A FRACTIONAL ORLICZ-SOBOLEV EIGENVALUE PROBLEM AND RELATED HARDY INEQUALITIES

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Abstract. In this article we define the first Dirichlet eigenvalue for the fractional $g-$Laplacian and we prove diverse properties on it, including isolation, positivity of its eigenfunctions and its behaviour as $s \to 1^+$. In the second part of this manuscript we prove some modular and norm Hardy inequalities in fractional Orlicz-Sobolev spaces, which provide for lower bounds of eigenvalues in certain configurations.

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1. Introduction

There are only few articles that have turned out to be as influential and truly important as Sergei Sobolev 1938 manuscript “On a theorem of functional analysis” [44], where he introduced his celebrated inequalities and a meaningful universe of applications was opened in benefit of several branches of analysis.

In the last years there has been an increasing interest in problems involving anomalous diffusion due to the wide spectrum of applications in several fields of science. Fractional derivatives are a powerful tool modelling such power law phenomena, and fractional spaces supply for the natural framework to handle them. Nonetheless, fractional order Sobolev spaces are not a novelty, in fact, they become object of study since they were introduced in the 1950s in the pioneering works of E. Gagliardo [16] and L. Slobodecki˘ı [43] as a generalization of the mentioned Sobolev’s article. Up to the date it is almost impossible to give a comprehensive list of references. We remit the interested reader to the survey [12] and references mentioned there.

On the other hand, in many contexts it is useful to consider asymptotic behaviours different than powers, or different behaviours near the origin and near infinity. Spaces allowing such non-standard growths are nowadays known as Orlicz spaces, named after the polish mathematician Wladyslaw Orlicz,

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who introduced them in [38] in the early 1930s. For an introductory survey we remit, for instance, the seminal textbook [25] and the books [27, 41].

The main difference between Orlicz and Sobolev spaces lies in the fact that modulars have some, but not all, of the properties of norms. See [11] for a nice monograph on modular spaces. In this context, the power growth is generalized by a so-called Young function, that is, an application \( G : \mathbb{R}_+ \to \mathbb{R}_+ \) admitting an integral representation \( G(t) = \int_0^t g(s) \, ds \), where \( g \) is assumed to satisfy the renowned natural growth Lieberman’s condition considered in [29]

\[
0 < g^- \leq \frac{tg'(t)}{g(t)} \leq g^+ < \infty \quad \text{for all } t > 0
\]

for certain constants \( g^\pm \) (condition which can sometimes be weakened, as it is clarified later). It is well-known that the upper boundedness in the last expression is equivalent to \( G \) (and \( g \)) to satisfy the \( \Delta_2 \) condition (see [27, Section 3.4]), i.e., there exists a positive \textit{doubling constant} \( C \geq 2 \) such that

\[
G(2t) \leq CG(t) \quad \forall t > 0.
\]

For such class of functions, given an open set \( \Omega \subseteq \mathbb{R}^N \), the modular \( \Phi_G \) is defined as

\[
\Phi_G(u) = \int_\Omega G(|u(x)|) \, dx,
\]

and the classic \textit{Orlicz-Sobolev space} \( W^{1,G}(\Omega) \) as the set of functions such that \( \Phi_G(|u|) + \Phi_G(|\nabla u|) \) is bounded. It is important to mention that we also ask the conjugate function \( G^* \) to fulfill the \( \Delta_2 \) condition in order to guarantee the reflexivity and separability of these spaces. The corresponding \textit{Luxemburg norms} is defined as

\[
\|u\|_G := \inf \left\{ \lambda > 0 : \Phi_G\left(\frac{u}{\lambda}\right) \leq 1 \right\}.
\]

In this settings, the \( g \)–Laplacian

\[
\Delta_g u = \text{div} \left( g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right)
\]

fits perfectly for modelling non-standard diffusion processes, and when \( pG(t) = t^p \) it leads to the well-known \( p \)–Laplacian operator. This kind of operators were widely studied. For a systematic treatment of its properties we address to the work [29] due to G. Liebermann.

A bridge between fractional order theories and Orlicz-Sobolev settings is provided in [14], where the authors define the fractional order Orlicz-Sobolev space associated to a Young function \( G \) and a fractional parameter \( s \in (0, 1) \) as

\[
W^{s,G}(\Omega) = \left\{ u \in L^G(\Omega) \text{ such that } \Phi_{s,G}(u) < \infty \right\},
\]

where \( \Omega \subseteq \mathbb{R}^N \) is an open set and the fractional modular \( \Phi_{s,G} \) is defined as

\[
\Phi_{s,G}(u) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} G\left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^N}
\]

and the corresponding Luxemburg’s norm is given by \( \|u\|_{s,G} := \|u\|_G + [u]_{s,G} \), where the \((s, G)\)-\textit{Gagliardo semi-norm} is defined as

\[
[u]_{s,G} := \inf \left\{ \lambda > 0 : \Phi_{s,G}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.
\]

In this context, the fractional \((s, G)\)–Laplacian

\[
(-\Delta_g)^s u := \text{p.v.} \int_{\mathbb{R}^N} g\left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{dy}{|x - y|^{N+s}}
\]

is a direct generalization of the fractional \( p \)–Laplacian (it corresponds to \( pG(t) = t^p \)) and the \( g \)–Laplacian (it is obtained as \( s \to 1^+ \), see [14]).
The main interest in this paper is to study the Dirichlet eigenvalue problem associated to \((-\Delta_g)^s\) in a bounded open set \(\Omega \subset \mathbb{R}^N\), namely,

\[
\begin{cases}
(-\Delta_g)^s u = \lambda g(u) & \text{in } \Omega, \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

The natural space in which consider this problem is \(W^{1,2}_0(\Omega, G)\), defined as the set of functions in \(W^{1,2}(\Omega)\) vanishing outside \(\Omega\).

Let us start with a brief review on the state of the art of this problem. Observe that for the particular case of \(G(t) = \frac{t^p}{p}\), (1.1) becomes the Dirichlet eigenproblem for the fractional \(p\)-Laplacian. In [30] Lindgren and Lindqvist studied its spectrum and put special attention to the principal eigenvalue, which is proved to inherit many properties of the \(p\)-Laplacian one: simplicity, isolation, constant sign of eigenfunctions, etc. This line of research was widely continued, only to mention a few works, we remit to [6, 7, 15] and references to these papers.

When the Young function does not obey necessarily a power law but a local configuration is considered, that is, when taking \(s \to 1^+\) in (1.1), many references can be found in the literature. We refer for instance the papers of Mustonen and Tienari [36], Tienari [45], Gossez and Manásevich [17] and García-Huidobro, Le, Manásevich and Schmit [37], where several results and generalizations were analyzed in this context. It is worthy to mention that the difficulty tackling this nonstandard growth problem can be considerably increased when the \(\Delta_2\) condition on \(G\) (or its conjugate function \(G^s\)) is not assumed to hold. Indeed, the doubling condition is intimately related with the reflexivity of the space, see [27, Theorem 3.13.9]. We remark that some of the cited works do not assume that condition in their hypothesis.

The main scope of the first part of this manuscript is to define a first eigenvalue of (1.1) and study their properties. In contrast with the prototypical \(p\)-Laplacian, the possible lack of homogeneity of the problem does not allow to normalize eigenfunctions with unit modular \(\Phi_G(u)\). In fact, eigenvalues may depend strongly on the energy level of the solution; for any \(\mu > 0\), we can consider the minimization problem

\[
\alpha_\mu = \inf_{u \in M_\mu} \frac{\Phi_{s,G}(u)}{\Phi_G(u)} \quad \text{with} \quad M_\mu = \{u \in W^{s,G}_0(\Omega) : \Phi_G(u) = \mu\}.
\]

However, in general this quantity is not an eigenvalue of of (1.1). Nevertheless, for each value of \(\mu > 0\) we prove that there exists an eigenvalue \(\lambda_\mu > 0\) of the Euler-Lagrange equation (1.1) with nonnegative eigenfunction \(v_\mu \in M_\mu\). In general \(\alpha_\mu\) may differ from \(\lambda_\mu\), although they are comparable. Hence, we consider the less quantity over all possible values of \(\lambda_\mu\), i.e.,

\[
\lambda_1 = \inf\{\lambda_\mu : \mu \in \mathbb{R} \setminus \{0\}\}.
\]

Since in Proposition 3.6 we prove that the spectrum of (1.1) is closed, the number \(\lambda_1\) is in turn an eigenvalue of (1.1) with nonnegative eigenfunction \(u_1\).

Analyzing properties of weak solutions of (1.1) can be a tough task. Fortunately, owing the ideas from [30], it can be proved that weak solutions of (1.1) are viscosity solutions provided that they are continuous and a lower growth control of \(G\) is considered, namely, when

\[
(C) \quad s(1-s)G^{-} > N
\]

is assumed to hold, see [13, Lema 3.7]. This observation allows to employ the machinery of the pointwise test functions of the viscosity theory.

We pointed out that \(u_1\) can be considered to be nonnegative. In fact, by a strong maximum type argument, in Corollary 4.5 we establish that \(u_1\) have constant sign in \(\Omega\) under the assumption (C). Furthermore, in Proposition 4.6 we prove that it is the unique eigenfunction satisfying that property.

When dealing with the isolation of first eigenvalues it is key the use of a critical exponent Sobolev type embedding. Due to (up to the date) a lack of a fractional Orlicz-Sobolev embedding, we restrict the study of the isolation of \(\lambda_1\) to domains where the measure is comparable with the diameter. When \(\Omega\) belongs to this class and (C) is assumed, Proposition 4.9 establishes the isolation of \(\lambda_1\) in the spectrum.
Since the Rayleigh quotient in the definition of $\lambda_1$ has not homogeneity properties in general, simplicity of the principal eigenvalue of (1.1) is not expected to hold. For simplicity of $p$–Laplacian type eigenvalues, we refer to the classical work of Anane [1].

Another issue we consider is the behaviour of $\alpha_{1,s} := \inf\{\alpha_\mu : \mu > 0\}$ as $s \to 1^+$. In Theorem 4.10, owing some tools of the $\Gamma$–convergence, it is proved that $\alpha_{1,s}$, the inverse of the best constant in the modular inclusion of $W_0^{s,G}(\Omega)$ into $L^G(\Omega)$ converges to $\bar{\alpha}_1$, the inverse of the best constant in the modular inclusion of $W_0^{1,G}(\Omega)$ into $L^{\widetilde{G}}(\Omega)$, where $\widetilde{G}$ is a function obtained in terms of $G$; more precisely, we prove that $\lim_{s \to 1^+} (1-s)\alpha_{1,s}(\Omega) = \bar{\alpha}_1(\Omega)$. See the statement of Theorem 4.10 for precise definitions.

The second part of this manuscript is devoted to analyze some Hardy type inequalities. Since G.H. Hardy proved in 1920 its nowadays classical inequality

$$[1-p-\alpha]^p \int_0^\infty |u(x)|^p |x|^{-p-\alpha} \, dx \leq p \int_0^\infty |u'(x)|^p |x|^{-\alpha} \, dx$$

for a suitable set of exponents and functions, a vast amount of generalizations were provided due to its relevance in analysis. In general $L^p$ spaces we refer, for instance, to [24, 34, 35, 42] and references therein. In fractional spaces see for instance [5, 26, 31] and references to these papers. It is worthy to note that in these cases, norm inequalities are equivalent to integral inequalities, up to a power. However, the situation is more subtle in Orlicz spaces. Two kind of results can be distinguished: modular Hardy inequalities and norm Hardy inequalities, being in general the second type stronger than the first one. Just to mention some references, we refer to [2, 3, 4, 20, 28] for a very general family of inequalities.

