Efficient Frequency-Domain Decoding Algorithms for Reed-Solomon Codes

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Abstract

This work develops frequency-domain decoding algorithms for \((n = 2^m, k)\) systematic Reed-Solomon (RS) codes over fields \(\mathbb{F}_{2^m}, m \in \mathbb{Z}^+\), where \(n - k\) is a power of two. The proposed algorithms are based on a new polynomial basis with a fast Fourier transform with computational complexity of order \(O(n \lg(n))\). First, the basis of syndrome polynomials is reformulated in the decoding procedure so that the new transforms can be applied to the decoding procedure. A fast extended Euclidean algorithm is developed to determine the error locator polynomial. The computational complexity of the proposed decoding algorithm is \(O(n \lg(n-k) + (n-k) \lg^2(n-k))\), improving upon the best currently available decoding complexity of \(O(n \lg^2(n) \lg \lg(n))\) and reaching the best known complexity bound that was established by Justesen in 1976, whose approach is for RS codes that operate only on some specified finite fields. As revealed by the computer simulations, the proposed decoding algorithm is 50 times faster than the conventional one for the \((2^{16}, 2^{15})\) RS code.

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I. Introduction

Reed-Solomon (RS) codes are a class of block error-correcting codes that were invented by Reed and Solomon [1] in 1960. An \((n, k)\) RS code is constructed over \(\mathbb{F}_q\), for \(n = q - 1\). An extended version, called extended Reed-Solomon codes [2], admits a codeword length of up to \(n = q\) or \(n = q + 1\). The systematic way of \((n, k)\) RS code appends \(n - k\) parity symbols to the \(k\) message symbols, forming a codeword of length \(n\). \((n, k)\) RS codes can correct up to \([(n - k)/2]\) erroneous symbols. Nowadays, RS codes have numerous important applications, including barcodes (such as the QR code), storage devices (such as DVD and Blu-ray Discs), digital television (such as DVB and ATSC), and data transmission technologies (such as DSL and WiMAX). RS codes are also be used to design other forward error correction codes, such as regenerating codes [3][4] and local reconstruction (such as DVB and ATSC), and data transmission technologies (such as DSL and WiMAX). RS codes are also

A conventional syndrome-based RS decoding algorithm requires \(O(n(n - k) + (n - k)^2)\) operations. Some approaches [8][9] achieve lower computational complexities by using fast Fourier transforms (FFT) or fast polynomial multiplication techniques. However, the complexities of FFTs over finite fields vary with the size of the fields \(\mathbb{F}_q\).

When \(q - 1\) is a smooth number, meaning that \(q - 1\) can be factorized into many small prime integers, an FFT over \(\mathbb{F}_q\) in \(O(n \log(n))\) additions and multiplications can be developed. A conventional case involves choosing Fermat primes \(q \in \{2^m + 1 | m = 1, 2, 4, 8, 16\}\). Based on the transforms, in 1976, Justesen [8] developed an \(O(n \log^2(n))\) approach for decoding \((n, k)\) RS code over \(\mathbb{F}_{2^m+1}\), where \(O(n \log^2(n))\) is the best known complexity bound.

However, if \(q - 1\) is not a smooth number, then the complexity of Justesen’s approach [8] may be far from \(O(n \log^2(n))\), such as when \(q = 2^m\). As mentioned above, RS codes are typically constructed over characteristic-2 fields and these have been comprehensively examined. In 2002, Gao [9] presented an algorithm for decoding RS codes over arbitrary finite fields with a complexity of \(O(n \log^2(n) \log \log(n))\), by utilizing fast polynomial multiplication algorithms [10]. He also suggested [9] that if RS codes are applied over \(\mathbb{F}_{2^m}\), then the additive FFT proposed by Gao and Mateer [11] that has \(O(n \log(n) \log \log(n))\) operations can be used to reduce the leading constant further. To the best of the authors’ knowledge, the FFT given in [11] is the fastest algorithm over \(\mathbb{F}_{2^m}\). (For more FFTs, please see Table II of Lin et al. [12]).

Thus, if \(q - 1\) is a smooth number, then decoding an RS codeword takes \(O(n \log^2(n))\) operations, or else \(O(n \log^2(n) \log \log(n))\) operations [8] and [9], that adds an extra \(\log \log(n)\) factor to the complexity of order. Consider that characteristic-2 fields are widely used in the development of RS codes, this raises a natural question about the existence of an \(O(n \log^2(n))\) RS decoding algorithm over \(\mathbb{F}_{2^m}\). This paper provides a positive answer to this question.

The aforementioned extra \(\log \log(n)\) factor comes from the FFTs over finite fields [9]. Accordingly, to design an algorithm for decoding RS codes with a complexity of \(O(n \log^2(n))\), an FFT with a complexity of \(O(n \log(n))\) is required. Recently, Lin et al. [12] showed a possible way to solve aforementioned FFT problem. In [12], the authors defined a non-standard polynomial basis based on subspace polynomials over \(\mathbb{F}_{2^m}\). For a polynomial of degree \(h - 1\) that is represented in this non-standard basis, \(h\)-point multipoint evaluations can be made in \(O(h \log(h))\) field operations. Based on the multipoint evaluation algorithm, encoding/erasure decoding algorithms with \(O(n \log(n))\) for \((n, k)\) RS codes were proposed [12]. However, the error-correction decoding algorithm for the RS codes was not provided [12].

This paper develops an error correction decoding algorithm for \((n = 2^m, k)\) RS codes over \(\mathbb{F}_{2^m}\), for \(k/n \geq 1/2\) and \((n-k)\) a power of two. The complexity of the proposed decoding algorithm is \(O(n \log(n-k) + (n-k) \log^2(n-k))\). Holding constant the code rate \(k/n\) yields a complexity \(O(n \log^2(n))\), which is better than the best existing complexity of \(O(n \log^2(n) \log \log(n))\), that was achieved by Gao [9] in 2002, and reaches the best known complexity \(O(n \log^2(n))\), that was achieved on certain fields, by Justesen [8] in 1976. The core of the technique is based on the non-standard polynomial basis, that was given by Lin et al. [12]. To exploit the advantages of the new basis, the syndrome-based RS decoding approach is reformed such that all polynomial arithmetics are implemented in the new basis. Additionally, the efficient polynomial division algorithm and Euclidean algorithm in this new basis are proposed. Based on these techniques, a fast error correction decoding algorithm for RS codes is proposed. The major contributions of this paper are summarized as follows.

1) An \(O(h \log(h))\) algorithm for polynomial division in the new basis is derived.
2) An $O(h \lg^2(h))$ fast half-GCD algorithm in the new basis is presented.
3) An $O(n \lg(n-k))$ RS encoding algorithm is presented, for $n-k$ a power of two.
4) A syndrome-based RS decoding algorithm that is based on the new basis is demonstrated.
5) An $O(n \lg(n-k) + (n-k) \lg^2(n-k))$ RS decoding algorithm is presented, for $n-k$ a power of two.

Notably, [12] gave the encoding algorithms for RS codes with the complexity $O(n \lg(k))$, for $k$ a power of two. The encoding algorithm [12] is suitable for coding rate $k/n \leq 0.5$, and the proposed version is suitable for $k/n \geq 0.5$.

The rest of this paper is organized as follows. Section II reviews the definitions of the non-standard polynomial basis and the corresponding multipoint polynomial evaluation algorithm. Section III provides an alternative polynomial basis that is constructed using monic polynomials. The polynomial operations that are used in the encoding/decoding of RS codes are explicated. Section IV presents the fast extended Euclidean algorithm that is based on the half-GCD method. Section V and Section VI introduce the algorithms for encoding and decoding RS codes. Section VII presents simulations and draws conclusions.

II. POLYNOMIAL BASIS BASED ON SUBSPACE POLYNOMIAL

This section reviews the subspace polynomials over $\mathbb{F}_{2^m}$, the polynomial basis defined in [12], and its multipoint evaluation algorithms.

