A SURVEY ON GEOMETRY OF WARPED PRODUCT SUBMANIFOLDS

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Abstract. The warped product $N_1 \times f N_2$ of two Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ is the product manifold $N_1 \times N_2$ equipped with the warped product metric $g = g_1 + f^2 g_2$, where $f$ is a positive function on $N_1$. The notion of warped product manifolds is one of the most fruitful generalizations of Riemannian products. Such notion plays very important roles in differential geometry as well as in physics, especially in general relativity. Warped product manifolds have been studied for a long period of time. In contrast, the study of warped product submanifolds was only initiated around the beginning of this century in a series of articles [36, 39, 40, 43]. Since then the study of warped product submanifolds has become a very active research subject.

In this article we survey important results on warped product submanifolds in various ambient manifolds. It is the author’s hope that this survey article will provide a good introduction on the theory of warped product submanifolds as well as a useful reference for further research on this vibrant research subject.

Table of Contents
1. Introduction ................................................................. 1
2. Preliminaries ..................................................................... 2
3. Warped product submanifolds of Riemannian manifolds ............ 6
4. Multiply warped product submanifolds .................................... 8
5. Arbitrary warped products in complex space forms ..................... 10
6. Warped products as Kählerian submanifolds ............................. 13
7. $CR$-products in Kähler manifolds ...................................... 14
8. Warped product Lagrangian submanifolds of Kähler manifolds ...... 15
9. Warped Product $CR$-submanifolds of Kähler manifolds ............ 16
10. $CR$-warped products with compact holomorphic factor ............. 18
11. Another optimal inequality for $CR$-warped products ............... 20
12. Warped Product $CR$-submanifolds and $\delta$-invariants ............ 21
13. Warped product real hypersurfaces in complex space forms ........ 22
14. Warped Product $CR$-submanifolds of nearly Kähler manifolds ...... 25
15. Warped product submanifolds in para-Kähler manifolds ............ 26
16. Contact $CR$-warped product submanifolds in Sasakian manifolds .. 28
17. Warped product submanifolds in affine spaces ........................ 30
18. Twisted product submanifolds ............................................ 35
19. Related articles ............................................................. 37
20. References ...................................................................... 38

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1. INTRODUCTION

Let $B$ and $F$ be two Riemannian manifolds equipped with Riemannian metrics $g_B$ and $g_F$, respectively, and let $f$ be a positive function on $B$. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

\begin{equation}
||X||^2 = ||\pi_*(X)||^2 + f^2(\pi(x))||\eta_*(X)||^2
\end{equation}

for any tangent vector $X \in T_xM$. Thus we have $g = g_B + f^2g_F$. The function $f$ is called the warping function (cf. [21]). The concept of warped products appeared in the mathematical and physical literature long before [21], e.g., warped products were called semi-reducible spaces in [92]. It is well-known that the notion of warped products plays important roles in differential geometry as well as in physics, especially in the theory of general relativity (cf. [54, 119]).

According to a famous theorem of J. F. Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension. In particular, Nash’s embedding theorem implies that every warped product manifold $N_1 \times_f N_2$ can be isometrically embedded as Riemannian submanifolds in Euclidean spaces with sufficiently high codimension.

In view of Nash’s theorem, the author asked in early 2000s the following fundamental question concerning warped product submanifolds (see [43]).

**Fundamental Question:** What can we conclude from an arbitrary isometric immersion of a warped product manifold into a Euclidean space with arbitrary codimension?

Or, more generally,

What can we conclude from an arbitrary isometric immersion of a warped product manifold into an arbitrary Riemannian manifold?

The study of this fundamental question on warped product submanifolds was not initiated until the beginning of this century by the author in a series of articles [36, 39, 40, 43]. Since then the study of warped product submanifolds has become a very active research topic in differential geometry of submanifolds.

In this article we survey the most important results on warped product submanifolds in various manifolds, including Riemannian, Kähler, nearly Kähler, para-Kähler and Sasakian manifolds. It is the author’s hope that this survey article will provide a good introduction on the theory of warped product submanifolds as well as a useful reference for further research on this subject.

2. PRELIMINARIES

In this section we provide some basic notations, formulas, definitions, and results for later use.

2.1. Basic notations and formulas. Let $M$ be an $n$-dimensional submanifold of a Riemannian $m$-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}^m$ such that, restricted to $M$, the vectors $e_1, \ldots, e_n$ are tangent to $M$ and hence $e_{n+1}, \ldots, e_m$ are normal to $M$. Let $K(e_i \wedge e_j)$ and $\tilde{K}(e_i \wedge e_j)$ denote respectively the sectional curvatures of $M$ and $\tilde{M}^m$ of the plane section spanned by $e_i$ and $e_j$. 
For the submanifold $M$ in $\tilde{M}^m$ we denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connections of $M$ and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [27, 54])

\begin{align}
(2.1) \quad \tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \\
(2.2) \quad \tilde{\nabla}_X \xi &= A_\xi X + D_X \xi,
\end{align}

for any vector fields $X, Y$ tangent to $M$ and vector field $\xi$ normal to $M$, where $\sigma$ denotes the second fundamental form, $D$ the normal connection, and $A$ the shape operator of the submanifold.

Let $\{\sigma^r_{ij}\}, i, j = 1, \ldots, n; r = n+1, \ldots, m,$ denote the coefficients of the second fundamental form $h$ with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$. Then we have

$$
\sigma^r_{ij} = \langle \sigma(e_i, e_j), e_r \rangle = \langle A_{e_r} e_i, e_j \rangle,
$$

where $\langle , \rangle$ denotes the inner product.

The mean curvature vector $-\vec{H}$ is defined by

\begin{equation}
(2.3) \quad -\vec{H} = \frac{1}{n} \text{trace } \sigma = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i),
\end{equation}

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle $TM$ of $M$. The squared mean curvature is then given by

$$
H^2 = \langle -\vec{H}, -\vec{H} \rangle.
$$

A submanifold $M$ is called minimal in $\tilde{M}^m$ if its mean curvature vector vanishes identically.

Denote by $R$ and $\tilde{R}$ the Riemann curvature tensors of $M$ and $\tilde{M}^m$, respectively. Then the equation of Gauss is given by

\begin{equation}
(2.4) \quad R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle
\end{equation}

for vectors $X, Y, Z, W$ tangent to $M$. In particular, for a submanifold of a Riemannian manifold of constant sectional curvature $c$, we have

\begin{equation}
(2.5) \quad R(X, Y; Z, W) = c \{ \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \} \\
+ \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle.
\end{equation}

Let $M$ be a Riemannian $p$-manifold and $e_1, \ldots, e_p$ be an orthonormal frame fields on $M$. For differentiable function $\varphi$ on $M$, the Laplacian $\Delta \varphi$ of $\varphi$ is defined by

\begin{equation}
(2.6) \quad \Delta \varphi = \sum_{j=1}^{p} \{ (\nabla_{e_j} \varphi) - e_j \varphi \}.
\end{equation}

For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_p M$, the scalar curvature $\tau$ of $M$ at $p$ is defined to be

\begin{equation}
(2.7) \quad \tau(p) = \sum_{i<j} K(e_i \wedge e_j),
\end{equation}

where $K(e_i \wedge e_j)$ denotes the sectional curvature of the plane section spanned by $e_i$ and $e_j$. 
Let $L$ be a subspace of $T_xM$ of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $L$. The scalar curvature $\tau(L)$ of the $r$-plane section $L$ is defined by

$$\tau(L) = \sum_{\alpha \prec \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \quad (2.8)$$

2.2. $\delta$-invariants. For a Riemannian $n$-manifold $M$, let $K(\psi)$ denote the sectional curvature associated with a 2-plane section $\psi \subset T_xM$, $x \in M$. For an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_xM$, the scalar curvature $\tau_M$ of $M$ at $x$ is defined to be

$$\tau_M(x) = \sum_{i < j} K(e_i, e_j). \quad (2.9)$$

Let $L$ be a $r$-subspace of $T_xM$ with $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $L$. We define the scalar curvature $\tau(L)$ of $L$ by

$$\tau(L) = \sum_{\alpha \prec \beta} K(e_\alpha, e_\beta), \quad 1 \leq \alpha, \beta \leq r. \quad (2.10)$$

For an integer $k \geq 0$ let $\mathcal{S}(n, k)$ denote the set consisting of unordered $k$-tuples $(n_1, \ldots, n_k)$ of integers $\geq 2$ such that $n > n_1$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of unordered $k$-tuples with $k \geq 0$ for a fixed $n$.

For each $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the $\delta$-invariant $\delta(n_1, \ldots, n_k)(x)$ is defined by

$$\delta(n_1, \ldots, n_k)(x) = \tau_M(x) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ run over all $k$ mutually orthogonal subspaces of $T_xM$ such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

Some other invariants of similar nature, i.e., invariants obtained from the scalar curvature by deleting certain amount of sectional curvatures, are also called $\delta$-invariants. For a general survey on $\delta$-invariants and their applications, see the recent book [33].

2.3. Complex extensors. We recall the notions of complex extensors and Lagrangian $H$-umbilical submanifolds introduced in [33].

Let $G : N^{p-1} \rightarrow \mathbb{E}^p$ be an isometric immersion of a Riemannian $(p-1)$-manifold into the Euclidean $p$-space $\mathbb{E}^p$ and let $F : I \rightarrow \mathbb{C}^*$ be a unit speed curve in $\mathbb{C}^* = \mathbb{C} - \{0\}$. We extend $G : N^{p-1} \rightarrow \mathbb{E}^p$ to an immersion of $I \times N^{p-1}$ into $\mathbb{C}^p$ as

$$F \otimes G : I \times N^{p-1} \rightarrow \mathbb{C} \otimes \mathbb{E}^p = \mathbb{C}^p, \quad (2.10)$$

where $(F \otimes G)(s, q) = F(s) \otimes G(q)$ for $s \in I$, $q \in N^{p-1}$. This extension $F \otimes G$ of $G$ via tensor product is called the complex extensor of $G$ via $F$.

A Lagrangian submanifold of $\mathbb{M}^p$ without totally geodesic points is called $H$-umbilical if its second fundamental form takes the following simple form (cf. [33]):

$$\sigma(\bar{e}_1, \bar{e}_1) = \lambda J\bar{e}_1, \quad \sigma(\bar{e}_j, \bar{e}_j) = \mu J\bar{e}_j, \quad j > 1,$$

$$\sigma(\bar{e}_1, \bar{e}_j) = \mu J\bar{e}_j, \quad \sigma(\bar{e}_j, \bar{e}_k) = 0, \quad 2 \leq j \neq k \leq p \quad (2.11)$$

for some functions $\lambda, \mu$ with respect to a suitable orthonormal local frame field $\{\bar{e}_1, \ldots, \bar{e}_p\}$. Such submanifolds are the simplest Lagrangian submanifolds next to the totally geodesic ones.
Example 2.1. (Lagrangian pseudo-sphere) For a real number $b > 0$, let $F : \mathbb{R} \to \mathbb{C}$ be the unit speed curve given by

$$F(s) = \frac{e^{2bsi} + 1}{2bi}.$$  

With respect to the induced metric, the complex extensor $\phi = F \otimes \iota$ of the unit hypersphere of $\mathbb{E}^n$ via $F$ is a Lagrangian isometric immersion of an open portion of an $n$-sphere $S^n(b^2)$ of sectional curvature $b^2$ into $\mathbb{C}^n$ which is simply called a Lagrangian pseudo-sphere.

A Lagrangian pseudo-sphere is a Lagrangian $H$-umbilical submanifold satisfying (2.11) with $\lambda = 2\mu$. Conversely, Lagrangian pseudo-spheres are the only Lagrangian $H$-umbilical submanifolds of $\mathbb{C}^n$ which satisfy (2.11) with $\lambda = 2\mu$.

Example 2.2. (Whitney sphere) Let $w : S^p(1) \to \mathbb{C}^p$ be the map of the unit $p$-sphere into $\mathbb{C}^p$ defined by

$$w(y_0, y_1, \ldots, y_p) = \frac{1 + iy_0}{1 + y_0^2}(y_1, \ldots, y_p), \quad y_0^2 + y_1^2 + \ldots + y_p^2 = 1.$$  

The map $w$ is a (non-isometric) Lagrangian immersion with one self-intersection point which is called the Whitney $p$-sphere. The Whitney $p$-sphere is a complex extensor of the unit hypersphere of $\mathbb{E}^n$ centered at the origin.

The following results from [33] are fundamental for complex extendors.

Proposition 2.1. Let $\iota : S^{p-1} \to \mathbb{E}^p$ be a hypersphere of $\mathbb{E}^p$ centered at the origin. Then every complex extensor $\phi = F \otimes \iota$ of $\iota$ via a unit speed curve $F : I \to \mathbb{C}^*$ is a Lagrangian $H$-umbilical submanifold of $\mathbb{C}^p$ unless $F$ is contained in a line through the origin (which gives a totally geodesic Lagrangian submanifold).

Theorem 2.1. Let $n \geq 3$ and $L : M \to \mathbb{C}^n$ be a Lagrangian $H$-umbilical isometric immersion. Then we have:

1. If $M$ is of constant sectional curvature, then either $M$ is flat or, up to rigid motions of $\mathbb{C}^n$, $L$ is a Lagrangian pseudo-sphere.
2. If $M$ contains no open subset of constant sectional curvature, then, up to rigid motions of $\mathbb{C}^n$, $L$ is a complex extensor of the unit hypersphere of $\mathbb{E}^n$.

2.4. Warped product immersion. Suppose that $M_1, \ldots, M_k$ are Riemannian manifolds and that

$$f : M_1 \times \cdots \times M_k \to E^N$$  

is an isometric immersion of the Riemannian product $M_1 \times \cdots \times M_k$ into Euclidean $N$-space. J. D. Moore [103] proved that if the second fundamental form $\sigma$ of $f$ has the property that $\sigma(X, Y) = 0$ for $X$ tangent to $M_i$ and $Y$ tangent to $M_j$, $i \neq j$, then $f$ is a product immersion, that is, there exist isometric immersions $f_i : M_i \to E^{m_i}, 1 \leq i \leq k$, such that

$$f(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k))$$  

(2.12)
when $x_i \in M_i$ for $1 \leq i \leq k$.

Let $M_0, \cdots, M_k$ be Riemannian manifolds, $M = M_0 \times \cdots \times M_k$ their product, and $\pi_i : M \rightarrow M_i$ the canonical projection. If $\rho_1, \cdots, \rho_k : M_0 \rightarrow \mathbb{R}_+$ are positive-valued functions, then

\[
\Delta f = \sum_{i=1}^{k} \rho_i \circ \pi_0^i \Delta f_i + \sum_{i=1}^{k} (\rho_i \circ \pi_0^i)^2 \langle \pi_i^* X, \pi_i^* Y \rangle
\]

defines a Riemannian metric on $M$, called a warped product metric. $M$ endowed with this metric is denoted by $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$.

A warped product immersion is defined as follows: Let $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product and let $f_i : N_i \rightarrow M_i$, $i = 0, \cdots, k$, be isometric immersions, and define $\sigma_i := \rho_i \circ f_0 : N_0 \rightarrow \mathbb{R}_+$ for $i = 1, \cdots, k$. Then the map

\[
f : N_0 \times_{\sigma_1} N_1 \times \cdots \times_{\sigma_k} N_k \rightarrow M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k
\]
given by $f(x_0, \cdots, x_k) := (f_0(x_0), f_1(x_1), \cdots, f_k(x_k))$ is an isometric immersion, which is called a warped product immersion.

S. Nölker [110] extended Moore’s result to the following.

**Theorem 2.2.** Let $f : N_0 \times_{\sigma_1} N_1 \times \cdots \times_{\sigma_k} N_k \rightarrow \mathbb{R}^n(c)$ be an isometric immersion into a Riemannian manifold of constant curvature $c$. If $h$ is the second fundamental form of $f$ and $h(X_i, X_j) = 0$, for all vector fields $X_i$ and $X_j$, tangent to $N_i$ and $N_j$ respectively, with $i \neq j$, then, locally, $f$ is a warped product immersion.

3. **WARPED PRODUCT SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS**

In this section, we present solutions from [42] to the Fundamental Question mentioned in Introduction.

