Heavy Baryon Molecules in Effective Field Theory

Jun-Xu Lu,1 Li-Sheng Geng,1∗ and M. Pavón Valderrama†

1School of Physics and Nuclear Energy Engineering,
International Research Center for Nuclei and Particles in the Cosmos and
Beijing Key Laboratory of Advanced Nuclear Materials and Physics,
Beihang University, Beijing 100191, China

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We discuss the effective field theory description of bound states composed of a heavy baryon and antibaryon. This framework is a variation of the ones already developed for heavy meson-antimeson states to describe the $X(3872)$ or the $Z_c$ and $Z_b$ resonances. We consider the case of heavy baryons for which the light quark pair is in S-wave and we explore how heavy quark spin symmetry constrains the heavy baryon-antibaryon potential. The one pion exchange potential mediates the low energy dynamics of this system. We determine the relative importance of pion exchanges, in particular the tensor force. We find that in general pion exchanges are probably perturbative for hidden charm baryonium and non-perturbative for the hidden bottom case. If we assume that the contact-range couplings of the effective field theory are saturated by the exchange of vector mesons, we can estimate for which quantum numbers it is more probable to find a heavy baryonium state.

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I. INTRODUCTION

Heavy hadron molecules – bound states composed of heavy hadrons – are a type of exotic hadron. The theoretical basis for their existence is robust: in analogy with the nuclear forces that bind the nucleons, heavy hadrons can exchange light mesons, generating exchange forces that might be strong enough to bind them 1–5. The discovery of the $X(3872)$ more than a decade ago 6 provided the first candidate for a molecular state, in fact the strongest candidate so far. The $X(3872)$ turned out not to be alone: a series of similarly puzzling hidden charm (hidden bottom) states that do not fit in the charmonium (bottomonium) spectrum have been found in different experiments since then. They are usually referred to as XYZ states. Among the latest discoveries a few are particularly good candidates for molecular states: in the charm sector we have the $Z_c(3900)$, $Z_c(4020)$ 7,8 which are suspected to be $D^*D^*$, $D^*DD^*$ molecules 9,10, and the $P_c(4380)^\pm$ and $P_c(4450)^\pm$ pentaquark states 11, which might contain $D\Sigma^*_c$, $D^*\Sigma_c$, $D^*\Sigma^*_c$ and even $D\Lambda_c(2590)$ molecular components 12,13. In the bottom sector we have the $Z_b(10610)$ and $Z_b(10610)$ 14,15, which might be $BB^*$, $B^*B^*$ molecules 16,17.

We expect molecular states to be relatively narrow for states happening above the open charm threshold. For the moment the masses of the experimentally discovered states has reached the heavy meson-meson and heavy meson-baryon threshold (3.7 and 4.1 GeV/4.3 GeV for $D\bar{D}$ and $\Lambda_c\bar{D}/\Sigma_c\bar{D}$ respectively), but barely the heavy baryon-baryon threshold (4.5, 4.7, 4.9 GeV for $\Lambda_c\Lambda_c$, $\Lambda_c\Sigma_c$ and $\Sigma_c\Sigma_c$). A narrow resonance near the heavy baryon-baryon threshold would be an excellent candidate for a heavy baryon-antibaryon bound state. Though these states have not been found yet, it is fairly straightforward to extend the available descriptions of heavy meson-antimeson molecules to them and explore the relevant dynamics behind these states. In a few instance it might be possible to predict the location of a heavy baryonium states, the $\Lambda_c(2590)\Sigma_c$ system being an illustrative example 18.

Heavy hadron-antihadron molecules are among the most interesting theoretical objects of hadronic physics. Owing to their heavy-light quark content, they are simultaneously subjected to isospin, SU(3)-flavour, chiral and heavy quark symmetry, a high degree of symmetry that can translate into a fairly regular spectrum 10,21,24–28. This spectrum will not be fully realized in nature: unless these states are shallow they will be a mixture of molecule, charmonium and other exotic components. Yet these potential regularities in the molecular spectrum can be successfully exploited to uncover the nature of a few of the XYZ states. The most clear example probably is the $Z_c$’s and $Z_b$’s resonances, which seem to be related by different realizations of heavy quark symmetry 10.

Heavy hadron molecules possess another interesting quality: they show a clear separation of scales. On the one hand we have the size of the hadrons, which is of the order of 0.5 fm, while on the other we have the size of the bound state, which is expected to be bigger than the individual hadrons within it. As a consequence heavy hadron molecules are amenable to an effective field theory (EFT) treatment, where all quantities can be expressed as an expansion of a light over a heavy energy scale. EFT descriptions of heavy hadron molecules have been exploited successfully in the past specifically to systems composed of heavy mesons and antimesons 24,29–31. In this manuscript we extend the heavy hadron EFT formulated in Ref. 25 and put in use in Refs. 10,26,27.
to the case of the heavy baryon-antibaryon molecules. As commented, these type of molecules might very well be discovered in the next few years. The purpose of this work is to explore the symmetry constrains and the kind of EFT that is to be expected in these systems, rather than to make concrete predictions of the possible location of these states. Yet we will speculate a bit about this later issue on the basis of the relative strength of the long range pion exchange and the saturation of the EFT low energy constants by $\rho$, $\omega$ and $\phi$ meson exchange.

The manuscript is structured as follows: in Section II we make a brief introduction to EFTs. In Section III we present the leading order EFT potential for heavy baryon-antibaryon states, which is composed of a series of contact four-baryon vertices plus the time-honoured pion exchange potential and in Appendix B we do the same for this type of hadron molecule. In Section IV we explore the question of whether pions are perturbative or not and for this type of molecule might very well be possible. Finally in Section V we discuss the possible power countings to describe molecular states. In Section VI we speculate about which heavy baryon-antibaryon molecules might be more probable. Finally in Section VII we present our conclusions. In Appendix A we present the complete derivation of the one pion exchange potential and in Appendix B we do the same for the four-baryon contact vertices.

II. EFFECTIVE FIELD THEORY FOR HEAVY BARYON MOLECULES

Effective field theories (EFTs) are generic and systematic descriptions of low energy processes. They can be applied to physical systems in which there is a distinct separation of scales, but where the underlying high energy theory for that system is unknown or unsolvable. The separation of scales can be used to express the low energy observables as expansions in terms of a small parameter. Now we will present a brief description of their formalism in the line of Ref. [25, 32].

A. Power Counting

We begin by considering a physical amplitude $\mathcal{A}$. At low energies $\mathcal{A}$ can be expressed as an infinite sum of diagrams involving the fields and symmetries of the EFT

$$ \mathcal{A} = \sum_D \mathcal{A}^{(D)} . $$

(1)

In turn $\mathcal{A}$ depends on the scales of the physical system it describes. Within EFT we divide all scales into light and hard scales, $Q$ and $M$, from which we write

$$ \mathcal{A} = \mathcal{A}(Q, M) . $$

(2)

This amplitude will have a series of scaling properties. The most obvious one is that physics is independent of the choice of units. If we make the transformation

$$ Q \to \lambda Q \quad \text{and} \quad M \to \lambda M , $$

(3)

we find that the amplitude transforms as

$$ \mathcal{A}(\lambda Q, \lambda M) = \lambda^d \mathcal{A}(Q, M) , $$

(4)

where $d$ is the canonical dimension of $\mathcal{A}$. If we use this property we can rewrite $\mathcal{A}$ as

$$ \mathcal{A}(Q, M) = M^d \sum_D \mathcal{A}^{(D)}(\frac{Q}{M}, 1) , $$

(5)

from which we can factorize $M$ out of the amplitudes. The second scaling property is about how amplitudes behave under a rescaling of $Q$

$$ Q \to \lambda Q , $$

(6)

though in this case the behaviour will be different depending on the diagram

$$ \mathcal{A}^{(D)}(\lambda Q, M) = \lambda^{\nu_D} \mathcal{A}^{(D)}(Q, M) , $$

(7)

where $\nu_D$ is an exponent that depends under the particular diagram under consideration. This property allows us to write the previous sum of diagrams as

$$ \mathcal{A}(Q, M) = \sum_D \mathcal{A}^{(D)}(Q, M) $$

$$ = M^d \sum_D \left( \frac{Q}{M} \right)^{\nu_D} \mathcal{A}^{(D)}(\frac{Q}{M}) $$

$$ = M^d \sum_{\nu = \nu_{\text{min}}}^{\nu_{\text{max}}} \left( \frac{Q}{M} \right)^{\nu} \mathcal{A}^{(\nu)}(\frac{Q}{M}) , $$

(8)

i.e. as a power series, where to go from the second to the third line we have switched from a sum over diagrams to a sum over diagrams with the same scaling properties

$$ \mathcal{A}^{(\nu)} = \sum_{D(\nu)} \mathcal{A}^{(D(\nu))} , $$

(9)

where $D(\nu)$ is the set of all the diagrams such that $\nu_D = \nu$.

This is indeed the most important feature of the EFT: we can organize all physical quantities as power series of the ratio $Q/M$. The advantage of this is that we do not have to calculate all the diagrams contributing to a process. We only have to choose those that have the smallest scaling $\nu$ and then cut the expansion

$$ \mathcal{A} = M^d \sum_{\nu = \nu_{\text{min}}}^{\nu_{\text{max}}} \left( \frac{Q}{M} \right)^{\nu} \mathcal{A}^{(\nu)}(\frac{Q}{M}) $$

$$ + M^d \mathcal{O} \left( \frac{Q}{M} \right)^{\nu_{\text{max}}+1} , $$

(10)

where we can appreciate that we can know in advance what is the theoretical error of the calculation after the truncation.
B. Iteration and Bound States

Now we will consider an EFT where the low energy degrees of freedoms are (heavy) hadrons and pions. For these systems the light and hard scales comprise the following

\[ Q = \{ p, m_\pi \}, \]
\[ M = \{ M_H, 4\pi f_\pi \}, \]

where \( p \) refers to any external momenta and \( m_\pi \) to the pion mass. The hard scales include the hadron masses \( M_H \) and \( 4\pi f_\pi \sim 1 \text{ GeV} \). For particular hadronic systems there might be additional light scales. To give an example, if we consider the deuteron in the neutron-proton system or the \( X(3872) \) as a \( DD^* \) molecule, the inverse of the scattering length will have to be included as a light scale. In the counting above the hadrons are non-relativisitic as a consequence of the light scale. In the counting above the hadrons are non-relativisitic as a consequence of \( p \sim Q \) and \( M_H \sim M \). This means that they interact through a standard quantum mechanical potential, where for simplicity we will limit the discussion to systems of two hadrons. The expansion of this potential reads

\[ V = \sum_\nu V^{(\nu)}. \]

There are two type of diagrams contributing to the potential: (i) diagrams involving hadrons and their local interactions, generating a contact-range force and (ii) diagrams involving the intermediate exchange of pions, generating a finite-range force. Thus we can write the EFT potential as

\[ V = V_C + V_F. \]

If we write the potential in momentum space, the contact range potential looks as a polynomial of the external hadron momenta \( \vec{p} \) and \( \vec{p}' \)

\[ \langle \vec{p}' | V_C | \vec{p} \rangle = \sum_\nu C_\nu \phi_\nu(\vec{p}, \vec{p}'), \]

where \( C_\nu \) is a coupling constant. It is trivial to check that the order of the polynomial is identical to the power counting index \( \nu \). As a consequence we can see that the EFT expansion for \( V_C \) naively begins at \( \nu = 0 \). For the finite-range potential the conclusion is similar: the lowest-dimensional pion-exchange potential we can write for a two hadron system is the one pion exchange (OPE) potential, which in general reads

\[ \langle \vec{q}' | V_F^{(0)} | \vec{q} \rangle \sim F_0 \frac{\bar{a}_1 \cdot \vec{q}' \bar{a}_2 \cdot \vec{q}}{q^2 + m_\pi^2}, \]

where \( \vec{q}' - \vec{p}' \) is the pion momentum and \( m_\pi \) is the pion mass. In general we have that \( q \sim Q \) and \( m_\pi \sim Q \), from which we get to the conclusion that the OPE potential is order \( \nu = 0 \).

However the previous analysis fails if there are bound states in the two hadron system. Bound states are solution of a dynamical equation, such as Schrödinger or Lippmann-Schwinger, and require non-perturbative physics, which in turn require \( \nu = -1 \) for the piece of the potential that needs to be iterated. We can see this from the Lippmann-Schwinger equation

\[ |\Psi_B\rangle = G_0 V |\Psi_B\rangle, \]

where \( |\Psi_B\rangle \) is the bound state wave function and \( G_0 \) the resolvent operator

\[ G_0(E) = \frac{1}{E - H_0}, \]

with \( H_0 \) the free hamiltonian. It is apparent that when \( G_0 \) appears in loops it can be counted as

\[ \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{E - \frac{E}{2\pi} \sim \mu Q}, \]

where \( \vec{l} \) is the momentum loop and \( \mu \) the reduced mass of the system, which is a hard scale \( \mu \sim M \). However if the lowest order piece of the potential behaves as \( V \sim Q^0 \), we have that the successive iterations of the potential should be more and more suppressed. Therefore to have a bound state in the EFT the only solution is that the lowest order piece of the potential behaves as \( V \sim Q^{-1} \), in which case

\[ \mathcal{O}(V) = \mathcal{O}(VG_0V) = \mathcal{O}(VG_0VG_0V) = \ldots. \]

The only problem is to explain why there is a \( V \sim Q^{-1} \) piece in the EFT potential. This is relatively easy: if we take into account that the canonical dimension of the potential in momentum space is \( d = -2 \), we can estimate the size of the coupling \( C_0 \) in \( V_C \) and \( F_0 \) in the OPE potential

\[ C_0 \sim \frac{1}{M^2}, \quad F_0 \sim \frac{1}{M^2}, \]

But if any of these two couplings is contaminated by a light scale

\[ C_0 \sim \frac{1}{MQ} \quad \text{or} \quad F_0 \sim \frac{1}{MQ}, \]

the potential will be promoted from \( Q^0 \) to \( Q^{-1} \), allowing for the existence of bound states\(^3\).

For two-hadron systems with a bound state near threshold, where the scattering length \( a_0 \) is much bigger than the Compton wave-length of the pion \( m_\pi a_0 \gg 1 \), the natural choice is to promote \( C_0 \) and choose the missing scale to be \( 1/a_0 \sim Q \)

\[ C_0 \sim \frac{a_0}{M}. \]

\(^3\) We mention in passing that the promotion can also be understood in terms of the anomalous dimension of the coupling \( C_0 \), i.e. to its scaling with respect to the cut-off \([52, 53]\).
This scaling is found in many two-body systems, all of which behave in the same way at low energies. This is why this behaviour is also referred to as universality [24].

The other possibility is the promotion of the OPE potential from $Q^0$ to $Q^{-1}$. This involves to recognize that the constant $F_0$ can be written as

$$F_0 \sim \frac{1}{\mu \Lambda_{\text{OPE}}},$$

where we have chosen the hard scale $M$ to be $\mu$. Here $\Lambda_{\text{OPE}}$ is the characteristic scale that sets the strength of the OPE potential. If $\Lambda_{\text{OPE}} \sim Q$, OPE will become non-perturbative. It is important to notice that in general the value of $F_0$ scales as $1/f_Q^2$ and is actually mostly independent of the mass of the hadrons. As a consequence the factorization above indicates that OPE will always become non-perturbative if the mass of the hadrons is high enough, a conclusion in the line of the previous findings of Ref. [5].

C. Coupled Channels

If we are considering heavy baryons, they come in multiplets related by HQSS. For heavy baryons where the light quarks are in s-wave, we have three types

$$\{B'_Q, B_Q, B'_B\}. \quad (25)$$

For $B'_Q$ the light quarks have spin 0, while for $B_Q$ and $B'_B$ the light quark spin is 1 and the total spin 1/2 and 3/2 respectively. The mass of the first type of heavy baryon — $B_Q$ — is not related to the mass of the other two types of heavy baryon. But the last two — $\{B_Q, B'_Q\}$ — form a HQSS multiplet and in the heavy quark limit their masses are identical

$$\Delta_Q = M_{B'_Q} - M_{B_Q} \sim \mathcal{O}(\Delta_{\text{OCD}}/m_Q), \quad (26)$$

where $\Delta_{\text{OCD}} \sim 200$ MeV and $m_Q$ is the mass of the heavy quark. We can provide an example by considering the spin-1/2 $\Sigma_c(2453)$ and $\Sigma_b(5811)$ and their spin-3/2 HQSS partners $\Sigma_c^*(2518), \Sigma'_b(5832)$:

$$M(\Sigma^*_c) - M(\Sigma_c) \simeq 65 \text{ MeV},$$
$$M(\Sigma^*_b) - M(\Sigma_b) \simeq 21 \text{ MeV},$$

where in the second case the mass splitting is much smaller because $m_b > m_c$. This might indicate that coupled channel dynamics could play a role in the description of heavy baryon molecules.

To evaluate the importance of coupled channels within EFT we begin by considering the two-body heavy baryon system

$$\{BQB_Q, BQB'_Q, B'_QB_Q, B'_QB'_Q\}. \quad (29)$$

We describe this system with a scattering equation of the type

$$T_{AB} = V_{AB} + \sum_C V_{AC} G_{0C}(E) T_{CB}, \quad (30)$$

where $A, B$ and $C$ each represent a particle channel. The idea is to check how to count $G_{0C}$ depending on the channel. For that we consider the loop integral

$$\int \frac{d^3l}{(2\pi)^3} \frac{1}{E + \Delta_C - \frac{i\gamma}{2\mu}}$$

where $\Delta_C$ is the energy shift of channel $C$ with respect to some reference channel. To give an example, if the reference channel is $B'_Q B_Q$ and channel $C$ is $B_Q B_Q$, we have that $\Delta_C = 2 \Delta_Q$. For the reference channel we have that $\sqrt{2\mu E} = k \sim Q$, where $k$ is the center-of-mass on-shell momentum of the external heavy baryon legs. The scale governing the coupled channels is $\Lambda_{\text{CC}} = \sqrt{2\mu \Delta_C}$, which for the $\Sigma_c (\Sigma_b)$ heavy baryons is 400/564 (350/495) MeV, depending on whether we are having one or two $B_Q \rightarrow B'_Q$ transitions. Notice that $\Lambda_{\text{CC}}$ scales as $m_Q^2$ in the heavy quark mass as a consequence of $\Delta_Q \sim 1/m_Q$ and $\mu \sim m_Q$. If we consider $\Lambda_{\text{CC}}$ to be a light scale, the loop integral scales as usual

$$\int \frac{d^3l}{(2\pi)^3} \frac{2\mu}{k^2 + \Lambda_{\text{CC}}^2 - l^2} \sim \mu Q.$$ \quad (32)

However if we take $\Lambda_{\text{CC}}$ to be a hard scale we have to reexpand the resolvent operator $G_{0C}$

$$\int \frac{d^3l}{(2\pi)^3} \frac{1}{\Lambda_{\text{CC}}^2 - l^2} =$$
$$\int \frac{d^3l}{(2\pi)^3} \frac{1}{\Lambda_{\text{CC}}^2 - l^2} =$$
$$\int \frac{d^3l}{(2\pi)^3} \frac{1}{\Lambda_{\text{CC}}^2} (1 + \frac{l^2 - k^2}{\Lambda_{\text{CC}}^2} + \ldots) =$$

$$\frac{\mu Q^3}{\Lambda_{\text{CC}}^4} + \mathcal{O}(Q^5), \quad (33)$$

and we arrive to the conclusion that the coupled channels are suppressed by a factor $(Q/\Lambda_{\text{CC}})^2$, i.e. by two orders in the EFT expansion.