In the fractional Orlicz-Sobolev settings, we prove in Theorem 5.3 a one dimensional modular inequality for continuous functions, i.e., given $s \in (0,1)$, there exists a constant $c_H > 0$ independent of $u \in W_0^{s,G}(0,\ell)$, such that

$$\int_0^\ell G \left( \frac{|u(x)|}{x^s} \right) \, dx \leq c_H \Phi_{s,G}(u),$$

where $sg^- > 1$. From this result we deduce in Corollary 5.4, under the same assumptions, the existence of a constant $\tilde{c}_H > 0$ independent of $u \in W_0^{s,G}(0,\ell)$ for which the following norm inequality holds

$$\left\| \frac{u}{x^s} \right\|_{G} \leq \tilde{c}_H \|u\|_{s,G}.$$

Moreover, in Theorem 5.5 we prove that the above norm inequality holds when dispensing with the continuity, but assuming the relation $sg^- > 1-s$. Finally, in Theorem 5.10, assuming a suitable lower behaviour of the Young function $G$ we establish the validity of a modular Hardy inequality in $\mathbb{R}^N$ having the form

$$\Phi_G \left( \frac{u(x)}{|x|^s} \right) \leq C_H \Phi_{s,G}(u)$$

for any function $u \in W_0^{s,G}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open and bounded set and $s \in (0,1)$. The correspondent norm inequality is provided in Corollary 5.12. Finally, as a direct application of these results, lower bounds of eigenvalues can be obtained.

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2. Preliminary results

2.1. Young functions. We say that $G : \mathbb{R}_+ \to \mathbb{R}_+$ is a Young function if it admits the integral formulation $G(t) = \int_0^t g(s) \, ds$, and the growth condition

$$\tag{L} 0 < g^- \leq \frac{tg'(t)}{g(t)} \leq g^+ < \infty \quad \forall t > 0$$
is satisfied for certain constants $g^\pm$. It is worthy to mention that condition (L) can be weakened in some cases by considering

$$(L') \quad 0 < g^- \leq \frac{tg(t)}{G(t)} \leq g^+ < \infty \quad \forall t > 0.$$  

In fact, a condition of this type (possibly with other constants) can be deduced from (L). In general, for results making use only of the fractional space framework, it is enough with asking for condition (L'). However, when the structure of the fractional operator is taken into account, more regularity on the Young function is needed, and (L) must be put into manifest. Indeed, for all the results in this section it is enough with (L') but for the remaining sections (L) is required.

**Remark 2.1.** Without loss of generality, we assume that $G$ is even in the whole real line.

Young functions can be proved to be convex and $C^2$. Moreover, they fulfill the following properties (see for instance, [33, Lemma 2.1]).

**Lemma 2.2.** Let $G$ be a Young function, $s, t \geq 0$ and define $G^+ := 1 + g^+$. Then

$$(g_1) \quad \min \{sG^-, sG^+\} g(t) \leq g(st) \leq \max \{sG^-, sG^+\} g(t),$$

$$(g_2) \quad g(2t) \leq cg(t) \quad \text{with} \quad c := 2g^+,$$

$$(g_3) \quad \frac{tg(t)}{G^+} \leq G(t) \leq tg(t).$$

**Lemma 2.3.** Let $G$ be a Young function, $s, t \geq 0$ and define $G^\pm := 1 + g^\pm$. Then

$$(G_1) \quad \min \{sG^-, sG^+\} G(t) \leq G(st) \leq G^+ \max \{sG^-, sG^+\} G(t),$$

$$(G_2) \quad G(s + t) \leq C(G(s) + G(t)) \quad \text{with} \quad C := G^+ 2g^+,$$

$$(G_3) \quad G \text{ is super-linear at zero, that is } \lim_{t \to 0} \frac{G(t)}{t} = 0.$$  

**Lemma 2.4.** [14, Lemma 2.6] Let $G$ be a Young function and $s, t \geq 0$. Then for any $\delta > 0$ there exists a positive constant $C_\delta$ such that

$$G(s + t) \leq C_\delta G(s) + (1 + \delta) G^+ G(t).$$

Condition $(g_2)$ is known as the \(\Delta_2\) condition or doubling condition and, as it is showed in [27, Theorem 3.4.4], it is equivalent to

$$\frac{tg'(t)}{g(t)} \leq g^+, \quad \forall t > 0.$$  

Without loss of generality $g$ can be normalized such that $g(1) = 1$.

The complementary Young function $G^*$ of a Young function $G$ is defined as

$$G^*(t) = \sup \{tw - G(w) : w > 0\}.$$  

From this definition the following Young-type inequality holds

$$st \leq G(s) + G^*(t) \quad \text{for all} \quad s, t \geq 0.$$  

It is not hard to see that $G^*$ can be written in terms of the inverse of $g$ as

$$(2.1) \quad G^*(t) = \int_0^t g^{-1}(s) \, ds,$$

see [41, Theorem 2.6.8], from where, by using [33, Lemma 2.2], we get

$$(G'_1) \quad \min \{sG^-, sG^+\} G^*(t) \leq G^*(st) \leq \max \{sG^-, sG^+\} G^*(t), \quad \forall s, t \geq 0.$$  

**Example 2.5.** A wide range of functions under these hypothesis can be considered.
(1) **Powers.** We have that \( g^- = g^+ = p - 1 \) when \( pG(t) = tp \), and conversely, if \( g^- = g^+ \) then \( G \) is a power.

(2) **Powers x logarithms.** We can consider \( G \) such that \( g(t) = t^a \log(bt + c) \) with \( a, b, c > 0 \) such that (L) holds for \( g^- = a \) and \( g^+ = a + 1 \).

(3) **Different power behaviour.** Another important example is the family of functions \( G \) allowing different power behaviour near 0 and infinity. \( G \) can be considered such that \( g \in C^1([0, \infty)) \), \( g(t) = c_1 t^{\gamma_1} \) for \( t \leq s \) and \( g(t) = c_2 t^{\gamma_2} + d \) for \( t \geq s \). In this case \( g^- = \min\{\gamma_1, \gamma_2\} \) and \( g^+ = \max\{\gamma_1, \gamma_2\} \).

(4) **Linear combinations.** If \( g_1 \) and \( g_2 \) satisfy (L) then \( a_1 g_1 + a_2 g_2 \) also satisfies (L) when \( a_1, a_2 \geq 0 \).

(5) **Products.** If \( g_1 \) and \( g_2 \) satisfy (L) with constants \( g_1^+ \) then \( g_1 g_2 \) also satisfies (L) with constants \( g^- = g_1^- g_2^- \) and \( g^+ = g_1^+ g_2^+ \).

(6) **Compositions.** If \( g_1 \) and \( g_2 \) satisfy (L) with constants \( g_i^+ \), then \( g_1 \circ g_2 \) also satisfies (L) with constants \( g^- = g_1^- g_2^- \) and \( g^+ = g_1^+ g_2^+ \).

For our purposes, we will always assume Young functions \( G \) for which \( L^G(\Omega) \) is a reflexive space. This is equivalent to ask \( G \) and \( G^* \) to satisfy the \( \Delta_2 \) condition, see [27, Theorem 3.13.9]. More precisely, \( G \) always inherits the doubling property from \( G^* \), but inverse implication holds only when the reflexivity of \( L^G(\Omega) \) is assumed.

### 2.2. Fractional Orlicz-Sobolev spaces.

Given a Young function \( G \), a fractional parameter \( s \in (0, 1) \) and an open set \( \Omega \subseteq \mathbb{R}^N \), we consider the Orlicz spaces \( L^G(\Omega) \) and the fractional Orlicz-Sobolev space \( W^{s,G}(\Omega) \) defined as

\[
L^G(\Omega) := \{ u : \mathbb{R}^N \to \mathbb{R} \text{ Lebesgue measurable, such that } \Phi_G(u) < \infty \},
\]

\[
W^{s,G}(\Omega) := \{ u \in L^G(\Omega) \text{ such that } \Phi_{s,G}(u) < \infty \},
\]

where the modulars \( \Phi_G \) and \( \Phi_{s,G} \) are defined as

\[
\Phi_G(u) := \int_{\Omega} G(u(x)) \, dx,
\]

\[
\Phi_{s,G}(u) := \int_{\mathbb{R}^N \times \mathbb{R}^N} G\left( \frac{u(x) - u(y)}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^N}.
\]

These spaces are endowed with the so-called **Luxemburg norms** defined as

\[
\|u\|_G := \inf \left\{ \lambda > 0 : \Phi_G\left( \frac{u}{\lambda} \right) \leq 1 \right\},
\]

\[
\|u\|_{s,G} := \|u\|_G + [u]_{s,G},
\]

where the \((s,G)-Gagliardo semi-norm\) is defined as

\[
[u]_{s,G} := \inf \left\{ \lambda > 0 : \Phi_{s,G}\left( \frac{u}{\lambda} \right) \leq 1 \right\}.
\]

We also define the following space

\[
W^{s,G}_0(\Omega) := \{ u \in W^{s,G}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.
\]

Alternatively, one can consider the space \( \widetilde{W}^{s,G}(\Omega) := C^\infty_c(\Omega)^{||\cdot||_{s,G}} \). In the classical case, i.e. when \( pG(t) = tp \), these spaces \( W^{s,p}_0(\Omega) \) and \( \widetilde{W}^{s,p}(\Omega) \) are known to coincide when \( sp < 1 \) or when \( 0 < s < 1 \) and \( \Omega \) has Lipschitz boundary. For the purposes of this paper we consider the space \( W^{s,G}_0(\Omega) \). Observe that the following inclusions hold

\[
W^{s,G}_0(\Omega) \subset W^{s,G}(\mathbb{R}^N) \subset L^G(\mathbb{R}^N).
\]

Hereafter, \( \Omega \) will always stand for a bounded open set in \( \mathbb{R}^N \) whose diameter is denoted as

\[
d = \text{diam}(\Omega) = \sup\{|x - y| : x, y \in \Omega\}.
\]

Moreover, every function \( u \in L^G(\Omega) \) will be assumed to be extended by 0 to \( \mathbb{R}^N \setminus \Omega \).

We finish this section recalling some results on fractional Orlicz-Sobolev spaces.

Proposition 2.6 ([14], Proposition 2.10). Let $G$ be a Young function such that $G$ and $G^*$ satisfy the $\Delta_2$ condition, and consider $s \in (0,1)$. Then $W^{s,G}(\mathbb{R}^N)$ is a reflexive and separable Banach space. Moreover, $C^\infty_c(\mathbb{R}^N)$ is dense in $W^{s,G}(\mathbb{R}^N)$.

A variant of the well-known Fréchet-Kolmogorov compactness theorem gives the compactness of the inclusion of $W^{s,G}$ into $L^G$.

Theorem 2.7 ([14], Theorem 3.1). Let $s \in (0,1)$ and $G$ a Young function. Then $W^{s,G}(\Omega) \subseteq L^G(\Omega)$.

Another useful result regarding strong convergence is the following.

Proposition 2.8 ([41], Theorem 12). Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in $L^G$ and $u \in L^G$. If $G^*$ satisfies the $\Delta_2$ condition, $\Phi_G(u_n) \to \Phi_G(u)$ and $u_n \to u$ a.e., then $u_n \to u$ in the $L^G$ norm.

