A COUNTEREXAMPLE TO GUILLEMIN’S ZOLLFREI CONJECTURE

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Abstract. We construct Zollfrei Lorentzian metrics on every nontrivial orientable circle bundle over a orientable closed surface. Further we prove a weaker version of Guillemin’s conjecture assuming global hyperbolicity of the universal cover.

In [8] a pseudo-Riemannian metric $g$ on a compact manifold $M$ is called Zollfrei if the geodesic flow of $(M, g)$ induces the structure of a fibration by circles on the set of lightlike nonzero vectors $\{g(v, v) = 0, v \neq 0\} \subseteq TM$. In particular every lightlike geodesic of a Zollfrei metric is closed. We will call manifolds that admit a Zollfrei metric Zollfrei as well.

It is not difficult to classify all Zollfrei surfaces. [8] shows that every Zollfrei surface $(M, g)$ admits a finite cover $(M', g')$, which is globally conformal to $(\mathbb{R}^2/\mathbb{Z}^2, dx dy)$ where $x$ and $y$ are the canonical coordinates on $\mathbb{R}^2$ and $dx dy$ denotes the metric induced by $dx dy := \frac{1}{2}(dx \otimes dy + dy \otimes dx)$. However already for 3-manifolds the questions of determining the diffeomorphism type of a manifold admitting a Zollfrei metric is wide open. Besides the examples described in [8], called standard examples, none were known so far. The manifolds of the standard examples all have one of the four diffeomorphism types of compact manifolds with universal cover $S^2 \times \mathbb{R}$ (see [13]), i.e. $S^2 \times S^1$ and the three manifolds with double cover $S^2 \times S^1$ corresponding to the three involutions of $S^2 \times S^1$:

(i) $(x, y, z, t) \mapsto (-x, -y, -z, t)$
(ii) $(x, y, z, t) \mapsto (-x, -y, -z, t + \pi)$
(iii) $(x, y, z, t) \mapsto (-x, -y, z, t + \pi)$.

Here the 2-sphere is considered as the submanifold $\{x^2 + y^2 + z^2 = 1\}$ of $\mathbb{R}^3$. The standard examples all lift to a metric $g_{\text{can}} - \lambda dt^2$ on $S^2 \times \mathbb{R}$, where $g_{\text{can}}$ is the canonical round sphere metric of radius one and $\lambda > 0$.

In 1989 V. Guillemin conjectured in his book [8], page 8:

Conjecture ([8]). Every Zollfrei manifold in dimension three has the same diffeomorphism type as one of the standard examples.

In this article we will give a counterexample to the Conjecture.

Theorem 1. Every nontrivial orientable circle bundle over a closed and orientable surface admits Zollfrei metrics.

Note that the Gysin sequence for the integral cohomology implies that the different circle-bundles over different surfaces are nonhomeomorphic.

It is well known that the lightlike geodesics of a pseudo-Riemannian manifold are invariant, as unparameterized curves, under global conformal changes of the metric. Therefore Theorem in fact yields an infinite-dimensional family of Zollfrei metrics.

\vspace{1em}

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\vspace{1em}153C22, 53C50, closed geodesics, Lorentzian manifolds
2. Proof

2.1. The Theorem of Boothby and Wang \[3\]. Following \[7\], section 7.2., we call a vector field regular if around every point there exists a flow box that every flow line intersects at most once. We call a contact form $\alpha$ on an odd dimensional manifold regular if the Reeb vector field $\mathcal{R}$ of $\alpha$ is regular.

Let $\pi: M \to B$ be a principal circle bundle and $\mathcal{R} \in \Gamma(TM)$ tangent to the fibres with $2\pi$-periodic flow, i.e. $\mathcal{R}$ generates the circle action on $M$. Then, following \[7\], a differential 1-form $\alpha \in \Omega^1(M)$ is a connection 1-form if (1) $\mathcal{L}_\mathcal{R}(\alpha) \equiv 0$ and (2) $\alpha(\mathcal{R}) = 1$. Note that here we explicitly assume the identification $i : \mathbb{R} \cong \mathbb{R}$.

