Designing Binary Sequence Set with Optimized Correlation Properties via ADMM Approach

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Abstract—In this paper, we design low correlation binary sequences favorable in wireless communication and radar applications. First, we formulate the designing problem as a nonconvex combination optimization problem with flexible correlation interval; second, by relaxing constraints and introducing auxiliary variables, the original minimization problem is equivalent to a consensus continuous optimization problem; third, to achieve its good approximate solution efficiently, we propose the distributed executable algorithms based on alternating direction method of multipliers (ADMM); fourth, we prove that the proposed ADMM algorithms can converge to some stationary point of the approximate problem. Moreover, the computational complexity analysis is considered. Simulation results demonstrate that the proposed ADMM approaches outperform state-of-the-art ones in either computational cost or selection of correlation interval of the designed binary sequences.

Index Terms—Binary sequence, auto/cross-correlation, Box ADMM, convergence/complexity analysis.

I. INTRODUCTION

CONSTANT modulus sequences with good correlation play a fundamental role in modern communication systems [1]–[3]. These sequences are widely used in a variety of applications, such as power control [4], channel estimation [5], synchronization [7], signal detection and separation [8], and mitigation of interference [9], etc. Among these sequences, binary sequences, such as m-sequences [10], Kasami code [11], Gold code [12], Barker code [13], Bent code [14], etc., have been widely studied due to their simplicity of implementation and high-energy efficiency.

Generally speaking, perfect correlation sequences are desired in communication systems when their autocorrelation values are zero for all shifts different from zero and cross-correlation values are zero for every shift. It is known that these perfect unitary sequences do not exist and this is the main challenge to construct zero-correlation sequences limited by bounds [3]. For Baker codes, Storer and Turyn proved that there are no sequences for odd $N \geq 13$ [15] and except $N = 4$, no other perfect period binary sequences exist with $N < 54,896,4900$ [16]. Therefore, zero correlation zone (ZCZ) sequence, whose correlation values within a zone are zero, are proposed in [17] and so do low correlation zone (LCZ) sequences in [18]. In fact, the construction of low correlation binary sequences is a well-known computational problem.

At the early stage, exhaustive search method is the main tool in the construction of short sequence with ideal correlation properties. The authors in [19] used the trace function to construct cascaded Gordon-Mills-Welch (GMW) sequence with low autocorrelation and three valued cross-correlation. The authors in [20] presented five new classes of binary sequences of period $2^N - 1$ with ideal autocorrelation by extensive computer search. Based on the framework of exhaustive search algorithm, authors in [21] customized the fast method to construct binary sequences with low autocorrelations. In [22], the author adopted a hybrid approach combining simulated annealing approach with a traditional iterative code selection algorithm to design orthogonal polyphase sequence sets. The authors in [23] proposed an iterated variable depth search algorithm to search binary sequences with integrated sidelobe level (ISL) and low peak sidelobe level (PSL). Owing to the exponential size $O(2^N)$ of the configuration space, above exhaustive search methods are limited to design short sequences.

Pseudo-Noise (PN) sequences generated from Feedback Shift Registers (FSR) can be designed as long sequences. A kind of these sequences, m-sequences, are easily generated using linear FSR with length $2^N - 1$. Many sequences, such as Gold code, Kasami code, GMW sequence are derived from m-sequences. But the disadvantage of these sequences is that they are relatively small in number [24]. Interleaved technique is another method used to analyse and design sequences with good correlation [25]. Its key idea is construct long sequences from short ones [26], [27]. More literature on this area can be found in [28]–[30] and the references therein. This kind of computational design method is simple to implement, but it can only be designed for sequences with length of $2^N - 1$ or $2^N$, which leads to its lack of flexibility.

The limitations motivated the researchers to perform analytical construction method to design sequences. In recent years, a large number of literatures related to constant modulus sequences design have emerged (see [31]–[35] and references therein). As a special case of that, the study of discrete sequence also attracts many researchers. The authors in [36] bridged the gap between the exhaustive search method and the analytical constructions method, and proposed a construction method which can be accomplished in polynomial time. The authors in [37] formulated the sequence sets design as a quadratically constrained quadratic program problem and proposed an algorithm based on semidefinite program (SDP) relaxation and randomized projection technique to tackle it. An efficient coordinate-descent framework method was proposed to design sequences with good ISL/PSL for MIMO radars and communication systems in [38], [39]. Due to element-wise optimization, this method has the drawback of low efficiency. In [40], [41], the authors formulated the sequence design with similarity constraint as quadratic optimization problem, the Serial Iterative Algorithm (SIA) [40] and Alternating Direction Penalty Method (ADPM) algorithm [41] are proposed successively. The authors in [42] adopt the effective...
II. PROBLEM FORMULATION

A. System Model

Let consider a binary sequence set \( X = [x_1, x_2, \cdots, x_M] \), and each element in sequences \( \{x_i\}_{i=1}^M \) is \(-1\) or \(1\). The aperiodic and periodic correlation functions of sequence \( x_i \) and \( x_j \) at shift \( n \) are defined as

\[
\rho_{ij,n}^A = \sum_{k=n+1}^{N} x_{i,k} x_{j,k-n}, \quad \rho_{ij,n}^P = \sum_{k=n+1}^{N} x_{i,k} x_{j,k-n \mod(N)},
\]

where \( i, j = 1, \cdots, M; n = -N + 1, \cdots, N - 1 \). When \( i = j \), \( \rho_{ii,n}^A \) and \( \rho_{ii,n}^P \) represent the aperiodic/periodic auto-correlation function of \( \{x_i\}_{i=1}^M \). Otherwise, they are cross-correlation functions.

There are some metrics that are used to evaluate the goodness of the correlation properties of binary sequences. The most commonly used ones are the ISL [43], PSL [44], and Peak to Average Power Ratio (PAPR) [45], etc. ISL gives the relationship among the sequence and its shift version. It is often used to measure the synchronization performance between the received signal and all the interference signals caused by multipath in the wireless communication system. PSL is a metric derived from the autocorrelation function which describes the relationship among the maximum of the side lobes (SL). It means the worst case of interference caused by SL to main lobe. Therefore binary sequence sets with small PSL value are desirable.

