Second-order amplitudes in loop quantum gravity

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Abstract
We explore some second-order amplitudes in loop quantum gravity. In particular, we compute some second-order contributions to diagonal components of the graviton propagator in the large distance limit, using the old version of the Barrett–Crane vertex amplitude. We illustrate the geometry associated with these terms. We find some peculiar phenomena in the large distance behavior of these amplitudes, related to the geometry of the generalized triangulations dual to the Feynman graphs of the corresponding group field theory. In particular, we point out a possible further difficulty with the old Barrett–Crane vertex: it appears to lead to flatness instead of Ricci flatness, at least in some situations. The observation raises the question whether this difficulty remains with the new version of the vertex.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The problem of quantum gravity is—in a sense—a double problem: first, to find the appropriate theory, which we expected to be a background-independent quantum field theory, and second, to learn how to extract physics from such a background-independent quantum field theory. This second problem is highly non-trivial, because most, if not all, of the conventional tools for extracting physics from a quantum field theory rely heavily on the existence of an external metric background.

A tentative solution to this second problem has been developed in the last years [1–3] and is based on two ingredients. The first is the boundary formalism [4, 5], which we briefly summarize below. The second is the idea of computing transition amplitudes order by order in a background-independent expansion, where the order is given by the number of interaction-vertices, or, equivalently, in the number of $n$-simplices of the associated dual cellular complex. If the theory is expressed as a group field theory (GFT) [6], this expansion amounts to a
perturbative expansion in the GFT coupling constant $\lambda$ [5]. We denote this expansion the ‘vertex expansion’, and we discuss below its physical viability.

With only a few exceptions [3, 7], so far most of the literature has been concentrated on first-order terms in this expansion [8–12]. Here we explore the structure, geometry and physical meaning of some second-order terms. The importance of studying higher order terms is multifold. First, it is not yet clear which is the physical regime where the vertex expansion is good; the easiest way to address the problem is to compare the first-order terms with higher ones. Second, the structure of the expansion is still far from being fully settled: there are open questions concerning the correct normalization of the amplitudes, and similar. Again, we think that the best way of addressing these issues is concretely, by studying the terms of the expansion.

The different terms in the vertex expansion that we study have a simple geometrical interpretation. Roughly speaking, the lowest order term can be viewed as describing an approximation to general relativity where the geometry of the spacetime region under consideration is approximated by a single 4-simplex, of variable shape and size. Higher order terms give the approximations where the geometry of the region is approximated by a larger number of glued 4-simplices, each of varying shape and size. Here we illustrate in detail the geometry of some cellular complexes contributing to the second-order approximation, obtained by gluing 4-simplices in this way.

We restrict ourselves to computing the diagonal part of the propagator, instead of writing its full tensorial structure [10]. Also, we use the dynamics defined by the old Barrett–Crane vertex [13]; in particular, the specific model we use is the theory GFT/B (see [5]), introduced in [14, 15]; the result extends immediately also to the theory GFT/C, introduced in [16], which is characterized by particularly good finiteness properties [17–19]. For this reason, the results presented here are a bit out-of-date: they need to be extended to the new-vertex models introduced recently [20–25], which have far better properties. Nothing seems to prevent such extension, and the work done here should open the way for analyzing the theory with more interest. Similarly, the semiclassical behavior considered here needs to be compared with the results on the semiclassical behavior of these new models [26, 27].

We find a certain number of features of the amplitudes, which we summarize in the conclusion section. Of particular interest is the fact that the amplitude appears to be suppressed, at least in some cases, unless the triangulation admits a flat metric. This is not what we expect for the classical limit, which should be dominated by Ricci flatness. This problem can be a further sign of the difficulties of the old Barrett–Crane model: we think that it needs to be seriously addressed in the context of the new models.

2. Preliminaries

We briefly recall the basis of the formalism that we use. This is not a self-sufficient introduction: we refer the reader to [3] for complete definitions and details, and to [5] for a general introduction. On the other hand, we address here and we offer some clarification on some general questions that have been raised concerning the approach.

2.1. The boundary formalism

The key idea for extracting physics from a background-independent formulation of quantum field theory is to compute transition amplitudes associated with a finite spacetime region, as

\footnote{For an analysis of higher order corrections to the quantum gravity propagator in three dimensions see [8, 9].}
functions of the quantum state on the boundary $\Sigma$ of the region [1, 4, 5]. In particular, the boundary state will include the quantum state of the gravitational field, namely the quantum state of the boundary geometry. Physically, this means that we are describing a region of quantum spacetime, as it is observed by apparatuses that take measurements at its boundary—the key point being that these measurements include (quantum) measurements of distances.

If we do so, the information about the background geometry of the region is provided dynamically by the (measured) boundary quantum state itself. Formally, if the boundary geometry determines a classical solution of Einstein’s equations in the bulk, then we expect the Feynman integral in the region to be dominated by configurations around the classical one. In this way, the interior background geometry is determined by the boundary quantum state: this allows us to define background-dependent quantities in the context of a fully background-independent bulk theory.

The main tool of this approach is the boundary functional, formally defined by the functional integral over all fields $\phi$ in the interior region, at fixed boundary value $\phi$:

$$W[\phi] = \int_{\phi|_{\Sigma} = \phi} D\phi \ e^{iS[\phi]}.$$  \hspace{1cm} (1)

In a background-independent theory, where measure $D\phi$ and action $S[\phi]$ are diff-invariant, this quantity does not depend on the spacetime location of $\Sigma$. By contracting this quantity with a state $\Psi[\phi]$, we obtain a probability amplitude associated with this state

$$\langle W|\Psi \rangle \equiv \int D\phi \ W[\phi] \Psi[\phi].$$  \hspace{1cm} (2)

This amplitude can then be compared, say, with the amplitude

$$\langle W|\phi(x)\phi(y)\rangle,$$  \hspace{1cm} (3)

where $\phi(x)$ is a field operator creating a quantum excitation over $\Psi$. The quantity

$$W(x, y; \Psi) = \langle W|\phi(x)\phi(y)\rangle,$$  \hspace{1cm} (4)

where the boundary state $\Psi$ satisfies $\langle \Psi|\Psi \rangle = 1$ and

$$\langle W|\Psi \rangle = 1,$$  \hspace{1cm} (5)

gives the probability amplitude for a field’s quantum, or a ‘particle’, to propagate from $x$ to $y$ in the background defined by $\Psi$ (for the meaning of ‘particle’ in this context, see [28]).

This formalism reduces to the standard quantum mechanical formalism on a flat space, if we take $\Sigma$ to be the union of the two hypersurfaces $t = 0$ and $t = T$, and $\Psi$ to be the element $\Psi_{00} = |0\rangle \otimes |0\rangle$ of the tensor product $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$ of the initial and final state spaces [29–31]. Then, taking $x = (\vec{x}, T)$, $y = (\vec{y}, 0)$,

$$W(x, y; \Psi_{00}) = \langle 0|\phi(\vec{x}) e^{-iHT} \phi(\vec{y})|0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle,$$  \hspace{1cm} (6)

while the normalization condition (5) is clearly satisfied:

$$\langle 0| e^{-iHT} |0\rangle = 1.$$  \hspace{1cm} (7)

But the formalism remains meaningful in a diffeomorphism invariant context because the positions $y$ and $x$ of the incoming and outgoing particles are well defined with respect to the boundary geometry specified by $\Psi$. That is, $W(x, y; \Psi)$ is not invariant under a coordinate transformation on $x$ and $y$ alone, but it is invariant under a coordinate transformation acting on $x$, $y$, as well as $\Psi$. The relative position of the particle with respect to the boundary geometry is diffeomorphism-invariant and the amplitude is therefore well defined and physically meaningful.
If the background geometry defined by the state $\Psi_1$ is flat, then the quantity (4) should reduce to the conventional quantum field theoretical two-point function in the weak field limit. In particular, if we compute this quantity in general relativity for the gravitational field, then (4) should reduce to the weak-field graviton propagator

$$W_{\mu\nu\rho\sigma}(x, y; \Psi_1) \rightarrow \langle 0 | h_{\mu}(x) h_{\nu}(y) | 0 \rangle$$

in the large distance limit. The dictionary of the translation between the 3-geometry and 3-geometry transition-amplitude language, and the graviton scattering language, is studied in detail in [32].