Conditions (1) and (2) imply that there exists a well defined closed 2-form $\omega$, called the curvature form, defined by $\pi^* \omega = \text{d}\alpha$. The class $\left[\frac{\text{d}\alpha}{2\pi}\right] \in H^2(B, \mathbb{Z})$ is called the Euler class of the bundle $\pi: M \to B$.

**Theorem 2** (\[7\], Theorem 7.2.4). Let $(B, \omega)$ be a closed symplectic manifold with integral symplectic form $\omega/2\pi$. Let $\pi: M \to B$ be the principal $S^1$-bundle with Euler class $\left[\frac{\text{d}\alpha}{2\pi}\right] \in H^2(B, \mathbb{Z})$. Then there is a connection 1-form $\alpha$ on $M$ with the following properties:

- $\alpha$ is a regular contact form,
- the curvature form of $\alpha$ is $\omega$,
- the vector field $\mathcal{R}$ defining the principal circle action on $M$ coincides with the Reeb vector field of $\alpha$.

Let $(B, g)$ be a smooth closed surface with constant curvature $K$ and volume form $\text{d}\text{vol}$ such that $\text{vol}(B, g) \in 2\pi\mathbb{Z}$. Then $\frac{\text{d}\text{vol}}{2\pi}$ is an integral symplectic form. Denote with $\pi: M \to B$ the $S^1$-principle bundle over $B$ with Euler class $\left[\frac{\text{d}\text{vol}}{2\pi}\right] \in H^2(B, \mathbb{Z})$. Define for $\varphi \in (0, \frac{\pi}{2})$ the Lorentzian metric

$$h_\varphi := \pi^* g - \cot^2 \varphi \cdot \alpha \otimes \alpha$$

on $M$. Note that $\pi - 2\varphi$ is the opening angle of the light cones of $h_\varphi$ around $\mathcal{R}$.

The Reeb vector field $\mathcal{R}$ of $\alpha$ is a timelike Killing vector field of $h_\varphi$, i.e. $(M, h_\varphi)$ is stationary spacetime. Denote with $\Phi$ the flow of $\mathcal{R}$.

Note that in the case of $B \cong S^2$ and $K = 4$ we can describe $h_\varphi$ as the pseudo-Riemannian analogues to the Berger spheres \[1\]. Consider the canonical embedding $i : S^2 \to \mathbb{H} \cong \mathbb{R}^4$ into the Quaternions and the orthonormal frame field $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ with $\mathcal{I} : x \mapsto i \cdot x$, $\mathcal{J} : x \mapsto j \cdot x$ and $\mathcal{K} : x \mapsto k \cdot x$. By $(\mathcal{I}^*, \mathcal{J}^*, \mathcal{K}^*)$ denote the dual frame field. Let $(\langle \cdot, \cdot \rangle)$ be the canonical scalar product on $\mathbb{R}^4$. Then we know that $\mathcal{T}^* = \alpha$ and $\mathcal{R} = \mathcal{I}$ for $(B, g) = (\mathbb{CP}^1, g_{FS})$ where $g_{FS}$ denotes the Fubini-Studii metric. Further it follows that

$$h_\varphi(x) := \langle i^* \langle \cdot, \cdot \rangle, \frac{1}{\sin^2 \varphi} \mathcal{T}^* \otimes \mathcal{T}^* \rangle = \mathcal{J}^* \otimes \mathcal{J}^* + \mathcal{K}^* \otimes \mathcal{K}^* - \cot^2(\varphi) \mathcal{T}^* \otimes \mathcal{T}^*.$$

2.2. The Arrival Time. Following \[1\] we will call a Lorentzian manifold $(\mathcal{M}, g)$ standard stationary iff $\mathcal{M}$ splits into a product $\mathcal{M}_0 \times \mathbb{R}$ with

$$g(x, t)[(v, \tau), (v, \tau)] = g_0(x)[v, v] + 2g_0(x)[\delta(x), v] \tau - \beta(x) \tau^2$$

where $(x, t) \in \mathcal{M}_0 \times \mathbb{R}$, $(v, \tau) \in T_x \mathcal{M}_0 \times \mathbb{R}$, $g_0$ is a Riemannian metric on $\mathcal{M}_0$, $\delta$ a smooth vector field on $\mathcal{M}_0$ and $\beta$ a positive smooth function on $\mathcal{M}_0$.

