Full Characterization of Minimal Linear Codes as Cutting Blocking Sets

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Abstract

In this paper, we first study in detail the relationship between minimal linear codes and cutting blocking sets, which were recently introduced by Bonini and Borello, and then completely characterize minimal linear codes as cutting blocking sets. As a direct result, minimal projective codes of dimension 3 and $t$-fold blocking sets with $t \geq 2$ in projective planes are identical objects. Some bounds on the parameters of minimal codes are derived from this characterization. This confirms a recent conjecture by Alfarano, Borello and Neri in \cite{Alfarano2019} about a lower bound of the minimum distance of a minimal code. Using this new link between minimal codes and blocking sets, we also present new general primary and secondary constructions of minimal linear codes. As a result, infinite families of minimal linear codes not satisfying the Aschikhmin-Barg’s condition are obtained. In addition to this, the weight distributions of two subfamilies of the proposed minimal linear codes are established. Open problems are also presented.

Keywords: Linear code, minimal code, hyperplane, blocking set, secret sharing.

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1. Introduction

Throughout the paper, we will assume the reader to have familiarity with linear codes (see for instance \cite{MacWilliams}). A $q$-ary linear code of length $n$ and dimension $k$ will be referred to as an $[n, k]_q$ code. Further, if the code has minimum distance $d$, it will be referred to as an $[n, k, d]_q$ code. When the alphabet size $q$ is clear from the context, we omit the subscript. Let $C$ be an $[n, k, d]_q$ linear code. $C$ is called projective if any two of its coordinates are linearly independent, or in other words, if the minimum distance $d^\perp$ of its dual code $C^\perp$ is at least three.

The Hamming weight (for short, weight) of a vector $\mathbf{v}$ is the number of its nonzero entries and is denoted $\text{wt}(\mathbf{v})$. The minimum (respectively, maximum) weight of the code $C$ is the minimum (respectively, maximum) nonzero weight among all codewords of $C$, $w_{\text{min}} = \min(\text{wt(}\mathbf{c}\text{)))$ (respectively, $w_{\text{max}} = \max(\text{wt(}\mathbf{c}\text{)))$.

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Let \( c = (c_0, \cdots, c_{n-1}) \) be a codeword in \( C \). The support \( \text{Supp}(c) \) of the codeword \( c \) is the set of indices of its nonzero coordinates:

\[
\text{Supp}(c) = \{i : c_i \neq 0\}.
\]

A codeword \( c \) of the linear code \( C \) is called minimal if its support does not contain the support of any other linearly independent codeword. \( C \) is called a minimal linear code if all codewords of \( C \) are minimal. Minimal codes are a special class of linear codes. They have applications in secret sharing schemes [24, 25]. A sufficient condition for a linear code to be minimal is given in the following lemma [1].

**Lemma 1** (Aschikhmin-Barg). A linear code \( C \) over \( \text{GF}(q) \) is minimal if \( \frac{w_{\min}}{w_{\max}} > \frac{q-1}{q} \).

Many minimal linear codes satisfying the condition \( \frac{w_{\min}}{w_{\max}} > \frac{q-1}{q} \) are obtained from linear codes with few weights [16, 17, 27, 28, 29, 31].

The sufficient condition in Lemma 1 is not necessary for minimal codes. Recently, searching for minimal linear codes with \( \frac{w_{\min}}{w_{\max}} \leq \frac{q-1}{q} \) has been an interesting research topic. Chang and Hyun [13] made a breakthrough and constructed an infinite family of minimal binary linear codes with \( \frac{w_{\min}}{w_{\max}} < \frac{1}{2} \). Ding, Heng and Zhou [18] gave a necessary and sufficient condition for a binary linear code to be minimal. Three infinite families of minimal binary linear codes with \( \frac{w_{\min}}{w_{\max}} \leq \frac{1}{2} \) were obtained using this condition. They also constructed an infinite family of minimal ternary linear codes with \( \frac{w_{\min}}{w_{\max}} \leq \frac{2}{3} \) in [19]. Bartoli and Bonini [9] generalized the construction of minimal linear codes in [19] from ternary case to odd characteristic case. In [11], an inductive construction of minimal codes was presented. Li and Yue [22] obtained some minimal binary linear codes with Boolean functions. Xu and Qu [33] constructed minimal \( q \)-ary linear codes from some special functions. Lu, Wu and Cao [23] studied the existence of minimal linear codes. Bonini and Borello [10] presented a family of codes arising from cutting blocking sets. Infinitely many of these codes do not satisfy Aschikhmin-Barg’s condition.

In this paper, we mainly study further the characterizations and constructions of minimal codes. First, we investigate in detail the relationship between minimal linear codes and blocking sets, and completely characterize minimal linear codes as cutting blocking sets. In particular, minimal projective codes of dimension 3 and \( t \)-fold blocking sets with \( t \geq 2 \) in projective planes are identical objects. A tight lower bound for the minimum distance of a minimal code was derived using the geometric characterization of minimal codes. This settles the conjecture in [2]. By the new characterization of minimal linear codes, we present a primary construction and a general secondary construction of minimal codes. Some new infinite classes of minimal \( q \)-ary linear codes with \( \frac{w_{\min}}{w_{\max}} \leq \frac{q-1}{q} \) are derived. Finally we determine the weight distributions of two subfamilies of the proposed minimal codes. Open problems are also presented.

The rest of this paper is organized as follows. In Section 2 we present some basic results on linear codes from defining sets and blocking sets. In Section 3 we study in detail the relationship between minimal linear codes and blocking sets. It enables us to identify a minimal code as a cutting blocking set. In Section 4 we present a primary construction of minimal codes from hyperplanes and a general secondary construction of minimal codes, and establish the weight distributions of two subfamilies of the proposed minimal codes. In section 5 we conclude this paper.
2. Background

2.1. Linear codes from multisets in vectorial spaces

Let $V$ be a vector space over $\text{GF}(q)$ and let $\langle \cdot, \cdot \rangle$ be an inner product in $V$. For $v \in V \setminus \{0\}$ we will denote by $\langle v \rangle$ the one dimensional subspace generated by $v$. A multiset $D$ in $V$ is $k$-dimensional if the linear subspace $\text{Span}(D)$ over $\text{GF}(q)$ generated by $D$ has dimension $k$. A subset $D$ of $V \setminus \{0\}$ is called a projection of $D$ if for any nonzero $v \in D$ there exists a unique $v' \in D$ such that $v = \lambda v'$ where $\lambda \in \text{GF}(q)^*$. And $D$ is called projective if $\#D = \#D^*$. Let $D := \{\{g_1, \cdots, g_n\}\}$ be a multiset in $k$-dimensional vector space $V$. A classic generic construction of linear codes from multisets of vector spaces is described as $C_D = \{(\langle v, g_0 \rangle, \cdots, \langle v, g_{n-1} \rangle) : v \in V\}$. (1)

Employing this general scheme, Ding et al. constructed many families of linear codes with few weights $[14, 15]$. We call $D$ the defining set of $C_D$. By definition, the dimension of the code $C_D$ is at most $k$. Although different orderings of the elements of $D$ give different linear codes, these codes are permutation equivalent. Hence we do not consider these codes obtained by different orderings of the elements in $D$. Further, $C_D$ is a projective code if and only if $D$ is projective. Xiang proved that any $q$-ary linear code $C$ may be generated with a defining set $D$ via this construction $[32]$. The familiar Griesmer bound says that for an $[n,k,d]$ code $C$ over $\text{GF}(q)$,

$$n \geq d + \left\lceil \frac{d}{q} \right\rceil + \cdots + \left\lceil \frac{d}{q^{k-1}} \right\rceil.$$ (2)

The bound was proved by Griesmer in 1960 for binary linear codes and generalized by Solomon and Stiffler in 1965.

2.2. Blocking sets and blocking multisets

In $[10]$, Bonini and Borello introduced the concept of cutting $s$-blocking sets for sets in affine, projective and vector spaces. In order to study minimal codes, we will consider the corresponding concept for multisets. For any multiset $D$ in a vector space $V$, we will denote the multiset $D \setminus \{0\}$ by $D^*$. In geometry, a blocking set is a set of points in a projective plane which every line intersects and which does not contain an entire line. Instead of talking about projective planes and lines, one could deal with high dimensional spaces and their subspaces.

**Definition 2.** Let $D$ be a multiset of an $n$-dimensional vector space $V$. Then $D$ is called a vectorial $s$-blocking multiset if every subspace of codimension $s$ of $V$ has a non-empty intersection with $D^*$. Furthermore, if $D$ is a set, $D$ is also called a vectorial $s$-blocking set. A vectorial 1-blocking multiset is also referred as vectorial blocking multiset.

A generalization of blocking sets, called multiple blocking sets, was introduced by Bruen in $[6]$.

**Definition 3.** A vectorial $s$-blocking multiset $D$ of a vector space $V$ is called $t$-fold if every $(n-s)$-dimensional subspace contains at least $t$ elements of $D^*$ and some $(n-s)$-dimensional subspace contains exactly $t$ elements of $D^*$.
In order to investigate minimal codes, Bonini and Borello introduced the crucial concept of cutting blocking sets \([10]\).

