HAUSDORFF DIMENSION OF UNIONS OF AFFINE SUBSPACES AND OF FURSTENBERG-TYPE SETS

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Abstract. We prove that for any $1 \leq k < n$ and $s \leq 1$, the union of any nonempty $s$-Hausdorff dimensional family of $k$-dimensional affine subspaces of $\mathbb{R}^n$ has Hausdorff dimension $k + s$. More generally, we show that for any $0 < \alpha \leq k$, if $B \subseteq \mathbb{R}^n$ and $E$ is a nonempty collection of $k$-dimensional affine subspaces of $\mathbb{R}^n$ such that every $P \in E$ intersects $B$ in a set of Hausdorff dimension at least $\alpha$, then $\dim B \geq 2\alpha - k + \min(\dim E, 1)$, where $\dim$ denotes the Hausdorff dimension. As a consequence, we generalize the well-known Furstenberg-type estimate that every $\alpha$-Furstenberg set has Hausdorff dimension at least $2\alpha$; we strengthen a theorem of Falconer and Mattila [5]; and we show that for any $0 \leq k < n$, if a set $A \subseteq \mathbb{R}^n$ contains the $k$-skeleton of a rotated unit cube around every point of $\mathbb{R}^n$, or if $A$ contains a $k$-dimensional affine subspace at a fixed positive distance from every point of $\mathbb{R}^n$, then the Hausdorff dimension of $A$ is at least $k + 1$.

1. Introduction

There are several problems gathering around the general principle that an $s$-dimensional collection of $d$-dimensional sets in $\mathbb{R}^n$ must have positive measure if $s + d > n$ and Hausdorff dimension $s + d$ if $s + d \leq n$, unless the sets have large intersections. For example, Wolff [18, 19] proved that if a planar set $B$ contains a circle around every point of a Borel set $S \subset \mathbb{R}^2$ of Hausdorff dimension $s$ then $B$ has positive Lebesgue measure provided $s > 1$, and the Hausdorff dimension of $B$ is at least $s + 1$ when $s \leq 1$. Most of these problems are only partially solved. The most famous example is the Kakeya conjecture, which states that every Besicovitch set (a compact set that contains a unit line segment in every direction) in $\mathbb{R}^n$ has Hausdorff dimension $n$, see e.g [12]. Note that the directions of lines of $\mathbb{R}^n$ form a set of dimension $n - 1$, so the line segments of a Besicovitch set form a collection of Hausdorff dimension at least $n - 1$, so the above principle would indeed imply the Kakeya conjecture. On the other hand, the following trivial example shows that this principle cannot be applied for every $s$-dimensional collection of lines: for any collection of lines of a fixed plane of $\mathbb{R}^3$ the union clearly has Hausdorff dimension at most 2, which is less than $s + 1$ if $s > 1$. In this paper we show that the above principle holds for any $s$-dimensional collection of lines or even $k$-dimensional affine subspaces provided that $s \leq 1$.

Theorem 1.1. For any integers $1 \leq k < n$ and $s \in [0, 1]$ the union of any nonempty $s$-Hausdorff-dimensional family of $k$-dimensional affine subspaces of $\mathbb{R}^n$ has Hausdorff dimension $s + k$.

For the special case $k = n - 1$ this was proved by Oberlin [14] for compact (or analytic) families of hyperplanes. He also proved [15] that for any integers $1 \leq k < n$ and any $s \geq 0$, the union of any nonempty compact (or analytic) $s$-Hausdorff-dimensional family of $k$-dimensional affine subspaces of $\mathbb{R}^n$ has Hausdorff dimension

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at least \( \min\{n, 2k - k(n - k) + s\} \). Moreover, he proved that \( s > (k + 1)(n - k) - k \) implies positive Lebesgue measure for such unions, and the bound for \( s \) is sharp. His results are in harmony with the above heuristic principle in the case of hyperplanes.

Falconer and Mattila \cite{FM} proved a stronger statement both in the \( s \leq 1 \) and \( s > 1 \) cases for hyperplanes: instead of full \( n - 1 \)-dimensional affine subspaces it is enough to take a positive measure subset of each of them.

We can go even further for any \( k < n \): it is enough to take a \( k \)-Hausdorff dimensional subset of each \( k \)-dimensional subspace:

**Theorem 1.2.** Let \( 1 \leq k < n \) be integers and \( s \in [0,1] \). If \( E \) is a nonempty \( s \)-Hausdorff dimensional family of \( k \)-dimensional affine subspaces and \( B \) is a subset of \( \bigcup_{P \in E} P \) such that \( B \cap P \) has Hausdorff dimension \( k \) for every \( P \in E \) then

\[
(1) \quad \dim B = \dim \left( \bigcup_{P \in E} P \right) = s + k,
\]

where here and in the sequel \( \dim \) denotes Hausdorff dimension.

Note that this theorem does not assume any kind of measurability of \( E \) (or \( B \)), unlike the mentioned results of Oberlin and Falconer and Mattila.

As we explained above, the equality on the right hand side of (1) is not true without the restriction \( s \leq 1 \). But it is possible that the equality on the left hand side always holds. If it holds for \( k = 1 \) in \( \mathbb{R}^n \) for all \( n \geq 2 \), it would imply that Besicovitch sets in \( \mathbb{R}^n \) have Hausdorff dimension at least \( n - 1 \) and upper Minkowski dimension \( n \), see \cite{MolterRela}.

In Theorem 1.2 \( \dim B \leq \dim \left( \bigcup_{P \in E} P \right) \) is obvious, \( \dim \left( \bigcup_{P \in E} P \right) \leq s + k \) is easy (Lemma 2.4), the essence of the result is the estimate \( \dim B \geq s + k \).

It is natural to ask what happens if we go further and take the union of \( \alpha \)-Hausdorff dimensional subspaces of an \( s \)-dimensional family of \( k \)-dimensional affine subspaces of \( \mathbb{R}^n \) for some \( s \leq 1 \) and \( \alpha \in [0,k] \). Our most general result (Theorem 2.1) gives that in this case the union has Hausdorff dimension at least \( 2\alpha - k + s \).

Note that this result implies a known Furstenberg-type estimate. Let \( 0 < \alpha \leq 1 \) and suppose that \( F \subset \mathbb{R}^2 \) is a Furstenberg set: a compact set such that for every \( e \in S^1 \) there is a line \( L_e \) in direction \( e \) for which \( \dim L_e \cap F \geq \alpha \), see e.g. \cite{FK}. Since the sets \( L_e \) form an at least 1-dimensional collection of lines of \( \mathbb{R}^2 \) our above mentioned result gives \( \dim F \geq 2\alpha - 1 + 1 = 2\alpha \).

Molter and Rela \cite{MR} proved that if \( E \subset S^1 \) has Hausdorff dimension \( s \), \( F \subset \mathbb{R}^2 \) and for every \( e \in S^1 \) there is a line \( L_e \) in direction \( e \) for which \( \dim L_e \cap F \geq \alpha \) then \( \dim F \geq 2\alpha - 1 + s \) and \( \dim F \geq \alpha + \frac{1}{2} \). So our result is also a generalization of the first estimate of Molter and Rela.

