SELF-ADJOINTNESS OF THE GAFFNEY LAPLACIAN ON VECTOR BUNDLES

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Abstract. We study the Gaffney Laplacian on a vector bundle equipped with a compatible metric and connection over a Riemannian manifold that is possibly geodesically incomplete. Under the hypothesis that the Cauchy boundary is polar, we demonstrate the self-adjointness of this Laplacian. Furthermore, we show that negligible boundary is a necessary and sufficient condition for the self-adjointness of this operator.

1. Introduction

The study of the essential self-adjointness of the Laplace-Beltrami operator ∆ on a geodesically complete Riemannian manifold (M,g) in L^2(M) and the standard L^2-spaces of differential forms was initiated by Gaffney in [9]. Later, also in the setting of geodesically complete Riemannian manifolds, Cordes in [7] proved the essential self-adjointness of positive integer powers of the operator ∆ on C^∞(M). In contrast to Cordes’ “stationary” approach, Chernoff in [6] used a wave equation method to establish the essential self-adjointness of positive integer powers of the Laplace operator on differential forms.

In [2], the first author considered the case of a vector bundle V equipped with a metric h and connection ∇ that are compatible, over a geodesically complete manifold. On factorising certain “density problems” in terms of a first-order operator and applying the results of Chernoff from [6], he established the density of C^∞(V) in the Sobolev space W^{1,2}(V) (the closure of u ∈ C^∞ ∩ L^2(V) with ∇u ∈ C^∞ ∩ L^2(T^*M ⊗ V)) under the Sobolev norm ||·||_{W^{1,2}} = ||·|| + ||∇·||, the density of C^∞_c(T^*M ⊗ V) in the...
domain of the divergence operator (the adjoint of $\nabla$ in $L^2$), as well as the essential self-adjointness of the Bochner Laplacian $-\text{tr} \nabla^2$ on $C_c^\infty(\mathcal{V})$.

In the context of Riemannian manifolds $(\mathcal{M}, g)$ that are possibly geodesically incomplete, Masamune in [14] studied the essential self-adjointness of the Laplace-Beltrami operator on functions. In particular, he showed that the essential self-adjointness is equivalent to the negligible boundary property (see §5). In the same paper, he showed that this property is equivalent to the equality $W^{1,2}_0(\mathcal{M}) = W^{1,2}(\mathcal{M})$ (where the space $W^{1,2}_0(\mathcal{M})$ is the closure of $C_c^\infty(\mathcal{M})$ with respect to the Sobolev norm).

On a related note, it turns out that the zero capacity of the Cauchy boundary of a Riemannian manifold implies $W^{1,2}_0(\mathcal{M}) = W^{1,2}(\mathcal{M})$; see [13, 14] by Masamune and [10] by Grigor’yan and Masamune. A related study of the essential self-adjointness of the sub-Laplacian on a sub-Riemannian manifold with the pseudo-0-negligible boundary property can be found in [15] by Masamune.

In the present paper, we consider a vector bundle $\mathcal{V}$ with a compatible metric $h$ and connection $\nabla$ over a Riemannian manifold $(\mathcal{M}, g)$ (without boundary) that is possibly geodesically incomplete. First, by studying the effect of zero capacity of the Cauchy boundary on $(\mathcal{V}, h, \nabla)$, we show that $W^{1,2}_0(\mathcal{V}) = W^{1,2}(\mathcal{V})$ (Theorem 4.3).

We then proceed to extend the notion of negligible boundary to $(\mathcal{M}, \mathcal{V}, \nabla)$. In Theorem 5.1, we show that the self-adjointness of the Gaffney Laplacian (see §2) is equivalent to the negligible boundary property, which is, in turn, equivalent to the equality $W^{1,2}_0(\mathcal{V}) = W^{1,2}(\mathcal{V})$. Additionally, in Theorem 5.2, which is a Bochner Laplacian analogue of Theorem 3 in [14], we establish the equivalence between essential self-adjointness of the Bochner Laplacian and negligible boundary.

Our analysis rests upon an application of a version of integration by parts, which enable us to establish useful links among the first-order operators entering the definitions of Dirichlet, Neumann, and Gaffney Laplacians; see Theorem 3.3 below. Other useful links among those operators are obtained by extracting information from the negligible boundary property. Additionally, in the proof of the essential self-adjointness of the Bochner Laplacian (Theorem 5.2), we adapt the heat equation method of Masamune in [14, 15] to our setting.

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2. Preliminaries

Let $\mathcal{M}$ be a smooth manifold with a smooth metric $g$. We emphasise that $\mathcal{M}$ is a manifold without boundary and that $g$ is possibly geodesically incomplete. We denote
the induced volume measure by $\mu_\kappa$. We assume that $\mathcal{V}$ is a smooth vector bundle over $\mathcal{M}$, equipped with a smooth metric $h$.

Furthermore, let $\nabla$ be a connection on $\mathcal{M}$ that is compatible with the metric. Note that we do not assume this is the Levi-Civita connection. By the same symbol, we denote a connection on $\mathcal{V}$ which is compatible with $h$.

We define the Lebesgue spaces $L^p(\mathcal{V})$ for $p \geq 1$, although we only require this theory for the case $p = 2$. First, we note some measure theoretic notions. We say that a set $A \subset \mathcal{M}$ is measurable if, whenever $(U, \psi)$ is a chart and $U \cap A \neq \emptyset$, then the set $\psi(A \cap U)$ is Lebesgue measurable. Thus, we define the space of measurable sections $\Gamma(\mathcal{V})$, where we say that a section is measurable if its coefficients are measurable when seen through trivialisations. We remark that this notion of measurability is equivalent to measurability with respect to the induced measure $\mu_\kappa$. See [3] for details.