2.2. LQG implementation

The boundary formalism described above can be made concrete in the context of loop quantum gravity (LQG) [33, 34]. We take the state $\Psi_1$ to live in the LQG state space, and we take the boundary functional given by the spinfoam formalism [5, 35]. The compatibility between the canonical loop theory and the covariant spinfoam theory, still unclear when the boundary formalism was developing, has since been firmly established [25]. In particular, a spinfoam dynamics $W$ can be generated by group-field-theoretical methods, where it reproduces the spinfoam model amplitude on any given two-complex, at each order of a vertex expansion [5]. By choosing an $s$-knot basis $|s\rangle$ in the LQG state space [5], we have

$$W_{abcd}(x, y, q) = \sum_{s,s'} W[s] \langle s'|h_{ab}(x)h_{cd}(y)|s\rangle |\Psi_q[s]\rangle$$

(9)

for a state satisfying the normalization condition

$$\sum_s W[s] |\Psi_q[s]\rangle = 1.$$

Concretely, a boundary functional $W[s]$ is defined by any spinfoam model. In fact, a spinfoam model is precisely an algorithm that computes an amplitude $W[s]$ for each boundary spin network $s$. It has the intuitive interpretation as a regularization of the Misner–Wheeler integral-over-4-geometries

$$W[q] = \int_{\Sigma = q} Dg e^{\mathcal{S}_{EH}[g]}$$

(11)

intrinsically regularized by the discreteness of the geometry as established by LQG. Here $\mathcal{S}_{EH}[g]$ is the Einstein–Hilbert action and $q$ is the boundary 3-geometry.

One possible triangulation independent way to write $W[s]$ is to use GFT [6]. Here we use the group field theory B (GFT-B) spinfoam model

$$W[s] = \int D\phi f_s(\phi) e^{-\int S_{SO}[\phi, \phi']},$$

(12)

The field $\phi$ is a function on $SO(4)^4$; $s$ is an s-knot [5] (or, loosely speaking, a ‘spin network’) with $n$ nodes and $f_s$ is a polynomial of order $n$ in the fields, obtained contracting the indices of the field following the path defined by the s-knot. See [5] for the details and the notation. The choice of this spinfoam model here is dictated only by simplicity and convenience. In fact, the limitations of this model are well known, and the results here need to be extended to the more realistic models.

The Feynman rules of this theory are as follows. The field $\phi$ decomposes in modes $\phi_{\alpha n}^{ij}$ with $n = 1, 2, 3, 4$. Here $j_n$ are $SU(2)$ representations, $\alpha_n$ is an index in the $j_n$ representation space and $i$ labels the (elements of a basis in the space of the) intertwiners in the tensor product of the four representation $j_n$. The standard quantum field theoretical perturbation expansion
in \( \lambda \) of this amplitude generates a sum over Feynman graphs with five-valent vertices. The propagator is

\[
P_{js}^{\lambda, \alpha n} = \sum_s P_{js}^{s(\lambda), s(\alpha n)}
\]

where the sum is over all permutations \( s \) of four elements and

\[
P_{js}^{\lambda, \alpha n} = \delta_{ii'} \prod_n \delta_{\alpha n} \delta_{jns, jms} \delta_{\lambda ns, \lambda ms}.
\]

The vertex is five-valent, and is given by

\[
V_{\alpha n mn}^{s, i} = \lambda B(jnm) \prod_n \delta_{\alpha nm} \delta_{jnm, jmn}
\]

where \( B(jnm) \) is the vertex amplitude; see [5]. For large spins, the asymptotic behavior of the vertex amplitude is given by

\[
B(jnm) \sim e^{iS_{\text{Regge}}(jnm)} + e^{-iS_{\text{Regge}}(jnm)} + D(jnm)
\]

where \( S_{\text{Regge}}(jnm) \) is the Regge action associated with a 4-simplex with areas proportional to \( jnm \) and \( D(jnm) \) is a factor that appears when the areas \( jnm \) define degenerate configurations of the 4-simplex. The Regge action has the form

\[
S_{\text{Regge}}(jnm) = \sum_{mn} \phi_{nm}(jnm) j_{nm}
\]

where \( \phi_{nm}(jnm) \) are the dihedral angles of the 4-simplices with areas proportional to \( jnm \).

2.3. The vertex expansion

The GFT formulation suggests a perturbative expansion for \( W[s] \): the expansion in the GFT coupling constant \( \lambda \), namely the vertex expansion. The physical meaning of this expansion is clarified by noting that individual terms of this expansion can be equally obtained by truncating general relativity to the finite number of degrees of freedom system formed by a Regge triangulation on a given discretization of spacetime. Note that it is neither a short-scale, nor a large-scale expansion, since the individual 4-simplices can be large or small [22]. It is rather more similar to the approximation used very effectively in cosmology, where only some degrees of freedom of the geometry of the universe are left free [37].

Can such a truncation provide an interesting approximation to the quantum gravitational dynamics? The truncation is background-independent, in the sense in which Regge calculus is. But one may worry that on a fixed triangulation the theory has a finite number of degrees of freedom and therefore it cannot sufficiently capture the field-like behavior of gravity. This objection is wrong, since it would apply to the standard perturbative expansion of QED as well: if we compute a scattering process between a finite number \( m \) of particles and a finite order \( n \) in perturbative QED, we are restricting the QED Fock space to the subspace formed
by a finite number of particles (as many as $n$ vertices can produce from $m$ particles). Thus, we are de facto truncating QCD to a theory with a finite number of degrees of freedom (a particle has obviously a finite number of degrees of freedom). In other words, conventional QFT perturbation expansion includes a truncation of the field theory to a theory with a finite number of degrees of freedom. There is no reason for the same not be viable in gravity.

The correct question, then, is not if the truncation given by the vertex expansion yields a viable approximation, but rather in which regime this approximation is viable. Again, the Regge lattice analogy provides the answer: any gravitational physics that can be captured by the finite Regge triangulation. For instance if a phenomenon is characterized by a size (wavelength) $l$ and can be confined in a region of size $L$, then $L/l$ sets the scale of the relevant number of ‘cells’ needed to approximate the phenomenon.

Lattice QCD provides a good example of this: effective lattice QCD calculation yields the correct mass spectrum of the hadrons using lattices that have a rather small number of cells. This number is determined by the ratio between the size of the hadron and the minimal relevant wavelength. What is remarkable is that good quantum physics is obtained with cubic lattices with sides of only a few cells. Clearly there is no real need of infinite lattices to do physics.

2.4. The large distance expansion

We are interested in the graviton two-point function (4) at first order in $\lambda$, in the limit in which the boundary geometry is large. In this limit we are only looking at very large wavelengths, and it is therefore reasonable to expect that the vertex expansion is viable. The calculation of the graviton two-point function (4) in this limit at first order in $\lambda$ on the basis of the formalism described above was completed in [2] for the diagonal terms ($\mu = \nu$ and $\rho = \sigma$), and in [10] for the other terms. The fact that the non-diagonal terms of the propagator turned out to be wrong was a main reason for the replacement of the old version of the Barrett–Crane vertex with its new version [20, 22, 23, 25]. The correct $1/L^2$ dependence of the propagator on the distance $L$, obtained in [2], was confirmed in a next to leading order evaluation [3]. Several second-order terms are considered below.

