Conservation laws in the continuum $1/r^2$ systems

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Abstract

We study the conservation laws of both the classical and the quantum-mechanical continuum $1/r^2$ type systems. For the classical case, we introduce new integrals of motion along the recent ideas of Shastry and Sutherland (SS), supplementing the usual integrals of motion constructed much earlier by Moser. We show by explicit construction that one set of integrals can be related algebraically to the other. The difference of these two sets of integrals then gives rise to yet another complete set of integrals of motion. For the quantum case, we first need to resum the integrals proposed by Calogero, Marchioro and Ragnisco. We give a diagrammatic construction scheme for these new integrals, which are the quantum analogues of the classical traces. Again we show that there is a relationship between these new integrals and the quantum integrals of SS by explicit construction. Finally, we go to the asymptotic or low-density limit and derive recursion relations of the two sets
of asymptotic integrals.

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I. INTRODUCTION

The integrability of both the classical and the quantum one-dimensional problem of $N$ particles interacting via the two-body potentials $V_0(x) = g^2/x^2$, $V_t(x) = g^2\Phi^2 \sin^{-2}[\Phi x]$ and $V_h(x) = g^2\Phi^2 \sinh^{-2}[\Phi x]$ has been shown more than two decades ago by Moser \cite{1} (for the classical problem) and Calogero, Marchioro and Ragnisco \cite{CMR} (for the quantum problem), both groups exploiting a technique due to Lax \cite{2}. These early results have been reviewed, extended and collected nicely both for the classical and the quantum cases by Olshanetsky and Perelomov in Ref. \cite{3,4}.

For the classical systems, integrability restricts the motion in terms of action-angle variables onto a torus in phase space. However, for the quantum case, integrability leads to solvability only for those special cases which support scattering, i.e., systems which fly apart when the walls of the box are removed. In these cases, integrability implies conservation of individual momenta and thus the wave function is given asymptotically by Bethe’s Ansatz. For the above interaction potentials, Sutherland \cite{5} has exploited this fact to determine the properties of the quantum systems in the thermodynamic limit.

Recently, Shastry and Sutherland \cite{SS} have given an independent proof of integrability of the quantum many-body problem and constructed new integrals of motion. However, for any finite number of particles $N$, we know that in principle we have exactly $N$ conserved quantities. Therefore we expect the new integrals of motion to be related to the integrals constructed by CMR. It is the aim of the present work to elucidate some of the features of the new integrals of motion and to show their relation to the integrals of CMR. We emphasize that this new proof of integrability has also made possible the application of the ideas of the asymptotic Bethe Ansatz to the $1/r^2$ models with quantum exchange \cite{6}.

In section II we show that the new construction of SS gives integrals of motion also for the classical problem. We next explicitly calculate these new integrals up to $n = 4$ and compare them to the integrals of CMR. This then gives rise to yet another set of integrals $K_n$. Section III is devoted to a comparison of the two series of integrals of motion for the quantum case. The integrals given by CMR are not extensive quantities and we need to resum them via an application of the linked cluster theorem. In section IV, we take the asymptotic or low-density limit of the problem and section V summarizes and discusses our results.

II. THE CLASSICAL CASE

The Hamiltonian of primary interest for our present work is given as

$$H = \sum_i p_i^2 + \lambda(\lambda - 1)\Phi^2 \sum_{ij}' \sinh^{-2}[\Phi(x_i - x_j)].$$

(1)

The interaction term reduces to $V_0$ in the limit of high-densities (or $\Phi \to 0$) and the trigonometric interaction $V_t$ is just the analytic continuation of $\Phi \to i\Phi$. Here and in the following, we will use the primed sum $\sum'$ to indicate that the summation runs over unequal indices only.
A. Moser’s invariants

Let us briefly recall the method of \[1, 2\]: We introduce the Lax pair $L, M$,

\[ L_{jk} = p_j \delta_{jk} + i(1 - \delta_{jk}) \sqrt{\lambda(\lambda - 1)} \Phi \coth[\Phi(x_j - x_k)], \]  
(2)

\[ M_{jk} = 2 \sqrt{\lambda(\lambda - 1)} \Phi^2 \left[ \delta_{jk} \sum_l \sinh^{-2}[\Phi(x_j - x_l)] + (1 - \delta_{jk}) \sinh^{-2}[\Phi(x_j - x_k)] \right]. \]  
(3)

The classical equations of motion then imply the matrix equation

\[ \frac{dL}{dt} = \{L, H\} = i[ML - LM], \]  
(4)

where we define the Poisson brackets as $\{F, G\} = \sum_{j=1}^{N} \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial x_j}$. The time evolution of $L$ consequently is an isospectral deformation,

\[ L(t) = \exp[i \int_0^t M(\tau) d\tau] L(0) \exp[-i \int_0^t M(\tau) d\tau], \]  
and the integrals of motion are simply given as the traces

\[ T_n = \text{Tr} L^n(t) = \text{Tr} L^n. \]  
(5)

We also need to show that the $T_n$’s are in involution, e.g., $\{T_n, T_m\} = 0$. Using the Jacobi relation for Poisson brackets, we see that

\[ \{H, \{T_n, T_m\}\} = \{T_n, \{H, T_m\}\} - \{T_m, \{H, T_n\}\}, \]  
(6)

and thus $\{T_n, T_m\}$ is also an integral of motion. But, allowing the system the evolve in time, all particles scatter, the Lax matrix itself evolves into

\[ L \xrightarrow{t \to \infty} L^\infty = \begin{cases} L_{jj} = k_j, \\ L_{jk} = +i \sqrt{\lambda(\lambda - 1)}, & j > k; \\ L_{jk} = -i \sqrt{\lambda(\lambda - 1)}, & j < k. \end{cases} \]  
(7)

and so the coordinate dependence vanishes. Thus the Poisson bracket $\{T_n, T_m\}$ evaluates to zero. We remark that it is this procedure that we use to prove involution for all the integrals constructed in the following chapters.

