On one boundary control problem of string vibrations with given velocity of points at an intermediate moment of time

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Abstract. The boundary control problem for the string vibration equation with given initial and final conditions, with the given velocity of the string points at an intermediate moment of time is considered. It is controlled by displacement of one end while the other end is fixed. A constructive approach for constructing the boundary control action is proposed, which is applied for given functions.

1. Introduction

Numerous works, in particular \cite{1}-\cite{12}, are devoted to the study of control problems and optimal control of vibrational processes with both distributed and boundary actions. Modeling and control of dynamical systems described by both ordinary differential equations and partial differential equations with intermediate conditions are an actively developing area in modern control theory. In particular, papers \cite{6}, \cite{8}, \cite{9}, \cite{13}, \cite{14} are devoted to the study of such problems.

In the present work, the problem of boundary control of vibrations of a homogeneous string with given initial and final conditions, with the given velocity of the string points at an intermediate moment of time is solved. It is controlled by the displacement of the left end while the right end is fixed. The problem is reduced to the control problem of distributed actions with zero boundary conditions. Using the method of variables separation and methods of the theory of finite-dimensional systems control for arbitrary numbers of first harmonics, a boundary control is constructed under the influence of which the speed of the string points takes a given value (or close to a given value) at an intermediate moment of time. A computational experiment with the construction of graphs and their comparative analysis were carried out, which confirm the obtained theoretical results.
2. Problem statement
Consider small transverse vibrations of a stretched homogeneous string. They are described by the function \( Q(x,t), 0 \leq x \leq l, 0 \leq t \leq T \), which satisfies the wave equation
\[
\frac{\partial^2 Q}{\partial t^2} = a^2 \frac{\partial^2 Q}{\partial x^2}, \quad 0 < x < l, \quad t > 0
\]  
with the boundary conditions
\[
Q(0,t) = u(t), \quad Q(l,t) = 0, \quad 0 \leq t \leq T.
\]

Let the initial and final conditions be given
\[
Q(x,0) = \varphi_0(x), \quad \left. \frac{\partial Q}{\partial t} \right|_{t=0} = \psi_0(x), \quad 0 \leq x \leq l,
\]
\[
Q(x,T) = \varphi_T(x) = \varphi_2(x), \quad \left. \frac{\partial Q}{\partial t} \right|_{t=T} = \psi_T(x) = \psi_2(x), \quad 0 \leq x \leq l,
\]
where \( T \) is some given finite moment of time. In the equation (1) \( a^2 = \frac{T_0}{\rho} \), where \( T_0 \) is the string tension, \( \rho \) is the density of the homogeneous string, and function \( u(t) \) is the boundary control.

Let at some intermediate moment of time \( t_1 \) \((0 = t_0 < t_1 < t_2 = T)\) the values of the velocities of the string points are given in the form:
\[
\left. \frac{\partial Q}{\partial t} \right|_{t=t_1} = \psi_1(x), \quad 0 \leq x \leq l.
\]

We formulate the following boundary control problem of string vibrations.
Among the possible boundary controls for \( u(t), 0 \leq t \leq T \), (2) it is required to find such control under the influence of which the vibrational motion of the system (1) from a given initial state (3) goes into the final state (4), ensuring that condition (5) is satisfied at intermediate moment of time.

It is assumed that the function \( Q(x,t) \in C^2(\Omega_T) \), where the set \( \Omega_T = \{(x,t) : x \in [0,l], \ t \in [0,T]\} \), functions \( \varphi_0(x), \varphi_T(x) \) belong to the space \( C^2[0,l] \), and functions \( \psi_i(x) \ (i = 0, 1, 2) \) belong to the space \( C^1[0,l] \). The function \( u(t) \in C^2[0,T] \) is called admissible control. It is also assumed that all functions are such that the compatibility conditions below are satisfied.

