Global existence and decay estimates for the nonlinear wave equations with space-time dependent dissipative term

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Abstract

We study the global existence and decay estimates for nonlinear wave equations with the space-time dependent dissipative term in an exterior domain. The linear dissipative effect may vanish in a compact space region. Moreover the nonlinear terms need not divergence form. For getting the higher order energy estimates, we introduce an argument using the rescaling. The method is useful to control derivatives of the dissipative coefficient.

Key Words and Phrases. dissipative nonlinear wave equations, time-space variable coefficient, time decay estimates

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1 Introduction

Let $d \geq 2$ and $\Omega = \mathbb{R}^d / \mathcal{O}$, where $\mathcal{O}$ is a star-shaped domain with a smooth and compact boundary $\partial \Omega$. Moreover we assume that $\mathcal{O}$ contains the origin. In this paper, we consider the initial-boundary value problem for nonlinear wave equations with the space-time dependent dissipative term:

\[
\begin{aligned}
& (\partial_t^2 - \Delta + B(t, x)\partial_t)u(t, x) = F(\partial u, \partial^2 u) & (t, x) & \in [0, \infty) \times \Omega, \\
& u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x) & x & \in \Omega, \\
& u(t, x) = 0 & (t, x) & \in [0, \infty) \times \partial \Omega,
\end{aligned}
\]

where $u = (u^1, \cdots, u^d)$, $\nabla = (\partial_{x_1}, \cdots, \partial_{x_d})$ and $\partial = (\partial_t, \nabla)$. The initial data $(u_0, u_1)$ belongs to $H^L(\Omega) \times H^{L-1}(\Omega)$ and satisfies the compatibility condition of order $L - 1$. $H^L(\Omega)$ is the Sobolev space in $\Omega$. We make the following assumptions for the space-time dependent damping coefficient matrix $B(t, x) = (B_{pq}(t, x))_{p,q=1,\cdots,d}$:

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(B0) $B_{pq}$ belong to $\mathcal{B}^\infty([0, \infty) \times \Omega)$, where $\mathcal{B}^\infty$ is the function space of smooth functions with bounded derivatives.

(B1) $B(t, x)$ is nonnegative definite in $[0, \infty) \times \Omega$.

(B2) $\partial_t B(t, x)$ is nonpositive definite in $[0, \infty) \times \Omega$.

(B3) There exist $b_0 > 0$ and $R > 0$ such that

$$
\sum_{p, q=1}^{d} B_{pq}(t, x) \eta_p \eta_q \geq b_0 |\eta|^2 \quad (t \in [0, \infty), |x| \geq R, \eta \in \mathbb{R}^d).
$$

As is in (B3), dissipative term works on $|x| \geq R$. This means that dissipative effect may vanish in a compact space region.

We treat quadratic nonlinear terms. In what follows, $\partial_0$ means $\partial_t$ and $\partial_j (j = 1, 2, \cdots, d)$ means $\partial_x^j$. Assume that $F$ is of the form

$$
F(\partial u, \partial^2 u) = \left( \tilde{F}_i(\partial u) + \sum_{j=1}^{d} \sum_{0 \leq a, b \leq d} c^{ab}_{ij}(\partial u) \partial_a \partial_b u^j \right)_{i=1, \cdots, d},
$$

which satisfy

$$
c^{ab}_{ij} = c_{ji}^{ba}, \quad (1.1)
$$

$$
|D_\xi \tilde{F}_i(\xi)| \leq C_{\alpha, p_1} |\xi|^{\max\{0, p_1 - |\alpha|\}} \quad (\xi \in \mathbb{R}^d \times \mathbb{R}^{d+1}, |\alpha| \leq L - 1) \quad (1.2)
$$

and

$$
|D_\xi^2 c^{ab}_{ij}(\xi)| \leq C_{\alpha, p_2} |\xi|^{\max\{0, p_2 - 1 - |\alpha|\}} \quad (\xi \in \mathbb{R}^d \times \mathbb{R}^{d+1}, |\alpha| \leq L - 1) \quad (1.3)
$$

for some $p_l \geq 2$ ($l = 1, 2$). The main objective of this paper is to prove the global existence and decay estimate to (DW).

In the case of the coefficient function $B$ vanishes, (DW) become the nonlinear wave equations. Then it is well known that no matter how small the initial data, there do not exist globally defined smooth solutions in general (e.g.

(6), (10)). If $F$ has the “Null condition” then (DW) has a global smooth solution for sufficiently smooth and small the initial data (e.g.

(8), (15)).

In the case of the coefficient function $B \equiv \text{Const} > 0$, there are many results (6, 10 etc.). When linear or semilinear version, it is known that the asymptotic profile of the solution to (DW) is given by the corresponding solution of heat equation (e.g.

(13), (14) etc.). Such a property is called the diffusion phenomenon. Recently, there are many research concerning the diffusion phenomenon for the nonuniform dissipation. When the dissipation depends on space variable $B = B(x)$, Todorova and Yordanov consider like $B = (1 + |x|)^{-\gamma}$ to linear and semilinear version in (10), (17). They show that if $0 \leq \gamma < 1$ then the solution to (DW) have some decay estimates, indeed Wakasugi (18) confirms the diffusion phenomenon recently. When the dissipation depend on time variable $B = B(t)$, Wirth (20) proves that if $tB(t) \to +\infty$ and $B \in L^1$ then
the solution to (DW) satisfy the decay estimate like corresponding solution to heat equation. On the other hand, Mochizuki [11] consider the scattering for the free wave equation when $B$ depends time-space variable. He prove that if there exist $\xi, \eta \in L^1$ and small $\varepsilon$ such that

$$|B(t,x)| \leq \varepsilon \xi(|x|) + \eta(t), \quad \xi, \eta \geq 0, \quad \xi' \leq 0, \quad \xi'^{2} \leq 2\xi \xi''$$

then the solution to (DW) close to the free wave equation. We remark that $B(t,x) = (1 + t)^{-\alpha}(1 + |x|)^{-\beta}$, $\alpha + \beta > 1$ is a sufficient condition of above the conditions.

Now we consider the nonuniform dissipative term which doesn’t decay near infinity but vanishes in a compact region. Nakao [12] gets the energy decay estimates like $E(u(t)) = O((1 + t)^{-1})$ when $B$ depends on space variable only, where $E(u(t))$ is the standard energy of wave equations. Furthermore, Ikehata [2] get the decay estimates as $\|u(t)\|_{L^2(\Omega)} = O((1 + t)^{-2})$ with the additional condition for the initial data. Those results mostly deal with the linear and the semilinear problem. For quasilinear version with divergence form, Nakao [12] consider the equation

$$\partial_t^2 u - \text{div}\{\sigma(|\nabla u|)\nabla u\} + a(x)\partial_t u = 0, \quad (1.4)$$

where $\sigma$ is a smooth function like $\sigma(x) = (1 + |x|)^{-\frac{1}{2}}$. Then the nonlinearity order $p$ satisfy $p \geq 3$. Besides, the author of the present paper deals with $p = 2$ and prove the decay estimates in [19]. Moreover there is no result when $B$ depend space-time variables. In this paper, we assume the space-time dependent dissipation $B$ effective near the space infinity, even if the nonlinear terms $F$ have no "null condition" and "divergence form".

First, we get the global existence as follows:

**Theorem 1.1.** Let $L \geq \left[\frac{d}{2}\right] + 3$. Then there exists a small constant $\hat{\delta} > 0$ such that if the initial data $(u_0, u_1) \in H^L(\Omega) \times H^{L-1}(\Omega)$ satisfy the compatibility condition of order $L - 1$ and

$$\|(u_0, u_1)\|_{H^L(\Omega) \times H^{L-1}(\Omega)} \leq \delta, \quad (1.5)$$

then there exists a unique global solution to (DW) in $\bigcap_{j=0}^{L-1} C^j([0, \infty); H^{L-j}(\Omega) \cap H^1_0(\Omega)) \cap C^L([0, \infty); L^2(\Omega))$.

In the proof of Theorem 1.1 we use higher order energies (see for instance [5,15]) and the rescaling (see section 2). Note that if $B = \text{Const} > 0$, we can prove theorem 1.1 under the assumption $\|(\nabla u_0, u_1)\|_{H^{L-1}(\Omega) \times H^{L-1}(\Omega)} \leq \hat{\delta}$ instead of (1.5), i.e. the smallness of $\|u_0\|_{L^2(\Omega)}$ is also needed for the case of nonuniform dissipative term.

Next, we also prove the decay estimate as follows:

**Theorem 1.2.** In addition to the assumptions in Theorem 1.1, we assume (H1) and (H2).
(H1) \[ \|d_0(\cdot)[B(0)u_0 + u_1]\|_{L^2(\Omega)} < \infty. \]

(H2) \[ \int_0^\infty \|d_0(\cdot)\partial_t B(s)\|_{L^\infty(\Omega)} ds < \infty, \]

where \(d_0 : \mathbb{R}^d \to \mathbb{R}\) is defined by
\[
d_0(x) = \begin{cases} |x| \log(A|x|) & (d \geq 3), \\ |x| & (d = 2) \end{cases}
\]
with a constant \(A\) satisfying \(\inf_{x \in \Omega} A|x| \geq 2\). Furthermore if \(d = 2\), we also assume (H3).

(H3) There exists \(M\) such that \(\text{supp} u_0 \cup \text{supp} u_1 \subset \{x \in \Omega : |x| \leq M\}\).

Then the global solution \(u\) to (DW) satisfy following estimates:
\[
\sum_{\mu=0}^{L-1} \| (\partial_t^\mu u(t), \partial_t^{\mu+1} u(t) ) \|_{H^{L-\mu}(\Omega) \times H^{L-\mu-1}(\Omega)}^2 \leq E_0 (1 + t)^{-1}
\]
and
\[
\| (\nabla u(t), \partial_t u(t) ) \|_{L^2(\Omega) \times L^2(\Omega)}^2 \leq E_0 (1 + t)^{-2},
\]
where \(E_0\) is a constant depend on \((u_0, u_1)\) and \(M\).

Theorem 1.2 is the decay estimate which correspond to the result of Ikehata [2].

The paper is organized as follows. In section 2 we prepare some known lemmas and the rescaling function. In section 3 we prove the high energy estimates to (DW) and the theorem 1.1. In the proof, the rescaling argument plays an important role. In section 4 we prove the Theorem 1.2.

2 Preliminaries

We consider the rescaling to (DW). Let \(u\) be the solution to (DW). We define \(v(t, x) = \lambda^{-1} u(\lambda t, \lambda x) (\lambda > 0)\), then \(v\) satisfies
\[
\partial_t^2 v(t, x) - \Delta v(t, x) = \lambda \left\{ \partial_t^2 u(\lambda t, \lambda x) - \Delta u(\lambda t, \lambda x) \right\}
\]
\[
= -\lambda B(\lambda t, \lambda x) \partial_t u(\lambda t, \lambda x) + \lambda F(\partial u(\lambda t, \lambda x), \partial \nabla u(\lambda t, \lambda x))
\]
\[
= -\lambda B(\lambda t, \lambda x) \partial_t v(t, x) + F_\lambda(\partial v(t, x), \partial \nabla v(t, x)),
\]
where \((F_\lambda)_i(\partial v, \partial \nabla v) = (\bar{F}_\lambda)_i(\partial v) + \sum_{j=1}^d \sum_{0 \leq a, b \leq d} e^{ab}_{ij}(\partial v) \partial_a \partial_b v_j\) and \(\bar{F}_\lambda(\partial v) = \lambda \bar{F}(\partial v)\).

