ON THE NONEXISTENCE OF HARMONIC AND BI-HARMONIC MAPS

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ABSTRACT

In this paper, we study the existence of harmonic and bi-harmonic maps into Riemannian manifolds admitting a conformal vector field, or a nontrivial Ricci solitons.

Keywords Harmonic maps; Bi-harmonic maps; Ricci solitons; Conformal vector fields.

1 Preliminaries and Notations

We give some definitions. (1) Let \((M,g)\) be a Riemannian manifold. By \(R\) and \(\text{Ric}\) we denote respectively the Riemannian curvature tensor and the Ricci tensor of \((M,g)\). Thus \(R\) and \(\text{Ric}\) are defined by:

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

\[
\text{Ric}(X,Y) = g(R(X,e_i)e_i,Y),
\]

where \(\nabla\) is the Levi-Civita connection with respect to \(g\), \(\{e_i\}\) is an orthonormal frame, and \(X,Y,Z \in \Gamma(TM)\). The divergence of \((0,p)\)-tensor \(\alpha\) on \(M\) is defined by:

\[
(\text{div} \alpha)(X_1,\ldots,X_{p-1}) = (\nabla_{e_i} \alpha)(e_i,X_1,\ldots,X_{p-1}),
\]

where \(X_1,\ldots,X_{p-1} \in \Gamma(TM)\), and \(\{e_i\}\) is an orthonormal frame. Given a smooth function \(\lambda\) on \(M\), the gradient of \(\lambda\) is defined by:

\[
g(\text{grad} \lambda, X) = X(\lambda),
\]

the Hessian of \(\lambda\) is defined by:

\[
(\text{Hess} \lambda)(X,Y) = g(\nabla_X \text{grad} \lambda, Y),
\]

where \(X,Y \in \Gamma(TM)\) (for more details, see for example [13]).

(2) A vector field \(\xi\) on a Riemannian manifold \((M,g)\) is called a conformal if \(\mathcal{L}_\xi g = 2fg\), for some smooth function \(f\) on \(M\), where \(\mathcal{L}_\xi g\) is the Lie derivative of the metric \(g\) with respect to \(\xi\), that is:

\[
g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2fg(X,Y), \quad X,Y \in \Gamma(TM).
\]

The function \(f\) is then called the potential function of the conformal vector field \(\xi\). If \(\xi\) is conformal with constant potential function \(f\), then it is called homothetic, while \(f = 0\) it is Killing (see [11], [14], [18]).

(3) A Ricci soliton structure on a Riemannian manifold \((M,g)\) is the choice of a smooth vector field \(\xi\) satisfying the soliton equation:

\[
\text{Ric} + \frac{1}{2}\mathcal{L}_\xi g = \lambda g,
\]

for some constant \(\lambda \in \mathbb{R}\), where \(\mathcal{L}_\xi g\) is the Lie derivative of the metric \(g\) with respect to \(\xi\). The Ricci soliton \((M,g,\xi,\lambda)\) is said to be shrinking, steady or expansive according to whether the coefficient \(\lambda\) appearing in equation
that \( \xi = \mathrm{grad} f \), for some smooth function \( f \) on \( M \), we say that \((M, g, \mathrm{grad} f, \lambda)\) is a gradient Ricci soliton with potential \( f \). In this situation, the soliton equation reads:

\[
\mathrm{Ric} + \mathrm{Hess} f = \lambda g, \tag{8}
\]

(see [8], [9], [16]). If \( \xi = 0 \), we recover the definition of an Einstein metric with Einstein constant \( \lambda \). If \((M, g)\) is not Einstein, we call the soliton nontrivial.

(4) A vector field \( \xi \) on a Riemannian manifold \((M, g)\) is said to be a Jacobi-type vector field if it satisfies:

\[
\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi + R(\xi, X)Y = 0, \quad X \in \Gamma(TM),
\]

(9) Note that, there are Jacobi-type vector fields on a Riemannian manifold which are not Killing vector fields (see [5]).

