Unstable states in a model of nonrelativistic quantum electrodynamics: rate of decay, regeneration by decay products, sojourn time and irreversibility

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Abstract

We consider a family of states in a model of nonrelativistic quantum electrodynamics, with positive Hamiltonian $H$. For a given initial state $\Psi$, the return probability amplitude $R_\Psi(t) = (\Psi, \exp(-iHt)\Psi)$ may be written for positive times, as the sum of an exponentially decaying term and a correction $O(\frac{1}{t})$, for large times $t$ and small coupling constants (Theorem 4.1). The correction term is seen to be related both to the positivity of $H$ and to the existence of the virtual process of regeneration of the decaying state from the decay products, which is shown to be essentially quantum field theoretic, i.e., not present in nonrelativistic Schroedinger quantum mechanics. Some implications of this fact are analysed from the point of view of a general picture of irreversibility and the "arrow of time" in quantum field theory. Finally, we make a first application of a time-energy uncertainty theorem to a quantum field theoretic model, in order to find a lower bound to the energy fluctuation in the state $\Psi$ (Theorem 5.2). In the process, it is also suggested that the time of sojourn $\tau_H(\Psi) = \int_0^\infty |R_\Psi(t)|^2 dt$ is the most natural quantity to consider in connection with the decay of unstable atoms or particles: it is proved to coincide with the average lifetime of the decaying state, a standard quantity in quantum

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1 Introduction, motivation and synopsis. The model

1.1 Introduction and motivation

The problem of unstable states in quantum (field) theory has its origin in Gamow’s early treatment of alpha decay ([Gam28], see also [BH]). Its crucial importance to physics is due to two related facts: all atomic states - except for the ground state - are resonances, and, in elementary particle physics, all but the lightest particles are unstable. In the former case, we have to do with a bound state problem of an atom in the presence of the electromagnetic field, in the latter case the particles are not bound. The two situations present radically different problems: we summarize some of the (open) problems in the particle case in the conclusion.

The first treatment of unstable (decaying) states of atoms in interaction with the electromagnetic field was proposed by Weisskopf in his thesis, of which a lively account is given in [Wei74]. The ensuing paper by Weisskopf and Wigner [WW30] is the first paper where a divergent integral appeared! The assumptions made by Weisskopf and Wigner were carefully analysed and criticized by Davidovich and Nussenzveig ([DN80], see also Davidovich’s Ph.D. thesis [Dav]). A review of their work, with several improvements, was published by Nussenzveig in 1984 [Bar84]. We refer to [DN80] for further references on the previous literature on the subject.

Davidovich and Nussenzveig were primarily concerned with providing a theory of natural line shape of certain atomic levels, e.g., those concerned by the Lamb shift in the hydrogen atom [Sak67]. Their approach may be summarized as follows: to identify, initially, in the full Hamiltonian of interaction between the atoms and the electromagnetic field, a model for decaying states which incorporates as many realistic features as possible, while remaining exactly soluble. The omitted terms from the full Hamiltonian would then be dealt with by a suitable perturbation theory. When specialized to $N = 2$ atomic levels, their model coincides with the spin-Boson model in the rotating-wave approximation (but infinite number of photon modes), whose spectrum was fully analysed by Huebner and Spohn [HS95] in 1995. This is the model we shall revisit.

In this review we intend to clarify several points in their discussion, partly in view of a rigorous result due to Christopher King [Kin91], who revisited
this model in 1991. It is, however, important to mention that the problem of atomic resonances in nonrelativistic quantum electrodynamics has been treated at great length in an important series of papers by Bach, Froehlich and Sigal (see [BF100] and references given there, and [GS06] for a textbook account). They introduce, however, the electromagnetic vector potential field with an ultraviolet cutoff. Our model, in spite of several rather drastic approximations, has no ultraviolet cutoff. In addition, as in [Kin91], we do not adopt their concept of resonance, related to complex energies. The use of complex energies and frequencies, which is not a priori physically motivated, leads to pathologies, such as the well-known "exponential catastrophe" in both classical and quantum physics (see [Bar84] and section 3.1, and it seems therefore conceptually of great advantage to avoid them, as we do in this paper.

In the framework of the theory of complex dilatations in potential theory, a unified picture of resonances and exponential decay laws was given by W. Hunziker [Hun90]. He did not, however, establish the $O(|t|^{-3/2})$ behavior expected for the correction term in potential theory (see remark 2.1). The corresponding $O(|t|^{-1})$ term in the present model will be seen to be of crucial conceptual importance.

We now provide a brief synopsis of this review.

In section 2 we incorporate the problem of unstable states in the general framework of normalized, positive linear functionals over an algebra of observables [Sew86]. This is very explicit in the present model due to the fact that the Hamiltonian $H$ is well defined as a positive, self-adjoint operator on Fock space, but has a number of conceptual advantages. The first one (Theorem 2.1) is that we are able to introduce in a natural way a whole family of time-dependent ("decaying") states which tend asymptotically in time to a different family of states (the "decay product states"). For a given observable, we obtain the so-called return probability of the decaying state

$$|R_\Psi(t)|^2 = |(\Psi, \exp(-iHt)\Psi)|^2$$

where $\Psi$ is a specific (normalized to one) vector in the Fock space of atoms and field, and $(\cdot, \cdot)$ denotes the scalar product in this space. The corresponding amplitude $R_\Psi(t)$ is a basic quantity. The spectral measure of $H$ is absolutely continuous (see, e.g., [BB03] or [MW13]), and thus

$$R_\Psi(t) = \int_0^\infty d\lambda \exp(-i\lambda t)g_\Psi(\lambda)$$

(1)
for some locally (Lebesgue) integrable function $g_{\Psi}$.

Theorem 2.2 is essentially a basic result of Sinha [Sin72], relating positivity of the Hamiltonian and the rate of decay: the return probability amplitude cannot be a pure exponential, but must be corrected by a term $c(t) = o(t)$. Particularly important is theorem 2.5, which shows that this correction $c(t)$ is a purely quantum mechanical virtual process, the regeneration of the decaying state from the decay products. The proof of the theorem is not at all new: it is due to Sinha, Williams, and Fonda, Ghirardi and Rimini ([Sin72], [Wil71], [FGR78]). These important results remain somewhat forgotten, mostly due to the fact that the authors had, at the time, a somewhat unclear notion in mind, that of an ”unstable wave function” and a ”unstable particle Hilbert space”. Our sole contribution in this section was to show that their ideas and methods remain applicable within the usual formalism of quantum mechanics, at least in the present model. Even more important, however, was that Theorem 2.5 illuminates the physical interpretation of the correction term, which could in fact be shown to be of intrinsically field theoretic nature. This follows from the fact that the behaviour of $c(t)$ for large times is \textbf{not} determined by the free Hamiltonian, as is the case in potential theory ([Bar84], [FGR78]), but rather solely by the interaction Hamiltonian (Remark 2.1).

Section 3 is divided into three parts. In subsection 3.1, we briefly describe the method of ”decay without analyticity”, which we follow in this paper, and was initiated by King [Kin91].

Subsection 3.2 is devoted to the problem of irreversibility and the ”arrow of time” in the present model and in quantum field theory in general, following the lines of a recent paper [Wre], as well as [NW14].

Finally, we provide in section 4 the proof of the main theorem, Theorem 4.1, which states that the correction $c(t)$ is $O(\beta^2 t)$ for sufficiently small $\beta$ and large $t$. King [Kin91] made only minimal assumptions on the dipole-moment matrix element functions and obtained only the $O(\beta^2)$ part, although the Riemann-Lebesgue lemma implies that it is $o(t)$ (Theorem 2.1). We use the exact dipole-moment functions, which have been calculated by Nussenzveig in terms of hypergeometric functions [Bar84]. Our method of proof follows King [Kin91] and consists of comparing $g_{\Psi}(\lambda)$ in (1) with the Lorentzian or Breit-Wigner function

$$g_{\Psi}^L(\lambda) = \frac{\Gamma}{2\pi[(\lambda - \lambda_0)^2 + \frac{\Gamma^2}{4}]}$$

(2)
If we insert (2) into (1) and replace the integral from zero to infinity by one from $-\infty$ to $\infty$, we obtain

$$R_u^\Psi(t) = \exp(-i\lambda_0 t - \frac{\Gamma t}{2})$$

where the superscript $u$ stands for "unbounded", i.e., (3) corresponds to a non-semibounded Hamiltonian, for which the spectrum extends to $-\infty$. (3) results from writing (2) as a sum of two pole contributions, and further applying Cauchy’s theorem along a contour along the real line, closed by a large semi-circle in the lower half plane, the latter’s contribution vanishing if $t > 0$. This is done by King [Kin91], who proceeds from this point to estimate the remainder. We use (1) directly, with the splitting $g_\Psi = g^L_\Psi + (g_\Psi - g^L_\Psi)$: the integral (1) with $g_\Psi$ replaced by $g^L_\Psi$ is evaluated along a contour following the positive real line, a quarter circle at infinity in the lower half plane and coming back along the negative imaginary axis. The latter’s contribution yields a correction $c(t) = O(\frac{1}{t})$ to the residue at the pole (which coincides with the r.h.s. of (3)). The contribution of the remainder $g_\Psi - g^L_\Psi$ is shown to yield a correction of the same type. This is the content of theorem 4.1, some details of which are left to appendix B. Our result for the correction to the exponential behavior disagrees with Nussenzveig’s [Bar84], which is $O(\frac{1}{t^2})$. Remarks 4.1 and 4.2 summarize the reasons of our disagreements with [Kin91] and [Bar84].

Finally, we make (what we believe to be) a first application of a time-energy uncertainty theorem (theorem 3.17 of [MW13], reproduced with some slight corrections and improvements in appendix A) to a quantum field theoretic model, in order to find a lower bound to the energy fluctuation in the state $\Psi$ (Theorem 5.2). The significance of this theorem is better appreciated by observing that this fluctuation equals

$$\int_0^\infty d\lambda \lambda^2 |g_\Psi(\lambda)|^2$$

but the same quantity, evaluated for the Lorentzian $g^L_\Psi$, is infinite. In the process, it is also suggested that the time of sojourn $\tau_H(\Psi) = \int_0^\infty |R_\Psi(t)|^2 dt$ is the most natural quantity to consider in connection with the decay of unstable atoms or particles: it is proved to coincide with the the average lifetime of the decaying state, a standard quantity in quantum probability.

Section 6 is a conclusion, which also summarizes the open problems which remain to be treated in the analogous model for elementary particles [AMKG57].
As in [Kin91], no use is made of complex energies associated to analytic continuations of the resolvent operator to “unphysical” Riemann sheets. Thermal states are not treated in this review.

