On the automorphism groups of binary linear codes

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Abstract. Let $C$ be a binary linear code and suppose that its automorphism group contains a non trivial subgroup $G$. What can we say about $C$ knowing $G$? In this paper we collect some answers to this question in the cases $G \cong C_p$, $G \cong C_{2p}$ and $G \cong D_{2p}$ ($p$ an odd prime), with a particular regard to the case in which $C$ is self-dual. Furthermore we generalize some methods used in other papers on this subject. Finally we give a short survey on the problem of determining the automorphism group of a putative self-dual $[72,36,16]$ code, in order to show where these methods can be applied.

This paper is a presentation of some of the main results about the automorphism group of binary linear codes obtained by the author in his Ph.D. thesis. Part of the results are proved in joint papers with Wolfgang Willems, Francesca Dalla Volta and Gabriele Nebe.

The problem we want to investigate is the following: let $C$ be a (self-dual) binary linear code and suppose that $\text{Aut}(C)$ contains a non trivial subgroup $G$. What can we say about $C$ knowing $G$?

To face this problem, usually we want to find out “smaller pieces” which are easier to determine and then look at the structure of the whole code.

In Section 2 we present a classical decomposition of codes with automorphisms of odd prime order. In Section 3 we summarize the most significant results of [BW], about codes with automorphisms of order $2p$, where $p$ is an odd prime. Section 4 is a generalization of methods used in [FN] and [BDN], about codes whose automorphism groups contain particular dihedral groups. Finally, in Section 5 we point out and generalize some theoretical tools used in [Bor1], [BDN] and [Bor2].

Our methods can be applied

- to study the possible automorphism groups of extremal self-dual binary linear codes;
- to construct self-orthogonal binary linear codes with large minimum distance and relatively large dimension;
- to classify self-dual binary linear codes with certain parameters.

Obviously the last one is the most ambitious.

In the last section, which is a short survey on the problem of determining the automorphism group of a putative extremal self-dual $[72,36,16]$ code, we underline where these methods can be applied, showing their power.
1. Background and notations

We refer the reader to [HP] for basic notions of Coding Theory and to [HB] for basic notions of Representation Theory. In this section we just want to fix the notations we use.

Let \( \mathcal{C} \) be an \([n,k,d]\) code. Then we denote by \( G(\mathcal{C}) \) a generator matrix of \( \mathcal{C} \), i.e. a matrix in \( \text{Mat}_{k,n}(\mathbb{F}_2) \) whose rows generate \( \mathcal{C} \).

Let \( \sigma \in S_n \). Then we define \( \mathcal{C}^\sigma := \{ c^\sigma \mid c \in \mathcal{C} \} \). The automorphism group of \( \mathcal{C} \) is

\[
\text{Aut}(\mathcal{C}) := \{ \sigma \in S_n \mid \mathcal{C}^\sigma = \mathcal{C} \} \leq S_n.
\]

The fixed code of \( \sigma \) is defined as

\[
\mathcal{C}(\sigma) := \{ c \in \mathcal{C} \mid c^\sigma = c \},
\]

that is obviously a subcode of \( \mathcal{C} \).

If we call \( \Omega_1, \ldots, \Omega_{m_\sigma} \) the orbits of \( \sigma \) on the coordinates \( \{1, \ldots, n\} \), we have trivially that \( c = (c_1, \ldots, c_n) \in \mathcal{C} \) is in \( \mathcal{C}(\sigma) \) if and only if \( c_i = c_j \) for all \( i, j \in \Omega_k \), for every \( k \in \{1, \ldots, m_\sigma\} \). In this case we say that \( c \) is constant on the orbits of \( \sigma \). Thus we can define a natural projection associated to \( \sigma \)

\[
\pi_\sigma : \mathcal{C}(\sigma) \to \mathbb{F}_2^{m_\sigma}
\]

such that \( (\pi_\sigma(c))_k := c_h \) for any \( h \in \Omega_k \), which is clearly well-defined for \( c \in \mathcal{C}(\sigma) \).

If \( \sigma \) is a permutation of order \( p \) we say that \( \sigma \) is of type \( p\cdot(c,f) \) if it has \( c \) cycles of length \( p \) and \( f \) fixed points.

If \( \sigma \) is a permutation of order \( p \cdot q \) we say that \( \sigma \) is of type \( p \cdot q : (a,b,c,f) \) if it has \( a \) cycles of length \( p \), \( b \) cycles of length \( q \), \( c \) cycles of length \( p \cdot q \) and \( f \) fixed points.

Let \( \mathcal{C}, \mathcal{D} \leq \mathbb{F}_2^n \). We set \( \mathcal{C} + \mathcal{D} := \{ c + d \mid c \in \mathcal{C}, d \in \mathcal{D} \} \), sum of \( \mathcal{C} \) and \( \mathcal{D} \). If \( \mathcal{C} \cap \mathcal{D} = \{0\} \), we say that the sum is direct and we denote it by \( \mathcal{C} \oplus \mathcal{D} \). This should not be confused with another common concept of direct sum of codes, which we do not use in this paper.

We use the following notations for groups:

- \( C_n \) is the cyclic group of order \( n \);
- \( D_n \) is the dihedral group of order \( n \);
- \( S_n \) is the symmetric group of degree \( n \);
- \( A_n \) is the alternating group of degree \( n \).

Furthermore, for \( H, G \) groups, \( H \times G \) is the direct product of \( H \) and \( G \) while \( H \rtimes G \) is a semidirect product of \( H \) and \( G \). If \( H \leq G \), we denote the centralizer and the normalizer of \( H \) in \( G \) by \( \text{C}_G(H) \) and \( \text{N}_G(H) \) respectively.

