Correlation between clustering and degree in affiliation networks

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Abstract
We are interested in the probability that two randomly selected neighbors of a random vertex of degree (at least) \( k \) are adjacent. We evaluate this probability for a power law random intersection graph, where each vertex is prescribed a collection of attributes and two vertices are adjacent whenever they share a common attribute. We show that the probability obeys the scaling \( k^{-\delta} \) as \( k \to +\infty \). Our results are mathematically rigorous. The parameter \( 0 \leq \delta \leq 1 \) is determined by the tail indices of power law random weights defining the links between vertices and attributes.

Keywords: clustering coefficient, degree distribution, random intersection graph, affiliation network, complex network.

1 Introduction and Results
It looks plausible, that in a social network the chances of two neighbors of a given actor to be adjacent is a decreasing function of actor’s degree (the total number of its neighbors). Empirical evidence of this phenomenon has been reported in a number of papers, see, e.g., [7], [15], [13], [8]. Theoretical explanations have been derived in [6] and [13] with the aid of a hierarchical deterministic network model, and in [2] with the aid of a random intersection graph model of an affiliation network. We note that theoretical results [6], [13] and [2] only address the scaling \( k^{-1} \), i.e., \( \delta = 1 \). In particular, they do not explain empirically observed scaling \( k^{-\delta} \) with \( \delta \approx 0.75 \) reported in [13], see also [8]. In the present paper we develop further the approach of [2] and address the range \( 0 \leq \delta < 1 \). The development resorts to a more realistic fitness model of an affiliation network that accounts for variable activities of actors and attractiveness of attributes described below.

An affiliation network defines adjacency relations between actors by using an auxiliary set of attributes. Let \( V = \{v_1, \ldots, v_n\} \) denote the set of actors (vertices) and \( W = \{w_1, \ldots, w_m\} \) denote the set of attributes. Every actor \( v_i \) is prescribed a collection of attributes and two actors \( v_i \) and \( v_j \) are declared adjacent in the network if they share a common attribute. For example, in the film actor network two actors are adjacent if they have played in the same movie, in the collaboration network two scientists are adjacent if they have coauthored a publication, in the consumer copurchase network two consumers are adjacent if they have purchased similar products.

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A convenient model of a large affiliation network is obtained by linking (prescribing) attributes to actors at random \cite{9, 10, 12}. Furthermore, in order to model the heterogeneity of human activity, we assign every actor $v_j$ a random weight $Y_j$ reflecting its activity. Similarly, a random weight $X_i$ is assigned to an attribute $w_i$ to model its attractiveness. Now $w_i$ is linked to $v_j$ at random and with probability proportional to the attractiveness $X_i$ and activity $Y_j$. The random affiliation network obtained in this way is called a random intersection graph, see \cite{4}.

We assume in what follows that $X_0, X_1, \ldots, X_m, Y_0, Y_1, \ldots, Y_n$ are independent non-negative random variables. Furthermore, each $X_i$ (respectively $Y_j$) has the same probability distribution denoted $P_X$ (respectively $P_Y$). Given realized values $X = \{X_i\}_{i=1}^m$ and $Y = \{Y_j\}_{j=1}^n$ we define the random bipartite graph $H_{X,Y}$ with the bipartition $W \cup V$, where links $\{w_i, v_j\}$ are inserted with probabilities $p_{ij} = \min\{1, X_iY_j/\sqrt{nm}\}$ independently for each $(i, j) \in [m] \times [n]$. The random intersection graph $G = G(P_X, P_Y, n, m)$ defines the adjacency relation on the vertex set $V$: vertices $v', v'' \in V$ are declared adjacent (denoted $v' \sim v''$) whenever $v'$ and $v''$ have a common neighbor in $H_{X,Y}$. Such a neighbor belongs to the set $W$ and it is called a witness of the edge $v' \sim v''$. We note that for $n, m \to +\infty$ satisfying $m/n \to \beta$ for some $\beta > 0$, the random intersection graph $G$ admits a tunable global clustering coefficient and power law degree distribution \cite{5, 6}.

Next we introduce network characteristics studied in this paper. Given a finite graph $G$ and integer $k = 2, 3, \ldots$, define the clustering coefficients

$$c_G(k) = \mathbf{P}(v_2^* \sim v_3^* \mid v_1 \sim v_2, v_3 \sim v_1, d(v_1) = k),$$

and

$$C_G(k) = \mathbf{P}(v_2 \sim v_3^* \sim v_1^*, v_3 \sim v_1^*, d(v_1) \geq k).$$

Here $(v_1^*, v_2^*, v_3^*)$ is an ordered triple of vertices of $G$ drawn uniformly at random, $d(v)$ denotes the degree of a vertex $v$. Note that for a deterministic graph $G$, coefficients \[ and \] are the respective ratios of subgraph counts

$$\frac{\sum_{v: d(v) = k} N_\Delta(v)}{\sum_{v: d(v) = k} (d(v)/2)^2}$$

and

$$\frac{\sum_{v: d(v) \geq k} N_\Delta(v)}{\sum_{v: d(v) \geq k} (d(v)/2)^2}.$$

Here $N_\Delta(v)$ and $(d(v)/2)$ are the numbers of triangles and cherries incident to $v$. Differently, for the random graph $G$ the conditional probabilities \[ and \] refer to the two sources of randomness: the random sampling of vertices $(v_1^*, v_2^*, v_3^*)$ and the randomly graph generation mechanism. From the fact that the probability distribution of $G$ is invariant under permutation of its vertices we obtain that

$$c_G(k) = \mathbf{P}(v_2 \sim v_3 | v_2 \sim v_1, v_3 \sim v_1, d(v_1) = k),$$

and

$$C_G(k) = \mathbf{P}(v_2 \sim v_3 | v_2 \sim v_1, v_3 \sim v_1, d(v_1) \geq k).$$

An argument bearing on the law of large numbers suggests that for large $n, m$ the ratios \[ can be approximated by respective probabilities \[ and \].

Our Theorem 2 below establishes a first order asymptotics as $n, m \to +\infty$ of the probabilities \[ and \].

$$c_G(k) = \left(1 + \beta^{1/2}b(k)a^{-1}(k)\right)^{-1} + o(1),$$

and

$$C_G(k) = \left(1 + \beta^{1/2}B(k)A^{-1}(k)\right)^{-1} + o(1).$$

Here $a(k), b(k)$ and $A(k), B(k)$ are defined in Theorem 2 below. Our Theorem 1 describes the dependence on $k$ of the leading term of \[ . Namely, for a power law distributions $P_X$ and $P_Y$ the leading term of \[ obeys the scaling $k^{-\delta}$. 

