ABSTRACT: Vacuum polarisation in QED in a background gravitational field induces interactions which effectively violate the strong equivalence principle and affect the propagation of light. In the low frequency limit, Drummond and Hathrell have shown that this mechanism leads to superluminal photon velocities. To confront this phenomenon with causality, however, it is necessary to extend the calculation of the phase velocity \( v_{\text{ph}}(\omega) \) to high frequencies, since it is \( v_{\text{ph}}(\infty) \) which determines the characteristics of the effective wave equation and thus the causal structure. In this paper, we use a recently constructed expression, valid to all orders in a derivative expansion, for the effective action of QED in curved spacetime to determine the frequency dependence of the phase velocity and investigate whether superluminal velocities indeed persist in the high frequency limit.
1. Introduction

It has been known since the original work of Drummond and Hathrell [1] that quantum effects have important consequences for the propagation of light in curved spacetime. In the classical theory of electrodynamics coupled to general relativity, light propagates simply along null geodesics. In quantum electrodynamics, however, vacuum polarisation changes the picture and the background gravitational field becomes a dispersive medium for the propagation of photons. In itself, this is perhaps not surprising. The one-loop vacuum polarisation contribution to the photon propagator introduces a non-trivial length scale \( \lambda_c \) (the inverse electron mass) and it is natural that photon propagation will be significantly affected when the typical curvature scale \( L \) is comparable to \( \lambda_c \). However, the remarkable result found by Drummond and Hathrell is that in many cases the effect of vacuum polarisation is to induce a change in the velocity of light to ‘superluminal’ speeds, i.e. \( v > c \), where \( c \) is the fundamental constant.

Since this original discovery, many special cases have been studied in detail [1,2,3,4], including propagation in black hole spacetimes described by the Schwarzschild, Reissner-Nordström or Kerr metrics, the FRW metric of big bang cosmology, and gravitational wave backgrounds, in particular the Bondi-Sachs metric describing asymptotic radiation from an isolated source. The phenomenon of superluminal propagation has been observed in all these examples of gravitational fields and a number of general features have been identified. A notable result is the ‘horizon’ [14] or ‘touching’ [15] theorem, which shows that even in the presence of superluminal velocities, the effective black hole event horizon defined by physical photon propagation coincides precisely with the geometric horizon. Another important observation is that the speed of light increases rapidly (as \( 1/t^2 \) in the radiation dominated era) in the early stages of a FRW cosmology [1], with potential implications for the horizon problem and related issues.

All this work has been based on the initial Drummond-Hathrell analysis, in which they showed that the effect of one-loop vacuum polarisation is to induce the following effective action, generalising the free-field Maxwell theory:

\[
\Gamma = \int dx \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{m^2} \left( aR F_{\mu\nu} F^{\mu\nu} + bR_{\mu\nu} F^{\mu\lambda} F^{\nu}_{\lambda} + cR_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} + dD_{\mu} F^{\mu\lambda} D_{\nu} F^{\nu}_{\lambda} \right) \right]
\]

(1.1)

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1 Other examples and a selection of related work may be found in refs.[5, 6, 7, 8, 9, 10] and possible implications for time machines in [11, 12, 13].
Here, $a,b,c,d$ are constants of $O(\alpha)$ and $m$ is the electron mass. The notable feature is the direct coupling of the electromagnetic field to the curvature. This is an effective violation of the strong equivalence principle, which states that dynamical laws should be the same in the local inertial frames at each point in spacetime. More precisely, this requires the coupling of electromagnetism to gravity to be through the connections only, independent of curvature. Eq.(1.1) shows that while this principle may be consistently imposed at the classical level, it is necessarily violated in quantum electrodynamics.

Standard geometric optics methods applied to this effective action results in a modified light cone \[1,14\]:

\[
k^2 - \frac{8\pi}{m^2} (2b + 4c)T_{\mu\lambda}k^\mu k^\lambda + \frac{8c}{m^2} C_{\mu\nu\lambda\rho}k^\mu k^\lambda a^\nu a^\rho = 0 \quad (1.2)
\]

where $k^\mu$ is the wave vector and $a^\mu$ is the polarisation vector. In the second term, we have replaced the Ricci tensor by the energy-momentum tensor using the Einstein equations. This emphasises that this contribution is related to the presence of matter; indeed this takes the same form as the subluminal corrections to the speed of light in other scenarios, such as background magnetic fields, finite temperature, etc.[16,17]. The uniquely gravitational term involving the Weyl tensor depends explicitly on $a^\mu$ and gives a polarisation dependence of the speed of light (gravitational birefringence). Moreover, it is readily seen that this term changes sign for the two physical, transverse polarisations so that for vacuum (Ricci flat) spacetimes, if one photon polarisation is subluminal, the other is necessarily superluminal. This property, and many others, are most clearly seen by rewriting the light cone condition (1.2) in Newman-Penrose formalism [14].

The effective action (1.1) is, however, only the first term in a derivative expansion, with higher order terms in $O(D_m)$ omitted. The corresponding light-cone condition (1.2) is therefore valid a priori only in the low frequency approximation. The modified light velocity derived from it is the phase velocity $v_{ph} \equiv \frac{\omega}{k}$ (where $k_\mu = (\omega, k)$ in a local inertial frame) at $\omega \sim 0$. In order to discuss the obvious issues concerning causality, however, the relevant ‘speed of light’ is not $v_{ph}$ at $\omega \sim 0$ but $v_{ph}$ in the high frequency limit $\omega \to \infty$. (We discuss this point carefully in section 2.) In order to address causality, therefore, we first need to establish the effective light cone condition for high frequencies, which itself involves finding the ‘high frequency limit’ of the effective action (1.1).

We have recently evaluated the effective action for QED in curved spacetime to all orders in $O(D_m)$, keeping terms of the form $R FF$ as in (1.1), that is, the terms relevant to photon propagation to lowest order in $O(D_m)$, i.e. $O(\frac{\lambda^2}{L^2})$. The derivation and full details of this result are presented in ref. [18]. Here, we generalise the geometric optics derivation of photon propagation using this new effective action and discuss what we can learn about the critical high frequency behaviour of the phase velocity. The question we wish to answer
is whether the phenomenon of superluminal photon propagation is a curiosity of the low frequency approximation or whether it persists at high frequency, forcing us to confront the serious implications for causality associated with faster than light motion. In the end, our results are intriguing but not as yet conclusive. It appears that further field-theoretic developments may be needed to give a final resolution of the nature of dispersion in gravitational fields.

The paper is presented as follows. In order to clarify exactly what we mean by the “speed of light” and why it is \( v_{ph}(\infty) \) which controls the causal behaviour of the theory, we review various definitions and theorems concerning the propagation of light in section 2. In section 3, we review the fundamentals of geometric optics and its application to the Drummond-Hathrell action, clarifying some subtle points arising from earlier work. Our new result for the QED effective action in curved spacetime valid to all orders in the derivative expansion is presented in section 4, including some technical formulae for form factors. The implications for photon propagation in the high frequency limit are described in sections 5 and 6, leading to the apparent prediction that high frequency superluminal velocities are possible in certain spacetimes. This preliminary conclusion is challenged in section 7, where we compare our gravitational analysis with the closely related problem of photon propagation in a background magnetic field. Section 8 summarises our final conclusions.

2. The “Speed of Light”

Our fundamental interest is in whether the fact that the phase velocity at low frequencies can be superluminal (and therefore imply motion backwards in time in a class of local inertial frames) is in contradiction with established notions of causality. It has been discussed elsewhere that while in special relativity superluminal motion necessarily implies a causal paradox, in general relativity this is not necessarily so. The key question is whether a spacetime which is stably causal (see ref. [23], Proposition 6.4.9 for a precise definition) with respect to the original metric remains stably causal with respect to the effective metric defined by the modified light cones. Essentially, this means that the spacetime should still admit a foliation into a set of spacelike according to the effective light cones. This is an interesting global question, which is beyond the scope of this paper. It seems entirely possible, however, that the Drummond-Hathrell

\[ \text{2 To be precise we mean here: “in Minkowski spacetime with no boundaries”. An analysis of causality related to the phenomenon of superluminal propagation between Casimir plates has recently been given in ref. [19] (see also [21][22]).} \]
modifications to the light cones should not destroy stable causality.