We conclude giving the definition of a fundamental number: we denote by \( s(p) \) the multiplicative order of \( 2 \) in \( \mathbb{F}_p^\times \), i.e. the smallest \( m \in \mathbb{N} \) such that \( p \mid 2^m - 1 \).

2. Cyclic group of order \( p \) (\( p \) an odd prime)

In this section we introduce a well-known classical decomposition of codes with automorphisms of odd prime order. We want to present it for completeness, although it is just a particular reformulation of Maschke’s Theorem, and to fix some notations useful in the following.

Let \( \mathcal{V} := \mathbb{F}_2^n \) and \( \sigma \in S_n \) a permutation of odd prime order \( p \). Then, it is trivial to prove that

\[
\mathcal{V} = \mathcal{V}(\sigma) \oplus \mathcal{V}(\sigma)^\perp
\]
where $\mathcal{V}(\sigma)$ is the subspace fixed by $\sigma$ and $\mathcal{V}(\sigma)\perp$ is the dual of $\mathcal{V}(\sigma)$, that is clearly the subspace of even-weight vectors on the orbits of $\sigma$. We note that $\mathcal{C}(\sigma) = \mathcal{C} \cap \mathcal{V}(\sigma)$ and we define $\mathcal{E}(\sigma) := \mathcal{C} \cap \mathcal{V}(\sigma)\perp$. Then we have the following.

**Theorem 2.1.** Let $\mathcal{C}$ be a binary linear code and suppose $\sigma \in \text{Aut}(\mathcal{C})$ of odd prime order $p$. Then

$$\mathcal{C} = \mathcal{C}(\sigma) \oplus \mathcal{E}(\sigma),$$

where $\mathcal{C}(\sigma)$ is the fixed code of $\sigma$ and $\mathcal{E}(\sigma)$ is the subcode of even-weight codewords on the orbits of $\sigma$.

In order to get more information on the subcode $\mathcal{E}(\sigma)$, with a particular regard to the case in which $\mathcal{C}$ is self-dual, we investigate more closely the decomposition of $\mathcal{V}$. Firstly we consider the case in which $n = p$ and then the general case.

Let $n = p$, so that $\sigma$ is of type $p-(1,0)$. Thus

$$G(\mathcal{V}(\sigma)) = [1\ 1\ 1\ 1\ ...\ 1\ 1] \text{ and } G(\mathcal{V}(\sigma)\perp) = \begin{bmatrix} 1 & 1 & 0 & ... & 0 & 0 \\ 0 & 1 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... & 1 & 1 \end{bmatrix}$$

in Mat$_{1,p}(\mathbb{F}_2)$ and Mat$_{p-1,p}(\mathbb{F}_2)$ respectively.

There is a natural isomorphism of vector spaces

$$\varphi : \mathbb{F}_2^p \rightarrow \mathbb{F}_2[x]/(x^p + 1) := Q$$

which maps $(v_0, \ldots, v_{p-1}) \mapsto v_0 + \ldots + v_{p-1}x^{p-1}$.

Notice that $(x^p + 1) = (x + 1)(x^{p-1} + x^{p-2} + \ldots + x + 1)$, with $(x + 1)$ and $(x^{p-1} + x^{p-2} + \ldots + x + 1)$ coprime (since $p$ is odd). It is well-known that the polynomial $(x^{p-1} + x^{p-2} + \ldots + x + 1)$ is the product of $t := \frac{p-1}{s(p)}$ irreducible polynomials of degree $s(p)$. So, let $(x^p + 1) = q_0(x)q_1(x)\ldots q_t(x)$, where $q_0(x) := (x + 1)$ and the other terms are the $t$ irreducible polynomials of degree $s(p)$. By the Chinese Remainder Theorem we have

$$\mathbb{F}_2[x]/(x^p + 1) \cong \mathbb{F}_2[x]/(q_0(x)) \oplus \mathbb{F}_2[x]/(q_1(x)) \oplus \ldots \oplus \mathbb{F}_2[x]/(q_t(x)) \cong \mathbb{F}_2 \oplus \mathbb{F}_2^{1/p} \oplus \ldots \oplus \mathbb{F}_2^{1/p}$$

Furthermore, calling $Q_j := \frac{x^j + 1}{q_j(x)}$ we have $\mathbb{F}_2[x]/(q_j(x)) \cong (Q_j) =: \mathcal{I}_j$ which is a principal ideal of $\mathbb{F}_2[x]/(x^p + 1)$ generated by $Q_j$. Notice that $Q_j^2 = Q_j$ and $Q_iQ_j = 0$ if $i \neq j$ (the equalities are mod $x^p + 1$). Then

$$\mathcal{V} \cong \mathbb{F}_2[x]/(x^p + 1) = \mathcal{I}_0 \perp \mathcal{I}_1 \perp \ldots \perp \mathcal{I}_t$$

is an orthogonal sum of ideals (generated by orthogonal idempotents), such that $\mathcal{I}_0 \cong \mathbb{F}_2$ and $\mathcal{I}_1 \cong \ldots \cong \mathcal{I}_t \cong \mathbb{F}_2^{1/p}$.

