THE EIGENVECTORS OF A COMBINATORIAL MATRIX

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ABSTRACT. In this paper, we derive the eigenvectors of a combinatorial matrix whose eigenvalues studied by Kilic and Stanica. We follow the method of Cooper and Melham since they considered the special case of this matrix.

1. Introduction

In [7], Peele and Stănică studied \( n \times n \) matrices with the \( (i, j) \) entry the binomial coefficient \( \binom{i-1}{j-1} \), respectively, \( \binom{i-1}{n-j} \) and derived many interesting results on powers of these matrices. In [8], one of them found that the same is true for a much larger class of what he called netted matrices, namely matrices with entries satisfying a certain type of recurrence among the entries of all \( 2 \times 2 \) cells.

Let \( R_n \) be the matrix whose \( (i, j) \) entries are \( a_{i,j} = \binom{i-1}{n-j} \), which satisfy

\[
a_{i,j-1} = a_{i-1,j-1} + a_{i-1,j}. \tag{1.1}
\]

The previous recurrence can be extended for \( i \geq 0, j \geq 0 \), using the boundary conditions \( a_{1,n} = 1, a_{1,j} = 0, j \neq n \). Remark the following consequences of the boundary conditions and recurrence (1.1): \( a_{i,j} = 0 \) for \( i + j \leq n \), and \( a_{i,n+1} = 0, 1 \leq i \leq n \).

The matrix \( R_n \) was firstly studied by Carlitz [2] who gave explicit forms for the eigenvalues of \( R_n \). Let \( f_{n+1} (x) = \det (xI - R_n) \) be the characteristic polynomial of \( R_n \). Thus

\[
f_n (x) = \sum_{r=0}^{n+1} (-1)^{r(r+1)/2} \binom{n}{r} F_n x^{n-r}
\]

where \( \binom{n}{r}_F \) denote the Fibonomial coefficient, defined (for \( n \geq r > 0 \)) by

\[
\binom{n}{r}_F = \frac{F_1 F_2 \ldots F_n}{(F_1 F_2 \ldots F_r) (F_1 F_2 \ldots F_{n-r})}
\]

Received by the editors Feb. 15, 2011, Accepted: March 24, 2011.
2000 Mathematics Subject Classification. 11B39, 15A15, 15A18.

Key words and phrases. Fibonomial coefficients, binomial coefficients, eigenvector, Pascal matrix.

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Let $\phi, \bar{\phi} = (1 \pm \sqrt{5})/2$. Thus the eigenvalues of $R_n$ are $\phi^n, \phi^{n-1}\bar{\phi}, \ldots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n$.

In [7] it was proved that the entries of the power $R^n_c$ satisfy the recurrence

$$F_{n-1}a_{i,j}^{(c)} = F_0a_{i-1,j}^{(c)} + F_{i+1}a_{i-1,j-1}^{(c)} - F_ia_{i-1,j-1}^{(c)}$$

(1.2)

where $F_n$ is the Fibonacci sequence. Closed forms for all entries of $R^n_c$ were not found, but several results concerning the generating functions of rows and columns were obtained (see [7, 8]). Further, the generating function for the $(i,j)$-th entry of the $c$-th power of a generalization of $R_n$, namely

$$Q_n(a,b) = \left( a^{i+j-n} - n \right),$$

is

$$B_n^{(c)}(x,y) = \frac{(U_{c-1} + U_cy)(U_{c-1} + yU_c)^{n-1}}{U_{c-1} + U_cy - x(U_c + U_{c+1}y)}.$$

Regarding this generalization, in [3], the authors gave the characteristic polynomial of $Q_n(a,b)$ and the trace of $k$th power of $Q_n(a,b)$, that is, $\text{tr}(Q^k_n(a,b))$, by using the method of Carlitz [2]:

$$\text{tr} \left( Q^k_n(a,b) \right) = \frac{U_{kn}}{U_k},$$

(1.3)

where $\binom{n}{r}_U$ stands for the generalized Fibonacci number, defined by

$$\binom{n}{r}_U = \frac{U_1U_2\ldots U_n}{U_1U_2\ldots U_{r-1}U_{r+1}\ldots U_{n-r}},$$

for $n \geq i > 0$, where $\binom{n}{r}_U = \binom{n}{r}_{U^*} = 1$.