A. Subspace polynomial

Let $\mathbb{F}_{2^m}$ denote an extension finite field with dimension $m$ over $\mathbb{F}_2$. Let $\mathbf{v} = (v_0, v_1, \ldots, v_{m-1})$ denote a basis of $\mathbb{F}_{2^m}$. That is, all $v_i \in \mathbb{F}_{2^m}$ are linearly independent over $\mathbb{F}_2$. A $k$-dimensional space $V_k$ of $\mathbb{F}_{2^m}$ is defined as

$$V_k = \text{span}(\mathbf{v}_k) = \{i_0 \cdot v_0 + i_1 \cdot v_1 + \cdots + i_{k-1} \cdot v_{k-1} | \forall i_j \in \{0, 1\}\},$$

where $\mathbf{v}_k = (v_0, v_1, \ldots, v_{k-1})$ is a basis of space $V_k$, and $k \leq m$. We can form a strictly ascending chain of subspaces given by

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = \mathbb{F}_{2^m}.$$

Let $\{\omega_i\}_{i=0}^{2^m-1}$ denote the elements of $\mathbb{F}_{2^m}$. Each of which is defined as

$$\omega_i = i_0 \cdot v_0 + i_1 \cdot v_1 + \cdots + i_{m-1} \cdot v_{m-1},$$

where $i_j \in \{0, 1\}$ is the binary representation of $i$. That is,

$$i = i_0 + i_1 \cdot 2 + \cdots + i_{m-1} \cdot 2^{m-1}, \forall i_j \in \{0, 1\}.$$

It can be seen that $V_k = \{\omega_i\}_{i=0}^{2^k-1}$, $k = 0, 1, \ldots, m$.

The subspace polynomial [11, 13, 14] of $V_k$ is defined as

$$s_k(x) = \prod_{a \in V_k} (x - a).$$

Clearly, $\deg(s_k(x)) = 2^k$. The properties of $s_k(x)$ are given in [13, 15].

Lemma 1 ([13, 15]). (i) $s_k(x)$ is an $\mathbb{F}_2$-linearized polynomial for which

$$s_k(x) = \sum_{i=0}^{k} s_{k,i} x^i,$$

where each coefficient $s_{k,i} \in \mathbb{F}_{2^m}$. Furthermore,

$$s_k(x + y) = s_k(x) + s_k(y), \forall x, y \in \mathbb{F}_{2^m}[x].$$

(ii) The formal derivative of $s_k(x)$ is

$$s_k'(x) = \prod_{a \in V_k, a \neq 0} a.$$
B. Polynomial basis

Let \( \mathbb{K}(x) = (\mathbb{K}_0(x), \mathbb{K}_1(x), \ldots, \mathbb{K}_{2^m-1}(x)) \) denote a polynomial basis in \( \mathbb{F}_{2^m}[x]/(x^{2^m} - x) \). Each of which is defined as

\[
\mathbb{K}_i(x) = \frac{X_i(x)}{p_i},
\]

where

\[
X_i(x) = \prod_{j=0}^{m-1} (s_j(x))^{i_j}, \quad p_i = \prod_{j=0}^{m-1} (s_j(v_j))^{i_j}.
\]

Here, each \( i_j \in \{0,1\} \) is the binary representation of \( i \). Notice that \( (s_j(x))^0 = (s_j(v_j))^0 = 1 \). It can be seen that \( \deg(\mathbb{K}_i) = i \). A polynomial \( D_h(x) \) of degree \( h \) in the basis \( \mathbb{K}(x) \) is defined as

\[
D_h(x) = \sum_{i=0}^{h-1} d_i \mathbb{K}_i(x),
\]

where each \( d_i \in \mathbb{F}_{2^m} \). In this paper, we use \( D_h = (d_0, d_1, \ldots, d_{h-1}) \) to denote the vector of the coefficients of \( D_h(x) \). Based on the fact \( \deg(\mathbb{K}_i(x)) = i \), the new basis possesses the following properties.

Corollary 1. Let

\[
f(x) = \sum_{i=0}^{2^m-1} f_i(0) x^i = \sum_{i=0}^{2^m-1} f_i(1) \mathbb{K}_i(x)
\]

be a polynomial of degree \( h \) over \( \mathbb{F}_{2^m} \). The following properties hold.

1) \( f_i(0) = f_i(1) = 0 \), for \( i \geq h + 1 \).
2) \( f_h(1) = f_h(0) \cdot p_h \).
3) For \( 0 \leq j \leq h \), \( (f_j^{(1)}, f_{j+1}^{(1)}, \ldots, f_h^{(1)}) \) is determined by \( (f_j^{(0)}, f_{j+1}^{(0)}, \ldots, f_h^{(0)}) \), and vice versa.

C. Multipoint evaluation at \( V_k + \beta \)

For any polynomial \( f(x) \) and a space \( V \), let the notation \( f(V) \) denote a set of evaluation values \( f(V) = \{ f(a) | \forall a \in V \} \). In \([12]\), Lin et al. developed a multipoint evaluation algorithm to calculate \( D_{2^k}(V_k + \beta) \), where

\[
V_k + \beta = \{ a + \beta | a \in V_k \}.
\]

Algorithm \([\text{I}]\) shows the algorithm steps. In addition, let \( \text{ITr}(\bullet, \beta) \) denote the inverse algorithm. \( \text{ITr}(\bullet, \beta) \) is the procedure by backtracking the steps in Algorithm \([\text{I}]\) The details are described in Appendix \([\text{II}]\).

III. POLYNOMIAL BASIS WITH MONIC POLYNOMIALS AND ITS OPERATIONS

In this section, we define an alternative version of the polynomial basis, and its operations to perform polynomial multiplication, formal derivative, and division on the new basis. All these operations will be used in the design of encoding/decoding of RS codes. The polynomial basis is defined as

\[
\mathbb{X}(x) = (X_0(x), X_1(x), \ldots, X_{2^m-1}(x))
\]

in \( \mathbb{F}_{2^m}[x]/(x^{2^m} - x) \), where each \( X_i(x) \) is defined in \([\text{II}]\). As compared with \( \mathbb{K}(x) \), each \( X_i(x) \) in \( \mathbb{X}(x) \) is a monic polynomial. For any polynomial in \( \mathbb{X}(x) \), it take only \( h \) multiplication to convert the basis of the polynomial to \( \mathbb{K}(x) \):

\[
D_{h-1}(x) = \sum_{i=0}^{h-1} d_i \cdot X_i(x) = \sum_{i=0}^{h-1} d_i \cdot \mathbb{K}_i(x).
\]

Thus, the multipoint evaluation algorithm in \( \mathbb{K}(x) \) (Algorithm \([\text{II}]\) can also be applied on \( \mathbb{X}(x) \), by just taking \( h \) additional multiplications in the basis conversion between \( \mathbb{K}(x) \) and \( \mathbb{X}(x) \). Thus, the complexity of the multipoint evaluation algorithm in \( \mathbb{X}(x) \) is \( O(h \log(h)) \).
Algorithm 1 Multipoint evaluation algorithm for polynomials in \( \mathbb{F}(x) \)

**Input:** \( \text{Tr}(D_{2^k}, \beta) \), where \( D_{2^k} = (d_0, d_1, \ldots, d_{2^k-1}) \) is the coefficient vector of \( D_{2^k}(x) \) in \( \mathbb{F}(x) \), and \( \beta \in \mathbb{F}_{2^m} 

**Output:** A vector of evaluation values \((\tilde{D}_{2^k}(\omega_0 + \beta), \tilde{D}_{2^k}(\omega_1 + \beta), \ldots, \tilde{D}_{2^k}(\omega_{2^k-1} + \beta))\)