Let now $\sigma$ be of type $p-(c,f)$ and $n = pc + f$. Without lost of generality we can relabel the coordinates to have

$$\sigma = (1, \ldots, p)(p + 1, \ldots, 2p)\ldots,((c - 1)p + 1, \ldots, pc).$$

As $\mathcal{V}(\sigma)\perp$ is the set of all even-weight vectors on the orbits of $\sigma$, we have that $v_i = 0$, for all $i \in \{pc + 1, \ldots, n\}$, for every $v \in \mathcal{V}(\sigma)\perp$. Let us call $(\mathcal{V}(\sigma)\perp)^* \leq \mathbb{F}_2^{pc}$
In particular, with (2.1) to a map \( \varphi_p \) as follows

\[
\varphi_p := \varphi \times \cdots \times \varphi : \mathbb{F}_2^c \to \mathbb{Q},
\]

via the natural identification \((\mathbb{F}_2^c)^c = \mathbb{F}_2^c\).

Let \( \varphi_p' \) the map \( \varphi_p \times \text{id}_f \), where \( \text{id}_f := \mathbb{F}_2^f \to \mathbb{F}_2^f \) is the identity map, so that \( \varphi_p' : \mathbb{F}_2^n \to \mathbb{Q}^c \oplus \mathbb{F}_2^f \). This map gives an isomorphism of vector spaces

\[
\mathcal{V} = \mathbb{F}_2^n \cong \mathbb{F}_2^{c+f} \oplus \mathbb{F}_2^{c_2(p)} \oplus \cdots \oplus \mathbb{F}_2^{c_2(p)}.
\]

It is easy to observe that \( \varphi_p'(\mathcal{V}(\sigma)) \cong \mathbb{F}_2^{c+f} \) and \( \varphi_p((\mathcal{V}(\sigma))^{\perp}) \cong \mathbb{F}_2^{c_2(p)} \oplus \cdots \oplus \mathbb{F}_2^{c_2(p)} \). Furthermore \( \varphi_p'|_{\mathcal{V}(\sigma)} = \pi_\sigma \).

Let us come back to the subcode \( \mathcal{E}(\sigma) \).

Clearly, if \( s(p) < p - 1 \), so that \( t = 1 \), \( \mathcal{E}(\sigma) \) can be decomposed further. A very nice investigation of this case is contained in [FN].

Here we consider only the fundamental case in which \( s(p) = p - 1 \). Then

\[
\pi_\sigma(\mathcal{C}(\sigma)) \leq \mathbb{F}_2^{c+f} \quad \text{and} \quad \varphi_p(\mathcal{E}(\sigma)^*) \leq \mathbb{F}_2^{c_2(p)-1},
\]

where \( \mathcal{E}(\sigma)^* \) is the code obtained puncturing \( \mathcal{E}(\sigma) \) on the last \( f \) coordinates.

We conclude this section stating an important theorem, proved by Vassil I. Yorgov.

**Theorem 2.2** ([Yo1]). Let \( \mathcal{C} \) be a binary code with an automorphism \( \sigma \) of odd prime order \( p \), with \( s(p) = p - 1 \). Then the following are equivalent:

a) \( \mathcal{C} \) is self-dual.

b) \( \pi_\sigma(\mathcal{C}(\sigma)) \) is self-dual and \( \varphi_p(\mathcal{E}(\sigma)^*) \) is Hermitian self-dual.

**Remark 2.3.** “\( \pi_\sigma(\mathcal{C}(\sigma)) \)” is self-dual if \( \mathcal{C} \) is self-dual” holds for every odd prime \( p \) (see for example [CP]). Does it hold also for \( p = 2 \)? In general the answer is negative. For example, there are automorphisms of order 2 of the extended Hamming Code of length 8 for which it holds true and others for which it is false.

**3. Cyclic group of order 2p (p an odd prime)**

Throughout this section we consider \( \mathcal{C} \), a self-dual code of even length \( n \), and \( \sigma_{2p} \in \text{Aut}(\mathcal{C}) \) of order \( 2p \), where \( p \) is an odd prime. We show some module theoretical properties of such a code, assuming that the involution \( \sigma_2 := \sigma_{2p}^2 \) acts fixed point freely on the \( n \) coordinates.

Without loss of generality, we may assume that

\[
\sigma_2 = \sigma_{2p}^2 = (1, 2)(3, 4)\ldots(n - 1, n).
\]

We consider the natural projection \( \pi_{\sigma_2} : \mathcal{C}(\sigma_2) \to \mathbb{F}_2^\oplus \) and the map

\[
\phi : \mathcal{C} \to \mathbb{F}_2^\oplus,
\]

with \((c_1, c_2, \ldots, c_{n-1}, c_n)^\phi = (c_1 + c_2, \ldots, c_{n-1} + c_n)\).

Stefka Bouyuklieva proved [Bou1] that

\[
\phi(\mathcal{C}) \leq \pi_{\sigma_2}(\mathcal{C}(\sigma_2)) = \phi(\mathcal{C})^{\perp}.
\]

In particular,

\[
\phi(\mathcal{C}) = \pi_{\sigma_2}(\mathcal{C}(\sigma_2)) = \phi(\mathcal{C})^{\perp} \iff \dim \pi_{\sigma_2}(\mathcal{C}(\sigma_2)) = \dim \mathcal{C}(\sigma_2) = \frac{n}{4}.
\]
Starting from this easy observation, we proved the following result, that is the crucial theorem of our joint work with W. Willems.

**Theorem 3.1 (BW).** The code $C$ is a projective $\mathbb{F}_2\langle \sigma_{2p}\rangle$-module if and only if $\pi_{\sigma_2}(\mathcal{C}(\sigma_2))$ is a self-dual code.

