Abstract

Inverse problems are at the heart of many practical problems such as medical image reconstruction or non-destructive evaluation. A characteristic feature of inverse problems is their instability with respect to data perturbations. In order to stabilize the inversion process, regularization methods have to be developed and applied. In this paper, we introduce and analyze the concept of filtered diagonal frame decomposition, which extends the classical filtered singular value decomposition (or spectral filtering) to the case of frames. The use of frames as generalized singular systems allows for a better adaption to a given class of potential solutions of the inverse problem. This is also beneficial for problems where the SVD is not available analytically. We show that filtered diagonal frame decompositions provide convergent regularization methods. Moreover, we derive convergence rates under source conditions and prove order optimality when the frame under consideration is a Riesz basis. Our analysis applies to unbounded and bounded forward operators. As a practical application of our tools we study filtered diagonal frame decompositions for inverting the Radon transform as an unbounded operator on $L^2(\mathbb{R}^2)$.

Keywords: Inverse problems, frame decomposition, Moore-Penrose inverse, convergence analysis, convergence rates, Radon transform, computed tomography

1 Introduction

This paper is concerned with solving inverse problems of the form

$$ y = Kx + z, \quad (1.1) $$
where $\mathbf{K} : \text{dom}(\mathbf{K}) \subseteq \mathbb{X} \rightarrow \mathbb{Y}$ is a closed densely defined linear operator between Hilbert spaces $\mathbb{X}$ and $\mathbb{Y}$, and $z$ denotes the data distortion that satisfies $\|z\| \leq \delta$ for some noise level $\delta \geq 0$. A characteristic property of inverse problems is that they are ill-posed \[10, 24\]. This means that the solution of (1.1) is either not unique or is unstable with respect to perturbations of the right-hand side. Note that our treatment includes the case of unbounded forward operators. On the one hand this does not make proofs significantly more complicated than in the case of bounded forward operators, and on the other hand unbounded forward operators are important for practically relevant inverse problems. For example, the Radon transform is well known to be unbounded as an operator on $L^2(\mathbb{R}^2)$ which is the natural Hilbert space where wavelet frames are defined. Restricting to functions vanishing outside a bounded domain would make the Radon transform bounded but would also require to adjust the underlying wavelets to the boundary. Further, on bounded domains, main theoretical tools such as the Fourier slice identity are not directly applicable.

Arguably, the theory of solving inverse problems of the form (1.1) is quite well developed. Especially, the class of filter based methods gives a wide range of solution schemes. Assuming that $\mathbf{K}$ has a singular value decomposition (SVD) $\mathbf{K} = \sum_{n \in \mathbb{N}} \sigma_n \langle \cdot, u_n \rangle v_n$, these methods take one of the following equivalent forms

\begin{align}
F_\alpha y &= \sum_{n \in \mathbb{N}} g_\alpha(\sigma_n^2) \langle \mathbf{K}^* y, u_n \rangle u_n \quad (1.2) \\
F_\alpha y &= \sum_{n \in \mathbb{N}} f_\alpha(\sigma_n) \langle y, v_n \rangle u_n \quad (1.3)
\end{align}

Here $(g_\alpha)_{\alpha > 0}$ is a family of bounded functions converging pointwise to $1/\lambda$ as $\alpha \rightarrow 0$ and $f_\alpha(\sigma) := \sigma g_\alpha(\sigma^2)$. Note that the form (1.2) derives from functional calculus applied to $g_\alpha(\mathbf{K}^* \mathbf{K}) \mathbf{K}^*$ whereas (1.3) can be naturally generalized to frame decompositions instead of an SVD. The form (1.3) can be seen as regularized version of the SVD based formula $\mathbf{K}^\dagger y = \sum_{n \in \mathbb{N}} \sigma_n^{-1} \langle y, v_n \rangle u_n$ for the Moore-Penrose pseudo inverse $\mathbf{K}^\dagger$ of $\mathbf{K}$. The analysis of such regularization methods can be found, for example, in \[10, 15\] in the case of bounded $\mathbf{K}$; compare \[18\] for the case of unbounded forward operators. For general background on pseudo inverses, see, for example, \[2\].

The SVD cannot be adapted to the underlying signal class and therefore is not always a good representation for various kinds of inverse problems. Instead, certain diagonal frame decompositions generalizing the SVD are better suited because the defining frames can be adjusted to a particular application \[3, 8, 12\]. To the best of our knowledge, filter based methods based on diagonal frame decompositions have not been rigorously studied in the context of regularization theory. (Note that after initial submission of our manuscript a related analysis appeared in \[20\]. Most notably, opposed to that paper, our analysis allows unbounded forward operators and considers order optimality and characterization of ill-posedness via the frame decomposition. On the other hand, \[20\] additionally considers the discrepancy principle which we do not address.) This paper addresses this issue and develops a regularization theory for diagonalizing systems including the SVD based filter methods as special case.
1.1 Filtered diagonal frame decomposition

A diagonal frame decomposition (DFD) for the operator $K$ consists of a frame $(u_\lambda)_{\lambda \in \Lambda}$ of $(\ker K)^\perp$, a frame $(v_\lambda)_{\lambda \in \Lambda}$ of $\overline{\text{ran} K}$ and a sequence of positive numbers $(\kappa_\lambda)_{\lambda \in \Lambda}$ such that the pseudo inverse of $K$ has the form (see Section 2.2)

$$
\forall y \in \text{dom}(K^\dagger) = \text{ran}(K) \oplus \text{ran}(K)^\perp : \quad K^\dagger y = \sum_{\lambda \in \Lambda} \frac{1}{\kappa_\lambda} \langle y, v_\lambda \rangle \bar{u}_\lambda .
$$

(1.4)

Here $(\bar{u}_\lambda)_\lambda$ is any dual frame of $(u_\lambda)_{\lambda \in \Lambda}$ and $\kappa_\lambda > 0$ are the generalized singular values. Equation (1.4) is a generalization of the SVD allowing frames as non-orthogonal generalized singular systems $(u_\lambda)_\lambda$ and $(v_\lambda)_\lambda$. Moreover, both systems are in general overcomplete, which is another main reason for using frames. Opposed to the SVD, many different DFDs for a given operator can exist and the quasi-singular systems can be adapted to a particular signal class.

In the case of ill-posed problems where $K^\dagger$ is unbounded, regularization techniques have to be applied in order to approximately but stably solve (1.1). Based on a DFD of the forward operator, in this paper, we consider filtered DFDs defined as

$$
F_\alpha y := \sum_{\lambda \in \Lambda} f_\alpha(\kappa_\lambda) \langle y, v_\lambda \rangle \bar{u}_\lambda .
$$

Here $(f_\alpha)_{\alpha > 0}$ is a family of functions converging pointwise to $1/\kappa$ as $\alpha \to 0$ (more precisely, a regularizing filter; see Definition 3.1). In case we take the SVD as the DFD then the filtered DFD reduces to classical filter based regularization. However, the filtered DFD contains other interesting special cases. In particular, taking $(u_\lambda)_{\lambda \in \Lambda}$ as wavelet, curvelet and shearlet system yields DFDs for image reconstruction [3, 5, 8, 12]. We also point out that such systems are often used in variational regularization schemes [6, 7, 9, 13, 14, 21, 23] which are related but different from the approach followed in this paper. In the context of variational regularization, regularized solutions are constructed as minimizers of a generalized Tikhonov functional formed by adding a frame-dependent regularizer to the operator-dependent data fitting term.

1.2 Outline

In Section 2 we introduce and study the concept of a DFD and relate the ill-posedness of the inverse problem (1.1) to the decay of the quasi-singular values. In Section 3 we introduce filtered DFDs to account for the ill-posedness of (1.1). We show that filtered DFDs yield regularization methods and we derive convergence rates under source-type conditions on the unknowns to be recovered. In Section 4 we present and implement filtered DFDs for stable Radon transform inversion as practically relevant example from medical image reconstruction. The paper concludes with a short discussion and outlook given in Section 5.
2 Operator inversion by diagonal frame decomposition

Throughout this paper $X$ and $Y$ denote Hilbert spaces over $K \in \{\mathbb{R}, \mathbb{C}\}$ and $K: \text{dom}(K) \subseteq X \to Y$ a closed, densely defined linear operator. Note that we do not assume the operator $K$ to be bounded. For example, this allows to include the Radon transform on $L^2(\mathbb{R}^2)$ in our setting; see Section 4. In this section, we introduce diagonal frame decompositions (DFDs) which in the following sections will be used to regularize the inverse problem defined by the forward operator $K$.

2.1 Frames

We start by briefly recalling some basic facts about frame theory [4, 22, 1]. A family $u = (u_\lambda)_{\lambda \in \Lambda} \in \mathbb{U}^\Lambda$ where $\Lambda$ is an at most countable index set is called frame for the Hilbert space $\mathbb{U}$ if there are constants $A, B > 0$ such that
\[
\forall x \in \mathbb{U}: \quad A\|x\|^2 \leq \sum_{\lambda \in \Lambda} |\langle x, u_\lambda \rangle|^2 \leq B\|x\|^2.
\] (2.1)

The constants $A$ and $B$ are called lower and upper frame bounds of $u$, respectively. The frame is called tight if $A = B$ and exact if it fails to be a frame whenever any single element is deleted from the sequence $(u_\lambda)_{\lambda \in \Lambda}$. A frame that is not a Riesz basis is said to be overcomplete.

