A new class of entanglement measures

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Abstract We introduce new entanglement measures on the set of density operators on tensor product Hilbert spaces. These measures are based on the greatest cross norm on the tensor product of the sets of trace class operators on Hilbert space. We show that they satisfy the basic requirements on entanglement measures discussed in the literature, including convexity, invariance under local unitary operations and non-increase under local quantum operations and classical communication.

I Introduction

This paper is devoted to the study of entanglement of quantum states, which is one of the most decisively non-classical features in quantum theory. The question of quantifying entanglement in the case of mixed quantum states represented by density operators on finite dimensional Hilbert spaces has recently been studied extensively in the context of quantum information theory, see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein.

An entanglement measure is a real-valued function defined on the set of density operators on some tensor product Hilbert space subject to further physically motivated conditions, see, e.g., [3, 4, 5, 6, 7] and below. A number of entanglement measures have been discussed in the literature, such as the von Neumann reduced entropy, the relative entropy of entanglement [4], the entanglement of distillation and the entanglement of formation [2]. Several authors proposed physically motivated postulates to characterize entanglement measures, see below. These postulates (although they vary from author to author in the details) have in common that they are based on the concepts of the operational formulation of quantum mechanics [11]. We shall discuss one version of these operational characterizations of entanglement measures in Section IV.

In this paper we introduce new entanglement measures based on the greatest cross norm on the tensor product of the sets of trace class operators on Hilbert space (see Sections V and VI). We shall show that the measures introduced in this work satisfy all the basic requirements for entanglement measures. These include convexity, invariance under local unitary transformations, and non-increase under procedures composed of local quantum operations and classical communication.

Throughout this paper the set of trace class operators on some Hilbert space $\mathcal{H}$ is denoted by $\mathcal{T}(\mathcal{H})$ and the set of bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. A density operator is a positive trace class operator with trace one.
II Preliminaries

In this section we collect some basic definitions and results which are used in the course of this paper.

In the present paper we restrict ourselves mainly to the situation of a composite quantum system consisting of two subsystems with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the Hilbert spaces of the subsystems (except in Section VI). The states of the system are identified with the density operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

**Definition 1** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces of arbitrary dimension. A density operator $\rho$ on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called separable or disentangled if there exist a family $\{\omega_i\}$ of positive real numbers, a family $\{\rho_i^{(1)}\}$ of density operators on $\mathcal{H}_1$ and a family $\{\rho_i^{(2)}\}$ of density operators on $\mathcal{H}_2$ such that

$$\rho = \sum_i \omega_i \rho_i^{(1)} \otimes \rho_i^{(2)},$$

where the sum converges in trace class norm.

The set of states is a convex set and its extreme points, which are also called pure states, are the projection operators. Every pure state obviously correspond to a unit vector $\psi$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$. We denote the projection operator onto the subspace spanned by the unit vector $\psi$ by $P_\psi$.

The Schmidt decomposition is of central importance in the characterization and quantification of entanglement associated with pure states.

**Lemma 2** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces of arbitrary dimension and let $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then there exist a family of non-negative real numbers $\{p_i\}_i$ and orthonormal bases $\{a_i\}_i$ and $\{b_i\}_i$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively such that

$$\psi = \sum_i \sqrt{p_i} a_i \otimes b_i.$$ 

The family of positive numbers $\{p_i\}_i$ is called the family of Schmidt coefficients of $\psi$. For pure states the family of Schmidt coefficients of a state completely characterizes the amount of entanglement of that state. A pure state $\psi$ is separable if and only if $\psi = a \otimes b$ for some $a \in \mathcal{H}_1$ and $b \in \mathcal{H}_2$.

The von Neumann reduced entropy for density operators $\sigma$ on a tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as

$$S_{\text{vN}}(\sigma) := -\text{Tr}_{\mathcal{H}_1} (\text{Tr}_{\mathcal{H}_2} \sigma \ln(\text{Tr}_{\mathcal{H}_2} \sigma)), \quad (2)$$

where $\text{Tr}_{\mathcal{H}_1}$ and $\text{Tr}_{\mathcal{H}_2}$ denote the partial traces over $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. In the case of pure states $\sigma = P_\psi$, it can be shown that $-\text{Tr}_{\mathcal{H}_1} (\text{Tr}_{\mathcal{H}_2} P_\psi \ln(\text{Tr}_{\mathcal{H}_2} P_\psi)) = -\sum_i p_i \ln p_i$ where $\{p_i\}_i$ denotes the family of Schmidt coefficients of $\psi$. However, for a general mixed state $\sigma$ we have $\text{Tr}_{\mathcal{H}_1} (\text{Tr}_{\mathcal{H}_2} \sigma \ln(\text{Tr}_{\mathcal{H}_2} \sigma)) \neq \text{Tr}_{\mathcal{H}_2} (\text{Tr}_{\mathcal{H}_1} \sigma \ln(\text{Tr}_{\mathcal{H}_1} \sigma))$. 