One of the reasons which makes interesting to determine if the code is projective is explained in the following remark.

**Remark 3.2.** Let $G$ be a finite group and $\mathcal{M}$ a projective $KG$-submodule. Then for every decomposition $\text{soc}(\mathcal{M}) = V_1 \oplus \ldots \oplus V_m$ of the socle in irreducible $KG$-submodules, we have $\mathcal{M} = \text{P}(V_1) \oplus \ldots \oplus \text{P}(V_m)$, where $\text{P}(V_i)$ is the projective cover of $V_i$ in $\mathcal{M}$, for all $i \in \{1, \ldots, m\}$.

So, whenever we have a projective module, there are several restrictions on its structure and, in particular, the knowledge of its socle gives us a lot of information about the whole module.

### 3.1. Consequences on the structure of $C$

We deduce some properties of $C$ related to the action of the automorphism $\sigma_{2p}$.

Since $\sigma_2$ acts fixed point freely, $\sigma_{2p}$ is of type $2p$-$(w, 0, x; 0)$ for certain $x, w \in \mathbb{N}$ such that $n = 2px + 2w$. Thus we have the following decomposition of the $\mathbb{F}_2\langle \sigma_{2p}\rangle$-module $\mathbb{F}_2^n$:

$$\mathbb{F}_2^n \cong \mathbb{F}_2\langle \sigma_{2p}\rangle \oplus \ldots \oplus \mathbb{F}_2\langle \sigma_{2p}\rangle \oplus \mathbb{F}_2\langle \sigma_2\rangle \oplus \ldots \oplus \mathbb{F}_2\langle \sigma_2\rangle.$$  

By Section 2 recalling that $\mathbb{F}_2\langle \sigma_{2p}\rangle \cong \mathbb{F}_2\langle \sigma_{2p}^2\rangle \otimes \mathbb{F}_2\langle \sigma_{2p}^p\rangle$ we get

$$\mathbb{F}_2^n \cong \underbrace{V_0 \oplus \ldots \oplus V_0}_{x \text{ times}} \oplus \underbrace{V_i \oplus \ldots \oplus V_i}_{w \text{ times}} \oplus \underbrace{V_j \oplus \ldots \oplus V_j}_{x \text{ times}} \oplus \underbrace{V_j \oplus \ldots \oplus V_j}_{w \text{ times}},$$

where $t := \frac{p-1}{s(p)}$, $V_0 \cong \mathbb{F}_2$, $V_i$ is an irreducible module of dimension $s(p)$ for every $i \in \{1, \ldots, t\}$ and $V_j$ is a non-split extension of $V_j$ by $V_j$ for every $j \in \{0, \ldots, t\}$.

Then we get the following result for self-dual codes.

**Proposition 3.3 (BW).** Let $\mathcal{C}$ be a self-dual binary linear code of length $n$ and suppose $\sigma_{2p} \in \text{Aut}(\mathcal{C})$ of type $2p$-$(w, 0, x; 0)$. Then the code $\mathcal{C}$ has the following structure as an $\mathbb{F}_2\langle \sigma_{2p}\rangle$-module:

$$\mathcal{C} = \underbrace{V_0 \oplus \ldots \oplus V_0}_{y_0 \text{ times}} \oplus \underbrace{V_0 \oplus \ldots \oplus V_0}_{z_0 \text{ times}} \oplus \ldots \oplus \underbrace{V_i \oplus \ldots \oplus V_i}_{y_i \text{ times}} \oplus \underbrace{V_i \oplus \ldots \oplus V_i}_{z_t \text{ times}},$$

where
a) \[ 2y_0 + z_0 = x + w, \]
b_1) \[ 2y_i + z_i = x \quad \text{for all } i \in \{1, \ldots, t\}, \text{ if } s(p) \text{ is even}, \]
b_2) \[ z_i = z_{2i} \text{ and } y_i + y_{2i} + z_i = x \quad \text{for all } i \in \{1, \ldots, t\}, \text{ if } s(p) \text{ is odd}. \]

In particular \( x \equiv z_1 \equiv \ldots \equiv z_t \mod 2, \) if \( s(p) \) is even.

This quite technical proposition has a strong consequence in a particular case.

**Corollary 3.4 ([BW]).** Let \( C \) be a self-dual binary linear code of length \( n \equiv 0 \mod 4 \). Suppose \( \sigma_{2p} \in \text{Aut}(C) \) of type \( 2p-(w,0,x;0) \) with \( s(p) \) even. If \( w \) is odd, then

\[ \dim C(\sigma_2) = \dim \pi_{\sigma_2}(C(\sigma_2)) \geq \frac{n}{4} + \frac{s(p)t}{2} = \frac{n}{4} + \frac{p - 1}{2}, \]

where \( \sigma_2 = \sigma_{2p}^2 \).

In particular \( \pi_{\sigma_2}(C(\sigma_2)) \) is not self-dual so that \( C \) is not a projective \( \mathbb{F}_2(\sigma_{2p}) \)-module.

Other consequences of Proposition 3.3 can be found in [BW].

### 4. Dihedral group of order \( 2p \) (\( p \) an odd prime)

In this section we consider the structure of a self-dual binary linear code \( C \) with a dihedral group as subgroup of \( \text{Aut}(C) \). We try to generalize here the main idea used in [FN] by G. Nebe and Thomas Feulner to approach the case \( D_{10} \) for the extremal self-dual binary linear code of length 72. The assumptions we make are somehow too strong, but they make the notations simpler and they are sufficient for our purposes.