Let us define $\alpha_{jk} = \sqrt{\lambda(\lambda - 1)} \Phi \coth[\Phi(x_j - x_k)]$ and $\alpha_{jj} = 0$. Then a direct calculation of the integrals of motion up to $n = 4$ gives

\[ T_1 = \sum_i p_i = P, \]  
(8a)

\[ T_2 = \sum_i p_i^2 + \sum_{ij} \alpha_{ij}^2, \]  
(8b)

\[ T_3 = \sum_i p_i^3 + 3 \sum_{ij} \alpha_{ij}^2 p_i, \]  
(8c)

\[ T_4 = \sum_i p_i^4 + 2 \sum_{ij} \alpha_{ij}^2 (p_j^2 + p_i p_j + p_i^2) + \text{Tr} \alpha^4. \]  
(8d)
Using $\alpha_{ij}^2 = \Phi \sqrt{\lambda(\lambda - 1)} \left[ \sinh^{-2}[\Phi(x_j - x_k)] + 1 \right]$, we see that $T_2 = H + \Phi^2 \lambda(\lambda - 1)N(N-1)$. Note that due to the antisymmetry $\alpha_{ij} = -\alpha_{ji}$, only even powers of $\alpha$ — and thus integer powers of $\lambda$ — will appear in all these expressions.

Let us now define the classical down-boost \[ X = \sum_{j=1}^{N} x_j. \] (9)

We then find easily that \[ \{X, T_n\} = nT_{n-1}. \] (10)

Further, Jacobi’s identity gives \[ \{X, \{T_n, T_m\}\} = (n-1)\{T_{n-1}, T_m\} + (m-1)\{T_{m-1}, T_n\}. \] (11)

As a particular case, suppose $n = 2$, so $n - 1 = 1$ and $T_{n-1} = P$. Then by translation invariance $\{P, T_n\} = 0$, so we conclude that if $\{H, T_n\} = 0$, then $\{H, T_{n-1}\} = 0$. In particular, $\{H, T_N\} = 0$ implies that all $T_n$ are integrals. Finally, we may construct all integrals of motion from $T_N$ by repeatedly using the boost $X$ in the representation

\[ X = \sum_{j=1}^{N} \frac{\partial}{\partial p_j}. \] (12)

### B. Shastry’s invariants

In Ref. [SS], Shastry and Sutherland provide a set of integrals of motion for the quantum problem. However, mimicking their arguments, we can straightforwardly construct integrals of motion for the classical case, too. Let us introduce the singular matrix $\Delta_{jk} = 1$ for all $i, j$ and the vector $\eta_j = 1$ for all $j$. Then we define integrals of motion s.t.

\[ J_n = \text{Tr}[L^n(t)\Delta] = \eta^\dagger L^n(t)\eta = \sum_{i_1, i_2, \ldots, i_{n+1}} L_{i_1 i_2} L_{i_2 i_3} \cdots L_{i_{n-1} i_n} L_{i_n i_{n+1}}, \] (13)

with the Lax matrix $L$ given as before. We then have

\[ \frac{dJ_n}{dt} = \frac{d}{dt}\{\text{Tr}[\exp[iMt]L^n(t)\exp[-iMt]\Delta]\} \]
\[ = i\text{Tr}[ML^n(t)\Delta - L^n(t)M\Delta] \]
\[ = i[\text{Tr}[L^n(t)\Delta M] - \text{Tr}[L^n(t)M\Delta]] \]
\[ = 0, \]

since $M\Delta = \Delta M = 0$ as shown in SS. Involution for these integrals of motion is proven by the same asymptotic argument as before. A direct calculation of the conserved quantities of SS up to $n = 4$ gives:
\[ J_1 = \sum_i p_i, \]  
\[ J_2 = \sum_i p_i^2 + \sum_{ij} \alpha_{ij}^2 - \sum_{ijk} \alpha_{ijk} \alpha_{jk}, \]  
\[ J_3 = \sum_i p_i^3 + 3 \sum_{ij} \alpha_{ij}^2 p_i - \sum_{ijk} \alpha_{ij} \alpha_{jk} (p_i + p_j + p_k), \]  
\[ J_4 = \sum_i p_i^4 + 2 \sum_{ij} \alpha_{ij}^2 \left[ p_i^2 + p_i p_j + p_j^2 \right] + \text{Tr} \alpha^4 + \sum_{i \neq j \neq k \neq l \neq m \neq i} \alpha_{ij} \alpha_{jk} \alpha_{kl} \alpha_{lm} \]  
\[ - \sum_{ij} \alpha_{ij} \alpha_{jk} \left[ p_i^2 + p_j^2 + p_k^2 + p_i p_j + p_j p_k + p_k p_i \right]. \]  

Again the Hamiltonian can be found in the \( n = 2 \) term, \( J_2 = H + \Phi^2 \lambda (\lambda - 1) N (N^2 - 1)/3 \) and again only even powers of \( \alpha \) appear in the expressions of the \( J_n \)'s.

The action of the down-boost on these new integrals of motion is as in Eq. (10), e.g., \( \{ X, J_n \} = nJ_{n-1} \). Much more useful is the up-boost \( Y \) which we define as

\[ Y = \sum_i x_i p_i^2 + \sum_{ij} (x_i + x_j) \alpha_{ij}^2 / 2 \]  

in analogy with the up-boost operator \( \sum_n nS_n S_{n+1} \) in the Heisenberg model. Unfortunately, this up-boost only works, if we restrict ourselves to the potential \( V_0 \) such that \( \alpha_{ij}^2 = \lambda (\lambda - 1) / (x_i - x_j)^2 \). In this case, we find by explicit construction that \( \{ Y, J_n \} = (n + 1)J_{n+1} \). The Jacobi relation \( \{ J_m, \{ Y, J_n \} \} = \{ Y, \{ J_m, J_n \} \} - \{ J_n, \{ J_m, Y \} \} \) now gives

\[ (n + 1) \{ J_m, J_{n+1} \} = \{ Y, \{ J_m, J_n \} \} - (m + 1) \{ J_n, J_{m+1} \}. \]  

Thus, if \( \{ J_m, J_n \} = 0 \) and \( \{ J_{m+1}, J_n \} = 0 \), we also have \( \{ J_m, J_{n+1} \} = 0 \). We emphasize that the up-boost \( (10) \) seems to work only for the special potential \( V_0 \).

