Two lectures on Two-Dimensional Gravity

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In Memoriam: Carlos Aragone

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When Carlos Aragone visited the physics community in Boston, he became interested in our on-going research concerning lower-dimensional physics, and he contributed to our understanding of gravitational models in (2+1) dimensions. His death deprives us of a valued research colleague; Latin American compatriots will also miss his organizational activities, with which he endeavored to keep physics lively in Venezuela, and more widely in South America.

While Aragone was interested in and contributed to (2+1)-dimensional gravity — planar gravity — I shall here speak on gravity in (1+1)-dimensional space-time — lineal gravity. The purpose of studying lower dimensional theories, and specifically lower dimensional gravity, is to gain insight into difficult conceptional issues, which are present and even more opaque in the physical (3+1)-dimensional world. Perhaps lessons learned in the lower-dimensional setting can be used to explicate physical problems. Moreover, if we are lucky, the lower-dimensional theories can have a direct physical relevance to modelling phenomena that is actually dynamically confined to the lower dimensionality. This is what happened with (2+1)-dimensional gravity: gravitational physics in the presence of cosmic strings (infinitely long, perpendicular to a plane) is adequately described planar gravity. Indeed the recently discussed causality puzzles raised by “Gott time machines” were resolved with the help of the lower-dimensional model [1].

In my first lecture I shall describe how gravity can be formulated as a gauge theory, both classically and quantum mechanically. In the second lecture I shall discuss possible obstructions to quantizing gravity-matter theory.

I. GAUGE THEORIES OF GRAVITY

A diffeomorphism-invariant gravity theory is obviously invariant against transformations whose parameters are functions of space-time, just as in a local gauge theory. Consequently it has been long believed that gravity theory can be formulated as a gauge theory, but in four dimensions it has never been precisely clear how to do this. On the other hand in lower dimensions, it has been possible to give a clear and definite prescription for a gauge theoretic
formulation of the relevant gravity theories. Let me review the steps.

Step 1. Formulate gravity, not in terms of the metric tensor $g_{\mu\nu}$, but rather in terms of Einstein-Cartan variables: Vielbeine $e_a^\mu$ and spin-connections $\omega_{\mu}^{ab}$, which are viewed as independent quantities — the relation between them will be given by an equation of motion. [Notation: Greek letters denote space-time components; Roman letters denote components in a flat tangent space with metric $\eta_{ab} = \text{diag}(1, -1, \ldots)$; $g_{\mu\nu} = e_a^\mu e_b^\nu \eta_{ab}$; the anti-symmetric symbol $\epsilon^{ab\ldots}$ is normalized by $\epsilon^{01\ldots} = 1$.] In addition to $e_a^\mu$ and $\omega_{\mu}^{ab}$, it may be necessary to use further variables, see below. This formulation already gives rise to a gauge theory of the local Lorentz group $G_L$, with $\omega_{\mu}^{ab}$ being the “gauge potential,” associated with the Lorentz generator $J_{ab}, \omega_{\mu} \equiv \omega_{\mu}^{ab} J_{ab}$. The Vielbein $e_a^\mu$ transforms contravariantly under the Lorentz group — it is not a potential.

Step 2. Choose a gauge group $G$ that contains $G_L$, with generators $Q_A$, which include the Lorentz generators $J_{ab}$, but also comprise additional generators that are associated with variables other than $\omega_{\mu}^{ab}$. Specifically the Vielbeine are associated with translations $P_a, e_\mu = e_a^\mu P_a$, while other variables (if they exist) are associated with other generators as needed. In this way one arrives at Lie-algebra valued potentials.

$$A_\mu = A_\mu^A Q_A = \omega_\mu + e_\mu + \ldots \quad (1.1)$$

Step 3. Construct the gauge field curvature, in the usual way,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (1.2)$$

where the commutator in $[1.2]$ is evaluated from the Lie algebra of the group.

Step 4. It remains to construct a dynamical equation for these gauge fields, such that the gauge field equation is recognized as the relevant gravitational equation, when it is reexpressed in terms of the gravity variables $\omega_{\mu}^{ab}, e_\mu, \ldots$. To this end, we seek an action for the gauge variables, which is gauge invariant, and now we encounter the first novelty of lower-dimensional gravity: A gauge invariant action can indeed be given, but it is not of the Yang-Mills paradigm $\langle F_{\mu\nu}, F_{\mu\nu} \rangle$, rather use is made of novel structures available in lower-dimensions. [Here $\langle , \rangle$ denotes a symmetric, invariant bilinear form on the Lie algebra; when
the group is semi-simple this may be given by the Killing-Cartan metric, for non semi-simple
groups other invariants are used, see below.]

(i) As is well known \([2]\), in (2+1) dimensions one can form the Chern-Simons action.
When one uses the ISO\((2, 1)\) Poincaré group one arrives at a gauge theoretical formulation
of Einstein gravity. Similarly using the \(SO(3, 1)\) or \(SO(2, 2)\), de Sitter or anti-de Sitter
groups leads to gravity with cosmological constant of one or the other sign. Finally conformal
gravity arises from the \(SO(3, 2)\) group. We do not here continue presenting the theory in
this dimensionality, but turn to the (1+1) dimensional case.

(ii) When it comes to gravity in (1+1) dimensions, it is necessary to \textit{invent} a model,
because Einstein’s theory does not exist on a line: the two-dimensional Einstein tensor
\(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\) vanishes identically; the Einstein-Hilbert action \(\int d^2x \sqrt{-g} R\) is a surface term — it is the topological Euler invariant and does not lead to equations of motion.

A two-dimensional gravity model that has been widely studied by people interested in
string theory, conformal field theory and statistical mechanics is Polyakov’s induced gravity \([3]\), which is given by computing the partition function for a massless field interacting with an
external gravitational field. While this has engendered much research, nothing particularly
relevant to four-dimensional gravity has emerged — presumably because Polyakov’s induced
gravity action is non-local.

