A QR Decomposition Approach to Factor Modeling of Multivariate Time Series

A THESIS

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Dedicated to mom and dad for their care,

and grandparents for their love.
This is to certify that the thesis entitled **A QR Decomposition Approach to Factor Modeling of Multivariate Time Series**, submitted by **Immanuel David Rajan M.** to the Indian Institute of Technology Madras, for the award of the degree of **Master of Science (Research)**, is a bonafide record of the research work carried out by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

An observed $K$-dimensional series \( \{y_n\}_{n=1}^{N} \) is expressed in terms of a lower $p$-dimensional latent series called factors $f_n$ and random noise $\varepsilon_n$. The equation, \( y_n = Qf_n + \varepsilon_n \) is taken to relate the factors with the observation. The goal is to determine the dimension of the factors, $p$, the factor loading matrix, $Q$, and the factors $f_n$. Here, it is assumed that the noise co-variance is positive definite and allowed to be correlated with the factors. An augmented matrix, 
\[
\tilde{M} \triangleq \begin{bmatrix} \tilde{\Sigma}_{yy}(1) & \tilde{\Sigma}_{yy}(2) & \cdots & \tilde{\Sigma}_{yy}(m) \end{bmatrix}
\]
is formed using the observed sample autocovariances \( \tilde{\Sigma}_{yy}(l) = \frac{1}{N-l} \sum_{n=1}^{N-l} (y_{n+l} - \bar{y}) (y_{n} - \bar{y})^T \), \( \bar{y} = \frac{1}{N} \sum_{n=1}^{N} y_n \). Estimating $p$ is equated to determining the numerical rank of $\tilde{M}$. Using Rank Revealing QR (RRQR) decomposition, a model order detection scheme is proposed for determining the numerical rank and for estimating the loading matrix $Q$. The rate of convergence of the estimates, as $K$ and $N$ tends to infinity, is derived and compared with that of the existing Eigen Value Decomposition based approach. Two applications of this algorithm, i) The problem of extracting signals from their noisy mixtures and ii) modelling of the S&P index are presented.
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Terminology

Matrix Notations

1. Let \( a_n \in \mathbb{R}^K \) and \( b_n \in \mathbb{R}^K \), \( n = 1, 2, \ldots, N \) be two vector valued stochastic processes. The cross-covariance between them is denoted by,

\[
\Sigma_{ab}(k) \triangleq \mathcal{E} \{(A[n + k] - \mathcal{E}\{A\})(B[n] - \mathcal{E}\{B\})\},
\]

where \( \mathcal{E}\{\cdot\} \) is the expectation operator.

2. The sample covariance (finite time approximation) is denoted by,

\[
\tilde{\Sigma}_{ab}(k) \triangleq \frac{1}{N - k} \sum_{n=1}^{N-k} (a_{n+k} - \bar{a})(b_n - \bar{b})^\top,
\]

where \( \bar{a} \) and \( \bar{b} \) are the sample means of the stochastic series \( \{a_n\} \) and \( \{b_n\} \) given by \( \bar{a} = \frac{1}{N} \sum_{n=1}^{N} a_n, \bar{b} = \frac{1}{N} \sum_{n=1}^{N} b_n \) respectively.

3. The difference between the sample covariance and the ideal estimate is denoted by the perturbation

\[
\Delta \Sigma_{ab}(k) \triangleq \tilde{\Sigma}_{ab}(k) - \Sigma_{ab}(k).
\]

Order Notations

The following notations are used for denoting the asymptotic convergence rates:

1. Let \( \{a_n\} \) and \( \{b_n\} \) be deterministic sequences. \( a_n = O(b_n) \) implies, there exists a constant \( M > 0 \) such that, \( 0 < \lim_{n \to \infty} \frac{a_n}{b_n} \leq M \).

2. \( a_n \asymp b_n \) implies for a given integer \( n_0 > 0 \) there exists \( m, M > 0 \) such that \( m < |a_n| < M \), for all \( n > n_0 \).

3. \( a_n = o(b_n) \) implies, \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \).

4. A stochastic sequence \( \{a_n\} \) is said to be of smaller order (\( o \)) in probability to the deterministic sequence \( \{b_n\} \), denoted by \( a_n = o_P(b_n) \), if \( \frac{a_n}{b_n} \to_P 0 \) where \( \to_P \) denotes convergence in probability. That, given \( \epsilon > 0 \) there exists an \( \delta > 0 \) and an integer \( n_0 > 0 \) such that \( P\{|\frac{a_n}{b_n}| < \epsilon\} > 1 - \delta, \forall n > n_0 \). In other words \( P\{|\frac{a_n}{b_n}| < \epsilon\} \to 1 \) as \( n \to \infty \).

5. Similarly with \( \{a_n\} \), a stochastic sequence, and \( \{b_n\} \), a deterministic sequence, \( a_n = O_P(b_n) \) implies, given \( \epsilon > 0 \) there exists a constant \( M > 0 \) such that \( P\{|\frac{a_n}{b_n}| < \epsilon\} \geq 1 - \epsilon, \forall n \geq n_0 \)

6. \( a_n \asymp_P b_n \) implies, given \( \epsilon > 0 \) there exists constants \( m, M > 0 \) such that \( P\{m < |\frac{a_n}{b_n}| < M\} \geq 1 - \epsilon, \forall n \geq n_0 \)
For more details on the order notations refer ([24, pp 53-55]) and also Appendix 2.

**Miscellaneous Notations**

1. The MATLAB notations are used to address various blocks of the matrix:
   (a) $A(:, j)$ refers to the $j^{th}$ column of $A$
   (b) $A(j, :)$ refers to the $j^{th}$ row of $A$
   (c) $A(a : b, c : d)$ refers to the block of $A$ consisting of rows $a$ to $b$ and columns $c$ to $d$

2. The $\sigma_i (A)$ notation is used to indicate the $i^{th}$ highest singular value of the matrix $A \in \mathbb{R}^{m \times n}$.

3. $\lambda_i (A)$ represents the $i^{th}$ largest eigen value of the matrix $A$. 
Chapter 1

Introduction

Factor modeling refers to modeling observations in terms of its constituent factors. In Multivariate Statistics, factor modeling refers to modeling a given high dimensional series called “observations” in terms of lower dimensional time series called “factors”. A linear relationship

\[ y_n = Hx_n + \epsilon_n, \quad n = 1, 2, \ldots, N, \]

(1.1)
is assumed between the observations and the factors. Here \( y_n \in \mathbb{R}^K \) is the observed data, \( x_n \in \mathbb{R}^p \) the vector of factors, \( H \in \mathbb{R}^{K \times p} \) the factor loading matrix and \( \epsilon_n \in \mathbb{R}^{K \times 1} \) being the random noise. Without loss of generality, it could be assumed that \( y_n \) and \( \epsilon_n \) are of zero mean.

Factor models were first introduced by Spearman in 1904, [32], wherein the hidden factor “ability” of a set of students were estimated from the “points” obtained in the examinations (quizzes) conducted. Consider the following example:

**Example 1.1.** The scores of students in three different exams are as follows:

|          | Classics, \( y_1 \) | French, \( y_2 \) | English, \( y_3 \) |
|----------|----------------------|-------------------|-------------------|
| **Student 1** | 10                   | 20                | 15                |
| **Student 2** | 20                   | 40                | 30                |
| **Student 3** | 15                   | 30                | 22.5              |
In matrix form it could be written as,

\[
Y = \begin{bmatrix}
10 & 20 & 15 \\
20 & 40 & 30 \\
15 & 30 & 22.5
\end{bmatrix}.
\]

This matrix is rank one and could be written as \( Y = HF \) where \( H = \begin{bmatrix} 1 & 2 & 1.5 \end{bmatrix}^T \) and \( F = \begin{bmatrix} 10 & 20 & 15 \end{bmatrix} \). Normally, the students scores would not be exact as above, and all the smaller deviations from this single factor model will be bundled up into the matrix \( E \). Leading to the more general model,

\[
Y = HF + E.
\]

Many scenarios could be modelled using (1.1). For example, the cock-tail party problem where \( p \) number of speakers, present in a room, are recorded by \( K \) number of microphones, with \( K > p \). Here, each microphone records a mix of every person’s speech. The vector of observations from the microphones, \( y_n \in \mathbb{R}^K \), could be expressed as a linear combination of the speech signals, \( x_n \in \mathbb{R}^p \). In the stock market, the daily returns, \( y_n \), of a collection of stocks, referred to as a portfolio, can be interpreted as a linear combination of a few hidden factors that affect all the stocks.

Also, in the context of signal processing, factor models have been used in sensor array processing [27, 31], spectrum analysis [34] and time series analysis [3].

The problem chosen in this thesis is to estimate \( H \) and \( \{x_1 \ldots x_N\} \) given \( \{y_1 \ldots y_N\} \) in (1.1). Note the pair \( (H, x_n) \) cannot be uniquely determined as it can be replaced by \( (HA, A^{-1}x_n) \) for any non-singular \( A \). However, as \( \mathcal{R}(H) = \mathcal{R}(HA) \), where \( \mathcal{R}(.) \) denotes the range, the space
spanned by the columns of $H$ is unique. Hence, without loss of generality, assuming a QR decomposition for $H$, (1.1) can be replaced by

$$y_n = QRx_n + \varepsilon_n = Qf_n + \varepsilon_n,$$  \hspace{1cm} (1.2)$$

where

$$f_n = Rx_n$$ \hspace{1cm} (1.3)$$

and $Q$ is a $K \times p$ matrix with orthonormal columns. The goal here is to find a $Q$, such that its columns are orthonormal and span $\mathcal{R}(H)$. Using $Q$, an estimate of the factors $f_n$ can be obtained by setting, $f_n = Q^T y_n$.

### 1.1 Outline of the Proposed approach

In this thesis, a QR decomposition based approach is used to estimate $p$, $Q$ and $f_n$. First a matrix

$$M \triangleq \begin{bmatrix} \Sigma_{yy}(1) & \Sigma_{yy}(2) & \ldots & \Sigma_{yy}(m) \end{bmatrix}$$ \hspace{1cm} (1.4)$$

is constructed. Here, $\Sigma_{yy}(l)$ are as defined in (1). Note, due to (1.2),

$$\Sigma_{yy}(l) = \mathcal{E}\left\{ y_n + iy_n^T \right\} = Q\Sigma_{ff}(l)Q^T + Q\Sigma_{f\varepsilon}(l) + \Sigma_{\varepsilon f}(l)Q^T + \Sigma_{\varepsilon\varepsilon}(l).$$ \hspace{1cm} (1.5)$$
Assuming that the factors are not correlated with the future noise, \(i.e., \Sigma_{\varepsilon f}(l) = \mathcal{E}\{\varepsilon_{n+l}f_n^\top\} = 0\), and the noise not having correlations across time, \(i.e., \Sigma_{\varepsilon\varepsilon}(l) = \mathcal{E}\{\varepsilon_{n+l}\varepsilon_n^\top\} = 0\) leads to,

\[
\Sigma_{yy}(l) = Q\left[\Sigma_{ff}(l)Q^\top + \Sigma_{f\varepsilon}(l)\right].
\]

Thus,

\[
M = \begin{bmatrix} \Sigma_{ff}(1)Q^\top + \Sigma_{f\varepsilon}(1) \\ \vdots \\ \Sigma_{ff}(m)Q^\top + \Sigma_{f\varepsilon}(m) \end{bmatrix} \triangleq QP,
\]

(1.6)

where

\[
P = \begin{bmatrix} P_1 & P_2 & \ldots & P_m \end{bmatrix}
\]

(1.7)

with

\[
P_l = \Sigma_{ff}(l)Q^\top + \Sigma_{f\varepsilon}(l).
\]

(1.8)

Assuming all factors have a non-zero correlation at least in one of the lags \(l = 1, \ldots, m\), implies \(P\) is full row rank \(i.e., P\) is of rank \(p\). This further implies that \(M\) is also of rank \(p\).

Hence the QR decomposition \([14, 12]\),

\[
M \triangleq QR,
\]

(1.9)

of \(M\), (1.4), with \(Q\) being an orthonormal matrix would have an upper triangular \(R\) with last \(K - p\)
rows being zeros, i.e.,

\[
R = \begin{bmatrix}
R_{11} \\
0
\end{bmatrix}_{K \times mK}.
\]

Thus \( Q \) is obtained by setting

\[
Q \triangleq Q(:, 1 : p),
\]

the first \( p \) columns of \( Q \) (in MATLAB Notations) and an estimate of the factors is obtained by setting \( \hat{f}_n = Q^T y_n \). Note, for any permutation matrix \( \Pi \in \mathbb{R}^{mK \times mK} \) such that the first \( p \) columns of \( M\Pi \) are linearly independent, would have a QR decomposition of the form

\[
M\Pi \triangleq \hat{Q}\hat{R}, \tag{1.10}
\]

with \( \hat{R} \) having the last \( K - p \) rows as zeros. A different \( Q \) can be obtained by setting \( Q \triangleq \hat{Q}(:, 1 : p) \). As both \( Q(:, 1 : p) \) and \( \hat{Q}(:, 1 : p) \) span the same column space, there need not be any specific preference of one over the other. However, it would be prudent to choose a \( \Pi \) such that the \( p \) columns selected by \( \Pi \) have the best condition number.

In the finite data case \( M, (1.4), \) is replaced by

\[
\tilde{M} \triangleq \begin{bmatrix}
\tilde{\Sigma}_{yy}(1) & \tilde{\Sigma}_{yy}(2) & \ldots & \tilde{\Sigma}_{yy}(m)
\end{bmatrix}, \tag{1.11}
\]

where \( \tilde{\Sigma}_{yy}(l) \) is as in (2). Note \( \text{rank}\left(\tilde{M}\right) \neq p \), in fact it will be of full rank (\( \text{rank}\left(\tilde{M}\right) = K \)) with probability one. The determination of \( p \) and \( Q \) from \( \tilde{M} \) are not as trivial as before. The matrix \( \tilde{M} \) could be interpreted as a perturbed version of the matrix \( M \),

\[
\tilde{M} = M + \Delta M
\]
or in more general form,

$$\tilde{M}\Pi = M\Pi + \Delta M\Pi.$$ 

And the problem could be restated as finding an estimate of $M\Pi$ from $\tilde{M}\Pi$. A general QR decomposition,

$$\tilde{M}\Pi \triangleq \begin{bmatrix} p & mK - p \\ \tilde{M}_p & \tilde{M}_{mK-p} \end{bmatrix} \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{Q}_p & \tilde{Q}_{K-p} \end{bmatrix} \begin{bmatrix} \tilde{R}_{21} & \tilde{R}_{22} \\ 0 & \tilde{R}_{22} \end{bmatrix} = \tilde{Q}\tilde{R}, \quad (1.12)$$

where $\tilde{R}_{22} \neq 0$. Assuming rank $(M) = p$, an estimate of $M\Pi$ can be obtained by setting, $M\Pi = \tilde{Q}(:, 1 : p)\tilde{R}_{11}$. For a good approximation of $M\Pi$, it is desirable to have a $\Pi$ such that $\sigma_{\min}(\tilde{R}_{11}) \gg \sigma_{\max}(\tilde{R}_{22})$, where $\sigma_{\min}(. )$ denotes the minimum singular value and $\sigma_{\max}(. )$ denotes the maximum singular value. Rank Revealing QR (RRQR) algorithms are a class of algorithms that find a permutation matrix $\Pi$ such that either,

$$\max_\Pi \left( \sigma_{\min}(\tilde{R}_{11}) \right)$$

or,

$$\min_\Pi \left( \sigma_{\max}(\tilde{R}_{22}) \right)$$

or both are satisfied. Using such a $\Pi$, a good estimate of $M\Pi$ can be obtained by setting, $M\Pi = \tilde{Q}_p\tilde{R}_{11}$. An the estimate of the factor loading matrix and the factors can be got by setting, $\hat{Q} = \tilde{Q}_p$.
and \( \hat{f}_n = \hat{Q}^\top y_n \) respectively.

So far, rank \((M) = p\) was assumed to be known. Determination of rank \((M)\) from \(\tilde{M}\), is obtained by equating it to the determination of numerical rank, \([13, 33]\), of the matrix \(\tilde{M}\). In this thesis, an RRQR based approach is used to determine numerical rank. An estimate, \(\hat{p}\), of rank \((M)\), is set to,

\[
\hat{p} = \arg \max_i \frac{\gamma_i + \epsilon}{\gamma_{i+1} + \epsilon},
\]

(1.13)

where \(\gamma_i\) and \(\gamma_{i+1}\) are the \(i^{th}\) and \(i + 1^{th}\) diagonal values of \(\tilde{R}\), (1.12), got by assuming numerical rank to be \(i\) in the Hybrid-III RRQR algorithm \([7]\) with \(\epsilon = \frac{\gamma_1}{\sqrt{NK}}\).

Note, Hybrid-III RRQR algorithm satisfies the bounds in Hybrid-I algorithm and hence could also be used in estimation of factor loading matrix \(Q\). Here, Hybrid-I algorithm is used because of its lesser numerical complexity.

In this thesis, asymptotic analysis of the estimates, \(\hat{p}\), \(\hat{Q}\) and \(\hat{f}_n\) are presented. The rates of convergence of the estimates, when the dimension of the observations \(K\) and the duration of the observations \(N\) tends to infinity, under the constraint \(\frac{K^4}{\sqrt{N}} = o(1)\), are derived. It is shown that the proposed algorithm has better convergence rates than the Eigen Value Decomposition (EVD) based approach, [22], under certain conditions.

### 1.2 Chapter Outline

In Chapter 2, the regulatory conditions under which the proposed algorithm is designed are summarized along with a step-by-step implementation procedure for the proposed algorithm.

In Chapter 3, the two existing algorithms for factor modeling, namely the Principal Component Analysis Method \([20, 2, 3, 19, 26]\) and the EVD based method, are discussed in detail.
In Chapter 4, a detailed exposition of RRQR algorithms are presented along with derivation of bounds satisfied by the Hybrid algorithms.

In Chapter 5, the problem of finding the number of factors, \( p \) from \( \tilde{M} \) is discussed in the numerical rank perspective. The asymptotic rate of convergence for the estimate of the model order \( \hat{p} \) as \( K, N \to \infty \) is presented therein.

In Chapter 6, the Convergence of \( \hat{Q} \) obtained by using \( \tilde{M} \) to \( Q \) obtained from \( M \) is studied. The asymptotic rates are also got for the same.

In Chapter 7, the proposed algorithm is compared with the existing EVD based algorithm of [22].

In Chapter 8, the numerical results obtained by simulating the algorithm using Monte-Carlo trials is presented and compared with other existing algorithms. There is also the simulation of the famous cock-tail party problem wherein the proposed algorithm is used as pre-processing for the popular Independent Component Analysis (ICA) technique. A real life application of the proposed algorithm to the finance market is also presented.

Finally, in Chapter 9, concludes with a discussion of the entire thesis along with the prospect of further developments.
1.3 Contributions of this thesis

In this thesis, a new algorithm for factor modeling is proposed. While other popular algorithms like the PCA based algorithms have quite stringent constraints on noise \( \varepsilon_n \) \( i.e., \)

\[
\mathcal{E}\{f_n\varepsilon^\top_m\} = 0, \forall \ n, \ m \tag{1.14}
\]

and

\[
\mathcal{E}\{\varepsilon_n\varepsilon^\top_m\} = \sigma^2I_{n\times n}\delta(n-m), \tag{1.15}
\]

where \( I_{n\times n} \) is an Identity matrix and \( \delta(l) \) is the Kronecker delta function, the proposed algorithm has less stringent constraints allowing the factors to be correlated with the past noise and also that noise can have any-covariance \( \Sigma \).

The proposed algorithm is subjected to rigorous asymptotic analysis wherein

- Rate of convergence of the model order estimate \( \hat{p} \), (1.13), to \( p \) as \( K, N \to \infty \) are derived.

- Convergence rates of the factor loading matrix \( \hat{Q} \) and factors \( \hat{f}_n \) as \( K, N \to \infty \) are also derived.

From these asymptotic properties, it was found that the proposed algorithm performs better than the EVD based algorithm proposed in [22] in certain scenarios.

Two interesting applications of the proposed algorithm is also discussed.

- The Cock-tail party problem of separating multiple signals from their linear noisy mixtures.

- Modeling the S&P 500 index and predicting the daily returns of individual companies.
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Chapter 2

Data Model and Summary of Algorithm

In this chapter, the Data Model and assumptions upon which the proposed algorithm works are summarized in Section (2.1). Section (2.2) gives a step-by-step implementation procedure for the proposed algorithm.