Finally we recall that fractional Orlicz-Sobolev spaces are embedded into the usual fractional Sobolev spaces as well.

Proposition 2.9. [13, Corollary 2.10] Given $0 < t < s < 1$ and a Young function $G$, for any $q$ such that $1 \leq q < g^-$ it holds that $W^{s,G}_0(\Omega) \subset W^{t,q}_0(\Omega)$ with continuous inclusion.

As a consequence, since $W^{s,G}_0(\Omega)$ is continuously embedded into $C^{0,\alpha}(\Omega)$ for $\alpha = s - \frac{N}{g^-} > 0$, see [12, Section 8], we can characterize continuous functions in fractional Orlicz-Sobolev spaces.

Corollary 2.10. Let $\Omega$ be a bounded and open set $s \in (0,1)$. If $G$ is a Young function such that $sg^- > N$, then $W^{s,G}_0(\Omega) \subset C^{0,\alpha}(\Omega)$.

2.3. A Poincaré’s inequality. A modular and a norm Poincaré inequality for functions in $W^{s,G}_0(\Omega)$ is proved in this section, giving as a consequence that $\| \cdot \|_{s,G}$ is an equivalent norm in $W^{s,G}_0(\Omega)$.

Theorem 2.11. Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded and let $G$ be a Young function. Then for $s \in (0,1)$ it holds that

$$\Phi_G(u) \leq \Phi_{s,G}(C_p d^* u)$$

for all $u \in W^{s,G}_0(\Omega)$, where $C_p = \left( \frac{s(\frac{g^-}{N+g^+})^3}{N\omega_N} \right)^{\frac{1}{N}}$ with $\omega_N$ the volume of the unit ball in $\mathbb{R}^N$.

Proof. Let $C_p$ be a positive constant to determine. Given $x \in \Omega$, observe that when $|x - y| \geq d$, then $y \notin \Omega$. Hence, by using $(G_1)$ we get

$$\Phi_{s,G}(C_p d^* u) \geq \int_{|x-y| \geq d} G \left( \frac{d^*}{|x-y|^s} C_p u(x) \right) \frac{dydx}{|x-y|^N} \geq \frac{C_p^G}{(G^+)^2} \left( \int_{\Omega} G(u(x)) \right) d_{G^+} \left( \int_{|z| \geq d} \frac{dz}{|x-y|^{N+g^+}} \right)$$

since, without loss of generality we can assume that $C_p \geq 1$.

Now, by using polar coordinates we have that

$$\int_{|z| \geq d} \frac{dz}{|x-y|^{N+g^+}} = \frac{N\omega_N}{sG^+} d^{-sG^+},$$

and the result follows choosing properly the constant $C_p$.

In general norm inequalities are weaker than the corresponding modular ones. From the previous result it can be deduced the following norm Poincaré’s inequality.

Corollary 2.12. Under the same assumptions than in Theorem 2.11, it holds that

$$\| u \|_{s,G} \leq C_p d^*[u]_{s,G}$$

for every $s \in (0,1)$ and $u \in W^{s,G}_0(\Omega)$. 
Proof. Given $u \in W_0^{s,G}(\Omega)$, applying Theorem 2.11 to the function \( \frac{u}{C_p d^s[u]_{s,G}} \), we get
\[
\Phi_G \left( \frac{u}{C_p d^s[u]_{s,G}} \right) \leq \Phi_{s,G} \left( \frac{u}{|u|_{s,G}} \right) = 1
\]
by definition of the Luxemburg’s norm. Consequently,
\[
\|u\|_G = \inf \{ \lambda : \Phi_G \left( \frac{u}{\lambda} \right) \leq 1 \} \leq C_p d^s[u]_{s,G}
\]
as desired. \( \square \)

As a direct consequence \((G_1)\), one obtains the following.

**Corollary 2.13.** Under the same assumptions than in Theorem 2.11, it holds that
\[
\Phi_G(u) \leq C \max\{d^{G^+}, d^{G^-}\} \Phi_{s,G}(u)
\]
for all $s \in (0,1)$ and $u \in W_0^{s,G}(\Omega)$, where $C$ depends only on $s$, $N$ and $G^\pm$.

In section 3 we deal with the differences between the best constant in the modular Poincaré inequality for functions in $W_0^{s,G}(\Omega)$ and the principal eigenvalue of the Dirichlet fractional $g -$Laplacian as well as its behavior as $s \to 1$.

### 2.4. The fractional $g$–Laplacian operator.

Let $G$ be a Young function and $s \in (0,1)$ be a parameter. The fractional $g$–Laplacian operator is defined as
\[
(-\Delta_g)^s u := \text{p.v.} \int_{\mathbb{R}^N} g \left( \frac{u(x) - u(y)}{|x - y|^s} \right) \frac{dy}{|x - y|^{N+s}},
\]
where p.v. stands for *in principal value* and $g = G'$ is well defined in view of \((G_4)\).

This operator is well defined between $W^{s,G}(\mathbb{R}^N)$ and its dual space $W^{-s,G}^*(\mathbb{R}^N)$. In fact, in [14, Theorem 6.12] the following representation formula is provided
\[
\langle (-\Delta_g)^s u, v \rangle = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g \left( \frac{u(x) - u(y)}{|x - y|^s} \right) \frac{v(x) - v(y)}{|x - y|^{N+s}} \frac{dy}{|x - y|^{N}}
\]
for any $v \in W^{s,G}(\mathbb{R}^N)$.

Given a bounded open set $\Omega \subset \mathbb{R}^N$ and $f \in L^{G^*}(\Omega)$, we say that $u \in W_0^{s,G}(\Omega)$ is a *weak solution* of the Dirichlet type equation
\[
\begin{align*}
(-\Delta_g)^s u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \mathbb{R}^N \setminus \Omega
\end{align*}
\]
if for any $v \in W_0^{s,G}(\Omega)$ it holds that
\[
\langle (-\Delta_g)^s u, v \rangle = \int_{\Omega} f v \, dx.
\]
Observe that since $C_c^\infty(\Omega) \subset W_0^{s,G}(\Omega)$, weak solutions of \((2.3)\) are solutions in the sense of distributions.

The link between solutions to \((2.3)\) and minimum points of the functional
\[
\mathcal{F}_s(u) := \mathcal{J}_s(u) - \int_{\Omega} f u \, dx, \quad \mathcal{J}_s(u) := \begin{cases} 
\Phi_{s,G}(u) & \text{if } u \in W_0^{s,G}(\Omega) \\
+\infty & \text{otherwise},
\end{cases}
\]
is stated in the following result.

**Theorem 2.14** ([14], Theorem 6.15). Let $\Omega \subset \mathbb{R}^N$ be open and bounded and $f \in L^{G^*}(\Omega)$. Then $u \in W_0^{s,G}(\Omega)$ is a weak solution of \((2.3)\) if and only if
\[
\mathcal{F}_s(u) = \inf_{v \in L^{G^*}(\Omega)} \mathcal{F}_s(v),
\]
where the functional $\mathcal{F}_s : L^{G}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is given by \((2.4)\).
3. The eigenvalue problem

Given an open and bounded set $\Omega \subset \mathbb{R}^N$, a parameter $s \in (0,1)$ and a Young function $G$ such that $G^\prime = g$, we consider the following Dirichlet type problem

$$\begin{aligned}
(3.1) & \quad \int (-\Delta_g)^s u = \lambda g(u) \quad \text{in } \Omega, \\
& \quad u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.
\end{aligned}$$

We say that $\lambda$ is an eigenvalue of $(3.1)$ with eigenfunction $u \in W_0^{s,G}(\Omega) \setminus \{0\}$, provided that

$$\begin{aligned}
(3.2) & \quad \langle (-\Delta_g)^s u, v \rangle = \lambda \int_{\Omega} g(u)v \, dx
\end{aligned}$$

holds for all $v \in W_0^{s,G}(\Omega)$.

We define the spectrum $\Sigma$ as

$$\Sigma := \{ \lambda \in \mathbb{R} : \text{there exists } u \in W_0^{s,G}(\Omega), \text{ nontrivial solution to } (3.2) \}.$$ 

In order to establish existence of a principal eigenvalue of $(3.1)$, we consider the functionals $F, G : W_0^{s,G}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}
(3.3) & \quad F(u) = \Phi_{s,G}(u), \\
& \quad G(u) = \Phi_G(u).
\end{aligned}$$

In contrast with $p-$Laplacian type problems, the possible lack of homogeneity does not allow to normalize eigenfunctions to have unit modular $\Phi_G(u)$. Indeed, given $\mu > 0$, we can consider the minimization problem

$$\begin{aligned}
(3.4) & \quad \alpha_\mu = \inf_{u \in M_\mu} \frac{F(u)}{G(u)} \quad \text{with } M_\mu = \{ u \in W_0^{s,G}(\Omega) : \Phi_G(u) = \mu \}.
\end{aligned}$$

This quantity can be seen as the inverse of the best constant in the modular inclusion of $M_\mu$ into $L^G(\Omega)$ but, unlike the case of powers, in general $\alpha_\mu$ is not an eigenvalue of $(3.1)$.

We will see that for each election of $\mu$ there is a constant $\lambda_\mu$ (which is, in fact, comparable with $\alpha_\mu$) satisfying $(3.2)$ for some nonnegative $u_\mu \in M_\mu$. Therefore, we can consider the less quantity $\lambda_\mu$ over all possible choices of $\mu$, that is,

$$\lambda_1 = \inf \{ \lambda_\mu : \mu > 0 \}.$$ 

Since $\Sigma$ is closed, we will see that this election of $\lambda_1$ can be considered to be the first eigenvalue of $(3.1)$ and it shares similar properties with the principal eigenvalue of $p-$Laplacian type problems.