The question is whether the values that we obtain for $\Lambda_{\text{CC}}$ are either soft or hard. The lower range of the hard scale $M \sim 0.5 - 1.0$ GeV is actually close to $\Lambda_{\text{CC}}$. This indicates that in general it will be simpler to take $\Lambda_{\text{CC}} \sim M$. Molecular states are only easy to identify if the binding momentum is not too high. For binding momenta of $\gamma = \sqrt{-2\mu E_B} \sim \Lambda_{\text{CC}} \sim 350 - 400$ MeV, the bound state will probably contain sizeable non-molecular components. In this case we have $\gamma/M \sim 0.4 - 0.8$, signalling a poor convergence. For shallow bound states the addition of coupled channel dynamics seems however not justified from the EFT point of view.
III. THE LEADING ORDER POTENTIAL

A. The Heavy Baryon Superfields

Heavy baryons have the structure

\[ |Q(qq)\rangle, \]  

where \( Q \) is the heavy quark and \( qq \) the light quark pair. The light quark pair can couple their spin to \( S_L = 0, 1 \) and we will assume that they are in an S-wave. If the light spin is \( S_L = 0 \), we have a \( J^P = \frac{1}{2}^- \) baryon

\[ B_Q^0 = |Q(qq)_{S_L=0}\rangle. \]  

This heavy baryon belongs to the \( 3 \) representation of the \( SU(3) \) flavour group. If the light spin is \( S_L = 1 \), we have instead a \( J^P = \frac{1}{2}^+ \) or \( J^P = \frac{3}{2}^+ \) baryon

\[ B_Q = |Q(qq)_{S_L=1}\rangle_{J=1/2}, \]  

\[ B_Q' = |Q(qq)_{S_L=1}\rangle_{J=3/2}. \]

These heavy baryons belong to the 6 representation of \( SU(3) \)-flavour. If we are considering the heavy quark to be a charm quark \( Q = c \), the flavour components of the \( B_Q' \) heavy baryon read

\[ B'_c = \left( \begin{array}{c} \Xi^0_c \\ -\Xi^+_c \\ \Lambda^+_c \end{array} \right), \]  

while for the \( B_c \) heavy baryons they are

\[ B_c = \left( \begin{array}{c} \frac{1}{\sqrt{2}} \Sigma^+_c \\ \frac{1}{\sqrt{2}} \Sigma^0_c \\ \frac{1}{\sqrt{2}} \Xi^+_c \\ \frac{1}{\sqrt{2}} \Xi^0_c \\ \frac{1}{\sqrt{2}} \Lambda^+_c \\ \frac{1}{\sqrt{2}} \Omega^+_c \end{array} \right). \]  

For the \( B'_c \) baryons we have exactly the same components as for \( B_c \), only that with a star to indicate that they are spin-3/2 baryons. In most cases it will be more practical to simply consider the \( SU(2) \) isospin structure instead. Taking into account that \( SU(2) \) isospin is a subgroup of \( SU(3) \) flavour, we will in many instances consider the full flavour structure.

Instead of using the fields \( B_Q' \), \( B_Q \) and \( B'_Q \), it is more practical to define the superfields \( T \) and \( S \)

\[ T_Q(v) = \frac{1 + \gamma_5}{2} B_Q' \]  

\[ S_{Q\mu}(v) = \frac{1}{\sqrt{3}} (\gamma_\mu + v_\mu) \gamma_5 \frac{1 + \gamma_5}{2} B_Q + \frac{1 + \gamma_5}{2} B^*_Q, \]  

which have good transformation properties under rotations of the heavy quark spin. In the equation above we have labelled the spin-3/2 heavy baryon field with a Lorentz index: \( B_Q^{0 \mu} \). The reason is that this is a Rarita-Schwinger field, where the spin-3/2 nature of this field is taken into account by coupling a Lorentz index with a Dirac spinor. This procedure generates a spurious spin-1/2 component which we remove by imposing the additional condition \( \gamma_\mu B_Q^{0 \mu} = 0 \). For more details on the heavy baryon field see Appendix A.

B. C- and G-Parity

Here we are considering systems composed of a baryon and an antibaryon. When the heavy baryon-antibaryon pair is electrically neutral and does not have strangeness either, for instance \( \Sigma^+ \Xi^- \), \( \Sigma^0 \Xi^0 \) and \( \Xi^- \Xi^- \), C-parity is well defined and its eigenvalues are given by \( C = \left( -1 \right)^{L+S} \). For the \( B_Q B_Q' \), \( B_Q B_Q' \), and \( B_Q B_Q' \) cases the C-parity of the baryon-antibaryon pair is simply \( C = \left( -1 \right)^{L+S} \), where \( L \) and \( S \) are the orbital angular momentum and spin of the system. For the \( B_Q B_Q' / B_Q B_Q' \), \( B_Q B_Q' / B_Q B_Q' \) and \( B_Q B_Q' / B_Q B_Q' \) cases we define instead the following states

\[ |B' \bar{B}(\eta)\rangle = \frac{1}{\sqrt{2}} \left[ |B' B\rangle + \eta |B \bar{B'}\rangle \right], \]  

\[ |B' \bar{B}^*(\eta)\rangle = \frac{1}{\sqrt{2}} \left[ |B' B^*\rangle + \eta |B \bar{B'}^*\rangle \right], \]  

\[ |B \bar{B}^*(\eta)\rangle = \frac{1}{\sqrt{2}} \left[ |B B^*\rangle + \eta |B^* B\rangle \right], \]  

for which the C-parity is \( C = \eta \left( -1 \right)^{L+S} \), where \( L \) (\( S \)) is the total orbital angular momentum (spin) of the heavy baryon-antibaryon pair. This definition is valid for the convention \( C|B_Q'\rangle = +|B_Q\rangle, C|B_Q\rangle = +|B_Q\rangle \) and \( C|B_Q\rangle = -|B_Q^*\rangle \). Besides we have to include the rule \( C|B_Q^{0 \mu}\rangle = -i \eta \langle C|B_Q\rangle |C|B_Q^{0 \mu}\rangle \rangle \)\( (\eta) \)\( (\eta) \), independently of whether the heavy baryons are of the same or a different type. It can be checked that the C-parity of the previous combinations is consistent with the C-parity that can deduced from the heavy-light spin decomposition (see Appendix B). Finally if we decide to take the convention \( C|B_Q\rangle = +|B_Q\rangle \), we can simply change the sign of \( \eta \) in the two-hadron states above containing a \( B_Q \) baryon and the C-parity will remain to be \( C = \eta \left( -1 \right)^{L+S} \).

For heavy baryon-antibaryon states that are not electrically neutral but belong to the same \( SU(2) \) isospin representation as a neutral state, C-parity is not a well-defined quantum number. However we can define the related G-parity transformation in the following way

\[ G = C e^{i \pi I_2}, \]  

that is, a C-parity transformation combined with a rotation in isospin space. For the electrically charged states G-parity is well defined and its eigenvalues are given by \( G = C \left( -1 \right)^I \). For example \( \Sigma^+_I \Sigma^- \), which is an \( I = 2 \) state, has a G-parity \( G = \left( -1 \right)^2 C = \left( -1 \right)^{L+S} \).
C. General Structure of the Potential

The EFT potential at lowest order contains a contact-range and a finite-range piece

\[ V_{\text{EFT}}^{(0)} = V_C^{(0)} + V_F^{(0)}. \]  

(46)

The lowest order contact-range potential is a constant in momentum space, while the finite-range potential is the time-honored OPE potential. In principle one could also consider the exchange of the other SU(3) Goldstone bosons, but here we will ignore them as their masses are at least of order of 0.5 GeV and hence considered to be a hard scale. In the following lines we will write the explicit form of these pieces.

D. The Contact-Range Potential

The LO contact-range potential simply takes the form

\[ \langle p' | V_C^{(0)} | p \rangle = C, \]  

(47)

with C a coupling constant and where \( p \) (\( p' \)) is the center-of-mass momentum of the incoming (outgoing) heavy baryon-antibaryon pair. In principle there should be one independent coupling constant for each quantum number and type of heavy baryon molecule. However the contact-range potential is constrained by SU(3) flavour and heavy quark spin symmetry (HQSS). As a consequence the number of possible couplings will be greatly reduced. We will begin by considering the HQSS structure of the potential and later the SU(3) flavour one.

For the S-wave contact interactions of the heavy baryon-antibaryon system the meaning of HQSS is the following: the coupling does not depend on the total spin of the heavy quark pair \( Q \bar{Q} \). In addition the contacts of the \( T \) and \( S \) baryons are unrelated, as the wave function of the light components differs in these two cases.

The simplest case is the \( TT \) system, for which the light quark pair always couple to \( S_L = 0 \). The contact potential reads

\[ \langle TT | V_C^{(0)} | 0^- \rangle = A_0, \]  

\[ \langle TT | V_C^{(0)} | 1^- \rangle = A_0, \]  

(48)

where \( J^P \) is the total angular momentum and parity of the baryon-antibaryon pair. We call the coupling \( A_0 \), where the subscript 0 indicates that the coupling corresponds to total light spin \( S_L = 0 \). It is worth noticing that without HQSS we should have expected two different couplings (one for each value of \( J^P \)) instead of one.

The next case is the \( TS \) and \( ST \) system. At first sight it looks trivial because the total light spin is always \( S_L = 1 \). Yet one must take into account which light quark pair carries the spin, \( q\bar{q} \) or \( \bar{q}q \). The interaction is different depending on whether the transition involve a light quark spin-flip (\( TS \to ST \)) or does not (\( TS \to TS \)). That is, we have a direct and an exchange contact which we call \( B_{1D} \) and \( B_{1E} \). In the particle bases

\[ B_1 = \{ |B'\bar{B}\rangle, |B\bar{B}'\rangle \}, \]  

(50)

\[ B_2 = \{ |B'\bar{B}'\rangle, |B\bar{B}'\rangle \}, \]  

(51)

we have that the \( B'\bar{B}/B\bar{B}' \) potential reads

\[ V_{B_1}(0^-) = \left( \begin{array}{c} B_{1D} \\ -B_{1E} \end{array} \right), \]  

(52)

\[ V_{B_1}(1^-) = \left( \begin{array}{c} B_{1D} \\ \frac{1}{3} B_{1E} \end{array} \right), \]  

(53)

while for the \( B'\bar{B}^*/B^*\bar{B}' \) we have

\[ V_{B_2}(1^-) = \left( \begin{array}{c} B_{1D} \\ -\frac{1}{3} B_{1E} \end{array} \right), \]  

(54)

\[ V_{B_2}(2^-) = \left( \begin{array}{c} B_{1D} \\ -B_{1E} \end{array} \right). \]  

(55)

For states with will well-defined C-parity we can simply write

\[ \langle B'\bar{B}|V_C(0^-)|B\bar{B}\rangle = B_{1D} - B_{1E}, \]  

\[ \langle B'\bar{B}|V_C(0^-)|B\bar{B}\rangle = B_{1D} + B_{1E}, \]  

(56)

\[ \langle B'\bar{B}|V_C(1^-)|B\bar{B}\rangle = B_{1D} - \frac{1}{3} B_{1E}, \]  

\[ \langle B'\bar{B}|V_C(1^-)|B\bar{B}\rangle = B_{1D} + \frac{1}{3} B_{1E}, \]  

(58)

\[ \langle B'\bar{B}^*|V_C(1^-)|B^*\bar{B}'\rangle = B_{1D} + \frac{1}{3} B_{1E}, \]  

\[ \langle B'\bar{B}^*|V_C(1^-)|B^*\bar{B}'\rangle = B_{1D} - \frac{1}{3} B_{1E}, \]  

(60)

\[ \langle B'\bar{B}^*|V_C(2^-)|B^*\bar{B}'\rangle = B_{1D} - B_{1E}, \]  

\[ \langle B'\bar{B}^*|V_C(2^-)|B^*\bar{B}'\rangle = B_{1D} + B_{1E}, \]  

(62)

where we have dropped the \( Q \) subscripts for simplicity. Finally for states with well-defined G-parity, where \( G = (-1)^I C \), we use the potential corresponding to the C-parity of the neutral component of the isospin multiplet. We can appreciate how HQSS constrains the possible contact-range couplings from eight to two.

The \( SS \) system is the more complex one: the total light spin can couple to \( S_L = 0, 1, 2 \). Strong interactions conserve total spin, parity and C-parity: for this case we can have transitions in which the light spin changes by two units. As a consequence we have four possible couplings: \( C_0, C_1, C_2 \) and \( C_{02} \), where the last one couples \( S_L = 0 \) with \( S_L = 2 \). Owing to the complexity of the spin structure of the \( SS \) system we write again the potential in the \( B_Q \) and \( B_Q^* \) basis. If we consider the \( B_Q \bar{B}_Q \) case we have

\[ \langle B\bar{B}|V(0^-)|B\bar{B}\rangle = \frac{1}{3} C_0 + \frac{2}{3} C_1, \]  

\[ \langle B\bar{B}|V(0^-)|B\bar{B}\rangle = \frac{1}{27} C_0 + \frac{6}{27} C_1 \]  

\[ + \frac{20}{27} C_2 - \frac{4\sqrt{3}}{27} C_{02}. \]  

(65)
The $B_Qar{B}_Q$ case is more involved as there is the possibility of having $B_QB_Q^* \to B_Q^*B_Q$ transitions. If we consider $SU(2)$-isospin alone, it is apparent that isospin multiplets can be divided into SU(3)-flavour components. That is, for the basis defined in Eq. (44), the potential reads

$$V_B(1^-) = \left( \frac{16}{54} \frac{C_0 + 33 C_1 + 5 C_2}{54} + \frac{4\sqrt{5}}{27} C_{02}, \frac{16}{54} \frac{C_0 - 21 C_1 + 5 C_2}{54} + \frac{4\sqrt{5}}{27} C_{02}, \frac{16}{54} \frac{C_0 + 33 C_1 + 5 C_2}{54} + \frac{4\sqrt{5}}{27} C_{02} \right),$$

$$V_B(2^-) = \left( \frac{1}{6} C_1 + \frac{3}{4} C_2, \frac{1}{6} C_1 - \frac{3}{4} C_2, \frac{1}{6} C_1 + \frac{3}{4} C_2 \right).$$

If the $B_Qar{B}_Q$ system under consideration is either $\Sigma_c\bar{\Sigma}_c$ or $\Xi_c\bar{\Xi}_c$ in a neutral configuration (i.e. $\Sigma_c^+ \Sigma_c^-$, $\Sigma_c^0 \Sigma_c^-$, etc.), C-parity is well-defined. If we write the potential for the basis defined in Eq. (41), the potential reads

$$\langle B\bar{B}^*|V(1^-)|B\bar{B}^*\rangle = C_1,$$

$$\langle B\bar{B}^*|V(1^-)|B\bar{B}^*\rangle = \frac{1}{6} C_0 + \frac{6}{27} C_1 + \frac{2}{3} C_2,$$

$$\langle B\bar{B}^*|V(2^-)|B\bar{B}^*\rangle = \frac{1}{3} C_0 + \frac{2}{3} C_2,$$

$$\langle B\bar{B}^*|V(2^-)|B\bar{B}^*\rangle = C_2.$$

For states with well-defined G-parity, if we ignore isospin breaking effects, the potential is the same as the one for neutral component of the isospin multiplet with $C = (-1)^I G$.

The last case is the $B_Q^*\bar{B}_Q$ system, for which

$$\langle B^*\bar{B}^*|V(0^-)|B^*\bar{B}^*\rangle = \frac{2}{3} C_0 + \frac{1}{3} C_1,$$

$$\langle B^*\bar{B}^*|V(1^-)|B^*\bar{B}^*\rangle = \frac{10}{27} C_0 + \frac{15}{27} C_1 + \frac{2}{3} C_2,$$

$$\langle B^*\bar{B}^*|V(2^-)|B^*\bar{B}^*\rangle = \frac{2}{3} C_0 + \frac{1}{3} C_2,$$

$$\langle B^*\bar{B}^*|V(3^-)|B^*\bar{B}^*\rangle = C_2.$$

Besides the HQS structure, heavy baryon-antibaryon systems also have SU(2)-isospin and SU(3)-flavour structure. If we consider SU(2)-isospin alone, it is apparent that for $T\bar{T}$ and $T\bar{S}$ molecules there should be two independent isospin components. That is, $A_0$ and $B_1$ can be subdivided into

$$A_0 \to \{A_0^{(1=0)}, A_0^{(1=1)}\},$$

$$B_1 \to \{B_1^{(1=0)}, B_1^{(1=1)}\} \text{ (for } \Xi_c\bar{\Xi}_c\text{)},$$

$$B_1 \to \{B_1^{(1=2)}, B_1^{(1=3/2)}\} \text{ (for } \Sigma_c\bar{\Sigma}_c\text{)}.$$

Meanwhile for $S\bar{S}$ molecules, we have that $C_0$, $C_1$, $C_2$ and $C_02$ can be further divided into three components

$$C_{S_L(S'_L)} \to \{C_{S_L(S'_L)}^{(I=0)}, C_{S_L(S'_L)}^{(I=1)}, C_{S_L(S'_L)}^{(I=2)}\} \text{ (for } \Sigma_c\bar{\Sigma}_c\text{),}$$

plus similar decompositions (into two components) for the $\Xi_c\bar{\Xi}_c$ and $\Xi_c\bar{\Xi}'_c$. However the isospin structure is only useful if we are considering $\Xi_c\bar{\Xi}_c$ and $\Sigma_c\bar{\Sigma}_c$ molecules separately. If we are considering them together, it is evident that the couplings for $\Xi_c\bar{\Xi}'_c$ and $\Xi_c\bar{\Xi}_c$ (to give an example) will be related. For that we will have to consider the full SU(3)-flavour structure.

SU(3)-flavour indicates that $T\bar{T}$, $S\bar{T}$ molecules and $S\bar{S}$ molecules correspond to $3 \otimes 3 = 1 \oplus 8$, $6 \otimes 3 = 8 \oplus 10$ and $6 \otimes 6 = 1 \oplus 8 \oplus 27$ respectively. That is, for $T\bar{T}$ and $T\bar{S}$ we have that the HQS coupling can be divided into two SU(3)-flavour couplings

$$A_0 \to \{A_0^{(1)} , A_0^{(8)}\},$$

$$B_1 \to \{B_1^{(8)} , B_1^{(10)}\}$$

while for $S\bar{S}$ we will have three SU(3)-flavour couplings

$$C_{S_L(S'_L)} \to \{C_{S_L(S'_L)}^{(1)} , C_{S_L(S'_L)}^{(8)} , C_{S_L(S'_L)}^{(27)}\}.$$
The OPE potential in momentum space reads

\begin{align}
\langle T_1 T_2 | V_F^{(0)} | T_1' T_2' \rangle &= 0 , \quad (92) \\
\langle T_1 S_2 | V_F^{(0)} | S_1' S_2' \rangle &= -R_1 R_2 \frac{g_0^2}{f_\pi} \vec{I}_2 \cdot \vec{q} \bar{a}_1 \cdot \vec{q} \bar{a}_2 \cdot \vec{q} , \quad (93) \\
\langle S_1 S_2 | V_F^{(0)} | S_1' S_2' \rangle &= -R_1 R_2 \frac{g_0^2}{f_\pi} \vec{I}_2 \cdot \vec{q} \bar{a}_1 \cdot \vec{q} \bar{a}_2 \cdot \vec{q} , \quad (94)
\end{align}

where we have chosen a specific notation to cover all the possible combinations. The subscripts 1 and 2 are used to denote vertices 1 and 2 in the diagrams of Fig. 1. In the equation above \( R_1 \) and \( R_2 \) are numerical factors which depends on the transition we are considering (the bar indicates an antibaryon to antibaryon transition), \( g_2 \) is the axial coupling for the \( t_1, t_2 \) heavy baryon, \( g_3 \) is the coupling involved in \( T \rightarrow S \pi \) transitions, \( f_\pi \) the pion decay constant, \( \mu_\pi \) is the effective pion mass for the baryons involved in the particular channel considered. If the particles in the vertex 1 and 2 have the same mass, then \( \mu_\pi = m_\pi \). On the other hand if they have different masses (e.g. \( S_1 = B, S_1' = B' \)) and the mass splitting is given by \( \Delta Q \), then we have that \( \mu_\pi^2 = m_\pi^2 - \Delta Q^2 \). In the equation above \( \vec{I}_1 \) and \( \vec{I}_2 \) are the isospin matrices corresponding to vertex 1 and 2. If we have a heavy baryon with isospin 1/2 at vertex 1 we can simply make the substitution \( \vec{I}_1 = \vec{S}_1 \), where \( \vec{S}_1 \) are the Pauli matrices. If the heavy baryon at vertex 1 has spin one we use the notation \( \vec{I}_1 = \vec{S}_1 \). The spin operators \( a_1 \) and \( a_2 \) depend on which is the initial and final spin of the heavy baryons at vertex 1 and 2. If the initial and final heavy baryons at vertex 1(2) have spin 1/2 we have \( \vec{a}_1 = \vec{S}_1 \) (\( \vec{a}_2 = \vec{S}_2 \)). If the initial and final heavy baryons at vertex 1(2) have spin 3/2 we have \( \vec{a}_1 = \vec{S}_1^{(3/2)} \) (\( \vec{a}_2 = \vec{S}_2^{(3/2)} \)), where \( \vec{S}_1^{(3/2)} \) are the \( J = 3/2 \) spin matrices. If the initial and final heavy baryons at vertex 1(2) switch from spin 1/2 to spin 3/2 (or viceversa), then \( \vec{a}_1 = \vec{S}_1 \) (\( \vec{a}_2 = \vec{S}_2 \)), where \( \vec{S}_1 \) are special \( 2 \times 4 \) spin matrices that describe the transition from a different initial and final spin. Notice that we can also compute the heavy baryon-baryon potential by making the change

\begin{equation}
R_1 \vec{R}_2 \rightarrow R_1 R_2 , \quad (95)
\end{equation}

in Eqs. (93) and (94) and consulting the proper values in Table II.