(5) Let \( \varphi : (M, g) \to (N, h) \) be a smooth map between two Riemannian manifolds, \( \tau(\varphi) \) the tension field of \( \varphi \) given by:

\[
\tau(\varphi) = \mathrm{trace} \nabla d\varphi = \nabla^\varphi_i d\varphi(e_i) - d\varphi(\nabla^M_i e_i),
\]

(10) where \( \nabla^M \) is the Levi-Civita connection of \((M, g)\), \( \nabla^\varphi \) denote the pull-back connection on \( \varphi^{-1}TN \) and \( \{e_i\} \) is an orthonormal frame on \((M, g)\). Then \( \varphi \) is called harmonic if the tension field vanishes, i.e. \( \tau(\varphi) = 0 \) (see [11], [9], [17]). We define the index form for harmonic maps by (see [4], [15]):

\[
I(v, w) = \int_M h(J_\varphi(v), w)\nu_\varphi, \quad v, w \in \Gamma(\varphi^{-1}TN)
\]

(11) (or over any compact subset \( D \subset M \)), where:

\[
J_\varphi(v) = -\mathrm{trace} R^N(v, d\varphi) d\varphi - \mathrm{trace} (\nabla^\varphi)^2 v
= -R^N(v, d\varphi(e_i)) d\varphi(e_i) - \nabla^\varphi_{e_i} \nabla^\varphi v + \nabla^N_{\varphi e_i} v,
\]

(12) \( R^N \) is the curvature tensor of \((N, h)\), \( \nabla^N \) is the Levi-Civita connection of \((N, h)\), and \( \nu_\varphi \) is the volume form of \((M, g)\) (see [11]). If \( \tau_\varphi(\varphi) \equiv J_\varphi(\tau(\varphi)) \) is null on \( M \), then \( \varphi \) is called a bi-harmonic map (see [3], [10], [12]).

2 Main Results

2.1 Harmonic maps and conformal vector fields

**Proposition 1.** Let \((M, g)\) be a compact orientable Riemannian manifold without boundary, and \((N, h)\) a Riemannian manifold admitting a conformal vector field \( \xi \) with potential function \( f > 0 \) at any point. Then, any harmonic map \( \varphi \) from \((M, g)\) to \((N, h)\) is constant.

**Proof.** Let \( X \in \Gamma(TM) \), we set:

\[
\omega(X) = h(\xi \circ \varphi, d\varphi(X)),
\]

(13) let \( \{e_i\} \) be a normal orthonormal frame at \( x \in M \), we have:

\[
\mathrm{div}^M \omega = e_i [h(\xi \circ \varphi, d\varphi(e_i))],
\]

(14) by equation (14), and the harmonicity condition of \( \varphi \), we get:

\[
\mathrm{div}^M \omega = h(\nabla^\varphi_{e_i} (\xi \circ \varphi), d\varphi(e_i)),
\]

(15) since \( \xi \) is a conformal vector field, we find that:

\[
\mathrm{div}^M \omega = (f \circ \varphi) h(d\varphi(e_i), d\varphi(e_i)) = (f \circ \varphi)|d\varphi|^2,
\]

(16) the Proposition follows from equation (16), and the divergence theorem (see [11]), with \( f > 0 \) on \( N \).

**Remark 2.** (1) Proposition remains true if the potential function \( f < 0 \) on \( N \) (consider the conformal vector field \( \xi' = -\xi \)).

(2) If the potential function is non-zero constant, that is \( \mathcal{L}_h h = 2kh \) on \((N, h)\) with \( k \neq 0 \), then any harmonic map \( \varphi \) from a compact orientable Riemannian manifold without boundary \((M, g)\) to \((N, h)\) is necessarily constant (see [13]).