2.1 Assumptions on the Data

Regulatory Conditions

The data model assumed as (1.1). Now, the factors could be written as

\[
X = [x_1, x_2, \ldots, x_N] = [x^{(1)}, x^{(2)}, \ldots, x^{(p)}]^\top
\]

where \( x_n \in \mathbb{R}^p \) represents the values taken by the factors at the \( n^{th} \) instant and \( x^{(i)} \in \mathbb{R}^N \) represents the \( i^{th} \) factor which is a series in time \( n = 1, \ldots, N \). Now, assume that linear combination of all the factors could result white noise, then, the last factor \( x^{(p)} \) could be written as

\[
x^{(p)}_n = e_n + \sum_{i \neq p} \alpha_i x^{(i)}_n, \quad n = 1, \ldots, N
\]
where \( e_n \) is a white noise sequence. Thus, replacing the pair \((H, x_n)\) in (1.1) with the pair \((HA^{-1}, Ax_n)\) with 

\[
A = \begin{bmatrix}
1 \\
1 & 1 \\
& & \ddots \\
& & & 1 \\
& & & & -\alpha_1 \\
& & & & -\alpha_2 \\
& & & & \ldots \\
& & & & -\alpha_{p-1} \\
& & & & 1
\end{bmatrix}
\]

would result in the factors \( f = Ax_n = [x^{(1)}, x^{(2)}, \ldots, e]^\top \) where \( e \) is a white noise sequence. Thus, since \( e \) is white noise, it does not have correlation at any lags \( l = 1, \ldots, m \). Thus, the rank of \( M \) in (1.4) would be \((p - 1)\) and the last factor would not be detected by the proposed algorithm. Thus, the following condition is assumed.

\[\begin{align*}
(A1) \quad & \text{No linear combination of the factors } x^{(i)}, \text{ results in white noise } i.e., \\
& \begin{bmatrix}
\Sigma_{xx}(1) \\
\vdots \\
\Sigma_{xx}(m)
\end{bmatrix} \text{ is of rank } p \text{ (full row rank)}. 
\end{align*}\]

The following three conditions could be deduced from (1.5):

\[\begin{align*}
(A2) \quad & \text{The noise } \varepsilon \text{ is not correlated with past factors, } i.e., \Sigma_{\varepsilon x}(k) = \text{cov}(\varepsilon_{n+k}, x_n) = Z, \forall k > 0 \text{ where } Z \text{ is a zero matrix.} \\
(A3) \quad & \text{The factors } x_n \text{ can have correlation with past noise } \varepsilon_n, i.e., \text{cov}(x_{n+k}, \varepsilon_n) \text{ need not necessarily be } Z \text{ but bounded as } N, K \to \infty. \\
(A4) \quad & \text{The auto-covariance of the noise needs to be bounded, } i.e., \text{cov}(\varepsilon_n, \varepsilon_n) \text{ is bounded as } N, K \to \infty.
\end{align*}\]

The following mixing conditions, [10, pp 67-77], ensures that as \( N \to \infty, \tilde{M}, (1.11), \) tends to \( M, (1.4) \) for a fixed \( K, i.e., \) the finite sample estimates converge to the ensemble average.
The process \( \{y_n\} \) is a zero mean strictly stationary process with \( \psi \) mixing with mixing coefficients \( \psi_{ij} \) satisfying \( \sum_{n \geq 1} n \psi(n) < \infty \) also \( \mathbb{E}(|y_n|^4) < \infty \). Refer ([10]) for a comprehensive study of mixing properties. The \( \psi \) mixing could also be replaced with \( \alpha \) mixing with mixing coefficient \( \alpha_n = O(n^{-5}) \) and \( \mathbb{E}(|y_n|^{12}) < \infty \) element wise. For properties regarding the same, refer Theorem 27.4 in ([6]). Refer the Appendix 2 for more details regarding mixing properties.

**Factor Strength**

A factor is considered a strong factor if it affects all the observed outputs and a weak factor if not. To model this mathematically, the columns of \( H \) determine the strength with which the factors, \( x_n \), affect the observations, \( y_n \). The following two assumptions on the columns of \( H \) quantify the impact of the factors on the observations:

\[ (A6) \quad \text{The columns of the matrix } H = \begin{bmatrix} h_1 & h_2 & \ldots & h_p \end{bmatrix} \text{ are such that, } O\left(\|h_i\|_2^2\right) = O\left(K^{1-\delta}\right), i = 1, 2, ..., p \text{ and } 0 \leq \delta < 1. \]

\[ (A7) \quad \text{The columns of } H \text{ are independent of each other, i.e.,} \]

\[ O\left(\min_{\theta_j} \left\| h_i - \sum_{j \neq i} \theta_j h_j \right\|_2^2\right) = O\left(K^{1-\delta}\right) \quad (2.1) \]

The above assumptions \( A1-A7 \) are quite similar to the ones made in ([22]).

Finally, consider the following notations used to denote the strength of factor correlation with noise:
**Definition 2.1.** Define

\[
\kappa_{\text{max}} \triangleq \max_{1 \leq l \leq m} \| \Sigma_{f\varepsilon}(l) \|_2 \tag{2.2}
\]

\[
\kappa_{\text{min}} \triangleq \min_{1 \leq l \leq m} \sigma_{\text{min}}(\Sigma_{f\varepsilon}(l)) \tag{2.3}
\]

where \( \kappa_{\text{max}} \) and \( \kappa_{\text{min}} \) correspond to a measure of factor correlation with noise.

### 2.2 Summary of the Algorithm

In this section, the steps in implementation of the proposed algorithm are presented.

1. Let \( \{y_n\}_{n=1}^N \) be the observed data. And the data model is assumed as in (1.2).
2. Compute \( \tilde{\mathbf{M}} \) defined in (1.11), where \( \tilde{\Sigma}_{yy}(l) \) is got by (3.4).
3. To evaluate the Numerical Rank, \( p \),
   (a) Set \( i = 1 \)
   (b) First the permutation matrix \( \mathbf{\Pi} \) is got by using HYBRID-III algorithm assuming a numerical rank of \( i \).
   (c) Take Gram-Schmidt QR (GS-QR) decomposition of \( \tilde{\mathbf{M}} \mathbf{\Pi} \), i.e., \( \tilde{\mathbf{M}} \mathbf{\Pi} = \tilde{\mathbf{Q}} \tilde{\mathbf{R}} \). Compute \( r_i = \frac{\gamma_i + \epsilon}{\gamma_{i+1} + \epsilon} \) where \( \gamma_i \) and \( \gamma_{i+1} \) are the \( i^{th} \) and \( i+1^{th} \) diagonal values of \( \tilde{\mathbf{R}} \) and \( \epsilon = \frac{\gamma_i}{\sqrt{KN}} \).
   (d) Set \( i = i + 1 \)
   (e) Repeat Until \( i < P \) where, \( P \) is taken such that \( p < P \).
4. Now, estimate of numerical rank, \( \hat{p} \), is got by finding the \( i \) that maximizes \( r_i \) as in (1.13).
5. Evaluate RRQR decomposition of \( \tilde{\mathbf{M}} \) using HYBRID-I algorithm ([7]) presented in Algorithm (4.2). Let the RRQR Decomposition be as in (1.12).
6. Now, the finite sample estimate of \( \hat{\mathbf{Q}} \) is got by, \( \hat{\mathbf{Q}}_{N,K} = \tilde{\mathbf{Q}}(:,1:\hat{p}) \).
7. Finally the estimate of factors is got by, \( \hat{f}_n = \hat{\mathbf{Q}}_{N,K}^\dagger y_n \).

An exposition of RRQR-Algorithms are presented in Chapter (4).
Chapter 3

Existing Algorithms

In this chapter, two existing algorithms for factor modeling are analysed. Section 3.1 discusses the PCA based method of factor analysis and Section 3.2 covers the Eigen Value Decomposition based method.

3.1 PCA based Method

The PCA based method is dealt in many literature, notably [20, 2, 3, 19, 26]. Assume that the observed time series \( \{ y_n \in \mathbb{R}^K \}_{n=1}^N \) could be modelled as,

\[
y_n = Q f_n + \varepsilon_n
\]

where, \( f_n \in \mathbb{R}^p \) are factors and \( \varepsilon_n \in \mathbb{R}^K \) is random noise. The factors and noise are assumed to satisfy:

**Condition_1:** The factors and noise are uncorrelated with each other i.e., \( \Sigma_{f\varepsilon}(l) = \mathbb{E} \{ f_{n+l}\varepsilon_l^\top \} = 0, \ \forall l \).

**Condition_2:** The noise should have covariance given by \( \Sigma_{\varepsilon\varepsilon}(l) = \mathbb{E} \{ \varepsilon_{n+l}\varepsilon_l^\top \} = \delta(l)\sigma^2 I \), where \( \delta(l) \) is the Kronecker delta function and \( I \in \mathbb{R}^{K\times K} \) is an Identity matrix.

Under the above conditions, the autocovariance matrix of the observations is

\[
\Sigma_{yy}(0) = \mathbb{E} \{ y_n y_n^\top \} = Q \Sigma_{ff}(0) Q^\top + \sigma^2 I.
\]
Assuming that $\Sigma_{ff}(0)$ to be full rank, implies that $Q\Sigma_{ff}(0)Q^\top$ is of rank $p$ and will have $p$ non-zero positive Eigen Values. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ be the eigen values of $Q\Sigma_{ff}(0)Q^\top$. The Eigen Value Decomposition of $\Sigma_{yy}(0)$ is,

$$
\Sigma_{yy}(0) = U \begin{bmatrix}
\lambda_1 + \sigma^2 \\
& \ddots \\
& & \lambda_p + \sigma^2 \\
& & & \sigma^2 \\
& & & & \ddots \\
& & & & & \sigma^2 \\
\end{bmatrix} U^\top
$$

Thus, the number of factors equal $p$ and the factor loading matrix could be set to, $\hat{Q} = U(:, 1 : p)$, the eigen vectors that correspond to the largest $p$ Eigen values. The factors are then $\hat{f}_n = \hat{Q}^\top y_n$.

Henceforth, this technique will be referred to as PCA.

Now, the above procedure is ideal. In practice, $\Sigma_{yy}(0)$ will be replaced by its sample covariance, $\tilde{\Sigma}_{yy}(0) = \frac{1}{N} \sum_{n=1}^{N} y_n y_n^\top$. In which case, estimating the number of factors and the factor loading matrix becomes a bit more difficult. The problem of finding the number of factors is dealt as a model order selection problem using AIC (Akaike Information Criteria) and MDL (Minimum Code Length, [29]) in [26]. In [4], an alternative Information Theoretic Criteria is proposed and proved to be better than the AIC, [1], BIC (Bayesian Information Criteria) and all of their derivatives. In [4], the model order is taken as the $p$ that minimizes the Information Content,

$$
IC_p = \ln \left(V(p, \hat{f}_n)\right) + p \left(\frac{K + N}{KN}\right) \ln \left(\frac{KN}{K + N}\right)
$$
where
\[
V(p, \hat{f}_n) = \min_{\hat{Q}} \frac{1}{KN} \sum_{n=1}^{N} \left\| y_n - \hat{Q}^{(p)} \hat{f}_n \right\|_2^2
\]

Here, \( \hat{Q}^{(p)} \) and \( \hat{f}_n^{(p)} \) are obtained using PCA by assuming model order as \( p \). Thus, the estimate of model order is
\[
\hat{p}_{N,K} = \arg \min_p \{ IC_p \}.
\]

For more details regarding the same, refer [4].

To estimate the factor loading matrix, a Maximum Likelihood framework is used. The Maximum Likelihood framework for factor analysis was first introduced by Lawley in the year 1940, [23]. A detailed version of this ML approach is given in, [19, 28, 4, 20].

Let,
\[
\tilde{\Sigma}_{yy}(0) = \tilde{U} \hat{\Lambda} \tilde{U}^T
\]

be the eigen value decomposition of \( \tilde{\Sigma}_{yy}(0) \), where \( \hat{\Lambda} \) is a diagonal matrix of eigen values arranged in descending order, \( \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \ldots \geq \hat{\lambda}_K \). It was proved in [28], that the optimal solution to the ML estimate for the error co-variance, \( \sigma^2 I \), is given by \( \hat{\sigma}^2 = \sum_{i=p+1}^{K} \hat{\lambda}_i \) and the ML estimate for \( Q \) is given by \( \hat{Q} = \tilde{U} (:, 1 : p) \).

The disadvantages of this PCA based method are,

1. The ML estimate inherently assumes \( y_n \) is iid Gaussian (Though in real life simulations, it could not be ensured).
2. The requirement of conditions 1 and 2 regarding the noise \( \varepsilon_n \) and factors \( f_n \) as stated before.
3. No properties of the factors \( f_n \) are taken into consideration.
3.2 EVD Method

Recently, Lam et. al., had proposed a new technique for factor modeling, [22]. This technique helps in overcoming the disadvantages pointed out in the previous PCA based method.

In [22], the authors setup a matrix

\[ S \triangleq \sum_{l=1}^{m} \Sigma_{yy}(l)\Sigma_{yy}(l)^\top \]  
\[ = MM^\top, \]  

where \( M \) is as defined in (1.4). From (3.1) and (1.6),

\[ S = QPP^\top Q^\top. \]  

As \( S \) is an \( K \times K \) positive semi-definite matrix with rank \( p \) (assuming the \( p \times p \) matrix \( PP^\top \) to be full rank), its Eigen value decomposition

\[ S = U\Lambda U^\top \]  

would have a diagonal matrix \( \Lambda \) with the first \( p \) diagonal elements strictly positive and the rest \( K - p \) diagonal elements being zero. The factor loading matrix was chosen to be \( Q = U(:, 1:p) \), the first \( p \) columns of \( U \) corresponding to the non-zero Eigen values. Here, \( U(:, 1:p) \) refers to the MATLAB notation of selecting first \( p \) columns. An estimate \( \hat{f}_n \) of the factors was obtained by setting \( \hat{f}_n = Q^\top y_n \).

In practice, the auto-covariances \( \Sigma_{yy}(k), \Sigma_{ff}(k) \) and \( \Sigma_{fe}(k) \) \( k = 0, 1, 2, \ldots \) are approximated
by their corresponding finite sample equivalents, for example $\Sigma_{yy}(k)$ is replaced by

$$
\tilde{\Sigma}_{yy}(k) = \frac{1}{N-k} \sum_{n=1}^{N-k} (y_{n+k} - \bar{y}) (y_n - \bar{y})^\top,
$$

(3.4)

where $\bar{y} \triangleq \frac{1}{N} \sum_{t=1}^{N} y_t$. The Eigen values of the matrix $S$ formed using (3.4) will not have $K - p$ zero Eigen values. In fact all the Eigen values will be different with probability 1. Therefore, $p$ cannot be directly determined by computing the Eigen values of $S$. In [22], the model order, $p$, was estimated using

$$
\hat{p} = \arg \max_{1 < i < R} \frac{\lambda_i}{\lambda_{i+1}}
$$

(3.5)

where $\lambda_i > \lambda_{i+1}$ denotes the Eigen values of the matrix $S$ arranged in descending order. The properties of this ratio based estimator (3.5) are summarized in [21].

The Proposed QR decomposition based algorithm will be compared with this algorithm in Chapter 7 with regards to asymptotic properties of their estimates and in Chapter 8 through their simulations.
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Chapter 4

RRQR Decomposition

In this chapter, the RRQR algorithms used in this thesis are discussed in detail. The derivations of the results pertaining to the RRQR algorithm are presented in this chapter for the sake of completion.

Define a matrix $A$ of dimension $m \times n$ such that,

$$A \triangleq \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix},$$

where $a_i$ is an $m$ dimensional column vector. The QR decomposition of $A$ is given by:

$$A = QR,$$

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \ldots & q_n \end{bmatrix},$$

where $q_i$'s are the orthonormal columns of $Q$, $R$ is the upper triangular matrix and $<,>$ denotes inner product. There are many algorithms to achieve the above decomposition refer [12, 14] for more details.
In the above equation, the matrix $Q$ and $R$ could be segmented as

\[
Q = \begin{bmatrix}
p & m - p \\
Q_1 & Q_2
\end{bmatrix}
\]

\[
R \triangleq \begin{bmatrix}
p & n - p \\
p & R_{11} & R_{12} \\
m - p & 0 & R_{22}
\end{bmatrix}, \tag{4.2}
\]

for any $p < \min(n, m)$. From (4.1), it can be seen that the minimum singular value of $R_{11}$, $\sigma_{\min}(R_{11})$ is dependent on the columns $a_1, \ldots, a_p$ of $A$ and maximum singular value of $R_{22}$, $\sigma_{\max}(R_{22})$, depends on the columns $a_{p+1}, \ldots, a_n$ of $A$. Thus permuting the columns of $A$ and taking QR would result in different $\sigma_{\min}(R_{11})$ and $\sigma_{\max}(R_{22})$. Let $\Pi$ be a permutation matrix, the QR decomposition of $A\Pi$ be given by,

\[
A\Pi = \tilde{Q}\tilde{R}.
\]

Given a $p < \min(m, n)$, RRQR algorithms are a group of algorithms that find a $\Pi$ such that,

\[
\Pi \triangleq \arg \max_{\Pi \in \mathcal{P}} \sigma_{\min}(\tilde{R}_{11}) \tag{4.3}
\]

or

\[
\Pi \triangleq \arg \min_{\Pi \in \mathcal{P}} \sigma_{\max}(\tilde{R}_{22}) \tag{4.4}
\]

or both, where $\mathcal{P}$ is a set of all permutation matrices and $\tilde{R}_{11}, \tilde{R}_{22}$ are got by segmenting $\tilde{R}$ as in (4.2). The algorithms that solve (4.3), (4.4) or both (4.3) and (4.4) are are termed, Type I, Type II
and Type-III algorithms respectively, [7].

In this thesis, the Hybrid RRQR algorithms proposed by Chandrasekaran et. al., [7], are of particular interest. The Hybrid-I algorithm is a type I algorithm that satisfies,

$$
\sigma_{\text{min}} (R_{11}) \geq \frac{\sigma_p (A)}{\sqrt{p(n - p + 1)}},
$$

$$
\sigma_{\text{max}} (R_{22}) \leq \sigma_{\text{min}} (R_{11}) \sqrt{p(n - p + 1)}.
$$

The Hybrid-II is a type -II algorithm that satisfies

$$
\sigma_{\text{max}} (R_{22}) \leq \sigma_{p+1} (A) \sqrt{(p + 1)(n - p)}
$$

$$
\sigma_{\text{min}} (R_{11}) \geq \frac{\sigma_{\text{max}} (R_{22})}{\sqrt{(p + 1)(n - p)}}.
$$

Finally, the Hybrid-III algorithm combines the best of the previous two bounds to ensure,

$$
\sigma_{\text{max}} (R_{22}) \leq \sigma_{p+1} (A) \sqrt{(p + 1)(n - p)}
$$

$$
\sigma_{\text{min}} (R_{11}) \geq \frac{\sigma_p (A)}{\sqrt{p(n - p + 1)}}.
$$

Derivation of these bounds can be found in [7]. These derivations are also presented later in this chapter for the sake of completion.

To understand the working of the Hybrid algorithms, one needs to understand the working of two other algorithms: the QR decomposition with Column Pivoting (QR-CP) algorithm proposed by Golub, [14], which is a Type-I algorithm and its corresponding Type-II version proposed by Stewart (Stewart-II) in [33].
The following theorem relates to Interlacing Property of singular values. It is essential for understanding the rest of the thesis:

**Theorem 4.1.** Let \( \Sigma \in \mathbb{R}^{m \times n} \), \( m < n \), and \( \Sigma_{11}^{k \times l} \), \( k < l \), be any sub matrix of \( \Sigma \) with \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \) and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \) being their corresponding singular values. Then,

\[
\sigma_j \geq \lambda_j \geq \sigma_{n-k+j}, j = 1, 2, \ldots k.
\]

**Proof.** Refer to Appendix 1. \qed

**QR-CP Algorithm**

Let the matrix to be permuted be \( A \in \mathbb{R}^{m \times n} \). The algorithm is as follows:

**Initialize** \( s = 0; \Pi = I \in \mathbb{R}^{n \times n} \).

Let

\[
j = \arg \max_i \| A(:, i) \|_2.
\]

**if** \( j \neq 1 \), interchange 1\(^{st}\) and \( j^{th} \) column of \( \Pi \) and take QR decomposition\(^1\),

\[
A\Pi = Q^{(0)} R^{(0)},
\]

---

\(^1\)Householder or Gram Schmidt methods.
where $R^{(s)}$ denotes the matrix $R$ being segmented as,

\[
R^{(s)} \triangleq \begin{bmatrix}
R_{11}^{(s)} & R_{12}^{(s)} \\
R_{21}^{(s)} & R_{22}^{(s)}
\end{bmatrix}.
\]

**Repeat** $s = s + 1$;

\[j = \arg \max_i \| R_{22}^{(s)}(:, i) \|_2\]

**if** $j \neq 1$,

Interchange columns $s + 1$ and $s + j$ of $\Pi$ and retriangularize $R^{(s)}\Pi = QR^{(s+1)}$.