In this paper, we consider the optimization metric of PSL. The definitions are given in [4].

\[
PSL = \sum_{i=1}^{M} \max_{l \in L \setminus \{0\}} \left\{ \rho_{ii,l}^A \right\} + \sum_{i=1}^{M} \sum_{j=1}^{M} \max_{l \in L} \left\{ \rho_{ij,l}^A \right\},
\]

where \( \rho_{ij,l}^A = \rho_{ij,l}^A \) and \( \rho_{ij,l}^P = \rho_{ij,l}^P \) address the aperiodic/periodic function respectively, \( L \) denotes the shift set interval of interest. To facilitate the subsequent expression, we denote it as \( f(X) \). By utilizing off-line matrices \( S_i^A \) and cyclic shift matrices \( S_i^P \) defined in [3], [4] and denote \( x_i = Xs_i \), where \( i \)-th element in \( s_i \) is \( 1 \) and the rest elements...
are zeros.

\[
S^A_P = \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix},
\]

(3)

\[
S^P_l = \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix},
\]

(4)

\[f(X) = \sum_{i=1}^{M} \max_{l \in \mathcal{L} \setminus \emptyset} \left\{ (Xs_l)^T S_l Xs_l \right\} + \sum_{i=1}^{M} \max_{j=1, j \neq i} \left\{ (Xs_i)^T S_i Xs_j \right\}
\]

\[= \sum_{l=1}^{M} \sum_{j=1}^{M} \max_{i \in \mathcal{L}} \left\{ f_{i,j,l}(X) \right\},
\]

where

\[f_{i,j,l}(X) = \left\{ (Xs_i)^T S_l Xs_j - N \delta_{i-l,j+l} \right\}.
\]

In (6), \(\delta_{i-l,j+l}\) in (5) denotes the Dirac-\(\delta\) function and \(S_l = S^A_P\) and \(S_l = S^P_l\) are aperiodic and periodic cases respectively. To facilitate the subsequent expression, we define the notation set \(\mathcal{L} = \{ij, l, i, j = 1, \cdots, M, l \in \mathcal{L}\}\). Then, the problem for designing binary sequence set with good PSL can be formulated as

\[
\min_{X \in \mathcal{R}^{N \times M}} \sum_{i,j=1}^{M} \max_{l \in \mathcal{L}} \left\{ f_{o}(X) \right\},
\]

subject to \(X \in \mathcal{X}\),

(7)

where

\[\mathcal{X} = \{X | x_{i,n} \in \{-1, 1\}, i = 1, \cdots, M, n = 1, \cdots, N\}.\]

Since the constraint \(X \in \mathcal{X}\) is discrete, problem (7) is a combination optimization problem with respect to variable \(\pm 1\). That means the computational complexity of obtaining global optimal solution to problem (7) grows exponentially with the size of set \(\mathcal{X}\). Such a high complexity in practical application is unbearable. The usual way to solve this problem is to relax the binary constraint to continuous box constraints, i.e., \(X \in \mathcal{X}_B\), where

\[\mathcal{X}_B = \{X | x_i \in [-1, 1], i = 1, \cdots, M\}.
\]

The relaxed operation of binary constraint may lead the elements in \(X\) to be non-binary solution during the iteration process. To encourage binary solutions, some penalty method, e.g., adding the penalty term in the objective function can be introduced. However, the non-convexity of the added penalty term may lead to further issues, namely undesirable local minima and sensitivity to the initialization. To tackle it, we relax the binary constraint to \(\ell_p\)-Box intersection.

 Proposition: \(\ell_p\)-Box Intersection [49]: The binary set \(\{-1, 1\}^{N \times M}\) can be equivalently replaced by the intersection between a sphere \(X_S\) and a box \(X_B\), as follows:

\[X \in \{-1, 1\}^{N \times M} \Leftrightarrow X \in X_S \cap X_B.
\]

(8)

where

\[X_S = \{X : \|X\|_p = MN\}.
\]

Note that \(X_S\) can be seen as a \((MN - 1)\)-dimensional \(\ell_p\)-sphere centered at origin of axes with radius \((MN)^{\frac{1}{p}}\) and \(p \in (0, \infty)\). To illustrate this proposition, we present a 2-dimensional example with different \(p\) parameters in Fig. 1.

It is obvious from the figure that binary set \([-1, 1]^2\) is the intersection between the sphere \(X_S\) and the box \(X_B\).

By relaxing the binary constraint to \(\ell_p\)-Box and introducing auxiliary variables \(\{Z_o \in \mathcal{R}^{N \times M}\}_{o \in \mathcal{O}}\), problem (7) can be transformed into the following global consensus problem

\[
\min_{(X_1, X_2, Z_o) \in \mathcal{R}^{N \times M}} \sum_{i,j=1}^{M} \max_{l \in \mathcal{L}} \left\{ f_{o}(Z_o) \right\},
\]

subject to \(X_1 \in X_B, X_2 \in X_S, X_1 = Z_o, X_2 = Z_o, o \in \mathcal{O}\).

The introduction of auxiliary variables makes each subproblem have its local variables. Thus, each subproblem can be solved in parallel. Define the following matrices

\[\tilde{X} = [X_1; X_2], A = \left[I_N; I_N\right].\]

(11)

The linear constraints can be rewritten as \(AZ_o = \tilde{X}\).

III. Solving Algorithm

In this section, one algorithm named \(\ell_p\)-Box ADMM algorithm is developed to solve problem (10). In comparison with the state-of-the-art methods, one major benefit of proposed algorithm is the parallel execution structure. Another benefit is that the feasible region of the optimization variable is relaxed to the intersection of two continuous regions. The former one benefit can greatly improve the algorithm execution’s
efficiency. The latter one can promote the sequence with better correlation performance.

A. \( l_p \)-Box ADMM Algorithm

In (11), \( \mathbf{A}^T = [\mathbf{I}_N, \mathbf{I}_N] \in \mathbb{R}^{N \times 2N} \) is full row rank, i.e., \( \text{rank}(\mathbf{A}^T) = N < 2N \). Main difficulty during the convergence analysis is the constraint \( \mathbf{A}\mathbf{Z}_o = \mathbf{X} \). To tackle it, define \( \mathbf{A} = [\mathbf{A}, \sigma \mathbf{I}_N] \in \mathbb{R}^{2N \times 3N} \), \( \mathbf{Z}_o = [\mathbf{Z}_o, \hat{\mathbf{Z}}_o] \), where \( [\mathbf{Z}_{o1}, \mathbf{Z}_{o2}] = \hat{\mathbf{Z}}_o \) are introduced perturbed variable to construct constraint \( \mathbf{A}\mathbf{Z}_o = \mathbf{X} \), i.e.,

\[
\mathbf{A}\mathbf{Z}_o + \sigma\hat{\mathbf{Z}}_o = \mathbf{X},
\]

(12)

where \( \sigma > 0 \) is sufficiently small. Then, the associated perturbed problem is presented as

\[
\min_{\mathbf{X}, \mathbf{Z}_o, \mathbf{A}\mathbf{Z}_o, \mathbf{X}_B, \mathbf{X}_S} \sum_{i,j=1}^{M} \left( \max_{l \in L} \{ f_o(\mathbf{Z}_o) \} + \frac{\sigma^2}{2} \mathbf{Z}_o^T \mathbf{Z}_o \right),
\]

subject to \( \mathbf{X}_1 \in \mathcal{X}_B, \mathbf{X}_2 \in \mathcal{X}_S, \mathbf{Z}_o = \mathbf{X}, \mathbf{X}_o \in \mathcal{O} \).