C. Relation between invariants

We can again use the Jacobi relation to show that the Poisson bracket \( \{ T_n, J_m \} \) is an integral of motion which in the asymptotic limits evaluates to zero. The difference between the integrals of motion of Moser and SS then gives rise to yet another set of constants,

\[ K_n = J_n - T_n = \sum_{i_1 \neq i_{n+1}} L_{i_1 i_2} L_{i_2 i_3} \cdots L_{i_{n-1} i_n} L_{i_n i_{n+1}}. \]  

Various terms in the \( J_n \)'s can be simplified with the help of the coth-rule,

\[ \alpha_{ij} \alpha_{jk} + \alpha_{ij} \alpha_{ki} + \alpha_{jk} \alpha_{ki} = -\Phi^2 \lambda (\lambda - 1), \]

and hence we find
\[ K_1 = 0, \]  
\[ K_2 = \Phi^2 \lambda (\lambda - 1) N(\lambda - 1)(\lambda - 2)/3, \]  
\[ K_3 = \Phi^2 \lambda (\lambda - 1)(\lambda - 1)(\lambda - 2) P, \]  
\[ K_4 = \Phi^2 \lambda (\lambda - 1)(\lambda - 2) \left( (N - 2) T_2 + P^2 \right) + \left[ \Phi^2 \lambda (\lambda - 1)(\lambda - 1)(\lambda - 2) \right]^2 / 9. \]  

Note that \( K_3 \) is the first term that is not a simple constant, and in order to make the \( K_n \)'s a complete set of integrals of motion, we may simply use \( K_{N+1} \) and \( K_{N+2} \). Thus we conclude that by construction, we can express Shastry’s integrals of motion in terms of Moser’s and vice versa. We emphasize that this relationship is not linear, but only algebraic as seen from the existence of the \( P^2 \) term in \( K_4 \).

Taking the limit \( \Phi \to 0 \), we see that the \( K_n \)'s are zero. Thus only for the simplest case of the Calogero potential \( V_0(x) = g^2 / x^2 \) do we find that the Moser set of integrals of motion is identical to the set of SS.

### III. THE QUANTUM CASE

In the quantum case, the elements of the Lax \( L \) and \( M \) matrices become operators themselves, i.e., the momentum operator is \( p_j = -i\partial / \partial x_j \) and we have the commutation relation \([x_j, p_k] = i\delta_{jk}\). Since operator elements do not necessarily commute, we always mean an ordered product of elements when we multiply matrices in the following.

#### A. Calogero’s invariants

The early work of Calogero, Marchioro and Ragnisco [CMR] quantised the classical Lax equation, by antisymmetrising the right-hand side of Eq. (4). The proof of invariance of the traces then does no longer hold. However, CMR also showed that after replacing the classical variables with the corresponding quantum mechanical operators, we can define new integrals of motion \( I_n \) s.t.

\[ \Delta(\beta) \equiv \det[1 - \beta L] \equiv 1 + \sum_{n=1}^{N} (-\beta)^n I_n. \]  

CMR then go on to argue that these \( I_n \) are conserved, \([I_n, H] = 0\), and in involution, \([I_n, I_m] = 0\). The later result is again proved [7] by use of the asymptotic limit as in the last section. A direct calculation of the conserved quantities of CMR up to \( n = 5 \) for \( H \) yields:

\[ I_1 = \sum_i p_i, \]  
\[ I_2 = \frac{1}{2} I_1^2 - \frac{1}{2} \left[ \sum_i p_i^2 + \sum_{ij} \alpha_{ij}^2 \right], \]  
\[ I_3 = \frac{1}{6} \sum_{ijk} p_i p_j p_k - \frac{1}{2} \sum_{ijk} \alpha_{ijk}^2 p_i, \]
\[ I_4 = \frac{1}{4!} \sum_{ijkl} p_i p_j p_k p_l - \frac{1}{4} \sum_{ijkl} \alpha_{ij}^2 p_k p_l \]
\[ - \frac{1}{4} \sum_{ijkl} \alpha_{ij} \alpha_{jk} \alpha_{kl} \alpha_{li} + \frac{1}{8} \sum_{ijkl} \alpha_{ij}^2 \alpha_{kl}^2, \tag{25d} \]

\[ I_5 = \frac{1}{5!} \sum_{ijklm} p_i p_j p_k p_l p_m - \frac{1}{12} \sum_{ijklm} \alpha_{ij}^2 p_k p_l p_m \]
\[ - \frac{1}{4} \sum_{ijklm} \alpha_{ij} \alpha_{jk} \alpha_{kl} \alpha_{li} p_m + \frac{1}{8} \sum_{ijklm} \alpha_{ij}^2 \alpha_{kl}^2 p_m. \tag{25e} \]

Note that the Hamiltonian can be found in the term in parenthesis in \( I_2 \).

Let us define a quantum down-boost operator analogous to the classical boost \([4]\). With \( X = \sum_{j=1}^{N} x_j \) as before, we then find
\[ [X, I_m] = i(N - m + 1)I_{m-1}. \tag{26} \]

Using Jacobi’s identity for commutators, we can easily show that as previously, \([H, I_n] = 0\) implies \([H, I_{n-1}] = 0\) and thus \([H, I_n] = 0\) implies all \( I_n \) are integrals. A particularly nice result is to write \( I_N = \det L \), treat the momenta \( p_j \) as classical \( c \)-numbers since there are no ordering ambiguities, and use the representation
\[ X = \sum_{j=1}^{N} \frac{i}{\partial p_j} \tag{27} \]

to generate all \( I_n \) in the quantum case.

Of special importance in the following will be that as in the classical invariants by Moser, \( \alpha \) will only appear in even powers in the \( I_n \)’s. Therefore, \( \lambda \) will occur with integer powers only and terms such as \([\lambda(\lambda - 1)]^{3/2}\) do not exist.