3. Reducing the original problem to a problem with zero boundary conditions
As the boundary conditions (2) are non-homogeneous, the solution of the stated problem is reduced to a problem with zero boundary conditions. Thus, following [15], we seek for the solution of the equation (1) in the form of a sum:
\[
Q(x,t) = V(x,t) + W(x,t),
\]
where \( V(x,t) \) is an unknown function to be determined with the homogeneous boundary conditions
\[
V(0,t) = V(l,t) = 0,
\]
and the function \( W(x,t) \) is solution of the equation (1) with the non-homogeneous boundary conditions
\[
W(0,t) = u(t), \quad W(l,t) = 0.
\]
The function \( W(x,t) \) has the form
\[
W(x,t) = \left(1 - \frac{x}{l}\right) u(t) .
\] (8)

Substituting (6) into (1) and taking into account (8), we obtain the following equation for determining the function \( V(x,t) \):
\[
\frac{\partial^2 V}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial x^2} + F(x,t),
\] (9)

where
\[
F(x,t) = \left(\frac{x}{l} - 1\right) u''(t) .
\] (10)

From the initial, intermediate and final conditions (3)-(5) with considering (7) we obtain following compatibility conditions
\[8\], \[9\]
\[
u(0) = \varphi_0(0), \quad u'(0) = \psi_0(0), \quad \varphi_0(l) = \psi_0(l) = 0, \quad \Rightarrow (11)
\]
\[
u(t_1) = \varphi_1(0), \quad \psi_1(l) = 0, \quad \Rightarrow (12)
\]
\[
u(T) = \varphi_T(0), \quad u'(T) = \psi_T(0), \quad \varphi_T(l) = \psi_T(l) = 0. \quad \Rightarrow (13)
\]

The function \( V(x,t) \), by virtue of conditions (2)-(5) and with considering conditions (11)-(13), must satisfy the initial conditions
\[
V(x,0) = \varphi_0(x) + \left(\frac{x}{l} - 1\right) \varphi_0(0), \quad \frac{\partial V}{\partial t} \bigg|_{t=0} = \psi_0(x) + \left(\frac{x}{l} - 1\right) \psi_0(0), \quad \Rightarrow (14)
\]
intermediate condition
\[
\frac{\partial V}{\partial t} \bigg|_{t=t_1} = \psi_1(x) + \left(\frac{x}{l} - 1\right) \psi_1(0), \quad \Rightarrow (15)
\]
and final conditions
\[
V(x,T) = \varphi_T(x) + \left(\frac{x}{l} - 1\right) \varphi_T(0), \quad \frac{\partial V}{\partial t} \bigg|_{t=T} = \psi_T(x) + \left(\frac{x}{l} - 1\right) \psi_T(0). \quad \Rightarrow (16)
\]

Thus, the solution to the problem is reduced to the following problem: it is required to find such a control \( u(t) \), \( 0 \leq t \leq T \), under the influence of which the vibrational motion (9) with boundary conditions (7) from the given initial state (14) through the intermediate state (15) goes to the final state (16).

4. The problem solution

Taking into account that the boundary conditions (7) are homogeneous and the compatibility conditions are satisfied, according to the theory of Fourier series, we seek the solution of equation (9) in the form
\[
V(x,t) = \sum_{k=1}^{\infty} V_k(t) \sin \frac{\pi k}{l} x.
\] (17)

We represent the functions \( F(x,t) \), \( \varphi_0(x) \), \( \varphi_T(x) \) and \( \psi_i(x) \), \( (i = 0, 1, 2) \) in the form of Fourier series and, substituting their values together with \( V(x,t) \) in (9), (10) and in conditions (14)-(16), we obtain
\[
\ddot{V}_k(t) + \lambda_k^2 V_k(t) = F_k(t), \quad \lambda_k^2 = \left(\frac{a \pi k}{L}\right)^2, \quad F_k(t) = -\frac{2a}{\lambda_k l} u''(t), \quad \Rightarrow (18)
\]
\[ V_k^{(0)} = \varphi_k^{(0)} - \frac{2a}{\lambda_k} \varphi_0(0), \quad \dot{V}_k^{(0)} = \psi_k^{(0)} - \frac{2a}{\lambda_k} \psi_0(0), \]  
(19)  
\[ \dot{V}_k(t_1) = \psi_k^{(1)} - \frac{2a}{\lambda_k} \psi_1(0), \]  
(20)  
\[ V_k(T) = \varphi_k^{(T)} - \frac{2a}{\lambda_k} \varphi_T(0), \quad \dot{V}_k(T) = \psi_k^{(T)} - \frac{2a}{\lambda_k} \psi_T(0), \]  
(21)

where \( F_k(t), \varphi_k^{(0)}, \varphi_k^{(T)} \) and \( \psi_k^{(i)} (i = 0, 1, 2) \) denote the Fourier coefficients, corresponding to the functions \( F(x, t), \varphi_0(x), \varphi_T(x) \) and \( \psi_i(x) \) (\( i = 0, 1, 2 \)).