The most explicit way to construct the potential for one particular channel is to make use of Table II where all the factors are listed. For instance if we are considering the \( \Xi_c^* \rightarrow \Xi_c^* \) transition in vertex 1 and a \( \Sigma_c^* \rightarrow \Sigma_c^* \) transition in vertex 2. If we use Table II we find \( R_1 = 1/\sqrt{6} \), \( I_1 = \vec{I}_1 / 2 \), \( \vec{a}_1 = \vec{S}_1 \) for vertex 1 and \( R_2 = 1/\sqrt{6} \), \( I_2 = \vec{I}_2 \), \( \vec{a}_2 = \vec{S}_2 \) for vertex 2. Putting the pieces together the
potential reads
\[
\langle \Xi_c^* \Sigma_c | V_1^{(0)} | \Xi_c^* \Sigma_c^\prime \rangle = -\frac{1}{6} \frac{g_3^2}{f^2} \frac{\bar{r}_1 \cdot T_1^c S_1^c \cdot \bar{q} S_2^c \cdot \bar{q}}{q^2 + \mu_\pi^2}. \tag{96}
\]

The other cases can be obtained analogously.

If we Fourier-transform the potential into coordinate space we obtain
\[
\langle T_1 \bar{S}_2 | V_1^{(0)}(\vec{r}) | S_1^c R_1^c \rangle =
- R_1 \bar{R}_2 \bar{T}_1 \cdot \bar{T}_2 \frac{g_3^2}{3 f^2} C_{12}(\bar{a}_1, \bar{a}_2) \delta^2(\vec{r})
+ R_1 \bar{R}_2 \bar{T}_1 \cdot \bar{T}_2 \left[ C_{12}(\bar{a}_1, \bar{a}_2) W_C(\vec{r}) + S_{12}(\bar{a}_1, \bar{a}_2, \vec{r}) W_T(\vec{r}) \right], \tag{97}
\]
\[
\langle S_1 \bar{S}_2 | V_1^{(0)}(\vec{r}) | S_1^c R_1^c \rangle =
- R_1 \bar{R}_2 \bar{T}_1 \cdot \bar{T}_2 \frac{g_3^2}{3 f^2} C_{12}(\bar{a}_1, \bar{a}_2) \delta^2(\vec{r})
+ R_1 \bar{R}_2 \bar{T}_1 \cdot \bar{T}_2 \left[ C_{12}(\bar{a}_1, \bar{a}_2) W_C(\vec{r}) + S_{12}(\bar{a}_1, \bar{a}_2, \vec{r}) W_T(\vec{r}) \right], \tag{98}
\]
where \(C_{12}\) and \(S_{12}\) are the central and tensor operators, defined as
\[
C_{12}(\bar{a}_1, \bar{a}_2) = \bar{a}_1 \cdot \bar{a}_2, \tag{99}
\]
\[
S_{12}(\bar{a}_1, \bar{a}_2, \vec{r}) = 3 \bar{a}_1 \cdot \vec{r} \bar{a}_2 \cdot \vec{r} - \bar{a}_1 \cdot \bar{a}_2. \tag{100}
\]

The central and tensor piece of the potential \(W_C\) and \(W_T\) can be written as
\[
W_C = \frac{g_2^2 \mu_\pi^3}{12 \pi f^2} \frac{e^{-\mu_\pi r}}{\mu_\pi r}, \tag{101}
\]
\[
W_T = \frac{g_2^2 \mu_\pi^3}{12 \pi f^2} \frac{e^{-\mu_\pi r}}{\mu_\pi r} \left( 1 + \frac{3}{\mu_\pi r} + \frac{3}{(\mu_\pi r)^2} \right), \tag{102}
\]
where \(g_2 = g_3\) or \(g_3\) depending on the case and \(\mu_\pi\) is the effective pion mass for the channel under consideration.

\section{F. Partial Wave Projection}

Now we project the coordinate space potential into partial waves. For that we have to work with baryon-antibaryon states with well-defined total angular momentum and parity \(J^P\). If we are considering the \(\Sigma_c^{(*)}\) or the \(\Xi_c^{(*)}\) systems, we also have to work with states with well-defined C-parity. We will only consider states containing an S-wave, as they are the more likely to have a bound state. If we use the spectroscopic notation \(2S+1L_J\) to denote the partial waves, we have the following combinations
\[
|B'(-)\rangle = \{1S_0\}, \tag{103}
|B'(1-)\rangle = \{3S_1, 3D_1\}, \tag{104}
|B'(-)\rangle = \{3S_3, 3D_3\}, \tag{105}
|B'(0-)\rangle = \{1S_0, 5D_0\}, \tag{106}
|B'(1-)\rangle = \{3S_3, 3D_3, 7D_3\}, \tag{107}
|B'(1-)\rangle = \{3D_2, 5S_2, 5D_2, 5G_2\}, \tag{108}
|B'(2-)\rangle = \{3D_2, 5S_2, 5D_2, 5G_2\}, \tag{109}
|B'(3-)\rangle = \{3S_3, 3G_3, 7S_3, 7D_3, 7G_3, 7I_3\}, \tag{110}
\]
where \(B'\) denotes either a \(S_L = 0\) heavy baryon \((B')\) or a \(S_L = 1, S = 1/2\) heavy baryon \((B)\).

The calculation of the matrix elements is in general straightforward except for the \(BB^*\) case, for which we have two different versions of the central and tensor operators. If the vertex 1 (2) is of the type \(BB^*\) \((B^*\bar{B}\pi)\) or viceversa, the central and tensor operators are
\[
C_{12}^{D_{12}} = \hat{\sigma}_1 \cdot \hat{S}_2^{(3/2)}, \tag{111}
C_{12}^{D_{21}} = \vec{S}_1^{(3/2)} \cdot \hat{\sigma}_2, \tag{112}
\]
\[
S_{12}^{D_{12}} = 3 \hat{\sigma}_1 \cdot \hat{\vec{S}}_2^{(3/2)} \cdot \vec{r} - \hat{\sigma}_1 \cdot \vec{S}_2^{(3/2)}, \tag{113}
S_{12}^{D_{21}} = 3 \vec{S}_1^{(3/2)} \cdot \vec{r} \hat{\sigma}_2 - \vec{S}_1^{(3/2)} \cdot \hat{\sigma}_2, \tag{114}
\]
where we use the superscript \(D\) to indicate “direct”, meaning that the spin of the heavy baryons is not exchanged. There are two versions of the direct operators, \(D_{12}\) and \(D_{21}\), where each one refers to the transition
\[
D_{12} = \{BB^* \rightarrow BB^*\}, \tag{115}
D_{21} = \{B^*\bar{B} \rightarrow B^*\bar{B}\}. \tag{116}
\]
This distinction is only important for the matrix elements of the tensor force in which the total spin of the heavy baryon-antibaryon system is switched by one unit, i.e. \(|s - s'| = 1\), in which there will be a change of sign between the 12 and 21 operators. For the rest of the cases, i.e. \(|s - s'| = 0\), the distinction is of no consequence. On the contrary if the 1 (2) is of the type \(BB^*\pi\) \((B^*\bar{B}\pi)\), i.e. if the heavy baryons switch spin, we have
\[
C_{12}^{E_{12}} = \vec{S}_1 \cdot \vec{S}_2, \tag{117}
C_{12}^{E_{21}} = \vec{S}_1 \cdot \vec{S}_2, \tag{118}
\]
\[
S_{12}^{E_{12}} = 3 \vec{S}_1 \cdot \vec{r} \vec{S}_2 \cdot \vec{r} - \vec{S}_1 \cdot \vec{S}_2, \tag{119}
S_{12}^{E_{21}} = 3 \vec{S}_1 \cdot \vec{r} \vec{S}_2 \cdot \vec{r} - \vec{S}_1 \cdot \vec{S}_2, \tag{120}
\]
where the superscript \(E\) indicates “exchange”. We also make the distinction among \(E_{12}\) and \(E_{21}\)
\[
E_{12} = \{BB^* \rightarrow B^*\bar{B}\}, \tag{121}
E_{21} = \{B^*\bar{B} \rightarrow BB^*\}. \tag{122}
\]
In this case we have to take into account that the effective mass of the pion changes from \( m_\pi \) to \( m_\pi'^2 = m_\pi^2 - \Delta Q^2 \). If we are dealing with the \( \Sigma_c^{(*)}\Sigma_c^{(*)} \) and \( \Xi_{cc}^{(*)}\Xi_{cc}^{(*)} \) systems in the heavy quark limit, where \( \Delta Q \to 0 \), we can simply average the direct and exchange central and tensor operators for states with well-defined C-parity.

For the \( B'\bar{B}^* \) cases the situation is similar to the one in the \( B\bar{B}^* \) case, except that there is no direct operator: in this latter case there is only the \( E_{12} \) and \( E_{21} \) operators.

The central operator is the easiest one to evaluate: it conserves parity, C- and G-parity, spin, orbital angular momentum and total angular momentum. As a consequence their matrix elements are diagonal in the partial wave basis. For the \( B'\bar{B}^* \) case we have

\[
C^{B'\bar{B}^*}_{12}(J^-) = \frac{1}{2} S(S+1) - \frac{15}{4} \tag{125}
\]

where \( S \) is the spin of the partial wave considered. For the \( B\bar{B}^* \) case the evaluation depends on whether we are considering the direct or the exchange central operator

\[
C^{D_{12}}_{12}(J^-) = C^{D_{21}}_{12}(J^-) = S(S+1) - \frac{9}{2} \tag{126}
\]

\[
C^{E_{12}}_{12}(J^-) = C^{E_{21}}_{12}(J^-) = \frac{1}{6} S(S+1) \tag{127}
\]

Finally for the \( B'\bar{B}^* \) case we simply take the \( C^{E_{12}}_{12} \) written above.

Now we consider the tensor operator, which conserves parity, total angular momentum plus C- and G-parity when it applies, but not spin or orbital angular momentum. As a consequence the value of the \( L \) quantum number can change by an even number of units, where a change of an odd number of units is forbidden owing to parity conservation. The \( S \) quantum number can change by either an odd or even number of units, but odd changes are forbidden in system with good C- or G-parity (as well as in systems where \( \bar{a}_1 \) and \( \bar{a}_2 \) are the same operator, as happens when both initial and heavy baryons have the same spin). If we use the \( \Sigma^{*+1}\bar{I}_2 \) basis that we defined before, for the \( B'\bar{B}^* \) case we have

\[
S^{B'\bar{B}^*}_{12}(0^-) = 0, \tag{128}
\]

\[
S^{B'\bar{B}^*}_{12}(1^-) = \eta \begin{pmatrix} 0 & 2\sqrt{2} \\ 2 \sqrt{2} & -2 \end{pmatrix}, \tag{129}
\]

where again \( \eta \) only applies to the \( |B'\bar{B}(\eta)\rangle \) states of Eq. (122), while otherwise \( \eta = 1 \). For the \( B'\bar{B}^* \) case we have

\[
S^{B'\bar{B}^*}_{12}(0^-) = \begin{pmatrix} 0 & -3 \\ -3 & -3 \end{pmatrix}, \tag{130}
\]

\[
S^{B'\bar{B}^*}_{12}(1^-) = \begin{pmatrix} 0 & \frac{17}{\sqrt{2}} & -\frac{3\sqrt{7}}{2} & \frac{17}{2} \\ \frac{17}{\sqrt{2}} & \frac{51\sqrt{3}}{16} & -\frac{17}{16} & 0 \\ -\frac{3\sqrt{7}}{2} & \frac{51\sqrt{3}}{16} & \frac{36}{16} & -\frac{3\sqrt{6}}{16} \\ \frac{17}{2} & 0 & \frac{36}{16} & 0 \end{pmatrix}, \tag{131}
\]

\[
S^{B'\bar{B}^*}_{12}(2^-) = \begin{pmatrix} 0 & -3 \sqrt{3} & 3 \sqrt{7} & -9 \sqrt{3} \\ -3 \sqrt{3} & 0 & 3 \sqrt{7} & 0 \\ 3 \sqrt{7} & 3 \sqrt{7} & 0 & 18 \sqrt{7} \\ -9 \sqrt{3} & 0 & 18 \sqrt{7} & 0 \end{pmatrix}, \tag{132}
\]

For the \( B\bar{B}^* \) direct tensor operators we have

\[
S^{D_{12}}_{12}(1^-) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \end{pmatrix}, \tag{134}
\]

\[
S^{D_{21}}_{12}(1^-) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}, \tag{135}
\]

\[
S^{D_{12}}_{12}(2^-) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -3 \sqrt{10} & \frac{3}{2} \sqrt{7} & 6 \sqrt{10} \\ -3 \sqrt{10} & 0 & 3 \sqrt{10} & 0 \\ \frac{3}{2} \sqrt{7} & 3 \sqrt{10} & 0 & \frac{18}{7} \sqrt{5} \\ 6 \sqrt{10} & 0 & \frac{18}{7} \sqrt{5} & -\frac{15 \sqrt{2}}{7} \end{pmatrix}, \tag{136}
\]

\[
S^{D_{21}}_{12}(2^-) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 3 \sqrt{10} & -\frac{3}{2} \sqrt{7} & -6 \sqrt{10} \\ 3 \sqrt{10} & 0 & 3 \sqrt{10} & 0 \\ \frac{3}{2} \sqrt{7} & 3 \sqrt{10} & 0 & \frac{18}{7} \sqrt{5} \\ 6 \sqrt{10} & 0 & \frac{18}{7} \sqrt{5} & -\frac{15 \sqrt{2}}{7} \end{pmatrix}, \tag{137}
\]
while for the $B\bar{B}$ and $B'\bar{B}'$ exchange tensor operators we have

$$S_{12}^{E^{12}}(1^-) = \begin{pmatrix} 0 & -\frac{5}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{5}{3\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \quad (138)$$

$$S_{12}^{E^{21}}(1^-) = \begin{pmatrix} 0 & -\frac{5}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{5}{3\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \quad (139)$$

Finally for the central operators

$$S_{12}^{E^{12}}(2^-) = \begin{pmatrix} -\frac{5}{8} & \sqrt{\frac{3}{10}} & -\frac{1}{2} \sqrt{\frac{3}{10}} & -2 \sqrt{\frac{3}{35}} \\ \frac{1}{2} \sqrt{\frac{3}{10}} & 0 & -\sqrt{\frac{7}{10}} & 0 \\ -\sqrt{\frac{3}{10}} & -\sqrt{\frac{7}{10}} & -\frac{3}{14} & -\frac{6}{\sqrt{5}} \\ -2 \sqrt{\frac{3}{35}} & 0 & -\frac{6}{\sqrt{5}} & \frac{5}{7} \end{pmatrix}. \quad (140)$$

$$S_{12}^{E^{21}}(2^-) = \begin{pmatrix} -\frac{5}{8} & -\sqrt{\frac{3}{10}} & \frac{1}{2} \sqrt{\frac{3}{10}} & 2 \sqrt{\frac{3}{35}} \\ \frac{1}{2} \sqrt{\frac{3}{10}} & 0 & -\sqrt{\frac{7}{10}} & 0 \\ -\sqrt{\frac{3}{10}} & -\sqrt{\frac{7}{10}} & -\frac{3}{14} & -\frac{6}{\sqrt{5}} \\ -2 \sqrt{\frac{3}{35}} & 0 & -\frac{6}{\sqrt{5}} & \frac{5}{7} \end{pmatrix}. \quad (141)$$

For channels with well-defined C-parity, the direct tensor operator reads

$$S_{12}^{D^{2}}(1^{\pm}) = S_{12}^{D^{12}}(1^-), \quad (143)$$

$$S_{12}^{D^{2}}(2^{\pm}) = S_{12}^{D^{12}}(2^-), \quad (144)$$

while the exchange tensor operator reads

$$S_{12}^{E^{12}}(1^+) = \begin{pmatrix} 0 & 5 \frac{3\sqrt{2}}{2} & 1 \frac{1}{2} \sqrt{2} \\ \frac{5}{3\sqrt{2}} & -\frac{5}{6} & \frac{1}{2} \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \frac{1}{2} \end{pmatrix}, \quad (145)$$

$$S_{12}^{E^{21}}(2^+) = \begin{pmatrix} 0 & 5 \frac{3\sqrt{2}}{2} & 2 \frac{1}{2} \sqrt{2} \\ \frac{5}{3\sqrt{2}} & -\frac{5}{6} & \frac{1}{2} \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \frac{1}{2} \end{pmatrix}, \quad (146)$$

while for the opposite C-parity we simply have

$$S_{12}^{E^{12}}(1^-) = -S_{12}^{E^{12}}(1^+), \quad (147)$$

$$S_{12}^{E^{21}}(2^-) = -S_{12}^{E^{21}}(2^+). \quad (148)$$

For Fig. 2 can be expressed as a ratio of scales

$$\frac{\langle p|V G_0 V|p\rangle}{\langle p|V|p\rangle} = \frac{Q}{\Lambda_C}, \quad (155)$$

where $Q$ is a light scale (either the external momentum $p$ or the pion mass $m$) and $\Lambda_C$ is a scale that characterizes central OPE. The evaluation of this ratio at $p = 0$ leaves the pion mass as the only light scale left, in which case we obtain the following value for the central scale

$$\Lambda_C = \frac{1}{|\sigma_\tau|} \frac{12\pi f_\pi^2}{\mu_{BB'}|R_1 R_2| q_0^3}, \quad (156)$$

2 Recently a more sophisticated method for determining the perturbativeness of OPE has been developed in Ref. [32] for peripheral waves with $L \geq 1$. Unfortunately it has not been extended yet to S-waves.
with \( \mu_{B\bar{B}} \) the reduced mass of the system and where \( \sigma \) and \( \tau \) are the evaluation of the spin and isospin operators corresponding to the particular case under consideration. For the charm and bottom sectors the value is respectively

\[
\Lambda_C(Q = c) = \frac{530 \text{ MeV}}{|\sigma\tau||R_1 R_2|g_c^2}, \quad (157)
\]

\[
\Lambda_C(Q = b) = \frac{225 \text{ MeV}}{|\sigma\tau||R_1 R_2|g_b^2}, \quad (158)
\]

which depend on the value of the couplings \( g_2 \) and \( g_3 \). The value of \( g_3 \) can extracted from the \( \Sigma^{++} \rightarrow \Lambda^{+} \pi^+ \) decay measured by Belle in Ref. \[38\], yielding \( g_3 = 0.973^{+0.019}_{-0.042} \) \[39\] (notice that the previous reference uses the convention of Yan et al. \[40\] to define the axial couplings, instead of the one by Cho \[41, 42\] that we employ here). In contrast \( g_2 \) is experimentally unavailable, but on the basis of quark model relationships one can estimate it to be \( g_2 = -\sqrt{2}g_3 = -1.38 \). If we consider the values of \( g_2 \) and \( g_3 \) from the lattice QCD calculation of Ref. \[43\], we obtain instead \( g_2 = -0.84 \pm 0.20 \) and \( g_3 = 0.71 \pm 0.13 \), where it is important to mention that they are calculated in the \( m_Q = \infty \) limit (notice again that our convention for \( g_2 \) differs from the definition used in Ref. \[43\] by a sign, which has been taken into account). Had we applied the quark model relationship to the lattice QCD value of \( g_3 \), we would have obtained \( g_2 = -1.00 \), which is considerably larger than the lattice QCD determination but yet within its error bar.