**Definition 4.** A vectorial \(s\)-blocking multiset \(D\) in a vector space \(V\) is cutting if its intersection with every linear subspace of codimension \(s\) of \(V\) is not contained in any other linear subspace of codimension \(s\).

With some effort, we can give similarly the definitions of affine cutting \(s\)-blocking sets and projective cutting \(s\)-blocking sets; and the details for these can be found in \([10]\).

According to the previous definitions, \(D\) is a vectorial cutting \(s\)-blocking set if and only if \(\overline{D}\) is a vectorial cutting \(s\)-blocking set, where \(\overline{D}\) is a projection of \(D\). In fact, if \(\overline{D}\) is a vectorial cutting \(s\)-blocking set in an \(n\)-dimensional vector space \(V\) over \(GF(q)\), then \(\overline{D}\) can be identified as a projective cutting \(s\)-blocking set in the \((n−1)\)-dimensional projective space \(PG(n−1,q)\). If \(D\) is a projective cutting \(s\)-blocking set in \(PG(n−1,q)\), then so is \(D'\), where \(D \subseteq D'\). Some authors refer to a \(t\)-fold blocking set of cardinality \(f\) in \(PG(n−1,q)\) as an \(\{f,t;n−1,q\}\)-minihyper.

Bose and Burton determined the smallest point sets of \(PG(n,q)\) that meet every subspace of \(PG(n,q)\) of a given dimension \((n−s)\).

**Theorem 5** (Bose and Burton, \([3]\)). An \(s\)-blocking set of \(PG(n,q)\) has at least \(\Theta_s := \frac{q^{s+1}−1}{q−1}\) points. In the case of equality the \(s\)-blocking set is an \(s\)-dimensional subspace.

An \(s\)-blocking set containing an \(s\)-dimensional subspace is called trivial. For a non-trivial blocking set in the projective plane \(PG(2,q)\) the theorem gives an improved lower bound.

**Theorem 6** (Bruen, \([4,5]\)). In \(PG(2,q)\) a non-trivial blocking set has size at least \(q + \sqrt{q} + 1\). In the case of equality the blocking set is a Baer subplane.

A Baer subplane of \(PG(2,q)\) is an embedded \(PG(2,\sqrt{q})\) subgeometry.

The following theorem follows form a simple counting argument.

**Theorem 7** (Harrach. Proposition 1.5.3, \([20]\)). If \(D\) is any blocking set of the projective plane \(PG(2,q)\), then for any line \(\ell\) not contained in \(D\), we have \(\#(D \setminus \ell) \geq q\).

The following theorems give some general lower bounds on the size of a \(t\)-fold blocking set in \(PG(2,q)\).

**Theorem 8** (Ball, \([8]\)). Let \(D\) be a \(t\)-fold blocking set in \(PG(2,q)\). If \(D\) contains no line, then it has at least \(tq + \sqrt{tq} + 1\) points.

**Theorem 9** (Bruen, \([7]\)). Let \(D\) be a \(t\)-fold blocking set in \(PG(2,q)\) that contains a line. If \(t \geq 2\), then \(\#D \geq tq + q + t + 2\).

**Theorem 10** (Ball, \([8]\)). Let \(D\) be a nontrivial \(2\)-fold blocking set in \(PG(2,q)\). Then the following are true.

1. If \(q < 9\) then \(D\) has at least \(3q\) points.
2. If \(q = 11, 13, 17\) or \(19\) then \(\#D \geq \frac{5q+7}{2}\).
3. If \(q = p^{2e+1} > 19\) then \(\#D \geq 2q + p^e \left[ \frac{p^{e+1}+1}{p^e+1} \right] + 2\).
4. If \(q > 4\) is a square then \(\#D \geq 2q + 2\sqrt{q} + 2\).
The following result gives a depiction of 2-fold blocking sets in projective plane.

**Theorem 11** (Blokhuis, Storme and Szönyi, [12]). Let $D$ be a $t$-fold blocking set in $\text{PG}(2, q)$, $q = p^e$, $p$ prime, of size $t(q + 1) + c$. Let $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$. If $q > 4$ is a square, $t < q^{1/4}/2$ and $c < c_p q^{2/3}$, then $c \geq t \sqrt[4]{q}$ and $D$ contains the union of $t$ disjoint Baer subplanes, except for $t = 1$, in which case $D$ contains a line or a Baer subplane.

The following theorem presents a lower bound on the size of an affine blocking set in affine space.

**Theorem 12** (Jamison, [21]). If $\mathcal{V}$ is a vector space of dimension $k$ over the finite field $\text{GF}(q)$, then any subset of $\mathcal{V}$ which meets every hyperplane of $\mathcal{V}$ contains at least $k(q - 1) + 1$ points.

### 3. Characterization of minimal codes using cutting blocking sets

First of all, we present an equivalent description of cutting blocking sets.

**Proposition 13.** Let $D$ be a projective subset of a $n$-dimensional vector space $\mathcal{V}$. Then $D$ is a cutting blocking set if and only if for any $(n - 1)$-dimensional linear subspace $H$, the intersection $D \cap H$ is $(n - 1)$-dimensional.

**Proof.** Let $D$ be a cutting blocking set. Assume that $D \cap H$ is $k$-dimensional, where $H$ is $(n - 1)$-dimensional linear subspace and $k < (n - 1)$. Then there exists a basis $\{v_1, \ldots, v_n\}$ of $\mathcal{V}$ such that $\text{Span}(D \cap H) = \text{Span}(v_1, \ldots, v_k)$ and $H = \text{Span}(v_1, \ldots, v_{n-2}, v_{n-1})$. Let $H'$ be the $(n - 1)$-dimensional vectorial subspace spanned by $v_1, \ldots, v_{n-2}$ and $v_n$. Since $k < (n - 2)$, $H$ and $H'$ are different $(n - 1)$-dimensional subspaces. On the other hand, it’s easily observed that

$$D \cap H = D \cap \text{Span}(v_1, \ldots, v_k) \subseteq D \cap H'.$$

This is contrary to the definition of cutting blocking sets.

Conversely, assume that for any $(n - 1)$-dimensional linear subspace $H$, the intersection $D \cap H$ is $(n - 1)$-dimensional. Then, $D$ is a blocking set. We only need to prove that $D$ is cutting. Suppose $D$ is not cutting. Then, there are two different $(n - 1)$-dimensional subspaces $H$ and $H'$ such that $D \cap H \subseteq D \cap H'$. Thus, $D \cap H$ is contained in the $(n - 2)$-dimensional vectorial subspace $H \cap H'$. This is contrary to the assumption that $D \cap H$ is $(n - 1)$-dimensional. This completes the proof. 

We can now state the main theorem, i.e., the characterization of minimal codes in terms of cutting blocking sets, which shows that minimal codes and vectorial cutting blocking multisets are identical objects. Similar results have been obtained independently and simultaneously by Alfarano, Borello and Neri in [2].

**Theorem 14.** Let $C$ be a $q$-ary $[n, k]$ linear code with generator matrix $G = [g_0, \ldots, g_{n-1}]$, where $g_i \in \text{GF}(q)^k$. Let $\overline{D}$ denote any projection of the multiset $D = \{g_0, \ldots, g_{n-1}\}$. Then, $C$ is a minimal code if and only if $\overline{D}$ is a vectorial cutting blocking set in the $k$-dimensional vector space $\text{GF}(q)^k$, in other words, $\overline{D}$ is a projective cutting blocking set in $\text{PG}(k - 1, q)$. 


Proof. Since \( \mathcal{C} \) is a minimal code if and only if \( \overline{\mathcal{C}} \) is a minimal code, we only need to consider the case that \( \mathcal{C} \) is projective. When \( \mathcal{C} \) is a projective code, we can choose \( \overline{\mathcal{D}} = D \). Notice that for any codeword \( \mathbf{c} \in \mathcal{C} \), there exists a unique \( \mathbf{v} \in \text{GF}(q)^k \) such that
\[
\mathbf{c} = \mathbf{c}_\mathbf{v} := (\langle \mathbf{v}, g_0 \rangle, \cdots, \langle \mathbf{v}, g_{n-1} \rangle).
\] (3)

Then
\[
\text{Supp}(\mathbf{c}) = \{0,1,\cdots,n-1\} \setminus \{i : g_i \in H_\mathbf{v} \cap D\},
\] (4)
where \( H_\mathbf{v} \) is the \((k-1)\)-dimensional vectorial subspace \( \{ \mathbf{w} \in \text{GF}(q)^k : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \} \).

Let \( \mathcal{C} \) be a minimal code. Assume that \( D \) is not a cutting blocking set. Then there exist two nonzero vectors \( \mathbf{v}, \mathbf{v}' \) such that \( H_\mathbf{v} \cap D \subseteq H_\mathbf{v}' \cap D \) and \( \mathbf{v} \notin \langle \mathbf{v}' \rangle \). Hence \( \text{Supp}(\mathbf{c}') \subseteq \text{Supp}(\mathbf{c}) \) and \( \mathbf{c} = \lambda \mathbf{c}' \) for any \( \lambda \in \text{GF}(q) \), where \( \mathbf{c} = (\langle \mathbf{v}, g_0 \rangle, \cdots, \langle \mathbf{v}, g_{n-1} \rangle) \) and \( \mathbf{c}' = (\langle \mathbf{v}', g_0 \rangle, \cdots, \langle \mathbf{v}', g_{n-1} \rangle) \). This is contrary to the assumption that \( \mathcal{C} \) is a minimal code.

