ETA-PRODUCT $\eta(7\tau)^7/\eta(\tau)$

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ABSTRACT. Let $L_{\Phi_7}(s)$ be the Dirichlet series associated to the eta-product $\eta(7\tau)^7/\eta(\tau) \in M_3(\Gamma_0(7), \varepsilon)$ (here $\varepsilon(n) := (\frac{n}{7}) = (\frac{-7}{n})$ is the Dirichlet character defined by the residue symbol). We show that $L_{\Phi_7}(s)$ decomposes into the difference of two $L$-functions:

$$L_{\Phi_7}(s) = \frac{1}{8}(L(s, \varepsilon)L(s-2, 1) - L(s-1, \xi)),$$

where i) $L(s, \varepsilon)$ and $L(s, 1)$ are Dirichlet $L$-functions for the characters $\varepsilon$ and 1 modulo 7, respectively, and ii) $L(s, \xi)$ is the $L$-function for a Hecke character $\xi$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$.

This expression of $L_{\Phi_7}(s)$ gives a new proof of the non-negativity of the Fourier coefficients of the product $\eta(7\tau)^7/\eta(\tau)$, conjectured in [S3] and proven by Ibukiyama [I]. We also prove the uniqueness of the above decomposition of $L_{\Phi_7}(s)$ in a suitable sense.

1. INTRODUCTION

Let $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$, $q = \exp(2\pi \sqrt{-1}\tau)$ be the Dedekind eta-function (e.g. [R]). A product $\prod_{i \in I} \eta(i\tau)^{e(i)}$, where $I$ is a finite set of positive integers and $e : I \rightarrow \mathbb{Z}$ is any map, is called an eta-product. The eta-product can be developed in a Laurent series in powers of $q$, whose coefficients are called the Fourier coefficients.

Ibukiyama [I] has shown the following result, which answers to a part of a conjecture given by the author [S3] (see the next paragraph).

Theorem 1.1. Let $p$ be a rational prime number. Then the Fourier coefficients of the eta product $\eta_{\Phi_p} := \eta(p\tau)^p/\eta(\tau)$ are non-negative.

The proof in [I] is given by expressing the eta-product as a difference of two generating functions of two arithmetically constructed lattices.

More in general than the theorem, for any positive integer $h$ which may not be prime, we have the following non-negativity conjecture.

Conjecture ([S3]). Define a sequence $\Phi_h(\lambda) (h \in \mathbb{Z}_{>0})$ of cyclotomic polynomials by the recursive relation: $
abla^{\lambda} = \prod_{d|h} \Phi_d(\lambda^{h/d}).$. Explicitly, $\Phi_h(\lambda) = \frac{\phi(h)\mu(h)}{\prod_{d|h}(1-\lambda^d)^{\mu(d)}}$ where $\phi$ and $\mu$ are the Euler function and the Möbius function. Then the Fourier coefficients of the eta-product $\eta_{\Phi_h}(\tau) := \frac{\eta(h\tau)^{\phi(h)}}{\prod_{d|h}(1-\lambda^{d\mu(d)})}$ are non-negative integers.

This was proven for $h = 2, 3, 4, 5, 6$ [S1,2,3] by a use of the Dirichlet series $L_{\Phi_h}(s)$ associated to the eta-products $\eta_{\Phi_h}$. Precisely, we show that
$L_{\Phi_h}(s)$ admits either an Euler product for $h = 2, 3, 5$ or a decomposition into a difference of two Euler products for $h = 4, 6$, and that these expressions lead to a direct proof of the positivity of the coefficients.

In the present note, we prove in section 2 that the Dirichlet series $L_{\Phi_h}(s)$ decomposes into a difference of two $L$-functions, which admit Euler products, as stated in Abstract. In section 3, we show that this expression implies the non-negativity of the Dirichlet coefficients of $L_{\Phi_h}(s)$. In section 4, we prove a general lemma on the uniqueness of the decomposition of Dirichlet series into a difference of two Euler products, and apply it to $L_{\Phi_7}(s)$ (and also to $L_{\Phi_4}(s)$ and $L_{\Phi_6}(s)$). Finally, we remark in section 5 that such difference decomposition of $L_{\Phi_p}(s)$ for the prime $p \geq 11$ does not exist. If $h$ is a composite number, we do not know when $L_{\Phi_h}(s)$ admits such a difference decomposition.