**Until** $s < \min(n, m)$.

Note that $R_{11}^{(s)}$ and $R_{22}^{(s)}$ changes in dimension with every increase in $s$.

The above QR-CP algorithm is an approximation of the greedy algorithm which selects columns such that $\sigma_{\text{min}} \left( R_{11}^{(s)} \right)$ is maximized. For further details regarding this approximation, refer [7, pp 596-598].

Now, to look at the properties of the QR-CP algorithm, let

\[A\Pi = \tilde{Q}\tilde{R}\]

where $\Pi$ is obtained using QR-CP algorithm. The following two lemmas summarize the properties of QR-CP algorithm:

\[\text{Let } A\Pi = QR. \text{ If } R\Pi = \tilde{Q}\tilde{R}, \text{ Then, } A\Pi\Pi = Q\tilde{Q}\tilde{R}. \text{ Thus, finding permutations to columns of } R \text{ is equivalent to finding permutations to columns of } A.\]
**Lemma 4.1.** The diagonal elements in $\tilde{R}$ satisfy

$$
\gamma_i \geq \frac{\sigma_i (A)}{\sqrt{n-i+1}}, \quad i = 1, \ldots, \min(n, m).
$$

(4.5)

where $\gamma_i$ is the $i^{th}$ diagonal element of $\tilde{R}$.

**Proof.** From (4.1) and the above algorithm, it can be seen that $\gamma_i$ corresponds to the highest $2$ - norm among the columns of $R^{(i-1)}_{22}$. Since $R^{(i-1)}_{22}$ has $n-i+1$ columns,

$$
\gamma_i \geq \frac{\|R^{(i-1)}_{22}\|_F}{\sqrt{n-i+1}}.
$$

Since $\|R^{(i-1)}_{22}\|_F \geq \|R^{(i-1)}_{22}\|_2$,

$$
\gamma_i \geq \frac{\|R^{(i-1)}_{22}\|_2}{\sqrt{n-i+1}}.
$$

(4.6)

Interlacing property\(^3\) of Singular Values [14, 25] states that:

$$
\sigma_s (\tilde{R}) \geq \|\tilde{R}^{(s)}_{22}\|_2 \geq \sigma_{s+1}(\tilde{R}).
$$

(4.7)

Thus, combining (4.6) and (4.7) leads to (4.5). \hfill \square

**Lemma 4.2.** When using QR-CP and segmenting $\tilde{R}$ as in (4.2),

$$
\sigma_{\text{min}} (R_{11}) \geq \frac{\sigma_p (A)}{\sqrt{pm}2^p}.
$$

(4.8)

**Proof.** Note $R_{11}$ could be written as

$$
R_{11} = DW.
$$

\(^3\)Refer Appendix 1 on page 93 for a more elegant proof on Interlacing property.
where $D = \text{diag}(R_{11})$ is a diagonal matrix with diagonal elements of $R_{11}$. By the property of QR-CP algorithm, the diagonal elements of $R_{11}$ are the highest elements. Thus, $W$ is an upper triangular matrix such that the diagonal elements are equal to 1 and the rest of the elements are less than 1. Thus,

$$\sigma_{\text{min}}(R_{11}) \geq \sigma_{\text{min}}(D) \sigma_{\text{min}}(W)$$

Since, for a diagonal matrix, the singular values equal the diagonal elements, from (4.5),

$$\sigma_{\text{min}}(D) \geq \frac{\sigma_{p}(A)}{\sqrt{n - p + 1}}.$$ 

Thus, since $\sigma_{\text{min}}(W) = \frac{1}{\|W^{-1}\|_2}$,

$$\sigma_{\text{min}}(R_{11}) \geq \frac{\sigma_{p}(A)}{\sqrt{n - p + 1} \|W^{-1}\|_2}.$$ 

Now, $\|W^{-1}\|_2 \leq \sqrt{p}^2p$ refer [7, pp 606], thus, we get (4.8).

The above proof is paterned after the proof provided in [7] for the same.

An implementation scheme for the QR-CP algorithm when applied to $\tilde{M}$, (1.11), is summarized in Algorithm 4.1.

**Stewart’s Type-II Algorithm**

The Stewart’s Type-II algorithm proposed first in [33] is discussed here. In, [7], this Stewart’s algorithm is framed as a type-II equivalent of the QR-CP algorithm. In, [7], both Type-I and Type-II algorithms were unified in a single framework. This unification enables derivation of the properties of the Type-II algorithm from its Type-I equivalents. Note Stewart-II algorithm is
Algorithm 4.1 QR Decomposition with Column Pivoting

1. Let the matrix to be decomposed be $\tilde{M} \in \mathbb{R}^{K,mK}$.
2. Let the permutation matrix $\Pi$ be initialized as an Identity matrix of order $mK$.
3. Take a value of $P$ such that, $p < P < \min(K, N)$
4. Take Gram-Schmidt QR (GS-QR) decomposition of $\tilde{M}$ i.e., $\tilde{M} = Q^{(0)} R^{(0)}$, where $R^{(s)} \triangleq$

$$
\begin{bmatrix}
R_{11}^{(s)} & R_{12}^{(s)} \\
0 & R_{22}^{(s)}
\end{bmatrix}
$$

and set $s = 0$.
5. Repeat
   
   (a) Let $e_i$ be the $i^{th}$ column of an identity matrix of order $mK - s$.
   (b) $j \triangleq \arg \max_{1 \leq i \leq mK-s} \| R_{22}^{(s)} e_i \|_2$
   (c) Interchange the columns $j$ and $s + 1$ in $\Pi$.
   (d) Compute, GS-QR of $\tilde{M}\Pi$ as $\tilde{M}\Pi \triangleq Q^{(s)} R^{(s)}$
   (e) $s = s + 1$
6. Until $s > P$
7. Return $\Pi$
applicable only for Invertible matrices.

**Unification**

Assume \( A \in \mathbb{R}^{n \times n} \) is invertible and of numerical rank \( p \). For details regarding Numerical rank refer Chapter 5. For now, just take \( p \) as the block size used for segmenting \( R \) in (4.1).

Note, the Type-II formulation is equivalent to,

\[
\min_{\Pi} \sigma_{\text{max}} (R_{22}) = \min_{\Pi} \frac{1}{\sigma_{\text{min}} (R_{22}^{-1})} = \frac{1}{\max_{\Pi} \left( \sigma_{\text{min}} (R_{22}^{-1}) \right)}.
\]  \hspace{1cm} (4.9)

Let,

\[ R\Pi \triangleq \bar{Q} \bar{R}, \quad \bar{R} \triangleq \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \]

Assuming \( R \) to be invertible, inverting both sides of the above equation leads to,

\[ \Pi^\top R^{-1} = \begin{bmatrix} R_{11}^{-1} & -R_{11}^{-1}R_{12}R_{22}^{-1} \\ 0 & R_{22}^{-1} \end{bmatrix} \bar{Q}^\top. \]

Now, taking transpose on both sides,

\[ R^{-\top} \Pi = \bar{Q} \begin{bmatrix} R_{11}^{-\top} & 0 \\ -R_{22}^{-\top}R_{12}^{-\top} & R_{22}^{-\top} \end{bmatrix}. \]  \hspace{1cm} (4.10)

Thus from (4.10), (4.9) and the fact that \( \sigma_{\text{min}} (R_{22}^{-1}) = \sigma_{\text{min}} (R_{22}^{-\top}) \); It can be seen that applying Type-II algorithm to \( R \) is equivalent to Type-I algorithm applied to \( R^{-\top} \). To make it more clearer,
applying Type-I algorithm to $R^{-\top}$ gives,

$$R^{-\top}\Pi = \tilde{Q}P,$$

where

$$P = \begin{bmatrix} n-p & p \\ P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}.$$  \hspace{1cm} (4.11)

Note in Type-I algorithm $\sigma_{\min}(P_{11})$ is maximized. With some block permutations the above $P$ could be written as,

$$\begin{array}{c}
p & n-p \\
p & P_{22} & 0 \\
n-p & P_{12} & P_{11} \end{array}$$

Now, permuting the rows and columns of the individual blocks,

$$\tilde{P} = \begin{bmatrix} p & n-p \\ P_{p} & 0 \\ J_{n-p}P_{12}J_{p} & J_{n-p}P_{11}J_{n-p} \end{bmatrix}$$

where $J_{l}$ are permutation matrices of order $l$ with ones along the anti-diagonal. This ensures $\tilde{P}$ is lower triangular. Now, the above $\tilde{P}$ is equivalent to the structure in (4.10), thus $J_{n-p}P_{11}J_{n-p}$ can be equated to $R_{22}^{-\top}$ and from (4.9), maximizing $\sigma_{\min}(R_{22}^{-\top})$ is equivalent to minimizing $\sigma_{\max}(R_{22})$ which is the original Type -II problem. Thus, from the above it is evident that applying a Type-I algorithm to the rows of the inverse of a matrix $A$ is equivalent to applying Type-II algorithm directly to $A$. 

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Algorithm

Thus, the Stewart’s algorithm could be written as applying QR-CP algorithm to rows of the Inverse, i.e.,

Initialize

\[ s = 0; \ \Pi = I \in \mathbb{R}^{n \times n}; A = QR; R^{(0)} = R, \text{ where } R^{(s)} = \begin{bmatrix} n - s & s \\ s & R_{11}^{(s)} & R_{12}^{(s)} \\ 0 & R_{22}^{(s)} \end{bmatrix}. \]

Repeat \( j = \arg \max_i \left\| R^{(s)-1}(i, :) \right\|_2 \)

if \( j \neq n - s, \)

Interchange columns \( n - s \) and \( j \) of \( \Pi \) and retriangularize \( R^{(s)} \Pi = QR^{(s+1)}. \)

endif

\[ s = s + 1; \]

Until \( s < n - p. \)

Note that \( R_{11}^{(s)} \) and \( R_{22}^{(s)} \) changes in dimension with every increase in \( s \)

The following lemma states the bounds satisfied by \( R_{22} \) got using Stewart-II decomposition.

Lemma 4.3. Using Stewart’s Type-II algorithm ensures the following:

\[
\sigma_{\max}(R_{22}) \leq \sigma_{p+1}(A) \sqrt{n(n-p)2^{n-p}} \quad (4.12)
\]
**Proof.** Using Type-I, QR-CP algorithm (4.11) ensures (ref (4.8)),

\[
\sigma_{\min}(P_{11}) \geq \frac{\sigma_{n-p}(A)}{\sqrt{n - pn2^{n-p}}}. \tag{4.13}
\]

Using Unification principle, \(P_{11} = R_{22}^{-\top}\) also, using (4.9) leads to,

\[
\sigma_{\max}(R_{22}) = \frac{1}{\left(\sigma_{\min}(P_{11})\right)} \tag{4.14}
\]

Thus from (4.13) and (4.14), (4.12) is got. \(\square\)

Given a matrix \(A \in \mathbb{R}^{n \times n}\). Applying Stewart-II algorithm assuming a numerical rank of \(n - 1\) would ensure that, the column with the lowest 2-norm in \(A\) gets permuted with the last column. Note, the weakest column in \(A\) is the column that has the lowest 2-norm or the row with the highest 2-norm in \(A^{-1}\).

**Hybrid Algorithms**

The Hybrid Algorithms are got by combining QR-CP and the Stewart-II algorithm. The first Hybrid algorithm we’ll be dealing with is the Hybrid-I algorithm.

Consider the matrix \(A \in \mathbb{R}^{m \times n}\) with numerical rank \(p\). The QR decomposition of \(A\) be given
by $A = QR$. Now, segment $R$ in two different ways,

$$
R \triangleq \begin{bmatrix}
    p & n - p \\
    m - p & \begin{bmatrix}
        R_{11} & R_{12} \\
        0 & R_{22}
    \end{bmatrix}
\end{bmatrix},
$$

(4.15)

$$
\triangleq \begin{bmatrix}
    p - 1 & n - p + 1 \\
    m - p + 1 & \begin{bmatrix}
        \bar{R}_{11} & \bar{R}_{12} \\
        0 & \bar{R}_{22}
    \end{bmatrix}
\end{bmatrix}.
$$

(4.16)

The Hybrid-I algorithm works by iterating between the QR-CP algorithm applied to $\bar{R}_{22}$ and Stewart-II algorithm applied to $R_{11}$, until no permutations occur. The Stewart-II applied to $R_{11}$ ensures that the $p^{th}$ column in $R$ is the weakest column (the column with the lowest 2-norm) in $R_{11}$. Also QR-CP applied to $\bar{R}_{22}$, ensures that the $p^{th}$ column of $R$ is the best column (column with the highest 2-norm in $\bar{R}_{22}$) in $\bar{R}_{22}$ (4.16).

Hybrid-I satisfies both the QR-CP and Stewart-II’s conditions. Note, QR-CP to $\bar{R}_{22}$ to get the $p^{th}$ column of $R$ ensures that,

$$
r_{pp} \geq \frac{\|\bar{R}_{22}\|_2}{\sqrt{n - p + 1}},
$$

(4.17)

where $r_{pp}$ is the $p^{th}$ diagonal element of $R$ (refer to (4.6)). Since $R_{22}$ (4.15) is a sub matrix to $\bar{R}_{22},$

$$
r_{pp} \geq \frac{\|R_{22}\|_2}{\sqrt{n - p + 1}}.
$$

(4.18)

Also, Stewart-II applied to $\bar{R}_{11}$ (4.15) ensures that

$$
r_{pp} \leq \sigma_{\text{min}}(R_{11}) \sqrt{p}.
$$

(4.19)
combining (4.18) and (4.19) leads to,
\[
\| R_{22} \|_2 \leq \sigma_{\text{min}}(R_{11}) \sqrt{p(n - p + 1)}.
\]

Using Interlacing property, \( \sigma_{\text{max}}(R_{22}) \geq \sigma_{p}(A) \) applying this fact in (4.17) and using (4.19) leads to,
\[
\sigma_{\text{min}}(R_{11}) \geq \frac{\sigma_{p}(A)}{\sqrt{p(n - p + 1)}}.
\]

Thus the required bounds for Hybrid-I are derived. The detailed step-by-step procedure for implementation of Hybrid-I algorithm for \( \tilde{M}, 1.11 \), is summarized in Algorithm (4.2).

While Hybrid-I algorithm uses QR-CP and Stewart-II algorithm to ensure the \( p^{th} \) column is the best in \( \tilde{R}_{22} \) and the worst in \( R_{11} \) respectively, segmenting \( R \) as
\[
R \triangleq \begin{bmatrix}
p + 1 & n - p - 1 \\
p + 1 & 0 & R_{22} \\
m - p - 1 & \end{bmatrix},
\]
\[
\triangleq \begin{bmatrix}
p & n - p \\
p & \tilde{R}_{11} & \tilde{R}_{12} \\
m - p & 0 & \tilde{R}_{22} \\
\end{bmatrix}.
\]

and using QR-CP and Stewart-II to ensure that the \( p + 1^{th} \) column of \( R \) is the best possible in \( \tilde{R}_{22} \) and worst of \( R_{11} \) respectively results in the Hybrid-II decomposition.
Algorithm 4.2 HYBRID-I algorithm of RRQR

1. The Matrix to be decomposed is got as $\tilde{M} \in \mathbb{R}^{K \times Km}$. Let the numerical rank be assumed to be $\hat{p}$.

2. Using the $\Pi$ from Algorithm (4.1), apply QR decomposition to $\tilde{M}\Pi$ as $\tilde{M}\Pi = \tilde{Q}\tilde{R}$ and set $s = 0$;

3. Repeat
   (a) QR-CP Block
      i. Set $\text{permuted} = 0$.
      ii. Let $\tilde{R}^{(s)} \triangleq \begin{bmatrix} \hat{p} - 1 & mK - \hat{p} + 1 \\ 1 & 0 \\ \hat{p} - 1 & \hat{p} \end{bmatrix}$
      iii. Let $e_i$ be the $i^{th}$ column of an identity matrix of order $mK - \hat{p} + 1$.
      iv. Let $j = \arg\max_{1 \leq i \leq mK-\hat{p}+1} \| \tilde{R}^{(s)}_{22} e_i \|_2$
      v. If $j \neq 1$ then
         • $\text{permuted} = 1$
         • Interchange columns $\hat{p} + j - 1$ and $\hat{p}$ in $\Pi$
         • apply QR-GS to $\tilde{M}\Pi$ to get new $\tilde{Q}\tilde{R}$.

   (b) Stewart-II Block
      i. Now divide $\tilde{R}$ as $\tilde{R}^{(s)} \triangleq \begin{bmatrix} \hat{p} & 0 \\ \hat{p} & \hat{p} - 1 \\ 1 & 0 \\ \hat{p} - 1 & \hat{p} \end{bmatrix}$ and Let $e_i$ be the $i^{th}$ column
         of an identity matrix of order $\hat{p}$.
      ii. Let $j = \arg\max_{1 \leq i \leq \hat{p}} \| e_i^\top \tilde{R}^{(s)}_{11} \|$
      iii. If $j \neq \hat{p}$ then
         • $\text{permuted} = 1$
         • Exchange columns $j$ and $\hat{p}$ in $\Pi$
         • apply QR-GS to $\tilde{M}\Pi$ to get new $\tilde{Q}^{(s)}\tilde{R}^{(s)}$.
         • $s = s + 1$

4. Until $\text{permuted} \neq 0$

5. Now RRQR decomposition is given by taking GS-QR of $\tilde{M}\Pi$ as, $\tilde{M}\Pi = \tilde{Q}\tilde{R}$
The Hybrid-II equivalent of (4.17) is,

\[ r_{p+1,p+1} \geq \sigma_{\text{max}} \left( \bar{R}_{22} \right) \frac{\sigma_{\text{max}} \left( \bar{R}_{22} \right)}{\sqrt{n-p}}, \]

where \( r_{p+1,p+1} \) is the \( p + 1 \)th diagonal value of \( R \). Also, the equivalent of (4.19) is given by

\[ r_{p+1,p+1} \leq \sigma_{\text{min}} (R_{11}) \sqrt{p+1}. \]  \( (4.22) \)

Combining the above two inequalities results in

\[ \sigma_{\text{min}} (R_{11}) \geq \frac{\sigma_{\text{max}} \left( \bar{R}_{22} \right)}{\sqrt{(p+1)(n-p)}}. \]

Due to Interlacing property, \( (\sigma_{\text{min}} \left( \bar{R}_{11} \right) \geq \sigma_{\text{min}} (R_{11})) \),

\[ \sigma_{\text{min}} (R_{11}) \geq \frac{\sigma_{\text{max}} \left( \bar{R}_{22} \right)}{\sqrt{(p+1)(n-p)}}. \]

Also, using Interlacing Property and the fact that \( \sigma_i (A) = \sigma_i (R) \),

\[ \sigma_{\text{min}} (R_{11}) \leq \sigma_{k+1} (A) \]

Thus,

\[ \sigma_{\text{max}} \left( \bar{R}_{22} \right) \leq \sigma_{k+1} (A) \sqrt{(p+1)(n-p)}. \]

Which completes the bounds for Hybrid-II algorithm. Thus, it can be seen that applying Hybrid-II algorithm with the assumption of numerical rank \( p \) is equivalent to applying Hybrid-I algorithm assuming a numerical rank of \( p + 1 \).
The Hybrid-III algorithm is as follows:

Apply Hybrid-I algorithm on the matrix $A$ followed by the application of Hybrid-II. Repeat this procedure until no permutations occur.