Conversely, let \( \overline{\mathcal{D}} \) be a cutting blocking set in the \( k \)-dimensional vector space \( \text{GF}(q)^k \). Suppose that \( \mathcal{C} \) is not a minimal code. Then there exist two nonzero \( \mathbf{v}, \mathbf{v}' \in \text{GF}(q)^k \) such that \( \text{Supp}(\mathbf{c}_\mathbf{v}) \subseteq \text{Supp}(\mathbf{c}_\mathbf{v}') \) and \( \mathbf{v} \neq \lambda \mathbf{v}' \) for any \( \lambda \in \text{GF}(q) \), where \( \mathbf{c}_\mathbf{v} \) and \( \mathbf{c}_\mathbf{v}' \) are given by (3). Applying (4), one obtains \( H_\mathbf{v}' \cap D \subseteq H_\mathbf{v} \cap D \). This is contrary to the assumption that \( D \) is cutting. This completes the proof.

Example 15. Let \( D_{\leq h} \) be the subset of \( \text{GF}(2)^k \) defined by
\[
D_{\leq h} = \left\{ \mathbf{x} \in \text{GF}(2)^k : 1 \leq \text{wt}(\mathbf{x}) \leq h \right\},
\]
where \( k \) and \( h \) are integers such that \( k \geq 4 \) and \( 2 \leq h \leq k \). Then, it was shown that \( \mathcal{C}_{D_{\leq h}} \) is a minimal linear code \([34]\). From Theorem 14, \( D_{\geq 2} \) is a vectorial cutting blocking set in \( \text{GF}(2)^k \). Hence, for any \( h \geq 2 \), \( D_{\leq h} \) is also a vectorial cutting blocking set in \( \text{GF}(2)^k \) as \( D_{\geq 2} \subseteq D_{\leq h} \). By Theorem 14 again, \( \mathcal{C}_{D_{\leq h}} \) is a minimal linear code.

Example 16. For \( q \) odd, let \( D_{\geq h} \) be the subset of \( \text{GF}(q)^k \) defined by
\[
D_{\geq h} = \left\{ \mathbf{x} \in \text{GF}(q)^k : \text{wt}(\mathbf{x}) \geq h \right\},
\]
where \( k \) and \( h \) are integers such that \( k \geq 4 \) and \( 1 \leq h \leq k - 1 \). Then, it was shown that \( D_{\geq h} \) is a vectorial cutting blocking set in \( \text{GF}(q)^k \) \([10]\). It follows from Theorem 14 that \( \mathcal{C}_{D_{\geq h}} \) is a minimal linear code and \( \overline{\mathcal{C}}_{D_{\geq h}} \) is a minimal projective code.

Example 17. Let \( k = h\ell \) be a positive integer and consider the subset of \( \text{GF}(q)^k \) defined by
\[
D_{h,\ell} = \left\{ (x_1, \cdots, x_n) \in \text{GF}(q)^k \setminus \{0\} : \sum_{j=0}^{\ell-1} x_{h(j+1)-1}x_{h(j+2)} \cdots x_{h(j+h)} = 0 \right\},
\] (5)
where \( h \) and \( \ell \) are integers such that \( h \geq 2 \) and \( \ell \geq 2 \). It follows from \([10, \text{Theorem 5.1}] \) and Theorem 14 that \( \mathcal{C}_{D_{h,\ell}} \) is a minimal linear code and \( \overline{\mathcal{C}}_{D_{h,\ell}} \) is a minimal projective code.
Combining Proposition[13] and Theorem[14] yields the following results.

**Corollary 18.** Let $C$ be a $q$-ary $[n,k]$ minimal linear code with generator matrix $G = [g_0, \ldots, g_{n-1}]$, where $g_i \in \text{GF}(q)^k$. Let $\overline{D}$ denote any projection of the multiset $D = \{g_0, \ldots, g_{n-1}\}$. Then $\overline{D}$ is a $t$-fold blocking set in $\text{PG}(k-1,q)$ with $t \geq (k-1)$.

**Theorem 19.** Let $C$ be a $q$-ary $[n,3]$ linear code with generator matrix $G = [g_0, \ldots, g_{n-1}]$, where $g_i \in \text{GF}(q)^3$. Let $\overline{D}$ denote any projection of the multiset $D = \{g_0, \ldots, g_{n-1}\}$. Then $C$ is a minimal code, if and only if, $\overline{D}$ is a $t$-fold blocking set in $\text{PG}(2,q)$ with $t \geq 2$.

Theorem[19] shows that minimal projective codes of dimension 3 and $t$-fold blocking sets with $t \geq 2$ in projective planes are identical objects. It is worth noting that, in the projective space $\text{PG}(k-1,n)$ with $k \geq 4$, the concept of cutting blocking sets is stronger than the concept of $(k-1)$-fold blocking sets. Therefore, the result of Theorem[19] cannot be extended to the case $k \geq 4$.

**Corollary 20.** Let $C$ be a $q$-ary $[n,k]$ minimal projective linear code with maximum weight $w_{\text{max}}$ and generator matrix $G = [g_0, \ldots, g_{n-1}]$, where $g_i \in \text{GF}(q)^k$. Then $D = \{g_0, \ldots, g_{n-1}\}$ is a $t$-fold blocking set in $\text{PG}(k-1,q)$ with $t = n - w_{\text{max}}$.

**Corollary 21.** Let $C$ be a $q$-ary $[n,k]$ minimal linear code of maximum weight $w_{\text{max}}$. Then $w_{\text{max}} \leq n-k+1$.

Using the link between minimal codes and blocking sets, we then deduce our lower bound on the minimum distance of a $q$-ary linear code with prescribed dimension. This confirms a recent conjecture by Alfarano, Borello and Neri in [2].

**Theorem 22.** Let $C$ be a minimal linear codes of dimension $k$ over $\text{GF}(q)$ and let $d$ be the minimum distance of $C$. Then

$$d \geq (q-1)(k-1) + 1.$$  

**Proof.** It suffices to prove the theorem in the case when $C$ is a projective code. Without loss of generality we can assume that the vector $(0,0,\ldots,0,1,1,\ldots,1)$ is a minimum-weight codeword in $C$. Thus there exists a generator matrix of $C$ under the form:

$$G = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\end{bmatrix},$$

where $n$ is the length of the linear code $C$. Write $u_i = (u_{i,1}, \ldots, u_{i,n-k+1})$ and $v_j = (v_{j,1}, \ldots, v_{j,n-k+1})$, where $1 \leq i \leq n-d$ and $1 \leq j \leq d$. Let $D_0 = \{(0,u_i) : 1 \leq i \leq n-d\}$, $D_1 = \{(1,v_j) : 1 \leq j \leq d\}$ and $B = \{v_j : 1 \leq j \leq d\}$. From Theorem[14] it follows that $D_0 \cup D_1$ is a vectorial cutting blocking set in the $k$-dimensional vector space $\text{GF}(q)^k$. We next claim that $B$ is an affine cutting blocking set in $\text{GF}(q)^{k-1}$. Let $H$ be any affine hyperplane in the affine space $\text{GF}(q)^{k-1}$ defined by $a_1 x_1 + \cdots + a_{k-1} x_{k-1} = b$, where $(a_1, \ldots, a_{k-1}) \in \text{GF}(q)^{k-1} \setminus \{0\}$ and $b \in \text{GF}(q)$. Let us denote by $\tilde{H}$ the hyperplane $\{(x_0, \ldots, x_{k-1}) \in \text{GF}(q)^k : -bx_0 + a_1 x_1 + \cdots + a_{k-1} x_{k-1} = 0\}$. Combining Proposition[13] with Theorem[14] we see that the set $\tilde{H} \cap (D_0 \cup D_1)$ is $(k-1)$-dimensional. It is evident...
that \( \dim \left( \text{Span} \left( \tilde{H} \cap D_0 \right) \right) \leq k - 2 \). Consequently, the intersection of \( \tilde{H} \) and \( D_1 \) is not empty. In particular, there exists a vector \((x_1, \cdots, x_{k-1}) \in B \) such that \((1, x_1, \cdots, x_{k-1}) \in \tilde{H} \cap D_1\), that is, \(a_1x_1 + \cdots + a_{k-1}x_{k-1} = b\). Thus \((x_1, \cdots, x_{k-1}) \in H \cap B\). According to the above discussion, it follows that \( B \) is an affine blocking set in \( \text{GF}(q)^{k-1} \). The desired conclusion then follows from Theorem 12.

Combining Theorem 22 with Griesmer bound in (2) we get the following theorem, which improves the results in [2] and [23].