In [S2, Conjecture 13.5], we give a wide class of eta-products whose Fourier coefficients are conjecturally non-negative and are of interest.

2. Hecke $L$-function $L(s, \xi)$ for a character $\xi$ on $\mathbb{Q}(\sqrt{-7})$

We recall Hecke’s $L$-function for a character $\xi$ on the imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$, and, then, decompose $L_{\Phi_{7}}(s)$ by a use of it. For a background on analytic number theory, one is referred to [M] and [R].

Since the class number of $\mathbb{Q}(\sqrt{-7})$ is equal to 1, we can introduce the Hecke character $\xi$ for the non-zero ideals of $K := \mathbb{Q}(\sqrt{-7})$ by

$$\xi((a)) := \left( \frac{a}{|a|} \right)^2 \quad (a \in K \setminus \{0\}).$$

Then, the $L$-function for $\xi$ is defined by the following Dirichlet series, which, as a result of definition, has the Euler product:

$$L(s, \xi) := \sum_{a \in \mathcal{O}_K} \xi(a)N_K(a)^{-s} = \prod_{p \text{ prime}} \left(1 - \xi(p)N_K(p)^{-s} \right)^{-1}.$$

Here, $a$ (resp. $p$) runs over all non-zero integral (resp. prime) ideals of $\mathcal{O}_K$, and $N_K(a)$ is the absolute norm of $a$ (i.e. $N_K(a) = |\mathcal{O}_K/a|$).

The first main result of the present note is the following.

**Lemma 2.1.** The Dirichlet series $L_{\Phi_{7}}(s)$ associated to the eta-product $\eta(7\tau)^7/\eta(\tau)$ decomposes into a difference of two $L$-functions as follows:

$$L_{\Phi_{7}}(s) = \frac{1}{8} \left(L(s, \varepsilon)L(s - 2, 1) - L(s - 1, \xi)\right),$$

where we recall that $\varepsilon = \left( \frac{7}{\tau} \right) = \left( \frac{\tau}{7} \right)$ is the residue symbol modulo 7.
Proof. The L-function $L(s-1, \xi)$ is associated to a Fourier series

$$f(\tau) := \sum_a \xi(a) N_K(a) e^{2\pi \sqrt{-N(a)} \tau}.$$  \hspace{1cm} (4)

According to Hecke [H1][H2], $f(\tau)$ is an automorphic form belonging in $S_3(\Gamma_0(7), \varepsilon)$ (see [M, Th.4.8.2]). Similarly, $L(s, \varepsilon)L(s-2, 1)$ and $L(s-2, \varepsilon)L(s, 1)$ are associated to Eisenstein series, say $E(\tau)$ and $E'(\tau)$, in $M_q(\Gamma_0(7), \varepsilon)$. Since $\Gamma_0(7) \setminus \mathcal{H}$ has two cusps and $\dim_{\mathbb{C}} S_3(\Gamma_0(7), \varepsilon) = 1$, $M_q(\Gamma_0(7), \varepsilon)$ is spanned by $E, E'$ and $f$. To show the equality: $\eta_{\Phi, l}(\tau) = \frac{1}{8}(E(\tau) - f(\tau))$, it suffices to show that $n$th Fourier coefficients $c(n)$ of $\eta_{\Phi, l}(\tau)$ coincide with $n$th Dirichlet coefficients of $\frac{1}{8}(L(s, \varepsilon)L(s-2, 1) - L(s-1, \xi))$ for $1 \leq n \leq 3$. We give an explicit integral description (which we shall use in the next section) of the coefficients of $L(s-1, \xi)$. For the end, we factorize $L(s-1, \xi)$ w.r.t. rational primes $p, q$ in $\mathbb{Z}_{>0}$:

$$L(s-1, \xi) := \frac{1}{1 + 7^{-s+1}} \prod_{\varepsilon(q) = -1} \frac{1}{1 - q^{-2s+2}} \prod_{\varepsilon(p) = 1} \frac{1}{P_p(p^{-s})},$$  \hspace{1cm} (5)

where $P_p(\lambda) \in \mathbb{Z}[\lambda]$ for a prime $p$ with $\varepsilon(p) = 1$ is defined in next (6).