For $p \in [1, \infty)$, the set $L^p(\mathcal{M}; \mathcal{V})$ (or $L^p(\mathcal{V})$ for short) is defined as the set of sections $\xi \in \Gamma(\mathcal{V})$ such that

$$\int_{\mathcal{M}} |\xi(x)|_{h(x)}^p \, d\mu_\kappa(x) < \infty.$$  

We bestow $L^p(\mathcal{V})$ with the norm $\|\xi\|_p = (\int_{\mathcal{M}} |\xi|_{h(x)}^p \, d\mu_\kappa)^{\frac{1}{p}}$. In the case of $p = \infty$, we define $L^\infty(\mathcal{V})$ to consist of sections $\xi \in \Gamma(\mathcal{V})$ such that there exists $C > 0$ with $|\xi(x)|_{h(x)} \leq C$ for $x$-a.e. The norm $\|\xi\|_\infty$ is the infimum over all such constants $C$. Each of these spaces is a Banach space (strictly speaking, modulo sections which differ on a set of measure zero). The space $L^2(\mathcal{V})$ is a Hilbert space with inner product

$$\langle \xi, \zeta \rangle = \int_{\mathcal{M}} h_x(\xi(x), \zeta(x)) \, d\mu_\kappa(x)$$

for $\xi, \zeta \in L^2(\mathcal{V})$. We assume these spaces are complex valued by identifying a real space with its complexification. The local $L^p$ spaces are denoted by $L^p_{\text{loc}}(\mathcal{V})$ and they contain sections $\xi \in \Gamma(\mathcal{V})$ satisfying: $\xi \in L^p(K, \mathcal{V})$ for every open $K \subset \mathcal{M}$ with $\overline{K}$ compact.

Let

$$S^p_k(\mathcal{V}) = \{ u \in C^\infty \cap L^p(\mathcal{V}) : \nabla^i u \in C^\infty \cap L^p((\otimes_{j=1}^i T^* \mathcal{M}) \otimes \mathcal{V}), \ i = 1, \ldots, k \},$$

and define $W^{k,p}(\mathcal{V})$ as the closure of $S^p_k(\mathcal{V})$ under the norm

$$\|u\|_{W^{k,p}} = \|u\| + \sum_{i=1}^k \|\nabla^i u\|.$$  

Furthermore, let $W^{k,p}_0(\mathcal{V})$ denote the closure of $C^\infty_c(\mathcal{V})$ under the same norm. We define $W^{k,p}(\mathcal{V})$ to be sections $\xi \in W^{k,p}_0(\mathcal{V})$ with support $\text{spt} \xi$ compact. The local Sobolev spaces $W^{k,p}_{\text{loc}}(\mathcal{V})$ consist of sections $\xi \in \Gamma(\mathcal{V})$ such that $\xi \in W^{k,p}(K, \mathcal{V})$ for every open $K \subset \mathcal{M}$ with $\overline{K}$ compact. From here on, unless otherwise stated, we will be solely concerned with the case $p = 2$. Hence, we denote $S^2_k(\mathcal{V})$ by $S_k(\mathcal{V})$.  

We also note the following characterisation:

\[ W^{1,2}(\mathcal{V}) = \{ u \in L^2(\mathcal{V}) : \nabla u \in L^2(T^*\mathcal{M} \otimes \mathcal{V}) \} \, . \]

For this, we cite Theorem 2 (i) in [14], which is a version of (W) for functions. We note that the proof generalises with only superficial modifications to the vector bundle setting as it relies purely upon the properties of Friedrich mollification.

Define \( \nabla_c : \mathcal{C}_c^\infty(\mathcal{V}) \rightarrow \mathcal{C}_c^\infty(T^*\mathcal{M} \otimes \mathcal{V}) \) by \( \nabla_c = \nabla \) with domain \( \mathcal{D}(\nabla_c) = \mathcal{C}_c^\infty(\mathcal{V}) \) and \( \nabla_2 = \nabla \) with domain \( \mathcal{D}(\nabla_2) = S_1(\mathcal{V}) \).

In particular, the compatibility of \( \nabla \) with \( h \) induces the following adjoint formulae:

\[ \langle u, -\text{tr} \nabla_c v \rangle = \langle \nabla_2 u, v \rangle, \quad u \in S_1(\mathcal{V}), \ v \in \mathcal{C}_c^\infty(\mathcal{V}), \]
\[ \langle w, -\text{tr} \nabla_2 z \rangle = \langle \nabla_c w, z \rangle, \quad w \in \mathcal{C}_c^\infty(\mathcal{V}), \ z \in S_1(\mathcal{V}). \]

Since \( \mathcal{C}_c^\infty(\mathcal{V}) \) is dense in \( L^2(\mathcal{V}) \), we obtain by standard theory (i.e. Theorem III.5.28 in [11] by Kato) that the operators \( \nabla_c \), \( \nabla_2 \), \( -\text{tr} \nabla_c \) and \( -\text{tr} \nabla_2 \) are densely-defined and closable. Hence, we define the following operators:

\[
\nabla_D := \nabla_c, \quad \nabla_N := \nabla_2, \\
\text{div}_D := -\nabla_c^*, \quad \text{div}_N := -\nabla_2^*, \\
\text{div}_{D,-} := \text{tr} \nabla_2, \quad \text{div}_{N,-} := \text{tr} \nabla_c. 
\]

First, observe the following.

**Proposition 2.1.** The following operator inclusions hold: \( \nabla_D \subset \nabla_N \), \( \text{div}_N \subset \text{div}_D \), \( \text{div}_{D,-} \subset \text{div}_D \), \( \text{div}_{N,-} \subset \text{div}_N \).

