Learning to Find Hard Instances of Graph Problems

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Abstract

Finding hard instances, which need a long time to solve, of graph problems such as the graph coloring problem and the maximum clique problem, is important for (1) building a good benchmark for evaluating the performance of algorithms, and (2) analyzing the algorithms to accelerate them. The existing methods for generating hard instances rely on parameters or rules that are found by domain experts; however, they are specific to the problem. Hence, it is difficult to generate hard instances for general cases. To address this issue, in this paper, we formulate finding hard instances of graph problems as two equivalent optimization problems. Then, we propose a method to automatically find hard instances by solving the optimization problems. The advantage of the proposed algorithm over the existing rule based approach is that it does not require any task specific knowledge. To the best of our knowledge, this is the first non-trivial method in the literature to automatically find hard instances. Through experiments on various problems, we demonstrate that our proposed method can generate instances that are a few to several orders of magnitude harder than the random based approach in many settings. In particular, our method outperforms rule-based algorithms in the 3-coloring problem.

1 Introduction

Given an algorithm for a combinatorial problem, how do we find instances that require a long time to be solved? We call such instances hard instances. Finding hard instances is important for the following reasons:

Reason 1 Hard instances help us analyze and accelerate the algorithm.

Reason 2 We can use hard instances as a benchmark for evaluating the performance of algorithms.

The following simple example shows how hard instances help analyze an algorithm: Consider a sorting problem and quicksort, which uses the first element as a pivot. Suppose we do not know that the worst time complexity of quicksort is $\Theta(n^2)$. If we try running quicksort on some random sequences, quicksort seems to solve all instances in $O(n \log n)$ time. However, once we find a hard instance $[n, n-1, \ldots, 3, 2, 1]$, it can be seen that the worst time complexity is $\Theta(n^2)$. Moreover, such an observation shows that choosing an elaborate pivot improves the algorithm (Reason 1). In addition, such an extreme case is important for benchmarks because it reveals whether the algorithms are robust to worst cases or are efficient only for random cases (Reason 2).

We will consider Reasons 1 and 2 in detail in the experiments section.

Moreover, finding hard instances is important not only for academic subjects but also for practical and industrial subjects. For example, a task scheduler program solves the vertex coloring problem to optimize schedules. However, if a user inputs a malicious schedule (\textit{i.e.}, a hard instance), the solver takes a large amount of time to solve the problem and may hang up. If the developers have such inputs beforehand, they can cope with the problem by setting the appropriate timeout period or maximum size of the input. Another example is the preparation of competitive programming contests. If we create hard instances for each problem, we can accurately check whether the submission is correct or not, which is useful for preparing contests.

The naive method for finding a hard instance is to generate many instances at random and select the hardest instance by simply evaluating them in a brute force manner. Though it is generic and simple, it is not efficient because many algorithms solve random instances far more quickly than the worst case (\textit{e.g.}, quicksort). Therefore, to find hard instances efficiently, we need to develop a method that can capture the structure of the problem and generate rare instances. We focus on graph problems because they appear in many important problems including the graph coloring problem and the maximum clique problem.

In this paper, we first formulate the task of finding a hard instance of a graph problem as two equivalent optimization problems and then propose the use of meta-heuristic search methods and reinforcement learning to solve them. To the best of our knowledge, this is the first work to generate hard instances by using optimization. Furthermore, we show that our method can be used not only for generating instances that
require a long time to solve but also for estimating the approximation ratio of approximation algorithms and for generating hard instances in terms of amortized time and maximum delay for enumerating algorithms. Through experiments on various problems, we show that our proposed method can generate instances that are a few to several orders of magnitude harder than the random based approach in many settings, and that our method outperforms rule-based algorithms in the 3-coloring problem.

The major contributions of this paper are as follows:

1. **Formulation** We formulate the task of finding a hard instance of a graph problem as two equivalent optimization problems.

2. **Novel method** We propose effective methods for finding a hard instance by solving the optimization problems.

3. **Effectiveness** We demonstrate the effectiveness of our methods through experiments with four problems and six algorithms.

## 2 Proposed Method

In this section, we first formulate the hard instance generation problem as two equivalent optimization problems. Then, we propose hard instance generation algorithms by optimization.

### 2.1 Problem Setting

In this section, we specify the task of finding hard instances of graph problems that we tackle in this paper. In particular, we focus on problems with undirected, unweighted, and simple graphs, because they appear in many important problems such as the graph coloring problem, the minimum vertex cover problem, and the maximum clique problem.