The discussion about the values of the axial couplings, and in particular \( g_2 \), is important because it can change the value of \( \Lambda_C \) by a large factor. While for the \( T\bar{S} \) molecules the value of \( \Lambda_C \) in the charm sector can be expected to be

\[
\Lambda_C^{T\bar{S}}(Q = c) \sim \frac{560^{+50}_{-20} \text{ MeV}}{|\sigma\tau||R_1 R_2|}, \quad (159)
\]

for the \( S\bar{S} \) molecules in the charm sector we have a huge spread

\[
\Lambda_C^{S\bar{S}}(Q = c) \sim \frac{(280 - 750) \text{ MeV}}{|\sigma\tau||R_1 R_2|}, \quad (160)
\]

where \( \Lambda_C \) can change almost by a factor of 3 owing to the uncertainty of \( g_2 \). In the bottom sector it is instead more advisable to use the lattice QCD determination for \( g_2 \) and \( g_3 \), leading to

\[
\Lambda_C^{T\bar{S}}(Q = b) \sim \frac{450^{+220}_{-130} \text{ MeV}}{|\sigma\tau||R_1 R_2|}, \quad (161)
\]

\[
\Lambda_C^{S\bar{S}}(Q = b) \sim \frac{330^{+220}_{-110} \text{ MeV}}{|\sigma\tau||R_1 R_2|}. \quad (162)
\]

The previous values have to be combined with the \( |\sigma\tau||R_1 R_2| \) factor. The maximum value of this factor happens for the channels with lowest spin and isospin. If we only consider cases for which central OPE is attractive, a few representative values are

\[
\Xi^+_c \Xi'^+_c(0^+ - 0^+) \rightarrow \frac{1}{2}, \quad (163)
\]

\[
\Xi^+_c \Xi'^+_c(1^+ - 0^+) \rightarrow \frac{11}{48}, \quad (164)
\]

\[
\Sigma^+_c \Sigma'^+_c(0^+ - 0^+) \rightarrow \frac{5}{8}, \quad (165)
\]

\[
\Sigma^+_c \Sigma'^+_c(0^+ - 0^+) \rightarrow \frac{4}{3}, \quad (166)
\]

\[
\Sigma^+_c \Sigma'^+_c(1^+ - 0^+) \rightarrow \frac{11}{18}, \quad (167)
\]

\[
\Sigma^+_c \Sigma'^+_c(0^+ - 0^+) \rightarrow \frac{5}{3}, \quad (168)
\]

where the values in brackets are the \( J^{PC} (I^G) \) quantum numbers. We can see that with the exception of the \( S = 0, I = 0, \Sigma^+_c \Sigma'^+_c \) and \( \Sigma^+_c \Sigma'^+_c \) molecules, the \( |\sigma\tau||R_1 R_2| \) factor pushes the baseline estimation for the scale \( \Lambda_C \) to the harder side. That is, we expect central OPE to be in general perturbative. The exceptions might be \( \Sigma_Q \Sigma_Q \) and \( \Sigma^*_Q \Sigma^*_Q \) molecules. In the charm sector this requires the absolute value of the axial coupling \( |g_2| \) to lie on the high end, i.e. the value \( g_2 = -1.38 \) from the quark model relationship. In the bottom sector the situation is similar: central OPE will only be non-perturbative for the higher values of \( |g_2| \), i.e. \( |g_2| > 1.0 \).

### B. The Tensor Potential

The tensor piece of the OPE potential requires a more involved analysis. A direct comparison of the diagrams
in Fig. 2 is not possible. The reason is that the iteration of the tensor piece of OPE diverges, see for instance Refs. [25, 37] for a detailed explanation. Thus we must resort to a method that does not involve the direct evaluation of iterated tensor OPE diagrams.

The type of power-law behaviour of the tensor OPE potential is analogous to the behaviour of a few physical systems studied in atomic physics. The potential between two dipoles is of the $1/r^3$ type, just like the tensor force. The failure of standard perturbation theory for these systems is well-known in atomic physics, where alternative techniques have been developed to deal with this type of potentials. The work of Cavagnero [44] explains that the divergences of perturbation theory in these type of systems is similar to the role of secular perturbations in classical mechanics, i.e. a type of perturbation that is small at short time scales but ends up diverging at large time scales. The solution is to redefine (or renormalize) some quantity in order to obtain a finite result again. For the perturbative series of $1/r^3$ potentials the quantity we renormalize is the angular momentum. The zero-th order term in the perturbative expansion of the wave function is now

$$\Psi^{(0)}(r; k) = \frac{J_{\nu}(kr)}{r},$$

(169)

instead of the standard $\frac{J_{\nu+1/2}(kr)}{r}$. In the expression above $\nu$ is the renormalized angular momentum, which happens to be a function of the momenta $k$ and a length scale $a$ that is related to the strength of the potential. The secular series is built not only by adding higher order terms but also by making $\nu(x)$ to depend on $x = ka$. If we switch off the potential and take $ka = 0$, we have $\nu(0) = l + \frac{1}{2}$ and we recover the free wave function. For small enough values of $x = ka$ we expect $\nu(x)$ to be expandable in powers of $x$, i.e. to be perturbative. By expanding the secular series and the renormalized angular momentum we can recover the original perturbative series. However the interesting feature of the series above is that we can determine the values of $x$ for which $\nu(x)$ is analytic. When $\nu(x)$ is not analytic, it does not admit a power series in $x$ anymore. This in turn means that there is no way to rearrange the secular series into the standard perturbative series, leading to its failure.

For the $1/r^3$ potential, which is equivalent to the tensor force for distances $m_\pi r < 1$, the secular series has been studied in detail by Gao [45] for the uncoupled channel case. Birse [46] extended the previous techniques for the coupled channel case and particularized them for the nucleon-nucleon system. In a previous publication by one of the authors [25] the analysis of Birse was applied to the heavy baryon-antibaryon system. In this work we extend it to the heavy baryon-antibaryon system.

We will consider the tensor force in the limit $m_\pi \to 0$, for which the OPE potential can be written as

$$2\mu_{BB} V_{BB}(r) = \frac{a_3}{r^3} S_j,$$

(170)

where the potential is a matrix in the coupled channel space and $S_j$ is the tensor operator. We have that $\mu_{BB}$ is the reduced mass of the heavy baryon-antibaryon system and $a_3$ is the length scale that determines the strength of the tensor piece of the potential. The potential in this limit is amenable to the secular perturbative series developed in Refs. [25, 45, 46]. The corrections stemming from the finite value of $m_\pi$ where considered in Ref. [25] and will be discussed later on in this section.

### C. The Renormalized Angular Momentum

Now we explain how the secular perturbation theory looks like and most importantly, how to calculate the renormalized angular momenta $\nu$. We begin by writing the reduced Schrödinger equation for coupled channels (the uncoupled channel can be consulted in Ref. [25])

$$-\mathbf{u}_{k,j}'' + \left[ S_j \frac{a_3}{r^3} + \frac{L_j^2}{r^2} \right] \mathbf{u}_{k,j}(r) = \gamma^2 \mathbf{u}_{k,j}(r),$$

(171)

where we are considering $N$ angular momentum channels. In the equation above $S_j$ is the tensor operator matrix (as written in Sect. [III]), while $L_j$ is a diagonal matrix representing the angular momentum operator $L^2$

$$L_j^2 = \text{diag}(l_1(l_1 + 1), \ldots, l_N(l_N + 1)).$$

(172)

The reduced wave function $\mathbf{u}_{k,j}$ is an $N \times N$ matrix, where column $j$ represents a solution that behaves as a free wave with angular momentum $l_j$ when we take $a_3 \to 0$. The solution of the Schrödinger equation is a linear combination of the functions $\xi$ and $\eta$:

$$\mathbf{u}_{k,j}(r) = \sum_{\{l_j\}} \left[ \alpha_{l_j} \xi_{l_j}(r; k) + \beta_{l_j} \eta_{l_j}(r; k) \right],$$

(173)

![Table II. Reduced critical momenta $\kappa_c$ for the different types of coupled channels containing an S-wave that appear in the heavy baryon-antibaryon system.](image)
where we sum over the possible values of the angular momenta and with $\xi$ and $\eta$ $N$-component vectors that can be written as sums of Bessel functions

\begin{align}
\xi_j(r; k) &= \sum_{m=-\infty}^{\infty} b_m(\nu_j) \sqrt{r} J_{m+\nu_j}(kr), \\
\eta_j(r; k) &= \sum_{m=-\infty}^{\infty} (-1)^m b_{-m}(\nu_j) \sqrt{r} J_{-m-\nu_j}(kr).
\end{align}

We have $N$ different solutions for $\xi$ and $\eta$ that we have labelled with the subscript $i_j$ to indicate that for $\kappa = 0$ they behave as a free wave of angular momentum $l_j$. The recursive relationship from which one can compute $b_m(\nu_j)$ can be consulted in Ref. [10] but are of no concern if we are only interested in the $\nu_i$’s.

The renormalized angular momenta $\nu_i = \nu_i(\kappa)$ can be calculated by finding the zeros of

\begin{equation}
\det (F_j(\nu, \kappa)) = 0,
\end{equation}

where $F_j(\nu, \kappa)$ is an $N \times N$ matrix. This equation admits $N$ solutions, one for each value of the angular momentum. For $\kappa = 0$ these solutions behave as

\begin{equation}
\nu_i(\kappa = 0) = l_i + \frac{1}{2},
\end{equation}

with $i = 1, \ldots, N$. We define $F_j(\nu, \kappa)$ as

\begin{equation}
F_j(\nu, \kappa) \equiv f_j(\nu) - \frac{\kappa^2}{\nu} [R_1(\nu) - R_1(-\nu)],
\end{equation}

where $f_j(\nu)$ is a diagonal matrix defined by

\begin{equation}
\frac{f_j(\nu)}{2\nu} = \text{diag}(\nu^2 - (l_1 + \frac{1}{2})^2, \ldots, \nu^2 - (l_N + \frac{1}{2})^2).
\end{equation}

In turn, $R_1(\nu)$ is given by the recursive relation

\begin{equation}
R_n(\nu) = [f_{\nu}(n + \nu) - \kappa^2 S_j R_{n+1}(\nu) S_j]^{-1},
\end{equation}

which can be accurately solved with between 20 and 30 iterations [10] (that is, one takes $R_N = 0$ for large enough $N$, e.g. $20 - 30$, and solves the recursion relation backwards).

The analytical properties of $\nu_i(\kappa)$ are the following: for $\kappa = 0$ we begin at $\nu_i = l_i + \frac{1}{2}$ and as $\kappa$ increases $\nu_i(\kappa)$ moves slowly downwards. Once we reach $\nu_i = l_i$, at $\kappa = \kappa_c$, we have that $\nu_i$ splits into the complex conjugate solutions $\nu_i(\kappa) = l_i \pm i\rho_i(\kappa)$. This is a non-analyticity which marks the point above which $\nu_i(\kappa)$ cannot be expressed as a perturbative series. This in turn defines $\kappa_c$, the critical value of $\kappa$ we were looking for. Usually the first $\nu_i$ to split is the one that corresponds to the smallest angular momentum and also the one that determines the breakdown of the perturbative series.

For computing the critical momenta we need first the matrix elements of the tensor operator in the channel under consideration. For the $B^0\bar{B}^0$ and $B^+\bar{B}^*$ molecules this is trivial as we only have to take the matrices in Eq. (129) and Eqs. (130,133) respectively. The $B^0\bar{B}^*$ molecules only contain the exchange tensor operator, which means we only have to look at Eqs. (115) and (116). The $BB^*$ molecules are the most complicated because they contain a direct and exchange tensor operator. In addition the effective pion masses are different for the direct and exchange tensor operators. Here we ignore this effect: in the present calculation we are making the approximation that HQSS is exact and therefore there is no mass splitting between the $B$ and $B^*$ heavy baryons. However the length scale $a_3$ is different for the direct and exchange operators, i.e.

\begin{equation}
2\rho_{BB} V_{BB^*}(r) = \frac{a_3^D}{r^3} S_j^D + \frac{a_3^E}{r^3} S_j^E.
\end{equation}

This problem is easy to overcome for the reason that both scales are proportional to each other

\begin{equation}
a_3^E = -\frac{3}{4} a_3^D,
\end{equation}

as can be checked either by inspecting Table III or the coupled-channel form of the potential in Eq. (A105) from Appendix A. Thus in the $BB^*$ system we will be computing the critical values of the matrix

\begin{equation}
S_j = S_j^D - \frac{3}{4} S_j^E.
\end{equation}

D. Critical Momenta

The critical $\kappa_c$ for the different possible heavy baryon-antibaryon bound states containing an $S$-wave are listed in Table III. The previous values have been obtained under the assumption that the effective pion mass can be taken to be zero. The effect of finite pion mass was considered in Ref. [25], where it was found that it increases the range of momenta where tensor OPE is perturbative by the following factor

\begin{equation}
\kappa_c(m_\pi) = \kappa_c e^{+m_\pi R_c},
\end{equation}

where $R_c$ is the radius below which we do not expect the OPE potential to be valid. The value of this radius is rather ambiguous. In Ref. [25] the estimation $R_c = 0.5 - 0.8 \text{ fm}$ was proposed, yielding

\begin{equation}
\kappa_c(m_\pi) \simeq 1.5 \kappa_c.
\end{equation}

Higher values might be more appropriate indeed, but here we will stick to the value above.

To obtain the critical momenta we multiply $\kappa_c(m_\pi)$ by the relationship $k_\pi = \kappa_c(m_\pi)/\left|a_3\right|$, where $a_3$ is the tensor length scale. In general we have

\begin{equation}
\left|a_3\right| = \left|R_1 \tilde{T}_2 \tau\right| \frac{\mu_{BB^*} g_3^2}{2\pi f_3^2},
\end{equation}

where $\tau = \tilde{T}_1 \cdot \tilde{T}_2$ and $g_i = g_2, g_3$ depending on whether we are considering the $SS$ or the $TS$ potential. The
If we particularize for the $T$ and $\bar{T}$ we can use the inverse of that we used for writing their tensor matrices. We can use these scales seems rather soft but the factors $R_1$, $R_2$ and $\tau$ will increase the values of $\Lambda_3$ considerably. A few representative values are

$$\Lambda_3 \Sigma_c \rightarrow 279^{+55}_{-30} \text{MeV},$$

$$\Xi_c^- \Xi_c^-(I = 0) \rightarrow 372^{+34}_{-31} \text{MeV},$$

$$\Xi_c^0 \Xi_c^0(I = 0) \rightarrow 250 - 700 \text{MeV},$$

$$\Sigma_c^+(\Sigma_c^0) (I = 0) \rightarrow 100 - 300 \text{MeV},$$

where the most important spread is for the $\Sigma_c^-(\Sigma_c^0)$ family of bound states. Most of the critical momenta are of the order of $2/3$, while the finite pion mass multiplies this value by $3/2$. In short, the numbers above represent pretty well the actual momenta below which tensor OPE is perturbative. Most of this momenta are above the coupled channel threshold, which means that we can simply treat tensor OPE as a perturbation. The exception seems to be the isoscalar $\Sigma_c^-(\Sigma_c^0)$ states, but again this comes with the catch that we are not that sure about the value of $g_2$ in the charm sector. As a consequence the expectation is that the critical momenta will lie somewhere in the middle of the spread, above 200 MeV at least. For the bottom sector the scale $\Lambda_3$ is

$$\Lambda_3 \Sigma_b \rightarrow 160^{+130}_{-60} \text{MeV},$$

$$\Xi_b^- \Xi_b^- (I = 0) \rightarrow 210^{+160}_{-70} \text{MeV},$$

$$\Xi_b^0 \Xi_b^0 (I = 0) \rightarrow 450^{+220}_{-130} \text{MeV},$$

$$\Sigma_b^+(\Sigma_b^0) (I = 0) \rightarrow 170^{+80}_{-50} \text{MeV},$$

and the conclusions are pretty similar to those of the charm sector, the difference being that in this latter case we are more sure about the probable non-perturbative character of the isoscalar $\Sigma_b^+(\Sigma_b^0)$ molecules.

### V. Power Counting for Heavy Baryon Molecules

In this section we discuss the different possible power countings for the heavy baryon-antibaryon states. We are interested in the case where there are bound states. This excludes naive dimensional analysis, namely

$$V_C^{(0)}(\bar{q}q) \sim Q^0, \quad V_F^{(0)} \sim Q^0,$$

as this counting leads to purely perturbative heavy baryon-antibaryon interactions. The existence of bound states requires that at least one of the components of the potential is promoted to $Q^{-1}$. There are different choices depending on which piece of the interaction to promote. We will here consider three options: (i) promotion of the contacts (with perturbative pions), (ii) promotion of central OPE and (iii) promotion of tensor OPE. For a more in depth discussion of the power counting of two-body systems we refer the reader to Ref. [52].

#### A. Counting with Perturbative Pions

The first possibility is that the binding mechanism for heavy baryon-antibaryon is of short-range nature. Within the EFT language this amounts to the promotion of the contact-range potential from $Q^0$ to $Q^{-1}$. Within this power counting the leading order (LO) potential will be composed of contacts, while the next-to-leading order potential (NLO) will contain the OPE potential plus a few additional contacts

$$V^{LO} = V_C^{(-1)}$$

$$V^{NLO} = V_C^{(0)} + V_{OPE}.$$ We do not have to promote all the possible contact interactions that we obtain from the heavy-light spin decomposition: in general a subset of it will be enough.

There is one important detail with this counting. If we consider the S-wave contact-range interactions in EFT, they admit the momentum expansion

$$\langle p' | V_C | p \rangle = C + D(p^2 + p'^2) + \ldots$$

where the dots denote couplings involving more derivatives of the baryon fields. The naive expectation for the scaling of the $C$ and $D$ couplings is

$$C \sim \frac{1}{M^2}, \quad D \sim \frac{1}{M^4},$$

However if we promote the coupling without derivatives, the term with derivatives also gets promoted

$$C \sim \frac{1}{MQ}, \quad D \sim \frac{1}{M^2Q^2}.$$

As a consequence the ordering of the contact-range potential will be

$$\langle p' | V_C^{(-1)} | p \rangle = C,$$

$$\langle p' | V_C^{(0)} | p \rangle = D(p^2 + p'^2).$$
That is, the NLO potential will contain a contact-range interaction with two-derivatives on the baryon fields. As a consequence if we promote a particular $C_{\alpha\beta}$ coupling to $Q^{-1}$, the corresponding coupling with two derivative will also have to be promoted to $Q^0$.

B. Counting with Non-Perturbative Central OPE

The second possibility is that the binding mechanism depends on the attraction provided by central OPE too. We can distinguish two cases: (i) the binding depends on central OPE alone and (ii) the binding depends on the interplay of the contacts and central OPE.

In the first case we have a relatively simple power counting in which

$$V^{LO} = V_{OPE(C)} , \quad (208)$$

where by $OPE(C)$ it is meant the central piece of OPE. The NLO potential contains tensor OPE and the contacts

$$V^{NLO} = V_{OPE(T)} + V^{(0)}_C , \quad (209)$$

where

$$\langle p'| V^{(0)}_C | p \rangle = C_0 . \quad (210)$$

Contacts with $2n$ derivatives on the baryon fields will enter at order $Q^{2n}$.

The second case is identical to the counting in which the binding depends on the contacts alone, except for the fact that we include OPE in the LO:

$$V^{LO} = V^{(-1)}_C + V_{OPE(C)} , \quad (211)$$

$$V^{NLO} = V^{(0)}_C + V_{OPE(T)} , \quad (212)$$

where $V_{OPE(T)}$ is the tensor piece of OPE, while $V^{(-1)}_C$ and $V^{(0)}_C$ are just as in Eqs. (206) and (207).

C. Counting with Non-Perturbative Tensor OPE

The most complex power counting arises when tensor OPE is non-perturbative. Tensor OPE is a singular potential, i.e. it diverges at least as fast as $1/r^2$

$$\lim_{r \to 0} r^2 V(r) \neq 0 . \quad (213)$$

These potentials require a special treatment in EFT [47–53], as we will explain here. For the specific case of the $1/r^3$ potential we begin by defining its characteristic length scale

$$\lim_{r \to 0} r^3 2\mu V(r) = \pm a_3 , \quad (214)$$

where $\mu$ is the reduced mass of the two-body system we are considering.