Moreover, we obtain a characterisation of the Sobolev spaces in terms of \( \nabla_D \) and \( \nabla_N \).

**Proposition 2.2.** The Sobolev space \( W^{1,2}(\mathcal{V}) = \mathcal{D}(\nabla_N) \) and the Sobolev space \( W^{1,2}_0(\mathcal{V}) = \mathcal{D}(\nabla_D) \).

We remark that the operators \( \text{div}_D \) and \( \text{div}_N \) can be obtained as closed, densely-defined operators even if the compatibility assumption on \( h \) and \( \nabla \) is dropped. This is a consequence of the well known fact that operators \( \nabla_c \) and \( \nabla_2 \) are always densely-defined and closable (c.f. Proposition 2.2 in [4]). In particular, this means that Proposition 2.2 is valid even in this more general context. However, the inclusions \( \text{tr} \nabla_c \subset \text{div}_N \) and \( \text{tr} \nabla_2 \subset \text{div}_D \) may no longer hold. In particular, we cannot assert that \( \text{div}_D \) and \( \text{div}_N \) are differential operators.

Dropping the compatibility requirement becomes crucial when attempting to study Sobolev spaces in the setting of low-regularity metrics, for the simple reason that the metric may not be differentiable. Some initial progress in this direction can be found in [3] for the special case of Sobolev spaces of functions under so-called “rough metrics.” These considerations are beyond the scope of this paper and we will always assume compatibility between the metric and connection unless otherwise stated.
Define the following two self-adjoint operators, which we respectively call the Dirichlet and Neumann Laplacian:

\[ \Delta_D := - \text{div}_D \nabla_D \] and \[ \Delta_N := - \text{div}_N \nabla_N. \]

On writing the energy associated to our Sobolev spaces, namely,
\[ E_D[u] = \int_M |\nabla_D u|^2 \, d\mu_g \] and \[ E_N[v] = \int_M |\nabla_N v|^2 \, d\mu_g, \]
for \( u \in W^{1,2}_0(V) \) and \( v \in W^{1,2}(V) \), it is immediate that \( W^{1,2}_0(V) = \mathcal{D}(\sqrt{\Delta_D}) \) and \( W^{1,2}(V) = \mathcal{D}(\sqrt{\Delta_N}) \). Note that when \( M = U \subset \mathbb{R}^n \), where \( U \) is the interior of a bounded Lipschitz domain and \( V = M \times \mathbb{C} \), the bundle of functions, then \( \Delta_D \) and \( \Delta_N \) denote the classical Dirichlet and Neumann Laplacians respectively. This justifies our notation and nomenclature in this more general setting.

We also consider the composition of operators \(- \text{tr} \nabla_2 \) and \( \nabla_2^* \):

\[ \Delta_s := (- \text{tr} \nabla_2) \nabla_2^*, \]
with the induced domain \( \mathcal{D}(\Delta)_s = \{ u \in S_1(V) : \nabla u \in S_1(T^* \otimes V) \} \). We call this operator the Bochner Laplacian. Let \( \Delta_{D,s} \) and \( \Delta_{N,s} \) be the restrictions of \( \Delta_D \) and \( \Delta_N \) to \( \mathcal{D}(\Delta_D) \cap C^\infty(V) \) and \( \mathcal{D}(\Delta_N) \cap C^\infty(V) \) respectively. The following operator relations easily follow from definitions and Proposition 2.1:

\[(L) \quad \Delta_{D,s} \subset \Delta_s \quad \text{and} \quad \Delta_{N,s} \subset \Delta_s. \]

Furthermore, a fundamental object of our study will be the following (not necessarily self-adjoint) operator:

\[ \Delta_G := - \text{div}_D \nabla_N. \]

In the context of scalar-valued functions, \( \Delta_G \) appears in [10] by Grigor’yan and Masamune under the name Gaffney Laplacian. To maintain consistency with the literature, we will retain this nomenclature in our more general setting.

### 3. General results

Let us consider an additional \( L^2 \)-differential operator \( \nabla_L := (- \text{div}_N_-)^* \). Recall the containment \( \text{div}_N_- \subset \text{div}_N \subset \text{div}_D \) from Proposition 2.1. On taking adjoints we obtain that \( \nabla_D \subset \nabla_N \subset \nabla_L \). We will show that \( \nabla_L = \nabla_N \), but first, we present the following abstract lemma inspired by this endeavour.

**Lemma 3.1.** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces, with inner products \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) respectively. Let \( \mathcal{R}_s, \mathcal{R} \subset \mathcal{H}_2 \), be dense subspaces satisfying \( \mathcal{R}_s \subset \mathcal{R} \). Suppose that:

(i) \( \mathcal{R} \) is equipped with a norm \( \| \cdot \|_\mathcal{R} \) satisfying \( \| u \|_\mathcal{R} \leq \| u \|_\mathcal{R}_s \) for all \( u \in \mathcal{R} \),
(ii) the inner product \( \langle \cdot, \cdot \rangle_2 \) extends continuously from \( \mathcal{R}_s \) to the pairing (separably continuous bilinear map)

\[ \langle \cdot, \cdot \rangle : \mathcal{R} \times \mathcal{R}' \rightarrow \mathbb{C}, \]

where \( \mathcal{R}' \) is the dual space of \( \mathcal{R} \),
(iii) \( T : \mathcal{D}(T) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_2 \), densely-defined,
(iv) $\tilde{T} : \mathcal{H}_1 \rightarrow \mathcal{H}$ is a continuous map (with respect to the norm on $\mathcal{H}$), and $T|_{D(T)} = T$.

