Threefold extremal curve germs with one non-Gorenstein point

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Abstract. An extremal curve germ is the analytic germ of a threefold with terminal singularities along a reduced complete curve admitting a contraction whose fibres have dimension at most one. The aim of the present paper is to review the results concerning contractions whose central fibre is irreducible and contains only one non-Gorenstein point.

Keywords: extremal curve germ, terminal singularity, canonical divisor, birational map, blow-up, flip, Q-conic bundle.

§ 1. Introduction

One of the most important problems in three-dimensional birational geometry is to describe explicitly all the steps of the Minimal Model Program (MMP). These steps consist of certain maps, called divisorial and flipping contractions as well as fibre-type contractions (Mori contractions). The structure of these maps is still unknown in complete generality, though much progress has been made in this direction. We refer to [1] for an introduction to the subject. The aim of the present paper is to review the results concerning contractions whose fibres have dimension at most 1. The project was started in the original paper [2], where the minimal model problem was solved in the three-dimensional case. To study Mori contractions in this situation, one needs to work in the analytic category, and analytic counterparts of the relevant notions are needed. The central objects of this paper were the so-called extremal curve germs.

An extremal curve germ is the analytic germ of a threefold with terminal singularities along a reduced complete curve admitting a contraction whose fibres have dimension at most one. The present paper is a survey of known results on the classification of objects of this type. We mainly concentrate on the case of an irreducible central fibre with only one non-Gorenstein point. In this case, the results are complete but scattered in the literature. This is the main reason for writing the present survey.

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The classification of extremal curve germs is done in terms of a general element $H$ of the linear system $|\mathcal{O}_X|$ of trivial Cartier divisors containing $C$. In many cases, $H$ is a normal surface and then the threefold can be viewed as the total space of a one-parameter deformation of $H$.

A birational extremal curve germ $f: (X, C) \to (Z, o)$ is said to be semistable if, for a general member $D \in |-K_Z|$, the germ $D_Z := \text{Spec}_Z f_* \mathcal{O}_D$ is Du Val of type $A$; see [3]. The semistable case is divided into two subcases (k1A) and (k2A) according to the number of non-Gorenstein points of $X$ on $C$. The other (non-semistable) cases are said to be exceptional. It turns out that the study of semistable and exceptional germs requires different approaches. For example, in the exceptional flipping case, [3] provides relatively simple computations of the flipped variety. These computations become more explicit for semistable germs. In the case (k2A), considering a general element $H \in |\mathcal{O}_X|$, one can decide whether $(X, C)$ is flipping or divisorial ([4], Corollary 4.1) and describe the resulting flipped variety ([4], Theorem 4.7) or $Z$ ([4], Theorem 4.5) respectively. The case (k1A) was similarly treated in [5] under the additional assumption “$b_2(X_s) = 1$” (see 11.4.6 below). By the local classification (see Propositions 5.4 and 5.5), a semistable extremal curve germ of type (k1A) can be of type (IA$^\vee$) or (IA). They are treated in §§ 8, 9 and 11.

We summarize some results.

1.1. Theorem. Let $f: (X, C) \to (Z, o)$ be a flipping extremal curve germ with irreducible central fibre $C$, and let $H_Z \in |\mathcal{O}_Z|$ be a general hyperplane section containing $C$. Then the surfaces $H_Z$ and $f^{-1}(H_Z)$ are normal and have rational singularities. The singularity $(H_Z, o)$ is log terminal except for the cases described in 11.5.5 and 12.3.3. Moreover, $(H_Z, o)$ is a cyclic quotient singularity if and only if $(X, C)$ is semistable.

The case of a semistable $(X, C)$ follows from Lemma 9.3.1. When $(X, C)$ has only one non-Gorenstein point (that is, it belongs to type (k1A)), we can use an explicit classification (8.2.1, 9.1.6, 10.4, 11.3, 11.5.2, 12.3). We refer to [3], §9, for the remaining case (kAD).

1.2. Theorem. Let $f: (X, C) \to (Z, o)$ be a divisorial extremal curve germ with irreducible central fibre $C$, and let $H_Z \in |\mathcal{O}_Z|$ be a general hyperplane section containing $o$. Then $(H_Z, o)$ is either a Du Val point, a rational log canonical point of type $\tilde{D}$ (in the case 9.1.5), or a cyclic quotient singularity of class $T$. Moreover, the last two possibilities occur only if $(X, C)$ has a locally imprimitive point or $(X, C)$ is semistable and has two non-Gorenstein points whose indices are not coprime.

Moreover, the singularity $(Z, o)$ is terminal by Theorem 6.4. If $(X, C)$ has no covers étale in codimension 1, then $(Z, o)$ is of index 1, and $H_Z$ is Du Val. In the semistable case, the assertion of the theorem follows from Lemma 9.3.1. It remains to consider the locally imprimitive cases (IA$^\vee$) and (II$^\vee$) (see 9.1.5 and 9.1.6).

Note that the surface $H := f^{-1}(H_Z)$ can be non-normal for a divisorial curve germ (see Example 12.6).

For the $\mathbb{Q}$-conic bundle $f: (X, C) \to (Z, o)$ we can show that the base is Du Val of type $A$ (Corollary 7.7.1). The proof uses the existence of a Du Val member $D \in |-K_X|$; see [6], [7] and Theorem 7.3.
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§ 2. Preliminaries

2.1. Threefold terminal singularities. Recall that a three-dimensional terminal singularity of index $m$ is a quotient of an isolated hypersurface singularity by the cyclic group $\mu_m$ of order $m$. More precisely, let $(X, P)$ be an analytic germ of a three-dimensional terminal singularity of index $m$. Then there is a terminal singularity $(X^\sharp, P^\sharp)$ of index 1 and a cyclic $\mu_m$-cover

$$(X^\sharp, P^\sharp) \longrightarrow (X, P)$$

which is étale outside $P$; see [8]. Moreover, the singularity $(X^\sharp, P^\sharp)$ can be embedded in $(\mathbb{C}^4, 0)$ so that its general hyperplane section is a surface Du Val singularity (thus $(X^\sharp, P^\sharp)$ is the so-called cDV-singularity). A detailed classification of all possibilities for the equations of $X^\sharp \subset \mathbb{C}^4$ and the actions of $\mu_m$ was obtained in [9] (see also [10], [11]).

Assume that $m > 1$. Then the $\mu_m$-action on $(X^\sharp, P^\sharp)$ will be analyzed. We fix a character $\chi$ generating $\text{Hom}(\mu_m, \mathbb{C}^*) = \mathbb{Z}/m\mathbb{Z}$. Given a $\mu_m$-semi-invariant $z$, we write

$$\text{wt}(z) \equiv a \mod m$$

if $g(z) = \chi(g)az$ for all $g \in \mu_m$.

2.1.1. Theorem (see [9]). In this notation, the singularity $(X^\sharp, P^\sharp)$ is $\mu_m$-isomorphic to a hypersurface $\phi = 0$ in $(\mathbb{C}^4_{x_1,\ldots,x_4}, 0)$ such that one of the following assertions holds for some $a, b \in \mathbb{Z}$ prime to $m$:

(i) $\text{wt}(x, \phi) \equiv (a, b, -a, 0, 0) \mod m$;
(ii) $m = 4$ and $\text{wt}(x, \phi) \equiv (a, b, -a, 2, 2) \mod m$.

In case (ii) we say that $(X, P)$ is a point of type cAx/4.

Thus the locus $\mathcal{Y} \subset \mathbb{C}^4$ of the points at which the $\mu_m$-action is not free is a coordinate axis not lying in $X^\sharp$. The number

$$\text{aw}(X, P) := \text{mult}_0(\phi|_\mathcal{Y})$$

is well defined and is called the axial multiplicity of $(X, P)$.

2.2. Recall that a contraction is a proper surjective morphism $f: X \rightarrow Z$ of normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$.

2.2.1. Definition. Let $(X, C)$ be the analytic germ of a threefold with terminal singularities along a reduced complete curve. We say that $(X, C)$ is an extremal curve germ if there is a contraction

$$f: (X, C) \rightarrow (Z, o)$$

such that $C = f^{-1}(o)_{\text{red}}$ and $-K_X$ is $f$-ample. Furthermore, $f$ is said to be flipping if its exceptional locus coincides with $C$, and divisorial if its exceptional locus is two-dimensional. If $f$ is not birational, then $Z$ is a surface and $(X, C)$ is called a Q-conic bundle germ [6].
In general, we do not assume that $X$ is $\mathbb{Q}$-factorial. This is because $\mathbb{Q}$-factoriality is not a local condition in the analytic category (see [12], §1).

The following easy example is needed for future reference.

2.3. Example. Consider the following action of $\mu_m$ on $\mathbb{P}^1 \times \mathbb{C}^2_{u,v}$:

$$(x; u, v) \mapsto (\epsilon^a x; \epsilon u, \epsilon^{-1} v),$$

where $\epsilon$ is a primitive $m$th root of unity and $\gcd(m, a) = 1$. Put

$$X := \mathbb{P}^1 \times \mathbb{C}^2 / \mu_m, \quad Z := \mathbb{C}^2 / \mu_m$$

and let $f: X \to Z$ be the natural projection. Since $\mu_m$ acts freely in codimension one, the divisor $-K_X$ is $f$-ample. The images of two fixed points on $\mathbb{P}^1 \times \mathbb{C}^2$ are terminal cyclic quotient singularities of types $(1/m)(\pm a, 1, -1)$ on $X$. Hence, $f$ is a $\mathbb{Q}$-conic bundle. Any $\mathbb{Q}$-conic bundle germ biholomorphic to $f$ as above is said to be toroidal.

The following key fact is an immediate corollary of the Kawamata–Viehweg vanishing theorem.

2.4. Theorem. Let $f: (X, C) \to (Z, o)$ be an extremal curve germ. Then $R^i f_* \mathcal{O}_X = 0$ when $i > 0$.

2.4.1. Corollary (compare with [2], Remark 1.2.1, Corollary 1.3).

(i) If $\mathcal{I}$ is an ideal such that $\text{Supp}(\mathcal{O}_X / \mathcal{I}) \subset C$, then $H^1(\mathcal{O}_X / \mathcal{I}) = 0$.

(ii) $p_a(C) = 0$ and $C$ is a union of smooth rational curves.

(iii) $\text{Pic} X \cong H^2(C, \mathbb{Z}) \cong \mathbb{Z}^\rho$, where $\rho$ is the number of irreducible components of $C$.

2.4.2. Remark. When $C$ is reducible, the germ $(X, C')$ is also an extremal curve germ for every proper curve $C' \subset C$.

2.5. Lemma. Let $f: (X, C) \to (Z, o)$ be an extremal curve germ.

(i) If $f$ is birational, then there is an effective $\mathbb{Q}$-divisor $B$ on $Z$ such that the pair $(Z, B)$ has only a canonical singularity at $o$. If, moreover, $f$ is flipping, then the singularity of $(Z, B)$ at $o$ is terminal.

(ii) If $f$ is a $\mathbb{Q}$-conic bundle, then $Z$ has a log terminal singularity at $o$.

Proof. Take $n \gg 0$ so that the divisor $nK_X$ is Cartier and the linear system $|-nK_X|$ is base-point-free. Let $H \in |-nK_X|$ be a general member. Then $H$ is a smooth surface meeting the components of $C$ transversally. For (i), put $D := (1/n)H$ and $B := f_* D$. Then the singularities of the pair $(X, D)$ are terminal. Since $f$ is crepant with respect to $K_X + D$ and does not contract components of $D$, we see that the singularities of $(Z, B)$ are canonical ([13], Lemma 3.38). To show (ii), we note that the restriction $f_H: H \to Z$ is a finite morphism. Thus $(Z, o)$ is a log terminal singularity ([13], Proposition 5.20). □

However, we do not assert in (i) that the point $(Z, o)$ is $\mathbb{Q}$-Gorenstein, even in the divisorial case; see Theorem 6.4. The result of (ii) is significantly improved in 4.6.5 and [6], 1.2.7.
2.6. A general member of $|\mathcal{O}_X|$. Let $f : (X, C) \to (Z, o)$ be an extremal curve germ. When $f$ is a $\mathbb{Q}$-conic bundle, we assume that $(Z, o)$ is smooth. We write $|\mathcal{O}_Z|$ for the linear system of Cartier divisors (hyperplane sections) passing through $o$ and put $|\mathcal{O}_X| := f^*|\mathcal{O}_Z|$. Let $H$ be a general member of $|\mathcal{O}_X|$, and put $H_Z = f(H)$. Let $H^n \to H$ be the normalization (we put $H^n = H$ if $H$ is normal). By [13], §5.25, and Lemma 2.5 both surfaces $H_Z$ and $H$ are Cohen–Macaulay. Hence, by Bertini’s theorem, $H_Z$ is normal. Then the composite map $H^n \to H_Z$ has connected fibres. Moreover, it is a fibration into rational curves when $\dim Z = 2$, and a birational contraction of a divisor to a point $(H_Z, o)$ when $f$ is birational. Thus $H^n$ has only rational singularities in the case of $\mathbb{Q}$-conic bundles. The same holds in the birational case if the singularity $(H_Z, o)$ is rational.

2.7. Notation for dual graphs. Let $S$ be a normal surface and let $C \subset S$ be a curve. Suppose that the exceptional divisors and the proper transform of $C$ on the minimal resolution of $S$ form a normal-crossing divisor, call it $R$. We use the standard notation for the dual graph $\Delta(S, C)$ of $R$: each vertex $\diamond$ corresponds to an irreducible component of $C$, each vertex $\circ$ corresponds to an exceptional divisor, and we may use $\bullet$ instead of $\diamond$ to emphasize that it is a complete $(-1)$-curve. A number attached to a vertex denotes minus the self-intersection number. For brevity, we may omit 2 when the self-intersection number is equal to $-2$.

2.7.1. Proposition (see [14]). (i) Let $f : X \to Z$ be a $\mathbb{Q}$-conic bundle. If $X$ is Gorenstein (and terminal), then $Z$ is smooth and there is a vector bundle $\mathcal{E}$ of rank 3 on $Z$ and an embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ such that every scheme fibre $X_z, z \in Z$, is a conic in $\mathbb{P}(\mathcal{E})_z$.

(ii) (See also [3], §4.7.2.) Let $f : (X, C) \to (Z, o)$ be a birational curve germ such that $X$ is Gorenstein. Then $f$ is divisorial, $(Z, o)$ is smooth, $C$ is irreducible, and $f$ is the blow-up of a curve $B \subset Z$ having only planar singularities. Moreover, $X$ has exactly one singular point, which is of type $cA$, and the graph $\Delta(H, C)$ has the following form for a general element $H \in |\mathcal{O}_X|$: 

\[
\bullet - \circ - \cdots - \circ \quad \stackrel{m}{\text{.}}
\]

The following fact is a particular case of Theorem 4.9 in [3].

2.8. Theorem. Let $f : (X, C) \to (Z, o)$ be a divisorial extremal curve germ. Let $E$ be its exceptional locus (with reduced structure) and put $B := f(E)_{\text{red}}$. Assume that $K_Z$ is a $\mathbb{Q}$-Cartier divisor (this holds automatically when $C$ is irreducible; see 6.4). Then the following assertions hold.

(i) The exceptional set $E$ of $f$ is purely two-dimensional and is a $\mathbb{Q}$-Cartier divisor, and the singularity $(Z, o)$ is terminal.

(ii) The variety $X$ is the symbolic blow-up of $B$, that is,

$$X = \text{Proj}_Z \bigoplus_{m=0}^{\infty} \mathcal{I}_B^{(m)},$$

where $\mathcal{I}_B$ is the ideal sheaf of $B$ and $\mathcal{I}_B^{(m)}$ denotes its symbolic power. In particular, $X$ is uniquely determined by $B \subset Z$. 

It is possible to study divisorial curve germs algebraically, by a scrupulous analysis of the curve \( B \) and its embedding \( B \subset X \) (see [15]–[18], [19], §6.1, [20]). This method is completely different from our approach.

§ 3. Basic techniques

3.1. Let \( \mathcal{I}_C \subset \mathcal{O}_X \) be the ideal sheaf of \( C \) and let \( \mathcal{I}_C^{(n)} \) be its symbolic \( n \)th power, that is, the saturation of \( \mathcal{I}_C^n \) in \( \mathcal{O}_X \). We put

\[
\text{gr}^n_C \mathcal{O} := \mathcal{I}_C^{(n)}/\mathcal{I}_C^{(n+1)}.\]

Furthermore, let \( F^n \omega_X \) be the saturation of \( \mathcal{I}_C^n \omega_X \) in \( \omega_X \). We put

\[
\text{gr}^n_C \omega := F^n \omega_X / F^{n+1} \omega_X.
\]

Let \( m \) be the index of \( K_X \). We have natural homomorphisms

\[
\alpha_1: \bigwedge^2 \text{gr}^1_C \mathcal{O} \rightarrow \mathcal{H}om_{\mathcal{O}_C}(\Omega^1_C, \text{gr}^0_C \omega),
\]

\[
\beta_0: (\text{gr}^0_C \omega)^{\otimes m} \rightarrow (\omega_X^{\otimes m})^{\vee \vee} \otimes \mathcal{O}_C.
\]

Write

\[
i_P(1) := \text{len}_P \text{Coker}(\alpha_1), \quad w_P(0) := \text{len}_P \text{Coker}(\beta_0)/m. \quad (3.1.1)
\]

To study extremal germs more carefully, Mori [2] also introduced local invariants \( i_P(n) \), \( w_P(n) \), \( w^*_P(n) \) similar to \( i_P(1) \) and \( w_P(0) \). We do not define them here.

Assume that \( C \simeq \mathbb{P}^1 \). Then, by § 2.3.1 in [2], we have

\[
- \deg \text{gr}^0_C \omega = -K_X \cdot C + \sum_P w_P(0), \quad (3.1.2)
\]

\[
2 + \deg \text{gr}^0_C \omega - \deg \text{gr}^1_C \mathcal{O} = \sum_P i_P(1). \quad (3.1.3)
\]

Since \( \text{rk} \text{gr}^1_C \mathcal{O} = 2 \), we obtain from Theorem 2.4 that

\[
\deg \text{gr}^1_C \mathcal{O} \geq -2, \quad (3.1.4)
\]

\[
4 \geq - \deg \text{gr}^0_C \omega + \sum_P i_P(1) = -K_X \cdot C + \sum_P w_P(0) + \sum_P i_P(1). \quad (3.1.5)
\]

3.1.6. Remark. When \( f \) is birational, we have \( \text{gr}^0_C \omega = \mathcal{O}_C(-1) \) by the Grauert–Riemenschneider vanishing theorem (see [2], §2.3). This is not the case for \( \mathbb{Q} \)-conic bundles: easy computations show that \( \deg \text{gr}^0_C \omega = -2 \) in the toroidal example (Example 2.3; see (3.1.2)). We similarly have \( \deg \text{gr}^0_C \omega = -2 \) in the case 9.1.2. Below, we shall prove that these two examples are the only exceptions (see Corollaries 4.7.8 and 5.5.1).
3.2. Let \((X, P)\) be the germ of a threefold terminal singularity. Throughout the paper, \((X^2, P^2) \rightarrow (X, P)\) denotes the cover of index 1. Given any object \(V\) on \(X\), we denote the pullback of \(V\) on \(X^2\) by \(V^2\).

3.3. **Lemma** ([2], §2.16). In this notation, assume that the curve \(C^2\) is smooth. Put
\[
\ell(P) := \text{len}_P T_C^{(2)}/I_C^{(2)},
\]
where \(I_C^{(2)}\) is the ideal of \(C^2\) in \(X^2\). Then
\[
i_P(1) = \begin{cases} 
\ell(P) & \text{if } m=1, \\
[\ell(P) + 6]/4 & \text{if } (X, P) \text{ is of type } \text{cAx}/4, \\
\ell(P)/m + 1 & \text{if } (X, P) \text{ is not as above.}
\end{cases}
\]

3.4. **Lemma** ([2], §§2.10, 2.15). If \((X, P)\) is singular, then \(i_P(1) \geq 1\). If \((X, P)\) is not Gorenstein, then \(w_P(0) > 0\).

Then we obtain the following corollary from (3.1.5).

3.4.1. **Corollary.** An extremal curve germ \((X, C \simeq \mathbb{P}^1)\) has at most three singular points.

3.5. Let \((X, C)\) be an extremal curve germ. By Lemma 2.4.1(i), we have \(H^1(\text{gr}^1_C \mathcal{O}) = 0\). Then the standard exact sequence
\[
0 \longrightarrow I_C^{(n+1)} \longrightarrow I_C^{(n)} \longrightarrow \text{gr}^n_C \mathcal{O} \longrightarrow 0
\]
yields the following easy but useful fact.

3.5.1. **Lemma.** The following assertions hold.

(i) If \(H^1(\text{gr}^n_C \mathcal{O}) = 0\) and the map \(H^0(I_C^{(n)}) \rightarrow H^0(\text{gr}^n_C \mathcal{O})\) is surjective, then \(H^1(I_C^{(n+1)}) \simeq H^1(I_C^{(n)})\). In particular, \(H^1(I) = 0\) from the case \(n = 0\).