Proof. Recall a well-known fact (e.g. [T]) on the prime ideals in $\mathbb{Q}(\sqrt{7})$:

i) $(q)$ is a prime ideal for any rational prime $q$ with $\varepsilon(q) = -1$,

ii) $p = x_p^2 + 7 \cdot y_p = (x_p + y_p \sqrt{7})(x_p - y_p \sqrt{7})$ for any odd rational prime number $p$ with $\varepsilon(p) = 1$,

iii) $2 = \frac{7 + \sqrt{7}}{4} \cdot \frac{1 - \sqrt{7}}{2}$ and $7 = - (\sqrt{7})^2$.

Put $\pi_2 := \frac{1 + \sqrt{7}}{2}$ and $\pi_p := x_p + y_p \sqrt{7}$ for an odd rational prime number $p$ with $\varepsilon(p) = 1$ and, define the quadratic polynomials

$$P_2(X) := (1 - \pi_2^2 X)(1 - \pi_2^2 X) = 1 + 3X + 2X^2 \text{ and }$$

$$P_p(X) := (1 - \pi_p^2 X)(1 - \pi_p^2 X) = 1 - 2(x_p^2 - 7y_p^2)X + p^2X^2.$$  \hspace{1cm} (6)

Then (5) follows from the Euler product in (2) and

i) $\xi((\pi_p)) = \pi_p^2/p$ and $N_K((\pi_p)) = p$ for $\varepsilon(p) = 1$,

ii) $\xi((q)) = 1$ and $N_K((q)) = q^2$ for $\varepsilon(q) = -1$,

iii) $\xi((\sqrt{7})) = -1$ and $N_K((\sqrt{7})) = 7$.  \hspace{1cm} $\Box$

Put $L(s, \varepsilon)L(s-2, 1) = \sum_{n=1}^{\infty} a(n)n^{-s}$ and $L(s-1, \xi) = \sum_{n=1}^{\infty} b(n)n^{-s}$, and we give expicite expressions of the coefficients $a(n)$ and $b(n)$. Let

$$n = 7^k \prod_{i \in I} p_i^{k_i} \prod_{j \in J} q_j^{m_j}$$

be the prime decomposition of $n \in \mathbb{Z}_{>0}$ where $\{p_i \mid i \in I\}$ and $\{q_j \mid j \in J\}$ are finite set of distinct prime numbers with $\varepsilon(p_i) = 1$ and $\varepsilon(q_j) = -1$. \hspace{1cm} (7)
Then, by a use of (5) together with (6), one obtains the formulae:

\[(7) \quad a(n) = 7^{2k} \prod_{i \in I} p_i^{2(l_i+1)} - 1 \prod_{j \in J} q_j^{2(m_j+1)} - (-1)^{m_j+1} \]

\[(8) \quad b(n) = (-7)^k \prod_{i \in I} \left( \sum_{t=0}^{l_i} \pi_{p_i}^{2t} \pi_{i-1}^{2(l_i-t)} \right) \prod_{j \in J} \frac{1 - (-1)^{m_j+1}}{2} q_j^{m_j} \]

Finally, we give the Fourier expansion of \(\eta_{7\tau}\) up to degree 50.

