Conformal Sigma Models
Corresponding to Gauged
Wess-Zumino-Witten Theories

A.A. Tseytlin

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

Abstract
We develop a field-theoretical approach to determination of the background target space fields corresponding to general $G/H$ coset conformal theories described by gauged WZW models. The basic idea is to identify the effective action of a gauged WZW theory with the effective action of a sigma model. The derivation of the quantum effective action in the gauged WZW theory is presented in detail, both in the bosonic and in the supersymmetric cases. We explain why and how one can truncate the effective action by omitting most of the non-local terms (thus providing a justification for some previous suggestions). The resulting metric, dilaton and the antisymmetric tensor are non-trivial functions of $1/k$ (or $\alpha'$) and represent a large class of conformal sigma models. The exact expressions for the fields in the supersymmetric case are equal to the leading order ('semiclassical') bosonic expressions (with no shift of $k$). An explicit form in which we find the sigma model couplings makes it possible to prove that the metric and the dilaton are equivalent to the fields which are obtained in the operator approach, i.e. by identifying the $L_0$-operator of the conformal theory with a Klein-Gordon operator in a background. The metric can be considered as a 'deformation' of an invariant metric on the coset space $G/H$ and the dilaton can be in general represented in terms of the logarithm of the ratio of the determinants of the 'deformed' and 'round' metrics.
1. Introduction

Given a conformal theory formulated in the operator approach [1] it is not \textit{a priori} clear which is a 2\textit{d} field theory (‘sigma model’) which corresponds to it (if such correspondence is possible at all). To have a sigma-model interpretation (at least in some limit) \textit{is} important in the case when a conformal theory is used to represent a solution of string theory. A large class of (super)conformal theories based on the \textit{G/H} coset construction [2] can be described in terms of gauged Wess-Zumino-Witten theories [3][4]. Both compact [5] and non-compact [6][7] coset models describe most of the known Euclidean (‘internal’ space) and Minkowski (black hole and cosmological) string solutions.

The first example of a sigma model interpretation of a gauged WZW model was given in [8][9] for the case of the \textit{SU}(2)/\textit{U}(1) or \textit{SL}(2,\mathbb{R})/\textit{U}(1) theory. It was found that the sigma model metric is different [8] from the standard invariant metric on the coset space and that the sigma model contains also a non-trivial dilaton coupling [9]. These two facts are, of course, not unrelated: the dilaton must be present in order to satisfy the conformal invariance condition given that the invariant metric on a homogeneous space has a non-trivial Ricci tensor [9]. The idea in [8][9] was to eliminate the 2\textit{d} gauge field from the classical action of gauged WZW model. Since this was done at the semiclassical level only, the resulting sigma model was conformal only in the leading order (in \(\alpha'\) or \(1/k\)) approximation. It was suggested in [10] that the exact expressions for the metric and the dilaton can be obtained by using the ‘operator approach’, i.e. by interpreting the \(L_0\)-operator of the corresponding coset theory as a Klein-Gordon operator in a background. The expressions proposed in [10] (which explicitly depended on \(\alpha'\)) were, in fact, found to be solutions of the sigma model conformal invariance conditions up to four loop orders [11][12].

Since the operator approach to determination of the background fields is rather heuristic, being based on a number of implicit assumptions (and do not allowing one to compute the antisymmetric tensor coupling in a straightforward way) it is desirable to have a direct
field-theoretic method of derivation of a sigma model corresponding to a coset theory which generalizes the idea of [9] to all orders in $1/k$. Such a method was recently suggested in [13] and considered also in [14]. The main point is first to replace the classical gauged WZW action by the exact effective one and then eliminate the gauge field. Since the gauged WZW theory is ‘exactly soluble’, its effective action can be found explicitly [13]. The aim of the present paper is to clarify and develop further this approach. In particular, we shall generalize the analysis of [13] to the case of supersymmetric WZW theory and derive the general expressions for the metric, dilaton and antisymmetric tensor for an arbitrary $G/H$ model (justifying and extending the results of [13] [14]). The explicit form in which we shall find the sigma model couplings will make it possible to check that the resulting metric and dilaton are the same that appear (in more abstract form) in the operator approach [10] [15] [16]. We shall also prove in general that the dilaton can be expressed essentially in terms the logarithm of the determinant of the metric (confirming previous suggestions [17] [15] [14]).

Let us first explain the basic idea of our approach. To give a sigma model interpretation to a gauged WZW theory one, should, in principle, should fix a gauge and integrate out the gauge field. In practice, this is rather difficult to do since questions of regularisation, measure, preservation of conformal invariance, etc should be properly taken into account. These issues are easy to resolve at the semiclassical level [9] but they become quite subtle once one tries to obtain exact results. A way to by-pass these complications [13] is to find first the effective action $\Gamma_{gwzw}$ in the gauged WZW theory and then identify it with the effective action $\Gamma_{sm}$ of the corresponding sigma model. Let $S(g, A, \gamma)$ be the classical action of a gauged WZW theory defined on a curved 2d background ($g$ is an element of a group $G$, $A_m$ is the 2d gauge field taking values in the algebra of a subgroup $H$). The fact that the product $\sqrt{G} e^{-2\phi}$ is $k$-independent was observed in the $SL(2, R)/U(1)$ case in [10] [17]. It was further checked on a number of non-trivial $G/H$ models, formulated as a general statement and argued for using path integral measure considerations in [13] [14].
$H$, and $\gamma_{mn}$ is a 2d metric). The quantum effective action $\Gamma_{gwzw}(g, A, \gamma)$ for the fields $g, A$ is given by

$$\Gamma_{gwzw} = S(g, A, \gamma) + \text{(quantum corrections)}$$

$$= S(g, A, \gamma) + \int d^2z \sqrt{\gamma} R\phi(A, g) + \ldots .$$  \hspace{1cm} (1.1)

Let $S(x^\mu, \gamma)$ be the classical action of a sigma model which should correspond to the gauged WZW theory

$$S(x, \gamma) = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{\gamma} \left[ \partial_m x^\mu \partial^m x^\nu G_{\mu\nu}(x) + i\epsilon^{mn} \partial_m x^\mu \partial_n x^\nu B_{\mu\nu}(x) + \alpha' R\phi(x) \right].$$  \hspace{1cm} (1.2)

Here $x^\mu (\mu = 1, \ldots, \text{dim}G/H)$ are some coordinates on $G/H$. The quantum effective action in the theory (1.2) has the following symbolic form

$$\Gamma_{sm} = S(x, \gamma) + \text{(nonlocal terms)} .$$  \hspace{1cm} (1.3)

The arguments in (1.1) and (1.3) are already classical (background) fields. The idea is to find the sigma model action $S(x, \gamma)$ by comparing the effective actions (1.1) and (1.3). That means one should solve for the gauge field in (1.1), fix a gauge and then identify the local second-derivative part of the result as $S(x, \gamma)$. It is clear that in deriving the sigma model action one can ignore all possible non-local terms which may appear in $\Gamma_{gwzw}(g, A, \gamma)$ or in the process of solving for the gauge field.\footnote{The issue of non-local terms was a matter of some confusion in [13].} The derivation of the effective action $\Gamma_{gwzw}$ which is the basic element of this approach \cite{13} will be further clarified and extended to the supersymmetric case below.

We shall begin (Sec.2) with a discussion of the operator approach. After a review of refs.\cite{10,15} we shall point out that the answers to the questions why the resulting background metric is a ‘deformed’ one (i.e. is different from the standard $G$-invariant metric on $G/H$) and why one gets a non-constant dilaton with $\sqrt{G} \ e^{-2\phi}$ being $k$-independent are closely related. There exists a close connection with the discussion \cite{18} of the analogy...
between the structure of Hamiltonians of quasi-exactly-solvable quantum mechanics systems [19] and that of the stress tensor of conformal theories based on generalised [20][18] Sugawara or affine-Virasoro construction. The scalar $\phi$ which was interpreted as an ‘imaginary phase’ in [18] is, in fact, the dilaton of the sigma model corresponding to a given conformal theory. As a result, we show in general that the dilaton which appears in the operator approach is given by the logarithm of the ratio of the determinants of the sigma model metric and an invariant metric on $G/H$, so that the combination $\sqrt{G} e^{-2\phi}$ does not depend on the matrix in the bilinear form of the currents in the stress energy tensor (in particular, it is $k$-independent). We also comment on possible application of a similar approach to providing a sigma model interpretation to conformal theories based on more general solutions of the ‘master equation’ constructed in [20][18][21] (it is clear that the dilaton field will in general be non-trivial as in the case of the coset models).

Secs.3,4,5 are devoted to the description of the field-theoretic approach to derivation of the sigma model couplings corresponding to $G/H$ coset conformal theories. As a preparation for the analysis of the gauged WZW theory case, in Sec.3 we present the expression for the effective action in the ungauged WZW theory [13]. We explicitly include the effect of field renormalisation which makes the effective action non-local. We compare our result ($S(g) = kI(g) \rightarrow \Gamma(g) = (k + \frac{1}{2}c_G)I(g')$) with other ‘effective actions’ in WZW theory which appeared in the literature [22][23][24][25]. In Sec.3.2 we generalise the analysis to the case of the $N = 1$ supersymmetric WZW theory emphasising that there is no shift of $k$ in the resulting effective action $\Gamma(g) = kI(g')$ (the shifts of $k$ due to fermionic and bosonic contributions cancel each other).

In Sec.4.1 we consider the derivation of the effective action in the bosonic gauged WZW theory. The quantisation of gauged $G/H$ WZW theory [3][4] is based on representing the corresponding path integral in terms of the path integrals in ungauged WZW theories for the group and subgroup. This makes it possible to use the analysis of the ungauged WZW theory carried out in Sec.3. We clarify the previous discussion in [13] by pointing out
that for the purpose of deriving the corresponding local sigma model action the non-local
terms in $\Gamma$ introduced by field renormalisations can be ignored. For the same reason, it
is possible to ignore the non-local terms which are of cubic and higher order in the gauge
potential. As a result, we obtain a ‘truncated’ effective action $\Gamma_{tr}(g,A)$ (4.17),(4.18) which
is quadratic in $A_m$ but still non-local. We compare our approach with that of [14] (where
a dimensionally reduced $d = 1$ form of the effective action was used) explaining that while
the sigma model metric and dilaton we should get should be equivalent to that of [14] our
direct approach makes possible also to compute the antisymmetric tensor coupling.3

In analogy with the case of the ungauged supersymmetric WZW theory (Sec.3.2) the
derivation of the effective action in gauged supersymmetric WZW theory in Sec.4.2 can
be effectively reduced to the discussion of the bosonic case. As in the bosonic case and the
case of ungauged supersymmetric WZW theory our treatment of the gauged supersym-
metric WZW theory is in correspondence with the results of the operator approach to the
superconformal coset theory [2][26][5]. We use manifestly supersymmetric approach which
is parallel to the one of ref. [1] in the bosonic case with the fields replaced by superfields
(our approach is different from the previous path integral analysis of this theory in [27]).
We find that up to the non-local corrections introduced by the field renormalisations the
effective action of gauged supersymmetric WZW theory is equal to the classical action of
the bosonic gauged WZW theory. We note that the absence of a shift of $k$ is consistent
with perturbation theory. As a consequence, the exact form in the corresponding sigma
model will be the same as the ‘semiclassical’ form of the sigma model in the bosonic theory.
This conclusion is in agreement with the one obtained in the operator approach in [12] (in
the case of the $SL(2,R)/U(1)$ supersymmetric theory) and in [13] (in the case of a general
$G/H$ supersymmetric theory).

3 The general expression for the antisymmetric tensor coupling we shall get is equivalent to
the expression already suggested in [14] using the analogy with the result for the metric. Ref. [14]
contains also the derivation (without assuming the 1$d$ reduction) of the antisymmetric tensor in
a particular case of the $SL(2,R) \times SO(1,1)/SO(1,1)$ ($D = 3$ “black string”) model.
In Sec. 5 we start with the ‘truncated’ effective action of the gauged WZW theory and eliminate the gauge field (Sec. 5.1). The local part of the resulting action is then put in the sigma model form (Sec. 5.2) and the corresponding metric, antisymmetric tensor and dilaton couplings are determined (the general form of the sigma model couplings we find is equivalent to the expressions presented in [14]). The background fields are non-trivial functions of the parameter $b = \frac{c_H - c_G}{2(k + \frac{3}{4}c_G)} = \frac{1}{4}(c_H - c_G)\alpha'$ and describe a large class of conformal sigma models. In Sec. 5.3 we put the metric into a more explicit form by making the key observation that as a consequence of the gauge invariance (before gauge fixing) the metric defined on the full group space has $\text{dim}H$ null vectors. We represent the metric in terms of a particular basis orthogonal to the null vectors and compute its inverse and determinant. The metric can be considered as a deformation of a ‘standard’ invariant metric on the coset space $G/H$.

Finally, in Sec. 6 we establish the equivalence between the results for the metric and dilaton found in the operator and field-theoretic approaches. In particular, we explicitly prove that dilaton and the metric found in Sec. 5 satisfy relation $\sqrt{G} e^{-2\phi} = \sqrt{G^{(0)}}$ where $G^{(0)}_{\mu\nu}$ is the standard metric on $G/H$ (and is $k$-independent).

2. Operator Approach to Derivation of Background Geometry Corresponding to Coset Conformal Theories

2.1. Basic ideas

A possible strategy of determining the geometry corresponding to a given conformal theory is to try to interpret the Virasoro condition $(L_0 + \bar{L}_0 - 2)F = 0$ on states as linear field equations in some background and to extract the expressions for the background fields from the explicit form of the differential operators involved. The marginal operators $F$ of conformal theory serve as ‘probes’ of geometry, so that one may be able to determine the corresponding metric, etc. from their equations just as from geodesic equations or field equations in a curved space.
In order to implement this program one is to make a number of important assumptions.
First, one should specify which configuration ('target') space $M$ (with coordinates $x^\mu$, $\mu = 1, ..., D$) should be used, so that $F$ will be parametrised by fields on $M$, and $L_0$ acting on $F$ will reduce to differential operators on $M$. Next, one should understand how to represent the resulting equations in terms of background fields. The main assumption is that the conformal theory should correspond to a sigma model

$$S = \frac{1}{4\pi\alpha'} \int d^2 z \sqrt{\gamma} \left[ \partial_m x^\mu \partial^m x^\nu G_{\mu\nu}(x) + i \epsilon^{mn} \partial_m x^\mu \partial_n x^\nu B_{\mu\nu}(x) \right. $$

$$\left. + \alpha' R\phi(x) + T(x) + ... \right]. \quad (2.1)$$

If this assumption is true (for example, if a Lagrangian formulation of a conformal theory is known and the existence of a sigma model representation can be checked in the 'semi-classical' approximation) then the 'anomalous dimension operator', i.e. the derivative of the $\beta$-functions at the conformal point $(\frac{\partial \beta_i}{\partial \phi})_*$ should be equivalent to the 'Klein-Gordon' operator $L_0 + \bar{L}_0$ for the corresponding marginal perturbations of conformal theory. One is thus to invoke the knowledge of the structure of the sigma model conformal anomaly coefficients ('$\beta$-functions'), or the effective action which generates them

$$S = \int d^D x \sqrt{G} e^{-2\phi} \left\{ \frac{2}{3}(D - 26) - \alpha'[R + 4(\partial_\mu \phi)^2 - \frac{1}{12} H_{\lambda\mu\nu}^2] \right. $$

$$\left. + \frac{1}{16} [\alpha'(\partial_\mu T)^2 - 4T^2] + ... \right\} . \quad (2.2)$$

One should start with this background-independent action, linearise the corresponding equations near an arbitrary background, and compare them with the equations for the corresponding states in conformal theory. The equations for the tachyon, graviton, dilaton and the antisymmetric tensor perturbations ($T$, $h = G - G_*$, $\varphi = \phi - \phi_*$, $b = B - B_*$; in what follows we shall omit the superscript $*$ indicating background fields) take the following symbolic form ($\alpha' = 1$)

$$(-\Delta + 2G^{\mu\nu} \partial_\mu \phi \partial_\nu)T - 4T + ... = 0 , \quad \Delta \equiv \frac{1}{\sqrt{G}} \partial_\mu (\sqrt{G} G^{\mu\nu} \partial_\nu) , \quad (2.3)$$
Given a second order differential equation which follows from the $L_0$-condition for the lowest scalar ‘tachyonic’ state it should be possible to determine the corresponding background metric and dilaton by looking at the coefficients of the terms which are second and first order in derivatives and comparing them with (2.3). To determine the antisymmetric tensor field strength one should compare the first-derivative terms in the equations for ‘massless’ perturbations with the corresponding terms in (2.4), (2.5), (2.6).

It should be emphasised that if this approach works at all, its consistency should not be surprising. If the correspondence between a conformal sigma model and a conformal field theory exists in a given case, then solutions of the conformal invariance conditions that follow from (2.2) should represent a conformal point; their perturbations should correspond to marginal perturbations of a conformal theory, i.e. the equations for the latter must have the form of (2.3)–(2.6).

