Clustered Graph Coloring 
and Layered Treewidth*

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Abstract

A graph coloring has bounded clustering if each monochromatic component has bounded size. This paper studies clustered coloring, where the number of colors depends on an excluded complete bipartite subgraph. This is a much weaker assumption than previous works, where typically the number of colors depends on an excluded minor. This paper focuses on graph classes with bounded layered treewidth, which include planar graphs, graphs of bounded Euler genus, graphs embeddable on a fixed surface with a bounded number of crossings per edge, amongst other examples. Our main theorem says that for fixed integers $s, t, k$, every graph with layered treewidth at most $k$ and with no $K_{s,t}$ subgraph is $(s+2)$-colorable with bounded clustering. In the $s = 1$ case, which corresponds to graphs of bounded maximum degree, we obtain polynomial bounds on the clustering. This greatly improves a corresponding result of Esperet and Joret for graphs of bounded genus. The $s = 3$ case implies that every graph with a drawing on a fixed surface with a bounded number of crossings per edge is 5-colorable with bounded clustering. Our main theorem is also a critical component in two companion papers that study clustered coloring of graphs with no $K_{s,t}$-subgraph and excluding a fixed minor, odd minor or topological minor.

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1 Introduction

This paper considers graph colorings where the condition that adjacent vertices are assigned distinct colors is relaxed. Instead, we require that every monochromatic component has bounded size (for a given graph class). More formally, a coloring of a graph $G$ is a function that assigns one color to each vertex of $G$. A monochromatic component with respect to a coloring of $G$ is a connected component of a subgraph of $G$ induced by all the vertices assigned the same color. A coloring has clustering $\eta$ if every monochromatic component has at most $\eta$ vertices. Our focus is on minimizing the number of colors, with small monochromatic components as a secondary goal.

The clustered chromatic number of a graph class $\mathcal{G}$ is the minimum integer $k$ such that for some integer $\eta$, every graph in $\mathcal{G}$ is $k$-colorable with clustering $\eta$. There have been several recent papers on this topic [1, 8, 13, 15, 16, 22–26, 30, 31, 34, 35, 45]; see [46] for a survey.

This paper is the first of three companion papers [32, 33]. The unifying theme that distinguishes this work from previous contributions is that the number of colors is determined by an excluded subgraph. Excluding a subgraph is a much weaker assumption than excluding a minor, which is a typical assumption in previous results. In particular, we consider graphs with no $K_{s,t}$ subgraph plus various other structural properties, and prove that every such graph is colorable with bounded clustering, where the number of colors only depends on $s$. All the dependence on $t$ and the structural property in question is hidden in the clustering function.

Note that the case $s = 1$ is already interesting, since a graph has no $K_{1,t}$ subgraph if and only if it has maximum degree less than $t$. Many of our theorems generalize known results for graphs of bounded maximum degree to the setting of an excluded $K_{s,t}$ subgraph.

Note that for $s \geq 2$, no result with bounded clustering is possible for graphs with no $K_{s,t}$ subgraph, without making some extra assumption. In particular, for every graph $H$ that contains a cycle, and for all $k, \eta \in \mathbb{N}$, if $G$ is a graph with chromatic number greater than $k\eta$ and girth greater than $|V(H)|$ (which exists [14]), then $G$ contains no $H$ subgraph and $G$ is not $k$-colorable with clustering $\eta$, for otherwise $G$ would be $k\eta$-colorable.

1.1 Main Result

In this paper, the “other structural property” mentioned above is “bounded layered treewidth”. First we explain what this means. A tree-decomposition of a graph $G$ is a pair $(T, \mathcal{X} = (X_x : x \in V(T)))$, where $T$ is a tree, and for each node $x \in V(T)$, $X_x$ is a non-empty subset of $V(G)$ called a bag, such that for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in X_x\}$ induces a non-empty (connected) subtree of $T$, and for each edge $vw \in E(G)$ there is a node $x \in V(T)$ such that $\{v, w\} \subseteq X_x$. The width of a tree-decomposition $(T, \mathcal{X})$ is $\max\{|X_x| - 1 : x \in V(T)\}$. The treewidth of a graph $G$ is the minimum width of a tree-decomposition of $G$. Treewidth is a key parameter in algorithmic and structural graph theory; see [3, 21, 38, 39] for surveys.

A layering of a graph $G$ is an ordered partition $(V_1, \ldots, V_n)$ of $V(G)$ into (possibly empty) sets such that for each edge $vw \in E(G)$ there exists $i \in [1, n - 1]$ such that $\{v, w\} \subseteq V_i \cup V_{i+1}$. The layered treewidth of a graph $G$ is the minimum nonnegative integer $\ell$ such that $G$ has a tree-decomposition $(T, \mathcal{X} = (X_x : x \in V(T)))$ and a layering $(V_1, \ldots, V_n)$, such that $|X_x \cap V_i| \leq \ell$ for each bag $X_x$ and layer $V_i$. This says that the subgraph induced by each layer has bounded treewidth, and moreover, a single tree-decomposition of $G$ has bounded treewidth when restricted to each layer. In fact, these properties hold when considering a bounded sequence of consecutive layers.

Layered treewidth was independently introduced by Dujmović, Morin, and Wood [12] and Shahrrokh [43]. Dujmović et al. [12] proved that every planar graph has layered treewidth at most
3; more generally, that every graph with Euler genus\(^1\) at most \(g\) has layered treewidth at most \(2g + 3\); and most generally, that a minor-closed class has bounded layered treewidth if and only if it excludes some apex graph as a minor. Layered treewidth is of interest beyond minor-closed classes, since as described below, there are several natural graph classes that have bounded layered treewidth but contain arbitrarily large complete graph minors.

Consider coloring a graph \(G\) with layered treewidth \(w\). Say \((V_1, \ldots, V_n)\) is the corresponding layering. Then \(G[\bigcup_{i \text{ odd}} V_i]\) has treewidth at most \(w - 1\), and is thus properly \(w\)-colorable. Similarly, \(G[\bigcup_{i \text{ even}} V_i]\) is properly \(w\)-colorable. Thus \(G\) is properly \(2w\)-colorable. This bound is best possible since \(K_{2w}\) has layered treewidth \(w\). In fact, for every integer \(d\) and integer \(w\), there is a graph \(G\) with treewidth \(2w - 1\), such that every \((2w - 1)\)-coloring of \(G\) has a vertex of monochromatic degree at least \(d\), implying there is a monochromatic component with more than \(d\) vertices [46]. Bannister, Devanny, Dujmović, Eppstein, and Wood [2] observed that every graph with treewidth \(k\) has layered treewidth at most \(\lceil \frac{k+1}{2} \rceil\) (using two layers), so \(G\) has layered treewidth at most \(w\). This says that the clustered chromatic number of the class of graphs with layered treewidth \(w\) equals \(2w\), and indeed, every such graph is properly \(2w\)-colorable.

On the other hand, we prove that for clustered coloring of graphs excluding a \(K_{s,t}\)-subgraph in addition to having bounded layered treewidth, only \(s + 2\) colors are needed (no matter how large the upper bound on layered treewidth). This is the main result of the paper.

**Theorem 1.** For all \(s, t, w \in \mathbb{N}\) there exists \(\eta \in \mathbb{N}\) such that every graph with layered treewidth at most \(w\) and with no \(K_{s,t}\) subgraph is \((s + 2)\)-colorable with clustering \(\eta\).

Theorem 1 is proved in Section 6. The case \(s = 1\) in Theorem 1 is of particular interest, since it applies for graphs of maximum degree \(\Delta < t\). It implies that graphs of bounded Euler genus and of bounded maximum degree are \(3\)-colorable with bounded clustering, which was previously proved by Esperet and Joret [15]. The clustering function proved by Esperet and Joret [15] is roughly \(O(\Delta^{32\Delta^2})\), for graphs with Euler genus \(g\) and maximum degree \(\Delta\). While Esperet and Joret [15] made no effort to reduce this function, their method will not lead to a sub-exponential clustering bound. In the case \(s = 1\) we give a different proof of Theorem 1 with the following polynomial clustering bound.

**Theorem 2.** For every \(w, \Delta \in \mathbb{N}\), every graph with layered treewidth at most \(w\) and maximum degree at most \(\Delta\) is \(3\)-colorable with clustering \(O(w^{19}\Delta^{37})\). In particular, every graph with Euler genus at most \(g\) and maximum degree at most \(\Delta\) is \(3\)-colorable with clustering \(O(g^{19}\Delta^{37})\).

Note that the proof of Theorem 2 (presented in Section 2) is relatively simple, avoiding many technicalities that arise when dealing with graph embeddings. This proof highlights the utility of layered treewidth as a general tool.

The number of colors in Theorem 2 is best possible since the Hex Lemma [20] says that every 2-coloring of the \(n \times n\) planar triangular grid (which has layered treewidth 2 and maximum degree 6) contains a monochromatic path of length at least \(n\). The planar triangular grid and its generalizations are a key lower bound in the theory of clustered colorings [46].

Further motivation for Theorem 1 is that it is a critical ingredient in the proofs of results in our companion papers [32, 33] about clustered colorings of graphs excluding a minor, odd minor, or topological minor. For these proofs we actually need the following stronger result proved in Section 8 (where the \(\xi = 0\) case is Theorem 1).

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\(^1\)The *Euler genus* of an orientable surface with \(h\) handles is \(2h\). The *Euler genus* of a non-orientable surface with \(c\) cross-caps is \(c\). The *Euler genus* of a graph \(G\) is the minimum Euler genus of a surface in which \(G\) embeds (with no crossings).
Theorem 3. For all $s, t, w \in \mathbb{N}$ and $\xi \in \mathbb{N}_0$, there exists $\eta \in \mathbb{N}$ such that if $G$ is a graph with no $K_{s,t}$ subgraph and $G - Z$ has layered treewidth at most $w$ for some $Z \subseteq V(G)$ with $|Z| \leq \xi$, then $G$ is $(s + 2)$-colorable with clustering $\eta$.

Note that the number of colors in Theorem 3 is best possible for $s \leq 2$: Suppose that the theorem holds with $s = 2$, $t = 7$, $w = 2$ and $\xi = 1$, but with only 3 colors. Let $n \gg \eta^2$. Let $G$ be obtained from the $n \times n$ triangular grid by adding one dominant vertex $v$. Then $G$ contains no $K_{2,7}$-subgraph, and $G - v$ has layered treewidth 2. By assumption, $G$ is 3-colorable with clustering $\eta$. Say $v$ is blue. Since $v$ is dominant, at most $\eta$ rows or columns contain a blue vertex. The non-blue induced subgraph contains an $\eta \times \eta$ triangular grid (since $n \gg \eta^2$). This contradicts the Hex Lemma mentioned above.

We remark that the class of graphs mentioned in Theorem 3 is more general than the class of graphs with bounded layered treewidth. For example, the class of graphs that can be made planar (and hence bounded layered treewidth) by deleting one vertex contains graphs of arbitrarily large layered treewidth.

1.2 Examples

We now give several examples of graph classes with bounded layered treewidth, for which Theorems 1 and 2 give interesting results.

$(g, k)$-Planar Graphs

A graph is $(g, k)$-planar if it can be drawn in a surface of Euler genus at most $g$ with at most $k$ crossings on each edge (assuming no three edges cross at a single point). Such graphs can contain arbitrarily large complete graph minors, even in the $g = 0$ and $k = 1$ case [10]. On the other hand, Dujmović et al. [10] proved that every $(g, k)$-planar graph has layered treewidth at most $(4g + 6)(k + 1)$. Theorem 2 then implies:

**Corollary 4.** For all $g, k, \Delta \in \mathbb{N}$, every $(g, k)$-planar graph with maximum degree at most $\Delta$ is $3$-colorable with clustering $O(g^{19}k^{19}\Delta^{37})$.

Now consider $(g, k)$-planar graphs without the assumption of bounded maximum degree. Wood [46] proved that such graphs are 12-colorable with clustering bounded by a function of $g$ and $k$. Ossona de Mendez, Oum, and Wood [36] proved that every $(g, k)$-planar graph contains no $K_{3,t}$ subgraph for some $t = O(kg^2)$. Thus Theorem 1 with $s = 3$ implies that this bound of 12 can be reduced to 5:

**Corollary 5.** For all $g, k \in \mathbb{N}_0$, there exists $\eta \in \mathbb{N}$, such that every $(g, k)$-planar graph is $5$-colorable with clustering $\eta$.

Corollary 5 highlights the utility of excluding a $K_{s,t}$ subgraph. It also generalizes a theorem of Esperet and Ochem [16] who proved the $k = 0$ case, which says that every graph with bounded Euler genus is 5-colorable with bounded clustering. Note that Dvořák and Norin [13] recently proved that every graph with bounded Euler genus is 4-colorable (in fact, 4-choosable) with bounded clustering. It is open whether every $(g, k)$-planar graph is 4-colorable with bounded clustering.
Map Graphs

Map graphs are defined as follows. Start with a graph $G_0$ embedded in a surface of Euler genus $g$, with each face labelled a “nation” or a “lake”, where each vertex of $G_0$ is incident with at most $d$ nations. Let $G$ be the graph whose vertices are the nations of $G_0$, where two vertices are adjacent in $G$ if the corresponding faces in $G_0$ share a vertex. Then $G$ is called a $(g,d)$-map graph. A $(0,d)$-map graph is called a (plane) $d$-map graph; such graphs have been extensively studied [5–7, 9, 17]. The $(g,3)$-map graphs are precisely the graphs of Euler genus at most $g$ (see [7, 10]). So $(g,d)$-map graphs provide a natural generalization of graphs embedded in a surface that allows for arbitrarily large cliques even in the $g=0$ case (since if a vertex of $G_0$ is incident with $d$ nations then $G$ contains $K_d$). Dujmović et al. [10] proved that every $(g,d)$-map graph has layered treewidth at most $(2g+3)(2d+1)$ and is $(g,O(d^2))$-planar. Thus Corollary 4 implies:

**Corollary 6.** For all $g,d,\Delta \in \mathbb{N}$, every $(g,d)$-map graph with maximum degree at most $\Delta$ is 3-colorable with clustering $O(g^{19}d^{38}\Delta^{37})$.

Similarly, Corollary 5 implies:

**Corollary 7.** For all $g,d \in \mathbb{N}$, there exists $\eta \in \mathbb{N}$, such that every $(g,d)$-map graph is 5-colorable with clustering $\eta$.

It is straightforward to prove that, in fact, if a $(g,d)$-map graph $G$ contains a $K_{3,t}$ subgraph, then $t \leq 6d(g+1)$. Thus Theorem 1 can be applied directly with $s=3$, slightly improving the clustering bounds.

It is open whether every $(g,d)$-map graph is 4-colorable with clustering bounded by a function of $g$ and $d$.

String Graphs

A string graph is the intersection graph of a set of curves in the plane with no three curves meeting at a single point [18, 19, 29, 37, 41, 42]. For an integer $k \geq 2$, if each curve is in at most $k$ intersections with other curves, then the corresponding string graph is called a $k$-string graph. A $(g,k)$-string graph is defined analogously for curves on a surface of Euler genus at most $g$. Dujmović, Joret, Morin, Norin, and Wood [11] proved that every $(g,k)$-string graph has layered treewidth at most $2(k-1)(2g+3)$. By definition, the maximum degree of a $(g,k)$-string graph is at most $k$ (and might be less than $k$ since two curves might have multiple intersections). Thus Theorem 2 implies:

**Corollary 8.** For all integers $g \geq 0$ and $k \geq 2$, there exists $\eta \in \mathbb{N}$ such that every $(g,k)$-string graph is 3-colorable with clustering $O(g^{19}k^{56})$.

1.3 Notation

Let $G$ be a graph with vertex-set $V(G)$ and edge-set $E(G)$. For $v \in V(G)$, let $N_G(v) := \{w \in V(G) : tw \in E(G)\}$ be the neighborhood of $v$, and let $N_G[v] := N_G(v) \cup \{v\}$. For $X \subseteq V(G)$, let $N_G(X) := \bigcup_{v \in X}(N_G(v) \setminus X)$ and $N_G[X] := N_G(X) \cup X$. Denote the subgraph of $G$ induced by $X$ by $G[X]$. Let $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{N}_0 := \{0,1,2,\ldots\}$. For $m,n \in \mathbb{N}$, let $[m,n] := \{m,m+1,\ldots,n\}$ and $[n] := [1,n]$. 

2 Bounded Layered Treewidth and Bounded Degree

This section proves Theorem 2, which says that graphs of bounded layered treewidth and bounded maximum degree are 3-colorable with bounded clustering. We need the following analogous result by Alon et al. [1] for bounded treewidth graphs.

**Lemma 9** ([1]). There is a function \( f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that every graph with treewidth \( w \) and maximum degree \( \Delta \) is 2-colorable with clustering \( f(w, \Delta) \in O(k\Delta) \).

For a graph \( G \) with bounded maximum degree and bounded layered treewidth, if \((V_1, \ldots, V_n)\) is the corresponding layering of \( G \), then Lemma 9 is applicable to \( G[V_i] \), which has bounded treewidth. The idea of the proof of Theorem 2 is to use colors 1 and 2 for all layers \( V_i \) with \( i \equiv 1 \pmod{3} \), use colors 2 and 3 for all layers \( V_i \) with \( i \equiv 2 \pmod{3} \), and use colors 3 and 1 for all layers \( V_i \) with \( i \equiv 3 \pmod{3} \). Then each monochromatic component is contained within two consecutive layers. The key to the proof is to control the growth of monochromatic components between consecutive layers. The next lemma is useful for this purpose.

**Lemma 10.** Let \( w, \Delta, d, k, h \in \mathbb{N} \). Let \( G \) be a graph with maximum degree at most \( \Delta \). Let \((T, X)\) be a tree-decomposition of \( G \) with width at most \( w \), where \( X = (X_t : t \in V(T)) \). For \( i \geq 1 \), let \( Y_i \) be a subset of \( V(T) \), \( T_i \) a subtree of \( T \) containing \( Y_i \), and \( E_i \) a set of pairs of vertices in \( \bigcup_{x \in Y_i} X_x \) with \( |E_i| \leq k \). Let \( G' \) be the graph with \( V(G') = V(G) \) and \( E(G') = E(G) \cup \bigcup_{i \geq 1} E_i \). If every vertex of \( G \) appears in at most \( d \) pairs in \( \bigcup_{i \geq 1} E_i \), and every vertex of \( T \) is contained in at most \( h \) members of \( \{T_1, T_2, \ldots\} \), then \( G' \) has maximum degree at most \( \Delta + d \) and has a tree-decomposition \((T, X')\) of width at most \( w + 2hk \).

**Proof.** Since every vertex of \( G \) appears in at most \( d \) pairs in \( \bigcup_{i \geq 1} E_i \), \( G' \) has maximum degree at most \( \Delta + d \). For every \( i \geq 1 \), let \( Z_i \) be the set of the vertices appearing in some pair of \( E_i \). Note that \( Z_i \subseteq \bigcup_{x \in Y_i} X_x \) and \( |Z_i| \leq 2|E_i| \leq 2k \). For every \( t \in V(T) \), let \( X'_t := X_t \cup \bigcup_{i \in t \in V(T_i)} Z_i \). Let \( X' := (X'_t : t \in V(T)) \).

We claim that \((T, X')\) is a tree-decomposition of \( G' \). It is clear that \( \bigcup_{t \in V(T)} X'_t \supseteq V(G') \). For each \( i \in \mathbb{N} \), for every \( t \in V(T_i) \), since \( X'_t \supseteq Z_i \), \( X'_t \) contains both ends of each edge in \( E_i \). For each \( v \in V(G') \),

\[
\{t \in V(T) : v \in X'_t\} = \{t \in V(T) : v \in X_t\} \cup \bigcup_{i : v \in Z_i} V(T_i).
\]

Note that for every \( v \in V(G) \) and \( i \geq 1 \), if \( v \in Z_i \) then \( v \in X_t \) for some \( t \in Y_i \subseteq V(T_i) \). Hence \( \{t : v \in X'_t\} \) induces a subtree of \( T \). This proves that \((T, X')\) is a tree-decomposition of \( G' \).

Since for every \( t \in V(T) \), \( |X'_t| \leq |X_t| + \sum_{i : t \in V(T_i)} |Z_i| \leq w + 1 + 2hk \), the width of \((T, X')\) is at most \( w + 2hk \).

We now prove Theorem 2.

**Theorem 11.** Let \( \Delta, w \in \mathbb{N} \). Then every graph \( G \) with maximum degree at most \( \Delta \) and with layered treewidth at most \( w \) is 3-colorable with clustering \( g(w, \Delta) \), for some function \( g(w, \Delta) \in O(w^{19}\Delta^{37}) \).

**Proof.** Let \( f \) be the function from Lemma 9. Define

\[
\begin{align*}
f_1 &:= f(w, \Delta) \\
\Delta_2 &:= \Delta + f_1^2 \\
w_2 &:= w + 2(w + 1)f_1^2 \Delta^2
\end{align*}
\]

\( \in O(w\Delta) \)  
\( \in O(w\Delta^3) \)  
\( \in O(w^3\Delta^4) \),
Let $G$ be a graph of maximum degree at most $\Delta$ and layered treewidth at most $w$. Let $(T, \mathcal{X})$ and $(V_i : i \geq 1)$ be a tree-decomposition of $G$ and a layering of $G$ such that $|X_t \cap V_i| \leq w$ for every $t \in V(T)$ and $i \geq 1$, where $\mathcal{X} = (X_t : t \in V(T))$. For $j \in [3]$, let $U_j = \bigcup_{i=0}^{\infty} V_{3i+j}$.

By Lemma 9, there exists a coloring $c_1 : U_1 \to \{1, 2\}$ such that every monochromatic component of $G[U_1]$ contains at most $f_1$ vertices. For each $i \in \mathbb{N}_0$, let $\mathcal{C}_i$ be the set of monochromatic components of $G[U_1]$ contained in $V_{3i+1}$ with color 2 with respect to $c_1$. For each $i \in \mathbb{N}_0$ and $C \in \mathcal{C}_i$, define the following:

- Let $Y_{i,C}$ be a minimal subset of $V(T)$ such that for every edge $e$ of $G$ between $V(C)$ and $N_G(V(C)) \cap V_{3i+2}$, there exists a node $t \in Y_{i,C}$ such that both ends of $e$ belong to $X_t$.
- Let $E_{i,C}$ be the set of all pairs of distinct vertices in $N_G(V(C)) \cap V_{3i+2}$.
- Let $T_{i,C}$ be the subtree of $T$ induced by $\{t \in V(T) : X_t \cap V(C) \neq \emptyset\}$.

Note that there are at most $|V(C)|\Delta \leq f_1 \Delta$ edges of $G$ between $V(C)$ and $N_G(V(C))$. So $|Y_{i,C}| \leq f_1 \Delta$ and $|E_{i,C}| \leq f_2 \Delta^2$ for every $i \in \mathbb{N}_0$ and $C \in \mathcal{C}_i$. In addition, $Y_{i,C} \subseteq V(T_{i,C})$ for every $i \in \mathbb{N}_0$ and $C \in \mathcal{C}_i$. Since $(T, (X_t \cap U_1 : t \in V(T)))$ is a tree-decomposition of $G[U_1]$ with width at most $w$, for every $t \in V(T)$ and $i \in \mathbb{N}_0$, there exist at most $w+1$ different members $C$ of $\mathcal{C}_i$ such that $t \in V(T_{i,C})$. Furthermore, $N_G(V(C)) \cap V_{3i+2} \subseteq \bigcup_{x \in Y_{i,C}} X_x$, so each pair in $E_{i,C}$ consists of two vertices in $\bigcup_{x \in Y_{i,C}} X_x$. Since every vertex $v$ in $U_2$ is adjacent in $G$ to at most $\Delta$ members of $\bigcup_{i \in \mathbb{N}_0} \mathcal{C}_i$ and every member $C$ of $\bigcup_{i \in \mathbb{N}_0} \mathcal{C}_i$ creates at most $f_1 \Delta$ pairs in $E_{i,C}$ involving $v$, where $i_C$ is the index such that $C \in \mathcal{C}_{i_C}$, every vertex in $U_2$ appears in at most $f_1 \Delta \cdot \Delta = f_1 \Delta^2$ pairs in $\bigcup_{i \in \mathbb{N}_0, C \in \mathcal{C}_i} E_{i,C}$.

Let $G_2$ be the graph with $V(G_2) := U_2$ and

$$E(G_2) := E(G[U_2]) \cup \bigcup_{i \in \mathbb{N}_0, C \in \mathcal{C}_i} E_{i,C}.$$ 

We have shown that Lemma 10 is applicable with $k = f_1 \Delta^2$ and $d = f_1 \Delta^2$ and $h = w+1$. Thus $G_2$ has maximum degree at most $\Delta_2$ and a tree-decomposition $(T, \mathcal{X}(2))$ of width at most $w_2$. Say $\mathcal{X} = (X_t : t \in V(T))$. By Lemma 9, there exists a coloring $c_2 : U_2 \to \{2, 3\}$ such that every monochromatic component of $G_2$ with respect to $c_2$ contains at most $f_2$ vertices.

Note that we may assume that $(T, \mathcal{X}(2))$ is a tree-decomposition of $G[U_1 \cup U_2]$ by redefining $X_t$ to be the union of $X_t$ and $X_t \cap U_1$, for every $t \in V(T)$.

For each $i \in \mathbb{N}_0$, let $\mathcal{C}'_i$ be the set of the monochromatic components either of $G[U_1]$ with color 1 with respect to $c_1$ or of $G_2$ with color 3 with respect to $c_2$. For each $i \in \mathbb{N}_0$ and $C \in \mathcal{C}'_i$, define the following:

- Let $Y_{i,C}'$ be a minimal subset of $V(T)$ such that for every edge $e$ of $G$ between $V(C)$ and $N_G(V(C)) \cap V_{3i+3}$, there exists a node $t \in Y_{i,C}'$ such that both ends of $e$ belong to $X_t$.
- Let $E_{i,C}'$ be the set of all pairs of distinct vertices of $N_G(V(C)) \cap V_{3i+3}$.
Let $T'_{i,C}$ be the subtree of $T$ induced by $\{t \in V(T) : X_{i}^{(2)} \cap V(C) \neq \emptyset\}$. Note that there are at most $|V(C)|\Delta \leq f_{2}\Delta$ edges of $G$ between $V(C)$ and $N_{G}(V(C)) \cap V_{3i+3}$ for every $i \in \mathbb{N}_{0}$ and $C \in \mathcal{C}_{i}'$. So $|Y_{i,C}'| \leq f_{2}\Delta$ and $|E_{i,C}'| \leq f_{2}^{2}\Delta^{2}$ for every $i \in \mathbb{N}_{0}$ and $C \in \mathcal{C}_{i}'$. In addition, $Y_{i,C}' \subseteq V(T'_{i,C})$ for every $i \in \mathbb{N}_{0}$ and $C \in \mathcal{C}_{i}'$. Since $(T, (X_{i}^{(2)} \cap U_{2} : t \in V(T)))$ is a tree-decomposition of $G$ with width at most $w_{2}$ and $(T, (X_{i}^{(2)} \cap U_{1} : t \in V(T)))$ is a tree-decomposition of $G[U_{1}]$ with width at most $w_{1}$, for every $t \in V(T)$ and $i \in \mathbb{N}_{0}$, there exist at most $2(w_{2} + 1)$ different members $C \in \mathcal{C}_{i}$ such that $t \in V(T'_{i,C})$. Furthermore, each pair in $E_{i,C}'$ consists of two vertices in $\bigcup_{x \in Y_{i,C}'} X_{x}^{(2)}$. Since every vertex $v$ in $U_{3}$ is adjacent in $G$ to at most $\Delta$ members of $\bigcup_{i \in \mathbb{N}_{0}} \mathcal{C}_{i}'$, and every member $C$ of $\bigcup_{i \in \mathbb{N}_{0}} \mathcal{C}_{i}'$ creates at most $f_{2}\Delta$ pairs in $E_{i,C}'$ involving $v$, where $i_{C}$ is the index such that $C \in \mathcal{C}_{i_{C}}$, every vertex in $U_{3}$ appears in at most $f_{2}\Delta^{2}$ pairs in $\bigcup_{i \in \mathbb{N}_{0}, C \in \mathcal{C}_{i}'} E_{i,C}'$.

Let $G_{3}$ be the graph with $V(G_{3}) := U_{3}$ and $E(G_{3}) := E(G[U_{3}]) \cup \bigcup_{i \in \mathbb{N}_{0}, C \in \mathcal{C}_{i}'} E_{i,C}'$. We have shown that Lemma 10 is applicable with $k = f_{2}^{2}\Delta^{2}$ and $a = f_{2}\Delta^{2}$ and $h = 2(w_{2} + 1)$. Hence $G_{3}$ has maximum degree at most $\Delta_{3}$ and a tree-decomposition $(T, \mathcal{X}^{(3)})$ with width at most $w_{3}$.

Define $c : V(G) \to \{1, 2, 3\}$ such that for every $v \in V(G)$, we have $c(v) := c_{j}(v)$, where $j$ is the index for which $v \in U_{j}$. Now we prove that every monochromatic component of $G$ with respect to $c$ contains at most $g(w, \Delta)$ vertices.

Let $D$ be a monochromatic component of $G$ with respect to $c$ with color 2. Since $D$ is connected, for every pair of vertices $u, v \in V(D) \cap U_{2}$, there exists a path $P_{uv}$ in $D$ from $u$ to $v$. Since $V(D) \subseteq U_{1} \cup U_{2}$, for every maximal subpath $P$ of $P_{uv}$ contained in $U_{1}$, there exists $i \in \mathbb{N}_{0}$ and $C \in \mathcal{C}_{i}$ such that there exists a pair in $E_{i,C}$ consisting of the two vertices in $P$ adjacent to the ends of $P_{uv}$. That is, there exists a path in $G_{2}[V(D) \cap U_{2}]$ connecting $u, v$ for every pair $u, v \in V(D) \cap U_{2}$. Hence $G_{2}[V(D) \cap U_{2}]$ is connected. So $G_{2}[V(D) \cap U_{2}]$ is a monochromatic component of $G_{2}$ with color 2 with respect to $c_{2}$ and contains at most $f_{2}$ vertices. Hence there are at most $f_{2}\Delta$ edges of $G$ between $V(D) \cap U_{2}$ and $N_{G}(V(D) \cap U_{2})$. So $D[V(D) \cap U_{1}]$ contains at most $f_{2}\Delta$ components. Since each component of $D[V(D) \cap U_{1}]$ is a monochromatic component of $G[U_{1}]$ with respect to $c_{1}$, it contains at most $f_{1}$ vertices. Hence $D[V(D) \cap U_{1}]$ contains at most $f_{2}\Delta f_{1}$ vertices. Since $D$ has color 2, $V(D) \cap U_{3} = \emptyset$. Therefore, $D$ contains at most $(1 + f_{1}\Delta) f_{2} \leq g(w, \Delta)$ vertices.

Let $D'$ be a monochromatic component of $G$ with respect to $c$ with color $b$, where $b \in \{1, 3\}$. Since $D'$ is connected, by an analogous argument to that in the previous paragraph, $G_{3}[V(D') \cap U_{3}]$ is connected. So $G_{3}[V(D') \cap U_{3}]$ is a monochromatic component of $G_{3}$ with color $b$ with respect to $c_{3}$ and contains at most $f_{3}$ vertices. Hence there are at most $f_{3}\Delta$ edges of $G$ between $V(D') \cap U_{3}$ and $N_{G}(V(D') \cap U_{3})$. So $D[V(D') \cap U_{b'}]$ contains at most $f_{3}\Delta$ components, where $b' = 1$ if $b = 1$ and $b' = 2$ if $b = 3$. Since each component of $D[V(D') \cap U_{b'}]$ is a monochromatic component of $G[U_{b'}]$ with respect to $c_{b'}$, it contains at most $f_{b}$ vertices. Hence $D[V(D') \cap U_{b'}]$ contains at most $f_{3}\Delta f_{b}$ vertices. Since $D$ has color $b$, $V(D) \cap U_{b+1} = \emptyset$, where $U_{4} = U_{1}$. Therefore, $D$ contains at most $(1 + f_{2}\Delta) f_{3} \leq g(w, \Delta)$ vertices. This completes the proof. \hfill \square

### 3 Bounded Neighborhoods

We now set out to prove Theorem 1 for all $s$. A key step in the above proof for the $s = 1$ case says that if $X$ is a bounded-size set of vertices in a graph, then $N_{G}(X)$ also has bounded size (since
$G$ has bounded maximum degree). This fails for graphs with no $K_{s,t}$ subgraph for $s \geq 2$, since
vertices in $X$ might have unbounded degree. To circumvent this issue we focus on those vertices
with at least $s$ neighbors in $X$, and then show that there are a bounded number of such vertices.
The following notation formalizes this simple but important idea. For a graph $G$, a set $X \subseteq V(G)$,
and $s \in \mathbb{N}$, define

$$
N^{\geq s}_G(X) := \{ v \in V(G) \setminus X : |N_G(v) \cap X| \geq s \} \text{ and }
N^{< s}_G(X) := \{ v \in V(G) \setminus X : 1 \leq |N_G(v) \cap X| < s \}.
$$

When the graph $G$ is clear from the context we write $N^{\geq s}(X)$ instead of $N^{\geq s}_G(X)$ and $N^{< s}(X)$
instead of $N^{< s}_G(X)$.

**Lemma 12.** For all $s, t \in \mathbb{N}$, there exists a function $f_{s,t} : \mathbb{N}_0 \to \mathbb{N}_0$ such that for every graph $G$
with no $K_{s,t}$ subgraph, if $X \subseteq V(G)$ then $|N^{\geq s}(X)| \leq f_{s,t}(|X|)$.

**Proof.** Define $f_{s,t}(x) = \left(\frac{x}{s}\right)(t - 1)$ for every $x \in \mathbb{N}_0$. For every $y \in N^{\geq s}(X)$, let $Z_y$ be a subset
of $N_G(y) \cap X$ with size $s$. Since there are exactly $\binom{|X|}{s}$ subsets of $X$ with size $s$, if $|N^{\geq s}(X)| > f_{s,t}(|X|)$,
then there exists a subset $Y$ of $N^{\geq s}(X)$ with size $t$ such that $Z_y$ is identical for all $y \in Y$, implying
$G[Y \cup \bigcup_{y \in Y} Z_y]$ contains a $K_{s,t}$ subgraph, which is a contradiction. Thus $|N^{\geq s}(X)| \leq f_{s,t}(|X|)$. \qed

The function $f$ in Lemma 12 can be improved if we know more about the graph $G$, which helps
to get better upper bounds on the clustering. A **1-subdivision** of a graph $H$ is a graph obtained
from $H$ by subdividing every edge exactly once. For a graph $G$, let $\nabla(G)$ be the maximum average
degree of a graph $H$ for which the 1-subdivision of $H$ is a subgraph of $G$. The following result is
implicit in [36]. We include the proof for completeness.

**Lemma 13.** For all $s, t \in \mathbb{N}$ and positive $\nabla \in \mathbb{R}$ there is a number $c := \max\{t - 1, \frac{s}{2} + (t - 1)\left(\frac{\nabla}{s-1}\right)\}$, such that for every graph $G$ with no $K_{s,t}$ subgraph and with $\nabla(G) \leq \nabla$, if $X \subseteq V(G)$ then $|N^{\geq s}(X)| \leq c|X|$.

**Proof.** In the case $s = 1$, the proof of Lemma 12 implies this lemma since $c \geq t - 1$. Now assume
that $s \geq 2$. Let $H$ be the bipartite graph with bipartition $\{N^{\geq s}_G(X), \binom{|X|}{s}\}$, where $v \in N^{\geq s}_G(X)$ is
adjacent in $H$ to $\{x, y\} \in \binom{|X|}{s}$ whenever $x, y \in N_G(v) \cap X$. Let $M$ be a maximal matching in
$H$. Let $Q$ be the graph with vertex-set $X$, where $xy \in E(Q)$ whenever $\{x, y\} \in M$ for some
vertex $v \in N^{\geq s}_G(X)$. Thus, the 1-subdivision of every subgraph of $Q$ is a subgraph of $G$. Hence
$|M| = |E(Q)| \leq \frac{s}{2}|V(Q)| = \frac{s}{2}|X|$. Moreover, $Q$ is $\nabla$-degenerate, implying $Q$ contains at most
$\binom{|X|}{s-1}$ cliques of size exactly $s$. Exactly $|M|$ vertices in $N^{\geq s}_G(X)$ are incident with an edge in $M$.
For each vertex $v \in N^{\geq s}_G(X)$ not incident with an edge in $M$, by maximality, $N_G(v) \cap X$ is a clique
in $Q$ of size at least $s$. Define a mapping from each vertex $v \in N^{\geq s}_G(X)$ to a clique of size exactly $s$
in $Q[N_G(v) \cap X]$. Since $G$ has no $K_{s,t}$ subgraph, at most $t - 1$ vertices $v \in N^{\geq s}_G(X)$ are mapped
to each fixed $s$-clique in $Q$. Hence $|N^{\geq s}_G(X)| \leq \frac{s}{2}|X| + (t - 1)\binom{|X|}{s-1}|X| \\ \leq c|X|$. \qed

Lemma 13 is applicable in many instances. In particular, every graph $G$ with treewidth $k$ has
$\nabla(G) \leq 2k$ (since if a 1-subdivision of some graph $G'$ is a subgraph of $G$, then $G'$ has treewidth
at most $k$, and every graph with treewidth at most $k$ has average degree less than $2k$). By an
analogous argument, using bounds on the average degree independently due to Thomason [44]
and Kostochka [28], every $H$-minor-free graph $G$ has $\nabla(G) \leq O\big(|V(H)|\sqrt{\log |V(H)|}\big)$. Similarly,
using bounds on the average degree independently due to Komlós and Szemerédi [27] and Bollobás
and Thomason [4], every $H$-topological-minor-free graph $G$ has $\nabla(G) \leq O\big(|V(H)|^2\big)$. Finally,
Dujmović et al. [12, Lemmas 8,9] proved that $\nabla(G) \leq 5w$ for every graph with layered treewidth
$w$. In all these cases, Lemma 13 implies that the function $f$ in Lemma 12 can be made linear.
Corollary 14. For all \( s,t,w \in \mathbb{N} \), let \( f_{s,t,w} : \mathbb{N}_0 \to \mathbb{Q} \) be the function defined by \( f_{s,t,w}(x) := \left( \frac{2w}{2} + (t-1)\left(\frac{w}{2}\right) \right)x \) for every \( x \in \mathbb{N}_0 \). Then for every graph \( G \) with no \( K_{s,t} \) subgraph and with layered treewidth at most \( w \), if \( X \) is a subset of \( V(G) \), then \( |N^{\geq s}(X)| \leq f_{s,t,w}(|X|) \).

Proof. Dujmović et al. [12, Lemmas 8,9] showed that \( \nabla(G) \leq 5w \) for every graph with layered treewidth \( w \). So this result immediately follows from Lemma 13.

4 Fences in a Tree-decomposition

This section introduces the notion of a fence, which will be used throughout. We start with a variant of a well-known result about separators in graphs of bounded treewidth.

Lemma 15. Let \( w \in \mathbb{N}_\text{v} \). Let \( G \) be a graph and \((T,\mathcal{X})\) a tree-decomposition of \( G \) of width at most \( w \), where \( \mathcal{X} = (X_t : t \in V(T)) \). If \( Q \) is a subset of \( V(G) \) with \( |Q| \geq 12w + 13 \), then there exists \( t^* \in V(T) \) such that for every component \( T' \) of \( T - t^* \), \( |(Q \cap (\bigcup_{t \in V(T')} X_t)) \cup X_{t^*}| < \frac{3}{4}|Q| \).

Proof. Suppose that there exists an edge \( xy \) of \( T \) such that \( |(Q \cap (\bigcup_{t \in V(T_i)} X_t)) \cup X_x \cup X_y| \geq \frac{2}{3}|Q| \) for each \( i \in [2] \), where \( T_1,T_2 \) are the components of \( T - xy \). Then

\[
|Q| + 2|X_x \cup X_y| \geq \sum_{i=1}^{2} |(Q \cap (\bigcup_{t \in V(T_i)} X_t)) \cup X_x \cup X_y| \geq \frac{4}{3}|Q|.
\]

Since the width of \((T,\mathcal{X})\) is at most \( w \), \( 4(w+1) \geq 2|X_x \cup X_y| \geq \frac{4}{3}|Q| \geq 4w + \frac{12}{3} \), a contradiction.

First assume that there exists an edge \( xy \) of \( T \) such that \( |(Q \cap (\bigcup_{t \in V(T_i)} X_t)) \cup X_x \cup X_y| < \frac{3}{4}|Q| \) for each \( i \in [2] \), where \( T_1,T_2 \) are the components of \( T - xy \). Let \( t^* := x \). Then for every component \( T' \) of \( T - t^* \), \( (Q \cap (\bigcup_{t \in V(T_i)} X_t)) \cup X_{t^*} \subseteq (Q \cap (\bigcup_{t \in V(T_i)} X_t)) \cup X_x \cup X_y \) for some \( i \in [2] \), and hence \( |(Q \cap (\bigcup_{t \in V(T')} X_t)) \cup X_{t^*}| < \frac{3}{4}|Q| \). So the lemma holds.

Now assume that for every edge \( xy \) of \( T \), there exists a unique \( r \in \{x,y\} \) such that \( |(Q \cap (\bigcup_{t \in V(T_r)} X_t)) \cup X_x \cup X_y| \geq \frac{2}{3}|Q| \), where \( T_x,T_y \) are the components of \( T - xy \) containing \( x,y \), respectively. Orient the edge \( xy \) so that \( r \) is the head of this edge. We obtain an orientation of \( T \). Note that the sum of the out-degree of the nodes of \( T \) equals \( |E(T)| = |V(T)| - 1 \). So some node \( t^* \) has out-degree 0.

For each component \( T' \) of \( T - t^* \), let \( t_{T'} \) be the node in \( T' \) adjacent in \( T \) to \( t^* \). By the definition of the direction of \( t_{T'},t^* \),

\[
|(Q \cap (\bigcup_{t \in V(T_{t_{T'}})} X_t)) \cup X_{t_{T'}}| \leq |(Q \cap (\bigcup_{t \in V(T_{T'})} X_t)) \cup X_{t^*}| < \frac{2}{3}|Q|.
\]

Note that \( T_{t_{T'}} = T' \) for every component \( T' \) of \( T - t^* \). This proves the lemma.

Let \( T \) be a tree and \( F \) a subset of \( V(T) \). An \( F \)-part of \( T \) is an induced subtree of \( T \) obtained from a component of \( T - F \) by the adding nodes in \( F \) adjacent in \( T \) to this component. For an \( F \)-part \( T' \) of \( T \), define \( \partial T' \) to be \( F \cap V(T') \).

Lemma 16. Let \((T,\mathcal{X})\) be a tree-decomposition of a graph \( G \) of width at most \( w \in \mathbb{N}_\text{v} \), where \( \mathcal{X} = (X_t : t \in V(T)) \). Then for every \( \epsilon \in \mathbb{R} \) with \( 1 \geq \epsilon \geq \frac{1}{w+1} \) and every \( Q \subseteq V(G) \), there exists a subset \( F \) of \( V(T) \) with \( |F| \leq \max\{\epsilon(|Q| - 3w - 3),0\} \) such that for every \( F \)-part \( T' \) of \( T \),

\[
|\left(\bigcup_{t \in V(T')} X_t \cap Q\right) \cup \bigcup_{t \in \partial T'} X_t| \leq \frac{1}{\epsilon} \cdot (12w + 13).
\]
Proof. We shall prove this lemma by induction on \(|Q|\). If \(|Q| \leq \frac{1}{2}(12w+13)\), then let \(F := \emptyset\); then for every \(F\)-part \(T'\) of \(T\), \(|\bigcup_{t \in V(T')} X_t \cap Q\) \(\cup \bigcup_{t \in \partial T'} X_t| = |Q| \leq \frac{1}{2}(12w+13)\). So we may assume that \(|Q| > \frac{1}{2}(12w+13) \geq 12w+13\) and the lemma holds for all \(Q\) with smaller size.

By Lemma 15, there exists \(t^* \in V(T)\) such that for every component \(T'\) of \(T - t^*\), \(|(Q \cap \bigcup_{t \in V(T')} X_t) \cup X_{t^*}| < \frac{2}{3}|Q|\). That is, for every \(\{t^*\}\)-part \(T'\) of \(T\), \(|\bigcup_{t \in V(T')} X_t \cap Q\) \(\cup \bigcup_{t \in \partial T'} X_t| < \frac{2}{3}|Q|\).

For each \(\{t^*\}\)-part \(T'\) of \(T\), let \(Q_{T'} := (\bigcup_{t \in V(T')} X_t \cap Q) \cup X_{t^*}\), so \(|Q_{T'}| < \frac{2}{3}|Q|\). Let \(T_1, T_2, \ldots, T_k\) be the \(\{t^*\}\)-parts \(T'\) of \(T\) with \(|Q_{T'}| \geq \frac{1}{3}(12w+13)\).

Let \(f\) be the function defined by \(f(x) := \max\{\epsilon(x - 3w - 3), 0\}\) for every \(x \in \mathbb{R}\). For each \(i \in [k]\), since \(|Q_{T_i}| < \frac{2}{3}|Q| < |Q|\), the induction hypothesis implies that there exists \(F_i \subseteq V(T_i)\) with \(|F_i| \leq f(|Q_{T_i}|)\) such that for every \(F_i\)-part \(T'\) of \(T_i\), \(|\bigcup_{t \in V(T')} X_t \cap Q_{T_i}\) \(\cup \bigcup_{t \in \partial T'} X_t| \leq \frac{1}{3}(12w+13)\)

Define \(F = \{t^*\} \cup \bigcup_{i=1}^k F_i\). Note that for every \(F\)-part \(T'\) of \(T\), either \(T'\) is a \(\{t^*\}\)-part of \(T\) with \(|Q_{T'}| \leq \frac{1}{3}(12w+13)\), or there exists \(i \in [k]\) such that \(T'\) is an \(F_i\)-part of \(T_i\). In the former case, \(|\bigcup_{t \in V(T')} X_t \cap Q\) \(\cup \bigcup_{t \in \partial T'} X_t| = |Q_{T'}| \leq \frac{1}{3}(12w+13)\). In the latter case, \(|\bigcup_{t \in V(T')} X_t \cap Q\) \(\cup \bigcup_{t \in \partial T'} X_t| \leq |\bigcup_{t \in V(T')} X_t \cap Q_{T_i}\) \(\cup \bigcup_{t \in \partial T'} X_t| \leq \frac{1}{3}(12w+13)\) since \(X_{t^*} \subseteq Q_{T_i}\). Hence \(|\bigcup_{t \in V(T')} X_t \cap Q\) \(\cup \bigcup_{t \in \partial T'} X_t| \leq \frac{1}{3}(12w+13)\) for every \(F\)-part \(T'\) of \(T\).

To prove this lemma, it suffices to prove that \(|F| \leq f(|Q|)\). Note that \(|F| \leq 1 + \sum_{i=1}^k |F_i| \leq 1 + \sum_{i=1}^k f(|Q_{T_i}|)\). Since \(|Q_{T_i}| \geq \frac{1}{3}(12w+13) \geq 12w+13\) for every \(i \in [k]\), \(|F| \leq 1 + \sum_{i=1}^k \epsilon(|Q_{T_i}| - 3w - 3)\).

If \(k = 0\), then \(f(|Q|) = \epsilon(|Q| - 3w - 3) = \epsilon|Q| - \epsilon(3w + 3) \geq 12w + 13 - (3w + 3) > 1 = |F|\) since \(\epsilon|Q| \geq 12w + 13\) and \(\epsilon \leq 1\). If \(k = 1\), then \(|F| \leq 1 + \epsilon(|Q_{T_1}| - 3w - 3) \leq 1 + \frac{\epsilon}{3}|Q| - 3w - 3) = \epsilon(|Q| - 3w - 3) + 1 + \epsilon(kw + w - (k-1)w + 1)) \leq f(|Q|)\) since \(\epsilon|Q| \geq 12w + 13\). Hence we may assume that \(k \geq 2\). Then

\[
|F| \leq 1 + \sum_{i=1}^k \epsilon(|Q| \cap \bigcup_{t \in V(T_i) - \{t^*\}} X_t|) + |X_{t^*}| - 3w - 3
\]

\[
\leq 1 + \epsilon|Q| + k\epsilon|X_{t^*}| - 3w - 3
\]

\[
= \epsilon(|Q| - 3w - 3) + 1 + \epsilon|X_{t^*}| - (k-1)\epsilon(3w + 3)
\]

\[
\leq f(|Q|) + 1 + \epsilon(kw + w - 3(k-1)(w + 1))
\]

\[
\leq f(|Q|)
\]

since \(k \geq 2\). This proves the lemma.

Lemma 17. Let \((T, \mathcal{X})\) be a tree-decomposition of a graph \(G\) of width at most \(w \in \mathbb{N}_0\), where \(\mathcal{X} = \{X_t : t \in V(T)\}\). Then for every \(Q \subseteq V(G)\), there exists a subset \(F\) of \(V(T)\) with \(|F| \leq \max\{|Q| - 3w - 3, 1\}\) such that:

1. for every \(F\)-part \(T'\) of \(T\), \(|\bigcup_{t \in V(T')} X_t \cap Q| \leq 12w + 13\), and

2. if \(Q \neq \emptyset\), then for every \(t^* \in F\), there exists at least two \(F\)-parts \(T'\) of \(T\) satisfying \(t^* \in \partial T'\) and \(Q \cap \bigcup_{t \in V(T')} X_t \neq Q \cap X_{t^*} \neq \emptyset\).

Proof. Let \(Q \subseteq V(G)\). By Lemma 16 with \(\epsilon = 1\), there exists \(F \subseteq V(T)\) with \(|F| \leq \max\{|Q| - 3w - 3, 1\}\) such that for every \(F\)-part \(T'\) of \(T\), \(|\bigcup_{t \in V(T')} X_t \cap Q| \leq 12w + 13\). We further assume that \(F\) is minimal.

Suppose that \(Q \neq \emptyset\) and there exists \(t^* \in F\) such that there is at most one \(F\)-part \(T'\) of \(T\) satisfying \(t^* \in \partial T'\) and \(Q \cap \bigcup_{t \in \partial T'} X_t \neq Q \cap X_{t^*} \neq \emptyset\). Let \(F^* := F - \{t^*\}\). Note that for every \(F\)-part
W of $T$ with $t^* \not\in \partial W$, $W$ is an $F^*$-part of $T$ and $\partial W \subseteq F - \{t^*\}$, so $|\bigcup_{t \in V(W)} X_t \cap Q| \leq 12w + 13$. In addition, there is exactly one $F^*$-part $T^*$ of $T$ with $t^* \in V(T^*)$, and every $F^*$-part of $T$ other than $T^*$ is an $F$-part of $T$. Since there is at most one $F$-part $T'$ of $T$ satisfying $t^* \in \partial T'$ and $Q \cap \bigcup_{t \in \partial T'} X_t - X_{t^*} \neq \emptyset$, we know $|\bigcup_{t \in V(T')} X_t \cap Q| = |\bigcup_{t \in V(T^')} X_t \cap Q| \leq 12w + 13$. This contradicts the minimality of $F$ and proves the lemma.

We call the set $F$ mentioned in Lemma 17 a $(T, \mathcal{X}, Q)$-fence.

5 Evolution and Fans

This section introduces the notions of evolution, fans and gap. For $k \in \mathbb{N}$, a $k$-evolution is an infinite sequence $(s_1, s_2, \ldots)$ such that:

(EVO1) for each $i \in \mathbb{N}$, $s_i$ is an infinite sequence over nonnegative integers, and the sum of all entries of $s_i$ is at most $k$,

(EVO2) for each $i \in \mathbb{N}$, $s_i \neq s_{i+1}$ unless $s_i$ is the zero sequence,

(EVO3) for each $i \in \mathbb{N}$ in which $s_i$ is not a zero sequence, and for each $j \in [\ell_i]$, where $\ell_i$ is the largest index such that the $\ell_i$-th entry of $s_i$ is nonzero, the $j$-th entry of $s_{i+1}$ is at most the $j$-th entry of $s_i$,

(EVO4) for each $i \in \mathbb{N}$ and each $j \in \mathbb{N}$ with $j \geq 2$, if the $j$-th entry of $s_{i+1}$ is $0$ and the $j$-th entry of $s_i$ is nonzero, then either $s_{i+1}$ is a zero sequence, or there exists $j' \in [j - 1]$ such that the $j'$-th entry of $s_{i+1}$ is smaller than the $j'$-th entry of $s_i$.

Lemma 18. There exists a function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $k$-evolution $(s_1, s_2, \ldots)$, if $s_{i,j}$ is the $j$-th entry of $s_i$, then there exists $\eta' \in [g(k, s_{1,1})]$ such that the first entry of $s_{\eta'}$ is $0$.

Proof. Define $g : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{N}$ such that for every $y \in \mathbb{N}_0$, $g(1, y) := 2$, and for every $x \in \mathbb{N}$, $g(x + 1, 1) := g(x, x)$ and $g(x + 1, y + 1) := g(x + 1, y) + g(x, x)$.

Let $k \in \mathbb{N}$. Let $(s_1, s_2, \ldots)$ be a $k$-evolution. We prove this lemma by induction on the lexicographic order of $(k, s_{1,1})$.

When $k = 1$, $s_1$ is either the zero sequence or the sequence with $s_{1,1} = 1$ and $s_{i,j} = 0$ for every $j \in \mathbb{N} - \{1\}$, so $s_{2,1} = 0$ by (EVO2) and (EVO3). This proves the case $k = 1$ since $g(1, 1) \geq 2$.

When $s_{1,1} = 0$, we are done since $g(1, 0) \geq 1$. So we may assume that $s_{1,1} \geq 1$, $k \geq 2$, and this lemma holds for every $(k', \ell') \in \mathbb{N} \times \mathbb{N}_{<k}$ with $(k', \ell')$ lexicographically smaller than $(k, s_{1,1})$.

Suppose to the contrary that $s_{i,1} \geq 1$ for every $i \in [g(k, s_{1,1})]$.

Suppose there exists $i^* \in \{g(k - 1, k - 1)\}$ such that $s_{i^*,1} < s_{1,1}$. Since $g(k - 1, k - 1) \leq g(k, s_{1,1})$, $s_{1,1} \geq 2$. So $(s_{i^*,1}, s_{i^*+1,1}, s_{i^*+2,1}, \ldots)$ is a $k$-evolution with $s_{i^*,1} \leq s_{1,1} - 1$. By the minimality of $(k, s_{1,1})$, there exists $\eta'_i \in [g(k, s_{i^*,1})]$ such that $s_{i^*,1} + \eta'_i = 0$. This is a contradiction since $i^* + \eta'_i \leq g(k - 1, k - 1) + g(k, s_{1,1} - 1) \leq g(k, s_{1,1})$.

So $s_{i,1} = s_{1,1}$ for every $i \in [g(k - 1, k - 1)]$.

For every $i \in \mathbb{N}$, let $m_i$ and $\ell_i$ be the smallest and largest indices of nonzero entries in $s_i$; for $\alpha, \beta \in \mathbb{N}$ with $\alpha$, let $s_i[\alpha, \infty) = s_i \cap [\alpha, \infty]$. If $s_i[\alpha, \infty)$ is the subsequence of $s_i$ starting at the $\alpha$-th entry. For every $i \in [g(k - 1, k - 1)]$, let $a_i \in s_i[0, \infty]$; for every $i \in \mathbb{N} - [g(k - 1, k - 1)]$, let $a_i$ be the infinite zero sequence. Since $a_{i,1} = s_{i,1}$ for every $i \in [g(k - 1, k - 1)]$, $(a_1, a_2, \ldots)$ is a $(k - s_{1,1})$-evolution. Since $(k - s_{1,1}, a_{1,1})$ is lexicographically smaller than $(k, s_{1,1})$, there exists $\eta'_i \in [g(k - s_{1,1}, a_{1,1})] \subseteq [g(k - 1, k - 1)]$ such that $a_{i,1} = 0$. By (EVO3), $s_{i,j} = 0$ for every $i \in [\eta'_i]$ and $j \in [2, m_i]$. So by (EVO4), $s_{\eta'_i,1} \leq s_{1,1}$. But $\eta'_i \leq g(k - 1, k - 1)$, a contradiction. This proves the lemma.
Now we prove another auxiliary lemma that will be used in the rest of the paper.

Let $(T, \mathcal{X})$ be a rooted tree-decomposition of a graph $G$ of width $w$, where $\mathcal{X} = (X_t : t \in V(T))$. Let $t, t'$ be distinct nodes of $T$ with $t' \in V(T_t)$. Let $T_t$ be the subtree of $T$ rooted at $t$. For $k \in [0, w + 1]$ and $m \in \mathbb{N}_0$, a $(t, t', k)$-fan of size $m$ is a sequence $(t_1, t_2, \ldots, t_m)$ of nodes such that

(FAN1) for every $j \in [m - 1]$, $t_{j+1} \in V(T_{t_j}) - \{t_j\}$ and $t' \in V(T_{t_j})$,

(FAN2) for every $j \in [m]$, $|X_{t_j} \cap X_t| = k$; and

(FAN3) $X_{t_j} - X_t$ are pairwise disjoint for all $j \in [m]$.

The $(t, t')$-gap in $(T, \mathcal{X})$ is the sequence $(a_0, a_1, \ldots, a_{w+1})$ such that for every $i \in [0, w + 1]$, $a_i$ is the maximum $m$ such that there exists a $(t, t', i)$-fan of size $m$.