1. **if** \( k = 0 \) **then return** \( \tilde{d}_0 \)
2. **end if**
3. Compute \( \tilde{D}_{2^k-1} = (g_0^{(0)}, g_1^{(0)}, \ldots, g_{2^k-1}^{(0)}) \) and \( \tilde{D}_{2^k-1}^{(1)} = (g_0^{(1)}, g_1^{(1)}, \ldots, g_{2^k-1}^{(1)}) \), where
   \[
   g_i^{(0)} = \tilde{d}_i + \frac{s_{k-1}(\beta)}{s_{k-1}(v_{k-1})} \tilde{d}_{i+2^{k-1}} - i,
   \]
   \[
   g_i^{(1)} = g_i^{(0)} + \tilde{d}_{i+2^{k-1}} - i.
   \]
4. **Call**
   \[
   V_0 = \text{Tr}(\tilde{D}_{2^k-1}^{(0)}, \beta)
   \]
   \[
   V_1 = \text{Tr}(\tilde{D}_{2^k-1}^{(1)}, v_{k-1} + \beta)
   \]
5. **return** \((V_0, V_1)\)

In order to simplify the notations, in this rest of this paper, the polynomials are represented in \( \mathbb{F}(x) \). The notation
\[
\text{FFT}_\beta(D_{2^k}) = (D_{2^k}(\omega_0 + \beta), D_{2^k}(\omega_1 + \beta), \ldots, D_{2^k}(\omega_{2^k-1} + \beta))
\]
denotes the evaluation of \( D_{2^k}(x) \) in \( \mathbb{F}(x) \) at \( V_k + \beta = \{\omega_i + \beta\}_{i=0}^{2^k-1} \). Moreover, the inverse transform is denoted as \( \text{IFFT}_\beta(\cdot) \). Based on Algorithm [1], the transforms are defined as
\[
\text{FFT}_\beta(D_{2^k}) = \text{Tr}(D_{2^k} \otimes P_{2^k}, \beta), \quad \text{IFFT}_\beta(D_{2^k}) = \text{ITr}(D_{2^k}, \beta) \otimes P_{2^k},
\]
where \( P_{2^k} = (p_0, p_1, \ldots, p_{2^k-1}) \). The operation \( \otimes \) is the pairwise multiplication on two vectors, and the operation \( \odot \) is the pairwise division. The polynomial multiplication and formal derivative in \( \mathbb{F}(x) \) are shown in Appendix [C]. The polynomial division is given below.

### A. Polynomial Division based on new basis

Let \( S(A(x), i) \) denote the quotient via dividing \( A(x) \) by \( s_i(x) \), where \( A(x) \) is expressed in basis \( \mathbb{X}(x) \) and \( \deg(A(x)) < 2^{i+1} \). Precisely, for any polynomial \( A(x) = \sum_{l=0}^{h-1} a_l X_l(x) \) and \( h < 2^{i+1} \), we have
\[
S(A(x), i) = \sum_{l=0}^{h-1-2^i} a_{l+2^i} X_l(x).
\]

Given a dividend \( a(x) \) and a divisor \( b(x) \), the polynomial division is to determine the quotient \( Q(x) \) and the remainder \( r(x) \) such that
\[
a(x) = Q(x) \cdot b(x) + r(x), \tag{12}
\]
where \( \deg(r(x)) \leq \deg(b(x)) - 1 \). In this work, we consider
\[
\deg(a(x)) > \deg(b(x)) \geq 0. \tag{13}
\]
The cases outside the region of \((13)\) are trivial so that we do not consider it in this work.

Assume the quotient \( Q(x) \) is known, the remainder can be calculated by
\[
r(x) = a(x) - Q(x) \cdot b(x). \tag{14}
\]
Thus, the objective is to determine \( Q(x) \). We multiply \((12)\) by \( X_y(x) \), for a positive integer \( y \), to obtain
\[
a(x) \cdot X_y(x) = Q(x) \cdot b(x) \cdot X_y(x) + r(x) \cdot X_y(x). \tag{15}
\]
reformulating \((15)\), we have
\[
A(x) = Q(x) \cdot B(x) + R(x). \tag{16}
\]
Algorithm 2. Polynomial divisions in \( X(x) \)

**Input:** A dividend \( a(x) \) and a divisor \( b(x) \), where \( \deg(a(x)) > \deg(b(x)) \geq 0 \)

**Output:** A quotient \( Q(x) \) and a remainder \( r(x) \), such that

\[
a(x) = Q(x) \cdot b(x) + r(x).
\]

1: Compute

\[
A(x) = a(x) \cdot X_y(x),
\]

\[
B(x) = b(x) \cdot X_y(x),
\]

where \( y \) is defined as \( (17) \).

2: Find \( \Lambda(x) \) such that \( (18) \) holds.

3: Compute \( Q(x) \) by \( (24) \).

4: Compute \( r(x) \) by \( (14) \).

5: return \( Q(x) \) and \( r(x) \).

Next we present a method to determine \( Q(x) \) in \( (15) \) via \( (16) \). Here, the value of \( y \) is defined as

\[
y = 2^{D_\ell} - \deg(b(x)) - 1,
\]

where

\[
D_\ell = \lceil \lg(\deg(a(x)) + 1) \rceil.
\]

Assume that there exists a polynomial \( \Lambda(x) \) such that

\[
\Lambda(x) \cdot s_1(x) \cdot B(x) = s_{D_a}(x) + H(x),
\]

where \( \deg(H(x)) \leq \deg(B(x)) + 1 = 2^{D_\ell} \), and

\[
D_a = \lceil \lg(\deg(A(x)) + 1) \rceil.
\]

The algorithm to determine \( \Lambda(x) \) is shown in Section III-B

**Lemma 2.**

\[
D_a = D_\ell + 1.
\]

From \( (18) \) and Lemma 2, the degree of \( \Lambda(x) \) is thus

\[
\deg(\Lambda(x)) = 2^{D_a} - \deg(B(x)) - 2 = 2^{D_\ell} - 1.
\]

After obtaining \( \Lambda(x) \), \( (16) \) is multiplied by \( \hat{\Lambda}(x) = \Lambda(x) \cdot s_1(x) \) to get

\[
A(x) \cdot \hat{\Lambda}(x) = Q(x) \cdot B(x) \cdot \hat{\Lambda}(x) + R(x) \cdot \hat{\Lambda}(x)
\]

\[\Rightarrow A(x) \cdot \hat{\Lambda}(x) = Q(x) \cdot s_{D_a}(x) + Q(x) \cdot H(x) + R(x) \cdot \hat{\Lambda}(x).\] (22)

**Lemma 3.** In \( (22) \), we have

\[
\deg(Q(x) \cdot H(x) + R(x) \cdot \hat{\Lambda}(x)) \leq 2^{D_a} - 1.
\] (23)

In \( (22) \), \( Q(x) \cdot s_{D_a}(x) \) is a polynomial where the coefficients of \( Q(x) \) starts from \( X_{2^{D_a}}(x) = s_{D_a}(x) \). By Lemma 3, other terms have degrees less than \( 2^{D_a} - 1 \). Thus, the quotient can be obtained by

\[
Q(x) = S(A(x) \cdot \Lambda(x) \cdot s_1(x), D_a).
\] (24)

Algorithm 2 shows the details of the polynomial division. The complexity is analyzed below. In Step 1, as \( \deg(A(x)) = \deg(a(x)) + \deg(y(x)) < 2^{D_\ell + 1} \) and \( \deg(B(x)) = \deg(b(x)) + \deg(y(x)) = 2^{D_\ell} - 1 \), the complexity is \( O(2^{D_\ell} \lg(2^{D_\ell})) \). In Step 2, the algorithm to determine \( \Lambda_x(x) \) given in Section III-B is with the complexity of order \( O(2^{D_\ell} \lg(2^{D_\ell})) \). In Step 3, the complexity to compute \( A(x) \cdot \Lambda(x) \) is \( O(2^{D_a} \lg(2^{D_a})) \). In Step 4, the complexity is \( O(2^{D_a} \lg(2^{D_a})) \). In summary, Algorithm 2 has the complexity \( O(2^{D_a} \lg(2^{D_a})) = O(\deg(a(x)) \lg(\deg(a(x))) \).
Algorithm 3 $\Lambda(x)$ computation

**Input:** A polynomial $B(x)$

**Output:** A polynomial $\Lambda(x)$ such that (18) holds, where $\deg(\Lambda(x)) = \deg(B(x)) = 2^{D_\ell} - 1$.

1: Let $\Lambda_0(x) = b_{d_B}^{-1}$.
2: for $i = 1, 2, \ldots, D_\ell$ do
3:  Compute (28).
4:  Compute (29) (or (31), equivalently).
5: end for
6: return $\Lambda_{D_\ell}(x)$.

