New Conditions of The Existence of Fixed Point in $\Delta$ – Ordered Banach Algebra

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Abstract

The main idea is to construct a new algebra and find new necessary and sufficient conditions equivalent to the existence of fixed point. In this work, an algebra is constructed, called $\Delta$- ordered Banach algebra, we define convergent in this new space, Topological structure on $\Delta$ – ordered Banach Algebra and prove this as Housdorff space. Also, we define new conditions as $\Delta$ – lipschitz , $\Delta$ – contraction conditions in this algebra construct, we prove this condition is the existence and uniqueness results of the fixed point. In this paper, we prove a common fixed point if the self-functions satisfy the new condition which is called $\phi$ – contraction.

Keywords: Fixed point, Ordered Banach algebra, lipschitz mapping, and contraction mapping

Introduction

It is known Banach contraction principle and a number of generality in background of metric spaces play a fundamental role for several complications of functional analysis, differential and integral equations.

Gahler (1963) [6] presented the notion of 2-metric spaces as a generalization of an usual metric space. Gahler proved that geometrically $d(a,b,c)$ represents the region of a triangle formed by the point $a,b,c \in \mathbb{R}$ equally its vertices.

An usual metric space is a continuous function, but Ha, Cho and While (1988) [15] examined that a 2- metric space is not a continuous mapping. Dhage (1984) [5] introduced the notion of a $D$- metric space as a generality of a 2- metric space; and studied the topological properties of $D$- metric space

Mustafa and Sim (2006) [17] introduced a newfangled metric called $G$-metric space. They show the topological constructions of Dhages [4] work unacceptable, after Sedghi, Shobe and Zhom (2007) [20] presented concept, which is named $D^*$- metric space, but Fernardcz, Sle, Saxena, Malviya and Kuman (2017)[13] generalized an $S$-metric space to $\Delta$-metric space.

Many researchers have their consideration to generalizing metric (see Yan and Shao Yuan on (2011) [25], Sastry, Srinivas, Chandra and Balaiah (2011) [14], Kim and Soo (2012)[20], Dey and Saha (2013) [4], Liu and Xu (2013) [8] introduced some concepts of a cone metric space over Banach algebra. Some researchers then developed many concepts as, Nashine and Altun, (2012)[9], Tiwari and Dubey (2013) [22], Arun and Zaheer (2014) [3]. But Nashine and Altun (2012) [10] defined cone metric spaces and proved some fixed point theorems of contractive maps in such a space using the normality condition. Also, Rahimi & Soleimani (2014) [12] used the notion ordered cone metric space. But some scholars have attention about fixed point theorem such as Badshah, Bhagatand and Shukla(2016)[23] how introduced some fixed point theorem for $\alpha-\phi$- metric mapping in 2- metric spaces and Ma, Jiang and Hongkaisun (2014) [24] state fixed point theorem on $C^*$-algebra valued metric spaces.

The point $x$- that satisfies the equation $x = T(x)$ is called a fixed point of the function $T$ which is considered the root of the equation above. To find this root, we first find an initial holding value of $x_0$. Then, we calculate the value of the function $T$ in $x_0$ to get another root called $x_1$ that is $x_1 = T(x_0)$; and then repeat the process can get
a new approximate value \( x_2 = T (x_1) \). Thus, a sequence of root values can be generated by applying the formula \( x_{n + 1} = T (x_n) \) for \( n = 0, 1, 2, \ldots \).

The fixed point \( a \) in the equation above represents the distance of the intersection point of the curves of \( y = x \), \( y = T (x) \) for each axis \( x \), \( y \). If \( x_0 \) is the initial fixed point, then \( T(x_0) \) is the length of the column from \( x_0 \) on the \( x \) axis until it intersects the curve of the \( T \)-function and since the points on the rectangle \( y = x \) are equal to the distance from both axes \( y \) and \( x \), so the line passing at the point \((x_0, T(x_0))\) rectangle the \( x \)-axis will intersect the line \( y=x \) in the \( x \)-axis, represent \( x_1 \) where

\[ x_1 = T(x_0) \]

In a similar way, we find the remaining points where \( x_{n + 1} = T (x_n) \). Here, we ask the following question: How do we choose the function \( T \) to ensure that the generated values are converged from the repeated formula \( x_{n + 1} = T (x_n) \)?

To answer the question, we can prove the existence and uniqueness of fixed point under some new conditions by constructing a new algebra called \( \Delta \)-ordered Banach algebra.

### 1- \( \Delta \)-Ordered Banach Algebra

We start this section by a definition of Banach algebra.

**Definition (2.1)**[2]: Let \( E \) is a linear space over field of real numbers. \( E \) is called Banach algebra if \( E \) is Banach space with an operation of multiplication is defined as following: for \( x, y, z \in A \), for all \( \alpha \in \mathbb{R} \)

1) \((xy)z = x(yz)\)
2) \(x(y + z) = xy + xz \) and \((x + y)z = xz + yz\)
3) \(\alpha(xy) = (\alpha x)y = x(\alpha y)\)
4) \(\| xy \| \leq \| x \| \| y \|\)

We consider a Banach algebra has an identity, that is \( ex = xe \) for all \( xeE \). (Multiplicative identity)

If there is an element \( yeA \) such that \( yx = e \), \( yeE \) is called inverse of \( x \) and denoted by \( x^{-1} \).