**Theorem 23.** Let \( k \geq 2 \) and let \( C \) be a \( q \)-ary \([n,k]\) minimal linear code. Then,

\[
n \geq (q - 1)(k - 1) + 1 + \sum_{i=1}^{k-1} \left\lceil \frac{(q - 1)(k - 1) + 1}{q^i} \right\rceil.
\]

In particular, there does not exist a \( q \)-ary minimal linear code of length \( n \) and dimension larger than \( \frac{n}{q} + 1 \).

The lower bound \((q - 1)(k - 1) + 1 + \sum_{i=1}^{k-1} \left\lceil \frac{(q - 1)(k - 1) + 1}{q^i} \right\rceil\) on the length \( n \) of minimal \([n,k,d]_q\) linear codes is not very tight. It would be nice if the following problem can be settled.

**Open Problem 24.** Determine the minimum length of minimal \( q \)-ary linear codes with dimension \( k \).

### 4. New constructions of minimal linear codes

In this section, we present new primary and secondary constructions of minimal linear codes.

#### 4.1. Primary construction of minimal codes via unions of hyperplanes

Let \( H_a \) denote the hyperplane \( \{x \in \text{GF}(q)^k : \langle a, x \rangle = 0\} \), where \( a \in \text{GF}(q)^k \) is nonzero. For any nonempty subset \( S \) of \( \text{GF}(q)^k \setminus \{0\} \), let \( D_S \) be the subset of \( \text{GF}(q)^k \) defined by

\[
D_S = (\cup_{a \in S} H_a) \setminus \{0\}.
\]

**Proposition 25.** Let \( S \) be a nonempty proper subset of \( \text{GF}(q)^k \setminus \{0\} \) where \( k \geq 3 \). Then the set \( D_S \) defined by (6) is a vectorial cutting blocking set in \( \text{GF}(q)^k \), if and only if, the dimension of \( \text{Span}(S) \) is at least 3.

**Proof.** Assume that \( \dim_{\text{GF}(q)} \left( \text{Span}(S) \right) = 1 \). Then \( D_S = H_a \setminus \{0\} \) for some \( a \in \text{GF}(q)^k \). Choose any \( a' \in \text{GF}(q)^k \setminus \{a\} \), then \( H_{a' + a} \cap D_S \subseteq H_{a'} \). Note that \( H_{a' + a} \) and \( H_{a'} \) are two distinct hyperplanes, which shows that \( D_S \) is not a vectorial cutting blocking set in \( \text{GF}(q)^k \) from Definition 4.

For the case \( \dim_{\text{GF}(q)} \left( \text{Span}(S) \right) = 2 \), there exist \( a, a' \in \text{GF}(q)^k \) such that

\[
D_S = \left( \bigcup_{i=1}^{h} H_{a + \lambda_i a'} \right) \cup (H_a \cup H_{a'}),
\]

where \( \lambda_i \in \text{GF}(q)^* \). Since \( D_S \) is a proper subset of \( \text{GF}(q)^k \) and \( \text{GF}(q)^k = \left( \bigcup_{\lambda \in \text{GF}(q)^*} H_{a + \lambda a'} \right) \cup (H_a \cup H_{a'}) \), there exists a \( \lambda \in \text{GF}(q)^* \) such that \( D_S \) does not contain the hyperplane \( H_{a + \lambda a'} \). It is
observed that $H_{a_1 + \lambda a'} \cap H_{a''} = H_a \cap H_{a''}$ for any $a'' \in S$, which implies that $H_{a_1 + \lambda a'} \cap D_S \subseteq H_a$. By Definition 24, $D_S$ is not a vectorial cutting blocking set in GF($q$)$^k$.

Suppose that $\dim_{GF(q)}(\text{Span}(S)) \geq 3$. Let $H_{a_1}$ and $H_{a_2}$ be two distinct hyperplanes. Since the dimension of $\text{Span}(S)$ is at least 3, there exists $a \in S$ such that $\{a_1, a_2, a\}$ is 3-dimensional. Then, there exists a solution $x \in GF(q)^k$ for the linear system

$$
\begin{align*}
\langle a_1, x \rangle &= 1, \\
\langle a_2, x \rangle &= 0, \\
\langle a, x \rangle &= 0.
\end{align*}
$$

From $x \in H_a \subseteq D_S$, one obtains $x \in H_{a_2} \cap D_S$ and $x \notin H_{a_1}$, that is, $H_{a_2} \cap D_S$ is not contained in the hyperplane $H_{a_1}$. It follows that $D_S$ is a vectorial cutting blocking set in GF($q$)$^k$ from Definition 24. This completes the proof.

The following theorem is derived from Theorem 26 and Proposition 25, which describes minimal codes constructed by unions of hyperplanes.

**Theorem 26.** Let $S$ be a nonempty proper subset of GF($q$)$^k \setminus \{0\}$ where $k \geq 3$. Let $D_S$ be the set and $C_{D_S}$ be the code defined by (4) and (7), respectively. Then $C_{D_S}$ is a minimal code of dimension $k$, if and only if, the dimension of Span(S) is at least 3.

As a result of Theorem 26, we have the following for minimal projective codes.

**Corollary 27.** Let $S$ be a nonempty proper subset of GF($q$)$^k \setminus \{0\}$ where $k \geq 3$. Let $D_S$ be the set defined by (6) and let $\overline{D}_S$ be any projection of $D_S$. Then $\overline{C}_{D_S}$ is a minimal projective code of dimension $k$, if and only if, the dimension of Span(S) is at least 3.

It is easily observed that the following lemma holds.

**Lemma 28.** Let $T$ be a subset of cardinality $t$ of $\{1, 2, \cdots, k\}$ and $a \in GF(q)^k$. Let $N(a, T)$ denote the number of solutions $x = (x_1, \cdots, x_k) \in GF(q)^k$ of the following system of linear equations

$$
\begin{align*}
x_j &= 0, \quad \text{for } j \in T, \\
\langle a, x \rangle &= 0.
\end{align*}
$$

Then

$$
N(a, T) = \begin{cases} 
q^{k-t}, & \text{if } \text{Supp}(a) \subseteq T, \\
q^{k-t-1}, & \text{otherwise}.
\end{cases}
$$

**Theorem 29.** Let $h$ and $k$ be two integers with $3 \leq h \leq k$ and $D = \{(x_1, \cdots, x_k) \in GF(q)^k \setminus \{0\} : x_1 \cdots x_h = 0\}$. Then the code $C_D$ in Equation (1) is a $([q^{h} - (q - 1)^h]q^{k-h} - 1, k, w_{\min}]$ minimal linear code with the weight distribution in Table 1 where $w_{\min} = q^{k-h}([q^{h} - (q - 1)^h]q^{h-1} - (q - 1)^h)$.

Furthermore,

$$
\frac{w_{\min}}{w_{\max}} \leq \frac{q - 1}{q},
$$

if and only if $h \leq 1 + \frac{1}{\log_2(q - 1)}$. 

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By Theorem 26, \( C_D \) is a minimal code of dimension \( k \). Clearly, the length of \( C_D \) is \( \#D = (q^h - (q - 1)^h)q^{k-h} - 1 \).

Next, we consider the weight distribution of \( C_D \). Let \( \mathbf{c}_a = (\langle \mathbf{a}, \mathbf{x} \rangle)_{\mathbf{x} \in D} \) be a codeword in \( C_D \) corresponding to \( \mathbf{a} \in GF(q)^k \setminus \{0\} \). Let \( \text{zr}(\mathbf{c}_a) \) be the number of solutions of the system of equations

\[
\begin{align*}
x_1 x_2 \cdots x_h &= 0, \\
\langle \mathbf{a}, \mathbf{x} \rangle &= 0.
\end{align*}
\]

Applying the inclusion-exclusion principle, we see that

\[
\text{zr}(\mathbf{c}_a) = \sum_{i=1}^{h} (-1)^{i-1} \sum_{T \subseteq \{1, \ldots, h\}, \#T = i} N(\mathbf{a}, T)
\]

\[
= \sum_{T \subseteq \{1, \ldots, h\}} (-1)^{\#T-1} N(\mathbf{a}, T)
\]

\[
= \sum_{\text{Supp}(\mathbf{a}) \not\subseteq T \subseteq \{1, \ldots, h\}} (-1)^{\#T-1} N(\mathbf{a}, T)
\]

\[
= \sum_{\text{Supp}(\mathbf{a}) \subseteq T \subseteq \{1, \ldots, h\}} (-1)^{\#T-1} N(\mathbf{a}, T).
\]

where \( N(\mathbf{a}, T) \) is defined as in Lemma 28 if \( T \neq \emptyset \) and \( N(\mathbf{a}, T) = 0 \) if \( T = \emptyset \). By Lemma 28, we deduce

\[
\text{zr}(\mathbf{c}_a) = \sum_{\text{Supp}(\mathbf{a}) \not\subseteq T \subseteq \{1, \ldots, h\}, T \neq \emptyset} (-1)^{\#T-1} q^{k-\#T-1}
\]

\[
+ \sum_{\text{Supp}(\mathbf{a}) \subseteq T \subseteq \{1, \ldots, h\}} (-1)^{\#T-1} q^{k-\#T}
\]

\[
= \sum_{T \subseteq \{1, \ldots, h\}, T \neq \emptyset} (-1)^{\#T-1} q^{k-\#T-1}
\]

\[
+ (q-1) \sum_{\text{Supp}(\mathbf{a}) \subseteq T \subseteq \{1, \ldots, h\}} (-1)^{\#T-1} q^{k-\#T-1}
\]

\[
= \left\{ \begin{array}{ll}
\sum_{i=1}^{h} (-1)^{i+1} \binom{h}{i} q^{k-i-1}, & \text{if } \text{Supp}(\mathbf{a}) \not\subseteq \{1, \ldots, h\}, \\
\sum_{i=1}^{h} (-1)^{i+1} \binom{h}{i} q^{k-i-1} + (q-1) \sum_{i=3}^{h} (-1)^{i+1} \binom{h-2}{i-3} q^{k-i-1}, & \text{otherwise},
\end{array} \right.
\]