For a warped product $N_1 \times f N_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibers, respectively. Thus $\mathcal{D}_1$ is obtained from tangent vectors of $N_1$ via the horizontal lift and $\mathcal{D}_2$ obtained by tangent vectors of $N_2$ via the vertical lift.

Let $\phi : N_1 \times f N_2 \rightarrow R^m(c)$ be an isometric immersion of a warped product $N_1 \times f N_2$ into a Riemannian manifold with constant sectional curvature $c$. Denote by $\sigma$ the second fundamental form of $\phi$.

The immersion $\phi : N_1 \times f N_2 \rightarrow R^m(c)$ is called mixed totally geodesic if $\sigma(X, Z) = 0$ for any $X$ in $\mathcal{D}_1$ and $Z$ in $\mathcal{D}_2$.

The following result was proved in [42].

**Theorem 3.1.** For any isometric immersion $\phi : N_1 \times f N_2 \rightarrow R^m(c)$ of a warped product $N_1 \times f N_2$ into a Riemannian manifold of constant curvature $c$, we have

\[
\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 c,
\]

where $n_1 = \text{dim } N_1$, $n = n_1 + n_2$, $H^2$ is the squared mean curvature of $\phi$, and $\Delta f$ is the Laplacian on $\phi$.

The equality sign of (3.1) holds identically if and only if $i : N_1 \times f N_2 \rightarrow R^m(c)$ is a mixed totally geodesic immersion satisfying $\text{trace } h_1 = \text{trace } h_2$, where $\text{trace } h_1$ and $\text{trace } h_2$ denote the trace of $\sigma$ restricted to $N_1$ and $N_2$, respectively.
The classification of isometric immersions from warped products into a real space form $R^m(c)$ satisfying the equality case of $\eqref{3.1}$ have been obtained in $[51]$. Theorem $3.1$ has many applications $[42]$. 

**Corollary 3.1.** Let $N_1$ and $N_2$ be two Riemannian manifolds and $f$ be a nonzero harmonic function on $N_1$. Then every minimal isometric immersion of $N_1 \times_f N_2$ into any Euclidean space is a warped product immersion.

**Remark 3.1.** There exist many minimal immersion of $N_1 \times_f N_2$ into Euclidean spaces. For example, if $N_2$ is a minimal submanifold of $S^{m-1}(1) \subset \mathbb{E}^m$, then the minimal cone $C(N_2)$ over $N_2$ with vertex at the origin is a warped product $\mathbb{R}_+ \times_s N_2$ whose warping function $f = s$ is a harmonic function. Here $s$ is the coordinate function of the positive real line $\mathbb{R}_+$.

**Corollary 3.2.** Let $f \neq 0$ be a harmonic function on $N_1$. Then for any Riemannian manifold $N_2$ the warped product $N_1 \times_f N_2$ does not admits any minimal isometric immersion into any hyperbolic space.

**Corollary 3.3.** Let $f$ be a function on $N_1$ with $\Delta f = \lambda f$ with $\lambda > 0$. Then for any Riemannian manifold $N_2$ the warped product $N_1 \times_f N_2$ admits no minimal isometric immersion into any Euclidean space or hyperbolic space.

**Remark 3.2.** In views of Corollaries $3.2$ and $3.3$ we point out that there exist minimal immersions from $N_1 \times_f N_2$ into hyperbolic space such that the warping function $f$ is an eigenfunction with negative eigenvalue. For example, $\mathbb{R} \times_e \mathbb{E}^{n-1}$ admits an isometric minimal immersion into the hyperbolic space $H^{n+1}(-1)$ of constant curvature $-1$.

**Corollary 3.4.** If $N_1$ is a compact, then $N_1 \times_f N_2$ does not admit a minimal isometric immersion into any Euclidean space or hyperbolic space.

**Remark 3.3.** For Corollary $3.4$ we point out that there exist many minimal immersions of $N_1 \times_f N_2$ into $\mathbb{E}^m$ with compact $N_2$. For examples, a hypercartenoid in $\mathbb{E}^{n+1}$ is a minimal hypersurfaces which is isometric to a warped product $\mathbb{R} \times_f S^{n-1}$. Also, for any compact minimal submanifold $N_2$ of $S^{m-1} \subset \mathbb{E}^m$, the minimal cone $C(N_2)$ is a warped product $\mathbb{R}_+ \times_s N_2$ which is also a such example.

**Remark 3.4.** Contrast to Euclidean and hyperbolic spaces, $S^m$ admits minimal warped product submanifolds $N_1 \times_f N_2$ with $N_1$ and $N_2$ being compact. The simplest such examples are minimal Clifford tori defined by

$$M_{k,n-k} = S^k \left( \sqrt{\frac{k}{n}} \right) \times S^{n-k} \left( \sqrt{\frac{n-k}{n}} \right) \subset S^{n+1}, \ 1 \leq k < n.$$  

**Remark 3.5.** Suceava constructs in $[137]$ a family of warped products of hyperbolic planes which do not admit any isometric minimal immersion into Euclidean space, by applying some $\delta$-invariants introduced in $[35]$.

By making a minor modification of the proof of Theorem $3.1$ in $[42]$, using the method of $[50]$, we have the following general solution from $[64]$ to the Fundamental Question.

**Theorem 3.2.** If $\tilde{M}^m$ is a Riemannian manifold with sectional curvatures bounded from above by a constant $c$, then for any isometric immersion $\phi : N_1 \times_f N_2 \to \tilde{M}^m$
from a warped product $N_1 \times_f N_2$ into $\hat{M}_1^n$ the warping function $f$ satisfies

$$
\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 c,
$$

where $n_1 = \dim N_1$ and $n_2 = \dim N_2$.

Another general solution to the Fundamental Question is the following result from [48].

**Theorem 3.3 ([53]).** For any isometric immersion $\phi : N_1 \times_f N_2 \rightarrow R^n(c)$, the scalar curvature $\tau$ of the warped product $N_1 \times_f N_2$ satisfies

$$
\tau \leq \frac{\Delta f}{n_1 f} + \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)c.
$$

If $n = 2$, the equality case of (3.3) holds automatically.

If $n \geq 3$, the equality sign of (3.3) holds identically if and only if either

1. $N_1 \times_f N_2$ is of constant curvature $c$, the warping function $f$ is an eigenfunction with eigenvalue $c$, i.e., $\Delta f = cf$, and $N_1 \times_f N_2$ is immersed as a totally geodesic submanifold in $R^n(c)$, or

2. locally, $N_1 \times_f N_2$ is immersed as a rotational hypersurface in a totally geodesic submanifold $R^{n+1}(c)$ of $R^n(c)$ with a geodesic of $R^{n+1}(c)$ as its profile curve.

**Remark 3.6.** Every Riemannian manifold of constant curvature $c$ can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$. For examples, $S^n(1)$ is locally isometric to $(0, \infty) \times_{\cos x} S^{n-1}(1)$; $\mathbb{E}^n$ is locally isometric to $(0, \infty) \times_x S^{n-1}(1)$; and $H^n(-1)$ is locally isometric to $\mathbb{R} \times_e \mathbb{E}^{n-1}$.

There are other warped product decompositions of $R^n(c)$ whose warping function satisfies $\Delta f = cf$. For example, let $\{x_1, \ldots, x_n\}$ be a Euclidean coordinate system of $\mathbb{E}^n$ and let $\rho = \sum_{j=1}^{n_1} a_j x_j + b$, where $a_1, \ldots, a_{n_1}, b$ are real numbers satisfying $\sum_{j=1}^{n_1} a_j^2 = 1$. Then the warped product $\mathbb{E}^{n_1} \times_{\rho} S^{n_2}(1)$ is a flat space whose warping function is a harmonic function. In fact, those are the only warped product decompositions of flat spaces whose warping functions are harmonic functions.

4. MULTIPLY WARPED PRODUCT SUBMANIFOLDS

Let $(N_i, g_i), i = 1, \ldots, k$, be Riemannian manifolds. For a multiply warped product manifold $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$, let $D_i$ denote the distributions obtained from the vectors tangent to $N_i$ (or more precisely, vectors tangent to the horizontal lifts of $N_i$). Assume that

$$
\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \rightarrow \hat{M}
$$

is an isometric immersion of a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian manifold $\hat{M}$. Denote by $\sigma$ the second fundamental form of $\phi$. Then the immersion $\phi$ is called *mixed totally geodesic* if $\sigma(D_i, D_j) = \{0\}$ holds for distinct $i, j \in \{1, \ldots, k\}$.

Let $\psi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \rightarrow \hat{M}$ be an isometric immersion of a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into an arbitrary Riemannian manifold $\hat{M}$.
Denote by \( \text{trace} \ h_i \) the trace of \( \sigma \) restricted to \( N_i \), that is

\[
\text{trace} \ h_i = \sum_{\alpha=1}^{n_i} \sigma(e_{\alpha}, e_{\alpha})
\]

for some orthonormal frame fields \( e_1, \ldots, e_{n_i} \) of \( D_i \).

An extension of Theorems 3.1 and 3.2 is the following result from [57].

**Theorem 4.1.** Let \( \phi : N_1 \times f_2 N_2 \times \cdots \times f_k N_k \to \tilde{M}^m \) be an isometric immersion of a multiply warped product \( N := N_1 \times f_2 N_2 \times \cdots \times f_k N_k \) into an arbitrary Riemannian \( m \)-manifold. Then we have

\[
\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \leq \frac{n^2(k-1)}{2k} H^2 + n_1(n - n_1) \max \tilde{K},
\]

where \( n = \sum_{i=1}^{k} n_i \) and \( \max \tilde{K}(p) \) denotes the maximum of the sectional curvature function of \( \tilde{M}^m \) restricted to 2-plane sections of the tangent space \( T_p N \) of \( N \) at \( p = (p_1, \ldots, p_k) \).

The equality sign of (4.1) holds identically if and only if the following two statements hold:

1. \( \phi \) is a mixed totally geodesic immersion satisfying \( \text{trace} \ h_1 = \cdots = \text{trace} \ h_k \);
2. at each point \( p \in N \), the sectional curvature function \( \tilde{K} \) of \( \tilde{M}^m \) satisfies

\[
\tilde{K}(u,v) = \max \tilde{K}(p)
\]

for each unit vector \( u \) in \( T_{p_1}(N_1) \) and unit vector \( v \) in \( T_{(p_2, \ldots, p_k)}(N_2 \times \cdots \times N_k) \).

The following example shows that inequalities (4.1) is sharp.

**Example 4.1.** Let \( M_1 \times p_{2} M_2 \times \cdots \times p_{k} M_k \) be a multiply warped product representation of a real space form \( R^m(c) \). Assume that \( \psi_1 : N_1 \to M_1 \) is a minimal immersion of \( N_1 \) into \( M_1 \) and let \( f_2, \ldots, f_k \) be the restrictions of \( p_2, \ldots, p_k \) on \( N_1 \). Then the following warped product immersion:

\[
\psi = (\psi_1, id, \ldots, id) : N_1 \times f_2 M_2 \times \cdots \times f_k M_k \to M_1 \times p_{2} M_2 \times \cdots \times p_{k} M_k \subset R^m(c)
\]

is a mixed totally geodesic warped product submanifold of \( R^m(c) \) which satisfies the condition:

\[
\text{trace} \ h_1 = \cdots = \text{trace} \ h_k = 0.
\]

Thus \( \psi \) satisfies the equality case of (4.1) according to Theorem 4.1. Therefore inequality (4.1) is optimal.

By applying Theorem 4.1 we have the following corollaries.

**Corollary 4.1.** Let \( \phi : N_1 \times f_2 N_2 \times \cdots \times f_k N_k \to R^m(c) \) be an isometric immersion of the multiply warped product \( N_1 \times f_2 N_2 \times \cdots \times f_k N_k \) into a Riemannian \( m \)-manifold of constant curvature \( c \). If we have

\[
\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} = \frac{n^2(k-1)}{2k} H^2 + n_1(n - n_1)c,
\]

then \( \phi \) is a warped product immersion.
Corollary 4.2. If $f_2, \ldots, f_k$ are harmonic functions on $N_1$ or eigenfunctions of the Laplacian on $N_1$ with positive eigenvalues, then the multiply warped product $N_1 \times f_2 N_2 \times \cdots \times f_k N_k$ cannot be isometrically immersed into every Riemannian manifold of negative sectional curvature as a minimal submanifold.

Corollary 4.3. If $f_2, \ldots, f_k$ are eigenfunctions of the Laplacian $\Delta$ on $N_1$ with nonnegative eigenvalues and at least one of $f_2, \ldots, f_k$ is non-harmonic, then the multiply warped product manifold $N_1 \times f_2 N_2 \times \cdots \times f_k N_k$ cannot be isometrically immersed into every Riemannian manifold of non-positive sectional curvature as a minimal submanifold.

Corollary 4.4. If $f_2, \ldots, f_k$ are harmonic functions on $N_1$, then every isometric minimal immersion of the multiply warped product manifold $N_1 \times f_2 N_2 \times \cdots \times f_k N_k$ into a Euclidean space is a warped product immersion.

Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds and let $\sigma_1 : N_1 \rightarrow (0, \infty)$ and $\sigma_2 : N_2 \rightarrow (0, \infty)$ be differentiable functions. The doubly warped product $N = \sigma_2 N_1 \times \sigma_1 N_2$ is the product manifold $N_1 \times N_2$ endowed with the metric

$$g = \sigma_2^2 g_1 + \sigma_1^2 g_2.$$ 

The following result is obtained in [118].

Theorem 4.2. Let $\phi : \sigma_2 N_1 \times_{\sigma_1} N_2 \rightarrow \hat{M}^m$ be an isometric immersion of a doubly warped product $\sigma_2 N_1 \times_{\sigma_1} N_2$ into an arbitrary Riemannian $m$-manifold. Then we have

$$(4.2) \quad n_2 \frac{\Delta_1 \sigma_1}{\sigma_1} + n_1 \frac{\Delta_2 \sigma_2}{\sigma_2} \leq \frac{n^2}{4} H^2 + n_1 n_2 \max \hat{K},$$

where $n_i = \dim N_i$, $n = n_1 + n_2$, $\Delta_i$ denotes the Laplacian of $N_i$.

The equality sign of (4.2) holds identically if and only if the following two statements hold:

1. $\phi$ is a mixed totally geodesic immersion satisfying trace $h_1 = \text{trace} \, h_2$ and
2. at each point $p = (p_1, p_2) \in N$, the sectional curvature function $\hat{K}$ of $\hat{M}^m$ satisfies $\hat{K}(u, v) = \max \hat{K}(p)$ for each unit vector $u$ in $T_{p_1}(N_1)$ and unit vector $v$ in $T_{p_2}(N_2)$.

5. ARBITRARY WARPED PRODUCTS IN COMPLEX SPACE FORMS

Now, we present the following results form [45, 47] for arbitrary warped products submanifolds in non-flat complex space forms.

For arbitrary warped products submanifolds in complex hyperbolic spaces, we have the following general results from [45].

Theorem 5.1. Let $\phi : N_1 \times f N_2 \rightarrow CH^m(4c)$ be an arbitrary isometric immersion of a warped product $N_1 \times f N_2$ into the complex hyperbolic $m$-space $CH^m(4c)$ of constant holomorphic sectional curvature $4c$. Then we have

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 c.$$ 

The equality sign of (5.1) holds if and only if the following three conditions hold:

1. $\phi$ is mixed totally geodesic,
2. trace $h_1 = \text{trace} \, h_2$, and
(3) $JD_1 \perp D_2$, where $J$ is the almost complex structure of $CH^m$.

Some interesting applications of Theorem 5.1 are the following three non-immersion theorems.

**Theorem 5.2.** Let $N_1 \times_f N_2$ be a warped product whose warping function $f$ is harmonic. Then $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.

**Theorem 5.3.** If $f$ is an eigenfunction of Laplacian on $N_1$ with eigenvalue $\lambda > 0$, then $N_1 \times_f N_2$ does not admits an isometric minimal immersion into any complex hyperbolic space.

**Theorem 5.4.** If $N_1$ is compact, then every warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.

For arbitrary warped products submanifolds in the complex projective $m$-space $CP^m(4c)$ with constant holomorphic sectional curvature $4c$, we have the following results from [47].