B. Determining $\Lambda(x)$

Given $B(x) = \sum_{j=0}^{d_B} b_j x^j$ with $b_{d_B} \neq 0$, this subsection presents a method to determine $\Lambda(x)$ in (18). Notice that $d_B = \deg(\Lambda) = 2^{D_\ell} - 1$. The method is a modified version of Newton division algorithm presented in [10][16].

The method iteratively computes the coefficients of $\Lambda(x)$ from high degrees to low degrees. In each iteration, we calculate the updated polynomial $\Lambda_i(x)$ with degree $\deg(\Lambda_i) = 2^i - 1$, for $i = 0, 1, \ldots, D_\ell$. The initial polynomial is defined as

$$\Lambda_0(x) = b_{d_B}^{-1}. \tag{25}$$

We define a set of polynomials $\{B_i(x)\}_{i=0}^{D_\ell}$ as follows. Let $B_{D_\ell}(x) = B(x)$, and

$$B_i(x) = S(B_{i+1}(x), i) \quad i = 0, 1, \ldots, D_\ell - 1. \tag{26}$$

It is clear that $\deg(B_i) = 2^i - 1$. (26) can be rewritten as

$$B_{i+1}(x) = B_i(x) \cdot s_i(x) + \bar{B}_i(x), \tag{27}$$

where $\bar{B}_i(x)$ is the residual polynomial and $\deg(\bar{B}_i) \leq 2^i - 1$.

In each iteration, we compute

$$\bar{\Lambda}_i(x) = (\Lambda_{i-1}(x))^2 \cdot B_i(x) \cdot s_1(x), \tag{28}$$

and hence $\deg(\bar{\Lambda}_i(x)) = 2^{i+1} - 1$. Then $\Lambda_i(x)$ is defined as

$$\Lambda_i(x) = S((s_{i-1}(x))^2 \cdot \bar{\Lambda}_i(x), i + 1). \tag{29}$$

The algorithm repeats performing (28) and (29) to obtain $\Lambda(x) = \Lambda_{D_\ell}(x)$. The validity of the iterative method is proved as follows.

**Lemma 4.** $\Lambda_i(x)$ possesses the following equality:

$$\Lambda_i(x) \cdot B_i(x) \cdot s_1(x) = s_{i+1}(x) + \bar{r}_i(x), \tag{30}$$

where $\deg(\bar{r}_i(x)) \leq 2^i$.

The details of the algorithm is given in Algorithm 3. For the complexity, each iteration (lines 3-4) calculates (28) and (29). In (28), since $\deg(\Lambda_{i-1}) = 2^{i-1} - 1$, $\deg(B_i) = 2^i - 1$ and $\deg(s_i) = 2$, the multiplications in (28) requires $O(2^i \lg(2^i))$. In (29), the computation can be simplified without polynomial multiplications as follows:

$$\Lambda_i^{\text{imp}}(x) = S(\bar{\Lambda}_i(x), i),$$

$$\Lambda_i(x) = \Lambda_i^{\text{imp}}(x) + S(\Lambda_i^{\text{imp}}(x), i - 1) \cdot s_{i-1}(v_{i-1}). \tag{31}$$

The complexity of calculating (31) is $O(2^i)$, so that each iteration takes $O(2^i \lg(2^i))$ operations. Thus, the computation complexity for the loop (line 2-5) takes

$$\sum_{i=1}^{D_\ell} O(2^i \lg(2^i)) = O(2^{D_\ell} \lg(2^{D_\ell})).$$
IV. Extended Euclidean Algorithm Based on Half-GCD Approach

This section introduces the extended Euclidean algorithm that will be used in the decoding RS codes. Given two polynomials \( a(x) = r_{-1}(x) \), \( b(x) = r_0(x) \), and

\[
\deg(b(x)) < \deg(a(x)) < 2^g.
\]  

Euclidean algorithm is a procedure to recursively divide \( r_{k-2}(x) \) by \( r_{k-1}(x) \) to obtain

\[
r_{k-2}(x) = q_k(x) \cdot r_{k-1}(x) + r_k(x),
\]

where \( \deg(r_k) < \deg(r_{k-1}) \). The procedure stops at \( r_N(x) = 0 \), and \( r_{N-1}(x) \) is the greatest common divisor (gcd) of \( a(x) \) and \( b(x) \). An extension version, called extended Euclidean algorithm, calculates \( r_k(x) \) with the coefficients \((u_k(x), v_k(x))\) in each iteration such that

\[
a(x) \cdot u_k(x) + b(x) \cdot v_k(x) = r_k(x).
\]

In matrix form, the \((k-1)\)-th step of extended Euclidean algorithm can be expressed as

\[
\begin{bmatrix}
  r_{k-2}(x) \\
  r_{k-1}(x)
\end{bmatrix} =
\begin{bmatrix}
  u_{k-2}(x) & v_{k-2}(x) \\
  u_{k-1}(x) & v_{k-1}(x)
\end{bmatrix}
\begin{bmatrix}
  a(x) \\
  b(x)
\end{bmatrix},
\]

and the next step is given by

\[
\begin{bmatrix}
  r_{k-1}(x) \\
  r_k(x)
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -q_k(x) \end{bmatrix}\begin{bmatrix}
  u_{k-2}(x) & v_{k-2}(x) \\
  u_{k-1}(x) & v_{k-1}(x)
\end{bmatrix}
\begin{bmatrix}
  a(x) \\
  b(x)
\end{bmatrix}.
\]

In this section, we present an approach, called half-GCD algorithm \([17][10]\), that calculates the temporal result of extended Euclidean algorithm at \(s\)-th step such that

\[
\deg(r_s(x)) \leq h/2 - 1.
\]

This approach will be performed to solve the error locator polynomial (see \(49\)) in the decoding procedure of RS codes.

For polynomials in the monomial basis \( \left\{ 1, x, x^2, \ldots \right\} \), there exist asymptotically fast approaches such as Algorithm 11.6 of \([10]\) and Figure 8.3 of \([16]\), in the complexity of order \(O(M(h)\log(h))\), where \(M(h)\) is the complexity to multiply two polynomials of degrees \(h/2\). The idea is based on an observation that the quotient only depends on the upper degree coefficients of the dividend and the divisor. From Corollary \([1]\) it can be seen that this idea is applicable to the new basis \(X(x)\), and the details are shown in Algorithm \(4\). Algorithm \(4\) is similar to that given in \([10]\) and \([16]\) except the polynomial basis is \(X(x)\). In Algorithm \(4\), we partition the inputs \(a(x)\) and \(b(x)\) into several portions, so that the procedure can be recursively applied on each portion. For the monomial basis, it is quite simple to make such partition. For \(X(x)\), we need to choose some fixed points at \(s_{2^{g-2}}(x)\) and \(s_{2^{g-1}}(x)\) to partition the polynomial. That is,

\[
a(x) = a_{LL}(x) + s_{g-2}(x)a_{LH}(x) + s_{g-1}(x)a_{H}(x) = a_{LL}(x) + s_{g-2}(x)a_{LH}(x) + s_{g-2}(x)s_{g-2}(x) + s_{g-2}(v_{g-2})a_{H}(x)
\]

where \(a_M(x) = a_{LH}(x) + (s_{g-2}(x) + s_{g-2}(v_{g-2}))a_{H}(x)\). Notice that \(\deg(a_{LL}(x)) < 2^{g-2}, \deg(a_{LH}(x)) < 2^{g-2}, \deg(a_{H}(x)) < 2^{g-1}, \) and \(\deg(a_M(x)) < 2^{g-2} + 2^{g-2}\). Also, \(b(x)\) can be partitioned in the same manner:

\[
b(x) = b_{LL}(x) + s_{g-2}(x)b_{LH}(x) + s_{g-1}(x)b_{H}(x) = b_{LL}(x) + s_{g-2}(x)b_{LH}(x) + s_{g-2}(v_{g-2})b_{H}(x).
\]