Given an algorithm for a particular graph problem, our goal is to automatically obtain hard instances, which the algorithm requires an extremely large number of steps to solve. We do not assume any problem-specific property to generate the hard instances; instead, we use only the hardness measures of the problem instances: $\text{hardness}(A, L)$, where $L$ is the given algorithm, $A \in \{0, 1\}^{n(n-1)/2}$ is the adjacent matrix, and $n$ is the number of vertices. The definition of the hardness value is arbitrary if it can be obtained by actually running the algorithm on the instance. For example, in our experiment, the hardness of an instance of the 3-coloring problem is measured by the number of recursive calls of Brélaz’s algorithm, and the instance hardness of the isomorphism problem is measured by the time Nauty needs to solve the problem.

To avoid obtaining trivial instances, we aim to solve the problems by satisfying the following two assumptions.

**Assumption1:** Small instance. We fix the number of vertices $n$ because we can generate arbitrarily hard instances just by increasing the number of vertices, which is not practical. Moreover, since small instances can be visualized and are easy to interpret and analyze, it is important to generate a hard instance without increasing the size of the instance.

**Assumption2:** Sample efficiency. Evaluating the hardness value is time consuming, especially when $x$ is a hard instance. Therefore, we cannot evaluate too many instances, which motivates us to find hard instances more efficiently. To overcome this problem, we set the budget of evaluation $B$. In other words, we have to find an instance that is as hard as possible with no more than $B$ evaluations. This prevents us from using a method that evaluates too many objective functions (e.g., brute-force search and generic algorithm).

### 2.2 Formulation as Two Equivalent Optimization Problems

We formulate the task of finding a hard instance as two equivalent optimization problems and directly optimize them. The instance can be represented as an adjacent matrix $A \in \{0, 1\}^{n(n-1)/2}$ because we consider problems with undirected, unweighted, and simple graphs. Therefore, the task of finding a hard instance can be written as follows:

**Problem 1**

\[
\begin{align*}
\text{maximize} & \quad \text{hardness}(A, L) \\
\text{subject to} & \quad A \in \{0, 1\}^{n(n-1)/2}.
\end{align*}
\]

This problem can be solved by the hill climbing or simulated annealing methods, which are well-known meta-heuristic search algorithms. In an early experiment, we found that the simulated annealing method did not improve the efficiency (i.e., temperature $= 0$ was the best). Therefore, we only use the hill climbing method in the experiment. We use a graph (i.e., an instance) as the state of searching and add or delete an edge to the graph to generate a neighbor solution. Since there are a number of neighbor solutions, calculating the hardness values of all of them is not cost effective; for example, when the number of vertices is $n = 50$, the number of neighborhoods of the graph is $n(n-1)/2 = 1225$. Therefore, we resort to a randomized approach. More specifically, we first randomly choose a neighborhood of the current graph, and evaluate its hardness value. If the hardness value is higher than that of the current graph, we set the neighbor graph as the next state. Otherwise, we stay at the current state.

However, Problem 1 is difficult to solve because $A$ consists of discrete variables. Therefore, we transform this problem into a continuous space.

**Problem 2**

\[
\begin{align*}
\text{maximize} & \quad \mathbb{E}_{A \sim \text{Bernoulli}(P)}[\text{hardness}(A, L)] \\
\text{subject to} & \quad P \in [0, 1]^{n(n-1)/2}.
\end{align*}
\]

Here, $A \sim \text{Bernoulli}(P)$ denotes that each $A_i$ is sampled from Bernoulli($P_i$) independently. This modification does not change the optimal value.

**Theorem 1.** The optimal values of Problem 1 and Problem 2 are the same.

**Proof.** Let $M$ be the optimal value of Problem 1 and $\text{hardness}(A^*, L) = M$. Let $M'$ be the optimal value of Problem 2. For any graph $A$, $\text{hardness}(A, L) \leq M$. Therefore, for any $P$, $\mathbb{E}_{A \sim \text{Bernoulli}(P)}[\text{hardness}(A, L)] \leq M$. $M' \leq M$.

On the other hand, $M' \geq \mathbb{E}_{A \sim \text{Bernoulli}(A^*)}[\text{hardness}(A, L)] = \text{hardness}(A^*, L) = M$. Therefore, $M = M'$ holds.

