Flexing infinite frameworks with applications to braced Penrose tilings

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A planar framework — a graph together with a map of its vertices to the plane — is flexible if it allows a continuous deformation preserving the distances between adjacent vertices. Extending a recent previous result, we prove that a connected graph with a countable vertex set can be realized as a flexible framework if and only if it has a so-called NAC-coloring. The tools developed to prove this result are then applied to frameworks where every 4-cycle is a parallelogram, and countably infinite graphs with $n$-fold rotational symmetry. With this, we determine a simple combinatorial characterization that determines whether the 1-skeleton of a Penrose rhombus tiling with a given set of braced rhombi will have a flexible motion, and also whether the motion will preserve 5-fold rotational symmetry.

1 Introduction

A realization of a (simple) graph $G = (V_G, E_G)$ in the plane is a map $\rho : V_G \rightarrow \mathbb{R}^2$ associating to each vertex a point in the plane such that adjacent vertices are mapped to distinct points, i.e. $\rho(u) \neq \rho(v)$ for each edge $uv \in E_G$. A graph together with one of its realizations is called a planar framework, or simply a framework for short. If a framework can be continuously deformed in the plane keeping the distances between adjacent vertices, then we say that such framework is flexible; otherwise, the framework is called rigid (see Definition 2.1). An example of a flexible framework with a countably infinite set of vertices can be seen in Figure 1.

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Rigidity and flexibility of finite frameworks are well-studied topics with a vast amount of literature between them. For example, the finite graphs that have a rigid generic realization* (and equivalently, only have rigid generic realizations) are combinatorially characterized by the so-called Laman’s theorem, first proven by Pollaczek-Geiringer in 1924 [25] and independently proven by Laman over forty years later [19]. However, not much is known on rigidity for countably infinite graphs, i.e. graph with a countably infinite amount of vertices. The combinatorial characterization of Pollaczek-Geiringer and Laman was extended to countably infinite frameworks by Kitson and Power in [17], where it was shown that a countably infinite graph will have only rigid generic realizations† if and only if it contains an increasing sequence of finite graphs with rigid realizations. Countably infinite graphs can, however, be constructed with both rigid and flexible generic realizations; for example, see [5, Figure 4.2]. This raises the question: which rigidity and flexibility results for finite graphs can be extended to infinite graphs, and which cannot? This question has been looked into in recent years, with particular interest regarding periodic frameworks [2, 21].

In the case of finite graphs, it was proven by Grasegger, Legeršký and Schicho in [13] that the existence of a flexible (possibly not generic) realization of a connected finite graph is equivalent to the existence of a particular type of red-blue coloring of the edges of the graph called a NAC-coloring (see Definition 2.5). We shall prove that this characterization can indeed be extended to countably infinite graphs.

**Theorem 1.1.** A countably infinite connected graph admits a flexible realization if and only if it has a NAC-coloring.

The proof of the main theorem for finite graphs (see [13, Theorem 3.1]) relies on two different techniques. The easier implication involves constructing a flexible framework from a NAC-coloring, which can be extended to countably infinite graphs with relative ease (Proposition 3.1). However, the algebraic methods required to obtain a NAC-coloring from a finite flexible framework break down for infinite frameworks, although the technique was adapted to infinite frameworks with periodic symmetry by Dewar in [6]. We will bypass this obstacle by instead considering an ascending tower of finite

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* A realization in which the coordinates of the points are algebraically independent.
† A realization in which every restriction of the realization to a finite vertex set is generic.
subgraphs that covers the whole graph and then recursively applying the known result about NAC-colorings for finite graphs. In particular, once a set of NAC-colorings with some specific properties is found for each subgraph of the tower, there is an infinite chain of NAC-colorings extending each other by König’s lemma (Lemma 3.3). This infinite chain will give us a NAC-coloring of the infinite graph. Theorem 1.1 was originally presented at [10], however the proof technique was vastly different and utilized much more heavy machinery from algebraic geometry; see Remark 3.5.

There is a particular type of infinite framework that we shall wish to apply Theorem 1.1 and its methods to: the Penrose frameworks, i.e. frameworks that are the 1-skeletons of Penrose rhombus tilings (see Section 5 for more details). The eponymous tiling was first formulated by Penrose in the 1970s [24] and is one of the first examples of a quasi-crystal, an infinite structure that has order but is not periodic. Ideally, we would like to be able to determine which rhombi we should brace to obtain rigidity solely from the ribbon graph of the Penrose framework (Definition 4.6). A major hurdle when working with Penrose frameworks is that they are neither periodic nor generic, so most previous tools cannot be used with them.

To determine which braced Penrose frameworks are rigid, we shall study the much larger class of P-frameworks (Definition 4.3). First defined in [12] as the generalization of 1-skeletons of parallelogram tilings, it was shown by Grasegger and Legerský for a finite P-framework $(G, \rho)$ with some parallelograms braced, the following properties are equivalent: (i) $(G, \rho)$ is rigid, (ii) $G$ has no cartesian NAC-coloring (Definition 4.8), and (iii) the bracing graph (Definition 4.6) of $(G, \rho)$ is connected. We shall extend this result to countably infinite P-frameworks, allowing us to give a characterization for rigid braced Penrose frameworks.

**Theorem 1.2.** Let $(G, \rho)$ be a countably infinite braced P-framework. Then the following are equivalent:

(i) $(G, \rho)$ is rigid.

(ii) $G$ has no cartesian NAC-coloring.

(iii) The bracing graph of $G$ is connected.

As some Penrose tilings have 5-fold rotational symmetry, we would also like to be able to determine whether there exists a flex of a Penrose framework that maintains the 5-fold rotational symmetry. To do this, we shall first apply the methods used in the proof of Theorem 1.1 to infinite graphs with $k$-fold rotational symmetry to give a combinatorial characterization for the existence of $k$-fold rotationally symmetric flexible realizations. This will use methods employed by Dewar, Grasegger and Legerský, who gave a combinatorial characterization of finite rotationally symmetric graphs with flexible symmetric realizations [7]. We shall then combine this result with the methods used to prove Theorem 1.2 to obtain a combinatorial characterization of P-frameworks with rotationally symmetric flexible motions. With this we will be able to prove the following result.
Figure 2: A 5-fold rotationally symmetric deformation of a Penrose framework. The filled rhombi preserve their shapes during the flex.

Corollary 1.3. Any Penrose framework with 5-fold rotational symmetry has a non-trivial flex (Definition 2.1) that preserves 5-fold rotational symmetry.

See Figure 2 for an example of one of the 5-fold rotationally symmetric Penrose frameworks with a 5-fold rotationally symmetric flex\(^1\).

The paper shall be organized as follows. Section 2 will cover all the required background for rigid and flexible frameworks, NAC-colorings, and all of the algebraic and combinatorial tools we shall require throughout. In Section 3, we will prove our main result of Theorem 1.1. Section 4 will introduce P-frameworks and will provide a proof for Theorem 1.2. Section 5 will formally define Penrose tilings and frameworks, and will also contain the corollaries we can obtain by applying Theorem 1.2. Section 6 will introduce rotational symmetry for frameworks in general, and will also extend the main result of [7] to frameworks with a countably infinite amount of vertices, and Section 7 will combine the results of Sections 4 and 6 to give a characterization of \(k\)-fold rotationally symmetric braced P-frameworks with flexes that preserve the symmetry. Finally, in Section 8 we shall describe all the possible realizations of the countably infinite bipartite framework that are both injective and flexible. This is notable as our construction detailed in Proposition 3.1 cannot form injective realizations of this graph.

\(^1\)An animation of the flex together with the code producing it using package FLEXRiLoG [11] can be found at https://jan.legersky.cz/PenroseFrameworks
2 Preliminaries

We recall the notions of rigidity and flexibility, which are usually given for finite graphs, but extend naturally to graphs with countably many vertices and edges. We denote by \((\lambda_e)_{e \in E_G}\), where \(\lambda_{uv} = \|\rho(u) - \rho(v)\|\), the edge lengths induced by a framework \((G, \rho)\). For \(K \in \{\mathbb{R}, \mathbb{C}\}\), we will denote the set of maps \(\rho : V_G \rightarrow K^2\) by \((K^2)^V_G\). When \(G\) is finite, we will make no distinction between \((\mathbb{K}^2)^V_G\) and \(\mathbb{K}^{2|V_G|}\).

**Definition 2.1.** A flex of the framework \((G, \rho)\) is a continuous path \(t \mapsto \rho_t, t \in [0, \delta)\) for some \(\delta > 0\), in the space of realizations of \(G^0\) such that \(\rho_0 = \rho\) and each \((G, \rho_t)\) induces the same edge lengths as \((G, \rho)\). The flex is called trivial if for all \(t \in [0, \delta)\) the realization \(\rho_t\) is congruent to \(\rho\), namely, there exists a Euclidean isometry \(M_t\) of \(\mathbb{R}^2\) such that \(M_t \rho_t(v) = \rho(v)\) for all \(v \in V_G\).

We define a framework to be flexible if there is a non-trivial flex in \(\mathbb{R}^2\). Otherwise, it is called rigid.

Due to the nature of our methods, we will want to transform the information about the existence of a flex into the existence of a positive-dimensional algebraic set consisting of realizations. To do so, we will need to remove all but finitely many congruent copies of every realization. The usual method of doing this is to add extra algebraic constraints that stop any rigid body motions of the framework, such as requiring that a single edge is fixed in place. In order to deal with various special cases like symmetry, we introduce the following more general notion.

**Definition 2.2.** Let \(H\) be a finite graph. Let \(X\) be an algebraic set in \((\mathbb{C}^2)^V_H\) given by polynomials with real coefficients such that for every \(\rho \in (\mathbb{R}^2)^V_H \cap X\) there are only finitely many realizations \(\rho' \in (\mathbb{R}^2)^V_H \cap X\) congruent to \(\rho\). We call any such set \(X\) a restricting set. We use variables \((x_v, y_v)\) for the coordinates of vertex \(v \in V_H\).