(ii) If for all \(i < n\) one has \(H^1(\text{gr}^i_C \mathcal{O}) = 0\) and the map \(H^0(I_C^{(i)}) \rightarrow H^0(\text{gr}^i_C \mathcal{O})\) is surjective, then \(H^1(I_C^{(n)}) \simeq H^1(\text{gr}^n_C \mathcal{O}) = 0\).

(iii) If \(H^0(\text{gr}^1_C \mathcal{O}) = 0\), then \(H^1(I_C^{(2)}) = H^1(\text{gr}^2_C \mathcal{O}) = 0\).

In particular, if a general member \(H \in |\mathcal{O}_X|\) is normal, then \(H^0(\text{gr}^1_C \mathcal{O}) \neq 0\).

However, we note that this condition is necessary but not sufficient for the normality of \(H\); see [21].

3.6. The sheaves \(\text{gr}^n_C \omega\).

3.6.1. **Lemma.** Let \(f: (X, C) \rightarrow (Z, o)\) be an extremal curve germ.

(i) (See [2], §1.2.) If \(f\) is birational, then \(R^i f_* \omega_X = 0\) for \(i > 0\).

(ii) (See [6], Lemma 4.1.) If \(f\) is a \(\mathbb{Q}\)-conic bundle and \(Z\) is smooth, then there is a canonical isomorphism
\[
R^1 f_* \omega_X \simeq \omega_Z.
\]

**Proof.** Part (i) follows from the Grauert–Riemenschneider vanishing theorem. We now prove (ii). Let \(g: W \rightarrow X\) be a resolution. By Proposition 7.6 in [22] we have \(R^1(f \circ g)_* \omega_W = \omega_Z\). Since \(X\) has only terminal singularities, we have \(g_* \omega_W = \omega_X\) and, by the Grauert–Riemenschneider vanishing theorem, \(R^i g_* \omega_W = 0\) for \(i > 0\). Then the Leray spectral sequence gives us \(R^1 f_* \omega_X = R^1(f \circ g)_* \omega_W = \omega_Z\).
We also have the following useful fact.

3.7. **Corollary.** Let \( f: (X, C) \simeq \mathbb{P}^1 \to (Z, o) \) be an extremal curve germ.

(i) If \( f \) is birational, then \( \deg \text{gr}_C^0 \omega = -1 \).

(ii) Assume that \( f \) is a \( \mathbb{Q} \)-conic bundle with smooth base. If \( \deg \text{gr}_C^0 \omega \neq -1 \), then \( f^{-1}(o) = C \) (as a scheme).

**Sketch of proof.** To prove (i), we note from 3.6.1(i) that \( H^1(\omega_X/\mathcal{J}\omega_X) = 0 \) for an arbitrary ideal \( \mathcal{J} \) such that \( \text{Supp}(\mathcal{O}_X/\mathcal{J}) \subset C \). Hence, \( H^1(\text{gr}_C^0 \omega) = 0 \) in this case. On the other hand, \( \deg \text{gr}_C^0 \omega < 0 \) by (3.1.2). To prove (ii), we use Theorem 4.4 in [6] with \( J = \mathcal{I}_C \). □

3.7.1. **Lemma** ([2], Corollary 1.15, [23], Proposition 4.2, [6], Lemma 4.4.2). Let \( f: (X, C) \to (Z, o) \) be an extremal curve germ. Suppose that \( C \) is reducible and let \( P \) be a singular point of \( C \). If \( X \) is Gorenstein at \( P \), then \( f \) is a \( \mathbb{Q} \)-conic bundle and \( C \) has two components meeting at \( P \). If, moreover, \( (Z, o) \) is smooth, then \( X \) is Gorenstein (see Proposition 2.7.1(i)).

We will show in Corollary 4.7.6 that \( (Z, o) \) is automatically smooth under these assumptions.

**Proof.** By Corollary 2.4.1, there are at least two components (say, \( C_1, C_2 \subset C \)) passing through \( P \). Replacing \( (X, C) \) by \( (X, C_1 \cup C_2) \), we may assume that \( C = C_1 \cup C_2 \) (see Remark 2.4.2). Since the point \( P \in X \) is Gorenstein, the sheaf \( \text{gr}_C^0 \omega = \omega_X \otimes \mathcal{O}_C \) is invertible at \( P \). Consider the injection

\[
\varphi: \text{gr}_C^0 \omega \hookrightarrow \text{gr}_{C_1}^0 \omega \oplus \text{gr}_{C_2}^0 \omega.
\]

Recall that \((X, C_i)\) is a (birational) extremal curve germ by Remark 2.4.2. Then \( \text{gr}_{C_i}^0 \omega = \mathcal{O}_{C_i}(-1) \) by Corollary 3.7(i). Hence \( H^0(\text{Coker}(\varphi)) = H^1(\text{gr}_C^0 \omega) \). On the other hand, \( \text{Coker}(\varphi) \) is a sheaf of finite length supported at \( P \). Since \( \text{gr}_C^0 \omega \) is invertible, \( \text{Coker}(\varphi) \) is non-trivial. Therefore \( H^1(\text{gr}_C^0 \omega) \neq 0 \) and, by Corollary 3.7, the contraction \( f \) is a \( \mathbb{Q} \)-conic bundle. Moreover, if the base \((Z, o)\) is smooth, then again by Corollary 3.7, we have \( C = f^{-1}(o) \) (scheme-theoretically). Hence \( P \) is the only singular point of \( X \) and we are done. □

§ 4. **Topological observations**

Let \( \text{Cl}^{\text{sc}}(X) \) be a subgroup of the divisor class group \( \text{Cl}(X) \) consisting of Weil divisor classes which are \( \mathbb{Q} \)-Cartier. The following easy corollary of the classification of terminal singularities will be used without special reference.

4.1. **Proposition** ([12], Lemma 5.1). Let \( (X, P) \) be the (analytic) germ of a three-dimensional terminal singularity of index \( m \). Then

\[
\text{Cl}^{\text{sc}}(X, P) \simeq \pi_1(X \setminus \{P\}) \simeq \mathbb{Z}/m\mathbb{Z}.
\]  

(4.1.1)

4.2. **Definition** ([2], (0.4.16), (1.7)). Let \( (X, P) \) be a terminal three-dimensional singularity of index \( m \) and let \( C \subset X \) be a smooth curve passing through \( P \). We say that \( C \) is (locally) primitive at \( P \) if the natural map

\[
\varrho: \mathbb{Z} \simeq \pi_1(C \setminus \{P\}) \longrightarrow \pi_1(X \setminus \{P\}) \simeq \mathbb{Z}/m\mathbb{Z}
\]
We have an exact sequence
\[ \theta \]
following assertions hold
\[ \theta \\ d \neq \text{Cl} \]
\[ P \]
4.4. Let \( (X, C) \) be an extremal curve germ with irreducible central fibre \( C \). Let \( P_1, \ldots, P_n \) be the non-Gorenstein points of \( X \) and let \( m_1, \ldots, m_n \) be their indices. We have an exact sequence
\[ 0 \longrightarrow \text{Pic}(X) \longrightarrow \text{Cl}^\text{sc}(X) \longrightarrow \bigoplus \text{Cl}^\text{sc}(X, P_i) \longrightarrow 0. \]

4.4.2. Corollary. In this notation, assume that \( C \) is irreducible. For every \( i = 1, \ldots, n \) let \( D_i \) be an effective Weil \( \mathbb{Q} \)-Cartier divisor whose class generates \( \text{Cl}^\text{sc}(X, P_i) \), and let \( H \) be an effective Cartier divisor such that \( H \cdot C = 1 \). Then the following assertions hold.

(i) The group \( \text{Cl}^\text{sc}(X) \) is generated by the classes of \( H, D_1, \ldots, D_n \).

(ii) If the point \( P_i \) is imprimitive of splitting degree \( s_i \) and subindex \( m_i \), then the class of \( H - m_i D_i \) is an \( s_i \)-torsion element of \( \text{Cl}^\text{sc}(X) \).

(iii) If \( (X, C) \) is locally primitive at distinct points \( P_i, P_j \in C \) and \( \gcd(m_i, m_j) = d \neq 1 \), then the class of \( \frac{m_i}{d} D_i - \frac{m_j}{d} D_j \) is a \( d \)-torsion element of \( \text{Cl}^\text{sc}(X) \).

4.5. Construction. Let \( f : (X, C) \to (Z, o) \) be an extremal curve germ and let \( \theta : (X^b, C^b) \to (X, C) \) be a finite cover which is étale in codimension 1. Clearly, \( \theta \) must be étale over the Gorenstein locus of \( X \). The Stein factorization gives us the diagram
\[ \begin{array}{ccc}
(X^b, C^b) & \xrightarrow{\theta} & (X, C) \\
\downarrow f^b & & \downarrow f \\
(Z^b, o^b) & \longrightarrow & (Z, o),
\end{array} \]
where \( (Z^b, o^b) \to (Z, o) \) is a finite cover which is étale over \( Z \setminus \{o\} \). We have \( K_{X^b} = \theta^* K_X \) and the singularities of \( X^b \) are terminal. In particular, \( (X^b, C^b) \) is an extremal curve germ. Note that in our situation \( X^b \) coincides with the normalization of \( X \times_Z Z^b \) and \( C^b := f^{b-1}(C)_{\text{red}} \).
Conversely, if \( f: (X, C) \to (Z, o) \) is an extremal curve germ and \((Z^b, o^b) \to (Z, o)\) is a finite cover which is étale over \( Z \setminus \{ o \} \), then the base change produces the diagram \((4.5.1)\) where \( X^b \) is the normalization of \( X \times_Z Z^b \), \((X^b, C^b)\) is an extremal curve germ and \( \theta \) is étale in codimension 1.

4.6. Definition. Let \((X, C)\) be an extremal curve germ. By Construction 4.5, the torsion part \( \text{Cl}(X)_{\text{tors}} \subset \text{Cl}(X) \) determines an abelian Galois cover

\[
\tau: (X', C') \longrightarrow (X, C),
\]

which is étale over the Gorenstein locus of \( X \). We call this map the torsion-free cover of \((X, C)\). The degree of this cover is called the topological index of \((X, C)\).

Similarly to \((4.5.1)\), we have a diagram

\[
\begin{array}{ccc}
(X', C') & \xrightarrow{\tau} & (X, C) \\
\downarrow f' & & \downarrow f \\
(Z', o') & \longrightarrow & (Z, o).
\end{array}
\]

Hence \((X', C')\) is also an extremal curve germ. Clearly, \( \text{Cl}(X') \) is torsion-free.

4.6.3. Lemma. Let \((X, C)\) be an extremal curve germ and let \( \theta: (X^b, C^b) \to (X, C) \) be a finite cover which is étale in codimension two. Then \( \theta \) is a cyclic cover.

Proof. We may assume that \( \theta \) is a Galois cover with group \( G \). It suffices to show that \( G \) is cyclic. Taking the composite with the torsion-free cover, we may also assume that \( \text{Cl}^{\text{sc}}(X^b) \) is torsion-free. By construction, \( G \) acts effectively on \( C^b = \bigcup C_i^b \) (because \( X \) has only isolated singularities). Since \( C^b \) is a tree of smooth rational curves, it is easy to prove by induction on the number of components of \( C^b \) that \( G \) has either an invariant component \( C_i^b \subset C^b \) or a fixed point \( P^b \in \text{Sing}(C^b) \).

In the latter case, let \( P = \theta(P^b) \). There is a surjection \( \pi_1(X \setminus \{ P \}) \to G \). Since \( \pi_1(U \setminus \{ P \}) \) is cyclic (see 4.1), we are done.

In the former case, let \( C_i := \theta(C_i^b) \). By Remark 2.4.2 we may replace \((X^b, C^b)\) by \((X^b, C_i^b)\) and \((X, C)\) by \((X, C_i)\). Thus \( C = C_i, C^b = C_i^b \) and \( C^b / G = C \cong \mathbb{P}^1 \).

Assume that \( G \) is not cyclic. Then there are no fixed points on \( C^b \). If \( X^b \) has a point of index \( m > 1 \), then its orbit contains at least two points of the same index. By \((4.4.1)\), the torsion part of \( \text{Cl}^{\text{sc}}(X^b) \) is non-trivial. This contradicts the assumption above. Thus \( X^b \) is Gorenstein.

Let \( P_1, \ldots, P_n \in C \) be all the branch points of the morphism \( C^b \to C \) and let \( m_1, \ldots, m_n \) be their ramification indices. By the Hurwitz formula we can write

\[
\frac{1}{|G|} (2g(C_i^b) - 2) = 2g(C_i) - 2 + \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right).
\]

Hence, \( \sum 1/m_i > n - 2 \). Since the group \( G \) is not cyclic, we have \( n > 2 \). The index of \( P_i \in X \) is equal to \( m_i \). By \((3.1.1)\) and Lemma 3.4 we have \( w_{P_i}(0) \geq 1/m_i \) and \( i_{P_i}(1) \geq 1 \). Therefore, \( \sum w_{P_i}(0) > 1 \) and \( \deg \text{gr}_C^0 \omega = -1 \) by \((3.1.5)\). Then we get a contradiction by \((3.1.2)\). □
4.6.4. Corollary. Let \((X, C)\) be an extremal curve germ. Then the torsion part \(\text{Cl}(X)_{\text{tors}} \subset \text{Cl}(X)\) is a cyclic group. Hence the torsion-free cover (4.6.1) is cyclic. Moreover, \((X', C')\) has no finite cover which is étale in codimension 1.

4.6.5. Corollary ([24], Lemma 1.10). If \(f: (X, C) \to (Z, o)\) is a \(\mathbb{Q}\)-conic bundle germ, then \((Z, o)\) is a cyclic quotient singularity.

Proof. This follows from Lemma 4.6.3 and Construction 4.5. \(\square\)

4.7. From now on, we assume that \(f: (X, C) \to (Z, o)\) is an extremal curve germ with \(C \simeq \mathbb{P}^1\). Assume that the torsion part \(\text{Cl}(X)_{\text{tors}} = \mathbb{Z}/d\mathbb{Z}\) is non-trivial and consider the torsion-free cover (4.6.1). Thus, \((X, C) = (X', C')/G\) and \((Z, o) = (Z', o')/G\), where \(G = \mu_d\) acts freely on \(Z' \setminus \{o'\}\) and \(X' \setminus \tau^{-1}(\text{Sing}(X))\). We distinguish two cases (compare with [2], formula (1.12)).

4.7.1. Case: \(C'\) is irreducible. Then \(G = \mu_d\) has exactly two fixed points \(P_1'\) and \(P_2'\) on \(C' \simeq \mathbb{P}^1\). They give us two points \(P_i := \tau(P_i')\) on \(C\) whose indices are divisible by \(d\). The germ \((X, C)\) is locally primitive along \(C\).

4.7.2. Case: \(C' = \bigcup_{i=1}^s C_i'\), where \(s > 1\) and \(C_i' \simeq \mathbb{P}^1\). In this case, \(G\) acts on \(\{C_1', \ldots, C_s'\}\) transitively. Since \(p_a(C') = 0\), each component \(C_i'\) meets the closure of \(C' \setminus C_i'\) at one point. Therefore, in this case, all the irreducible components \(C_i'\) pass through a single point \(P'\) and do not meet each other elsewhere. In this case \((X, C)\) is imprimitive at \(\tau(P')\) of splitting degree \(s\) and has no other locally imprimitive points.

In the case 4.7.2, it is worthwhile to mention that \(\tau(P')\) is the only non-Gorenstein point of \(X\) and \(d = s\) (see [2], Theorems 6.7, 9.4, and [6], §7).

4.7.3. Corollary ([2], (1.10)). Let \((X, C \simeq \mathbb{P}^1)\) be an extremal curve germ. Let \(P_1, \ldots, P_n\) be all the non-Gorenstein points of \(X\). Then the following conditions are equivalent:

(i) \(D \cdot C = 1/m_1 \cdots m_n\) for some \(D \in \text{Cl}^{\text{sc}}(X)\);
(ii) \(\text{Cl}^{\text{sc}}(X) \simeq \mathbb{Z}\);
(iii) \(\text{Cl}^{\text{sc}}(X)\) is torsion-free;
(iv) \((X, C)\) is locally imprimitive and \(\gcd(m_i, m_j) = 1, i \neq j\).

Proof. This follows from Lemma 4.6.3 and (4.4.1). \(\square\)

4.7.4. Corollary (compare with [6], Lemma 2.8). Let \((X, C \simeq \mathbb{P}^1)\) be an extremal curve germ, \(d\) the topological index of \((X, C)\), and \(m_1, \ldots, m_r\) the indices of all the non-Gorenstein points. Assume that \((X, C)\) is either a divisorial germ or a \(\mathbb{Q}\)-conic bundle which is not toroidal (Example 2.3). Then

\[-K_X \cdot C = d/m_1 \cdots m_r.\]  \(4.7.5\)

Proof. It follows from (4.4.1) that the ample generator \(D\) of the group \(\text{Cl}^{\text{sc}}(X)/\equiv\) satisfies \(D \cdot C = d/m_1 \cdots m_r\). Write \(-K_X \equiv aD\) for some \(a \in \mathbb{Z}\). Taking the intersection of \(D\) and \(K_X\) with a general one-dimensional fibre \(L\), we obtain \(-K_X \cdot L = D \cdot L\) and \(a = 1\). \(\square\)

We can now strengthen Lemma 3.7.1.
4.7.6. **Corollary.** Let \( f: (X, C) \to (Z, o) \) be an extremal curve germ. Suppose that \( C \) is reducible and let \( P \) be a singular point of \( C \). If \( X \) is Gorenstein at \( P \), then \((Z,o)\) is smooth and \( f \) is a Gorenstein conic bundle.

**Proof.** By Lemma 3.7.1, \( f \) is a \( \mathbb{Q} \)-conic bundle and \((Z,o)\) is a singular point. We recall that \((Z,o)\) is a quotient singularity (see Lemma 2.5). Thus there is a finite Galois cover \((Z^p, o^p) \to (Z, o)\) étale over \( Z \setminus \{ o \} \), where the point \((Z^p, o^p)\) is smooth. Then we can consider the base change (see (4.5.1)). Thus \( X = X^p / G \), where \( G \) is a finite group acting freely on \( X^p \) outside a finite set of points. Since \( X \) is Gorenstein at \( P \), so is \( X^p \) at all the points \( P_i^p \in \theta^{-1}(P) \). Moreover, \( \theta \) is étale over \( P \) by (4.1.1). Hence, the central curve \( C^p \) is singular at \( P_i^p \). The variety \( X^p \) is Gorenstein by Lemma 3.7.1, and the contraction \( f^p: X^p \to Z^p \) is a standard Gorenstein conic bundle by Corollary 2.7.1. In particular, \( C^p \) is a plane conic. Thus \( C^p \) has two components meeting at a single point \( \theta^{-1}(P) \), which must be fixed by \( G \). Again by (4.1.1), the group \( G \) is trivial, a contradiction. \( \square \)

4.7.7. **Corollary.** Let \( f: (X, C \simeq \mathbb{P}^1) \to (Z, o) \) be an extremal curve germ. Assume that \((X, C)\) is locally imprimitive at \( P \). If the subindex of \( P \) is equal to 1, then \( f \) is a \( \mathbb{Q} \)-conic bundle and the contraction \( f' \) in the diagram (4.6.2) is a Gorenstein conic bundle.

\( \mathbb{Q} \)-conic bundles that are quotients of Gorenstein conic bundles over a finite group were described in [24], §2. It turns out that such a \( \mathbb{Q} \)-conic bundle is locally imprimitive if and only if it is of type 9.1.2.

4.7.8. **Corollary** (compare with Proposition 1.14 in [2]). Let \( f: (X, C) \to (Z, o) \) be an extremal curve germ. Assume that \( C \) is irreducible. If \( \text{gr}_C^0 \omega \not\cong \mathcal{O}_C(-1) \), then \( f \) is a \( \mathbb{Q} \)-conic bundle and, in the notation of (4.6.2), we have \( f'^{-1}(o') = C' \). If, furthermore, \((X, C)\) is locally primitive, then it is toroidal (Example 2.3).

**Proof.** By Remark 3.1.6, the contraction \( f \) is a \( \mathbb{Q} \)-conic bundle. We apply the construction (4.6.2). Since

\[
H^1(\text{gr}_C^0 \omega) = H^1(\text{gr}_C^0 \omega)^{\mu_d},
\]

we have \( H^1(\text{gr}_C^0 \omega) \neq 0 \). By Corollary 3.7, \( C' = f'^{-1}(o') \). If \( f \) is locally primitive, then \( C' \) is irreducible (see 4.7.1). Therefore \( C' \simeq \mathbb{P}^1 \) and \( X' \) is smooth. Up to an analytic isomorphism, we may assume that there is a \( \mu_d \)-equivariant decomposition \( X' \simeq Z' \times \mathbb{P}^1 \). Hence, \( f \) is toroidal as in Example 2.3 ([24], §2). \( \square \)

4.8. **Proposition** ([6], Lemma 9.2.3, [2], §0.13.3). An extremal curve germ \((X, C \simeq \mathbb{P}^1)\) has at most two non-Gorenstein points.

