The Leray-Schauder Degree as Topological Method Solution of Nonlinear Elliptic Equations

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1. Introduction

The infinite dimensional space under consideration in this section are normed linear vector spaces are their subsets. Extensions will be discussed in the next section. Let X be a real linear vector space. On X we define a norm ||·|| or ||·|| for short which satisfies the following hypotheses:

\[ \|x + y\| \leq \|x\| + \|y\|, \text{for all } x, y \in X \] (1)
\[ \|\lambda x\| \leq |
\lambda\|\|x\|, \text{for all } x \in X, \text{and for all } \lambda \in \mathbb{R} \] (2)
\[ \|x\| = 0 \text{ if and only if } x = 0 \] (3)

The combination is called a normed linear vector space. If an addition is X complete it is called a Banach space. A normed linear space is complete if every Cauchy sequence has a limit in X, \( \{x^n\} \subset X \) with \( \|x^n - x^m\| \to 0 \) as \( n, m \to \infty \), implies that there exists a \( x \in X \) such that \( \|x^n - x\| \to 0 \) as \( n \to \infty \). Normed vector space and Banach space are examples of metric and complete metric spaces respectively, where the metric is given by: \( d(x, y) = \|x - y\| \).

Continuity: Definition (1-1): A mapping \( f:X \to Y \) is continuous if \( x_n \to x \) (in X) implies that \( f(x_n) \to f(x) \) (in Y). A map is uniformly continuous on X, if for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( \|x - y\| < \delta \) implies that \( \|f(x) - f(y)\| < \epsilon \). The latter can be defined with respect to a closed subset \( A \subset X \).

A continuous function \( f:X \to Y \) is bounded if \( f(\hat{U}) \subset X \) is bounded for any bounded subset \( \hat{U} \subset X \). Continuous mapping on \( \mathbb{R}^n \) are necessarily bounded, i.e. bounded sets in \( \mathbb{R}^n \) are mapped to bounded set under \( f \). This is however not the case in general Banach spaces.

Lemma (1-2): A uniformly continuous map is bounded.

Proof: We need to show that for any bounded set the image \( A \subset X \) the image \( f(A) \subset Y \) is also bounded. Choose \( R > 0 \) such that \( A \subset B_R(0) \), and let \( n > \frac{2R}{\delta} \). Then for any two points \( x, y \in A \) it holds that \( \|x - y\| \leq 2R \), and one can define the line segment \( t = x + t(y - x), t \in [0,1], \text{in } B_R(0) \).

For \( t_i = \frac{i}{n} \) we obtain point \( x^{i} \in B_R(0) \), with \( \|x^{i} - x^{i+1}\| < \delta \), by the choice of \( n \). Since \( f \) is uniformly continuous it follows that \( \|f(x^{i}) - f(x^{i+1})\| < \epsilon, \text{for all } i \).

Form the triangle inequality we then get: \( \|f(x) - f(y)\| \leq \sum \|f(x^{i}) - f(x^{i+1})\| < n \epsilon \).

Which prove the boundedness of \( f \)

Differentiability: Definition (1-3): A mapping \( f:C(X, Y) \) is called Gateaux differentiable in the direction \( h \in X \), at a point \( x_0 \), if there exists a \( y \in Y \) such that \( \lim_{t \to 0} -\|f(x_0 + th) - f(x_0) - ty\| = 0 \),

With \( x_0 + th \) defined in a neighborhood \( N \) of \( x_0 \).

Compact and finite rank maps: An important subspace of continuous mappings are the compact mappings from \( f:X \to X \). A mapping \( f:X \to X \) is compact if \( \hat{f}(\hat{U}) \) is compact for any bounded subset \( \hat{U} \subset X \). Compact mapping are bounded since \( \hat{f}(\hat{U}) \) is bounded for any bounded set \( \hat{U} \subset X \). The space of compact mappings on \( X \) is denoted by...
$K(X)$.

Definition (1-4): A continuous map $K: \Omega \subset X \to X$ is called compact if $f(\overline{\Omega})$ is compact.

Lemma (1-5): Let $k \in K(\overline{\Omega})$, then for any $\epsilon$, there exists a finite rank map $k^{\epsilon} \in C(\overline{\Omega})$ such that $\| k - k^{\epsilon} \|_{C^0} < \epsilon$ if $\epsilon \in C(\overline{\Omega})$ and $C^0(\overline{\Omega})$.