So \(v\) is the solution to the following initial-boundary value problem (DW)\(_\lambda\):
\[
(DW)_\lambda \begin{cases}
(\partial_t^2 - \Delta + B_\lambda(t, x) \partial_t) v = F_\lambda(\partial v, \partial \nabla v) & [0, \infty) \times \Omega_\lambda, \\
v(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x) & x \in \Omega_\lambda, \\
v(t, x) = 0 & (t, x) \in [0, \infty) \times \partial \Omega_\lambda,
\end{cases}
\]
where Ωλ = \{x : λx ∈ Ω\}, Bλ(t, x) = \lambda B(\lambda t, λx), v_0(x) = \lambda^{-1} u_0(\lambda x), v_1(x) = u_1(\lambda x). Then Bλ satisfy following \((B1)_λ\) - \((B3)_λ\) instead of \((B1)\) - \((B3)\).

\((B1)_λ\) \(B_λ(t, x)\) is nonnegative in \([0, ∞) \times Ωλ\),

\((B2)_λ\) \(∂_t B_λ(t, x)\) is nonpositive in \([0, ∞) \times Ωλ\),

\((B3)_λ\) There exist \(b_0 > 0\) and \(R > 0\) such that

\[
\sum_{p,q=1,...,d} (B_λ(t, x))_{pq} η_p η_q \geq \lambda b_0 |η|^2 \quad (t ∈ [0, ∞), |x| ≥ \frac{R}{λ}, η ∈ \mathbb{R}^d).
\]

Furthermore \(B_λ\) satisfies

\[(B4)_λ \|\partial_α B_λ\|_{L^∞((0,∞) × Ωλ)} ≤ \lambda^{α + 1} \|\partial_α B\|_{L^∞((0,∞) × Ω)}\].

We consider \((DW)_λ\) instead of \((DW)\). Throughout this paper, \(\|\cdot\|_p = \|\cdot\|_{L^p(Ωλ)}\), \(\|\cdot\|_{H^1} = \|\cdot\|_{H^1(Ωλ)}\) and \((\cdot, \cdot)\) stands for \(L^2(Ωλ)\)-inner product.

First, we introduce some known results. First we prepare following lemmas for estimating nonlinear terms.

**Lemma 2.1** (Sobolev’s lemma). There exists a constant \(C_λ > 0\) such that

\[
\|f\|_∞ \leq C_λ \|f\|_{H^1(Ωλ)}^{\frac{1}{2}}, \quad (f ∈ H^1(Ωλ)).
\]

**Lemma 2.2** (Elliptic estimate). There exists \(C_λ > 0\) such that for any \(φ ∈ H^m(Ωλ) \cap H^1_0(Ωλ)\) with an integer \(m ≥ 2\), we have

\[
\sum_{|α|=m} \|\nabla^α φ\|_2 \leq C_λ (\|Δ φ\|_{H^{m-2}} + \|\nabla φ\|_2).
\]

Next, we prepare the Poincare type inequality associated with \(B_λ\).

**Lemma 2.3** (Poincare type inequality). There exists a constant \(C_1 ≥ 1/4\) such that

\[
\|f\|_2^2 \leq \lambda^{-1} C_1 (f, B_λ f) + \lambda^{-2} C_1 \|∇ f\|_2^2 \quad (f ∈ H^1_0(Ωλ), \lambda > 0) \quad (2.1)
\]

**Proof.** We define \(U_r = \{x ∈ Ωλ : |x| ≤ r\}\). Using Poincare inequality, we obtain the following estimate:

\[
\int_{U_r} |f(x)|^2 dx ≤ r^2 \int_{U_r} |∇ f(x)|^2 dx \quad (f ∈ H^1_0(U_r)).
\]

Let \(f ∈ H^1_0(Ωλ)\) and \(ρ ∈ C^∞_0(\mathbb{R}^d)\) be a function satisfying \(0 ≤ ρ ≤ 1, ρ(x) = 1(|x| ≤ 1), ρ(x) = 0(|x| ≥ \frac{3}{2})\). We define \(ρ_λ(x) = ρ(\frac{λx}{4})\) for \(λ > 0\), then because of \(ρ_λ f ∈ H^1_0(U_{\frac{4}{λ}})\) we have

\[
\|f\|_2^2 = \int_{Ωλ} |ρ_λ(x)f(x)|^2 dx + \int_{Ωλ} (1 - |ρ_λ(x)|^2)|f(x)|^2 dx
\]
\[
\frac{4R^2}{\lambda^2} \int_{\Omega} |\nabla \rho f(x)|^2 \, dx + \frac{4R^2}{\lambda^2} \int_{\Omega} |\rho f(x)|^2 \, dx + \frac{4R^2}{\lambda^2} \int_{\Omega} |\rho \lambda(x)f(x)|^2 \, dx
\]
\[
\leq \left( 4R^2 \|\nabla \rho\|_\infty^2 + 1 \right) \int_{|x| \geq \frac{R}{\lambda}} |f(x)|^2 \, dx + \frac{4R^2}{\lambda^2} \int_{\Omega} |\nabla f(x)|^2 \, dx.
\]

Hence we get (2.1). \(\square\)

Finally, we introduce Hardy inequality and Gagliardo-Nirenberg inequality. We need these in the proof of Theorem 1.2 in section 4.

**Lemma 2.4** (Hardy inequality; see for instance [1]). Let \(d \geq 2\). There exists a constant \(C_\lambda > 0\) such that any \(f \in H^1_0(\Omega_\lambda)\) satisfies
\[
\left\| \frac{f}{d_0} \right\|_2 \leq C_\lambda \|\nabla f\|_2,
\]
where \(d_0\) is defined by (1.6).

**Lemma 2.5** (Gagliardo-Nirenberg inequality). Assume \(1 \leq q < d\) and \(\frac{1}{d} = \frac{1}{q} - \frac{1}{d}\). Then there exists a constant \(C_\lambda > 0\) such that
\[
\|f\|_r \leq C_\lambda \|\nabla f\|_q \quad (f \in C^\infty_0(\Omega_\lambda)).
\]

### 3 Energy estimates

In this section, we give a proof of the high order energy estimates.

First we introduce some notations. The energy \(E(v(t))\) and the higher order energies \(Z_m(v(t))\) are defined by
\[
E(v(t)) = \frac{1}{2} \left( \|\partial_t v(t)\|_2^2 + \|\nabla v(t)\|_2^2 \right),
\]
\[
Z_m(v(t)) = \sum_{\mu=0}^{L-1-m} \left\{ \|\nabla \partial_t^\mu v(t)\|_{H^m}^2 + \|\partial_t^{\mu+1} v(t)\|_{H^m}^2 \right\} \quad (0 \leq m \leq L-1)
\]
and
\[
Z(v(t)) = \sum_{m=0}^{L-1} Z_m(v(t)).
\]
Proposition 3.1 holds. The following result concerning local existence is standard: We sometimes omit $t$ or $v(t)$. Note that $\|v\|_2^2 + Z_{L-1}(v) = \|(v, \partial_t v)\|_{H^L \times H^{L-1}}^2$ holds. The following result concerning local existence is standard:

**Proposition 3.1** (Local existence; for instance [4] or [12]). Let $L \geq \left[\frac{d}{2}\right] + 3$ and assume that $\partial \Omega_{\lambda}$ is smooth. Furthermore we assume the initial date $(v_0, v_1) \in H^L(\Omega_{\lambda}) \times H^{L-1}(\Omega_{\lambda})$ satisfies the compatibility condition of order $L-1$ associated with the $(\text{DW})_{\lambda}$. Then there exists a unique local solution to $(\text{DW})_{\lambda}$ in

$$X_T := \bigcap_{j=0}^{L-1} C^{j}([0, T); H^L_j(\Omega_{\lambda}) \cap H^1_{1,j}(\Omega_{\lambda})) \bigcap C^L([0, T); L^2(\Omega_{\lambda})),$$

where $T$ depend on $\|(v_0, v_1)\|_{H^L \times H^{L-1}}$.

We define the function spaces $X^T_\delta$ and $X_\delta$ as follows:

$$X^T_\delta = \{ v \in X^T : \|v(t)\|_2^2 + Z(v(t)) \leq \delta^2 \quad (0 \leq t \leq T) \}$$

and

$$X_\delta = \{ v \in X^\infty : \|v(t)\|_2^2 + Z(v(t)) \leq \delta^2 \quad (0 \leq t < \infty) \}.$$

This section, we prove the following proposition:

**Proposition 3.2.** There exist $0 < \lambda < 1$ and $\delta = \delta(\lambda)$ such that the local solution $v \in X^T_\delta$ to $(\text{DW})_{\lambda}$ satisfies

$$\|v(t)\|_2^2 + Z(v(t)) + \int_0^t Z(v(s))ds \leq C_\lambda \|(v_0, v_1)\|_{H^L \times H^{L-1}}^2. \quad (3.4)$$

If Proposition 3.2 holds, then we can prove Theorem 1.1. Indeed, using Proposition 3.1 and Proposition 3.2, we can prove the unique global existence theorem to $(\text{DW})_{\lambda}$ by standard continuation argument. Furthermore we put $u(t, x) := \lambda v(\lambda^{-1} t, \lambda^{-1} x)$, it is easy to see $u$ satisfies the statement of Theorem 1.1.

In order to show Proposition 3.2 we only need the estimates of $Z_\delta$. Indeed we can prove the next lemma (see for instance [5]).

**Lemma 3.3.** For any $\lambda > 0$ there exist $\delta = \delta(\lambda)$ and $C_\lambda > 0$ such that a local solution $v \in X^T_\delta$ to $(\text{DW})_{\lambda}$ satisfy the following estimates:

$$Z(v(t)) \leq C_\lambda Z_0(v(t)) \quad (t \in [0, T)) \quad (3.5)$$

and

$$Z(v(t)) \leq C_\lambda Z_{L-1}(v(t)) \quad (t \in [0, T)). \quad (3.6)$$

**Proof.** Let $v \in X^T_\delta$ is a solution to $(\text{DW})_{\lambda}$. First, we prove (3.5). For $1 \leq m \leq L - 1$, it hold that

$$Z_m = \sum_{\mu=0}^{L-1-m} \left\{ \| \nabla \partial^\mu_t v \|_2^2 + \sum_{2 \leq |\alpha| \leq m+1} \| \partial^\alpha \nabla^\mu v \|_2^2 + \| \partial^\mu v \|_{H^m}^2 \right\}$$

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\[ \leq C \left( Z_0 + \sum_{\mu=0}^{L-1-m} \sum_{2 \leq |a| \leq m+1} \| \partial_t^\mu \nabla^a v \|_2^2 + Z_{m-1} \right). \]

Using Lemma 2.2 and (DW)\( \lambda \), we get
\[ \sum_{\mu=0}^{L-1-m} \sum_{2 \leq |a| \leq m+1} \| \partial_t^\mu \nabla^a v \|_2^2 \leq C_\lambda \sum_{\mu=0}^{L-1-m} \left( \| \partial_t^\mu \Delta v \|_{H^{m-1}}^2 + \| \partial_t^\mu \nabla v \|_2^2 \right) \]
\[ \leq C_\lambda \sum_{\mu=0}^{L-1-m} \left( \| \partial_t^{\mu+2} v \|_{H^{m-1}}^2 + \| \partial_t^\mu (B_\lambda \partial_t v) \|_{H^{m-1}}^2 + \| \partial_t^\mu F_\lambda \|_{H^{m-1}}^2 + \| \partial_t^\mu \nabla v \|_2^2 \right). \]

It is easy to see that
\[ \sum_{\mu=0}^{L-1-m} \| \partial_t^{\mu+2} v \|_{H^{m-1}}^2 \leq Z_{m-1}, \quad \sum_{\mu=0}^{L-1-m} \| \partial_t^\mu \nabla v \|_2^2 \leq Z_0 \]

and
\[ \sum_{\mu=0}^{L-1-m} \| \partial_t^\mu (B_\lambda \partial_t v) \|_{H^{m-1}}^2 \leq C_\lambda Z_{m-1}. \]

Furthermore applying Lemma A.1 and Lemma A.2 to the nonlinear terms, we get
\[ Z_m(v(t)) \leq C_\lambda Z_0(v(t)) + C_\lambda Z_{m-1}(v(t)) + C_\lambda Z(v(t))^2 \]
\[ \leq C_\lambda Z_0(v(t)) + C_\lambda Z_{m-1}(v(t)) + C_\lambda \delta^2 Z(v(t)). \]

Therefore we obtain inductively that
\[ Z(v(t)) = \sum_{m=0}^{L-1} Z_m(v(t)) \leq C_\lambda Z_0(v(t)) + C_\lambda \delta^2 Z(v(t)). \]

Choosing \( \delta \) small enough depending on \( \lambda \), we get (3.5).