**Proposition 2.2** [19]: Let \( E \) be Banach algebra has a unite \( e \), \( xeE \). If the condition spectral radius \( \sigma_e(x) < 1 \) (for all \( \varepsilon > 0 \)), then

\[
(e - x)^{-1} = \sum_{i=0}^{\infty} a^i
\]

**Remark 2.3** [19]: Let \( E \) be Banach algebra with spectral radius \( \sigma_e(x) \) of \( x \) satisfy \( \sigma_e(x) \leq ||x||.\)

**Remark 2.4** [2]: If \( \sigma_e(x) < 1 \), then \( \| x^n \| \rightarrow 0 \) as \( n \rightarrow \infty \).

**Lemma 2.5** [2]: If \( E \) is a real Banach algebra with cone \( C \) and if \( 0 \leq u \leq c \) for each \( 0 \leq c \), therefore \( u = 0 \).

**Lemma 2.6** [2]: Let \( C \) be a cone and \( a \leq b + c \) for \( ccC \), then \( a \leq b \).

A sub set \( C \) of \( E \) is called a algebra cone of \( E \) if

1) \( C \) non-empty closed and \( \{ o, e \} \subset C \)
2) $aa + \beta b \in C$ for all $a, \beta > 0$

3) $x, y \in C$

4) $C \cap (-C) = \{0\}$.

"We can define a preference ordering $\leq$ with respect to $C$ by $x \leq y$ ifff $x \in C$. $x < y$ with stand for $x \leq y$ and $x \neq y$ the cone $C$ is called normal if there exist $N > 0$ such that, for all $x, y \in E$

$0 \leq x \leq y \implies \| x \| \leq N \| y \|$.

Now, we define a new construction called $\Delta$- ordered Banach algebra.

**Definition 2.7:** Let $X$ be a non-empty. A function $\Delta: [0, \infty) \times X \times X \to E$ is called an $\Delta$- metric on $X$ if

1) $\Delta(\lambda, x, y) \geq 0$ for $x, y \in X, \lambda \geq 0$

2) $x = y$ if and only if $\Delta(\lambda, x, y) = 0$

3) $\Delta(\lambda, x, y) = \Delta(\lambda, y, x)$

4) $\Delta(\lambda, x, y) \leq \Delta(\mu, y, a)$ for $\mu > \lambda > 0$ and $x, y, a \in X$

5) $\Delta(\lambda + \mu, x, y) \leq \Delta(\lambda, x, y) + \Delta(\mu, y, a)$

The triple $(X, E, \Delta)$ is called $\Delta$- ordered Banach algebra.

**Example 2.8:** Let $X$ be locally compact Hausdorff space, $C(X) = \{ f: X \to R, \text{continuous function} \}$, and $C^*(X) = \{ f \in C(X): f(x) \geq 0 \ for \ all \ x \in X \}$, define multiplication in the natural way. Therefore $C(X)$ with superstition norm is ordered Banach algebra. It is obvious that $(C(X), X, \Delta)$ is $\Delta$- ordered Banach algebra where

$\Delta: [0, \infty) \times X \times X \to C(X)$ by $\Delta(\lambda, a, b) = \sup | f(a) - f(b) | e^{\lambda}$

2- **Topological structure on $\Delta$- ordered Banach Algebra**

**Definition 3.1:** Let $(X, E, C)$ be $\Delta$- ordered Banach algebra. For all $xeX$, for all $c > 0$, the set $B_\Delta(\lambda, x, c) = \{ y \in X: \Delta(\lambda, x, y) < c \}$ is called $\Delta$-ball with and radius $c > 0$ and admits $x$.

And put $\beta = \{ B_\Delta(\lambda, x, c): x \in X, and c > 0 \}$.

**Theorem 3.2:** Let $(E, C)$ be ordered Banach algebra, then $(X, E, \Delta)$ is a Hausdorff space.

**Proof:** Let $(E, X, \Delta)$ be a $\Delta$- ordered Banach algebra. Let $x, y \in X$ with $x \neq y$, $\lambda, \mu \geq 0$, we take $c = \Delta(\lambda + \mu, x, y) \cup B(\lambda, x, c), V = B(\mu, y, c)$.

Then $xeU$ and $y \in V$. We support $U \cap V \neq \emptyset$. There exist $a \in U \cap V$.

But $\Delta(\lambda + \mu, x, a) \leq \Delta(\lambda, x, a) + \Delta(\mu, y, a) \leq \frac{c}{2} + \frac{c}{2} = c$.

That is $c < c$ and this contradiction.