Proof: Let $k(\overline{\Omega})$ is compact it can be covered by finitely many balls $B(x^l)$, with $x^l \in k(\overline{\Omega})$. Define $\mu^l(x) = \delta(x - x^l)$.

Where $\delta(x) = \max(0, -\| k(x) - x^l \|)$. This maximum is zero whenever $k(x) \notin B(x^l)$ and therefore $\mu^l(x) = 0$, unless $\| k(x) - x^l \| < \epsilon$.

Set $k^{\epsilon}(x) = \sum \mu^l(x). x^l$. Now $k^{\epsilon}(\Omega) \subset \text{span} \{x^l\}$. As for the approximation we obtain: $\| k - k^{\epsilon} \|_{C^0} = \| k - \sum \mu^l(x). (x - x^l) \|_{C^0}$.

$\| k - k^{\epsilon} \|_{C^0} = \sup_{x \in \Omega} \left| \sum \mu^l(x) (k(x) - x^l) \right| < \sum \mu^l(x) \epsilon = \epsilon$

Which complete the proof #

2. Preliminaries

Definition (1-4): Let $f$ be a continuous map of the form $f \in C^0(\overline{\Omega})$, and let $f \notin (\partial \Omega)$. Let $k^{\epsilon}$ be a finite rank perturbation with $\| k - k^{\epsilon} \|_{C^0} < \epsilon$ and $\epsilon < \delta/2$ ($\delta$ as given above) and with $k^{\epsilon}(\overline{\Omega}) \subset Y^{\epsilon} \subset X$ (subspaces). Then for any finite dimensional subspace $X^{\epsilon}$ containing both $Y^{\epsilon}$ and $p$, define the Leray Schauder degree as

$$\deg(f, \Omega, p) = \deg(f^{\epsilon}, \Omega \cap X^{\epsilon}, p)$$

Where $f^{\epsilon} = Id - k^{\epsilon}$. If there is no ambiguity about the context we mostly omit the subscript for the notation.

Lemma (2-2): Let $X^{\epsilon} \subset X$ be any finite dimensional subspace such that $Y^{\epsilon} \subset X^{\epsilon}$ and $p \in X^{\epsilon}$. Then $\deg(f, \Omega \cap X^{\epsilon}, p) := \deg(f^{\epsilon}, \Omega \cap X^{\epsilon}, p)$

Proof: Step (i): Consider mappings of the form $g = Id - h; D \subset \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^n \oplus \mathbb{R}^m$, with $h(D) \subset \mathbb{R}^n$. Suppose $p \in \mathbb{R}^n$ and $p \notin g(\partial D)$. Then $\deg(g, D, P) = \deg(g_n, D \cap \mathbb{R}^n, p)$, where $g_n = g|_{\mathbb{R}^n}$

We prove the above statement in the case that $h$ is $C^1$, since the degree is defined via $C^1$ approximations and with $P = 0$. Let $\omega_1$ and $\omega_2$ be top forms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively with $\int_{\mathbb{R}^n} \omega_1 = \int_{\mathbb{R}^n} \omega_2 = 1$ and their supports contained in a sufficiently small neighborhood of the origin. In terms of coordinates we write $x = x_1 + x_2, x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. For the degree this yields

$$\int g^* (\omega_1 \oplus \omega_2) = \int (\omega_1(x_1 - h(x_1 + x_2)) \omega_2(x_2)) dx_1 dx_2$$

By the specific form of we have that $J_g(x_1 + x_2) = \det \left( Id - \frac{\partial h}{\partial x_1} \right) \text{since the expression for the degree is independent of } \omega_1 \text{ and } \omega_2$ we can choose $\omega_2$ to approximate a density function that peaks at 0 and has integral equal to 1 approximating a delta distribution. Due to the independence on $\omega_2$ this give

$$\deg(g, D, P) = \int_{\mathbb{R}^n} \omega_1(x_1 - h(x_1 + x_2)) det \left( Id - \frac{\partial h}{\partial x_1} \right) dx_1 = \deg(g_n, D \cap \mathbb{R}^n, p)$$