Lemma 19. Let $w \in \mathbb{N}$ and let $(T, \mathcal{X})$ be a rooted tree-decomposition of a graph $G$ of width at most $w$, where $\mathcal{X} = (X_t : t \in V(T))$. Let $t, t'$ be distinct nodes of $T$ with $t' \in V(T_t)$. Let $t''$ be a node in $V(T) - \{t, t'\}$ such that $t''$ belongs to the path in $T$ from $t$ to $t'$. Then the $(t'', t')$-gap in $(T, \mathcal{X})$ is lexicographically at most the $(t', t'')$-gap in $(T, \mathcal{X})$.

Proof. For every $i \in [0, w + 1]$, let $A_i = (p_{i,1}, p_{i,2}, \ldots)$ and $B_i = (q_{i,1}, q_{i,2}, \ldots)$ be a $(t, t', i)$-fan and a $(t', t'', i)$-fan of largest size, respectively. Let $(a_0, a_1, \ldots, a_{w+1})$ and $(b_0, b_1, \ldots, b_{w+1})$ be the $(t, t')$-gap and the $(t'', t')$-gap in $(T, \mathcal{X})$, respectively.

Clearly, every $(t', t'', 0)$-fan is a $(t, t', 0)$-fan, so $b_0 \leq a_0$. We are done if $b_0 < a_0$. So we may assume that $a_0 = b_0$. Note that either $A_0 = B_0 = \emptyset$ or $X_{q_{0,1}} \cap X_{t''} = \emptyset \subseteq X_t$. So there exists a maximum $k^*$ such that for every $i \in [0, k^*]$, $a_i = b_i$ and either $A_i = B_i = \emptyset$ or $X_{q_{i,1}} \cap X_{t''} \subseteq X_t$. Note that it implies that $B_i$ is a $(t, t', i)$-fan for every $i \in [0, k^*]$. We are done if $k^* = w + 1$. So we may assume that $k^* \leq w$. And we may assume that $b_{k^*+1} \geq a_{k^*+1}$, for otherwise we are done. If $B_{k^*+1} = \emptyset$, then $0 = b_{k^*+1} \geq a_{k^*+1}$, so $A_{k^*+1} = B_{k^*+1} = \emptyset$, contradicting the maximality of $k^*$. So $B_{k^*+1} \neq \emptyset$.

Suppose $X_{q_{k^*+1, i}} \cap X_{t''} \nsubseteq X_t$. Let $k := |X_{q_{k^*+1, i}} \cap X_{t''} \cap X_t|$. So $k \leq |X_{q_{k^*+1, i}} \cap X_{t''}| - 1 = k^*$. For every $q \in B_k$, since $(T, \mathcal{X})$ is a tree-decomposition and $|X_q \cap X_{t''}| = k < k^* + 1 = |X_{q_{k^*+1, i}} \cap X_{t''}|$, $q \in V(T_{q_{k^*+1, i}}) - \{q_{k^*+1, i}\}$. Since $k \leq k^*$, $X_q \cap X_{t''}$ is a $k$-element subset of $X_t$ for every $q \in B_k$ by the definition of $k^*$. So $(q_{k^*+1, i}, q_{k, j} : j \in [\|B_k\|])$ is a $(t, t', k)$-fan of size $|B_k| + 1$, contradicting the maximality of $|B_k|$. Hence $X_{q_{k^*+1, i}} \cap X_{t''} \subseteq X_t$. Since $(T, \mathcal{X})$ is a tree-decomposition and $B_{k^*+1}$ is a $(t'', t', k^*+1)$-fan, $X_{q_{k^*+1, j}} \cap X_{t''} = X_{q_{k^*+1, 1}} \cap X_{t''} = X_{q_{k^*+1, i}} \cap X_{t''} = X_{q_{k^*+1, 1}} \cap X_{t'}$ for every $j \in [\|B_{k^*+1}\|]$. So $B_{k^*+1}$ is a $(t, t', k^*+1)$-fan. This implies that $b_{k^*+1} \geq a_{k^*+1}$. Since we assume $b_{k^*+1} \geq a_{k^*+1}$, it implies that $a_{k^*+1} = b_{k^*+1}$, contradicting the maximality of $k^*$.

Given $k \in \mathbb{Z}$ and a sequence $s = (s_1, s_2, \ldots, s_n)$ over $\mathbb{Z}$ for some $n \in \mathbb{N}$, the $k$-cap of $s$ is the sequence $(s'_1, s'_2, \ldots, s'_n)$ such that $s'_i = \min\{s_i, k\}$ for every $i \in [n]$.

Lemma 20. Let $w \in \mathbb{N}$ and let $(T, \mathcal{X})$ be a rooted tree-decomposition of a graph $G$ of width at most $w$, where $\mathcal{X} = (X_t : t \in V(T))$. Let $t, t'$ be distinct nodes of $T$ with $t' \in V(T_t)$. Let $\xi, \xi^* \in \mathbb{N}_0$ with $\xi^* \geq \xi$. Let $t''$ be a node in $V(T) - \{t, t'\}$ such that $t''$ belongs to the path in $T$ from $t$ to $t'$. If for every $i \in [0, w + 1]$ and every $(t, t', i)$-fan $(t_1, t_2, \ldots, t_m)$ (for some $m \in \mathbb{N}_0$), $|\{j \in [m] : t_j \in V(T_{t''})\}| \leq \xi - 1$, then the $\xi^*$-cap of the $(t'', t')$-gap in $(T, \mathcal{X})$ is lexicographically smaller than the $\xi^*$-cap of the $(t, t')$-gap in $(T, \mathcal{X})$. 

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Proof. For every $i \in [0, w + 1]$, let $A_i = (p_{i,1}, p_{i,2}, \ldots)$ and $B_i = (q_{i,1}, q_{i,2}, \ldots)$ be a $(t, t', i)$-fan and a $(t'', t', i)$-fan of largest size, respectively. Let $(a_0, a_1, \ldots, a_{w+1})$ and $(b_0, b_1, \ldots, b_{w+1})$ be the $\xi^*$-cap of the $(t, t')$-gap and the $\xi^*$-cap of the $(t'', t')$-gap in $(T, \mathcal{X})$, respectively. Note that for every $i \in [0, w + 1]$, $a_i = \min\{|A_i|, \xi^*\}$ and $b_i = \min\{|B_i|, \xi^*\}$.

Clearly, every $(t'', t', 0)$-fan is a $(t, t', 0)$-fan, so $b_0 \leq a_0$. We are done if $b_0 < a_0$. So we may assume that $a_0 = b_0$. Note that either $A_0 = B_0 = \emptyset$ or $q_{0,1} \cap X_{t'} = \emptyset \subseteq X_t$. So there exists a maximum $k^*$ such that for every $i \in [0, k^*]$, $a_i = b_i$, and either $A_i = B_i = \emptyset$ or $X_{q_{i,1}} \cap X_{t'} \subseteq X_t$. Note that this implies that $B_i$ is a $(t, t', i)$-fan for every $i \in [0, k^*]$.

First assume that $k^* \leq w$. We may assume that $b_{k^*+1} \geq a_{k^*+1}$, otherwise we are done. If $B_{k^*+1} = \emptyset$, then $0 = b_{k^*+1} \geq a_{k^*+1}$, so $B_{k^*+1} = B_{k^*+1} = \emptyset$, contradicting the maximality of $k^*$. So $B_{k^*+1} \neq \emptyset$.

Suppose $X_{q_{k^*+1,1}} \cap X_{t'} \not\subseteq X_t$. Let $k := |X_{q_{k^*+1,1}} \cap X_{t'}|$. So $k \leq |X_{q_{k^*+1,1}} \cap X_{t'}| - 1 = k^*$. For every $q \in B_k$, since $(T, \mathcal{X})$ is a tree-decomposition and $|X_q \cap X_{t'}| = k^* = 1 = |X_{q_{k^*+1,1}} \cap X_{t'}|$, $q \in V(T_{q_{k^*+1,1}})$.

Since $k \leq k^*$, $X_q \cap X_{t'}$ is a $k$-element subset of $X_t$ for every $q \in B_k$. So $(q_{k^*+1,1}, q_{j,k} : j \in [B_k])$ is a $(t, t', k)$-fan of size $|B_k| + 1$. If $|B_k| \leq \xi^* - 1$, then $\xi^* \geq |B_k| + 1 = b_k + 1 = a_k + 1 = a_k$, contradicting the maximality of $a_k$. So $|B_k| \geq \xi^*$. Since $B_k$ is a $(t, t', k)$-fan, $q_{j,k} \in V(T_{t'})$ for every $j \in [B_k]$. However, since $k \leq k^*$, $B_k$ is also a $(t, t', k)$-fan, so $|\{j \in [B_k] : q_{j,k} \in V(T_{t'})\}| \leq \xi^* - 1$, a contradiction.

Hence $X_{q_{k^*+1,1}} \cap X_{t'} \subseteq X_t$. So $B_{k^*+1}$ is a $(t, t', k^* + 1)$-fan. Hence $a_{k^*+1} \geq b_{k^*+1}$. But we assumed that $b_{k^*+1} \geq a_{k^*+1}$. So $B_{k^*+1} \neq \emptyset$, contradicting the maximality of $k^*$.

Therefore, $k^* = w + 1$. Let $\ell := |X_t \cap X_{t'}|$. So $\ell \leq w + 1$ since the width of $(T, X)$ is at most $w$. Since $(t'')$ is a $(t, t', \ell)$-fan, $b_\ell = a_\ell \geq 1$, so $B_\ell \neq \emptyset$. Hence $X_{q_{\ell,1}} \cap X_{t'} \subseteq X_t$. So $(t'', t_{q_{\ell,j}}, j \in [|B_\ell|])$ is a $(t, t', \ell)$-fan of size $|B_\ell| + 1$. If $|B_\ell| \leq \xi^* - 1$, then $\xi^* \geq |B_\ell| + 1 = b_\ell + 1 = a_\ell + 1$, a contradiction. So $|B_\ell| \geq \xi^*$. Since $B_\ell$ is a $(t, t', \ell)$-fan, $q_{j,k} \in V(T_{t'})$ for every $j \in [|B_\ell|]$. However, since $\ell \leq k^*$, $B_\ell$ is also a $(t, t', \ell)$-fan, so $|\{j \in [|B_\ell|] : q_{j,k} \in V(T_{t'})\}| \leq \xi^* - 1$, a contradiction. □

6 Setup for Main Proof

This section proves Theorem 1. The proof uses a list-coloring argument, where we assume that color $i$ does not appear in the lists of vertices in layers $V_j$ with $j \equiv i$ (mod $s + 2$). This ensures that each monochromatic component is contained within at most $s + 1$ consecutive layers.

For our purposes a color is an element of $\mathbb{Z}$. A list-assignment of a graph $G$ is a function $L$ with domain containing $V(G)$ such that $L(v)$ is a non-empty set of colors for each vertex $v \in V(G)$. For a list-assignment $L$ of $G$, an $L$-coloring of $G$ is a function $c$ with domain $V(G)$ such that $c(v) \in L(v)$ for every $v \in V(G)$. So an $L$-coloring has clustering $\eta$ if every monochromatic component has at most $\eta$ vertices. A list-assignment $L$ of a graph $G$ is an $\ell$-list-assignment if $|L(v)| \geq \ell$ for every vertex $v \in V(G)$.

Let $G$ be a graph and $Z \subseteq V(G)$. A $Z$-layering $\mathcal{V}$ of $G$ is an ordered partition $(V_1, V_2, \ldots)$ of $V(G) - Z$ into (possibly empty) sets such that for every edge $e$ of $G - Z$, there exists $i \in \mathbb{N}$ such that both ends of $e$ are contained in $V_i \cup V_{i+1}$. Note that a layering is equivalent to an $\emptyset$-layering. For a tree-decomposition $(T, \mathcal{X})$ of $G$, with $\mathcal{X} = (X_t : t \in V(T))$, the $\ell$-width of $(T, \mathcal{X})$ is

$$\max_{i \in \mathbb{N}} \max_{t \in V(T)} |X_t \cap V_i|.$$

Let $G$ be a graph and let $Z \subseteq V(G)$. For every $s \in \mathbb{N}$ and $\ell \in [s + 2]$, an $s$-segment of a $Z$-layering $(V_1, V_2, \ldots)$ of level $\ell$ is $\bigcup_{j=a}^{a+s} V_j$ for some (possibly non-positive) integer $a$ with $a \equiv \ell + 1$.
(mod $s + 2$), where $V_a = \emptyset$ if $a \leq 0$. When the integer $s$ is clear from the context, we write *segment* instead of $s$-segment.

Let $G$ be a graph and let $s \in \mathbb{N}$. Let $Z \subseteq V(G)$ and $\mathcal{V} = (V_1, V_2, \ldots)$ be a $Z$-layering of $G$. A list-assignment $L$ of $G$ is $(s, \mathcal{V})$-compatible if the following conditions hold:

- $L(v) \subseteq [s + 2]$ for every $v \in V(G)$.
- $i \not\in L(v)$ for every $i \in [s + 2]$ and $v \in \bigcup\{V_j : j \equiv i \pmod{s + 2}\}$.

Note that there is no condition on $L(v)$ for $v \in Z$.

We remark that for every $i \in [s + 2]$, if $v \in V(G)$ with $i \in L(v)$, then either $v \in Z$, or $v$ belongs to a segment of $\mathcal{V}$ with level $i$. This leads to the following easy observation that we frequently use.

**Proposition 21.** Let $G$ be a graph and let $s \in \mathbb{N}$. Let $\mathcal{V} = (V_1, V_2, \ldots)$ be a layering of $G$, and let $L$ be an $(s, \mathcal{V})$-compatible list-assignment. If $k \in \mathbb{N}$ and $c$ is an $L$-coloring such that for every $i \in [s + 2]$ and every $s$-segment $S$ with level $i$, every monochromatic component with respect to $c$ contained in $G[S]$ with color $i$ contains at most $k$ vertices, then $c$ has clustering at most $k$.

**Proof.** Since for every $i \in [s + 2]$, if $v \in V(G)$ with $i \in L(v)$, then $v$ belongs to a segment of $\mathcal{V}$ with level $i$, so every monochromatic component with color $i$ with respect to $c$ is contained in some segment of $\mathcal{V}$ with level $i$. \qed

Let $G$ be a graph, $Z \subseteq V(G)$, $s$ a positive integer, $\mathcal{V}$ a $Z$-layering of $G$, and $L$ an $(s, \mathcal{V})$-compatible list-assignment of $G$. For $Y_1 \subseteq V(G)$, we say that $(Y_1, L)$ is a $\mathcal{V}$-standard pair if the following conditions hold:

- (L1) $Y_1 = \{v \in V(G) : |L(v)| = 1\}$.
- (L2) For every $y \in N^{<s}(Y_1)$, $|L(y)| = s + 1 - |N_G(y) \cap Y_1|$, and $L(y) \cap L(u) = \emptyset$ for every $u \in N_G(y) \cap Y_1$. (Note that $|L(y)| \geq 2$.)
- (L3) For every $v \in V(G) - N_G[Y_1]$, we have $|L(v)| = s + 1$.

Let $G$ be a graph, $Z \subseteq V(G)$, $\mathcal{V}$ a $Z$-layering, and $(Y_1, L)$ a $\mathcal{V}$-standard pair. For a subset $W$ of $V(G)$ and color $i$ (not necessarily belonging to $\bigcup_{v \in V(G)} L(v)$), a $(W, i)$-progress of $(Y_1, L)$ is a pair $(Y'_1, L')$ defined as follows:

- Let $Y'_1 := Y_1 \cup W$.
- For every $y \in Y_1$, let $L'(y) := L(y)$.
- For every $y \in Y'_1 - Y_1$, let $L'(y)$ be a 1-element subset of $L(y) - \{i\}$, which exists by (L1).
- For each $v \in N^{<s}(Y'_1)$, let $L'(v)$ be a subset of $L(v) - \bigcup\{L'(w) : w \in N_G(v) \cap (W - Y_1)\}$

of size $|L(v)| - |N_G(v) \cap (W - Y_1)|$, which exists since $|L'(w)| = 1$ for $w \in N_G(v) \cap (W - Y_1)$.

- For every $v \in V(G) - (Y'_1 \cup N^{<s}(Y'_1))$, define $L'(v) := L(v)$. 

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It is straightforward to show that \((Y'_1, L')\) is well-defined and is an \((s, V)\)-compatible \(V\)-standard pair; see [32] for a more general notion of “progress”, where the analogous claim is proved in full. The intuition here is that in the \((W, i)\)-progress, vertices in \(W - Y_1\) are precolored by a color different from \(i\). Also note that every \(L'\)-coloring is an \(L\)-coloring.

Let \((Y_1, L)\) be a \(V\)-standard pair of a graph \(G\). For \(y \in Y_1\), a gate for \(y\) (with respect to \((Y_1, L)\)) is a vertex \(v \in N_G(y) - Y_1\) such that \(L(v) \cap L(y) \neq \emptyset\). For \(W \subseteq Y_1\), let

\[
A_{(Y_1, L)}(W) := \{v \in V(G) - Y_1 : v \text{ is a gate for some } y \in W \text{ with respect to } (Y_1, L)\}.
\]

The next lemma implies Theorem 1 by taking \(\eta = 0\) and \(Y_1 = \emptyset\). Most of the work in proving Lemma 22 is done by Lemma 23 in the next section. So we prove Lemma 22 first, assuming Lemma 23.

**Lemma 22.** For every \(s, t, w \in \mathbb{N}\) and \(\eta \in \mathbb{N}_0\), there exists \(\eta^* \in \mathbb{N}\) such that if \(G\) is a graph with no \(K_{s,t}\) subgraph, \(V\) is a layering of \(G\), \(L\) is a \((s, V)\)-compatible list-assignment, \(Y_1\) is a subset of \(V(G)\), \((Y_1, L)\) is a \(V\)-standard pair, and \((T, X)\) is a tree-decomposition of \(G\) of \(V\)-width at most \(w\) such that \(|Y_1 \cap S| \leq \eta\) for every \(s\)-segment \(S\) of \(V\), then there exists an \(L\)-coloring of \(G\) with clustering \(\eta^*\).

**Proof of Lemma 22 assuming Lemma 23.** Let \(s, t, w \in \mathbb{N}\) and \(\eta \in \mathbb{N}_0\). Let \(\eta^* (s, t, w + \eta)\) from Lemma 23. Let \(G\) be a graph with no \(K_{s,t}\) subgraph, \(V\) a layering of \(G\), \(L\) an \((s, V)\)-compatible list-assignment, \(Y_1\) a subset of \(V(G)\), \((Y_1, L)\) a \(V\)-standard pair, and \((T, X)\) a tree-decomposition of \(G\) of \(V\)-width at most \(w\) such that \(|Y_1 \cap S| \leq \eta\) for every \(s\)-segment \(S\) of \(V\). Say \(X = (X_t : t \in V(T))\).

For each \(t \in V(T)\), let \(X^*_t := X_t \cup Y_1\). Let \(X^* = (X^*_t : t \in V(T))\). Let \(t^*\) be a node of \(T\). Then \((T, X^*)\) is a tree-decomposition of \(G\) such that \(X^*_{t^*}\) contains \(Y_1\). Since for every \(s\)-segment \(S\) of \(V\), \(|Y_1 \cap S| \leq \eta\), the \(V\)-width of \((T, X^*)\) is at most \(w + \eta\). Therefore, by Lemma 23, there exists an \(L\)-coloring of \(G\) with clustering \(\eta^*\). 

\[\square\]

7 Main Lemma

The next lemma is the heart of the proof of Theorem 1.

**Lemma 23.** For every \(s, t, w \in \mathbb{N}\), there exists \(\eta^* := \eta^*(s, t, w) \in \mathbb{N}\) such that if \(G\) is a graph with no \(K_{s,t}\) subgraph, \(V\) is a layering of \(G\), \(L\) is an \((s, V)\)-compatible list-assignment, \(Y_1\) is a subset of \(V(G)\), \((Y_1, L)\) is a \(V\)-standard pair, and \((T, X)\) is a tree-decomposition of \(G\) of \(V\)-width at most \(w\) such that some bag contains \(Y_1\), then there exists an \(L\)-coloring of \(G\) with clustering \(\eta^*\).

**Proof.** Let \(s, t, w \in \mathbb{N}\). We start by defining several values used throughout the proof:

- Let \(f\) be the function \(f_{s,t,w}\) in Corollary 14.
- Let \(f_0\) be the identity function \(f\) on \(\mathbb{N}_0\); for every \(i \geq 1\), let \(f\) be the function from \(\mathbb{N}_0\) to \(\mathbb{N}_0\) such that \(f_i(x) = f_{i-1}(x) + f(f_{i-1}(x))\) for every \(x \in \mathbb{N}_0\).
- Let \(s^* := 12(s + 2)\).
- Let \(w_0 := 12w s^* + 13\).
- Let \(g_0 : \mathbb{N}_0 \to \mathbb{N}_0\) be the function defined by \(g_0(0) := w_0\) and \(g_0(x) := f_1(g_0(x - 1)) + 2w_0\) for every \(x \in \mathbb{N}\).
• Let $g_1 : \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_1(0) := g_0(w_0)$ and $g_1(x) := f_1(g_1(x - 1) + 3w_0)$ for every $x \in \mathbb{N}_0$.

• Let $g_2 : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_2(0, y) := y + w_0$ for every $y \in \mathbb{N}_0$, and $g_2(x, y) := f_2(g_2(x - 1, y)) + 3w_0$ for every $x \in \mathbb{N}_0$.

• Let $g_3 : \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_3(0) := g_1(s + 2)$, and $g_3(x) := g_2(s + 2, x - 1)$ for every $x \in \mathbb{N}_0$.

• Let $g_4 : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_4(0, y) := y$ and $g_4(x, y) := (s + 2) \cdot f_1(g_4(x - 1, y) + 2\eta_1)$ for every $x \in \mathbb{N}$ and $y \in \mathbb{N}_0$.

• Let $g_5 : \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_5(x) := f_{s+2}(g_4(w_0, x) + 2\eta_1) + 2g_3(4w_0)$ for every $x \in \mathbb{N}_0$.

• Let $g_6 : \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_6(0) := w_0$, and $g_6(x) := g_5(g_6(x - 1)) + w_0$ for every $x \in \mathbb{N}_0$.

• Let $\eta_2 := g_6(8w_0)$.

• Let $\eta_3 := (\eta_1 + 2\eta_2 + 1)w_0$.

• Let $g_7 : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_7(0, y) := y$ and $g_7(x, y) := f_1(g_7(x - 1, y)) + w_0$ for every $x \in \mathbb{N}$ and $y \in \mathbb{N}_0$.

• Let $g_8 : \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_8(0) := \eta_1 + 2\eta_2$ and $g_8(x) := g_7(s + 2, g_8(x - 1))$ for every $x \in \mathbb{N}_0$.

• Let $g_9 : \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_9(0) := 2g_8(w_0) + \eta_2$ and $g_9(x) := f_1(g_9(x - 1)) + w_0$ for every $x \in \mathbb{N}$.

• Let $\eta_4 := g_9(s + 2)$.

• Let $g_{10} : \mathbb{N}_0 \to \mathbb{N}_0$ be the function defined by $g_{10}(0) := 2\eta_4$ and $g_{10}(x) := f_{s+2}(g_{10}(x - 1) + 3\eta_3)$ for every $x \in \mathbb{N}_0$.

• Let $\eta_5 := g_{10}(s + 2)$.

• Let $\eta_6 := 2\eta_4 \cdot f(2\eta_4)$.

• Let $\xi : \mathbb{N}_0 \to \mathbb{N}$, $h_0 : \mathbb{N}_0 \to \mathbb{N}$ and $h : \mathbb{N}_0 \to \mathbb{N}$ be the functions such that $\xi(0) := 2\eta_5 + 5w_0 + 1$, and for every $x \in \mathbb{N}_0$,

\[- h_0(x) := g_{18}((x\eta_6 + 1)(\xi(x) + 1)^{w_0 + 1}\eta_6, (x\eta_6 + 1)(\xi(x) + 1)^{w_0 + 1}\eta_6), \text{ where } g_{18} \text{ is the function } g \text{ mentioned in Lemma 18},
\]

\[- h(0) := \eta_6h_0(0),
\]

\[- h(x + 1) := (x + 2)h(x)^{x+3} + \eta_6h_0(x + 2), \text{ and}
\]

\[- \xi(x + 1) := (f(\eta_5) + 1) \cdot 4(\eta_5 + 2w_0 + 2)^2\sum_{\alpha=0}^{x} h(\alpha) (\eta_5 + 4w_0 + 1) + 2\eta_5 + 5w_0 + 1.
\]

• Let $\eta_7 := (w_0 \cdot (h(w_0) + 1)(w_0 + 2) + h(w_0) \cdot (w_0 + 2) + w_0 + 1) \cdot w_0$.

• Let $\eta_8 := g_{18}(\eta_7, \eta_7) + 1$, where $g_{18}$ is the function $g$ mentioned in Lemma 18.

• Let $\eta_9 := (w_0 + 2)\eta_8^\eta_5$.

• Let $\eta^* := \eta_4 + \eta_5\eta_9$. 

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Let $G$ be a graph with no $K_{s,t}$ subgraph, $V = (V_1, V_2, \ldots)$ a layering of $G$, $L$ an $(s, V)$-compatible list-assignment, $Y_1$ a subset of $V(G)$, $(Y_1, L)$ a $V$-standard pair, and $(T, \mathcal{X})$ a tree-decomposition of $G$ of $V$-width at most $w$ such that some bag contains $Y_1$, where $\mathcal{X} = (X_t : t \in V(T))$. For every $R \subseteq V(T)$, define $X_R := \bigcup_{t \in R} X_t$. For every $U \subseteq V(G)$, define $\mathcal{X}|_U = (X_t \cap U : t \in V(T))$.

Let $r^*$ be the node of $T$ such that $X_{r^*}$ contains $Y_1$. Consider $T$ to be rooted at $r^*$. Orient the edges of $T$ away from $r^*$. For each node $q$ of $T$, define the height of $q$ to be the length of the path in $T$ from $r^*$ to $q$, and let $T_q$ be the subtree of $T$ induced by $q$ and all the descendants of $q$. For $i \in N_0$, define $T^{(i)}$ to be the subtree of $T$ induced by the nodes of height at most $i$. For each vertex $v$ of $G$, let $r_v$ be the node of $T$ closest to $r^*$ with $v \in X_{r_v}$; define the height of $v$ to be the height of $r_v$. For $i \in N_0$, define $\partial X_{V(T^{(i)})}$ to be the union of the bags of the nodes of height exactly $i$.

We will construct a desired $L$-coloring of $G$ by an algorithm. We now give an informal description of several notions that will be used in the algorithm below. The layers are partitioned into pairwise disjoint belts, where each belt consists of a very large (but still bounded) set of consecutive layers. An interface consists of roughly the last one third of the layers within one belt, along with the first roughly one third of the layers in the next belt. Then the interior of an interface is the last few layers of the first belt, along with the first few layers of the next belt. The algorithm also uses a linear ordering of $V(G)$ that never changes. We associate with each subgraph of $G$ the first vertex in the ordering that is in the subgraph. We use this vertex to order subgraphs, whereby a subgraph that has a vertex early in the ordering is considered to be “old”. We now formalize these ideas.

Define $\sigma$ to be a linear order of $V(G)$ such that for any distinct vertices $u, v$, if the height of $u$ is less than the height of $v$, then $\sigma(u) < \sigma(v)$. For every subgraph $H$ of $G$, define $\sigma(H) := \min\{\sigma(v) : v \in V(H)\}$. Note that for any $k \in N_0$, any set $Z$ consisting of $k$ consecutive layers, any $t \in V(T)$, any $V$-standard pair $(Y', L')$, any monochromatic component $M$ with respect to any $L'$-coloring in $G[Y']$ intersecting $Z \cap X_t$, there exist at most $|Z \cap X_t| \leq kw$ monochromatic components $M'$ with respect to any $L'$-coloring in $G[Y']$ intersecting $Z \cap X_t$ such that $\sigma(M') < \sigma(M)$.

For any collection $E$ of 2-element subsets of $V(G)$, an $E$-pseudocoloring of $G$ is a connected subgraph of $G + E$, where $G + E$ is the graph with vertex-set $V(G)$ and edge-set $E(G) \cup E$.

A belt of $(T, \mathcal{X})$ is a subset of $V(G)$ of the form $\bigcup_{i=a+1}^{a+s} V_i$ for some nonnegative integer $a$ with $a \equiv 0 \pmod{s}$. Note that each belt consists of $s^*$ layers. So for every belt $B$ of $(T, \mathcal{X})$, $(T, \mathcal{X}|_B)$ is a tree-decomposition of $G[B]$ of width at most $s^*w$.

For every $j \in [|[V] - 1|]$, let

$$I_j := \bigcup_{i=(j-\frac{1}{2})s^*}^{(j+\frac{1}{2})s^*} V_i \quad \text{and} \quad \overline{I}_j := \bigcup_{i=(j-\frac{1}{2})s^*-2s+5}^{(j+\frac{1}{2})s^*+2s+5} V_i.$$

$I_j$ is called the interface at $j$. (Note that $s^*$ is a multiple of 3, so the indices in the definition of $I_j$ and $\overline{I}_j$ are integers.) For every $j \in [|[V] - 1|]$, define

$$I_{j, 0} := \bigcup_{i=js^*-(s+2)+1}^{js^*} V_i \quad \text{and} \quad \overline{I}_{j, 0} := \bigcup_{i=js^*-(s+2)+1}^{js^*+2(s+2)+1} V_i,$$

and define

$$I_{j, 1} := \bigcup_{i=js^*+s+2}^{js^*+s+2} V_i \quad \text{and} \quad \overline{I}_{j, 1} := \bigcup_{i=js^*+1}^{js^*+2(s+2)+1} V_i.$$
Also define \( I_j^0 := I_{j,0} \cup I_{j,1} \) and \( T_j^0 := \overline{T_{j,0}} \cup \overline{T_{j,1}} \). For every \( j \in [|\mathcal{V}|] \), define \( S_j^0 \) to be the set of all \( s \)-segments \( S \) intersecting \( I_j^0 \). Note that \( I_j^0 \subseteq \bigcup_{S \in S_j^0} S \subseteq T_j^0 \leq I_j \) for every \( j \in [|\mathcal{V}|] \). In addition, for every \( j \in [|\mathcal{V}|] \), \( I_j \) is contained in a union of two consecutive belts, so for every \( t \in V(T) \), \( |I_j \cap X_t| \leq 2s^*w \leq w_0 \).

We now give some intuition about the algorithm that follows. The input to the algorithm is a tree-decomposition of \( G \) with bounded layered width, a set \( Y_1 \) of precolored vertices, and a list assignment \( L \) of \( G \). Throughout the algorithm, \( Y^{(i,*,*)} \) refers to the current set of colored vertices, where the superscript \((i,*,*)\) indicates the stage of the algorithm. This vector is incremented in lexicographic order as the algorithm proceeds. The algorithm starts with stage \((0,-1,0)\), during which time we initialize several variables. Throughout the algorithm, a monochromatic component refers to a component of the subgraph of \( G \) induced by the current precolored set of vertices (or to be more precise, vertices with one color in their list).

The algorithm does a BFS search of the tree \( T \) indexing the tree-decomposition, considering the nodes \( t \) of \( T \) of height \( i \) in turn (for \( i = 0, 1, 2, \ldots \)). The algorithm first builds a fence around the subgraph of vertices in bags rooted at node \( t \) (relative to belts). At stage \((i,-1,*)\), the algorithm tries to isolate the \( k \)-th oldest component intersecting some segment intersecting \( I_j^0 \). Then at stage \((i,0,*)\), the algorithm isolates the other monochromatic components intersecting \( X_t \). In both these stages, \(*\) refers to the color given to vertices around the component that we are trying to isolate. Then in stage \((i,*,*)\) we isolate the fences associated with subtree \( T_{j,t} \). The next stage adds fake edges between two current monochromatic components if in the future it is possible that they get joined by a monochromatic path. Now these components behave like one component (called a pseudo-component). Finally, the algorithm moves to the next height in the tree, and builds new fences with respect to the nodes at height \( i \) in the tree.

**Stage \((0,-1,0)\): Initialization**

- Let \((Y^{(0,-1,0)}, L^{(0,-1,0)})\) be an \((X_{r*}, 0)\)-progress of \((Y_1, L)\).
- Let \( U^{(0,-1)} := \emptyset \) and \( D^{(0,0,0)} := \emptyset \).
- Let \( F_{j,0} := \emptyset \) for every \( j \in [|\mathcal{V}|] \).
- Let \( S_{j,t}^{(0,0)} := \emptyset \), \( S_{j,t}^{(0,1)} := \emptyset \) and \( S_{j,t}^{(0,2)} := \emptyset \) for every \( j \in [|\mathcal{V}|] \) and \( t \in V(T) \).
- Let \( E_{j,t}^{(-1,w_0+1)} := \emptyset \) for every \( j \in [|\mathcal{V}|] \) and \( t \in V(T) \).

For \( i = 0, 1, 2, \ldots \), define the following:

**Stage: Building Fences**

- For every node \( t \) of \( T \) of height \( i \), define the following:
  - For every \( j \in [|\mathcal{V}|] \),
    * Let \( F'_{j,p} := F_{j,0} \) if \( t = r^* \), and let \( p \) be the parent of \( t \) if \( t \neq r^* \).
    * For every \( F'_{j,p} \cap V(T_t) \)-part \( T' \) of \( T_t \) containing \( t \), define \( F_{j,T'} \) to be a \((T', X|_{X(T')} \cap \overline{T_{j,0}}, (Y^{(i,-1,0)} \cup X_{r^*}) \cap X_{V(T')} \cap \overline{T_{j}})-fence.\)
    * Let \( F'_{j,p} := F'_{j,p} \cup \bigcup_{T'} F_{j,T'} \), where the union is over all \( F'_{j,p} \cap V(T_t) \)-parts \( T' \) of \( T_t \) containing \( t \).
    * Let \( T_{j,t} := \bigcup_{T'} T' \), where the union is over all \( F_{j,p} \cap V(T_t) \)-parts \( T' \) of \( T_t \) containing \( t \).
* Let $\partial T_{j,t} := \bigcup_{T'} (\partial T' - \{t\})$, where the union is over all $F_{j,p} \cap V(T_t)$-parts $T'$ of $T_t$ containing $t$.

- Let $Z_t := (X_V(T_t) - \bigcup_{j=1}^{[V]} I_j^*) \cup (\bigcup_{j=1}^{[V]} (I_j \cap X_V(T_{j,i+1})))$.

- Let $U^{(i,-1,0)} := U^{(i,-1)}$.

**Stage $(i, -1, \star)$: Isolate the $k$-th Oldest Component Intersecting a Segment in $S_j^*$:**

- For each $k \in [0, w_0 - 1]$, define the following:
  - For every $j \in [\{V\} - 1]$ and node $t$ of $T$ of height $i$, define the following:
    * Let $E_{j,t} := E_{j,t}^{(i-1,w_0+1)}$.
    * Let $M_{j,k}^{(t)}$ be the monochromatic $E_{j,t}$-pseudocomponent in $G[Y^{(i,-1,k)}]$ such that $\sigma(M_{j,k}^{(t)})$ is the $(k+1)$-th smallest among all monochromatic $E_{j,t}$-pseudocomponents in $G[Y^{(i,-1,k)}]$ intersecting $X_t$ with $A_{L_i^{(i,-1,k)}} (V(M_{j,k}^{(t)})) \cap X_V(T_t) - X_t \neq \emptyset$ and contained in some $s$-segment $S \in S_j^*$ whose level equals the color of $M_{j,k}^{(t)}$.
    - Let $W_0^{(i,-1,k)} := \{v \in Y^{(i,-1,k)} \cap X_V(T_t) : t \in V(T^{(i)}) - V(T^{(i-1)})\}$, there exists a monochromatic path $P$ in $G[Y^{(i,-1,k)} \cap X_V(T_t)]$ from $v$ to $X_V(T^{(i)}) \cap V(M_{j,k}^{(t)})$.
    - Let $(Y^{(i,-1,k,0)}, L^{(i,-1,k,0)}) = (Y^{(i,-1,k)}, L^{(i,-1,k)})$.
    - For every $q \in [0, s + 1]$, define the following:
      * Let $W_1^{(i,-1,k,q)} := \{v \in W_0^{(i,-1,k)} : L^{(i,-1,k)}(v) = q + 1\}$.
      * Let $W_2^{(i,-1,k,q)} := A_{L_i^{(i,-1,k,q)}} (W_1^{(i,-1,k,q)}) \cap (\bigcup_{t \in V(T^{(i)}) - V(T^{(i-1)})} Z_t)$.
      * Let $(Y^{(i,-1,k,q+1)}, L^{(i,-1,k,q+1)})$ be a $(W_2^{(i,-1,k,q)}, q + 1)$-progress of $(Y^{(i,-1,k,q)}, L^{(i,-1,k,q)})$.
      * For each $t \in V(T^{(i)}) - V(T^{(i-1)})$, if $(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap \overline{T_j} \cap Z_t \neq \emptyset$ and $c_j^{(t)}$ is undefined, then let $c_j^{(t)} := q + 1$.
      - Let $(Y^{(i,-1,k,q+1)}, L^{(i,-1,k,q+1)}) := (Y^{(i,-1,k,s+2)}, L^{(i,-1,k,s+2)})$.
      - Let $U^{(i,-1,k+1)} := U^{(i,-1,k)} \cup W_0^{(i,-1,k)}$.

**Stage $(i, 0, \star)$: Isolate the other components intersecting $X_t$**

- Let $(Y^{(i,0,0)}, L^{(i,0,0)}) := (Y^{(i,-1,w_0)}, L^{(i,-1,w_0)})$.
- Let $U^{(i,0)} := U^{(i,-1,w_0)}$.

- For each $k \in [0, s + 1]$, define the following:
  - Let $W_0^{(i,0,k)} := \{v \in Y^{(i,0,k)} - (X_V(T^{(i)})) - \partial X_V(T^{(i)})) : L^{(i,0,k)}(v) = \{k + 1\}$ and there exists a path $P$ in $G[Y^{(i,0,k)} - (X_V(T^{(i)})) - \partial X_V(T^{(i)}))$ from $v$ to $X_V(T^{(i)})$ such that $L^{(i,0,k)}(q) = \{k + 1\}$ for all $q \in V(P)$.
  - Let $W_2^{(i,0,k)} := A_{L_i^{(i,0,k)}} (W_0^{(i,0,k)}) \cap (\bigcup_{t \in V(T^{(i)}) - V(T^{(i-1)})} Z_t)$.
  - Let $(Y^{(i,0,k+1)}, L^{(i,0,k+1)})$ be a $(W_2^{(i,0,k)}, k + 1)$-progress of $(Y^{(i,0,k)}, L^{(i,0,k)})$.
  - Let $U^{(i,k+1)} := U^{(i,k)} \cup W_0^{(i,0,k)}$.
Stage \((i, \star, \star)\): Isolating the Fences of \(T_{j,t}\)

- Let \(W_3^{(i,-1)} := \bigcup_{t \in V(T^{(i)}) - V(T^{(i-1)})} X_t \cap I_j\).
- Let \(W_4^{(i)} := \bigcup_{t \in V(T^{(i)}) - V(T^{(i-1)})} X_{V(T_{j,t})} \cap I_j\).
- For each \(\ell \in [0, |V(T)|]\), define the following:
  - Let \(W_3^{(i,\ell)} := W_3^{(i,\ell-1)} \cup \bigcup_{t \in V(T^{(i)}) - V(T^{(i-1)})} \bigcup_{q=1}^{V|-1} \bigcup \bigcup_{q} (X_q \cap I_j)\), where the last union is over all nodes \(q \in \partial T_{j,t}\) for which there exists a monochromatic path in \(G[Y^{(i,\ell,0)} \cap I_j \cap W_4^{(i)}]\) from \(\bigcup_{q=1}^{V|-1} W_3^{(i,\ell)}\) to \(X_q \cap I_j \cap \{v \in D^{(i,\ell,0)} \cap X_t \cap X_q \cap I_j : q' \in V(T_t) - \{t\}\}
  - Let \((Y^{(i,\ell,0)}, L^{(i,\ell,0)})\) be a \((W_3^{(i,\ell)}, 0)\)-progress of \((Y^{(i,\ell,s+2)}, L^{(i,\ell,s+2)})\).
- For each \(k \in [0, s + 1]\), define the following:
  * Let \(W_0^{(i,\ell+1,k)} := \{v \in Y^{(i,\ell+1,k)} \cap W_4^{(i)} : L^{(i,\ell+1,k)}(v) = \{k+1\}\}
  - and there exists a path \(P\) in \(G[Y^{(i,\ell+1,k)} \cap W_4^{(i)}]\) from \(v\) to \(W_3^{(i,\ell)}\)
  * Let \(W_2^{(i,\ell+1,k)} := A_{L^{(i,\ell+1,k)}}(W_0^{(i,\ell+1,k)}) \cap W_4^{(i)}\)
  * Let \((Y^{(i,\ell+1,k+1)}, L^{(i,\ell+1,k+1)})\) be the \((W_2^{(i,\ell+1,k+1)}, k+1)\)-progress of \((Y^{(i,\ell+1,k)}, L^{(i,\ell+1,k+1)})\).
  * Let \(D^{(i,\ell,k+1)} := D^{(i,\ell,k)} \cup W_0^{(i,\ell+1,k)}\).
  - Let \(D^{(i,\ell+1,0)} := D^{(i,\ell,s+2)}\).
  - Let \((Y^{(i,\ell+2,0)}, L^{(i,\ell+2,0)}) := (Y^{(i,\ell+1,s+2)}, L^{(i,\ell+1,s+2)})\).

Stage: Adding Fake Edges

- For every \(j \in [|V| - 1] \cap I' \in V(T_t)\), define the following:
  - Let \(E_{j,t'}^{(i,0)} := E^{(i,-1,w_0+1)}\).
  - For every \(k \in [0, w_0 - 1]\), let \(M_k\) be the monochromatic \(E_{j,t'}^{(i,k)}\)-pseudocomponent in \(G[Y^{(i,|V(T)|+1,s+2)}]\) such that \(\sigma(M_k)\) is the \((k+1)\)-th smallest among all monochromatic \(E_{j,t'}^{(i,k)}\)-pseudocomponents in \(G[Y^{(i,|V(T)|+1,s+2)}]\) intersecting \(X_t\) and contained in an \(s\)-segment in \(S_j^\circ\) whose level equals the color of \(M_k\), and define \(E_{j,t'}^{(i,k,0)} := E^{(i,k)}\) and \(E_{j,t'}^{(i,k+1)} := E_{j,t'}^{(i,k,|V(G)|)}\), where for every \(\ell \in [0, |V(G)| - 1]\), \(E_{j,t'}^{(i,k,\ell-1)} := E^{(i,k,\ell)}\) and for every \(\ell' \in [-1, |V(G)| - 1]\), \(E_{j,t'}^{(i,k,\ell,\ell'+1)}\) is defined to be the union of \(E_{j,t'}^{(i,k,\ell,\ell')}\) and the set of dangerous pairs \(\{u, v\}\), where \(\{u, v\}\) is dangerous if the following conditions hold:
    * \(u, v\) are distinct vertices of \(Y^{(i,|V(T)|+1,s+2)}\)
    * \(L^{(i,|V(T)|+1,s+2)}(u) = L^{(i,|V(T)|+1,s+2)}(v)\)
    * \(\{u, v\}\) is contained in an \(s\)-segment \(S' \in S_j^\circ\) whose level is the unique color in \(L^{(i,|V(T)|+1,s+2)}(u)\).
* there exists \( t'' \in \partial T_{j,t} \) with \( V(M_k) \cap X_{t''} \neq \emptyset \) and with \( A_{L(i,|V(T)|+1,s+2)}(V(M_k)) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \) such that \( \{u,v\} \subseteq X_{t''} \), \( t'' \) is a witness for \( X_{t''} \cap I_j \subseteq W^{(i,\alpha)}_3 \) for some \( \alpha \in [0,|V(T)|] \), and for every \( k' \in [0, k-1] \), either \( V(M_{k'}) \cap X_{t''} = \emptyset \) or \( A_{L(i,|V(T)|+1,s+2)}(V(M_{k'})) \cap X_{V(T_{t''})} - X_{t''} = \emptyset \),

(we call \( t'' \) is a witness for \( \{u,v\} \in E_{j,t'',t''}^{(i,k+1)} \))

* for every \( x \in \{u,v\} \), if \( M_x \) is the monochromatic \( E_{j,t',t'}^{(i,k,k',\ell)} \)-pseudocomponent in \( G[Y(i,|V(T)|+1,s+2)] \) containing \( x \), then \( A_{L(i,|V(T)|+1,s+2)}(V(M_x)) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \), and \( \sigma(M_x) \) is not the monochromatic \( E_{j,t',t'}^{(i,k,k',\ell)} \)-pseudocomponent \( M'' \) in \( G[Y(i,|V(T)|+1,s+2)] \) such that \( \sigma(M'') \) is minimum among all monochromatic \( E_{j,t',t'}^{(i,k,k',\ell)} \)-pseudocomponents in \( G[Y(i,|V(T)|+1,s+2)] \) contained in some \( s \)-segment in \( S_j \) whose level equals its color such that \( V(M'') \cap X_{t''} \neq \emptyset \) and \( A_{L(i,|V(T)|+1,s+2)}(V(M'')) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \),

* \( M_u \neq M_v, \sigma(M_u) < \sigma(M_v), \sigma(M_u) \) is the \((\ell+1)\)-th smallest among all monochromatic \( E_{j,t',t'}^{(i,k,k',\ell)} \)-pseudocomponents \( M'' \) in \( G[Y(i,|V(T)|+1,s+2)] \) such that \( V(M'') \cap X_{t''} \neq \emptyset \) and \( A_{L(i,|V(T)|+1,s+2)}(V(M'')) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \), and the \( s \)-segment containing \( V(M'') \) whose level equals its color such that \( V(M'') \cap X_{t''} \neq \emptyset \) and \( A_{L(i,|V(T)|+1,s+2)}(V(M'')) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \),

* there does not exist \( O \subseteq V(T_{t''}) - \{t''\} \) with \(|O| = \xi(\ell)\) such that \( O = \{o_1, o_2, \ldots, o_{|O|}\} \), where \( o_{\alpha+1} \in V(T_{o_{\alpha}}) - \{o_{\alpha}\} \) and \( X_{o_{\alpha+1}} \cap X_{o_{\alpha}} \cap I_j - X_{t''} = \emptyset \) for every \( \alpha \in [|O| - 1], A_{L(i,|V(T)|+1,s+2)}(V(M_v)) \cap X_{V(T_{t''})} - X_{t''} \subseteq X_{V(T_{t''})} - (X_{V(T_{o_{\alpha}})} - X_{o_{\alpha}}) \) and \( A_{L(i,|V(T)|+1,s+2)}(V(M_u)) \cap X_{V(T_{t''})} - X_{t''} \subseteq X_{V(T_{t''})} - X_{o_{\alpha}} \) for every monochromatic \( E_{j,t',t'}^{(i,k,k',\ell)} \)-pseudocomponent \( M'' \) in \( G[Y(i,|V(T)|+1,s+2)] \) with \( \sigma(M'') \leq \sigma(M_u) \) contained in some \( s \)-segment in \( S_j \) whose level equals its color such that \( V(M'') \cap X_{t''} \neq \emptyset \) and \( A_{L(i,|V(T)|+1,s+2)}(V(M'')) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \),

* for every node \( t'' \in V(T) \) of height \( i_{t''} \) with \( i_{t''} < i \) and \( T_{j,t''} \subseteq T_{j,t''} \), if there exists a monochromatic path in \( G[Y(i_{t''},|V(T)|+1,s+2) \cap X_{V(T_{j,t''})}] \) from \( \bigcup_{t''=1}^{t''} W^{(i_{t''},t'')}_3 \) to \( \{u,v\} \), and there does not exist any monochromatic path in \( G[Y(i_{t''},|V(T)|+1,s+2) \cap X_{V(T_{t''})}] \) from \( u \) to \( v \), then there exist \( t'' \in \partial T_{j,t''} \cap V(T_{t''}) \) with \( \{u',v'\} \subseteq X_{t''} \) such that \( \{u',v'\} \neq \{u,v\}, \{u',v'\} \in E_{j,t',t''} \) and there exist a monochromatic path in \( G[Y(i_{t''},|V(T)|+1,s+2)] \) from \( u \) to \( u' \) and a monochromatic path in \( G[Y(i_{t''},|V(T)|+1,s+2)] \) from \( v \) to \( v' \).

Let \( E_{j,t',t''}^{(i,k,k',\ell)} : = E_{j,t',t''}^{(i,k,k',|V(G)|)} \cup \{ (e \cap e') : (e \in E_{j,t',t''}^{(i,k,k',|V(G)|)}), (e \cap e') = 1 \} \).

Let \( E_{j,t',t''}^{(i,w_0)} \) be the union of \( E_{j,t',t''}^{(i,w_0)} \) and the set of \( w_0 \)-dangerous pairs \( \{u,v\}, \) where \( \{u,v\} \in w_0 \)-dangerous if the following conditions hold:

* \( u, v \) are distinct vertices of \( Y(i,|V(T)|+1,s+2) \),
* \( L(i,|V(T)|+1,s+2)(u) - L(i,|V(T)|+1,s+2)(v) \),
* \( \{u,v\} \) is contained in an \( s \)-segment \( S_j \) whose level is the unique color in \( L(i,|V(T)|+1,s+2)(u) \),
* there exists \( t'' \in \partial T_{j,t'} \) with \( \bigcup_{k=0}^{w_0-1} V(M_{k}) \cap X_{t''} = \emptyset \) such that \( \{u,v\} \subseteq X_{t''} \), \( t'' \) is a witness for \( X_{t''} \cap I_j \subseteq W^{(i,\alpha)}_3 \) for some \( \alpha \in [0,|V(T)|] \),

(we call \( t'' \) is a witness for \( \{u,v\} \in E_{j,t'',t''}^{(i,w_0+1)} \))

* for every \( x \in \{u,v\} \), if \( M_x \) is the monochromatic \( E_{j,t',t'}^{(i,k)} \)-pseudocomponent in \( G[Y(i,|V(T)|+1,s+2)] \) containing \( x \), then \( A_{L(i,|V(T)|+1,s+2)}(V(M_x)) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \),

* for every node \( t'' \in V(T) \) of height \( i_{t''} \) with \( i_{t''} < i \) and \( T_{j,t''} \subseteq T_{j,t''} \), if there exists
a monochromatic path in $G[Y^i,T]+1,s+2) \cap X_V(T_{i,t})$ to
\{u,v\}, and there exists no monochromatic path in $G[Y^i,T]+1,s+2) \cap X_V(T_{i,t})$ from $u$ to $v$, then there exist $t''$, $\partial T_{i,t''} \cap V(T')$ and $\{u',v'\} \subseteq X_{i,t''}$ such that
\{u',v'\} \neq \{u,v\}$, \{u',v'\} \subseteq F_{i,t''}$ and there exist a monochromatic path in $G[Y^i,T]+1,s+2)]$ from $u$ to $u'$ and a monochromatic path in $G[Y^i,T]+1,s+2)]$ from $v$ to $v'$.

Stage: Moving to Next Height in Tree

- Let $(Y^{i+1,1,0}, L^{i+1,1,0})$ be a $(\partial X_V(T^{i+1,1}), 0)$-progress of $(Y^i,T)+1,s+2), L^i,T)+1,s+2))$.
- Let $U^{i+1,1} := U^{i,s+2}$.
- Let $D^{i+1,0,0} := D(i,|V|,s+2)$.

For every node $t'$ of $T$ of height at least $i+1$ and $j \in [|V| − 1]$,

- If $t' \in V(T_{j,t})$, where $t$ is the node of $T$ of height $i$ with $t' \in V(T_t)$, then define
  * $S^{i+1,0}_{j,t'} := S^{i+1,0}_j \cup \bigcup_{v=0}^{|V(T)|} \bigcup_{c,k=0}^{s+1} W_0^{i,\ell,k}$, and
  * $S^{i+1,1}_{j,t'} := S^{i+1,1}_j \cup \bigcup_{v=1}^{|V(T)|} \bigcup_{c,k=0}^{s+1} (X_q \cap I_j)$, where the last union is over all nodes $q$ of $T$ such that $q \in V(T_{t'}) \setminus \{t'\}$ and $q$ is a witness for $X_q \cap I_j \subseteq W_3^{i,t'}$ for some $\ell' \in [0,|V(T)|]$;
  * $S^{i+1,2}_{j,t'} := S^{i+1,2}_j \cup \bigcup_{w=0}^{|w_0|} \bigcup_{c,k=0}^{s+1} W_0^{i,\ell,k} \cup \bigcup_{c,k=0}^{s+1} W_0^{i,0,0}$.
- otherwise, define $S^{i+1,0}_{j,t'} := S^{i+1,0}_j$, $S^{i+1,1}_{j,t'} := S^{i+1,1}_j$ and $S^{i+1,2}_{j,t'} := S^{i+1,2}_j$.

Stage: Building a New Fence

For every node $t \in V(T^i) \setminus V(T^{i−1})$ and every $j \in [|V| − 1]$, define the following:

- For every $F_{j,p} \cap V(T_t)$-part $T'$ of $T_t$ containing $t$, let $F'_{j,t'}$ be a
  $(T', X|_{V(T') \cap T_j}, Y^i,T)+1,s+2), L^i,T)+1,s+2))$-fence.
- Let $F'_{j,t'} := F_{j,p} \cup T'_t$, where the union is over all $F_{j,p} \cap V(T_t)$-parts $T'$ of $T_t$ containing $t$.

It is clear that if $(i',j',k')$ is a triple lexicographically smaller than a triple $(i,j,k)$, then $L^{i,j,k}(v) \subseteq L^{i',j',k'}(v)$ for every $v \in V(G)$, so $Y^{i,j,k} \supseteq Y^{i',j',k'}$. In addition, $X_{V(T^i)} \subseteq Y^i$ for every $i \in \mathbb{N}_0$, since $(Y^{0−1,0}, L^{0−1,0})$ is an $(Xₚ, 0)$-progress of $(Y, L)$ and $(Y^{0−1,0}, L^{0−1,0})$ is a $(\partial X_V(T^{i+1}), 0)$-progress of $(Y^i,T)+1,s+2), L^i,T)+1,s+2))$ for every $i \in \mathbb{N}_0$. In particular, we have $|L^{i,T}−1,0)(v)| = 1$ for every $v \in V(G)$, so there exists a unique $L^{i,T}−1,0$-coloring $c$ of $G$. By construction, $c$ is an L-coloring. We prove below that $c$ has clustering $q^*$.

Claim 23.1. Let $i \in \mathbb{N}_0$, and let $t \in V(T^i) \setminus V(T^{i−1})$. Let $j \in [|V| − 1]$ and $\ell \in [−1,|V(T)|]$. If $k \in [0,|V(T)|]$ and $P$ is a monochromatic path with respect to $c$ of color $k + 1$ contained in $G[W_4^{i,\ell}]$ intersecting $W_3^{i,\ell}$, then $V(P) \subseteq Y^{i,\ell+1,k}$ and $A_{L^{i,\ell+1,k+1}}(V(P)) \cap W_4^{i,\ell} = \emptyset$.

Proof. First suppose that $V(P) \subseteq Y^{i,\ell+1,k}$. Since $W_3^{i,\ell} \subseteq Y^{i,\ell+1,k}$, $V(P) \cap W_3^{i,\ell}$ is a nonempty subset of $Y^{i,\ell+1,k}$. So there exists a longest subpath $P'$ of $P$ contained in $G[Y^{i,\ell+1,k}]$ intersecting $W_3^{i,\ell}$. Since $V(P') \subseteq V(P) \subseteq W_4^{i,\ell}$, $V(P') \subseteq W_0^{i,\ell+1,k}$. In addition, $P \neq P'$, for otherwise $V(P) = V(P') \subseteq Y^{i,\ell+1,k}$, a contradiction. So there exist $v \in V(P')$ and $u \in$
$N_P(v) - V(P')$. That is, $c(u) = k + 1$ and $u \not\in Y^{(i,\ell+1,k)}$. So $u \in A_{L,(i,\ell+1,k)}(\{v\}) \cap V(P) - W_0^{(i,\ell+1,k)} \subseteq A_{L,(i,\ell+1,k)}(W_0^{(i,\ell+1,k)}) \cap W_4^{(i)} \subseteq W_2^{(i,\ell+1,k)}$. But $(Y^{(i,\ell+1,k+1)}, L^{(i,\ell+1,k+1)})$ is a $(W_2^{(i,\ell+1,k)}, k+1)$-progress, so $k + 1 \not\in L^{(i,\ell+1,k+1)}(u)$. Hence $c(u) \neq k + 1$, a contradiction.

Now we suppose that $A_{L,(i,\ell+1,k)}(V(P)) \cap W_4^{(i)} = \emptyset$. So there exists $z \in A_{L,(i,\ell+1,k+1)}(V(P)) \cap W_4^{(i)}$.

Note that we have shown that $V(P) \subseteq Y^{(i,\ell+1,k)}$, so $V(P) \subseteq W_0^{(i,\ell+1,k)}$ and $A_{L,(i,\ell+1,k+1)}(V(P)) \subseteq A_{L,(i,\ell+1,k)}(V(P))$. In addition, $z \in A_{L,(i,\ell+1,k+1)}(V(P))$, so $z \not\subseteq Y^{(i,\ell+1,k+1)} \supseteq W_0^{(i,\ell+1,k)}$.