In [7], the authors proposed a conjecture on the eigenvalues of matrix $R_n$, which was proven independently in [1] and the unpublished manuscript [9]. Also, in [6], they found the eigenvectors of $R_n$.

In [7, 8], it was shown that the inverse of $R_n$ is the matrix

$$R_n^{-1} = \left( (1)^{i+j+1}n \right),$$

and, in general, the inverse of $Q_n(a,b)$ is

$$Q_n^{-1}(a,b) = \left( (1)^{n+i+j+1}a^{n-i-j} \right),$$

(1.4)

Let $\phi = \frac{1+\sqrt{5}}{2}, \bar{\phi} = \frac{1-\sqrt{5}}{2}$ be the golden section and its conjugate. The eigenvalues $R_n$ are $\phi^n, \phi^{n-1}\bar{\phi}, \ldots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n.$
Let the sequences \( \{u_n\}, \{v_n\} \) be defined by
\[
\begin{align*}
u_n &= au_{n-1} + bu_{n-2} \\
v_n &= av_{n-1} + bv_{n-2},
\end{align*}
\]
for \( n > 1 \), where \( a_0 = 0, u_1 = 1, \) and \( v_0 = 2, v_1 = a, \) respectively. Let \( \alpha, \beta \) be the roots of the associated equation \( x^2 - ax - b = 0 \). The next lemma can be found in [4].

**Lemma 1.1.** For \( k \geq 1 \) and \( n > 1 \),
\[
\begin{align*}
u_{kn} &= v_k u_{k(n-1)} + (-1)^{k+1} b^k u_{k(n-2)} \\
v_{kn} &= v_k v_{k(n-1)} + (-1)^{k+1} b^k v_{k(n-2)}. \\
\end{align*}
\]

In [5], using the sequence \( v_k \), they defined the \( n \times n \) matrix \( H_n(v_k, b^k) \) as follows:
\[
H_n(v_k, b^k) = \begin{pmatrix} v_i^j + (i - n)(-b)^j & \cdots & v_i^j (n - 1)(-b)^j \\
\vdots & \ddots & \vdots \\
v_i^j (n - 1)(-b)^j & \cdots & v_i^j + (i - n)(-b)^j \end{pmatrix}_{1 \leq i, j \leq n}.
\]

As in equation (1.4), they also found the inverse of the matrix \( H_n \), namely
\[
H_n^{-1}(v_k, b^k) = \begin{pmatrix} (-1)^{i+j}(n-i)(-b)^j & \cdots & (-1)^{i+j}(n-i)(-b)^j \end{pmatrix}_{i,j}.
\]

It is well known that for \( n \geq -1 \),
\[
u_{n+1} = \sum_r \binom{n}{r} a^{2r-n} b^{n-r}.
\]

Thus the authors [5] generalized this identity as well as they gave the following results:

**Lemma 1.2.** For \( k > 0 \) and \( n \geq -1 \),
\[
\frac{u_k(n+1)}{u_k} = \sum_r \binom{n}{r} v_i^j (n-i)(-b)^j.
\]

**Lemma 1.3.** For all \( m > 0 \),
\[
\text{tr}(H_n^m(v_k, b^k)) = \frac{u_k n m}{u_k}.
\]

**Theorem 1.4.** The eigenvalues of \( H_n(v_k, b^k) \) are
\[
\alpha^{kn}, \alpha^{k(n-1)} \beta, \ldots, \alpha^k \beta^{k(n-1)}, \beta^{kn}.
\]
In [6], the authors considered the matrix $Q_n(a, b)$ and gave its eigenvectors. In this section, we consider the generalization of matrix $Q_n(a, b)$ namely $H_n(v_k, b^k)$ and then determine its eigenvectors by using the method given in [6].