Next, we prove (3.6). Using the same way as for the proof of (3.5), for \( 0 \leq m \leq L - 3 \), we have
\[ Z_m = \sum_{\mu=0}^{L-1-m} \left\{ \| \nabla \partial_t^\mu v \|_{H^m}^2 + \| \partial_t^{\mu+1} v \|_{H^m}^2 \right\} \]
\[ = \sum_{\mu=0}^{1} \| \nabla \partial_t^\mu v \|_{H^m}^2 + \sum_{\mu=2}^{L-1-m} \| \nabla \partial_t^\mu v \|_{H^m}^2 + \| \partial_t v \|_{H^m}^2 + \sum_{\mu=1}^{L-1-m} \| \partial_t^{\mu+1} v \|_{H^m}^2 \]
\[ \leq C Z_{L-1}(v) \]
\[ + C \sum_{\mu=2}^{L-1-m} \left( \| \nabla \partial_t^{\mu-2} \Delta v \|_{H^m}^2 + \| \nabla \partial_t^{\mu-2} (B_\lambda \partial_t v) \|_{H^m}^2 + \| \nabla \partial_t^{\mu-2} F_\lambda \|_{H^m}^2 \right) . \]
Lemma 3.4. Let $\sigma$ where $\lambda$ satisfy a local solution $v$ and for any $K > 1$ we can get $Z_{L-2} \leq C_\lambda Z_{L-1} + C_\delta^Z$, thus it follows that
\[
Z(v(t)) = \sum_{m=0}^{L-1} Z_m(v(t)) \leq C_\lambda Z_{L-1}(v(t)) + C_\lambda \delta^Z(v(t)).
\]
Choosing $\delta$ small enough depend on $\lambda$, we get (3.7). This completes the proof of Lemma 3.3. \hfill \Box

We consider the estimates of $Z_0(v(t))$.

**Lemma 3.4.** Let $\mu \leq L - 1$ and $\lambda \leq 1$. Then there exists a constant $C > 0$ such that a local solution $v$ to (DW)$_\lambda$ satisfy
\[
\frac{d}{dt} E(\partial_t^\mu v) + \langle \partial_t^{\mu+1}v, B_\lambda \partial_t^{\mu+1}v \rangle \leq C\lambda^2 Z_0(v) + \langle \partial_t^{\mu+1}v, \partial_t^\mu F_\lambda \rangle, \tag{3.7}
\]
and for any $K > 0$, it holds that
\[
\frac{d}{dt} \langle \partial_t^{\mu+1}v, [h, \nabla \partial_t^\mu v] \rangle + \int_{\Omega_\lambda} \frac{d\phi + |x|\phi'}{2} |\partial_t^{\mu+1}v|^2 dx + \int_{\Omega_\lambda} \frac{(2 - d)\phi + |x|\phi'}{2} |\nabla \partial_t^\mu v|^2 dx \leq \frac{1}{2} \int_{\partial\Omega_\lambda} \| B \|_{L^\infty([0,\infty) \times \Omega)} \beta_0 \Delta R^2 \lambda K \| \nabla \partial_t^\mu v \|^2 + C \lambda Z_0(v) + \langle \partial_t^\mu F_\lambda, [h, \nabla \partial_t^\mu v] \rangle, \tag{3.9}
\]
where $\sigma$ is the outward pointing unit normal vector of $\partial\Omega_\lambda$,
\[
\phi(r) = \begin{cases} b_0, & (r \leq \frac{R}{K}) \\ \frac{b_0 R}{\lambda r}, & (r \geq \frac{R}{K}) \end{cases}, \quad h(x) = x\phi(|x|) \tag{3.10}
\]
and $[h; \nabla g] : \mathbb{R}^d \to \mathbb{R}^d$ is defined by
\[
([h; \nabla g])^i(x) = h(x) \cdot \nabla g^i(x) \quad (i = 1, 2, \ldots, d)
\]
for any $g : \mathbb{R}^d \to \mathbb{R}^d$. 

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Proof. Let \(0 \leq \mu \leq L - 1\) and \(\lambda \leq 1\). First, we prove (3.7). Applying \(\partial_t^\mu\) to (DW) and taking inner product it by \(\partial_t^{\mu+1}u\), we have

\[
\frac{d}{dt} E(\partial_t^\mu v) + \langle \partial_t^{\mu+1}v, B_\lambda \partial_t^{\mu+1}v \rangle = - \sum_{1 \leq \nu \leq \mu} \left( \mu \right) \langle \partial_t^{\mu+1}v, \partial_t^{\nu} B_\lambda \partial_t^{\mu-\nu+1}v \rangle + \langle \partial_t^{\mu+1}v, \partial_t^{\mu} F_\lambda \rangle,
\]

where we use

\[
\partial_t^\mu v = 0 \quad \text{on} \quad \partial \Omega_\lambda.
\]

Note that even if \(\mu = 0\), (3.11) holds in the sense of the first term in the right-hand side to be zero. Since \(\lambda \leq 1\), the definition of \(Z_0\) and condition (B4)\(\lambda\) imply that

\[
- \sum_{1 \leq \nu \leq \mu} \left( \mu \right) \langle \partial_t^{\mu+1}v, \partial_t^{\nu} B_\lambda \partial_t^{\mu-\nu+1}v \rangle \leq CZ_0 \sum_{1 \leq \nu \leq \mu} \| \partial_t^{\nu} B_\lambda \|_{\infty} \leq C\lambda^2 Z_0,
\]

thus we obtain (3.7).

Second, we prove (3.8). Applying \(\partial_t^\mu\) to (DW) and taking inner product it by \(\partial_t^{\mu} u\), we have

\[
\frac{d}{dt} \left\{ \langle \partial_t^{\mu} v, \partial_t^{\mu+1}v \rangle + \frac{1}{2} \langle \partial_t^{\mu} v, B_\lambda \partial_t^{\mu+1}v \rangle \right\} + \| \nabla \partial_t^{\mu} v \|^2_2 - \| \partial_t^{\mu+1}v \|^2_2
\]

\[
= \langle \partial_t^{\mu} v, \partial_t^{\mu+2}v \rangle + \langle \partial_t^{\mu} v, B_\lambda \partial_t^{\mu+1}v \rangle + \| \nabla \partial_t^{\mu} v \|^2_2 + \frac{1}{2} \langle \partial_t^{\mu} v, \partial_t B_\lambda \partial_t^{\mu}v \rangle
\]

\[
= - \sum_{1 \leq \nu \leq \mu} \left( \mu \right) \langle \partial_t^{\mu} v, \partial_t^{\nu} B_\lambda \partial_t^{\mu-\nu+1}v \rangle + \frac{1}{2} \langle \partial_t^{\mu} v, \partial_t B_\lambda \partial_t^{\mu}v \rangle + \langle \partial_t^{\mu} v, \partial_t^{\mu} F_\lambda \rangle
\]

\[
\leq C\lambda^2 Z_0 + \langle \partial_t^{\mu} v, \partial_t^{\mu} F_\lambda \rangle,
\]

where we use assumption (B2)\(\lambda\). It means that (3.8) holds.

Finally, we prove (3.9). Applying \(\partial_t^\mu\) to (DW)\(\lambda\) and taking inner product (DW) by \([h; \nabla \partial_t^{\mu} v]\) we obtain

\[
\langle \partial_t^{\mu+2} v, [h; \nabla \partial_t^{\mu} v] \rangle = \langle \Delta \partial_t^{\mu} v, [h; \nabla \partial_t^{\mu} v] \rangle + \langle B_\lambda \partial_t^{\mu+1} v, [h; \nabla \partial_t^{\mu} v] \rangle
\]

\[
= - \sum_{1 \leq \nu \leq \mu} \left( \mu \right) \langle \partial_t^{\mu} B_\lambda \partial_t^{\mu-\nu+1}v, [h; \nabla \partial_t^{\mu} v] \rangle + \langle \partial_t^{\mu} F_\lambda, [h; \nabla \partial_t^{\mu} v] \rangle
\]

Noting

\[
\nabla \partial_t^{\mu} v^k = \sigma \cdot \nabla \partial_t^{\mu} v^k \quad \text{on} \quad \partial \Omega_\lambda,
\]

we obtain

\[
\langle \partial_t^{\mu+2} v, [h; \nabla \partial_t^{\mu} v] \rangle - \frac{d}{dt} \langle \partial_t^{\mu+1} v, [h; \nabla \partial_t^{\mu} v] \rangle = - \sum_{i,k=1}^d \int_{\Omega_\lambda} \partial_t^{\mu+1} v^k \partial_i \partial_t^{\mu+1} v^k h^i \, dx
\]
\[-\frac{1}{2} \int_{\Omega} h \cdot \nabla |\partial_t^{\mu+1} v|^2 dx = \frac{1}{2} \int_{\Omega} \text{div} h |\partial_t^{\mu+1} v|^2 dx\]

and

\[- \langle \triangle \partial_t^{\mu} v, [h; \nabla \partial_t^{\mu} v] \rangle\]

\[= \sum_{k=1}^{d} \int_{\Omega} \nabla \partial_t^{\mu} v^k \cdot \nabla (h \cdot \nabla \partial_t^{\mu} v^k) dx - \sum_{k=1}^{d} \int_{\partial \Omega} \sigma \cdot \nabla \partial_t^{\mu} v^k h \cdot \nabla \partial_t^{\mu} v^k dS\]

\[= \sum_{i,j,k=1}^{d} \int_{\Omega} \partial_j \partial_t^{\mu} v^k \partial_i \partial_t^{\mu} v^k \partial_i h^j dx + \frac{1}{2} \int_{\Omega} h \cdot \nabla |\nabla \partial_t^{\mu} v|^2 dx\]

\[- \int_{\partial \Omega} h \cdot \sigma |\nabla \partial_t^{\mu} v|^2 dS\]

\[= \sum_{i,j,k=1}^{d} \int_{\Omega} \partial_j \partial_t^{\mu} v^k \partial_i \partial_t^{\mu} v^k \partial_i h^j dx - \frac{1}{2} \int_{\Omega} \text{div} h |\nabla \partial_t^{\mu} v|^2 dx\]

\[- \frac{1}{2} \int_{\partial \Omega} h \cdot \sigma |\nabla \partial_t^{\mu} v|^2 dS.\]