Step (ii): Since $Y^{\epsilon} \subset X^{\epsilon}$ and $p \in X^{\epsilon}$ we may assume without loss of generality that $X^{\epsilon} \subset X^{\epsilon}$. By construction it holds that $f^{\epsilon}; \overline{\Omega} \subset X^{\epsilon}$ and $f^{\epsilon}; \overline{\Omega} \subset X^{\epsilon}$. By construction it holds that $f^{\epsilon}; \overline{\Omega} \subset X^{\epsilon}$ and $f^{\epsilon}; \overline{\Omega} \subset \mathbb{R}^n$. Consider the linear change of variables $y = q(x)$ such that $q(X^{\epsilon}) = \mathbb{R}^n \oplus 0$ and $\mathbb{R}^{(X^{\epsilon})} = \mathbb{R}^n \oplus \mathbb{R}^m$. From Step (i) it follows that $\text{deg}(g, D, P) = \text{deg}(g_n, D \cap \mathbb{R}^n, p)$. It remains to prove that the degree is invariant under the change of coordinates. Using differential characterization of the degree we obtain:

$$\int \omega^* \omega = \int (\omega_0 f^* \omega^{\epsilon})^* \omega$$

$$= \int (\omega_0 (f^* \omega^{\epsilon}))^* (f^* \omega^{\epsilon})$$

$$= \text{sign} \left( J_{q^{\epsilon}}(x) \right) \int_{\mathbb{R}^n} \omega^* \omega$$

$$= \text{sign} \left( J_{q^{\epsilon}}(x) \right) \text{deg}(f^*, \Omega \cap X^{\epsilon}, p) \int_{\mathbb{R}^n} \omega$$

$$= \text{deg}(f^*, \Omega \cap X^{\epsilon}, p) \int_{\mathbb{R}^n} \omega$$

Since $\text{deg}(g, D, q(p)) = \text{deg}(f^*, \Omega \cap X^{\epsilon}, p)$ it follows that:

$$\text{deg}(g, D, q(p)) = \text{deg}(f^*, \Omega \cap X^{\epsilon}, p)$$

Which proves that the degree is invariant under coordinate changes.

By restricting to the subspace $\epsilon$ we obtain $\text{deg}(g_n, D \cap \mathbb{R}^n, q(p)) = \text{deg}(f^*, \Omega \cap X^{\epsilon}, p)$. The proof follows now from Step (i).

Lemma (2-3): Let $k^{\epsilon}$ and $\overline{k^{\epsilon}}$ both be finite rank approximations for with $\| k - k^{\epsilon} \|_{C^0} < \epsilon \| k - \overline{k^{\epsilon}} \|_{C^0} < \epsilon$
and \( \epsilon < \delta / 2 \). Then

\[
\text{deg}(I_d - k^\epsilon, \Omega \cap X^\epsilon, p) = \text{deg}(I_d - \bar{k}^\epsilon, \Omega \cap \bar{X}^\epsilon, p)
\]

For any subspace \( X^\epsilon \) and \( \bar{X}^\epsilon \) containing both \( p \) and the
ranges of \( k^\epsilon \) and \( \bar{k}^\epsilon \).

Respectively.

Proof:

Let \( Z^\epsilon \subset X \) be a finite dimensional linear subspace
containing both \( X^\epsilon \) and \( \bar{X}^\epsilon \). From Lemma (2-3) it follows that

\[
\text{deg}(I_d - k^\epsilon, \Omega \cap X^\epsilon, p) = \text{deg}(I_d - k^\epsilon, \Omega \cap Z^\epsilon, p)
\]

\[
\text{deg}(I_d - \bar{k}^\epsilon, \Omega \cap \bar{X}^\epsilon, p) = \text{deg}(I_d - \bar{k}^\epsilon, \Omega \cap Z^\epsilon, p)
\]

Consider the compact homotopy \( k_t^\epsilon = (1 - t)k^\epsilon + t \bar{k}^\epsilon \),
which yields a homotopy \( f_t = I_d - k_t^\epsilon \) and is a proper homotopy.

Then \( \text{deg}(I_d - k^\epsilon, \Omega \cap Z^\epsilon, p) = \text{deg}(I_d - \bar{k}^\epsilon, \Omega \cap Z^\epsilon, p) \)
which then prove that \( \text{deg}(I_d - k^\epsilon, \Omega \cap X^\epsilon, p) = \text{deg}(I_d - \bar{k}^\epsilon, \Omega \cap \bar{X}^\epsilon, p) \).

The Leray–Schauder degree is well defined.