1.2 The model

As mentioned in the previous section, in our account, we shall consider a prototypical model for the Lyman $\alpha$ transition in hydrogen: this will imply no qualitative restriction regarding the final results. We follow [Bar84] and choose his units $\hbar = c = 1$; this still allows to set a unit of length, which is chosen as the Bohr radius $a_B = (me^2)^{-1} = (\alpha m)^{-1} = 1$, from which

$$\beta = \frac{e}{m} = \alpha^{3/2}$$

with

$$e = \alpha^{1/2}$$

Above, $e, m$ denote charge and mass of the electron, and $\alpha$ the fine-structure constant, approximately equal to $\frac{1}{137}$. The ground state energy is

$$E_{01} = -\frac{\alpha}{2}$$

and the resonant level (e.g., one of the two Lamb-shifted levels, degenerate in the Dirac theory [Sak67]) will have the energy $E_{0r}$; we denote

$$E_0 = E_{0r} - E_{01}$$

The model considered in ([Bar84, DNS80, Dav]), when specialized to $N = 2$ atomic levels, amounts to the spin-Boson model in the rotating-wave approximation ([HS95], section 6, pg. 317), which we write

$$H = H_0 + H_I$$

with

$$H_0 = E_0 \frac{1 + \sigma_z}{2} + 1 \otimes \int d^3k |k|a^\dagger(k)a(k)$$

and

$$H_I = \beta[\sigma_- \otimes a^\dagger(g) + \sigma_+ \otimes a(g)]$$
The operators act on the Hilbert space

\[ \mathcal{H} \equiv \mathcal{C}^2 \otimes \mathcal{F} \]  

where \( \mathcal{F} \) denotes symmetric (Boson) Fock space on \( L^2(\mathbb{R}^3) \) (see, e.g., [MR04]), which describes the photons. We shall denote by \( \langle \cdot, \cdot \rangle \) the scalar product in \( \mathcal{H} \). Formally, \( a(g) \equiv \int d^3k g(k) a(k) \), and \( k \) denotes a three-dimensional vector. The \( \dagger \) denotes adjoint, \( \sigma_{\pm} = \frac{\sigma_x \pm \sigma_y}{2} \), and \( \sigma_{x,y,z} \) are the usual Pauli matrices. \( E_0 \) is given by (8) in the concrete case of the Lyman \( \alpha \) transition, and

\[ g(k) = g(|k|) = \sqrt{|k|} f(|k|) \]  

where

\[ f(k) = (|k|^2 + a^2)^{-2} \]  

with

\[ a = \frac{3}{2} \]  

with the choice of units (5), (6): the above functions \( f \) are special dipole-moment matrix-element functions for hydrogen, which may be computed explicitly in terms of hypergeometric functions ([Bar84], (8.21)). As mentioned, we take the above example as a prototype: consideration of the other cases in [Bar84] bring no qualitative alterations in the forthcoming results. The operator

\[ N = \frac{1 + \sigma_z}{2} + \int d^3k a^\dagger(k) a(k) \]  

commutes with \( H \). We write

\[ N = \sum_{l=0}^{\infty} l P_l \]  

and introduce the notation

\[ H_l \equiv P_l H P_l \]  

\( H_l \) is the restriction of \( H \) to the subspace \( P_l \mathcal{H} \). The subspace \( P_0 \mathcal{H} \) is one-dimensional and consists of the ground state vector

\[ \Phi_0 \equiv |\rightarrow\rangle \otimes |\Omega\rangle \]  

with energy zero, where

\[ \sigma_z |\pm\rangle = |\pm\rangle \]
denote the upper |+) and lower |−) atomic levels, and |Ω) denotes the zero-photon state in \( \mathcal{F} \). Note that \( \Phi_0 \) is also eigenstate of the free Hamiltonian \( H_0 \), with energy zero, and we say therefore that the model has a persistent zero particle state. Thus, by a theorem in ([Wig67], pg. 250) - which is logically independent from Haag’s theorem ([Wig67], pg. 249), the model is well-defined in Fock space, and \( \mathcal{H} \), defined by (12), is, indeed, the adequate Hilbert space - a fact which we know, of course, directly (see, e.g., [HS95], appendix A).

We shall confine ourselves to the subspace \( P_1 \mathcal{H} \). Let

\[
\Phi_1 \equiv |+\rangle \otimes |\Omega\rangle
\]

(21)

and

\[
\Phi_2(f) \equiv |−\rangle \otimes a^\dagger(f)|\Omega\rangle \text{ with } f \in L^2(\mathbb{R}^3)
\]

(22)

The subspace \( P_1 \mathcal{H} \) consists of linear combinations

\[
\Phi_{a,b} \equiv a\Phi_1 + b\Phi_2(f)
\]

(23)

where \( a, b \) are complex coefficients, and we take \( f \) normalized in \( L^2(\mathbb{R}^3) \):

\[
\langle f, f \rangle = 1
\]

(24)

where \( \langle ., . \rangle \) denotes the scalar product in \( L^2(\mathbb{R}^3) \). Thus, \( P_1 \mathcal{H} \) is isomorphic to the space

\[
\mathcal{H}_1 \equiv \mathbb{C} \oplus L^2(\mathbb{R}^3)
\]

(25)

with \( H_1 \equiv P_1 H P_1 \) is isomorphic to \( H_1 \) (using the same symbol) given by

\[
H_1 = \begin{bmatrix}
E_0 & \beta \langle g, . \rangle \\
\beta g & |k|
\end{bmatrix}
\]

(26)

where \( g \) is given by (13), (14). This is the famous Friedrichs model, see [HS95] and references given there. The following theorem follows from [Kim91] or ([How75], Proposition 1, pg. 417):

**Theorem 1.1.** For the model (24), (25), (26), let

\[
E_0 > \beta^2 \int_0^\infty dk f(k)^2
\]

(27)

Then:
\( a. \) \( H_1 \) has spectrum
\[
\sigma(H_1) = [0, \infty)
\] (28)

which is purely absolutely continuous and, furthermore, for all \( z \in \mathbb{C} \) not in the positive real axis:
\[
r_1(z) \equiv (\Phi_1, (H_1 - z)^{-1}\Phi_1)
\]
\[
= (E_0 - z - \beta^2 \int_0^\infty dk \frac{g(k)^2}{k - z})^{-1}
\] (29)

We have absorbed in the quantity \( \beta^2 \) in (26) the factor \( 4\pi \) coming from integration over the solid angle, and denote by \((.,.)\) the scalar product in \( \mathcal{H}_1 \). Given the spectral family \( \{E(\lambda)\}_{\lambda \in [0, \infty)} \) associated to \( \mathcal{H}_1 \) (see, e.g., [BB03]), statement b.) of theorem 1.1 means that the Stieltjes measure (for the definition, see, e.g., [Sew86], pg. 41):
\[
d\mu_{\Phi_1}(\lambda) = (\Phi_1, E(\lambda)\Phi_1) = \int_0^\lambda g_{\Phi_1}(u)du
\] (30)

where
\[
g_{\Phi_1}(u) = \frac{d\mu_{\Phi_1}(u)}{du}
\] (31)

exists almost everywhere (a.e.) in \( u \) and defines a (locally) \( L^1 \) function. By a.) of theorem 1.1 we may express \( g_{\Phi_1} \) in terms of \( r_{\Phi_1}(z) \) by ([RS72], [Jak06]):
\[
g_{\Phi_1}(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} [r_{\Phi_1}(\lambda + i\epsilon) - r_{\Phi_1}(\lambda - i\epsilon)]
\] (32)

In spite of the exact result b.) of theorem 1.1 the time evolution of the initial state \( \Phi_1 \) is not explicitly known - a symptom of the complexity of the time evolution of quantum systems even in the simplest situations, and one must rely on suitable estimates. Before we do so, however, we wish to describe the problem in a more general framework, that of states, defined as positive, normalized linear functionals over an algebra of observables. The fact that the model is defined on Fock space plays a crucial role in this description.
2 States and dynamical instability

2.1 Dynamical instability of the Weisskopf-Wigner state

A quantum dynamical system is most generally described by a triple \((A, \tau_t, \omega)\) - a C*- or W*- dynamical system \([BR87\text{, pg. 136}]\), where \(A\) is a *-algebra of observables, \(t \in \mathbb{R} \rightarrow \tau_t\) a group of automorphisms of \(A\), associated to the dynamical evolution, and \(\omega\) a state on \(A\), that is, a positive, normalized linear functional on \(A\), i.e., satisfying

\[
\omega(A^*A) \geq 0
\]

as well as

\[
\omega(1) = 1
\]

and

\[
\omega(\lambda A + \mu B) = \lambda \omega(A) + \mu \omega(B)
\]

where \(A, B \in A\) and \(\lambda, \mu \in \mathbb{C}\). In the present case, \(A\) will be identified with the algebra of bounded operators on \(H\), given by \([12]\), i.e.,

\[
A = A_1 \otimes \tilde{A}
\]

where \(A_1\) is the algebra generated by the Pauli matrices and the identity, and \(\tilde{A}\) may be taken as the Weyl algebra, generated by the operators \(W(f)\) of, e.g., \([BR97\text{, Proposition 5.2.4}]\). They are bounded functionals of the operators \(a^*(g), a(g)\). If \(A \in A\),

\[
\tau_t(A) \equiv \exp(iH_1 t)A \exp(-iH_1 t)
\]

Corresponding to \([21], [22]\), two states \(\omega_1, \omega_2\) are relevant:

\[
\omega_1 \equiv (\Phi_1, \cdot \Phi_1)
\]

\[
\omega_2 \equiv (\Phi_2(f), \cdot \Phi_2(f))
\]

Note that the normalization of \(\omega_2\) is due to \([24]\). The state \(\omega_1\) is called the **Weisskopf-Wigner state**, see \([Bar84, DN80, Dav]\) for a discussion of the physical limitations of this concept and references. We shall view it as a mathematical caricature of a more realistic initial, decaying state, which
takes into account the impossibility of separating the atom from the radiation field.