Table 1: The weight distribution of the code \( C_D \) of Theorem 29

| Weight \( w \) | No. of codewords \( A_w \) |
|----------------|-----------------------------|
| \((q-1)q^{k-h-1}(q^h - (q - 1)^h)\) | \(q^k - q^h\) |
| \((q-1)q^{k-h-1}(q^h - (q - 1)^h)\) | \((q-1)^{s(h)}\) |
| \(+(-1)^s q^{k-h-1}(q - 1)^{h-s+1}\) for \( s = 1, 2, \ldots, h \) | |

\[
= \left\{ \begin{array}{ll}
\sum_{i=1}^{h} (-1)^{i+1} \binom{h}{i} q^{k-i-1}, & \text{if } \text{Supp}(\mathbf{a}) \not\subseteq \{1, \ldots, h\}, \\
\sum_{i=1}^{h} (-1)^{i+1} \binom{h}{i} q^{k-i-1} + (q-1) \sum_{i=3}^{h} (-1)^{i+1} \binom{h-2}{i-3} q^{k-i-1}, & \text{otherwise},
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\sum_{i=1}^{h} (-1)^{i+1} \binom{h}{i} q^{k-i-1}, & \text{if } \text{Supp}(\mathbf{a}) \not\subseteq \{1, \ldots, h\}, \\
\sum_{i=1}^{h} (-1)^{i+1} \binom{h}{i} q^{k-i-1} + (q-1) \sum_{i=3}^{h} (-1)^{i+1} \binom{h-2}{i-3} q^{k-i-1}, & \text{otherwise},
\end{array} \right.
\]
where \( s = \text{wt}(a) \). Then, one obtains

\[
\text{wt}(c_a) = \# D - (zr(c_a) - 1) = \begin{cases} 
(q^h - (q-1)^h)q^{k-h} - \sum_{i=1}^{h}(-1)^{i+1} \binom{h}{i} q^{k-i-1} \\
-q^h \sum_{i=s}^{h}(-1)^{i+1} \binom{h-s}{i} q^{k-i-1}, & \text{if } \text{Supp}(a) \subseteq \{1, \ldots, h\}, \\
(q^h - (q-1)^h)q^{k-h} - \sum_{i=1}^{h}(-1)^{i+1} \binom{h}{i} q^{k-i-1}, & \text{otherwise}. 
\end{cases}
\]

By the identities \( \sum_{i=s}^{h}(-1)^{i+1} \binom{h-s}{i} q^{k-i-1} = (-1)^{s+1} q^k - q^h - q^{-s} \) and \( \sum_{i=1}^{h}(-1)^{i+1} \binom{h}{i} q^{k-i+1} = q^{k-1} - q^{-h-1}(q-1)^h \), one gets

\[
\text{wt}(c_a) = \begin{cases} 
(q-1)q^{k-h-1}(q^h - (q-1)^h) \\
+(-1)^s q^k - (q-1)^{h-s+1}, & \text{if } \text{Supp}(a) \subseteq \{1, \ldots, h\}, \\
(q-1)q^{k-h-1}(q^h - (q-1)^h), & \text{otherwise}, 
\end{cases}
\]

where \( a \neq \mathbf{0} \) and \( s = \text{wt}(a) \). The weight distribution in Table 1 then follows from Equation (7).

From the weight distribution of \( C_D \) in Table 1, one gets

\[
\text{w}_{\text{min}} = (q-1)q^{k-h-1}(q^h - (q-1)^h - (q-1)^{h-1})
\]

and

\[
\text{w}_{\text{max}} = (q-1)q^{k-h-1}(q^h - (q-1)^h + (q-1)^{h-2}).
\]

Then

\[
\frac{\text{w}_{\text{min}}}{\text{w}_{\text{max}}} = \frac{q^h - (q-1)^h}{q^h - 2(q-1)^{h-2}}. 
\]

Note that

\[
(q-1)B - qA = (q-1)^{h-1} \left( 2 - \left( \frac{q}{q-1} \right)^{h-1} \right),
\]

where \( A = q^{h-1} - (q-1)^{h-1} \) and \( B = q^h - (q-2)(q-1)^{h-2} \). Combining Equations (8) and (9) gives that \( \frac{\text{w}_{\text{min}}}{\text{w}_{\text{max}}} \leq \frac{q-1}{q} \), if and only if \( h \leq 1 + \frac{1}{\log_2 \frac{q}{q-1}} \). This completes the proof.

Corollary 30. Let \( h \) and \( k \) be two integers with \( 3 \leq h \leq k \) and \( D = \{(x_1, \ldots, x_k) \in \text{GF}(q)^k \setminus \{0\} : x_1 \cdots x_h = 0\} \). Let \( \overline{D} \) be any projection of \( D \). Then the code \( C_{\overline{D}} \) in Equation (7) is a minimal projective code with parameters \( \left[ \frac{q^h - 1}{q-1} - q^{-h} (q-1)^{h-1}, k, q^h (q^h - (q-1)^{h-1}) \right] \), whose weight distribution is listed in Table 2. Furthermore,

\[
\frac{\text{w}_{\text{min}}}{\text{w}_{\text{max}}} \leq \frac{q-1}{q},
\]

if and only if \( h \leq 1 + \frac{1}{\log_2 \frac{q}{q-1}} \).
Corollary 31. Let $D = \{(x_1, \cdots, x_k) \in \text{GF}(q)^k \setminus \{0\} : x_1x_2x_3 = 0\}$, where $q \geq 4$ and $k \geq 3$. Let $\overline{D}$ be any projection of $D$. Then the codes $C_D$ and $C_{\overline{D}}$ in Equation (7) are minimal codes with $\frac{w_{\text{min}}}{w_{\text{max}}} < \frac{q-1}{q}$.

Proof. The conclusion follows from Theorem 29, Corollary 30 and the inequality $1 + \frac{1}{\log_2(\frac{q}{q-1})} > 1 + \frac{1}{\log_2(\frac{q}{q-1})} \approx 3.41 > 3$.

The following numerical data is consistent with the conclusion of Theorem 29.

Example 32. Let $q = 3$, $k = 4$ and $h = 3$. Then the set $C_D$ in Theorem 29 is a minimal code with parameters $[56, 4, 30]$ and weight enumerator $1 + 6z^{30} + 8z^{36} + 54z^{38} + 12z^{42}$. Obviously, $h > 1 + \frac{1}{\log_2(\frac{q}{q-1})} \approx 2.71$ and $\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{5}{7} > \frac{3}{4}$.

Example 33. Let $q = 4$, $k = 4$ and $h = 3$. Then the set $C_D$ in Theorem 29 is a minimal code with parameters $[147, 4, 84]$ and weight enumerator $1 + 9z^{84} + 27z^{108} + 192z^{111} + 27z^{120}$. Obviously, $h < 1 + \frac{1}{\log_2(\frac{q}{q-1})} \approx 3.41$ and $\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{7}{15} < \frac{3}{4}$.

Lemma 34. Let $h \geq 2$ and $b \in \text{GF}(q)$. Let $N_{h,b}$ denote the cardinality of the set $\{(x_1, \cdots, x_h) \in \text{GF}(q)^h : x_1 + \cdots + x_h = b, x_i \neq 0 \text{ for } i = 1, \cdots, h\}$. Then

$$N_{h,b} = \begin{cases} 
(q - 1)^h + (-1)^h(q - 1), & \text{if } b = 0, \\
\frac{(q - 1)^h - (-1)^h}{q}, & \text{if } b \neq 0.
\end{cases}$$

Proof. By the definition of $N_{h,b}$, one has $N_{h,b} = N_{h,1}$ for $b \in \text{GF}(q)^*$ and $N_{h,0} + (q - 1)N_{h,1} = (q - 1)^h$.

Plugging $N_{h,0} = \sum_{a \in \text{GF}(q)^*} N_{h-1,a} = (q - 1)N_{h-1,1}$ into Equation (10), we deduce that

$$N_{h-1,1} + N_{h,1} = (q - 1)^{h-1}.$$ 

That is

$$(N_{h,1} - q^{-1}(q - 1)^h) + (N_{h-1,1} - q^{-1}(q - 1)^{h-1}) = 0.$$ 

Then

$$N_{h,1} = q^{-1}(q - 1)^h + (-1)^{h-1}(N_{h-1,1} - q^{-1}(q - 1)^{2-1})$$

$$= \frac{(q - 1)^h - (-1)^h}{q}.$$
Proof.

