Quadratic Poisson algebras for two dimensional classical superintegrable systems and quadratic associative algebras for quantum superintegrable systems.

C. Daskaloyannis*

Physics Department,
Aristotle University of Thessaloniki,
54006 Thessaloniki, Greece

February 2000

Abstract

The integrals of motion of the classical two dimensional superintegrable systems with quadratic integrals of motion close in a restrained quadratic Poisson algebra, whose the general form is investigated. Each classical superintegrable problem has a quantum counterpart, a quantum superintegrable system. The quadratic Poisson algebra is deformed to a quantum associative algebra, the finite dimensional representations of this algebra are calculated by using a deformed parafermion oscillator technique. It is shown that, the finite dimensional representations of the quadratic algebra are determined by the energy eigenvalues of the superintegrable system. The calculation of energy eigenvalues is reduced to the solution of algebraic equations, which are universal for all two dimensional superintegrable systems with quadratic integrals of motion.

Running title: Quadratic algebras for superintegrable systems
PACS Numbers: 03.65.Fd; 02.10.Tq; 45.20.Jj;

*e:mail address: daskalo@auth.gr
I Introduction

In classical mechanics, integrable system is a system possessing more constants of motion in addition to the energy. A comprehensive review of the two-dimensional integrable classical systems is given by Hietarinta [1], where the space was assumed to be flat. The case of non flat space is under current investigation [2, 3]. An interesting subset of the totality of integrable systems is the set of systems, which possess a maximum number of integrals, these systems are termed as superintegrable ones. The Coulomb and the harmonic oscillator potentials are the most familiar classical superintegrable systems, whose their quantum counterpart has nice symmetry properties, which are described by the \( so(N + 1) \) and \( su(N) \) Lie algebras respectively.

The Hamiltonian of a classical system is generally a quadratic function of the momenta. In the case of the flat space, all the known two dimensional superintegrable systems with quadratic integrals of motion are simultaneously separable in more than two orthogonal coordinate systems [4]. The integrals of motion of a two dimensional superintegrable system in flat space close in a restrained classical Poisson algebra [4, 5, 6]. The study of the quadratic Poisson algebras is a matter under investigation, related to several branches of physics as: the solution of the classical Yang - Baxter equation [8], the two dimensional superintegrable systems in flat space or on the sphere [7], the statistics [9] or the case of "exactly solvable" classical problems [10].

The quantization of a classical integrable system corresponds generally to a quantum integrable system. The mechanism of quantum deformation of a classical system to a quantum one is not fully understood. Initially the problem of quantization of classical superintegrable system was viewed as a relatively simple and somehow trivial problem [11], but several authors have proved that this quantization procedure has to add correction terms to the integrals of motion or to the Hamiltonian, these correction terms are of order \( \mathcal{O}(\hbar^2) \) [12, 13]. The result of the quantum deformation of a superintegrable system is realized by the shift of the classical Poisson algebra to some quantum polynomial associative algebra. The same fact is true in the case of quadratic Poisson algebra corresponding to the Yang - Baxter equation [8], which is turned to a quantum quadratic associative algebra [14] with four generators. The same idea was discussed in reference [10], where the classical problems, which are expressed by a quadratic Poisson algebra are mapped to quantum ones described by the corresponding quantum operator quadratic algebra. The same shift is indeed true for the superintegrable systems, where
the classical ones correspond to the quantum ones and the classical quadratic Poisson algebra is mapped to a quadratic associative algebra[15]–[19].

In this paper we show that the deformation of the classical Poisson algebra to a quadratic associative algebra implies a deformation of the parameters of the quadratic algebra. The general form of the quadratic algebras, which are encountered in the case of the two dimensional quantum superintegrable systems, is investigated. In references [5, 6, 17, 16, 17, 20, 21, 22, 23] was conjectured that, the energy eigenvalues correspond to finite dimensional representations of the latent quadratic algebras. Granovskii et al in [10] studied the representations of the quadratic Askey-Wilson algebras \(QAW(3)\). Using the proposed ladder representation, the finite dimensional representations are calculated and this method was applied to several superintegrable systems [15, 16, 17, 21, 23]. Another method [5, 6] for calculating the finite dimensional representations is the use of the deformed oscillator algebra [24] and their finite dimensional version which are termed as generalized deformed parafermionic algebras [25]. The main task of this paper is to reduce the calculations of eigenvalues to a system of two algebraic equations with two parameters to be determined. These equations are universal equations, which are valid of all superintegrable systems, with quadratic integrals of motion.

This paper is organized as follows: In section [II] the general form of the quadratic Poisson algebra for a two dimensional system with quadratic integrals of motion is derived. In section [III] the special form of the Poisson algebra of the known two dimensional superintegrable systems in flat space is written. In section [IV] the quantum version of the Poisson quadratic algebra is studied. The deformed parafermionic oscillator algebra is reviewed and the oscillator realization of the quadratic algebras is realized. The finite dimensional representations of the quadratic algebras are generated by using the technique of deformed parafermionic algebras. The problem is reduced to the solution of a system of two algebraic equations in section [V]. In section [VI] the energy eigenvalues of all the known superintegrable systems in the flat two dimensional space are determined by solving the appropriate algebraic equations. Finally in section [VII] there is a discussion of the results of this paper.
II Quadratic Poisson Algebras

Let consider a two dimensional superintegrable system. The general form of
the Hamiltonian is:

\[ H = a(q_1, q_2)p_1^2 + 2b(q_1, q_2)p_1p_2 + c(q_1, q_2)p_2^2 + V(q_1, q_2) \]  (1)

this Hamiltonian is a quadratic form of the momenta. The system is superintegrable, therefore there are two additional integrals of motion \( A \) and \( B \). In that section, we consider that, these integrals of motion are quadratic functions of the momenta, i.e. they are given by the general forms:

\[ A = A(q_1, q_2, p_1, p_2) = c(q_1, q_2)p_1^2 + 2d(q_1, q_2)p_1p_2 + e(q_1, q_2)p_2^2 + f(q_1, q_2)p_1 + g(q_1, q_2)p_2 + Q(q_1, q_2) \]

The integral of motion \( B \) is assumed to be indeed a quadratic form, which is analogous to above one.

\[ B = B(q_1, q_2, p_1, p_2) = h(q_1, l_2)p_1^2 + 2k(q_1, q_2)p_1p_2 + l(q_1, q_2)p_2^2 + m(q_1, q_2)p_1 + n(q_1, q_2)p_2 + S(q_1, q_2) \]

By definition the following relations are satisfied:

\[ \{ H, A \}_p = \{ H, B \}_p = 0 \]  (2)

where \( \{ \ldots \} \) is the usual Poisson bracket.

From the integrals of motion \( A, B \), we can construct the integral of motion:

\[ C = \{ A, B \}_p \]  (3)

The integral of motion \( C \) is not a new independent integral of motion, which is a cubic function of the momenta. The integral \( C \) is not independent from the integrals \( H, A \) and \( B \) as it will be shown later. The fact that, the integral \( C \) is a cubic function of momenta, implies the impossibility of expressing \( C \) as a polynomial function of the other integrals, which are quadratic functions of momenta. Starting from the integral of motion \( C \), we can construct the (non independent) integrals \( \{ A, C \}_p \) and \( \{ B, C \}_p \). These integrals are quartic functions of the momenta, i.e. functions of fourth order. Therefore these integrals could be expressed as quadratic combinations of the integrals \( H, A, \) and \( B \). Therefore the following relations are assumed to be valid:

\[ \{ A, C \}_p = \alpha A^2 + \beta B^2 + 2\gamma AB + \delta A + \epsilon B + \zeta \]  (4)
and
\[ \{B, C\}_P = aA^2 + bB^2 + 2cAB + dA + eB + z \]  

We can take appropriate a linear combination of the integrals \(A\) and \(B\) and we can always consider the case \(\beta = 0\).