**B. Shastry’s invariants**

In Ref. [SS], Shastry and Sutherland provide a proof of integrability in the quantum case via an entirely different method: The Hamiltonian \( H \) is given as before but the Lax matrices now read
\[ L_{jk}^{SS} = p_j \delta_{jk} + i(1 - \delta_{jk}) \lambda \Phi \coth[\Phi(x_j - x_k)] \]
\[ \equiv p_j \delta_{jk} + i(1 - \delta_{jk}) \chi_{jk}, \tag{28} \]
\[ M_{jk}^{SS} = 2 \lambda \Phi^2 \left[ \delta_{jk} \sum_l \sinh^{-2}[\Phi(x_j - x_l)] + (1 - \delta_{jk}) \sinh^{-2}[\Phi(x_j - x_k)] \right]. \tag{29} \]

with \( \chi_{ii} = 0 \). SS define their conserved quantum integrals of motion as in Eq. (18), e.g., \( J_n = \eta^{\dagger}(L^{SS})^n \eta \). The new Lax matrices obey the ordered Lax equation
\[ [L^{SS}, H] = M^{SS} L^{SS} - L^{SS} M^{SS}, \tag{30} \]
and we may easily prove invariance via

\[ [L^{SS}, H] = M^{SS} L^{SS} - L^{SS} M^{SS}, \]
\[ [J_n, H] = \eta^\dagger [(L^{SS})^n, H] \eta = \eta^\dagger \left[ M^{SS}(L^{SS})^n - (L^{SS})^n M^{SS} \right] \eta = 0, \quad (32) \]

since as before \( \eta^\dagger M^{SS} = M^{SS} \eta = 0 \). A direct calculation of the conserved quantities of SS up to \( n = 4 \) yields:

\[
J_1 = \sum_i p_i, \quad (33a)
\]

\[
J_2 = \sum_i p_i^2 + \sum_{ij} (\chi_{ij}^2 + \chi_{ij}^\prime) - \sum_{ijk} \chi_{ij} \chi_{jk}, \quad (33b)
\]

\[
J_3 = \sum_i p_i^3 + 3 \sum_{ij} (\chi_{ij}^2 + \chi_{ij}^\prime) p_i - \sum_{ijk} \chi_{ij} \chi_{jk}(p_i + p_j + p_k), \quad (33c)
\]

\[
J_4 = \sum_i p_i^4 + 2 \sum_{ij} (\chi_{ij}^2 + \chi_{ij}^\prime) [p_i^2 + p_j p_j + p_j^2] + \sum_{i \neq j \neq k \neq l \neq m \neq i} \chi_{ij} \chi_{jk} \chi_{kl} \chi_{lm}
+ \text{Tr} \chi^4 - \sum_{ijk} \chi_{ij} \chi_{jk} [p_i^2 + p_j^2 + p_k^2 + p_i p_j + p_j p_k + p_k p_i]
+ 2i \sum_{ij} \chi_{ij}'' p_j + 4i \sum_{ij} \chi_{ij} \chi_{ij}^\prime p_j + i \sum_{ijk} \chi_{ij} \chi_{jk}^\prime (p_j - p_k)
- \sum_{ij} \chi_{ij}'' - 2 \sum_{ij} \chi_{ij} \chi_{ij}'' + 2 \sum_{ijk} \chi_{ij} \chi_{jk}''
- \sum_{ij} (\chi_{ij}^\prime)^2 + \sum_{ijk} \chi_{ij} \chi_{jk}^\prime + 3 \sum_{ijk} \chi_{ij}^2 \chi_{jk}^\prime + 2 \sum_{ij} \chi_{ij}^2 \chi_{ij}^\prime
- \sum_{ijkl} \chi_{ij} \chi_{jk} \chi_{kl} + 2 \chi_{ij} \chi_{jk} \chi_{kl}^\prime + \sum_{ijkl} \chi_{ij} \chi_{jk} \chi_{kl}^\prime \quad (33d)
\]

The derivative \( \chi_{jk}^\prime \) is defined by the commutator \([p_j, \chi_{jk}^{(n)}] \equiv -i \chi_{jk}^{(n+1)}\). See the appendix for an explicit list of derivatives.

Using \( \chi_{ij}^\prime = -\Phi^2 \lambda \sinh^{-2}[\Phi(x_i - x_j)] \), we see that just as in the classical case, \( J_2 \) contains the Hamiltonian, i.e., \( J_2 = H + \Phi^2 \lambda^2 N(N^2 - 1)/3 \). However, the interaction strength \( \lambda(\lambda - 1) \) in the Hamiltonian could only be obtained with the modified form of the Lax matrix \( L^{SS} \). Also, the \( \lambda \) dependence of the constant term in the above equation is different from its classical counterpart. We remark that the last terms in Eq. (33b) and (33c) can again be written as const. and const. \( \times \sum_i p_i \) by the coth-rule of Eq. (22).

The down-boost operator acts as before, e.g., \([X, J_n] = i n J_{n-1}\). In case of the potential \( V_0 \), we may also use the up-boost of Eq. (19) in operator form as

\[
Y = \sum_i (x_i p_i^2 + p_i^2 x_i)/2 + \sum_{ij} (x_i + x_j) \alpha_{ij}^2/2. \quad (34)
\]

Then \([Y, J_n] = i(n + 1)J_{n+1}\) and we again have from the Jacobi identity

\[
i(n + 1)[J_m, J_{n+1}] = [Y, [J_m, J_{n+1}]] - i(m + 1)[J_n, J_{m+1}], \quad (35)
\]
so if \([J_m, J_n] = 0\) and \([J_{m+1}, J_n] = 0\), this then implies \([J_m, J_{n+1}] = 0\). We remark that an operator similar to our up-boost operator \(Y\), which we constructed in analogy to the boost in the Heisenberg model, has been found previously by Wadati, Hikami and Ujino in the context of an investigation of the systems with algebraic potential \(V_0\) [9].

Finally, we note another interesting property of these integrals of motion: Let \(\Psi_0\) denote the ground state of the model, then it has been shown in Ref. [8] that \(\sum_j L_{ij}^{SS} \Psi_0 = 0\) for all \(i = 1, \ldots, N\). Therefore, we see that

\[
\Psi_0^\dagger J_n \Psi_0 = 0
\]

for all \(n\). Thus all the \(J_n\)’s somehow know about the ground state and subtract the appropriate expectation values, e.g., the ground state expectation value of the Hamiltonian is just the above constant \(\Phi^2 \lambda^2 N(N^2 - 1)/3\).