We determine the computational complexity as follows. The algorithm complexity is denoted as \(T(h)\) of polynomial degrees \(h = 2^g\). In Steps 3 and 8, the algorithm shall call the routine twice, and it takes \(2 \cdot T(h/2)\). \((41)\) is the polynomial division, where \(\deg(z_{M0}(x)) < h\). This requires \(O(h\log(h))\) by using the algorithm addressed in Sec. \([III]\). In \((39)\) and \((42)\), the computations involve polynomial additions and polynomials multiplications. As those polynomials have degrees less than \(h\), the complexity is \(O(h\log(h))\) by the results given in Appendix \([C]\). In summary, the overall complexity is

\[
T(h) = 2T(h/2) + O(h\log(h)), \quad \text{and} \quad T(h) = O(h\log^2(h)).
\]
Algorithm 4 Half-GCD algorithm

Input: HGCD$(a(x), b(x), 2^g)$, where $a(x), b(x) \in \mathbb{F}_{2^m}[x]$, $\deg(b(x)) < \deg(a(x)) < 2^g$

Output: $Z = \begin{bmatrix} z_0(x) \\ z_1(x) \end{bmatrix}$ and $M = \begin{bmatrix} u_0(x) & v_0(x) \\ u_1(x) & v_1(x) \end{bmatrix}$ such that $Z = M \cdot \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}$, that is the temporal result of the Extended Euclidean Algorithm to meet $\deg(z_1(x)) \leq 2^{g-1} - 1$.

1: if $\deg(b(x)) < 2^{g-1}$ then return

$$Z = \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (38)$$

2: end if

3: Call HGCD$(a_H(x), b_H(x), 2^{g-1})$ to obtain $(Z_H, M_H)$.

4: Compute

$$\left[ \begin{array}{c} z_{M0}(x) \\ z_{M1}(x) \end{array} \right] = Z_H \cdot (s_{g-2}(x) + s_{g-2}(v_{g-2})) + M_H \cdot \begin{bmatrix} a_{LL}(x) \\ b_{LL}(x) \end{bmatrix}. \quad (39)$$

5: if $\deg(z_{M1}(x)) < 2^{g-2}$ then return

$$Z = \begin{bmatrix} z_{M0}(x) \\ z_{M1}(x) \end{bmatrix} \cdot s_{g-2}(x) + M_H \cdot a_{LL}(x), \quad M = M_H. \quad (40)$$

6: end if

7: $z_{M0}(x)$ is divided by $z_{M1}(x)$ to get

$$z_{M0}(x) = q_M(x) \cdot z_{M1}(x) + r_M(x), \quad (41)$$

where $\deg(r_M(x)) < \deg(z_{M1}(x)) < 2^{g-1}$.

8: Call HGCD$(z_{M1}(x), r_M(x), 2^{g-1})$ to obtain $(Y_M, M_M)$.

9: return

$$M = M_M \cdot \begin{bmatrix} 0 & 1 \\ 1 & -q_M(x) \end{bmatrix} \cdot M_H, \quad Z = Y_M \cdot s_{g-2}(x) + M \cdot \begin{bmatrix} a_{LL}(x) \\ b_{LL}(x) \end{bmatrix}. \quad (42)$$

V. REED-SOLOMON ENCODING ALGORITHM

This section develops an $O(n \lg(n-k))$ encoding algorithm of $(n = 2^m, k)$ RS code over $\mathbb{F}_{2^m}$, for $T = n - k$ a power of two. There exist two viewpoints for the RS codes, termed as polynomial evaluation approach and generator polynomial approach. In the viewpoint of polynomial evaluation approach, the message is interpreted as a polynomial $u(x)$ of degree less than $k$ over $\mathbb{F}_{2^m}$. The codeword $v = (v_0, v_1, \ldots, v_{n-1})$, where each $v_i = u(\omega_i)$, is defined as the evaluations of $u(x)$ at $n$ distinct points. Assume $u(x) = \sum_{i=0}^{k-1} u_i \bar{X}_i(x)$ is in the basis $\bar{X}(x)$. Let

$$\bar{u}_n = (u, 0) = (u_0, u_1, \ldots, u_{k-1}, 0, 0, \ldots, 0)$$

denote the vector of coefficients $u$ with appending $T$ zeros. By using Algorithm 1, $v$ can be computed via $v = \text{Tr}(\bar{u}_n, 0)$, and its inversion is expressed as

$$\bar{u}_n = \text{ITr}(v, 0). \quad (43)$$

As $\bar{u}_n$ consists of two parts $u$ and 0, the computation for (43) can be divided into two parts accordingly. The computation for $u$ is unnecessary, so that we can remove it to reduce the complexity. To begin with, $v$ is divided into a number of individual vectors

$$v = (v_0, v_1, \ldots, v_{n-1}) = (v_0, v_1, \ldots, v_{n/T-1}),$$

where each $v_i$ is a $T$-element vector defined as

$$v_i = (v_{i:T}, v_{i+1:T}, \ldots, v_{i+T-1:T}).$$

By removing the formula for $u$ in (43), the formula for 0 is given by

$$0 = \text{ITr}(v_0, 0) + \text{ITr}(v_1, T) + \cdots + \text{ITr}(v_{n/T-1}, k), \quad (44)$$
where \(+\) is the addition for vectors. 

(44) is the core transform for the proposed RS encoding algorithm. For example, assume \(v_0\) is the parity, and others \(\{v_i\}_{i=1}^{n/T-1}\) are the message symbols. From (44), the parity is computed via

\[
\begin{align*}
\mathbf{v}_0' &= \text{ITr}(\mathbf{v}_1, T) + \text{ITr}(\mathbf{v}_2, 2T) + \cdots + \text{ITr}(\mathbf{v}_{n/T-1}, k), \\
\mathbf{v}_0 &= \text{Tr}(\mathbf{v}_0', 0).
\end{align*}
\]

(45)

In the same way, it is possible to choose another \(v_i\) as the parity. This algorithm requires a \(T\)-point FFT and \((n/T - 1)\) times of \(T\)-point IFFT. The overall complexity is then equal to

\[O(T \log(T)) + (n/T - 1)O(T \log(T)) = O(n \log(n - k)).\]

VI. REED-SOLOMON DECODING ALGORITHM

This section shows a decoding algorithm for \((n = 2^m, k)\) RS codes over \(\mathbb{F}_{2^m}\), where the codeword \(\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})\) is generated by the approach in Section V. The proposed decoding algorithm is based on the syndrome-based decoding process. Let

\[
\mathbf{r} = (r_0, r_1, \ldots, r_{n-1}) = \mathbf{v} + \mathbf{e}
\]

denote the received codeword, where \(\mathbf{e}\) is the error pattern containing \(v \leq (n - k)/2 = T/2\) errors. Assume errors occur at \(\{\lambda(\lambda_i)|\lambda_i \in E\}\), and thus the error-locator polynomial is expressed as \(\lambda(x) = \prod_{\lambda_i \in E}(x - \lambda_i)\).