ADMM, a popular technique, is suitable for solving problem (10) with multiple separable subproblems. The corresponding augmented Lagrangian function can be expressed as

\[
\mathcal{L}(\mathbf{X}, \mathbf{Z}_o, \mathbf{A}\mathbf{Z}_o, \omega, \mathcal{O}) \equiv \sum_{i,j=1}^{M} \left( \max_{l \in L} \{ f_o(\mathbf{Z}_o) \} + \frac{\sigma^2}{2} \mathbf{Z}_o^T \mathbf{Z}_o \right) + \sum_{\omega \in \mathcal{O}} \left( \langle \mathbf{A}_o, \mathbf{X} - \mathbf{A}\mathbf{Z}_o \rangle + \rho_o \left\| \mathbf{X} - \mathbf{A}\mathbf{Z}_o \right\|_F^2 \right).
\]

(14)

where \( \{ \mathbf{A}_o \in \mathbb{R}^{N \times M}, \rho_o > 0 \}_{\omega \in \mathcal{O}} \) are Lagrangian multipliers and penalty parameters respectively. Thus, the \( l_p \)-Box ADMM framework to solve problem (10) can be described as

\[
\mathbf{X}_o^{k+1} = \arg \min_{\mathbf{X}_1 \in \mathcal{X}_B, \mathbf{X}_2 \in \mathcal{X}_S} \mathcal{L}(\mathbf{X}_o, \mathbf{X}_o^k, \mathbf{A}\mathbf{Z}_o, \omega, \mathcal{O}),
\]

(15a)

\[
\hat{\mathbf{Z}}_o^{k+1} = \arg \min_{\hat{\mathbf{Z}}_o \in \mathcal{O}} \mathcal{L}(\mathbf{X}_o, \mathbf{X}_o^k, \mathbf{A}\mathbf{Z}_o, \omega, \mathcal{O}),
\]

(15b)

\[
\mathbf{A}_o^{k+1} = \mathbf{A}_o^k + \rho_o(\mathbf{X}_o^{k+1} - \mathbf{A}\mathbf{Z}_o^{k+1}), \omega \in \mathcal{O},
\]

(15c)

where \( k \) denotes the iteration number. Since function \( \mathcal{L}(\cdot) \) is nonconvex related to variable \( \mathbf{Z}_o \). The challenge of implementing Penalty Box ADMM is how to solve (15b).

1) Solving Subproblem (15a): Problem (15a) can be equivalent to the following problem

\[
\min_{\mathbf{X}_1 \in \mathcal{X}_B, \mathbf{X}_2 \in \mathcal{X}_S} \sum_{\omega \in \mathcal{O}} \left( \langle \mathbf{A}_o, \mathbf{X} - \mathbf{A\mathbf{Z}_o} \rangle + \frac{\rho_o}{2} \left\| \mathbf{X} - \mathbf{A\mathbf{Z}_o} \right\|_F^2 \right),
\]

subject to \( \mathbf{X}_1 \in \mathcal{X}_B, \mathbf{X}_2 \in \mathcal{X}_S \).

Then, we obtain its solution

\[
\hat{\mathbf{X}}_o^{k+1} = \left[ \begin{array}{c} \hat{\mathbf{X}}_o^{k+1} \\ \hat{\mathbf{X}}_o^{k+1} \end{array} \right] = \sum_{\omega \in \mathcal{O}} \left( \rho_o \hat{\mathbf{Z}}_o^k - \mathbf{A}_o \right) \sum_{\omega \in \mathcal{O}} \rho_o.
\]

(18)

Projecting \( \hat{\mathbf{X}}_o^{k+1} \) and \( \hat{\mathbf{X}}_o^{k+1} \) onto \( \mathcal{X}_B \) and \( \mathcal{X}_S \) respectively, we can obtain

\[
\mathbf{X}_o^{k+1} = \prod_{\mathbf{x} \in \mathcal{X}_B} \left( \hat{\mathbf{X}}_o^{k+1} \right), \mathbf{X}_o^{k+1} = \prod_{\mathbf{x} \in \mathcal{X}_S} \left( \hat{\mathbf{X}}_o^{k+1} \right),
\]

(19)

where \( \prod_{\mathbf{x} \in \mathcal{X}_B} \) project every entry of the input variable onto \([-1, 1]\) and

\[
\prod_{\mathbf{x} \in \mathcal{X}_S} \left( \mathbf{X} = \frac{\mathbf{X}}{\left\| \mathbf{X} \right\|_F} \right)^{(M+1)/p}.
\]

(20)

Problem (15a)’s solution is \( \mathbf{X}_o^{k+1} = [\mathbf{X}_1^{k+1}; \mathbf{X}_2^{k+1}] \).

2) Solving Subproblem (15b): Since (15b) is an unconstrained problem, the major challenge to solve is how to handle \( \max_{l \in L} \{ f_o(\mathbf{Z}_o) \} \). Define the following vector

\[
f_{ij, l}(\{ \mathbf{Z}_o \}) = \left[ \cdots, f_{ij, l}(\mathbf{Z}_{ij, l}), f_{ij, l+1}(\mathbf{Z}_{ij, l+1}), \cdots \right] \in \mathbb{R}^{[L]},
\]

Then, \( \max_{l \in L} \{ f_o(\mathbf{Z}_o) \} \) is equivalent to \( \| f_{ij, l}(\{ \mathbf{Z}_o \}) \|_q \). Given that from an analytical point of view, the \( \ell_{\infty} \)-norm is not a well-behaved function, \( \ell_{q} \)-norms will be used instead, i.e.,

\[
\max_{l \in L} \{ f_o(\mathbf{Z}_o) \} = \lim_{q \to \infty} \| f_{ij, l}(\{ \mathbf{Z}_o \}) \|_q = \left( \sum_{l \in L} f_{ij, l}(\{ \mathbf{Z}_o \}) \right)^{\frac{1}{q}},
\]

(21)

where \( q > 2 \) is a integer. Minimizing \( \left( \sum_{l \in L} f_{ij, l}(\{ \mathbf{Z}_o \}) \right)^{\frac{1}{q}} \) is equivalent to minimizing \( \sum_{l \in L} f_{ij, l}(\{ \mathbf{Z}_o \}) \). Assuming the current iteration index is \( k \), \( f_{ij, l}(\{ \mathbf{Z}_o \}) \) can be majorated at point \( \mathbf{Z}_o^k \)

\[
f_{ij, l}(\mathbf{Z}_o) \leq \frac{q}{2} f_{ij, l}(\mathbf{Z}_o^k) f_{ij, l}(\mathbf{Z}_o) + f_{cons}.
\]

(22)

where \( f_{cons} \) is the constant term. Plugging (21) and (22) into problem (15b) and dropping the constant term, we get the following approximate problem

\[
\min_{\mathbf{X}, \mathbf{Z}_o, \mathbf{A}_o} \mathcal{L}(\mathbf{X}, \{ \mathbf{Z}_o, \mathbf{A}_o \}, \omega, \mathcal{O}),
\]

(23)

where

\[
\mathcal{L}(\mathbf{X}, \{ \mathbf{Z}_o, \mathbf{A}_o \}, \omega, \mathcal{O}) \equiv \sum_{\omega \in \mathcal{O}} \left( w_o f_{ij, l}(\mathbf{Z}_o^k) + \frac{\rho_o}{2} \left\| \mathbf{X} - \mathbf{A}\mathbf{Z}_o \right\|_F^2 \right)
\]