**Proof.** Assume that \( P_1, P_2, P_3 \in X \) are singular points of indices \( m_1, m_2, m_3 > 1 \). If \((X, C)\) is locally imprimitive at some point, then the torsion-free cover \( \tau: (X', C') \to (X, C) \) takes the form 4.7.2, that is, \( C' \) is a union of \( s \) components \( C_1', \ldots, C_s' \) passing through a single point, say \( P' \), and \( \tau \) is étale over \( X' \setminus \{ P' \} \). By Corollary 4.7.7, the point \( P' \in X' \) is not Gorenstein. Thus, for any component \( C_i' \), the germ \((X, C_i')\) has at least three non-Gorenstein points.

Replacing \((X, C)\) by \((X', C_i')\), we may assume that \((X, C)\) is locally primitive, that is, the maps \( \pi_1(C \setminus \{ P_i \}) \to \pi_1(U_i \setminus \{ P_i \}) \) are surjective, where \( U_i \subset X \) is
a small neighbourhood of \( P_i \). Then one can easily compute the fundamental group of the open set \( X \setminus \{ P_1, P_2, P_3 \} \) using Van Kampen’s theorem and (4.1.1):

\[
\pi_1(X \setminus \{ P_1, P_2, P_3 \}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle / \{ \sigma_1^{m_1} = \sigma_2^{m_2} = \sigma_3^{m_3} = 1 \}.
\]

This group has a finite quotient group \( G \) in which the images of \( \sigma_1, \sigma_2, \sigma_3 \) are of orders exactly \( m_1, m_2, m_3 \) respectively (see, for example, [25]). This quotient determines a finite Galois cover \( \tau: (X', C') \to (X, C) \) with non-abelian Galois group \( G \). This contradicts Lemma 4.6.3. \( \square \)

§ 5. Local description

5.1. Notation. Let \( (X, P) \) be a threefold terminal singularity of index \( m \) and let \( C \subset (X, P) \) be a smooth curve such that \( P \) has subindex \( m \) and splitting degree \( s \) (see Definition 4.2). We use the notation in 2.1. Put \( C^s := \pi^{-1}(C) \). Then \( C^s \) has \( s \) irreducible components.

Let \( (C^i, P^i) \) be the normalization of an irreducible component \( C^s(i) \subset C^s \), \( 1 \leq i \leq s \), and let \( \tau: (X', C') \to (X, C) \) be the torsion-free cover (see (4.6.2)). Then \( \mu_m \) acts naturally on \( (X^s, P^s) \) and \( (C^s, P^s) \), and so does \( \mu_{m^*} \) on \( (C', P') \). Let

\[
\eta: \mathcal{O}_{X^s, P^s} \to \mathcal{O}_{X', P'}
\]

be the natural map. Since \( (X, P) \) and \( (C, P) \) are normal, one has

\[
\mathcal{O}_{X, P} = (\mathcal{O}_{X^s, P^s})^{\mu_m} \quad \text{and} \quad \mathcal{O}_{C, P} = (\mathcal{O}_{C^s, P^s})^{\mu_{m^*}}.
\]

Since \( \mu_{m^*} \) acts freely on \( X^s \setminus \{ P^s \} \), so it does on \( C^s \setminus \{ P^s \} \) and hence \( \mu_{m^*} \) on \( C'^s \setminus \{ P'^s \} \). Therefore \( \mathcal{O}_{C^s, P^s} \) has a uniformizing parameter (say, \( t \)) which is a \( \mu_{m^*} \)-semi-invariant. Let \( \chi \) be a generator of the group \( \text{Hom}(\mu_{m^*}, \mathbb{C}^*) = \mathbb{Z}/m\mathbb{Z} \) whose restriction \( \bar{\chi} \) to \( \mu_{m^*} \) is the character associated with \( t \). Then \( \mathcal{O}_{C, P} = \mathbb{C}\{t\}^{\mu_{m^*}} \).

Given a semi-invariant \( z \neq 0 \), we write \( C^s\text{-}wt(z) \) (or simply \( wt(z) \) when no confusion is possible) for the element \( n \in \mathbb{Z}/m\mathbb{Z} \) such that \( n\chi \) is the character associated with \( z \). Given a \( \mu_{m^*} \)-semi-invariant \( z \in \mathcal{O}_{X^s, P^s} \), we put

\[
C^s\text{-ord}(z) := \sup \{ n \in \mathbb{Z}_{\geq 0} \mid \eta(z) \in t^n\mathbb{C}\{t\} \}.
\]

We also write \( \text{ord}(z) \) when no confusion is possible. Put

\[
\text{ow}(z) := (\text{ord}(z), \text{wt}(z)).
\]

We define semigroups

\[
\text{ord}(C^s) := \{ \text{ord}(z) \mid z \in \mathcal{O}_{C^s, P^s}, \ z \neq 0 \} \subset \mathbb{Z}_{\geq 0},
\]

\[
\text{ow}(C^s) := \{ (\text{ord}(z), \text{wt}(z)) \mid z \in \mathcal{O}_{C^s, P^s}, \ z \neq 0 \} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}/m\mathbb{Z}.
\]

One can show that the curve \( C^s \) admits a monomial parametrization in certain coordinates (see Lemma 2.7 in [2] for a precise statement).
5.2. **Notation.** Let $f : (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be an extremal curve germ and let $P \in C$ be a point of index $m \geq 1$. Let $s$ (resp. $m$) be the splitting degree (resp. the subindex). Consider the $\mu_m$-cover

$$\pi : (X^s, P^s) \to (X, P)$$

of index 1 and put $C^s := \pi^{-1}(C)$. Take normalized $\ell$-coordinates $(x_1, \ldots, x_4)$ and let $\phi$ be an $\ell$-equation of $X \supset C \ni P$ (see [2], §2.6). Put $a_i := \text{ord}(x_i)$. Note that $a_i < \infty$ and $\text{wt}(x_i) \equiv a_i \mod m$.

The following assertion is a key fact in the local classification of possible singularities of extremal curve germs.

5.3. **Lemma** ([2], §§3.8, 4.2; [6], §5). In the above notation, assume that $P$ is not of type (IE$^\vee$) below. Then $\text{ow}(C^s)$ is generated by $\text{ow}(x_1)$ and $\text{ow}(x_2)$. In particular, $C^s$ is a planar curve.

This lemma enables us to obtain a local classification of possible singularities. We reproduce this classification below. We start with the primitive case.

5.4. **Proposition** ([2], Proposition 4.2, [6], Proposition 5.2.1). Let $f : (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be an extremal curve germ and let $P \in C$ be a primitive point of index $m \geq 1$. Then the semigroup $\text{ord}(C^s)$ is generated by $a_1$ and $a_2$ modulo a permutation of the coordinates $x_i$. Moreover, exactly one of the following assertions holds:

- (IA) $a_1 + a_3 \equiv 0 \mod m$, $a_4 = m$, $m \in \mathbb{Z}_{>0}a_1 + \mathbb{Z}_{>0}a_2$, where we may still permute $x_1$ and $x_3$ if $a_2 = 1$;
- (IB) $a_1 + a_3 \equiv 0 \mod m$, $a_2 = m$, $a_1 \geq 2$;
- (IC) $a_1 + a_2 = a_3 = m$, $a_4 \neq a_1, a_2 \mod m$, $2 \leq a_1 < a_2$, $m \geq 5$;
- (IIA) $m = 4$, $P$ is of type cAx/4 and $\text{ord}(x) = (1, 1, 3, 2)$;
- (IIB) $m = 4$, $P$ is of type cAx/4 and $\text{ord}(x) = (3, 2, 5, 5)$;
- (III) $m = 1$, $X = X^s$, $C = C^s$ and $P \in X$ is a point of type cDV.

We now consider the locally imprimitive case.

5.5. **Proposition** ([2], Proposition 4.2, [6], Proposition 5.3.1). Let $f : (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be an extremal curve germ and let $P \in C$ be an imprimitive point of index $m$. Then the semigroup $\text{ow}(C^s)$ is generated by $\text{ow}(x_1)$ and $\text{ow}(x_2)$ modulo permutation of the coordinates $x_i$ and changes of $\ell$-characters, except for the case (IE$^\vee$) below. Moreover, exactly one of the following assertions holds:

- (I$A^\vee$) $m > 1$, $\text{wt}(x_1) + \text{wt}(x_3) \equiv 0 \mod m$, $\text{ow} x_4 = (m, 0)$, $\text{ow}(C^s)$ is generated by $\text{ow}(x_1)$ and $\text{ow}(x_2)$, and $w_P(0) \geq 1/2$;
- (IC$^\vee$) $s = 2$, $m$ is an even integer $\geq 4$, and

$$
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  \text{wt} & 1 & -1 & 0 & m+1 \mod m; \\
  \text{ord} & 1 & m-1 & m & m+1
\end{array}
$$

- (II$^\vee$) $m = s = 2$, $P$ is of type cAx/4, and

$$
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  \text{wt} & 1 & 3 & 3 & 2 \mod 4; \\
  \text{ord} & 1 & 1 & 1 & 2
\end{array}
$$
series expansion. Since classification of terminal singularities that either

\begin{align*}
\text{(ID)} \quad m = 1, s = 2, P & \text{ is of type } cA2 \text{ or } cAx2, \text{ and} \\
& x_1 \ x_2 \ x_3 \ x_4 \\
& \text{wt} \ 1 \ 1 \ 1 \ 0 \ \text{mod} \ 2; \\
& \text{ord} \ 1 \ 1 \ 1 \ 1 \\
\text{(IE)} \quad m = 2, s = 4, P & \text{ is of type } cA8, \text{ and} \\
& x_1 \ x_2 \ x_3 \ x_4 \\
& \text{wt} \ 5 \ 1 \ 3 \ 0 \ \text{mod} \ 8. \\
& \text{ord} \ 1 \ 1 \ 1 \ 2
\end{align*}

Moreover, the cases (ID) and (IE) occur if and only if \( f \) is a \( \mathbb{Q} \)-conic bundle
and \( C' = f'^{-1}(o') \). In these cases, \( P \) is the only non-Gorenstein point.

Proofs are based on very careful local computations. We do not present them
here. See Example 5.8 for a sample of such computations.

5.5.1. Corollary. Let \( (X, C \simeq \mathbb{P}^1) \) be an extremal curve germ. Assume that
\( \text{gr}_C^0 \omega \not\in \mathcal{O}(-1) \). Then \( (X, C) \) is a \( \mathbb{Q} \)-conic bundle germ, and either this germ is
toroidal, or the only non-Gorenstein point of \( (X, C) \) is of type (ID).

Proof. By Corollary 4.7.8, \( (X, C) \) is a \( \mathbb{Q} \)-conic bundle and \( C' = f'^{-1}(o') \). Assume
that it is not toroidal. Then it is locally imprimitive by Corollary 4.7.8, and the
proposition above yields that \( (X, C) \) has a unique non-Gorenstein point, say \( P \),
which is of type (ID) or (IE). In the case (IE) we have \(-K_X \cdot C = 1/2\) (see
4.7.4). Easy computations show that \( w_P(0) = 1/2 \) and, therefore, \( \text{deg gr}^0_C \omega = -1 \)
by (3.1.2). \( \square \)

5.6. By Lemma 5.3, there are monomials \( \lambda_3 \) and \( \lambda_4 \) in \( x_1, x_2 \) such that
\( x_3 = \lambda_3(x_1, x_2) \) and \( x_4 = \lambda_4(x_1, x_2) \) on \( C^2 \). Then
\( x_1^{s_2} - x_2^{s_1}, \ x_3 - \lambda_3, \ x_4 - \lambda_4 \)
generate the defining ideal \( I^2 \) of the curve \( C^2 \subset \mathbb{C}^4 \). Then the equation of \( X^2 \) can
be written in the form
\[
\phi = (x_1^{s_2} - x_2^{s_1}) \phi_2 + (x_3 - \lambda_3) \phi_3 + (x_4 - \lambda_4) \phi_4
\]
for some semi-invariants \( \phi_i \in \mathbb{C}\{x_1, \ldots, x_4\} \) with suitable weights.

5.7. Lemma. The following assertions hold in the notation of 5.4 and 5.5.

(i) If \( P \) is of type (IC) or (IC'), then \( (X^2, P^2) \) is smooth and \( I^2 = (x_1^{s_2} - x_2^{s_1}, x_4 - \lambda_4, \phi) \).

(ii) If \( P \) is of type (IB) or (II'), then \( I^2 = (x_3 - \lambda_3, x_4 - \lambda_4, \phi) \).

Proof. For example, consider the case (IC). Then \( \lambda_3 \) must be equal to \( x_1x_2 \) and, therefore,
\[
\phi = (x_1^{s_2} - x_2^{s_1}) \phi_2 + (x_3 - x_1x_2) \phi_3 + (x_4 - \lambda_4) \phi_4.
\]
Since \( P \) is of type (IC), we see that \( m = a_1 + a_2 > 4, \phi_2 \in (x) \) and that \( \phi_4, \lambda_4 \in (x)^2 \)
because \( \text{wt}(x_4) \not\equiv 0, \pm \text{wt}(x_1), \pm \text{wt}(x_2) \) mod \( m \). Since \( m \geq 5 \), it follows from the
classification of terminal singularities that either \( x_1x_2 \) or \( x_3 \) must occur in the power
series expansion. Since \( a_1, a_2 \geq 2 \), this is possible only when \( \phi_3 \) is a unit. \( \square \)
5.8. Example. By Lemma 5.7 (i), a point \( P \in (X, C) \) of type \((IC^\vee)\) can be written in the form
\[
(X, C, P) = (\mathbb{C}^3_{x_1, x_2, x_4}, \{x_4 = x_2^2 - x_1^{2m-2} = 0\}, 0)/\mu_{2m}(1, -1, m + 1).
\]
We have \( C \simeq \{x_4 = x_2 - x_1^{m-1} = 0\}/\mu_m \) and \( x_1^{m} \) is a local uniformizing parameter on \( C \). Hence, \( \mathcal{O}_{C, P} = C\{x_1^{m}\} \). Furthermore,
\[
\mathcal{O}_{C}(mK_X) = \mathcal{O}_{C}(dx_1 \wedge dx_2 \wedge dx_4)^m,
\]
\[
\text{gr}_C^0 \omega = \mathcal{O}_{C}(x_1^{m-1}dx_1 \wedge dx_2 \wedge dx_4),
\]
\[
\text{gr}_C^1 \mathcal{O} = \mathcal{O}_{C}(x_4^m_{1}x_4) \oplus \mathcal{O}_{C}(x_2^2 - x_1^{2m-2}),
\]
\[
w_P(0) = (m-1)/m, \quad i_P(1) = 2.
\]

§ 6. Deformations

In this section we discuss deformations of extremal curve germs. It is known that a small deformation of a terminal singularity is again terminal (see, for example, [26], Theorem 9.1.14). Moreover, any three-dimensional terminal singularity admits a \( \mathbb{Q} \)-smoothing, that is, a deformation to a collection of cyclic quotient singularities ([10], §6.4A).

To study extremal curve germs, it is very convenient to deform the original germ to a more general one. For example, this procedure sometimes increases the number of singular points, which can be used to derive a contradiction.

6.1. Definition. Let \( X \) be a threefold with at worst terminal singularities. We say that \( X \) is ordinary at \( P \) (or \( P \) is an ordinary point) if \((X, P)\) is either a cyclic quotient singularity or an ordinary double point.

We note that deformations of extremal curve germs are unobstructed.

6.2. Proposition ([2], §1b.8.2, [3], §11.4.2, [6], 6.1). Suppose that \( f: (X, C) \to (Z, o) \) is an extremal curve germ and \( P \in C \). Then every deformation of germs \((X, P) \supset (C, P)\) can be extended to a deformation of \((X, C)\) in such a way that the deformation is trivial outside a small neighbourhood of \( P \).

Proof. Let \( P_i \in X \) be singular points. Consider the natural morphism
\[
\Psi: \text{Def}(X) \longrightarrow \prod \text{Def}(X, P_i).
\]
It suffices to show that \( \Psi \) is smooth (in particular, surjective). The obstruction to globalizing the deformation in \( \prod \text{Def}(X, P_i) \) lies in \( R^2f_*T_X \). Since \( f \) has only one-dimensional fibres, \( R^2f_*T_X = 0 \). An alternative (more explicit) proof may be found in [2], §1b.8.2. \( \square \)

The following theorem was proved for divisorial \( f \) in [21], Theorem 3.2, for flipping \( f \) in [3], (11.4), and for \( \mathbb{Q} \)-conic bundles in [6], (6.2).

6.3. Theorem ([6], (6.2)). Let \( f: (X, C) \to (Z, o) \) be a divisorial (resp. flipping, \( \mathbb{Q} \)-conic bundle) extremal curve germ, where \( C \) is not necessarily irreducible. Let \( \pi: \mathfrak{X} \to (\mathbb{C}^1_{\lambda}, 0) \) be a flat deformation of \( X = \mathfrak{X}_0 := \pi^{-1}(0) \) over the germ \((\mathbb{C}^1_{\lambda}, 0)\).
with a flat closed subspace \( \mathcal{C} \subset \mathcal{X} \) such that \( C = \mathcal{C}_0 \). Then there is a flat deformation \( \mathcal{I} \to (\mathcal{C}_\lambda, 0) \) and a proper \( \mathbb{C}_\lambda \)-morphism \( f: \mathcal{X} \to \mathcal{I} \) such that \( f = f_0 \) and

\[
f_\lambda: (\mathcal{X}_\lambda, f_\lambda^{-1}(\lambda))_{\text{red}} \to (\mathcal{I}_\lambda, o_\lambda)
\]

is a divisorial (resp. flipping, \( \mathbb{Q} \)-conic bundle) curve germ for every small \( \lambda \), where \( o_\lambda := f_\lambda(\mathcal{C}_\lambda) \).

Note however that deformations do not preserve irreducibility of the central fibre: one can easily construct an example of an extremal curve germ \( (X, C) \cong \mathbb{P}^1 \) whose deformation \( (\mathcal{X}_\lambda, f_\lambda^{-1}(\lambda))_{\text{red}} \) has reducible central fibre. In practice, we often choose an appropriate irreducible component of \( f_\lambda^{-1}(\lambda)_{\text{red}} \) and obtain an extremal curve germ whose central fibre is irreducible (see Remark 2.4.2).

**6.3.1.** Let \( f: (X, C) \to (Z, o) \) be an extremal curve germ with a singular point \( P \in C \) of index \( m \), and let \( P_1, \ldots, P_r \) be the other singular points of \( X \) on \( C \). Let \( (X^\sharp, P_1^\sharp) \to (X, P) \) be the cover of index 1, \( (X^\sharp, P_1^\sharp) \subset (\mathbb{C}_A^4, x_1, \ldots, x_4, 0) \) an equivariant embedding as in 2.1.1, and \( \phi = 0 \) an equation of \( X^\sharp \). We choose a semi-invariant \( \psi \in \mathbb{C}\{x_1, \ldots, x_4\} \) with \( \text{wt}(\psi) \equiv \text{wt}(\phi) \mod m \) such that

\[
X_{\lambda, \epsilon} := \{(x_1, \ldots, x_4) \mid \phi + \lambda \psi = 0, \ |x_i| < \epsilon\}/\mu_m \subset \mathbb{C}^4/\mu_m
\]

has only terminal singularities for \( |\lambda| \ll \epsilon \ll 1 \).

**6.3.2. Proposition** ([2], §4.7). For an appropriate choice of \( \psi \), each nearby extremal curve germ \( X^\circ_{\lambda, \epsilon} \supset C_{\lambda, \epsilon} \cong \mathbb{P}^1 \) contains \( P, P_1, \ldots, P_r \), so that \( (X^\circ_{\lambda, \epsilon}, P_i) \supset (C_{\lambda, \epsilon}, P_i) \) is naturally isomorphic to \( (X, P_i) \supset (C, P_i) \) for all \( i \) and \( X_{\lambda, \epsilon} \supset C_{\lambda, \epsilon} \) contains all the singularities \( \in C_{\lambda} \) of \( X_{\lambda} \supset C_{\lambda} \) other than \( P_1, \ldots, P_r \). All the singularities of \( X_{\lambda, \epsilon} \supset C_{\lambda, \epsilon} \) are ordinary.