Let \(\tilde{\mathbf{r}}(x), \deg(\tilde{\mathbf{r}}) < 2^m\), denote a polynomial \(\tilde{\mathbf{r}}(\omega_i) = r_i\), for each \(\omega_i \in \mathbb{F}_{2^m}\). It is clear to see that

\[
\tilde{\mathbf{r}}(\omega_i) \cdot \lambda(\omega_i) = \tilde{\mathbf{r}}(\omega_i) \cdot \lambda(\omega_i)\]

for all \(\omega_i \in \mathbb{F}_{2^m}\). Thus, by Chinese Remainder theorem, we have

\[
\mathbf{u}(\omega_i) \cdot \lambda(\omega_i) = \tilde{\mathbf{r}}(\omega_i) \cdot \lambda(\omega_i),
\]

where \(\deg(q(x)) < v \leq T/2\). (46) is the key equation used in the decoding algorithms given in [18] and [9]. From (46), we can extract the higher part of the coefficients of \(\tilde{\mathbf{r}}(x)\), resulting in the key equation of the syndrome based approach. Precisely, \(\tilde{\mathbf{r}}(x)\) is divided into two parts given by

\[
\tilde{\mathbf{r}}(x) = \tilde{\mathbf{r}}_0(x) + X_k(x)s(x),
\]

(47)

where \(\deg(\tilde{\mathbf{r}}_0(x)) \leq k - 1\). Notably, if no error occurs, \(\tilde{\mathbf{r}}(x) = \mathbf{u}(x)\) and hence \(s(x) = 0\). Thus, \(s(x)\) is the syndrome polynomial. Since \(k = 2^t\) for some \(t < m\), by (2) and (3), we have

\[
x^{2^m} - x = \prod_{i=0}^{2^m-1} (x - \omega_i) = \prod_{i=0}^{k-1} (x - \omega_i) \prod_{i=k}^{2^m-1} (x - \omega_i) = X_k(x) \cdot s_t(x - \omega_k) = X_k(x) \cdot (s_t(x) - s_t(\omega_k)).
\]

(48)

By plugging (47) and (48) into (46), we have

\[
\begin{align*}
\mathbf{u}(x)\lambda(x) &= (\tilde{\mathbf{r}}_0(x) + X_k(x)s(x))\lambda(x) + q(x)(x^{2^m} - x) \\
\Rightarrow \frac{(\mathbf{u}(x) - \tilde{\mathbf{r}}_0(x))\lambda(x)}{X_k(x)} &= s(x)\lambda(x) + q(x)(s_t(x) - s_t(\omega_k)) \\
\Rightarrow \frac{(\mathbf{u}(x) - \tilde{\mathbf{r}}_0(x))\lambda(x)}{X_k(x)} + q(x)s_t(\omega_k) &= s(x)\lambda(x) + q(x)s_t(x) \\
\Rightarrow q_0(x) &= s(x)\lambda(x) + q(x)s_t(x),
\end{align*}
\]

(49)

where \(\deg(q_0(x)) = \max\{\deg(\mathbf{u}(x)) + \deg(\lambda(x)) - k, \deg(q(x))\} \leq \max\{v, v - 1\} = v \leq T/2\). (49) is the key equation to find the error locator polynomial.

To find \(\lambda(x)\), extended Euclidean algorithm is applied on \(s_t(x)\) and \(s(x)\). The extended Euclidean algorithm stops when the remainder has degree less than \(T/2\). After obtaining \(\lambda(x)\), the next step is to finding the locations of errors \(E = \{\omega_i|\lambda(\omega_i) = 0, \forall i \in \mathbb{F}_{2^m}\}\), that is the set of roots of \(\lambda(x)\). Notice that, if the degree of \(\lambda(x)\) is
The fast decoding algorithm can also be used in shorter RS codes. The approach under the parameter configurations described above. In our simulations, the proposed RS algorithm is performed on Intel Xeon X5650 and Windows 7 platform. When \( n = 2^{16}, k/n = 1/2 \), and the codes is over \( \mathbb{F}_{2^{16}} \), the program took about \( 2.22 \times 10^{-3} \) second to generate a codeword. When a codeword suffers \( (n-k)/2 \) errors, the decoding takes about 0.401 seconds. As for the comparisons, we also ran the standard RS decoding algorithm \([13]\), that took about 22.014 seconds to decode a codeword. Thus, the proposed decoding is around 50 times faster than the traditional approach under the parameter configurations described above. In our simulations, the proposed RS algorithm is suitable for long RS codes. We feel that it is possible to further reduce the leading constant of the algorithm such that the fast decoding algorithm can also be used in shorter RS codes.

VII. CONCLUDING REMARKS

In the simulations, we implement the algorithm in C and compile the program in 64-bit GCC compiler on Intel Xeon X5650 and Windows 7 platform. When \( n = 2^{16}, k/n = 1/2 \), and the codes is over \( \mathbb{F}_{2^{16}} \), the program took about \( 2.22 \times 10^{-3} \) second to generate a codeword. When a codeword suffers \( (n-k)/2 \) errors, the decoding takes about 0.401 seconds. As for the comparisons, we also ran the standard RS decoding algorithm \([13]\), that took about 22.014 seconds to decode a codeword. Thus, the proposed decoding is around 50 times faster than the traditional approach under the parameter configurations described above. In our simulations, the proposed RS algorithm is suitable for long RS codes. We feel that it is possible to further reduce the leading constant of the algorithm such that the fast decoding algorithm can also be used in shorter RS codes.
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APPENDIX A
PROOFS OF LEMMAS 2, 3 AND 4

A. Lemma 2
Proof. From (19), we have
\[ D_a = \lceil \lg(\text{deg}(a(x)) + y + 1) \rceil \]
\[ = \lceil \lg(\text{deg}(a(x)) + 2^{D_\ell} - \text{deg}(b(x))) \rceil. \]  
(53)

From (13), we have
\[ [\lg(\text{deg}(a(x)) + 2^{D_\ell} - \text{deg}(b(x)))] \]
\[ \geq [\lg(2^{D_\ell} + 1)] = D_\ell + 1. \]  
(54)

Moreover,
\[ [\lg(\text{deg}(a(x)) + 2^{D_\ell} - \text{deg}(b(x)))] \]
\[ \leq [\lg(2^{D_\ell} + 2^{D_\ell})] = D_\ell + 1. \]  
(55)

This concludes that (20) holds.

B. Lemma 3
Proof. (23) is a summation of two terms. For the first term, we have
\[ \text{deg}(Q(x) \cdot H(x)) \]
\[ \leq \text{deg}(Q(x)) + \text{deg}(B(x)) + 1 \]
\[ = \text{deg}(A(x)) + 1 \]
\[ = \text{deg}(a(x)) + y + 1 \]
\[ = \text{deg}(a(x)) + 2^{D_\ell} - \text{deg}(b(x)) \]
\[ \leq \text{deg}(a(x)) + 2^{D_\ell} \]
\[ \leq 2^{D_\ell} - 1 + 2^{D_\ell} \]
\[ = 2^{D_a} - 1. \]  
(56)

For the second term, we have
\[ \text{deg}(R(x) \cdot \Lambda(x)) \]
\[ = \text{deg}(r(x)) + y + 2^{D_\ell} + 1 \]
\[ = \text{deg}(r(x)) + (2^{D_\ell} - \text{deg}(b(x)) - 1) + 2^{D_\ell} + 1 \]
\[ = 2^{D_a} + \text{deg}(r(x)) - \text{deg}(b(x)) \]
\[ \leq 2^{D_a} - 1. \]  
(57)

This concludes that (23) holds.