Let us now suppose that

- \( p \) is an odd prime with \( s(p) = p - 1 \);
- \( C \) is a self-dual binary linear code of length \( n \) (\( n \) divisible by \( 2p \));
- \( \sigma_{p} \in \text{Aut}(C) \) of order \( p \) is fixed point free (so that the number of cycles is \( c = \frac{n}{p} \));
- \( \sigma_2 \in \text{Aut}(C) \) of order \( 2 \) is fixed point free;
- \( \langle \sigma_p \rangle \rtimes \langle \sigma_2 \rangle \cong D_{2p} \) is a dihedral group of order \( 2p \).

As we have seen in Section 2, \( C = C(\sigma_p) \oplus E(\sigma_p) \). The action of the involution \( \sigma_2 \) and the results of Theorem 2.2 give strong restrictions on the structure to \( C \), as we will prove.

Without lost of generality we can set

\[ \sigma_p := (1, \ldots, p)(p + 1, \ldots, 2p) \ldots (n - p + 1, \ldots, n) \]

and

\[ \sigma_2 := (1, p + 1)(2, 2p) \ldots (p, p + 2) \ldots (n - p, n - p + 2). \]

#### 4.1. Preliminaries

We need to understand better the structure of the field \( \mathbb{F}_{2^p-1} \) in its realization as an ideal \( \mathcal{I} \) of \( \mathbb{F}_2[x]/(x^p + 1) \), presented in Section 2.

**Remark 4.1.** In the following we indicate with \( a \mod b \) the remainder of the division of \( a \) by \( b \).

Furthermore, we indentify the cosets of \( \mathbb{F}_2[x]/(x^p + 1) \) with their representatives.
Remember that the ideal $\mathcal{I}$ is generated by $(1 + x)$. It is straightforward to observe that $(x + x^2 + \ldots + x^{p-1}) \in \mathcal{I}$ is the identity of the field.

Since $s(p) = p - 1$ we have that

$$(1 + x), (1 + x)^2, (1 + x)^4, \ldots, (1 + x)^{2^{p-2}}$$

is an $\mathbb{F}_2$-basis of $\mathbb{F}_{2p-1}$. Furthermore

$$a_0(1 + x) + a_1(1 + x)^2 + \ldots + a_{p-2}(1 + x)^{2^{p-2}} =$$

$$= (a_0 + \ldots + a_{p-2}) + a_0x + a_1x^2 + \ldots + a_{p-2}x^{2^{p-2}}.$$

Let $\psi : i \mapsto i + \frac{c}{p} \mod p - 1$ and $\Phi_{\frac{c}{p}}$ the Frobenius automorphism of $\mathbb{F}_{2p-1}$.

$$\Phi_{\frac{c}{p}}((a_0 + \ldots + a_{p-1}) + a_0x + a_1x^2 + \ldots + a_{p-2}x^{2^{p-2}}) =$$

$$= (a_0 + \ldots + a_{p-1}) + a_{\psi^{-1}(0)}x + a_{\psi^{-1}(1)}x^2 + \ldots + a_{\psi^{-1}(p-2)}x^{2^{p-2}}.$$

If we identify every polynomial with the ordered vector of $\mathbb{F}_p$ of its coefficients, the Frobenius automorphism corresponds to a permutation of $S_p$.

Since $[\frac{2^{p-2}}{2}]_p = [-1]_p$, the permutation

$$\prod_{i=1}^{\frac{c}{p}} (2^i \mod p, 2^{\psi(i)} \mod p)$$

is equal to

$$(1, p-1)(2, p-2)(3, p-3) \ldots \left(\frac{p-1}{2}, \frac{p+1}{2}\right)$$

so that the Frobenius automorphism corresponds to the following permutation on the coefficients of polynomials

$$(2, p)(3, p-1)(4, p-2) \ldots \left(\frac{p+1}{2}, \frac{p+3}{2}\right)$$

that inverts the order of the last $p - 1$ coordinates of the cycle of length $p$.

Let us consider now the direct product of two copies of $\mathbb{F}_{2p-1}$, so that the coefficients live in $\mathbb{F}_{2p}^2$. The permutation

$$(1, p+1)(2, 2p)(3, 2p-1)(4, 2p-2) \ldots (p, p+2) \in S_{2p}$$

corresponds to $(\alpha, \beta) \mapsto \left(\Phi_{\frac{c}{2p}}(\beta), \Phi_{\frac{c}{2p}}(\alpha)\right)$ over $\mathbb{F}_{2p-1}^2$.

Let us set $\pi := \Phi_{\frac{c}{2p}}(\alpha) = \alpha^{\frac{2^{p-1}}{2}}$.

It follows easily that the permutation

$$\sigma_2 = (1, p+1)(2, 2p) \ldots (p, p+2) \ldots (n - p, n - p + 2)$$

acts as follows

$$(\alpha_1, \alpha_2, \ldots, \alpha_{c-1}, \alpha_c) \mapsto (\overline{\alpha_2}, \overline{\alpha_1}, \ldots, \overline{\alpha_c}, \overline{\alpha_{c-1}})$$

on $\mathbb{F}_{2p-1}^c$ ($c$ even).
4.2. Main theorem. We can now state the main result. The notations are those fixed in the introduction of this section.