Before addressing such issues, however, we need to be clear what exactly we mean by the “speed of light” which determines the light cones to be used in determining the causal structure of spacetime. In this section, we therefore review briefly some basic definitions and results from classical optics in order to motivate our subsequent analysis.

A particularly illuminating discussion of wave propagation in a simple dispersive medium is given in the classic work by Brillouin [24]. This considers propagation of a sharp-fronted pulse of waves in a medium with a single absorption band, with refractive index $n(\omega)$:

$$n^2(\omega) = 1 - \frac{a^2}{\omega^2 - \omega_0^2 + 2i\omega \rho}$$

where $a, \rho$ are constants and $\omega_0$ is the characteristic frequency of the medium. Five distinct velocities are identified: the phase velocity $v_{ph} = c\frac{\omega}{|k|} = \Re \frac{1}{n(\omega)}$, group velocity $v_{gp} = \frac{d\omega}{dk}$, signal velocity $v_{sig}$, energy-transfer velocity $v_{en}$ and wavefront velocity $v_{wf}$, with precise definitions related to the behaviour of contours and saddle points in the relevant Fourier integrals in the complex $\omega$-plane. Their frequency dependence is illustrated in Fig. 1.

![Fig.1 Sketch of the behaviour of the phase, group and signal velocities with frequency in the model described by the refractive index (2.1). The energy-transfer velocity (not shown) is always less than $c$ and becomes small near $\omega_0$. The wavefront speed is identically equal to $c$.](image)

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3 In fact, if we take into account the distinction discussed in section 3 between the phase velocity and the ray velocity $v_{ray}$, and include the fundamental speed of light constant $c$ from the Lorentz transformations, we arrive at 7 distinct definitions of “speed of light”.
As the pulse propagates, the first disturbances to arrive are very small amplitude waves, ‘frontrunners’, which define the wavefront velocity \( v_{wf} \). These are followed continuously by waves with amplitudes comparable to the initial pulse; the arrival of this part of the complete waveform is identified in \[24\] as the signal velocity \( v_{sig} \). As can be seen from Fig. 1, it essentially coincides with the more familiar group velocity for frequencies far from \( \omega_0 \), but gives a much more intuitively reasonable sense of the propagation of a signal than the group velocity, whose behaviour in the vicinity of an absorption band is relatively eccentric.\(^4\) As the figure also makes clear, the phase velocity itself simply does not represent a ‘speed of light’ relevant for considerations of signal propagation or causality.

The appropriate velocity to define light cones and causality is in fact the wavefront velocity \( v_{wf} \). (Notice that in Fig. 1, \( v_{wf} \) is a constant, equal to \( c \), independent of the frequency or details of the absorption band.) This is determined by the boundary between the regions of zero and non-zero disturbance (more generally, a discontinuity in the first or higher derivative of the disturbance field) as the pulse propagates. Mathematically, this definition of wavefront is identified with the characteristics of the partial differential equation governing the wave propagation \[27\]. Our problem is therefore to determine the velocity associated with the characteristics of the wave operator derived from the modified Maxwell equations of motion appropriate to the new effective action.

A very complete and rigorous discussion of the wave equation in curved spacetime is given in the monograph by Friedlander \[28\], in which it is proved (Theorem 3.2.1) that the characteristics are simply the null hypersurfaces of the spacetime manifold, in other words that the wavefront always propagates with the fundamental speed \( c \). However, this discussion assumes the standard form of the (gauge-fixed) Maxwell wave equation (cf. ref. \[28\], eq.(3.2.1)) and explicitly does not cover the modified wave equation derived from the action (1.1), precisely because of the extra curvature couplings.

Instead, the key result for our purposes, which allows a derivation of the wavefront velocity, is derived by Leontovich \[29\]. In this paper, an elegant proof is presented for a very general set of PDEs that the wavefront velocity associated with the characteristics is identical to the \( \omega \rightarrow \infty \) limit of the phase velocity, i.e.

\[
v_{wf} = \lim_{\omega \rightarrow \infty} \frac{\omega}{|k|} = \lim_{\omega \rightarrow \infty} v_{ph}(\omega)
\]

(2.2)

This proof appears to be of sufficient generality to apply to our discussion of photon propagation using the modified effective action of section 4.

\(^4\) Notice that it is the group velocity which is measured in quantum optics experiments which find light speeds of essentially zero \[25\] or many times \( c \) \[26\]. A particularly clear description in terms of the effective refractive index is given in \[23\].
The wavefront velocity in a gravitational background is therefore not given \textit{a priori} by \(c\). Taking vacuum polarisation into account, there is no simple non-dispersive medium corresponding to the vacuum of classical Maxwell theory in which the phase velocity represents a true speed of propagation; in curved spacetime QED, even the vacuum is dispersive. In order to discuss causality, we therefore have to extend the original Drummond-Hathrell results for \(v_{\text{ph}}(\omega \sim 0)\) to the high frequency limit \(v_{\text{ph}}(\omega \rightarrow \infty)\), as already emphasised in ref.[1]. This is why the effective action (1.1) to lowest order in the derivative expansion is not sufficient and we require the all-orders effective action of section 4.

A final twist emerges if we write the standard dispersion relation for the refractive index \(n(\omega)\) in the limit \(\omega \rightarrow \infty\):
\[
n(\infty) = n(0) - \frac{2}{\pi} \int_0^{\infty} \frac{d\omega}{\omega} \Im n(\omega)
\] (2.3)
For a standard dispersive medium, \(\Im n(\omega) > 0\), which implies that \(n(\infty) < n(0)\), or equivalently \(v_{\text{ph}}(\infty) > v_{\text{ph}}(0)\). Evidently this is satisfied by Fig. 1. The key question though is whether the usual assumption of positivity of \(\Im n(\omega)\) holds in the present situation of the QED vacuum in a gravitational field. If so, then (as already noted in ref.[1]) the superluminal Drummond-Hathrell results for \(v_{\text{ph}}(0)\) would actually be \textit{lower bounds} on the all-important wavefront velocity \(v_{\text{ph}}(\infty)\). However, it is not so clear that positivity of \(\Im n(\omega)\) is reliable in the gravitational context. Indeed it has been explicitly criticised by Dolgov and Khriplovich in refs.[30,31], who point out that since gravity is an inhomogeneous medium in which beam focusing as well as diverging can happen (see next section), a growth in amplitude corresponding to \(\Im n(\omega) < 0\) is possible. It therefore seems best to set the dispersion relation (2.3) aside for the moment and concentrate instead on a direct attempt to determine \(v_{\text{ph}}(\infty)\). This is the goal of this paper.

3. Low Frequency Photon Propagation

The simplest way to deduce the light-cone condition implied by the effective action is to use geometric optics. In this section, we review the approach introduced in ref.[1] (see also refs.[2,14]) emphasising some points which will be important later.