(3) An harmonic map from a compact orientable Riemannian manifold without boundary to a Riemannian manifold admitting a Killing vector field is not necessarily constant (for example the identity map on the unit \((2n + 1)\)-dimensional sphere on \( \mathbb{R}^{2n+1} \), note that the unit odd-dimensional sphere admits a Killing vector field (see [22]).
From Proposition 1, we get the following result:

**Corollary 3.** Let \((\mathcal{N}, \mathcal{H})\) be an \(n\)-dimensional Riemannian manifold which admits a Killing vector field \(\mathcal{F}\). Consider \((\mathcal{N}, h)\) a Riemannian hypersurface of \((\mathcal{N}, \mathcal{H})\) such that \(h\) is the induced metric of \(\mathcal{H}\) on \(\mathcal{N}\). Suppose that:

- \((\mathcal{N}, h)\) is totally umbilical, that is:
  \[
  B(X, Y) = \rho h(X, Y)\eta, \quad \forall X, Y \in \Gamma(T\mathcal{N}),
  \]
  for some smooth function \(\rho\) on \(\mathcal{N}\), where \(B\) is the second fundamental form of \(\mathcal{N}\) on \(\mathcal{H}\) given by \(B(X, Y) = (\nabla_X Y)\), \(\nabla\) is the Levi-Civita connection on \(\mathcal{H}\), and \(\eta\) is the unit normal to \(\mathcal{N}\);
- the function \(\tilde{h}(\mathcal{F}, H) \neq 0\) everywhere on \(\mathcal{N}\), where \(H\) is the mean curvature of \((\mathcal{N}, h)\) given by the formula:
  \[
  H = \frac{1}{n-1} \text{trace}_h B.
  \]

Then, any harmonic map from a compact orientable Riemannian manifold without boundary to \((\mathcal{N}, h)\) is constant.

**Proof.** It is possible to express \(\mathcal{F}\) as \(\mathcal{F} = \mathcal{F} + f\eta\), where \(\mathcal{F}\) is tangent to \(\mathcal{N}\) and \(f\) is a smooth function on \(\mathcal{N}\). Thus we have:

\[
(\mathcal{L}_\mathcal{F} \mathcal{H})(X, Y) = (\mathcal{L}_\mathcal{F} h)(X, Y) + f(\tilde{h}(\nabla_X \eta, Y) + \tilde{h}(\nabla_Y \eta, X)),
\]

where \(X, Y \in \Gamma(T\mathcal{N})\) (see [6]), by equation (17) with \(\mathcal{L}_\mathcal{F} \mathcal{H} = 0\), we get:

\[
(\mathcal{L}_\mathcal{F} h)(X, Y) = 2f\tilde{h}(\eta, B(X, Y)),
\]

since \(\mathcal{N}\) is totally umbilical, (18) becomes:

\[
(\mathcal{L}_\mathcal{F} h)(X, Y) = 2f\rho h(X, Y),
\]

the Corollary follows from Proposition 1 and equation (19) with:

\[
f\rho = \tilde{h}(\mathcal{F}, \eta)\tilde{h}(H, \eta) = \tilde{h}(\mathcal{F}, H).
\]

\[\square\]

In the case of non-compact Riemannian manifold, we obtain the following results:

**Theorem 4.** Let \((\mathcal{M}, g)\) be a complete non-compact Riemannian manifold, and \((\mathcal{N}, h)\) a Riemannian manifold admitting a conformal vector field \(\mathcal{F}\) with potential function \(f > 0\) at any point. If \(\varphi : (\mathcal{M}, g) \to (\mathcal{N}, h)\) is harmonic map, satisfying:

\[
\int_M \frac{|\varphi|}{f} \geq \infty,
\]

then \(\varphi\) is constant.