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III Effects and Operations

In this section we recall some of the fundamental concepts and definitions in the operational approach to quantum theory and in quantum measurement theory [11, 12, 13, 14, 15]. The quantum mechanical state of a quantum system is described by a density operator \( \rho \) on the system’s Hilbert space \( \mathcal{H} \), i.e., by a positive trace class operator with trace one. Let \( \mathcal{K} \) be another Hilbert space. An operation is a positive linear map \( T : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K}) \) such that \( T \) is trace non-increasing for positive trace class operators, i.e., \( 0 \leq \text{Tr}(T(\sigma)) \leq \text{Tr}(\sigma) \) for all positive \( \sigma \in \mathcal{T}(\mathcal{H}) \). Following [14] we adopt the point of view that allowed operations in a laboratory are (O1) adding an ancilla, (O2) tracing out part of the system, (O3) performing unitary operations, and (O4) performing possibly selective yes-no experiments. It can be shown (for a detailed proof see, e.g., [16]) that the class of operations \( T : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K}) \) composed out of operations of the form (O1)-(O4) coincides with the class of trace non-increasing completely positive operations, i.e., has the property that for all \( n \geq 0 \) the map \( T_n \) on \( \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^n) \) defined by \( T_n := T \otimes 1_n \), where \( 1_n \in \mathcal{B}(\mathbb{C}^n) \) denotes the unit matrix, is positive. For a further physical motivation of the requirement of complete positivity see, e.g., [11]. In the sequel it is always understood that all operations are completely positive. If \( \mathcal{H} \) and \( \mathcal{K} \) are both finite dimensional Hilbert spaces, then it follows from the Choi-Kraus representation theorem for operations [11, 13, 17] that for every operation \( T : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K}) \) there exists a family of bounded operators \( \{A_k : \mathcal{H} \to \mathcal{K}\}_k \) with \( \sum_k A_k A_k^\dagger \leq 1_{\mathcal{K}} \) such that \( T \) can be expressed as

\[
T(\sigma) = \sum_k A_k^\dagger \sigma A_k
\]  

(3)

for all \( \sigma \in \mathcal{T}(\mathcal{H}) \). If \( \mathcal{H} = \mathcal{K} \), the Choi-Kraus representation is also valid for infinite dimensional Hilbert spaces (all sums converge in trace class norm). The family \( \{A_k\}_k \) is not unique. However, the operator \( E := \sum_k A_k A_k^\dagger = T^*(1) \) is independent of the family \( \{A_k\}_k \) chosen and is called the effect corresponding to the operation \( T \) and its associated yes-no measurement \( T^* \) denotes the adjoint of \( T \), [11]). Generally, an operator \( E \) is called an effect operator if \( E \) is bounded and Hermitean and if \( 0 \leq E \leq 1 \). Effect operator valued measures are then the most general observables in the theory [14]. They are also called positive operator valued (POV) measures. A Lüders-von Neumann operation is an operation of the form \( T_L(\sigma) = \sum_k P_k \sigma P_k \) where \( \{P_k\}_k \) is a set of mutually orthogonal projection operators on \( \mathcal{H} \). Lüders-von Neumann operations are repeatable. In the case of a general operation, it is possible to view the terms in its Choi-Kraus representation as representing different possible measurement outcomes. In the terminology of operational quantum theory the individual terms in the Choi-Kraus representation [3] form a set of operations corresponding to coexistent effects, see [11, 14]: two effect operators \( E_1 \) and \( E_2 \) are called coexistent if there exist effect operators \( F, G, H \) with \( F + G + H \leq 1 \) such that \( E_1 = F + G \) and \( E_2 = F + H \) (in general \( F, G \) and \( H \) will not be unique however). Therefore in general two coexistent effect operators \( E_1 \) and \( E_2 \) do not correspond to mutually complementary measurement outcomes but instead may have some ‘overlap’ represented by the operator \( F \) even if \( E_1 + E_2 \leq 1 \). Coexistent effect operators need not commute.
IV Entanglement measures