**Example 2.3.** Let \((H, \rho)\) be a finite framework. Let \(\bar{u}\bar{v}\) be an edge of \(H\) and \((\lambda_e)_{e \in E_H}\) be the edge lengths induced by \(\rho\). The zero set of the equations

\[
\begin{align*}
x_u &= 0, & y_u &= 0, & x_{\bar{u}} &= \lambda_{\bar{u}\bar{v}}, & y_{\bar{u}} &= 0.
\end{align*}
\]

is a restricting set. Since these four equations impose that the vertices of the edge \(\bar{u}\bar{v}\) have a fixed position, there is only one realization congruent to a given one, namely, its reflection w.r.t. the \(x\)-axis.

Later in the paper, we use other restricting sets to deal with rotationally symmetric frameworks and frameworks with parallelograms.

**Definition 2.4.** Let \(H\) be a finite graph and let \((H, \rho)\) be a flexible framework with induced edge lengths \((\lambda_e)_{e \in E_H}\). Let \(X\) be a restricting set. We consider the following polynomial system in \((\mathbb{C}^2)^V_H\) for unknown coordinates \((x_u, y_u)\) where \(u \in V_H\):

\[
(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda_{uv}^2 \quad \text{for all } uv \in E_H.\tag{1}
\]

The topology we will gift the set \((\mathbb{R}^2)^V_G\) will always be the product topology, hence a path \(t \mapsto \rho_t\) will be continuous if and only if each path \(t \mapsto \rho_t(v)\) is continuous.
An irreducible complex algebraic curve $C$ in the intersection of $X$ and the zero set of (1) is called an algebraic motion of $(H, \rho)$ w.r.t. $X$. If the restricting set $X$ is the one obtained as in Example 2.3, we call it an algebraic motion of $(H, \rho)$ w.r.t. $\bar{w}\bar{v}$.

Note that if $(H, \rho)$ has a non-trivial flex, then the system given by (1) and Example 2.3 has infinitely many solutions and so $(H, \rho)$ admits an algebraic motion w.r.t. $\bar{w}\bar{v}$.

The techniques employed in [13] prove that a finite connected graph has a flexible framework if and only if it admits a so-called NAC-coloring. We extend the notion of NAC-colorings to countable infinite graphs in a straightforward manner.

**Definition 2.5.** Let $G$ be a graph (finite or countable). A coloring of edges $\delta: E_G \to \{\text{blue, red}\}$ is called a NAC-coloring, if it is surjective and for every cycle* in $G$, either all edges have the same color, or there are at least two edges in each color. A NAC-coloring $\delta$ induces two subgraphs of $G$:

$$G_{\text{red}}^\delta = (V_G, \{e \in E_G: \delta(e) = \text{red}\}) \quad \text{and} \quad G_{\text{blue}}^\delta = (V_G, \{e \in E_G: \delta(e) = \text{blue}\}).$$

The set of NAC-colorings of $G$ is denoted by $\text{NAC}_G$. See Figure 3 for an example.

![Figure 3: A NAC-coloring of the graph in Figure 1](image)

As was shown in [13], all NAC-colorings can be constructed starting from valuations on the function field (see [28]) of an algebraic motion $C$. The following functions are used:

$$W_{u,v}(\rho) := (x_v - x_u) + i(y_v - y_u), \quad Z_{u,v}(\rho) := (x_v - x_u) - i(y_v - y_u),$$

where $uv$ is an edge of the graph and $(x_u, y_u) = \rho(u)$ for $\rho \in C$. This allows the assignment of some NAC-colorings to an algebraic motion (see also [14, Definition 2.9]).

**Definition 2.6.** Let $H$ be a finite graph and let $(H, \rho)$ be a flexible framework. Let $C$ be an algebraic motion of $(H, \rho)$ w.r.t. a restricting set $X$. A NAC-coloring $\delta$ of $H$ is called active w.r.t. $C$ if there exists a valuation $\nu$ on the function field of $C$ and $\beta \in \mathbb{Q}$ such that $\delta(uv) = \text{red} \iff \nu(W_{u,v}) > \beta$ for $uv \in E_H$.

We finish the section with an important technical lemma we will require throughout the paper.

*Although the graph can be infinite, cycles are always finite.
Lemma 2.7. Let \((H, \rho)\) be a finite framework and \(u_1v_1, u_2v_2 \in E_H\). Let \(X\) be a restricting set. Let \(\alpha\) be a non-trivial flex of \((H, \rho)\) such that for every realization in the flex there is a congruent realization in \(X\). If the angle between the edges \(u_1v_1\) and \(u_2v_2\) is non-constant along the flex \(\alpha\), then there exists an algebraic motion \(C\) of \((H, \rho)\) w.r.t. \(X\) and an active NAC-coloring \(\delta\) w.r.t. \(C\) such that \(\delta(u_1v_1) \neq \delta(u_2v_2)\).

Proof. Let \(A \subseteq X\) be the algebraic set of frameworks with the same induced edge lengths as \(\rho\). For every realization in the flex \(\alpha\), there is a congruent one contained in \(A\) by assumption. Define \(f : A \rightarrow C\) to be the map

\[
f(\rho') = (\rho'(u_1) - \rho'(v_1)) \cdot (\rho'(u_2) - \rho'(v_2)).
\]

As algebraic sets only have finitely many irreducible components and there are infinitely many values of the angle between \(u_1v_1\) and \(u_2v_2\) along the flex \(\alpha\), there exists an irreducible component \(A_0\) of \(A\) which contains realizations \(\rho_1, \rho_2 \in A_0\) where \(f(\rho_1) \neq f(\rho_2)\).

By [22, Lemma pg. 56], there exists an algebraic curve \(C\) in \(A_0\) that contains \(\rho_1\) and \(\rho_2\). It is immediate that \(W_{u,v} Z_{u,v} = \lambda_{uv}^2\) for all \(uv \in E_H\). Since

\[
2f = W_{u_1,v_1} Z_{u_2,v_2} + W_{u_2,v_2} Z_{u_1,v_1} = \lambda_{u_1,v_1}^2 \frac{W_{u_2,v_2}^2}{W_{u_2,v_2}} + W_{u_2,v_2} \frac{W_{u_1,v_1}^2}{W_{u_1,v_1}},
\]

the map \(W_{u_1,v_1} W_{u_2,v_2}\) is not constant on \(C\). By Chevalley's theorem, there is a valuation \(\nu\) of the complex function field of \(C\) such that \(\nu(W_{u_1,v_1} W_{u_2,v_2}) > 0\), hence, \(\nu(W_{u_1,v_1}) > \nu(W_{u_2,v_2})\). We now apply the methods outlined in [13, Theorem 3.1] (see also [14, Theorem 2.8]) to construct an active NAC-coloring \(\delta\) w.r.t. \(C\) according to Definition 2.6 with \(\beta : = \nu(W_{u_2,v_2})\). By construction, we have \(\delta(u_1v_1) \neq \delta(u_2v_2)\). \(\square\)

3 Flexible realizations of infinite graphs

In this section we shall prove Theorem 1.1. To begin with, we shall prove that every NAC-coloring yields a flexible framework. The following statement was proved for the case of finite graphs in [13, Theorem 3.1].

Proposition 3.1. Let \(G\) be a countably infinite graph. If \(G\) has a NAC-coloring, then there is a flexible framework \((G, \rho)\).

Proof. Let \(\delta\) be a NAC-coloring of \(G\). Let \(R_1, R_2, \ldots\) be the sets of vertices of connected components of the graph \(G^\text{red}\) and \(B_1, B_2, \ldots\) be the sets of vertices of connected components of the graph \(G^\text{blue}\). Let \(r_1, r_2, \ldots\) and \(b_1, b_2, \ldots\) be pairwise distinct vectors in \(\mathbb{R}^2\) such that \(r_1 = b_1 = (0,0)\). For \(t \in [0, 2\pi)\), we define a map \(\rho_t : V_G \rightarrow \mathbb{R}^2\) by

\[
\rho_t(v) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \cdot b_j + r_i,
\]

where \(i\) and \(j\) are the unique indices such that \(v \in R_i \cap B_j\). It can now be checked that \((G, \rho) = (G, \rho_0)\) is a realization (i.e. \(\rho(u) \neq \rho(v)\) for all \(uv \in E\)) and \(t \mapsto \rho_t\) is a flex of \((G, \rho)\); see [13, Theorem 3.1] for more details. \(\square\)
A useful technique when working with countably infinite graphs is to consider increasing sequences of subgraphs. We define this more explicitly as follows.

**Definition 3.2.** Let $G$ be a connected graph. A sequence $(G_n)_{n \in \mathbb{N}}$ of induced finite connected subgraphs of $G$ is a *subgraph tower* in $G$ if $G_n$ is a proper subgraph of $G_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} V_{G_n} = V_G$.\footnote{Our definition differs slightly from that given in [17], as we shall always assume that all subgraphs in a subgraph tower are induced and the tower is vertex spanning.}

The construction of a NAC-coloring of an infinite graph from a sequence of NAC-colorings of a subgraph tower is inspired by the proof of De Bruijn-Erdős theorem for countable graphs, see for instance [23]. We use the following version of König’s infinity lemma.

**Lemma 3.3 ([18]).** Let $(S_n)_{n \in \mathbb{N}}$ be an infinite sequence of disjoint non-empty finite sets. Let $<$ be a relation in $\bigcup_{n \in \mathbb{N}} S_n$ such that for every positive integer $n$ and $x \in S_{n+1}$, there exists $y \in S_n$ with $y < x$. There exists an infinite sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in S_n$ and $x_n < x_{n+1}$ for all positive integers $n$.

**Lemma 3.4.** Let $G$ be a countably infinite graph with a subgraph tower $(G_n)_{n \in \mathbb{N}}$. Let $e_1, e_2$ be two edges of $G_1$. If for all $n \in \mathbb{N}$, there is a NAC-coloring $\delta_n$ of $G_n$ such that $\delta_n(e_1) \neq \delta_n(e_2)$, then $G$ has a NAC-coloring.