Our original motivation comes from the following question: What is the minimal Hausdorff dimension of a set in \( \mathbb{R}^n \) that contains the \( k \)-skeleton of a rotated unit cube centered at every point of \( \mathbb{R}^n \)? In \cite{M} it is proved that for every \( 0 \leq k < n \) there exist such sets of Hausdorff dimension at most \( k + 1 \). As a fairly quick application of the above results we show (Theorem 2.6) that for every \( 0 \leq k < n \) such a set must have Hausdorff dimension at least \( k + 1 \), so \( k + 1 \) is the minimal Hausdorff dimension.

We remark that if we have \( k \)-skeletons of rotated and scaled cubes centered at every point then the minimal Hausdorff dimension is \( k \), see \cite{M}, and if we allow only scaled axis-parallel cubes then the minimal Hausdorff dimension is \( n - 1 \), see \cite{H2} for \( k = 1, n = 2 \) and \cite{H3} for the general case. We also show (Theorem 2.5) that if \( A \) contains a \( k \)-dimensional affine subspace at a fixed positive distance from every point of \( \mathbb{R}^n \), then the Hausdorff dimension of \( A \) is at least \( k + 1 \).

The paper is organized as follows: In Section 2 we state our most general result (Theorem 2.1), and prove its corollaries. In Section 3 we prove Theorem 2.1 subject
to a lemma (Lemma 2.7), which will be proved in Sections 4 and 5. In Section 4 we prove a purely geometrical lemma, which will be used during the $L^2$ estimation procedure in Section 5 to prove Lemma 2.7.

**Notation 1.3.** For any integers $1 \leq k < n$, let $A(n,k)$ denote the space of all $k$-dimensional affine subspaces of $\mathbb{R}^n$. For any $s \geq 0$, $\delta \in (0,\infty]$ and $A \subset \mathbb{R}^n$, the $s$-dimensional Hausdorff $\delta$-premeasure of $A$ will be denoted by $\mathcal{H}_s^\delta(A)$, the $s$-dimensional Hausdorff measure by $\mathcal{H}^s(A)$, and the Hausdorff dimension of $A$ by $\dim A$. The open ball of center $x$ and radius $r$ will be denoted by $B(x,r)$ or $B_\rho(x,r)$ if we want to indicate the metric $\rho$. For a set $U \subset \mathbb{R}^n$, $U_\delta = \cup_{x \in U} B(x,\delta)$ denotes the open $\delta$-neighborhood of $U$. We will use the notation $a \lesssim b$ if $a \leq Cb$ where $C$ is a constant depending on $\alpha$. If it is clear from the context what $C$ should depend on, we may write only $a \lesssim b$.

2. The most general theorem and its corollaries

Our most general result is the following:

**Theorem 2.1.** Let $1 \leq k < n$ be integers, let $A(n,k)$ denote the space of all $k$-dimensional affine subspaces of $\mathbb{R}^n$ and consider any natural metric on $A(n,k)$. Let $0 < \alpha \leq k$ be any real number. Suppose that $B \subset \mathbb{R}^n, \emptyset \neq E \subset A(n,k)$ and for every $k$-dimensional affine subspace $P \in E$, $\dim (P \cap B) \geq \alpha$. Then

$$\dim B \geq 2\alpha - k + \min(\dim E, 1).$$

**Remark 2.2.** An example for such a metric on $A(n,k)$ is defined in [11], p. 53. Let $\rho$ denote the given metric on $A(n,k)$. We say that $\rho$ is a natural metric if $\rho$ and the metric $d$ defined in [11] are strongly equivalent; that is, there exist positive constants $K_1$ and $K_2$ such that, for every $P, P' \in A(n,k)$, $K_1 \cdot d(P, P') \leq \rho(P, P') \leq K_2 \cdot d(P, P')$.

**Remark 2.3.** For $\alpha = k$ and $\dim E \leq k + 1$ the estimate (2) is sharp in the sense that for any $s \in [0, k + 1]$ there exist sets $E$ and $B$ with the above property and $\dim E = s$ such that we have equality in (2); it is easy to see using Theorem 1.2 that we obtain such an example by letting $E$ to be any $s$-Hausdorff dimensional collection of $k$-dimensional affine subspaces of a fixed $k + 1$-dimensional subspace of $\mathbb{R}^n$ and $B = \cup_{P \in E} P$.

Clearly, (2) can be a good estimate only when $\alpha$ is close to $k$: for $\alpha < k - 1$ the right-hand side of (2) is less than $\alpha$ but trivially, $\dim B \geq \alpha$. Since finding the best estimate for the $n = 2, k = 1, \dim E = 1, \alpha < 1$ case is essentially equivalent to finding the minimal Hausdorff dimension of a Furstenberg set, this cannot be easy and it is unlikely that our estimate is sharp for any $\alpha < k$.

By combining the $\alpha = k$ case of Theorem 2.1 and the following lemma, we obtain Theorem 1.2 and its special case Theorem 1.1.

**Lemma 2.4.** For any $1 \leq k < n$ integers and $\emptyset \neq E \subset A(n,k)$ we have

$$\dim \left( \bigcup_{P \in E} P \right) \leq k + \dim E.$$

**Proof.** By taking a finite decomposition of $E$ if necessary, we can assume that there exists a $P_0 \in A(n,k)$ such that the orthogonal projection of $P_0$ onto any $P \in E$ is $P$. Fix such a $P_0$. For any $P \in E$ and $t \in P_0$ let $h(P,t)$ be the orthogonal projection of $t$ onto $P$. Then $h((P \times P_0)) = P$ for any $P \in E$, so $h(E \times P_0) = \bigcup_{P \in E} P$. It is not hard to check that $h : E \times P_0 \to \mathbb{R}^n$ is locally Lipschitz, therefore we obtain

$$\dim \left( \bigcup_{P \in E} P \right) = \dim (h(E \times P_0)) \leq \dim (E \times P_0) = \dim E + k.$$
Now we show a simple direct application of Theorem 2.1.

**Corollary 2.5.** Let $0 \leq k < n$ be integers, $0 \leq \alpha \leq k$, $\emptyset \neq C \subset \mathbb{R}^n$, and $B \subset \mathbb{R}^n$ such that for every $x \in C$ there exists a $k$-dimensional affine subspace $P$ containing $x$ such that $P$ intersects $B$ in a nonempty set of Hausdorff dimension at least $\alpha$. Then $\dim B \geq 2\alpha - k + \min(\dim C - k, 1)$.

Specially, if $1 \leq k$ and a set $A \subset \mathbb{R}^n$ contains a $k$-dimensional punctured affine subspace through every point of a set $C$ with $\dim C \geq k + 1$, then $\dim A \geq k + 1$.

**Proof.** If $k = 0$, or $k \geq 1$ and $\alpha = 0$, then the statement clearly holds. Suppose now $k \geq 1, \alpha > 0$. Let $E \subset A(n,k)$ be the set of those $k$-dimensional affine subspaces that intersect $B$ in a nonempty set of Hausdorff dimension at least $\alpha$. Then $C \subset \bigcup_{P \in E} P$, thus $\dim C \leq \dim E + k - \alpha$ by Lemma 2.4, which means, $\dim E \geq \dim C - k$. Applying Theorem 2.1 for $B$ and $E$, we obtain $\dim B \geq 2\alpha - k + \min(\dim C - k, 1)$. □

Our next goal is to show that if a set $B \subset \mathbb{R}^n$ contains the $k$-skeleton of a rotated unit cube around every point of $\mathbb{R}^n$ then $\dim B \geq k + 1$, as it was already stated in the Introduction. Instead of the $k$-skeleton of the unit cube we will prove (Corollary 2.8) the analogous result for any $k$-Hausdorff dimensional set $S \subset \mathbb{R}^n$ that can be covered by countably many $k$-dimensional affine subspaces. This result will follow from the following theorem.