An entanglement measure is a functional $E$ defined on the set of density operators on the Hilbert space of a composite quantum system measuring the degree of entanglement of every given density operator. Every measure of entanglement $E$ should satisfy the following requirements [2, 3, 4, 6, 7]

(E0) An entanglement measure is a positive real-valued functional $E$ which for any given two systems is well-defined on the set $D(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of density operators on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the Hilbert spaces of the two systems. Moreover, $E$ is expansible, i.e., whenever $\rho \in D(\mathcal{H}_1 \otimes \mathcal{H}_2) \subset D(H_1 \otimes H_2)$ with embeddings $\mathcal{H}_1 \hookrightarrow H_1$ and $\mathcal{H}_2 \hookrightarrow H_2$ of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ into larger Hilbert spaces $H_1$ and $H_2$ respectively, then $E|_{\mathcal{H}_1 \otimes \mathcal{H}_2}(\rho) = E|_{H_1 \otimes H_2}(\rho)$.

(E1) If $\sigma$ is separable, then $E(\sigma) = 0$.

(E2) Local unitary transformations leave $E$ invariant, i.e.,

$$E(\sigma) = E\left( (U_1 \otimes U_2)^\dagger \sigma (U_1^\dagger \otimes U_2^\dagger) \right)$$

for all unitary operators $U_1$ and $U_2$ acting on $\mathcal{H}_1$ or $\mathcal{H}_2$ respectively.

(E3) Entanglement cannot increase under procedures consisting of local operations on the two quantum systems and classical communication. If $T$ is an operation which is trace-preserving on positive operators and can be realized by means of local operations and classical communication, i.e., is composed out of local operations of the form (O1) - (O4) and classical communication, then

$$E(T(\sigma)) \leq E(\sigma)$$ (4-a)

for all $\sigma \in D(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It is clear that every procedure acting on an individual single quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2$ composed only of local operations and classical communication can formally be represented as a finite sequence of operations of the form $T_1 \otimes T_2$, where $T_1$ and $T_2$ are local operations on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The requirement that entanglement cannot increase under local operations and classical communication is thus equivalent to

$$E((T_1 \otimes T_2)(\sigma)) \leq E(\sigma),$$ (4-b)

for all $\sigma \in D(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Remark 3 Equation (4-b) stipulates that local operations cannot increase entanglement. In the quantum information literature most authors replace Equations (4-a) and (4-b) by the stronger requirement

$$\sum_i p_i E(\sigma_i) \leq E(\sigma).$$ (5)
Equation 5 stipulates that after the measurement the entanglement (as measured by $E$) averaged over the possible output states $\sigma_i$ is less than or equal to the original entanglement. Here $p_i$ denotes the probability that the final state $\sigma_i$ occurs. In the literature Equation 5 is normally taken as the formal expression for the paradigm that it is impossible to create or increase entanglement by performing procedures composed of local quantum operations and classical communication alone. A disadvantage of Equation (5) is that it makes sense only in measurement situations and that the ‘possible output states’ corresponding to a given operation $T$ are not uniquely defined. Mathematically this corresponds to the fact that the Choi-Kraus representation of an operation $T$ is in general not unique. The difference between Equations (4-a) and (5) is that Equation (5) stipulates that entanglement cannot increase on average under local operations and classical communications (for a detailed discussion of this point see [7]). In contrast Equation (4-a) says that entanglement cannot increase for any operation which acts on individual systems and is composed of local operations and classical communication. If one takes up the former (ensemble) point of view of Equation (5), then Equation (4-a) does no longer represent the most general condition because from the ensemble point of view the most general operations composed out of local operations and classical communications can contain correlations between terms of the Choi-Kraus representations of subsequent local operations. A precise definition can be found, e.g., in [16]. Some authors consider also other classes of local operations, most prominently the class of separable operations considered in [3].

(E4) Mixing of states does not increase entanglement, i.e., $E$ is convex

$$E(\lambda \sigma + (1-\lambda)\tau) \leq \lambda E(\sigma) + (1-\lambda)E(\tau)$$

for all $0 \leq \lambda \leq 1$ and all $\sigma, \tau \in D(H_1 \otimes H_2)$.