The Jacobi equality for the Poisson brackets induces the relation
\[ \{A, \{B, C\}_P\}_P = \{B, \{A, C\}_P\}_P \]

The following relations
\[ b = -\gamma, \quad c = -\alpha \quad \text{and} \quad e = -\delta \]

must be satisfied.

The integrals \(A, B\) and \(C\) satisfy the quadratic Poisson algebra:
\[ \begin{align*}
\{A, B\}_P &= C \\
\{A, C\}_P &= aA^2 + 2\gamma A, B + \delta A + \epsilon B + \zeta \\
\{B, C\}_P &= aA^2 - \gamma B^2 - 2\alpha AB + dA - \delta B + z
\end{align*} \]  

where \(\alpha, \gamma, a\) are constants and
\[ \begin{align*}
\delta &= \delta(H) = \delta_0 + \delta_1 H \\
\epsilon &= \epsilon(H) = \epsilon_0 + \epsilon_1 H \\
\zeta &= \zeta(H) = \zeta_0 + \zeta_1 H + \zeta_2 H^2 \\
d &= d(H) = d_0 + d_1 H \\
z &= z(H) = z_0 + z_1 H + z_2 H^2
\end{align*} \]

where \(\delta_i, \epsilon_i, \zeta_i, d_i\) and \(z_i\) are constants. The associative algebra, whose the generators satisfy equations (6), is the general form of the closed Poisson algebra of the integrals of superintegrable systems with integrals quadratic in momenta.

The quadratic Poisson algebra (6) possess a Casimir which is a function of momenta of degree 6 and it is given by:
\[ K = C^2 - 2\alpha A^2 B - 2\gamma AB^2 - 2\delta AB - \\
- \epsilon B^2 - 2\zeta B + \frac{2}{3}aA^3 + dA^2 + 2zA = \\
= k_0 + k_1 H + k_2 H^2 + k_3 H^3 \]  

Obviously
\[ \{K, A\}_P = \{K, B\}_P = \{K, C\}_P = 0 \]
Therefore the integrals of motion of a superintegrable two dimensional system with quadratic integrals of motion close a constrained classical quadratic Poisson algebra (6), corresponding to a Casimir equal at most to a cubic function of the Hamiltonian (7).

In the general case of a superintegrable system the integrals are not necessarily quadratic functions of the momenta, but rather polynomial functions of the momenta. The case of the systems with a quadratic and a cubic integral of motion are recently studied by Tsiganov [26]. The general form of the Poisson algebra of generators $A$, $B$, and $C$ is characterized by a polynomial function $h(A, B)$, which satisfy the following equations:

$$
\{A, B\}_P = C \\
\{A, C\}_P = \partial h / \partial B \\
\{B, C\}_P = -\partial h / \partial A
$$

(8)

and the Casimir of the algebra is given by

$$
K = K(H) = C^2 - 2h(A, B), \quad \{K, A\}_P = \{K, B\}_P = 0
$$

(9)

where $h(A, B)$ is a polynomial function of the integrals of motion $A$ and $B$. In the case of the quadratic Poisson algebra (6) the form of the function $h(A, B)$ is given by equation (7):

$$
h(A, B) = -\frac{\alpha}{2} A^3 + \alpha A^2 B + \gamma AB^2 \\
-\frac{\delta}{2} A^2 + \delta AB + \frac{\gamma}{2} B^2 \\
-\frac{\zeta}{2} A + \zeta B
$$

In the general case of a two dimensional superintegrable system, with quadratic Hamiltonian, one integral $A$ of order $m$ in momenta and one integral $B$ of order $n$ ($n \geq m$), the general form of the function $h(A, B)$ can be given by the general form:

$$
h(A, B) = h_0(A) + h_1(A)B + h_2(A)B^2
$$

where $h_i(A)$ are polynomials of the integrals $A$ and $H$. The proof of this assumption is based on the dependence of the integrals of motion on the momenta. For simplicity reasons, the proof of this proposition will not be given here.
III Poisson algebras for superintegrable systems

Let consider the superintegrable systems with quadratic integrals of motion, these potentials are given by several authors starting from different but comparable points of view. In references [1, 2] the integrals of motion are generated by solving the Darboux conditions for integrability of quadratic integrals. In [3] the Hamilton - Jacobi equation is solved by separation of variables and the two dimensional Hamiltonians which are separable in more than one coordinate system are obtained. The separation of variables is essential for solving the quantum counterpart of the superintegrable system and the solution of the associate Schrödinger equations is given in [7]. Using this method the quantum superintegrable systems have been solved on the sphere [3] and the hyperboloid [19]. From classical point of view the super integrable are given in [3], while the case of a pseudo Euclidean kinetic term has been studied in [2]. The extension on the systems with a quadratic and a cubic integral of motion is systematized in [26].

In this section we consider the case superintegrable systems given in ref [7], because in the next sections we study the quantum counterparts of these potentials. In this paper the following superintegrable systems are considered: Potential i):

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 + \omega^2 r^2 + \frac{\mu_1}{x^2} + \frac{\mu_2}{y^2} \right) \]

This potential has the following independent integrals of motion:

\[ A = p_x^2 + \omega^2 x^2 + \frac{\mu_1}{x^2} \]

and

\[ B = (xp_y - yp_x)^2 + r^2 \left( \frac{\mu_1}{x^2} + \frac{\mu_2}{y^2} \right) \]

The constants, which characterize the corresponding quadratic algebra (6), are given by:

\[ \alpha = -8, \quad \gamma = 0, \quad \delta = 16H, \]

\[ \epsilon = -16\omega^2, \quad \zeta = 16(\mu_1 + \mu_2)\omega^2 \]

\[ a = 0, \quad d = 0, \quad z = 16(\mu_2 - \mu_1)\omega^2 \]
the value of the Casimir (7) is:

\[ K = -16 \left( (\mu_2 - \mu_1)^2 \omega^2 + 4\mu_1 H^2 \right) \]

Potential ii):

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 + \omega^2 \left( 4x^2 + y^2 \right) + \frac{\mu}{y^2} \right) \]

This potential has the following independent integrals of motion:

\[ A = p_x^2 + 4\omega^2 x^2 \]

and

\[ B = (xp_y - yp_x) p_y + \frac{\mu x}{y^2} - \omega^2 xy^2 \]

The constants, which characterize the corresponding quadratic algebra (8), are given by:

\[ \alpha = 0, \quad \gamma = 0, \quad \delta = 0, \]
\[ \epsilon = -16\omega^2, \quad \zeta = 0, \]
\[ a = -6, \quad d = 16H, \quad z = 8\mu_2\omega^2 - 8H^2 \]

the value of the Casimir (7) is:

\[ K = 0 \]

Potential iii):

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 + \frac{k}{r} + \frac{1}{r} \left( \frac{\mu_1}{r + x} + \frac{\mu_2}{r - x} \right) \right) \]

This potential has the following independent integrals of motion:

\[ A = (xp_y - yp_x)^2 + r \left( \frac{\mu_1}{r + x} + \frac{\mu_2}{r - x} \right) \]

and

\[ B = (xp_y - yp_x) p_y - \frac{\mu_1 r - x}{2r r + x} + \frac{\mu_2 r + x}{2r r - x} + \frac{kx}{2r r} \]
The constants, which characterize the corresponding quadratic algebra \((\mathfrak{f})\), are given by:

\[
\begin{align*}
\alpha &= 0, \quad \gamma = -2, \quad \delta = 0, \\
\epsilon &= 0, \quad \zeta = k(\mu_1 - \mu_2), \\
a &= 0, \quad d = -8H, \\
z &= 4(\mu_1 + \mu_2)H - k^2/2
\end{align*}
\]

the value of the Casimir (7) is:

\[K = 2(\mu_1 - \mu_2)^2 H - k^2(\mu_1 + \mu_2)\]

Potential iv):

\[H = \frac{1}{2} \left( p_x^2 + p_y^2 + \frac{k}{r} + \mu_1 \frac{\sqrt{r + x}}{r} + \mu_2 \frac{\sqrt{r - x}}{r} \right)\]