**Proof.** Let \(\rho\) be a smooth function with compact support on \(\mathcal{M}\), we set:

\[
\omega(X) = h(\mathcal{F} \varphi, \rho^2 d\varphi(X)), \quad X \in \Gamma(TM).
\]

Let \(\{e_i\}\) be a normal orthonormal frame at \(x \in \mathcal{M}\), we have:

\[
\text{div}^\mathcal{M} \omega = e_i [h(\mathcal{F} \varphi, \rho^2 d\varphi(e_i))],
\]

by equation (22), and the harmonicity condition of \(\varphi\), we get:

\[
\text{div}^\mathcal{M} \omega = h(\nabla^\mathcal{M}_e (\mathcal{F} \varphi), \rho^2 d\varphi(e_i)) + h(\mathcal{F} \varphi, \nabla^\mathcal{M}_e \rho^2 d\varphi(e_i))
\]

\[
= \rho^2 h(\nabla^\mathcal{M}_e (\mathcal{F} \varphi), d\varphi(e_i)) + 2\rho \varepsilon_i (\rho) h(\mathcal{F} \varphi, d\varphi(e_i)),
\]

since \(\mathcal{F}\) is a conformal vector field with potential function \(f\), we find that:

\[
\rho^2 h(\nabla^\mathcal{M}_e (\mathcal{F} \varphi), d\varphi(e_i)) = (f \varphi) \rho^2 h(d\varphi(e_i), d\varphi(e_i)),
\]

(24)
by Young’s inequality we have:
\[-2ρe_1(p)h(ξ ∘ φ, dφ(e_i)) \leq \lambda ρ^2|dφ|^2 + \frac{1}{\lambda}e_i(p)^2|ξ ∘ φ|^2,\]  
(25)

for all function $λ > 0$ on $M$, because of the inequality:
\[|\sqrt{\lambda}ρdφ(e_i) + \frac{1}{\sqrt{\lambda}}e_i(p)(ξ ∘ φ)|^2 \geq 0.\]

From (23), (24) and (25) we deduce the inequality:
\[(f ∘ φ)ρ^2|dφ|^2 - \text{div}^M ω \leq \lambda ρ^2|dφ|^2 + \frac{1}{\lambda}e_i(p)^2|ξ ∘ φ|^2,\]  
(26)

let $λ = (f ∘ φ)/2$, by (26) we have:
\[\frac{1}{2}(f ∘ φ)ρ^2|dφ|^2 - \text{div}^M ω \leq \frac{2}{f ∘ φ}e_i(p)^2|ξ ∘ φ|^2,\]  
(27)

by the divergence theorem, and (27) we have:
\[\frac{1}{2} \int_M (f ∘ φ)ρ^2|dφ|^2v^g \leq 2 \int_M e_i(p)^2\frac{|ξ ∘ φ|^2}{f ∘ φ}v^g.\]  
(28)

Consider the smooth function $ρ = ρ_R$ such that, $ρ ≤ 1$ on $M$, $ρ = 1$ on the ball $B(p, R)$, $ρ = 0$ on $M\setminus B(p, 2R)$ and $|\text{grad}^M ρ| ≤ \frac{1}{R}$ (see [19]). From (28) we get:
\[\frac{1}{2} \int_M (f ∘ φ)ρ^2|dφ|^2v^g \leq \frac{8}{R^2} \int_M \frac{|ξ ∘ φ|^2}{f ∘ φ}v^g,\]  
(29)

since $\int_M \frac{|ξ ∘ φ|^2}{f ∘ φ}v^g < ∞$, when $R → ∞$, we obtain:
\[\int_M (f ∘ φ)|dφ|^2v^g = 0.\]  
(30)

Consequently, $|dφ| = 0$, that is $φ$ is constant. □

From Theorem 4 we get the following:

Corollary 5. Let $(M, g)$ be a complete non-compact Riemannian manifold and let $ξ$ a conformal vector field on $(M, g)$ with potential function $f > 0$ at any point. Then:
\[\int_M \frac{|ξ|^2}{f}v^g = ∞.\]

2.2 Bi-harmonic maps and conformal vector fields

Theorem 6. Let $(M, g)$ be a compact orientable Riemannian manifold without boundary, and let $ξ$ a conformal vector field with non-constant potential function $f$ on a Riemannian manifold $(N, h)$ such that $\text{grad}^N f$ is parallel. Then, any bi-harmonic map $φ$ from $(M, g)$ to $(N, h)$ is constant.