**Proof.** Let

$$S_n = \{ \delta \in \text{NAC}_{G_n} : \delta(e_1) = \text{blue} \neq \delta(e_2) \}$$

for every $n \in \mathbb{N}$. Each $S_n$ is non-empty since either $\delta_n$ is in $S_n$, or the NAC-coloring obtained by swapping the colors from $\delta_n$ is in $S_n$. Let $<$ be the relation in $\bigcup_{n \in \mathbb{N}} S_n$ given by restriction to edges of the smaller graph. Clearly, a restriction of a NAC-coloring to a subgraph is a NAC-coloring, provided that the obtained coloring is surjective, which is guaranteed by the condition $\delta(e_1) = \text{blue} \neq \delta(e_2)$. Hence, the assumption of Lemma 3.3 holds. Therefore, we get a sequence of NAC-colorings of $(G_n)_{n \in \mathbb{N}}$ extending each other.

This sequence of NAC-colorings gives a well-defined coloring for $G$, since each edge in $G$ is eventually an edge of a subgraph in the tower. Such a coloring is a NAC-coloring since every cycle of $G$ is contained in some $G_j$ in the subgraph tower. \square

**Proof of Theorem 1.1.** If $G$ has a NAC-coloring then there is a flexible realization of $G$ by Proposition 3.1. Now suppose $G$ has a flexible realization $\rho$ with a non-trivial flex $\alpha$. Since $\alpha$ is non-trivial, there are edges $e_1, e_2 \in E_G$ whose angle changes along the flex. We choose any subgraph tower $(G_n)_{n \in \mathbb{N}}$ such that $e_1, e_2 \in E_{G_n}$ for each $n \in \mathbb{N}$. Clearly, the restriction of the flex $\alpha$ to $V_{G_n}$ is a flex of $G_n$ such that the angle between $e_1$ and $e_2$ changes. Hence by Lemma 2.7 with the restricting set given in Example 2.3, there is a NAC-coloring $\delta_n$ of $G_n$ such that $\delta_n(e_1) \neq \delta_n(e_2)$ for all $n \in \mathbb{N}$. A NAC-coloring of $G$ is now provided by Lemma 3.4. \square
Remark 3.5. As we mentioned in the introduction, Theorem 1.1 was originally presented at [10] with the following proof technique. A chain of algebraic motions restricting to each other was constructed for a subgraph tower. With this, an active NAC-coloring of the first motion can be extended to an active NAC-coloring of the second etc. and this is used to obtain a NAC-coloring of the infinite graph.

4 Bracing of infinite P-frameworks

We will apply the methods used in the previous section to a specific class of frameworks that was first introduced in [12]. We begin with the following definitions.

Definition 4.1. Let $G$ be a graph. Consider the relation on the set of edges, where two edges are in relation if they are opposite edges of a 4-cycle subgraph of $G$. An equivalence class of the reflexive-transitive closure of the relation is called a ribbon. A ribbon $r$ is simple if the subgraph induced by $r$ does not contain any 4-cycle.

Definition 4.2. Let $G$ be a connected graph. A realization $\rho : V_G \to \mathbb{R}^2$ for $G$ such that $\rho$ is injective and each 4-cycle in $G$ forms a non-degenerate (i.e., not all vertices are collinear) parallelogram in $\rho$ is called a parallelogram realization.

Definition 4.3. A graph is called ribbon-cutting graph if it is connected and every ribbon is an edge cut, namely, removal of the edges of the ribbon disconnects the graph. If $\rho$ is a parallelogram realization of a ribbon-cutting graph $G$, we call the framework $(G, \rho)$ a P-framework.

Remark 4.4. Although the parallelograms in a P-framework are required to be non-degenerate, we do not impose that a non-trivial flex of the P-framework consists of parallelogram realizations.

As we can see with the next result, the existence of a parallelogram realization of a ribbon-cutting graph will imply certain structural properties for the graph.

Proposition 4.5. Let $(G, \rho)$ be a P-framework. All ribbons of $G$ are simple and every two vertices of $G$ are separated by a ribbon.

Proof. The proof is analogous to the one for the finite case, see [12, Remark 3.5, Theorem 3.9].

Definition 4.6. A braced ribbon-cutting graph is a graph $G = (V_G, E_c \cup E_d)$ where $E_c$ and $E_d$ are two non-empty disjoint sets such that the graph $(V_G, E_c)$ is a ribbon-cutting graph and the edges in $E_d$ correspond to diagonals of some 4-cycles of $(V_G, E_c)$. These diagonals are also called braces. If $r$ is a ribbon of $(V_G, E_c)$, then

$$r \cup \{u_1u_3 \in E_d : \exists 4\text{-cycle } (u_1, u_2, u_3, u_4) \text{ of } (V_G, E_c) \text{ s.t. } u_1u_2, u_3u_4 \in r\}$$

is a ribbon of the braced ribbon-cutting graph $G$.

The framework $(G, \rho)$ is called braced P-framework if $G$ is a braced ribbon-cutting graph and $\rho$ is a parallelogram realization for $(V_G, E_c)$. 

Definition 4.7. Let $G$ be a braced ribbon-cutting graph. The ribbon graph $\Gamma$ of $G$ is the graph with the set of vertices being the set of ribbons of $G$ and two ribbons $r_1, r_2$ are adjacent if and only if there is a 4-cycle $(u_1, u_2, u_3, u_4)$ in the underlying unbraced graph of $G$ such that $u_1u_2, u_3u_4 \in r_1$ and $u_1u_4, u_2u_3 \in r_2$. The subgraph $(V_\Gamma, E_b)$ of $\Gamma$, where

$$E_b = \{r_1r_2 \in E_\Gamma : r_1 \cap r_2 \text{ is a non-empty subset of braces of } G\} ,$$

is called the bracing (sub)graph; see Figure 4 for some examples.

Similar to the last section, we wish to link the existence of flexible motions of P-frameworks to the existence of NAC-colorings with certain properties. The result about P-frameworks is even stronger, namely, we can decide about flexibility of a given parallelogram realization, not just its existence, depending on the existence of this special NAC-coloring.

Definition 4.8. A NAC-coloring $\delta$ of a graph $G$ is called cartesian if no two distinct vertices are connected by a red and blue path simultaneously.

Equivalently, a NAC-coloring $\delta$ is cartesian if and only if for every connected component $R$ of $G^\text{red}_\delta$ and $B$ of $G^\text{blue}_\delta$, the intersection of the vertex sets of $R$ and $B$ contains at most one vertex. Notice that if the construction of a flex in Proposition 3.1 is applied to a cartesian NAC-coloring, then the realization is injective.

Lemma 4.9. Let $(G, \rho)$ be a braced P-framework. A NAC-coloring of $G$ is cartesian if and only if each ribbon of $G$ is monochromatic.

Proof. The proof is analogous to the finite case, see [12, Theorem 3.9, Lemma 4.1] □
We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. $\neg(ii) \Rightarrow \neg(i)$: If $G$ has a cartesian NAC-coloring, then $(G, \rho)$ is flexible since the constructive method given by [12, Theorem 4.4] can be extended to countably infinite P-frameworks; see also the construction of flex $\rho_1$ in the proof of Lemma 7.4. It exploits the construction described in the proof of Proposition 3.1 with carefully chosen vectors $r_i$ and $b_j$ so that the obtained flex starts at $\rho$.

$\neg(i) \Rightarrow \neg(ii)$: Let $(G, \rho)$ be a flexible infinite P-framework. Let $\alpha$: $[0, 1) \rightarrow (\mathbb{R}^2)^{V_G}$ be the corresponding non-trivial flex. Since the flex is non-trivial, there exist edges $e_1$ and $e_2$ such that the angle between these two edges changes along the flex. Let $(G_n)_{n \in \mathbb{N}}$ be a subgraph tower of $G$ such that $e_1, e_2 \in E_{G_n}^{**}$. For any $n \in \mathbb{N}$, projecting the flex $\alpha$ to $G_n$ gives a flex $\alpha_n$ of $G_n$. We would like to use Lemma 2.7 with the restricting set $X$ given by the equations from Example 2.3 and the following
\begin{align*}
x_{u_2} - x_{u_1} &= x_{u_3} - x_{u_4}, \quad x_{u_4} - x_{u_1} = x_{u_3} - x_{u_2}, \\
y_{u_2} - y_{u_1} &= y_{u_3} - y_{u_4}, \quad y_{u_4} - y_{u_1} = y_{u_3} - y_{u_2},
\end{align*}
for each 4-cycle $(u_1, u_2, u_3, u_4)$ in $G_n$; this forces all 4-cycles to be realized as parallelograms. It is possible that some realizations in the flex $\alpha_n$ will not correspond to a congruent copy in $X$, since a parallelogram can fold to an antiparallelogram. Nevertheless, if a 4-cycle is at some point in the flex realized as an antiparallelogram instead of a congruent copy in $X$, there is a minimum value of parameter $t$ such that all vertices of the 4-cycle are collinear. By the assumption that the parallelograms in a P-framework are non-degenerate, this value $t$ is positive. Let $\varepsilon_n$ be the minimum of such values over all 4-cycles in $G_n$ having this behavior; since $G_n$ is finite, there are only finitely many 4-cycles, hence $\varepsilon_n > 0$. Hence, the realizations in the restriction of the flex to $[0, \varepsilon_n)$ have a congruent realization in $X$. By Lemma 2.7, for every $n \in \mathbb{N}$, there is a NAC-coloring $\delta_n$ of $G_n$ such that $\delta_n(e_1) = \text{blue}$ and $\delta_n(e_2) = \text{red}$. Moreover, each ribbon in the constructed active NAC-coloring is monochromatic since $W_{u_1, u_2} = W_{u_3, u_4}$ for a 4-cycle $(u_1, u_2, u_3, u_4)$ by (3). Now let
\[ S_n = \{ \delta \in \text{NAC}_{G_n} : \text{all ribbons are monochromatic in } \delta \text{ and } \delta(e_1) = \text{blue} \neq \delta(e_2) \} \]
for every $n \in \mathbb{N}$. By the discussion above, $\delta_n \in S_n$, namely, $S_n$ is non-empty for every $n \in \mathbb{N}$. Let $<$ be the relation in $\bigcup_{n \in \mathbb{N}} S_n$ given by restriction to edges of the smaller graph. Clearly, a restriction of a NAC-coloring to a subgraph is a NAC-coloring, provided that the obtained coloring is surjective, which is guaranteed by the condition $\delta(e_1) = \text{blue} \neq \delta(e_2)$. Since a ribbon in a subgraph is a subset of a ribbon of the supergraph, the property that ribbons are monochromatic holds for restrictions as well. Hence, the assumption of Lemma 3.3 holds. Therefore, we get a sequence of NAC-colorings of $(G_n)_{n \in \mathbb{N}}$ extending each other.