In geometric optics (see [32] for a thorough discussion) the electromagnetic field is written as the product of a slowly-varying amplitude and a rapidly-varying phase, i.e.
\[
F_{\mu\nu} = \Re \left( \tilde{f}_{\mu\nu} + i\epsilon h_{\mu\nu} + \ldots \right)e^{i\varphi}
\] (3.1)
Here, \(\epsilon\) is a parameter introduced purely as a book-keeping device to keep track of the relative order of magnitude of terms. The field equations and Bianchi identities are solved order by order in \(\epsilon\).
The wave vector is defined as the derivative of the phase, i.e. \( k_\mu = \partial_\mu \vartheta \). The leading order term in the Bianchi identity \( D[\lambda F_{\mu\nu}] = 0 \) is of \( O(\frac{1}{\varepsilon}) \) and constrains \( f_{\mu\nu} \) to have the form

\[
f_{\mu\nu} = k_\mu A_\nu - k_\nu A_\mu
\]  

where \( A_\mu = A a_\mu \). \( A \) represents the amplitude while \( a^\mu \) (normalised so that \( a^\mu a_\mu = -1 \)) specifies the polarisation. For physical polarisations, \( k_\mu a^\mu = 0 \).

Conventional curved spacetime QED is based on the usual Maxwell action, so the equation of motion is simply

\[
D_\mu F^{\mu\nu} = 0
\]  

At leading order, \( O(\frac{1}{\varepsilon}) \), this becomes

\[
k_\mu f^{\mu\nu} = 0
\]  

and since this implies

\[
k^2 a^\nu = 0
\]  

we immediately deduce that \( k^2 = 0 \), i.e. \( k^\mu \) is a null vector. Then, from its definition as a gradient, we see

\[
k^\mu D_\mu k^\nu = k^\mu D^\nu k_\mu = \frac{1}{2} D^\nu k^2 = 0
\]  

Light rays (photon trajectories) are defined as the integral curves of \( k^\mu \), i.e. the curves \( x^\mu(s) \) where \( \frac{dx^\mu}{ds} = k^\mu \). These curves therefore satisfy

\[
0 = k^\mu D_\mu k^\nu = \frac{d^2 x^\nu}{ds^2} + \Gamma^\nu_{\mu\lambda} \frac{dx^\mu}{ds} \frac{dx^\lambda}{ds}
\]  

which is the geodesic equation. So in the conventional theory, light rays are null geodesics.

The subleading, \( O(1) \), term in the equation of motion gives

\[
k^\mu D_\mu A^\nu = -\frac{1}{2} (D_\mu k^\mu) A^\nu
\]  

which decomposes into

\[
k^\mu D_\mu a^\nu = 0
\]  

and

\[
k^\mu D_\mu (\ln A) = -\frac{1}{2} D_\mu k^\mu
\]  

The first shows that the polarisation vector is parallel transported along the null geodesic rays while the second, whose r.h.s. is simply minus the optical scalar \( \theta \), shows how the amplitude varies as the beam of rays focuses or diverges.
We now apply the same methods to the modified effective action (1.1). This gives rise to a new equation of motion which, under the approximations listed below, simplifies to:

\[ D_{\mu} F^{\mu\nu} - \frac{1}{m^2} \left[ 2b R_{\mu\lambda} D^{\mu} F^{\lambda\nu} + 4c R_{\mu}^{\ \nu} \lambda_{\rho} D^{\mu} F^{\lambda\rho} \right] = 0 \]  

(3.11)

Here, we have neglected derivatives of the curvature tensor, which would be suppressed by powers of \( O(\lambda/L) \), where \( \lambda \) is the photon wavelength and \( L \) is a typical curvature scale, and we have omitted the new contributions involving \( D_{\mu} F^{\mu\nu} \) : since this term is already \( O(\alpha) \) using the equations of motion, these contributions only affect the light cone condition at \( O(\alpha^2) \) and must be dropped for consistency. Making the standard geometric optics assumptions described above, we then find the new light cone condition:

\[ k_{\mu} f^{\mu\nu} - \frac{1}{m^2} \left[ 2b R^{\mu}_{\ \lambda} k_{\mu} f^{\lambda\nu} + 4c R^{\mu\nu}_{\ \lambda\rho} k_{\mu} f^{\lambda\rho} \right] = 0 \]  

(3.12)

Eq.(3.12) can now be rewritten as an equation for the polarisation vector \( a^{\mu} \), and re-expressing in terms of the Weyl tensor we find

\[ k^2 a^{\nu} - \frac{(2b + 4c)}{m^2} R_{\mu\lambda} \left( k^{\mu} k^{\lambda} a^{\nu} - k^{\mu} k^{\nu} a^{\lambda} \right) - \frac{8c}{m^2} C_{\mu}^{\space \nu}_{\ \lambda\rho} k^{\mu} k^{\lambda} a^{\rho} = 0 \]  

(3.13)

The solutions of this equation describe the propagation for a photon of wave vector \( k_{\mu} \) and polarisation \( a^{\mu} \). Contracting with \( a_{\nu} \), we find the effective light cone

\[ k^2 - \frac{(2b + 4c)}{m^2} R_{\mu\lambda} k^{\mu} k^{\lambda} + \frac{8c}{m^2} C_{\mu\nu}^{\space \lambda\rho} k^{\mu} k^{\lambda} a^{\nu} a^{\rho} = 0 \]  

(3.14)

from which (1.2) follows immediately.

Notice that in the discussion of the free Maxwell theory, we did not need to distinguish between the photon momentum \( p^{\mu} \), i.e. the tangent vector to the light rays, and the wave vector \( k_{\mu} \) since they were simply related by raising the index using the spacetime metric, \( p^{\mu} = g^{\mu\nu} k_{\nu} \). In the modified theory, there is an important distinction. The wave vector, defined as the derivative of the phase, is a covariant vector or 1-form, whereas the photon momentum/ tangent vector to the rays is a true contravariant vector. The relation is non-trivial. In fact, when as in (3.14) we can write the light cone condition for the wave vector as the homogeneous form

\[ G^{\mu\nu} k_{\mu} k_{\nu} = 0 \]  

(3.15)

we should define the corresponding ‘momentum’ as

\[ p^{\mu} = G^{\mu\nu} k_{\nu} \]  

(3.16)
and the light rays as curves $x^\mu(s)$ where $\frac{dx^\mu}{ds} = p^\mu$. This definition of momentum satisfies

$$G_{\mu\nu}p^\mu p^\nu = G^\mu\nu k_\mu k_\nu = 0$$

(3.17)

where $G \equiv G^{-1}$ therefore defines a new effective metric which determines light cones mapped out by the geometric optics light rays. (Indices are always raised or lowered using the true metric $g_{\mu\nu}$.) The ray velocity $v_{\text{ray}}$ corresponding to the momentum $p^\mu$, which is the velocity with which the equal-phase surfaces advance, is given by

$$v_{\text{ray}} = \left| \frac{p^\mu}{p^0} \right| = \left| \frac{d|x|}{dt} \right|$$

(3.18)

along the ray, and is in general different from the phase velocity

$$v_{\text{ph}} = \frac{k^0}{|k|}$$

(3.19)

A nice example of this is given in ref. 13, which analyses certain aspects of superluminal propagation in Minkowski spacetime with Casimir plates. The discrepancy between $v_{\text{ph}}$ and $v_{\text{ray}}$ (called $c_{\text{light}}$ in 13) in that example is due to a difference between the direction of propagation along the rays and the wave 3-vector. Otherwise, it follows directly from (3.17) that $v_{\text{ray}}$ and $v_{\text{ph}}$ are identical.