Apart from the requirements (E0) - (E4) on entanglement measures many authors add further requirements to the definition of entanglement measures but we are not going to discuss them in this paper. For a details, see [16]. In the sequel we exclude the trivial functional $E \equiv 0$ which also satisfies (E0) - (E4).

**Remark 4** Postulate (E2) is an immediate consequence of (E3).

**Remark 5** It has been argued in [8] that the entanglement of distillation $E_D$ introduced in [2] does vanish for certain non-separable states (so called bound entangled states). Therefore it has been pointed out in [8] that replacing (E1) by the stronger requirement that for every entanglement measure $E(\sigma) = 0$ if and only if $\sigma$ is separable might exclude interesting entanglement measures. For more information the reader is referred to the references.

**Example 6** Post selection of a subensemble means selecting a (non-normalized) output state of a quantum operation and normalizing its trace to 1. This procedure can lead to an increase in entanglement. This can be seen by considering a very simple example. Consider a composite quantum system composed of two 3-level quantum systems and the state

$$\rho_\varepsilon = (1-\varepsilon)|00\rangle\langle 00| + \frac{\varepsilon}{2}(|12\rangle - |21\rangle)(\langle 12| - \langle 21|).$$
For $\varepsilon$ small it is intuitively obvious that this state does not contain ‘much’ entanglement and every entanglement measure should reflect this. Indeed, consider for example the relative entropy of entanglement introduced in [3] defined by

$$E_S(\sigma) := \inf_\rho (\text{Tr}(\sigma \ln \sigma - \sigma \ln \rho))$$  \hspace{1cm} (6)

where the infimum runs over all separable states $\rho$ for which $\text{Tr}(\sigma \ln \rho)$ is well-defined and finite. Elementary estimates using the results of [3] show that

$$E_S(\rho_e) \leq \varepsilon \ln 2.$$

If we subject the system to an operation testing whether or not the system is in the state $|00\rangle$ and select after the measurement the subensemble corresponding to the negative outcome (system is not in the state $|00\rangle$), then clearly the final state after the operation and post selection is given by $\frac{1}{2}(|12\rangle - |21\rangle)(|12\rangle - |21\rangle)$. Notice that this operation can be achieved by local operations and classical communication. We find

$$E_S\left(\frac{1}{2}(|12\rangle - |21\rangle)(|12\rangle - |21\rangle)\right) = \ln 2 > E_S(\rho_e).$$

Similarly it can be shown that the entanglement measure $\| \cdot \|_\gamma$ to be introduced below may increase under post selection of subensembles. Therefore we see that we must not replace the operation in Equation (4-a) by some normalized non-linear operation $\rho \mapsto \frac{T_{(\rho)}}{\text{Tr}(T_{(\rho)})}$ corresponding to post selection of a subensemble.

V A new class of entanglement measures

Consider the situation that the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are both finite dimensional and consider the spaces $\mathcal{T}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2)$ of trace class operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Both spaces are Banach spaces when equipped with the trace class norm $\| \cdot \|_1^{(1)}$ or $\| \cdot \|_1^{(2)}$ respectively, see, e.g., Schatten [18]. In the sequel we shall drop the superscript and write $\| \cdot \|_1$ for both norms, slightly abusing the notation; it will always be clear from the context which norm is meant. The algebraic tensor product $\mathcal{T}(\mathcal{H}_1) \otimes_{\text{alg}} \mathcal{T}(\mathcal{H}_2)$ of $\mathcal{T}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2)$ is defined as the set of all finite linear combinations of elementary tensors $u \otimes v$, i.e., the set of all finite sums $\sum_{i=1}^n u_i \otimes v_i$ where $u_i \in \mathcal{T}(\mathcal{H}_1)$ and $v_i \in \mathcal{T}(\mathcal{H}_2)$ for all $i$.