**Theorem 5.5.** Let $\phi : N_1 \times_f N_2 \to CP^m(4c)$ be an arbitrary isometric immersion of a warped product into the complex projective $m$-space $CP^m(4c)$ of constant holomorphic sectional curvature $4c$. Then we have

$$\Delta f \leq \frac{(n_1 + n_2)^2}{4n_2}H^2 + (3 + n_1)c.$$  

(5.2)

The equality sign of (5.2) holds identically if and only if we have

1. $n_1 = n_2 = 1$,
2. $f$ is an eigenfunction of the Laplacian of $N_1$ with eigenvalue $4c$, and
3. $\phi$ is totally geodesic and holomorphic.

From now on, we denote $S^n(1)$, $RP^n(1)$, $CP^n(4)$ and $CH^n(-4)$ simply by $S^n$, $RP^n$, $CP^n$ and $CH^n$, respectively.

An immediate application of Theorem 5.5 is the following non-immersion result.

**Theorem 5.6.** If $f$ is a positive function on a Riemannian $n_1$-manifold $N_1$ such that $\Delta f/f > 3 + n_1$ at some point $p \in N_1$, then, for any Riemannian manifold $N_2$, the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into $CP^m$ for any $m$.

For totally real minimal immersions, Theorem 5.6 can be sharpen as the following.

**Theorem 5.7.** If $f$ is a positive function on a Riemannian $n_1$-manifold $N_1$ such that $(\Delta f)/f > n_1$ at some point $p \in N_1$, then, for any Riemannian manifold $N_2$, the warped product $N_1 \times_f N_2$ does not admit any isometric totally real minimal immersion into $CP^m$ for any $m$.

The following examples show that Theorems 5.5, 5.6 and 5.7 are sharp.

**Example 5.1.** Let $I = (-\pi/4, \pi/4)$, $N_2 = S^1$ and $f = \frac{1}{2} \cos 2s$. Then the warped product

$$N_1 \times_f N_2 = I \times_{(\cos 2s)/2} S^1$$

has constant sectional curvature 4. Clearly, we have $\Delta f/f = 4$. If we define the complex structure $J$ on the warped product by $J \left(\frac{\partial}{\partial \tau}\right) = 2(\sec 2s)\frac{\partial}{\partial \sigma}$, then $(I \times_{(\cos 2s)/2} S^1, g, J)$ is holomorphically isometric to a dense open subset of $CP^1$. 
Let $\phi : CP^1 \to CP^m$ be a standard totally geodesic embedding of $CP^1$ into $CP^m$. Then the restriction of $\phi$ to $I \times_{(cos\, 2\alpha)/2} S^1$ gives rise to a minimal isometric immersion of $I \times_{(cos\, 2\alpha)/2} S^1$ into $CP^m$ which satisfies the equality case of (5.2) on $I \times_{(cos\, 2\alpha)/2} S^1$ identically.

**Example 5.2.** Consider the same warped product $N_1 \times f N_2 = I \times_{(cos\, 2\alpha)/2} S^1$ as given in Example 5.1. Let $\phi : CP^1 \to CP^m$ be the totally geodesic holomorphic embedding of $CP^1$ into $CP^m$. Then the restriction of $\phi$ to $N_1 \times f N_2$ is an isometric minimal immersion of $N_1 \times f N_2$ into $CP^m$ which satisfies $(\Delta \phi)/\phi = 3 + n_1$ identically. This example shows that the assumption “$(\Delta f)/f > n_1$ at some point in $N_1$” given in Theorem 5.6 is best possible.

**Example 5.3.** Let $g_1$ be the standard metric on $S^{n-1}$. Denote by $N_1 \times f N_2$ the warped product given by $N_1 = (-\pi/2, \pi/2)$, $N_2 = S^{n-1}$ and $f = \cos s$. Then the warping function of this warped product satisfies $\Delta f/f = n_1$ identically. Moreover, it is easy to verify that this warped product is isometric to a dense open subset of $S^n$.

Let

$$\phi : S^n \xrightarrow{\text{projection}} RP^m \xrightarrow{\text{totally geodesic}} CP^n$$

be a standard totally geodesic Lagrangian immersion of $S^n$ into $CP^n$. Then the restriction of $\phi$ to $N_1 \times f N_2$ is a totally real minimal immersion. This example illustrates that the assumption “$(\Delta f)/f > n_1$ at some point in $N_1$” given in Theorem 5.6 is also sharp.

A generalized complex space form of constant curvature $c$ and constant type $\alpha$, denoted by $M(c, \alpha)$, is an almost Hermitian manifold $(M, J, g)$ whose curvature tensor $R$ satisfies

$$R(X, Y, Z, W) = \frac{1}{4}(c + 3\alpha)\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}$$

$$+ \frac{1}{4}(c - \alpha)\{g(X, JZ)g(Y, JW) - g(X, JW)g(Y, JZ) + 2g(X, JY)g(Z, JW)\}$$

for some $c$ and $\alpha$. The class of generalized complex space forms contains all complex space forms and the nearly Kähler manifold $S^6$.

For warped products submanifold of a generalized complex space form, the following result is obtained in [97].

**Theorem 5.8.** Let $N = N_1 \times f N_1$ a warped product submanifold of a generalized complex space form $M(c, \alpha)$. Then

1. For $c \leq \alpha$, one has

$$\frac{\Delta f}{f} \geq \frac{n_1^2}{4n_2} H^2 + n_1 \frac{c + 3\alpha}{4}$$

where $n_i = \dim N_i$ and $n = n_1 + n_2$.

2. For $c > \alpha$, one has

$$\frac{\Delta f}{f} \geq \frac{n_1^2}{4n_2} H^2 + n_1 \frac{c + 3\alpha}{4} + \frac{3c - \alpha}{8} ||P||^2,$$

where $PX$ denotes the part of $JX$ which is tangent to the submanifold.
The equality case of (5.4) and (5.5) were discussed in [97].
For CR-warped product submanifold of a generalized complex space form, one has the following result from [4].

**Theorem 5.9.** Let \( N = N_T \times_f N_\perp \) a CR-warped product submanifold of a generalized complex space form \( M(c, \alpha) \). Then we have

\[
||\sigma||^2 \geq 2p \left( ||\nabla (\ln f)||^2 + \frac{1}{2} \Delta (\ln f) + \frac{h(c - \alpha)}{4} \right),
\]

where \( \sigma \) is the second fundamental form) and \( 2h \) and \( p \) are the real dimensions of \( N_T \) and \( N_\perp \), respectively.

6. WARPED PRODUCTS AS KÄHLERIAN SUBMANIFOLDS

Let \((z^i_1, \ldots, z^i_s), 1 \leq i \leq s\), be homogeneous coordinates of \( CP^{\alpha_i} \). Define a map:

\[
S_{\alpha_1 \cdots \alpha_s} : CP^{\alpha_1} \times \cdots \times CP^{\alpha_s} \to CP^N, \quad N = \prod_{i=1}^s (\alpha_i + 1) - 1,
\]

which maps a point \(((z^1_1, \ldots, z^1_{\alpha_1}), \ldots, (z^s_1, \ldots, z^s_{\alpha_s}))\) in \( CP^{\alpha_1} \times \cdots \times CP^{\alpha_s} \) to the point \((z^1_{i_1} \cdots z^s_{i_s})_{1 \leq i_1 \leq \alpha_1, \ldots, 1 \leq i_s \leq \alpha_s}\) in \( CP^N \). The map \( S_{\alpha_1 \cdots \alpha_s} \) is a Kähler embedding which is known as the Segre embedding. The Segre embedding was constructed by C. Segre in 1891.

The following results from [28, 58] obtained in 1981 can be regarded as “converse” to Segre embedding constructed in 1891.

**Theorem 6.1.** Let \( M^{\alpha_1}_1, \ldots, M^{\alpha_s}_s \) be Kähler manifolds of dimensions \( \alpha_1, \ldots, \alpha_s \), respectively. Then every holomorphically isometric immersion

\[
f : M^{\alpha_1}_1 \times \cdots \times M^{\alpha_s}_s \to CP^N, \quad N = \prod_{i=1}^s (\alpha_i + 1) - 1,
\]

of \( M^{\alpha_1}_1 \times \cdots \times M^{\alpha_s}_s \) into \( CP^N \) is locally the Segre embedding, i.e., \( M^{\alpha_1}_1, \ldots, M^{\alpha_s}_s \) are open portions of \( CP^{\alpha_1}, \ldots, CP^{\alpha_s} \), respectively. Moreover, \( f \) is congruent to the Segre embedding.

Let \( \nabla^k \sigma, k = 0, 1, 2, \ldots \), denote the \( k \)-th covariant derivative of the second fundamental form. Denoted by \( ||\nabla^k \sigma||^2 \) the squared norm of \( \nabla^k \sigma \).

The following result was proved in [28, 58].

**Theorem 6.2.** Let \( M^{\alpha_1}_1 \times \cdots \times M^{\alpha_s}_s \) be a product Kähler submanifold of \( CP^N \). Then

\[
||\nabla^{k-2} \sigma||^2 \geq k! \cdot 2^k \sum_{i_1 < \cdots < i_k} \alpha_1 \cdots \alpha_k,
\]

for \( k = 2, 3, \ldots \).

The equality sign of (6.2) holds for some \( k \) if and only if \( M^{\alpha_1}_1, \ldots, M^{\alpha_s}_s \) are open parts of \( CP^{\alpha_1}, \ldots, CP^{\alpha_s} \), respectively, and the immersion is congruent to the Segre embedding.

In particular, if \( k = 2 \), Theorem 6.2 reduces to the following result of [28].
Theorem 6.3. Let $M^h_1 \times M^p_2$ be a product Kähler submanifold of $CP^N$. Then we have

\[(6.3) \quad ||\sigma||^2 \geq 8hp.\]

The equality sign of (6.3) holds if and only if $M^h_1$ and $M^p_2$ are open portions of $CP^h$ and $CP^p$, respectively, and the immersion is congruent to the Segre embedding $S_{h,p}$.

We may extend Theorem 6.3 to the following for warped products.

Theorem 6.4. Let $(M^h_1, g_1)$ and $(M^p_2, g_2)$ be two Kähler manifolds of complex dimension $h$ and $p$ respectively and let $f$ be a positive function on $M^h_1$. If $\phi : M^h_1 \times f M^p_2 \rightarrow CP^N$ is a holomorphically isometric immersion of the warped product manifold $M^h_1 \times f M^p_2$ into $CP^N$. Then $f$ is a constant, say $c$. Moreover, we have

\[(6.4) \quad ||\sigma||^2 \geq 8hp.\]

The equality sign of (6.4) holds if and only if $(M^h_1, g_1)$ and $(M^p_2, cg_2)$ are open portions of $CP^h$ and $CP^p$, respectively, and the immersion $\phi$ is congruent to the Segre embedding.

Proof. Under the hypothesis, the warped product manifold $M^h_1 \times f M^p_2$ is a Kähler manifold. Therefore, the warping function must be a positive constant. Now, the theorem follows from Theorem 6.3. \[\square\]

7. CR-PRODUCTS IN KÄHLER MANIFOLDS

A submanifold $N$ in a Kähler manifold $\tilde{M}$ is called a totally real submanifold if the almost complex structure $J$ of $\tilde{M}$ carries each tangent space $T_xN$ of $N$ into its corresponding normal space $T_x^{\perp}N$ [61]. The submanifold $N$ is called a holomorphic submanifold (or Kähler submanifold) if $J$ carries each $T_xN$ into itself.

A submanifold $N$ of a Kähler manifold $\tilde{M}$ is called a CR-submanifold [20] if there exists on $N$ a holomorphic distribution $D$ whose orthogonal complement $D^{\perp}$ is a totally real distribution, i.e., $JD^{\perp} \subset T_x^{\perp}N$.

A CR-submanifold of a Kähler manifold $\tilde{M}$ is called a CR-product [28] if it is a Riemannian product $N_T \times N_\perp$ of a Kähler submanifold $N_T$ and a totally real submanifold $N_\perp$. It is called mixed totally geodesic if the second fundamental form of the CR-submanifold satisfying $\sigma(X, Z) = 0$ for any $X \in D$ and $Z \in D^{\perp}$.

For CR-products in complex space forms, the following result from [28] are known.

Theorem 7.1. We have

(i) A CR-submanifold in the complex Euclidean $m$-space $\mathbb{C}^m$ is a CR-product if and only if it is a direct sum of a Kähler submanifold and a totally real submanifold of linear complex subspaces.

(ii) There do not exist CR-products in complex hyperbolic spaces other than Kähler submanifolds and totally real submanifolds.

CR-products $N_T \times N_\perp$ in $CP^{h+p+hp}$ are always obtained from the Segre embedding $S_{h,p}$; namely, we have the following results from [28].
Theorem 7.2. Let $N^h_T \times N^p_L$ be the CR-product in $\mathbb{C}P^m$ with constant holomorphic sectional curvature 4. Then

\begin{equation}
    m \geq h + p + hp.
\end{equation}

The equality sign of (7.1) holds if and only if

(a) $N^h_T$ is a totally geodesic Kähler submanifold,
(b) $N^p_L$ is a totally real submanifold, and
(c) the immersion is given by

\[ N^h_T \times N^p_L \rightarrow \mathbb{C}P^h \times \mathbb{C}P^p \xrightarrow{S_{hp}} \mathbb{C}P^{h+p+hp}. \]

Theorem 7.3. Let $N^h_T \times N^p_L$ be the CR-product in $\mathbb{C}P^m$. Then the squared norm of the second fundamental form satisfies

\begin{equation}
    ||\sigma||^2 \geq 4hp.
\end{equation}

The equality sign of (7.2) holds if and only if

(a) $N^h_T$ is a totally geodesic Kähler submanifold,
(b) $N^p_L$ is a totally geodesic totally real submanifold, and
(c) the immersion is given by

\[ N^h_T \times N^p_L \text{ totally geodesic} \rightarrow \mathbb{C}P^h \times \mathbb{C}P^p \xrightarrow{S_{hp}} \mathbb{C}P^{h+p+hp} \subset \mathbb{C}P^m. \]

8. WARPED PRODUCT LAGRANGIAN SUBMANIFOLDS OF KÄHLER MANIFOLDS

A totally real submanifold $N$ in a Kähler manifold $\tilde{M}$ is called a Lagrangian submanifold if $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} \tilde{M}$. For the most recent survey on differential geometry of Lagrangian submanifolds, see [37, 41].

For Lagrangian immersions into complex Euclidean $n$-space $\mathbb{C}^n$, a well-known result of M. Gromov [74] states that a compact $n$-manifold $M$ admits a Lagrangian immersion (not necessary isometric) into $\mathbb{C}^n$ if and only if the complexification $TM \otimes \mathbb{C}$ of the tangent bundle of $M$ is trivial. In particular, Gromov’s result implies that there exists no topological obstruction to Lagrangian immersions for compact 3-manifolds in $\mathbb{C}^3$, because the tangent bundle of a 3-manifold is always trivial.

Not every warped product $N_1 \times_f N_2$ can be isometrically immersed in a complex space form as a Lagrangian submanifold. Therefore, from Riemannian point of view, it is natural to ask the following basic question.

Problem 8.1. When a warped product $n$-manifold admits a Lagrangian isometric immersion into $\mathbb{C}^n$?

For warped products of curves and the unit $(n-1)$-sphere $S^{n-1}$, we have the following existence theorem from [33, 48].

Theorem 8.1. Every simply-connected open portion of a warped product manifold $I \times_f S^{n-1}$ of an open interval $I$ and a unit $(n-1)$-sphere admits an isometric Lagrangian immersion into $\mathbb{C}^n$.

The Lagrangian immersions given in Theorem 8.1 are expressed in terms of complex extensors in the sense of [33]. In particular, Theorem 8.1 implies the following.

Corollary 8.1. Every warped product surface does admit a Lagrangian isometric immersion into $\mathbb{C}^2$.

Consequently, we have a complete solution to Problem 8.1 for $n = 2$.

Since all rotation hypersurfaces and real space forms can be expressed, at least locally, as warped products of some curves and a unit sphere, therefore Theorem 8.1 implies the following

Corollary 8.2. Every rotation hypersurface of $E^{n+1}$ can be isometrically immersed as Lagrangian submanifold in $\mathbb{C}^n$.