**Theorem 4.2.** Let \( C \) be a self-dual code of length \( n \) such that \( \langle \sigma_p \rangle \times \langle \sigma_2 \rangle \) is a subgroup of \( \text{Aut}(C) \). If \( \pi_{\sigma_p}(C(\sigma_2)) \) is self-dual, then there exist

- \( A \subseteq \mathbb{F}_2^{e_1} \), which is a self-dual binary linear code,
- \( B \subseteq \mathbb{F}_{2^{p-1}}^f \), which is a \( \mathbb{F}_{2^{p-1}} \)-linear trace-Hermitian self-dual code,

such that

\[
C = \pi^{-1}(A) \oplus \varphi^{-1}_p (\langle \pi^{-1}(B) \rangle_{\mathbb{F}_{2^{p-1}}})
\]

where \( \pi_{\sigma_p} \) is the natural projection associated to \( \sigma_p \), \( \varphi_p \) is the map defined in Section 2, and

\[
\pi := \mathbb{F}_{2^{p-1}}^c \to \mathbb{F}_{2^{p-1}}^c
\]

maps \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{c-1}, \varepsilon_c) \mapsto (\varepsilon_1, \ldots, \varepsilon_{c-1}) \).

**Proof.** As we have proved in Section 2

\[
C = C(\sigma_p) \oplus \mathcal{E}(\sigma_p).
\]

Put \( A := \pi_{\sigma_p}(C(\sigma_p)) \leq \mathbb{F}_2^{e_1+f} \). This is self-dual by Theorem 2.2.

Let us consider \( \varphi_p(\mathcal{E}(\sigma_p)) \leq \mathbb{F}_{2^{p-1}}^c \). This is an Hermitian self-dual code, again by Theorem 2.2.

As we have just shown the action of \( \sigma_2 \) on \( \varphi_p(\mathcal{E}(\sigma_p)) \) is the following

\[
(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{c-1}, \varepsilon_c)^{\sigma_2} = (\varepsilon_2, \varepsilon_1, \varepsilon_3, \varepsilon_4, \ldots, \varepsilon_{c-1})
\]

Note that this action is only \( \mathbb{F}_{2^{p-1}} \)-linear. Furthermore, the fixed code of \( \sigma_2 \) is

\[
\varphi_p(\mathcal{E}(\sigma_p))(\sigma_2) := \{ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{c-1}, \varepsilon_c) \in \varphi_p(\mathcal{E}(\sigma_p)) \}.
\]

Put \( B := \pi(\varphi_p(\mathcal{E}(\sigma_p))(\sigma_2)) \).

For \( \gamma, \epsilon \in B \) the Hermitian inner product of their preimages in \( \varphi_p(\mathcal{E}(\sigma_p))(\sigma_2) \) is

\[
\sum_{i=1}^{\frac{f}{2}} (\varepsilon_i \overline{\gamma_i} + \overline{\varepsilon_i} \gamma_i)
\]

which is 0 since \( \varphi_p(\mathcal{E}(\sigma_p)) \) is Hermitian self-dual. Therefore \( B \) is trace-Hermitian self-orthogonal. We have

\[
\dim_{\mathbb{F}_2}(B) = \dim_{\mathbb{F}_2}(\varphi_p(\mathcal{E}(\sigma_p))(\sigma_2)) = \frac{1}{2} \dim_{\mathbb{F}_2}(\varphi_p(\mathcal{E}(\sigma_p)))
\]

since \( \varphi_p(\mathcal{E}(\sigma_p)) \) is a projective \( \mathbb{F}_2(\sigma_2) \)-module (since \( \pi_{\sigma_2}(C(\sigma_2)) \) is self-dual), and so \( B \) is self-dual.

Since \( \dim_{\mathbb{F}_2}(B) = \dim_{\mathbb{F}_{2^{p-1}}}(\varphi_p(\mathcal{E}(\sigma_p))) \), the \( \mathbb{F}_{2^{p-1}} \)-linear code \( \varphi_p(\mathcal{E}(\sigma_p)) \leq \mathbb{F}_{2^{p-1}} \) is obtained from \( B \) as stated. \( \square \)

5. Interaction between fixed codes

In this section we investigate the interaction between fixed codes of different automorphisms. In particular, we want to give an idea of what can be said in the case that the automorphism group of a binary linear code (not necessarily self-dual) contains a subgroup \( H \) that is a semidirect product (abelian or not) of two subgroups, say \( H = A \rtimes B \).
5.1. Non-abelian semidirect products of two subgroups. Let us start from the non-abelian case.

Actually, in this case we have an action of $H$ on the normal subgroup $A$ and in particular on the fixed codes of the automorphisms belonging to $A$. We restrict our attention to a particular case. However, this case gives some flavor of what can be done in general.

**Notation.** For $\tau, \sigma \in S_n$ we denote by $\tau^\sigma$ the conjugate of $\tau$ by $\sigma$.

Let us start with a basic and trivial lemma.

**Lemma 5.1.** Let $C$ be a linear code of length $n$ and take $\tau \in \text{Aut}(C)$. If $\sigma$ is a permutation of $S_n$ then $\tau^\sigma \in \text{Aut}(C^\sigma)$ and $C(\tau)\sigma = C(\tau^\sigma)$.

**Proof.** The first assertion is clear. Then, for $c \in C$ we have $c \in C(\tau)^\sigma \iff c^\sigma^{-1} \in C(\tau) \iff c^\sigma^{-1} \tau = c^\sigma^{-1} \iff c^\sigma = c \iff c \in C(\tau^\sigma)$, which proves the second assertion. \hfill $\square$

This easy observation suggests a construction for codes with semidirect automorphism subgroups.