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Theorem 1. Let $\alpha, \gamma > 5$ and $\beta, c_X, c_Y > 0$. Let $m, n \to \infty$. Assume that $m/n \to \beta$. Suppose that as $t \to +\infty$

$$
P(X > t) = (c_X + o(1))t^{-\alpha}, \quad P(Y > t) = (c_Y + o(1))t^{-\gamma}.
$$

Then for $\delta = ((\alpha - \gamma - 1) \wedge 1) \vee (-1)$ we have as $k \to +\infty$

$$
\frac{B(k)}{A(k)} = (c + o(1))k^\delta.
$$

The constant $c = c(\alpha, \gamma, \beta, c_X, c_Y) > 0$ admits an explicit expression in terms of $\alpha, \gamma, \beta, c_X, c_Y$.

It follows from [9] that for large $n$ and $m$ the clustering coefficient $C_G(k)$ obeys the scaling $k^{-\delta}$, where $0 \leq \delta \leq 1$. A related result establishing $k^{-1}$ scaling for $c_G(k)$ has been shown in [2] in the case where $P_Y$ is heavy tailed and $P_X$ is degenerate ($P(X_1 = x) = 1$ for some $x > 0$).

We note the “phase transition” in the scaling $k^{-\delta}$ at $\alpha = \gamma + 2$: for $\alpha \geq \gamma + 2$ we have $\delta = 1$ and for $\alpha < \gamma + 2$ we have $\delta < 1$. Our explanation of this phenomenon is as follows. Every attribute $w_i$ forms a clique in $G$ induced by vertices linked to $w_i$. Given the weight $X_i$ (of $w_i$), the expected size of the clique is proportional to $X_i$. Now, for relatively small $\alpha$ (namely, $\alpha < \gamma + 2$) the sequence $X_1, X_2, \ldots, X_m$ contains sufficiently many large weights so that the corresponding large cliques (formed by attributes) have a tangible effect on the probability of

Indeed, large cliques may increase the value of the sequence $X_1, X_2, \ldots, X_m$.

The proof of Theorem 1 uses known results about the tail asymptotics of randomly stopped sums of heavy tailed independent random variables in the case where the random number of summands is heavy tailed [1]. Similar results are likely to be true also for the local probabilities of randomly stopped sums (work in progress [2]). They would extend Theorem 1 to $c_G(k)$ as well.

Before formulating Theorem 2 we introduce some more notation. We denote $a_r = E X_r^\gamma$, $b_r = EY_r^\gamma$. Let $\beta \in (0, +\infty)$. Let $\Lambda_k$, $k = 0, 1, 2$ be mixed Poisson random variables with the distributions

$$
P(\Lambda_k = s) = E e^{-\lambda_k} \lambda_k^s / s!, \quad s = 0, 1, \ldots.
$$

Here $\lambda_0 = Y_1 \beta^{1/2} a_1$ and $\lambda_k = X_k \beta^{-1/2} b_1$ for $k = 1, 2$. Furthermore, for $r = 0, 1, 2, \ldots$ and $k = 0, 1, 2$, let $\Lambda_k^{(r)}$ be a non-negative integer valued random variable with the distribution

$$
P(\Lambda_k^{(r)} = s) = (E \lambda_k^{(r)})^{-1} E\left(e^{-\lambda_k \lambda_k^{s+r} / s!}\right), \quad s = 0, 1, 2, \ldots.
$$

Note that $\Lambda_k^{(0)}$ have the same probability distribution as $\Lambda_k$. Let $\tau_i$, $i \geq 1$ be random variables with the probability distribution

$$
P(\tau_i = s) = \frac{s + 1}{E \lambda_1} P(\Lambda_1 = s + 1), \quad s = 0, 1, 2, \ldots.
$$

Assuming that random variables $\{\tau_i, i \geq 1\}$ are independent of $\Lambda_0^{(r)}$ we introduce the random variables

$$
d^{(r)}_s = \sum_{j=1}^{\Lambda_0^{(r)}} \tau_j, \quad r = 0, 1, 2, \quad
$$

We denote for short $d_s = d^{(0)}_s = \sum_{j=1}^{\Lambda_0} \tau_j$.

\(^1\)Update: Technical report “Local probabilities of randomly stopped sums of power law lattice random variables” available at [http://arxiv.org/abs/1801.01035](http://arxiv.org/abs/1801.01035)
Theorem 2. Let \( m,n \to \infty \). Assume that \( m/n \to \beta \) for some \( \beta \in (0, +\infty) \). Suppose that \( \mathbb{E}X_1^4 < \infty \) and \( \mathbb{E}Y_1^4 < \infty \). Then for each integer \( k \geq 2 \) relations \([6]\) and \([7]\) hold with

\[
\begin{align*}
  a(k) &= a_1^3 \beta P(d_i^{(1)} + \Lambda_1^{(3)} = k - 2), \\
  b(k) &= a_2^3 b_2 P(d_i^{(2)} + \Lambda_2^{(2)} = k - 2), \\
  A(k) &= a_3^2 \Lambda_1 (d_i^{(3)} \geq k - 2), \\
  B(k) &= a_2^2 b_2 P(d_i^{(2)} + \Lambda_2^{(2)} \geq k - 2).
\end{align*}
\]

Here we assume that random variables \( d_i^{(1)} \) and \( \Lambda_1^{(3)} \) are independent. Furthermore, we assume that random variables \( d_i^{(2)} \), \( \Lambda_1^{(2)} \) and \( \Lambda_2^{(2)} \) are independent and \( \Lambda_2^{(2)} \) has the same distribution as \( \Lambda_1^{(2)} \).

2 Proof

We first prove Theorem 2 and then Theorem I. Before the proof we introduce some notation. We denote \( \{1, 2, \ldots, r\} = [r] \) and \( (x)_k = x(x - 1) \cdots (x - k + 1) \). We denote by \( \{w_i \to v_j\} \) the event that \( w_i \) and \( v_j \) are neighbors in the bipartite graph \( H = H_{X,Y} \). We denote

\[
\mathbb{I}_{ij} = \mathbb{I}_{w_i = v_j}, \quad \lambda_{ij} = \frac{X_i Y_j}{\sqrt{mn}}.
\]

Let \( P^* = P_{X_1,Y_1} \) and \( P^{**} = P_{X_1,X_2,Y_1} \) denote the conditional probabilities given \( X_1, Y_1 \) and \( X_1, X_2, Y_1 \) respectively. Furthermore, for \( i = 1, 2 \), we denote by \( P_{X_i} \) and \( P_{Y_i} \) the conditional probabilities given \( X_i \) and \( Y_i \) respectively.