Definition 2.1 (Analysis and synthesis operator). Let $u = (u_\lambda)_{\lambda \in \Lambda}$ be a frame for the Hilbert space $\mathbb{U}$. The analysis and synthesis operator of $u$, respectively, are defined by
\[
T_u: \mathbb{U} \to \ell^2(\Lambda): x \mapsto (\langle x, u_\lambda \rangle)_{\lambda \in \Lambda}
\] (2.2)
\[
T_u^*: \ell^2(\Lambda) \to \mathbb{U}: (c_\lambda)_{\lambda \in \Lambda} \mapsto \sum_{\lambda \in \Lambda} c_\lambda u_\lambda.
\] (2.3)

One easily verifies that $T_u$ and $T_u^*$ are linear bounded operators and the synthesis operator $T_u^*$ is the adjoint of the analysis operator $T_u$.

Definition 2.2 (Dual frame). Let $u = (u_\lambda)_{\lambda \in \Lambda}$ be a frame for the Hilbert space $\mathbb{U}$. A frame $\bar{u} = (\bar{u}_\lambda)_{\lambda \in \Lambda}$ for $\mathbb{U}$ is called a dual frame of $u$ if the following duality condition holds:
\[
\forall x \in \mathbb{U}: \quad x = \sum_{\lambda \in \Lambda} \langle x, u_\lambda \rangle \bar{u}_\lambda = T_u^* T_u x.
\] (2.4)

Every frame has at least one dual frame and if the frame $u$ is over-complete, then there exist infinitely many dual frames of $u$.

Definition 2.3 (Norm bounded frames). Let $\mathbb{U}$ be a Hilbert space and $u$ a frame for $\mathbb{U}$. We call $u$ norm bounded from below if there exists a constant $a > 0$ such that $\inf_{\lambda \in \Lambda} \|u_\lambda\| \geq a$. 


Note that every frame is already norm bounded from above. In fact, the upper frame condition implies
\[ \|u_\lambda\|^4 = |\langle u_\lambda, u_\lambda \rangle|^2 \leq \sum_{\mu \in \Lambda} |\langle u_\lambda, u_\mu \rangle|^2 \leq B \|u_\lambda\|^2 \] which gives \[ \sup_{\lambda \in \Lambda} \|u_\lambda\| \leq \sqrt{B}. \] On the other hand one easily constructs examples of frames that are not norm bounded from below.

### 2.2 Diagonal frame decomposition

We use the following notion extending the wavelet-vaguelette decomposition (WVD) and biorthogonal curvelet decomposition to more general frames. It will allow us to unify and extend existing filter based regularization methods to the frame case.

**Definition 2.4** (Diagonal frame decomposition, DFD). Let \( K : \text{dom}(K) \subseteq X \to Y \) be a closed and densely defined linear operator, and \( \Lambda \) an at most countable index set. We call \((u, v, \kappa) = (u_\lambda, v_\lambda, \kappa_\lambda)_{\lambda \in \Lambda}\) a diagonal frame decomposition (DFD) for the operator \( K \) if the following holds:

(D1) \((u_\lambda)_{\lambda \in \Lambda}\) is a frame for \((\ker K)^\perp \subseteq X\),

(D2) \((v_\lambda)_{\lambda \in \Lambda}\) is a frame for \(\overline{\text{ran} K} \subseteq Y\),

(D3) \((\kappa_\lambda)_{\lambda \in \Lambda} \in (0, \infty)^\Lambda\) satisfies the quasi-singular relations

\[ \forall \lambda \in \Lambda : \; K^* v_\lambda = \kappa_\lambda u_\lambda. \quad (2.5) \]

We call \((\kappa_\lambda)_{\lambda \in \Lambda}\) the quasi-singular values and \((u_\lambda)_{\lambda \in \Lambda}, (v_\lambda)_{\lambda \in \Lambda}\) the corresponding quasi-singular systems.

In the case \( u \) is an orthonormal wavelet basis, then the DFD reduces to the WVD introduced in [8]. A WVD decomposition has been constructed for the classical computed tomography modeled by the two-dimensional Radon transform see [8]. In the case of the two-dimensional Radon transform, a biorthogonal curvelet decomposition was constructed in [3]. In [3], the authors derived biorthogonal shearlet decompositions for two- and three-dimensional Radon transforms. The limited data case has been studied in [11].

Note that the quasi-singular relations in (2.5) imply that \( v_\lambda \in \text{dom}(K^*) \) and \( u_\lambda \in \text{ran}(K^*) \) which in the unbounded case are abstract smoothness requirements. Interestingly, opposed to the SVD case, a DFD does not require \( v_\lambda \in \text{ran}(K) \) in general.

**Remark 2.5** (DFDs in the ONB case). Consider the special case where \( u \) is an orthonormal basis (ONB) and let \((\tilde{v}_\lambda)_{\lambda}\) be a dual frame of \( v \). The quasi-singular relations in this case imply \( Ku_\lambda = \kappa_\lambda \tilde{v}_\lambda \) and thus \( \tilde{v}_\lambda \in \text{ran}(K) \) and \( u_\lambda \in \text{dom}(K) \) for all \( \lambda \in \Lambda \). Further, one can also check that the frames \( v \) and \( \tilde{v} \) are biorthogonal, \( \langle v_\lambda, \tilde{v}_\mu \rangle = \delta_{\lambda\mu}. \) This in turn implies that \( v \) is a Riesz basis (see [2]) and that \( \tilde{v} \) is the dual Riesz basis uniquely determined by \( v \).
Remark 2.6 (Multiplication operators on \(l^2\)). For any sequence \(a = (a_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda\) define the pointwise multiplication operator

\[ M_a : \text{dom}(M_a) \subseteq l^2(\Lambda) \rightarrow l^2(\Lambda) : (c_\lambda)_{\lambda \in \Lambda} \mapsto (a_\lambda c_\lambda)_{\lambda \in \Lambda} \]

with domain \(\text{dom}(M_a) := \{(c_\lambda)_{\lambda \in \Lambda} \in l^2(\Lambda) \mid (a_\lambda c_\lambda)_{\lambda \in \Lambda} \in l^2(\Lambda)\}\). Then \(M_a\) is closed and densely defined, and bounded if and only if \(a\) is bounded.

Remark 2.7 (DFD as frame-based factorization). Let \((u, v, \kappa)\) be a DFD for \(K\). Then (2.5) is equivalent to \((K^*u_\lambda, x) = \kappa_\lambda(u_\lambda, x)\) for all \(\lambda \in \Lambda\) and all \(x \in \text{dom}(K)\). Moreover, \(\text{ran}(T_u|_{\text{dom}(K)}) = \{(u_\lambda, x)_{\lambda \in \Lambda} \mid x \in \text{dom}(K)\} \subseteq \text{dom}(M_\kappa)\). Hence (2.5) is equivalent to \(T_uK = M_\kappa T_u|_{\text{dom}(K)}\).

Remark 2.8 (Moore-Penrose inverse). Recall that \(K\) is closed and densely defined but potentially unbounded. For such operators, the Moore-Penrose inverse \(K^\dagger\): \(\text{dom}(K^\dagger) \subseteq Y \rightarrow X\) with \(\text{dom}(K^\dagger) := \text{ran}(K) \oplus \text{ran}(K)^\perp\) is defined as in the case of bounded forward operators, and is closed with dense domain [10, Theorem 2.12]. For \(y \in \text{dom}(K^\dagger)\), \(K^\dagger y\) is uniquely characterized either as the unique solution of \(Kx = P_{\text{ran}(K)}y\) in \(\text{dom}(K) \cap \text{ker}(K)^\perp\) or the unique least-squares solution of \(Kx = y\) having minimal norm.

Theorem 2.9 (Moore-Penrose inverse via DFD). Let \((u, v, \kappa)\) be a DFD for \(K\) and \(\bar{u} = (\bar{u}_\lambda)_{\lambda \in \Lambda}\) be a dual frame of \(u\). Then

\[ \forall y \in \text{dom}(K^\dagger) : \quad K^\dagger y = \sum_{\lambda \in \Lambda} \frac{1}{\kappa_\lambda} \langle y, v_\lambda \rangle \bar{u}_\lambda. \]  

(2.6)

Equivalently, \(K^\dagger = T_{\bar{u}}^* M_1/\kappa T_v|_{\text{dom}(K^\dagger)}\) where \(1/\kappa\) denotes the pointwise inverse of \(\kappa\).

Proof. For any \(y \in \text{dom}(K^\dagger) = \text{ran}(K) \oplus \text{ran}(K)^\perp\) define \(B y := \sum_{\lambda \in \Lambda} \frac{1}{\kappa_\lambda} \langle y, v_\lambda \rangle \bar{u}_\lambda\). We will show that the mapping \(B : \text{dom}(K^\dagger) \subseteq Y \rightarrow X : y \mapsto By\) equals the Moore-Penrose inverse. For that purpose note that any element in \(\text{dom}(K^\dagger)\) has the unique representation \(y = Kx^\dagger + y^\perp\) where \(x^\dagger \in \text{ker}(K)^\perp \cap \text{dom}(K)\) and \(y^\perp \in \text{ran}(K)^\perp\). The identity \(\kappa_\lambda^{-1} \langle y, v_\lambda \rangle = \langle x^\dagger, u_\lambda \rangle\) shows that \(B y\) is well defined as absolutely convergent sum. Further,

\[ B y = \sum_{\lambda \in \Lambda} \frac{1}{\kappa_\lambda} \langle y, v_\lambda \rangle \bar{u}_\lambda = \sum_{\lambda \in \Lambda} \frac{1}{\kappa_\lambda} \langle Kx^\dagger, v_\lambda \rangle \bar{u}_\lambda = \sum_{\lambda \in \Lambda} \frac{1}{\kappa_\lambda} \langle x^\dagger, K^*v_\lambda \rangle \bar{u}_\lambda = \sum_{\lambda \in \Lambda} \langle x^\dagger, u_\lambda \rangle \bar{u}_\lambda = x^\dagger = K^\dagger y. \]  

(2.7)

Here we used the definition of \(B\), the fact that \(v_\lambda \in \overline{\text{ran}(K)}\), the quasi-singular relation [2.5], and the fact that \(\bar{u}\) is a dual frame of \(u\) for \((\text{ker}(K))^\perp\).