**Theorem 2.6.** Let $0 \leq k < n$ be integers, $0 \leq \alpha \leq k$ and $0 \leq r$ be real numbers, $\emptyset \neq C \subset \mathbb{R}^n$, and $B \subset \mathbb{R}^n$ be such that for every $x \in C$ there exists a $k$-dimensional affine subspace $P$ at distance $r$ from $x$ such that $P$ intersects $B$ in a nonempty set of Hausdorff dimension at least $\alpha$. Then $\dim B \geq 2\alpha - k + \dim C - (n - 1)$.

Specially, if $B$ contains a $k$-dimensional affine subspace at a fixed positive distance from every point of $\mathbb{R}^n$, or if $B$ contains the $k$-skeleton of a rotated unit cube around every point of $\mathbb{R}^n$, then $\dim B \geq k + 1$.

**Proof.** If $r = 0$, then we can apply Corollary 2.5 and thus we get $\dim B \geq 2\alpha - k + \min(\dim C - k, 1) \geq 2\alpha - k + \dim C - (n - 1)$.

Suppose now that $r > 0$. If $k = 0$, then the condition of Theorem 2.6 means that for every $x \in C$ there exists a point contained in $B$ at distance $r$ from $x$. Then $\bigcup_{p \in P} (p + rS^{n-1}) \supset C$, where $S^{n-1}$ denotes the unit sphere of center 0 in $\mathbb{R}^n$. Let $g : \mathbb{R}^n \times S^{n-1} \to \mathbb{R}^n$, $(p,e) \mapsto p + re$. Clearly, $g$ is Lipschitz and $g(B \times S^{n-1}) = \bigcup_{p \in B} (p + rS^{n-1})$. Thus we have

\[
\dim C \leq \dim \bigcup_{p \in B} (p + rS^{n-1}) \leq \dim (B \times rS^{n-1}) = \dim B + n - 1,
\]

thus $\dim B \geq \dim C - (n - 1)$.

If $k \geq 1$ and $\alpha = 0$, then the statement is trivially true, so suppose now that $k \geq 1, \alpha > 0$. We will use a similar argument as in the case $k = 0$, but we use Theorem 2.1. Let $E \subset A(n,k)$ be the set of those $k$-dimensional affine subspaces that intersect $B$ in a set of Hausdorff dimension at least $\alpha$. By Theorem 2.1 it is enough to prove that $\dim E \geq \dim C - (n - 1)$. For each $P \in E$ let $D(P) \subset \mathbb{R}^n$ be the union of those $k$-dimensional affine subspaces that are parallel to $P$ and are at distance $r$ from $P$ (in the Euclidean distance of $\mathbb{R}^n$). Clearly, $D(P)$ is exactly the set of those points of $\mathbb{R}^n$ that are at distance $r$ from $P$, thus by assumption, $\bigcup_{P \in E} D(P) \supset C$. It is easy to see that $\dim D(P) = n - 1$ for any $P \in E$.

For any $P \in A(n,k)$, let $V_P$ denote the translate of $P$ containing 0 and let $V_P^\perp$ denote the orthogonal complement of $V_P$. It is easy to see that there is a finite
decomposition $E = \bigcup_{i=1}^{N} E_i$ such that for all $i$ there exists a $P_i \in A(n,k)$ with the following properties: the orthogonal projection of $P_i$ onto any $P \in E_i$ is $P$, and the orthogonal projection of the $(n-k-1)$-sphere $V_P^\perp \cap S^{n-1}$ onto $V_P^\perp$ is contained in the $\frac{\epsilon}{r}$-neighborhood of the $(n-k-1)$-sphere $V_P^\perp \cap S^{n-1}$, for any $P \in E_i$.

Using the above properties, one can easily define for all $i$ a locally Lipschitz map $h_i : E_i \times D(P_i) \to \mathbb{R}^n$ such that $h_i(\{P\} \times D(P_i)) = D(P)$ for all $P \in E_i$. We obtain
\[
\dim C \leq \dim \bigcup_{P \in E} D(P) = \max_i \dim \bigcup_{P \in E_i} D(P) = \max_i \dim h_i(E_i \times D(P_i)) \leq \max_i \dim (E_i \times D(P_i)) = \max_i \dim E_i + n - 1 = \dim E + n - 1,
\]
and thus $\dim E \geq \dim C - (n - 1)$ and we are done. \hfill \qed

**Remark 2.7.** In the special cases mentioned in Theorem 2.6 the estimate is sharp. It is easy to see that $B = \mathbb{R}^{k+1} \times Q^{n-k-1}$ contains a $k$-dimensional affine subspace at every positive distance from every point of $\mathbb{R}^n$ and clearly $\dim B = k + 1$. The construction given in [11] for a set $B$ with $\dim B = k + 1$ containing the $k$-skeleton of a rotated unit cube centered at every positive distance from every point of $\mathbb{R}^n$ is also based on this example.

**Corollary 2.8.** Let $0 \leq k < n$ be integers, $S \subset \mathbb{R}^n$ with $\dim S = k$ that can be covered by a countable union of $k$-dimensional affine subspaces. Let $0 \neq C \subset \mathbb{R}^n$, $A \subset \mathbb{R}^n$ such that for all $x \in C$ there exists a rotation $T \in SO(n)$ such that $A$ contains $x + T(S)$. Then $\dim A \geq \max(k, k + \dim C - (n - 1))$.

**Proof.** Clearly, $\dim A \geq k$. Let $S_i \subset \mathbb{R}^n$, $i \geq 1$, be $k$-dimensional affine subspaces such that $S \subset \bigcup_{i \geq 1} S_i$. Let $r_i = d(0, S_i)$, and $\alpha_i = \dim (S_i \cap S)$. Then $\sup_{i \geq 1} \alpha_i = k$ by $\dim S = k$. The set $A$ has the property that for all $x \in \mathbb{R}^n$, there exists an affine subspace $P = x + T(S)$ at distance $r_i$ from $x$ such that $\dim (A \cap P \cap S_i) \geq \alpha_i$, thus we can apply Theorem 2.6 for each $i$. We obtain that $\dim A \geq 2\alpha_i - k + \dim C - (n - 1)$ for all $i \geq 1$, and thus $\dim A \geq k + \dim C - (n - 1)$.$\hfill \qed$

**Remark 2.9.** The authors in [11] show that the estimate in Corollary 2.8 is sharp if $\dim C = n$ and $S$ can be covered by a countable union of $k$-dimensional affine subspaces that do not contain the origin.