The solution of the equation (18) with conditions (19) and its derivative with respect to time have the form

\[ V_k(t) = V_k(0) \cos \lambda_k t + \frac{1}{\lambda_k} \dot{V}_k(0) \sin \lambda_k t + \frac{1}{\lambda_k} \int_0^t F_k(\tau) \sin \lambda_k (t - \tau) d\tau, \]

\[ \dot{V}_k(t) = -\lambda_k V_k(0) \sin \lambda_k t + \dot{V}_k(0) \cos \lambda_k t + \int_0^t F_k(\tau) \cos \lambda_k (t - \tau) d\tau. \]

Now, considering the intermediate (20) and final (21) conditions, using the approaches given in [14], according to conditions (11)-(13), from (22) it is obtained, that the function \( u(\tau) \) for each \( k \) must satisfy the following relations:

\[ \int_0^T \dot{H}_k(\tau) u(\tau) d\tau = C_k(t_1, T), \quad k = 1, 2, \ldots, \]  
(23)  
\[ \dot{H}_k(\tau) = \begin{pmatrix} \sin \lambda_k (T - \tau) \\ \cos \lambda_k (T - \tau) \end{pmatrix}, \quad C_k(t_1, T) = \begin{pmatrix} C_{1k}(T) \\ C_{2k}(T) \end{pmatrix}, \quad h^{(1)}_k(\tau) = \begin{cases} \cos \lambda_k (t_1 - \tau), & 0 \leq \tau \leq t_1, \\ 0, & t_1 < \tau \leq T, \end{cases} \]

\[ C_{1k}(T) = \frac{1}{\lambda_k^2} \left[ \frac{\lambda_k}{2a} \tilde{C}_{1k}(T) + X_{1k} \right], \quad \tilde{C}_{1k}(T) = \lambda_k V_k(T) - \lambda_k V_k(0) \cos \lambda_k T - \dot{V}_k(0) \sin \lambda_k T, \]

\[ C_{2k}(T) = \frac{1}{\lambda_k^2} \left[ \frac{\lambda_k}{2a} \tilde{C}_{2k}(T) + X_{2k} \right], \quad \tilde{C}_{2k}(T) = \dot{V}_k(T) + \lambda_k V_k(0) \sin \lambda_k T - \dot{V}_k(0) \cos \lambda_k T, \]

\[ C_{2k}(t_1) = \frac{1}{\lambda_k} \left[ \frac{\lambda_k}{2a} \tilde{C}_{2k}(t_1) + X^{(1)}_{2k} \right], \quad \tilde{C}_{2k}(t_1) = \dot{V}_k(t_1) + \lambda_k V_k(0) \sin \lambda_k (t_1 - \dot{V}_k(0) \cos \lambda_k t_1, \]

\[ X_{1k} = \lambda_k \varphi_T(0) - \psi_0(0) \sin \lambda_k T - \lambda_k \varphi_0(0) \cos \lambda_k T, \]

\[ X_{2k} = \psi_T(0) - \psi_0(0) \cos \lambda_k T + \lambda_k \varphi_0(0) \sin \lambda_k T, \]

\[ X^{(1)}_{2k} = \psi_1(0) - \psi_0(0) \cos \lambda_k t_1 + \lambda_k \varphi_0(0) \sin \lambda_k t_1. \]

Thus, to find the function \( u(\tau), \tau \in [0, T] \), infinite integral relations (23) are obtained. In practice, the first \( n \) harmonics of the vibrations are chosen and the control synthesis problem is solved using the methods of control theory for finite-dimensional systems. We introduce the following notation of block matrices

\[ H_n(\tau) = \begin{pmatrix} \tilde{H}_1(\tau) \\ \tilde{H}_2(\tau) \\ \vdots \\ \tilde{H}_n(\tau) \end{pmatrix}, \quad \eta_n = \begin{pmatrix} C_1(t_1, T) \\ C_2(t_1, T) \\ \vdots \\ C_n(t_1, T) \end{pmatrix} \]

