TYPES OF SERRE SUBCATEGORIES OF GROTHENDIECK CATEGORIES

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Abstract. Every Serre subcategory of an abelian category is assigned a unique type. The type of a Serre subcategory of a Grothendieck category is in the list:

$$(0, 0), (0, -1), (1, -1), (0, -2), (1, -2), (2, -1), (+\infty, -\infty);$$

and for each $(m, -n)$ in this list, there exists a Serre subcategory such that its type is $(m, -n)$. This uses right (left) recollements of abelian categories, Tachikawa-Ohtake [TO] on strongly hereditary torsion pairs, and Geigle-Lenzing [GL] on localizing subcategories. If all the functors in a recollement of abelian categories are exact, then the recollement splits. Quite surprising, any left recollement of a Grothendieck category can be extended to a recollement; but this is not true for a right recollement. Thus, a colocalizing subcategory of a Grothendieck category is localizing; but the converse is not true. All these results do not hold in triangulated categories.

Key words and phrases. the type of a Serre subcategory, right recollement, strongly hereditary torsion pair, quotient functor, localizing subcategory, Grothendieck category

1. Introduction

Given a Serre subcategory $S$ of an abelian category $\mathcal{A}$ with inclusion functor $i : S \to \mathcal{A}$ and quotient functor $Q : \mathcal{A} \to \mathcal{A}/S$, it is fundamental to know when it is localizing (resp. colocalizing), i.e., $Q$ has a right (resp. left) adjoint $(\mathcal{S}, \mathcal{G})$. By W. Geigle and H. Lenzing [GL], $S$ is localizing if and only if there exists an exact sequence $0 \to S_1 \to A \to C \to S_2 \to 0$ with $S_1 \in S$, $S_2 \in S$, and $C \in S^{\perp} \leq 1$; and if and only if the restriction $Q : S^{\perp} \leq 1 \to \mathcal{A}/S$ is an equivalence of categories. In this case the right adjoint of $Q$ is fully faithful. There is a corresponding work for a thick triangulated subcategory of a triangulated category (A. Neeman [N, Chap. 9]).

It is then natural to describe Serre subcategories of a fixed abelian category via the length of two adjoint sequences where $i$ and $Q$ lie. A finite or an infinite sequence $(\cdots, F_1, F_0, F_{-1}, \cdots)$ of functors between additive categories is an adjoint sequence, if each pair $(F_i, F_{i-1})$ is an adjoint pair. Each functor in an adjoint sequence is additive.

Let $S$ be a Serre subcategory of an abelian category $\mathcal{A}$ with the inclusion functor $i : S \to \mathcal{A}$ and the quotient functor $Q : \mathcal{A} \to \mathcal{A}/S$. The pair $(S, i)$ is of type $(m, -n)$, or in short, the Serre subcategory $S$ is of type $(m, -n)$, where $m$ and $n$ are in the set $N_0 \cup \{+\infty\}$, and $N_0$ is the set of non-negative integers, provided that there exist adjoint sequences

$$(F_m, \cdots, F_1, F_0 = i, F_{-1}, \cdots, F_{-n}) \quad \text{and} \quad (G_m, \cdots, G_1, G_0 = Q, G_{-1}, \cdots, G_{-n})$$

such that $F_m$ and $G_m$ can not have left adjoints at the same time, and that $F_{-n}$ and $G_{-n}$ can not have right adjoints at the same time.

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We stress that the type of $S$ depends on the abelian category $A$ in which $S$ is a Serre subcategory. Since in an adjoint pair one functor uniquely determines the other, every Serre subcategory is of a unique type $(m, -n)$. We will see a Serre subcategory $S$ of type $(1, -2)$, but with adjoint sequences $(F_1, i, F_{-1}, F_{-2}, F_{-3}, F_{-4}, F_{-5})$, $(G_4, G_3, G_2, G_1, Q, G_{-1}, G_{-2})$.

See Remark 5.5

A right recollement $(B, A, C, i_*, i^!, j^*, j_*)$ (see e.g. [P, Kö]) of abelian categories is a diagram

$$
\begin{array}{ccc}
B & \overset{i_*}{\longrightarrow} & A \\
 i^! & \overset{j^*}{\longrightarrow} & j_* \\
\end{array}
$$

of functors between abelian categories such that

(i) $i_*$ and $j^*$ are exact functors;
(ii) $i_*$ and $j_*$ are fully faithful;
(iii) $(i_*, i^!)$ and $(j^*, j_*)$ are adjoint pairs; and
(iv) $\text{Im} i_* = \text{Ker} j^*$ (and thus $i^!j_* = 0$).

In a right recollement (1.1) the functor $i^!$ and $j_*$ are left exact but not exact, in general. A right recollement is also called a localization sequence e.g. in [S], [G], [IKM], and [Kr], and a step in [BGS].

A left recollement $(B, A, C, i^!, i_*, j^*, j_*)$ of abelian categories is a diagram

$$
\begin{array}{ccc}
B & \overset{i^!}{\longrightarrow} & A \\
 i_* & \overset{j^*}{\longrightarrow} & j_* \\
\end{array}
$$

of functors between abelian categories such that

(i) $i^!$ and $j^*$ are exact;
(ii) $i_*$ and $j_!$ are fully faithful;
(iii) $(i^!, i_*)$ and $(j_!, j^*)$ are adjoint pairs; and
(iv) $\text{Im} i^! = \text{Ker} j^*$ (and thus $i^!j_! = 0$).

Note that in a left recollement (1.2) the functor $i^!$ and $j_!$ are right exact but not exact, in general. Thus, given a right recollement $(B, A, C, i_*, i^!, j^*, j_*)$, the data $(C, A, B, j^*, j_*, i_*, i^!)$ is not a left recollement in general (similar remark for a left recollement).

A recollement is first introduced for triangulated categories by A. Beilinson, J. Bernstein, and P. Deligne [BBD]. A recollement of abelian categories appeared in [Ku] and [CPS]. A recollement $(B, A, C, i^!, i_*, j^!, j_*)$ of abelian categories is a diagram

$$
\begin{array}{ccc}
B & \overset{i^!}{\longrightarrow} & A \\
 i_* & \overset{j^!}{\longrightarrow} & j_* \\
\end{array}
$$

of functors between abelian categories such that

(i) $(i^!, i_*)$, $(i_*, i^!)$, $(j^!, j_*)$ and $(j^*, j_*)$ are adjoint pairs;
(ii) $i_*$, $j_!$ and $j_*$ are fully faithful; and
(iii) $\text{Im} i^! = \text{Ker} j^*$.
Thus in a recollement (1.3) the functors $i_*$ and $j^*$ are exact. So (1.3) is a recollement if and only if the upper two rows is a left recollement and the lower two rows is a right recollement. By V. Franjou and T. Pirashvili [FP], recollements of abelian categories have some different properties from recollement of triangulated categories. For example, $\text{Ker} i^* \neq \text{Im} j_!$ and $\text{Ker} i^! \neq \text{Im} j_*$. In general, and Parshall-Scott’s theorem on comparison between two recollements of triangulated categories ([PS, Thm. 2.5]) does not hold in general. See also [Ps, PV, GYZ].

In a recollement of abelian categories, if $i^*$ and $i^!$ are exact, then $j_!$ and $j^*$ are also exact (see Prop. 3.1 and 3.2). The following result describes recollements of abelian categories with exact functors.

**Theorem 1.1.** Given a recollement (1.3) of abelian categories. If $i^*$ and $i^!$ are exact, then $i^* \cong i^!$ and $j_! \cong j^*$, and $A \cong B \oplus C$.

Quite surprising, we have

**Theorem 1.2.** Assume that $A$ is a Grothendieck category. Then any left recollement (1.2) of abelian categories can be extended to a recollement of abelian categories.

As a consequence, a colocalizing subcategory of a Grothendieck category is localizing. We stress that a right recollement of abelian categories does not necessarily extend to a recollement, and that a localizing subcategory of a Grothendieck category is not necessarily colocalizing. See Subsection 5.2. On the other hand, W. Geigle and H. Lenzing [GL, Prop. 5.3] have proved that any Serre subcategory $S$ of the finitely generated module category of an Artin algebra is always localizing and colocalizing.

**Theorem 1.3.** The type of a Serre subcategory of a Grothendieck category $A$ is in the list

$$(0,0), (0,-1), (1,-1), (0,-2), (1,-2), (2,-1), (+\infty,-\infty);$$

and for each $(m,-n)$ in this list, there exists a Serre subcategory such that its type is $(m,-n)$; and if a Serre subcategory $S$ is of type $(+\infty,-\infty)$, then $A \cong S \oplus (A/S)$ as categories.

The main tools for proving Theorem 1.3 are the work of strongly hereditary torsion pairs by H. Tachikawa and K. Ohtake [TO; O], the work of localizing subcategories by Geigle-Lenzing [GL], and the argument on right (left) recollements of abelian categories, especially Theorems 1.1 and 1.2. This result could also be reformulated in terms of the height of a ladder of a Grothendieck category (see [BGS], [AHKLY], [ZZZZ]). Theorems 1.1, 1.2 and 1.3 do not hold in triangulated categories.

## 2. Preliminaries

Throughout $A$ is an abelian category. A subcategory means a full subcategory closed under isomorphisms. We will use the properties of a Grothendieck category $A$: it is well-powered ([M]) in the sense that for each object $A \in A$, the class of the subobjects of $A$ forms a set; $A$ has coproducts and products, enough injective objects; and the canonical morphism from a coproduct to the corresponding product is a monomorphism (see [F], [Mit]).
2.1. **Serre subcategories.** For Serre subcategories and quotient categories we refer to [G], [Pop], and [GL]. A subcategory \( S \) of \( A \) is a *Serre subcategory* if \( S \) is closed under subobjects, quotient objects, and extensions. If \( S \) is a Serre subcategory of \( A \) with the inclusion functor \( i : S \to A \), then we have the quotient category \( A/S \) which is abelian, and the quotient functor \( Q : A \to A/S \) is exact with \( Qi = 0 \), and \( Q \) has the universal property in the sense that if \( F : A \to C \) is an exact functor between abelian categories with \( Fi = 0 \), then there exists a unique exact functor \( G : A/S \to C \) such that \( F = GQ \).