It is well known that we can define a cross norm on $\mathcal{T}(\mathcal{H}_1) \otimes_{\text{alg}} \mathcal{T}(\mathcal{H}_2)$ by [19]

$$\| t \|_\gamma := \inf \left\{ \sum_{i=1}^n \| u_i \|_1 \| v_i \|_1 \mid t = \sum_{i=1}^n u_i \otimes v_i \right\},$$

(7)

where $t \in \mathcal{T}(\mathcal{H}_1) \otimes_{\text{alg}} \mathcal{T}(\mathcal{H}_2)$ and where the infimum runs over all finite decompositions of $t$ into elementary tensors. It is also well known that $\| \cdot \|_\gamma$ majorizes any subcross seminorm on $\mathcal{T}(\mathcal{H}_1) \otimes_{\text{alg}} \mathcal{T}(\mathcal{H}_2)$. We denote the completion of $\mathcal{T}(\mathcal{H}_1) \otimes_{\text{alg}} \mathcal{T}(\mathcal{H}_2)$ with respect to $\| \cdot \|_\gamma$ by $\mathcal{T}(\mathcal{H}_1) \otimes_\gamma \mathcal{T}(\mathcal{H}_2)$. $\mathcal{T}(\mathcal{H}_1) \otimes_\gamma \mathcal{T}(\mathcal{H}_2)$ is a Banach algebra [19].
As both $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, $\mathcal{T}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_2)$ and $\mathcal{B}(\mathcal{H}_1) \otimes \text{alg} \mathcal{B}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, see, e.g., [20], Example 11.1.6. In finite dimensions all Banach space norms on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, in particular the operator norm $\| \cdot \|$, the trace class norm $\| \cdot \|_1$, and the norm $\| \cdot \|_{\text{tr}}$, are equivalent, i.e., generate the same topology on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

For later reference we compute the value of $\| P_\psi \|_\gamma$ for one dimensional projection operators $P_\psi = |\psi\rangle \langle \psi|$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ in terms of the coefficients in the Schmidt representation of $|\psi\rangle$. In this section we make extensive use of the Dirac bra-ket notation.

**Proposition 7** Let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be a unit vector and $|\psi\rangle = \sum_i \sqrt{p_i} |\phi_i\rangle \otimes |\chi_i\rangle$ its Schmidt representation, where $\{|\phi_i\rangle\}_i$ and $\{|\chi_i\rangle\}_i$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and where $p_i \geq 0$ and $\sum_i p_i = 1$. Let $P_\psi$ denote the one dimensional projection operator onto the subspace spanned by $|\psi\rangle$. Then

$$\| P_\psi \|_\gamma = \sum_{ij} \sqrt{p_i} p_j = \left( \sum_i \sqrt{p_i} \right)^2.$$

**Proof**: Without loss of generality we assume that $\mathcal{H}_1 = \mathcal{H}_2$ which can always be achieved by possibly suitably enlarging one of the two Hilbert spaces. Further, we identify $\mathcal{H}_1 = \mathcal{H}_2$ with $\mathbb{C}^n$, where $n = \dim \mathcal{H}_1$, i.e., we fix an orthonormal basis in $\mathcal{H}_1$ which we identify with the canonical real basis in $\mathbb{C}^n$. With respect to this canonical real basis in $\mathbb{C}^n$ we can define complex conjugates of elements of $\mathcal{H}_1$ and the complex conjugate as well as the transpose of a linear operator acting on $\mathcal{H}_1$. From the Schmidt decomposition it follows that

$$P_\psi = |\psi\rangle \langle \psi| = \sum_{ij} \sqrt{p_i} p_j |\phi_i\rangle \langle \phi_j| \otimes |\chi_i\rangle \langle \chi_j|.$$

From the definition of $\| \cdot \|_\gamma$ it is thus obvious that $\| P_\psi \|_\gamma \leq \sum_{ij} \sqrt{p_i p_j}$. Now consider the Hilbert space $\mathcal{H}$ of Hilbert-Schmidt operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ equipped with the Hilbert-Schmidt inner product $\langle f \mid g \rangle = \text{Tr}(f^\dagger g)$. Equation (8) induces an operator $\mathcal{A}_\psi$ on $\mathcal{H}$ as follows. Every element $\zeta$ in $\mathcal{H}$ can be written $\zeta = \sum_k x_k \otimes y_k$ where $x_k$ and $y_k$ are trace class operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $\mathcal{A}_\psi$ is defined on $\zeta$ as $\mathcal{A}_\psi(\zeta) := \sum_{jk} \sqrt{p_i p_j} \langle \chi_j^* | x_k | \phi_i \rangle \langle \phi_j| \otimes \chi_k | y \rangle$ where $|\chi_j\rangle$ denotes the complex conjugate of the vector $|\chi_i\rangle$ with respect to the canonical real basis in $\mathbb{C}^n$. Proposition 11.1.8 in [20] implies that $\mathcal{A}_\psi(\zeta)$ is independent of the representation of $\zeta$. Consider a representation $P_\psi = \sum_{ij} u_i \otimes v_i$ of $P_\psi$ as sum over simple tensors. Denote the transpose of $v_i$ by $v_i^T$. Then the operator defined by

$$\mathcal{A}_\psi(\zeta) := \sum_{i,k=1}^r \text{Tr}(v_i^T x_k) u_i \otimes y_k$$