If \( P \) is a primitive (resp. imprimitive) point, then \( X_{\lambda, \epsilon} \supset C_{\lambda, \epsilon} \) is locally primitive (resp. \( P \) is an imprimitive point of \( X_{\lambda, \epsilon} \supset C_{\lambda, \epsilon} \) whose subindex and splitting degree are equal to those of \( X \supset C \supset P \)). Depending on the type of \( X \supset C \supset P \), one has

| type | \( X_{\lambda, \epsilon} \supset C_{\lambda, \epsilon} \) |
|------|----------------------------------|
| (IA) | \( P \) \( (\text{IA}) \) \( m \) the same as for \( X \) |
| (IA\(^\vee\)) | \( P \) \( (\text{IA}'') \) \( m \) the same as for \( X \) |
| (IIA) | \( P \) \( (\text{IA}) \) \( m \) the same as for \( X \) |
| (II\(^\prime\)) | \( P \) \( (\text{IA}'') \) \( m \) the same as for \( X \) |
| (IB) | \( a_1 \) points \( (\text{IA}) \) \( m \) the same as for \( X \) |
| (IB) | \( P \) and \( Q \) \( (\text{IA}) \) \( 4 \) and \( 2 \) |
| (III) | \( i_\lambda \) points \( (\text{III}) \) \( 1 \) 0 |

In the case (III), one can also make \( X_{\lambda, \epsilon} \) smooth by choosing another suitable function \( \psi \).

**6.3.3. Corollary.** An arbitrary extremal curve germ \( (X, C) \) can be deformed to an extremal curve germ \( (X^\circ, C^\circ) \) with only ordinary points.

**6.3.4. Corollary.** A flipping extremal curve germ \( (X, C) \) has at least one non-Gorenstein point.
Proof. Assume that \((X, C)\) has only singular points of type (III). Repeatedly applying the smoothing of Proposition 6.3.2 at the points of type (III), we obtain a flipping extremal curve germ \((X^o, C^o)\) such that \(X^o\) is smooth. Thus, by (3.1.3) and Corollary 4.7.8, for the normal bundle of the curve \(C^o\) we have
\[
\deg N_{C^o/X^o} = -\deg gr_{C^o}^1 \mathcal{O} = -1.
\]
Hence the space of deformations of \(C^o\) in \(X^o\) has dimension \(\geq 1\). This means that \(C^o\) is movable inside \(X^o\), contrary to the hypothesis that \((X, C)\) is flipping. □

If \(f : X \to Z\) is a \(K\)-negative extremal divisorial contraction from a variety \(X\) with terminal \(\mathbb{Q}\)-factorial singularities, then the target variety \(Z\) is also terminal. This is not the case for divisorial extremal curve germs. The problem is that the exceptional locus of \(f\) is not necessarily a divisor in this case (because \(\mathbb{Q}\)-factoriality is not assumed). Nevertheless we have the following theorem.

6.4. Theorem ([21], Theorem 3.1). Let \(f : (X, C) \to (Z, o)\) be a three-dimensional divisorial extremal curve germ, where the curve \(C\) is not necessarily irreducible, and let \(E\) be the exceptional locus of \(f\). Then the divisorial part of \(E\) is a \(\mathbb{Q}\)-Cartier divisor. If, furthermore, \(C\) is irreducible, then \(E\) is a \(\mathbb{Q}\)-Cartier divisor and \((Z, o)\) is a terminal singularity.

6.5. Corollary. Let \(f : (X, C \simeq \mathbb{P}^1) \to (Z, o)\) be a three-dimensional birational extremal curve germ. Then \(f\) is divisorial if and only if \((Z, o)\) is a terminal singularity.

The proof uses deformation techniques.

6.5.1. Corollary ([2], Theorem 6.3, [6], Proposition 8.3). An extremal curve germ \((X, C)\) cannot have a point of type (IB).

Proof. Assume that \((X, C)\) has a point \(P\) of type (IB). We can apply deformations to \((X, C)\) and obtain an extremal curve germ \((X^o, C^o \simeq \mathbb{P}^1)\) with only ordinary singular points and which has at least two points \(P^o\) and \(Q^o\) of type (IA) and of the same index \(m > 1\). By Proposition 4.8, \(P^o\) and \(Q^o\) are the only non-Gorenstein points of \(X^o\), and \(P\) is the only non-Gorenstein point of \(X\). Thus \(\text{Cl}^{\text{sc}}(X^o) \simeq \mathbb{Z} \oplus \mathbb{Z}/m,\) by (4.4.1) and, therefore, one can find a cyclic cover \((X', C') \to (X^o, C^o)\) of degree \(m\) such that this cover is étale outside \((P^o, Q^o)\), and \((X', C')\) is again an extremal curve germ with \(C' \simeq \mathbb{P}^1\). By Corollary 6.3.4, the germs \((X', C')\) and \((X^o, C^o)\) cannot be flipping.

Assume that \((X^o, C^o)\) is divisorial and let \(f^o : (X^o, C^o) \to (Z^o, o^o)\) be the corresponding contraction. By Theorem 6.4, the point \((Z^o, o^o)\) is terminal and Construction 4.5 shows that \((Z^o, o^o)\) is of index \(m\). By [27] there is an exceptional divisor, say \(E\), with centre \(o^o\) whose discrepancy is equal to \(a(E, Z^o) = 1/m\). On the other hand, since \(E\) is not \(f^o\)-exceptional and the contraction \(f^o\) is \(K\)-negative, we have \(a(E, Z^o) > a(E, Z^o) = 1/m\), a contradiction.

Finally, assume that \((X^o, C^o)\) is a \(\mathbb{Q}\)-conic bundle. Since \((X, C)\) has exactly one non-Gorenstein point which is locally primitive, the base \((S, o)\) is smooth. Since \((X^o, C^o)\) has two points of the same index \(> 1\), the base \((Z^o, o^o)\) is singular. By Theorem 6.3, there is a deformation family with general fibre \((Z^o, o^o)\) and special fibre \((S, o)\) (in this case, a general fibre \(f_{\lambda}^{-1}(o_{\lambda})_{\text{red}}\) must be irreducible). This is impossible. □
6.6. Deformation arguments can also be used to show the existence of extremal curve germs. Suppose that we are given a normal surface germ \((H,C)\) along a curve \(C \simeq \mathbb{P}^1\) and a contraction \(f_H: H \to H_Z\) such that \(C\) is a fibre. Let \(P_1, \ldots, P_r \in H\) be singular points. Assume also that near each point \(P_i\) there is a small one-parameter deformation \(H^i\) of \(H \cap U_{P_i}\) (where \(U_{P_i}\) is a neighbourhood of \(P_i\)) such that the total space \(V^i = \bigcup H^i\) has a terminal singularity at \(P_i\). Arguing as in the proof of Proposition 6.2, we see that the natural morphism

\[
\text{Def} H \to \prod \text{Def}(H, P_i)
\]

is smooth. Hence there is a global one-parameter deformation \(H_t\) of \(H\) inducing a local deformation of \(H^i\) near each \(P_i\). Then we construct a threefold \(X\) as the total space of the one-parameter deformation \(X = \bigcup H_t\). This shows the existence of a variety \(X \supset C\) with \(H \in |\mathcal{O}_X|\) such that \(P_i \in C \cap U_{P_i} \subset U_{P_i}\) has the desired structure. Note however that \(H\) need not be general in \(|\mathcal{O}_X|\). The contraction \(f: X \to Z\) exists by arguments similar to those in Theorem 6.3 and [3], §11.4.1. The contraction is birational (resp. a \(\mathbb{Q}\)-conic bundle) if \(H_Z\) is a surface (resp. a curve).

§ 7. General member of \(|-K_X|\)

7.1. Let \(X\) be a threefold with only terminal singularities and let \(D\) be an effective integral \(\mathbb{Q}\)-Cartier divisor on \(X\). Then \(D\) is a Cohen-Macaulay variety ([13], Corollary 5.25). Hence the following assertions hold.

- If \(D\) has only isolated singularities, then \(D\) is normal.
- If \(D \sim -K_X\), then \(D\) is Gorenstein.

In certain situations we can say more.

7.2. Theorem ([10], (6.4B)). Let \((X, P)\) be a three-dimensional terminal singularity. Then a general member of \(|-K_X|\) has at most a Du Val singularity at \(P\).

7.2.1. Depending on the types of terminal singularities, a general member \(D \in |-K_X|\) and its pre-image \(D^\sharp\) under the cover of index 1 are described below (see [10], (6.4B)).

| name | equation of \(D^\sharp\) | \(\mu_m\)-action | cover \(D^\sharp \to D\) | aw \((X, P)\) |
|------|------------------|-----------------|-------------------|-------------|
| cA/m | \(xy + z^k\)     | \((1, -1, 0)\)  | \(A_k-1 \rightarrow A_{km-1}\) | \(k\)   |
| cAx/4| \(x^2 + y^2 + z^{2k-1}\) | \((1, 3, 2)\)  | \(A_{2k-2} \rightarrow D_{2k+1}\) | \(2k - 1\) |
| cD/3 | \(x^2 + y^3 + z^3\) | \((0, 1, 2)\)  | \(D_4 \rightarrow E_6\) | \(2\)   |
| cAx/2| \(x^2 + y^2 + z^{2k}\) | \((0, 1, 1)\)  | \(A_{2k-1} \rightarrow D_{k+2}\) | \(2\)   |
| cD/2 | \(x^2 + y^2 z + z^k\) | \((1, 1, 0)\)  | \(D_{k+1} \rightarrow D_{2k}\) | \(k\)   |
| cE/2 | \(x^2 + y^3 + z^4\) | \((1, 0, 1)\)  | \(E_6 \rightarrow E_7\) | \(3\)   |

M. Reid conjectured that an analogue of Theorem 7.2 holds for any \(K\)-negative contraction of terminal threefolds (the ‘general elephant’ conjecture). This conjecture is very important in birational geometry. The following theorem shows that this conjecture is true for extremal curve germs. Various parts of this theorem were proved in [2], [3], [6], [7].

7.3. Theorem. Let \((X, C \simeq \mathbb{P}^1)\) be an extremal curve germ. Then a general member of the linear system \(|-K_X|\) is normal and has only Du Val singularities.
Note that by inversion of adjunction ([28], §3, [29], Ch. 17), a general member of \( D \in |{-K_X}| \) is Du Val if and only if the pair \( (X, D) \) is purely log terminal.

All possibilities for general members of \( |{-K_X}| \) have been classified; see [3] and [7].

We reproduce this classification in the case when the germ \( (X, C \simeq \mathbb{P}^1) \) has only one non-Gorenstein point.

7.4. Theorem. Let \( f : (X, C \simeq \mathbb{P}^1) \to (Z, o) \) be an extremal curve germ. Assume that \( (X, C) \) has only one non-Gorenstein point \( P \). Let \( D \in |{-K_X}| \) be a general member. When \( f \) is birational, we put \( D_Z := f(D) \) (this is a general member of \( |{-K_Z}| \)). When \( f \) is a \( \mathbb{Q} \)-conic bundle, we put \( D_Z := \text{Spec}_Z f_*O_D \). Then \( D \) and \( D_Z \) have only Du Val singularities and the morphism \( f_D : D \to D_Z \) is birational and crepant. Moreover, only one of the following possibilities holds.

7.4.1. (See [3], (2.2.1), (2.2.1’), [6], (1.2.3)–(1.2.6)). We have \( D \cap C = \{P\} \) and \( f_D : D \to D_Z \) is an isomorphism. In this case, \( D \) induces a general member of \( |{-K_{X,P}}| \) and \( \Delta(D) \) is described in Table 7.2.1.

7.4.2. (See [3], (2.2.2), [7], 1.3.1). \( P \in (X, C) \) is of type (IC),

\[
\Delta(D, C): \quad \circ - \cdots - \circ \quad \blacktriangleleft \quad \circ \quad \circ \quad \circ ,
\]

where \( m \) (the index of the singularity \( (X, P) \)) is odd and \( m \geq 5 \).

7.4.3. (See [3], (2.2.2’), [7], 1.3.2.) \( P \in (X, C) \) is of type (IIB),

\[
\Delta(D, C): \quad \circ \quad \circ \quad \circ \quad \blacktriangleleft \quad \circ \quad \circ \quad \circ .
\]

In some cases of 7.4.1, there are additional restrictions on the general member \( D \in |{-K_X}| \). For example, in the case when \( f \) is birational and \( (X, P) \) is of type \( c_{AX}/2 \), the general member \( D \in |{-K_X}| \) is of type \( D_4 \); see [3], §4.8.5.7. A lot of restrictions are imposed on imprimitive \( \mathbb{Q} \)-conic bundles (see 9.1).

7.5. We outline the main ideas of the proof in the case when \( (X, C) \) has only one non-Gorenstein point. Thus, let \( (X, C) \) be an extremal curve germ with a single non-Gorenstein point \( P \).

7.5.1. Lemma ([2], Theorem 7.3, [6], §7, 8.6.1). Using the notation of Theorem 7.4 and the symbols for primitive (resp. imprimitive) points as in Proposition 5.4 (resp. Proposition 5.5), we have the following assertions.

(i) If \( P \in (X, C) \) is of type (IA), (IIA), (IA'), (IIA') or (IE'), then \( D \cap C = \{P\} \) for a general member \( D \in |{-K_X}| \).

(ii) If \( P \in (X, C) \) is of type (IC) or (IIB), then \( S \cap C = \{P\} \) for a general member \( S \in |{-2K_X}| \). Moreover, the pair \( (X, \frac{1}{2}S) \) is Kawamata log terminal.

Sketch of proof. Consider the case when \( P \) is of type (IA). Take \( \psi := x_2 + \psi_\bullet \), where \( \psi_\bullet \in \mathbb{C}\{x_1, \ldots, x_4\} \) is a sufficiently general semi-invariant with \( \text{wt}(\psi_\bullet) \equiv \text{wt}(x_2) \), and put \( D := \{\psi = 0\}/\mu_m \). Then \( D \cap C = \{P\} \) and \( \text{wt}(\psi) \equiv \text{wt}(\Omega) \), where
\[ \Omega = \text{Res}(\phi^{-1}dx_1 \wedge \cdots \wedge dx_4). \] Therefore, \( \psi \Omega^{-1} \in \mathcal{O}_X(-K_X) \) in a neighbourhood of \( P \). Since \( P \) is the only non-Gorenstein point, it follows that \( K_X + D \) is a Cartier divisor globally by (4.4.1). On the other hand, \( D \cdot C = (1/m) \text{ord} \psi = (a_2/m) < 1 \) and \( -K_X \cdot C < 1 \) (see (3.1.3) and 5.5.1). Therefore, \( K_X + D \equiv 0 \). Then Corollary 2.4.1(iii) implies that \( K_X + D \sim 0 \). The other cases can be treated similarly. \( \square \)

Thus we are done in the cases 7.5.1(i). The cases (IC) and (IIB) are much more delicate. A rough idea of their proof is to use the surjectivity of the restriction map

\[ H^0(X, \mathcal{O}_X(-K_X)) \to H^0(S, \mathcal{O}_S(-K_X)) \]

and extend a ‘good’ member of \( |-K_X|_S \) to \( X \).

7.6. Kawamata \[12\] had shown that Theorem 7.3 in the flipping case is a sufficient condition for the existence of flips. Indeed, using a Bertini-type argument (see \[13\], Corollary 2.33), one can show that the pair \( (X, \frac{1}{2}S) \) is Kawamata log terminal for a general member \( S \in |-2K_X| \). Consider the double cover \( (X^o, C^o) \to (X, C) \) branched over \( S \). Then \( X^o \) has only canonical singularities (see \[13\], §5.20, \[2\], §7.2) and admits a flopping contraction of \( C^o \). Then the existence of a flip for \( (X, C) \) follows from the existence of a flop for \( (X^o, C^o) \).

7.7. As a corollary of Theorem 7.3, we have the following fact which was conjectured by Iskovskikh \[30\].

7.7.1. Corollary ([24], [6]). Let \( f: (X, C) \to (Z, o) \) be a \( \mathbb{Q} \)-conic bundle germ. Then \( (Z, o) \) is either a smooth point or a Du Val singularity of type A.

This corollary has important applications in the birational geometry of conic bundles (see \[30\], \[31\]).

§ 8. Germs of index 2

In this section we discuss extremal curve germs with only points of index two. The methods are different from those used in the other sections. Throughout this section we do not assume that the central curve of the extremal curve germ is irreducible.

8.1. Proposition \([3], \S 4.6\). Let \( (X, C) \) be an extremal curve germ of index two. When \( (X, C) \) is a \( \mathbb{Q} \)-conic bundle germ we assume that the base surface is smooth. Then the following assertions hold.

(i) If \( P \) is a point of index two, then \( P \) is the only non-Gorenstein point, and all components of \( C \) pass through \( P \) and do not meet each other elsewhere.

(ii) Each germ \( (X, C_i) \) is of type (IA) at \( P \).

(iii) A general member \( F \in |-K_X| \) satisfies \( F \cap C = \{P\} \) and has merely a Du Val singularity at \( P \).

Proof. By Lemma 3.7.1, it suffices to show that every irreducible component of \( C \) has at most one non-Gorenstein point. Assume the opposite: a component \( C_i \subset C \) contains two points \( P \) and \( Q \) of index two. If \( \text{gr}_{C_i}^0 \not= \mathcal{O}_C(-1) \), then \( (X, C_i) \) is a \( \mathbb{Q} \)-conic bundle germ by Corollary 3.7(i). In this case, \( C = C_i \) and \( (X, C_i) \) is
primitive (because the base is smooth). This contradicts Corollary 4.7.8. Thus, $\text{gr}_{C_i}^0 \simeq \mathcal{O}_C(-1)$. Since the numbers $K_X \cdot C_i$, $w_P(0)$ and $w_Q(0)$ are strictly positive and belong to $\frac{1}{2}\mathbb{Z}$, we get a contradiction by (3.1.2).

Part (ii) follows from Proposition 5.4 (the case (IB) is excluded by Corollary 6.5.1).

Part (iii) can be proved as in §7. □

We first consider the birational case following [3], §4.

**8.2. Theorem** ([3], §4.7). Let $(X, C)$ be a birational extremal curve germ of index two. Let $P \in X$ be a non-Gorenstein point. Then a general member $H \in |\mathcal{O}_X|$ is normal and has only rational singularities. Only the following possibilities may occur for the dual graph $\Delta(H, C)$, with $\underbrace{\circ-\cdots-\circ}_{n-2}$ replaced by $\bullet$ if $n = 1$.

The first case is the only flipping case.

**8.2.1.** Here $C \simeq \mathbb{P}^1$, the singularity $(X, P)$ is of type $\text{cA}/2$, $(H_Z, o)$ is of type $\frac{1}{2n+1}(1, 2n-1)$ and

$$\Delta(H, C): \quad \bullet - \underbrace{\circ-\cdots-\circ}_{n-2} - \circ.$$

In the remaining cases, $(X, C)$ is divisorial. Then $(Z, o)$ is a cDV point and $(H_Z, o)$ is a Du Val singularity. By saying that the point $(H_Z, o)$ is of type $\Lambda_0$, we mean that it is smooth.

| No. | $(X, P)$ | $(H_Z, o)$ | $\Delta(H, C)$ |
|-----|----------|------------|----------------|
|     |          | $C$ has one component |                     |
| 8.2.2 | cA/2 | $\text{A}_1$ | $\circ - \bullet - \underbrace{\circ-\cdots-\circ}_{n-2} - \circ$ |
| 8.2.3 | cA/2 | $\text{A}_0$ | $\circ - \circ - \bullet - \bullet$ |
| 8.2.4 | cA/2 | $\text{A}_2$ | $\underbrace{\circ-\circ}_{3} - \bullet$ |
| 8.2.5 | cA/2 | $\text{A}_0$ | $\underbrace{\circ-\circ}_{3} - \circ - \circ - \circ$ |
|     |          | $C$ has two components |                     |
| 8.2.6 | cA/2 | $\text{A}_m$ | $\bullet - \underbrace{\circ-\cdots-\circ}_{n-2} - \circ$ |
| 8.2.7 | cA/2 | $\text{A}_0$ | $\circ - \bullet - \underbrace{\circ-\cdots-\circ}_{n-2} - \circ - \bullet$ |
| 8.2.8 | cA/2 | $\text{A}_1$ | $\underbrace{\circ-\circ}_{3} - \bullet - \underbrace{\circ-\cdots-\circ}_{n-2} - \circ$ |
| 8.2.9 | cA/2 | $\text{A}_0$ | $\underbrace{\circ-\circ}_{3} - \bullet$ |
Threefold extremal curve germs

| No.  | \((X, P)\) | \((H_Z, o)\) | \(\Delta(H, C)\) |
|------|-------------|--------------|-----------------|
| 8.2.10 | cA/2 | A_0          | C has three components |
|       |           |              | ![Diagram](image) |

**C has one component**

| 8.2.11 | cAx/2, cD/2 or cE/2 | D_4 | ![Diagram](image) |
|--------|---------------------|-----|------------------|
| 8.2.12 | cD/2 or cE/2        | D_{n+4} | ![Diagram](image) |

where \(n \geq 1\) and \(n = 1\) if \((X, P)\) is of type cE/2

| 8.2.13 | cE/2 | E_6 | ![Diagram](image) |

Note that the singularities of \(H\) are log terminal in all cases except for 8.2.11, 8.2.12, 8.2.13. In cases 8.2.11 and 8.2.12, the singularities of \(H\) are log canonical.