So $z \in A_{L,(i,\ell+1,k+1)}(V(P)) \cap W_4^{(i)} - W_0^{(i,\ell+1,k)} \subseteq A_{L,(i,\ell+1,k)}(V(P)) \cap W_4^{(i)} - W_0^{(i,\ell+1,k)} = W_2^{(i,\ell+1,k)}$. Since $(Y^{(i,\ell+1,k+1)}, L^{(i,\ell+1,k+1)})$ is a $(W_2^{(i,\ell+1,k)}, k+1)$-progress of $(Y^{(i,\ell+1,k)}, L^{(i,\ell+1,k)})$, we know $k + 1 \not\in L^{(i,\ell+1,k+1)}(z)$, so $z \not\subseteq A_{L,(i,\ell+1,k+1)}(V(P))$, a contradiction. This proves the claim.

**Claim 23.2.** Let $i, i' \in \mathbb{N}_0$ with $i' < i$, and let $t \in V(T(i)) - V(T(i-1))$. Let $j \in [\lceil V \rceil - 1]$ and $\ell \in [0, \lceil V(T) \rceil]$. If $Y^{(i',\ell+1,s+2)} \cap X_{V(T)} \cap I_j - X_t \neq Y^{(i',\ell+1,s+2)} \cap X_{V(T)} \cap I_j - X_t$, then $(W_3^{(i',\ell)} - W_3^{(i',\ell-1)}) \cap I_j \neq \emptyset$, and either

- $\ell > 0$, or
- $|X_t \cap I_j \cap D^{(i',\ell+1,0)}| > |X_t \cap I_j \cap D^{(i',\ell,0)}|$, or
- $|X_t \cap I_j \cap S^{(i',\ell+1,1)}| > |X_t \cap I_j \cap S^{(i',\ell,0)}|$, or
- $|X_t \cap I_j \cap S^{(i',\ell+1,1)}| > |X_t \cap I_j \cap S^{(i',\ell,1)}|$. 

**Proof.** Suppose that $(W_3^{(i',\ell)} - W_3^{(i',\ell-1)}) \cap I_j = \emptyset$. Then for every $k \in [0, s+1]$ and every monochromatic path $P$ of color $k$ + 1 in $G[Y^{(i',\ell+1,k)} \cap W_4^{(i')}]$ intersecting $W_3^{(i',\ell)} \cap I_j$, $P$ is a monochromatic path with respect to $c$ in $G[W_4^{(i')}]$ intersecting $W_3^{(i',\ell)} \cap I_j \subseteq W_3^{(i',\ell-1)} \cap I_j$, so Claim 23.1 implies that $V(P) \subseteq Y^{(i',\ell,k)} \cap W_4^{(i')}$, and hence $V(P) \subseteq W_0^{(i',\ell,k)}$. Hence $W_0^{(i',\ell+1,k)} \cap I_j \subseteq W_0^{(i',\ell,k)} \cap I_j$ for every $k \in [0, s+1]$. Since no edge of $G$ is between $I_j$ and $I_{j'}$ for any $j' \neq j$, for every $k \in [0, s+1]$, 

$$W_2^{(i',\ell+1,k)} \cap I_j = A_{L,(i',\ell+1,k)}(W_0^{(i',\ell+1,k)}) \cap W_4^{(i)} \cap I_j$$

$$= A_{L,(i',\ell+1,k)}(W_0^{(i',\ell+1,k)} \cap I_j) \cap W_4^{(i)} \cap I_j$$

$$\subseteq A_{L,(i',\ell+1,k)}(W_0^{(i',\ell,k)} \cap I_j) \cap W_4^{(i)} \cap I_j$$

$$\subseteq A_{L,(i',\ell+1,k)}(W_0^{(i',\ell,k)} \cap W_4^{(i)}) \cap I_j$$

$$\subseteq A_{L,(i',\ell+1,k)}(W_0^{(i',\ell,k)}) \cap W_4^{(i)} \cap I_j$$

$$= W_2^{(i',\ell,k)} \cap I_j$$

Hence $(Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap I_j \subseteq \bigcup_{k=0}^{s+1} (W_2^{(i',\ell+1,k)} \cap I_j) \subseteq \bigcup_{k=0}^{s+1} (W_2^{(i',\ell,k)} \cap I_j) \subseteq Y^{(i',\ell,s+2)} \cap I_j$.

So $Y^{(i',\ell,s+2)} \cap I_j \subseteq Y^{(i',\ell,s+2)} \cap I_j$, a contradiction.

Hence $(W_3^{(i',\ell)} - W_3^{(i',\ell-1)}) \cap I_j \neq \emptyset$.

Suppose to the contrary that $\ell = 0$, $|X_t \cap I_j \cap D^{(i',\ell,s+2)}| \leq |X_t \cap I_j \cap D^{(i',\ell,0)}|$, $|X_t \cap I_j \cap S^{(i',\ell+1,0)}| \leq |X_t \cap I_j \cap S^{(i',\ell,0)}|$, and $|X_t \cap I_j \cap S^{(i',\ell+1,1)}| \leq |X_t \cap I_j \cap S^{(i',\ell,1)}|$. Since $D^{(i',\ell,s+2)} \geq D^{(i',\ell,0)}$ and $S^{(i',\ell+1,0)} \geq S^{(i',\ell,0)}$ and $S^{(i',\ell+1,1)} \geq S^{(i',\ell,1)}$, we know $X_t \cap I_j \cap D^{(i',\ell,s+2)} = X_t \cap I_j \cap D^{(i',\ell,0)}$, $X_t \cap I_j \cap S^{(i',\ell+1,0)} = X_t \cap I_j \cap S^{(i',\ell,0)}$, and $X_t \cap I_j \cap S^{(i',\ell+1,1)} = X_t \cap I_j \cap S^{(i',\ell,1)}$.

Since $(W_3^{(i',\ell)} - W_3^{(i',\ell-1)}) \cap I_j \neq \emptyset$, there exist $t' \in V(T^{(i')}) - V(T^{(i'-1)})$ and $q \in \partial T_{j,t}$ such that $q$ is a witness for $X_q \cap I_j \subseteq W_3^{(i',\ell)}$ and $X_q \cap I_j \not\subseteq W_3^{(i',\ell-1)}$. So $X_q \cap I_j$ is a nonempty subset of $W_3^{(i',\ell)) \cap I_j$.
with \( X_q \cap I_j \not\subseteq W_3^{(i',-1)} \), and there exists a monochromatic path \( P \in G[Y^{(i',\ell,s+2)} \cap I_j \cap W_4^{(i)}] \) with respect to \( c \) from \( \bigcup_{i'=1}^{t-1} W_3^{(i',\ell)} \) to \( \{X_q \cap I_j \} - ((D^{(i',\ell,0)} - X_P) \cup \{v \in D^{(i',\ell,0)} \cap X_P \}) \) and \( X_q \cap I_j : q' \in V(T_v) \} - \{t' \} \). \( q' \) is a witness for \( X_q' \cap I_j \subseteq W_3^{(i',\ell)} \) for some \( i' \in [0,i' - 1] \) and \( t' \in [0,|V(T)|] \) and \( | \nabla_{I^{t',i'}} | \) internally disjoint from \( X^{V(T_v)} \cap X_{\partial T_v} \). We further choose such node \( q \) such that \( q \in V(T_v) \) if possible. Note that there exists \( k \in [0,s+1] \) such that the color of \( P \) is \( k+1 \).

We first suppose that \( q \in V(T_v) \). Since \( V(P) \subseteq Y^{(i',\ell,s+2)} \cap W_3^{(i',\ell)} \subseteq Y^{(i',\ell,1+k)} \cap W_4^{(i)} \), \( V(P) \subseteq W_3^{(i',\ell,1+k)} \subseteq D^{(i',\ell,k+1)} \subseteq D^{(i',\ell,s+2)} \). Since \( X_t \cap I_j \cap D^{(i',\ell,s+2)} = X_t \cap I_j \cap D^{(i',\ell,0)}, V(P) \cap I_{t} \subseteq D^{(i',\ell,0)} \).

Let \( z \) be the end of \( P \) belonging to \( X_t \cap I_j \) - \( (D^{(i',\ell,0)} - X_P) \cup \{v \in D^{(i',\ell,0)} \cap X_P \} \). We further choose such node \( q \) such that \( q \in V(T_v) \) if possible.

Suppose \( z \notin D^{(i',\ell,0)} \). Since \( X_t \cap V(P) \subseteq D^{(i',\ell,0)}, z \notin X_t \). Since \( \ell = 0 \), \( P \) has one end in \( W_3^{(i',-1)} \). Since \( i' < i \) and \( q \in V(T_v) \), \( V(P) \) intersects \( X_t \), and there exists a subpath \( P' \subseteq P \) from \( X_t \cap V(P) \subseteq D^{(i',\ell,0)}, z \in X_{V(T_v)} -(X_t \cup D^{(i',\ell,0)}) \) and \( | \nabla_{I^{t',i'}} | \) internally disjoint from \( X_t \). So there exist \( a \in V(P') \cap D^{(i',\ell,0)} \) and \( b \in N_{P'}(a) \notin D^{(i',\ell,0)} \). Since \( b \notin V(P) \subseteq D^{(i',\ell,0)} \subseteq D^{(i',\ell,\ell,0)} \). Since \( X_t \cap V(P') \subseteq D^{(i',\ell,0)}, b \in N_{P'}(a) \notin D^{(i',\ell,0)} \cap X_t \). Let \( I^* = \{i'' \mid \exists a \in \bigcup_{i''=0}^{t-1} W_3^{(i'',\ell)} \} \). Since \( \ell = 0 \) and \( a \in D^{(i'',\ell,0)} \), \( a \in D^{(i'',\ell,0)} = D^{(i'-1,|V(T)|+2)} \). Since \( c(a) = k+1 \), \( a \in \bigcup_{i''=0}^{t-1} W_3^{(i'',\ell)} \) if possible.

Suppose \( z \notin D^{(i',\ell,0)} \). Since \( X_t \cap V(P) \subseteq D^{(i',\ell,0)}, z \notin X_t \). Since \( \ell = 0 \), \( P \) has one end in \( W_3^{(i',-1)} \). Since \( i' < i \) and \( q \in V(T_v) \), \( V(P) \) intersects \( X_t \), and there exists a subpath \( P' \subseteq P \) from \( X_t \cap V(P) \subseteq D^{(i',\ell,0)}, z \in X_{V(T_v)} -(X_t \cup D^{(i',\ell,0)}) \) and \( | \nabla_{I^{t',i'}} | \) internally disjoint from \( X_t \). So there exist \( a \in V(P') \cap D^{(i',\ell,0)} \) and \( b \in N_{P'}(a) \notin D^{(i',\ell,0)} \). Since \( b \notin V(P) \subseteq D^{(i',\ell,0)} \subseteq D^{(i',\ell,\ell,0)} \). Since \( X_t \cap V(P') \subseteq D^{(i',\ell,0)}, b \in N_{P'}(a) \notin D^{(i',\ell,0)} \cap X_t \). Let \( I^* = \{i'' \mid \exists a \in \bigcup_{i''=0}^{t-1} W_3^{(i'',\ell)} \} \). Since \( \ell = 0 \) and \( a \in D^{(i'',\ell,0)} \), \( a \in D^{(i'',\ell,0)} = D^{(i'-1,|V(T)|+2)} \). Since \( c(a) = k+1 \), \( a \in \bigcup_{i''=0}^{t-1} W_3^{(i'',\ell)} \) if possible.

On the other hand, since \( t \in V(T_{j,v}) \cap \partial T_{j,v} \) and \( a \in V(P) \subseteq W_3^{(i',\ell,1+k)} \), we know \( a \in S^{(i',\ell,0)} \). Since \( X_t \cap I_j \cap S^{(i',\ell,0)} = X_t \cap I_j \cap S^{(i',\ell,0)} \), we know \( a \notin X_t \). Recall that \( b \notin W_4^{(i')} \) for every \( i' \in I^* \). Since \( ab \in E(G) \) and \( V(P) \subseteq I_j \), we know for every \( i' \in I^* \), \( a \in X_{\partial T_{j,v}} \) for some node \( t_v \) of \( T \) of height \( i'' \). So for every \( i'' \in I^* \), there exists \( q_{i''} \in \partial T_{j,v} \) such that \( a \in X_{q_{i''}} \cap I_j \), where \( t_v \) is a node of \( T \) of height \( i'' \). If there exists \( i'' \in I^* \) with \( q_{i''} \in V(T_{v}) \} - \{t' \} \), then \( t_v \) is the node of \( T \) of height \( i'' \) with \( t' \in V(T_{v}) \), and since \( q \notin \partial T_{j,v} \) and \( q_{i''} \in V(T_v) \} - \{t' \} \), no node in \( \partial T_{j,v} \) is contained in the path in \( T \) from \( t' \) to the parent of \( q \) for every \( i'' \in I^* \), so we have \( b \notin W_4^{(i')} \) since \( b \notin W_4^{(i')} \), a contradiction. Hence \( q_{i''} \notin V(T_v) \} - \{t' \} \) for every \( i'' \in I^* \). Since \( I^* \neq \emptyset \), \( q_{i''} \notin V(T_v) \} - \{t' \} \) for some \( i'' \in I^* \), so \( a \in X_{q_{i''}} \cap X_{V(T_v)} \). This contradicts that \( a \notin X_t \).

Hence \( z \in D^{(i',\ell,0)} \). Since \( z \notin D^{(i',\ell,0)} \), \( X_v \), \( z \notin X_v \). So there does not exist \( q' \in V(T_v) \} - \{t' \} \) such that \( z \in X_{q'} \cap I_j \), \( q' \) is a witness for \( X_{q'} \cap I_j \subseteq W_3^{(i',\ell,0)} \) for some \( i'' \in [0,i'' - 1] \) and \( t' \in [0,|V(T)|] \). Since \( \ell = 0 \), \( \bigcup_{i'=1}^{t-1} W_3^{(i',\ell)} = W_3^{(i',-1)} \subseteq X_{V(T_v)} \). Since \( i' < i \) and \( z \in X_q \),

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$z \in X_t \cap X_t'$. Since $z \in X_q \cap I_j$ and $q \in V(T_t)$ and $q$ is a witness for $X_q \cap I_j \subseteq W_3^{(t', t)}$, and $t \in V(T_{t', t}) - \partial T_{t', t}$, we know $z \in s^{(t', t', t', t, 1)}$. Since $X_t \cap I_j \cap S_{t', t}^{(t', t', t', t', 1)} = X_t \cap I_j \cap S_{t', t}^{(t', t')}$, $z \in s^{(t', t')}$.

So there exist $i_z \in [0, i' - 1]$, a node $t_z$ of $T$ of height $i_z$ with $t \in V(T_{t_z, t}) - \partial T_{t_z, t}$, $\ell_z \in [0, |V(T)|]$, and $q_z \in V(T_t)$ such that $z \subset X_{q_z} \cap I_j$ and $q_z$ is a witness for $X_{q_z} \cap I_j \subseteq W_3^{(t, t')}$ for some $i' \in [0, i' - 1]$ and $t' \in [0, |V(T)|]$, a contradiction.

Therefore $q \notin V(T_t)$. So $(W_3^{(t', t')} - W_3^{(t', t', t', t', 1)}) \cap I_j \cap X(V(T_t)) \subseteq (W_3^{(t', t')} - W_3^{(t', t', t', t', 1)}) \cap I_j \cap X_t$ by the choice of $q$. Hence $Y^{(t', t', t', t', t', t')} \cap X(V(T_t)) \cap I_j \cap X_t = Y^{(t', t', t', t', t', t')} \cap X(V(T_t)) \cap I_j \cap X_t$. Since $Y^{(t', t', t', t', t', t')} \cap X(V(T_t)) \cap I_j \cap X_t$, there exists $k' \in [0, s + 1]$ such that $Y^{(t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t', t'
$\partial T_{j,t'} \cap V(T_t) - \{ t \}$ for $X_q \cap I_j \subseteq W_3^{(i',+1)}$ and $X_q \cap I_j \not\subseteq W_3^{(i',0)}$. Then either:

- $|X_t \cap I_j \cap S_{j,t}^{(i',1,0)}| > |X_t \cap I_j \cap S_{j,t}^{(i',0)}|$, or
- $|X_t \cap I_j \cap S_{j,t}^{(i'+1,1)}| > |X_t \cap I_j \cap S_{j,t}^{(i',1)}|$, or
- for every monochromatic path $P$ in $G[Y^{(i',+1,1,s+2)} \cap I_j \cap W_4^{(i',0)}]$ with respect to $c$ from $\bigcup_{t'=-1}^t W_3^{(i',0)}$ to

$$X_q \cap I_j - ((D^{(i',+1,0)} - X_{t'}) \cup \{ v \in D^{(i',+1,0)} \cap X_{t'} \cap X_{t'} \cap I_j : q' \in V(T_{t'}) - \{ t' \}, q' \text{ is a witness for } X_{t'} \cap I_j \subseteq W_3^{(i',0)} \text{ for some } i'' \in [0, i' - 1] \text{ and } t' \in [0, |V(T)|] \})$$

and internally disjoint from $X_{V(T(t'))} \cup X_{\partial T_{j,t'}}$, $V(P) \subseteq X_{V(T_t)} - X_t$.

**Proof.** We may assume that $|X_t \cap I_j \cap S_{j,t}^{(i',0)}| \leq |X_t \cap I_j \cap S_{j,t}^{(i',1)}|$ and $|X_t \cap I_j \cap S_{j,t}^{(i'+1,1)}| \leq |X_t \cap I_j \cap S_{j,t}^{(i',1)}|$, otherwise we are done. So $X_t \cap I_j \cap S_{j,t}^{(i',0)} = X_t \cap I_j \cap S_{j,t}^{(i',1)}$ and $X_t \cap I_j \cap S_{j,t}^{(i'+1,1)} = X_t \cap I_j \cap S_{j,t}^{(i',1)}$.

Since there exists a witness $q \in \partial T_{j,t'} \cap V(T_t) - \{ t \}$ for $X_q \cap I_j \subseteq W_3^{(i',+1)}$ and $X_q \cap I_j \not\subseteq W_3^{(i',0)}$, there exists a monochromatic path $P$ in $G[Y^{(i',+1,1,s+2)} \cap I_j \cap W_4^{(i',0)}]$ with respect to $c$ from $\bigcup_{t'=-1}^t W_3^{(i',0)}$ to a vertex

$$z \in X_q \cap I_j - ((D^{(i',0)} - X_{t'}) \cup \{ v \in D^{(i',0)} \cap X_{t'} \cap X_{t'} \cap I_j : q' \in V(T_{t'}) - \{ t' \}, q' \text{ is a witness for } X_{t'} \cap I_j \subseteq W_3^{(i',0)} \text{ for some } i'' \in [0, i' - 1] \text{ and } t' \in [0, |V(T)|] \})$$

and internally disjoint from $X_{V(T(t'))} \cup X_{\partial T_{j,t'}}$. We further choose $P$ such that $V(P) \subseteq X_{V(T_t)} - X_t$ if possible. We may assume that $V(P) \subseteq X_{V(T_t)} - X_t$, otherwise we are done.

Let $k \in [0, s + 1]$ be the number such that $P$ is of color $k + 1$. By Claim 23.1, $(V(P) \subseteq Y^{(i',+1,k)}$. Since $V(P) \subseteq X_{V(T_t)} - X_t$ and $q \in V(T_t) - \{ t \}$, there exists a vertex $z'$ in $V(P) \cap X_t$ such that the subpath of $P$ between $z$ and $z'$ is contained in $G[X_{V(T_t)}]$ and internally disjoint from $X_t$. If $\ell \geq 0$, then let $\ell_1 = \ell$; if $\ell = -1$, then let $\ell_1 = 0$. Since $\ell_1 \geq \ell$ and $V(P) \subseteq Y^{(i',+1,k)}$, $V(P) \subseteq Y^{(i',0)}$. Since $\ell_1 \geq 0$, $V(P) \subseteq W_0^{(i',+1,k)}$. So $z' \in S_{j,t}^{(i',0)} \cap X_t \cap I_j$. Since $X_t \cap I_j \cap S_{j,t}^{(i',0)} = X_t \cap I_j \cap S_{j,t}^{(i',0)}$, $z' \in S_{j,t}^{(i',0)} \cap X_t \cap I_j$. Hence there exists $i_1 \in [0, i' - 1]$ such that $z' \in S_{j,t}^{(i',0)} - S_{j,t}^{(i',0)}$. Note that such $i_1$ exists since $S_{j,t}^{(i,0)} = \emptyset$. Let $t_1$ be the node of $T$ of height $i_1$ such that $t \in V(T_{t_1})$. Since $z' \in S_{j,t}^{(i+1,0)} - S_{j,t}^{(i',0)}$, $t \in V(T_{t_1}) - \partial T_{j,t_1}$, and there exists $t_2 \in [0, |V(T)|] \cap X_{V(T(t'))} \cup X_{\partial T_{j,t'}}$. Hence there exists a monochromatic path $P_{t',t}$ with respect to $c$ in $G[Y^{(i',+1,1,k)} \cap W_4^{(i',1)}]$ from $z'$ to $W_3^{(i'+1,0)}$.

Since $t \in V(T_{t_1}) - \partial T_{j,t_1}$, $\partial T_{j,t_1}$ is disjoint from the path in $T$ from $t_1$ to $t'$. So $V(T_{j,t'}) \subseteq V(T_{t_1})$ as $i_1 < i'$. So $P \cup P_{t',t}$ is a monochromatic subpath with respect to $c$ of color $k + 1$ contained in $W_3^{(i',0)}$ containing $z'$ and intersecting $W_3^{(i'+1,0)}$. By Claim 23.1, the path $P_{t'}$ contained in $P \cup P_{t'}$ from $W_3^{(i'+1,0)}$ to $z$ satisfies $V(P_{t}) \subseteq Y^{(i'+1,0)}$. Hence $V(P_{t}) \subseteq W_0^{(i'+1,0)}$. So $z \in D^{(i',+1,1)} \subseteq D^{(i',0)}$. Since $z \in X_{t} \cap I_j - (D^{(i',+1,0)} - X_{t'})$, $z \in X_{t'}$. So $z \in X_t$. Since $q \in \partial T_{j,t'} \cap V(T_t) - \{ t \}$, $t \in V(T_{T_{j,t}}) - \partial T_{j,t}$. Since $z \in X_{q} \cap I_j$, $z \in S_{j,t}^{(i'+1,1)}$. Since $S_{j,t}^{(i'+1,1)} \cap X_t \cap I_j = S_{j,t}^{(i',1)} \cap X_t \cap I_j$, $z \in S_{j,t}^{(i',1)}$. So there exists $i_2 \in [0, i' - 1]$ such that $z \in S_{j,t}^{(i'+1,1)} - S_{j,t}^{(i',0)}$. Note that $i_2$ exists since $S_{j,t}^{(i,0)} = \emptyset$. Hence there exists $t' \in V(T)$ of height $i_2$ with $t \in V(T_{t_1})$ and $t \in V(T_{T_{j,t'}}) - \partial T_{j,t'}$. Therefore, $z \in X_{q} \cap I_j$, where $q_z \in V(T_t) - \{ t \}$ is a witness for $X_{q_z} \cap I_j \subseteq W_3^{(i'+1,0)}$ for some
\[ \ell_1' \in [0, |V(T)|]. \]
Note that \( q_x \in V(T_{t'}) - \{t'\} \) since \( q_x \in V(T_t) - \{t\} \). However, the existence of \( q_x \) contradicts that \( z \in X_q \cap I_j - \{v \in D^{(i'+1,0)} \cap X_{t'} \cap X_{t'}' \cap I_j : q' \in V(T_{t'}) - \{t'\}, q' \) is a witness for \( X_{q'} \cap I_j \subseteq W_3^{(i'+1,0)} \) for some \( i'' \in [0, i' - 1] \) and \( \ell' \in [0, |V(T)|] \). This proves the claim.

**Claim 23.4.** Let \( i, i' \in \mathbb{N}_0 \) with \( i' < i \). Let \( t \in V(T^{(i)}) - V(T^{(i-1)}) \) and let \( t' \in V(T^{(i')}) - V(T^{(i'-1)}) \) with \( t \in V(T_t) \). Let \( j \in [|V| - 1] \). Let \( \ell \in [-1, |V(T_t)|] \) such that there exists no witness \( q \in \partial T_{j,t} \cap V(T_t) - \{t\} \) for \( X_q \cap I_j \subseteq W_3^{(i',i+1,0)} \) and \( X_q \cap I_j \not\subseteq W_3^{(i',i,0)} \). If \( Y^{(i',i+1,0)} \cap X_{V(T_t)} \cap I_j - X_t \neq Y^{(i',i+1,0,2)} \cap X_{V(T_t)} \cap I_j - X_t \), then either:

- \( |X_t \cap I_j \cap S_{j,t}^{(i',i+1,0)}| > |X_t \cap I_j \cap S_{j,t}^{(i',i,0)}|, \) or
- \( |X_t \cap I_j \cap S_{j,t}^{(i',i+1,0)}| > |X_t \cap I_j \cap S_{j,t}^{(i',i,0)}| \).

**Proof.** Suppose to the contrary that \( |X_t \cap I_j \cap S_{j,t}^{(i',i+1,0)}| \leq |X_t \cap I_j \cap S_{j,t}^{(i',i,0)}| \) and \( |X_t \cap I_j \cap S_{j,t}^{(i',i+1,0)}| \leq |X_t \cap I_j \cap S_{j,t}^{(i',i,0)}| \). So \( X_t \cap I_j \cap S_{j,t}^{(i',i+1,0)} = X_t \cap I_j \cap S_{j,t}^{(i',i,0)} \) and \( X_t \cap I_j \cap S_{j,t}^{(i',i+1,0)} = X_t \cap I_j \cap S_{j,t}^{(i',i,0)} \). Since there exists no witness \( q \in \partial T_{j,t'} \cap V(T_t) \) for \( X_q \cap I_j \subseteq W_3^{(i'+1,0)} \) and \( X_q \cap I_j \not\subseteq W_3^{(i',i,0)} \), we know \( X_{q_0} \cap I_j \subseteq W_3^{(i',i+1,0)} \). Suppose that \( q_{O} \notin V(T_t) \). Since there exists no witness \( q \in \partial T_{j,t'} \cap V(T_t) \) for \( X_q \cap I_j \subseteq W_3^{(i'+1,0)} \) and \( X_q \cap I_j \not\subseteq W_3^{(i',i,0)} \), we know \( X_{q_0} \cap I_j \subseteq W_3^{(i',i,0)} \). So by Claim 23.1, \( V(Q) \subseteq W_0^{(i',i+1,1)} \) and \( x' \in W_2^{(i',i+1,1)} \subseteq Y^{(i',i+1,1)} \subseteq Y^{(i',i+2,2)} \), a contradiction.

So \( q_{O} \notin V(T_t) \). Hence there exists a vertex \( x^* \in X_t \cap I_j \) such that the subpath of \( Q \) between \( x^* \) and \( x \) is contained in \( G[X_{V(T_t)}] \). Since \( x^* \in X_t \cap I_j \cap W_0^{(i',i,0,2+2)} \) and \( t \in V(T_{t'}) - \partial T_{j,t'} \), \( x^* \in S_{j,t}^{(i'+1,0)} \cap X_t \cap I_j \). Since \( X_t \cap I_j \cap S_{j,t}^{(i'+1,0)} = X_t \cap I_j \cap S_{j,t}^{(i',i,0)} \), there exists \( i_2 \in [0, i' - 1] \) such that \( x^* \in S_{j,t}^{(i'+1,0)} - S_{j,t}^{(i',i,0)} \). So the node \( t' \) of \( T \) of height \( i_2 \) with \( t' \in V(T_{t_2}) \) satisfies that \( t \in V(T_{j,t_2}) - \partial T_{j,t_2} \), and there exists \( \ell_x \in [0, |V(T)|] \) such that \( x^* \in W_0^{(i',i_2,1+k)} \), since \( c(x^*) = k + 1 \). So there exists a monochromatic path \( Q_{x^*} \) in \( G[Y^{(i',i_2,1+k)} \cap W_4^{(i_2)}] \) from \( x^* \) to \( W_3^{(i_2,1+k)} \).

Since \( t \in V(T_{j,t_2}) - \partial T_{j,t_2}, \partial T_{j,t_2} \) is disjoint from the path in \( T \) from \( t_2 \) to \( t' \). Since \( i_2 < i' \), \( V(T_{j,t'}) \subseteq V(T_{j,t_2}) \). So \( Q \cup Q_{x^*} \) is a monochromatic subgraph with respect to \( c \) of color \( k + 1 \) contained in \( W_4^{(i_2)} \) containing \( x^* \) and intersects \( W_3^{(i_2,1+k)} \). By Claim 23.1, the path \( Q_x \) contained in \( Q \cup Q_{x^*} \) from \( W_3^{(i_2,1+k)} \) to \( x \) satisfies \( V(Q_x) \subseteq Y^{(i_2,1+k)} \) and \( A_{L_{(i_2,1+k+1)}}(V(Q_x)) \cap W_4^{(i_2)} = \emptyset \). But \( x' \in A_{L_{(i_2,1+k+1)}}(V(Q_x)) \cap W_4^{(i_2)} \subseteq L_{(i_2,1+k+1)}(V(Q_x)) \cap W_4^{(i_2)} = \emptyset \), a contradiction. This proves the claim.

**Claim 23.5.** Let \( i, i' \in \mathbb{N}_0 \) with \( i' < i \), and let \( t \in V(T^{(i)}) - V(T^{(i-1)}) \). Let \( j \in [|V| - 1] \). If \( Y^{(i',0,s+2)} \cap X_{V(T_t)} \cap I_j - X_t = Y^{(i',0,s+2)} \cap X_{V(T_t)} \cap I_j - X_t \), then either:

- \( Y^{(i',0,s+2)} \cap X_{V(T_t)} \cap I_j - X_t = Y^{(i',|V(T)|+1,s+2)} \cap X_{V(T_t)} \cap I_j - X_t \), or
- \( |X_t \cap I_j \cap S_{j,t}^{(i'+1,0)}| > |X_t \cap I_j \cap S_{j,t}^{(i',|V(T)|+1,s+2)}| \).

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Claim 23.6.  

\[ |X_t \cap I_j \cap S_{j,t}^{(i'+1,1)}| > |X_t \cap I_j \cap S_{j,t}^{(i',1)}|. \]

Proof. We may assume that \( Y^{(i',0,s+2)} \cap X_{V(T)}(i) \cap I_j - X_t \neq Y^{(i',|V(T)|+1,s+2)} \cap X_{V(T)}(i) \cap I_j - X_t \), for otherwise we are done. Since \( Y^{(i',0,s+2)} \cap X_{V(T)}(i) \cap I_j - X_t = Y^{(i',1,s+2)} \cap X_{V(T)}(i) \cap I_j - X_t \), and so there exists a minimum \( \ell \in [0, |V(T)| - 1] \) such that \( Y^{(i',\ell+1,1)} \cap X_{V(T)}(i) \cap I_j - X_t \neq Y^{(i',\ell+2,2)} \cap X_{V(T)}(i) \cap I_j - X_t \). By the minimality of \( \ell \), \( Y^{(i',\ell+1)} \cap X_{V(T)}(i) \cap I_j - X_t = Y^{(i',\ell+1,1)} \cap X_{V(T)}(i) \cap I_j - X_t \).

Let \( i' \) be the node of \( T \) of height \( i' \) with \( t \in V(T_{i'}) \). Since \( Y^{(i',0,s+2)} \cap X_{V(T)}(i) \cap I_j - X_t \neq Y^{(i',|V(T)|+1,s+2)} \cap X_{V(T)}(i) \cap I_j - X_t \), we have that \( |X_t \cap I_j \cap S_{j,t}^{(i'+1,1)}| \leq |X_t \cap I_j \cap S_{j,t}^{(i',0)}| \) and \( |X_t \cap I_j \cap S_{j,t}^{(i'+1,1)}| \leq |X_t \cap I_j \cap S_{j,t}^{(i',1)}| \). So \( X_t \cap I_j \cap S_{j,t}^{(i'+1,0)} = X_t \cap I_j \cap S_{j,t}^{(i',0)} \) and \( X_t \cap I_j \cap S_{j,t}^{(i'+1,1)} = X_t \cap I_j \cap S_{j,t}^{(i',1)} \).

We first suppose that there exist \( \ell_0 \in [-1, \ell] \) and a witness \( q \in \partial T_{j',t} \cap V(T_{i'}) \) for \( X_q \cap I_j \subset W_3^{(i',0,\ell_0+1)} \) and \( X_q \cap I_j \not\subset W_3^{(i',\ell_0)} \). Choose \( \ell_0 \) so that \( \ell_0 \) is as small as possible. So there exists a monochromatic path \( P \in G[Y^{(i',0+1,\ell_0+1)} \cap I_j \cap W_t] \) with respect to \( c \) from \( \bigcup_{\ell_0} W_3^{(i',\ell_0+1)} \) to a node \( z \in X_q \cap I_j - \{ (D^{(i',\ell_0+1)} - X_t) \cup \{ v \in D^{(i',\ell_0+1)} \cap X_{V(T_{i'})} \cap X_q \cap I_j : q' \in V(T_{i'}) - \{ t \}, q' \} \} \). By Claim 23.3, \( V(P) \subset X_{V(T)}(i) - X_t \). In particular, \( \ell_0 = 0 \).

Let \( k \in [0, s + 1] \) be the number such that \( P \) is of color \( k + 1 \). If \( V(P) \) intersects \( W_3^{(i',0,\ell_0-1)} \), then by Claim 23.1, \( V(P) \subset Y^{(i',0,k)} \), so \( q \) is a witness for \( X_q \cap I_j \subset W_3^{(i',\ell_0)} \), a contradiction. So \( V(P) \cap W_3^{(i',0,\ell_0-1)} = \emptyset \). Then there exists a witness \( q' \in V(T_{i'}) - \{ t \} \) for \( X_q' \cap I_j \subset W_3^{(i',\ell_0)} \) such that \( V(P) \) is from \( X_q' \) to \( X_q \), contradicting the minimality of \( \ell_0 \).

Therefore, there do not exist \( \ell_0 \in [-1, \ell] \) and a witness \( q \in \partial T_{j',t} \cap V(T_{i'}) \) for \( X_q \cap I_j \subset W_3^{(i',0,\ell_0+1)} \) and \( X_q \cap I_j \not\subset W_3^{(i',\ell_0)} \). In particular, there exists no witness \( q \in \partial T_{j',t} \cap V(T_{i'}) \) for \( X_q \cap I_j \subset W_3^{(i',0,\ell_0+1)} \) and \( X_q \cap I_j \not\subset W_3^{(i',\ell_0)} \). By Claim 23.4, either \( |X_t \cap I_j \cap S_{j,t}^{(i'+1,0)}| > |X_t \cap I_j \cap S_{j,t}^{(i',0)}| \) or \( |X_t \cap I_j \cap S_{j,t}^{(i'+1,1)}| > |X_t \cap I_j \cap S_{j,t}^{(i',1)}| \), a contradiction. This proves the claim. \( \square \)

Claim 23.6. Let \( i \in \mathbb{N}_0 \), and let \( t \in V(T^{(i)}) - V(T^{(i-1)}) \). Let \( j \in [|V| - 1] \) and \( \ell \in [0, |V(T)|] \). Let \( S := \{ i' \in [0, i-1] : Y^{(i',|V(T)|+1,\ell+2)} \cap X_{V(T)}(i) \cap I_j - X_t \neq Y^{(i',0,\ell+2)} \cap X_{V(T)}(i) \cap I_j - X_t \} \). Then \( |S| \leq 3w_0 \).

Proof. Let \( S_1 := \{ i' \in [0, i-1] : Y^{(i',1,\ell+2)} \cap X_{V(T)}(i) \cap I_j - X_t \neq Y^{(i',0,\ell+2)} \cap X_{V(T)}(i) \cap I_j - X_t \} \) and let \( S_2 := \{ i' \in [0, i-1] : |X_t \cap I_j \cap S_{j,t}^{(i'+1,0)}| + |X_t \cap I_j \cap S_{j,t}^{(i'+1,1)}| + |X_t \cap I_j \cap D^{(i',0,\ell+2)}| > \}

\[ |X_t \cap I_j \cap S_{j,t}^{(i',0)}| + |X_t \cap I_j \cap S_{j,t}^{(i',1)}| + |X_t \cap I_j \cap D^{(i',0,\ell+2)}| \}. \] By Claim 23.5, \( S = S_1 \cup S_2 \). By Claim 23.2, \( S_1 \subseteq S_2 \). So \( S = S_2 \). Since \( |X_t \cap I_j| \leq w_0 \), \( |S| = |S_2| \leq 3w_0 \).

\( \square \)

Claim 23.7. Let \( t \in V(T) \) and \( j \in [|V| - 1] \). Let \( (Y', L') \) be a \( V \)-standard pair and let \( Z \subset Y' \). Then \( A_{L'}(Z) \subseteq N_G^Z[Z], A_{L'}(Z) \cap X_{V(T,j,t)} \subseteq N_G^Z(Z \cap X_{V(T,j,t)}) \cup X_{\partial T_{j,t}} \cup X_t \), and \( A_{L'}(Z) \cap X_{V(T,j,t)} \subseteq f(|Z \cap X_{V(T,j,t)}|) \cup |X_{\partial T_{j,t}}| + |X_t| \).

Proof. It is obvious that \( A_{L'}(Z) \subseteq N_G^Z[Z] \) since \( (Y', L') \) is a \( V \)-standard pair. Since \( (Y', L') \) is a \( V \)-standard pair, \( A_{L'}(Z) \cap X_{V(T,j,t)} \subseteq N_G^Z(Z \cap X_{V(T,j,t)}) \cup (X_{\partial T_{j,t}} \cup X_t) \). So \( |A_{L'}(Z) \cap X_{V(T,j,t)}| \leq \{ N_G^Z(Z \cap X_{V(T,j,t)}) \} + |X_{\partial T_{j,t}}| + |X_t| \leq f(|Z \cap X_{V(T,j,t)}|) + |X_{\partial T_{j,t}}| + |X_t| \) by Corollary 14. \( \square \)
Claim 23.8. Let $i, i' \in \mathbb{N}_0$ with $i' < i$, and let $t \in V(T^{(i)}) - V(T^{(i-1)})$ and $t' \in V(T^{(i')}) - V(T^{(i'-1)})$ with $t \in V(T_{i'})$. Let $j \in [[|V| - 1]]$. Then the following hold:

- $|Y^{(i',-1,0)} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})}| \leq w_0$.
- $|X_{\partial T_{j,t'}} \cap X_{V(T_i)} \cap T_j| \leq w_0$.

Proof. Let $F_{j,p} := F_{j,0}$ if $t' = r^*$; let $p$ be the parent of $t'$ if $t' \neq r^*$. By definition, $F_{j,p} = F_{j,p} \cup \bigcup_{T'} F_{j,0}$, where the union is over all $F_{j,p} \cap V(T_{i'})$-parts $T'$ of $T_{i'}$ containing $t'$, and each $F_{j,p}$ is a $(T', X_{V(T_{i'})} \cap T_j, (Y^{(i',-1,0)} \cup X_{\partial T'}) \cap X_{V(T_{i'})} \cap T_j)$-hence. So for each $F_{j,p} \cap V(T_{i'})$-part $T'$ of $T_{i'}$, $|(Y^{(i',-1,0)} \cap T_j \cap X_{V(T_{i'})}) \cup (X_{\partial T'} \cap T_j)| \leq w_0$. By definition, $T_{j,t'} = \bigcup_{T'} T'$, where the union is over all $F_{j,p} \cap V(T_{i'})$-parts $T'$ of $T_{i'}$ containing $t'$. Since $t \in V(T_{i'}) \setminus \{t'\}$, there exists at most one $F_{j,p} \cap V(T_{i'})$-part of $T_{i'}$ containing both $t'$ and $t$. If there exists an $F_{j,p} \cap V(T_{i'})$-part of $T_{i'}$ containing both $t'$ and $t$, then denote it by $T^*$; otherwise let $T^* := \emptyset$. So $X_{V(T_i)} \cap X_{V(T_{j,t'})} \subseteq X_{V(T^*)}$. Hence $|Y^{(i',-1,0)} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})}| \leq |Y^{(i',-1,0)} \cap T_j \cap X_{V(T_{i'})}| \leq w_0$. Similarly, $|X_{\partial T_{j,t'}} \cap X_{V(T_i)} \cap T_j| \leq |X_{\partial T_{j,t'}} \cap X_{V(T_{i'})} \cap T_j| \leq |X_{\partial T^*} \cap T_j| \leq w_0$. □

Claim 23.9. Let $i, i' \in \mathbb{N}_0$ with $i' < i$, and let $t \in V(T^{(i)}) - V(T^{(i-1)})$ and $t' \in V(T^{(i')}) - V(T^{(i'-1)})$ with $t \in V(T_{i'})$. Let $j \in [[|V| - 1]]$. Then $|Y^{(i',0,0)} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})}| \leq g_0(w_0)$.

Proof. For every $k \in [w_0]$, let $q_k \in [0, s + 1]$ be the number such that the color of $M_{j,k}^{(i',1)}$ is $q_k + 1$. So for every $k \in [w_0]$,

$$(Y^{(i',-1,k+1)} - Y^{(i',-1,k)}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})} - X_t$$

$$\subseteq A_{i',-1,k,q_k}(W^{(i',-1,k,q_k)}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})} - X_t$$

$$\subseteq (N_{G}^{s}(W^{(i',-1,k,q_k)}) \cap X_{V(T_{j,t'})} \cup X_{\partial T_{j,t'}} \cup X_{t'}) \cap Z_{t'} \cup T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})} - X_t,$$

where the last inclusion follows from Claim 23.7. Note that for every $k \in [w_0]$, since every vertex in $W^{(i',-1,k,q_k)} \cap T_j$ is connected by a monochromatic path to $I_{j'}$, we know $N_{G}[W^{(i',-1,k,q_k)} \cap T_j] \subseteq \overline{T_j}$. So for every $k \in [w_0]$,

$$(N_{G}^{s}(W^{(i',-1,k,q_k)}) \cap X_{V(T_{j,t'})} \cup X_{\partial T_{j,t'}} \cup X_{t'}) \cap Z_{t'} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})} - X_t$$

$$\subseteq (N_{G}^{s}(W^{(i',-1,k,q_k)}) \cap X_{V(T_{j,t'})} \cup X_{\partial T_{j,t'}} \cap X_{t'}) \cap Z_{t'} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})} - X_t \cup (X_{\partial T_{j,t'}} \cap T_j \cap X_{V(T_i)})$$

$$\subseteq (N_{G}^{s}(W^{(i',-1,k,q_k)}) \cap X_{V(T_{j,t'})} \cap T_j \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})} - X_t \cup (X_{\partial T_{j,t'}} \cap T_j \cap X_{V(T_i)})$$

$$\subseteq N_{G}^{s}(W^{(i',-1,k,q_k)}) \cap X_{V(T_{j,t'})} \cap T_j \cap X_{V(T_i)} \cup (X_{\partial T_{j,t'}} \cap T_j \cap X_{V(T_i)}).$$

Hence for every $k \in [w_0]$,

$$|Y^{(i',-1,k+1)} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})}|$$

$$\leq |Y^{(i',-1,k)} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})}|$$

$$+ |X_t \cap T_j| + |(Y^{(i',-1,k+1)} - Y^{(i',-1,k)}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})} - X_t|$$

$$\leq |Y^{(i',-1,k)} \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t'})}|$$

$$+ |X_t \cap T_j| + f((Y^{(i',-1,k)} \cap T_j \cap X_{V(T_i)} \cap T_j \cap X_{V(T_i)})) + |X_{\partial T_{j,t'}} \cap T_j \cap X_{V(T_i)}|.$$

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where the last inequality follows from Claim 23.8.

By Claim 23.8, \( |Y^{i',-1,0}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_j,t')}| \leq w_0 = g_0(0) \). So it is easy to verify that for every \( k \in [w_0] \), \( |Y^{i',-1,k}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_j,t')}| \leq g_0(k) \) by induction on \( k \). Therefore, \( |Y^{i',0,0}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_j,t')}| = |Y^{i',-1,w_0}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_j,t')}| \leq g_0(w_0). \)

**Claim 23.10.** Let \( i, i' \in \mathbb{N}_0 \) with \( i' \leq i \), and let \( t \in V(T^{i}) - V(T^{i-1}) \) and \( t' \in V(T^{i'}) - V(T^{i'-1}) \) with \( t \in V(T_{t'}) \). Let \( j \in [\mathbb{V} - 1] \). Denote by \( I_j = \bigcup_{a=0}^{b} V_a \). Then the following hold:

- For every \( k \in [0, s+1] \),
  \[
  |Y^{i',0,k+1}) \cap ( \bigcup_{a=a-(s+3)+(k+1)}^{b+(s+3)-(k+1)} V_a \cap X_{V(T_i)} \cap X_{V(T_j,t')}| 
  \leq f_1(|Y^{i',0,k}) \cap ( \bigcup_{a=a-(s+3)+k}^{b+(s+3)-k} V_a \cap X_{V(T_i)} \cap X_{V(T_j,t')}|) + |X_{\partial T_{j,t'}} \cap X_{V(T_i)} \cap T_j| + 2w_0. 
  \]

- If \( i' < i \), then \( |Y^{i',0,s+2}) \cap ( \bigcup_{a=a-1}^{b+1} V_a \cap X_{V(T_i)} \cap X_{V(T_j,t')}| \leq g_1(s + 2). \)

- If \( i' < i \), then for every \( \ell \in [0, |V(T)| + 1] \),
  \[
  |Y^{i',\ell+1,s+2}) \cap ( \bigcup_{a=a-1}^{b+1} V_a \cap X_{V(T_i)} \cap X_{V(T_j,t')}| 
  \leq g_2(s + 2, |Y^{i',\ell,s+2}) \cap ( \bigcup_{a=a-1}^{b+1} V_a \cap X_{V(T_i)} \cap X_{V(T_j,t')}|). 
  \]

- If \( i' < i \), then \( |Y^{i',|V(T)|+1,s+2}) \cap ( \bigcup_{a=a-1}^{b+1} V_a \cap X_{V(T_i)} \cap X_{V(T_j,t')}| \leq g_3(4w_0). \)

**Proof.** We first prove Statement 1. Since \( t' \) is the node of \( T \) of height \( i' \) such that \( t \in V(T_{t'}) \), for every \( k \in [0, s+1] \),

\[
(Y^{i',0,k+1}) - Y^{i',0,k}) \cap T_j \cap X_{V(T_i)} \subseteq W_2^{i',0,k}) \cap T_j \cap X_{V(T_i)} 
= A_L(v',0,k)) \cap ( \bigcup_{t' \in V(T^{i'} - V(T^{i'-1})} Z_{t'}) \cap T_j \cap X_{V(T_i)} 
\subseteq A_L(v',0,k)) \cap Z_{t'} \cap T_j \cap X_{V(T_i)} 
\subseteq A_L(v',0,k)) \cap Z_{t'} \cap T_j \cap X_{V(T_i)}. 
\]

So \( (Y^{i',0,k+1}) - Y^{i',0,k}) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t})} \subseteq A_L(v',0,k)) \cap T_j \cap X_{V(T_i)} \cap X_{V(T_{j,t})}. \) By Claim 23.7,

\[
A_L(v',0,k)) \cap ( \bigcup_{a=a-(s+3)+(k+1)}^{b+(s+3)-(k+1)} V_a \cap X_{V(T_i)} - X_t 
\]

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\[ \subseteq A_L^{(i',0,k)} (Y^{(i',0,k)} \cap X_{V(T_i)}) \cap (\bigcup_{a=a-(s+3)+k}^{b+(s+3)-k} V_a) \cap (\bigcup_{a=a-(s+3)+k+1}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} - X_t, \]

and

\[ A_L^{(i',0,k)} (Y^{(i',0,k)} \cap X_{V(T_i)}) \cap X_{V(T_{i',t'})} \subseteq N_G^s (Y^{(i',0,k)} \cap X_{V(T_i)} \cap X_{V(T_{i',t'})}) \cup X_{\partial T_{i',t'}} \cup X_{t'}. \]

Therefore,

\[ (Y^{(i',0,k+1)} - Y^{(i',0,k)}) \cap (\bigcup_{a=a-(s+3)+k+1}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} \cap X_{V(T_{i',t'})} \]

\[ \subseteq (A_L^{(i',0,k)} (Y^{(i',0,k)} \cap X_{V(T_i)}) \cap (\bigcup_{a=a-(s+3)+k+1}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} - X_t) \cup (X_t \cap \overline{T_j}) \]

\[ \subseteq \left( (N_G^s (Y^{(i',0,k)} \cap (\bigcup_{a=a-(s+3)+k+1}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} \cap X_{V(T_{i',t'})}) \cup X_{\partial T_{i',t'}} \cup X_{t'} \right) \]

\[ \cap (\bigcup_{a=a-(s+3)+k+1}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} - X_t \right) \cup (X_t \cap \overline{T_j}), \]

where the last inequality follows from Claim 23.7. Hence

\[ |(Y^{(i',0,k+1)} - Y^{(i',0,k)}) \cap (\bigcup_{a=a-(s+3)+(k+1)}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} \cap X_{V(T_{i',t'})}| \]

\[ \leq |N_G^s (Y^{(i',0,k)} \cap (\bigcup_{a=a-(s+3)+k}^{b+(s+3)-k} V_a) \cap X_{V(T_i)} \cap X_{V(T_{i',t'})}) \cap (\bigcup_{a=a-(s+3)+k+1}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} - X_t| \]

\[ + |(X_{\partial T_{i',t'}} \cup X_{t'}) \cap (\bigcup_{a=a-(s+3)+k+1}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} - X_t| + |X_t \cap \overline{T_j}| \]

\[ \leq f|Y^{(i',0,k)} \cap (\bigcup_{a=a-(s+3)+k}^{b+(s+3)+k} V_a) \cap X_{V(T_i)} \cap X_{V(T_{i',t'})}| + |X_{\partial T_{i',t'}} \cap X_{V(T_i)} \cap \overline{T_j}| + |X_{t'} \cap \overline{T_j}| + |X_t \cap \overline{T_j}| + 2w_0. \]

So

\[ |Y^{(i',0,k+1)} \cap (\bigcup_{a=a-(s+3)+(k+1)}^{b+(s+3)-(k+1)} V_a) \cap X_{V(T_i)} \cap X_{V(T_{i',t'})}| \]
\begin{equation}
= |Y^{(i',0,k)} \cap (\bigcup_{\alpha=a-(s+3)+(k+1)}^{b+(s+3)-(k+1)} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}| \\
+ |(Y^{(i',0,k+1)} - Y^{(i',0,k)}) \cap (\bigcup_{\alpha=a-(s+3)+(k+1)}^{b+(s+3)-(k+1)} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}| \\
\leq |Y^{(i',0,k)} \cap (\bigcup_{\alpha=a-(s+3)+k}^{b+(s+3)-k} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}| \\
+ f(|Y^{(i',0,k)} \cap (\bigcup_{\alpha=a-(s+3)+k}^{b+(s+3)-k} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}|) + |X_{\partial T_{j',t'}} \cap X_{V(T_i)} \cap \overline{T_j}| + 2w_0 \\
= f_1(|Y^{(i',0,k)} \cap (\bigcup_{\alpha=a-(s+3)+k}^{b+(s+3)-k} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}|) + |X_{\partial T_{j',t'}} \cap X_{V(T_i)} \cap \overline{T_j}| + 2w_0.
\end{equation}

This proves Statement 1.

Now we prove Statement 2 of this claim. Assume $i' < i$. By Claim 23.8 and Statement 1 of this claim, for every $k \in [0, s + 1]$, 
\begin{equation}
|Y^{(i',0,k+1)} \cap (\bigcup_{\alpha=a-(s+3)+(k+1)}^{b+(s+3)-(k+1)} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}| \\
\leq f_1(|Y^{(i',0,k)} \cap (\bigcup_{\alpha=a-(s+3)+k}^{b+(s+3)-k} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}|) + |X_{\partial T_{j',t'}} \cap X_{V(T_i)} \cap \overline{T_j}| + 2w_0 \\
\leq f_1(|Y^{(i',0,k)} \cap (\bigcup_{\alpha=a-(s+3)+k}^{b+(s+3)-k} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}|) + 3w_0.
\end{equation}

Then it is easy to verify that for every $k \in [0, s+2]$, $|Y^{(i',0,k)} \cap (\bigcup_{\alpha=a-(s+3)+k}^{b+(s+3)-k} V_\alpha) \cap X_{V(T_i)} \cap X_{V(T_{j',t'})}| \leq g_1(k)$ by induction on $k$. (Note that the base case $k = 0$ follows from Claim 23.9.) Then Statement 2 follows from the case $k = s + 2$.

Now we prove Statement 3 of this claim. Note that for every $\ell \in [0, |V(T)|]$, 
\begin{equation}
|Y^{(i',\ell+1,0)} \cap (\bigcup_{\alpha=a-1}^{b+1} V_\alpha) \cap X_{V(T_i)}| \\
= |Y^{(i',\ell+1,0)} \cap (\bigcup_{\alpha=a-1}^{b+1} V_\alpha) \cap X_{V(T_i)}| + |W_3^{(i',\ell,0)} \cap (\bigcup_{\alpha=a-1}^{b+1} V_\alpha) \cap X_{V(T_i)}| \\
\leq |Y^{(i',\ell+1,0)} \cap (\bigcup_{\alpha=a-1}^{b+1} V_\alpha) \cap X_{V(T_i)}| + |X_{\partial T_{j',t'}} \cap \overline{T_j} \cap X_{V(T_i)}| \\
\leq |Y^{(i',\ell+1,0)} \cap (\bigcup_{\alpha=a-1}^{b+1} V_\alpha) \cap X_{V(T_i)}| + w_0 \\
= g_2(0, |Y^{(i',\ell+1,0)} \cap (\bigcup_{\alpha=a-1}^{b+1} V_\alpha) \cap X_{V(T_i)}|)
\end{equation}
by Claim 23.8. For every \( \ell \in [0, |V(T)|] \) and \( k \in [0, s + 1] \), by Claim 23.7, we know

\[
W_2^{(i',\ell+1,k)} \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_i)
\]

\[
= A_L^{(i',\ell+1,k)}(W_0^{(i',\ell+1,k)}) \cap X_V(T_j) \cap I_j
\]

\[
\subseteq A_L^{(i',\ell+1,k)}(W_0^{(i',\ell+1,k)}) \cap X_V(T_j,t') \cap I_j
\]

\[
\subseteq (A_L^{(i',\ell+1,k)}(W_0^{(i',\ell+1,k)}) \cap X_V(T_j,t') \cup X_{\partial T_j,t'} \cup X_{t'}) \cap X_V(T_j,t') \cap I_j
\]

\[
\subseteq (A_L^{(i',\ell+1,k)}(W_0^{(i',\ell+1,k)}) \cap I_j \cap X_V(T_j,t') \cup X_{\partial T_j,t'} \cup X_{t'}) \cap X_V(T_j,t') \cap I_j
\]

\[
\subseteq (N_G^{s}(W_0^{(i',\ell+1,k)}) \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cup X_{\partial T_j,t'} \cup X_{t'}) \cap X_V(T_j,t') \cap I_j.
\]

So for every \( \ell \in [0, |V(T)|] \) and \( k \in [0, s + 1] \),

\[
(Y^{(i',\ell+1,k+1)} - Y^{(i',\ell+1,k)}) \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_i)
\]

\[
\subseteq W_2^{(i',\ell+1,k)} \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_i)
\]

\[
\subseteq (N_G^{s}(W_0^{(i',\ell+1,k)}) \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cup X_{\partial T_j,t'} \cup X_{t'}) \cap X_V(T_j,t') \cap I_j \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_i)
\]

\[
\subseteq (N_G^{s}(W_0^{(i',\ell+1,k)}) \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cap X_V(T_i) \cup X_{t} \cup X_{\partial T_j,t'} \cup X_{t'}) \cap X_V(T_j,t') \cap I_j \cap X_V(T_i)
\]

\[
\subseteq N_G^{s}[Y^{(i',\ell+1,k)} \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cap X_V(T_i)] \cup (X_t \cap I_j) \cup (X_{\partial T_j,t'} \cap I_j \cap X_V(T_i)) \cup (X_{t'} \cap I_j).
\]

Hence for every \( \ell \in [0, |V(T)|] \) and \( k \in [0, s + 1] \),

\[
||Y^{(i',\ell+1,k+1)} - Y^{(i',\ell+1,k)}|| \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_i)
\]

\[
\leq |N_G^{s}[Y^{(i',\ell+1,k)} \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cap X_V(T_i)]| + |X_t \cap I_j| + |X_{\partial T_j,t'} \cap I_j \cap X_V(T_i)| + |X_{t'} \cap I_j|
\]

\[
\leq f_1(Y^{(i',\ell+1,k)} \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cap X_V(T_i)) + 3w_0
\]

by Claim 23.8. So for every \( \ell \in [0, |V(T)|] \) and \( k \in [0, s + 1] \),

\[
|Y^{(i',\ell+1,k+1)} \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cap X_V(T_i)| \leq f_2(Y^{(i',\ell+1,k)} \cap ( \bigcup_{a=a-1}^{b+1} V_a ) \cap X_V(T_j,t') \cap X_V(T_i)) + 3w_0.
\]
Then for every $\ell \in [0, |V(T)|]$, it is easy to verify by induction on $k \in [0, s + 2]$ that
\[
|Y^{(i',\ell+1,k)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_{V(T)}| \leq g_2(k, |Y^{(i',\ell,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_{V(T)}|).
\]

Hence
\[
|Y^{(i',\ell+1,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_{V(T)}| \leq g_2(s+2, |Y^{(i',\ell,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_{V(T)}|).
\]

Now we prove Statement 4 of this claim. Let
\[
S := \{ \ell \in [0, |V(T)|] : |Y^{(i',\ell+1,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T)}| > |Y^{(i',\ell,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T)}| \}
\]
For every $\ell \in S$, either $(Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap X_t \cap (\bigcup_{a=a-1}^{b+1} V_a) \neq \emptyset$, or $(Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap X_t \neq \emptyset$. Note that $(Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap (\bigcup_{a=a-1}^{b+1} V_a \subseteq (Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap W_{4'}(\bigcup_{a=a-1}^{b+1} V_a \subseteq (Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap I_j$. So for every $\ell \in S$, either $(Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap X_t \cap (\bigcup_{a=a-1}^{b+1} V_a) \neq \emptyset$, or $(Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap I_j \cap X_{V(T)} - X_t \neq \emptyset$. Let
\[
S_1 := \{ \ell \in [0, |V(T)|] : (Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap X_t \cap (\bigcup_{a=a-1}^{b+1} V_a) \neq \emptyset \}
\]

\[
S_2 := \{ \ell \in [0, |V(T)|] : (Y^{(i',\ell+1,s+2)} - Y^{(i',\ell,s+2)}) \cap I_j \cap X_{V(T)} - X_t \neq \emptyset \}
\]
So $S = S_1 \cup S_2$. Note that $|S_1| \leq |X_t \cap (\bigcup_{a=a-1}^{b+1} V_a)| \leq w_0$. Let
\[
S_3 := \{ \ell \in [0, |V(T)|] : (W_{3}(^{i',\ell}) - W_{3}(^{i',\ell-1})) \cap I_j \cap X_{V(T)} \neq \emptyset \}
\]
Since $W_{3}(^{i',\ell}) \cap I_j \cap X_{V(T)} \subseteq X_{\partial T_{j',t'}} \cap I_j \cap X_{V(T)}$ for every $\ell \in [0, |V(T)|]$, we know $|S_3| \leq |X_{\partial T_{j',t'}} \cap I_j \cap X_{V(T)}| \leq w_0$ by Claim 23.8 since $i' < i$. Let
\[
S_4 := \{ \ell \in [0, |V(T)|] : |X_t \cap I_j \cap S_{j,t}(^{i',1,0})| + |X_t \cap I_j \cap S_{j,t}(^{i',1,1})| > |X_t \cap I_j \cap S_{j,t}(^{i',0})| + |X_t \cap I_j \cap S_{j,t}(^{i,1})| \}
\]
For every $\ell \in S_2 - S_3$, we know $(W_{3}(^{i',\ell}) - W_{3}(^{i',\ell-1})) \cap I_j \cap X_{V(T)} = \emptyset$, so there exists no witness $q \in \partial T_{j',t'} \cap V(T) \setminus \{t\}$ for $X_q \cap I_j \cap W_{3}(^{i',\ell})$ and $X_q \cap I_j \cap W_{3}(^{i',\ell-1})$, and hence $\ell \in S_4$ by Claim 23.4. So $S = S_1 \cup S_2 \subseteq S_1 \cup S_3 \cup S_4$. Therefore, $|S| \leq |S_1| + |S_3| + |S_4| \leq w_0 + w_0 + 2w_0 = 4w_0$.