(9)

is equal to $\mathcal{A}_\psi$ (by virtue of Proposition 11.1.8 in [20]). We denote the trace on $\mathcal{T}(\mathcal{H})$ by $\tau(\cdot)$. The operator $\mathcal{A}_\psi$ is of trace class and the right hand side of Equation 8 is the so-called polar representation of $\mathcal{A}_\psi$ which implies $\tau(\mathcal{A}_\psi) = \sum_{ij} \sqrt{p_i p_j}$, see [18]. $\mathcal{A}_\psi$ admits also many other
 representations $\mathcal{A}_\psi = \sum_i f_i \otimes g_i$ with families of operators $\{f_i\}$ and $\{g_i\}$ acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. It is well known that

$$\tau(\mathcal{A}_\psi) = \inf \left\{ \sum_i \| f_i \|_2 \| g_i \|_2 \left| \mathcal{A}_\psi = \sum_i f_i \otimes g_i \right. \right\} \leq \| P_\psi \|_\gamma,$$

where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm and where the latter inequality follows from $\| z \|_2 \leq \| z \|_1$ and from the fact that each decomposition of $\mathcal{A}_\psi$ corresponds in an obvious one-to-one fashion to a decomposition of $P_\psi$. This proves the proposition. □

**Corollary 8** Let $\rho$ be a density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional Hilbert spaces. If $\rho = \sum_{i,j} a_{ij} \langle \phi_i | \otimes | \chi_j \rangle$, then $\| \rho \|_\gamma = \sum_{i,j} | a_{ij} |$.

An immediate corollary of Proposition 7 is that a pure state $| \psi \rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is separable if and only if $\| P_\psi \|_\gamma = 1$. In [1] it has been proven that more generally all separable density matrices can be characterized by $\| \cdot \|_\gamma$.

**Theorem 9** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional Hilbert spaces and $\rho$ be a density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\rho$ is separable if and only if $\| \rho \|_\gamma = 1$.

In [1] it has been tentatively suggested that $\| \cdot \|_\gamma$ can be considered as a quantitative measure of entanglement. In the present work we substantiate this claim by proving

**Proposition 10** The function

$$E_\gamma(\sigma) := \| \sigma \|_\gamma \log \| \sigma \|_\gamma$$

satisfies the criteria (E0) - (E4) for entanglement measures.

**Proof:** (E1) is an immediate consequence of Theorem 9 and (E0) and (E2) are clear. (E3): Let $T$ be an operation composed of local operations, and classical communication. As we have argued above every such $T$ can be realized as a sequence of operations of the form $T_1 \otimes T_2$ where $T_1$ and $T_2$ are local operations on system 1 and 2 respectively. We show that $\| (T_1 \otimes T_2)(\sigma) \|_\gamma \leq \| \sigma \|_\gamma$. By linearity of $T_1 \otimes T_2$ every decomposition of $\sigma$ into finite sums of simple tensors $\sigma = \sum_{i=1}^r x_i \otimes y_i$, where $x_i$ and $y_i$ are trace class operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, induces a decomposition of $(T_1 \otimes T_2)(\sigma)$ into a sum of simple tensors $(T_1 \otimes T_2)(\sigma) = \sum_{i=1}^r T_1(x_i) \otimes T_2(y_i)$. Thus

$$\| (T_1 \otimes T_2)(\sigma) \|_\gamma = \inf \left\{ \sum_{i=1}^r \| X_i \|_1 \| Y_i \|_1 \left| (T_1 \otimes T_2)(\sigma) = \sum_{i=1}^r X_i \otimes Y_i \right. \right\}$$

$$\leq \inf \left\{ \sum_{i=1}^r \| T_1(x_i) \|_1 \| T_2(y_i) \|_1 \left| \sigma = \sum_{i=1}^r x_i \otimes y_i \right. \right\}$$

