Hecke category actions via Smith–Treumann theory

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Abstract
Let $G$ be a simply connected semisimple algebraic group over a field of characteristic greater than the Coxeter number. We construct a monoidal action of the diagrammatic Hecke category on the principal block $\text{Rep}_0(G)$ of $\text{Rep}(G)$ by wall-crossing functors. This action was conjectured to exist by Riche and Williamson. Our method uses constructible sheaves and relies on Smith–Treumann theory.

1. Introduction
A motivating problem in the study of reductive algebraic groups over a field $k$ of positive characteristic $\ell > 0$ is the determination of characters for important classes of modules. First conjectured by Lusztig, there exists a character formula for irreducible representations in terms of Kazhdan–Lusztig polynomials. This formula is known to be true for almost all $\ell$, but was shown in [Wil17] not to hold under the original hypothesis $\ell > h$.

In response to this and other questions, Riche and Williamson [RW18] conjectured new formulas for simple and indecomposable tilting modules, applying to any $\ell > h$ and (after variation) perhaps all $\ell$. These formulas replace Kazhdan–Lusztig polynomials with $\ell$-Kazhdan–Lusztig polynomials, which are suggested to be better suited to modular representation theory. The new conjectures are derived in [RW18] as a consequence of a more categorical proposition: the existence of an action of the diagrammatic Hecke category $\mathcal{H}$ (defined in [EW16]) on the principal block $\text{Rep}_0(G)$ by wall-crossing functors, categorifying the action of the affine Weyl group on its antispherical module. Using methods from the theory of 2-Kac–Moody actions, Riche and Williamson proved their categorical conjecture for $\text{GL}_n$, but the general statement has remained open until recently. After the tilting and irreducible character formulas were established by other means [AMRW19] for $\ell > 2h - 2$, the following year saw two major developments.

(i) The expansion of Smith–Treumann theory (as initiated by Treumann [Tre19] and Leslie and Lonergan [LL21]) by Riche and Williamson [RW22], yielding a geometric proof of the linkage principle and establishing the tilting character formulas in all characteristics.

(ii) The resolution of the categorical conjecture in full generality by Bezrukavnikov and Riche [BR22]. Their approach is essentially coherent, making use of localisation theorems in positive characteristic and a new bimodule-theoretic realisation of $\mathcal{H}$ found by Abe [Abe21].

Our objective is to provide an alternative proof of Riche and Williamson’s conjecture, using the machinery of constructible sheaves and Smith–Treumann theory. Assume now that $G$ is a semisimple algebraic group of adjoint type over an algebraically closed field $F$ of characteristic greater than the Coxeter number.

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The starting point of our approach is a realisation of the Hecke category via parity complexes on the neutral component $\mathcal{F}^0$ of the affine flag variety $\mathcal{F}$ of $G$, first proved in [RW18]. Here we need a mild modification to incorporate loop rotation $G_m$-equivariance. This realisation induces a graded right $\mathcal{H}$-module equivalence between the antispherical quotient of $\mathcal{H}$ and $\text{Parity}_{IW,G_m}(\mathcal{F}^0,k)$, the category of Iwahori–Whittaker parity complexes on $\mathcal{F}^0$. Through an understanding of the morphism spaces between parity objects, as provided by [RW22, §7], we show that the graded action of $\mathcal{H}$ descends further to the Smith quotient $\text{Sm}_{\text{par}}^0(\mathcal{F}^0,k)$ of $\text{Parity}_{IW}(\mathcal{F}^0,k)$. Up to graded shift, the indecomposable object $B_s \in \mathcal{H}$ acts on $\text{Sm}_{\text{par}}^0(\mathcal{F}^0,k)$ by the functor induced by the composite $(q_s^*)^*(q_s^*)$, where $s$ is an affine simple reflection and $q_s : \mathcal{F} \to \mathcal{F}^s$ is the natural morphism between certain partial affine flag varieties.

We now use three ingredients (already available in the literature) to transfer the Hecke category action to $\text{Rep}_0(G)$. Let $\mathcal{G}$ denote the affine Grassmannian of $G$, and let $\mathcal{G}^{\nu}$ denote the fixed points of $\mathcal{G}$ under loop rotation by the $\ell$th roots of unity $\omega \leq G_m$. The first ingredient is the miraculous decomposition $\mathcal{G}^{\nu} = \bigsqcup_{\nu} \mathcal{G}^{\nu}(\nu)$, where the $\mathcal{G}^{\nu}(\nu)$ are the neutral components of (thin) partial affine flag varieties. The second ingredient is the main theorem of [RW22], which shows that Smith restriction from Iwahori–Whittaker perverse sheaves on $\mathcal{G}$ to the Smith category of $\mathcal{G}^{\nu}$ is fully faithful on the tilting subcategory, with the following well-understood essential image.

The categories in the lower row admit a decomposition into ‘blocks’, preserved by the equivalence and ultimately tied to the linkage principle for $G$; our choices determine principal blocks $\text{Tilt}_{IW}^0$ and $\text{Sm}_{IW}^0$ on the left- and right-hand sides, respectively.

Our final ingredient is a version of the geometric Satake equivalence due to [BGM+19]:

$$\text{Rep}(G) \xrightarrow{\text{eq}} \text{Perv}_{IW}(\mathcal{G},k).$$

Through the identification of $\mathcal{G}^0$ with a component of $\mathcal{G}^{\nu}$ in (1.2), $\text{Sm}_{IW}^0$ inherits a right $\mathcal{H}$-module action from $\text{Sm}_{IW}^0(\mathcal{G}^0,k)$. Using (1.3) and (1.4), we are able to transfer the action first to $\text{Tilt}_{IW}^0$ and hence to $\text{Tilt}(\text{Rep}_0(G))$, the tilting subcategory of the principal block of $G$. As explained in [RW18, Remark 5.1.2(1)], we may now deduce the existence of a right $\mathcal{H}$-action on $\text{Rep}_0(G)$, and it remains only to verify that the push–pull action of $B_s$ maps across to the wall-crossing functor $\theta_s$.

We in fact show something more precise: functors of pushing and pulling in Smith theory correspond to translation functors onto and off walls in representation theory. In the case of pushforward, suppose $\gamma$ is the dominant coweight labelling an indecomposable tilting module
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$T(\gamma)$ which affords the translation functor $T^s$ for $G$. On categories of tilting objects, the geometric Satake equivalence sends $T^s$ to the composite of convolution with a tilting sheaf $\mathcal{F}(\gamma)$ and a projection. Our first observation is that Smith localisation erases the difference between convolution with $\mathcal{F}(\gamma)$ and the functor $(\phi_2)_*(\phi_1)^*$, where the $\phi_i$ are the projections associated with a certain correspondence $\mathcal{Y} \subseteq \mathcal{G}_r \times \mathcal{G}_r$. Our second observation is that since ‘Smith localisation commutes with everything’ (in the words of Treumann), the following commutative diagram of geometric morphisms corresponds to a diagram of functors commuting up to natural isomorphism in Smith theory.

Here $\mathcal{Z}$ is just the graph of $q^*$, so $(q^*)_* \cong z^*_n z^*$ and we are done; the case of pullback can be approached similarly or by citing properties of adjunctions.

The structure of the paper is as follows. In §2 we fix notation and cover algebraic preliminaries. In §3, we provide context on the geometry of affine Grassmannians and flag varieties, (equivariant) derived categories, versions of the geometric Satake correspondence, and parity complexes, among other topics. Section 4 recapitulates some of the main results of the third part of [RW18] with additional $\mathbb{G}_m$-equivariance. Section 5 recalls the foundations of Smith–Treumann theory before constructing the above-described push–pull action of $\mathcal{H}$. Everything is tied together in §6, where the action is transported to $\text{Rep}_{0}(G)$ and the main results are stated (Theorems 6.6 and 6.13).

2. Algebraic preliminaries

2.1 Notation

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$, where $p$ does not divide the order of the fundamental group $\pi_1(G)$; throughout, all schemes will have ground ring $\mathbb{F}$. Put $\mathcal{O}_n = \mathbb{F}[z^n]$ and $\mathcal{K}_n = \mathbb{F}(z^n)$, with $\mathcal{O} = \mathcal{O}_1$ and $\mathcal{K} = \mathcal{K}_1$. More generally, the omission of a subscript $n$ in previously defined notation will refer to the case $n = 1$, unless otherwise noted.

Let $k$ be a finite field of characteristic $\ell \neq p$. We will write $\mathcal{O} = W(k)$ for the ring of Witt vectors over $k$, or in other words the unique unramified extension of $\mathbb{Z}_\ell$ of degree $[k : \mathbb{F}_\ell]$, and set $\mathcal{K}$ to be the quotient field of $\mathcal{O}$. We obtain an $\ell$-modular system:

\[
\begin{array}{c}
\kappa < \mathcal{O} < \mathcal{K}.
\end{array}
\] (2.1)

Define a coefficient ring to be one of those in (2.1), or a finite extension of $k$ or $\mathcal{K}$; these will provide the coefficients for étale sheaves. Geometric functors on derived categories will be assumed to be derived.

For $n \geq 1$, $\varpi_n \subseteq \mathbb{G}_m$ denotes the subgroup of $n$th roots of unity; put $\varpi = \varpi_\ell$.

2.2 Category theory

We assume knowledge of core concepts and constructions from category theory, including adjunctions and counit–unit pairs [Bor94, §3], triangulated categories [Nee14], derived categories and
Let $A$ be an abelian group. We will require the notion of an $A$-graded additive category $C$ when $A = \mathbb{Z}$ or $A = \mathbb{Z}/n\mathbb{Z}$, as well as the ‘de-grading’ $C_{\text{deg}}$ of $C$, which has the same objects as those of $C$ and $A$-graded Hom spaces

$$\text{Hom}_{C_{\text{deg}}}(X,Y) = \bigsqcup_{a \in A} \text{Hom}_{C}(X,Y(a)).$$

See [AJS94, §E.3] for a careful account of the aforementioned.

Given a field extension $k'/k$ and a $k$-linear category $C$, we write $k' \otimes_k C$ for the category $C'$ whose objects are the same as those of $C$ and whose Hom spaces are

$$\text{Hom}_{C'}(X,Y) = k' \otimes_k \text{Hom}(X,Y),$$

with the obvious composition rule. This notion of scalar extension is adequate for additive categories, but does not generally preserve the property of being abelian; see, e.g., [Sta18] for a discussion of the subtler case of abelian categories.

The formation of direct limits of systems of categories will be essential, particularly for derived categories over directed families of varieties related by pushforward functors. A framework for describing such direct limits is given in [Was04, Appendix A]; see also [Bry93, §5.2].

### 2.3 Roots and Weyl groups

Fix $T \subseteq B \subseteq G$ a maximal torus and Borel subgroup of $G$, with $U$ the unipotent radical of $B$; let $B^+$ denote the opposite Borel subgroup to $B$, with unipotent radical $U^+$. Associated to these data is a root system $(\Phi \subseteq X, \Phi^V \subseteq X^V)$; we write $\Phi_+ \subseteq \Phi$ for the positive roots opposite to $B$ and $\Sigma \subseteq \Phi_+$ for the simple roots. These give rise to a Coxeter group $(W_t, S_t)$, the finite Weyl group generated by finite simple reflections $s_\alpha$, $\alpha \in \Sigma$, which has a longest element $w_0 \in W_t$. Our assumptions on $G$ ensure the existence of an element $\rho^\vee \in X^V$ such that $\langle \alpha, \rho^\vee \rangle = 1$ for all $\alpha \in \Sigma$; indeed, $\rho^\vee$ is necessarily the half-sum of the positive coroots. We assume from now on that $\ell > h$, the Coxeter number of $\Phi$.

Considering the extended torus $\tilde{T} = T \times \mathbb{G}_m$ gives us access to affine roots of the form $\alpha + m\delta \in X^*(\tilde{T})$ for $m \in \mathbb{Z}$, where $\delta \in X^*(\tilde{T})$ is the projection onto the factor $\mathbb{G}_m$. The affine Weyl group is $(W = W_t \ltimes \mathbb{Z}\Phi^V, S)$, while the extended affine Weyl group is $W_t \ltimes X^V$; here the affine simple reflections $S$ comprise $S_t$ along with elements $t_{\beta^V}s_\beta$, where $\beta \in \Phi$ is maximal in the ordering determined by $\Phi_+$ and $t_{\beta^V}$ denotes the image of $\beta^V \in \mathbb{Z}\Phi^V$ in $W$. Let $^iW$ (respectively, $W^i$) be the set of $w \in W$ which are Bruhat-minimal in $W_tw$ (respectively, $wW_t$), and similarly define $^jW$ (respectively, $W^j$) for a subset $J \subseteq S$. We consider the standard action of $W$ on $V = X^V \otimes_{\mathbb{Z}} \mathbb{R}$, along with the centred and dilated actions

$$(wt_\nu) \cdot_n x = w(x + n\nu + \rho^\vee) - \rho^\vee, \quad (wt_\nu) \circ_n x = w(x + n\nu).$$

For future reference, recall that an expression of $w \in W$ (with respect to $S$) is a tuple $\underline{w} = (s_1, s_2, \ldots, s_m)$, $s_i \in S$, such that $w = s_1s_2\cdots s_m$. We say $\underline{w}$ is a reduced expression (or rex) if the length $m$ of $\underline{w}$ is minimal among expressions of $w$.

**Remark 2.1.** Our focus on $X^V$ arises from our intention to state representation-theoretic results for $G$, after making geometric arguments on the side of $G$.

### 2.4 Blocks, translations, and tilting objects

Write $\text{Rep}(G)$ for the abelian category of finite-dimensional algebraic representations of $G$. It is monoidal with the tensor product $\otimes$ over $\mathbb{k}$, and its simple objects $L(x)$ are parametrised...
by $x \in X^\vee_+$. The linkage principle provides a decomposition of $\text{Rep}(G)$ into a direct sum of abelian subcategories, 

$$\text{Rep}(G) = \bigoplus_{c \in X^\vee/(W, \bullet)} \text{Rep}_c(G),$$

where the block $\text{Rep}_c(G)$ is the Serre subcategory generated by the $L(x)$ with $x \in c \cap X^\vee_+$. (Note: while blocks of a category are often understood to be indecomposable, $\text{Rep}_c(G)$ need not be.) Now, let

$$C_\ell = \{ x \in X^\vee : 0 < \langle \alpha, x + \rho^\vee \rangle < \ell \text{ for all } \alpha \in \Phi_+ \},$$

$$\overline{C}_\ell = \{ x \in X^\vee : 0 \leq \langle \alpha, x + \rho^\vee \rangle \leq \ell \text{ for all } \alpha \in \Phi_+ \}.$$ 

Since $\ell > h$, there is a regular weight $\lambda_0 \in C_\ell \neq \emptyset$; we could take $\lambda_0 = 0$, but do not insist on it. For each $s \in S$, let $\mu_s$ denote a subregular weight lying on the reflection hyperplane of $s$ in $\overline{C}_\ell$, with respect to the $\bullet_\ell$-action of $W$, but no other such hyperplanes; these exist in our setting by [Jan03, §II.6.3]. We then have the principal block $\text{Rep}_0(G) = \text{Rep}_{[\lambda_0]}(G)$ and the subregular blocks $\text{Rep}_s(G) = \text{Rep}_{[\mu_s]}(G)$, where $[\lambda] \in X^\vee/(W, \bullet)$ is the image of $\lambda \in X^\vee$.

Furthermore, $\text{Rep}(G)$ has the structure of a highest weight category, descending to all of its blocks, in the sense described in [RW18] and originally in [CPS88]. In particular, there are standard and costandard objects $\Delta(x)$, $\nabla(x)$ in $\text{Rep}(G)$ (respectively, $\text{Rep}_0(G)$; respectively, $\text{Rep}_s(G)$) for $x \in X^\vee_+$ (respectively, $x \in [\lambda_0] \cap X^\vee_+$; respectively, $x \in [\mu_s] \cap X^\vee_+$), admitting morphisms $\Delta(x) \rightarrow L(x) \hookrightarrow \nabla(x)$. Objects which possess a filtration by standard objects and a filtration by costandard objects are said to be tilting. These form additive (but not abelian) subcategories

$$\text{Tilt} \subseteq \text{Rep}(G), \quad \text{Tilt}_0 \subseteq \text{Rep}_0(G), \quad \text{Tilt}_s \subseteq \text{Rep}_s(G);$$

the first of these is closed under $\otimes$. These categories are Krull–Schmidt, with the indecomposable tilting objects $T(x)$ in $\text{Tilt}$ (respectively, $\text{Tilt}_0$; respectively, $\text{Tilt}_s$) parametrised by $x \in X^\vee_+$ (respectively, $x \in [\lambda_0] \cap X^\vee_+$; respectively, $x \in [\mu_s] \cap X^\vee_+$).