Denote with $F$ the Finsler metric on $\mathcal{M}_0$ given by

$$F(x, v) := \sqrt{g_0(x)[v, v] + \tilde{g}_0(x)[\delta(x), v]^2 + \tilde{g}_0(x)[\delta(x), v]}$$

where $\tilde{g}_0 := g_0/\beta$. We have the following Fermat’s principle:

**Theorem 3** (\[1\] Theorem 4.1). Let $(\mathcal{M}, g)$ be a standard stationary spacetime and $(x_0, t_0) \in \mathcal{M}$, $x \in \mathbb{R} \mapsto \gamma(s) = (x_1, s) \in \mathcal{M}$, $x_1 \in \mathcal{M}_0$. A curve $s \in [0, 1] \to$
z(s) = (x(s), t(s)) ∈ M is a future pointing lightlike geodesic of (M, g/β) if and only if x(s) is a geodesic for the Fermat metric F, parameterized to have constant Riemannian speed h(x)[x, x] = g_{0}(x)[δ(x), x][2] + g_{0}[x, x], and
\begin{equation}
(2) \quad t(s) = \int_{0}^{s} \left( g_{0}(x)[δ(x), x] + \sqrt{g_{0}(x)[δ(x), x][2] + g_{0}[x, x]} \right) \, dv.
\end{equation}

Recall that the lightlike geodesics of (M_0 × ℝ, g) are reparameterizations of the lightlike geodesics of (M_0 × ℝ, g).

2.3. The Global Construction. We want to apply the reasoning of Theorem 4 to the lightlike geodesics of (M, h_φ). It is clear that these Lorentzian manifold are not standard stationary. But we can consider the problem locally. Choose a h_φ-spacelike immersion I: D^2 → M, i.e. I^* h_φ > 0. Then the map Φ_I : D^2 × ℝ → M, (p, t) ↦ Φ(I(p), t) is an immersion of D^2 × ℝ. It follows that (D^2 × ℝ, (Φ_I)^* h_φ) is a standard stationary Lorentzian spacetime in the sense of [4].

Remark 4. Consider the arrival time t_x for a closed curve x : [a, b] → D^2 as defined by (2). Then [t_x(b) − t_x(a)]mod 2π quantifies the obstruction of the lightlike curve s ↦ Φ_I(x(s), t_x(s)) to being closed in M. This follows from the fact that the map Φ_I is 2π-periodic in the ℝ-factor. So if the arrival time is a rational multiple of 2π, e.g. t_x(b) − t_x(a) = 2πk/2π, then s ↦ Φ(I^0(s), t_x(s)) is closed.

We will now give a coordinate invariant description of the arrival time, i.e. a functional for curves in the orbit space B. Let I : D^2 → M be a spacelike immersion as before. We express the components of (1) in terms of objects defined on B. We immediately see that \( \bar{g}_0 = -\frac{I^* h_φ}{h_φ} \). Next define \( Y := h_φ(Y, R) \) for \( Y ∈ TM \). Then on one side we have \( h_φ(Y, Y) = g(\pi_*, Y, \pi_* Y) \). One the other side we have
\[
 h_φ(Y, Y) = -h_φ(R, R) \left( \frac{h_φ(Y, Y)}{-h_φ(R, R)} + \frac{h_φ(Y, R)²}{h_φ(R, R)} \right).
\]
From (1) follows that \( \bar{g}_0(δ(x), .) = -I^* α \). Combining the equations we get
\[
 F(x, v) = \sqrt{\frac{(π ∘ I)^* g(v, v)}{-h_φ(R, R)}} - I^* α(v) = \tan φ \cdot \sqrt{(π ∘ I)^* g(v, v)} - I^* α(v).
\]

Spacelike immersions I : D^2 → M are induced for example by certain local section of π : M → B. Therefore for U ⊆ B contractible and a section s : U → M such that the image of s is spacelike, we can define \( F_U \) via the previous formula. Note that a global version of \( F_U \) does not exists, as M is nontrivial. In our case this comes from the fact that α has no well defined counterpart on B. But do has a well defined counterpart on B in the form of \( dvol^φ \). So the global version of the arrival time functional, we are looking for, is no longer well defined on curves \( γ : I → B \), but instead is defined for (e.g.) smooth maps \( f : S → B \) where S is a compact oriented 2-manifold with nonempty boundary. From this point of view the arrival time functional for a map \( f : S → B \) is
\[
 cp_φ(f) := \tan φ \cdot L^φ(f|_S) + \int_S f^*(-dvol^φ)
\]
where \( L^φ \) denotes the length functional of g. Following [12] we will call functionals like \( cp_φ \) charged particles.