If the potential is repulsive, i.e. we have a plus sign in the equation above, there is no apparent problem: we can apply the counting rules for non-perturbative central OPE. Most probably however tensor OPE will be perturbative if it is repulsive. We mention in passing that the power counting of repulsive tensor forces is not yet properly understood and that what we have offered here is just the most practical solution to it [32].

If the potential is attractive, i.e. we have a minus sign in the equation above, the solutions of the reduced Schrödinger equation for $r \leq a_3$ and $m_r r \leq 1$ will take the form [49, 50]

$$u(r) \propto \left( \frac{r}{a_3} \right)^{3/4} \sin \left( \sqrt{\frac{a_3}{r}} + \phi_3 \right) , \quad (215)$$

where $u$ is the reduced wave function and $\phi_3$ a phase that specifies the solution. This solution is correct regardless of the angular momentum quantum number, the reason being that the centrifugal barrier $l(l+1)/r^2$ does not diverge as fast as the tensor force for $r \to 0$. The problem with the reduced wave function above is that independently of the phase $\phi_3$ the reduced wave function will be regular at short distances: $u(0) = 0$. Therefore any solution will be an acceptable solution of the reduced Schrödinger equation. Yet there should be only one physical solution. The conclusion is that the phase $\phi_3$ can only be determined from the short range physics.

The interpretation of this fact from the EFT point of view is that a non-perturbative tensor force requires a non-perturbative contact potential. That is, the LO potential will be

$$V^{LO} = V_{OPE} + V^{(-1)}_C , \quad (216)$$

with $V^{(-1)}_C$ the lowest order contact-range potential. The counting of the contacts will be modified as follows

$$C \sim \frac{1}{M Q} , \quad D \sim \frac{1}{M^{7/2} Q^{3/2}} , \quad (217)$$

or equivalently we can write

$$\langle p'| V^{(-1)}_C | p \rangle = C , \quad (218)$$

$$\langle p'| V^{(3/2)}_C | p \rangle = D (p^2 + p'^2) , \quad (219)$$

where the contacts with derivatives get promoted by half an order.

This power counting is rather simple except for two complications: angular momentum coupled channels and HQSS. The first of these problems has been already treated in the nucleon-nucleon case [49, 52]. Here we present a brief derivation. The tensor force mixes partial waves with different angular momentum, but the analysis above is valid for a single partial wave channel. We have seen that the tensor force can be written in matrix form, where for $m_r r \leq 1$ we have

$$2\mu V(r) \to S_j \frac{a_3}{r^j} . \quad (220)$$

Besides there are only significant changes in the power counting when the tensor force is attractive. As the
tensor force overcomes the centrifugal barrier at short distances, the coupled channel equivalent of attractive/repulsive tensor forces is to find the eigenvalues of the matrix $S_j$

$$D_j S_j D_j^{-1} = \text{diag}(s_1, s_2, \ldots, s_n),$$

(221)

where $D_j$ is a change of basis matrix and diag is a diagonal matrix with the eigenvalues of the matrix $S_j$. If we have $n$ attractive channels, we will be required to have $n(n+1)/2$ contact-range interactions to be able to fix the physical solution \cite{49, 50}. This is regardless of what is the angular momentum structure of these contacts. From the point of view of fixing the solutions it does not really matter if the contacts are purely S-wave or whether they mix different partial wave.

However if we consider HQSS instead it becomes really important to know the angular momentum structure of the contacts. Otherwise the HQSS of the contact-couplings will be apparently broken by non-perturbative tensor forces. We can explain this problem with a specific example: the $2^{-} - B B^*$ and $3^{-} - B^* B^*$ channels. The contact-range potential in these two channels is identical owing to HQSS

$$V^{LO}_c(2^{-}) = V^{LO}_c(3^{-}).$$

(222)

For the central operator $C_{12}$ this identity is not necessarily for $J^{PC} = 2^{-}$ and $3^{-}$: it is only obeyed if $S = 2$ and 3 respectively. The $J^{PC} = 2^{-}$ and $3^{-}$ channels contain components with $L \neq 0$ and $S = 1$ (a $3D_2$ and $3D_3 G_3$ partial wave respectively). In fact the matrices have different dimensions

$$C_{12}(2^{-}) = \text{diag}(-\frac{11}{4}, \frac{9}{4}, \frac{9}{4}, \frac{9}{4}),$$

(223)

$$C_{12}(3^{-}) = \text{diag}(-\frac{11}{4}, \frac{11}{4}, \frac{9}{4}, \frac{9}{4}, \frac{9}{4}),$$

(224)

where diag represents the diagonal matrix in the bases defined by Eqs. \ref{106} and \ref{110}. The value $-11/4$ corresponds to $S = 1$ both in the $2^{-}$ and $3^{-}$ cases (which requires $L \neq 0$), while $9/4$ corresponds to $S = 2$ in $2^{-}$ and $S = 3$ in $3^{-}$. For the tensor operator the previous identity is not respected at all: besides the fact that the tensor matrices have different dimensions, the eigenvalues are not identical either

$$S_{12}(2^{-}) \rightarrow \left\{ \frac{1}{4}(-2 \pm 5\sqrt{7}), \frac{9}{4} \right\},$$

(225)

$$S_{12}(3^{-}) \rightarrow \left\{ \frac{1}{2}(-7 \pm \sqrt{13}), \frac{1}{2}(-1 \pm 2\sqrt{7}), \frac{9}{2}, 0 \right\}.$$

(226)

At first sight this seems to require HQSS-breaking contact interactions if we want to renormalize the tensor forces properly, but this is not necessarily true. Tensor forces mix partial waves with different angular momentum and they require contact interactions that behave in an analogous way. In particular they require contacts with SD-, SG-... wave mixing or contacts mixing different spin channels. These contacts will have a different HQSS structure than the S-wave contacts we have derived here. Yet the particular structure of these contacts is beyond the scope of this work. It is sufficient to point out that the structure can be quite involved and will require a detailed analysis of which are the partial waves that render tensor OPE non-perturbative in each case.

**VI. CAN WE PREDICT HEAVY BARYON MOLECULES?**

Here we explore the possibility of predicting the existence of heavy baryon molecules. In general this possibility lies outside the capabilities of EFT. The reason is that EFTs are generic frameworks, they cover all the different situations that can happen in a low energy system. If we think of heavy hadron molecules, that includes the case in which there are no bound states as well as the case in which there are. The expression of this flexibility can be seen in these two facts: (i) the contact-range couplings are free parameters, (ii) the contact-range couplings can either scale as $Q^{-1}$ or $Q^0$. There are exceptions though to this rule of thumb. If the states are a consequence of long range physics, EFTs can predict them. In atomic physics the Efimov states are a very illustrative example of this \cite{53, 54}. The prediction of a baryonium involving a $\Lambda_c(2590)$ and $\Sigma_c$ baryons is also possible because the long range physics takes the form of a Coulomb-like potential \cite{22}, notice that this involves the P-wave heavy baryons and thus it is a different type of baryon-antibaryon system than the one discussed in the present work). If the states are a consequence of short range physics, EFT can predict new resonances provided that there is previous experimental information available. For instance a molecular $1^{++} X(3872)$ requires the existence of a $2^{++} X(4012)$ partner \cite{23, 24}, though the reliability of this prediction depends on how well-preserved are the symmetries of the system and on how robust are our assumptions about the power counting \cite{55}, the relevant low energy degrees of freedom \cite{56} and so on.

Theoretical studies of hadronic molecules adscribe the binding mechanism to either short- or long-range causes. In the pioneering works about the existence of molecular states \cite{1, 2} the exchange of light mesons is the factor responsible for the binding of charmed meson-antimeson pairs. Conversely other works \cite{57, 58} consider that the short-range force is what binds the molecules. There is also a series of work which explain the properties of molecular states solely on the basis of short-range interactions \cite{24, 29, 61}, but without making explicit assumptions about the binding mechanism.

If we consider the charm sector within EFT, there are good reasons to expect that the mechanism binding heavy hadron molecules is of a short-range nature. In this case the OPE potential is probably too weak to be the binding mechanism \cite{23, 24}. Of course there are interesting examples in the literature on how to use
the OPE potential to predict new states, the most illustrative one being the work by Törnqvist [2, 3]. This type of works regularize the OPE potential with a multipolar form factor, where the cut-off is relatively hard \((\Lambda \geq 1.0 \text{ GeV})\). Besides, older works usually rely on a value of the \(DD^*\pi/DD^*\pi\) coupling that lies on the high range of the experimental value that is extracted from the \(D^+ \to D\pi\) decay, \(g_1 = 0.59 \pm 0.01 \pm 0.07\) \([22, 53]\). From the point of view of EFT the previous idea amounts to an educated guess of the short-range physics, independently of whether the basis for the predictions is the long range OPE potential or not.

Here we will explore first the short-range binding mechanism then the long range one. For the short-range case we will saturate the contact-range coupling constants with \(\rho\) - and \(\omega\)-exchange (plus \(\phi\)-exchange for the case of \(\Xi_c^{(*)}\Xi_c^{(*)}\) molecules) and speculate whether there is enough attraction to bind for a cut-off of natural size. That is, we will use phenomenological input to attempt to constrain the EFT parameters. The reliability of the predictions that can be made with this approach is rather limited, but it is worth a try. We can illustrate the strong and weak points of this idea with the \(D^{(*)}D^{(*)}\) system. In that case if we consider only interactions that do not involve derivatives over the light vector meson fields, \(\rho\)- and \(\omega\)-exchange give rise to a potential of the type

\[
V(q) = g^2 \frac{((\hat{T}_1 \cdot \hat{T}_2 - 1))}{q^2 + m^2_V},
\]

where we are assuming in a first approximation \(m_\rho = m_\omega = m_V\), and \(|g_\rho| = |g_\omega| = |g_V|\), which come from SU(3)-flavour and chiral SU(3) symmetries. From this potential we can predict that the meson vector exchange will be most attractive for isoscalar molecules, while for isovector ones it will be zero. We notice that the \(X_c(3872)\) is indeed an isoscalar (if we ignore the long distance isospin breaking due to the different masses of the \(D_0D^*_0\) and \(D_+D^*_+\) heavy meson pairs). This seems to indicate that the previous argument is on the right track. Yet there is a problem with the \(Z_c(3900)\) and \(Z_c(4025)\) states, which are presumed to be \(DD^*\) and \(D^*D^*\) isovector molecules. In this second case we notice that the strength of the contact-range coupling for the \(Z_c\’s\) is about 30\% of that of the \(X_c(3872)\) \([10]\), which is actually compatible with the expected size of the SU(3) symmetry breaking for the relationship \(g_\rho = -g_\omega\). However the biggest short-coming of the previous idea is that the saturation of the contact-range couplings provided by \(\rho\)- and \(\omega\)-exchange does not fully realize the rich light-spin structure that it is found after considering HQSS. As a consequence it does a poor job in explaining why there seems to be only one \(D^{(*)}D^{(*)}\) isoscalar molecule (the \(X_c(3872)\)), instead of the six possible ones.

For the heavy baryon-antibaryon case the starting point is the lagrangian for the SSV and TSV vertices, where \(V\) is a vector meson. If we consider interactions with no derivatives, which allow for saturation of the lowest order EFT couplings, we find

\[
\mathcal{L}_{SSV} = \lambda_2 \bar{S}v_{\mu}(V^\mu)j_{Q}^j, \quad (228)
\]

\[
\mathcal{L}_{TSV} = \lambda_3 \left[ \bar{Q}_{\mu}(V^\mu)j_{Q}^{i}\right]_{\gamma_5} + \epsilon^{ijk} \bar{Q}_{\mu}(V^\mu)j_{Q}^{i}T_{Q}^{jk}, \quad (229)
\]

where the latin indices indicate the sum over the SU(3) components. The vector-meson field is given by

\[
V = \begin{pmatrix}
\rho^0 \\
\omega^3 \\
\rho^+ \\
K^+ \\
K^* \\
\phi
\end{pmatrix}, \quad (230)
\]

where the Lorentz index \(\mu\) is implicitly understood. After working out the details along the lines of Appendix A we arrive to the potentials

\[
\langle \Xi_c \Xi_c' | V | \Xi_c' \Xi_c \rangle = 3\lambda_2^2 \frac{1}{q^2 + m_V^2} \left( \hat{T}_1 \cdot \hat{T}_2 - 1 \right), \quad (231)
\]

\[
\langle \Xi_c \Xi_c' | V | \Xi_c' \Xi_c \rangle = \lambda_2^2 \frac{1}{q^2 + m_V^2} \left( \hat{T}_1 \cdot \hat{T}_2 - 1 \right), \quad (232)
\]

\[
\langle \Sigma_c (\Sigma_{c'} | V | \Sigma_{c'} \rangle = \lambda_2^2 \frac{1}{q^2 + m_V^2} \left( \hat{T}_1 \cdot \hat{T}_2 - 1 \right), \quad (233)
\]

with the additional rules

\[
\langle B_Q \bar{B}_Q | V | B_Q \bar{B}_Q \rangle = 0, \quad (234)
\]

\[
\langle B_Q \bar{B}_Q | V | B_Q \bar{B}_Q \rangle = 0, \quad (235)
\]

\[
\langle B_Q \bar{B}_Q | V | B_Q \bar{B}_Q \rangle = 0, \quad (236)
\]

for the \(SS\) potential, where \(B_Q(B_Q^*)\) is the spin-1/2 (3/2) \(S_L = 1\) heavy baryon. Notice that in the \(\Xi_c^{(*)}\Xi_c^{(*)}\) system we write with the peculiar notation

\[
\hat{T}_1 \cdot \hat{T}_2 - 1 \quad (237)
\]

This is to keep track of the contributions coming from the \(\rho\), \(\omega\) and \(\phi\) mesons separately \((\hat{T}_1 \cdot \hat{T}_2, -1\) and \(-2\) respectively). Equivalently, if we assume saturation of the contact couplings, for the \(TS\) case we get

\[
B_{1D}^{(8)} = -\frac{3}{2} \frac{\lambda_2^2}{m_V^2}, \quad (238)
\]

\[
B_{1D}^{(10)} = -\frac{6}{\sqrt{2}} \frac{\lambda_2^2}{m_V^2}, \quad (239)
\]

while for the \(SS\) case we get

\[
C_0^{(1)} = -2 \frac{\lambda_2^2}{m_V^2}, \quad (240)
\]

\[
C_0^{(8)} = \frac{4 \lambda_2^2}{5 m_V^2}, \quad (241)
\]

\[
C_0^{(27)} = 0, \quad (242)
\]
where the superscript represent the SU(3)-flavour decomposition. The other couplings do not receive contributions from the vector mesons.

The conclusion is basically the same as with the $D^*(\bar{D}^*)$ system: bound states will be more probable in the isoscalar sector, where the piece of the vector meson exchange potential with no derivatives is most attractive. In addition we expect the isoscalar $\Sigma_Q^{(*)}\Sigma_Q^{(*)}$ molecule to be the best candidate, because in such a case the relative attraction of the vector meson exchange potential is twice the corresponding one for the $\Xi_Q^{(*)}\Xi_Q^{(*)}$ case.

The second binding mechanism we will explore is OPE itself. We assess the relative strength of the OPE potential for each molecule considered here in the following way: we will calculate the critical radius for which OPE is valid till the critical radius and (ii) no short-range physics. It is worth noticing that if the tensor force is attractive, it can always bind the system (in the hypothetical case where we consider the tensor force to be valid at all distances). The reason is that it is a singular potential. For the partial wave coupled channels considered here the tensor matrix always has an attractive eigenchannel, which means that it can always bind. The question is of course up to which distance we can believe the OPE potential to be valid. We list the critical radii for the $\Sigma_Q^{(*)}\Sigma_Q^{(*)}$ molecules in Table III. The choice of the $\Sigma_Q^{(*)}\Sigma_Q^{(*)}$ is clear: it is the channel where the potential is stronger to the higher isospin of the $\Sigma_Q$’s. The isoscalar molecules are the ones showing more attraction and higher critical radii, reaching in a few cases 1 fm. For the hidden charm molecules the uncertainty is really big as the value of $g_2$ is not experimentally known. In general a high critical radius does not necessarily imply a molecular state: if the short-range potential is also attractive the bound state might be deep, leading to a compact structure with a high admixture of non-molecular components. To give a sense of scale, we can compare with the critical radius for which OPE binds a few systems

- Deuteron ($np$): 1.00 fm
- $X(3782)$ ($D\bar{D}^*$): 0.30 fm
- $Z_c(3900)$ ($D\bar{D}^*$): 0.10 fm
- $Z_b(10610)$ ($B\bar{B}^*$): 0.26 fm
- $P_c(4450)^+$ ($\Sigma_c\bar{D}^*$): 0.58 – 0.85 fm

where we write in brackets the hadron content of each system (the reason for the $P_c(4450)^+$ value range is the uncertainty in $g_2$). As can be appreciated the spread is relatively big but most of the radii are rather small indicating that the heavy hadron molecules depend more on the short-range attraction than on the long-range OPE to bind, though this does not exclude that OPE might be responsible for the final touch. The deuteron case is interesting: the deuteron binds in $S = 1$, while the $S = 0$ np configuration there is no binding (neither there is a tensor force). At first this seems to suggest that the tensor force is essential to bind. Yet on a closer look the $S = 0$ np channel has a virtual state almost at threshold ($E_V \simeq 0.07\text{ MeV}$ [64]), i.e. it almost binds. Though the difference between binding ($S = 1$) and not binding ($S = 0$) can be explained from the tensor force, its contribution is modest. If we try to explain the deuteron as bound by OPE, the difference should be far greater. The previous indicates that the short-range interaction is in general the most important factor in the binding of hadron systems. One is also tempted to conclude that for the combinations requiring a quark content that can lead to short-range repulsion [8] (because of Fermi-Dirac statistics), such as the deuteron (six light quarks) and the $P_c(4450)^+$ (three light quarks), more attraction from OPE is required, which translates into higher critical radii in those two cases in comparison with the heavy meson-antimeson molecules. This suggests that for the heavy baryon-antibaryon case binding should be easier, as the light quark content $qqq\bar{q}$ is not expected to lead to a particularly strong short-range repulsion. Yet the evidence is merely circumstantial. Be it as it may, the outlook for heavy baryon-antibaryon molecules is that of a relatively rich spectrum. The best bet for a bound state are the isoscalar $\Sigma_Q^{(*)}\Sigma_Q^{(*)}$ molecules, though that will depend on the short-range interaction not leading to excessive binding, which would imply compact states with a complex admixture of molecular and non-molecular components (instead of a “pure” molecular state).

VII. DISCUSSION AND CONCLUSIONS

In this work we have presented a general discussion of the EFT treatment of heavy baryon-antibaryon bound states. EFTs exploit the existence of a clear separation of scales to express the observable quantities of a low energy system as a power series. The size of a hadron molecule is expected to be larger than the hadrons forming it. As a consequence we can build EFTs to describe this type of systems. Besides, heavy hadron molecules are subjected to chiral, isospin, flavour and heavy quark spin symmetry. This degree of symmetry translates into a few interesting regularities in their spectrum. For the moment the practical use of heavy hadron EFT is limited to the lowest order. The reason is the scarcity of experimental data regarding the scattering of heavy hadrons and their bound states. In most cases we barely have enough data to determine the LO couplings. That is, we expect EFT predictions for hadron molecules to have a relative error of $Q/M$, i.e. between $1/6$ and $1/3$. In this

\[ \text{Notice that in the meson-exchange description of the two-nucleon system this short-range repulsion is achieved from a large breaking of the expected SU(3) coupling for the } \omega \text{ meson.} \]
situation the value of expanding the description to subleading orders lies in the assessment of which dynamics are expected to be relevant, rather than in improving the accuracy of the theory. For the heavy baryon-antibaryon system there are no molecular candidates yet, but the fast pace at which experiments are discovering new states makes it interesting to discuss what type of physics is important in their description, a task for which EFTs are the ideal analysis tool.