3.1. The first eigenvalue $\lambda_1$. We first prove that $\alpha_\mu$ is attained $\forall \mu > 0$. 

**Proposition 3.1.** The minimization problem $(3.4)$ has a solution $\alpha_\mu$ for any $\mu > 0$.

**Proof.** Let $\{u_k\}_{k \in \mathbb{N}} \subset M_\mu$ be a minimizing sequence for $\alpha_\mu$. Let us see that $\|u_k\|_{s,G}$ is bounded independently of $k$. If $\|u_k\|_{s,G} \leq 1$ there is nothing to prove. Assume that $\|u\|_{s,G} \geq 1 + \varepsilon$ for some $\varepsilon > 0$, then by using $(G_1)$ we obtain that

$$\begin{aligned}
\Phi_{s,G}(u_k) & \geq \frac{1}{G^+} \Phi_{s,G} \left( \frac{u_k(1 + \varepsilon) \|u_k\|_{s,G}}{1 + \varepsilon} \right) \|u_k\|_{s,G}^{G^+} \\
& \geq \frac{1}{G^+} \Phi_{s,G} \left( \frac{u_k}{\|u_k\|_{s,G}} \right) \left( \frac{\|u_k\|_{s,G}}{1 + \varepsilon} \right)^{G^+} \left( \frac{\|u_k\|_{s,G}}{1 + \varepsilon} \right)^{G^+} = \frac{1}{G^+} \left( \frac{\|u_k\|_{s,G}}{1 + \varepsilon} \right)^{G^+},
\end{aligned}$$

where the last equality follows from the definition of the Luxemburg norm. Hence, when $\|u_k\|_{s,G} > 1$ the sequence $\{u_k\}_{k \in \mathbb{N}} \subset M_\mu$ is uniformly bounded:

$$\|u_k\|_{s,G} \leq (G^+ \Phi_{s,G}(u_k))^{1\over G^+} = (G^+ \mu \alpha_\mu)^{1\over G^+}.$$
Hence, due to the Hölder’s inequality for Orlicz spaces (see [27, Theorem 3.7.5]), \( g(|\theta_u| + |\theta_v|) \theta_v \in L^1(\mathbb{R}^{2N}, d\mu) \). Thus, by the dominated convergence theorem, 

\[
\langle \mathcal{F}'(u), v \rangle = \lim_{t \to 0} \frac{\Phi_{s,G}(u + tv) - \Phi_{s,G}(v)}{t} = \frac{d}{dt} \Phi_{s,G}(u + tv) \bigg|_{t=0} = \int_{\mathbb{R}^{2N}} g(\theta_u) \theta_v \, d\mu = \langle (-\Delta_g)^s u, v \rangle.
\]
Let us see that $F'$ is continuous. Let $\{u_{k}\}_{k \in \mathbb{N}} \subset W_{0}^{s,G}(\Omega)$ be a such that $u_{k} \rightarrow u$ and observe that

$$
|\langle F'(u_{k}) - F'(u), v \rangle| = \left| \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} (g(\theta_{u}) - g(\theta_{u_{k}})) \theta_{v} \, d\mu \right|
$$

then, by Egoroff's Theorem, there exists a positive sequence $\delta_{k} \rightarrow 0$ such that

$$
\sup_{\|v\|_{L^{\infty}} \leq 1} \left| \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} (g(\theta_{u}) - g(\theta_{u_{k}})) \theta_{v} \, d\mu \right| \leq \|g(\theta_{u}) - g(\theta_{u_{k}})\|_{L^{\infty}(\mathbb{R}^{2N},d\mu)} + \delta_{k},
$$

where we have used again the Hölder's inequality, and according to the Luxemburg norm definition, we have denoted

$$
\|g(\theta_{u})\|_{L^{\infty}(\mathbb{R}^{2N},d\mu)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^{N}} G^\ast \left( \frac{g(\theta_{u})}{\lambda} \right) \, d\mu \leq 1 \right\}.
$$

Now, since $G^\ast$ satisfies the $\Delta_{2}$ condition, by Proposition 2.8 we get

$$
\|g(\theta_{u}) - g(\theta_{u_{k}})\|_{L^{\infty}(\mathbb{R}^{2N},d\mu)} \rightarrow 0,
$$

and therefore $|\langle F'(u_{k}) - F'(u), v \rangle|_{(W_{0}^{s,G}(\Omega))'} \rightarrow 0$ as required.

A similar reasoning allows us to claim that $G \in C^{1}$ and

$$
\lim_{t \rightarrow 0} \frac{G(u + tv) - G(v)}{t} = \frac{d}{dt} \Phi_{G}(u + tv) \big|_{t = 0} = \int_{\Omega} g(u)v \, dx.
$$

The proof is concluded. \qed

As a consequence, we get the following result.

**Theorem 3.3.** For every $\mu > 0$ there exists a number positive $\lambda_{\mu}$ bounded by below independently of $\mu$ such that (3.2) holds for a nonnegative solution $u_{\mu}$ such that $G(u_{\mu}) = \mu$.

**Proof.** In view of Proposition 3.2, from the Lagrange multiplier rule there exist $\lambda_{\mu}$ and $u_{\mu} \in W_{0}^{s,G}(\Omega)$ such that $G(u_{\mu}) = \mu$ and

$$
\langle (-\Delta_{g})^{s} u_{\mu}, v \rangle = \lambda_{\mu} \int_{\Omega} g(u_{\mu})v
$$

for all $v \in W_{0}^{s,G}(\Omega)$, that is, $u_{\mu}$ is a weak solution of (3.1). Choosing $v = u_{\mu}$ in the last expression, we obtain that $\lambda_{\mu} > 0$. Moreover, since the functionals $F$ and $G$ are invariant by replacing $u_{\mu}$ with $|u_{\mu}|$ we may assume that $u_{\mu}$ is one-signed in $\Omega$.

From the above expression we get

$$
\lambda_{\mu} = \frac{\langle (-\Delta_{g})^{s} u_{\mu}, u_{\mu} \rangle}{\int_{\Omega} g(u_{\mu})u_{\mu} \, dx},
$$

and hence, from (L) and Corollary 2.13 we obtain that the lower bound of $\lambda_{\mu}$ is in fact independent on $\mu:

$$
0 < C \max\{d^{sG^+}, d^{sG^-}\} \leq \frac{G^{-} \Phi_{G}(u_{\mu})}{G^{+} \Phi_{G}(u_{\mu})} \leq \lambda_{\mu},
$$

where $C$ depends only on $s$, $N$ and $G^{\pm}$.

\qed

In view of the previous result, it has sense to define the quantity

$$
\lambda_{1} = \inf \{ \lambda_{\mu} : \mu > 0 \}
$$

where $\lambda_{\mu}$ be the number defined in Theorem 3.3, and the following holds.

**Corollary 3.4.** The quantity $\lambda_{1}$ defined in (3.5) is strictly positive and

$$
\lambda_{1} \geq C \min\{d^{-sG^-}, d^{-sG^+}\},
$$

where $C$ is a constant depending only on $s$, $N$ and $G^{\pm}$.\]
Remark 3.5. Observe that from (L) it follows that \( \alpha_\mu \) and \( \lambda_\mu \) are comparable for any \( \mu > 0 \). Indeed, if we define
\[
(3.6) \quad \alpha_1 := \inf \{ \alpha_\mu : \mu > 0 \}
\]
we have that
\[
\frac{g^-}{g^+} \alpha_1 \leq \lambda_1 \leq \frac{g^+}{g^-} \alpha_1.
\]

Finally, we prove that \( \Sigma \) is closed, from where we deduce that the quantity \( \lambda_1 \) defined in (3.5) is an eigenvalue of (3.1) as well.

**Proposition 3.6.** The spectrum of (3.1) is closed.

**Proof.** The fact that \( \Sigma \) is closed follows from the monotonicity of \((-\Delta)^s\). Indeed, let \( \lambda_j \to \lambda \) and let \( u_j \in W^{s,G}_0(\Omega) \) be an eigenfunction associated to \( \lambda_j \). Arguing as in the proofs of Proposition 3.1 and 3.2, there exists \( u \in W^{s,G}_0(\Omega) \) and \( \eta \in L^G(\mathbb{R}^{2N}, |x-y|^{-N}dxdy) \) such that
\[
u_k \to u \quad \text{weakly in } W^{s,G}_0(\Omega),
u_k \to u \quad \text{strongly in } L^G(\Omega),\quad g(u_j) \to \eta \quad \text{weakly in } L^G(\Omega),
\]
and hence we obtain that
\[
\int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{v(x) - v(y)}{|x-y|^s} \frac{dx \, dy}{|x-y|^N} = \lambda \int_{\Omega} \xi v \quad \text{for all } v \in W^{s,G}_0(\Omega).
\]
Consequently, the proof will finish if we show that for any \( v \in W^{s,G}_0(\Omega) \)
\[
\int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} g\left(\frac{u(x) - u(y)}{|x-y|^s}\right) \frac{v(x) - v(y)}{|x-y|^N} \frac{dx \, dy}{|x-y|^N} = \int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{v(x) - v(y)}{|x-y|^s} \frac{dx \, dy}{|x-y|^N},
\]
\[
\int_{\Omega} g(u)v = \int_{\Omega} \xi v.
\]
For this purpose we use the fact that \( g = G^\prime \) is monotone since \( G \) is a Young function, that is,
\[(g(a) - g(b))(a - b) \geq 0 \quad \text{for all } a, b \in \mathbb{R}.
\]
For every \( w \in W^{s,G}_0(\Omega) \) we get
\[
0 \leq \int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \left\{ g(\theta_{u_j}(x,y)) - g(\theta_w(x,y)) \right\} \{ \theta_{u_j}(x,y) - \theta_w(x,y) \} \frac{dx \, dy}{|x-y|^N} = \lambda_j \int_{\Omega} \xi (u - w) \, dx - \int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} g(\theta_w(x,y)) \{ \theta_{u_j}(x,y) - \theta_w(x,y) \} \frac{dx \, dy}{|x-y|^N}.
\]
where, for any \( v \in W^{s,G}_0(\Omega) \) we have denoted by simplicity \( \theta_v(x,y) = \frac{v(x) - v(y)}{|x-y|^s} \). So, taking limit as \( j \to \infty \) in the inequality above we get
\[
0 \leq \lambda \int_{\Omega} \xi (u - w) \, dx - \int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \eta(\theta_u(x,y) - \theta_w(x,y)) \frac{dx \, dy}{|x-y|^N} - \int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} g(\theta_w(x,y)) \{ \theta_u(x,y) - \theta_w(x,y) \} \frac{dx \, dy}{|x-y|^N}.
\]
So, if we take \( w = u - tv \) with \( v \in W^{s,G}_0(\Omega) \) given \( t > 0 \), we obtain that
\[
0 \leq \int\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \left( \eta - g(\theta_{u-tv}(x,y)) \right) \frac{v(x) - v(y)}{|x-y|^s} \frac{dx \, dy}{|x-y|^N}.
\]
and taking $t \to 0^+$, we arrive at
\[
0 \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \eta - \frac{u(x) - u(y)}{|x-y|^s} \right) v(x) - v(y) \frac{dx dy}{|x-y|^s |x-y|^N}.
\]
From this inequality it is easy to see that (3.7) follows and the proof is concluded. \hfill \Box

**Corollary 3.7.** The number $\lambda_1$ defined in (3.5) is an eigenvalue of (3.1).

### 4. Some properties on the principal eigenvalue

Although eigenfunctions have been defined in a weak sense, they can be proved to be also viscosity solutions of $(-\Delta_g)^s u - \lambda g(u) = 0$ provided that they are continuous. This convenient observation provides for a powerful tool in order to understand their behaviour. Following the approach of [30] we establish some properties on the first eigenpair of (3.1), namely, isolation of $\lambda_1$ in the spectrum and constant sign of their eigenfunctions. Since our eigenproblem is inhomogeneous in general, simplicity of $\lambda_1$ is not expected to hold.

#### 4.1. Weak solutions vs viscosity solutions

Let us start defining weak and viscosity solutions in this settings. Given a bounded domain $\Omega \subset \mathbb{R}^N$ and $f \in L^G(\Omega)$, we consider the following Dirichlet equation
\[
\begin{cases}
(-\Delta_g)^s u = f & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]
We say that $u \in W^{s,G}_0(\Omega)$ is a weak sub-solution (super-solution) to (4.1) if
\[
((-\Delta_g)^s u, v) \leq (\geq) \int_{\Omega} f v \quad \text{for all } v \in W^{s,G}_0(\Omega).
\]
If $u$ is simultaneously a weak super and sub-solution, then we say that $u$ is a weak solution to (4.1).

We say that an upper (lower) semi-continuous function $u$ such that $u \leq 0$ ($u \geq 0$) in $\mathbb{R}^N \setminus \Omega$ is a viscosity sub-solution (super-solution) to (4.1)) if whenever $x_0 \in \Omega$ and $\varphi \in C^1(\mathbb{R}^N)$ are such that
\[
(i) \quad \varphi(x_0) = u(x_0), \quad (ii) \quad u(x) \leq (\geq) \varphi(x) \quad \text{for } x \neq x_0
\]
then $(-\Delta_g)^s \varphi(x_0) \leq (\geq) f(x_0)$.