2.2. $SL(2, \mathbb{R})/U(1)$ model

This ‘operator’ approach was used in [10] to determine the exact metric and dilaton backgrounds corresponding to the $SL(2, \mathbb{R})/U(1)$ model (see also [12] for the supersymmetric case) and was later applied to more general $G/H$ coset models in [15]. Related ideas were discussed in [28] [18]. Let us first briefly recall the argument from [10]. The stress tensor of the model can be represented in the form

$$T_{zz} = \frac{1}{k-2} \eta^{AB} J_A(z) J_B(z) - \frac{1}{k} (J_3(z))^2,$$

where $\eta_{AB}$ is the metric on the Lie algebra of $SL(2, \mathbb{R})$. It is sufficient to consider only the zero modes. The equations for the physical scalar state (tachyon) $T(g) = T(r, \theta_L, \theta_R)$ ($r, \theta_L, \theta_R$ are coordinates on $SL(2, \mathbb{R})$) are

$$\left(L_0 + \bar{L}_0 - 2\right) T = 0, \quad (2.7)$$
From the expressions for the zero modes of the left (and right) currents (we shall use the same notation $J_A$ for the zero modes of the currents $J_A(z)$)

$$J_{\pm} = e^{\pm i\theta_L} \left[ \frac{\partial}{\partial r} \pm i(\sinh r)^{-1} \left( \frac{\partial}{\partial \theta_R} - \cosh r \frac{\partial}{\partial \theta_L} \right) \right], \quad J_3 = i \frac{\partial}{\partial \theta_L}$$

one finds that

$$L_0 = -\frac{1}{k-2} \Delta_0 - \frac{1}{k} \frac{\partial^2}{\partial \theta_L^2}, \quad \bar{L}_0 = -\frac{1}{k-2} \Delta_0 - \frac{1}{k} \frac{\partial^2}{\partial \theta_R^2},$$

$$\Delta_0 = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + (\sinh r)^{-2} \left( \frac{\partial^2}{\partial \theta_L^2} - 2 \cosh r \frac{\partial}{\partial \theta_R \partial \theta_L} + \frac{\partial^2}{\partial \theta_R^2} \right),$$

so that (2.8) is satisfied if

$$T = T(r, \theta) + \bar{T}(r, \bar{\theta}), \quad \theta \equiv \frac{1}{2}(\theta_L - \theta_R), \quad \bar{\theta} \equiv \frac{1}{2}(\theta_L + \theta_R).$$

Restricting $L_0$ to $T(r, \theta)$ we get

$$L_0 = -\frac{1}{k-2} \left[ \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + (\coth^2 \frac{r}{2} - \frac{2}{k}) \frac{\partial^2}{\partial \theta^2} \right].$$

As a result, eq.(2.7) can be represented as a covariant Laplace equation

$$2(L_0 - 1)T(r, \theta) = \left[ -\frac{1}{e^{-2\phi} \sqrt{G}} \partial_\mu (e^{-2\phi} \sqrt{G} G^{\mu\nu} \partial_\nu) - 2 \right] T = 0,$$

where $x^\mu = (r, \theta)$ and

$$G_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2} (k-2) [dr^2 + f(r) d\theta^2], \quad f(r) = \frac{4 \tanh^2 \frac{r}{2}}{1 - \frac{2}{k} \tanh^2 \frac{r}{2}},$$

$$\phi = \phi_0 - \frac{1}{2} \ln \sinh r + \frac{1}{4} \ln f(r).$$

The metric (2.10) can be rewritten also as follows

$$G_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2} (k-2) [dr^2 + e^{4(\phi - \phi_0)} \sinh^2 r d\theta^2].$$
Hence the deformation of the metric from the canonical metric on the homogeneous space $SL(2, R)/U(1)$ (or the standard metric on $S^2$ in the case of the $SU(2)/U(1)$ model where $r = i\varphi$) can be attributed to the presence of a non-constant dilaton background. One can also give an opposite interpretation: the fact that the metric which is extracted from the $L_0$-operator of conformal theory turns out to be different from the invariant metric on the coset space implies the presence of a non-constant dilaton background. Note also that the “measure” combination

$$\sqrt{G} \ e^{-2\phi} = a \ \sinh r, \quad a = \frac{1}{2} e^{-2\phi_0}(k - 2) = \text{const}$$

is proportional to the canonical measure on the coset and thus is essentially independent of $k$ (one can make $a = 1$ by an appropriate choice of the constant $\phi_0$). All these observations are not accidental and will be given a systematic explanation below.

The leading-order form of the expansion of the metric (2.10) and the dilaton (2.11) in powers of $1/k$ solves the one-loop Weyl invariance conditions for the corresponding $D = 2$ sigma model [29][9]. To satisfy the “beta-function” equations at higher loop orders one must include corrections to the metric and the dilaton which (up to a field redefinition) are in agreement with the exact representation (2.10),(2.11) as was checked up to three- and four-loop orders in [11] and [12]. A similar argument in the case of supersymmetric model (where $1/(k - 2)$ in $T_{zz}$ is replaced by $1/k$ and thus there is no $2/k$ term in the brackets in $L_0$) shows that there are no $1/k$ corrections to the leading-order metric and the dilaton of the bosonic Euclidean black-hole background [12][13] (this is again consistent with the sigma model perturbation theory up to five loops [12]).

2.3. **General expressions for the metric and dilaton in the case of affine-Virasoro construction**

It is straightforward to give a formal generalisation of the above analysis to the case of an arbitrary $G/H$ coset model (see [15][16]). Let $T_A = (T_a, T_i)$ be the generators of $G$ where
$T_a$ are the generators of $H \ (A = 1, \ldots, D_G; \ a = 1, \ldots, D_H; \ i = 1, \ldots, D; \ D = D_G - D_H)$.

Let us consider again the equations (2.7),(2.8) for the lowest level physical $H$-invariant scalar state $T(x^\mu)$, where $x^\mu (\mu = 1, \ldots, D)$ are coordinates on $G/H$ ($D$ combinations of coordinates $x^M$ on $G$ which are invariant under $H$). Now the zero mode part of $L_0$ is given by

$$L_0 = \frac{1}{2} \left( \frac{1}{k + \frac{1}{2}c_G} \eta^{AB} J_A J_B - \frac{1}{k + \frac{1}{2}c_H} \eta^{ab} J_a J_b \right),$$

where

$$f^{ACD} f^{B}_{CD} = c_G \eta^{AB}, \quad f^{acd} f^{b}_{cd} = c_H \eta^{ab}$$

and $J_0^A$ and $J_0^a$ are the zero modes of the (left) currents corresponding to the group and the subgroup. Representing the currents in terms of differential operators

$$J_A = E^M_A(x) \frac{\partial}{\partial x^M}, \quad J_a = E^M_a(x) \frac{\partial}{\partial x^M},$$

and considering only $H$-invariant states, i.e. $(J_a - \bar{J}_a) T = 0$, we can identify $L_0 - 1$ acting on $T(x^\mu)$ with the covariant Laplace operator (2.9). Then the background metric and the dilaton are determined by (we restrict all the fields to depend on $x^\mu$ only)

$$G^{\mu\nu} = -\left( \frac{1}{k + \frac{1}{2}c_G} E^{A\mu} E^\nu_A - \frac{1}{k + \frac{1}{2}c_H} E^{a\mu} E^\nu_a \right), \quad (2.12)$$

$$\partial_\mu \ln (\sqrt{G} e^{-2\phi}) = -G_{\mu\nu}(\frac{1}{k + \frac{1}{2}c_G} E^{A\lambda} \partial_\lambda E^\nu_A - \frac{1}{k + \frac{1}{2}c_H} E^{a\lambda} \partial_\lambda E^\nu_a).$$

This metric and the dilaton were computed explicitly for a number of models (in particular, for $SO(D - 1, 2)/SO(D - 1, 1)$ with $D = 3, 4$) in [13] and it was observed that as in the $D = 2$ case the combination $\sqrt{G} e^{-2\phi}$ is $k$-independent.

One is naturally led to the following questions: why is the resulting metric different from the standard $G$-invariant metric on $G/H$, why do we get a non-constant dilaton and why is $\sqrt{G} e^{-2\phi}$ $k$-independent? These questions turn out to be closely related. Our approach is partly inspired by the discussion [18] of the analogy between the structure of Hamiltonians of quasi-exactly-solvable (QES) quantum-mechanical systems [19] and that
of the stress tensor of conformal theories based on affine-Virasoro construction \cite{20,18}. Given a compact finite-dimensional (simple) Lie group $G$ and a set of its generators $J_A$, the Hamiltonian of the QES system can be put in the form (we shall ignore possible terms which are linear in $J_A$)

$$\mathcal{H} = C^{AB} J_A J_B \ ,$$

where $C_{AB}$ is a constant symmetric matrix. At the same time, the holomorphic stress tensor of a conformal theory based on generalised Sugawara construction \cite{20,18} is represented by

$$T_{zz} = C^{AB} J_A(z) J_B(z) \ ,$$

where $J^A(z)$ are the generators of an affine (Kac-Moody) algebra determined by the structure constants $f_{BC}^A$ and the Killing metric $\eta_{AB}$ of $G$ (with the central term proportional to $\kappa_{AB} = k \eta_{AB}$). The condition that $T_{zz}$ should satisfy the Virasoro algebra imposes a ‘master equation’ on $C^{AB}$ \cite{20,18}

$$C^{AB} = 2 C^{AC} \kappa_{CD} C^{DB} - C^{CD} C^{KL} f_{CK}^A f_{DL}^B - C^{CD} f_{CL}^K f_{DK}^{(A} C^{B)L} \ ,$$

(2.15)

(the central charge of the Virasoro algebra is $C = 2\kappa_{AB} C^{AB}$). The standard Sugawara-GKO solution of (2.15) is

$$C^{AB} = \frac{1}{k + \frac{1}{2}C_G} \eta^{AB} - \frac{1}{k + \frac{1}{2}C_H} \eta_{\bar{H}}^{AB} \ ,$$

(2.16)

where $\eta_{\bar{H}}^{AB}$ denotes the projector on the Lie algebra of $H$.

As is clear from the above discussion, the background metric corresponding to a conformal theory based on the stress tensor (2.14) can be determined by looking only at the zero mode part of (2.14), i.e. at its ‘dimensionally reduced’ analogue (2.13). As was discussed in \cite{19,18}, if the generators $J_A$ of $G$ can be realised as vector fields on a manifold $M$ (which will play the role of a configuration space of a quantum mechanical system) and are anti-Hermitian with respect to a scalar product defined by some metric $G^{(0)}$ on $M$ then
the operator (2.13) reduces to a covariant Laplacian on $M$ with the metric determined by $C^{AB}$ and $G^{(0)}$ and an extra scalar field (the scalar field term can be traded for a potential by a phase transformation). In the case when the Hamiltonian (2.13) originates from a conformal theory $C^{AB}$ is not arbitrary but is a solution of the ‘master equation’ (2.15). Also, the choice of the configuration space $M$ is implicitly dictated by the conformal theory.

In general, unless we consider particular solutions of (2.15) which may have non-trivial extra symmetries (commuting operators), the only natural choice for a configuration space is the group space $G$ itself. Representing the zero modes of the currents $J^A(z)$ and $\bar{J}^A(z)$ as differential operators on $G$ (with coordinates $x^M$)

$$J_A = E^M_A(x)\partial_M, \quad \bar{J}_A = \bar{E}^M_A(x)\partial_M,$$

$$G_{0MN} = \eta_{AB}E^A_ME^B_N = \eta_{AB}\bar{E}^A_M\bar{E}^B_N,$$

$$[J_A, J_B] = f^C_{AB}J_C, \quad [\bar{J}_A, \bar{J}_B] = f^C_{BA}\bar{J}_C, \quad [J_A, \bar{J}_B] = 0,$$

($E^M_A$ and $E^M_A$ are the left-invariant and right-invariant vielbeins on $G$; indices are raised and lowered with $\eta_{AB}$ and $G_{0MN}$) we get from the zero mode part of $(L_0 + \bar{L}_0 - 2)F = 0$ the following equation for the lowest scalar state $T(x)$

$$[-(G^{MN}\partial_M\partial_N + G^N\partial_N) - 2]T(x) = 0,$$

$$G^{MN} = -\frac{1}{2}C^{AB}(E^M_A E^N_B + \bar{E}^M_A \bar{E}^N_B), \quad G^N = -\frac{1}{2}C^{AB}(E^M_A \partial_M E^N_B + \bar{E}^M_A \partial_M \bar{E}^N_B).$$

Eq.(2.20) becomes equivalent to the sigma model equation (2.3) if there exists a scalar $\phi$ such that

$$G^N = G^{MN}\partial_M \ln (\sqrt{G} e^{-2\phi}).$$

In fact, such $\phi$ can be found explicitly by using the obvious properties of $E^A_M$, or, equivalently, by observing that that since $J^A$ and $\bar{J}^A$ are anti-Hermitian with respect to the invariant scalar product on the group defined by $G_{0MN}$, $(f,g) = \int d^Dx\sqrt{G}f^*(x)g(x)$.

One has

$$\partial_M E^M_A = -E^M_A \partial_M \ln \sqrt{G_0}, \quad E^A_M \partial_M E^M_A = -E^M_A \partial_M E^A_N,$$
\[ \partial_M E^A_N - \partial_N E^A_M = f^A_{BC} E^B_M E^C_N . \]

As a consequence,
\[ \phi = \frac{1}{2} \ln \sqrt{G/G_0} , \quad \text{i.e.} \quad \sqrt{G} e^{-2\phi} = \sqrt{G_0} . \]

The ‘measure factor’ in (2.22) is thus universal, i.e. is independent of \( C^{AB} \). To understand a simple origin of this result one should compare the term in the effective action (2.2) leading to (2.3)
\[ \int d^D x \sqrt{G} e^{-2\phi} G^{MN} \partial_M T \partial_N T \]
with the ‘expectation value’ of the zero-mode ‘Hamiltonian’
\[ (T, \mathcal{H}T) = \int d^D x \sqrt{G_0} T(x) \mathcal{H}T(x) , \quad \mathcal{H} = \frac{1}{2} C^{AB} (J_A J_B + \bar{J}_A \bar{J}_B) \]
and use the antihermiticity of the currents with respect to the left-right symmetric measure defined by \( G_0 \). The dilaton’s role is to compensate for the fact that the two scalar products have different measures.

The dilaton field is non-trivial because in general the metric \( G_{MN} \) is not equal to the canonical Killing metric \( G_{0MN} \) on \( G \). The dilaton is constant only in the simplest case of the standard Sugawara solution \( C^{AB} \sim \eta^{AB} \) corresponding to the group \( G \). A ‘deformation’ of the metric is directly related to the non-triviality of \( C^{AB} \) which is dictated by the conformal invariance (Virasoro) condition (2.15). If there exists the corresponding Lorentz-invariant sigma model it should also contain the antisymmetric tensor coupling.

Note that the determinant of \( G^{MN} \) in (2.21) can not be factorised into the product of the determinant of \( C^{AB} \) and the rest because \( \tilde{E}_A^M \) is different from \( E_A^M \).

For an attempt to construct a field-theoretic realisation of the affine-Virasoro construction see [30], where, in particular, the presence of the antisymmetric tensor coupling was pointed out. One may expect that it may be possible to reinterpret the ‘master equation’ (2.15) as the (one-loop) Weyl invariance condition \( \bar{R}_{MN} + 2 D_M D_N \phi = 0 \) for the metric \( G_{MN} \), antisymmetric tensor and the dilaton (the operator product relation from which (2.16) is derived probably corresponds to using (a non-local) renormalisation scheme in which only a one-loop contribution is present in the sigma model Weyl anomaly coefficients; the condition of Weyl invariance at higher loop orders will be automatically satisfied as a consequence of the 1-loop relation and the Kac-Moody algebra). This is to be compared with the approach of [30] where (2.16) was interpreted as an Einstein-like equation on the group space but the dilaton was not introduced.
There seems to emerge an interesting connection between algebraic and geometric aspects of such conformal theories (and corresponding string solutions). The geometry is determined by a choice of the group and a choice of a particular solution of the ‘master equation’. The question about a relation between group-theoretic and geometric aspects of a similar construction was raised independently in the quantum mechanical context in \cite{18} \cite{19} and in \cite{30}. Now we see that once the condition of conformal invariance is satisfied, the geometry which appears is that of the corresponding string solutions described by conformal sigma models.

2.4. Explicit expressions for the (inverse) metric and dilaton in the case of the $G/H$ coset conformal theory

Let us now return to the $G/H$ case, i.e. specialise the general expressions (2.21),(2.22) to the solution of (2.15) corresponding to the $G/H$ coset conformal theory with $C^{AB}$ given by (2.17). Here the main assumption of an existence of the sigma model description is satisfied in view of the existence of the Lagrangian formulation in terms of gauged WZW models (this assumption can be checked explicitly, e.g., in the semiclassical approximation).