\[\eta_{7\tau} = q + q^7 + 2q^8 + 3q^9 + 5q^{10} + 7q^{11} + 11q^{12} + 8q^{13} + 15q^{14} + 16q^{15} + 21q^{16} + 21q^{17} + 28q^{18} + 24q^{19} + 44q^{20} + 36q^{21} + 49q^{22} + 45q^{23} + 63q^{24} + 49q^{25} + 74q^{26} + 64q^{27} + 85q^{28} + 72q^{29} + 105q^{30} + 82q^{31} + 133q^{32} + 112q^{33} + 120q^{34} + 120q^{35} + 165q^{36} + 122q^{37} + 180q^{38} + 147q^{39} + 186q^{40} + 176q^{41} + 225q^{42} + 255q^{43} + 255q^{44} + 21q^{45} + 245q^{46} + 224q^{47} + 324q^{48} + 219q^{49} + 338q^{50} + 276q^{51} + 341q^{52} + 294q^{53} + 385q^{54} + \cdots \]

By inspection, we check the equality \(c(n) = \frac{1}{8}(a(n) - b(n))\) for \(n\) with \(1 \leq n \leq 3\). This completes a proof of Lemma 2.1.

Remark 1. As we see in the above proof, once one guess a correct formula (3), then its proof is straightforward. However, we do not know yet what is a “correct formula” for \(L_{\Phi_h}(s)\) for \(h > 7\) (see §5).

3. Positivity of Fourier coefficients of \(\eta(7\tau)^7/\eta(\tau)\)

As an immediate consequence of Lemma 2.1. together with the explicit formulae (6) and (7), we obtain the following positivity.

Corollary. All Fourier coefficients of \(\eta(7\tau)^7/\eta(\tau)\) are positive.

Proof. Lemma 2.1. says \(c(n) = \frac{1}{8}(a(n) - b(n))\) for all \(n \in \mathbb{Z}_{\geq 1}\). To show \(a(n) > b(n)\) for all \(n \in \mathbb{Z}_{\geq 1}\), it is sufficient to show \(a(p^k) > |b(p^k)|\) for any primary number \(p^k\) (i.e. \(p\) is a prime number and \(k \in \mathbb{Z}_{>0}\)) because of the multiplicativity of \(a(n)\) and \(b(n)\). We separate cases:

Case \(p = 7\). \(a(7^k) = 7^{2k} > 7^k = |b(7^k)|\).

Case \(\varepsilon(p) = 1\). \(a(p^k) > p^{2k} \geq (k+1)p^k = \sum_{i=0}^{k} \pi_p^{2i} \pi_p^{2(k-i)} \geq |b(p^k)|\).

Case \(\varepsilon(q) = -1\). \(a(q^k) - |b(q^k)| \geq \frac{q^{2(k+1)-1}}{q^2+1} - q^k = \frac{q^{k+2}-1}{q^2+1} > 0\). □

4. Uniqueness of decomposition of Dirichlet series

We show the second main result of the present note: Under a mild assumption on a Dirichlet series \(L(s) = \sum_{n \in \mathbb{Z}_{\geq 1}} c(n)n^{-s}\), we show the uniqueness of the decomposition of \(L(s)\) into the form:

\[(9) \quad L(s) = aM(s) + bN(s)\]

where \(M(s)\) and \(N(s)\) are Dirichlet series which admit Euler product and \(a, b\) are constants. For our applications, we assume that \(c(1) = 0\) so
that one automatically has $a+b = 0$ (since the first Dirichlet coefficients of $M(s)$ and $N(s)$ are automatically equal to 1).

**Lemma 4.1.** Let $L(s) = \sum_{n \in \mathbb{Z}_{\geq 1}} c(n)n^{-s}$ be a Dirichlet series such that i) $c(1) = 0$ and ii) there are five relatively prime integers $l, m, n, u, v \in \mathbb{Z}_{\geq 1}$ such that $c(l)c(m)c(n)c(u)c(v) \neq 0$. If there exists a decomposition (9), where $M(s)$ and $N(s)$ are Dirichlet series having Euler products, then it is unique up to the transposition of $M(s)$ and $N(s)$.