The critical points of \( cp_φ \) describe the periodic orbits of a charged particle on \((B, tan(φ) ∙ g)\) moving under the influence of the “magnetic field” \(-dvol^φ\).
2.4. Extremals of \( cp_\varepsilon \). The general framework can be described as follows: The definitions and results are taken from [11]. Let \( \omega \) be an arbitrary 2-form on \( B \). Consider for the class of contractible open subsets \( U \subseteq B \) the family of functionals

\[
cp_U : C^\infty(I,U) \to \mathbb{R}, \quad \gamma \mapsto cp_U(\gamma) := L^g(\gamma) + \int_\gamma \sigma,
\]

where \( \sigma \in \Omega^1(U) \) is a primitive of \( \omega|_U \). The physical intuition behind \( cp_U \) is that the critical points of \( cp_U \) model the motion of a charged particle moving under the influence of a magnetic field \( \omega \). In physics terms the 1-form \( \sigma \) represents a vector potential of the magnetic field \( \omega|_U \).

**Remark 5.** Let \( \gamma : [a,b] \to U \) be a regular curve parameterized w.r.t. constant \( g \)-arclength. Then

\[
\frac{1}{|\gamma|_g} \nabla_g \dot{\gamma} = -\omega(\gamma,.)^#
\]

are the Euler-Lagrange equations of \( cp_U \) (Here we set \( |v|_g := \sqrt{g(v,v)} \)). Especially the Euler-Lagrange equations are independent of \( U \) and \( \sigma \). We will refer to the solutions as **extremals**.

Recall that there exist constant \( \varepsilon, \delta > 0 \) such that any pair of points \( x, y \in B \), with distance at most \( \delta \), can be joined by a unique solution \( \gamma : [0,1] \to B \) of \( \dot{\gamma} = \nabla_{\gamma} \omega \) lying completely in the ball of radius \( \varepsilon \) around \( x \), compare [11].

Next we describe the relation between the critical points of \( cp_\varepsilon \) and \( cp_U \). The following definition is taken from [11]. **A film** \( \Pi \subseteq B \) is an oriented surface with nonempty boundary embedded into \( B \) such that the boundary is a union of finitely many closed curves \( \gamma_\alpha \) with the following properties:

1) \( \gamma_\alpha \) has the boundary orientation,
2) every \( \gamma_\alpha \) is a finite polygon of extremal segments whose lengths do not exceed \( \delta \) and
3) the curves \( \gamma_\alpha \) are disjoint, i.e. \( \gamma_\alpha \cap \gamma_\beta = \emptyset \) if \( \alpha \neq \beta \).

The space of films on \( B \) is denoted with \( L(B) \). Define

\[
cp : L(B) \to \mathbb{R}, \quad \Pi \mapsto L^g(\Pi_{\|S}) + \int_{\Pi} \omega.
\]

**Lemma 6** ([11], Lemma 1). Let \( \Pi \in L(B) \). Then \( \delta \cp(\Pi) = 0 \), i.e. \( \Pi \) is a critical point of \( \cp \), if and only if the boundary of \( \Pi \) consists of a union of smooth closed extremals.