Theorem (2-4): Let \( F \) be \( C^m \) – function \( \Omega \times R^m \) which
defined a strongly elliptic nonlinear partial differential
operator \( F(u) \) of order \( m \) on \( \Omega \). Let \( 0 < \lambda < 1 \) and for
\( 0 \leq t \leq 1 \), let \( F(x, p, t) = t F(x, p) + (1 - t) \sum_{|a|=2m} \|p_a\|^2 \),
\( F_t \) the corresponding partial differential operator of order.

Suppose that all of the following conditions are satisfied:

For each \( \geq 0 \), there exists constant \( \mu \) with \( 0 < \mu < \lambda < 1 \) and
a differential operator \( H \) of order \( m - 1 \) "possibly nonlinear" on \( \Omega \) such that for each \( u \in C_0^{2m-1,\lambda}(\Omega) \), \( v \in C_0^{2m,\mu}(\Omega) \) with \( \|u\|_{C_0^{2m-1,\lambda}} \leq R \) and

\[
\{p_{\lambda}(u, v)\}_\lambda = \left\{ \begin{array}{ll}
D^a u(x), & |a| = m \\
D^a v(x), & |a| = m
\end{array} \right\}
\]

\[
\sum_{\lambda=2m} \sum_{|a|=2m} \|p_a\| D^a \eta + \sum_{\lambda=m-1} \sum_{|a|=\lambda} \|p_a\| D^a \eta = 0
\]

\[
\text{Has only } \eta = 0 \text{ as a solution in } C_0^{2m,\mu}(\Omega), 0 \leq t \leq 1.
\]

For given \( R > 0 \) and the corresponding function \( H \) of condition
(1), there exists a function \( R_t(s) \) such that for \( u \in C_0^{2m-1,\lambda}(\Omega) \) with \( \|u\|_{C_0^{2m-1,\lambda}} \leq R \) and any
\( v \in C_0^{2m,\lambda}(\Omega) \) such that \( p_{\lambda}(u, v) + t H(v, p(u)) = f \), for
some \( t \in [0,1] \) and \( f \in C_0^{\lambda,\lambda}(\Omega) \) with \( \|f\|_{C_0^{\lambda,\lambda}} \leq s \), \( \|v\|_{C_0^{2m,\lambda}} \leq R_t(s) \).

There exists a constant \( R_0 > 0 \) such that for any \( t \in [0,1] \)
\[
F_t = F_t(p(u, v)) + t H_t(p(v)) - t H_t(p(u))
\]

and in \( v \in C_0^{2m,\mu}(\Omega) \), if

\[
F_t(v) = 0 \text{ we have } \|v\|_{C_0^{2m-1,\lambda}} \leq R_o.
\]

Then the equation \( F(u) = 0 \) has a solution \( u \in C_0^{2m,\mu}(\Omega) \).

Proof of Theorem: Let \( R = R_o, B = \{u \in C_0^{2m,\mu}(\Omega), \|u\| \leq R \} \). Let \( H \) be
the function corresponding to \( R \) by condition (1) of the hypothesis of Theorem (2-1).

For each \( u \in R \), we consider the equation:

\[
(i) \quad F_t(p(u, v)) + t H_t(p(v)) = t H_t(p(u))
\]

For \( \in C_0^{2m,\mu}(\Omega) \). The linearized form of equation (i) is

\[
(ii) \quad \sum_{\lambda=2m} \sum_{|a|=2m} F_t(p(u, v)) D^a \eta + \sum_{\lambda=m-1} \sum_{|a|=\lambda} t H_t(p(u, v)) D^a \eta = 0
\]

Which by condition (1) has only \( \eta = 0 \) as a solution in \( C_0^{2m,\mu}(\Omega) \) for a fixed \( \mu \) with \( 0 < \mu < \lambda < 1 \). Moreover by
condition (ii) of the hypothesis, the solution \( v \) of the equation

\[
(iii) \quad F_t(p(u, v)) + t H_t(p(u, v)) = f
\]

For \( \|u\|_{C_0^{2m-1,\lambda}(\Omega)} \leq R \) for \( f \in C_0^{2m,\mu}(\Omega) \) with
\( \|f\|_{C_0^{\lambda,\lambda}} \leq s \), where \( v \) lies in \( C_0^{2m,\lambda}(\Omega) \), must satisfy the inequality
\( \|v\|_{C_0^{2m-1,\lambda}(\Omega)} \leq R_t(s) \). Hence the hypotheses of
Theorem (2-1) are satisfied for the family of equations (ii) and in particular, equation (i) has one and only one solution
\( v_t \) for each \( t \in [0,1] \).