$$\leq \| T_1 \| \| T_2 \| \inf \left\{ \sum_{i=1}^r \| x_i \|_1 \| y_i \|_1 \left| \sigma = \sum_{i=1}^r x_i \otimes y_i \right. \right\}$$

$$\leq \inf \left\{ \sum_{i=1}^r \| x_i \|_1 \| y_i \|_1 \left| \sigma = \sum_{i=1}^r x_i \otimes y_i \right. \right\}$$

$$= \| \sigma \|_\gamma,$$
where we have used that both $T_1$ and $T_2$ are bounded maps on the space of trace class operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and that

$$
\|T_i\| = \sup\{\text{Tr}(T_i(\rho)) \mid \rho \in \mathcal{T}(\mathcal{H}_i), \rho \geq 0 \text{ and } \text{Tr}(\rho) = 1\} \leq 1,
$$

see, e.g., Lemma 2.2.1 in [12]. (E3) follows from the fact that $[1, \infty[ \ni s \mapsto s \log s$ is monotone. Finally, (E4) follows from the facts that $\|\cdot\|_\gamma$ is subadditive and that $[1, \infty[ \ni s \mapsto s \log s$ is monotone and convex. \hfill \Box

**Remark 11** It follows from the proof of Proposition 10 that if $f$ is a convex, monotonously increasing function on $[1, \infty[$ with $f(1) = 0$, then

$$E_f(\sigma) := f(\|\sigma\|_\gamma)$$

is an entanglement measure satisfying the requirements (E0) - (E4). A possible choice is $f_1(x) = x - 1$ leading to the entanglement measure $E_{f_1}(\sigma) = \|\sigma\|_\gamma - 1$. This shows that indeed (as claimed in [1]) the function $\|\sigma\|_\gamma - 1$ is an entanglement measure on the space of density operators. Other possible choices for $f$ are $f_2(x) = x \ln x - x + 1$, $f_3(x) = e^{a(x-1)}$, $a > 0$ and so forth.

**Corollary 12** The entanglement measures constructed in Remark 11 (including the measure $E_\gamma$ from Proposition 10) satisfy that $E_f(\sigma) = 0$ if and only if $\sigma$ is separable.

**Proof:** This is an immediate consequence of Theorem 3. \hfill \Box

**Proposition 13** Let $T_1$ and $T_2$ be two trace-preserving Lüders-von Neumann operations on finite dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and let $T_L = T_1 \otimes T_2$ denote the corresponding Lüders-von Neumann operation acting locally on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let $T_1(\sigma_1) = \sum_i P_i \sigma_1 P_i$ and $T_2(\sigma_2) = \sum_j Q_j \sigma_2 Q_j$ be Choi-Kraus representations of $T_1$ and $T_2$ respectively in terms of families $\{P_i\}$ and $\{Q_j\}$ of, respectively, mutually orthogonal projection operators. Then the entanglement measure $E_{f_1} = \|\cdot\|_\gamma - 1$ as in Remark 11 satisfies

$$
\sum_{ij} p_{ij} \left(\|\sigma_{ij}\|_\gamma - 1\right) \leq \|\sigma\|_\gamma - 1
$$

where $p_{ij} := \text{Tr}(\sigma_{ij}(P_i \otimes Q_j)\sigma(P_i \otimes Q_j))$ and $\sigma_{ij} = \frac{(P_i \otimes Q_j)\sigma(P_i \otimes Q_j)}{p_{ij}}$ and where $\sigma$ is a density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

**Proof:** Let $P$ and $P'$ be orthogonal projection operators. Then $\|PxP + P'yP'\|_1 = \|PxP\|_1 + \|P'yP'\|_1$ for all operators $x,y$. This follows from considering the spectral resolutions of $PxP$ and $P'yP'$. Hence $\sum_i \|Px_k P_i\|_1 = \|\sum_i P_i x_k P_i\|_1 \leq \|x_k\|_1$. A similar argument shows that $\sum_j \|Q_j z Q_j\|_1 \leq \|z\|_1$ for all $z \in \mathcal{B}(\mathcal{H}_2)$. Hence

$$
\sum_{ij} p_{ij} \left(\|\sigma_{ij}\|_\gamma - 1\right) \leq \inf\left\{\sum_{ijk} \|P_i x_k P_i\|_1\|Q_j y_k Q_j\|_1 \left| \sigma = \sum_k x_k \otimes y_k \right\} - 1
$$