(24)

where \( w_o \) is the normalized weight

\[
w_o = \left( \frac{f_{ij, l}(\mathbf{Z}_o^k)}{\max_{\omega \in \mathcal{O}} \{ f_{ij, l}(\mathbf{Z}_o) \}} \right)^{q-2}.
\]

(25)
For all $o \in \mathcal{O}$, subproblems in (23) are independent of each other and each subproblem is unconstrained w.r.t. variable $Z_o$. That means subproblems in (23) can be implemented in parallel. However, solving problem (23) is still difficult, since $f^2(Z_o)$ is nonconvex related to $Z_o$ (see (6)). To tackle it, we have the following lemma, which indicates that $\{f^2(Z_o)\}_{o \in \mathcal{O}}$ are continuous, differentiable and have Lipschitz continuous gradients in the finite domain $\mathcal{X}_o = \{x_i \mid x_i \in [-c, c], i = 1, \ldots, M\}$ of the point $\tilde{X} \in \mathcal{X}_B$ (see proof in Appendix A).

Lemma 1: gradients $\nabla f_o(X)$ are Lipschitz continuous, i.e.,
$$\|\nabla f_o^2(\tilde{X}) - \nabla f_o^2(\hat{X})\|_F \leq L_o \|\tilde{X} - \hat{X}\|_F, \ o \in \mathcal{O},$$
where $\tilde{X} \in \mathcal{X}_o, \hat{X} \in \mathcal{X}_B$ and constants $L_o \geq 2(N + 1)c^2$, $\hat{c} = \max(c, 1)$.

Based on Lemma 1 and Decent Lemma in [48], we have
$$\mathcal{L}_o(\bar{X}_o + k, Z_o, \Lambda^k_o) \leq w_o f_o^2(\bar{X}_o + k) + \langle w_o \nabla f_o(\bar{X}_o + k), Z_o - \bar{X}_o \rangle + \frac{w_o L_o}{2} \|\bar{X}_o + k - Z_o\|_F^2 + \langle \Lambda^k_o, \bar{X}_o + k - \bar{X}_o \rangle$$
$$+ \frac{P_o}{2} \|\bar{X}_o + k - \bar{X}_o\|_2^2 + \sigma^2 \bar{Z}_o^T \bar{Z}_o.$$ (28)

Define the right-hand side of the above inequality as $\mathcal{U}_o(Z_o)$. We customize the $\ell_p$-Box ADMM by minimizing it instead of (23). Since $\mathcal{U}_o(Z_o)$ is convex quadratic function w.r.t. $Z_o$, the optimal solution can be obtained by setting $\nabla Z_o \mathcal{U}_o(Z_o) = 0$. Through solving the equation, we get the solution
$$\bar{Z}_o^k = \Lambda^k_o + \rho_o (\bar{X}_o + k - \bar{X}_o Z_o),$$
$$Z_o^k = X_o^{k+1} + \frac{\Lambda^T_o (\bar{X}_o - \rho_o \sigma \bar{Z}_o^k - w_o \nabla f_o(\bar{X}_o + k))}{\rho_o + w_o L_o},$$ (29a, 29b)
where
$$X_o^{k+1} = \frac{\Lambda^T_o (\bar{X}_o + k - \sigma \bar{Z}_o^k)}{2}.$$ (30)

Combining (15c), (19) and (29), we summarize the customized penalty Box ADMM algorithm in Table II.

| TABLE II |
| --- |
| **THE CUSTOMIZED PENALTY BOX ADMM ALGORITHM** |
| **Initialization**: Compute Lipschitz constants $\{L_o, i \in \mathcal{L}\}$ according to (27). Set iteration index $k=1$, initialize $\hat{X}^1 \in \mathcal{A} \hat{Z}_o^1$, and $\{\lambda^k_o, o \in \mathcal{O}\}$ randomly, and let $\{\lambda^k_o = A \hat{Z}_o^1, o \in \mathcal{O}\}$. |
| **repeat** |
| S.1 Compute $\hat{X}^{k+1}$ via (18) and (19), i.e., |
| $\hat{X}^{k+1} = \prod_{X \in \mathcal{X}_B} \left[ \prod_{X \in \mathcal{X}_S} \left( \hat{X}^{k+1}_1 \right) \right]$. |
| S.2 Compute $\{Z_o^{k+1}, o \in \mathcal{O}\}$ via (29) in parallel, i.e., |
| $Z_o^{k+1} = X_o^{k+1} + \frac{\Lambda^T_o (\bar{X}_o - \rho_o \sigma \bar{Z}_o^{k+1}) - \bar{w}_o \nabla f_o (\bar{X}_o + k)}{\rho_o + \bar{w}_o L_o}$. |
| S.3 Compute $\{\lambda^k_o, o \in \mathcal{O}\}$ via (15c) in parallel, i.e., |
| $\lambda^k_o = \Lambda^k_o + \rho_o (\bar{X}_o + k - \bar{X}_o Z_o^{k+1})$. |
| until some preset termination criterion is satisfied. |
| Let $X_o^{k+1}$ be the output. |

In the following section.

IV. ALGORITHM ANALYSIS

In this section, we show several analyses on the proposed ADMM algorithm, such as convergence and computational complexity. The proposed algorithms are convergent to some stationary point of the approximate problem. To reduce the algorithm’s computational complexity, we exploit the inherent sparsity characteristic of the problem.

A. Convergence Issue

Before presenting the convergence conclusion, we first give a few corresponding lemmas.

**Lemma 2**: For the proposed algorithm, the augmented Lagrangian function has the following inequality
$$\mathcal{L}(X, \{\bar{Z}_o, \Lambda^k_o\}_{o \in \mathcal{O}}) - \mathcal{L}(X^{k+1}, \{\bar{Z}_o, \Lambda^k_o\}_{o \in \mathcal{O}})$$
$$\geq \sum_{o \in \mathcal{O}} \frac{\epsilon_o}{\rho_o} \|\bar{Z}_o^{k+1} - \bar{Z}_o^k\|_2^2 + \epsilon_o \|Z_o^{k+1} - Z_o^k\|_2^2.$$ (31)

In each $\ell_p$-Box ADMM iteration, if $\epsilon_o, \epsilon_o, \epsilon_o \geq 0$, $\mathcal{L}(X, \{\bar{Z}_o, \Lambda^k_o\}_{o \in \mathcal{O}})$ decreases sufficiently.

**Lemma 3**: If $\rho_o \geq 5L_o$, the augmented Lagrangian function is lower bounded, i.e.,
$$\mathcal{L}(X, \{\bar{Z}_o, \Lambda^k_o\}_{o \in \mathcal{O}}) \geq 0, \forall k.$$ (32)

The proof of Lemma 2 and 3 is given in Appendix B.

Lemma 2 and 3 show that augmented Lagrangian function $\mathcal{L}(\cdot)$ decreases sufficiently and has a lower bound, which...
indicates it is convergent. We have the following theorem to characterize the proposed \( \ell_p \)-Box ADMM algorithm.