On the other hand, if the covering subspaces contain the origin, then the estimate is not always sharp. Indeed, if $S$ is a punctured line through the origin and $C = \mathbb{R}^n$, then $A$ is a Nikodym set, thus the conjecture is $\dim A = n$. The lower bounds obtained for the dimension of Besicovitch sets give lower bounds for the dimension of Nikodym sets, thus for $\dim A$ as well. A survey of the currently best lower bounds can be found in [12]. As an example, by [17], $\dim A \geq \frac{n+2}{2}$ which is better than the bound 2 given by Corollary 2.8 provided $n > 2$.

3. The proof of Theorem 2.1

In this section we prove Theorem 2.1 subject to a lemma (Lemma 3.7), which will be proved in Sections 4 and 5.

We start with addressing measurability issues. For the definition of analytic sets, see e.g. [9].

**Lemma 3.1.** For $X \subset \mathbb{R}^n$, $\alpha > 0$ and $c \geq 0$ let
\[ E_{\alpha,c,X} = \{ P \in A(n,k) : \mathcal{H}_{\alpha,c}^n(P \cap X) > c \}. \]
If $X \subset \mathbb{R}^n$ is bounded $G_\delta$, then $E_{\alpha,c,X}$ is analytic.

Lemma 3.1 is an unpublished result of M. Elekes and Z. Vidnyánszky. Similar statements were also proved in [4]. For completeness, we include a proof.
Remark 3.2. It is easy to see that if \( X \subset \mathbb{R}^n \) is compact, then \( E_{\alpha,c,X} \) is \( F_\sigma \), thus also analytic. Therefore, to prove Theorem 2.1 (or any of the above mentioned results) with the extra assumption that \( B \subset \mathbb{R}^n \) is compact, the following argument could be skipped.

Proof. Let
\[
T = \{(P,x) \in A(n,k) \times \mathbb{R}^n : x \in P\},
\]
this is the natural vector bundle of rank \( k \) over \( A(n,k) \). Let \( \varphi : T \to \mathbb{R}^n \) be defined by \( \varphi((P,x)) = x \), and let \( \pi : T \to A(n,k) \) be defined by \( \pi((P,x)) = P \). On \( T \) we can consider the metric inherited from a product metric on \( A(n,k) \times \mathbb{R}^n \) so that \( \varphi \) is isometry on all fibres.

Let \( \mathcal{K} \) be the space of those non-empty compact subsets of \( T \) which lie in one fibre, that is,
\[
\mathcal{K} = \{K \subset T : K \text{ is non-empty compact, and } \pi(K) \text{ is a singleton}\}.
\]
This is a complete metric space in the Hausdorff metric. Not to mix up singletons and their unique elements, let \( \pi' : \mathcal{K} \to A(n,k) \) be defined by \( \pi'(K) = \pi(K) \).

Since \( X \) is \( G_\delta \), \( \varphi^{-1}(X) \) is \( G_\delta \) in \( T \). It is easy to check that
\[
\mathcal{K}(\varphi^{-1}(X)) \overset{\text{def}}{=} \{K \in \mathcal{K} : K \subset \varphi^{-1}(X)\}
\]
is also \( G_\delta \) in \( \mathcal{K} \).

For \( \alpha > 0 \) and \( d > 0 \), let
\[
\mathcal{K}_d^\alpha = \{K \in \mathcal{K} : \mathcal{H}_\infty^\alpha(\varphi(K)) \geq d\}.
\]
It is easy to see that these are closed sets in \( \mathcal{K} \).

Let
\[
\mathcal{K}_{\alpha,c,X} = \mathcal{K}(\varphi^{-1}(X)) \cap \bigcup_n \mathcal{K}_c^{\alpha+1/n}.
\]
Clearly, this is a Borel set in \( \mathcal{K} \). We claim that
\[
(3) \quad E_{\alpha,c,X} = \pi'(\mathcal{K}_{\alpha,c,X}).
\]
Clearly, the right hand side consists of those \( P \in A(n,k) \) for which \( P \cap X \) contains a compact subset \( K \) with \( \mathcal{H}_\infty^\alpha(K) > c \). We will show that for any \( P \in A(n,k) \),
\[
(4) \quad \exists K \subset P \cap X \text{ compact with } \mathcal{H}_\infty^\alpha(K) > c \iff \mathcal{H}_\infty^\alpha(P \cap X) > c, \quad \text{which implies } (3).
\]
To prove (4), we use the concept of capacities (see e.g. [8], Section 30).

Definition. Let \( Y \) be a Hausdorff topological space. A capacity on \( Y \) is a map \( \gamma : \mathcal{P}(Y) \to [0,\infty] \) such that
\[
\begin{align*}
\text{(i)} & \quad A \subset B \implies \gamma(A) \leq \gamma(B), \\
\text{(ii)} & \quad A_0 \subset A_1 \subset \cdots \implies \gamma(A_n) \to \gamma(\bigcup_n A_n), \\
\text{(iii)} & \quad \text{for any compact } K \subset Y \text{ we have } \gamma(K) < \infty, \text{ and if } \gamma(K) < r, \text{ then for some open } U \supset K, \gamma(U) < r.
\end{align*}
\]
We claim that \( \gamma = \mathcal{H}_\infty^\alpha \) is a capacity on \( \overline{B(0,R)} \) for any \( R > 0 \). Indeed, it is clear that \( \mathcal{H}_\infty^\alpha \) satisfies properties (i) and (iii) in any metric space, and it follows from the results in [8] that (iii) holds for \( \mathcal{H}_\infty^\alpha \) in any compact metric space.

Since \( X \) is bounded \( G_\delta \) (thus also analytic), and \( \mathcal{H}_\infty^\alpha \) is a capacity on the compact metric space \( \overline{B(0,R)} \) with \( X \subset \overline{B(0,R)} \), the Choquet Capacitability Theorem ([8], (30.13)) can be applied, and it gives precisely (4).

Finally, (3) implies that \( E_{\alpha,c,X} \) is a continuous image of a Borel set, thus analytic, and we are done. \( \square \)
Lemma 3.4. We can make the following further assumptions in Theorem 2.1.

(i) $B$ is a $G_δ$ set, that is, a countable intersection of open sets;
(ii) $\mathcal{H}^α(P \cap B) > 0$ for every $P \in E$;
(iii) $B$ is bounded;
(iv) $E \subset A(n,k)$ is compact, and $\mathcal{H}^α(E) > 0$. Moreover, there is $ε > 0$ such that for every $P \in E$, 
$$\mathcal{H}^α(P \cap B) \geq ε.$$ 

Statement (i) is clearly weaker than (iv); it is stated to guide the proof.

Proof. First we remark that if $E$ is replaced by any subset $\bar{E} \subset E$, or $B$ is replaced by any superset $\bar{B} \supset B$, then the condition $\dim(P \cap B) \geq \dim(P \cap B) \geq α$ in Theorem 2.1 is trivially satisfied for all $P \in \bar{E} \subset E$.

(i) Let $\bar{B} \supset B$ be a $G_δ$ set with $\dim B = \dim \bar{B}$; the existence of such set is proved for example in [3]. Clearly, it is enough to prove Theorem 2.1 for $\bar{B}$ replacing $B$.

(ii) Clearly, replacing $α$ with a slightly smaller value, we may assume, without loss of generality, that $\mathcal{H}^α(P \cap B) > 0$ for every $P \in E$.