### C. Perturbation theory in the Lax matrices

Looking at Eq. (25), we see that each \(I_n, n > 1\) in fact contains various powers of \(I_1\). Furthermore, in the thermodynamic limit, the \(I_n\)’s are not extensive quantities. Thus the situation seems to be similar to the usual problem of connected and disconnected pieces of diagrams encountered in perturbation theory. In brief, CMR’s \(I_n\) seems to contain disconnected pieces and we hope that by a linked cluster expansion, we can write new integrals of motion with connected graphs only.

Let us be specific: With the help of the fermionic coherent path integral [10], we may rewrite the determinant

\[
\Delta(\beta) = \det[1 - \beta L],
\]

(37)

\[
= \int \prod_a dc_a^* dc_a \exp[- \sum_{jk} c_j^*[\delta_{jk} - \beta L_{jk}] c_k],
\]

(38)

\[
= \int \prod_a dc_a^* dc_a \exp[-\beta \sum_{jk} c_j^*[\delta_{jk}/\beta] c_k - c_j^* L_{jk} c_k],
\]

(39)

where \(c_a^*, c_a, a = 1, \ldots, N\) are Grassmann variables. Note first that we may write this expression both for a classical \(L\) and a quantum \(L\). The fact that the elements of a quantum matrix will not necessarily commute with each other is been taken care of by the Grassmann nature of the integration: each momentum \(p_i\) will only encounter indices \(j \neq i\), otherwise the integration measure will have expressions like \(c_i c_i\) or \(c_i^* c_i^*\) which are zero.

When we now include a dummy time dependence for the Grassmann variables, i.e. \(c_a^{(*)} = c_a^{(*)}(t)\), we can write

\[
\Delta(\beta) = \int \prod_a dc_a^*(\tau) dc_a(\tau) \exp[- \int_0^\beta dt(\sum_{jk} c_j^*(t)(\delta_{jk}/\beta) c_k(t) - c_j^*(t)L_{jk} c_k(t))],
\]

(40)

\[
= \Delta_0 \langle \exp[- \int_0^\beta dt(\sum_{jk} -c_j^*(t)L_{jk} c_k(t))\rangle_0,
\]

(41)

where the average is defined as
\[ \langle F(c^*_n(t_i)c^*_j(t_j) \ldots c_g(t_k)c_h(t_l) \ldots) \rangle_0 \]
\[ = \frac{1}{\Delta_0} \int \prod_{\alpha} dc^*_\alpha(\tau)dc^*_\alpha(\tau) \exp[-\int_0^\beta dt \sum_{j'} c^*_j(t)(1/\beta)c_{j'}(t)] \times \]
\[ F(c^*_a(t_i)c^*_b(t_j) \ldots c_g(t_k)c_h(t_l) \ldots). \]  
\[ (42) \]

This is very much like a path integral description of a many-body partition function \( Z \). We further note that the interaction part \( V = \sum_{ijk} c^*_j(t)L_{ijk}c_k(t) \) is just the super Lax operator \( \mathcal{L} \) of \( \text{SS} \).

The perturbation expansion is obtained by expanding Eq. (40) in a power series

\[ \Delta(\beta)/\Delta_0 = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^\beta dt_1dt_2 \ldots dt_n \]
\[ \langle \sum_{i_1j_1} c^*_i(t_1)L_{i_1j_1}c_j(t_1) \cdots \sum_{i_nj_n} c^*_i(t_n)L_{i_nj_n}c_j(t_n) \rangle_0; \]  
\[ (43) \]
\[ \equiv \sum_{n=0}^\infty \frac{(-1)^n}{n!} \Delta_n, \]  
\[ (44) \]

and \( \Delta_n \sim \beta^n I_n \). The last equation is obtained by comparison with Eq. (24) and \( \Delta_0 = 1 \). Note that \( I_n = 0 \) in Eq. (14) for all \( n > N \). E.g. for \( N = 2 \), we have \( I_3 \sim \sum_{i_1j_1i_2j_2} \sum_{i_3j_3} L_{i_1j_1}L_{i_2j_2}L_{i_3j_3} (c^*_i c^*_j c^*_k c_j c_k c_j) \) \( 0 \) and clearly \( i_3, j_3 \) always take index values already covered by \( \{i_1, j_1, i_2, j_2\} \). Thus the bracket \( \langle \rangle \) is zero by the Grassmann character of the \( c \)'s.

Let us now calculate the first few orders of \( \Delta(\beta) \). With \( g_i \) being a dummy propagator, we find

\[ \Delta_1 = -\beta \sum_i L_{ii}g_i \]  
\[ (45a) \]
\[ \Delta_2 = \frac{1}{2} \beta^2 \sum_{ij} (L_{ij}L_{ji} - L_{ii}L_{jj}) g_i g_j \]  
\[ (45b) \]
\[ \Delta_3 = -\frac{1}{3!} \beta^3 \sum_{i_1i_2j_1,j_2j_3} L_{i_1j_1}L_{i_2j_2}L_{i_3j_3} (c^*_i c^*_j c^*_k c_j c_k c_j) \]
\[ = -\frac{1}{3!} \beta^3 \sum_{ijk} (L_{ik}L_{jj}L_{ki} - L_{ik}L_{ij}L_{kj} - L_{ij}L_{jk}L_{ki} \]
\[ + L_{ii}L_{jijk} + L_{ij}L_{jiLkk} - L_{ii}L_{jijk}L_{kk} g_i g_j g_k. \]  
\[ (45c) \]

Introducing the diagrammatic notation \( i \rightarrow j \equiv L_{ij} \), we can represent these expressions by their graphs as in Fig. [1]. Note that only in Eq. (45c) do we need to worry about the ordering of the matrix products. If we ignore that ordering for the moment — the classical case — we have

\[ \Delta_3 = -\frac{1}{3!} \beta^3 \sum_{ijk} [3L_{ii}L_{jijk}L_{kj} - 2L_{ij}L_{jik}L_{ki} - L_{ij}L_{jijk}L_{kk}] g_i g_j g_k. \]  
\[ (46) \]

We see that the second term in Eq. (46), representing the fully connected diagram, is actually just \( \text{const.} \times \text{Tr}(L^3) \).
Let us see what these expressions tell us: (i) We see that in fact the $\Delta_n$’s are just the $I_n$’s of CMR with $g_i = 1$. (ii) We observe that the fully connected diagrams give the traces of powers of $L$ in the classical case as was expected from the well-known matrix formula $\ln \det A = \text{Tr} \ln A$. Thus these connected diagrams are the quantum analogue of the classical integrals of motion. (iii) The ordering of the matrix products becomes important for $n > 2$, thus necessitating order labeling of diagrams.