**Proof.** Put $M(s) = \sum_{n \in \mathbb{Z}_{\geq 1}} a(n)n^{-s}$, $N(s) = \sum_{n \in \mathbb{Z}_{\geq 1}} b(n)n^{-s}$ and $c := a = -b$ so that one has the relation among the Dirichlet coefficients:

$$c(n) = c(a(n) - b(n)) \quad (n \in \mathbb{Z}_{\geq 1}).$$

Clearly $c \neq 0$, else $L(s) = 0$ contradicting to the assumption on $L(s)$.

We first remark that one sees from (10) that if $c(n) = c(m) = 0$ for relatively prime positive integers $n$ and $m$ then $c(nm) = 0$. Consequently, if $c(n) \neq 0$, then there exists a primary factor $p^k$ of $n$ (i.e. $p$ is a prime number and $k$ is a positive integer s.t. $p^k|n$) such that $c(p^k) \neq 0$.

Suppose there exist another decomposition $L(s) = c'(M'(s) - N'(s))$. Using Dirichlet coefficients $a'(n), b'(n)$ of $M'(s), N'(s)$, this means

$$c(n) = c'(a'(n) - b'(n)) \quad (n \in \mathbb{Z}_{\geq 1}).$$

Let $n, m \in \mathbb{Z}_{\geq 1}$ be relatively prime to each other, then the multiplicativities of the Dirichlet coefficients $a, b, a'$ and $b'$ implies

$$c(mn) = c(a(n)a(m) - b(n)b(m)) = c'(a'(n)a'(m) - b'(n)b'(m))$$

Substituting $b(n) = a(n) - c(n)/c$, $b'(n) = a'(n) - c(n)/c'$ and $b(m) = a(m) - c(m)/c$, $b'(m) = a'(m) - c(m)/c'$ in *

$$E(m, n) : c(n)(a(m) - a'(m)) + c(m)(a(n) - a'(n)) = (\frac{1}{c} - \frac{1}{c'})c(n)c(m).$$

Let $k, m, n \in \mathbb{Z}_{\geq 1}$ be relatively prime to each other and $c(m)c(n) \neq 0$, then

$$(c(k)E(m, n) - c(m)E(n, k) - c(n)E(k, m))/c(m)c(n)$$

is the equality

$$a(k) - a'(k) = \frac{1}{2}(\frac{1}{c} - \frac{1}{c'})c(k).$$

This, together with (10) and (11), can be rewritten as the linear relations among $a(k), b(k)$ and $a'(k), b'(k)$ for all $k$ prime to $mn$:

$$a'(k) = (1 - \lambda)a(k) + \lambda b(k) \quad \text{and} \quad b'(k) = \lambda a(k) + (1 - \lambda)b(k),$$

where $\lambda := \frac{1}{2}(\frac{1}{c} - \frac{1}{c'})$ so that $\lambda = 0$ or 1 if and only if $c = c'$ or $c = -c'$, respectively. Summing two relations, we also obtain the relation:

$$a(k) + b(k) = a'(k) + b'(k).$$
If \( c = c' \) (i.e. \( \lambda = 0 \)), then the proof of Lemma 4.1. is already achieved as follows: by substituting \( c = c' \) in * and using **, one has

\[
\ast \ast \ast \quad a(k) = a'(k) \quad \text{and} \quad b(k) = b'(k)
\]

for any \( k \in \mathbb{Z}_{\geq 1} \) prime to \( m, n \). By replacing the role of \( m, n \) by \( u, v \), the equalities \( \ast \ast \ast \) hold for any primary numbers \( k \). The \( \ast \ast \ast \) extends, further, for any positive integers \( k \) due to the multiplicativity of \( a, a', b \) and \( b' \). This means \( M(s) = M'(s) \) and \( N(s) = N'(s) \).