Then, $(X, E, \Delta)$ is a Hausdorff space.
Definition 3.3: Let \((X,E,\Delta)\) be a \(\Delta\)–ordered Banach algebra. A sequence \(\{x_n\}\) in \((X, \Delta)\) converges to a point \(x\) if for every \(c \in E\) with \(c > 0\), there exist a positive integer \(N_0\) such that \(\Delta(\lambda, x_n, x) < c\) for \(n \geq N_0\). We denote by \(\lim_{n \to \infty} x_n = x\) if \(x_n \to x\) as \(n \to \infty\).

Definition 3.4: Let \((C,E,\Delta)\) be a \(\Delta\)–ordered Banach algebra. A sequence \(\{x_n\}\) is said to be Cauchy sequence if for each \(c > 0\) there exists a positive integer \(N_0\) such that \(\Delta(\lambda, x_n, x_m) < c\) for all \(n, m \geq N_0\).

Examples 3.5: Let \((X,C(X), \Delta)\) a \(\Delta\)–ordered Banach algebra in example (3.2), take the set of rational numbers \(\mathbb{Q}\).

Define \(\Delta = \{0, \infty\} \times X \times X \to C(X)\) is in example. Let \(\{x_t\}\) be a sequence defined by \(\alpha_t = (1 + \frac{1}{t})^t\). We note that \(x_t \in \mathbb{Q}\) for each \(t \in \mathbb{Z}\), note that \(\Delta(\lambda, x_t, x_k) = \left| f(x_t) - f(x_k) \right| e^{-\lambda}\)

\[= \left| \left(1 + \frac{1}{t}\right)^t - (1 + \frac{1}{k})^k \right| e^\lambda \quad \text{as} \quad t, k \to \infty\]

\(\Delta(\lambda, x_t, x_k) \to 0\)

That is for each \(c > \theta\), there is \(N_0 \in \mathbb{Z}^+\) such that \(\Delta(\lambda, a_t, a_k) < c\) for all \(t, k \geq N_0\).

Thus, \(\{a_t\}\) is a Cauchy sequence, but \(a_t \to e\) as \(t \to \infty\), \(e \notin \mathbb{Q}\). Hence, \(\{a_t\}\) is not convergent.

Definition 3.7: Let \((X,E,\Delta)\) and \((X', E', \Delta')\) are \(\Delta\)– ordered Banach algebra. A mapping \(f: X \to X'\) is said to be continuous at \(x \in X\) when ever \(\{x_n\}\) convergent to \(x\), then \(\{f(x_n)\}\) is convergent to \(f(x)\).

Definition 3.8: Let \((X,E,\Delta)\) be \(\Delta\)–ordered Banach algebra, \((X,E,\Delta)\) is called complete if for each Cauchy sequence is convergent in \(X\).

Definition 3.9: Let \((X,E,\Delta)\) be \(\Delta\)–ordered Banach algebra. A map \(T: X \to X\) is called Lipchitz if for all \(c > 0\), there exist a vector \(N \in \mathbb{C}\) with \(\sigma_c(N) < 1\) for each \(x, y \in X\),

\(\Delta(\mu, T_x, T_y) \leq N. \Delta(\mu, a, b)\) for all \(x, y \in X\) and \(\lambda \leq \mu\)

Example 3.10: Let \((\{0, \infty\}, \mathbb{C}(X), \Delta)\) be a \(\Delta\)– ordered Banach algebra. Define \(T: X \to X\) as follows \(T(a) = \frac{a}{2}\)

\(\Delta(\mu, T_a, T_b) = \sup \left| f(T_a) - f(T_b) \right| e^{\lambda}\)

\(= \sup \left| f \circ T(a) - f \circ T(b) \right| e^{\lambda} = \sup \left| f \left(\frac{a}{2}\right) - f \left(\frac{b}{2}\right) \right| e^{\lambda}\)

\(= \frac{1}{2} \sup \left| f(a) - f(b) \right| e^{\lambda}\)

\(\Delta(\mu, a, b) = \sup \left| f(a) - f(b) \right| e^{\mu}\)

That is \(T\) is a Lipschitz map in \(X\)

Definition 3.11: Let \((X,E,\Delta)\) be \(\Delta\)–ordered Banach algebra. A sequence \(\{x_t\}\) is said to be \(m\)– sequence if for all \(m > 0\), there exists \(t \in X_t\) such that \(x_t < m\) for all \(n \geq t\).

Lemma 3.12: Let \((X,E,\Delta)\) be \(\Delta\)– ordered Banach algebra. \(\{m x_t\}\) is a \(m\)– sequence for all \(c > 0\) if the sequence \(\{x_t\}\) is a \(m\)– sequence in \(C\).

