AN EXPLICIT FORMULA FOR THE NATURAL AND
CONFORMALLY INVARIANT QUANTIZATION

F. RADOUX

Abstract. In [5], P. Lecomte conjectured the existence of a natural
and conformally invariant quantization. In [7], we gave a proof of this
theorem thanks to the theory of Cartan connections. In this paper,
we give an explicit formula for the natural and conformally invariant
quantization of trace-free symbols thanks to the method used in [7] and
to tools already used in [8] in the projective setting. This formula is
extremely similar to the one giving the natural and projectively invariant
quantization in [8].

1. Introduction

A quantization can be defined as a linear bijection from the space \( \mathcal{S}(M) \) of
symmetric contravariant tensor fields on a manifold \( M \) (also called the space
of Symbols) to the space \( \mathcal{D}_{1/2}(M) \) of differential operators acting between
half-densities.

It is known that there is no natural quantization procedure. In other
words, the spaces of symbols and of differential operators are not isomorphic
as representations of \( \text{Diff}(M) \).

The idea of equivariant quantization, introduced by P. Lecomte and V.
Ovsienko in [9] is to reduce the group of local diffeomorphisms in the fol-
lowing way: if a Lie group \( G \) acts (locally) on a manifold \( M \), the action
can be lifted to tensor fields and to differential operators and symbols. A
\( G \)-equivariant quantization is then a quantization that exchanges the actions
of \( G \) on symbols and differential operators.

In [2], the authors considered the group \( SO(p + 1, q + 1) \) acting on the
space \( \mathbb{R}^{p+q} \) or on a manifold endowed with a flat pseudo-conformal structure
of signature \( (p, q) \). They showed the existence and uniqueness of a \( SO(p + 1, q + 1) \)-equivariant quantization in non-critical situations.

The problem of the \( so(p + 1, q + 1) \)-equivariant quantization on \( \mathbb{R}^m \) has a
counterpart on an arbitrary manifold \( M \). In [5], P. Lecomte conjectured the
existence of a natural and conformally invariant quantization, i.e. a quanti-
ization procedure depending on a pseudo-riemannian metric, that would be
natural (in all arguments) and that would be left invariant by a conformal change of metric.

We proved in [7] the existence of such a quantization using Cartan connections theory.

The goal of this paper is to obtain an explicit formula on $M$ for the natural and conformally invariant quantization of trace-free symbols. This task can be realized using tools exposed in [7] and [1].

The paper is organized as follows. In the first section, we recall briefly the notions exposed in [7] necessary to understand the article. In the second part, we calculate the explicit formula giving the natural and conformally invariant quantization for trace-free symbols on the Cartan fiber bundle using the method exposed in [7]. In the third section, we develop as in [8] this formula in terms of natural operators on the base manifold $M$, using tools explained in [1], in order to obtain the announced explicit formula. It constitutes the generalization to any weight of density of the formula given by M. Eastwood in [3] thanks to a completely different method. Moreover, Eastwood formula is given under a different form.

2. Fundamental tools

Throughout this work, we let $M$ be a smooth manifold of dimension $m \geq 3$.

2.1. Tensor densities. Denote by $\Delta^\lambda(\mathbb{R}^m)$ the one dimensional representation of $GL(m, \mathbb{R})$ given by

$$
\rho(A)e = |\det A|^{-\lambda}e, \quad \forall A \in GL(m, \mathbb{R}), \forall e \in \Delta^\lambda(\mathbb{R}^m).
$$

The vector bundle of $\lambda$-densities is then defined by

$$
P^1M \times_{\rho} \Delta^\lambda(\mathbb{R}^m) \to M,
$$

where $P^1M$ is the linear frame bundle of $M$.

Recall that the space $\mathcal{F}_\lambda(M)$ of smooth sections of this bundle, the space of $\lambda$-densities, can be identified with the space $C^\infty(P^1M, \Delta^\lambda(\mathbb{R}^m))_{GL(m, \mathbb{R})}$ of functions $f$ such that

$$
f(uA) = \rho(A^{-1})f(u) \quad \forall u \in P^1M, \forall A \in GL(m, \mathbb{R}).
$$

2.2. Differential operators and symbols. As usual, we denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$.

The space $\mathcal{D}_{\lambda,\mu}$ is filtered by the order of differential operators. The space of symbols is then the associated graded space of $\mathcal{D}_{\lambda,\mu}$. It is also known that the principal operators $\sigma_l$ ($l \in \mathbb{N}$) allow to identify the space of symbols with the space of contravariant symmetric tensor fields with coefficients in $\delta$-densities where $\delta = \mu - \lambda$ is the shift value.