2.3 Ill-posedness and quasi singular values

Typical inverse problems are unstable in the sense that the Moore-Penrose inverse is unbounded. It is well known that the Moore-Penrose inverse of an operator having a SVD is bounded if and only if the singular values do not accumulate at zero. Below we show that a similar characterization holds for the quasi-singular values in a DFD.
Theorem 2.10 (Characterization of ill-posedness via DFD). Let \((u, v, \kappa)\) be a DFD of \(K\). Then the following assertions hold.

(a) \(\inf_{\lambda \in \Lambda} \kappa_\lambda > 0 \Rightarrow K^\dagger\) is bounded.

(b) \(v\) norm bounded from below \& \(K^\dagger\) bounded \(\Rightarrow \inf_{\lambda \in \Lambda} \kappa_\lambda > 0\).

Proof. (a) Let \(\bar{u}\) be a dual frame of \(u\). Then, for every \(y \in \text{dom}(K^\dagger)\) we have

\[
\|K^\dagger y\|^2 = \left\| \sum_{\lambda \in \Lambda} \kappa_\lambda^{-1} \langle y, v_\lambda \rangle \bar{u}_\lambda \right\|^2 \leq \|T_{\bar{u}}^\dagger\|^2 \sum_{\lambda \in \Lambda} |\kappa_\lambda^{-1} \langle y, v_\lambda \rangle|^2 \\
\leq \frac{\|T_{\bar{u}}^\dagger\|^2}{(\inf_{\lambda \in \Lambda} \kappa_\lambda)^2} \sum_{\lambda \in \Lambda} \|\langle y, v_\lambda \rangle\|^2 \leq \frac{\|T_{\bar{u}}^\dagger\|^2 \|T_v\|^2}{(\inf_{\lambda \in \Lambda} \kappa_\lambda)^2} \|y\|^2,
\]

which implies \(K^\dagger\) is bounded.

(b) Let \(K^\dagger\) be bounded with norm \(\|K^\dagger\|\) and suppose \(\inf_{\lambda \in \Lambda} \kappa_\lambda = 0\). Then the family \((\kappa_\lambda^{-1} v_\lambda)_{\lambda \in \Lambda}\) has no upper frame bound. This can be shown by contradiction: Suppose it has an upper frame bound \(B\) we know that \(\sup_{\lambda \in \Lambda} \|\kappa_\lambda^{-1} v_\lambda\| \leq \sqrt{B}\), but since \(v\) is norm bounded from below we have \(\sup_{\lambda \in \Lambda} \|\kappa_\lambda^{-1} v_\lambda\| = \infty\). Hence we have that for all constants \(B > 0\) there exists \(y \in \text{ran}K\) such that

\[
\sum_{\lambda \in \Lambda} |\langle y, \kappa_\lambda^{-1} v_\lambda \rangle|^2 > B \|y\|^2. \tag{2.8}
\]

Now choose \(B = \|T_{\bar{u}}^\dagger\|^2 \|K^\dagger\|^2\), where \(\bar{u}\) is an arbitrary dual frame of \(u\), and let \(y\) be such that (2.8) is satisfied. It is well known that if \(K^\dagger\) is bounded, \(K\) has closed range \([16]\). Thereby, \(y \in \text{dom}(K^\dagger)\). Moreover, it has the unique representation \(y = Kx^\dagger\) with \(x^\dagger \in \ker(K)^\perp \cap \text{dom}(K)\) and by \(\langle Kx^\dagger, \kappa_\lambda^{-1} v_\lambda \rangle = \langle x^\dagger, v_\lambda \rangle\) follows that \((\langle y, \kappa_\lambda^{-1} v_\lambda \rangle)_{\lambda \in \Lambda} \in l^2(\Lambda)\). Then we have

\[
\|K^\dagger y\|^2 = \left\| \sum_{\lambda \in \Lambda} \kappa_\lambda^{-1} \langle y, v_\lambda \rangle \bar{u}_\lambda \right\|^2 \geq \frac{1}{\|T_{\bar{u}}^\dagger\|^2} \sum_{\lambda \in \Lambda} |\langle y, \kappa_\lambda^{-1} v_\lambda \rangle|^2 \\
> \frac{1}{\|T_{\bar{u}}^\dagger\|^2} B \|y\|^2 = \|K^\dagger\|^2 \|y\|^2,
\]

which leads to a contradiction. \(\square\)

Compact operators with infinite dimensional range are typical examples of linear operators with non-closed range. Moreover, the spectral theorem for compact operators states that zero is the only accumulation point of the singular values \((\sigma_\lambda)_{\lambda \in \Lambda}\). This means that we can find a bijection \(\pi: \mathbb{N} \to \Lambda\) such that \((\kappa_{\pi(n)})_{n \in \mathbb{N}}\) is a decreasing null-sequence. Below we show that the same holds for a DFD if \(u\) is norm bounded from below.

Theorem 2.11 (Quasi-singular values for compact operators). Suppose that \(K: \mathbb{X} \to \mathbb{Y}\) is a compact linear operator and assume that \((u, v, \kappa)\) is a DFD for \(K\), where \(u\) is norm bounded from below. Then, zero is the only accumulation point of \(\kappa\).
Proof. Without loss of generality consider the case \( \Lambda = \mathbb{N} \). Aiming for a contradiction, we assume that \( \kappa \) has an accumulation point different from zero (\( \infty \) is allowed). Therefore we can find a subsequence \( (\kappa_n(k))_{k \in \mathbb{N}} \) with \( \inf_{k \in \mathbb{N}} \kappa_n(k) := c > 0 \). Consequently \( \|v_n(k)/\kappa_n(k)\| \leq c^{-1}\sqrt{B_\varphi} \), where \( B_\varphi \) is the upper frame bound of \( v \). In particular, the sequence \( (v_n(k)/\kappa_n(k))_{k \in \mathbb{N}} \) is bounded. Because \( K^* \) is compact, there exists another subsequence \( (v_n(k(k))/\kappa_n(k(k)))_{k(k) \in \mathbb{N}} \) such that \( u_{n(k(k))} = K^*(v_{n(k(k))/\kappa_n(k(k)))} \) strongly converges to some \( x \in \text{ran}(K^*) \subseteq \text{ker}(K)^\perp \). Because \( u \) is norm bounded from below we have \( x \neq 0 \). Choose \( \epsilon > 0 \) such that \( \|x\|^2 \geq 2\epsilon \). Since \( u_{n(k(k))} \rightarrow x \) we can choose \( N \in \mathbb{N} \) such that for all \( \ell \geq N \): \( \|u_n(k(k)) - x\|^2 < \epsilon \). From this it follows \( 2\Re(\langle u_{n(k(k))}, x \rangle) > \|u_{n(k(k))}\|^2 + \|x\|^2 - \epsilon > \epsilon \). Consequently,

\[
\sum_{n \in \mathbb{N}} |\langle x, u_n \rangle|^2 \geq \sum_{\ell = N}^{\infty} |\langle x, u_{n(k(k))} \rangle|^2 \geq \sum_{\ell = N}^{\infty} \frac{\epsilon^2}{4} = \infty.
\]

This contradicts the frame condition of \( u \).

If \( u \) is not norm bounded from below, \( (\kappa_\lambda)_{\lambda \in \Lambda} \) can have one or more accumulation points as the following elementary example shows. Note that this example is not intended as representative forward operator we are interested in, but rather indicates to be careful when the frames are not bounded from below.

Example 2.12. Let \( \mathbb{K} = \mathbb{Y} = \ell^2(\mathbb{N}) \) and consider the diagonal multiplication operator \( K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) : (x_i)_{i \in \mathbb{N}} \mapsto (x_i/\sqrt{i + 1})_{i \in \mathbb{N}} \). Clearly \( K \) is self-adjoint and compact with SVD given by \( (\varepsilon_i)_{i \in \mathbb{N}}, \varepsilon_i)_{i \in \mathbb{N}}, (1/\sqrt{i + 1})_{i \in \mathbb{N}} \) where \( (\varepsilon_i)_{i \in \mathbb{N}} \) denotes the standard basis of \( \ell^2(\mathbb{N}) \). Define

\[
\begin{align*}
\mathbf{u} & := \begin{pmatrix} e_0, e_0 \mid e_1, e_1 \mid e_2, e_2 \mid e_3, e_3 \mid \ldots \end{pmatrix} \\
\mathbf{v} & := \begin{pmatrix} e_0, e_0 \mid e_1, e_1 \mid e_2, e_2 \mid e_3, e_3 \mid \ldots \end{pmatrix} \\
\kappa & := \begin{pmatrix} 1, 1 \mid 1/\sqrt{2}, 1/\sqrt{2} \mid 1/\sqrt{3}, 1/\sqrt{3} \mid \ldots \end{pmatrix}.
\end{align*}
\]