Proof: Suppose \(\{x_t\}\) is a \(m\)– sequence for all \(c > 0\), there exists \(t \in \mathbb{E}^+\) such that \(x_t < c\) for \(n > t\). For all \(c > 0\), \(m x_t \leq mc\) by take \(\frac{c}{m} = t\).
3- Main Results

**Definition 4.1:** Let \((X, E, \Delta)\) be a \(\Delta\)-ordered Banach algebra. \(T : X \to X\) holds the contradiction condition if
\[
\Delta(\lambda, T \xi, T \eta) \leq t^n \Delta(\frac{\lambda}{2^n}, x_1, x_0)
\]

**Theorem 4.2:** Let \((X, E, \Delta)\) be a \(\Delta\)-ordered Banach algebra. Suppose \(T : X \to X\) holds the \(\Delta\)-contradiction condition
\[
\Delta(\lambda, T \xi, T \eta) \leq m_1 \Delta\left(\frac{\lambda}{4}, x_0, T \eta\right) + m_2 \Delta\left(\frac{\lambda}{4}, m_3 \Delta\left(\frac{\lambda}{4}, T \xi, T \eta\right) + m_4 \Delta\left(\frac{\lambda}{4}, y, T \eta\right)
\]
where \(0 < \sum_{i=1}^{4} m_i = 1\), for \(i = 1, 2, 3, 4\) Then \(T\) is a unique fixed point in \(X\).

**Proof:** choose \(x_0 \in X\), \(x_1 = T x_0\) and \(x_{n+1} = T x_n\)

Take \(0 < m_i \leq 1\), for \(i = 1, 2, 3, 4\)

First we see,
\[
\Delta(\lambda, x_{n+1}, x_n) = \Delta(\lambda, T x_n, T x_{n-1}) \leq m_1 \Delta\left(\frac{\lambda}{4}, x_n, T x_{n}\right) + m_2 \Delta\left(\frac{\lambda}{4}, x_{n-1}, T x_{n}\right) + m_3 \Delta\left(\frac{\lambda}{4}, x_n, T x_{n-1}\right) + m_4 \Delta\left(\frac{\lambda}{4}, x_{n-1}, T x_{n-1}\right)
\]
\[
\leq m_1 \Delta\left(\frac{\lambda}{4}, x_n, x_{n+1}\right) + m_2 \Delta\left(\frac{\lambda}{4}, x_{n-1}, x_n\right) + m_3 \Delta\left(\frac{\lambda}{4}, x_n, x_{n}\right) + m_4 \Delta\left(\frac{\lambda}{4}, x_{n-1}, x_n\right)
\]
\[
\leq m_1 \Delta\left(\frac{\lambda}{4}, x_n, x_{n+1}\right) + m_2 \Delta\left(\frac{\lambda}{4}, x_{n-1}, x_n\right) + m_3 \Delta\left(\frac{\lambda}{4}, x_n, x_{n}\right) + m_4 \Delta\left(\frac{\lambda}{4}, x_{n-1}, x_n\right)
\]
\[
\leq (m_1 + m_3) \Delta\left(\frac{\lambda}{4}, x_n, x_{n+1}\right) + (m_2 + m_4) \Delta\left(\frac{\lambda}{4}, x_{n-1}, x_n\right)
\]
\[
\leq t_1 \Delta\left(\frac{\lambda}{4}, x_n, x_{n+1}\right) + t_2 \Delta\left(\frac{\lambda}{4}, x_{n-1}, x_n\right)
\]
\[
\leq \cdots \leq t_1^n \Delta\left(\frac{\lambda}{2^n}, x_0, x_1\right) + t_2^n \Delta\left(\frac{\lambda}{2^n}, x_2, x_0\right)
\]
if follows that
\[
\Delta(\lambda, x_{n+1}, x_n) \leq t_1^n \Delta\left(\frac{\lambda}{2^n}, x_0, x_1\right) + t_2^n \Delta\left(\frac{\lambda}{2^n}, x_1, x_0\right)
\]
\[
\Delta(\lambda, x_{n+1}, x_n) = (t_1^n + t_2^n) \Delta\left(\frac{\lambda}{4}, x_0, x_1\right)
\]

Put \(k = (t_1^n + t_2^n)\),
\[
\frac{1 - \varepsilon}{1 + \varepsilon} \leq |k| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \text{ for } 0 < \varepsilon < 1
\]

It is clearly see that \(\sigma_{\varepsilon}(k) < 1\)

\[
\Delta(\lambda, x_{n+1}, x_n) \leq k \Delta\left(\frac{\lambda}{2^n-1}, x_{n}, x_{n-1}\right) + \cdots + k^n \Delta\left(\frac{\lambda}{2^n-1}, x_{1}, x_{0}\right)
\]
\[
\Delta(\lambda, x_{n}, x_{n+m}) \leq k^m \Delta\left(\frac{\lambda}{2^n-1}, x_{n}, x_{n-1}\right) + \cdots + k^{n+m} \Delta\left(\frac{\lambda}{2^n-1}, x_{1}, x_{0}\right)
\]
When \( n, m \to \infty \) we have \( \lim_{n,m \to \infty} \Delta(\lambda, x_n, x_{n+m}) = 0 \)

Thus \( \{x_n\} \) is Cauchy sequence in \( (X, E, \Delta) \)

Since \( (X, E, \Delta) \) is Banach algebra

That is \((X, E, \Delta)\) is complete.