Finally, a continuous function $u$ is a viscosity solution to (4.1) if it is a viscosity super-solution and a viscosity sub-solution.

**Remark 4.1.** Since $(-\Delta_g)^s (\varphi + C) = (-\Delta_g)^s \varphi$, the previous definitions are equivalent if the function $\varphi(x) + C$ (or $\varphi(x) - C$) touches $u$ from below (from above, respectively) at $x_0$.

Furthermore, in the previous definitions we may assume that the test function touches $u$ strictly. Indeed, for a test function $\varphi$ touching $u$ from below, consider the function $h(x) = \varphi(x) - \eta(x)$, where $\eta \in C^\infty(\mathbb{R}^N)$ satisfies $\eta(x_0) = 0$ and $\eta(x) > 0$ for $x \neq x_0$. Notice that $h$ touches $u$ strictly. Moreover, since the function $q$ is increasing it holds that $(-\Delta_g)^s h(x_0) \leq (-\Delta_g)^s \varphi(x_0)$. For further details about general theory of viscosity solutions we refer, for instance, to the classical monographs [9, 21].

As it is observed in the definitions above, the theory of viscosity solutions is based on a pointwise testing. By [13, Lemma 2.17], the fractional $g^{-}$-Laplacian is well defined pointwisely for any test function $\varphi \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and for every $x \in \mathbb{R}^N$ provided that $g^{-} > \frac{N}{s(1-s)}$. Furthermore, continuity of weak solutions can be deduced from Corollary 2.10 as long as $sg^{-} > N$. Therefore, in order to deal with viscosity solutions coming from continuous weak solutions, we will reiteratively consider the condition
\[
g^{-} > \frac{N}{s(1-s)}.
\]

**Proposition 4.2.** [13, Lemma 3.7] Assume that (4.2) holds. Let $u \in W^{s,G}_0(\Omega)$ be a weak solution of (4.1). Then $u$ is a viscosity solution of (4.1).
The following comparison principle holds for weak solution of (4.1).

**Lemma 4.3.** [13, Lemma 3.10] Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and let $u, v \in W^{s,G}(\Omega)$ be two continuous functions satisfying

i) $v \geq u$ in $\mathbb{R}^N \setminus \Omega$,

ii) for any non-negative continuous function $\varphi \in W^{s,G}_0(\Omega)$

$$
\iint_{\mathbb{R}^N \times \mathbb{R}^N} g\left(\frac{u(y) - v(x)}{|y - x|^s}\right)(\varphi(x) - \varphi(y)) \frac{dx dy}{|x - y|^{s+N}} \geq \iint_{\mathbb{R}^N \times \mathbb{R}^N} g\left(\frac{u(y) - u(x)}{|y - x|^s}\right)(\varphi(x) - \varphi(y)) \frac{dx dy}{|x - y|^{s+N}}.
$$

Then $v \geq u$ in $\mathbb{R}^N$.

### 4.2. The sign of eigenfunctions.

Observe that the functional $\mathcal{F}$ defined in (3.3) is non-decreasing when replacing $u$ by $|u|$. Therefore, a function minimizing $\lambda_1$ can be considered to be non-negative. From the following strong maximum type result is derived that continuous eigenfunctions are, in fact, strictly positive in $\Omega$.

**Proposition 4.4.** Assume (4.2) and let $u \in W^{s,G}_0(\Omega)$ be nonnegative. If $u$ is a viscosity super-solution of $(-\Delta)^s u = 0$ in $\Omega$, then either $u > 0$ in $\Omega$ or $u \equiv 0$.

**Proof.** At a point $x_0 \in \Omega$ where $u(x_0) = 0$ we have that, for any test function $\varphi$ touching $u$ from below,

$$
0 \geq -(-\Delta)^s \varphi(x_0) = \text{p.v.} \int_{\mathbb{R}^N} g\left(\frac{\varphi(y)}{|x - y|^s}\right) \frac{dy}{|x - y|^{N+s}}
$$

since $\varphi(x_0) = 0$. If $v \geq 0$ then $\varphi \equiv 0$. If $u \equiv 0$, by using the continuity of $u$, we can select a test function $\varphi$ such that $0 \leq \varphi \leq u$ which is positive in some point. Consequently $u \equiv 0$ or $u > 0$ in $\Omega$. \qed

**Corollary 4.5.** Eigenfunctions of $\lambda_1$ have constant sign in $\Omega$.

Following the ideas of [39] for $p$-Laplacian type operators (and then extended to the non-local case in [30]), the following result claims that eigenfunctions corresponding to eigenvalues different than $\lambda_1$ must change sign in $\Omega$.

**Proposition 4.6.** Assume that (4.2) holds. Then, the only eigenfunctions of (3.1) with constant sign are those corresponding to $\lambda_1$.

**Proof.** The result follows by a contradiction argument. Assume that $u \geq 0$, $v \neq 0$, is a weak solution of (3.1) with eigenvalue $\lambda > \lambda_1(\Omega)$. From Proposition 4.4 we have that $v > 0$ in $\Omega$. By the exhaustion property of the domain $\Omega$ (see [22] p.317) there exists a smooth domain $\Omega' \subset \Omega$ such that $\lambda_1' = \lambda_1(\Omega') < \lambda_1(\Omega) + \varepsilon < \lambda$ for $\varepsilon$ arbitrarily small depending on $\Omega'$. Let us denote $v'$ the eigenfunction corresponding to $\lambda_1'$. Since Corollary 2.10 holds, $v' \in C(\overline{\Omega'})$ and $v' = 0$ in $(\mathbb{R}^N \setminus \Omega') \cup \partial \Omega'$. Since $v > 0$ in $\Omega$ it follows that $\min\{v(x) : x \in \overline{\Omega'}\} > 0$. Hence, multiplying $v$ by a suitable constant if needed, we have that $v \geq v'$ in $\mathbb{R}^N$.

For any non-negative test function $\varphi \in C_0^\infty(\Omega')$ it holds that

$$
\iint_{\mathbb{R}^N \times \mathbb{R}^N} g\left(\frac{v'(x) - v'(y)}{|x - y|^s}\right) \varphi(x) - \varphi(y) \frac{dx dy}{|x - y|^{s+N}} = \lambda_1' \int_{\Omega'} g(v') \varphi dx \leq \lambda_1' \int_{\Omega} g(v) \varphi dx \leq \lambda \int_{\Omega} g(\Lambda v) \varphi dx
$$

$$
= \iint_{\mathbb{R}^N \times \mathbb{R}^N} g\left(\frac{\Lambda v(x) - \Lambda v(y)}{|x - y|^s}\right) \varphi(x) - \varphi(y) \frac{dx dy}{|x - y|^{s+N}}.
$$

Therefore, we conclude that $\lambda_1' < \lambda_1(\Omega)$, and finally, there is no eigenfunction with constant sign in $\Omega$.
where we used property (\(g_1\)) and denoted \(0 < \Lambda := \left(\frac{N}{\lambda^*}\right)^{1/g^*} < 1\). Therefore, the comparison principle stated in Lemma 4.3 yields that \(\Lambda v \geq v'\).

The same procedure can be repeated by starting with \(\Lambda v\) instead of \(v\) to obtain that \(\Lambda^2 v \geq v'\). By iterating we arrive at a contradiction since \(\Lambda^j v \geq v'\) for all \(j \in \mathbb{N}\) but \(\Lambda^j \to 0\) as \(j \to \infty\), giving that \(v' = 0\), a contradiction. \(\square\)

4.3. **Isolation of \(\lambda_1\)**. Classical arguments for proving isolation of the first eigenvalue of \(p\)-Laplacian type operators involve lower bound estimates for the Rayleigh quotient of \(\lambda_1(\Omega)\) in terms of \(|\Omega|\), see [1, 30]. A powerful tool leading to such estimates is given by the Sobolev inequalities. Although immersion for the classic Orlicz-Sobolev space are known (see for instance the works of Donaldson and Trudinger, and Adams, for instance in [27, Sections 7.2]), unfortunately, up to the date, the critical space for the fractional Orlicz-Sobolev immersion is not completely understood. To overcome that obstacle we restrict ourselves to the class of domains \(\mathcal{O}_\beta\) where the measure is comparable with its diameter, i.e., for \(\beta > 0\),

\[
\mathcal{O}_\beta = \{\Omega \subset \mathbb{R}^N : |\Omega| \geq \beta d^N\}.
\]

As a consequence of the Poincaré’s inequality, Proposition 3.4 gives a lower bound of \(\lambda_1(\Omega)\) in terms of a negative power of \(\text{diam}(\Omega)\). Hence, assuming that \(\Omega \in \mathcal{O}_\beta\),

\[
\lambda_1(\Omega) \geq C|\Omega|^{-\gamma},
\]

where \(C = C(N, \beta)\) and \(\gamma = \gamma(s, g^\pm, N) \geq 0\).

The following auxiliary lemma is useful for our next result.

**Lemma 4.7.** Let \(G\) be a Young function such that \(g = G'\) satisfying \((L)\) with \(g^- \geq 1\). Then, the function \(h(t) = \frac{1}{t}g\left(\frac{t}{c}\right)\) is increasing for any \(c > 0\).

**Proof.** Since \(h'(t) = g'\left(\frac{t}{c}\right) \frac{1}{ct} - g\left(\frac{t}{c}\right) \frac{1}{ct^2}\), \(h\) is increasing if \(\frac{g'(t/c)}{g(t/c)} \geq \frac{1}{c}\) which is true since \(g^- \geq 1\). \(\square\)

Observe that condition (4.2) on \(G\) is assumed when considering viscosity solutions, and it automatically gives \(g^- \geq 1\).

**Proposition 4.8.** Given \(\Omega \in \mathcal{O}_\beta\), let \(\lambda(\Omega)\) be an eigenvalue of \((3.1)\) with continuous sign-changing eigenfunction \(u\). Then

\[
G^+ \lambda(\Omega) > \lambda_1(\Omega^+), \quad G^+ \lambda(\Omega) > \lambda_1(\Omega^-)
\]

holds for the open sets \(\Omega^+ = \{u > 0\}\) and \(\Omega^- = \{u < 0\}\). Moreover

\[
\lambda(\Omega) \geq C(N, s, g^\pm)|\Omega^\pm|^{-\gamma}
\]

where \(\gamma\) is the constant given in (4.3).