Corollary 8.3. Every Riemannian $n$-manifold of constant sectional curvature $c$ can be locally isometrically immersed in $\mathbb{C}^n$ as a Lagrangian submanifold.

Remark 8.1. Not every Riemannian $n$-manifold of constant sectional curvature can be globally isometrically immersed in $\mathbb{C}^n$ as a Lagrangian submanifold. For instance, it is known from [35] that every compact Riemannian $n$-manifold with positive sectional curvature (or with positive Ricci curvature) does not admit any Lagrangian isometric immersion into $\mathbb{C}^n$.

Remark 8.2. For further results on warped products Lagrangian submanifolds in complex space forms, see [23].

9. WARPED PRODUCT CR-SUBMANIFOLDS OF KÄHLER MANIFOLDS

In this section we present results concerning warped product $CR$-submanifold in an arbitrary Kähler manifold. First, we mention the following result from [39].

Theorem 9.1. If $N_\perp \times f N_T$ is a warped product $CR$-submanifold of a Kähler manifold $\tilde{M}$ such that $N_\perp$ is a totally real and $N_T$ a Kähler submanifold of $\tilde{M}$, then it is a $CR$-product.

Theorem 9.1 shows that there does not exist warped product $CR$-submanifolds of the form $N_\perp \times f N_T$ other than $CR$-products. So, we shall only consider warped product $CR$-submanifolds of the form: $N_T \times f N_\perp$, by reversing the two factors $N_T$ and $N_\perp$. We simply call such $CR$-submanifolds $CR$-warped products [39].

$CR$-warped products are characterized in [39] as follows.

Proposition 9.1. A proper $CR$-submanifold $M$ of a Kähler manifold $\tilde{M}$ is locally a $CR$-warped product if and only if the shape operator $A$ satisfies

\begin{equation}
A_{JZ}X = ((JX)\mu)Z, \quad X \in \mathcal{D}, \quad Z \in \mathcal{D}^\perp,
\end{equation}

for some function $\mu$ on $M$ satisfying $W\mu = 0$, $\forall W \in \mathcal{D}^\perp$.

A fundamental general result on $CR$-warped products in arbitrary Kähler manifolds is the following theorem from [39].

Theorem 9.2. Let $N_T \times f N_\perp$ be a $CR$-warped product submanifold in an arbitrary Kähler manifold $M$. Then the second fundamental form $\sigma$ satisfies

\begin{equation}
||\sigma||^2 \geq 2p ||\nabla (\ln f)||^2,
\end{equation}

where $\nabla (\ln f)$ is the gradient of $\ln f$ on $N_T$ and $p = \dim N_\perp$. 
If the equality sign of (9.2) holds identically, then \( N_T \) is a totally geodesic Kähler submanifold and \( N_\perp \) is a totally umbilical totally real submanifold of \( \tilde{M} \). Moreover, \( N_T \times_f N_\perp \) is minimal in \( \tilde{M} \).

When \( M \) is anti-holomorphic, i.e., when \( JD_x^\perp = T_x^\perp N \), and \( p > 1 \). The equality sign of (9.2) holds identically if and only if \( N_\perp \) is a totally umbilical submanifold of \( \tilde{M} \).

If \( M \) is anti-holomorphic and \( p = 1 \), then the equality sign of (9.2) holds identically if and only if the characteristic vector field \( J_\xi \) of \( M \) is a principal vector field with zero as its principal curvature. (Notice that in this case, \( M \) is a real hypersurface in \( \tilde{M} \).) Also, in this case, the equality sign of (9.2) holds identically if and only if \( M \) is a minimal hypersurface in \( \tilde{M} \).

CR-warped products in complex space forms satisfying the equality case of (9.2) have been completely classified in [39, 40].

**Theorem 9.3.** A CR-warped product \( N_T \times_f N_\perp \) in \( \mathbb{C}^m \) satisfies
\[
\|\sigma\|^2 = 2p\|\nabla (\ln f)\|^2
\]
identically if and only if the following four statements hold:

(i) \( N_T \) is an open portion of a complex Euclidean \( h \)-space \( \mathbb{C}^h \),

(ii) \( N_\perp \) is an open portion of the unit \( p \)-sphere \( S^p \),

(iii) there exists \( a = (a_1, \ldots, a_h) \in S^{h-1} \subset \mathbb{E}^h \) such that \( f = \sqrt{\langle a, z \rangle^2 + \langle ia, z \rangle^2} \)

for \( z = (z_1, \ldots, z_h) \in \mathbb{C}^h \), \( w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1} \), and

(iv) up to rigid motions, the immersion is given by
\[
\mathbf{x}(z, w) = \left( z_1 + (w_0 - 1)a_1 \sum_{j=1}^{h} a_j z_j, \ldots, z_h + a(w_0 - 1)a_h \sum_{j=1}^{h} a_j z_j, \right.
\]
\[
\left. w_1 \sum_{j=1}^{h} a_j z_j, \ldots, w_p \sum_{j=1}^{h} a_j z_j, 0, \ldots, 0 \right).
\]

A CR-warped product \( N_T \times_f N_\perp \) is said to be trivial if its warping function \( f \) is constant. A trivial CR-warped product \( N_T \times_f N_\perp \) is nothing but a CR-product \( N_T \times N'_\perp \), where \( N'_\perp \) is the manifold with metric \( f^2 g_{N'_\perp} \) which is homothetic to the original metric \( g_{N_\perp} \) on \( N_\perp \).

The following result from [40] completely classifies CR-warped products in complex projective spaces satisfying the equality case of (9.2) identically.

**Theorem 9.4.** A non-trivial CR-warped product \( N_T \times_f N_\perp \) in \( CP^m \) satisfies the basic equality \( \|\sigma\|^2 = 2p\|\nabla (\ln f)\|^2 \) if and only if we have

(1) \( N_T \) is an open portion of complex Euclidean \( h \)-space \( \mathbb{C}^h \),

(2) \( N_\perp \) is an open portion of a unit \( p \)-sphere \( S^p \), and

(3) up to rigid motions, the immersion \( \tilde{\mathbf{x}} \) of \( N_T \times_f N_\perp \) into \( CP^m \) is the composition \( \pi \circ \tilde{\mathbf{x}} \), where
\[
\tilde{\mathbf{x}}(z, w) = \left( z_0 + (w_0 - 1)a_0 \sum_{j=0}^{h} a_j z_j, \ldots, z_h + (w_0 - 1)a_h \sum_{j=0}^{h} a_j z_j, \right.
\]
\[
\left. w_1 \sum_{j=0}^{h} a_j z_j, \ldots, w_p \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \right),
\]
\[ \pi \text{ is the projection } \pi : \mathbb{C}^{m+1} \to \mathbb{C}P^m, \ a_0, \ldots, a_h \text{ are real numbers satisfying } a_0^2 + a_1^2 + \cdots + a_h^2 = 1, \ \zeta = (z_0, z_1, \ldots, z_h) \in \mathbb{C}^{h+1} \text{ and } w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}. \]

The following result from [40] completely classifies CR-warped products in complex hyperbolic spaces satisfying the equality case of (9.2) identically.

**Theorem 9.5.** A CR-warped product \( N_T \times f \ N_\perp \) in \( CH^m \) satisfies the basic equality

\[ ||\sigma||^2 = 2p||\nabla(\ln f)||^2 \]

if and only if one of the following two cases occurs:

1. \( N_T \) is an open portion of complex Euclidean \( h \)-space \( \mathbb{C}h \), \( N_\perp \) is an open portion of a unit \( p \)-sphere \( S^p \) and, up to rigid motions, the immersion is the composition \( \pi \circ \mathbf{x} \), where \( \pi \) is the projection \( \pi : \mathbb{C}^{m+1} \to \mathbb{C}h^m \) and

\[ \mathbf{x}(z, w) = \left( z_0 + a_0 (1 - w_0) \sum_{j=0}^{h} a_j z_j, z_1 + a_1 (w_0 - 1) \sum_{j=0}^{h} a_j z_j, \ldots, \right. \]

\[ z_h + a_h (w_0 - 1) \sum_{j=0}^{h} a_j z_j, w_1 \sum_{j=0}^{h} a_j z_j, \ldots, w_p \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \]

for some real numbers \( a_0, \ldots, a_h \) satisfying \( a_0^2 - a_1^2 - \cdots - a_h^2 = -1 \), where \( z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1} \) and \( w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1} \).

2. \( p = 1 \), \( N_T \) is an open portion of \( \mathbb{C}h \) and, up to rigid motions, the immersion is the composition \( \pi \circ \mathbf{x} \), where

\[ \mathbf{x}(z, t) = \left( z_0 + a_0 (\cosh t - 1) \sum_{j=0}^{h} a_j z_j, z_1 + a_1 (1 - \cosh t) \sum_{j=0}^{h} a_j z_j, \right. \]

\[ \ldots, z_h + a_h (1 - \cosh t) \sum_{j=0}^{h} a_j z_j, \sinh t \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \]

for some real numbers \( a_0, a_1, \ldots, a_{h+1} \) satisfying \( a_0^2 - a_1^2 - \cdots - a_h^2 = 1 \).

A multiply warped product \( N_T \times f_2 N_2 \times \cdots \times f_k N_k \) in a Kähler manifold \( \tilde{M} \) is called a multiply CR-warped product if \( N_T \) is a holomorphic submanifold and \( N_\perp = T_z N_2 \times \cdots \times T_z N_k \) is a totally real submanifold of \( \tilde{M} \).

Theorem 9.2 was extended in [50] to multiply CR-warped products in the following.

**Theorem 9.6.** Let \( N = N_T \times f_2 N_2 \times \cdots \times f_k N_k \) be a multiply CR-warped product in an arbitrary Kähler manifold \( \tilde{M} \). Then the second fundamental form \( \sigma \) and the warping functions \( f_2, \ldots, f_k \) satisfy

\[ ||\sigma||^2 \geq 2 \sum_{i=2}^{k} n_i ||\nabla(\ln f_i)||^2. \]  

The equality sign of (9.3) holds identically if and only if the following four statements hold:

1. \( N_T \) is a totally geodesic holomorphic submanifold of \( \tilde{M} \);
2. For each \( i \in \{2, \ldots, k\} \), \( N_i \) is a totally umbilical submanifold of \( \tilde{M} \) with \( -\nabla(\ln f_i) \) as its mean curvature vector;
Warped product submanifolds

(c) \( f_2 N_2 \times \cdots \times f_k N_k \) is immersed as mixed totally geodesic submanifold in \( \tilde{M} \);
and
(d) For each point \( p \in N \), the first normal space \( \text{Im} h_p \) is a subspace of \( J(T_p N_\perp) \).

10. CR-WARPED PRODUCTS WITH COMPACT HOLOMORPHIC FACTOR

When the holomorphic factor \( N_T \) of a CR-warped product \( N_T \times f N_\perp \) is compact, we have the following sharp results from [19].

Theorem 10.1. Let \( N_T \times f N_\perp \) be a CR-warped product in the complex projective \( m \)-space \( CP^m \) of constant holomorphic sectional curvature \( 4 \). If \( N_T \) is compact, then
\[
m \geq h + p + hp.
\]

Theorem 10.2. If \( N_T \times f N_\perp \) is a CR-warped product in \( CP^{h+p+hp} \) with compact \( N_T \), then \( N_T \) is holomorphically isometric to \( CP^h \).

Theorem 10.3. For any CR-warped product \( N_T \times f N_\perp \) in \( CP^m \) with compact \( N_T \) and any \( q \in N_\perp \), we have
\[
\int_{N_T \times \{q\}} ||\sigma||^2 dV_T \geq 4hp \text{vol}(N_T), \tag{10.1}
\]
where \( ||\sigma|| \) is the norm of the second fundamental form, \( dV_T \) is the volume element of \( N_T \), and \( \text{vol}(N_T) \) is the volume of \( N_T \).

The equality sign of \( \text{(10.1)} \) holds identically if and only if we have:
1. The warping function \( f \) is constant.
2. \( (N_T, g_{N_T}) \) is holomorphically isometric to \( CP^h \) and it is isometrically immersed in \( CP^m \) as a totally geodesic complex submanifold.
3. \( (N_\perp, f^* g_{N_\perp}) \) is isometric to an open portion of the real projective \( p \)-space \( RP^p \) of constant sectional curvature one and it is isometrically immersed in \( CP^m \) as a totally geodesic totally real submanifold.
4. \( N_T \times f N_\perp \) is immersed linearly fully in a linear complex subspace \( CP^{h+p+hp} \) of \( CP^m \); and moreover, the immersion is rigid.

Theorem 10.4. Let \( N_T \times f N_\perp \) be a CR-warped product with compact \( N_T \) in \( CP^m \). If the warping function \( f \) is non-constant, then, for each \( q \in N_\perp \), we have
\[
\int_{N_T \times \{q\}} ||\sigma||^2 dV_T \geq 2p\lambda_1 \int_{N_T} (\ln f)^2 dV_T + 4hp \text{vol}(N_T), \tag{10.2}
\]
where \( \lambda_1 \) is the first positive eigenvalue of the Laplacian \( \Delta \) of \( N_T \).

Moreover, the equality sign of \( \text{(10.2)} \) holds identically if and only if we have
1. \( \Delta \ln f = \lambda_1 \ln f \).
2. The CR-warped product is both \( N_T \)-totally geodesic and \( N_\perp \)-totally geodesic.

The following example shows that Theorems 10.3 and 10.4 are sharp.

Example 10.1. Let \( \iota_1 \) be the identity map of \( CP^h \) and let
\[
\iota_2 : RP^p \to CP^p
\]
be a totally geodesic Lagrangian embedding of \( RP^p \) into \( CP^p \). Denote by
\[
\iota = (\iota_1, \iota_2) : CP^h \times RP^p \to CP^h \times CP^p
\]
the product embedding of $t_1$ and $t_2$. Moreover, let $S_{h,p}$ be the Segre embedding of $CP^h \times CP^p$ into $CP^{h+p+h+p}$. Then the composition $\phi = S_{h,p} \circ \iota:

\begin{align*}
CP^h \times RP^p & \xrightarrow{(t_1, t_2)} CP^h \times CP^p \xrightarrow{S_{h,p}} CP^{h+p+h+p}
\end{align*}

is a CR-warped product in $CP^{h+p+h+p}$ whose holomorphic factor $N_T = CP^h$ is a compact manifold. Since the second fundamental form of $\phi$ satisfies the equation: $||\sigma||^2 = 4hp$, we have the equality case of (11.1) identically.

The next example shows that the assumption of compactness in Theorems 10.3 and 10.4 cannot be removed.

**Example 10.2.** Let $C^* = \mathbb{C} - \{0\}$ and $\mathbb{C}^{m+1} = \mathbb{C}^{m+1} - \{0\}$. Denote by $\{z_0, \ldots, z_h\}$ a natural complex coordinate system on $\mathbb{C}^{m+1}$.

Consider the action of $C^*$ on $\mathbb{C}^{m+1}$ given by

$$\lambda \cdot (z_0, \ldots, z_m) = (\lambda z_0, \ldots, \lambda z_m)$$

for $\lambda \in C^*$. Let $\pi(z)$ denote the equivalent class containing $z$ under this action. Then the set of equivalent classes is the complex projective $m$-space $CP^m$ with the complex structure induced from the complex structure on $\mathbb{C}^{m+1}$.

For any two natural numbers $h$ and $p$, we define a map:

$$\bar{\phi} : \mathbb{C}^{h+1}_* \times S^p \to \mathbb{C}^{h+p+1}_*$$

by

$$\bar{\phi}(z_0, \ldots, z_h; w_0, \ldots, w_p) = (w_0z_0, w_1z_0, \ldots, w_pz_0, z_1, \ldots, z_h)$$

for $(z_0, \ldots, z_h)$ in $\mathbb{C}^{h+1}_*$ and $(w_0, \ldots, w_p)$ in $S^p$ with $\sum_{j=0}^p w_j^2 = 1$.