We set \( C_t(u) = v_t \). Then \( C_t \) is a well-defined mapping on
\( B_R \) whose range we consider as a subset of \( C_0^{2m-1,\lambda}(\Omega) \). Since
the map \( u \rightarrow H(p(u)) \) carries bounded sets of
\( C_0^{2m-1,\lambda}(\Omega) \) into bounded sets of \( C_0^{2m,\lambda}(\Omega) \) it follows from the argument of the preceding paragraph that
\( \|C_t u\|_{C_0^{2m-1,\lambda}(\Omega)} \leq R_t(s) \) for all \( u \in B_R \) and all \( t \in [0,1] \) with a fixed constant
\( R_t > 0 \). Since \( C_0^{2m,\lambda}(\Omega) \) has a compact injection into
\( C_0^{2m-1,\lambda}(\Omega) \), it follows that \( u_{\text{ext}} C_t(B_R) \) is precompact in
\( C_0^{2m-1,\lambda}(\Omega) \).

We wish now to verify that mapping \( [t, u] \rightarrow C_t(u) \) is a continuous mapping of
\( [0,1] \times B_R \) into \( C_0^{2m-1,\lambda}(\Omega) \). Let \( t_o \) be a fixed number in
\( [0,1], u_o \), we have

\[
F_t(p(u, v)) + t H_t(p(v)) - t H_t(p(u))
\]

Furthermore

\[
F_t(p(u, v)) = F_t(p(u, v)) + F_t(p(u, v))(u - v_0) + R(u, t_o v)
\]
Where
\[ v_0 = C_{t_0}(v_0) \text{ and } \| R_t(u_0, v_0, t_0, v) \|_{\rho, 0, t_0} = 0 (\| u - v_0 \|) \text{ as } \| u - v_0 \|_{\rho, 0, t_0} \to 0 \]
The norm of \( u - v_0 \) will be taken in \( C_{0, 2m, \lambda} (\Omega) \) throughout this argument.

Similarly
\[ t_0 H_{t_0}(p(v)) = t_0 H_{t_0}(p(v_0)) + t_0 H_{t_0}(p(v_0)) (u - v_0) + R_t(u_0, v_0, t_0, v) \]
Where \( R_t(u_0, v_0, t_0, v) = 0 (\| u - v_0 \|) \text{ as } \| u - v_0 \|_{\rho, 0, t_0} \to 0 \)
It follows that for \( v \) near \( v_0 \) in \( C_{0, 2m, \lambda} (\Omega) \) we have
\[ F_t(p(u, v)) + t H_t p(u) - t H_t p(u) = \{ F_t(p(u_0, v_0)) + t_0 H_{t_0}(p(u_0)) - t_0 H_{t_0}(p(v_0)) [F_t(p(u_0, v_0)) + t_0 H_{t_0}(p(u_0)) (u - v_0) + R_2(u_0, v_0, t_0, u, v) \]
Where \( \| R_2(u_0, v_0, t_0, u, v) \|_{\rho, 0, t_0} \to 0 \)
\[ \leq \sigma(u - v_0) \| u - v_0 \|_{\rho, 0, t_0} + \sigma_2(u_0, v_0, t_0, u, v, v) \]
The operator in square brackets is an isomorphism of \( C_{0, 2m, \lambda} (\Omega) \) with \( C_{0, \lambda} (\Omega) \) by

- Condition (1) of the hypothesis. Hence for \( \| u - v_0 \| + |t - t_0| \) sufficiently small, we may find a solution \( v \) of equation (i) in a prescribed neighborhood of \( v_0 \) in \( C_{0, 2m, \lambda} (\Omega) \) with \( \| u - v_0 \|_{\rho, 0, t_0} \leq \rho (\| u - v_0 \| + |t - t_0|) \)

Where \( (s) \to 0 \) as \( s \to 0 \). Since the solution of (i) is unique, \( v = C_t(u) \). Hence \( C_t \) maps \([0, 1] \times B_R \) continuously into \( C_{0, 2m, \lambda} (\Omega) \) and a fortiori into \( C_{0, 2m, \lambda - 1, \lambda} (\Omega) \).