3. Joining two 4-simplices

3.1. One internal propagator: $4 \rightarrow 1 \rightarrow 4$ Pachner’s move

Consider the second-order Feynman diagram.

Call $v^u$ and $v^d$ (for up and down) the two vertices of this graph, $e$ the internal propagator and $e^u_n$ and $e^d_n$, where $n = 1, 2, 3, 4$ the external legs.

Such a graph can appear in computing the amplitude associated with the observable $f_{s_8}(\phi)$ determined by the spin network $s_8$ illustrated in figures 1 and 2. The graph of this spin network is $\Gamma_8$. It consists of two tetrahedral spin networks connected by four links. The spin network $s_8$ is obtained by coloring the links and nodes of $\Gamma_8$. We denote the spins and intertwiners associated with nodes and links of $\Gamma_8$ as in figure 1, panel b. That is, we denote by $j^U_n, j^A_n, j^L_n$. 

6
the 20 representations associated with the 20 links $l_{un}^a, l_{ud}^d, l_{nu}^i$ of $\Gamma_8$, and $i_{u}^a, i_{d}^d$ the eight intertwiners associated with the eight nodes $n_u^u$ and $n_d^d$. The set $s_8 = (\Gamma_8, j_{un}^u, j_{ud}^d, j_{nu}^i, i_{u}^a, i_{d}^d)$ defines a boundary spin network.
The observable $f_{n}(\phi)$ determined by this spin network is

$$f_{n}(\phi) = \sum_{\alpha_{n}14} \prod_{a=1,4} \phi_{\alpha_{nm}a}^{\alpha_{nm}a} \phi_{\alpha_{nm}a}^{\alpha_{nm}a} \tag{18}$$

where we have used the notation $j_{mn}^{a} := j_{mn}^{a}$. This is a monomial of order 8 in the field.

The expansion of its expectation value at order $\lambda^{2}$ gives

$$W[s_{8}] = \frac{\lambda^{2}}{2(5!)^{2}} \int D\phi \ f_{n}(\phi) \left( \phi \right)^{2} e^{-\phi^{2}}. \tag{19}$$

The Wick expansion of this integral gives two vertices and nine propagators. (If $d$ is the order of $\phi$ in $f_{n}$, and $n_{s}$ is the number of vertices, then the number of propagators is clearly $n_{p} = \frac{d^{2}s_{8}^{2}}{2}$. ) In particular, consider the term where the four up (resp. down) legs of the graph are connected to the upper (resp. lower) tetrahedral spin network, as shown in figure 6; the corresponding amplitude is

$$W[s_{8}] = \left( \prod_{n=1,4} \mathcal{P}_{\alpha_{nm}a}^{\alpha_{nm}a} \right) \phi_{\alpha_{nm}a}^{\alpha_{nm}a} \mathcal{P}_{\beta_{nm}a}^{\beta_{nm}a} \mathcal{V}_{\gamma_{nm}a}^{\gamma_{nm}a} \left( \prod_{n=1,4} \mathcal{P}_{\beta_{nm}a}^{\beta_{nm}a} \right). \tag{20}$$

In the first parenthesis we have the contribution from the up boundary (contractions among the ‘up’ four-boundary tetrahedra $u_{m}$), then we have the contraction between these and the fifth (taken as ‘internal’), then the internal propagator, and then the down vertex and the contractions in the ‘down’ boundary.

The sums over permutations (13) in the propagators give rise to a sum over two-complexes having the Feynman graph as two-skeleton. Since we are interested in the large $j$ behavior of the amplitude, and since each face of the two-complex carries some powers of $j$, the dominant term will be the one with the maximum number of faces. It is not hard to see that this term is given by the term

$$W[s_{8}] = \left( \prod_{n=1,4} \mathcal{P}_{\alpha_{nm}a}^{\alpha_{nm}a} \right) \phi_{\alpha_{nm}a}^{\alpha_{nm}a} \mathcal{P}_{\beta_{nm}a}^{\beta_{nm}a} \mathcal{V}_{\gamma_{nm}a}^{\gamma_{nm}a} \left( \prod_{n=1,4} \mathcal{P}_{\beta_{nm}a}^{\beta_{nm}a} \right) = \left( \prod_{n=1,4} \dim \left( f_{n}^{a} \left( j_{nm} \right) \right) \dim \left( j_{nm}^{a} \right) \right) \left( \prod_{n<m, n=1,4} \dim \left( f_{nm}^{a} \left( j_{nm} \right) \right) \right) B(j_{nm}) B(j_{nm}). \tag{21}$$

of the sum over permutations. Let us analyze this term. It is obtained by adding 16 faces to the Feynman graph: four faces $f_{n}$ bounded by the three edges: $e_{n}^{a}$, $e_{n}$, six faces $f_{nm}^{a}$ bounded by the two edges: $e_{n}^{a}$ and $e_{m}^{a}$ and six faces $f_{nm}^{a}$ bounded by the two edges: $e_{n}^{a}$ and $e_{m}^{a}$. The nodes $n_{n}^{a}$ and $n_{m}^{a}$ of $\Gamma_{8}$ bound the edges $e_{n}^{a}$ and $e_{m}^{a}$ respectively. The links $l_{n}^{a}$, $l_{m}^{a}$ and $l_{nm}^{a}$ of this graph bound the faces $f_{n}$, $f_{nm}$ and $f_{nm}$, respectively.

The structure of this two-complex becomes transparent by noting that it is the complex dual to a rather simple triangulation, which we call $\Delta_{8}$. This is obtained by gluing two 4-simplices by one tetrahedron $T$. The four-dimensional triangulation $\Delta_{8}$ is illustrated in the first panel of figure 3. This is the 4D analog of the 3D and 2D cases illustrated in the other two panels of the figure.

The triangulation $\Delta_{8}$ is formed by 6 points, which we label as 1, 2, 3, 4, $u$, $d$; the 14 edges $(n, m), (n, u), (n, d)$; the 12 faces $(n, m, n), (n, m, u), (n, m, d)$; the 9 tetrahedra $(1, 2, 3, 4), (n, m, p, u), (n, m, p, d)$ and the two 4-simplices $(1, 2, 3, 4, u), (1, 2, 3, 4, d)$.

Here $n = 1, 2, 3, 4$ and $n \neq m \neq p$.

The two vertices $v^{a}$ and $v^{b}$ are dual to the two 4-simplices of this triangulation. The four triangles that bound $T$ are the dual to the faces $f_{n}$; the other six triangles of the upper (resp. lower) tetrahedron $T$ are those of the 4D analog of the 3D and 2D cases illustrated in the other two panels of the figure.
Figure 3. The spacetime triangulation $\Delta_4$ and the 3D and 2D analogs.

| Tetrahedra | Triangles |
|------------|-----------|
| $u_1$      | $t_{23}$  | $t_{24}$  | $t_{34}$ |
| $u_2$      | $t_{13}$  | $t_{14}$  | $t_{124}$|
| $u_3$      | $t_{12}$  | $t_{14}$  | $t_{124}$|
| $u_4$      | $t_{12}$  | $t_{13}$  | $t_{124}$|

| $p_1$ | $t_{234}$ | $t_{134}$ | $t_{124}$ | $t_{123}$ |

| $d_1$ | $t_{35}$  | $t_{34}$  | $t_{234}$ |
| $d_2$ | $t_{13}$  | $t_{14}$  | $t_{124}$ |
| $d_3$ | $t_{12}$  | $t_{14}$  | $t_{124}$ |
| $d_4$ | $t_{12}$  | $t_{13}$  | $t_{124}$ |

Figure 4. (a) Tetrahedral decomposition: the up 4-simplex (tetrahedra $u_1 u_2 u_3 u_4 p_1$) is glued to the down 4-simplex ($p_1 d_1 d_2 d_3 d_4$) through the shared tetrahedron $p_1$. (b) GFT diagram: every line corresponds to a triangle, four lines grouped to a tetrahedron. (c) Relation between tetrahedra and triangles.

lower) 4-simplex are dual to the faces $f^u_{nm}$ (resp. $f^d_{nm}$). Note that all these triangles belong to the boundary of the triangulation, as is clear from the 3D analog. The graph dual to this boundary is clearly the $\Gamma_8$ graph of figure 1: the four upper nodes of $\Gamma_8$ correspond to the four upper tetrahedra; the four lower nodes of $\Gamma$ correspond to the four lower tetrahedra. The six links joining the upper (lower) nodes correspond to the six upper (lower) vertical triangles $t^u_{nm}$ ($t^d_{nm}$); the four vertical links correspond to the four triangles $t_n$ bounding $T$. Therefore, $\Gamma_8$ is the dual of a triangulation of a compact 3D surface, with the topology of a three-sphere, which can be viewed as the boundary of the spacetime region formed by two adjacent 4-simplices.