Then \( \{x_n\} \) is convergent to \( x^* \in X \) such that \( x_n \to x^* \)

Next, we claim that \( x^* \) is a fixed point of \( T \)

Actually,

\[
\Delta\left(\frac{\lambda}{4}, T x^*, x^*\right) \leq \Delta(\lambda, T x^*, x^*) \leq K [\Delta\left(\frac{\lambda}{2}, x^*, T x_n\right) + \Delta\left(\frac{\lambda}{2}, T x_n, T x^*\right)]
\]

\[
= k \Delta\left(\frac{\lambda}{2}, x^*, x_{n+1}\right) + k \Delta\left(\frac{\lambda}{2}, T x_n, T x^*\right)
\]

\[
\leq k [m_1 \Delta\left(\frac{\lambda}{4}, x^*, T x^*\right) + m_2 \Delta\left(\frac{\lambda}{4}, x_{n+1}, T x^*\right)]
\]

\[
+ m_3 \Delta\left(\frac{\lambda}{4}, x^*, T x_{n+1}\right) + m_4 \Delta\left(\frac{\lambda}{4}, x_{n+1}, T x^*\right)
\]

\[
\leq k m_1 \Delta\left(\frac{\lambda}{4}, x^*, T x^*\right) + k^2 m_2 \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) + k^2 m_3 \Delta\left(\frac{\lambda}{4}, x_{n+1}, x^*\right)
\]

\[
= (k m_1 + k^2 m_2 + k^2 m_3) \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) + k \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right)
\]

then

\[
(1 - k m_1 - k^2 m_4) \Delta\left(\frac{\lambda}{4}, x^*, T x^*\right) \leq (k^2 m_2 + k^2 m_3) \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) + (k^2 m_2 + k^2 m_3) \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) + k \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right)
\]

\[
\Delta\left(\frac{\lambda}{4}, x^*, T x^*\right) \leq \frac{(k^2 m_2 + k^2 m_3) \Delta\left(\frac{\lambda}{4}, x^*, T x^*\right) + (k^2 m_2 + k^2 m_3 + k) \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right) + k \Delta\left(\frac{\lambda}{4}, x^*, x_{n+1}\right)}{(1 - k m_1 - k^2 m_4)} \leq c
\]

We can see easily \( \Delta(\lambda, x^*, T x^*) = 0 \) is the mapping \( T \) which has a fixed point \( x^* \)

At last, for uniqueness, if there is \( y^* \) other fixed point, then

\[
\Delta(\lambda, x^*, y^*) = \Delta(\lambda, T x^*, T y^*)
\]

\[
\leq m_1 \Delta\left(\frac{\lambda}{4}, x^*, T x^*\right) + m_2 \Delta\left(\frac{\lambda}{4}, y^*, T x^*\right) + m_3 \Delta\left(\frac{\lambda}{4}, x^*, T y^*\right) + m_4 \Delta\left(\frac{\lambda}{4}, y^*, x^*\right)
\]

\[
\leq m_1 \Delta\left(\frac{\lambda}{4}, x^*, T x^*\right) + m_2 \Delta\left(\frac{\lambda}{4}, x^*, y^*\right) + m_3 \Delta\left(\frac{\lambda}{4}, y^*, x^*\right) + m_4 \Delta\left(\frac{\lambda}{4}, y^*, T y^*\right)
\]

\[
= (m_2 + m_4) \Delta(\lambda, x^*, y^*)
\]

Since \( 0 < (m_2 + m_4) < 1 \), we deduce from lemma that \( x^* = y^* \)
Definition 4.3: Let \((E, C)\) be ordered Banach algebra with algebra cone \(C\). Take \(\Phi\) be the set of all functions \(\varphi: E^3 \to E\) satisfying the following properties:

1) \(\varphi(e, e, e) = e\)

2) Let \(a, b \in E\) be such that if either \(a \leq \varphi(a, b, b)\) or \(a \leq \varphi(b, a, b)\) or \(a \leq \varphi(b, a, a)\)

Definition 4.3: A self-mapping \(T\) on \(\Delta\)-ordered Banach algebra \((X, E, \Delta)\) is called \(\varphi\)-contraction, if there exists a map \(\varphi \in \Phi\) satisfy

\[
\Delta(\lambda, x, T x) \leq \varphi(\Delta(\lambda, x, y), \Delta(x, T x), \Delta(\lambda, y, T y)) \leq \Delta(\lambda, x, y)
\]

Theorem 4.4: Let \((X, E, \Delta)\) be \(\Delta\)-ordered Banach algebra and \(T\) a \(\varphi\)-contraction. If there exists \(\lambda > 0\) such that for all \(x \in X\)

\[
\Delta(\lambda, x, T x) = \sup \{\Delta(\lambda, x, T x) : x \in X\}
\]

Hence \(T x_0 = x_0\)