An important theoretical role is played by translation functors [Jan03, §II.7] between the blocks of $\text{Rep}(G)$. Specifically, we may fix translation functors onto and off the $s$-walls, $s \in S$:

$$T^s = T^s_{\mu_s}, \quad \text{respectively } T_s = T^s_{\lambda_0};$$

these are defined in terms of a finite-dimensional module $M$ with $\dim M_\gamma = 1$ and such that $\gamma' \leq \gamma$ for all its weights $\gamma'$, where $\gamma$ is the unique dominant element of $W(\mu_s - \lambda_0)$; respectively, $W(\lambda_0 - \mu_s)$. For concreteness and convenience, we will take $M = T(\gamma)$. Our choices of $M$, $\lambda_0$, and the $\mu_s$ do not matter up to natural isomorphism, as follows from [Jan03, Remark 7.6(1)] and [Jan03, Proposition 7.9]. The functors $(T^s, T_s)$ are left and right adjoint to each other (hence, exact), preserve tilting modules, and restrict to the principal and subregular (tilting) blocks:

$$\text{Rep}_0(G) \xrightarrow{T^s_{\lambda_0}} \text{Rep}_s(G), \quad \text{Tilt}_0(G) \xrightarrow{T^s_{\lambda_0}} \text{Tilt}_s(G).$$

Translation onto the $s$-wall then off it yields the wall-crossing functor $\theta_s = T_s T^s$, a self-adjoint endofunctor of the principal block.

3. Geometric ingredients

3.1 Loop groups and the affine Grassmannian

Detailed treatments of the objects introduced in this subsection can be found in [Gör10, Kum02, Zhu17]; much of our notation follows [RW22, §4]. For $n \geq 1$, the $n$th positive loop group $L^+_n G$ of
affine root subgroups rise to and the family of affine reflections $s$ we can identify (the affine function $f$)

The zero sets of the affine functions, or equivalently the form a hyperplane arrangement giving rise to a system of facets.

For $n$, it is a subfunctor of the loop group $L_nG$, an ind-affine group ind-scheme representing the functor from $\mathbb{F}$-algebras to sets given by

$$A \mapsto G(A[\lbrack z^n \rbrack]).$$

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We suppress $n$ from notation in case $n = 1$. Root subgroups $u_\alpha : \mathbb{G}_{a} \sim U_\alpha \subseteq G$, for $\alpha \in \Phi$, give rise to affine root subgroups $U_{\alpha + m\delta} \subseteq LG$, the images of morphisms described by the formula $x \mapsto u_\alpha(xz^m)$.

Let $A$ be an $\mathbb{F}$-algebra. For any $a \in A^\times$, there is a map of $\mathbb{F}$-algebras

$$A(\lbrack z \rbrack) \rightarrow A(\lbrack z \rbrack), \quad z^m \mapsto a^m z^m.$$

This yields $G(A(\lbrack z \rbrack)) \rightarrow G(A(\lbrack z \rbrack))$ and hence a loop rotation action of $\mathbb{G}_m$ on $LG$ stabilising $L^+G$. Since

$$(\text{Spec } A(\lbrack z \rbrack))/\mathbb{G}_m \cong \text{Spec } A(\lbrack z^n \rbrack),$$

we can identify $(LG)^{\mathbb{G}_m} \cong L_nG$ and $(L^+G)^{\mathbb{G}_m} \cong L^+_nG$; see [RW22, Lemma 4.2].

The affine Grassmannian of $G$ is the ind-projective ind-scheme $G_r$ representing the fppf sheafification of the functor

$$A \mapsto (LG)(A)/(L^+G)(A).$$

For $\lambda \in X^\vee$, the image of $z$ under the mapping $G(\lbrack z \rbrack)^\times \rightarrow G(\lbrack z^n \rbrack)$ induced by $\lambda$ yields a point $z^\lambda$ in $LG$. If we denote by $L_\lambda$ the coset of $z^\lambda$ in $G_r$, then every $L^+G$-orbit on $G_r$ has the form $gr^\lambda = L^+G \cdot L_\lambda$ for $\lambda \in X^\vee$.

### 3.2 Partial affine flag varieties

Recall $V = X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ and fix $n \geq 1$ a positive integer. To each affine root $a = \alpha + m\delta$ we associate the affine function

$$f_a^n : V \rightarrow \mathbb{R}, \quad f_a^n(v) = \langle a, v \rangle + mn$$

and the family of affine reflections $s_a = t_{ma} \circ s_\alpha$, whose $\mathfrak{c}_n$-action on $V$ is given by

$$s_a \circ_n v = v - f_a^n(v)\alpha^\vee.$$  

The zero sets of the affine functions, or equivalently the $\mathfrak{c}_n$-fixed points of the affine reflections, form a hyperplane arrangement giving rise to a system of facets. In particular, we have

$$a_n = \{ \lambda \in V : -n < \langle \alpha, \lambda \rangle < 0 \text{ for all } \alpha \in \Phi_+ \},$$

an alcove whose closure is a fundamental domain for the $\mathfrak{c}_n$-action of $W$ on $V$. For every facet $f \subseteq \overline{a_n}$, Bruhat–Tits theory (as described in [RW22, §4.2]) provides a parahoric group scheme $P^f$ over the ring $\mathcal{O}_n = \mathbb{F}[z^n]$, such that $P^{a_n} \subseteq P^f$. Let $L_n^+P^f \subseteq L_nG$ be the affine subgroup scheme representing the functor

$$A \mapsto P^f(A[\lbrack z \rbrack]).$$

We then have the partial affine flag variety $G_r^{\mathbb{G}_m}$, the ind-projective ind-scheme representing the fppf sheafification of the functor

$$A \mapsto (L_nG)(A)/(L^+_nP^f)(A).$$

For $\lambda \in -\overline{a_n}$, denote by $f_\lambda \subseteq \overline{a_n}$ the facet containing $-\lambda$. Then the maps

$$L_nG \rightarrow G_r^{\mathbb{G}_m}, \quad g \mapsto g \cdot L_\lambda$$

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factor through embeddings $\mathcal{F}_{\mathfrak{m}}^{\ell} \hookrightarrow \mathcal{G}^{\omega_n}$. Their images $\mathcal{G}^{\omega_n}_{(\nu)}$ feature in the beautiful and crucially important decomposition of fixed points [RW22, Proposition 4.6]:

$$\mathcal{G}^{\omega_n} \cong \bigsqcup_{\nu \in \mathfrak{X} \setminus \mathfrak{X}^{\omega_n}} \mathcal{G}^{\omega_n}_{(\nu)}, \quad \text{where } \mathcal{G}^{\omega_n}_{(\nu)} \cong \mathcal{F}^{\ell}_{\mathfrak{m}} \hookrightarrow \mathcal{G}^{\omega_n}.$$

In the context of $\omega_n$-fixed points, omission of $n$ from notation will correspond to the case $n = \ell$, so we will write, e.g., $\mathcal{G}^{\ell}_{(\nu)} = \mathcal{G}^{\ell}_{(\nu)}$ and $j^{\ell}_{(\nu)} = j^{(\nu)}$. Our main interest will be in the following special affine flag varieties. First, note that evaluation at zero $A[z^n] \to A$ yields a morphism $\text{ev}_n : L^+_nG \to G$. The $n$th Iwahori group is $I_n = \text{ev}^{-1}_n(B)$; it coincides with the positive loop group $L^+_n P^n$. The corresponding partial affine flag variety is written $\mathcal{F}_n$ and known simply as the $n$th affine flag variety of $G$; it is isomorphic to $\mathcal{F}$ via the map induced by the change of variables $z \mapsto z^n$, and it admits a decomposition

$$\mathcal{F}_n = \bigsqcup_{x \in W} \mathcal{F}_{n,x}, \quad \text{where } \mathcal{F}_{n,x} = I_n \cdot xI_n/I_n.$$

Here by $xI_n/I_n$ we mean the coset of $z^\lambda w \in N_G(T)(\mathcal{K}_n)$, assuming $x = t\lambda w$ and $w \in N_G(T)$ is a lift of $w \in W$. We let $I_{n,u} = \text{ev}^{-1}_n(U)$ denote the pro-unipotent radical of $I_n$. Replacing $B$ (respectively, $U$) with $B^+$ (respectively, $U^+$) yields opposite groups $I^+_n = w_0 I_n w_0$, $I^+_{n,u} = w_0 I_{n,u} w_0$.

Note that the $L^+_n$-orbits on $\mathcal{G}_n$ decompose into finitely many Iwahori orbits,

$$\mathcal{G}^{\nu} = \bigsqcup_{\nu' \in W(\lambda)} \mathcal{G}^{\nu'}, \quad \text{where } \mathcal{G}^{\nu'} = I \cdot L^{\nu'} = I_u \cdot L^{\nu'}, \quad (3.1)$$

and an analogous statement holds for $I^+$ and $I^+_n$.

Second, observe that $\mu = \mu_n + \rho^+ \in -\mathfrak{a}_I \cap \mathfrak{x}^+$, so we have a parahoric group scheme $P^\mu$ and positive loop group $P^\mu_n = L^+_n P^\mu$ for the facet $\mathfrak{f}_n = \mathfrak{f}_{\mu_n + \rho^+}$; we write $\mathcal{F}^\mu_n = \mathcal{F}^\mu_n$. If $s \in W_I$ is a finite simple reflection, then $P^\mu_s$ is the inverse image $\text{ev}^{-1}_n(P_s)$ of the standard parabolic subgroup $P_s \subseteq G$ containing $B$. In any case, $I_n \subseteq \mathcal{F}^\mu_n$ and there is a natural proper morphism $q_n^\mu : \mathcal{F}_n \to \mathcal{F}^\mu_n$. This morphism is a $\mathbb{P}^1_G$-bundle and hence such that $(q_n^\mu)^{-1} \cong (q_n^\mu)^{-1}[2]$; see [PR08, 8.6.1, Proposition 8.7].

Because $L^+_nG$ and $I_n$ are stable under the action of $\mathbb{G}_m$ on $L_nG$, we obtain a $\mathbb{G}_m$-action on $\mathcal{F}_n$. It stabilises $I_n$, so the action also preserves $I_n$-orbits. In particular, if $X \subseteq \mathcal{F}_n$ is a locally closed finite union of orbits, then $X$ admits an action of $I_n \times \mathbb{G}_m$.

### 3.3 Equivariant derived categories on partial affine varieties

In the following, our schemes will be defined over $\mathbb{F}$ and we will work in the ‘étale context’ described in [RW18, §9.3(2)], referring to (possibly equivariant) derived categories of étale sheaves over a coefficient ring $\mathbb{L}$. Such equivariant derived categories were introduced in [BL06] for the topological setting; the necessary adjustments for étale sheaves are provided in [Wei13].

In particular, if $X$ is a locally closed finite union of $I$-orbits (respectively, $\mathcal{F}_n$), then one considers the action of an appropriate finite-type quotient of $I$ on $X$ in order to construct $D^b_I(X, \mathbb{L})$. Inclusions $X \hookrightarrow Y$ of locally closed subsets induce fully faithful pushforward functors $D^b_I(X, \mathbb{L}) \to D^b_I(Y, \mathbb{L})$. Thus we can take a direct limit over those $X$ which are closed to obtain a triangulated equivariant derived category $D^b_I(\mathcal{F}_n, \mathbb{L})$ (respectively, $D^b_I(\mathcal{F}_n, \mathbb{L})$). We may similarly obtain $D^b_{I \times \mathbb{G}_m}(\mathcal{F}_n, \mathbb{L})$ (respectively, $D^b_{I \times \mathbb{G}_m}(\mathcal{F}_n, \mathbb{L})$). For more on such ind-constructions, see [BR18, §A.4] or [Nad04, §2.2].
3.4 (Loop rotation equivariant) Iwahori–Whittaker derived categories

The material in this section draws from [AR18, Appendix A] and [RW22, §§5.1–5.2]. For our description of Iwahori–Whittaker categories, we specialise coefficients to $L = k$; greater generality is possible (see, e.g., [RW18, §11]) but unnecessary for us.

Let $\mathcal{X}$ denote a partial affine flag variety and assume there exists a non-trivial $p$th root of unity $\zeta \in k$. Then let

$$\tau : \mathbb{G}_a \to \mathbb{G}_a, \quad x \mapsto x^p - x,$$

be the Artin–Schreier map; this is a Galois covering with Galois group $\mathbb{F}_p$. We define the associated Artin–Schreier local system $\mathcal{L}_{AS}$ to be the summand of $\tau_*\mathcal{L}_{\mathbb{G}_a}$ on which $\mathbb{F}_p$ acts by powers of $\zeta$. Finally, let

$$\chi = \chi_0 \circ \text{ev} : I_u^+ \to U^+ \to \mathbb{G}_a,$$

where $\chi_0 : U^+ \to \mathbb{G}_a$ is a fixed morphism of algebraic groups which is non-trivial on any simple root subgroup of $U^+$. If $X \subseteq \mathcal{X}$ is a locally closed finite union of $I^+$-orbits, then $I_u^+$ acts on $X$ through some finite quotient $J$ of $I_u^+$, which can be chosen in such a way that $\chi$ factors through a morphism $\chi_J : J \to \mathbb{G}_a$. We can then consider the $(J, \chi_J^*, \mathcal{L}_{AS})$-equivariant derived category $D_{IW}^b(X, k) = D_{J, \chi}^b(X, k)$: this is the full subcategory of $D_{c}^b(X, k)$ whose objects $F$ are such that

$$a_J^* F \cong \chi_J^* \mathcal{L}_{AS} \boxtimes F,$$

where $a_J : J \times X \to X$ is the action map. As with previous constructions, this category is independent of our choice of the quotient $J$ (up to equivalence). Importantly, we have an essentially surjective averaging functor

$$D_c^b(X, k) \to D_{IW}^b(X, k), \quad F \mapsto (a_J)_!(\chi_J^* \mathcal{L}_{AS} \boxtimes F)$$

by [AR18, Lemma A.4]. Taking a direct limit over closed $X$, we obtain a triangulated Iwahori–Whittaker category $D_{IW}^b(\mathcal{X}, k)$. This category has a natural perverse $t$-structure with heart $\text{Perv}_{IW}(\mathcal{X}, k)$, and admits a fully faithful forgetful functor $D_{IW}^b(\mathcal{X}, k) \to D_c^b(\mathcal{X}, k)$.

Note that the action of $I^+$ on $\mathcal{X}$ extends to an action of $I^+ \times \mathbb{G}_m$. Then, for each $X$ as above, the finite-type quotient $J$ of $I_u^+$ can be chosen in such a way that the action of $\mathbb{G}_m$ on $I^+$ descends to an action on $J$. We can hence consider $D_{IW, \mathbb{G}_m}^b(X, k)$, the full subcategory of $D_{c, \mathbb{G}_m}^b(X, k)$ whose objects $F$ satisfy the analogue of the condition (3.2) in $D_{c, \mathbb{G}_m}^b(J \times X, k)$; here $\mathbb{G}_m$ is assumed to act on $J \times X$ diagonally.

As with the previous construction, $D_{IW, \mathbb{G}_m}^b(X, k)$ is triangulated with a natural $t$-structure, and is independent of our choices up to equivalence. Its heart will be written $\text{Perv}_{IW, \mathbb{G}_m}(X, k)$. Taking a direct limit over those $X$ which are closed, we obtain a triangulated loop rotation equivariant Iwahori–Whittaker category $D_{IW, \mathbb{G}_m}^b(\mathcal{X}, k)$, with heart $\text{Perv}_{IW, \mathbb{G}_m}(\mathcal{X}, k)$ with respect to the inherited $t$-structure. There is a functor of forgetting $\mathbb{G}_m$-equivariance,

$$D_{c, \mathbb{G}_m}^b(\mathcal{X}, k) \to D_{IW}^b(\mathcal{X}, k).$$

More generally, if $\mathcal{Y}$ is a finite product of affine flag varieties, let $\mathcal{U} \subseteq \mathcal{Y}$ denote an ind-subscheme which is the direct limit of closed finite unions of $I^+$-orbits. Then all of the foregoing constructions are valid for $\mathcal{U}$, so we have categories $D_{IW, \mathbb{G}_m}^b(\mathcal{U}, k)$ and $D_{IW}^b(\mathcal{U}, k)$, as well as a forgetful functor between them.