A Serre subcategory \( S \) is *localizing*, if the quotient functor \( Q : A \to A/S \) has a right adjoint \( j_* \). In this case, \( j_* \) is fully faithful ([GL, Prop. 2.2]). Dually, a Serre subcategory \( S \) is *colocalizing*, if \( Q \) has a left adjoint \( j_! \). In this case, \( j_! \) is fully faithful (the dual of [GL, Prop. 2.2]).

2.2. **Exact sequences of abelian categories.** A sequence \( 0 \to B \overset{i_*}{\to} A \overset{j^*}{\to} C \to 0 \) of exact functors between abelian categories is an *exact sequence*, provided that there exists a Serre subcategory \( S \) of an abelian category \( A' \) such that there is a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & B & \overset{i_*}{\to} & A & \overset{j^*}{\to} & C & \to & 0 \\
\vert & & \nearrow & & \nearrow & & \searrow & & \vert \\
0 & \to & S & \overset{i}{\to} & A^t & \overset{Q}{\to} & A'/S & \to & 0.
\end{array}
\]

A sequence \( 0 \to B \overset{i_*}{\to} A \overset{j^*}{\to} C \to 0 \) of exact functors between abelian categories is an exact sequence if and only if \( i_* \) is fully faithful, \( i_*B \) is a Serre subcategory of \( A \), \( j^*i_* = 0 \), and \( j^* \) has also the universal property. In this case, we have \( \text{Im}i_* = \text{Ker}j^* \).

2.3. **Torsion pairs.** For torsion pairs in an abelian category we refer to [D], [J], and [TO]. A pair \((T, F)\) of subcategories of \( A \) is a *torsion pair* ([D]), if \( \text{Hom}(T, F) = 0 \) for \( T \in T \) and \( F \in F \), and for each object \( A \in A \), there is an exact sequence \( 0 \to T \to A \to F \to 0 \) with \( T \in T \) and \( F \in F \). In this case, the exact sequence is called the *t-decomposition* of \( A \) with respect to \((T, F)\).

A subcategory \( T \) (resp. \( F \)) of a torsion pair \((T, F)\) is a *torsion class* (resp. a *torsionfree class*) if there exists a subcategory \( F \) (resp. \( T \)) such that \((T, F)\) is a torsion pair. If \((T, F)\) is a torsion pair, then \( F = T^{\perp_0} \) and \( T = T^{\perp_0} F \), where \( T^{\perp_0} := \{ A \in A \mid \text{Hom}(T, A) = 0, \ \forall T \in T \} \), and \( T^{\perp_0} \) is dually defined. By S. E. Dickson [D, Thm.2.3], if \( A \) is a well-powered abelian category with coproducts and products, then a subcategory \( T \) (resp. \( F \)) is a torsion class (resp. a torsionfree class) if and only if \( T \) (resp. \( F \)) is closed under quotient objects, extensions, and coproducts (resp. under subobjects, extensions, and products).

A subcategory \( B \) is *weakly localizing*, provided that for each object \( A \) of \( A \), there exists an exact sequence

\[
0 \to B_1 \to A \to C \to B_2 \to 0
\]

with \( B_1 \in B \), \( B_2 \in B \), and \( C \in B^{\perp_{\leq 1}} := \{ A \in A \mid \text{Hom}(B, A) = 0 = \text{Ext}^1(B, A), \ \forall B \in B \} \). By W. Geigle and H. Lenzing [GL, Prop. 2.2], a Serre subcategory \( S \) is localizing if and only if it is weakly localizing. Dually, a subcategory \( B \) is *weakly colocalizing*, provided that for each object \( A \) of \( A \), there exists an exact sequence

\[
0 \to B_1 \to C \to A \to B_2 \to 0
\]
with $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, and $C \in \downarrow_{\leq 1} B := \{ A \in \mathcal{A} \mid \text{Hom}(A, B) = 0 = \text{Ext}^1(A, B), \forall B \in \mathcal{B}\}$.

By the dual of [GL, Prop. 2.2], a Serre subcategory $\mathcal{S}$ is colocalizing if and only if it is weakly colocalizing.

Following H. Tachikawa and K. Ohtake [TO], a torsion pair $(\mathcal{T}, \mathcal{F})$ is hereditary (resp. cohereditary), if $\mathcal{T}$ (resp. $\mathcal{F}$) is closed under subobjects (resp. quotient objects); and it is strongly hereditary (resp. strongly cohereditary), if $\mathcal{T}$ (resp. $\mathcal{F}$) is weakly localizing (resp. weakly colocalizing). Every strongly hereditary (resp. strongly cohereditary) torsion pair is hereditary (resp. cohereditary) (see [TO, Prop. 1.7, 1.8]). K. Ohtake [O, Thm. 2.6, 1.6] has proved that if $\mathcal{A}$ has enough injective objects (resp. enough projective objects), then every hereditary (resp. cohereditary) torsion pair is also strongly hereditary (resp. strongly cohereditary) (see also [TO, Thm. 1.8*, 1.8]).

2.4. Let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a fully faithful functor between abelian categories. We say that $F$ is Giraud if $F$ has a left adjoint which is an exact functor. Dually, $F$ is coGiraud if $F$ has a right adjoint which is exact. By [TO, Coroll. 3.8, 2.8], $F$ is Giraud (resp. coGiraud) if and only if $FC$ is a Giraud subcategory (a coGiraud subcategory) of $\mathcal{A}$ in the sense of [TO].

**Lemma 2.1.** Let $\mathcal{A}$ be an abelian category.

1. Given a Giraud functor $j_* : \mathcal{C} \rightarrow \mathcal{A}$ with an exact left adjoint $j^* : \mathcal{A} \rightarrow \mathcal{C}$, there exists a functor $i^! : \mathcal{A} \rightarrow \text{Ker} j^*$, such that $(\text{Ker} j^*, \mathcal{A}, i, i^!, j^*, j_*)$ is a right recollement, where $i : \text{Ker} j^* \rightarrow \mathcal{A}$ is the inclusion functor.

1' Given a coGiraud functor $j_! : \mathcal{C} \rightarrow \mathcal{A}$ with exact right adjoint $j^* : \mathcal{A} \rightarrow \mathcal{C}$, there exists a functor $i_* : \mathcal{A} \rightarrow \text{Ker} j^*$, such that $(\text{Ker} j^*, \mathcal{A}, i, i^!, j^*, j_*)$ is a left recollement, where $i : \text{Ker} j^* \rightarrow \mathcal{A}$ is the inclusion functor.

2. Let $0 \rightarrow B \overset{i}{\rightarrow} A \overset{j}{\rightarrow} \mathcal{C} \rightarrow 0$ be an exact sequence of abelian categories. If $j^*$ has a right adjoint $j_*$, then $j_*$ is fully faithful, and there exists a functor $i^! : A \rightarrow \text{Ker} j^*$ such that $(\text{Ker} j^*, \mathcal{A}, i, i^!, j^*, j_*)$ is a right recollement, where $i : \text{Ker} j^* \rightarrow \mathcal{A}$ is the inclusion functor.

2' Let $0 \rightarrow B \overset{i}{\rightarrow} A \overset{j^*}{\rightarrow} \mathcal{C} \rightarrow 0$ be an exact sequence of abelian categories. If $j^*$ has a left adjoint $j_!$, then $j_!$ is fully faithful, and there exists a functor $i_* : A \rightarrow \text{Ker} j^*$ such that $(\text{Ker} j^*, \mathcal{A}, i, i^!, j^*, j_*)$ is a left recollement, where $i : \text{Ker} j^* \rightarrow \mathcal{A}$ is the inclusion functor.

3. If $(\mathcal{B}, \mathcal{C})$ is a strongly hereditary torsion pair in $\mathcal{A}$. Then $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ and there is a right recollement of abelian category

$$\begin{array}{ccc}
\mathcal{B} & \xleftarrow{i} & \mathcal{A} & \xrightarrow{Q} & \mathcal{A}/\mathcal{B}
\end{array}$$

with $\text{Im} j_* = \downarrow_{\leq 1} B$, where $i$ is the inclusion functor and $Q$ is the quotient functor.

3' If $(\mathcal{B}, \mathcal{C})$ is a strongly cohereditary torsion pair in $\mathcal{A}$. Then $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$ and there is a left recollement of abelian category

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i^*} & \mathcal{A} & \xleftarrow{Q} & \mathcal{A}/\mathcal{C}
\end{array}$$

with $\text{Im} j_! \cong \downarrow_{\leq 1} C$ as categories, where $i$ is the inclusion functor and $Q$ is the quotient functor.
Proof. We only prove (1), (2) and (3). The assertions (1'), (2') and (3') can be dually proved.

(1) By assumption $j^*$ is exact, thus $\text{Ker}j^*$ is an abelian category and the inclusion functor $i : \text{Ker}j^* \to A$ is exact. We claim that $i$ admits a right adjoint $i^! : A \to \text{Ker}j^*$. In fact, for any $A \in \mathcal{A}$, there is an exact sequence $0 \to \text{Ker}j_A \to A \xrightarrow{\zeta} j_*j^*A$, where $\zeta : \text{Id}_A \to j_*j^*$ is the unit of the adjoint pair $(j^*, j_*)$. Put $i^!A := \text{Ker}j_A$. Since $j^*$ is exact and $j_*j^*$ is an isomorphism, $\text{Ker}j_A \in \text{Ker}j^*$. Thus, $i^! : A \to \text{Ker}j^*$ defines a functor. For $B \in \text{Ker}j^*$ and $A \in \mathcal{A}$, since $\text{Hom}_A(B, j_*j^*A) \cong \text{Hom}_C(j^*B, j^*A) = 0$, by applying the left exact functor $\text{Hom}_A(B, -)$ to the exact sequence we get an isomorphism $\text{Hom}(B, \text{Ker}j_A) \cong \text{Hom}(iB, A)$. This proves the claim, and hence $(\text{Ker}j^*, A, C, i, i^!, j^*, j_*)$ a right recollement.