Let $0 \leq p \leq n - 1$ be a fixed integer,

$$f(x) = (x - \alpha_k)^p(x - \beta_k)^{n-1-p} = \sum_{r=0}^{n-1} s_r x^r,$$

and

$$s = (s_0, s_1, ..., s_{n-1})^T.$$

**Theorem 2.1.** For $m \geq 0$

$$f^{(m)}(x) = m! \frac{f(x)}{(x - \alpha_k)^m(x - \beta_k)^m} \sum_{j=0}^{m} \binom{p}{m-j} \binom{n-1-p}{j} (x-\alpha_k)^j(x-\beta_k)^{m-j}.$$

**Proof.** It can be proved with the use of Leibniz’s formula for the $m$-th derivative of a product of two functions. We recall the Leibniz’s formula: For $m \geq 0$

$$\frac{d^m}{dx^m} g(x)h(x) = \sum_{j=0}^{m} \binom{m}{j} g^{(m-j)}(x)h^{(j)}(x). \quad (2.1)$$

We use the notation $x^n$ to denote the falling factorial, and hence

$$f^{(m)}(x) = \sum_{j=0}^{m} \binom{m}{j} (x-\alpha_k)^{p-m+j}(n-1-p)\frac{1}{\beta_k^{n-1-p-j}}.$$ 

$$= m! \frac{f(x)}{(x - \alpha_k)^m(x - \beta_k)^m} \sum_{j=0}^{m} \binom{p}{m-j} \binom{n-1-p}{j} (x-\alpha_k)^j(x-\beta_k)^{m-j},$$

as claimed. \qed

**Lemma 2.2.** Suppose that $0 \leq m \leq n - 1$ be a fixed integer. Then,

$$s_{n-1-m} = \sum_{j=0}^{m} (-1)^j \binom{p}{m-j} \binom{n-1-p}{j} \alpha_k^{(m-j)} \beta_k^j$$

and

$$(H_n(v_k, b^k) s)_{n-1-m} = \sum_{r=m}^{n-1} (-1)^m \binom{k}{r} b^k \binom{r}{m} v_k^{r-m} s_r.$$

**Proof.** The proof of the first equation can be followed from computing the coefficient of $x^{n-1-m}$ of $f(x)$ by multiplying $(x - \alpha_k)^p$ times $(x - \beta_k)^{n-1-p}$. The second proof can be seen by computing the product of $H_n(v_k, b^k)$ and $s$. \qed
Theorem 2.3. \[ H_n(v_k, b^k) s = (\alpha^k)^{n-1-p} \beta^{kp}s. \] (2.2)

Proof. Consider
\[
(H_n(v_k, b^k) s)_{n-1-m} = \sum_{r=m}^{n-1} (-1)^m (-b)^{km} \binom{r}{m} v_k^{r-m} s_r
\]
\[
= \frac{(-1)^m (-b)^{km}}{m!} \sum_{r=m}^{n-1} s_r r^m v_k^{r-m}
\]
\[
= \frac{(-1)^m (-b)^{km}}{m!} f(m)(v_k)
\]
\[
= \frac{(-1)^m (-b)^{km}}{m!} (v_k - \alpha^k)^{p}(v_k - \beta^k)^{n-1-p} m!
\]
\[
= \frac{(-1)^m (-b)^{km}}{m!} \frac{(v_k - \alpha^k)^{m}(v_k - \beta^k)^{n-1-p} m!}{m!} \times
\]
\[
\sum_{j=0}^{m} \binom{p}{m-j} \binom{n-1-p}{j} (v_k - \alpha^k)^{j}(v_k - \beta^k)^{m-j}
\]
\[
= \alpha^k(n-1-p) \beta^{kp} \sum_{j=0}^{m} (-1)^m \binom{p}{m-j} \binom{n-1-p}{j} \alpha^{k(m-j)} \beta^{kj}
\]
\[
= \alpha^k(n-1-p) \beta^{kp} s_{n-1-m}. \]

Thus the proof is complete. \(\square\)

ÖZET: Bu çalışmada, Kilic ve Stanica tarafından verilen bir kombinatoryal matrisin özvektörleri verilen bir kombinatoryal matrisin özvektörleri, Cooper ve Melham’ın metodu takip edilerek elde edilmiştir.

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