(27)
with dimensions $H_n(\tau) - (3n \times 1)$, $\eta_n - (3n \times 1)$. Therefore, for the first $n$ harmonics taking into account (27) from (23) it is obtained:

$$
\int_0^T H_n(\tau)u_n(\tau)d\tau = \eta_n. \tag{28}
$$

For arbitrary numbers of first harmonics, the boundary control action $u_n(t)$, satisfying the integral relation (28), has the form [14], [16]

$$
u_n(t) = H_n^T(t)S_n^{-1}\eta_n + f_n(t), \tag{29}$$

where $H_n^T(t)$ is a transposed matrix, $f_n(t)$ is some vector-function, such that

$$
\int_0^T H_n(t)f_n(t)dt = 0, \quad S_n = \int_0^T H_n(t)H_n^T(t)dt. \tag{30}
$$

Here $H_n(t)H_n^T(t)$ is the outer product, $S_n$ is a known matrix of dimension $(3n \times 3n)$ and it is assumed that det $S_n \neq 0$. Substituting (29) into (18), and the expression obtained for $F_k(t)$, into (22), we obtain the function $V_k(t)$, $t \in [0, T]$. Further, from formula (17) we have:

$$V_n(x,t) = \sum_{k=1}^{n} V_k(t) \sin \frac{\pi k}{T} x, \tag{31}$$

and from (6) the deflection function of the string $Q_n(x,t)$ for the first $n$ harmonics can be written in the form

$$Q_n(x,t) = V_n(x,t) + W_n(x,t),$$

where

$$W_n(x,t) = \left(1 - \frac{x}{l}\right) u_n(t). \tag{32}$$

5. Constructing a solution for case $n = 1$

Applying the above proposed approach, we construct a boundary control for $n = 1$ (i.e. $k = 1$). Then, according to (27), $H_1(\tau) = \hat{H}_1(\tau)$, $\eta_1 = \hat{C}_1(t_1, T)$, and from (30) we obtain

$$S_1 = \int_0^{t_2} H_1(\tau)H_1^T(\tau)d\tau = \left(\begin{array}{ccc}
s_{11}^{(1)} & s_{12}^{(1)} & s_{13}^{(1)} \\
(s_{21}^{(1)}) & s_{22}^{(1)} & s_{23}^{(1)} \\
(s_{31}^{(1)}) & s_{32}^{(1)} & s_{33}^{(1)}
\end{array}\right) =$$

$$= \int_0^{t_2} \begin{pmatrix}
sin^2\lambda_1(t_2 - \tau) & \sin\lambda_1(t_2 - \tau) \cos\lambda_1(t_2 - \tau) & h_{21}^{(1)}(\tau) \sin\lambda_1(t_2 - \tau) \\
\sin\lambda_1(t_2 - \tau) \cos\lambda_1(t_2 - \tau) & \cos^2\lambda_1(t_2 - \tau) & h_{21}^{(1)}(\tau) \cos\lambda_1(t_2 - \tau) \\
h_{21}^{(1)}(\tau) \sin\lambda_1(t_2 - \tau) & h_{21}^{(1)}(\tau) \cos\lambda_1(t_2 - \tau) & (h_{21}^{(1)}(\tau))^2
\end{pmatrix} d\tau.
$$

Elements of matrix $S_1$, according to the notation (24), have the following form

$$s_{11}^{(1)} = \frac{t_2}{2} - \frac{1}{4\lambda_1} \sin 2\lambda_1 t_2, \quad s_{22}^{(1)} = \frac{t_2}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 t_2,$$

$$s_{13}^{(1)} = s_{31}^{(1)} = \frac{1}{2\lambda_1} \prod_{i=1}^{2} \sin \lambda_1 t_i + \frac{t_1}{2} \sin \lambda_1(t_2 - t_1), \quad s_{12}^{(1)} = s_{21}^{(1)} = \frac{\sin^2\lambda_1 t_2}{2\lambda_1},$$
$$s_{23}^{(1)} = s_{32}^{(1)} = \frac{1}{2\lambda_1} \cos \lambda_1 t_2 \sin \lambda_1 t_1 + \frac{t_1}{2} \cos \lambda_1 (t_2 - t_1), \quad s_{33}^{(1)} = \frac{t_1}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 t_1,$$

wherein $\Delta = \det S_1 \neq 0$. Elements of matrix $S_1^{-1}$ have the following form