(v) $S : \mathcal{D}(S) \subset \mathcal{H}_2 \rightarrow \mathcal{H}_1$, is densely-defined and $\mathcal{R}_s \subset \mathcal{D}(S) \subset \mathcal{R}$,

(vi) $\langle u, Sv \rangle_1 = \langle Tu, v \rangle$ for all $u \in \mathcal{H}_1$ and $v \in \mathcal{D}(S)$.

Then, $\mathcal{D}(S^*) = \{u \in \mathcal{H}_1 : \tilde{T}u \in \mathcal{H}_2\}$.

**Proof.** Let $\mathcal{D} = \{u \in \mathcal{H}_1 : \tilde{T}u \in \mathcal{H}_2\}$. It is trivial from the fact that $S$ and $T$ are adjoint to each other that $\mathcal{D} \subset \mathcal{D}(S^*)$.

To prove the converse, first recall that $\mathcal{D}(S^*)$ can be characterised as the set of $u \in \mathcal{H}_1$ for which $v \mapsto \langle u, Sv \rangle_1$ is continuous. Here, continuity is measured with respect to the topology induced by $\| \cdot \|_2$ on $\mathcal{H}_2$.

Fix $u \in \mathcal{D}(S^*)$. Then, $\langle u, Sv \rangle_1 = \langle \tilde{T}u, v \rangle$ and $v \mapsto \langle \tilde{T}u, v \rangle$ is continuous in $\mathcal{H}_2$. Letting $f_u$ be this map, continuity is equivalent to $\| f_u(v) \| = \| \langle \tilde{T}u, v \rangle \| \leq C \| v \|_2$ for some $C > 0$ and whenever $v \in \mathcal{D}(S)$. Since $\mathcal{D}(S)$ is dense in $\mathcal{H}_2$, we have that $f_u(v)$ extends to the whole of $\mathcal{H}_2$ as a bounded map. That is, $f_u \in \mathcal{H}'_2$ and hence, by the Riesz representation theorem, there exists $z \in \mathcal{H}_2$ such that $f_u(v) = \langle z, v \rangle_2$ for all $v \in \mathcal{H}_2$. Since $\mathcal{H}_2 \subset \mathcal{R}'$, we have that $z \in \mathcal{R}'$ and since $\langle \cdot, \cdot \rangle$ extends $\langle \cdot, \cdot \rangle_2$ from $\mathcal{R}_s$ via continuity, we obtain that $\langle \tilde{T}u, v \rangle = \langle z, v \rangle_2$ for all $v \in \mathcal{R}$. Thus, $\tilde{T}u = z \in \mathcal{H}_2$. This proves $u \in \mathcal{D}$.

We remark that the formulation of this lemma is to provide a tool to compute the domain of an operator when there are distributional tools in hand. In application, we will see that $\mathcal{R}'$ represents the space of distributional sections, $\mathcal{R}$ an appropriate Sobolev space, and $\mathcal{R}_s$, the space of compactly supported smooth sections. We have phrased this lemma in this generality in the hope that it will be useful beyond the scope of our immediate applications in this paper.

Let $\mathcal{E}'(\mathcal{V})$ be the dual space of $\mathcal{C}^\infty(\mathcal{V})$ and $\mathcal{D}'(\mathcal{V})$ the dual space of $\mathcal{C}_c^\infty(\mathcal{V})$. Define $W_{\text{comp}}^{1,2}(\mathcal{V}) := W_{\text{loc}}^{1,2}(\mathcal{V}) \cap \mathcal{E}'(\mathcal{V})$. The choice of the notation “comp” in the definition is because $W_{\text{comp}}^{1,2}(\mathcal{V})$ is exactly the $W^{1,2}(\mathcal{V})$ sections with compact support, which is a consequence of the fact that $\mathcal{E}'(\mathcal{V})$ is the space of compactly supported distributional sections; see Exercise 2.3.5 in [17] by van den Ban and Crainic.

**Lemma 3.2.** The $L^2(\mathcal{V})$ inner product $\langle \cdot, \cdot \rangle$ extends continuously to a pairing

$$\langle \cdot, \cdot \rangle : W_{\text{comp}}^{1,2}(\mathcal{V}) \times W^{-1,2}_{\text{loc}}(\mathcal{V}) \rightarrow \mathbb{C}$$

from $C_c^\infty(\mathcal{V})$ by continuity. Furthermore, the equality

$$(\nabla u, v) = \langle u, - \text{tr} \nabla v \rangle$$

holds if one of $\text{spt} u$ or $\text{spt} v$ is compact, and either

(i) $u \in L^2_{\text{loc}}(\mathcal{V})$ and $v \in W^{1,2}_{\text{loc}}(T^*\mathcal{M} \otimes \mathcal{V})$, or

(ii) $u \in W^{1,2}_{\text{loc}}(\mathcal{V})$ and $v \in L^2_{\text{loc}}(T^*\mathcal{M} \otimes \mathcal{V})$. 
For the statement about the pairing in Lemma 3.2, see Lemma 9.2.9 in [17]. For the equality (P), see Lemma 8.8 in the paper [5] by Braverman, the second author and Shubin. A good reference for similar results is §7 (particularly §7.7) and Theorem 7.7 in [16] by Shubin.