EFT indicates that heavy baryon molecules can be described in terms of contact interactions and pion exchanges. At LO we expect the contact four-baryon interactions to be essential. The OPE potential is however expected to be subleading for hidden-charm and hidden-bottom baryon molecules. The exception are the isoscalar $\Sigma_Q\Sigma_C$, $\Sigma_Q^*\Sigma_C$ and $\Sigma_Q^*\Sigma_C^*$ molecules, where the tensor OPE potential might very well appear at LO, but this conclusion is only strong in the bottom sector where we expect the value of the axial coupling $g_2$ to be close to the once obtained in lattice QCD calculations. The role of particle coupled channels, i.e. transitions in which at least one heavy baryon changes from the fundamental to the excited state ($B_Q \to B_Q'$) or viceversa, is expected to be subleading too, at least if the molecule is shallow enough (with a binding momentum $\gamma = \sqrt{M_{B_Q'}|E_{B'}|}$ smaller than 350 MeV). These findings resemble previous ones for heavy meson molecules [23].

HQSS strongly constrains the contact-range interaction. This is the most important piece of the EFT potential: symmetries in the contacts are very likely to translate into symmetries in the spectrum. For the $T\bar{T}$ and $T\bar{S}$ molecules the contact potential does not depend on the total spin of the system

$$\langle T\bar{T}|V(0^-)|T\bar{T}\rangle = \langle T\bar{T}|V(1^-)|T\bar{T}\rangle,$$  

(243)

that is, these heavy baryon molecules are expected to come in pairs, as happens with the $Z_c$’s and $Z_b$’s heavy meson molecule candidates. For the $T\bar{S}/S\bar{T}$ molecules if we consider states with well-defined C-parity, there is the symmetry

$$\langle T\bar{S}|V(0^-)|T\bar{S}\rangle = \langle T\bar{S}|V(2^-)|T\bar{S}\rangle,$$  

(244)

while for the $1^-$ molecules we have to take into account whether the $S$ is a $B$ or a $B^*$ heavy baryon

$$\langle B^*\bar{B}|V(1^-)|B^*\bar{B}\rangle = \langle B^*\bar{B}|V(1^-)|B^*\bar{B}\rangle.$$  

(245)

The conclusion is that $T\bar{S}$ molecules also appear in pairs. For $S\bar{S}$ molecules the contacts have a far richer structure. For finite heavy quark mass it is worth noticing the relationship

$$\langle S\bar{S}|V(2^-)|S\bar{S}\rangle = \langle S\bar{S}|V(3^-)|S\bar{S}\rangle,$$  

(246)

which implies that these two molecules also appear in pairs. There is a similar relationship for heavy meson molecules that connects the $1^{++} X(3872)$ with a $2^{++} D^*\bar{D}^*$ partner. This molecule — the $X(4012)$ — has not been found yet in experiments, which in turn has prompted theoreticians to speculate about possible mechanisms for its disappearance. The most prosaic explanation is that the size of $1/m_Q$ corrections to HQSS might very well make the $X_2$ unbound, but other options have been explored. Three other possibilities are: (i) the decays $X_2 \to D\bar{D}$ and $X_2 \to D^*\bar{D}^*$; (ii) the effect of coupled channel dynamics [53]; and (iii) the effect of nearby quarkonia [54]. If we consider the $2^-$ and $3^-$ molecules, these effects might also play a role and spoil the HQSS pattern. We point out that (i) and (ii) are expected to be small from the point of view of the power counting developed in this work. The last of these effects, i.e. $Q\bar{Q}$ states near a heavy hadron-antihadron threshold, is not trivial to assess owing to the vagaries of the quarkonium spectrum, but we expect it to be subleading for molecules.

The interesting question remains of whether there is a way to predict heavy baryon molecules. EFTs are generic frameworks that in most settings are unable to
make predictions without preexisting experimental information. For instance EFT can predict the existence of a $2^−−$ heavy baryon molecule if the $3^−−$ molecule is already known to exist, but not necessarily on its own. There are exceptions though, where a good recent example is provided by the $Λ_c(2590)\bar{Σ}_c$ system for which a bound state prediction is possible [23]. But in general we will have to rely on phenomenological arguments up to a certain extent, which are expected to be a very tentative guide at best. In this regard if we assume the saturation of the contact-range coupling with $ρ$, $ω$- and $ϕ$-meson exchange, the most probable candidates for a heavy baryon-antibaryon bound state are the isoscalar $Σ_cΣ_c$, $Σ_c^*Σ_c$ and $Σ_c^*Σ_c^*$ molecules, located at 4906, 4970 and 5035 MeV respectively. The application of this argument to the heavy meson-antimeson case points towards the existence of six S-wave states (instead of just one confirmed so far) and misses the existence of the molecular candidates $Z_c(3900)$ and $Z_c(4020)$. But the argument is not incompatible either with the previous observations owing to the size of SU(3) flavour symmetry breaking. The saturation argument can be supplemented with the relative strength of the OPE potential in heavy baryon-antibaryon molecules, which we assess by calculating the radius for which OPE would be able to bind by itself the system. This second argument leads to similar conclusions as saturation, pointing to isoscalar molecules as the most probable, but without excluding the possibility of isovector and isotensor ones.

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Appendix A: The One Pion Exchange Potential in Heavy Hadron Chiral Perturbation Theory

The OPE potential is the finite-range piece of the leading order potential in the EFT that we have developed here. In this appendix we explain how to compute it. The idea is to obtain non-relativistic amplitudes for processes involving an incoming and outgoing heavy baryon and a pion, which might be incoming or outgoing

$$A(B \rightarrow B'\pi; \vec{q}),$$  \hspace{1cm} (A1)

where $B(B')$ are the initial/final baryon and $\vec{q}$ the momentum of the pion if outgoing (if incoming we change the momentum to $-\vec{q}$). These amplitudes can be then combined to compute the OPE potential (or for that matter any one boson exchange potential) as follows

$$\langle B'_1 B'_2 | V | B_1 B_2 \rangle = \frac{A_1(\vec{q}) A_2(-\vec{q})}{\vec{q}^2 + \mu_\pi^2},$$  \hspace{1cm} (A2)

where 1 and 2 refer to the pion vertices 1 and 2 and $\mu_\pi$ is the effective pion mass for this particular transition (which is not necessarily the physical pion mass because $m_{B'_1} - m_{B_1} \neq 0$ and gives the pion a non-zero zero-th component to its 4-momentum). The amplitudes $A_1$ and $A_2$ may refer to baryons or antibaryons indistinctively. In the following lines we will explain how to do the derivation in detail.

1. The Heavy Baryon Field

Heavy baryons contain a heavy quark and two light quarks, i.e. they have the structure

$$|Q qq\rangle.$$  \hspace{1cm} (A3)

The combined spin of the light quark pair is $S_L = 0, 1$. If the light quark spin is $S_L = 0$, we have the spin-1/2 heavy baryons $B_Q^j$

$$|B_Q^j\rangle = |Q (qq)s_L=0\rangle.$$  \hspace{1cm} (A4)

If the light quark spin is $S_L = 1$, we have spin-1/2 and spin-3/2 heavy baryons $B_Q$ and $B_Q^*$

$$|B_Q\rangle = |Q (qq)s_L=1\rangle|_{j=1/2},$$  \hspace{1cm} (A5)

$$|B_Q^*\rangle = |Q (qq)s_L=1\rangle|_{j=3/2}.$$  \hspace{1cm} (A6)

Within Heavy Hadron EFT it is customary to use the fields $T(v)$ and $S_{\mu}(v)$ instead, which have good transformation properties under rotations of the spin of the heavy quark. For the $B_Q'$ heavy baryon the definition is trivial

$$T_Q(v) = \frac{1 + \frac{\gamma_5}{2}}{2} B_Q',$$  \hspace{1cm} (A7)

$$\bar{T}_Q(v) = \frac{1 + \frac{\gamma_5}{2}}{2}.$$  \hspace{1cm} (A8)

The field $T_Q(v)$ transforms as

$$T_Q(v) \rightarrow e^{-i\vec{S}_v \cdot \vec{S}_v} T_Q(v),$$  \hspace{1cm} (A9)

where $\vec{S}_v$ is a transformation of $SU(2)_v$, the $SU(2)$ spin group of the heavy quark $Q$ moving at velocity $v$. For the $B_Q$ and $B_Q^*$ heavy baryon pair the definition is more involved

$$S_{Q\mu}(v) = \frac{1}{\sqrt{3}}(\gamma_\mu + v_\mu) \gamma_5 \frac{1 + \frac{\gamma_5}{2}}{2} B_Q$$

$$+ \frac{1 + \frac{\gamma_5}{2}}{2} B_{Q\mu}^*,$$  \hspace{1cm} (A10)

$$\bar{S}_{Q\mu}(v) = \frac{1}{\sqrt{3}} \frac{1}{\gamma_5} B_Q \frac{1 + \frac{\gamma_5}{2}}{2} (\gamma_\mu + v_\mu)$$

$$+ B_{Q\mu}^* \frac{1 + \frac{\gamma_5}{2}}{2},$$  \hspace{1cm} (A11)
where $\gamma_\mu$, $\gamma_5$ are the Dirac matrices. The $S_{Q\mu}(v)$ field transforms as

$$ S_{Q\mu}(v) \rightarrow e^{-i\vec{S}_T \cdot S_{Q\mu}(v)}. \quad (A12) $$

The $S_{Q\mu}(v)$ also contains a Lorentz index, which comes from the spin-3/2 field $B_{Q\mu}$. The usual way to have a spin-3/2 field is to couple a Lorentz index with a Dirac spinor. The coupling contains a spurious spin-1/2 component that is removed with the condition

$$ \gamma_\mu B_{Q\mu} = 0. \quad (A13) $$

For the fields $S_{Q\mu}(v)$ there are the analogous constains

$$ v^\mu S_{Q\mu}(v) = 0 \quad \text{and} \quad \not{v} S_{Q\mu}(v) = S_{Q\mu}(v). \quad (A14) $$

Finally, we will take the velocity parameter to be $v = (1, 0)$.

We are interested in heavy baryon-antibaryon molecules, i.e. we need the antibaryon fields. Here it is important to notice that

$$ \bar{T}_Q(v) = T_Q^\dagger(v) \gamma_0, \quad (A15) $$

$$ \bar{S}_{Q\mu}(v) = S_{Q\mu}(v) \gamma_0, \quad (A16) $$

are the operators for creating heavy baryons. Heavy antibaryons require the definition of new $\bar{T}_Q(v)$, $\bar{T}_Q(v)$, $\bar{S}_{Q\mu}(v)$ and $\bar{S}_{Q\mu}(v)$ fields. Here we will not need to define them explicitly because we will use C- and G-parity transformations to deduce the interactions involving the heavy antibaryons.

Other important aspect of the heavy baryon fields is their SU(2)-isospin and SU(3)-flavour structure. If we add SU(3)-flavours for the heavy baryons with $Q = c$ and $S_L = 0$, we have

$$ B_c = \begin{pmatrix} \Xi_c^0 \\
\Xi_c^+ \\
h_c^+ \end{pmatrix} \quad (A17) $$

while for $Q = c$, $S_L = 1$ we have

$$ B_c = \begin{pmatrix} \Sigma_c^+ \\
\Sigma_c^0 \frac{1}{\sqrt{2}} \\
h_c^0 \frac{1}{\sqrt{2}} \end{pmatrix} \quad (A18) $$

plus the corresponding expressions for the spin-3/2 heavy baryons $B_{c}^*$. However in the following we will only consider isospin indices: when we talk about the $T_Q$ we could either be referring to $\Lambda_c$ (isoscalar) or $\Xi_c$ (isospinor), while when we talk to $S_Q$ it could either be $\Sigma_c$ (isovector), $\Xi'_c$ (isospinor) or $\Omega_c$ (isoscalar).

It is important to notice that we are interested in the OPE potential: the isoscalar $\Omega_c$ cannot exchange single pions and hence will not be further considered here. The isoscalar $\Lambda_c$ and the isospinor $\Xi_c$ can only exchange pions in vertices involving a $\Sigma_c$ and a $\Sigma_c^*$ respectively, i.e. there is no $\Xi_c^+ \Xi c^0 \pi$ or $\Lambda_c \Lambda c^0 \pi$ vertex but there are $\Xi_c^{(\pm)} \pi$ and $\Lambda_c \Sigma_c^{(\pm)} \pi$ vertices. This factor limits the set of possible bound states that we will consider.

2. The Chiral Lagrangian at Lowest Order

The interaction of heavy baryons and pions can be written as

$$ L_{TT\pi} = 0, \quad (A19) $$

$$ L_{ST\pi} = g_3 \left[ \epsilon_{ijk} \bar{T}_Q (A^\mu)^j_{\bar{k}} S_{Q\mu}^k + \epsilon_{ijk} \bar{S}_{Q\mu}^k (A^\mu)^j_{\bar{k}} T_Q, \right] \quad (A20) $$

$$ L_{SS\pi} = i g_2 \epsilon_{\mu\nu\lambda\sigma} \bar{S}_{Q\mu}^j (A^\nu)^j_{\bar{k}} (S^\lambda)^j_{\bar{k}} \eta_{\sigma}, \quad (A21) $$

where the latin indices $i,j,k,l$ indicate either the SU(2)-isospin or the SU(3)-flavour components, $g_2$, $g_3$ are coupling constants and $\epsilon_{\mu\nu\lambda\sigma}$ is the 4-dimensional Levi-Civita symbol. In the equation above $A^\mu$ is the pseudo Goldstone-boson field,

$$ A^\mu = \frac{i}{2} (\xi^l \partial^\mu \xi - \xi^l \partial^\mu \xi)^l, \quad (A22) $$

where $\xi$ is defined as

$$ \xi = e^{\frac{i}{f_\pi} M, \quad (A23) $$

with the matrix $M$

$$ M = \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{2}} & \pi^+ & K^+ \\
-\pi^0 + \frac{\eta}{\sqrt{2}} & K^0 & K^0 \\
\eta & K^0 & -\sqrt{\frac{\eta}{2}} \eta \end{pmatrix}, \quad (A24) $$

which entails that we are taking the normalization choice $f_\pi \approx 132$ MeV. For SU(2) the $A^\mu$ field reduces to the following expansion in the pion field

$$ A^\mu = -\frac{1}{f_\pi} \partial^\mu \pi + \frac{1}{6f_\pi^3} [\pi, [\pi, \partial^\mu \pi]] + \ldots, \quad (A25) $$

where $\pi$ refers to the SU(2) submatrix in the equation above (after removing the contribution from the $\eta$), i.e.

$$ \pi = \begin{pmatrix} \pi^0 \\
\pi^+ \\
-\sqrt{\frac{\eta}{2}} \eta \end{pmatrix}, \quad (A26) $$

3. The Non-Relativistic Limit

The potential is a well-defined quantity in the non-relativistic limit. In this limit the heavy baryon fields reduce to

$$ B_Q' \rightarrow \sqrt{2MB'} \begin{pmatrix} \chi_s \\
0 \end{pmatrix}, \quad (A27) $$

$$ B_Q \rightarrow \sqrt{2MB} \begin{pmatrix} \chi_s \\
0 \end{pmatrix}, \quad (A28) $$

$$ B_{Q\mu}^* \rightarrow \begin{pmatrix} \tilde{B}_Q^* \\
0 \end{pmatrix} \rightarrow \sqrt{2MB'} \begin{pmatrix} 0 \\
\tilde{\chi}_s \end{pmatrix), \quad (A29) $$
where $\chi^i_s, \chi_s$ are standard spinor, while $\bar{\chi}_s$ is a 3-component spinor that fulfills the condition

$$\sigma \cdot \bar{\chi}_s = 0.$$  \hfill (A30)

This is the non-relativistic version of $\gamma^\mu B_{Q\mu} = 0$, which ensures that $B_{Q\mu}^* \bar{\chi}_s$ is a genuine spin-3/2 field. We have that $M_B'$ is the $S_L = 0$ heavy baryon mass, while $M_B, M_{B^*}$ are the $S_L = 1$ heavy baryon masses. We have that in the heavy quark limit the $S_L = 1$ baryon masses are identical: $M_B = M_{B^*}$. However this does not happen with the $M_B'$ mass, which remains different from $M_B$ and $M_{B^*}$ in the heavy quark limit. Taking into account the expressions above, in the non-relativistic limit the $S_L = 0$ heavy field reduces to

$$T_Q = \begin{pmatrix} B_Q^\dagger \\ 0 \end{pmatrix}, \quad (A31)$$

$$\bar{T}_Q = \begin{pmatrix} B_Q^\dagger \\ 0 \end{pmatrix}, \quad (A32)$$

while the $S_L = 1$ heavy fields read

$$\tilde{S}_Q = \begin{pmatrix} \sqrt{1 \over 3} \bar{\sigma} B_{NR} + \bar{B}_{NR}^* \\ 0 \end{pmatrix}, \quad (A33)$$

$$\bar{\tilde{S}}_Q = \begin{pmatrix} \sqrt{1 \over 3} \bar{\sigma} B_{NR}^* + \bar{B}_{NR} \end{pmatrix}, \quad (A34)$$

where $B_{NR}$ is merely the spinor $\chi_s$ we were considering before and $\bar{B}_{NR}$ is the vector spinor $\bar{\chi}_s$. The notation can be further simplified by noticing that (i) there is no difference between the $T_Q / \tilde{S}_Q$ and $T_Q^\dagger / \bar{\tilde{S}}_Q$ fields in the non-relativistic limit and (ii) by ignoring the antibaryon components. In this case we simply end up with

$$T_Q = B_{NR}^\dagger, \quad (A35)$$

$$\bar{T}_Q = B_{NR}^\dagger, \quad (A36)$$

$$\tilde{S}_Q = \sqrt{1 \over 3} \bar{\sigma} B_{NR} + \bar{B}_{NR}^*, \quad (A37)$$

$$\bar{\tilde{S}}_Q = \sqrt{1 \over 3} \bar{\sigma} B_{NR}^* + \bar{B}_{NR}. \quad (A38)$$

The pion field $A_{\mu}$ reduces in the non-relativistic to

$$\tilde{A} = - \frac{1}{f_\pi} \nabla \pi + \mathcal{O}(\frac{1}{f_\pi^2}), \quad (A39)$$

from which we can rewrite the lagrangian as

$$\mathcal{L}_{ST\pi} = - g_3 \left[ \epsilon_{ijk} T_Q^{i\dagger} (\bar{A})_j^\dagger \cdot \tilde{S}_Q^k + \epsilon_{ijk} \bar{S}_Q^{i\dagger} (\bar{A})_j^\dagger T_Q^k \right], \quad (A40)$$

$$\mathcal{L}_{SS\pi} = - i g_2 \text{Tr} \left[ \bar{S}_Q^{i\dagger} \cdot (\bar{A} \times \tilde{S}_Q) \right], \quad (A41)$$

where the trace is over isospin space. We can also expand in terms of the fields $B_{NR}', B_{NR}$ and $B_{NR}^*$:

$$\mathcal{L}_{ST\pi} = \frac{g_3}{\sqrt{3} f_\pi} B_{NR}^\dagger \bar{\sigma} \cdot \nabla \pi B_{NR}$$

$$\mathcal{L}_{SS\pi} = \frac{g_2}{f_\pi} B_{NR}^\dagger \bar{\sigma} \cdot \nabla \pi B_{NR}$$

$$\mathcal{L}_{SS\pi} = \frac{g_2}{f_\pi} \bar{B}_{NR}^\dagger \cdot \nabla \pi B_{NR} \quad (A42)$$

$$\mathcal{L}_{SS\pi} = \frac{g_3}{\sqrt{3} f_\pi} B_{NR}^\dagger \bar{\sigma} \cdot \nabla \pi B_{NR}$$

$$\mathcal{L}_{SS\pi} = \frac{g_3}{\sqrt{3} f_\pi} \bar{B}_{NR}^\dagger \cdot \nabla \pi B_{NR} \quad (A43)$$

where the isospin / flavour indices have been removed to make the expressions shorter. We notice that the derivatives of the pion field can be substituted by the pion momentum: $\nabla \rightarrow i \partial$.