By the definition of the set $C_D$ in Equation (1) is an $[n,k,w_{\text{min}}]$ minimal linear code, where $n = q^{k-h-1} (q^{h+1} - (q-1)^{h+1} + (-1)^h(q-1)) - 1$ and

$$w_{\text{min}} = q^{k-h-1} \left((q-1)q^h - (q-1)^{h+1} + (-1)^h(q-1)\right).$$
Furthermore,\[ \frac{w_{\text{min}}}{w_{\text{max}}} \leq \frac{q - 1}{q}, \]

provided that \(2(q - 1)^h - q^h + (-1)^h(q - 2) \geq 0.\)

**Proof.** By Theorem 26, \(C_D\) is a minimal code of dimension \(k\). The length of the code \(C_D\) follows from Lemma 35. Let \(c_a = (\langle a, x \rangle)_{x \in D}\) be a codeword in \(C_D\) corresponding to \(a \in \text{GF}(q)^k \setminus \{0\}\).

Note that
\[
\begin{align*}
\text{wt}(c) &= n + 1 - q^{k-1}, \quad \text{where} \quad c = \{a \in \text{GF}(q)^k \setminus \{0\} \mid \langle a, x \rangle = 0 \}.
\end{align*}
\]

The desired value of \(w_{\text{min}}\) then follows from wt\((c)\) = \(n + 1 - q^{k-1}\), where \(c = \{a \in \text{GF}(q)^k \setminus \{0\} \mid \langle a, x \rangle = 0 \}.

Next, consider the weight \(w_2\) of the codeword \(c = (ax_{h-1} - x_h)_{(x_1, \ldots, x_k) \in D}\), where \(a \neq 0\) or \(-1\). Then
\[
\begin{align*}
w_2 &= n + 1 - \#\{(x_1, \ldots, x_k) \in D : ax_{h-1} - x_h = 0\}
\end{align*}
\]

where \(D_2\) denotes the set of solutions \((x_1, \ldots, x_{h}) \in \text{GF}(q)^h\) of the system of linear equations
\[
\begin{align*}
x_1 \cdots x_h(x_1 + \cdots + x_h) &= 0, \\
ax_{h-1} - x_h &= 0.
\end{align*}
\]

Thus, \(\#D_2\) equals the number of solutions \((x_1, \ldots, x_{h-1}) \in \text{GF}(q)^{h-1}\) of the equation
\[
x_1 \cdots x_{h-1}(x_1 + \cdots + x_{h-2} + (1 + a)x_{h-1}) = 0.
\]

Using the linear transformation
\[
\begin{align*}
x_j' &= x_j, \\
x_j' &= (1 + a)x_j, \quad \text{for} \quad 1 \leq j \leq h - 2, \\
x_j' &= (1 + a)x_j, \quad \text{for} \quad j = h - 1.
\end{align*}
\]

over \(\text{GF}(q)^{h-1}\), we deduce that the equations \(x_1 \cdots x_{h-1}(x_1 + \cdots + (1 + a)x_{h-1}) = 0\) and \(x_1 \cdots x_{h-1}(x_1 + \cdots + x_{h-1}) = 0\) have the same number of solutions over \(\text{GF}(q)^{h-1}\). By Lemma 35, \(\#D_2 = q^{k-h-1}(q^h - (q-1)^{h} + (-1)^{h}(q-2))\). It’s easily checked that
\[
(q-1)w_2 - qw_{\text{min}} = (q-1)q^{k-h-1} \left(2(q-1)^h - q^h + (-1)^h(q-2)\right).
\]

Hence, if \(2(q-1)^h - q^h + (-1)^h(q-2) \geq 0\), then \(\frac{w_{\text{min}}}{w_{\text{max}}} \leq \frac{w_{\text{min}}}{w_2} \leq \frac{q-1}{q}\). This completes the proof. \(\square\)

Let \(h \geq 3\) be an integer. Theorem 36 shows that there exists a constant \(q_0\) such that the code \(C_D\) in Theorem 36 is a minimal code with \(\frac{w_{\text{min}}}{w_{\text{max}}} \leq \frac{q-1}{q}\) for any power \(q \geq q_0\) of prime. Finally, we settle the weight distribution of the codes in Theorem 36 for the case \(h = 3\).
**Theorem 37.** Let $k$ be an integer with $k \geq 3$ and $D = \{(x_1, \cdots, x_k) \in \text{GF}(q)^k \setminus \{0\} : x_1x_2x_3(x_1 + x_2 + x_3) = 0\}$. Then the set $C_D$ in Equation (7) is an $[n, k, w_{\min}]$ minimal linear code with the weight distribution in Table 3 where $n = 4q^{k-1} - 6q^{k-2} + 3q^{k-3} - 1$ and $w_{\min} = 3q^{k-1} - 6q^{k-2} + 3q^{k-3}$. Furthermore,

$$\frac{w_{\min}}{w_{\max}} \leq \frac{q - 1}{q},$$

if and only if $q \geq 4$.

| Weight $w$ | No. of codewords $A_w$ |
|------------|-------------------------|
| 0          | 1                       |
| $3q^{k-1} - 6q^{k-2} + 3q^{k-3}$ | 4($q - 1$) |
| $4q^{k-1} - 10q^{k-2} + 6q^{k-3}$ | $(q - 1)(q - 2)(q - 3)$ |
| $4q^{k-1} - 10q^{k-2} + 9q^{k-3} - 3q^{k-4}$ | $q^k - q^3$ |
| $4q^{k-1} - 9q^{k-2} + 5q^{k-3}$ | $6(q - 1)(q - 2)$ |
| $4q^{k-1} - 8q^{k-2} + 4q^{k-3}$ | $3(q - 1)$ |

**Proof.** By Theorem 36, $C_D$ is a minimal code of dimension $k$ with length $n = 4q^{k-1} - 6q^{k-2} + 3q^{k-3} - 1$ and minimum weight $w_{\min} = 3q^{k-1} - 6q^{k-2} + 3q^{k-3}$.

Next, we consider the weight distribution of $C_D$. Let $c_a = (\langle a, x \rangle)_{x \in D}$ be a codeword in $C_D$ corresponding to $a \in \text{GF}(q)^k \setminus \{0\}$. Let $\text{zr}(c_a)$ be the number of solutions of the system of equations

$$\begin{cases}
    x_1x_2x_3(x_1 + x_2 + x_3) = 0 \\
    \langle a, x \rangle = 0
\end{cases}$$

Let $N'(a, T)$ denote the number of solutions $x = (x_1, \cdots, x_k) \in \text{GF}(q)^k$ of the following system of linear equations

$$\begin{cases}
    x_j = 0, & \text{for } j \in T, \\
    x_1 + x_2 + x_3 = 0, \\
    \langle a, x \rangle = 0
\end{cases}$$

Employing the inclusion-exclusion principle, we have

$$\text{zr}(c_a) = \sum_{i=0}^{3} (-1)^{i-1} \sum_{T \subseteq \{1, 2, 3\}, \#T = i} (N(a, T) - N'(a, T))$$

$$= \sum_{T \subseteq \{1, 2, 3\}} (-1)^{\#T-1} (N(a, T) - N'(a, T)), \quad (12)$$

where $N(a, T)$ is defined as in Lemma 28 if $T \neq 0$ and $N(a, T) = 0$ if $T = 0$. Then, we deduce

$$\text{zr}(c_a) = \begin{cases}
    q^{k-1}, & \text{if } \langle a, x \rangle = ax_i, a(x_1 + x_2 + x_3), \\
    2q^{k-2} - q^{k-3}, & \text{if } \langle a, x \rangle = a(x_1 + x_j), \\
    3q^{k-2} - 2q^{k-3}, & \text{if } \langle a, x \rangle = ax_i + bx_j, ax_i + ax_j + bx_k, \\
    4q^{k-2} - 3q^{k-3}, & \text{if } \langle a, x \rangle = ax_1 + bx_2 + cx_3, \\
    4q^{k-2} - 6q^{k-3} + 3q^{k-4}, & \text{if } \text{Supp}(a) \not\subseteq \{1, 2, 3\},
\end{cases}$$
where \(i, j, k\) are pairwise distinct integers in \(\{1, 2, 3\}\) and \(a, b, c \in \text{GF}(q)^*\) are pairwise distinct. Then, one obtains

\[
\text{wt}(c_a) = n - (\text{zd}(c_a) - 1) = \begin{cases} 
3q^{k-1} - 6q^{k-2} + 3q^{k-3}, & \text{if } \langle a, x \rangle = ax_i a(x_1 + x_2 + x_3), \\
4q^{k-1} - 8q^{k-2} + 4q^{k-3}, & \text{if } \langle a, x \rangle = a(x_i + x_j), \\
4q^{k-1} - 9q^{k-2} + 5q^{k-3}, & \text{if } \langle a, x \rangle = ax_i + bx_j a(x_i + ax_j + bx_k), \\
4q^{k-1} - 10q^{k-2} + 6q^{k-3}, & \text{if } \langle a, x \rangle = ax_1 + bx_2 + cx_3, \\
4q^{k-1} - 10q^{k-2} + 9q^{k-3} - 3q^{k-4}, & \text{if } \text{Supp}(a) \not\subseteq \{1, 2, 3\},
\end{cases}
\]

(13)

where \(i, j, k\) are pairwise distinct integers in \(\{1, 2, 3\}\) and \(a, b, c \in \text{GF}(q)^*\) are pairwise distinct. The weight distribution in Table 3 then follows from Equation (13).