**Theorem 3.3.** The following operator equalities hold:

(i) \( \nabla_L = \nabla_N \)

(ii) \( \text{div}_N = \text{div}_{N,-} \)

(iii) \( \text{div}_D = \text{div}_{D,-} \)

**Proof.** Recall that \( \nabla_L = (-\text{tr} \nabla_c)^* = (-\text{tr} \nabla_s)^* \). First, we use Lemma 3.2 and Lemma 3.1 and show that \( \mathcal{D}(\nabla_L) = \{ u \in L^2(\mathcal{V}) : \nabla u \in L^2(T^*\mathcal{M} \otimes \mathcal{V}) \} \). To that end, let \( \mathcal{H}_1 = L^2(\mathcal{V}), \mathcal{H}_2 = L^2(T^*\mathcal{M} \otimes \mathcal{V}), S = -\text{tr} \nabla_c \) with \( \mathcal{R}_s = \mathcal{D}(S) = C^\infty_c(T^*\mathcal{M} \otimes \mathcal{V}) \) and \( T = \nabla_2 \). Now, let \( \mathcal{R} = W^{1,2}_\text{comp}(T^*\mathcal{M} \otimes \mathcal{V}) \) so that \( \mathcal{R}' = W^{-1,2}_\text{comp}(T^*\mathcal{M} \otimes \mathcal{V}) \) since \( W^{1,2}_\text{comp} \) is the dual of \( W^{-1,2}_\text{comp} \) (see Lemma 9.2.9 in [17]). On noting that \( \nabla : L^2(\mathcal{V}) \to W^{-1,2}_\text{loc}(T^*\mathcal{M} \otimes \mathcal{V}) \), we define \( \tilde{T} = \nabla \), with the continuity given by pseudo-differential theory. Also, \( \mathcal{D}(S) = C^\infty_c(T^*\mathcal{M} \otimes \mathcal{V}) \subset W^{1,2}_\text{comp}(T^*\mathcal{M} \otimes \mathcal{V}) = \mathcal{R} \) and \( \mathcal{R} = W^{1,2}_\text{comp}(T^*\mathcal{M} \otimes \mathcal{V}) \subset L^2(T^*\mathcal{M} \otimes \mathcal{V}) = \mathcal{H}_2 \). Thus, we have shown that the hypotheses (i), (ii), (iii), (iv) and (v) of Lemma 3.1 are satisfied. Furthermore, hypothesis (ii) of Lemma 3.1 is satisfied by Lemma 3.2. Finally, the fulfilment of hypothesis (vi) of Lemma 3.1 follows from (P) with \( u \in L^2(\mathcal{V}) \) and \( v \in C^\infty_c(T^*\mathcal{M} \otimes \mathcal{V}) \). Thus, we conclude that \( \mathcal{D}(\nabla_L) = \{ u \in L^2(\mathcal{V}) : \nabla u \in L^2(T^*\mathcal{M} \otimes \mathcal{V}) \} \), and the equality \( \mathcal{D}(\nabla_L) = W^{1,2}(\mathcal{V}) \) follows from (W). This proves property (i).

Property (i) implies \( \nabla_L^* = \nabla_N^* \), which immediately gives property (ii). For property (iii), we use the L^2-space notations \( \mathcal{H}_1 = L^2(T^*\mathcal{M} \otimes \mathcal{V}), \mathcal{H}_2 = L^2(\mathcal{V}), \) operators \( S = -\nabla_c \) and \( T = \text{tr} \nabla_2 \), Sobolev space \( \mathcal{R} = W^{1,2}_\text{comp}(\mathcal{V}) \), and the set \( \mathcal{R}_s = \mathcal{D}(S) = C^\infty_c(\mathcal{V}) \). With these notations, property (iii) follows from Lemma 3.1 and Lemma 3.2 by using the same arguments as in the proof of property (i). \( \square \)

The following proposition is a vector bundle analogue of Lemma 3 from [14].

**Proposition 3.4.** The following equalities hold:

\[ \Delta_{D,s} = \Delta_D \quad \text{and} \quad \Delta_{N,s} = \Delta_N. \]

**Proof.** As the same kind of proof applies to both equalities, we will only prove the first one. We recall (L) and apply the heat-equation method of Masamune. Since \( -\Delta_D \) is a non-positive self-adjoint operator, it generates a strongly continuous contraction semigroup \( (e^{-t\Delta_D})_{t\geq 0} \) on \( L^2(\mathcal{V}) \). Let \( u \in \mathcal{D}(\Delta_D) \) be arbitrary, and consider the family \( u_t := e^{-t\Delta_D} u \). By computing \( u_t \) via the functional calculus for sectorial operators, we can easily see that \( e^{-t\Delta_D} \in \mathcal{D}(\Delta_D) \) (see §D in [1]). Furthermore, since \( \Delta_D \) is an elliptic operator, using elliptic regularity (see Corollary 7.1(b) and Corollary 7.4 in [16]) we have \( u_t \in C^\infty(\mathcal{V}) \), and, therefore, \( u_t \in \mathcal{D}(\Delta_{D,s}) \). Moreover, we have

\[ u_t \to u, \quad \text{as } t \to 0+, \]

for the statement about the pairing in Lemma 3.2, see Lemma 9.2.9 in [17]. For the equality (P), see Lemma 8.8 in the paper [5] by Braverman, the second author and Shubin. A good reference for similar results is §7 (particularly §7.7) and Theorem 7.7 in [16] by Shubin.
and since $e^{-t\Delta_D}$ commutes with $\Delta_D$ (the functional calculus commutes with the operator on its domain), we have
\[
\Delta_{D,s} u_t = \Delta_D u_t = e^{-t\Delta_D} \Delta_D u \to \Delta_D u, \quad \text{as } t \to 0+,
\]
where both convergence relations are understood in the $L^2$-sense. This proves the equality $\Delta_{D,s} = \Delta_D$. □

The next proposition is a vector bundle analogue of Lemma 3.6 (i) and (ii) from [10].

**Proposition 3.5.** The following are equivalent:

(i) $W^{1,2}_0(V) = W^{1,2}(V)$,

(ii) $\Delta_D = \Delta_N$,

(iii) $\Delta_G$ is self-adjoint.