(iii) If $B$ is not bounded then consider $B = \bigcup_n B_n$, where $B_n$ is bounded $G_δ$ and define $E_n = \{P \in E : \mathcal{H}^α(P \cap B_n) > 0\}$. Clearly, $E = \bigcup_n E_n$ thus $\dim E = \sup\{\dim E_n : n \in \mathbb{N}\}$. If Theorem 2.1 holds for the bounded set $B_n$ and $E_n \subset A(n,k)$ for every $n$ then it holds for $B$ and $E$ as well. Thus we can assume that $B$ is bounded.

(iv) By [1], we may assume that $B$ is $G_δ$. By [3], for every $P \in E$, $\mathcal{H}^α(P \cap B) > 0$, and thus $\mathcal{H}^α(E \cap P) > 0$. Thus $E \subset \bigcup_{i=1}^∞ E_{α,i,B}$, where the sets $E_{α,i,B}$ are the analytic sets given by Lemma 3.4. For every $δ > 0$, $\mathcal{H}^{∞-δ}(E) = ∞$ and therefore there is $i = i(δ)$ with $\mathcal{H}^{∞-δ}(E_{α,i,B}) > 0$. By Howroyd’s theorem [7], there is a compact set $E^ δ \subset E_{α,i,B}$ with $\mathcal{H}^{∞-δ}(E^ δ) > 0$. If Theorem 2.1 holds for these compact sets $E^ δ$, then $\dim \bar{E} \geq 2α - k + s - δ$ for every $δ > 0$, which finishes the proof. □

Let $e_0 = (0,\ldots,0)$; let $e_i = (1,0,\ldots,0),\ldots,e_n = (0,\ldots,0,1)$ be the standard basis vectors of $\mathbb{R}^n$, and let $V$ be the $k$-dimensional linear space generated by $e_1,\ldots,e_k$. Put $H_0 = V^⊥$, and $H_i = e_i + H_0$. Then $H_i$ is an $n - k$-dimensional affine subspace for all $i = 1,\ldots,k$. We use the sets $H_i (i = 0,\ldots,k)$ to describe the structure of $E$ by investigating the intersection of the elements of $E$ with them.

Let $C$ denote the convex hull of the vectors $e_0,e_1,\ldots,e_k$ in $V$, $Q \subset H_0$ the $n - k$-dimensional closed unit cube of center $e_0$ in $H_0$, and $S = C × Q \subset \mathbb{R}^n$. Fix $δ_0 > 0$ and an open set $S'$ such that

$$S'_{δ_0} \subset S,$$

where $S'_{δ_0}$ denotes the open $δ_0$-neighborhood of $S'$.

Lemma 3.4. We can make the following further assumptions in Theorem 2.1.

(i) For every $P \in E$, $P \cap H_i$ is a singleton and contained in $S$ for all $i = 0,1,\ldots,k$;
(ii) $B \subset S'$.

Proof.
(I) We can cover $E$ by finitely many compact subsets for which (I) holds after applying a suitable similarity transformation.

(II) Since we may assume that $B$ is bounded, this can be obtained after applying a homothety. □

Let us now fix $B, E, \varepsilon, S', \delta_0$ (and $s$ and $\alpha$) with properties given by Lemma 3.3 and such that Lemma 3.4 is satisfied. That is, $B$ is bounded and $G_{\delta_0} E$ is compact and $\mathcal{H}^s(E) > 0$, and

\[ \mathcal{H}^s_\infty(P \cap B) \geq \varepsilon \]

for all $P \in E$ for a fixed $\varepsilon > 0$.

We apply Frostman’s lemma (see e.g. [11]) to obtain a probability measure $\mu$ on $A(n, k)$ (for which Borel and analytic sets are measurable) supported on $E$ for which

\[ \mu(B(P, r)) \lesssim r^s \]

for all $r > 0$ and all $P \in E$.

Now we turn to estimating the dimension of the set $B$. Our aim is to show that

\[ \mathcal{H}^{2\alpha - k + s - \gamma}(B) > 0 \]

for any $\gamma > 0$. Fix $\gamma > 0$, and let

\[ u = 2\alpha - k + s - \gamma. \]

Let $M$ be a positive integer such that

\[ \sum_{k=1}^{\infty} 1/k^2 < \varepsilon \quad \text{and} \quad 2^{-M+1} \leq \delta_0. \]

Let $B \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$ be any countable cover with $2r_i \leq 2^{-M}$ for all $i$. For any $l \geq M$, let

\[ J_l = \{ i : 2^{-l} < r_i \leq 2^{-l+1} \}. \]

Let $R_l = \bigcup_{i \in J_l} B(x_i, r_i)$, and $B_l = R_l \cap B$. Then $B = \bigcup_{l=M}^{\infty} B_l$.

Our aim is to find a big enough subset of $B$ that is covered by balls of approximately the same radii and such that many of the affine subspaces of $E$ have big intersection with it.

Remark 3.5. In the subsequent proofs, applications of Lemma 3.1 imply that the sets we take $\mu$-measure of are $\mu$-measurable, since they are in the $\sigma$-algebra generated by analytic sets.

Lemma 3.6. There exists an integer $l \geq M$ such that

\[ \mu \left( P \in E : \mathcal{H}^s_\infty(P \cap B_l) \geq \frac{1}{l^2} \right) \geq \frac{1}{l^2}. \]

Proof. Let

\[ A_l = \left\{ P \in E : \mathcal{H}^s_\infty(P \cap B_l) \geq \frac{1}{l^2} \right\}, \]

and assume that $\mu(A_l) < 1/l^2$ for all $l \geq M$. Since $\sum_{l=M}^{\infty} 1/l^2 < 1$ (we may assume $\varepsilon \leq 1$), these sets $A_l$ cannot cover $E$. Therefore, there exists $P \in E$ such that $\mathcal{H}^s_\infty(P \cap B_l) < 1/l^2$, and thus $\mathcal{H}^s_\infty(P \cap B) < \sum_{l=M}^{\infty} 1/l^2 < \varepsilon$, which contradicts (6). □
Fix the integer $l$ obtained by Lemma 3.6 and let

$$\tilde{E} = A_l = \left\{ P \in E : \mathcal{H}^\infty_\alpha (P \cap B_l) \geq \frac{1}{l^2} \right\}.$$  

We will use the notation $\tilde{P} = P \cap B_l$ for any $P \in \tilde{E}$. We have

$$\mu(\tilde{E}) \geq \frac{1}{l^2} \text{ and } \mathcal{H}^\infty_\alpha (\tilde{P}) \geq \frac{1}{l^2}$$

for every $P \in \tilde{E}$ by Lemma 3.6. Note also that

$$\tilde{P}_{\delta_0} \subset S$$

for every $P \in \tilde{E}$ by (11) of Lemma 3.4 and the definition of $S'$ and $\delta_0$.

Let

$$F = \bigcup_{P \in \tilde{E}} \tilde{P} \subset B_l \subset B.$$

Our aim is to find a lower estimate for $L^n(F_\delta)$. We will prove the following.