Proof of Theorem 2. We only prove \([6]\). The proof of \([7]\) is much the same. Introduce events

\[
A = \{v_1 \sim v_2, v_1 \sim v_3, v_2 \sim v_3\}, \quad B = \{v_1 \sim v_2, v_1 \sim v_3\}, \quad K = \{d(v_1) = k\}.
\]

We derive \([6]\) from the identity

\[
P(v_2 \sim v_3 \mid v_1 \sim v_2, v_1 \sim v_3, d(v_1) = k) = \frac{P(A \cap K)}{P(B \cap K)}
\]

combined with the relations shown below

\[
\begin{align*}
  P(A \cap K) &= n^{-2} \beta^{-1/2} a(k) + o(n^{-2}), \\
  P(B \cap K) &= n^{-2} \beta^{-1/2} a(k) + n^{-2} b(k) + o(n^{-2}).
\end{align*}
\]

Proof of \([12]\) and \([13]\). Introduce the sets of indices

\[
C_1 = [m], \quad C_2 = \{(i,j) : i \neq j; \ i, j \in [m]\}, \\
C_3 = \{(i,j,k) : i \neq j \neq k \neq i; \ i, j, k \in [m]\}
\]

and split

\[
B = B_1 \cup B_2, \quad A = B_1 \cup B_3, \quad B_k = \bigcup_{x \in C_k} B_{k,x}, \quad k = 1, 2, 3,
\]

where

\[
\begin{align*}
  B_{1,i} &= \{w_i \to v_1, w_i \to v_2, w_i \to v_3\}, \\
  B_{2,ij} &= \{w_i \to v_1, w_j \to v_2, w_j \to v_3\}, \\
  B_{3,ijk} &= \{w_i \to v_1, w_j \to v_2, w_k \to v_3\}.
\end{align*}
\]
We write
\[ P(A \cap K) = P(B_1 \cap K) + P((B_3 \cap K) \setminus B_1), \] (14)
\[ P(B \cap K) = P(B_1 \cap K) + P(B_2 \cap K) - P(B_1 \cap B_2 \cap K) \] (15)
and evaluate \( P(B_k \cap K) \), for \( k = 1, 2 \), using inclusion-exclusion,
\[ \sum_{x \in C_k} P(B_{k,x} \cap K) - \sum_{\{x,y\} \subseteq C_k} P(B_{k,x} \cap B_{k,y}) \leq P(B_k \cap K) \leq \sum_{x \in C_k} P(B_{k,x} \cap K). \] (16)

We show in Lemma 2 below that the quantities
\[ R_k := \sum_{\{x,y\} \subseteq C_k} P(B_{k,x} \cap B_{k,y}), \quad k = 1, 2, \]
\[ R_3 := P((B_3 \cap K) \setminus B_1), \quad R_4 := P(B_1 \cap B_2 \cap K) \]
are negligibly small. More precisely, we establish the bounds \( R_i = O(n^{-3}) \), \( 1 \leq i \leq 4 \). Invoking these bounds in (14), (15), (16) we obtain
\[ P(A \cap K) = P(B_1 \cap K) + o(n^{-2}) = mP(B_{1,1} \cap K) + o(n^{-2}), \]
\[ P(B \cap K) = P(B_1 \cap K) + P(B_2 \cap K) + o(n^{-2}) \]
\[ = mP(B_{1,1} \cap K) + (m)_2P(B_{2,(1,2)} \cap K) + o(n^{-2}). \]

In the remaining part of the proof we evaluate the probabilities
\[ p_1 := P(B_{1,1} \cap K) \quad \text{and} \quad p_2 := P(B_{2,(1,2)} \cap K). \]

We shall show that
\[ (nm)^{3/2}/p_1 = a(k) + o(1) \quad \text{and} \quad (nm)^2/p_2 = b(k) + o(1). \] (20)

Finally, invoking (20) in (18), (19) we obtain (12), (13) thus proving (6).

It remains to prove (20). For convenience we divide the proof into three steps. For this part of the proof we need some more notation. Let \( d_1^* \) (respectively \( d_2^* \)) denote the number of neighbors of \( v_1 \) in \( V^* = \{v_1, v_3, \ldots, v_n\} \) witnessed by the attribute \( w_1 \) (respectively \( w_2 \)). Let \( d_1' \) (respectively \( d_2' \)) denote the number of neighbors of \( v_1 \) in \( V^* \) witnessed by some attributes from \( W_1' = \{w_2, w_3, \ldots, w_m\} \) (respectively \( W_2' = \{w_3, w_4, \ldots, w_m\} \)).

**Step 1.** We firstly show that
\[ p_1 = P(B_{1,1} \cap \{d_1^* + d_1' = k - 2\}) + O(n^{-4}), \]
\[ p_2 = P(B_{2,(1,2)} \cap \{d_1^* + d_2 + d_2' = k - 2\}) + O(n^{-5}). \]

To show (21) we count neighbors of \( v_1 \) in \( V^* \). The number of such neighbors is denoted \( d^*(v_1) \). We have \( d^*(v_1) = d_1^* + d_1' - d_0 \), where \( d_0 \) is the number of neighbors of \( v_1 \) witnessed by \( w_1 \) and by some attribute(s) \( w_i \in W_1' \) simultaneously. Combining the inequality
\[ d_0 \leq \sum_{j=1}^n \left( I_{1j} I_{11} \sum_{i=2}^m I_{ij} I_{1i} \right) \]

with Markov’s inequality we obtain
\[ P(B_{1,1} \cap \{d_0 \geq 1\}) \leq E(I_{11} I_{11} I_{1i} I_{21} I_{21}) \leq (n - 3)(m - 1)E(I_{11} I_{11} I_{1i} I_{21} I_{21}). \]
Furthermore, invoking the inequality
\[ \mathbb{E} \mathbb{I}_{B_{1.1}} \mathbb{I}_{14 \mathbb{I}_{14 \mathbb{I}_{24} \mathbb{I}_{21}}} = \mathbb{E} p_{11} p_{12} p_{13} p_{14} p_{21} p_{24} \leq a_2 a_4 b_1^2 b_2^3 (nm)^{-3} \]
we obtain \( \mathbb{P}(B_{1.1} \cap \{ d_0 \geq 1 \}) = O(n^{-4}) \). Now (21) follows from the fact that the event \( B_{1.1} \) implies \( d(v_1) = d^*(v_1) + 2 \).