Recognising the distinction between $p$ and $k$ also clarifies a potentially confusing point in the important example of propagation in a FRW spacetime 1. Since the FRW metric is Weyl flat, the modified light cone condition (1.2) reads simply

$$k^2 = \zeta T_{\mu\nu}k^\mu k^\nu$$

(3.20)

where $\zeta = \frac{8\pi}{m^2}(2b + 4c)$ and the energy-momentum tensor is

$$T_{\mu\nu} = (\rho + P)n_\mu n_\nu - Pg_{\mu\nu}$$

(3.21)

with $n^\mu \equiv (e_i)^\mu$ specifying the time direction in a comoving orthonormal frame. $\rho$ is the energy density and $P$ is the pressure, which in a radiation-dominated era are related by $\rho - 3P = 0$. The phase velocity is independent of polarisation and is found to be superluminal:

$$v_{\text{ph}} = \frac{k^0}{|k|} = 1 + \frac{1}{2} \zeta(\rho + P)$$

(3.22)

At first sight, this looks surprising given that $k^2 > 0$. However, if instead we consider the momentum along the rays, we find

$$p^2 = g_{\mu\nu}p^\mu p^\nu = -\zeta(\rho + P)(k^0)^2$$

(3.23)
and

$$v_{\text{ray}} = \frac{|p|}{p^0} = 1 + \frac{1}{2} \zeta (\rho + P)$$  \hspace{1cm} (3.24)

The effective metric $G = G^{-1}$ is

$$G_{\mu \nu} = \begin{pmatrix}
1 + \zeta (\rho + P) & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}$$  \hspace{1cm} (3.25)

In this case, therefore, we find equal, superluminal velocities $v_{\text{ph}} = v_{\text{ray}}$ and $p^2 < 0$ is manifestly spacelike as required.

In the radiation dominated era, where $ho(t) = \frac{3}{32\pi} t^{-2}$, we have

$$v_{\text{ph}} = 1 + \frac{1}{16\pi} \zeta t^{-2}$$  \hspace{1cm} (3.26)

which, as already observed in [1], increases towards the early universe. Although this expression is only reliable in the perturbative regime where the correction term is small, it is intriguing that QED predicts a rise in the speed of light in the early universe. It is a matter of speculation whether this superluminal effect persists for high curvatures ($L \sim \lambda_c$) and whether it could be important in the context of the horizon problem.

4. The Effective Action for QED in a Gravitational Field

The effective action presented in this section has recently been derived [18] by adapting the more general background field calculations of [33], this latter paper being the culmination of the theoretical development described in [34]. The result is an effective action which incorporates the one-loop vacuum polarisation contributions to the photon propagator in an arbitrary, weak gravitational field. We therefore keep terms of type $RFF$, i.e. quadratic in the electromagnetic field but only of first order in the curvature. This neglects terms of higher orders in $O(\lambda^2 \gamma^2)$. However, the new feature compared to the Drummond-Hathrell action (1.1) is that terms to all orders in derivatives are kept. This allows a discussion of the frequency dependence of the modifications to photon propagation.

The effective action is given by

$$\Gamma = \Gamma_{(0)} + \ln \det S(x, x')$$  \hspace{1cm} (4.1)

where $\Gamma_{(0)}$ is the free Maxwell action and $S(x, x')$ is the Green function of the Dirac operator in the background gravitational field, i.e.

$$(i\slashed{D} - m)S(x, x') = \frac{i}{\sqrt{-g}} \delta(x, x')$$  \hspace{1cm} (4.2)
In fact it is more convenient to introduce the scalar Green function $G(x, x')$ defined by
\[ S(x, x') = (i\mathcal{D} + m)G(x, x') \] (4.3)
so that
\[ \left(D^2 + ie\sigma^{\mu\nu}F_{\mu\nu} - \frac{1}{4}R + m^2\right)G(x, x') = -\frac{i}{\sqrt{-g}}\delta(x, x') \] (4.4)
Then we evaluate $\Gamma$ from the heat kernel, or proper time, representation
\[ \Gamma = \Gamma(0) - \frac{1}{2} \int_0^\infty ds \frac{e^{-is^2}}{s} \text{Tr}G(x, x'; s) \] (4.5)
where
\[ \mathcal{D}G(x, x'; s) = i\frac{\partial}{\partial s}G(x, x'; s) \] (4.6)
with $G(x, x'; 0) = G(x, x')$. Here, $\mathcal{D}$ is the differential operator in eq.(4.4) at $m = 0$.

The details of the derivation of the effective action are given in ref.[18] and here we simply quote the result:
\[ \Gamma = \int dx\sqrt{-g} \left[ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{m^2} \left(D_{\mu}F^{\mu\lambda} \mathcal{G}^0_{\lambda} D_{\nu}F^{\nu}_{\lambda} \right. \right. \right. \]
\[ + \mathcal{G}^1_{1} RF_{\mu\nu}F^{\mu\nu} + \mathcal{G}^2_{2} R_{\mu\nu}F^{\mu\lambda}F^{\nu}_{\lambda} + \mathcal{G}^3_{3} R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho} \right. \]
\[ + \frac{1}{m^4} \left( \mathcal{G}^4_{4} RD_{\mu}F^{\mu\lambda}D_{\nu}F^{\nu}_{\lambda} \right. \right. \]
\[ + \mathcal{G}^5_{5} R_{\mu\nu}D_{\lambda}F^{\lambda\mu}D_{\rho}F^{\rho\nu} + \mathcal{G}^6_{6} R_{\mu\nu}D^{\mu}F^{\lambda\rho}D^{\nu}F_{\lambda\rho} \]
\[ + \mathcal{G}^7_{7} R_{\mu\nu}D^{\mu}D^{\nu}F^{\lambda\rho}F_{\lambda\rho} + \mathcal{G}^8_{8} R_{\mu\nu}D^{\mu}D^{\lambda}F_{\lambda\rho}F^{\rho\nu} \]
\[ + \mathcal{G}^9_{9} R_{\mu\nu\lambda\rho}D_{\sigma}F^{\sigma\rho}D^{\lambda}F^{\mu\nu} \left. \right) \right] \]
(4.7)
In this formula, the $\mathcal{G}^n_{n}$ ($n \geq 1$) are form factor functions of three operators:
\[ \mathcal{G}^n_{n} \equiv G_n \left( \frac{D_{(1)}^2}{m^2}, \frac{D_{(2)}^2}{m^2}, \frac{D_{(3)}^2}{m^2} \right) \] (4.8)
where the first entry ($D_{(1)}^2$) acts on the first following term (the curvature), etc. $\mathcal{G}^0_{0}$ is similarly defined as a single variable function. The $G_n$ are themselves expressed as proper time integrals:
\[ G_n(x_1, x_2, x_3) = -\frac{1}{2\pi} \alpha \int_0^\infty ds \frac{e^{-is}}{s} g_n(-isx_1, -isx_2, -isx_3) \] (4.9)
where \( p = 1 \) for \( n = 0, \ldots, 3 \) and \( p = 2 \) for \( n = 4, \ldots, 9 \), and we have rescaled \( s \) by a factor of \( m^2 \) so as to be a dimensionless variable.

A crucial feature of this form of the effective action is that it is local, in the sense that the form factors \( \widetilde{G}_n \) have an expansion in positive powers of the \( D^2_{(i)} \). This depends on making the choice of basis operators above, in contrast to the original form quoted in ref. [33].

The values of \( G_n(0,0,0) \) for \( n = 0, \ldots, 3 \) reproduce the Drummond-Hathrell results. For these, we have:

\[
\begin{align*}
a &= -\frac{1}{2} \frac{\alpha}{\pi} g_1(0,0,0) & b &= -\frac{1}{2} \frac{\alpha}{\pi} g_2(0,0,0) = \frac{13}{360} \frac{\alpha}{\pi} \\
c &= -\frac{1}{2} \frac{\alpha}{\pi} g_3(0,0,0) & d &= -\frac{1}{2} \frac{\alpha}{\pi} g_0(0) = -\frac{1}{30} \frac{\alpha}{\pi}
\end{align*}
\]