**Proof.** As usual, we decompose \(u = u^+ - u^-\) where \(u^\pm = \max\{\pm u, 0\}\) denote the positive and negative part of \(u\), respectively. Observe that

\[
(u(x) - u(y))(u^+(x) - u^+(y)) = |u^+(x) - u^+(y)|^2 + u^+(x)u^-(y) + u^+(y)u^-(x).
\]

Hence, choosing \(u^+\) as a test in (3.2) and taking into account Remark 2.1, from the identity above we find that

\[
\lambda \int_{\Omega} g(u)u^+ \, dx = \int_{\mathbb{R}^N \times \mathbb{R}^N} g\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{u^+(x) - u^+(y)}{|x - y|^N} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^N \times \mathbb{R}^N} g\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{|u^+(x) - u^+(y)|^2}{|u(x) - u(y)|} \frac{|u(x) - u(y)|}{|x - y|^{N+s}} \, dx \, dy
\]

\[
+ 2 \int_{\mathbb{R}^N \times \mathbb{R}^N} g\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{u^+(x)u^-(y)}{|u(x) - u(y)|} \frac{u^+(x)u^-(y)}{|u(x) - u(y)|} \, dx \, dy := I + II.
\]
Now, since the following identity holds
\[ |u(x) - u(y)|^2 = |u^+(x) - u^+(y)|^2 + |u^-(x) - u^-(y)|^2 + 2u^+(x)u^-(y) + 2u^+(y)u^-(x), \]
we get that \(|u(x) - u(y)| \geq |u^+(x) - u^+(y)|\). Therefore, from Lemma 4.7 we get
\[ I \geq \int_{\mathbb{R}^N} g \left( \frac{|u^+(x) - u^+(y)|}{|x-y|^s} \right) \frac{|u^+(x) - u^+(y)|}{|x-y|^s} \, dx \, dy := I'. \]
Moreover, the identity above also gives that \(|u(x) - u(y)| \geq \sqrt{2u^+(x)u^-(y)}\). Hence
\[ II \geq \int_{\mathbb{R}^N} g \left( \frac{\sqrt{2u^+(x)u^-(y)}}{|x-y|^s} \right) \frac{u^+(x)u^-(y)}{|u(x) - u(y)|} \, dx \, dy := II' > 0, \]
since \(g\) is increasing. Consequently, we find that
\[ \lambda(\Omega) \int_{\Omega^+} g(u^+) \, dx \geq I' + II' > I'. \]
from where, since \(u^+ \in W_0^{s, G}(\Omega^+)\) the last expression yields \(\lambda(\Omega) > \lambda_1(\Omega^+)\). Finally, from (4.3) we conclude that
\[ \lambda(\Omega) > \lambda_1(\Omega^+) \geq C(N, s, \mathfrak{F}^s) |\Omega^+|^{-\gamma}. \]
The proof for \(\Omega^-\) is analogous. \(\square\)

Proposition 4.9. Given \(\Omega \in \Omega_\beta\), we have that \(\lambda_1(\Omega)\) is isolated.

Proof. Assume that eigenfunctions corresponding to \(\lambda_1\) have energy \(\mu\) for some \(\mu > 0\), and suppose that there exists a sequence \(\{\tilde{\lambda}_k\}_{k \in \mathbb{N}}\) of eigenvalues with energy \(\mu\) tending to \(\lambda_1\), with \(\tilde{\lambda}_k \neq \lambda_1\). Let \(u_k\) be an eigenfunction of \(\tilde{\lambda}_k\). Hence we have
\[ \Phi_G(u_k) = \mu, \quad \mu \tilde{\lambda}_k = \Phi_{s, G}(u_k). \]
With the same arguments that in the proof of Proposition 3.1, there exists a function \(u \in W_0^{s, G}(\Omega)\) such that \(u_k \to u\) in \(L^G(\Omega)\) and \(\Phi_G(u) = \mu\). Moreover, up to a subsequence, we can assume that \(u_k \to u\) a.e., and from Fatou’s Lemma
\[ \Phi_{s, G}(u) \leq \liminf_{k \to \infty} \Phi_{s, G}(u_k) = \liminf_{k \to \infty} \mu \tilde{\lambda}_k = \mu \lambda_1. \]
Therefore \(u\) is an eigenfunction of \(\lambda_1\). Hence, from Proposition 4.4, either \(u > 0\) or \(u < 0\) in \(\Omega\). But, if \(\tilde{\lambda}_k > \lambda_1\), then \(u_k\) must change its sign in \(\Omega\) in view of Proposition 4.6. Then \(\Omega_k^+ = \{u_k > 0\}\) and \(\Omega_k^- = \{u_k < 0\}\) are non empty sets. Indeed, from Proposition 3.4, one obtains that \(\tilde{\lambda}_k \geq C_k |\Omega_k^+|^{-\gamma}\), and then the sets
\[ \Omega_k^\pm = \limsup_{k \to \infty} \Omega_k^\pm \]
have positive measure. Passing to a subsequence if needed, we obtain that \(u > 0\) in \(\Omega^+\) and \(u < 0\) in \(\Omega^-\), which is impossible for a first eigenfunction. \(\square\)

4.4. Behaviour of \(\alpha_1\) as \(s \to 1\). As a direct implication of the \(\Gamma\)-convergence result for modulars stated in [14], the behaviour of the Poincaré constant (3.6) as \(s \to 1^+\) can be characterized. For definitions and an introduction to the \(\Gamma\)-convergence theory, see [10].

Let \(\Omega \subset \mathbb{R}^N\) be an open and bounded set and let \(G\) be a Young function. For any \(s \in (0, 1)\), we define the functionals \(\mathcal{J}_s, \mathcal{J} : L^G(\Omega) \to \mathbb{R} \cup \{+\infty\}\) by
\[ \mathcal{J}_s(u) = \begin{cases} (1-s) \Phi_{s, G}(u) & \text{if } u \in W_0^{s, G}(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \quad \mathcal{J}(u) = \begin{cases} \Phi_G(|\nabla u|) & \text{if } u \in W_0^{1, G}(\Omega), \\ +\infty & \text{otherwise}. \end{cases} \]
where
\[ \tilde{G}(t) := \lim_{s \uparrow 1} (1-s) \int_0^1 \int_{\mathbb{S}^{N-1}} G(t|z_N|r^{1-s}) \, dS_z \, \frac{dr}{r}. \]
In [14, Theorem 6.5] it is showed that \( J_s \) is lower semi-continuous in \( L^G(\Omega) \) (see [11, Remark 2.3.7]), from [10, Proposition 6.21] it follows that \( J_s \) converges to \( F \) as \( s \to 1^+ \).

Recall that the main feature of the \( \Gamma \)-convergence is that it implies the convergence of minima (see [10, Theorem 7.4]), that is,

\[
\lim_{s \to 1^+} \min_{L^G(\Omega)} J_s(u) = \min_{L^G(\Omega)} F(u).
\]

Observe that if \( u \in W^1,\tilde{G}(\Omega) \) then \( u \in L^G(\Omega) \) since \( G \) and \( \tilde{G} \) are comparable as stated in [14, Proposition 2.16]. Therefore, from the last above expression one clearly obtains the following result.

**Theorem 4.10.** Let \( \Omega \subset \mathbb{R}^N \) be bounded and open and \( G \) a Young function. To stress the dependence on \( s \), denote \( \alpha_{1,s}(\Omega) \) the number defined in (3.6). Then,

\[
\lim_{s \to 1^+} (1 - s)\alpha_{1,s}(\Omega) = \bar{\alpha}_1(\Omega),
\]

where \( \bar{\alpha}_1(\Omega) = \inf\{\bar{\alpha}_\mu : \mu \in \mathbb{R} \setminus \{0\}\} \), and

\[
\bar{\alpha}_\mu = \min_{u \in M_\mu} \frac{\Phi_G(|\nabla u|)}{\Phi_G(u)} \quad \text{with} \quad M_\mu = \{u \in W^1,\tilde{G}(\Omega) : \Phi_G(u) = \mu\}
\]

is the inverse of the best constant in the modular inclusion of \( M_\mu \) into \( L^\tilde{G}(\Omega) \).

5. **Hardy type inequalities**

This section is devoted to prove some modular and norm Hardy inequalities for fractional Orlicz-Sobolev spaces.

The classical Hardy inequality comes back to the early 1920’s, and it is named after the English mathematician Godfrey H. Hardy, who in [18, 19] proved that

\[
\int_0^\infty \frac{|u(x)|^p}{x^{p+\alpha}} \, dx \leq \left( \frac{p^p}{1 - p - \alpha \alpha} \right) \int_0^\infty \frac{|u'(x)|^p}{x^{\alpha}} \, dx
\]

for \( 0 < \alpha p - 1 \), where \( u \) is an absolutely continuous function in \((0, \infty)\) such that \( \int_0^\infty \frac{|u'(x)|}{x^\alpha} \, dx < \infty \) and \( \lim_{x \to 0} u(x) = 0 \) for \( \alpha > 1 - p \) and \( \lim_{x \to \infty} u(x) = 0 \) for \( \alpha < 1 - p \).

Evidently, the above expression is equivalent to a *norm inequality*, up to a power,

\[
\left\| \frac{u(x)}{x} \right\|_{L^p(\mathbb{R}^+,x^{-\alpha})} \leq \frac{p}{1 - p - \alpha \alpha} \|u'\|_{L^p(\mathbb{R}^+,x^{-\alpha})}.
\]

However, in general Orlicz spaces will be distinguished *modular Hardy inequalities* and *norm Hardy inequalities*, being the first type stronger than the second one.

Hardy type inequalities were studied in very general settings, including extensions to general \( L^p \) spaces involving Radon measures by authors such as Muckenhoupt [35], Maz’ya and Rozin [34], Kokilashvili [24], Sawyer [42] and many others. Several extension to non-local settings were made, see for instance [5, 31, 26].

The natural generalization to one-dimensional Orlicz spaces was tackled by several authors, contributing to a complete characterization of admissible weights \( w, r, \rho, v \) and non-decreasing functions \( P, Q \) for which the following expression holds

\[
Q^{-1} \left( \int_0^\infty Q(w(x)|Tu(x)|r(x)) \, dx \right) \leq P^{-1} \left( \int_0^\infty P(c_H \rho(x)|u(x)|v(x)) \, dx \right),
\]

with \( Tu(x) = \int_0^x K(x,y)u(y) \, dy \) being the generalized Hardy operator with Kernel \( K \), and \( c_H \) a positive constant independent of \( u \). Furthermore, in some extent, these kind of results can be translated to the
higher dimensional case. To cite just a few references, we remit to the papers of Bloom and Kerman [3, 4], Lai [28], Heining and Maligrada [20], Cianchi [8] and references therein.

Coming up next in this section we provide for some modular and norm Hardy inequalities in fractional Orlicz-Sobolev spaces.

5.1. One-dimensional Hardy inequalities. As we will note in a moment, a key ingredient in our proof is the following local modular Hardy inequality.