In this case there is an extra symmetry which makes it possible to subject the states to the ‘$H$-invariance’ condition $(J_a - \bar{J}_a)F = 0$. In particular,

$$(J_a - \bar{J}_a)T = 0 , \quad Z^M_a \partial_M T = 0 , \quad Z^M_a \equiv E^M_a - \bar{E}^M_a .$$  \hspace{0.5cm} (2.23)

As a result, $T$ can be restricted to depend only on $D = D_G - D_H$ coordinates $x^\mu$ of the coset space $G/H$ which will thus play the role of the configuration space of the corresponding sigma model. The presence of the constraint (2.23) implies that the metric we will get from (2.20),(2.21) will be a ‘projected’ one. Let us define the projection operator on the subspace orthogonal (with respect to $G_{0MN}$) to $Z^M_a$

$\Pi^N_M \equiv \delta^N_M - Z^N_a (ZZ)^{-1ab} Z_{Mb} , \quad (ZZ)_{ab} = G_{0MN} Z^M_a Z^N_b , \quad \Pi^2 = \Pi . \hspace{0.5cm} (2.24)$
Then

\[ G^{MN} = \Pi^M_K \hat{G}^{KL} \Pi^N_L, \quad \hat{G}^{MN} = \frac{1}{k + \frac{1}{2} c_G} \eta^{AB} E^M_A E^N_B - \frac{1}{k + \frac{1}{2} c_H} \eta^{ab} E^M_a E^N_b, \quad (2.25) \]

\[ \hat{G}^{MN} = \frac{1}{k + \frac{1}{2} c_G} (E^M_A E^{AN} - \gamma E^M_a E^{aN}) = \frac{1}{k + \frac{1}{2} c_G} [E^M_i E^iN - (\gamma - 1) E^M_a E^{aN}], \quad (2.26) \]

\[ \gamma = \frac{k + \frac{1}{2} c_G}{k + \frac{1}{2} c_H}, \quad \gamma - 1 = \frac{c_G - c_H}{2(k + \frac{1}{2} c_H)}. \quad (2.27) \]

We have split the indices \( A = (a, i), \ i = 1, \ldots, D \) on the indices corresponding to the subalgebra and the indices corresponding to the tangent space to \( G/H \). If one solves (2.23) explicitly, replacing \( x^M \) by the coset space coordinates \( x^\mu \), which are some \( D \) invariant combinations of \( x^M \) such that

\[ Z^M_a H^\mu_M = 0, \quad H^\mu_M = \frac{\partial x^\mu}{\partial x^M}, \quad (2.28) \]

then the metric (2.25) takes the form

\[ G^{\mu\nu} = H^\mu_M \hat{G}^{MN} H^\nu_N = H^\mu_M \hat{G}^{MN} H^\nu_N , \]

\[ G^{\mu\nu} = \frac{1}{k + \frac{1}{2} c_G} (E^{\mu A} E^\nu_A - \gamma E^{\mu a} E^\nu_a) , \quad E^\mu_A \equiv H^\mu_M E^M_A. \quad (2.29) \]

In the simplest case \( H^\mu_M = \delta^\mu_M \). More generally, one can choose any set of vectors \( H^\mu_M \) which are orthogonal to \( Z^M_a \). As a result, we will get again eqs. (2.20)–(2.22) with the tensor indices \( M, N, \ldots \), restricted to \( G/H \), i.e. replaced by \( \mu, \nu, \ldots \). Since in the present case (under the constraint (2.23)) the operators \( J_A \) are anti-Hermitian with respect to the invariant metric on the coset

\[ G^{(0)}_{\mu\nu} = 0, \quad \phi = \frac{1}{2} \ln \sqrt{\frac{G}{G^{(0)}}}. \quad (2.31) \]

we find as in (2.22)

\[ \phi = \frac{1}{2} \ln \sqrt{\frac{G}{G^{(0)}}}. \quad (2.31) \]

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Similar expression for the scalar $\phi$ was given in [18] where $\phi$ was interpreted in as an ‘imaginary phase’ since one can replace the scalar term $G^{\mu\nu}\partial_\mu \phi \partial_\nu$ by a potential, performing a similarity transformation.

The above expressions (2.29)–(2.31) lead us again to the following conclusions:

1. the metric $G_{\mu\nu}$ is different from the standard $G$-invariant metric $G^{(0)}_{\mu\nu}$ on $G/H$ because the matrix $C^{AB}$, in general, is different from the Killing metric $\eta^{AB}$ of $G$;

2. the presence of a non-trivial dilaton is a consequence of $G_{\mu\nu} \neq G^{(0)}_{\mu\nu}$, i.e. of $C^{AB} \neq \text{const} \, \eta^{AB}$;

3. $\sqrt{G} \, e^{-2\phi}$ is equal to the $G$-invariant measure factor $\sqrt{G^{(0)}}$ on $M = G/H$ and thus automatically does not depend on $C^{AB}$. In particular, it does not depend on the parameter $k$ of the coset conformal theory (2.17). This provides a simple explanation of the fact of $k$-independence of $\sqrt{G} \, e^{-2\phi}$ anticipated (on the basis of explicit examples and path integral measure considerations) in [17][31].

It should certainly be possible to compute also the antisymmetric tensor background by comparing the equation for a massless $(1, 1)$ state with (2.4)–(2.6). We shall find the antisymmetric tensor and also reproduce the expressions for the metric (2.29) and the dilaton (2.31) in Secs.5,6 by using the direct field-theoretical approach starting with the gauged WZW theory (which provides a Lagrangian formulation of the coset conformal theory).

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6 It was assumed in [18] that the generators $J_A$ can be realised as vector fields on the homogeneous space $M = G/H$. In the context of quantum mechanical applications, the choice of $H$ need not be related to the choice of $C^{AB}$. However, from the point of view of determining the geometrical background corresponding to a coset conformal field theory the possibility to realise the generators $J_A$ as vector fields on $G/H$ is related to the fact that it is sufficient to restrict consideration to $H$-invariant states. A one-dimensional quantum mechanical system with the configuration space $M$ can be considered at the same time as a dimensional reduction of a 2d sigma model corresponding to the conformal theory.

7 Then $\mathcal{H} = e^\phi \mathcal{H}' e^{-\phi}$, $\mathcal{H}' = -\Delta + V$, $V = G^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - \Delta \phi$. It is interesting to note that when $G_{\mu\nu}$ and $\phi$ satisfy the sigma model Weyl invariance conditions the leading order term in the potential $V$ is equal to $\frac{1}{4} R$ (plus a constant central charge deficit).
Let us mention that one can also use the operator approach to determine the background geometry in the case of superconformal coset models. For example, one can give a sigma model interpretation to the $N = 2$ Kazama-Suzuki models $[5]$ based on Kähler $G/H$ spaces. The corresponding metrics will be different from the invariant Kähler metrics on $G/H$ and the dilaton will be non-trivial (see $[15]$).

It is clear, at the same time, that the operator approach has a number of obvious shortcomings. It is indirect and based on a number of implicit assumptions. If a given conformal theory admits a Lagrangian formulation (and there exists a weak coupling limit) then it should be possible to derive the corresponding exact sigma model action using field-theoretical methods. This will be demonstrated below for the case of $G/H$ model.

3. Effective Action in WZW Theory

3.1. Bosonic WZW theory

Below we shall first review the argument which leads to the expression for the effective action in WZW theory suggested in $[13]$ and then comment on relations to other approaches. Up to a field renormalisation, our effective action is essentially equal to the classical WZW action with the shifted $k$, $k \rightarrow k + \frac{1}{2} c_G$ ($c_G$ = the eigenvalue of the second Casimir operator in the adjoint representation). From one point of view, the shift is related to the Legendre transformation involved; from another, it is a one-loop phenomenon originating from a determinant (cf.$[32][33]$).

The WZW theory $[34][35]$ is defined by the action

$$S = kI(g) \, , \quad I \equiv \frac{1}{2\pi} \int d^2z \, \text{Tr} \left( \partial g^{-1} \tilde{\partial} g \right) + \frac{i}{12\pi} \int d^3z \, \text{Tr} \left( g^{-1} dg \right)^3 \, . \quad (3.1)$$

As it is easy to see, using the Polyakov-Wiegmann identity $[36]$,

$$I(ab) = I(a) + I(b) - \frac{1}{\pi} \int d^2z \, \text{Tr} \left( a^{-1} \partial a \, \tilde{\partial} b^{-1} \right) \, , \quad (3.2)$$
the generating functional \( W(B) \) of the correlators of the currents

\[
e^{-W(B)} = \int [dg] \ e^{-S(g)+B\bar{j}(g)} ,
\]

(3.3)

\[
B\bar{j}(g) \equiv \frac{k}{\pi} \int d^2z \ \text{Tr} (B\bar{J}), \quad \bar{J} \equiv \bar{g}g^{-1},
\]

(3.4)
is given by \[34\] \[22\]

\[
W(B) = -kI(u) , \quad B = u^{-1}\partial u .
\]

(3.5)

\( W \) does not receive quantum corrections being equal to the classical action evaluated on the classical solution depending on \( B [32] \).

We would like to determine the quantum effective action \( \Gamma \) for the original chiral field \( g \) itself. As discussed in \[13\] it can be represented as a ‘quantum’ Legendre transform of \( W(B) \)

\[
e^{-\Gamma(g)} = \int [dB] \ e^{-W(B)+B\bar{j}(g)} = \int [dg'] \ e^{-S(g')} \ \delta[\bar{J}(g') - \bar{J}(g)] .
\]

(3.6)

To compute (3.6) we change the variable from \( B = u^{-1}\partial u \) to \( u \) and define the resulting Jacobian in the left-right symmetric way as the square root of the non-chiral determinant \[36\] \[37\] \[38\]

\[
\det (\partial + [B, ]) \det (\bar{\partial} + [\bar{B}, ]) = \exp [c_GI(v^{-1}u)] \det \partial \det \bar{\partial} ,
\]

(3.7)

\[
B = u^{-1}\partial u , \quad \bar{B} = v^{-1}\bar{\partial}v
\]

with \( \bar{B} = 0 \), i.e.

\[
[dB] = [du] \ \text{“det(\partial + [B, ]”)} = [du] \ \exp [\frac{1}{2}c_GI(u)] \ (\det\partial\bar{\partial})^{1/2} .
\]

(3.8)
The extra \( \frac{1}{2} \) in front of \( c_G \) in (3.8) which is absent in the standard expression for the chiral determinant is very important, being a way of implementing the left-right symmetry of the theory. It is necessary in order to get the correct shift of \( k \) in the final expression for the effective action. We find

\[
e^{-\Gamma(g)} = N \int [du] \ \exp [(k + \frac{1}{2}c_G)I(u) + B(u)\bar{j}(g)] .
\]

(3.9)
This integral is computed by the same method as (3.3), i.e. by using the ‘non-abelian
generalisation of the gaussian integral’ ([36] (see (3.2),(3.4),(3.5)) and we get

\[ \Gamma(g) = (k + \frac{1}{2}c_G)I(g') , \]  

(3.10)

where the field renormalization is understood in the following sense

\[ \bar{\partial}g'g'^{-1} = \frac{k}{k + \frac{1}{2}c_G} \bar{\partial}gg^{-1} . \]

(3.11)

Eq.(3.11) corresponds to the renormalisation of the current \( k\bar{J} = (k + \frac{1}{2}c_G)\bar{J}' \).

This action has the right symmetries (conformal and chiral \( G \times G \) invariance) one would like to preserve at the quantum level.\(^8\) Note that because of the field renormalisation (3.11) the effective action becomes non-local when expressed in terms of the original field \( g \). Using the parametrisation \( \bar{\partial}gg^{-1} = T_A \bar{E}_M^A(x)\bar{\partial}x^M \) (\( T_A \) are the generators of \( G \)) one can solve (3.11) within the \( 1/k \) - expansion,

\[ x'^M = x^M + \sum_{n=1}^{\infty} \frac{1}{k^n} y_n^M (x) . \]

The corrections \( y_n^M \) contain inverse powers of differential operators, i.e. are non-local functionals of \( x^M(z) \). For example,

\[ y_1^M = -\frac{1}{2}c_G P_K^M \bar{\partial}x^K , \quad (P^{-1})_K^M \equiv \delta_K^M \bar{\partial} + \bar{E}_A^M \partial_K \bar{E}_N^A \bar{\partial}x^N . \]

The functional \( \Gamma(g) \) (3.6) should be equivalent to the standard generating functional of 1-PI correlators of the field \( g \) itself. Although this is not obvious, the resulting action (3.10) is perfectly consistent with the presence of the shifted \( k \) in the quantum equations of motion and the stress tensor in the operator approach to WZW model as conformal theory [40].

\[ (k + \frac{1}{2}c_G)\bar{\partial}g(z, \bar{z}) = :J_A(z)g(z, \bar{z})T^A : , \quad (k + \frac{1}{2}c_G)\bar{\partial}g(z, \bar{z}) = :\bar{J}_A(\bar{z})T^Ag(z, \bar{z}) : , \]

\(^8\) It may be possible to prove that these symmetries fix \( \Gamma \) uniquely up to the two (k- and current-) renormalisation constants using, e.g., the ‘quantum action principle’ and BRST cohomology method, cf. [33].
\[ T_{zz} = \frac{1}{k + \frac{1}{2} c_G} \eta^{AB} J_A(z) J_B(z) : . \]

The action (3.10) can be considered as a ‘classical’ representation of these quantum relations with the normal ordering suppressed.\(^9\)

Alternatively, one may start with the assumption that the effective action \( \Gamma(g) \) in the WZW theory must satisfy conformal and chiral \( G \times G \) invariance conditions. Then a natural (and probably unique) choice for such \( \Gamma \) is the classical action itself, up to the renormalisations of \( k \) and the current, \( \Gamma(g) = k' I(g'), \ \bar{\partial} g' g'^{-1} = Z \bar{\partial} g g^{-1} \). Correspondence with the c.f.t. approach then fixes \( k' = k + \frac{1}{2} c_G \) (and probably fixes also \( Z = \frac{k}{k + \frac{1}{2} c_G} \)). A possibility to find an exact expression for the effective action of the WZW theory should not be surprising, given its solubility in the operator approach.

Computed on a curved 2d background the quantum effective action contains also the usual Weyl anomaly term (see e.g. [32])\(^10\)

\[ \Gamma_{anom}(\gamma) = -\frac{C}{96\pi} \int R \Delta^{-1} R, \quad C = \frac{k D_G}{k + \frac{1}{2} c_G}. \quad (3.12) \]

As is well known [36], the WZW action (3.1) can also be represented as a non-local functional of the current

\[ I(g) = \omega(J), \quad J = g^{-1} \bar{\partial} g, \quad (3.13) \]

\[ \omega(J) = -\frac{1}{\pi} \int d^2 z \ Tr \left\{ J \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 2} (\frac{1}{\bar{\partial} J, J})^n \bar{\partial} J \right\} \]

\[ = -\frac{1}{\pi} \int d^2 z \ Tr \left\{ \frac{1}{2} \bar{\partial} J \frac{1}{\bar{\partial} J} - \frac{1}{3} J [\bar{\partial} J, \frac{1}{\bar{\partial} J} J] + ... \right\}, \quad (3.14) \]

\(^9\) Note that the background value of the stress tensor should be given by the variation of the effective action over the 2d metric. As follows from (3.10),(3.11), \( T_{zz} = (k + \frac{1}{2} c_G)(\bar{\partial} g g^{-1})^2 = \frac{1}{k + \frac{1}{2} c_G} (k \bar{\partial} g g^{-1})^2. \) The field renormalisation in (3.10) is actually a renormalisation of the currents, equivalent to the one in the quantum equations of motion.

\(^{10}\) There is no extra term which is a non-trivial functional of both \( g \) and the 2d metric \( \gamma_{mn} \) since the vanishing of the \( \beta \)-function of the WZW model implies that the operator of the trace of the stress tensor is proportional to \( CR \).
or,
\[ I(g) = \tilde{\omega}(\bar{J}) , \quad \bar{J} = \tilde{\partial}g^{-1}, \quad (3.15) \]
\[
\tilde{\omega}(\bar{J}) = -\frac{1}{\pi} \int d^2 z \text{ Tr} \left\{ \sum_{n=0}^{\infty} \frac{1}{n+2} (\frac{1}{\partial} \bar{J}, [\frac{1}{\partial} \bar{J}, \bar{J}]) \right\} \]
\[ = -\frac{1}{\pi} \int d^2 z \text{ Tr} \left\{ \frac{1}{2} \bar{J} \frac{\partial}{\partial} \bar{J} + \frac{1}{3} \bar{J} \frac{\partial}{\partial} \bar{J} + ... \right\}. \quad (3.16) \]

This suggests an alternative way of computing the path integral \( (3.6) \). Changing first the quantum variable from \( g' \) to \( \bar{J}' \) and taking into account the contribution of the Jacobian as in \( (3.8),(3.9) \) we get
\[
e^{-\Gamma(g)} = \int [d\bar{J}'] \exp\left\{ -(k + \frac{1}{2}c_G)\tilde{\omega}(\bar{J}') \right\} \delta[\bar{J}' - \bar{J}(g)] . \quad (3.17) \]

The formal use of the \( \delta \)-function gives
\[
\Gamma(g) = (k + \frac{1}{2}c_G)\tilde{\omega}(\bar{J}) = (k + \frac{1}{2}c_G)I(g) , \quad (3.18) \]
i.e. we reproduce \( (3.10) \) but without a field renormalisation. The reason for an apparent paradox is that a regularisation prescription implicit in the formalism based on eqs. \( (3.2),(3.7) \) is consistent with the formal \( \delta \)-function identities for the composite currents only if an extra field renormalisation is included (see in this connection eqs. \( [41],[42],[24] \)). A related paradox is found if one differentiates \( (3.6) \) over the 2-metric. Since the classical stress tensor \( \bar{T} \) is proportional to \( \frac{1}{k} \bar{J}^2 \), naively using the \( \delta \)-function identity one finds the same expression (with unshifted \( k \)) for the derivative of \( \Gamma \). The correct of the \( \delta \)-function identity in the present context is:
\[
< F[\bar{J}(g')]\delta[\bar{J}(g') - \bar{J}(g)] >= F[Z\bar{J}(g)], \quad Z = \frac{k}{\sqrt{k + \frac{4}{3}c_G}}. \]
As we noted already, such an identity is try only under a particular choice of the regularisation scheme.