For the proof of Theorem 6 we need the following lemma.

Lemma 7. Let $(M, g)$ be a compact orientable Riemannian manifold without boundary and $(N, h)$ a Riemannian manifold admitting a proper homothetic vector field $ζ$, i.e. $\mathcal{L}_ζ h = 2kh$ with $k ∈ \mathbb{R}^*$. Then, any bi-harmonic map $φ$ from $(M, g)$ to $(N, h)$ is constant.

Proof of Theorem 6 We set $ζ = [\text{grad}^N f, ξ]$, since $\text{grad}^N f$ is parallel on $(N, h)$, then $ζ$ is an homothetic vector field satisfying $\nabla^N_U ζ = [\text{grad}^N f]^2 U$ for any $U ∈ \Gamma(TN)$ (see [11]). The Theorem 6 follows from Lemma 7. □

From Theorem 6 we deduce:

Corollary 8. Let $(M, g)$ be a compact orientable Riemannian manifold without boundary, and let $ξ$ a conformal vector field with non-constant potential function $f$ on $(M, g)$. Then, $\text{grad} f$ is not parallel.
2.3 Harmonic Maps to Ricci Solitons

**Proposition 9.** Let \((M,g)\) be a compact orientable Riemannian manifold without boundary, and \((N,h,\xi,\lambda)\) a non-trivial Ricci soliton with:

\[
\text{Ric}^N > \lambda h \quad \text{or} \quad \text{Ric}^N < \lambda h.
\]

Then any harmonic map \(\phi\) from \((M,g)\) to \((N,h)\) is constant.

**Proof.** Let \(X \in \Gamma(TM)\), we set:

\[
\omega(X) = h(\xi \circ \phi, d\phi(X)),
\]

and let \(\{e_i\}\) be a normal orthonormal frame at \(x \in M\), we have:

\[
\text{div}^M \omega = e_i [h(\xi \circ \phi, d\phi(e_i))],
\]

by equation (32), the harmonicity condition of \(\phi\), we get:

\[
\text{div}^M \omega = h(\nabla_{e_i}^\phi(\xi \circ \phi), d\phi(e_i)) = \frac{1}{2}(\mathcal{L}_\xi h)(d\phi(e_i), d\phi(e_i)),
\]

from the soliton equation, we find that:

\[
\text{div}^M \omega = \lambda h(d\phi(e_i), d\phi(e_i)) - \text{Ric}^N(d\phi(e_i), d\phi(e_i))
\]

the Proposition 9 follows from equation (34), and the divergence theorem.

\(\square\)

**Remark 10.** The condition \(\text{Ric}^N > \lambda h\) (resp. \(\text{Ric}^N < \lambda h\)) is equivalent to \(\text{Ric}^N(v,v) > \lambda h(v,v)\) (resp. \(\text{Ric}^N(v,v) < \lambda h(v,v)\)), for any \(v \in T_p N - \{0\}\), where \(p \in N\).

It is known that the cigar soliton:

\[
(\mathbb{R}^2, \frac{dx^2 + dy^2}{1 + x^2 + y^2}),
\]

is steady with strictly positive Ricci tensor (see [8]), according to Proposition 9, we have the following:

**Corollary 11.** Any harmonic map \(\phi\) from a compact orientable Riemannian manifold without boundary to the cigar soliton is constant.

In the case of non-compact Riemannian manifold, we obtain the following results:

**Theorem 12.** Let \((M,g)\) be a complete non-compact Riemannian manifold, and \((N,h,\xi,\lambda)\) a nontrivial Ricci soliton with \(\text{Ric}^N < \mu h\), for some constant \(\mu < \lambda\). If \(\phi : (M,g) \rightarrow (N,h)\) is harmonic map, satisfying:

\[
\int_M |\xi \circ \phi|^2 v^g < \infty,
\]

then \(\phi\) is constant.