In fact, the state $\omega_1$ is not general enough even in the very restricted framework of the model (26), which allows spontaneous emission of a photon. We thus define

$$\Phi_{r,s}^1 \equiv r\Phi_1 + s\Phi_2(f)$$

(40)

as well as

$$\Phi_{r,s}^2 \equiv s\Phi_1 - r\Phi_2(f)$$

(41)

where $r, s$ are complex numbers such that

$$|r|^2 + |s|^2 = 1$$

(42)

and the corresponding states

$$\omega_{r,s}^1 \equiv (\Phi_{r,s}^1, \cdot \Phi_{r,s}^1)$$

(43)

together with

$$\omega_{r,s}^2 \equiv (\Phi_{r,s}^2, \cdot \Phi_{r,s}^2)$$

(44)

For the Weisskopf-Wigner state $\omega_1$, we have, by (37),

$$(\omega_1 \circ \tau_t)(A) = (\exp(-iH_1t)\Phi_1, A \exp(-iH_1t)\Phi_1)$$

(45)

In case of the special observable

$$A = |\Phi_1)(\Phi_1|$$

(46)

we obtain

$$(\omega_1 \circ \tau_t)(A) = |R_{\Phi_1}(t)|^2$$

(47)

where

$$R_{\Phi_1}(t) \equiv (\Phi_1, \exp(-iH_1t)\Phi_1)$$

(48)

is called the return probability amplitude of the vector $\Phi_1$; $|R_{\Phi_1}(t)|^2$ is the corresponding return probability. a.) of the next theorem is a mathematical expression of the dynamical instability of the state $\omega_{r,s}^1$:

**Theorem 2.1.**

a.) $\lim_{t \to \infty}(\omega_{r,s}^1(t)(A) = \omega_{r,s}^2(A) \forall A \in \mathcal{A}$;

b.) $R_{\Phi_1}(t) = o(t)$ as $t \to \infty$
Proof. By (37),
\[
(\omega_{r,s}^1 \circ \tau_t)(A) = (\exp(-iH_1 t)\Phi_{r,s}^1, A \exp(-iH_1 t)\Phi_{r,s}^1)
\]
using in \(P_1 H\) the orthonormal basis consisting of the vectors \(\Phi_{r,s}^1\), \(\Phi_{r,s}^2\); we find
\[
\exp(-iH_1 t)\Phi_{r,s}^1 = \alpha(t)\Phi_{r,s}^1 + \beta(t)\Phi_{r,s}^2
\]
where
\[
|\alpha(t)|^2 + |\beta(t)|^2 = 1
\]
By a.) of theorem 1.1, (50) and (30), (31),
\[
\alpha(t) = \int_0^\infty g_\Psi(\lambda) \exp(-i\lambda t) d\lambda
\]
where we wrote, for brevity, \(\Psi = \Phi_{r,s}^1\). Due to normalization \((\Psi, \Psi) = 1\), \(g_\Psi \in L^1(0,\infty)\), and thus, by the Riemann-Lebesgue lemma ([MW13], [Kat76], Theorem 1.7, pg. 123), \(\alpha(t) = o(t)\) as \(t \to \infty\). Together with (51), we get
\[
\lim_{t \to \infty} |\beta(t)|^2 = 1
\]
which proves a.). Case b.) corresponds to setting \(r = 1\) and \(s = 0\) in (40), and using (48), (30) and (31), together with the Riemann-Lebesgue lemma.

2.2 Connection between positivity of the Hamiltonian and the rate of decay

One way of satisfying b.) of theorem 2.1 is by assuming exponential decay. In this connection we have the following theorem, due to Sinha ([Sin72], Lemma 5, pg. 628):

**Theorem 2.2.** If, for all \(t \geq 0\),
\[
R_{\Phi_1}(t) = \exp(-i\lambda_0 t) \exp(-\frac{\Gamma t}{2}) + c(t)
\]
for some \(\lambda_0 \in \mathbb{R}\) and \(\Gamma > 0\), then
\[
c(t) \neq 0
\]
and
\[
c(t) = o(t)\text{ but it is not } O(\exp(-at))\text{ for any } a > 0
\]
as \(t \to \infty\).
Proof. Assuming (54), with \(c(t) \equiv 0\), it follows that \(R_{\Phi_1} \in L^1(\mathbb{R})\), under which hypothesis the inversion formula ([Chu74], Theorem 6.2.1) yields

\[
\mu_{\Phi_1}(\lambda) - \mu_{\Phi_1}(0) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} R_{\Phi_1}(t) \frac{\exp(-it\lambda) - 1}{-it} dt
\]

The r.h.s. of the above formula may be extended to a function of a complex variable \(z = \lambda + i\rho\) by the relation

\[
\mu_{\Phi_1}(z) - \mu_{\Phi_1}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{\Phi_1}(t) \frac{\exp(-itz) - 1}{-it} dt
\]

defining a function \(\mu_{\Phi_1}(z)\) analytic in the strip \(-a < \rho < a\), whose boundary value, as \(\rho \to 0\), is the original Stieltjes measure \(\mu_{\Phi_1}(\lambda)\). It follows that the support of \(\mu_{\Phi_1}\) must be the whole real line, which contradicts a.) of theorem 1.1; thus, (55), as well as (56), must hold.

\(\square\)

**Definition 2.3.** When (54) holds, \(\lambda_0\) is called the **level shift** and \(\Gamma\) is called the **half-width** of the state \(\Phi_1\).

Theorem 2.1 lies at the root of the connection between the rate of of decay and positivity of the Hamiltonian. Another important approach to this connection, also believed to be quite general, but which will only be established within the present model in our main result in section 4, proceeds by comparing \(g_{\Phi_1}\) in (32) with the Lorentzian or Breit-Wigner function (2).

The next section focuses on a different aspect of relation (55): it is shown to be due to the regeneration of the unstable system from the decay products. This aspect will be shown to have important consequences regarding the explicit rate of decay of \(c(t)\), which distinguish field theory from potential theory (see remark 2.1).

### 2.3 The regeneration of the unstable state from the decay products

We now consider \(r = 1\) and \(s = 0\) in (40), i.e., \(\Phi_1\) as initial state, and \(r = -1, s = 0\) in (41), i.e., \(\Phi_2(f) \equiv \Phi_2\) as final state in (50), with \(\alpha(0) = 1\).
and $\beta(0) = 0$, ignoring possible phases, and assume $t \geq 0$ (see the next subsection for the modifications due to very small times). We thus have

$$\Psi(t) = \exp(-itH_1)\Psi(0) = R_{\Phi_1}(t)\Phi_1 + S(t)\Phi_2 \tag{57}$$

where $R_{\Phi_1}$ is defined by (48), and

$$S(t) \equiv (\Phi_2, \exp(-itH_1)\Phi_1) \tag{58}$$

We take $\Phi_1$ to describe the "unstable state" at $t = 0$; $\Phi_2$ describes the "decay products".

**Definition 2.4.** If, either, $S(t) \neq 0$ in (57), for some $t > 0$, or $(\exp(-itH_1)\Phi_2, \Phi_1) \neq 0$, for some $t > 0$, we say regeneration of the unstable state from the decay products has taken place.

Assume, now, that

$$R_{\Phi_1}(t) = \exp(-\alpha t) \text{ with } \Re \alpha > 0 \tag{59}$$

The following theorem is essentially due to Sinha, Williams and Fonda, Ghirardi and Rimini ([Sin72], [Wil71], [FGR78]; we adapt parts of an argument of [FGR78] to the present model):

**Theorem 2.5 (Regeneration of the unstable state from the decay products in case of non-pure exponential decay).** Equation (59) is true if and only if regeneration of the unstable system from the decay products (definition 2.4) occurs.

**Proof.** Apply $\exp(-it'H_1)$ to both sides of (57), obtaining

$$\exp[-i(t + t')H_1]\Phi_1 = R_{\Phi_1}(t)\exp(-it'H_1)\Phi_1 + + S(t)\exp(-it'H_1)\Phi_2$$

whose scalar product with $\Phi_1$ yields

$$R_{\Phi_1}(t + t') - R_{\Phi_1}(t)R_{\Phi_1}(t') = S(t)S(-t') \tag{60}$$

If (59) holds, the r.h.s. of (60) is zero, $\forall t, t' \geq 0$, and therefore one of the options in definition 2.4 takes place. On the other hand, if one of the options in definition 2.4 occurs, the left hand side of (60) is identically zero, $\forall t, t' \geq 0$, i.e.,

$$R_{\Phi_1}(t + t') = R_{\Phi_1}(t)R_{\Phi_1}(t') \tag{61}$$
By (48), $R_{\Phi_1}$ is a continuous function satisfying $R_{\Phi_1}(0) = 1$ and (2.1), that is, b.) of theorem 2.1. The only solution of (61) satisfying these conditions is the function given by the r.h.s. of (59).

\[ \square \]

**Remark 2.1.** Theorem 2.5 provides the physical justification of (55): regeneration of the unstable state from the decay products, a purely quantum mechanical virtual process, depending, in addition, on the field-theoretic nature of the model. Indeed, either of the two options in definition 2.4, depends on the interaction term (11), because, if $H$, in (9), is replaced by the free Hamiltonian $H_0$ in (10), it is immediately seen that none of the options in definition 2.4 holds: both $S(t)$ and $\bar{S}(-t')$ are identically zero for all $t, t' \geq 0$.

The forthcoming theorem 4.1 demonstrates that the above assertion remains true for the whole domain of asymptotic values of the time variable, because the correction term corresponding to (55), is there shown to be $O(\frac{1}{t})$. In contrast, the behavior for large times of the return probability amplitude associated to the limiting state $\Phi_2$, given by theorem 2.1, if it were determined by the free Hamiltonian, would be

\[
(\Phi_2(f), \exp(-itH_0)\Phi_2(f)) =
= (f, \exp(-it\omega(k))f) = O(t^{-2})
= \langle f, f \rangle
= \int \frac{d^3k}{2\omega(k)} |f(k)|^2
\]

the latter being the relativistic scalar product for the photon wave-functions; this corresponds to the correction term found in ([Bar84], [DN80], [Dav]), and there claimed to be a consequence of causality.

The asymptotic behavior of the return probability amplitude differs, therefore, qualitatively from that found in potential theory, where it is indeed due to the free evolution, i.e., $O(t^{-3/2})$ in three dimensions, whenever the potential falls off at least as fast as $|\vec{x}|^{-1-\epsilon}$, for some $\epsilon > 0$, as $|\vec{x}| \to \infty$, i.e., faster than Coulomb, see [RS78].
2.3.1 Renormalization of $R_{\Phi_1}$: time shift

The forthcoming theorem 4.1 will replace (54) by

$$R_{\Phi_1}(t) = (1 - \beta^2 A) \exp(-i\lambda_0 t) \exp(-\frac{\Gamma t}{2}) + c(t)$$  \hspace{1cm} (62)

where $A$ is a complex quantity which will be defined there. We may write (62) in the form

$$R_{\Phi_1}(t) = \exp(\alpha(t + t_1)) + c(t)$$  \hspace{1cm} (63)

where $\Re \alpha = -\frac{\Gamma}{2}$ and $\Im \alpha = -\lambda_0$. We define $t_1$ and $t_2$ by

$$\exp\left(-\frac{\Gamma t_1}{2}\right) = |1 - \beta^2 A| < 1$$  \hspace{1cm} (64)

and

$$\exp(-i\lambda_0 t_2) = \exp(i\phi)$$  \hspace{1cm} (65)

with

$$1 - \beta^2 A = |1 - \beta^2 A| \exp(i\phi)$$  \hspace{1cm} (66)

We trivially renormalize $R_{\Phi_1}(t) \rightarrow \exp[-i\lambda_0(t_1 - t_2)]R_{\Phi_1}(t)$ as well, and further define the time-shift

$$\tau = t + t_1 \text{ with } \tau \geq t_1$$  \hspace{1cm} (67)

In theorem 2.5 we use the variable $\tau \geq t_1 > 0$ instead of $t \geq 0$: the proof proceeds as given, imposing the initial condition at $t_1$, i.e., $R_{\Phi_1}(\tau = t_1) = \exp(\alpha t_1)$. The condition $\tau \geq t_1$ in (67), as well as the time-shift, are quite instrumental to accomodate for the fact that the exponential behavior (54) cannot be valid for $t = 0$ because of the condition

$$\frac{d}{dt}R_{\Phi_1}(t) = 0 \text{ at } t = 0$$  \hspace{1cm} (68)

(68) may be immediately verified from (26). Since the first term in (63) does not satisfy (68), but has derivative equal to $\alpha$ at $\tau = t_1$, the second term must satisfy $\frac{d}{d\tau}(\tau = t_1) = -\alpha$. 