**Lemma 3.7.** There is a constant $c > 0$ depending on $E$, $n$, and $k$ but independent of $l$, $\epsilon$, $\gamma$ and the covering of $B$ such that, for every $0 < \delta \leq \delta_0$,

$$L^n(F_\delta) \geq \frac{\delta^{n-(2\alpha-k+s)}}{l^8 \log 2}.$$

**Remark 3.8.** Note that the integer $l$, the sets $\tilde{E}$, $\tilde{P}$ for every $P \in \tilde{E}$, and $F$ depend on the cover $B \subset \bigcup_{i=1}^\infty B(x_i, r_i)$. We prove Lemma 3.7 in Sections 4 and 5.

**Remark 3.9.** As it happens often, it would be easier to prove the lower bound for the box dimension of $B$. For that purpose, we would not need the previous steps, it would be enough to estimate $L^n(B_\delta)$ from below. To prove the lower bound for the Hausdorff dimension, we sorted out a big enough part of $B$ that can be covered by balls of approximately the same radius.

Recall that $B \subset \bigcup_{i=1}^\infty B(x_i, r_i)$, $2r_i \leq 2^{-M} \leq \frac{\delta_0}{3}$ for all $i$. We will use that the balls with indices from $J_l$ have approximately the same radius. We have that

$$\sum_{i=1}^{\infty} (2r_i)^u = \sum_{l=M}^{\infty} \sum_{i \in J_l} (2r_i)^u \geq \sum_{i \in J_l} (2r_i)^u \geq \sum_{i \in J_l} (2^{-l})^u.$$

Let $\delta = 2^{-l+1}$. Then $\delta \leq 2^{-M+1} \leq \delta_0$.

The set $F$ was constructed to satisfy

$$F \subset B_l \subset \bigcup_{i \in J_l} B(x_i, r_i) \subset \bigcup_{i \in J_l} B(x_i, \delta),$$

and thus

$$F_\delta \subset \bigcup_{i \in J_l} B(x_i, 2\delta).$$

Using

$$\frac{\delta^u}{\delta^{n-(2\alpha-k+s-\gamma)}} \geq \frac{L^n(B(x_i, 2\delta))}{\delta^{n-(2\alpha-k+s-\gamma)}}$$
and Lemma 5.7 we get
\[ \sum_{i \in J} (2^{-i})^u \geq \sum_{i \in J} \frac{L^n(B(x_i, 2\delta))}{\delta^{2(2a-k+s-\gamma)} n} \geq \frac{L^n(F_k)}{\delta^{-(2a-k+s)+\gamma}} \geq \frac{1}{\delta^{(2a-k+s)+\gamma} \log \frac{\delta}{\rho}} \geq \frac{1}{2^{-i} |\rho|^9}. \]
Thus we obtain
\[ \inf_{B \subseteq \bigcup_{i \in J} B(x_i, r_i)} \sum_{i=1}^\infty (2r_i)^u \geq \inf_{i \geq 1} \frac{1}{2^{-i} |\rho|^9} \geq 1, \]
proving that \( H^u(B) > 0 \) and we are done.

4. Geometric arguments

Now we start proving Lemma 5.7. In this section we prove a purely geometric lemma using only the set \( \tilde{E} \subset A(n, k) \). This part is independent of the set \( B \) and the number \( \alpha \).

**Lemma 4.1.** For any \( P, P' \in \tilde{E} \),
\[ L^n(P_0 \cap P'_0 \cap S) \lesssim \frac{\delta^{n-k+1}}{\rho(P, P')} + \delta \]
for all \( 0 < \delta \leq \delta_0 \), where \( \rho \) denotes the metric on \( A(n, k) \), and \( \delta_0 \) is from \( \rho \).

To prove Lemma 4.1 we will define a new metric on \( \tilde{E} \) by making use of \( \rho \) of Lemma 5.3. We will assign a code to each \( k \)-dimensional affine subspace in \( \tilde{E} \). For a given \( P \in \tilde{E} \), let \( (0, a^0) = (0, \ldots, 0, a^0_1, \ldots, a^0_{n-k}) \) denote the standard \( \mathbb{R}^n \)-coordinates of \( P \cap H_0 \). Similarly, let \( (l^i, a^i) = (0, \ldots, 1, \ldots, 0, a^i_1, \ldots, a^i_{n-k}) \) denote the standard \( \mathbb{R}^n \)-coordinates of \( P \cap H_l \) for each \( l = 1, \ldots, k \). Let \( b^l = a^l - a^0 \in \mathbb{R}^{n-k} \) for each \( l = 1, \ldots, k \). We refer to \( a^l \) as the vertical intercept, and to \( \{b^l\}_{l=1}^k \) as the slopes of \( P \).

We say that the point \( x = x(P) = (a^0, b^1, \ldots, b^k) = (a, b) \in \mathbb{R}^{(k+1)(n-k)} \) is the code of the \( k \)-dimensional affine subspace \( P \in \tilde{E} \). By (1) of Lemma 3.4 one can see that \( P \to x(P) \) is well defined and injective on \( \tilde{E} \).

We will use the maximum metric on the code space \( \mathbb{R}^{(k+1)(n-k)} \). This means, \( \|x - x'\| = \max(\|a - a'\|, \|b - b'\|) \), where
\[ \|a - a'\| = \max_{j=1, \ldots, n-k} |a_j^0 - a_j^0|, \]
\[ \|b - b'\| = \max_{j=1, \ldots, n-k} \left( \max_{l=1, \ldots, k} |b_{j}^l - b_{j}^l| \right). \]

**Remark 4.2.** Put \( d(P, P') = \|x(P) - x(P')\| \), then \( d \) is a natural metric on \( \tilde{E} \). Thus the metrics \( d \) and \( \rho \) are strongly equivalent, this means, there exist positive constants \( K_1 \) and \( K_2 \) such that, for every \( P, P' \in \tilde{E}, K_1 \cdot d(P, P') \leq \rho(P, P') \leq K_2 \cdot d(P, P') \).

In order to prove Lemma 4.1 we show that if \( P \) and \( P' \) are translated along \( H_0 \) far enough from each other compared to their slopes, then the intersection of their \( \delta \)-tubes is empty in \( S \), and if the slopes of \( P \) and \( P' \) are far enough from each other, then the intersection of their \( \delta \)-tubes is small enough in \( S \).
Lemma 4.3. (a) There is a constant $D > 0$ (depending only on $n$ and $k$) such that if
\[ \|a - a'\| > \|b - b'\| + D\delta \]
then $P_\delta \cap P'_\delta \cap S = \emptyset$ for all $0 < \delta \leq \delta_0$.
(b) If $\|b - b'\| > 0$, then $L^n(P_\delta \cap P'_\delta \cap S) \leq \frac{\delta_{n-k+1}}{\|b - b'\|}$ for all $0 < \delta \leq \delta_0$.