D. Constructing connected diagrams

We now want to rewrite the perturbation expansion (43) such that we only use fully connected diagrams. And we want to do this such that we can minimise the ordering problems coming from the quantum character of the Lax matrix. The basic program is due to Thiele and known as the linked cluster theorem. It can be summarised as follows: We resum the series (44) as

$$\Delta(\beta) = \exp[-\sum_{n=1}^{\infty} \frac{\beta^n}{(n-1)!} T_n],$$

(47)

and use it to define the $T_n$’s. Comparing Eq. (44) and (47), we find up to $n = 4$,

$$T_1 = I_1,$$

(48a)

$$T_2 = I_1^2 - 2I_2,$$

(48b)

$$T_3 = I_1^3 - 3I_1I_2 + 3I_3,$$

(48c)

$$T_4 = I_1^4 - 4I_1I_2^2 + 2I_2^2 + 4I_3I_1 - 4I_4,$$

(48d)

Since CMR have already proven $[I_n, I_m] = 0$, there is no ordering problem for the $I_n$’s in the construction of the $T_n$’s and we furthermore have $[T_n, T_m] = 0$. Since $T_2$ is, up to a constant, the Hamiltonian, this implies both involution and invariance.

As expected, we find that each $T_n$ corresponds to the fully connected diagrams of the series (44). We can now directly use the diagrammatic approach to construct the $T_n$’s. However, for a given $n$, there are $(n-1)!$ different labeled diagrams and thus different matrix orderings. Each diagram itself is an ordered operator expression and it is quite tedious to get them into a form as in Eq. (47) with all momenta to the right. As an example, we give the diagrams for $T_4$ in Fig. 2.
Ignoring matrix and quantum ordering, the resultant expressions for the $T_n$’s are equal to the classically invariant traces of Eq. (3). Thus we may hope that due to the special form of the quantum Lax matrix $L$, the matrix product order somehow is unimportant and the $T_n$’s are just the quantum traces $Tr L^n$. Explicitly calculating the quantum traces, we find that indeed up to $n = 3$, we have $T_n = Tr L^n$. However, for $n = 4$, the quantum trace includes the nonzero term $-2 \sum_{ij} \alpha_{ij} \alpha_{ik} \alpha_{kj}$. Note that this term has a factor $| \lambda(\lambda - 1)|^{2/3}$ [11]. But as shown in section III A, such a term does not arise in the $T_n$’s and consequently also not in the $T_n$’s. Therefore, the $T_n$’s are not simply the quantum traces.

Note that these expressions are again very close to the ones obtained for $J_n$. However, as before in the classical case, we see that already for $n = 2, 3$, there are the same constants in the $J$ expressions which do not appear in the $T$ expressions.

E. Relation between invariants

We again would like to see if we can express Shastry’s integrals in terms of our $T_n$’s. As we have seen in section [11], for the classical case, we expect this relation to be algebraic. Fortunately, as shown in the last section, fractional powers of $\lambda$ neither appear in the $J_n$’s nor in the $T_n$’s so that no a priori reasons forbid an algebraic relationship in the quantum case.

Furthermore, given two sets of integrals of motion $\{T_n\}$ and $\{J_n\}$, we know that commutators of integrals are themselves integrals of motion, and since asymptotically these integrals evaluate to zero, the two sets of integrals can be simultaneously diagonalized. A relationship between asymptotic integrals of the form $J_n = A_n[\{T_m\}]$ can always be found, since either set of integrals gives an algebraically complete set of symmetric polynomials of increasing degree. Suppose we have such a relationship. Then, the operators $J_n$ and $A_n[\{T_m\}]$ have the same eigenvalues in the same basis, hence must be the same operator, and so there must exist a relationship $J_n = A_n[\{T_m\}]$ between the operators themselves.

Replacing $\alpha_{ij}$ and $\chi_{ij}$ by their appropriate definitions, using the explicit form for the derivatives as given in the appendix and counting powers of $\lambda$ and $p$’s, we then have

\begin{align}
J_1 &= T_1, \\
J_2 &= \Phi^2 \lambda [3 + \lambda (N - 2)] N(N - 1)/3 + T_2, \\
J_3 &= \Phi^2 \lambda [3 + \lambda (N - 2)] (N - 1)T_1 + T_3, \\
J_4 &= \left[ -75 + 120 \lambda - 48 \lambda^2 + (55 - 110 \lambda + 52 \lambda^2) N + (-5 + 25 \lambda - 18 \lambda^2) N^2 + 2 \lambda^2 N^3 \right] \times \\
& \quad \lambda^2 N(N - 1) \Phi^4/15 + [2 + \lambda (N - 2)] \lambda \Phi^2 T_2^2 + \\
& \quad \left[ 10 - 12 \lambda + 11 (\lambda - 1) N + (2 \lambda - 1) N^2 \right] \lambda \Phi^2 T_1 + T_4. \tag{49d}
\end{align}

Hence we have succeeded in writing Shastry’s quantum integrals in terms of the $T_n$’s which in turn are derived from Calogero’s quantum integrals for up to $n = 4$. Again, as in the classical case, this relationship is not linear, since we observe the $T_1^2$ term in Eq. (49d). And only if we restrict ourselves to the potential $V_0(x) = g^2/2 x^2$ by taking the limit $\Phi \to 0$, do we find that both sets of integrals of motion are identical.