Even before Polyakov’s proposal, Teitelboim and I \([4]\) suggested that a way to obtain non-
trivial gravitational dynamics in (1+1) dimensions is to introduce an additional gravitational
variable, a world scalar Lagrange multiplier field \(\eta\), with which, together with the Riemann
scalar, one can construct non-trivial actions, \textit{e.g.}

\[
\mathcal{L}_1 = \sqrt{-g} \eta (R - \lambda) \tag{1.3}
\]

where \(\lambda\) is a cosmological constant. Initially, only sporadic research was performed on such
models, but more recently it was realized that similar theories arise in a limit of string
theory. Consequently many people, who have become weary in the string program, have
begun analyzing these “string-inspired” theories. A particularly popular model is given by
the Callan-Giddings-Harvey-Strominger Lagrange density \([5]\)
\[ L_2 = \sqrt{-g}(\eta R - \lambda) \]  

which is like (1.3), except that the Lagrange multiplier field \( \eta \) does not multiply the cosmological constant. [Actually CGHS present their model in terms of a “dilaton” field \( \phi \), related to \( \eta \) by \( \eta = e^{-2\phi} \), and they use a rescaled metric tensor \( \bar{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu} = g_{\mu\nu}/\eta \). Thus in terms of the CGHS variables, their Lagrange density, equivalent to (1.4), reads \( L_{CGHS} = \sqrt{-\bar{g}} e^{-2\phi}(\bar{R} + 4\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda). \)]

A gauge theoretical formulation of these models uses the gauge curvature tensor \( F^A_{\mu\nu} \) for some appropriately chosen group, indexed by \( A \); evidently \( F^A_{\mu\nu} \) transforms according to the adjoint representation. Next we take a multiplet of world scalar Lagrange multipliers \( \eta_A \), transforming according to the coadjoint representation, with which an invariant Lagrange density can be constructed, without use of a group metric.

\[ \mathcal{L} = \frac{1}{2}\eta_A \epsilon^{\mu\nu} F^A_{\mu\nu} \]  

(1.5)

Gauge theories with a Lagrange density as in (1.5) are also called B-F models, where “B” represents the Lagrange multiplier multiplet. One shows that with the three-parameter \( SO(2,1) \) de Sitter or anti-de Sitter groups, (1.3) produces the dynamics of \( L_1 \) [8], while use of the centrally extended Poincaré group \( ISO(1,1) \), gives rise to the dynamics of \( L_2 \) [7]. [The (1+1)-dimensional, centrally extended Poincaré group possesses a single Lorentz generator \( J_{ab} = \epsilon_{ab}J \), two translation generators \( P_a, a = \{0,1\} \), which unconventionally close on a central element \( I: [P_a, P_b] = \epsilon_{ab}I \). The spin connection \( \omega^{ab} = \epsilon^{ab}\omega_\mu \) is associated with \( J \), the Zweibein \( \epsilon^a_\mu \) with \( P_a \), and a further potential \( a_\mu \) is needed for \( I \). Thus the centrally extended Poincaré group contains four parameters; it is an unconventional contraction of \( SO(2,1) \).]

Step 5. Finding classical solutions in the gauge theoretic formalism is especially easy. Varying \( \eta_A \) in (1.3) gives

\[ F_{\mu\nu} = 0 \]  

(1.6)

Consequently \( A_\mu \) is a pure gauge,

\[ A_\mu = U^{-1}\partial_\mu U \]  

(1.7a)
where $U$ is an element of the group. When the gauge $U = \text{constant}$ is chosen, we get

$$A_\mu = 0$$

(1.7b)

Varying $A_\mu$ in (1.5) requires $\eta_A$ to be covariantly constant.

$$D_\mu \eta = 0$$

(1.8)

In the gauge (1.7b) this means that $\eta$ is the constant $\eta^0$, while more generally

$$\eta = U^{-1} \eta^0 U$$

(1.9)

The solution (1.7b) is especially provocative when it is remembered that $A_\mu$ collects the gravitational variables $\omega_\mu$, $e_\mu^a$, ... in terms of which the geometry of space-time is encoded. Evidently when these vanish, a space-time cannot be constructed. But the problem is overcome simply by choosing any gauge function $U$, such that (1.7a) is non-singular and then the geometry can be reconstructed. The invariant characteristics of space-time are contained in the constant past of $\eta_A$, as is seen from (1.9). When a group metric is available one may move indices and construct an invariant.

$$C = \langle \eta, \eta \rangle = \eta_A \eta^A$$

(1.10)

[For $SO(2, 1)$ the metric is the Cartan-Killing metric; for centrally extended $ISO(1, 1)$, which is not semi-simple, the Cartan-Killing bilinear form is singular and cannot serve as a metric, but an alternate metric can be constructed, so that $\eta_A \eta^A = \eta_a \eta^a - 2\eta_2 \eta_3$. Because this group is solvable, there also exists an invariant vector $V^A$, so that $V^A \eta_A = \eta_3$ is also an invariant.] Note that the extended $ISO(1, 1)$ model $\approx$ CGHS theory does not possess a cosmological constant in its gauge theoretical formulation: $\lambda$ emerges as a solution to the equations of motion, which require, inter alia, that $\eta_3 = 0$, $\eta_3 = \lambda$. (The dash signifies differentiation with respect to the spatial variable $\sigma$.)