Therefore we get from (3.13) that

\[\frac{d}{dt} \langle \partial_t^{\mu+1} v, [h; \nabla \partial_t^{\mu} v] \rangle + \frac{1}{2} \int_{\Omega} (|\partial_t^{\mu+1} v|^2 - |\nabla \partial_t^{\mu} v|^2) \text{div} h dx \quad (3.15)\]

\[+ \sum_{i,j,k=1}^{d} \int_{\Omega} \partial_j \partial_t^{\mu} v^k \partial_i \partial_t^{\mu} v^k \partial_i h^j dx\]

\[= \frac{1}{2} \int_{\partial \Omega} h \cdot \sigma |\nabla \partial_t^{\mu} v|^2 dS - \langle B_{\lambda} \partial_t^{\mu+1} v, [h; \nabla \partial_t^{\mu} v] \rangle\]

\[- \sum_{1 \leq \nu \leq \mu} \left( \begin{array}{c} \mu \\ \nu \end{array} \right) \langle \partial_t^{\nu} B_{\lambda} \partial_t^{\mu-\nu+1} v, [h; \nabla \partial_t^{\mu} v] \rangle + \langle \partial_t^{\mu} F_{\lambda}, [h; \nabla \partial_t^{\mu} v] \rangle.\]

Now we remark that

\[\frac{\partial h^i}{\partial x_j} = \delta_{ij} \phi(|x|) + \phi'(|x|) \frac{x_i x_j}{|x|},\]

\[\text{div} h(x) = d\phi(|x|) + \phi'(|x|)|x|,\]

\[||h||_\infty \leq \frac{b_0 R}{\lambda} \quad \text{and} \quad ||\nabla h||_\infty \leq 2b_0.\]

Using (3.16) and \(\phi'(r) \leq 0\), we obtain

\[\sum_{i,j,k=1}^{d} \int_{\Omega} |\partial_i \partial_t^{\mu} v^k| \partial_j \partial_t^{\mu} v^k \partial_i h^j dx\]

\[= \sum_{i,k=1}^{d} \int_{\Omega} |\partial_i \partial_t^{\mu} v^k|^2 dx + \sum_{i,j,k=1}^{d} \int_{\Omega} \partial_j \partial_t^{\mu} v^k \partial_i \partial_t^{\mu} v^k \phi' \frac{x_i x_j}{|x|} dx.\]
\[ \int_{\Omega_t} |\nabla \partial_t^\mu v|^2 \phi dx + \sum_{k=1}^d \int_{\Omega_t} |x \cdot \nabla \partial_t^\mu v|^2 \frac{1}{|x|} dx \]
\[ \geq \int_{\Omega_t} \{\phi + |x|\phi'\} |\nabla \partial_t^\mu v|^2 dx. \]

These estimates and (3.15) imply that

\[ \frac{d}{dt} \langle \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v] \rangle \]
\[ + \int_{\Omega_t} \left( \frac{(d\phi + |x|\phi')}{2} \right) |\partial_t^{\mu+1} v|^2 dx + \int_{\Omega_t} \left( \frac{(2 - d)\phi + |x|\phi'}{2} \right) |\nabla \partial_t^\mu v|^2 dx \]
\[ \leq \frac{1}{2} \int_{\partial\Omega_t} h \cdot \sigma \cdot \nabla \partial_t^\mu v \, dS - \langle B_\lambda \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v] \rangle \]
\[- \sum_{1 \leq \nu \leq \mu} \left( \frac{\mu}{\nu} \right) \langle \partial_t^\nu B_\lambda \partial_t^{\mu-\nu+1} v, [h; \nabla \partial_t^\mu v] \rangle + \langle \partial_t^\mu F_\lambda, [h; \nabla \partial_t^\mu v] \rangle. \]

Let us estimate for the right side of (3.17). We calculate

\[ \langle B_\lambda \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v] \rangle = \langle \sqrt{B_\lambda} \partial_t^{\mu+1} v, \sqrt{B_\lambda} [h; \nabla \partial_t^\mu v] \rangle \]
\[ \leq \frac{K}{4} \| \sqrt{B_\lambda} \partial_t^{\mu+1} v \|_2^2 + \frac{1}{K} \| \sqrt{B_\lambda} [h; \nabla \partial_t^\mu v] \|_2^2 \]
\[ \leq \frac{K}{4} \langle \partial_t^{\mu+1} v, B_\lambda \partial_t^{\mu+1} v \rangle + \| B \|_{L^\infty([0,\infty) \times \Omega)} \| B_0 \|_{L^2} \| \nabla \partial_t^\mu v \|_2. \]

\[ \leq C \sum_{1 \leq \nu \leq \mu} \left( \frac{\mu}{\nu} \right) \| \partial_t^\nu B_\lambda \|_{\infty} \| \partial_t^{\mu-\nu+1} v \|_2 \| \nabla \partial_t^\mu v \|_2 \leq C\lambda Z_0. \]

where we use (B1)_\lambda, (B4)_\lambda and (3.16). Combining estimates (3.17) - (3.19), we get (3.9). This completes the proof of Lemma 3.4.

We define \( G(v(t)) \) by

\[ G(v(t)) = \frac{C_0}{2\lambda} Z_0(v(t)) + \frac{b_0(2d - 1)}{4} \sum_{\mu=0}^{L-1} \langle \partial_t^{\mu+1} v(t), \partial_t^\mu v(t) \rangle \]
\[ + \frac{b_0(2d - 1)}{8} \sum_{\mu=0}^{L-1} \langle \partial_t^\mu v(t), B_\lambda(t) \partial_t^\mu v(t) \rangle + \sum_{\mu=0}^{L-1} \langle \partial_t^{\mu+1} v(t), [h; \nabla \partial_t^\mu v(t)] \rangle, \]

where

\[ C_0 = \max \left\{ 4b_0R + \frac{C_1 b_0(2d - 1)}{2}, d, \| B \|_{L^\infty([0,\infty) \times \Omega)} b_0^2 R^2 \times \frac{8}{b_0} \right\}. \]
Lemma 3.5. There exists a constant $0 < \lambda < 1$ such that the local solution $v$ to $(\text{DW})_\lambda$ satisfies

$$
\frac{d}{dt} G(v(t)) + \frac{b_0}{16} Z_0(v(t)) + \sum_{\mu=0}^{L-1} h \cdot \sigma \cdot \nabla \partial^\mu_t v^2 dS + C_\lambda \sum_{\mu=0}^{L-1} (\partial^\mu_t F_\lambda, \partial^{\mu+1}_t v + \partial^\mu_t v + [h; \nabla \partial^\mu_t v]),
$$

where $C_\lambda = \frac{C_0}{\lambda} + \frac{b_0(2d-1)}{4} + 1$.

Proof. Let $\mu \leq L-1$ and $K \geq \frac{d}{\lambda}$. Calculating $K \times (3.7) + \frac{b_0(2d-1)}{4} \times (3.7) + (3.7)$, we have

$$
\frac{d}{dt} \left\{ K E(\partial^\mu_t v) + \frac{b_0(2d-1)}{4} (\partial^{\mu+1}_t v, \partial^\mu_t v) \right\} + \frac{b_0(2d-1)}{8} (\partial^\mu_t v, B_\lambda \partial^\mu_t v) + (\partial^{\mu+1}_t v, [h; \nabla \partial^\mu_t v])
$$

$$
+ K (\partial^{\mu+1}_t v, B_\lambda \partial^\mu_t v) - \frac{b_0(2d-1)}{4} \|\partial^{\mu+1}_t v\|_2^2 + \int_{\Omega_\lambda} \left\{ \frac{d\alpha + |x| \phi'}{2} \right\} |\partial^\mu_t v| dx
$$

$$
+ \frac{b_0(2d-1)}{4} \|\nabla \partial^\mu_t v\|_2^2 + \int_{\Omega_\lambda} \left\{ \frac{(2-d)\phi + |x| \phi'}{2} \right\} |\nabla \partial^\mu_t v| dx
$$

$$
\leq \frac{K}{4} (\partial^{\mu+1}_t v, B_\lambda \partial^\mu_t v) + \frac{\|B\|_{L \rightarrow (0, \infty) \times \Omega} b_0^2 R^2}{\lambda K} \|\nabla \partial^\mu_t v\|_2^2
$$

$$
+ \frac{1}{2} \int_{\partial \Omega_\lambda} h \cdot \sigma \cdot \nabla \partial^\mu_t v^2 dS + C \lambda^2 Z_0 \left( K + \frac{b_0(2d-1)}{4} + \frac{1}{\lambda} \right)
$$

$$
+ \left( K + \frac{b_0(2d-1)}{4} + 1 \right) (\partial^\mu_t F_\lambda, \partial^{\mu+1}_t v + \partial^\mu_t v + [h; \nabla \partial^\mu_t v]).
$$

Using (B3)$_\lambda$, $K \geq \frac{4}{\lambda}$, $\phi \geq 0$ and

$$
\eta \phi'(r) = \begin{cases} 0, & (r < \frac{2}{\lambda}), \\ -\phi(r), & (r \geq \frac{2}{\lambda}), \end{cases}
$$

we obtain

$$
K (\partial^{\mu+1}_t v, B_\lambda \partial^\mu_t v) - \frac{b_0(2d-1)}{4} \|\partial^{\mu+1}_t v\|_2^2 + \int_{\Omega_\lambda} \left\{ \frac{d\phi + |x| \phi'}{2} \right\} |\partial^\mu_t v|^2 dx
$$

$$
\geq \frac{K}{2} (\partial^{\mu+1}_t v, B_\lambda \partial^\mu_t v) + \int_{\Omega_\lambda} \left\{ \frac{-b_0(2d-1)}{4} + \frac{db_0}{2} \right\} |\partial^{\mu+1}_t v|^2 dx
$$

$$
+ \int_{|x| \geq \frac{2}{\lambda}} \left\{ \frac{\lambda b_0 K}{2} - \frac{b_0(2d-1)}{4} + \frac{d-1}{2} \phi \right\} |\partial^{\mu+1}_t v|^2 dx
$$

$$
\geq \frac{K}{2} (\partial^{\mu+1}_t v, B_\lambda \partial^\mu_t v) + \frac{b_0}{4} \|\partial^{\mu+1}_t v\|_2^2
$$

(3.24)
and
\[ \frac{b_0(2d-1)}{4} \| \nabla \partial_t^\mu v \|_2^2 + \int_{\Omega_\lambda} \left( \frac{2 - d}{2} \phi + |x| \phi' \right) |\nabla \partial_t^\mu v|^2 dx \]
(3.25)
\[ = \int_{U_\lambda} \left( \frac{b_0(2d-1)}{4} + \frac{2 - d}{2} b_0 \right) |\nabla \partial_t^\mu v|^2 dx \\
+ \int_{|x| \geq \frac{R}{2}} \left( \frac{b_0(2d-1)}{4} + \frac{1 - d}{2} b_0 R \frac{1}{\lambda|x|} \right) |\nabla \partial_t^\mu v|^2 dx \geq \frac{b_0}{4} \| \nabla \partial_t^\mu v \|_2^2. \]