By Statements 2 and 3 of this claim, it is easy to verify by induction on $k \in [|S|]$ that
\[
|Y^{(i',\ell,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_{V(T)}| \leq g_3(|S|),
\]
where we denote the elements of $S$ by $\ell_1 < \ell_2 < \cdots < \ell_{|S|}$. Therefore, $|Y^{(i',V(T))^{i'+1,s+2}}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_{V(T)}| \leq g_3(|S|) \leq g_3(4w_0)$. □

\textbf{Claim 23.11.} Let $i \in \mathbb{N}_0$ and let $t \in V(T^{(0)}) - V(T^{(i-1)})$. Let $j \in [|V| - 1]$. If $i'$ is an integer in $[0, i - 1]$ such that either $|Y^{(i',0,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_t| \neq |Y^{(i',0,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_t|$ or $|Y^{(i',-1,0)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_t| \neq |Y^{(i',-1,s+2)}(\bigcup_{a=a-1}^{b+1} V_a) \cap X_{V(T_1,t')} \cap X_t|$, then $|S_{j,t}(^{i'+1,2}) \cap X_t \cap I_j| > |S_{j,t}(^{i'+1,2}) \cap X_t \cap I_j|$. 

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Proof. Assume that $i'$ is an integer in $[0, i - 1]$ such that either $|Y(i',0,0) \cap I_j \cap X_{V(T_i)} - X_i| \neq |Y(i',0,0+2) \cap I_j \cap X_{V(T_i)} - X_i|$ or $|Y(i',-1,0) \cap I_j \cap X_{V(T_i)} - X_i| \neq |Y(i',0,-1,0+a) \cap I_j \cap X_{V(T_i)} - X_i|$. Let $t'$ be the node of $T$ of height $i'$ with $t \in V(T_{i'})$. Then either $W_2(i',-1,k,q) \cap I_j \cap X_{V(T_i)} - X_i \neq \emptyset$ or some $k \in [0, w_0 - 1]$ and $q \in [0, s + 1]$, or $W_2(2,0,k) \cap I_j \cap X_{V(T_i)} - X_i \neq \emptyset$ for some $k \in [0, s + 1]$. For the latter, define $\ell_0 = -1$; for the latter, define $\ell_0 = 0$. So there exists a monochromatic path $P$ with respect to $c$ of color $\alpha + 1$ contained in $G[W_0(i',0,\alpha)]$ intersecting $X_{V(T_{i'})}$ such that either $\ell_0 = -1$ and $A_{L(i',-1,k,q)}(V(P)) \cap Z_t \cap X_{V(T_i)} - X_i \neq \emptyset$ for some $k \in [0, w_0 - 1]$, or $\ell_0 = 0$ and $A_{L(i',0,\alpha)}(V(P)) \cap Z_t \cap X_{V(T_i)} - X_i \neq \emptyset$. If $\ell_0 = -1$, then let $\beta(\ell_0) := (-1, k, \alpha)$ and $\beta'(\ell_0) := (-1, k, \alpha + 1)$; if $\ell_0 = 0$, then let $\beta(\ell_0) := (0, \alpha)$ and $\beta'(\ell_0) := (0, \alpha + 1)$. So $A_{L(i',\alpha)}(V(P)) \cap Z_t \cap X_{V(T_i)} - X_i \neq \emptyset$.

Note that if $\ell_0 = -1$, then $V(P) \subseteq W_1(i',\alpha)$. If $\ell_0 = -1$, then let $W_0(i',\alpha) := W_1(i',\alpha)$. Hence $V(P) \subseteq W_0(i',\alpha)$.

Since $i' < i$, $V(P) \cap X_{i'} \neq \emptyset$. Since $A_{L(i',\beta(\ell_0))}(V(P)) \cap I_j \neq \emptyset$, $V(P) \subseteq I_j$. Let $P'$ be a subpath of $P$ contained in $G[W_0(i',\alpha)]$ from $X_{i'}$ to a vertex $v$ internally disjoint from $X_{i'}$, where there exists $u \in A_{L(i',\beta(\ell_0))}(V(P)) \cap Z_t \cap X_{V(T_i)} - X_i = \emptyset$. Let $x$ be an end of $P'$ in $X_{i'}$. Since $Z_t \cap I_j \cap X_{V(T_i)} - X_i \neq \emptyset$, $t \in V(T_{j,t'}) - \partial T_{j,t'}$. Since $V(P') \subseteq W_0(i',\alpha)$, we know $x \in S_{j,t'}(i'+2)$.

We may assume that $x \in S_{j,t'}(i'+2)$, for otherwise we are done. So there exist $i'' \in [0, i' - 1]$ with $t \in V(T_{j,t''}) - \partial T_{j,t''}$, where $t''$ is the node of $T$ of height $i''$ with $t \in V(T_{i''})$, such that $x \in S_{j,t''}(i'+2) - S_{j,t'}(i'+2)$. Hence there exist $\ell'' \in \{0, -1\}$ and $k'' \in [0, w_0 - 1]$ with $i'' \neq i'$ such that $x \in W_0(i'',k'',\alpha)$.

So there exists a monochromatic path $Q$ in $G[Y(i'',\ell'',k'')]$ from $X_{V(T_{i''})}$ to $x$ internally disjoint from $X_{V(T_{i''})}$. Since $x \in V(P') \cap Y(i'',\ell'',k'')$, there exists a maximal subpath $P^*$ of $P$ contained in $G[Y(i'',\ell'',k'')]$ containing $x$. So $Q \cup P^*$ is a monochromatic connected subgraph in $G[Y(i'',\ell'',k'')]$ intersecting $X_{V(T_{i''})}$. Hence $P^* \subseteq W_0(i'',\ell'',k'')$.

Suppose $P^* \neq P'$. Then there exists $a \in V(P^*)$ and $b \in N_{P^*}(a) \cap V(P') - V(a)$. So $b \notin Y(i'',\ell'',k'')$ by the maximality of $P^*$. Hence $b \in Y(i',\beta(\ell_0)) - Y(i',\ell'',k'')$. For each $i'' \in [i'', \ell'']$, let $t'' \in T_i$ be the node of $T$ of height $i''$ with $t'' \in V(T_{i''})$. If $b \in Z_{t''}$, then $b \in W_2(i'',-1,k'',\alpha) \cup W_2(i'',0,\alpha)$, and hence $c(a) \neq c(b)$ since $a \in W_0(i'',\ell'',k'')$, a contradiction. So $b \notin Z_{t''}$. In particular, $b \in I_j$. Since $b \in I_j$ and $b \in Y(i',\beta(\ell_0))$, there exists a minimum $i^* \in [i'', i']$ such that $b \in V(T_{i^*})$. So $b \in Z_{t_{i^*}}$ and $i^* > i'' + 1$. Therefore, $b \notin Z_{t_{i'}}$ for every $i'' \in [i'' + 1, i^* - 1]$. Hence $b \notin Y(i',-1,0)$.

Since $V(P^*) \subseteq Y(i'',\ell'',k'') \subseteq Y(i',-1,0)$, $a \in W_0(i',-1,k') \cup W_0(0,\alpha)$ for some $k' \in [0, w_0 - 1]$, so $b \in W_2(i',-1,k',\alpha) \cup W_2(0,\alpha)$, and hence $c(b) \neq c(a)$, a contradiction.

Hence $P^* = P'$. So $P' = P^* \subseteq W_0(i'',\ell'',k'')$. Since $i'' < i'$ and $t \in V(T_{j,t'}) - \partial T_{j,t''}$, $Z_{t''} \cap X_{V(T_i)} \subseteq Z_{t_{i'}} \cap X_{V(T_i)}$. So $u \in A_{L(i',\beta(\ell_0))}(V(P')) \cap Z_{t_{i'}} \cap X_{V(T_i)} \subseteq A_{L(i',\beta(\ell_0))}(V(P')) \cap Z_{t_{i'}} \cap X_{V(T_i)} \subseteq (W_0(i',-1,k'',\alpha) \cup W_2(i',0,\alpha)) \cap Z_{t_{i'}} \cap X_{V(T_i)}$ for some $k'' \in [0, w_0 - 1]$. Hence $\alpha + 1 \notin L(i',\beta(\ell_0))(u)$.

That is, $u \notin A_{L(i',\beta(\ell_0))}(V(P'))$, a contradiction. □

Claim 23.12. Let $i \in \mathbb{N}$ and let $t \in V(T(i)) - V(T(i-1))$. Let $j \in [\{|V| - 1]. Let S := \{i' \in [0, i - 1] : |Y(i',-1,0) \cap I_j \cap X_{V(T_i)} - X_i| \neq |Y(i',0,0,2) \cap I_j \cap X_{V(T_i)} - X_i|\}. Then $|S| \leq w_0$.

Proof. Let $i' \in S$. Since $|Y(i',-1,0) \cap I_j \cap X_{V(T_i)} - X_i| \neq |Y(i',0,0,2) \cap I_j \cap X_{V(T_i)} - X_i|$, we know either $|Y(i',-1,0) \cap I_j \cap X_{V(T_i)} - X_i| \neq |Y(i',0,0,2) \cap I_j \cap X_{V(T_i)} - X_i|$ or $|Y(i',0,0) \cap I_j \cap X_{V(T_i)} - X_i| \neq $
Claim 23.13. Let \( i \in \mathbb{N}_0 \) and let \( t \in V(T(i)) - V(T(i-1)) \). Let \( j \in [\|Y\| - 1] \). Then \( |Y^{(i,-1,0)} \cap I_j \cap X_{V(T_i)}| \leq \eta_i \).

Proof. Let \( S_0 := \{ i' \in [0, i - 1] : |Y^{(i',0,s+2)} \cap I_j \cap X_{V(T_i)} - X_i| > |Y^{(i',-1,0)} \cap I_j \cap X_{V(T_i)} - X_i| \} \).

By Claim 23.12, \( |S_0| \leq w_0 \). Let \( S_2 := \{ i' \in [0, i - 1] : |Y^{(i',|V(T)|)+1,s+2} \cap I_j \cap X_{V(T_i)} - X_i| > |Y^{(i',0,s+2)} \cap I_j \cap X_{V(T_i)} - X_i| \} \). Since \( S_j \subset I_j \), \( |S_2| \leq 3w_0 \) by Claim 23.6.

For every \( i' \in [0, i] \), let \( t_{i'} \) be the node of \( V \) of height \( i' \) with \( t \in V(T_{i'}). \)

For every \( i' \in S_0 \), \( (Y^{(i',0,s+2)} - Y^{(i',-1,0)}) \cap I_j \cap X_{V(T_i)} - X_i \subset X_{V(T_j,t_{i'}r)} \cap I_j \cap X_{V(T_i)} - X_i \). So for every \( i' \in S_0 \), \( |(Y^{(i',0,s+2)} - Y^{(i',-1,0)}) \cap I_j \cap X_{V(T_i)} - X_i| \leq |(Y^{(i',0,s+2)} \cap I_j \cap X_{V(T_i)} \cap X_{V(T_j,t_{i'}r)})| \leq g_1(s+2) \)

by Statement 2 in Claim 23.10.

For every \( i' \in S_2 \), \( (Y^{(i',|V(T)|)+1,s+2} - Y^{(i',0,s+2)}) \cap I_j \cap X_{V(T_i)} - X_i \subset X_{V(T_j,t_{i'}r)} \cap I_j \cap X_{V(T_i)} - X_i \). So for every \( i' \in S_2 \), \( |(Y^{(i',|V(T)|)+1,s+2} - Y^{(i',0,s+2)}) \cap I_j \cap X_{V(T_i)} - X_i| \leq |(Y^{(i',|V(T)|)+1,s+2} \cap I_j \cap X_{V(T_i)} \cap X_{V(T_j,t_{i'}r)})| \leq 3g_3(4w_0) \) by Statement 4 in Claim 23.10.

Note that for every \( i' \in [0, i - 1] \), \( (Y^{(i',1,-1,0)} - Y^{(i',|V(T)|)+1,s+2}) \cap X_{V(T_i)} \subset X_i \). Hence \( |Y^{(i,1,-1,0)} \cap I_j \cap X_{V(T_i)} - X_i| \leq |Y^{(0,0)} \cap I_j \cap X_{V(T_i)} - X_i| + |S_0| \cdot g_1(s + 2) + |S_2| \cdot g_3(4w_0) \leq \theta + w_0g_1(s + 2) + 3w_0g_3(4w_0) \). Therefore, \( |Y^{(i,1,-1,0)} \cap I_j \cap X_{V(T_i)}| \leq w_0g_1(s + 2) + 3w_0g_3(4w_0) + |X_i \cap I_j| \leq w_0g_1(s + 2) + 3w_0g_3(4w_0) + \eta_i \).

Claim 23.14. Let \( i \in \mathbb{N}_0 \), and let \( t \) be a node of \( V \) of height \( i \). Let \( j \in [\|Y\| - 1] \). Let \( Z_j \) be the set obtained from the \( j \)-th belt by deleting \( I_{j-1} \cup I_{j,0} \), where \( I_{0,1} = \emptyset \). Let \( S := \{ i' \in [0, i - 1] : |Y^{(i',1,-1,0)} \cap X_{V(T_i)} \cap Z_j - X_i| < |Y^{(i',0,s+2)} \cap X_{V(T_i)} \cap Z_j - X_i| \} \). Then \( |S| \leq w_0 \).

Proof. Let \( Z_j := Z_j \cup I_{j-1} \cup I_j \). For any \( i \in \mathbb{N}_0 \) and \( k \in [0, s + 2] \), denote \( U^{(i,k)} \) by \( U^{(i,0,k)} \). We shall show that for every \( i' \in S \), \( |U^{(i',1,-1,0)} \cap X_i \cap \overline{Z_j}| \leq |U^{(i',0,s+2)} \cap X_i \cap \overline{Z_j}| \).

Suppose to the contrary that there exists \( i' \in S \) such that \( |U^{(i',1,-1,0)} \cap X_i \cap \overline{Z_j}| \geq |U^{(i',0,s+2)} \cap X_i \cap \overline{Z_j}| \). Since \( U^{(i',1,-1,0)} \subset U^{(i',0,s+2)} \), \( U^{(i',1,-1,0)} \cap X_i \cap \overline{Z_j} = U^{(i',0,s+2)} \cap X_i \cap \overline{Z_j} \). Since \( |Y^{(i',1,-1,0)} \cap X_{V(T_i)} \cap Z_j - X_i| < |Y^{(i',0,s+2)} \cap X_{V(T_i)} \cap Z_j - X_i| \), there exist \( \ell \in \{ -1, 0 \} \) and \( k \in [0, w_0 - 1] \) such that \( W_{0^{(i',1,-k)}} \neq \emptyset \) and \( A_{L^{(i',1,-k)}}(W_{0^{(i',1,-k)}}) \cap Z_j \cap X_{V(T_i)} - X_i \neq \emptyset \). Since \( i' < i \), there exists a monochromatic path \( P \in G[Y^{(i',0,s+1,j)}] \) from \( X_{V(T_{i'})} \) to \( X_{V(T_i)} \) intersecting \( N_G[Z_j] \), and \( A_{L^{(i',1,\ell,k)}}(V(P)) \cap (Y^{(i',0,s+2)} - Y^{(i',1,-k)}) \cap Z_j \neq \emptyset \). Since \( P \) is a monochromatic path, \( V(P) \subset \overline{Z_j} \). Let \( x \in V(P) \cap X_i \cap \overline{Z_j} \). So \( x \in U^{(i',0,s+2)} \cap X_i \cap \overline{Z_j} = U^{(i',1,-1,0)} \cap X_i \cap \overline{Z_j} \). Hence there exist \( i'' \in [0, i' - 1] \), \( \ell'' \in \{ -1, 0 \} \) and \( k'' \in [0, w_0 - 1] \) such that \( x \in W_{0^{(i'',\ell'',k'')}} \). So there exists a monochromatic path \( P'' \) contained in \( G[Y^{(i'',\ell'',k'')}}] \) from \( X_{V(T_{i''})} \) to \( x \). Hence \( P \cup P'' \) is a monochromatic connected subgraph in \( G[Y^{(i'',\ell'',k'')}}] \) intersecting \( x \) and \( X_{V(T_{i''})} \). Let \( Q \) be a maximal connected subgraph of \( P \cup P'' \) contained in \( G[Y^{(i'',\ell'',k'')}}] \) containing \( x \) and intersecting \( X_{V(T_{i''})} \). So \( Q \) contains \( P'' \), and \( V(Q) \subset W_{0^{(i'',\ell'',k'')}} \). If \( V(P) \subset V(Q) \), then \( A_{L^{(i'',\ell'',k'')}}(V(P)) \cap Z_j \subset Y^{(i'',1,-1,0)} \subset Y^{(i'',-1,0)} \), so \( A_{L^{(i',1,\ell,k)}}(V(P)) \cap (Y^{(i',0,s+2)} - Y^{(i',1,-k)}) \cap Z_j = \emptyset \), a contradiction. So \( V(P) \not\subset V(Q) \). Hence there exist \( u \in V(Q) \) and \( v \in N_G(u) - V(Q) \) such that \( c(u) = c(v) \). But \( c(v) \neq c(u) \not\in L^{(i'',\ell'',k'')}}(v) \) by the definition of \( L^{(i'',\ell'',k'')}}(v) \), a contradiction.

Hence \( |U^{(i',1,-1,0)} \cap X_i \cap \overline{Z_j}| < |U^{(i',0,s+2)} \cap X_i \cap \overline{Z_j}| \) for every \( i' \in S \). Therefore, \( |S| \leq |X_i \cap \overline{Z_j}| \leq w_0 \).
Claim 23.15. Let $i, i' \in \mathbb{N}_0$ with $i' < i$, and let $t$ be a node of $T$ of height $i$. Let $j \in [|V| - 1]$. Let $Z_j$ be the set obtained from the $j$-th belt by deleting $I_{j-1,1} \cup I_{j,0}$, where $I_{0,1} = \emptyset$. Then $|Y^{(i',|V(T)|+1,s+2)} \cap Z_j \cap X_{V(T_i)}| \leq g_4(\{Y^{(i'-1,0)} \cap Z_j \cap X_{V(T_i)}\})$.

**Proof.** Note that $N_G[Z_j] \subseteq Z_j \cup I_{i-1}^0 \cup I_i^0$. For every $k \in [0, w_0 - 1]$ and $q \in [0, s + 1]$,

$$\begin{align*}
|Y^{(i'-1,k+1)} \cap Z_j \cap X_{V(T_i)}| &= |X_t \cap Z_j| + |Y^{(i'-1,k,s+2)} \cap Z_j \cap X_{V(T_i)} - X_t| \\
&\leq |X_t \cap Z_j| + |Y^{(i'-1,k,0)} \cap Z_j \cap X_{V(T_i)} - X_t| + (s + 2) \cdot f(|Y^{(i'-1,k)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1) \\
&\leq w_0 + |Y^{(i'-1,k)} \cap Z_j \cap X_{V(T_i)} - X_t| + (s + 2) \cdot f(|Y^{(i'-1,k)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1) \\
&\leq (s + 2) \cdot f(|Y^{(i'-1,k)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1).
\end{align*}$$

Hence it is easy to verify that for every $k \in [0, w_0]$, $|Y^{(i'-1,k)} \cap Z_j \cap X_{V(T_i)}| \leq g_4(k, |Y^{(i'-1,0)} \cap Z_j \cap X_{V(T_i)}|)$ by induction on $k$. Therefore $|Y^{(i',0,0)} \cap Z_j \cap X_{V(T_i)}| = |Y^{(i'-1,w_0)} \cap Z_j \cap X_{V(T_i)}| \leq g_4(w_0, |Y^{(i'-1,0)} \cap Z_j \cap X_{V(T_i)}|)$.

For every $k \in [0, s + 1]$,

$$\begin{align*}
|Y^{(i',0,k+1)} - Y^{(i',0,k)} \cap Z_j \cap X_{V(T_i)} - X_t| &\leq |A_{L^{(i',0,k)}}(W_0^{(i',0,k)}) \cap Z_j \cap X_{V(T_i)} - X_t| \\
&\leq |A_{L^{(i',0,k)}}(W_0^{(i',0,k)}) \cap (Z_j \cup I_{j-1}^0 \cup I_j^0) \cap X_{V(T_i)}) \cap Z_j \cap X_{V(T_i)} - X_t| \\
&\leq |N^{G_0}[W_0^{(i',0,k)} \cap (Z_j \cup I_{j-1}^0 \cup I_j^0) \cap X_{V(T_i)}]| \\
&\leq |N^{G_0}[Y^{(i',0,k)} \cap (Z_j \cup I_{j-1}^0 \cup I_j^0) \cap X_{V(T_i)}]| \\
&\leq f(|Y^{(i',0,k)} \cap (Z_j \cup I_{j-1}^0 \cup I_j^0) \cap X_{V(T_i)}|) \\
&\leq f(|Y^{(i',0,k)} \cap Z_j \cap X_{V(T_i)}| + |Y^{(i'-1,0)} \cap (I_{j-1}^0 \cup I_j^0) \cap X_{V(T_i)}| \\
&\leq f(|Y^{(i',0,k)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1),
\end{align*}$$

where the last inequality follows from Claim 23.13. So for every $k \in [0, s + 1]$,

$$|Y^{(i',0,k+1)} \cap Z_j \cap X_{V(T_i)}|$$
\[
\leq |X_t \cap Z_j| + |Y^{(i',0,k+1)} \cap Z_j \cap X_{V(T_i)} - X_t| \\
\leq |X_t \cap Z_j| + |Y^{(i',0,k)} \cap Z_j \cap X_{V(T_i)} - X_t| + f(|Y^{(i',0,k)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1) \\
\leq f_1(|Y^{(i',0,k)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1).
\]

Therefore, \(|Y^{(i',0,s+2)} \cap Z_j \cap X_{V(T_i)}| \leq f_{s+2}(|Y^{(i',0,0)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1).\)

Note that

\[
(Y^{(i',|V(T)|+1,s+2)} - Y^{(i',0,s+2)}) \cap Z_j \cap X_{V(T_i)} \\
\subseteq (Y^{(i',|V(T)|+1,s+2)} - Y^{(i',0,s+2)}) \cap (I_{j-1} \cup I_j) \cap X_{V(T_{j',T})} \cap X_{V(T_i)} \\
\subseteq Y^{(i',|V(T)|+1,s+2)} \cap (I_{j-1} \cup I_j) \cap X_{V(T_{j',T})} \cap X_{V(T_i)},
\]

where \(t'\) is the node of \(T\) of height \(i'\) with \(t \in V(T_{i'}).\) So \(|(Y^{(i',|V(T)|+1,s+2)} - Y^{(i',0,s+2)}) \cap Z_j \cap X_{V(T_i)}| \leq |Y^{(i',|V(T)|+1,s+2)} \cap (I_{j-1} \cup I_j) \cap X_{V(T_{j',T})} \cap X_{V(T_i)}| \leq 2g_3(4w_0)\) by Statement 4 in Claim 23.10.

Therefore,

\[
|Y^{(i',|V(T)|+1,s+2)} \cap Z_j \cap X_{V(T_i)}| \\
\leq |Y^{(i',0,s+2)} \cap Z_j \cap X_{V(T_i)}| + 2g_3(4w_0) \\
\leq f_{s+2}(|Y^{(i',0,0)} \cap Z_j \cap X_{V(T_i)}| + 2\eta_1) + 2g_3(4w_0) \\
\leq f_{s+2}(g_4(w_0), |Y^{(i',-1,0)} \cap Z_j \cap X_{V(T_i)}|) + 2\eta_1) + 2g_3(4w_0) \\
= g_5(|Y^{(i',-1,0)} \cap Z_j \cap X_{V(T_i)}|).
\]

Claim 23.16. Let \(i \in \mathbb{N}_0,\) and let \(t\) be a node of \(T\) of height \(i.\) Let \(j \in [|V| - 1].\) Let \(Z_j\) be the set obtained from the \(j\)-th belt by deleting \(I_{j-1} \cup I_{j,0},\) where \(I_{0,1} = \emptyset.\) Then \(|Y^{(i,-1,0)} \cap Z_j \cap X_{V(T_i)}| \leq \eta_2.\)

Proof. Let

\[
S_1 := \{i' \in [0, i-1]: |Y^{(i',-1,0)} \cap X_{V(T_i)} \cap Z_j - X_t| < |Y^{(i',0,s+2)} \cap X_{V(T_i)} \cap Z_j - X_t|\}.
\]

By Claim 23.14, \(|S_1| \leq w_0.\) Let

\[
S_2 := \{i' \in [0, i-1]: |Y^{(i',0,s+2)} \cap X_{V(T_i)} \cap Z_j - X_t| < |Y^{(i',|V(T)|+1,s+2)} \cap X_{V(T_i)} \cap Z_j - X_t|\}.
\]

Note that for every \(i' \in S_2,\) \((Y^{(i',|V(T)|+1,s+2)} - Y^{(i',0,s+2)}) \cap X_{V(T_i)} \cap Z_j - X_t \subseteq (Y^{(i',|V(T)|+1,s+2)} - Y^{(i',0,s+2)}) \cap X_{V(T_i)} \cap Z_j - X_t,\) where \(I_0 = \emptyset.\) So

\[
|S_2| \leq \{|i' \in [0, i-1]: |Y^{(i',0,s+2)} \cap X_{V(T_i)} \cap I_{j-1} - X_t| \\
\leq |Y^{(i',|V(T)|+1,s+2)} \cap X_{V(T_i)} \cap I_{j-1} - X_t|\} + \{|i' \in [0, i-1]: |Y^{(i',0,s+2)} \cap X_{V(T_i)} \cap I_j - X_t| \\
\leq |Y^{(i',|V(T)|+1,s+2)} \cap X_{V(T_i)} \cap I_j - X_t|\} \\
\leq 6w_0.
\]

by Claim 23.6. Let

\[
S_3 := \{i' \in [0, i-1]: |Y^{(i',-1,0)} \cap X_{V(T_i)} \cap Z_j - X_t| < |Y^{(i',|V(T)|+1,s+2)} \cap X_{V(T_i)} \cap Z_j - X_t|\}.
\]

So \(|S_3| \leq |S_1| + |S_2| \leq 7w_0.\) Let

\[
S := \{i' \in [0, i-1]: |Y^{(i',-1,0)} \cap X_{V(T_i)} \cap Z_j - X_t| < |Y^{(i'+1,-1,0)} \cap X_{V(T_i)} \cap Z_j - X_t|\}.
\]

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Since for every $i' \in [0, i - 1]$, $Y^{(i'+1,-1,0)} \cap Z_j \cap X_{V(T_i)} \subseteq (Y^{(i',i',V(T))}+1, s+2) \cap Z_j \cap X_{V(T_i)} \cup (X_t \cap Z_j)$, we know $|S| \leq |S_{\eta^2}| + |X_t \cap Z_j| \leq 8w_0$.

By Claim 23.15, for every $i' \in S$, $|Y^{(i'+1,-1,0)} \cap Z_j \cap X_{V(T_i)}| \leq |Y^{(i',i',V(T))}+1, s+2) \cap Z_j \cap X_{V(T_i)}| + |X_t \cap Z_j| \leq g_{60}(|Y^{(i',-1,0)} \cap Z_j \cap X_{V(T_i)}|) + w_0$.

Note that $|Y^{(0,-1,0)} \cap Z_j \cap X_{V(T_i)}| \leq |X_t \cap Z_j| \leq w_0 = g_{60}(0)$. Then it is easy to verify that $|Y^{(i,-1,0)} \cap Z_j \cap X_{V(T_i)}| \leq g_{60}(|S|)$ by induction on the elements in $S$. Since $|S| \leq 8w_0$, $|Y^{(i,-1,0)} \cap Z_j \cap X_{V(T_i)}| \leq g_{60}(8w_0) = \eta_2$. \hfill $\square$

Claim 23.17. Let $i \in \mathbb{N}_0$ and let $t \in V(T^{(i)}) - V(T^{(i-1)})$. Let $j \in \{|V|-1\}$. Then $|X_{\partial T_{j,t}} \cap T_j| \leq \eta_3$.

Proof. By the definition of a fence (Statement 2 in Lemma 17), for every $t^* \in \partial T_{j,t} - \{t\}$, there exist at least two $F_{j,p} \cap V(T_i)$ parts $T'$ with $X_{V(T_j)} \cap (Y^{(i,-1,0)} \cap X_{\partial T^*}) \cap \partial T_j \cap \partial T_j \neq \emptyset$. So some $F_{j,p} \cap V(T_i)$-part $T''$ is disjoint from $t$ and $X_{V(T''')} \cap (Y^{(i,-1,0)} \cap X_{\partial T^*}) \cap T_j \cap \partial T_j \cap \partial X_{t'} \neq \emptyset$. By induction, we know $|\partial T_{j,t} - \{t\}| \leq |Y^{(i,-1,0)} \cap \partial T_j \cap \partial T_j - \partial X_{t}|$. Hence $|\partial T_{j,t}| \leq |Y^{(i,-1,0)} \cap \partial T_j \cap \partial T_j - \partial X_{t}| + 1 \leq |Y^{(i,-1,0)} \cap \partial T_j \cap \partial X_{t} + 1| \leq |Y^{(i,-1,0)} \cap (Z_j \cup Z_{j+1} \cup T_j)| + 1 \leq \eta_1 + 2\eta_2 + 1$ by Claims 23.13 and 23.16, where $Z_j$ and $Z_{j+1}$ are the sets obtained from the $j$-th belt by deleting $I_{j-1,1} \cup I_{j,0}$ and from the $(j+1)$-th belt by deleting $I_{j,1} \cup I_{j+1,0}$, respectively. Therefore, $|X_{\partial T_{j,t}} \cap T_j| \leq |\partial T_{j,t}| \cdot w_0 \leq (\eta_1 + 2\eta_2 + 1)w_0 = \eta_3$. \hfill $\square$

Claim 23.18. Let $i \in \mathbb{N}_0$ and let $t \in V(T^{(i)}) - V(T^{(i-1)})$. Let $j \in \{|V|-1\}$. Then $|Y^{(i,0,0)} \cap I_j \cap X_{V(T_i)}| \leq g_{60}(w_0)$.

Proof. Note that for every $k \in [0, w_0 - 1]$, $W_{0}^{(i,-1,k)} \cap X_{V(T_i)}$ intersects

$$\bigcup_{j'=1}^{\{|V|-1\}} V(M_{j,k}^{(i)}) \subseteq \bigcup_{j'=1}^{\{|V|-1\}} \bigcup_{S \in S_{\eta^2}} S.$$

Since $W_{0}^{(i,-1,k)}$ consists of monochromatic components, $W_{0}^{(i,-1,k)} \cap X_{V(T_i)} \subseteq \bigcup_{j'=1}^{\{|V|-1\}} \bigcup_{S \in S_{\eta^2}} S$. So

$$(Y^{(i,-1,k+1)} - Y^{(i,-1,k)}) \cap X_{V(T_i)} \subseteq W_{2}^{(i,-1,k)} \cap X_{V(T_i)} \subseteq N_G\left[\bigcup_{j'=1}^{\{|V|-1\}} \bigcup_{S \in S_{\eta^2}} S \right] \subseteq \bigcup_{j'=1}^{\{|V|-1\}} \bigcup_{j'=1}^{\{|V|-1\}} I_{j'}.$$

So for every $k \in [0, w_0 - 1]$ and $q \in [0, s + 1]$,

$$(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_i)} - X_t \subseteq A_{L^{(i,-1,k,s)}}(Y^{(i,-1,k,q)} \cap T_j) \cap I_j \cap X_{V(T_i)} - X_t \subseteq N_G^{(s)}(Y^{(i,-1,k,q)} \cap T_j) \cap I_j \cap X_{V(T_i)} - X_t \subseteq N_G^{(s)}(Y^{(i,-1,k,q)} \cap T_j) \cap I_j \cap X_{V(T_i)}.$$

Hence for every $k \in [0, w_0 - 1]$ and $q \in [0, s + 1]$, $|Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_i)} - X_t| \leq |N_G^{(s)}(Y^{(i,-1,k,q)} \cap T_j) \cap X_{V(T_i)}| \leq f(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_i)})|$. Therefore, for every $k \in [0, w_0 - 1]$ and $q \in [0, s + 1]$,

$$|(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_i)}| \leq |(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_i)} - X_t| + |X_t \cap I_j| \leq f(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_i)})| + w_0.$$
\[ \leq f(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_i)}|) + w_0 + |Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_i)}| = f_1(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_i)}|) + w_0. \]

Note that for every \( k \in [0, w_0 - 1] \), \(|Y^{(i,-1,k,0)} \cap I_j \cap X_{V(T_i)}| \leq g_7(0, |Y^{(i,-1,k,0)} \cap I_j \cap X_{V(T_i)}|). \) So it is easy to verify that for every \( k \in [0, w_0 - 1] \), for every \( q \in [0, s + 2] \), \(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_i)}| \leq g_7(q, |Y^{(i,-1,k,0)} \cap I_j \cap X_{V(T_i)}|) \) by induction on \( q \). That is, for every \( k \in [0, w_0 - 1] \), \(|Y^{(i,-1,k+1)} \cap I_j \cap X_{V(T_i)}| = |Y^{(i,-1,k,s+2)} \cap I_j \cap X_{V(T_i)}| \leq g_7(s + 2, |Y^{(i,-1,k,0)} \cap I_j \cap X_{V(T_i)}|) = g_7(s + 2, |Y^{(i,-1,k)} \cap I_j \cap X_{V(T_i)}|). \)

Note that \(|Y^{(i,-1,1)} \cap I_j \cap X_{V(T_i)}| \leq \eta_1 + 2\eta_2 = g_8(0) \) by Claims 23.13 and 23.16. So it is easy to verify that for every \( k \in [0, w_0] \), \(|Y^{(i,-1,k)} \cap I_j \cap X_{V(T_i)}| \leq g_8(k) \) by induction on \( k \). Therefore, \(|Y^{(i,0,0)} \cap I_j \cap X_{V(T_i)}| = |Y^{(i,-1,w_0)} \cap I_j \cap X_{V(T_i)}| \leq g_8(w_0) \).

Claim 23.19. Let \( i \in \mathbb{N}_0 \) and let \( t \in V(T^{(i)}) - V(T^{(i-1)}) \). Let \( j \in [|V| - 1] \). Then \(|Y^{(i,0,s+2)} \cap B_j \cap X_{V(T_i)}| \leq \eta_4 \), where \( B_j \) is the \( j \)-th belt.

Proof. Let \( B_j := \bigcup_{a=a_j} V_a \). For every \( k \in [0, s + 2] \), let \( R_k := \bigcup_{a=a_j-(s+2)+k} V_a \). Let \( Z_j := B_j - (I_{j-1} \cup I_{j,0}) \). Note that \( B_j = R_{k+1} \cap I_{j+1} \cup Z_j \cap I_{j} = R_0 \) for every \( k \in [0, s + 1] \).

Note that for every \( k \in [0, s + 1] \),
\[
(Y^{(i,0,k+1)} - Y^{(i,0,k)}) \cap R_{k+1} \cap X_{V(T_i)} = X_t \subseteq A_L^{(i,0,k)}(W_0^{(i,0,k)}) \cap R_{k+1} \cap X_{V(T_i)} - X_t
\[
\subseteq A_L^{(i,0,k)}(W_0^{(i,0,k)}) \cap R_k \cap X_{V(T_i)} \cap X_{V(T_i)} - X_t
\[
\subseteq A_L^{(i,0,k)}(Y^{(i,0,k)}) \cap R_k \cap X_{V(T_i)} \cap X_{V(T_i)} - X_t
\[
\subseteq A_L^{(i,0,k)}(Y^{(i,0,k)}) \cap R_k \cap X_{V(T_i)}.
\]

So for every \( k \in [0, s + 1] \), \(|(Y^{(i,0,k+1)} - Y^{(i,0,k)}) \cap R_{k+1} \cap X_{V(T_i)} - X_t| \leq |N_G^{\geq s}(Y^{(i,0,k)} \cap R_k \cap X_{V(T_i)})| \leq f(|Y^{(i,0,k)} \cap R_k \cap X_{V(T_i)}|). \) Hence for every \( k \in [0, s + 1] \),
\[
|Y^{(i,0,k+1)} \cap R_k \cap X_{V(T_i)}|
\[
\leq |Y^{(i,0,k)} \cap R_k \cap X_{V(T_i)}| + f(|Y^{(i,0,k)} \cap R_k \cap X_{V(T_i)}|) + w_0 = f_1(|Y^{(i,0,k)} \cap R_k \cap X_{V(T_i)}|) + w_0.
\]

Recall that \((Y^{(i,0,0)} - Y^{(i,-1,0)}) \cap B_j \cap X_{V(T_i)} \subseteq N_G \bigcup_{S \subseteq \bigcup_{i \in \mathbb{N}_0} S_j} X_{V(T_i)} \subseteq (I_{j-1} \cup I_j) \cap X_{V(T_i)}\). So
\[
|Y^{(i,0,0)} \cap R_0 \cap X_{V(T_i)}|
\[
= |Y^{(i,0,0)} \cap (I_{j-1} \cup Z_j \cup I_j) \cap X_{V(T_i)}|
\[
\leq |Y^{(i,0,0)} \cap I_{j-1} \cap X_{V(T_i)}| + |Y^{(i,0,0)} \cap (Z_j - (I_{j-1} \cup I_j)) \cap X_{V(T_i)}| + |Y^{(i,0,0)} \cap I_j \cap X_{V(T_i)}|
\[
\leq g_8(w_0) + g_8(w_0)
\[
\leq 2g_8(w_0) + \eta_2
\[
= g_9(0)
\]

by Claims 23.16 and 23.18. Hence it is easy to verify that for every \( k \in [0, s + 2] \), \(|Y^{(i,0,k)} \cap R_k \cap X_{V(T_i)}| \leq g_9(k) \). In particular, \(|Y^{(i,0,s+2)} \cap B_j \cap X_{V(T_i)}| = |Y^{(i,0,s+2)} \cap R_{s+2} \cap X_{V(T_i)}| \leq g_9(s + 2) = \eta_4 \).
Claim 23.20. Let $M$ be a monochromatic component with respect to $c$. If $V(M) \cap \bigcup_{j=1}^{i-1} I_j \neq \emptyset$, then $|V(M)| \leq \eta_4$.

Proof. Let $i$ be the minimum such that $V(M) \cap X_{V(T(i))} \neq \emptyset$. By the minimality of $i$, there exists $t \in V(T)$ of height $i$ such that $V(M) \subseteq X_{V(T)}$ and $V(M) \cap X_t \neq \emptyset$. We claim that $V(M) \subseteq Y^{(i,0,s+2)} \cap X_{V(T)}$.

Suppose to the contrary that $V(M) \nsubseteq Y^{(i,0,s+2)}$. Let $k \in [0, s+1]$ such that $c(v) = k+1$ for every $v \in V(M)$. Since $X_t \subseteq Y^{(i-1,0), k}$, $V(M) \cap Y^{(i-1,0) \cap X_{V(T)} = \emptyset}$. So $V(M) \cap Y^{(i,k)} \cap X_{V(T)} = \emptyset$. Let $M'$ be the union of all components of $M[Y^{(i,k)} \cap X_{V(T)}]$ intersecting $V(T(i))$. Since $V(M) \cap X_{V(T(i))} = \emptyset$, $V(M') \subseteq W^{(i,k)}$. Since $V(M) \not\subseteq Y^{(i,0,s+2)}$, $V(M) \not\subseteq Y^{(i,k)}$, so there exists $v \in V(M) - V(M')$ adjacent in $G$ to $V(M')$. So $v \in A_{i,(i,k)}(V(M'))$. Since $v \in V(M)$, and $V(M)$ is disjoint from $\bigcup_{j=1}^{i-1} I_j$, $v \in Z_t$. So $v \in W^{(i,k)}$. Since $(Y^{(i,k+1)}, L^{(i,k+1)})$ is a $(W^{(i,k)}, k+1)$-progress of $(Y^{(i,k)}, L^{(i,k)})$, $k+1 \notin L^{(i,k+1)}(v)$. But $c(v) = k+1$, a contradiction.

Therefore, $V(M) \subseteq Y^{(i,0,s+2)} \cap X_{V(T)}$. Since $M$ is a monochromatic component disjoint from $\bigcup_{j=1}^{i-1} I_j$, there exists $j^* \in [|V| - 1]$ such that $V(M) \subseteq Z_{j^*}$, where $Z_{j^*}$ is the set obtained from the $j^*$-th belt by deleting $I_{j^*-1} \cup I_{j^*}$, so $|V(M)| \leq |Y^{(i,0,s+2)} \cap X_{V(T)} \cap Z_{j^*}| \leq \eta_4$ by Claim 23.19.

\[ \square \]

Claim 23.21. Let $i \in \mathbb{N}_0$ and let $t \in V(T(i)) - V(T(i-1))$. Let $j \in [|V| - 1]$. Then $|Y^{(i)|V(T)|+1,s+2} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| \leq \eta_4 = g_10(0)$.

For every $\ell \in [0, |V(T)|]$, by Claim 23.17,

\[
|Y^{(i,0,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| \leq |Y^{(i,0,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| + |X_{\partial T, j} \cap (I_j \cup I_{j-1} \cup I_{j+1})| \\
\leq |Y^{(i,0,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| + 3\eta_3.
\]

For every $\ell \in [0, |V(T)|]$ and $k \in [0, s+1]$, $|Y^{(i,k+1)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| \leq f_1(|Y^{(i,k)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}|)$. So for every $\ell \in [0, |V(T)|],$

\[
|Y^{(i,0,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| \leq f_{s+2}(|Y^{(i,0,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}|) \\
= f_{s+2}(|Y^{(i,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| + 3\eta_3).
\]

By Claim 23.17, $|X_{\partial T, j} \cap (I_j \cup I_{j-1} \cup I_{j+1})| \leq 3\eta_3$, so there are at most $3\eta_3$ numbers $\ell \in [0, |V(T)|]$ such that $Y^{(i,0,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)} \neq Y^{(i,0)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}$. Hence it is straightforward to verify by induction on $\ell$ that $|Y^{(i,|V(T)|+1,s+2)} \cap (B_j \cup B_{j+1}) \cap X_{V(T)}| \leq g_{10}(s+2) = \eta_5$.

Given any monochromatic component $M$ with respect to $c$, there exists minimum $i_M \in \mathbb{N}_0$ such that $V(M) \cap X_{V(T(i_M))} \neq \emptyset$, and there uniquely exists $r_M \in V(T(i_M))$ such that $V(M) \cap X_{r_M} \neq \emptyset$ and $V(M) \subseteq X_{V(T(r_M))}$. Recall that every monochromatic component is contained in some $s$-segment. For every monochromatic component $M$ with respect to $c$, let $S_M$ be the $s$-segment containing $V(M)$ whose level equals the color of $M$.

\[ \square \]

Claim 23.22. Let $j \in [|V| - 1]$, and let $M$ be a monochromatic component with respect to $c$ such that $S_M \cap I_j \neq \emptyset$. Let $i \in \mathbb{N}_0$ and $t$ be a node of $T$ of height $i$ such that $V(M) \cap X_t \neq \emptyset$ for some witness $t^* \in \partial T, j \cup \{t\}$ for $X_r \cap I_j \subseteq W^{(i+1)} \cap \{t\}$ for every $\ell \in [-1, |V(T)|]$. Then $V(M) \cap X_{V(T)} \subseteq Y^{(i,|V(T)|+1,s+2)}$ and $A_{i,(i,|V(T)|+1,s+2)}(V(M) \cap X_{V(T(j,t))}) \cap X_{V(T(j,t))} = \emptyset$.
Proof. We may assume that there exists no \( t' \in V(T) \) with \( t' \in V(T_{t'}) \) and \( V(T_{t'}) \cap \partial T_{j,t'} \neq \emptyset \) such that \( V(M) \cap X_{t'} \neq \emptyset \) for some witness \( t' \in \partial T_{j,t'} \cup \{t'\} \) for \( X_{t'} \cap I_j \subseteq W_3^{(i',\ell')} \) for some \( \ell' \in [-1,|V(T)|] \), where \( t' \) is the height of \( t' \), for otherwise we may replace \( t \) by \( t' \) due to the facts that \( T_{j,t} \subseteq T_{j,t'} \) and \( Y^{(i',|V(T)|+1,1,\ell'+2)} \subseteq Y^{(i',|V(T)|+1,1,\ell+2)} \).

Suppose to the contrary that either \( V(M) \cap X_{V(T_{j,t})} \subseteq Y^{(i',|V(T)|+1,1,\ell+2)} \), or \( A_{L_{\ell(i',|V(T)|+1,1,\ell+2)}}(V(M) \cap X_{V(T_{j,t})}) \cap X_{V(T_{j,t})} \neq \emptyset \). Note that \( V(M) \cap X_{V(T_{j,t})} \subseteq W_4^{(i)} \). By Claim 23.1, some component \( Q \) of \( G[V(M) \cap X_{V(T_{j,t})}] \) is disjoint from \( \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \). Since \( V(M) \) intersects \( X_{t'} \subseteq X_{\partial T_{j,t}} \), \( V(Q) \) intersects \( X_{\partial T_{j,t}} \cup X_t \). Since \( X_t \cap I_j \subseteq W_4^{(i-1)} \), \( V(Q) \cap X_t = \emptyset \). So \( V(Q) \) intersects \( X_{\partial T_{j,t}} \). Since \( X_{t'} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), there exist a node \( t'' \in \partial T_{j,t} \) with \( X_{t''} \cap V(Q) \neq \emptyset \) and \( X_{t''} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \) and a path \( P \) in \( M \) internally disjoint from \( \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \cup V(Q) \) passing through a vertex in \( V(M) \cap \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), a vertex in \( X_{t'} \cap V(Q) \) and a vertex in \( X_{t''} \cap V(Q) \) in the order listed. We further choose \( Q \) such that \( P \) is as short as possible.

Let \( P_0, P_1, \ldots, P_m \) (some some \( m \in \mathbb{N}_0 \)) be the maximal subpaths of \( P \) contained in \( G[X_{V(T_{j,t})}] \) internally disjoint from \( X_{\partial T_{j,t}} \cup X_t \), where \( P_0 \) intersects \( \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \) and \( P \) passes through \( P_0, P_1, \ldots, P_m \) in the order listed. So \( P_m \) intersects \( Q \). Since \( P \) is internally disjoint from \( \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), there exist (not necessarily distinct) \( t_1, t_2, \ldots, t_{m+1} \in \partial T_{j,t} \) such that for every \( \ell \in [m] \), \( P_t \) is from \( X_{t_j} \) to \( X_{t_{j+1}} \). So \( V(Q) \cap X_{t_{m+1}} \neq \emptyset \).

Since \( V(Q) \cap \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \) is the height of \( t \), there exists a minimum \( \ell^* \in [m+1] \) such that \( X_{t_{\ell^*}} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \). Let \( t_0 = t' \). So \( X_{t_{\ell^*}-1} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \). Note that if there exists \( q \in [-1,|V(T)|] \) such that \( I_j \cap \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} = I_j \cap \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), then \( I_j \cap \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \). This implies that there exists \( Q \) such that \( I_j \cap \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), then there exist at least \( q \) nodes \( t'' \in \partial T_{j,t} \) such that \( X_{t''} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \). This together with \( X_{t_{\ell^*}} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \) imply that \( X_{t_{\ell^*}-1} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \).

Since \( P_{\ell'-1} \) is contained in \( G[W_4^{(i,0)}] \) and intersects \( X_{t_{\ell*}-1} \), and \( X_{t_{\ell*}-1} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), Claim 23.1 implies that \( V(P_{\ell'-1}) \subseteq Y^{(i',|V(T)|,k)} \), where \( k+1 \) is the color of \( M \). Let \( v \) be the vertex in \( V(P_{\ell'-1}) \). Since \( X_{t_{\ell*}} \cap I_j \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), \( v \in (D^{(i',|V(T)|,0)} - X_t) \cup \{u \in D^{(i',|V(T)|,0)} \cap X_t \cap X_{q'} \cap I_j : q' \in V(T_t) - \{t\}, q' \text{ is a witness for } X_{q'} \cap I_j \subseteq W_3^{(i',\ell')} \text{ for some } i' \in [0, i - 1] \} \). Suppose there exists \( q' \) such that \( v \in D^{(i',|V(T)|,0)} \cap X_t \cap X_{q'} \cap I_j \). Then there exists a node \( t'' \) of height \( i' \in [0, i - 1] \) with \( t' \in V(T_{t''}) \) such that \( q' \in D^{(i',|V(T)|,0)} \cap X_t \cap I_j \subseteq W_3^{(i',\ell')} \) for some \( \ell' \in [0, |V(T)|] \). Since \( q' \in (V(T_t) - \{t\}) \cap \partial T_{j,t''} \), we know \( \partial T_{j,t''} \) is disjoint from the path in \( T \) between \( t' \) and \( t'' \). So \( t_{\ell'} \in \partial T_{j,t''} \cap V(T_{t''}) \). Hence \( t'' \) is a node of \( t'' \in V(T_{t''}) \) such that \( v \in V(M) \cap X_{q'} \neq \emptyset \) and \( q' \) is a witness for \( X_{q'} \cap I_j \subseteq W_3^{(i',\ell')} \) for some \( t'' \in [0, |V(T)|] \), a contradiction.

Therefore no such \( q' \) exists, and hence \( v \in D^{(i',|V(T)|,0)} - X_t \). So \( v \in W_3^{(i',\ell',\ell')} \) for some \( i' \in [0, i], \ell' \in [1, |V(T)|+1] \) with \((i', \ell')\) lexicographically smaller than \((i, |V(T)|)\). We assume that \((i', \ell')\) is lexicographically minimum. So there exists a monochromatic path \( P' \) in \( G[Y^{(i',\ell',k)} \cap W_4^{(i',\ell')}] \) from \( v \) to \( W_3^{(i',\ell'-1)} \) internally disjoint from \( W_3^{(i',\ell'-1)} \). The minimality of \((i', \ell')\) implies that \( v \notin D^{(i',\ell',0)} \). So if \( i' = i \), then \( t_{\ell'} \) is a witness for \( X_{t_{\ell'}} \cap I_j \subseteq W_3^{(i',\ell')} \subseteq \bigcup_{\ell=1}^{\ell_{T_{j,t}}} W_3^{(i,\ell)} \), a contradiction. Hence \( i' < i \).

Let \( u \) be the end of \( P' \) in \( W_3^{(i',\ell'-1)} \). Since \( P' \) is in \( G[Y^{(i',\ell',k)} \cap W_4^{(i',\ell')}] \) from \( v \) to \( W_3^{(i',\ell'-1)} \)
internally disjoint from $W_{3}^{(i',t'-1)}$, there exists $q^* \in \partial T_{j,t'} \cup \{t''\}$ for some node $t''$ of height $i'$ such that

$u \in X_{q^*} \cap I_j$ and $q^*$ is a witness for $X_{q^*} \cap I_j \subseteq W_{3}^{(i',t'-1)}$. Since $v \in (X_{V(T_j)} - X_t) \cap W_{4}^{(i)}$, $t \in V(T_{j,t'}) - \{t''\}$ and $\partial T_{j,t'}$ is disjoint from the path in $T$ between $t''$ and $t$, so $V(T_{j,t'}) \cap \partial T_{j,t} \neq \emptyset$. But $u \in V(M) \cap X_{q^*}$ and $q^* \in \partial T_{j,t'} \cup \{t''\}$ is a witness for $X_{q^*} \cap I_j \subseteq W_{3}^{(i',t'-1)}$ with $i' < i$, a contradiction. This proves the claim.

For every node $q$ of $T$, let $i_q$ be the height of $q$. For every $i \in \mathbb{N}_0$, node $t \in V(T)$ of height $i$ and $j \in \{|\mathcal{V}| - 1\}$, define the following:

- Let the $(t, j)$-stamp be the sequence $(b_1, b_2, \ldots, b_{|V(G)|})$, such that for each $\ell \in \{|V(G)|\}$:
  - if the vertex $u \in V(G)$ with $\sigma(u) = \ell$ belongs to a monochromatic component $M$ in $G[Y^{(i,1,0)}]$ with respect to $c$ such that $S_M \cap I^o_j \neq \emptyset$ and $\sigma(u) = \sigma(M)$, then define $b_\ell := |A_{L^{(i,1,0)}}(V(M)) \cap X_{V(T_j)} - X_t|$
  - otherwise $b_\ell := 0$.

- For each monochromatic $E_{j,t}$-pseudocomponent $M$ in $G[Y^{(i,1,0)}]$ with respect to $c$ such that $S_M \cap I^o_j \neq \emptyset$, let $b_M$ be the sequence whose $\ell$-th entry, for each $\ell \in \{|V(G)|\}$:
  - equals $b_\ell$ if there exists a monochromatic component in $G[Y^{(i,1,0)}]$ with respect to $c$ with $\sigma$-value $\ell$, and it is contained in $M$, and
  - equals 0 otherwise.

- Let the $(t, j)$-signature be the sequence $(a_1, a_2, \ldots, a_{|V(G)|})$, where for each $\ell \in \{|V(G)|\}$:
  - if the vertex $u \in V(G)$ with $\sigma(u) = \ell$ belongs to a monochromatic $E_{j,t}$-pseudocomponent $M$ in $G[Y^{(i,1,0)}]$ with respect to $c$ such that $S_M \cap I^o_j \neq \emptyset$ and $\sigma(u) = \sigma(M)$, then let $a_\ell := b_M$
  - otherwise, let $a_\ell$ be the zero sequence with $|V(G)|$ entries.

Furthermore, if $M$ is the monochromatic $E_{j,t}$-pseudocomponent that defines $a_\ell$ for some $\ell \in \{|V(G)|\}$ mentioned above, then the $(t, M)$-signature is the sequence $(a_1, a_2, \ldots, a_\ell)$. For any $a \in \ell$ and $\beta \in \{|V(G)|\}$, define the $(\alpha, \beta)$-entry of the $(t, M)$-signature to be the $\beta$-th entry in $a_\ell$.

Claim 23.23. Let $i \in \mathbb{N}_0$, $t$ a node of $T$ of height $i$, and $j \in \{|\mathcal{V}| - 1\}$. Then the sum of the $(\beta, \gamma)$-entries of the $(t, j)$-signature over all $\beta, \gamma \in \{|V(G)|\}$ is at most $\eta_6$.