Step 6: Quantization of pure gravity is effected by taking the gravitational Lagrange density $\mathcal{L}_g$ to be proportional to (1.5).

$$(4\pi G)\mathcal{L}_g = \eta_A F_{01}^A = \eta_A \dot{A}^A_1 + (4\pi G)A_0^A G_A$$

(1.11)
$G_A = \frac{1}{(4\pi G)}(D_1\eta)_A$ (1.12)

$G$ is “Newton’s” gravitational constant, the over-dot signifies differentiation with respect to time $t$. The last equality in (1.11) follows from the previous after a spatial integration by parts. The phase-space form of (1.11) shows that $\eta_A$ and $A_1^A$ are a canonical pair: momentum and coordinate respectively, satisfying the quantum commutator

$$[A_1^A(\sigma), \eta_B(\tilde{\sigma})] = 4\pi G \delta^A_B \delta(\sigma - \tilde{\sigma})$$ (1.13)

The quantities $G_A$ in (1.12) are the “Gauss Law” constraints, whose algebra can be computed with the help of (1.13) and is found to follow the Lie algebra of the relevant group, with structure constants $f_{ABC}$.

$$[Q_A, Q_B] = f_{AB}^C Q_C$$ (1.14)

$$[G_A(\sigma), G_B(\tilde{\sigma})] = i f_{AB}^C G_C(\sigma) \delta(\sigma - \tilde{\sigma})$$ (1.15)

These constraints are first-class, they can be imposed on states

$$G_A(\sigma)|\Psi\rangle = 0 \Rightarrow (\eta'_A + f_{AB}^C A_1^B \eta_C)|\Psi\rangle = 0$$ (1.16)

Although no group metric is needed to present (1.16), when a group metric does exist (as in our two examples) $f_{ABC}$ is totally anti-symmetric and contracting (1.16) with $\eta^A$ leaves

$$\eta^A \eta'_A |\Psi\rangle = \frac{1}{2} (\eta^A \eta_A)' |\Psi\rangle = 0$$ (1.17)

This means that physical states satisfying (1.17) possess support only for the constant portion of the invariant $\eta^A \eta_A$. Moreover, when there exists an invariant vector $V^A$, e.g. when the group is the centrally extended $ISO(1,1)$, and $V^A f_{AB}^C = 0$, it follows from (1.16) that

$$V^A \eta'_A |\Psi\rangle = 0$$ (1.18)

so that $|\Psi\rangle$ has support only for the constant part of the invariant $V^A \eta_A$.

The abstract states $|\Psi\rangle$ may be explicitly presented in the Schrödinger representation with momentum polarization, such that states are wave functionals of $\eta_A$ and $A_1^A$ is realized as $4\pi G i \delta^A_{\eta_A}$. Thus Eq. (1.16) becomes
\[
\left( \eta^I_A + 4\pi G i f_{AB}^C \eta_C \frac{\delta}{\delta \eta_B} \right) \Psi(\eta) = 0 \tag{1.19}
\]

Before discussing solutions to this equation, let me record a peculiar and not-very-well-known fact about gauge theories in the Schrödinger representation. Let us recall that on the canonical variables, coordinate \( A \) and momentum \( \Pi \), gauge transformations act as

\[
A^U = U^{-1} A U + U^{-1} dU \tag{1.20a}
\]
\[
\Pi^U = U^{-1} \Pi U \tag{1.20b}
\]

The above holds for Yang-Mills theories (where \( \Pi \) is the electric field) and B-F theories where \( \Pi \) is \( \eta \) — both are gauge covariant. (It does not hold in theories with a Chern-Simons term, because then \( \Pi \) is not gauge covariant.) In the coordinate representation, where wave functionals \( \Phi \) depend on \( A \), Gauss’ law ensures that they are gauge invariant (we ignore the topological vacuum angle).

\[
\Phi(A^U) = \Phi(A) \tag{1.21}
\]

The question then arises, how do the momentum space wave functionals \( \Psi(\Pi) \) respond to gauge transformations? The answer is gotten by considering a functional Fourier transform representation for \( \Psi(\Pi) \).

\[
\Psi(\Pi) = \int \mathcal{D} A e^{-i \int (\Pi, A) \Phi(A)} \tag{1.22a}
\]

Using (1.21) and performing various changes of integration variables, with unit Jacobian, leaves

\[
\Psi(\Pi) = \int \mathcal{D} A e^{-i \int (\Pi, A) \Phi(U^{-1} A U + U^{-1} dU)} \\
= \int \mathcal{D} A e^{-i \int (\Pi U U^{-1}, A - U^{-1} dU)} \Phi(A) \\
= e^{i \int (\Pi dU U^{-1})} \Psi(\Pi^U) \tag{1.22b}
\]

Thus, unlike in the coordinate polarization, with the momentum polarization wave functionals are not gauge invariant, rather they satisfy the 1-cocycle transformation law \[8\].
\[ \Psi(\Pi^U) = e^{-i \int (\Pi, dU U^{-1})} \Psi(\Pi) \]  

(1.23)

One may extract a phase from \( \Psi \) and work with a gauge invariant functional \( \psi \), by defining

\[ \Psi(\Pi) = e^{-i \Omega(\Pi)} \psi(\Pi) \]  

(1.24)

\( \psi(\Pi) \) is gauge invariant, consistent with (1.23), provided \( \Omega \) satisfies

\[ \Omega(U^{-1} \Pi U) - \Omega(\Pi) = \int \langle \Pi, dU U^{-1} \rangle \]  

(1.25)

In particular, for the B-F theories that we have been considering in one spatial dimension, where \( \Pi = \frac{1}{4\pi G} \eta \), we seek a functional \( \Omega(\eta) \), such that

\[ \Omega(U^{-1} \eta U) - \Omega(\eta) = \frac{1}{4\pi G} \int \langle \eta, dU U^{-1} \rangle \]  

(1.26a)

The solution to this is

\[ \Omega(\eta) = \frac{1}{4\pi G} \int K(\eta) \, d\sigma \]  

(1.26b)

where \( K(\eta)d\sigma \) is the Kirillov-Kostant 1-form, evaluated on the co-adjoint orbit of the relevant group.