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3 The method of decay without analyticity, irreversibility and the arrow of time

3.1 The method of decay without analyticity

In this section we investigate the validity of (54) - or, rather, of (62) for suitable \( A \). We thereby avoid the use of complex energies and frequencies, which are associated to the analytic continuation of the resolvent ((b.) of theorem 1.1) to "unphysical" Riemann sheets. We describe this procedure by the shorthand "the method of decay without analyticity", which should not be confused with the wish to avoid any particular method of treating the problem of resonances.

As remarked by Nussenzveig [Bar84], the pathologies associated to the use of "complex eigenfrequencies" \( \omega_n = \omega'_n - i\gamma_n \), with \( \omega'_n \) real and \( \gamma_n \) positive, appeared already in J.J. Thomson’s treatment [Tho83] of the free modes of oscillation of the electromagnetic field around a perfectly conducting sphere. Although \( \exp(-i\omega_n t) = \exp(-i\omega'_n t)\exp(-\gamma_n t) \) decays exponentially as \( t \to \infty \), as expected from radiation damping, the corresponding radial behavior of free outgoing electromagnetic waves is of the form \( \exp[-i\omega_n (t - r/c)] \), which blows up exponentially as \( r \to \infty \). A similar behavior occurs in quantum theory, associated to the so-called Gamow vectors (see, e.g., [MW13], section 5). Such behavior imposes the use of a space-cutoff in the Green functions, showing that the - a priori not physically motivated - concept of complex energies and frequencies is delicate, and it would be conceptually of great advantage to avoid them. We attempt to do so in this paper, following [Kin91], who initiated this method in 1991.

3.2 Irreversibility and the problem of the arrow of time in quantum field theory

In his paper, Christopher King [Kin91] assumed everywhere that \( t \geq 0 \), without mentioning it explicitly. It happens, however, that the decay of unstable systems - atoms or particles - presents a prototypical example of the existence of a time arrow: choosing an initial time, the decay has precisely the same behavior whatever time direction is chosen. The problem of the arrow of time is: is there an objective way to distinguish a "future" direction, in agreement with our general psychological perception that "time passes"?
In his paper, Nussenzveig [Bar84] proposes that the solution of the “exponential catastrophe” mentioned in the previous subsection lies in the fact that the decay should be necessarily treated together with the preparation of the state, which must have cost a finite amount of energy and have occurred at some finite time in the past. Our method avoids, however, the use of complex energies, and we therefore do not find any “exponential catastrophe”.

We retain, however, Nussenzveig’s suggestion as a natural and physically compelling explanation of the asymmetry between past and future, i.e., of the arrow of time, which has been proposed in thermodynamics [Wre], and will now be briefly sketched for the present model. For this purpose we use the framework of section 2.1.

**Definition 3.1.** Let a $C^*$-dynamical system $(A, \tau_t)$ be given. An *adiabatic transformation* consists of two successive steps. The first step, called preparation of the state, starts at some $t = -r$, with $r > 0$, when the state $\omega_{-r}$ is invariant under $\tau_{-r}$ and the Hamiltonian is $H_{-r}$, and ends at $t = 0$, when the Hamiltonian is $H_0$, such that

$$H_{-r} = H_0$$

and the state is $\omega_0 = \omega$. The second step is a dynamical evolution of the state of the form

$$t \in \mathbb{R} \to \omega_t \equiv \omega \circ \tau_t$$

**Remark 3.1.** Definition 3.1 is nontrivial only if the preparation of the state yields $\omega \circ \tau_t \neq \omega$, i.e., if the initial state $\omega$ is not invariant under the automorphism $\tau_t$.

In this section we show that the time-arrow problem is a consequence of the preparation of the state in definition 3.1. Firstly, we identify the quantities occurring in that definition. $A$ is the algebra defined in section 2.1, and $\tau_t(A) \equiv \exp(iHt)A \exp(-iHt)$ instead of (37) (only in this section), where $H_{-r} = H$ is given by (17), $\omega_{-r} = (\Phi_0, \cdot \Phi_0)$, where $\Phi_0$ is given by (19) is the ground state of the Hamiltonian $H$. The state $\omega_0 = \omega$ is taken to be $\omega = \omega_1$, the Weisskopf-Wigner state defined by (37). If we further define

$$\bar{\omega}_t \equiv \omega_{-t} \text{ for } t \geq 0$$

and specialize to the observable $A$ defined by (45), we obtain

$$\bar{\omega}_t(A) = |R_{\Phi_1}(-t)|^2 = |R_{\Phi_1}(t)|^2 = \omega_t(A)$$

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Of course, (71) is a direct consequence of the self-adjointness of $H$, and, in a more general formulation, of the invariance of the state $\omega$ under the time-reversal automorphism (see, in this connection, [Sew02], section 4.1.1). In this specific situation, (71) expresses the time-arrow problem if $|R_{\Phi_1}(t)|^2$ decays, as $t \to \infty$: the decay occurs in any of the two time directions.

According to definition 3.1 the system is closed from $t = 0$ to $t = \infty$, but not from $t = -r$ to $t = 0$, where it is subject to external conditions, but is still thermally isolated. The work $W$ done by the time-dependent external forces on the system satisfies the, under assumption (69),

$$W \geq 0$$

by the Kelvin-Planck statement of the second law. We shall assume that

$$0 < W = U < \infty$$

where $U$ is the energy imparted to the system. Under assumption (73), the case $r = 0$ in definition 3.1 i.e., an instantaneous preparation of the state at $t = 0$, a ”$\delta(t)$” pulse, is excluded. This obvious physical requirement has a far-reaching consequence:

**Time arrow theorem** Under an adiabatic transformation (definition 3.1 with $r \neq 0$), there is a breakdown of time-reversal symmetry ((73)) and therefore, in general, a time-arrow exists.

It is not easy to provide concrete tractable models of the preparation of the state, according to definition 3.1. For the Hamiltonian considered in this paper, an adequate model satisfying (69) is given by the Hamiltonian

$$H(t) = H + f(t)\pi\sigma_y$$

where $f(t) = 0$ if $t \leq -r$ or $t \geq 0$ (74)

where $H$ is given by (9) and

$$f(t) = g(t + r/2)$$

Above, $g$ denotes a smooth approximation to the delta function. We take for the initial state at $t = -r$ the ground state (19). We take further the limit of a ”$\delta$” pulse, acting on $\Phi_0$. The result for the evolution of this vector is

$$\exp(iHr/2)\exp(i\pi\sigma_y)\Phi_0 = \exp(iE_0r/2)\Phi_1$$

(75)

where $E_0$ denotes the (renormalized) resonant state energy (8). The last phase disappears upon construction of the state. The work done by the
time-dependent external forces is $W = E_0$, which satisfies (72) and (73). In this way, we obtain the new state $\omega = \omega_1$, but with time translated by the quantity $r/2$. If $r$ is taken much smaller than the half-life (inverse half-width) of the state $\omega_1$ (see definition 2.3 and section 4), the half-width and the level shift will change by a negligible amount in comparison with the state $\omega$ evolving from $t = 0$ with Hamiltonian $H$. It should be emphasized that the ”sudden” interaction we considered was done solely for technical reasons, as a caricature of the adequate model (74), which complies with the requirements of definition 3.1.

It should be remarked that the problem of irreversibility is distinct from the time-arrow problem, see [Wre]; in general, it is expected that the mean entropy increases, starting from a given state, in any of the two time directions [Wre]. Unfortunately, in the present model, due to the conservation laws, the entropy remains identically zero in the course of time, although, nevertheless, a) of theorem 2.1 remains as an expression of irreversibility.

It is, however, possible to conjecture that a unified picture of irreversibility in quantum field theory arises from some of the results of the present paper, in particular theorem 2.5, the regeneration of the unstable state from the decay products, which may be expected to hold in more realistic quantum field theories. One example might be the model including the vacuum-polarizing term

$$H'_I = \beta[\sigma_+ \otimes a^\dagger(g) + \sigma_- \otimes a(g)]$$

Generalizing the model further in order to include a number $N$ of identical two-level atoms, the vectors corresponding to (21) and (22) would be $\Phi_1^{(N)} = \otimes_{i=1}^N |+\rangle_i \otimes |\Omega\rangle$, the Weisskopf-Wigner state, and $\Phi_2(f)^{(N)} = \otimes_{i=1}^N |-\rangle_i \otimes a^\dagger(f)|\Omega\rangle$, the decay-product state.

Starting from a Weisskopf Wigner-state at $t = 0$, we expect to have, by theorem 2.5 subsequently a superposition of the type (40), but now for the states $\Phi_1^{(N)}$ and $\Phi_2(f)^{(N)}$, which, for $N \to \infty$, may be expected to tend, by Hepp’s lemma 3 ([Hep72], see also [NW14] for a complete proof), to an incoherent superposition, as states on the quasi-local algebra, leading to an increase of the specific entropy, as befits an entangled state. This would agree with the theory proposed in [Wre]. The picture is completed by viewing the previous description as a part of a whole, inscribed into a history ([Gri84], [GMH91], [Omn94]), the Weisskopf-Wigner state being (a caricature of) a state obtained by interaction with the environment, in the theory formulated by Narnhofer and Thirring [HAMH99] and Thirring [Thi96], and briefly re-
The necessity of an "event-enhanced" quantum theory has been emphasized by Blanchard and Jadczik [B195]. One point should be mentioned: the times between interactions with the environment are macroscopic, since they refer to classical (macroscopic) observables, which are associated to the projectors building up the histories, and thus much larger than the microscopic times associated to the half-life of atomic states described in this paper (see also the discussion in [Wre]). In addition the proof of lemma 3, page 23, of [NW14], that the state collapse of a decoherent state produces, in the average, a reduction of the quantum Boltzmann entropy, applies to the Gibbs-von Neumann entropy considered in [Wre], and, in fact, the proof given there was done, initially, for the latter, and then specialized to the quantum Boltzmann entropy. As mentioned in the conclusion of [NW14], the expected (average) increase of the entropy leads to the expectation that the non-automorphic events (collapses) are rare in the time scale given by thermodynamics.

The picture described above seems to be complementary, in several aspects, to two interesting approaches: the (relativistic) ETH approach, presented in [Frob], and the approach by Buchholz and Buchholz and Roberts [Buc77], [BR14]. In the latter, a basic point is that, by Huyghens’ principle, outgoing massless particles created in the past of a given light cone will never enter that cone, and thus an associated loss of information is inevitable in the case of massless particles. For this purpose, a preferred time direction is assumed, which may be justified by the present "time-arrow" theorem, noting that the occurrence of a state collapse as it happens in a measuring process or preparation process is not restricted to the observation by an observer who is not a part of the physical system, in agreement with the ideas of Haag [Haa14] and Froehlich [Frob], [Froa]. Concerning the former [Frob], the basic complementary aspect is the (conjectured) formation of a decoherent state, in the form of a "Schroedinger cat", and the subsequent (conjectured) increase of the specific entropy. This fact would have important consequences. As shown by Narnhofer and Thirring (HAMH99, [NT]) and Thirring [Thi96], if the dynamics between interactions with the environment (collapses) is described by a quantum K system ([NT90], [Emc76]), each state collapse purifies any mixed state in the classical quantities for almost all histories. This explains the macroscopic purity found in nature. In other words, Schroedinger’s cat is most likely to be either alive or dead [NT].