**8.2.14. Theorem** ([3], §4.2). In the notation of Theorem 8.2 assume that \(f\) is flipping (see 8.2.1) and let \((X, C) \dashrightarrow (X^+, C^+)\) be the corresponding flip. Then the following assertions hold.

(i) In appropriate coordinates, the point \((X \ni P)\) is given by

\[
\{x_1x_2 + p(x_3^2, x_4) = 0\}/\mu_2(1, 1, 1, 0)
\]

and \(C\) is the \(x_1\)-axis.

(ii) \(X^+\) has at most one singular point. It must be isolated, of type cDV, with equation \(x_1x_2 + p(x_3, x_4) = 0\), and \(C^+\) is the \(x_1\)-axis.

(iii) \((Z, o)\) is a rational triple point given by the \(2 \times 2\)-minors of the matrix

\[
\begin{pmatrix}
  z_1 & z_2 & z_3 \\
  z_2 & z_5 & p(z_1, z_4)
\end{pmatrix}.
\]

The proofs use the following standard construction.

**8.2.15. Construction.** Let \(C_i \subset C\) be irreducible components of \(C\). Since \(X\) has only points of index 1 and 2, the number \(m_i = -2K_X \cdot C_i\) is a positive integer. Let \(E_i \subset X\) be the union of \(m_i\) disjoint discs transversal to \(C_i\). We put \(E = \sum E_i\). Then \(E \subset |{-2K_X}|\). Hence we can take the corresponding double cover \(X' \rightarrow X\) branched over \(E\). Here \(X'\) has only terminal singularities of index 1. Let \(E' \subset X'\) be the pre-image of \(E\). The natural map \(E' \rightarrow E\) is an isomorphism. The Stein factorization induces a diagram

\[
\begin{array}{ccc}
E \subset X & \rightarrow & C \\
\downarrow f & & \downarrow f' \\
D \subset Z & \rightarrow & C'
\end{array}
\]
where $D := f(E)$ and $D' := f(E')$. Here $Z' \to Z$ is a double cover branched over $D$. By construction, the morphism $f'$ is crepant with respect to $K_X'$, and the fibres of $f'$ have dimension $\leq 1$. Therefore $Z'$ has only cDV points (if $f$ is divisorial, then $Z'$ also has a double curve).

**Sketch of the proof of 8.2 and 8.2.14.** The above construction defines a $\mu_2$-action on $X'/Z'$ and the quotient is equal to $X/Z$. The set of fixed points of this action on $Z'$ is precisely $D'$. Since $Z' \ni o'$ is a point of type cDV, it is a hypersurface in $\mathbb{C}^4$, and thus it can be written down explicitly. This enables us to get equations for $X$ and $Z$. We have a $\mu_2$-equivariant embedding $(Z', o') \subset ((\mathbb{C}^4_{y_1, \ldots, y_4}, 0)$ and we may assume that the coordinates are eigenvectors and $y_1, \ldots, y_j$ are the coordinates of weight 1. Thus, $D' = \{y_1 = \cdots = y_j = 0\} \cap Z'$. Hence $j = 1$ or 2.

**8.2.16. Assertion.** (i) If $D'$ is a Cartier divisor, then $f$ is divisorial and $D$ is singular along $f(E)$, where $E$ is the $f$-exceptional divisor.

(ii) If $D'$ is not Cartier, then $f$ is flipping, $D$ is smooth and $C$ is irreducible.

**Proof.** If $j = 1$, then $D'$ is Cartier. In this case, $f$ must be divisorial. Indeed, otherwise $D'$ cannot be Cartier because $f'$ is an isomorphism outside the origin and $E'$ is $f'$-ample. Hence $f$ contracts an exceptional divisor $E \subset X$. Then we have $K_X \cdot l = -1$, where $l$ is a general fibre of $E$. Hence $E \cdot F = 2$. Therefore, $D$ has a double curve along the image of $E$ and is smooth elsewhere. If $E$ is chosen generically, then $D$ has an ordinary double curve along the image of $E$.

Assume that $j = 2$. Then the plane $\{y_1 = y_2 = 0\}$ must be contained in $Z'$. Furthermore, the divisor $D'$ is irreducible and this implies that the curve $C$ is irreducible. Then $E \to D$ is an isomorphism outside the origin and, in fact, it turns out to be an isomorphism everywhere. In particular, $D$ is smooth. This implies that $f$ is flipping. $\square$

We first consider the flipping case. Since $\{y_1 = y_2 = 0\} \subset Z'$, the equation of $Z'$ can be written in the form $y_1\phi_1 + y_2\phi_2 = 0$. If $\text{wt}(\phi_1) = \text{wt}(\phi_2) = 1$, then $y_1\phi_1 + y_2\phi_2 \in (y_1, y_2)^2$. It follows that $Z'$ is singular along $\{y_1 = y_2 = 0\}$. This is impossible. Thus $\text{wt}(\phi_1) = \text{wt}(\phi_2) = 1$. Since $Z'$ is a double point, either $\phi_1$ or $\phi_2$ must contain a linear term. Assume that $\phi_1$ contains $y_j$. For reasons of weight, $j = 3$ or 4. We can now rewrite the equation in the form

$$y_1y_3 + y_2y_4^2 = 0.$$  

Using this explicit equation, we can easily compute everything. The variety $X$ is obtained by blowing up $\{y_2 = y_3 = 0\}$ and taking quotient by the group action. This gives us one singular point with the required equation. The flipped variety $X^+$ is obtained by blowing up $\{y_1 = y_2 = 0\}$ and taking the quotient by the group action. To get equations for $(Z, o)$, we note that the invariants of the $\mu_2$-action on $\mathbb{C}\{y_1, \ldots, y_4\}$ are

$$z_1 = y_2^2, \quad z_2 = y_1y_2, \quad z_3 = y_3, \quad z_4 = y_4, \quad z_5 = y_1^2.$$  

We get exactly the equations given by the minors of the matrix in the assertion of the theorem. The equation $z_4 = c \cdot z_1$ determines a hyperplane section.
We now consider the divisorial case. Then $D' \subset Z'$ is a Cartier divisor and the $\mu_2$-action is given by $\text{wt}(y) = (0, 0, 0, 1)$. Let $D'$ be given by the equations $y_4 = \psi(y_1, y_2, y_3) = 0$. Thus we can write the equation of $Z'$ in the form

$$y_4^2 \phi(y_1, \ldots, y_4) + \psi(y_1, y_2, y_3) = 0.$$  

Since $f'$ is crepant, $Z'$ cannot be smooth. In particular, $\text{mult}_0(\psi) \geq 2$. The equation of $Z$ is now given by

$$t \phi(y_1, y_2, y_3, t) + \psi(y_1, y_2, y_3) = 0 \quad (t = y_4^2). \quad (8.2.17)$$

In particular, this shows that $(Z, o)$ is an (isolated) point of type cDV (compare with 6.5). The proof proceeds by a careful analysis of the equations; see §4 in [3] for details. □

8.3. We now consider $\mathbb{Q}$-conic bundles. The case of a singular base surface is easy.

8.3.1. Proposition ([24], §3, [32]). A $\mathbb{Q}$-conic bundle of index two over a singular base is either of type 9.1.2 or toroidal (Example 2.3).

$\mathbb{Q}$-conic bundles of index two over a smooth base were classified in [24], §3, [6]. As in the birational case, these are quotients of certain elliptic fibrations by an involution. On the other hand, note that there is an embedding in a relative weighted projective space.

8.4. Theorem. Let $f: (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ of index two. Assume that $(Z, o)$ is smooth. Fix an isomorphism $(Z, o) \simeq (\mathbb{C}^2, 0)$. Then there is an embedding

$$X \xleftarrow{f} \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2 \quad (8.4.1)$$

such that $X$ is given by two equations,

$$q_1(y_1, y_2, y_3) - \psi_1(y_1, \ldots, y_4; u, v) = 0,$$
$$q_2(y_1, y_2, y_3) - \psi_2(y_1, \ldots, y_4; u, v) = 0, \quad (8.4.2)$$

where $\psi_i$ and $q_i$ are weighted quadratic polynomials in the variables $y_1, \ldots, y_4$ with respect to the weights

$$\text{wt}(y_1, \ldots, y_4) = (1, 1, 1, 2) \quad \text{and} \quad \psi_i(y_1, \ldots, y_4; 0, 0) = 0.$$

The only non-Gorenstein point of $X$ is $(0, 0, 0, 1; 0, 0)$. Here is the list of all possibilities for $q_1$ and $q_2$ up to projective transformations.

| no. | $q_1$ | $q_2$ | $f^{-1}(o)$ |
|-----|-------|-------|------------|
| 8.4.3 | $y_1^2 - y_2^2$ | $y_1 y_2 - y_3^2$ | $C_1 + C_2 + C_3 + C_4$ |
| 8.4.4 | $y_1 y_2$ | $(y_1 + y_2) y_3$ | $2C_1 + C_2 + C_3$ |
| 8.4.5 | $y_1 y_2 - y_3^2$ | $y_1 y_3$ | $3C_1 + C_2$ |
| 8.4.6 | $y_1^2 - y_2^2$ | $y_3^2$ | $2C_1 + 2C_2$ |
| 8.4.7 | $y_1 y_2 - y_3^2$ | $y_1^2$ | $4C_1$ |
| 8.4.8 | $y_2^2$ | $y_2^2$ | $4C_1$ |
Conversely, suppose that $X \subset \mathbb{P}(1,1,1,2) \times \mathbb{C}^2$ is given by equations of the form (8.4.2) and the singularities of $X$ are terminal. Then the projection

$$f : (X, f^{-1}(0)_{\text{red}}) \to (\mathbb{C}^2, 0)$$

is a $\mathbb{Q}$-conic bundle of index two.

8.4.9. Remark. A general member $H \in |\mathcal{O}_X|$ is normal in the case 8.4.7 and non-normal in the case 8.4.8.

Sketch of proof. We first prove the latter assertion. By hypothesis, $X$ has only terminal singularities. Then $X$ does not contain the surface $\{y_1 = y_2 = y_3 = 0\} = \text{Sing}(\mathbb{P} \times \mathbb{C}^2)$ (otherwise both polynomials $\psi_1$ and $\psi_2$ are independent of $y_4$). By the adjunction formula, $K_X = -L|_X$, where $L$ is a Weil divisor on $\mathbb{P} \times \mathbb{C}^2$ such that the restriction $L|_P$ coincides with $\mathcal{O}_P(1)$. Therefore, $X \to \mathbb{C}^2$ is a $\mathbb{Q}$-conic bundle. It is easy to see that the only non-Gorenstein point of $X$ is $(0,0,0,1;0,0)$, and it is of index two.

Now let $f : (X, C) \to (Z, o) \simeq (\mathbb{C}^2, 0)$ be a $\mathbb{Q}$-conic bundle germ of index two, $P \in X$ a point of index two, and $\pi : (X^2, P^2) \to (X, P)$ the cover of index 1. We need the following lemma.

8.4.10. Lemma ([6], 12.1.9). Let $F^\sharp = \pi^{-1}(F)_{\text{red}}$ be the pullback of $F$. We write $\Gamma := f^{-1}(o)$ for the scheme fibre and put $\Gamma^\sharp = \pi^{-1}(\Gamma)$. Then we have

$$\mathcal{O}_{F^\sharp \cap \Gamma^\sharp} \simeq \mathbb{C}[x, y]/(xy, x^2 + y^2).$$

Furthermore, the $\mu_2$-action is given by $\text{wt}(x, y) \equiv (1,1) \mod 2$.

Using this lemma, one can apply the arguments in [33], pp.631–633, to obtain the desired embedding $X \subset \mathbb{P}(1,1,1,2) \times Z$. This can be done by considering the graded anti-canonical $\mathcal{O}_Z$-algebra

$$\mathcal{R} := \bigoplus_{i \geq 0} \mathcal{R}_i, \quad \text{where} \quad \mathcal{R}_i := H^0(\mathcal{O}_X(-iK_X)).$$

We sketch the main idea.

Let $w$ be a local generator of $\mathcal{O}_{X^\sharp}(-K_X)$ at $P^\sharp$, let $u$, $v$ be coordinates on $Z = \mathbb{C}^2$ and let $z = 0$ be the local equation of $F^\sharp$ in $(X^\sharp, P^\sharp)$. Using the vanishing of $H^1(\mathcal{O}_X(-K_X))$ for $i > 0$ and the exact sequence

$$0 \to \mathcal{O}_X(-(i-1)K_X) \to \mathcal{O}_X(-iK_X) \to \mathcal{O}_F(-iK_X) \to 0,$$

one can see that

$$\mathcal{R}_i/(zw)\mathcal{R}_{i-1} \simeq H^0(\mathcal{O}_F(-iK_X)), \quad i > 0.$$ 

Therefore,

$$\mathcal{R}_i/(zw)\mathcal{R}_{i-1} + (u,v)\mathcal{R}_i = (\mathcal{O}_{F^\sharp \cap \Gamma^\sharp}(-iK_X))^{\mu_2}.$$

By Lemma 8.4.10, we have an embedding

$$\mathcal{R}/(zw, u, v)\mathcal{R} \hookrightarrow (\mathbb{C}[x, y, w]/(xy, x^2 + y^2))^{\mu_2}. $$
Using the fact that $\mathcal{R}_0/(u, v)\mathcal{R}_0 = C$, one can easily see that

$$\mathcal{R}/(zw, u, v)\mathcal{R} = C[y_1, y_2, y_4]/(y_1y_2, y_1^2 + y_2^2),$$

where $y_1 = lw, y_2 = wv, y_4 = w^2$. Put $y_3 := zw$. Then, as on pp. 631–633 in [33], we obtain

$$\mathcal{R} \simeq \mathcal{O}_Z[y_1, y_2, y_3, y_4]/\mathcal{I},$$

where $\mathcal{I}$ is generated by the regular sequence

$$y_1y_2 + y_3\ell_1(y_1, \ldots, y_3) + \psi_1(y_1, \ldots, y_4; u, v),$$

$$y_1^2 + y_2^2 + y_3\ell_2(y_1, \ldots, y_3) + \psi_2(y_1, \ldots, y_4; u, v)$$

with $\psi_i(y_1, \ldots, y_4; 0, 0) = 0$. $\square$

Note also that Construction 8.2.15 in the $\mathbb{Q}$-conic bundle case produces an elliptic fibration. This can be used for the classification (see [24], §3).

8.4.11. Example. Let $X \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2_{u, v}$ be given by the equations

$$y_1y_2 = (au + bu^2 + cvu)y_4,$$

$$(y_1 + y_2 + y_3)y_3 = vy_4,$$

where $a, b, c \in \mathbb{C}$ are constants. It is easy to check that the projection $X \to \mathbb{C}^2$ is a $\mathbb{Q}$-conic bundle as in 8.4.3. The only singular point is of type $cA/2$. If $a \neq 0$, then this point is a cyclic quotient of type $\frac{1}{2}(1, 1, 1)$.

8.4.12. Example. Let $X \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2_{u, v}$ be given by the equations

$$y_1^2 = uy_2^2 + vy_4,$$

$$y_2^2 = uy_4 + vy_3^2.$$  

Then the projection $X \to \mathbb{C}^2$ is a $\mathbb{Q}$-conic bundle of type 8.4.8 containing one singular point of type $\frac{1}{2}(1, 1, 1)$ and two ordinary double points.

More examples are given in [21], §7 and Remark 6.7.1, and [24], §3. One can show (see [21], §7) that every type of terminal singularity of index two can occur in some $\mathbb{Q}$-conic bundle of index two as in 8.4.7 or 8.4.8.

§ 9. Locally imprimitive germs

In this section we collect results concerning extremal curve germs with a locally imprimitive point. Note that the imprimitive point is unique in this case and the splitting cover is locally primitive along any irreducible component of the central curve (see Corollary 4.6.4). Moreover, one can show that the imprimitive point is the only non-Gorenstein point; see [2], Theorems 6.7, 9.4, and [6], §7.

The following theorem summarizes the results contained in [2], [3], [6], [21].

9.1. Theorem. Let $f : (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be an extremal curve germ such that $(X, C)$ is locally imprimitive. Suppose that $P \in X$ is an imprimitive point, and let $m, s,$ and $m$ be its index, splitting degree, and subindex respectively. In this case, $P$ is the only non-Gorenstein point and $X$ has at most one point of type (III). Moreover, one of the following assertions holds.
9.1.1. (See [6], 1.2.3.) The germ $f$ is a $\mathbb{Q}$-conic bundle, $(X, C)$ is of type $(\text{IE}^\vee)$ at $P$, the point $(Z, o)$ is Du Val of type $A_3$, and $X$ has a cyclic quotient singularity of type $\frac{1}{8}(5, 1, 3)$ at $P$ and has no other singular points. Furthermore, $(X, C)$ is the quotient of the $\mathbb{Q}$-conic bundle germ of index 2 given by the following two equations in $\mathbb{P}(1, 1, 1, 2)_{y_1, \ldots, y_4} \times \mathbb{C}^2_{u,v}$:

\[
y_1^2 - y_2^2 = u\psi_1(y_1, \ldots, y_4; u, v) + v\psi_2(y_1, \ldots, y_4; u, v),
y_1y_2 - y_3^2 = u\psi_3(y_1, \ldots, y_4; u, v) + v\psi_4(y_1, \ldots, y_4; u, v),
\]

by the $\mu_4$-action

\[
y_1 \mapsto -i y_1, \quad y_2 \mapsto i y_2, \quad y_3 \mapsto -y_3, \quad y_4 \mapsto i y_4, \quad u \mapsto i u, \quad v \mapsto -i v
\]

(for example, one can take $\psi_1 = \psi_4 = y_1$, $\psi_2 = \psi_3 = 0$).

9.1.2. (See [6], 1.2.4.) The germ $f$ is a $\mathbb{Q}$-conic bundle, $(X, C)$ is of type $(\text{ID}^\vee)$ at $P$, the point $(Z, o)$ is Du Val of type $A_1$, and $(X, C)$ is the quotient of a Gorenstein conic bundle given by the following equation in $\mathbb{P}^2_{y_1, y_2, y_3} \times \mathbb{C}^2_{u,v}$:

\[
y_1^2 + y_2^2 + \psi(u, v)y_3^2 = 0, \quad \psi(u, v) \in \mathbb{C}\{u^2, v^2, uv\},
\]

by the $\mu_2$-action

\[
u \mapsto -u, \quad v \mapsto -v, \quad y_1 \mapsto -y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto y_3.
\]

Here $\psi(u, v)$ has no multiple factors. In this case, $(X, P)$ is the only singular point and it is of type $cA/2$ or $cAx/2$.

9.1.3. (See [6], 1.2.5.) The germ $f$ is a $\mathbb{Q}$-conic bundle, $(X, C)$ is of type $(\text{IA}^\vee)$ at $P$ with $\overline{m} = 2$, $s = 2$, the point $(Z, o)$ is Du Val of type $A_1$, $(X, P)$ is a cyclic quotient singularity of type $\frac{1}{4}(1, 1, 3)$ and $(X, C)$ is the quotient of the $\mathbb{Q}$-conic bundle germ of index 2 given by the following two equations in $\mathbb{P}(1, 1, 1, 2)_{y_1, \ldots, y_4} \times \mathbb{C}^2_{u,v}$:

\[
y_1^2 - y_2^2 = u\psi_1(y_1, \ldots, y_4; u, v) + v\psi_2(y_1, \ldots, y_4; u, v),
y_3^2 = u\psi_3(y_1, \ldots, y_4; u, v) + v\psi_4(y_1, \ldots, y_4; u, v),
\]

by the $\mu_2$-action

\[
y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto -y_4, \quad u \mapsto -u, \quad v \mapsto -v.
\]

For example, one can take $\psi_1 = \psi_4 = y_1$, $\psi_2 = 0$, $\psi_3 = uy_2^2 + \lambda y_1y_2$, where $\lambda$ is a constant. If $\lambda \neq 0$, then $P$ is the only singular point. If $\lambda = 0$, then $X$ also has a point of type $(\text{III})$.

9.1.4. (See [6], 1.2.6.) The germ $f$ is a $\mathbb{Q}$-conic bundle, $(X, C)$ is of type $(\text{II}^\vee)$ at $P$, the point $(Z, o)$ is Du Val of type $A_1$, and $(X, C)$ is a quotient of the same form as in 9.1.3. For example, one can take $\psi_1 = u^2y_4$, $\psi_2 = \psi_4 = y_4$, $\psi_3 = uy_2^2 + \lambda y_1y_2$, where $\lambda$ is a constant.
9.1.5. (See [3], Theorem 4.11.2.) The germ \( f \) is divisorial, \((X, C)\) is of type \((\Pi^\vee)\) at \( P \) and a general member \( H \in |\mathcal{O}_X| \) is normal. The graph \( \Delta(H, C) \) is of the form

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \cdots & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \cdots & \circ & \circ & \circ \\
\end{array}
\]

In this case, \((X, C)\) is the quotient of a divisorial extremal curve germ \((\overline{X}, \overline{C})\) of index two by \( \mu_2 \) which acts freely outside \( P \) and switches the two components of \( \overline{C} \). The point \((Z, o)\) is terminal of index two. It is given by

\[
\{ t\phi(y_1, t) + y_3^2 - y_2^2 = 0 \}/\mu_2(1, 1, 0, 1)
\]

(compare with (8.2.17)), where the image of the exceptional divisor is the curve \( \{ y_2 = y_3 = t = 0 \}/\mu_2 \).