Returning now to our specific gravity models, we conclude by recording the solution to the constraints for the CGHS model, based on centrally extended \( ISO(1,1) \). With the explicit structure constants, the four equations of (1.19) are solved by

\[ \Psi(\eta) = e^{-i \Omega} \delta(\eta') \delta((\eta_{a} \eta^{a} - 2\eta_{2} \eta_{3})') \psi \]  

(1.27)

where \( \Omega = \frac{1}{8\pi G \lambda} \int \epsilon^{ab} d\eta_{a} \eta_{b} \) is the relevant Kirillov-Kostant form. The functional \( \delta \) functions ensure that \( \psi \) depends only on the constant parts of the two invariants \( V^{A} \eta_{A} = \eta_{3} \) and \( \eta_{A} \eta^{A} = \eta_{a} \eta^{a} - 2\eta_{2} \eta_{3} \); \( \lambda \) is (the constant part of) \( \eta_{3} \) — it is the cosmological constant [9].
II. OBSTRUCTIONS TO QUANTIZING GRAVITY THEORY AND QUANTAL MODIFICATION OF WHEELER DE WITT EQUATION

In a canonical, Hamiltonian approach to quantizing a theory with local symmetry — a theory that is invariant against transformations whose parameters are arbitrary functions on space-time — there occur constraints, which are imposed on physical states. Typically these constraints correspond to time components of the Euler-Lagrange equations, and familiar examples arise in gauge theories. The time component of the gauge field equation is the Gauss law.

\[ G_A \equiv D \cdot E^A - \rho^A = 0 \] (2.1)

Here \( E^A \) is the (non-Abelian) electric field, \( \rho^A \) the matter charge density, and \( D \) denotes the gauge-covariant derivative. When expressed in terms of canonical variables, \( G_A \) does not involve time-derivatives — it depends on canonical coordinates and momenta, which we denote collectively by the symbols \( X \) and \( P \) respectively (\( X \) and \( P \) are fields defined at fixed time): \( G_A = G_A(X, P) \). Thus in a Schrödinger representation for the theory, the Gauss law condition on physical states

\[ G_A(X, P) | \psi \rangle = 0 \] (2.2)

corresponds to a (functional) differential equation that the state functional \( \Psi(X) \) must satisfy.

\[ G_A \left( X, \frac{1}{i} \frac{\delta}{\delta X} \right) \Psi(X) = 0 \] (2.3)

In fact, Eq. (2.3) represents an infinite number of equations, one for each spatial point \( r \), since \( G_A \) is also the generator of the local symmetry: \( G_A = G_A(r) \). Consequently, questions of consistency (integrability) arise, and these may be examined by considering the commutator of two constraints. Precisely because the \( G_A \) generate the symmetry transformation, one expects their commutator to follow the Lie algebra with structure constants \( f_{AB}^C \).

\[ [G_A(r), G_B(\tilde{r})] = i f_{AB}^C G_C(r) \delta(r - \tilde{r}) \] (2.4)
If (2.4) holds, the constraints are consistent — they are first class — and the constraint equations are integrable, at least locally.

However, it is by now well-known that Eq. (2.4), which does hold classically with Poisson bracketing, may acquire a quantal anomaly. Indeed when the matter charge density is constructed from fermions of a definite chirality, the Gauss law algebra is modified by an extension — a Schwinger term — the constraint equations become second-class and Eq. (2.3) is inconsistent and cannot be solved. We call such gauge theories “anomalous.”

This does not mean that a quantum theory cannot be constructed from an anomalous gauge theory. One can adopt various strategies for overcoming the obstruction, but these represent modifications of the original model. Moreover, the resulting quantum theory possesses physical content that is very far removed from what one might infer by studying the classical model. All this is explicitly illustrated by the anomalous chiral Schwinger model, whose Gauss law is obstructed, while a successful construction of the quantum theory leads to massive excitations, which cannot be anticipated from the un-quantized equations [10].

With these facts in mind, we turn now to gravity theory, which obviously is invariant against local transformations that redefine coordinates of space-time.

Indeed over the years there have been many attempts to describe gravity in terms of a gauge theory. That program is entirely successful in three- and two-dimensional space-time, where gravitational models are formulated in terms of Einstein–Cartan variables (spin-connection, Vielbein) as non-Abelian gauge theories, based not on the Yang-Mills paradigm, but rather on topological Chern-Simons and B-F structures.

But even remaining with the conventional metric-based formulation, it is recognized that the time components of Einstein’s equation comprise the constraints.

\[
\frac{1}{8\pi G} \left( R^0_\nu - \frac{1}{2} \delta^0_\nu R \right) - T^0_\nu = 0
\]  

(2.5)

The gravitational part is the time component of the Einstein tensor \( R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R \); weighted by Newton’s constant \( G \), this equals the time component of the matter energy-momentum tensor, \( T^\mu_\nu \). In the quantized theory, the collection of canonical operators on the left side in (2.5) annihilates physical states. The resulting equations may be presented as
\[ \mathcal{E} | \psi \rangle = 0 \quad \text{(2.6)} \]
\[ \mathcal{P}_i | \psi \rangle = 0 \quad \text{(2.7)} \]

where \( \mathcal{E} \) is the energy constraint

\[ \mathcal{E} = \mathcal{E}^{\text{gravity}} + \mathcal{E}^{\text{matter}} \quad \text{(2.8)} \]

and \( \mathcal{P}_i \) is the momentum constraint.

\[ \mathcal{P}_i = \mathcal{P}_i^{\text{gravity}} + \mathcal{P}_i^{\text{matter}} \quad \text{(2.9)} \]

Taking for definiteness matter to be described by a massless, spinless field \( \varphi \), with canonical momentum \( \Pi \), we have

\[ \mathcal{E}^{\text{matter}} = \frac{1}{2} \left( \Pi^2 + \gamma^{ij} \partial_i \varphi \partial_j \varphi \right) \quad \text{(2.10)} \]
\[ \mathcal{P}_i^{\text{matter}} = \partial_i \varphi \Pi \quad \text{(2.11)} \]

Here \( \gamma_{ij} \) is the spatial metric tensor; \( \gamma \), its determinant; \( \gamma^{ij} \), its inverse.

The momentum constraint in Eq. (2.7) is easy to unravel. In a Schrödinger representation, it requires that \( \Psi(\gamma_{ij}, \varphi) \) be a functional of the canonical field variables \( \gamma_{ij}, \varphi \) that is invariant against reparameterization of the spatial coordinates and such functionals are easy to construct.