In the case of a fully relativistic quantum field theory, the vacuum state has been proved to satisfy the property of time-like clustering [Mai68], i.e.,
to be a mixing state. In a generalized form it is equivalent, for factor states, to the properties of return to equilibrium and weak asymptotic abelianness, by lemma 3.5 of [NT91]. Weak asymptotic abelianness has been shown for a class of relativistic quantum field theories in [JNW10], as a consequence of covariance under the Poincaré group: for K systems the correspondent histories decohere for long times [Thi96]. In addition, such systems are "memoryless" in the sense of forgetting all causal links [Thi96]. In this sense, the picture suggested above may even be strengthened by relativistic invariance. It must, however, be stressed that the notions of entropy and entanglement in relativistic quantum field theory are delicate and must be handled with great care, see [NT12], [Yng].

4 Decay without analyticity: the correction $c(t)$ to exponential decay. The main theorem and its proof

We refer to (13) and (14), (15) and define the functions $G$ and $F$, which will play a key role in the following:

$$G(\lambda) \equiv g(\lambda)^2$$

$$F(\lambda) \equiv \text{vp} \int_\delta^\infty \frac{G(k)}{k - \lambda} dk \text{ for all } \rho > 0 \text{ and } 0 < \lambda < \infty$$

where $\text{vp}$ denotes the Cauchy principal value ([BB03], chapter 3.2, pg. 33). Note that for $\lambda = 0$ the principal value in (78) is not defined, but we add to (78)

$$F(0) \equiv \lim_{\lambda \downarrow 0} F(\lambda) = \int_0^\infty \frac{G(k)}{k} dk$$

(79) is proved in appendix B. By (13) and (17), it follows that $G$ satisfies:

$$\sup_{x \in [0, \infty)} |G'(k)(1 + k^2)| < \infty$$

(80)

The following Sokhotski-Plemelj formula ([BB03], chapter 3.3, page 37) will be used:
\[
\lim_{\epsilon \to 0} \frac{1}{x \pm i \epsilon} = \mp i \pi \delta + v p \frac{1}{x} \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad (81)
\]

From the proof of (81), e.g., in [BE03, loc.cit., it is immediately apparent that (81) holds as a functional on test-functions \( G \) which need not belong to the Schwartz space \( \mathcal{D}(\mathbb{R}) \) but need only satisfy (80). Using this fact, we obtain from (29), (32), (77) and (78) the equation

For all \( \lambda > 0 \)
\[
g_{\Phi_1}(\lambda) = \frac{d\mu_{\Phi_1}(\lambda)}{d\lambda} = \frac{\beta^2 G(\lambda)}{(E_0 - \lambda - \beta^2 F(\lambda))^2 + (\pi \beta^2 G(\lambda))^2} \quad (82)
\]

From (48), (30) and (31), we obtain

\[
R_{\Phi_1}(t) = \int_0^\infty g_{\Phi_1}(\lambda) \exp(-i\lambda t) d\lambda \quad (83)
\]

(82), properties a.) and b.) of \( F \) in appendix B, (77), (13), and (14) imply that \( g_{\Phi_1}(\lambda) \) in (83) is uniformly bounded in \( \lambda \) near \( \lambda = 0 \) and of decay \( O(\lambda^{-7}) \) for large \( \lambda \), so that the integral on the r.h.s. of (83) is well defined. We may now state our main theorem:

**Theorem 4.1.** There exists a constant \( b > 0 \) such that, if
\[
\beta < b \quad (84)
\]
then (62) holds in the form
\[
R_{\Phi_1}(t) = (1 + O(\beta^2)) \exp(-i\lambda_0 t) \exp\left(-\frac{\Gamma t}{2}\right) + c(t) \quad (85)
\]

with the level shift \( \lambda_0 \) given by the unique solution in a sufficiently small neighbourhood of \( E_0 > 0 \) of the equation
\[
E_0 - \lambda_0 - \beta^2 F(\lambda_0) = 0 \quad (86)
\]
and the half-width \( \Gamma \) is given by
\[
\Gamma = 2\pi \beta^2 G(E_0) \quad (87)
\]
Furthermore, in (85), $c(t)$ is given by

$$c(t) = c_1(t) + c_2(t)$$  \hspace{1cm} (88)$$

where

$$\lim_{t \to \infty} t c_1(t) = \frac{\beta^2 d}{E_0}$$  \hspace{1cm} (89)$$

for some constant $d > 0$ independent of $\beta$ and

$$|c_2(t)| \leq \frac{c \beta^2}{t}$$  \hspace{1cm} (90)$$

for all $t > 0$ and $c > 0$ independent of $t$.

**Proof.** As in [Kin91], the strategy of the proof will be to approximate $g_{\phi_1}$, given by (82), by a Lorentzian (or Breit-Wigner) function: this will yield (85), with (88) and $c_2 = 0$, and $c_1$ satisfying (89). An estimate of the remainder provides then (88), with $c_2$ satisfying (90).

We expand, as in [Kin91], (82) around $\lambda = \lambda_0$ (the solution of (86) under assumption (84), which exists by the implicit function theorem under our assumptions on $G$ and $F$, in particular the continuous differentiability of $F$ in a neighborhood of $E_0$) to second order. Define

$$\kappa \equiv -1 - \beta^2 F'(\lambda_0) - i \pi \beta^2 G'(\lambda_0)$$  \hspace{1cm} (91)$$

where the prime indicates differentiation. Then

$$E_0 - \lambda - \beta^2 F(\lambda) - i \pi \beta^2 G(\lambda) = \kappa(\lambda - \lambda_0) - i \pi \beta^2 G(\lambda_0) + w(\lambda)$$  \hspace{1cm} (92)$$

where the remainder $w(\lambda)$ in (92) is equal to

$$w(\lambda) = -\beta^2 [F(\lambda) - F(\lambda_0) - F'(\lambda_0)(\lambda - \lambda_0)]$$

$$- i \pi \beta^2 [G(\lambda) - G(\lambda_0) - G'(\lambda_0)(\lambda - \lambda_0)]$$  \hspace{1cm} (93)$$

From (91),

$$(\kappa)^{-1} = (-1 - \beta^2 F'(\lambda_0) - i \pi \beta^2 G'(\lambda_0))^{-1}$$

$$= -(1 + \beta^2 A)^{-1}$$  \hspace{1cm} (94)$$
where

\[ A \equiv F'(\lambda_0) + i\pi G'(\lambda_0) \quad (95) \]

From (94)

\[ (\kappa)^{-1} = -[1 - \beta^2 A + B(\beta^2 A)^2] \quad (96) \]

where

\[ |B| \leq 2 \quad (97) \]

if

\[ \beta^2 \sqrt{[F'(\lambda_0)^2 + \pi^2 \beta^2 G'(\lambda_0)^2]} < \frac{1}{2} \quad (98) \]

Thus, a Lorentzian (or Breit-Wigner) approximation to \( g_{\Phi_1} \), given by (31) or (32), is

\[ L(\lambda) \equiv \frac{1}{\pi \Im} \frac{1}{\kappa} \left( \lambda - \lambda_0 - i\pi \beta^2 \kappa^{-1} G(\lambda_0) \right)^{-1} \quad (99) \]

where

\[ \kappa^{-1} G(\lambda_0) = -G(\lambda_0) + O(\beta^2) \quad (100) \]

by (96)-(98). By (99) and (100), the point

\[ \bar{\lambda} \equiv \lambda_0 + i\pi \beta^2 \kappa^{-1} G(\lambda_0) = \lambda_0 - i\pi \beta^2 G(\lambda_0) + O(\beta^4) \quad (101) \]

lies on the lower half of the complex plane. Accordingly, we write

\[ R_{\Phi_1}(t) = I_L(t) + D_L(t) \quad (102) \]

where

\[ I_L(t) \equiv \int_0^\infty \exp(-it\lambda)L(\lambda) d\lambda \quad (103) \]

and

\[ D_L(t) \equiv \int_0^\infty \exp(-it\lambda)(g_{\Phi_1}(\lambda) - L(\lambda)) d\lambda \quad (104) \]

We apply Cauchy’s theorem to the complex integral of

\[ f(z) \equiv \exp(-it z)L(z) \quad (105) \]

along the clockwise circuit \( \Gamma \equiv C_1 \cup C_2 \cup (-C_3) \), where \( C_1 \equiv \{ iy; -R \leq y \leq 0 \} \), \( C_2 \equiv [0, R] \), and \( C_3 = \{ \exp(i\theta); -\frac{\pi}{2} \leq \theta \leq 0 \} \), and let \( R \to \infty \), avoiding
the pole $\bar{\lambda}$. The contribution of $C_3$ tends to zero due to the term $\exp(-itz)$ in (103) (recall that $t > 0$). We now estimate that of $C_1$, writing first

$$L(\lambda) = \frac{1}{\pi} \frac{1}{\kappa(\lambda - \lambda_0) - i\pi \beta^2 G(\lambda_0)} =$$

$$= \frac{1}{2\pi} \frac{1}{(\kappa(\lambda - \lambda_0) - i\pi \beta^2 G(\lambda_0)) - \frac{1}{(\kappa(\lambda + \lambda_0) - i\pi \beta^2 G(\lambda_0)}}$$

(106)

Therefore, by (103),

$$I_L(t) = -2\pi \text{res}(\bar{\lambda}) - \frac{\beta^2 G(\lambda_0)}{t} \int_0^\infty dy \exp(-y)f(t, y)$$

(107)

where

$$f(t, y) \equiv \left[\kappa\left(\frac{-iy}{t} - \lambda_0\right) - i\pi \beta^2 G(\lambda_0)\right][\kappa\left(\frac{-iy}{t} - \lambda_0\right) + i\pi \beta^2 G(\lambda_0)]$$

(108)

By (101) and (105),

$$\text{res}(\bar{\lambda}) = \exp(-it\lambda_0) \exp(-\pi \beta^2 G(\lambda_0)t)[1 + O(\beta^2)]$$

(109)

We have

$$|\kappa\left(\frac{-iy}{t} - \lambda_0\right) - i\pi \beta^2 G(\lambda_0)|$$

$$\geq |\kappa|\frac{-iy}{t} - \lambda_0| - \pi \beta^2 G(\lambda_0)$$

$$\geq (1 - O(\beta^2))\lambda_0 - \pi \beta^2 G(\lambda_0) \geq \lambda_0 - O(\beta^2)$$

and similarly for the other denominator in (108), by (94)-(98). Hence, by (108)

$$|f(t, y)| \leq (\lambda_0 - O(\beta^2))^{-2}$$

(110)

By (107), (108), (110) and the Lebesgue dominated convergence theorem, we obtain the (89)-part of (85) of theorem 4.1.