As an example we calculate the extremals of \((\mathbb{C}P^1, \tan \varphi \cdot g_{FS}, \dvol_{FS})\). The Fubini-Study metric is a Riemannian metric on \( \mathbb{C}P^1 \cong S^2 \) of constant curvature equal to 4 and diameter \( \pi/2 \). Hence \((\mathbb{C}P^1, g^{FS})\) is isometric to \((S^2, \frac{1}{4} g_{can})\) where \( g_{can} \) denotes the canonical metric on \( S^2 \) with curvature equal to one. Fix the “polar” parameterization

\[
P : U := (0, \pi) \times (0, 2\pi) \to S^2, \quad (\theta, \psi) \mapsto (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta).
\]

We claim that one solution to the Euler-Lagrange equations of \( cp_U \) is the curve \( t \mapsto (\sin \theta \cos t, \sin \theta \sin t, \cos \theta) \) with \( \theta = \arctan(2 \tan \varphi) \). Note that this curve is simply closed. Observe that \( \Gamma^\theta_{\psi\psi} = 0 \) and \( \Gamma^\psi_{\varphi\psi} = -\sin(\theta) \cos(\theta) \). Consequently we have

\[
\frac{\tan \varphi}{|\gamma|} \nabla_{\partial_\theta} \partial_\psi = -2 \cos(\theta) \tan(\varphi) \partial_\theta
\]

and

\[-\omega(\dot{\gamma},.)^# = \dvol_{FS}(\partial_\psi,.)^# = -\sin(\theta) \partial_\theta.
\]
Then the curve $t \mapsto (\sin \theta \cos t, \sin \theta \sin t, \cos \theta)$ is a solution of the Euler Lagrange equation iff $-2 \tan \varphi \cos \theta = -\sin \theta$ or equivalently $\theta = \arctan(2 \tan \varphi)$.

**Remark 7.** Recall that if $(B, g)$ has constant curvature, the isometry group of the universal cover $(\tilde{B}, \tilde{g})$ acts transitively on $T^1 B$. Further it preserves $\tilde{c}_p$, the lift of $c_p$ if $\omega$ is a multiple of $d\text{vol}^g$. This implies that the extremals of $\tilde{c}_p$ are either all closed or all non-closed. In the case of nonnegative curvature all extremals of $\tilde{c}_p$ and therefore $c_p$ are closed. For curvature equal to $-1$ the extremals are closed iff $|\omega(\gamma,.)|_g > 1$. Note that it is sufficient to consider the case $K = -1$.

2.5. **Lightlike Geodesics of $(M, h_\varphi)$**. Since $c_\varphi$ is constant on all simply closed extremals, Remark 4 readily implies:

**Proposition 8.** The lightlike geodesics of $(M, h_\varphi)$ are either all closed or all non-closed depending on whether the extremals of $c_\varphi$ are closed and $c_\varphi(\gamma) \in \mathbb{Q}$ for any extremal $\gamma$ of $c_\varphi$.

Again as an example we determine the condition for $(CP^1, \tan \varphi \cdot g_{FS}, d\text{vol}^{FS})$. In the polar parameterization $P$ we can choose the primitive $\sigma = -\frac{\cos \theta}{4} d\psi$ of $P^*(\ast d\text{vol}^{FS})$. Thus we have $\int, \sigma = -\frac{1}{2} \cos(\arctan(2 \tan \varphi))$. The length of every simply closed solution of (3) is $\pi \sin(\arctan(2 \tan \varphi))$. Bringing everything together we get

$$c_{\nu}(\gamma) = \pi \left( \sin(\arctan(2 \tan \varphi)) - \frac{\cos(\arctan(2 \tan \varphi))}{2} \right)$$

$$= \frac{\pi}{2} \cos(\arctan(2 \tan \varphi))(4 \tan(\varphi) - 1)$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{1 + 4 \tan^2 \varphi}}(4 \tan(\varphi) - 1)$$

From Remark 3 we know that $c_{\nu}(\gamma) \mod 2\pi$ quantifies the obstruction for the lightlike geodesics over $\gamma$ to being closed. Therefore the lightlike geodesics of $(S^3, h_\varphi)$ will eventually close iff $\frac{4 \tan(\varphi) - 1}{\sqrt{1 + 4 \tan^2 \varphi}} \in \mathbb{Q}$.