$$\sim s_{11}^{(1)} = \frac{1}{\Delta} \left[ \prod_{i=1}^{2} \left( \frac{t_1}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 t_i \right) - \left( \frac{1}{2\lambda_1} \cos \lambda_1 T \sin \lambda_1 t_1 + \frac{t_1}{2} \cos \lambda_1 (T - t_1) \right)^2 \right], \quad \sim s_{12}^{(1)} =$$

$$= \sim s_{21}^{(1)} = \frac{1}{\Delta} \left[ \prod_{i=1}^{2} \sin \lambda_1 t_i + \frac{t_1}{2} \sin \lambda_1 (t_2 - t_1) \right] \left( \frac{1}{2\lambda_1} \cos \lambda_1 t_2 \sin \lambda_1 t_1 + \frac{t_1}{2} \cos \lambda_1 (t_2 - t_1) \right) -$$

$$- \frac{\sin^2 \lambda_1 t_2}{2\lambda_1} \left( \frac{t_1}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 t_1 \right)^2,$$

$$\sim s_{13}^{(1)} = s_{31}^{(1)} = \frac{1}{\Delta} \left[ \sin^2 \frac{\lambda_1 T}{2\lambda_1} \left( \frac{1}{2\lambda_1} \cos \lambda_1 T \sin \lambda_1 t_1 + \frac{t_1}{2} \cos \lambda_1 (T - t_1) \right) -$$

$$- \left( \frac{1}{2\lambda_1} \prod_{i=1}^{2} \sin \lambda_1 t_i + \frac{t_1}{2} \sin \lambda_1 (t_2 - t_1) \right) \left( \frac{t_2}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 t_2 \right)^2 \right],$$

$$\sim s_{22}^{(1)} = \frac{1}{\Delta} \left[ \left( \frac{t_2}{2} - \frac{1}{4\lambda_1} \sin 2\lambda_1 t_2 \right) \left( \frac{1}{2\lambda_1} \frac{t_1}{2} \sin 2\lambda_1 t_1 \right) - \left( \frac{1}{2\lambda_1} \prod_{i=1}^{2} \sin \lambda_1 t_i + \frac{t_1}{2} \sin \lambda_1 (t_2 - t_1) \right)^2 \right],$$

$$\sim s_{23}^{(1)} = s_{32}^{(1)} = \frac{1}{\Delta} \left[ \sin^2 \frac{\lambda_1 t_2}{2\lambda_1} \left( \frac{1}{2\lambda_1} \prod_{i=1}^{2} \sin \lambda_1 t_i + \frac{t_1}{2} \sin \lambda_1 (t_2 - t_1) \right) -$$

$$- \left( \frac{t_2}{2} - \frac{1}{4\lambda_1} \sin 2\lambda_1 t_2 \right) \left( \frac{1}{2\lambda_1} \cos \lambda_1 t_2 \sin \lambda_1 t_1 + \frac{t_1}{2} \cos \lambda_1 (t_2 - t_1) \right) \right],$$

$$\sim s_{33}^{(1)} = \frac{1}{\Delta} \left[ \frac{t_2}{4} - \left( \frac{1}{4\lambda_1} \sin 2\lambda_1 t_2 \right)^2 - \left( \sin^2 \frac{\lambda_1 t_2}{2\lambda_1} \right)^2 \right].$$

From formula (29) follows, that $u_1(\tau) = H_1^T(\tau)S_1^{-1}Tl + f_1(\tau)$. Assuming, that $f_1(\tau) = 0$, we obtain:

at $\tau \in [0, t_1]$

$$u_1(\tau) = \sin \lambda_1 (T - \tau) \left[ \sim s_{11}^{(1)} C_11(T) + \sim s_{12}^{(1)} C_{21}(T) + \sim s_{13}^{(1)} C_{21}(t_1) \right] +$$

$$+ \cos \lambda_1 (T - \tau) \left[ \sim s_{12}^{(1)} C_11(T) + \sim s_{22}^{(1)} C_{21}(T) + \sim s_{23}^{(1)} C_{21}(t_1) \right] +$$

$$+ \cos \lambda_1 (t_1 - \tau) \left[ \sim s_{13}^{(1)} C_11(T) + \sim s_{23}^{(1)} C_{21}(T) + \sim s_{33}^{(1)} C_{21}(t_1) \right],$$