**Proof.** Let \(\rho\) be a smooth function with compact support on \(M\), we set:

\[
\omega(X) = h(\xi \circ \phi, \rho^2 d\phi(X)), \quad X \in \Gamma(TM).
\]

Let \(\{e_i\}\) be a normal orthonormal frame at \(x \in M\), we have:

\[
\text{div}^M \omega = e_i [h(\xi \circ \phi, \rho^2 d\phi(e_i))],
\]

by equation (37), the harmonicity condition of \(\phi\), we get:

\[
\text{div}^M \omega = h(\nabla_{e_i}^\phi(\xi \circ \phi), \rho^2 d\phi(e_i)) + h(\xi \circ \phi, \nabla_{e_i}^\phi \rho^2 d\phi(e_i))
\]

\[
= \rho^2 h(\nabla_{e_i}^\phi(\xi \circ \phi), d\phi(e_i)) + 2\rho e_i(\rho) h(\xi \circ \phi, d\phi(e_i)),
\]

by the soliton equation, we find that:

\[
\rho^2 h(\nabla_{e_i}^\phi(\xi \circ \phi), d\phi(e_i)) = \lambda \rho^2 h(d\phi(e_i), d\phi(e_i))
\]

\[
- \rho^2 \text{Ric}^N(d\phi(e_i), d\phi(e_i)),
\]

(39)
by Young’s inequality we have:
\[-2\rho e_i(\rho) h(\xi \circ \varphi, d\varphi(e_i)) \leq \varepsilon \rho^2 |d\varphi|^2 + \frac{1}{\varepsilon} e_i(\rho)^2 |\xi \circ \varphi|^2,\]  
(40)

for all \(\varepsilon > 0\). From (38), (39) and (40) we deduce the inequality:
\[
\lambda \rho^2 |d\varphi|^2 - \rho^2 \text{Ric}^N (d\varphi(e_i), d\varphi(e_i)) - \text{div}^M \omega \\
\leq \varepsilon \rho^2 |d\varphi|^2 + \frac{1}{\varepsilon} e_i(\rho)^2 |\xi \circ \varphi|^2,
\]
(41)

let \(\varepsilon = \lambda - \mu\), by (41) we have:
\[
\rho^2 [\mu |d\varphi|^2 - \text{Ric}^N (d\varphi(e_i), d\varphi(e_i))] - \text{div}^M \omega \\
\leq \frac{1}{\lambda - \mu} e_i(\rho)^2 |\xi \circ \varphi|^2,
\]
(42)

by the divergence theorem, and (42) we have:
\[
\int_M \rho^2 [\mu |d\varphi|^2 - \text{Ric}^N (d\varphi(e_i), d\varphi(e_i))] \, \nu^g \\
\leq \frac{1}{\lambda - \mu} \int_M e_i(\rho)^2 |\xi \circ \varphi|^2 \, \nu^g.
\]
(43)

Consider the smooth function \(\rho = \rho_R\) such that, \(\rho \leq 1\) on \(M\), \(\rho = 1\) on the ball \(B(p, R)\), \(\rho = 0\) on \(M \setminus B(p, 2R)\) and \(|\text{grad}^M \rho| \leq \frac{2}{R}\) (see [19]). From (43) we get:
\[
\int_M \rho^2 [\mu |d\varphi|^2 - \text{Ric}^N (d\varphi(e_i), d\varphi(e_i))] \, \nu^g \\
\leq \frac{4}{(\lambda - \mu) R^2} \int_M |\xi \circ \varphi|^2 \, \nu^g,
\]
(44)

since \(\int_M |\xi \circ \varphi|^2 \, \nu^g < \infty\), when \(R \to \infty\), we obtain:
\[
\int_M [\mu |d\varphi|^2 - \text{Ric}^N (d\varphi(e_i), d\varphi(e_i))] \, \nu^g = 0.
\]
(45)

Consequently, \(d\varphi(e_i) = 0\), for all \(i\) (because \(\mu h - \text{Ric}^N > 0\)), that is \(\varphi\) is constant.