Combining (3.23), (3.24) and (3.25), we have
\[ \frac{d}{dt} \left\{ KE(\partial_t^\mu v) + \frac{b_0(2d-1)}{4} \langle \partial_t^{\mu+1} v, \partial_t^\mu v \rangle \right. \\
+ \frac{b_0(2d-1)}{8} \langle \partial_t^\mu v, B_\lambda \partial_t^\mu v \rangle + \langle \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v] \rangle \right. \\
+ \frac{b_0}{4} \left( \| \partial_t^{\mu+1} v \|_2^2 + \| \nabla \partial_t^\mu v \|_2^2 \right) \right. \\
\leq \left. \frac{-B}{\lambda K} \| \nabla \partial_t^\mu v \|_2^2 + \frac{1}{2} \int_{\partial\Omega_\lambda} h \cdot \sigma |\nabla \partial_t^\mu v|^2 dS \right. \\
+ C \lambda^2 Z_0(v) \left( K + \frac{b_0(2d-1)}{4} + \frac{1}{\lambda} \right) \\
+ \left( K + \frac{b_0(2d-1)}{4} + 1 \right) \langle \partial_t^\mu F_\lambda, \partial_t^{\mu+1} v + \partial_t^\mu v + [h; \nabla \partial_t^\mu v] \rangle \right. . \]

Let \( K = \frac{C_0}{\lambda} \) and sum up \( \mu \) from 0 to \( L - 1 \). Then we get
\[ \frac{d}{dt} G(v) + \frac{b_0}{8} Z_0(v) \leq C \lambda Z_0(v) \]
\[ + \frac{1}{2} \sum_{\mu=0}^{L-1} \int_{\partial\Omega_\lambda} h \cdot \sigma |\nabla \partial_t^\mu v|^2 dS + \tilde{C}_\lambda \sum_{\mu=0}^{L-1} \langle \partial_t^\mu F_\lambda, \partial_t^{\mu+1} v + \partial_t^\mu v + [h; \nabla \partial_t^\mu v] \rangle, \]
where we use (3.21). We choose \( \lambda \) small such that \( \lambda \) satisfies \( C \lambda \leq \frac{b_0}{16} \). Then we obtain
\[ \frac{d}{dt} G(v) + \frac{b_0}{16} Z_0(v) \]
\[ \leq \frac{1}{2} \sum_{\mu=0}^{L-1} \int_{\partial\Omega_\lambda} h \cdot \sigma |\nabla \partial_t^\mu v|^2 dS + \tilde{C}_\lambda \sum_{\mu=0}^{L-1} \langle \partial_t^\mu F_\lambda, \partial_t^{\mu+1} v + \partial_t^\mu v + [h; \nabla \partial_t^\mu v] \rangle. \]
This completes the proof of Lemma 3.5. □

We choose the \( \lambda \) sufficiently small to hold the Lemma 3.5. Next Lemma are the estimates of the nonlinear terms.
Lemma 3.6. Let $\mu \leq L - 1$. Then there exists a constant $C_\lambda > 0$ such that the local solution $v \in X^T_\omega$ to $(DW)_\lambda$ satisfies

$$\langle \partial_t^{\mu + 1} v, \partial_t^\mu F_\lambda \rangle \leq C_\lambda \delta Z_0 - \frac{1}{2} \sum_{i,j=1}^d \sum_{\a,\b \leq d} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^{\mu + 1} v^i \partial_t^\mu \partial_{\a} v^j \partial_{\b} v^j c_{ij}^{\a b} (\partial v) dx$$

(3.26)

$$+ \sum_{i,j=1}^d \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^{\mu + 1} v^i \partial_t^\mu v^j c_{ij}^{00} (\partial v) dx,$$

$$\langle \partial_t^\mu v, \partial_t^\mu F_\lambda \rangle \leq C_\lambda \delta Z_0 + \sum_{i,j=1}^d \sum_{\a,\b \leq d} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^\mu v^i \partial_t^\mu \partial_{\a} v^j \partial_{\b} v^j c_{ij}^{0b} (\partial v) dx$$

(3.27)

and

$$\langle [h; \nabla \partial_t^\mu v], \partial_t^\mu F_\lambda \rangle \leq C_\lambda \delta Z_0 + C_\lambda \delta \int_{\partial \Omega_\lambda} |h \cdot \sigma| |\sigma \cdot \nabla \partial_t^\mu v|^2 dS$$

(3.28)

Proof. We may assume that $\delta < 1$. First, we prove (3.26). Using Lemma 3.3 and Lemma A.1, we have

$$\langle \partial_t^{\mu + 1} v, \partial_t^\mu F_\lambda \rangle \leq \| \partial_t^{\mu + 1} v \|_2 \| \partial_t^\mu F_\lambda \|_2 \leq C_\lambda \delta Z \leq C_\lambda \delta Z_0.$$

Furthermore we calculate that

$$\sum_{i,j=1}^d \sum_{\a,\b \leq d} \int_{\Omega_\lambda} \partial_t^{\mu + 1} v^i \partial_t^\mu (c_{ij}^{\a b} (\partial v) \partial_a \partial_b v^j) dx$$

$$= \sum_{\a,\b \leq d} \sum_{\a,\b \leq d} \int_{\Omega_\lambda} \partial_t^{\mu + 1} v^i \partial_t^\mu \partial_a v^j \partial_b v^j c_{ij}^{\a b} (\partial v) dx$$

$$- \sum_{i,j=1}^d \sum_{\a,\b \leq d} \int_{\Omega_\lambda} \partial_t^{\mu + 1} v^i \partial_t^\mu \partial_{\a} v^j \partial_{\b} v^j c_{ij}^{\a b} (\partial v) dx$$

$$- \sum_{i,j=1}^d \sum_{\a,\b \leq d} \int_{\Omega_\lambda} \partial_t^{\mu + 1} \partial_{\a} v^i \partial_t^\mu \partial_{\b} v^j c_{ij}^{\a b} (\partial v) dx$$

$$+ \sum_{i,j=1}^d \sum_{\a,\b \leq d} \int_{\Omega_\lambda} \partial_a (\partial_t^{\mu + 1} v^i \partial_t^\mu \partial_{\b} v^j c_{ij}^{\a b} (\partial v)) dx$$

$$= J_1 + J_2 + \sum_{i,j=1}^d \frac{1}{2} \int_{\Omega_\lambda} \partial_t^\mu \partial_{\a} v^i \partial_t^\mu \partial_{\b} v^j \partial_a (c_{ij}^{\a b} (\partial v)) dx.$$
\[- \sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \frac{1}{2} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^\mu i \partial_t^\mu j \partial_a v^i \partial_b v^j c_{ij}^a (\partial v) dx \]
\[+ \sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^{\mu + 1} i \partial_t^\mu j \partial_a v^i \partial_b v^j c_{ij}^a (\partial v) dx \]
\[= J_1 + J_2 + J_3 - \sum_{i,j=1}^{d} \frac{1}{2} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^\mu i \partial_t^\mu j \partial_a v^i \partial_b v^j c_{ij}^a (\partial v) dx \]
\[+ \sum_{i,j=1}^{d} \frac{1}{2} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^{\mu + 1} i \partial_t^\mu j \partial_a v^i \partial_b v^j c_{ij}^a (\partial v) dx .\]

We can estimate \(|J_k| \leq C_\lambda \delta Z_0 (k = 1, 2, 3)\) from Lemma 3.3 and Lemma A.2.

Therefore we get (3.26).

Second, we prove (3.27). Using Lemma 3.3 and Lemma A.1, we have
\[
\langle \partial_t^\mu i, \partial_t^\mu j \delta Z_\lambda \rangle \leq \| \partial_t^\mu i \|_2 \| \partial_t^\mu j \delta Z_\lambda \|_2 \leq C_\lambda \delta Z \leq C_\lambda \delta Z_0.
\]

Furthermore we calculate that
\[
\sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \int_{\Omega_\lambda} \partial_t^\mu i \partial_t^\mu j (c_{ij}^a (\partial v) \partial_a \partial_b v^j) dx
\]
\[= \sum_{0 \leq \mu \leq \mu} \left( \begin{array}{c} \mu \\ \mu \end{array} \right) \sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \int_{\Omega_\lambda} \partial_t^\mu i \partial_t^\mu j \partial_a \partial_b v^j (c_{ij}^a (\partial v)) dx \]
\[+ \sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \int_{\Omega_\lambda} \partial_a \partial_t^\mu i \partial_t^\mu j \partial_b v^j c_{ij}^a (\partial v) dx \]
\[+ \sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \int_{\Omega_\lambda} \partial_t^\mu i \partial_t^\mu j \partial_a (c_{ij}^a (\partial v)) dx \]
\[+ \sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^\mu i \partial_t^\mu j \partial_a \partial_b v^j c_{ij}^a (\partial v) dx .\]

We can estimate \(|J_k| \leq C_\lambda \delta Z_0 (k = 4, 5, 6)\) from Lemma 3.3 and Lemma A.2.

Therefore we get (3.27).

Finally, we prove (3.28). Using Lemma 3.3 and Lemma A.1, we have
\[
\langle \| h \|_\infty, \partial_t^\mu i \delta Z_\lambda \rangle \leq \| h \|_\infty \| \partial_t^\mu i \delta Z_\lambda \|_2 \leq C_\lambda \delta Z \leq C_\lambda \delta Z_0.
\]

We calculate that
\[
\sum_{i,j=1}^{d} \sum_{0 \leq a, b \leq d} \int_{\Omega_\lambda} h \cdot \nabla \partial_t^\mu i \partial_t^\mu j (c_{ij}^a (\partial v) \partial_a \partial_b v^j) dx
\]
Therefore we get (3.28). This completes the proof of Lemma 3.6.

We can estimate

\[ |G(\tilde{v}(v))| = \left\| \partial_v J_t \partial_t v \partial_t^2 v \partial_h v^2 \partial_t \varepsilon_{ij}^a (\partial v) \right\| \leq C \lambda Z(v) \]

We can estimate \(|J_k| \leq C_\lambda \delta Z_0 (k = 7, 8, 9, 10, 11)\) from Lemma 3.3 and Lemma 3.4. Moreover, using (3.14) and

\[ \| \partial v \|_{L^\infty(\partial \Omega)} \leq C_\lambda \| \partial v \|_{H^1(\Omega)} \leq C_\lambda \| \partial v \|_{H^1(\Omega)} + C \lambda Z(v) \]

(The second inequality is the trace theorem, see for instance [9]), we get

\[ J_k \leq C \sum_{i,j=1}^d \sum_{a,b \leq d} \| \varepsilon_{ij}^a (\partial v) \|_{L^\infty(\partial \Omega)} \int_{\partial \Omega} |h \cdot \sigma| ||\sigma \cdot \nabla \partial_t^a v|^2 \, dS \]

\[ \leq C_\lambda \delta \int_{\partial \Omega} |h \cdot \sigma| ||\sigma \cdot \nabla \partial_t^a v|^2 \, dS \quad (k = 12, 13). \]

Therefore we get (3.28). This completes the proof of Lemma 3.6. \qed

We define \( \tilde{G} \) as follows:

\[ \tilde{G}(v(t)) = G(v(t)) + C\lambda \sum_{\mu=0}^{L-1} \sum_{i,j=1}^d \left( \frac{1}{2} \sum_{1 \leq a,b \leq d} \int_{\Omega} \partial_t^a \partial_a v^i \partial_t^a \partial_h v^j \varepsilon_{ij}^a (\partial v) \right) \]

\[ + \sum_{i,j=1}^d \sum_{1 \leq a,b \leq d} \frac{d}{dt} \int_{\Omega} h \cdot \nabla \partial_t^a v^i \partial_t^a \partial_h v^j \varepsilon_{ij}^a (\partial v). \]
Then the following lemma holds.