Of course it is (2.6), the Wheeler-DeWitt equation, that is highly non-trivial and once again one asks about its consistency. If the commutators of \( \mathcal{E} \) with \( \mathcal{P} \) follow their Poisson brackets one would expect that the following algebra holds.

\[ [\mathcal{P}_i(r), \mathcal{P}_j(\tilde{r})] = i \mathcal{P}_j(r) \partial_i \delta(r - \tilde{r}) + i \mathcal{P}_i(\tilde{r}) \partial_j \delta(r - \tilde{r}) \quad \text{(2.12a)} \]
\[ [\mathcal{E}(r), \mathcal{E}(\tilde{r})] = i \left( \mathcal{P}^i(r) + \mathcal{P}^i(\tilde{r}) \right) \partial_i \delta(r - \tilde{r}) \quad \text{(2.12b)} \]
\[ [\mathcal{E}(r), \mathcal{P}_i(\tilde{r})] = i \left( \mathcal{E}(r) + \mathcal{E}(\tilde{r}) \right) \partial_i \delta(r - \tilde{r}) \quad \text{(2.12c)} \]

Here \( \mathcal{P}^i \equiv \gamma^{ij} \mathcal{P}_j \). If true, Eqs. (2.12) would demonstrate the consistency of the constraints, since they appear first-class. Unfortunately, establishing (2.12) in the quantized theory is highly problematical. First of all there is the issue of operator ordering in the gravitational
portion of $\mathcal{E}$ and $\mathcal{P}$. Much has been said about this, and I shall not address that difficulty here.

The problem that I want to call attention to is the very likely occurrence of an extension in the $[\mathcal{E}, \mathcal{P}]$ commutator (2.12c). We know that in flat space, the commutator between the matter energy and momentum densities possesses a triple derivative Schwinger term [11]. There does not appear any known mechanism arising from the gravity variables that would effect a cancelation of this obstruction.

A definite resolution of this question in the full quantum theory is out of reach at the present time. Non-canonical Schwinger terms can be determined only after a clear understanding of the singularities in the quantum field theory and the nature of its Hilbert space are in hand, and this is obviously lacking for four-dimensional quantum gravity.

Faced with the impasse, we turn to a gravitational model in two-dimensional space-time — a lineal gravity theory — where the calculation can be carried to a definite conclusion: an obstruction does exist and the model is anomalous. Various mechanisms are available to overcome the anomaly, but the resulting various quantum theories are inequivalent and bear little resemblance to the classical model.

In two dimensions, Einstein’s equation is vacuous because $R^\mu_\nu = \frac{1}{2} \delta^\mu_\nu R$; therefore gravitational dynamics has to be invented afresh. The models that have been studied recently posit local dynamics for the “gravity” sector, which involves as variables the metric tensor and an additional world scalar (“dilaton” or Lagrange multiplier) field. Such “scalar-tensor” theories, introduced a decade ago [4], are obtained by dimensional reduction from higher-dimensional Einstein theory [4,12]. They should be contrasted with models where quantum fluctuations of matter variables induce gravitational dynamics [4], which therefore are non-local and do not appear to offer any insight into the questions posed by the physical, four-dimensional theory.

The model we study is the so-called “string-inspired dilaton gravity” – CGHS theory [5]. The gravitational action involves the metric tensor $g_{\mu\nu}$, the dilaton field $\phi$, $\eta \equiv e^{-2\phi}$, and a cosmological constant $\lambda$. The matter action describes the coupling of a massless, spinless field $\varphi$. 
\[ I_{\text{gravity}} = \int d^2x \sqrt{-g} \eta (R - \lambda) \quad (2.13) \]
\[ I_{\text{matter}} = \frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \quad (2.14) \]

The total action is the sum of (2.13) and (2.14), weighted by “Newton’s” constant \( G \):
\[ I = \frac{1}{4\pi G} I_{\text{gravity}} + I_{\text{matter}} \quad (2.15) \]

Before embarking on the quantal analysis let us remark that classically the theory can be solved completely. Indeed, varying \( \eta \) shows that \( R \) vanishes, space-time is flat, and \( I_{\text{matter}} \) describes the flat-space motion of free massless scalar field, to be sure in a diffeomorphism invariant fashion. One would hope to regain this simple dynamics after quantization. For point particles, where \( I_{\text{matter}} = -m \int \sqrt{g} dx^\mu dx^\nu \), the quantization is successful: one finds a diffeomorphism invariant, quantum description of free particles \[13\]. With matter fields, the quantum development encounters difficulties.

In fact this theory can be given a gauge-theoretical “B-F” description based on the centrally extended Poincaré group in (1+1) dimensions \[14\]. This formulation aided us immeasurably in the subsequent analysis/transformations. However, I shall not discuss this here, because in retrospect it proved possible to carry the analysis forward within the metric formulation (2.13–2.15).

After a remarkable sequence of redefinitions and canonical transformations on the dynamical variables in (2.13–2.14), one can present \( I \) in terms of a first-order Lagrange density \( \mathcal{L} \) that is a sum of quadratic terms \[14\].
\[ \mathcal{L} = \pi_a r^a + \Pi \dot{\varphi} - \alpha \mathcal{E} - \beta \mathcal{P} \quad (2.16) \]
\[ \mathcal{E} = -\frac{1}{2} \left( \frac{1}{\Lambda} \pi_a \pi_a + \Lambda r^a r^a \right) + \frac{1}{2} \left( \Pi^2 + \varphi'^2 \right) \quad (2.17) \]
\[ \mathcal{P} = -r^a \pi_a - \varphi' \Pi \quad (2.18) \]