We now apply the theorem of the Leray – Schauder degree to the family of mappings
\[ I - C_t, 0 \leq t \leq 1 \]
For \( t = 1, C_t = 0 \) since then \( v \) is a solution of
\[ \sum_{|a|=m} D^a \partial^{a} v = 0 \]
Hence the degree of \( T_0 \) over \( B_R \) with respect to \( 0 \) is equal to \( +1 \). For each \( t \in [0, 1], C_t \)
Is a compact map and the degree of \( T_t \) over \( B_R \) with respect to \( 0 \) is well – defined since
For \( u \in B_R \) with \( \| u \|_{\rho, 2m, \lambda(t)} = R, T_t u = 0 \) implies that
\[ F_t(p(u)) + t H_t p(u) = t H_t p(u) \text{ i.e } F_t(p(u)) = 0 \]
And for solutions of the latter equation. Condition (3) of the hypothesis assures that
\[ \| u \|_{\rho, 2m, \lambda(t)} \leq R_R = R \]
The degree of \( T_t \) over \( B_R \) with respect to \( 0 \) is constant in \( t \) by the continuity and compactness of \( C_t \) in the pair \( [t_0, u] \). Hence the degree of \( T_t \) over \( B_L \) with respect \( t \) ois equal to \( +1 \) and there exists a solution \( u \) in \( B_R \) of \( T_t u = 0 \).

This is equivalent, however to \( F(p(u)) = 0 \)

The problem with degree theory in infinite dimensional spaces is that homotopy
Invariance, a basic property of the degree, prevents the existence of a nontrivial
Degree theory. We can alter the notion of homotopy invariance in order to a degree theory, or limit the types of maps for which a degree is well defined the Leray Schauder degree does both by considering specific types of mappings. Namely mappings of the form \( f = I - k \), where \( I d \) is the identity map on \( X \) and
\[ k \in K(\Omega) \]
Homotopies are considered in the same class, denote the function class by \( C_{0, \lambda}^0(\Omega) \}
Properties of the Leray – Schauder degree:

Theorem (2-5): For Leray –Schauder degree we have the following properties:
(A1) if \( p \in \Omega, \) then \( deg_{LS}(I d - k, \Omega, p) = 1 \)
(A2) For \( \Omega^1, \Omega^2 \subset \Omega, \) disjoint open subsets of \( \Omega, \) and then \( p \in \Omega^1 \cup \Omega^2, \) it holds that
\[ deg_{LS}(f, \Omega, p) = deg_{LS}(f, \Omega^1, p) + deg_{LS}(f, \Omega^2, p) \]
(A3) For any continuous path \( t \to f_t = I t - k, t \in K(\Omega) \)
And \( t \to p_t, p_t \notin f_t(\partial \Omega) \) it holds that \( deg_{LS}(f_t, \Omega, p_t) \) is independent of \( \in [0, 1]; deg_{LS} \) and is called a degree theory.
As in the case of the Brouwer degree the essential properties of the Leray –Schauder degree follow from (A1) – (A2).

Property (Validity of the degree) (2-6): If \( p \notin \Omega, \) the \( deg_{LS}(f, \Omega, p) = 0 \).
Conversely, if then \( deg_{LS}(f, \Omega, p) = 0 \), then there exists a \( x \in \Omega \) such that \( f(x) = p \).

Property (Continuity of the degree) (2-7): The degree \( deg_{LS}(f, \Omega, p) \) is continuous in \( f = I d - k \), i.e there exists a \( \delta = (\delta(f), \Omega)) > 0 \), such that for all \( g = I d - k \) satisfying
\[ \| k - k \|_{\Omega} < \delta , \]
It holds that \( p \notin g(\partial \Omega) \) and \( deg_{LS}(g, \Omega, p) = deg_{LS}(f, \Omega, p) \).

Property (Dependence on path components) (2-8): The degree only depends on the path components \( D \subset X \setminus f(\partial \Omega) \)
I.e for any two points \( p, q \in D \subset X \setminus f(\partial \Omega) \) it holds that
\[ deg_{LS}(f, \Omega, p) = deg_{LS}(f, \Omega, D) \]
invariant under translation, i.e. for any \( q \in X \) it holds that
\[ deg_{L^2}(f - q, \Omega, p - q) = deg_{L^2}(f, \Omega, p) \]

Property (Excision) (2-10): Let \( \Omega \subset \subset \Omega \) be a closed subset in \( \Omega \) and \( p \neq f(\lambda) \). Then \( deg_{L^2}(f, \Omega, p) = deg_{L^2}(f, \Omega, \lambda, p) \).