From the weight distribution of \(C_D\) in Table 3 one gets

\[w_{\text{max}} = 4q^{k-1} - 8q^{k-2} + 4q^{k-3}.\]

Then

\[
\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{3}{4}.
\]

(14)

Equation (14) implies that \(\frac{w_{\text{min}}}{w_{\text{max}}} \leq \frac{q-1}{q}\), if and only if \(q \geq 4\). This completes the proof.

The following numerical data is consistent with the conclusion of Theorem 37.

**Example 38.** Let \(q = 3, k = 4\) and \(h = 3\). Then the set \(C_D\) in Theorem 37 is a minimal code with parameters \([62, 4, 36]\) and weight enumerator \(1 + 8z^{36} + 66z^{42} + 6z^{48}\). Obviously, \(\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{3}{4} > \frac{2}{3}\).

**Example 39.** Let \(q = 4, k = 4\) and \(h = 3\). Then the set \(C_D\) in Theorem 37 is a minimal code with parameters \([171, 4, 108]\) and weight enumerator \(1 + 12z^{108} + 6z^{120} + 192z^{129} + 36z^{132} + 9z^{144}\). Obviously, \(\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{3}{4}\).

The following open problems would be challenging.

**Open Problem 40.** Determine the weight distribution of the minimal code \(C_D\) for the case that

\[
D := \left\{(x_1, \cdots, x_k) \in \text{GF}(q)^k \setminus \{0\} : \left(\sum_{i=1}^{h} x_i\right) \prod_{i=1}^{h} x_i = 0\right\},
\]

where \(4 \leq h \leq k\).

**Open Problem 41.** Determine the parameters of the linear code \(C_D\) for the case that

\[
D := \left\{(x_1, \cdots, x_k) \in \text{GF}(q)^k \setminus \{0\} : \prod_{1 \leq i < j \leq h} (x_i + x_j) = 0\right\},
\]

where \(3 \leq h \leq k\).
**Open Problem 42.** Determine the parameters of the minimal code $C_D$ for the case that

$$D := \left\{ (x_1, \cdots, x_k) \in \mathbb{GF}(q)^k \setminus \{0\} : \prod_{i=1}^{h} x_i \prod_{1 \leq i < j \leq h} (x_i + x_j) = 0 \right\},$$

where $3 \leq h \leq k$.

**Open Problem 43.** Determine the parameters of the minimal code $C_D$ for the case that

$$D := \left\{ (x_1, \cdots, x_k) \in \mathbb{GF}(q)^k \setminus \{0\} : \left( \sum_{i=1}^{h} x_i \right) \prod_{i=1}^{h} x_i \prod_{1 \leq i < j \leq h} (x_i + x_j) = 0 \right\},$$

where $3 \leq h \leq k$.

Other good minimal codes may be produced by choosing an appropriate union of hyperplanes.

### 4.2. Secondary constructions of minimal codes

We introduce now a secondary construction of minimal linear codes, which allows constructing minimal codes of dimension $(k + 1)$ from $k$-dimensional minimal codes.

**Lemma 44.** Let $k \geq 2$. Let $D$ be a vectorial cutting blocking set in $\mathbb{GF}(q)^k$ such that $D = a \cdot D$ for any $a \in \mathbb{GF}(q)^*$. Then, for any $a \in \mathbb{GF}(q)^*$ and $c \in \mathbb{GF}(q)$, there exists $x \in D$ such that $\langle a, x \rangle = c$.

**Proof.** Suppose that there exists $a \in \mathbb{GF}(q)^*$ and $c \in \mathbb{GF}(q)$ such that $\langle a, x \rangle \neq c$ for any $x \in D$. Let $H_a$ be the hyperplane corresponding to $a$. If $c = 0$, then $H_a \cap D = \emptyset$, which is contrary to Proposition 13. If $c \neq 0$, then $D \subseteq H_a$. Thus, for any other hyperplane $H'$, $H' \cup D \subseteq H_a$, which is contrary to the definition of cutting blocking set. This completes the proof.

The following follows from the definition of cutting blocking sets directly.

**Lemma 45.** Let $k \geq 2$. Let $D$ be a vectorial cutting blocking set in $\mathbb{GF}(q)^k$. Let $a_1, a_2 \in \mathbb{GF}(q)^k$ such that they are linearly independent.

1. There exist $x, x' \in D$ such that $\langle a_1, x \rangle = 0$ and $\langle a_1, x' \rangle \neq 0$.
2. There exists $x \in D$ such that $\langle a_1, x \rangle \neq 0$ and $\langle a_2, x \rangle \neq 0$.

We now present the secondary construction of minimal linear codes in the next theorem.

**Theorem 46.** Let $k \geq 2$. Let $D_1$ and $D_2$ be two vectorial cutting blocking sets in $\mathbb{GF}(q)^k$ such that $D_1 = a \cdot D_1$ for any $a \in \mathbb{GF}(q)^*$. Define the subset of $\mathbb{GF}(q)^{k+1}$ by

$$[\widehat{D_1}, \widehat{D_2}] := \left\{ (x, 1) \in \mathbb{GF}(q)^{k+1} : x \in D_1 \right\} \cup \left\{ (x, 0) \in \mathbb{GF}(q)^{k+1} : x \in D_2 \right\}. \tag{15}$$

Then, $[\widehat{D_1}, \widehat{D_2}]$ is a vectorial cutting blocking set in $\mathbb{GF}(q)^{k+1}$. In particular, $C_{[\widehat{D_1}, \widehat{D_2}]}$ is a minimal code of length $(\#D_1 + \#D_2)$ and dimension $(k + 1)$.
Proof. By Theorem 44 the definition of cutting blocking sets and the assumption that $D_1 = a \cdot D_1$ for any $a \in \text{GF}(q)^*$, we only need to prove that for any $(a_i, b_i) \in \text{GF}(q)^k \times \text{GF}(q)$ ($i = 1, 2$), where $(a_1, b_1)$ and $(a_2, b_2)$ are linearly independent over $\text{GF}(q)$, there exists $(x, y) \in D_1 \times \text{GF}(q)^* \cup D_2 \times \{0\}$ such that

\[
\begin{align*}
\langle a_1, x \rangle + b_1y &\neq 0, \\
\langle a_2, x \rangle + b_2y &= 0.
\end{align*}
\] (16)

The proof is carried out by considering the following three cases.

If $a_1$ and $a_2$ are linearly independent, by the definition of cutting blocking sets, there exists an $x \in D_2$ such that $\langle a_1, x \rangle \neq 0$ and $\langle a_2, x \rangle = 0$. Then, $(x, 0) \in D_2 \times \{0\}$ satisfies (16).

If $a_1 = \lambda a_2$ and $a_2 \neq 0$, by Lemma 44 there exists an $x \in D_1$ such that $\langle a_2, x \rangle + b_2 = 0$. Then $(x, 1) \in D_1 \times \text{GF}(q)^*$ satisfies (16) from the fact $b_1 \neq \lambda b_2$.

If $a_2 = 0$, then $a_1 \neq 0$. By Lemma 45 there exists an $x \in D_2$ such that $\langle a_1, x \rangle \neq 0$. Thus, $(x, 0) \in D_2 \times \{0\}$ satisfies (16).

This completes the proof. □

For any cutting blocking set $D$, if $D$ does not satisfy the condition $D = aD$ for any $a \in \text{GF}(q)^*$, then the cutting blocking set $D'$ satisfies the previous condition, where $D' = \{ax : a \in \text{GF}(q), x \in D\}$.

Let $D_1$ and $D_2$ be two vectorial cutting blocking sets in $\text{GF}(q)$. Define

\[
\widetilde{[D_1, D_2]} := \left\{ (x, y) \in \text{GF}(q)^{k+1} : x \in D_1 \right\} \cup \left\{ (x, 0) \in \text{GF}(q)^{k+1} : x \in D_2 \right\},
\]

where $y_x \in \text{GF}(q)^*$ and $y_{ax} = y_x$ for any $a \in \text{GF}(q)^*$. Then, the code $C_{[\tilde{D_1}, \tilde{D_2}]}$ is also a minimal code as $C_{[\tilde{D_1}, \tilde{D_2}]}$ and $C_{[\tilde{D_1}, \tilde{D_2}]}$ are equivalent up to a monomial transformation.

**Corollary 47.** Let $D$ be a vectorial cutting blocking set in $\text{GF}(q)^k$ such that $D = a \cdot D$ for any $a \in \text{GF}(q)^*$. Then $C_{[\tilde{D_1}, \tilde{D_2}]}$ is a minimal code of length $2\#D$ and dimension $(k + 1)$, where $[D, \tilde{D}]$ is given by (18). Furthermore, if $C_D$ satisfies the condition $\frac{w_{\min}}{w_{\max}} \leq \frac{q-1}{q}$, then so does $C_{\tilde{D_1}, \tilde{D_2}}$.