More precisely, we denote by $S^l_\delta(\mathbb{R}^m)$ or simply $S^l_\delta$ the space $S^l\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m)$ endowed with the natural representation $\rho$ of $GL(m, \mathbb{R})$. Then the
vector bundle of symbols of degree \( l \) is
\[
P^1 M \times_\rho S^l_\delta(\mathbb{R}^m) \to M.
\]
The space \( S^l_\delta(M) \) of symbols of degree \( l \) is then the space of smooth sections of this bundle, which can be identified with \( C^\infty(P^1 M, S^l_\delta(\mathbb{R}^m))_{GL(m,\mathbb{R})} \).

Finally, the whole space of symbols is
\[
S_\delta(M) = \bigoplus_{l=0}^{\infty} S^l_\delta(M),
\]
endowed with the classical actions of diffeomorphisms and of vector fields.

2.3. Natural and equivariant quantizations. A quantization on \( M \) is a linear bijection \( Q_M \) from the space of symbols \( S_\delta(M) \) to the space of differential operators \( D_{\lambda,\mu}(M) \) such that
\[
\sigma_l(Q_M(S)) = S, \quad \forall S \in S^l_\delta(M), \quad \forall l \in \mathbb{N},
\]
where \( \sigma_l \) is the principal symbol operator on the space of operators of order less or equal to \( l \).

In the conformal sense, a natural quantization is a collection of quantizations \( Q_M \) depending on a pseudo-Riemannian metric such that
- For all pseudo-Riemannian metric \( g \) on \( M \), \( Q_M(g) \) is a quantization,
- If \( \phi \) is a local diffeomorphism from \( M \) to \( N \), then one has
\[
Q_M(\phi^* g)(\phi^* S) = \phi^*(Q_N(g)(S)),
\]
for all pseudo-Riemannian metrics \( g \) on \( N \), and all \( S \in S_\delta(N) \).

Recall now that two pseudo Riemannian metrics \( g \) and \( g' \) on a manifold \( M \) are conformally equivalent if and only if there exists a positive function \( f \) such that \( g' = fg \).

A quantization \( Q_M \) is then conformally equivariant if one has \( Q_M(g) = Q_M(g') \) whenever \( g \) and \( g' \) are conformally equivalent.

2.4. Conformal group and conformal algebra. These tools were presented in detail in [7, Section 3]. We give here the most important ones for this paper to be self-contained.

Given \( p \) and \( q \) such that \( p + q = m \), we consider the conformal group \( G = SO(p + 1, q + 1) \) and its following subgroup \( H \):
\[
H = \left\{ \begin{pmatrix} a^{-1} & 0 & 0 \\ a^{-1}A \xi^2 & A & 0 \\ \frac{1}{\mathbb{R}^m|\xi|^2} & \xi & a \end{pmatrix} : A \in O(p, q), a \in \mathbb{R}_0, \xi \in \mathbb{R}^{m*} \right\}/\{\pm I_{m+2}\}.
\]
The subgroup \( H \) is a semi-direct product \( G_0 \ltimes G_1 \). Here \( G_0 \) is isomorphic to \( CO(p, q) \) and \( G_1 \) is isomorphic to \( \mathbb{R}^{m*} \).

The Lie algebra of \( G \) is \( \mathfrak{g} = so(p + 1, q + 1) \). It decomposes as a direct sum of subalgebras:
\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1
\]
(1)
where \( \mathfrak{g}_{-1} \cong \mathbb{R}^m \), \( \mathfrak{g}_0 \cong co(p, q) \), and \( \mathfrak{g}_1 \cong \mathbb{R}^{m*} \).
This correspondence induces a structure of Lie algebra on $\mathbb{R}^m \oplus \text{co}(p,q) \oplus \mathbb{R}^{m\ast}$. It is easy to see that the adjoint actions of $G_0$ and of $\text{co}(p,q)$ on $\mathfrak{g}_{-1} = \mathbb{R}^m$ and on $\mathfrak{g}_1 = \mathbb{R}^{m\ast}$ coincides with the natural actions of $CO(p,q)$ and of $\text{co}(p,q)$. It is interesting for the sequel to note that:

$$[v, \xi] = v \otimes \xi + \xi^\sharp(v)I_m - \xi^\flat \otimes v^\flat,$$

if $v \in \mathfrak{g}_{-1}$, $\xi \in \mathfrak{g}_1$ and if $I_m$ denotes the identity matrix of dimension $m$. The applications $\flat$ and $\sharp$ represent the classical isomorphisms between $\mathbb{R}^m$ and $\mathbb{R}^{m\ast}$ detailed in [7].