C. Lemma 4
Proof. The proof follows the mathematical induction. From (25), the base case \( i = 0 \) is given by
\[ \Lambda_0(x) \cdot B_0(x) \cdot s_1(x) = s_1(x), \]
and thus \( \tilde{r}_0(x) = 0. \)

Assume (30) holds at \( i = j \). That is,
\[ \Lambda_j(x) \cdot B_j(x) \cdot s_1(x) = s_{j+1}(x) + \tilde{r}_j(x). \]  
(58)

Then we multiply \( (s_j(x))^2 \) to (58) to obtain
\[ (s_j(x))^2 \cdot \Lambda_j(x) \cdot B_j(x) \cdot s_1(x) \]
\[ = (s_j(x))^2 \cdot s_{j+1}(x) + (s_j(x))^2 \cdot \tilde{r}_j(x). \]  
(59)
From (7), we have
\[(s_j(x))^2 \cdot s_{j+1}(x) + (s_j(x))^2 \cdot \bar{r}_j(x)\]
\[= (s_{j+1}(x))^2 + s_j(v_j)s_j(x)s_{j+1}(x) + (s_j(x))^2 \cdot \bar{r}_j(x)\]
\[= s_{j+2}(x) + s_{j+1}(v_{j+1})s_{j+1}(x) + s_j(v_j)s_j(x)s_{j+1}(x)\]
\[+ (s_j(x))^2 \cdot \bar{r}_j(x).\tag{60}\]

By (60) and (27), (59) can be rewritten as
\[s_j(x) \cdot \Lambda_j(x) \cdot B_{j+1}(x) \cdot s_1(x) = s_{j+2}(x) + \hat{r}_j(x),\tag{61}\]
where
\[\hat{r}_j(x) = s_{j+1}(v_{j+1})s_{j+1}(x) + s_j(v_j)s_j(x)s_{j+1}(x)\]
\[+ (s_j(x))^2 \cdot \bar{r}_j(x) + s_j(x) \cdot \Lambda_j(x) \cdot B_j(x) \cdot s_1(x).\tag{62}\]

The degree of each term of \(\hat{r}_j(x)\) is given by
\[\deg(s_{j+1}(v_{j+1})s_{j+1}(x)) = 2^{j+1},\]
\[\deg(s_j(v_j)s_j(x)s_{j+1}(x)) = 2^j + 2^j + 1,\]
\[\deg((s_j(x))^2 \cdot \bar{r}_j(x)) \leq 2^j + 2^j,\]
\[\deg(s_j(x) \cdot \Lambda_j(x) \cdot B_j(x) \cdot s_1(x)) \leq 2^j + (2^j - 1) + (2^j - 1) + 2.\]

Thus, we have \(\deg(\hat{r}_j(x)) \leq 2^{j+1} + 2^j\).

When \(i = j + 1\), from (28) and (29), we have
\[\Lambda_{j+1}(x) = S((s_j(x) \cdot \Lambda_j(x))^2 \cdot B_{j+1}(x) \cdot s_1(x), j + 2).\]

The above equation can be rewritten as
\[\Lambda_{j+1}(x) \cdot s_{j+2}(x) + \tilde{r}_{j+2}(x)\]
\[= (s_j(x) \cdot \Lambda_j(x))^2 \cdot B_{j+1}(x) \cdot s_1(x),\tag{63}\]
where \(\deg(\tilde{r}_{j+2}(x)) \leq 2^{j+2} - 1\). We then multiply (63) by \(B_{j+1}(x) \cdot s_1(x)\) to obtain
\[\Lambda_{j+1}(x) \cdot B_{j+1}(x) \cdot s_1(x) \cdot s_{j+2}(x)\]
\[+ \tilde{r}_{j+2}(x) \cdot B_{j+1}(x) \cdot s_1(x)\]
\[= (s_j(x) \cdot \Lambda_j(x) \cdot B_{j+1}(x) \cdot s_1(x))^2\]
\[= (s_{j+2}(x) + \hat{r}_j(x))^2\]
\[= (s_{j+2}(x))^2 + (\hat{r}_j(x))^2.\tag{64}\]

(64) is divided by \(s_{j+2}(x)\) to get
\[\Lambda_{j+1}(x) \cdot B_{j+1}(x) \cdot s_1(x) = s_{j+2}(x) + \tilde{r}_{j+1}(x),\tag{65}\]
where
\[\tilde{r}_{j+1}(x) = \frac{(\hat{r}_j(x))^2 - \tilde{r}_{j+2}(x) \cdot B_{j+1}(x) \cdot s_1(x)}{s_{j+2}(x)}.\]

In (65), the degree of each term of \(\deg(\tilde{r}_{j+1}(x))\) is as follows:
\[\deg((\hat{r}_j(x))^2) \leq 2 \cdot (2^{j+1} + 2^j),\]
\[\deg(\tilde{r}_{j+2}(x) \cdot B_{j+1}(x) \cdot s_1(x)) \leq (2^{j+2} - 1) + (2^{j+1} - 1) + 2,\]
\[\deg(s_{j+2}(x)) = 2^{j+2}.\]

Thus, \(\deg(\tilde{r}_{j+1}(x)) \leq 2^{j+1}\). This completes the proof. \(\square\)
APPENDIX B

MULTIPOINT EVALUATION AT $V_k + \beta$

In [12], Lin et al. developed a $O(2^k \log^2(2^k))$ algorithm to evaluate $\bar{D}_{2^k}(x)$ at $x \in V_k + \beta$. This section showed another viewpoint to describe the algorithm [12]. The algorithm is based on the following lemma.

**Lemma 5.** Given the coefficients of $\bar{D}_{2^k}(x)$ in the basis $\bar{x}(x)$, we have

$$\bar{D}_{2^k}(a + \gamma) = \sum_{i=0}^{2^{k-1}-1} \left( \tilde{d}_i + \frac{s_{k-1}(a)}{s_{k-1}(v_{k-1})} \tilde{d}_{i+2^{k-1}} \right) \bar{x}_i(a + \gamma) \quad \forall a \in V_{k-1}, \gamma \in \mathbb{F}_{2^m}. \quad (66)$$

**Proof.** From the definition of $\bar{x}(x)$, $\bar{D}_{2^k}(x)$ can be reformulated as

$$\bar{D}_{2^k}(x) = \sum_{i=0}^{2^{k-1}-1} \tilde{d}_i \bar{x}_i(x)$$

$$= \sum_{i=0}^{2^{k-1}-1} \tilde{d}_i \bar{x}_i(x) + \sum_{i=2^{k-1}}^{2^{k-1}-1} \tilde{d}_i \bar{x}_i(x)$$

$$= \sum_{i=0}^{2^{k-1}-1} \tilde{d}_i \bar{x}_i(x) + \sum_{i=0}^{2^{k-1}-1} \frac{s_{k-1}(x)}{s_{k-1}(v_{k-1})} \sum_{i=0}^{2^{k-1}-1} \tilde{d}_{i+2^{k-1}} \bar{x}_i(x)$$

$$= \sum_{i=0}^{2^{k-1}-1} \left( \tilde{d}_i + \frac{s_{k-1}(x)}{s_{k-1}(v_{k-1})} \tilde{d}_{i+2^{k-1}} \right) \bar{x}_i(x). \quad (67)$$

From Lemma [1] we have

$$s_{k-1}(a + \gamma) = s_{k-1}(a) + s_{k-1}(\gamma) = s_{k-1}(\gamma) \quad \forall a \in V_{k-1}. \quad (68)$$

From (67) and (68), we have

$$\bar{D}_{2^k}(a + \gamma) = \sum_{i=0}^{2^{k-1}-1} \left( \tilde{d}_i + \frac{s_{k-1}(a + \gamma)}{s_{k-1}(v_{k-1})} \tilde{d}_{i+2^{k-1}} \right) \bar{x}_i(a + \gamma)$$

$$= \sum_{i=0}^{2^{k-1}-1} \left( \tilde{d}_i + \frac{s_{k-1}(\gamma)}{s_{k-1}(v_{k-1})} \tilde{d}_{i+2^{k-1}} \right) \bar{x}_i(a + \gamma) \quad \forall a \in V_{k-1}, \gamma \in \mathbb{F}_{2^m}. \quad (69)$$

This completes the proof.