**Proof.** The equivalence of (i) and (ii) is easy. If $\Delta_D = \Delta_N$, then $\nabla_N = \nabla_D$. Hence, $-\text{div}_D = \nabla_N^*$, and, therefore, $\Delta_G = \Delta_N$. Thus, $\Delta_G$ is self-adjoint. As for the remaining implication, note that $\Delta_D \subset \Delta_G$ and $\Delta_N \subset \Delta_G$. Taking adjoints and using the self-adjointness of $\Delta_D$, $\Delta_N$, and $\Delta_G$, we get the equality (i). □

**Remark 3.6.** Note that this proposition holds even if metric compatibility between $\nabla$ and $h$ is dropped, upon defining $\text{div}_D$ and $\text{div}_N$ abstractly as adjoints of $-\nabla_D$ and $-\nabla_N$ respectively.

4. **Polar Boundary**

Let $\overline{M}$ be the metric completion of $M$ with respect to the Riemannian distance. We define the Cauchy boundary of $M$ as
\[
\partial_C M := \overline{M} \setminus M.
\]

Following [10], we now give the definition of 1-capacity on $M$. Let $\mathcal{O}$ be the collection of all open sets of $\overline{M}$. For a set $\Omega \in \mathcal{O}$, we define
\[
\mathcal{L}(\Omega) := \{ u \in W^{1,2}(M) : 0 \leq u \leq 1 \text{ and } u|_{\Omega \cap M} = 1 \}.
\]
We define the 1-capacity of $\Omega \in \mathcal{O}$ as follows:
\[
\text{Cap}(\Omega) := \inf_{u \in \mathcal{L}(\Omega)} \int_M (|u|^2 + |du|^2) \, \mu_\gamma \quad \text{if } \mathcal{L}(\Omega) \neq \emptyset.
\]
Furthermore, define $\text{Cap}(\emptyset) = \infty$ if $\mathcal{L}(\emptyset) = \emptyset$ and $\text{Cap}(\emptyset) = 0$. For an arbitrary set $\Sigma \subset M$, define
\[
\text{Cap}(\Sigma) := \inf_{\Omega \in \mathcal{L}(\Omega)} \text{Cap}(\Omega).
\]
We call $\Sigma$ polar if $\text{Cap}(\Sigma) = 0$. If $\Sigma = \emptyset$, we set $\text{Cap}(\Sigma) = 0$.

We prove that polarity of the Cauchy boundary is a sufficient condition to establish $W^{1,2}_0(V) = W^{1,2}(V)$. But first, we prove the following approximation lemma which is noteworthy in its own right.
Lemma 4.1. The set $L^\infty \cap W^{1,2}(\mathcal{V})$ is dense in $W^{1,2}(\mathcal{V})$.

Proof. We use the truncation procedure of Lemma 2 in [12] by Leinfelder and Simader. By definition of $W^{1,2}(\mathcal{V})$, it is enough to show that $L^\infty(\mathcal{V}) \cap W^{1,2}(\mathcal{V})$ is dense in $S^1(\mathcal{V})$ with respect to $W^{1,2}$-norm. By Lemma 1.16 in [8] by Eichhorn, for all $v \in C^\infty(\mathcal{V})$ we have the following diamagnetic inequality

$$|d|v|_h \leq |\nabla v|, \quad \mu_{\gamma}\text{-a.e. } x \in \mathcal{M},$$

where $|\cdot|_h$ is the norm with respect to the metric $h$ of $\mathcal{V}$. In particular, (†) holds for all $u \in S^1(\mathcal{V})$. Now using (†) and (W) together, we conclude that $|u|_h \in W^{1,2}(\mathcal{M})$. For simplicity we will suppress $h$ in $|u|_h$ for the remainder of the proof.

For $R > 0$ define the following family of Lipschitz functions:

$$\psi_R(t) = \begin{cases} 
1, & \text{if } t \leq R; \\
\frac{R}{t}, & \text{if } t > R.
\end{cases}$$

Note that $\psi'_R(t) = 0$ if $t < R$, $0 \leq \psi_R(t) \leq 1$, $|t\psi_R(t)| \leq R$ and $|t\psi'_R(t)| \leq 1$.

We now apply Theorem A in [12] to the composition $\psi_R \circ |u|$. We remark that Theorem A in [12] was proven for the composition $f \circ w$, where $f : \mathbb{R}^k \to \mathbb{R}$ is a Lipschitz function of class $C^1(\mathbb{R}^k \setminus \Gamma)$ with a closed countable set $\Gamma \subset \mathbb{R}^k$, and where $w : \Omega \to \mathbb{R}^k$ belongs to the Sobolev space $W^{1,2}(\Omega, \mathbb{C})$ with $\Omega$ an open set in $\mathbb{R}^m$. However, the corresponding arguments can be carried over without change to the case of functions $w$ defined on a Riemannian manifold. Hence, we obtain

$$d(\psi_R \circ |u|) = \psi'_R(|u|)d|u| \quad \mu_{\gamma}\text{-a.e. } x \in \mathcal{M}.$$  \hfill (1)

As in the proof of Lemma 2 in [12] we set

$$u_R := (\psi_R \circ |u|)u.$$  

Clearly, $u_R \in L^\infty(\mathcal{V}) \cap W^{1,2}(\mathcal{V})$. Using Leibniz rule and (1), we have

$$\nabla u_R = (\nabla u)\psi_R(|u|) + u \otimes (\psi'_R(|u|)d|u|),$$

in the sense of distributional sections of $\mathcal{V}$.

