ON THE RESIDUALITY A FINITE \( p \)-GROUP
OF \( HNN \)-EXTENSIONS

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A criterion for the \( HNN \)-extension of a finite \( p \)-group to be residually a finite \( p \)-group is obtained and based on this criterion the sufficient condition for residuality a finite \( p \)-group of \( HNN \)-extension with arbitrary base group is proved. Then these results are applied to give for groups from two classes of one-relator groups the necessary and sufficient condition to be residually a finite \( p \)-group.

1. Introduction. Statement of results

Almost all known results on the residual finiteness of generalized free products of groups are obtained by use of methods offered by G. Baumslag in work [2]. This methods is based on the assertion (proved in this work) that the generalized free product of two finite groups is residual finite and makes use of the notion of compatible subgroups (introduced there as well). Then this methods was transferred to the construction of \( HNN \)-extension of groups: the residual finiteness of \( HNN \)-extension of finite group was established independently in works [1] and [5], and in [1] the sufficient condition of residual finiteness of \( HNN \)-extension analogous to corresponding condition from [2] was given. A criterion of residuality a finite \( p \)-group of generalized free product of two finite \( p \)-groups was obtained by G. Higman [6] and based on this criterion certain modification of notion of compatible subgroups leads to analogous methods of investigation of residuality a finite \( p \)-group of generalized free products (see [9]).

The aim of the present paper is to obtain a criterion for \( HNN \)-extension of a finite \( p \)-group to be residually a finite \( p \)-group (Theorem 1) and to prove the based on this criterion a sufficient condition (and necessary one) for residuality a finite \( p \)-group of any \( HNN \)-extension (Theorem 2). As illustration of these results, we give the necessary and sufficient conditions for groups from two famous classes of one-relator groups to be residually a finite \( p \)-group (Theorems 3 and 4).

To formulate results of the paper more explicitly, recall that the chief series of some group \( G \) is a normal series which does not admit a non-trivial normal supplements. It is easy to see that a normal series of a finite \( p \)-group is the chief series if and only if all of its factors are of order \( p \). Now, our first result is

**Theorem 1.** Let \( G \) be a finite \( p \)-group, let \( A \) and \( B \) be subgroups of \( G \) and let \( \varphi : A \to B \) be isomorphism. Then \( HNN \)-extension \( G^* = (G, t; t^{-1}At = B, \varphi) \) is the residually a finite \( p \)-group if and only if there exists a chief series

\[
1 = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G
\]
of group $G$ satisfying the following conditions:

1. $(A \cap G_i) \varphi = B \cap G_i$ $(i = 0, 1, \ldots, n)$;
2. for any $i = 0, 1, \ldots, n - 1$ and for every $a \in A \cap G_i$ elements $a \varphi$ and $a$ are congruent modulo subgroup $G_i$.

This Theorem was announced in [11]. It should be noted that a criterion of residuality a finite $p$-group of $HNN$-extension of finite $p$-group in another terms was also obtained in work [12]. Nevertheless, the criterion in Theorem 1 appears to be more suitable for investigation of residuality a finite $p$-group of $HNN$-extensions with infinite base group (see Theorem 2 below). Let us remark also that the proof of Theorem 1 is quite elementary since it makes use of only ordinary properties of construction of $HNN$-extension. By the similar arguments one can also prove Higman’s Theorem mentioned above. It is relevant to recall that original proof of Higman, as well as the proof of respective result in [12], exploits the construction of wreath product.

Higman’s Theorem implies, in particular, that the generalized free product of two finite $p$-groups is residually a finite $p$-group provided that amalgamated subgroups are cyclic. The following simple example demonstrates that, unlike of this, the assumption that subgroups $A$ and $B$ of a finite $p$-group $G$ are cyclic does not guarantee the residuality a finite $p$-group of group $G^* = (G, t; t^{-1} A t = B, \varphi)$.

Let $H^* = (H, t; t^{-1} a t = b^p)$ be the $HNN$-extension of group

$$H = \langle a, b; a^{-1} b a = b^{1+p}, a^p = 1, b^{p^2} = 1 \rangle$$

of order $p^3$, where associated subgroups $A$ and $B$ are generated by elements $a$ and $b^p$ respectively. If we suppose that the group $H^*$ is residually a finite $p$-group then intersection of all members $\gamma_n(H^*)$ of its lower central series must coincide with identity subgroup. Therefore, we can choose the number $n$ such that $a \in \gamma_n(H^*) \setminus \gamma_{n+1}(H^*)$. But then congruences $b \equiv b^{1+p} \pmod{\gamma_{n+1}(H^*)}$ and $a \equiv b^p \pmod{\gamma_{n+1}(H^*)}$ will imply that $a \in \gamma_{n+1}(H^*)$. (Conditions of Theorem 1 are not satisfied here since $A \cap B = 1$ and the first non-identity member of any chief series of group $H$ must coincide with its centre $B$.)