If \(M = N\) and \(\varphi = \text{Id}_M\), from Theorem [12] we deduce:

Corollary 13. Let \((M, g, \xi, \lambda)\) be a complete non-compact nontrivial Ricci soliton with \(\text{Ric} < \mu h\) for some constant \(\mu < \lambda\). Then:
\[
\int_M |\xi|^2 \, \nu^g = \infty.
\]

2.4 Bi-harmonic Maps to Ricci Solitons

Theorem 14. Let \((M, g)\) be a compact orientable Riemannian manifold without boundary, and \((N, h, \xi, \lambda)\) a nontrivial Ricci soliton with:
\[
\text{Ric}^N > \lambda h \quad \text{or} \quad \text{Ric}^N < \lambda h.
\]

Suppose that \(\xi\) is Jacobi-type vector field. Then any bi-harmonic map \(\varphi\) from \((M, g)\) to \((N, h)\) is constant.

Proof. We set:
\[
\eta(X) = h(\xi \circ \varphi, \nabla^g_X \tau(\varphi)), \quad X \in \Gamma(TM),
\]
(46)

calculating in a normal frame at \(x \in M\), we have:
\[
\text{div}^M \eta = e_i [h(\xi \circ \varphi, \nabla^g_{e_i} \tau(\varphi))] \\
= h(\nabla^g_{e_i} (\xi \circ \varphi), \nabla^g_{e_i} \tau(\varphi)) + h(\xi \circ \varphi, \nabla^g_{e_i} \nabla^g_{e_i} \tau(\varphi)),
\]
(47)
from equation (47), and the bi-harmonicity condition of \( \varphi \), we get:

\[
\text{div}^M \eta = h(\nabla^g \xi \circ \varphi, \nabla^g \tau(\varphi)) - h(R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i), \xi \circ \varphi),
\]

the first term on the left-hand side of (48) is

\[
h(\nabla^g \xi \circ \varphi, \nabla^g \tau(\varphi)) = e_i[h(\nabla^g \xi \circ \varphi, \tau(\varphi))] - h(\nabla^g \xi \circ \varphi, \tau(\varphi)),
\]

by equations (48), (49), and the following property:

\[
h(R^N(X,Y)Z,W) = h(R^N(W,Z)Y,X),
\]

where \( X, Y, Z, W \in \Gamma(TM) \), we conclude that:

\[
\text{div}^M \eta = \text{div}^M h(\nabla^g \xi \circ \varphi, \tau(\varphi)) - h(\nabla^g \xi \circ \varphi, \tau(\varphi)) - h(R^N(\xi \circ \varphi, d\varphi(e_i))d\varphi(e_i), \tau(\varphi)),
\]

since \( \xi \) is a Jacobi-type vector field, we have:

\[
\text{div}^M \eta = \text{div}^M h(\nabla^g \xi \circ \varphi, \tau(\varphi)) - h(\nabla^N(\xi \circ \varphi, \tau(\varphi)),
\]

by the soliton equation, we get:

\[
\text{div}^M \eta = \text{div}^M h(\nabla^g \xi \circ \varphi, \tau(\varphi)) - \lambda |\tau(\varphi)|^2 + \text{Ric}^N(\tau(\varphi), \tau(\varphi)),
\]

from equation (52), and the divergence theorem, with \( \text{Ric}^N < \lambda h \) (or \( \text{Ric}^N > \lambda h \), we get \( \tau(\varphi) = 0 \), i.e. \( \varphi \) is harmonic map, so by the Proposition\[9\] \( \varphi \) is constant. \[\square\]

From Theorem\[13\] we deduce:

**Corollary 15.** Let \((M,g,\xi,\lambda)\) be a compact nontrivial Ricci soliton with:

\[
\text{Ric} > \lambda g \quad \text{or} \quad \text{Ric} < \lambda g.
\]

Then \( \xi \) is not Jacobi-type vector field.

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