Proof. Fix $P, P' \in \bar{E}$, and put $f, g : \mathbb{R}^k \to \mathbb{R}^{n-k}$.
\[ t = (t_1, \ldots, t_k) \mapsto a^0 + t_1 b^1 + \cdots + t_k b^k = f(t), \]
\[ t = (t_1, \ldots, t_k) \mapsto a'^0 + t_1 b'^1 + \cdots + t_k b'^k = g(t), \]
where $a^0, b^1, \ldots, b^k$ and $a'^0, b'^1, \ldots, b'^k$ are the code coordinates of $P$ and $P'$, respectively. Then
\[ P \cap S = \{(t, f(t)) : t \in C\}, \quad P' \cap S = \{(t, g(t)) : t \in C\}. \]

One can easily prove using (1) of Lemma 3.4 and the compactness of $S$, that there is a constant $c > 0$ independent of $\delta$ such that for all $Q \in \bar{E}$,
\[ Q_\delta \subset Q + \{(0) \times (-c\delta, c\delta)^{n-k}\}, \]
where for $A, B \subset \mathbb{R}^n$, $A + B = \{a + b : a \in A, b \in B\}$. Fix such a constant $c$.

Applying (15) for $P$ and $P'$, we have
\[ P_\delta \subset \{(t, u) \in \mathbb{R}^n : |f(t) - u| < c\delta\}, \quad P'_\delta \subset \{(t, u) \in \mathbb{R}^n : |g(t) - u| < c\delta\}, \]
and
\[ P_\delta \cap P'_\delta \cap S \subset \{(t, u) \in \mathbb{R}^n : u \in (B(f(t), c\delta) \cap B(g(t), c\delta)), t \in C\}. \]

Clearly, $|f(t) - g(t)| > 2c\delta$ implies $B(f(t), c\delta) \cap B(g(t), c\delta) = \emptyset$, thus $P_\delta \cap P'_\delta \cap S = \emptyset$. Put $D = 2c$, then $\|a - a'\| > \|b - b'\| + D\delta$ implies $|f(t) - g(t)| > 2c\delta$, thus we are done with the proof of (16) of Lemma 4.3.

By (16) and Fubini’s theorem we also have
\[ L^n(P_\delta \cap P'_\delta \cap S) \leq \int_C L^{n-k}(B(f(t), c\delta) \cap B(g(t), c\delta))d\mathcal{L}^k(t). \]

If $B(f(t), c\delta) \cap B(g(t), c\delta) \neq \emptyset$, we will use the trivial estimate
\[ L^{n-k}(B(f(t), c\delta) \cap B(g(t), c\delta)) \lesssim_{n,k} (c\delta)^{n-k} \lesssim \delta^{n-k}. \]

Put $N = \{t \in C : |f(t) - g(t)| \leq 2c\delta\}$, then
\[ L^n(P_\delta \cap P'_\delta \cap S) \lesssim \int_N \delta^{n-k}d\mathcal{L}^k(t). \]

Clearly, we have
\[ \bigcap_{j=1}^{n-k} \left\{ t \in C : \left| a_j^0 - a_j' \right| + \sum_{i=1}^{k} t_i \left| b_i^j - b_i'^j \right| \leq 2c\delta \right\}. \]

By the definition of $\|\cdot\|$, there are indices $i, j$ such that $0 < \|b - b'\| = |b_j^i - b_j'^i|$. Fix such an $i$ and $j$, we can assume that $i = k$ without loss of generality. Then we get using (17) that
\[ N \subset \{t \in C : p_-(t) \leq t_k \leq p_+(t)\}, \]
where
\[ p_-(t) = p_-(t_1, \ldots, t_{k-1}) = \frac{-2c\delta - (a^0 - a'^0) - \sum_{i=1}^{k-1} t_i (b_i^j - b_i'^j)}{b_k^j - b_k'^j}, \]
and
\[ p_+(t) = p_+(t_1, \ldots, t_{k-1}) = \frac{2c\delta - (a^0 - a'^0) + \sum_{i=1}^{k-1} t_i (b_i^j - b_i'^j)}{b_k^j - b_k'^j}. \]
The set \( \{ t \in C : p_-(t) \leq t_k \leq p_+(t) \} \) is obtained as the intersection of the simplex \( C \) and the strip between the parallel hyperplanes \( \{ t_k = p_+(t) \}, \{ t_k = p_-(t) \} \). One can easily calculate the distance of these hyperplanes, using the normal vector \( n = (b_j^1 - b_j^2, \ldots , b_j^k - b_j^2) \). One gets

\[
d = d(\{ t_k = p_+(t) \}, \{ t_k = p_-(t) \}) = \frac{2c\delta - (a^0 - a^0') - \sum_{i=1}^{k-1} t_i(b_j^i - b_j^i)}{b_j^k - b_j^2}.
\]

Thus the set \( N \) is contained in a rectangular box, where the shortest side length is \( d \) and the others are \( \text{diam} (C) = \sqrt{2} \). Then

\[
\mathcal{L}^k(N) \leq \frac{\delta}{\sqrt{\sum_{i=1}^{k} (b_j^i - b_j^i)^2}} \leq \frac{\delta}{|b_j^k - b_j^2|} = \frac{\delta}{\|b - b'\|}.
\]

thus

\[
\mathcal{L}^n(P_k \cap P_0' \cap S) \lesssim \frac{\delta^{n-k+1}}{\|b - b'\|}
\]

and we are done with the proof of Lemma 4.3. \( \square \)

Now we prove Lemma 4.1.

Proof. Using (13) of Lemma 4.3 we obtain that \( P_k \cap P_0' \cap S = \emptyset \) for all \( 0 < \delta \leq \delta_0 \) if \( \|a - a'\| > \|b - b'\| + D\delta \), so (13) is clearly satisfied.

Assume now that \( \|a - a'\| \leq \|b - b'\| + D\delta \), and \( \|b - b'\| \leq \delta \). By Remark 1.2

\[
\rho(P, P') \leq K_2 \|x - x'\| \leq K_2(D + 1)\delta \lesssim \delta,
\]

and then since \( S \) is bounded, we have

\[
\mathcal{L}^n(P_k \cap P_0' \cap S) \leq \mathcal{L}^n(P_k \cap S) \lesssim \delta^{n-k} = \frac{\delta^{n-k+1}}{\delta} \lesssim \frac{\delta^{n-k+1}}{\rho(P, P') + \delta}.
\]

Thus we are done in this case.

If \( \|a - a'\| \leq \|b - b'\| + D\delta \) and \( \|b - b'\| \geq \delta \), we have that

\[
\rho(P, P') + \delta \leq K_2 \|x - x'\| + \delta \leq K_2(\|b - b'\| + D\delta) + \delta \lesssim \|b - b'\|
\]

using Remark 1.2 again. Applying (13) of Lemma 4.3 we obtain that

\[
\mathcal{L}^n(P_k \cap P_0' \cap S) \lesssim \frac{\delta^{n-k+1}}{\|b - b'\|} \lesssim \frac{\delta^{n-k+1}}{\rho(P, P') + \delta},
\]

which is (13). \( \square \)