Let $\chi_G$ denote the characteristic function of a set $G$. From the properties of $\psi_R$ it follows that

$$|\nabla u_R - \nabla u| \leq (|\nabla u| + |d|u|)|\chi_{|u| \geq R}|$$

and

$$|u_R - u| \leq |u|\chi_{|u| \geq R}.$$  

Therefore, as $R \to \infty$, we have $\|u_R - u\|_{W^{1,2}} \to 0$. \hfill $\square$

Remark 4.2. We note that the diamagnetic inequality (†) as proved by Eichhorn in [8] assumes the compatibility of $h$ and $\nabla$. It would be interesting to know whether this inequality still holds without this assumption.

The following is then a vector-bundle analogue of Lemma 2.2 (a) in [10].

Theorem 4.3. If $\partial C M$ is polar, then $W^{1,2}_0(\mathcal{V}) = W^{1,2}(\mathcal{V})$. 
Proof. The proof mimics that of Lemma 2.2(a) in [10]. For an arbitrary \( u \in W^{1,2}_0(V) \) we construct a sequence \( u_j \in W^{1,2}_0(V) \) such that \( \|u - u_j\|_{W^{1,2}} \to 0 \) as \( j \to \infty \). By Lemma 4.1, we assume that \( u \in L^\infty(V) \cap W^{1,2}(V) \).

Since \( \text{Cap}(\partial C, M) = 0 \) there exists a sequence of open sets \( \Omega_k \subset \overline{M} \), \( k \geq 1 \), such that \( \partial C, M \subset \Omega_{k+1} \subset \Omega_k \) and \( \text{Cap}(\Omega_k) \to 0 \) as \( k \to \infty \). For each \( k \geq 1 \), let \( \{\varphi_j^{(k)}\}_{j \geq 1} \) be a sequence of functions with the following properties: \( \varphi_j^{(k)} \in \mathcal{L}(\Omega_k) \) and \( \|\varphi_j^{(k)}\|_{W^{1,2}} \to \text{Cap}(\Omega_k) \), as \( j \to \infty \). Define

\[
\varphi_j := \varphi_j^{(j)} \quad \text{and} \quad u_j := (1 - \varphi_j)u.
\]

Fix a point \( x_0 \in M \) and define

\[
\sigma_r(x) := 1 \wedge (r^{-1}(2r - d(x, x_0)))_+ \quad r > 0,
\]

where \( d(\cdot, \cdot) \) is the distance with respect to the metric on \( M \), \( f_+ \) is the positive part of a function \( f \), and \( a \wedge b := \min\{a, b\} \).

Clearly, \( \sigma_r \in W^{1,2}(M) \) for all \( r > 0 \), \( \sigma_r \to 1 \), and \( d\sigma_r \to 0 \) as \( r \to \infty \) in the \( \mu_g \)-a.e. sense. We define \( v_{r,j} := \sigma_r u_j \) and observe that \( v_{r,j} \in W^{1,2}_0(V) \) and \( \|v_{r,j} - u_j\|_{W^{1,2}} \to 0 \) as \( r \to \infty \). Thus, we may assume (without loss of generality) that \( u_j \in W^{1,2}_0(V) \).

Since \( (1 - \varphi_j) \to 1 \) \( \mu_g \)-a.e. as \( j \to \infty \), it follows that

\[
\|u - u_j\|_2 \to 0, \quad \text{as} \quad j \to \infty.
\]

Finally, using the properties \( \varphi_j \to 0 \) \( \mu_g \)-a.e., \( \nabla u \in L^2(T^*M \otimes V) \), \( u \in L^\infty(V) \) and \( \|d\varphi_j\|_2 \to 0 \), we obtain

\[
\|\nabla u_j\|_2 = \|(1 - \varphi_j)\nabla u - d\varphi_j \otimes u\|_2 \to \|\nabla u\|_2,
\]

as \( j \to \infty \). \( \square \)

The following corollary then follows directly from Proposition 3.5 and Theorem 4.3.

**Corollary 4.4.** If \( \partial C, M \) is polar, then \( \Delta_G \) is self-adjoint.

5. **Negligible Boundary**

Let \( (M, V, \nabla) \) be as in §2 and recall the set

\[
S_1(V) = \{ u \in C^\infty \cap L^2(V) : \nabla u \in C^\infty \cap L^2(T^*M \otimes V) \}.
\]

We say that \( (M, V, \nabla) \) has negligible boundary if

\[
\langle \nabla_2 u, v \rangle = \langle u, -\text{tr} \nabla_2 v \rangle, \quad \text{for all} \quad u \in S_1(V), \quad v \in S_1(T^*M \otimes V).
\]

This definition is analogous to the one used by Masamune in [13, 14] in the study of the self-adjointness of the Laplacian acting on functions. The term “negligible boundary” goes back to the work of Gaffney in [9]. In this section, we will illustrate the link between this geometric condition (NB) and the equality \( W^{1,2}_0(V) = W^{1,2}(V) \).
First, we consider the relationship of (NB) to operators that we have introduced previously. The following is the link between negligible boundary and the Gaffney Laplacian.

**Theorem 5.1.** The operator $\Delta_G$ is self-adjoint if and only if $(\mathcal{M}, \mathcal{V}, \nabla)$ has negligible boundary.

**Proof.** If $\Delta_G$ is self-adjoint, by Proposition 3.5 we get $W^{1,2}_0(\mathcal{V}) = W^{1,2}(\mathcal{V})$. Now Proposition 2.2 implies $\nabla_D = \nabla_N$. To verify the property (NB), we first approximate $u \in S_1(\mathcal{V})$ by a sequence $u_j \in C_c^\infty(\mathcal{V})$ in $W^{1,2}$-norm. Next, we recall the property (D):

$$\langle \nabla_c u_j, v \rangle = \langle u_j, -\text{tr} \nabla_2 v \rangle, \quad \text{for all } v \in S_1(\mathcal{T}^*\mathcal{M} \otimes \mathcal{V}).$$

Finally, we take the limit as $j \to \infty$ on both sides, and this shows (NB).