**Theorem 1:** \( \forall \phi \in \mathbb{D} \) if penalty parameters \( \rho_o \) and Lipschitz constants \( L_o \) satisfy some mild conditions, the proposed ADMM algorithms converge to some stationary point \( X^* \) of approximate problem \( 34 \), i.e.,

\[
\langle \nabla F(X^*), X - X^* \rangle \geq 0, \quad (33)
\]

where

\[
F(X) = \sum_{o \in \mathbb{O}} w_o f_o^2(X). \quad (34)
\]

The detail of the proof and convergence conditions are given in Appendix C.

**B. Implementation Analysis**

Observing the proposed ADMM algorithm, the computational cost is mainly multiplication of solving the gradient \( \nabla f_o^2(X) \). Function \( f_o^2(X) \) is defined as

\[
f_o^2(X) = \| s_i^T X^T S_i x_s - N \delta_{|i-j|+|l|} \|^2. \quad (35)
\]

For gradient \( \nabla f_o^2(X) \), we have

\[
\nabla f_o^2(X) = \begin{bmatrix}
\frac{\partial f_o^2(X)}{\partial x_{i,1}} & \cdots & \frac{\partial f_o^2(X)}{\partial x_{m,1}} & \cdots & \frac{\partial f_o^2(X)}{\partial x_{M,l}} \\
\frac{\partial f_o^2(X)}{\partial x_{i,2}} & \cdots & \frac{\partial f_o^2(X)}{\partial x_{m,2}} & \cdots & \frac{\partial f_o^2(X)}{\partial x_{M,l}} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial f_o^2(X)}{\partial x_{i,N}} & \cdots & \frac{\partial f_o^2(X)}{\partial x_{m,N}} & \cdots & \frac{\partial f_o^2(X)}{\partial x_{M,l}}
\end{bmatrix},
\]

where

\[
\frac{\partial f_o^2(X)}{\partial x_{i,n}} = 2 \text{Tr} \left( \frac{\partial (s_i^T X^T S_i x_s)}{\partial x_{i,n}} \right)^T \left( s_i^T X^T S_i x_s - N \delta_{|i-j|+|l|} \right). \quad (37)
\]

For \( \frac{\partial (s_i^T X^T S_i x_s)}{\partial x_{i,n}} \), when \( l = 0 \), we have

\[
\frac{\partial X^T X}{\partial x_{i,n}} = \begin{bmatrix}
n-1 \text{ zeros} & x_{i,1} & \cdots & 0 \\
x_{i,1} & \cdots & x_{i,n-1} & x_{i,n+1} & \cdots & x_{i,M} \\
0 & x_{i,n+1} & \cdots & x_{i,M} \\
0 & \vdots & \ddots & \ddots & \ddots & \ddots \\
x_{i,1} & \cdots & x_{i,n-1} & 0 \\
0 & x_{i,n+1} & \cdots & x_{i,M}
\end{bmatrix}. \quad (38)
\]

From (38), it can be found that there are \( 2(M - 1) \) nonzero elements in \( \frac{\partial X^T X}{\partial x_{i,n}} \). It implies that obtaining \( \frac{\partial f_o^2(X)}{\partial X} \) needs \( 2N \) multiplications at most. Since \( \mathbb{O} = \{ij,l|i,j = 1, \cdots, M, l \in \mathbb{L}\} \), there are \( M^2 |\mathbb{L}| \) entries in set \( \mathbb{O} \), obtaining all \( \{\nabla f_o^2(X), o \in \mathbb{O}\} \) needs \( 2M^2 N |\mathbb{L}| \) multiplications (|\mathbb{L}| denotes set \( \mathbb{L} \)’s size). Observing Table II, we can see that the computational cost of other terms is far less than \( \nabla f_o^2(X) \).

Therefore, we can conclude that the computational cost of the proposed ADMM algorithm is \( \mathcal{O}(M^2 N |\mathbb{L}|) \).

**V. Simulation Results**

In this section, several numerical examples are presented to illustrate the performance of the proposed ADMM algorithm. The simulation parameters are set as follows: For the proposed ADMM algorithm, we define primal/dual residuals [50] at the \( k \)-th iteration as

\[
R^k = \sum_{o \in \mathbb{O}} \| X^{k+1} - Z_o^{k+1} \|^2_F, \quad D^k = \sum_{o \in \mathbb{O}} \| Z_o^{k+1} - Z_o^k \|^2_F.
\]

Then, the termination criterion in the simulations is set as \( R^k + D^k \leq 10^{-3} \), or the maximum iteration number 500 is reached. In the simulations, we set \( M = 1, 2 \), \( L \in [6, 10] \), \( N = 2^L \) (aperiodic case) or \( 2^L - 1 \) (periodic case), and \( L = [0, N/2] \) or \( [0, N - 1] \).

In comparison, two state-of-the-art methods, BiST [39] and MM-WeCorr [32], are carried out here. All approaches are initialized with the random binary sequence. Besides, all experiments are performed in MATLAB 2019b/Windows 7.

1 It should be noted that the proposed algorithm can be implemented in parallel, the computational time will not increase proportionally with \( |\mathbb{L}| \).
environment on a computer with 2.1GHz Intel 4100 × 2 CPU and 64GB RAM.

A. Convergence Performance

Fig. 2 and 3 show the convergence characteristics of the proposed ADMM algorithm. All results are obtained over 50 independent trials. From Fig. 2 we can see that both the ISL and PSL values of Penalty Box/ℓ_p-Box ADMM converge faster than the Box ADMM. While for the residual values, the result in Fig. 3 shows just the opposite. This is mostly because during the iteration process, compared with ℓ_p-Box ADM algorithm, the variable elements of X in the Box ADMM algorithm more easily tend to be non-binary solutions. Specifically, the existence of non-binary solutions makes the residual in each Box ADMM iteration smaller than other methods (see Fig. 2). Also, it verifies that introducing extra processing methods (penalty terms or ℓ_p-Box) to encourage binary solutions is feasible and effective.

Here, it should be noted that the proposed ADMM algorithm converge to the stationary point of the approximate problem. The exact convergence analysis is presented in Theorem 1.

B. Aperiodic correlation Performance

To evaluate the correlation properties of the binary sequences, we use the normalized ISL/PSL in dB

\[ \text{ISL} = 10 \log_{10} \frac{\text{ISL}}{M N^2}, \quad \text{PSL} = 10 \log_{10} \frac{\text{PSL}^2}{M N^2}. \]

Fig. 4-7 compare the correlation performance between the proposed ADMM algorithm and the MM-WeCorr and BiST approach. The sequences obtained via quantization of MM-WeCorr, named MM-WeCorr (Binary), is also considered.

From Fig. 4 it can be seen that the aperiodic ISLR/PSLR of the proposed methods is similar to BiST, about 5 ~ 6 dB lower than that of the initialization version. For MM-WeCorr with continuous phase, this difference value is more than 10 dB, which enjoys the best correlation performance for both ISLR and PSLR. But the ISLR/PSLR of its quantified version (MM-WeCorr (Binary)) is only 2 dB lower than the initialization version, even less. Thus it can be seen that direct quantization of the designed continuous phase sequence has a large loss of performance, which fully explains the necessity of designing binary sequence.