9.1.6. (See [21], Theorem 1.9.) The germ \( f \) is birational and a general member \( H \in |\mathcal{O}_X| \) is normal and has only log terminal singularities of class \( T \) (see 9.1.8 below). The graph \( \Delta(H, C) \) is of the form

\[
\begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \cdots & \circ & \circ \\
\circ & \circ & \circ & \circ & \cdots & \circ & \circ \\
\end{array}
\]

(9.1.7)

Here \( r \neq 1, n \) and the chain \([c_1, \ldots, c_n]\) corresponds to the non-Du Val singularity \((H, P)\) of class \( T \). The chain of \((-2)\)-vertices in the bottom row corresponds to a Du Val point \((H, Q)\). This chain may possibly be empty (then \( (H, Q) \) is smooth). The germ \((X, C)\) is of type \((\text{IA}^\vee)\) and the curve \( C^\sharp \) (defined in 9.4) is reducible. The contraction \( f \) is either divisorial or flipping, depending on whether the point \((H'_Z, o')\) (introduced in 9.5) is Gorenstein or not.

9.1.8. Recall that a log terminal surface singularity \((H, P)\) is called a singularity of class \( T \) if it admits a one-parameter smoothing \( \{ H_t \} \) (with \( H_0 = H \)) whose total space \( X = \bigcup H_t \) is \( \mathbb{Q} \)-Gorenstein [34], [11]. By inversion of adjunction, this total space must be terminal. Any singularity of class \( T \) is either a Du Val point or a cyclic quotient

\[
\frac{1}{m^2d}(1, mdt - 1), \quad \gcd(m, t) = 1.
\]

There is an explicit characterization of such singularities in terms of minimal resolutions; see [11], §3.

9.2. The rough idea of the proof of Theorem 9.1 is to apply the construction (4.6.2). Then \((X, C)\) may be regarded as the quotient of an extremal curve germ \((X', C')\) with a reducible central fibre by \( \mu_s \). In the case \((\text{ID}^\vee)\) we have \( \overline{m} = 1 \). Hence, \((X', C')\) is a Gorenstein conic bundle germ 2.7.1 (i). Then the action can easily be written down explicitly (see [24], §2). In the cases \((\text{IE}^\vee)\) and \((\Pi^\vee)\) we have \( \overline{m} = 2 \). Then \((X', C')\) is an extremal curve germ of index two and we can apply the results in §8. The case \((\text{IC}^\vee)\) does not occur ([2], Theorem 6.1 (i), [6], 7.3).
9.3. Consider the case (IA'). We need the following useful observation, which enables one to study general hyperplane sections $H \in |\mathcal{O}_X|$. This will also be used in the case (IA) below.

9.3.1. Lemma. Let $(Z, o)$ be a normal threefold singularity, $D_Z \in -K_X$ a general member, and $H_Z$ a general hyperplane section. Assume that $(D_Z, o)$ is a Du Val singularity of type A. Then the pair $(X, H_Z + D_Z)$ is log canonical. In particular, $(H_Z, o)$ is a cyclic quotient singularity.

Proof. Clearly, $H_Z \cap D_Z$ is a general hyperplane section of $(D_Z, o)$ and, therefore, $H_Z \cap D_Z = \Gamma_1 + \Gamma_2$ for some irreducible curves $\Gamma_i$ such that the pair $(D_Z, \Gamma_1 + \Gamma_2)$ is log canonical. By inversion of adjunction, so is the pair $(Z, D_Z + H_Z)$; see [28], §3, [35]. Hence $(H_Z, \Gamma_1 + \Gamma_2)$ is log canonical and $(H_Z, o)$ is a cyclic quotient singularity (see, for example, [29], Ch. 3). □

9.3.2. Proposition. Let $f : (X, C) \to (Z, o)$ be an extremal curve germ ($C$ is not necessarily irreducible). Let $D \in -K_X$ and $H \in |\mathcal{O}_X|$ be general members, $\Lambda$ the non-normal locus of $H$, and $\nu : H^n \to H$ the normalization (when $H$ is normal, we put $\Lambda = \emptyset$ and $\nu = \id$).

Assume that $D \cap C$ is a point $P$ such that $(D, P)$ is a Du Val singularity of type A. Then the pairs $(X, D + H)$ and $(H^n, \nu^{-1}(D) + \nu^{-1}(\Lambda))$ are log canonical. In particular, $H$ has only normal crossings in codimension one. If $f$ is birational, then the pair $(Z, D_Z + H_Z)$ is also log canonical, where $D_Z = f(D) \in -K_Z$ and $H_Z := f(H) \in |\mathcal{O}_Z|$. In this case, $(H_Z, o)$ is a cyclic quotient singularity.

Proof. We first consider the case when $f$ is birational. Then $(D_Z, o) \simeq (D, P)$ is a Du Val singularity of type A. The pair $(X, H_Z + D_Z)$ is log canonical by Lemma 9.3.1. Take $H := f^*H_Z$. Then $K_X + D + H = f^*(K_Z + D_Z + H_Z)$, that is, the contraction $f$ is $K_X + D + H$-crepant. Hence the pair $(X, D + H)$ is log canonical, and so is the pair $(H^n, \nu^{-1}(D) + \nu^{-1}(\Lambda))$, again by inversion of adjunction.

We now consider the case when $Z$ is a surface. We claim that $(X, D + H)$ is log canonical near $D$. Indeed, consider the restriction $\varphi = f_D : (D, P) \to (Z, o)$. Let $\Xi \subset Z \simeq \mathbb{C}^2$ be the branch divisor of $\varphi$. By the Hurwitz formula, we can write $K_D = \varphi^*(K_Z + \frac{1}{2}\Xi)$. Hence,

$$K_D + H|_D = \varphi^*\left(K_Z + \frac{1}{2}\Xi + H_Z\right).$$

Using this equality and the inversion of adjunction, we obtain the following equivalences: $(X, D + H)$ is log canonical near $D$ $\iff (D, H|_D = \varphi^*H_Z)$ is log canonical $\iff (Z = \mathbb{C}^2, \frac{1}{2}\Xi + H_Z)$ is log canonical. Thus it suffices to show that $(Z, \frac{1}{2}\Xi + H_Z)$ is log canonical. Let $\xi(u, v) = 0$ be the equation of $\Xi \subset \mathbb{C}^2$. Then $(D, P)$ is given by the equation $w^2 = \xi(u, v)$ in $\mathbb{C}^3_{u,v,w}$. By the classification of Du Val singularities, we can choose coordinates $u, v$ such that $\xi = u^2 + v^{n+1}$. Take $H_Z := \{v-u = 0\}$. Then $\text{ord}_Q \xi(u, v)|_{H_Z} = 2$. By inversion of adjunction, the pair $(Z, H_Z + \frac{1}{2}\Xi)$ is log canonical. Thus we have shown that $(X, D + H)$ is log canonical near $D$. Assume that $(X, D + H)$ is not log canonical at some point $Q \in C$. By what was said above, $Q \notin D$. Note that $H$ is smooth outside $C$ by
Bertini’s theorem. If $H$ is normal, then we have an immediate contradiction by a connectedness result ([28], Theorem 6.9) applied to $(H, D|_H)$. If $H$ is not normal, then we can apply the same result to the normalization. □

9.3.3. We claim that $H$ is normal. Indeed, assume the opposite: $H$ is singular along $C$. By the lemma above, $C$ is the minimal log canonical centre of $(X, H)$; see [36]. Now let $\tau: (X', C') \to (X, C)$ be the torsion-free cover 4.6 and put $H' := \tau^* H$. Then the pair $(X', C')$ is log canonical and $C'$ is its minimal log canonical centre ([29], §20.4). Since the minimal log canonical centre is normal [36], we conclude that $C'$ is irreducible. This contradicts the imprimitivity of $(X, C)$ at $P$.

9.3.4. Thus, $H$ is normal and then $P$ is the only log canonical centre of the pair $(X, H + D)$. It follows that the pair $(X, H)$ is purely log terminal. Since $H$ is a Cartier divisor, the singularities of $H$ are of class $T$ (see 9.1.8). This imposes a very strong restriction on the dual graph of the minimal resolution. If $(X, C)$ is a $\mathbb{Q}$-conic bundle germ, then one can show by a completely combinatorial technique that there is only one possibility for $\Delta(H, C)$ (compare with [37]):

$$
\begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ \\
\end{array}
$$

But in this case the pair $(H, C)$ is purely log terminal and, therefore, so is $(H', C')$. Hence $C'$ cannot split and this case does not occur. In the birational case, we obtain (9.1.7). Since $(H, C)$ cannot be purely log terminal (as above), we have $r \neq 1, n$. This concludes our explanation of the proof of Theorem 9.1. □

9.4. To decide whether an extremal curve germ $(X, C)$ is locally imprimitive at $P$, one needs to compute the inverse image $C^\sharp$ of $C$ in the cover $(X^\sharp, P^\sharp)$ of index 1. It can be computed within the pullback $H^\sharp$ of $H$ as in Example 9.6 once a diagram like (9.1.7) has been exhibited. Indeed, $P$ is imprimitive if and only if the splitting degree $s$ is greater than 1 (we recall that $s$ is the number of irreducible components of $C^\sharp$; see Definition 4.2).

9.5. To distinguish divisorial and flipping contractions in the case 9.1.6, one can use the following arguments. Let $f': (X', C') \to (Z', o')$ be the torsion-free cover (4.6.2). By Corollary 6.5, the germ $(X, C \simeq \mathbb{P}^1)$ is divisorial if and only if the point $(Z, o)$ is terminal and if and only if the point $(Z', o')$ is terminal of index 1 (that is, isolated cDV). Note that in our case $(H_Z, o)$ is a cyclic quotient singularity, and so is its pullback $(H'_Z, o')$. Hence $(X, C)$ is divisorial if and only if the point $(H'_Z, o')$ is Gorenstein, that is, a Du Val singularity in our case. Once the germ $(H, C \simeq \mathbb{P}^1)$ is given, one can find its splitting cover $(H', C')$ and, therefore, the surface germ $(H'_Z, o')$ can be computed.

9.6. Example. Consider the quotient surface singularity

$$(H, P) = (\mathbb{C}^2_{u,v}, 0)/\mu_{m^2}(1, m - 1), \quad m \geq 3.$$ 

It is of class T, and for its cover of index 1 we have

$$(H^\sharp, P^\sharp) = (\mathbb{C}^2, 0)/\mu_m(1, m - 1).$$
Hence it is Du Val of type $A_{m-1}$. Consider the $\mu_m$-equivariant curve
\[ C^g = \{ u^{m-2} - v^{m+2} = 0 \}/\mu_m \subset H^g \]
and $C = C^g/\mu_m$. If $m$ is odd (resp. even), then $C^g$ is irreducible (resp. has two irreducible components) and it is easy to see that the quotient curve $C$ is smooth. We now consider the weighted $\frac{1}{m}(1, m - 1)$-blow-up of the surface $(H, P)$. In the chart $v \neq 0$, the origin is a Du Val point $\mathbb{C}^2/\mu_{m-1}(-1, m^2)$ of type $A_{m-2}$, the exceptional divisor $\Lambda$ is $v' = 0$, and the proper transform $\tilde{C}$ of $C$ is given by $v' = u^{m-2}$. Hence, on the minimal resolution of the $A_{m-2}$-point, both curves $\Lambda$ and $\tilde{C}$ meet the same end of the chain. Therefore, the dual graph $\Delta(H, C)$ is of the form
\[
\overset{m+2}{\circ} \quad \overset{m-3}{\circ} \quad \overset{\cdots}{\circ} \quad \cdot
\]

We now suppose that $C$ is a compact curve, $C \simeq \mathbb{P}^1$, and consider a surface germ $(H, C)$ whose minimal resolution is of the above form. It is easy to see that $K_H \cdot C = -2/m$ and $C$ can be contracted to a cyclic quotient singularity $(H_Z, o)$ of type $\frac{1}{4}(1, 1)$. There is a Gorenstein threefold germ $X^g$ with a $\mu_m$-action containing $H^g$ as a $\mu_m$-invariant hypersurface. By 6.6 (and [11], §3), the germ $(H, C)$ has a smoothing in a $\mathbb{Q}$-Gorenstein threefold $X$ containing $H$ as a Cartier divisor. By inversion of adjunction (see [28], §3, [29], Ch. 17), $X$ has only terminal singularities. Arguing as in Theorem 6.3, we see that there is a birational contraction $f : X \to Z$ extending $H \to H_Z$. Note that $C^g$ can be identified with the pullback of $C$ under the splitting cover of $(X, C)$ at $P$. We now distinguish two cases according to the parity of $m$.

a) $m$ is even. Then $(X, C)$ is imprimitive of splitting degree $2$ at $P$. Since $(H_Z, o)$ is a type-$T$ singularity of index $2$, its pullback $(H'_Z, o')$ in the torsion-free (degree $2$) cover (4.6.2) is Du Val and, therefore, $(Z', o')$ is a point of type cDV. Hence, both contractions $f'$ and $f$ are divisorial.

b) $m$ is odd. Then $(X, C)$ is primitive and the contraction is flipping by (4.7.5). Note that the singularity $(Z, o)$ is not $\mathbb{Q}$-Gorenstein in this case. On the other hand, since $(H_Z, o)$ is a singularity of class $T$, it has a $\mathbb{Q}$-Gorenstein smoothing. This smoothing belongs to a component of the versal deformation space which is different from that corresponding to $(Z, o)$; see [11], §3.9.

§ 10. Cases (IC) and (IIB)

In this section we consider curve germs of types (IC) and (IIB).

10.1. Case (IIB). Let $(X, P)$ be the germ of a three-dimensional terminal singularity and let $C \subset (X, P)$ be a smooth curve. We recall that the triple $(X, C, P)$ is of type (IIB) if $(X, P)$ is a terminal singularity of type cAx/4 and there are analytic isomorphisms
\[
(X, P) \simeq \{ y_1^2 - y_2^3 + \alpha = 0 \}/\mu_4 \subset \mathbb{C}^4_{y_1, \ldots, y_4}/\mu_4(3, 2, 1, 1),
\]
\[
C = \{ y_1^2 - y_2^3 = y_3 = y_4 = 0 \}/\mu_4,
\]
where \( \alpha = \alpha(y_1, \ldots, y_4) \in (y_3, y_4) \) is a semi-invariant with \( \text{wt}(\alpha) \equiv 2 \mod 4 \) and the quadratic part \( \alpha_2 \) of the function \( \alpha(0, 0, y_3, y_4) \) is not zero (see [2], §A.3). We say that \((X, P)\) is a simple (resp. double) cAx/4-point if \( \text{rk} \alpha_2 = 2 \) (resp. \( \text{rk} \alpha_2 = 1 \)).

10.2. Theorem ([38]). Let \( f: (X, C \simeq \mathbb{P}^1) \to (Z, o) \) be an extremal curve germ. Suppose that \( X \) contains a point \( P \) of type (IIB). Then \((X, C)\) is not flipping ([3], Theorem 4.5) and \( P \in X \) is the unique singular point of \( X \) on \( C \). Furthermore, a general member \( H \in |\mathcal{O}_X| \) is normal, smooth outside \( P \), and has only rational singularities. Only the following possibilities may occur for the dual graphs \((H, C)\) and \( H_Z := f(H) \).

| No.  | cAx/4-point | \( \Delta(H, C) \) | \( \Delta(H_Z, o) \) |
|------|-------------|-------------------|-------------------|
| 10.2.1 | simple | \( \begin{array}{c} 3 \\ \circ \\ 4 \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \) | A_2 |
| 10.2.2 | simple | \( \begin{array}{c} 3 \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \) | A_0 |
| 10.2.3 | double | \( \begin{array}{c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \) | D_4 |
| 10.2.4 | double | \( \begin{array}{c} \circ \\ \circ \\ \circ \\ \circ \end{array} \) | c |

The last column indicates whether the germ is divisorial (d) or a \( \mathbb{Q} \)-conic bundle (c). The column \( \Delta(H_Z, o) \) is not used in the last case (c).

An example of a divisorial contraction of type 10.2.1 was given in [3], §4.12. The case 10.2.2 was also studied by Ducat [20], Theorem 4.1(2b), in terms of symbolic blow-ups of smooth threefolds.

Sketch of proof. In our case, a general member \( D \in |-K_X| \) contains \( C \) and has only Du Val singularities and the graph \( \Delta(D, C) \) is of the form 7.4.3. Under the identifications 10.1, a general member \( D \in |-K_X| \) near \( P \) is given by \( \lambda y_3 + \mu y_4 = 0 \) for some \( \lambda, \mu \in \mathcal{O}_X \) such that the constants \( \lambda(0), \mu(0) \) are general in \( \mathbb{C}^* \) ([3], §2.11, [7], §4). Put \( \Gamma := H \cap D \).

By Theorem 4.5 in [3], the contraction \( f \) is not flipping. When \( f \) is divisorial, we put \( D_Z := f(D) \) and \( \Gamma_Z := f(\Gamma) \). Then \( D_Z \in |-K_Z| \), \( H_Z \) is a general hyperplane section of \((Z, o)\), and \( \Gamma_Z \) is a general hyperplane section of \( D_Z \). When \( f \) is a \( \mathbb{Q} \)-conic bundle, we put \( D_Z := \text{Spec} \mathbb{Z} f_* \mathcal{O}_D \) (the Stein factorization) and let \( \Gamma_Z \subset D_Z \) be the image of \( \Gamma \). In both cases, \( D_Z \) is a Du Val singularity of type \( E_6 \) by 7.4.3.

We claim that \( \Gamma = C + \Gamma_1 \) (as a scheme), where \( \Gamma_1 \) is a reduced irreducible curve, and the surface \( H \) is normal, smooth outside \( P \), and has only rational singularities. Indeed, consider two cases, 10.2.5 and 10.2.7.

10.2.5. Case: \( f \) is divisorial. Since the point \((Z, o)\) is terminal of index 1, the germ \((H_Z, o)\) is a Du Val singularity. Since \( \Gamma_Z \) is a general hyperplane section
of $D_Z$, we see that the graph $\Delta(D, \Gamma)$ is of the form

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \\
1 \quad 2 \quad 3 \quad 4
\end{array}
\]

where $\Delta$ corresponds to the proper transform of $\Gamma_Z$ and the numbers attached to the vertices are the coefficients of the corresponding exceptional curves in the pullback of $\Gamma_Z$. By Bertini’s theorem, $H$ is smooth outside $C$. Since the coefficient of $C$ is equal to 1, we have $D \cap H = C + \Gamma_1$ (as a scheme). Hence $H$ is smooth outside $P$. In particular, $H$ is normal. Since $f_H: H \to H_Z$ is a birational contraction and $(H_Z, o)$ is a Du Val singularity, the singularities of $H$ are rational.

10.2.7. Case: $f$ is a $Q$-conic bundle. We may assume that, in an appropriate coordinate system, the germ $(D_Z, o_Z)$ is given by $x^2 + y^3 + z^4 = 0$ and the double cover $(D_Z, o_Z) \to (Z, o)$ is just the projection to the $(y, z)$-plane. Then $\Gamma_Z$ is given by $z = 0$. As in the case above, we see that $\Delta(D, \Gamma)$ is of the form (10.2.6). Therefore, $H$ is smooth outside $P$. The restriction $f_H: H \to H_Z$ is a fibration into rational curves. Hence $H$ has only rational singularities. This proves our claim.