Now assume that $(\mathcal{M}, \mathcal{V}, \nabla)$ has negligible boundary. Our goal is to show that $-\text{div}_D = \nabla_2^*$. From (NB) it follows that $-\text{tr} \nabla_2 \subset \nabla_2^*$, which, after taking closures, leads to $-\text{tr} \nabla_2 \subset \nabla_2^*$. By Theorem 3.3 (iii), we can rewrite the last inclusion as $-\text{div}_D \subset \nabla_2^*$. Additionally, by Proposition 2.1 we have $-\text{div}_N \subset -\text{div}_D$, that is, $\nabla_2^* \subset -\text{div}_D$. Thus, we have shown that $-\text{div}_D = \nabla_2^*$. Noting that $\nabla_2^* = \nabla_N^*$, we have

$$\Delta_G = \Delta_G = -\text{div}_D \nabla_N = \nabla_2^* \nabla_N = \nabla_N^* \nabla_N.$$

Now the self-adjointness of $\Delta_G$ follows by von Neumann’s Theorem; see Theorem V.3.24 in [11].

The following theorem then links (NB) to the Bochner Laplacian $\Delta_s$.

**Theorem 5.2.** The operator $\Delta_s$ is essentially self-adjoint if and only if $(\mathcal{M}, \mathcal{V}, \nabla)$ has negligible boundary.

**Proof.** Assume that $(\mathcal{M}, \mathcal{V}, \nabla)$ has negligible boundary. We will first show that $\Delta_s$ is symmetric. For $w, z \in \mathcal{D}(\Delta_s)$, we have

$$\langle -\text{tr} \nabla_2(\nabla_2 w), z \rangle = \langle \nabla_2 w, \nabla_2 z \rangle = \langle w, -\text{tr} \nabla_2(\nabla_2 z) \rangle,$$

where in the first equality we used (NB) with $u = z$ and $v = \nabla_2 w$, and in the second equality we used (NB) with $u = w$ and $v = \nabla_2 z$. This shows that $\Delta_s$ is symmetric. We now show that $\overline{\Delta_s} = \Delta_G$. Taking closures in (L) and using Proposition 3.4 we obtain

$$\Delta_D \subset \overline{\Delta_s} \quad \text{and} \quad \Delta_N \subset \overline{\Delta_s}.$$

Since $\Delta_s$ is symmetric, so is $\overline{\Delta_s}$. Since $\Delta_D$ and $\Delta_N$ are self-adjoint and $\overline{\Delta_s}$ is symmetric, from (2) we get $\Delta_D = \overline{\Delta_s} = \Delta_N$. Therefore, $\overline{\Delta_s}$ self-adjoint, that is, $\Delta_s$ is essentially self-adjoint.

Now assume that $\Delta_s$ is essentially self-adjoint. Taking closures in (L) and using Proposition 3.4 we obtain (2), which leads to $\Delta_D = \overline{\Delta_s} = \Delta_N$, that is, $W^{1,2}_0(\mathcal{V}) = W^{1,2}(\mathcal{V})$. Now by Theorem 5.1 and Proposition 3.5, it follows that $(\mathcal{M}, \mathcal{V}, \nabla)$ has negligible boundary. 

$\square$
To summarise, we present the following list of equivalences. We note that this easily follows from Theorem 5.1, 5.2 and Proposition 3.5.

**Corollary 5.3.** The following equivalences hold:

(i) the triplet \((\mathcal{M}, \mathcal{V}, \nabla)\) has negligible boundary,
(ii) \(W^{1,2}_0(\mathcal{V}) = W^{1,2}(\mathcal{V})\),
(iii) the operator \(\Delta_G\) is self-adjoint,
(iv) the operator \(\Delta_s\) is essentially self-adjoint.

We conclude this paper with the following natural questions that have risen out of our analysis.

**Question 1.** Does there exist a manifold \((\mathcal{M}, g)\) and two vector bundles \((\mathcal{V}_1, h_1, \nabla_1)\) and \((\mathcal{V}_2, h_2, \nabla_2)\) with \(h_i\) and \(\nabla_i\) compatible \((i = 1, 2)\), so that \(W^{1,2}_0(\mathcal{V}_1) = W^{1,2}(\mathcal{V}_1)\) but \(W^{1,2}_0(\mathcal{V}_2) \neq W^{1,2}(\mathcal{V}_2)\)?

**Question 2.** Does there exist a manifold \((\mathcal{M}, g)\) and a vector bundle \((\mathcal{V}, h, \nabla)\) with \(\nabla\) and \(h\) compatible such that either: \(W^{1,2}_0(\mathcal{M}) = W^{1,2}(\mathcal{M})\) and \(W^{1,2}_0(\mathcal{V}) \neq W^{1,2}(\mathcal{V})\), or, \(W^{1,2}_0(\mathcal{V}) = W^{1,2}(\mathcal{V})\) and \(W^{1,2}_0(\mathcal{M}) \neq W^{1,2}(\mathcal{M})\)?

We remark that the answers to both questions are negative if \((\mathcal{M}, g)\) is a geodesically complete manifold; see, for instance [2] or [8]. Thus, it is necessary that these questions be considered only in the case that \((\mathcal{M}, g)\) is geodesically incomplete. In this situation, we do not expect the negligible boundary property for one vector bundle to necessarily follow from another. Hence, by Corollary 5.3, we at least expect Question 1 to have an affirmative answer.

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