The path of the indices in (21) gives the geometrical decomposition of the triangulation illustrated in figures 4 and 5, where each line corresponds to a face of the two-complex, or, equivalently, a triangle of the triangulation.
If we interpret the vertical axis as a ‘time’ axis, the triangulation $\Delta_8$ represents the world history of a point $d$ opening up to a tetrahedron $T$ and then recollapsing to a point $u$. (In the 3D case, we have a point opening up to a triangle and then recollapsing; in the 2D case, we have a point opening up to a segment and then recollapsing. See figure 3.) The process described by this amplitude can therefore be interpreted as a creation and annihilation of an ‘atom of space’ [3].

Following [3], let us now use the amplitude (21) for computing a contribution to the graviton two-point function, at a given boundary state. At this order, the relevant component of the boundary state is on the graph $\Gamma_8$:

$$\Psi_q[s_8] = \Psi_q(j_{nm}^u, j_n^d).$$

Let us assume that the boundary state describes a regular semiclassical geometry for the boundary of the triangulation $\Delta_8$. Let this be peaked on the geometry $q_8$ defined by the boundary of the region of $R^4$ formed by the two pyramidal 4-simplices having for basis a regular tetrahedron $T$ with side of length $L'$, and height $T$. Since we are interested in the large $j$ regime, we peak the state on the values

$$j_n^{(8)} = \frac{\sqrt{3}}{32\pi \hbar G} L'^2 \equiv j_L,$$

$$j_{nm}^{(8)} = j_{nm}^{(d)} = \frac{1}{8\pi \hbar G} \left(2L' \sqrt{L'^2/8 + T^2} + \frac{\sqrt{3}}{2} L'^2 \right) \equiv j_{TL}.$$  

We do not give here the explicit value of the background dihedral angles $\Phi^{(8)}$, which can be obtained by elementary geometry: for details see the appendix in [3]. We choose a boundary state given by a Gaussian peaked on $q_8$. Writing all spins in a single vector $j_l = (j_{nm}^u, j_{nm}^d, j_n^s)$, we have

$$\Psi_q[s_8] = C_8 \ e^{-\alpha_{ij}(j_l - j_l^{(8)})(j_l - j_l^{(8)})^\ast \Phi^{(8)} j_l}.$$  

Following [2, 3], we contract the four indices of (9) with normals to boundary triangles, and choose in particular, say, to look at the ‘diagonal’ term determined by the triangles $\nu_{12}$ and $\nu_{13}$, obtaining

$$G_{s_8}(L', T) \equiv \langle \nu_{12}^u \nu_{13}^d \rangle \langle \nu_{13}^u \nu_{12}^d \rangle \ W^{abcd}(x, y, q_8) = \sum_{s'} W[s] W[s'] \langle \delta j_{12} \delta j_{13}^d \rangle \Psi_q[s].$$

where $\delta j = j - j^{(8)}$. All the terms in this expression are now well defined.
We now use the asymptotic expression for each B, as in [2]. This is given by the cosine of the Regge action plus the degenerate term. The phase in the boundary state suppresses the sum unless it is matched by a corresponding phase of a term in \( W \). This happens for only one of the exponentials in the cosine, as can be seen as follows. The sum of the Regge actions for the two 4-simplices, \( S_{\text{Regge}} = S_{\text{Regge}}^a + S_{\text{Regge}}^b \), can be expanded around \( j_{F} \) and \( j_{L} \):

\[
S_{\text{Regge}}(j_{nm}, j_{\lambda}) = \gamma_{nmj}^{(u)} + \phi_{nmj}^{(u)} j_{\lambda} + \phi_{nmj}^{(u)} j_{\lambda}^{d} + \frac{1}{2} G_{ll}^{(u)} \delta j_{l} \delta j_{l}^{d},
\]

where \( \phi_{nm}^{(u)} \) and \( \phi_{nm}^{(u)} \) are the dihedral angles of flat 4-simplices with the given boundary intrinsic geometry; the linear terms in the expansion of the Regge action sum up, giving the dihedral angle of the boundary of the 4D region, which is precisely the sum of the dihedral angles of the two 4-simplices at the faces of \( T \). That is, \( \phi_{nm}^{(u)} = \Phi_{nm}^{(u)} \), but \( 2 \phi_{nm}^{(u)} = \Phi_{nm}^{(u)} \). The second-order term in (27) is the ‘discrete derivative’ [3]

\[
G_{ll}^{(u)} = \left( \frac{\delta^2 S_{\text{Regge}}}{\delta j_{l} \delta j_{l}^{d}} \right)_{j_{nm} = j_{\lambda}^{(u)}}.
\]

This matrix can be computed from elementary geometry. Being a derivative of an angle with respect to an area, \( G_{ll}^{(u)} \) should scale as the inverse of \( \sqrt{j_{u} j_{\lambda}^{(u)}} \). It is therefore convenient to define the scaled quantity

\[
\Gamma_{ll}^{(u)} = \frac{G_{ll}^{(u)}}{\sqrt{j_{u} j_{\lambda}^{(u)}}}.
\]

Thus, we obtain

\[
G_{a}(L', T) = \frac{4 \lambda^2 N_{8}}{J_{T}} \sum_{\delta j_{h}} \delta j_{12}^{a} \delta j_{13}^{a} P_{2} e^{-i S_{\text{Regge}}^{a} + S_{\text{Regge}}^{b} i} \delta j_{\alpha}(j_{\lambda} - j_{\lambda}^{(u)})(j_{\lambda} - j_{\lambda}^{(u)}) \Phi_{1}^{(u)} j_{l}
\]

where \( N_{8} \) is fixed by the normalization condition. If only the Feynman graph that we are considering enters it, we have

\[
\frac{1}{N_{8}} = \frac{4 \lambda^2}{J_{T}} \sum_{\delta j_{h}} P_{2} e^{-i S_{\text{Regge}}^{a} + S_{\text{Regge}}^{b} i} \delta j_{\alpha}(j_{\lambda} - j_{\lambda}^{(u)})(j_{\lambda} - j_{\lambda}^{(u)}) \Phi_{1}^{(u)} j_{l}.
\]

The Gaussian peaks the sums around the background values. We can therefore expand the summand around these values. The first-order term of the expansion of the Regge action around these values cancels the phases in the state, leaving

\[
G_{a}(L', T) = \frac{4 \lambda^2 N_{8}}{J_{T}} \sum_{\delta j_{h}} \delta j_{12}^{a} \delta j_{13}^{a} e^{-i \delta j_{\alpha} J_{A^{a}}^{\prime}},
\]

where we have introduced the matrix

\[
A_{ll}^{\prime} = 2 \delta A_{ll}^{\prime} + i G_{ll}^{\prime} = \sqrt{J_{T}}(J_{T}^{(0)} J_{T}^{(0)} (2 \alpha + i \Gamma))_{ll}^{\prime} = \sqrt{J_{T}}(J_{T}^{(0)} J_{T}^{(0)} A_{ll}^{\prime})
\]

and the vector \( \delta j = (\delta j_{a}, \delta j_{d}, \delta j_{s}) \). Approximating the sum with Gaussian integrals gives

\[
G_{a}(L', T) = \frac{16 \pi}{J_{T}} (A)_{J_{T}}^{-1} \left( J_{T}^{(0)} J_{T}^{(0)} A_{ll}^{\prime} \right)_{J_{T}}^{(0)} (A)_{J_{T}}^{-1} J_{T}^{(0)}
\]

which is proportional to \( 1/j_{T} \), as in the first-order calculation [2]. Thus, we recover the expected \( 1/j_{T} \) behavior of the linearized theory.
3.2. Two internal propagators: $3 \rightarrow 2 \rightarrow 3$ Pachner's move

Consider the Feynman diagram.