10.2.8. Furthermore, $\text{gr}^1_C \mathcal{O} \simeq \mathcal{O}_{P^1}(d_1) \oplus \mathcal{O}_{P^1}(d_2)$ for some $d_1 \geq d_2$. Since $H^1(\text{gr}^1_C \mathcal{O}) = 0$ by Corollary 2.4.1 (i), we have $d_2 \geq -1$. Since $H$ is normal, $d_1 \geq 0$ (see Lemma 3.5.1). On the other hand, $\deg \text{gr}^1_C \mathcal{O} = 1 - i_P(1)$ by (3.1.3). One can compute $i_P(1) = 2$ from [2], (2.12), for the points $P$ of type (IIB) described in 10.1. Therefore,

$$\text{gr}^1_C \mathcal{O} \simeq \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-1)$$

and $\mathcal{O}_C(-H) = \mathcal{O} \subset \text{gr}^1_C \mathcal{O}$, that is, the local equation of $H$ must be a generator of $\mathcal{O} \subset \text{gr}^1_C \mathcal{O}$. In the notation 10.1, the surface $H \subset X$ is given locally by the equation $y_3 y_4 + y_4 y_4 = 0$ near $P$, where $y_3, y_4 \in \mathcal{O}_{P^1, X}$ are semi-invariants with $\text{wt}(y_4) \equiv 3$ and at least one of $y_3, y_4$ contains a linear term in $y_1$. Therefore, the surface germ $(H, P)$ can be given in $\mathbb{C}^4 / \mu_4(3, 2, 1, 1)$ by two equations,

$$y_1^2 - y_2^2 + \eta(y_3, y_4) + \phi(y_1, y_2, y_3, y_4) = 0,$$

$$y_1 l(y_3, y_4) + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi(y_1, y_2, y_3, y_4) = 0,$$

where $\eta, l, q$ and $\xi$ are homogeneous polynomials of degree 2, 1, 2 and 4 respectively, $\eta \neq 0, l \neq 0, \phi, \psi \in (y_3, y_4)$, $\sigma$-ord $\phi \geq 3/2$, and $\sigma$-ord $\psi \geq 2$. Moreover, $\text{rk} \eta = 2$ (resp. $\text{rk} \eta = 1$) if $(X, P)$ is a simple (resp. double) cAx/4-point. Then, considering the weighted $1/4(3, 2, 1, 1)$-blow-up and using the rationality of $(H, P)$ and the generality of $H$ in $|\mathcal{O}_X|$, we obtain all the possibilities in Theorem 10.2. See [38], §3, for details.

10.3. Case (IC). Let $(X, C \simeq \mathbb{P}^1)$ be an extremal curve germ. Assume that $(X, C)$ has an (IC)-point $P$ of index $m$. Then $P$ is the only singular point of $X$, the index $m$ is odd $\geq 5$, and $w_P(0) = (m - 1)/m$. Moreover, $i_P(1) = a_1 = 2$ and

$$(X, C, P) \simeq (\mathbb{C}^3_{y_1, y_2, y_4}, \{y_1^{m-2} - y_2^2 = y_4 = 0\}, 0)/\mu_m(2, m - 2, 1),$$

(see Lemma 5.7 (i) and [2], §§5.5, 6.5, A.3). Here is a complete classification of extremal curve germs of type (IC).
10.4. **Theorem** ([3], §8, [38]). Let \( f : (X, C \simeq \mathbb{P}^1) \to (Z, o) \) be an extremal curve germ of type (IC), \( P \in X \) a (unique) singular point, and \( m \) its index. Then a general member \( H \in |\mathcal{O}_X| \) is normal, smooth outside \( P \), and has only rational singularities. Moreover, \((X, C)\) is not divisorial and one of the following cases holds.

10.4.1. \((X, C)\) is flipping and only the following possibilities may occur for the dual graphs of \((H, C)\) and \( H_Z = f(H) \):

\[
\begin{align*}
\text{\bullet} & - \circ - \cdots - \circ - \circ \\
3 & \quad \quad (m+3)/2 \quad \quad 3 \quad \quad (m-7)/2 \\
\circ & - \cdots - \circ - \circ \\
3 & \quad \quad (m+3)/2 \\
\end{align*}
\]

where \( \circ - \circ - \cdots - \circ - \circ \) must be replaced by \( \circ \) in the case \( m = 5 \).

10.4.2. \((X, C)\) is a \( \mathbb{Q}\)-conic bundle, \( m = 5 \), and \( \Delta(H, C) \) is of the form

\[
\begin{align*}
\text{\bullet} & - \circ - \cdots - \circ - \circ \\
3 & \quad \quad 3 \\
\circ & - \circ - \cdots - \circ - \circ \\
3 & \quad \quad 3 \\
\end{align*}
\]

In the flipping case, the general member \( H^+ \in |\mathcal{O}_{X^+}| \) on the flipped variety is also computed. Here \( X^+ \) is either of index two or Gorenstein; see [3], §A.3.

§11. **Case (IA)**

11.1. An extremal curve germ \((X, C \simeq \mathbb{P}^1)\) is said to be of type (IA) if it contains exactly one non-Gorenstein point \( P \), which is of type (IA). For the reader’s convenience, we mention the following characterization of extremal curve germs \((X, C \simeq \mathbb{P}^1)\) of type (IA), where \( D \in |-K_X| \) is a general member (see Proposition 5.4 and Theorem 7.4):

\((X, C)\) is of type (IA) if and only if (i) \( P \) is locally primitive, (ii) \( D \cap C \) is a singleton, and (iii) \((X, P)\) is not of type cAx/4.

11.1.1. From now on, we assume that the germ \((X, C \simeq \mathbb{P}^1)\) satisfies the assumptions of 11.1. Only the following possibilities may occur for \((X, P)\):

(i) \((X, P)\) is of type cA/m, and in this case \((X, C)\) is said to be of type (k1A) according to [3];

(ii) \((X, P)\) is of type cD/3;

(iii) \((X, P)\) is of type cAx/2, cD/2 or cE/2.
Thus, in our case of type (IA), the germ \((X, C)\) is semistable if and only if it is of type (k1A). Extremal curve germs of index two are classified in §8. Thus we discuss here the cases (k1A) and cD/3. We begin with \(\mathbb{Q}\)-conic bundles.

11.2. Theorem ([21], 1.6). Let \((X, C \simeq \mathbb{P}^1)\) be a \(\mathbb{Q}\)-conic bundle germ of index \(m > 2\) and of type (IA). Let \(P \in X\) be the non-Gorenstein point. Then \((X, P)\) is a point of type \(cA/m\) and a general member \(H \in |\mathcal{O}_X|\) is non-normal. Furthermore, the dual graph of \((H^n, C^n)\), where \(H^n\) is the normalization and \(C^n\) is the inverse image of \(C\), is of the form

\[
a \circ \cdots \circ \otimes \circ \quad \bullet \quad b \circ \cdots \circ \otimes \circ
\]

(in particular, the curve \(C^n\) is irreducible). Here the chain \(\Delta_1\) (resp. \(\Delta_2\)) corresponds to the singularity of type \(\frac{1}{m}(1, a)\) (resp. \(\frac{1}{m}(1, -a)\)) for some integer \(a\), \(1 \leq a < m\), relatively prime to \(m\). The germ \((H, C)\) is analytically isomorphic to the germ along the line \(y = z = 0\) in \(\mathbb{P}(1, a, m - a, m)\) of the hypersurface given by the weighted polynomial

\[
\phi := x^{2m - 2a} y^2 + x^{2a} z^2 + yzu
\]

of degree \(2m\) in the variables \(x, y, z, u\). Then \((X, C)\) can be represented as the analytic germ along \(C \times 0\) of the subvariety of \(\mathbb{P}(1, a, m - a, m) \times \mathbb{C}_t\) given by

\[
\phi + \alpha_1 x^{2m - a} y + \alpha_2 x^{m - a} uy + \alpha_3 x^{2m} + \alpha_4 x^m u + \alpha_5 u^2 = 0
\]

for some \(\alpha_1, \ldots, \alpha_5 \in t\mathcal{O}_{0, C^1}\). The second projection \(X \to \mathbb{C}_t\) factors through a certain \(\mathbb{Q}\)-conic bundle structure \(X \to \mathbb{C}^2\).

11.3. Theorem ([21], 1.9; see also [39]). Let \((X, C)\) be a birational extremal curve germ of type (k1A). Let \(P \in X\) be the point of index \(m \geq 2\).

11.3.1. If a general element \(H\) is normal, then the graph \(\Delta(H, C)\) is of the same form as in (9.1.7), but the cases \(r = 1\) and \(r = n\) are not excluded.

11.3.2. If every member of \(|\mathcal{O}_X|\) is non-normal, then the dual graph of the normalization \((H^n, C^n)\) is of the form

\[
a \circ \cdots \circ \otimes \circ \quad \bullet \quad c \circ \cdots \circ \otimes \circ \quad \circ \quad b \circ \cdots \circ \otimes \circ
\]

(in particular, \(C^n\) is reducible). The chain \(\Delta_1\) (resp. \(\Delta_2\)) corresponds to the singularity of type \(\frac{1}{m}(1, a)\) (resp. \(\frac{1}{m}(1, -a)\)) for some \(a\) with \(\gcd(m, a) = 1\), and the chain \(\Delta_3\) corresponds to the point \((H^n, Q^n)\), where \(Q^n = C^n_1 \cap C^n_2\). Moreover,

\[
\sum (c_i - 2) \leq 2 \quad \text{and} \quad \tilde{C}_1^2 + \tilde{C}_2^2 + 5 - \sum (c_i - 2) \geq 0,
\]

where \(\tilde{C} = \tilde{C}_1 + \tilde{C}_2\) is the proper transform of \(C\) on the minimal resolution \(\tilde{H}\). Both components of \(\tilde{C}\) are contracted on the minimal model of \(\tilde{H}\). In this case,

\[
(X, C, P) \simeq \langle \{ \alpha(x_1, \ldots, x_4) = 0 \}, x_1\text{-axis}, 0 \rangle / \mathbb{Z}_m(1, a, -a, 0),
\]

\(\mathbb{Z}_m\) being the cyclic group of order \(m\).
where \( \gcd(m,a) = 1 \), and \( \alpha = 0 \) is the equation of a terminal point of type \( cA/m \) in \( \mathbb{C}^4/\mathbb{m} \). (In particular, \( (X,C) \) is of type (IA).)

Conversely, for any germ \( (H,C \simeq \mathbb{P}^1) \) of the form 11.3.1 or 11.3.2 admitting a birational contraction \( (H,C) \rightarrow H_{Z}, o \), there is a threefold birational contraction \( f: (X,C) \rightarrow (Z,o) \) of type (IA) as in 11.1 such that \( H \in |\mathcal{O}_X| \).

11.3.4. To study a general member \( H \in |\mathcal{O}_X| \), we can use Lemma 9.3.2. However, we cannot assert as in 9.3.3 that \( H \) is normal. In fact, arguments similar to 9.3.4 show that the case of normal \( H \) does not occur when \( (X,C) \) is a \( \mathbb{Q} \)-conic bundle.

11.4. Let us outline the proofs of Theorems 11.2 and 11.3. The case when \( H \) is normal can be treated in the same way as 9.1.6 (see 9.3.4), and \( X \) can be recovered as a one-parameter deformation space by 6.6. Examples will be given in 11.4.7.

Suppose that \( H \) is not normal. Let \( \nu: H_{n} \rightarrow H \) be the normalization and let \( C_{n} \subset H_{n} \) be the inverse image of \( C \). By inversion of adjunction, the pair \( (H,C) \) is semi-log canonical, the pair \( (H_{n},C_{n}) \) is log canonical and the point \( P \in (H,C) \) is semi-log terminal; see [29], §16.9. In particular, \( H \) has generically normal crossings. At (finitely many) “dissident” points, \( H \) may have singularities worse than just normal crossing points.

11.4.1. Since \( H \) has a \( \mathbb{Q} \)-Gorenstein smoothing, only the following possibilities may occur in accordance with [11], Theorem 4.24, §5.2:

- a pinch point: \( \{x^2 - y^2z = 0\} \subset \mathbb{C}^3 \);
- a degenerate cusp of embedding dimension at most 4, where a degenerate cusp is a non-normal Gorenstein singularity having a semi-resolution whose exceptional divisor is a cycle of smooth rational curves or a rational nodal curve (see [40]);
- a semi-log canonical singularity of the form

\[ \{xy = 0\}/\mu_m(a,-a,1), \quad \gcd(a,n) = 1 \]

(this point corresponds to \( P \in H \)).

11.4.2. The restriction \( \nu_C: C_{n} \rightarrow C \) of the normalization to the inverse image of \( C \) is a double cover. We distinguish two possibilities:

(i) \( C_{n} \) is smooth irreducible and \( \nu_C \) is branched at two points;

(ii) \( C_{n} \) has two irreducible components meeting at one point and the restriction of \( \nu_C \) to each of them is an isomorphism.

A detailed analysis (see [21] and also [39]) shows that (i) gives rise to the \( \mathbb{Q} \)-conic bundle case (11.2.1) while (ii) gives rise to the birational case (11.3.3). In both cases, the subgraphs \( \Delta_1 \) and \( \Delta_2 \) correspond to points \( P_{1,n}, P_{2,n} \in H_{n} \) lying over \( P \in H \).

11.4.3. To recover \( X \) as a one-parameter deformation space, we can also apply the arguments used in 6.6. However, in the case of a non-normal surface \( H \), this requires restrictions on the singularities and some additional technical tools [41]. Fortunately, the results in [41] are applicable when \( H \) has singularities as described above. Moreover, the universal deformation family of \( (H,C) \) has been computed explicitly ([21], 6.8.3) in the case of \( \mathbb{Q} \)-conic bundles.
11.4.4. To check divisoriality, one can use the criterion in Corollary 6.5. Indeed, if \( f \) is divisorial, then \((Z, o)\) is a terminal point and its index is equal to 1 because \((X, C)\) is primitive (see 4.5). Therefore, its general hyperplane section \((H_Z, o)\) must be a Du Val singularity. If, on the contrary, \( f \) is flipping, then \((Z, o)\) is not \(\mathbb{Q}\)-Gorenstein and \((H_Z, o)\) cannot be Du Val. Given a graph \(\Delta(H, C)\) of type (9.1.7), one can easily obtain the graph \(\Delta(H_Z)\) by successively contracting the black vertices. Thus the Du Val property of \((H_Z, o)\) can be checked in purely combinatorial terms.

11.4.5. Remark. Under the hypotheses of 11.3.1 and (9.1.7), assume that we have \(r = 1\) or \(r = n\). Then the graph \(\Delta(H, C)\) is a chain. In this case there is an element \(D \in |-K_X|\) containing \(C\) and having Du Val singularities only. This is a particular case of the situation considered in [4], where a powerful algorithm for constructing \((X, C)\) was obtained.

11.4.6. A special case of Theorem 11.3 was studied in detail in [5], where the authors assumed that \(b_2(H_t) = 1\) for the nearby fibre \(H_t\) of the one-parameter deformation \(\bigcup H_t = X\). This strong assumption is equivalent to saying that \(H\) is normal and has a so-called Wahl singularity at \(P\): \((H, P) \simeq \mathbb{C}^2/\mu_m^2(1, ma - 1)\). Under this condition, the authors showed that birational germs of this type belong to the same deformation family as those of type \((k2A)\) studied in [4]. They also constructed a universal family and generalized the algorithm of [4] for computing flips.

11.4.7. Examples. We consider some examples of extremal germs of type 11.3.1.

(i) Germs of index two (8.2.1–8.2.5) are of type \((IA)\). By using the arguments in 11.4.4, one can conclude that the germ in 8.2.1 is flipping and those in 8.2.2–8.2.5 are divisorial.

(ii) Let \(\Delta(H, C)\) be of the form

\[
\bullet \quad \overset{c_1}{\circ} \quad \ldots \quad \overset{c_n}{\circ},
\]

where the white vertices form the dual graph of a non-Du Val T-singularity (see 9.1.8). It is easy to see that \(C\) can be contracted to a cyclic quotient non-Du Val point. Therefore, the one-parameter deformation produces a flipping contraction. Since \((H, C)\) is purely log terminal, the contraction is primitive.

(iii) Let \(\Delta(H, C)\) be of the form

\[
\begin{array}{c}
\overset{3}{\circ} ~ \overset{5}{\circ} ~ \circ \\
\overset{\bullet}{\circ} ~ \circ ~ \circ ~ \circ ~ \circ
\end{array}
\]

This is an example of a divisorial contraction to a smooth point.

(iv) A series of examples was given in 9.6 (b).

For completeness, we give an example of a birational curve germ with non-normal \(H\).

11.4.8. Example ([39], Example 2, [21], Example 6.10.3). Consider a surface \(\tilde{H}\) containing a configuration with the graph

\[
\begin{array}{c}
\overset{\bullet}{\circ} ~ \overset{4}{\circ} ~ \overset{4}{\circ} \\
\overset{\bullet}{\circ} ~ \overset{\circ}{\circ} ~ \overset{\circ}{\circ} ~ \overset{\circ}{\circ} ~ \overset{\circ}{\circ} ~ \overset{3}{\circ}
\end{array}
\]
Contracting all curves except those marked by $C_1$ and $C_2$, we obtain a normal surface $H^n$ with two cyclic quotient singularities $P_1$ and $P_2$ of types $\frac{1}{7}(1,2)$ and $\frac{1}{7}(1,-2)$. Identifying the curves $C_1$ and $C_2$, we obtain a non-normal surface $H$ such that the map $\nu: H^n \to H$ is the normalization. The dissident singularities of $H$ are a degenerate cusp of multiplicity 2 and embedding dimension 3 (located at $\nu(C_1 \cap C_2)$) and a point of type $\{xy = 0\}/\mu_7(2,-2,1)$. The results of [41] are applicable here and yield the existence of a one-parameter smoothing $X \supset H \supset C$ which is a divisorial curve germ, where $H$ is general in $|\mathcal{O}_X|$; see [21], Proposition 6.3, Theorem 6.10.

11.5. Points of type cD/3. Let $(X, C, P)$ be a triple of type (IA), where $(X, P)$ is a singularity of type cD/3; see [9], [10]. These triples are described as follows ([3], §6.5). Put $\sigma := (1,1,2,3)$. Up to a change of coordinates, the point $(X, C, P)$ is given in $\mathbb{C}^4_{y_1, \ldots, y_4}$ by

\begin{equation}
(X, C, P) = \{ \{ \alpha = 0 \}, \{ y_1\text{-axis} \}, 0 \}/\mu_3(1,1,2,0), \quad \alpha = y_1^2 + y_3^3 + \delta_3(y_1, y_2) + \text{(terms of degree } \geq 4),
\end{equation}

where $\delta_3 \neq 0$ is a homogeneous polynomial of degree 3 and $\alpha$ is invariant. Moreover,

$$\alpha \equiv y_1^\ell y_i \mod (y_2, y_3, y_4)^2,$$

where $\ell = \ell(P)$ and $i = 2$ (resp. 3, 4) if $\ell \equiv 2$ (resp. 1, 0) mod 3; see [2], (2.16). If the polynomial $\delta_3(y_1, y_2)$ is square-free (resp. has a double factor, is the cube of a linear form), then $(X, P)$ is called a simple (resp. double, triple) point of type cD/3.

Extremal curve germs containing a terminal singular point of type cD/3 are described by the following theorem.

11.5.2. Theorem ([3], Theorems 6.2, 6.3, [21], Theorems 4.5, 4.8). Let $f: (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be an extremal curve germ having a point $P$ of type cD/3. Then $f$ is a birational contraction (not a $\mathbb{Q}$-conic bundle). General members $H \in |\mathcal{O}_X|$ and $H_Z = f(H) \in |\mathcal{O}_Z|$ are normal and have only rational singularities. We have the following possibilities for the graphs $\Delta(H, C)$ and $\Delta(H_Z, o)$ and local invariants.

| No. | $\ell(P)$ | $i_P(1)$ | $\Delta(H, C)$ | $\Delta(H_Z, o)$ |
|-----|-----------|-----------|----------------|-----------------|
| Cases of a simple cD/3-point $P$ |
| 11.5.3 | 2 | 1 | $\bullet$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ | $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ | f |
| 11.5.4 | 2 | 1 | $\bullet$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ | $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ | $A_2$ | d |

| Cases of a double cD/3-point $P$ |
| 11.5.5 | 2 | 1 | $\bullet$ $\circ$ | $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ | $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ | f |
No. \(\ell(P)\) \(i_P(1)\) \(\Delta(H, C)\) \(\Delta(H_Z, o)\)  
11.5.6 2 1 - - - - D\(_4\) d  
11.5.7 3, 4 2 - - - - f  
11.5.8 3, 4 2 - - - - E\(_6\) d  

Case of a triple cD/3-point \(P\)

The variety \(X\) is smooth outside \(P\) in cases 11.5.3, 11.5.5, 11.5.7, 11.5.8, and may have at most one point of type (III) in cases 11.5.4, 11.5.6. The last column indicates whether the germ is flipping (f) or divisorial (d).

Note that [3], §6 and [21], §4 provide much more information about these contractions: the infinitesimal structure, a criterion for an arbitrary germ to be of the corresponding type, and computations of the flipped varieties ([3], §A.1). Flipping contractions can be constructed explicitly by patching certain open subsets.

11.5.9. Example ([3], §6.11). Let \(V \supset C\) be the germ of a smooth threefold along a curve \(C \simeq \mathbb{P}^1\) such that \(\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C\). Pick a point \(P \in C\) and let \((v_1, v_2, v_3)\) be coordinates at \((V, P)\) such that \((C, P) = \{v_1\text{-axis}\}\). Let \((X, C, P)\) be a point of type cD/3 as in (11.5.1) with \(\ell = 2\). For suitable \(\varepsilon_1\) and \(\varepsilon_2\) with \(0 < \varepsilon_1 < \varepsilon_2 \ll 1\), the functions \((y_3^3, y_4, y_1y_3)\) determine a coordinate system in \(U = (X, P) \cap \{\varepsilon_1 < |y_3^3| < \varepsilon_2\}\) by the implicit function theorem. Thus the identifications \(v_1 = y_3^3, v_2 = y_4\) and \(v_3 = y_1y_3\) patch \((X, P)\) and \(V \setminus (V, P) \cap \{|v_1| < \varepsilon_1\}\) along \(U\). By [3], §6.2.4, the germ \((X, C)\) is a flipping curve germ of type cD/3 as in 11.5.3 or 11.5.5 (depending on the choice of \(\delta_3\) in (11.5.1)).