Thus, when Hybrid-III halts, it satisfies the bounds for both Hybrid-I and Hybrid-II.
Chapter 5

Model order: A Numerical Rank Perspective

In this chapter, the determination of model order (i.e., the number of factors \( p \)) is framed as the problem of estimating the numerical rank of the matrix \( \tilde{M} \) (1.11). Recall that the matrix \( \tilde{M} \) could be considered as the perturbed version of the matrix \( M \) given by:

\[
\tilde{M} = M + \Delta M.
\]

While \( M \) defined by (1.4) is of rank \( p \), \( \tilde{M} \) is of full row rank \( K \) with probability one. When \( M \) is known, determining the model order is just to equate it to the rank of the matrix \( M \). Since in practice only \( \tilde{M} \) is available, we resort to the estimation of the numerical rank of \( \tilde{M} \). Consider the following definition taken from [13]:

**Definition 5.1.** [13]: A matrix \( A \) has numerical rank \((\mu, \varepsilon, p)\) with respect to norm \( \| . \| \) if \( \mu, \varepsilon, p \) satisfy the following two conditions:

1. \( p = \inf \{ rank(B) : \| A - B \| \leq \varepsilon \} \)
2. \( \varepsilon < \mu \leq \sup \{ \eta : \| A - B \| \leq \eta \Rightarrow rank(B) \geq p \} \)

For ease of notation, \((\mu, \varepsilon, p)_p\) is used to denote the numerical rank w.r.t norm \( \| . \|_p \). The following example is used to illustrate the above definition:
**Example 5.1.** Consider the matrix $A \triangleq \begin{bmatrix} 18 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$. Let $\varepsilon, \mu$ be bounded by,

$$0.8 < \varepsilon < \mu < 5$$

with the above bounds for $\varepsilon, \mu$, the numerical rank of $A$ is $p = 2$ w.r.t $\| \cdot \|_2$.

It can be explained as follows: Consider the matrix,

$$B = \begin{bmatrix} 18 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note, $\inf \{ \text{rank}(B) \} = 1 \implies \varepsilon = 0$. Which in turn implies $\|A - B\|_2 = 5$. This means, $\varepsilon \geq 5$ if $B$ is to take rank of 1 and since it is outside the bound, (5.1), $\inf \{ \text{rank}(B) \} = 2 \forall \varepsilon < 5$. Now, consider the matrix

$$B = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

Note, $\inf \{ \text{rank}(B) \} = 3 \implies \varepsilon > 0$ which in turn implies $\|A - B\|_2$ should not be greater than 0.8. But the bound, (5.1), ensures that $\varepsilon > 0.8$. Thus, numerical rank cannot be 3 implying under the selected bound, (5.1), numerical rank of $A$ is 2.

Note, the above example can be extended to any matrix, $A$, as stated in the following theorem:
Theorem. [13] Let $\sigma_1 > \sigma_2 > \ldots > \sigma_K$ denote the singular values of the $K \times Km$ matrix $\tilde{M}$.

Then, $\tilde{M}$ is of numerical rank $(\mu, \varepsilon, p)$ w.r.t $\|\cdot\|_2$ if

$$\sigma_{p+1}(\tilde{M}) \leq \varepsilon < \mu \leq \sigma_p(\tilde{M})$$

(5.2)

Proof. Refer [13] for a proof.

Now, Consider a matrix $A$ perturbed by another matrix $\Delta A$. The following theorem shows how this disturbance of $\Delta A$ will affect the numerical rank of the matrix $\tilde{A} = A + \Delta A$.

Theorem. Let $\varepsilon, \mu$ be bounded by:

$$\sigma_{p+1}(A) + \sigma_{\max}(\Delta A) \leq \varepsilon < \mu \leq \sigma_p(A) - \sigma_{\max}(\Delta A).$$

Under this bound, both $A$ and $A + \Delta A$ is of same numerical rank $(\mu, \varepsilon, p)_2$.

Proof. Assume

$$\sigma_{p+1}(A) + \sigma_{\max}(\Delta A) \leq \varepsilon < \mu \leq \sigma_p(A) - \sigma_{\max}(\Delta A)$$

(5.3)

holds true. Since,

$$\sigma(A + \Delta A) < \sigma_{p+1}(A) + \sigma_{\max}(\Delta A)$$

and

$$\sigma_p(A + \Delta A) > \sigma_p(A) - \sigma_{\max}(\Delta A)$$

holds true,

$$\sigma_{p+1}(A + \Delta A) \leq \varepsilon < \mu \leq \sigma_p(A + \Delta A)$$

(5.4)
is satisfied which implies that \( A + \Delta A \) is of numerical rank \((\mu, \varepsilon, p)\). Also, since \( \sigma_{p+1}(A) + \sigma_{\text{max}}(\Delta A) \geq \sigma_{p+1}(A) \) and \( \sigma_p(A) - \sigma_{\text{max}}(\Delta A) \leq \sigma_p(A) \), (5.3) implies

\[
\sigma_{p+1}(A) \leq \varepsilon < \mu \leq \sigma_p(A)
\]

which in turn implies \( A \) is also of numerical rank \((\mu, \varepsilon, p)\).

The above theorem gives an idea of bounds on \( \mu, \varepsilon \) based on the amount of perturbation \( \Delta A \) such that the numerical rank remains constant.

In the current scenario, we have the matrix \( M \) of rank \( p \) perturbed by a matrix \( \Delta M \) of full rank. Thus for \( \tilde{M} = M + \Delta M \) to have numerical rank \((\mu, \varepsilon, p)\),

\[
\sigma_{\text{max}}(\Delta M) \leq \varepsilon < \mu \leq \sigma_p(M) - \sigma_{\text{max}}(\Delta M)
\]

In our scenario, \( \sigma_{\text{max}}(\Delta M) \to 0 \) when \( N \to \infty \) because of assumption (A5) hence \( \sigma_{\text{max}}(\Delta M) \ll \sigma_p(M) \) as \( N \to \infty \). Hence the separation of \( \sigma_{p+1}(\tilde{M}) \) and \( \sigma_p(\tilde{M}) \) grows with increase in \( N \) which enabled the ratio based estimate [21], given by

\[
\hat{p} = \arg \max_{1 \leq i \leq R} \frac{\sigma_i(\tilde{M})}{\sigma_{i+1}(\tilde{M})}, \quad \text{(5.5)}
\]

to work.

The above determination of numerical rank was with the help of using Singular Value Decomposition (SVD), for asymptotic properties of (5.5), refer to [21]. Now, to analyze how to determine the numerical rank using RRQR decomposition, consider the following theorem proposed by
Theorem. Let

\[
\tilde{M}\Pi = \begin{bmatrix}
\tilde{Q}_\rho & \tilde{Q}_{K-\rho} \\
\tilde{R}^{(\rho)}_{11} & \tilde{R}_{12} \\
0 & \tilde{R}^{(K-\rho)}_{22}
\end{bmatrix}
\]  

(5.6)

be the QR decomposition of \(\tilde{M}\Pi\). If there exists, \(\hat{\mu}, \hat{\epsilon} > 0\) such that

\[
\sigma_{\min}\left(\tilde{R}^{(\hat{\rho})}_{11}\right) = \hat{\mu} > \hat{\epsilon} = \left\|\tilde{R}^{(K-\hat{\rho})}_{22}\right\|_2, \quad \text{ (5.7)}
\]

then, \(\tilde{M}\) has numerical rank \((\hat{\mu}, \hat{\epsilon}, \hat{\rho})\).

Proof. Refer to [13] for a proof. \(\square\)

As the sample correlations are assumed to converge to their respective expected values with increasing sample size \(N\) (for a fixed \(K\)), (5.7) must hold with a high probability for large values of \(N\) at \(\hat{\rho} = p\). This is proved in the following theorem.

Theorem 5.1. Assume that

\[
M \triangleq \begin{bmatrix}
\Sigma_{yy}(1) & \Sigma_{yy}(2) & \ldots & \Sigma_{yy}(m)
\end{bmatrix}
\]  

(5.8)

is of rank \('p'\) where \(\Sigma_{yy}(l)\) denotes the autocovariance matrix of the observations \(y_n\). Let \(\Pi\) be the...
permutation matrix obtained using HYBRID-III and let $\tilde{M}\Pi$ be decomposed as in

$$
\tilde{M}\Pi = \begin{bmatrix}
\tilde{M}_p & \tilde{M}_{mK-p}
\end{bmatrix}
= \begin{bmatrix}
\tilde{Q}_p & \tilde{Q}_{K-p}
\end{bmatrix}
\begin{bmatrix}
\tilde{R}_{11} & \tilde{R}_{12} \\
0 & \tilde{R}_{22}
\end{bmatrix}
= \tilde{Q}\tilde{R},
$$

(5.9)

It can be shown that under the assumptions (A1-A7), refer Chapter 3 regarding assumptions, with $K, N \rightarrow \infty$ under the constraint\(^1\)

$$
\frac{K^{1+\delta}}{\sqrt{N}} = o(1),
$$

(5.10)

it holds that,

$$
o_{p}\left(\sigma_{\min}\left(\tilde{R}^{(p)}_{11}\right)\right) = \left\|\tilde{R}^{(p)}_{22}\right\|_2.
$$

(5.11)

Proof. The following three lemmas are required to prove the above theorem:

\[ \square \]

**Lemma 5.1.** Let $\gamma_i$ and $\gamma_{i+1}$ be the $i^{th}$ and $(i+1)^{th}$ diagonal value of $\tilde{R}$, (5.9), got by applying HYBRID-III algorithm with assumption of numerical rank to be $i$. The Hybrid - III algorithm ensures that $\gamma_i$ satisfies the following property:

$$
\sigma_i\left(\tilde{M}\right)\sqrt{(i+1)(mK-i)} \geq \gamma_i\left(\tilde{R}\right) \geq \frac{\sigma_i\left(\tilde{M}\right)}{\sqrt{mK-i}},
$$

(5.12)

where $\gamma_i$ is the $i^{th}$ diagonal element of $\tilde{R}$ got by applying HYBRID-III algorithm assuming a numerical rank of $i$ and $\sigma_i\left(\tilde{M}\right)$ is the $i^{th}$ highest singular value of $\tilde{M}$.

**Proof.** Let $\gamma_{s+1}$ be the $s + 1^{th}$ diagonal value of $\tilde{R}$ in, (5.9), where $\Pi$ is got using HYBRID-III

\[ \text{Note, (5.10) implies that } \sqrt{N} \text{ grows faster than } K^{1+\delta}. \]
algorithm assuming numerical rank \( s \). Segment \( \tilde{R} \) as,

\[
\tilde{R} \triangleq \begin{bmatrix} s & mK - s \\ s & \tilde{R}_{11}^{(s)} & \tilde{R}_{12}^{(s)} \\ K - s & 0 & \tilde{R}_{22}^{(s)} \end{bmatrix}.
\]

Let \( \alpha_i \) be the 2-norm of the \( i \)th column of \( \tilde{R}_{22}^{(s)} \), i.e.,

\[
\alpha_i = \| \tilde{R}_{22}^{(s)}(:, i) \|_2.
\]

The column pivoting part of HYBRID-III algorithm ensures \( \gamma_{s+1} \) to be,

\[
\gamma_{s+1} = \max_{1 \leq i \leq mK - s} \alpha_i.
\]

Note that,

\[
\| \tilde{R}_{22}^{(s)} \|_2 \geq \gamma_{s+1} \geq \| \tilde{R}_{22}^{(s)} \|_F / \sqrt{mK - s}.
\] (5.13)

Since \( \| \tilde{R}_{22}^{(s)} \|_F \geq \| \tilde{R}_{22}^{(s)} \|_2 \),

\[
\| \tilde{R}_{22}^{(s)} \|_2 \geq \gamma_{s+1} \geq \| \tilde{R}_{22}^{(s)} \|_2 / \sqrt{mK - s}.
\] (5.14)

Interlacing property of Singular Values [14, 25] leads to

\[
\sigma_s \left( \tilde{R} \right) \geq \| \tilde{R}_{22}^{(s)} \|_2 \geq \sigma_{s+1}(\tilde{R});
\]
and the *Hybrid - III* algorithm [7] guarantees,
\[
\|R_{22}\|_2 \leq \sigma_{s+1}(\tilde{R}) \sqrt{(s + 1)(mK - s)}.
\]

Thus,
\[
\sigma_{s+1}(\tilde{R}) \sqrt{(s + 1)(mK - s)} \geq \|\tilde{R}^{(s)}_{22}\|_2 \geq \sigma_{s+1}(\tilde{R}). \tag{5.15}
\]

Since, \(\sigma_i(M) = \sigma_i(\tilde{R})\), substituting (5.15) in (5.14) we get (5.12).

**Lemma 5.2.** The following rates of convergence hold as \(K, N \to \infty\) under the assumptions (A1-A3):
\[
\|\Sigma_{ff}(l)\|_F \leq \|H\|_F^2 \|\Sigma_{xx}(l)\|_F = O(K^{1-\delta}) = \|\Sigma_{ff}(l)\|_2 \asymp \sigma_{\text{min}}(\Sigma_{ff}(l)) \tag{5.16}
\]

**Proof.** Note
\[
y_n = Hx_n + \varepsilon_n
\]
\[
= Qf_n + \varepsilon_n,
\]
where, \(H = QR\) is the QR decomposition of \(H\) and \(f_n = Rx_n\). Therefore,
\[
\begin{bmatrix}
h_1 \\
h_2 \\
\vdots \\
h_p
\end{bmatrix}
= \begin{bmatrix}
Q_1 & Q_2
\end{bmatrix}
\begin{bmatrix}
R_1 \\
0
\end{bmatrix},
\]
where

$$Q_1 = \begin{bmatrix} q_1 & \cdots & q_p \end{bmatrix}$$

and

$$R_1 = \begin{bmatrix} <h_1, q_1> & <h_2, q_1> & \cdots & <h_p, q_1> \\ 0 & <h_2, q_2> & \cdots & <h_p, q_2> \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & <h_p, q_p> \end{bmatrix}.$$  

Here $<a, b>$ represents the inner product of $a$ and $b$. Thus the $i^{th}$ diagonal element of $R$ given by

$$|<h_i, q_i>| = \left\| h_i - \sum_{k=1}^{i-1} <h_i, q_k> q_k \right\|_2$$  \hspace{1cm} (5.17)

has a maximum value $|<h_i, q_i>|$ can take is $\|h_i\|_2 = O \left( K^{1-\delta} \right)$. Since in equation (5.17) each $q_k$ could be expressed as a linear combination of $h_1, \ldots, h_k$, the minimum value of (5.17), got under the assumption (2.1), is $|<h_i, q_i>|_{min} = O \left( K^{1-\delta} \right)$. Hence, the diagonal elements of $R$ are of the order, $O \left( K^{1-\delta} \right)$. And since the off-diagonal elements are of lesser or same order, it can be inferred that

$$\|R\|_F = \sqrt{p} O \left( K^{1-\delta} \right) = O \left( K^{1-\delta} \right).$$

Since $\Sigma_{xx}(l)$ is a constant matrix independent of $K$, $(A1)$,

$$\|\Sigma_{xx}(l)\|_F = O(1).$$

As

$$\Sigma_f(k) = R \Sigma_{xx}(k) R^T, \forall k = 1, 2, \ldots,$$
\[ \| \Sigma_f(k) \|_F \leq \| R \|_F \| \Sigma_x(k) \|_F \| R^T \|_F = O \left( K^{1-\delta} \right). \] (5.18)

The proof for
\[ \| \Sigma_{ff}(l) \|_2 \approx \sigma_{\min}( \Sigma_{ff}(l) ) \approx O \left( K^{1-\delta} \right) \] (5.19)

is given in [22]. Thus, from (5.18) and (5.19), (5.16) is got.

**Lemma 5.3.** The following rate of convergences hold under the assumptions (A1-A7):

\[ \| \Delta \Sigma_{fe}(l) \|_2 \approx_P \| \Delta \Sigma_{fe}(l) \|_F = O_P \left( K^{1-\frac{\delta}{2}} N^{-\frac{1}{2}} \right) \] (5.20)
\[ \| \Delta \Sigma_{ff}(l) \|_2 \approx_P \| \Delta \Sigma_{ff}(l) \|_F = O_P \left( K^{1-\delta} N^{-\frac{1}{2}} \right) \] (5.21)
\[ \| \Delta \Sigma_{ef}(l) \|_2 \approx_P \| \Delta \Sigma_{ef}(l) \|_F = O_P \left( K^{1-\frac{\delta}{2}} N^{-\frac{1}{2}} \right) \] (5.22)
\[ \| \Delta \Sigma_{ee}(l) \|_2 \approx_P \| \Delta \Sigma_{ee}(l) \|_F = O_P \left( KN^{-\frac{1}{2}} \right) \] (5.23)
\[ \| F \|_2^2 = O_P \left( K^{1-\delta} N \right) \] (5.24)

where \( F \) is as in (6.21) and \( X \triangleq \left[ x_1 \ x_2 \ \ldots \ x_N \right] \).

**Proof.** \( \tilde{\Sigma}_{fe}(l) = R \tilde{\Sigma}_{xe}(l) \) and \( \Sigma_{fe}(l) = R \Sigma_{xe}(l) \). Hence,

\[ \| \Delta \Sigma_{fe}(l) \|_F = \| R [ \Delta \Sigma_{xe}(l) ] \|_F \]
\[ \leq \| R \|_F \| \Delta \Sigma_{xe}(l) \|_F. \] (5.25)

Each element in \( \left[ \tilde{\Sigma}_{xe}(l) - \Sigma_{xe}(l) \right] \) converges at the rate of \( O_P \left( N^{-\frac{1}{2}} \right) \) According to assumption
$A5$ in Chapter (2), and there are $p_lK$ elements. Therefore,

$$\|\Delta \Sigma_{xx}(l)\|_F = O_P \left( \sqrt{\frac{p_lK}{N}} \right).$$  \hfill (5.26)

Using (5.26) and the fact that $\|R\|_F = O \left( K^{1-\delta} \right)$, along with (5.2) and substituting in (5.25) leads to (5.20). A similar derivation yields, (5.22).

Note that

$$[\Delta \Sigma_{ff}(l)] = R [\Delta \Sigma_{xx}(l)] R^T.$$  \hfill (5.27)

Since, every element in $[\Delta \Sigma_{xx}(l) - \Sigma_{xx}(l)]$ converges at $O_P \left( N^{-\frac{1}{2}} \right)$ and there are in total $p_l^2$ elements

$$\|\Delta \Sigma_{xx}(l)\|_F = O_P \left( p_lN^{-\frac{1}{2}} \right).$$

Therefore,

$$\|\Delta \Sigma_{ff}(l)\|_F \leq \|R\|_F \|\Delta \Sigma_{xx}(l)\|_F \|R\|_F$$

$$= O \left( K^{1-\delta} \right) O_P \left( p_lN^{-\frac{1}{2}} \right) O \left( K^{1-\delta} \right)$$

$$= O_P \left( K^{1-\delta} N^{-\frac{1}{2}} \right).$$

Now we prove, (5.23). As, every element in the matrix $[\Delta \Sigma_{ee}(l) - \Sigma_{ee}(l)]$ converges as $O_P \left( N^{-\frac{1}{2}} \right)$ (Note condition $A5$ ) and there are a total of $K^2$ elements, (5.23) is obtained. Finally, to prove (5.24), Note $\|F\|_2 \leq \|R\|_F \|X\|_2 = O_P \left( K^{1-\delta} N^{\frac{1}{2}} \right)$. Squaring which, (5.24), is obtained.