Fig. 5 shows the example of correlation comparison with parameter \( N = 512, M = 1, L = [1, N - 1] \).
Fig. 5. Aperiodic correlation levels with $N = 512, M = 1, L = [1, N-1]$.

Fig. 6. Aperiodic correlation levels with $N = 1024, M = 1, L = [1, N/2]$.

the proposed algorithms, MM-WeCorr (Binary)’s correlation performance is only better than the initialization which As for Penalty Box/$\ell_p$-Box ADMM, their performance is similar to that of the BiST approach.

Fig. 6 presented a set of experiments with shift interval $L = [1, N/2], N = 1024, M = 1$. It should be noted that the shift interval in comparison algorithm BiST cannot be flexibly selected. Therefore, the experiment of its algorithm is missing. From the figure, it can be seen that compared with the random initialization, the correlation level of the proposed algorithm with $L = [1, N/2]$ is lower about 4 dB, but for MM-WeCorr (Binary), only 1 dB. The correlation comparison with parameter $N = 2048, M = 1, L = \{[1, N/4], [N/2, 3N/4]\}$ is given in Fig. 7. Two shift intervals in L lead MM-WeCorr (Binary) is difficult to implemented, therefore the results is missing. The normalized correlation level in L of proposed algorithm is lower about 4 ~ 5 dB than that of random initialization. Fig.

\[ L = \{1, N/4, [N/2, 3N/4]\} \]

shows the autocorrelation and cross-correlation level with $L = [0, N/2]$ of two sequences respectively. From the figures, we can see that the correlation level of the sequence generated by the proposed ADMM algorithm is lower on average 2 dB than that of the initialization sequence. However, the same results in Fig. 6 is 4 ~ 5 dB with $M = 1$. The inconsistency between the results of Fig. 6 and Fig. 8 is probably caused by the lack of design degrees of freedom with the increase of $M$.

All the results in Fig. 6 ~ 8 illustrate the flexibility selectable correlation interval of the model proposed in this paper.

Table III

| Computational Complexity | BiST | MM-WeCorr | proposed ADMM algorithm |
|--------------------------|------|-----------|-------------------------|
| $\mathcal{O}(MN^3)$      | $\mathcal{O}(M^2N^2)$ | $\mathcal{O}(M^2N|L|)$ |

C. Periodic correlation Performance

Fig. 10 shows the comparison of the averaged periodic ISLR/PSLR values for different algorithms. Reference [51] presented that $2\sqrt{N}$ is the PSL of the best known set of the structured sequences (i.e., Gold, Kasami, $m$-sequences, etc.). In this paper, we consider $2\sqrt{N}$ as the periodic PSL comparison benchmark. From the figure, it can be seen that the periodic ISLR/PSLR of the proposed $\ell_p$-Box ADMM is similar to BiST, about $7 \sim 8$ dB lower than that of the initialization version and $3 \sim 4$ dB lower than $2\sqrt{N}$. If the weight of penalty term in Penalty Box ADMM is reasonably chosen, the periodic PSL correlation performance of the generated sequence is a litter bit lower than that of $\ell_p$-Box ADMM. Since there is no penalty weight, $\ell_p$-Box ADMM is easier to implement.

D. Computational Complexity

The computational complexity in each iteration of the proposed ADMM algorithm and two state-of-the-art algorithms
Fig. 8. Aperiodic correlation levels with $N = 512, M = 2, L = [0, N/2]$.

Fig. 9. Comparison of the averaged periodic ISLR/PSLR values for different algorithms over 50 independent trails.

is listed in Table III. Since $|L| \leq N$ and $M < N$, we can conclude that the computational complexity of the proposed algorithm is cheaper than BiST and MM-WeCorr. Besides, MM-WeCorr can not be used directly to design binary sequence. Moreover, notice that, unlike BiST and MM-WeCorr, the proposed algorithm can be performed in parallel, which means that they are more suitable for large-scale applications from a practical viewpoint of implementation.

VI. Conclusion

In this paper, we formulated the binary sequences with low correlation properties design problem as a nonconvex combination optimization model. Then, an efficient algorithm, named by Box ADMM, was proposed to solve the formulated problem. To encourage binary solutions, ADMM algorithm, $\ell_p$-Box ADMM algorithm is proposed based on Box ADMM algorithm. We proved that, if proper parameters are chosen, the proposed ADMM algorithm converge to some stationary point of the approximate problem. Moreover, we also provided the computational complexity of the proposed approaches. Numerical experiments showed that, compared to the state-of-the-art methods, the proposed algorithm can choose the optimized correlation interval flexibly and obtain good correlation
performance. Besides, the parallel implementation structure makes the proposed algorithm be more suitable for large-scale applications.

APPENDIX A

PROOF OF LEMMA 1

The idea of proving lemma 1 is based on the definition of Lipschitz continuity.

A. For \( \nabla f_{i,0}^2(X) \)

We have the derivations in (39).

\[
\frac{\| \nabla f_{i,0}^2(X) - \nabla f_{i,0}^2(\hat{X}) \|^2_F}{\| X - \hat{X} \|^2_F} \leq \max_{i,n} \left\{ \left| \frac{\partial f_{i,0}^2(X)}{\partial x_{i,n}} - \frac{\partial f_{i,0}^2(\hat{X})}{\partial x_{i,n}} \right| \right\}^2.
\]

(39)

Since \( f_{i,0}^2(X) \) is continuous and differentiable (see (6)), according to the Lagrangian mean value theorem, there exists some point \( \tilde{x}_{i,n} \in (\hat{x}_{i,n}, x_{i,n}) \) which satisfies

\[
\frac{\partial f_{i,0}^2(X)}{\partial x_{i,n}} - \frac{\partial f_{i,0}^2(\hat{X})}{\partial x_{i,n}} = \frac{\partial^2 f_{i,0}^2(X)}{\partial x^2_{i,n}} \tilde{x}_{i,n} - \hat{x}_{i,n}.
\]

(40)

Plugging (40) into (39), we get

\[
\frac{\| \nabla f_{i,0}^2(X) - \nabla f_{i,0}^2(\hat{X}) \|^2_F}{\| X - \hat{X} \|^2_F} \leq \max_{i,n} \left\{ \left| \frac{\partial^2 f_{i,0}^2(X)}{\partial x^2_{i,n}} \right| \right\}.
\]

(41)

According to the definition of Lipschitz continuity, we prove Lemma 1 through bounding the right-hand side of (41). Based on (37), we have the derivations in (42).

\[
\frac{\| \partial^2 f_{i,0}^2(X) \|_F}{\| X \|_F} \leq 2 \left\{ \left| \frac{\partial f_{i,0}^2(X)}{\partial x_{i,n}} \right| \right\}^2 + 2 \left| \frac{\partial f_{i,0}^2(X)}{\partial x_{i,n}} \right| \left| \frac{\partial f_{i,0}^2(X)}{\partial x_{i,n}} \right|.
\]

(42)

B. For \( \nabla f_{i,0}^2(X) \)

Through similar above derivations, we have

\[
\frac{\| \partial^2 f_{i,0}^2(X) \|_F}{\| X \|_F} < 2(N + 1)c^2,
\]

(46)

which results in gradients \( \nabla f_{i,0}(X), l \in L \) being Lipschitz continuous with constant \( L_o \geq 2(N + 1)c^2 \).