For \( x \in \text{ker}(K)^\perp = \mathbb{K} \) and \( y \in \text{ran}(K) = \mathbb{Y} \) we have

\[
\sum_{\lambda \in \Lambda} |\langle x, \lambda \rangle|^2 = \sum_{n \in \mathbb{N}} (n + 2)|\langle x, e_n/\sqrt{n + 1} \rangle|^2
\]

\[
= \sum_{n \in \mathbb{N}} |\langle x, e_n \rangle|^2 + \sum_{n \in \mathbb{N}} \frac{1}{n + 1}|\langle x, e_n \rangle|^2
\]

\[
= \sum_{n \in \mathbb{N}} |\langle y, v_\lambda \rangle|^2 = \sum_{n \in \mathbb{N}} |\langle y, e_n \rangle|^2 + \sum_{n \in \mathbb{N}} (n + 1)|\langle y, e_n/\sqrt{n + 1} \rangle|^2
\]

\[
= \|y\|^2 + \|y\|^2 = 2\|y\|^2.
\]

Hence \( \mathbf{u} \) is a frame with frame bounds \( A = 1 \) and \( B = 2 \) and \( \mathbf{v} \) is a frame with bounds \( A = B = 2 \). Moreover, the quasi-singular value relation \( K^*v_\lambda = \kappa_\lambda u_\lambda \) holds. Therefore \( (\mathbf{u}, \mathbf{v}, \kappa) \) is a DFD for the compact operator \( K \). However, the sequence \( \kappa \) has accumulation points 0 and 1.
Note that we can easily modify example 2.12 such that $\infty$ is an accumulation point of $\kappa$. To see this consider $u$ and $v$ from the example above and change $u$ and $\kappa$ to

\[ u = \left( e_0, e_0 \mid \frac{e_1}{2}, \frac{e_1}{2}, \frac{e_2}{3}, \frac{e_2}{3}, \frac{e_2}{3}, \frac{e_2}{3} | \ldots \right) \]

\[ \kappa = \left( 1, 1 \mid \sqrt{2}, \frac{1}{\sqrt{2}}, \sqrt{3}, \frac{1}{\sqrt{3}}, \sqrt{3}, \frac{1}{\sqrt{3}} | \ldots \right). \]

Then $(u, v, \kappa)$ is still a valid DFD of $K$ where $u$ has frame bounds $A = 1$ and $B = 2$, and $\kappa$ has accumulation points 0 and $\infty$.

### 3 Regularization by filtered DFD

Throughout this section, let $(u, v, \kappa)$ be a DFD of the operator $K: \text{dom}(K) \subseteq X \to Y$ and $\bar{u}$ a dual frame of $u$. Recall that we allow the forward operator $K$ to be unbounded. For typical inverse problems, the Moore Penrose inverse $K^\dagger$ is unbounded and has to be regularized. In this section we develop a regularization concept by filtered DFDs.

#### 3.1 Filtered DFD

A wide class of classical regularization methods can be constructed by spectral filtering. Below we extend these concepts to regularization by filtering a DFD. We start by defining regularizing filters using properties similar to [10, Theorem 4.2]. Be aware that our filter functions $f_\alpha$ correspond to $\kappa g_\alpha(\kappa^2)$ where $g_\alpha$ are the filters commonly used in spectral filtering. In spectral filtering, the operators $g_\alpha(K^*K)K^*(y) = \sum_{\lambda \in \Lambda} \kappa_\lambda g_\alpha(\kappa_\lambda^2) \langle y, u_\lambda \rangle \bar{u}_\lambda$ are derived from functional calculus. Further note that our assumptions (F1)-(F3) with $f_\alpha(\kappa) = \kappa g_\alpha(\kappa^2)$ are weaker than the ones [10, Theorem 4.2] where $g_\alpha$ is assumed to be bounded.

**Definition 3.1** (Regularizing filter). We call a family $(f_\alpha)_{\alpha > 0}$ of piecewise continuous functions $f_\alpha: (0, \infty) \to \mathbb{R}$ a regularizing filter if,

- (F1) $\forall \alpha > 0: \|f_\alpha\|_\infty < \infty$.
- (F2) $\exists C > 0: \sup\{\|\kappa f_\alpha(\kappa)\| \mid \alpha > 0 \wedge \kappa \geq 0\} \leq C$.
- (F3) $\forall \kappa \in (0, \infty): \lim_{\alpha \to 0} f_\alpha(\kappa) = 1/\kappa$.

Using a regularizing filter we define the following central concept of this paper.

**Definition 3.2** (Filtered DFD). Let $(f_\alpha)_{\alpha > 0}$ be a regularizing filter and define

\[ \forall \alpha > 0: \quad F_\alpha: Y \to X: y \mapsto \sum_{\lambda \in \Lambda} f_\alpha(\kappa_\lambda) \langle y, v_\lambda \rangle \bar{u}_\lambda. \quad (3.1) \]

We call the family $(F_\alpha)_{\alpha > 0}$ the filtered diagonal frame decomposition (filtered DFD) according to $(f_\alpha)_{\alpha > 0}$ based on the DFD $(u, v, \kappa)$ and the dual frame $\bar{u}$.

As mentioned above, our filter functions $f_\alpha$ correspond to $\kappa g_\alpha(\kappa^2)$ where $g_\alpha$ are the filters commonly used in spectral filtering. In spectral filtering, the operators $g_\alpha(K^*K)K^*(y) = \sum_{\lambda \in \Lambda} \kappa_\lambda g_\alpha(\kappa_\lambda^2) \langle y, u_\lambda \rangle \bar{u}_\lambda$ are derived from functional calculus. Further note that our assumptions (F1)-(F3) with $f_\alpha(\kappa) = \kappa g_\alpha(\kappa^2)$ are weaker than the ones [10, Theorem 4.2] where $g_\alpha$ is assumed to be bounded.
3.2 Convergent regularization methods

Below we show that filtered DFD yields a well defined convergent regularization method. To that end, we recall the definition of a regularization method taken from [10, Definition 3.1] for case of bounded \( K \) and adopted to the unbounded case considered here. For regularization with unbounded forward operators see, for example, [18, 16].

**Definition 3.3** (Regularization method). Let \( (R_\alpha)_{\alpha>0} \) be a family of continuous operators \( R_\alpha: \mathbb{Y} \to \mathbb{X}, \ y \in \text{dom}(K^\dagger) \) and \( \alpha^*: (0, \infty) \times \mathbb{Y} \to (0, \infty) \). Then the pair \( ((R_\alpha)_{\alpha>0}, \alpha^*) \) is a regularization method for the solution of \( Kx = y \), if

\[
\lim_{\delta \to 0} \sup \{ \alpha^*(\delta, y^\delta) \mid y^\delta \in \mathbb{Y} \land \| y^\delta - y \| \leq \delta \} = 0 \\
\lim_{\delta \to 0} \sup \{ \| K^\dagger y - R_{\alpha^*(\delta, y^\delta)} y^\delta \| \mid y^\delta \in \mathbb{Y} \land \| y^\delta - y \| \leq \delta \} = 0.
\]

In this case we call \( \alpha^* \) an admissible parameter choice. If for any \( y \in \text{dom}(K^\dagger) \) there exists an admissible parameter choice, then we call \( (R_\alpha)_{\alpha>0} \) a regularization of the Moore Penrose inverse \( K^\dagger \).

Given an SVD \((u_n, v_n, \sigma_n)_{n \in \mathbb{N}}\) of \( K \) and a regularizing filter \((f_\alpha)_{\alpha>0}\), it is well known (at least if \( \kappa \mapsto \kappa^{-1} f_\alpha(\kappa) \) is bounded) that the family

\[
\sum_{n \in \mathbb{N}} g_\alpha(\sigma_n^2) (K^\ast y, u_n) u_n = \sum_{n \in \mathbb{N}} f_\alpha(\sigma_n) (y, v_n) u_n = F_\alpha(y)
\]

with \( f_\alpha(\sigma_n) = \sigma_n g_\alpha(\sigma_n^2) \) defines a regularization method [10, Theorem 8] together with convergence rates. Two prominent examples of filter-based regularization methods are classical Tikhonov regularization and truncated SVD. In truncated SVD, the regularizing filter is given by \( f_\alpha(\sigma) = \sigma^{-1} \chi_{[\alpha, \infty)}(\sigma^2) \). In Tikhonov regularization, the regularizing filter is given by \( f_\alpha(\sigma) = \sigma/(\sigma^2 + \alpha) \). In this paper we generalize such results by allowing a DFD instead of the SVD. To that end we use the following well known result.

**Lemma 3.4** (Characterization of linear regularizations). Let \( (R_\alpha)_{\alpha>0} \) be a family of linear bounded operators which pointwise converge to \( K^\dagger \) on \( \text{dom}(K^\dagger) \) and let \( y \in \text{dom}(K^\dagger) \). If the parameter choice \( \alpha^*: (0, \infty) \to (0, \infty) \) satisfies \( \lim_{\delta \to 0} \alpha^*(\delta) = \lim_{\delta \to 0} \delta \| R_{\alpha^*}(\delta) \| = 0 \), then the pair \( ((R_\alpha)_{\alpha>0}, \alpha^*) \) is a regularization method for \( Kx = y \).

**Proof.** For the case of bounded forward operators see, for example, [10, Proposition 3.7]. The simple proof is based on the estimate \( \| K^\dagger y - R_\alpha y^\delta \| \leq \| K^\dagger y - R_\alpha y \| + \delta \| R_\alpha \| \) and applies to case of unbounded \( K \).

\[\square\]

3.3 Well-posedness and convergence

Let \((f_\alpha)_{\alpha>0}\) be a regularizing filter and \((F_\alpha)_{\alpha>0}\) be the filtered DFD defined by [3.1].