The proof of (22) is almost the same. We color \( w_1 \) red, \( w_2 \) green and all \( w_i \in W_3^2 \) we color yellow. Let \( d' \) denote the number of neighbors of \( v_i \) witnessed by at least two attributes of different colors. Note that the number \( d^*(v_1) \) of neighbors of \( v_1 \) in \( V^* \) satisfies, by inclusion-exclusion,
\[
d_1^* + d_2^* + d_3^* - 2d' \leq d^*(v_1) \leq d_1^* + d_2^* + d_3'.
\]
We combine the inequality
\[
d'_0 \leq \sum_{j=1}^{n} \left( I_{11} I_{1j} I_{21} I_{2j} + (I_{11} I_{1j} + I_{21} I_{2j}) I_{j1} I_{jj} \right)
\]
with the identity \( I_{B_{2.1.2}} \mathbb{I}_{14 \mathbb{I}_{14 \mathbb{I}_{24} \mathbb{I}_{21}} = I_{B_{2.1.2}} \) and obtain, by Markov’s inequality and symmetry, that
\[
\mathbb{P}(B_{2.1.2} \cap \{ d'_0 \geq 1 \}) \leq \mathbb{E} \mathbb{I}_{B_{2.1.2}} d'_0 \leq (n - 3) \mathbb{E} I_{B_{2.1.2}} \left( I_{14 \mathbb{I}_{24} + 2(m - 2) I_{14 \mathbb{I}_{31} \mathbb{I}_{34}} \right).
\]

Furthermore, invoking the inequalities
\[
\mathbb{E} \mathbb{I}_{B_{2.1.2}} \mathbb{I}_{14 \mathbb{I}_{24} = \mathbb{E} p_{11} p_{12} p_{14} p_{21} p_{23} p_{24} \leq a_2 a_4 b_1^2 b_2^3 (mn)^{-3},
\mathbb{E} \mathbb{I}_{B_{2.1.2}} \mathbb{I}_{14 \mathbb{I}_{31} \mathbb{I}_{34} = \mathbb{E} p_{11} p_{12} p_{14} p_{21} p_{23} p_{31} p_{34} \leq a_2 a_3 b_1^2 b_3 (mn)^{-7/2}
\]
we obtain \( \mathbb{P}(B_{2.1.2} \cap \{ d'_0 \geq 1 \}) = O(n^{-5}) \). Now (22) follows from (23) and the identity \( d(v_1) = d^*(v_1) + 2 \).

**Step 2.** We secondly show that
\[
(nm)^{3/2} p_1 = b_1^2 \mathbb{E} \left( X_1^2 Y_1 \mathbb{P}(\Lambda_1 + d_1 = k - 2 \mid X_1, Y_1) \right) + o(1),
\]
\[
(nm)^{3} p_2 = b_1^2 \mathbb{E} \left( X_1^2 Y_1 \mathbb{P}(\Lambda_1 + A_2 + d_1 = k - 2 \mid X_1, X_2, Y_1) \right) + o(1).
\]
Let us prove (24). We have
\[
\mathbb{P}(B_{1.1} \cap \{ d_1^* + d_1' = k - 2 \} = \mathbb{E}(p_{11} p_{12} p_{13} \mathbb{P}^*(d_1^* + d_1' = k - 2))
\]
\[
= \mathbb{E}(\lambda_1 \lambda_2 \lambda_3 \mathbb{P}^*(d_1^* + d_1' = k - 2)) + o((nm)^{-3/2})
\]
and
\[
(nm)^{3/2} \mathbb{E}(\lambda_1 \lambda_2 \lambda_3 \mathbb{P}^*(d_1^* + d_1' = k - 2)) = b_1^2 \mathbb{E}(X_1^2 Y_1 \mathbb{P}^*(d_1^* + d_1' = k - 2))
\]
\[
= \mathbb{E}(X_1^2 Y_1 \mathbb{P}^*(d_1 + A_1 = k - 2)) + o(1).
\]
Here (27) follows from Lemma 1 by Lebesgue’s dominated convergence theorem. Furthermore, (26) follows from the inequalities
\[
\lambda_1 \lambda_2 \lambda_3 \geq p_{11} p_{12} p_{13} \geq \lambda_1 \lambda_2 \lambda_3 (1 - \mathbb{I}_{[\lambda_{11}>1]} - \mathbb{I}_{[\lambda_{12}>1]} - \mathbb{I}_{[\lambda_{13}>1]})
\]
combined with the simple bound
\[
\mathbb{E}\left(\lambda_1 \lambda_2 \lambda_3 (\mathbb{I}_{[\lambda_{11}>1]} + \mathbb{I}_{[\lambda_{12}>1]} + \mathbb{I}_{[\lambda_{13}>1]})\right) = O\left(\mathbb{E}(\lambda_1 \lambda_2 \lambda_3)\right) = o((nm)^{-3/2}).
\]
Note that (21), (26), (27) imply (24).
The proof of (25) is much the same. We have
\[
P(B_{2,(1,2)} \cap \{d_i^* + d_2^* + d_2' = k - 2\})
= E(p_{11}p_{12}p_{21}p_{23}P^{**}(d_1^* + d_2^* + d_2' = k - 2))
= E(\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{23}P^{**}(d_1^* + d_1' = k - 2)) + o((nm)^{-2})
\]
and
\[
(nm)^2 E(\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{23}P^{**}(d_1^* + d_2^* + d_2' = k - 2))
= b_1^2 E(X_1^2X_2^2Y_1^2P^{**}(d_1^* + d_2' = k - 2))
= b_1^2 E(X_1^2X_2^2Y_1^2P^{**}(d_1 + \Lambda_1 + \Lambda_2 = k - 2)) + o(1).
\]
Step 3. In this final step we show that
\[
E(X_1^2Y_1P^{*}(d_1 + \Lambda_1 = k - 2)) = a_3 b_1 P(d_1^{(1)} + \Lambda_1^{(3)} = k - 2). \tag{28}
\]
\[
E(X_1^2X_2^2Y_1P^{**}(d_1 + \Lambda_1 + \Lambda_2 = k - 2))
= a_3^2 b_2 P(d_1^{(2)} + \Lambda_1^{(2)} + \Lambda_2^{(2)} = k - 2). \tag{29}
\]
In the proof we use the observation that
\[
E\left(Y_1^iP(Y_1 = d_1)\right) = E\sum_{i \geq 0} \left(Y_1^iP(Y_1 = \Lambda_0 = i)P\left(\sum_{j = 0}^i \tau_j = s\right)\right)
= \sum_{i \geq 0} \left(b_i P(\Lambda_0^{(r)} = i)P\left(\sum_{j = 0}^i \tau_j = s\right)\right)
= b_i P(d_i^{(r)} = s).
\]
To show (28) we write the quantity on the left in the form
\[
E\left(X_1^2Y_1\sum_{s + t = k - 2} P(Y_1 = d_1) \cdot P(X_1 = \Lambda_1 = t)\right)
= \sum_{s + t = k - 2} b_1 P(d_1^{(1)} = s) \cdot a_3 P(\Lambda_1^{(3)} = t)
= b_1 a_3 P(d_1^{(1)} + \Lambda_1^{(3)} = k - 2).
\]
To show (29) we write the quantity on the left in the form
\[
E\left(X_1^2X_2^2Y_1\sum_{s + t + u = k - 2} P(Y_1 = d_1) \cdot P(X_1 = \Lambda_1 = t) \cdot P(X_2 = \Lambda_2 = u)\right)
= \sum_{s + t + u = k - 2} b_2 P(d_1^{(2)} = s) \cdot a_2 P(\Lambda_1^{(2)} = t) \cdot a_2 P(\Lambda_2^{(2)} = u)
= b_2 a_2^2 P(d_1^{(2)} + \Lambda_1^{(2)} + \Lambda_2^{(2)} = k - 2).
\]
\[\square\]
Proof of Theorem 1. In the proof we use shorthand notation $\tilde{A}(k) = \mathbb{P}(d_1^{(1)} + \Lambda_1^{(3)} \geq k)$ and $\tilde{B}(k) = \mathbb{P}(d_2^{(2)} + \Lambda_1^{(3)} + \Lambda_2^{(2)} \geq k)$. Given two positive functions $f(t)$ and $g(t)$ we denote $f(t) \simeq g(t)$ whenever $f(t)/g(t) \to 1$ as $t \to +\infty$.