This will appear in the amplitude of an observable $f_{s_6}$ defined by a spin network with graph $\Gamma_6$, illustrated in figure 6, consisting of two triangular spin networks connected by six links: three 'up' nodes $u_n$, and three 'down' nodes $d_n$, $n = 1, 2, 3$.

It is convenient to denote the links of this spin network as follows. Call $l_{u_{nm}}^u$ (resp. $l_{d_{nm}}^u$) with $n \neq m$ and $n, m = 1, 2, 3$ the three upper (resp. lower) links, and denote $l_{s_{nv}}^u$ with $v = 4, 5$ the two links joining $u_n$ and $d_n$. Denote by $j_{u_{nm}}$, $j_{d_{nm}}$, $j_{s_{nv}}$ the representations associated with the 12 links $l_{u_{nm}}^u$, $l_{d_{nm}}^u$, $l_{s_{nv}}^u$ of $\Gamma_6$, and $i_u$, $i_d$ the six intertwiners associated with the six nodes $n_u$ and $n_d$. The set $s_6 = (\Gamma_6, j_{u_{nm}}^u, j_{d_{nm}}^u, j_{s_{nv}}^u, i_u, i_d)$ is the boundary spin network we consider in this section.

The boundary function $f_{s_6}(\phi)$ for this spin network is a monomial of order 6 in the field:

$$f_{s_6}(\phi) = \sum_{[\alpha]} \prod_{n=1,2,3} \rightphi^{\alpha_{n_m} \alpha_{n_i}}_{\beta_{n_m} \beta_{n_i}} \rightphi^{\beta_{n_m} \beta_{n_i}}_{\alpha_{n_m} \alpha_{n_i}}$$

(35)

where $n \neq m = 1, \ldots, 5$. At order $\lambda^2$, the corresponding amplitude

$$W[s_6] = \frac{\lambda^2}{2(5!)} \int D\phi f_{s_6}(\phi) \left( \int \phi^5 \right)^2 e^{-\int \phi^2}$$

(36)

gives two vertices and eight propagators:

$$W[s_6] = \prod_{n=1,2,3} \mathcal{P}_{\Gamma_6}^{i_{u_{nm}} i_{d_{nm}} i_{s_{nv}}} \mathcal{P}_{\beta_{n_m} \beta_{n_i} \alpha_{n_m} \alpha_{n_i}} \times \prod_{n=1,2,3} \mathcal{P}_{\Gamma_6}^{i_{u_{nm}} i_{d_{nm}} i_{s_{nv}}} \mathcal{P}_{\beta_{n_m} \beta_{n_i} \alpha_{n_m} \alpha_{n_i}} \left( \prod_{n=1,2,3} \mathcal{P}_{\Gamma_6}^{i_{u_{nm}} i_{d_{nm}} i_{s_{nv}}} \mathcal{P}_{\beta_{n_m} \beta_{n_i} \alpha_{n_m} \alpha_{n_i}} \right).$$

(37)
As before, the two-complex with the highest number of faces is the one with \( \mathcal{P} \) replaced by \( P \). This is dual to the (generalized) triangulation \( \Delta_6 \), obtained by gluing two 4-simplices via two tetrahedra. This is schematically indicated in figure 7.

The triangulation \( \Delta_6 \) is formed by 5 points, which we label as 1, 2, 3, 4, 5; the 11 edges \((n, m), (n, 4), (n, 5), (4, 5)_u, (4, 5)_d \) (here \( n = 1, 2, 3 \); note that there are two distinct edges connecting the points 4 and 5); the 13 faces \((1, 2, 3, 4), (1, 2, 3, 5), (n, m, 4, 5)_u, (n, m, 4, 5)_d \); the 8 tetrahedra \((1, 2, 3, 4), (1, 2, 3, 5), (n, m, 4, 5)_u, (n, m, 4, 5)_d \) and the two 4-simplices \((1, 2, 3, 4, 5)_u, (1, 2, 3, 4, 5)_d \).

We use also the notation \( p_4 \equiv (1, 2, 3, 4) \) and \( p_5 \equiv (1, 2, 3, 5) \) for these two tetrahedra, \( T_2 = p_4 \cup p_5 \) their union, and \( \tau = (1, 2, 3) \) the triangle that separates them. The triangulation can be interpreted as representing the world history of the line \((4, 5)_d \) opening up to the volume \( T_2 \), and then recollapsing to the line \((4, 5)_u \). The initial and final lines join at both ends. See figures 8 and 9.

We use also the notation \( t_{nm4} \equiv (n, m, 4), t_{nm5} \equiv (n, m, 5) \), and the notation \( t_1^{u,d} \equiv (2, 3, 4, 5)_{u,d}, t_2^{u,d} \equiv (3, 1, 4, 5)_{u,d}, t_3^{u,d} \equiv (1, 2, 4, 5)_{u,d} \) (cyclically in 1, 2, 3). The tetrahedra in the triangulation \( \Delta_6 \) and their relations are represented in figure 10.

For this term, we have

\[
W[s_6] = \sum_j \dim(j) \prod_{n < m} \dim(j_{nm}^u) \dim(j_{nm}^d) \dim(j_{nm}^s) B(j, j_{nm}^u, j_{nm}^d) B(j, j_{nm}^d, j_{nm}^u).
\]

(38)

Note the sum over the spin \( j \) of the internal face. This sum is finite, because it is controlled by the Clebsch–Gordan relations between spins at the edges.

The vacuum boundary state \( \Psi_0[s_6] \) for spin network \( s_6 \) will be a function \( \Psi_0[s_6] \) peaked on background values \((j_{nm}^{u,d}, j_{nm}^s)\) which represent a given background geometry \( q_6 \). Note that we cannot imbed two flat non-degenerate 4-simplices glued along two faces into \( \mathbb{R}^4 \) (for the same reason for which two non-degenerate triangles in \( \mathbb{R}^2 \) cannot be glued along two sides).
Figure 8. (a) The spacetime triangulation \( \Delta_6 \). The points 5u and 5d must be identified, and so the points 4u and 4d. The triangle 2 − 3 − 5u must be identified with the triangle 2 − 3 − 5d, and so on. The line 4u − 5u must not be identified with the line 4d − 5d. (b) The 3D analog of \( \Delta_4 \); here as well the two ends of the upper horizontal line must join the two ends of the lower horizontal line, and the two side triangles of the upper tetrahedron must be identified with the two side triangles of the lower tetrahedron.

Figure 9. The labeling of the vertices of the central tetrahedra (a) and the lateral tetrahedra (b).

Thus, we cannot fix the geometry \( q_6 \) as we did in the previous case. Instead, let us proceed as follows.