I shall not derive this, but merely explain it. The index “a” runs over flat 2-dimensional \((t, \sigma)\) space, with signature \((1, -1)\). Dot (dash) signify differentiation with respect to time \( t \) (space \( \sigma \)). The four variables \( \{r^a, \alpha, \beta\} \) correspond to the four gravitational variables \((g_{\mu\nu}, \eta)\), where only \( r^a \) is dynamical with canonically conjugate momentum \( \pi_a \), while \( \alpha \) and \( \beta \) act
as Lagrange multipliers. Notice that regardless of the sign \( \Lambda \equiv \lambda/8\pi G \), the gravitational contribution to \( \mathcal{E} \), is quadratic with indefinite sign.

\[
\mathcal{E}^{\text{gravity}} = -\frac{1}{2} \left( \frac{1}{\Lambda} \pi^a \pi_a + \Lambda a' a' \right) \\
= -\frac{1}{2} \left( \frac{1}{\Lambda} (\pi_0)^2 - \frac{1}{\Lambda} (\pi_1)^2 + \Lambda (r_0')^2 - \Lambda (r_1')^2 \right) \\
= -\mathcal{E}_0 + \mathcal{E}_1 
\]  

\( \mathcal{E}_0 = \frac{1}{2} \left( \frac{1}{\Lambda} (\pi_0)^2 + \Lambda (r_0')^2 \right) \)  

\( \mathcal{E}_1 = \frac{1}{2} \left( \frac{1}{\Lambda} (\pi_1)^2 + \Lambda (r_1')^2 \right) \)  

(2.19a)

(2.19b)

(2.19c)

On the other hand, the gravitational contribution to the momentum does not show alteration of sign.

\[
\mathcal{P}^{\text{gravity}} = -r^a \pi_a \\
= -r_0' \pi_0 - r_1' \pi_1 \\
= \mathcal{P}_0 + \mathcal{P}_1
\]

\( \mathcal{P}_0 = -r_0' \pi_0 \)  

\( \mathcal{P}_1 = -r_1' \pi_1 \)  

(2.20a)

(2.20b)

(2.20c)

One may understand the relative negative sign between the two gravitational contributors \((a = 0, 1)\) as follows. Pure metric gravity in two space-time dimensions is described by three functions collected in \( g_{\mu\nu} \). Diffeomorphism invariance involves 2 functions, which reduce the number of variables by \( 2 \times 2 \), \textit{i.e.} pure gravity has \( 3 - 4 = -1 \) degrees of freedom. Adding the dilaton \( \phi \) gives a net number of \(-1 + 1 = 0\), as in our final gravitational Lagrangian.

The matter contribution is the conventional expression for massless and spinless fields:

\[
\mathcal{E}^{\text{matter}} = \frac{1}{2} (\Pi^2 + \varphi'^2) \\
\mathcal{P}^{\text{matter}} = -\varphi' \Pi
\]

(2.21)

(2.22)

It is with the formulation in Eqs. (2.16)–(2.22) of the theory (2.13)–(2.15) that we embark upon the various quantization procedures.

The transformed theory appears very simple: there are three independent dynamical fields \( \{r^a, \varphi\} \) and together with the canonical momenta \( \{\pi_a, \Pi\} \) they lead to a quadratic
Hamiltonian, which has no interaction terms among the three. Similarly, the momentum comprises non-interacting terms. However, there remains a subtle “correlation interaction” as a consequence of the constraint that $\mathcal{E}$ and $\mathcal{P}$ annihilate physical states, as follows from varying the Lagrange multipliers $\alpha$ and $\beta$ in (2.16).

$$\mathcal{E} | \psi \rangle = 0 \quad (2.23)$$

$$\mathcal{P} | \psi \rangle = 0 \quad (2.24)$$

Thus, even though $\mathcal{E}$ and $\mathcal{P}$ each are sums of non-interacting variables, the physical states $| \psi \rangle$ are not direct products of states for the separate degrees of freedom. Note that Eqs. (2.23), (2.24) comprise the entire physical content of the theory. There is no need for any further “gauge fixing” or “ghost” variables — this is the advantage of the Hamiltonian formalism.

As in four dimensions, the momentum constraint (2.24) enforces invariance of the state functional $\Psi(r^a, \varphi)$ against spatial coordinate transformations, while the energy constraint (2.23) — the Wheeler-DeWitt equation in the present lineal gravity context — is highly non-trivial.

Once again one looks to the algebra of the constraints to check consistency. The reduction of (2.12) to one spatial dimension leaves (after the identification $\mathcal{P}_i \to -\mathcal{P}, \gamma \gamma^{ij} \to 1$),

$$i[\mathcal{P}(\sigma), \mathcal{P}(\tilde{\sigma})] = (\mathcal{P}(\sigma) + \mathcal{P}(\tilde{\sigma})) \delta'(\sigma - \tilde{\sigma}) \quad (2.25a)$$

$$i[\mathcal{E}(\sigma), \mathcal{E}(\tilde{\sigma})] = (\mathcal{P}(\sigma) + \mathcal{P}(\tilde{\sigma})) \delta'(\sigma - \tilde{\sigma}) \quad (2.25b)$$

$$i[\mathcal{E}(\sigma), \mathcal{P}(\tilde{\sigma})] = (\mathcal{E}(\sigma) + \mathcal{E}(\tilde{\sigma})) \delta'(\sigma - \tilde{\sigma}) - \frac{c}{12\pi \delta'''}(\sigma - \tilde{\sigma}) \quad (2.25c)$$

where we have allowed for a possible central extension of strength $c$, and it remains to calculate this quantity.

The gained advantage in two dimensional space-time is that all operators are quadratic, see (2.19)-(2.22); the singularity structure may be assessed and $c$ computed; obviously it is composed of independent contributions.

$$c = c^{\text{gravity}} + c^{\text{matter}}, \quad c^{\text{gravity}} = c_0 + c_1 \quad (2.26)$$

Surprisingly, however, there is more than one way of handling infinities and more than one
answer for $c$ can be gotten. This reflects the fact, already known to Jordan in the 1930s [15], that an anomalous Schwinger term depends on how the vacuum is defined.