Lemma 5.1. Let $G$ be a Young function. Consider $\Omega \subset \mathbb{R}$ and $u \in L^G(\Omega)$ such that $u = 0$ in $\mathbb{R} \setminus \Omega$ and $x^{-\alpha}u' \in L^G(\Omega)$. Then, for $\alpha > -\frac{\varepsilon}{1+\gamma}$ there exists a constant $c_H > 0$ depending only on $\alpha$ and $G$ such that
\begin{equation}
\tag{5.1}
\int_0^\infty G\left(\frac{u(x)}{x^{1+\alpha}}\right) \, dx \leq c_H \int_0^\infty G\left(\frac{u'(x)}{x^{\alpha}}\right) \, dx
\end{equation}
or equivalently,
\begin{equation}
\tag{5.2}
\int_0^\infty G\left(\frac{1}{x^{\alpha+1}} \int_0^x \frac{u(t)}{t} \, dt\right) \, dx \leq c_H \int_0^\infty G\left(\frac{u(x)}{x^{\alpha+1}}\right) \, dx.
\end{equation}

Proof. First, observe that property (L) implies that $G$ satisfies the inequality
\[ G^- \leq \frac{tG^+(t)}{G(t)} \leq G^+ \quad \forall t > 0. \]
Moreover, for any $\varepsilon > 0$, the function $G_{\varepsilon} := \varepsilon G$ fulfills the same bounds, i.e., $G_{\varepsilon}^\pm = G^\pm$. From property $(G_1^*)$, it holds that
\[ G_{\varepsilon}^*(rt) \leq rG_{\varepsilon}^*(t), \quad t \geq 0, r \geq 1 \]
where $G_{\varepsilon}^\pm := \frac{G^\pm}{G_{\varepsilon}}$. In fact, we have that $G_{\varepsilon}^\pm = \frac{1+\varepsilon}{1+\frac{1}{\varepsilon}} > 1$. Hence, applying [40] (or [32, Corollary 4]) we get that for any $\alpha > -1/G_{\varepsilon}^+$ it holds
\begin{equation}
\tag{5.3}
\left\| \frac{1}{x^{1+\alpha}} \int_0^x \frac{u(t)}{t} \, dt \right\|_{G_{\varepsilon}} \leq \frac{G_{\varepsilon}^\pm}{1+\alpha G_{\varepsilon}^\pm} \left\| \frac{u(x)}{x^{\alpha+1}} \right\|_{G_{\varepsilon}} = \hat{c} \left\| \frac{u(x)}{x^{\alpha+1}} \right\|_{G_{\varepsilon}}
\end{equation}
where $\hat{c} = \frac{G_{\varepsilon}^-}{(1+\alpha)G_{\varepsilon}^+ - 1}$ is a positive constant independent on $\varepsilon$.

In [2], Bloom and Kerman show that integral inequalities are equivalent to the uniform boundedness of a family of norm inequalities. Indeed, taking $Tf(x) = x^{-(\alpha+1)} \int_0^x f(t)t^\alpha \, dt$ and $f(x) = \frac{u(x)}{x^{\alpha+1}}$ in [2, Proposition 2.5], we find that the validity of (5.3) for all $\varepsilon > 0$ with a constant $\hat{c}$ independent on $\varepsilon$ implies that
\[ \int_0^\infty G\left(\frac{1}{x^{\alpha+1}} \int_0^x \frac{u(t)}{t} \, dt\right) \, dx \leq \int_0^\infty G\left(\frac{\hat{c} \cdot u(x)}{x^{\alpha+1}}\right) \, dx. \]
Finally, by using condition $(G_1)$ we obtain that the lemma follows with constant $c_H = \gamma \hat{c}^+$, where $\gamma = \gamma(G^\pm)$.

Remark 5.2. We will be interested in the case in which $\alpha = s - 1$. If we assume the condition $sg^- > 1$ on $G$, it is automatically fulfilled the condition $\alpha > -\frac{s}{1+\gamma}$ in the previous lemma.

In the spirit of [26], our first result states a one-dimensional modular Hardy inequality for continuous functions in the fractional Orlicz-Sobolev space $W_0^{s,G}(\Omega)$, with $\Omega = (0, \ell) \subset \mathbb{R}$, which, in view of Corollary 2.10, means when assuming $sg^- > 1$.

Theorem 5.3. Let $s \in (0, 1)$ and $G$ be a Young function such that $sg^- > 1$. Let $\Omega = (0, \ell) \subset \mathbb{R}$ be an interval. Then for all $u \in W_0^{s,G}(\Omega)$ it holds that
\[ \int_0^\ell G\left(\frac{u(x)}{x^s}\right) \, dx \leq C(1 + c_H) \Phi_{s,G}(u), \]
where $c_H$ is the constant in (5.1) and $C$ the doubling constant for $G$.

Proof. Given $u \in W_0^{s,G}(\Omega)$ consider the function

$$v(x) = u(x) - \frac{1}{x} \int_0^x u(t) \, dt.$$  

For $0 < a < b < \infty$, by using integration by parts, it is straightforward to see that

$$\int_a^b \frac{v(x)}{x} \, dx = \int_a^b \frac{u(x)}{x} \, dx - \int_a^b \frac{1}{x^2} \int_0^x u(s) \, ds \, dx$$  

$$= \int_a^b \frac{u(x)}{x} \, dx - \frac{1}{x} \int_0^x u(s) \, ds \bigg|_a^b - \int_a^b \frac{u(x)}{x} \, dx$$  

$$= \frac{1}{b} \int_0^b u(x) \, dx - \frac{1}{a} \int_0^a u(x) \, dx.$$  

Since $u$ is a continuous, by the Lebesgue differentiation theorem we have that

$$\lim_{a \to 0} \frac{1}{a} \int_0^a u(x) \, dx = 0.$$  

Hence, taking $b = t$ and letting $a \to 0$ in (5.4), we get

$$\int_0^t \frac{v(x)}{x} \, dx = \frac{1}{t} \int_0^t u(x) \, dx,$$  

from where $u$ can be written in terms of $v$ as

$$u(x) = v(x) + \int_0^x \frac{v(t)}{t} \, dt.$$  

From the last expression, by using $(G_2)$ and the modular Hardy inequality stated in (5.2) (which can be applied in view of Remark 5.2) we find that

$$\int_0^\infty G\left(\frac{u(x)}{x^s}\right) \, dx \leq C \int_0^\infty G\left(\frac{v(x)}{x^s}\right) \, dx + C \int_0^\infty G\left(\frac{1}{x^s} \int_0^x \frac{v(t)}{t} \, dt\right) \, dx$$  

$$\leq C(1 + c_H) \int_0^\infty G\left(\frac{v(x)}{x^s}\right) \, dx$$  

$$= C(1 + c_H) \int_0^\infty G\left(\frac{u(x) - \frac{1}{x} \int_0^x u(t) \, dt}{x^s}\right) \, dx.$$  

Now, by using Jensen’s inequality we get

$$\int_0^\infty G\left(\frac{u(x) - \frac{1}{x} \int_0^x u(t) \, dt}{x^s}\right) \, dx \leq \int_0^\infty G\left(\frac{1}{x} \int_0^x \frac{|u(x) - u(t)|}{x^s} \, dt\right) \, dx$$  

$$\leq \int_0^\infty \frac{1}{x} \int_0^x G\left(\frac{|u(x) - u(t)|}{x^s}\right) \, dt \, dx.$$  

Observe that the previous expression can be rewritten and bounded as

$$\int_0^\infty \int_0^x G\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \, dy \, dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \, dy \, dx$$  

and hence

$$\int_0^\infty G\left(\frac{u(x) - \frac{1}{x} \int_0^x u(y) \, dy}{x^s}\right) \, dx \leq \Phi_{s,C}(u),$$  

from where the proof follows. 

From the modular Hardy inequality it is straightforward to get a norm inequality as shows the following corollary.
Corollary 5.4. Under the same assumptions that in Theorem 5.3, 
\[ \| \frac{u}{x^s} \|_G \leq C^\gamma (1 + c_H)^\gamma [u]_{s,G}, \]
where \( \gamma = \gamma (G^\pm) \).

Proof. First, observe that using \((G_1)\), Theorem 5.3 can be written as 
\[ \Phi_G \left( \frac{u}{[u]_{s,G} C x^s} \right) \leq \Phi_G \left( \frac{u}{[u]_{s,G}} \right) \leq 1. \]
Hence, the definition of the Luxemburg norm yields 
\[ \| \frac{u}{x^s} \|_G = \inf \left\{ \lambda : \Phi_G \left( \frac{u}{\lambda x^s} \right) \leq 1 \right\} \leq \hat{C}[u]_{s,G} \]
as desired. \( \square \)

This same technique does not allow to obtain a modular Hardy inequality without assuming continuity on \( u \). However, thanks to the estimates from \([32, 40] \) on the local constant \( c_H \), it is possible to improve the range of \( g^- \) in the norm Hardy inequality.

Theorem 5.5. Given \( s \in (0,1) \), assume that 
\[ \frac{1-s}{s} < g^-. \]
Let \( \Omega = (0, \ell) \) be a subset of \( \mathbb{R} \). Then, for all \( u \in W^{s,G}_0(\Omega) \) there exists \( \hat{C} = \frac{s(1+g^-) - g^-}{s(1+g^-) - 1} \) such that 
\[ \| \frac{u}{x^s} \|_G \leq \hat{C}[u]_{s,G}. \]

Proof. Observe that inequality \((5.5)\) applied to \( u/[u]_{s,G} \) gives that 
\[ \int_0^\infty G \left( \frac{|u(x) - \frac{1}{x} \int_0^x u(y) dy|}{x^s[u]_{s,G}} \right) dx \leq \Phi_G \left( \frac{u}{[u]_{s,G}} \right) \leq 1, \]
from where, from the Luxemburg’s norm definition, we get 
\[ \left\| \frac{u - \frac{1}{x} \int_0^x u(y) dy}{x^s} \right\|_G \leq [u]_{s,G}. \]
Recall that \( G^* \) satisfies the \( \Delta_2 \) condition. Indeed, from property \((G_1^*)\), it holds 
\[ G^*(st) \leq sG^*(t), \quad t \geq 0, s \geq 1 \]
where \( g^+_s := \frac{G^-}{s} \). Hence, due to \((5.7)\), we can invoke \([32, \text{Corollary 4}] \) to get that 
\[ \left\| \frac{1}{x^{1+s}} \int_0^x u(y) dy \right\|_G \leq \frac{g^+_s}{1 + sg^+_s} \left\| \frac{u}{x^s} \right\|_G. \]
By using the triangular inequality for the Luxemburg norm together with the last two above inequalities, we find that 
\[ \left\| \frac{u}{x^s} \right\|_G \leq \left\| \frac{u - \frac{1}{x} \int_0^x u(y) dy}{x^s} \right\|_G + \left\| \frac{1}{x^{1+s}} \int_0^x u(y) dy \right\|_G \leq [u]_{s,G} + \frac{g^+_s}{1 + sg^+_s} \left\| \frac{u}{x^s} \right\|_G. \]
In view of \((5.6)\), the condition \( g^+_s/(1 + sg^+_s) < 1 \) holds, and the desired inequality follows. \( \square \)
5.2. A Hardy inequality in $\mathbb{R}^N$. In this section we prove a Hardy type inequality for fractional Orlicz-Sobolev functions in $\mathbb{R}^N$ whose Young functions $G$ satisfies the following growth behavior

$$
\int_0^x \frac{g(t)}{t^{1+s}} \, dt \leq c_g \frac{g(x)}{x^s}, \quad 0 < s < 1, \quad x > 0,
$$

for a suitable positive constant $c_g$ independent on $u$, where $g = G'$.

Example 5.6. In some important cases the constant $c_g$ can be explicitly computed to be $c_g = \frac{1}{p-s}$.

Indeed, when $g(t) = t^p$, $p > 1$, we get

$$
\int_0^x \frac{g(t)}{t^{1+s}} \, dt = \int_0^x t^{p-1-s} \, dt = \frac{x^{p-s}}{p-s} = c_g \frac{g(x)}{x^s}.
$$

When $g(t) = t^p \log(t)$, $p > 1$, one obtains that

$$
\int_0^x \frac{g(t)}{t^{1+s}} \, dt = \frac{x^{p-s}(\log(s) - 1)}{p-s} \leq c_g \frac{g(x)}{x^s}.
$$

More generally, when $g(t) = t^p \log(a + t)$, $p > 1$, $a > 0$ we get

$$
\int_0^x \frac{g(t)}{t^{1+s}} \, dt = \frac{x^{p-s} \log(1+x)}{p-s} + (2F_1(p-s,1,p-s+1,-\frac{x}{a}) - 1) \frac{x^{p-s}}{(p-s)^2} \leq c_g \frac{g(x)}{x^s}
$$

since this hyper-geometric function satisfies that $0 < 2F_1(p-s,1,p-s+1,-\frac{x}{a}) < 1$.