Suppose \( c \neq c' \) (i.e. \( \lambda \neq 0 \)). Then, * means another decomposition:

\[
(11)' \quad c(k) = \frac{c}{\lambda}(a(k) - a'(k))
\]

for any \( k \in \mathbb{Z}_{\geq 1} \) prime to \( m, n \). Replacing (11) by (11)', we can repeat the previous discussions to induce * and **, where we replace the role of \( m, n \) by \( u, v \), and consider integers \( k \) which is prime to \( m, n \) and also to \( u, v \). Then, in addition to * and **, we obtain: \( \ast ': \quad 0 = a(k) - a(k) = \frac{1}{2}\lambda c(k) \) and \( \ast \ast ': \quad a(k) + b(k) = a(k) + a'(k) \) for all \( k \) prime to \( m, n, u, v \). Taking \( k = l \) with \( c(l) \neq 0 \), which exists by the assumption of Lemma, we obtain \( c = -c' \). By the similar argument for the case \( c = c' \), we obtain: \( \ast \ast \ast ': \quad a(k) = b'(k) \), \( b(k) = a'(k) \) for all \( k \in \mathbb{Z}_{\geq 1} \) and, therefore, \( M(s) = N'(s) \) and \( N(s) = M'(s) \).

**Corollary.** The Dirichlet series \( L_{\Phi_4}(s) \) satisfies the assumptions i) and ii) so that the decomposition (3) is unique in the sense of Lemma 4.1.

**Remark 2.** Lemma 4.1. can be formulated more precisely according to the # of relatively prime \( n \)'s with \( c(n) \neq 0 \). The case \# = 5 of Lemma 4.1. is the strongest case. Since the other cases for \# < 5 are involved but not used in the present note, they are omitted.

**Remark 3.** There are a few more known Dirichlet series associated to eta-products, which decompose as (9) and satisfy the assumption of Lemma 4.1, namely, \( \eta(48\tau)^3/\eta(24\tau) \), \( \eta_{\Phi_4}(8\tau) = \eta(32\tau)^2/\eta(16\tau)/\eta(8\tau) \) and \( \eta_{\Phi_6}(12\tau) = \eta(72\tau)/\eta(36\tau)/\eta(24\tau)/\eta(12\tau) \). They have an origin in a study of elliptic root systems (see [S1]).

5. **Non-decomposability of \( L_{\Phi_4}(s) \) for \( p \geq 11 \)**

We finally give the following remark, which can be shown trivially.

**Fact.** The Dirichlet series \( L_{\Phi_4}(s) \) associated to a eta-product \( \eta(p\tau)^p/\eta(\tau) \) for a prime number \( p \) with \( p \geq 11 \) does not admit a decomposition (9).

**Proof.** Suppose a decomposition (9) exists, i.e. there is a Dirichlet series \( M(s) \) and a constant \( c \neq 0 \) such that \( M(s) - \frac{L_{\Phi_4}(s)}{c} \) is a Dirichlet series admitting an Euler product. Let \( c(n), a(n) \) and \( b(n) \) be the Dirichlet
coefficients of $L_{\Phi_p}(s)$, $M(s)$ and $M(s) - \frac{1}{c} L_{\Phi_p}(s)$. The following fact follows from the explicit expression of the eta product $\eta(p\tau)^p/\eta(\tau)$:

1. $c(n) = 0$ for $1 \leq n < (p^2 - 1)/24$ ($\geq 5$),
2. $c(n) \neq 0$ for $(p^2 - 1)/24 \leq n < (p^2 - 1)/24 + p$.

Thus, we can find an odd integer $m$ such that $1 < m < (p^2 - 1)/24$ and $(p^2 - 1)/24 \leq 2m < (p^2 - 1)/24 + p$. Then, $a(2)a(m) = b(2)b(m) = b(2m) = a(2m) - \frac{1}{c}(2m) = a(2)a(m) - \frac{1}{c}(2m)$ should imply $\frac{1}{c}(2m) = 0$. Since $c(2m) \neq 0$ (due to ii)), one has $\frac{1}{c} = 0$ which is impossible. □

Acknowledgement: The author is grateful to Professor Hiroshi Saito for his help to identify $L(s, \xi)$ with Hecke $L$-function.

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