Maintaining equivalence between the local field theory and operator conformal theory results is rather subtle and depends on a choice of a particular regularisation prescription (which should correspond to a normal ordering prescription in c.f.t.). As in the case of the 3d Chern-Simons theory the one-loop shift of \( k \) in the effective action may happen
in one regularisation and not happen in another one (see e.g. [33]). The absence of a renormalisation of \( k \) in the standard Legendre transform of the generating functional for correlators of currents (which does not receive loop corrections [32]) and its presence in the ‘quantum’ Legendre transform (34) seems related to an observation [44] that similar ‘quantum’ Legendre transform in \( SL(2, R) \) Chern-Simons theory relates two representations (in terms of affine and Virasoro conformal blocks) with ‘bare’ and renormalised values of \( k \).

Let us now compare (3.10) with other ‘effective actions’ in WZW theory which appeared in the literature. Similar effective actions were recently discussed in the context of ‘induced’ gauge theory (and 2d gravity) [22] [45] [23] [24]. Since the fermionic determinants in a background gauge field are expressed in terms of \( \omega(A) \) or \( \bar{\omega}(\bar{A}) \) (with the coefficient \(-k, \text{cf.}(3.7)\)) one can take the ‘induced’ action \( S(A) = -k\omega(A) \) as a classical action and find the corresponding quantum effective action for \( A \). Introducing the source \( \bar{J} \) for \( A \) one can compute the generating functional \( W(\bar{J}) \) by integrating \( \exp(-S(A) + \bar{J}A) \) over \( A \) (cf. (3.3)). Using (3.2) and the standard expression [36] for the determinant (3.8) (without \( \frac{1}{2} \) in front of \( c_G \)) one obtains: \( W(\bar{J}) = -(k + c_G)I(u), \bar{J} = -(k + c_G)\bar{\partial}uu^{-1} \). If the effective action \( S_{eff}(A) \) is defined as the Legendre transform of \( W(\bar{J}) \), then

\[
S_{eff}(A) = -(k + c_G)\omega(A) .
\] (3.19)

To have a consistency with the one-loop perturbative computation of \( S_{eff} \) [22] [23] one needs also to include a field renormalisation factor so that the conjecture for the exact form of the effective action reads [22] [23] [24]

\[
S_{eff}(A) = -(k + c_G)\omega(ZA) , \quad Z = 1 - \frac{c_G}{2k} + O\left(\frac{1}{k^2}\right) .
\] (3.19')

A non-trivial expression for \( Z \) reflects a choice of a specific regularisation prescription made in perturbative computations (see also the above remarks concerning (3.18)). It is

\[11\] In general, the Legendre transform of the functional \( W(A) = a\omega(bA) = aI(g), bA = g^{-1}\partial g \)
is given by \( \bar{W}(B) = a\omega(bA) - \frac{1}{\pi} \int d^2 z BA = -a\bar{\omega}(cB) = -aI(u) , \quad cB = \partial uu^{-1} , \quad c = (ab)^{-1} \)
(the coefficients are easily checked in the abelian case).
not clear, however, that this choice is consistent with the conformal field theory approach since straightforward application of the Polyakov-Wiegmann identity and the expression \[36\] for the chiral determinant (which is derived using Ward identity) gives (3.19) with \(Z = 1\).

The expression (3.19), though similar to our action (3.10), (3.11) in structure, is different in two respects. The minus sign in (3.19) is due to the fact that one has started with the induced WZW action (with coefficient \(-k\)). The factor of \(\frac{1}{2}\) difference in the shifts of \(k\) in the overall coefficients in (3.10) and in (3.19) is related to the fact that (3.10) is an effective action (in the left-right symmetric WZW theory) for the field \(g\) itself (cf.(3.6)) while (3.19) was derived using the “chiral” field \(A\) as the main quantum variable.

One may also draw an analogy between the quantum effective action (3.10) and the free field action which was suggested in \[25\] as a basis for the free field formulation of the WZW conformal theory. Using the Gauss decomposition in upper triangular, diagonal and lower triangular matrices \(g = g_U(\psi)g_D(\varphi)g_L(\chi)\) to parametrise \(g\) in terms of the fields \(\chi_\alpha\) and \(\psi_\alpha\) (labelled by all positive roots \(\alpha\) of the algebra of \(G\)) and the field \(\varphi_n\) (taking values in the Cartan torus, \(n = 1, ..., r; \ r = \text{rank} \ G\)) one finds for the classical action (3.1) \[25\]

\[
S = \frac{k}{2\pi} \int d^2 z \left[ \operatorname{Tr} (\partial g_D^{-1} \bar{\partial} g_D) + 2 \operatorname{Tr} (J'(\varphi, \chi) \bar{\partial}(g_L^{-1})) \right], \quad J'(\varphi, \chi) \equiv (g_U g_D)^{-1} \partial (g_U g_D) .
\]

Introducing instead of \(\psi_\alpha\) the new field \(W_\alpha\) such that

\[
k \operatorname{Tr} [J'(\varphi, \chi) \bar{\partial} g_L^{-1}](\chi) = W_\alpha \bar{\partial} \chi_\alpha , \quad (3.20)
\]

one can represent (3.12) in the form \[25\]

\[
S = \frac{1}{\pi} \int d^2 z (W_\alpha \bar{\partial} \chi_\alpha - \frac{1}{2} k \bar{\partial} \varphi_n \bar{\partial} \varphi_n) . \quad (3.21)
\]

The change of variables \(\psi_\alpha \rightarrow W_\alpha = Y^\beta_\alpha (\chi, \varphi, \psi) \bar{\partial} \psi_\beta\) is accompanied by a Jacobian. The latter should be defined in such a way that the resulting free theory is consistent with the operator approach to WZW conformal model based the affine algebra and Sugawara
representation. In particular, the Jacobian should depend only on \( \varphi_n \) and the 2d metric \( \gamma \) \[^{25}\]. Its logarithm contains three terms: \( c_G \partial \varphi_n \bar{\partial} \varphi_n \) (which leads to a shift of the value of the coefficient \( k \) in (3.21)); \( \rho_n \varphi_n R \) (\( \rho_n = \) one half of the sum of all positive roots); and a pure Weyl anomaly term. The final ‘quantum WZW action’ on a curved 2d background then takes the form (in the conformal gauge) \[^{25}\]

\[
S_q = \frac{1}{\pi} \int d^2 z \left[ W_\alpha \bar{\partial} \chi_\alpha - \frac{1}{2} (k + \frac{1}{2} c_G) \partial \varphi_n \bar{\partial} \varphi_n - \frac{1}{4} \rho_n \varphi_n \sqrt{\gamma} R \right]
- \frac{(D_G - r)}{192 \pi} \int R \Delta^{-1} R .
\]  (3.22)

The shift of \( k \) in (3.22) is the same as in (3.10) (one can of course rescale \( W_\alpha \) to make the shifted \( k \) appearing in front of the whole action) and, as in (3.10), originates from a 1-loop determinant. Note that the fields in (3.15) are still quantum; in particular, they need to be integrated out to get the correct central charge term (3.12) \[^{25}\]:

\[
C = \frac{1}{2} (D_G - r) + r - \frac{12 \rho^2}{k + \frac{1}{2} c_G} + \frac{1}{2} (D_G - r) = \frac{kD_G}{k + \frac{1}{2} c_G} .
\]

The background values of the fields \( W_\alpha, \chi_\alpha, \varphi_n \) should correspond to the argument \( g \) of the effective action (3.10).

3.2. Supersymmetric WZW theory

Computation of the effective action in supersymmetric WZW theory can be reduced to that in the bosonic WZW theory by observing \[^{46}\][^{47}\][^{48}\] that by a formal field redefinition the action of the supersymmetric WZW theory can be represented as a sum of the bosonic WZW action and the action of the free Majorana fermions in the adjoint representation of the group \( G \). As was noted in \[^{48}\][^{49}\][^{27}\], the transformation of fermions which is needed to decouple them from \( g \) is chiral (the interaction term is \( \bar{\psi} (1 + \gamma_5) \gamma^a \partial_a gg^{-1} \psi \)) and therefore produces a non-trivial Jacobian. The logarithm of the fermionic determinant gives a contribution proportional to the bosonic WZW action. The net result is the shift of the coefficient \( k \) in the bosonic part of the action \[^{49}\]:

\[
kI(g, \psi) \rightarrow \hat{k}I(g) + I_0(\psi) , \quad \hat{k} \equiv k - \frac{1}{2} c_G .
\]  (3.23)
This gives, in particular, the following expression for the central charge

\[ C_{\text{susy}} = C(\hat{k}) + \frac{1}{2} D_G = \frac{k - \frac{1}{2} c_G}{k} D_G + \frac{1}{2} D_G = \left( \frac{3}{2} - \frac{c_G}{2k} \right) D_G , \]  

(3.24)

implying that the central charge contains only the leading \( \frac{1}{k} \)-correction (i.e. the two-loop correction in the perturbative expansion). The expression (3.24) is consistent with the absence of the 3- and 4-loop corrections to the dilatonic \( \beta \)-function in the corresponding sigma model [50].

Note that though the question about the shift of \( k \) in (3.23) may look ambiguous, the correct choice is actually fixed by the condition of correspondence with the perturbation theory. Had one started with the often-used component form of the supersymmetric WZW action in which bosons and fermions are decoupled from the very beginning, one would get no shift of \( k \) in (3.23). A prescription in which there is no shift of \( k \) (and hence the central charge, is given by the naive expression [51] \( C_{\text{susy}} = C(G, k) + \frac{1}{2} D_G \) containing terms of all orders in \( 1/k \) is inconsistent with the standard renormalisation scheme employed in perturbative sigma model computations (such a prescription is effectively non-local when considered from the sigma model point of view). The quantum equivalence of the supersymmetric WZW theory to the bosonic WZW theory and the set of free fermions was proved also at the level of the full conformal field theory in the operator approach [26] (in particular, it was shown that, in agreement with (3.23),(3.24), the super Kac-Moody algebra with the central parameter \( k \) reduces to the direct product of the bosonic Kac-Moody algebra with the parameter \( \hat{k} = k - \frac{1}{2} c_G \) and the free fermionic algebra).

12 As discussed in [50], a consistency of the shift of \( k \) in (3.23) and hence of (3.24) with expectations about higher loop (\( L \geq 5 \)) corrections to the dilaton \( \beta \)-function presumably relies on a choice of a specific regularisation scheme (in which a version of the Adler-Bardeen theorem is true).

13 The shift of \( k \) is a direct consequence of a manifestly supersymmetric approach. Once one makes a chiral rotation to decouple fermions from bosons, the new supersymmetric transformation laws will involve \( \gamma_5 \) and probably will be anomalous at higher loop orders.
To find the (bosonic part of the) effective action corresponding to the supersymmetric WZW theory one needs to repeat the argument which led to the expression (3.10), (3.11) for the effective action in the bosonic WZW theory. Instead of (3.3) we get

\[ e^{-W(B)} = \int [dg][d\psi] \exp\left\{ -kI(g, \psi) + B \bar{\psi}(g) \right\} \]

\[ = N \int [dg] \exp\left\{ -(k - \frac{1}{2}c_G)I(g) + B \bar{\psi}(g) \right\}. \quad (3.25) \]

Though the source term in (3.25) contains the original unshifted parameter \( k \) this does not matter at the end since \( B \) is integrated out in \( \Gamma \) (3.6). As a result, the effective action in the supersymmetric WZW theory is obtained by replacing \( k \) by \( \hat{k} = k - \frac{1}{2}c_G \) in (3.10), (3.11),

\[ \Gamma(g) = kI(g'), \quad (3.26) \]

\[ \bar{\partial}g'g'^{-1} = (1 - \frac{c_G}{2\hat{k}})\bar{\partial}gg^{-1}, \quad (3.27) \]

i.e., it is equal to the classical WZW action with unshifted \( k \). The shift of \( k \) in \( \Gamma \) produced by integrating out fermions in (3.25) is exactly cancelled out by the contribution of the bosonic determinant (3.8) in (3.9). Note also that the field renormalisation relation (3.27) contains only a one-loop correction.

4. Effective Action in gauged WZW Theory

4.1. Bosonic gauged WZW theory

In this section we shall first consider a derivation of the effective action in the bosonic gauged WZW theory (clarifying the discussion in [13]) and then generalise it to the supersymmetric gauged WZW theory case. The quantisation of the gauged \( G/H \) WZW theory is based on representing the corresponding path integral in terms of path integrals in ungauged WZW theories for the group and the subgroup. That makes it possible to use the analysis of the ungauged WZW theory carried out in the previous section.
The classical gauged WZW action \cite{34,37}

\[
I(g, A) = I(g) + \frac{1}{\pi} \int d^2 z \, \text{Tr} \left( -A \partial g g^{-1} + A g^{-1} \partial g + g^{-1} A g A - A \bar{A} \right)
\equiv I_0(g, A) - \frac{1}{\pi} \int d^2 z \, \text{Tr} \left( A \bar{A} \right) \tag{4.1}
\]

is invariant under the standard vector $H$-gauge transformations ($A, \bar{A}$ take values in the algebra of $H$)

\[
g \to u^{-1} g u \, , \, A \to u^{-1} (A + \partial) u \, , \, \bar{A} \to u^{-1} (\bar{A} + \bar{\partial}) u \, , \, u = u(z, \bar{z}) .
\]

The two terms in (4.1) $I_0$ and $\int \text{Tr} \left( A \bar{A} \right)$ are separately invariant under the holomorphic vector gauge transformations $g \to u^{-1}(z) g u(\bar{z})$.

Parametrising $A$ and $\bar{A}$ in terms of $h$ and $\bar{h}$ which take values in $H$ and transform as

\[
h \to u^{-1} h , \, \bar{h} \to u^{-1} \bar{h} ,
\]

one can use the Polyakov-Wiegmann identity (3.2) to represent the gauged action as the difference of the two gauge-invariant terms: the ungauged WZW actions corresponding to the group $G$ and subgroup $H$,

\[
I(g, A) = I(h^{-1} g \bar{h}) - I(h^{-1} \bar{h}) . \tag{4.3}
\]

Since

\[
I(h^{-1} g \bar{h}) = I_0(g, A) + I(h^{-1}) + I(\bar{h}) , \quad I(h^{-1} \bar{h}) = \frac{1}{\pi} \int d^2 z \, \text{Tr} \left( A \bar{A} \right) + I(h^{-1}) + I(\bar{h}) ,
\]

the non-local terms $I(h^{-1}) + I(\bar{h})$ cancel out in the classical action (4.3) (but will survive in the effective one since the coefficients of the two terms in (4.3) will get different quantum corrections). Changing the variables in the path integral

\[
Z = \int [dg][dA][d\bar{A}] \, e^{-kI(g, A)} \tag{4.4}
\]
and using the expression (3.7) for the non-chiral determinant
\[ \det (\partial + [A, ]) \det (\bar{\partial} + [\bar{A}, ]) = \exp [c_H I(h^{-1}\bar{h})] \det \partial \det \bar{\partial} \],
we get
\[ Z = N \int [dg][dh][d\bar{h}] \exp [-kI(h^{-1}g\bar{h}) + (k + c_H)I(h^{-1}\bar{h})] \]
\[ = N' \int [d\tilde{g}][d\tilde{h}] \exp [-kI(\tilde{g}) - (k - c_H)I(\tilde{h})] , \]
\[ \tilde{g} \equiv h^{-1}g\bar{h}, \quad \tilde{h} \equiv h^{-1}\bar{h}, \]
where a gauge fixing was assumed (\(N'\) is proportional to the product of \(D_H\) free scalar determinants originating from the Jacobian of the change of variables). One concludes that \(kI(g, A)\) can be quantised as the ‘product’ of the two WZW theories for the groups \(G\) and \(H\) with the levels \(k\) and \(-(k + c_H)\), or, equivalently, as the ‘ratio’ \(G_k/H_k\) of the WZW theories with the equal levels \(k\). This is clear, for example, from the resulting expression for central charge \([3,4]\)
\[ C(G/H) = C(G, k) + C(H, -k - c_H) - 2D_H = C(G, k) - C(H, k) , \]
\[ C(G, k) = \frac{kD_G}{k + \frac{1}{2}c_G} , \]
where \(-2D_H\) corresponds to the contribution of the free determinants in \(N\).