More examples of flipping contractions can be found in [3], §§6.17 and 6.21. To show that all the possibilities in Theorem 11.5.2 occur, one can also use the deformation arguments in 6.6.

11.5.10. Example. Consider a surface contraction \(f_H: H \to H_Z\) with dual graph 11.5.8 and consider the triple of germs

\[(X, H, P) = \{\{y_2^3 + y_3^3 + y_3y_1^4 + y_4^2\}, \{y_4 = y_1y_3\}, 0\}/\mu_3(1, 1, 2, 0),\]

where \(H\) is cut out by \(y_4 = y_1y_3\). Here \((X, P)\) is a triple singularity of type cD/3 (see (11.5.1)). By [21], 4.12, the dual graph of the minimal resolution of \((H, P)\) is the same as in 11.5.8. By 6.6, we obtain a birational contraction \(f: X \to Z\).
extending \( f_H : H \rightarrow H_Z \) as in 11.5.8. Examples similar to 11.5.4 and 11.5.6 were given in [21], 4.14.

Divisorial contractions of type 11.5.8 were also studied in [18], §5.1(2), by another method.

§ 12. Case (IIA)

12.1. Let \((X, C)\) be an extremal curve germ and let \( f : (X, C) \rightarrow (Z, o) \) be the corresponding contraction. Assume that \((X, C)\) has a point \( P \) of type (IIA). Then it follows from [2], §§6.7, 9.4 and [6], 8.6, 9.1, 10.7 that \( P \) is the only non-Gorenstein point of \( X \) and \((X, C)\) has at most one Gorenstein singular point \( R \); see [2], §6.2, [6], 9.3. Since \( P \in (X, C) \) is locally primitive, the topological index of \((X, C)\) is equal to 1. Hence, the base \((Z, o)\) is smooth in the case of \( \mathbb{Q}\)-conic bundles and is a point of type cDV (or a smooth point) in the divisorial case (compare with 6.4).

12.2. By [2], §A.3, we can write a point of type (IIA) in the form

\[
(X, P) = \{\alpha = 0\}/\mu_4(1, 1, 3, 2) \subset \mathbb{C}^4_{y_1, \ldots, y_4}/\mu_4(1, 1, 3, 2),
\]

where \( \alpha = \alpha(y_1, \ldots, y_4) \) is a semi-invariant such that

\[
\text{wt} \alpha \equiv 2 \mod 4, \quad \alpha \equiv y_1^{\ell(P)} y_j \mod (y_2, y_3, y_4)^2,
\]

with \( j = 2 \) (resp. 3, 4), if \( \ell(P) \equiv 1 \) (resp. 3, 0) mod 4 (see (3.3.1)) and \((I_C^4)^{(2)} = (y_j) + (I_C^4)^2\). Moreover, \( y_2^2, y_3^2 \in \alpha \) (because \((X, P)\) is terminal point of type cAx/4). Note that \( \ell(P) \not\equiv 2 \mod 4 \) because there are no variables of weight \( \text{wt} \equiv 0 \mod 4 \).

12.3. Theorem ([3], §§7.2–7.4, [42], [43]). Let \( f : (X, C \simeq \mathbb{P}^1) \rightarrow (Z, o) \) be an extremal curve germ having a point \( P \) of type (IIA). We have the following possibilities for the graphs \( \Delta(H, C) \) and \( \Delta(H_Z, o) \) and the local invariants.

| No. | \( i_P(1) \) | \( \ell(P) \) | \( \Delta(H, C) \) | \( \Delta(H_Z, o) \) |
|-----|-------------|-------------|----------------|----------------|
| Cases: \( H \) is normal |
| 12.3.1 | 1 | 1 | \begin{array}{c} 4 \circ \\
\bullet 4 \circ \circ \circ \circ \circ \\
\end{array} | \begin{array}{c} 4 \circ \\
\circ 3 \circ \circ \circ \circ \\
\end{array} | f |
| 12.3.2 | 1 | 1 | \begin{array}{c} 4 \circ \\
\bullet 4 \circ \circ \circ \circ \circ \\
\end{array} | \begin{array}{c} \circ \circ \circ \circ \circ \\
\circ 3 \circ \circ \circ \circ \\
\end{array} | f |
| 12.3.3 | 2 | 3, 4 | \begin{array}{c} \circ \circ \circ \circ \\
\frac{4}{4} \circ \circ \circ \circ \\
\end{array} | \begin{array}{c} \circ \circ \circ \circ \\
\circ 3 \circ \circ \circ \circ \\
\end{array} | f |
| 12.3.4 | 1 | 1 | \begin{array}{c} \circ \circ \circ \circ \\
\bullet 4 \circ \circ \circ \circ \\
\end{array} | \begin{array}{c} A_1 \\
\end{array} | d |
The variety $X$ can have (at most) one further singular point of type (III) in all cases except for 12.3.1, 12.3.3, 12.3.6 and 12.3.7, where the singular point is unique.

Examples of flipping contractions can be constructed as in 11.5.9.

**12.4. Example** ([3], §7.6.4). Let $V \supset C$ be the germ of a smooth threefold along $C \simeq \mathbb{P}^1$ such that $\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$. Pick a point $P \in C$ and let $(v_1, v_2, v_3)$ be coordinates at $(V, P)$ such that $(C, P) = \{v_1\text{-axis}\}$. Let $(X, C, P)$ be a point of type (IIA) as in (12.2.1), (12.2.2) with $\alpha \equiv y_1 y_2 \mod(y_2, y_3, y_4)^2$. For suitable $\varepsilon_1$ and $\varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, the functions $(y_1^3, y_1^2 y_4, y_1 y_3)$ form a coordinate system in $U = (X, P) \cap \{\varepsilon_1 < |y_1^3| < \varepsilon_2\}$ by the implicit function theorem. Thus, the identifications $v_1 = y_1^3$, $v_2 = y_1^2 y_4$, $v_3 = y_1 y_3$ patch $(X, P)$ and $V \setminus (V, P) \cap \{|v_1| < \varepsilon_1\}$ along $U$. By [3], §7.2.4, the germ $(X, C)$ is a flipping curve germ of type (IIA) as in 12.3.1. See [3], §§7.9.4, 7.12.5, for more examples of flipping contractions.

In the case when $H$ is normal, the existence in the above theorem can be established using the arguments in 6.6. We also have the following explicit example in the case 12.3.6.

**12.5. Example** ([42], 6.6). Let $Z \subset \mathbb{C}_5$ be defined by two equations,

\[
0 = z_2^2 + z_3 + z_4 z_5^k + z_1^3, \quad k \geq 1,
\]

\[
0 = z_1^2 z_2^2 + z_4^2 - z_3 z_5 + z_1^3 z_2 + c z_1^2 z_4.
\]

Eliminating $z_3$ by means of the first equation, we easily see that $(Z, 0)$ is a threefold singularity of type $cD_5$. Let $B \subset Z$ be the $z_5$-axis and let $f : X \to Z$ be the

\[\text{Cases: } H \text{ is not normal}\]

| No. | $i_P(1)$ | $\ell(P)$ | $\Delta(H, C)$ | $\Delta(H_Z, \circ)$ |
|-----|---------|----------|----------------|-------------------|
| 12.3.5 | 1 | 1 | $\circ \ 
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\circ \ | $\Delta \ 
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| 12.3.6 | 2 | 3, 5 | $\bullet \ 
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| 12.3.7 | 2 | 4, 5 | $\bullet \ 
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| 12.3.8 | 1 | | | $\Delta \ 
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| 12.3.9 | | | | $\Delta \ 
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\[1\text{This case was erroneously omitted in [16], Theorem 3.6 and Corollary 3.8.}\]
Threefold extremal curve germs

weighted blow-up of \( B \) with weights \((1, 1, 4, 2, 0)\). An easy computation shows that \( C := f^{-1}(0)_{\text{red}} \simeq \mathbb{P}^1 \) and \( X \) is covered by two charts: the \( z_1 \)-chart and the \( z_3 \)-chart. The origin of the \( z_3 \)-chart is a point \( P \) of type (IIA) with \( \ell(P) = 3 \):

\[
\{ y_1^3 y_3 + y_2^2 + y_3^2 + y_4 (y_1^2 y_2^2 + y_4^2 + y_1^3 y_2 + c y_1^2 y_4)^k = 0 \} / \mu_4(1, 1, 3, 2),
\]

where \((C, P)\) is the \( y_1 \)-axis. Moreover, \( X \) is smooth outside \( P \). Thus \( X \to Z \) is a divisorial contraction of type 12.3.6. See [42], 8.3.3, for an example with \( \ell(P) = 5 \).

The case 12.3.6 was also studied by Tziolas [16], Theorem 3.6.

The existence of a germ of type 12.3.8 can be established as in Example 12.5. This is done in the following example.

12.6. Example ([43], 3.6). Let \( Z \subset \mathbb{C}^5_{z_1, \ldots, z_5} \) be defined by

\[
0 = z_2^2 + z_3 + z_4 z_5^k - z_1^3, \quad k \geq 1,
0 = z_1^2 z_2^2 + z_4^2 - z_3 z_5.
\]

Then \((Z, 0)\) is a threefold singularity of type \( cD_5 \). Let \( B \subset Z \) be the \( z_5 \)-axis and let \( f: X \to Z \) be the weighted \((1, 1, 4, 2, 0)\)-blow-up. The origin of the \( z_3 \)-chart is a point \( P \) of type (IIA) with \( \ell(P) = 3 \):

\[
\{ -y_1^3 y_3 + y_2^2 + y_3^2 + y_4 (y_1^2 y_2^2 + y_4^2)^k = 0 \} / \mu_4(1, 1, 3, 2),
\]

where \((C, P)\) is the \( y_1 \)-axis. The \( z_1 \)-chart contains a point of type (III). See also [43], 3.7, for an example of a divisorial germ as in 12.3.8 whose singular locus consists of a single point \( P \) of type (IIA) with \( \ell(P) = 7 \).

12.7. Example ([43], 4.8). Let \( X \) be a hypersurface of degree 10 in the weighted projective space \( \mathbb{P}(1, 1, 3, 2, 4)_{x_1, \ldots, x_4, w} \) given by the equation

\[
w \phi_6 - x_1^6 \phi_4 = 0, \quad \text{where} \quad \phi_6 := x_1^4 x_4 + x_3^2 + x_2^2 w + \delta x_4^3,
\phi_4 := x_1^2 + \nu x_2 x_3 + \eta x_1 x_4 + \mu x_1^3 x_2
\]

(we assume for simplicity that the coefficients \( \delta, \nu, \eta \) are general). We regard \( X \) as a small analytic neighbourhood of a curve \( C \). In the affine chart \( U_w := \{ w \neq 0 \} \simeq \mathbb{C}^4 / \mu_4(1, 1, 3, 2) \), the variety \( X \) is given by the equation

\[
\phi_6(y_1, y_2, y_3, y_4, 1) - y_1^6 \phi_4(y_1, y_2, y_3, y_4, 1) = 0
\]

and \( C \) is the \( y_1 \)-axis. Clearly, it is of the form (12.2.2). Thus the origin \( P \in (X, C) \) is a point of type (IIA) with \( \ell(P) = 4 \).

In the affine chart \( U_1 := \{ x_1 \neq 0 \} \simeq \mathbb{C}^4 \), the variety \( X \) is given by

\[
w \phi_6(1, z_2, z_3, z_4, w) - \phi_4(1, z_2, z_3, z_4, w) = 0.
\]

If \( \mu \neq 0 \), then \( X \) is smooth outside \( P \), whence \((X, C)\) is as in the case 12.3.7. If \( \mu = 0 \), then \((X, C)\) has a point of type (III) at \((0, 0, 0, \eta)\).

We claim that \((X, C)\) admits the structure of a \( \mathbb{Q} \)-conic bundle as in 12.3.9 (resp. 12.3.7) when \( \mu = 0 \) (resp. \( \mu \neq 0 \)).
Proof. Consider the surface $H = \{ \phi_6 = \phi_4 = 0 \} \subset X$. Let $\psi : H^n \to H$ be the normalization (we put $H^n = H$ if $H$ is normal) and let $C^n := \psi^{-1}(C)$. One can check explicitly that $H$ is normal and smooth outside $P$ if $\mu \neq 0$. But if $\mu = 0$, then $H$ is singular along $C$, the curve $C^n$ is irreducible and rational, and $\psi_C := C^n \to C$ is a double cover. Moreover, the singularities of $H^n$ are rational. Note that $H$ is a fibre of the fibration $\pi : X \to D$ over a small disc around the origin, given by the rational function $\phi_4/w = \phi_6/x_1^6$, which is regular in a neighbourhood of $C$. Analyzing the minimal resolution, one can show that there is a fibration $f_H : H \to B$ into rational curves (where $B \subset \mathbb{C}$ is a small disc around the origin) such that $C = f_H^{-1}(0)_{\text{red}}$. Then the existence of a contraction is a consequence of the following assertion. □

12.7.1. Assertion. (i) $H^1(\hat{X}, \mathcal{O}_\hat{X}) = 0$, where $\hat{X}$ denotes the completion of $X$ along $C$.

(ii) The contraction $f_H : H \to B$ extends to a contraction $\hat{f} : \hat{X} \to \hat{Z}$.

(iii) There is a contraction $f : X \to Z$ that approximates $\hat{f} : \hat{X} \to \hat{Z}$.

Proof. For a proof of part (i), we refer to [43], 4.8.4.

(ii) Since $H^1(\mathcal{O}_\hat{X}) = 0$, the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{O}_H(H) \longrightarrow 0$$

shows that the map $H^0(\mathcal{O}_\hat{X}(\hat{H})) \to H^0(\mathcal{O}_\hat{H}(\hat{H}))$ is surjective. Hence there is a divisor $\hat{H}_1 \subset |\mathcal{O}_\hat{X}|$ such that $\hat{H}_1|_{\hat{H}} = \hat{\mathcal{O}}$. Then the divisors $\hat{H}$ and $\hat{H}_1$ determine a contraction $\hat{f} : \hat{X} \to \hat{Z}$.

(iii) Let $F$ be the scheme fibre of $f_H : H \to B$ over the origin. The above arguments show that deformations of $F$ are unobstructed. Therefore the corresponding component of the Douady space is smooth and two-dimensional. This enables us to produce a contraction on $X$. □

12.8. Example ([43], 4.9). As in Example 12.7, let $X \subset \mathbb{P}(1,1,3,2,4)$ be a small analytic neighbourhood of $C = \{(x_1,w)\text{-line}\}$ given by the equation $x_1^6\phi_4 - w\phi_6 = 0$, where

$$\phi_6 := x_3^2 + x_2^2w + \delta x_4^3 + cx_1^2x_4^2,$$

$$\phi_4 := x_4^2 + \nu x_2x_3 + \eta x_1^2x_4.$$

It is easy to check that $P := (0 : 0 : 0 : 0 : 1)$ is the only singular point of $X$ on $C$ and it is a point of type (IIA) with $\ell(P) = 8$. The rational function $\phi_4/w = \phi_6/x_1^6$ near $C$ defines a fibration whose central fibre $H$ is given by the equation $\phi_4 = \phi_6 = 0$ and $\Delta(H,C)$ is of the form 12.3.9. The existence of a contraction $f : X \to Z$ can be proved in the same way as Assertion 12.7.1.

12.9. Example ([43], 4.9.1). We can similarly construct an example of a $\mathbb{Q}$-conic bundle with $\ell(P) = 5$ and normal $H$ as in 12.3.7. Let $X \subset \mathbb{P}(1,1,3,2,4)$ be given by $w\phi_6 - x_1^6\phi_4 = 0$, where

$$\phi_6 := x_1^5x_2 + x_2^2w + x_3^2 + \delta x_4^3 + cx_1^2x_4^2.$$
and $\phi_4$ is as in Example 12.7. The origin $P \in (X, C)$ of the affine chart $U_w \simeq \mathbb{C}^4/\mu_4(1,1,3,2)$ is a point of type (IIA) with $\ell(P) = 5$. It is easy to see that $X$ is smooth outside $P$. The rational function $\phi_4/w = \phi_6/x_1^6$ determines a fibration on $X$ near $C$ with central fibre $H = \{\phi_4 = \phi_6 = 0\}$.

§13. A remark on divisorial contractions

13.1. Proposition. Let $(X, C \simeq \mathbb{P}^1)$ be a divisorial curve germ with a single non-Gorenstein point which is not of type $\text{cA}/m$ with $m > 2$. Let $f : (X, C) \to (Z, o)$ be the corresponding contraction, $E \subset X$ the exceptional divisor, and $B := f(E)$ the blow-up curve. Then the multiplicity $\text{mult}_o(B)$ is given by the following table:

| $(X, C)$ | $\Delta(H, C)$ | $\text{mult}_o(B)$ | $H_Z$ | $D_Z$ |
|----------|-----------------|---------------------|-------|-------|
| (IIA)    | 12.3.4, 12.3.5  | 3                    | $A_1$ | $D_{2n+1}$ |
| (IIA)    | 12.3.6, 12.3.8  | 1                    | $D_5$ | $D_{2n+1}$ |
| (IIB)    | 10.2.1, 10.2.3  | 2                    | $A_2, D_4$ | $E_6$ |
| (IIB)    | 10.2.2          | 5                    | $A_0$ | $E_6$ |
| cD/3     | 11.5.4, 11.5.6  | 2                    | $A_2, D_4$ | $E_6$ |
| cD/3     | 11.5.8          | 1                    | $E_6$ | $E_6$ |
| cA/2     | 8.2.2           | $n$                  | $A_1$ | $A$ |
| cA/2     | 8.2.3           | 3                    | $A_0$ | $A$ |
| cA/2     | 8.2.4           | 1                    | $A_2$ | $A$ |
| cA/2     | 8.2.5           | 4                    | $A_0$ | $A$ |
| cAx/2, cD/2 | 8.2.11, 8.2.12 | 1                    | $D$   | $D$ |
| cE/2     | 8.2.11, 8.2.13  | 1                    | $D, E_6$ | $E_7$ |

where $H_Z$ is a general hyperplane section of $(Z, o)$ and $D_Z$ is a general hyperplane section of $(Z, o)$ passing through $B$. The meaning of $m$ in the case $\text{cA}/2$ is the same as in 8.2.2.

The cases with $\text{mult}_o(B) = 1$ (that is, with smooth $B$) were studied in detail by Tziolas [15], [39], [16], [18].

Proof. Recall that $Z$ is $\mathbb{Q}$-Gorenstein and $E$ is a $\mathbb{Q}$-Cartier divisor (Theorem 6.4). By the classification, $Z$ is actually Gorenstein in all our cases (that is, $H_Z$ has at worst a Du Val singularity). Hence, $E \in |K_X|$. Put $H := f^*(H_Z)$. Let $D \in |-K_X|$ be a general member. We have $-K_X \cdot C = 1/m$, where $m$ is the index of the non-Gorenstein point (see 4.7.4). Assume for simplicity that $H$ is normal. The case 12.3.8 can be treated in a similar way.

13.2. Lemma ([39], Lemma 5.1). In the notation above, if $H$ is normal, then

$$\text{mult}_o(B) = -\frac{(K_C \cdot C)^2}{(C^2)_H} = -\frac{1}{m^2(C^2)_H}. \tag{13.2.1}$$

Now let $\psi : \hat{H} \to H$ be the minimal resolution. Write $\psi^*C = \hat{C} + \Theta$, where $\text{Supp}(\Theta) \subset \text{Exc}(\psi)$ and $\Theta = \sum \theta_i \Theta_i$. Since $\hat{C}^2 = -1$, we have

$$C^2 = -1 + \hat{C} \cdot \Theta = -1 + \sum' \theta_i, \tag{13.2.2}$$
where $\sum'$ is taken over all components $\Theta_i$ that meet $\hat{C}$. The coefficients $\theta_i$ can be computed from the standard system of linear equations

$$0 = -\Theta_j \cdot \psi^* C = \Theta_j \cdot \hat{C} + \sum_i \theta_i \Theta_j \cdot \Theta_i.$$ 

Now $\text{mult}_o(B)$ can be computed using (13.2.2). Consider, for example, the cases 12.3.4, 12.3.5 and 12.3.6 of Theorem 12.3 (the other cases are similar). In the following graphs, we attach the coefficients $\theta_i$ of the divisor $\Theta = \psi^* C - \hat{C}$ to the appropriate vertices and indicate the values of $C^2$. This immediately gives us the values of $\text{mult}_o(B)$, as desired. □

12.3.4

$C^2 = -1/48;

12.3.5

$C^2 = -1/48;

12.3.6

$C^2 = -1/16.$

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