This potential has the following independent integrals of motion:

\[A = (yp_x - xp_y) p_y + \frac{\mu_1 (r - x) \sqrt{r + u}}{2r} - \frac{\mu_2 (r + x) \sqrt{r - u}}{2r} - \frac{kx}{2r}\]

and

\[B = (xp_y - yp_x) p_x - \frac{\mu_1 x \sqrt{r - u}}{2r} + \frac{\mu_2 x \sqrt{r + u}}{2r} - \frac{ky}{2r}\]

The constants, which characterize the corresponding quadratic algebra \((\mathfrak{f})\), are given by:

\[
\begin{align*}
\alpha &= 0, \quad \gamma = 0, \quad \delta = 0, \\
\epsilon &= 2H, \quad \zeta = -\mu_1 \mu_2/2, \\
a &= 0, \quad d = -2H, \\
z &= \frac{\mu_1^2 - \mu_2^2}{4}
\end{align*}
\]

the value of the Casimir (7) is:

\[K = -k^2 H/2 - k(\mu_1^2 + \mu_2^2)/4\]

**IV  The quadratic associative algebra**

The quantum counterparts of the classical systems, which have been studied in section II, are quantum superintegrable systems. The quadratic classical Poisson algebra \((\mathfrak{f})\) possesses a quantum counterpart, which is a quadratic associative algebra of operators. The form of the quadratic algebra is similar
to the classical Poisson algebra, the involved constants are generally functions of $\hbar$ and they should coincide with the classical constants in the case $\hbar \to 0$.

Let consider the quadratic associative algebra generated by the generators $\{A, B, C\}$, which satisfy the commutation relations

\[
\begin{align*}
[A, B] &= C \\
[A, C] &= \alpha A^2 + \beta B^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta \\
[B, C] &= a A^2 + b B^2 + c \{A, B\} + d A + e B + z
\end{align*}
\] (10)

After rotating the generators $A$ and $B$, we can always consider the case $\beta = 0$.

The Jacobi equality for the commutator induces the relation

\[
[A, [B, C]] = [B, [A, C]]
\]

the following relations

\[
b = -\gamma, \quad c = -\alpha \quad \text{and} \quad e = -\delta
\]

must be satisfied, and consequently the general form of the quadratic algebra (10) can be explicitly written as follows:

\[
\begin{align*}
[A, B] &= C \\
[A, C] &= \alpha A^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta \quad \text{(11)} \\
[B, C] &= a A^2 - \gamma B^2 - \alpha \{A, B\} + d A - \delta B + z \quad \text{(12)}
\end{align*}
\]

The Casimir of this algebra is given by:

\[
K = C^2 - \alpha \{A^2, B\} - \gamma \{A, B^2\} + (\alpha \gamma - \delta) \{A, B\} + \\
+ (\gamma^2 - \epsilon) B^2 + (\gamma \delta - 2 \zeta) B + \\
+ \frac{2a}{3} A^3 + (d + \frac{\alpha \gamma}{3} + \alpha^2) A^2 + (\frac{\alpha \epsilon}{3} + \alpha \delta + 2 z) A
\] (14)

another useful form of the Casimir of the algebra is given by:

\[
K = C^2 + \frac{2a}{3} A^3 - \frac{2\alpha}{3} \{A, A, B\} - \frac{2\gamma}{3} \{A, B, B\} + \\
+ \left(\frac{2\alpha^2}{3} + d + \frac{2\alpha \gamma}{3}\right) A^2 + \left(-\epsilon + \frac{2\alpha \epsilon}{3}\right) B^2 + \\
+ \left(-\delta + \frac{4 \gamma}{3}\right) \{A, B\} + \left(\frac{2a \delta}{3} + \frac{\alpha \epsilon}{3} + \frac{4 \gamma}{3} + 2 z\right) A + \\
+ \left(-\frac{\alpha \epsilon}{3} + \frac{2 \delta \gamma}{3} - 2 \zeta\right) B + \frac{7 \gamma}{3} - \frac{\alpha \epsilon}{3}
\] (15)

where

\[
\{A, B, C\} = ABC + ACB + BAC + BCA + CAB + CBA
\]
This quadratic algebra has many similarities to the Racah algebra $QR(3)$, which is a special case of the Askey - Wilson algebra $QAW(3)$. The algebra $(11-13)$ does not coincide with the Racah algebra $QR(3)$, if $a \neq 0$ in the relation $(13)$. Unless this difference between $(10)$ and $QR(3)$ algebra a representation theory can be constructed by following the same procedures as they were described by Granovskii, Lutzenko and Zhedanov in ref. $(10, 13, 14)$. In this paper we shall give a realization of this algebra using the deformed oscillator techniques$[24]$. The finite dimensional representations of the algebra $(10)$ will be constructed by constructing a realization of the algebra with the generalized parafermionic algebra introduced by Quesne$[25]$.

V Deformed Parafermionic Algebra

Let now consider a realization of the algebra $(11-13)$, by using of the deformed oscillator technique, i.e. by using a deformed oscillator algebra$[24] \{b^\dagger, b, N\}$, which satisfies the

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad b^\dagger b = \Phi (N), \quad b b^\dagger = \Phi (N + 1)$$

(16)

where the function $\Phi(x)$ is a ”well behaved” real function which satisfies the boundary condition:

$$\Phi(0) = 0, \quad \text{and} \quad \Phi(x) > 0 \quad \text{for} \quad x > 0$$

(17)

As it is well known$[24]$ this constraint imposes the existence a Fock type representation of the deformed oscillator algebra, which is bounded by bellow, i.e. there is a Fock basis $|n>$, $n = 0, 1, \ldots$ such that

$$\begin{align*}
N |n> & = n |n> \\
b^\dagger |n> & = \sqrt{\Phi (n + 1)} |n + 1> \\
b |0> & = 0 \\
b |n> & = \sqrt{\Phi (n)} |n - 1> 
\end{align*}$$

(18)

The Fock representation $(18)$ is bounded by bellow. The generalized deformed algebra given in ref $[24]$ is equivalent to several deformed oscillator schemes as the Odaka- Kishi - Kamefuchi unification scheme $[27]$, the Beckers- Debergh unification scheme $[28]$, The Fibonacci oscillator $[29]$, for a discussion of deformation schemes see $[30]$.
In the case of nilpotent deformed oscillator algebras, there is a positive integer p, such that
\[ b^{p+1} = 0, \quad (b^\dagger)^{p+1} = 0 \]
the above equations imply that
\[ \Phi(p + 1) = 0, \quad (19) \]
In that case the deformed oscillator (13) has a finite dimensional representation, with dimension equal to p+1, this kind of oscillators are called deformed parafermion oscillators of order p.

An interesting property of the deformed parafermionic algebra is that the existence of a faithful finite dimensional representation of the algebra implies that:
\[ N(N - 1) (N - 2) \cdots (N - p) = 0 \quad (20) \]
The above restriction and the constraints (17) and (19) imply that the general form of the structure function \( \Phi(N) \) has the general form [25]:
\[ \Phi(N) = N(N - 1)(a_0 + a_1N + a_2N^2 + \cdots + a_pN^{p-1}) \]
A systematic study and applications of the parafermionic oscillator is given in references [25, 31, 32, 33].