Explicit analytic forms for all the form factors \( \widetilde{G}_n \) are known and are given in detail in ref. [18]. Here, we simply quote the expressions for the form factors relevant for Ricci flat spaces, \( g_3(x_1, x_2, x_3) \) and \( g_9(x_1, x_2, x_3) \). Moreover, since terms involving derivatives acting on the curvature are identified as higher order in \( \lambda^2 \) in the present context, we restrict to the special case \( x_1 = 0 \). Then, from ref. [18], we find:

\[
g_3(0, x_2, x_3) = -F(0, x_2, x_3) \frac{1}{\Delta} \left[ 4 + \frac{1}{4} (x_2 + x_3) (x_2 + x_3) \right] \\
+ f(x_2) \frac{1}{\Delta} \left[ \frac{3x_2 - x_3}{2x_2} + \frac{1}{4} (x_2 - x_3) \left( 1 - \frac{2(x_2 + x_3)^2}{\Delta} \right) \right] \\
+ f(x_3) \frac{1}{\Delta} \left[ \frac{3x_3 - x_2}{2x_3} + \frac{1}{4} (x_3 - x_2) \left( 1 - \frac{2(x_2 + x_3)^2}{\Delta} \right) \right] \\
+ \frac{1}{\Delta} \left[ -1 + \frac{1}{2} \left( \frac{x_3}{x_2} + \frac{x_2}{x_3} + x_2 + x_3 \right) \right] + \frac{1}{12} \left( \frac{1}{x_2} + \frac{1}{x_3} \right) + \frac{1}{4} \left( \frac{1}{x_2^2} + \frac{1}{x_3^2} \right)
\]

(4.11)

and

\[
g_9(0, x_2, x_3) = -F(0, x_2, x_3) \frac{1}{\Delta} \left[ 4 + x_2 + x_3 \right] \\
+ f(x_2) \left[ \frac{2}{\Delta^2} (x_2^2 - x_3^2) - \frac{1}{2x_2^2} - \frac{2}{x_2^3} \right] \\
+ f(x_3) \left[ -\frac{2}{\Delta^2} (x_2^2 - x_3^2) \right] + f'(x_2) \left( \frac{1}{x_2^2} + \frac{1}{2x_2} \right) - \frac{2}{\Delta} + \frac{1}{3x_2^2} + \frac{2}{x_2^3}
\]

(4.12)
where $\Delta = (x_2 - x_3)^2$. It can be checked that all the inverse powers of $x_2$ and $x_3$ cancel leaving a finite $x_2 = x_3 = 0$ limit, as required for the form factors to be local. Here,

$$f(x) = \int_0^1 d\alpha e^{-\alpha(1-\alpha)x} \quad (4.13)$$

and

$$F(x_1, x_2, x_3) = \int_{\alpha \geq 0} d^3\alpha \, \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \, e^{-\alpha_1 \alpha_2 x_2 - \alpha_2 \alpha_3 x_1 - \alpha_3 \alpha_1 x_2} \quad (4.14)$$

It will also be useful to note the simpler expression

$$F(0, x_2, x_3) = \frac{1}{x_2 - x_3} \int_0^1 d\alpha \, \frac{1}{\alpha} \left[ e^{-\alpha(1-\alpha)x_3} - e^{-\alpha(1-\alpha)x_2} \right] \quad (4.15)$$

The behaviour of these functions is illustrated in the following plots of $g_3(0, x_2, x_3)$ and $g_9(0, x_2, x_3)$. Notice that along the diagonals $x_2 = x_3$, both functions tend asymptotically to zero. However, if one argument is set to zero, then the functions may tend to a finite limit. The values at the origin are $g_3(0, 0, 0) = \frac{1}{180}$ and $g_9(0, 0, 0) = \frac{5}{504}$.

Fig.2 3D plots of $g_3(0, x_2, x_3)$ over different ranges.

Fig.3 Graphs of $g_3(0, x_2, x_3)$ along the $x_2$ or $x_3$ axes (left) and the diagonal $x_2 = x_3$ (right).
Fig. 4 3D plots of $g_9(0, x_2, x_3)$ over different ranges.

Fig. 5 Graphs of $g_9(0, x_2, x_3)$ along the $x_2$ axis (top left), the diagonal $x_2 = x_3$ (top right) and the $x_3$ axis (lower).
5. Geometric Optics with the New Effective Action

In this section, we apply the geometric optics methods introduced in section 3 to the effective action (4.7). In order to simplify the following discussion, we restrict here to the special case of Ricci flat spacetimes. This will be sufficient to extract the most important information, viz. the influence of the purely gravitational Weyl tensor contributions to high frequency photon propagation.

It is easiest to present our results by building up from simplified cases. So we consider first the modifications to the equation of motion \( \frac{\delta F}{\delta A} = 0 \) from the \( \tilde{G}_3 \) and \( \tilde{G}_9 \) terms neglecting their non-trivial derivative dependence. We then find:

\[
D_\mu F^{\mu\nu} + \frac{1}{m^2} \left[ -4G_3(0,0,0) R_{\mu\nu\lambda\rho} D^\mu F^{\lambda\rho} + G_9(0,0,0) D^2 (R_{\mu\nu\lambda\rho} D^\mu F^{\lambda\rho}) \right] = 0 \quad (5.1)
\]

Here, we have discarded all the \( O(\alpha) \) terms involving \( D_\mu F^{\mu\nu} \) for the reason explained in section 3, together with some terms involving derivatives of the curvature which are always suppressed. Notice that we are free to interchange covariant derivatives at will, since a commutator produces a further power of curvature and therefore a further \( O(\frac{\lambda^2}{L^2}) \) suppression. We have also used the Bianchi identities for the curvature and omitted terms involving the Ricci tensor. The important observation, however, is that the only effect on the equation of motion of the derivatives in the structure of the \( G_9 \) term in the effective action is to produce an extra \( D^2 \) in the form factor.

We now come to the implementation of the geometric optics approximation taking into account the non-trivial \( D^2 \) dependence in the form factors. Again, we illustrate this with a simple example. Consider a term

\[
\frac{1}{m^4} \int dx \sqrt{-g} \, R_{\mu\nu\lambda\rho} F^{\mu\nu} D^2 F^{\lambda\rho} \quad (5.2)
\]

in the effective action. This gives the following contribution to the equation of motion for \( D_\mu F^{\mu\nu} \):

\[
\frac{1}{m^4} \left[ -2(D_\mu R_{\mu\nu\lambda\rho}) D^\mu F^{\lambda\rho} - 4(D_{\sigma} R_{\mu\nu\lambda\rho}) D^\sigma D^\mu F^{\lambda\rho} - 4R_{\mu\nu\lambda\rho} D^2 D^\mu F^{\lambda\rho} \right] \quad (5.3)
\]

Compared with the basic \( G_3 \) term above, the relative orders of these terms are as follows. The first is suppressed by \( O(\frac{\lambda^2}{L^2}) \), since it involves extra derivatives of the curvature. The second involves an extra derivative of the curvature tensor but a compensating extra power of \( k \) from the associated derivative acting on the field strength, so overall this term is of

\[\text{Some early related analysis can be found in ref.}^{[5]}\]

\[\text{15}\]
relative order $O(\frac{\lambda^2}{\lambda L})$. In the low frequency region $\lambda > \lambda_c$ considered in \[1\] this would be neglected, but here we are interested in extending the results to $\lambda < \lambda_c$. At first glance, the third term appears to be dominant because of the extra two powers of $k$ coming from $D^2$ acting on $F^{\lambda\rho}$. However, this appears contracted as $k^2$ which is not large but rather zero at leading order, and $O(\alpha)$ in the full theory. So in fact this term must be discarded for consistency with the perturbative expansion in $\alpha$.

Returning to the second term, after substituting the geometric optics ansatz for $F^{\lambda\rho}$, we find that we recover the same structural form for the light cone condition but with a factor now involving $\frac{1}{m^2} k \cdot D R_{\mu\nu\lambda\rho}$. It is clear that after we generalise from (5.2) to the complete form factors, these terms will sum up and we will find the light cone modified by functions of $k \cdot D$ acting on the curvature.