3.5 Parity complexes

Again let $X \subseteq \mathcal{X}$ be a locally closed finite union of $I$-orbits (respectively, $I^+$-orbits). Using forgetful functors to $D_c^b(X, \mathbb{L})$, we have a notion of parity complexes [JMW12] in $D_{c, \mathbb{G}_m}^b(X, \mathbb{L})$. 

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(respectively, $D^b_{\text{IW}, \mathbb{G}_m}(X, k)$). In the direct limit, we obtain full subcategories

$$\text{Parity}_{I \times \mathbb{G}_m}(\mathcal{X}, \mathbb{L}) \subseteq D^b_{I \times \mathbb{G}_m}(\mathcal{X}, \mathbb{L}),$$

$$\text{Parity}_{\text{IW}}(\mathcal{X}, k) \subseteq D^b_{\text{IW}}(\mathcal{X}, k), \quad \text{Parity}_{I \times \mathbb{G}_m}(\mathcal{X}, k) \subseteq D^b_{I \times \mathbb{G}_m}(\mathcal{X}, k).$$

The partial affine flag variety $\mathcal{X}$ has a Bruhat decomposition

$$\mathcal{X} = \bigsqcup_{\alpha \in A} \mathcal{X}_\alpha,$$

(3.4)

where the $\mathcal{X}_\alpha$ are $I$-orbits of dimensions $d_\alpha$. For reasons articulated in [RW22, §5.3] and building on the general theory of [JMW12], there is (up to isomorphism and shift) a unique indecomposable parity complex

$$\mathcal{E}_\mathcal{X}(\alpha) \in \text{Parity}_{I \times \mathbb{G}_m}(\mathcal{X}, \mathbb{L})$$

which is supported on $\mathcal{X}_\alpha$ and whose restriction to $\mathcal{X}_\alpha$ is $\mathbb{L}|_{\mathcal{X}_\alpha}[d_\alpha]$. In the cases where $\mathcal{X}$ is $\mathcal{F}I$ (respectively, $\mathcal{F}I^s$), we have $A = W$ (respectively, $A = W^s = \{w \in W : w < ws\}$) and we write $\mathcal{E}_{\mathcal{F}I}(w) = \mathcal{E}(w)$ (respectively, $\mathcal{E}_{\mathcal{F}I^s}(w) = \mathcal{E}(w)$); sometimes a subscript $\mathbb{L}$ will be included to emphasise the ground ring.

On the other hand, $\mathcal{X}$ admits a similar decomposition

$$\mathcal{X} = \bigsqcup_{\alpha \in A^+} \mathcal{X}^+_\alpha,$$

(3.5)

where the $\mathcal{X}^+_\alpha$ are $I^+$-orbits of dimensions $d^+_\alpha$. Write $A_+ \subseteq A^+$ for the subset parametrising the orbits $\mathcal{X}_\alpha$ that support a (non-zero) simple Iwahori–Whittaker local system $\mathcal{L}_{\mathbb{G}_m}(\alpha) \in D^b_{\text{IW}, \mathbb{G}_m}(\mathcal{X}^+_\alpha, k)$, which is necessarily unique up to isomorphism. In a similar fashion to the previous case, there is (up to isomorphism and shift) a unique parity complex

$$\mathcal{E}_{\text{IW}, \mathbb{G}_m}(\alpha) \in \text{Parity}_{\text{IW}, \mathbb{G}_m}(\mathcal{X}, k), \quad \mathcal{E}_{\mathcal{X}}(\alpha) \in \text{Parity}_{\text{IW}}(\mathcal{X}, k),$$

supported on $\mathcal{X}^+_\alpha$ and extending $\mathcal{L}_{\mathbb{G}_m}[d^+_\alpha]$, for each $\alpha \in A_+$. Moreover, the former of these is sent to the latter under the forgetful functor from $\text{Parity}_{\text{IW}, \mathbb{G}_m}(\mathcal{X}, k)$ to $\text{Parity}_{\text{IW}}(\mathcal{X}, k)$.

The following will be our main examples:

(i) $\mathcal{X} = \mathcal{F}r$, where $(A_+, A^+) = (X_+, X^+));

(ii) $\mathcal{X} = \mathcal{F}I$, where $(A_+, A^+) = (W, W)$ and we write

$$\mathcal{E}_{\text{IW}, \mathbb{G}_m}(w) := \mathcal{E}_{\text{IW}, \mathbb{G}_m}(w), \quad \mathcal{E}_{\text{IW}}(w) := \mathcal{E}_{\text{IW}}(w);$$

(iii) $\mathcal{X} = \mathcal{F}I^s$, where $(A_+, A^+) = (W^s, W^s)$ for $W^s = \{w \in W^s \cap W : ws \in W\}$, and we write

$$\mathcal{E}_{\text{IW}, \mathbb{G}_m}(w) := \mathcal{E}_{\text{IW}, \mathbb{G}_m}(w), \quad \mathcal{E}_{\text{IW}}(w) := \mathcal{E}_{\text{IW}}(w).$$

3.6 Highest weight objects, convolution, and averaging

Consider now the affine embeddings $j_\alpha : \mathcal{X}^+_\alpha \rightarrow \mathcal{X}$ for $\alpha \in A^+$. We then have standard and costandard objects in $D^b_{\text{IW}, \mathbb{G}_m}(\mathcal{X}, k)$,

$$\Delta_{\text{IW}, \mathbb{G}_m}(\alpha) = (j_{\alpha})_* \mathcal{L}_{\mathbb{G}_m}(\alpha)[d^+_\alpha], \quad \nabla_{\text{IW}, \mathbb{G}_m}(\alpha) = (j_{\alpha})_+ \mathcal{L}_{\mathbb{G}_m}(\alpha)[d^+_\alpha],$$

whose images under the forgetful functor to $D^b_{\text{IW}}(\mathcal{X}, k)$ are written $\Delta_{\text{IW}}(\alpha), \nabla_{\text{IW}}(\alpha)$, respectively. We adopt notational abbreviations as in § 3.5, writing, e.g.,

$$\Delta_{\text{IW}}(\alpha) = \Delta_{\text{IW}}(\alpha) \quad \text{and} \quad \Delta_{\text{IW}}(\alpha) = \Delta_{\text{IW}}(\alpha).$$

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As described in sources such as [BR18, § 6.2] and [dCM09, § 4.5], there is a convolution product \( \ast \) on \( D^b_{I \times G_{\mathfrak{m}}} (\mathcal{X}, L) \) when \( \mathcal{X} = \mathcal{F} \). Similarly (and taking care with coefficient rings), we have

\[
D^b_{L^+G} (Sr, L) \ast D^b_{L^+G} (Sr, k), \quad D^b_{IW} (Sr, k) \ast D^b_{L^+G} (Sr, k),
\]

where the symbol \( \ast \) indicates the second category is right-acting on the first category via convolution. For \( \mathcal{X} = \mathcal{F} \), we have an averaging functor

\[
Av : D^b_{I \times G_{\mathfrak{m}}} (\mathcal{F}, k) \rightarrow D^b_{IW, G_{\mathfrak{m}}} (\mathcal{F}, k), \quad F \mapsto \Delta_{IW, G_{\mathfrak{m}}} (1) \ast F;
\]

this definition agrees with (3.3) on locally closed finite unions of orbits.

3.7 Extension of scalars
Recall from [RW18, § 10.2] that any extension \( L \rightarrow L' \) of coefficient rings induces a monoidal extension of scalars functor

\[
L'(-) = L' \otimes_L (-) : D^b_{I \times G_{\mathfrak{m}}} (\mathcal{X}, L) \rightarrow D^b_{I \times G_{\mathfrak{m}}} (\mathcal{X}, L')
\]

which preserves parity subcategories; see [JMW12, Lemma 2.36]. Extension of scalars also affords isomorphisms

\[
L' \otimes_L \text{Hom}_{\text{Parity}_{I \times G_{\mathfrak{m}}}} (\mathcal{X}, L) (\mathcal{E}, \mathcal{F}) \cong \text{Hom}_{\text{Parity}_{I \times G_{\mathfrak{m}}}} (\mathcal{X}, L') (L'(\mathcal{E}), L'(\mathcal{F})),
\]

and \( L'(-) \) is compatible with pushforward and pullback along morphisms such as \( q^\alpha \). In the case of \( \mathcal{O} \rightarrow \mathcal{K} \), we deduce an \( \mathcal{O} \)-module injection

\[
\text{Hom}_{\text{Parity}_{I \times G_{\mathfrak{m}}}} (\mathcal{X}, \mathcal{O}) (\mathcal{E}, \mathcal{F}) \hookrightarrow \text{Hom}_{\text{Parity}_{I \times G_{\mathfrak{m}}}} (\mathcal{X}, \mathcal{K}) (\mathcal{E}(\mathcal{O}), \mathcal{K}(\mathcal{F})),
\]

since the source is torsion free over \( \mathcal{O} \); see [MR18, Lemma 2.2(2)]. In the case of \( \mathcal{O} \rightarrow k \), it holds that \( k(\mathcal{E}) \) is a parity complex if and only if \( \mathcal{E} \) is; moreover, by [JMW12, Proposition 2.39], extension to \( k \) respects indecomposable parity objects:

\[
k(\mathcal{E}_{\mathfrak{m}} (\alpha)) \cong \mathcal{E}_{\mathfrak{m}} (\alpha).
\]

3.8 Geometric Satake equivalence
We briefly summarise the contents of [RW22, 8.1], to fix notation and recall key results. Denote by \( \text{Perv}(Sr, k) \subseteq D^b_{L^+G} (Sr, k) \) the Satake category of \( L^+G \)-equivariant perverse \( k \)-sheaves on \( Sr \); this category inherits an exact monoidal convolution product \( \ast \) from \( D^b_{L^+G} (Sr, k) \), while its simple objects \( IC(\lambda) \) are parametrised by \( \lambda \in X^+_+ \). Our main application for perverse sheaves derives from the following theorem, due to Mirković and Vilonen [MV07].

**Theorem 3.1.** There is an equivalence of monoidal categories,

\[
\text{Sat} : (\text{Rep}(G), \otimes) \cong (\text{Perv}(Sr, k), \ast),
\]

where \( G \) denotes the \( k \)-group from (1.1).

In fact, even more important for us will be a formulation of this theorem featuring the Iwahori–Whittaker derived category; this is made possible by the following result of [BGM+19].

**Theorem 3.2.** There is an equivalence of abelian categories,

\[
\text{Perv}(Sr, k) \cong \text{Perv}_{IW} (Sr, k), \quad \mathcal{F} \mapsto \Delta_{IW}^r (\rho^\vee) \ast \mathcal{F}.
\]

Also key is that the forgetful functor \( F_{G_{\mathfrak{m}}} : \text{Perv}_{IW, G_{\mathfrak{m}}} (Sr, k) \rightarrow \text{Perv}_{IW} (Sr, k) \) is an equivalence of categories; this is shown in [RW22, Lemma 5.2].
Since \( \text{Perv}(G, k) \) has the structure of a highest weight category, we may refer to its tilting subcategory \( \text{Tilt}(G, k) \subseteq \text{Perv}(G, k) \), whose indecomposable tilting objects \( \mathcal{T}(x) \) are indexed by \( x \in X^\vee_+ \). We likewise have subcategories

\[
\text{Tilt}_{IW}(G, k) \subseteq \text{Perv}_{IW}(G, k), \quad \text{Tilt}_{IW, \mathbb{C}_m}(G, k) \subseteq \text{Perv}_{IW}(G, k),
\]

using the equivalence \( \text{For}_{\mathbb{C}_m} \) to make sense of the latter; their indecomposable objects are written \( \mathcal{T}_{IW}(x) \) and \( \mathcal{T}_{IW, \mathbb{C}_m}(x), x \in X^\vee_+ \).

4. The loop antispherical module

The constructions and results discussed in the previous section also apply to the simply connected cover \( \tilde{G} \) of \( G \). By definition, \( \tilde{G} \) has the same Bruhat–Tits building as \( G \), but its extended Weyl group and affine Weyl group coincide and are isomorphic to \( W \). In particular, there are loop groups \( L_n \tilde{G} \) and \( L_n^+ \tilde{G} \); Iwahori subgroups \( \tilde{I}_n \) and \( \tilde{I}_n^+ \); parahoric subgroups \( \tilde{P}_n^f \), for each facet \( f \subseteq \tilde{\mathcal{R}}_1 \) and affine flag varieties \( \tilde{F}_n^f \). Importantly, \( \xi : \tilde{G} \to G \) induces a map \( \tilde{F}_n^f \to P_n^f \) and hence a closed immersion

\[
\tilde{F}_n^f \to F_n^f,
\]

which identifies \( \tilde{F}_n^f \) with the neutral component \( F_n^f \) of \( F_n^f \), since \( \tilde{F}_n^f \) does not divide the order of the fundamental group \( \pi_1(G) \); see [PR08, §6].

We will work with \( \tilde{G} \) in much of what follows because it is essential for the geometric Hecke category. Analogues for \( G \) of objects we have constructed for \( \tilde{G} \) will be denoted by decorating the appropriate symbol with a tilde, e.g. the affine flag variety \( \tilde{F} \) of \( \tilde{G} \).

4.1 Bott–Samelson varieties

For any expression \( \bm{w} = (s_1, s_2, \ldots, s_m) \) in \( S \), there is a Bott–Samelson variety (or Demazure variety)

\[
\nu_{\bm{w}} : \text{BS}(\bm{w}) = \tilde{P}^{s_1} \times \tilde{I} \times \tilde{P}^{s_2} \times \tilde{I} \cdots \times \tilde{I} \times \tilde{P}^{s_m} / \tilde{I} \to \tilde{F}.
\]

By definition, this variety is the quotient \( (\tilde{P}^{s_1} \times \cdots \times \tilde{P}^{s_m}) / \tilde{I}^m \), where the right action of \( \tilde{I}^m \) is given on points by

\[
(x_1, \ldots, x_m) \cdot (g_1, \ldots, g_m) = (x_1g_1, g_1^{-1}x_2g_2, \ldots, g_m^{-1}x_mg_m).
\]

As explained in [PR08, §8.8], this is a smooth projective \( \mathbb{F} \)-variety of dimension \( m \). The morphism \( \nu_{\bm{w}} \) is equivariant for the natural \( \tilde{I} \)-action on its source and target [RW18, §9.1]; since \( \nu_{\bm{w}} \) arises from the multiplication

\[
\tilde{P}^{s_1} \times \cdots \times \tilde{P}^{s_k} \to L\tilde{G},
\]

it is also \( G_m \)-equivariant and thus equivariant for the action of \( \tilde{I} \times G_m \). In fact, \( \nu_{\bm{w}} \) factors through a projective subvariety of \( \tilde{F} \), so \( \nu_{\bm{w}} \) preserves bounded \( \tilde{I} \times G_m \)-equivariant sheaves. Consider then the object

\[
\mathcal{E}(\bm{w}) = (\nu_{\bm{w}})_* \|_{\text{BS}(\bm{w})} [\ell(\bm{w})] \in D^b_{\tilde{I} \times G_m}(\tilde{F}, \mathbb{L}).
\]

Remark 4.1. For the sake of convolution, we should consider the Bott–Samelson and flag varieties to be constructed from the groups \( \widehat{P}^s = \tilde{P}^s \times G_m, \; \tilde{I} = \tilde{I} \times G_m \), etc. This does not change the foregoing constructions, up to isomorphism, but we must take account of it in the following proof.
LEMMA 4.2. For any expressions $w, v$, there is an isomorphism

$$\mathcal{E}(w) \ast \mathcal{E}(v) \cong \mathcal{E}(wv).$$

Proof. We follow the proof of [RW18, Lemma 10.2.1]. Write $w = s \cdots t$, and choose closed finite unions of $\bar{l}$-orbits $X_w, X_v$ through which $\nu_w, \nu_v$ factor and upon which $\mathcal{E}(w), \mathcal{E}(v)$ are supported. The action of $\bar{l}$ on $X_w$ factors through a finite-type quotient $J = \bar{l}/N$ appearing in a pro-algebraic representation of $\bar{l}$; let $\tilde{X}_w$ be the preimage of $X_w$ in $(\mathbb{L}G)/N$. We then have the following diagram.

$$X_w \times X_v \xrightarrow{p} \tilde{X}_w \times X_v \xrightarrow{q} \tilde{X}_w \times \mathfrak{I} X_v \xrightarrow{m} \mathfrak{I} l$$

By construction, the convolution product is $m_* (\mathcal{E}(w) \boxtimes \mathcal{E}(v))$, where $\mathcal{E}(w) \boxtimes \mathcal{E}(v)$ satisfies

$$q^* (\mathcal{E}(w) \boxtimes \mathcal{E}(v)) \cong p^* (\mathcal{E}(w) \boxtimes \mathcal{E}(v)).$$

Proper, respectively, smooth, base change along the cartesian squares

$$\tilde{\mathfrak{P}}^s \times \mathfrak{I} \cdots \times \mathfrak{I} t/N \xrightarrow{\mu} \tilde{X}_w, \quad \tilde{\mathfrak{P}}^s \times \mathfrak{I} \cdots \times \mathfrak{I} t/N \times \text{BS}(v) \xrightarrow{\mu \times \nu_v} \tilde{X}_w \times X_v \xrightarrow{\mu \times \nu_v} \tilde{X}_w \times \mathfrak{I} X_v$$

shows $\mathcal{E}(w) \boxtimes \mathcal{E}(v) \cong (\mu \times \nu_v)_* [\mathcal{E}(w) \boxtimes \mathcal{E}(v)]$; now use $m(\mu \times \nu_v) = \nu_w$. \hfill \Box

The smoothness of $\tilde{\mathfrak{P}}^s / \mathfrak{I}$ implies $\mathcal{E}(s)$ is a parity object, and convolution by $\mathcal{E}(s)$ preserves parity objects for the reasons given in [RW18, §9.4], so due to (4.1) all the $\mathcal{E}(w)$ are parity objects.