Since the image of $\bar{\phi}$ is invariant under the action of $C^*$, the composition:

$$\pi \circ \bar{\phi} : \mathbb{C}^{h+1}_* \times S^p \xrightarrow{\bar{\phi}} \mathbb{C}^{h+p+1}_* \xrightarrow{\pi} CP^{h+p}$$

induces a CR-immersion of the product manifold $N_T \times S^p$ into $CP^{h+p}$, where

$$N_T = \{(z_0, \ldots, z_h) \in CP^h : z_0 \neq 0\}$$

is a proper open subset of $CP^h$. Clearly, the induced metric on $N_T \times S^p$ is a warped product metric and the holomorphic factor $N_T$ is non-compact.

Notice that the complex dimension of the ambient space is $h + p$; far less than $h + p + hp$.

### 11. Another Optimal Inequality for CR-Warped Products

All CR-warped products in complex space forms also satisfy another general optimal inequality obtained in [54].

**Theorem 11.1.** Let $N = N_T^h \times_f N^p$ be a CR-warped product in a complex space form $\overline{M}(4c)$ of constant holomorphic sectional curvature $c$. Then we have

$$||\sigma||^2 \geq 2p\{||\nabla (\ln f)||^2 + \Delta (\ln f) + 2hc\}. \tag{11.1}$$

If the equality sign of (11.1) holds identically, then $N_T$ is a totally geodesic submanifold and $N^p$ is a totally umbilical submanifold. Moreover, $N$ is a minimal submanifold in $\overline{M}(4c)$. 
The following three theorems from [54] completely classify all CR-warped products which satisfy the equality case of (11.1) identically.

**Theorem 11.2.** Let \( \phi : N^h_T \times_f N^p_\perp \to \mathbb{C}^m \) be a CR-warped product in \( \mathbb{C}^m \). Then we have

\[
||\sigma||^2 \geq 2p\{(||\nabla (\ln f)||^2 + \Delta (\ln f))\}.
\]

The equality case of (11.2) holds identically if and only if the following four statements hold.

1. \( N_T \) is an open portion of \( \mathbb{C}^h_h := \mathbb{C}^h - \{0\} \);
2. \( N_\perp \) is an open portion of \( S^p \);
3. There is \( \alpha, 1 \leq \alpha \leq h \), and complex Euclidean coordinates \( \{z_1, \ldots, z_h\} \) on \( \mathbb{C}^h \) such that
\[
f = \sqrt{\sum_{j=1}^h z_j \bar{z}_j}.
\]
4. Up to rigid motions, the immersion \( \phi \) is given by
\[
\phi = (w_0 z_1, \ldots, w_p z_1, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_\alpha+1, \ldots, z_h, 0, \ldots, 0)
\]
for \( z = (z_1, \ldots, z_h) \in \mathbb{C}^h_h \) and \( w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{P}^{p+1} \).

**Theorem 11.3.** Let \( \phi : N_T \times_f N_\perp \to CP^m \) be a CR-warped product with \( \dim_{\mathbb{C}} N_T = h \) and \( \dim_{\mathbb{R}} N_\perp = p \). Then we have

\[
||\sigma||^2 \geq 2p\{(||\nabla (\ln f)||^2 + \Delta (\ln f) + 2h)\}.
\]

The CR-warped product satisfies the equality case of (11.3) identically if and only if the following three statements hold.

(a) \( N_T \) is an open portion of complex projective \( h \)-space \( CP^h \);
(b) \( N_\perp \) is an open portion of unit \( p \)-sphere \( S^p \); and
(c) There exists a natural number \( \alpha \leq h \) such that, up to rigid motions, \( \phi \) is the composition \( \pi \circ \bar{\phi} \), where
\[
\bar{\phi}(z, w) = (w_0 z_0, \ldots, w_p z_0, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_\alpha+1, \ldots, z_h, 0, \ldots, 0)
\]
for \( z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1}_h \) and \( w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{P}^{p+1} \), where \( \pi \) is the projection \( \pi : C^{m+1}_m \to CP^m \).

**Theorem 11.4.** Let \( \phi : N_T \times_f N_\perp \to CH^m \) be a CR-warped product with \( \dim_{\mathbb{C}} N_T = h \) and \( \dim_{\mathbb{R}} N_\perp = p \). Then we have

\[
||\sigma||^2 \geq 2p\{(||\nabla (\ln f)||^2 + \Delta (\ln f) - 2h)\}.
\]

The CR-warped product satisfies the equality case of (11.4) identically if and only if the following three statements hold.

(a) \( N_T \) is an open portion of complex hyperbolic \( h \)-space \( CH^h \);
(b) \( N_\perp \) is an open portion of unit \( p \)-sphere \( S^p \) (or \( \mathbb{R} \), when \( p = 1 \)); and
(c) up to rigid motions, \( \phi \) is the composition \( \pi \circ \bar{\phi} \), where either \( \bar{\phi} \) is given by
\[
\bar{\phi}(z, w) = (z_0, \ldots, z_\beta, w_0 z_{\beta+1}, \ldots, w_p z_{\beta+1}, \ldots, w_0 z_h, \ldots, w_p z_h, 0, \ldots, 0)
\]
for \( 0 < \beta \leq h \), \( z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1}_h \) and \( w = (w_0, \ldots, w_p) \in S^p \), or \( \bar{\phi} \) is given by
\[
\bar{\phi}(z, u) = (z_0 \cosh u, z_0 \sinh u, z_1 \cos u, z_1 \sin u, \ldots, z_\alpha \cos u, z_\alpha \sin u, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0)
\]
for \( z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1} \), where \( \pi \) is the projection \( \pi : \mathbb{C}^{m+1} \to CH^m(-4) \).

12. WARPED PRODUCT \( CR \)-SUBMANIFOLDS AND \( \delta \)-INVARIANTS

Let \( N \) be a \( CR \)-submanifold of a Kähler manifold with holomorphic distribution \( D \) and totally real distribution \( D^\perp \). We define the \( CR \) \( \delta \)-invariant \( \delta(D) \) of \( N \) by

\[
\delta(D)(x) = \tau(x) - \tau(D_x),
\]

where \( \tau \) and \( \tau(D) \) are the scalar curvature of \( N \) and the scalar curvature of the holomorphic distribution \( D \subset TN \), respectively (see [51, 55] for details).

The following result from [55] provides a general optimal inequality involving the \( CR \) \( \delta \)-invariant for \( CR \)-warped submanifolds in complex space forms.

**Theorem 12.1.** Let \( N = NT \times_f N^\perp \) be a \( CR \)-warped product in a complex space form \( M^{h+p}(4c) \) with \( h = \dim_C NT \geq 1 \) and \( p = \dim N^\perp \geq 2 \). Then

\[
H^2 \geq \frac{2(p+2)}{(2h+p)^2(p-1)} \left\{ \delta(D) - \frac{p\Delta f}{f} - \frac{p(p-1)c}{2} \right\},
\]

where \( \Delta f \) is the Laplacian of the warping function \( f \) and \( H^2 \) is the squared mean curvature.

The equality sign of (12.2) holds at a point \( x \in N \) if and only if there exists an orthonormal basis \( \{e_{2h+1}, \ldots, e_n\} \) of \( D^\perp_x \) such that the coefficients of the second fundamental \( \sigma \) with respect to \( \{e_{2h+1}, \ldots, e_n\} \) satisfy

\[
\sigma^r_s = 3\sigma^r_s, \quad \text{for} \quad 2h+1 \leq r \neq s \leq 2h+p,
\]
\[
\sigma^r_s = 0, \quad \text{for distinct} \quad r, s, t \in \{2h+1, \ldots, 2h+p\}.
\]

All \( CR \)-warped products in \( \mathbb{C}^{h+p} \) satisfying the equality case of (12.2) identically are completely classified in [55] as follows.

**Theorem 12.2.** Let \( \psi : NT \times_f N^\perp \to \mathbb{C}^{h+p} \) be a \( CR \)-warped product in \( \mathbb{C}^{h+p} \) with \( h = \dim_C NT \geq 1 \) and \( p = \dim N^\perp \geq 2 \). Then

\[
H^2 \geq \frac{2(p+2)}{(2h+p)^2(p-1)} \left\{ \delta(D) - \frac{p\Delta f}{f} \right\},
\]

The equality sign of (12.3) holds identically if and only if, up to dilations and rigid motions of \( \mathbb{C}^{h+p} \), one of the following three cases occurs:

(a) The \( CR \)-warped product is an open part of the \( CR \)-product \( \mathbb{C}^1 \times WP \subset \mathbb{C}^h \times \mathbb{C}^p \), where \( WP \) is the Whitney \( p \)-sphere in \( \mathbb{C}^p \);

(b) \( NT \) is an open part of \( \mathbb{C}^h \), \( N^\perp \) is an open part of the unit \( p \)-sphere \( S^p \), \( f = |z_1| \) and \( \psi \) is the minimal immersion defined by

\[
(z_1w_0, \ldots, z_1w_p, z_2, \ldots, z_h),
\]

where \( z = (z_1, \ldots, z_h) \in \mathbb{C}^h \) and \( w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{C}^{p+1} \);

(c) \( NT \) is an open part of \( \mathbb{C}^h \), \( N^\perp \) is the warped product of a curve and an open part of \( S^{p-1} \) with warping function \( \varphi = \frac{e^{(\sqrt{c-1})t}}{\sqrt{c-1}} \text{cn}(\sqrt{c-1}t, \sqrt{c-1}) \), \( c > 1 \), \( f = |z_1| \), and \( \psi \) is the non-minimal immersion defined by

\[
(z_1e^{\frac{\varphi(z_1^2+\varphi^2)}{\varphi^2-1}}dt, z_1\varphi e^{ik\varphi dt}w_1, \ldots, z_1\varphi e^{ik\varphi dt}w_p, z_2, \ldots, z_h),
\]
with $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $(w_1, \ldots, w_p) \in S^{p-1}(1) \subset \mathbb{E}^p$, and $k = \sqrt{c^4 - 1}/2$.

13. WARPED PRODUCT REAL HYPERSURFACES IN COMPLEX SPACE FORMS

We have the following non-existence theorem from [59] for Riemannian product real hypersurfaces in complex space forms.

Theorem 13.1. There do not exist real hypersurfaces in complex projective and complex hyperbolic spaces which are Riemannian products of two or more Riemannian manifolds of positive dimension.

In other words, every real hypersurface in a nonflat complex space form is irreducible.

A contact manifold is an odd-dimensional manifold $M^{2n+1}$ with a 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$. A curve $\gamma = \gamma(t)$ in a contact manifold is called a Legendre curve if $\eta(\beta'(t)) = 0$ along $\beta$. Let $S^{2n+1}(c)$ denote the hypersphere in $\mathbb{C}^{n+1}$ with curvature $c$ centered at the origin. Then $S^{2n+1}(c)$ is a contact manifold endowed with a canonical contact structure which is the dual 1-form of the characteristic vector field $J\xi$, where $J$ is the complex structure and $\xi$ the unit normal vector on $S^{2n+1}(c)$.

Legendre curves are known to play an important role in the study of contact manifolds, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves.

Contrast to Theorem 13.1, there exist many warped product real hypersurfaces in complex space forms as given in the following three theorems from [44].

Theorem 13.2. Let $a$ be a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ be a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbb{C}^2$ defined on an open interval $I$. Then

$$x(z_1, \ldots, z_n, t) = (a\Gamma_1(t)z_1, a\Gamma_2(t)z_1, z_2, \ldots, z_n), \quad z_1 \neq 0$$

defines a real hypersurface which is the warped product $\mathbb{C}^n_1 \times_{a|z_1|} I$ of a complex $n$-plane and $I$, where $\mathbb{C}^n_1 = \{(z_1, \ldots, z_n) : z_1 \neq 0\}$.

Conversely, up to rigid motions of $\mathbb{C}^{n+1}$, every real hypersurface in $\mathbb{C}^{n+1}$ which is the warped product $N \times_I I$ of a complex hypersurface $N$ and an open interval $I$ is either obtained in the way described above or given by the product submanifold $\mathbb{C}^n \times C \subset \mathbb{C}^n \times C^1$ of $\mathbb{C}^n$ and a real curve $C$ in $C^1$.

Let $S^{2n+3}$ denote the unit hypersphere in $\mathbb{C}^{n+2}$ centered at the origin and put

$$U(1) = \{\lambda \in \mathbb{C} : \lambda\bar{\lambda} = 1\}.$$

Then there is a $U(1)$-action on $S^{2n+3}$ defined by $z \mapsto \lambda z$. At $z \in S^{2n+3}$ the vector $V = iz$ is tangent to the flow of the action. The quotient space $S^{2n+3}/\sim$, under the identification induced from the action, is a complex projective space $\mathbb{C}P^{n+1}$ which endows with the canonical Fubini-Study metric of constant holomorphic sectional curvature 4.

The almost complex structure $J$ on $\mathbb{C}P^{n+1}$ is induced from the complex structure $J$ on $\mathbb{C}^{n+2}$ via the Hopf fibration: $\pi : S^{2n+3} \to \mathbb{C}P^{n+1}$. It is well-known that the Hopf fibration $\pi$ is a Riemannian submersion such that $V = iz$ spans the vertical subspaces.
Let \( \phi : M \to CP^{n+1} \) be an isometric immersion. Then \( \hat{M} = \pi^{-1}(M) \) is a principal circle bundle over \( M \) with totally geodesic fibers. The lift \( \hat{\phi} : \hat{M} \to S^{2n+3} \) of \( \phi \) is an isometric immersion so that the diagram:

\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{\phi}} & S^{2n+3} \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{\phi} & CP^{n+1}
\end{array}
\]

commutes.

Conversely, if \( \psi : \hat{M} \to S^{2n+3} \) is an isometric immersion which is invariant under the \( U(1) \)-action, then there is a unique isometric immersion \( \psi_\pi : \pi(\hat{M}) \to CP^{n+1} \) such that the associated diagram commutes. We simply call the immersion \( \psi_\pi : \pi(\hat{M}) \to CP^{n+1} \) the projection of \( \psi : \hat{M} \to S^{2n+3} \).

For a given vector \( X \in T_z(CP^{n+1}) \) and a point \( u \in S^{2n+2} \) with \( \pi(u) = z \), we denote by \( X_u \) the horizontal lift of \( X \) at \( u \) via \( \pi \). There exists a canonical orthogonal decomposition:

\[
T_uS^{2n+3} = (T_{\pi(u)}CP^{n+1})^*_u \oplus \text{Span}\{V_u\}
\]

Since \( \pi \) is a Riemannian submersion, \( X \) and \( X_u \) have the same length.

We put

\[
S^{2n+1}_* = \left\{ (z_0, \ldots, z_n) : \sum_{k=0}^n z_k \bar{z}_k = 1, z_0 \neq 0 \right\}, \quad CP^n_0 = \pi(S^{2n+1}_*).
\]

The following theorem from [44] classifies all warped products hypersurfaces of the form \( N \times_f I \) in complex projective spaces.

**Theorem 13.3.** Suppose that \( a \) is a positive number and \( \gamma(t) = (\Gamma_1(t), \Gamma_2(t)) \) is a unit speed Legendre curve \( \gamma : I \to S^3(a^2) \subset \mathbb{C}^2 \) defined on an open interval \( I \). Let \( x : S^{2n+1}_* \times I \to \mathbb{C}^{n+2} \) be the map defined by

\[
x(z_0, \ldots, z_n, t) = (a\Gamma_1(t)z_0, a\Gamma_2(t)z_0, z_1, \ldots, z_n), \quad \sum_{k=0}^n z_k \bar{z}_k = 1.
\]

Then

1. \( x \) induces an isometric immersion \( \psi : S^{2n+1}_* \times a[z_0] I \to S^{2n+3} \).
2. The image \( \psi(S^{2n+1}_* \times a[z_0] I) \) in \( S^{2n+3} \) is invariant under the action of \( U(1) \).
3. The projection \( \psi_\pi : \pi(S^{2n+1}_* \times a[z_0] I) \to CP^{n+1} \) of \( \psi \) via \( \pi \) is a warped product hypersurface \( CP^n_0 \times a[z_0] I \) in \( CP^{n+1} \).

Conversely, if a real hypersurface in \( CP^{n+1} \) is a warped product \( N \times_f I \) of a complex hypersurface \( N \) of \( CP^{n+1} \) and an open interval \( I \), then, up to rigid motions, it is locally obtained in the way described above.