\[**It is not clear whether we can choose each $G_n$ to be a ribbon-cutting graph. However, our proof does not require this is true.\]
By the same reason as in Lemma 3.4, this sequence gives a NAC-coloring for $G$. Moreover, each ribbon of the NAC-coloring is monochromatic, otherwise two edges of the ribbon with different colors together with all edges certifying that they are in the same ribbon are eventually contained in some $G_n$ in the same ribbon, which contradicts the assumption imposed in the definition of $S_n$. Hence, the NAC-coloring is cartesian by Lemma 4.9.

(ii) $\Leftrightarrow$ (iii): The proof of this for finite P-frameworks (see [12, Theorem 4.5]) extends to countably infinite P-frameworks.

In [26], Power proved that a parallelogram framework $(G, \rho)$ (a P-framework that is the 1-skeleton of a parallelogram tiling of the plane) has a connected bracing graph if and only if it is infinitesimally rigid; i.e. for every map $\mu : V_G \to \mathbb{R}^2$ with

$$(\rho(u) - \rho(v)) \cdot (\mu(u) - \mu(v)) = 0$$

for all $uv \in E_G$,

there exists a real-valued $2 \times 2$ skew-symmetric matrix $S$ and a point $q \in \mathbb{R}^2$ so that $\mu(v) = S\rho(v) + q$ for all $v \in V_G$. While infinitesimal rigidity is a sufficient condition for rigidity for finite frameworks (see [1]), and is also a sufficient condition for rigidity for generic countably infinite frameworks in the plane (see [17, Theorem 4.1]), this is not the case for non-generic frameworks; see Figure 5 for an example of an infinitesimally rigid but flexible framework. Since all parallelogram frameworks are non-generic, a weaker version of Theorem 1.2 for countably infinite parallelogram frameworks does not directly follow from Power’s result. However, we can obtain the following result by combining Power’s result with Theorem 1.2.

**Corollary 4.10.** A parallelogram framework is rigid if and only if it is infinitesimally rigid.

![Figure 5](image-url)

Figure 5: An infinite framework that is infinitesimally rigid but flexible, as shown in [16, Section 5]. Any generic realization of the graph is infinitesimally rigid. The graph has infinitely many NAC-colorings, however the unique cartesian NAC-coloring (up to switching the colors of all of the edges) is the coloring shown. With the construction given in Proposition 3.1 applied to the unique cartesian NAC-coloring, we can obtain an infinite P-framework.

A major obstacle to extending Corollary 4.10 to P-frameworks is that Power’s proof requires that the union of any two ribbons contains at most one 4-cycle. While this extra condition holds for parallelogram frameworks, it does not for all P-frameworks; for example, the union of any two distinct ribbons of 1-skeleton of the cube is exactly two disjoint 4-cycles; see Figure 6.
5 Bracing Penrose frameworks

We shall apply the results of Section 4 to the family of infinite P-frameworks generated by the Penrose (rhombus) tilings; see Figure 7. Any Penrose tiling is formed from two types of rhombi with sides marked by arrows (see Figure 8): a “fat” rhombus tile $F$ with angles $72^\circ$ and $108^\circ$, and a “thin” rhombus tile $T$ with angles $36^\circ$ and $144^\circ$. To construct a Penrose tiling, one simply needs to match copies of the tiles together so that the arrows they meet at are oriented in the same direction and have the same type (i.e. dashed or not dashed). The copies of the two types of tiles can be rotated, reflected and translated however you like. For more details on the described construction, we refer the reader to [3]. Interestingly, while there exist uncountably many distinct tilings (modulo rotations, reflections and translations) that can be constructed by this method, there exist exactly two which have 5-fold rotational symmetry. Both of these properties can be proven by de Bruijn’s method of implementing pentagrids; see [3, 4] for more information on the topic.

By considering the shared sides of tiles in a Penrose tiling as edges and the shared corners as vertices, it is easy to see how we can transform any Penrose tiling into an infinite P-framework. Any such P-framework that we obtain from this method shall be called a Penrose framework.

Let $(G, \rho)$ be a P-framework with ribbon $r$. As every edge in a ribbon is parallel, we can choose a unit vector $u_r$ so that $u_r$ is orthogonal to $\rho(u) - \rho(v)$ for all $uv \in r$; we call any such unit vector a direction of $r$. It is immediate that every ribbon has exactly
two directions: $u_r$ and $-u_r$. We say two ribbons $r, r'$ with directions $u_r, u_{r'}$ respectively are parallel if $u_r$ and $u_{r'}$ are parallel. The following result can be proven by de Bruijn’s method of implementing pentagrids [3, 4].

**Lemma 5.1.** Let $(G, \rho)$ be a Penrose framework with ribbon graph $\Gamma$. Then the following properties hold.

(i) Two ribbons are not adjacent in $\Gamma$ if and only if they are parallel.

(ii) There exist 5 cyclically ordered unit vectors $u_1, \ldots, u_5$ where the angle between $u_i \mod 5$ and $u_{i+1} \mod 5$ is $2\pi/5$, so that any ribbon $r$ has direction $u_i$ for some $i$.

(iii) For each unit vector $u_i$, there exists a countably infinite amount of ribbons with direction $u_i$.

(iv) If $r$ is a ribbon with direction $u_i \mod 5$ and $r'$ is a ribbon with direction $u_{i+1} \mod 5$, then there exists a unique 4-cycle in $r \cup r'$, and its image under $\rho$ is a copy of the tile $F$.

(v) If $r$ is a ribbon with direction $u_i \mod 5$ and $r'$ is a ribbon with direction $u_{i+2} \mod 5$, then there exists a unique 4-cycle in $r \cup r'$, and its image under $\rho$ is a copy of the tile $T$.

Similar to the infinite braced grid, we can achieve rigidity by simply bracing every parallelogram in two ribbons that are adjacent in the ribbon graph.

**Corollary 5.2.** Let $(G, \rho)$ be a braced Penrose framework formed by bracing every rhombus in any two ribbons. Then $(G, \rho)$ is flexible if and only if the two ribbons are parallel.

**Proof.** Let $r_1, r_2$ be the braced ribbons and let $\Gamma_b$ be the bracing graph of $(G, \rho)$. By Lemma 5.1, each ribbon $r_i$ is adjacent in $\Gamma_b$ to every ribbon it is not parallel to. If $r_1, r_2$ are not parallel, then $r_1 r_2$ is an edge of $\Gamma_b$ and every other vertex in $\Gamma_b$ is adjacent to $r_1$ or $r_2$, hence $(G, \rho)$ is rigid by Theorem 1.2. If $r_1, r_2$ are parallel, then every ribbon parallel to both $r_1, r_2$ will be an isolated vertex in $\Gamma_b$, hence $(G, \rho)$ is flexible by Theorem 1.2.

The following result was proven by Power for infinitesimal rigidity [26]. It follows then that we can achieve the following result by combining Power’s result with Corollary 4.10, however we have included a proof for the sake of completeness.
Corollary 5.3. Let \((G, \rho)\) be a braced Penrose framework formed by bracing every rhombus that corresponds to the tile \(F\), or by bracing every rhombus that corresponds to the tile \(T\). Then \((G, \rho)\) is rigid.

Proof. Define the graph \(H = (V_H, E_H)\), where \(V_H\) is the disjoint union of countably infinite sets \(V_1, \ldots, V_5\), and \(uv \in E_H\) for \(u \in V_i, v \in V_j\) if and only if \((i - j) \equiv \pm 1 \mod 5\). By Lemma 5.1, the bracing graph of \((G, \rho)\) is isomorphic to \(H\), irrespective of whether we are bracing every rhombus corresponding to the tile \(F\) or to the tile \(T\). As the graph \(H\) is connected it follows that \((G, \rho)\) is rigid by Theorem 1.2. \(\Box\)

In fact, as we shall see with the following two results, we do not need to brace all the rhombi corresponding to a given type of tile to obtain a rigid braced Penrose framework.

Corollary 5.4. Let \(P\) be a Penrose tiling. Choose a tile type \(S \in \{F, T\}\) and a rotational orientation \(\sigma\) so that \(P\) contains a translate of the tile \(\sigma S\). Let \((G, \rho)\) be a braced Penrose framework formed by bracing every rhombus that corresponds to the tile \(S\) except those that are translates of \(\sigma S\). Then \((G, \rho)\) is rigid.

Proof. Define the graph \(H = (V_H, E_H)\), where \(V_H\) is the disjoint union of countably infinite sets \(V_1, \ldots, V_5\), and \(uv \in E_H\) for \(u \in V_i, v \in V_j\) if and only if \(|i - j| = 1\). By Lemma 5.1, the bracing graph of \((G, \rho)\) is isomorphic to \(H\). As the graph \(H\) is connected it follows that \((G, \rho)\) is rigid by Theorem 1.2. \(\Box\)

Corollary 5.5. Fix a tile \(S \in \{F, T\}\). Let \((G_p, \rho)\) be a braced Penrose framework formed by randomly bracing every rhombus that corresponds to the tile \(S\) with probability \(p \in (0, 1]\). Then \((G_p, \rho)\) is almost surely rigid.