DEFINITION 4.3. We consider a category $\text{Parity}^\text{BS}_{\mathfrak{I} \times G_m} (\mathfrak{I} l, L)$ whose objects are pairs $(w, m)$ for $w$ an expression and $m$ an integer, with morphisms

$$\text{Hom}((w, m), (v, n)) = \text{Hom}(\mathcal{E}(w)[m], \mathcal{E}(v)[n]).$$

It is monoidal with $(w, m) \ast (v, n) = (wv, m + n)$ on objects, and the product of morphisms defined through (4.1).

The natural functor $\text{Parity}^\text{BS}_{\mathfrak{I} \times G_m} (\mathfrak{I} l, L) \to \text{Parity}_{\tilde{\mathfrak{I}} \times G_m} (\tilde{\mathfrak{I}} l, L)$ realises the latter category as the Karoubi envelope of the additive hull of the former category [RW18, Lemma 10.2.3]. Note that $\text{Parity}_{\tilde{\mathfrak{I}} \times G_m} (\tilde{\mathfrak{I}} l, L)$ is Krull–Schmidt, as a full subcategory of the Krull–Schmidt category $D^b_{\tilde{\mathfrak{I}} \times G_m} (\tilde{\mathfrak{I}} l, L)$; see [JMW12, Remark 2.1] and [LC07].

4.2 Realisations and Bott–Samelson Hecke categories

We begin by describing two realisations of the Coxeter system $(W, S)$, in the sense of [EW16, §3.1].

Consider first the $L$-module

$$\mathfrak{h}_L = L \otimes_{\mathbb{Z}} \mathfrak{h}_Z, \quad \text{where} \quad \mathfrak{h}_Z = \mathbb{Z} \Phi \oplus \mathbb{Z} d.$$

For $s \in S_f$, $\mathfrak{h}_L$ contains roots $\alpha_s \in \Phi$ and $\mathfrak{h}_L^\vee$ contains coroots $\alpha_s^\vee \in \Phi^\vee$. For $s \in S - S_f$, the image of $s$ under the quotient map $W \to W_f$ is a reflection $s_g$; we take $\alpha_s = -\theta \in \mathfrak{h}_L$ and $\alpha_s^\vee = \delta - \theta^\vee \in \mathfrak{h}_L^\vee$, where $\delta$ is dual to $d$. We define $\langle d, \alpha_s^\vee \rangle$ to be 0 or 1 according to whether $s \in S_f$ or $s \in S - S_f$.
and hence reflections
\[ \sigma_s : \mathfrak{h}_L \to \mathfrak{h}_L, \quad \sigma_s(v) = v - \langle v, \alpha_s^\vee \rangle \alpha_s. \]

Now, the data \((\mathfrak{h}_L, \{\alpha_s\}, \{\alpha_s^\vee\})\), together with the assignment of the simple reflections in \(W\) to the \(\sigma_s\), define the loop realisation of \((W, S)\). In fact, this realisation is a quotient of the ‘traditional’ realisation associated to the affinisation of the Lie algebra of \(G\), discussed in [RW18, Remark 10.7.2(2)].

Second, consider the realisation of \((W, S)\) described in [RW18, §4.2]; there it is denoted \(h\), but we will denote it \(h'_L\) and call it the standard realisation. We have
\[ h'_L = L \otimes_{\mathbb{Z}} \mathbb{Z} \Phi \hookrightarrow h_L, \]
with the same simple coroots and simple roots as \(h_L\), except with \(-\theta^\vee\) in place of \(\delta - \theta^\vee\) for \(\alpha_s^\vee\) when \(s \in S - S_f\). Clearly the standard and loop realisations have the same Cartan matrix; hence, by [RW18, §4.2], both realisations satisfy the technical conditions in [EW16, Definition 3.6] and [EW16, Assumption 3.7] if \(\ell > h\).

Now let
\[ R'_L = \text{Sym}(((h'_L)^*)), \quad R_L = \text{Sym}(h^*_L) = R'_L[d], \]
which are generated as \(L\)-algebras by the simple roots in \(h_L\) (respectively, \(h'_L\)). The algebra \(R'_L\) (respectively, \(R_L\)) is essential to define the graded \(L\)-linear monoidal Bott–Samelson Hecke category \(\mathcal{H}_{BS}\) (respectively, \(\mathcal{H}_{BS}'\)) and diagrammatic Hecke category \(\mathcal{H}'\) (respectively, \(\mathcal{H}'\)) associated to the standard realisation (respectively, the loop realisation). For these definitions we refer the reader to [EW16, Definition 5.2], which features the notation \(D\) (suppressing mention of the realisation). The standard generating objects, shift of grading, and monoidal product in Hecke categories will be denoted by the symbols \(B_s\), \(\langle 1 \rangle\), and \(\star\), in turn. As usual, for an expression \(w = (s_1, \ldots, s_m)\) in \(S\), we write
\[ B_w = B_{s_1} \star \cdots \star B_{s_m}. \]

4.3 Hecke category equivalences

The following theorem is a \(G_m\)-equivariant version of the main result in [RW18, §10].

**Theorem 4.4.** There is an equivalence of \(L\)-linear graded monoidal categories,
\[ \Delta_{BS} : \mathcal{H}_{BS} \cong \text{Parity}_{I \times G_m}^{BS} (\tilde{F}I, L), \]
lifting to an equivalence \(\Delta : \mathcal{H} \cong \text{Parity}_{I \times G_m} (\tilde{F}I, L)\).

**Proof.** The proof is almost identical to that given in [RW18, §§10.3–10.6] for [RW18, Theorem 10.3.2], so for efficiency we merely annotate the meaningful points of difference in specific sections of that paper.

In modifying the proof of [RW18], we must universally replace the ind-varieties \(\mathcal{X}\) and \(\mathcal{X}^s\) by \(\tilde{F}I\) and \(\tilde{F}I^s\), respectively, and the Borel subgroup \(B\) by \(\tilde{I} \times G_m\) (or by \(\tilde{I}\), when working with Bott–Samelson resolutions); compare with [RW18, Theorem 10.7.1] and the remarks preceding it. Recall that we are working in the étale context [RW18, §9.3(2)], so the ring \(\mathbb{Z}'\) should be chosen as \(\mathbb{O}\) and then the appropriate analogue of [RW18, Lemma 10.2.2] holds. No other changes are necessary through to the end of [RW18, §10.3]; in the subsequent section we have the following.

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(10.4.2) We replace the indented isomorphism with
\[
\Hom_{\mathcal{D}_I^{\mathbb{Z}\times \mathbb{G}_m}(\tilde{\mathcal{F}}, \mathcal{F})} (\mathcal{E}(\emptyset), \mathcal{E}(\emptyset)[2m]) \cong H^{2m}_{\tilde{I}\times \mathbb{G}_m}(\pt, \mathbb{Z})
\cong H^{2m}_{\tilde{I}\times \mathbb{G}_m}(\pt, \mathbb{Z}')
\cong \text{Sym}^m(\mathbb{Z}' \otimes_{\mathbb{Z}} \mathbb{H}_\mathbb{Z}),
\]
where the second-last isomorphism is due to the existence of a surjection $\tilde{I} \times \mathbb{G}_m \to \tilde{T} \times \mathbb{G}_m$ with a pro-unipotent kernel.

(10.4.3) No modification is needed for the description of the image of the upper dot morphism. For the lower dot morphism, note that in the classical setting the identification
\[
\mathcal{E}(\mathbb{Z}) = \mathbb{Z}_{\mathbb{B}(s)}[1] = \mathbb{D} \mathbb{Z}_{\mathbb{B}(s)}[-1]
\]
is canonical after a fixed choice of orientation of $\mathbb{C}$, i.e. of $\sqrt{-1} \in \mathbb{C}$. This can be rephrased as the choice of a continuous isomorphism between the groups $\mathbb{Q}/\mathbb{Z}$ and the roots of unity $\mu_\infty \leq \mathbb{C}^\times$. In the étale context on the one-dimensional $\mathbb{F}$-variety $\mathbb{P}^1$, we replace this by a fixed choice of an isomorphism between $\mathbb{Z}_\ell$ and $H^2(\mathbb{P}^1, \mathbb{Z}_\ell)$; see [DIST13, §§ 7.1, 7.4]. This base changes to an isomorphism between $\mathbb{O}$ and $H^2(\mathbb{P}^1, \mathbb{O})$.

(10.4.4) The statement of Lemma 10.7 goes through without modification, since we still have
\[
\mathbb{B} \mathbb{S}(ss) \cong \mathbb{P}_s/\tilde{I} \times \mathbb{P}_s/\tilde{I} \cong \mathbb{P}^1 \times \mathbb{P}^1.
\]
Modulo the selection of adjunction morphisms $a_s, a_t$ (in 10.4.3), Lemma 10.8 and the remainder of 10.4.4 are formal consequences.

(10.4.5) For the statement of Lemma 10.9, we must move from $\mathbb{Q}$ to $\mathbb{K}$, and from $\mathcal{B}$-equivariant derived categories on $\mathcal{X}$ to $\tilde{I} \times \mathbb{G}_m$-equivariant categories on $\tilde{\mathcal{F}}$. The former change is handled by our analogue of Lemma 10.3. For the latter change, recall that the $\tilde{I}$-equivariant setting on $\tilde{\mathcal{F}}$ is already verified by [RW18, § 10.7], so it remains to replace $\tilde{I}$ with $\tilde{I} \times \mathbb{G}_m$. For this, observe that by [MR18, Lemma 2.2],
\[
R_{\mathbb{K}} \otimes_{R'_{\mathbb{K}}} \Hom_{\mathcal{D}_I^{\mathbb{G}_m}(\tilde{\mathcal{F}}, \mathcal{F})} (\mathbb{K}(F_s), \mathbb{K}(F_t))
\cong \Hom_{\mathcal{D}_I^{\mathbb{G}_m}(\tilde{\mathcal{F}}, \mathcal{F})} (\mathbb{K}(F_s), \mathbb{K}(F_t)).
\]
The decomposition of $\mathbb{K}(F_s)$ and $\mathbb{K}(F_t)$ into IC-sheaves given by the proof of [RW18, Lemma 10.4.3] shows that the graded module (4.3) is concentrated in non-negative degrees. Since $R_{\mathbb{K}} = R'_{\mathbb{K}}[d]$ is a polynomial ring over $R'_{\mathbb{K}}$ and
\[
\Hom_{\mathcal{D}_I^{\mathbb{G}_m}(\tilde{\mathcal{F}}, \mathcal{F})} (\mathbb{K}(F_s), \mathbb{K}(F_t)) = \mathbb{K},
\]
the same is true for $\Hom_{\mathcal{D}_I^{\mathbb{G}_m}(\tilde{\mathcal{F}}, \mathcal{F})} (\mathbb{K}(F_s), \mathbb{K}(F_t))$.

Lemma 10.10 hinges on the assertion that $\nu_{\mathbb{W}}$ is birational with connected fibers, as proven in [Mat88, Lemme 32]. The birationality is clear, considering the open Schubert cell $\tilde{I}wI/\tilde{I} \subseteq \tilde{\mathcal{F}}_{\mathbb{W}}$. Concerning connectedness, note that, by definition, the target of $\nu_{\mathbb{W}}$ is a normal variety, so we can apply Zariski’s main theorem.

In addition to the universal changes indicated previously, there are no meaningful alterations to note for [RW18, §§ 10.5-10.6] (or the proofs and results those sections reference from earlier in the paper, such as [RW18, Proposition 9.8.1]).
By definition, any object in $\text{Parity}_{I \rtimes \mathbb{G}_m}(\tilde{F}, k)$ arises as a direct sum of graded shifts of sum-mands of objects obtained from $\mathcal{E}(\varnothing)$ by convolution with various $\mathcal{E}(s)$. Now $\text{Av}$ commutes with the convolution products discussed in §3.6 (working now with $\tilde{G}$ rather than $G$), i.e. $\text{Av}(E \star F) = \text{Av}(E) \star F$, and

$$\text{Av}(\mathcal{E}(1)) = \Delta_{IW, \mathbb{G}_m}(1) \cong \nabla_{IW, \mathbb{G}_m}(1)$$

is a parity object, so if $\mathcal{E} \in \text{Parity}_{I \rtimes \mathbb{G}_m}(\tilde{F}, k)$, then $\text{Av}(\mathcal{E}) \in \text{Parity}_{IW, \mathbb{G}_m}(\tilde{F}, k)$. These same observations are made in the proof of [RW18, Corollary 11.2.3], and they enable us to define $\text{Parity}^{BS}_{IW, \mathbb{G}_m}(\tilde{F}, k)$ to be the essential image of $\text{Av} : \text{Parity}^{BS}_{I \rtimes \mathbb{G}_m}(\tilde{F}, k) \to \text{Parity}_{IW, \mathbb{G}_m}(\tilde{F}, k)$.

As in [RW18, §11], the functors $\Delta_{BS}$ and $\Delta$ induce equivalences of certain right Hecke category modules.

**Theorem 4.5.** We have diagrams of categories with horizontal equivalences, commuting up to natural isomorphism:

$$\begin{array}{ccc}
\mathcal{H}_{BS} & \xrightarrow{\Delta_{BS}} & \text{Parity}^{BS}_{I \rtimes \mathbb{G}_m}(\tilde{F}, k) \\
\downarrow q & & \downarrow \text{Av} \\
\mathcal{H}_{BS} & \xrightarrow{\Delta_{BS}} & \text{Parity}^{BS}_{IW, \mathbb{G}_m}(\tilde{F}, k).
\end{array}$$

Here overlines denote what we will refer to as loop antispherical quotients. In particular, $\mathcal{H}_{BS}$ is obtained from $\mathcal{H}_{BS}$ by killing morphisms factoring through objects $B_w$, whenever $w$ begins with some element $s \in S_f$. By furthermore killing the morphism $d \in \text{End}(B_\varnothing) = \mathbb{R}_k$, we obtain the (standard) antispherical quotients $\mathcal{H}_{BS}'$ and $\mathcal{H}'$ discussed in sources such as [RW18, §1.3] and [LW22]. The snake arrow denotes passage to Karoubi envelopes of additive hulls.