We now prove that $D_L(t)$, defined by (105), satisfies the bound

$$|D_L(t)| \leq \frac{c\beta^2}{t} \text{ for all } t > 0$$

(111)
where \( c \) is a constant, independent of \( \beta \) and \( t \). Together with (102), this proves (90). By definition (104), (82), (92) and (93), we find

\[
D_L(t) = D_L^{(1)}(t) + D_L^{(1)}(t)
\]

(112)

where

\[
D_L^{(1)}(t) = \frac{1}{2\pi} \int_0^\infty \exp(-it\lambda)(-w(\lambda)) \frac{1}{|\kappa(\lambda - \lambda_0)|} h(\lambda, \beta)
\]

(113)

In (112), the bar denotes complex conjugate and

\[
h(\lambda, \beta) \equiv E_0 - \lambda - \beta^2 F(\lambda - i\pi \beta^2 G(\lambda)
\]

(114)

By (112) and (113), in order to prove (90), it suffices to prove

\[
|D_L^{(1)}(t)| \leq \frac{c\beta^2}{t} \text{ for all } t > 0
\]

(115)

The proof of (115) is done in appendix B.

Remark 4.1. Instead of the splitting (102), King [Kin91] defines (in our notation)

\[
I_L(t) \equiv \int_{-\infty}^{\infty} \exp(-it\lambda) L(\lambda) d\lambda
\]

(116)

He thereby adds to \( R_{\Phi_1}(t) \) a term

\[
I_L'(t) \equiv \int_{-\infty}^{0} \exp(-it\lambda) L(\lambda) d\lambda
\]

Since, by (91) and (99), \( L(\lambda) \) is \( O(1) \) and not \( O(\beta^2) \), it is not clear to us which quantity cancels this added term in his result. It happens that it is just the fact that \( I_L(t) \) is given by (103) - and not (116) - which is responsible for the \( c_1(t) = O(\frac{1}{t}) \) there in theorem 4.1. The rest of the proof of theorem 4.1 is devoted to establishing that the correction to the Lorentzian term does not alter this conclusion qualitatively, as demonstrated by (85), (88), (89) and (90) of that theorem.

Remark 4.2. In [Bar84], instead of (79), \( r_{\Phi_1}(t) \) is evaluated by a complex integral of \( r_{\Phi_1}(z) \) along the circuit \( C_R \equiv C_{1,R} \cup C_\delta \cup C'_{1,R} \), where \( C_{1,R} \) is a slight deformation of \( [R, \delta] \), \( C_\delta \) is the (anticlockwise) circle of radius \( \delta \) around the
origin, and $C'_1R$ is the reflection of $C_{1,R}$ through the positive real axis. Due to the existence of the pole of the function $f$, given by (13), in the negative imaginary axis, he deforms the contour to $\tilde{C}_R \equiv C_{1,R} \cup C_{\delta} \cup C_{-\pi/4,R} \cup C_R \cup C'_1, R$.

where $C_{-\pi/4,R} \equiv \{\rho \exp(-i\pi/4); 0 \leq \rho \leq R\}$ and $C_R \equiv \{R \exp(i\theta); -\pi/4 \leq \theta \leq 0\}$. It is understood that $C_{-\pi/4}$ belongs to the first Riemann sheet, and there is a slight “step” when passing from $C_{-\pi/4}$ to $C_R$; the latter is assumed to lie on the second Riemann sheet. By Cauchy’s theorem, as $R \to \infty$ and $\delta \to 0$, the complex integral of $r\Phi_1(z)$ along $C_R$ tends to the sum of the contributions of the pole $\bar{\lambda}$, given by (101), and of the quantity $A \equiv \lim_{R \to \infty} \int_{C_{-\pi/4,R}} r\Phi_1(z)dz$ (117)

(the contribution of $C_R$ tends to zero as $R \to \infty$). By arguments similar to those of theorem 4.1, $A = O(1/t)$ as $t \to \infty$. We do not, however, find the additional contribution of the reflected contour $C'_{-\pi/4,R}$ (analogous to $C'_{1,R}$), lying on the second Riemann sheet, which, as claimed by Nussenzveig, would imply a correction term $O(1/t^2)$, instead of $O(1/t)$, in theorem 4.1.

The complex-analytic method sketched above is more difficult to justify mathematically, because it relies on the global behavior of $r\Phi_1(z)$ as a function of the complex variable $z$: in particular, the poles of the function $f$ in (13) play a role in this behavior, but are not expected to be relevant to the r.h.s. of (85).

5 Sojourn time, its physical interpretation and a time-energy uncertainty relation

The characterization of the half-width $\Gamma$ by (54) (definition 2.3) becomes awkward in practice, see (62). Since $\Gamma$ is the most fundamental physical quantity characterizing decay, it would both more elegant and conceptually more advantageous to characterize it by a global quantity - i.e., not relying on pointwise estimates in the time variable. This subject has a very long history, well summarized in the introduction to the article of Gislason, Sabelli and Wood [GSW85], with various important references: it is known under the general heading of ”time-energy uncertainty relation”. More recent reviews of the topic, which also added significant new results, are the articles by Brunetti and Fredenhagen [BF02] and Pfeifer and Fröhlich [PF95], as well as the book [Bus02], to which we also refer for additional references.
An initial relevant remark is that the early version of the time-energy uncertainty relation, stating that, if the energy of a system is measured during a time $\Delta t$, the corresponding uncertainty $\Delta E$ in the energy variable $E$ must satisfy $\Delta E \Delta t \geq \frac{1}{2} \hbar$, is physically untenable, because, as reviewed in the introduction to [GSW85], it seems generally accepted that the energy of a system can be measured with arbitrary precision and speed. This was first pointed out by Aharonov and Bohm [AB61]. The point we wish to make is that the very designation "time-energy uncertainty relation" is inadequate, because the quantity multiplying $\Delta E$ in the would-be inequality is of entirely different nature from "$\Delta t$". Our results in this section bring a new light on this matter.

We assume a slightly more general setting than in previous sections. Let $H$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and, for $\Psi \in \mathcal{H}$, define

$$ R_\Psi(t) = (\Psi, \exp(-itH)\Psi) \quad (118) $$

This is just the return probability amplitude for the vector $\Psi$, given by (48). For some $\Psi_0 \in \mathcal{H}$, assume that

$$ R_{\Psi_0} \in L^2(-\infty, \infty) \quad (119) $$

and define the sojourn time of the system in the state $\Psi_0$ ([Sin77], [BDFSL10]) by

$$ \tau_H(\Psi_0) \equiv \int_{0}^{\infty} |R_{\Psi_0}(t)|^2 dt \quad (120) $$

By a theorem of Sinha [Sin77], (119) requires that $H$ have purely absolutely continuous (a.c.) spectrum. A lower bound to the sojourn time is given by the rigorous version of the Gislason-Sabelli-Wood time-energy uncertainty relation proved in ([MW13], Theorem 3.17, page 81):

**Theorem 5.1** (rigorous version of the theorem of Gislason-Sabelli-Wood [GSW85]). Let (119) hold and

$$ \Psi_0 \in D(H) \text{ i.e., } ||H\Psi_0|| < \infty \quad (121) $$

Then

$$ I_H(\Psi_0) \equiv \tau_H(\Psi_0)\Delta E \geq \frac{3\pi}{25} \sqrt{(5)} \quad (122) $$

where

$$ (\Delta E)^2 \equiv (\Psi_0, H^2\Psi_0) - (\Psi_0, H\Psi_0)^2 \quad (123) $$

is the energy variance (uncertainty) in the state $\Psi_0$. 

This theorem has been applied to estimate the half-widths of negative ion resonances in [DGS85].

In order to assess the physical meaning of $\tau_H(\Psi_0)$, let, following [GSW85],

$$Q(t) \equiv |R_{\Psi_0}(t)|^2$$

(124)
denote the (quantum) probability that the system has not decayed up to the time $t$. Then the quantity

$$Q(t) - Q(t + \Delta t) = -Q'(t)\Delta t + o(\Delta t)$$
equals the quantum probability that the system has decayed in the interval $[t, t + \Delta t)$, and thus the average lifetime $\tau$ of the decaying state is

$$\tau = -\int_0^\infty dt Q'(t) = [tQ(t)]_0^\infty + \int_0^\infty dt Q(t) = \tau_H(\Psi_0)$$

(125)
as long as

$$\lim_{t \to \infty} tQ(t) = 0$$

(126)

Due to the fact that Gislason, Sabelli and Wood use the theoretical physicist’s lore, rather than the mathematical physicist’s (in particular due to their systematic use of improper (infinite norm) states, as is also done in [Bar84]), it is crucial to prove their result mathematically, which we do in appendix A: this is essentially theorem 3.17 of [MW13], up to some minor corrections, and is included for the reader’s convenience, because [MW13] is not readily available. It must, however, be said that the central part of [GSW85] is very ingenious: in fact, the authors correctly guessed the exact form of the minimum uncertainty functional (the truncated parabola), see the proof of theorem 5.1 in appendix A.

Our main result in this section is the following theorem, which seems to be the first application of a time-energy uncertainty relation to any quantum field theoretic model:

**Theorem 5.2.** For model (22), (119), as well as (121), are true, if $\Psi_0 = \Phi_1$, the Weisskopf-Wigner state. Moreover:

a.)

$$\Delta E \geq 0.843\Gamma$$

(127)

b. Equation (126) holds, and therefore the time of sojourn has the interpretation of an average lifetime.
Proof. \((119)\) follows directly from theorem 4.1. By the spectral theorem,
\[||H\Psi_0||^2 = \int_{-\infty}^{\infty} d\lambda \lambda^2 g_{\Phi_1}(\lambda) \] (128)

In (82), by (13), (14), (77), the numerator \(G(\lambda)\) decays as \(|\lambda|^{-7}\) for large \(|\lambda|\), and
\[\frac{\lambda^2}{(E_0 - \lambda - \beta^2 F(\lambda))^2 + (\pi \beta^2 G(\lambda))^2} \leq c\]
where the constant \(c\) indeps of \(\lambda\) and the other parameters, by property \(a.)\) of \(F(\lambda)\) proved in appendix B. Thus,
\[\int_{-\infty}^{\infty} d\lambda \lambda^2 g_{\Phi_1}(\lambda) < \infty\]
which, together with (128), proves (121).