Moreover, the objective function of Problem 2 is smooth.
Theorem 2. The objective function of Problem 2 $f(P) = \mathbb{E}_{A \sim \text{Bernoulli}(P)}[\text{hardness}(A, L)]$ is an analytic function. Especially, this function is infinitely differentiable.

Proof.

$$f(P) = \mathbb{E}_{A \sim \text{Bernoulli}(P)}[\text{hardness}(A, L)]$$

$$= \sum_A \prod_{i=1}^{n(n-1)/2} (P_i^{A_i}(1 - P_i)^{1-A_i}) \cdot \text{hardness}(A, L),$$

where the sum is taken over $A \in \{0, 1\}^{n(n-1)/2}$. Here, $f(P)$ is a polynomial of the elements of $P$. Therefore, $f(P)$ is an analytic function and is infinitely differentiable.

Thanks to Theorems 1 and 2, we can solve the maximization problem efficiently by using the continuous optimization technique, which is much easier than discrete optimization. Hence, in this paper, we propose an efficient continuous optimization algorithm for solving Problem 2.

2.3 Probabilistic Graph Model

To solve Problem 2, we employ the machine learning approach based on immediate-reinforcement learning with a probabilistic graph generative model. Note that we may be able to solve the problem by using a sequential model that outputs nodes and/or edges one by one [You et al., 2018a; You et al., 2018b]. However, since these methods model molecular graphs, and sequential models are hard to train for large graphs [Ma et al., 2018], we do not adopt this scheme.

In the immediate-reinforcement learning framework, the action of an agent corresponds to the generation of a graph $A$ (i.e., an instance), which is expected to be a hard instance, and the reward $r$ of the action corresponds to the cost of solving the instance calculated by running the algorithm on $A$ (i.e., $r = \text{hardness}(A, L)$). The reward $r$ is used to improve the accuracy of reward prediction. In this setting, we assume that there is no input from the environment, and use noise $z$ as the input. Each action is independent because the episode contains only one action.

To decide an action, a neural network model outputs the probability $P_i$ that the edge $i$ appears for each edge $i$ ($i = 1, 2, \ldots, n(n-1)/2$). We sample each edge independently according to the probability $P_i$ to construct a graph $A$ (i.e., an action). Next, we evaluate the hardness $h(L, A)$ to get a reward. Then, we update the weight of the neural network model $w_i$ using REINFORCE algorithm [Williams, 1992]:

$$w_i \leftarrow w_i + \alpha \frac{\partial}{\partial w_i} \sum_{i=1}^{n(n-1)/2} (\log P_i^{A_i} + \log(1 - P_i)^{1-A_i}),$$

where $\alpha$ is the learning rate. Note that $A_i$ and $A_j$ ($i \neq j$) are independent when $P$ is observed, but are dependent without any condition because $P$ is not independent. Therefore, the neural network model can model nonlinear relationships between edges. The procedure used to train the neural network model is shown in Algorithm 1.

The advantage of the neural network model is that it can output a variety of instances once trained, whereas the hill climbing method outputs only one instance. Generating many hard instances is useful not only for creating a benchmark but also for performing algorithm analysis by comparing the difference or extracting the common structures. As we will see later in the experiments, the neural network model and the hill climbing method can generate far harder instances than the uniformly random graph model. In particular, the neural network model performs the best.

3 Extensions

In our proposed method, the choice of the hardness function is arbitrary. Therefore, our method can find not only instances that need a long time to solve but also hard instances in other meanings. We introduce two important examples.

3.1 Estimating Approximation Ratio

Let $L$ be an approximation algorithm, $A$ be an instance of the problem, $L(A)$ be the value that $L$ outputs for $A$, and $\text{OPT}(A)$ be the optimal value of $A$. The approximation ratio of $L$ is defined by $r(L) = \max_{A \text{ is an instance of the problem}} \frac{L(A)}{\text{OPT}(A)}$ for a minimizing problem and by $r(L) = \max_{A \text{ is an instance of the problem}} \frac{\text{OPT}(A)}{L(A)}$ for a maximizing problem. Estimating the approximation ratios is important for investigating the performance of the approximation algorithms. However, it is not trivial what instance maximizes the term. Here, we use $\frac{L(A)}{\text{OPT}(A)}$ or $\frac{\text{OPT}(A)}{L(A)}$ as the hardness value of the instance $A$; then, we can search the maximizer with our methods. We will show an illustrative example with the well-known minimum vertex cover algorithm in Section 5.5.