---

\footnote{Refer Appendix 2 (mixing properties) for details}
The convergence rates of \( \| \Delta \Sigma_{f\varepsilon}(l) \|_2, \| \Delta \Sigma_{ff}(l) \|_2, \| \Delta \Sigma_{\varepsilon f}(l) \|_2 \) and \( \| \Delta \Sigma_{\varepsilon\varepsilon}(l) \|_2 \) are proved in [22]. It is found that these rates are same as that obtained for Frobenius norm derived above. \( \square \)

**Lemma 5.4.** Consider the matrices \( M \) as in (5.8). Let \( \tilde{M} \) and \( \Delta M \) be given by:

\[
\tilde{M} \triangleq \begin{bmatrix}
\tilde{\Sigma}_{yy}(1) & \tilde{\Sigma}_{yy}(2) & \cdots & \tilde{\Sigma}_{yy}(m)
\end{bmatrix}
\]  

and

\[
\Delta M \triangleq \tilde{M} - M = \begin{bmatrix}
\Delta \Sigma_{yy}(1) & \Delta \Sigma_{yy}(2) & \cdots & \Delta \Sigma_{yy}(m)
\end{bmatrix}
\]  

The following results hold under the assumptions (A1-A7) as \( K, N \to \infty \):

\[
\sigma_p(M) \asymp \begin{cases}
K^{1-\delta}, & \text{if } \kappa_{\text{max}} = o(K^{1-\delta}) \\
\kappa_{\text{min}}, & \text{if } o(\kappa_{\text{min}}) = K^{1-\delta}
\end{cases}
\]  

(5.30)

\[
\sigma_1(M) \asymp \max(\kappa_{\text{max}}, K^{1-\delta}).
\]  

(5.31)

\[
\| \Delta M \|_2 \asymp_P \| \Delta M \|_F = O_P \left( KN^{-\frac{1}{2}} \right)
\]  

(5.32)

\[
\sigma_p(\tilde{M}) \asymp_P \begin{cases}
K^{1-\delta}, & \text{if } \kappa_{\text{max}} = o(K^{1-\delta}), \forall K^{\delta} \sqrt{N} = o(1)
\\
\kappa_{\text{min}}, & \text{if } o(\kappa_{\text{min}}) = K^{1-\delta}
\end{cases}
\]

(5.33)

\[
\sigma_1(\tilde{M}) \asymp_P \max(\kappa_{\text{max}}, K^{1-\delta}), \forall \frac{K^{\delta}}{\sqrt{N}} = o(1).
\]  

(5.34)

**Proof.** \( \sigma_p(M) \) is given by\(^3\)

\(^3\)Note: The following inequality holds true: \( \sigma_n(A) - \sigma_{\text{max}}(B) \leq \sigma_n(A+B) \leq \sigma_n(A) + \sigma_{\text{max}}(B) \) (refer pp
\[ \sigma_p(M) = \sigma_p \left[ \Sigma_{yy}(1) \ldots \Sigma_{yy}(m) \right] \]
\[ \geq \sigma_p \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right] - \sigma_{\text{max}} \left[ \Sigma_{f\varepsilon}(1) \ldots \Sigma_{f\varepsilon}(m) \right] \]

(or)

\[ \sigma_p(M) \geq \sigma_p \left[ \Sigma_{f\varepsilon}(1) \ldots \Sigma_{f\varepsilon}(m) \right] - \sigma_{\text{max}} \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right] \]

Also,

\[ \sigma_p(M) \leq \sigma_p \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right] + \sigma_{\text{max}} \left[ \Sigma_{f\varepsilon}(1) \ldots \Sigma_{f\varepsilon}(m) \right] \]

(or)

\[ \sigma_p(M) \leq \sigma_p \left[ \Sigma_{f\varepsilon}(1) \ldots \Sigma_{f\varepsilon}(m) \right] + \sigma_{\text{max}} \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right] \]

If \( \kappa_{\text{max}} = o(K^{1-\delta}) \), we get,

\[ O \left( \sigma_p \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right] \right) \geq \sigma_p(M) \geq O \left( \sigma_p \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right] \right) \]
\[ \sigma_p(M) \approx K^{1-\delta}. \] (5.37)

If \( K^{1-\delta} = o(\kappa_{\text{min}}) \),

\[ O \left( \sigma_p \left[ \Sigma_{f\varepsilon}(1) \ldots \Sigma_{f\varepsilon}(m) \right] \right) \geq \sigma_p(M) \geq O \left( \sigma_p \left[ \Sigma_{f\varepsilon}(1) \ldots \Sigma_{f\varepsilon}(m) \right] \right) \]
\[ \sigma_p(M) \approx \kappa_{\text{min}}. \] (5.38)
Thus, from (5.37) and (5.38) we get (5.30). Also,

\[
\begin{align*}
\sigma_1(M) &= \sigma_1 \left[ \Sigma_{yy}(1) \ldots \Sigma_{yy}(m) \right] \\
&\geq \sigma_1 \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right] - \sigma_{\max} \left[ \Sigma_{f\epsilon}(1) \ldots \Sigma_{f\epsilon}(m) \right] \quad (5.39)
\end{align*}
\]

(or)

\[
\sigma_1(M) \geq \sigma_1 \left[ \Sigma_{f\epsilon}(1) \ldots \Sigma_{f\epsilon}(m) \right] - \sigma_1 \left[ \Sigma_{ff}(1) \ldots \Sigma_{ff}(m) \right]
\]

and following the same arguments as above, we get

\[
\sigma_1(M) \asymp \max \left( \kappa_{\max}, K^{1-\delta} \right).
\]

Now, Frobenius norm of \( \Delta M \) is given by,

\[
\|
\begin{align*}
\Delta M \|_F &= \left\| \Delta \Sigma_{yy}(1) \ldots \Delta \Sigma_{yy}(m) \right\|_F \\
&\leq \| \Delta \Sigma_{yy}(1) \|_F + \ldots + \| \Delta \Sigma_{yy}(m) \|_F \\
\end{align*}
\]

(5.40)

where

\[
\begin{align*}
\| \Delta \Sigma_{yy}(l) \|_F &\leq \sqrt{p} \| \Delta \Sigma_{f\epsilon}(l) \|_F + p \| \Delta \Sigma_{ff}(l) \|_F + \sqrt{p} \| \Delta \Sigma_{\epsilon f}(l) \|_F + \| \Delta \Sigma_{\epsilon \epsilon}(l) \|_F \\
&= O_P \left( K^{1-\frac{\delta}{2}} N^{-\frac{1}{2}} \right) + O_P \left( K^{1-\delta} N^{-\frac{3}{2}} \right) \\
&\ldots + O_P \left( K^{1-\frac{\delta}{2}} N^{-\frac{1}{2}} \right) + O_P \left( KN^{-\frac{1}{2}} \right) \\
&= O_P \left( KN^{-\frac{1}{2}} \right).
\end{align*}
\] (5.41)
Thus, substituting (5.41) in (5.40),

\[ \| \Delta M \|_F = O_P \left( KN^{-\frac{1}{2}} \right). \]  

(5.42)

Note that \( \| \Delta M \|_2 \) also has the same convergence rate as shown in [22]. Now,

\[ \sigma_p(M) + \sigma_{\text{max}}(\Delta M) \geq \sigma_p(\tilde{M}) \geq \sigma_{p}(M) - \sigma_{\text{max}}(\Delta M) \]  

(5.43)

and

\[ \sigma_1(M) - \sigma_{\text{max}}(\Delta M) \geq \sigma_1(\tilde{M}) \geq \sigma_1(M) - \sigma_{\text{max}}(\Delta M) \]  

(5.44)

upon substituting (5.30), (5.31) and (5.32) in (5.43) and (5.44). (5.33) and (5.34) is got. \( \square \)

Now proceeding to prove the Theorem:

**Proof.** When using HYBRID-III on \( \tilde{M} \), (5.28), to get a decomposition as given in (5.9), the following inequalities are satisfied:

\[ \sigma_{\text{min}}(\tilde{R}_{11}^{(p)}) \geq \frac{\sigma_p(\tilde{M})}{\sqrt{p(K - p + 1)}} \]  

(5.45)

and

\[ \sigma_{\text{max}}(\tilde{R}_{22}^{(p)}) \leq \sigma_{p+1}(\tilde{M}) \sqrt{(p + 1)(K - p)}. \]  

(5.46)

Refer to Section 11 in [7] for a justification of these inequalities. Note HYBRID-III is applied on \( \tilde{M} \) assuming a numerical rank \( p \). From (5.33) in lemma 5.4, under the constraint \( \frac{K^5}{\sqrt{N}} = o(1) \),

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\[
\sigma_{\min}\left(R_{11}^{(p)}\right) \geq \begin{cases} 
O_P\left(\frac{K_1^{1-\delta}}{p(K-p+1)}\right), & \text{if } \kappa_{\max} = o\left(K^{1-\delta}\right) \\
O_P\left(\frac{\kappa_{\min}}{\sqrt{p(K-p+1)}}\right), & \text{if } o\left(\kappa_{\min}\right) = K^{1-\delta} 
\end{cases} 
\] (5.47)

\[
\geq \begin{cases} 
O_P\left(K^{0.5-\delta}\right), & \kappa_{\max} = o\left(K^{1-\delta}\right) \\
O_P\left(\frac{\kappa_{\min}}{\sqrt{K}}\right), & o\left(\kappa_{\min}\right) = K^{1-\delta} 
\end{cases} 
\] (5.48)

Equation (5.46) and

\[
\sigma_{p+1}\left(\hat{M}\right) \leq \sigma_{p+1}\left(M\right) + \|\Delta M\|_2, \\
\leq \|\Delta M\|_2 \quad \text{as } \sigma_{p+1}\left(M\right) = 0
\]

implies

\[
\left\|\phi_{22}^{(p)}\right\|_2 \leq \|\Delta M\|_2 O\left(\sqrt{K}\right)
\]

Using (5.32),

\[
\left\|\phi_{22}^{(p)}\right\|_2 = O_P\left(K^{3/2}N^{-1/2}\right) 
\] (5.49)

Hence from (5.48) and (5.49), under the constraint

\[
\frac{K^{1+\delta}}{\sqrt{N}} = o(1), \\
o_P\left(\sigma_{\min}\left(\hat{R}_{11}^{(p)}\right)\right) = \left\|\phi_{22}\right\|_2
\]
Note for the matrix $M$, $\sigma_{\min} \left( R_{11}^{(p)} \right) > \left\| R_{22}^{(K-p)} \right\|_2 = 0$, implying that $p$ is a valid numerical rank. Nevertheless, it is quite possible that $\sigma_{\min} \left( R_{11}^{(p-1)} \right) > \left\| R_{22}^{(K-p-1)} \right\|_2$, implying the non-uniqueness of a numerical rank. In the case of $M$ determining $p$ is simple as the last $K - p$ rows of the $R$ matrix are zeros. In the case of $\tilde{M}$, an estimate $\hat{p}$ that maximizes the ratio

$$r_i = \frac{\gamma_i + \epsilon}{\gamma_{i+1} + \epsilon}, \quad i = 1, 2, \ldots, K$$

(5.50)

where $\epsilon = \frac{\gamma_1}{\sqrt{KN}}$ is as mentioned in (1.13) is used. The following theorem shows that for large values of $N$, $\hat{p}$ is equal to $p$ with a very high probability.

**Theorem 5.2.** Let $M$ as in (1.4) be of rank $'p'$ and let QR decomposition of $\tilde{M}\Pi$ be as in (1.12) where $\Pi$ is the permutation matrix got by HYBRID-III decomposition. Define $\gamma_i$ as the $i^{th}$ diagonal value of $\tilde{R}$ in (1.12). Under the assumptions (A1-A7), refer Chapter 3 regarding assumptions, the following properties hold true as $K, N \to \infty$ under the constraint $\frac{K^\delta}{\sqrt{N}} = o(1)$ regarding the ratio (5.50).

Let $\text{UB} \left( r_i \right)$ and $\text{LB} \left( r_i \right)$ denote the upper and lower bounds for rate of growth of $r_i$ as $K, N \to \infty$. 

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Case 1. for $i = 1, p \neq 1$,

$$UB(r_1) = \begin{cases} 
O_P\left(\sqrt{K}\right), & \kappa_{\text{max}} = o(K^{1-\delta}) \\
O_P\left(\frac{\kappa_{\text{max}}}{\kappa_{\text{min}}}\sqrt{K}\right), & o(\kappa_{\text{min}}) = K^{1-\delta}
\end{cases}$$

$$LB(r_1) = \begin{cases} 
O\left(\frac{1}{\sqrt{K}}\right), & \kappa_{\text{max}} = o(K^{1-\delta}) \\
O\left(\frac{\kappa_{\text{max}}}{\kappa_{\text{min}}\sqrt{K}}\right), & o(\kappa_{\text{min}}) = K^{1-\delta}
\end{cases}$$

(5.51)

Case 2. for $2 < i < p$,

$$UB\left(\{r_i\}_{i=2}^{p-1}\right) = O_P(K) ,$$

$$LB\left(\{r_i\}_{i=2}^{p-1}\right) = O_P(K^{-1}) ,$$

(5.52)

Case 3. for $i = p$,

$$UB(r_p) = O_P\left(NK^2\right)$$

$$LB(r_p) = \begin{cases} 
O_P\left(K^{-(1+\delta)}N^{\frac{1}{2}}\right), & \kappa_{\text{max}} = o(K^{1-\delta}) \\
O_P\left(\kappa_{\text{min}}N^{\frac{1}{2}}K^{-2}\right), & o(\kappa_{\text{min}}) = K^{1-\delta}
\end{cases}$$

(5.53)
**Case 4.** for \( i > p \),

\[
UB\left(\{r_i\}_{i>p}\right) = \begin{cases} 
O_P\left(K^{\frac{3}{2}+\delta}\right), & \kappa_{max} = o(K^{1-\delta}) \\
O_P\left(K_{min}\right), & o(\kappa_{min}) = K^{1-\delta}
\end{cases}
\]

\[
LB\left(\{r_i\}_{i>p}\right) = \begin{cases} 
O_P\left(K^{-(\frac{3}{2}+\delta)}\right), & \kappa_{max} = o(K^{1-\delta}) \\
O_P\left(K_{min}K^{-\frac{3}{2}}\right), & o(\kappa_{min}) = K^{1-\delta}
\end{cases}
\]

(5.54)

**Proof.** The rate of convergences for \( \gamma_i + \epsilon \) for different \( i \) is derived under the constraint \( \frac{K^\delta}{\sqrt{N}} = o(1) \) as follows:

For \( i = 1 \),

\[
\frac{\sigma_1\left(\tilde{M}\right)}{\sqrt{mK}} \leq \gamma_1 \leq \sigma_1\left(\tilde{M}\right)
\]

Thus from (5.34),

\[
UB\left(\gamma_1\right) = O_P\left(\max\left(\kappa_{max}, K^{1-\delta}\right)\right)
\]

\[
LB\left(\gamma_2\right) = O_P\left(\frac{O_P\left(\max\left(\kappa_{max}, K^{1-\delta}\right)\right)}{O\left(\sqrt{K}\right)}\right)
\]

Since, \( \epsilon = \frac{\gamma_1}{\sqrt{KN}} \),

\[
UB\left(\epsilon\right) = \frac{O_P\left(\max\left(\kappa_{max}, K^{1-\delta}\right)\right)}{\sqrt{KN}}
\]

\[
LB\left(\epsilon\right) = \frac{O_P\left(\max\left(\kappa_{max}, K^{1-\delta}\right)\right)}{O\left(\sqrt{KN}\right)}
\]
Thus,

\[
\begin{align*}
UB(\gamma_1 + \epsilon) &= O_P \left( \max(\kappa_{max}, K^{1-\delta}) \right) \\
LB(\gamma_1 + \epsilon) &= O_P \left( \max(\kappa_{max}, K^{1-\delta}) \right) / O \left( \sqrt{K} \right)
\end{align*}
\] (5.55)

For \( p > i > 2 \), from (5.12) and (5.33),

\[
UB \left( \{ \gamma_i + \epsilon \}_{i=2}^p \right) = \begin{cases} 
O_P \left( K^{1.5-\delta} \right), & \forall \kappa_{max} = o \left( K^{1-\delta} \right) \\
O_P \left( \kappa_{min} \sqrt{K} \right), & \forall o \left( \kappa_{min} \right) = K^{1-\delta}
\end{cases}
\]

\[
LB \left( \{ \gamma_i + \epsilon \}_{i=2}^p \right) = \begin{cases} 
O_P \left( K^{1-\delta} \right) / O \left( \sqrt{K} \right), & \forall \kappa_{max} = o \left( K^{1-\delta} \right) \\
O_P \left( \kappa_{min} \right) / O \left( \sqrt{K} \right), & \forall o \left( \kappa_{min} \right) = K^{1-\delta}
\end{cases}
\] (5.56)

For \( i > p \), from (5.12) and (5.32),

\[
\sigma_{p+1} \left( \tilde{M} \right) / \sqrt{mK - p} \leq \{ \gamma_i \}_{i=p} \leq \sigma_{p+1} \left( \tilde{M} \right) \sqrt{mK - p},
\]

\[
0 \leq \{ \gamma_i \}_{i=p} \leq \| \Delta M \|_2 \sqrt{mK - p}.
\]

thus,

\[
UB \left( \{ \gamma_i + \epsilon \}_{i=p+1} \right) = O_P \left( K^{3/2} N^{-\frac{1}{2}} \right)
\]

\[
LB \left( \{ \gamma_i + \epsilon \}_{i=p+1} \right) = LB(\epsilon) = O_P \left( \max(\kappa_{max}, K^{1-\delta}) \right) / O \left( K \sqrt{N} \right)
\] (5.57)

Let \( \mathcal{U} (\cdot) \) and \( \mathcal{L} (\cdot) \) denote upper-bound and lower-bound respectively. The ratio (5.50), is given
by,

\[ UB(r_i) = \frac{UB(\gamma_i + \epsilon)}{LB(\gamma_{i+1} + \epsilon)}. \]
\[ LB(r_i) = \frac{LB(\gamma_i + \epsilon)}{UB(\gamma_{i+1} + \epsilon)}. \]

For \( i = 1 \), From (5.55), and (5.56) we get (5.51).

For \( p > i > 1 \), from 5.56, we get (5.52).

For \( i = p \),

\[ \frac{LB(\gamma_p + \epsilon)}{UB(\gamma_{p+1} + \epsilon)} \leq r_p \leq \frac{UB(\gamma_p + \epsilon)}{LB(\gamma_{p+1} + \epsilon)} \]

Thus, from (5.56) and (5.57), we get (5.53) For \( i > p \),

\[ \frac{LB\left(\{\gamma_i + \epsilon\}_{i>p}\right)}{UB\left(\{\gamma_i + \epsilon\}_{i>p}\right)} \leq \{r_i\}_{i>p} \leq \frac{UB\left(\{\gamma_i + \epsilon\}_{i>p}\right)}{LB\left(\{\gamma_i + \epsilon\}_{i>p}\right)} \]

Thus using (5.57), leads to (5.54).

As per Theorem (5.2), the ratio \( r \) will be maximized at \( \hat{p} = p \) with high probability if the lower bounds of Case 3 grows faster than the upper bound of Case 1, Case 2 and Case 4. This is satisfied when the rate of increase of \( K, N \rightarrow \infty \) is constrained by,

\[ \frac{K^{\frac{2}{5} + 2\delta}}{N^{\frac{1}{2}}} = o(1). \quad (5.58) \]

Note that the QR decomposition based factor modeling is better than the EVD based approach as the term, \( \epsilon = \frac{\gamma_1}{\sqrt{KN}} \), ensures that in the maximization of the ratio \( r_i, i = p \) is a peak for \( r_i \).
whereas \( r_i \) at \( i = p \) is just a knee point in the EVD based model order determination.
Chapter 6

Perturbation Analysis of $\hat{Q}$

In this Chapter, $\tilde{M}$ as defined in (1.11) is treated as the perturbed version of $M$ (1.4). Let

$$\Delta M \triangleq \tilde{M} - M = \begin{bmatrix} \Delta \Sigma_{yy}(1) & \Delta \Sigma_{yy}(2) & \ldots & \Delta \Sigma_{yy}(m) \end{bmatrix} \quad (6.1)$$

and $\Delta \Sigma_{yy}(l)$ denote the perturbation in $M$ and $\Sigma_{yy}(l)$ respectively. Also, $\hat{Q}$ got from $\tilde{M}\Pi$ is taken as the perturbed version of $Q$ got from $M\Pi$, where $\Pi$ is any permutation matrix. Assumption (A5) implies that $\tilde{M}\Pi$ converges to $M\Pi$ as $N \to \infty$ (for a fixed $K$), refer to Theorem 27.4 in [6]. Thus $\hat{Q}$ should go to $Q$ as $N \to \infty$. Therefore, the search is for a permutation $\Pi$ that renders $\|\hat{Q} - Q\|_F$ reasonably small for a finite $N$. As mentioned earlier, here a unique $\Pi$ is determined using HYBRID-I RRQR algorithm. This choice is justified in Section 6.1. In Section 6.2, the convergence rate of $\|\hat{Q} - Q\|_F$ for the proposed algorithm along with a measure of the convergence rate of factors, $\hat{f}_n$ is presented.

