a', \\
    d_{ac}' &= \left| \frac{X_2Z_2 + N_2^2}{N_2(X_2 + Z_2)} \right| \cdot a', \\
    d_s' &= \left| \frac{X_2Z_2 - N_2^2}{N_2(Z_2 - X_2)} \right| \cdot a',
\end{align*}
\]

where \((X_2, Y_2)\) and \((Z_2, W_2)\) are arbitrary nontrivial different rational solutions of arbitrary \(C_{N_2}\) congruent number equation, the product \(X_2Z_2\) is a square.

Condition of the existence of PC is the rationality of

\[
d_{ab}' = \sqrt{1 + \left( \frac{2Y_2W_2}{N_2(Z_2^2 - X_2^2)} \right)^2} \cdot a'.
\]

### 8 Invariant Parametrization of NPC. PC Invariant Condition

From the Theorem 3
\[
\left(1 - \frac{\gamma_3^2}{\gamma_3}\right)^2 = \frac{\left(\frac{\alpha_3}{\beta_3}\right)^2(\alpha_3\beta_3)(1 - \left(\frac{\alpha_3}{\beta_3}\right)^2)}{\alpha_3^4} \left(1 - (\alpha_3\beta_3)^2\right)
\]

Denote the numerator by \(\eta_3^2\)

\[
\eta_3^2 = \left(\frac{\alpha_3}{\beta_3}\right)^2(\alpha_3\beta_3)(1 - \left(\frac{\alpha_3}{\beta_3}\right)^2) \left(1 - (\alpha_3\beta_3)^2\right)
\]

By Property 2

\[
\frac{\alpha_3}{\beta_3} = \frac{X}{N}, \quad \alpha_3\beta_3 = \frac{Z}{N},
\]

where \((X, Y)\) and \((Z, W)\) are arbitrary nontrivial different solutions of arbitrary \(C_N\) congruent number equation. Because solutions the product (ratio) of which is a square are infinitely many (Property 1), for these solutions

\[
\alpha_3 = \frac{\sqrt{XZ}}{N}, \quad \beta_3 = \frac{Z}{X}, \quad \eta_3 = \frac{YW}{N^3},
\]

also from (7)

\[
\frac{1 - \gamma_3^2}{\gamma_3} = \frac{YW}{XZN}.
\]

System (4) gives the fifth parametrization formulae of sides and diagonals of NPC.

**Theorem 6.** Complete parametrization of NPC is given by formulae:
\[ b = \frac{|YW|}{2XZN} \cdot a, \]
\[ c = \frac{|X - Z|}{2\sqrt{XZ}} \cdot a, \]
\[ d_{bc} = \frac{|N^2 - XZ|}{2N\sqrt{XZ}} \cdot a, \]
\[ d_{ac} = \frac{|X + Z|}{2\sqrt{XZ}} \cdot a, \]
\[ d_s = \frac{N^2 + XZ}{2N\sqrt{XZ}} \cdot a, \]

where \((X, Y)\) and \((Z, W)\) are arbitrary nontrivial different rational solutions of arbitrary \(C_N\) congruent number equation, the product \(XZ\) is a square.

Condition of the existence of PC is the rationality of
\[ d_{ab} = \sqrt{1 + \left(\frac{YW}{2N\sqrt{XZ}}\right)^2} \cdot a. \]

As preceding parametrizations the last one is invariable under the first reflected transformation. Though second reflected transformation gives

\[ a' = \frac{|YW|}{2XZN} \cdot b', \]
\[ c' = \frac{|X - Z|}{2\sqrt{XZ}} \cdot b'. \]
So, the second reflected transformation interchanges sides $a$ and $b$, though side $c$ is left unchanged. Due to this, we call the last parametrization an invariant parametrization. Invariant parametrization is the most convenient for construction of concrete NPC.

Take $N = 5$ and following solutions $[12]$:

\[
X = \frac{25}{2^2}, \quad Y = \frac{75}{2^3}, \quad Z = \frac{1681}{12^2}, \quad W = \frac{62279}{12^3},
\]

By the first parametrization and its reflection:

\[
\begin{align*}
a &= 5079408, & b &= 1762717, & c &= 2242044, \\
d_{ac} &= 5552220, & d_{bc} &= 2852005, & d_s &= 5825317;
\end{align*}
\]

\[
\begin{align*}
a' &= 5035485, & b' &= 7050868, & c' &= 8968176, \\
d'_{ac} &= 10285149, & d'_{bc} &= 11408020, & d'_s &= 12469925.
\end{align*}
\]

By the second parametrization and its reflection:

\[
\begin{align*}
a &= 863005, & b &= 2242044, & c &= 1537008, \\
d_{ac} &= 1762717, & d_{bc} &= 2718300, & d_s &= 2852005;
\end{align*}
\]

\[
\begin{align*}
a' &= 8063044, & b' &= 11210220, & c' &= 3559017,
\end{align*}
\]
$d_{ac}' = 8813585, \quad d_{bc}' = 11761617, \quad d_s' = 14260025.$

By invariant parametrization:

$$a = 9840, \quad b = 4557, \quad c = 3124,$$

$$d_{ac} = 10324, \quad d_{bc} = 5525, \quad d_s = 11285.$$

There have been attempts to connect the solutions of congruent number equation with NPC earlier, we should mention [8, 11, 13, 14]. These researches show the existence of such relationship, but due to the complexity of their formulae this relationship is not expressed explicitly. That is why NPC parametrizations and PC existence equivalency conditions have not been produced. Elliptic curves associated with PC are considered in [15, 16].

9 Finding Congruent Number Equation and its Solutions by Given NPC

As shown in preceeding part, using invariant parametrization NPC is constructed by each of four pairs of solutions of congruent number equation. Place solution pairs in square brackets.