Using asymptotic formulas for the tail probabilities of randomly stopped sums $d^{(r)}$ reported in [1], and the formulas for the tail probabilities of $\Lambda_k^{(r)}$ shown in Lemma 3 we obtain

\[
\begin{align*}
\mathbb{P}(d_1^{(1)} \geq k) & \simeq c_Y \alpha \gamma^{-1} a_1^{-1} b_1^{-2} k^{1-\gamma}, \\
\mathbb{P}(d_2^{(2)} \geq k) & \simeq c_Y \alpha \gamma^{-2} a_2^{-1} b_2^{-2} k^{2-\gamma}, \\
\mathbb{P}(\Lambda_1^{(r)} \geq k) & \simeq c_X \alpha \beta^{(r-\alpha)/2} a_r^{-1} b_r^{-r} k^{\gamma-\alpha}, \quad r = 2, 3.
\end{align*}
\]

Next we combine these asymptotic formulas with the aid of Lemma 4. We have

\[
\begin{align*}
\tilde{A}(k) & \simeq \mathbb{P}(d_1^{(1)} \geq k), \quad \text{for } \alpha > \gamma + 2, \\
\tilde{A}(k) & \simeq \mathbb{P}(d_1^{(1)} \geq k) + \mathbb{P}(\Lambda_1^{(3)} \geq k), \quad \text{for } \alpha = \gamma + 2, \\
\tilde{A}(k) & \simeq \mathbb{P}(\Lambda_1^{(3)} \geq k), \quad \text{for } \alpha < \gamma + 2
\end{align*}
\]

and

\[
\begin{align*}
\tilde{B}(k) & \simeq \mathbb{P}(d_2^{(2)} \geq k), \quad \text{for } \alpha > \gamma, \\
\tilde{B}(k) & \simeq \mathbb{P}(d_2^{(2)} \geq k) + \mathbb{P}(\Lambda_1^{(2)} \geq k) + \mathbb{P}(\Lambda_2^{(2)} \geq k), \quad \text{for } \alpha = \gamma, \\
\tilde{B}(k) & \simeq \mathbb{P}(\Lambda_1^{(2)} \geq k) + \mathbb{P}(\Lambda_2^{(2)} \geq k), \quad \text{for } \alpha < \gamma.
\end{align*}
\]

Finally, from (30), (31), (32) we derive (9).

\[\square\]

3 Auxiliary lemmas

Let $d_1^*$ (respectively $d_2^*$) denote the number vertices in $V^* = \{v_4, v_5, \ldots, v_n\}$ linked to the attribute $w_1$ (respectively $w_2$). Let $x_1, x_2, y_1 \geq 0$. For $k = 1, 2$, let $d_k, \Lambda_k$ denote the random variables $d_k^*, \Lambda_k$ conditioned on the event $X_k = x_k$ (to get $d_k, \Lambda_k$ we replace $X_k$ by a non-random number $x_k$ in the definition of $d_k^*, \Lambda_k$). Let $d_1, d_2$ and $d_*$ denote the random variables $d_1^*, d_2^*$ and $d_*$ conditioned on the event $Y_1 = y_1$ (to get $d_1, d_2$ and $d_*$ we replace $Y_1$ by a non-random number $y_1$ in the definition of $d_1^*, d_2^*$ and $d_*^*$).

Lemma 1. Let $\beta > 0$. Let $n, m \to +\infty$. Assume that $m/n \to \beta$. Assume that $\mathbf{E}X_i^2 < \infty$ and $\mathbf{E}Y_j < \infty$. For any $x_1, x_2, y_1 \geq 0$ and $s, t, u, 0, 1, 2, \ldots$, we have

\[
\mathbb{P}(d_1 = s, d_1 = t) \to \mathbb{P}(d_1 = s, \Lambda_1 = t) = \mathbb{P}(d_* = s)\mathbb{P}(\Lambda_1 = t), \quad (33)
\]

\[
\mathbb{P}(d_2 = s, d_2 = t, d_2 = u) \to \mathbb{P}(d_2 = s, \Lambda_1 = t, \Lambda_2 = u) = \mathbb{P}(d_* = s)\mathbb{P}(\Lambda_1 = t)\mathbb{P}(\Lambda_2 = u). \quad (34)
\]