The Lie algebras corresponding to $G_0$, $G_1$ and $H$ are respectively $\mathfrak{g}_0$, $\mathfrak{g}_1$, and $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

2.5. Cartan fiber bundles. It is well-known that there is a bijective and natural correspondence between the conformal structures on $M$ and the reductions of $P^1M$ to the structure group $G_0 \cong CO(p,q)$. The representations $(V, \rho)$ of $GL(m, \mathbb{R})$ defined so far can be restricted to the group $CO(p,q)$. Therefore, once a conformal structure is given, i.e. a reduction $P_0$ of $P^1M$ to $G_0$, we can identify tensors fields of type $V$ as $G_0$-equivariant functions on $P_0$.

In [4], one shows that it is possible to associate at each $G_0$-structure $P_0$ a principal $H$-bundle $P$ on $M$, this association being natural and obviously conformally invariant. Since $H$ can be considered as a subgroup of $G_2^{m\ast}$, this $H$-bundle can be considered as a reduction of $P^2M$. The relationship between conformal structures and reductions of $P^2M$ to $H$ is given by the following proposition.

**Proposition 1.** There is a natural one-to-one correspondence between the conformal equivalence classes of pseudo-Riemannian metrics on $M$ and the reductions of $P^2M$ to $H$.

Throughout this work, we will freely identify conformal structures and reductions of $P^2M$ to $H$.

2.6. Cartan connections. Let $L$ be a Lie group and $L_0$ a closed subgroup. Denote by $\mathfrak{l}$ and $\mathfrak{l}_0$ the corresponding Lie algebras. Let $N \to M$ be a principal $L_0$-bundle over $M$, such that $\dim M = \dim L/L_0$. A Cartan connection on $N$ is an $\mathfrak{l}$-valued one-form $\omega$ on $N$ such that

1. If $R_a$ denotes the right action of $a \in L_0$ on $N$, then $R_a^\ast \omega = \text{Ad}(a^{-1})\omega$,
2. If $k^\ast$ is the vertical vector field associated to $k \in \mathfrak{l}_0$, then $\omega(k^\ast) = k$,
3. $\forall u \in N$, $\omega_u : T_u N \to \mathfrak{l}$ is a linear bijection.

When considering in this definition a principal $H$-bundle $P$, and taking as group $L$ the group $G$ and for $L_0$ the group $H$, we obtain the definition of Cartan conformal connections.

If $\omega$ is a Cartan connection defined on an $H$-principal bundle $P$, then its curvature $\Omega$ is defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

(2)
The notion of Normal Cartan connection is defined by natural conditions imposed on the components of the curvature.

Now, the following result ([4, p. 135]) gives the relationship between conformal structures and Cartan connections:

**Proposition 2.** A unique normal Cartan conformal connection is associated to every conformal structure $P$. This association is natural.

The connection associated to a conformal structure $P$ is called the normal conformal connection of the conformal structure.

2.7. Lift of equivariant functions. In a previous subsection, we recalled how to associate an $H$-principal bundle $P$ to a conformal structure $P_0$. We now recall how the densities and the symbols can be regarded as equivariant functions on $P$.

If $(V, \rho)$ is a representation of $G_0$, then we may extend it to a representation $(V, \rho')$ of $H$ (see [7]). Now, using the representation $\rho'$, we can recall the relationship between equivariant functions on $P_0$ and equivariant functions on $P$ (see [1]): if we denote by $p$ the projection $P \to P_0$, we have

**Proposition 3.** If $(V, \rho)$ is a representation of $G_0$, then the map
\[ p^*: C^\infty(P_0, V) \to C^\infty(P, V) : f \mapsto f \circ p \]
defines a bijection from $C^\infty(P_0, V)_{G_0}$ to $C^\infty(P, V)_H$.