Nevertheless, it is relevant to expect that in the case when subgroups $A$ and $B$ are cyclic one can find a more simple criterion for group $G^*$ to be residually a finite $p$-group. As some confirmation of that, in the case when subgroups $A$ and $B$ are equal we have

**Corollary.** Let $G$ be a finite $p$-group, let $A$ be non-identity cyclic subgroup of $G$ with generator $a$ and let $\varphi$ be automorphism of $A$ such that $a \varphi = a^k$ for some integer $k$ (coprime to $p$). Then group $G^* = (G, t; t^{-1} a t = a^k)$ is residually a finite $p$-group if and only if $k \equiv 1 \pmod{p}$.
Indeed, if $1 = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$ is a chief series of group $G$ then distinct members of sequence $(A \cap G_i)$ ($i = 0, 1, \ldots, n$) constitute the unique chief series of group $A$. Hence $(A \cap G_i)\varphi = A \cap G_i$ for all $i = 0, 1, \ldots, n$. If $a \in G_{i+1} \setminus G_i$ then, since element $aG_i$ of quotient group $G/G_i$ is of order $p$, the equality $(a\varphi)G_i = aG_i$ implies that $k \equiv 1 \pmod{p}$. Conversely, if this congruence is fulfilled then it is obvious that any chief series of group $G$ satisfies the condition (2) of Theorem 1.

In what follows we need some notions that came from the work [2] and are utilized now in almost all investigations of residual properties of $HNN$-extensions.

A family $(N_\lambda)_{\lambda \in \Lambda}$ of normal subgroups of a group $G$ is said to be filtration if $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. If $H$ is a subgroup of group $G$ and if $\bigcap_{\lambda \in \Lambda} HN_\lambda = H$ then filtration $(N_\lambda)_{\lambda \in \Lambda}$ is called $H$-filtration. If $H$ and $K$ are two subgroups of $G$ then filtration will be called $(H,K)$-filtration if it is $H$-filtration and $K$-filtration simultaneously.

Let now $G$ be a group with subgroups $A$ and $B$ and let $\varphi : A \to B$ be isomorphism. Subgroup $H$ of group $G$ is called $(A,B,\varphi)$-compatible if $(A \cap H)\varphi = B \cap H$. (So, the condition (1) in Theorem 1 means that all subgroups $G_i$ are $(A,B,\varphi)$-compatible.) It is easy to see that if $H$ is a normal $(A,B,\varphi)$-compatible subgroup of $G$ then the mapping $\varphi_H : AH/H \to BH/H$ (well) defined by the rule $(aH)\varphi_H = (a\varphi)H$ (where $a \in A$) is an isomorphism of subgroup $AH/H$ of quotient group $G/H$ onto subgroup $BH/H$. Furthermore, the natural homomorphism of group $G$ onto quotient group $G/H$ can be extended to homomorphism $\rho_H$ (sending $t$ to $t$) of $HNN$-extension $G^* = (G,t; t^{-1}At = B,\varphi)$ onto $HNN$-extension

$$G^*_H = (G/H,t; t^{-1}AH/H t = BH/H, \varphi_H).$$

Let $\mathcal{F}_G(A,B,\varphi)$ denote the family of all $(A,B,\varphi)$-compatible normal subgroups of finite index of group $G$. Mentioned above sufficient condition of residual finiteness of $HNN$-extension $G^*$ of group $G$ in [1] consists of requirement for the family $\mathcal{F}_G(A,B,\varphi)$ to be $(A,B)$-filtration. To formulate the analogous condition for residuality a finite $p$-group of $HNN$-extension we give here based on the Theorem 1 corresponding modification of notion $(A,B,\varphi)$-compatibility.

Let, as before, $G$ be a group with subgroups $A$ and $B$ and $\varphi : A \to B$ be isomorphism. Let $p$ be a prime number. Subgroup $H$ of group $G$ will be called $(A,B,\varphi,p)$-compatible if there exists a sequence

$$H = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$$

of subgroups of group $G$ such that

1) for any $i = 0, 1, \ldots, n$ subgroup $G_i$ is $(A,B,\varphi)$-compatible and normal in $G$, and
2) for any \( i = 0,1,\ldots,n-1 \) the quotient group \( G_{i+1}/G_i \) is of order \( p \) and for every \( a \in A \cap G_{i+1} \) elements \( a \varphi \) and \( a \) are congruent modulo subgroup \( G_i \).