3.2 Hard Instances for Enumerating Algorithms

Enumerating algorithms output all the elements that satisfy some property. When evaluating the efficiency of enumerating algorithms, the amortized time and maximum delay are sometimes investigated instead of the total amount of time. Our method can generate hard instances in terms of the amortized time and maximum delay by setting these measures as the hardness value.
4 Related Work

In this section, we review the existing hard instance generation algorithms and deep graph generative models.

Constructing Hard Instances: There have been several researches on constructing hard instances of combinatorial problems. Hard instance generation was first studied in relation to the phase transition phenomena [Cheeseman et al., 1991; Hogg and Williams, 1994], which utilizes order parameters to generate hard instances. The three coloring instance generation by Mizuno and Nishihara (2008) and the graph isomorphism instance generation by Neuen et al. (2017) generate hard instances with rule-based algorithms. However, these works depend on problem specific knowledge, whereas our method is independent of the problem. Another approach is to generate instances by solving other related problems. For example, the Latin square problem is found useful for constructing a benchmark for the graph-coloring problem [Gomes and Shmoys, 2002] and SAT [Achlioptas et al., 2000]. However, the conversion of the solution of the Latin square problem into those of other problems also requires problem-specific knowledge. When there is no knowledge about constructing a benchmark for a problem, random instances tend to be used as a benchmark.

Deep Generative Graph Models: Recently, several generative graph models utilizing deep learning techniques were proposed. The variational graph auto-encoder [Kipf and Welling, 2016] is one of the first models of this kind. It is a variant of the Variational Auto Encoder (VAE), which outputs a probabilistic adjacent matrix. This model was used for the link prediction of citation networks. Then, VAE [Simonovsky and Komodakis, 2018; Kusner et al., 2017; Grover et al., 2018; Ma et al., 2018], Generative Adversarial Networks (GAN) [Wang et al., 2018; Bojchevski et al., 2018; Pan et al., 2018], and sequential generation [You et al., 2018b; Li et al., 2018; Liu et al., 2018; You et al., 2018a] based generating models were proposed. In particular, they succeeded in generating various de-novo chemical materials and modeling real-world networks. ORGAN [Guimaraes et al., 2017] utilizes SeqGAN [Yu et al., 2017] and reinforcement learning to generate molecular graphs with the desired properties. It uses SMILES [Weininger, 1988] to represent a molecular graph because SeqGAN generates a sequence of symbols rather than a graph itself. MolGAN [Cao and Kipf, 2018] is another graph generative model utilizing GAN and reinforcement learning. It models the probabilistic adjacent matrix and attributes of graphs directly instead of using SMILES. The generative graph model of this work belongs to this stream. To the best of our knowledge, there is no deep learning based hard instance generating algorithm.

5 Experiments

Through experiments with a variety of problems and algorithms, we will confirm the following facts:

Observation 1: The neural network model and the hill climbing method can generate far harder instances than the uniformly random models. In particular, the neural network model performs the best (Sections 5.1 to 5.4).

Observation 2: The neural network model is useful for estimating approximation ratios of approximation algorithms (Section 5.5).

Observation 3: The neural network model can generate a variety of hard instances, and it helps develop benchmarks and analyze algorithms further (Section 5.6).

Observation 4: The neural network model is computationally efficient for generating hard instances (Section 5.7).

We carry out experiments with the 3-coloring problem, the minimum vertex cover problem, the maximum clique problem, and the graph isomorphism problem. We use the neural network model and the hill climbing method to generate hard instances for these problems. We restart these methods when they are trapped by the local maxima. We adopt Erdős-Rényi models as trivial baselines and several rule-based algorithms in the 3-coloring problem and the graph isomorphism problem as strong baselines. When we use these baselines, we generate $B$ (i.e., the budget) graphs using the algorithm and output the hardest instance among them. We set $B = 100000$ and stop when it takes more than three days. We set the number of nodes as $n = 50$ except for the maximum clique problem. We set $n = 32$ for the maximum clique problem because many methods generate instances that are too hard to make a comparison. These choices meet Assumption 1. Graphs with $n \leq 50$ nodes can be easily visualized (see Figure 1).