**Proof.** The argument is analogous to the proof of [RW18, Theorem 11.7.1], with modifications and annotations as follows. We first make the same universal notational changes as in the proof of Theorem 4.4, along with the specialisation $J = S_f$. The next step is to verify the analogue of [RW18, Lemma 11.2.5], namely that if $w \in W - W_f$, then

$$\text{Av}(\mathcal{E}(w)) = 0.$$  \hspace{1cm} (4.4)

For this, note that we have a square of categories commuting up to natural isomorphism,

$$\begin{array}{ccc}
\text{Parity}_{I \rtimes \mathbb{G}_m}(\tilde{F}, k) & \xrightarrow{\text{For}} & \text{Parity}_{I}(\tilde{F}, k) \\
\downarrow \text{Av} & & \downarrow \text{Av} \\
\text{Parity}_{IW, \mathbb{G}_m}(\tilde{F}, k) & \xrightarrow{\text{For}} & \text{Parity}_{IW}(\tilde{F}, k)
\end{array}$$

this is evident from the definition of $\text{Av}$ as in (3.3). Since $\text{Parity}_{IW, \mathbb{G}_m}(\tilde{F}, k)$ is Krull–Schmidt and $\text{For}(\mathcal{E}_{IW, \mathbb{G}_m}(u)) = \mathcal{E}_{IW}(u)$ for $u \in W$ by the general theory of parity complexes [MR18, Lemma 2.4], we can conclude that $\text{For}(\mathcal{F}) = 0$ forces $\mathcal{F} = 0$ for $\mathcal{F} \in \text{Parity}_{I \rtimes \mathbb{G}_m}(\tilde{F}, k)$. But indeed,

$$\text{For}(\text{Av}(\mathcal{E}(w))) = \text{Av}(\text{For}(\mathcal{E}(w))) = 0$$

by [RW18, Lemma 11.2.5], so (4.4) follows and, in fact, $\text{Av}(\mathcal{E}(w)) = 0$ for any expression $w$ starting with an element of $S_f$. This implies the existence of the functor $\Delta_{BS}$. The proof that
it is fully faithful proceeds just as in [RW18], with only notational alterations, because [RW18, Lemma 11.1.1–2], the theory of [RW18, §11.3], and [RW18, Proposition 11.4.1] are adapted immediately.

\[ \square \]

**Corollary 4.6.** The category $\text{Parity}_{IW,G_m}(\mathcal{F}, k)$ admits an action of $\mathcal{H}$ by graded functors, such that $\Delta$ is an equivalence of graded right $\mathcal{H}$-modules.

### 5. Geometric action on the Smith quotient

In this section, we begin by recalling a number of key results and constructions from [RW22], particularly the Smith categories of $\mathbb{F}$-varieties with trivial $\varpi$-action and some associated functors. Leveraging an understanding of the morphism spaces between parity objects in the Iwahori–Whittaker Smith category on $\mathcal{F}$, we are then able to define and study an action of $\mathcal{H}$ on the parity Smith quotient.

#### 5.1 Smith categories

Suppose $X$ is an $\mathbb{F}$-variety with an action of $G_m$. Recall from [RW22] the equivariant Smith category, defined as the Verdier quotient

$$\text{Sm}(X^\varpi, k) = D_{G_m}^b(X^\varpi, k)/D_{G_m}^b(X^\varpi, k)_\varpi,$$

where $D_{G_m}^b(X^\varpi, k)_\varpi$ is the full subcategory of objects $\mathcal{F}$ for which $\text{Res}_m^G(\mathcal{F})$ has perfect geometric stalks in the sense of [RW22, §3.3]. Our main interest will be in the following variant: if $\mathcal{X}$ is as in §3.5 and $Y \subseteq \mathcal{X}^\varpi$ is a finite union of orbits of $I^+_\ell = (I^+)^\varpi$, we can consider a category $D_{IW,G_m}(Y, k)$, constructed as in §3.4 via a restriction of $\chi$ to $(I^+_u)^\varpi$. (In [RW22], this modified construction is denoted $D_{IW,G_m}(Y, k)$, but we will slightly abuse notation and suppress the subscript $\ell$.) There is then an Iwahori–Whittaker Smith quotient

$$Q_Y : D_{IW,G_m}(Y, k) \to \text{Sm}_{IW}(Y, k) = D_{IW,G_m}^b(Y, k)/D_{IW,G_m}^b(Y, k)_\varpi.$$

**Proposition 5.1.** Assume that $\mathcal{X}$, $\mathcal{X}_1$, and $\mathcal{X}_2$ are partial affine flag varieties. Let $Y \subseteq \mathcal{X}_1^\varpi$ and $Z \subseteq \mathcal{X}_2^\varpi$ be locally closed finite unions of $I^+_\ell$-orbits.

1. If $f : Y \to Z$ is a quasi-separated morphism of $I^+_\ell \ltimes G_m$-varieties, then for $\dagger \in \{!, *\}$ there exist functors

$$f^\dagger_{\text{Sm}} : \text{Sm}_{IW}(Y, k) \to \text{Sm}_{IW}(Z, k), \quad f^\dagger_{\text{Sm}} : \text{Sm}_{IW}(Z, k) \to \text{Sm}_{IW}(Y, k)$$

such that the following diagrams commute.

\[
\begin{array}{ccc}
D_{IW,G_m}(Z, k) & \xrightarrow{f^!} & D_{IW,G_m}(Y, k) \\
\downarrow q_Z & & \downarrow q_Y \\
\text{Sm}_{IW}(Z, k) & \xrightarrow{f^!_{\text{Sm}}} & \text{Sm}_{IW}(Y, k)
\end{array}
\quad \quad
\begin{array}{ccc}
D_{IW,G_m}(Y, k) & \xrightarrow{f^*} & D_{IW,G_m}(Z, k) \\
\downarrow q_Y & & \downarrow q_Z \\
\text{Sm}_{IW}(Y, k) & \xrightarrow{f^*_{\text{Sm}}} & \text{Sm}_{IW}(Z, k)
\end{array}
\]

We have that $(f^!_{\text{Sm}}, f^!)$ and $(f^*_{\text{Sm}}, f^*)$ are adjoint pairs of functors, so, in particular, if $f$ is a closed embedding, then $f^!_{\text{Sm}} = f^!$ is fully faithful.

2. Up to a change from $\zeta$ to $\zeta^{-1}$, Verdier duality $\mathbb{D}_Y$ preserves $D_{IW,G_m}^b(Y, k)_\varpi$ and therefore descends to a functor on $\text{Sm}_{IW}(Y, k)$.

3. If $X \subseteq \mathcal{X}$ is a locally closed finite union of $I^+$-orbits and $i_X : X^\varpi \hookrightarrow X$ is the inclusion, then the cone of the natural morphism $i_X^! \to i_X^*$ is killed by the Smith quotient functor,
yielding a Smith restriction functor

\[ i_X^* : D^b_{IW,G_m}(X, \kappa) \to \text{Sm}_{IW}(X^\varpi, \kappa). \]

(4) Let \( X_1 \subseteq \mathcal{X}_1, X_2 \subseteq \mathcal{X}_2 \) be locally closed finite unions of \( I^+ \)-orbits, with \( f : X_1 \to X_2 \) a quasi-separated morphism of \( I^+ \times \mathbb{G}_m \)-varieties inducing \( f^\varpi : X_1^\varpi \to X_2^\varpi \). For \( \dagger \in \{!, \ast\}\),

\[ i_{X_1}^! \circ f^\dagger \cong (f^\varpi)^\dagger \circ i_{X_2}^! . \]

When \( f \) is the inclusion \( j : X_1 \hookrightarrow X_2 \) for \( X_1 \subseteq X_2 \subseteq \mathcal{X} \), we also have

\[ i_{X_1}^* \circ j^! \cong (j^\varpi)^* \circ i_{X_2}^* . \]

(5) There is a canonical natural isomorphism,

\[ e_Y : \text{id} \cong [2] : \text{Sm}_{IW}(Y, \kappa) \to \text{Sm}_{IW}(Y, \kappa). \]

**Proof.** Most of these statements are proven in [RW22, § 6]; the exceptions are statement (1), which is a generalised version of [RW22, Lemma 6.1], and statement (2). It is evident from the proof of the latter lemma that our claim holds for \((f^*, f_*)\), so it will suffice to prove statement (2). First, we have that

\[ \text{Av}_\zeta(\mathbb{D} F) \cong \mathbb{D} \text{Av}_{\zeta^{-1}}(F), \]

using obvious notation to keep track of the \( p \)-th root of unity chosen for the construction; this follows from the discussion preceding [BGM+19, Lemma 3.8] and shows that \( \mathbb{D} \) respects Iwahori–Whittaker sheaves in the required sense. It remains to prove that \( \mathbb{D}(F) \) lies in \( D^b_{G_m}(Y, \kappa)_{\varpi}\text{-perf} \) if \( F \) does. In view of the identification

\[ D^b_{\varpi}(Y, \kappa) \cong D^b(Y, \kappa[\varpi]) \]

and by a standard dévissage argument, we reduce to proving that if \( L = j_! L \) is the extension by zero of a locally constant sheaf on an open stratum \( j : U \hookrightarrow Y \), with stalks which are free \( \kappa[\varpi]\)-modules, then the stalks of \( \mathbb{D} L \) are likewise. Now, \( \mathbb{D} L \) is a shift and Tate twist of the dual local system \( L^\vee \) over \( \kappa[\varpi] \), so in particular its stalks are free \( \kappa[\varpi]\)-modules. This shows that \( \mathbb{D} L \) lies in \( D^b_{G_m}(U, \kappa)_{\varpi}\text{-perf} \). Then \( \mathbb{D} L = j_* \mathbb{D} L \) lies in \( D^b_{G_m}(Y, \kappa)_{\varpi}\text{-perf} \) by [RW22, Lemma 3.6].

Suppose now that \( \mathcal{Y} \) is a partial affine flag variety or a finite product of such ind-schemes, and consider an ind-subscheme \( \mathcal{U} \subseteq \mathcal{Y}^\varpi \), the direct limit of closed finite unions of \( I^+_\ell\)-orbits. Using Proposition 5.1(1), we define \( \text{Sm}_{IW}(\mathcal{U}, \kappa) \) to be the direct limit of the categories \( \text{Sm}_{IW}(Y, \kappa) \), for \( Y \subseteq \mathcal{U} \) a closed finite union of \( I^+_\ell\)-orbits. The Smith functors in Proposition 5.1(1) and the natural isomorphism in Proposition 5.1(5) similarly extend through such direct limits; we write

\[ Q_{\mathcal{U}} : D^b_{IW,G_m}(\mathcal{U}, \kappa) \to \text{Sm}_{IW}(\mathcal{U}, \kappa), \]

constructing the source of \( Q_{\mathcal{U}} \) as at the end of § 3.4 (with \( I^+ \) replaced by \( I^+_\ell \)). When \( \mathcal{U} = \mathcal{Y}^\varpi \) for \( \mathcal{Y} \subseteq \mathcal{Y} \) the direct limit of closed finite unions of \( I^+\)-orbits, we obtain a Smith restriction functor

\[ i_{\mathcal{U}}^* : D^b_{IW,G_m}(\mathcal{Y}, \kappa) \to \text{Sm}_{IW}(\mathcal{Y}^\varpi, \kappa). \]

In particular, there are functors

\[ (q^s)^s_{\text{Sm}} : \text{Sm}_{IW}(\mathcal{F}_\ell^+, \kappa) \to \text{Sm}_{IW}(\mathcal{F}_\ell, \kappa), \quad (q^s)^s_{\text{Sm}} : \text{Sm}_{IW}(\mathcal{F}_\ell^+, \kappa) \to \text{Sm}_{IW}(\mathcal{F}_\ell^+, \kappa), \]

the direct limits of the Smith functors associated to projections between compatible closed finite unions of \( I^+_\ell\)-orbits on \( \mathcal{F}_\ell \) and \( \mathcal{F}_\ell^+ \).

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With $\mathcal{X}$ as in §3.5 and $\alpha \in A_+$, let
\[ \mathcal{L}_\text{Sm}^\mathfrak{X}_\alpha(\alpha) = i_{\mathfrak{X}_\alpha}^* + (\mathcal{L}_\mathfrak{AS}^\mathfrak{X}(\alpha)) \in \text{Sm}_{\mathbb{F}}(\mathfrak{X}_\alpha, k) \]
and note that $(\mathcal{X}_\alpha^\mathfrak{X})^\mathfrak{X}$ is an orbit of both $I_\ell^+$ and $I_{u,\ell}^+$. The next result is obtained by adapting the proofs of [RW22, Lemma 6.3] and [RW22, Proposition 6.5].

**Proposition 5.2.** We have
\[
\text{Hom}_{\text{Sm}_{\mathbb{F}}((\mathcal{X}_\alpha^\mathfrak{X})^\mathfrak{X}), k}(\mathcal{L}_\text{Sm}^\mathfrak{X}(\alpha), \mathcal{L}_\text{Sm}(\alpha)[n]) = \begin{cases} k & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}
\]

More generally, if $Y \subseteq \mathcal{X}^\mathfrak{X}$ is a locally closed finite union of $I_\ell^+$-orbits, then $\text{Sm}_{\mathbb{F}}(Y, k)$ has finite-dimensional Hom spaces.

**Lemma 5.3.** Let $f : Y \to Z$ be a quasi-separated morphism between $\mathbb{F}$-varieties as in Proposition 5.1. Then
\[
f_{\text{Sm}^Z}^* = e_Y f_{\text{Sm}}^*. 
\]

**Proof.** By the construction in [RW22, Lemma 6.4], there is $c : k_{pt} \to k_{pt}[2]$ in $D_{G_m,c}^b(\text{pt}, k)$ such that
\[
e_Y Q_Y = Q_Y((-) \otimes p_Y^*(c)), \quad e_Z Q_Z = Q_Z((-) \otimes p_Z^*(c)),
\]
where $p_Y$ and $p_Z$ are the projections to pt. Since $p_Y = p_Z \circ f$, we have
\[
f_{\text{Sm}^Z}^* e_Z Q_Z = f_{\text{Sm}}^* Q_Z((-) \otimes p_Z^*(c)) = Q_Y(f^*((-) \otimes p_Y^*(c))) = Q_Y(f^*(-) \otimes p_Y^*(c))
\]
which is simply $e_Y Q_Y f^* = e_Y f_{\text{Sm}}^* Q_Z$. Using the universal property of the quotient functor $Q_Z$, we then deduce the claimed equality. □

### 5.2 Parity Smith categories

**Definition 5.4.** Let $\mathcal{X}$ be partial affine flag variety and suppose $Y \subseteq \mathcal{X}^\mathfrak{X}$ is a locally closed finite union of $I_\ell^+$-orbits. Consider an ind-scheme $\mathcal{U} \subseteq \mathcal{X}^\mathfrak{X}$ as in §5.1 (for $\mathcal{Y} = \mathcal{X}$).

1. For $\dagger \in \{!, *\}$, we say $\mathcal{F} \in \text{Sm}_{\mathbb{F}}(Y, k)$ is $\dagger$-even (respectively, $\dagger$-odd) if for any inclusion
\[
f_\alpha : \mathcal{F}^\mathfrak{X}_\alpha \hookrightarrow Y, \quad \alpha \in A_+, 
\]
the object $(f_\alpha^\mathfrak{X})^\mathfrak{X}_\alpha \mathcal{F}$ decomposes as a direct sum of copies of $\mathcal{L}_\text{Sm}^\mathfrak{X}_\alpha(\alpha)$ (respectively, $\mathcal{L}_\text{Sm}^\mathfrak{X}_\alpha(\alpha)[1]$). An object is even (respectively, odd) if it is both !-even and *-even (respectively, !-odd and *-odd).

2. We write $\text{Sm}_{\mathbb{F}}^0(Y, k)$ for the full subcategory consisting of even objects (respectively, $\text{Sm}_{\mathbb{F}}^1(Y, k)$ for the full subcategory of odd objects). The parity subcategory is then
\[
\text{Sm}_{\mathbb{F}}^\text{par}(Y, k) = \text{Sm}_{\mathbb{F}}^0(Y, k) \oplus \text{Sm}_{\mathbb{F}}^1(Y, k).
\]

Taking direct limits over closed $Y \subseteq \mathcal{U}$, we obtain $\text{Sm}_{\mathbb{F}}^0(\mathcal{U}, k)$, $\text{Sm}_{\mathbb{F}}^1(\mathcal{U}, k)$, and $\text{Sm}_{\mathbb{F}}^\text{par}(\mathcal{U}, k)$.