**Theorem 5.2.** Let $C$ be a binary linear code. Suppose that $G = E_m \rtimes H$ is a subgroup of $\text{Aut}(C)$, where $E_m$ is an elementary abelian $p$-group and $H$ acts transitively on $E_m$. Then

$$\sum_{c \in C(\tau)} C(c) = \sum_{\kappa \in H} C(c_0)^\kappa$$

for any $c_0 \in E_m^\times$.

**Proof.** It follows directly from Lemma 5.1. \hfill $\square$

Then we have the following.

**Corollary 5.3.** Let $p$ be a Mersenne prime, that is $p = 2^r - 1$ for a certain $r \in \mathbb{N}$. Let $E_2^r$ be an elementary abelian group of order $2^r$ and let $G = E_2^r \rtimes \langle \sigma_p \rangle$, where $\sigma_p$ is an automorphism of order $p$ ($G$ non abelian).

Suppose that $C$ is a binary linear code such that $G$ is a subgroup of $\text{Aut}(C)$. Then for any involution $\varepsilon_0 \in E_2^r$ it holds that

$$\sum_{c \in C(\varepsilon)} C(c) = \sum_{i=0}^{p-1} C(\varepsilon_0)^{\sigma_p^i}.$$

**Proof.** $|E_2^r| = 2^r - 1$. The cyclic group $\langle \sigma_p \rangle$ acts on it. The orbits for this action have order $p$ or order 1. Since $p = |E_2^r|$ there is only one orbit of order $p$: supposing the contrary we have $G$ abelian, a contradiction. So the action is transitive and the assertion follows from Theorem 5.2. \hfill $\square$

Obviously, similar results can be deduced for other groups. Notice that $A_4$ satisfies the hypothesis of Corollary 5.3 with $p = 3$.

Let us conclude this subsection, underlining a very useful tool to investigate further a code with such an automorphism group.
Let $D := \sum_{\epsilon \in E_m^2} C(\epsilon)$. The group $G$ acts on $Q := D^\perp / D$ with kernel containing $E_m$. The space $Q$ is hence a $F_2(\sigma_p)$-module. On this space we still have a decomposition in the part fixed by $\sigma_p$ and its complement and we can repeat arguments totally analogous to the ones in Section 4. This gives again a very restrictive structure.

5.2. Direct products of cyclic groups. Let us conclude with a few considerations on the interaction between fixed codes of different automorphisms in the abelian case. The results of this subsection can be generalized to any abelian finite group, but the notation would become too complex.

We consider in particular the group $C_p \times C_q$ with $p, q$ not necessarily distinct primes. This case gives an idea of what can be said in a general context.

Let us suppose that $C$ is a code (not necessarily self-dual) such that $C_p \times C_q$ is a subgroup of $Aut(C)$ with $C_p = \langle \sigma_p \rangle$, $C_q = \langle \sigma_q \rangle$, cyclic groups of prime (not necessarily distinct) order.

Let $\sigma_p$ be of type $p-(c, f)$. Then

$$\pi_{\sigma_p}(C(\sigma_p)) \leq F_2^{c+f}.$$ 

Every element of $C_{S_n}(\sigma_p)$ (the centralizer of $\sigma_p$ in $S_n$) acts on the orbits of $\sigma_p$. So we can define naturally a projection

$$\eta_{\sigma_p} : C_{S_n}(\sigma_p) \to S_{c+f}$$

that maps $\tau \in C_{S_n}(\sigma_p)$ on the permutation corresponding to the action of $\tau$ on the orbits of $\sigma_p$.

If $\sigma_q$ is of type $q-(c', f')$ we can define in a completely analogous way

$$\eta_{\sigma_q} : C_{S_n}(\sigma_q) \to S_{c'+f'}.$$ 

We collect in the following some observations.

**Remark 5.4.** Let $C$ be a code such that $C_p \times C_q \leq Aut(C)$ with $C_p = \langle \sigma_p \rangle$, $C_q = \langle \sigma_q \rangle$, cyclic groups of prime (not necessarily distinct) order. Then

a) $\eta_{\sigma_p}(\sigma_q) \in Aut(\pi_{\sigma_p}(C(\sigma_p)))$;

b) $\eta_{\sigma_q}(\sigma_p) \in Aut(\pi_{\sigma_q}(C(\sigma_q)))$;

c) $\eta_{\sigma_p}(\pi_{\sigma_p}(C(\sigma_p)))(\eta_{\sigma_q}(\sigma_q)) = \eta_{\sigma_p}(\sigma_p)(\pi_{\sigma_q}(C(\sigma_q))(\eta_{\sigma_q}(\sigma_q)))$;

d) if $p, q$ are distinct and $\sigma_p \sigma_q$ is of type $pq-(a, b; c; f)$ then $\eta_{\sigma_p}(\sigma_q)$ is of type $q-(c + b, a + f)$ and $\eta_{\sigma_q}(\sigma_p)$ is of type $p-(c + a, b + f)$.

Notice that a) and b) are strong conditions on the fixed codes.

6. The automorphism group of an extremal self-dual code of length 72

The existence of an extremal self-dual code of length 72 is a long-standing open problem of classical Coding Theory [5].

We give here a brief overview of the investigation of its possible automorphism groups. We do not follow a chronological order, nor we mention all the papers related to the topic. Our aim is to outline all the steps necessary to prove the final theorem and to underline where the methods presented in the previous sections can be applied.

For all this section let $C$ be an extremal self-dual [72, 36, 16] code.
6.1. Cycle-structure of the automorphisms. In order to get information on the whole group $\text{Aut}(C)$, we begin to investigate the cycle-structure of the possible automorphisms.