**Proposition 3.5** (Existence and stability). For any \( \alpha > 0 \) the operator \( F_\alpha \) is well defined, linear and bounded. Moreover, \( \| F_\alpha \| \leq \| f_\alpha \|_\infty (B_\alpha B_v)^{1/2} \), where \( B_\alpha \) and \( B_v \) are the upper frame bounds of \( \bar{u} \) and \( v \), respectively.
Proof. Let $\alpha > 0$, $y \in \mathbb{Y}$. According to (F2), $f_\alpha$ is bounded and therefore $(f_\alpha(\kappa\lambda)\langle y, v_\lambda \rangle)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$. Further, $\|F_\alpha y\|^2 = \|\sum_{\lambda \in \Lambda} f_\alpha(\kappa\lambda)\langle y, v_\lambda \rangle u_\lambda\|^2 \leq \|f_\alpha\|_\infty^2 B_\alpha B_v \|y\|^2$ which shows that $F_\alpha y$ is well defined and bounded with $\|F_\alpha\| \leq \|f_\alpha\|_\infty(B_\alpha B_v)^{1/2}$. □

Proposition 3.6 (Pointwise convergence). For all $y \in \text{dom}(K^\dagger)$: $\lim_{\alpha \to 0} F_\alpha y = K^\dagger y$.

Proof. Let $y = y^\dagger + y^\perp \in \text{ran}(K) \oplus \text{ran}(K)^\perp$ and set $x^\dagger := K^\dagger y \in \ker(K)^\perp \cap \text{dom}(K)$. Then $Kx^\dagger = P_{\text{ran}(K)} y = y^\dagger$ and therefore

$$
\|x^\dagger - F_\alpha y\|^2 = \left\|x^\dagger - \sum_{\lambda \in \Lambda} f_\alpha(\kappa\lambda)\langle y, v_\lambda \rangle \bar{u}_\lambda\right\|^2 \\
= \left\|x^\dagger - \sum_{\lambda \in \Lambda} f_\alpha(\kappa\lambda)\langle y^\dagger, v_\lambda \rangle \bar{u}_\lambda\right\|^2 \\
= \left\|x^\dagger - \sum_{\lambda \in \Lambda} f_\alpha(\kappa\lambda)\langle Kx^\dagger, v_\lambda \rangle \bar{u}_\lambda\right\|^2 \\
= \left\|\sum_{\lambda \in \Lambda} \langle x^\dagger, u_\lambda \rangle \bar{u}_\lambda - \sum_{\lambda \in \Lambda} \kappa_\lambda f_\alpha(\kappa\lambda)\langle x^\dagger, u_\lambda \rangle \bar{u}_\lambda\right\|^2 \\
\leq B_\alpha \sum_{\lambda \in \Lambda} |1 - \kappa_\lambda f_\alpha(\kappa\lambda)|^2 |x^\dagger, u_\lambda|^2 \\
\leq \sup_{\lambda \in \Lambda} |1 - \kappa_\lambda f_\alpha(\kappa\lambda)|^2 B_\alpha B_\delta \|x^\dagger\|^2.
$$

According to (F2), (F3) we have $\sup_{\alpha, \lambda}|1 - \kappa_\lambda f_\alpha(\kappa\lambda)|^2 < \infty$ and $\lim_{\alpha \to 0} |1 - \kappa_\lambda f_\alpha(\kappa\lambda)| = 0$ pointwise. Therefore, application of the dominated convergence theorem to the series in the second last line yields $\|x^\dagger - F_\alpha y\|^2 \to 0$ for $\alpha \to 0$. □

By collecting the above results we obtain the following convergence theorem for filtered DFD.

Theorem 3.7 (Convergence). Let $(f_\alpha)_{\alpha > 0}$ be a regularizing filter, $(u, v, \kappa)$ be a DFD of $K$: $\text{dom}(K) \subseteq X \to Y$ and $\bar{u}$ a dual frame of $u$. Then $((F_\alpha)_{\alpha > 0}, \alpha^*)$ is a regularization method for $Kx = y$ provided that the parameter choice $\alpha^*: (0, \infty) \to (0, \infty)$ satisfies $0 = \lim_{\delta \to 0} \alpha^*(\delta) = \lim_{\delta \to 0} \delta \|f_{\alpha^*}\|_\infty$.

Proof. According to Propositions 3.5 and 3.6, $(F_\alpha)_{\alpha > 0}$ is a family of bounded linear operators that converges pointwise to $K^\dagger$ on $\text{dom}(K)$. According to Lemma 3.4 the pair $((F_\alpha)_{\alpha > 0}, \alpha^*)$ is a regularization method if $\alpha^*(\delta), \delta \|F_{\alpha^*}\| \to 0$ as $\delta \to 0$. The estimate $\|F_\alpha\| \leq \|f_\alpha\|_\infty \sqrt{B_\alpha B_\delta}$ of Proposition 3.5 finally yields the claim. □
### 3.4 Convergence rates

Next we derive convergence rates which give quantitative estimates on the reconstruction error \( \| x^\dag - x^\dag_\alpha \| \).

**Theorem 3.8 (Convergence rates).** Let \((f_\alpha)_{\alpha > 0}\) be a regularizing filter, \((u, v, \kappa)\) be a DFD of \(K\), \(\bar{u}\) a dual frame of \(u\) and \((F_\alpha)_{\alpha > 0}\) be the filtered DFD defined by (3.1). For given numbers \(\rho, \mu > 0\) and some constant \(C_\mu\) suppose

\[
(\text{R1}) \quad \|f_\alpha\|_\infty = O(\alpha^{-1/2}) \quad \text{as} \quad \alpha \to 0,
\]

\[
(\text{R2}) \quad \forall \alpha > 0: \quad \sup\{\kappa^2\mu \mid \kappa f_\alpha(\kappa) \mid \kappa \in (0, \infty)\} \leq C_\mu \alpha^\mu,
\]

\[
(\text{R3}) \quad \alpha = \alpha^*(\delta, y^\delta) \asymp (\delta/\rho)^{2/(2\mu+1)}.
\]

Suppose \(x^\dag \in X\) satisfies the following source-type condition

\[
\exists \omega \in L^2(\Lambda): \left(\|\omega\|_2 \leq \rho \land \forall \lambda \in \Lambda: \langle x^\dag, u_\lambda \rangle = \kappa^2\mu \omega_\lambda \right)
\]

(3.2)

Then, for some constant \(c = c_\mu\) and all \(y^\delta \in Y\) with \(\|y^\delta - Kx^\dag\| \leq \delta\) with sufficiently small \(\delta\), the following convergence rate result holds:

\[
\|x^\dag - F_\alpha(y^\delta)\| \leq c_\mu \delta^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}}.
\]

**Proof.** Let \(x^\dag, \omega, y^\delta\) satisfy \(\langle x^\dag, u_\lambda \rangle = \kappa^2\mu \omega_\lambda\), \(\|\omega\|_2 \leq \rho, \|y^\delta - Kx^\dag\| \leq \delta\). Then

\[
\|F_\alpha(y^\delta) - x^\dag\| \leq \|F_\alpha(y^\delta - Kx^\dag)\| + \|F_\alpha(Kx^\dag) - x^\dag\|
\]

\[
\leq \|F_\alpha\| \delta + \left( \sum_{\lambda \in \Lambda} (1 - \kappa f_\alpha(\kappa_\lambda)) \langle x^\dag, u_\lambda \rangle \bar{u}_\lambda \right)
\]

\[
\leq \sqrt{B_a B_v} \|f_\alpha\|_\infty \delta + \left( B_a \sum_{\lambda \in \Lambda} |1 - \kappa f_\alpha(\kappa_\lambda)|^2 |\langle x^\dag, u_\lambda \rangle|^2 \right)^{\frac{1}{2}}
\]

\[
\leq c_1 \alpha^{-1/2} \delta + \left( B_a \sum_{\lambda \in \Lambda} |1 - \kappa f_\alpha(\kappa_\lambda)|^2 |\kappa^2\mu \omega_\lambda|^2 \right)^{\frac{1}{2}}
\]

\[
\leq c_1 \alpha^{-1/2} \delta + \sqrt{B_a C_\mu \alpha^\mu \rho}.
\]

Now choose \(\alpha = \alpha^*(\delta, y^\delta) \asymp (\delta/\rho)^{2/(2\mu+1)}\). Then the above estimate implies

\[
\|F_\alpha^{\alpha^*(\delta, y^\delta)}(y^\delta) - x^\dag\| \leq c_2 \left( \delta^{1-\frac{1}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} + \delta^{\frac{2\mu}{2\mu+1}} \rho^{1-\frac{2\mu}{2\mu+1}} \right) = O \left( \delta^{2\mu/(2\mu+1)} \right),
\]

and completes the proof.

**Remark 3.9 (Qualification of a filter).** For a given regularizing filter, the condition \((\text{R2})\) may only hold for \(\mu \in (0, \mu_0]\) but not for \(\mu > \mu_0\). The index \(\mu_0\) is often called the qualification of the regularizing filter (see the discussion on [77, page 76]). If \((\text{R2})\) holds for all \(\mu > 0\), the qualification is said to be infinite. It is known that the qualification of \(f_\alpha(\kappa) = \kappa/(\kappa^2 + \alpha)\) is \(\mu_0 = 1\) and that \(f_\alpha(\kappa) = \kappa^{-1} \chi_{[\alpha, \infty)}(\kappa^2)\) has infinite qualification.
Remark 3.10 (Source conditions and generalizations). In Theorem 3.8 we derived convergence rates for elements satisfying the source-type condition (3.2) which can be written as $T_u x^\dagger \in \text{ran}(M_\kappa^{2\mu})$. This may be seen as an abstract smoothness condition for $x^\dagger$. As in the case of classical spectral filtering one could study source conditions of the form $T_u x^\dagger \in \text{ran}(\phi(M_\kappa))$ for more general index functions $\phi$. For example, logarithmic source conditions are useful for exponentially ill-posed problems; see [19]. Another generalization is the use of approximate source conditions based on distance functions [17]. Investigating such concepts in the context of DFDs seems very interesting but beyond the scope of the present article.