As we continue, we will use the representation $\rho'_e$ of the Lie algebra of $H$ on $V$. If we recall that this algebra is isomorphic to $g_0 \oplus g_1$, then we have
\[ \rho'_e(A, \xi) = \rho_e(A), \quad \forall A \in g_0, \xi \in g_1. \quad (3) \]

2.8. The application $Q_\omega$. The construction of the application $Q_\omega$ is based on the concept of invariant differentiation developed in [1]. Let us recall the definition:

**Definition 1.** If $f \in C^\infty(P, V)$ then $(\nabla^\omega)^k f \in C^\infty(P, \otimes^k \mathbb{R}^m \otimes V)$ is defined by
\[ (\nabla^\omega)^k f(u)(X_1, \ldots, X_k) = L_{\omega^{-1}(X_1)} \circ \cdots \circ L_{\omega^{-1}(X_k)} f(u) \]
for $X_1, \ldots, X_k \in \mathbb{R}^m$.

**Definition 2.** The map $Q_\omega$ is defined by its restrictions to $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$, $(k \in \mathbb{N})$ : we set
\[ Q_\omega(T)(f) = \langle T, (\nabla^\omega)^k f \rangle, \quad (4) \]
for all $T \in C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ and $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^m))$.

Explicitly, when the symbol $T$ writes $tA \otimes h_1 \otimes \cdots \otimes h_k$ for $t \in C^\infty(P)$, $A \in \Delta^\delta(\mathbb{R}^m)$ and $h_1, \ldots, h_k \in \mathbb{R}^m \cong \mathfrak{g}_{-1}$ then one has
\[ Q_\omega(T)f = tA \circ L_{\omega^{-1}(h_1)} \circ \cdots \circ L_{\omega^{-1}(h_k)} f, \]
where $t$ is considered as a multiplication operator.
2.9. The map $\gamma$.

Definition 3. We define $\gamma$ on $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ by
\[
\gamma(h)(x_1 \otimes \cdots \otimes x_k \otimes A) = \lambda \sum_{i=1}^k \text{tr}([h, x_i]) x_1 \otimes \cdots \otimes (i) \cdots \otimes x_k \otimes A
+ \sum_{i=1}^k \sum_{j>i} x_1 \otimes \cdots \otimes ([h, x_i], x_j) \otimes \cdots \otimes x_k \otimes A.
\]
for every $x_1, \ldots, x_k \in \mathfrak{g}_{-1}$, $A \in \Delta^\delta(\mathbb{R}^m)$ and $h \in \mathfrak{g}_1$. Then we extend it to $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ by $C^\infty(P)$-linearity.

Definition 4. A trace-free symbol $S$ is a symbol such that $i(g)S = 0$ if $g$ is a metric belonging to the conformal structure $P$.

If $S$ is an equivariant function representing a trace-free symbol, $i(g_0)S = 0$ if $g_0$ represents the canonical metric on $\mathbb{R}^m$ corresponding to the conformal structure $P$ (see [7], section 3). It is then easy to show that

Proposition 4. If $S$ is a trace-free symbol of degree $k$, $\gamma(h)S = -k(\lambda m + k - 1)i(h)S$. In particular, $\gamma(h)S$ is trace-free.

2.10. Casimir-like operators. Recall that we can define an operator called the Casimir operator $C^\circ$ on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ (see [7]). This operator $C^\circ$ is semi-simple. The vector space $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ can be decomposed as an $o(p, q)$-representation into irreducible components (since $o(p, q)$ is semi-simple):
\[
\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m) = \bigoplus_{s=1}^{n_k} I_{k, s}.
\]
The restriction of $C^\circ$ to $C^\infty(P, I_{k, s})$ is then a scalar multiple of the identity.

We defined in [7] two other operators. If we denote respectively by $(e_1, \ldots, e_m)$ and $(e^1, \ldots, e^m)$ a basis of $\mathfrak{g}_{-1}$ and a basis of $\mathfrak{g}_1$ which are dual with respect to the Killing form of $so(p+1, q+1)$, then

Definition 5. The operator $N^\omega$ is defined on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ by
\[
N^\omega = -2 \sum_{i=1}^m \gamma(e^i)L_{\omega^{-1}}(e_i),
\]
and we set
\[
C^\omega := C^\circ + N^\omega.
\]

2.11. Construction of the quantization. Recall that $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ is decomposed as a representation of $o(p, q)$ as the direct sum of irreducible components $I_{k, s}$ with $0 \leq s \leq \frac{k}{2}$ (see [2]). Remark that if $S$ is a trace-free symbol of degree $k$, then $S \in I_{k, 0}$. Denote by $E_{k, s}$ the space $C^\infty(P, I_{k, s})$ and by $\alpha_{k, s}$ the eigenvalue of $C^\circ$ restricted to $E_{k, s}$.