We remark that (33) tells us that random vector $(d_1, d_1)$ converges in distribution to the random vector $(d_1, \Lambda_1)$. Similarly, (34) tells us that random vector $(d_1, d_1, d_2)$ converges in distribution to the random vector $(d_1, \Lambda_1, \Lambda_2)$. In particular, (33) implies for any $r = 0, 1, 2, \ldots$ that

\[
\mathbb{P}(d_1 + d_1 \geq r) \to \mathbb{P}(d_* + \Lambda_1 \geq r) \quad \text{and} \quad \mathbb{P}(d_1 + d_1 = r) \to \mathbb{P}(d_* + \Lambda_1 = r).
\]
as \( n, m \to +\infty \). \( \{ \}) \) implies that
\[
\mathbf{P}(\tilde{d}_2 + \tilde{d}_1 + \tilde{d}_2 \geq r) \to \mathbf{P}(\tilde{d}_s + \tilde{\lambda}_1 + \tilde{\lambda}_2 \geq r),
\]
\[
\mathbf{P}(\tilde{d}_2 + \tilde{d}_1 + \tilde{d}_2 = r) \to \mathbf{P}(\tilde{d}_s + \tilde{\lambda}_1 + \tilde{\lambda}_2 = r).
\]

Proof of Lemma 4. Before the proof we introduce some notation. Let \( \mathbf{P}_1 \) denote the conditional probability given \( \{Y_4, Y_5, \ldots, Y_n\} \). For \( a > 0 \) and \( s = 0, 1, 2 \ldots \) we denote by \( f_s(a) = a^s e^{-a} / s! \) the Poisson probability. Below we use the fact that \( |f_s(a) - f_s(b)| \leq |a - b| \). Furthermore we denote
\[
\tilde{\lambda}_k = x_k \beta^{-1/2} b_1 \quad \text{and} \quad \tilde{\lambda}_{3/k} = \sum_{j=4}^{n} \tilde{\lambda}_{kj}, \quad \tilde{\lambda}_{4/k} = \sum_{j=4}^{n} \tilde{p}_{kj}, \quad k = 1, 2.
\]

Here \( \tilde{p}_{kj}, \tilde{\lambda}_{kj} \) are defined in the same way as \( p_{kj}, \lambda_{kj} \), but with \( X_k \) replaced by \( x_k \), for \( k = 1, 2 \).

Proof of \( \{ \}) \). We have
\[
\mathbf{P}(\tilde{d}_1 = s, \tilde{d}_1 = t) = \mathbf{E}\mathbf{P}_1(\tilde{d}_1 = s) \mathbf{P}_1(\tilde{d}_1 = t).
\]

Given \( \{Y_4, Y_5, \ldots, Y_n\} \), the random variable \( \tilde{d}_1 \) is a sum of independent Bernoulli random variables. We invoke Le Cam’s inequality, see, e.g., [3],
\[
|\mathbf{P}_1(\tilde{d}_1 = t) - f_t(\tilde{\lambda}_{4/1})| \leq \sum_{j=4}^{n} \tilde{p}_{1j}^2 =: R_1^t
\]

and use simple inequalities
\[
|f_t(\tilde{\lambda}_{4/1}) - f_t(\tilde{\lambda}_{3/1})| \leq |\tilde{\lambda}_{4/1} - \tilde{\lambda}_{3/1}| \leq \sum_{j=4}^{n} \tilde{\lambda}_{lj} I_{\{\tilde{\lambda}_{lj} > 1\}} =: R_2^t,
\]
\[
|f_t(\tilde{\lambda}_{3/1}) - f_t(\tilde{\lambda}_1)| \leq |\tilde{\lambda}_{3/1} - \tilde{\lambda}_1| = x_1 \sqrt{n/m} \left( n^{-1} \sum_{j=4}^{n} Y_j \right) - \beta^{-1/2} b_1.
\]

Note that \( |f_t(\tilde{\lambda}_{3/1}) - f_t(\tilde{\lambda}_1)| \to 0 \) almost surely, by the law of large numbers. Furthermore,
\[
\mathbf{E}R_2^t = (n - 4)(nm)^{-1/2} x_1 \mathbf{E}Y_4 \mathbf{I}_{\{Y_4 > \sqrt{nm}\}} = o(1),
\]

because \( \mathbf{E}Y_4 \mathbf{I}_{\{Y_4 > \sqrt{nm}\}} = o(1) \). We similarly show that \( \mathbf{E}R_1^t = o(1) \). For any \( \varepsilon \in (0, 1) \) the inequality \( \tilde{p}_{1j}^2 \leq \tilde{\lambda}_{lj} \left( \varepsilon + \mathbf{I}_{\{\tilde{\lambda}_{lj} > \varepsilon\}} \right) \) implies
\[
\mathbf{E}R_1^t = (n - 4)\mathbf{E}p_{14}^2 \leq (n - 4)\mathbf{E}\tilde{\lambda}_{4} \left( \varepsilon + \mathbf{I}_{\{\tilde{\lambda}_{lj} > \varepsilon\}} \right) \leq (n - 4)(nm)^{-1/2} (x_1 b_1 \varepsilon + o(1)).
\]

We obtain the bound \( \mathbf{E}R_1^t \leq \beta^{-1/2} x_1 b_1 \varepsilon + o(1) \), which implies \( \mathbf{E}R_1^t = o(1) \). Now it follows from \( \{ \}) \), \( \{ \}) \), \( \{ \}) \) that
\[
\mathbf{E}(\mathbf{P}_1(\tilde{d}_1 = s) \mathbf{P}_1(\tilde{d}_1 = t)) = \mathbf{E}(\mathbf{P}_1(\tilde{d}_1 = s) f_t(\tilde{\lambda}_1)) + o(1)
\]
\[
= \mathbf{P}(\tilde{d}_1 = s) f_t(\tilde{\lambda}_1) + o(1).
\]

Next we use the fact that \( \mathbf{P}(\tilde{d}_1 = s) \to \mathbf{P}(\tilde{d}_s = s) \). The proof of this fact repeats literally the proof of statement (ii) of Theorem 1 of \([3]\). Finally, from \( \{ \}) \), \( \{ \}) \) we obtain \( \{ \}) \):
\[
\mathbf{P}(\tilde{d}_1 = s, \tilde{d}_1 = t) = \mathbf{E}(\mathbf{P}_1(\tilde{d}_1 = s) \mathbf{P}_1(\tilde{d}_1 = t)) \to \mathbf{P}(\tilde{d}_s = s) f_t(\tilde{\lambda}_1).
\]
Proof of [34]. It is similar to that of [33]. We have
\[
P(\hat{d}_2 = s, \hat{d}_1 = t, \hat{d}_2 = u) = E(P_1(\hat{d}_2 = s)P_1(\hat{d}_1 = t)P_1(\hat{d}_2 = u)).
\] (40)
By the same argument as above (see (36), (37), (38)), we obtain
\[
E(P_1(\hat{d}_2 = s)P_1(\hat{d}_1 = t)P_1(\hat{d}_2 = u)) = E(P_1(\hat{d}_2 = s)f_1(\hat{\lambda}_1)P_1(\hat{d}_2 = u)) + o(1)
= E(P_1(\hat{d}_2 = s)f_1(\hat{\lambda}_1)f_u(\hat{\lambda}_2)) + o(1)
= f_1(\hat{\lambda}_1)f_u(\hat{\lambda}_2)P(\hat{d}_2 = s) + o(1).
\] (41)
Finally, we use the fact that \(P(\hat{d}_2 = s) \rightarrow P(\hat{d}_s = s)\). The proof of this fact repeats literally the proof of statement (ii) of Theorem 1 of [33]. Now from [40], [41] we obtain [34]. \(\Box\)