Property (Additivity) (2-11): Suppose that \( \Omega_1 \subset \subset \Omega_i = 1, \ldots, i \) are disjoint open subsets of \( \Omega \), and \( p \not\in f(\Omega_1 \setminus (U_i \Omega_i)) \), then \( deg_{L^2}(f, \Omega, p) = \sum_i deg_{L^2}(f, \Omega_i, p) \).

Definition of the Compact homotopy (2-12): Two mappings
\[ f, g : \Omega \subset X \rightarrow Y \]
are said to be compactly homotopic relative of \( F \) if there exists a family of compact mappings \( k(t, \cdot) : \Omega \subset X \rightarrow Y, t \in [0,1] \), such that
\[ h(t, x) = F(x) + k(t, x) \]
\[ h(0, x) = F(x) + k(0, x) = f(x) \]
\[ h(1, x) = F(x) + k(1, x) = g(x) \]

The associated compact homotopy classes are denoted by \([f, .]_c\).

3. Main Result: Semi Linear Elliptic Equations and a Priori Estimates

In this section we will give Application of the Leray - Schauder degree in the context of nonlinear elliptic equations. We follow the notes by L. Nirenberg.

The methods that we discuss apply in general for elliptic differential operator of any order. Let \( D \subset \mathbb{R}^n \) be a bounded domain with smooth boundary \( \partial D \).

Consider the problem \(-\Delta u = g(x, u, \nabla u), u = 0, x \in \partial D \).

For the nonlinearity \( g \) we assume that \( C^m \) - function of arguments, i.e. \( g \in C^m(\overline{D} \times \mathbb{R} \times \mathbb{R}^n), \) and \( |g(x, u, \nabla u)| \leq C + C |\nabla u|^r, r < 1 \)

Uniformly in \( x \in \overline{D} \) and \( u \in \mathbb{R} \). Under these conditions we can prove the following result.

Theorem (3-1): Under the assumptions on \( g \) the above elliptic equations has a solution \( u \in C^m(\overline{D}) \). Moreover, if \( g(x, 0, 0) \neq 0 \), then the solution \( u \) is not identically zero.

Proof: The idea behind the proof is the formulate the above elliptic equation as a problem of finding zeroes of an appropriate function \( f \) on (infinite dimensional ) Banach space.

Let start with choosing an appropriate space in which to work. Define \( X = H^2 \cap \mathcal{H}_0^1(D) \) to be the intersection of two Sobolev space.

The space \( H^2 \cap \mathcal{H}_0^1(D) \) is a Hilbert space with norm
\[ ||u||_X = \int_D |\Delta u| \, dx \]

Due to the Dirichlet boundary conditions the Laplace operator
\[ -\Delta : H^2 \cap \mathcal{H}_0^1(D) \subset L^2(D) \rightarrow L^2(D) \]
has a compact inverse \((-\Delta)^{-1} : L^2(D) \rightarrow L^2(D) \).

We rewrite the elliptic equation as:
\[ u - (-\Delta)^{-1} g(x, u, \nabla u) = 0 \]

The above equation can be regarded as a seeking zeroes of the (Nemytskii) mapping \( f(u) = u - (-\Delta)^{-1} g(x, u, \nabla u) \) on \( H^2 \cap \mathcal{H}_0^1(D) \). By the estimate on \( g \) we have that
\[ \left| \int_B |g(x, u(x), \nabla u(x))|^2 \, dx \right| \leq C \int_B \left| \nabla u(x) \right|^2 \, dx \leq \frac{C}{\lambda_1} \left( \int_B \left| \nabla u(x) \right|^2 \, dx \right)^\gamma \]

Which proves that for \( u \in X, g(x, u, \nabla u) \) is an \( L^2 \) function. Consequently, \( f \) is continuous and \( \mathcal{F} = \mathcal{F}(X) \subset X \).

The space \( \mathcal{F} \) is a compact map. Indeed, \( \mathcal{F} \) is a composition of the Nemytskii map \( u \rightarrow g(x, u, \nabla u) (\text{from } Y \rightarrow L^2) \), the inverse Laplacian \((-\Delta)^{-1} \) on \( L^2 \rightarrow X \) and the compact embedding \( X \rightarrow Y \), which proves the compactness of \( \mathcal{F} \).