With $\Psi_0$ the $N$-particle ground state as before, we may use Eq. (38) and hence relate the expectation values of various $T_n$’s. E.g., $\Psi_0^\dagger T_2 \Psi_0 = \Phi^2 \lambda [3 + \lambda (N - 2)] N(N - 1)/3$ and
\(\Psi_0^\dagger T_3 \Psi_0 = -\Phi^2 \lambda (3 + \lambda (N - 2))(N - 1)\Psi_0^\dagger T_1 \Psi_0 \sim \Psi_0^\dagger P \Psi_0 = 0.\) We further note that as in the classical case, we may define new non trivial constants of motion \(K_n = J_n - T_n\) for \(\Phi \neq 0.\) We then have \(\Psi_0^\dagger K_n \Psi_0 = -\Psi_0^\dagger T_n \Psi_0.\) Unfortunately, we can not give a simple formula directly in terms of the Lax matrices for the construction of the \(K_n\)’s analogous to Eq. (21).

**IV. THE ASYMPTOTIC LIMIT**

In the asymptotic \(t \to \infty\) limit — equivalent to the low-density limit — the elements of the Lax matrix are no longer operators, but numbers. Thus explicit calculations are much easier and we hope that we can study the connection between asymptotic Calogero and Shastry integrals of motion in more detail than in the last section.

**A. Asymptotics of Calogero’s integrals**

The asymptotic form for the Lax matrix \(L\) gives a corresponding asymptotic form for the Calogero integrals \(I_n \to I_n.\) We define a generating function of the asymptotic Calogero integrals,

\[
D[z, \lambda] = \frac{1}{2} \left[ \prod_{j=1}^{N} (1 - z p_j - i \lambda z) + \prod_{j=1}^{N} (1 - z p_j + i \lambda z) \right],
\]

and then have

\[
\det[1 - z L^\infty] = D[z, \sqrt{\lambda(\lambda - 1)}].
\]

We also define the elementary symmetric functions of the \(N\) variables \(p_j\) as

\[
a_r[p] = \sum_{1 \leq j_1 < \ldots < j_r} p_{j_1} \cdots p_{j_r},
\]

and for convenience \(a_0 = 1,\) and \(a_{-r} = 0.\) Then

\[
D[z, \lambda] = \sum_{r=0}^{\infty} (-\lambda^2 z^2)^r a_{N-2r} [1 - z p].
\]

In particular,

\[
I_N = a_N[p] - \lambda(\lambda - 1) a_{N-2}[p] + \lambda^2(\lambda - 1)^2 a_{N-4}[p] - \ldots,
\]

and the other integrals can be constructed via

\[
\left[ \sum_{k=1}^{N} \frac{\partial}{\partial p_k}, I_j \right] = (N - j + 1) I_{j-1}.
\]

Since the elementary symmetric functions \(a_j\) obey the same relationship, this also gives us the expression for \(I_j\) as a linear combination of \(a_j, a_{j-2}, a_{j-4}, \ldots.\) The general expression for \(I_j\) in terms of the \(a_r[p]\) can be obtained using
\[ a_r[1 - zp] = \sum_{j=0}^{N} (-z)^{r-j} \binom{N+j-r}{j} a_{r-j}[p], \]  

(56)

and we find

\[ \mathcal{I}_j = \sum_{j=0}^{N} \left[ -\lambda(\lambda - 1) \right]^r \binom{N+2r-j}{2r} a_{2r-j}[p]. \]  

(57)

B. Some generating functions

Let us define the quantity

\[ \mathcal{Z}[z, \lambda] = \prod_{j=1}^{N} \left[ 1 - z(p_j + i\lambda) \right] \equiv \mathcal{D}[z, \lambda] - i\lambda z\mathcal{N}[z, \lambda]. \]  

(58)

The symmetric part of \( \mathcal{Z} \) is

\[ \mathcal{D} = \frac{(\mathcal{Z}[z, \lambda] + \mathcal{Z}[z, -\lambda])}{2} \] as in the previous section, while the antisymmetric part \( (\mathcal{Z}[z, \lambda] - \mathcal{Z}[z, -\lambda])/2 \) is given by

\[ i\lambda z\mathcal{N}[z, \lambda] = i\lambda z \sum_{r=0}^{\infty} (-\lambda^2 z^2)^r a_{N-2r-1}[1 - zp]. \]  

(59)

For \( z \) real, these are the real and imaginary parts of \( \mathcal{Z} \), respectively. This expression for \( \mathcal{Z} \) is the standard generating function for the elementary symmetric functions, so that

\[ \mathcal{Z}[z, \lambda] = \sum_{j=0}^{N} (-z)^j a_j[p + i\lambda]. \]  

(60)

Clearly, then, we have \( \mathcal{I}_j = \text{Re}\{a_j[p + i\sqrt{\lambda(\lambda - 1)}]\} \). Taking the logarithm of \( \mathcal{Z} \),

\[ \ln \mathcal{Z}[z, \lambda] = \sum_{j=1}^{N} \ln[1 - z(p_j + i\lambda)] \]  

(61)

one advantage of using the generating function \( \mathcal{Z} \) is clear when one anticipates the thermodynamic limit. We now consider

\[ \frac{\partial}{\partial z} \ln \mathcal{Z} = \sum_{j=1}^{N} \frac{-p_j + i\lambda}{1 - z(p_j + i\lambda)}. \]  

(62)

This, however, is the generating function for the symmetric power sums

\[ b_r[p] = \sum_{j=1}^{N} p_j^r \]  

(63)

since

\[ P[z] = \sum_{r=0}^{\infty} z^r b_{r+1}[p] = \sum_{j=1}^{N} \frac{p_j}{1 - zp_j} \]  

(64)

and we see that

\[ \frac{\partial}{\partial z} \ln \mathcal{Z}[z, \lambda] = \sum_{r=0}^{\infty} z^r b_{r+1}[p + i\lambda] \equiv P[z, \lambda]. \]  

(65)
C. Asymptotics of Shastry’s integrals

Shastry’s integrals also approach an asymptotic form $J_j \to J_j$, and similarly we define an asymptotic generating function

$$G[z, \lambda] \equiv \frac{\mathcal{N}[z, \lambda]}{\mathcal{D}[z, \lambda]}.$$  \hfill (66)

then, we see that $\mathcal{N}[z, \lambda] = G[z, \lambda]\mathcal{D}[z, \lambda]$, so expanding, we have

$$\sum_{k=1}^{N} (-z)^k \text{Im}\{a_k[p + i\lambda]\} = \lambda \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} J_{j-1} \text{Re}\{a_{k-j}[p + i\lambda]\}. \hfill (67)$$