The next proposition records some of the main features of parity Smith categories established in [RW22, §7]. Note that for $\mathcal{F} \in \text{Sm}_{\mathbb{F}}(Y, k)$, its support is
\[
\text{supp} \mathcal{F} = \bigcup \{(\mathcal{X}_\alpha^\mathfrak{X})^\mathfrak{X} \subseteq Y : (f_\alpha^\mathfrak{X})^\mathfrak{X}_\alpha \mathcal{F} \neq 0 \text{ or } (f_\alpha^\mathfrak{X})^\mathfrak{X}_\alpha \mathcal{F} \neq 0\}.
\]
PROP. 5.5. Assume that \( \mathcal{X} \) is a partial affine flag variety with \( Y \subseteq \mathcal{X}^{\infty} \) a locally closed finite union of orbits.

1. All of the Smith categories mentioned in Def. 5.4(2) are Krull–Schmidt.
2. If \( \mathcal{F} \in \text{Sm}_{IW}(Y, k) \) is \( \ast \)-even and \( \mathcal{G} \in \text{Sm}_{IW}(Y, k) \) is \( ! \)-odd, then
   \[
   \text{Hom}_{\text{Sm}_{IW}(Y, k)}(\mathcal{F}, \mathcal{G}) = 0.
   \]
3. If \( Z \subseteq Y \) is an open union of \( I^+_\ell \)-orbits, then indecomposable parity Smith objects on \( Y \) are either indecomposable or zero upon restriction to \( Z \).
4. If \( \mathcal{F} \in \text{Sm}^{\text{IW}}_{IW}(Y, k) \) is indecomposable, there is exactly one \( \alpha \in A_+ \) such that \( (\mathcal{X}^+)_\alpha^{\infty} \) is open in \( \text{supp}(\mathcal{F}) \). Conversely, given \( \alpha \in A_+ \), there is (up to isomorphism) a unique even object \( \mathcal{F} \) (respectively, odd object \( \mathcal{F} \)) in \( \text{Sm}^{\text{IW}}_{IW}(Y, k) \) containing \( (\mathcal{X}^+)_\alpha^{\infty} \) as an open subset of its support and restricting to \( \mathcal{L}_{\text{Sm}}^\mathcal{F}(\alpha) \) (respectively, to \( \mathcal{L}_{\text{Sm}}^\mathcal{F}(\alpha)[1] \)).

3. CONSTRUCTING THE ACTION

We need two more results in preparation, for which we specialise to the case \( \mathcal{X} = \mathcal{G} \). As in § 3.2, for \( \lambda = \lambda_0 + \rho^\vee \) and \( \mu = \mu_\omega + \rho^\vee \) we freely use identifications
\[
\mathcal{G}_\rho(\lambda) = \mathcal{F}_{\ell_\ell}^{\rho^\omega} = \mathcal{F}_\ell^\omega = \mathcal{F}_\ell^\omega, \quad \mathcal{G}_\rho(\mu) = \mathcal{F}_{\ell_\ell}^{\rho^\omega} = \mathcal{F}_\ell^\omega = \mathcal{F}_\ell^\omega.
\]

First, an important fact: if \( \kappa, \kappa' \in \mathbf{X}^\vee \) are such that \( \mathcal{G}_\rho^\kappa, \mathcal{G}_\rho^\kappa' \) lie in the same connected component of \( \mathcal{G} \), then their dimensions are of the same parity [Zhu17, § 2.1.11]. Further, if \( \kappa, \kappa' \in \mathbf{X}^\vee_++ \), then the \( I^+ \)-orbits \( \mathcal{G}_\rho^\kappa, \mathcal{G}_\rho^\kappa' \) are of codimension 0 in \( \mathcal{G}_\rho^\kappa, \mathcal{G}_\rho^\kappa' \), respectively; one can see this by combining [Zhu17, § 2.1.5] with the proof of [RW22, Lemma 4.4]. Hence, the aforementioned fact also holds for dominant regular Iwahori orbits: if \( \mathcal{G}_\rho^\kappa, \mathcal{G}_\rho^\kappa' \) are in the same connected component, then their dimensions are of the same parity. For \( \nu \in -\mathbf{a}^\vee \cap \mathbf{X}^\vee \), let \( \mathcal{G}(\nu) \in \mathcal{F}_2 \) be the parity of the dimensions of the dominant regular Iwahori orbits lying in the same connected component of \( \mathcal{G} \) as \( \mathcal{G}_\rho(\nu) \).

Following [RW22, § 7.3], we define \( \text{Sm}^\mathcal{H}_{IW}(\mathcal{G}_\rho^{\infty}, k) \) to be the full subcategory of \( \text{Sm}_{IW}(\mathcal{G}_\rho^{\infty}, k) \) generated by objects whose restriction to \( \mathcal{G}_\rho(\nu) \) is even (respectively, odd) if \( \mathcal{G}(\nu) = 0 \) (respectively, \( \mathcal{G}(\nu) = 1 \)). We then have [RW22, Theorem 7.4] a diagram commuting up to natural isomorphism as follows.

\[
\begin{array}{ccc}
\text{Perv}_{IW}(\mathcal{G}_\rho, k) & \xrightarrow{\cong} & \text{Perv}_{IW, \mathcal{G}_\rho}(\mathcal{G}_\rho, k) \\
\downarrow & & \downarrow \\
\text{Tilt}_{IW}(\mathcal{G}_\rho, k) & \xrightarrow{\cong} & \text{Tilt}_{IW, \mathcal{G}_\rho}(\mathcal{G}_\rho, k)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Sm}_{IW}^{\mathcal{G}_\rho}(\mathcal{G}_\rho^{\infty}, k) & \xrightarrow{\cong} & \text{Sm}_{IW}^{\mathcal{G}_\rho}(\mathcal{G}_\rho^{\infty}, k) \\
\text{Sm}_{IW}^{\mathcal{G}_\rho}(\mathcal{G}_\rho^{\infty}, k) & \xrightarrow{\cong} & \text{Sm}_{IW}^{\mathcal{G}_\rho}(\mathcal{G}_\rho^{\infty}, k)
\end{array}
\]

The equivalences in the first square are the inverse of \( \text{For}_{\mathcal{G}_\rho} \) and its restriction; we denote the composite equivalence along the bottom row by \( \delta^\mathcal{G}_\rho \). The existence of this diagram, and particularly the equivalence \( \delta^\mathcal{G}_\rho \), is a central and ‘miraculous’ result in [RW22]. It implies the existence of the (even or odd) indecomposable objects
\[
\mathcal{E}^\mathcal{G}_\rho(\nu) = \delta^\mathcal{G}_\rho(\mathcal{E}^\mathcal{G}_\rho_{IW, \mathcal{G}_\rho}(\nu)) \in \text{Sm}^{\mathcal{G}_\rho}(\mathcal{G}_\rho^{\infty}, k), \quad \nu \in \mathbf{X}^\vee_++
\]
with \( (\mathcal{G}_\rho^+)^{\infty} \) open in \( \text{supp}(\mathcal{E}^\mathcal{G}_\rho_{IW, \mathcal{G}_\rho}(\nu)) \).

Second, as stated in [RW22, § 7.4], the functor \( Q_{\mathcal{G}_\rho^{\infty}} \) preserves parity objects. This may be proven in a similar fashion to the following lemma, which verifies the analogous fact for the \( \ell \)-thin affine flag variety.

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Lemma 5.6. The functor \( Q = Q_{\tilde{\mathcal{F}}} : D^b_{IW,G_m}(\tilde{\mathcal{F}}, k) \to \text{Sm}_{IW}(\tilde{\mathcal{F}}, k) \) restricts to a functor
\[
\text{Parity}_{IW,G_m}(\tilde{\mathcal{F}}, k) \to \text{Sm}_{IW}^{\text{par}}(\tilde{\mathcal{F}}, k)
\]
preserving even and odd objects.

Proof. Up to shift, an indecomposable object in \( \text{Parity}_{IW,G_m}(\tilde{\mathcal{F}}, k) \) has the form \( \mathcal{F} = \mathcal{E}_{IW,G_m}(w) \), where \( w \in \mathfrak{t}_W \); its parity agrees with that of \( \ell(w) \). The \( \mathfrak{l}^+ \)-orbit \( \tilde{\mathcal{F}}_x \) associated to any \( x \in \mathfrak{t}_W \) supports a nonzero Iwahori–Whittaker local system, and corresponds to the \( \mathfrak{l}^+ \)-orbit \( (\mathfrak{g}^r_{\mathfrak{l},x} \lambda)_{\mathfrak{w}} \) under (5.1). Now \( (j_{x}^{\mathfrak{w}})^{\dagger}_{\text{Sm}} Q_{\mathcal{F}} = Q(j_{x}^{\mathfrak{w}})^{\dagger}_{\mathcal{F}} \), where by assumption \( (j_{x}^{\mathfrak{w}})^{\dagger}_{\mathcal{F}} \) is a direct sum of graded shifts of copies of \( \mathcal{L}_{\mathcal{AS}}^{\ell}(x) \), the parity of those shifts agreeing with that of \( \mathcal{F} \). In view of the relevant uniqueness statement, we have
\[
\mathcal{L}_{\mathcal{AS}}^{\ell}(x) \cong i_{\mathfrak{l} \cap \mathfrak{w}}^* \mathcal{L}_{\mathcal{AS}}^{\mathfrak{g}^r_{\mathfrak{l}}}(x \cdot \mathfrak{w}),
\]
so \( Q(j_{x}^{\mathfrak{w}})^{\dagger}_{\mathcal{F}} \) is a direct sum of graded shifts of copies of \( \mathcal{L}_{\text{Sm}}^{\mathfrak{g}^r}(x \cdot \mathfrak{w}) \). Thus, we see \( Q \) restricts as described, preserving evenness and oddness. □

The proof of [RW22, Proposition 7.6] shows that if \( \mathcal{E}, \mathcal{F} \in \text{Parity}_{IW,G_m}(\mathfrak{g}^r_{\mathfrak{w}}, k) \) have the same parity, then there exist canonical isomorphisms
\[
\text{Hom}_{\text{Sm}_{IW}(\mathfrak{g}^r_{\mathfrak{w}}, k)}(Q_{\mathfrak{g}^r_{\mathfrak{w}}}(\mathcal{E}), Q_{\mathfrak{g}^r_{\mathfrak{w}}}(\mathcal{F})) \cong \mathfrak{k}' \otimes \text{H}_{G_m}(pt,k) \text{Hom}^\bullet_{D^b_{IW,G_m}(\mathfrak{g}^r_{\mathfrak{w}}, k)}(\mathcal{E}, \mathcal{F})
\]
compatible with composition of morphisms in each category, where \( \mathfrak{k}' \) denotes \( k \) viewed as a \( \text{H}_{G_m}(pt,k) \)-module under the map
\[
\text{H}_{G_m}(pt,k) \cong k[x] \rightarrow k, \quad x \mapsto 1.
\]
Moreover, \( Q_{\mathfrak{g}^r_{\mathfrak{w}}} \) is shown to preserve indecomposability of parity objects. For \( \lambda \in -\mathfrak{a}_W \cap X^\vee \), the inclusions \( j_{(\lambda)} : \mathfrak{g}^r(\lambda) \hookrightarrow \mathfrak{g}^r \) induce fully faithful pushforward functors, so for every \( \mathcal{E}, \mathcal{F} \) in \( \text{Parity}_{IW,G_m}(\mathfrak{g}^r(\lambda), k) \) of the same parity, there are canonical isomorphisms
\[
\text{Hom}_{\text{Sm}_{IW}(\mathfrak{g}^r(\lambda), k)}(Q_{\mathfrak{g}^r(\lambda)}(\mathcal{E}), Q_{\mathfrak{g}^r(\lambda)}(\mathcal{F})) \cong \mathfrak{k}' \otimes \text{H}_{G_m}(pt,k) \text{Hom}^\bullet(\mathcal{E}, \mathcal{F}). \quad (5.3)
\]

Proposition 5.7. The category \( \text{Sm}_{IW}^{\text{par}}(\tilde{\mathcal{F}}, k) \) admits a graded right action of \( \mathcal{H} \), such that \( Q = Q_{\tilde{\mathcal{F}}} \) is a graded right \( \mathcal{H} \)-module functor.

Proof. Let \( \widetilde{\text{Parity}}(\tilde{\mathcal{F}}, k) \) be the category whose objects are those of \( \text{Parity}_{IW,G_m}(\tilde{\mathcal{F}}, k) \), with Hom spaces
\[
\text{Hom}_{\widetilde{\text{Parity}}(\tilde{\mathcal{F}}, k)}(\mathcal{E}, \mathcal{F}) = \mathfrak{k}' \otimes \text{H}_{G_m}(pt,k) \bigoplus_n \text{Hom}^{2n}_{\text{Parity}_{IW,G_m}(\tilde{\mathcal{F}}, k)}(\mathcal{E}, \mathcal{F}).
\]
Note \( \widetilde{\text{Parity}}(\tilde{\mathcal{F}}, k) \) is naturally equipped with an autoequivalence [1] whose square is naturally isomorphic to the identity functor. Combining Proposition 5.5(1) and (2), (5.3), and Lemma 5.6, we see that \( Q \) factors on the parity subcategory as follows.
\[
\begin{array}{ccc}
\text{Parity}_{IW,G_m}(\tilde{\mathcal{F}}, k) & \xrightarrow{p} & \widetilde{\text{Parity}}(\tilde{\mathcal{F}}, k) \\
Q & \downarrow & \downarrow Q \\
\text{Sm}_{IW}^{\text{par}}(\tilde{\mathcal{F}}, k) & \xrightarrow{\tilde{Q}} & \text{Sm}_{IW}^{\text{par}}(\tilde{\mathcal{F}}, k)
\end{array}
\]
(5.4)

Here \( P \) is trivial on objects and maps morphisms \( f \) to simple tensors \( 1 \otimes f \), while \( \tilde{Q} \) is essentially surjective and fully faithful, so an equivalence of (graded) categories. By Corollary 4.6, there
is a graded right action of \( \mathcal{H} \) on \( \text{Parity}_{IW, \mathcal{G}_m}(\mathcal{F}_l, k) \) descending to \( \hat{\text{Parity}}(\mathcal{F}_l, k) \); this implies the claim, by transport of structure along \( \hat{Q} \).

Some notation from Proposition 5.1(5): let \( e = e_{\hat{F}l} \) and \( e^s = e_{\hat{F}l}^s \), and let \( e^n \) and \( e^{s,n} \) denote the induced natural isomorphisms \( \text{id} \cong [2n] \), \( n \in \mathbb{Z} \). Note for future reference that if \( \varphi : \mathcal{E} \to \mathcal{F} [2n] \) is a morphism between parity objects, then

\[
\hat{Q}(1 \otimes \varphi) = (e^n_{\mathcal{E}, \mathcal{F}})^{-1}Q(\varphi).
\]

The functor \( Q^s = Q_{\hat{F}l}^s \) also respects parity categories and factors through a similar equivalence \( \hat{Q}^s : \hat{\text{Parity}}(\hat{\mathcal{F}}_l, k) \to \text{Sm}_{IW}^{\text{par}}(\hat{\mathcal{F}}_l^s, k) \).

Our understanding of the \( \mathcal{H} \)-module structure of \( \text{Sm}_{IW}^{\text{par}}(\mathcal{F}_l, k) \) will hinge on the action of objects \( B_s \in \mathcal{H} \), for \( s \in S \). In view of Corollary 4.6, \( \mathcal{A}v \) is a right \( \mathcal{H} \)-module functor, so

\[
((-) \cdot B_s) \circ \mathcal{A}v = \mathcal{A}v \circ ((-) \star \mathcal{E}(s))
\]

and

\[
(q^s)^*(q^s)_* [1] = (q^s)^*(q^s)_*[1] \quad (5.5)
\]

and

\[
(q^s)^*(q^s)_*[1] \circ \mathcal{A}v; \quad (5.6)
\]

here in (5.5) we rely on [RW18, Lemma 9.4.2] and in (5.6) we use that \( \mathcal{A}v \) commutes with the functor \( (q^s)^*(q^s)_* \) (see the proof of [RW18, Corollary 11.2.3]). But \( \mathcal{A}v \) is a quotient functor, so we conclude that \( -(\cdot) \cdot B_s = (q^s)^*(q^s)_*[1] \), i.e. that \( B_s \) acts on \( \text{Parity}_{IW, \mathcal{G}_m}(\mathcal{F}_l, k) \) through the push–pull composite \( (q^s)^*(q^s)_*[1] \). This can also be checked directly from the definition of convolution.