(2) Since $0 \to B \xrightarrow{j_*} A \xrightarrow{j^*} C \to 0$ is an exact sequence of abelian categories, without loss of generality, one may regard $j^*$ just as the quotient functor $Q : A \to A/i_*\mathcal{B}$. By [GL, Prop. 2.2], $j_*$ is fully faithful. Thus $j_* : C \to A$ is a Giraud functor with the exact left adjoint $j^* : A \to C$, and hence the assertion follows from (1).

(3) By [TO, Prop. 1.7'], $(B, C)$ is a hereditary torsion pair. Thus $B$ is a Serre subcategory. Since by assumption $\mathcal{B}$ is weakly localizing, by [GL, Prop. 2.2], the quotient functor $Q : A \to A/B$ admits a right adjoint $j_* : A/B \to A$ which is fully faithful, and $\text{Im}j_* = B^{\perp \leq 1}$. So $j_*$ is a Giraud functor with an exact left adjoint $Q$. By (1), there exists a functor $i^! : A \to \text{Ker}Q = B$, such that $(B, A, A/B, i, i^!, j_*)$ is a right recollement. \hfill \Box

3. Recollements of abelian categories with exact functors

3.1. The following proposition gives the properties of a right recollement of abelian categories we need. Some of them are well-known for recollements of abelian categories (see [FP], [Ps], [PV]).

Proposition 3.1. Let (1.1) be a right recollement of abelian categories. Then

(1) $\text{Im}i_* \rightarrow$ a weakly localizing subcategory. Explicitly, for each object $A \in \mathcal{A}$, there is an exact sequence $0 \to i_*i^!A \xrightarrow{\omega} A \xrightarrow{\zeta} j_*j^*A \to i_*B \to 0$ for some object $B \in \mathcal{B}$, with $j_*j^*A \in (\text{Im}i_*)^{\perp \leq 1}$, where $\omega$ the counit and $\zeta$ is the unit.

(2) $0 \to B \xrightarrow{i_*} A \xrightarrow{j_*} C \to 0$ is an exact sequence of abelian categories.

(3) $\text{Ker}i^! = (\text{Im}i_*)^{\perp \omega}$; $(\text{Im}i_*, \text{Ker}i^!)$ is a strongly hereditary torsion pair in $\mathcal{A}$, and $0 \to i_*i^!A \xrightarrow{\omega} A \to \text{Coker}i_A \to 0$ is the $t$-decomposition of $A$.

(4) The following are equivalent:

(i) $i^!$ is exact;

(ii) $i^!$ and $j_*$ are exact;

(iii) $0 \to C \xrightarrow{j_*} A \xrightarrow{i_*} B \to 0$ is an exact sequence of abelian categories;

(iv) the sequence $0 \to i_*i^!A \xrightarrow{\omega} A \xrightarrow{\zeta} j_*j^*A \to 0$ is exact for each object $A \in \mathcal{A}$;

(v) $\text{Im}j_* = \text{Ker}i^!$, $(\text{Im}i_*, \text{Im}j_*)$ is a cohereditary torsion pair in $\mathcal{A}$, and $0 \to i_*i^!A \xrightarrow{\omega} A \xrightarrow{\zeta} j_*j^*A \to 0$ is the $t$-decomposition of $A$;

(vi) $(\text{Im}i_*, \text{Im}j_*)$ is a hereditary and cohereditary torsion pair in $\mathcal{A}$;

(vii) $(\text{Im}i_*, \text{Im}j_*)$ is a strongly hereditary and strongly cohereditary torsion pair in $\mathcal{A}$;

(viii) $(\text{Im}i_*, \text{Im}j_*)$ is a strongly cohereditary torsion pair in $\mathcal{A}$. 

Proof. (1) Applying the exact functor \( j^* \) to the exact sequence \( 0 \rightarrow \text{Ker} \xi_A \rightarrow A \xrightarrow{\xi_A} j_* j^* A \rightarrow \text{Coker} \xi_A \rightarrow 0 \), we get an exact sequence \( 0 \rightarrow j^* \text{Ker} \xi_A \rightarrow j^* A \xrightarrow{\xi_A \circ j_*} j^* j_* j^* A \rightarrow j^* \text{Coker} \xi_A \rightarrow 0 \). Since \( j_* \) is fully faithful, \( j^* \xi_A \) is an isomorphism. So \( j^* \text{Ker} \xi_A = 0 = j^* \text{Coker} \xi_A \), and then \( \text{Ker} \xi_A \cong i_* B \) and \( \text{Coker} \xi_A \cong i_* B \) for some \( B' \in \mathcal{B} \) and \( B \in \mathcal{B} \). Applying the left exact functor \( i_* i^! \) to the exact sequence \( 0 \rightarrow i_* B' \rightarrow A \xrightarrow{\xi_A} j_* j^* A \), we get a commutative diagram

\[
\begin{array}{ccc}
i_* i^* B' & \xrightarrow{\approx} & i_* i^! A \\
\downarrow \omega_{i_* B'} & & \downarrow \omega_A \\
i_* B' & \rightarrow & A.
\end{array}
\]

Since \( i_* \) is fully faithful, \( \omega_{i_* B'} \) is an isomorphism, and hence \( \text{Ker} \xi_A \cong i_* B' \cong i_* i^! A \). So we get the desired exact sequence.

To see \( \text{Im} i_* \) is weakly localizing, by the exact sequence just established, it suffices to show \( \text{Coker} \xi_A \cong i_* B \) for each object \( A \in \mathcal{A} \). We only need to show \( \text{Ext}^1(i_* B, j_* j^* A) = 0 \) for \( B \in \mathcal{B} \). Let \( 0 \rightarrow j_* j^* A \xrightarrow{\alpha} X \rightarrow i_* B \rightarrow 0 \) be an exact sequence. Applying the exact functor \( j^* \) we get an isomorphism \( j^* \text{a} : j^* j_* j^* A \cong j^* X \). By the commutative diagram

\[
\begin{array}{cccc}
\text{Hom}(j^* X, j^* A) & \xrightarrow{(j^* a, -)} & \text{Hom}(j^* j_* j^* A, j^* A) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}(X, j_* j^* A) & \xrightarrow{(a, -)} & \text{Hom}(j_* j^* A, j_* j^* A)
\end{array}
\]

we see that \( \text{Hom}(a, j_* j^* A) : \text{Hom}(X, j_* j^* A) \rightarrow \text{Hom}(j_* j^* A, j_* j^* A) \) is an isomorphism, which implies that the exact sequence \( 0 \rightarrow j_* j^* A \xrightarrow{\alpha} X \rightarrow i_* B \rightarrow 0 \) splits.

(2) Since \( \text{Im} i_* = \text{Ker} j^* \) and \( j^* \) is exact, \( i_* \mathcal{B} \) is a Serre subcategory of \( \mathcal{A} \). Since \( j^* i_* = 0 \), by the universal property of the quotient functor \( Q : \mathcal{A} \rightarrow \mathcal{A}/i_* \mathcal{B} \) we get a unique exact functor \( F : \mathcal{A}/i_* \mathcal{B} \rightarrow \mathcal{C} \) such that \( FQ \cong j^* \). We claim that \( Q j_* \) is a quasi-inverse of \( F \). In fact, \( FQ j_* \cong j^* j_* \cong \text{Id}_C \); on the other hand, for each object \( A \in \mathcal{A} \), by (1) there is a functorial isomorphism \( QA \cong Q j_* j^* A \), and hence for each object \( QA \in \mathcal{A}/i_* \mathcal{B} \) there are functorial isomorphisms

\[
Q j_* F(QA) \cong Q j_* F(Q j_* j^* A) \cong Q j_* (j^* j_* j^* A) \cong Q j_* j^* A \cong QA
\]

in \( \mathcal{A}/B \), i.e., \( Q j_* F \cong \text{Id}_{\mathcal{A}/i_* \mathcal{B}} \).

(3) For \( A \in (\text{Im} i_*)^{-1} \circ \), we have \( \text{Hom}_B(i^! A, i^! A) \cong \text{Hom}_B(i_* i^! A, A) = 0 \), thus \( i^! A = 0 \). So \( (\text{Im} i_*)^{-1} \subseteq \text{Ker} i^! \). Conversely, if \( A \in \text{Ker} i^! \), then for each \( B \in \mathcal{B} \) we have \( \text{Hom}_A(i_* B, A) \cong \text{Hom}_B(B, i^! A) = 0 \). So \( \text{Ker} i^! \subseteq (\text{Im} i_*)^{-1} \). This shows \( \text{Ker} i^! = (\text{Im} i_*)^{-1} \).

For \( A \in \mathcal{A} \), considering the exact sequence \( 0 \rightarrow i_* i^! A \xrightarrow{\alpha} A \xrightarrow{\xi_A} j_* j^* A \rightarrow \text{Coker} \omega_A \rightarrow 0 \), we get an exact sequence \( 0 \rightarrow i_* i^! A \xrightarrow{\alpha} A \rightarrow \text{Coker} \omega_A \rightarrow 0 \). To see \( (\text{Im} i_*, \text{Ker} i^!) \) is a torsion pair in \( \mathcal{A} \), it suffices to show \( \text{Coker} \omega_A \in \text{Ker} i^! \). We see this by applying the left exact functor \( i^! \) to the exact sequence \( 0 \rightarrow \text{Coker} \omega_A \rightarrow j_* j^* A \), and using \( i^! j_* = 0 \). By (1), \( \text{Im} i_* \) is a weakly localizing subcategory. Thus \( (\text{Im} i_*, \text{Ker} i^!) \) is a strongly hereditary torsion pair.