The tree-like subspace $T_\gamma(I_{k, s})$ associated to $I_{k, s}$ is defined by
\[
T_\gamma(I_{k, s}) = \bigoplus_{l \in \mathbb{N}} T_\gamma^l(I_{k, s}),
\]
where $T_t^0(I_{k,s}) = I_{k,s}$ and $T_t^{l+1}(I_{k,s}) = \gamma(g_1)(T_t^l(I_{k,s}))$, for all $l \in \mathbb{N}$. The space $T_t^l(E_{k,s})$ is then defined in the same way. Since $\gamma$ is $C^\infty(P)$-linear, this space is equal to $C^\infty(P, T_t^l(I_{k,s}))$.

**Definition 6.** A value of $\delta$ is critical if there exists $k, s$ such that the eigenvalue $\alpha_{k,s}$ corresponding to an irreducible component $I_{k,s}$ of $\otimes^k g_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ belongs to the spectrum of the restriction of $C^\delta$ to $\bigoplus_{l \geq 1} T_t^l(E_{k,s})$.

Recall now the following result:

**Theorem 5.** If $\delta$ is not critical, for every $T$ in $C^\infty(P, I_{k,s})$, (where $I_{k,s}$ is an irreducible component of $\otimes^k g_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$) there exists a unique function $\hat{T}$ in $C^\infty(P, T_t^l(I_{k,s}))$ such that

$$
\begin{aligned}
\hat{T} &= T_k + \cdots + T_0, \\
C^\omega(\hat{T}) &= \alpha_{k,s} T.
\end{aligned}
$$

(5)

This result allows to define the main ingredient in order to define the quantization: The “modification map”, acting on symbols.

**Definition 7.** Suppose that $\delta$ is not critical. Then the map

$$
R : \bigoplus_{k=0}^\infty C^\infty(P, S_k^\delta(M)) \rightarrow \bigoplus_{k=0}^\infty C^\infty(P, \otimes^k g_{-1} \otimes \Delta^\delta(\mathbb{R}^m))
$$

is the linear extension of the association $T \mapsto \hat{T}$.

And finally, the main result:

**Theorem 6.** If $\delta$ is not critical, then the formula

$$
Q_M : (g, T) \mapsto Q_M(g, T)(f) = (p^*)^{-1}[Q_{\omega}(R(p^*T))(p^* f)],
$$

(6)

(where $Q_{\omega}$ is given by (4)) defines a natural and conformally invariant quantization.

3. The First Explicit Formula

Define now the numbers $\gamma_{2k-l}$:

$$
\gamma_{2k-l} = \frac{m + 2k - l - m\delta}{m}.
$$

We will say that a value of $\delta$ is critical if there are $k, l \in \mathbb{N}$ such that $2 \leq l \leq k + 1$ and $\gamma_{2k-l} = 0$.

We can then give the formula giving the natural and conformally invariant quantization in terms of the normal Cartan connection for the trace-free symbols (see [8] for the definitions of $\nabla^\omega_s$ and $\text{Div}^\omega_s$):

**Theorem 7.** If $\delta$ is not critical, then the collection of maps

$$
Q_M : S^2 T^* M \times S_k^\delta(M) \rightarrow \mathcal{D}_{\lambda, \mu}(M)
$$

defined by

$$
Q_M(g, S)(f) = p^*^{-1} \left( \sum_{l=0}^k C_{k,l}(\text{Div}^\omega_s p^* S, \nabla^\omega_{\delta, k-1} p^* f) \right)
$$

(6)
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defines a conformally invariant natural quantization for the trace-free symbols if

\[ C_{k,l} = \frac{(\lambda + \frac{k-1}{m}) \cdots (\lambda + \frac{k-1}{m})}{\gamma_{2k-2} \cdots \gamma_{2k-l-1}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1. \]

**Proof.** Thanks to Theorem 5 to the definition of \( N_\omega \) and to Proposition 4, one has

\[ S_l = 2 \sum_{i=1}^{m} \gamma(\epsilon_i) L_{\omega^{-1}}(e_i) S_{l+1}, \quad 0 \leq l \leq k - 1. \]

One concludes using Proposition 4 and the fact that (see [2]):

\[ \alpha_{k,0} = 2k(1 - k + m(\delta - 1)) - m^2 \delta(\delta - 1). \]