We shall show, that there is a realization of the quadratic algebra, such that
\[ A = A(N) \quad (21) \]
\[ B = b(N) + b^\dagger \rho(N) + \rho(N)b \quad (22) \]
where the \( A[x], \ b[x] \) and \( \rho(x) \) are functions, which will be determined. In that case (14) implies:
\[ C = [A, B] \Rightarrow C = b^\dagger \Delta A(N) \rho(N) - \rho(N) \Delta A(N)b \quad (23) \]
where
\[ \Delta A(N) = A(N + 1) - A(N) \]
Using equations (21), (22) and (12) we find:
\[ [A, C] = [A(N), b^\dagger \Delta A(N) \rho(N) - \rho(N) \Delta A(N)b] = b^\dagger (\Delta A(N))^2 \rho(N) + \rho(N) (\Delta A(N))^2 b = \]
\[ \alpha A^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta = \]
\[ b^\dagger (\gamma (A(N + 1) + A(N)) + \epsilon) \rho(N) + \]
\[ + \rho(N) (\gamma (A(N + 1) + A(N)) + \epsilon) b + \]
\[ + \alpha A(N)^2 + 2\gamma A(N) b(N) + \delta A(N) + \epsilon B(N) + \zeta \quad (24) \]
therefore we have the following relations:

\[(\Delta A (\mathcal{N}))^2 = \gamma (A (\mathcal{N} + 1) + A (\mathcal{N})) + \epsilon \]  
\[\alpha A (\mathcal{N})^2 + 2\gamma A (\mathcal{N}) b (\mathcal{N}) + \delta A (\mathcal{N}) + \epsilon B (\mathcal{N}) + \zeta = 0 \]

while the function \(\rho (\mathcal{N})\) can be arbitrarily determined. In fact this function can be fixed, in order to have a polynomial structure function \(\Phi(x)\) for the deformed oscillator algebra \([10]\).

The solutions of equation (25) depend on the value of the parameter \(\gamma\), while the function \(b(\mathcal{N})\) is uniquely determined by equation (26) (provided that almost one among the parameters \(\gamma\) or \(\epsilon\) is not zero). At this stage, the cases \(\gamma \neq 0\) or \(\gamma = 0\), should be treated separately. We can see that:

**Case 1: \(\gamma \neq 0\)**

In that case the solutions of equations (25) and (26) are given by:

\[A (\mathcal{N}) = \frac{\gamma}{2} \left((\mathcal{N} + u)^2 - 1/4 - \frac{\epsilon}{\gamma^2}\right) \]  
\[b (\mathcal{N}) = -\frac{\alpha ((\mathcal{N} + u)^2 - 1/4)}{4 \gamma^4} + \frac{\alpha \epsilon - \delta \gamma}{2 \gamma^4} - \frac{\alpha \epsilon^2 - 2 \delta \epsilon \gamma + 4 \gamma^2 \zeta}{4 \gamma^4} \]

**Case 2: \(\gamma = 0\), \(\epsilon \neq 0\)**

The solutions of equations (25) and (26) are given by:

\[A(\mathcal{N}) = \sqrt{\epsilon} (\mathcal{N} + u) \]  
\[b(\mathcal{N}) = -\alpha (\mathcal{N} + u)^2 - \frac{\delta}{\sqrt{\epsilon}} (\mathcal{N} + u) - \frac{\zeta}{\epsilon} \]

The constant \(u\) will be determined later.

Using the above definitions of equations \(A(\mathcal{N})\) and \(b(\mathcal{N})\), the left hand side and right hand side of equation (13) gives the following equation:

\[2 \Phi (\mathcal{N} + 1) \left(\Delta A (\mathcal{N}) + \frac{\delta}{2}\right) \rho(\mathcal{N}) - 2 \Phi (\mathcal{N}) \left(\Delta A (\mathcal{N} - 1) - \frac{\delta}{2}\right) \rho(\mathcal{N} - 1) =
\]
\[= a A^2 (\mathcal{N}) - \gamma b^2 (\mathcal{N}) - 2 \alpha A (\mathcal{N}) b(\mathcal{N}) + d A (\mathcal{N}) - \delta b(\mathcal{N}) + z \]  

\[\]
Equation (14) gives the following relation:

\[
K = \Phi(N + 1) (\gamma^2 - \epsilon - 2\gamma A(N) - \Delta A^2(N)) \rho(N) + \\
+ \Phi(N) (\gamma^2 - \epsilon - 2\gamma A(N) - \Delta A^2(N - 1)) \rho(N - 1) - \\
-2\alpha A^2(N) b(N) + (\gamma^2 - \epsilon - 2\gamma A(N)) b^2(N) + \\
+ 2(\alpha\gamma - \delta) A(N) b(N) + (\gamma\delta - 2\zeta) b(N) + \\
+ \frac{3}{2} A^3(N) + \left(d + \frac{3}{2} a\gamma + \alpha^2\right) A^2(N) + \\
+ \left(\frac{1}{3} a\alpha + \alpha\delta + 2\zeta\right) A(N) 
\]  

Equations (31) and (32) are linear functions of the expressions \(\Phi(N)\) and \(\Phi(N + 1)\), then the function \(\Phi(N)\) can be determined, if the function \(\rho(N)\) is given. The solution of this system, i.e. the function \(\Phi(N)\) depends on two parameters \(u\) and \(K\) and it is given by the following formulae:

**Case 1:** \(\gamma \neq 0\)

\[
\rho(N) = \frac{1}{3 \cdot 2^{12} \cdot \gamma^8(N + u)(1 + N + u)(1 + 2(N + u))^2}
\]

and

\[
\Phi(N) = -3072\gamma^6 K(-1 + 2(N + u))^2 - \\
-48\gamma^6(\alpha^2\epsilon - \alpha\delta\gamma + a\epsilon\gamma - d\gamma^2) \cdot (-3 + 2(N + u))(-1 + 2(N + u))^4(1 + 2(N + u)) + \\
+\gamma^8(3\alpha^2 + 4a\gamma)(-3 + 2(N + u))^2(-1 + 2(N + u))^4(1 + 2(N + u))^2 + \\
+ 768(\alpha\epsilon^2 - 2\delta\epsilon\gamma + 4\gamma^2\zeta)^2 + \\
+ 32\gamma^4(-1 + 2(N + u))^2(-1 - 12(N + u) + 12(N + u)^2). \\
\cdot (3\alpha^2\epsilon^2 - 6\alpha\delta\epsilon\gamma + 2a\epsilon^2\gamma + 2\delta^2\gamma^2 - 4d\epsilon\gamma^2 + 8\gamma^3 z + 4\alpha\gamma^2\zeta) - \\
- 256\gamma^2(-1 + 2(N + u))^2. \\
\cdot (3\alpha^2\epsilon^3 - 9\alpha\delta\epsilon^2\gamma + a\epsilon^3\gamma + 6\delta^2\epsilon\gamma^2 - 3d\epsilon\gamma^2 + 2\delta^2\gamma^4 + \\
+ 2d\epsilon\gamma^4 + 12\epsilon\gamma^3 z - 4\gamma^5 z + 12\alpha\epsilon\gamma^2\zeta - 12\delta\gamma^3\zeta + 4\alpha\gamma^4\zeta) 
\] 

**Case 2:** \(\gamma = 0, \epsilon \neq 0\)

\[
\rho(N) = 1
\]
\[ \Phi(\mathcal{N}) = \frac{1}{4} \left( -\frac{K}{\epsilon} - \frac{\gamma}{\sqrt{\epsilon}} - \frac{\delta}{\sqrt{\epsilon}} \epsilon + \frac{\eta^2}{\epsilon^2} \right) - \frac{1}{12} \left( 3d - a\sqrt{\epsilon} - 3\alpha \frac{\delta}{\sqrt{\epsilon}} + 3 \left( \frac{\delta}{\sqrt{\epsilon}} \right)^2 - 6\alpha \frac{\delta}{\sqrt{\epsilon}} + 6\alpha^2 \epsilon - 6\frac{\delta^2}{\epsilon} \right) (\mathcal{N} + u) \]

\[ + \frac{1}{4} \left( \alpha^2 + d - a\sqrt{\epsilon} - 3\alpha \frac{\delta}{\sqrt{\epsilon}} + \left( \frac{\delta}{\sqrt{\epsilon}} \right)^2 + 2\alpha \frac{\delta}{\epsilon} \right) (\mathcal{N} + u)^2 - \frac{1}{6} \left( 3\alpha^2 - a\sqrt{\epsilon} - 3\alpha \frac{\delta}{\sqrt{\epsilon}} \right) (\mathcal{N} + u)^3 + \frac{1}{4} \alpha^2 (\mathcal{N} + u)^4 \]

(34)

The above formula is valid for \( \epsilon > 0 \).