Observe that condition (5.8) is related with the validity of $\frac{g(t)}{t^s} \leq \frac{d}{dt} \left( c_g \frac{g(t)}{t^s} \right)$, which basically prescribes a lower bound for the value of $g^-$ in (L).

The key point in our proof is to replace the average function used in the one-dimensional case with the maximal of the function. Recall that given a function $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$
\mathcal{M}u(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |u(y)| \, dy
$$

where the supremum is taken over all open cubes $Q \subset \mathbb{R}^N$ with $x \in Q$ and $|Q|$ denotes the Lebesgue measure of $Q$. A cube will mean a compact cubic interval with nonempty interior.

Observe that the maximal operator is well-defined for functions in $L^G(\mathbb{R}^N)$ since this space in contained in $L^1_{\text{loc}}(\mathbb{R}^N)$.

First, observe that Young functions have the following representation.

**Lemma 5.7.** Let $G$ be a Young function such that $G' = g$. If $u \in L^G(\mathbb{R}^N)$ then

$$
\Phi_G(u) = \int_0^\infty |\{ u > t \}| g(t) \, dt.
$$

**Proof.** It holds from the fact that $G$ is a strictly increasing function and $G(t) \to \infty$ as $t \to \infty$. \qed

The following auxiliary result was showed by Torchinsky in [46, p.92].

**Lemma 5.8.** Let $u \in L^1(\mathbb{R}^N)$. Then

$$
|\{ \mathcal{M}u > t \}| \leq \frac{c_N}{t} \int_{t/2}^\infty |\{ |u| > x \}| \, dx \quad \text{for all } t > 0,
$$

where $c_N$ is a constant depending only on $N$.

Following the ideas of [23] we obtain the next inequality.
Lemma 5.9. Consider \( s \in (0,1) \) and let \( u \in L^G(\mathbb{R}^N) \) such that \( \Phi_G\left(\frac{u(x)}{|x|^s}\right) < \infty \). Then

\[
\Phi_G\left(\frac{Mu(x)}{|x|^s}\right) \leq Cc_N\Phi_G\left(\frac{u(x)}{|x|^s}\right)
\]

where \( C \) is the doubling constant, \( c_\theta \) is given in (5.8) and \( c_N \) in Lemma 5.8.

Proof. From Lemma 5.7 we can write

\[
I := \int_{\mathbb{R}^N} G\left(\frac{Mu(x)}{|x|^s}\right) \, dx = \int_0^\infty \left| \left\{ \frac{Mu(t)}{t^s} > t \right\} \right| g(t) \, dt = \int_0^\infty \left| \{Mu > t^{1+s}\} \right| g(t) \, dt.
\]

By using Lemma 5.8, \( I \) can be bounded as

\[
I \leq c_N \int_0^\infty \frac{g(t)}{t^{1+s}} \int_{t^{1+s}/2}^{\infty} |\{|u| > x\}| \, dx \, dt.
\]

Moreover, interchanging the integration order we get

\[
I \leq c_N \int_0^\infty |\{|u| > x\}| \left( \int_0^{(2x)^{\frac{1}{1+s}}} g(t) \frac{dt}{t^{1+s}} \right) \, dx.
\]

Now, condition (5.8) leads to

\[
I \leq c_\theta c_N \int_0^\infty \left| \{|u| > x\} \right| \frac{g(t)}{t^{1+s}} \, dt.
\]

Denoting \((2x)^{\frac{1}{1+s}} = t\) we have that \( x = \frac{t^{1+s}}{2} \) and \( dx = \frac{1+s}{2} t^s \, dt \), from where the last expression can be written as

\[
\frac{1}{2} (1+s) c_\theta c_N \int_0^\infty \left| \left\{ \frac{2|u|}{t^s} > t \right\} \right| g(t) \, dt,
\]

and using Lemma 5.7 this expression is, in turn, equal to

\[
\frac{1}{2} (1+s) c_\theta c_N \int_{\mathbb{R}^N} G\left(\frac{2u(x)}{|x|^s}\right) \, dx.
\]

Finally, by using the \( \Delta_2 \) condition and the fact that \( 0 < s < 1 \) we arrive at

\[
I \leq Cc_\theta c_N \int_{\mathbb{R}^N} G\left(\frac{u(x)}{|x|^s}\right)
\]

as desired. \( \square \)

Theorem 5.10. Let \( \Omega \subset \mathbb{R}^N \) be an open and bounded set. Assume that (5.8) holds for a constant \( c_\theta \) sufficiently small, let us say

\[
(5.9) \quad c_\theta \leq \frac{\omega^\gamma_N}{Cc_N(1+\delta)G^+ 2^N \gamma}
\]

where \( C \) is the doubling constant, \( c_N \) is given in Lemma 5.8, \( \delta \) is a positive number, \( \omega_N \) is the volume of the unit ball in \( \mathbb{R}^N \) and \( \gamma = G^+ \) when \( 2^N/\omega_N \leq 1 \), otherwise \( \gamma = G^- \). Then the following inequality holds:

\[
\Phi_G\left(\frac{u(x)}{|x|^s}\right) \leq C_H \Phi_{s,G}(u)
\]

for every \( u \in W^s_0, G(\Omega) \), where \( C_H \) is independent on \( u \) and it is given by

\[
C_H = \frac{2^s G^+ C_\delta d^s \omega^\gamma_N}{\omega_N - (1+\delta) G^+ 2^N \gamma Cc_N}
\]

where \( C_\delta \) is given in Lemma 2.4.

Remark 5.11. When considering the functions in Example 5.6, condition (5.9) means to take \( p \) big enough, for instance, \( p > s + \frac{2N}{\omega_N} 2^s G^+ + 2^N \gamma \).

\[5.8\]
Proof. Given $x \in \mathbb{R}^N$ consider the set $B_x = \{ y \in \mathbb{R}^N : |y| \leq |x| \}$ and observe that $|B_x| = \omega_N |x|^N$, where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$. If $y \in B_x$ then $|x - y| \leq 2|x|$ that is, $\frac{1}{2|x-y|} \geq \frac{1}{2|x|}$, hence

$$\Phi_{s,G}(u) \geq \int_{\mathbb{R}^N} \int_{B_x} G \left( \frac{u(x) - u(y)}{2|x|^s} \right) \frac{dy}{|x|^{N+s}} \geq \frac{1}{2^{s+G^+}} \int_{\mathbb{R}^N} \int_{B_x} G \left( \frac{u(x) - u(y)}{|x|^s} \right) \frac{dy}{|x|^{N+s}} dx,$$

where we have used (G1). By using Jensen’s inequality the last expression can be bounded by below as

$$\frac{1}{2^{s+G^+}} \int_{\mathbb{R}^N} G \left( \frac{1}{|B_x|} \int_{B_x} u(x) - u(y) \frac{dy}{|x|^s} \right) \frac{|B_x|}{|x|^{N+s}} dx,$$

and observe that this expression can be written as

$$\frac{\omega_N}{2^{s+G^+}} \int_{\mathbb{R}^N} G \left( \frac{1}{|B_x|} \int_{B_x} u(y) \frac{dy}{|x|^s} \right) \frac{dx}{|x|^s},$$

Now, by using Lemma 2.4 and the fact that $\Omega$ is bounded we get

$$\int_{\Omega} G \left( \frac{u(x)}{|x|^s} \right) dx \leq C_\delta d^s \int_{\Omega} G \left( \frac{u(x)}{|x|^s} \right) \frac{dx}{|x|^s} + (1 + \delta)^{G^+} \int_{\Omega} G \left( \frac{1}{|B_x|} \int_{B_x} u(y) \frac{dy}{|x|^s} \right) dx,$$

where $\delta > 0$ is an arbitrary number. Consider $Q_x := [-|x|, |x|]^N$, the $N$-cube of side $|x|$. Observe that $B_x \subset Q_x$, and moreover, their measures are comparable. Indeed, $|B_x| = \omega_N |x|^N$ and $|Q_x| = 2^N |x|^N$. Consequently,

$$\frac{1}{|B_x|} \int_{B_x} |u(y)| dx \leq \frac{2^N}{\omega_N} \frac{1}{|Q_x|} \int_{Q_x} |u(y)| dx \leq \frac{2^N}{\omega_N} \mathcal{M} u(x)$$

and then

$$\int_{\Omega} G \left( \frac{u(x)}{|x|^s} \right) dx \leq \frac{2^{G^+}}{\omega_N} C_\delta d^s \Phi_{s,G}(u) + (1 + \delta)^{G^+} \left( \frac{2^N}{\omega_N} \mathcal{M} u(x) \right) + \frac{1}{|B_x|} \int_{B_x} u(y) \frac{dy}{|x|^s} \int_{Q_x} u(y) \frac{dy}{|x|^s} \leq \frac{2^N}{\omega_N} \mathcal{M} u(x)$$

where $\gamma = \gamma(G^\pm)$. Moreover, by using Lemma 5.9,

$$\Phi_{s,G} \left( \frac{u(x)}{|x|^s} \right) \leq \frac{2^{G^+}}{\omega_N} C_\delta d^s \Phi_{s,G}(u) + (1 + \delta)^{G^+} C c_G c_N \left( \frac{2^N}{\omega_N} \right) \Phi_{s,G} \left( \frac{u(x)}{|x|^s} \right),$$

and in view of (5.9) the result follows.

As was before showed, the modular inequality implies a norm inequality.

**Corollary 5.12.** Under the same assumptions that in Theorem 5.10,

$$\left\| \frac{u}{|x|^s} \right\|_G \leq C_\delta^\gamma |u|_{s,G},$$

where $\delta = \gamma(G^\pm)$.

### 5.3. Final remarks

Given an open and bounded set $\Omega \subset \mathbb{R}^N$, a Young function $G$ and $s \in (0,1)$, consider the weighted eigenvalue problem

$$(-\Delta_g)^s u = \frac{\lambda}{|x|^s} g \left( \frac{u}{|x|^s} \right) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.$$

As a direct consequence of Theorems 5.3 and 5.10, we have that, under their respective hypothesis, the inverse of the Hardy constants $C(1 + c_H)$ (when $N = 1$) and $C_H$ (for $N \geq 1$) are lower bounds of the first eigenvalue of the Euler-Lagrange equation (5.10).

Observe that, under their respective hypothesis on $G$, Theorems 2.11, 5.3 and 5.10 give a lower bounds for the first eigenvalue $\lambda_1$ of (1.1) in terms of the diameter of $\Omega$, namely

$$\lambda_1(\Omega) \geq C \min\{d(\Omega)^{-sG^-}, d(\Omega)^{-sG^+}\}.$$
for a suitable constant $C$ depending on $s$, $N$ and $G^\pm$. In particular, when $G$ satisfies the $\Delta'_2$ condition (or sub-multiplicative condition) (see [25, Chapter I.5] for further details and examples), i.e., there exists $K > 0$ such that

$$G(ab) \leq KG(a)G(b) \quad \text{for all } a, b \in \mathbb{R}^+,$$

a more accurate lower bound of $\lambda_1$ can be obtained, namely,

$$\lambda_1(\Omega) \geq \frac{C}{G(d(\Omega)^s)}$$

where $C$ depends on $s$, $N$, $K$ and $G^\pm$.

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