Equating powers of $z$, this gives

$$\text{Im}\{a_k[p + i\lambda]\} = \lambda \sum_{j=1}^{k} (-1)^{j-1} J_{j-1} \text{Re}\{a_{k-j}[p + i\lambda]\}. \hfill (68)$$

More explicitly,

$$\lambda^{-1} \text{Im}\{a_k[p + i\lambda]\} = N \text{Re}\{a_k[p + i\lambda]\} - \lambda_1 \text{Re}\{a_{k-1}[p + i\lambda]\} + \ldots + (-1)^k J_k. \hfill (69)$$

This allows us to iterate and find $\mathcal{J}_k$ in terms of the other $\mathcal{J}_j$,

$$\mathcal{J}_k = \mathcal{J}_{k-1} \text{Re}\{a_1[p + i\lambda]\} - \ldots - (-1)^k N \text{Re}\{a_k[p + i\lambda]\} + (-1)^k \lambda^{-1} \text{Im}\{a_{k+1}[p + i\lambda]\}. \hfill (70)$$

Since $I_j = \text{Re}\{a_j[p + i\lambda]\}$, this recursion relation also relates the asymptotic forms of the CMR and SS integrals. However, this relation is between asymptotic integrals with the same parameter, and such integrals do not even commute.

V. CONCLUSIONS

Our original intent and hope at taking up the present work was to give a simple connection between the integrals of motion of CMR and the recently discovered integrals of SS. In fact, we were speculating that due to the special structure of the quantum Lax matrix, we would simply find $T_n \sim J_n$.

Quite the opposite has happened: In the quantum case, though we have succeeded in rewriting the $I_n$ integrals of CMR into extensive quantities $T_n$, we have however failed to give a simple formula for the connection of these $T_n$’s to the $J_n$ integrals of SS for all but the simplest potential $V_0(x) = g^2/x^2$. In general, we do find a complicated algebraic relationship which gives rise to yet another set of non-trivial integrals of motion $K_n$.

In the classical case, we show that the quantum definition of SS may also be used to construct classical integrals of motion. Hence here the situation now is just as in the quantum case and again we show that we may reexpress the integrals of SS in terms of the classical integrals of Moser. Again, the difference of these two sets of integrals vanishes only for the potential $V_0$ and otherwise may be used to define new constants and this time, we can give
an explicit formula for the $K_n$'s directly in terms on the Lax $L$ matrix. Indeed, we may even define a one-parameter family of integrals of motion in the classical case, e.g.,

$$R_n(\delta) = \text{Tr}[L^n(1 + \delta \Delta')]$$

with $\Delta'_{ij} = 1$ if $i \neq j$ and zero otherwise. Then $R_n$ interpolates between Moser’s integrals $R_n(0) = T_n$ and the integrals of SS $R_n(1) = J_n$.

Most of the previous formulas are given in terms of $\alpha$’s and $\chi$’s and are thus valid not only for the hyperbolic pair potential $V_h$, but also for the trigonometric $V_t$ and the rational $V_0$ after taking the appropriate limits as mentioned in the beginning of section II. However, we have found an up-boost only for the classical and the quantum many-body system with algebraic potential $V_0(r) = g^2/r^2$. Also, the elliptic potential $V_e = 1/\text{sn}^2$, is not included in this study: Although the classical integrals of Moser and the quantum integrals of CMR are valid for this potential, the proof of SS does no longer hold both for the classical and the quantum case. This is due to the fact that the row and column sums of the elliptic Lax $M$ matrix are no longer zero. The Ansatz $J'_n = \text{Tr}L^n g[\{x_j\}]$ gives an equation for the coordinate dependent matrix $g$ as $\frac{\partial}{\partial t} g = M g - g M$. Unfortunately, we have not found a non trivial such $g$ such that it gives the SS integrals of motion in the elliptic case.

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APPENDIX A: DERIVATIVES

We have defined $\gamma_{ij} = \coth[\Phi(x_i - x_j)]$ and so $\chi_{ij} = \Phi \lambda \gamma_{ij}$ and $\alpha_{ij} = \Phi \sqrt{\lambda(\lambda - 1) \gamma_{ij}}$. Then the derivatives are

$$\frac{\partial \chi_{ij}}{\partial x_i} = \chi'_{ij} = \Phi^2 \lambda [1 - \gamma_{ij}^2]$$

$$\chi''_{ij} = -2 \Phi^3 \lambda \gamma_{ij} (1 - \gamma_{ij}^2)$$

$$\chi'''_{ij} = -2 \Phi^4 \lambda (1 - 4 \gamma_{ij}^2 + 3 \gamma_{ij}^4).$$

The same relations hold for $\alpha_{ij}$, i.e. $\alpha'_{ij} = \Phi^2 \sqrt{\lambda(\lambda - 1)[1 - \gamma_{ij}^2]}$, $\alpha''_{ij} = -2 \Phi^3 \sqrt{\lambda(\lambda - 1) \gamma_{ij} (1 - \gamma_{ij}^2)}$, and $\alpha'''_{ij} = -2 \Phi^4 \sqrt{\lambda(\lambda - 1)(1 - 4 \gamma_{ij}^2 + 3 \gamma_{ij}^4)}$, and, lastly, $\gamma_{ij} = \Phi [1 - \gamma_{ij}^2]$, $\gamma''_{ij} = -2 \Phi \gamma_{ij} (1 - \gamma_{ij}^2)$, and $\gamma'''_{ij} = -2 \Phi (1 - 4 \gamma_{ij}^2 + 3 \gamma_{ij}^4)$.
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FIGURES

FIG. 1. Diagrammatic representation of Eq. [13]. Each line is labeled by a number indicating precedence in the corresponding matrix product. 1 corresponds to the right-most matrix.

FIG. 2. The 6 diagrams associated with $T_4$. 
Figure 1, R.A. Römer et al., “Conservation laws in the continuum $1/r^2$ systems.”
Figure 2, R.A. Römer et al., “Conservation laws in the continuum $1/r^2$ systems”.