at $\tau \in (t_1, T]$

$$u_1(\tau) = \sin \lambda_1 (T - \tau) \left[ \sim s_{11}^{(1)} C_11(T) + \sim s_{12}^{(1)} C_{21}(T) + \sim s_{13}^{(1)} C_{21}(t_1) \right] +$$

$$+ \cos \lambda_1 (T - \tau) \left[ \sim s_{12}^{(1)} C_11(T) + \sim s_{22}^{(1)} C_{21}(T) + \sim s_{23}^{(1)} C_{21}(t_1) \right].$$
6. Numerical experiment
Let $t_1 = 4 \frac{2}{a}$, $t_2 = 8 \frac{1}{a}$. Then, with considering $\lambda_1 = \frac{4\pi}{7}$ we obtain $t_1\lambda_1 = 4\pi$, $t_2\lambda_1 = 8\pi$, $\lambda_1(t_2 - t_1) = 4\pi$. For matrices $S_1$ and $S_1^{-1}$ we have:

$$S_1 = \begin{pmatrix} \frac{4\pi}{7} & 0 & 0 \\ 0 & \frac{4\pi}{7} & \frac{4\pi}{7} \\ \frac{4\pi}{7} & \frac{4\pi}{7} & \frac{4\pi}{7} \end{pmatrix}, \quad S_1^{-1} = \begin{pmatrix} \frac{\lambda_1}{4\pi} & 0 & 0 \\ 0 & \frac{\lambda_1}{4\pi} & \frac{\lambda_1}{4\pi} \\ 0 & \frac{\lambda_1}{4\pi} & \frac{\lambda_1}{4\pi} \end{pmatrix}. $$

Note, that $\det S_1 = \frac{16\pi^3}{\lambda_1^3}$. From formulas (25) and (26) we obtain

$$C_{11}(T) = \frac{l}{2a} (V_1(T) - V_1(0)) + \frac{\varphi_T(0) - \varphi_0(0)}{\lambda_1}, \quad C_{21}(T) = \frac{l}{2a\lambda_1} \left( V_1(T) - V_1(0) \right) + \frac{\psi_T(0) - \psi_0(0)}{\lambda_1^2},$$

$$C_{21}(t_1) = \frac{l}{2a\lambda_1} \left( V_1(t_1) - V_1(0) \right) + \frac{\psi_T(0) - \psi_0(0)}{\lambda_1^2}. $$

For control $u_1(\tau)$ we have:

at $\tau \in [0, t_1]$

$$u_1(\tau) = \cos \lambda_1 \tau \left( \frac{\dot{V}_1(t_1) - \dot{V}_1(0)}{4\lambda_1} + \frac{\psi_0(0) - \varphi_0(0)}{2\lambda_1 \pi} \right) + \sin \lambda_1 \tau \left( \frac{V_1(0) - V_1(T)}{8} + \frac{\varphi_0(0) - \varphi_T(0)}{4\pi} \right),$$

at $\tau \in (t_1, T]$

$$u_1(\tau) = \cos \lambda_1 \tau \left( \frac{\dot{V}_1(T) - \dot{V}_1(t_1)}{4\lambda_1} + \frac{\psi_T(0) - \psi_0(0)}{2\lambda_1 \pi} \right) + \sin \lambda_1 \tau \left( \frac{V_1(0) - V_1(T)}{8} + \frac{\varphi_0(0) - \varphi_T(0)}{4\pi} \right).$$

From formula (22) with considering, that $F_1(\tau) = -\frac{2a}{\lambda_1^2} u''_1(\tau)$, we have:

at $\tau \in [0, t_1]$

$$V_1(\tau) = \left( V_1(0) + \frac{a\tau(V_1(T) - V_1(0))}{8l} + \frac{a\tau(\varphi_T(0) - \varphi_0(0))}{4\pi l} \right) \cos \lambda_1 \tau + \left( \frac{\dot{V}_1(0)}{\lambda_1} + \frac{\tau(\dot{V}_1(t_1) - \dot{V}_1(0))}{4\pi} + \frac{V_1(0) - V_1(T)}{8\pi} + \frac{\tau(\psi_0(0) - \psi_T(0))}{2\pi^2} + \frac{\varphi_0(0) - \varphi_T(0)}{4\pi^2} \right) \sin \lambda_1 \tau,$$