Let \( v_u \) and \( v_d \) be two regular 4-simplices, of side \( L \). Identify two tetrahedra \( (p_4 \) and \( p_5) \) of these two 4-simplices. This defines a conical space that is flat except on the triangle \( \tau \) that separates \( p_4 \) and \( p_5 \), where the deficit angle is \( 2\pi \) minus twice the dihedral angle of a regular 4-simplex. This space has a fixed boundary geometry, which we take as the definition of \( q_6 \).

We take the boundary state to be a Gaussian peaked on \( q_6 \):

\[
\Psi_q[q_6] = C_6 \exp[-\alpha_\ell(x-y)^2 + \phi_\ell^2 y].
\]

(39)

Following the same steps as above, we now find that the Regge action is also a function of the summation variable \( j_{45} \), which represents the area of the internal triangle \( \tau \):

\[
S_{\text{Regge}}(j_{nm,j_{45}}, j_{n,v}, j_{n,v}, j_{45}) = \tilde{\phi}_{nm,j_{45}}^n + \tilde{\phi}_{n,v,j_{45}}^n + \tilde{\phi}_{nm,j_{45}}^n + \tilde{\phi}_{n,v,j_{45}}^n + \frac{1}{2} G_\ell j_{45} j_{45} + \frac{1}{2} G_\ell j_{45} j_{45} + \left( \tilde{\phi}_{n,v,j_{45}}^n + \tilde{\phi}_{n,v,j_{45}}^n \right) j_{45} + \frac{1}{2} G_\ell j_{45} j_{45},
\]

(40)
where \( \tilde{\phi}_{n}^{(6)} \) and \( \tilde{\phi}_{nm}^{(6)} \) are the dihedral angles of flat 4-simplices with the given boundary geometry and are supposed to be a function of the reference background value \( j_{45}^{(6)} \), such as the ‘discrete derivative’ and the fluctuation \( \delta j_{45} = j_{45} - j_{45}^{(6)} \). Since we have an internal loop we sum over \( \delta j_{45} \):

\[
G_{n}(L', T) = \frac{4A^{2}N}{f_{T}L} \sum_{\delta j_{45}} \sum_{\delta j_{45}} \delta j_{45}^{u} \delta j_{45}^{d} P_{1}^{2} e^{-i(S_{\text{Regge}} + K_{T} \frac{\pi}{2})} e^{i \tilde{\phi}_{u}^{(6)}(\delta j_{45})(\delta j_{45}) + i \tilde{\phi}_{d}^{(6)}(\delta j_{45})} \tag{41}
\]

where \( N \) is fixed by the normalization. The first-order term of the expansion of the Regge action cancels the phases in the boundary state but gives an extra phase \( i (\tilde{\phi}_{u}^{(6)} + \tilde{\phi}_{d}^{(6)}) \delta j_{45} \) and two discrete derivative terms \( G_{(45)u} \) and \( G_{(45)(45)} \) related to the internal loop, leaving

\[
G_{n}(L', T) = N \sum_{\delta j_{45}} \delta j_{45}^{u} \delta j_{45}^{d} \exp \left( -\tilde{\phi}_{u}^{(6)} \delta j_{45}^{u} \delta j_{45}^{d} - \frac{1}{2} G_{u} \delta j_{45}^{u} \delta j_{45}^{d} + i \left( \tilde{\phi}_{u}^{(6)} + \tilde{\phi}_{d}^{(6)} \right) \delta j_{45}^{u} + G_{(45)u} \delta j_{45}^{u} \delta j_{45}^{d} + \frac{1}{2} G_{(45)(45)} \delta j_{45}^{u} \delta j_{45}^{d} \right) \tag{42}
\]

The dihedral angle \( \tilde{\phi}_{45}^{(6)u} \) are the angles between the normals of the two internal tetrahedra \( (p_{1}, p_{2}) \) in the two 4-simplices. Their sum is the deficit angle at the triangle \( T_{2} \). As mentioned, this deficit angle cannot be zero (or a multiple of \( 2\pi \)), because there is no imbedding of two non-degenerate 4-simplices glued by two tetrahedra into \( R^{4} \). Therefore,

\[
\tilde{\phi}_{45}^{(6)u} + \tilde{\phi}_{45}^{(6)d} \neq 0. \tag{43}
\]

But it follows from this that the sum over \( \delta j_{45} \) is a sum of a rapidly oscillating function, and is therefore suppressed. Thus, this term is strongly suppressed in the large \( j \) limit. Note that the
denominator might also be suppressed at this order in $\lambda$, but is not going to be suppressed at all orders in $\lambda$; therefore, the suppression is effective.

3.3. Three internal propagators: $2 \to 3 \to 2$ Pachner’s move

Consider the Feynman graph

![Feynman Diagram]

The boundary graph $\Gamma_4$ is illustrated in figure 11: two theta spin networks connected by two links.

Denote the nodes of the first link as $u_1$ and $u_2$ and the nodes of the second one as $d_1$ and $d_2$. The generalized triangulation $\Delta_4$ that gives the maximal contribution is obtained by gluing two 4-simplices via three tetrahedra. See figure 13.

The triangulation $\Delta_4$ is formed by 5 points, which we label as 1, 2, 3, 4, 5; the 10 edges $(n, m), (n, 1), (n, 2), (1, 2)$ (here $n = 3, 4, 5$); the 11 faces $(n, m, 1), (n, m, 2), (n, 1, 2), (3, 4, 5)_{u}, (3, 4, 5)_{d}$ (note that there are two distinct triangles connecting the points 3, 4 and 5); the 7 tetrahedra $(n, m, 1, 2), (3, 4, 5, 1)_{u}, (3, 4, 5, 2)_{u}, (3, 4, 5, 1)_{d}, (3, 4, 5, 2)_{d}$; and the two 4-simplices $(1, 2, 3, 4, 5)_{u}, (1, 2, 3, 4, 5)_{d}$. See figure 12.
The labels $u$ and $d$ refer to 345.

Figure 13. Tetrahedral decomposition: two 4-simplices share three tetrahedra $p_1, p_2$ and $p_3$.

We use the notation $p_3 = (4, 5, 1, 2)$, $p_4 = (5, 3, 1, 2)$, $p_5 = (3, 4, 1, 2)$ and we call $T_3 = p_3 \cup p_4 \cup p_5$ the union of the three central tetrahedra. The triangulation $\Delta_4$ can be seen as the world history of the triangle $(3, 4, 5)_u$ evolving into $T_3$ and then collapsing into the triangle $(3, 4, 5)_u$. The initial and final triangles share their perimeters, and in particular their vertices.

The representations associated with the eight links $l_u^{12}, l_d^{12}, l_{sv}$ of $\Gamma_{14}$ are $j_u^{12}, j_d^{12}, j_{sv}$ while $i_u^{12}, i_d^{12}$ are the four intertwiners associated with the four nodes $u_n$ and $d_n$ ($n = 1, 2$). The set $s_4 = (\Gamma_{14}, j_u^{12}, j_d^{12}, j_{sv}, i_u^{12}, i_d^{12})$ is the boundary spin network we consider in this section. The boundary function $f_{s_4}(\phi)$ is of order 4:

$$f_{s_4}(\phi) = \sum_{a_{uu}, a_{dd}} \prod_{n=1,2} \Phi_{j_{uu}}^{i_{uu}} \Phi_{j_{dd}}^{i_{dd}}$$

and the expansion gives two vertices and seven propagators:

$$W[s_4] = \left( \prod_{n=1,2} \mathcal{P}^{j_{uu}^{a_{uu}}, j_{dd}^{a_{dd}}} \sigma_{uu} \tau_{dd} \right) \mathcal{V}^{j_{uu}^{a_{uu}}, j_{dd}^{a_{dd}}, j_{sv}^{a_{sv}}} \mathcal{P}^{j_{sv}^{a_{sv}}, j_{uu}^{a_{uu}}, j_{dd}^{a_{dd}}} \mathcal{P}^{j_{uu}^{a_{uu}}, j_{dd}^{a_{dd}}, j_{sv}^{a_{sv}}} \mathcal{P}^{j_{sv}^{a_{sv}}, j_{uu}^{a_{uu}}, j_{dd}^{a_{dd}}} \sum_{j_{12}} \left( \prod_{n=1,2} \mathcal{P}^{j_{uu}^{a_{uu}}, j_{dd}^{a_{dd}}} \sigma_{uu} \tau_{dd} \right) \mathcal{B}(j_{12}^{a_{12}}) \mathcal{B}(j_{12}^{a_{12}})$$

where the $I$ label of $j_{12}$ refers to the three internal faces shared by the tetrahedra, along which they are glued together (loops in the GFT diagram).
We now chose a boundary geometry with non-degenerate areas and dihedral angles, defining non-degenerate 4-simplices. Let \( j_i = (j_{12}^i, j_{13}^i, j_{14}^i, j_{15}^i, j_{23}^i, j_{24}^i, j_{25}^i, j_{34}^i, j_{35}^i, j_{45}^i) \) be the background values on which the boundary function \( \Psi_q[x] \) is peaked:

\[
\Psi_q[x] = C_4 \, e^{-\langle a_{\mu \nu} \rangle j_{\mu \nu} + b_{\mu \nu} (j_{\mu \nu} - j_{\mu \nu}^0) + c \delta j_{\mu \nu}^0} \tag{46}
\]

and \( \langle j_{\mu \nu} \rangle, \langle j_{\mu \nu}^0 \rangle \) the areas of the internal faces determined by the boundary geometry. The Regge action, as before, will be written as an expansion over the boundary \( j_I \) but also over the internal background reference \( j_{34}^i, j_{45}^i \) and \( j_{45}^{(4)} \):

\[
S_{\text{Regge}}(j_{34}^u, j_{45}^{(4)}, j_{35}^v, j_{45}^w, j_{11}) = \phi_{nm}^u j_{nm} + \phi_{n}^u j_{n}^v + b \phi_{nm}^u j_{nm} + b_{n}^u j_{n}^v + \frac{1}{2} G_{ll}^u \delta j_{l}^u
\]

\[
+ i(\tilde{\phi}_{ij}^u + \tilde{\phi}_{ij}^{(4)}) j_{ij} + G_{ij}^{(4)} \delta j_{ij} + \frac{1}{2} G_{ij},(KL) \delta j_{ij} \delta j_{KL} \tag{47}
\]

where \( I, J, K, L = 3, 4, 5 \), \( I < J, K < L \), \( \tilde{\phi}_{ij}^u \) and \( \tilde{\phi}_{nm}^{(4)} \) are the dihedral angles.

The phases in the external angles cancel the phases in the boundary state. The phases in the internal angles are

\[
(\phi_{34}^u + \phi_{34}^{(4)}) j_{34}^u + (\phi_{35}^u + \phi_{35}^{(4)}) j_{35}^u + (\phi_{45}^u + \phi_{45}^{(4)}) j_{45}^u = 0. \tag{48}
\]

The sum over the three independent variables \( j_{34}, j_{45}, j_{35} \) again suppresses the amplitude, because the deficit angles in the parenthesis cannot vanish, for the same reason as in the previous section.

However, this result raises a problem. We expect contributions in a Feynman sum to be suppressed in the semiclassical approximation if there is no classical trajectory that gives a saddle point in the sum. Here the classical trajectories should reproduce the Einstein equations, and these demand the Ricci tensor to vanish, and not the Riemann tensor. But the vanishing of all deficit angles above corresponds to flatness, namely to the vanishing of the Riemann tensor. Why do not (non-flat) Ricci-flat configurations contribute in the semiclassical limit?

The origin of the problem can be traced to the oversimplification of the dynamics which characterizes the old Barrett–Crane vertex. In this model, the spins are the sole dynamical variables. In the semiclassical limit, they correspond to areas of triangles of Regge-like triangulations. The vertex approximates correctly the Regge action, but this is not sufficient to reproduce the Regge-calculus dynamics, because the variables are areas instead of lengths. On this, see [36]. Let us see how this problem reflects here.

We are considering a (generalized) triangulation obtained by gluing two 4-simplices by three tetrahedra. The segments of this triangulation are 10, because all segments are shared by the two 4-simplices. But the areas are 11, because only 9 triangles are shared, and each 4-simplex has a triangle which is not shared with the other 4-simplex. Therefore, there should be one relation among the areas to be satisfied if the areas have to define a geometrical (Regge-like) generalized triangulation. Of these 11 areas, 8 are boundary areas, and 3 are internal.

If we fix the eight external areas, there should be a relation between the three internal areas. Suppose we linearize this relation

\[
a \delta j_{34} + b \delta j_{45} + c \delta j_{35} = 0 \tag{49}
\]

and impose this in the integral of \( \delta j_{nm} \). Then the integral is no longer suppressed, provided that the deficit angle angles satisfy relations such as

\[
(\phi_{34}^u + \phi_{34}^{(4)}) = a \phi, \quad (\phi_{45}^u + \phi_{45}^{(4)}) = b \phi, \quad (\phi_{35}^u + \phi_{35}^{(4)}) = c \phi. \tag{50}
\]

That is, the amplitude may fail to be suppressed even if the triangulation is not flat. It is reasonable to expect that the above condition reflects Ricci flatness.

In the model we are considering, a relation between the fluctuations of the spins does not seem to be implemented; this is perhaps one additional sign of the problems of the old Barrett–Crane model. Do the new models correct this problem?
3.4. Four internal propagators: \(1 \rightarrow 4 \rightarrow 1\) Pachner’s move

Finally, consider the Feynman graph

which looks like a self-energy correction for the GFT propagator. The potential divergences of this graph have been analyzed in [7] in the simple case of vanishing boundary areas.

The boundary graph \(\Gamma_2\) is dual to the spin network of figure 14: a tetrahedral spin network with two nodes \(u_1\) and \(d_1\). The links of \(\Gamma_2\) are the four side links \(l^n_s\), which connect \(u_1\) with \(d_1\).

The corresponding maximal four-dimensional triangulation \(\Delta_2\) is made by two 4-simplices glued by four tetrahedra.

The triangulation \(\Delta_2\) is formed by five points, which we label as 1, 2, 3, 4, 5; the ten edges \((n, m), (n, 5)\) (here \(n = 1, 2, 3, 4\)); the ten faces \((n, m, n), (n, m, 5)\); the six tetrahedra \((n, m, p, 5), (1, 2, 3, 4)_u, (1, 2, 3, 4)_d\) (note that there are two distinct tetrahedra connecting the points 1, 2, 3 and 4); and the two 4-simplices \((1, 2, 3, 4, 5)_u, (1, 2, 3, 4, 5)_d\). See figure 15.

This can be seen as the world history of the tetrahedron \(d \equiv (1, 2, 3, 4)_d\) evolving into a set of four tetrahedra, having the same 3D boundary as the original one, and then evolving back to a single tetrahedron \(u \equiv (1, 2, 3, 4)_u\).

The boundary of \(\Delta_2\) is made by two tetrahedra (figure 3.4): ‘up’ tetrahedron \(u_1\) and ‘down’ \(d_1\). They share all their four faces, that is, the triangles \(t_{234}, t_{235}, t_{245}, t_{345}\). See figures 16–18.
Figure 15. The spacetime triangulation $\Delta_2$ and the lower dimensional analogies. Upper and lower lateral side must be identified.

Figure 16. The labeling of the vertices of the triangulation $\Delta_2$.