**Proposition 9.** If the lightlike geodesics of $(M, h_\varphi)$ are closed then $(M, h_\varphi)$ is Zollfrei.

**Proof.** First recall that (3) defines an Euler-Lagrange flow on $TB$. The point of the proof is to note the relation between the Euler-Lagrange flow of $c_\varphi$ on $TB^\times := TB \setminus \{ \text{zero section} \}$ and the geodesic flow of $h_\varphi$ on the smooth manifold $\text{Light}_h$ of lightlike tangent vectors $v \neq 0$ of $(M, h_\varphi)$. More precisely the Euler-Lagrange flow $\Phi_B$ of $\cot \varphi \cdot c_\varphi$ on $TB^\times$ and the geodesic flow $\Phi_M$ on $\text{Light}_h$ are conjugated via $\pi_*$, i.e. $\pi_* \circ \Phi_M = \Phi_B \circ \pi_*$, where $\pi: M \to B$ denotes the bundle projection.

1) $\Phi_B$ induces an circle fibration on $TB^\times$: Consider the pullback $\tilde{c}_p$ of $c_\varphi$ to the universal cover $\tilde{\pi}: \tilde{B} \to B$. $\tilde{c}_p$ is invariant under the action of $\text{Isom}(\tilde{B}, \tilde{g})$, the isometry group of $(\tilde{B}, \tilde{g})$. Recall that since $\tilde{g}$ has constant curvature, $\text{Isom}(\tilde{B}, \tilde{g})$ acts transitively on $T^r \tilde{B}$ for every $r > 0$ and commutes with $\Phi_{\tilde{B}}$, the Euler-Lagrange flow of $\tilde{c}_p$. The isotropy group of every flowline is closed. This defines a fibre bundle structure on every $T^r \tilde{B}$ with 1-dimensional fibre. Since $\Phi_{\tilde{B}}(v, t) = \Phi_B(\lambda \cdot v, \frac{t}{\lambda})$, these fibrations extend to a fibration of $T\tilde{B}^\times$. The assumption that the lightlike geodesics of $h_\varphi$ are closed implies that the extremals of $c_\varphi$ are closed as well. From Remark 7 we know that in this case the extremals of $\tilde{c}_p$ and with it the flowlines of $\Phi_{\tilde{B}}$ are closed as well. Therefore the isotropy groups are compact, i.e. the fibres are diffeomorphic to $S^1$. 

The fibration structure is of course invariant under the induced action of $\pi_1(B)$. Therefore it descends to a fibration of $TB^\times$ over the smooth manifold of flowlines of $\Phi_B$.

2) $\Phi_M$ induces a circle fibration on Light $\varphi$: Since $\pi_\ast$ conjugates $\Phi_M$ with $\Phi_B$, every flowline of $\Phi_M$ induces a finite covering of the respective flowline of $\Phi_B$. We have seen in Remark 7 that $cp_\varphi$ is constant on every extremal of minimal period. Then Remark 8 implies that these coverings all have the same number of leaves. Together with part 1) this yields that $\Phi_M$ induces the structure of a circle fibration on Light $\varphi$, i.e. $(M, h_\varphi)$ is Zollfrei.

3. A weaker conjecture

Despite Theorem 1 one can hope to prove a weaker version of the conjecture, e.g. assuming additional properties of the pseudo-Riemannian universal cover. Note that for 3-manifolds, up to sign, every pseudo-Riemannian metric, that is not Riemannian or anti-Riemannian, is Lorentzian. This opens for us the possibility to use notions of causality theory from Lorentzian geometry. 

Theorem 10. If the 3-manifold $M$ admits a Zollfrei metric $g$ such that the universal Lorentzian cover is globally hyperbolic, $M$ is covered by $S^2 \times S^1$, i.e. diffeomorphic to either $S^2 \times S^1$, $\mathbb{R}P^2 \times S^1$, $\mathbb{R}P^3 \mathbb{R}P^3$ or the nonorientable 2-sphere bundle over $S^1$.

Recall that according to [2] we can define a Lorentzian manifold $(M, g)$ to be globally hyperbolic if $(M, g)$ is isometric to $(N \times \mathbb{R}, g_0 + g_0(\delta, .) - \beta dt^2)$, where $\beta$ is a smooth positive function on $N \times \mathbb{R}$, $g_0$ is a Riemannian metric on $N$ and $\delta$ is a smooth vector field on $N$ both i.g. depending on the $t$-coordinate. Note that global hyperbolicity implies causality for a Lorentzian manifold.