One possible approach to derivation of the effective action in the gauge theory (4.4) is to introduce sources for all fields \((g, A, \bar{A})\) and to use the background field method. However, we prefer to couple sources only to gauge-invariant combinations of fields \(\tilde{g}\) and \(\tilde{h}\) in (4.7). Then the problem is reduced to computation of the effective actions in the two decoupled ungauged WZW theories. According to our discussion in Sec.3 to find the effective action of the WZW theory for group \(G\) one is to replace its parameter \(k\) by \(k + \frac{1}{2}c_G\) and to renormalise the field (see (3.10),(3.11)). This implies that the effective action in the gauged WZW theory is given by (note that \(-(k + c_H) + \frac{1}{2}c_H = -(k + \frac{1}{2}c_H)\))
\[ \Gamma(g, A) = (k + \frac{1}{2}c_G)I(\tilde{g}') - (k + \frac{1}{2}c_H)I(\tilde{h}') , \]
\[ \bar{\partial} \tilde{g}' \tilde{g}'^{-1} = \frac{k}{k + \frac{1}{2} c_G} \bar{\partial} \tilde{g} \tilde{g}^{-1}, \quad \bar{\partial} \tilde{h}' \tilde{h}'^{-1} = \frac{k + c_H}{k + \frac{1}{2} c_H} \bar{\partial} \tilde{h} \tilde{h}^{-1}. \] (4.9)

As in the ungauged case, the structure of the effective action (4.8) is in a natural correspondence with that of the stress tensor in the conformal field theory approach \[ T_{zz} = \frac{1}{k + \frac{1}{2} c_G} : J_G^2 : - \frac{1}{k + \frac{1}{2} c_H} : J_H^2 : \].

The field renormalisations (4.9) complicate the problem of representing the action (4.8) in terms of the original fields \( g, A \) and \( \bar{A} \). However, as explained in the Introduction, for the purpose of finding a sigma model which corresponds to the gauged WZW theory it is sufficient to consider only the \textit{local} part of the effective action (4.8). In what follows we shall drop out various non-local parts of (4.8) in several stages. First, we shall ignore the field renormalisations (4.9) (since they introduce additional non-localities, see Sec.3), i.e. we shall replace (4.8) by the following action (which is gauge-invariant and, in general, still non-local)

\[
\Gamma'(g, A) = (k + \frac{1}{2} c_G) I(\tilde{g}) - (k + \frac{1}{2} c_H) I(\tilde{h}) . \quad (4.10)
\]

Using (4.2),(4.3),(4.7) we can represent (4.10) in terms of \( g \) and the gauge field \[ \Gamma'(g, A) = (k + \frac{1}{2} c_G) I(g, A) + \frac{1}{2} (c_G - c_H) \Omega(A) . \quad (4.11)\]

Here \( \Omega(A) \) is a non-local gauge invariant functional of \( A \) and \( \bar{A} \),

\[
\Omega(A) \equiv I(h^{-1} \tilde{h}) = \omega(A) + \bar{\omega}(-\bar{A}) + \frac{1}{\pi} \int d^2 z \ Tr (A \bar{A}) , \quad (4.12)
\]

where the functionals \( \omega \) and \( \bar{\omega} \) have already appeared in (3.13)–(3.16)

\[
\omega(A) = I(h^{-1}) = -\frac{1}{\pi} \int d^2 z \ Tr \left\{ \frac{1}{2} A \frac{\partial}{\partial A} A - \frac{1}{3} A [\frac{1}{2} A, \frac{\partial}{\partial A}] + O(A^4) \right\}, \quad (4.13)
\]

\[
\bar{\omega}(-\bar{A}) = I(\bar{h}) = -\frac{1}{\pi} \int d^2 z \ Tr \left\{ \frac{1}{2} \bar{A} \frac{\partial}{\partial \bar{A}} \bar{A} - \frac{1}{3} \bar{A} [\frac{1}{2} \bar{A}, \frac{\partial}{\partial \bar{A}}] + O(\bar{A}^4) \right\} . \quad (4.14)
\]

The ‘quantum correction’ \( \Omega(A) \) (equivalent to the induced action corresponding to Dirac fermions) contains both local and non-local terms. As it is clear from (4.13),(4.14) (see also (3.14),(3.16)) all terms in \( \Omega \) which are \( O(A^n, \bar{A}^m) \) with \( n, m \geq 3 \) are non-local, i.e.

\[
\Gamma'(g, A) = (k + \frac{1}{2} c_G) I(g, A) + \frac{1}{2} (c_G - c_H) \Omega_0(A) + (\text{non-local}) , \quad (4.15)
\]

30
\[ \Omega_0(A) \equiv \frac{1}{\pi} \int d^2z \text{Tr} \left( A \bar{A} - \frac{1}{2} A \bar{\partial} A - \frac{1}{2} \bar{A} \partial \bar{A} \right) \]  
\[ = \frac{1}{2\pi} \int \text{Tr} F \frac{1}{\partial \bar{\partial}} F, \quad F \equiv \partial A - \bar{\partial} \bar{A}. \]

In the case when the subgroup \( H \) is abelian, the higher order non-local terms in (4.15) automatically cancel out, i.e. \( \Omega_0 = \Omega \) \([3]\]. In contrast to the full \( \Omega \) its quadratic part \( \Omega_0 \) is invariant only under the abelian gauge transformations, \( A \rightarrow A + \partial \epsilon, \; \bar{A} \rightarrow \bar{A} + \bar{\partial} \epsilon \). Dropping out the non-local \( O(A^n, \bar{A}^m) \) terms in \( \Gamma' \) (i.e. replacing \( \Omega \) in (4.11) by \( \Omega_0 \)) we reduce it to the following action

\[ \Gamma_{tr}(g, A) = (k + \frac{1}{2} c_G) \left[ I(g) + \Delta I(g, A) \right], \]  
\[ \Delta I(g, A) \equiv \frac{1}{\pi} \int d^2z \text{Tr} \left[ (-A \bar{\partial}g^{-1} + \bar{A} g^{-1} \partial g + g^{-1}A g \bar{A} - A \bar{A}) \right. \]  
\[ + \frac{1}{2} b \left( AQ + \bar{A}Q^{-1} \bar{A} - 2A \bar{A} \right) \]  
\[ b \equiv -\frac{(c_G - c_H)}{2(k + \frac{1}{2} c_G)}, \quad Q \equiv \bar{\partial} \frac{\partial}{\partial}, \quad Q^{-1} \equiv \partial \frac{\partial}{\partial}. \]  

The absence of the full non-abelian gauge invariance of the quantum correction term proportional to \( b \Omega_0 \) should not present a problem, since it is clear that the gauge invariance can always be restored by re-introducing the higher order non-local terms.

Let us note that in terms of the redefined gauge fields

\[ A' = Q^{1/2} A, \quad \bar{A}' = Q^{-1/2} \bar{A}, \]  
we can represent (4.18) in the form

\[ \Delta I(g, A) = \frac{1}{\pi} \int d^2z \text{Tr} \left[ (-Q^{-1/2} A' \bar{\partial}g^{-1} + Q^{1/2} \bar{A}' g^{-1} \partial g + g^{-1}Q^{-1/2} A' g Q^{1/2} \bar{A}' - A' \bar{A}') \right] \]  
\[ + \frac{1}{2} b \left( A' - \bar{A}' \right)^2 \]  
\[ \]  
(4.21)

If we were to naively ignore the \( Q \)-insertions, we would obtain the following action (omitting primes on \( A, \bar{A} \))

\[ \check{\Gamma}(g, A) = (k + \frac{1}{2} c_G) \left[ I(g) + \Delta I(g, A) \right], \]  
\[ \]  
(4.22)
\[
\Delta \bar{I}(g, A) = \frac{1}{\pi} \int d^2z \, \text{Tr} \left[ (-A \bar{\partial} g g^{-1} + A g^{-1} \partial g + g^{-1} A g \bar{A} - A \bar{A}) + \frac{1}{2} b (A - \bar{A})^2 \right].
\] (4.23)

The dimensionally reduced form of the action (4.22),(4.23) was recently proposed (for the same purpose of deriving the couplings of the corresponding sigma model) in [14]. The main idea in [14] was to concentrate only on the zero mode dynamics, i.e. to consider a reduction of the effective action (4.11) to one dimension taking all the fields to depend on the time coordinate only (which is effectively equivalent to setting \( \partial = \bar{\partial} \), i.e. \( Q = 1 \)). It was observed that the resulting 1d action (which is invariant under the 1d gauge transformation) becomes local and quadratic in the gauge field. While this ansatz makes possible to determine the metric (and dilaton) of the corresponding sigma model (since the original gauged WZW theory is Lorentz invariant, the sigma model action should be given by \( \int d^2z \, G_{\mu \nu} \partial_m x^\mu \partial^n x^\nu + ... \) so that to find the target space metric it is sufficient to compute the term in the effective action which is quadratic in the time derivatives of the sigma model fields), it has an obvious deficiency. After all, we are dealing with a 2d theory which, in general, contains more couplings than its dimensionally reduced analogue. In particular, the 1d ansatz does not allow one to compute the antisymmetric tensor coupling in a systematic way.

The truncated effective action (4.17) will be our starting point in the subsequent derivation of the sigma model couplings. To extract a local part of (4.18) is somewhat non-trivial. It is not correct to omit the terms with the operator \( Q \) insertions (since \( Q \) has dimension zero and since this would break the gauge invariance of (4.17)); it is also not correct just to replace \( Q \) by 1 (this would break the Lorentz invariance). As we shall see in Sec.5, one should first integrate out the gauge fields and then discard all the non-local terms. As is clear from the above discussion, our result for the sigma model metric and dilaton should be the same as in the 1d approach of [14].
4.2. Supersymmetric gauged WZW theory

As in the case of the ungauged supersymmetric WZW theory considered in Sec.3.2, the derivation of the effective action in gauged supersymmetric WZW theory can be effectively reduced to the discussion of the bosonic case. The path integral quantisation of the supersymmetric theory follows the same pattern as in the previous subsection. As in the bosonic case and the case of ungauged supersymmetric WZW theory (Sec.3.2) our treatment of the gauged supersymmetric WZW theory will be in correspondence with the results of the operator approach to the superconformal coset theory \[3 \, 20 \, 5\]. To guarantee the \( N = 1 \) supersymmetry we shall use the superfield formulation of the theory as a starting point (see e.g. \[52\]). Our approach is parallel to the one followed in \[4\] in the bosonic case (with the fields replaced by superfields).

The supersymmetric version of the ungauged WZW action (3.1) is obtained by replacing \( g \) by the corresponding superfield and making other standard replacements \((z^a \rightarrow (z^a, \theta, \bar{\theta}), \partial \rightarrow D, \text{etc.})\) \[46\]

\[
\hat{g} = \exp(T_A X^A), \quad X^A = x^A + \theta \psi^A + \bar{\theta} \bar{\psi}^A + \bar{\theta} \theta f^A, \quad D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z},
\]

\[
\hat{S} = k \hat{I}(\hat{g}), \quad \hat{I}(\hat{g}) \equiv \frac{1}{2\pi} \int d^2 z d^2 \theta \operatorname{Tr} (D\hat{g}^{-1} \bar{D}\hat{g}) + (WZ - \text{term}).
\]

The Polyakov-Wiegmann identity (3.2) also has a straightforward supersymmetric generalisation. The supersymmetric version of the gauged WZW action (4.1) is given by

\[
\hat{I}(\hat{g}, \hat{A}) = \hat{I}(\hat{g}) + \frac{1}{\pi} \int d^2 z d^2 \theta \operatorname{Tr} (-\hat{A} \bar{D}\hat{g}\hat{g}^{-1} - \hat{A} \hat{g}^{-1}\bar{D}\hat{g} + \hat{g}^{-1}\hat{A}\bar{g}\hat{A} - \hat{A}\hat{A})
\]

\[
= \hat{I}(\hat{g}) - \hat{I}(\hat{h}),
\]

\[14\] Our approach is different from the previous path integral analysis of gauged supersymmetric WZW theory in \[27\] where component formalism was used and only the case of \( G = H \) was considered. It was mentioned \[27\], however, that using a superfield in place of the gauge field one should be able to give a manifestly supersymmetric treatment of the problem.
where the gauge superfields $\hat{A}, \hat{A}$ take values in the algebra of the subgroup $H$ and (cf. (4.2),(4.3),(4.7))

$$\hat{A} = \hat{h}D\hat{h}^{-1}, \quad \hat{A} = \hat{h}D\hat{h}^{-1},$$

(4.28)

$$\tilde{\hat{g}} \equiv \hat{h}^{-1}\hat{g}\hat{h}, \quad \tilde{\hat{h}} \equiv \hat{h}^{-1}\hat{h}.$$  

(4.29)

In view of (4.27) the quantisation of the theory can be reduced to that of the two un- 

gauged supersymmetric WZW theories corresponding to the group and the subgroup 

cf.(4.6),(4.7)),

$$Z = \int [d\tilde{\hat{g}}][d\hat{A}][d\hat{A}] \exp\{-kI(\tilde{\hat{g}}, \hat{A})\}$$

$$= \int [d\tilde{\hat{g}}][d\tilde{\hat{h}}] J \exp\{-kI(\tilde{\hat{g}}) + kI(\tilde{\hat{h}})\}.$$  

(4.30)

Here $J$ stands for the product of Jacobians of the change of superfield variables from $\hat{A}$

to $\hat{h}$ and from $\hat{A}$ to $\hat{h}$ (and includes also a gauge fixing factor). While in the bosonic

case the corresponding product (regularised in the left-right symmetric way as in (3.7))

is non-trivial and leads to the shift of the coefficient of the $H$-term in the action (see

(4.6)), in the superfield case each of the Jacobians is proportional to a field-independent

factor. This happens because the non-trivial contribution of the bosonic determinant is

cancelled out by a contribution of the fermionic one (this cancellation is similar to that of

the bosonic and fermionic contributions to the coefficient $k$ of the effective action in the

ungauged supersymmetric WZW theory, see (3.25),(3.26)). In fact, as in the bosonic case,

the Jacobian of the change $\hat{A} \rightarrow \hat{h}$ can be expressed in terms of the path integral with

the action $\int d^2zd^2\theta U(DV + [\hat{A}, V])$, where $U$ and $V$ are superfields of opposite statistics.

Re-writing this action in component fields and integrating them out it is easy to see that

this Jacobian is $\hat{A}$-independent.

The theory can thus be represented as a ‘product’ of the two supersymmetric WZW

theories for the groups $G$ and $H$ with levels $k$ and $-k$. Since we know already the expression

(3.26),(3.27) for the effective action in the ungauged supersymmetric WZW model, it is

now easy to write down the resulting effective action in the theory (4.26). In particular, we
conclude that there are no shifts of \( k \). To see this in detail at the component level, let us first return to the component notation and make the chiral rotation to decouple fermions from bosons as discussed in Sec.3.2. According to (3.23),(3.25) we get the following result

\[
Z = N' \int [d\tilde{g}] [d\tilde{h}] [d\psi_G] [d\psi_H] \exp\left\{ -(k - \frac{1}{2}c_G)I(\tilde{g}) + (k + \frac{1}{2}c_H)I(\tilde{h}) - I_0(\psi_G) + I_0(\psi_H) \right\}. \tag{4.31}
\]

The factor \( N' \) contains determinants of the free fermions in the adjoint representations of \( G \) and \( H \) as well as the contribution \((\det \bar{\partial} \partial)^{3DH}\) originating from \( J \) in (4.30) (which provides the correct count of the free degrees of freedom). Up to these free-theory factors, we can represent the resulting theory as the ‘ratio’ \( G_{k - \frac{1}{2}c_G}/H_{k - \frac{1}{2}c_H} \) of the bosonic WZW theories for the groups \( G \) and \( H \) with levels \( k_G = k - \frac{1}{2}c_G \) and \( k_H = k - \frac{1}{2}c_H \) (we separate the shift \( c_H \) corresponding to the bosonic change of variable in order to identify (4.31) with (4.6))

\[
Z = N' \int [d\tilde{g}] [d\tilde{h}] \exp\left\{-k_GI(\tilde{g}) + (k_H + c_H)I(\tilde{h})\right\}. \tag{4.32}
\]

In particular, we get the following expression for the central charge (cf. (3.24),(4.7))

\[
C_{\text{susy}}(G/H) = C(G, k - \frac{1}{2}c_G) + C(H, -k - \frac{1}{2}c_G) + \frac{1}{2}D_G + \frac{1}{2}D_H - 3D_H
\]

\[
= [C(G, k - \frac{1}{2}c_G) + \frac{1}{2}D_G] - [C(H, -k - \frac{1}{2}c_G) + \frac{1}{2}D_H] = C_{\text{susy}}(G) - C_{\text{susy}}(H), \tag{4.33}
\]

\[
C_{\text{susy}}(G) = (1 - \frac{c_G}{2k})D_G + \frac{1}{2}D_G, \quad C_{\text{susy}}(H) = (1 - \frac{c_H}{2k})D_H + \frac{1}{2}D_H
\]

(we have included the contributions of the free fermions and the factor contained in \( N' \) in (4.31)).