**Lemma 2.** The quantities \(R_i, 1 \leq i \leq 4\) defined in [17] satisfy \(R_4 = O(n^{-3})\).

**Proof of Lemma 2.** The bound \(R_1 = O(n^{-3})\) is obtained from the identity \(R_1 = \binom{m}{2}P(B_1 \cap B_2)\) and inequalities
\[
P(B_{1,1} \cap B_{1,2}) = E_{11}I_{12}E_{21}E_{22} \leq E\lambda_{11}\lambda_{12}\lambda_{13}\lambda_{21}\lambda_{22}\lambda_{23} = a_2^3b_3^2(mn)^{-3}.
\]

The bound \(R_2 = O(n^{-3})\) follows from inequalities
\[
R_2 = \sum_{\{(i,j),(i,r)\} \in C_2, j \neq r} P(B_{2,i} \cap B_{2,i}) + \sum_{\{(i,j),(k,j)\} \in C_2, i \neq k} P(B_{2,i} \cap B_{2,k}) + \sum_{\{(i,j),(i,i)\} \in C_2} P(B_{2,i} \cap B_{2,i}) + \sum_{\{(i,j),(k,i)\} \in C_2, i \neq k \neq j \neq r \neq i} P(B_{2,i} \cap B_{2,k})
= 2^{-1}(m)_3P(B_{2,1,2} \cap B_{2,1,3}) + 2^{-1}(m)_3P(B_{2,1,3} \cap B_{2,2,3}) + (m)_2P(B_{2,1,2} \cap B_{2,2,1}) + 2^{-1}(m)_4P(B_{2,1,2} \cap B_{2,3,4}) + 2^{-1}(m)_3E_{p_{11}p_{12}p_{21}p_{23}p_{31}p_{33}} + 2^{-1}(m)_3E_{p_{11}p_{12}p_{21}p_{23}p_{31}p_{33}} + (m)_2E_{p_{11}p_{12}p_{21}p_{23}p_{31}p_{33}} + 2^{-1}(m)_4E_{p_{11}p_{12}p_{21}p_{23}p_{31}p_{33}}
\leq (m)_3 \frac{a_2^3b_3^2b_2^3}{(nm)^3} + (m)_2 \frac{a_2^3b_3^2b_2^3}{(nm)^3} + 2^{-1}(m)_4 \frac{a_2^3b_3^2b_4}{(nm)^4}.
\]

The bound \(R_3 = O(n^{-3})\) is obtained from the inequalities
\[
P(B_3) \leq \sum_{x \in C_3} P(B_{3,x}) = (m)_3P(B_{3,1,2,3})
= \frac{(m)_3E_{p_{11}p_{12}p_{21}p_{23}p_{31}p_{33}}}{(nm)^3} \leq \frac{(m)_3a_2^3b_3^3}{(nm)^3}.
\]

The bound \(R_4 = O(n^{-3})\) is obtained from the inequalities
\[
P(B_1 \cap B_2) \leq \sum_{y \in C_2} P(B_1 \cap B_{2,y}) = m(m - 1)P(B_1 \cap B_{2,1,2}),
\] (42)
\[
P(B_1 \cap B_{2,1,2}) \leq P(B_{1,1} \cap B_{2,1,2}) + P(B_{1,2} \cap B_{2,1,2}) + (m - 2)P(B_{1,3} \cap B_{2,1,2})
\]
and bounds
\[ P(B_{1,1} \cap B_{2,(1,2)}) = P(B_{1,2} \cap B_{2,(1,2)}) = \mathbb{E}p_{11}p_{12}p_{13}p_{21}p_{23} \leq a_2a_3b_1b_2^2(nm)^{5/2}, \]
\[ P(B_{1,3} \cap B_{2,(1,2)}) = \mathbb{E}p_{11}p_{12}p_{21}p_{23}p_{31}p_{32}p_{33} \leq a_2^2a_3b_2^3b_3(nm)^{-7/2}. \]

Lemma 3. Let \( \alpha, c > 0 \). Let \( r \) be an integer and \( 0 \leq r < \alpha \). Let \( t \to +\infty \). For a non-negative random variable \( Z \) satisfying \( P(Z > t) = (c + o(1))t^{-\alpha} \) we have
\[ \mathbb{E}(Z^r I_{\{Z > t\}}) = (c + o(1))\alpha(\alpha - r)^{-1}t^{\alpha - r}. \] (43)

Denote \( h_r = \mathbb{E}Z^r \). For a random variable \( \Lambda_Z \) with the distribution \( P(\Lambda^r_Z = k) = h_r^{-1}\mathbb{E}(e^{-Z} Z^{k+r}/k!) \), \( k = 0, 1, 2, \ldots \), we have
\[ P(\Lambda^r_Z > t) = (1 + o(1))h_r^{-1}\mathbb{E}(Z^r I_{\{Z > t\}}) = (1 + o(1))h_r^{-1}\alpha(\alpha - r)^{-1}t^{\alpha - r}. \] (44)

Proof of Lemma 3
Denote \( F(x) = P(Z \leq x) = 1 - \bar{F}(x) \). To show (43) for \( r = 1, 2, \ldots \) we apply integration by parts formula for the Lebesgue-Stieltjes integral
\[ \mathbb{E}(Z^r I_{\{Z > t\}}) = \int_t^{+\infty} x^r dF(x) = -\int_t^{+\infty} x^r d\bar{F}(x) \]
\[ = t^r \bar{F}(t) + \int_t^{+\infty} x^{r-1} \bar{F}(x) dx \]
and invoke \( \bar{F}(x) = P(Z > x) = (c + o(1))x^{-\alpha} \).