Proof. Let $a, a'$ be distinct entries of the $(t, j)$-signature. So $a, a'$ are determined by two different $E_{j,t}$-pseudocomponents $M_a, M_{a'}$ in $G[Y^{(i,1,0)}]$. Since $M_a \neq M_{a'}$, $V(M_a) \cap V(M_{a'}) = \emptyset$. So there exists no $\ell \in \{|V(G)|\}$ such that both the $\ell$-th entry of $a$ and the $\ell$-th entry of $a'$ are nonzero. Hence $\sum_{\beta=1}^{\{|V(G)|\}} a_\beta$ equals the $(t, j)$-stamp, where we denote the $(t, j)$-signature by $(a_1, a_2, \ldots, a_{|V(G)|})$. Therefore, the sum of all entries of the $(t, j)$-signature equals the sum of the entries of the $(t, j)$-stamp.

Note that there are at most $|Y^{(i,1,0)} \cap X_{V(T_j)} \cap T_j|$ monochromatic components $M$ in $G[Y^{(i,1,0)}]$ with respect to $c$ with $S_M \cap I^o_j \neq \emptyset$ and $A_{L^{(i,1,0)}}(V(M)) \cap X_{V(T_j)} - X_t \neq \emptyset$. So by Claim 23.19, there are at most $2\eta_4$ nonzero entries in the $(t, j)$-stamp. And for each monochromatic component $M$ in $G[Y^{(i,1,0)}]$ with respect to $c$ with $S_M \cap I^o_j \neq \emptyset$ and $A_{L^{(i,1,0)}}(V(M)) \cap X_{V(T_j)} - X_t \neq \emptyset$, $|A_{L^{(i,1,0)}}(V(M)) \cap X_{V(T_j)} - X_t| \leq f(2\eta_4)$ by Claim 23.19. Hence each entry of the $(t, j)$-stamp is at most $f(2\eta_4)$. Therefore, the sum of the entries of the $(t, j)$-stamp is at most $2\eta_4 \cdot f(2\eta_4) = \eta_6$. □
Claim 23.24. Let $j \in [|\mathcal{V}| - 1]$. Let $t$ and $t'$ be nodes of $T$. Let $\ell \in [0, w_0 - 1]$. Assume that for every $t'' \in V(T)$ with $V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t)$, $M^{(t'')}_{j,\ell} \supseteq M^{(t)}_{j,\ell}$, and for every $\alpha \in [0, \ell - 1]$, $M^{(t''\alpha)}_{j,\ell} = M^{(t)}_{j,\ell}$ and the $(t'', M^{(t)}_{j,\ell})$-signature equals the $(t, M^{(t)}_{j,\ell})$-signature. Assume that there exist $\beta^* \in \mathbb{N}$ and nodes $t_0, t_1, \ldots, t_{\beta^*}$ such that $t_0 = t$, $t_{\beta^*} = t'$, and for every $\alpha \in [\beta^*]$, $t_\alpha \in V(T_{j,\alpha_{\alpha-1}})$.

Let $\Phi := \{ t_\alpha : \alpha \in [\beta^*], \alpha \neq \beta^*, \text{either } M^{(t_\alpha)}_{j,\ell} \neq M^{(t_{\alpha-1})}_{j,\ell} \text{ or the } (t_\alpha, M^{(t_\alpha)}_{j,\ell}) \text{-signature is different from the } (t_{\alpha-1}, M^{(t_{\alpha-1})}_{j,\ell}) \text{-signature} \}$.

If $|\Phi| \geq h_0(\ell)$, then there exists $t_{\alpha^*} \in \Phi$ such that the $(t_{\alpha^*}, M^{(t_{\alpha^*})}_{j,\ell})$-signature is lexicographically smaller than the $(t, M^{(t)}_{j,\ell})$-signature.

Proof. Suppose to the contrary that for every $t_\alpha \in \Phi$, the $(t_\alpha, M^{(t_\alpha)}_{j,\ell})$-signature is not lexicographically smaller than the $(t, M^{(t)}_{j,\ell})$-signature.

For every $t'' \in V(T) - \{ t \}$, let $p_{t''}$ be the parent of $t''$. Let

$$\Phi' := \{ t'' \in V(T) : V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t), \text{ either } M^{(t''\alpha)}_{j,\ell} \neq M^{(p_{t''})}_{j,\ell} \text{ and the } (p_{t''}, M^{(p_{t''})}_{j,\ell}) \text{-signature is different from the } (t'', M^{(t'')}_{j,\ell}) \text{-signature} \}.$$ 

Note that for every $t_\alpha \in \Phi$, there exists $t'' \in \Phi'$ such that $V(T_{t_{\alpha-1}}) \subseteq V(T_{t''}) \subseteq V(T_{t_{\alpha-1}}) - \{ t_{\alpha-1} \}$. Hence $|\Phi'| \geq |\Phi| \geq h_0(\ell)$.

Since for every $t'' \in V(T)$ with $V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t)$, $M^{(t''\alpha)}_{j,\ell} = M^{(t_{\alpha-1})}_{j,\ell}$, and for every $\alpha \in [0, \ell - 1]$, $M^{(t''\alpha)}_{j,\ell} = M^{(t_{\alpha-1})}_{j,\ell}$ and the $(t'', M^{(t''\alpha)}_{j,\ell})$-signature equals the $(t, M^{(t_{\alpha-1})}_{j,\ell})$-signature, we know that:

(i) for every $t'' \in V(T)$ with $V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t)$, every $\beta \in [\ell - 1]$, and $\gamma \in [|V(G)|]$, the $(\beta, \gamma)$-entry of the $(t'', M^{(t''\alpha)}_{j,\ell})$-signature equals the $(\beta, \gamma)$-entry of the $(t, M^{(t_{\alpha-1})}_{j,\ell})$-signature,

(ii) for every $\beta \in [\ell - 1]$ and $t_\alpha \in \Phi$, $A_{L(t_{\alpha-1}, t, 0)}(V(M^{(t_{\alpha})}_{j,\beta})) \cap X_{V(T_{t_{\alpha}})} - X_{t_{\alpha}} = A_{L(t_{\alpha-1}, t, 0)}(V(M^{(t_{\alpha-1})}_{j,\beta})) \cap X_{V(T_t)} - X_t \subseteq X_{V(T_{t''})} - X_{t''}$,

(iii) for every $\beta \in [\ell - 1]$ and every $t'' \in V(T_{t''}) \subseteq V(T_t)$, $M^{(t''\beta)}_{j,\ell} = M^{(t\beta)}_{j,\ell}$.

(iv) for every $t'' \in V(T_{t''}) \subseteq V(T_t)$, $\sigma(M^{(t''\alpha)}_{j,\ell}) = \sigma(M^{(t_{\alpha-1})}_{j,\ell})$,

(v) for every $t'' \in V(T_{t''}) \subseteq V(T_t) - \{ t \}$ and every $\gamma \in [|V(G)|]$, if the $(\ell, \gamma)$-entry of the $(t'', M^{(t''\alpha)}_{j,\ell})$-signature is greater than the $(\ell, \gamma)$-entry of $(p_{t''}, M^{(p_{t''})}_{j,\ell})$-signature, then the $(\ell, \gamma)$-entry of the $(p_{t''}, M^{(p_{t''})}_{j,\ell})$-signature equals 0, and there exist $e_{t''\gamma} \in (E_{j,t''}^{(p_{t''},\gamma,\omega_0+1)} - E_{j,t''}^{(p_{t''},\gamma,0)}) \cap E(M^{(t''\alpha)}_{j,\ell}) - E(M^{(p_{t''})}_{j,\ell})$ and $c_{t''\gamma} \in \partial T_{p_{t''}\beta}$ such that $c_{t''\gamma}$ is a witness for $e_{t''\gamma}$.

For every $t'' \in V(T_{t''}) \subseteq V(T_t) - \{ t \}$ and every $\gamma \in [|V(G)|]$, in which $e_{t''\gamma}$ is defined, let $\alpha_{t''} \in [\beta^*]$, be the element such that $V(T_{t_{\alpha_{t''}-1}}) \subseteq V(T_{t''}) \subseteq V(T_{t_{\alpha_{t''}-1}}) - \{ t_{\alpha_{t''}-1} \}$, so $e_{t''\gamma} \in V(T_{t_{\alpha_{t''}-1}}) - V(T_{t_{\alpha_{t''}-1}}) - \{ t_{\alpha_{t''}-1} \}$, and hence by (ii), $z_{t''\gamma} \in \partial T_{t_{\alpha_{t''}-1}}$, which belongs to the path in $T$ between $p_{t''\gamma}$ and $t_{\alpha_{t''}}$ and $\ell \neq 0$.

(b) by the $\sigma$-value condition for $e_{t''\gamma} \in E_{j,t''}^{(p_{t''},\gamma,\omega_0+1)}$, and hence by (iii), $e_{t''\gamma} \notin E_{j,t''}^{(p_{t''},\gamma,\omega_0+1)} - E_{j,t''}^{(p_{t''},\gamma,0)}$.

Therefore, $e_{t''\gamma} \in (E_{j,t''}^{(p_{t''},\gamma,\omega_0+1)} - E_{j,t''}^{(p_{t''},\gamma,0)}) \cap E(M^{(t''\alpha)}_{j,\ell}) - E(M^{(p_{t''})}_{j,\ell})$.

For every $t'' \in V(T)$ with $V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t)$, every $\beta \in [0, \ell]$ and every $\gamma \in [|V(G)|]$, if the $(\beta, \gamma)$-entry $a$ of the $(t'', M^{(t''\alpha)}_{j,\ell})$-signature is nonzero, then we define the rank of $a$ as follows:

- If $t'' = t$, then the rank of $a$ is defined to be 0.
If \( t'' \neq t \) and the \((\beta, \gamma)\)-entry of the \((p_{t''}, M_{j, \ell}^{(p_{t''})})\)-signature is zero, then define the rank of \( a \) to be \( m + 1 \), where \( m \) is the maximum rank of the \((\beta', \gamma')\)-entries of the \((p_{t''}, M_{j, \ell}^{(p_{t''})})\)-signature among all \( \beta' \in [0, \ell] \) and \( \gamma' \in [[V(G)]] \), and define \( t_{\beta, \gamma} \) to be the node \( t'' \) of \( T \) of largest height such that \( A_{L(t_{\beta, \gamma} - 1, 0)}(V(M)) \cap X_{V(T_{t_{\beta, \gamma}})} - X_{t''} \subseteq X_{V(T_{t_{\beta, \gamma}})} \) for every monochromatic component \( M \) that defines a nonzero \((\beta'', \gamma'')\)-entry with rank at most \( m \) in the \((t'', M_{j, \ell}^{(t'')})\)-signature over all \( \beta'' \in [0, \ell] \) and \( \gamma'' \in [[V(G)]].

If \( t'' \neq t \) and the \((\beta, \gamma)\)-entry of the \((p_{t''}, M_{j, \ell}^{(p_{t''})})\)-signature is nonzero, then define the rank of \( a \) to be the rank of the \((\beta, \gamma)\)-entry of the \((p_{t''}, M_{j, \ell}^{(p_{t''})})\)-signature.

Recall that for every \( t'' \) with \( V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t) - \{t\} \) and every \( \gamma \in [[V(G)]] \) in which \( e_{t'', \gamma} \) is defined, \( e_{t'', \gamma} \in (E_{j, t''}(w_{0}) - E_{j, t''}(0)) \cap E(M_{j, \ell}^{(t''}) - E(M_{j, \ell}^{(p_{t''})}) \). So if we denote the monochromatic component corresponding to the \((\ell, \gamma)\)-entry of the \((t'', M_{j, \ell}^{(t''})\)-signature by \( M' \),

then \( A_{L_{t'', \gamma}(V(T_{t''}))(\ell, \gamma)}(V(M')) \cap X_{V(T_{t_{\beta, \gamma}})} - X_{t''} \neq \emptyset \), and there does not exist \( O \subseteq V(T_{z_{\ell, \gamma}}) - \{z_{\ell, \gamma}\} \) with \( |O| = \xi(\ell) \) such that \( O = \{o_1, o_2, \ldots, o_{|O|}\} \), where \( o_{\alpha+1} \in V(T_{o_{\alpha}}) - \{o_{\alpha}\} \) and \( X_{o_{\alpha+1}} \cap X_{o_{\alpha}} - X_{z_{\ell, \gamma}} = \emptyset \) for every \( \alpha \in [|O| - 1] \), \( A_{L_{t'', \gamma}(V(T_{t''}))(\ell, \gamma)}(V(M')) \cap X_{V(T_{z_{\ell, \gamma}})} - X_{t''} \subseteq X_{V(T_{z_{\ell, \gamma}})} - (X_{V(T_{o_{\alpha}})} - X_{o_{\alpha}}) \) and \( A_{L_{t'', \gamma}(V(T_{t''}))(\ell, \gamma)}(V(M')) \cap X_{V(T_{z_{\ell, \gamma}})} - X_{t''} \subseteq X_{V(T_{o_{\alpha}})} - X_{o_{\alpha}} \) for every monochromatic component \( M'' \) corresponding to a nonzero \((\beta'', \gamma'')\)-entry (for some \( \beta'' \in [0, \ell] \) and \( \gamma'' \in [[V(G)]]) \) of the \((p_{t''}, M_{j, \ell}^{(p_{t''})})\)-signature with rank smaller than the rank of the \((\ell, \gamma)\)-entry of the \((t'', M_{j, \ell}^{(p_{t''})})\)-signature. So if \( t^* \) is a node in \( V(T_{t''}) \) for some \( t'' \) with \( V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t) - \{t\} \) such that the \((\ell, \gamma)\)-entry of the \((t', M_{j, \ell}^{(t')})\)-signature is 0, then for every \( \alpha \in [w_0 + 1] \) and every \((z_{\ell, \gamma}, r_{\ell, \gamma}, \alpha, \ldots, q_m)\) for some \( m \in N_0 \) in \((T, X|I_{\ell})\), \(|\{\alpha' \in [m] : q_{\alpha'} \in V(T_{t''})\}| \leq \xi(\ell) - 1 \) due to the non-existence of the set \( O \). By Lemma 20, the \( \xi(\ell)\)-cap of the \((t^*, r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\) is lexicographically smaller than the \( \xi(\ell)\)-cap of the \((z_{\ell, \gamma}, r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\). Note that since \( z_{\ell, \gamma} \in V(T_{t''}) \), the \((t'', r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\) is not lexicographically smaller than the \((z_{\ell, \gamma}, r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\) by Lemma 19. Hence the \( \xi(\ell)\)-cap of the \((t^*, r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\) is lexicographically smaller than the \( \xi(\ell)\)-cap of the \((t'', r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\).

Note that (ii) implies that it is impossible that there exists \( t'' \in V(T) \) with \( V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t) \) such that there exist no \( \beta'' \in [0, \ell] \) and \( \gamma'' \in [[V(G)]]) \) such that the \((\beta'', \gamma'')\)-entry in the \((t'', M_{j, \ell}^{(t'')})\)-signature is nonzero and is with rank 0.

For every \( t'' \in V(T) \) with \( V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t) \), define \( a_{t''} \) to be an infinite sequence \((a_{t''}, a_{t''}, 1, \ldots)\) such that for every \( \alpha \in N_0 \), \( a_{t''} = (a_{t''}, 1, a_{t''}, 2) \), where:

- \( a_{t''}, \alpha, 1 \) is the sum of all \((\beta, \gamma)\)-entries with rank \( \alpha \) in the \((t'', M_{j, \ell}^{(t'')})\)-signature over all \( \beta \in [0, \ell] \) and \( \gamma \in [[V(G)]], \) and

- if \( a_{t''}, \alpha, 1 = 0 \), then \( a_{t''}, \alpha, 2 \) is the zero sequence of length \( w_0 \); otherwise, if \( z_{\ell, \gamma} \in V(T_{t''}) \), then define \( a_{t''}, \alpha, 2 \) to be the \( \xi(\ell)\)-cap of the \((z_{\ell, \gamma}, r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\); otherwise, define \( a_{t''}, \alpha, 2 \) to be the \( \xi(\ell)\)-cap of the \((t'', r_{\ell, \gamma})\)-gap in \((T, X|I_{\ell})\).

By Claim 23.23, for every \( \alpha \in N \), \( a_{t''}, 0, a_{t''}, 1, [0, \ell_{t''} \times [0, \xi(\ell)]]) \rightarrow [0, (\ell_{t''} + 1) \xi(\ell) + 1] \) such that \( \ell \) maps zero sequences to zero, and for all elements \( x, y \) in the domain of \( \ell \), \( \ell(x) < \ell(y) \) if and only if \( x \) that is lexicographically smaller than \( y \).

For every \( t'' \in V(T) \) with \( V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t) \), define \( b_{t''} \) to be an infinite sequence \((b_{t''}, 1, b_{t''}, 2, \ldots)\) such that for every \( \alpha \in N \), \( b_{t''}, \alpha = \ell(a_{t''}, \alpha) \).
Denote $\Phi'$ by $\{\phi'_1, \phi'_2, \ldots, \phi'_{|\Phi'|}\}$, where for every $\alpha \in [1, \Phi']$, $\phi'_\alpha + 1 \in V(T_{\phi'_\alpha}) - \{\phi'_\alpha\}$. For every $\alpha \in [1, \Phi']$, let $c_\alpha := b_{\phi'_\alpha}$; for every $\alpha \in \mathbb{N} - [1, \Phi']$, let $c_\alpha$ be the infinite sequence. Then $(c_1, c_2, \ldots)$ is a $((\ell_\eta + 1)(\xi(\ell) + 1)^{\omega+1}\eta_0)$-evolution by Claim 23.23.

Since $|\Phi'| \geq h_0(\ell)$, the first entry of $c_{h_0(\ell)} = 0$ by Lemma 18. Therefore, there do not exist $\beta' \in [0, \ell]$ and $\gamma' \in [|V(G)|]$ such that the $(\beta', \gamma')$-entry in the $(\phi'_{h_0}, M_{j,\ell}^{(\phi'_{h_0})})$-signature is nonzero and with rank 0, a contradiction. This proves the claim.

\begin{claim}
Let $j \in [|V| - 1]$. Let $t$ and $t'$ be nodes of $T$. Let $\ell \in [0, w_0 - 1]$. Assume that for every $t'' \in V(T)$ with $V(T_{t'}) \subseteq V(T_{t''}) \subseteq V(T_t)$,
\begin{itemize}
  \item either $M_{j,\ell}^{(t'') \alpha} \supseteq M_{j,\ell}^{(t)}$, or there exists no monochromatic $E_{j,\ell'}$-pseudocomponent $M$ in $G[Y^{(t''\alpha, 1,0)}]$ such that $M \supseteq M_{j,\ell}^{(t)}$ and $A_{L^{(t''\alpha, 1,0)}}(V(M)) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset$, and
  \item for every $\alpha \in [0, \ell - 1]$, $M_{j,\alpha}^{(t'') \alpha} = M_{j,\alpha}^{(t)}$ and the $(t'', M_{j,\alpha}^{(t''\alpha)})$-signature equals the $(t, M_{j,\alpha}^{(t\alpha)})$-signature.
\end{itemize}
Assume there exist $\alpha^* \in \mathbb{N}$ and nodes $t_0, t_1, \ldots, t_{\beta^*}$ such that $t_0 = t$, $t_{\beta^*} = t'$, and for every $\alpha \in [\beta^*]$, $t_\alpha \in V(T_{j,\alpha^* - 1})$. Let $\Phi = \{t_\alpha : \alpha \in [\beta^*]\}$, either:
\begin{itemize}
  \item $M_{j,\ell}^{(t_\alpha\alpha)} \supseteq M_{j,\ell}^{(t)}$, and either $M_{j,\alpha}^{(t_\alpha\alpha)} \neq M_{j,\alpha}^{(t_{\alpha-1}\alpha)}$ or the $(t_\alpha, M_{j,\alpha}^{(t_\alpha\alpha)})$-signature is different from the $(t_{\alpha-1}, M_{j,\alpha}^{(t_{\alpha-1}\alpha)})$-signature, or
  \item there exists no monochromatic $E_{j,\ell'}$-pseudocomponent $M$ in $G[Y^{(t_{\alpha\alpha}, 1,0)}]$ such that $M \supseteq M_{j,\ell}^{(t)}$ and $A_{L^{(t_{\alpha\alpha}, 1,0)}}(V(M)) \cap X_{V(T_{t_{\alpha\alpha}})} - X_{t_{\alpha\alpha}} \neq \emptyset$.
\end{itemize}
If $|\Phi| \geq \eta_0 \cdot h_0(\ell)$, then there exists $t_{\alpha^*} \in \Phi$ such that there exists no monochromatic $E_{j,\ell'}$-pseudocomponent $M$ in $G[Y^{(t_{\alpha^*\alpha}, 1,0)}]$ such that $M \supseteq M_{j,\ell}^{(t)}$ and $A_{L^{(t_{\alpha^*\alpha}, 1,0)}}(V(M)) \cap X_{V(T_{t_{\alpha^*\alpha}})} - X_{t_{\alpha^*\alpha}} \neq \emptyset$.

\begin{proof}
We may assume that $M_{j,\ell}^{(t_\alpha\alpha)} \supseteq M_{j,\ell}^{(t)}$ for every $t_\alpha \in \Phi$, for otherwise we are done. For every $\beta \in [\eta_0]$, let $\Phi_\beta := \{t_{(\beta - 1)h_0(\ell) + \gamma} : \gamma \in [h_0(\ell)]\}$. Note that $|\Phi_\beta| = h_0(\ell)$ for every $\beta \in [\eta_0]$. By Claim 23.24, there exist $t^*_{\beta} \in [\Phi_1]$ such that the $(t^*_{\beta}, M_{j,\ell}^{(t^*_{\beta})})$-signature is lexicographically smaller than the $(t, M_{j,\ell}^{(t)})$-signature. For each $\beta \in [2, \eta_0]$, there exist $t^*_{\beta} \in (\bigcup_{\beta = 1}^{\eta_0} \Phi_\beta) \cap V(T_{t_{(\beta - 1)h_0(\ell)}}) - \{t^*_{\beta - 1}\}$ such that the $(t^*_{\beta}, M_{j,\ell}^{(t^*_{\beta})})$-signature is lexicographically smaller than the $(t^*_{\beta - 1}, M_{j,\ell}^{(t^*_{\beta - 1})})$-signature. Since for every $t'' \in V(T)$ with $V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t)$ and $\alpha \in [0, \ell - 1]$, $M_{j,\alpha}^{(t''\alpha)} = M_{j,\alpha}^{(t)}$ and the $(t'', M_{j,\alpha}^{(t''\alpha)})$-signature equals the $(t, M_{j,\alpha}^{(t\alpha)})$-signature, we know for every $\beta \in [2, \eta_0]$, that $(\gamma, \gamma')$-entry of the $(t^*_{\beta}, M_{j,\ell}^{(t^*_{\beta})})$-signature is lexicographically smaller than the $(t^*_{\beta - 1}, M_{j,\ell}^{(t^*_{\beta - 1})})$-signature. And $\gamma' \in [|V(G)|]$, the $(\gamma, \gamma')$-entry of the $(t^*_{\beta}, M_{j,\ell}^{(t^*_{\beta})})$-signature equals the $(\gamma, \gamma')$-entry of the $(t^*_{\beta - 1}, M_{j,\ell}^{(t^*_{\beta - 1})})$-signature. In addition, by the definition of the ordering $\sigma$, for every $\beta \in [2, \eta_0]$, $\gamma \in [0, \sigma(M_{j,\ell}^{(t)} - 1] \text{ and } \gamma' \in [|V(G)|]$, if the $(\gamma, \gamma')$-entry of the $(t^*_{\beta}, M_{j,\ell}^{(t^*_{\beta})})$-signature is zero, then the $(\gamma, \gamma')$-entry of the $(t^*_{\beta - 1}, M_{j,\ell}^{(t^*_{\beta - 1})})$-signature is zero. Therefore, by Claim 23.23, there exists $\beta^* \in [\eta_0]$ such that for every $\gamma \in [|V(G)|]$, the $(\sigma(M_{j,\ell}^{(t)}), \gamma)$-entry of the $(t_{\beta^*}, M_{j,\ell}^{(t_{\beta^*})})$-signature is zero. So $M_{j,\ell}^{(t_{\beta^*\alpha})} \not\supseteq M_{j,\ell}^{(t)}$, a contradiction.
\end{proof}
Claim 23.26. Let \( j \in \{\|V\| - 1\} \). Let \( t \) and \( t' \) be nodes of \( T \). Let \( \ell \in [0, w_0 - 1] \). For every \( t'' \in V(T) \) with \( V(T_\ell) \subseteq V(T_{t''}) \subseteq V(T_t) \), let

\[
C_{t''} := \{ \alpha \in [0, \ell] : \text{there exists a monochromatic } E_{j, t''} \text{-pseudocomponent } M \text{ in } G[Y^{(i, t'')_\ell, 1, 0}] \\
\text{such that } M \supseteq M_{j, \alpha}^{(t)} \text{ and } A_{E_{j, t''}, 1, 0}(V(M)) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset \}.
\]

Assume that for every \( t'' \in V(T) - \{t\} \) with \( V(T_\ell) \subseteq V(T_{t''}) \subseteq V(T_t) \) and every \( \ell_0 \in [0, \ell] \), if \( M_1 \) is a monochromatic \( E_{j, t''} \)-pseudocomponent in \( G[Y^{(i, t'')_\ell, 1, 0}] \) with \( M_1 \supseteq M_{j, \alpha}^{(t)} \) and \( M_2 \) is a monochromatic \( E_{j, t''} \)-pseudocomponent in \( G[Y^{(i, t'')_\ell, 1, 0}] \) intersecting \( X_{t''} \) with \( M_2 \supseteq M_{j, \alpha}^{(t)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), then \( M_2 \supseteq M_{j, \alpha'}^{(t)} \) for some \( \alpha' \leq \ell_0 \). Assume there exist \( \beta^* \in \mathbb{N} \) and nodes \( t_0, t_1, \ldots, t_{\beta^*} \) such that \( t_0 = t, t_{\beta^*} = t', \) and for every \( \alpha \in [\beta^*], t_\alpha \in V(T_{j, t_{\alpha} - 1}). \) For every \( \alpha \in [0, \beta^*] \) and \( \alpha' \in C_{t_\alpha}, \) let \( k_{\alpha'} \) be the index such that \( M_{j, k_{\alpha'}}^{(t_\alpha)} \supseteq M_{j, \alpha'}^{(t)} \). Let \( \Phi = \{ t_\alpha : \alpha \in [\beta^*] \} \) such that either:

- \( |C_{t_\alpha}| < |C_{t_{\alpha - 1}}|, \) or
- \( |C_{t_\alpha}| = |C_{t_{\alpha - 1}}|, \) and there exists \( \alpha' \in C_{t_\alpha} \) such that either \( M_{j, k_{\alpha'}}^{(t_\alpha)} \neq M_{j, k_{\alpha'}}^{(t_{\alpha - 1})} \) or the \((t_\alpha, M_{j, k_{\alpha'}}^{(t_\alpha)})\)-signature is different from the \((t_{\alpha - 1}, M_{j, k_{\alpha'}}^{(t_{\alpha - 1})})\)-signature.

If \( |\Phi| \geq h(\ell) \), then there exists \( t_{\alpha'} \in \Phi \) such that \( \ell \notin C_{t_{\alpha'}}. \)

Proof. We prove this claim by induction on \( \ell \). The case \( \ell = 0 \) follows from Claim 23.25 since \( h(0) \geq h_0 h_0(0) \).

Now assume that \( \ell \geq 1 \). Suppose to the contrary that \( \ell \in C_{t_\alpha} \) for every \( t_\alpha \in \Phi \).

Since \( |C_{t_{\alpha}}| : t_\alpha \in \Phi \) is non-increasing on \( \alpha \), there exists \( \Phi_0 = \{ \phi_{0,1}, \ldots, \phi_{0,|\Phi_0|} \} \subseteq \Phi \) with \( |\Phi_0| \geq |\Phi|/(\ell + 1) \) such that there exists \( N_0 \in \{\beta^*\} \) such that for every \( \alpha \in |\Phi_0|, \phi_{0,\alpha} = t_{N_0 + \alpha} \) and \( |C_{\phi_{0,\alpha}}| = |C_{t_{\alpha}}| \) since \( \ell \in C_{t_\alpha} \) for every \( t_\alpha \in \Phi, C_{\phi_{0,\alpha}} \neq \emptyset. \)

For every \( \alpha \in |\Phi_0| - 1 \), let \( R_\alpha = \{ \beta \in C_{\phi_{0,\alpha}} : M_{j, \beta}^{(t_\alpha)} = M_{j, \beta}^{(t_{\alpha - 1})} \) and the \((t_\alpha, M_{j, \beta})\)-signature equals the \((t_{\alpha - 1}, M_{j, \beta}^{(t_{\alpha - 1})})\)-signature. Since \( C_{\phi_{0,\alpha}} \neq \emptyset \), by the definition of \( \Phi, R_\alpha \neq C_{\phi_{0,\alpha}} \) for any \( \alpha \in |\Phi_0|. \)

Recall for for every \( t'' \in V(T) - \{t\} \) with \( V(T_\ell) \subseteq V(T_{t''}) \subseteq V(T_t) \) and every \( \ell_0 \in [0, \ell] \), if \( M_1 \) is a monochromatic \( E_{j, t''} \)-pseudocomponent in \( G[Y^{(i, t'')_\ell, 1, 0}] \) with \( M_1 \supseteq M_{j, \alpha}^{(t)} \) and \( M_2 \) is a monochromatic \( E_{j, t''} \)-pseudocomponent in \( G[Y^{(i, t'')_\ell, 1, 0}] \) with \( M_2 \supseteq M_{j, \alpha}^{(t)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), then \( M_2 \supseteq M_{j, \alpha'}^{(t)} \) for some \( \alpha' \leq \ell_0 \). So for every \( t'' \in V(T) - \{t\} \) with \( V(T_\ell) \subseteq V(T_{t''}) \subseteq V(T_t) \), there exists a linear order \( \pi_{t''} \) on \( C_{t''} \) such that \( \pi_{t''}(x) < \pi_{t''}(y) \) for every \( x, y \in C_{t''} \) with \( k_x < k_y \). Note that the element of \( C_{t''} \) with smallest \( \pi_{t''}\)-value must be \( \min C_{t''} \).

Since \( \ell \geq 1 \), by the induction hypothesis, if there exist at least \( h(0) \) different indices \( \alpha \in |\Phi_0| \) such that \( \min C_{\phi_{0,\alpha}} \notin R_\alpha \), then \( \min C_{\phi_{0,1}} \cap C_{\phi_{0,1} - C_{\Phi_0,|\Phi_0|}}, \) a contradiction. So there exists \( \Phi_1 = \{ \phi_{1,1}, \ldots, \phi_{1,|\Phi_1|} \} \subseteq \Phi_0 \) with \( |\Phi_1| \geq |\Phi_0| / \frac{h(0)}{h(0)} \geq \frac{|\Phi|}{(\ell + 1)h(\ell - 1)} \cdot \frac{|\Phi|}{(\ell + 1)h(\ell - 1)} \) such that there exists \( N_1 \in [\beta^*] \) such that for every \( \alpha \in |\Phi_1|, \phi_{1,\alpha} = t_{N_1 + \alpha} \) and \( \min C_{\phi_{1,\alpha}} \in R_{N_1 + \alpha}. \) Since \( \min C_{\phi_{1,1}} \in R_{N_1 + \alpha} \) for every \( \alpha \in |\Phi_1| \), we know if \( |C_{\phi_{1,1}}| \geq 2 \), then for every \( t'' \in \Phi_1 \), the second-smallest element in \( C_{t''} \) with respect to \( \pi_{t''} \) is the second-smallest element in \( C_{\phi_{1,1}} \) with respect to \( \pi_{\phi_{1,1}} \).

Hence there exists a maximum \( \ell^* \in [1, |C_{\phi_{1,1}}|] \) such that there exist \( \Phi_0 \supseteq \Phi_1 \supseteq \cdots \supseteq \Phi_{\ell^*} \) such that \( |\Phi_{\ell^*}| \geq \frac{|\Phi|}{(\ell + 1)h(\ell - 1)^{\ell^* - 1}} \) and there exists \( N_{\ell^*} \in [\beta^*] \) such that:

- for every \( \alpha \in |\Phi_{\ell^*}|, \phi_{\ell^*, \alpha} = t_{N_{\ell^*} + \alpha} \) and \( R_{N_{\ell^*} + \alpha} \) contains the \( \beta \)-th smallest element of \( C_{\phi_{\ell^*}} \) for every \( \beta \in [\ell^* - 1], \) and
• if $|C_{\phi_0,1}| \geq \ell^* + 1$, then for every $t'' \in \Phi_{t^*}$ and $\alpha \in [0, \ell^*]$, the $(\alpha + 1)$-th smallest element in $C_{\ell^*}$ with respect to $\pi_{\ell^*}$ is the $(\alpha + 1)$-th smallest element in $C_{\phi_0,1}$ with respect to $\pi_{\phi_*,1}$.

Since $R_\alpha \neq C_{\phi_0,1}$ for any $\alpha \in [[\Phi_1]]$, so $\ell^* \leq |C_{\phi_0,1}| - 1 \leq \ell - 1$.

By the maximality of $\ell^*$, there exist at least $h(\ell - 1) \geq \eta_\ell h_\ell(\ell)$ different indices $\alpha \in [[\Phi_{t^*}]]$ such that the $(\ell^* + 1)$-th smallest element of $C_{\phi_*,1}$ with respect to $\pi_{\phi_*,\alpha}$ is not in $R_{N_{t^*} + \alpha}$. Since $R_{N_{t^*} + \alpha}$ contains the $\beta$-th smallest element of $C_{\phi_0,1}$ with respect to $\pi_{\phi_*,1}$ for every $\beta \in [\ell^* - 1]$, by Claim 23.25, there exists $t^* \in \Phi_{t^*}$ such that the $\ell^*$-th smallest element of $C_{\phi_0,1}$ with respect to $\pi_{\phi_*,1}$ belongs to $C_{\phi_0,1} - C_{\phi_*,1} = \emptyset$, a contradiction. \[\square\]

**Claim 23.27.** Let $j \in [[\mathcal{V}] - 1]$. Let $i \in \mathbb{N}_0$, and let $t \in V(T)$ a node of $T$ of height $i$. Assume that $M_{j,0}^{(i)}$ is of color $k + 1$ for some $k \in [0, s + 1]$, and the s-segment $S$ of level $k + 1$ containing $V(M_{j,0}^{(i)})$ intersecting $I_j$. Assume that for every $t' \in V(T_i)$, if there exists a monochromatic $E_{j}^{(t',-1,a,0)}$-pseudocomponent $M$ in $G[Y^{(t',-1,0)}]$ with $M \supseteq M_{j,0}^{(i)}$ such that that $V(M) \cap X_{t'} \neq \emptyset$ and $A_{L(t',-1,0)}(V(M)) \cap X_{V(T_{t'})} - X_{t'} \neq \emptyset$, then $M = M_{j,0}^{(i)}$.

If $x$ is a vertex in $X_{V(T_i)}$ such that there exists a monochromatic path $P$ with respect to $c$ contained in $G[X_{V(T_i)}]$, then $x \in Y^{(i,-1,0)}$.

**Proof.** Suppose to the contrary that $x \not\in Y^{(i,-1,0)}$.

Since $X_i \subseteq Y^{(i,-1,0)}$, $x \not\in X_i$. Since $P$ is a monochromatic path in $G[X_{V(T_i)}]$ with respect to $c$ from $x$ to $V(M_{j,0}^{(i)}) \cap X_i \subseteq Y^{(i,-1,0)}$, there exist $u \in V(P) \cap Y^{(i,-1,0)}$ and $v \in N_P(u) - Y^{(i,-1,0)}$ such that there exists a monochromatic path in $G[X_{V(T_i)}] \cap Y^{(i,-1,0)}$ with respect to $c$ from $u$ to $V(M_{j,0}^{(i)}) \cap X_i$. So $u \in W_0^{(i,-1,0)}$. Since $c(u) = c(v)$ and $v \not\in Y^{(i,-1,0)}$, $v \not\in Z_i$. So $v \in I_j - X_{V(T_i)}$. Hence there exist $t_0, t_1, \ldots, t_q \in V(T)$ for some $q \in \mathbb{N}$, where $t_0 = t$, such that $t_{q+1} \in \partial T_{i,t'}$ for every $q' \in [0, q - 1]$, and $v \in X_{V(T_{t_i})} - X_{V(T_{t_{i+1}})}$. So $v \in X_{V(T_{t_i})} - X_{t_q}$ and $u \in X_{V(T_{t_i})}$.

Since $u \in X_{V(T_{t_i})}$ and $P$ is from $X_i$ to $u$, for every $q' \in [q]$, there exists a monochromatic $E_{j,t'}$-pseudocomponent $M' = G[Y^{(it',-1,0)}]$ with respect to $c$ intersecting $X_{t_{q'}}$ such that $M' \supseteq M_{j,0}^{(i)}$. Since $v \in I_j \cap X_{V(T_{t_i})} - X_{t_q}$, $v \not\in Y^{(it',-1,0)}$. For every $q' \in [q]$, since $u \in V(M_{j,0}^{(i)}) \subseteq V(M')$, $v \in A_{L^{(it',-1,0)}}(V(M'))$, and hence there exists $k_{q'} \in [0, w_0 - 1]$ such that $M_{q'} \subseteq M_{j,k_{q'}}$. By assumption, $k_{q'} = 0$ for each $q' \in [q]$. In particular, $u \in V(M_{j,0}^{(i)}) \subseteq V(M_{j,0}^{(i)})$. Then $u \in W_0^{(i,-1,0)}$. Since $v \in I_j \cap X_{V(T_{t_i})} - X_{V(T_{t_{i+1}})}$, $v \not\in Y^{(it',-1,0)}$ and $c(u) \not\in L^{(it',0,0)}(v)$, a contradiction. \[\square\]

**Claim 23.28.** Let $j \in [[\mathcal{V}] - 1]$. Let $t^* \in V(T)$ and $t \in V(T_{j,t^*}) - \{t^*\}$. Let $z \in \partial T_{j,t^*}$ such that $z$ is a witness for $X_z \cap I_j \subseteq \bigcup_{\ell = 1}^{[[\mathcal{V}]]} W_3^{(it*,\ell)}$. Assume that the following hold:

- There exists a monochromatic path $P_z$ in $G[X_{V(T_z)}]$ with respect to $c$ internally disjoint from $X_z$ with distinct ends $u_z \in X_z$ and $v_z \in X_z$.
- $\{u_z, v_z\} \not\subseteq E_{j,t}$.
- The s-segment containing $V(P_z)$ whose level equals the color of $P_z$ belongs to $S_z^0$.

- There exist the minimum $k^* \in [0, w_0 - 1]$ and a monochromatic $E_{j,t}^{(it^*,k^*)}$-pseudocomponent in $G[Y^{(it^*,|V(T)|+1,s+2)}]$ with respect to $c$ such that:

  - $\sigma(M_{k^*})$ is the $(k^* + 1)$-th smallest among all monochromatic $E_{j,t}^{(it^*,k^*)}$-pseudocomponents in $G[Y^{(it^*,|V(T)|+1,s+2)}]$ with respect to $c$ intersecting $X_{t^*}$ and contained in some s-segment in $S_z^0$ whose level equals its color, and
\[ V(M_k^*) \cap X_z \neq \emptyset \text{ and } A_{L(i^{\ast},[V(T)]+1,s+2)}(V(M_k^*)) \cap X_V(T_z) - X_z \neq \emptyset. \]

- There exists \( \ell^* \in [0, w_0 - 1] \) such that for every \( x \in \{u_z, v_z\} \), if \( M_k \) is the monochromatic \( E_{j,t}^{(i^{\ast},k^*,\ell^*)} \)-pseudocomponent in \( G[Y(i^{\ast},[V(T)]+1,s+2)] \) with respect to \( c \) containing \( x \), then:

\[ - A_{L(i^{\ast},[V(T)]+1,s+2)}(V(M_k)) \cap X_V(T_z) - X_z \neq \emptyset, \]
\[ - M_{u_z} \neq M_{v_z}, \text{ and } \sigma(M_{u_z}) < \sigma(M_{v_z}), \]
\[ - \sigma(M_{u_z}) \text{ is the } (\ell^*+1)-\text{th smallest among all monochromatic } E_{j,t}^{(i^{\ast},k^*,\ell^*)} \text{-pseudocomponents } M'' \text{ in } G[Y(i^{\ast},[V(T)]+1,s+2)] \text{ with respect to } c \text{ such that } V(M'') \cap X_z \neq \emptyset, \]
\[ A_{L(i^{\ast},[V(T)]+1,s+2)}(V(M'')) \cap X_V(T_z) - X_z \neq \emptyset, \text{ and the } s\text{-segment containing } V(M'') \text{ whose level equals the color of } M'' \text{ belongs to } S_j. \]

- There exists no node \( t'' \in V(T) \) with \( i_{t''} < i_{t^*} \) and \( T_{j,t''} \subseteq T_{j,t^*} \) such that there exists a monochromatic path in \( G[Y(i_{t''},[V(T)]+1,s+2)] \cap X_V(T_{j,t''}) \) from \( \bigcup_{i=-1}^{[V(T)]} W_{3}^{(i_{t''},\ell)} \) to \( \{u_z, v_z\} \).

Assume that if \( M_1 \) is a monochromatic \( E_{j,z}^{(i^{\ast},k^*,\ell^*)} \)-pseudocomponent in \( G[Y(i^{\ast},[V(T)]+1,s+2)] \) with \( M_1 \supseteq M_{u_z} \) and \( M_2 \) is a monochromatic \( E_{j,z}^{(i^{\ast},k^*,\ell^*)} \)-pseudocomponent in \( G[Y(i^{\ast},[V(T)]+1,s+2)] \) intersecting \( X_z \) with \( M_2 \supseteq M_{j,\alpha}^{(z)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), then \( M_2 \supseteq M_{j,\alpha}^{(z)} \) for some \( \alpha' \) with \( \sigma(M_{j,\alpha}') \leq \sigma(M_{u_z}) \). Assume that there exists \( k^*_z \in [0, \ell^*] \) such that \( M_{j,k^*_z}^{(z)} \supseteq M_{u_z} \). Assume that for every \( t' \in V(T_z) - \{z\} \) and every \( \ell_0 \in [0, k^*_z] \), if \( M_1 \) is a monochromatic \( E_{j,t'}^{(i^{\ast},k^*,\ell^*)} \)-pseudocomponent in \( G[Y(i^{\ast},[V(T)]+1,s+2)] \) with \( M_1 \supseteq M_{j,\ell_0}^{(z)} \) and \( M_2 \) is a monochromatic \( E_{j,t'}^{(i^{\ast},k^*,\ell^*)} \)-pseudocomponent in \( G[Y(i^{\ast},[V(T)]+1,s+2)] \) intersecting \( X_{t'} \) with \( M_2 \supseteq M_{j,\alpha}^{(z)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), then \( M_2 \supseteq M_{j,\alpha}^{(z)} \) for some \( \alpha' \) with \( \sigma(M_{j,\alpha}') \leq \sigma(M_{u_z}) \).

Then \( V(P_z) \subseteq Y(i^{\ast},[V(T)]+1,s+2). \)

**Proof.** Since \( t, z \in V(T_{j,t^*}) \), there exists no \( t'' \in V(T) \) with \( t'' \in V(T_{t''}) \) such that \( V(T_{t''}) \) contains exactly one of \( t, z \). Since \( \{u_z, v_z\} \notin E_{j,t}, \{u_z, v_z\} \notin E_{j,z}^{(i^{\ast},k^*,\ell^*)} \). So either:

- \( \ell^* = 0 \), or
- there exists \( O \subseteq V(T_z) - \{z\} \) with \( |O| = \xi(\ell^*) \) such that:

\[ - O = \{o_1, o_2, \ldots, o_{\xi(\ell^*)}\}, \text{ where } o_{\alpha+1} \in V(T_{o_\alpha}) - \{o_\alpha\} \text{ and } X_{o_{\alpha+1}} \cap X_{o_\alpha} \cap I_j - X_z = \emptyset \text{ for every } \alpha \in [\xi(\ell^*), 1], \]
\[ - A_{L(i^{\ast},[V(T)]+1,s+2)}(V(M_{o_\alpha})) \cap X_V(T_z) - X_z \subseteq X_V(T_z) - (X_V(T_{o_\alpha}) - X_{o_\alpha}), \text{ and} \]
\[ - A_{L(i^{\ast},[V(T)]+1,s+2)}(V(M'')) \cap X_V(T_z) - X_z \subseteq X_V(T_{o_{\xi(\ell^*)}}) - X_{o_{\xi(\ell^*)}} \text{ for every monochromatic } E_{j,z}^{(i^{\ast},k^*,\ell^*)} \text{-pseudocomponent } M'' \text{ in } G[Y(i^{\ast},[V(T)]+1,s+2)] \text{ with respect to } c \text{ contained in some } s\text{-segment in } S_j \text{ whose level equals its color with } \sigma(M'') \leq \sigma(M_{u_z}) \text{ such that } V(M'') \cap X_z \neq \emptyset \text{ and } A_{L(i^{\ast},[V(T)])}(V(M'')) \cap X_V(T_z) - X_z \neq \emptyset. \]

We first assume that \( \ell^* = 0 \). So \( E_{j,z}^{(i^{\ast},k^*,\ell^*)} = E_{j,z}^{(i^{\ast},k^*)} \). It implies that \( \sigma(M_{u_z}) \leq \sigma(M_{k^*}) \). Since \( V(M_{k^*}) \cap X_t \neq \emptyset, V(M_{u_z}) \cap X_t \neq \emptyset \). Hence the minimality of \( k^* \) implies that \( M_{k^*} = M_{u_z} \).

Since \( \ell^* = 0 \) and we assume that if \( M_1 \) is a monochromatic \( E_{j,z}^{(i^{\ast},k^*,\ell^*)} \)-pseudocomponent in \( G[Y(i^{\ast},[V(T)]+1,s+2)] \) with \( M_1 \supseteq M_{u_z} \) and \( M_2 \) is a monochromatic \( E_{j,z}^{(i^{\ast},k^*,\ell^*)} \)-pseudocomponent in \( G[Y(i^{\ast},[V(T)]+1,s+2)] \) with \( M_2 \supseteq M_{j,\alpha}^{(z)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), then \( M_2 \supseteq M_{j,\alpha}^{(z)} \) for some \( \alpha' \) with \( \sigma(M_{j,\alpha}') \leq \sigma(M_{u_z}) \), we know that \( M_{j,0}^{(z)} \supseteq M_{u_z} \). So \( k^*_z = 0 \). Since we assume that for every \( t' \in V(T_z) - \{z\} \) and
every $\ell_0 \in [0, k^*_z]$, if $M_1$ is a monochromatic $E_{j,\ell^*}$-pseudocomponent in $G[Y^{(i_{\ell^*},-1, 0)]$ with $M_1 \supseteq M^{(z)}_{j,\ell_0}$ and $M_2$ is a monochromatic $E_{j,\ell^*}$-pseudocomponent in $G[Y^{(i_{\ell^*},-1, 0)]$ with $M_2 \supseteq M^{(z)}_{j,\ell_0}$ for some $\alpha \in [0, w_0 - 1]$ such that $\sigma(M_2) \leq \sigma(M_1)$, then $M_2 \supseteq M^{(z)}_{j,\ell_0}$ for some $\alpha' \leq \ell_0$, we know for every $t' \in V(T_2) - \{z\}$, if there exists a monochromatic $E_{j,\ell^*}$-pseudocomponent $M_1$ in $G[Y^{(i_{\ell^*},-1, 0)]$ with $M_1 \supseteq M^{(z)}_{j,\ell_0}$ such that $V(M) \cap X_{t'} \neq \emptyset$ and $A_{L^{(i_{\ell^*},-1, 0)}}(V(M)) \cap X_{t_1} - t_2 \neq \emptyset$, then $M_1 = M^{(z)}_{j,\ell_0}$.

Therefore, by Claim 23.27, since for every vertex $v \in V(P_z)$, $P_z$ contains a monochromatic path with respect to $c$ from $v$ to $u_z \in V(M^{(z)}_{j,\ell_0}) \cap X_z$, $v \in V^{(i_{\ell^*},-1, 0)}$. Hence $V(P_z) \subseteq Y^{(i_{\ell^*},-1, 0)}$. Suppose that $V(P_z) \supseteq Y^{(i_{\ell^*},-1, 0)}$. Then there exists $u^* \in V(P_z) - Y^{(i_{\ell^*},-1, 0)}$ such that all internal vertices of the subpath of $P_z$ from $u$ to $u^*$ are in $Y^{(i_{\ell^*},-1, 0)}$. Recall that $M_{k^*} = M_{u_z}$, so by the process in Stages $(t^*, -1, *)$ and $(t^*, 0, *)$, if $u^* \in Z_{t^*}$, then $c(u^*) \notin L(i_{t^*}, Y^{(i_{\ell^*},-1, 0)}, t^*, \alpha) + 1$, a contradiction. So $u^* \notin Z_{t^*}$. Since $u^* \in X_{V(T_2)}$ and $z \in \partial T_{j,t}$, $u^* \in X_{V(T_2)} \cap I^*_j - X_z$. That is, $u^* \notin Z_{t^*}$ for every node $t'' \in V(T_{t^*})$ with $z \in V(T_{t''}) - \{t''\}$. So $u^* \notin Y^{(i_{\ell^*},-1, 0)}$, a contradiction. Therefore, $V(P_z) \subseteq Y^{(i_{\ell^*},-1, 0)} \subseteq Y^{(i_{\ell^*},-1, 0)}$.

Hence we may assume that $t^* \neq 0$. Hence there exists $O_0 \subseteq V(T_2) - \{z\}$ with $|O_0| \geq \xi(\ell^*)$ such that:

- $O_0 = \{o_{\ell_0, 1}, o_{\ell_0, 2}, \ldots, o_{\ell_0, |O_0|}\}$, where $o_{\ell_0, \alpha + 1} \in V(T_{o_{\ell_0, \alpha}}) - \{o_{\ell_0, \alpha}\}$ and $X_{o_{\ell_0, \alpha + 1}} \cap X_{o_{\ell_0, \alpha}} \cap I_j - X_z = \emptyset$ for every $\alpha \in [|O_0| - 1]$,
- $A_{L^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0})}, t^*, \alpha}}(V(M_{e_z})) \cap X_{V(T_2)} - X_z \subseteq X_{V(T_2)} - (X_{V(T_{o_{\ell_0, \alpha}})} - X_{o_{\ell_0, \alpha}})$, and
- $A_{L^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0})}, t^*, \alpha}}(V(M_{e_z})) \cap X_{V(T_2)} - X_z \subseteq X_{V(T_{o_{\ell_0, \alpha}})} - X_{o_{\ell_0, \alpha}}$ for every monochromatic $E_{j,\ell^*}$-pseudocomponent $M''$ in $G[Y^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0})}, t^*, \alpha} + 1, s + 2)]$ with respect to $c$ contained in some $s$-segment in $S^*_j$ whose level equals its color with $\sigma(M'') \leq \sigma(M_{u_z})$ such that $V(M'') \cap X_z \neq \emptyset$ and $A_{L^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0})}, t^*, \alpha}}(V(M'')) \cap X_{V(T_2)} - X_z \neq \emptyset$.

Note that $O_0$ exists as an O. We choose $O_0$ such that $|O_0|$ is as large as possible.

For every $t'' \in V(T_2)$, let $C_{t''} := \{\alpha \in [0, k^*_z] :$ there exists a monochromatic $E_{j,\ell^*}$-pseudocomponent $M$ in $G[Y^{(i_{\ell^*},-1, 0)}]$ such that $M \supseteq M^{(z)}_{j,\ell_0}$ and $A_{L^{(i_{\ell^*},-1, 0)}}(V(M)) \cap X_{V(T_{t''})} - X_{t''} \neq \emptyset\}$. Recall that we assume that for every $t'' \in V(T_2) - \{z\}$ and $\alpha \in [0, k^*_z]$, if $\alpha$ is the $(k^* + 1)$-th smallest element in $C_{t''}$, then $M_{j,\ell_0} \supseteq M^{(z)}_{j,\ell_0}$ for some $k^* \in [0, k^*]$. So there exist at most $k^*_z \leq \ell^*$ monochromatic $E_{j,\ell^*}$-pseudocomponents $M$ in $G[Y^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0)}, t^*, \alpha}} + 1, s + 2)]$ with $V(M) \cap X_z \neq \emptyset$ and with $A_{L^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0)}, t^*, \alpha}}(V(M)) \cap X_{V(T_2)} - X_z \neq \emptyset$ such that the $s$-segment containing $M$ whose level equals its color is in $S^*_j$ and $\sigma(M) < \sigma(M_{u_z})$; furthermore, each such $M$ contains some $E_{j,\ell^*}$-pseudocomponent $M'$ in $G[Y^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0)}, t^*, \alpha}} + 1, s + 2)]$ with $\sigma(M') < \sigma(M_{u_z})$.

For every $y \in \{u_z, v_z\}$ and node $t'' \in V(T)$ with $V(T_{o_{\ell^*}(t^*)}) \subseteq V(T_{t''}) \subseteq V(T_2)$, let $M_{y, t''}$ be the monochromatic $E_{j,\ell^*}$-pseudocomponent in $G[Y^{(i_{\ell^*},-1, 0)}]$ with respect to $c$ containing $y$; note that it implies that $M_{y, t''}$ is the monochromatic $E_{j,\ell^*}$-pseudocomponent in $G[Y^{(i_{\ell^*},-1, 0)}]$ with respect to $c$ containing $y$ by Claim 23.22.

Since $z \in \partial T_{j,t}$ and $X_z \subseteq Y^{(i_{\ell^*}, Y^{(i_{\ell^*},-1, 0)}, t^*, \alpha}} + 1, s + 2)$, by the existence of $O_0$, there exists $O_1 \subseteq O_0 \subseteq V(T_2) - \{z\}$ with $|O_1| = |O_0| - 2w_0$ such that:

- $O_1 = \{o_{1,1}, o_{1,2}, \ldots, o_{1,|O_1|}\}$, where $o_{1, \alpha + 1} \in V(T_{o_{1, \alpha}}) - \{o_{1, \alpha}\}$ and $X_{o_{1, \alpha + 1}} \cap X_{o_{1, \alpha}} \cap I_j - X_z = \emptyset$ for every $\alpha \in [|O_1| - 1]$,
- $A_{L^{(i_{\ell^*},-1, 0)}}(V(M_{o_{1, \alpha}})) \cap X_{V(T_2)} - X_z \subseteq X_{V(T_2)} - (X_{V(T_{o_{1, \alpha}})} - X_{o_{1, \alpha}})$, and
\[ A_{L^{(t,\xi,-1,0)}}(V(M'')) \cap X_{V(T_z)} - X_z \subseteq X_{V(T_{\alpha_1,\xi,\epsilon}^* - w_0)} - X_{\alpha_1,\xi,\epsilon}^* - w_0 \text{ for every monochromatic } E_{\gamma} \text{-pseudo component } M'' \text{ in } G[Y^{(t,\xi,-1,0)}] \text{ with respect to } c \text{ contained in some } s\text{-segment in } S_j^0 \text{ whose level equals its color with } \sigma(M'') \leq \sigma(M_{u\epsilon,\gamma}^*) \text{ such that } V(M'') \cap X_z \neq \emptyset \text{ and } A_{L^{(t,\xi,-1,0)}}(V(M'')) \cap X_{V(T_z)} - X_z \neq \emptyset.\]

Suppose \( V(P_z) \subseteq Y^{(t,\xi,-1,0)} \). Recall that for every \( y \in \{u_z, v_z\}, A_{L^{(t,\xi,|V(T)|+1,\xi,\epsilon}^* - w_0)}(V(M_y)) \cap X_{V(T_z)} - X_z \neq \emptyset, \) so \( A_{L^{(t,\xi,|V(T)|+1,\xi,\epsilon}^* - w_0)}(V(M_y)) \cap V(P_z) \cap X_{V(T_z)} - X_z \neq \emptyset, \) for otherwise \( P_z \subseteq Y^{(t,\xi,|V(T)|+1,\xi,\epsilon}^* - w_0)} \subseteq Y^{(t,\xi,-1,0)} \) and we are done. By the existence of \( O_\gamma \), there exists a path \( Q \subseteq P_z - \{u_z, v_z\} \) in \( G[Y^{(t,\xi,-1,0)} \cap I_j] \) from \( X_{O_\gamma} \cap I_j - X_z \) to \( X_{O_\gamma} \cap I_j - X_z \) disjoint from \( X_z \). Since \( X_{O_\gamma} \cap I_j - X_z \) are pairwise disjoint among all \( \alpha \in [\xi, \epsilon) \), \( |V^{(t,\xi,-1,0)} \cap I_j - X_{V(T)}| \geq |V(Q)| \geq \xi(\epsilon) \). But by Claim 23.19, \( |V^{(t,\xi,-1,0)} \cap X_{V(T)} \cap I_j| \leq 2\eta_4 < \xi(\epsilon), \) a contradiction.