Now, \( (q^s)_* \) and \( (q^s)^* \equiv (q^s)^*[-2] \) respect parity objects, so \( (q^s)^*_{\text{Sm}} \) and \( (q^s)^*_{\mathcal{G}_m} \) restrict to functors between the parity Smith categories associated to \( \mathcal{F}_l \) and \( \mathcal{F}_l^s \). Since they respect gradings, \( (q^s)_* \) and \( (q^s)^* \) also induce functors

\[
\hat{(q^s)_*} : \hat{\text{Parity}}(\hat{\mathcal{F}}_l, k) \to \hat{\text{Parity}}(\hat{\mathcal{F}}_l^s, k), \quad \hat{(q^s)^*} : \hat{\text{Parity}}(\hat{\mathcal{F}}_l^s, k) \to \hat{\text{Parity}}(\hat{\mathcal{F}}_l, k).
\]

**Proposition 5.8.** The object \( B_s \in \mathcal{H} \) acts on \( \text{Sm}_{IW}^{\text{par}}(\mathcal{F}_l, k) \) by \( (q^s)^*_{\text{Sm}} (q^s)^*_{\mathcal{G}_m} [1] \).

**Proof.** Given the action’s construction, we just need to verify that the following two squares commute up to natural isomorphism.

\[
\begin{array}{ccc}
\text{Parity}(\mathcal{F}_l, k) & \overset{(q^s)^*}{\longrightarrow} & \text{Parity}(\mathcal{F}_l^s, k) \\
\downarrow \hat{Q} & & \downarrow \hat{Q} \\
\text{Sm}_{IW}^{\text{par}}(\mathcal{F}_l, k) & \overset{(q^s)^*_{\text{Sm}}}{\longrightarrow} & \text{Sm}_{IW}^{\text{par}}(\mathcal{F}_l^s, k)
\end{array}
\]

The adjunction \( (q^s)^* \dashv (q^s)_* \) yields adjunctions \( \hat{(q^s)^*} \dashv (q^s)_* \) and \( (q^s)^*_{\text{Sm}} \dashv (q^s)^*_{\mathcal{G}_m} \), so by the uniqueness properties of adjoints it suffices to check the case of \( (q^s)^* \), i.e. to check that both ways of traversing the second diagram agree. This is clear on objects, so take a morphism \( f \in \text{Hom}_{\text{Parity}(\mathcal{F}_l, k)}(\mathcal{E}, \mathcal{F}) \), which by linearity we may assume to be a simple tensor of the form \( f = 1 \otimes \varphi \), with \( \varphi : \mathcal{E} \to \mathcal{F} [2n] \). By repeated application of Lemma 5.3, we see that

\[
(q^s)^*_{\text{Sm}} e^{s,n} = e^n (q^s)^*_{\mathcal{G}_m} \iff (e^n)^{-1} (q^s)^*_{\text{Sm}} = (q^s)^*_{\mathcal{G}_m} (e^{s,n})^{-1}.
\]
Then
\[
\hat{Q}^s((q^n)^*(f)) = \hat{Q}^s(1 \otimes (q^n)^* \varphi) = (e_{\hat{Q}((q^n)^* \varphi)[2n]}^n)^{-1}Q^s((q^n)^* \varphi) \\
= (e_{\hat{Q}((q^n)^* \varphi)[2n]}^n)^{-1}(q^n)^*\text{Sm}Q(\varphi) \\
= (q^n)^*\text{Sm}(e_{\hat{Q}((q^n)^* \varphi)[2n]}^n)^{-1}Q(\varphi) = (q^n)^*\text{Sm}\hat{Q}(f),
\]
as was required to be shown. \qed

6. Bridge to representation theory

6.1 Blocks and their functors

Our goal is now to investigate how the \(H\)-action on \(\text{Sm}_{\text{Pr}}^\text{par}((\mathcal{F}, \mathbb{k}))\) transfers across the equivalence \(\hat{i}^2\) from (5.2) and the Iwahori–Whittaker version of the geometric Satake equivalence.

Recall from the proof of [RW22, Theorem 8.5] that there is a decomposition
\[
\text{Tilt}_{\text{Iw}}(\mathcal{G}_r, \mathbb{k}) = \bigoplus_{\nu \in \mathcal{X}} \text{Tilt}_{\text{Iw}}^\nu,
\]
where \(\text{Tilt}_{\text{Iw}}^\nu\) consists of direct sums of objects \(\mathcal{F}_{\text{Iw}}(\kappa)\) for \(\kappa \in (\mathcal{W} \cap \nu) \cap \mathcal{X}_{+}\). Given [RW22, (5.2)], we have an isomorphism \(\mathcal{F}_{\text{Iw}}(\kappa) \cong \mathcal{E}^\text{Gr}_{\text{Iw}}(\kappa)\) in \(D^b_{\text{Iw}}(\mathcal{G}_r, \mathbb{k})\), so
\[
i^2(\mathcal{F}_{\text{Iw}}(\kappa)) = i^2(\mathcal{E}^\text{Gr}_{\text{Iw}}(\kappa)) = \mathcal{E}^\text{Gr}_{\text{Sm}}(\kappa)
\]
by [RW22, (7.1)]; hence, \(\text{Sm}^\nu_{\text{Iw}}(\mathcal{G}_r^\infty, \mathbb{k})\) decomposes into blocks \(\text{Sm}^\nu_{\text{Iw}}\) consisting of the direct sums of objects \(\mathcal{E}^\text{Gr}_{\text{Sm}}(\kappa)\) for \(\kappa \in (\mathcal{W} \cap \nu) \cap \mathcal{X}_{+}\).

Now, \(\text{Sm}^\nu_{\text{Iw}} = \text{Sm}^\nu_{\text{Iw}}(\mathcal{G}_r(\nu), \mathbb{k})\) by the discussion in §5.3, so
\[
\text{Sm}^\text{par}_{\text{Iw}}(\mathcal{G}_r(\nu), \mathbb{k}) = \text{Sm}^\nu_{\text{Iw}} \oplus \text{Sm}^\nu_{\text{Iw}}[1].
\]

We also have the following commutative diagram with \(j(\nu) : \mathcal{G}_r(\nu) \hookrightarrow \mathcal{G}_r^\infty\).
\[
\begin{array}{ccc}
\text{Sm}^\nu_{\text{Iw}}(\mathcal{G}_r^\infty, \mathbb{k}) & \xrightarrow{\mathbf{pr}_\nu} & \text{Sm}_{\text{Iw}}(\mathcal{G}_r^\infty, \mathbb{k}) \\
\text{Sm}^\nu_{\text{Iw}} & \xleftarrow{j(\nu)^*} & \text{Sm}_{\text{Iw}}(\mathcal{G}_r(\nu), \mathbb{k})
\end{array}
\]

This is an immediate consequence of the fact that \(\text{Sm}^\nu_{\text{Iw}}(\mathcal{G}_r^\infty, \mathbb{k})\) is a Krull–Schmidt category whose indecomposable objects are each supported in a single connected component of \(\mathcal{G}_r^\infty\).

At the same time, the Iwahori–Whittaker version of geometric Satake equivalence sends \(\mathcal{T}(w \circ i \nu')\) to \(\mathcal{F}_{\text{Iw}}(w \circ i \nu')\), where \(w \in \mathcal{W}, \nu' \in \mathcal{C}_e\), and \(\nu = \nu' + \varsigma\); see the proof of [RW22, Theorem 8.8]. In particular, for \(s \in S, \lambda = \lambda_0 + \varsigma, \) and \(\mu = \mu_s + \varsigma,\) if we let \(\text{Sm}_{\text{Iw}} = \text{Sm}^\lambda_{\text{Iw}}\) and \(\text{Sm}^\nu_{\text{Iw}} = \text{Sm}^\nu_{\text{Iw}}\), then
\[
\text{Tilt}_0(G) \cong \text{Sm}_{\text{Iw}}, \quad \text{Tilt}_s(G) \cong \text{Sm}^s_{\text{Iw}}.
\]
The functors \((q^n)^*\) and \((q^n)^* \cong (q^n)^*[-2]\) preserve even objects [RW18, Lemma 9.4.2(2)], so Lemma 5.6 implies the existence of an endofunctor
\[
(q^n)^*_{\text{Sm}}(q^n)^*_{\text{Sm}} : \text{Sm}_{\text{Iw}} \to \text{Sm}_{\text{Iw}}.
\]
Now, (6.2) with \(\nu = \lambda\) shows that the degrading of \(\text{Sm}^\text{par}_{\text{Iw}}(\mathcal{F}, \mathbb{k})\) is equivalent to \(\text{Sm}_{\text{Iw}}\). Moreover, the operation of de-grading is 2-functorial in the sense that it sends graded functors between graded categories to functors between their de-gradings in a way that respects functor composition and appropriate natural transformations. In particular, since an action of \(H\) can be
described by a monoidal functor to an endofunctor category, we obtain the following corollary to Proposition 5.8.

**Proposition 6.1.** There is a right action of \( \mathcal{H} \) on \( \mathcal{S}_{IW} \), with \( B_s(n) \) acting by \( (q^s)_*\mathcal{S}_m(q^s)_*\mathcal{S} \).

A major part of what remains is to compare this endofunctor to the wall-crossing functor \( \theta_s = T_sT^s : \text{Tilt}_0(G) \to \text{Tilt}_s(G) \to \text{Tilt}_0(G) \).

### 6.2 Translation functors and fixed points on the affine Grassmannian

Recall that for \( \lambda_0, \mu_s \) as above, there is a unique \( \gamma \in X'_+ \cap W(\mu_s - \lambda_0) \), and we can define

\[
T^s : \text{Tilt}_0(G) \to \text{Tilt}_s(G), \quad M \mapsto \text{pr}_{\mu_s}(M \otimes T(\gamma)).
\]

Remembering that convolution affords an action of \( D^b_{L+G}(G, k) \) on \( D^b_{IW}(G, k) \), we see that the \([BGM+19]\) version of the Satake equivalence and the lower left equivalence in (5.2) yield a diagram of categories (commuting up to natural isomorphism),

\[
\begin{array}{ccc}
\text{Tilt}_0(G) & \xrightarrow{\cong} & \text{Tilt}_0^{IW, \mathcal{G}_m} \\
{\downarrow}T^s & & \downarrow{T_s} \\
\text{Tilt}_s(G) & \xrightarrow{\cong} & \text{Tilt}_s^{IW, \mathcal{G}_m}
\end{array}
\]

with \( T^s_0 = \text{pr}_{\mu_s}((-) \ast \mathcal{F}(\gamma)) \) and \( T^s_0 \mathcal{G}_m \) defined similarly, with respect to the action of \( \text{Tilt}(G, k) \) on \( \text{Tilt}_{IW, \mathcal{G}_m}(G, k) \) arising from the aforementioned equivalences. (In this diagram, the categories in the second and third columns are defined to be the essential images of the categories in the first column under the equivalences.) Naturally, then, our interest turns to the functor \((-) \ast \mathcal{F}(\gamma)\). To analyse it, we need the following technical lemma connecting objects in constructible derived categories with intersection cohomology complexes on their supports. In its proof, we encounter the *recollement* (‘gluing’) situation, as explained in \([hLa20]\).

**Lemma 6.2.** Assume \( X \) is a stratified ind-variety,

\[
X = \bigsqcup_{\zeta \in \Lambda} X^\zeta,
\]

where the strata \( X^\zeta \) are locally closed and simply connected varieties, such that

\[
\overline{X^\zeta} = \bigsqcup_{\zeta' \leq \zeta} X^{\zeta'}.
\]

Let \( J \subseteq \Lambda \) be a finite downward-closed partially ordered subset and write \( X^J \) for the disjoint union of the \( X_\zeta, \zeta \in J \). If \( \mathcal{F} \in D^b_\Lambda(X) \) is supported on \( X^J \), then \( \mathcal{F} \in D(J) = \langle \mathcal{I}(\zeta) : \zeta \in J \rangle_\Delta \), the full triangulated subcategory generated by the \( \mathcal{I}(\zeta) \). This coincides with the full triangulated subcategory generated by the tilting objects \( \mathcal{F}(\zeta), \zeta \in J \).

**Proof.** In the following, the symbols \( p_\tau \) and \( p_\mathcal{H} \) refer to perverse truncation and perverse cohomology functors on the constructible derived category \( D^b_\Lambda(X) \), respectively. The second claim follows from the first by \([Ric16, \text{Proposition 7.17}]\), so we need only prove the first claim. For this, we use induction on \( n = |J| \).

If \( n = 1 \), then \( J = \{ \zeta \} \) with

\[
X^\zeta = \overline{X^\zeta}.
\]

Let \( i \) be maximal with \( p_\mathcal{H}^i(\mathcal{F}) \neq 0 \), so there is a distinguished triangle in \( D^b_\Lambda(X) \),

\[
p_{\tau_{i-1}} \mathcal{F} \to \mathcal{F} \to p_\mathcal{H}^i(\mathcal{F})[-i] \xrightarrow{[1]}.
\]
Now if \( i' \) is maximal such that \( p_{\mathcal{X}^i}(p_{\tau < i} \mathcal{F}) \neq 0 \), then by construction \( i' < i \); induction on \( i \) now settles the case \( n = 1 \).

Suppose now that \( n > 1 \) and choose \( \zeta \in \mathcal{I} \) maximal. Then

\[
X^j - X^\zeta = \bigsqcup_{\zeta' \in \mathcal{I}} X^\zeta' - X^\zeta = \bigsqcup_{\zeta' \neq \zeta' \in \mathcal{I}} X^\zeta';
\]

since \( \overline{X^{\zeta'}} \) is a union of strata \( X^{\zeta''} \) with \( \zeta'' < \zeta' \), the maximality of \( \zeta \) shows that

\[
\bigsqcup_{\zeta' \neq \zeta' \in \mathcal{I}} X^\zeta' = \bigsqcup_{\zeta' \neq \zeta' \in \mathcal{I}} \overline{X^{\zeta'}}
\]

is closed (by finiteness of \( \mathcal{I} \)). Hence we have a recollement situation,

\[
X^\zeta \hookrightarrow X^j \hookrightarrow X^3 - X^\zeta;
\]

call these inclusions \( j \) and \( i \), respectively. Now, by assumption, \( \mathcal{F} \) is the pushforward of some \( \mathcal{F}' \in D^b_{X^j}(X^3) \). Accordingly, we can form a distinguished triangle

\[
j_! j^! \mathcal{F}' \to \mathcal{F}' \to i_* i^* \mathcal{F}' \left[ 1 \right].
\]

Note that \( i_* i^* \mathcal{F}' \) is supported on \( X^j \), where \( \mathcal{J} = \{ \zeta' \in \mathcal{J} : \zeta' < \zeta \} \) is a proper subset of \( \mathcal{I} \), so by induction \( i_* i^* \mathcal{F}' \in D(\mathcal{J}) \). We reduce to showing \( j_! j^! \mathcal{F}' \in D(\mathcal{J}) \). Note that \( j_! j^! \mathcal{F}' \) is a sheaf on \( X^\zeta \), so by the \( n = 1 \) case above (with \( X \) replaced by \( X^\zeta \) and \( \Lambda \) and \( \mathcal{J} \) by \( \{ \zeta' \} \) we can conclude that

\[
j_! j^! \mathcal{F}' \in (\mathbb{K}_{X^\zeta}[d_\zeta])_\Delta,
\]

and hence \( j_! j^! \mathcal{F}' \in (j_! \mathbb{K}_{X^\zeta}[d_\zeta])_\Delta \). Thus, it will suffice to prove that \( j_! \mathbb{K}_{X^\zeta}[d_\zeta] \in D(\mathcal{J}) \). Since \( j_! \) is right \( t \)-exact, \( j_! \mathbb{K}_{X^\zeta}[d_\zeta] \in D^b \Delta^0 \). Now we have a distinguished triangle,

\[
p_{\tau < 0}(j_! \mathbb{K}_{X^\zeta}[d_\zeta]) \to j_! \mathbb{K}_{X^\zeta}[d_\zeta] \to p_{\tau \geq 0}(j_! \mathbb{K}_{X^\zeta}[d_\zeta]) \left[ 1 \right].
\]