at $\tau \in (t_1, T]$

$$V_1(\tau) = \left( V_1(0) + \frac{a\tau(V_1(T) - V_1(0))}{8l} + \frac{a\tau(\varphi_T(0) - \varphi_0(0))}{4\pi l} \right) \cos \lambda_1 \tau + \left( \frac{\tau(\dot{V}_1(T) - \dot{V}_1(t_1))}{4\pi} + \frac{2\dot{V}_1(t_1) - \dot{V}_1(T)}{\lambda_1} + \frac{V_1(0) - V_1(T)}{8\pi} - \frac{2(\psi_0(0) - 2\psi_1(0) + \psi_T(0))}{\lambda_1 \pi} + \frac{\tau(\psi_T(0) - \psi_0(0))}{2\pi^2} + \frac{\varphi_0(0) - \varphi_T(0)}{4\pi^2} \right) \sin \lambda_1 \tau.$$

For $Q_1(x, t)$ from (32) we have

$$Q_1(x, t) = V_1(t) \sin \frac{\pi}{l} x + \left( 1 - \frac{x}{l} \right) u_1(t).$$
Now assume, that \(a = \frac{1}{3}, \lambda = 1\), then \(t_1 = 12, T = 24\), \(\lambda_1 = \frac{\pi}{3}\) and let

\[
\varphi_0(x) = \frac{1}{2}x^2 - \frac{2x}{5} - \frac{1}{10}, \quad \psi_0(x) = -x^2 + x,
\]

\[
\psi_1(x) = \frac{x^3}{3} - x^2 + \frac{2x}{3}, \quad \varphi_T(x) = 0, \quad \psi_T(x) = 0.
\]

The Fourier coefficients for the functions \(\varphi_0(x), \psi_0(x), \varphi_1(x), \varphi_T(x), \psi_T(x)\) are equal:

\[
\varphi_1^{(0)} = -\frac{4}{\pi^4} - \frac{1}{5\pi^3}, \quad \psi_1^{(0)} = \frac{8}{\pi^4}, \quad \psi_1^{(1)} = \frac{4}{\pi^4}, \quad \varphi_1^{(T)} = 0, \quad \psi_1^{(T)} = 0,
\]

respectively. The values of these functions at the edges of the string are as follows:

\[
\varphi_0(0) = -\frac{1}{10}, \quad \psi_0(0) = \psi_1(0) = \varphi_T(0) = \psi_T(0) = \varphi_0(1) = \psi_0(1) = \varphi_1(1) = \varphi_T(1) = \psi_T(1) = 0.
\]

Then

\[
V_1(0) = -\frac{4}{\pi^3}, \quad \dot{V}_1(0) = \frac{8}{3\pi^3}, \quad V_1(t_1) = \frac{4}{\pi^3}, \quad V_1(T) = 0, \quad \dot{V}_1(T) = 0,
\]

\[
C_{11}(T) = \frac{6}{\pi^3} + \frac{3}{5\pi}, \quad C_{21}(T) = -\frac{36}{\pi^3}, \quad C_{21}(t_1) = -\frac{18}{\pi^3}.
\]

Therefore, for the functions \(u_1(\tau), V_1(\tau)\) and \(Q_1(x, \tau)\), we have

\[
u_1(\tau) = -\frac{3}{\pi^4} \cos\frac{\pi}{3} \tau - \left(\frac{1}{2\pi^3} + \frac{1}{20\pi}\right) \sin\frac{\pi}{3} \tau, \quad \tau \in [0, T],
\]

\[
V_1(\tau) = \left(-\frac{4}{\pi^3} - \frac{1}{5\pi} + \frac{1}{6\pi} \left(\frac{\tau}{\pi^2} + \frac{\tau}{10}\right)\right) \cos\frac{\pi}{3} \tau + \left(\frac{47}{2\pi^4} - \frac{1}{20\pi^2} - \frac{1}{\pi}\right) \sin\frac{\pi}{3} \tau, \quad \tau \in [0, T],
\]

\[
Q_1(x, \tau) = \left(-\frac{4}{\pi^3} - \frac{1}{5\pi} + \frac{1}{6\pi} \left(\frac{\tau}{\pi^2} + \frac{\tau}{10}\right)\right) \cos\frac{\pi}{3} \tau + \left(\frac{47}{2\pi^4} - \frac{1}{20\pi^2} - \frac{1}{\pi}\right) \sin\frac{\pi}{3} \tau \sin\pi x +
\]

\[
+ \left(-\frac{3}{\pi^4} \cos\frac{\pi}{3} \tau - \left(\frac{1}{2\pi^3} + \frac{1}{20\pi}\right) \sin\frac{\pi}{3} \tau\right) (1 - x), \quad \tau \in [0, T].
\]