I. \[X; Z] \rightarrow (a, b, c),

II. \[-\frac{N^2}{X}; -\frac{N^2}{Z}] \rightarrow (a, b, c),

III. \[N \frac{X+N}{X-N}; N \frac{Z+N}{Z-N}] \rightarrow (b, a, c),
The aim of this part is the consideration of inverse problem. For given NPC, find corresponding congruent number equation (i.e. congruent number $N$) and its four solution pairs.

From invariant parametrization (Theorem 6)

$$\frac{c}{a} = \frac{|X - Z|}{2\sqrt{XZ}},$$
$$\frac{d_s}{a} = \frac{N^2 + XZ}{2N\sqrt{XZ}}.$$

By solving the system:

$$\frac{X}{Z} = \left(\frac{d_{ac} \pm c}{a}\right)^2, \quad \frac{XZ}{N^2} = \left(\frac{d_s \pm d_{bc}}{a}\right)^2;$$
$$\frac{X}{N} = \pm \frac{(d_{ac} \pm c)(d_s \pm d_{bc})}{a^2},$$
$$\frac{Z}{N} = \pm \frac{d_s \pm d_{bc}}{d_{ac} \pm c}.$$

In total there are eight solution pairs, only four of them satisfy congruent number equation solution conditions:

$$-1 < \frac{X}{N} < 0 \text{ or } \frac{X}{N} > 1.$$

Two more solution pairs are obtained by interchange of $X$ and $Z$ so, finally there are only two pairs of solutions connected by the first reflected transformation:
I. \[
\frac{X}{N} = \frac{(d_{ac} + c)(d_s + d_{bc})}{a^2}; \quad \frac{Z}{N} = \frac{d_s + d_{bc}}{d_{ac} + c}.
\]

II. \[
- \frac{N}{X} = -\frac{(d_{ac} - c)(d_s - d_{bc})}{a^2}; \quad -\frac{N}{Z} = -\frac{d_s - d_{bc}}{d_{ac} - c}.
\]

Third and fourth pairs are obtained by replacing \(a \leftrightarrow b\). By solving the system:

\[
c = \frac{|X - Z|}{2\sqrt{XZ}},
\]

\[
d_s = \frac{N^2 + XZ}{2N\sqrt{XZ}};
\]

obtain:

III. \[
\frac{X + N}{X - N} = \frac{d_s + d_{ac}}{d_{bc} + c}; \quad \frac{Z + N}{Z - N} = \frac{(d_{bc} + c)(d_s + d_{ac})}{b^2}.
\]

IV. \[
\frac{N - X}{N + X} = -\frac{d_s - d_{ac}}{d_{bc} - c}; \quad \frac{N - Z}{N + Z} = -\frac{(d_{bc} - c)(d_s - d_{ac})}{b^2}.
\]

Again III and IV pairs are connected by the first reflected transformation when, I \(\leftrightarrow\) III, and II \(\leftrightarrow\) IV are connected by the second reflected transformation.

To find congruent number \(N\), consider squares

\[
X(X^2 - N^2) \quad \text{and} \quad Z(Z^2 - N^2)
\]

and following is a square as well:

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\[
N^3 \cdot \frac{X}{N} \left( \left( \frac{X}{N} \right)^2 - 1 \right) \longrightarrow N \cdot \frac{X}{N} \left( \left( \frac{X}{N} \right)^2 - 1 \right),
\]
\[
N \cdot \frac{Z}{N} \left( \left( \frac{Z}{N} \right)^2 - 1 \right).
\]

By these conditions, congruent number \( N \) is obtained.

Discuss concrete numerical case for NPC:

\[
a = 672, \quad c = 104, \quad b = 153,
\]
\[
d_{ac} = 680, \quad d_{bc} = 185,
\]
\[
d_s = 697.
\]

Receive:

\[
\frac{X}{N} = \frac{49}{32} \quad \text{and} \quad \frac{Z}{N} = \frac{9}{8}.
\]

\( N \) is found by removing squares

\[
N \cdot \frac{49}{32} \left( \left( \frac{49}{32} \right)^2 - 1 \right) \longrightarrow N \cdot \frac{17}{2},
\]

because congruent number \( N \) is squarefree:

\[
N = 34.
\]

The same \( N \) is obtained by second solution. The third and fourth pairs of solutions are obtained from:
\[
\frac{X + N}{X - N} = \frac{81}{17} \quad \text{and} \quad \frac{Z + N}{Z - N} = 17.
\]

Summarizing the results we conclude: NPC with sides

\[672, 104, 153\]

is obtained by each of four pairs of solutions:

\[
\frac{833}{16} \quad \text{and} \quad \frac{153}{4},
\]

\[
-\frac{1088}{49} \quad \text{and} \quad -\frac{272}{9},
\]

\[162 \quad \text{and} \quad 578,
\]

\[-\frac{578}{81} \quad \text{and} \quad -2;
\]

of following congruent number equation

\[y^2 = x(x^2 - 34^2)\]

by invariant parametrization.

The first and second parametrizations give other congruent number equations and corresponding pairs of solutions. For given NPC:

by first parametrization \(N_1 = 4305\)

\[[X_1; Z_1] = \left[\frac{452025}{64}; \frac{18081}{4}\right],\]
\[
\left[ -\frac{N_1^2}{X_1}; -\frac{N_1^2}{Z_1}\right] = \left[ -2624; -4100\right].
\]

By second parametrization \(N_2 = 1717170\)

\[
[X_2; Z_2] = [165191754; 3016650],
\]

\[
\left[ -\frac{N_2^2}{X_2}; -\frac{N_2^2}{Z_2}\right] = \left[ -17850; -977466\right].
\]

Given examples show once again that invariant parametrization is the most convinient for calculation of concrete numerical cases.

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