Let \( \mathcal{F}_G^p(A, B, \varphi) \) denote the family of all \((A, B, \varphi, p)\)-compatible subgroups of group \( G \). Thus, Theorem 1 asserts, actually, that if \( G \) is a finite \( p \)-group then \( HNN \)-extension \( G^* = (G, t; t^{-1}At = B, \varphi) \) is residually a finite \( p \)-group if and only if the family \( \mathcal{F}_G^p(A, B, \varphi) \) contains identity subgroup.

The following Theorem giving sufficient condition for \( HNN \)-extension to be residually a finite \( p \)-group can be considered, as well, as a confirmation of the fact that the notion of \((A, B, \varphi, p)\)-compatibility indeed can be used as \( p \)-analog of notion of \((A, B, \varphi)\)-compatibility.

**Theorem 2.** Let \( G \) be a group with subgroups \( A \) and \( B \) and let \( \varphi : A \to B \) be isomorphism. Let \( G^* = (G, t; t^{-1}At = B, \varphi) \) be the \( HNN \)-extension of \( G \). Then

1. if group \( G^* \) is residually a finite \( p \)-group then the family \( \mathcal{F}_G^p(A, B, \varphi) \) is a filtration;
2. if the family \( \mathcal{F}_G^p(A, B, \varphi) \) is an \((A, B)\)-filtration then group \( G^* \) is residually a finite \( p \)-group.

In the case when group \( G \) is abelian and \( A \) and \( B \) are proper subgroups of \( G \) one can say somewhat more. Let \( g \in G \setminus A \) and \( h \in G \setminus B \). Then commutator \( u = [t^{-1}gt, h] \) is not equal to identity since its expression \( u = t^{-1}g^{-1}th^{-1}t^{-1}gt \) is reduced in group \( G^* \). If the group \( G^* \) is residually a finite \( p \)-group then some normal subgroup \( N \) of finite \( p \)-index of \( G^* \) does not contain element \( u \). If \( M = G \cap N \) then since quotient group \( G/M \) is abelian it follows that \( g \notin AM \). Thus, since subgroup \( M \) is \((A, B, \varphi, p)\)-compatible (see Lemma 2.2 below), we have obtained the

**Corollary.** If group \( G \) is abelian and \( A \) and \( B \) are proper subgroups of \( G \) then group \( G^* \) is residually a finite \( p \)-group if and only if the family \( \mathcal{F}_G^p(A, B, \varphi) \) is \((A, B)\)-filtration.

Let us consider now two classes of one-relator groups. The first is the class of Baumslag–Solitar groups, i.e. class of groups with presentation

\[
G(l, m) = \langle a, b; b^{-1}a^lb = a^m \rangle,
\]

where without loss of generality one can assume that \(|m| \geq l > 0\). It is well-known (see [3, 10]) that the group \( G(l, m) \) is residually finite if and only if either \( l = 1 \), or \(|m| = l \).

The second class consists of certain \( HNN \)-extensions of Baumslag–Solitar groups, namely, of groups with presentation

\[
H(l, m; k) = \langle a, t; t^{-1}a^{-k}ta^l t^{-1}ak t = a^m \rangle = \langle a, b, t; b^{-1}a^l b = a^m, t^{-1}ak t = b \rangle,
\]
where (again without loss of generality) it can be supposed that $|m| \geq l > 0$ and $k > 0$. Some properties of these groups were established by A. M. Brunner [4] (see also [8]). In particular, it is known that the group $H(l, m; k)$ is residually finite if and only if $|m| = l$.

Theorems 1 and 2 will be applied here to prove following assertions:

**Theorem 3.** Group $G(l, m) = \langle a, b; b^{-1}a^lb = a^m \rangle$ (where $|m| \geq l > 0$) is residually a finite $p$-group if and only if either $l = 1$ and $m \equiv 1 \pmod{p}$, or $|m| = l = p^r$ for some $r \geq 0$ and also if $m = -l$ then $p = 2$.

**Theorem 4.** Group $H(l, m; k) = \langle a, t; t^{-1}a^{-kt}a^lt^{-1}a^{kt} = a^m \rangle$ (where $|m| \geq l > 0$ and $k > 0$) is residually a finite $p$-group if and only if $|m| = l = p^r$ and $k = p^s$ for some integers $r \geq 0$ and $s \geq 0$ and also if $m = -l$ then $p = 2$ and $s \leq r$.

2. Proof of Theorems 1 and 2

To prove Theorem 1 we begin with simple and well-known (see, e. g., [12, Proposition 1]) remark:

**Lemma 2.1.** Let $G$ be a finite $p$-group, $A, B \leq G$ and let $\varphi : A \to B$ be isomorphism. Group $G^* = (G, t; t^{-1}At = B, \varphi)$ is residually a finite $p$-group if and only if there exists a homomorphism $\rho$ of group $G^*$ onto some finite $p$-group $X$ such that $\text{Ker } \rho \cap G = 1$.