In the present context, there is no argument about $c^{\text{matter}}$, the answer is

$$c^{\text{matter}} = 1 \quad (2.27)$$

The same holds for the positively signed gravity variable (assume $\Lambda > 0$, so that $r^1$ enters positively).

$$c_1 = 1 \quad (2.28)$$

But the negatively signed gravitational variable can be treated in more than one way, giving different answers for $c_0$. The different approaches may be named “Schrödinger representation quantum field theory” and “BRST string/conformal field theory,” and the variety arises owing to the various ways one can quantize a theory with a negative kinetic term, like the $r^0$ gravitational variable. (This variety is analogous to what is seen in Gupta-Bleuler quantization of electrodynamics: the time component potential $A_0$ enters with negative kinetic term.)

In the Schrödinger representation quantum field theory approach one maintains positive norm states in a Hilbert space, and finds $c_0 = -1$, $c^{\text{gravity}} = c_0 + c_1 = 0$, $c = c^{\text{gravity}} + c^{\text{matter}} = 1$. Thus pure gravity has no obstructions, only matter provides the obstruction. Consequently the constraints of pure gravity can be solved, indeed explicit formulas have been gotten by many people [16]. In our formalism, according to (2.19) and (2.20) the constraints read

$$\mathcal{E}^{\text{gravity}}|\psi\rangle^{\text{gravity}} \sim \frac{1}{2} \left\{ \frac{1}{\Lambda} \frac{\delta^2}{\delta r^a \delta r_a'} - \Lambda r^a r_a' \right\} \Psi^{\text{gravity}}(r^a) = 0 \quad (2.29)$$

$$\mathcal{P}^{\text{gravity}}|\psi\rangle^{\text{gravity}} \sim i r^a \frac{\delta}{\delta r_a} \Psi^{\text{gravity}}(r^a) = 0 \quad (2.30)$$

with two solutions

$$\Psi^{\text{gravity}}(r^a) = \exp \pm i \frac{\Lambda}{2} \int d\sigma \epsilon_{ab} r^a r^{b'} \quad (2.31a)$$

This may also be presented by an action of a definite operator on the Fock vacuum state $|0\rangle$. 

\[ \Psi_{\text{gravity}}(r^a) \propto \left[ \exp \pm \int dk \, a_0^\dagger(k) \, \epsilon(k) \, a_1^\dagger(-k) \right] |0\rangle. \]  

(2.31b)

with \( a_0^\dagger(k) \) creating field oscillations of definite momentum.

\[ a_0^\dagger(k) = -\frac{i}{\sqrt{4\pi |k|}} \int d\sigma e^{ik\sigma} \, \pi_a(\sigma) + \sqrt{\frac{\Lambda |k|}{4\pi}} \int d\sigma e^{ik\sigma} \, r^a(\sigma) \]  

(2.31c)

As expected, the state functional is invariant against spatial coordinate redefinition, \( \sigma \to \tilde{\sigma}(\sigma) \); this is best seen by recognizing that integrand in the exponent of (2.31a) is a 1-form:

\[ d\sigma \epsilon_{ab} \, r^a \, r^b = \epsilon_{ab} \, r^a \, dr^b. \]

Although this state is here presented for a gravity model in the Schrödinger representation field theory context, it is also of interest to practitioners of conformal field theory and string theory. The algebra (2.25), especially when written in decoupled form,

\[ \Theta_{\pm} = \frac{1}{2}(\mathcal{E} \mp P) \]  

(2.32)

\[ [\Theta_{\pm}(\sigma), \Theta_{\pm}(\tilde{\sigma})] = \pm i \left( \Theta_{\pm}(\sigma) + \Theta_{\pm}(\tilde{\sigma}) \right) \delta'(\sigma - \tilde{\sigma}) \mp \frac{ic}{24\pi} \delta''''(\sigma - \tilde{\sigma}) \]  

(2.33a)

\[ [\Theta_{\pm}(\sigma), \Theta_{\mp}(\tilde{\sigma})] = 0 \]  

(2.33b)

is recognized as the position-space version of the Virasoro algebra and the Schwinger term is just the Virasoro anomaly. Usually one does not find a field theoretic non-ghost realization without the Virasoro center; yet the CGHS model, without matter provides an explicit example. Usually one does not expect that all the Virasoro generators annihilate a state, but in fact our states (2.31) enjoy that property.

Once matter is added, a center appears, \( c = 1 \), and the theory becomes anomalous. In the same Schrödinger representation approach used above, one strategy is the following modification of a method due to Kuchař \[\{14,17\}. \] The Lagrange density (2.16) is presented in terms of decoupled constraints.

\[ \mathcal{L} = \pi_a \dot{r}^a + \Pi \dot{\varphi} - \lambda^+ \Theta^+ - \lambda^- \Theta^- \]  

(2.34a)

\[ \lambda^\pm = \alpha \pm \beta \]  

(2.34b)
Then the gravity variables \( \{ \pi_a, r^a \} \) are transformed by a linear canonical transformation to a new set \( \{ P_\pm, X^\pm \} \), in terms of which (2.34a) reads

\[
\mathcal{L} = P_+ \dot{X}^+ + P_- \dot{X}^- + \Pi \dot{\phi} - \lambda^+ \left( P_+ X'^+ + \theta^\text{matter}_+ \right) - \lambda^- \left( -P_- X'^- + \theta^\text{matter}_- \right)
\]

(2.35a)

\[
\theta^\text{matter}_\pm = \frac{1}{4} (\Pi \pm \phi')^2
\]

(2.35b)

The gravity portions of the constraints \( \Theta_\pm \) have been transformed to \( \pm P_\pm X'^\pm \) — expressions that look like momentum densities for fields \( X^\pm \), and thus satisfy the \( \Theta_\pm \) algebra (2.33) without center, as do also momentum densities, see (2.12a).