$$
\leq \inf\left\{\sum_k \|x_k\|_1\|y_k\|_1 \left| \sigma = \sum_k x_k \otimes y_k \right\} - 1
$$

$$
= \|\sigma\|_\gamma - 1. \Box
$$
It is known that some physically interesting entanglement measures coincide with the von Neumann reduced entropy on pure states, for instance the relative entropy of entanglement [4]. However, it follows immediately from Proposition 7 that \( \gamma \) does not coincide with the von Neumann reduced entropy on pure states: it follows from [4] that the entropy of entanglement for a pure state of the form \( |\phi\rangle = \alpha |00\rangle + \beta |11\rangle \) is equal to \(-|\alpha|^2 \ln |\alpha|^2 - |\beta|^2 \ln |\beta|^2\), whereas it follows from Proposition 7 that \( \gamma (|\phi\rangle \langle \phi|) = 2 (|\alpha| + |\beta|)^2 \ln (|\alpha| + |\beta|) \). Therefore we have explicitly constructed an entanglement measure satisfying a physically reasonable set of requirements which is not equal to the von Neumann reduced entropy on pure states. We have proven

**Proposition 14** \( \gamma \) and \( S_{vN} \) do not coincide on pure states.

In [16] necessary and sufficient conditions for an entanglement measure to coincide with \( S_{vN} \) on pure states were derived. It is easy to see that, e.g., \( \gamma \) does not satisfy the additivity condition (P4) considered in [16].

**VI Higher Tensor Product Hilbert Spaces**

So far we restricted ourselves to tensor product Hilbert spaces of two finite dimensional Hilbert spaces. It is straightforward, however, to generalize our results to the situation of tensor products of more than two, but at most finitely many, finite dimensional Hilbert spaces. To this end consider the tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \) of \( n \) finite dimensional Hilbert spaces \( \mathcal{H}_1, \cdots, \mathcal{H}_n \). The obvious generalization of the definition of \( \| \cdot \|_\gamma \) is

\[
\| \gamma \|^{(n)} := \inf \left\{ \sum_{i=1}^r \left\| u_i^{(1)} \right\|_1 \cdots \left\| u_i^{(n)} \right\|_1 \left| t = \sum_{i=1}^r u_i^{(1)} \otimes \cdots \otimes u_i^{(n)} \right. \right\},
\]

where \( t \) is a trace class operator on \( \mathcal{H} \).

It is straightforward to generalize the main result of [1] to \( n \)-fold tensor product Hilbert spaces \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \).

**Definition 15** Let \( \mathcal{H}_1, \cdots, \mathcal{H}_n \) be Hilbert spaces of arbitrary dimension. A density operator \( \rho \) on the tensor product \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \) is called disentangled or separable (with respect to \( \mathcal{H}_1, \cdots, \mathcal{H}_n \)) if there exist a family \( \{ \omega_i \} \) of positive real numbers, and families \( \{ \rho_i^{(k)} \} \) of density operators on \( \mathcal{H}_k \) respectively, where \( 1 \leq k \leq n \), such that

\[
\rho = \sum \omega_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(n)},
\]

where the sum converges in trace class norm.

**Theorem 16** Let \( \mathcal{H}_1, \cdots, \mathcal{H}_n \) be finite dimensional Hilbert spaces and \( \rho \) be a density operator on \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \). Then \( \rho \) is separable if and only if \( \| \rho \|^{(n)}_\gamma = 1 \).
We now consider the situation that $\mathcal{H}$ is the $m$-fold tensor product of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with two finite dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The functional $E_\gamma$ from Proposition 10 admits an obvious extension

$$E_\gamma(\sigma) := \|\sigma\|_\gamma^{(n)} \ln \|\sigma\|_\gamma^{(n)}$$

(12)

for all trace class operators $\sigma$ on $\mathcal{H}$.

**Proposition 17** The functional defined by Equation (12) satisfies the criteria (E0)-(E4) for entanglement measures.

**VII Conclusion**

To conclude, in this paper we have introduced a new class of entanglement measures on the space of density operators on tensor product Hilbert spaces. Our entanglement measures are based on the greatest cross norm $\| \cdot \|_\gamma$ on the set of trace class operators on the tensor product Hilbert space. We showed that our entanglement measures satisfy a number of physically desirable requirements, in particular that they do not increase under local quantum operations.

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