5. The proof of Lemma 5.7

L^2 argument

In this section we prove Lemma 5.7 with help on an \( L^2 \) estimation technique. It resembles the technique that Córdoba used in his proof for the Kakeya maximal inequality in the plane, see [2].
By Fubini’s theorem we have the following:
\[
\int_{\tilde{E}} \mathcal{L}^n(\tilde{P}_\delta) d\mu(P) = \int_{\tilde{E}} \int_{\mathbb{R}^n} \chi_{\tilde{P}_\delta}(y) dy d\mu(P)
\]
\[
= \int_{\mathbb{R}^n} \int_{\tilde{E}} \chi_{\tilde{P}_\delta}(y) d\mu(P) dy = \int_{\mathbb{R}^n} \chi_{\tilde{P}_\delta}(y) \cdot \int_{\tilde{E}} \chi_{\tilde{P}_\delta}(y) d\mu(P) dy,
\]
where \( F = \bigcup_{P \in \tilde{E}} \tilde{P} \) from (12). Now we apply the Cauchy-Schwarz inequality for the \( L^2 \) functions \( y \mapsto \chi_{F_\delta}(y) \) and \( y \mapsto \int_{\tilde{E}} \chi_{\tilde{P}_\delta}(y) d\mu(P) \). We get
\[
\int_{\mathbb{R}^n} \chi_{F_\delta}(y) \cdot \left( \int_{\tilde{E}} \chi_{\tilde{P}_\delta}(y) d\mu(P) \right) dy \leq \left( \int_{\mathbb{R}^n} \chi^2_{F_\delta}(y) dy \right)^{1/2} \cdot \left( \int_{\tilde{E}} \left( \int_{\mathbb{R}^n} \chi_{\tilde{P}_\delta}(y) d\mu(P) \right)^2 dy \right)^{1/2}
\]
\[
= (\mathcal{L}^n(F_\delta))^{1/2} \cdot \left( \int_{\mathbb{R}^n} \int_{\tilde{E}} \chi_{\tilde{P}_\delta}(y) \cdot \chi_{\tilde{P}_\delta}(y) d\mu(P) d\mu(P') dy \right)^{1/2}
\]
\[
= (\mathcal{L}^n(F_\delta))^{1/2} \cdot \left( \int_{\tilde{E} \times \tilde{E}} \mathcal{L}^n \left( \tilde{P}_\delta \cap \tilde{P}_{\delta}' \right) d\mu(P) d\mu(P') \right)^{1/2}.
\]
We proved that
\[
\int_{\tilde{E}} \mathcal{L}^n(\tilde{P}_\delta) d\mu(P) \leq (\mathcal{L}^n(F_\delta))^{1/2} \cdot \left( \int_{\tilde{E} \times \tilde{E}} \mathcal{L}^n \left( \tilde{P}_\delta \cap \tilde{P}_{\delta}' \right) d\mu(P) d\mu(P') \right)^{1/2}.
\]
On the other hand, there is a lower bound for the left hand side.

For any \( U \subset \mathbb{R}^n \) and \( \varepsilon > 0 \), let \( N(U, \varepsilon) \) denote the smallest number of \( \varepsilon \)-balls needed to cover \( U \). It is well known (see e.g. (11)) that \( \mathcal{L}^n(U_{\varepsilon}) \gtrsim_n N(U, 2\varepsilon)\varepsilon^n \) for every \( U \subset \mathbb{R}^n \) and \( \varepsilon > 0 \). Since \( H^\alpha_{\infty}(\tilde{P}) \geq \frac{1}{\varepsilon^2} \) by (10), we have \( N(\tilde{P}, \varepsilon) \cdot (2\varepsilon)^n \geq \frac{1}{\varepsilon^n} \) for every \( P \in \tilde{E} \) and \( \varepsilon > 0 \), thus
\[
(18) \quad \mathcal{L}^n(\tilde{P}_\delta) \gtrsim N(\tilde{P}, 2\varepsilon) \cdot \delta^n \gtrsim \delta^{n-\alpha} \cdot \frac{1}{l^2}.
\]
Then
\[
\int_{\tilde{E}} \mathcal{L}^n(\tilde{P}_\delta) d\mu(P) \gtrsim \delta^{n-\alpha} \cdot \frac{1}{l^2} \cdot \mu(\tilde{E}) \gtrsim \delta^{n-\alpha} \cdot \frac{1}{l^2}
\]
by (10). Thus we get
\[
(19) \quad \frac{\delta^{2n-2\alpha}}{l^8} \lesssim \mathcal{L}^n(F_\delta) \cdot \left( \int_{\tilde{E} \times \tilde{E}} \mathcal{L}^n \left( \tilde{P}_\delta \cap \tilde{P}_{\delta}' \right) d\mu(P) d\mu(P') \right).
\]
Thus we need to find an upper estimate for
\[
(20) \quad \int_{\tilde{E} \times \tilde{E}} \mathcal{L}^n(\tilde{P}_\delta \cap \tilde{P}_{\delta}') d\mu(P) d\mu(P').
\]
This means, we have to investigate, how the different \( \delta \)-tubes intersect each other.

We will estimate the integral (20) by dividing the set \( \tilde{E} \) into parts. One can easily check using Remark 3,5 that the elements of this partition will be measurable.
Fix a $P' \in \tilde{E}$. Put
\[ E_0 = \{ P \in \tilde{E} : \rho(P, P') \leq \delta \} \]
and
\[ E_j = \{ P \in \tilde{E} : 2^{j-1} \delta < \rho(P, P') \leq 2^j \delta \} \]
for $j = 1, \ldots, N$, where $N \lesssim \log \frac{1}{\delta}$. Clearly, we have $\tilde{E} = \bigcup_{j=0}^{N} E_j$, so

\[ \int_{E_j} L^n(\tilde{P}_5 \cap \tilde{P'}_5) d\mu(P) \]

By (11) and Lemma 4.1, we obtain

\[ \int_{E_j} L^n(\tilde{P}_5 \cap \tilde{P'}_5) d\mu(P) \lesssim \frac{\delta^{n-k+1}}{\delta} \mu(E_0) \lesssim \delta^{n-k} \cdot \delta^s \]

by (22) and (7). For $j \in \{1, \ldots, N\}$, we get

\[ \int_{E_j} L^n(\tilde{P}_5 \cap \tilde{P'}_5) d\mu(P) \lesssim \frac{\delta^{n-k+1}}{2^{j-1} \delta + \delta} \mu(E_j) \lesssim \frac{\delta^{n-k}}{2^j (2^j \delta)^s} = \frac{\delta^{n-k+s} \cdot 2^j}{2^j} \]

by (22) and (7) again. Applying these estimates for (21) and using $s \leq 1$, we get

\[
\int_{E} L^n(\tilde{P}_5 \cap \tilde{P'}_5) d\mu(P) \lesssim \delta^{n-k+s} \left( 1 + \sum_{j=1}^{N} \frac{2^j}{2^j} \right) \lesssim \delta^{n-k+s} N \lesssim \delta^{n-k+s} \log \frac{1}{\delta}.
\]

Finally we integrate with respect to $P'$ and obtain by $\mu(\tilde{E}) \leq 1$ that

\[
\int \int_{E \times \tilde{E}} L^n(\tilde{P}_5 \cap \tilde{P'}_5) d\mu(P) d\mu(P') \lesssim \delta^{n-k+s} \log \frac{1}{\delta}.
\]

Recalling (19), we obtain that

\[
\frac{\delta^{2n-2s}}{18} \lesssim L^n(F_3) \cdot \delta^{n-k+s} \log \frac{1}{\delta},
\]

thus

\[
L^n(F_3) \gtrsim \frac{\delta^{n-2s+k-s}}{l^8 \log \frac{1}{\delta}}
\]

and we are done with the proof of Lemma 3.7 and so also with the proof of Theorem 2.4.

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