6.1 Perturbation analysis of QR decomposition:

Let, $\tilde{M}_p$ denote the first $p$ columns selected from $\tilde{M}\Pi$ as mentioned in (1.12) and $M_p$, $\Delta M_p$ denote the first $p$ columns selected from $M\Pi$ and $\Delta M\Pi$ respectively. Thus,

$$\tilde{M}_p = M_p + \Delta M_p. \quad (6.2)$$
Consider the following two lemmas taken from [8]:

**Lemma 6.1.** Let \( M_p \in \mathbb{R}^{K \times p} \) be of full column rank, \( p \), then: \( M_p + \Delta M_p \) is also of rank \( p \) if,

\[
\sigma_{\min}(M_p) > \sigma_{\max}(P_1 \Delta M_p)
\]

(6.3)

where, \( P_1 \) is the projection matrix that projects \( \Delta M_p \) to \( \mathcal{R}(M_p) \).

**Proof.** Refer to [8] for a proof.

**Lemma 6.2.** Let \( M_p \in \mathbb{R}^{K \times p} \) be a matrix of rank \( p \), with the QR decomposition \( M_p = Q_1 R \) and \( \Delta M_p \) be the perturbation on \( M_p \) such that \( M_p + \Delta M_p \) is also of rank \( p \). \( M_p + \Delta M_p \) has unique QR factorization:

\[
M_p + \Delta M_p = (Q_1 + \Delta Q_1) (R + \Delta R)
\]

where,

\[
\|\Delta Q_1\|_F \leq \sqrt{2} \frac{\|\Delta M_p\|_F}{\sigma_{\min}(M_p)} + O(\epsilon^2)
\]

(6.4)

and

\[
\epsilon = \frac{\|\Delta M_p\|_F}{\|M_p\|_2}.
\]

**Proof.** Refer to [8] for a proof.

Note that in the current context \( \|\Delta Q_1\|_F = \|\hat{Q} - Q\|_F \). Lemmas 1 and 2 suggest choosing a permutation matrix \( \Pi \) such that \( \sigma_{\min}(M_p) \) is maximized. Maximizing \( \sigma_{\min}(M_p) \) is equivalent to
maximizing $\sigma_{\text{min}}(\tilde{M}_p)$ as

$$\sigma_{\text{min}}(M_p) \geq \sigma_{\text{min}}(\tilde{M}_p) - \sigma_{\text{max}}(\Delta M_p).$$

HYBRID-I RRQR algorithm [7] selects a $\Pi$ by trying\(^1\) to maximize $\sigma_{\text{min}}(\tilde{M}_p)$. The effect of using HYBRID-I algorithm of RRQR is presented in Theorem 4.

**Theorem 6.1.** Let $\tilde{M} \in \mathbb{R}^{K \times nK}$. Then the HYBRID-I RRQR algorithm could be used to select a $\Pi$ such that the first $p$ columns from $\tilde{M}\Pi$, $\tilde{M}_p$, satisfies

$$\sigma_p(\tilde{M}) \geq \sigma_{\text{min}}(\tilde{M}_p) \geq \frac{\sigma_p(\tilde{M})}{\sqrt{p(K - p + 1)}}$$

(6.5)

**Proof.** Applying Hybrid I algorithm [7] to $\tilde{M}$, leads to a permutation matrix $\Pi$ given by (1.12), where,

$$\sigma_{\text{min}}(\tilde{R}_{11}) \geq \frac{\sigma_p(\tilde{R})}{\sqrt{p(K - p + 1)}}$$

(6.6)

Refer to [7] for a complete proof. Select $\tilde{M}_p$ as the first $p$ columns of $\tilde{M}\Pi$. QR decomposition of $\tilde{M}_p$ gives,

$$\tilde{M}_p = Q_p \tilde{R}_{11}.$$ 

Hence,

$$\sigma_{\text{min}}(\tilde{M}_p) = \sigma_{\text{min}}(\tilde{R}_{11})$$

(6.7)

\(^1\)To get a $\Pi$ that maximizes $\sigma_{\text{min}}(\tilde{M}_p)$ is a NP-hard problem and applying a brute force method would require finding $\sigma_{\text{min}}(\tilde{M}_p)$ for $KmC_p$ matrices where $Km$ is the total number of columns in $\tilde{M}$. HYBRID-I RRQR is a computationally simpler technique that generates a permutation $\Pi$ which satisfies (6.5)
Interlacing property of singular values, [14, 25], ensures,

\[ \sigma_p (\tilde{M}) \geq \sigma_{\min} (\tilde{M}_p). \]  

(6.8)

Thus, (6.6), (6.7) and (6.8) leads to (6.5).

Henceforth, the term RRQR is used to refer the HYBRID-I RRQR algorithm for simplicity. In the next subsection, convergence of \( \hat{Q} \) to \( Q \), obtained by using the RRQR on \( \tilde{M} \) and \( M \) respectively, as \( N, K \to \infty \) is investigated.

### 6.2 Convergence Analysis of \( \hat{Q} \)

The rate of convergence of \( \hat{Q} \) to \( Q \) as \( N, K \to \infty \) is summarized in the following theorem:

**Theorem 6.2.** Under the conditions (A1-A7), refer Chapter 3 regarding assumptions, with the optimistic assumption that the upper bound is achieved in (6.5) and with rate of increase of \( K \) and \( N \) constrained by \( \frac{K^1}{N} = o(1) \), the following convergence rate for \( \hat{Q} \) is got:

\[
\| \hat{Q} - Q \|_F \asymp \begin{cases} 
O_P \left( K^{\frac{1}{2}} N^{-\frac{1}{2}} \right), & \kappa_{\max} = o \left( K^{1-\delta} \right) \\
O_P \left( \kappa_{\min}^{-1} K^{1-\frac{3}{2}} N^{-\frac{1}{2}} \right), & K^{1-\delta} = o \left( \kappa_{\min} \right)
\end{cases}
\]

(6.9)

and in the worst case scenario of (6.5), where the lower-bound of (6.5) is taken, under the constraint \( \frac{K^{1+\delta}}{N} = o(1) \),

\[
\| \hat{Q} - Q \|_F \asymp \begin{cases} 
O_P \left( K^{\frac{1+\delta}{2}} N^{-\frac{1}{2}} \right), & \kappa_{\max} = o \left( K^{1-\delta} \right) \\
O_P \left( \kappa_{\min}^{-1} K^{\frac{3+\delta}{2}} N^{-\frac{1}{2}} \right), & K^{1-\delta} = o \left( \kappa_{\min} \right)
\end{cases}
\]

(6.10)
Proof. The perturbation on $Q$ is given by (6.4). Let, $\tilde{M}_p$ denote the first $p$ columns selected from $\tilde{M}\Pi$ as mentioned in (1.12) and $M_p, \Delta M_p$ denote the first $p$ columns selected from $M\Pi$ and $\Delta M\Pi$ respectively. Then,

$$\tilde{M}_p = M_p + \Delta M_p. \quad (6.11)$$

According to (6.3), The following equation needs to be satisfied:

$$\sigma_{min}(\tilde{M}_p) > \sigma_{max}(\Delta M_p).$$

According to (6.5),

$$\sigma_p(\tilde{M}) \geq \sigma_{min}(\tilde{M}_p) \geq \frac{\sigma_p(\tilde{M})}{\sqrt{p(K-p+1)}} \quad (6.12)$$

Thus according to (5.33) and (6.12), The best estimate of $\sigma_p(\tilde{M}_p)$ is given by,

$$\sigma_{min}(\tilde{M}_p) \asymp_p \begin{cases} 
K^{1-\delta}, & \text{if } \kappa_{max} = o\left(K^{1-\delta}\right), \forall \frac{K^{\delta}}{\sqrt{N}} = o(1), \\
\kappa_{min}, & \text{if } o(\kappa_{min}) = o\left(K^{1-\delta}\right)
\end{cases} \quad (6.13)$$

and in the worst case scenario,

$$\sigma_{min}(\tilde{M}_p) \asymp_p \begin{cases} 
\frac{K^{1-\delta}}{\sqrt{K}}, & \text{if } \kappa_{max} = o\left(K^{1-\delta}\right), \forall \frac{K^{\delta}}{\sqrt{N}} = o(1), \\
\frac{\kappa_{min}}{\sqrt{K}}, & \text{if } o(\kappa_{min}) = o\left(K^{1-\delta}\right)
\end{cases} \quad (6.14)$$

Note that $\Delta M_p$ are $p$ columns selected from $\Delta M\Pi$. Using (5.40) and (5.41) it can be inferred that
\[ O(\|\Delta M\|_F) = O(\Delta \Sigma_{yy}(l)) \]

Let \( \Delta \Sigma_{yy}(l) \) be \( p \) columns selected from \( \Delta \Sigma_{yy}(l) \). Then,

\[
\|\Delta M_p\|_F \approx O(\Delta \Sigma_{yy}(l)) \\
\leq O\left(\|\Delta \Sigma^{(p)}_{f\varepsilon}(l)\|_F\right) + O\left(\|\Delta \Sigma^{(p)}_{f\varepsilon}(l)\|_F\right) + \ldots \\
\ldots + O\left(\|\Delta \Sigma^{(p)}_{\varepsilon\varepsilon}(l)\|_F\right) + O\left(\|\Delta \Sigma^{(p)}_{\varepsilon\varepsilon}\|_F\right) \tag{6.15}
\]

Note that \( \|\Delta \Sigma^{(p)}_{f\varepsilon}(l)\|_F \leq \|\Delta \Sigma_{f\varepsilon}(l)\|_F \leq \|\Delta \Sigma_{f\varepsilon}(l)\|_F \leq \|\Delta \Sigma_{f\varepsilon}(l)\|_F \leq \|\Delta \Sigma_{f\varepsilon}(l)\|_F \).

From the proof of (5.23), it can be seen that \( \Delta \Sigma_{\varepsilon\varepsilon}(l) \) is an uniform perturbation in all columns.

Thus, let \( \Delta \Sigma^{(p)}_{\varepsilon\varepsilon} \) be the perturbation in \( p \) columns due to \( \Delta \Sigma_{\varepsilon\varepsilon}(l) \) then

\[
\|\Delta \Sigma^{(p)}_{\varepsilon\varepsilon}\|_F \approx_p (Kp)^{\frac{1}{2}} N^{\frac{1}{2}} \\
\approx_p K^{\frac{1}{2}} N^{\frac{1}{2}}. \tag{6.16}
\]

Thus substituting (5.20), (5.21), (5.22) and (6.16) in (6.15) leads to

\[
\|\Delta M_p\|_F \leq O_p\left(\left(K^{1-\delta} N^{-\frac{1}{2}}\right)\right) + O_p\left(\left(K^{1-\delta} N^{-\frac{1}{2}}\right)\right) + O_p\left(\left(K^{\frac{1}{2}} N^{-\frac{1}{2}}\right)\right) \\
= O_p\left(\left(K^{1-\delta} N^{-\frac{1}{2}}\right)\right) \tag{6.17}
\]

Thus in the best case scenario (6.13), under the constraint \( \frac{K^\delta}{\sqrt{N}} = o(1) \), (6.3) is satisfied with high probability as

\[
o\left(\sigma_{\min}\left(\tilde{M}_p\right)\right) = \|\Delta M_p\|_F.
\]

Similarly in the worst case scenario, from (6.14) and (6.17), it can be shown that under the condition \( \frac{K^{1+\delta}}{\sqrt{N}} = o(1) \), lemma 1 is satisfied with high probability.
In order to estimate $\|\Delta Q\|_F$ as given by (6.4), $\sigma_{\min}(M_p)$ has to be estimated. Note

$$\sigma_{\min}(\hat{M}_p) + \sigma_{\max}(\Delta M_p) \geq \sigma_{\min}(M_p) \geq \sigma_{\min}(\hat{M}_p) - \sigma_{\max}(\Delta M_p).$$  

(6.18)

Now, $\sigma_{\max}(\Delta M_p) \leq \|\Delta M_p\|_F$ thus substituting, (6.13) and (6.17) in (6.18), for the optimistic scenario,

$$\sigma_{\min}(M_p) \simeq_P \begin{cases} K^{1-\delta}, & \text{if } \kappa_{\max} = o\left(K^{1-\delta}\right), \forall \frac{K^{\delta}}{N} = o(1). \\ \kappa_{\min}, & \text{if } o(\kappa_{\min}) = o\left(K^{1-\delta}\right) \end{cases}$$

(6.19)

and in the worst case scenario substituting (6.14) and (6.17) in (6.18),

$$\sigma_{\min}(M_p) \simeq_P \begin{cases} \frac{K^{1-\delta}}{\sqrt{K}}, & \text{if } \kappa_{\max} = o\left(K^{1-\delta}\right), \forall \frac{K^{1+\delta}}{N} = o(1). \\ \frac{\kappa_{\min}}{\sqrt{K}}, & \text{if } o(\kappa_{\min}) = o\left(K^{1-\delta}\right) \end{cases}$$

(6.20)

In the optimistic viewpoint, using (6.19) and (6.17) in (6.4) leads to (6.9). Similarly for the worst case scenario, substituting (6.17) and (6.20) in (6.4), leads to (6.10).

The following theorem determines the convergence rate of the estimated factors. This is similar to Theorem 3 of [22]. Let

$$Y \triangleq \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}$$

$$F \triangleq \begin{bmatrix} f_1 & f_2 & \cdots & f_N \end{bmatrix}$$

$$E \triangleq \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_N \end{bmatrix}.$$  

(6.21)

The convergence of the factors are got as a measure of the convergence of the Root Mean Square Error (RMSE) given by, $(KN)^{-0.5}\|\hat{Q}\hat{F} - QF\|_F$, where $\hat{F}$ is the estimated factors.
Theorem 6.3. Under the conditions (A1-A7), refer Chapter 3 regarding assumptions,

\[(KN)^{-0.5} \| \hat{Q} \hat{F} - QF \|_F \leq O_F \left( K^{-\frac{\delta}{2}} \| \hat{Q} - Q \|_F + K^{-\frac{1}{2}} \right) \] (6.22)

Proof. The RMSE error of the estimated \( \hat{Y} = \hat{Q} \hat{F} \) is given by, \( KN^{-\frac{1}{2}} \| \hat{Q} \hat{F} - QF \|_F \). Now,

\[
\begin{align*}
[ \hat{Q} \hat{F} - QF ] &= [ \hat{Q} \hat{Q}^T QF - \hat{Q} \hat{Q}^T E - QF ] \\
&= [ \hat{Q} \hat{Q}^T - I ] QF - \hat{Q} [ \hat{Q} - Q ]^T E + \hat{Q} \hat{Q}^T E \\
&= K_1 + K_2 + K_3
\end{align*}
\]

where, \( K_1 = [ \hat{Q} \hat{Q}^T - I ] QF \), \( K_2 = \hat{Q} [ \hat{Q} - Q ]^T E \), \( K_3 = \hat{Q} \hat{Q}^T E \). Now,

\[
K_1 = [ \hat{Q} \hat{Q}^T - I + QQ^T - QQ^T ] QF
\]

\[
= [ \hat{Q} \hat{Q}^T - QQ^T ] QF - [ I - QQ^T ] QF
\]

Note, \( [ I - QQ^T ] \) is a projection matrix onto the null space of \( Q \) hence, \( [ I - QQ^T ] QF = 0 \). Hence,

\[
K_1 = [ \hat{Q} \hat{Q}^T - QQ^T ] QF
\] (6.23)

Now using \( \hat{Q} = Q + \Delta Q \),

\[
[ \hat{Q} \hat{Q}^T - QQ^T ] = [ \Delta Q \Delta Q^T + Q \Delta Q^T + \Delta QQ^T ]
\]
hence,

$$\| \hat{Q}\hat{Q}^T - QQ^T \|_F \leq \| \Delta Q \|_F^2 + \| Q\Delta Q^+ \|_F$$

Thus,

$$\| K_1 \|_F = O (\| \Delta Q \|_F \| F \|_F)$$

Substituting (5.24),

$$\left\| \hat{Q}\hat{Q}^T - QQ^T \right\|_F \leq O \left( \| \Delta Q \|_F \right) O \left( K^{1+\frac{\delta}{2}} N^{\frac{1}{2}} \right) \quad (6.24)$$

Now,

$$\| K_2 \|_F = \left\| \hat{Q} \left[ \hat{Q} - Q \right]^T E \right\|_F$$

$$\leq \| \Delta Q^T E \|_F \leq \| \Delta Q^T \|_F \| E \|_2$$

also,

$$\| K_3 \|_F = \left\| \hat{Q}Q^T E \right\|_F$$

$$\leq \| Q^T E \|_F$$

$$\leq \left( \sum_{n=1}^{N} \sum_{j=1}^{p} (q_j^T \varepsilon_n)^2 \right)^{0.5}$$

Now consider the Random variable $q_j^T \varepsilon_n$ where $\mathcal{E} \{ q_j^T \varepsilon_n \} = 0$ and $var \left( q_j^T \varepsilon_n \right) = q_j \Sigma_{\varepsilon\varepsilon} q_j^T \leq \sigma_{\text{max}} (\Sigma_{\varepsilon\varepsilon}) = c < \infty$ where $c$ is a constant independent of $K, N$ according to assumption (A4).
Thus,

\[ \|K_3\|_F \leq p^\frac{3}{2} N^\frac{1}{2} O_F(1). \]

Since \( \|\Delta Q^T\|_F = o_P(1) \) and \( \|Q\|_F = \sqrt{\bar{p}}, \|K_2\|_F \) is dominated by \( \|K_3\|_F \) in probability. Hence,

\[ (KN)^{-\frac{1}{2}} \|\hat{Q} - Q\|_F \leq O_P \left(K^{-\frac{1}{2}} \|\hat{Q} - Q\|_F + K^{-\frac{1}{2}}\right). \]
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Chapter 7

Comparison with the EVD Method

In this Chapter, the proposed algorithm is compared against the EVD based algorithm proposed in [22] with regards to the asymptotic properties derived in the previous Chapter.

The convergence of the estimate, $\hat{Q}^{EVD}$ of $Q$ got using the EVD algorithm and the corresponding expressions for rate of convergence are presented in [22]. The following theorem is taken from [22].

**Theorem 7.1.** Under the conditions A2-A7 and (A8),

(A8) There should be at least one $\Sigma_{xx}(l)$ of full rank $p$.

and all the Eigen values of $S$ (3.1) being distinct the following convergence rates hold,

$$
\| \hat{Q}^{EVD} - Q \|_2 \asymp \begin{cases} 
O_P \left( K^\delta N^{-\frac{1}{2}} \right), & \kappa_{\max} = o \left( K^{1-\delta} \right) \\
O_P \left( k_{\min}^{-2} \kappa_{\max} K N^{-\frac{1}{2}} \right), & \kappa_{\min}^{-2} \kappa_{\max} K N^{-\frac{1}{2}} = o(1) 
\end{cases}
$$

(7.1)

**Proof.** Refer to [22] for a proof.

In the best case scenario, (6.9), the ratio of the convergence rates of the EVD algorithm vs. the
The proposed RRQR algorithm is given by,

\[
\frac{\|\hat{Q}^{(EVD)} - Q\|_2}{\|\hat{Q}^{(RRQR)} - Q\|_F} \begin{cases} \kappa_{\text{max}}^{-\frac{\delta}{2}}, & \kappa_{\text{max}} = o\left(K^{1-\delta}\right) \\ \kappa_{\text{max}} \kappa_{\text{min}}^{-\frac{\delta}{2}}, & o\left(\kappa_{\text{min}}\right) = K^{1-\delta} \end{cases},
\]

and in the worst case scenario, (6.10), the ratio is given by,

\[
\frac{\|\hat{Q}^{(EVD)} - Q\|_2}{\|\hat{Q}^{(RRQR)} - Q\|_F} \begin{cases} K^{-\frac{1-\delta}{2}}, & \kappa_{\text{max}} = o\left(K^{1-\delta}\right) \\ \frac{\kappa_{\text{max}}}{\kappa_{\text{min}} \sqrt{K}}, & o\left(\kappa_{\text{min}}\right) = K^{1-\delta} \end{cases}.
\]

It is evident from (7.2), the proposed algorithm performs better than the EVD algorithm in the best case scenario. However in the worst case scenario (7.3), it is evident that EVD Algorithm would perform better. To probe this comparison further, consider the following example:

**Example 7.1.** Consider a data model as in (1.1), with \( p = 2 \) and the factors given by

\[
x_n \triangleq \begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \end{bmatrix}^\top
\]

where,

\[
x_n^{(1)} = \tilde{e}_n^{(1)} + \alpha_1 \tilde{e}_{n-1}^{(1)}, \quad x_n^{(2)} = \tilde{e}_n^{(2)} + \alpha_2 \tilde{e}_{n-2}^{(2)}
\]

with \( \tilde{e}_n^{(1)} \) and \( \tilde{e}_n^{(2)} \) being \( N(0, 1) \) iid process independent of the measurement noise, \( \varepsilon_n \sim N(0, I) \).