We further estimate \(\tau_H(\Phi_1)\):
\[\tau_H(\Phi_1) = \int_0^\infty dt |R_{\Phi_1}(t)|^2 = I_1 + I_2 + I_3 \] (129)
where
\[I_1 \equiv \int_0^{t_\epsilon} dt |R_{\Phi_1}(t)|^2 \] (130)
\[I_2 \equiv \int_{t_\epsilon}^{t_0} dt |R_{\Phi_1}(t)|^2 \] (131)
\[I_3 \equiv \int_{t_0}^{\infty} dt |R_{\Phi_1}(t)|^2 \] (132)

We choose
\[t_\epsilon = 10^{-4} \frac{1}{\Gamma} \] (133)
whereby
\[I_1 \leq 10^{-4} \frac{1}{\Gamma} \] (134)
We further choose \(t_0\) such that
\[\exp(-\frac{t}{2\tau}) \geq |c| \frac{\beta^2}{E_0 t} \text{ if } t_\epsilon \leq t \leq t_0 \] (135)
Let
\[ \tau = \frac{1}{\Gamma} = (2\pi\beta^2 E_0)^{-1} \]
Then,
\[ \frac{|c|\beta^2 \tau}{E_0 \tau^t} = \frac{|c|\beta^2}{E_0} (2\pi\beta^2 E_0) \frac{\tau}{t} \]
The r.h.s. above equals \( y \frac{\tau}{t} \), where
\[ y = |c|\beta^4 2\pi = |c|\alpha^6 2\pi = 10^{-12} \]
If \( \frac{t_0}{\tau} = 48 \), \( \frac{t_0}{2\tau} = 24 \), and \( 135 \) becomes the inequality
\[ \exp(-24) \geq 10^{-12} \frac{1}{48} \]
which does hold. \( 135 \) is, therefore, satisfied for \( t = t_0 \). It is straightforward to verify \( 135 \) for \( t_c \). Due to \( 135 \) it is also easy to prove that
\[ |\tau H(\Phi_1) - \frac{1}{\Gamma}| \leq 10^{-4} \frac{1}{\Gamma} \]
(136)
Indeed, the dominating term in \( 129 \) is \( \frac{\exp(-\Gamma t_c) - \exp(-\Gamma t_0)}{\Gamma} \), which is very close to \( \frac{1}{t} \) by the choice \( 133 \), which is consistent with \( 68 \). The dominating term in the remainder is given by the contribution of the r.h.s. of \( 135 \) in the integral \( I_3 \), which equals
\[ \int_{t_0}^{\infty} dt \frac{|c|^2 \beta^4}{E_0^2 t^2} = \frac{|c|^2 \beta^4}{E_0^2} 2\pi \beta^2 E_0 \frac{t_0^2}{48} \]
The latter quantity is of order \( \alpha^8 \approx 10^{-16} \) for \( |c| \) of order one. By \( 122 \) of theorem 5.1, and \( 136 \), we obtain \( 127 \). By theorem 4.1 and \( 125 \), it follows that \( Q(t) = O(\frac{1}{t^2}) \) for large \( t \), so that \( 126 \) holds, and thus b.).

\[ \blacksquare \]

Remark 5.1. The interest of \( 127 \) is better appreciated by realizing that the method of proof of theorem 4.1, i.e., comparison with the Lorentzian \( L(\lambda) \), fails for \( \Delta E \), because the r.h.s. of \( 128 \), when \( g_{\omega_1}(\lambda) \) is replaced by \( L(\lambda) \), is infinite.
Further, (136) shows that the sojourn time equals indeed, to a very good approximation, the inverse half-width of the state. This is due to the apparently general fact that, both in atomic and particle physics, the Lorentzian (Breit-Wigner) approximation is excellent - as seen from (135) and the fact that, after 48 lifetimes, the atom "has decayed for all practical purposes", as remarked by Nussenzveig in [Bar84].

**Remark 5.2.** In order that the level shift \( \lambda_0 - E_0 \) may be measured with great precision, as is the case of the Lamb shift, it is crucial that it is of lower order than the width. It seems remarkable that this is so even in this simple model, where \( \lambda_0 - E_0 = O(\beta^2) = O(\alpha^3) \), and \( \Gamma = 2\pi\beta^2G(E_0) \approx 2\pi\beta^2E_0 = O(\alpha^3)\alpha = O(\alpha^4) \), since \( E_0 = O(\alpha) \).

6 Conclusion

In theorem 4.1 we provided a first proof of the fact that positivity of the Hamiltonian \( H \) implies (62), with \( c(t) = O(1/t) \). This correction, although very small and negligible for the computation of the half-width \( \frac{1}{\Gamma} \) (theorem 5.2), plays nevertheless a basic conceptual role. By theorem 2.5, it is due to the regeneration of the decaying state from the decay products, a virtual process which is of the same nature of the tunneling which plays a crucial role in the Gamow theory of alpha decay ([Gam28], [BH]) but, unlike the latter, is characteristic of a quantum field theory (see remark 2.1). This connection is due to Sinha, Williams, and Fonda, Girardi and Rimini ([Sin72], [Wil71], [FGR78]), in connection with the forgotten concept of "unstable wave-function", but is shown to be a sound one in the usual framework of quantum mechanics and/or quantum field theory.

The connection between positivity of the Hamiltonian and the existence of some \( (o(t)) \) correction to exponential decay is well-known, an in its most general form due to Sinha [Sin72], viz., Theorem 2.2. The proof of theorem 4.1 relates directly the correction \( c(t) = O(1/t) \) to the positivity of the Hamiltonian.

Also due to Sinha [Sin77] and Lavine [Lav78] is the concept of sojourn time \( \tau_H(\Psi) \) given by (120). As a functional over a particular set of elements \( \Psi \) of the Hilbert space \( H \), on which the self-adjoint operator is defined (e.g., in potential theory, the set of Kato-smooth vectors, see [RS78] and [Lav78], the problem was posed by the late Pierre Duclos (see also [BDFSL10]) of obtaining lower bounds to \( \tau_H(\Psi) \), motivated by the expectation that, near
resonances, $\tau_H$ assumes very large values; one lower bound was given by Lavine’s form of the time-energy uncertainty relation [Lav78], another by the rigorous form of the Gislason-Sabelli-Wood time-energy uncertainty relation, theorem 3.17 of [MW13], reproduced in appendix A. The application to the present model (theorem 5.2) shows that the sojourn time is the physically most natural concept describing decay, because it coincides with the average lifetime of the state, a standard concept in quantum probability.

In spite of its simplicity, the present model has some surprisingly realistic features (see, e.g., remark 5.2). Its most unrealistic aspect is, of course, the lack of vacuum polarization, which allows us to work in Fock space and yields an unphysical conservation law, which is, however, responsible for the relatively easy estimates of the time evolution, viz., of the return probability amplitude of the Weisskopf-Wigner state. When the ”counterrotating” term given by (76) is added to $H$, this is no longer the case, but a perturbative treatment (Dav, see also [DN80]) is available: the final results for the Lamb shift, as well as for the line shape, are in good agreement with experiment [DN80].

On the other hand, the application of this model to particle physics [AMKG57] yields a completely different problem: (13) and (14) are no longer true, but play the role of cutoffs, which must be eliminated, by fixing the level-shift and the half-width at their physical values in the limit when the cutoffs are removed. In this process, however, ”ghosts” appear, or, alternatively, the coupling constant becomes complex [AMKG57]. As discussed elsewhere [JW18], this is an open problem of crucial importance: it would represent the first quantum field theoretic model of an unstable particle. Parenthetically, we have always been referring to models with no ultraviolet cutoffs, or where they have been eliminated, because they still display some of the group theoretic symmetries of a bona-fide field theory; after all, the very concept of particles arises from representations of the Poincaré group [Wei96]. There do exist, however, important models of unstable particles with a cutoff, see [ABFG11].

As a final remark, the conservation laws in the present model reduce its potential application to the study of irreversibility [Wre]. It allows, however, a discussion of the problem of the arrow of time (see section 3.2), and to formulate a general conceptual framework for the theory of irreversibility in quantum field theory. We have done so in section 3.2, which also discusses in which way this proposed framework may be complementary to other theories [Froa, Frob, Buc77, BR14, BJ95]. We believe that the fact that we use
the "method of decay without analyticity" described in the introduction and in greater detail in section 3.1, which does not add any new or controversial issues involved in some special notion of "resonance", is of particular importance in discussion of such conceptual issues.

7 Appendix A: a time-energy uncertainty relation

In this appendix we prove theorem 5.1, which is theorem 3.17 of [MW13] up to some minor corrections. The notation is the same as in section 5.

Proof of theorem 5.1
As in (31) we write (omitting the subscript $\Psi_0$ in $g$ for brevity):

$$g(u) = \frac{d\mu_{\Psi_0}(u)}{du}$$  \hfill (A.1)

By the spectral theorem and (A.1),

$$g(\lambda) \geq 0 \text{ for a.e. } \lambda \in \mathbb{R}$$  \hfill (A.2)

and by normalization of $\Psi_0$,

$$\int_{-\infty}^{\infty} d\lambda g(\lambda) = 1$$  \hfill (A.3)

By (83), (120), and Parseval’s formula we obtain

$$\tau_H(\Psi_0) = \pi \int_{-\infty}^{\infty} d\lambda \lambda^2 g(\lambda)$$  \hfill (A.4)

By the spectral theorem, and assumption (121),

$$(\Psi_0, H^2\Psi_0) = \int_{-\infty}^{\infty} d\lambda \lambda^2 g(\lambda) < \infty$$  \hfill (A.5)

By (A.4) and (A.5), $I_H(\Psi_0)$, defined by (122), is a functional of the sole function $g(.)$. Define now