In fact, the part “only if” of Lemma is obvious since group $G$ is finite. Conversely, if $\text{Ker } \rho \cap G = 1$ then (see [7]) subgroup $\text{Ker } \rho$ is free. So, group $G^*$ is free-by-(finite $p$-group) and therefore is residually a finite $p$-group.

Suppose now that the $HNN$-extension $G^* = (G, t; t^{-1}At = B, \varphi)$ of finite $p$-group $G$ is residually a finite $p$-group. Then by Lemma 2.1 we can consider the group $G$ as a subgroup of some finite $p$-group $X$ with element $x$ such that $x^{-1}ax = a\varphi$ for all $a \in A$. Let

$$1 = X_0 \leq X_1 \leq \cdots \leq X_{n-1} \leq X_n = X$$

be a chief series of group $X$ and $G_i = G \cap X_i$ ($i = 0, 1, \ldots, n$). Then distinct members of sequence $G_0, G_1, \ldots, G_{n-1}, G_n$ form the chief series of group $G$. Since $A \cap G_i = A \cap X_i$ and $B \cap G_i = B \cap X_i$,

$$(A \cap G_i)\varphi = (A \cap X_i)\varphi = (A \cap X_i)^x = A^x \cap X_i = B \cap X_i = B \cap G_i.$$ 

Let $\psi$ be the embedding of quotient group $G/G_i$ into quotient group $X/X_i$ which takes coset $gG_i$ to coset $gX_i$. Since subgroup $(G_{i+1}/G_i)\psi$ is contained in central subgroup $X_{i+1}/X_i$ of $X/X_i$, for any element $a \in A \cap G_{i+1}$ we have

$$(aG_i)\psi = aX_i = a^xX_i = (a\varphi)X_i = ((a\varphi)G_i)\psi.$$

Since the mapping $\psi$ is injective, this implies that $(a \varphi)G_i = aG_i$. Thus, we have the chief series of group $G$ satisfying the conditions (1) and (2) of Theorem 1.

Conversely, suppose that some chief series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$$

of group $G$ satisfies the conditions (1) and (2) of Theorem 1. By induction on $n$ we shall show that then there exists a homomorphism of group $G^*$ onto some finite $p$-group $X$ which acts injectively on subgroup $G$ (and thus, in view of Lemma 2.1, we shall prove that the group $G^*$ is residually a finite $p$-group).

It is easy to see that if $n = 1$ then for group $X$ we can take the direct product of group $G$ and cyclic group of order $p$ with generator $x$; the desired mapping acts on group $G$ identically and sends element $t$ to element $x$.

Let $n > 1$. Since $(A \cap G_1) \varphi = B \cap G_1$ and subgroup $G_1$ is of order $p$, we must consider only two following cases:

a) $G_1 \triangleleft A, G_1 \triangleleft B$

b) $A \cap G_1 = B \cap G_1 = 1$.

In the case a) we set $\overline{G} = G/G_1$, $\overline{G}_i = G_i/G_1$ ($i = 1, 2, \ldots, n$), $\overline{A} = A/G_1$ and $\overline{B} = B/G_1$. Then

$$1 = \overline{G}_1 \leq \overline{G}_2 \leq \cdots \leq \overline{G}_{n-1} \leq \overline{G}_n = \overline{G}$$

is the chief series of group $\overline{G}$. Since subgroup $G_1$ is $(A, B, \varphi)$-compatible, the mapping $\overline{\varphi} = \varphi_{G_1}$ is an isomorphism of subgroup $\overline{A}$ onto subgroup $\overline{B}$. Moreover, since $\overline{A} \cap \overline{G}_i = (A \cap G_1)/G_1 \overline{B} \cap \overline{G}_i = (B \cap G_1)/G_1$, we have $(\overline{A} \cap \overline{G}_i)\overline{\varphi} = \overline{B} \cap \overline{G}_i$. Also, it can be immediately checked that for any $i = 1, 2, \ldots, n-1$ and for every element $aG_1 \in \overline{A} \cap \overline{G}_{i+1}$ cosets $aG_1$ and $(aG_1)\overline{\varphi}$ are congruent modulo subgroup $\overline{G}_i$. Consequently, by induction there exists a homomorphism $\sigma$ of group $\overline{G}^* = (\overline{G}, t; t^{-1}A\overline{t} = \overline{B}, \overline{\varphi})$ onto finite $p$-group $X$ such that $\text{Ker } \sigma \cap \overline{G} = 1$.