APPENDIX B

PROOF OF LEMMA 2\textsuperscript{[2,3]}

A. Proof of Lemma 2

To facilitate the subsequent derivations, we define the following quantities

\[
\Delta^k_{X} = \mathcal{L}_0\left( X^{k+1}, \{ \hat{Z}_o^{k+1}, \hat{A}_o^{k+1} \} \right) - \mathcal{L}_0\left( X^{k}, \{ \hat{Z}_o^{k}, \hat{A}_o^{k} \} \right),
\]

\[
\Delta^k_{Z_o} = \mathcal{L}_o\left( X^{k+1}, \{ \hat{Z}_o^{k+1}, \hat{A}_o^{k+1} \} \right) - \mathcal{L}_o\left( X^{k}, \{ \hat{Z}_o^{k}, \hat{A}_o^{k} \} \right),
\]

\[
\Delta^k_{\Lambda_o} = \mathcal{L}_o\left( X^{k+1}, \{ \hat{Z}_o^{k+1}, \hat{A}_o^{k+1} \} \right) - \mathcal{L}_o\left( X^{k}, \{ \hat{Z}_o^{k+1}, \hat{A}_o^{k+1} \} \right).
\]

Then, the successive difference in Lemma 2 can be rewritten as

\[
\mathcal{L}_o\left( X^{k+1}, \{ \hat{Z}_o^{k+1}, \hat{A}_o^{k+1} \} \right) - \mathcal{L}_o\left( X^{k}, \{ \hat{Z}_o^{k+1}, \hat{A}_o^{k+1} \} \right) = \Delta^k_{X} + \sum_{o \in O} \left( \Delta^k_{Z_o} + \Delta^k_{\Lambda_o} \right).
\]

(47)

1) For \( \Delta^k_{X} \): According to \( \mathcal{L}_o(\cdot) \)'s strong convexity related to \( X \), we get

\[
\Delta^k_{X} \geq \left\langle \nabla X \mathcal{L}_o\left( X^{k+1}, \{ Z_o^{k+1}, A_o^{k+1} \}, X, X^{k+1} \right) \right\rangle + \left( \sum_{o \in O} \frac{\rho_o}{2} \right) \left\| X^{k+1} - X^k \right\|^2_F.
\]

(48)

Since \( X^{k+1} = \arg \min_{X \in X^o} \mathcal{L}_o\left( X, \{ Z_o^{k+1}, A_o^{k+1} \} \right) \), we have

\[
\left\langle \nabla X \mathcal{L}_o\left( X^{k+1}, \{ Z_o^{k+1}, A_o^{k+1} \} \right) \right\rangle \geq 0.
\]

(49)

Plugging it into (48), we obtain

\[
\Delta^k_{X} \geq \left( \sum_{o \in O} \frac{\rho_o}{2} \right) \left\| X^{k+1} - X^k \right\|^2_F.
\]

(50)

2) For \( \Delta^k_{Z_o} \):

\[
\Delta^k_{Z_o} \geq \mathcal{L}_o\left( X^{k+1}, \{ Z_o^{k+1}, A_o^{k+1} \} \right) - \mathcal{L}_o\left( X^{k+1}, \{ Z_o^{k+1}, A_o^{k+1} \} \right)
\]

\[
= \mathcal{L}_o\left( X^{k+1}, \{ Z_o^{k+1}, A_o^{k+1} \} \right) - \mathcal{L}_o\left( X^{k+1}, \{ Z_o^{k+1}, A_o^{k+1} \} \right)
\]

\[
= \Delta^k_{Z_o} + \Delta^k_{\Lambda_o}.
\]

(51)
\[
\Delta_{\mathcal{UU}}^k = w_o \left[ f_o^2(Z_o^k) - f_o^2(X^{k+1}) + \langle \nabla f_o^2(X^{k+1}), X^{k+1} - Z_o^k \rangle \right] - \frac{L_o}{2} \| X^{k+1} - Z_o^k \|_F^2 \\
\geq w_o \left[ \langle \nabla f_o^2(Z_o^k) - \nabla f_o^2(X^{k+1}), X^{k+1} - Z_o^k \rangle \right] - \frac{L_o}{2} \| X^{k+1} - Z_o^k \|_F^2
\]

For \( \| X^{k+1} - X^k \|_F^2 \), we have
\[
\| X^{k+1} - X^k \|_F^2 = \frac{1}{2} \left( A^T(X^{k+1} - X^k) + A^T(X^k - X^k) \right)^2 F \\
\leq \frac{1}{4} \left( 2 \| A^T(X^{k+1} - X^k) \|_F^2 + 2\| A^T(X^k - X^k) \|_F^2 \right) \\
\leq \| X^{k+1} - X^k \|_F^2 + \sigma^2 \| Z_o^{k+1} - Z_o^k \|_F^2,
\]

where (a) and (b) come from Lemma \[1\].

For Box ADMM algorithm
\[
\text{lower bounded.}
\]

If \( \tilde{\rho}_o \) and (b) come from the Lipschitz continuity of the gradient of \( \nabla f_o^2(X) \).

\section*{Appendix C

\textbf{Proof of Theorem 1}

\section*{A. For Box ADMM algorithm}

We desire the conditions in Lemmas \[23\] hold, i.e., \( \tilde{\epsilon}_o, \tilde{\epsilon}_o \geq 0 \), and \( \tilde{\rho}_o \geq 5\tilde{w}_o L_o \).

For Box ADMM formula \[52\], we can get that when \( \tilde{\rho}_o \geq 7.8\tilde{w}_o L_o \) and \( \tilde{\rho}_o \geq 5.3\tilde{w}_o L_o, \tilde{\epsilon}_o, \tilde{\epsilon}_o \geq 0 \) hold. Therefore, we conclude that when \( \tilde{\rho}_o \geq 7.8\tilde{w}_o L_o \), all conditions in Lemmas \[23\] hold. To simplify the description, we choose \( \forall \mathcal{U} \in \mathcal{E}, \tilde{\rho}_o \geq 8\tilde{w}_o L_o \).