Figure 17. The tetrahedral decomposition of the $1 \rightarrow 4 \rightarrow 1$ diagram.

Denote by $j^v_i$ ($v = 1, 4$) the spins associated with the four links $l^v_i$ of $\Gamma_2$, and $i^u_i, i^d_i$ the two intertwiners associated with the two nodes $u_1$ and $d_1$. The set $s_2 = (\Gamma_2, j^v_i, i^u_i, i^d_i)$ with $v = 1, 2, 3, 4$ is the boundary spin network. The boundary function $f_{s_2}(\phi)$ is of order 2:

$$f_{s_2}(\phi) = \sum_{a_{uv} b_{uv}} \phi^{a_{uv} i_{uv}} \phi^{b_{uv} i_{uv}}.$$  (51)
The Wick expansion of the highest dimensional contribution gives two vertices and six propagators

\[ W[s_2] = \psi_{(1)}^{f_{10}} \phi_{(2)}^{l_{10}} \phi_{(3)}^{l_{10}} \phi_{(4)}^{l_{10}} \phi_{(5)}^{l_{10}} \phi_{(6)}^{l_{10}} \times \psi_{(7)}^{l_{10}} \phi_{(8)}^{l_{10}} \phi_{(9)}^{l_{10}} \phi_{(10)}^{l_{10}} \phi_{(11)}^{l_{10}} \phi_{(12)}^{l_{10}} \]

\[ = \left( \dim i^d \dim j^d \prod_{i=1}^{4} \dim(j_{1i}) \right) \sum_{j_{1k}} \left( \prod_{K=2,3,4,5; K < L} \dim(j_{KL}) B(j_{1m}) B(j_{mn}) \right). \]

(52)

Let us choose again the boundary geometry that defines the boundary state \( \Psi[s_2] \) as the one obtained by gluing regular simplices:

\[ \Psi[s_2] = C_2 e^{-\frac{(\phi_{(2)}^{l_{10}}(j_{10} - j_{20}))(j_{10} - j_{30}))(j_{10} - j_{40})}{2(\Phi_1)}}. \]

(53)

Note that something new happens with this amplitude, which did not happen in the previous cases: in general, the amplitude is suppressed unless the first-order expansion of the Regge action matches the phases of the boundary state. This gives a certain number of conditions on the internal spins (since these determine the dihedral angle appearing in the Regge action). Now, in the previous cases these conditions were sufficient to determine the value around which to expand the internal spins uniquely. But this is no longer true in the present case. Indeed, there are four external triangles, and therefore four conditions for the cancellation of the four phases \( \Phi_2 \) in (53), while there are six internal faces. Therefore, we can expect a two-parameter set of internal spins, whose contribution to the amplitude is not suppressed by boundary-phase cancellation.

On the other hand, there are six internal deficit angles that appear in the expansion of the Regge action around the boundary \( j_i \) and the internal references \( j_{ij}^{(2)} \), with \( I < J \) and \( I, J = 2, 3, 4, 5 \), where now \( j_i = (j_{iv}) \), with \( v = 1, 4 \) and \( I, J, K, L = 2, 3, 4, 5, I < J, K < L \)

\[ S_{\text{Regge}}(j_i^{(2)}, j_{ij}) = \phi_n^{(2)} j_{1v} + \frac{1}{2} G_{ll}^{ij} \delta j_i \delta j_i + \phi_n^{(2)} j_{1v} + \phi_{IJ}^{(2)} + \phi_{IJ}^{(2)d} j_{ij} + G_{(12)}^{ij} \delta j_{1i} \delta j_i + \frac{1}{2} G_{(12),(KL)} \delta j_{1i} \delta j_{1j}, \]

(54)

and they give unmatched phases

\[ \sum_{I,J=2,3,4,5; I < J} \left( \phi_{IJ}^{(2)} + \phi_{IJ}^{(2)d} \right) j_{IJ}^2. \]

(55)

Thus, it appears that this term is suppressed as well. However, we expect this term to have a divergence, due to the presence of a bubble; does the divergence show in this large \( j \) limit?
3.5. Subleading corrections

Remarkably, none of the $\lambda^2$ terms considered here appear to give just a second-order correction in $1/j$ to the propagator. The first case we have considered, namely the $4 \rightarrow 1 \rightarrow 4$ case, which is not suppressed, gives a contribution at the leading $1/j$ order. (This would happen for the other terms as well if we disregard the exponential suppression factor.) Note that in this case there is no first-order term for a boundary state that has only support on the graph $\Gamma_8$.

Of course terms of higher orders in $1/j$ abound as contributions to the amplitude. First, we have expanded the vertex amplitude in $1/j$: all the vertex subleading terms will contribute to the amplitude. More interestingly, recall that the propagator of the GFT includes the sum over permutations (13). We have systematically considered only the term with the highest power in $j$ of this sum. The other terms give lower powers in $j$.

Note in fact that the same sum over permutations appears in the normalization of the amplitude which is fixed by (11). The dominant term of the normalization will always be the one with highest power in $j$, and therefore the other terms in the amplitude will contribute as increasing powers in $1/j$. It is tempting to speculate that these terms are probably to be interpreted as related to the corrections of the Newton potential, in agreement with standard corrections obtained with quantum-field-theoretical techniques discussed in the literature (see [38, 39])

\[ V(r) = a_1 \frac{Gm_1 m_2}{r} \left( 1 + a_2 \frac{G(m_1 + m_2)}{c^2} \frac{1}{r} + a_3 \left( \frac{Gh}{c^3} \right)^2 \frac{1}{r^2} \right) + \cdots \]  

(56)

with $a_1$, $a_2$ and $a_3$ numerical coefficients. The actual calculation of these coefficients using the techniques developed here, however, requires more work, of which the one presented here is only a first step.

4. Conclusions

We have explored some second-order contributions to the loop quantum gravity scattering amplitudes. Our results are preliminary, and a more extensive study of these terms is needed. Some general considerations appear nevertheless to be possible, and some interesting phenomena have appeared.

(i) Second-order terms do not appear to spoil the correct large distance behavior of the two-point function.

(ii) The dominant contribution for the terms considered here appears to come from the two-complex with the maximal number of faces, which minimizes the complexity of the topology of its dual triangulation. In other words, triangulations with very funny topologies appear to contribute less, at large distance.

(iii) Terms of higher order in $\lambda$ contribute to the dominant $1/j$ term of the propagator, a result perhaps unexpected, but that was already pointed out in [3]. (This is the result of the $1 \rightarrow 4 \rightarrow 1$ case.)

(iv) The amplitude is suppressed unless the triangulation admits configurations where internal deficit angles appropriately vanish. Thus, triangulations that admit only geometries that cannot solve the (discretized) Einstein equations do not contribute in the large $j$ limit. (This is the result of the $2 \rightarrow 3 \rightarrow 2$ case.)

(v) Apparently, however, only triangulations that admit a flat geometry seem not to be suppressed. This appears to be a problem, since it is Ricci flatness that seems to be the physically reasonable requirement. This might be a problem of the model that we
have used, and we think that it should be clarified in the context of the new models [20, 22, 23, 25]. (This is the result of the 3 → 2 → 3 case.)

(vi) When the bulk of the triangulation is sufficiently more complex than the boundary, the internal spins are not fixed by the boundary geometry. The consequences of this case on the semiclassical limit are not yet clear to us. (This is the result of the 4 → 1 → 4 case.)

(vii) Finally, it is not clear to us what happens in the large \( j \) limit to the bubble divergence which is expected in the 4 → 1 → 4 case.

Many other aspects of the problem remain unclear. Among the most important are how to work with general boundary states with components on different graphs, and to understand which one is the physical regime where the expansion in powers of \( \lambda \) is viable. We think that continuing a concrete systematical exploration of the amplitudes may be a useful path for addressing these questions.

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