In the complex pseudo-Euclidean space \( \mathbb{C}^{n+2}_1 \) endowed with pseudo-Euclidean metric

\[
g_0 = -dz_0d\bar{z}_0 + \sum_{j=1}^{n+1} dz_j d\bar{z}_j,
\]

we define the anti-de Sitter space-time by

\[
H_1^{2n+3} = \{ (z_0, z_1, \ldots, z_{n+1}) : \langle z, z \rangle = -1 \}.
\]
It is known that $H_1^{2n+3}$ has constant sectional curvature $-1$. There is a $U(1)$-action on $H_1^{2n+3}$ defined by $z \to \lambda z$. At a point $z \in H_1^{2n+3}$, $iz$ is tangent to the flow of the action. The orbit is given by $z_t = e^{it}z$ with $\frac{dz_t}{dt} = iz_t$ which lies in the negative-definite plane spanned by $z$ and $iz$.

The quotient space $H_1^{2n+3}/\sim$ is the complex hyperbolic space $CH^{n+1}$ which endows a canonical Kähler metric of constant holomorphic sectional curvature $-4$. The complex structure $J$ on $CH^{n+1}$ is induced from the canonical complex structure $J$ on $C^{n+2}$ via the totally geodesic fibration: $\pi : H_1^{2n+3} \to CH^{n+1}$.

Let $\phi : M \to CH^{n+1}$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over $M$ with totally geodesic fibers. The lift $\hat{\phi} : \hat{M} \to H_1^{2n+3}$ of $\phi$ is an isometric immersion such that the diagram:

$$
\begin{array}{ccc}
\hat{M} & \xrightarrow{\phi} & H_1^{2n+3} \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{\phi} & CH^{n+1}
\end{array}
$$

commutes.

Conversely, if $\psi : \hat{M} \to H_1^{2n+3}$ is an isometric immersion which is invariant under the $U(1)$-action, there is a unique isometric immersion $\psi_n : \pi(\hat{M}) \to CH^{n+1}$, called the projection of $\psi$ so that the associated diagram commutes.

We put

$$
H_{1*}^{2n+1} = \{(z_0, \ldots, z_n) \in H_1^{2n+1} : z_n \neq 0\},
$$

$$CH_{n*} = \pi(H_{1*}^{2n+1}).
$$

The next theorem from [44] classifies all warped products hypersurfaces of the form $N \times I$ in complex hyperbolic spaces.

**Theorem 13.4.** Suppose that $a$ is a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbb{C}^2$. Let $y : H_{1*}^{2n+1} \times I \to C^{n+2}$ be the map defined by

$$
y(z_0, \ldots, z_n, t) = (z_0, \ldots, z_{n-1}, a\Gamma_1(t)z_n, a\Gamma_2(t)z_n),
$$

$$z_0\bar{z}_0 - \sum_{k=1}^{n} z_k\bar{z}_k = 1.
$$

Then we have

1. $y$ induces an isometric immersion $\psi : H_{1*}^{2n+1} \times_{a|z_n|} I \to H_1^{2n+3}$.
2. The image $\psi(H_{1*}^{2n+1} \times_{a|z_n|} I)$ in $H_1^{2n+3}$ is invariant under the $U(1)$-action.
3. The projection $\psi_\pi : \pi(H_{1*}^{2n+1} \times_{a|z_n|} I) \to CH^{n+1}$ of $\psi$ via $\pi$ is a warped product hypersurface $CH_{n*} \times_{a|z_n|} I$ in $CH^{n+1}$.

Conversely, if a real hypersurface in $CH^{n+1}$ is a warped product $N \times I$ of a complex hypersurface $N$ and an open interval $I$, then, up to rigid motions, it is locally obtained in the way described above.

### 14. Warped Product CR-Submanifolds of Nearly Kähler Manifolds

Let $M$ be an almost Hermitian manifold with metric tensor $g$ and almost complex structure $J$. Then, according to A. Gray [73] in 1970, $M$ is called a nearly Kähler manifold.
manifold provided that
\[(\nabla_X J)X = 0, \; \forall X \in TM.\]

Historically speaking, nearly Kähler manifolds are exactly the Tachibana manifolds initially studied in S. Tachibana [141] around 1959.

Nearly Kähler manifolds form an interesting class of manifolds admitting a metric connection with parallel totally antisymmetric torsion (see [2]).

The best known example of nearly Kähler manifolds, but not Kählerian, is \(S^6(1)\) with the nearly Kählerian structure induced from the vector cross product on the space of purely imaginary Cayley numbers \(O\). More general examples of nearly Kähler manifolds are the homogeneous spaces \(G/K\), where \(G\) is a compact semisimple Lie group and \(K\) is the fixed point set of an automorphism of \(G\) of order 3 [155].

Strict nearly Kähler manifolds obtained a lot of consideration in the 1980s due to their relation to Killing spinors. Th. Friedrich and R. Grunewald showed in [72] that a 6-dimensional Riemannian manifold admits a Riemannian Killing spinor if and only if it is nearly Kähler.

The only known 6-dimensional strict nearly Kähler manifolds are
\[S^6 = G_2/SU(3), \; Sp(2)/SU(2) \times U(1), \; SU(3)/U(1) \times U(1), \; S^3 \times S^3.\]

In fact, these are the only homogeneous nearly Kähler manifolds in dimension six [26]. P.-A. Nagy proved in [109] that indeed any strict and complete nearly Kähler manifold is locally a Riemannian product of homogeneous nearly Kähler spaces, twistor spaces over Kähler manifolds and 6-dimensional nearly Kähler manifolds.

The non-existence result for warped products \(N_\perp \times_f N_T\) in Kähler manifolds, Theorem 9.1, was extended in [87, 130] to warped products in nearly Kähler manifolds in the following.

**Theorem 14.1.** There does not exist a proper warped product CR-submanifold of the form \(N_\perp \times_f N_T\) with \(N_\perp\) a totally real submanifold and \(N_T\) a holomorphic submanifold, in a nearly Kähler manifold.

This theorem was further extended in [86] to warped products \(N \times_f N_T\) in a nearly Kähler manifold with \(N\) and \(N_T\) being Riemannian and holomorphic submanifolds of \(\tilde{M}\), respectively.

Similarly, Theorem 9.3 was also extended in [4, 87, 130] to nearly Kähler manifold as follows.

**Theorem 14.2.** Let \(M = N_T \times_f N_\perp\) be a CR-warped submanifold of a nearly Kähler manifold \(\tilde{M}\). Then we have

1. The squared norm of the second fundamental form \(\sigma\) satisfies
   \[||\sigma||^2 \geq 2p||\nabla \ln f||^2,\]
   where \(p\) is the dimension of \(N_\perp\).

2. If the equality in (14.2) holds identically, then \(N_T\) is a totally geodesic submanifold, \(N_\perp\) a totally umbilical submanifold, and \(M\) is a minimal submanifold of \(\tilde{M}\).

It was shown in [84] that there does not exist a proper doubly warped product submanifold of a nearly Kähler manifold \(\tilde{M}\) with one of the factors a holomorphic submanifold. It also shows that there do not exist doubly twisted product generic submanifolds of nearly Kähler manifolds in the form \(g_t N_T \times_f N_1\) such that \(N_T\) is holomorphic and \(N_1\) is an arbitrary submanifold in \(\tilde{M}\).
15. WARPED PRODUCT SUBMANIFOLDS IN PARA-KÄHLER MANIFOLDS

An almost para-Hermitian manifold is a manifold $\widetilde{M}$ equipped with an almost product structure $\mathcal{P} \neq \pm I$ and a pseudo-Riemannian metric $\tilde{g}$ such that

$$\mathcal{P}^2 = I, \quad \tilde{g}(\mathcal{P}X, \mathcal{P}Y) = -\tilde{g}(X, Y),$$

for vector fields $X, Y$ tangent to $M$, where $I$ is the identity map. Clearly, it follows from (15.1) that the dimension of $\widetilde{M}$ is even and the metric $\tilde{g}$ is neutral. An almost para-Hermitian manifold is called para-Kähler if it satisfies $\nabla \mathcal{P} = 0$ identically, where $\nabla$ is the Levi Civita connection of $\widetilde{M}$. We define $||X||_2$ associated with $\tilde{g}$ on $\widetilde{M}$ by $||X||_2 = \tilde{g}(X, X)$.

A pseudo-Riemannian submanifold $M$ of a para-Kähler manifold $\widetilde{M}$ is called invariant if the tangent bundle of $M$ is invariant under the action of $\mathcal{P}$. $M$ is called anti-invariant if $\mathcal{P}$ maps each tangent space $T_pM$, $p \in M$, into the normal space $T^\perp_p M$ (cf. [54]).

A pseudo-Riemannian submanifold $M$ of a para-Kähler manifold $\widetilde{M}$ is called a $\mathcal{P}R$-submanifold if the tangent bundle $TM$ of $M$ is the direct sum of an invariant distribution $\mathcal{D}$ and an anti-invariant distribution $\mathcal{D}^\perp$, i.e.,

$$T(M) = \mathcal{D} \oplus \mathcal{D}^\perp, \quad \mathcal{P}\mathcal{D} = \mathcal{D}, \quad \mathcal{P}\mathcal{D}^\perp \subseteq T^\perp_p (M).$$

A $\mathcal{P}R$-submanifold is called a $\mathcal{P}R$-warped product if it is a warped product $N_T \times_f N_\perp$ of an invariant submanifold $N_T$ and an anti-invariant submanifold $N_\perp$.

The notion of $\mathcal{P}R$-warped product submanifolds in para-Kähler manifolds are introduced and studied in [60]. In particular, the following results were obtained in [60].

**Theorem 15.1.** Let $N_T \times_f N_\perp$ be a $\mathcal{P}R$-warped product in a para-Kähler manifold $\widetilde{M}$. If $N_\perp$ is space-like (respectively, time-like) and $\dim \widetilde{M} = \dim N_T + 2 \dim N_\perp$, then the second fundamental form $\sigma$ of $N_T \times_f N_\perp$ satisfies

$$||\sigma||^2 \leq 2\rho ||\nabla \ln f||_2$$

(respectively, $||\sigma||^2 \geq 2\rho ||\nabla \ln f||_2$).

If the equality sign of (15.2) holds identically, we have

$$\sigma(\mathcal{D}, \mathcal{D}) = \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = \{0\}.$$

Para-complex numbers were introduced by J. T. Graves in 1845 as a generalization of complex numbers. Such numbers have the expression $v = x + jy$, where $x, y$ are real numbers and $j$ satisfies $j^2 = 1$, $j \neq 1$. The conjugate of $v$ is $\bar{v} = x - jy$. The multiplication of two para-complex numbers is defined by

$$(a + jb)(s + jt) = (as + bt) + j(at + bs).$$

For each natural number $m$, we put

$$\mathbb{D}^m = \{(x_1 + jy_1, \ldots, x_m + jy_m) : x_i, y_i \in \mathbb{R}\}.$$
In the following we denote by $\mathbb{S}^p, \mathbb{E}^p$ and $\mathbb{H}^p$ the unit $p$-sphere, the Euclidean $p$-space and the unit hyperbolic $p$-space, respectively.

The following theorem from [H] completely classifies space-like $\mathcal{P}R$-warped products in $\mathcal{P}^m$ which satisfy the equality case of (15.2) identically.

**Theorem 15.2.** Let $N_\top \times_f N_\bot$ be a space-like $\mathcal{P}R$-warped product in the para-Kähler $(h+p)$-plane $\mathcal{P}^{h+p}$ with $h = \frac{1}{2} \dim N_\top$ and $p = \dim N_\bot$. Then we have

$$||\sigma||^2 \leq 2p||\nabla \ln f||_2.$$  

The equality sign of (15.3) holds identically if and only if $N_\top$ is an open part of a para-Kähler $h$-plane, $N_\bot$ is an open part of $\mathbb{S}^p, \mathbb{E}^p$ or $\mathbb{H}^p$, and the immersion is given by one of the following:

1. $\Phi : D_1 \times_f \mathbb{S}^p \rightarrow \mathcal{P}^{h+p};$

$$\Phi(z, w) = \left( z_1 + \bar{v}_1(w_0 - 1) \sum_{j=1}^h v_j z_j, \ldots, z_h + \bar{v}_h(w_0 - 1) \sum_{j=1}^h v_j z_j, \right.$$  

$$w_1 \sum_{j=1}^h jv_j z_j, \ldots, w_p \sum_{j=1}^h jv_j z_j \right), \> h \geq 2,$$


with warping function $f = \sqrt{(\bar{v}, z)^2 - (j\bar{v}, z)^2}$, where $v = (v_1, \ldots, v_h) \in \mathbb{S}^{2h-1} \subset \mathbb{D}^h$, $w = (w_0, w_1, \ldots, w_p) \in \mathbb{S}^p$, $z = (z_1, \ldots, z_h) \in D_1$ and $D_1 = \{ z \in \mathbb{D}^h : (\bar{v}, z)^2 > (j\bar{v}, z)^2 \}$.

2. $\Phi : D_1 \times_f \mathbb{H}^p \rightarrow \mathcal{P}^{h+p};$

$$\Phi(z, w) = \left( z_1 + \bar{v}_1(w_0 - 1) \sum_{j=1}^h v_j z_j, \ldots, z_h + \bar{v}_h(w_0 - 1) \sum_{j=1}^h v_j z_j, \right.$$  

$$w_1 \sum_{j=1}^h jv_j z_j, \ldots, w_p \sum_{j=1}^h jv_j z_j \right), \> h \geq 1,$$


with the warping function $f = \sqrt{(\bar{v}, z)^2 - (j\bar{v}, z)^2}$, where $v = (v_1, \ldots, v_h) \in \mathbb{H}^{2h-1} \subset \mathbb{D}^h$, $w = (w_0, w_1, \ldots, w_p) \in \mathbb{H}^p$ and $z = (z_1, \ldots, z_h) \in D_1$.

3. $\Phi(z, u) : D_2 \times_f \mathbb{E}^p \rightarrow \mathcal{P}^{h+p};$

$$\Phi(z, u) = \left( z_1 + \bar{v}_1 \left( \sum_{a=1}^p u_a \right) v_0, \ldots, z_h + \bar{v}_h \left( \sum_{a=1}^p u_a \right) v_0, \right.$$  

$$u_1 \sum_{j=1}^h jv_j z_j, \ldots, u_p \sum_{j=1}^h jv_j z_j \right), \> h \geq 2,$$


with the warping function $f = \sqrt{(\bar{v}, z)^2 - (j\bar{v}, z)^2}$, where $v = (v_1, \ldots, v_h)$ is a light-like vector in $\mathbb{D}^h$, $z = (z_1, \ldots, z_h) \in D_1$ and $u = (u_1, \ldots, u_p) \in \mathbb{E}^p$.

Moreover, in this case, each leaf $\mathbb{E}^p$ is quasi-minimal in $\mathcal{P}^{h+p}$.

4. $\Phi(z, u) : D_2 \times_f \mathbb{E}^p \rightarrow \mathcal{P}^{h+p};$

$$\Phi(z, u) = \left( z_1 + \bar{v}_1 \left( \sum_{a=1}^p u_a \right) v_0, \ldots, z_h + \bar{v}_h \left( \sum_{a=1}^p u_a \right) v_0, \right.$$  

$$u_1 \sum_{j=1}^h jv_j z_j, \ldots, u_p \sum_{j=1}^h jv_j z_j \right), \> h \geq 1,$$


with the warping function $f = -\sqrt{(\bar{v}, z)}$, where $v_0 = \sqrt{b_1 + \epsilon j\sqrt{b_1}}$ with $b_1 > 0$, $D_2 = \{ z \in \mathbb{D}^h : (\bar{v}, z) < 0 \}$, $v = (v_1, \ldots, v_h) = (b_1 + \epsilon j b_1, \ldots, b_h + \epsilon j b_h)$, $\epsilon = \pm 1$, $z = (z_1, \ldots, z_h) \in D_2$ and $u = (u_1, \ldots, u_p) \in \mathbb{E}^p$.

In each of the four cases the warped product is minimal in $\mathbb{E}^{2(h+p)}$. 