**Lemma 3.7.** Let \( v \in X^T_\delta \) be the solution to \((DW)_\lambda\) and \( \lambda \) is sufficiently small to hold the Lemma 3.5. Then there exist a \( \delta = \delta(\lambda) \) such that \( v \) satisfy

\[
\frac{d}{dt} \tilde{G}(v(t)) + \frac{b_0}{32} Z_0(v(t)) \leq 0 \quad (t \in [0, T]) \tag{3.29}
\]

and

\[
\tilde{G}(v(t)) \geq \|v(t)\|^2 + Z_0(v(t)) \quad (t \in [0, T]), \tag{3.30}
\]

where comparability constant is independent of \( \delta, t \) and \( T \).

**Proof.** First, we prove (3.29). Using Lemma 3.5 and Lemma 3.6, we obtain

\[
\begin{align*}
- \frac{1}{2} \int_{\Omega_\lambda} \partial_t^{\mu+1} v^t \partial_t^{\nu+1} v^t \partial_{ij}^{\mu} (\partial v) dx & \quad - \sum_{0 \leq b \leq d} \left\{ \int_{\Omega_\lambda} \partial_t^{\mu} v^t \partial_t^{\nu} \partial_b v^t \partial_{ij}^{\mu} (\partial v) dx + \int_{\Omega_\lambda} h \cdot \nabla \partial_t^{\mu} v^t \partial_t^{\nu} \partial_b v^t \partial_{ij}^{\mu} (\partial v) dx \right\} .
\end{align*}
\]

Because of \( \mathbb{R}^d / \Omega_\lambda \) is star shaped, it holds that \( h \cdot \sigma \leq 0 \) on \( \partial \Omega_\lambda \). Then we can choose \( \delta \) sufficiently small depend on \( \lambda \) such that (3.29) holds.

Next, we prove (3.30). It follows from (2.1) that

\[
\left| \frac{b_0(2d - 1)}{4} \sum_{\mu=0}^{L-1} \langle \partial_t^{\mu} v, \partial_t^{\mu+1} v \rangle \right| \leq \frac{b_0(2d - 1)}{4} \sum_{\mu=0}^{L-1} \left\{ \frac{\lambda}{4C_1} \| \partial_t^{\mu} v \|^2 + \frac{C_1}{\lambda} \| \partial_t^{\mu+1} v \|^2 \right\}
\]

\[
\leq \frac{b_0(2d - 1)}{4} \sum_{\mu=0}^{L-1} \left\{ \frac{1}{4} \langle \partial_t^{\mu} v, B_\lambda \partial_t^{\mu} v \rangle + \frac{1}{4\lambda} \| \nabla \partial_t^{\mu} v \|^2 + \frac{C_1}{\lambda} \| \partial_t^{\mu+1} v \|^2 \right\}
\]

\[
\leq \frac{b_0(2d - 1)}{16} \sum_{\mu=0}^{L-1} \langle \partial_t^{\mu} v, B_\lambda \partial_t^{\mu} v \rangle + \frac{C_1b_0(2d - 1)}{4\lambda} Z_0
\]

and

\[
\left| \sum_{\mu=0}^{L-1} \langle \partial_t^{\mu+1} v, [h, \nabla \partial_t^{\mu} v] \rangle \right| \leq \sum_{\mu=0}^{L-1} \| \partial_t^{\mu+1} v \|_2 \| \nabla \partial_t^{\mu} v \|_2 \| h \|_\infty \leq \frac{b_0 R}{\lambda} Z_0. \tag{3.32}
\]
Using (3.31), (3.32), (3.21) and Lemma 2.3, we have

\[
G(v) \geq \frac{1}{\lambda} \left( \frac{C_0}{2} - b_0 R - \frac{C_1 b_0 (2d - 1)}{4} \right) Z_0 + \frac{b_0 (2d - 1)}{16} \sum_{\mu=0}^{L-1} \langle \partial_t^\mu v, B_\lambda \partial_t^\mu v \rangle
\]

\[
\geq \frac{b_0 R}{\lambda} Z_0 + \frac{b_0 (2d - 1)}{16} \sum_{\mu=0}^{L-1} \langle \partial_t^\mu v, B_\lambda \partial_t^\mu v \rangle \geq C_\lambda \langle v \rangle_2^2 + Z_0.
\]

On the other hand, Lemma 2.1, Lemma 3.3 and Lemma A.2 imply that

\[
\left| \int_{\Omega} \partial_t^\mu \partial_a v^i \partial_t^\mu \partial_b v^j c^{ab}_{ij} (\partial v) dx \right| \leq \| \partial_a \partial_t^\mu v^i \|_2 \| \partial_t^\mu \partial_b v^j c^{ab}_{ij} (\partial v) \|_2 \leq C_\lambda \delta Z_0,
\]

\[
\left| \int_{\Omega} \partial_t^{\mu+1} v^i \partial_t^{\mu+1} v^j c^{00}_{ij} (\partial v) dx \right| \leq \| \partial_t^{\mu+1} v^i \|_2 \| \partial_t^{\mu+1} v^j c^{00}_{ij} (\partial v) \|_2 \leq C_\lambda \delta Z_0,
\]

\[
\left| \int_{\Omega} \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c^{0b}_{ij} (\partial v) dx \right| \leq \| \partial_t^\mu v^i \|_2 \| \partial_t^\mu \partial_b v^j c^{0b}_{ij} (\partial v) \|_2 \leq C_\lambda \delta Z_0
\]

and

\[
\left| \int_{\Omega} h \cdot \nabla \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c^{0b}_{ij} (\partial v) dx \right| \leq \| h \|_\infty \| \nabla \partial_t^\mu v^i \|_2 \| \partial_t^\mu \partial_b v^j c^{0b}_{ij} (\partial v) \|_2 \leq C_\lambda \delta Z_0.
\]

Since these estimates and (3.33) imply that we can choose \( \delta = \delta(\lambda) \) to hold \( \hat{G}(v) \geq C_\lambda \langle v \rangle_2^2 + Z_0 \). It is clear that \( \hat{G}(v(t)) \leq C_\lambda \langle v \rangle_2^2 + Z_0 \) is true. Thus it holds that (3.30). This completes the proof of Lemma 3.7.

**Proof of Proposition 3.2**

Let \( \lambda \) and \( \delta \) be sufficiently small to hold Lemma 3.3 and Lemma 3.7. Integrating (3.29) over \([0, t]\), we have

\[
\hat{G}(v(t)) + \frac{b_0}{32} \int_0^t Z_0(v(s)) ds \leq \hat{G}(v(t))_{t=0}.
\]

Since (3.30) imply that

\[
\| v(t) \|_2^2 + Z_0(v(t)) + \int_0^t Z_0(v(s)) ds \leq C_\lambda \langle v_0 \rangle_2^2 + Z_0(v(t))_{t=0},
\]

furthermore using Lemma 3.3 we get (3.4). This completes the proof of Proposition 3.2.
4 Decay Estimate

In this section, we prove Theorem 1.2. In what follows, \( \lambda \) and \( \delta \) be sufficiently small to hold Theorem 1.1. Let \( v \in \mathcal{X}_\delta \) be the solution to (DW)\( \lambda \). Since (3.29) implies

\[
\frac{d}{dt}\{(1 + t)\tilde{G}(v(t))\} = \tilde{G}(v(t)) + (1 + t)\frac{d}{dt}\tilde{G}(v(t)) \leq \tilde{G}(v(t)) - \frac{b_0}{32}(1 + t)Z_0(v(t)).
\]

Integrating the above estimate over \([0, t]\) and using (3.30), Lemma 2.3 and Proposition 3.2, we obtain

\[
(1 + t)\{\|v(t)\|_2^2 + Z(v(t))\} + \int_0^t (1 + s)Z(v(s))ds \leq C\lambda\|v_0, v_1\|_{H^{L \times H^{L-1}}} + C\lambda\int_0^t \langle v(s), B_\lambda(s)v(s)\rangle ds.
\]

We want to the estimate for the second term in the right-hand side in (4.1). As is in Ikehata [2], we consider indefinite integral of \( v \). We define

\[
w(t, x) = \int_0^t v(s, x)ds.
\]

Then \( w \) satisfies

\[
\begin{cases}
(\partial_t^2 - \Delta + B_\lambda(t, x)\partial_t)w = \int_0^t (\partial_t B_\lambda v + F_\lambda)ds + B_\lambda(0)v_0 + v_1 & (t, x) \in [0, \infty) \times \Omega_\lambda, \\
w(0, x) = 0, \partial_t w(0, x) = v_0(x) & x \in \Omega_\lambda, \\
w(t, x) = 0 & (t, x) \in [0, \infty) \times \partial\Omega_\lambda.
\end{cases}
\]

We remark that \( \partial_t w = v \) and \( E(w(t)) \) is well-defined in \([0, \infty)\).

**Lemma 4.1.** We assume that following (H1)\( \lambda \) and (H2)\( \lambda \) hold:

(H1)\( \lambda \) \( \|d_0(\cdot)\{B_\lambda(0)v_0 + v_1\}\|_2 < \infty \),

(H2)\( \lambda \) \( \int_0^\infty \|d_0(\cdot)\partial_t B_\lambda(s)\|_\infty ds < \infty \),

where \( d_0 \) is defined in (1.6). Then it holds following (i) and (ii).

(i) When \( d \geq 3 \), there exists a constant \( E_0 = E_0(v_0, v_1) \) such that

\[
\int_0^t \langle v, B_\lambda v \rangle ds \leq E_0
\]

(ii) When \( d = 2 \), we assume also that (H3)\( \lambda \) holds.
(H3) \( \lambda \) There exists \( M > 0 \) such that \( \supp v_0 \cup \supp v_1 \subset \{ x \in \Omega \lambda : |x| < \frac{M}{\lambda} \} \).