The entire obstruction in the full gravity plus matter constraints comes from the commutator of the matter contributions \( \theta^\text{matter}_\pm \). In order to remove the obstruction, we modify the theory by adding \( \Delta \Theta_\pm \) to the constraint \( \Theta_\pm \), such that no center arises in the modified constraints. An expression for \( \Delta \Theta_\pm \) that does the job is

\[
\Delta \Theta_\pm = \frac{1}{48\pi} \left( \ln X'^\pm \right)''
\]

(2.36)

Hence \( \tilde{\Theta}_\pm = \Theta_\pm + \Delta \Theta_\pm \) possess no obstruction in their algebra, and can annihilate states. Explicitly, the modified constraint equations read in the Schrödinger representation (after dividing by \( X'^\pm \))

\[
\left( \frac{1}{i \delta X^\pm} \pm \frac{1}{48\pi X'^\pm} \left( \ln X'^\pm \right)'' \pm \frac{1}{X'^\pm} \theta^\text{matter}_\pm \right) \psi(X^\pm, \phi) = 0
\]

(2.37)

It is recognized that the anomaly has been removed by introducing functional \( U(1) \) connections in \( X^\pm \) space, whose curvature cancels the anomaly. In the modified constraint there still is no mixing between gravitational variables \( \{ P_\pm, X^\pm \} \) and matter variables \( \{ \Pi, \varphi \} \). But the modified gravitational contribution is no longer quadratic — indeed it is non-polynomial — and we have no idea how to solve (2.37). We suspect, however, that just as its matter-free version, Eq. (2.37) possesses only a few solutions — far fewer than the rich spectrum that emerges upon BRST quantization, which we now examine.

In the BRST quantization method, extensively employed by string and conformal field theory investigators, one adds ghosts, which carry their own anomaly of \( c_{\text{ghost}} = -26 \). Also one improves \( \Theta_\pm \) by the addition of \( \Delta \Theta_\pm \) so that \( c \) is increased; for example, with
\[
\Delta \Theta_{\pm} = \frac{Q}{\sqrt{4\pi}} (\Pi \pm \varphi')'
\]  
\[c \rightarrow c + 3Q^2\]  
\[(2.38)\]

[The modification (2.38) corresponds to “improving” the energy momentum tensor by \((\partial_\mu \partial_\nu - g_{\mu\nu} \Box)\varphi\). The “background charge” \(Q\) is chosen so that the total anomaly vanishes.]

\[c + 3Q^2 + c_{\text{ghost}} = 0\]  
\[(2.40)\]

Moreover, the constraints are relaxed by imposing that physical states are annihilated by the “BRST” charges, rather than by the bosonic constraints. This is roughly equivalent to enforcing “half” the bosonic constraints, the positive frequency portions. In this way one arrives at a rich and well known spectrum.

Within BRST quantization, the negative signed gravitational field \(r^0\) is quantized so that negative norm states arise — just as in Gupta-Bleuler electrodynamics. [Negative norm states cannot arise in a Schrödinger representation, where the inner product is explicitly given by a (functional) integral, leading to positive norm.] One then finds \(c_0 = 1\); the center is insensitive to the signature with which fields enter the action. As a consequence, \(c_{\text{gravity}} = c_0 + c_1 = 2\) so that even pure gravity constraints possess an obstruction.

Evidently, pure gravity with \(c_{\text{gravity}} = 2\) requires \(Q = 2\sqrt{2}\). The rich BRST spectrum is much more plentiful than the two states (2.31) found in the Schrödinger representation and does not appear to reflect the fact that the classical pure gravity theory is without excitations.

Gravity with matter carries \(c = 3\), and becomes quantizable at \(Q = \sqrt{23/3}\). Once again a rich spectrum emerges, but it shows no apparent relation to a flat-space particle spectrum.

We conclude that without question, the CGHS model, and other similar two-dimensional gravity models, are afflicted by anomalies in their constraint algebras, which become second-class and frustrate straightforward quantization. While anomalies can be calculated and are finite, their specific value depends on the way singularities of quantum field theory are resolved, and this leads to a variety of procedures for overcoming the problem and to a variety of quantum field theories, with quite different properties.
Two methods were discussed: (i) a Schrödinger representation with Kuchař-type improvement as needed, i.e. when matter is present, and (ii) BRST quantization. (Actually several other approaches are also available [14].) Only in the first method for pure matterless gravity, with positive norm states and vanishing anomaly, does the quantum theory bear any resemblance to the classical theory, in the sense that the classical gravity theory has no propagating degrees of freedom, while the quantum Hilbert space has only the two states in (2.31), neither of which contains any further degrees of freedom. In other cases, e.g. with matter, the classical picture of physics seems irrelevant to the behavior of the quantum theory.

Presumably, if anomalies were absent, the different quantization procedures (Schrödinger representation, BRST, . . . ) would produce the same physics. However, the anomalies are present and interfere with equivalence.

Finally, we remark that our investigation has exposed an interesting structure within Virasoro theory: there exists a field theoretic realization of the algebra without the anomaly, in terms of spinless fields and with no ghost fields. Moreover, there are states that are annihilated by all the Virasoro generators.

What does any of this teach us about the physical four-dimensional model? We believe that an extension in the constraint algebra will arise for all physical, propagating degrees of freedom: for matter fields, as is seen already in two dimensions, and also for gravity fields, which in four dimensions (unlike in two) carry physical energy. How to overcome this obstruction to quantization is unclear to us, but we expect that the resulting quantum theory will be far different from its classical counterpart. Especially problematic is the fact that flat-space calculations of anomalous Schwinger terms in four dimensions yield infinite results, essentially for dimensional reasons. Moreover, it should be clear that any announced “solutions” to the constraints that result from formal analysis must be viewed as preliminary: properties of the Hilbert space and of the inner product must be fixed first in order to give an unambiguous determination of any obstructions.

We believe that our two-dimensional investigation, although in a much simpler and unphysical setting, nevertheless contains important clues for realistic theories. Certainly that
was the lesson of gauge theories: anomalies and vacuum angle have corresponding roles in the Schwinger model and in QCD!
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