VI Finite dimensional representations of quadratic algebras

Let consider a representation of the quadratic algebra, which is diagonal to the generator \( A \) and the Casimir \( K \). Using the parafermionic realization defined by equations (21) and (22), we see that this a representation diagonal to the parafermionic number operator \( \mathcal{N} \) and the Casimir \( K \). The basis of a such representation corresponds to the Fock basis of the parafermionic oscillator, i.e. the vectors \( |k, n> \), \( n = 0, 1, \ldots \) of the carrier Fock space satisfy the equations

\[ \mathcal{N}|k, n >= n|k, n >, \quad K|k, n >= k|k, n > \]

The structure function (33) (or respectively (33) ) depend on the eigenvalues of the of the parafermionic number operator \( \mathcal{N} \) and the Casimir \( K \). The vectors \( |k, n> \) are also eigenvectors of the generator \( A \), i.e.

\[ A|k, n >= A(k, n)|k, n > \]

In the case \( \gamma \neq 0 \) we find from equation (27)

\[ A(k, n) = \frac{\gamma}{2} \left( (n + u)^2 - 1/4 - \frac{\epsilon}{\gamma^2} \right) \]

In the case \( \gamma = 0, \epsilon \neq 0 \) we find from equation (29)

\[ A(k, n) = \sqrt{\epsilon} (n + u) \]
If the deformed oscillator corresponds to a deformed Parafermionic oscillator of order $p$ then the two parameters of the calculation $k$ and $u$ should satisfy the constrints (17) and (19) the system:

$$\Phi(0, u, k) = 0$$

$$\Phi(p + 1, u, k) = 0$$

(35)

then the parameter $u = u(k, p)$ is a solution of the system of equations (35).

Generally there are many solutions of the above system, but a unitary representation of the deformed parafermionic oscillator is restrained by the additional restriction

$$\Phi(x) > 0, \text{ for } x = 1, 2, \ldots, p$$

We must point out that the system (35) corresponds to a representation with dimension equal to $p + 1$.

The proposed method of calculation of the representation of the quadratic algebra is an alternative to the method given by Granovskii et al. [10, 15, 16, 17] and reduces the search of the representations to the solution of a system of polynomial equations (35). Also its is applied to an algebra not included in the cases of the algebras, which are treated in the above references. We must point out, that there are several papers on the representations of quadratic (or generally polynomial algebras) [34, 35, 36, 37, 38, 39, 40], these algebras are deformations of the su(2) or osp(1/2) algebras. The general form of the quadratic algebra, which is studied in this paper, is different by definition from the deformed versions of su(2) or osp(1/2).

VII Quadratic algebras for the quantum superintegrable systems

In this section, we shall give an example of the calculation of eigenvalues of a superintegrable two-dimensional system, by using the methods of the previous section. The calculation by an empirical method was performed in [3] and the solution of the same problem by using separation of variables was studied in [4]. Here in order to show the effects of the quantization procedure we don’t use $\hbar = 1$ as it was considered in references [3] and [4]. That means that the following commutation relations are taken in consideration:

$$[x, p_x] = i\hbar, \quad [y, p_y] = i\hbar$$
VII-a Potential i)

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 + \omega^2 r^2 + \frac{\mu_1}{x^2} + \frac{\mu_2}{y^2} \right) \]

This potential has the following independent integrals of motion:

\[ A = p_x^2 + \omega^2 x^2 + \frac{\mu_1}{x^2} \]

and

\[ B = (xp_y - yp_x)^2 + r^2 \left( \frac{\mu_1}{x^2} + \frac{\mu_2}{y^2} \right) \]

The constants, which characterize the corresponding quadratic algebra (10), are given by:

\[ \alpha = 8\hbar^2, \quad \gamma = 0, \quad \delta = -16\hbar^2 H, \]
\[ \epsilon = 16\hbar^2 \omega^2, \quad \zeta = -16\hbar^2 (\mu_1 + \mu_2) \omega^2 + 8\hbar^4 \omega^2 \]
\[ a = 0, \quad d = 16\hbar^4, \quad z = -16\hbar^2 (\mu_2 - \mu_1) \omega^2 - 16\hbar^4 H \]

the value of the Casimir (14) is:

\[ K = 16\hbar^2 \left( (\mu_2 - \mu_1)^2 \omega^2 + 4\mu_1 H^2 \right) - 16\hbar^4 \left( 3H^2 + 2\hbar^2 \omega^2 - 2(\mu_1 + \mu_2) \right) \]

For simplicity reasons we introduce the positive parameters \( k_1 \) and \( k_1 \), which are related to the potential parameters \( \mu_1 \) and \( \mu_2 \) by the relations:

\[ \mu_1 = \left( k_1^2 - \frac{1}{4} \right) \hbar^2 \quad \mu_2 = \left( k_2^2 - \frac{1}{4} \right) \hbar^2 \]

This quadratic algebra corresponds to the case \( \gamma = 0 \) and \( \epsilon > 0 \) of the algebra given by equations (11–13). In that case, the structure function (34) of the deformed parafermionic algebra of Section [V] can be given by the simple form:

\[ \Phi(x) = 16\hbar^4 \left( x + u - \frac{1}{2} - \frac{k_1}{2} - \frac{1}{2\hbar \omega} - \frac{E^2}{2\hbar \omega} \right) \left( x + u - \frac{1}{2} + \frac{k_1}{2} - \frac{1}{2\hbar \omega} - \frac{E^2}{2\hbar \omega} \right) \]

In the above formula \( E \) is the eigenvalue of the energy. The values of the parameters \( u \) and \( E \) corresponding to the representation of the parafermionic
algebra of dimension $p + 1$ are determined by the restrictions (35), which are transcribed as:

$$\Phi(0) = 0, \quad \Phi(p + 1) = 0$$

One should notice, that only the solutions which correspond to positive eigenvalues of the integral $A$ must be retained. The acceptable solutions are four and correspond to the following values of the parameters $u$ and $E$:

$$u = \frac{1}{2} + \frac{\epsilon_1 k_1}{2}, \quad E = 2\hbar\omega \left( p + 1 + \frac{\epsilon_1 k_1 + \epsilon_2 k_2}{2} \right)$$

where $\epsilon_i = \pm 1$. The corresponding structure function is

$$\Phi(x) = 16\hbar \frac{4}{x} (p + 1 - x) (x + \epsilon_1 k_1) (p + 1 - x + \epsilon_2 k_2)$$

The corresponding eigenvalues of the operator $A$ are given by:

$$A(m) = 4\hbar \omega \left( m + \frac{\epsilon_1 k_1 + \epsilon_2 k_2}{2} \right), \quad m = 0, 1, \ldots, p$$

The structure function $\Phi(x)$ should be a positive function, for $x = 1, 2, \ldots, p$ therefore the constants $k_i$ are restricted by the relations:

$$\epsilon_1 k_1 > -1, \quad \epsilon_2 k_2 > -1$$

**VII-b Potential ii)**

$$H = \frac{1}{2} \left( p_x^2 + p_y^2 + \omega^2 \left( 4x^2 + y^2 \right) + \frac{\mu}{y^2} \right)$$

This potential has the following independent integrals of motion:

$$A = p_x^2 + 4\omega^2 x^2$$

and

$$B = \frac{1}{2} \{xp_y - yp_x, p_y\} + \frac{\mu x}{y^2} - \omega^2 xy^2$$

The constants, which characterize the corresponding quadratic algebra (10), are given by:

$$\alpha = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = 16\hbar^2 \omega^2, \quad \zeta = 0, \quad a = 6\hbar^2, \quad d = -16\hbar^2 H, \quad z = -8\hbar^2 (\mu \omega^2 - H^2) + 6\hbar^4 \omega^2$$

18
the value of the Casimir (14) is:

\[ K = 64\hbar^4\omega^2H \]