Then there exists \( C_{\lambda,M} > 0 \) such that
\[
\int_0^t (v, B_\lambda v) ds \leq C_{\lambda,M} \| (v_0, v_1) \|^2_{H^2(H^2)} + C_{\lambda,M} \left\{ \int_0^t (1 + s)Z ds \right\}^2.
\]

(4.5)

Proof. Taking inner product \((4.3)\) by \( \partial_t w \), we have
\[
\frac{d}{dt} E(w(t)) + \langle v(t), B_\lambda(t)v(t) \rangle = \langle \partial_t w(t), B_\lambda(0)v_0 + v_1 \rangle + \langle \partial_t w(t), \int_0^t \partial_\lambda B_\lambda(s)v(s) ds \rangle + \langle \partial_t w(t), \int_0^t F_\lambda ds \rangle.
\]

Integrating above equality over \([0, t]\), we obtain
\[
E(w(t)) + \int_0^t \langle v(s), B_\lambda v(s) \rangle ds \leq \frac{1}{2} \| v_0 \|^2 + \langle w(t), B_\lambda(0)v_0 + v_1 \rangle
\]
\[
+ \int_0^t \langle \partial_t w(s), \int_0^s \partial_\lambda B_\lambda(r)v(r) dr \rangle ds + \int_0^t \langle \partial_t w(s), \int_0^s F_\lambda dr \rangle ds
\]
\[
= \frac{1}{2} \| v_0 \|^2 + (A) + (B) + (C).
\]

First, we estimate \((A)\). Using Lemma \(2.3\) we get
\[
(A) = \langle w, B_\lambda(0)v_0 + v_1 \rangle \leq \left\| \frac{w}{d_0(\cdot)} \right\|_{L^2} \| d_0(\cdot) \{ B_\lambda(0)v_0 + v_1 \} \|_{L^2}
\]
\[
\leq C_\lambda \| \nabla w \|_{L^2} \| d_0(\cdot) \{ B_\lambda(0)v_0 + v_1 \} \|_{L^2} \leq \frac{1}{4} E(w(t)) + C_\lambda \| d_0(\cdot) \{ B_\lambda(0)v_0 + v_1 \} \|^2_{L^2}.
\]

In particular, if \( \supp v_0 \cup \supp v_1 \subset \{ x \in \Omega \lambda : |x| < M/\lambda \} \) then we have
\[
\| d(\cdot) \{ B_\lambda(0)v_0 + v_1 \} \|^2_{L^2} \leq C_{\lambda,M} \| (v_0, v_1) \|^2_{H^2(H^2)}.
\]

Second, we estimate \((B)\). Using \((H2)\) and \(3.4\) we calculate
\[
(B) = \int_0^t \langle \partial_t w(s), \int_0^s \partial_\lambda B_\lambda(r)v(r) dr \rangle ds
\]
\[
= \langle w(t), \int_0^t \partial_\lambda B_\lambda(s)v(s) ds \rangle - \int_0^t \langle w(s), \partial_\lambda B_\lambda(s)v(s) \rangle ds
\]
\[
\leq C_\lambda \sup_{0 \leq s \leq t} \left\| \frac{w(s)}{d_0(\cdot)} \right\|_{L^2} \sup_{0 \leq s \leq t} \| v(s) \|^2_{L^2} \int_0^t \| d_0(\cdot) \partial_\lambda B_\lambda(s) \|_{L^\infty} ds
\]
\[
\leq C_\lambda \sup_{0 \leq s \leq t} \| \nabla w(s) \|_{L^2} \sup_{0 \leq s \leq t} \| v(s) \|^2_{L^2} \int_0^t \| d_0(\cdot) \partial_\lambda B_\lambda(s) \|_{L^\infty} ds
\]
\[
\leq \frac{1}{4} E(w(s)) + C_\lambda \sup_{0 \leq s \leq t} \| v(s) \|^2_{L^2} \left\{ \int_0^t \| d_0(\cdot) \partial_\lambda B_\lambda(s) \|_{L^\infty} ds \right\}^2.
\]
Furthermore using Lemma 2.4 and Hölder inequality, we obtain
\[ \leq \frac{1}{4} \sup_{0 \leq s \leq t} E(w(s)) + C_\lambda \|(v_0, v_1)\|_{H^t \times H^{t-1}}^2. \]

Finally, we estimate (C). When \( d \geq 3 \), using Lemma 2.5 we have
\[
\int_0^t \langle \partial_t w, \int_0^s F_\lambda \, dr \rangle \, ds = \langle w, \int_0^t F_\lambda \, ds \rangle - \int_0^t \langle w, F_\lambda \rangle \, ds
\]
\[
\leq 2 \sup_{0 \leq s \leq t} \|w(s)\|_2 \|F_\lambda\|_{L^\infty_t(X)} \leq \frac{1}{4} \sup_{0 \leq s \leq t} E(w(s)) + C_\lambda \left\{ \int_0^t \|F_\lambda\|_{L^\infty_t(X)} \, ds \right\}^2.
\]

Using Lemma 2.1 (3.4) and \( p_l \geq 2 \) (\( l = 1, 2 \)), we obtain
\[
\int_0^t \|F_\lambda\|_{L^4_t} \, ds \leq C_\lambda \int_0^t Z_0 \, ds \leq C_\lambda \|(v_0, v_1)\|_{H^t \times H^{t-1}}^2. \tag{4.7}
\]

Therefore we get
\[
(C) \leq \frac{1}{4} \sup_{0 \leq s \leq t} E(w(s)) + C\|(v_0, v_1)\|_{H^t \times H^{t-1}}^2.
\]

On the other hand when \( d = 2 \), using Lemma 2.5 and Hölder inequality, we have
\[
(C) = \int_0^t \langle \partial_t w, \int_0^s F_\lambda \, dr \rangle \, ds = \langle w, \int_0^t F_\lambda \, ds \rangle - \int_0^t \langle w, F_\lambda \rangle \, ds
\]
\[
\leq C_\lambda \sup_{0 \leq s \leq t} \left\{ \left\| \frac{w(s)}{1 - |t|} \right\|_r \int_0^t \left\| \right\|_r \right\} \leq C_\lambda \sup_{0 \leq s \leq t} \left\{ \left\| \nabla \left( \frac{w(s)}{1 - |t|} \right) \right\|_q \right\} \int_0^t \left\| \right\|_r \, ds,
\]

where \( r \in (2, \infty) \), \( \frac{1}{r} + \frac{1}{r'} = 1 \) and \( \frac{1}{r} = \frac{1}{2} - \frac{1}{2} \). From the assumption (H3)_\lambda and the finite speed of propagation, it holds that
\[
\text{supp}(t) \cup \text{supp} \partial_t v(t) \subset \{ x \in \Omega_\lambda : |x| \leq M/\lambda + 2t \} \quad (t \in [0, \infty)). \tag{4.8}
\]

Thus using (4.8) and considering the same way of (4.7), we get
\[
\int_0^t \left\| \right\|_r \, ds \leq C_\lambda, M \int_0^t (1 + s) Z_0 \, ds.
\]

Furthermore using Lemma 2.4 and Hölder inequality, we obtain
\[
\left\| \nabla \left( \frac{w}{1 - |t|} \right) \right\|_q \leq \left\| \nabla w \right\|_q + \left\| \frac{w}{1 - |t|} \right\|_q \leq C_\lambda \left\| \nabla w \right\|_2 \left\| \frac{1}{1 - |t|} \right\|_r + C_\lambda \left\| \frac{w}{d_0} \right\|_2 \left\| \frac{d_0}{1 - |t|} \right\|_r
\]
\[
\leq C_\lambda \left\| \nabla w \right\|_2 \left( \left\| \frac{1}{1 - |t|} \right\|_r + \left\| \frac{d_0}{1 - |t| + \varepsilon_0} \right\|_\infty \left\| \frac{1}{1 - |t| - \varepsilon_0} \right\|_r \right),
\]

where \( \varepsilon_0 = \frac{r - 2}{2r} \). Remember \( 0 \notin \Omega_\lambda \) and \( r > r(1 - \varepsilon_0) > 2 \), we get
\[
\left\| \nabla w \right\|_2 \left( \left\| \frac{1}{1 - |t|} \right\|_r + \left\| \frac{d_0}{1 - |t| + \varepsilon_0} \right\|_\infty \left\| \frac{1}{1 - |t| - \varepsilon_0} \right\|_r \right) \leq C_\lambda \left\| \nabla w \right\|_2.
\]
Above estimates and Lemma 3.3 imply that
\[
(C) \leq C_{\lambda,M} \sup_{0 \leq s \leq t} \|\nabla w(s)\|_2 \int_0^t (1 + s)Z(v(s))ds \\
\leq \frac{1}{4} \sup_{0 \leq s \leq t} E(w(s)) + C_{\lambda,M} \left\{ \int_0^t (1 + s)Z(v(s))ds \right\}^2.
\]
Combining estimates for (A), (B), (C) and (4.6), we get (4.4) and (4.5). This completes the proof of Lemma 4.1.

Remark 4.2. If $F_\lambda$ has divergence form, we can prove (i) if that $d = 2$. Then we do not need assume (H3)$_\lambda$ (See for instance: [12]).

Proof of Theorem 1.2
It is easy to see that if \{(v_0, u_1), B\} satisfy (H1), (H2) and (H3), then \{(v_0, v_1), B_\lambda\} satisfy (H1)$_\lambda$, (H2)$_\lambda$ and (H3)$_\lambda$ respectively. Therefore when $d \geq 3$, combining (4.1) and (4.4), we get
\[
(1 + t)\|v(t)\|^2_2 + Z(v(t)) + \int_0^t (1 + s)Z(v(s))ds \leq E_0(v_0, v_1). \tag{4.9}
\]
The above estimate means (1.7).
When $d = 2$, combining (4.1) and (4.5), we get
\[
(1 + t)\|v(t)\|^2_2 + Z(v(t)) + \int_0^t (1 + s)Z(v(s))ds \\
\leq C_{\lambda,M}\|v_0, v_1\|^2_{H^2 \times H^{-1}} + C_{\lambda,M} \left\{ \int_0^t (1 + s)Z(v(s))ds \right\}^2.
\]
The above estimate and $\|v_0, v_1\|^2_{H^2 \times H^{-1}} \leq \delta^2$ imply
\[
H(t) \leq C_{\lambda,M}\delta^2 + (H(t))^2,
\]
where $H(t) = \int_0^t (1 + s)Z(v(s))ds$. Because of $H(0) = 0$, we can choose a small $\delta$ depend on $\lambda$ and $M$ such that $H(t) \leq C_{\lambda,M}$ ($t \in [0, \infty)$). Therefore we obtain
\[
(1 + t)\|v(t)\|^2_2 + Z(v(t)) + \int_0^t (1 + s)Z(v(s))ds \leq C_{\lambda,M}\|v_0, v_1\|^2_{H^2 \times H^{-1}} + C_{\lambda,M}^2.
\]
This means (1.7). This completes the proof of (1.7).

Next, we prove (1.8). We calculate
\[
\frac{d}{dt} \{(1 + t)^2 E(v(t))\} = 2(1 + t)E(v(t)) - (1 + t)^2 \langle \partial_t v, B_\lambda \partial_t v \rangle + (1 + t) \langle \partial_t v, F_\lambda \rangle.
\]
Integrating the above equality over $[0, t]$, we get
\[
(1 + t)^2 E(v(t)) + \int_0^t (1 + s)^2 \langle \partial_t v, B_\lambda \partial_t v \rangle ds
\]
\[
= 2(1 + t)E(v(t)) - (1 + t)^2 \langle \partial_0 v, B_\lambda \partial_0 v \rangle + (1 + t) \langle \partial_0 v, F_\lambda \rangle.
\]
\[
\int_0^t (1 + s)^2 \langle \partial_t v, B_\lambda \partial_t v \rangle ds
\]
\[
= 2(1 + t)E(v(t)) - (1 + t)^2 \langle \partial_0 v, B_\lambda \partial_0 v \rangle + (1 + t) \langle \partial_0 v, F_\lambda \rangle.
\]

where $D_v \leq E$ such that $\lambda = \sup_{0 \leq s \leq t} (1 + s)^2 E(v(s)) \geq \frac{1}{2} \sup_{0 \leq s \leq t} (1 + s)^2 E(v(s))$.