16. CONTACT CR-WARPED PRODUCT SUBMANIFOLDS IN SASAKIAN MANIFOLDS

An odd-dimensional Riemannian manifold \((M,g)\) is called an almost contact metric manifold if there exist on \(M\) a \((1,1)\)-tensor field \(\phi\), a vector field \(\xi\) and a 1-form \(\eta\) such that

\[
\begin{align*}
\phi^2 X &= -X + \eta(X)\xi, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y)
\end{align*}
\]

for vector fields \(X, Y\) on \(M\). On an almost contact metric manifold, we also have \(\phi\xi = 0\) and \(\eta \circ \phi = 0\). The vector field \(\xi\) is called the structure vector field.

By a contact manifold we mean a \((2n+1)\)-manifold \(M\) together with a global 1-form \(\eta\) satisfying \(\eta \wedge (d\eta)^n \neq 0\) on \(M\). If \(\eta\) of an almost contact metric manifold \((M, \phi, \xi, \eta, g)\) is a contact form and if \(\eta\) satisfies \(d\eta(X, Y) = g(X, \phi Y)\) for all vectors \(X, Y\) tangent to \(M\), then \(M\) is called a contact metric manifold. A contact metric manifold is called \(K\)-contact if its characteristic vector field \(\xi\) is a Killing vector field. It is well-known that a contact metric manifold is a \(K\)-contact manifold if and only if

\[
\nabla_X \xi = -\phi X
\]

holds for all vector fields \(X\) on \(M\). In fact, an almost contact metric manifold satisfying condition [16.3] is also a \(K\)-contact manifold. Condition [16.3] is equivalent to

\[
K(X, \xi) = 1
\]

for every unit tangent vector \(X\) orthogonal to \(\xi\).

An almost contact metric structure of \(M\) is called normal if the Nijenhuis torsion \([\phi, \phi]\) of \(\phi\) equals to \(-2d\eta \otimes \xi\). A normal contact metric manifold is called a Sasakian manifold. It can be proved that an almost contact metric manifold is Sasakian if and only if the Riemann curvature tensor \(R\) satisfies

\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y
\]

for any vector fields \(X, Y\) on \(M\). A Sasakian manifold is also \(K\)-contact but the converse is not true in general if \(\dim M \geq 5\).

A plane section \(\pi\) in \(T_pM\) is called a \(\phi\)-section if it is spanned by \(X\) and \(\phi X\), where \(X\) is a unit tangent vector orthogonal to \(\xi\). The sectional curvature of a \(\phi\)-section is called a \(\phi\)-sectional curvature. A Sasakian manifold with constant \(\phi\)-sectional curvature \(c\) is said to be a Sasakian space form and is denoted by \(\tilde{M}(c)\).

A submanifold \(N\) normal to \(\xi\) in a Sasakian manifold is said to be a \(C\)-totally real submanifold. In this case, it follows that \(\phi\) maps any tangent space of \(N\) into the normal space, that is, \(\phi(T_x^\perp N) \subset T_x^\perp N\), for every \(x \in N\). For submanifolds tangent to the structure vector field \(\xi\), there are different classes of submanifolds. We mention the following.

(i) A submanifold \(N\) tangent to \(\xi\) is called an invariant submanifold if \(\phi\) preserves any tangent space of \(M\), that is, \(\phi(T_x N) \subset T_x N\), for every \(x \in N\).

(ii) A submanifold \(N\) tangent to \(\xi\) is called an anti-invariant submanifold (or totally real submanifold) if \(\phi\) maps any tangent space of \(N\) into the normal space, that is, \(\phi(T_x N) \subset T_x^\perp N\), for every \(x \in N\).
(iii) A submanifold $N$ tangent to $\xi$ is called a contact \textit{CR-submanifold} if it admits an invariant distribution $D$ whose orthogonal complementary distribution $D^\perp$ is anti-invariant, that is, $TN = D \oplus D^\perp$ with $\phi(D_x) \subset D_x$ and $\phi(D^\perp_x) \subset T^\perp_x N$, for every $x \in N$.

(iv) A contact \textit{CR-submanifold} $N$ of a Sasakian manifold $\tilde{M}$ is called \textit{(contact) CR-product} if it is locally a Riemannian product of a $\phi$-invariant submanifold $N_T$ tangent to $\xi$ and a totally real submanifold $N_\perp$ of $\tilde{M}$.

First, we present the following two results from \cite{76, 104}.

\textbf{Theorem 16.1.} Let $\tilde{M}$ be a Sasakian manifold and let $N = N_\perp \times_f N_T$ be a warped product \textit{CR-submanifold} such that $N_\perp$ is a totally real submanifold and $N_T$ an invariant submanifold of $\tilde{M}$. Then $N$ is a \textit{CR-product}.

\textbf{Theorem 16.2.} Let $N = N_T \times_f N_\perp$ be a \textit{CR-warped products} in a Sasakian manifold $\tilde{M}$. Then we have

1. The squared norm of the second fundamental form of $N$ satisfies

\begin{equation}
||\sigma||^2 \geq 2p(||\nabla(\ln f)||^2 + 1), \quad p = \dim N_\perp.
\end{equation}

2. If the equality sign of (16.6) holds identically, then $N_T$ is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\tilde{M}$. Further, $N$ is a minimal submanifold of $\tilde{M}$.

3. For the case $T^\perp N = \phi D^\perp$, if $p > 1$ then the equality sign in (16.6) holds identically if and only if $N_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

For contact \textit{CR-warped products} in Sasakian space forms we have the following from \cite{104} (see also \cite{76}).

\textbf{Proposition 16.1.} Let $N = N_T \times_f N_\perp$ be a non-trivial (i.e., $f$ non constant) complete, simply-connected, contact \textit{CR-warped product} those second fundamental form $\sigma$ satisfies

\begin{equation}
||\sigma||^2 \geq 2p(||\nabla(\ln f)||^2 + 1)
\end{equation}

in a Sasakian space form $\tilde{M}^{2m+1}(c)$. Then we have

1. $N_T$ is a totally geodesic Sasakian submanifold of $\tilde{M}^{2m+1}(c)$. Thus $N_T$ is a Sasakian space form $N_T^{2m+1}(c)$.

2. $N_\perp$ is a totally umbilical totally real submanifold of $\tilde{M}^{2m+1}(c)$. Hence $N_\perp$ is a Riemannian manifold of constant sectional curvature. Denote it by $\epsilon$.

3. If $p > 1$, the function $f$ satisfies

\begin{equation}
||\nabla f||^2 = \epsilon - \frac{c + 3}{4} f^2.
\end{equation}

The next result from \cite{76} determines the minimum codimension of a contact \textit{CR-warped product} with compact invariant factor in an odd-dimensional sphere endowed with the standard Sasakian structure.

\textbf{Theorem 16.3.} Let $N = N_T \times_f N_\perp$ be a contact \textit{CR-warped product} in the $(2m+1)$-dimensional unit sphere $S^{2m+1}$. If $N_T$ is compact, then

\begin{equation}
m \geq s + p + sp,
\end{equation}

where $\dim N_T = 2s + 1$ and $\dim N_\perp = p$. 
An easy consequence of Theorem 16.3 is the following.

**Corollary 16.1.** Let $\tilde{M}(c)$ be a Sasakian space form with $c < 3$. Then there do not exist contact CR-warped products $N_T \times f N_\perp$, with $N_T$ a compact invariant submanifold tangent to $\xi$ and $N_\perp$ a $C$-totally real submanifold of $\tilde{M}(c)$.

The following results from [76, 104] are the contact version of Theorems 10.1 and 10.3.

**Theorem 16.4.** Let $N = N_T \times f N_\perp$ a contact CR-warped product of a Sasakian space form $\tilde{M}^{2m+1}(c)$ such that $N_T$ is a $(2s+1)$-dimensional invariant submanifold tangent to $\xi$ and $N_\perp$ a $p$-dimensional $C$-totally real submanifold of $\tilde{M}$. Then

1. The squared norm of the second fundamental form satisfies

\[
||\sigma||^2 \geq 2p \left( ||\nabla(\ln f)||^2 - \Delta(\ln f) + \frac{c+3}{2}s + 1 \right).
\]  

2. The equality sign of (16.7) holds identically if and only if we have:

   (2.1) $N_T$ is a totally geodesic invariant submanifold of $\tilde{M}(c)$. Hence $N_T$ is a Sasakian space form of constant $\phi$-sectional curvature $c$.

   (2.2) $N_\perp$ is a totally umbilical totally real submanifold of $\tilde{M}$. Hence $N_\perp$ is a real space form of sectional curvature $\epsilon \geq (c+3)/4$.

An example of contact CR-warped submanifold satisfying the equality case of (16.7), but not the equality case of (16.6) was constructed in [104]. Also, a contact version of Theorem 3.2 for warped product submanifolds in Sasakian space forms was given in [95].

17. WARPED PRODUCT SUBMANIFOLDS IN AFFINE SPACES

If $M$ is an $n$-dimensional manifold, let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface of the affine $(n+1)$-space whose position vector field is nowhere tangent to $M$. Then $f$ can be regarded as a transversal field along itself. We call $\xi = -f$ the centroaffine normal. The $f$ together with this normalization is called a centroaffine hypersurface.

The centroaffine structure equations are given by

\[
D_X f_* (Y) = f_* (\nabla_X Y) + h(X,Y)\xi,
\]

\[
D_X \xi = -f_*(X),
\]  

where $D$ denotes the canonical flat connection of $\mathbb{R}^{n+1}$, $\nabla$ is a torsion-free connection on $M$, called the induced centroaffine connection, and $h$ is a non-degenerate symmetric $(0,2)$-tensor field, called the centroaffine metric.

From now on we assume that the centroaffine hypersurface is definite, i.e., $h$ is definite. In case that $h$ is negative definite, we shall replace $\xi = -f$ by $\xi = f$ for the affine normal. In this way, the second fundamental form $h$ is always positive definite. In both cases, (17.1) holds. Equation (17.2) change sign. In case $\xi = -f$, we call $M$ positive definite; in case $\xi = f$, we call $M$ negative definite.

Denote by $\nabla$ the Levi-Civita connection of $h$ and by $\hat{R}$ and $\hat{k}$ the curvature tensor and the normalized scalar curvature of $h$, respectively. The difference tensor $K$ is defined by

\[
K(X,Y) = K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y,
\]
which is a symmetric \((1, 2)\)-tensor field. The difference tensor \(K\) and the cubic form \(C\) are related by
\[
C(X, Y, Z) = -2h(K_X Y, Z).
\]
Thus, for each \(X\), \(K_X\) is self-adjoint with respect to \(h\). The Tchebychev form \(T\) and the Tchebychev vector field \(T^\#\) of \(M\) are defined respectively by
\[
\begin{align*}
T(X) &= \frac{1}{n} \text{trace} K_X, \\
h(T^\#, X) &= T(X).
\end{align*}
\]
If \(T = 0\) and if we consider \(M\) as a hypersurface of the equiaffine space, then \(M\) is a so-called proper affine hypersphere centered at the origin.

If the difference tensor \(K\) vanishes, then \(M\) is a quadric, centered at the origin, in particular an ellipsoid if \(M\) is positive definite and a two-sheeted hyperboloid if \(M\) is negative definite.

An affine hypersurface \(\phi : M \to \mathbb{R}^{n+1}\) is called a graph hypersurface if the transversal vector field \(\xi\) is a constant vector field. A result of [111] states that a graph hypersurface \(M\) is locally affine equivalent to the graph immersion of a certain function \(F\). Again in case that \(h\) is nondegenerate, it defines a semi-Riemannian metric, called the Calabi metric of the graph hypersurface. If \(T = 0\), a graph hypersurface is a so-called improper affine hypersphere.

Let \(M_1\) and \(M_2\) be two improper affine hyperspheres in \(\mathbb{R}^{p+1}\) and \(\mathbb{R}^{q+1}\) defined respectively by the equations:
\[
x_{p+1} = F_1(x_1, \ldots, x_p), \quad y_{q+1} = F_2(y_1, \ldots, y_q).
\]
Then one can define a new improper affine hypersphere \(M\) in \(\mathbb{R}^{p+q+1}\) by
\[
z = F_1(x_1, \ldots, x_p) + F_2(y_1, \ldots, y_q),
\]
where \((x_1, \ldots, x_p, y_1, \ldots, y_q, z)\) are the coordinates on \(\mathbb{R}^{p+q+1}\). The Calabi normal of \(M\) is \((0, \ldots, 0, 1)\). Obviously, the Calabi metric on \(M\) is the product metric. Following [69] we call this composition the Calabi composition of \(M_1\) and \(M_2\).

For a Riemannian \(n\)-manifold \((M, g)\) with Levi-Civita connection \(\nabla\), É. Cartan and A. P. Norden studied nondegenerate affine immersions \(f : (M, \nabla) \to \mathbb{R}^{n+1}\) with a transversal vector field \(\xi\) and with \(\nabla\) as the induced connection.

The well-known Cartan-Norden theorem states that if \(f\) is such an affine immersion, then either \(\nabla\) is flat and \(f\) is a graph immersion or \(\nabla\) is not flat and \(\mathbb{R}^{n+1}\) admits a parallel Riemannian metric relative to which \(f\) is an isometric immersion and \(\xi\) is perpendicular to \(f(M)\) (cf. for instance, [112, p. 159]) (see, also [68]).

In [52, 53], the author studied Riemannian manifolds in affine geometry from a view point different from Cartan-Norden. More precisely, he investigated the following.

**Realization Problem:** Which Riemannian manifolds \((M, g)\) can be immersed as affine hypersurfaces in an affine space in such a way that the fundamental form \(h\) (e.g. induced by the centroaffine normalization or a constant transversal vector field) is the given Riemannian metric \(g\)?

We say that a Riemannian manifold \((M, g)\) can be realized as an affine hypersurface if there exists a codimension one affine immersion from \(M\) into some affine space in such a way that the induced affine metric \(h\) is exactly the Riemannian metric \(g\) of \(M\) (notice that we do not put any assumption on the affine connection).
Warped product submanifolds in affine hypersurfaces was investigated in [52, 53]. In particular, it was shown in [52] that there exist many warped product Riemannian manifolds which can be realized either as graph or centroaffine hypersurfaces. More precisely, we have the following results from [52].

**Theorem 17.1.** Let \( f = f(s) \) be a positive function defined on an open interval \( I \). Assume that \( R, S^n(a^2), H^n(-a^2), \) and \( E^n \) are equipped with their canonical metrics. Then we have:

(a) Every warped product surface \( I \times f R \) can be realized as a graph surface in the affine 3-space \( R^3 \).

(b) For each integer \( n > 2 \), the warped product manifold \( I \times f H^{n-1}(-a^2) \) can be realized as a centroaffine hypersurface in \( R^{n+1} \).

(c) If \( f'(s) \neq 0 \) on \( I \), then the warped product manifold \( I \times f E^{n-1}, n > 2 \), can be realized as a graph hypersurface in \( R^{n+1} \).

(d) If \( f'(s)^2 > a^2 \) on \( I \) for some positive number \( a \), then the warped product manifold \( I \times f S^{n-1}(a^2), n > 2 \), can be realized as a graph hypersurface in \( R^{n+1} \).

**Theorem 17.2.** The following results hold.

(a) If \( n > 2 \) and \( f = f(s) \) is a positive function defined on an open interval \( I \), then we have:

(a.1) If \( f'(s)^2 > f^2(s) - a^2 \) on \( I \) for some positive number \( a \), then \( I \times f H^{n-1}(-a^2) \) can be realized as a centroaffine hypersurface in \( R^{n+1} \).

(a.2) If \( f'(s)^2 > f(s)^2 \) on \( I \), then \( I \times f E^{n-1} \) can be realized as a centroaffine hypersurface in \( R^{n+1} \).

(a.3) If \( f'(s)^2 > f(s)^2 + a^2 \) on \( I \) for some positive number \( a \), then \( I \times f S^{n-1}(a^2) \) can be realized as a graph hypersurface in \( R^{n+1} \).

(b) If \( n = 2 \) and \( f = f(s) \) is a positive function defined on a closed interval \( [\alpha, \beta] \), then the warped product surface \( J \times f R, J = (\alpha, \beta) \), can always be realized as a centroaffine surface in \( R^3 \).