Proof. Define the graph \(H = (V_H, E_H)\), where \(V_H\) is the disjoint union of countably infinite sets \(V_1, \ldots, V_5\), and \(uv \in E_H\) for \(u \in V_i, v \in V_j\) if and only if \((i - j) \equiv \pm 1 \mod 5\). By Lemma 5.1, the bracing graph of \((G_1, \rho)\) is isomorphic to \(H\). Now define \(H_p\) be a spanning subgraph of \(H\) that corresponds to the bracing graph of \((G_p, \rho)\) under the isomorphism between the ribbon graph and \(H\). It follows from Theorem 1.2 that we need only show that \(H_p\) is connected. Note that the probability that an edge of \(H\) is also an edge of \(H_p\) is \(p\).

Choose two vertices \(u, v \in V_H\). We may assume by relabelling if necessary that \(u \in V_1\) and \(v \in V_t\) for some \(t \in \{1, 2, 3\}\). First suppose that \(t \in \{1, 3\}\). The probability of a vertex \(w \in V_2\) being adjacent to both \(u\) and \(v\) is \(p^2\), hence the probability that there exists a path from \(u\) to \(v\) in \(H_p\) is at least \(\lim_{n \to \infty} 1 - (1 - p^2)^n = 1\). Now suppose that \(t = 2\). Given vertices \(w_1 \in V_1\) and \(w_2 \in V_2\), the probability that \((u, w_2, w_1, v)\) is a path in \(H_p\) is \(p^3\), hence the probability that there exists a path from \(u\) to \(v\) in \(H_p\) is at least \(\lim_{n \to \infty} 1 - (1 - p^3)^n = 1\). \(\Box\)

6 Rotationally symmetric infinite frameworks

In this section, we extend the main result of [7] to infinite graphs. We recall the definitions of \(k\)-fold rotational symmetry, where \(k \geq 2\).
Definition 6.1. Let $G$ be a graph and let the group $C_k := \langle \omega : \omega^k = 1 \rangle$ act on $G$, namely, there exists an injective group homomorphism $\varphi : C_k \to \text{Aut}(G)$. We define $\gamma v := \varphi(\gamma)(v)$ for $\gamma \in C_k$ and $\gamma e := \gamma v \gamma v$ for any edge $e = uv \in E_G$. A vertex $v \in V_G$ is an invariant vertex if $\gamma v = v$ for all $\gamma \in C_k$, and partially invariant if $\gamma v = v$ for some $\gamma \in C_k$, $\gamma \neq 1$. The graph $G$ is called $C_k$-symmetric if every partially invariant vertex is invariant, and the set of invariant vertices of $G$ forms an independent set.

Definition 6.2. A realization $\rho$ of a $C_k$-symmetric graph $G$ in $\mathbb{R}^2$ is called $C_k$-symmetric if $\rho(\gamma v) = \Theta_\gamma \rho(v)$ for each $v \in V_G$ and $\gamma = \omega^i \in C_k$, where $\Theta_\omega$ is the $2\pi/k$ clockwise rotation matrix and $\Theta_\gamma = \Theta_\omega^i$. The framework $(G, \rho)$ is $C_k$-symmetric if $\rho$ is $C_k$-symmetric.

Notice that the requirements on invariant vertices to form an independent set is justified by the fact that they must be mapped to the origin in a $C_k$-symmetric framework and we disallow edges to have zero length.

Definition 6.3. If there is a non-trivial flex $\rho_t$ of a $C_k$-symmetric framework $(G, \rho)$ such that each $\rho_t$ is $C_k$-symmetric, then $(G, \rho)$ is $C_k$-symmetric flexible (or $k$-fold rotation symmetric flexible), and $C_k$-symmetric rigid otherwise.

Definition 6.4. Let $\delta$ be a NAC-coloring of a $C_k$-symmetric graph $G$. A red component $H$, i.e., a connected component of $G^\delta_{\text{red}}$, is partially invariant if there exists $\gamma \in C_k \setminus \{1\}$ such that $\gamma H = H$, and invariant if $\gamma H = H$ for all $\gamma \in C_k$. We define blue partially invariant components and blue invariant components analogously. The NAC-coloring $\delta$ is called $C_k$-symmetric if $\delta(\gamma e) = \delta(e)$ for all $e \in E_G$ and $\gamma \in C_k$ and no two distinct blue, resp. red, partially invariant components are connected by an edge.

Similarly to standard NAC-colorings, $C_k$-symmetric NAC-colorings can be used to determine whether a $C_k$-symmetric graph has a $C_k$-symmetric flexible realization.

Theorem 6.5 ([7]). A finite $C_k$-symmetric connected graph has a $C_k$-symmetric NAC-coloring if and only if it has a $C_k$-symmetric flexible realization in $\mathbb{R}^2$.

We shall now extend this result to infinite $C_k$-symmetric graphs. We begin with the following construction that was originally outlined in [7] for finite $C_k$-symmetric graphs.

Proposition 6.6. Let $G$ be a countably infinite $C_k$-symmetric graph. If $G$ has a $C_k$-symmetric NAC-coloring, then there is a $C_k$-symmetrically flexible realization of $G$.

Proof. Let $R^0_1, \ldots, R^{k-1}_1, R^0_2, \ldots, R^{k-1}_2, \ldots$ (resp. $B^0_1, \ldots, B^{k-1}_1, \ldots$) be the red (resp. blue) components of $G^\delta_{\text{red}}$ (resp. $G^\delta_{\text{blue}}$) that are not partially invariant. We can assume that $R^i_j = \omega^i R^0_j$ and $B^i_j = \omega^i B^0_j$ for $0 \leq i < k$ and $1 \leq j$.

Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be points in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $a_j \neq \Theta_\omega a_j$, and $b_j \neq \Theta_\omega b_j$ for $j \neq j'$ and $0 \leq i < k$ arbitrary. We define functions $\overline{a}, \overline{b} : V_G \to \mathbb{R}^2$ by

$$\overline{a}(v) = \begin{cases} \Theta_\omega a_j & \text{if } v \in R^i_j \\ (0,0) & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{b}(v) = \begin{cases} \Theta_\omega b_j & \text{if } v \in B^i_j \\ (0,0) & \text{otherwise} \end{cases}$$
Notice that a vertex is mapped to \((0,0)\) if and only if it is in a partially invariant component. We now construct realizations \(p_t\) of \(G\) by setting
\[
p_t(v) := \Theta(t)\overrightarrow{a}(v) + \overrightarrow{b}(v),
\]
where \(\Theta(t)\) is the clockwise rotation matrix by \(t\) radians. By the same arguments as in [7, Lemma 2], we can see that this gives a \(C_k\)-symmetric flex.

We will now need to define a special type of subgraph tower.

**Definition 6.7.** Let \(G\) be a \(C_k\)-symmetric graph with a subgraph tower \((G_n)_{n\in\mathbb{N}}\). We say the subgraph tower is \(C_k\)-symmetric if for every subgraph \(G_n\), the restriction of every automorphism \(\varphi(\gamma)\) in the image of \(\varphi\) to \(G_n\) is also an automorphism of \(G_n\).

**Lemma 6.8.** Let \(G\) be a countably infinite \(C_k\)-symmetric graph with a \(C_k\)-symmetric subgraph tower \((G_n)_{n\in\mathbb{N}}\). Let \(e_1, e_2\) be two edges of \(G_1\). If there is a \(C_k\)-symmetric NAC-coloring \(\delta_n\) of \(G_n\) such that \(\delta_n(e_1) \neq \delta_n(e_2)\) for all \(n \in \mathbb{N}\), then \(G\) has a \(C_k\)-symmetric NAC-coloring.

**Proof.** Let
\[
S_n = \{\delta \in \text{NAC}_{G_n} : \delta \text{ is } C_k\text{-symmetric and } \delta(e_1) = \text{blue } \neq \delta(e_2)\}
\]
for every \(n \in \mathbb{N}\). Each \(S_n\) is non-empty since either \(\delta_n\) is in \(S_n\), or the NAC-coloring obtained by swapping the colors from \(\delta_n\) is in \(S_n\). Let \(\prec\) be the relation in \(\bigcup_{n\in\mathbb{N}} S_n\) given by restriction to edges of the smaller graph. Clearly, a restriction of a NAC-coloring that is invariant under the \(C_k\) group action to a subgraph is a NAC-coloring that is invariant under the \(C_k\) group action, provided that the obtained coloring is surjective, which is guaranteed by the condition \(\delta(e_1) = \text{blue } \neq \delta(e_2)\). Notice that a monochromatic component \(H\) is partially invariant if and only if the orbit of each non-invariant vertex in \(H\) contains at least two elements from \(V_H\). Since the restriction removes all vertices in the same orbit, the restriction of a monochromatic component cannot be partially invariant unless the unrestricted component is partially invariant. This guarantees that no two partially invariant monochromatic components in the restriction are connected by an edge, namely, the NAC-coloring obtained by the restriction is \(C_k\)-symmetric. Hence, the assumption of Lemma 3.3 holds. Therefore, we get a sequence of \(C_k\)-symmetric NAC-colorings of \((G_n)_{n\in\mathbb{N}}\) extending each other.

This sequence of \(C_k\)-symmetric NAC-colorings gives a NAC-coloring for \(G\) by the same reasons as in Lemma 3.4. It is immediate that the NAC-coloring will also be invariant under the \(C_k\) group action. Any such NAC-coloring will also be \(C_k\)-symmetric, since for every two vertices \(u, v\) contained in partially invariant monochromatic components, we can restrict to a \(G_n\) containing both \(u\) and \(v\) to see that \(u\) and \(v\) are non-adjacent.