Let $\rho = \rho_{G_1}$ be the homomorphism of group $G^*$ onto the group $\overline{G}^*$ which extends natural mapping of $G$ onto quotient group $\overline{G}$. Let $L = \text{Ker } (\rho \sigma)$. Then $\text{Ker } \rho = G_1$, $G^*/L \simeq X, G \cap L = G_1$. Furthermore, since $L/G_1 \simeq \text{Ker } \sigma$ and the group $\text{Ker } \sigma$ is free there exists a free subgroup $F$ of $L$ such that $L = FG_1$ and $F \cap G_1 = 1$. As the condition (2) implies that subgroup $G_1$ belongs to the centre of $G^*$, we have $L = F \times G_1$. Let $N$ denote intersection of all normal subgroups of index $p$ of group $L$. Then $N$ is contained in $F$ and is normal in $G$ because it is characteristic in $L$. Moreover, quotient group $L/N$ is a finite $p$-group since group $L$ is finitely generated as a subgroup of finite index of finitely generated group $G^*$. At last, $N \cap G = N \cap L \cap G = N \cap G_1 = N \cap F \cap G_1 = 1$. Thus, the natural homomorphism of group $G^*$ onto quotient group $G^*/N$ is desired.

In the case b) we set $A_1 = AG_1$ and $B_1 = BG_1$. Since $G_1$ is a central subgroup of $G$, $A_1 = A \times G_1, B_1 = B \times G_1$. Therefore, the mapping $\varphi_1 : A_1 \rightarrow B_1$ which
for any $i \in F$ from item c) in Lemma 2.2 it follows that there exists subgroup $H \in F$ such that for any non-identity element $g \in H \subset F$ we have $g \varphi \in F$. Furthermore, if element $g = ax$ (where $a \in A$ and $x \in G_1$) belongs to subgroup $A_1 \cap G_{i+1} = (A \cap G_{i+1})G_1$ then $a \in A \cap G_{i+1}$ and, therefore, $(g \varphi)G_i = (a \varphi)xG_i = \varphi G_i \cdot xG_i = \varphi G_i \cdot xG_i = \varphi G_i$. Hence, by the case a), treated above, there exists a homomorphism $\sigma$ of group

$$G_1^* = (G, t; t^{-1}A_1 t = B_1, \varphi_1)$$

onto some finite $p$-group $X$ which acts injectively on subgroup $G$. Since the isomorphism $\varphi$ coincides with restriction to subgroup $A$ of isomorphism $\varphi_1$, the identity mapping of group $G$ can be extended to homomorphism $\rho : G^* \to G^*_1$. Then the homomorphism $\rho \sigma$ maps group $G^*$ onto group $X$ and acts injectively on $G$. Thus, inductive step is completed and Theorem 1 is proved.

Let us proceed to prove Theorem 2. Let $G$ be a group with subgroups $A$ and $B$ and let $\varphi : A \to B$ be isomorphism.

**Lemma 2.2.** a) Any normal $(A, B, \varphi)$-compatible subgroup $H$ of group $G$ belongs to the family $F^p_G(A, B, \varphi)$ if and only if the group $G^*_H$ is residually a finite $p$-group.

b) Let $N$ be a normal subgroup of finite $p$-index of group $G^* = (G, t; t^{-1}A_1 t = B, \varphi)$ and $M = G \cap N$. Then $M \in F^p_G(A, B, \varphi)$.

c) The family $F^p_G(A, B, \varphi)$ is closed under intersections of finite collections of subgroups.

The proof of all assertions in Lemma 2.2 does not evoke a special difficulties. The validity of item ) follows immediately from Theorem 1 applied to group $G/H$ and from definition of $(A, B, \varphi, p)$-compatibility. As well, item b) is direct consequence of item a) and of Lemma 2.1. To prove item c) it is sufficient to remark that if subgroups $H$ and $K$ belong to family $F^p_G(A, B, \varphi)$ and $L = H \cap K$ then there exists a homomorphism $\rho$ of group $G^*_L$ into direct product $G^*_H \times G^*_K$ which acts on subgroup $G/L$ injectively.

The assertion (1) in Theorem 2 is an obvious consequence of item b) in Lemma 2.2. It follows from the item ) in Lemma 2.2 that to prove the assertion (2) it is enough to show that for any non-identity element $w$ of group $G^*$ there exists subgroup $H \in F^p_G(A, B, \varphi)$ such that $w \rho_H \neq 1$.

If element $w$ belongs to subgroup $G$ then the existence of such subgroup is ensured by assumption that the family $F^p_G(A, B, \varphi)$ is filtration.