Using the above estimate and (1.7), we obtain (1.8). This completes the proof of Theorem 1.2.

\[ \square \]

## A Estimates of nonlinear terms

We show the estimates of the nonlinear terms $F_\lambda$.  

**Lemma A.1.** Let $v \in X_\lambda$, $\delta \leq 1$ and $|\alpha| \leq L - 1$. Then there exists $C_\lambda > 0$ such that  
\[ \| \partial^\alpha F_\lambda(\partial v(t)) \|_2 \leq C_\lambda Z(v(t)) \quad (t \in [0, T]). \]  
\[ \text{(A.1)} \]

**Proof.** If $|\alpha| = 0$, it is easy to see from Lemma 2.1 that we prove (A.1). Let $1 \leq |\alpha| \leq L - 1$. From the chain rule, we have  
\[ \partial^\alpha F_\lambda(\partial v) \]

\[ = \lambda \sum_{\alpha_1 + \cdots + \alpha_l = \alpha} C_{\alpha_1, \cdots, \alpha_l} \sum_{0 \leq j_1, \cdots, j_l \leq d} D_{i_1 j_1} \cdots D_{i_l j_l} \tilde{F}(\partial v) \partial^{\alpha_1} \partial_{j_1} v^{i_1} \cdots \partial^{\alpha_l} \partial_{j_l} v^{i_l}, \]

where $D_{i j} = D_{i j}$. When $l \geq 2$, we choose $2 < q_l \leq \infty$ ($i = 1, 2, \cdots, l$) satisfying  
\[ \sum_{k=1}^l \frac{1}{q_k} = \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq \frac{1}{q_k} + \frac{L - 1 - |\alpha_k|}{d}. \]
\[ \text{(A.2)} \]

Then it holds that  
\[ \| \partial^\alpha F_\lambda(\partial v) \|_2 \]

\[ \leq C_\lambda \sum_{\alpha_1 + \cdots + \alpha_l = \alpha} \sum_{0 \leq j_1, \cdots, j_l \leq d} \| D_{i_1 j_1} \cdots D_{i_l j_l} \tilde{F}(\partial v) \|_\infty \| \partial^{\alpha_1} \partial_{j_1} v^{i_1} \cdots \partial^{\alpha_l} \partial_{j_l} v^{i_l} \|_2 \]

\[ \leq C_\lambda \sum_{\alpha_1 + \cdots + \alpha_l = \alpha} \sum_{0 \leq j_1, \cdots, j_l \leq d} \| \partial v \|_{H^{\frac{\max\{0, p_1 - l\}}{d} + 1}} \prod_{k=1}^l \| \partial^{\alpha_k} \partial_{j_k} v^{i_k} \|_{q_k} \]

\[ \leq C_\lambda \sum_{\alpha_1 + \cdots + \alpha_l = \alpha} \| \partial v \|_{H^{\frac{\max\{0, p_1 - l\}}{d} + 1}} \prod_{k=1}^l \| \partial^{\alpha_k} v \|_{H^{L - 1 - |\alpha_k|}} \]

\[ \leq C_\lambda(Z(v))^{\frac{1}{2}} \leq C_\lambda Z(v), \]
where we use the generalized Hölder inequality and a well known embedding lemma like \( L^q \subset H^s \) \((2 < q < \infty, \frac{1}{2} \leq \frac{1}{q} + \frac{s}{d})\). Indeed it holds from \( L \geq \lceil d/2 \rceil + 3 \) that

\[
\sum_{k=1}^{l} \left\{ \frac{1}{2} - \frac{L - 1 - \lvert \alpha_k \rvert}{d} \right\} - \frac{1}{2} \leq (l - 1) \left\{ \frac{1}{2} + \frac{1 - L}{d} \right\} \leq - \frac{l - 1}{d} < 0.
\]

thus we can choose \( q_k \) satisfying (A.2). When \( l = 1 \), we should choose \( q_1 = 2 \). This completes the proof of Lemma A.1 \( \square \)

**Lemma A.2.** Let \( v \in X_\delta^T \), \( \delta \leq 1 \), \( 1 \leq \lvert \alpha \rvert \leq L \), \( \lvert \beta \rvert \leq L - 1 \) and \( \lvert a + \beta \rvert \leq L + 1 \). For any \( i, j, a \) and \( b \), it holds that

\[
\lVert \partial^\alpha v^i \partial^\beta (c_{ij}^{ab}(\partial v)) \rVert_2 \leq C_L Z(v)
\]

**Proof.** First, we assume \( \lvert \beta \rvert = 1 \). Then we have

\[
\lVert \partial^\alpha v^i \partial^\beta (c_{ij}^{ab}(\partial v)) \rVert_2 \leq \lVert \partial^\alpha v \rVert_2 \sum_{0 \leq \beta \leq d} \lVert D_{ij} c_{ij}^{ab}(\partial v) \partial^\beta \partial v^k \rVert_\infty
\]

\[
\leq C \lVert \partial^\alpha v \rVert_2 \lVert \partial v \rVert_\infty^{p^2 - 2} \lVert \partial^\beta \partial v \rVert_\infty \leq C_L Z(v)^{\frac{2}{p^2}} \leq C_L Z(v).
\]

In the same way, we can prove when \( \beta = 0 \).

Next, we assume \( \lvert \beta \rvert \geq 2 \). In the same way as the proof of Lemma A.2, we obtain

\[
\lVert \partial^\alpha v^i \partial^\beta (c_{ij}^{ab}(\partial v)) \rVert_2 \leq C \sum_{\beta_1 + \cdots + \beta_l = \beta} \sum_{\alpha_1, \ldots, \alpha_\ell \leq d} \lVert \partial^\alpha v^i D_{i_1 j_1} \cdots D_{i_\ell j_\ell} c_{ij}^{ab}(\partial v) \partial^{\beta_1} \partial^{\beta_2} \cdots \partial^{\beta_l} \partial^{\beta_l} \partial v^i \partial v \rVert_2
\]

\[
\leq C \sum_{\beta_1 + \cdots + \beta_l = \beta} \sum_{\alpha_1, \ldots, \alpha_\ell \leq d} \lVert D_{i_1 j_1} \cdots D_{i_\ell j_\ell} c_{ij}^{ab}(\partial v) \rVert_\infty \lVert \partial^\alpha v^i \partial^{\beta_1} \partial^{\beta_2} \cdots \partial^{\beta_l} \partial v^i \rVert_2
\]

\[
\leq C \sum_{\beta_1 + \cdots + \beta_l = \beta} \sum_{\alpha_1, \ldots, \alpha_\ell \leq d} \lVert \partial v \rVert_\max_\{0, p_2 - 1 - l\} \lVert \partial^\alpha v \rVert_\infty \prod_{k=1}^{l} \lVert \partial^{\beta_k} \partial v^k \rVert_2 \rVert_{\lVert r \rVert}
\]

\[
\leq C_L \sum_{\beta_1 + \cdots + \beta_l = \beta} \lVert \partial v \rVert_\max_0 \{0, p_2 - 1 - l\} \lVert \partial^\alpha v \rVert_{H^{L - |\alpha|}} \prod_{k=1}^{l} \lVert \partial^{\beta_k} \partial v^k \rVert_{H^{L - 1 - |\beta_k|}}
\]

\[
\leq C_L Z^{\max(l + 1, p_2 + 1)} \leq C_L Z(v),
\]

where we choose \( 2 < r_k < \infty \) \((k = 0, 1, \ldots, l)\) satisfying

\[
\sum_{k=0}^{l} \frac{1}{r_k} = \frac{1}{2}, \quad \frac{1}{2} \leq \frac{1}{r_0} + \frac{L - |\alpha|}{d} \quad \text{and} \quad \frac{1}{2} \leq \frac{1}{r_k} + \frac{L - |\beta_k| - 1}{d}.
\]
Indeed it holds from $L \geq \lfloor d/2 \rfloor + 3$ that
\[
\frac{1}{2} \frac{L - |\alpha|}{d} + \sum_{k=1}^{l} \left\{ \frac{1}{2} \frac{L - 1 - |\beta_k|}{d} \right\} - \frac{1}{2} \frac{l}{d} + \frac{l(1 - L)}{d} + 1 \leq -\frac{3l + 2}{2d} < 0,
\]
thus we can choose $q_k$ satisfying (A.3). This completes the proof of Lemma A.2.

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References

[1] W. Dan, Y. Shibata, On a local energy decay of solutions of a dissipative wave equation, Funkcial. Ekvac., 38 (1995), 545-568.
[2] R. Ikehata, Fast decay of solutions for linear wave equations with dissipation localized near infinity in an exterior domain, J. Differential Equations 188 (2003), 390-405.
[3] F. John, Blow-up for quasilinear wave equations in three space dimensions, Comm. Pure Appl. Math. 34 (1981), no. 1, 29-51.
[4] T. Kato, Abstract differential equations and nonlinear mixed problems, Lezioni Fermiane: Accademie Nazionale dei Lincei. Scuola Normale Superiore, 1988.
[5] H. Kubo, Almost global existence for nonlinear wave equations in an exterior domain in two space dimensions, arXiv:1204.3725
[6] S. Kawashima, M. Nakao and K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan 47 (1995), no. 4, 617-653.
[7] M. Keel, H. Smith and C. D. Sogge, Almost global existence for quasilinear wave equations in three space dimensions, J. Amer. Math. Soc. 17 (2004), 109-153.
[8] S. Klainerman, The null condition and global existence to nonlinear wave equations, Lectures in Applied Math., 23 (1986), 293-326.
[9] J. L. Lions and E. Magenes, Nonhomogeneous boundary value problems and applications, vol. 1, Springer-Verlag, New York, 1972. MR50:2670
[10] A. Matsumura, On the Asymptotic Behavior of Solutions of Semi-linear Wave Equations, Publ. RIMS, Kyoto Univ.12 (1976), 169-189.
[11] K. Mochizuki, On scattering for wave equations with time dependent coefficients, Tsukuba J. Math, Vol. 31 No. 2 (2007), 327-342.
[12] M. Nakao, Decay and Global Existence for Nonlinear Wave Equations with Localized Dissipations in General Exterior Domains, New Trends in the Theory of Hyperbolic Equations Operator Theory: Advances and Applications Volume 159, 2005, pp 213-299.
[13] T. Narazaki, $L^p - L^q$ estimates for damped wave equations and their applications to semi-linear problem, J. Math. Soc. Japan, 56 (2004), 585-626.

[14] K. Nishihara, $L^p - L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application. Math. Z. 244 (2003), no. 3, 631-649.

[15] T. Sideris, Nonresonance and global existence of prestressed nonlinear elastic waves, Ann. of Math. (2), 151 (2000), 849-874.

[16] G. Todorova, B. Yordanov, Weighted $L^2$-estimates of dissipative wave equations with variable coefficients, J. Differential Equations 246 (2009), no. 12, 4497-4518.

[17] G. Todorova, B. Yordanov, Nonlinear dissipative wave equations with potential, Control methods in PDE-dynamical Systems, 317-337, Contemp. Math., 426, Amer. Math. Soc., Providence, RI, 2007.

[18] Y. Wakasugi, On diffusion phenomena for the linear wave equation with space-dependent damping. [arXiv:1309.3377]

[19] T. Watanabe, Global existence and decay estimates for quasilinear wave equations with nonuniform dissipative term. [arXiv:1311.6573]

[20] J. Wirth, Wave equations with time-dependent dissipation II. Effective dissipation, J. Differential Equations, 232 (2007), 74-103.