For \( k = 1, 2, \cdots, +\infty \), summing both sides of \[31\], we can obtain
\[
\mathcal{L}_o(X^k, \{Z_o^k, A_o^k\}_{o \in \mathcal{O}}) - \mathcal{L}_o(X^{k+1}, \{Z_o^{k+1}, A_o^{k+1}\}_{o \in \mathcal{O}}) \\
\geq \sum_{k=1}^{+\infty} \sum_{o \in \mathcal{O}} \frac{1}{2\tilde{\rho}_o^2} \left( \epsilon_o \| X^{k+1} - X^k \|_F^2 + \epsilon_o \| Z_o^{k+1} - Z_o^k \|_F^2 \right)
\]

\[
+ \tilde{\epsilon}_o \sigma^2 \| Z_o^{k+1} - Z_o^k \|_F^2.
\]
\[
\begin{align*}
\mathcal{L}_o(\mathbf{X}^{k+1}, \{\mathbf{Z}_o^{k+1}, A_o^{k+1}\}_{o \in O}) \\
= w_o f_o^2(\mathbf{Z}_o^{k+1}) + \frac{\sigma^2}{2} \mathbf{Z}_o^T \mathbf{Z}_o + \left( \frac{1}{2} w_o \left[ \nabla f_o^2(\mathbf{X}^{k+1}) + L_o(\mathbf{Z}_o^{k+1} - \mathbf{X}^{k+1}) \right] \right) \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \\
+ \frac{\rho_o}{2} \| \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \|^2_F \\
= \mathcal{L}_x + \mathcal{L}_{\tilde{\mathbf{Z}}_o} + \frac{\rho_o}{2} \| \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \|^2_F 
\end{align*}
\]

(60)

\[
\begin{align*}
\mathcal{L}_{\tilde{\mathbf{Z}}_o} = w_o f_o^2(\mathbf{Z}_o^{k+1}) + \left( \frac{1}{2} w_o \left[ \nabla f_o^2(\mathbf{X}^{k+1}) + L_o(\mathbf{Z}_o^{k+1} - \mathbf{X}^{k+1}) \right] \right) \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \\
= w_o f_o^2(\mathbf{Z}_o^{k+1}) + \left( \frac{1}{2} w_o \left[ \nabla f_o^2(\mathbf{X}^{k+1}) + L_o(\mathbf{Z}_o^{k+1} - \mathbf{X}^{k+1}) \right] \right) \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \\
\geq w_o f_o^2(\mathbf{Z}_o^{k+1}) + \left( \frac{1}{2} w_o \left[ \nabla f_o^2(\mathbf{X}^{k+1}) + L_o(\mathbf{Z}_o^{k+1} - \mathbf{X}^{k+1}) \right] \right) \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \\
\geq w_o f_o^2(\mathbf{Z}_o^{k+1}) + \frac{5w_oL_o}{4} \| \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \|^2_F \\
\geq w_o f_o^2(\mathbf{Z}_o^{k+1}) + \frac{5w_oL_o}{4} \| \mathbf{X}^{k+1} - \tilde{\mathbf{A}} \tilde{\mathbf{Z}}_o^{k+1} \|^2_F 
\end{align*}
\]

(61)

According to the boundness of \(\mathcal{L}_o(\mathbf{X}^{k+1}, \{\mathbf{Z}_o^{k+1}, A_o^{k+1}\}_{o \in O})\), above equation can rewritten as

\[
\begin{align*}
\mathcal{L}_o(\mathbf{X}^{1}, \{\mathbf{Z}_o^{1}, A_o^{1}\}_{o \in O}) \\
\geq \sum_{k=1}^{+\infty} \sum_{o \in O} \frac{1}{2} \left( \epsilon_o \| \mathbf{X}^{k+1} - \mathbf{X}_o^{k+1} \|^2_F + \epsilon_o \| \mathbf{Z}_o^{k+1} - \mathbf{Z}_o^k \|^2_F \\
+ \epsilon_o \sigma^2 \| \mathbf{Z}_o^{k+1} - \mathbf{Z}_o^k \|^2_F \right) 
\end{align*}
\]

(64)

Since \(\epsilon_o, \epsilon_o, \epsilon_o \geq 0\) and \(\mathcal{L}_o(\mathbf{X}^{k+1}, \{\mathbf{Z}_o^{k+1}, A_o^{k+1}\}_{o \in O})\) is finite, we conclude that

\[
\lim_{k \to +\infty} \| \mathbf{X}^{k+1} - \mathbf{X}_o^{k+1} \|^2_F = 0, \quad \lim_{k \to +\infty} \| \mathbf{Z}_o^{k+1} - \mathbf{Z}_o^k \|^2_F = 0.
\]

(65)

Combining (67) and (15c), we get

\[
\lim_{k \to +\infty} \| \boldsymbol{A}_o^{k} - \hat{\boldsymbol{A}}_o \|^2_F = 0, \quad \lim_{k \to +\infty} \| \hat{\mathbf{X}}^{k+1} - \tilde{\mathbf{A}} \hat{\mathbf{Z}}_o^{k+1} \|^2_F = 0.
\]

(68) and (69) indicate that \(\hat{\mathbf{Z}}_o^*\) is a stationary point of approx-

\(\hat{\mathbf{X}}^*\) converges to some limit point as \(k \to +\infty\), i.e.,

\[
\lim_{k \to +\infty} \mathbf{X}^k = \mathbf{X}^*.
\]

(66)

Combining it with (65), we can see that \(\{\mathbf{Z}_o^k\}\) converges. From (56), we can obtain

\[
\tilde{\mathbf{A}}^T \mathbf{A}_o^{k+1} = \left[ \frac{w_o \left( \nabla f_o^2(\mathbf{X}^{k+1}) + L_o(\mathbf{Z}_o^{k+1} - \mathbf{X}^{k+1}) \right)}{\sigma^2 \mathbf{Z}_o^{k+1}} \right].
\]

(67)

Since \(\lim_{k \to +\infty} \mathbf{Z}_o^k - \mathbf{X}^k = 0\) and \(\nabla f_o^2(\mathbf{X}^k)\) is Lipschitz continuous, (67) can be rewritten as

\[
\tilde{\mathbf{A}}^T \mathbf{A}_o^* = \left[ \frac{w_o \nabla f_o^2(\mathbf{X}^*)}{\sigma^2 \mathbf{Z}_o^*} \right].
\]

(68)

which means \(\{\mathbf{A}_o^k\}\) converges. Based on (65), it can be obtained

\[
\mathbf{X}^* = \boldsymbol{A}_o \hat{\mathbf{Z}}_o^*.
\]

(69)
Combining it with (65), we can obtain
\[
\left\| X^{k+1} - AZ^{k+1}_o \right\| = \left\| \frac{1}{\rho_o} (\Lambda_o^{k+1} - \Lambda_o^k) + \sigma Z^{k+1}_o \right\| \leq \frac{1}{\rho_o} \left\| (\Lambda_o^{k+1} - \Lambda_o^k) \right\| + \left\| \sigma Z^{k+1}_o \right\|.
\]
(70)

Combining it with (65), we get
\[
\lim_{k \to +\infty} \left\| X^{k+1} - AZ^{k+1}_o \right\| = \left\| X^* - AZ^*_o \right\| \leq \left\| \sigma Z^*_o \right\| = \mathcal{O}(\sigma),
\]
(71)

which means that the stationary point \( Z^*_o \) is close to the stationary point of the original problem (10) within the range \( \mathcal{O}(\sigma) \).

\[\Box\]

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