The factor loading matrix \( H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \) is a \( K \times 2 \) matrix, where \( h_1 \) and \( h_2 \) are orthogonal with \( \|h_1\|_2^2 \sim K^{1-\delta_1} \) and \( \|h_2\|_2^2 \sim K^{1-\delta_2} \) where \( 0 \leq \delta_1 < \delta_2 \leq 1 \). This is similar to the factor
strength assumption taken in [21].

Let the QR decomposition of $H$ be given by,

$$H = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_{11} \\ 0 \end{bmatrix}$$

where $Q_1$ is the $K \times 2$ matrix that spans the column space of $H$, $Q_2$ is the $K \times (K - 2)$ matrix which spans the perpendicular space of $H$ and

$$R_{11} = \begin{bmatrix} \| h_1 \|_2 & 0 \\ 0 & \| h_2 \|_2 \end{bmatrix}.$$

Note that, since $x_n$ is independent of $\varepsilon_n$,

$$\Sigma_{yy}(l) = Q_1 R_{11} \Sigma_{xx}(l) R_{11}^T Q_1^T$$

Thus,

$$\Sigma_{yy}(1) = Q_1 \begin{bmatrix} \| h_1 \|_2^2 & 0 \\ 0 & 0 \end{bmatrix} Q_1^T \quad (7.6)$$

$$\Sigma_{yy}(2) = Q_1 \begin{bmatrix} 0 & 0 \\ 0 & \| h_2 \|_2^2 \end{bmatrix} Q_1^T \quad (7.7)$$
Further more, it is assumed that $o(K^{1-\delta_2}) = KN^{-\frac{1}{2}}$. It can be shown that

\[
\left\| \hat{Q}^{EVD} - Q \right\|_2 \asymp \frac{\|h_1\|_2^2 O_P \left( KN^{-\frac{1}{2}} \right)}{\|h_2\|_2^4} \quad (7.8)
\]

\[
O_P \left( KN^{-\frac{1}{2}} \right) \leq \left\| \hat{Q}^{RRQR} - Q \right\|_F \leq \frac{O_P \left( (\frac{K}{2}) N^{-\frac{1}{2}} \right)}{\|h_2\|_2^4} \quad (7.9)
\]

**Proof.** In [22], it is seen that when using the EVD method,

\[
\left\| \hat{Q} - Q \right\|_2 \propto \frac{\|\Delta (MM^\top)\|_2}{\sigma_p (MM^\top)} \quad (7.10)
\]

For the current example, from (7.6) and (7.7),

\[
\sigma_p (MM^\top) = \|h_2\|_2^4 \quad (7.11)
\]

Also it holds that,

\[
\left\| \Delta (MM^\top) \right\|_2 \leq \|M\Delta M^\top\|_2 + \|\Delta MM^\top\|_2 + \|\Delta M\Delta M^\top\|_2 \leq O \left( \|M\Delta M^\top\|_2 \right) = O \left( \|M\|_2 \|\Delta M\|_2 \right) \quad (7.12)
\]

and substituting from (5.32) and (7.6) in (7.12) leads to,

\[
\left\| \Delta (MM^\top) \right\|_2 \asymp \|h_1\|_2^2 O_P \left( KN^{-\frac{1}{2}} \right) \quad (7.13)
\]

Thus, substituting (7.11) and (7.13) in (7.10) the convergence rate for the EVD method is got as
The convergence rate for QR decomposition is given by (6.4). Note from (7.7),

\[ \sigma_2(M) = \|h_2\|_2^2. \]

Now, from (5.32) and (5.43), under the condition \(o(\|h_2\|_2^2) = K N^{-\frac{1}{2}},\)

\[ \sigma_2(\tilde{M}) \preceq_P \|h_2\|_2^2. \]

and from (6.5),

\[ \|h_2\|_2^2 \geq \sigma_2(M_2) \geq \frac{\|h_2\|_2^2}{\sqrt{K}}. \]

Thus, from (6.17) and (6.18),

\[ \|h_2\|_2^2 \geq \sigma_2(M_2) \geq \frac{\|h_2\|_2^2}{\sqrt{K}} \]

finally, applying (6.17) and the upper and lower bounds for \(\sigma_2(M_2)\) in (6.4), (7.9) is obtained.

Thus the ratio of the convergence rates is given by,

\[ O\left(K^{\delta_2 - \delta_1}\right) \leq \frac{\|Q^{TED} - Q\|_2}{\|Q_{RRQR} - Q\|_F} \leq O_P\left(K^{\delta_2 - \delta_1 - \frac{1}{2}}\right) \quad (7.14) \]

Thus, in the best case scenario, the proposed algorithm is better by a factor of \(O\left(K^{\delta_2 - \delta_1}\right)\) and in the worst case scenario the convergence rate is \(O_P\left(K^{\delta_2 - \delta_1 - \frac{1}{2}}\right)\), where \(\delta_2 > \delta_1\). Thus if, \(\delta_2 - \delta_1 > \frac{1}{2}\), then even in the worst case, convergence rate of the proposed algorithm is better than that of the EVD algorithm.
Numerical illustrations of the proposed algorithm are presented in this Chapter. Here, four numerical examples are considered. As the original motivation for this work comes from ([22]), a simulation example presented therein is considered first. The results obtained by using the proposed algorithm is compared with the EVD algorithm proposed in ([22]). The second problem considered here is the simulation of Example 7.1 presented in Chapter 7. The results of the performance of the proposed algorithm, the EVD and the PCA ([4]) are compared. In the third example, a new algorithm is proposed for the separation of multiple audio signals from noisy mixtures. The new algorithm is a conjunction of the proposed RRQR algorithm and the Independent Component Analysis (ICA) technique. The presented algorithm has better noise characteristics when compared to the conventional Noisy ICA technique ([17]). Finally, the proposed algorithm is applied to model the US stock market data in comparison with the PCA based model.

8.1 Simulation 1

This simulation is used to illustrate the rate of convergence of \( \hat{Q} \) to \( Q \) for the RRQR algorithm in comparison with the EVD method. Consider (1.1) with a factor loading matrix

\[
H = \begin{bmatrix}
2 \cos \left( \frac{2\pi}{K} \right) & 2 \cos \left( \frac{4\pi}{K} \right) & \ldots & 2 \cos (2\pi)
\end{bmatrix}^T,
\]

(8.1)

the factors \( x_n \) being generated by

\[
x_n = 0.9x_{n-1} + \eta_n,
\]

(8.2)
where process noise $\eta_n$ and measurement noise $\epsilon_n$ are Gaussian noises with mean zero and variance 4. Data sets were generated using (1.1), (8.1) and (8.2) for $K = 20, 180, 400$ and $1000$ and data length $N = 200$ and $500$.

Matrix $\tilde{M}$, (1.11), is generated for each pair $(K, N)$ with $m = 5$. The $L_2$ norm of the error $\| \hat{Q}_i - Q \|_i, i = 1, 2$, is tabulated in Table 8.1. Here $\hat{Q}_1$ and $\hat{Q}_2$ are the estimates obtained using EVD and RRQR methods respectively. Figure (8.1) shows the convergence of $\| \hat{Q} - Q \|_F$ for RRQR and EVD methods with $K = 60$ while varying $N$ for two different deltas, $\delta = 0$ given by (8.1), and $\delta = 0.5$ given by

$$H_1 = \left[ H(1 : \frac{K}{2}) \ Z \right]^T.$$ 

where $Z$ is a vector of $\frac{K}{2}$ zeros. It was noted that there was not much change in the observation when varying $K$ hence was neglected.

Assuming a fixed $N = 100$ and $K = 180$ in the above model, using 100 monte carlo trials, the mean and standard deviation of the ratio, (3.5),

$$r_i = \frac{\lambda_i}{\lambda_{i+1}}$$

used in detecting the model order in the EVD method is plotted in figure 8.2. While the mean and standard deviation of the ratio, (1.13),

$$r_i = \frac{\gamma_i + \epsilon}{\gamma_{i+1} + \epsilon},$$

proposed for determining the model order using RRQR decomposition is plotted in figure 8.3.
Table 8.1: Summary of the results of Simulation 1, all values multiplied by 1000.

| $K$ | $\|Q_1 - Q\|$ | $\|Q_2 - Q\|$ | $\|Q_1 - Q\|$ | $\|Q_2 - Q\|$ |
|-----|----------------|----------------|----------------|----------------|
|     | $N = 200$     | $N = 500$     | $N = 200$     | $N = 500$     |
| 20  | 15.9           | 11.8           | 14.9           | 10.9           |
| 180 | 17.4           | 12.3           | 15.9           | 11.3           |
| 400 | 18             | 12.1           | 16.5           | 11.2           |
| 1000| 18             | 12.3           | 11.6           | 11.4           |

From figure 8.2 and 8.3, it can be seen that the RRQR based model order detection scheme works as good as EVD even when $K \approx N$. Also, it can be seen from the error bar that the proposed RRQR method has lesser standard deviation and hence more consistent when compared to the EVD estimate.
Figure 8.2: Plot of ratio of eigen values of $MM^T$, $r_i$ (3.5), for fixed $K = 180; N = 100$.

Figure 8.3: Plot of ratio of diagonal values of $R$ in $M\Pi = QR$, $r_i$, (1.13), where $\Pi$ is got using RRQR Method for fixed $K = 180; N = 100$. 
8.2 Simulation 2

Consider the data model given by (1.1) with the number of factors, \( p = 2 \) as in Example 1 and the factors \( x_n \) taken as in (7.5) with \( \alpha_1 = 0.5 \) and \( \alpha_2 = 0.5 \). The factor strengths are taken as \( \delta_1 = 0 \) and \( \delta_2 = 0.5 \) with the elements of the columns of \( H \) given by \( h_1 \sim U(-4, 4) \) and \( h_2 (1: \frac{K}{2}) \sim U(-4, 4) \) and \( h_2 (\frac{K}{2} + 1: K) = 0 \). Two scenarios are considered for the random noise \( \varepsilon_n \) in (1.1). Firstly, \( \varepsilon_n \) is taken to be generated by \( N(0, I) \), \( I \in \mathbb{R}^{K \times K} \) is an identity matrix. In the second scenario, \( \varepsilon_n \) is taken as \( N(0, 0.1\Sigma) \), where the elements of \( \Sigma \in \mathbb{R}^{K \times K} \) are given by

\[
\sigma_{i,j} = \frac{1}{2} \left(|i|^{2w} - |i-j|^{2w} + |j|^{2w}\right), \tag{8.3}
\]

where \( w \), the Hurst parameter, is taken as 0.6.

The data sets were simulated for all pairs of \( (K, N) \) where \( K \) takes values of 100, 200, 400 and \( N \) takes, 100, 200. The simulated data set was run against the QR-PC algorithm, the EVD algorithm and also the PCA algorithm proposed in ([4]). In ([4]), the model order is the \( p \) that minimizes the Information Content,

\[
IC_p = \ln \left(V(p, \hat{f}_n)\right) + p \left(\frac{K + N}{KN}\right) \ln \left(\frac{KN}{K + N}\right)
\]

where

\[
V(p, \hat{f}_n) = \min_{\hat{Q}} \frac{1}{KN} \sum_{n=1}^{N} \left\| y_n - \hat{Q}^{(p)} \hat{f}_n^{(p)} \right\|_2^2
\]

Here, \( \hat{Q}^{(p)} \) and \( \hat{f}_n^{(p)} \) are obtained using PCA by assuming model order as \( p \). Thus, the estimate of
model order is

\[ \hat{p}_{N,K} = \arg \min_p \{ IC_p \}. \]

The Root Mean Square Error (RMSE) and Forecast Error (FE) were observed for all pairs \((K, N)\) and the mean over 100 montecarlo runs are tabulated. The RMSE error is

\[ \epsilon_{\text{rmse}} = \left( \frac{\sum_{n=1}^{N} \left\| \hat{Q}_{N,K}\hat{f}_n - Hx_n \right\|_2}{KN} \right)^{0.5} \]  \hspace{1cm} (8.4)  

and the mean FE given by

\[ \epsilon_{fe} = \frac{1}{N} \sum_{n=1}^{N} \left( K^{-0.5} \left\| \hat{y}_n^{(1)} - y_n \right\|_2 \right), \]  \hspace{1cm} (8.5)  

where \( \hat{y}_n^{(1)} \) is the one step ahead forecast of \( y_n \) got by using \( \hat{y}_n^{(1)} = \hat{Q}_{N,K}\hat{f}_n^{(1)} \) and \( \hat{f}_n^{(1)} \) is the one step ahead forecast of the factors got by fitting AR model to factors using Yule-Walker algorithm.

The results are summarized in TABLE 8.2.

It could be noted that while the PCA method is better at estimating the RMSE error, the EVD and proposed methods perform better when FE is concerned. Also, the proposed method performs better than the EVD in certain cases. It could be noted that when \( \epsilon \sim \mathcal{N}(0, I) \), all the three algorithms were able to detect the correct model order. However when \( \epsilon \sim \mathcal{N}(0, \Sigma) \), only the EVD and RRQR methods were able to detect the model order correctly, while PCA overestimated the model order by a huge difference. Finally it could also be noted that the number of lags, \( m \), considered is increased, the performance of the EVD and RRQR methods increased.
Table 8.2: Summary of Observations from Simulation 2. The RMSE and FE values are multiplied by 1000.

| Noise | I. $\varepsilon_n \sim N(0, I)$ | II. $\varepsilon_n \sim N(0, 0.1\Sigma)$ |
|-------|----------------------------------|----------------------------------------|
|       | $K$ | 100 | 100 | 100 | 200 | 200 | 200 | 300 | 100 | 100 | 100 | 200 | 200 |
| $N$   | 100 | 200 | 300 | 100 | 200 | 200 | 300 | 100 | 200 | 300 | 200 |
| RRQR, $m = 2$ | $\hat{\beta}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
|       | $\varepsilon_{fe}$ | 1474 | 1486 | 1460 | 1465 | 1481 | 1480 | 1244 | 1230 | 1226 | 1250 | 1238 |
|       | $\varepsilon_{rmse}$ | 52.8 | 23.0 | 16.2 | 37.9 | 22.7 | 15.4 | 138 | 80 | 16 | 103 | 26.0 |
| RRQR, $m = 9$ | $\hat{\beta}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
|       | $\varepsilon_{fe}$ | 1479 | 1479 | 1482 | 1481 | 1489 | 1482 | 1229 | 1229 | 1238 | 1227 | 1231 |
|       | $\varepsilon_{rmse}$ | 45.2 | 23.5 | 15.9 | 31.1 | 22.3 | 15.4 | 214 | 126 | 53.2 | 15.6 | 45.3 |
| EVD, $m = 2$ | $\hat{\beta}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
|       | $\varepsilon_{fe}$ | 1477 | 1480 | 1487 | 1476 | 1491 | 1484 | 1226 | 1230 | 1238 | 1229 | 1233 |
|       | $\varepsilon_{rmse}$ | 34.5 | 19.6 | 14.1 | 22.9 | 18.2 | 13.0 | 189 | 105 | 27.5 | 186 | 94.8 |
| EVD, $m = 2$ | $\hat{\beta}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
|       | $\varepsilon_{fe}$ | 1470 | 1484 | 1461 | 1455 | 1477 | 1479 | 1243 | 1229 | 1238 | 1249 | 1238 |
|       | $\varepsilon_{rmse}$ | 19.9 | 12.2 | 9.4 | 11.2 | 10 | 7.4 | 107.2 | 75.8 | 61.9 | 114.7 | 93.6 |

8.3 Simulation 3 (ICA Pre-processing)

The Independent Component Analysis, ([9, 18]), is a well known technique for estimating the factors. The basic ICA technique is used to extract $p$ sources from $p$ mixtures. When the mixtures are noise corrupted and the number of such mixtures being larger than the total number of sources, the Noisy ICA algorithm is used. The Noisy ICA ([17]) algorithm considers a model of the form,

$$y_n = Ws_n + \epsilon_n$$

(8.6)

where,

1. The factors, $s_n$, are independent of each other.
2. Not more than one factor is Gaussian.
3. The factors $s_n$ are independent of $\epsilon_n$.  

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4. The noise $\epsilon_n$ is Gaussian with co-variance $\sigma^2 I$, where $I$ is an Identity matrix.

Given $\{y_n\}_{n=1}^{N}$, the Noisy ICA Algorithm estimates $W$ and $s_n$ uniquely.

Here, an algorithm is developed by combining the proposed RRQR technique with the basic ICA algorithm. This helps in relaxing the conditions 3,4 presented above. The algorithm is summarized below:

1. Obtain $Q$ and $f_n$ from the mixtures, $y_n$, as in data model (1.2) using the RRQR based technique presented in Section 5.

2. Determine $s_n$, $A$ from $f_n$ such that,

$$s_n \triangleq A^{-1}f_n$$

$$\triangleq \begin{bmatrix} s_n^{(1)} & \ldots & s_n^{(p)} \end{bmatrix}^\top$$

where $\tilde{A}$ is a matrix obtained by using the basic ICA, such that the components of $\tilde{s}_n$ are independent.

Thus,

$$y_n = QA^{-1} f_n + \epsilon_n$$

Assuming $W = QA$,

$$y_n = Ws_n + \epsilon_n$$

(8.7)

Thus, $W$ and $s_n$ could be extracted as in (8.7) under the following assumptions:

1. The components of $s_n$ are independent with each other.

2. Not more than one factor has Gaussian distribution.

3. All $s_n$ should have correlations with the past. i.e., $\mathcal{E}\{s_{n+l}s_n\} \neq 0$ for at least one $l = 1, \ldots, m$.

4. The random noise $\epsilon_n$ can be correlated with the factors $s_n$, i.e., $\mathcal{E}\{s_{n+l}\epsilon_n\}$ need not necessarily be zero for all $l \geq 0$. 
5. The auto co-variance of random noise, $\varepsilon_n$, can be $\Sigma$, where $\Sigma$ is any positive semi definite matrix.

The graphical representation of the proposed algorithm is depicted in Figure (8.4).

In this thesis, an approximation of the popular *cocktail-party* problem is simulated. The cocktail party problem is to separate multiple sound sources using recordings from multiple microphones. Let the number of sound sources be two, given by $s = \begin{bmatrix} s^{(1)} \\ s^{(2)} \end{bmatrix}^T$ with $N$ samples (different pairs of speech and music were considered as sources). The sources are mixed by a random matrix $W \in \mathbb{R}^{K \times 2}$ where each element of $W$ is $|N(0, 1)|$. Finally measurement noise $\varepsilon_n$ is added. Two scenarios are considered where in the first case, each element of $\varepsilon_n$, is taken to be $N(0, 0.1)$ and in the second case, $\varepsilon_n$, is taken to be $N(0, \Sigma)$. Here $\Sigma \in \mathbb{R}^{K \times K}$ are as described in (8.3). Note, this is an approximation of the cocktail party problem as it does not include reverberations from the room or time delay between different recordings or multiple noise sources which
might exceed the number of microphones used. For a more detailed analysis of the cocktail party problem refer, ([15]). The performance of the algorithm is evaluated using PESQ score ([30]). The PESQ is a tool that compares the original signal $s_n$ with the recovered signal $\hat{s}_n$ and it gives a best score of 4.5 when both match exactly. The ICA algorithm used is from ([16]). Since the speech and music sources have a super Gaussian distribution, The kurtosis is assumed positive and $tanh$ non-linearity is used as a measure of non gaussianity in the implementation of fast ICA.