3.5 Examples of regularizing filters

In this subsection we study examples of filtered DFDs, namely truncated DFD and Tikhonov-filtered DFD. We verify that the corresponding filters satisfy the requirement for being regularizing and also that the convergence rate conditions in Theorem 3.8 are satisfied for all $\mu > 0$ in case of truncated DFD and for $\mu \leq 1$ in case of Tikhonov-filtered DFD.

**Truncated DFD:** For any $\alpha > 0$ consider the cut-off functions

$$f^{(1)}_\alpha : (0, \infty) \to \mathbb{R} : \kappa \mapsto \begin{cases} 1/\kappa & \text{if } \kappa^2 \geq \alpha \\ 0 & \text{if } \kappa^2 < \alpha. \end{cases}$$

Obviously conditions (F1)-(F3) in Definition 3.1 are satisfied with $C = 1$ which implies that $(f^{(1)}_\alpha)_{\alpha > 0}$ is a regularizing filter. Furthermore, $\sup\{\kappa^{2\mu}|1-\kappa f^{(1)}_\alpha(\kappa)| \mid \kappa > 0\} = \sup\{\kappa^{2\mu}|1-\kappa f^{(1)}_\alpha(\kappa)| \mid \kappa^2 < \alpha\} = \alpha^\mu$ for all $\alpha, \mu > 0$. Hence the convergence rates conditions (R1) of Theorem 3.8 are satisfied. The corresponding filtered DFD becomes

$$F^{(1)}_\alpha(y) := \sum_{\kappa^2 \geq \alpha} \frac{1}{\kappa^2} \langle y, v_\lambda \rangle \bar{u}_\lambda.$$  

In the special case where $(u, v, \kappa)$ is an SVD for $K$, this is well-known truncated SVD. In analogy, for general DFDs we name (3.3) truncated DFD.

The considerations above allow application of Propositions 3.5, 3.6 and Theorem 3.7 showing well-posedness, stability and convergence of (3.3). Moreover Theorem 3.8 can be applied for any $\mu > 0$. Thus for $x^\dagger$ with $T_u x^\dagger \in \text{ran}(M_\kappa^{2\mu})$, the parameter choice $\alpha \simeq \delta^{2/(2\mu+1)}$ yields the convergence rate $\|x^\dagger - F^{(1)}_\alpha\|y^\delta = \mathcal{O}(\delta^{2\mu/(2\mu+1)}).

**Tikhonov type DFD:** For any $\alpha > 0$ consider the Tikhonov filter

$$f^{(2)}_\alpha : (0, \infty) \to \mathbb{R} : \kappa \mapsto \frac{\kappa}{\kappa^2 + \alpha}.$$  

For all $\kappa, \alpha > 0$ we have $|\kappa f^{(2)}_\alpha(\kappa)| = |\kappa^2/(\kappa^2 + \alpha)| \leq 1$ and $\lim_{\alpha \to 0} f^{(2)}_\alpha(\kappa) = 1/\kappa$. Further, $f^{(2)}_\alpha$ is bounded, takes its maximum at $\kappa^2 = \alpha$ and $\|f^{(2)}_\alpha\|_\infty = \alpha^{-1/2}/2$. Hence conditions...
Moreover, for $\mu \in (0,1]$ the function $\kappa \mapsto \kappa^{2\mu}|1 - \kappa f^{(2)}(\kappa)| = \kappa^{2\mu}/(\kappa^2 + \alpha)$ has its supremum at $\kappa = \sqrt{\alpha \mu/(1 - \mu)}$. Thus $\sup_{\kappa} \kappa^{2\mu}|1 - \kappa f^{(2)}(\kappa)| = ((2\mu)^{\mu}(2 - 2\mu)^{1 - \mu})/2\alpha^{\mu}$. Hence the filter $(f^{(2)}_\alpha)_{\alpha > 0}$ also satisfies the convergence rates conditions of Theorem 3.8.

According to the above considerations, Propositions 3.5, 3.6 and Theorem 3.7 show well-posedness and convergence of the filtered DFD

$$F^{(2)}_\alpha(y) := \sum_{\lambda \in \Lambda} \frac{\kappa \lambda \alpha}{\kappa^2 + \alpha} \langle y, v_\lambda \rangle \bar{u}_\lambda. \quad (3.4)$$

Moreover, for $\mu \in (0,1]$, the parameter choice $\alpha \simeq \delta^{2/(2\mu + 1)}$ yields the convergence rate $\|x^\dagger - F\alpha y^\delta\| = O(\delta^{2/(2\mu + 1)})$. In the special case where $(u, v, \kappa)$ is a SVD then (3.4) reduces to Tikhonov regularization as in this case $F^{(2)}_\alpha y$ equals the minimizer of the Tikhonov functional $\|Kx - y\|^2 + \alpha \|x\|^2$. For general DFDs this relation does not hold true.

Notice that the Tikhonov filter $(f^{(2)}_\alpha)_{\alpha > 0}$ does not satisfy (R2) for $\mu > 1$, which means that the Tikhonov filter has qualification $\mu_0 = 1$; see Remark 3.9. This is one motivation for considering regularization methods with higher qualification that can also be implemented without knowledge of the SVD, such as iterated Tikhonov regularization. Anyway, in this work we allow more general DFDs which provides an alternative strategy to avoid numerically costly SVD computation.

### 3.6 Order optimality

In the following we prove that the convergence rates obtained in Theorem 3.8 are order optimal for the source set defined by (3.2) in the special case that the frame $u$ admits a biorthogonal sequence $\bar{u} = (u_\lambda)_{\lambda \in \Lambda}$ with $\forall \lambda, \nu \in \Lambda: \langle u_\lambda, \bar{u}_\nu \rangle = \delta_{\lambda\nu}$. The requirement that $u$ has a biorthogonal sequence is equivalent to $u$ being a Riesz-basis of $\ker(K)^\perp$. To do this, define

$$U_{\mu, \rho} := \left\{ x \in \text{dom}(K) \mid \langle x, u_\lambda \rangle = \kappa^{2\mu} w_\lambda \wedge \sum_{\lambda \in \Lambda} |w_\lambda|^2 = \rho^2 \right\} \quad (3.5)$$

and for any set $\mathcal{M} \subseteq \text{dom}(K)$ define $\epsilon(\mathcal{M}, \delta) := \sup\{ \|x\| \mid x \in \mathcal{M} \wedge \|Kx\| \leq \delta \}$.

We have that $\epsilon(\mathcal{M}, \delta)$ is a lower bound for the worst case reconstruction error

$$E(\mathcal{M}, \delta, R) := \sup\{ \|Ry - x\| \mid x \in \mathcal{M} \wedge y^\delta \in \mathcal{Y} \wedge \|Kx - y^\delta\| \leq \delta \}, \quad (3.6)$$

for an arbitrary mapping $R: \mathcal{Y} \to \mathcal{X}$ (in this context called reconstruction method) with $R(0) = 0$; see [10]. A family $(R^\delta)_{\delta > 0}$ of reconstruction methods is called order optimal on $\mathcal{M}$, if $E(\mathcal{M}, \delta, R^\delta) \leq c \epsilon(\mathcal{M}, \delta)$ for all sufficiently small $\delta$ and some constant $c > 0$.

To show that the convergence rate of Theorem 3.8 is order optimal therefore amounts to bound $\epsilon(U_{\mu, \rho}, \delta)$.

**Theorem 3.11.** Let $(u, v, \kappa)$ be a DFD of $K$ such that $u$ has a biorthogonal sequence $\bar{u}$ and $0$ is an accumulation point of $\kappa$. Then for the source sets $U_{\mu, \rho}$ defined by (3.5) and
some sequence \((\delta_n)_{n \in \mathbb{N}}\) converging to 0, we have
\[
\epsilon(U_{\mu, \rho}, \delta_n) \geq \sqrt{\frac{B_v}{A_u}} \delta_n^{\frac{2\mu}{2\mu + 1}} \rho^{\frac{1}{2\mu + 1}}.
\]

In particular, under the assumptions of Theorem 3.8, the family \((F_{\alpha^*(\delta, \cdot)})_{\delta > 0}\) is an order
optimal reconstruction method for the source set \(U_{\mu, \rho}\).