Clearly, the set of evaluation points can be divided into two individual subsets

$$V_k + \beta = (V_{k-1} + \beta) \cup (V_{k-1} + v_{k-1} + \beta),$$

where $(V_{k-1} + v_{k-1} + \beta)$ is the coset of $(V_{k-1} + \beta)$ by adding $v_{k-1}$. Accordingly, the set of polynomial evaluations can be divided into two subsets

$$\bar{D}_{2^k}(V_k + \beta) = \bar{D}_{2^k}(V_{k-1} + \beta) \cup \bar{D}_{2^k}(V_{k-1} + v_{k-1} + \beta). \quad (70)$$

The formulas for the two subsets are given as follows. By substituting $\gamma = \beta$ into (66), we obtain

$$\bar{D}_{2^k}(a + \beta) = \sum_{i=0}^{2^{k-1}-1} \left( \tilde{d}_i + \frac{s_{k-1}(\beta)}{s_{k-1}(v_{k-1})} \tilde{d}_{i+2^{k-1}} \right) \bar{x}_i(a + \beta)$$

$$= \sum_{i=0}^{2^{k-1}-1} g_i \bar{x}_i(a + \beta) = \bar{D}^{(0)}_{2^{k-1}}(a + \beta) \quad \forall a \in V_{k-1}. \quad (71)$$
This converts $\tilde{D}_{2^k}(V_{k-1} + \beta)$ into $D_{2^k-1}^{(0)}(V_{k-1} + \beta)$. Furthermore, by substituting $\gamma = v_{k-1} + \beta$ into (66), we obtain

$$
\tilde{D}_{2^k}(a + v_{k-1} + \beta) \\
= \sum_{i=0}^{2^{k-1}-1} (\tilde{d}_i + \frac{s_{k-1}(v_{k-1} + \beta)}{s_{k-1}(v_{k-1})} \tilde{d}_{i+2^k-1}) \tilde{X}_i(a + v_{k-1} + \beta)
$$

(72)

This converts $\tilde{D}_{2^k}(V_{k-1} + v_{k-1} + \beta)$ into $D_{2^k-1}^{(1)}(V_{k-1} + v_{k-1} + \beta)$.

From (71) and (72), the set of evaluation points (70) can be expressed as

$$
\tilde{D}_{2^k}(V_k + \beta) = \tilde{D}_{2^k-1}^{(0)}(V_{k-1} + \beta) \cup \tilde{D}_{2^k-1}^{(1)}(V_{k-1} + v_{k-1} + \beta).
$$

The complexity of obtaining the coefficients of the two polynomials are discussed below. In (71), each coefficient $g_i$ can be calculated by taking an addition and a multiplication, except if $s_{k-1}(\gamma) = 0$, then $g_i = \tilde{d}_i$ without any arithmetic operations. However, we do not consider the exception here, because the reduction from the exception is limited. As $D_{2^k-1}^{(0)}(x)$ has $2^{k-1}$ coefficients, it takes a total of $2^{k-1}$ additions and $2^{k-1}$ multiplications to obtain them. In (72), calculating each coefficient $g_i + \tilde{d}_{i+2^k-1}$ takes an addition, so it should take a total of $2^{k-1}$ additions to obtain the coefficients of $\tilde{D}_{2^k-1}^{(1)}(x)$.

This procedure can be applied recursively to each set $\tilde{D}_{2^k-1}^{(0)}(V_{k-1} + \beta)$ and $\tilde{D}_{2^k-1}^{(1)}(V_{k-1} + v_{k-1} + \beta)$ until the size of each set is 1. Algorithm 1 shows the details. Based on the divide-and-conquer strategy, the additive complexity and the multiplicative complexity are respectively written as

$$
A(h) = 2 \cdot A(h/2) + h, \quad M(h) = 2 \cdot M(h/2) + h/2,
$$

and the result is $A(h) = h \log(h)$ and $M(h) = h/2 \log(h)$.

By backtracking the transform steps, we can obtain its inversion algorithm accordingly. In this case, we have the coefficients of $D_{2^k-1}^{(0)}(x)$ and $D_{2^k-1}^{(1)}(x)$, and the objective is to obtain the coefficients of $\tilde{D}_{2^k}(x)$. From (72), the coefficients $\{\tilde{d}_{i+2^k-1}\}_{i=0}^{2^{k-1}-1}$ are calculated by taking $2^{k-1}$ additions. From (71), the coefficients $\{\tilde{d}_i\}_{i=0}^{2^{k-1}-1}$ are calculated by taking $2^{k-1}$ additions and $2^{k-1}$ multiplications. It is clear to see that the inversion has the same computational complexity as the transform.

APPENDIX C

POLYNOMIAL MULIPLICATION AND FORMAL DERIVATIVE ON NEW BASIS

[12] showed the polynomial multiplication and formal derivative in $\tilde{X}(x)$. We take the similar procedure to show the corresponding operations in $X(x)$.

A. Multiplication

To multiply two polynomials, there exists a well-known fast approach based on FFT techniques. This approach can also be applied on the basis $X(x)$ over finite fields $\mathbb{F}_{2^h}$. Let $a(x) = \sum_{i=0}^{h-1} a_i \cdot X_i(x)$ and $b(x) = \sum_{i=0}^{h-1} b_i \cdot X_i(x)$ denote the two polynomials with degrees less than $h$. Its product $a(x) \cdot b(x)$ can be computed as

$$
\text{IFFT}_\beta(\text{FFT}_\beta(a_{2h}) \otimes \text{FFT}_\beta(b_{2h})),
$$

where $a_{2h} = (a_0, a_1, \ldots, a_{h-1}, 0, \ldots, 0)$ is a $2h$-point vector represents the coefficients of $a(x)$ up to degree $2h-1$. Similarly, $b_{2h}$ is defined accordingly. The operation $\otimes$ performs pairwise multiplication on two vectors. This requires one $2h$-point IFFT, two $h$-point FFTs and $2h$ multiplications such that the complexity is $O(h \log(h))$. 

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B. Formal derivative

For a polynomial \( D_{2^k}(x) \) in \( \mathbb{R}(x) \), we have

\[
D_{2^k}(x) = \sum_{i=0}^{2^{k-1}-1} d_i X_i(x) = \sum_{i=0}^{2^{k-1}-1} d_i X_i(x) + \sum_{i=2^k-1}^{2^{k-1}-1} d_i X_i(x)
\]

\[
= \sum_{i=0}^{2^{k-1}-1} d_i X_i(x) + s_{k-1}(x) \sum_{i=2^k-1}^{2^{k-1}-1} d_i X_i(x) = D_{2^{k-1}}^{(0)}(x) + s_{k-1}(x) D_{2^{k-1}}^{(1)}(x).
\] (73)

The formal derivative of \( D_{2^k}(x) \) is given by

\[
D'_{2^k}(x) = [D_{2^{k-1}}^{(0)}]'(x) + s'_{k-1}(x) D_{2^{k-1}}^{(1)}(x) + s_{k-1}(x) [D_{2^{k-1}}^{(1)}]'(x).
\] (74)

From Lemma \( s'_{k-1}(x) \) is a constant. \([D_{2^{k-1}}^{(0)}]'(x)\) and \( s_{k-1}(x) [D_{2^{k-1}}^{(1)}]'(x)\) can be computed recursively. Let \( h = 2^k \), and the recursive form of the complexity is written by \( T(h) = 2 \cdot T(h/2) + \mathcal{O}(h) \) and then \( T(h) = \mathcal{O}(h \log(h)) \).