For uniqueness, let \(y_0\) be other fixed point of \(T\) that is \(T y_0 = y_0\)

Now, \(\Delta(\lambda, x_0, y_0) = \varphi(\Delta_3(x_0, y_0), \Delta_3(x_0, T x_0), \Delta_3(y_0, y_0))\n
\Delta(\lambda, x_0, y_0) = \varphi(\Delta_3(x_0, y_0), \Delta_3(x_0, x_0), \Delta_3(y_0, y_0))\n
\Delta(\lambda, x_0, y_0) = \varphi(\Delta_3(x_0, y_0) , 0, 0)\n
There for \(\Delta_3(\lambda, x_0, y_0) \leq 0\) or \(\Delta_3(x_0, y_0) = 0\). Implies \(x_0 \neq y_0\)

That is the fixed point is unique and this complete the proof

Theorem 4.5: Let \(S\) and \(T\) be self-mapping on \(\Delta\)-Banach algebra \((X, E, \Delta)\) satisfy the condition

\[
\Delta_2(\lambda, x, y) = \varphi(\Delta(x, y), \Delta(x, T x), \Delta(\lambda, y, S y)) \quad \text{for all } x, y \in X
\]

If there exists \(y \in X\) such that

\[
\Delta(\lambda, y, y) = \Delta(\lambda, z, S y)
\]

Then there exist a unique common fixed point of \(S\) and \(T\)

Proof: Let \(T y_0 = x_0\), put \(x = x_0, y = T x_0\), we obtain
\[ \Delta_{2}(\lambda, x_{0}, S(T_{x_{0}})) \leq \varphi(\Delta(\lambda, x_{0}, T_{x_{0}}), \Delta(\lambda, x_{0}, T_{x_{0}}), \Delta(\lambda, x_{0}, S(T_{x_{0}})) \]

By (3) we get
\[ \Delta(\lambda, x_{0}, S(T_{x_{0}})) \leq k \Delta(\lambda, x_{0}, T_{x_{0}}) \leq \Delta(\lambda, x_{0}, S(T_{x_{0}})) \]

This contradicts (4.1)

To prove that \( x_{0} \) is also a fixed point of \( S \), let \( S_{x_{0}} = x_{0} \), therefore.
\[ \Delta(\lambda, x_{0}, S_{x_{0}}) = \Delta(\lambda, T_{x_{0}}, S_{x_{0}}) \leq \varphi[\Delta(\lambda, x_{0}, x_{0}), \Delta(\lambda, x_{0}, T_{x_{0}}), \Delta(\lambda, x_{0}, S_{x_{0}})] \]

Or \( \Delta(\lambda, x_{0}, S_{x_{0}}) \leq \varphi(0, 0, 0) \) \( \Delta(\lambda, x_{0}, S_{x_{0}}) \leq 0 \) or \( S_{x_{0}} = x_{0} \)

For uniqueness, let \( y_{0} \) be another fixed point of \( S \) and \( T \) that is
\[ T_{y_{0}} = S_{y_{0}} = y_{0} \]
then
\[ \Delta(\lambda, x_{0}, y_{0}) = \Delta(\lambda, T_{y_{0}}, T_{y_{0}}) \leq \varphi(\Delta(\lambda, x_{0}, y_{0}), \Delta(\lambda, x_{0}, T_{y_{0}}), \Delta(\lambda, x_{0}, T_{y_{0}})) \]

Or \( \Delta_{2}(\lambda, x_{0}, y_{0}) \leq \varphi(\Delta(\lambda, x_{0}, y_{0}), \Delta(\lambda, x_{0}, y_{0}), \Delta(\lambda, y_{0}, y_{0})) \)
\[ \varphi(\Delta(\lambda, x_{0}, y_{0}), 0, 0) \]
That is \( \Delta(\lambda, x_{0}, y_{0}) \leq 0 \) implies \( x_{0} = y_{0} \).

**Corollary 4.6**: Let \( S \) and \( T \) be self-mapping of \( \Delta \)-ordered Banach algebra \((X, E, \Delta)\) satisfying the following conditions:

1) There exists integer \( n \) and \( m \) such that
\[ \Delta(\lambda, T^{n}x, S^{m}y) \leq \varphi[\Delta(\lambda, x, y), \Delta(\lambda, x, T^{n}x), \Delta(\lambda, y, S^{m}y)] \] for some \( \varphi \in \Phi \)

2) If there exists a point \( y \in X \) such that \( \Delta(\lambda, y, T^{n}x) \leq \Delta(\lambda, x, S^{m}y) \)

Then there exists a unique common fixed point of \( S \) and \( T \)

**Theorem 4.7**: Let \((X, E, \Delta)\) be a \( \Delta \)-ordered Banach algebra such that
\[ \Delta(\lambda, T_{x}, T_{y}) \leq \min \{\lambda \Delta(\lambda, x, T_{y}), \mu \Delta(\mu, y, T_{x})\} \]