**Theorem 6.9.** A \(C_k\)-symmetric connected countably infinite graph has a \(C_k\)-symmetric NAC-coloring if and only if it has \(C_k\)-symmetric flexible realization in \(\mathbb{R}^2\).
Proof. If $G$ has a $C_k$-symmetric NAC-coloring then there is a $C_k$-symmetric flexible realization of $G$ by Proposition 6.6. Now suppose $G$ has a $C_k$-symmetric realization with a non-trivial $C_k$-symmetric flex $\alpha$. Since $\alpha$ is non-trivial, there are edges $e_1, e_2 \in E_G$ whose angle changes along the flex. We choose any $C_k$-symmetric subgraph tower $(G_n)_{n \in \mathbb{N}}$ such that $e_1, e_2 \in E_{G_n}$. Clearly, the restriction of the flex $\alpha$ to $V_{G_n}$ is a $C_k$-symmetric flex of $G_n$ such that the angle between $e_1$ and $e_2$ changes. Hence by Lemma 2.7 with the restricting set given by $y_u = 0$ for some non-invariant vertex $u \in V_G$ and

$$x_{uv} = \cos(2\pi/k)x_v + \sin(2\pi/k)y_v, \quad y_{uv} = -\sin(2\pi/k)x_v + \cos(2\pi/k)y_v \quad (4)$$

for every $v \in V_G$, there is a NAC-coloring $\delta_n$ of $G_n$ such that $\delta_n(e_1) \neq \delta_n(e_2)$ for all $n \in \mathbb{N}$. By the technique of [7, Lemma 1], every $\delta_n$ is $C_k$-symmetric. A NAC-coloring of $G$ is now provided by Lemma 6.8. \qed

7 Rotationally symmetric P-frameworks

We now wish to combine the results of Sections 4 and 6. There are, however, some technical results we need to first cover. We begin with the two following lemmas which describe how ribbon edge cuts behave. The results can seen to be direct extensions of Lemma 3.2 and Lemma 3.6 of [12] to the setting of infinite P-frameworks.

Lemma 7.1. Let $(G, \rho)$ be a (countably infinite) P-framework. If $r$ is a ribbon of $G$, then $G \setminus r$ has exactly two connected components.

Lemma 7.2. Let $(G, \rho)$ be a (countably infinite) P-framework with a ribbon $r$. Let $V_1 \cup V_2$ be the vertex set of $r$, where all the vertices of $V_i$ lie in the same connected component of $G \setminus r$. Then for every pair of edges $u_1u_2, v_1v_2 \in r$ with $u_i, v_i \in V_i$, we have

$$\rho(u_1) - \rho(u_2) = \rho(v_1) - \rho(v_2).$$

Our next key result tells us that $C_k$-symmetric NAC-colorings can be defined much more simply for the graphs of P-frameworks.

Lemma 7.3. Let $(G, \rho)$ be a $C_k$-symmetric connected (countably infinite) P-framework, and let $\delta$ be a cartesian NAC-coloring of $G$. If $\delta(\gamma e) = \delta(e)$ for all $e \in E_G$ and $\gamma \in C_k$, then $\delta$ is a $C_k$-symmetric NAC-coloring of $G$.

Proof. Suppose without loss of generality that there exists two red partially invariant components $A_1$ and $A_2$ which are connected by a blue edge $u_1u_2$. Define $r$ to be the ribbon containing $u_1u_2$. Define $\gamma_i$ to be a group element that $A_i$ is invariant under. Since $u_1$ and $\gamma_1u_1$ lie in the same component, there exists a path $P_i \subset E_G$ in the induced subgraph on $A_i$ from $u_1$ to $\gamma_1u_1$. We note that for every integer $j$, we have $\gamma_2(\gamma_1\gamma_2)^{-1}A_1 = \gamma_2^j A_1$ connected to $(\gamma_1\gamma_2)^jA_2 = \gamma_2^j A_2$ by the edge $((\gamma_1\gamma_2)^ju_1)((\gamma_1\gamma_2)^ju_2)$, and we have $(\gamma_1\gamma_2)^jA_2 = \gamma_2^j A_2$ connected to $\gamma_2(\gamma_1\gamma_2)^jA_1 = \gamma_2 A_1$ by the edge $(\gamma_2(\gamma_1\gamma_2)^ju_1)(\gamma_2(\gamma_1\gamma_2)^ju_2)$. As the order of $\gamma_1\gamma_2$ is finite, there exists a circuit $C$ given by the sequence

$$u_1u_2, P_2, (\gamma_2u_2)(\gamma_2u_1), \gamma_2P_1, (\gamma_1\gamma_2u_1)(\gamma_1\gamma_2u_2), \gamma_1\gamma_2P_2 \ldots$$

$$\ldots, (\gamma_2(\gamma_1\gamma_2)^{m-1}u_2)(\gamma_2(\gamma_1\gamma_2)^{m-1}u_1), \gamma_2(\gamma_1\gamma_2)^{m-1}P_1, u_1u_2$$
for some positive integer \( m \). By Lemma 7.1, \( G \setminus r \) has exactly two connected components, one containing \( u_1 \) and the other containing \( u_2 \). Because \( u_1 u_2 \) is contained in the circuit \( C \), the ribbon \( r \) must contain an edge of \( C \setminus \{u_1 u_2\} \). As each edge in each path \( \gamma_2(\gamma_1 \gamma_2)\delta P_1 \) and \( (\gamma_1 \gamma_2)\delta P_2 \) is red and each edge of \( r \) is blue (Lemma 4.9), the ribbon \( r \) must contain an edge \( (\gamma u_1)(\gamma u_2) \) for some \( \gamma \in \{\gamma_1, \gamma_2\} \setminus \{1\} \). As distinct ribbons do not intersect each other, we have \( \gamma r = r \). The only possible two symmetries that a ribbon of a P-framework can have are \( C_2 \)-symmetric rotational symmetry and reflectional symmetry. Hence \( \Theta_\gamma \) is the \( \pi \) rotation matrix, i.e. \( \gamma^2 = 1 \) and \( \gamma \neq 1 \).

If both \( \gamma_1 \) and \( \gamma_2 \) have odd orders then, since \( C_k \) is a cyclic group, we have \( \langle \gamma_1, \gamma_2 \rangle \) has an odd order\(^1\), contradicting that \( \gamma \in \langle \gamma_1, \gamma_2 \rangle \). Without loss of generality, we may assume the order of \( \gamma_1 \) is even. Hence, \( \langle \gamma_1 \rangle \) contains an element which is its own inverse, namely, it corresponds to rotation by \( \pi \). Hence \( \gamma \in \langle \gamma_1 \rangle \) and \( \gamma A_1 = A_1 \); in particular, \( u_1, \gamma u_1 \in A_1 \). Since \( \rho(\gamma u_1) - \rho(\gamma u_2) = \rho(u_2) - \rho(u_1) \), we have by Lemma 7.2 that \( u_1 \) and \( \gamma u_1 \) lie in separate connected components of \( G \setminus r \). However, this contradicts that every pair of vertices in \( A_1 \) are connected by a path in \( G \setminus r \); this follows from noting that no edge of \( r \) can lie in \( A_1 \), since every edge in \( r \) is blue by Lemma 4.9.

The most technical part of this section is the construction of a \( C_k \)-symmetric flex.

**Lemma 7.4.** If a braced \( C_k \)-symmetric (countably infinite) P-framework has a cartesian \( C_k \)-symmetric NAC-coloring, then it is \( C_k \)-symmetrically flexible.

**Proof.** We extend the construction introduced in [12] to countably infinite frameworks and show that it yields a \( C_k \)-symmetric flex for both finite and infinite case. Let \( (G, \rho) \) be a braced \( C_k \)-symmetric (countably infinite) P-framework. Suppose that \( G \) has a cartesian \( C_k \)-symmetric NAC-coloring \( \delta \). Let \( G' \) be the underlying unbraced subgraph of \( G \).

Let \( \bar{u} \) be any vertex of \( G \). Let \( R_1, R_2, \ldots \) (resp. \( B_1, B_2, \ldots \)) be the vertex sets of the connected components of \( G_{\text{red}}^\delta \) (resp. \( G_{\text{blue}}^\delta \))\(^2\), and let \( \bar{R}, \bar{B}, \) be the red, resp. blue component containing \( \bar{u} \). Note that the vertex sets of these components are identical in \( G \) and \( G' \). We define a map \( \rho_{\text{red}} : \{R_1, R_2, \ldots\} \rightarrow \mathbb{R}^2 \) as follows: for \( R_\ell \), let \( W \) be any walk in the unbraced graph \( G' \) from \( \bar{u} \) to a vertex in \( R_\ell \) and

\[
\rho_{\text{red}}(R_\ell) = \sum_{(u_1, u_2) \in W} (\rho(u_2) - \rho(u_1)).
\]

Notice that [12, Lemma 4.3] (or its infinite analogue) guarantees that \( \rho_{\text{red}} \) is well-defined, namely, the sum is independent of the choice of \( W \) and the vertex in \( R_\ell \). We define \( \rho_{\text{blue}} : \{B_1, \ldots, B_n\} \rightarrow \mathbb{R}^2 \) analogously by swapping red and blue.

\(^1\)As \( C_k \) is cyclic, there exists \( \ell \) such that \( \langle \gamma_1, \gamma_2 \rangle = \langle \omega^\ell \rangle \). If we choose \( \ell \) to be the smallest positive integer with this property, then \( k/\ell \) is the order of \( \langle \gamma_1, \gamma_2 \rangle \). Given \( k \), is the smallest positive integer so that \( \langle \gamma_i \rangle = \langle \omega^{k_i} \rangle \) for each \( i \), we note that \( k/k_i \) is the order of \( \langle \gamma_i \rangle \) and, by Bézout’s identity, \( \ell = \gcd(k_1, k_2) \). If we denote \( k = a2^b \) for some odd \( a \) then, as \( k_1, k_2 \) have odd order, we must have \( k_1 = a_12^{b_1} \) and \( k_2 = a_22^{b_2} \) for odd \( a_1, a_2 \), since both \( k/k_1 \) and \( k/k_2 \) are odd. Hence \( \ell = c2^c \) for some odd \( c \) and \( k/\ell \) is odd.