First, consider the order of magnitude of these corrections, $O(\frac{\lambda^2}{\lambda L})$. This is interesting because it is closely related to the condition for direct observability of the Drummond-Hathrell effect. Consider differently polarised light propagating with a velocity difference of $O(\alpha^2 \frac{\lambda^2}{\lambda L})$, as predicted by eq.(1.2), over a time $L$ characteristic of the spacetime curvature. This produces the biggest length difference between the rays which can be realised in the spacetime. To be observable, this should (as a rough order of magnitude) be greater than the wavelength $\lambda$. We therefore arrive at the following criterion for direct observability:

$$\frac{\alpha \lambda^2}{L^2} L > \lambda \quad (5.4)$$

that is

$$\alpha \frac{\lambda^2}{\lambda L} > 1 \quad (5.5)$$

We see that to access the frequency range where superluminal effects could in principle be observable, we need to satisfy the criterion $\frac{\lambda^2}{\lambda L} \gg 1$. But since, as we have just shown, this is the parameter governing the corrections to the light cone from the new form factor terms in the effective action, we find that these terms cannot be neglected as perturbatively small but must instead be summed to all orders.

Implementing this strategy, it can now be shown that the final light cone condition following from the equations of motion derived from (4.7) is:

$$k^2 + \frac{2}{m^2} \left[ 4 G_3 \left( 0, \frac{2ik \cdot D}{m^2}, 0 \right) - 2i k \cdot D G_9 \left( 0, \frac{2ik \cdot D}{m^2}, 0 \right) \right] R_{\mu\nu\lambda\rho} k^\mu k^\lambda a'^\nu a'^\rho = 0 \quad (5.6)$$

In the derivation, we have used the symmetry of $G_3(x_1, x_2, x_3)$ under $x_2 \leftrightarrow x_3$ (not present in $G_9$ of course), and omitted the $O(\frac{\lambda^2}{\lambda L})$ suppressed terms from the operator $D^2_{(1)}$ acting on $R_{\mu\nu\lambda\rho}$. 

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Eq. (5.6) is therefore the generalisation of the light cone (1.2), in the Ricci flat case, where we extend the effective action to include the non-trivial form factor operators $\overrightarrow{G}_n$. In the light cone condition, these form factors reduce to single-variable functions of the operator $k \cdot D$ acting on the curvature tensor $R_{\mu\nu\lambda\rho}$. Although the effective parameter $\frac{1}{m^2}k \cdot D$ is $O(\frac{\lambda^2}{\chi^2})$ and therefore not small in the region of interest, knowledge of the analytic expressions for the form factors should now enable these corrections to be exactly summed. We claim that this is the correct treatment of the equation of motion with extra derivatives in the perturbative corrections in a self-consistent geometric optics expansion.

In the next section, we study the condition (5.6) in more detail and discuss what we can learn from it about the high frequency behaviour of the phase velocity $v_{ph}$. Before that, we make a few comments on the relation of our result to the analysis of Khriplovich [31]. In a very interesting contribution to the debate on dispersion and the Drummond-Hathrell mechanism, Khriplovich considered the general structure of the three-particle vertex for the scattering of an on-shell photon by a graviton. The 3 possible Lorentz structures are shown to be equivalent to the 3 terms in the Drummond-Hathrell action, acted on by form factors which are functions of the invariant momenta for the photon and graviton legs. The essential observation of Khriplovich is that for on-shell (i.e. $k^2 = 0$) photons, the form factors reduce to constants, given by the coefficients $a, b, c$ of section 3. The conclusion is then that there is no dispersion: the light cone shift at $\omega = 0$ is unchanged for all frequencies.

Clearly, this differs from our conclusions here. The reason is not entirely clear. However, the argument in [31] for the triviality of the form factors in the vertex looks remarkably similar to the discussion below (5.3). In our case, we also had form factors which were functions of $D^2$ acting on the electromagnetic field, and applying geometric optics these at first sight would appear to give contributions only of $O(k^2)$ which would vanish. However, the more complete analysis showed that they nevertheless can give rise to contributions involving $k \cdot D R$ which do not vanish. The most likely explanation for the discrepancy between our results and ref. [31] is probably the omission of the analogous terms in the analysis of the photon-graviton scattering vertices.

6. High Frequency Photon Propagation

The result of the previous section is the modified light cone formula for Ricci flat spacetimes:

$$k^2 + \frac{2}{m^2} G(\frac{2k \cdot D}{m^2}) C_{\mu\nu\lambda\rho}k^\mu k^\nu a^\lambda a^\rho = 0 \quad (6.1)$$
where $G(x)$ is the known function:

$$G(x) = -\frac{1}{2\pi} \int_0^\infty \frac{ds}{s} e^{-is} g(sx)$$

(6.2)

where

$$g(x) = 4g_3(0, x, 0) + x g_9(0, x, 0)$$

(6.3)

In terms of the functions $f(x)$ and $F(x_1, x_2, x_3)$ defined in the last section, we can write the explicit form for $g(x)$ as:

$$g(x) = -\frac{16}{x^2} F(0, x, 0) + \frac{3}{x^2} f(x) + \left(\frac{1}{2} + \frac{1}{x}\right)f'(x) + \frac{5}{x^2} + \frac{1}{15}$$

(6.4)

This is plotted below. Analytically, we can show $g(0) = \frac{2}{90}$, ensuring agreement with the Drummond-Hathrell coefficient $c$, while numerically we find that $g(x)$ approaches an asymptotic value of 0.067 as $x \to \infty$.

![Plot of the function $g(x)$ which enters the formula for the modified light cone.](image)

With this explicit knowledge of the form factor, in principle we have control over the high frequency limit of the light cone and the phase velocity. However, the next difficulty is that in eq. (6.1), $G$ is a function of the operator $k \cdot D$ acting on the Weyl tensor. If $C_{\mu\nu\lambda\rho}$ is an eigenfunction then $G$ will reduce to a simple function of the eigenvalue and then Fig. 6 determines its asymptotic behaviour. However, in general this not true and consequently the interpretation of (6.1) is far from obvious.

At this point, the problem is reduced to differential geometry. The encouraging feature of (6.1) is that the operator $k \cdot D$ simply describes the variation along a null geodesic. (Notice that because the second term is already $O(\alpha)$, we can freely use the usual Maxwell relations for the quantities occurring there, e.g. the results $k \cdot Dk^\nu = 0$ and $k \cdot Da^\nu = 0$
derived in section 3.) The question then becomes what is known about the derivative of the Weyl tensor along a null geodesic.

As in our previous work, it is convenient to re-cast the light cone condition in Newman-Penrose formalism (see, e.g. refs. [35], [36] for reviews). This involves introducing a null tetrad \( e_a^\mu \) based on a set of complex null vectors \( (\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu) \). The components of the Weyl tensor in this basis are denoted by five complex scalars \( \Psi_0, \ldots, \Psi_4 \). For example, \( \Psi_0 = -C_{abcd} \ell^a m^b \ell^c m^d \) and \( \Psi_4 = -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d \).

Consider propagation along the null direction \( \ell^\mu \), i.e. choose \( k^\mu = \ell^\mu \). The two spacelike polarisation vectors \( a^\mu \) and \( b^\mu \) are related to the \( m^\mu \) and \( \bar{m}^\mu \) null vectors by
\[
\begin{align*}
    m^\mu &= \frac{1}{\sqrt{2}} (a^\mu + ib^\mu), \\
    \bar{m}^\mu &= \frac{1}{\sqrt{2}} (a^\mu - ib^\mu).
\end{align*}
\]
In this case, the light cone condition can be simply written as
\[
k^2 + \frac{\omega^2}{m^2} G \left( \frac{2\omega}{m^2} \ell^\mu D_\mu \right) (\Psi_0 + \Psi_0^*) = 0 \quad (6.5)
\]
We have searched the relativity literature for theorems on the behaviour of \( \ell^\mu D_\mu \Psi_0 \) without finding any results of general validity, although it seems plausible to us that some general properties may exist. (Notice, for example, that naturally enough this is not one of the combinations that are constrained by the Bianchi identities in Newman-Penrose form, as displayed for example in ref. [36], ch.1, eq.(321).) To try to build some intuition, we have therefore looked at particular cases. The most interesting is the example of photon propagation in the Bondi-Sachs metric [37,38] which we recently studied in detail [4].