(4) (i) \( \Rightarrow \) (ii) : Applying the left exact functor \( j_* \) to a given exact sequence \( 0 \rightarrow C_1 \xrightarrow{f} C \xrightarrow{g} C_2 \rightarrow 0 \) in \( \mathcal{C} \), we get an exact sequence \( 0 \rightarrow j_* C_1 \xrightarrow{j_* f} j_* C \xrightarrow{j_* g} j_* C_2 \rightarrow \text{Coker}(j_* g) \rightarrow 0 \); then
by applying the exact functor $i^!$ we see $i^! \text{Coker}(j_*g) = 0$ (since $i_*j_*C_2 = 0$). Applying the exact functor $j^*$ we get an exact sequence

$$0 \rightarrow j^*j_*C_1 \cong C_1 \rightarrow j^*j_*C \cong C \rightarrow j^*j_*C_2 \cong C_2 \rightarrow j^* \text{Coker}(j_*g) \rightarrow 0,$$

and thus $j^* \text{Coker}(j_*g) = 0$. So $\text{Coker}(j_*g) = i_!B$ for some $B \in \mathcal{B}$, and hence $0 = i^! \text{Coker}(j_*g) = i^!i_*B \cong B$. Thus $\text{Coker}(j_*g) = i_!B = 0$, which proves the exactness of $j_*$. 

(ii) $\Rightarrow$ (iii): We first claim $\text{Im}j_* = \text{Ker}i^!$. It is clear that $\text{Im}j_* \subseteq \text{Ker}i^!$. For each object $A \in \text{Ker}i^!$, by (1) we have an exact sequence $0 \rightarrow A \rightarrow j_*j^*A \rightarrow i_*B \rightarrow 0$; applying the exact functor $i^!$ we see that $i^!i_*B = 0$, and hence $B \cong i^!i_*B = 0$. So $A \cong j_*j^*A \in \text{Im}j_*$. This proves the claim. Thus $\text{Im}j_* = \text{Ker}i^!$ is a Serre subcategory. It remains to prove that $i^!$ has the universal property. For this, assume that $F : \mathcal{A} \rightarrow \mathcal{B}'$ is an exact functor such that $Fj_* = 0$. Applying $F$ to the exact sequence in (1) we get a functorial isomorphism $F(i_*i!A) \cong F(A)$ for each object $A \in \mathcal{A}$, i.e., $(Fi_*)i^! \cong F$. If $G : \mathcal{B} \rightarrow \mathcal{B}'$ is an exact functor such that $Gi^! \cong F$, then $Gi^! \cong (Fi_*)i^!$ and hence $G \cong F i_*$ since $i^!$ is dense.

(iii) $\Rightarrow$ (iv): For each object $A \in \mathcal{A}$, applying the exact functor $i^!$ to the exact sequence in (1), we get an exact sequence $0 \rightarrow i^!i_*i^!A \rightarrow i^!A \rightarrow i^!i_*j^*A \rightarrow i^!i_*B \rightarrow 0$. Since $i^!j_*j^*A = 0$, $i^!i_*B = 0$. Thus $B \cong i^!i_*B = 0$ and hence we get the desired exact sequence.

(iv) $\Rightarrow$ (v): From the given exact sequence one easily see $\text{Im}j_* = \text{Ker}i^!$, and hence $(\text{Im}i_*, \text{Im}j_*)$ is a torsion pair by (3). It remains to prove that $\text{Im}j_* = \text{Ker}i^!$ is closed under quotient objects. For this, let $0 \rightarrow A_1 \rightarrow A \rightarrow A_2 \rightarrow 0$ be an exact sequence with $A \in \text{Ker}i^!$. Then we get a commutative diagram with exact rows and columns:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
i_*i^!A_1 & \omega_{A_1} & j_*j^*A_1 & \zeta_{A_1} & A_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
i_*i^!A & \omega_A & A & \zeta_A & j_*j^*A & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
i_*i^!A_2 & \omega_{A_2} & A & \zeta_{A_2} & j_*j^*A_2 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Applying the Snake Lemma to the two columns on the right, we get an exact sequence $0 \rightarrow i_*i^!A_1 \rightarrow i_*i^!A \rightarrow i_*i^!A_2 \rightarrow 0$, and hence $i_*i^!A_2 = 0$. Since $i_*$ is fully faithful, $i^!A_2 = 0$.

(v) $\Rightarrow$ (vi): Since $\text{Im}i_* = \text{Ker}j^*$ and $j^*$ is exact, $\text{Im}i_*$ is closed under subobjects. So $(\text{Im}i_*, \text{Im}j_*)$ is a hereditary and cohereditary torsion pair.

(vi) $\Rightarrow$ (vii) follows from [TO, Thm. 4.1] (we stress that this step does not need that $\mathcal{A}$ has enough injective objects).

(vii) $\Rightarrow$ (viii) is clear.

(viii) $\Rightarrow$ (i): For each object $A \in \mathcal{A}$, by (1) we have an exact sequence $0 \rightarrow i_*i^!A \xrightarrow{\omega_A} A \xrightarrow{\zeta_A} j_*j^*A \rightarrow i_*B \rightarrow 0$ for some $B \in \mathcal{B}$. By assumption $(\text{Im}i_*, \text{Im}j_*)$ is a strongly cohereditary torsion pair, it follows from [TO, Prop. 1.7] that $(\text{Im}i_*, \text{Im}j_*)$ is a cohereditary torsion pair. Thus $\text{Im}j_*$ is
closed under quotient objects. So \( i_*B \in \text{Im} i_* \cap \text{Im} j_* = \{0\} \), and hence we get the exact sequence 
\[ 0 \to i_*i^!A \to j_*j^*A \to 0. \]

Let \( 0 \to A_1 \to A \to A_2 \to 0 \) be an exact sequence in \( \mathcal{A} \). Then we get a commutative diagram (3.1) with exact rows and columns. Applying the Snake Lemma to the two columns on the right, we get an exact sequence 
\[ 0 \to i_*i^!A_1 \to i_*i^!A \to \text{Coker}(i^!g) \to 0. \]
Applying the left exact functor \( i^! \) to \( 0 \to A_1 \to A \to A_2 \to 0 \) we get the exact sequence 
\[ 0 \to i^!A_1 \to i^!A \to \text{Coker}(i^!g) \to 0, \]
and hence we have the exact sequence \( 0 \to i_*i^!A_1 \to i_*i^!A \to \text{Coker}(i^!g) \to 0 \). Thus \( i_*\text{Coker}(i^!g) = 0 \). Since \( i_* \) is fully faithful, \( \text{Coker}(i^!g) = 0 \). This proves the exactness of \( i^! \).

### Proposition 3.2

Let (1.2) be a left recollement of abelian categories. Then

1. \( \text{Im} i_* \) is a weakly colocalizing subcategory. Explicitly, for each object \( A \in \mathcal{A} \), there is an exact sequence 
   \[ 0 \to i_*B \to j_*j^*A \to i_*i^!A \to 0 \]
   for some \( B \in \mathcal{B} \), with \( j_*j^*A \in \mathcal{C} \).
2. \( 0 \to B \xrightarrow{i_*} A \xrightarrow{j_*} \mathcal{C} \to 0 \) is an exact sequence of abelian categories.
3. \( \text{Ker}i^* = i^0(\text{Im} i_*); \) and \( (\text{Ker}i^*, \text{Im} i_*) \) is a strongly cohereditary torsion pair, and \( 0 \to \text{Ker}i^* \to i_*i^*A \to 0 \) is the \( t \)-decomposition of \( A \).
4. The following are equivalent:
   (i) \( i^* \) is exact;
   (ii) \( i^* \) and \( j_* \) are exact;
   (iii) \( 0 \to C \xrightarrow{i^*} A \xrightarrow{j^*} B \to 0 \) is an exact sequence of abelian categories;
   (iv) the sequence \( 0 \to j_*j^*A \to i_*i^*A \to 0 \) is exact for each object \( A \in \mathcal{A} \);
   (v) \( \text{Im}j_* = \text{Ker}i^*; \) \( \text{(Im}j_, \text{Im}i_*) \) is a hereditary torsion pair of \( \mathcal{A} \), and \( 0 \to j_*j^*A \to i_*i^*A \to 0 \) is the \( t \)-decomposition of \( A \);
   (vi) \( \text{(Im}j_, \text{Im}i_*) \) is a hereditary torsion pair of \( \mathcal{A} \);
   (vii) \( \text{(Im}j_, \text{Im}i_*) \) is a strongly hereditary and strongly cohereditary torsion pair of \( \mathcal{A} \);
   (viii) \( \text{(Im}j_, \text{Im}i_*) \) is a strongly hereditary torsion pair of \( \mathcal{A} \).

### Remark 3.3

By Lemma 2.1 (resp. Lemma 2.1'), there is a bijective correspondence between right (resp. left) recollements and Giraud functors (resp. coGiraud functors).

By Lemma 2.1 (3) and Proposition 5.1 (resp. Lemma 2.1 (3') and Proposition 5.1 (3)), there is a bijective correspondence between right (resp. left) recollements and strongly hereditary (resp. strongly cohereditary) torsion pairs.

### 3.3

To prove Theorem 1.1 we use the following fact, in which the first assertion is just [TO, Lemmas 4.2, 4.2*]. For the use of the second assertion, we include a proof.