Indeed, if \((e_1, \ldots, e_m)\) and \((\epsilon^1, \ldots, \epsilon^m)\) denote respectively the canonical bases of \( \mathbb{R}^m \) and \( \mathbb{R}^m^* \), \((e_1, \ldots, e_m)\) and \((-\epsilon^1, \ldots, -\epsilon^m)\) are Killing-dual with respect to the Killing form given in [2]. One applies eventually Theorem 6. \( \square \)

4. THE SECOND EXPLICIT FORMULA

In order to obtain an explicit formula for the quantization, we need to know the developments of the operators \( \nabla^\omega_{l} \) and \( \text{Div}^\omega_{l} \) in terms of operators on \( M \).

Let \( \gamma \) be a connection on \( P_0 \) corresponding to a covariant derivative \( \nabla \) and belonging to the underlying structure of a conformal structure \( P \). Recall that \( \gamma \) is the Levi-Civita connection of a metric belonging to \( P \). We denote by \( \tau \) the corresponding function on \( P \) with values in \( g_1 \), by \( \Gamma \) the corresponding deformation tensor (see [1]) and by \( \omega \) the normal Cartan connection on \( P \).

Let \((V, \rho)\) be a representation of \( G_0 \) inducing a representation \((\mathfrak{g}^*, \rho^*)\) of \( g_0 \). If we denote by \( \rho^*_s \) the canonical representation on \( \otimes^l \mathfrak{g}^*_1 \otimes V \) and if \( s \in C^\infty(P_0, V), G_0 \), then the development of \( \nabla^\omega_{l}(p^*s)(X_1, \ldots, X_l) \) is obtained inductively as follows (see [1, 3]):

\[
\begin{align*}
\nabla^\omega_{l}(p^*s)(X_1, \ldots, X_l) &= \rho^*_s((X_l, \tau))(\nabla^\omega_{l-1}(p^*s))(X_1, \ldots, X_{l-1}) \\
+ S_\tau(\nabla^\omega_{l-1}(p^*s))(X_1, \ldots, X_{l-1}) \\
+ S_\nabla(\nabla^\omega_{l-1}(p^*s))(X_1, \ldots, X_{l-1}) \\
+ S_\Gamma(\nabla^\omega_{l-1}(p^*s))(X_1, \ldots, X_{l-1}).
\end{align*}
\]

Recall that \( S_\tau \) replaces successively each \( \tau \) by \(-\frac{1}{2}[\tau, [\tau, X_l]]\), that \( S_\nabla \) adds successively a covariant derivative on the covariant derivatives of \( \Gamma \) and \( s \) and that \( S_\Gamma \) replaces successively each \( \tau \) by \( \Gamma.X_l \).

Recall too that \( \Gamma \) is equal in the conformal case to (see [1]):

\[
-\frac{1}{m-2}(\text{Ric} - \frac{g_0}{2} \frac{\text{R}}{(m-1)}),
\]
where Ric and R denote the equivariant functions on P representing respectively the Ricci tensor and the scalar curvature of the connection $\gamma$.

**Proposition 8.** If $f \in C^\infty(P_0, \Delta^\lambda(\mathbb{R}^m))_{G_0}$, then $\nabla^\omega^j(p^*f)(X,\ldots,X)$ is equal to $g_0(X,X)T(X,\ldots,X)$, where $T \in C^\infty(P, \otimes^j \mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$, plus a linear combination of terms of the form

$$(\otimes^{n-1} \otimes p^* (\otimes^{n-2} \nabla^l \Gamma \otimes \ldots \otimes \otimes \nabla^q f))(X,\ldots,X).$$

If we denote by $T(n_{-1},\ldots,n_{l-2},q)$ such a term, then $\nabla^\omega^j(p^*f)(X,\ldots,X)$ is equal to the corresponding linear combination of the following sums

$$(-\lambda m - 2l + n_{-1})T(n_{-1} + 1,\ldots,n_{l-2},q) + T(n_{-1},\ldots,n_{l-2},q + 1)$$

$$+ \sum_{j=1}^{l-2} n_j T(n_{-1},\ldots,n_j - 1,n_{j+1} + 1,\ldots,n_{l-2},q)$$

plus $g_0(X,X)T'(X,\ldots,X)$, where $T' \in C^\infty(P, \otimes^{l-1} \mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$.