Let $w = g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 \cdots t^{\varepsilon_n} g_n$ be a reduced form of element $w$, where $n \geq 1$. Then for any $i = 1, 2, \ldots, n - 1$ such that $\varepsilon_i + \varepsilon_{i+1} = 0$ we have $g_i \notin A$ if $\varepsilon_i = -1$ and $g_i \notin B$ if $\varepsilon_i = 1$. Now, from the assumption that the family $F^p_G(A, B, \varphi)$ is $(A, B)$-filtration and from item c) in Lemma 2.2 it follows that there exists subgroup $H \in F^p_G(A, B, \varphi)$ such that for any $i = 1, 2, \ldots, n - 1$ $g_i \notin AH$ if $g_i \notin A$ and $g_i \notin BH$ if $g_i \notin B$. This means
that the expression \((g_0H)t^{e_1}(g_1H)t^{e_2}(g_2H)\cdots t^{e_n}(g_nH)\) is a reduced form of element \(wp_{\mu}\) and therefore this element is not equal to 1. The proof of Theorem 2 is complete.

3. Proof of Theorems 3 4

The group \(G(l, m) = \langle a, b; b^{-1}a^lb = a^m \rangle\) is an HNN-extension of infinite cyclic group \(A\) generated by element \(a\) with stable letter \(b\), associated subgroups \(A^l\) and \(A^m\) and with isomorphism \(\varphi\) which sends element \(a^l\) to element \(a^m\). If this group is residually a finite \(p\)-group then it is residually finite and therefore, as was remarked above, \(l = 1\) or \(|m| = l\) (recall that we assume that \(|m| \geq l > 0\)).

Suppose, at first, that \(l = 1\). Let \(\rho\) be the homomorphism of group \(G(1, m)\) onto finite \(p\)-group \(X\) such that \(\alpha \nu \neq 1\). If \(p^s\) denote the order of element \(\alpha \nu\) then \(\rho\) passes through the group

\[G_s(1, m) = \langle a, b; b^{-1}a^m = a^m, a^{p^s} = 1 \rangle\]

which is an HNN-extension of finite cyclic group \(A/A^{p^s}\). Since there exists a homomorphism of group \(G_s(1, m)\) onto finite \(p\)-group \(X\) which acts injectively on base group of this HNN-extension, by Lemma 2.1 group \(G_s(1, m)\) is residually a finite \(p\)-group. The Corollary from Theorem 1 implies now that \(m \equiv 1 \pmod{p}\).

Conversely, if the congruence \(m \equiv 1 \pmod{p}\) holds and therefore (by the same Corollary) any group \(G_s(1, m)\) is residually a finite \(p\)-group then the group \(G(1, m)\) is residually a finite \(p\)-group too, since it, as easy to see, is residually groups \(G_s(1, m)\) \((s = 1, 2, \ldots)\).

If \(l > 1\) (and \(|m| = l\)) then by Corollary from Theorem 2 group \(G(l, m)\) is residually a finite \(p\)-group if and only if the family \(\mathcal{F}_A^p(A^l, A^m, \varphi)\) is \((A^l, A^m)\)-filtration.

Let \(l = l^1p^s\) where \(r > 0\) and \((l^1, p) = 1\). It is evident that if subgroup \(A^k\) of group \(A\) is \((A^l, A^m, p)\)-compatible then \(k\) is a \(p\)-number. Also, it is easy to see that if \(m = l\) then any subgroup of form \(A^{p^s}\) is \((A^l, A^m, p)\)-compatible and if \(m = -l\) and \(s > r\) then subgroup \(A^{p^s}\) is \((A^l, A^m, p)\)-compatible if and only if \(p = 2\). Let us note, at last, that if integers \(x\) and \(y\) are such that \(l_1x + p^sy = 1\) then the equality \(a^{p^r} = (a^l)^x \cdot (a^{p^s})^{p^s}y\) holds and therefore \(a^{p^r} \in A^l \cdot A^{p^s}\). Thus, the family \(\mathcal{F}_A^p(A^l, A^m, \varphi)\) is \((A^l, A^m)\)-filtration if and only if \(l = p^r\) for some \(r > 0\) and also if \(m = -l\) then \(p = 2\). Theorem 3 is proved.

Let us proceed now to prove the Theorem 4. Suppose that the group

\[H(l, m; k) = \langle a, b, t; b^{-1}a^l b = a^m, t^{-1}a^k t = b \rangle\]

(where \(|m| \geq l > 0\) \((k > 0)\)) is residually a finite \(p\)-group. Then (see [1, 7]) \(|m| = l\) and since group \(G(l, m)\) is a subgroup of \(H(l, m; k)\) the Theorem 3 implies that \(|m| = l = p^r\) for some \(r > 0\) and also if \(m = -l\) then \(p = 2\).