Proof. After extracting a subsequence we assume without loss of generality that \(\Lambda = \mathbb{N}\) and that \(\kappa\) converges to 0. For any \(\nu \in \mathbb{N}\) set \(x_{\nu} := \rho \kappa_\nu \bar{u}_\nu\) such that
\[
\langle x_{\nu}, u_\lambda \rangle = \kappa_\lambda^{2\mu} \rho \kappa_\lambda, \quad w_\lambda = \begin{cases} 
\rho, & \text{if } \lambda = \nu \\
0, & \text{else}.
\end{cases}
\]

By definition we have \(\|w\|_2 = \rho\) and \(x_{\nu} \in U_{\mu, \rho}\). If we consider the decreasing null-sequence of noise levels \(\delta_\nu = \rho \kappa_\nu^{2\mu + 1} / \sqrt{A_v}\) we get
\[
\|x_{\nu}\|^2 \geq \frac{1}{B_u} \sum_{\lambda \in \Lambda} |\langle u_\lambda, x_{\nu} \rangle|^2 = \frac{1}{B_u} \kappa_\nu^{4\mu} \rho^2 = A_v^{2\mu/(2\mu + 1)} \frac{1}{B_u} (\delta_\nu^{2\mu/(2\mu + 1)} \rho^{1/(2\mu + 1)})^2
\]
and
\[
\|K x_{\nu}\|^2 \leq \frac{1}{A_v} \sum_{\lambda \in \Lambda} |\langle v_\lambda, K x_{\nu} \rangle|^2 = \frac{1}{A_v} \sum_{\lambda \in \Lambda} \kappa_\lambda^2 |\langle u_\lambda, x_{\nu} \rangle|^2 = \frac{1}{A_v} \kappa_\nu^{2(2\mu + 1)} \rho^2 = \delta_\nu^2.
\]

Thus, \(\|K x_{\nu}\| \leq \delta_\nu\) and \(\epsilon(U_{\mu, \rho}, \delta_\nu) \geq \|x_{\nu}\| \geq \sqrt{B_u/A_v} \delta_\nu^{2\mu/(2\mu + 1)} \rho^{1/(2\mu + 1)}\).

Note that if \(\kappa\) does not accumulate at zero, then \(K^\dagger\) is bounded (see Theorem 2.10). In this case the inverse problem is well posed and
\[
\epsilon(U_{\mu, \rho}, \delta) = \sup\{\|x\| \mid x \in U_{\mu, \rho} \land \|K x\| \leq \delta\} = \sup\{\|K^\dagger y\| \mid y \in K(U_{\mu, \rho}) \land \|y\| \leq \delta\} = \|K^\dagger\|.
\]

This reflects that in the well-posed case, where \(K^\dagger\) is bounded the optimal convergence rate is \(O(\delta)\) independent of particular prior information. In the ill-posed case according to Theorem 3.11 this rate is not achievable.

4 Application to X-ray tomography

In this section we apply the concept of filtered DFDs to X-ray tomography as a prime example of an inverse problem in medical image reconstruction. In two spatial dimensions, X-ray tomography can be modeled by the 2D Radon transform. In this section we study filtered DFDs for the Radon transform on \(L^2(\mathbb{R}^2)\). Throughout this section, the Fourier transform of a function \(f \in L^1(\mathbb{R}^n)\) is defined by \(\mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i \langle \xi, x \rangle} \, dx\) and extended to functions in \(L^2(\mathbb{R}^n)\) by continuity. Its inverse transform is denoted by \(\mathcal{F}^{-1}\). For functions \(g\) defined on \(S^1 \times \mathbb{R}\) we write \(\mathcal{F}_2 g\) for the Fourier transform in the second argument.
4.1 The Radon transform on $L^2(\mathbb{R}^2)$

Wavelet frames are naturally defined on $L^2(\mathbb{R}^2)$. Therefore we will study the Radon transform as an operator on $L^2(\mathbb{R}^2)$, where it is known to be unbounded, closed, and densely defined. See [25] for further background. In this subsection we collect main ingredients for constructing DFDs and filtered DFDs for the Radon transform.

**Radon transform:** Let $L^2_0(\mathbb{R}^2) := \{ f \in L^2(\mathbb{R}^2) \mid \text{supp}(f) \text{ compact} \}$ denote the space of all square integrable functions on $\mathbb{R}^2$ that vanish outside a bounded domain. The 2D Radon transform $Rf$ of $f \in L^2_0(\mathbb{R}^2)$ is defined by

$$
\forall (\theta, s) \in \mathbb{S}^1 \times \mathbb{R}: \quad Rf(\theta, s) = \int_{\mathbb{R}} f(s\theta + t\theta^1) \, dt. \quad (4.1)
$$

The value $Rf(\theta, s)$ is the integral of $f$ over the affine line with normal vector $\theta \in \mathbb{S}^1$ and signed distance $s \in \mathbb{R}$. Given $f \in L^2_0(\mathbb{R}^2)$ these integrals are well defined for almost all $(\theta, s)$ and yield an element in $L^2(\mathbb{S}^1 \times \mathbb{R})$. The Radon transform can and is extended to a densely defined closed operator $R: \text{dom}(R) \subseteq L^2(\mathbb{R}^2) \to L^2(\mathbb{S}^1 \times \mathbb{R})$ with domain $\text{dom}(R) := \{ f \in L^2(\mathbb{R}^2) \mid \| \cdot \|^{-1/2} \mathcal{F} f \in L^2(\mathbb{S}^1 \times \mathbb{R}) \}$. Note that the form $\langle 4.1 \rangle$ of $Rf(\theta, s)$ as line integral does not hold for all $f \in \text{dom}(R)$.

**Adjoint Radon transform:** The adjoint $R^*: \text{dom}(R^*) \subseteq L^2(\mathbb{S}^1 \times \mathbb{R}) \to L^2(\mathbb{R}^2)$ of the Radon transform has domain $\text{dom}(R^*) := \{ g \in L^2(\mathbb{S}^1 \times \mathbb{R}) \mid \| \cdot \|^{-1/2} \mathcal{F}_2 g \in L^2(\mathbb{S}^1 \times \mathbb{R}) \}$, where $\sigma$ is the second argument of $\mathcal{F}_2 g$. One verifies that $\text{dom}(R^*)$ consists of all $g \in L^2(\mathbb{S}^1 \times \mathbb{R})$ such that $R^* g(x) := \int_{\mathbb{S}^1} g(\theta, \langle x, \theta \rangle) \, d\theta$ gives a square integrable function in which case $R^* g = R^2 g$. Operator $R^2$ is known as backprojection operator.

**Fourier slice theorem:** The Fourier slice theorem for $f \in \text{dom}(R)$ reads

$$
\mathcal{F}_2 Rf(\theta, \sigma) = \mathcal{F} f(\sigma\theta) \quad \text{for a.e.} \ (\theta, \sigma) \in \mathbb{S}^1 \times \mathbb{R}. \quad (4.2)
$$

The Fourier slice identity is commonly stated for functions $f \in L^1(\mathbb{R}^2) \supseteq L^2_0(\mathbb{R}^2)$ in which case $\mathcal{F} f$ is a continuous function and $\langle 4.2 \rangle$ holds point-wise as an easy consequence of the definition of the Radon and Fourier transforms. Let us verify that $\langle 4.2 \rangle$ indeed also holds on $\text{dom}(R)$. For that purpose note that $f \in \text{dom}(R)$ iff $|\mathcal{F} f|^2$ and $\| \cdot \|^{-1}|\mathcal{F} f|^2$ are integrable. The latter property together with a change of variable and Fubinis theorem shows

$$
\int_{\mathbb{R}^2} |\mathcal{F} f(\xi)|^2 \|\xi\|^{-1} \, d\xi = \int_{\mathbb{S}^1} \int_{\mathbb{R}} |\mathcal{F} f(\sigma\theta)|^2 \, d\sigma d\theta. \quad (4.2)
$$

Hence the right hand side in $\langle 4.2 \rangle$ is well defined as an element of $L^2(\mathbb{S}^1 \times \mathbb{R})$. The same holds true for the left hand side $\mathcal{F}_2 Rf$. In order that $\langle 4.2 \rangle$ holds true on $\text{dom}(R)$ one has to assure that $\mathcal{F}_2^{-1}[[(\theta, \sigma) \mapsto \mathcal{F} f(\sigma\theta)]]$ defines a closed operator on $\text{dom}(R)$ which is verified in straight forward manner.

**Normal operator:** The normal operator $R^* R$ for the Radon transform is again densely defined and closed with domain $\text{dom}(R^* R) = \{ f \in L^2(\mathbb{R}^2) \mid \| \cdot \|^{-1}|\mathcal{F} f| \in L^2(\mathbb{R}^2) \}$. The
Fourier slice identity (4.2) and Fubini’s theorem yield the isometry property

$$\forall f, g \in \text{dom}(R): \quad \int_{S^1} \int_{\mathbb{R}} Rf(\theta, s) Rg(\theta, s) \, ds \, d\theta = 2 \int_{\mathbb{R}^2} \frac{\mathcal{F}f(\xi)}{\|\xi\|} \mathcal{F}g(\xi) \, d\xi. \quad (4.3)$$

The left-hand side in (4.3) is the $L^2$-inner product $(Rf, Rg)$ which is equal to $(R^*Rf, g)$ provided that $Rf \in \text{dom}(R^*)$, or equivalently that $f \in \text{dom}(R^*R)$. Therefore (4.3) gives the Fourier representation $R^*Rf = 2\mathcal{F}^{-1}(\| \cdot \| \mathcal{F}f)$.

### 4.2 DFDs for the Radon transform

We now study DFDs $(u, v, \kappa)$ for the Radon transform on $L^2(\mathbb{R}^2)$. We first derive necessary properties for $v$ and $\kappa$ in the general case and subsequently derive the DFD for the case that $u$ is a wavelet ONB.