It could be noted from the observations in Table 8.3 that, the RRQR based method performs better in almost all cases. It could also be noted that in the first scenario of noise, where the same magnitude of noise was included in each observation, as the number of observed mixtures ($K$) increased, the separation was more accurate implying that information from the increased number of mixtures was used to separate out the components. The second scenario is quite different, as the number of observed mixtures increased, the noise variance also increased $i.e.$, the noise variance was proportional to the number of observations as can be seen from (8.3). Thus, with the increase in the number of observed mixtures, the increase in the noise magnitude offset the increase in the performance thus, with increase in $K$, decrease in the performance of the algorithm is seen. It was also noted that the PCA based method of ([4]) was unable to determine the number of sources under the second scenario and its estimate for the number of sources were between 20-30. Thus, in the simulation, the number of sources were assumed to be 2 in this case while using PCA. The EVD and the proposed method did not suffer from this problem and were able to estimate the number of sources as 2 quite consistently. It is also interesting to note that while speech-music separation was the best, speech-speech separation was the worst. Thus, the separation characteristic also depends on the type of source to be separated.
| Source Type       | speech-speech | speech-music | music-music |
|-------------------|---------------|--------------|-------------|
| K - Mixtures      | 10 40 100     | 10 40 100    | 10 40 100   |

(Average PESQ Score ) with \( \varepsilon_{i,j} \sim N(0, 0.1) \)

| Method | 10 40 100 | 10 40 100 | 10 40 100 |
|--------|-----------|-----------|-----------|
| RRQR   | 2.282     | 2.813     | 3.1300    |
| EVD    | 2.275     | 2.780     | 3.1277    |
| PCA    | 2.285     | 2.812     | 3.1312    |

(Average PESQ Score ) with \( \varepsilon_{n} \sim N(0, \Sigma) \)

| Method | 10 40 100 | 10 40 100 | 10 40 100 |
|--------|-----------|-----------|-----------|
| RRQR   | 3.515     | 3.088     | 2.670     |
| EVD    | 3.507     | 3.079     | 2.665     |
| PCA    | 4.064     | 4.086     | 3.956     |

Table 8.3: Errors Associated when used as pre-processing with ICA

8.4 Application on a finance portfolio

The proposed RRQR method, EVD and PCA were applied to model the returns of the S&P 500 index constituents. The S&P 500 is a diversified index that represents large companies in the US. It is often taken as a representative index of the US market as a whole. We consider the 500 stocks that are members of the S&P Index as of July 31, 2013. We collect the data for these 500 stocks from January 1, 2010 - July 31, 2013. The prices are adjusted for splits and dividends. The daily logarithmic returns are calculated from the adjusted prices. If \( P(n) \) and \( P(n - 1) \) are the prices at time at \( n \) and \( n - 1 \), then the logarithmic return is:

\[
y_n = \log(P(n)) - \log(P(n - 1)).
\]

Sixteen stocks did not have data for all days and hence were omitted from the study. The following steps are applied to develop the factor models that are used to forecast the returns.

1. Let the total number of days be \( N \) and the number of stocks be \( K \).
2. The mean across time is set to zero for all stocks, \( i.e., \)

\[
\bar{y}_n = y_n - \bar{y}, \; n = 1, 2, \ldots, N.
\]

where \( \bar{y} = \frac{1}{N} \sum_{n=1}^{N} y_n \) is the mean.
Table 8.4: Results from finance data application

|               | EVD-Method | Proposed Method | PCA  |
|---------------|------------|-----------------|------|
| \( q \)      | 1          | 1               | 6    |
| \( e_{\text{rmse}} \) | 0.0144     | 0.0146          | 0.0129 |
| \( e_{\text{fe}} \) | 0.0163     | 0.0163          | 0.0164 |

3. Now, the demeaned data is modeled as \( \tilde{y}_n = Qf_n + \varepsilon_n \) where the factor loading matrix, \( Q \) and factors \( f_n \) are estimated using data from \( n = 1, \ldots, 499 \). The following three algorithms were used for estimating \( Q \) and \( f_n \):
   (a) The EVD algorithm
   (b) The RRQR algorithm
   (c) The PCA based algorithm

4. An AR model of order 10 was fit for the factors \( f_n \) using Yule - Walker equations.

5. One step ahead out sample prediction \( \tilde{y}_{n+1} = Qf_{n+1} \), are computed for days, \( n = 500, \ldots, 509 \). Here \( f_{n+1} \) is the one step ahead prediction got using the AR model. Wherein, for estimation of \( n = 501, 502, \ldots, \) the true value of \( n = 500, 501, \ldots \) is used.

6. \( Q \) and \( f_n \) are re-estimated using data from \( n = 10, \ldots, 509 \) and an AR model is estimated. One step predictions are made for \( n = 510 - 519 \). This process is repeated till the one step ahead predictions are obtained for the last 400 days.

Using the above methodology, the following observations were taken:

1. Model order \( (q) \) is the mean of the model order estimates over every update of \( Q \) and \( f_n \).
2. RMSE error \( (e_{\text{rmse}}) \) is the mean of the \( e_{\text{rmse}} \) estimates over every update of \( Q \) and \( f_n \). Note, the \( e_{\text{rmse}} \) error for each 500 day block estimates of \( Q \) and \( f_n \) is computed using (8.4).
3. Forecast error \( (e_{\text{fe}}) \) as in (8.5) for last 400 days.

The results observed are tabulated in Table 8.4

It is a known fact that S&P index is in such a way that the stocks depend only on a single factor, as opposed to the six factors estimated by PCA. Here both EVD and RRQR have same prediction errors while PCA has a higher prediction error.
In this thesis, factor modelling of an observed \( K \)-dimensional multivariate series, \( y_n \), of length \( N \) \((n = 1, 2, \ldots, N)\) was considered. The goal was to estimate a low \( p \)-dimensional series called factors, \( f_n \) \((p < K)\) and a constant matrix \( Q \) such that \( y_n = Qf_n + \varepsilon_n \), where \( \varepsilon_n \) was random noise. The random noise was assumed to have covariance, \( \Sigma_{\varepsilon\varepsilon}(l) = \Sigma \delta(l) \), where \( \Sigma \) is a positive definite matrix. It was also assumed that the factors can be correlated with past noise and not with future noise i.e., \( \Sigma_{f\varepsilon}(l) \neq 0 \) for \( l \geq 0 \) and \( \Sigma_{f\varepsilon}(l) = 0 \) for \( l < 0 \). In principle a “fat matrix”, \( M \), can be formed by augmenting the autocovariance matrices \( \{\Sigma_{yy}(l)\}_{l=1}^{m} \). This matrix \( M \) would be of rank \( p \) and span the same column space as \( Q \) i.e., \( \mathcal{R}(M) = \mathcal{R}(Q) \). Hence, determining \( p \) and \( Q \) from SVD/QR decompositions of \( M \) is straight forward. In practice, only the finite time estimates of the autocovariances are observed and hence only a perturbed version \( \tilde{M} \) of \( M \) is observable. This matrix, \( \tilde{M} \), would be full row rank with probability one. In this paper, the model order \( p \) was determined by equating it to the “numerical rank” of the matrix \( \tilde{M} \). The numerical rank of \( \tilde{M} \) was determined by maximizing a ratio, (1.13), formed using the diagonal values of the \( R \) matrix of the HYBRID-III RRQR decomposition, [7]. With the model order \( p \) determined, an estimate of \( Q \) was obtained using HYBRID-I RRQR decomposition of \( \tilde{M} \). The factors, \( f_n \), were obtained by projecting the observations \( y_n \) on the space spanned by the estimate of \( Q \).

The HYBRID-III algorithm ensures that

\[
\sigma_p \left( \tilde{M} \right) \sqrt{(p + 1)(K - p)} \geq \sigma_p \left( \tilde{M}_p \right) \geq \frac{\sigma_p \left( \tilde{M} \right)}{\sqrt{p(K - p + 1)}},
\]  

(9.1)
were $\tilde{M}_p$ denotes the first $p$ columns selected from $\tilde{M} \Pi$ with $\Pi$ being the permutation matrix generated by the HYBRID-III algorithm. HYBRID-I and HYBRID-II algorithms satisfy the upper and lower bounds, respectively, of (9.1). The above bounds were crucial for proving the asymptotic convergence of the model order. In the simulations, QR-CP as well the other Hybrid algorithms were found to give similar results. Nevertheless, convergence results could be proved only for Hybrid algorithms. In particular, convergence of the model order scheme could be proved only for the Hybrid-III algorithm.

Conventional PCA based factor modelling cannot allow noise to be correlated with factors and that noise should have covariance of $\Sigma_{ee}(0) = \sigma^2 I$. The proposed algorithm relaxes the above and allows noise to have any covariance $\Sigma$. Thus it was seen from Simulations 8.2 and 8.3 that when the noise had covariance anything other than $\sigma^2 I$, the PCA based model order detection schemes [4] failed to detect the correct number of factors $p$, the proposed RRQR based scheme remained robust.

The method of using $\tilde{M}$ to determine the parameters in the factor model was first approached by [22] wherein EVD of $\tilde{M} \tilde{M}^T$ was used to obtain estimates for $p$, $Q$ and $f_n$. It was proved that in certain cases, the proposed RRQR based methodology is better than the EVD based method w.r.t the asymptotic properties (as both the dimension and duration of observations, $K$ and $N$, tends to infinity). The numerical Simulations 8.1, 8.2 and 8.3 also demonstrate that RRQR betters EVD in certain cases.

Using Example 1, it was shown that the factors might not have correlation at all lags and hence, in order to acquire all factors, lags, $l = 1, \ldots, m$ needs to be used with $m$ high enough. Conversely, it could be stated that using $l = a, \ldots, b$ where $\min (K, N) \geq b \geq a \geq 1$ to construct $M$ would result in the estimation of factors having correlation within those particular lags alone. This is
useful when, say, an $a$-step ahead prediction is desired (persistent factors) or if the undesirable factors have correlation only in lags less than $a$.

The use of the proposed method of factor modelling as pre-processing to ICA was demonstrated in Simulation 3. Finally, w.r.t the finance data application, the EVD method and the proposed method perform similarly and identify only one dominant factor. This is expected since the S&P 500 is a well diversified index and the only common factor among all the constituent stocks is the US market factor.

This work can be extended to devise a recursive algorithm to estimate the parameters of the factor model based on RRQR decomposition.
Appendix 1

Interlacing Property

The following lemma relates to the interlacing property of eigen values for symmetric matrices:

**Lemma 9.1.** [25, 11] Let $A$ be real symmetric matrix of dimension $n$ and $B \in \mathbb{R}^{k \times k}$ be a principal sub matrix of $A$ with $1 \leq k < n$. Let $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_k$ be the respective eigenvalues. Then

$$a_j \geq b_j \geq a_{n-k+j}, j = 1, 2, \ldots, k$$

\[ (9.2) \]

**Proof.** Refer [25] for a proof.

**Theorem 9.1.** Let $\Sigma \in \mathbb{R}^{m \times n}$, $m < n$, and $\Sigma^{k \times l}_{11}$, $k < l$, be any sub matrix of $\Sigma$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ being their corresponding singular values. Then,

$$\sigma_j \geq \lambda_j \geq \sigma_{n-k+j}, j = 1, 2, \ldots, k.$$  

**Proof.** First we start by establishing the connection between the eigen values and singular values so as to connect the result in the previous lemma to pertain to singular values. The following properties are used:
1. Singular values of a matrix $\Sigma \in \mathbb{R}^{m \times n}$, $m < n$, are equal to the square roots of the eigenvalues of the matrix $A = \Sigma^\top \Sigma$ (or) $\Sigma \Sigma^\top$.

2. Also, let QR decomposition of $\Sigma$ be, $\Sigma = QR$. Then singular values of $R$ equals that of $\Sigma$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$, since $Q$ is an orthogonal matrix.

Let $\Sigma_{11} \in \mathbb{R}^{k \times l}$, $k < l$, be a principal sub matrix of $\Sigma$. Using the second property, without loss of generality, $\Sigma$ can be written as upper triangular, for the purpose of dealing with singular values, i.e.,

$$\Sigma = \begin{bmatrix} l & n - l \\ k & \Sigma_{11} \\ m - k & 0 & \Sigma_{22} \end{bmatrix}.$$ 

Also, let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ be the singular values of $\Sigma_{11}$. Now, Consider the symmetric matrix, $A \triangleq \Sigma^\top \Sigma$ given by,

$$A = \Sigma^\top \Sigma = \begin{bmatrix} k & n - k \\ \Sigma_{11}^\top \Sigma_{11} & \Sigma_{12}^\top \Sigma_{12} \\ \Sigma_{12}^\top \Sigma_{12} & \Sigma_{12}^\top \Sigma_{22} & \Sigma_{22} \end{bmatrix}.$$ 

Note, the symmetric matrix $\Sigma_{11}^\top \Sigma_{11}$ is a principal sub matrix of $A$. Now, Lemma 9.1 can be applied assuming $B = \Sigma_{11}^\top \Sigma_{11}$. Thus, the eigen values of $\Sigma^\top \Sigma$ and $\Sigma_{11}^\top \Sigma_{11}$ are related as in (9.2). Thus taking the positive square root of which would correspond to the singular values of $\Sigma$ and $\Sigma_{11}$.

Thus,

$$\sigma_j \geq \lambda_j \geq \sigma_{n-k+j}, j = 1, 2, \ldots, k.$$
This above theorem is used in Chapters 4, 5 and 6.
Appendix 2

Landau notations and Mixing Properties

Properties of Order Notations (landau notations)

Here the properties of little $o$, big $O$, little $o_P$ and big $O_P$ are summarized. Refer also [24].

Properties of little $o$ and big $O$:

1. $a_n = O(a_n)$
2. If $a_n = o(b_n)$, then $a_n = O(b_n)$
3. If $a_n = O(b_n)$, then $O(a_n) + O(b_n) = O(a_n)$. 
4. If $a_n = O(b_n)$, then $o(a_n) + o(b_n) = o(a_n)$
5. Let $c$ be a constant, then $cO(a_n) = O(a_n)$ and $c o(b_n) = o(b_n)$.
6. $O(a_n)O(b_n) = O(a_n b_n)$; $o(a_n) o(b_n) = o(a_n b_n)$

The properties of $O_P$ and $o_P$ are summarized as follows:

1. Let $a_n$ be any stochastic sequence and $b_n$ be a deterministic sequence then, $o(a_n) = b_n$ implies the stochastic sequence $\frac{b_n}{a_n} \xrightarrow{P} 0$. 

while almost all the properties pertaining to $O$ and $o$ could be applicable to $O_P$ and $o_P$, the following two properties are not so obvious:

2. Let $a_n$ and $b_n$ deterministic sequences, then, $O(a_n) + O_P(b_n) = O_P(a_n + b_n)$.

Proof. Let,

$$C_n = A_n + X_n$$

where $A_n$ is a deterministic sequence and $X_n$ is a random sequence. Let the rate of convergence of $A_n$ and $X_n$ be as follows:

$$A_n = O(a_n)$$

$$X_n = O_P(b_n)$$

where $a_n$ and $b_n$ are deterministic sequences. By definition of $O(.)$ and $O_P(.)$,

$$\left| \frac{A_n}{a_n} \right| \leq M_1 \forall n > n_0$$

$$P \left( \left| \frac{X_n}{b_n} \right| \leq M_2 \right) \geq 1 - \epsilon_{n_0} \forall n > n_0$$

Thus,

$$C_n = O(a_n) + O_P(b_n)$$

$$\Rightarrow P \left( \left| \frac{A_n}{a_n} \right| + \left| \frac{X_n}{b_n} \right| \leq M_1 + M_2 \right) \geq 1 - \epsilon_{n_0} \forall n > n_0$$

$$\Rightarrow C_n = O_P(a_n + b_n)$$
if \( a_n > b_n \) then,

\[
C_n = O_P (a_n).
\]

\[\square\]

3. Let \( a_n \) and \( b_n \) deterministic sequences, then, \( O (a_n) O_P (b_n) = O_P (a_n b_n) \).

**Proof.** Let,

\[
C_n = A_n X_n
\]

where \( A_n \) is a deterministic sequence and \( X_n \) is a random sequence. Let the rate of convergence of \( A_n \) and \( X_n \) be as follows:

\[
A_n = O (a_n)
\]

\[
X_n = O_P (b_n)
\]

where \( a_n \) and \( b_n \) are deterministic sequences. By definition of \( O(A) \) and \( O_P (B) \),

\[
\left| \frac{A_n}{a_n} \right| \leq M_1 \forall n > n_0
\]

\[
P \left\{ \left| \frac{X_n}{b_n} \right| \leq M_2 \right\} \geq 1 - \epsilon_{n_0} \forall n > n_0
\]

Thus,

\[
C_n = O (a_n) O_P (b_n)
\]

\[
\implies P \left( \left| \frac{A_n}{a_n} \right| \left| \frac{X_n}{b_n} \right| \leq M_1 M_2 \right) \geq 1 - \epsilon_{n_0} \forall n > n_0
\]

\[
\implies C_n = O_P (a_n b_n)
\]
Mixing Conditions

In Chapter 2, the stationarity of the time series was due to the assumptions on the mixing conditions namely, \(\alpha(.)\) and \(\psi(.)\). Here an explanation pertaining to the same are summarized. For more details refer [6, pp 363 - 367] and [10, sec 2.6].

To express dependency across time for an observed stochastic sequence, one could use the Moving Average (MA) model. But it is too simplistic and cannot comply with complicated dependence structures. Hence, mixing conditions are used. Mixing conditions state that that, across time, two well separated instances are independent of each other. This enables the use of law of Large Numbers and the Central Limit Theorem for a non-independent stochastic series.

Let \(X_1, X_2, \ldots\) be a sequence of random variables. Now, \(\alpha\) mixing is defined as,

\[
\alpha_n = \sup_{A \in \sigma(X_1, \ldots, X_k); B \in \sigma(X_k + n, X_{k+n+1}, \ldots)} |P(A \cap B) - P(A)P(B)|
\]

where \(\sigma(X_1, \ldots, X_k)\) be the sigma algebra generated by \(X_1, \ldots, X_k\). Also, the \(\psi\) mixing condition is given by

\[
\psi_n = \sup_{A \in \sigma(X_1, \ldots, X_k); B \in \sigma(X_k + n, X_{k+n+1}, \ldots)} \left| 1 - \frac{P(B|A)P(B)}{P(B)} \right|
\]

Now, if \(\alpha_n \to 0\), then the random variables \(X_i\) and \(X_{i+n}\) are taken to be independent for large \(n\) and the sequence is addressed as to have \(\alpha\)-mixing. Note, if the mean and co-variance of \(X_n\) is independent of \(n\), then \(X_n\) is assumed to be stationary.

The following theorem is taken from Billingsley, [6, Thm 27.4]:

**Theorem 9.2.** Suppose \(X_1, X_2, \ldots\) is stationary with \(\alpha_n = O(n^{-5})\), \(E(X_n) = 0\) and \(E(X_n^{12}) < \)
Let
\[ S \triangleq \sum_{n=1}^{N} X_n, \]
then,
\[ N^{-1} \text{Var}(S_n) \to \sigma^2 = \mathcal{E}(X_1^2) + 2 \sum_{k=1}^{\infty} \mathcal{E}(X_1X_{1+k}), \]
where the series converges absolutely. If \( \sigma > 0 \), then \( \frac{S_n}{\sigma \sqrt{N}} \sim N(0,1) \).

**Proof.** Refer [6, Thm 27.4] for proof

This above theorem is important for numerous aspects as it implies that our \( \tilde{M} \), (1.11), tends to \( M \), (1.4), as \( N \to \infty \) for fixed \( K \). It also gives the rate of growth for a sum of a series satisfying the above conditions to be \( O_p(\sqrt{N}) \). This fact is used in deriving lemma (5.3).

In lemma (5.3), we obtained the matrix \( \tilde{\Sigma}_{xx}(l) - \Sigma_{xx}(l) \) where \( \tilde{\Sigma}_{xx}(l) \) is a sample covariance matrix between \( x_n \) and \( \varepsilon_n \) and \( \Sigma_{xx}(l) \) is their ideal co-variance matrix. Therein, \( x_n \) and \( \varepsilon_n \) are given by
\[ y_n = Hx_n + \varepsilon_n. \]
where \( y_n \) is the stochastic sequence, which satisfies the properties in Theorem 9.2 (Refer Condition (A5) of Chapter 2). Since \( x_n \) and \( \varepsilon_n \) are also linearly related to \( y_n \), they too satisfy those properties ensuring that the every element in the matrix \( x_{n+l}^\top \varepsilon_n - \mathcal{E}(x_{n+l}^\top \varepsilon_n^\top) \) also satisfy the conditions for Theorem 9.2. Since every element in the matrix \( \tilde{\Sigma}_{xx}(l) - \Sigma_{xx}(l) = \frac{1}{N} \sum_{n=1}^{N} (x_{n+l}^\top \varepsilon_n^\top - \mathcal{E}[x_{n+l}^\top \varepsilon_n^\top]) \) correspond to the sum of the above sequence from \( n = 1 \ldots N \) divided by \( N \), they all converge at the rate \( O_p\left(\frac{1}{\sqrt{N}}\right) \).

Note that \( \alpha - mixing \), also known as strong mixing, is weaker than \( \psi - mixing \). Hence
ψ - mixing implies α - mixing. For more details regarding the property of ψ mixing, refer [10].
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