For simplicity reasons we introduce the positive parameters \( k \), which is related to the potential parameter \( \mu \) by the relation:

\[ \mu = \left( k^2 - \frac{1}{4} \right) \hbar^2 \]

This quadratic algebra corresponds to the case \( \gamma = 0 \) and \( \epsilon > 0 \) of the algebra given by equations (11–13). In that case, the structure function (34) of the deformed parafermionic algebra of Section V can be given by the simple form:

\[ \Phi(x) = 8\hbar^3\omega \left( x + u - \frac{1}{2} \right) \left( x + u - \frac{1}{2} - \frac{k}{2} - \frac{E}{2\hbar\omega} \right) \left( x + u - \frac{1}{2} + \frac{k}{2} - \frac{E}{2\hbar\omega} \right) \]

In the above formula \( E \) is the eigenvalue of the energy. The values of the parameters \( u \) and \( E \) corresponding to the representation of the parafermionic algebra of dimension \( p + 1 \) are determined by the restrictions (35), which are transcribed as:

\[ \Phi(0) = 0, \quad \Phi(p + 1) = 0 \]

One should notice, that only the solutions which correspond to positive eigenvalues of the integral \( A \) must be retained. The acceptable solutions are four and correspond to the following values of the parameters \( u \) and \( E \):

\[ u = \frac{1}{2}, \quad E = 2\hbar\omega \left( p + 1 + \frac{\epsilon k}{2\hbar} \right) \]

where \( \epsilon = \pm 1 \). The corresponding structure function is

\[ \Phi(x) = 4\hbar^3x(p + 1 - x)(p + 1 - x + \epsilon k) \]

The structure function should be a positive function, therefore the values of the parameter \( k \) are restrained by

\[ \epsilon k > -1 \]

The eigenvalues of the operator \( A \) are given by:

\[ A(m) = 4\hbar\omega\left( m + \frac{1}{2} \right), \quad m = 0, 1, \ldots, p \]
VII-c Potential iii)

\[
H = \frac{1}{2} \left( p_x^2 + p_y^2 + \frac{k}{r} + \frac{1}{r} \left( \frac{\mu_1}{r+x} + \frac{\mu_2}{r-x} \right) \right)
\]

In ref [7] the parabolic coordinates have been used:

\[
x = \frac{1}{2} (\xi^2 - \eta^2), \quad p_x = \frac{\xi}{\xi^2 + \eta^2} p_\xi - \frac{\eta}{\xi^2 + \eta^2} p_\eta,
\]

\[
y = \xi \eta, \quad p_y = \frac{\eta}{\xi^2 + \eta^2} p_\xi + \frac{\xi}{\xi^2 + \eta^2} p_\eta,
\]

\[
[\xi, p_\xi] = i\hbar, \quad [\eta, p_\eta] = i\hbar
\]

For comparison reasons we quote all the relations in both, cartesian and parabolic systems, so

\[
H = \frac{1}{\xi^2 + \eta^2} \left( \frac{1}{2} \left( p_\xi^2 + p_\eta^2 \right) + k + \frac{\mu_1}{\xi^2} + \frac{\mu_2}{\eta^2} \right)
\]

This potential has the following independent integrals of motion:

\[
A = (xp_y - yp_x)^2 + r \left( \frac{\mu_1}{r+x} + \frac{\mu_1}{r-x} \right) = \frac{1}{2} \left( \frac{1}{2} (\eta p_\xi - \xi p_\eta)^2 + (\xi^2 + \eta^2) \left( \frac{\mu_1}{\xi^2} + \frac{\mu_2}{\eta^2} \right) \right)
\]

and

\[
B = \frac{1}{2} \left( \{xp_y - yp_x, p_y\} - \frac{\mu_1}{r} \frac{r-x}{r+x} + \frac{\mu_2}{r} \frac{r+x}{r-x} + \frac{k}{r} \right)
\]

\[
= \frac{1}{\xi^2 + \eta^2} \left( \frac{1}{2} (\xi^2 p_\xi^2 - \eta^2 p_\eta^2) + \mu_2 p_\eta^2 - \mu_1 p_\xi^2 + k \frac{\xi^2 - \eta^2}{2 \xi^2 + \eta^2} \right)
\]

The constants, which characterize the corresponding quadratic algebra (10), are given by:

\[
\alpha = 0, \quad \gamma = 2\hbar^2, \quad \delta = 0,
\]

\[
\epsilon = -\hbar^4, \quad \zeta = -\hbar^2 k(\mu_1 - \mu_2),
\]

\[
a = 0, \quad d = 8\hbar^2 H, \quad z = -\hbar^2 \left( 4(\mu_1 + \mu_2)H - k^2/2 \right) + \hbar^4 H
\]

the value of the Casimir (14) is:

\[
K = -\hbar^2 \left( 2(\mu_1 - \mu_2)^2 H - k^2(\mu_1 + \mu_2) \right) - 2\hbar ^4 \left( (\mu_1 + \mu_2)H - \frac{k^2}{4} \right) + \hbar^6 H
\]

For simplicity reasons we introduce the positive parameters \(k_1\) and \(k_2\), which are related to the potential parameters \(\mu_1\) and \(\mu_2\) by the relations:

\[
\mu_1 = \frac{\hbar^2}{2} \left( k_1^2 - \frac{1}{4} \right), \quad \mu_2 = \frac{\hbar^2}{2} \left( k_2^2 - \frac{1}{4} \right)
\]
This quadratic algebra corresponds to the case $\gamma \neq 0$ of the algebra given by equations (11–13). In that case, the structure function (33) of the deformed parafermionic algebra of Section V can be given by the simple form:

$$\Phi(x) = 3 \cdot 2^{14} h^{16} \cdot (2x - 1 + k_1 + k_2) (2x - 1 + k_1 - k_2) (2x - 1 - k_1 + k_2) \cdot (2x - 1 - k_1 - k_2) \left(8h^2 H x^2 - 8h^2 H x + 2h^2 H + k^2\right)$$

In the above formula $E$ is the eigenvalue of the energy. The values of the parameters $u$ and $E$ corresponding to the representation of the parafermionic algebra of dimension $p + 1$ are determined by the restrictions (35), which are transcribed as:

$$\Phi(0) = 0, \quad \Phi(p + 1) = 0$$

One should notice, that only the solutions which correspond to positive eigenvalues of the integral $A$ must be retained. The acceptable solutions are four and correspond to the following values of the parameters $u$ and $E$:

$$u = \frac{1}{2} (2 + \epsilon_1 k_1 + \epsilon_2 k_2), \quad E = -\frac{k^2}{2h^2 (2(p + 1) + \epsilon_1 k_1 + \epsilon_2 k_2)^2}$$

where $\epsilon_i = \pm 1$. The corresponding structure function is

$$\Phi(x) = 3 \cdot 2^{20} \cdot k^2 h^{16} \cdot x(p + 1 - x) (x + \epsilon_1 k_1) (x + \epsilon_2 k_2) \cdot (x + \epsilon_1 k_1 + \epsilon_2 k_2) \frac{(x + p + 1 + \epsilon_1 k_1 + \epsilon_2 k_2)}{(2(p + 1) + \epsilon_1 k_1 + \epsilon_2 k_2)^2}$$

The eigenvalues of the operator $A$ are given by the formula:

$$A(m) = h^2 \left(m + \epsilon_1 k_1 + \epsilon_2 k_2 + \frac{3}{2}\right)^2, \quad m = 0, 1, \ldots, p$$

The positive sign of the structure function for $x = 1, 2, \ldots, p$ is obtained when:

$$\epsilon_1 k_1 > -1, \quad \epsilon_2 k_2 > -1, \quad \text{and} \quad \epsilon_1 k_1 + \epsilon_2 k_2 > -1$$