5.1 3-coloring Problem

The 3-coloring problem is the problem of determining whether or not each vertex in a graph can be assigned one of the three colors so that no two adjacent vertices are assigned the same color. The 3-coloring problem is important because many practical problems such as the scheduling assignment problem and the register allocation problem of compilers are formalized as its variants. In this experiment, we use backtracking search based on Brélaz’s heuristics [Brélaz, 1979], which is the most popular algorithm for this problem. It assigns colors to the vertices one by one. At each step, it chooses one of the uncolored vertices that have the least number of candidate colors. If there are many such vertices, it chooses the vertex with the maximum degree. If the color assignment becomes inconsistent, it backtracks until it finds a consistent assignment. The hardness value is the number of recursive calls the algorithm needs to solve an instance. In addition to the uniformly random graph models, we use four rule-based algorithms in this experiment. The first column of Table 1 shows the result. It shows that the neural network model generates instances that are several orders of magnitude harder than uniformly random graph models and rule-based algorithms. This result also shows that we cannot reveal the worst complexity by evaluating only the random instances and that it is important to use hard instances to investigate the robustness of the algorithm (Reason 2).

Example Analysis: Figure 1 shows an example of the instance the neural network generates, which gives an insight into making a better search algorithm. There are obviously no solutions for this instance, because it has a 4-clique $C$ (highlighted in orange). However, it is not apparent to Brélaz’s
algorithm because the algorithm cannot explicitly detect 4-cliques. It first tries to assign colors to \( V \setminus C \). Every time it finds a solution for \( V \setminus C \), it is immediately rejected when the algorithm starts to color the 4-clique \( C \). Then, the search is back-tracked and the algorithm starts to find other assignments of \( V \setminus C \). However, it does not obtain any result because any assignment will be rejected by the 4-clique constraint. Finally, the algorithm finds a solution for \( V \setminus C \), and reports that there is no solution for this instance. The key point is that the 4-clique \( C \) is connected to \( V \setminus C \) by a path (highlighted in blue). When the algorithm is coloring \( V \setminus C \), the number of color candidates of Vertex 1 is not less than two, and the degree is only two. Therefore, Brélaz’s algorithm is reluctant to color this vertex. From this analysis, we can improve the backtracking search by preprocessing: deleting vertices whose degree is not more than two. If we delete such vertices, the answer does not change because we can color the vertices whose degree is not more than two whatever the coloring assignment of the other vertices is: just color the vertex with the color that is not the same as the colors of adjacent nodes. This improvement helps avoid the problem described above. This discussion is a good example to show that an algorithm with pivoting, the Bron Kerbosch algorithm without pivoting, and Fast Max-Clique, and Nauty, respectively.

Table 1: The experimental results: Each value is the mean of the maximum hardness in five runs. Brélaz, BB, BK, BKNP, FMC, and Nauty denote the backtracking search with Brélaz’s heuristics, branch and bound search with maximal matching lower bound, Bron Kerbosch algorithm with pivoting, Bron Kerbosch algorithm without pivoting, Fast Max-Clique, and Nauty, respectively.

| Problem                        | 3-coloring | Vertex Cover | Clique | Isomorphism |
|-------------------------------|-----------|--------------|--------|-------------|
| Algorithm                     | Brélaz    | BB           | BK     | BKNP        | FMC         | Nauty      |
| Neural Network                | 115177/0980.6 | 12771.8      | 74499.6 | 4706426.4   | 5882966.0   | 9400.0     |
| Hill Climbing                 | 212276998.8  | 46659.8      | 160091.8 | 2615672.0   | 195117.4    | 120.6      |
| Erdős-Rényi \( p = 0.1 \)   | 937.4      | 761.0        | 86.8    | 105.8       | 27966.0     | 122.2      |
| Erdős-Rényi \( p = 0.5 \)   | 2.0        | 635.6        | 562.4   | 1503.6      | 36657.6     | 32.6       |
| Erdős-Rényi \( p = 0.9 \)   | 2.0        | 489.8        | 9365.4  | 327251.4    | 107445.0    | 118.0      |
| Cheeseeman et al. (1991)     | 1567.6     | N/A          | N/A     | N/A         | N/A         | N/A        |
| Hogg and Williams (1994)     | 31708.2    | N/A          | N/A     | N/A         | N/A         | N/A        |
| Vlasie (1995)                | 85353.6    | N/A          | N/A     | N/A         | N/A         | N/A        |
| Mizuno and Nishihara (2008)  | 219342.2   | N/A          | N/A     | N/A         | N/A         | N/A        |
| \( R(B(G_n, \sigma)) \) (2017) | N/A       | N/A          | N/A     | N/A         | N/A         | 2700.0     |
| shrunken multipedes (2017)   | N/A       | N/A          | N/A     | N/A         | N/A         | 2182.0     |

5.2 Minimum Vertex Cover Problem

The minimum vertex cover problem is the problem of finding a minimum set of vertices such that each edge in a graph is incident to at least one vertex of that set. In this experiment, we use the branch and bound search, which uses the well-known maximal matching upper-bound and checks the vertex that can reduce the most number of constraints earlier. We use the number of recursive calls as the hardness value. Table 1 shows that the neural network model and hill climbing method succeed in consistently generating harder instances than the Erdős-Rényi models.