This conclusion is in agreement with the conformal algebra approach \([2]\) (note that in terms of the shifted level \( \hat{k} = k - \frac{1}{2}c_G \) we get the \( G_{\hat{k}}/H_{\hat{k} + \frac{1}{2}c_G - \frac{1}{2}c_H} \) - theory). Let us now compare the above approach with the one based on starting with the component
formulation of the gauged supersymmetric WZW theory in which the fermions are coupled only to $A, \bar{A}$ (see e.g. [53],[54])

$$Z = \int [dg][dA][d\bar{A}][d\psi] \exp\{-kI(g,A) - I_0(\psi,A)\}. \quad (4.34)$$

Here $I(g,A)$ is the bosonic action (4.1) and $I_0(\psi,A)$ is the action of Majorana fermions (taking values in the orthogonal complement of the algebra of $H$ in the algebra of $G$) minimally coupled to $A, \bar{A}$. One can obtain the action in (4.34) from the classical action (4.26) by fixing the fermionic part of the gauge invariance by the gauge condition $\psi_H = 0$ and solving for the fermionic components of the gauge superfields. Integrating over the fermions and going through the same steps as in (4.4)–(4.7) we finish with

$$Z = N' \int [d\tilde{g}][d\tilde{h}] \exp\{-kI(\tilde{g}) + \left[k + \frac{1}{2}(c_G - c_H) + c_H|I(\tilde{h})]\right\}, \quad (4.35)$$

where the contribution of fermions is proportional to $c_G - c_H$. This expression becomes equivalent to (4.31),(4.32) if $k$ in (4.35) is replaced by $\tilde{k} = k - \frac{1}{2}c_G$. As in the case of the ungauged supersymmetric WZW theory discussed in Sec.3.2, the approach based on the naive component action loses the shift of $k$ and this is inconsistent with manifestly supersymmetric perturbation theory.

Applying the results of Sec.3.2 and Sec.4.1 we are now ready to write down the expression for the effective action in the gauged supersymmetric WZW theory. Using either the representation in terms of the ungauged supersymmetric WZW theories (4.30) and (3.26),(3.27) or the equivalent formulation in terms of the ungauged bosonic WZW theories (4.31) and (3.10),(3.11) we get the following expression for the (bosonic part of) effective action

$$\Gamma_{\text{susy}}(g,A) = kI(\tilde{g}') - kI(\tilde{h}') \quad (4.36)$$

$$\tilde{\partial} \tilde{g}' \tilde{g}'^{-1} = (1 - \frac{c_G}{2k}) \tilde{\partial} \tilde{g} \tilde{g}^{-1}, \quad \tilde{\partial} \tilde{h}' \tilde{h}'^{-1} = (1 - \frac{c_H}{2k}) \tilde{\partial} \tilde{h} \tilde{h}^{-1}. \quad (4.36)$$

As in the ungauged supersymmetric WZW theory but in contrast to the result (4.8),(4.9) for the bosonic gauged WZW theory there are no shifts in the overall coefficients of the
\(G\)- and \(H\)- terms in \(\Gamma_{\text{susy}}\). Ignoring the non-local corrections introduced by the field renormalisations in (4.36), we arrive at the conclusion that the local part of the effective action of the gauged supersymmetric WZW theory is equal to the \textit{classical} action of the bosonic gauged WZW theory

\[
\Gamma_{\text{susy}}^{(\text{loc})}(g, A) = kI(g, A)
\]

\[
= k[I(g) + \frac{1}{\pi} \int d^2z \text{ Tr} (-A \bar{\partial}gg^{-1} + \bar{A} g^{-1}\partial g + g^{-1}Ag\bar{A} - A\bar{A})],
\]

(4.37)
i.e., in contrast to the bosonic case (4.17),(4.18), it does not contain the quantum correction term proportional to \(b = -\frac{(c_G - c_H)}{2(k + \frac{1}{2}c_G)}\).

As a consequence, the exact form of the corresponding sigma model will be equivalent to the ‘semiclassical’ form of the sigma model corresponding to the bosonic theory. This conclusion is the same as the one obtained in the operator approach in \[12\] (in the case of the \(SL(2, R)/U(1)\) supersymmetric theory) and in \[15\] (in the case of a general \(G/H\) supersymmetric theory).\[13\] In particular, there is no shift in the overall coefficient \(k\). As we have already emphasized above, it is \(k\) and not \(\hat{k}\) that is the coefficient in both classical and effective actions and hence is the parameter that should be used in a discussion of a correspondence with the bosonic model and in a computation of the sigma model couplings.

One can find the manifestly supersymmetric form of the corresponding sigma model by directly solving for the gauge superfields in (4.26). The result will take the general form of the (1,1) supersymmetric sigma model \(\int d^2z d^2\theta (G_{MN} + B_{MN}) DX^M \bar{D}X^N\). Fixing the gauge \((X^M \rightarrow X^\mu)\) and using component notation it is easy to read off the corresponding connection with torsion and (from the quartic fermionic term) its curvature.

\[\text{Note that ref. [15] used } \hat{k} = k - \frac{1}{2}c_G \text{ instead of } k.\]
5. Sigma Model Corresponding to Gauged WZW Theory

5.1. Elimination of the gauge field

In this section we are going to derive the couplings of the sigma model which corresponds to the gauged WZW theory. Later in Sec. 6 we shall establish the equivalence between the results of the field-theoretical approach and the expressions for the metric and dilaton obtained in the operator approach in Sec. 2. We shall also determine the antisymmetric tensor coupling which is difficult to find in the operator approach.

To give a sigma model interpretation to a gauged WZW theory one, should, in principle, fix a gauge and ‘integrate out’ the gauge field. As was explained in Introduction, our method of derivation of the sigma model corresponding to a gauged WZW theory is based on first finding the effective action $\Gamma_{gwzw}$ in the gauged WZW theory and then identifying it with the effective action for the sigma model. Both effective actions, in principle, contain local as well as non-local terms. In order to determine the classical sigma model action (i.e. the local second-derivative part of $\Gamma_{sm}$) it is sufficient to consider only the truncated part $\Gamma_{tr}$ of $\Gamma_{gwzw}$ (4.17),(4.18) in which some of the non-local terms have already been dropped. Since $\Gamma_{tr}$ is quadratic in the gauge field it is possible to treat it as in the semiclassical approximation, i.e. to integrate over the gauge field, to fix a gauge, etc. One may then again ignore all non-local terms which may appear in the process of elimination of the gauge field.

Let us start with the following form (4.21) of the truncated effective action (4.17),(4.18)

$$\Gamma_{tr}(g, A) = (k + \frac{1}{2}c_G)[I(g) + \Delta I(g, A)], \quad (5.1)$$

$$\Delta I(g, A) = \frac{1}{\pi} \int d^2 z \ Tr \left[ (-A' \ddot{J}' + \ddot{A}' J' + g^{-1}Q^{-1/2}A'gQ^{1/2}\dddot{A}' - A' \dddot{A}') \right. \left. + \frac{1}{2}b (A' - \dddot{A}')^2 \right]. \quad (5.2)$$

$$J' = Q^{1/2}(g^{-1}\partial g), \quad \ddot{J}' = Q^{-1/2}(\bar{\partial}gg^{-1}), \quad A' = Q^{1/2}A, \quad \dddot{A}' = Q^{-1/2}\ddot{\bar{A}}.$$
\[ Q = \frac{\partial}{\partial}, \quad Q^{-1} = \frac{\partial}{\partial}, \quad b = -\frac{c_G - c_H}{2(k + \frac{1}{2}c_G)}. \]

To integrate over \( A, \bar{A}, \) or, equivalently, over \( A', \bar{A}' \) let us first note that it is possible to ignore the factors \( Q^{\pm 1/2} \) in the \( O(A\bar{A}) \)-term in (5.2): as it is easy to understand, they produce only extra non-local terms which we are not interested in. Then (5.2) takes the form

\[
\Delta I'(g, A) = \frac{1}{\pi} \int d^2 z \left[ L_1(g, A) + L_2(g, A) \right],
\]

\[
L_1 = \text{Tr} \left( -A'J' + \bar{A}'J' \right), \quad L_2 = \text{Tr} \left[ (g^{-1}A'g\bar{A}' - A'\bar{A}') + \frac{1}{2}b (A' - \bar{A}')^2 \right].
\]

In spite of its form this action is still Lorentz invariant since \( A', \bar{A}' \) in (5.3) are the redefined fields of (4.20),(4.21).

It is useful first to diagonalise \( L_2 \), the quadratic in \((A, \bar{A})\), in (5.3). Let us follow the notation of Sec.2 \( (T_A = (T_a, T_i) \) are the generators of the algebra of \( G; \) \( T_a \) are the generators of the algebra of \( H; \) \( A = 1, ..., D_G; \) \( a = 1, ..., D_H; \) \( i = 1, ..., D \)\) and define the matrices \( C_{ab}, M_{ab}, K_{ab}, P_{ab} \) which are functions of \( g \)

\[
C_{ab} = \text{Tr} \left( T_agT_bg^{-1} \right), \quad \text{Tr} \left( T_aT_b \right) = \eta_{ab}, \quad M_{ab} = C_{ab} - \eta_{ab}, \quad (5.4)
\]

\[
K = M^{-1}M^T, \quad P = \frac{1}{2}(1 + K), \quad M_S = \frac{1}{2}(M + M^T) = MP, \quad M^T = M_{ba}. \quad (5.5)
\]

Though \( M \) may not be invertible on \( H \), it is non-degenerate for a generic \( g \) (at the end we shall fix a gauge restricting \( g \) to \( G/H \)). Introducing the following combinations \( B_a, \bar{B}_a \) of the gauge fields \( A_a, \bar{A}_a \) \( (A = T^aA_a, \bar{A} = T^a\bar{A}_a) \)

\[
B = \frac{1}{2}P^{-1}(A' - \bar{A}') , \quad \bar{B} = \frac{1}{2}P^{-1}(KA' + \bar{A}'), \quad (5.6)
\]

\[
A' = \bar{B} - B , \quad \bar{A}' = \bar{B} + KB ,
\]

we find

\[
L_2 = A'M\bar{A}' + \frac{1}{2}b(A' - \bar{A}')^2 = -BM\bar{B} + \bar{B}M_S\bar{B}. \quad (5.7)
\]

\[16 \] In our notation \( \eta_{AB} \) is negative definite for a compact group. Indices \( a, b, ... \) are raised and lowered with \( \eta_{ab} \). We shall sometimes use 1 to denote a unit matrix or \( \eta_{ab} \).
\[ M \equiv M_S - 2bP^T P, \quad M^T = M. \] (5.8)

Note that \( B \) is invariant under the abelian gauge transformations, \( A' \rightarrow A' + Q^{1/2} \partial \epsilon, \ A' \rightarrow \bar{A'} + Q^{-1/2} \bar{\partial} \epsilon \), i.e. it is the analogue of the transverse part of \( A_m^a \) (the \( b \)-correction term is contained only in the \( O(B^2) \) part of (5.7); cf. also (5.11) below). The current term \( L_1 \) in (5.3) takes the form

\[ L_1 = B \mathcal{J} + \bar{B} \bar{\mathcal{J}}, \] (5.9)

\[ \mathcal{J}_a = \text{Tr} \left( T_a \mathcal{J}' + T^b K_{ba} \mathcal{J}' \right), \quad \bar{\mathcal{J}}_a = \text{Tr} \left[ T_a (\mathcal{J}' - \bar{\mathcal{J}}') \right]. \] (5.10)

Integrating over \( B, \bar{B} \) we get from (5.7),(5.9)

\[
L_3(g) = \frac{1}{4} \mathcal{J} \mathcal{M}^{-1} \mathcal{J} - \frac{1}{4} \bar{\mathcal{J}} \bar{\mathcal{M}}^{-1} \bar{\mathcal{J}}
\]

\[
= \left[ -\frac{1}{4} \text{Tr} \left( T_a \partial g g^{-1} \right) (\mathcal{M}^{-1})^{ab} \text{Tr} \left( T_b \partial g g^{-1} \right) + \frac{1}{4} \text{Tr} \left( T_a g^{-1} \partial g \right) (K \mathcal{M}^{-1} K^T)^{ab} \text{Tr} \left( T_b g^{-1} \bar{\partial} g \right) \right.
\]

\[
+ \frac{1}{2} \text{Tr} \left( T_a g^{-1} \partial g \right) (K \mathcal{M}^{-1})^{ab} \text{Tr} \left( T_b \partial g g^{-1} \right) \] \quad \]

\[
- \left[ \frac{1}{4} \text{Tr} \left( T_a \partial g g^{-1} \right) (M^{-1})_{\bar{S}}^{ab} \text{Tr} \left( T_b \partial g g^{-1} \right) + \frac{1}{4} \text{Tr} \left( T_a g^{-1} \partial g \right) (M_{\bar{S}}^{-1})^{ab} \text{Tr} \left( T_b g^{-1} \bar{\partial} g \right) \right.
\]

\[
- \frac{1}{2} \text{Tr} \left( T_a g^{-1} \partial g \right) (M_{\bar{S}}^{-1})^{ab} \text{Tr} \left( T_b \partial g g^{-1} \right) \left] + (\text{non-local}) \right. \] (5.11)

We have used the expressions for the currents \( \mathcal{J}', \bar{\mathcal{J}}' \) in (5.2) and rearranged the derivatives contained in the \( Q \)-insertions to separate the local part of (5.11) (the operators \( Q^{\pm 1/2} \) either cancel out or double, producing \( \bar{\partial} / \partial \) or \( \partial / \bar{\partial} \) which effectively interchange the derivatives \( \partial \) and \( \bar{\partial} \) in the currents). As a result, we obtain from (5.1) the following local Lorentz invariant action

\[
\Gamma_{loc}(g) = \frac{k + \tfrac{1}{2} CG}{2\pi} \int d^2 z \left\{ \left[ \text{Tr} \left( \partial g^{-1} \bar{\partial} g \right) + (\text{WZ - term}) \right] \right.
\]

\[
+ \left[ \frac{1}{2} \text{Tr} \left( T_a \partial g g^{-1} \right) (\mathcal{M}^{-1} - \mathcal{M}_{\bar{S}}^{-1})^{ab} \text{Tr} \left( T_b \partial g g^{-1} \right) \right.
\]

\[
+ \frac{1}{2} \text{Tr} \left( T_a g^{-1} \partial g \right) (K \mathcal{M}^{-1} K^T - \mathcal{M}_{\bar{S}}^{-1})^{ab} \text{Tr} \left( T_b g^{-1} \bar{\partial} g \right) \]

\[
+ \text{Tr} \left( T_a g^{-1} \partial g \right) (K \mathcal{M}^{-1} + \mathcal{M}_{\bar{S}}^{-1})^{ab} \text{Tr} \left( T_b \partial g g^{-1} \right) \left] \right. \} . \] (5.12)
5.2. Sigma model representation

The gauge invariance of the effective action makes it possible to fix a gauge and express $g$ in terms of $D$ coordinates $x^\mu$ on $G/H$. It is useful, however, to start with the full set of $D_G$ coordinates $x^M$ on the group $G$ space and restrict to $G/H$ only at a later stage. We shall see that as a reflection of the gauge invariance of the original action, the resulting metric on $G$ will be degenerate, having $D_H$ null vectors. In what follows, by the sigma model fields we shall understand the expressions which are found after imposing the gauge condition, i.e. after replacing $x^M$ by $x^\mu$ and making the corresponding replacements of the tensor indices, $M,N,... \rightarrow \mu,\nu,...$, etc (see the end of Sec.5.3).