**Proof.** The conclusions follow from Theorem 46 and the fact that if $c \in C_D$ then $(c, c) \in C_{[\tilde{D_1}, \tilde{D_2}]}$. □

Similarly, one can prove the following corollary.

**Corollary 48.** Let $D$ be a vectorial cutting blocking set in $\text{GF}(q)^k \setminus \{0\}$ such that $D = a \cdot D$ for any $a \in \text{GF}(q)^*$. Let $\tilde{D}$ be any projection of $D$. Then $C_{[\tilde{D}, \tilde{D}]}$ is a minimal projective code of length $\frac{q\#D}{q-1}$ and dimension $(k + 1)$, where $[\tilde{D}, \tilde{D}]$ is given by (18). Furthermore, if $C_D$ satisfies the condition $\frac{w_{\min}}{w_{\max}} \leq \frac{q-1}{q}$, then so does $C_{[\tilde{D}, \tilde{D}]}$.

The following is a consequence of Proposition 25 and Theorem 46.

**Theorem 49.** Let $S_1$ and $S_2$ be two nonempty proper subsets of $\text{GF}(q)^k \setminus \{0\}$ such that the dimension of $\dim_{\text{GF}(q)}(\text{Span}(S_i))$ ($i = 1, 2$) is at least 3. Let $D_{S_i}$ be the set defined by (10). Then $C_{[\tilde{D_{S_1}}, \tilde{D_{S_2}}]}$ is a minimal linear code of dimension $(k + 1)$, where $[D_{S_1}, D_{S_2}]$ is given by (18).
Example 50. Let \( q = 4, k = 5 \) and \( D = \{ (x_1, \ldots, x_5) \in \text{GF}(q)^5 \setminus \{0\} : x_1x_2x_3 = 0 \} \). Then the set \( \widetilde{C}_{[D,D]} \) in Theorem 49 is a minimal code with parameters \([1182, 6, 591]\) and maximum weight \( w_{\text{max}} = 960 \). Obviously, \( \frac{w_{\text{min}}}{w_{\text{max}}} = \frac{197}{320} < \frac{3}{4} \).

It would be interesting to settle the following problem.

Open Problem 51. Determine the weight distribution of the minimal code \( \widetilde{C}_{[D,D]} \) for the case that

\[
D := \left\{ (x_1, \ldots, x_k) \in \text{GF}(q)^k \setminus \{0\} : \prod_{i=1}^h x_i = 0 \right\},
\]

where \( 3 \leq h \leq k \).

Theorem 52. Let \( h, \ell \) and \( k \) be integers such that \( h \geq 2 \) and \( \ell \leq (k - 2) \). Let \( D_{\leq h} \) and \( D_{\geq \ell} \) be the subsets of \( \text{GF}(q)^k \) defined by

\[
D_{\leq h} = \left\{ x \in \text{GF}(q)^k : 1 \leq \text{wt}(x) \leq h \right\},
\]

and

\[
D_{\geq \ell} = \left\{ x \in \text{GF}(q)^k : \ell \leq \text{wt}(x) \leq k \right\}.
\]

Then the codes \( \widetilde{C}_{[D_{\leq h}, D_{\leq h}]} \), \( \widetilde{C}_{[D_{\geq \ell}, D_{\geq \ell}]} \) and \( \widetilde{C}_{[D_{\leq h}, D_{\geq \ell}]} \) are minimal linear codes of dimension \((k + 1)\),

Proof. From [10] and [30], \( D_{\leq h} \) and \( D_{\geq \ell} \) are cutting blocking sets. The desired conclusion then follows from Theorem 46.

As a special case, the minimal codes \( \widetilde{C}_{[D_{\leq h}, D_{\leq h}]} \) have appeared in [18, 19] and [9]. Thus, Theorem 52 generalizes some previously known constructions of minimal codes.

Example 53. Let \( q = 3, k = 6 \). Then the set \( \widetilde{C}_{[D_{\leq h}, D_{\leq h}]} \) in Theorem 52 is a minimal code with parameters \([144, 7, 44]\) and maximum weight \( w_{\text{max}} = 104 \). Obviously, \( \frac{w_{\text{min}}}{w_{\text{max}}} = \frac{116}{26} < \frac{2}{3} \).

The following open problem would be interesting.

Open Problem 54. Determine the weight distributions of the minimal codes \( \widetilde{C}_{[D_{\leq h}, D_{\leq h}]} \), \( \widetilde{C}_{[D_{\geq \ell}, D_{\geq \ell}]} \) and \( \widetilde{C}_{[D_{\leq h}, D_{\geq \ell}]} \) in Theorem 52.

The following theorem presents a general approach to constructing minimal codes via cutting blocking sets and affine blocking sets.

Theorem 55. Let \( k \geq 3 \). Let \( D_1 \) be a subset of \( \text{GF}(q)^{k-1} \) and let \( D_2 \) be a vectorial cutting blocking sets in \( \text{GF}(q)^{k-1} \). Define the subset of \( \text{GF}(q)^k \) by

\[
[\widetilde{D}_1, \widetilde{D}_2] := \left\{ (x, 1) \in \text{GF}(q)^k : x \in D_1 \right\} \cup \left\{ (x, 0) \in \text{GF}(q)^k : x \in D_2 \right\}.
\]

Then, the code \( \widetilde{C}_{[\widetilde{D}_1, \widetilde{D}_2]} \) is a minimal code if and only if the set \( D_1 \) is an affine blocking set in \( \text{GF}(q)^{k-1} \).
Proof. We first suppose that $C_{[D_1, D_2]}$ is a minimal code. In the same manner as in the proof of Theorem 22 we can see that $D_1$ is an affine blocking set.

Conversely, suppose that $D_1$ is an affine blocking set in $GF(q)^{k-1}$. Analysis similar to that in the proof of Theorem 46 shows that $C_{[D_1, D_2]}$ is a minimal code. □

Corollary 56. Let $k \geq 3$. Let $D_1 = \{ a e_i : a \in GF(q), 1 \leq i \leq k-1 \}$, where $e_1, \ldots, e_{k-1}$ is a basis for $GF(q)^{k-1}$ over $GF(2)$. Let $D_2$ be a vectorial cutting blocking sets in $GF(q)^{k-1}$. Define the subset of $GF(q)^{k}$ by

$$[D_1, D_2] := \left\{ (x, 1) \in GF(q)^k : x \in D_1 \right\} \cup \left\{ (x, 0) \in GF(q)^k : x \in D_2 \right\}. \quad (18)$$

Then, the code $C_{[D_1, D_2]}$ is a minimal code with parameters $[\#D_2 + (k-1)(q-1) + 1, k, (k-1)(q-1) + 1]$. Proof. Let $H$ be an affine hyperplane given by $a_1 x_1 + \cdots + a_{k-1} x_{k-1} = b$. Since any nonzero linear functional must be nonzero on at least one basis vector, it is clear that multiplying that basis vector by an appropriate scalar yields a solution in $D_1$ to $a_1 x_1 + \cdots + a_{k-1} x_{k-1} = b$. Hence, $D_1$ is an affine blocking set containing $(k-1)(q-1) + 1$ vectors. The desired conclusion then follows from Theorems 22 and 55. □

Remark 57. Corollary 56 presents a generic construction of minimal codes with minimum distance achieving the lower bound in Theorem 22. In particular, the lower bound in Theorem 22 is tight.

Remark 58. Let $e_i$ be the vector in $GF(q)^{k-1}$ with all elements equal to zero, except ith element which is equal to one. Denote $D_1 = \{ a e_i : a \in GF(q), 1 \leq i \leq k-1 \}$. Let $D_2$ be the set given by

$$D_2 = \{ e_i + a e_j : a \in GF(q)^*, 1 \leq j < k-1 \} \cup (D_1 \setminus \{0\}).$$

Then the code $C_{[D_1, D_2]}$ is a minimal code. It is easily seen that $C_{[D_1, D_2]}$ is equivalent to the code in Theorem 5.4 of [2]. Thus Corollary 56 gives a new interpretation of this code.

5. Summary and concluding remarks

The main contributions of this paper are the following.

- A link between minimal linear codes and blocking sets was established and documented in Theorem 14, which says that projective minimal codes and cutting blocking sets are identical objects. Adopting this geometric perspective a tight lower bound on the minimum distance of minimal codes is yielded, which confirms a recent conjecture by Alfarano, Borello and Neri. The link also played an important role for the construction of minimal codes in later sections.

- A general primary construction of minimal linear codes from hyperplanes was derived in Theorem 26. With these general results, minimal codes with new parameters and explicit weight distributions were obtained.

- A general secondary construction of minimal linear codes was presented and documented in Theorem 46. From this construction, many minimal codes with $\frac{w_{\min}}{w_{\max}} \leq \frac{q-1}{q}$ were produced.
As observed, the geometric depiction of minimal codes via cutting blocking sets is very effective for analyzing and constructing minimal linear codes. Other good minimal codes may be obtained from the generic constructions proposed in this paper, a lot of work can be done in this direction. It would be nice if the open problems presented in this paper could be settled. The reader is cordially invited to attack these problems.

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