VII-d Potential iv)

$$H = \frac{1}{2} \left(p_x^2 + p_y^2 + \frac{1}{\hbar^2} + \mu_1 \frac{\sqrt{\mu_1 \xi + \mu_2 \eta}}{\hbar} + \mu_2 \frac{\sqrt{\mu_1 \xi + \mu_2 \eta}}{\hbar}\right) = \frac{1}{\xi + \eta} \left(\frac{1}{2} \left(p_\xi^2 + p_\eta^2\right) + k + \mu_1 \xi + \mu_2 \eta\right)$$
This potential has the following independent integrals of motion:

\[
A = \frac{1}{2} \left\{ (yp_x - xp_y), p_y \right\} + \frac{\mu_1 (r-x)\sqrt{r+u}}{r} - \frac{\mu_2 (r+x)\sqrt{r-u}}{r} - \frac{kx}{r} = \\
= \frac{1}{2(\xi^2 + \eta^2)} \left( \eta^2 p_\xi^2 - \xi^2 p_\eta^2 + k (\eta^2 - \xi^2) + 2\xi\eta (\mu_1\eta - \mu_2\xi) \right)
\]

and

\[
B = \frac{1}{2} \left\{ xp_y - yp_x, p_x \right\} - \frac{\mu_1 \sqrt{r-u}}{r} + \frac{\mu_2 \sqrt{r+u}}{r} - \frac{ku}{r} = \\
= -\frac{1}{2(\xi^2 + \eta^2)} \left( \xi\eta \left( p_\xi^2 + p_\eta^2 \right) - (\xi^2 + \eta^2) p_\xi p_\eta + 2k\xi\eta + (\mu_2\xi - \mu_1\eta) (\eta^2 - \xi^2) \right)
\]

The constants, which characterize the corresponding quadratic algebra (10), are given by:

\[
\alpha = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = -2\hbar^2 H, \quad \zeta = \hbar^2 \mu_1\mu_2/2, \\
a = 0, \quad d = 2\hbar^2 H, \quad z = -\hbar^2 (\mu_1^2 - \mu_2^2)/4
\]

the value of the Casimir (14) is:

\[
K = \hbar^2 k^2 H/2 + \hbar^2 k(\mu_1^2 + \mu_2^2)/4 + \hbar^4 H^2
\]

This quadratic algebra corresponds to the case \( \gamma = 0 \) and \( \epsilon > 0 \) of the algebra given by equations (11–13). It is worth noticing that the algebra is extremely simple, which can be reduced to the usual \( su(2) \). We prefer to treat this algebra with the proposed methods in this paper for pedagogical reasons.

The existence of the finite dimensional representations of this algebra implies that, the coefficient \( \epsilon \) in equation (12) should be positive, therefore the energy operator \( H \) must have energy eigenvalues \( E < 0 \). For simplicity reasons we introduce the new parameters:

\[
\varepsilon = \sqrt{-2E}/\hbar, \quad \lambda = k/\hbar^2, \quad \nu_1 = \mu_1/\hbar^2, \quad \nu_2 = \mu_2/\hbar^2, \quad \nu^2 = \nu_1^2 + \nu_2^2
\]

The structure function (34) of the deformed parafermionic algebra of Section V can be given by the form:

\[
\Phi(x) = \frac{\hbar^4}{16\varepsilon^4} \left( \nu_1^2 - \lambda\varepsilon^2 + 2(x + u - \frac{1}{2})\varepsilon^3 \right) \left( \nu_2^2 - \lambda\varepsilon^2 - 2(x + u - \frac{1}{2})\varepsilon^3 \right)
\]

In the above formula the parameter \( \varepsilon \) is related to the the eigenvalue \( E \) of the energy. The values of the parameters \( u \) and \( \varepsilon \), corresponding to the a
representation of the parafermionic algebra of dimension $p+1$, are determined
by the restrictions (35), which are transcribed as:

$$
\Phi(0) = 0, \quad \Phi(p + 1) = 0
$$

The first condition can be used for determining the acceptable values of the
parameter $u$. Two possible solutions are found:

$$
u_1 = \frac{\nu_2^2 - \lambda \varepsilon^2 + \varepsilon^3}{2 \varepsilon^3}$$

$$
u_2 = \frac{-\nu_1^2 - \lambda \varepsilon^2 - \varepsilon^3}{2 \varepsilon^3}$$

Using these solutions and the condition $Phi(p + 1) = 0$, we find that the $\varepsilon$
must satisfy two possible cubic equations:

$$
u_1 \rightarrow 2(p + 1) \varepsilon^3 - 2 \lambda \varepsilon^2 + \nu_2^2 = 0 \quad (38)$$

$$
u_2 \rightarrow 2(p + 1) \varepsilon^3 + 2 \lambda \varepsilon^2 - \nu_2^2 = 0 \quad (39)$$

If $\varepsilon$ is a solution of equation (38) then $-\varepsilon$ is the solution of the other equation (39), therefore there is almost one solution which is positive. This solution
leads to the structure function:

$$
\Phi(x) = \frac{\varepsilon^2}{4} x (p + 1 - x)
$$

which is positive for $x = 1, 2, \ldots, p$.

### VIII Discussion

If we compare the quadratic associative algebra, introduced in section [IV]
with the corresponding Poisson algebra given in section [II], we see that in
general the quantum constants are similar to the classical ones up to an factor
equal to $-h^2$, but there are quantum corrections of order $h^4$ and $h^6$. The
knowledge of the classical constants of the Poisson algebra is not sufficient to
reproduce the rules of quantum associative operator algebra. Therefore, the
passage from the classic Poisson algebra to the non commutative quantum
algebra can not be realized by simple replacements of the Poisson brackets
by commutators and by a symmetrization procedure.
The energy eigenvalues of section VII corroborate the results of reference [7] (the differences in the case of the potential iv are due to some misprints in that reference). The calculation of the energy eigenvalues in reference [7] was achieved by solving the corresponding Schroedinger differential equations, while in this paper the energy eigenvalues are obtained by algebraic methods. The advantage of the proposed method is that, the energy eigenvalues are reduced to simple algebraic calculations of the roots of polynomial equations, whose the form is universally determined by the the structure functions (33), (33) and the system (35). These equations are valid for any two dimensional superintegrable system with integrals of motion, which are quadratic functions of the momenta. The same equations should be valid in the case of two dimensional superintegrable systems in curved space [11]. The superintegrable systems bring up for discussion the open problem of the quantization of a Poisson algebra in a well determined context, because these systems and their quantum counterparts are explicitly known.

From the above discussion several open problems are risen:

- The calculation of the classical Poisson algebras and their quantum counterparts for the totality of the two dimensional problems in curved space. This study will lead to the calculation of the energy eigenvalues by algebraic methods.

- The two dimensional superintegrable systems are classified by the values of the constants of the Darboux conditions [3]. The relation of these constants with the constants of the quadratic Poisson algebra is not yet known.

- The Poisson algebras for the Drach superintegrable systems with a cubic integral of motion were written by using a classical analogue of the deformed parafermionic algebra [26]. Their quantum counterparts and the calculation of their energy eigenvalues are is a topic under investigation.

- The Poisson algebras and the associated quantum counterparts for the three dimensional superintegrable systems are not yet fully studied. Recently [12] the quantum quadratic algebras have been written down, but a systematic calculation of energy eigenvalues was not yet performed.
The above points show that, the study of non linear Poisson algebras and their quantum counterparts is a topic of interest.