Using the coordinate parametrisation

$$g^{-1} \partial g = T_A E^A_M (x) \partial x^M , \quad g^{-1} \bar{\partial} g = T_A E^A_M (x) \bar{\partial} x^M ,$$

$$\partial g g^{-1} = T_A \tilde{E}^A_M (x) \partial x^M , \quad \bar{\partial} g g^{-1} = T_A \tilde{E}^A_M (x) \bar{\partial} x^M ,$$

$$\tilde{E}^A_M = C^A_B (x) E^B_M , \quad C_{AB} = \text{Tr} (T_A g T_B g^{-1}) ,$$

we can represent (5.13) in the sigma model form

$$\Gamma_{loc} (g) = S(x) = -\frac{1}{\pi \alpha'} \int d^2 z \ G_{MN} (x) \partial x^M \bar{\partial} x^N , \quad \alpha' = \frac{2}{k + \frac{1}{2} c_G} ,$$

$$G_{MN} \equiv G_{(MN)} = G_{0MN} - \frac{1}{2} [K (M^{-1} - S^{-1}) K^T]_{ab} E^a_M E^b_N$$

$$- \frac{1}{2} (M^{-1} - S^{-1})_{ab} \tilde{E}^a_M \tilde{E}^b_N - (K M^{-1} + S^{-1})_{ab} E^a_{(M} \tilde{E}^b_{N)} ,$$

$$B_{MN} \equiv G_{[MN]} = B_{0MN} - (K M^{-1} + S^{-1})_{ab} E^a_{[M} \tilde{E}^b_{N]} .$$

We have used that $K M^{-1} K^T = M^{-1}$. Here $G_{0MN}$ stands for the original WZW coupling,

$$G_{0MN} = E^A_M \eta_{AB} E^B_N = \tilde{E}^A_M \eta_{AB} \tilde{E}^B_N ,$$

\footnote{Note that we rescale the metric by the factor $- (k + \frac{1}{2} c_G)$ with respect to the standard definition (used in Sec.2). In particular, $G_{MN}$ is negative definite in the compact case.}
\[ 3\partial_B [K] = E^A_R E^B_M E^C_N f_{ABC} = \tilde{E}^A_R \tilde{E}^B_M \tilde{E}^C_N f_{ABC} . \]

The result for the metric (5.18) is equivalent to the one found in [14] using the ‘one-dimensional’ ansatz. Our direct 2d approach makes it possible to derive also the antisymmetric tensor coupling (5.19).\footnote{An equivalent expression was also suggested in [14] using the analogy with the result for the metric.} It is useful to repeat the procedure of the elimination of the gauge field starting directly with the original form of the truncated effective action (4.17),(4.18). Representing (4.18) as
\[ \Delta I(g,A) = \frac{1}{\pi} \int d^2 z \left[ (-A\bar{J} + \bar{A}J) + AN\bar{A} + \frac{1}{2} b (QA + \bar{A}Q^{-1}\bar{A}) \right] , \]
(5.21)
\[ J_a = \text{Tr} (T_a g^{-1}\partial g) , \quad \bar{J}_a = \text{Tr} (T_a \partial gg^{-1}) , \quad N_{ab} \equiv M_{ab} - b\eta_{ab} , \]
and solving for \( A, \bar{A} \) we obtain (after omitting the explicitly non-local terms where \( Q \) or \( Q^{-1} \) are acting on \( N \))
\[ \Delta I(g) = \frac{1}{2\pi} \int d^2 z \left[ J\tilde{V}^{-1}(N^T \bar{J} + bQJ) + \bar{J}V^{-1T}(NJ + bQ^{-1}\bar{J}) \right] , \]
(5.22)
\[ V \equiv NN^T - b^2 = MM^T - 2bM_S , \quad \tilde{V} \equiv N^T N - b^2 = M^T M - 2bM_S , \]
(5.23)
\[ \tilde{V}^{-1} N^T = N^T V^{-1} , \quad M = M_S (MM^T)^{-1} V , \quad KV^{-1} K^T = \tilde{V}^{-1} . \]

Ignoring the non-local terms we can replace \( Q^{-1}\bar{J} \) by \( \text{Tr} (T_a \partial gg^{-1}) \) and \( QJ \) by \( \text{Tr} (T_a g^{-1}\partial g) \). Comparing (4.17),(5.22) with (5.17) and using (5.14),(5.15) we find
\[ G_{MN} = G_{0MN} - b(\tilde{V}^{-1})_{ab} E^a_M E^b_N - b(V^{-1})_{ab} \tilde{E}^a_M \tilde{E}^b_N - 2(\tilde{V}^{-1} N^T)_{ab} E^a_M \tilde{E}^b_N , \]
(5.24)
\[ B_{MN} = B_{0MN} - 2(\tilde{V}^{-1} N^T)_{ab} E^a_M \tilde{E}^b_N . \]
(5.25)

These expressions [14] can be transformed into the form (5.18),(5.19) with the help of (5.23).

Given the effective action (4.17),(4.18) it is possible also to derive the expression for the dilaton coupling. As was suggested in [4] and discussed in detail in [13] the dilaton
contribution can be found from the regularised determinant which appears after one integrates over the gauge field. It should be emphasized that we are not actually treating the arguments $A, \bar{A}$ of the effective action as quantum fields. The correct point of view is that the dilaton coupling term should be already contained in the quantum effective action (1.1) computed on a curved $2d$ background. The reason why the exact expression for the dilaton can be found from the determinant of the $A, \bar{A}$ - bilinear form is related to the fact that the dilaton term can be interpreted be as an anomaly-type ('semiclassical') contribution. If

$$Z = \int [dA] \exp \left( -\frac{1}{2\pi} \int d^2z \sqrt{\gamma} \, F_{\alpha\beta} A_\alpha A_\beta \right), \quad (5.26)$$

where $F_{\alpha\beta}(z)$ is a given matrix function and $\alpha, \beta$ stand for both internal and two-dimensional indices, then

$$Z = \exp \left[ -\frac{1}{2} (\text{Tr} \ln F)_{\text{reg}} \right]$$

$$= \exp \left[ -\frac{1}{16\pi} \int d^2 z \sqrt{\gamma} \left( c_0 \Lambda^2 \ln \det F + c_1 (\partial_m \ln \det F)^2 + c_2 R \ln \det F \right) \right]. \quad (5.27)$$

Here $\Lambda$ is an UV cut-off, $R$ is the curvature of the $2d$ metric $\gamma_{mn}$ and $c_i$ are finite coefficients. To preserve the conformal invariance of the theory one should define $Z$ in such a way that $c_1 = 0$ and $c_2 = -1$ \cite{55,56,57,58}. The quadratically divergent term can be interpreted as a contribution to the local measure while the coefficient of $R$ is the dilaton coupling of the corresponding sigma model, i.e.

$$Z = \prod_z \det F^{-1/2} \exp \left[ -\frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \, R \phi \right], \quad (5.28)$$

$$\phi = -\frac{1}{4} \ln \det F. \quad (5.29)$$

Taking into account the Jacobian of the transformation (5.6) (equal to $\det P$) we get from (5.7)

$$\det F = \det \mathcal{M} \det M_S (\det P)^{-2}. \quad (5.30)$$

$^{19}$ The $c_1$-term would have given a correction to the sigma model metric. In the semiclassical approximation one could argue that such term should be dropped since the corresponding correction to the metric would contain an extra power of $\alpha'$. 43
Using that
\[ M = M_S(MM^T)^{-1}V , \quad \det M_S = \det M \det P , \quad (5.31) \]
we find
\[ \phi = -\frac{1}{2} \ln \det M + \frac{1}{4} \ln \det M_S - \frac{1}{4} \ln \det \mathcal{M} \]
\[ = -\frac{1}{4} \ln \det V . \quad (5.33) \]
A similar representation for the exact dilaton \[ \text{in the } SL(2,R)/U(1) \text{ model was given in } [13], \]
while the expression equivalent to (5.33) has also appeared in \[ [14]. \]

The sigma model with the metric (5.18), antisymmetric tensor (5.19) and dilaton (5.32) should be conformal invariant to all orders in the sigma model loop expansion, i.e. should represent a large class of exact solutions of string equations. Depending on \( b \), the fields \( G_{MN}, B_{MN} \) and \( \phi \) are non-trivial functions of parameter \( k \) or \( \alpha' \) (see (5.17))
\[ k + \frac{1}{2}c_G = \frac{2}{\alpha'} , \quad b = -\frac{1}{4}(c_G - c_H)\alpha' , \quad (5.34) \]
i.e. the semiclassical limit corresponds to \( b \to 0 \).

As we have found in Sec.4.2, the local part of the bosonic term in the effective action in the gauged supersymmetric WZW theory is equal to the classical bosonic gauged WZW action (4.37) with unshifted \( k \) and no ‘quantum’ \( b \)-term. Thus \( \alpha' = \frac{2}{k} \) and the corresponding exact sigma model couplings are given by the ‘semiclassical’ bosonic expressions (5.18),(5.19),(5.32) with \( b = 0 \), i.e.
\[ G^{(s)}_{MN} = G_{0MN} - 2(M^{-1})_{ab}E^a_{(M}\tilde{E}^b_{N)} , \quad (5.35) \]
\[ B^{(s)}_{MN} = B_{0MN} - 2(M^{-1})_{ab}E^a_{[M}\tilde{E}^b_{N]} , \quad \phi^{(s)} = -\frac{1}{2} \ln \det M . \quad (5.36) \]
For example, in the case when \( G/H \) is Kähler, (5.35) and (5.36) give the couplings of the sigma model corresponding to \( N = 2 \) Kazama-Suzuki superconformal theories (both compact \[ [3] \] and non-compact \[ [4][5][4][13] \]). As was already anticipated in \[ [3] \] the sigma model metric is different from the invariant metric on \( G/H \). It is now clear that this can be attributed to the presence of a non-trivial dilaton coupling.
5.3. Explicit form of the sigma model metric

To be able to study the properties of (5.18), (5.19), (5.32), and, in particular, to establish the correspondence with the results found in the operator formalism in Sec. 2 (e.g. to prove that the dilaton can be expressed in terms of the ratio of the determinants of the ‘deformed’ and invariant metrics on $G/H$ (2.22)) it is important to transform the metric (5.18) into a more explicit form. This, in fact, can be done in general (without specifying $G$ and $H$). First, let us note that $\tilde{E}_A^M$ can be expressed in terms of $E_A^M$ with the help of the matrix $C_{AB}$ (5.16),

$$
\tilde{E}_a^M = C_{ab} E_b^M + C_{ai} E_i^M , \quad \tilde{E}^M_a = C_{ab} E^b_M + C_{ai} E^i_M ,
$$

(5.37)

$$
E^A_M E^B_N = \delta^A_B , \quad E^A_M E^N_A = \delta^N_M , \quad \tilde{E}_M^A \tilde{E}^B_M = \delta^A_B , \quad \tilde{E}_M^A \tilde{E}^N_A = \delta^N_M , \quad \tilde{E}_M^A E^M_B = C^A_B .
$$

The indices are raised and lowered with $\eta_{AB}$ and $G_{0MN} = E^A_M E_{AN} = E_a^M E_{aN} + E_i^M E_{iN}$. The matrix $C_{AB}$ satisfies the orthogonality relation $C^T \eta C = \eta$, i.e., in particular,

$$
C_{ad} C^d_b + C_{ai} C^i_b = \eta_{ab} .
$$

(5.38)

Using these relations we can put (5.18), (5.24) into the form

$$
G_{MN} = h_{AB} E^A_M E^B_N = h_{ij} E^i_M E^j_N + h_{ab} E^a_M E^b_N + 2 h_{ai} E^a_{(M} E^i_{N)} ,
$$

(5.39)

where

$$
h_{ij} = \eta_{ij} + f_{ab} C^a_i C^b_j , \quad f_{ab} = -b(V^{-1})_{ab} = -\frac{1}{2} (M^{-1} - M^{-1}_S)_{ab} ,
$$

(5.40)

$$
h_{ab} = \eta_{ab} - b \tilde{V}_{ab} - b V^{-1}_{cd} C^c_a C^d_b - (\tilde{V}^{-1} N^T)_{ad} C^d_b - (\tilde{V}^{-1} N^T)_{bd} C^d_a ,
$$

$$
h_{ai} = -b V^{-1}_{bd} C^b_a C^d_i - (\tilde{V}^{-1} N^T)_{ad} C^d_i .
$$

The key observation is that this metric is degenerate, having $D_H$ null vectors

$$
G_{MN} Y^N = 0 , \quad Y^N = E^N_A Y^A , \quad Y^A = \{ y^a , \quad y^a M^{-1}_{ab} C^{bi} \} ,
$$

(5.41)

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where \( y^a \) are free parameters (this is true, of course, for an arbitrary value of \( b \), i.e. also for the ‘semiclassical’ metric (5.35)). In view of (5.37) and the relation \( C_{ab} = M_{ab} + \eta_{ab} \),

\[
Y^N = -y^a M^{-1b}_a (E^N_b - \tilde{E}^N_b),
\]

i.e. the equivalent set of null vectors is represented by

\[
Z^N_a = E^N_a - \tilde{E}^N_a = -M_{ab} E^{Nb} - C_{ai} E^{Ni}.
\]

These vectors are recognised as being the generators of the vector subgroup \( H \) of the \( G \times G \) symmetry of the WZW action which was gauged in (4.1). They are Killing vectors of the group metric \( G_{0MN} \) and are also the Killing vectors of \( G_{MN} \) (see in this connection [58]). Note that it is the ‘Z-component’ of the current \( \tilde{J}_a \) (5.10) which is coupled to the ‘longitudinal’ part \( \tilde{B} \) of the gauge potential in (5.11).

It is possible to check that \( G_{MN} Z^N_a = 0 \) directly using (5.43), the representation (5.24) for the metric and (5.37), (5.38). As a simple illustration, let us consider a special case of symmetric \( C_{ab} \) (\( C \) is symmetric, for example, when \( \dim H = 1 \)). Then

\[
M_S = M, \quad K = P = 1, \quad M = M - 2b, \quad V = \tilde{V} = M(M - 2b)
\]

so that the background fields (5.18), (5.19), (5.32) take the form

\[
G_{MN} = G_{0MN} - 2(M^{-1})_{ab} E^a_{(M} E^b_{N)} - b[M(M - 2b)]^{-1} (E^a_M + \tilde{E}^a_M)(E^b_N + \tilde{E}^b_N),
\]

\[
B_{MN} = B_{0MN} - 2[[M(M - 2b)]^{-1} (M - b)]_{ab} E^a_{[M} E^b_{N]} ,
\]

\[
\phi = -\frac{1}{4} \ln \det [M(M - 2b)].
\]

In (5.44) we have separated the ‘semiclassical’ part from the higher-order correction. Multiplying (5.44) by \( E^N_a - \tilde{E}^N_a \) and using that

\[
(E_{aN} + \tilde{E}_{aN})(E^N_b - \tilde{E}^N_b) = C_{ab} - C_{ba}
\]
vanishes by assumption, one concludes that both the ‘semiclassical’ and ‘quantum’ parts of (5.44) give zero contributions to the product.

To get a non-degenerate metric we should restrict $G_{\alpha\beta}$ to the subspace orthogonal to the null vectors $Z^\alpha_a$. In general, given a set of null vectors and another non-degenerate ‘canonical’ metric (which we shall choose to be equal to the invariant metric $G_{0\alpha\beta} = E_{\alpha\beta}^0 E_{\gamma\delta}^0$ on $G$) one can define the projection operator on the subspace orthogonal (with respect to $G_{0\alpha\beta}$) to $Z^\alpha_a$

$$\Pi^\alpha_a \equiv \delta^\alpha_a - Z^\alpha_a (ZZ)^{-1} Z_a^b , \quad (ZZ)_{ab} = G_{0\alpha\beta} Z^\alpha_a Z^\beta_b , \quad \Pi^2 = \Pi . \quad (5.48)$$

One can change the original basis $E_{\alpha\beta}^i = (E_{\alpha\beta}^i, E_{\alpha\beta}^a)$ for the new one $(H^i_\alpha, Z^\alpha_a)$ with $H^i_\alpha$ being orthogonal to $Z^\alpha_a$

$$G_{0\alpha\beta} H^\alpha_i Z^\beta_a = 0 , \quad \Pi^\alpha_i H^\alpha_i = H^\alpha_i . \quad (5.49)$$

Then the degenerate metric (5.39) takes the form $(H^i_\alpha \equiv H^\alpha_j \eta^{ij} G_{0\alpha\beta})$

$$G_{\alpha\beta} = \Pi^\alpha_i H^\alpha_i H^\beta_j \eta^{ij} . \quad (5.50)$$

i.e.

$$G_{\alpha\beta} = g_{\alpha\beta} H^\alpha_i H^\beta_j . \quad (5.51)$$

The choice of $H^\alpha_i$ is not unique. We can take

$$H^\alpha_i = \Lambda^i_j \tilde{H}^\alpha_j , \quad \tilde{H}^\alpha_i = \Pi^\alpha_i E^\alpha_j , \quad (HH)_{ij} = G_{0\alpha\beta} H^\alpha_i H^\beta_j , \quad \Pi^\alpha_i = H^\alpha_i (HH)^{-1} H^\beta_j \eta^{ij} . \quad (5.52)$$

where $\Lambda^i_j$ can be fixed, for example, by the condition of orthonormality of $H^\alpha_i$, i.e. $(HH)_{ij} = \eta_{ij}$. The inverse to $G_{\alpha\beta}$ is given by

$$G^{-1M\beta} G_{N\beta} = \Pi^M_K , \quad G^{-1M\beta} = g^{(-1)ij} H^\alpha_i H^\beta_j , \quad \Pi^N_M = H^N_i (HH)^{-1} H^\beta_j \eta^{ij} . \quad (5.53)$$

$$g^{(-1)ik} (HH)_k g^{jl} = (HH)_k^{(-1)} . \quad (5.54)$$
In our case of (5.43), 

\[ (ZZ)_{ab} = -2(M_S)_{ab} \], i.e.,

\[ \Pi^N_M = \delta^N_M + \frac{1}{2} (F^N_a - \tilde{E}^N_a) M_{ab}^{-1} (E_{Ma} - \tilde{E}_{Ma}) . \]