**Theorem 17.3.** If a warped product manifold \( N_1 \times f N_2 \) can be realized as a graph hypersurface in \( R^{n+1} \), then the warping function satisfies

\[
(17.6) \quad \frac{\Delta f}{f} \geq -\frac{(n_1 + n_2)^2}{4n_2} h(T^#, T^#),
\]

where \( T^# \) is the Tchebychev vector field, \( n = n_1 + n_2, \) \( n_1 = \dim N_1 \) and \( n_2 = \dim N_2. \)

The following result characterizes affine hypersurfaces which verify the equality case of inequality (17.6).

**Theorem 17.4.** Let \( \phi : N_1 \times f N_2 \to R^{n+1} \) be a realization of a warped product manifold as a graph hypersurface. If the warping function satisfies the equality case of (17.6) identically, then we have:

(a) The Tchebychev vector field \( T^# \) vanishes identically.

(b) The warping function \( f \) is a harmonic function.

(c) \( N_1 \times f N_2 \) is realized as an improper affine hypersphere.

An application of Theorem 17.3 is the following.
Corollary 17.1. If \( N_1 \) is a compact Riemannian manifold, then every warped product manifold \( N_1 \times_f N_2 \) cannot be realized as an improper affine hypersphere in \( \mathbb{R}^{n+1} \).

As an application of Theorems 17.3 and 17.4 we have the following.

Theorem 17.5. If the Calabi metric of an improper affine hypersphere in an affine space is the Riemannian product metric of \( k \) Riemannian manifolds, then the improper affine hypersphere is locally the Calabi composition of \( k \) improper affine spheres.

Theorem 17.3 also implies the following.

Corollary 17.2. If the warping function \( f \) of a warped product manifold \( N_1 \times_f N_2 \) satisfies \( \Delta f < 0 \) at some point on \( N_1 \), then \( N_1 \times_f N_2 \) cannot be realized as an improper affine hypersphere in \( \mathbb{R}^{n+1} \).

Similarly, for centro-affine hypersurfaces we have the following results from [52].

Theorem 17.6. If a warped product manifold \( N_1 \times_f N_2 \) can be realized as a centroaffine hypersurface in \( \mathbb{R}^{n+1} \), then the warping function satisfies the equality case of (17.7) identically, then we have:

1. The Tchebychev vector field \( T^\# \) vanishes identically.
2. The warping function \( f \) is an eigenfunction of the Laplacian \( \Delta \) with eigenvalue \( n_1 \epsilon \).
3. \( N_1 \times_f N_2 \) is realized as a proper affine hypersphere centered at the origin.

Two immediate consequences of Theorem 17.6 are the following.

Corollary 17.3. If the warping function \( f \) of a warped product manifold \( N_1 \times_f N_2 \) satisfies \( \Delta f \leq 0 \) at some point on \( N_1 \), then \( N_1 \times_f N_2 \) cannot be realized as an elliptic proper affine hypersphere in \( \mathbb{R}^{n+1} \).

Corollary 17.4. If the warping function \( f \) of a warped product manifold \( N_1 \times_f N_2 \) satisfies \( (\Delta f)/f < -\dim N_1 \) at some point on \( N_1 \), then \( N_1 \times_f N_2 \) cannot be realized as a hyperbolic proper affine hypersphere in \( \mathbb{R}^{n+1} \).

Some other interesting applications of Theorem 17.6 are the following.

Corollary 17.5. If \( N_1 \) is a compact Riemannian manifold, then every warped product manifold \( N_1 \times_f N_2 \) with arbitrary warping function cannot be realized as an elliptic proper affine hypersphere in \( \mathbb{R}^{n+1} \).

Corollary 17.6. If \( N_1 \) is a compact Riemannian manifold, then every warped product manifold \( N_1 \times_f N_2 \) cannot be realized as an improper affine hypersphere in an affine space \( \mathbb{R}^{n+1} \).

The following examples show that the results of this section are optimal.
Example 17.1. Let $M = N_1 \times_{\cos s} N_2$ be the warped product of the open interval $N_1 = (-\pi, \pi)$ and an open portion $N_2$ of the unit $(n-1)$-sphere $S^{n-1}(1)$ equipped with the warped product metric:

\[(17.8) \quad h = ds^2 + \cos^2 s \left( du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \prod_{j=2}^{n-1} \cos^2 u_j du_n^2 \right).\]

Consider the immersion of $M$ into the affine $(n+1)$-space defined by

\[(17.9) \quad \left( \sin s, \sin u_2 \cos s, \ldots, \sin u_n \cos s \prod_{j=2}^{n-1} \cos^2 u_j, \cos s \prod_{j=2}^{n} \cos u_j \right).\]

Then $M$ is a centroaffine elliptic hypersurface whose centroaffine metric is the warped product metric (17.8) and it satisfies $T^\# = 0$. Moreover, the warping function $f = \cos s$ satisfies

$$\frac{\Delta f}{f} = 1 = \varepsilon n_1.$$  

Hence, this centroaffine hypersurface satisfies the equality case of (17.7) identically. Consequently, the estimate given in Theorem 17.6 is optimal for centroaffine elliptic hypersurfaces.

Example 17.2. Let $M = \mathbb{R} \times_{\cosh s} H^{n-1}(-1)$ be the warped product of the real line and the unit hyperbolic space $H^{n-1}(-1)$ equipped with warped product metric:

\[(17.10) \quad h = ds^2 + \cosh^2 s \left( du_2^2 + \cosh^2 u_2 du_3^2 + \cdots + \prod_{j=2}^{n-1} \cosh^2 u_j du_n^2 \right).\]

Consider the immersion of $M$ into the affine $(n+1)$-space defined by

$$\left( \sinh s, \sinh u_2 \cosh s, \ldots, \sinh u_n \cosh s \prod_{j=2}^{n-1} \cosh^2 u_j, \cosh s \prod_{j=2}^{n} \cosh u_j \right).$$

Then $M$ is a centroaffine hyperbolic hypersurface whose centroaffine metric is the warped product metric (17.10) and it satisfies $T^\# = 0$. Moreover, the warping function $f = \cosh s$ satisfies

$$\frac{\Delta f}{f} = -1 = \varepsilon n_1.$$  

Therefore, this centroaffine hypersurface satisfies the equality case of (17.7) identically. Consequently, the estimate given in Theorem 17.6 is optimal for centroaffine hyperbolic hypersurfaces as well.

Example 17.3. Let $M = \mathbb{R} \times_{s} N_2$ be the warped product of the real line and an open portion $N_2$ of $S^{n-1}(1)$ equipped with the warped product metric:

\[(17.11) \quad h = ds^2 + s^2 \left( du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \prod_{j=2}^{n-1} \cos^2 u_j du_n^2 \right).\]

Consider the immersion of $M$ into the affine $(n+1)$-space defined by

$$\left( \sin u_2, \sin u_3 \cos u_2, \ldots, \sin u_n \prod_{j=2}^{n-1} \cos^2 u_j, \prod_{j=2}^{n} \cos u_j, \frac{s}{2} \right).$$
Then $M$ is a graph hypersurface with Calabi normal given by $\xi = (0, \ldots, 0, 1)$ and it satisfies $T^# = 0$. Moreover, the Calabi metric of this graph hypersurface is given by the warped product metric (17.11). Clearly, the warping function is a harmonic function. Therefore, this warped product graph hypersurface satisfies the equality case of (17.6) identically. Consequently, the estimate given in Theorem 17.3 is also optimal.

Remark 17.1. Example 17.1 shows that the conditions $\Delta f \leq 0$ in Corollary 17.3 and the “harmonicity” in Corollary 17.4 are both necessary.

Remark 17.2. Example 17.1 implies that the condition “$N_1$ is a compact Riemannian manifold” given in Corollary 17.6 is necessary.

Remark 17.3. Example 17.2 illustrates that the condition $(\Delta f)/f < - \dim N_1$ given in Corollary 17.5 is sharp.

Remark 17.4. Example 17.3 shows that the condition $\Delta f < 0$ in Corollary 17.1 is optimal as well.

18. TWISTED PRODUCT SUBMANIFOLDS

Twisted products $B \times_\lambda F$ are natural generalizations of warped products, namely the function may depend on both factors (cf. [30]). When $\lambda$ depends only on $B$, the twisted product becomes a warped product. If $B$ is a point, the twisted product is nothing but a conformal change of metric on $F$.

The study of twisted product submanifolds was initiated in 2000 (see [36]). In particular, the following results are obtained in [36].

**Theorem 18.1.** If $M = N_\perp \times_\lambda N_T$ is a twisted product CR-submanifold of a Kähler manifold $\tilde{M}$ such that $N_\perp$ is a totally real submanifold and $N_T$ is a holomorphic submanifold of $\tilde{M}$, then $M$ is a CR-product.

**Theorem 18.2.** Let $M = N_T \times_\lambda N_\perp$ be a twisted product CR-submanifold of a Kähler manifold $\tilde{M}$ such that $N_\perp$ is a totally real submanifold and $N_T$ is a holomorphic submanifold of $\tilde{M}$. Then we have

1. The squared norm of the second fundamental form of $M$ in $\tilde{M}$ satisfies

   $$||\sigma||^2 \geq 2p \|\nabla^T (\ln \lambda)\|^2,$$

   where $\nabla^T (\ln \lambda)$ is the $N^T$-component of the gradient $\nabla (\ln \lambda)$ of $\ln \lambda$ and $p$ is the dimension of $N_\perp$.

2. If $||\sigma||^2 = 2p \|\nabla^T (\ln \lambda)\|^2$ holds identically, then $N_T$ is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

3. If $M$ is anti-holomorphic in $\tilde{M}$ and $\dim N_\perp > 1$, then $||\sigma||^2 = 2p \|\nabla^T (\ln \lambda)\|^2$ holds identically if and only if $N_T$ is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

For mixed foliate twisted product CR-submanifolds of Kähler manifolds, we have the following result from [36].

**Theorem 18.3.** Let $M = N_T \times_\lambda N_\perp$ be a twisted product CR-submanifold of a Kähler manifold $\tilde{M}$ such that $N_\perp$ is a totally real submanifold and $N_T$ is a holomorphic submanifold of $\tilde{M}$. If $M$ is mixed totally geodesic, then we have

1. The twisted function $\lambda$ is a function on $N_\perp$. 

(2) $N_T \times N^1_\perp$ is a CR-product, where $N^1_\perp$ denotes the manifold $N_\perp$ equipped with the metric $g^{N_\perp}_\perp = \lambda^2 g_{N_\perp}$.

Next, we provide ample examples of twisted product $CR$-submanifolds in complex Euclidean spaces which are not $CR$-warped product submanifolds.

Let $z : N_T \rightarrow \mathbb{C}^m$ be a holomorphic submanifold of a complex Euclidean $m$-space $\mathbb{C}^m$ and $w : N^1_\perp \rightarrow \mathbb{C}^\ell$ be a totally real submanifold such that the image of $N_T \times N^1_\perp$ under the product immersion $\psi = (z, w)$ does not contain the origin $(0, 0)$ of $\mathbb{C}^m \oplus \mathbb{C}^\ell$. Let $j : N^2_\perp \rightarrow S^{q-1} \subset \mathbb{E}^q$ be an isometric immersion of a Riemannian manifold $N^2_\perp$ into the unit hypersphere $S^{q-1}$ of $\mathbb{E}^q$ centered at the origin.

Consider the map

$$\phi = (z, w) \otimes j : N_T \times N^1_\perp \times N^2_\perp \rightarrow (\mathbb{C}^m \oplus \mathbb{C}^\ell) \otimes \mathbb{E}^q$$

defined by

$$(18.1) \quad \phi(p_1, p_2, p_3) = (z(p_1), z(p_2)) \otimes j(p_3),$$

for $p_1 \in N_T$, $p_2 \in N^1_\perp$, $p_3 \in N^2_\perp$.

On $(\mathbb{C}^m \oplus \mathbb{C}^\ell) \otimes \mathbb{E}^q$ we define a complex structure $J$ by

$$J((B, E) \otimes F) = (iB, iE) \otimes F, \quad i = \sqrt{-1},$$

for any $B \in \mathbb{C}^m$, $E \in \mathbb{C}^\ell$ and $F \in \mathbb{E}^q$. Then $(\mathbb{C}^m \oplus \mathbb{C}^\ell) \otimes \mathbb{E}^q$ becomes a complex Euclidean $(m + \ell)q$-space $\mathbb{C}^{(m+\ell)q}$.

Let us put $N_\perp = N^1_\perp \times N^2_\perp$. We denote by $|z|$ the distance function from the origin of $\mathbb{C}^m$ to the position of $N_T$ in $\mathbb{C}^m$ via $z$; and denote by $|w|$ the distance function from the origin of $\mathbb{C}^\ell$ to the position of $N^1_\perp$ in $\mathbb{C}^\ell$ via $w$. We define a function $\lambda$ by $\lambda = \sqrt{|z|^2 + |w|^2}$. Then $\lambda > 0$ is a differentiable function on $N_T \times N_\perp$, which depends on both $N_T$ and $N_\perp = N^1_\perp \times N^2_\perp$.

Let $M$ denote the twisted product $N_T \times_\lambda N_\perp$ with twisted function $\lambda$. Clearly, $M$ is not a warped product.

For such a twisted product $N_T \times_\lambda N_\perp$ in $\mathbb{C}^{(m+\ell)q}$ defined above we have the following.

**Proposition 18.1.** The map

$$\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \rightarrow \mathbb{C}^{(m+\ell)q}$$

defined by (18.1) satisfies the following properties:

1. $\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \rightarrow \mathbb{C}^{(m+\ell)q}$ is an isometric immersion.
2. $\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \rightarrow \mathbb{C}^{(m+\ell)q}$ is a twisted product $CR$-submanifold such that $N_T$ is a holomorphic submanifold and $N_\perp$ is a totally real submanifold of $\mathbb{C}^{(m+\ell)q}$.

Proposition 18.1 shows that there are many twisted product $CR$-submanifolds $N_T \times_\lambda N_\perp$ such that $N_T$ are holomorphic submanifolds and $N_\perp$ are totally real submanifolds. Moreover, such twisted product $CR$-submanifolds are not warped product $CR$-submanifolds.
19. RELATED ARTICLES

After the publication of [36, 39, 40, 43], there are more than 100 articles on warped product submanifolds appeared during the last 10 years. To help further study in this vibrant field of research, we divide those articles into 16 categories according to their main results as follows.

1. Warped products in Riemannian manifolds: [38, 42, 43, 48, 51, 54, 57, 64, 66, 67, 118, 137].

2. Warped products in (generalized) complex space forms: [23, 32, 33, 34, 39, 40, 43, 44, 45, 46, 47, 49, 54, 55, 56, 57, 59, 63, 97, 100, 122, 123].

3. Warped products in Kähler manifolds: [10, 39, 43, 54, 56, 57, 78, 78, 88, 125, 128, 129].

4. Warped products in nearly Kähler manifolds: [4, 84, 86, 87, 130, 146].

5. Warped products in locally conformal Kähler manifolds: [24, 25, 77, 83, 91, 94, 106, 157].

6. Warped products in para-Kähler manifolds: [54, 60, 130].

7. Warped products in Sasakian manifolds: [1, 9, 76, 101, 95, 104, 108, 113, 139, 143, 148].

8. Warped products in generalized Sasakian manifolds: [3, 79, 81, 87, 93, 108, 116, 139, 143, 148].

9. Warped products in Kenmotsu manifolds: [7, 11, 16, 85, 88, 90, 107, 115, 136, 139, 152].

10. Warped products in cosymplectic manifolds: [18, 75, 80, 82, 121, 132, 142, 149, 147, 148, 151, 153, 156, 157].

11. Warped products in other contact metric manifolds: [5, 12, 15, 17, 70, 79, 81, 108, 120, 133, 138, 141].

12. Warped products in Riemannian product manifolds: [8, 12, 13, 14, 19, 120, 127].

13. Warped products submanifolds in quaternionic manifolds: [98, 99, 154].

14. Warped products in affine space: [52, 53, 54].

15. Twisted products submanifolds: [36, 84, 145].

16. Warped products submanifolds in other spaces: [102, 121, 124, 135, 144].

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