### Lemma 3.4

Let \( (\mathcal{U}, \mathcal{V}) \) and \( (\mathcal{V}, \mathcal{W}) \) be torsion pairs in \( \mathcal{A} \). Assume that \( \mathcal{U} \) is closed under subobjects and \( \mathcal{W} \) is closed under quotient objects. Then \( \mathcal{U} = \mathcal{W} \).
For each object $A \in \mathcal{A}$, let $0 \to A_U \to A \xrightarrow{\varphi} A^V \to 0$ be the $t$-decomposition of $A$ with respect to $(U, V)$, and $0 \to A_V \to A \xrightarrow{\psi} A^W \to 0$ the $t$-decomposition of $A$ with respect to $(V, W)$. Then $A^V \cong A^V$ and $A^W \cong A_U$, $A \cong A_U \oplus A^V$, and $A \mapsto (A_U, A^V)$ gives an equivalence $\mathcal{A} \cong U \oplus V$ of categories.

**Proof.** Consider the push-out of $g$ and $h$, we get a commutative diagram with exact rows and columns

\[ \begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & & & \downarrow \\
0 & G & \to & A^V \\
\downarrow & & & \downarrow \\
A_U & A \xrightarrow{g} A^V & 0 \\
\downarrow & & & \downarrow \\
0 & D & \to & A^W \\
\downarrow & & & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array} \]

Since $V$ and $W$ are closed under quotient objects, $B \in V \cap W = \{0\}$. Since $U$ and $V$ are closed under subobjects, $E \in U \cap V = \{0\}$.

Thus $A^V \cong C \cong A^V$, and hence $g$ is a splitting epimorphism $A \cong A_U \oplus A^V$. Also, $A_U \cong D \cong A^W$, and hence $h$ is a splitting epimorphism. Taking $A \in U$ and $A \in W$, respectively, we see that $U = W$.

It is routine to verify that $A \mapsto U \oplus V$ given by $A \mapsto (A_U, A^V)$ is an equivalence of categories. ■

**Proof of Theorem 1.1** By Proposition 3.2(4)(v), $(\text{Im} j_i, \text{Im} i_*)$ is a torsion pair with $t$-decomposition $0 \to j_i^* A \xrightarrow{\omega_A} A \xrightarrow{\eta_A'} \text{Im} i_* A \to 0$ and $\text{Im} j_i = \text{Ker} i^*$ is closed under subobjects. By Proposition 3.3(4)(v), $(\text{Im} j_i, \text{Im} i_* j_i)$ is a torsion pair with $t$-decomposition $0 \to i_* j_i^* A \xrightarrow{\omega_A} A \xrightarrow{\eta_A'} j_i j_i^* A \to 0$ and $\text{Im} j_i = \text{Ker} i^*$ is closed under quotient objects. It follows from Lemma 3.4 that $\text{Im} j_i = \text{Im} j_i$ and $A \cong \text{Im} i_* \oplus \text{Im} j_i \cong B \oplus C$. For $A \in \mathcal{A}$, by Lemma 3.3, $i_* j_i^* A \cong i_* i^* A$ and $j_i j_i^* A \cong j_i j_i^* A$. Since $i_*$ is fully faithful, $i^* A \cong i^* A$ and hence $i^* \cong i^*$. Since $j_i$ is dense, it follows that $j_i \cong j_i$. ■

4. **Proof of Theorem 1.2**

**Lemma 4.1.** Let $\mathcal{A}$ be a Grothendieck category, and (1.2) a left recollement of abelian categories. Then $(i_* B, (i_* B)^{1 \omega})$ is a torsion pair.

**Proof.** By Proposition 3.2(3), $(\text{Ker} i^*, i_* B)$ is a torsion pair. So the torsionfree class $i_* B$ is closed under subobjects and products ([D, Thm. 2.3]). Since $\mathcal{A}$ is a Grothendieck category, any coproduct is a subobject of the corresponding product, $i_* B$ is also closed under coproducts. On the other hand, since $i_* B = \text{Ker} j_i^*$ and $j_i^*$ is exact, $i_* B$ is closed under quotient objects and extensions. Thus by [D, Thm. 2.3] $i_* B$ is a torsion class, and hence $(i_* B, (i_* B)^{1 \omega})$ is a torsion pair. ■

**Proof of Theorem 1.2** Given a left recollement (1.2), by Lemma 4.1 $(i_* B, (i_* B)^{1 \omega})$ is a torsion pair. Since $i_* B = \text{Ker} j_i^*$ and $j_i^*$ is exact, $i_* B$ is closed under subobjects, i.e., $(i_* B, (i_* B)^{1 \omega})$ is a hereditary torsion pair. Since $\mathcal{A}$ is a Grothendieck category, $A$ has enough injective objects. By [TO, Thm. 1.8], $(i_* B, (i_* B)^{1 \omega})$ is a strongly hereditary torsion pair. Applying Lemma 2.1(3) we get a right recollement.
where \( i \) is the inclusion functor and \( Q \) is the quotient functor. By Proposition 3.2(2), \( 0 \to B \xrightarrow{i} A \xrightarrow{Q} A/i_* B \) \( \xrightarrow{i} C \to 0 \) is an exact sequence of abelian categories. By the universal property of the functors \( j^* \) and \( Q \), we get a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{i_*} & A & \xrightarrow{Q} & A/i_* B \\
\downarrow{F} & & \downarrow{i} & & \downarrow{Q} \\
i_* B & \xrightarrow{i} & A & \xrightarrow{Q} & A/i_* B
\end{array}
\]

and hence we get a recollement

\[
\begin{array}{ccc}
B & \xrightarrow{i_*} & A & \xrightarrow{j_*} & C \\
\end{array}
\]

with \( i^! = F^{-1}i_! \) and \( j_* = jG \).

**Corollary 4.2.** A colocalizing subcategory of a Grothendieck category is localizing.

**Proof.** Let \( S \) be a colocalizing subcategory of a Grothendieck category \( A \). That is, the quotient functor \( Q : A \to A/S \) has a left adjoint, denoted by \( j_1 : A/S \to A \). By the dual of [GL, Prop. 2.2], \( j_1 \) is fully faithful. So \( j_1 \) is a coGiraud functor with exact right adjoint \( Q \). By Lemma 2.1(1') there exists a functor \( i_* : A \to \text{Ker}Q = S \), such that \((S, A, A/S, i^*, i, j_1, Q)\) is a left recollement, where \( i : S \to A \) is the inclusion functor. Then by Theorem 1.2 this left recollement can be extended to be a recollement, so \( Q \) has a right adjoint, i.e., \( S \) is localizing.

5. **Proof of Theorem 1.3**

5.1. **Serre subcategories of type** \((0, 0)\). For a ring \( R \), let \( \text{Mod}R \) be the category of right \( R \)-modules. If \( R \) is a right noetherian, let \( \text{mod}R \) be the category of finitely generated right \( R \)-modules.

**Lemma 5.1.** Let \( R \) be a right noetherian ring. Then \( \text{mod}R \) is a Serre subcategory of \( \text{Mod}R \).

**Proof.** It is clear that \( \text{mod}R \) is a Serre subcategory of \( \text{Mod}R \). Let \( i : \text{mod}R \to \text{Mod}R \) and \( Q : \text{Mod}R \to \text{Mod}R/\text{mod}R \) be the inclusion functor and the quotient functor, respectively. Assume that the type of \( \text{mod}R \) is not \((0, 0)\). Then there exist either adjoint pairs \((i_1, i)\) and \((j_1, Q)\), or adjoint pairs \((i, i_{-1})\) and \((Q, j_{-1})\).

In the first case, by Lemma 2.1(2') we get a left recollement

\[
\begin{array}{ccc}
\text{mod}R & \xrightarrow{i_1} \text{Mod}R & \xrightarrow{j_1} \text{Mod}R/\text{mod}R \\
\end{array}
\]

By Proposition 3.2(3) we have a torsion pair \((\text{Ker}i_1, \text{mod}R)\) in \( \text{Mod}R \). Thus the torsionfree class \( \text{mod}R \) is closed under products, which is absurd.

The dual argument shows that the second case is also impossible. We give a direct proof. For each \( X \in \text{Mod}R/\text{mod}R \) we have \( \text{Hom}_R(M, j_{-1}X) \cong \text{Hom}_{\text{Mod}R/\text{mod}R}(QM, X) = 0 \) for all \( M \in \text{mod}R \). So \( j_{-1}X \) has no non-zero finitely generated submodule. Thus \( j_{-1}X = 0 \). Since \( j_{-1} \) is fully faithful, \( \text{Mod}R/\text{mod}R = 0 \), i.e., \( \text{mod}R = \text{Mod}R \), which is absurd.
5.2. Serre subcategories of type \((0, -1)\).

**Lemma 5.2.** Let \(\text{Ab}_t\) be the category of the torsion abelian groups. Then \(\text{Ab}_t\) is a Serre subcategory of type \((0, -1)\).

**Proof.** Let \(\text{Ab}_f\) be the category of the abelian groups in which every non-zero element is of infinite order. Then \((\text{Ab}_t, \text{Ab}_f)\) is a torsion pair in \(\text{Mod}\mathbb{Z}\). Let \(i: \text{Ab}_t \to \text{Mod}\mathbb{Z}\) and \(Q: \text{Mod}\mathbb{Z} \to \text{Mod}\mathbb{Z}/\text{Ab}_t\) be the inclusion functor and the quotient functor, respectively. Since \(\text{Ab}_t\) is closed under submodules, it follows from [TO, Thm. 1.8*] (also [O, Thm. 2.6]) that \((\text{Ab}_f, \text{Ab}_t)\) is a strongly hereditary torsion pair. By Proposition 3.1(3) we get a right recollement

\[
\begin{array}{ccc}
\text{Ab}_t & \xrightarrow{i} & \text{Mod}\mathbb{Z} & \xrightarrow{j} & \text{Mod}\mathbb{Z}/\text{Ab}_t \\
\end{array}
\]

Thus \((\text{Ab}_t, \text{Keri}_1)\) is a torsion pair, by Proposition 3.1(3). Comparing with the torsion pair \((\text{Ab}_t, \text{Ab}_f)\) we get \(\text{Ab}_f = \text{Keri}_1\).