If there exists function \( F \) defined by \( F(x) = \lambda \Delta_{1}(\lambda, x, T_{x}) \) for each \( x \in X \) such that \( F(x) \neq F(T(x)) \), then \( T \) has a unique fixed point

**Proof**: Suppose for some \( x_{0}, x_{0} \neq T_{x_{0}} \). Then
\[ F(T_{x_{0}}) = \Delta(\lambda, x_{0}, S(T_{x_{0}})) \leq \min \{\lambda \Delta(\lambda, x_{0}, T_{x_{0}}), \mu \Delta(\mu, T_{x_{0}}, T_{x_{0}})\} \]
\[ \Delta(T_{x_{0}}, T(T_{x_{0}})) = \theta \]
\[ \Delta(\lambda, T_{x_{0}}, T(T_{x_{0}})) \leq \lambda \Delta(\lambda, x_{0}, T_{x_{0}}) \]
\[ F(T_{x_{0}}) \leq F(x_{0}) \] which is contradiction

Hence \( T_{x_{0}} = x_{0} \)

For uniqueness, let \( y \) be another point of \( X \) different from \( x_{0} \) such that \( y_{0} = T_{y_{0}} \), then
\[\Delta(\lambda, x_0, y_0) = \Delta(\lambda, T_{x_0}, T_{y_0}) \leq \min \{\lambda \Delta(\lambda, x_0, T_{x_0}), \lambda \Delta(\lambda, y_0, T_{y_0})\}\]
\[= \min \{\lambda \Delta(\lambda, x_0, x_0), \lambda \Delta(\lambda, y_0, y_0)\} = \min \{\theta, \theta\}\]
\[\Delta(\lambda, x_0, y_0) \leq \theta\]

Hence \(\Delta(\lambda, x_0, y_0) \leq 0\) which implies that \(\Delta(\lambda, x_0, y_0) = 0\) or \(y_0 = x_0\)

The proof is complete.

**Theorem 4.8:** Let \(T\) be a self-map on a compact \(\Delta\)-ordered Banach algebra \((E, \Lambda, C)\) satisfy Lipschitz condition

Then, \(T\) has a unique fixed point.

**Proof:** Suppose \(T\) satisfy Lipschitz condition. Then, \(T\) is a continuous map on \(X\) we define a function from \(X\) into \(x\) as \(F(x) = \Delta(\lambda, x, T_x)\) for all \(x \in X\).

Since \(T\) and \(\Delta\) are continuous, it follow \(F\) is continuous on \(X\). Since \(X\) is compact there exists a point \(y \in X\) such that \(F(y) = \inf \{\Delta(\lambda, x, T_x) : x \in X\}\).

We support that \(y \neq T_y\).

Otherwise, that a fixed point by Lipschitz condition

We have\(\Delta(\lambda, T_y, T^2_y) \leq k \Delta(\mu, y, T_y), 0 < \lambda \leq \mu\)

So that \(F(T_y) \leq T(y)\) which contradiction.

Then, \(y = T_y\)

Uniqueness follows from Lipschitz condition.

**Proposition 4.9:** Let \((X, E, \Delta)\) be a complete \(\Delta\)-ordered Banach algebra. Assume that the mapping \(T : X \to X\) satisfy

\[\Delta(\lambda, T^n x, T^n y) \leq k \Delta(\lambda, x, y), \text{ for each } x, y \in X, \text{ for } n \in \mathbb{Z}^+, \text{ where } k \text{ a vector with is } \sigma_x(k) < 1.\]

Then, \(T\) has a unique fixed point

**Proof:** \(T^n(T^*_x) = T(T^n x^*) = T^n x^* = T(T^{n-1} x^*) = T^{n-1}(x^*) = \cdots = T x^*\).

So, \(T x^*\) is also has fixed point of \(T^n\) then \(T x^* = x^*\)

\(x^*\) is a fixed point of \(T\).

**Theorem 4.10:** Let \((X, E, \Delta)\) be a compact \(\Delta\)-ordered Banach algebra. Suppose the mapping satisfy \(\Delta\)–Lipschitz condition in the following:

\[\Delta(\lambda, T_x, T_y) \leq k[\beta \Delta(\beta, T_x, y) + \mu \Delta(\mu, T_y, x)]\]

for all \(x, y \in X\), where \(k\) is a vector with \(k \epsilon (0, 1)\). Then, \(T\) has a unique fixed point in \(X\). Another sequence \(\{T^*_x\}\) converge to the fixed point.