**Proof.** The proof is similar to the proof of Proposition 7 in [8].

One deduces easily from Proposition 8 the following corollary (see [8] for the definition of $\nabla_s$):

**Proposition 9.** If $f \in C^\infty(P_0, \Delta^\lambda(\mathbb{R}^m))_{G_0}$, then $\nabla^j_s(p^*f)$ is equal to $g_0 \nabla T$, where $T \in C^\infty(P, S^{l-2} \mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$, plus a linear combination of terms of the form

$$(\tau^{n-1} \nabla^l \Gamma \otimes \ldots \otimes \nabla^q f)).$$

If we denote by $T(n_{-1},\ldots,n_{l-2},q)$ such a term, then $\nabla^\omega^j_s(p^*f)$ is equal to the corresponding linear combination of the following sums

$$(-\lambda m - 2l + n_{-1})T(n_{-1} + 1,\ldots,n_{l-2},q) + T(n_{-1},\ldots,n_{l-2},q + 1)$$

$$+ \sum_{j=1}^{l-2} n_j T(n_{-1},\ldots,n_j - 1,n_{j+1} + 1,\ldots,n_{l-2},q)$$

plus $g_0 \nabla T'$, where $T' \in C^\infty(P, S^{l-1} \mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$.

**Proof.** The proof is similar to the proof of Proposition 8 in [8].

Remark that the action of the algorithm on the generic term of the part of the development of $\nabla^j_s(p^*f)$ that does not contain factors $g_0$ can be summarized. Indeed, this action gives first

$$(-\lambda m - 2l + n_{-1})T(n_{-1} + 1,\ldots,n_{l-2},q).$$

It gives next

$$n_{-1} T(n_{-1} - 1,n_0 + 1,\ldots,n_{l-2},q).$$

Finally, it makes act the covariant derivative $\nabla_s$ on

$$(\nabla^{l-2} \Gamma \otimes \ldots \otimes \nabla^q f).$$
From now, we will denote by $r$ the following multiple of the tensor $\text{Ric}$ (recall that $\text{Ric}$ is symmetric for a metric connection):

$$r := \frac{1}{(2 - m)} \text{Ric}.$$ 

In the following proposition, $\text{Div}$ denotes the divergence operator:

**Proposition 10.** If $S \in C^\infty(P_0, \Delta^k \delta \otimes S^k \delta)_{G_0}$ is trace-free, then $\text{Div}^\omega(p^* S)$ is a linear combination of terms of the form

$$\langle r^{n-1} \vee p^*((\nabla_s^{k-2} r)^{n_k-2} \vee \ldots \vee r^{n_0}), p^*(\text{Div}^q S) \rangle.$$

If we denote by $T(n_{-1}, \ldots, n_{l-2}, q)$ such a term, then $\text{Div}^\omega T(n_{-1}, \ldots, n_{l-2}, q)$ is equal to

$$(\gamma_{2(k-l)-2m} + n_{-1})T(n_{-1} + 1, \ldots, n_{l-2}, q) + T(n_{-1}, \ldots, n_{l-2}, q + 1)$$

$$+ \sum_{j=-1}^{l-2} n_j T(n_{-1}, \ldots, n_{j-1}, n_{j+1} + 1, \ldots, n_{l-2}, q).$$

**Proof.** The proof is exactly similar to the one of Proposition 9 in [8], using the fact that $S$ and its divergences are trace-free. □

Remark that the action of the algorithm on the generic term of the development of $\text{Div}^\omega(p^* S)$ can be summarized. Indeed, this action gives first

$$(\gamma_{2(k-l)-2m} + n_{-1})T(n_{-1} + 1, \ldots, n_{l-2}, q).$$

It gives next

$$n_{-1} T(n_{-1} - 1, n_0 + 1, \ldots, n_{l-2}, q).$$

Finally, it makes act the divergence $\text{Div}$ on

$$\langle \langle (\nabla_s^{k-2} r)^{n_k-2} \vee \ldots \vee r^{n_0}, \text{Div}^q S \rangle, p^*(\nabla_s^l f) \rangle.$$ 

Because of the previous propositions, the quantization can be written as a linear combination of terms of the form

$$\langle r^{n-1} \vee p^*((\nabla_s^{k-2} r)^{n_k-2} \vee \ldots \vee r^{n_0}), p^*(\text{Div}^q S) \rangle, p^*(\nabla_s^l f) \rangle.$$

In this expression, recall that it suffices to consider the terms for which $n_{-1} = 0$ (see [8]).