This brings us into the realm of the Leray - Schauder degree.

Suppose \( u \in X \) is a solution of the equation (3-1-1), then the estimate on \( g(x, u, \nabla u) \) can be used now to obtain an a priori estimate on the solutions.

\[ ||u||_{\mathcal{F}} \leq C ||u||_X = C ||g(x, u, \nabla u)||_{L^2} \leq (1 + ||u||_X^2)^\gamma \]

Which, since \( \gamma < 1 \), implies that \( ||u||_{\mathcal{F}} \leq R \)

Define the domain \( \Omega = B_{2\rho}(0) \subset X \). Clearly, \( f \) is a continuous map from \( \overline{\Omega} \) into \( X \), which is of the from identity minus compact. Due to the above a priori estimate \( f^{-1}(0) \subset B_{\rho}(0) \) and \( 0 \not\in f_*(\partial \Omega) \) and therefore the Leary –Schauder degree \( d_0(f, \partial \Omega, 0) \) is well defined.

In order to compute this degree we consider the following homotopy:
\[ f_t(u) = u - t(-\Delta)^{-1} g(x, u, \nabla u) \]

Notice, for \( t \in [0,1] \) we have via the same a priori estimates, that \( f^{-1}(0) \subset B_{\rho}(0) \) and therefore \( 0 \not\in f_*(\partial \Omega) \) for all \( t \in [0,1] \). Homotopy invariance of the Leary –Schauder degree then yields
\[ d(f, \Omega, 0) = d(id, \Omega, 0) = 1 \]

Which implies, by validity property of the Leary – Schauder degree, that \( f^{-1}(0) \neq 0 \). Equation (3-1-1) thus has a solution \( u \in Y \). The equation yields \( u = (-\Delta)^{-1} g(x, u, \nabla u) \) on \( X \), which that the solution also lies in \( X \). To prove regularity we use a bootstrap argument. The integral estimates on \( g \) can be adjusted to \( L^p \) estimates. This gives, by the Sobolev embeddings that:
\[ u \in H^{1,p} \Rightarrow g(x, u, \nabla u) \in L^p \Rightarrow u \in H^{2,p} \Rightarrow u \in H^{1,p} \]
Where \( \frac{1}{p} = \frac{1}{2} - \frac{1}{n} \), provide \( n > p \). This yields the recurrence relation

\[
\frac{1}{p^{k+1}} = \frac{1}{p^k} - \frac{1}{n}
\]

We can repeat these recurrent steps until \( k \) time until \( 2(k + 1) > n > 2k \)

And then \( u \in H^{2, p^k} \), where \( p^k = \frac{2n}{n-2k} \). Gain by the Sobolev embeddings, we have that \( H^{2, p^k}(D) \hookrightarrow C^{1, \alpha}(\overline{D}) \).

Where \( \alpha = 1 - \frac{n}{p^k} \), since \( \frac{n}{p^k} = \frac{n}{2} \), and \( k + 1 > \frac{n}{2} > k \), it holds \( 0 < \alpha < 1 \). We now repeat the bootstrapping in the Holder space:

\[
u \in C^{1, \alpha} \Rightarrow g(x, u, \nabla u)u \in C^{0, \gamma \alpha} \Rightarrow u \in C^{2, \dot{\alpha}}
\]

Where \( \dot{\alpha} = \gamma \alpha \). The idea now is the use the elliptic regularity theory for the Laplacian be differentiation the equation. Let \( \nu_i = \frac{\partial u}{\partial x_i} \) then

\[
-\Delta \nu_i = \partial_x g + (\partial_u g) \nu_i + \sum_j \partial_{x_j} g \frac{\partial \nu_j}{\partial x_i}
\]

Singe \( g \) is \( C^{\infty} \) function of its arguments, and \( u \in C^{2, \dot{\alpha}} \), the right hand side is in \( C^{0, \alpha} \), implying that \( \nu_i \in C^{2, \alpha} \), and thus \( \nu \in C^{3, \dot{\alpha}} \). We can repeat this process indefinitely, which proves that \( u \in C^{\infty}(\overline{D}) \)

If \( g(x, 0, 0) \neq 0 \), then \( u = 0 \) cannot be a solution, and thus \( u \neq 0 \# \)

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