The Bondi-Sachs metric describes the gravitational radiation from an isolated source. The metric is
\[
ds^2 = -W du^2 - 2e^{2\beta} du dr + r^2 h_{ij} (dx^i - U^i du)(dx^j - U^j du) \quad (6.6)
\]
where
\[
h_{ij} dx^i dx^j = \frac{1}{2} (e^{2\gamma} + e^{2\delta}) d\theta^2 + 2 \sinh(\gamma - \delta) \sin \theta d\theta d\phi + \frac{1}{2} (e^{-2\gamma} + e^{-2\delta}) \sin^2 \theta d\phi^2 \quad (6.7)
\]
The metric is valid in the vicinity of future null infinity \( I^+ \). The family of hypersurfaces \( u = \text{const} \) are null, i.e. \( g^{\mu\nu} \partial_\mu u \partial_\nu u = 0 \). Their normal vector \( \ell_\mu \) satisfies
\[
\ell_\mu = \partial_\mu u \quad \Rightarrow \quad \ell^2 = 0, \quad \ell^\mu D_\mu \ell^\nu = 0 \quad (6.8)
\]
The curves with tangent vector \( \ell^\mu \) are therefore null geodesics; the coordinate \( r \) is a radial parameter along these rays and is identified as the luminosity distance.

The six independent functions \( W, \beta, \gamma, \delta, U^i \) characterising the metric have expansions in \( \frac{1}{r} \) in the asymptotic region near \( I^+ \), the coefficients of which describe the various features of the gravitational radiation. (See [4] for a brief summary.)
In the low frequency limit, the light cone is given by the simple formula (1.2). The velocity shift is quite different for the case of outgoing and incoming photons [4]. For outgoing photons, \( k^\mu = \ell^\mu \), and the light cone is

\[
    k^2 = \pm \frac{4\omega^2}{m^2} \left( \Psi_0 + \Psi_0^* \right) \sim O\left( \frac{1}{r^5} \right) \tag{6.9}
\]

while for incoming photons, \( k^\mu = n^\mu \),

\[
    k^2 = \pm \frac{4\omega^2}{m^2} \left( \Psi_4 + \Psi_4^* \right) \sim O\left( \frac{1}{r} \right) \tag{6.10}
\]

Now, it is a special feature of the Bondi-Sachs spacetime that the absolute derivatives of each of the Weyl scalars \( \Psi_0, \ldots, \Psi_4 \) along the ray direction \( \ell^\mu \) vanishes, i.e. \( \Psi_0, \ldots, \Psi_4 \) are parallel transported along the rays [38,35]. In this case, therefore, we have in particular:

\[
    \ell \cdot D \Psi_0 = 0 \quad \ell \cdot D \Psi_4 = 0 \tag{6.11}
\]

but there is no equivalent simple result for either \( n \cdot D \Psi_4 \) or \( n \cdot D \Psi_0 \).

These results can be easily checked for the simpler related example of a weak-field plane gravitational wave [1]. In this case, the first identity is trivial since \( \Psi_0 = 0 \), but in particular we can confirm that \( n \cdot D \Psi_4 \neq 0 \) and \( \Psi_4 \) is not an eigenfunction. The important result \( \ell \cdot D \Psi_0 = 0 \) therefore appears to be a very special property of the Bondi-Sachs metric and not an example of a general theorem on derivatives of the Weyl tensor.

Although it seems to be a special case, (6.11) is nevertheless important and leads to a remarkable conclusion. The full light cone condition (6.5) applied to outgoing photons in the Bondi-Sachs spacetime now reduces to

\[
    k^2 + \frac{\omega^2}{m^2} G(0) \left( \Psi_0 + \Psi_0^* \right) = 0 \tag{6.12}
\]

since \( \ell \cdot D \Psi_0 = 0 \). In other words, the low frequency Drummond-Hathrell prediction of a superluminal phase velocity \( v_{ph}(0) \) is exact for all frequencies. There is no dispersion, and the wavefront velocity \( v_{ph}(\infty) \) is indeed superluminal.

This is potentially a very strong result. According to the analysis presented here, we have found at least one example in which the wavefront truly propagates with superluminal velocity. Quantum effects have indeed shifted the light cone into the geometric spacelike region, with all the implications that brings for causality.
7. Propagation in Background Magnetic Fields

Before accepting the results of the last section as definitive, however, it is instructive to compare what we have done in this paper for the case of a background gravitational field to previously known results on the propagation of light in a background magnetic field.

The refractive index for photons moving transverse to a homogeneous magnetic field $B$ has been calculated explicitly as a function of frequency in two papers by Tsai and Erber \[39,40\]. They derive an effective action

$$\Gamma = -\int dx \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} A^\mu M_{\mu\nu} A^\nu \right]$$

(7.1)

where $M_{\mu\nu}$ is a differential operator acting on the electromagnetic field $A^\nu$. Writing the corresponding equation of motion and making the geometric optics ansatz described in section 3, we find the light cone condition

$$k^2 - M_{\mu\nu}(k) a^\mu a^\nu = 0$$

(7.2)

In terms of the refractive index $n(\omega) = c|k|/\omega = cv_{\text{ph}}(\omega)^{-1}$, this implies

$$n(\omega) = 1 - \frac{1}{2\omega^2} M_{\mu\nu}(k) a^\mu a^\nu$$

(7.3)

Denoting $M_{\mu\nu}a^\mu a^\nu$ by $M_{||}, M_{\perp}$ for the two polarisations, the complete expression for the birefringent refractive index is

$$n_{||,\perp}(\omega) = 1 - \frac{1}{2\omega^2} M_{||,\perp}$$

$$= 1 - \frac{\alpha}{4\pi} \left( \frac{eB}{m^2} \right)^2 \int_{-1}^{1} du \int_{0}^{\infty} ds \ N_{||,\perp}(u, z) e^{-is\left(1+s^2\Omega^2 P(u,z)\right)}$$

(7.4)

where $z = \frac{eB}{m^2}s$ and $\Omega = \frac{eB}{m^2} \frac{\omega}{m}$. The functions $N$ and $P$ are given by:

$$P(u, z) = \frac{1}{z^2} \left( \frac{\cos zu - \cos z}{2z \sin z} - \frac{1 - u^2}{4} \right) = \frac{1}{12} (1 - u^2) + O(z^2)$$

(7.5)

---

6 The extra power of $(is)$ in the prefactor relative to eq.23 arises because the lowest order terms in the effective action contributing to (7.1) are of 4th order in the background fields, i.e. $O(F^4)$, compared to the 3rd order terms of $O(RFF)$ in the gravitational case. For background magnetic fields, the analogue of the Drummond-Hathrell low frequency action is the familiar Euler-Heisenberg action (see, e.g. ref.[3]).
and
\[ \begin{align*}
N_\parallel &= -\frac{\cot z}{2z} \left( 1 - u^2 + \frac{u \sin zu}{\sin z} \right) + \frac{\cos zu}{2z \sin z} = \frac{1}{4}(1 - u^2)(1 - \frac{1}{3}u^2) + O(z^2) \\
N_\perp &= -\frac{z \cos zu}{2 \sin z} + \frac{zu \cot z \sin zu}{2 \sin z} + \frac{z(\cos zu - \cos z)}{\sin^2 z} = \frac{1}{8}(1 - u^2)(1 + \frac{1}{3}u^2) + O(z^2)
\end{align*} \] (6.9)

In the weak field, low frequency limit, we can disregard the function \( P \) and consider only the lowest term in the expansion of \( N \) in powers of \( z \). This reproduces the well-known results (see also [41,2]):

\[ n_\parallel, \perp \sim \omega \to 0 \quad 1 + \frac{\alpha}{4\pi} \left( \frac{eB}{m} \right)^2 \left[ \frac{14}{45}_\parallel, \frac{8}{45}_\perp \right] \] (7.7)

The weak field, high frequency limit is analysed in ref.[40]. It is shown that

\[ n_\parallel, \perp \sim \omega \to \infty \quad 1 - \frac{\alpha}{4\pi} \left( \frac{eB}{m} \right)^2 \left[ c_\parallel, c_\perp \right] \Omega^{-\frac{4}{3}} \] (7.8)

where the numerical constants are \( [c_\parallel, c_\perp] = \left[ \frac{3^{\frac{3}{4}}}{\pi} \sqrt{\frac{\pi}{\Gamma\left(\frac{2}{3}\right)}}, \Gamma\left(\frac{1}{3}\right)^{-1} \right] [3_\parallel, 2_\perp] \).