Proof of (44). Fix \( r \). For \( s, t, x > 0 \) and \( k = 0, 1, 2, \ldots \) we denote
\[ S_x^{(k)}(s) := \sum_{i<s} e^{-x}x^{i+k}/i!, \quad \tilde{S}_x^{(k)}(t) := \sum_{i \geq t} e^{-x}x^{i+k}/i! \]
For \( 0 < s < x < t \) we will use the inequalities (see [11])
\[ S_x^{(0)}(s) \leq e^{-x}(x/s)^s \quad \text{and} \quad S_x^{(0)}(t) \leq e^{-x}(x/t)^t. \] (45)

Given \( 0 < \varepsilon < 1 \) we write for short \( t_1 = t(1 - \varepsilon) \), \( t_2 = t(1 + \varepsilon) \) and split the probability
\[ P(\Lambda^r_Z > t) = h_r^{-1}\mathbb{E}S_x^{(0)}(t) = h_r^{-1}(I_1 + I_2 + I_3), \quad I_k = \mathbb{E}S_x^{(0)}(t)I_{\{Z \in A_k\}}, \]
\( A_1 = [0, t_1), \quad A_2 = [t_1, t_2), \quad A_3 = (t_2, +\infty) \).

We let \( \varepsilon = t^{-1/3} \) and evaluate \( I_1, I_2 \) and \( I_3 \). The second inequality of (45) implies
\[ I_1 = \mathbb{E}(Z^r S_x^{(0)}(t)I_{\{Z < t_1\}}) \leq \mathbb{E}(e^{-Z} Z^{t+r}(e/t)^{t}I_{\{Z < t_1\}}) \leq e^{-t_1}(t_1^t r^r(e/t)^t). \] (46)
In the last step we used the fact that \( z \to e^{-z} z^{t+r} \) is an increasing function on \( (0, t_1) \). Furthermore, the quantity on the right of (46) is less than
\[ t^r e^{t_1}(t_1/t)^t = t^r e^{t_1}(1 - \varepsilon) = t^r e^{t_1(1-\varepsilon)} \leq t^r e^{-t^2/2} = o(t^r). \]

Hence \( I_1 = o(t^{r-\alpha}) \). While estimating \( I_2 \) we use the inequalities \( t_1^{\alpha} - t_2^{\alpha} \leq c' \varepsilon t^{\alpha-1} \) and \( \tilde{S}_x^{(0)}(t) \leq 1 \). We obtain
\[ I_2 \leq \mathbb{E}Z^r I_{\{t_1 \leq Z \leq t_2\}} \leq t_2^r P(t_1 \leq Z \leq t_2) = t_2^r (t_2^{\alpha} - t_1^{\alpha}) c(1 + o(1)) = o(t^{r-\alpha}). \]
We finally evaluate $I_3$. From the identity $S_2^{(0)}(t) + S_x^{(0)}(t) = 1$ we obtain $S_x^{(0)}(t) = x(t - S_x^{(0)}(t))$.

Using this expression we write $I_3$ in the form

$$I_3 = E(Z^* 1_{(Z > t_2)}) + R,$$

where $R = E(Z^* S_x^{(0)}(t) 1_{(Z > t_3)})$.

Note that (43) implies

$$E(Z^* 1_{(Z > t_2)}) = (c + o(1))\alpha(\alpha - r)^{-1}t^{r - \alpha}.$$ 

We complete the proof by showing that $R = o(t^{r - \alpha})$. The first inequality of (45) implies

$$R \leq E(Z^{t + r} e^{z} (e/t)^{z_{(Z > t_3)})} \leq t_2^{1+r} e^{-t_2 (e/t)^t} = t_2^{e - t(1 + \varepsilon)^t}.$$

In the second inequality we used the fact that the function $z \rightarrow z^{t + r} e^{z}$ decreases on $(t_2, +\infty)$. In the last inequality we estimated $\ln(1 + \varepsilon) = t \ln(1 + \varepsilon) \leq t(\varepsilon - \varepsilon^2/4)$.

In the next lemma we collect several simple facts used in the proof of Theorem 1.

**Lemma 4.** Let $\alpha \geq \beta > 0$ and $a, b > 0$. Let $t \to +\infty$. Let $\eta, \xi$ be independent non-negative random variables. Assume that

$$P(\eta > t) = (a + o(1))t^{-\alpha} \quad \text{and} \quad P(\xi > t) = (b + o(1))t^{-\beta}.$$ 

Put $c = a + b$ for $\alpha = \beta$, and $c = b$ for $\alpha \geq \beta$. We have

$$P(\eta + \xi > t) = (c + o(1))t^{-\beta}. \quad (47)$$

For completeness, we present the proof Lemma 4.

**Proof of Lemma 4.** We first prove (47) for $\alpha > \beta$. Fix $1 > \gamma > \beta/\alpha$ and split the probability

$$P(\eta + \xi > t) = P(\eta + \xi > t, \eta < t^\gamma) + P(\eta + \xi > t, \eta \geq t^\gamma) =: P_1 + P_2. \quad (48)$$

Here

$$P_1 = E(P(\eta + \xi > t|\eta) 1_{(\eta < t^\gamma)}) = E((b + o(1))(t - \eta)^{-\beta} 1_{(\eta < t^\gamma)})$$

and

$$P_2 \leq P(\eta \geq t^\gamma) = O(t^{-\gamma \alpha}) = o(t^{-\beta}).$$

Let us prove (47) for $\alpha = \beta$. Fix $0.5 < \gamma < 1$. Denote $\phi = \max\{\eta, \xi\}$, $\tau = \min\{\eta, \xi\}$. We have

$$P(\phi > t) = 1 - (1 - P(\eta > t))(1 - P(\xi > t)) = (a + b + o(1))t^{-\beta} + O(t^{-2\beta}), \quad (49)$$

and

$$P(\tau > t) = P(\eta > t)P(\xi > t) = O(t^{-2\beta}).$$

We write $P(\eta + \xi > t) = P(\tau + \phi > t)$ and proceed similarly as in (48):

$$P(\tau + \phi > t) = P(\tau + \phi > t, \tau < t^\gamma) + P(\tau + \phi > t, \tau \geq t^\gamma) =: P_1^* + P_2^*. \quad (50)$$

Here

$$P_1^* \leq P(t^\gamma + \phi > t, \tau < t^\gamma) \leq P(t^\gamma + \phi > t) = P(\phi > t(1 - o(1))) = (a + b + o(1))t^{-\beta}$$

and

$$P_2 \leq P(\tau \geq t^\gamma) = O(t^{-2\beta \gamma}) = o(t^{-\beta}).$$

Finally, (49) and (50) imply

$$(a + b + o(1))t^{-\beta} = P(\phi > t) \leq P(\tau + \phi > t \leq (a + b + o(1))t^{-\beta}. \quad \square$$
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