**Proof:** choose \(x_0 \in X\) and set \(x_t = T^*_x, t \geq 1\), we have for \(t < m\)

\[\Delta(\lambda, x_t, x_{t+1}) \leq \beta \Delta(\beta, x_t, x_{t+1}) + \mu \Delta(\mu, x_{t+1}, x_m)\]
\[ \leq \beta \Delta(\lambda, x_t, x_{t+1}) + \mu[\beta \Delta(\lambda, x_{t+1}, x_{t+2}) + \mu \Delta(\lambda, x_{t+2}, x_{m})] \]
\[ \leq \beta \Delta(\lambda, x_t, x_{t+1}) + \mu \beta \Delta(\lambda, x_{t+1}, x_{t+2}) + \mu^2[\beta \Delta(\lambda, x_{t+2}, x_{t+1}) + \mu \Delta(\lambda, x_{t+3}, x_{m})] \]
\[ \leq \beta \Delta(\lambda, x_t, x_{t+1}) + \mu \beta \Delta(\lambda, x_{t+1}, x_{t+2}) + \mu^2 \beta \Delta(\lambda, x_{t+1}, x_{t+2}) + \mu^3 \Delta(\lambda, x_{t+3}, x_{t+1}) + \mu^4 \Delta(\lambda, x_{t+4}, x_{m}) \]

\[ \leq \beta [\Delta(\lambda, x_t, x_{t+1}) + \mu \Delta(\lambda, x_{t+1}, x_{t+2}) + \mu^2 \Delta(\lambda, x_{t+2}, x_{t+3}) + \ldots + \mu^n \Delta(\lambda, x_{t+m}, x_{m})] \]
\[ \leq \beta [k^t \Delta(\lambda, x_1, x_0) + \mu k^{t+1} \Delta(\lambda, x_1, x_0) + \ldots + \mu^m k^{t+m} \Delta(\lambda, x_1, x_0) + \mu^{t+1}[k^{t+m+1} \Delta(\lambda, x_1, x_0)] \]
\[ \leq \beta k^t[1 + \mu k + \ldots + \mu^m k^m] \Delta(\lambda, x_1, x_0) + \mu k^{t+m} \Delta(\lambda, x_1, x_0) \]
\[ \leq \beta k^t[\sum_{i=1}^{m+1} \mu^i k^i] \Delta(\lambda, x_1, x_0) + \mu^{m+1} k^{t+m} \Delta(\lambda, x_1, x_0) \]
\[ \leq \beta k^t[\sum_{i=1}^{m+1} \mu^i k^i] \Delta(\lambda, x_1, x_0) \]
\[ \leq \beta k^t(\epsilon - \mu k)^{-1} \Delta(\lambda, x_1, x_0) \]
\[
\| \Delta(\lambda, x_{n+1}, x_m) \| \leq \| \beta k^t \|. (\epsilon - \mu k)^{-1} \|. \Delta(\lambda, x_1, x_0) \|
\]
Since \( k^n \to 0 \) as \( n \to \infty \), where \( \| \Delta(\lambda, x_{m}, x_m) \| \to 0 \) as \( n \to \infty \)

Which implies \( \Delta(\lambda, x_t, x_m) \to 0 \) as \( (t, m \to 0) \)

Hence, \( \{x_t\} \) is a Cauchy sequence. Since \( X \) is complete, there exists \( x^* \in X \) such that \( x_t \to x^* \) as \( n \to \infty \), therefore

\[ \lim \Delta(\lambda, T_{x^*}, x^*) \leq k[\beta \Delta(\lambda, T_{x^*}, T_{x^*}) + \mu \Delta(\lambda, T_{x^*}, x_t)] \]
\[ \leq \beta k[\Delta(\lambda, x_0, x^*) + \Delta(\lambda, x^*, x_{t+1})] + \mu \Delta(\lambda, T_{x^*}, x_t) \]
\[ \| \Delta(\lambda, T_{x^*}, x^*) \| \]
\[ \leq \| \lambda \|. (k \|. \Delta(\lambda, x_t, x^*) + \Delta(\lambda, x^*, x_{t+1}) \|. + \| \mu \|. \Delta(\lambda, x_0, x^*) \|. \]

Which implies \( T_{x^*} = x^* \) and so \( x^* \) is fixed point.

To prove uniqueness, let \( b \) be another fixed point of \( T \).

Then \( \Delta(\lambda, x^*, b) = \Delta(\lambda, T_{x^*}, T_{b}) \leq k[\beta \Delta(\lambda, T_{x^*}, b) + \mu \Delta(\lambda, T_{b}, x^*)] \]
\[ = k[\beta \Delta(\lambda, x^*, b) + \mu \Delta(\lambda, b, x^*)] = k[\lambda + m] \Delta(\lambda, x^*, b) \]
Then, $[1 - k(\beta + \mu)]\Delta(\lambda, x^*, b) \leq 0$.

Since $k \epsilon (0,1)$ and $\beta, \mu > 0 \implies \Delta(\lambda, x^*, b) = 0$ so $x^* = b$.

The proof is complete.

5-Conclusion

In this paper, we introduce a new concept which is called $\Delta$- ordered Banach algebra. Also, we define \textit{lipshtiz condition in this pace ,} $\varphi$ – contraction, $\Delta$ – contraction and $\Delta$ – lipshtiz condition. In the new work, we prove fixed point theorems satisfying these maps in $\Delta$- ordered Banach algebra. Our conditions and results are new in comparison with those of the results of cone metric space. These results can be extended to other spaces.

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