Assume that the type of \(\text{Ab}_t\) is not \((0, -1)\). Then there exist either adjoint pairs \((i_1, i)\) and \((j_1, Q)\), or adjoint pairs \((i_1, i_2)\) and \((j_1, j_2)\).

In the first case, by Lemma 2.1(2') we get a left recollement \((\text{Ab}_t, \text{Mod}\mathbb{Z}, \text{Mod}\mathbb{Z}/\text{Ab}_t, i_1, i, j_1, Q)\), and hence \((\text{Keri}_1, \text{Ab}_t)\) is a torsion pair, by Proposition 3.2(3). So the torsionfree class \(\text{Ab}_t\) is closed under products, which is absurd.

In the second case, the functor \(i_1\) is exact, and hence \(\text{Ab}_f = \text{Keri}_1\) is closed under quotient groups, which is absurd. \(\blacksquare\)

**Remark.** The above argument also shows that there is a right recollement of abelian categories which cannot be extended to a recollement (cf. Theorem 1.22), and that a localizing subcategory is not necessarily colocalizing (cf. Corollary 4.2).

5.3. Serre subcategories of type \((0, -2)\) and \((1, -1)\).

**Lemma 5.3.** Let \(\mathcal{A}\) be a Grothendieck category. Assume that both \((\mathcal{T}, \mathcal{G})\) and \((\mathcal{G}, \mathcal{F})\) are hereditary torsion pairs in \(\mathcal{A}\), such that \(\mathcal{T}\) is not a torsionfree class. Then \(\mathcal{T}\) is a Serre subcategory of type \((0, -2)\), and \(\mathcal{G}\) is a Serre subcategory of type \((1, -1)\).

**Proof.** It is clear that \(\mathcal{T}\) and \(\mathcal{G}\) are Serre subcategories. Since \(\mathcal{A}\) is a Grothendieck category, \(\mathcal{A}\) has enough injective objects. Since \((\mathcal{T}, \mathcal{G})\) is a hereditary torsion pair, it follows from [TO, Thm. 1.8*] that \((\mathcal{T}, \mathcal{G})\) is strongly hereditary. By Lemma 2.1(3) there is a right recollement

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{i_\mathcal{T}} & \mathcal{A} & \xrightarrow{j_\mathcal{T}} & \mathcal{A}/\mathcal{T} \\
\end{array}
\]

with \(\text{Im}j_\mathcal{T} = \mathcal{T}^{\perp_{\leq 1}}\), where \(i_\mathcal{T}\) and \(j_\mathcal{T}\) are respectively the inclusion functor and the quotient functor. We claim \(\mathcal{G} = \mathcal{T}^{\perp_{\geq 1}}\). In fact, \(\mathcal{T}^{\perp_{\leq 1}} \subseteq \mathcal{T}^{\perp_{\geq 0}} = \mathcal{G}\). For each object \(G \in \mathcal{G}\), since \(\mathcal{T}\) is a weakly localizing subcategory, by definition there exists an exact sequence

\[
0 \to T_1 \xrightarrow{a} G \to C \xrightarrow{b} T_2 \to 0
\]

such that \(T_1 \in \mathcal{T}\), \(T_2 \in \mathcal{T}\), and \(C \in \mathcal{T}^{\perp_{\leq 1}}\). But \((\mathcal{T}, \mathcal{G})\) is a torsion pair, \(a = 0\) and \(T_1 = 0\). Since \(\mathcal{G}\) is closed under quotient objects, \(\text{Im}b \in \mathcal{G}\). Since by assumption \(\mathcal{T}\) is closed under subobjects, \(\text{Im}b \in \mathcal{T} \cap \mathcal{G} = \{0\}\). So \(G \cong C \in \mathcal{T}^{\perp_{\leq 1}}\). This proves the claim.
Since \((\mathcal{T}, \mathcal{G})\) is a hereditary and cohereditary torsion pair, it follows from [TO, Thm. 4.1] that \((\mathcal{T}, \mathcal{G})\) is a strongly cohereditary torsion pair. By Lemma 2.1(3) there is a left recollement

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{i_\mathcal{G}} & A \\
\xrightarrow{j_\mathcal{G}} & & \xrightarrow{Q_\mathcal{G}} A/\mathcal{G}
\end{array}
\]

with \(\text{Im}j_\mathcal{G} = \frac{1}{\mathcal{G}}, \) where \(i_\mathcal{G}\) and \(Q_\mathcal{G}\) are respectively the inclusion functor and the quotient functor.

Since we have shown \(\mathcal{G} = \mathcal{T}^{\perp_1}, \) it follows that \(\mathcal{T} \subseteq \frac{1}{\mathcal{G}} \subseteq \frac{1}{\mathcal{T}} = \mathcal{T}. \) Thus \(\mathcal{T} = \frac{1}{\mathcal{G}}. \)

Put \(\tilde{j}_1\) to be the equivalence \(A/\mathcal{G} \to j_1(A/\mathcal{G}) = \mathcal{T}, \) and \(\tilde{j}_{-1}\) to be the equivalence \(A/\mathcal{T} \to j_{-1}(A/\mathcal{T}) = \mathcal{G}. \) We claim the diagram of functors

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{i_\mathcal{T}} & A \\
\xrightarrow{j_\mathcal{T}} & & \xrightarrow{Q_\mathcal{T}} A/\mathcal{T}
\end{array}
\]

is a right recollement. In fact, since \(j_{-1} = i_\mathcal{T} j_{-1}, \) \((Q_\mathcal{T}, i_\mathcal{T} j_{-1})\) is an adjoint pair. By Proposition 3.2(1), for each object \(A \in \mathcal{A}\) there is an exact sequence

\[
0 \to i_\mathcal{G} G \to j_1 Q_\mathcal{G} A \to A \to i_\mathcal{G} i^*_A \to 0
\]

for some \(G \in \mathcal{G}. \) Since \(j_1 Q_\mathcal{G} A \in \mathcal{T}\) and \(\mathcal{T}\) is closed under subobjects, \(i_\mathcal{G} G \in \mathcal{T} \cap \mathcal{G} = \{0\}. \) Thus for \(T \in \mathcal{T}\) we have \(\text{Hom}(T, j_1 Q_\mathcal{G} A) \cong \text{Hom}(T, A). \) This shows that \((i_\mathcal{T}, \tilde{j}_1 Q_\mathcal{G})\) is an adjoint pair. This justifies the claim.

Again by [TO, Thm. 1.8*], \((\mathcal{G}, \mathcal{F})\) is a strongly hereditary torsion pair. By Lemma 2.1(3) there is a right recollement

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{i_\mathcal{G}} & A \\
\xrightarrow{j_\mathcal{G}} & & \xrightarrow{Q_\mathcal{G}} A/\mathcal{G}
\end{array}
\]

Rewrite this we get a diagram of functors

\[
\begin{array}{ccc}
A/\mathcal{G} & \xrightarrow{Q_\mathcal{G}} & A \\
j_{-2} & & i_{-2}
\end{array}
\]

(note that this is not a left recollement, since \(i_{-2} \) and \(j_{-2}\) are not exact). Hence we have a diagram of functors

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{j_\mathcal{T}} & A \\
\xrightarrow{j_{-2} j_{-1}} & & \xrightarrow{i_{-2}} A/\mathcal{T}
\end{array}
\]

Putting (5.1) and (5.2) together we get a diagram of functors

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{j_\mathcal{T}} & A \\
\xrightarrow{j_{-2} j_{-1}} & & \xrightarrow{i_{-2}} A/\mathcal{T}
\end{array}
\]

such that \((i_\mathcal{T}, \tilde{j}_1 Q_\mathcal{G}, j_{-2} j_{-1})\) and \((Q_\mathcal{T}, i_\mathcal{T} j_{-1}, j_{-1} i_{-2})\) are adjoint sequences.

Assume that the type of \(\mathcal{T}\) is not \((0, -2). \) Then there exist either adjoint pairs \((i_1, i_\mathcal{T})\) and \((j_1, Q_\mathcal{T}),\) or adjoint pairs \((j_{-1} j_{-1}^{-1}, i_{-3})\) and \((j_{-1} j_{-1}^{-1}, i_{-2}, j_{-3}). \)

In the first case, by Lemma 2.1(2') we get a left recollement \((\mathcal{T}, \mathcal{A}, \mathcal{A}/\mathcal{T}, i_1, i_\mathcal{T}, j_1, Q_\mathcal{T}),\) and hence \((\text{Ker}i_1, \mathcal{T})\) is a torsion pair, by Proposition 3.2(3). This contradicts the assumption that \(\mathcal{T}\) is not a torsionfree class.
In the second case, all the functors in (5.3) are exact, and hence (5.3) is a recollement $\langle \mathcal{A}/\mathcal{T}, \mathcal{A}, \mathcal{T} \rangle$. By Theorem 1.1 we have $i_T \cong \mathcal{J}_{j-2}^{-1}$ and $Q_T \cong \mathcal{J}_{j-1}^{-1}$, and hence both $i_T$ and $Q_T$ have left adjoints. This goes to the first case.

Thus the type of $\mathcal{T}$ is $(0, 2)$. This also proves the type of $\mathcal{G}$ is $(1, -1)$. □

Example 5.4. Let $K$ be a field, $R := \prod_{i=1}^{\infty} K_i$ and $I := \bigoplus_{i=1}^{\infty} K_i$ with each $K_i = K$. Then $R$ is a commutative ring and $I$ is an idempotent ideal of $R$. Put $G := \{M \in \text{Mod}R \mid MI = 0\}$. Then $G$ is a TTF-class in $\text{Mod}R$, i.e., $G$ is a torsion and torsion-free class, since $G$ is subobjects, quotient objects, extensions, coproducts and products. So we have a TTF-triple $(\mathcal{T}, \mathcal{G}, \mathcal{F})$.