To express the metric in terms of \( \Pi^N_M \) the form (5.18) of \( G_{MN} \) is most useful: as is obvious from (5.18),

\[ \Pi^N_M = \delta^N_M + 2 (E^N_a - \tilde{E}^N_a) M_{ab}^{-1} (E_{Ma} - \tilde{E}_{Ma}) . \]

Since \( \Pi^N_M \) \( E^M_a \) \( E^M_a = \frac{1}{2} (\tilde{E}^N_a + E^N_a K^c_a ) M_{ab}^{-1} (E_{Ma} - \tilde{E}_{Ma}) \), we find (5.50) with

\[ \hat{G}_{MN} = G_{0MN} - 2 E^a_M M_{ab}^{-1} E^b_N = E^i_M \eta_{ij} E^j_N + E^a_M (\eta_{ab} - 2 M_{ab}^{-1}) E^b_N . \]  (5.55)

A simple choice of (non-orthonormal) \( H^M_i \) is (we shall use bars to denote objects corresponding to this basis)

\[ \bar{H}^i_M = E^i_M - M_{ab}^{-1} C^{bi} E^a_M = p^i_M E^j_M + M_{ab}^{-1} C^{bi} M_{ac}^{-1} Z^c_M, \quad \bar{H}^i_M Z^a_M = 0 , \]  (5.56)

\[ (\bar{H} \bar{H})_{ij} = \bar{H}^i_M \bar{H}^j_N = p_{ij} , \quad p_{ij} = \eta_{ij} + (M M^T)^{-1} C^a_i C^b_j . \]  (5.57)

Expressing \( E^a_M, E^a_M \) in terms of \( \bar{H}^i_M, Z^a_M \) is effectively equivalent to dropping out terms with \( E^a_M \) in (5.39), i.e., we find

\[ G_{MN} = \bar{g}_{ij} \bar{H}^i_M \bar{H}^j_N, \quad \bar{g}_{ij} = h_{ij} = \eta_{ij} - b V_{ab}^{-1} C^a_i C^b_j . \]  (5.58)

Note that the metric \( \bar{g}_{ij} \) becomes trivial in the ‘semiclassical’ limit \( b = 0 \). If instead we use the basis \( \tilde{H}^i_M \) defined in (5.52)

\[ \tilde{H}^i_M = \Pi^M_N E^i_N = \Pi^M_N E^a_M p_{ij}^{-1} C^{aj} (M M^T)^{-1} Z^b_M, \]  (5.59)

then

\[ E^a_M = - M^{-1ab} C^{bi} \tilde{H}^i_M - (\eta^{ab} - M^{-1ac} M^{-1bd} C^{ci} p_{ij}^{-1} C^{dj} ) Z_{bM} , \]
and thus from (5.55)

\[ G_{MN} = \bar{g}_{ij} \bar{H}^i_M \bar{H}^j_N , \quad \bar{g}_{ij} = \eta_{ij} + [M^{-T} (1 - 2M^{-1})M^{-1}]_{ab} C^a_i C^b_j . \quad (5.60) \]

The representations (5.58) and (5.60) are related by the transformation

\[
\bar{H}^i_M = (p^{-1})^i_j \bar{H}^j_M , \quad \bar{g}_{ij} = \bar{g}_{kl} (p^{-1})^k_i (p^{-1})^l_j , \\
(\bar{H} \bar{H})_{ij} = \eta_{ij} + \frac{1}{2} (M_S^{-1})_{ab} C^a_i C^b_j . \quad (5.61)
\]

The $D \times D$ matrices of generic form $h_{ij} = \eta_{ij} + f_{ab} C^a_i C^b_j$ have the following multiplication law:

\[
(\eta_{ik} + f_{1ab} C^a_i C^b_k) \eta^{kl} (\eta_{lj} + f_{2ab} C^a_i C^b_j) = (\eta_{ij} + f_{3ab} C^a_i C^b_j) , \\
f_3 = f_1 + f_2 + f_1 (1 - CC^T) f_2 , \quad (5.62)
\]

where we have used (5.38), i.e. that $C^a_i C^b_i = (1 - CC^T)_{ab}$. The inverse of such a matrix is given by

\[
h^{-1}_{ij} = \eta_{ij} + f_{ab} C^a_i C^b_j , \quad f^{(-1)} = -[f^{-1} + (1 - CC^T)]^{-1} . \quad (5.63)
\]

The determinant of $h_{ij}$ can be computed by taking the variation with respect to $f_{ab}$

\[
\delta \ln \det h = \text{Tr} (h^{-1} \delta h) = h^{-1ij} C_{ai} C_{bj} \delta f^{ab} = \{(1 - CC^T)[1 + f^{(-1)} (1 - CC^T)]\}^{ab} \delta f^{ab} , \\
\det h = \det [1 + f (1 - CC^T)] . \quad (5.64)
\]

Applying (5.64) to $p_{ij}$, $\bar{g}_{ij}$ and $\bar{g}_{ij}$ in (5.57),(5.58) and (5.60) we get

\[
\det \bar{g}_{ij} = \det [(1 + b) V^{-1} M^2] , \quad \det p_{ij} = \det [-2M^{-2} M_S] , \quad (5.65) \\
\det \bar{g}_{ij} = \det [4(1 + b) V^{-1} M_S^2 M^{-2}] = \det \bar{g}_{ij} \det [p_{ij}]^2 . \quad (5.66)
\]

The inverse of $G_{MN}$ defined according to (5.53),(5.54) is found to be

\[
G^{-1}_{MN} = \bar{g}^{-1}_{ij} \bar{H}^i_M \bar{H}^j_N , \quad \bar{g}^{-1}_{ij} \equiv (p \bar{g} p)^{-1} , \quad (5.67)
\]

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\[
\hat{g}^{(-1)}_{ij} = \eta_{ij} + \frac{1}{2}[M_S^{-1} - \frac{1}{2(b+1)}M_S^{-1}MM^TM_S^{-1}]_{ab}C^a_iC^b_j. \tag{5.68}
\]

The metric \(5.67\) can be represented in the following form

\[
G^{-1MN} = \Pi^M_K \hat{G}^{-1KL}\Pi^N_L,
\]

\[
\hat{G}^{-1KL} = E^M_A E^{AN} - \gamma E^M_a E^{aN} = E^M_i E^{iN} - (\gamma - 1)E^M_a E^{aN}, \tag{5.69}
\]

\[
\gamma = (b + 1)^{-1} = \frac{k + \frac{1}{2}c_G}{k + \frac{1}{2}c_H}. \tag{5.70}
\]

This is readily checked using \(5.43\), \(5.56\), i.e.

\[
E^i_M = p^{-1i} \hat{H}^j_M + O(Z), \quad E^a_M = -M^{-1ab}C_{bi}p^{-1i} \hat{H}^j_M + O(Z),
\]

\[
E^M_i E^{iN} - (\gamma - 1)E^M_a E^{aN} = (p^{-1}g'p^{-1})_{ij} \hat{H}^M_i \hat{H}^{iN},
\]

\[
g'_ij = \eta_{ij} - (\gamma - 1)(MM^T)^{-1}C^a_iC^b_j, \quad g' = \hat{g}^{-1}, \tag{5.71}
\]

where we have noted that the matrix \(g'\) is nothing but the inverse of \(\hat{g}\) in \(5.58\).

To obtain a similar representation for the antisymmetric tensor coupling in \(5.25\) one is to express \(\hat{E}^a_M\) in terms of \(E^i_M, E^a_M\) (or \(\hat{H}^i_M, Z^a_M\)). We get

\[
B_{MN} = B_{0MN} - 2(\hat{V}^{-1}N^T C)_{ab}E^a_{[M}E^b_{N]} - 2(\hat{V}^{-1}N^T)_{ab}C^b_iE^a_{[M}E^i_{N]} \tag{5.72}
\]

It should be possible to find a more explicit representation for the field strength of \(B_{MN}\) using the expression for \(\partial_i K_{0MN}\). The gauge invariance of the action (before gauge fixing) implies that the Lie derivative of \(H^M_{KMN}\) along \(Z^a_M\) should vanish and that \(B_{MN}Z^a_M = 0\), i.e. like \(G_{MN}\) the antisymmetric tensor can be represented in terms of \(H^i_M\).

Since the sigma model \(\int d^2zG_{MN}(x)\partial x^M \bar{\partial} x^N + ...\) has a gauge invariance (generated by \(Z^a_M\)) the final step is to fix a gauge, e.g. to restrict coordinates \(x^M\) on \(G\) to coordinates \(x^\mu\) on \(G/H\). Let

\[
R^a(x^M) = 0, \quad R^a_M \delta x^M = 0, \quad R^a_M \equiv \partial_M R^a \tag{5.73}
\]
be a gauge condition (the corresponding ghost determinant which should be included in the measure is $\det R^a_M Z^b_N$). One may either add a gauge term into the sigma model action (which will then depend on all $x^M$ coordinates) or explicitly solve the gauge condition expressing $x^M = x^M(x^\mu)$ in terms of $D$ coordinates $x^\mu$ on $G/H$. In the latter case we will get a sigma model on the $D$-dimensional space with $H^i_M$ replaced by the $D \times D$ matrix $H^i_M \equiv H^i_M \partial_\mu x^M$ (i.e. a vielbein), $G_{MN}$ by $G_{\mu\nu}$, etc. The expressions for the sigma model metric and antisymmetric tensor then are

$$G_{\mu\nu} = G_{MN} \partial_\mu x^M \partial_\nu x^N, \quad G_{\mu\nu} = G_{MN} \partial_\mu x^M \partial_\nu x^N,$$

(5.74)

where $G_{MN}$ and $B_{MN}$ are given by (5.58),(5.56) and (5.72).

6. Equivalence of Results of Operator and Field–Theoretical Approaches

Let us now compare the expressions for the background metric and the dilaton corresponding to a gauged WZW model obtained in the field-theoretical approach (5.58),(5.73),(5.33)

$$G_{\mu\nu} = \bar{g}_{ij} \bar{H}^i_\mu \bar{H}^j_\nu, \quad \bar{g}_{ij} = \eta_{ij} - bV^{-1} \epsilon^a_i \epsilon^b_j.$$

(6.1)

$$\phi = \phi_0 - \frac{1}{4} \ln \det V, \quad V = MM^T - b(M + M^T),$$

(6.2)

$$M_{ab} = \text{Tr} (T_a g T_b g^{-1} - T_a T_b), \quad C_{ai} = \text{Tr} (T_a g T_i g^{-1}),$$

If one uses the formulation in terms of all $D_G$ coordinates $x^M$ one should also impose as usual the gauge invariance (BRST invariance) condition on the observables. Adding a gauge-fixing term in the action one obtains the following non-degenerate metric on $G$: $\bar{G}_{MN} = G_{MN} + q_{ab} R^a_M R^b_N$, where $q_{ab}$ can be chosen on the basis of convenience. The determinant of the degenerate metric $G_{MN}$ is then formally defined as follows

$$(\det G_{MN})^{-1/2} = (\det \bar{G}_{MN})^{-1/2} \det (R^a_M Z^b_N)(\det q_{ab})^{1/2}.$$
with the results (2.12),(2.25),(2.31) which were found in Sec.2 by identifying the operator
$L_0$ of conformal $G/H$ theory with a Klein-Gordon operator in a background. The restriction
of $L_0$ to $H$-invariant states implies imposing the condition $Z_a^M \partial_M = 0$ (where $Z_a^M$ is
given by (5.43)). Then the metric (2.12) is the ‘projected’ one (2.25) and therefore is equal
(up to the overall factor $-\frac{k+\frac{1}{2}c_G}{2}$ which we separated from the sigma model metric in
(5.17)) to the inverse of the metric (5.73) (see (5.69),(5.70)). We have thus demonstrated
that both the operator and the field-theoretical approaches lead to the same expression
for the target space metric. While the operator approach gave the inverse of the metric
in a simply looking but rather abstract form, following the direct sigma model approach
provided us with more explicit representation for the metric itself and clarified its general
structure.

The metric (6.1) can be considered as a ‘deformation’ of the ‘round’ metric on $G/H$.
The latter corresponds to the sigma model which is found by integrating out the gauge field
(taking values in the algebra of $H$) in the action invariant under the gauge transformations
$g' = gu$ generated by $E_a^M$

$$I = \int d^2z \, \text{Tr} \left( (g^{-1} \partial_m g + A_m)^2 \right). \quad (6.4)$$

This action (and hence the resulting sigma model metric) has also global $G$-invariance
which is absent in the gauged WZW action (4.1) (unless $H$ is an invariant subgroup of
non-simple $G$) being broken by the $g^{-1}Ag\bar{A}$-term). As a result, in contrast to the metric
in (5.74), the sigma model metric corresponding to (6.4) has global $G$ invariance. Before
gauge fixing we get a degenerate metric $G^{(0)}_{MN}$ on the full $G$ (with null vectors $E_a^M$).
Solving a gauge condition (5.73) and expressing $x^M$ in terms of $x^\mu$ we get the metric $G^{(0)}_{\mu\nu}$
on $D$-dimensional space $G/H$,

$$G^{(0)}_{MN} = \eta_{ij} E_i^M E_j^N, \quad G^{(0)}_{\mu\nu} = \eta_{ij} E_i^\mu E_j^\nu, \quad E_i^\mu = E_i^M \partial_\mu x^M. \quad (6.5)$$
Note that since according to (5.56) \( \bar{H}^i_M = E^i_M + O(E^a_M) \), choosing the gauge condition in (5.73),(5.74) such that \( \partial_M R^a = E^a_M \) it is easy to see that

\[
\det G_{\mu\nu} = \det G^{(0)}_{\mu\nu} \det \bar{g}_{ij} (\det M)^{-2},
\]

or (cf. (5.35))

\[
\det G^{(s)}_{\mu\nu} = \det G^{(0)}_{\mu\nu} (\det M)^{-2}, \quad G^{(s)}_{\mu\nu} = \eta_{ij} \bar{H}^i_\mu \bar{H}^j_\nu,
\]

where \( (\det M)^2 \) is the square of the corresponding ghost determinant \( (E^a_M Z^M_b = -M^a_b \), see (5.43)).

A similar equivalence is established between the results of the two approaches for the dilaton (2.31) and (5.33). As we have shown, the operator approach implies that the dilaton is given by the logarithm of the ratio (2.31) of the determinant of the metric and the determinant of an invariant metric on the coset space. Using (6.1),(6.2) and the expression (5.65) for the determinant of \( \bar{g}_{ij} \) and we get

\[
\sqrt{\det G^{(s)}_{\mu\nu}} e^{-2\phi} = \sqrt{\det G^{(s)}_{\mu\nu}} \sqrt{\det \bar{g}_{ij}} e^{-2\phi}
\]

\[
= a \sqrt{\det G^{(s)}_{\mu\nu}} \det M = a \sqrt{\det G^{(0)}_{\mu\nu}},
\]

\[
a = (\sqrt{1 + b})^D e^{-2\phi_0},
\]

i.e. as in (2.22) the ‘measure factor’ (6.8) is nothing but the canonical measure on the \( G/H \).

Since \( G^{(0)}_{\mu\nu} \) is \( b \)-independent we have thus proved in general that \( \sqrt{G} e^{-2\phi} \) is essentially \( k \)-independent, in agreement with [17] [13] [14].

It would be interesting to study the general properties of the metric (6.1), relating them to the properties of the basic matrix \( C_{AB} \) (i.e. of \( M_{ab}, C_{ai}, \) etc) and the value of \( b \) (for example, in the semiclassical limit the singularities of (5.35),(5.36) correspond to the fixed points of the transformation \( g \rightarrow hgh^{-1} \) where \( M \rightarrow 0 \) [59]; see also [31] [15]).

The sigma model couplings in the supersymmetric case (5.35),(5.36) does not depend on \( \alpha' \). That means they solve the sigma model conformal invariance conditions at each
order of the loop expansion. Namely, once the one-loop conditions are satisfied, all higher loop corrections to the $\beta$-functions vanish (up to a field redefinition ambiguity) on the corresponding background. This is interesting, given that these models have, in general, only $N = 1$ (and not $N = 2$ or $N = 4$) supersymmetry. There exists another class of $N = 1$ supersymmetric sigma models which have the same property.

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21 This was checked explicitly up to 5-loop order in the supersymmetric $SL(2, R)/U(1)$ model. While the two- and three-loop terms in the $\beta$-function of the $N = 1$ supersymmetric sigma model are known to vanish (in the minimal subtraction scheme), the four-loop term does not vanish in general. However, there exists such a renormalisation scheme in which it vanishes for the ‘one-loop’ $D = 2$ background of.

22 The corresponding $2 + D$-dimensional target space metric has null Killing vector and the ‘transverse’ $D$-dimensional part of the space is represented by the homogeneous $G/H$ Kähler-Einstein manifolds (the $N = 2$ sigma models representing the ‘transverse’ space have only one-loop $\beta$-function being non-zero).
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