In the sequel, we will need two operators that we will call $T_1$ and $T_2$.

If $T$ is a tensor of type $egin{pmatrix} 0 \\ j \end{pmatrix}$ with values in the $\lambda$-densities, then

$$T_1 T = (-\lambda m - j)(j + 1) \Gamma \vee T.$$ 

If $S$ is a trace-free symbol of degree $j$, then

$$T_2 S = (m\gamma_{k-2} - k + j)(k - j + 1)i(r) S.$$ 

The following results give the explicit developments of $\nabla^\omega_s(p^* f)$ and of $\text{Div}^\omega(p^* S)$:
Proposition 11. The term of degree $t$ in $\tau$ in the part of the development of $\nabla\omega^d(p^*f)$ that does not contain factors $g_0$ is equal to

$$
\left(\begin{array}{l}l \\ t \end{array}\right) \prod_{j=1}^{t} (-\lambda m - l + j)p^*(\pi_{t-l}(\sum_{j=0}^{l-t}(\nabla_s + T_1)^j)f),
$$

where $\pi_{t-l}$ denotes the projection on the operators of degree $l-t$ (the degree of $\nabla_s$ is 1 whereas the degree of $T_1$ is 2). We set $\prod_{j=1}^{t} (-\lambda m - l + j)$ equal to 1 if $t = 0$.

Proof. The proof is exactly similar to the one of Proposition 10 in [8]. □

Proposition 12. If $S$ is trace-free, the term of degree $t$ in $\tau$ in the development of $\text{Div}\omega^d(p^*S)$ is equal to

$$
\left(\begin{array}{l}l \\ t \end{array}\right) \prod_{j=1}^{t} (\gamma_{2k-2m} - l + j)p^*(\pi_{t-l}(\sum_{j=0}^{l-t}(\text{Div} + T_2)^j)S),
$$

where $\pi_{t-l}$ denotes the projection on the operators of degree $t-l$ (the degree of $\text{Div}$ is $-1$ whereas the degree of $T_2$ is $-2$). We set the product $\prod_{j=1}^{t} (\gamma_{2k-2m} - l + j)$ equal to 1 if $t = 0$.

Proof. The proof is completely similar to the one of Proposition 11 in [8]. □

We can now write the explicit formula giving the natural and conformally invariant quantization for the trace-free symbols:

Theorem 13. The quantization $Q_M$ for the trace-free symbols is given by the following formula:

$$
Q_M(g, S)(f) = \sum_{l=0}^{k} C_{k,l}(\pi_l(\sum_{j=0}^{l}(\text{Div} + T_2)^j)S, \pi_{k-l}(\sum_{j=0}^{k-l}(\nabla_s + T_1)^j)f).
$$

Remark that as $S$ and its divergences are trace-free, one can replace in the definition of the operators $T_1$ the deformation tensor $\Gamma$ by $r$. One can easily derive from this formula the formula at the third order. Indeed, if we denote by $D, T, \partial T$ the operators $\nabla_s, r \nabla$ and $(\nabla_s r) \nabla$ (resp. $\text{Div}, i(r)$ and $i(\nabla_s r)$) and if we denote by $\beta$ the number $-\lambda m$ (resp. $\gamma_4 m$), one obtains:

$$
\pi_1(\sum_{j=0}^{1}(D + T)^j) = D, \quad \pi_2(\sum_{j=0}^{2}(D + T)^j) = D^2 + \beta T,
$$

$$
\pi_3(\sum_{j=0}^{3}(D + T)^j) = D^3 + \beta DT + 2(\beta - 1)TD = D^3 + (3\beta - 2)TD + \beta(\partial T).
$$

We can then write the formula at the third order:

$$
(S, (\nabla^3_s - (3m\lambda + 2)r \nabla_s - \lambda m(\nabla_s r))f)
$$

$$
+ C_{3,1}(\text{Div}S, (\nabla^2_s - m\lambda r)f) + C_{3,2}((\text{Div}^2 + m\gamma_4 i(r))S, \nabla_s f)
$$
\[ + C_{3,3}((\text{Div}^3 + (3\gamma_4 m - 2)i(r)\text{Div} + m\gamma_4 i(\nabla_s r))S, f). \]

At the second order, the formula is simply:

\[ (S, (\nabla_s^2 - m\lambda r)f) + C_{2,1}(\text{Div}S, \nabla_s f) + C_{2,2}((\text{Div}^2 + m\gamma_2 i(r))S, f). \]

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