Let \(k = p^sk_1\), where \(s \geq 0\) and \((k_1, p) = 1\). If \(k_1 > 1\) then a \(t\)-reduced form of element \(w = [t^{-1}a^{-p^s} t a^l t^{-1}a^{p^s} t, a]\) of group \(H(l, m; k)\) is of length 8 and therefore
Suppose that \( m = -l \) (and therefore \( p = 2 \)). Let \( \sigma \) be homomorphism of group \( H(2^r, -2^r; 2^s) \) onto finite \( p \)-group \( X \) such that \( b\sigma \neq 1 \). Let

\[
1 = X_0 \leq X_1 \leq \cdots \leq X_n = X
\]

be a chief series of group \( X \) and \( b\sigma \in X_{i+1} \setminus X_i \). Since \( X_{i+1}/X_i \) is a central subgroup of group \( X/X_i \), we have \( (a\sigma)^{2^r} \equiv (a\sigma)^{2^r} \mod (X_i) \) and \( (a\sigma)^{2^r} \equiv b\sigma \mod (X_i) \). Therefore, element \((a\sigma)^{2^r+1}\) belongs to subgroup \( X_i \) and element \((a\sigma)^{2^r}\) does not belong to that subgroup. This implies the inequality \( s \leq r \).

Conversely, let us show that for any prime number \( p \) and for any integers \( r \geq 0 \) and \( s \geq 0 \) the group \( H(p^r, \varepsilon p^s; p^s) \) (where \( \varepsilon = \pm 1 \) and if \( \varepsilon = -1 \) then \( p = 2 \) and \( s < r \)) is residually a finite \( p \)-group.

Let \( G = \langle a, b; b^{-1}a^{p^r}b = a^{\varepsilon p^r} \rangle \) and let \( A \) and \( B \) be cyclic subgroups of \( G \) generated by elements \( a \) and \( b \) respectively and \( A_1 = A^{p^s} \). Then the group \( H(p^r, \varepsilon p^s; p^s) \) is \( HNN \)-extension \( (G, t; t^{-1}A_1t = B, \varphi) \) where isomorphism \( \varphi \) is defined by the equality \( a^{p^r}\varphi = b \).

By Theorem 2 it is enough to show that the family \( \mathcal{F}_G^p(A_1, B, \varphi) \) is \( (A_1, B) \)-filtration. Nevertheless, we begin with somewhat weaker assertion.

**Lemma 3.1.** The family of all normal \((A_1, B)\)-compatible subgroups of finite \( p \)-index of group \( G \) is \((A_1, B)\)-filtration.

**Proof.** It is evident that the group \( G \) is amalgamated free product

\[
G = (A \ast K; a^{p^r} = c)
\]

of groups \( A \) and \( K = \langle b, c; b^{-1}cb = c^\varepsilon \rangle \). For any integer \( n > \max(r, s) \) let \( L_n \) be subgroup of \( K \) generated by elements \( b^{p^{n-s}} \) and \( c^{p^{n-r}} \). Since when \( \varepsilon = -1 \) then \( p = 2 \), in any case \( L_n \) is a normal subgroup of \( K \). Also, it is obvious that the quotient group \( K/L_n \) is of order \( p^{2n-r-s} \) and its elements \( bL_n \) and \( cL_n \) are of order \( p^{n-s} \) and \( p^{n-r} \) respectively. Therefore, we can construct amalgamated free product \( G_n = (A/A^{p^n} \ast K/L_n; (aA^{p^n})^{p^r} = cL_n) \).

Let \( \rho_n \) be homomorphism of group \( G \) onto group \( G_n \) which extends the natural mappings of group \( A \) onto quotient \( A^{p^n} \) and of group \( K \) onto quotient \( K/L_n \). We
claim that for any element \( g \in G \) if \( g \neq 1 \) or if \( g \notin A_1 \) or if \( g \notin B \) then there exists an integer \( n \) such that \( gp_n \neq 1 \) or \( gp_n \notin A_1p_n \) or \( gp_n \notin Bp_n \) respectively. In fact, this is evident if \( g \in A \) or \( g \in K \). Moreover, in these cases the following assertion is obvious too: if element \( g \) does not belong to amalgamated subgroup in above decomposition of group \( G \), then for all numbers \( n \) that are large enough element \( gp_n \) does not belong to amalgamated subgroup in decomposition of group \( G_n \). This implies that if reduced form of element \( g \) is of length \( > 1 \) then for suitable \( n \) the reduced form of element \( gp_n \) is of the same length. Therefore, this element does not belong to both subgroups \( A_1p_n \) and \( Bp_n \).