**Necessary conditions:** Let $(u, v, \kappa)$ be a DFD for $R$ and assume $v_\lambda \in \text{ran}(R)$. Then $v_\lambda = \kappa_\lambda R\sigma_\lambda$ for some $\sigma_\lambda \in \text{dom}(R)$. By (D3), $R^*v_\lambda = \kappa_\lambda u_\lambda$ which shows that $u_\lambda = R^*R\sigma_\lambda$ and $\sigma_\lambda \in \text{dom}(R^*R)$. Equation (4.3) implies $\sigma_\lambda = \mathcal{F}^{-1}(\| \cdot \| \mathcal{F}u_\lambda)/2$ and therefore

$$v_\lambda = \frac{\kappa_\lambda}{2} R\mathcal{F}^{-1}(\| \cdot \| \mathcal{F}u_\lambda) = \frac{\kappa_\lambda}{2} R\Omega u_\lambda, \quad (4.4)$$

where $\Omega u := \mathcal{F}^{-1}(\| \cdot \| \mathcal{F}u)$.

Next assume that the frame $u$ has a multiscale structure

$$\forall (j, k, \ell) \in \Lambda = \mathbb{Z} \times \mathbb{Z}^2 \times L: \quad u_{j,k,\ell}(x) = 2^j u_{0,0,\ell}(2^j x - k). \quad (4.5)$$

Using (4.4), (4.3) and the scaling and translation property of $\mathcal{F}$ show

$$\|v_{j,k,\ell}\|^2 = \frac{\kappa_{j,k,\ell}^2}{(2\pi)^2} \int_{S^1} \int_{\mathbb{R}} |R\Omega(u_{j,k,\ell})(\theta, \sigma)|^2 \, d\sigma \, d\theta$$

$$= \frac{\kappa_{j,k,\ell}^2}{(2\pi)^2} \int_{\mathbb{R}^2} \|\xi\| |\mathcal{F}u_{j,k,\ell}(\xi)|^2 \, d\xi$$

$$= \frac{2^j \kappa_{j,k,\ell}^2}{(2\pi)^2} \int_{\mathbb{R}^2} \|\xi\| |\mathcal{F}u_{0,0,\ell}(\xi)|^2 \, d\xi$$

$$= \frac{2^j \kappa_{j,k,\ell}^2}{\kappa_{0,0,\ell}^2} \|v_{0,0,\ell}\|^2.$$ 

Assuming the frame elements to be bounded away from zero (as it is, for example, in the case of Riesz bases) this implies that the quasi-singular values satisfy $\kappa_{j,k,\ell} \asymp 2^{-j/2}$.

The considerations above show how to construct a DFD starting with a frame $u$ of the form (4.5). That such a construction actually results in a DFD in the case of wavelet ONB has been first shown in the seminal work of Donoho [8] and is outlined below.

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Wavelet vaguelette decomposition: Now let $u = (u_{\lambda})_{\lambda \in \Lambda}$ be a 2D (tensor product) wavelet ONB for $L^2(\mathbb{R}^2)$ of the form (4.5) where $\lambda \in \Lambda = \mathbb{Z} \times \mathbb{Z}^2 \times \{1, 2, 3\}$ consists of a triples $(j, k, \ell)$, where $j \in \mathbb{Z}$ is the scale index, $k = (k_1, k_2) \in \mathbb{Z}^2$ is the shift index and $\ell \in \{1, 2, 3\}$ indicates the chosen mother wavelet (horizontal, vertical or diagonal).

Theorem 4.1 (Wavelet-vaguelette decomposition [8]). Let $u \in L^2(\mathbb{R}^2)^{\Lambda}$ with $\Lambda = \mathbb{Z} \times \mathbb{Z}^2 \times \{1, 2, 3\}$ be a 2D wavelet ONB of the form (4.5) such that $u_{0,0,\ell}$ has compact support and $\| \cdot \|_{L^2(\mathbb{R}^2)} \in L^2(\mathbb{R}^2)$ for $\ell = 1, 2, 3$. Define $v$ by (4.4) and set $\kappa_{\lambda} := 2^{-j/2}$. Then $(u, v, \kappa)$ is a diagonal frame decomposition of $\mathbb{R}$ with a Riesz basis $v$ of $\overline{\text{ran} \mathcal{R}}$.

Proof. Following the construction of the previous paragraph, the quasi-singular value relations (D3) are satisfied. It remains to verify that $v$ forms a frame of $\overline{\text{ran} \mathcal{R}}$. For the proof we refer to the original work of Donoho [8]. He used wavelet-like functions, so-called vaguelettes, which were first introduced by Meyer, for his proof. Therefore he called this particular DFD the wavelet-vaguelette decomposition (WVD).

Inspired by the WVD related frame decompositions for the Radon transform have been derived where $u$ is a curvelet [3] or a shearlet frame [5].

Constructing frame coefficients: An essential ingredient in the actual implementation of the filtered DFD, is the efficient computation of the frame coefficients $\langle g, v_{j,k,\ell} \rangle$. For that purpose we make use of the explicit expression (4.4) which implies

$$
\langle g, v_{\lambda} \rangle = \frac{\kappa_{\lambda}}{2} \langle g, \Omega \mathcal{R} u_{\lambda} \rangle = \frac{\kappa_{\lambda}}{2} \langle \Omega \mathcal{R}^2 g, u_{\lambda} \rangle.
$$

(4.6)

Here $\Omega \mathcal{R}^2$ is the filtered backprojection (FBP) inversion formula for the Radon transform. Since the wavelet transform as well as $\Omega \mathcal{R}^2$ can be computed efficiently, this gives also an efficient algorithm for evaluating the coefficients $\langle g, v_{\lambda} \rangle$.

**Figure 4.1:** Reconstructions using FBP (left), truncated DFD (middle), and Tikhonv-filtered DFD (right) both with $\alpha = 0.15^2$. 

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4.3 Numerical results

Using the WVD \((u, v, \kappa)\) as in Theorem 4.1 as DFD together with the regularizing filters of Subsection 3.5 we obtain the following two filtered DFD reconstructions

\[
F^{(1)}_\alpha g = \sum_{2^{-j} \geq \alpha} \langle \Omega R^j g, u_{j,k,\ell} \rangle u_{j,k,\ell} \tag{4.7}
\]

\[
F^{(2)}_\alpha g = \sum_{j,k,\ell} 2^{-j/2} \frac{2^{-j} + \alpha}{2^{-j} + \alpha} \langle \Omega R^j g, u_{j,k,\ell} \rangle u_{j,k,\ell}. \tag{4.8}
\]

We refer to (4.7) as truncated WVD and to (4.8) as Tikhonov-filtered WVD. All ingredients for evaluating (4.7), (4.8) can be implemented in a straight forward and efficient manner: The FBP inversion formula \(\Omega R^j g\), the forward and inverse wavelet transform and the coefficient filtering.

Figure 4.1 shows reconstructions of the Shepp Logan phantom applied to Radon transform data with added Gaussian white noise using the FBP reconstruction, truncated WVD and Tikhonov-filtered WVD, respectively. Table 1 displays the \(\ell^2\)-error, the peak-signal-to-noise ratio (PSNR), and the structural similarity index measure (SSIM) of all reconstructions for various regularization parameters.

| Reconstruction method     | Parameter | \(\ell^2\)-error | PSNR  | SSIM  |
|---------------------------|-----------|------------------|-------|-------|
| FBP                       |           | 0.110            | 63.698| 0.314 |
| WVD truncated             | \(\alpha = 0.08^2\) | 0.109            | 63.765| 0.315 |
|                           | \(\alpha = 0.15^2\) | 0.104            | 71.263| 0.709 |
|                           | \(\alpha = 0.25^2\) | 0.223            | 68.426| 0.765 |
| WVD Tikhonov              | \(\alpha = 0.08^2\) | 0.086            | 67.473| 0.408 |
|                           | \(\alpha = 0.15^2\) | 0.125            | 69.75 | 0.573 |
|                           | \(\alpha = 0.25^2\) | 0.196            | 68.844| 0.706 |

Table 1: Evaluation of reconstruction results using common quality measures. The best results are marked in red (lowest-\(\ell^2\) error and highest PSNR and SSIM.)

5 Conclusion and outlook

In this work we analyzed the concept of diagonal frame decomposition (DFD) for the solution of linear inverse problems allowing potentially unbounded forward operators \(K\).
A DFD for the operator $K$ yields the explicit formula $K^\dagger = (T_{\bar{u}}^* M_{1/\kappa} T_{\bar{u}})|_{\text{dom}(K^\dagger)}$ for the Moore-Penrose inverse. In the ill-posed case, the Moore-Penrose generalized inverse $K^\dagger$ is unbounded as well as is the sequence $1/\kappa$. We showed that replacing the $1/\kappa_\lambda$ by a regularized filter (Definition 3.1) applied to the quasi-singular values $\kappa_\lambda$ results in a regularization method (Theorem 3.7). As another main result we derived convergence rates for filtered DFD in Theorem 3.8. By noting that the DFD reduced to the SVD in the case of orthogonal basis, we see that our results extend convergence and convergence rates results of filter based SVD regularization [10, 15] to the DFD case. We applied our theory to the inversion of the Radon transform by filtered DFD as practical application. The Radon transform is unbounded as an operator on $L^2(\mathbb{R}^2)$ highlighting benefits of including such operators in our theoretical analysis.

One advantage of filtered DFD regularization over variational regularization methods is their explicit form. Compared to SVD based regularization, benefits are that a DFD may be available even when no SVD is known or has to be computed numerically. Moreover, the associated analysis and synthesis operations can often be implemented efficiently for the DFD. The use of the DFD is of practical relevance as frames such as wavelets or curvelets have better approximation capabilities for typical images to be reconstructed [3] than singular systems. In order to fully exploit such properties a main aspect of future research is to extend the presented convergence analysis to non-linear filters. As a first step in this direction see the work [12] where a convergence analysis is presented using soft-thresholding defining a non-linear filtered DFD.

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