Hence \( V(P_z) - Y^{(t,\xi,-1,0)} \neq \emptyset \). In particular, for each \( y \in \{u_z, v_z\}, A_{L^{(t,\xi,-1,0)}}(V(M_{u\epsilon}^*)) \cap V(P_z) \cap X_{V(T_z)} - X_z \neq \emptyset. \)

There exist a maximum \( \beta* \in N \) and nodes \( z_1', z_2', \ldots, z_{\beta*} \) of \( T \) such that \( z_1' = z \), and for each \( \alpha \in [\beta* - 1], z_{\alpha+1}' \) is the node in \( \partial T_j, z_\alpha' \) with \( o_{1,\xi,\epsilon}^* - w_0 \in V(T_{z_\alpha + 1}') \). Hence \( V(P_z) - Y^{(t,\xi,-1,0)} \neq \emptyset. \)

Therefore, there exist \( \gamma \in N \) and \( O_{\gamma} \subseteq V(T_z') - \{z_\gamma'\} \) such that:

- \( |O_{\gamma}| \geq \xi(\ell_{\gamma}) - (\gamma - 1)(\eta_5 + 3w_0 + 1) - 2\gamma w_0, \) where \( \ell_{\gamma} \) is the number in \([0, w_0 - 1]\) such that \( \sigma(M_{u\epsilon,\gamma}^*) \) is the \((\ell_{\gamma} + 1)\)-th smallest among all monochromatic \( E_{\gamma} \)-pseudo components \( M'' \) in \( G[Y^{(t,\xi,-1,0)}] \) with respect to \( c \) such that \( V(M'') \cap X_{\gamma + 1}' \neq \emptyset \) and \( A_{L^{(t,\gamma,-1,0)}}(V(M'')) \cap X_{V(T_\gamma + 1)} - X_\gamma' \neq \emptyset \) and the \( s\)-segment containing \( V(M'') \) with level equal to \( c(M'') \) belongs to \( S_j^0, \)

- \( O_\gamma = \{o_{\gamma,1}, o_{\gamma,2}, \ldots, o_{\gamma,|O_\gamma|}\}, \) where \( o_{\gamma,\alpha + 1} \in V(T_{o_{\gamma,\alpha}}) - \{o_{\gamma,\alpha}\} \) and \( X_{O_{\gamma,\alpha + 1}} \cap X_{O_{\gamma,\alpha}} \cap I_j - X_\gamma' = \emptyset \) for every \( \alpha \in [0, |O_\gamma| - 1], \)

- there exists a monochromatic \( E_{\gamma} \)-pseudo component \( M_{\gamma}^* \) in \( G[Y^{(t,\xi,-1,0)}] \) with respect to \( c \) such that:
  - \( M_{\gamma}^* \neq M_{u\epsilon,\gamma}^* \)
  - \( V(M_{\gamma}^*) \cap X_{\gamma} \neq \emptyset, \)
  - there exists a subpath \( P_{z_\gamma}' \) of \( P_z \) from \( A_{L^{(t,\gamma,-1,0)}}(V(M_{\gamma}^*)) \cap X_{V(T_\gamma)} - X_\gamma' \) to \( A_{L^{(t,\gamma,-1,0)}}(V(M_{u\epsilon,\gamma}^*)) \cap X_{V(T_\gamma)} - X_\gamma' \) internally disjoint from \( X_\gamma' \), and \( A_{L^{(t,\gamma,-1,0)}}(V(M_{\gamma}^*)) \cap X_{V(T_\gamma)} - X_\gamma' \subseteq X_{V(T_\gamma)} - (X_{V(T_{\gamma - 1})} - X_{\gamma - 1}), \)
  - \( A_{L^{(t,\gamma,-1,0)}}(V(M'')) \cap X_{V(T_\gamma)} - X_\gamma' \subseteq X_{V(T_{\gamma - 1})} - X_{\gamma - 1}, \) for every monochromatic \( E_{\gamma} \)-pseudo component \( M'' \) in \( G[Y^{(t,\xi,-1,0)}] \) with respect to \( c \) contained in some \( s\)-segment in \( S_j^0 \) whose level equals its color with \( \sigma(M'') \leq \sigma(M_{u\epsilon,\gamma}^*) \) such that \( V(M'') \cap X_\gamma' \neq \emptyset \) and \( A_{L^{(t,\gamma,-1,0)}}(V(M'')) \cap X_{V(T_\gamma)} - X_\gamma' \neq \emptyset, \) and
  - \( V(M_{u\epsilon,\gamma}^*) \cap X_\gamma' \neq \emptyset \) and \( A_{L^{(t,\gamma,-1,0)}}(V(M_{u\epsilon,\gamma}^*)) \cap V(P_z) \cap X_{V(T_\gamma)} - X_\gamma' \neq \emptyset. \)

Note that \( \gamma \) exists since 1 is a candidate by choosing \( M_{\gamma}^* = M_{u\epsilon,\gamma}^* \). We choose \( \gamma \) and \( O_{\gamma} \) such that \( |X_{O_{\gamma,1}} \cap X_{O_{\gamma,2}} \cap I_j| \) is as large as possible, subject to this, \( \gamma \) is as large as possible, and subject to these, \( |O_{\gamma}| \) is as large as possible.

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Since $M^{*}_{z'_{\gamma}} \neq M^{*}_{u,z_{\gamma}+1}$, by the maximality of $|X_{o,\gamma-1} \cap X_{o,\gamma-2} \cap I_j|$, we know $|O_{\gamma}| \geq \xi(\ell_{\gamma}) \geq \xi(0)$.

Suppose $o_{\gamma,\gamma_{55}+w_0+1} \in V(T_{j,z'_{\gamma}})$. Let $k_{\gamma,v_{\gamma}} \in [0, w_{0}-1]$ such that $M^{*}_{j,k_{\gamma,v_{\gamma}}} \supseteq M^{*}_{z'_{\gamma}}$. Since $A_{L}^{(i_{\gamma}-1,1)}(V(M^{*}_{j,k_{\gamma,v_{\gamma}}})) \cap X_{V(T_{z'_{\gamma}})} - X_{z'_{\gamma}} \subseteq X_{V(T_{z'_{\gamma}})} - \delta_{o_{\gamma,k_{\gamma},v_{\gamma}+1}} - X_{o_{\gamma,1}}$, we know

$$A_{L}^{(i_{\gamma}-1,k_{\gamma},v_{\gamma})}(V(M^{*}_{j,k_{\gamma},v_{\gamma}})) \cap X_{V(T_{z'_{\gamma}})} - X_{z'_{\gamma}} \subseteq X_{V(T_{z'_{\gamma}})} - \delta_{o_{\gamma,k_{\gamma},v_{\gamma}+1}} - X_{o_{\gamma,1}}$$

Since $|O_{\gamma}| - w_{0} \geq \xi(0) - 3w_{0} > \eta_{5} + w_{0} + 1$ and $P'_{z'_{\gamma}}$ is a monochromatic path in $G$ respect to $c$ from $A_{L}^{(i_{\gamma}-1,0)}(V(M^{*}_{z'_{\gamma}})) \cap X_{V(T_{z'_{\gamma}})} - X_{z'_{\gamma}}$ to $A_{L}^{(i_{\gamma}-1,0)}(V(M^{*}_{u,z_{\gamma}+1})) \cap X_{V(T_{z'_{\gamma}})} - X_{z'_{\gamma}}$ internally disjoint from $X_{z'_{\gamma}}$, by Claim 23.22, there exists a collection of subpaths of $P'_{z_{\gamma}}$ contained in $G[Y^{(i_{\gamma},1)}(V)+1, s+2] \cap X_{V(T_{o_{\gamma},z'_{\gamma}})} \cap T_{j}$ from $X_{o_{\gamma},w_{0}}$ to $X_{o_{\gamma},w_{0}+1}$. Since $X_{o_{\gamma},1} \cap I_{j} - X_{z'_{\gamma}}$ are pairwise disjoint for $\alpha \in [O_{\gamma}]$, we know

$$Y^{(i_{\gamma},1)}(V) + 1, s+2 \cap X_{V(T_{o_{\gamma},z'_{\gamma}})} \cap T_{j} \geq (\eta_{5} + w_{0} + 1) - w_{0}$$

$$\geq \eta_{5} + 1$$

$$> Y^{(i_{\gamma},1)}(V) + 1, s+2 \cap X_{V(T_{o_{\gamma},z'_{\gamma}})} \cap T_{j}$$

by Claim 23.21, a contradiction.

Hence $o_{\gamma,\gamma_{55}+w_0+1} \notin V(T_{j,z'_{\gamma}})$. For every $\alpha \in [O_{\gamma}] - (\eta_{5} + w_{0} + 1)$, let $\alpha'_{\gamma,\alpha} := o_{\gamma,\gamma_{55}+w_0+1+\alpha}$. Let $O'_{\gamma} :=\{\alpha'_{\gamma,\alpha} : \alpha \in [O_{\gamma}] - (\eta_{5} + 2w_{0} + 1) - w_0\}$. Since $o_{\gamma,\gamma_{55}+w_0+1} \notin V(T_{j,z'_{\gamma}})$, $O'_{\gamma} \subseteq V(T_{o_{\gamma},z'_{\gamma}}) - \{z'_{\gamma}+1\}$. For every $\alpha \in [0, |O'_{\gamma}|-1]$, it is clear that $\alpha'_{\gamma,\alpha+1} \notin V(T_{o_{\gamma},z'_{\gamma}}) - \{\alpha'_{\gamma,\alpha}\}$, and $X_{o_{\gamma},1} \cap I_{j} - X_{z'_{\gamma}}$ are pairwise disjoint for $\alpha \in [O_{\gamma}]$, the existence of $P'_{z'_{\gamma}}$ and $O_{\gamma}$ implies that there exists a monochromatic $E_{j,z_{\gamma}+1}^{(i_{\gamma},0)}$-pseudo component $M^{*}_{z_{\gamma}+1,0}$ in $G[Y^{(i_{\gamma},1)}(V) + 1, s+2]$ respect to $c$ such that:

- $M^{*}_{z_{\gamma}+1,0} \neq M^{*}_{u,z_{\gamma}+1}$, where $M^{*}_{u,z_{\gamma}+1}$ is the $E_{j,z_{\gamma}+1}^{(i_{\gamma},0)}$-pseudo component in $G[Y^{(i_{\gamma},1)}(V) + 1, s+2]$ containing $M^{*}_{u,z_{\gamma}+1}$,
- $V(M^{*}_{z_{\gamma}+1,0}) \cap X_{z_{\gamma}+1} \neq \emptyset$,
- there exists a subpath $P'_{z_{\gamma}+1}$ of $P_{z'_{\gamma}} \subseteq P_{z_{\gamma}}$ from $A_{L}^{(i_{\gamma}-1,0)}(V(M^{*}_{z_{\gamma}+1,0})) \cap X_{V(T_{z_{\gamma}+1})} - X_{z_{\gamma}+1}$ to $A_{L}^{(i_{\gamma}-1,0)}(V(M^{*}_{u,z_{\gamma}+1})) \cap X_{V(T_{z_{\gamma}+1})} - X_{z_{\gamma}+1}$ internally disjoint from $X_{z_{\gamma}+1}$,
- $A_{L}^{(i_{\gamma}-1,0)}(V(M^{*}_{z_{\gamma}+1,0})) \cap X_{V(T_{z_{\gamma}+1})} - X_{z_{\gamma}+1} \subseteq X_{V(T'_{z_{\gamma}})} - X_{o_{\gamma},z'_{\gamma}}$ for every monochromatic $E_{j,z_{\gamma}+1}^{(i_{\gamma},0)}$-pseudo component $M''$ in $G[Y^{(i_{\gamma},1)}(V) + 1, s+2]$ with respect to $c$ contained in some $s$-segment in $S_{j}^{o}$ whose level equals its color with $\sigma(M'') \leq \sigma(M^{*}_{u,z_{\gamma}+1,0})$ such that $V(M'') \cap X_{z_{\gamma}+1} \neq \emptyset$ and $A_{L}^{(i_{\gamma}-1,0)}(V(M'')) \cap X_{V(T'_{z_{\gamma}})} - X_{z_{\gamma}+1} \neq \emptyset$, and
Let $M^*_{z_{\gamma+1}'}$ be the monochromatic $E_{j,z_{\gamma+1}'}$-pseudo-component in $G[Y^{(i,z_{\gamma+1}')}\cup (V(T)+1,s+2)]$ containing $M^*_{z_{\gamma+1}'}$.

Suppose $M^*_{z_{\gamma+1}'} \neq M^*_{u_{z_{\gamma+1}}}$. Let $p$ be the parent of $z_{\gamma+1}'$. Then there exists $O' \subseteq V(T_{z_{\gamma+1}'}) - \{z_{\gamma+1}'\}$ with $|O'| = \xi(\ell_p) - \gamma(\eta_p + 3w_0 + 1) - 2\gamma w_0$ such that $O' = \{o'_1, o'_2, \ldots, o'_{|O'|}\}$,

- $\ell_p$ is in $[0, w_0 - 1]$ such that $\sigma(M^*_{u_{z_{\gamma+1}}})$ is the $(\ell_p + 1)$-th smallest among all monochromatic $E_{j,z_{\gamma+1}'}$-pseudo-components $M''$ in $G[Y^{(i,z_{\gamma+1}')}\cup (V(T)+1,s+2)]$ with respect to $c$ such that $V(M'') \cap X_{z_{\gamma+1}'} \neq \emptyset$ and $A_{L^{(i,z_{\gamma+1}')}\cup (V(T)+1,s+2)} (V(M'')) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \neq \emptyset$, and the $s$-segment containing $V(M'')$ with level equals $c(M'')$ belongs to $S_{\gamma}'$,

- $o'_{\alpha+1} \in V(T_{o_{\alpha}}') - \{o'_{\alpha}\}$ and $X_{o'_{\alpha+1}} \cap X_{o_{\alpha}} \cap I_j - X_{z_{\gamma+1}'} = \emptyset$ for every $\alpha \in [0, |O'| - 1]$,

- $A_{L^{(i,z_{\gamma+1}')}\cup (V(T)+1,s+2)} (V(M^*)) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \subseteq X_{V(T_{z_{\gamma+1}'})} - (X_{V(T_{z_{\gamma+1}'})} - X_{o'_1})$, and

- $A_{L^{(i,z_{\gamma+1}')}\cup (V(T)+1,s+2)} (V(M'')) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \subset (X_{V(T_{z_{\gamma+1}'})} - X_{o'_1}) - X_{o'_{|O'|}}$ for every monochromatic $E_{j,z_{\gamma+1}'}$-pseudo-component $M''$ in $G[Y^{(i,z_{\gamma+1}')}\cup (V(T)+1,s+2)]$ with respect to $c$ contained in some $s$-segment in $S_{\gamma}'$ whose level equals its color with $\sigma(M'') \leq \sigma(M^*_{u_{z_{\gamma+1}'}})$ such that $V(M'') \cap X_{z_{\gamma+1}'} \neq \emptyset$ and $A_{L^{(i,z_{\gamma+1}')}\cup (V(T)+1,s+2)} (V(M'')) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \neq \emptyset$.

Using the argument for deriving $O_1$ from $O_0$, one obtains a set $O_{\gamma+1}$ from $O'$, contradicting the maximality of $\gamma$. (Note that the existence of the path $P_{z_{\gamma+1}'}$ is obtained by the path $P_{z_{\gamma+1}'}$.)

Hence $M^*_{z_{\gamma+1}'} = M^*_{u_{z_{\gamma+1}'}}$. For every $\alpha \in [|O'_1| - 2w_0]$, let $o^+_{\alpha,\alpha} = o_{\gamma,\alpha+3}$. Let $O_{\gamma}^+ = \{o^+_{\gamma,\alpha} : \alpha \in [0, |O'_1|]\}$. Using the argument for deriving $O_1$ from $O_0$, we know $O_{\gamma}^+$ satisfies the following:

- $V(M^1_{\gamma+1}) \cap X_{z_{\gamma+1}'} \neq \emptyset$, where $M^1_{\gamma+1}$ is the monochromatic $E_{j,z_{\gamma+1}'}(0)$-pseudo-component in $G[Y^{(i,z_{\gamma+1}'},-1.0}]$ containing $M^*_{z_{\gamma+1}'}$.

- there exists a subpath $Q_1$ of $P_2$ from $A_{L^{(i,z_{\gamma+1}'},-1.0}} (V(M^1_{\gamma+1})) \cap (X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'}) \cap (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}}) \cap (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}})$ to $A_{L^{(i,z_{\gamma+1}'},-1.0}} (V(M^1_{\gamma+1})) \cap (X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'}) \cap (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}})$ internally disjoint from $X_{z_{\gamma+1}'}$ intersecting $X_{o^+_{\gamma,1}} - X_{z_{\gamma+1}'}$ for every $\alpha \in [|O_{\gamma}^+|]$. 

- $A_{L^{(i,z_{\gamma+1}'},-1.0}} (V(M^1_{\gamma+1})) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \subseteq (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}}) \cup (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}})$

- $A_{L^{(i,z_{\gamma+1}'},-1.0}} (V(M'')) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \subseteq (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}}) \cup (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}})$

- $A_{L^{(i,z_{\gamma+1}'},-1.0}} (V(M'')) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \subseteq (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}}) \cup (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}})$

- $A_{L^{(i,z_{\gamma+1}'},-1.0}} (V(M'')) \cap X_{V(T_{z_{\gamma+1}'})} - X_{z_{\gamma+1}'} \subseteq (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}}) \cup (X_{V(T_{z_{\gamma+1}'})} - X_{o^+_{\gamma,1}})$

with $V(M'') \cap X_{z_{\gamma+1}'} \neq \emptyset$ and $\sigma(M'') > \sigma(M^1_{\gamma+1})$. 

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Note that \(|O^+_1| = |O^+_2| - 2w_0 = |O^-_1| - (\eta_5 + 3w_0 + 1) - 2w_0 \geq \xi(\ell_2) - (\eta_5 + 5w_0 + 1)\). In addition, recall that we assume that for every \(t' \in V(T_2 - \{z\})\) and \(\alpha \in [0, k']\), if \(\alpha\) is the \((k' + 1)\)-th smallest element in \(C_{\alpha'}\) for some \(k' \in [0, |C_{\alpha'}| - 1]\), then \(M^{(\alpha')}_{j, k_0} \supseteq M^{(\alpha)}_{j, \alpha}\) for some \(k_0 \in [0, k']\). So \(M^+_1 \subseteq M^{(\alpha')}_{j, k_0}\) for some \(\ell \in [0, \ell_2]\).

By Claim 23.21, \(|\bigcup_{M''} A_{L(\ell_{\gamma+1} + 1, 1)}(V(M'')) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1})| \leq f(\eta_5)\). So there exists \(O^+_2 = \{o^+_2, \alpha : \alpha \in |O^+_2|\} \subseteq O^+_2\) with \(|O^+_2| \geq (|O^+_1| - \eta_5)/(f(\eta_5) + 1)\) such that the following hold:

- \(V(M^+_1) \cap X_{\gamma+1} \neq \emptyset\).
- There exists a subpath \(Q_1\) of \(P\) from \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M^+_1)) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} = (X_{V(T_{\gamma+1})} \setminus X_{\gamma+1})\) to \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M^+_1)) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} = (X_{V(T_{\gamma+1})} \setminus X_{\gamma+1})\) internally disjoint from \(X_{\gamma+1}\) intersecting \(X_{\gamma+1} \neq \emptyset\) for every \(\alpha \in |O^+_2|\).

- \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M^+_1)) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} \subseteq (X_{V(T_{\gamma+1})} \setminus X) \cup (X_{V(T_{\gamma+1})} \setminus X_{\gamma+1})\) for every monochromatic \(E_{j, \gamma+1} \) pseudocomponent \(M'' \in G[Y(\ell_{\gamma+1} + 1, 1)]\) with respect to \(c\) contained in some \(s\)-segment in \(S_j\) whose level equals its color such that \(V(M'') \cap X_{\gamma+1} \neq \emptyset\) and \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M'')) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} \neq \emptyset\).

Therefore, there exist \(\theta \in \left[\frac{\xi(\ell_2) - (\eta_5 + 5w_0 + 1) - \eta_5}{f(\eta_5) + 1}\right]\) and \(\phi \in [\theta]\) such that there exists \(O^+_\theta \subseteq V(T_{\gamma+1}) - \{z'_{\gamma+1}\}\) with \(|O^+\theta| = \frac{\xi(\ell_2) - (\eta_5 + 5w_0 + 1) - \eta_5}{f(\eta_5) + 1} - (\phi - 1)\cdot(\eta_5 + 4w_0 + 1)\) such that:

- \(O^+_\theta = \{o^+_\theta, \alpha : \alpha \in |O^+_\theta|\}\),
- \(V(M^\theta) \cap X_{\gamma+1} \neq \emptyset\), where \(M^\theta\) is the monochromatic \(E_{j, \gamma+1} \) pseudocomponent in \(G[Y(\ell_{\gamma+1} + 1, 1)]\) containing \(M^*_{\gamma+1}\).
- There exists a subpath \(Q^\theta\) of \(P\) from \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M^\theta)) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} = (X_{V(T_{\gamma+1})} \setminus X_{\gamma+1})\) to \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M^\theta)) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} = (X_{V(T_{\gamma+1})} \setminus X_{\gamma+1})\) internally disjoint from \(X_{\gamma+1}\) intersecting \(X_{\gamma+1} \neq \emptyset\) for every \(\alpha \in |O^+_\theta|\).
- \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M'')) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} \subseteq (X_{V(T_{\gamma+1})} \setminus X) \cup (X_{V(T_{\gamma+1})} \setminus X_{\gamma+1})\) for every monochromatic \(E_{j, \gamma+1} \) pseudocomponent \(M'' \in G[Y(\ell_{\gamma+1} + 1, 1)]\) with respect to \(c\) contained in some \(s\)-segment in \(S_j\) whose level equals its color such that \(V(M'') \cap X_{\gamma+1} \neq \emptyset\) and \(A_{L(\ell_{\gamma+1} + 1, 1)}(V(M'')) \cap X_{V(T_{\gamma+1})} \cap X_{\gamma+1} \neq \emptyset\).
Note that such $\theta$ and $\phi$ exist as 1 and 1 are candidates for them, since

$$\frac{\xi(\ell_\gamma) - (2\eta_5 + 5w_0 + 1)}{(f(\eta_5) + 1) \cdot 4(\eta_5 + 2w_0 + 2)\sum_{a=0}^{\ell_\gamma} h(a) + h(0)(\eta_5 + 4w_0 + 1)} \geq 1.$$ 

We assume that $\theta$ is chosen to be as large as possible, and subject to this, $\phi$ is chosen to be as large as possible.

Note that $|O^\phi_\theta| \geq \frac{\xi(\ell_\gamma) - (2\eta_5 + 5w_0 + 1)}{(f(\eta_5) + 1) - (\theta - 1) \cdot 4(\eta_5 + 2w_0 + 2)\sum_{a=0}^{\ell_\gamma} h(a) + h(0)(\eta_5 + 4w_0 + 1)} \geq 4(\eta_5 + 2w_0 + 2)\sum_{a=0}^{\ell_\gamma} h(a) + h(0)(\eta_5 + 4w_0 + 1)$.

Let $U$ be the subset of $\mathbb{N}_0$, where we denote the elements of $U$ by $\theta = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu|U| \leq \beta^* - \gamma$, such that for each $\alpha \in |U| - 1$, $\mu_{\alpha+1}$ is the minimum in $[\mu_\alpha + 1, \beta^* - \gamma]$ such that there exist a monochromatic $E_{j_{\gamma+\mu_\alpha},(i_{\gamma+\mu_\alpha},0)}^{(0)}$-pseudocomponent $M''$ in $G[Y^{(i_{\gamma+\mu_\alpha},-1,0)}]$ with $\sigma(M'') \leq \sigma(M^\phi_\theta)$ and a monochromatic $E_{j_{\gamma+\mu_\alpha+1},\alpha}$-pseudocomponent $M'''$ in $G[Y^{(i_{\gamma+\mu_\alpha+1},-1,0)}]$ with $M'' \subseteq M'''$ such that either $M'' \neq M'''$, or

$$|A_L^{(i_{\gamma+\mu_\alpha+1},-1,0)}(V(M''')) \cap X_{V(T_{j_{\gamma+\mu_\alpha+1}})} - X_{z_{\gamma+\mu_\alpha+1}}| = |A_L^{(i_{\gamma+\mu_\alpha},-1,0)}(V(M'')) \cap X_{V(T_{j_{\gamma+\mu_\alpha}})} - X_{z_{\gamma+\mu_\alpha}}|.$$

Let $\kappa \in [0, \mu|U|]$ be the maximum such that $(|O^\phi_\theta| - (2\kappa + 1)w_0) - w_0 > \kappa(\eta_5 + 2w_0 + 1) + \eta_5 + w_0 + 1$, and for every $\alpha \in [0, \kappa]$,

(K1) $V(M^\phi_{\mu_\alpha}) \cap X_{z_{\gamma+\mu_\alpha}} \neq \emptyset$,

(K2) there exists a subpath $Q^\phi_{\mu_\alpha} \subseteq P_2$ from $A_L^{(i_{\gamma+\mu_\alpha},-1,0)}(V(M^\phi_{\mu_\alpha})) \cap (X_{V(T_{j_{\gamma+\mu_\alpha}})} - X_{z_{\gamma+\mu_\alpha}})$ to $A_L^{(i_{\gamma+\mu_\alpha},-1,0)}(V(M^\phi_{\mu_\alpha})) \cap (X_{V(T_{j_{\gamma+\mu_\alpha}})} - X_{z_{\gamma+\mu_\alpha}})$ and internally disjoint from $X_{z_{\gamma+\mu_\alpha}}$ intersecting $X_{o_{\gamma',\alpha'}} - X_{z_{\gamma+\mu_\alpha}}$ for every $\alpha' \in [(\eta_5 + 2w_0 + 1)\kappa + 1, |O^\phi_\theta| - 2\kappa w_0]$, and

(K3) $A_L^{(i_{\gamma+\mu_\alpha},-1,0)}(V(M'')) \cap X_{V(T_{j_{\gamma+\mu_\alpha}})} - X_{z_{\gamma+\mu_\alpha}}$

$$\subseteq (X_{V(T_{j_{\gamma+\mu_\alpha}})} - (X_{V(T_{j_{\gamma+\mu_\alpha}})}(\eta_5 + 2w_0 + 1) + 1) - X_{z_{\gamma+\mu_\alpha}} \cap (X_{V(T_{j_{\gamma+\mu_\alpha}}(\eta_5 + 2w_0 + 1) + 1)} - X_{z_{\gamma+\mu_\alpha}})$$

for every monochromatic $E_{j_{\gamma+\mu_\alpha},(i_{\gamma+\mu_\alpha},0)}^{(0)}$-pseudocomponent $M''$ in $G[Y^{(i_{\gamma+\mu_\alpha},-1,0)}]$ with respect to $c$ contained in some $s$-segment in $S_j$ whose level equals its color such that $V(M''') \cap X_{z_{\gamma+\mu_\alpha}} \neq \emptyset$ and $A_L^{(i_{\gamma+\mu_\alpha},-1,0)}(V(M'')) \cap X_{V(T_{j_{\gamma+\mu_\alpha}})} - X_{z_{\gamma+\mu_\alpha}} \neq \emptyset$, and

(K4) if $\alpha > 0$, then $o_{\gamma',(\alpha-1)}(\eta_5 + 2w_0 + 1) + \eta_5 + w_0 + 1 \not\subseteq V(T_{j_{\gamma+\mu_\alpha}})$.

Note that $\kappa$ exists since 0 is a candidate.

Suppose $o_{\gamma',(\alpha-1)}(\eta_5 + 2w_0 + 1) + \eta_5 + w_0 + 1 \subseteq V(T_{j_{\gamma+\mu_\alpha}})$. Let $k_\kappa \in [0, w_0 - 1]$ such that $M^{(i_{\gamma+\mu_\alpha},0)}_{j,k_\kappa} \supseteq M^\phi_{\mu_\alpha}$.

Since

$$A_L^{(i_{\gamma+\mu_\alpha},-1,0)}(V(M^\phi_{\mu_\alpha})) \cap X_{V(T_{j_{\gamma+\mu_\alpha}})} - X_{z_{\gamma+\mu_\alpha}} \subseteq (X_{V(T_{j_{\gamma+\mu_\alpha}})} - (X_{V(T_{j_{\gamma+\mu_\alpha}}(\eta_5 + 2w_0 + 1) + 1)} - X_{z_{\gamma+\mu_\alpha}}) \cap (X_{V(T_{j_{\gamma+\mu_\alpha}(\eta_5 + 2w_0 + 1) + 1)} - X_{z_{\gamma+\mu_\alpha}}),$$

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we know
\[A(L((z_1'+\mu,\kappa,-1,\kappa)) \cap XV(T_{\gamma' + \mu})) - X_{\gamma' + \mu}) \subseteq (XV(T_{\gamma' + \mu}) - (XV(T_{\gamma',\kappa,\eta_5+2\omega_0+1,\kappa,\kappa}) - X_{\gamma',\kappa,\eta_5+2\omega_0+1,\kappa,\kappa})) \cup (XV(T_{\gamma',\kappa,\eta_5-2\omega_0-\kappa})) - X_{\gamma',\kappa,\eta_5-2\omega_0-\kappa})\]
\[\subseteq (XV(T_{\gamma' + \mu}) - (XV(T_{\gamma',\kappa,\eta_5+2\omega_0+1,\kappa,\kappa}) - X_{\gamma',\kappa,\eta_5+2\omega_0+1,\kappa,\kappa}) \cup ((XV(T_{\gamma',\kappa,\eta_5-2\omega_0-\kappa})) - X_{\gamma',\kappa,\eta_5-2\omega_0-\kappa})).\]

Since \(|O_{\beta}|\geq (2\kappa+1)w_0 - w_0 > \kappa(\eta_5+2\omega_0+1) + \eta_5+w_0+1\) and \(Q_{\mu}^*\) is a monochromatic path in \(G\) respect to \(c\) from \(A(L((z_1'+\mu,\kappa,-1,\kappa)) \cap XV(T_{\gamma' + \mu})) - X_{\gamma' + \mu}\) to \(A(L((z_1'+\mu,\kappa,-1,\kappa)) \cap XV(T_{\gamma' + \mu})) - X_{\gamma' + \mu}\) internally disjoint from \(X_{\gamma' + \mu}\) for every \(\alpha \in [\kappa(\eta_5+2\omega_0+1)+1,|O_{\beta}|-2\omega_0w_0]\), by Claim 23.22, there exists a collection of subpaths of \(Q_{\mu}^*\)

In the path \(T\) from \(z_{\gamma',\alpha}\) to \(q\), the existence of \(Q_{\mu}^*\) implies that there exists a monochromatic \(E_{j,q}^*(z_{\gamma',\mu_\kappa})\)-pseudocomponent \(R_{\alpha}\) in \(G[Y((z_{\gamma',\mu_\kappa}) \cap XV(T_{\gamma'})) - X_{\gamma'}]\) containing a vertex \(y_{\kappa}\) in \(X_{\gamma'}\) such that the subpath of \(Q_{\mu}^*\) between \(y_{\kappa}\) and the end of \(Q_{\mu}^*\) is disjoint from \(XV(T_{\gamma',\mu_\kappa}) - X_{\gamma'}\). Suppose there exists such an \(\alpha\) such that \(R_{\alpha} \not\subseteq M_{\mu}^*\). Then there exists \(q^* \in \partial T_{\gamma',\mu_\kappa} - \{z_{\gamma',\alpha}\}\) such that some subpath \(Q''\) of \(Q_{\mu}^*\) is contained in \(XV(T_{\gamma'})\) and internally disjoint from \(XV(T_{\gamma',\mu_\kappa})\) and \(Y((z_{\gamma',\mu_\kappa}) \cap XV(T_{\gamma'})) - X_{\gamma'}\), where \(M'\) is the \(E_{j,q^*}^*(z_{\gamma',\alpha,\kappa})\)-pseudocomponent containing \(M_{\mu}^*\). Note that \(\sigma(M')\) is the smallest among all \(E_{j,q^*}^*(z_{\gamma',\kappa,0})\)-pseudocomponents \(M''\) with \(M'' = \emptyset\), \(A(L((z_1'+\mu,\kappa,-1,\kappa)) \cap XV(T_{\gamma'})) - X_{\gamma'}\neq \emptyset\) such that the smallest element in \(C_{\infty}\) for some \(k' \in [0,|C_{\infty}| - 1]\), then \(M_{\mu_\kappa} = M_{\mu_\kappa}^*\) for some \(k_\kappa \in [0, k']\). By Claim 23.27, \(Q'' \subseteq Y((z_{\gamma',\alpha,\kappa}) \cap XV(T_{\gamma'})) - X_{\gamma'}\), a contradiction. Therefore \(M_{\mu}^* \subseteq R_{\alpha}\) for every \(\alpha\).

Suppose \(\alpha < |U|\) and \(\left(|O_{\beta}| - (2\kappa+1+1)w_0 - w_0 < \kappa(\eta_5+2\omega_0+1) + \eta_5+w_0+1\right).\) Then \(z_{\gamma',\mu_\kappa}\) exists, and \((K1)-(K3)\) hold for \(\alpha = \kappa + 1\) since \((K1)-(K4)\) hold for \(\alpha = \kappa\). In addition, since \(O_{\beta}(\eta_5+2\omega_0+1) + \eta_5+w_0+1 \not\subseteq V(T_{\gamma',\mu_\kappa})\), we know \((K4)\) holds for \(\alpha = \kappa + 1\). This contradicts the maximality of \(\kappa\).

So either \(\kappa = |U|\), or \(\left(|O_{\beta}| - (2\kappa+1+1)w_0 - w_0 \leq \kappa(\eta_5+2\omega_0+1) + \eta_5+w_0+1\right)\). Hence, either \(\kappa = |U|\), or \(\kappa \geq \frac{|O_{\beta}|}{\eta_5+2\omega_0+1} - 2\). Since \(|U| \geq \frac{|O_{\beta}|}{\eta_5+2\omega_0+1}\), either case implies that \(\kappa \geq \frac{|O_{\beta}|}{\eta_5+2\omega_0+1} - 2\).

Let \(U'\) be the subset of \(U\) such that for every \(\mu_\kappa \in U'\), there exist a monochromatic \(E_{j,q^*}^*(z_{\gamma',\mu_\kappa})\)-pseudocomponent \(M''\) in \(G[Y((z_{\gamma',\mu_\kappa}) \cap XV(T_{\gamma'})) - X_{\gamma'}]\) with \(\sigma(M'') < \sigma(M_{\beta})\) and a monochromatic \(E_{j,q^*}^*(z_{\gamma',\mu_\kappa+1})\)-pseudocomponent \(M''\) in \(G[Y((z_{\gamma',\mu_\kappa+1}) \cap XV(T_{\gamma'})) - X_{\gamma'}]\) with \(M'' \subseteq M''\) such that either \(M'' \neq M''\), or
By Claim 23.26, \(|U'\) \leq \sum_{\alpha=0}^{\ell'} h(\alpha) + h(0).

Suppose there exist \(a^*_\circ \in [k]\) and \(b \in \mathbb{N}_0\) such that \(b\) is the minimum such that \(\mu_{a^*_\circ} < b\), \(\alpha^*_\circ | V(T_{j'_{\gamma + p_\alpha + 1}}) \in V(T_j)\) and \(\mu_{a^*_\circ} \not\subseteq U'\) for every \(\alpha'\) with \(\mu_{a^*_\circ} < \mu_{a^*_\circ} \leq b\). Note that \(\alpha^*_\circ (\eta_5 + 2w_0 + 1) + 1 < |O \mu_{a^*_\circ} - 2a^*_\circ w_0|\) by the definition of \(\alpha\). Then for every \(\alpha'\) with \(\mu_{a^*_\circ} < \mu_{a^*_\circ} \leq b\), since \(\mu_{a^*_\circ} \not\subseteq U'\), and by (K3), we know the end of \(Q_{a^*_\circ}^s\) in \(A (i', \gamma + p_\alpha + 1, 0) (V(M\mu_{a^*_\circ})) \cap X_n(T_{i', \gamma + p_\alpha + 1}) - X_w^{\circ \beta + p_\alpha} \cap (X_n(T_{i', \gamma + p_\alpha + 1}) - (X_n(T_{i', \gamma + p_\alpha + 1}) - X_w^{\circ \beta + p_\alpha + 1}))\) is not colored with \(c(P)\), a contradiction.

Hence for every \(\alpha' \in [k]\), there exists \(\mu_{a^*_\circ} \in U'\) with \(\mu_{a^*_\circ} > \mu_{a^*_\circ}\) such that \(\alpha^*_\circ | V(T_{j'_{\gamma + p_\alpha + 1}}) \in V(T_i)\). Hence \(|U'\) \geq \log_5 + 2w_0 + 2 \kappa \geq \log_5 + 2w_0 + 2(\frac{|O \mu_{a^*_\circ}|}{\eta_5 + 4w_0 + 1} - 2). Recall that \(|O \mu_{a^*_\circ}| \geq 4(\eta_5 + 2w_0 + 2)\Sigma_{\alpha=0}^{\ell'-1} h(\alpha) + h(0)(\eta_5 + 4w_0 + 1) > (\eta_5 + 2w_0 + 2)\Sigma_{\alpha=0}^{\ell'-1} h(\alpha) + h(0)(\eta_5 + 4w_0 + 1). So \(\Sigma_{\alpha=0}^{\ell'-1} h(\alpha) + h(0) \geq |U'\) \geq \log_5 + 2w_0 + 2(\frac{|O \mu_{a^*_\circ}|}{\eta_5 + 4w_0 + 1} - 2) > \Sigma_{\alpha=0}^{\ell'-1} h(\alpha) + h(0), a contradiction. This proves the claim. \(\Box\)

For every \(j \in [|V| - 1], t \in V(T)\) and \(E_j, t\)-pseudocomponent \(M\), let \(S_M\) be the \(s\)-segment containing \(V(M)\) whose level is \(\textcolor{blue}{\text{color}}\) of \(M\).

**Claim 23.29.** Let \(j \in [|V| - 1]. \) Let \(i \in \mathbb{N}_0, \) and let \(t\) be a node of \(T\) of height \(i\). Let \(t' \in V(T_i)\) be a node of height \(i'\) \(\in \mathbb{N}_0. \) Let \(M\) be a monochromatic \(E_j, t\)-pseudocomponent in \(G[Y_{(i, -1, 0)}]\) with respect to \(c\) such that \(S_M \cap I^j_{j'} \neq \emptyset. \) Assume there exists a monochromatic \(E_j, t\)-pseudocomponent \(M'\) with respect to \(c\) such that \(M \leq M'\) and \(V(M') \cap X_{t'} \neq \emptyset. \) If \(A_{L(i, -1, 0)} (V(M)) \cap X_n(T_i) - X_t = \emptyset, \) then \(V(M') = V(M). \)

**Proof.** Suppose to the contrary that \(V(M') \neq V(M). \) We assume that \(\sigma(M)\) is minimum among all such counterexamples. Then either there exists a vertex in \(V(M')\) \(\in V(M)\) adjacent in \(G\) to \(V(M')\), or there exists \(\epsilon \in E(M') \cap E_j, t' - E_{j, t}\) with \(|\epsilon \cap V(M)| = 1. \) Let \(v_M\) be the vertex in \(V(M)\) with \(\sigma(v_M) = \sigma(M).\)

Suppose there exists a vertex \(v \in (V(M') - V(M)) \cap X_n(T_i)\) adjacent in \(G\) to \(V(M)\). Then \(v \not\subseteq Y_{(i, -1, 0)}\). Since the height of \(t\) is \(i, v \in A_{L(i, -1, 0)} (V(M)) \cap X_n(T_i) - X_t, a contradiction.

So either there exists a vertex in \(V(M') - (V(M) \cup X_n(T_i))\) adjacent in \(G\) to \(V(M)\), or there exists \(e \in E(M') \cap E_{j, t'} - E_{j, t}\) with \(|e \cap V(M)| = 1. \)

For the former, there exists \(e''\) such that some vertex in \((Y_{(i', -1, 0)} - Y_{(i, -1, 0)}) \cap (V(M') - (V(M) \cup X_n(T_i)))\) adjacent in \(G\) to \(V(M)\), and we choose \(e''\) to be as small as possible. For the latter, there exist \(t_1 \in V(T_i)\) and \(t_2 \in \partial T_{j, t_1}\) such that \(t_2\) witnesses the membership of \(v \in E_{j, t'} - E_{j, t}, so e \in E_{j, t'} - E_{j, t}\) for some \(k_1 \in [0, w_0]\), and we choose \(e, t_1, t_2\) so that the pair \((i_1, k_1)\) is lexicographically minimal.

Suppose that \((i_1, k_1)\) was defined, and either \(i''\) was undefined or \(i_1 < i''\). Let \(x \in e \cap V(M)\). Let \(M_x\) be the monochromatic \(E_{j, t'}\)-pseudocomponent in \(G[Y_{(i_1, k_1, k_1)}]\) with respect to \(c\) containing \(x\). Since \(x \in V(M), V(M') \subseteq V(M_x), so V(M_x) \cap X_{t_1} \neq \emptyset \neq V(M_x) \cap X_{t_2}. Since t_2 is a witness for e and V(M_x) \cap X_{t_1} = \emptyset \neq V(M_x) \cap X_{t_2}, A_{L(i_1, k_1)} (V(M)) \cap V(T_{i_2} - X_{t_2} = \emptyset. But by the minimality of \((i_1, k_1)\) and the assumption that either \(i''\) does not exist or \(i_1 < i''\), we know \(V(M) \cap X_n(T_i) = V(M_x) \cap X_n(T_i), so A_{L(i_1, k_1)} (V(M)) \cap X_n(T_i) - X_{t_2} \subseteq A_{L(i, -1, 0)} (V(M)) \cap X_n(T_i) - X_{t_1} = \emptyset, a contradiction.

So \(i''\) was defined, and either \((i_1, k_1)\) was undefined or \(i_1 \geq i''\). Hence there exist nodes \(t^* \in V(T) - V(T_i), z \in \partial T_{j, t^*}, where z is a witness for X_{z} \cap I_{j} \subseteq U_{i-1} W_{3}^{(*, k_1, k_1)}\), and a path \(P_z\) in \(G[X_{n}(T_{i_1})]\) from a vertex \(u_z \in V(M) \cap X_{z}\) to a vertex \(v_z \in V(M) \cap X_{z}\) containing at least
one vertex in \( V(M') - V(M) \) and connecting two components of \( G[V(M)] \), where one of them contains \( v_M \) and \( u_z \). We choose such \( t^* \) such that \( i_{t^*} \) is as small as possible. The minimality of \( i_{t^*} \) and Claim 23.22 imply that there exists no node \( t''' \in V(T) \) with \( i_{t'''} < i_{t^*} \) and \( T_{j,t'''} \subseteq T_{j,t^*} \) such that there exists a monochromatic path in \( G[Y(i_{t'''}, |V(T)|+1,s+2) \cap X_{V(T_{j,t'''})}] \) from \( \bigcup_{i=1}^{b} W_{i}^{(t''',i)} \) to \( \{u_z, v_M\} \).

Since \( V(P_z) \not\subseteq V(M) \), \( V(P_z) \not\subseteq Y^{(i_{t^*},-1,0)} \). If \( t \in V(T_{j,t^*}) \), then let \( t_0 = t \); otherwise let \( t_0 \) be the node in \( \partial T_{j,t^*} \) with \( t \in V(T_{t_0}) \). So \( V(P_z) \not\subseteq Y^{(i_{t_0},-1,0)} \). Hence \( V(P_z) \not\subseteq Y^{(i_{t^*},-1,0)} \). So we have the following.

- There exist the minimum \( k^* \in [0, w_0 - 1] \) and a monochromatic \( E^{(i_{t^*}, k^*)}_{j,t} \)-pseudocomponent in \( G[Y(i_{t^*}, |V(T)|+1,s+2)] \) with respect to \( c \) such that:
  - \( \sigma(M_{k^*}) \) is the \( k^* \)-th smallest among all monochromatic \( E^{(i_{t^*}, k^*)}_{j,t} \)-pseudocomponents in \( G[Y(i_{t^*}, |V(T)|+1,s+2)] \) with respect to \( c \) intersecting \( X_{t^*} \) and contained in some \( s \)-segment in \( S_j^0 \) whose level equals its color, and
  - \( V(M_{k^*}) \cap X_z \neq \emptyset \) and \( A_{L(i_{t^*}, |V(T)|+1,s+2)}(V(M_{k^*})) \cap X_{V(T_z)} - X_z \neq \emptyset \).

- There exists \( \ell^* \in [0, w_0 - 1] \) such that for every \( x \in \{u_z, v_z\} \), if \( \ell^* \) is the monochromatic \( E^{(i_{t^*}, k^*, \ell^*)}_{j,t} \)-pseudocomponent in \( G[Y(i_{t^*}, |V(T)|+1,s+2)] \) with respect to \( c \) containing \( x \), then:
  - \( A_{L(i_{t^*}, |V(T)|+1,s+2)}(V(M_x)) \cap X_{V(T_z)} - X_z \neq \emptyset \),
  - \( M_u \neq M_{v_z} \) and \( \sigma(M_u) < \sigma(M_{v_z}) \), and
  - \( \sigma(M_{u_{\ell^*}}) \) is the \( k^* \)-th smallest among all monochromatic \( E^{(i_{t^*}, k^*, \ell^*)}_{j,t} \)-pseudocomponents \( M'' \) in \( G[Y(i_{t^*}, |V(T)|+1,s+2)] \) with respect to \( c \) such that \( V(M'') \cap X_z \neq \emptyset \), \( A_{L(i_{t^*}, |V(T)|+1,s+2)}(V(M'')) \cap X_{V(T_z)} - X_z \neq \emptyset \), and the \( s \)-segment containing \( V(M'') \) whose level equals the color of \( M'' \) belongs to \( S_j^0 \).

By Claim 23.28, one of the following holds.

(i) There exist a monochromatic \( E_{j,z} \)-pseudocomponent \( M_1 \) in \( G[Y(i_{z^*},-1,0)] \) with \( M_1 \supseteq M_{u_z} \) and a monochromatic \( E_{j,z} \)-pseudocomponent \( M_2 \) in \( G[Y(i_{z^*},-1,0)] \) intersecting \( X_z \) with \( M_2 \supseteq M_{j,\alpha}^{(z^*)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), but \( M_2 \nsubseteq M_{j,\alpha}^{(z^*)} \) for every \( \alpha' \) with \( \sigma(M_{j,\alpha}^{(z^*)}) \leq \sigma(M_{u_z}) \).

(ii) There exist \( k^*_\alpha \in [0, \ell^*] \) such that \( M_{j,k^*_\alpha} \supseteq M_{u_z} \), and there exist \( t' \in V(T_z) - \{z\} \) and \( \ell_0 \in [0, k^*_\alpha] \), a monochromatic \( E_{j,t'} \)-pseudocomponent \( M_1 \) in \( G[Y(i_{t'},-1,0)] \) with \( M_1 \supseteq M_{j,\alpha}^{(z)} \) and a monochromatic \( E_{j,t'} \)-pseudocomponent \( M_2 \) in \( G[Y(i_{t'},-1,0)] \) intersecting \( X_{t'} \) with \( M_2 \supseteq M_{j,\alpha}^{(z)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), but \( M_2 \nsubseteq M_{j,\alpha}^{(z)} \) for every \( \alpha' \leq \ell_0 \).

Suppose (i) holds. Then \( M_1 \neq M_2 \), so \( \sigma(M_2) < \sigma(M_1) \). Since \( M_1 \) contains \( M_{u_z} \) and \( M_{u_z} \) contains \( v_M \), \( \sigma(M_1) \leq \sigma(M_{u_z}) \leq \sigma(M) \). Let \( M'_2 \) be the monochromatic \( E_{j,t'} \)-pseudocomponent in \( G[Y(i_{t'},-1,0)] \) containing \( v_{M_2} \), where \( v_{M_2} \) is the vertex in \( M_2 \) with \( \sigma(v_{M_2}) = \sigma(M_2) \). Since \( \sigma(M_2) < \sigma(M_1) \), \( \sigma(M'_2) = \sigma(M_2) < \sigma(M_1) \leq \sigma(M_{u_z}) \). If \( A_{L(i_{t^*},-1,0)}(V(M'_2)) \cap X_{V(T_{z^*})} - X_{t^*} \neq \emptyset \), then \( M'_2 \) is contained in \( M_{j,\alpha}^{(z^*)} \) for some \( \alpha \) with \( \sigma(M_{j,\alpha}^{(z^*)}) \leq \sigma(M_{u_z}) \), but \( M_2 \) contains \( M'_2 \) and \( M_{j,\alpha}^{(z^*)} \) since \( i_{z^*} > i_{t^*} \), a contradiction. So \( A_{L(i_{t^*},-1,0)}(V(M'_2)) \cap X_{V(T_{z^*})} - X_{t^*} = \emptyset \). Since \( \sigma(M'_2) < \sigma(M_{u_z}) \leq \sigma(M) \), by the minimality of \( \sigma(M) \) for the counterexamples, \( V(M_2) = V(M'_2) \). But \( M_2 \nsubseteq M_{j,\alpha}^{(z^*)} \) for some
\[ \alpha \in [0, w_0 - 1], \text{ so } M_2' \supseteq M_j^{(t')} \supseteq \{v_{M_2}\} \text{ for some } \alpha, \text{ and hence } \sigma(M_j^{(t')}) \leq \sigma(v_{M_2}) \leq \sigma(M_{u_\alpha}), \text{ a contradiction.} \]

So (ii) holds. Then \( M_1 \neq M_2, \text{ so } \sigma(M_2) < \sigma(M_1). \text{ Since } M_1 \supseteq M_j^{(t_0)} \text{ and } M_j^{(t_0)} \supseteq M_{u_\alpha}, \text{ and } \ell_0 \leq k_2, \sigma(M_1) \leq \sigma(M_j^{(t_0)}) \leq \sigma(M_{u_\alpha}) \leq \sigma(v_{M_2}) = \sigma(M_2). \text{ Let } M_2'' \text{ be the } E_{j,z}-\text{pseudocomponent in } G[Y^{(i_\alpha,1,0)}] \text{ containing } v_{M_2}. \text{ Suppose } A_{L(i_\alpha,1,0)}(V(M_2'')) \cap X_{V(T_2)} = \emptyset. \text{ Then } M_2'' \text{ is contained in } M_j^{(t_0)} \text{ for some } \alpha^* \text{ with } \sigma(M_j^{(t_0)}) \leq \sigma(M_2'') = \sigma(M_2) < \sigma(M_1) \leq \sigma(M_j^{(t_0)}). \text{ So } \alpha^* \leq \ell_0. \text{ But } M_2 \text{ contains } M_2'' \text{ and } M_j^{(t_0)} \text{ since } i_{t'} > i_z, \text{ a contradiction. So } A_{L(i_\alpha,1,0)}(V(M_2'')) \cap X_{V(T_2)} = \emptyset. \text{ Since } \sigma(M_2'') < \sigma(M_1) \leq \sigma(M), \text{ by the minimality of } \sigma(M) \text{ for the counterexamples, } V(M_2) = V(M_2''). \text{ But } M_2 \supseteq M_j^{(t_0)} \text{ for some } \alpha' \in [0, w_0 - 1], \text{ so } M_2'' \supseteq M_j^{(t_0)} \supseteq \{v_{M_2}\} \text{ for some } \alpha'. \text{ Hence } \sigma(M_j^{(t_0)}) \leq \sigma(v_{M_2}) \leq \sigma(M_j^{(t_0)}), \text{ so } \alpha' \leq \ell_0, \text{ a contradiction. This proves the claim.} \]

**Claim 23.30.** Let \( j \in |V| - 1 \). Let \( t, t' \) be nodes of height \( i, i' \in N_0 \), respectively, where \( t' \in V(T_i) \). Let \( M \) be a monochromatic \( E_{j,t'}\)-pseudocomponent in \( G[Y^{(i',1,0)}] \) with \( t \in V(T_{r_M}) \). If there exists a path in \( M \) from \( v_M \) to a vertex \( v_t \in X_i \) internally disjoint from \( X_{V(T_t)} - X_t \), where \( \sigma(v_M) = \sigma(M) \), then there exists a monochromatic \( E_{j,t}\)-pseudocomponent in \( G[Y^{(i,1,0)}] \) containing both \( v_M \) and \( v_t \).

**Proof.** Suppose to the contrary that the claim is false. Choose \( M \) so that \( \sigma(M) \) is as small as possible among all counterexamples.

We first prove that \( t \in V(T_{r_M}) - \{r_M\} \). Suppose that \( t = r_M \). Since there exists a path in \( M \) from \( v_M \) to \( v_t \in X_i \) internally disjoint from \( X_{V(T_t)} - X_t \), this path is contained in \( G[X_i] + E_{j,t'} \).

Note that for every \( e \in E_{j,t'} \) with \( e \subseteq X_t \), there exists \( t'' \in V(T) \) with \( t \in \partial T_{r_M} \) such that \( t, t'' \) witness \( e \in E_{j,t'} \), so \( e \in E_{j,t} \). Since \( X_t \subseteq G[Y^{(i,1,0)}] \), there exists a monochromatic \( E_{j,t}\)-pseudocomponent in \( G[Y^{(i,1,0)}] \) with respect to \( c \) containing both \( v_M \) and \( v_t \), contradicting that \( M \) is a counterexample.

So \( t \in V(T_{r_M}) - \{r_M\} \).

Let \( P \) be a path in \( M \) from \( v_M \) to \( v_t \) internally disjoint from \( X_{V(T_t)} - X_t \). Note that \( E(P) \subseteq E_{j,t} \).

Since \( P \) is internally disjoint from \( X_{V(T_t)} - X_t \) and there exists no node \( t'' \in V(T) \) with \( t \in V(T_{r_M}) \) such that \( t, t'' \) contains exactly one of \( t, t' \). So if \( V(P) \subseteq Y^{(i-1,|V(T)|+1,s+2)} \), then some monochromatic \( E_{j,t}\)-pseudocomponent containing \( P \), a contradiction. Hence \( V(P) \not\subseteq Y^{(i-1,|V(T)|+1,s+2)} \). Further choose \( P \) so that:

(a) \( |V(P)| - Y^{(i-1,|V(T)|+1,s+2)} \) is as small as possible,

(b) subject to (a), \( |E(P) \cap E_{j,t}| \) is as small as possible, and

(c) subject to (a) and (b), \( |V(P)| \) is as large as possible.

Let \( t_1 := r_M \) and let \( i_1 \) be the height of \( t_1 \). Since \( t \in V(T_{r_M}) - \{r_M\}, i_1 \leq i - 1 \). Since \( v_M \in X_{r_M} \cap I_j \subseteq \bigcup_{i=-\infty}^{-1} W_3^{(i,1)} \cap I_j \cap Y^{(i,1,1,0)} \subseteq \bigcup_{i=-\infty}^{-1} W_3^{(i,1,1,0)} \cap I_j \cap Y^{(i,1,1,1,|V(T)|+1,s+2)}, \) we know \( V(P) \cap \bigcup_{i=-\infty}^{-1} W_3^{(i,1,1,1,|V(T)|+1,s+2)} \) is not empty. Let \( P_1 \) be the maximal subpath of \( P[V(P) \cap Y^{(i-1,|V(T)|+1,s+2)}] \) from \( v_M \in X_{t_1} \subseteq \bigcup_{i=-\infty}^{-1} W_3^{(i,1,1,1,|V(T)|+1,s+2)} \). Since \( V(P) \not\subseteq Y^{(i-1,|V(T)|+1,s+2)} \), there exists \( b_1 \) which is the end of \( P_1 \) adjacent in \( P \) to \( V(G) - Y^{(i-1,|V(T)|+1,s+2)} \). Let \( a_1 \) be the neighbor of \( b_1 \) in \( P \) but not in \( P_1 \). Note that \( a_1 \not\in Y^{(i-1,|V(T)|+1,s+2)} \).

Let \( p \) be the parent of \( t \). So the height \( i_p \) of \( p \) equals \( i - 1 \). Note that \( i_1 \leq i - 1 \). So there exist a maximum \( q \in \mathbb{N} \) and nodes \( t_2, \ldots, t_q \) of \( T \) such that for every \( q' \in [q - 1], t_q + 1 \) is the node in \( \partial T_{j,t'} \) such that \( t \in V(T_{t_q+1}) - \{t_q+1\} \). For every \( \ell \in [q], \) let \( i_{t_\ell} \) be the height of \( t_\ell \). The maximality of \( q \) implies that \( t \in V(T_{t_q+1}) - \{t_q\}, \) so \( i > i_{t_q} \) and \( p \in V(T_{j,t_q}) \).

Say that a tuple \((C, t', t'')\) is useful if:
Note that for some $C \subseteq \mathcal{X}$ and we may assume by symmetry that $X_{\tau}$ for $X_i$ where the choice of $(w, s, t) \in [0, |I_0|)$ subject to these, the distance in $\mathcal{X}$ is as large as possible, so there exists a monochromatic component $M_C$ in $G[Y^{(i, w)}|V(T)|+1, s+2] \cap X_{V(T_j, t)}$ with respect to $c$ intersecting $V(C)$ and $\bigcup_{i=1}^{V(T)} W_3^{(i, \tau)}$, where $i_0$ is the height of $t''$.