References

[1] J. Hietarinta, Direct Methods for the Search of the 2nd Invariant, Phys. Rep C147, 87 (1987)

[2] Rañada M. F., Superintegrable N=2 Systems, Quadratic Constants of Motion, and Potentials of Drach. J. Math Phys. 38 4165 (1997)

[3] Rañada M. F. and Santander M., Superintegrable systems on the two-dimensional sphere $S^2$ and the hypebolic plane $H^2$, J. Math. Phys. 40 5026 (1999)

[4] Friš J., Smorodinsky Ya. A., Uhlić M. and Winternitz P., On Higher Symmetries in Quantum Mechanics, Phys. Lett. 16 354 (1965); Friš J., Smorodinsky Ya. A., Uhlić M. and Winternitz P., Symmetry Groups in Classical and Quantum Mechanics, Sov. J. Nucl. Physics 4 444 (1967); Makarov A. A., Smorodinsky Ya. A., Valiev Kh. Winternitz P. A systematic Search for Nonrelativistic Systems with Dynamical Symmetries, Nuovo Cimento A 52 1061 (1967)

[5] Bonatsos D., Daskaloyannis C., Kokkotas K., Quantum-Algebraic Description of Quantum Superintegrable Systems in 2 Dimensions, Phys Rev A48 R3407 (1993)

[6] Bonatsos D., Daskaloyannis C., Kokkotas K., Deformed Oscillator Algebras for 2-Dimensional Quantum Superintegrable Systems, Phys. Rev Abf 50 3700 (1994)

[7] Kalnins E.G., Miller W., Pogosyan G.S., Superintegrability and Associated Polynomial Solutions - Euclidean-Space and the Sphere in 2 Dimensions. J. Math. Phys.37 6439 (1996)

[8] Sklyanin E. K., Some algebraic structures connected with the Yang - Baxter Equation Funct. Anal. Appl. 16 263 (1983); This paper can be found in Yang - Baxter equation in integrable Systems ed. M. Jimbo, World Scientific (1989), p.244.
[9] Essler FHL, Rittenberg V. *Representations of the Quadratic Algebra and Partially Asymmetric Diffusion with Open Boundaries*, J. Phys A:Math-Gen **29** 3375 (1996)

[10] Ya. I. Granovskii, I. M. Lutzenko and A. S. Zhedanov, *Mutual Integrability, Quadratic Algebras and Dynamical Symmetry*, Ann. Phys. NY **217**, 1 (1992)

[11] Korsch H. J., *On classical and quantum integrability*, Phys. Lett. **A90**, 113 (1982)

[12] Hietarinta J. and Grammaticos B., *On the $\hbar^2$ Correction Terms in Quantum Integrability*, J. Phys A:Math-Gen **22**, 1315-22 (1989)

[13] Hietarinta J., *Pure quantum integrability*, Phys. Lett. **A246**, 97 (1998)

[14] Sklyanin E. K., *Some algebraic structures connected with the Yang - Baxter Equation - Representations of quantum algebras* Funct. Anal. Appl. **17** 273 (1984); This paper can be found in *Yang - Baxter equation in integrable Systems* ed. M. Jimbo, World Scientific (1989), p.252.

[15] Granovskii Ya. I., Zhedanov A. S. and Lutzenko I. M., *Quadratic Algebras and Dynamics into the Curved Space, I. the oscillator*, Teor. Mat. Fiz. **91**, 207 (1992) (in russian)

[16] Granovskii Ya. I., Zhedanov A. S. and Lutzenko I. M., *Quadratic Algebras and Dynamics into the Curved Space, I. the Kepler Problem*, Teor. Mat. Fiz. **91**, 396 (1992) (in russian)

[17] Granovskii Ya. I., Zhedanov A.S. and Lutzenko I.M., *Quadratic Algebra as a Hidden Symmetry of the Hartmann Potential*, J. Phys. A:Math-Gen **24** 3887 (1991)

[18] Kalnins E.G., Miller W. and Pogosyan G.S., *Superintegrability on the 2-Dimensional Hyperboloid*. J. Math Phys **38** 5416 (1997)

[19] Kalnins E.G., Miller W., Hakobyan Y.M. and Pogosyan G.S., *Superintegrability on the 2-Dimensional Hyperboloid - II*, J. Math. Phys. **40**, 2291 (1999) and Preprint quant-ph/9907037

[20] Higgs, P. W., *Dynamical symmetries in a spherical geometry I*, J. Phys. A: Math. Gen. **12**, 309 (1979)
[21] Létourneau P. and Vinet L., *Superintegrable systems: Polynomial Algebras and Quasi-Exactly Solvable Hamiltonians*, Ann. Phys. (NY) **243**, 144 (1995)

[22] Gal’bert O. F., Granovskii Ya. I. and Zhedanov A. S., *Dynamical symmetry of anisotropic singular oscillator*, Phys. Lett. **A153**, 177 (1991)

[23] Zhedanov A. S., *The "Higgs algebra" as a "quantum" deformation of su(2)*, Mod. Phys. Lett. **A7**, 507 (1992)

[24] Daskaloyannis C., *Generalized Deformed Oscillator and Nonlinear Algebras*, J. Phys. **A24**, L789 (1991)

[25] Quesne C., *Generalized Deformed Parafermions, Nonlinear Deformations of so(3) and Exactly Solvable Potentials*, Phys. Lett. **A193**, 248 (1994)

[26] Tsiganov A. C., *On the Drach superintegrable systems*, Preprint [nlin.SI/0001053] (2000)

[27] Odaka K., Kishi T. and Kamefuchi S., *On Quantization of Simple Harmonic-Oscillators*, J. Phys. A: Math. Gen. **24**, L591 (1991)

[28] Beckers J. and Debergh N., *On a General Framework for Q-Particles, Paraparticles and Q-Paraparticles Through Deformations*, J. Phys. A: Math. Gen. **24**, L1277 (1991)

[29] Arik M., Dermican E., Turgut T., Ekinci L. and Mungan, *Fibonacci Oscillators*, Z. Phys. C **55**, 89 (1992)

[30] Bonatsos D. and Daskaloyannis C., *General deformed schemes and N=2 supersymmetric quantum mechanics*, Phys. Lett. **B307**, 100 (1993)

[31] Beckers J., Debergh N. and Quesne C., *Parasupesymmetric Quantum Mechanics with generalized deformed parafermions*, Helv.Phys.Acta **69**, 60 (1996)

[32] Klishевич S. and Plyushchay M., *Supersymmetry of parafermions*, Preprint (1999), [hep-th/9905149](http://arxiv.org/abs/hep-th/9905149)

[33] Debergh N., *On P=2 Generalized Deformed Parafermions and Exotic Statistics*, Journ. Phys. A: Math. Gen. **28**, 4945 (1995)
[34] Karassiov V. P. *G-Invariant Polynomial Extensions of Lie-Algebras in Quantum Many-Body Physics* Journ. Phys. A: Math. Gen. **27**, 153 (1994)

[35] Karassiov V. P., *Polynomial Lie-Algebras in Solving a Class of Integrable Models of Quantum Optics - Exact Methods and Quasiclassics*, Czech. Journ. Phys. **48**, 1381 (1998)

[36] Bonatsos D., Kolokotronis P., Daskaloyannis C. *Generalized Deformed SU(2) Algebras, Deformed Parafermionic Oscillators and Finite W-Algebras* Mod. Phys. Lett. A**10** 2197 (1995)

[37] Abdesselam A., Beckers J., Chakrabarti A., Debergh N., *On nonlinear angular momentum theories, their representations and associated Hopf structures*, J.Phys.A: Math Gen. **29**, 3075 (1996); Addendum-ibid. A **30**, 5239 (1997)

[38] Debergh N., *Supersymmetry of a Multi-Boson Hamiltonian Through the Higgs Algebra* Journ. Phys. A: Math. Gen. **31**, 4013 (1998);

[39] Beckers J., Brihaye Y. and Debergh N, *Supersymmetry of Multi-Boson Hamiltonians* Mod. Phys. Lett. A**14**, 1149 (1999);

[40] Vanderjeugt J. and Jagannathan R., *Polynomial Deformations of Osp(1/2) and Generalized Parabosons* J. Math. Phys. **36**, 4507 (1995)

[41] Daskaloyannis, C., *Finite Dimensional Representations of Quadratic Algebras with Three Generators and Applications*, Preprint (2000) [math-ph/0002001](http://arxiv.org/abs/math-ph/0002001)

[42] Kalnins E. G., Williams G. C., Miller W. Jr. and Pogosyan G. S., *Superintegrability in three-dimensional Euclidean space*, J. Math. Phys. **40**, 708 (1999)