It is clear that $\mathcal{T} = \{M \in \text{Mod}R \mid MI = M\}$. In fact, for any $R$-module $M_1$ with $M_1I = M_1$ and $M_2 \in \mathcal{G}$, we have $\text{Hom}(M_1, M_2) = 0$; and for any $M \in \text{Mod}R$, we have an exact sequence $0 \rightarrow MI \rightarrow M \rightarrow M/MI \rightarrow 0$ with $(M/I)I = MI$ and $M/MI \in \mathcal{G}$.

We claim that $\mathcal{T}$ is closed under subobjects. By [D, Thm. 2.9] this is equivalent to say that $\mathcal{G}$ is closed under taking injective envelopes. Thus, it suffices to prove that for any $M \in \mathcal{G}$, the injective envelope $E(M) = M$ satisfies $E(M)I = 0$. Otherwise, there is an $m \in E(M)$ with $mI \neq 0$. Set $L := \{b \in I \mid mb \neq 0\}$. Then $L \neq 0$. Choosing $b \in L$ such that the number of nonzero components is smallest. We may assume that each nonzero component of $b$ is $1_k$, the identity of $K$. In fact, if the nonzero components of $b$ are exactly $b_{i_1}, \cdots, b_{i_k}$, where $b_i$ is the $i$-th component of $b$, then we use $bb'$ to replace $b$, where the nonzero components of $b'$ are exactly $b_{i_1}^{-1}, \cdots, b_{i_k}^{-1}$ (note that $mbb' \neq 0$; otherwise $mb = mbb' = 0$). By the choice of $b$ we know that $mbI$ is a nonzero submodule of $E(M)$.

Since $E(M)$ is an essential extension of $M$, $mbI \cap M \neq 0$. So there is a nonzero element $r \in I$ with $mbr \neq 0$ and $mbr \in M$. Note that the support of $br \in I$ is contained in the support of $b$ (by definition the support of $b$ is the set of $i$ such that the $i$-th component of $b$ is not zero). By the choice of $b$, the support of $br \in I$ is just the support of $b$. Let $b_{i_1}', \cdots, b_{i_k}'$ be the nonzero components of $br$, and $d \in I$ the element with the nonzero components exactly $b_{i_1}', \cdots, b_{i_k}'$. Then we get the desired contradiction $0 \neq mb = mbrd \in MI = 0$. This proves the claim.

Since $K_i \in \mathcal{T}$ but $R \notin \mathcal{T}$, $\mathcal{T} = \{M \in \text{Mod}R \mid MI = M\}$ is not closed under products. Thus $\mathcal{T}$ is not a torsion-free class. By Lemma 5.3 the type of $\mathcal{T}$ is $(0, 2)$ and the type of $\mathcal{G}$ is $(1, -1)$.

5.4. Serre subcategories of type $(1, -2)$ and $(2, -1)$. Let $R$ and $S$ be rings, $S_M$ a non-zero $S$-$R$-bimodule, and $\Lambda = \left( \begin{smallmatrix} R & 0 \\ 0 & S \end{smallmatrix} \right)$ the triangular matrix ring. A right $\Lambda$-module is identified with a triple $(X, Y, f)$, where $X$ is a right $R$-module, $Y$ a right $S$-module, and $f : Y \otimes_S M \rightarrow X$ a right $R$-map; and a left $\Lambda$-module is identified with a triple $(V^\vee)_g$, where $U$ is a left $R$-module, $V$ a left $S$-module, and $g : M \otimes_R U \rightarrow V$ a left $S$-map ([ARS, p.71]). Put $e_1 = (1, 0)$ and $e_2 = (0, 1)$. It is well-known that there is a ladder of abelian categories (see [CPS, Sect. 2], [PV, 2.10]; also [H, 2.1], [AHKLY, Exam. 3.4])

\[
\begin{array}{ccc}
\text{Mod}R & \xrightarrow{i_0} & \text{Mod}\Lambda & \xrightarrow{j_0} & \text{Mod}S \\
\xrightarrow{i_1} & & \xrightarrow{j_1} & & \\
\xrightarrow{i_2} & & \xrightarrow{j_2} & & \\
\end{array}
\]
i.e., the upper three rows form a recollement of abelian categories, and \((i_{-1}, i_{-2})\) and \((j_{-1}, j_{-2})\) are adjoint pairs, where

\[
\begin{align*}
i_1 &= -\otimes A \Lambda/\Lambda e_2 \Lambda = -\otimes A (\frac{R}{0}) , \\
i_0 &= \text{Hom}_{\Lambda/\Lambda e_2 \Lambda}(\Lambda/\Lambda e_2 \Lambda, -) = \text{Hom}_R(R, -) \cong -\otimes_R R , \\
i_{-1} &= \text{Hom}_A(\Lambda/\Lambda e_2 \Lambda, -) = \text{Hom}_A(e_1 \Lambda, -) \cong -\otimes_A \Lambda e_1 , \\
i_{-2} &= \text{Hom}_{\Lambda/\Lambda e_2 \Lambda}(\Lambda e_1 , -) = \text{Hom}_R((\frac{R}{M}) , -) , \\
j_1 &= -\otimes_{e_2 \Lambda e_2} e_2 \Lambda = -\otimes_S (M, S) , \\
j_0 &= \text{Hom}_A(e_2 \Lambda, -) = \text{Hom}_A((M, S), -) \cong -\otimes_A \Lambda e_2 , \\
j_{-1} &= \text{Hom}_{e_2 \Lambda e_2}(\Lambda e_2, -) = \text{Hom}_S(S, -) \cong -\otimes_S S , \\
j_{-2} &= \text{Hom}_S(S, -) ,
\end{align*}
\]

where the right \(\Lambda\)-module \(R\) is given by \(r(\frac{r'}{m} s) := rr'\) and the right \(\Lambda\)-module \(S\) is given by \(s(\frac{r}{m} s') := ss'\). Note that \(\text{Mod}R\) is a Serre subcategory of \(\text{Mod}\Lambda\) and \(0 \to \text{Mod}R \xrightarrow{i_0} \text{Mod}\Lambda \xrightarrow{j_0} \text{Mod}S \to 0\) is an exact sequence of abelian categories.

We claim that the type of \(\text{Mod}R\) is \((1, -2)\).

In fact, since \(M \neq 0\), \((\frac{R}{0})\) is not flat as a left \(\Lambda\)-module, \(i_1 = -\otimes_A (\frac{R}{0})\) is not exact, and hence \(i_1\) has no left adjoint. Also, since \(M \neq 0\), \(S\) is not projective as a right \(\Lambda\)-module. So \(j_{-2} = \text{Hom}_A(S, -)\) is not exact, and hence \(j_{-2}\) has no right adjoint. This proves the claim. (We include another proof. If both \(i_1\) and \(j_1\) have left adjoints, then \(i_1\) and \(j_1\) are exact, and hence \(i_1 \cong i_{-1}\) by Theorem 11 i.e., \(-\otimes_A (\frac{R}{0}) \cong -\otimes_A (\frac{R}{M})\). But this is not true, since \(M \neq 0\). Similarly, if both \(i_{-2}\) and \(j_{-2}\) have right adjoints, then \(i_{-2}\) and \(j_{-2}\) are exact, and hence \(i_0 \cong i_{-2}\) by Theorem 11 i.e., \(\text{Hom}_R(R, -) \cong \text{Hom}_R((\frac{R}{M}) , -)\), which is absurd.)

The argument above also shows that \(0 \to \text{Mod}S \xrightarrow{j_{-1}} \text{Mod}\Lambda \xrightarrow{i_{-1}} \text{Mod}R \to 0\) is an exact sequence of abelian categories, and that the type of \(\text{Mod}S\) is \((2, -1)\).

**Remark 5.5.** Consider \(\Lambda = T_2(R) := (\frac{R}{0} \quad \frac{0}{R})\), then \(i_{-2} \cong j_1\) and hence we have adjoint sequences

\[
(i_1, i_0, i_{-1}, i_{-2} \cong j_1, j_0, j_{-1}, j_{-2})
\]

such that \(i_1 - \otimes_A (\frac{R}{0})\) has no left adjoint, and \(j_{-2} = \text{Hom}_A(R, -)\) has no right adjoint. Graphically we have

\[
\begin{array}{c}
\text{Mod}R & \xrightarrow{i_1} & \text{Mod} \Lambda & \xrightarrow{j_0} & \text{Mod}R \\
\xrightarrow{i_0} & & \xrightarrow{j_{-1}} & & \\
\xrightarrow{i_{-1}} & & \xrightarrow{j_1} & & \\
\xrightarrow{i_{-2} \cong j_1} & & \xrightarrow{j_{-2}} & & \\
\text{Mod}R & & \xrightarrow{j_{-1}} & & \text{Mod}R
\end{array}
\]

5.5. **Serre subcategories of type \((+\infty, -\infty)\).** Let \(\mathcal{S}\) and \(\mathcal{T}\) be abelian categories. Then as a subcategory of \(\mathcal{S} \oplus \mathcal{T}\), \(\mathcal{S}\) is a Serre subcategory of type \((+\infty, -\infty)\). In fact, it is clear that \((p_1, i_1, p_1)\) and \((i_2, p_2, i_2)\) are adjoint sequences, where \(i_1\) and \(i_2\) are embeddings, and \(p_1\) and \(p_2\) are
projections. On the other hand, if $S$ is a Serre subcategory of $A$ and the type of $S$ is $(+\infty, -\infty)$, then by Theorem 1.1 it is easy to see $A \cong S \oplus (A/S)$.