John H. Conway and Vera Pless, in a paper submitted in 1979 [CP], were the first who faced this problem. In particular they focused on the possible automorphisms of odd prime order. They proved that

- only 9 types of automorphism of odd prime order may occur in $\text{Aut}(C)$, namely 23-(3,3), 17-(4,4), 11-(6,6), 7-(10,2), 5-(14,2), 3-(18,18), 3-(20,12), 3-(22,6) and 3-(24,0).

They used arguments based on combinatorial properties of the codes.

Between 1981 and 1987, V. Pless, John G. Thompson, W. Cary Huffman and V.I. Yorgov [P, PT, HY] proved that

- automorphisms of orders 23, 17 and 11 cannot occur in $\text{Aut}(C)$.

Between 2002 and 2004, S. Bouyuklieva [Bou3, Bou2] proved that

- the eventual elements of order 2 and 3 in $\text{Aut}(C)$ are fixed point free.

More recently T. Feulner and G. Nebe [FN] showed that also

- automorphisms of orders 7 cannot occur in $\text{Aut}(C)$.

The techniques used are different case by case, but the main tool is the decomposition of codes with an automorphism of odd prime order discussed in Section 2. Let us summarize these results in the following.

**Proposition 6.1.** Let $\sigma$ be an automorphism of prime order of a self-dual $[72, 36, 16]$ code. Then $\sigma$ can be only of the following types: 2-(36,0), 3-(24,0) and 5-(14,2).

An immediate consequence of Proposition 6.1 is that $\text{Aut}(C)$ does not contain elements of order 15, 16, 25 and 27. Furthermore, the possible composite orders are 4, 6, 8, 9, 10, 12, 18, 36 and 72.

G. Nebe, Nikolay Yankov and the author [N, Ya, Bor1], excluded orders 10, 9 and 6, respectively. Order 10 can be excluded just looking at the automorphism groups of self-dual [36, 18, 8] codes, classified in [MG], which are the projection of possible fixed codes of involutions, and using Remark 5.4. The methods used for order 9 are a refinement of those in Section 2. For order 6 we used strongly the results contained in Section 3.

Finally, very recently V.I. Yorgov and Daniel Yorgov proved that automorphism of order 4 are not possible [YY]. So we have the following.

**Proposition 6.2.** Let $\sigma$ be a non-trivial automorphism of a self-dual $[72, 36, 16]$ code. Then its order is a prime among $\{2, 3, 5\}$.

6.2. Structure of the whole group. Once we have information on the cycle-structure of the automorphisms, we can investigate the structure of the whole group. By Proposition 6.1 we have immediately that

$$|\text{Aut}(C)| = 2^a 3^b 5^c$$

where $a$, $b$, $c$ are nonnegative integers.

S. Bouyuklieva was the first, in 2004, who studied the order of $\text{Aut}(C)$. She proved [Bou2] that

- 25 does not divide $|\text{Aut}(C)|$. 
This means that $|\text{Aut}(C)| = 2^a 3^b 5^c$ with $a, b$ nonnegative integers and $c = 0, 1$. If $c = 1$ then
- if $\sigma \in \text{Aut}(C)$ has order 5, $|N_{\text{Aut}(C)}(\sigma)| = 2^d 5$, with $d = 0, 1$ [Yo2].
- $|\{\text{aut. of order 5 in } \text{Aut}(C)\}| = 4 \cdot \frac{|\text{Aut}(C)|}{2^5}$.

So, by Burnside Lemma,
\[
\frac{1}{|\text{Aut}(C)|} \left( 72 + \gamma \cdot 2 \cdot 4 \cdot \frac{|\text{Aut}(C)|}{2^5} \right) = \frac{72}{2^a 3^b 5^\gamma} + \gamma \cdot \frac{8}{2^5} \in \mathbb{N}
\]

By Proposition [6.2] we have that
- $\text{Aut}(C)$ is trivial or isomorphic to one of the following: $C_2, C_3, C_2 \times C_2$, $C_5, S_3, C_2 \times C_2 \times C_2, C_3 \times C_3, D_{10}, A_4, (C_3 \times C_3) \rtimes C_2$ (the generalized dihedral group of order 18) or $A_5$,

since all other groups of order in $(\star)$ have elements of composite order (for a library of Small Groups see for example [BEQ]).

T. Feulner and G. Nebe [FN] proved that
- $\text{Aut}(C)$ does not contain a subgroup isomorphic to $C_3 \times C_3$ or $D_{10}$.

The author, in a joint paper [BDN] with F. Dalla Volta and G. Nebe, proved that
- $\text{Aut}(C)$ does not contain a subgroup isomorphic to $S_3$ or $A_4$.

Finally, the author proved [Bor2] that
- $\text{Aut}(C)$ does not contain a subgroup isomorphic to $C_2 \times C_2 \times C_2$.

The methods used for $C_3 \times C_3$ are a refinement of those presented in Section 2. The cases of $D_{10}$ and $S_3$ involve the methods of Section 3. For $A_4$ and $C_2 \times C_2 \times C_2$ we applied the methods of Section 4 with some more particular observations.

Let us summarize all these results in a theorem.

**Theorem 6.3.** Let $C$ be self-dual [72, 36, 16] code. Then $\text{Aut}(C)$ is trivial or isomorphic to $C_2, C_3, C_2 \times C_2$ or $C_5$.

**Remark 6.4.** The possible automorphism groups of a putative extremal self-dual code of length 72 are abelian and very small. So this code is almost a rigid object (i.e. without symmetries) and it might be very difficult to find it, if it exists.

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