5.3 Maximum Clique Problem

The maximum clique problem is the problem of finding the maximum set of vertices that have an edge between any pair of vertices in the set. In this experiment, we use the Bron Kerbosch algorithm with pivoting, the Bron Kerbosch algorithm without pivoting, and Fast Max-Clique [Pattabiraman et al., 2013]. For the Bron Kerbosch algorithm, we use the number of recursive calls as the hardness value, and for Fast Max-Clique, we use the CPU time consumption (10^{-6} sec) as the hardness value. Table 1 shows that the neural network model and the hill climbing method succeed in consistently generating harder instances than the Erdős-Rényi models.

5.4 Graph Isomorphism Problem

The graph isomorphism problem is the problem of assessing whether or not there exists a bijection \( f: V(G_1) \rightarrow V(G_2) \) such that there is an edge between \( v_1 \) and \( v_2 \) in \( G_1 \) if and only if there is an edge between \( f(v_1) \) and \( f(v_2) \) in \( G_2 \). The bijection \( f \) is called an isomorphism function, and if a pair of graphs has an isomorphism function, the two graphs are considered equivalent. It is common to use the graph canonization algorithm to solve this problem. We use one of the most popular software, Nauty [McKay and Piperno, 2014], in this experiment. We use the CPU time consumption (10^{-7} sec) as the hardness value. In addition to the uniformly random graph models, we use two rule-based algorithms in this problem. The sixth column of Table 1 shows the results. The neural network model performs better than all the other methods. However, the neural network model is trapped in the trivial local maximum, which is a clique. This is because the more the symmetry a graph has, the more the orbit that the graph has, and a clique has many orbits. Considering that the other methods cannot generate any meaningful structure, the neural network model is beneficial because it can generate such a local maximum graph without any task specific knowledge and can provide us some knowledge about the problem.

5.5 Estimating Approximation Ratio

In this section, we will show an illustrative example to estimate the approximation ratio using the greedy algorithm for the minimum vertex cover problem. It is known that the approximation ratio of the greedy algorithm is 2. We use the Erdős-Rényi model \( G_{50,0.1} \) and a neural network model. We set the number of vertices as \( n = 50 \) and the budget as
In this section, we confirm that the neural network model is computationally efficient for generating hard instances. The time complexity of the neural network model is dominated by matrix calculation of the last layer and edge sampling. Therefore, the time complexity of the neural network model is $O(n^2)$, which is the same as the time complexity of the Erdős-Rényi models and the hill climbing method. Through experiments, we show that the neural network model scales well. First, we initialize ten neural network models and each model generates 100 instances. We report the mean time of the generation in Figure 2 as the inference time. Then, we initialize ten neural network models; each model generates 100 instances, and we train each model for each instance with reward $r = 1$. We report the mean time of a single iteration in Figure 2 as the training time. Note that the time is independent of the network parameters and the reward. Figure 2 shows that the computation does not grow much even if the number of nodes increases. In particular, one iteration of the training takes only four seconds even with 1024 nodes. Considering that generating small instances is important and evaluating the hardness value takes more time than model training, the neural network model is highly efficient.

6 Conclusion

In this paper, we formulate finding a hard instance of graph problems as two equivalent optimization problems (Problems 1 and 2 in Section 2.2). To solve these problems, we propose two methods: hill climbing and reinforcement learning (Algorithm 1). Experiments with a variety of problems and algorithms show that our reinforcement learning method can find instances that are few to several orders of magnitude harder than Erdős-Rényi models in many settings. Especially in the 3-coloring problem, our method outperformed the rule-based method (Table 1).
7 Acknowledgments

This work was supported by JSPS KAKENHI Grant Number 15H01704. MY is supported by the JST PRESTO program JPMJPR165A. We thank Yuasuki Kobayashi and Alessio Conte for discussing about the extensions of our proposed method.

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