\(^2\)Although we use “...”, it might happen that there are only finitely many such components (for instance if \( G \) is finite).

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For $t \in [0, 2\pi]$ and $v \in V_G = V_{G'}$, where $v \in R_t \cap B_j$, let

$$\rho_t(v) = \Theta(t) \cdot \rho_{\text{red}}(R_t) + \rho_{\text{blue}}(B_j),$$

where $\Theta(t)$ is the clockwise rotation matrix by $t$ radians. The proof that $\rho_0(v) = \rho(v) - \rho(\bar{u})$ is the same as in [12, Lemma 7.5] as well as the fact that the lengths of edges in $E_G$ are non-zero and constant along the flex. If $\bar{u}$ is an invariant vertex, then one can show that $\rho_t$ is $C_k$-symmetric, but this is not the case in general. Hence, define the flex $\tilde{\rho}_t$ so that for each $v \in V_G$ we have

$$\tilde{\rho}_t(v) = \rho_t(v) - \frac{1}{k} \sum_{i=0}^{k-1} \rho_t(\omega^i \bar{u})$$

$$= \Theta(t) \cdot \rho_{\text{red}}(R_t) + \rho_{\text{blue}}(B_j) - \frac{1}{k} \sum_{i=0}^{k-1} \left( \Theta(t) \cdot \rho_{\text{red}}(\omega^i \bar{R}) + \rho_{\text{blue}}(\omega^i \bar{B}) \right).$$

Since the sum is independent of $v$, we have that the edge lengths are constant along the flex $\tilde{\rho}_t$ as it is so for $\rho_t$. Recall that $\Theta_\omega$ is the clockwise rotation matrix $\Theta(2\pi/k)$. As

$$\frac{1}{k} \sum_{i=0}^{k-1} \rho_0(\omega^i \bar{u}) = \frac{1}{k} \sum_{i=0}^{k-1} \left( \rho(\omega^i \bar{u}) - \rho(\bar{u}) \right) = -\rho(\bar{u}) + \left( \frac{1}{k} \sum_{i=0}^{k-1} \Theta_i^\omega \right) \rho(\bar{u}) = -\rho(\bar{u}),$$

and $\rho_t$ is a flex of the translation of $(G, \rho)$ by $\rho(\bar{u})$, we have that $\tilde{\rho}_t$ is a flex of $\rho$.

In order to prove that the flex $\tilde{\rho}_t$ is $C_k$-symmetric, we need the following identity (and analogous identity for $\rho_{\text{blue}}$):

$$k \cdot \rho_{\text{red}}(\omega \bar{R}) = \sum_{i=0}^{k-1} \left( \rho_{\text{red}}(\omega^i \bar{R}) - \Theta_\omega \cdot \rho_{\text{red}}(\omega^i \bar{R}) \right). \quad (5)$$

The 0-th summand is 0 since $\bar{u} \in \bar{R}$. If $W$ is a walk from $\bar{u}$ to $\omega \bar{u}$ in $G'$, then $\bar{W}_i = (W, \omega W, \ldots, \omega^{i-1} W)$ is a walk from $\bar{u}$ to $\omega^i \bar{u}$ in $G'$. Hence, the $i$-th summand on the right hand side for $i \geq 1$ equals

$$\sum_{(w_1, w_2) \in \bar{W}_i, \delta(w_1 w_2) = \text{blue}} (\rho(w_2) - \rho(w_1)) - \sum_{(w_1, w_2) \in \bar{W}_i, \delta(w_1 w_2) = \text{blue}} (\rho(\omega w_2) - \rho(\omega w_1))$$

$$= \sum_{(w_1, w_2) \in W, \delta(w_1 w_2) = \text{blue}} \sum_{j=0}^{i-1} \left( (\rho(\omega^j w_2) - \rho(\omega^j w_1)) - (\rho(\omega^{j+1} w_2) - \rho(\omega^{j+1} w_1)) \right)$$

$$= \sum_{(w_1, w_2) \in W, \delta(w_1 w_2) = \text{blue}} \left( (\rho(w_2) - \rho(w_1)) - (\rho(\omega^i w_2) - \rho(\omega^i w_1)) \right).$$
Notice that the last expression is zero for $i = 0$. Therefore, the right hand side of (5) is
\[
\sum_{i=0}^{k-1} \sum_{\substack{(u_1,u_2)\in W \delta(u_1 u_2) = \text{blue}}} \left( (\rho(w_2) - \rho(w_1)) - (\rho(\omega^i w_2) - \rho(\omega^i w_1)) \right) \\
= \sum_{i=0}^{k-1} \rho_{\text{red}}(\omega \bar{R}) - \sum_{\substack{(u_1,u_2)\in W \delta(u_1 u_2) = \text{blue}}} \sum_{i=0}^{k-1} (\rho(\omega^i w_2) - \rho(\omega^i w_1)),
\]
which concludes the proof of the identity. Finally, for $v \in R_\ell \cap B_j$ and walks $W_1$ from $\bar{u}$ to $\omega \bar{u}$ and $W$ from $\bar{u}$ to $v$ in $G'$ (hence $\omega W$ is from $\omega \bar{u}$ to $\omega v$) we have that $\rho_{\text{red}}(\omega R_\ell)$ is equal to
\[
\sum_{(u_1,u_2)\in W_1 \delta(u_1 u_2) = \text{blue}} (\rho(w_2) - \rho(w_1)) + \sum_{(u_1,u_2)\in \omega W \delta(u_1 u_2) = \text{blue}} (\rho(w_2) - \rho(w_1)) = \rho_{\text{red}}(\omega \bar{R}) + \Theta_\omega \rho_{\text{red}}(R_\ell),
\]
and similarly for $\rho_{\text{blue}}(\omega B_j)$. As
\[
\Theta(t) \left( \rho_{\text{red}}(\omega R_\ell) - \frac{1}{k} \sum_{i=0}^{k-1} \rho_{\text{red}}(\omega^i \bar{R}) \right) + \rho_{\text{blue}}(\omega B_j) - \frac{1}{k} \sum_{i=0}^{k-1} \rho_{\text{blue}}(\omega^i B)
\]
\[
= \Theta(t) \left( \Theta_\omega \rho_{\text{red}}(R_\ell) + \rho_{\text{red}}(\omega \bar{R}) - \frac{1}{k} \sum_{i=0}^{k-1} \rho_{\text{red}}(\omega^i \bar{R}) \right) \\
+ \Theta_\omega \rho_{\text{blue}}(B_j) + \rho_{\text{blue}}(\omega \bar{B}) - \frac{1}{k} \sum_{i=0}^{k-1} \rho_{\text{blue}}(\omega^i \bar{B})
\]
\[
(5) \Theta(t) \left( \Theta_\omega \rho_{\text{red}}(R_\ell) - \Theta_\omega \frac{1}{k} \sum_{i=0}^{k-1} \rho_{\text{red}}(\omega^i \bar{R}) \right) + \Theta_\omega \rho_{\text{blue}}(B_j) - \Theta_\omega \frac{1}{k} \sum_{i=0}^{k-1} \rho_{\text{blue}}(\omega^i \bar{B}),
\]
we have $\tilde{\rho}_t(\omega v) = \Theta_\omega \tilde{\rho}_t(v)$ as required. \qed

One aim of this section is to be able to determine whether a $C_4$-symmetric braced P-framework is $C_4$-flexible solely by its bracing graph. Our case example is the $2 \times 2$ grid shown in Figure 9. While it does have a unique NAC-coloring (up to switching all red and blue edge colors) that is invariant under the group action of $C_4$ as shown, the NAC-coloring is neither cartesian nor $C_4$-symmetric; the two blue components, the central vertex and the outer cycle, are both invariant but are connected by four red edges. It follows from Theorems 1.2 and 6.5 that it cannot be $C_4$-symmetrically flexible. However the bracing graph is disconnected as it is just four isolated vertices. As we shall soon see, this is actually not a problem, for while the bracing graph of the P-framework is disconnected, the quotient of the bracing graph by the symmetry of the ribbons is connected since it is a single vertex. To capitalize on this observation, we define the following concept.
Figure 9: The $2 \times 2$ grid with its unique NAC-coloring (up to switching all red and blue edge colors). Its four ribbons are shown as gray dashed lines.

**Definition 7.5.** Let $(G, \rho)$ be a $C_k$-symmetric (countably infinite) braced P-framework with bracing graph $\Gamma$. We define the *quotient bracing graph* $\tilde{\Gamma}$ to be the quotient graph of $\Gamma$ under the equivalence relation $r \sim r'$ if and only if $r' = \gamma r$ for some $\gamma \in C_k$.

With this definition we are finally ready for our main result of the section. It is worth noting that this result is new for both finite and countably infinite graphs.

**Theorem 7.6.** Let $(G, \rho)$ be a $C_k$-symmetric (countably infinite) braced P-framework. Then the following are equivalent:

(i) $(G, \rho)$ is $C_k$-symmetrically rigid.

(ii) $G$ has no cartesian $C_k$-symmetric NAC-coloring.

(iii) The quotient bracing graph of $G$ is connected.

**Proof.**

$\neg(ii) \Rightarrow \neg(i)$: The statement holds by Lemma 7.4.

$\neg(i) \Rightarrow \neg(ii)$: First suppose that $(G, \rho)$ is finite. By Lemma 2.7, there exists an algebraic motion $C$ w.r.t. the restricting set given by eqs. (3) and (4), and an active NAC-coloring $\delta$ w.r.t. $C$. It was shown in [7, Lemma 1] that $\delta$ is $C_k$-symmetric, and it follows immediately from eq. (3) that $\delta$ must also be cartesian.