Let $C_0$ be a component of $C^* - E_{j,t}$ such that some monochromatic component $M_{C_0}$ in $G[Y^{(i, w)}|V(T)|+1, s+2] \cap X_{V(T_j, t)}$ with respect to $c$ intersecting $V(C_0)$ and $\bigcup_{i=1}^{V(T)} W_3^{(i, \tau)}$. Recall that $p \in V(T_j, t)$, so $i_p \geq i_0 \geq i_t$. Since $E^* \subseteq E_{j,t}$, there exist $\tau \in V(T)$ and $\tau' \in \partial T_j, \tau$ such that $\tau'$ witnesses the membership of $e^*$ in $E_{j,t}$, and we may assume by symmetry that $u^* \in V(C_0)$. Since $e^* \in E_{j,t}$, there exist $\tau' \in V(T)$ and $\tau' \in \partial T_j, \tau$ such that $\tau'$ witnesses the membership of $e^*$ in $E_{j,t}$. In particular, $e^* \subseteq X_{\tau'}$ and $\tau'$ is a witness for $X_{\tau'} \cap I_j \subseteq \bigcup_{i=1}^{V(T)} W_3^{(i, \tau)}$, where $i_\tau$ is the height of $\tau$. Since $e^* \in E_{j,t}$ is witnessed by $\tau$ and $\tau', t \in V(T_j)$. Since $e^* \subseteq V(C^*) \subseteq X_{V(T_j, t)}$, either $T_{j,t}^* \subseteq T_{j,t}$ or $T_{j,t}^\ast \subseteq T_{j,t}^\ast$.

If $T_{j,t}^* \subseteq T_{j,t}$, then $i_\tau < i_t^*$ and $(C^*, \tau, t_q^*)$ is a useful tuple. Suppose $(C^*, \tau, t_q^*)$ is a useful tuple contradicts the choice of $(C^*, t^*, t_q^*)$, where $C^*$ is the component of $P_1[|V(P_1) \cap (V_{T_j, t})|] - \{e \in E_{j,t}: e \subseteq X_{e^*}, t'' \in (V(T_j, t)) \cup \partial T_j, t \cup \{t\}$ is a witness for $X_{\tau^*} \cap I_j \subseteq \bigcup_{i=1}^{V(T)} W_3^{(i, \tau)}$ for some $i^* \in [0, i_t^*]$ containing $u_M$.

Let $(C^*, t^*, t_q^*)$, where $q^* \in [q^*]$, be the useful tuple such that:

- $q^*$ is as large as possible,
- subject to this, $(C^*, t^*, t_q^*)$ is (inclusion) maximal, and
- subject to these, the distance in $\mathcal{P}$ between $C^*$ and $b_1$ is as small as possible.

So $T_{j,t} \subseteq T_{j,t}^*$ and $i_\tau > i_{t^*}$. Suppose $\tau' \in \partial T_{j,t^*}$. Since $T_{j,t} \subseteq T_{j,t}^*$ and $\tau'$ is a witness for $X_{\tau'} \cap I_j \subseteq \bigcup_{i=1}^{V(T)} W_3^{(i, \tau')}$, we know $\tau^* \neq t^*$ by the condition for the first entry of a useful tuple.

So $T_{\tau, \tau} \subseteq T_{j,t}$ and $i_\tau > i_{t^*}$, $\tau' \in \partial T_{j,t^*}$. Since $T_{\tau, \tau} \subseteq T_{j,t}$ and $\tau' \in \partial T_{j,t^*}$, we know $\tau^*$ is not a witness for $X_{\tau^*} \cap I_j \subseteq \bigcup_{i=1}^{V(T)} W_3^{(i, \tau^*)}$. So there exists $\ell_{t^*} \in [0, |V(T)|]$ such that either $u^* \in D^{(i_{t^*}, \ell_{t^*}, 0)} - X_{t^*}$, or $u^* \in D^{(i_{t^*}, \ell_{t^*}, k_{t^*}+1)} \cap X_{t^*} \cap X_{\tau^*} \cap I_j$ for some witness $t'' \in V(T_{\tau^*}) - \{t^*\}$ for $X_{\tau^*} \cap I_j \subseteq W_3^{(i_{t^*}, t''\tau)}$ for some $i'' \in [0, i_{t^*} - 1]$ and $\ell'' \in [0, |V(T)|]$. Either case implies that there exists $t'' \in V(T)$ with height at most $i_{t^*}$ such that $u^* \in D^{(i_{t^*}, \ell, k_{t^*}+1)}$ for some $\ell \in [0, |V(T)|]$ and $k_{t^*} \in [0, s+1]$, and $T_{j,t^*} \subseteq T_{j,t''}$, where $i_{t''\tau}$ is the height of $t''$. Choose $t'' \tau$ so that $i_{t''\tau} \in [0, i_{t^*}]$ as small as possible. Since $\tau^*$ is not a witness for $X_{\tau^*} \cap I_j \subseteq \bigcup_{i=1}^{V(T)} W_3^{(i, \tau^*)}$, $i_{t''\tau} < i_{t^*}$. So $T_{j,t} \subseteq T_{j,t''\tau}$. Hence $(C^*, t^*, t_{q^*})$ contradicts the choice of $(C^*, t^*, t_{q^*})$, where $C^*$ is the component of $P_1[|V(P_1) \cap X_{V(T_j, t''\tau)}|] - \{e \in E_{j,t}: e \subseteq X_{t''\tau},$
where $t'' \in (V(T_{i,t}) - V(T_{j,t})) \cup \partial T_{j,t} \cup \{t''\}$ is a witness for $X_{i,t} \cap I_j \subseteq \bigcup_{i=1}^{V(T)} W_3^{\ell_{i,t}}$ for some $i'' \in [0, i_{t})$ containing $u^*$.

Hence $\tau' \notin \partial T_{j,t}$. Note that there exists no monochromatic path in $G[X_{V(T_{i,t})} \cap Y^{(i_{t}, |V(T)|+1, s+2)}]$ from $u^*$ to $v^*$, for otherwise deleting $e^*$ from $P$ and adding this path into $P - e^*$ contradicts the choice of $P$. Since $i_{t} \leq i_{t'}$, there exists no monochromatic path in $G[X_{V(T_{i,t})} \cap Y^{(i_{t'}, |V(T)|+1, s+2)}]$ from $u^*$ to $v^*$. Recall that some monochromatic component in $G[Y^{(i_{t'}, |V(T)|+1, s+2)} \cap X_{V(T_{j,t})}]$ with respect to $c$ intersecting $V(C_0)$ and $\bigcup_{i=1}^{V(T)} W_3^{\ell_{i,t}}$ and $e^* \cap V(C_0) \neq \emptyset$. So there exists a monochromatic path in $G[Y^{(i_{t'}, |V(T)|+1, s+2)} \cap X_{V(T_{j,t})}]$ from $u^*$ to $v^*$ and a monochromatic path $P'_{u^*}$ from $u^*$ to $v^*$ and a monochromatic path $P'_{v^*}$ in $G[Y^{(i_{t'}, |V(T)|+1, s+2)}]$ from $u^*$ to $v^*$. Since $\{u', v'\} \subseteq E^{(i_{t'}, |V(T)|+1, s+2)}$ and there exist monochromatic path $P'_{u^*}$ in $G[Y^{(i_{t'}, |V(T)|+1, s+2)}]$ from $u^*$ to $v^*$ and a monochromatic path $P'_{v^*}$ in $G[Y^{(i_{t'}, |V(T)|+1, s+2)}]$ from $v^*$ to $v^*$. Since $\{u', v'\} \subseteq E^{(i_{t'}, |V(T)|+1, s+2)}$ and there exist monochromatic path $P'_{u^*}$ in $G[Y^{(i_{t'}, |V(T)|+1, s+2)}]$ from $u^*$ to $v^*$ and a monochromatic path $P'_{v^*}$ in $G[Y^{(i_{t'}, |V(T)|+1, s+2)}]$ from $v^*$ to $v^*$.

Therefore, $E(C^*) \cap E_{j,t} = \emptyset$. In particular, $C^*$ is a path in $G[X_{V(T_{j,t})}]$. Since the s-segment containing $C^*$ whose level equals the color of $C^*$ belongs to $\mathcal{S}_j$, $V(C^*) \subseteq W_{1}^{(i_{t})}$. Since $M_C$, is monochromatic with respect to $c$ intersecting $V(C^*)$ and $\bigcup_{i=1}^{V(T)} W_3^{\ell_{i,t}}$, by Claim 23.1, $V(C^*) \subseteq Y^{(i_{t}, |V(T)|+1, s+2)}$ and $A_{E_{j,t}}^{(i_{t}, |V(T)|+1, s+2)}(V(C^*) \cap X_{V(T_{j,t})}) \cap \hat{X}_{V(T_{j,t})} = \emptyset$. So $C^*$ is a path in $G[Y^{(i_{t}, |V(T)|+1, s+2)} \cap X_{V(T_{j,t})} \cap \hat{X}_{V(P_i)}]$. Since $i_{t} \geq i_{t'}$, $Y^{(i_{t}, |V(T)|+1, s+2)} \subseteq Y^{(i_{t'}, |V(T)|+1, s+2)}$. Hence, if some end of $C^*$ is not in $X_{i_{t}} \cup \partial T_{j,t}$, then this end is either $v_M$ or $b_1$. If $b_1$ is an end of $C^*$ and $b_1 \notin X_{i_{t}} \cup \partial T_{j,t}$, then $a_1 \in X_{V(T_{j,t})}$, and $c(b_1) \notin L^{(i_{t}, |V(T)|+1, s+2)}(a_1)$ (since $A_{E_{j,t}}^{(i_{t}, |V(T)|+1, s+2)}(V(C^*) \cap X_{V(T_{j,t})}) \cap \hat{X}_{V(T_{j,t})} = \emptyset$, but $c(a_1) = c(b_1)$, a contradiction. So all ends of $C^*$ except possibly $v_M$ are in $X_{i_{t}} \cup \partial T_{j,t}$.

Let $u, v$ be the ends of $C^*$, where the distance in $P$ between $b_1$ and $v$ is at most the distance in $P$ between $b_1$ and $u$. So $v \in X_{i_{t}} \cup \partial T_{j,t}$, and either $u = v_M$ or $u \in X_{i_{t}} \cup \partial T_{j,t}$. Suppose $v \in X_{i_{t}} - \partial T_{j,t}$. Since $t^* = 1\hat{a}$ is a witness for $X_{i_{t}} \cap I_j \subseteq W_{3}^{(i_{t}, -1)}$, $b_1 \notin X_{V(T_{i_{t}})} - X_{i_{t}}$, for otherwise $(C', t^*, q^*)$ contradicts the choice of $(C^*, t^*, q^*)$ for some $C'$ that is closer to $b_1$ in $P$. So by the definition of a useful tuple, $t^* = t_1$ and $b_1 \in X_{i_{t}}$. By the condition for the distance for $C^*$, $b_1 = v$. Since $b_1 = v \notin \partial T_{j,t}$, and $a_1 \in V(P) \subseteq X_{V(T_{i_{t}})}$, $a_1 \in X_{V(T_{j,t})} \cap Z_{t_1}$. Hence $a_1 \in Y^{(i_{t}, |V(T)|+1, s+2)} \cap Y^{(i_{t'}, |V(T)|+1, s+2)}$, a contradiction.

Hence there exists $z \in \partial T_{j,t}$ such that $v \in X_{z}$. Choose $z$ so that the subpath of $P$ between $v$ and $a_1$ interacts $X_{V(T_{z})} - X_{z}$ if possible. Suppose that the subpath of $P$ between $v$ and $a_1$ is disjoint from $X_{V(T_{z})} - X_{z}$. Then $v = b_1$ and $a_1 \in X_{V(T_{j,t})}$. But $a_1 \in A_{E_{j,t}}^{(i_{t}, |V(T)|+1, s+2)}(V(C^*) \cap X_{V(T_{j,t})}) \cap \hat{X}_{V(T_{j,t})} = \emptyset$, a contradiction.

Hence the subpath of $P$ between $b_1$ and $a_1$ interacts $X_{V(T_{z})} - X_{z}$. Since $P$ is internally disjoint from $X_{V(T_{z})} - X_{z}$, $t^* \neq z$. If $t \in V(T_{z}) - \{z\}$, then $t \notin V(T_{j,t})$, and since $T_{j,t} \subseteq T_{j,t}$ and $t \in V(T_{j,t})$, we know $q^* < q$ and $t_{q^*+1} \in V(T_{j,t})$, so there exists $q' \in \mathbb{N}$ such that $z \in V(T_{j,t_{q^*+q'}})$ and $V(T_{j,t_{q^*+q'}}) \cap V(T_{j,t}) - \{z\} \neq \emptyset$, and there exists another useful tuple $(C'', t_{q^*+q'}, t_{q^*+q'})$, where $C''$ is the component containing $v$, contradicting the choice of $(C^*, t^*, q^*)$.

So $t \notin V(T_{z})$. Hence there exists a maximal subpath $P_{z}$ of the subpath of $P$ between $v$ and $V(P) \cap X_{z}$ from $v$ to $X_{z}$ internally disjoint from $X_{z}$. Note that $P_{z}$ is contained in $G[X_{V(T_{z})}]$ with both ends of $P_{z}$ belonging to $X_{z}$, and $V(P_{z}) - X_{z} \neq \emptyset$.

Suppose that $z$ is not a witness for $X_{z} \cap I_j \subseteq \bigcup_{i=1}^{V(T)} W_3^{(i_{t}, \ell_{i,t})}$. Since $C^*$ is a connected subgraph in $G[V(P) \cap X_{V(T_{j,t})}]$ and $V(M_C) \cap \bigcup_{i=1}^{V(T)} W_3^{(i_{t}, \ell_{i,t})} \neq \emptyset$, there exists $\ell_{z} \in [0, |V(T)|)$ such that either $v \in D^{(i_{t}, \ell_{z}, 0)} - X_{z}$, or $v \in D^{(i_{t}, \ell_{z}, 0)} \cap X_{z} - X_{z} \cap I_j$ for some witness $q^* \in V(T_{i_{t}}) - \{t^*\}$ for
There exists the minimum \((z\in [0,i_t-1])\) and \(\ell'\in [0,|V(T)|]\). In either case, \(v\in D(i_t,\ell'.t')\). So let \(\ell''\) be the nodes of \(T\) of height \(\ell''\). In either case, there exists a monochromatic path in \(G[X(y''_v,|V(T)|+1,\ell'')\cap X(V(T))]\) connecting \(v\) and \(u_v\). If \(i''<i_t\) and \(v\not\in V(T)\), then \(T_{i,t'}\subset T_{i,t''}\). If \(v\in V(T)\), then \(i''<i_t\) and a node \(q'\in \partial T_{i,t'}\cap V(T_r)\) \(-\{t^*\} mentioned above exists, so \(T_{i,t'}\subset T_{i,t''}\). Hence \(T_{i,t'}\subset T_{i,t''}\) in either case, so \((C'',t'',t_{q^*})\) (for some \(C''\)) contradicts the choice of \((C^*,t^*,t_{q^*})\).

Hence \(z\) is a witness for \(X_z\cap I_j\subset \bigcup_{l=-1}^{t-1} W_3(i_{t}, \ell')\). If \(V(P)_z\subset Y(i_p,|V(T)|+1,\ell'',t'')\subset Y(i,|V(T)|+1,\ell''+1)\)(\(u_z\not\in V(P)_z\)), then \((C'',t'',t_{q^*})\) contradicts the distance condition in the choice of \((C^*,t^*,t_{q^*})\), where \(C''\) is the component containing the end of \(P\) other than \(v\).

So \(V(P)_z\subset Y(i_p,|V(T)|+1,\ell'',t'')\subset Y(i,|V(T)|+1,\ell''+1)\)(\(u_z\not\in V(P)_z\)), let \(u_z, v_z\) be the ends of \(P\). Not \(\{u_z, v_z\}\in X_z\cap I_j\subset \bigcup_{l=-1}^{t-1} W_3(i_{t}, \ell')\). By the existence of \(P\), \(L(i_{t},|V(T)|+1,\ell'',t'')\)(\(u_z\)
\(\bigcup_{l=-1}^{t-1} W_3(i_{t}, \ell')\)) \(\bigcup_{l=-1}^{t-1} W_3(i_{t}, \ell')\), and \(\{u_z, v_z\}\subset S\), where \(S\) is an s-segment whose level equals the element in \(L(i_{t},|V(T)|+1,\ell'',t'')(u_z)\) such that \(S\) intersects \(I_j\).

Note that \(\{u_z, v_z\}\not\in E_{i,t}\), for otherwise \((P - (V(P)_z - \{u_z, v_z\}))\)(\(u_z, v_z\)) contradicts the choice of \(P\) by (a). If \(p\not\in V(T_{j,t'})\), then define \(z'\) to be the node in \(\partial T_{j,t'}\) such that \(p\in V(T_{j,t'})\); otherwise, \(p\in V(T_{j,t'})\) and we define \(z' = p\). If \(u_z, v_z\in E_{i_{t},\ell''}\) for some \(k''\in [0, w_0 + 1]\), then either \(z' = t^*\) or \(u_z, v_z\in E_{j,t}\), so \(u_z, v_z\in E_{j,t}\), a contradiction. So \(u_z, v_z\not\in E_{i_{t},\ell''}\) for every \(k''\in [0, w_0 + 1]\). Since there exists no node \(t''\) of \(T\) of height at most \(i_t\), such that exactly one of \(z'\) and \(z\) belongs to \(T_{j,t''}\), we know \(E_{i_{t},\ell''}\subset E_{i_{t},\ell''}\) for every \(k''\in [0, w_0 + 1]\).

Hence \(\{u_z, v_z\}\not\in E_{i_{t},\ell''}\) for every \(k''\in [0, w_0 + 1]\). For every \(i''\in [i_t, i - 1]\) and \(k''\in [0, w_0 + 1]\), let \(M_{i_{t},i_{t},\ell''}\) be the monochromatic \(E_{i_{t},\ell''}\)-pseudocomponents with respect to \(c\) contained in \(G[Y(i''_v,|V(T)|+1,\ell'')]\) containing \(u_z, v_z\), respectively. Note that \(M_{i_{t},i_{t},\ell''}\neq M_{i_{t},i_{t},\ell''}\) for every \(i''\in [i_t, i - 1]\) and \(k''\in [0, w_0 + 1]\), by (a).

For every \(y\in \{u_z, v_z\}\), let \(P_y\) be the maximal subpath of \(P\) in \(G(V(P)_z\cap Y(i''_v,|V(T)|+1,\ell''))\) containing \(y\). Note that for every \(y\in \{u_z, v_z\}\), \(P_y\subset M_{i_{t},i_{t},\ell''}\) for every \(k''\in [0, w_0 + 1]\). Since \(V(P)_z\subset Y(i''_v,|V(T)|+1,\ell'')\), \(P_y\subset P_z\) for every \(y\in \{u_z, v_z\}\).

Suppose there exist \(y''\in \{u_z, v_z\}\) and \(k''\in [0, w_0 - 1]\) such that \(A_{L(i''_v,|V(T)|+1,\ell''+2)}(M_{i''_v,i''_v,\ell''})\cap V(P_z)\cap X(V(T)) - X_z = 0\). Since \(P_y\subset P_z\), there exists \(e_y\in E(P_z)\cap E_{i_{t},\ell''}\) such that \(e_y = \{y_1, y_2\}\) for some \(y_1\in V(P_y)\) and \(y_2\in V(P_y) - V(P_z)\), and the subpath of \(P_y\) between \(y^*\) and \(y_1\) is a path in \(G[Y(i''_v,|V(T)|+1,\ell'')]\). So there exist \(z_{e_y}\in V(T)\) with \(z\in V(T_{z_{e_y}})\) and \(z'_{e_y}\in V(T_{j_{z_{e_y}}})\) such that \(z'_{e_y}\) witnesses that \(e_y\in E_{i_{t},\ell''}\). So \(z'_{e_y}\) is a witness for \(X_{z_{e_y}}\cap I_j \subset W_3(x_{e_y},\alpha)\) for some \(\alpha\in [0,|V(T)|]\). Since \(V(P)_z\subset X(V(T))\) and \(P_z\) is internally disjoint from \(X_z\), \(z_{e_y}\in V(T) - \{z\}\). Since \(z\in V(T_{z_{e_y}})\), \(T_{j_{z_{e_y}}}\subset T_{j_{z_{e_y}}}\). Since the subpath of \(P_y\) between \(y^*\) and \(y_1\) is contained in \(P_y\subset P_z\cap X(V(T_{j_{z_{e_y}}}))\) and contains \(y_1\in \bigcup_{l=-1}^{t-1} W_3(i_{t}, \ell')\), we know \((C'', z_{e_y}, t_{q^*})\), for some \(C''\), is a useful pair contradicting the choice of \((C^*, t^*, t_{q^*})\).

Hence for every \(y\in \{u_z, v_z\}\) and \(k''\in [0, w_0 - 1]\), \(A_{L(i''_v,|V(T)|+1,\ell''+2)}(M_{i''_v,i''_v,\ell''})\cap V(P_z)\cap X(V(T)) - X_z = 0\). Furthermore, since \(\{u_z, v_z\}\subset V(P_1)\), by the choice of \((C^*, t^*, t_{q^*})\), there does not exist \(i''\in V(T)\) of height \(i''<i_t\) and \(T_{j,t}\subset T_{j,t''}\) such that there exists a monochromatic path in \(G[Y(i''_v,|V(T)|+1,\ell'')]\) from \(V(P)_z\) to \(\{u_z, v_z\}\).

For every \(i''\in [i_t, i - 1]\), \(k''\in [0, w_0 + 1]\), let \(M_{i_{t},i_{t},\ell''}\) and \(M_{i_{t},i_{t},\ell''}\) be the monochromatic \(E_{i_{t},\ell''}\)-pseudocomponents with respect to \(c\) contained in \(G[Y(i''_v,|V(T)|+1,\ell'')]\) containing \(u_z\) and \(v_z\), respectively.
Since \( \{u_z, v_z\} \not\in E_{j,z}^{(i,z,k)} \) for every \( k'' \in [0, w_0 + 1] \), \( z \) is not a witness for \( \{u_z, v_z\} \in E_{j,z}^{(i,z,w_0+1)} \). Recall that \( M_{s,i,z,k''} \neq M_{u_z,i,z,k''} \). By symmetry, we may assume that \( \sigma(M_{u_z,i,z,k''}) < \sigma(M_{v_z,i,z,k''}) \). Recall that \( z \) is a witness for \( X_z \cap I_j \subseteq \gamma_{3}^{(i,z)} \) for some \( \alpha \in [0, |V(T)|] \). Recall that \( E_{j,z}^{(i,z,k'')} = E_{j,z}^{(i,z,k'')} \) for every \( k'' \in [0, w_0 + 1] \). So there exist a minimum \( k^* \in [0, w_0 - 1] \) such that:

- the monochromatic \( E_{j,z}^{(i,z,k^*)} \)-pseudocomponent \( M_{k^*} \) in \( G[Y^{(i,z,-1,0)}] \) with respect to \( c \) such that \( \sigma(M_{k^*}) \) is the \((k^*+1)-\)smallest among all monochromatic \( E_{j,z}^{(i,z,k^*)} \)-pseudocomponents \( M' \) in \( G[Y^{(i,z,-1,0)}] \) with respect to \( c \) intersecting \( X_{t_r} \) and contained in some \( s \)-segment in \( S_j^{o} \) whose level equals the color of \( M' \),
- \( V(M_{k^*}) \cap X_z \neq \emptyset \) (by the existence of \( P_1 \)),
- \( A_{L,[i,z,|V(T)|/{1, s+2}]}(V(M_{k^*})) \cap X_{V(T_{s})} - X_z \neq \emptyset \),
- there exists \( \ell^* \in [0, w_0 - 1] \) such that \( \sigma(M_{u_z,i,z,k^*,\ell^*}) \) is the \((\ell^*+1)-\)th smallest among all monochromatic \( E_{j,z}^{(i,z,k^*,\ell^*)} \)-pseudocomponents \( M' \) in \( G[Y^{(i,z,-1,0)}] \) with respect to \( c \) contained in some \( s \)-segment in \( S_j^{o} \) whose level equals its color such that \( V(M') \cap X_z \neq \emptyset \) and \( A_{L,[i,z,|V(T)|/{1, s+2}]}(V(M')) \cap X_{V(T_{s})} - X_z \neq \emptyset \).

Denote \( M_{u_z,i,z,k^*,\ell^*} \) by \( M_{u_{z}} \).

Now we prove that if \( M_1 \) is a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) with \( M_1 \supseteq M_{u_z} \), and \( M_2 \) is a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) intersecting \( X_z \) with \( M_2 \supseteq M_{j,\alpha} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), then \( M_2 \supseteq M_{j,\alpha}^{(t_r)} \) for some \( \alpha' \) with \( \sigma(M_{j,\alpha}^{(t_r)}) \leq \sigma(M_{u_z}) \). Let \( M_1 \) be a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) with \( M_1 \supseteq M_{u_z} \), and \( M_2 \) be a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) with \( M_2 \supseteq M_{j,\alpha}^{(t_r)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \). Note that we may assume \( M_1 \neq M_2 \), so \( \sigma(M_2) < \sigma(M_1) \). Since \( \sigma(M_2) < \sigma(M_1) \), by the choice of the counterexamples of this claim, there exists a monochromatic \( E_{j,\alpha} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) containing \( v_{M_2} \) and some vertex \( u_{M_2} \) in \( X_{t_r} \), where \( v_{M_2} \) is the vertex in \( V(M_2) \) with \( \sigma(v_{M_2}) = \sigma(M_2) \). So there exists a monochromatic \( E_{j,\alpha} \)-pseudocomponent \( M_2' \) in \( G[Y^{(i,z,-1,0)}] \) with respect to \( c \) containing \( v_{M_2} \) (and hence intersecting \( X_{t_r} \)) such that \( S_{M_2'} \in S_j^{o} \). Since \( M_2 \) is a monochromatic \( E_{j,\alpha} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) containing \( u_{M_2} \) (and hence \( M_2 \supseteq M_2' \)) intersecting \( X_z \), by Claim 23.29, if \( A_{L,[i,z,|V(T_{s})|/{1, s+2}]}(V(M_2')) \cap X_{V(T_{s})} - X_{t_r} = \emptyset \), then \( V(M_2') = V(M_2) \), so for every \( \alpha \in [0, w_0 - 1] \) with \( M_{j,\alpha}^{(t_r)} \subseteq M_2 \), we have \( \sigma(M_{j,\alpha}^{(t_r)}) = \sigma(M_2) \leq \sigma(M_{u_z}) \), and we are done. So we may assume that \( A_{L,[i,z,|V(T_{s})|/{1, s+2}]}(V(M_2')) \cap X_{V(T_{s})} - X_{t_r} \neq \emptyset \). Hence there exists \( M_{j,\alpha'}^{(t_r)} \) containing \( M_2' \) such that \( M_2 \supseteq M_{j,\alpha'}^{(t_r)} \supseteq M_2' \) and \( \sigma(M_{j,\alpha'}^{(t_r)}) = \sigma(M_2') = \sigma(M_2) = \sigma(M_2') \leq \sigma(M_{u_z}) \).

Hence if \( M_1 \) is a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) with \( M_1 \supseteq M_{u_z} \), and \( M_2 \) is a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) intersecting \( X_z \) with \( M_2 \supseteq M_{j,\alpha}^{(t_r)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_2) \leq \sigma(M_1) \), then \( M_2 \supseteq M_{j,\alpha'}^{(t_r)} \) for some \( \alpha' \) with \( \sigma(M_{j,\alpha'}^{(t_r)}) \leq \sigma(M_{u_z}) \). Hence there exists \( k^*_z \in [0, \ell^*] \) such that \( M_{j,k^*_z} \supseteq M_{u_z} \).

Now we prove that for every \( z'' \in V(T_z) - \{z\} \) and every \( \ell_0 \in [0, k^*_z] \), if \( M_3 \) is a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) with \( M_3 \supseteq M_{j,\ell_0}^{(t_r)} \) and \( M_4 \) is a monochromatic \( E_{j,z} \)-pseudocomponent in \( G[Y^{(i,z,-1,0)}] \) intersecting \( X_{z''} \) with \( M_4 \supseteq M_{j,\alpha}^{(t_r)} \) for some \( \alpha \in [0, w_0 - 1] \) such that \( \sigma(M_4) \leq \sigma(M_3) \), then \( M_4 \supseteq M_{j,\alpha'}^{(t_r)} \) for some \( \alpha' \neq \ell_0 \). We may assume \( \sigma(M_3) = \sigma(M_4) \), for otherwise we are done. So \( \sigma(M_4) = \sigma(M_3) \). Since \( M_3 \supseteq M_{j,\ell_0}^{(t_r)} \) and \( \ell_0 \leq k^*_z \), \( \sigma(M_4) < \sigma(M_3) \leq \sigma(M_{j,k^*_z}^{(z)}) \leq \sigma(M_{j,k^*_z}^{(z)}) \leq \sigma(M_{u_z}) = \sigma(M) \). By the choice of \( M \) for the counterexamples, there exists
a monochromatic $E_{j,t}$-pseudocomponent $M_4'$ containing $v_{M_4'}$ and some vertex $u_{M_4}$ in $V(M_4) \cap X_z$, where $v_{M_4'}$ is the vertex in $M_4'$ with $\sigma(M_4') = \sigma(v_{M_4'})$, and hence $S_{M_4'} = S_{M_4} \in S_T$. Since $M_4$ is a monochromatic $E_{j,t}$-pseudocomponent in $G[Y^{(i_{j,t}',-1,0)}]$ with $M_4 \supseteq M_4'$ and $V(M_4) \cap X_{z''} \neq \emptyset$, by Claim 23.29, if $A_{L(t_{i_{j,t}'},-1,0)}(V(M_4')) \cap X_{V(T_z)} \neq \emptyset$, then $V(M_4') = V(M_4)$, so for every $\alpha \in [0, w_0 - 1]$ with $M_{\alpha,t} \subseteq M_4$, we have $\sigma(M_{\alpha,t}) = \sigma(M_4) \leq \sigma(M_{u,s})$, and we are done. So we may assume that $A_{L(t_{i_{j,t}'},-1,0)}(V(M_4')) \cap X_{V(T_z)} = \emptyset$. Hence there exists $M_{\alpha,t}$ containing $M_4'$ such that $M_4 \supseteq M_{\alpha,t} \supseteq M_4'$ and $\sigma(M_{\alpha,t}) = \sigma(M_4) \leq \sigma(M_{u,s})$.

Therefore, by Claim 23.28, $V(P_2) \subseteq Y^{(i_{j,t}',-1,0)} \subseteq Y^{(i_{j,t},-1)} \subseteq Y^{(i_{j,t},|V(T)| + 1, s+2)}$, a contradiction. This proves the claim.

Recall that for any monochromatic component $M$ with respect to $c$, $S_M$ is the unique $s$-segment $S_M$ whose level equals the color of $M$ with respect to $c$, and $r_M$ is the node $t$ of $T$ of the smallest height with $X_t \cap V(M) \neq \emptyset$.

Given any monochromatic component $M$ with respect to $c$ with $S_M \cap I_j \neq \emptyset$ for some $j \in [|V| - 1]$, let $K(M)$ be the subset of $V(T)$ constructed by repeatedly applying the following process until no more nodes can be added:

- $r_M \in K(M)$.
- For every $t \in K(M)$, if there exists $t' \in \partial T_{j,t}$ such that $V(M) \cap X_{V(T_{j,t})} = X_t \neq \emptyset$, then adding $t'$ into $K(M)$.

Claim 23.31. Let $j \in [|V| - 1]$. Let $M$ be a monochromatic component with respect to $c$ with $S_M \cap I_j \neq \emptyset$. Let $t \in V(T)$ be a node with $V(M) \cap X_{V(T_t)} = X_t \neq \emptyset$, and let $v \in V_T$. Then $v' \in V(M)$ such that $t \in V(T_{r_{v'}})$, where $r_{v'}$ is the node of $T$ of smallest height with $v' \in X_{r_{v'}}$. If there exists a path in $M$ from $v'$ to $v$ disjoint from $(X_{V(T_t)} - X_t) \cup (V(G) - X_{V(T_{r_{v'}})})$, then there exists a monochromatic $E_{j,t}$-pseudocomponent in $G[Y^{(i_{j,t},-1,0)} \cap V(M)]$ containing $v'$ and $v$, where $i$ is the height of $t$.

Proof. Suppose to the contrary that no $E_{j,t}$-pseudocomponent in $G[Y^{(i_{j,t},-1,0)} \cap V(M)]$ contains both $v'$ and $v$, where $i$ is the height of $t$. We further assume that the distance in $T$ between $t$ and $r_M$ is as small as possible among all counterexamples of this claim.

Suppose $t = r_M$. Then $r_{v'} = r_M$. Since there exists a path in $M$ from $v'$ to $v$ disjoint from $(X_{V(T_t)} - X_t) \cup (V(G) - X_{V(T_{r_{v'}})})$, this path is contained in $G[X_{r_M}]$. Since $X_t \subseteq Y^{(i_{j,t},-1)}$, every vertex in $X_t \cap V(M)$ is contained in some component of $G[Y^{(i_{j,t},-1,0)} \cap V(M)]$. In particular, $v$ and $v'$ belong to the same component of $G[Y^{(i_{j,t},-1,0)} \cap V(M)]$. So some $E_{j,t}$-pseudocomponent of $G[Y^{(i_{j,t},-1,0)} \cap V(M)]$ contains $v'$ and $v$.

So $t \neq r_M$. Hence there exists $t' \in K(M)$ such that $t \in V(T_{j,t'}) - \{t'\}$. Let $i'$ be the height of $t'$.

Since $t' \in K(M)$, $t' \in V(T_{r_M})$. Since $i > i'$ and there exists a path $P^*$ in $M$ from $v'$ to $v$ internally disjoint from $(X_{V(T_t)} - X_t) \cup (V(G) - X_{V(T_{r_{v'}})})$, $P^*$ contains a maximal subpath $P^{**}$ from $v'$ to $v$ internally disjoint from $X_{v'}$. Let $u$ be the end of $P^{**}$ other than $v$, if $|V(P^{**})| \geq 2$; let $u = v'$ if $|V(P^{**})| = 1$. Let $P$ be the subpath of $P^*$ from $u$ to $v$. Since $V(M) \cap X_{V(T_t)} - X_t \supseteq V(M) \cap X_{V(T_t)} - X_t \neq \emptyset$, the minimality of the distance between $t$ and $r_M$ implies that there exists a monochromatic $E_{j,t}$-pseudocomponent $M_0$ in $G[Y^{(i_{j,t}',-1,0)} \cap V(M)]$ containing $v'$ and $u$.

Since there exists no node $t''$ of $T$ of height at most $i'$ such that $T_{v'}$ contains exactly one of $t$ and $t'$, $E_{j,t} \supseteq E_{j,t}(t',0) \supseteq E_{j,t'}$. So if $V(P) \subseteq Y^{(i_{j,t},-1,0)}$, then $M_0 \cup P$ is contained some monochromatic
$E_{j,t'}$-pseudocomponent of $G[Y^{(i,-1,0)} \cap V(M)]$ containing $v$ and $v^*$, a contradiction. Hence $V(P) \ni Y^{(i,-1,0)}$.

Since $u \in X_{t'} \cap V(M_0)$ and $Y^{(s',-1,0)} \subseteq Y^{(i',[V(T)]+1,s+2)}$, there exists $k \in [0, u_0 - 1]$ such that $M_0$ is contained in $M'$, where $M'$ is the monochromatic $E_{j,t'}$-pseudocomponent with respect to $c$ in $G[Y^{(s',[V(T)]+1,s+2)}]$ such that $\sigma(M')$ is the $(k+1)$-th smallest among all monochromatic $E_{j,t'}$-pseudocomponents in $G[Y^{(s',[V(T)]+1,s+2)}]$ intersecting $X_{t'}$ and contained in some $s$-segment in $S_j$ whose level equals its color. Note that $M' \cup P \supseteq M_0 \cup P$ contains $v$ and $v^*$, and $E_{j,t'} \supseteq E_{j,t,0}^{(i',u_0+1)}$.

Recall that for every node $q$ of $T$, let $i_q$ be the height of $q$.

Let $t^*$ be the node of $T$ such that:

(a) $T_{j,t'} \subseteq T_{j,t^*}$,

(b) there exists a monochromatic path in $G[Y^{(i_1,-1,0)} \cap X_{V(T_{j,t^*})}]$ from $V(P)$ to $\bigcup_{t=1}^{[V(T)]} W_{3}^{(i_2,t)}$, subject to these, $T_{j,t^*}$ is (inclusion) maximal.

Note that $t^*$ exists since $t'$ is a candidate. Since $t \in V(T_{j,t'})$, then $t \in V(T_{j,t^*})$. Note that $i_{t^*} \leq i' < i$. Note that (b) implies that $V(M) \cap X_{t'} \neq \emptyset$ for some witness $t'' \in \partial T_{j,t'} \cup \{t^*\}$ for $X_{t'} \cap I_j \subseteq W_3^{(i_{t''},t)}$ for some $\ell \in [-1,[V(T)]$. By Claim 23.22, $V(M) \cap X_{V(T_{j,t^*})} \subseteq Y^{(i_1,[V(T)]+1,s+2)}$ and $X_{(i_1,t') \cap [V(T)](i_1+1,s+2)}(V(M) \cap X_{V(T_{j,t^*})}) \cap X_{V(T_{j,t^*})} = \emptyset$. In addition, (b), there exists a component $C_0$ of $M[V(M) \cap X_{V(T_{j,t^*})}]$ intersecting $V(P)$.

We first claim that every component of $M[V(M) \cap X_{V(T_{j,t^*})}]$ intersecting $V(P^*)$ intersects $X_{t''}$ for some witness $t''$ for $X_{t''} \cap I_j \subseteq W_3^{(i_{t''},t)}$ for some $\ell \in [-1,[V(T)]$. Suppose to the contrary that there exists a component $C^*$ of $M[V(M) \cap X_{V(T_{j,t^*})}]$ intersecting $V(P^*)$ disjoint from $X_{t''}$ for every witness $t''$ for $X_{t''} \cap I_j \subseteq W_3^{(i_{t''},t)}$ for some $\ell \in [-1,[V(T)]$. We further choose $C^*$ such that the subpath in $P^*$ from $V(C')$ to $V(C_0)$ is as short as possible. Let $C_1$ be the component of $M[V(M) \cap X_{V(T_{j,t^*})}]$ intersecting $V(P^*)$ such that $C_1 \neq C_0$ and the subpath in $P^*$ from $V(C')$ to $V(C_0)$ intersects $V(C_1)$, and subject to those, the subpath in $P^*$ from $V(C_1)$ to $V(C_0)$ is as short as possible. So there exists $t_{C^*} \in \partial T_{j,t^*} \cup \{t^*\}$ such that $X_{t_{C^*}} \cap V(C_1) \neq \emptyset \neq X_{t_{C^*}} \cap V(C'')$. Note that $t_{C^*}$ is not a witness $t''$ for $X_{t''} \cap I_j \subseteq W_3^{(i_{t''},t)}$ for any $\ell \in [-1,[V(T)]$. Since $t^*$ is a witness for $X_{t^*} \cap I_j \subseteq W_3^{(i_{t^*},-1)}$, $t_{C^*} \neq t^*$, so $t_{C^*} \in \partial T_{j,t^*}$. From the minimality of the distance between $C_0$ and $C'$ in $P^*$, we know that $V(C_1)$ intersects $X_{t_{C_1}}$ for some witness $t_{C_1}$ for $X_{t_{C_1}} \cap I_j \subseteq W_3^{(i_{t_{C_1}},t_{C_1})}$ for some $\ell_{C_1} \in [-1,[V(T)]$. Since $t_{C_1}$ is a witness but $t_{C^*}$ is not, we know either there exists $x_{C_1} \in X_{t_{C_1}} \cap V(C_1) \cap D(i_1,t_{C_1},0) - X_{t^*}$ for some $\ell_{C_1} \in [0,[V(T)])$, or there exists $x_{R} \in X_{t_{C^*}} \cap V(C_1) \cap D(i_1,t_{C_1},0) \cap X_{t_R} \cap I_j$ for some witness $q' \in V(T)$ for $X_{q'} \cap I_j \subseteq W_3^{(i_{C_1},t_{C_1})}$ for some $\ell_{C_1} \in [0,i^* - 1]$ and $\ell_{C_1} \in [0,[V(T)])$. In either case, there exists minimum $x_{C_1} \in D(i_1,t_{C_1},0)$ for some $\ell_{C_1} \in [0,[V(T)])$. If $i_{t^*} < i_{t_{C^*}}$, then the node $t''$ of height $i_{t^*}$ with $t^* \in V(T_{t''})$ contradicts the choice of $t^*$ by Claim 23.1. So $i_{t^*} = i_{t^*}$. It implies that $t_{C^*}$ is the witness for $X_{t_{C^*}} \cap I_j \subseteq W_3^{(i_{t_{C^*}},t_{C^*})}$ if we choose $\ell_{C^*}$ to be as small as possible, a contradiction.

Hence every component of $M[V(M) \cap X_{V(T_{j,t^*})}]$ intersecting $V(P^*)$ intersects $X_{t''}$ for some witness $t''$ for $X_{t''} \cap I_j \subseteq W_3^{(i_{t''},t)}$ for some $\ell \in [-1,[V(T)]$. Recall that $V(P) \ni Y^{(i,-1,0)}$. So $V(P) \ni Y^{(i-1,0)}$. Since $V(M) \cap X_{V(T_{j,t^*})} \subseteq Y^{(i-1,0)}$, $V(P) \cap X_{V(T_{j,t^*})} = \emptyset$. Note that for every subpath $Q$ of $P^*$ with $V(Q) \subseteq V(P)$, there exist a subpath $\overline{Q}$ of $P^*$ with $V(Q) \subseteq V(\overline{Q})$ internally disjoint from $X_{V(T_{j,t^*})}$ and there exists $t_{Q} \in \partial T_{j,t^*}$ such that $\overline{Q}$ is from a vertex $u_Q \in X_{t_Q}$ to a vertex $v_Q \in X_{t_Q}$, $V(\overline{Q}) \subseteq V(X_{t_Q})$ and $V(\overline{Q}) \cap X_{V(T_{j,t^*})} = V(Q) \neq \emptyset$.

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Suppose that there exists a path $Q$ of $P^*$ with $V(Q) \subseteq V(P) \cap X_{V(T_{i^*})} - X_{V(T_{j^*})}$ such that $V(Q) \not\subseteq Y^{(i_{i^*},|V(T)|+1,s+2)}$ and \{u_Q, v_Q\} \not\in E_{j,t}$. We choose $Q$ such that the distance in $P^*$ between $V(Q)$ and $v^*$ is as small as possible. Let $q$ be the child of $t^*$ such that $t_q \in V(T_q)$. For every $k' \in [0, w_0 + 1]$, let $M_{iQ,k'}$ and $M_{iQ,k'}$ be the monochromatic $E_{i^*}^{(i_{i^*},k')}\text{-}pseudocomponent in G[Y_{V(T)|+1,s+2}]$ with respect to $c$ containing $u_Q$ and $v_Q$, respectively. Since $Q$ is a path in $G$, for every $k' \in [0, w_0 - 1]$, $A_{G_{i^*},|V(T)+1,s+2}(V(M_{iQ,k'})) \cap \{X_{V(T_q)} - X_{T_q}\}$ is a monochromatic path in $G[Y_{V(T)|+1,s+2}]$ such that there exists a monochromatic path in $G[Y_{V(T)|+1,s+2}]$ from $\bigcup_{t=1}^{t=1} W_{3}^{(i_{i^*},k')}$ to \{u_Q, v_Q\}. Since \{u_Q, v_Q\} \not\in E_{j,t}, \{u_Q, v_Q\} \not\in E_{j,t}^{(i_{i^*},w_0+1)}$. Therefore, by the choice of $Q$, we know that $V(Q) \subseteq Y_{V(T)|+1,s+2}$ is a path of height $i_{i^*}$ with $i_{i^*} < i^*$ and $T_{Q,t} \subseteq T_{j,t+r}$ such that there exists a monochromatic path in $G[Y_{V(T)|+1,s+2}]$ with respect to $c$ containing $u$ and $v$. Therefore there exists a monochromatic $E_{j,t}^{-}\text{-}pseudocomponent in G[Y_{V(T)|+1,s+2}]$ with respect to $c$ containing $v^*$ and $v$. This proves the claim.

**Claim 23.32.** Let $j \in [|\mathcal{V}|-1]$. Let $i \in \mathbb{N}_0$, and let $t \in V(T)$ a node of $T$ of height $i$. Assume that $M_{j,0}^{(i)}$ is of color $k+1$ for some $k \in [0, s+1]$ and the $s$-segment $S$ of level $k+1$ containing $V(Q_{j,0})$ intersecting $I_j^s$. If $x$ is a vertex in $X_{V(T_j)}$ such that there exists a monochromatic path $P$ with respect to $c$ contained in $G[X_{V(T_j)}]$ from $x$ to $V(M_{j,0}) \cap X_t$, then $x \in Y_{V(T)|+1}^{(i)}$.

**Proof.** By Claim 23.27, it suffices to prove that for every $t' \in V(T^t)$, if there exists a monochromatic $E_{j,t'}^{(i_{i^*},w_0+1)}\text{-}pseudocomponent in G[Y_{V(T)|+1,s+2}]$ with $M \geq M_{j,0}^{(i_{i^*},w_0+1)}$ such that $V(M) \cap X_t \neq 0$ and $A_{G_{i^*},|V(T)|+1,s+2}(V(M)) \cap X_{V(T_t)} - X_{T_t} \neq 0$, then $M = M_{j,0}^{(i_{i^*},w_0+1)}$.

Let $t' \in V(T_t)$. Let $M$ be a monochromatic $E_{j,t'}^{(i_{i^*},w_0+1)}\text{-}pseudocomponent in G[Y_{V(T)|+1,s+2}]$ with $M \geq M_{j,0}^{(i_{i^*},w_0+1)}$ such that $V(M) \cap X_t \neq 0$ and $A_{G_{i^*},|V(T)|+1,s+2}(V(M)) \cap X_{V(T_t)} - X_{T_t} \neq 0$. Note that $E_{j,t'}^{(i_{i^*},w_0+1)} = E_{j,t'}$.

Suppose to the contrary that $M \neq M_{j,0}^{(i_{i^*},w_0+1)}$. Since $V(M) \cap X_t \neq 0$ and $A_{G_{i^*},|V(T)|+1,s+2}(V(M)) \cap X_{V(T_t)} - X_{T_t} \neq 0$, $\sigma(M) > \sigma(M_{j,0}^{(i_{i^*},w_0+1)})$. Since $M \geq M_{j,0}^{(i_{i^*},w_0+1)}$, we know $V(M) \cap X_t \neq 0$, so $t \in V(T_{T_m})$. Since $V(M) \cap X_t \neq 0$, $V(M) \cap X_t \neq 0$. By Claim 23.30, there exists a monochromatic $E_{j,t}^{-}\text{-}pseudocomponent in G[Y_{V(T)|+1,s+2}]$ containing $v_{M_{j,0}^{(i_{i^*},w_0+1)}}$ and some vertex $u_M \in V(M) \cap X_t$, where $v_{M_{j,0}^{(i_{i^*},w_0+1)}}$ is the vertex in $v_{M_{j,0}^{(i_{i^*},w_0+1)}}$ such that $\sigma(v_{M_{j,0}^{(i_{i^*},w_0+1)}}) = \sigma(M_{j,0}^{(i_{i^*},w_0+1)})$. So $\sigma(M') > \sigma(M_{j,0}^{(i_{i^*},w_0+1)}) < \sigma(M)$. Since $M \geq M_{j,0}^{(i_{i^*},w_0+1)}$ and $V(M') \cap X_t \neq 0$, we know $A_{G_{i^*},|V(T)|+1,s+2}(V(M')) \cap X_{V(T_t)} - X_{T_t} = 0$. By Claim 23.29, $V(M') = V(M_{j,0}^{(i_{i^*},w_0+1)})$. Therefore, $A_{G_{i^*},|V(T)|+1,s+2}(V(M')) \cap X_{V(T_t)} - X_{T_t} = 0$, a contradiction. This proves the claim.

**Claim 23.33.** Let $j \in [|\mathcal{V}|-1]$. Let $i \in \mathbb{N}_0$, and let $t$ be a node of $T$ of height $i$. Let $t' \in T_{j,t}$. Let $M$ be a monochromatic $E_{j,t}^{-}\text{-}pseudocomponent in G[Y_{V(T)|+1,s+2}]$ with respect to $c$ intersecting $X_t$ such that $S_M \cap X_t \neq 0$ and $V(M) \cap X_{t'} \neq 0$ for some $t'' \in V(T) - V(T)$. Assume there exists a monochromatic $E_{j,t}^{-}\text{-}pseudocomponent in G[Y_{V(T)|+1,s+2}]$ with respect to $c$ containing $M$ such that $V(M') \cap X_{t'} \neq 0$, where $t'$ is the height of $t'$. Then for every $(\beta, \gamma) \in [\sigma(M')] \times [|V(G)|]$, if
the $(\beta, \gamma)$-entry of the $(t', M')$-signature is nonzero, then $\beta \leq \sigma(M)$ and the $(\beta, \gamma')$-th entry in the $(t, M)$-signature is nonzero for some $\gamma' \in [V(G)]$.

**Proof.** Let $(\beta, \gamma)$ be an element of $[\sigma(M')] \times [V(G)]$ such that the $(\beta, \gamma)$-entry of the $(t', M')$-signature is nonzero. So there exists a monochromatic $E_{j,t}$-pseudocomponent $M_0$ in $G[Y^{(i',-1,0)}]$ with respect to $c$ such that $S_{M_0} \cap I_j \neq \emptyset$, $\sigma(M_0) = \beta$, and $A_{L_{(i',-1,0)}}(V(M_0)) \cap X_{V(T_{i'})} - X_i' \neq \emptyset$. By Claim 23.30, there exists an $E_{j,t}$-pseudocomponent $M_1$ in $G[Y^{(i',-1,0)}]$ with respect to $c$ such that $M_1$ contains $v_{M_0}$ and some vertex $u_{M_0} \in V(M_0) \cap X_i$, where $v_{M_0}$ is the vertex in $M_0$ such that $\sigma(v_{M_0}) = \sigma(M_0)$. Note that $\beta = \sigma(M_1) = \sigma(M_0)$ and $\gamma = \sigma(M') \leq \sigma(M)$. If $A_{L_{(i',-1,0)}}(V(M_1)) \cap X_{V(T_i)} - X_i = \emptyset$, then $V(M_1) = V(M_0)$ by Claim 23.29, so $A_{L_{(i',-1,0)}}(V(M_0)) \cap X_{V(T_{i'})} - X_i' \subseteq A_{L_{(i',-1,0)}}(V(M_1)) \cap X_{V(T_i)} - X_i \neq \emptyset$, a contradiction. So $A_{L_{(i',-1,0)}}(V(M_1)) \cap X_{V(T_i)} - X_i \neq \emptyset$. Hence the $(\beta, \gamma')$-th entry in the $(t, M)$-signature is nonzero for some $\gamma' \in [V(G)]$.

Given any monochromatic component $M$ with respect to $c$ with $S_M \cap I_j \neq \emptyset$ for some $j \in [|V| - 1]$, define $K^*(M) := \{r_M\} \cup \{t' \in K(M) - \{r_M\} : t' \in \partial T_{j,t},$ and $|V(M) \cap Y^{(i,t',|V(T)|+1, s+2)} \cap X_{V(T_{j't})}| > |V(M) \cap Y^{(i,t,|V(T)|+1, s+2)} \cap X_{V(T_{jt})}|-\}$. Choose such a vertex $k_t$ with height as small as possible. Thus $k_t \in K^*(M)$ for each $t \in K(M)$. Let $K' = \{r_M\} \cup \{t' \in V(T) : t' = k_t \text{ for some } t \in K(M)\}$. Note that $K' \subseteq K^*(M)$. So

$$|V(M)| = \sum_{t \in K(M)} |V(M) \cap Y^{(i,t,|V(T)|+1, s+2)} \cap X_{V(T_{j,t})}|$$

$$= \sum_{t \in K(M)} |V(M) \cap Y^{(i_{k_t},|V(T)|+1, s+2)} \cap X_{V(T_{j,t})}|$$

$$\leq \sum_{t \in K'} |V(M) \cap Y^{(i,t,|V(T)|+1, s+2)} \cap X_{V(T_i)}|$$

$$\leq \sum_{t \in K^*(M)} |V(M) \cap Y^{(i,t,|V(T)|+1, s+2)} \cap X_{V(T_i)}|$$

$$\leq |K^*(M)| \cdot \eta_5$$

by Claim 23.21. 

Given any monochromatic component $M$ with respect to $c$ with $S_M \cap I_j \neq \emptyset$ for some $j \in [|V| - 1]$, we define the following:

- Let $T_M$ be the rooted tree with $V(T_M) = K(M)$ with root $r_M$ such that $t \in V(T_M)$ is adjacent to $t' \in V(T_M)$ if and only if $t' \in \partial T_{j,t}$.
• For every $t \in V(T_M)$ and $k \in \mathbb{N}_0$, let $\mathcal{M}_M(t, k)$ be the set of all monochromatic $E_{j,t}-\text{pseudocomponents } M''$ in $G[Y^{(t_i,-1,0)}]$ with $S_M'' \cap I_j^k \neq \emptyset$ such that $k$ is the minimum integer such that there exists $t' \in V(T_M)$ of height $k$ in $T_M$ such that $V(M'') \cap X_t \neq \emptyset$, $V(M'') \cap X_{t'} \neq \emptyset$, and $A_{L((i,t_7,-1,0))}(V(M'')) \cap X_{V(T'_{t''})} - X_{t''} \neq \emptyset$.

Claim 23.35. Let $j \in [\lceil \mathcal{V} \rceil - 1]$. Let $M$ be a monochromatic component with respect to $c$ with $S_M \cap I_j^2 \neq \emptyset$. Let $t \in K(M) - \{r_M\}$.

If $\mathcal{M}_M(t, 0) = \emptyset$, then $K^*(M) \cap V(T_t) - \{t\} = \emptyset$. Furthermore, if $\mathcal{M}_M(t, 0) = \emptyset$ and $t \in K^*(M)$, then $V(M) \cap Y^{(t_i,-1,0)} \cap V(T_t) \neq \emptyset$ and $V(M) \cap Y^{(t_i,|V(T)|+1,\ldots,2)} \cap V(T_t)$, where $t''$ is the parent of $t$ in $T_M$.

Proof. Let $t' \in K^*(M) \cap V(T_t)$. Let $t''$ be the parent of $t'$ in $T_M$. That is, $t' \in \partial T_{j,t''}$. Note that $t'' \in V(T_t)$ unless $t'' = t$.

Since $t' \in K^*(M)$, there exists $q \in V(T_{t''})$ such that $V(M) \cap X_q \neq \emptyset$. Choose $q$ so that the height of $q$ is as large as possible.

Since $M$ is connected in $G$, for every vertex $u \in V(M) \cap X_q$, there exists a path $P_u$ in $M$ from $v_M$ to $u$, where $v_M$ is the vertex in $M$ with $\sigma(v_M) = \sigma(M)$. By the maximality of the height of $q$, $P_u$ is internally disjoint from $X_{V(T_{t''})} - X_q$. So by Claim 23.31, there exists a monochromatic $E_{j,q}$-pseudocomponent $M_1$ in $G[Y^{(q_i,-1,0)}]$ containing $v_M$ and $u$ for every $u \in V(M) \cap X_q$.

Since $t \in K(M)$, $P_u$ (for any $u \in V(M) \cap X_q$) contains a subpath from $v_M$ to some vertex in $V(M) \cap X_t$ internally disjoint from $X_{V(T_{t})}$, so by Claim 23.31, there exists a monochromatic $E_{j,t}$-pseudocomponent $M_0$ in $G[Y^{(t_i,-1,0)}]$ containing $v_M$ and some vertex in $V(M) \cap X_t$. Since $v_M \in V(M_0) \cap V(M_1)$, $M_0 \subseteq M_1$. Since $M_0$ contains $v_M$ and some vertex in $V(M) \cap X_t$, $V(M_0) \cap X_t \neq \emptyset \neq V(M_0) \cap X_{r_{M_0}}$. Since $\mathcal{M}_M(t, 0) = \emptyset$, $A_{L((i,t_7,-1,0))}(V(M_0)) \cap X_{V(T_{M_0})} - X_{r_{M_0}} = \emptyset$. So $A_{L((i,t_7,-1,0))}(V(M_0)) \cap X_{V(T_{M_0})} - X_t = \emptyset$. By Claim 23.29, $V(M_0) = V(M_1)$.

We claim that $V(M_1) \cap X_{V(T_{t})} \supseteq V(M) \cap X_{V(T_{t})}$. Suppose to the contrary that there exists $v \in V(M) \cap X_{V(T_{t})} - V(M_1)$. Then there exists a shortest path $R$ in $M$ from $V(M_1)$ to $v$. Choose $v$ to be the vertex such that $R$ is as short as possible. Since $A_{L((i,t_7,-1,0))}(V(M_1)) \cap X_{V(T_{t})} - X_t = A_{L((i,t_7,-1,0))}(V(M_0)) \cap X_{V(T_{t})} - X_t = \emptyset$, $R$ is disjoint from $X_{V(T_{t})} - X_t$ by the choice of $R$. Let $z$ be a vertex in $X_{r_{M_0}} \cap V(R)$, and let $a, b$ be the ends of $R$. Let $R_a$ be the subpath of $R$ from $a$ to $z$, and let $R_b$ be the subpath of $R$ from $b$ to $z$. Hence $R_a$ is disjoint from $(X_{V(T_{t})} - X_t) \cup (V(G) - V(X_{r_{M_0}}))$. So by Claim 23.31, there exists a monochromatic $E_{j,t}$-pseudocomponent $M_a$ in $G[Y^{(t_i,-1,0)}]$ containing $z$ and $a$. Similarly, there exists a monochromatic $E_{j,t}$-pseudocomponent $M_b$ in $G[Y^{(t_i,-1,0)}]$ containing $z$ and $a$. Since $z \in V(M_a) \cap V(M_b)$, $M_a = M_b$. But $V(M_a) \cap V(M_1) \neq \emptyset$, $M_1 \supseteq M_a$. So $v \in V(R) \subseteq V(M_1)$, a contradiction.