Let \( g \) be any non-identity element of group \( G \) and let integer \( n \) be chosen so that \( gp_n \neq 1 \). Since the group \( G_n \) (by Higman’s criterion mentioned above) is residually a finite \( p \)-group and its subgroups \( A/A^p \) and \( K/L_n \) are finite, there exists a normal subgroup \( M \) of finite \( p \)-index of group \( G_n \) such that \( gp_n \notin M \) and \( A/A^p \cap M = K/L_n \cap M = 1 \). Let \( N = M\rho_n^{-1} \). Then \( N \) does not contain element \( g \) and is a normal \((A_1, B, \varphi)\)-compatible subgroup of finite \( p \)-index of group \( G \). The similar arguments show that if element \( g \) does not belong to subgroup \( A_1 \) or to subgroup \( B \) then subgroup \( N \) can be chosen so that element \( g \) does not belong to subgroup \( A_1N \) or to subgroup \( BN \) respectively.

**Lemma 3.2.** For any integer \( n > s \) there exists a normal subgroup \( N \) of finite \( p \)-index of group \( G \) such that \( A \cap N = A^p \), \( B \cap N = B^{p^{n-s}} \) and \( a^{p^{n-1}} \equiv b^{p^{n-s-1}} \) (mod \( N \)).

*Proof.* Firstly suppose that \( n > r \). Assuming all notations from the proof of Lemma 3.1 let us consider also cyclic subgroup \( D \) of group \( K \), generated by element \( d = c^{p^{n-r-1}}b^{p^{n-s-1}} \). If in the case \( \varepsilon = -1 \) we shall suppose that \( n - s - 1 > 0 \) then in any case \( DL_n/L_n \) will be a central subgroup of order \( p \) of group \( K/L_n \) and elements \( c(DL_n) \) and \( b(DL_n) \) of quotient group \( K/DL_n \) will be of order \( p^{n-r} \) and \( p^{n-s} \) respectively. Since the group \( G_n' = (A/A^p \ast K/DL_n; (aA^p)^p = c(DL_n)) \) is residually a finite \( p \)-group, it contains a normal subgroup \( M \) of finite \( p \)-index such that \( A/A^p \cap M = K/DL_n \cap M = 1 \). Then the preimage \( N \) of subgroup \( M \) under the obvious homomorphism of group \( G \) onto group \( G_n' \) is desired subgroup.

If \( \varepsilon = -1 \) and \( n-s-1 = 0 \) then since \( n > r \) and \( s \leq r \) we have \( s = r \) and \( n = r+1 \). In this case desired subgroup \( N \) coincides with kernel of homomorphism \( \sigma \) of group \( G \) onto cyclic group \( X \) of order \( 2^{r+1} \) with generator \( x \), where \( a\sigma = x \) and \( b\sigma = x^{2^r} \).

At last, if \( n \leq r \) then group \( G \) can be mapped onto group

\[ T = \langle a, b; a^p = 1, b^{p^{n-s}} = 1, a^{p^{n-1}} = b^{p^{n-s-1}} \rangle. \]

Since group \( T \) is residually a finite \( p \)-group it has a normal subgroup of finite \( p \)-index whose intersections with (finite) free factors are trivial. The preimage of this subgroup is desired subgroup of \( G \).
Now we are able to prove the assertion mentioned above and thus to complete the proof of Theorem 4.

**Lemma 3.3.** The family $\mathcal{F}_G^p(A_1, B, \varphi)$ is $(A_1, B)$-filtration.

**Proof.** We shall show that any $(A_1, B)$-compatible subgroup $N$ of finite $p$-index of group $G$ such that $n > s$ where the integer $n$ is determined by the equality $A \cap N = A^p \cap n$ contains some subgroup $M$ from family $\mathcal{F}_G^p(A_1, B, \varphi)$. It is evident that then the desired assertion will be applied by Lemma 3.1.

Since inequalities $k < n - s$ and $n - k > s$ are equivalent, it follows from Lemma 3.2 that for every integer $k$, $0 \leq k < n - s$, in group $G$ there exists a normal subgroup $N_k$ of finite $p$-index such that $A \cap N_k = A^p \cap n - k$, $B \cap N_k = B^p \cap n - k - s \equiv b^p \cap n - k - s - 1$ (mod $N_k$). Let also $N_{n-s} = G$. For any $i = 0, 1, \ldots, n - s$ we set $M_i = \bigcap_{k=i}^{n-s} N_k$ and claim that subgroup $M = N \cap M_0$ is required. Indeed, since all members of the increasing sequence of normal subgroups $M, M_0, M_1, \ldots, M_{n-s-1}$, $M_{n-s} = G$ have finite $p$-index in group $G$ we can supplement it to such sequence of normal subgroups of $G$ all quotients of which are of order $p$. The immediately verification shows that the resulting sequence satisfies all requirements in definition of $(A_1, B, \varphi, p)$-compatible subgroup.

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