The complete function \( n(\omega) \) is sketched in Fig. 7. It shows exactly the features found in the simple absorption model described in section 2.

![Fig. 7](image)

Fig. 7 Sketch of the frequency dependence of the refractive index \( n(\omega) \) for light propagating in a background magnetic field. The crossover point is at \( \Omega \sim 1 \).

In particular, the phase velocity \( v_{ph}(\omega) \) begins less than 1 at low frequencies, showing birefringence but conventional subluminal behaviour. In the high frequency limit, however, the phase velocity approaches \( c \) from the superluminal side with a \( \omega^{-\frac{4}{3}} \) behaviour.
All this is by now standard and in line with our expectations. What we are interested in here, however, is the comparison between this calculation of $v_{\text{ph}}(\omega)$ via the effective action \((7.1)\) and the gravitational calculation of sections 5 and 6, in particular the origin of the non-analytic high frequency behaviour \((7.8)\).

The important observation is that the exponential term involving $\Omega^2 P$ is crucial in obtaining the high frequency limit. However, if we made a literal expansion of the effective action in powers of the magnetic field \((eB/m^2)\), this term would be regarded as higher order and discarded. It is important however, because it involves the product of the field and the frequency, \((eB\omega/m^3)\), and this is not small in the interesting high frequency region. The exponent in the effective action is vital in describing the high frequency propagation.

Now compare with our construction of the effective action in a background gravitational field. The action \((4.7)\) is obtained by expanding in the curvature and keeping only terms of $O(R)$. This appears to be analogous to the expansion of $N$ in \((7.4)\). We have found a non-trivial extension of the zero-frequency Drummond-Hathrell result by taking into account derivatives of the curvature. These are essential in the gravitational case because there is no change in the light cone for a constant curvature metric (since this is totally isotropic), whereas there is an interesting birefringent effect even for a constant magnetic field. By restricting the effective action rigorously to terms of first order in the curvature, however, we seem to have missed the analogue of the exponent terms in \((7.4)\), which would be characterised by the not necessarily small parameter \((R\omega/m^3)\).

The conclusion of this comparison is therefore discouraging. It seems that despite its complexity, the effective action \((4.7)\) may still not be sufficiently general to encode the high frequency behaviour of photon propagation.

8. Conclusions

In this final section, we attempt to synthesise what has been learnt from this investigation and identify what further work is required for a complete resolution of the dispersion problem for the propagation of photons in gravitational fields.

In view of our experience with the background magnetic field problem, it is natural to assume that the full effective action governing propagation in a background gravitational field takes an analogous form and the modified light cone can be written heuristically as

\[
k^2 + \frac{\alpha}{\pi} \int_0^\infty \frac{ds}{s} (is) \mathcal{N}(s, R) e^{-is(1+s^2\Omega^2 P(s, R))} = 0 \quad (8.1)
\]

where both $\mathcal{N}$ and $P$ can be expanded in powers of curvature, and derivatives of curvature,
with appropriate powers of $s$. The frequency dependent factor $\Omega$ would be $\Omega \sim \frac{R}{m^2} \frac{\omega}{m} \sim O(\frac{\lambda^3}{\chi L^2})$, where ‘$R$’ denotes some generic curvature component. If this is true, then an expansion of the effective action to first order in $O(\frac{R}{m^2})$ would not be sensitive to the $\mathcal{P}$ term in the exponent. The Drummond-Hathrell action would correspond to the leading order term in the expansion of $\mathcal{N}(s, R)$ in powers of $\frac{R}{m^2}$ neglecting derivatives, while our improved effective action sums up all orders in derivatives while retaining the restriction to leading order in curvature.

The omission of the $\mathcal{P}$ term would be justified only in the limit of small $\Omega$, i.e. for $\frac{\lambda^3}{\chi L^2} \ll 1$. Neglecting this therefore prevents us from accessing the genuinely high frequency limit $\lambda \to 0$ needed to find the asymptotic limit $v_{ph}(\infty)$ of the phase velocity. If eq.(8.1) is indeed on the right lines, it also looks inevitable that for high frequencies (large $\Omega$) the rapid variation in the exponent will drive the entire heat kernel integral to zero, restoring the usual light cone $k^2 = 0$ as $\omega \to \infty$. However, this is simply to assert that in this respect the gravitational problem behaves in the same way as the magnetic field background, and while this seems plausible in relation to the $\omega \to \infty$ limit we should be cautious: systematic cancellations or special identities (c.f. section 6) could change the picture, and we should remember that it was not foreseen that the low frequency behaviour of $v_{ph}(0)$ would differ so radically from other background field calculations as to produce superluminal velocities. So although the available evidence seems to point strongly towards a restoration of the usual light cone in the high frequency limit, this inference should be made with some caution.

Unfortunately, the quantum field theoretic calculation required to settle the issue by evaluating the ‘$\mathcal{P}$’ type correction to the exponent in (8.1) looks difficult in general, at least comparable to the evaluation of the effective action ‘$\mathcal{N}$’ terms in section 4. However, if we are only interested in the $\omega \to \infty$ limit, it may be sufficient just to perform some leading order type of calculation to establish the essential $(1 + s^2\Omega^2\mathcal{P})$ structure of the correction. Another possible simplification, which would be interesting in its own right, would be to reformulate the effective action calculation from the outset in Newman-Penrose form, thereby removing the plethora of indices arising in calculations involving the curvature tensors. Simple cases, such as black holes, in which only one Weyl scalar is non-vanishing might prove particularly tractable.

However, even if this picture is correct and the light cone is eventually driven back to $k^2 = 0$ in the high frequency limit, the analysis in this paper still represents an important extension of the domain of validity of the superluminal velocity prediction of Drummond and Hathrell. Recall from section 5 that the constraint on the frequency for which the superluminal effect is in principle observable is $\frac{\lambda^2}{\chi L} \gg 1$. Obviously this was not satisfied by the original $\omega \sim 0$ derivation. However, our extension based on the ‘all orders in
derivatives’ effective action does satisfy this constraint. Combining with the restriction \( \frac{\lambda^3}{\lambda L^2} \ll 1 \) in which the neglect of the \( \mathcal{P} \) type corrections is justified, we see that there is a frequency range

\[
\frac{\lambda_c}{L} \gg \frac{\lambda}{\lambda_c} \gg \frac{\lambda_c^2}{L^2}
\]

for which our expression (6.1)

\[
k^2 + \frac{2}{m^2} \frac{k \cdot D}{m^2} C_{\mu
u\lambda\rho} k^\mu k^\lambda a^\nu a^\rho = 0
\]

for the modified light cone is valid and predicts observable effects.

Since this formula allows superluminal corrections to the light cone, we conclude that superluminal propagation has indeed been established as an observable phenomenon even if, as seems likely, causality turns out to respected through the restoration of the standard light cone \( k^2 = 0 \) in the asymptotic high frequency limit.

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