Therefore, $V(M) \cap X_{V(T_{t})} \subseteq V(M_1) \cap X_{V(T_{t})} = V(M_0) \cap X_{V(T_{t})} \subseteq Y^{(t_i,-1,0)}$. Since $t' \in K^*(M)$, this implies that $t = t'$ and $V(M) \cap Y^{(t_i,-1,0)} \cap X_{V(T_{t''})} = V(M_0) \cap Y^{(t_i,|V(T)|+1,\ldots,2)} \cap X_{V(T_{t''})}$.

Since $t' \in K^*(M)$, $V(M) \cap Y^{(|V(T)|,\ldots,2)} \cap X_{V(T_{t''})} \neq V(M) \cap Y^{(|V(T)|,\ldots,2)} \cap X_{V(T_{t''})}$. Hence $V(M) \cap Y^{(t_i,-1,0)} \cap V(T_{t''}) \neq V(M) \cap Y^{(t_i,|V(T)|+1,\ldots,2)} \cap X_{V(T_{t''})}$. This proves the claim.

Claim 23.36. Let $j \in [\lceil \mathcal{V} \rceil - 1]$. Let $t \in \mathbb{N}_0$, and let $t$ be a node of $T$ of height $i$. Let $M$ be a monochromatic $E_{j,t}$-pseudocomponent in $G[Y^{(t_i,-1,0)}]$ with respect to $c$ intersecting $X_t$ such that $S_M \cap I_j^2 \neq \emptyset$ and $V(M) \cap X_{t''} \neq \emptyset$ for some $t'' \in V(T) - V(T_t)$. For every $q \in \partial T_{j,t}$, let $M_q$ be the $E_{j,q}$-pseudocomponent in $G[Y^{(q_i,-1,0)}]$ with respect to $c$ such that $M_q \supseteq M$. Then either:

• for every $t' \in \partial T_{j,t}$, the $(t', M_q)$-signature is lexicographically smaller than the $(t, M)$-signature, or

• there exists $t' \in \partial T_{j,t}$ such that:
the \((t', M_{t'})\)-signature equals the \((t, M)\)-signature, and

- for every \(\alpha\) with \(\sigma(M_{j,\alpha}^{(t)}) \leq \sigma(M)\), \(M_{j,\alpha}^{(t')} = M_{j,\alpha}^{(t)}\), and

- for every \(t'' \in \partial T_{j,t} - \{t'\}\) with \(V(M) \cap X_{t'} \neq \emptyset\), the \((t'', M_{t''})\)-signature has no nonzero entry and for every monochromatic component \(M''\) with respect to \(c\) with \(S_{M''} \cap I_j \neq \emptyset\) and \(V(M'') \cap X_t \neq \emptyset\), we have \(V(M'') \cap V(T_{t''}) = V(M'') \cap Y^{(i_1, \ell, |V(T)|, 1, s+2)} \cap V(T_{t''})\), or

- there exists \(t' \in \partial T_{j,t}\) such that:

  - either the \((t', M_{t'})\)-signature is different from the \((t, M)\)-signature, or there exists \(M_{j,\alpha}^{(t)}\) for some \(\alpha\) with \(\sigma(M_{j,\alpha}^{(t)}) \leq \sigma(M)\) such that \(M_{j,\alpha}^{(t')} \neq M_{j,\alpha}^{(t)}\), and

  - for every \(t'' \in \partial T_{j,t} - \{t'\}\) with \(V(M) \cap X_{t'} \neq \emptyset\), the \((\beta^*_{t''}, \gamma)\)-entry of the \((t'', M_{t''})\)-signature is zero for every \(\gamma \in [[V(G)]]\), where \(\beta^*_{t''}\) is the minimum \(\beta \in [[V(G)]]\) such that the \((\beta, \gamma)\)-entry of the \((t, M)\)-signature is nonzero for some \(\gamma' \in [[V(G)]]\).

\[\text{Proof.}\]

We may assume that the first statement does not hold, for otherwise we are done. So there exists \(t' \in \partial T_{j,t}\) such that the \((t', M_{t'})\)-signature is lexicographically at least the \((t, M)\)-signature.

Let \(\Lambda\) be the set such that for each \(\beta \in \Lambda\), there exists \(\gamma \in [[V(G)]]\) such that the \((\beta, \gamma)\)-entry of the \((t, M)\)-signature is nonzero. Let \(M_1, M_2, \ldots, M_{|\Lambda|}\) be the monochromatic \(E_{i,t}\)-pseudocomponents in \(G[Y^{(i_1, |V(T)|, 1, s+2)}]\) such that \(\sigma(M_1) < \sigma(M_2) < \ldots < \sigma(M_{|\Lambda|}) = \sigma(M)\). Let \(\Lambda'\) be the set such that for each \(\beta \in \Lambda\), there exists \(\gamma \in [[V(G)]]\) such that the \((\beta, \gamma)\)-entry of the \((t', M_{t'})\)-signature is nonzero.

Let \(M'_1, M'_2, \ldots, M'_{|\Lambda'|}\) be the monochromatic \(E_{i,t}\)-pseudocomponents in \(G[Y^{(i_1, |V(T)|, 1, s+2)}]\) such that \(\sigma(M'_1) < \sigma(M'_2) < \ldots < \sigma(M'_{|\Lambda'|}) = \sigma(M_{t'})\).

Since \(M_{t'} \supseteq M\) and \(V(M) \cap X_t \neq \emptyset\), we know \(V(M_{t'}) \cap X_t \neq \emptyset\). So by Claim 23.33, \(\Lambda' \subseteq \Lambda\). Since the \((t', M_{t'})\)-signature is lexicographically at least the \((t, M)\)-signature, \(m_{\text{min}} \in \Lambda'\), so \(M'_1 \supseteq M_1\). Note that \(M_1 = M_{t',0}\). By Claim 23.32, \(M'_1 = M_1\). So there exists a maximum \(k^* \in [|\Lambda'|]\) such that \(M'_k = M_{\alpha}^*\) for every \(\alpha \in [k^*]\). Since the \((t', M_{t'})\)-signature is lexicographically at least the \((t, M)\)-signature, for every \(\alpha \in [k^*]\), \(A_{L,1, |V(T)|, 1, s+2, 0}(V(M_{t'})) \cap X_{V(T_{t'})} - X_t \subseteq X_{V(T_{t'})} - X_{t'}\), so \(A_{L,1, |V(T)|, 1, s+2, 0}(V(M_{t'})) \cap X_{V(T_{t'})} - X_t \subseteq X_{V(T_{t'})} - X_{t'}\). We first assume that \(k^* = |\Lambda'|\). Since for every \(\alpha \in [|\Lambda'|]\), \(A_{L,1, |V(T)|, 1, s+2, 0}(V(M_{t'})) \cap X_{V(T_{t'})} - X_t \subseteq X_{V(T_{t'})} - X_{t'}\), we know the \((t, M)\)-signature equals the \((t', M_{t'})\)-signature. Furthermore, \(M_{j,\alpha}^{(t')} = M_{j,\alpha}^{(t',0)} = M_{j,\alpha}^{(t',0)} = M_{j,\alpha}^{(t)}\) for every \(\alpha \in [k^*] = [|\Lambda'|]\). In addition, for every \(t'' \in \partial T_{j,t} - \{t'\}\) with \(V(M) \cap X_{t'} \neq \emptyset\), the \((t'', M_{t''})\)-signature has no nonzero entry. Recall that for every \(\alpha \in [k^*] = [|\Lambda'|]\), \(A_{L,1, |V(T)|, 1, s+2, 0}(V(M_{t'})) \cap X_{V(T_{t'})} - X_t \subseteq X_{V(T_{t'})} - X_{t'}\) and \((A_{L,1, |V(T)|, 1, s+2, 0}(V(M_{t'})) \cap X_{V(T_{t'})} - X_t) \cap Z_t = \emptyset\). So for every \(t'' \in \partial T_{j,t} - \{t'\}\) and monochromatic component \(M''\) with respect to \(c\) with \(S_{M''} \cap I_j \neq \emptyset\) and \(V(M'') \cap X_t \neq \emptyset\), we know \(V(M'') \cap V(T_{t''}) = V(M'') \cap Y^{(i_1, |V(T)|, 1, s+2)} \cap V(T_{t''})\). This proves the second statement.

So we may assume that \(k^* < |\Lambda'|\). Recall that for every \(\alpha \in [k^*]\), \((A_{L,1, |V(T)|, 1, s+2, 0}(V(M_{t'})) \cap X_{V(T_{t'})} - X_t) \cap Z_t = \emptyset\). So by the maximality of \(k^*\), \(M_{j,\alpha}^{(t')} = M_{k^*,1}^{(t')} \neq M_{k^*,1}^{(t)}\). Since \(k^* \geq 1\), for every \(t'' \in \partial T_{j,t} - \{t'\}\) with \(V(M) \cap X_{t'} \neq \emptyset\), \((A_{L,1, |V(T)|, 1, s+2, 0}(V(M_{j})) \cap X_{V(T_{t''})} - X_{t''}) = \emptyset\), so the \((\beta^*_{t''}, \gamma)\)-entry of the \((t'', M_{t''})\)-signature is zero for every \(\gamma \in [[V(G)]]\), where \(\beta^*_{t''}\) is the minimum \(\beta \in [[V(G)]]\) such that the \((\beta, \gamma)\)-entry of the \((t, M)\)-signature is nonzero for some \(\gamma \in [[V(G)]]\). Hence the third statement holds.

\[\square\]

**Claim 23.37.** Let \(j \in [|V| - 1]\). Let \(t\) and \(t'\) be nodes of \(T\) with \(t' \in V(T_t)\). Let \(\ell \in [0, w_0 - 1]\). For every \(t'' \in V(T)\) with \(V(T_{t''}) \subseteq V(T_{t'}) \subseteq V(T_t)\), let \(C_{t''} = \{\alpha \in [0, \ell] : \text{there exists a monochromatic } E_{j,t''}\text{-pseudocomponent } M \text{ in } G[Y^{(i_{t''}, |V(T)|, 1, s+2, 0)}]\) such that \(M \supseteq M_{j,\alpha}^{(t)}\) and \(A_{L,1, |V(T)|, 1, s+2, 0}\)
Assume there exist $\beta^* \in \mathbb{N}$ and nodes $t_0, t_1, \ldots, t_{\beta^*}$ such that $t_0 = t, t_{\beta^*} = t'$, and for every $\alpha \in [\beta^*], \alpha_0 \in V(T_{j,0})$. For every $\alpha \in [0, \beta^*)$ and $\alpha' \in C_{t_\alpha}$, let $k_{\alpha'}$ be the index such that $M_{j,k_{\alpha'}} \supseteq M_{j,\alpha'}^{(t_\alpha)}$. Let $\Phi = \{t_\alpha : \alpha \in [\beta^*]\}$ such that either:

- $|C_{t_\alpha}| < |C_{t_{\alpha-1}}|$, or
- $|C_{t_\alpha}| = |C_{t_{\alpha-1}}|$, and there exists $\alpha' \in C_{t_\alpha}$ such that either $M_{j,k_{\alpha'}} = M_{j,k_{\alpha'}}^{(t_{\alpha-1})}$ or the $(t_\alpha, M_{j,k_{\alpha'}})$-signature is different from the $(t_{\alpha-1}, M_{j,k_{\alpha'}}^{(t_{\alpha-1})})$-signature).

If $|\Phi| \geq h(\ell)$, then there exists $t_{\alpha*} \in \Phi$ such that $\ell \notin C_{t_{\alpha*}}$.

**Proof.** By Claim 23.26, it suffices to prove that for every $t'' \in V(T) - \{t\}$ with $V(T_r) \subseteq V(T_{\alpha'}) \subseteq V(T_\ell)$ and every $\ell_0 \in [0, \ell]$, if $M_1$ is a monochromatic $E_{j,\ell'}$-pseudocomponent in $G[Y^{(t_{\ell'}, t', 0)]\}$ with $M_1 \supseteq M_{j,\ell_0}^{(t_{\ell'})}$ and $M_2$ is a monochromatic $E_{j,\ell'}$-pseudocomponent in $G[Y^{(t_{\ell'}, t', 0)]\}$ intersecting $X_{t'}$ with $M_{j,\ell'}(t') \subseteq M_{j,\alpha}$ for some $\alpha \in [0, w_0 - 1]$ such that $\sigma(M_2) \leq \sigma(M_1)$, then $M_2 \supseteq M_{j,\alpha}^{(t_{\ell'})}$ for some $\alpha' \leq \ell_0$.

We may assume that $M_1 \not\supseteq M_2$ for otherwise we are done. So $\sigma(M_2) < \sigma(M_1)$. By Claim 23.30, there exists a monochromatic $E_{j,\ell'}$-pseudocomponent in $G[Y^{(t_{\ell'}, t', 0)]\}$ containing $V(M_2)$ and some vertex $v_{M_2}$ in $X_{t'}$, where $v_{M_2}$ is the vertex in $V(M_2)$ with $\sigma(v_{M_2}) = \sigma(M_2)$. So there exists a monochromatic $E_{j,\ell'}$-pseudocomponent $M'_2$ in $G[Y^{(t_{\ell'}, t', 0)]\}$ with respect to $c$ containing $v_{M_2}$ and intersecting $X_{t'}$ such that $S_{M'_2} \subseteq S_{j,\ell'}$. Since $M_2$ is a monochromatic $E_{j,\ell'}$-pseudocomponent in $G[Y^{(t_{\ell'}, t', 0)]\}$ intersecting $V(M_2)$, by Claim 23.29, if $A_{L,(t_{\ell'}, t', 0)}(V(M_2)) \cap X_{V(T_\ell)} - X_\ell = \emptyset$, then $M'_2 = M_2$, so $M'_2 \supseteq M_2 \supseteq M_{j,\alpha}^{(t_{\ell'})}$ for some $\alpha \in [0, w_0 - 1]$, and hence $M'_2 = M_{j,\alpha}^{(t_{\ell'})}$, so $A_{L,(t_{\ell'}, t', 0)}(V(M_2)) \cap X_{V(T_\ell)} - X_\ell = A_{L,(t_{\ell'}, t', 0)}(V(M_{j,\alpha}^{(t_{\ell'})})) \cap X_{V(T_\ell)} - X_\ell \neq \emptyset$, a contradiction. So we may assume that $A_{L,(t_{\ell'}, t', 0)}(V(M_2)) \cap X_{V(T_\ell)} - X_\ell \neq \emptyset$. Hence there exists $\alpha' \in [0, w_0 - 1]$ such that $M_{j,\alpha'}^{(t_{\ell'})}$ containing $M'_2$. Since $t'' \in V(T_\ell) - \{t\}$, we know $M_2 \supseteq M_{j,\alpha'}^{(t_{\ell'})} \supseteq M'_2$ and $\sigma(M_{j,\alpha'}^{(t_{\ell'})}) \leq \sigma(M_2) = \sigma(M_2) \leq \sigma(M_1) \leq \sigma(M_{j,\alpha'}^{(t_{\ell'})})$, so $\alpha' \leq \ell_0$. This proves the claim. \qed

For every $j \in [|\mathcal{V}| - 1]$ and monochromatic component $M$ with respect to $c$ with $S_M \cap I_j = \emptyset$, define the following.

- Define $T'_M$ to be the rooted tree obtained from $T_M$ by deleting all nodes $t$ in which $K^*(M) \cap V(T_t) = \emptyset$. (Note that for every $t \in V(T'_M)$, $\mathcal{M}_M(t, 0) \neq \emptyset$ and contains some member intersecting $V(M)$ by Claims 23.29 and 23.31.)

- Define $\mu_M$ to be the function that maps each node $t$ of $T'_M$ to an infinite sequence $(a_{t_0}, a_{t_1}, \ldots)$, where each $a_{t_\alpha}$ is a triple $(a_{t_\alpha,0}, a_{t_\alpha,2}, a_{t_\alpha,3}) \in [0, w_0] \times [0, h(w_0)] \times [0, w_0 + 1]$, as follows.

  - Let $a_{r_M,0} := (|\mathcal{M}_M(r_M, 0)|, h(w_0), w_0 + 1)$ and $a_{r_M,\alpha} := (0, 0, 0)$ for every $\alpha \in \mathbb{N}$.

  - For every $t \in V(T'_M) - \{r_M\}$, where the parent of $t$ in $T_M$ is denoted by $p_t$ and height of $t$ in $T'_M$ is denoted by $k$, let $m \in [-1, k - 1]$ be the maximum such that for every $k' \in [0, m], \mathcal{M}_M(t, k') = \mathcal{M}_M(p_t, k')$ and for every $M' \in \mathcal{M}_M(p_t, k') = \mathcal{M}_M(t, k')$, the $(t, M')$-signature equals the $(p_t, M')$-signature,

    - let $m^* \geq m$ be the minimum in $[m + 1, k]$ such that $\mathcal{M}_M(t, m^*) \neq \emptyset$.

    - let $a_{t,m^*,1} := |\mathcal{M}_M(t, m^*)|$, and if $|\mathcal{M}_M(t, m^*)| \neq |\mathcal{M}_M(p_t, m^*)|$, then let $a_{t,m^*,2} := h(w_0)$; otherwise, let $a_{t,m^*,2} := a_{p,m^*,2} - 1$,
Note that every non-root leaf of $T_M$ with the largest degree of $T$ belongs to $K^*(M)$. By Claim 23.35, if $t$ is a node in $V(T_M)$ that has a child, then $M(t, 0) \neq \emptyset$. Note that $\mu_M(t) \leq \eta \ell / w_0$ for every $t \in V(T'_M)$.

**Claim 23.38.** Let $j \in [|V| - 1]$. Let $M$ be a monochromatic component with respect to $c$ with $S_M \cap I_j \neq \emptyset$. Let $t$ be a node of $T'_M$ that has a parent and a child in $T'_M$. Then either

- for every child $t'$ of $t$, $\mu'_M(t')$ is lexicographically smaller than $\mu'_M(t)$, or
- the degree of $t$ in $T'_M$ is two, and there exists a directed path $W_t$ in $T'_M$ from $t$ to a node $t_{W_t} \in V(T_t)$ such that

  - every internal node of $W_t$ is of degree two in $T'_M$,
  - either $t_{W_t}$ is a leaf of $T'_M$, or $\mu'_M(t_{W_t})$ is lexicographically smaller than $\mu'_M(t)$, or
  - there are at most $w_0 + 1$ internal nodes of $W_t$ belong to $K^*(M)$, and
  - if $t_{W_t}$ is not a leaf in $T'_M$ and there exists $\alpha \in \mathbb{N}_0$ such that the $\alpha$-th entry of $\mu'_M(t) \neq 0$ and the $\alpha$-th entry of $\mu'_M(t_{W_t}) = 0$, then there exists $\alpha' \in [0, \alpha - 1]$ such that the $\alpha'$-th entry of $\mu'_M(t_{W_t})$ is less than the $\alpha$-th entry of $\mu'_M(t)$.

**Proof.** For every $t' \in V(T_t) \cap V(T'_M)$, let $M^*_t$ be the member of $\bigcup_{\alpha=0}^k M_M(t, \alpha)$ intersecting $V(M)$ with the largest $\alpha$-value. For every $t' \in V(T_t) \cap V(T'_M)$ and every $\alpha \in \mathbb{N}_0$, let $s_{\alpha}$ be the subsequence of $\mu'_M(t')$ consisting of the entries with indices in $[0, \alpha]$. Let $k^*$ be the index such that $M^*_t \in M_M(t, k^*)$.

Apply Claim 23.36 to $M^*_t$. If the first statement of Claim 23.36 holds, then the first statement of this claim holds by Claims 23.29 and 23.31. Recall that since $t$ has a child in $T'_M$, $M_M(t, 0) \neq \emptyset$. If the third statement of Claim 23.36 holds, then since $M_M(t, 0) \neq \emptyset$, the first statement of this claim holds by Claims 23.29, 23.31 and 23.37. So we may assume that the second statement of Claim 23.36 holds. Hence $t$ has degree two in $T'_M$. In addition, there exists a directed path $W_t$ in $T'_M$ from $t$ to a node $t_{W_t} \in V(T_t)$ such that

(a) for every node $t' \in V(W_t) - \{t, t_{W_t}\}$, applying Claim 23.36 to $M^*_p$, the second statement of Claim 23.36 holds, where $p_{t'}$ is the parent of $t'$ in $T'_M$;

(b) for every node $t' \in V(W_t) - \{t_{W_t}\}$, $s_{k^*}(t')$ is not lexicographically smaller than $s_{k^*}(t)$, and

(c) subject to (a) and (b), $|V(W_t)|$ is as large as possible.
Note that such $W_t$ exists, since the path consists of $t$ is a candidate. Note that (a) implies that every internal node of $W_t$ is of degree two in $T_M$. By (c), if $t_{W_t}$ is not a leaf of $T'_M$, then either (a) or (b) is violated for taking $t' = t_{W_t}$; if (a) is violated for $t_{W_t}$, then the first or the third statement of Claim 23.36 holds when applying Claim 23.36 to $M_{p_{W_t}}^*$, where $p_{W_t}$ is the parent of $t_{W_t}$ in $T'_M$, so it together with (a) imply that $s_k(t_{W_t})$ is lexicographically smaller than $s_k(t)$, and hence $\mu'_M(t_{W_t})$ is lexicographically smaller than $\mu'_M(t)$; if (b) is violated for $t_{W_t}$, then $s_k(t_{W_t})$ is lexicographically smaller than $s_k(t)$, and hence $\mu'_M(t_{W_t})$ is lexicographically smaller than $\mu'_M(t)$.

Suppose there exist $w_0 + 2$ distinct internal nodes $t_1, t_2, ..., t_{w_0+2}$ of $W_t$ belonging to $K^*(M)$. We may assume that for every $\alpha \in [w_0 + 1]$, $t_{\alpha+1} \in V(T_\alpha)$, and $K^*(M) \cap V(W_t) \subseteq V(T_{w_0+2})$. For every $\alpha \in [w_0 + 2]$, let $p_\alpha$ and $t'_\alpha$ be the parent and child of $t_\alpha$ in $T'_M$, respectively.

We shall prove that for every $\alpha \in [w_0 + 1]$, there exist $t'_\alpha \in V(T_{p_\alpha}) \cap V(T_M)$ with $(V(M) \cap \bigcap_{\alpha \in [w_0 + 1]} \bigcap_{\beta \in [0, k]} X_{V(T'_M)}) - (Y(p_\alpha, V(T')_{(\alpha, 0, s+2)}) \cup X_{t'_\alpha}) \neq \emptyset$ and a monochromatic path $Q_\alpha$ in $G[Y^{(\alpha, 0, s+2)}]$ intersecting $X_{t''_\alpha}$ for every node $t''_\alpha$ in the path in $T$ from $t'_\alpha$ to $t_\alpha$ such that $Q_1, Q_2, ..., Q_{w_0+1}$ are disjoint. Note that if this statement is true, then (a), (b) and the definition of $\mu_M$ imply that $\{t'_\alpha : \alpha \in [w_0 + 1]\} \subseteq V(T_{w_0+2}) \cap V(T_M) - \{t_{w_0+2}\}$, so $X_{t_{w_0+2}}$ contains at least $w_0 + 1$ vertices, where each belongs to $Q_\alpha$ for some $\alpha \in [w_0 + 1]$, and hence those $w_0 + 1$ vertices belong to $X_{t_{w_0+1} \cap (B_{j-1} \cup B_j)}$ which has size at most $w_0$, a contradiction. Hence it suffices to prove this statement.

Let $\alpha^* \in [w_0 + 1]$. Assume that for every $\alpha \in [1, \alpha^* - 1]$, there exist $t'_\alpha \in V(T_{p_\alpha}) \cap V(T_M)$ with $(V(M) \cap \bigcap_{\alpha \in [1, \alpha^* - 1]} \bigcap_{\beta \in [0, k]} X_{V(T'_M)}) - (Y(p_\alpha, V(T')_{(\alpha, 0, s+2)}) \cup X_{t'_\alpha}) \neq \emptyset$ and a monochromatic path $Q_\alpha$ in $G[Y^{(\alpha, 0, s+2)}]$ intersecting $X_{t''_\alpha}$ for every node $t''_\alpha$ in the path in $T$ from $t'_\alpha$ to $t_\alpha$ such that $Q_1, Q_2, ..., Q_{w_0+1}$ are disjoint. Since $t_{\alpha^*} \in K^*(M)$, either $|V(M) \cap \bigcap_{\alpha \in [w_0]} \bigcap_{\beta \in [0, k]} X_{v_{T_{\alpha}}}| \neq |V(M) \cap \bigcap_{\alpha \in [w_0]} \bigcap_{\beta \in [0, k]} X_{v_{T_{\alpha}}}^*| \neq |V(M) \cap \bigcap_{\alpha \in [w_0]} \bigcap_{\beta \in [0, k]} X_{v_{T_{\alpha}}}| \neq |V(M) \cap \bigcap_{\alpha \in [w_0]} \bigcap_{\beta \in [0, k]} X_{v_{T_{\alpha}}}^*| \neq |V(M) \cap \bigcap_{\alpha \in [w_0]} \bigcap_{\beta \in [0, k]} X_{v_{T_{\alpha}}}^*|$. So there exist $t''_\alpha \in \partial T_{\alpha^*}$, and an $E_{j_{\alpha^*}}$-quasi-component $M_0$ in $G[Y^{(\alpha^*, 0), (0, \alpha^* + 2), (1, \alpha^* + 2)}]$ intersecting $X_{t''_\alpha} \cap V(M)$ but disjoint from $X_{t_{\alpha^*}} \cup Y^{(\alpha, 0, s+2)}$. Since $M$ is connected and $K^*(M) \cap V(T_{\alpha^*}) \neq \emptyset$, $V(M) \cap X_{t''_\alpha} \neq \emptyset$ by Claim 23.22. Let $M_1$ be the $E_{j_{\alpha^*}}$-quasi-component in $G[Y^{(\alpha^*, 0, \alpha^* - 1)}]$ containing $M_0$. If $A_{L_{(\alpha^*, 0, \alpha^* - 1)}}(V(M_1)) \cap X_{v_{T_{\alpha^*}}} - X_{t''_\alpha} \neq \emptyset$, then $M_1 \in M(t''_\alpha, k + 1)$ by (a), where $k$ is the height of $t''_\alpha$ in $T_M$, so by (a) and the definition of $t''_\alpha, k, \beta$ and $a'_{k', \beta}', k'$ for every $k' \in [k, k^*], (b)$ is violated, a contradiction. So $A_{L_{(\alpha^*, 0, \alpha^* - 1)}}(V(M_1)) \cap X_{v_{T_{\alpha^*}}} - X_{t''_\alpha} = \emptyset$. Let $q$ be the node in $V(T_{\alpha^*})$ of the largest height such that $V(M) \cap X_q \neq \emptyset$. So there exists a path $P$ in $M$ from $v_M$ to $V(M_1)$ disjoint from $X_{v_{T_{\alpha^*}}} - X_q$, where $v_M$ is the vertex in $M$ such that $\sigma(v_M) = \sigma(M)$. By Claim 23.31, there exists a monochromatic $E_{j_q}$-quasi-component $M_2$ in $G[Y^{(\alpha^*, 0, \alpha^* - 1)}]$ containing $v_M$ and $M_1$, since $q \in V(T_{\alpha^*})$. By Claim 23.29, since $A_{L_{(\alpha^*, 0, \alpha^* - 1)}}(V(M_1)) \cap X_{v_{T_{\alpha^*}}} - X_{t''_\alpha} = \emptyset$, $M_2 = M_1$. So $M_1$ contains $v_M$. If $A_{L_{(\alpha^*, 0, \alpha^* - 1)}}(V(M_1)) \cap X_{v_{T_{\alpha^*}}} - X_{v_{M}} \neq \emptyset$, then $K^*(M) \cap V(T_{\alpha^*}) - \{t''_\alpha\} \neq \emptyset$, but $\alpha^* \leq w_0 + 1$ and $W_t$ contains at least $w_0 + 2$ internal vertices, a contradiction. So $A_{L_{(\alpha^*, 0, \alpha^* - 1)}}(V(M_1)) \cap X_{v_{T_{\alpha^*}}} - X_{v_{M}} \neq \emptyset$. Since $M_1$ contains $v_M$, $M_1 \in M(t''_\alpha, 0)$. Since $M_1$ contains some vertex not in $Y^{(\alpha^*, 0, \alpha^* - 2)}$, contradicting (a).

So $|V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}| = |V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}|$. Hence either $|V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}| \neq |V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}|$, or $|V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}| \neq |V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}|$. Now we assume $|V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}| \neq |V(M) \cap Y^{(\alpha^*, 0, \alpha^* - 2)} \cap X_{v_{T_{\alpha^*}}}|$. Hence there
exist \( v \in V(M) \cap Y^{(i_{\alpha},-0,0)} - Y^{(i_{\alpha},-1,0)} \) and a monochromatic path \( Q \) in \( G[Y^{(i_{\alpha},-0,0)}] \) from \( X_{t_{\alpha}} \) to a neighbor \( u \) of \( v \) in \( G \), where \( v \) is a gate of \( u \). Note that \( v \) does not belong to any \( E_{j,\alpha} \)-pseudocomponent in \( G[Y^{(i_{\alpha},-0,0)}] \) by (a) and (b). Let \( M \) be the \( E_{j,\alpha} \)-pseudocomponent in \( G[Y^{(i_{\alpha},-0,0)}] \) containing \( v \). So \( M \cap X_{t_{\alpha}} = \emptyset \). Since \( v \in X_{V(T_{t_{\alpha}})} - X_{t_{\alpha}} \) and \( v_M \in X_{t_{\alpha \alpha}} \) and \( M \) is connected, \( A_{L(t_{\alpha},-0,0)}(M_M) \cap X_{V(T_{t_{\alpha}})} \neq \emptyset \). Note that there exists a path \( P_t \) in \( M[V(M) \cap X_{V(T_{t_{\alpha}})}] \) from \( (M_v) \) to \( \bigcup_{t=0}^{s_{\alpha}} M(t_{\alpha \alpha}) \) internally disjoint from \( V(M_v) \cup X_{t_{\alpha}} \) and internally disjoint from \( M(t_{\alpha \alpha}) \) for every \( \alpha \). By (a), \( V(P_t) \subseteq X_{V(T_{t_{\alpha \alpha}})} \cup X_{V(T_{t_{\alpha \alpha}})} \). Since \( |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| = |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| \), if \( V(M_v) \cap X_{t_{\alpha \alpha}} \neq \emptyset \), then \( \emptyset \in X_{t_{\alpha \alpha}} (t_{\alpha \alpha} \alpha) \cap X_{V(T_{t_{\alpha}})} | = \emptyset \) by the existence of \( P_t \), contradicting (b). So \( V(M_v) \cap X_{t_{\alpha \alpha}} = \emptyset \). Since \( |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| = |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| \), \( V(M_v) \subseteq X_{V(T_{t_{\alpha}})} - X_{t_{\alpha \alpha}} \). So there exists \( t_{\alpha \alpha} \in V(T_{t_{\alpha \alpha}}) - \{ t_{\alpha \alpha} \} \) such that \( v \in X_{V(T_{t_{\alpha \alpha}})} - X_{t_{\alpha \alpha}} \). Hence there exists a subpath \( Q_{\alpha \alpha} \) of \( Q \) from \( v \) to \( X_{t_{\alpha \alpha}} \) internally disjoint from \( X_{t_{\alpha \alpha}} \). So \( Q_{\alpha \alpha} \) contains a vertex in \( X_{t_{\alpha \alpha}} \) for every node \( t_{\alpha \alpha} \) in the path in \( T \) from \( t_{\alpha \alpha} \) to \( t_{\alpha \alpha} \). Since \( V(Q_{\alpha \alpha}) \subseteq X_{T_{t_{\alpha \alpha}}} \) and \( t_{\alpha \alpha} \in T_{t_{\alpha \alpha}} \), we know \( V(Q_{\alpha \alpha}) \cap I_j = \emptyset \). So \( Q_{\alpha \alpha} \) is disjoint from \( \bigcup_{\alpha=0}^{s_{\alpha}} Q_{\alpha} \) by the existence of \( v \). Hence Statement 2 of this claim holds.

So we may assume \( |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| = |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| \). Since \( |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| = |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha}})}| \), there exists \( q \in V(T_{t_{\alpha \alpha}}) - \{ p_{\alpha \alpha} \} \) with \( t_{\alpha \alpha} \in V(T_{t_{\alpha \alpha}}) - \{ q \} \) such that \( V(M) \cap Y^{(i_{\alpha},-0,0)} \cap X_{V(T_{t_{\alpha}})} - X_{t_{\alpha \alpha}} \neq V(M) \cap Y^{(i_{\alpha},-0,0)} \cap X_{V(T_{t_{\alpha}})} - X_{t_{\alpha \alpha}} \). So there exists \( y \in V(M) \cap Y^{(i_{\alpha},-0,0)} \cap X_{V(T_{t_{\alpha}})} - X_{t_{\alpha \alpha}} \) and a monochromatic path \( Q \) in \( G[Y^{(i_{\alpha},-0,0)}] \) from \( y \) to some vertex \( y' \) adjacent in \( G \) to \( y \). Note that \( V(Q) \cap X_{t_{\alpha \alpha}} \neq \emptyset \). Let \( M \) be the monochromatic \( \mathcal{E}_{j,\alpha} \)-pseudocomponent in \( G[Y^{(i_{\alpha},-0,0)}] \) containing \( Q \).

By (a) and (b), \( y \notin \bigcup_{\alpha=0}^{s_{\alpha}} V(M_{j,\alpha}) \). Let \( M_{j,\alpha}^{(t_{\alpha \alpha})} \) be the union of all members of \( \{ M_{j,\alpha}^{(t_{\alpha \alpha})} : \alpha \in [0, w_0 - 1] \} \). Since \( M \) is connected, there exists a path \( P_y \) in \( M[V(M) \cap X_{V(T_{t_{\alpha \alpha}})}] \) from \( y \) to \( V(M_{j,\alpha}^{(t_{\alpha \alpha})}) \) internally disjoint from \( X_{t_{\alpha \alpha}} \cup V(M_{j,\alpha}^{(t_{\alpha \alpha})}) \). Since \( |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha \alpha}})}| = |V(M) \cap Y^{(i_{\alpha},-0,0)}| \cap X_{V(T_{t_{\alpha \alpha}})}| \), by Claim 23.22, the neighbor in \( P_y \) of the vertex in \( V(P_y) \cap \bigcup_{\alpha=0}^{s_{\alpha}} V(M_{j,\alpha}^{(t_{\alpha \alpha})}) \) belongs to \( X_{V(T_{t_{\alpha \alpha}})} - X_{t_{\alpha \alpha}} \). Let \( M_y \) be the monochromatic \( E_{j,\alpha}^{(t_{\alpha \alpha})} \)-pseudocomponent in \( G[Y^{(i_{\alpha},-0,0)}] \) containing \( y \). By (a) and (b), we have \( y \notin \bigcup_{\alpha=0}^{s_{\alpha}} V(M_{j,\alpha}^{(t_{\alpha \alpha})}) \). By the existence of \( P_y \), \( A_{L(t_{\alpha},-0,0)}(V(M_y)) \cap X_{V(T_{t_{\alpha \alpha}})} - X_{t_{\alpha \alpha}} \neq \emptyset \). So if \( V(M_y) \cap X_{t_{\alpha \alpha}} \neq \emptyset \), then \( M_y \in \mathcal{M}_{j,\alpha}(t_{\alpha \alpha}, k) \) by (a), so (b) is violated, a contradiction. So \( V(M_y) \cap X_{t_{\alpha \alpha}} = \emptyset \). Then there exists \( t_{\alpha \alpha} \in V(T_{t_{\alpha \alpha}}) - \{ t_{\alpha \alpha} \} \) such that \( q \in X_{V(T_{t_{\alpha \alpha}})} - X_{t_{\alpha \alpha}} \). Since \( Q \) is from \( y \) to \( y' \), there exists a subpath \( Q_{\alpha \alpha} \) of \( Q \) from \( y \) to \( X_{t_{\alpha \alpha}} \) internally disjoint from \( X_{t_{\alpha \alpha}} \) and a monochromatic path \( Q \) in \( G[Y^{(i_{\alpha},-0,0)}] \) from \( y \) to some vertex \( y' \) adjacent in \( G \) to \( y \). Note that \( V(Q) \cap X_{t_{\alpha \alpha}} \neq \emptyset \). Let \( M_y \) be the monochromatic \( E_{j,\alpha}^{(t_{\alpha \alpha})} \)-pseudocomponent in \( G[Y^{(i_{\alpha},-0,0)}] \) containing \( Q \). This proves the desired statement.

It remains to prove the last statement of this claim. Assume \( t_{W} \) is not a leaf in \( T_{t_{W}} \) and there exists \( \beta \in N_0 \) such that the \( \beta \)-th entry of \( \mu_{j,\alpha}(t) \neq \emptyset \) and the \( \beta \)-th entry of \( \mu_{j,\alpha}(t_{W}) = \emptyset \). Recall that \( k^\star \) is the index such that \( M_k^\star \in \mathcal{M}_{j,\alpha}(t_k^\star) \), \( \mu_{j,\alpha}(t_{W}) \) is the parent of \( t_{W} \), and \( s_{k^\star}(t_{W}) \) is lexicographically smaller than \( s_{k^\star}(t) \). So we are done if \( \beta > k^\star \).

Hence we may assume \( \beta \in \{ 0, k^\star \} \). By Claim 23.29, some member of \( \mathcal{M}_{j,\alpha}(t_{W}, 0) \) intersects \( V(M) \) since some node belongs to \( K^\star(\alpha) \cap \cap V(T_{t_{W}} \cap V(T_{t_{W}}) - t_{W} \). In particular, \( \beta \geq 1 \). By (a),

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for every \( \alpha \in [0, \beta - 1] \subseteq [0, k^* - 1] \), the \( \alpha \)-th entry of \( \mu'_M(t) \) equals the \( \alpha \)-th entry of \( \mu'_M(p_{W_t}) \). So it suffices to prove that there exists \( \beta' \in [0, \beta - 1] \) such that the \( \beta' \)-th entry of \( \mu'_M(t_{W_t}) \) is smaller than the \( \beta' \)-th entry of \( \mu'_M(p_{W_t}) \).

By (a), \( M'_{p_{W_t}} \in M_M(p_{W_t}, k^*) \). Let \( \gamma \in [0, k^*] \) be an arbitrary element such that some member \( M_\gamma \) of \( M_M(p_{W_t}, \gamma) \) intersects \( V(M) \) and the \( \gamma \)-th entry of \( \mu'_M(t_{W_t}) \) equals zero. Let \( M'_\gamma \) be the monochromatic \( E_{t_{W_t}} \)-pseudoconnection in \( G[Y^{(t_{W_t}, -1, 0)}] \) containing \( M_\gamma \). If \( A_{L(V(t_{W_t}) - X_{t_{W_t}})}(M'_\gamma) \cap X_V(T_{W_t}) - X_{t_{W_t}} = \emptyset \), then by Claim 23.29, \( K^s(M) \cap V(T_{W_t}) - \{t_{W_t}\} = \emptyset \), contradicting that \( t_{W_t} \) is not a leaf of \( T_{M'} \). So \( A_{L(V(t_{W_t}) - X_{t_{W_t}})}(M'_\gamma) \cap X_V(T_{W_t}) - X_{t_{W_t}} \neq \emptyset \). In particular, \( V(M'_\gamma) \cap X_{t_{W_t}} \neq \emptyset \). So \( M'_{\gamma} \in M_M(t_{W_t}, \gamma') \) for some \( \gamma' \in [0, \gamma] \). Since the \( \gamma' \)-th entry of \( \mu'_M(t_{W_t}) \) equals zero, \( \gamma' \in [0, \gamma - 1] \). Hence there exists \( \gamma' \in [0, \gamma] \subseteq [0, \gamma - 1] \) such that the \( \gamma' \)-th entry of \( \mu'_M(t_{W_t}) \) is smaller than the \( \gamma' \)-th entry of \( \mu'_M(p_{W_t}) \), and some member \( M_M(t_{W_t}, \gamma') \) intersects \( V(M) \).

In particular, if some member of \( M_M(p_{W_t}, \beta) \) intersects \( V(M) \), then we are done by taking \( \gamma = \beta \). So we may assume that every member of \( M_M(p_{W_t}, \beta) \) is disjoint from \( V(M) \). Hence \( \beta \in [k^* - 1] \). By taking \( \gamma = k^* \), if the corresponding \( \gamma' \) is at most \( \beta \), then \( \gamma' < \beta \) (since the \( \beta \)-th entry of \( \mu'_M(t_{W_t}) \) is zero) and we are done. So we may assume that the corresponding \( \gamma' \) is greater than \( \beta \) when taking \( \gamma = k^* \). Hence some member of \( M_M(t_{W_t}, \rho) \) intersects \( V(M) \) for some \( \rho > \beta \). Recall that some member of \( M_M(t_{W_t}, 0) \) intersects \( V(M) \). So there exists \( \kappa \in [0, \beta - 1] \) such that some member of \( M_M(t_{W_t}, \kappa) \) intersects \( V(M) \). We choose \( \kappa \) to be as large as possible. Since \( M_M(t_{W_t}, \beta) = \emptyset \), the maximality of \( \kappa \) and the existence of \( \rho \) imply that there exists \( \beta' \in [0, \kappa] \) such that the \( \beta' \)-th entry of \( \mu'_M(t_{W_t}) \) is smaller than the \( \beta' \)-th entry of \( \mu'_M(p_{W_t}) \), by the definition of the third entry of the triples in \( \mu'_M(t_{W_t}) \). This proves the claim.

**Claim 23.39.** Let \( j \in [|V| - 1] \). Let \( M \) be a monochromatic component with respect to \( c \) with \( S_M \cap I^*_j \neq \emptyset \). Then \( |K^s(M)| \leq \eta_9 \).

**Proof.** By Claim 23.38, there exists a collection \( \mathcal{P} \) of paths in \( T_M' \) such that the following hold.

(i) Members of \( \mathcal{P} \) are pairwise internally disjoint.

(ii) For every \( P \in \mathcal{P} \), every internal node of \( P \) is of degree two in \( T_M' \), and there are at most \( w_0 + 1 \) internal nodes belong to \( K^s(M) \).

(iii) Let \( T_M^* \) be the rooted tree obtained from \( T_M' \) by suppressing all internal nodes of members of \( \mathcal{P} \). For every directed path \( Q = q_1q_2...q_{V(Q)} \) in \( T_M' \) not containing a non-root leaf of \( T_M' \), the infinite sequence \((\mu'_M(q_1), \mu'_M(q_2), ..., \mu'_M(q_{V(Q)}))\) is an \( \eta_7 \)-evolution, where \( \emptyset \) is the infinite sequence whose all entries are zero.

Since every leaf of \( T_M^* \) is a leaf of \( T_M' \) and hence belongs to \( K^s(M) \), by (iii), Lemma 18 and Claim 23.29, there exists no directed path in \( T_M^* \) on more than \( g_{1,8} \eta_7 + 1 = \eta_8 \) nodes.

By (ii), the maximum degree of \( T_M^* \) is at most the maximum degree of \( T_M' \). By the definition of fences, for every \( t \in V(T_M) \) and \( t' \in \partial T_{j,t} \), we have \( Y^{(t, -1, 0)} \cap T_j \cap X_{V(T_{j,t})} - X_{t'} \neq \emptyset \), so \( |\partial T_{j,t}| \leq \eta_5 \) by Claim 23.21. So the maximum degree of \( T_M \) (and hence the maximum degree of \( T_M^* \)) is at most \( \eta_5 \). Hence \( |V(T_M)\| \leq \eta_8^{\eta_5} \).

Note that each node in \( K^s(M) \) either belongs to \( V(T_M^*) \) or is an internal node of some member of \( \mathcal{P} \). By (ii), \( |K^s(M)| \leq |V(T_M)\| + (w_0 + 1)|E(T_M^*)\| \leq (w_0 + 2)|V(T_M)\| \leq (w_0 + 2)^{\eta_8^{\eta_5}} = \eta_9 \). This proves the claim.

Now we are ready to complete the proof of this lemma. Let \( M \) be a monochromatic component with respect to \( c \). If \( S_M \cap I^*_j = \emptyset \) for every \( j \in [|V| - 1] \), then \( V(M) \cap \bigcup_{j=1}^{[|V| - 1]} (I_{j,0} \cup I_{j,1}) = \emptyset \), so
If $S_M \cap I_j^p \neq \emptyset$ for some $j \in [V] - 1$, then $|V(M)| \leq \eta_0 \eta_5 \leq \eta^*$ by Claims 23.34 and 23.39. Therefore, $|V(M)| \leq \eta^*$. This proves Lemma 23.

\section{Allowing Apex Vertices}

The following lemma implies Theorem 3 by taking $k = 1$ and $Y_1 = \emptyset$.

**Lemma 24.** For all $s, t, w, k, \xi \in \mathbb{N}$, there exists $\eta^* \in \mathbb{N}$ such that for every graph $G$ containing no $K_{s,t}$ subgraph, if $Z$ is a subset of $V(G)$ with $|Z| \leq \xi$, $V = (V_1, V_2, \ldots, V_{|V|})$ is a $Z$-layering of $G$, $(T, \mathcal{X})$ is a tree-decomposition of $G - Z$ with $\mathcal{V}$-width at most $w$, $Y_1$ is a subset of $V(G)$ with $|Y_1| \leq k$, $L$ is an $(s, \mathcal{V})$-compatible list-assignment of $G$ such that $(Y_1, L)$ is a $\mathcal{V}$-standard pair, then there exists an $L$-coloring $c$ of $G$ such that every monochromatic component with respect to $c$ contains at most $\eta^*$ vertices.

**Proof.** Let $s, t, w, k, \xi \in \mathbb{N}$. Let $f$ be the function $f_{s,t}$ in Lemma 12. Let $h_0$ be the identity function. For $i \in \mathbb{N}$, let $h_i$ be the function defined by $h_i(x) := x + f(h_{i-1}(x))$ for every $x \in \mathbb{N}_0$. Let $\eta_1 := h_{s+1}(k + \xi)$. Let $\eta_2$ be the number $\eta^*$ in Lemma 22 taking $s = s$, $t = t$, $w = w + \xi$ and $\eta = h_{s+2}(k + \xi) + (s + 1)\xi$. Define $\eta^* := \max\{\eta_1, \eta_2\}$.

Let $G$ be a graph with no $K_{s,t}$ subgraph, $Z$ a subset of $V(G)$ with $|Z| \leq \xi$, $V = (V_1, V_2, \ldots, V_{|V|})$ a $Z$-layering of $G$, $(T, \mathcal{X})$ a tree-decomposition of $G - Z$ with $\mathcal{V}$-width at most $w$, $Y_1$ a subset of $V(G)$ with $|Y_1| \leq k$, $L$ an $(s, \mathcal{V})$-compatible list-assignment of $G$ such that $(Y_1, L)$ is a $\mathcal{V}$-standard pair. Say $\mathcal{X} = (X_p : p \in V(T))$.

We may assume that $|V| \geq 2$, since we may add empty layers into $\mathcal{V}$. Let $(Y^{(0)}, L^{(0)})$ be a $(Z \cup Y_1, 0)$-progress of $(Y_1, L)$. For each $i \in [s + 2]$, define $(Y^{(i)}, L^{(i)})$ to be an $(N_{s,t} G(Y^{(i-1)}), i)$-progress of $(Y^{(i-1)}, L^{(i-1)})$. Note that every $L^{(s+2)}$-coloring of $G$ is an $L$-coloring of $G$. It is clear that $|Y^{(i)}_1| \leq h_i(|Z \cup Y_1|) \leq h_i(\xi + k)$ for every $i \in [0, s + 2]$ by Lemma 12 and induction on $i$.

**Claim 24.1.** For every $L^{(s+2)}$-coloring of $G$, every monochromatic component intersecting $Z \cup Y_1$ is contained in $G[Y^{(s+1)}_1]$.

**Proof.** We shall prove that for each $i \in [s + 2]$ and for every $L^{(i)}$-coloring of $G$, every monochromatic component colored $i$ and intersecting $Z \cup Y_1$ is contained in $G[Y^{(i-1)}]$.

Suppose to the contrary that there exist $i \in [s + 2]$, an $L^{(i)}$-coloring of $G$, and a monochromatic component $M$ colored $i$ and intersecting $Z \cup Y_1$ such that $V(M) \not\subseteq Y^{(i-1)}$. Since $M$ intersects $Z \cup Y_1 \subseteq Y^{(i-1)}$, we have $V(M) \cap Y^{(i-1)} \neq \emptyset$. So there exist $y \in V(M) \cap Y^{(i-1)}$ and $v \in N_G(y) \cap V(M) - Y^{(i-1)}$. In particular, $L^{(i-1)}(y) = \{i\} \subseteq L^{(i)}(v) \subseteq L^{(i-1)}(v)$. Since $(Y^{(i-1)}, L^{(i-1)})$ is a $\mathcal{V}$-standard pair, (L2) implies that $v \not\in N_G \subseteq (Y^{(i-1)})$. Since $v \in N_G(y) \subseteq N_G(Y^{(i-1)})$, we have $v \in N_G(Y^{(i-1)})$. But since $(Y^{(i)}, L^{(i)})$ is an $(N_{s,t} G(Y^{(i-1)}), i)$-progress, $i \not\in L^{(i)}(v)$, a contradiction.

This proves that for each $i \in [s + 2]$ and for every $L^{(i)}$-coloring of $G$, every monochromatic component colored $i$ and intersecting $Z \cup Y_1$ is contained in $G[Y^{(i-1)}]$. Since every $L^{(s+2)}$-coloring of $G$ is an $L^{(i)}$-coloring of $G$ for every $i \in [s + 2]$, every monochromatic component with respect to $L^{(s+2)}$ intersecting $Z \cup Y_1$ is contained in $G[Y^{s+1}] \subseteq G[Y^{s+1}]$.

Let $G'$ be the graph obtained from $G - Z$ by adding a copy $z_i$ of $i$ into $V_i$ for every $z \in Z$ and $i \in [|V|]$ and adding an edge $z_i v$ for every edge $z v \in E(G)$ with $z \in Z$ and $v \in V_i$. For $i \in [|V|]$, define $V'_i := V_i \cup \{z_i : z \in Z\}$. Let $V' := (V'_1, V'_2, \ldots, V'_{|V|})$. Then $V'$ is a layering of $G'$. Let $(T, \mathcal{X}')$ be the tree-decomposition of $G'$, where $\mathcal{X}' := (X'_p : p \in V(T))$ and $X'_p := X_p \cup \{z_i : z \in Z, i \in [|V|]\}$ for every $q \in V(T)$. Then $(T, \mathcal{X}')$ is a tree-decomposition of $G'$ with $\mathcal{V}'$-width at most $w + \xi$.\qed
Let \( Y'_1 := Y^{(s+2)} \cup \{ z_i : z \in Z, i \in [|V|] \} \), let \( L'(v) := L^{(s+2)}(v) \) for every \( v \in V(G) - Z \), and let \( L'(z_i) := L^{(s+2)}(z) \) for every \( z \in Z \) and \( i \in [|V|] \). Note that \( (Y'_1, L') \) satisfies (L1)–(L3). But \( (Y'_1, L') \) is possibly not \((s, \mathcal{V}')\)-compatible since some \( L'(z_i) \) might contain a color that is not allowed for \( V_i \). For each \( i \in [|V'|] \) and \( z \in Z \) with \( L'(z_i) = \{ i' \} \) for some \( i' \in [s + 2] \) with \( i \equiv i' \) (mod \( s + 2 \)), delete \( z_i \) from \( G' \) and add edges \( z_i v \) for each edge \( z_i v \in E(G') \) if \( i > 1 \) (or delete \( z_i \) from \( G' \) and add edges \( z_i+1 v \) for each \( z_i v \in E(G') \) if \( i = 1 \)). Note that if \( z_i \) is deleted from \( G' \), then \( z_{i-1} \) and \( z_{i+1} \) are not deleted from \( G' \) (since \( L'(z_{i-1}) = L'(z_i) L'(z_{i+1}) \)). Let \( G'' \) be the resulting graph. For each \( i \in [|V'|] \), let \( V''_i := V'_i \cap V(G'') \). Let \( \mathcal{V}' := (V'_{i+1}, V'_i, \ldots , V'_1) \). So \( \mathcal{V}' \) is a layering of \( G'' \). Let \( Y''_1 := Y'_1 \cap V(G'') \) and \( L'' := L'(G'') \). Then \( L'' \) is an \((s, \mathcal{V}')\)-compatible list-assignment of \( G'' \), and \((Y''_1, L'')\) is a \( \mathcal{V}'\)-standard pair of \( G'' \). Let \( X''_q := X'_q \cap V(G'') \) for every \( q \in V(T) \), and let \( \mathcal{X}'' := (X''_q : q \in V(T)) \). Then \((T, \mathcal{X}'')\) is a tree-decomposition of \( G'' \) with \( \mathcal{V}'\)-width at most \( w + \xi \).

**Claim 24.2.** \( G'' \) does not contain \( K_{s,t} \) as a subgraph.

**Proof.** Suppose to the contrary that some subgraph \( H'' \) of \( G'' \) is isomorphic to \( K_{s,t} \). Note that for every \( z \in Z \), \( N_{G'}(z_i) \cap N_{G'}(z_j) = \emptyset \) for all distinct \( i, j \in [|V'|] \). So for every \( z \in Z \), no vertex of \( G'' \) is adjacent to two vertices in \( \{ z_i \in V(G'') : i \in [|V'|] \} \). Hence for every \( z \in Z \), each part of the bipartition of \( H'' \) contains at most one vertex in \( \{ z_i : z \in Z, i \in [|V'|] \} \). Furthermore, if \( z_i z_j \in E(H'') \) for some \( z, z' \in Z \) and \( i, j \in [|V'|] \), then \( z \neq z' \). Therefore, for every \( z \in Z \), \( H'' \) contains at most one vertex in \( \{ z_i : i \in [|V'|] \} \). Note that for every \( z \in Z \) and \( i \in [|V'|] \) with \( z_i \in V(G'') \), we have \( N_{G''}(z_i) \subseteq N_G(z) \).

Define \( H \) to be the subgraph of \( G \) obtained from \( H'' \) by replacing each vertex \( z_i \in V(H'') \) (for some \( z \in Z \) and \( i \in [|V'|] \)) by \( z \). Then \( H \) is isomorphic to \( K_{s,t} \), which is a contradiction. So \( G'' \) does not contain \( K_{s,t} \) as a subgraph.

Note that for every \( s \)-segment \( S \) of \( \mathcal{V}' \),

\[
|Y''_1 \cap S| \leq |Y_1^{(s+2)} \cap S| + (s + 1)|Z| \leq h_{s+2}(k + \xi) + (s + 1)\xi.
\]

By Lemma 22, there exists an \((s')\)-coloring \( c'' \) of \( G'' \) such that every monochromatic component with respect to \( c'' \) contains at most \( \eta_2 \) vertices. Let \( c \) be the function defined by \( c(v) := c''(v) \) for every \( v \in V(G) - Z \) and \( c(z) := c''(z_i) \) for every \( z \in Z \) and some \( i \in [|V'|] \). Note that for each \( z \in Z \), \( L'(z_i) \) is a constant 1-element subset of \( L^{(s+2)}(z) \) for all \( i \), so \( c \) is well-defined. Therefore \( c \) is an \((s+2)\)-coloring of \( G \) and hence an \( L\)-coloring of \( G \).

Let \( M \) be a monochromatic component with respect to \( c \). Since \( c \) is an \((s+2)\)-coloring of \( G \), if \( V(M) \cap (Z \cup Y_1) \neq \emptyset \), then \( M \) is contained in \( G[Y_1^{(s+1)}] \) by Claim 24.1, which contains at most \( h_{s+1}(k + \xi) = \eta_1 \leq \eta^* \) vertices. If \( V(M) \cap (Z \cup Y_1) = \emptyset \), then \( M \) is a monochromatic component with respect to \( c'' \) contained in \( G'' \), so \( M \) contains at most \( \eta_2 \leq \eta^* \) vertices. This proves that \( c \) has clustering at most \( \eta^* \).

\section{Open Problems}

The 4-Color Theorem [40] is best possible, even in the setting of clustered coloring. That is, for all \( c \) there are planar graphs for which every 3-coloring has a monochromatic component of size greater than \( c \); see [46]. These examples have unbounded maximum degree. This is necessary since Esperet and Joret [15] proved that every planar graph with bounded maximum degree is 3-colorable with bounded clustering. The examples mentioned in Section 1, in fact, contain large \( K_{2,1} \) subgraphs. The following question naturally arises: does every planar graph with no \( K_{2,1} \) subgraph have a 3-coloring with clustering \( f(t) \), for some function \( f \)? More generally, is every
graph with layered treewidth $k$ and with no $K_{2,t}$ subgraph 3-colorable with clustering $f(k,t)$, for some function $f$? For $s \geq 2$, is every graph with layered treewidth $k$ and with no $K_{s,t}$ subgraph $(s+1)$-colorable with clustering $f(k,s,t)$, for some function $f$? In our companion paper [32] we prove an affirmative answer to the weakening of this question with “layered treewidth” replaced by “treewidth” for $s \geq 1$.

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