At the time moments \(\tau = 0, 12, 24\), the values of the functions \(Q_1(x, \tau)\) and \(\dot{Q}_1(x, \tau)\) are equal:

\[
Q_1(x, 0) = -\left(\frac{4}{\pi^3} + \frac{1}{5\pi}\right) \sin\pi x - \frac{3}{\pi^4} (1 - x), \quad Q_1(x, 12) = -\frac{2}{\pi^3} \sin\pi x - \frac{3}{\pi^4} (1 - x),
\]

\[
Q_1(x, 24) = \frac{1}{5\pi} \sin\pi x - \frac{3}{\pi^4} (1 - x), \quad \dot{Q}_1(x, 0) = \frac{8}{\pi^3} \sin\pi x - (1 - x) \left(\frac{1}{60} + \frac{1}{6\pi^2}\right),
\]

\[
\dot{Q}_1(x, 12) = \frac{4}{\pi^3} \sin\pi x - (1 - x) \left(\frac{1}{60} + \frac{1}{6\pi^2}\right), \quad \dot{Q}_1(x, 24) = (x - 1) \left(\frac{1}{60} + \frac{1}{6\pi^2}\right).
\]

For a comparative analysis of the results obtained, we denote by

\[
\varepsilon_1(x, t_j) = |Q_1(x, t_j) - \varphi_j(x)|, \quad j = 0, 2;
\]

\[
\tilde{\varepsilon}_1(x, t_k) = |\dot{Q}_1(x, t_k) - \psi_k(x)|, \quad k = 0, 1, 2,
\]

then we obtain

\[
\max_{0 \leq x \leq 1} \varepsilon_1(x, 0) \approx 0.0692, \quad \max_{0 \leq x \leq 1} \varepsilon_1(x, 24) \approx 0.0490.
\]
\[
\int_0^1 \varepsilon_1(x,0) \, dx \approx 0.0255, \quad \int_0^1 \varepsilon_1(x,24) \, dx \approx 0.0293,
\]
\[
\max_{x \in [0,1]} \varepsilon_1(x,0) \approx 0.0405, \quad \max_{x \in [0,1]} \varepsilon_1(x,12) \approx 0.0486, \quad \max_{x \in [0,1]} \varepsilon_1(x,24) \approx 0.0336,
\]
\[
\int_0^1 \varepsilon_1(x,0) \, dx \approx 0.0192, \quad \int_0^1 \varepsilon_1(x,12) \, dx \approx 0.0204, \quad \int_0^1 \varepsilon_1(x,24) \, dx \approx 0.0168.
\]

The graphs of the functions \(Q_1(x,0)\) and \(\varphi_0(x)\), \(Q_1(x,0)\) and \(\psi_0(x)\), \(Q_1(x,12)\) and \(\psi_1(x)\) are illustrated in the figures 1, 2 and 3.

Graphical representations of functions \(Q_1(x,24)\) and \(\dot{Q}_1(x,24)\) are given in the figures 4 and 5 respectively.

**Figure 1.** Graphs of \(Q_1(x,0)\) (dashed line) and \(\varphi_0(x)\) (solid line).

**Figure 2.** Graphs of \(\dot{Q}_1(x,0)\) (dashed line) and \(\psi_0(x)\) (solid line).

**Figure 3.** Graphs of \(\dot{Q}_1(x,12)\) (dashed line) and \(\psi_1(x)\) (solid line).

**Figure 4.** The graph of \(Q_1(x,24)\).
Thus, the results of the analysis showed that under the influence of the constructed control, the behavior of the string deflection function and its derivative are sufficiently close to the given initial functions.

7. Conclusion
We proposed a constructive method for constructing the control of vibrations of a homogeneous string by displacement of one end with the other end fixed at the given velocity of the points of the string at an intermediate moment of time. The results can be used in the design of the boundary control of vibration processes. The proposed method can be extended to other not one-dimensional vibrational systems.

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