5.6. **Proof of Theorem 1.3.** Let $B$ be a Serre subcategory of type $(m, -n)$, where $m$ and $n$ are in the set $\{+\infty, 0, 1, 2, \cdots\}$. Denote by $i : B \rightarrow A$ the inclusion functor and $Q : A \rightarrow A/B$ the quotient functor. Put $h := m + n + 1$.

**Claim 1.** If $h \geq 5$, then $(m, -n) = (+\infty, -\infty)$.

Assume that there is a diagram of functors

\[
\begin{array}{ccc}
B & \xrightarrow{i_4} & A \\
\downarrow{i_3} & & \downarrow{j_3} \\
\downarrow{i_2} & & \downarrow{j_2} \\
\downarrow{i_1} & & \downarrow{j_1} \\
\downarrow{i} & & \downarrow{j} \\
A/B & \xrightarrow{Q} & A & \xrightarrow{i} & B
\end{array}
\]

such that $(i_4, i_3, i_2, i, i_1)$ and $(j_4, j_3, j_2, j_1, Q)$ are adjoint sequences. Then $i_1, i_2, i_3, j_1, j_2, j_3$ are exact. Since a left adjoint of $Q$ is fully faithful (the dual of [GL, Prop. 2.2]), $j_1$ is fully faithful. Thus the two rows at the bottom form a left recollement. By Proposition 2.14(iii), $0 \rightarrow A/B \xrightarrow{j} A \xrightarrow{i} B \rightarrow 0$ is an exact sequence of abelian categories and $\text{Ker}i_1 = \text{Im}j_1$. It follows from Lemma 2.1(2') that $i_2$ is fully faithful. Thus

\[
\begin{array}{ccc}
A/B & \xrightarrow{j_2} & A \\
\downarrow{Q} & & \downarrow{i} \\
\downarrow{j_1} & & \downarrow{i_2} \\
A & \xrightarrow{j} & B
\end{array}
\]

is recollement such that $j_2$ and $Q$ are exact. It follows from Theorem 1.1 that $j_2 \cong Q$ and $i_2 \cong i$, and hence the type is $(+\infty, -\infty)$.

Assume that there is a diagram of functors

\[
\begin{array}{ccc}
B & \xrightarrow{i_3} & A \\
\downarrow{i_2} & & \downarrow{j_2} \\
\downarrow{i_1} & & \downarrow{j_1} \\
\downarrow{i} & & \downarrow{j} \\
A/B & \xrightarrow{Q} & A & \xrightarrow{i} & B
\end{array}
\]

such that $(i_3, i_2, i_1, i, i_1)$ and $(j_3, j_2, j_1, Q, j_1)$ are adjoint sequences. Then the four functors $i_1, i_2, j_1, j_2$ are exact. By the dual of [GL, Prop. 2.2], $j_1$ is fully faithful. By the same argument as above we know that the type is $(+\infty, -\infty)$.

Assume that there is a diagram of functors

\[
\begin{array}{ccc}
B & \xrightarrow{i_2} & A \\
\downarrow{i_1} & & \downarrow{j_1} \\
\downarrow{i} & & \downarrow{j} \\
A/B & \xrightarrow{Q} & A & \xrightarrow{i} & B
\end{array}
\]

such that $(i_2, i_1, i, i_1, i_2)$ and $(j_2, j_1, Q, j_1, j_2)$ are adjoint sequences. Then $i_1, i_1, j_1, j_1$ are exact. By [GL, Prop. 2.2] and its dual, $j_1$ and $j_1$ are fully faithful. Thus the three rows in the middle form a recollement such that $i_1$ and $i_1$ are exact. It follows from Theorem 1.1 that $i_1 \cong i_1$ and $j_1 \cong j_1$, and hence the type is $(+\infty, -\infty)$.

Assume that there is a diagram of functors

\[
\begin{array}{ccc}
B & \xrightarrow{i_1} & A \\
\downarrow{i} & & \downarrow{j} \\
\downarrow{i_1} & & \downarrow{j_1} \\
\downarrow{i_2} & & \downarrow{j_2} \\
\downarrow{i_3} & & \downarrow{j_3} \\
\downarrow{i} & & \downarrow{j} \\
A/B & \xrightarrow{Q} & A & \xrightarrow{i} & B
\end{array}
\]

such that $(i_2, i_1, i, i_1, i_2)$ and $(j_2, j_1, Q, j_1, j_2)$ are adjoint sequences. Then $i_1, i_1, j_1, j_1$ are exact. By [GL, Prop. 2.2] and its dual, $j_1$ and $j_1$ are fully faithful. Thus the type is $(+\infty, -\infty)$.
such that \((i_1, i, i_{-1}, i_{-2}, i_{-3})\) and \((j_1, Q, j_{-1}, j_{-2}, j_{-3})\) are adjoint sequences. Then \(i_{-1}, i_{-2}, j_{-1}, j_{-2}\) are exact, and \(j_{-1}\) is fully faithful. So

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{i} & \mathcal{A} \\
\downarrow{i_{-1}} & & \downarrow{j_{-1}} \\
\mathcal{A}/\mathcal{B} & \xrightarrow{Q} & \mathcal{A}/\mathcal{B}
\end{array}
\]

is a right recollement. By Proposition \[8.1.4\](iii), \(0 \to \mathcal{A}/\mathcal{B} \xrightarrow{j_{-1}^{-1}} \mathcal{A} \xrightarrow{i_{-1}} \mathcal{B} \to 0\) is an exact sequence of abelian categories and \(\text{Ker} i_{-1} = \text{Im} j_{-1}\). It follows from Lemma \[2.1.2\] that \(i_{-1}\) is fully faithful. Thus

\[
\begin{array}{ccc}
\mathcal{A}/\mathcal{B} & \xrightarrow{i} & \mathcal{A} \\
\downarrow{j_{-1}^{-1}} & & \downarrow{j_{-1}} \\
\mathcal{B} & \xrightarrow{Q} & \mathcal{A}/\mathcal{B}
\end{array}
\]

is recollement such that \(Q\) and \(j_{-2}\) are exact. It follows from Theorem \[1.1\] that \(Q \cong j_{-2}\) and \(i \cong i_{-2}\), and hence the type is \((+\infty, -\infty)\).

Assume that there is a diagram of functors

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{i} & \mathcal{A} \\
\downarrow{i_{-1}} & & \downarrow{j_{-2}} \\
\mathcal{B} & \xrightarrow{Q} & \mathcal{A}/\mathcal{B}
\end{array}
\]

such that \((i, i_{-1}, i_{-2}, i_{-3}, i_{-4})\) and \((Q, j_{-1}, j_{-2}, j_{-3}, j_{-4})\) are adjoint sequences. Then the six functors \(i_{-1}, i_{-2}, i_{-3}, j_{-1}, j_{-2}, j_{-3}\) are exact. By the same argument as above we know that the type is \((+\infty, -\infty)\).

Up to now we have proved Claim 1. So, from now on we assume that \(h \leq 4\), i.e., \(m + n \leq 3\). Then the type \((m, -n)\) of \(\mathcal{B}\) is in the list

\[(3, 0), (2, -1), (1, -2), (0, -3), (2, 0), (1, -1), (0, -2), (1, 0), (0, -1), (0, 0).\]

Assume that there is a diagram of functors

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{i_{1}} & \mathcal{A} \\
\downarrow{i} & & \downarrow{j_{1}} \\
\mathcal{A}/\mathcal{B} & \xrightarrow{Q} & \mathcal{A}/\mathcal{B}
\end{array}
\]

such that \((i_1, i)\) and \((j_1, Q)\) are adjoint pairs. Then \(j_1\) is fully faithful, by the dual of [GL, Prop. 2.2]. So it is a left recollement, and hence by Theorem \[1.12\] it can be extended to be recollement. This shows that the type of \(\mathcal{B}\) is not in the set \\{(3, 0), (2, 0), (1, 0)\}.

Claim 2. The type of \(\mathcal{B}\) can not be \((0, -3)\).

Otherwise, there is a diagram of functors

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{i} & \mathcal{A} \\
\downarrow{i_{-1}} & & \downarrow{j_{3}} \\
\mathcal{A}/\mathcal{B} & \xrightarrow{Q} & \mathcal{A}/\mathcal{B}
\end{array}
\]

such that \((i, i_{-1}, i_{-2}, i_{-3})\) and \((Q, j_{-1}, j_{-2}, j_{-3})\) are adjoint sequences. Then \(i_{-1}, i_{-2}, j_{-1}, j_{-2}\) are exact, and \(j_{-1}\) is fully faithful. So, the upper two rows form a right recollement. Thus, by Proposition \[8.1.4\](iii), \(0 \to \mathcal{A}/\mathcal{B} \xrightarrow{j_{-1}^{-1}} \mathcal{A} \xrightarrow{i_{-1}} \mathcal{B} \to 0\) is an exact sequence of abelian categories, and hence \(i_{-2}\) is fully faithful, by Lemma \[2.1.2\]. So the upper three rows form a recollement \((\mathcal{A}/\mathcal{B}, \mathcal{A}, \mathcal{B}, Q, j_{-1}, j_{-2}, i, i_{-1}, i_{-2})\), and then \(i \cong i_{-2}\) and \(Q \cong j_{-2}\) by Theorem \[1.1\] Thus the type of \(\mathcal{B}\) is \((+\infty, -\infty)\), which is absurd.
It remains to prove that for each \((m, -n)\) in the list
\[ (+\infty, -\infty), (2, -1), (1, -2), (1, -1), (0, -2), (0, -1), (0, 0) \]
there exists a Serre subcategory of \(\mathcal{A}\) such that its type is \((m, -n)\). This is true by Subsections 5.1-5.5. This completes the proof of Theorem 1.3.

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