Theorem 10 follows from a result due to Low ([10], Theorem 5). The proof leans on the notion of refocussing spacetimes introduced in [9].

Definition 11 ([5], Definition 22). A strongly causal spacetime $(M, g)$ (that is not necessarily globally hyperbolic) is called refocussing at $p \in M$ if there exists a neighborhood $O$ of $p$ with the following property: For every open $U$ with $p \in U \subseteq O$ there exists $q \notin U$ such that all the lightlike geodesics through $q$ enter $U$. A spacetime $(M, g)$ is called refocussing if it is refocussing at some $p$, and it is called nonrefocussing if it is not refocussing at every $p \in M$.

Recall that global hyperbolicity implies strong causality for Lorentzian manifolds. So the definition is not needed in full generality, i.e. those who are not familiar with causality theory might as well substitute global hyperbolicity for strong causality in the definition.

Theorem 12 ([10], Theorem 5). Let $M$ be globally hyperbolic, with non-compact Cauchy hypersurface $N$. Then $M$ cannot be refocussing.

With this at hand we can state the following proposition for our purposes.

Proposition 13. Let $(M, g)$ be a Lorentzian manifold such that the universal Lorentzian cover $(\tilde{M}, \tilde{g})$ is globally hyperbolic. If there exists $p \in M$ such that all lightlike geodesics emanating from $p$ return to $p$ with uniformly bounded Riemannian arclength (w.r.t. a fixed complete Riemannian metric on $M$), then $M$ is compact and $\tilde{M}$ is spatially compact, i.e. $\tilde{M} \cong N \times \mathbb{R}$ with $N$ compact.
Proof. According to Theorem 12 the only points we have to prove are (1) $(\tilde{M}, \tilde{g})$ is refocusing and (2) $M$ is compact.

(1). Since the lightlike geodesic loops around $p$ have bounded Riemannian ar-
length, they share a common fundamental class $\eta \in \pi_1(M)$. Notice that $\eta$
isanontrivial, since else the universal cover would violate causality. Hence the uni-
versal cover is refocusing at any point $\tilde{p} \in \tilde{\pi}^{-1}(p)$, since all lightlike geodesics through
$\eta^{-1}(\tilde{p})$ meet $\tilde{p}$ and there exists a neighborhood of $\tilde{p}$ that does not contain $\eta^{-1}(\tilde{p})$.

(2). The deck transformation group of the universal cover acts properly discon-
tinuously on the universal cover. This implies that there exists a $k \in \mathbb{Z}$ such that
$\eta^k(N \times \{0\})$ is disjoint from $N \times \{0\}$, as $N$ is compact. This implies that the
quotient of $M \cong N \times \mathbb{R}$ by the group generated by $\eta^k$ is compact and moreover covers $M$. Then $M$ has to be compact.

Proof of Theorem 14. Theorem 12 implies for 3-manifolds that any Cauchy hyper-
surface in the universal cover has to be diffeomorphic to $S^2$. Thus $\tilde{M} \cong S^2 \times \mathbb{R}$. The compact quotients of $S^2 \times \mathbb{R}$ were classified in [13]. They are exactly $S^2 \times S^1$, $\mathbb{R}P^3 \times S^1$, $\mathbb{R}P^3 \times \mathbb{R}P^3$ and the nonorientable 2-sphere bundle over $S^1$.

At the end of these notes we want to post some questions in connection with
Zollfrei 3-manifolds. P. Mounoud asked if every Zollfrei 3-manifold is a Seifert
fibration. At this point the author does not have an idea how to prove such a claim
or how to give a counterexample. If a counterexample exists, then it cannot be
stationary, since [6] show that every compact stationary Lorentzian manifold is a
Seifert fibration.

On the other hand is the question if every Seifert fibration admits a Zollfrei
Lorentzian manifold. This question is especially interesting for the trivial bundles
over surfaces of genus greater than one. Again such examples cannot be stationary,
since then the Finsler metric $F$ (see section 2.2) will be globally well defined and the
lightlike geodesics correspond the geodesics of $F$ on the underlying surface. Since
the fundamental groups of the surfaces in question are nontrivial, not all lightlike
geodesics can be homotopic. This clearly contradicts the Zollfrei property.

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