4. The Spin and Isospin Factors

The next step is to calculate the matrix elements of the different vertices. For that we have to determine the spin and isospin factors. We begin by reminding that the relation between the isospin and particle basis is given by

$$B_{NR}(\Lambda_c) = \Lambda_c^+, \quad (A44)$$

$$B_{NR}(\Xi_c) = \begin{pmatrix} \Xi_c^+ \\ \Xi_c^- \end{pmatrix}, \quad (A45)$$

$$B_{NR}(\Xi'_c) = \begin{pmatrix} \Xi'_c^+ \\ \Xi'_c^- \end{pmatrix}, \quad (A46)$$

$$B_{NR}(\Sigma_c) = \begin{pmatrix} \Sigma_c^{++} \\ \Sigma_c^{--} \end{pmatrix}, \quad (A47)$$

$$B_{NR}(\Sigma'_c) = \begin{pmatrix} \Sigma'_c^{++} \\ \Sigma'_c^{--} \end{pmatrix}$$

for the $\Lambda_c, \Xi_c, \Xi'_c$ and $\Sigma_c$ baryons respectively. The relationship for the $\Xi'_c$ and $\Sigma'_c$ baryons is identical to that of the $\Xi_c$ and $\Sigma_c$ baryons. If we are interested in isospinors instead, we have

$$\Lambda_c^+ = \ket{00}_I, \quad (A48)$$

$$\{\Xi_c^+, \Xi_c^-\} = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}_I, \quad (A49)$$

$$\{\Xi'_c^+, \Xi'_c^-\} = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}_I, \quad (A50)$$

$$\{\Sigma_c^{++}, \Sigma_c^{-+}, \Sigma_c^{00}\} = \{1, 1\}_I, \{0, 0\}_I, \{1, 1\}_I \}.$$  \hfill (A51)
The isospin factors can be extracted by first expanding the isospin / flavour indices in the particle basis and later reinterpreting the result in terms of matrices in the isospin space. If we begin with the STπ lagrangian we find
\[
\Lambda^{\dagger}_{i} \pi^{a} \Sigma_{c} = t^{a}, \quad \text{(A52)}
\]
\[
\Xi^{\dagger}_{c} \pi^{a} \Xi_{c} = \frac{\tau^{a}}{2}, \quad \text{(A53)}
\]
where \(\pi^{a}\) is the pion field in the cartesian basis, \(\tau^{a}\) are the Pauli matrices and \(t^{a}\) are given by
\[
t^{1} = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
-\frac{1}{\sqrt{2}}
\end{pmatrix}, \quad \text{(A54)}
\]
\[
t^{2} = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{2}}
\end{pmatrix}, \quad \text{(A55)}
\]
\[
t^{3} = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}. \quad \text{(A56)}
\]
If we now go to the SSπ case
\[
\Xi^{\dagger}_{c} \pi^{a} \Xi^{\dagger}_{c} = \frac{\tau^{a}}{2 \sqrt{2}}, \quad \text{(A57)}
\]
\[
\Sigma^{\dagger}_{c} \pi^{a} \Sigma_{c} = \frac{T^{a}}{\sqrt{2}}, \quad \text{(A58)}
\]
where \(T^{a}\) are the \(J = 1\) angular momentum matrices in isospin space. The isospin factors for the \(\Xi^{\dagger}_{c}\) and \(\Sigma_{c}\) baryons are identical to the ones of \(\Xi_{c}\) and \(\Sigma_{c}\) baryons.

The next step is to factor out the spin in terms of angular momentum matrices or equivalent expressions. For the STπ vertices the factors are the following
\[
B^{\dagger}_{NR} \tilde{\sigma} \cdot \tilde{q} B_{NR} = \tilde{\sigma} \cdot \tilde{q}, \quad \text{(A59)}
\]
\[
B^{\dagger}_{NR} \tilde{q} \cdot B_{NR} = \tilde{S} \cdot \tilde{q}, \quad \text{(A60)}
\]
while for the SSπ vertices we have
\[
B^{\dagger}_{NR} \tilde{\sigma} \cdot (\tilde{q} \times \tilde{\sigma}) B_{NR} = -2 \tilde{\sigma} \cdot \tilde{q}, \quad \text{(A61)}
\]
\[
\tilde{B}^{\dagger}_{NR} \cdot (\tilde{q} \times \tilde{B}_{NR}) = -i \frac{2}{3} \tilde{S}^{(3/2)} \cdot \tilde{q}, \quad \text{(A62)}
\]
\[
\tilde{B}^{\dagger}_{NR} \tilde{\sigma} \cdot (\tilde{q} \times \tilde{B}_{NR}) = -i \tilde{S} \cdot \tilde{q}, \quad \text{(A63)}
\]
\[
\tilde{B}^{\dagger}_{NR} \cdot (\tilde{q} \times \tilde{\sigma}) B_{NR} = -i \tilde{S}^{(1/2)} \cdot \tilde{q}, \quad \text{(A64)}
\]
where \(\tilde{\sigma}\) are the Pauli matrices, \(\tilde{S}^{(3/2)}\) are the \(J = 3/2\) angular momentum matrices and \(\tilde{S}\) are 2x4 matrices that connect the spin-1/2 and spin-3/2 baryons. The \(\tilde{S}\) matrices read
\[
S_{1} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 1 & 0
\end{pmatrix}, \quad \text{(A65)}
\]
\[
S_{2} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0
\end{pmatrix}, \quad \text{(A66)}
\]
\[
S_{3} = \begin{pmatrix}
0 & -\sqrt{\frac{2}{3}} & 0 \\
0 & 0 & -\sqrt{\frac{2}{3}} \\
0 & 0 & 0
\end{pmatrix}, \quad \text{(A67)}
\]
which are normalized as follows
\[
S_{j} S_{j}^{\dagger} = \frac{2 \delta_{ij} - i \epsilon_{ijk} \sigma_{k}}{3}. \quad \text{(A68)}
\]
Now we can define the non-relativistic amplitudes as
\[
\mathcal{A}(B_{NR}^{(+)} \rightarrow B_{NR}^{(+)}) = -i (B_{NR}^{(+)} | \mathcal{L} | B_{NR}^{(+)}) \quad \text{(A69)}
\]
For the transitions involving \(\Lambda_{c}\) we have
\[
\mathcal{A}(\Lambda_{c} \rightarrow \Sigma_{c} \pi^{a}) = \frac{g_{3}}{\sqrt{3} f_{\pi}} t^{a} \tilde{\sigma} \cdot \tilde{q}, \quad \text{(A70)}
\]
\[
\mathcal{A}(\Lambda_{c} \rightarrow \Sigma_{c}^{*} \pi^{a}) = \frac{g_{3}}{f_{\pi}} t^{a} \tilde{\sigma} \tilde{S} \cdot \tilde{q}, \quad \text{(A71)}
\]
\[
\mathcal{A}(\Sigma_{c}^{*} \rightarrow \Lambda_{c} \pi^{a}) = \frac{g_{3}}{f_{\pi}} t^{a} \tilde{\sigma} \tilde{S} \cdot \tilde{q}. \quad \text{(A72)}
\]
For the transitions involving \(\Xi^{'}_{c}\) we have
\[
\mathcal{A}(\Xi^{'}_{c} \rightarrow \Xi^{'}_{c} \pi^{a}) = \frac{g_{3}}{\sqrt{3} f_{\pi}} \frac{t^{a}}{2} \tilde{\sigma} \cdot \tilde{q}, \quad \text{(A73)}
\]
\[
\mathcal{A}(\Xi^{'}_{c} \rightarrow \Xi^{'}_{c} \pi^{a}) = \frac{g_{3}}{f_{\pi}} \frac{t^{a}}{2} \tilde{S} \cdot \tilde{q}, \quad \text{(A74)}
\]
\[
\mathcal{A}(\Xi^{'}_{c} \rightarrow \Xi^{'}_{c} \pi^{a}) = \frac{g_{3}}{f_{\pi}} \frac{t^{a}}{2} \tilde{S} \cdot \tilde{q}, \quad \text{(A75)}
\]
For the ones with the \(\Xi^{'}_{c}\) and \(\Sigma^{*}_{c}\), the amplitudes read
\[
\mathcal{A}(\Xi^{'}_{c} \rightarrow \Xi^{'}_{c} \pi^{a}) = \frac{g_{2}}{\sqrt{3} f_{\pi}} \frac{T^{a}}{2} \tilde{\sigma} \cdot \tilde{q}, \quad \text{(A76)}
\]
\[
\mathcal{A}(\Xi^{'}_{c} \rightarrow \Xi^{'}_{c} \pi^{a}) = \frac{g_{2}}{\sqrt{3} f_{\pi}} \frac{T^{a}}{2} \tilde{S} \cdot \tilde{q}, \quad \text{(A77)}
\]
\[
\mathcal{A}(\Xi^{'}_{c} \rightarrow \Xi^{'}_{c} \pi^{a}) = \frac{g_{2}}{\sqrt{3} f_{\pi}} \frac{T^{a}}{2} \tilde{S} \cdot \tilde{q}, \quad \text{(A78)}
\]
\[
\mathcal{A}(\Xi^{'}_{c} \rightarrow \Xi^{'}_{c} \pi^{a}) = \frac{g_{2}}{\sqrt{3} f_{\pi}} \frac{T^{a}}{2} \tilde{S} \cdot \tilde{q}, \quad \text{(A79)}
\]
While finally for the ones with the \(\Sigma_{c}\) and \(\Sigma^{*}_{c}\):\[
\mathcal{A}(\Sigma_{c} \rightarrow \Sigma_{c} \pi^{a}) = \frac{2 g_{2}}{3 f_{\pi}} \frac{T^{a}}{2} \tilde{\sigma} \cdot \tilde{q}, \quad \text{(A80)}
\]
\[
\mathcal{A}(\Sigma_{c}^{*} \rightarrow \Sigma_{c}^{*} \pi^{a}) = \frac{2 g_{2}}{3 f_{\pi}} \frac{T^{a}}{2} \tilde{S} \cdot \tilde{q}, \quad \text{(A81)}
\]
\[
\mathcal{A}(\Sigma_{c} \rightarrow \Sigma_{c}^{*} \pi^{a}) = \frac{2 g_{2}}{\sqrt{3} f_{\pi}} \frac{T^{a}}{2} \tilde{S} \cdot \tilde{q}, \quad \text{(A82)}
\]
\[
\mathcal{A}(\Sigma_{c} \rightarrow \Sigma_{c}^{*} \pi^{a}) = \frac{2 g_{2}}{\sqrt{3} f_{\pi}} \frac{T^{a}}{2} \tilde{S} \cdot \tilde{q}. \quad \text{(A83)}
\]
5. G-parity and Heavy Antibaryons

The amplitudes we have obtained before are for baryons. The most explicit way to decude the amplitudes for antibaryons is to work in the isospin basis and apply a G-parity transformation. A G-parity transformation is a combination of a C-parity transformation and a rotation in isospin space

\[ G = C e^{i\pi I_2}, \]  

(A84)

with \( I_2 \) the second cartesian component of the isospin matrix. Now we have to determine how \( G \) operates on the different fields we are considering there. Pions have well-defined G-parity

\[ G|\pi\rangle = -|\pi\rangle. \]  

(A85)

We can write it in terms of the components of the pion field for completeness

\[ G\begin{pmatrix} |\pi^+\rangle \\ |\pi^0\rangle \\ |\pi^-\rangle \end{pmatrix} = -\begin{pmatrix} |\pi^+\rangle \\ |\pi^0\rangle \\ |\pi^-\rangle \end{pmatrix}. \]  

(A86)

If we consider baryons instead, \( G \) will transform a baryon into an antibaryon in the same isospin state. If we consider nucleons, which are probably the easiest example of isospin-1/2 baryons, the G-parity transformation will work as follows

\[ G\begin{pmatrix} |\bar{p}\rangle \\ |\bar{n}\rangle \end{pmatrix} = \begin{pmatrix} |\bar{n}\rangle \\ -|\bar{p}\rangle \end{pmatrix}. \]  

(A87)

As we have said before the G-parity transformation preserves isospin, which for antinucleons means that we can identify

\[ |\bar{n}\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \]  

(A88)

\[ |\bar{p}\rangle = -\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \]  

(A89)

where the subscript \( I \) is used to indicate that we are referring to isospins. From this we can appreciate that the idea of a G-parity transformation is to have an appropriate mapping between the isospin and the particle/antiparticle basis. For the heavy baryons the idea is the same as for the nucleons. There is one important difference though: if we are considering spin-3/2 heavy baryons there is an additional minus sign because of the different C-parity of spin-1/2 and spin-3/2 fields

\[ C|B_{NR}\rangle = |\bar{B}_{NR}\rangle, \]  

(A90)

\[ C|B_{SR}\rangle = -|\bar{B}_{SR}\rangle. \]  

(A91)

From now on we will consider the spin-1/2 heavy baryons: for the spin-3/2 ones the G-parity transformation will have an additional minus sign. The \( S_L = 0 \) and \( S_L = 1 \) heavy baryon isodoublets \( \{\Xi^+_c, \Xi^0_c\} \) and \( \{\Xi^+_c, \Xi^0_c\} \) transform as the nucleon

\[ G\begin{pmatrix} |\Xi^+_c\rangle \\ |\Xi^0_c\rangle \end{pmatrix} = \begin{pmatrix} -|\Xi^+_c\rangle \\ |\Xi^0_c\rangle \end{pmatrix}. \]  

(A92)

For the isotriplet baryons \( \{\Sigma^+_c, \Sigma^+_c, \Sigma^0_c\} \) we have instead

\[ G\begin{pmatrix} |\Sigma^+_c\rangle \\ |\Sigma^+_c\rangle \\ |\Sigma^0_c\rangle \end{pmatrix} = \begin{pmatrix} -|\Sigma^+_c\rangle \\ -|\Sigma^0_c\rangle \\ |\Sigma^0_c\rangle \end{pmatrix}. \]  

(A93)

With the G-parity transformation we can deduce the amplitudes for the antibaryons from the ones we already had for the baryons. To give a detailed example we can consider the amplitude

\[ \mathcal{A}(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a) = \begin{pmatrix} i(B_Q^{(i)} \pi^a)|L|\bar{B}_Q^{(i)} \end{pmatrix} \]  

\[ = -i\langle GB_Q^{(i)} \pi^a|L|\bar{B}_Q^{(i)} \rangle \]  

\[ = -i(B_Q^{(i)} \pi^a|G|L|\bar{B}_Q^{(i)} \rangle \]  

\[ = -i(B_Q^{(i)} \pi^a|L|\bar{B}_Q^{(i)} \rangle \]  

\[ = -A(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a), \]  

(A94)

where we have used that \( G^{-1} = G \) and that \( G|G = L \). Analogously we have

\[ \mathcal{A}(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a) = A(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a), \]  

(A95)

\[ \mathcal{A}(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a) = A(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a), \]  

(A96)

\[ \mathcal{A}(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a) = -A(\bar{B}_Q^{(i)} \rightarrow \bar{B}_Q^{(i)} \pi^a). \]  

(A97)

6. The OPE Potential

Now that we have the amplitudes in the correct normalization we can write the potential using Eq. (A2). For simplicity we will consider the heavy quark limit, in which the \( S_L = 1 \) \( B_Q \) and \( \bar{B}_Q \) heavy baryons are degenerate. For the \( T \bar{S} = \Lambda_c \Sigma_c, \Lambda_c \Sigma_c \) and \( T \bar{S} = \Xi_c \bar{\Xi}_c, \Xi_c \bar{\Xi}_c \) potentials we can write the potential in the bases

\[ B_{\Lambda_c \Sigma_c} = \left\{ \Lambda_c \Sigma_c, \Lambda_c \Sigma_c, \Lambda_c \Sigma_c, \Lambda_c \Sigma_c \right\}, \]  

(A98)

\[ B_{\Xi_c \bar{\Xi}_c} = \left\{ \Xi_c \bar{\Xi}_c, \Xi_c \bar{\Xi}_c, \Xi_c \bar{\Xi}_c, \Xi_c \bar{\Xi}_c \right\}. \]  

(A99)

The potential reads as follows
\[ V_{\Omega^{PE}}^{S}(\vec{q}) = \frac{g_3^2}{f_\pi^2} \tau \frac{1}{\vec{q}^2 + \mu_\pi^2} \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} & 0 \frac{1}{\sqrt{3}} \vec{\sigma}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & 0 \frac{1}{\sqrt{3}} \vec{S}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} \end{pmatrix} \]
\[ + \mathcal{O} \left( \frac{1}{m_\Omega} \right), \]

where the isospin factor \( \tau \) is
\[ \tau(\Lambda_c \Sigma_c) = 1, \quad (A101) \]
\[ \tau(\Xi_c \Xi_c') = \frac{\bar{\tau}_1 \bar{\tau}_2}{4}, \quad (A102) \]
The effective pion mass in the heavy quark limit is given by \( \mu_\pi^2 = m_\pi^2 - (m_{\Sigma_c} - m_{\Lambda_c})^2 \) or \( \mu_\pi^2 = m_\pi^2 - (m_{\Xi_c} - m_{\Xi_c'})^2 \)

\[ V_{\Omega^{PE}}^{SS}(\vec{q}) = \frac{2g_3^2}{9f_\pi^2} \tau \frac{1}{\vec{q}^2 + \mu_\pi^2} \begin{pmatrix} + \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} & -\lambda \vec{\sigma}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & +\lambda \vec{S}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} & -\lambda^2 \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} \\
-\lambda \vec{\sigma}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & +\lambda \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & -\lambda^2 \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & +\lambda \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} \\
+\lambda \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & -\lambda^2 \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & +\lambda \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & -\lambda^2 \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} \\
-\lambda^2 \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & +\lambda \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & -\lambda^2 \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & +\lambda \vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} \end{pmatrix} \]
\[ + \mathcal{O} \left( \frac{1}{m_\Omega} \right), \]

where \( \lambda = \frac{\sqrt{3}}{2} \) and where the isospin factor is
\[ \tau \left( \Xi_c \Xi_c' \right) = \frac{\bar{\tau}_1 \bar{\tau}_2}{4}, \quad (A106) \]
\[ \tau \left( \Sigma_c \Sigma_c \right) = \bar{T}_1 \cdot \bar{T}_2 . \quad (A107) \]

7. Coordinate Space

The general form of the \( ST \) and \( SS \) potential in momentum space can be written as
\[ V_{TS}^{(0)} = -R_1 \bar{R}_2 \frac{g_3^2}{f_\pi^2} \vec{T}_1 \cdot \vec{T}_2 \frac{\vec{a}_1 \cdot \vec{q} \vec{a}_2 \cdot \vec{q}}{\vec{q}^2 + \mu_\pi^2} , \quad (A108) \]
\[ V_{SS}^{(0)} = -R_1 \bar{R}_2 \frac{g_3^2}{f_\pi^2} \vec{T}_1 \cdot \vec{T}_2 \frac{\vec{a}_1 \cdot \vec{q} \vec{a}_2 \cdot \vec{q}}{\vec{q}^2 + \mu_\pi^2} , \quad (A109) \]
where \( R_1 \) and \( R_2 \) are numerical factors, \( \mu_\pi ) \) is the effective pion mass at the vertices (as explained in the previous section), \( \vec{T}_1, \vec{T}_2 \) the appropriate isospin matrices and \( \vec{a}_1, \vec{a}_2 \) are the spin matrices acting on vertex 1 and 2. The specific factors can be worked out easily from Eqs. \( (A100) \) and \( (A102) \) to obtain the results of Table II.