Now suppose $(G, \rho)$ is a flexible infinite $C_k$-symmetric P-framework. Let $\alpha : [0, 1) \rightarrow (R^2)^{V_G}$ be the corresponding non-trivial $C_k$-symmetric flex. Since the flex is non-trivial, there exist edges $e_1$ and $e_2$ such that the angle between these two edges changes along the flex. Let $(G_n)_{n \in \mathbb{N}}$ be a $C_k$-symmetric subgraph tower of $G$ such that $e_1, e_2 \in E_{G_n}$. For any $n \in \mathbb{N}$, projecting the flex $\alpha$ to $G_n$ gives a $C_k$-symmetric flex $\alpha_n$ of $G_n$. By a similar method to that used in Theorem 1.2, we may choose $\varepsilon_n \in (0, 1)$ for every $n \in \mathbb{N}$ so that $\alpha_n(t)$ is a $C_k$-symmetric P-framework for every $t \in [0, \varepsilon_n)$. For each $n \in \mathbb{N}$, apply Lemma 2.7 to the restriction of $\alpha_n$ to $[0, \varepsilon_n)$ and the restricting set given by eqs. (3) and (4) to obtain a NAC-coloring $\delta_n$ with $\delta(e_1) \neq \delta(e_2)$. Each $\delta_n$ is $C_k$-symmetric by [7, Lemma 1], and every ribbon of $G_n$ is monochromatic w.r.t. $\delta_n$ because of eq. (3). We can now implement the methods showcased in Theorem 1.2 and Lemma 6.8 with the sets

$$S_n = \{ \delta \in \text{NAC}_{G_n} : \delta \text{ is } C_k\text{-symmetric with all ribbons being monochromatic and } \delta(e_1) = \text{blue } \neq \delta(e_2) \}$$
to obtain the required cartesian $C_k$-symmetric NAC-coloring of $G$.

$(iii) \Rightarrow (ii)$: As the quotient bracing graph is connected, $\Gamma$ has at most $m$ connected components $\Gamma_1, \ldots, \Gamma_m$ for some $m \mid k$ so that $\Gamma_i = \gamma_i \Gamma_1$ for some $\gamma_i \in C_k$ for each $i \in \{1, \ldots, m\}$. By Lemma 4.9 all edges in the same ribbon have the same color, hence so do all the edges of ribbons in the same connected component of the bracing graph. Because of $C_k$ symmetry, the $C_k$-symmetric NAC-coloring must be invariant under the group symmetry. This implies every edge has the same color, contradicting that $G$ has a cartesian $C_k$-symmetric NAC-coloring.

$\neg (iii) \Rightarrow \neg (ii)$: Since the quotient NAC-coloring of $G$ is disconnected, there exists a red-blue vertex coloring $\delta_{\Gamma}$ of $\Gamma$ so that red (resp. blue) vertices are only connected to red (resp. blue) vertices, and $\delta_{\Gamma}(\gamma r) = \delta_{\Gamma}(r)$ for every $\gamma \in C_k$. Define $\delta$ to be the red-blue edge coloring where for every edge $e$ in a ribbon $r$ we have $\delta(e) = \delta_{\Gamma}(r)$. As noted in [12, Theorem 4.5], $\delta$ is a cartesian NAC-coloring of $G$, and it is immediate that $\delta(\gamma e) = \delta(e)$ for all $e \in E_G$ and $\gamma \in C_k$. By Lemma 7.3, $\delta$ is also $C_k$-symmetric as required.

It is immediate that, similar to Theorem 1.2, we can apply this result to either of the two rotationally symmetric Penrose frameworks. In particular, a non-braced 5-fold symmetric Penrose framework will have a flex that preserves its symmetry; see Figure 10.

Proof of Corollary 1.3. The quotient bracing graph of a Penrose framework with 5-fold symmetry is the countably infinite edgeless graph. The result now follows from Theorem 7.6.

Figure 10: A $C_k$-symmetric cartesian NAC-coloring yielding the flex depicted in Figure 2.

8 Flexibility of infinite complete bipartite frameworks

In [8], Dixon showed two constructions of flexible frameworks for the complete bipartite graph $K_{4,4}$. The first one (Dixon I) works for any finite bipartite graph and places the vertices of one part on $x$-axis and the vertices of the other part on the $y$-axis;
see Figure 11. The second flexible realization possesses the dihedral symmetry of a rectangle, see also [27]. More than a hundred years later, it was proved by Maehara and Tokushige [20] that an injective realization of the complete bipartite graph \( K_{m,n} \) vertices with \( m \geq 5 \) and \( n \geq 3 \) is flexible if and only if the vertices are placed as in the Dixon I construction (modulo rigid motions). Hence, Dixon I construction is the only one that can apply in the case of flexible injective realizations of the countably infinite complete bipartite graph. Contrary to the finite case, not all Dixon I realizations are flexible, as we will see in Theorem 8.2.

Let us remark that for \( K_{3,3} \), Walter and Husty [29] showed that the realizations obtained by restricting the two Dixon’s constructions for \( K_{4,4} \) are the only injective flexible ones, see also [15] for an alternative proof. However, there is also a third flexible injective realization on the sphere besides the two analogues of Dixon’s constructions [9].

**Definition 8.1.** Let \( A = \{a_n : n \in \mathbb{N}\} \) and \( B = \{b_n : n \in \mathbb{N}\} \) be vertex sets of the infinite complete bipartite graph \( K_{A,B} \). Let \( (K_{A,B}, \rho) \) be the framework with \( \rho(a_n) = (x_n, 0) \) and \( \rho(b_n) = (0, y_n) \), \( \rho \) being an injective realization. We define \( (K_{A,B}, \rho) \) to be an **infinite Dixon I linkage**.

We now illustrate that, unlike with the finite case, there exist rigid infinite Dixon I linkages.

**Theorem 8.2.** Let \( K_{A,B} \) be the complete bipartite graph with \( A \) and \( B \) countably infinite. Let \( \rho \) be an injective realization of \( K_{A,B} \). Then \( (K_{A,B}, \rho) \) is flexible if and only if it is congruent to an infinite Dixon I linkage with \( \inf_{n \in \mathbb{N}} |x_n| \neq 0 \neq \inf_{n \in \mathbb{N}} |y_n| \).

**Proof.** By the discussion above, a countably infinite complete bipartite framework can be flexible only if it is congruent to an infinite Dixon I linkage. We note that we may assume that \( x_n, y_n \geq 0 \) for all \( n \in \mathbb{N} \) since this will equate to reflecting points through the \( x- \) and \( y- \) axis. We may also assume without loss of generality that \( \inf_{n \in \mathbb{N}} x_n \geq \inf_{n \in \mathbb{N}} y_n \) by reflecting the framework with respect to the line \( y = x \). Hence our conditions to check are whether or not \( \inf_{n \in \mathbb{N}} x_n \) is positive.

First suppose that \( c := \inf_{n \in \mathbb{N}} x_n > 0 \). We show that Dixon I construction works also for the infinite case under this assumption. For each \( n \in \mathbb{N} \) and \( t \in [0, 1] \), the values \( x_n(t) := \sqrt{x_n^2 - c^2(1 - (1 - t)^2)} \) and \( y_n(t) := \sqrt{y_n^2 + c^2(1 - (1 - t)^2)} \) are well-defined.
With this, we define the continuous paths \( \alpha_t(a_n) := (x_n(t), 0) \) and \( \alpha_t(b_n) := (0, y_n(t)) \) for all \( n \in \mathbb{N} \). For any \( m, n \in \mathbb{N} \), the distance between \( \alpha_t(a_m) \) and \( \alpha_t(b_n) \) will remain constant as \( t \) varies, hence \( \alpha \) is a flex of \( (K_{A,B}, \rho) \). Since the vertices of \( A \) and \( B \) respectively are fixed to the axis, \( \alpha \) is non-trivial as required.

Now suppose that \( \inf_{n \in \mathbb{N}} x_n = 0 \). Let \( (a_{n_j})_{j \in \mathbb{N}} \) and \( (b_{n_k})_{k \in \mathbb{N}} \) be sequences where \( x_{n_j} \to 0 \) and \( y_{n_k} \to 0 \) as \( j, k \to \infty \). Suppose for contradiction that \( \alpha \) is a non-trivial flex of \( (K_{A,B}, \rho) \). We first note that by the result of Maehara and Tokushige mentioned above, we can assume that for all \( n \in \mathbb{N} \) and \( t \in [0, 1] \), we have \( \alpha_t(a_n) = (x_n(t), 0) \) and \( \alpha_t(b_n) = (0, y_n(t)) \) for some continuous functions \( x_n(t) \) and \( y_n(t) \). For any \( n \in \mathbb{N} \) we note that

\[
\sup_{t \in [0,1]} x_n(t)^2 \leq \| \rho(a_n) - \rho(b_n) \|^2 = x_n^2 + y_n^2 \to x_n^2 \quad \text{as } k \to \infty,
\]

\[
\sup_{t \in [0,1]} y_n(t)^2 \leq \| \rho(a_n) - \rho(b_n) \|^2 = x_n^2 + y_n^2 \to y_n^2 \quad \text{as } j \to \infty,
\]

thus \( x_n(t) \leq x_n \) and \( y_n(t) \leq y_n \) for all \( t \in [0,1] \). If we choose any \( m, n \in \mathbb{N} \) we see that for each \( t \in [0,1] \) we have

\[
x_m^2 + y_n^2 = \| \rho(a_m) - \rho(b_n) \|^2 = \| \alpha_t(a_m) - \alpha_t(b_n) \|^2 = x_m(t)^2 + y_n(t)^2 \leq x_m^2 + y_n^2.
\]

It follows that \( \alpha \) is constant, which contradicts non-triviality. \( \qed \)

We remark that all NAC-colorings of the complete countable infinite bipartite graph \( K_{A,B} \) are such that any flexible realization constructed using Proposition 3.1 will be non-injective. This can be seen from the fact that this is the case already for all NAC-colorings of the complete bipartite graph \( K_{3,3} \), see [15, Figure 10].

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