$$\tilde{H} = H - (\Psi_0, H\Psi_0)$$  \hfill (A.6)
which satisfies
\[(Ψ_0, HΨ_0) = 0 \quad (A.7)\]
The replacement \(H \to \tilde{H}\) does not change \(τ_H(Ψ_0)\) by definition (120). We may, further, by scaling \(g(λ) \to α^{-1}g(αλ)\), which preserves the normalization \((A.3)\), choose \(α\) such that
\[∫_{-∞}^{∞} dλλ^2g(λ) = 1 \quad (A.8)\]
which must be added to the condition
\[∫_{-∞}^{∞} dλλg(λ) = 0 \quad (A.9)\]
coming from \((A.6)\), noting that, now, \(g(\cdot)\) is defined by \((A.1)\), but with \(μ_{Ψ_0}\) the a.c. spectral measure associated to \(H\). The functional \(I_H(Ψ_0)\), defined by \((122)\), now becomes
\[I_H(Ψ_0) = I_g(Ψ_0) = ∫_{-∞}^{∞} dλg(λ)^2 \quad (A.10)\]
i.e., a functional on the set of real valued \(g \in L^2(\mathbb{R})\), satisfying, in addition, \((A.8)\), \((A.9)\), and the conditions \((A.2)\) and \((A.3)\) (continuity, as mentioned in \([GSW85]\), should not be assumed, as it is too restrictive). We now turn to the solution of this variational problem. Let
\[P_t(λ) = \begin{cases} \frac{3\sqrt(5)}{20}(1 - \frac{λ^2}{5}), & \text{if } |λ| ≤ 5 \\ 0 & \text{otherwise} \end{cases} \quad (A.11)\]
denote the truncated parabola. By explicit computation, it satisfies the conditions \((A.2)\) and \((A.3)\). We now prove that \(g_0 \in L^2(\mathbb{R})\) satisfying \((A.2)\) and \((A.3)\), as well as \((A.8)\) and \((A.9)\) is such that
\[I_{P_t}(Ψ_0) ≤ I_{g_0}(Ψ_0) \quad (A.12)\]
with equality only if \(P_t(λ) = g_0(λ)\) a.e. in \(\mathbb{R}\). In order to show \((A.12)\), let
\[g_1(λ) ≡ g_0(λ) - P_t(λ) \quad (A.13)\]
Then, we have
\[I_{g_0}(Ψ_0) = I_{P_t+g_1}(Ψ_0) = I_{P_t}(Ψ_0) + ∫_{-∞}^{∞} dλP_t(λ)g_1(λ) \quad (A.13)\]
Note that the last integral in (A.13) is finite by the Schwarz inequality. From (A.10), $I_{g_1}(\Psi_0) \geq 0$ and, indeed, $I_{g_1}(\Psi_0) > 0$ unless $g_1(\lambda) = 0$ a.e. in $\lambda \in \mathbb{R}$. Hence, we need only show that the last integral in (A.13) is non-negative. In order to do so, let
\[
\lambda \in \mathbb{R} \rightarrow g_2(\lambda) \equiv \frac{3\sqrt{5}}{20}(1 - \frac{\lambda^2}{5})
\]
(A.14)
denote the parabola without truncation. Since both $g_0$ and $P_t$ obey (A.2), (A.3), (A.8) and (A.9) require that
\[
\int_{-\infty}^{\infty} d\lambda g_1(\lambda)\lambda^n = 0 \text{ for } n = 0, 1, 2
\]
(A.15)
and, therefore,
\[
0 = \int_{-\infty}^{\infty} d\lambda g_1(\lambda)g_2(\lambda) = \\
= \int_{-5^{1/2}}^{5^{1/2}} d\lambda g_1(\lambda)g_2(\lambda) + \\
+ \int_{5^{1/2}}^{\infty} d\lambda g_1(\lambda)g_2(\lambda)
\]
(A.16)
The last two integrals on the r.h.s. of (A.16), which are finite by (A.13) and the assumption that $g_0$ satisfies (A.8), are non-positive because, for $|\lambda| \geq 5^{1/2}$, $g_2(\lambda) \leq 0$ by (A.14) and $g_1 = g_0 \geq 0$ a.e. by (A.2). Therefore, the first integral on the r.h.s. of (A.16) is non-negative. However, by definitions (A.11) and (A.14), $g_2(\lambda) = P_t(\lambda)$ if $|\lambda| \leq 5^{1/2}$, and, thus, the first integral on the r.h.s. of (A.16) equals the last integral in (A.13). This shows (A.12), which completes the proof. q.e.d.

8 Appendix B - completion of the proof of theorem 4.1

In this appendix we prove that (115) of theorem 4.1 holds. Together with (112), this proves (90), and thereby completes the proof of theorem 4.1.
We first write (113) as the limit, as $\delta \downarrow 0$, of the corresponding integral from $\delta > 0$ to $\infty$. By integration by parts on the latter, we find

$$D_1^L(t) = \lim_{\delta \downarrow 0} \left[ -\frac{w(\delta)}{it\alpha(\delta)\beta(\delta)} + \int_{\delta}^{\infty} d\lambda \exp(-it\lambda) \frac{d}{d\lambda} \left( \frac{w(\lambda)}{\alpha(\lambda)\beta(\lambda)} \right) \right]$$

where, for $\lambda > 0$,

$$\alpha(\lambda) \equiv E_j - \lambda - \beta^2 F(\lambda) - i\pi \beta^2 G(\lambda) \quad (B.2)$$

and

$$\beta(\lambda) \equiv \kappa(\lambda - \lambda_0) - i\pi \beta^2 G(\lambda_0) \quad (B.3)$$

We have, the prime denoting, as usual, the first derivative,

$$\alpha'(\lambda) = -1 - \beta^2 F'(\lambda) - i\pi \beta^2 G'(\lambda) \quad (B.4)$$

and

$$\beta'(\lambda) = \kappa \quad (B.5)$$

From (93),

$$w'(\lambda) = -\beta^2 (F'(\lambda) - F'(\lambda_0)) - i\pi \beta^2 (G'(\lambda) - G'(\lambda_0)) \quad (B.6)$$

From (13),(14),(77) and (78) we have

$$G(\lambda) = \lambda(\lambda^2 + a^2)^{-4} \text{ for } \lambda \geq 0 \quad (B.8.1)$$
\[ G'(\lambda) = (\lambda^2 + a^2)^{-4} - 8\lambda^2(\lambda^2 + a^2)^{-5} \quad (B.8.2) \]

When writing \( f(0) \) in the following, for some function \( f \), it will be meant the limit \( \lim_{\delta \downarrow 0} f(\delta) \). The finiteness of the resulting limits, for all the functions which follow, will result from (79), which will be proved later as part of the forthcoming property b.) of the function \( F \). We have, then:

\[ F(0) = \int_0^{\infty} (k^2 + a^2)^{-4} dk \quad (B.8.3) \]

\[ G(0) = 0 \quad (B.8.4) \]

\[ w(0) = -\beta^2[F(0) - F(\lambda_0) + \lambda_0 F'(\lambda_0)] - i\pi\beta^2[-G(\lambda_0 + \lambda_0 G'(\lambda_0)] \quad (B.8.5) \]

\[ \alpha(0) = E_0 - \beta^2 F(0) \quad (B.8.6) \]

\[ \beta(0) = -\kappa\lambda_0 - i\pi\beta^2 G(\lambda_0) \quad (B.8.7) \]

The first term in (B.1) satisfies, in the limit \( \delta \downarrow 0 \), the bound on the r.h.s. of (115), by (B.8.5), (B.8.6) and (B.8.7). Therefore, by (B.1) and (B.7), in order to conclude the proof of (115), we need only prove that

\[ | \int_0^{\infty} \frac{\alpha'(\lambda)}{\alpha(\lambda)^2} w(\lambda) d\lambda | < \infty \quad (B.9.1) \]

\[ | \int_0^{\infty} \frac{\beta'(\lambda)}{\alpha(\lambda) \beta(\lambda)^2} w(\lambda) d\lambda | < \infty \quad (B.9.2) \]

\[ | \int_0^{\infty} \frac{1}{\alpha(\lambda) \beta(\lambda)} w'(\lambda) d\lambda | < \infty \quad (B.9.3) \]

It follows from (B.2), (B.3), (B.4), (B.5), (B.8.1) and (B.8.2) and (93) that (B.9.1)-(B.9.3) hold if the two following assertions are true:

a.) For \( \lambda \) sufficiently large, \( F(\lambda) \) and \( F'(\lambda) \) are uniformly bounded in \( \lambda \);

b.) For \( \lambda \) in a sufficiently small right-neighbourhood of zero, \( F(\lambda) \) is uniformly bounded, (79) holds and

\[ F'(\lambda) = -\log \lambda + D \]

where \( 0 < D < \infty \) is independent of \( \lambda \).
Indeed, \( b. \) implies that \( \alpha', \) as well as \( w', \) are integrable in a neighbourhood of zero, which suffice to prove integrability of \( \frac{\alpha'(\lambda)}{\alpha(\lambda)^2 \beta(\lambda)} w(\lambda) \) and of \( \frac{1}{\alpha(\lambda) \beta(\lambda)} w'(\lambda) \), in a neighbourhood of zero, elements in the proof of (B.9.1) and (B.9.3). 

Convergence at infinity of the integrals on the left hand sides of (B.9.1)-(B.9.3) is an immediate consequence of the explicit formulae for \( \alpha, \beta \) and \( w, \) together with \( a. \).

In order to prove \( a. \) and \( b. \), we come back to (14), whereby, for any \( \lambda > 0, \)

\[
F(\lambda) = \lim_{r \to 0} \int_{|k-\lambda| \geq r} \frac{G(k)}{k - \lambda} dk
\]

We write

\[
\int_{|k-\lambda| \geq r} \frac{G(k)}{k - \lambda} = \int_{0}^{\lambda-r} \frac{G(k)}{k - \lambda} dk + \int_{\lambda+r}^{2\lambda} \frac{G(k)}{k - \lambda} + \int_{2\lambda}^{\infty} \frac{G(k)}{k - \lambda} dk
\]

but

\[
\int_{0}^{\lambda-r} \frac{G(k)}{k - \lambda} dk + \int_{\lambda+r}^{2\lambda} \frac{G(k)}{k - \lambda} = \int_{r}^{\lambda} \frac{1}{k} [G(k + \lambda) - G(k - \lambda)] dk
\]

Write

\[
G(k + \lambda) - G(k - \lambda) = k \int_{-1}^{1} dt G'(\lambda + kt)
\]

Thus,

\[
F(\lambda) = \int_{0}^{\lambda} dk \int_{-1}^{1} dt \left\{ [(\lambda + kt)^2 + a^2]^{-4} - 8(\lambda + kt)^2[(\lambda + kt)^2 + a^2]^{-5} \right\} + \int_{2\lambda}^{\infty} \frac{G(k)}{k - \lambda} dk
\]

\[(B.11)\]
We write

\[ F(\lambda) = -7 \int_0^\lambda dk \int_{-1}^1 dt[(\lambda + kt)^2 + a^2]^{-4} + \]
\[ + 8a^2 \int_0^\lambda dk \int_{-1}^1 dt[(\lambda + kt)^2 + a^2]^{-5} + \]
\[ + \int_{2\lambda}^\infty \frac{G(k)}{k - \lambda} dk \]

from which

\[ F'(\lambda) = -7 \int_{-1}^1 dt[\lambda^2(1 + t)^2 + a^2]^{-4} + \]
\[ + 8a^2 \int_{-1}^1 dt[\lambda^2(1 + t)^2 + a^2]^{-5} + \]
\[ + 28 \int_0^\lambda dk \int_{-1}^1 [(\lambda + kt)^2 + a^2]^{-5}(\lambda + kt) \]
\[ - 40a^2 \int_0^\lambda dk \int_{-1}^1 dt[(\lambda + kt)^2 + a^2]^{-6}(\lambda + kt) \]
\[ - 2 \int_{2\lambda}^\infty \frac{G(k)}{k - \lambda} dk - \int_{2\lambda}^\infty \frac{G(k)}{(k - \lambda)^2} dk \]

(B.12)

By (B.11), we obtain directly a.) for \( F(\lambda) \), as well as the statements in b.) which concern \( F(\lambda) \). Statement b.) for \( F'(\lambda) \) follows from (B.8.1) and the last term in (B.12). Statement a.) for \( F'(\lambda) \) is not entirely obvious from (B.12), but we use

\[ b(\lambda + kt) \leq (\lambda + kt)^2 + a^2 \]

which is true for \( b \) sufficiently small, to bound the third and fourth terms in (B.12) in absolute value by

\[ \text{const. } \int_0^\lambda dk((\lambda - k)^2 + a^2)^{-4} \text{ resp. const. } \int_0^\lambda dk(\lambda - k)^2 + a^2)^{-5} \]

which are trivially seen to be uniformly bounded in \( \lambda \) by a change of variable. This completes the proof of (115). q.e.d.
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