The coordinate space potential is obtained from Fourier transforming the momentum space one
\[ V^{(0)}(\vec{r}) = \int \frac{d^3 \vec{q}}{(2\pi)^3} V^{(0)}(\vec{q}) e^{-i\vec{q} \cdot \vec{r}} \]
\[ = R_1 \bar{R}_2 \frac{g_3^2}{f_\pi^2} \vec{T}_1 \cdot \vec{T}_2 (\vec{a}_1 \cdot \nabla) (\vec{a}_2 \cdot \nabla) \frac{e^{-\mu_\pi \vec{r}}}{4\pi r} , \quad (A110) \]
\[ C_{12}^{ST}(\vec{r}) = \begin{pmatrix}
0 & 1/3 \vec{s}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{S}_1 \cdot \vec{s}_2 \\
1/3 \vec{s}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{s}_1 \cdot \vec{S}_2 & 0 \\
0 & 1/3 \vec{s}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{S}_1 \cdot \vec{S}_2 \\
1/3 \vec{s}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{s}_1 \cdot \vec{S}_2 & 0
\end{pmatrix}, \]
\[ (A13) \]

\[ C_{12}^{S\tilde{S}}(\vec{r}) = \begin{pmatrix}
+ \vec{s}_1 \cdot \vec{s}_2 & -\lambda \vec{s}_1 \cdot \vec{S}_2 & +\lambda \vec{S}_1 \cdot \vec{s}_2 & -\lambda^2 \vec{S}_1 \cdot \vec{S}_2 \\
-\lambda \vec{s}_1 \cdot \vec{s}_2 & +\vec{s}_1 \cdot \vec{S}_2^{(3/2)} & -\lambda^2 \vec{S}_1 \cdot \vec{S}_2^{(3/2)} & +\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)} \\
+\lambda \vec{S}_1 \cdot \vec{s}_2 & -\lambda^2 \vec{S}_1 \cdot \vec{s}_2 & +\vec{S}_1 \cdot \vec{s}_2^{(3/2)} & -\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)} \\
-\lambda^2 \vec{S}_1 \cdot \vec{s}_2 & +\lambda \vec{S}_1 \cdot \vec{s}_2 & -\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)} & +\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)}
\end{pmatrix}, \]
\[ (A14) \]

\[ S_{12}^{ST}(\vec{r}) = \begin{pmatrix}
0 & 1/3 \vec{S}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{S}_1 \cdot \vec{S}_2 \\
1/3 \vec{S}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{S}_1 \cdot \vec{S}_2 & 0 \\
0 & 1/3 \vec{S}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{S}_1 \cdot \vec{S}_2 \\
1/3 \vec{S}_1 \cdot \vec{s}_2 & 0 & 1/3 \vec{S}_1 \cdot \vec{S}_2 & 0
\end{pmatrix}, \]
\[ (A15) \]

\[ S_{12}^{S\tilde{S}}(\vec{r}) = \begin{pmatrix}
+ \vec{S}_1 \cdot \vec{s}_2 & -\lambda \vec{S}_1 \cdot \vec{S}_2 & +\lambda \vec{S}_1 \cdot \vec{S}_2 & -\lambda^2 \vec{S}_1 \cdot \vec{S}_2 \\
-\lambda \vec{S}_1 \cdot \vec{s}_2 & +\vec{S}_1 \cdot \vec{S}_2^{(3/2)} & -\lambda^2 \vec{S}_1 \cdot \vec{S}_2^{(3/2)} & +\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)} \\
+\lambda \vec{S}_1 \cdot \vec{s}_2 & -\lambda^2 \vec{S}_1 \cdot \vec{s}_2 & +\vec{S}_1 \cdot \vec{s}_2^{(3/2)} & -\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)} \\
-\lambda^2 \vec{S}_1 \cdot \vec{s}_2 & +\lambda \vec{S}_1 \cdot \vec{s}_2 & -\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)} & +\lambda \vec{S}_1 \cdot \vec{S}_2^{(3/2)}
\end{pmatrix}. \]
\[ (A16) \]

8. The Partial Wave Projection

Heavy baryon-antibaryon bound states have well defined \( J^P \) quantum numbers. Hence we can simplify the OPE potential by projecting it into partial waves with well-defined parity and angular momentum. For this we define the states

\[ |S\tilde{S}(jm)\rangle = \sum_{lm_1sm_s} Y_{lm_1}(\hat{r})|sm_s\rangle \langle lm_1sm_s |jm\rangle, \]
\[ (A17) \]

where \( j, m \) is the total angular momentum and its third component for the heavy baryon-antibaryon pair, while \( l, m_1 \) and \( s, m_s \) refer to the angular momentum and spin of the pair. The product \( |lm_1sm_s\rangle \langle jm\rangle \) is the Clebsch-Gordan coefficient for that particular combination of total, orbital and spin angular momentum. The spin wave function can be further decomposed as

\[ |sm_s\rangle = \sum_{s_1s_2} |s_1m_1\rangle|s_2m_2\rangle \langle s_1m_1s_2m_2 |sm_s\rangle, \]
\[ (A18) \]

with \( s_1 \) and \( s_2 \) the spin of the heavy baryon 1 and 2 (either 1/2 or 3/2).

In this basis we can compute the partial wave projection of \( C_{12} \) and \( S_{12} \) as

\[
\langle (s' l')jm' |O_{12}(\hat{r})|sm\rangle = \\
\int d^2\hat{r}' \langle (s' l')jm' |O_{12}(\hat{r}')\rangle |(sl)jm\rangle = \\
\delta_{jj'} \delta_{m'm} \mathcal{O}_{ss'l'rr}^{ij},
\]
\[ (A19) \]

where the total angular momentum and its third component are conserved. The central force \( C_{12} \) conserves in addition the orbital angular momentum and the spin

\[ C_{ss'l'rr}^{ij} = \delta_{ss'} \delta_{l'l} \mathcal{C}_{1s}^{ij}. \]
\[ (A120) \]

The tensor force is more involved. Owing to conservation of parity \( |l - l'| \) must be zero or an even number. The spin transitions are a bit more complicated: nothing prevents \( |s - s'| \) to be an odd number in general. The expectation are the \( B^0B^\star \) and \( B^*B^* \) systems: in general these systems have well defined C- or G-parity, which implies that \( |s - s'| \) must be either zero or an even number. Even when C/G-parity is not well defined, as in the \( \Xi_1\Sigma_0 \) system, the tensor operator involves identical spin-matrices and is symmetrical under the exchange of particles 1 and 2:

\[ S_{12}^{B^0B^*} = 3 \vec{S}_1 \cdot \vec{s}_2 \cdot \hat{r} - \vec{s}_1 \cdot \vec{S}_2, \]
\[ (A121) \]
\[ S_{12}^{B^*B^*} = 3 \vec{s}_1 \cdot \vec{s}_2^{(3/2)} \cdot \hat{r} - \vec{S}_1 \cdot \vec{S}_2^{(3/2)} . \]
\[ (A122) \]

As a consequence the matrix elements for the tensor operator vanishes for odd \( |s - s'| \).

On the contrary if we are considering the \( B^0B^\star \), \( B^*B^\star \) and \( B^*B^* \) systems \( (\Xi_1\Xi_0^*, \Xi_1^\prime\Xi_0^* \) and \( \Sigma_1^\prime\Sigma_0^* \), it is perfectly possible to have a mix of even and odd spin. We will have the following set of partial waves

\[ |B^0(\vec{B}^\star(1^-))\rangle = \{3 \vec{S}_1 \cdot 3 D_1, 5 D_1\}, \]
\[ |B^0(\vec{B}^\star(2^-))\rangle = \{1 D_1, 5 S_2, 5 D_2, 5 G_2\}. \]
\[ (A123) \]
\[ (A124) \]

The odd \( |s - s'| \) transitions have a particularity that it is worth mentioning: the matrix element for the tensor operator for odd \( |s - s'| \) changes sign depending on whether

\[ |s - s'| = 1 \text{ or } 3 \]
it is the particle or the antiparticle the one which is in an excited state

\[
\langle B^\prime | B^* | S_{q_s}^{(s+1)J} | B^\prime | B^* \rangle = \langle B^\prime | B^* | S_{q_s}^{(s+1)J} | B^\prime | B^* \rangle, \quad (A125)
\]

\[
\langle B^\prime | B^* | S_{q_s}^{(s+1)J} | B^\prime | B^* \rangle = -\langle B^\prime | B^* | S_{q_s}^{(s+1)J} | B^\prime | B^* \rangle. \quad (A126)
\]

If we are considering states with well defined C-parity

\[
|B^\prime | B^* \rangle = \frac{1}{\sqrt{2}} \left[ |B^\prime | B^* \rangle + \eta |B^\prime | B^* \rangle \right], \quad (A127)
\]

where \( C = \eta (-1)^{L+J} \), we find the combinations

\[
|B^\prime | B^* (1^+) \rangle = \left\{ 3S_1(-), 3D_1(-), 5D_1(+) \right\}, \quad (A128)
\]

\[
|B^\prime | B^* (2^+) \rangle = \left\{ 3D_2(-), 5S_2(+), 5D_2(+), 5G_2(+) \right\}, \quad (A129)
\]

\[
|B^\prime | B^* (1^-) \rangle = \left\{ 3S_1(+), 3D_1(+), 5D_1(-) \right\}, \quad (A130)
\]

\[
|B^\prime | B^* (2^-) \rangle = \left\{ 3D_2(+), 5S_2(-), 5D_2(-), 5G_2(-) \right\}. \quad (A131)
\]

where the number in parentheses is the value of \( \eta = \pm 1 \).

**Appendix B: The Contact-Range Potential**

For the contact-range potential of the heavy baryon-antibaryon molecules we use a different approach. For the OPE potential we wrote the lowest-dimensional lagrangian compatible with HQS and then derive the potential from this lagrangian. Here we apply a different method. First we take into account that the lowest order contact-range potential is simply a constant in momentum space

\[
\langle p^\prime | V^{(0)}_C | 0 \rangle = C, \quad (B1)
\]

where the constant \( C \) can in principle depend on the quantum numbers, including in principle both the light and heavy quark spin of the heavy baryon-antibaryon pair involved. Then we constrain the previous dependency by taking into account that HQSS precludes \( C \) to depend on the heavy quark spin. We will explain in the following lines how to do that.

Heavy baryons are \( |Qqq \rangle \) states. The heavy and light spin structure of heavy baryons depends on what is the spin of the light quark pair. We have \( s_L = 0 \) or \( s_L = 1 \)

\[
T = B_Q = |Q(qq)_{s_L=0} \rangle, \quad (B2)
\]

\[
S = B_Q, B_{\bar{Q}} = |Q(qq)_{s_L=1} \rangle, \quad (B3)
\]

where in the second line \( B_Q \) have total spin of 1/2 and \( B_{\bar{Q}} \) 3/2. The non-relativistic reduction of the Rarita-Schwinger field \( B_Q^* \) is simply a 3-vector of spinors

\[
\bar{B}_Q^* = \chi = (\chi_1, \chi_2, \chi_3), \quad (B4)
\]

that fulfill the condition \( \sigma \cdot \chi = 0 \), which ensures that we are only taking the spin-3/2 component.

1. **The \( S\bar{S} \) Contact Potential**

For a heavy \( S\bar{S} \) baryon-antibaryon system (i.e. \( S_L = 1 \) for both baryons) we can decompose the spin wave function into heavy and light components as follows

\[
|B^\prime (s) | B^* (s\prime) \rangle = \sum_{S_H, S_L} D_{S_H, S_L} (J) S_H \otimes S_L | J \rangle, \quad (B5)
\]

where \( D \) are the coefficients for this change of basis. They fulfill the condition

\[
\sum_{S_H, S_L} |D_{S_H, S_L} (J)|^2 = 1. \quad (B6)
\]

From this decomposition we can calculate the light-spin components of the contact-range potential

\[
\langle S'_H | S'_L | V | S_H \otimes S_L \rangle = \delta_{S_H, S'_H} \langle S'_L | V | S_L \rangle, \quad (B7)
\]

where the light spin flip \( |S'_L - S_L \rangle \) is even because of C- or G-parity conservation in the \( q\bar{q} \) subsystem.

The general way to do the decomposition is to consider the spin wave function of the heavy hadrons

\[
|H_1 \rangle = |s_H s_{11} j_1 \rangle, \quad |H_2 \rangle = |s_H s_{12} j_2 \rangle, \quad (B8)
\]

where \( s_H, s_{12} \) is the heavy spin, \( s_{11}, s_{12} \) the light spin and \( j_1, j_2 \) the angular momenta of the two hadrons. When we couple the two hadrons together we have

\[
|H_1 H_2 \rangle = \sum_{s_{11}, s_{12}} D_{S_{11}, S_{12}} (J) |(s_H s_{12}) S_H (s_{11} s_{12}) S_L (j_1 j_2) J \rangle, \quad (B9)
\]

where the previous notation indicates that the heavy spins coupled to \( S_{11} \), the light spins to \( S_{12} \) and the angular momenta to \( J \). The coefficients \( D_{S_{11}, S_{12}} (J) \) can in fact be expressed in terms of 9-J symbols.
Finally if we are considering antihadrons, we should consider their behaviour under C-parity to define their spin wave functions consistently: they might differ by a sign from the ansatz $|s_H s_L J\rangle$.

If we go back to the $S\bar S$ heavy baryon-antibaryon system, for the $B\bar B$ case we find the following

\[ |B\bar B(0^-)\rangle = \frac{1}{\sqrt{3}} 0_H \otimes 0_L + \frac{1}{\sqrt{3}} 1_H \otimes 1_L |_{J=0}, \]  
\[ |B\bar B(1^-)\rangle = \frac{\sqrt{5}}{3} 0_H \otimes 1_L - \frac{1}{3\sqrt{3}} 1_H \otimes 0_L + \frac{2}{3} \frac{\sqrt{5}}{3} 1_H \otimes 2_L |_{J=1}. \]  

For the $B^*\bar B^*$ and $B^*\bar B$ cases, we include a minus sign if front of the states containing a $B^*$ to highlight the C-parity convention that we employ here:

\[ -|B^*\bar B(1^-)\rangle = \frac{1}{3} 1_H \otimes 0_L - \frac{2}{3} \frac{\sqrt{5}}{3} 1_H \otimes 0_L + \frac{1}{\sqrt{2}} 1_H \otimes 1_L |_{J=1} - \frac{1}{3} \sqrt{\frac{5}{6}} 1_H \otimes 2_L |_{J=1}, \]  
\[ + |B^*\bar B(1^-)\rangle = -\frac{1}{3} 1_H \otimes 0_L + \frac{2}{3} \frac{\sqrt{5}}{3} 1_H \otimes 0_L + \frac{1}{\sqrt{2}} 1_H \otimes 1_L |_{J=1} + \frac{1}{3} \sqrt{\frac{5}{6}} 1_H \otimes 2_L |_{J=1}. \]

Finally, for the $B^*\bar B^*$ case we have

\[ -|B^*\bar B^*(0^-)\rangle = \sqrt{\frac{2}{3}} 0_H \otimes 0_L - \frac{1}{\sqrt{3}} 1_H \otimes 1_L |_{J=0}, \]  
\[ -|B^*\bar B^*(1^-)\rangle = \frac{\sqrt{5}}{3} 0_H \otimes 1_L + \frac{1}{3} \sqrt{\frac{10}{3}} 1_H \otimes 0_L - \frac{1}{3} \sqrt{\frac{5}{3}} 1_H \otimes 2_L |_{J=1}, \]  
\[ -|B^*\bar B^*(2^-)\rangle = \frac{1}{\sqrt{3}} 0_H \otimes 2_L + \frac{2}{3} \frac{\sqrt{5}}{3} 1_H \otimes 1_L |_{J=2}, \]  
\[ -|B^*\bar B^*(3^-)\rangle = 1_H \otimes 2_L |_{J=3}, \]

where we have included the minus sign to stress the convention.

Finally for the $B^*\bar B^*$ we can also write the decomposition in the basis with well-defined C-parity for those cases where it applies

\[ |B^*\bar B(1^+)\rangle = 1_H \otimes 1_L |_{J=1}, \]  
\[ |B^*\bar B(1^-)\rangle = \frac{\sqrt{2}}{3} 0_H \otimes 1_L - \frac{4}{3\sqrt{3}} 1_H \otimes 0_L - \frac{1}{3} \sqrt{\frac{5}{3}} 1_H \otimes 2_L |_{J=1}, \]  
\[ |B^*\bar B(2^+)\rangle = \frac{\sqrt{2}}{3} 0_H \otimes 2_L - \frac{1}{3} \sqrt{\frac{5}{3}} 1_H \otimes 1_L |_{J=2}, \]  
\[ |B^*\bar B(2^-)\rangle = 1_H \otimes 2_L |_{J=2}. \]

From the previous decomposition and applying Eq. (B7) we obtain the contact-range potentials of Section III.

2. The $T\bar S/S\bar T$ Contact Potential

For a heavy $T\bar S/S\bar T$ baryon-antibaryon system (i.e. one baryon with $S_L = 0$ and the other with $S_L = 1$) the analysis requires special attention to the fact that the light quarks $q\bar q\bar q$ cannot be treated as identical particles in this case. The reason is that the spin of the light quarks and antiquarks is different. The two possible configurations

\[ |(Q\bar Q)_{S_H} (q\bar q)_{S_{L1}=0} (\bar q\bar q)_{S_{L2}=1}\rangle, \]  
\[ |(Q\bar Q)_{S_H} (q\bar q)_{S_{L1}=1} (\bar q\bar q)_{S_{L2}=0}\rangle, \]
generate different matrix elements. The heavy-light decomposition of the potential is in this case
\[
\langle S_H' \otimes S_L' \otimes S_L |V| S_H \otimes S_L \otimes S_L \rangle = \\
\delta_{S_H S_H'} \langle S_L' |S_L' \otimes S_L |V| S_L S_L \rangle, \quad (B26)
\]
where we take into account that the light quark spin from particles 1 and 2 is distinguishable. If we have a particle-antiparticle system, C-parity implies
\[
\langle S_L' |S_L |V| S_L S_L \rangle = \langle S_L' |S_L' \otimes S_L |V| S_L S_L \rangle. \quad (B27)
\]
As a consequence, for the $T\bar{S}/S\bar{T}$ case there are two contact couplings corresponding to
\[
(01|V|01) = \langle 10 |V| 10 \rangle, \quad (B28)
(01|V|10) = \langle 10 |V| 01 \rangle. \quad (B29)
\]
That is, a contact that conserves the spin of particles 1 and 2 and a contact that flips it. For the $B'\bar{B}$ and $BB'$ the heavy-light spin decomposition reads
\[
|B'\bar{B}(0^-)\rangle = +\frac{1}{\sqrt{3}}0_H \otimes 1_{\bar{q}\bar{q}} + \frac{\sqrt{2}}{3}1_H \otimes 1_{\bar{q}\bar{q}} |_{J=0}, \quad (B30)
\]
\[
|BB'(0^-)\rangle = -\frac{1}{\sqrt{3}}1_H \otimes 1_{\bar{q}\bar{q}} |_{J=0}, \quad (B31)
\]
\[
|B'\bar{B}(1^-)\rangle = -\frac{1}{\sqrt{3}}0_H \otimes 1_{\bar{q}\bar{q}} + \frac{\sqrt{2}}{3}1_H \otimes 1_{\bar{q}\bar{q}} |_{J=1}, \quad (B32)
\]
\[
|BB'(1^-)\rangle = +\frac{1}{\sqrt{3}}0_H \otimes 1_{\bar{q}\bar{q}} + \frac{\sqrt{2}}{3}1_H \otimes 1_{\bar{q}\bar{q}} |_{J=1}. \quad (B33)
\]
while for the $B'B^*$ and $B^*\bar{B}'$ we include the minus sign in front of the states to make the C-parity convention manifest
\[
-|B'B^*(1^-)\rangle = \sqrt{\frac{2}{3}}0_H \otimes 1_{\bar{q}\bar{q}} + \frac{1}{\sqrt{3}}1_H \otimes 1_{\bar{q}\bar{q}} |_{J=1}, \quad (B34)
\]
\[
+|B^*\bar{B}'(1^-)\rangle = \sqrt{\frac{2}{3}}0_H \otimes 1_{\bar{q}\bar{q}} - \frac{1}{\sqrt{3}}1_H \otimes 1_{\bar{q}\bar{q}} |_{J=1}. \quad (B35)
\]
\[
-|B'B^*(2^-)\rangle = 1_H \otimes 1_{\bar{q}\bar{q}} |_{J=2}, \quad (B36)
\]
\[
+|B^*\bar{B}'(2^-)\rangle = 1_H \otimes 1_{\bar{q}\bar{q}} |_{J=2}. \quad (B37)
\]
In the decomposition above only the quark pair with $S_z = 1$ is written. The other quark pair is implicitly understood, i.e.
\[
1_{\bar{q}\bar{q}} = 1_{\bar{q}\bar{q}} \otimes 0_{\bar{q}\bar{q}}, \quad (B38)
1_{\bar{q}\bar{q}} = 0_{\bar{q}\bar{q}} \otimes 1_{\bar{q}\bar{q}}. \quad (B39)
\]
From the decomposition and the definitions
\[
B_{D1} = (1_{\bar{q}\bar{q}}|V|1_{\bar{q}\bar{q}}) = (1_{\bar{q}\bar{q}}|V|1_{\bar{q}\bar{q}}), \quad (B40)
B_{E1} = (1_{\bar{q}\bar{q}}|V|1_{\bar{q}\bar{q}}) = (1_{\bar{q}\bar{q}}|V|1_{\bar{q}\bar{q}}), \quad (B41)
\]
we obtain the contact-range potentials of Section III.

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