Using the identification \( j^! \mathbb{K}_{X^\zeta} \cong \mathbb{K}_{X^\zeta} \) as well as the adjunctions \( (j_!, j^!) \) and \( (p_{\tau < 0}, j_!, j^!) \), we can see that

\[
j^* j_! \mathbb{K}_{X^\zeta} = j^! j_! \mathbb{K}_{X^\zeta} = j^! j^! \mathbb{K}_{X^\zeta} = j^! \mathbb{K}_{X^\zeta} = j^!(p_{\tau < 0})j_! \mathbb{K}_{X^\zeta} = j^!(p_{\tau < 0})\mathbb{K}_{X^\zeta}.
\]

The upshot is that \( p_{\tau < 0}(j_! \mathbb{K}_{X^\zeta}[d_\zeta]) \) is supported on \( \mathcal{J} - \{ \zeta \} \), so by the inductive assumption on \( n \) we reduce to considering \( p_{\tau < 0}(j_! \mathbb{K}_{X^\zeta}[d_\zeta]) \). This is a perverse sheaf, for which the result is well known. \( \square \)

From the restriction isomorphism \((j^\gamma)^* \mathcal{F}(\gamma) \cong \mathbb{K}_{Gr^\gamma}[\dim(Gr^\gamma)]\) and the adjunction isomorphism

\[
\text{Hom}((j^\gamma)^* \mathcal{F}(\gamma), \mathbb{K}_{Gr^\gamma}[\dim(Gr^\gamma)]) \cong \text{Hom}(\mathcal{F}(\gamma), \mathcal{K}_\gamma),
\]

for \( \mathcal{K}_\gamma = (j^\gamma)^* \mathbb{K}_{Gr^\gamma}[\dim(Gr^\gamma)] \), we produce a distinguished triangle in \( D^b_{L^+ G}(Gr, \mathbb{K}) \),

\[
C \to \mathcal{F}(\gamma) \to \mathcal{K}_\gamma \left[ 1 \right]. \quad (6.5)
\]

Evidently \((j^\gamma)^* C = 0 \), so \( C \) is supported on the union of the \( Gr^\zeta \) with \( \zeta < \gamma \) and by Lemma 6.2 we infer that \( C \in (\text{IC}(\zeta) : \zeta < \gamma)_\Delta \). This control on \( C \) will be shown to justify a final shift of attention, to the functor

\[
(-)^* \mathcal{K}_\gamma : D^b_{IW}(Gr, \mathbb{K}) \to D^b_{IW}(Gr, \mathbb{K}).
\]

For the remainder of this section, we let

\[
\mathcal{Y}_\gamma = LG \times L^+ G \to \mathcal{Y} = LG \times L^+ G, \quad \mathcal{Y} = Gr \times Gr.
\]
Proposition 6.3. Consider the following diagram:

\[
\mathcal{F} \xrightarrow{\pi} \mathcal{Y}^\gamma \xrightarrow{m} \mathcal{E}^r
\]

where \( \pi \) is a projection and \( m \) is induced by multiplication. There is a natural isomorphism,

\[
\mathcal{F} \star \mathcal{K}_\gamma \cong m_\ast \pi_\ast \mathcal{F}, \quad \mathcal{F} \in D^b_{IW}(\mathcal{E}^r, k).
\]

Proof. Suppose \( \mathcal{F} \) is supported on \( X \subseteq \mathcal{E}^r \) a locally closed finite union of \( L^+G \)-orbits, and let \( \tilde{X} \) denote the preimage of \( X \) under a suitable quotient \( p : (LG)/N \rightarrow \mathcal{E}^r \). Then we calculate \( \mathcal{F} \star \mathcal{K}_\gamma \) using the diagram

\[
\begin{array}{ccc}
X \times \mathcal{E}^r & \xrightarrow{p \times 1} & \tilde{X} \times L^+G \mathcal{E}^r \\
\downarrow{\text{pr}_1} & & \downarrow{\pi'} \\
\tilde{X} & \xrightarrow{p} & X
\end{array}
\]

where \( \pi' \) is the naturally induced map. Then

\[
q^* \pi^* \mathcal{F} \cong \text{pr}^*_1 p^* \mathcal{F} = p^* \mathcal{F} \boxtimes \mathcal{K}_\gamma,
\]

so that \((\pi')^* \mathcal{F} \cong \mathcal{F} \boxtimes \mathcal{K}_\gamma\). The following diagram also commutes, with a cartesian square.

\[
\begin{array}{ccc}
\tilde{X} \times L^+G \mathcal{E}^r & \xrightarrow{\epsilon} & \mathcal{Y}_\ast \\
\downarrow{\pi'} & & \downarrow{\pi} \\
X & \xrightarrow{\epsilon'} & \mathcal{E}^r
\end{array}
\]

Hence, \( m_\ast \pi_\ast \epsilon'_* \mathcal{F} \cong m_\ast j_\ast \pi_\ast \mathcal{F} \cong m_\ast (\pi')^* \mathcal{F} = \mathcal{F} \star \mathcal{K}_\gamma \), using smooth base change. \( \Box \)

It will become convenient to work with an untwisted version of the functor \( m_\ast \pi^* \). As mentioned in the proof of \[MV07, \text{Lemma 4.4}]\), the product of projection and multiplication yields an isomorphism of ind-schemes \( \pi \times m : \mathcal{Y}_\ast \xrightarrow{\sim} \mathcal{Y} \), in the notation of Proposition 6.3 and (6.6); it is \( \mathbb{G}_m \)-equivariant, as \( \pi \) and \( m \) are. Write \( \mathcal{Y}^\gamma \subseteq \mathcal{Y} \) for the image of \( \pi \times m \) restricted to \( \mathcal{Y}_\ast^\gamma \). Then we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}^r & \xrightarrow{\phi_1} & \mathcal{Y}_\ast^\gamma \\
\downarrow{\phi_2} & \xrightarrow{\phi_2} & \downarrow{\phi_2} \\
\mathcal{E}^r \times \mathcal{E}^r & \xrightarrow{m} & \mathcal{E}^r
\end{array}
\]

where the \( \phi_i \) are projection maps, and \( \mathcal{Y}^\gamma \) is stable under the natural diagonal action of \( I^+ \) (and even \( L^+G \)) on \( \mathcal{E}^r \times \mathcal{E}^r \).

6.3 Translation functors in Smith theory

Ultimately, we will show that \((\phi_2)_\ast \phi_1^* \) induces an action in Smith theory which coincides with \((q')_\ast^\gamma \mathcal{Y}_\ast^\gamma \mathcal{S} \); the next proposition facilitates the first step towards this goal.
Proposition 6.4. The following diagram commutes up to natural isomorphism.

\[
\begin{array}{ccc}
D^b_{IW, G_m}(\mathcal{S}) & \xrightarrow{i_*} & D^b_{IW, G_m}((\mathcal{G}^\gamma)) & \xrightarrow{(\phi_2)_*} & D^b_{IW, G_m}(\mathcal{S}) \\
\downarrow{i_*} & & \downarrow{i_*} & & \downarrow{i_*} \\
\text{Sm}_{IW}(\mathcal{S}^\omega) & \xrightarrow{(\phi_2)_*} & \text{Sm}_{IW}(\mathcal{G}^\gamma)^\omega & \xrightarrow{(\phi_2)_*} & \text{Sm}_{IW}(\mathcal{S}^\omega)
\end{array}
\]

Proof. The left square commutes due to functoriality as in Proposition 5.1(4):
\[
(\phi_1^\omega)_* i_{G_r}^* = (\phi_1^\omega)_* Q(i_{G_r} \circ \phi_1^\omega)^* = Q(\phi_1 \circ i_{G_r})^* = i_{G_r}^* \circ \phi_1^*.
\]

For the right square, we prove a general base change result. Let \( f : A \to B \) be a \( \varpi \)-equivariant quasi-separated morphism, where \( A \subseteq \mathcal{G}^\gamma \) and \( B \subseteq \mathcal{S} \) are closed finite unions of \( I^+ \)-orbits. We claim the following commutes.

\[
D^b_{IW, G_m}(A) \xrightarrow{f_*} D^b_{IW, G_m}(B)
\]

Indeed, suppose \( P \) is the pullback in a cartesian square as follows.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow{i} & & \uparrow{j} \\
P & \xrightarrow{f'} & B^\varpi
\end{array}
\]

The universal \( a : A^\varpi \to P \) is a closed immersion, so we have a recollement situation \( A^\varpi \to P \to \mathcal{U} \), where \( j : \mathcal{U} \to P \) has a free \( \varpi \)-action. Note that the quotient scheme \( U/\varpi \) exists, since \( U \) is quasi-projective over \( F \); see [RW22, Remark 2.2]. Now \( f' \circ j : U \to B^\varpi \) is \( \varpi \)-equivariant, so factors as \( j' \circ q_U : U \to U/\varpi \to B^\varpi \). By the proof of [RW22, Proposition 2.6], we see that \( (f' \circ j)_* H \) has perfect geometric stalks if \( H \) is an object in \( D^b_{\mathcal{U}, G_m}(U) \). After composing with \( f'_* \), a distinguished triangle in \( D^b_{IW, G_m}(P) \),

\[
a_i a_i^1 F \to F \to j_* j^* F \simeq [1],
\]

becomes a distinguished triangle in \( D^b_{IW, G_m}(B^\varpi) \),

\[
f'_* a_i a_i^1 F \to f'_* F \to (f' \circ j)_* j^* F \simeq [1],
\]

where \( f'_* a_i a_i^1 F = f'_* a_i^1 F \). Take Smith quotients, thinking of \( H = \text{Res}^G_{G_m}(j_* F) \):

\[
(f^\varpi)^*_a Q(a_i^1 F) \cong Q_B(f'_* F).
\]

If we assume \( F = (i')^! E \), so that \( a_i^1 F = i_i^! E \), we are left with

\[
(f^\varpi)^*_a i_i^! E \cong Q_B(f'_* i_i^! E) \cong Q_B(i_B f_* E) = i_B^! f_* E
\]

as desired; hence, (6.7) commutes. Now the projection \( \phi_2 \) is approximated by quasi-separated morphisms between closed finite unions of \( I^+ \)-orbits, so we can take a direct limit over fixed points to obtain the result. \( \square \)
HECKE CATEGORY ACTIONS VIA SMITH–TREUMANN THEORY

To bring the Smith categories associated with partial affine flag varieties into our calculation, we work with certain pullback ind-schemes \( \mathcal{W} \) and \( \mathcal{Z} \):

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\eta} & (\mathcal{Y}^\gamma)^\omega \\
\downarrow^u & & \downarrow^w \\
\mathcal{F}_H^x & \xleftarrow{h} & \mathcal{G}_u^\omega,
\end{array}
\quad
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\theta} & \mathcal{W} \\
\downarrow^z & & \downarrow^w \\
\mathcal{F}_L^x & \xleftarrow{h} & \mathcal{G}_r^\omega,
\end{array}
\quad (6.8)
\]

here \( h_\ell, h_\ell^s \) correspond to \( j(\lambda), j(\mu) \) when identifying \( \mathcal{F}_H^x \cong \mathcal{G}_r(\lambda), \mathcal{F}_L^x \cong \mathcal{G}_r(\mu) \).

**Proposition 6.5.** The graph \( \mathcal{Z} \) identifies with the graph of \( q^a : \mathcal{F}_H^x \to \mathcal{F}_L^x \).

**Proof.** It is easy to verify formally that

\[ \mathcal{Z} = (\mathcal{Y}^\gamma)^\omega \cap \mathcal{F}_H^x \times \mathcal{F}_L^x. \]  

(6.9)

Now, a typical point of \( \mathcal{Y}^\gamma \) has the form \([g, L_\gamma]\), since \([h, h'L_\gamma] = [hh', L_\gamma]\) for \( h \in LG, h' \in L^+G \). Then the points of \( \mathcal{Y}^\gamma \) have the form \((\pi, m)[g, L_\gamma] = (gL_0, gL_\gamma)\), or equivalently the form \((gL_\lambda, gL_\mu)\) with \( g \in LG \). Accordingly, by (6.9), we see that the points of \( \mathcal{Z} \) have the form \((fL_\lambda, hL_\mu)\) for \( f, h \in L_\ell G \), where there is some \( g \in LG \) such that

\[ fL_\lambda = gL_\lambda, \quad hL_\mu = gL_\mu. \]

Phrased differently, we require \( fz^\lambda \alpha z^{-\lambda} = hz^\mu \beta z^{-\mu} \) for some \( \alpha, \beta \in L^+G \) (the common value being \( g \in LG \)). Hence, the fiber of the first projection \( \pi : \mathcal{Z} \to \mathcal{F}_L^x \) over a fixed \( fL_\lambda \), for \( f \in L_\ell G \), can be identified with the points \( hL_\mu \in \mathcal{F}_H^x \) for \( h \in fz^\lambda L^+Gz^{-\lambda} L^+Gz^{-\mu} \cap L_\ell G \), or equivalently with the quotient

\[ \frac{fH_\lambda H_\mu \cap L_\ell G}{H_\mu \cap L_\ell G} \cong \frac{H_\lambda H_\mu \cap L_\ell G}{H_\lambda \cap H_\mu} = F, \]

writing \( H_\nu = z^\nu L^+Gz^{-\nu} \) and letting \( H_\mu \cap L_\ell G \) act by multiplication on the right. Of course, it will be sufficient to show that the fiber \( F = \{1\} \). Note first that we have \( \mathbb{G}_m \)-equivariant maps

\[ F \hookrightarrow \frac{H_\lambda H_\mu}{H_\mu} \cong \frac{H_\lambda}{H_\lambda \cap H_\mu} = H_\lambda \cdot L_\mu \subseteq \mathcal{G}_r. \]

Thus, \( F \hookrightarrow (H_\lambda \cdot L_\mu)^\omega \). To calculate the latter, note first that by (3.1),

\[ H_\lambda \cdot L_\mu = \bigsqcup_{\nu \in W(\mu - \lambda) + \lambda} I_\lambda \cdot L_\nu, \quad I_\lambda = z^\lambda I z^{-\lambda}. \]

For \( \alpha \in \Phi \), let \( \delta_\alpha \in \mathbb{Z} \) be 1 or 0 according to whether \( \alpha \in \Phi_+ \) or not. Then recall [RW22, Lemma 4.8] that we have a \( \mathbb{G}_m \)-equivariant isomorphism

\[ I_\nu^\omega = \prod_{\alpha} \prod_{\delta_\alpha \leq m < \langle \alpha, \nu \rangle} U_{\alpha + m \delta} \xrightarrow{\sim} \mathcal{G}_r, \quad x \mapsto xL_\nu. \]

Accordingly, the same formula defines an equivariant isomorphism

\[ I_\nu^{\lambda, \mu} = \prod_{\alpha} \prod_{\delta_\alpha + \langle \alpha, \lambda \rangle \leq m < \langle \alpha, \nu \rangle} U_{\alpha + m \delta} \xrightarrow{\sim} I_\lambda \cdot L_\nu. \]

Hence, taking \( \omega \)-fixed points,

\[ (I_\nu^{\lambda, \mu})_w = \prod_{\alpha} \prod_{\delta_\alpha + \langle \alpha, \lambda \rangle \leq m < \langle \alpha, \nu \rangle} U_{\alpha + m \delta} \xrightarrow{\sim} (I_\lambda \cdot L_\nu)_w. \]  

(6.10)
Now, let \( \nu = w(\mu - \lambda) + \lambda \) for \( w \in W_f \), and suppose 
\[
u L_\nu = uz^\nu L^+ G = uz^{\nu - \mu} L_\mu \in F, \quad \text{for } u \in (I_u^{\lambda,\nu})^\omega \subseteq L_\ell G.
\]
Then there is \( h_\lambda h_\mu \in H_\lambda H_\mu \cap L_\ell G \) with \( z^{\mu - \nu} h_\lambda h_\mu \in H_\mu \). This element also belongs to \( z^{\mu - \nu} L_\ell G \) and hence to the intersection \( H_\mu \cap z^{\mu - \nu} L_\ell G \). However, if \( \nu \neq \mu \), then we claim \( H_\mu \cap z^{\mu - \nu} L_\ell G \) is empty. By translation, it suffices to prove that 
\[
z_\nu L^+ G \cap L_\ell G z_\mu = \emptyset.
\]
If we suppose otherwise, then by acting on \( L_0 \in \mathcal{G}r \) we find that \( L_\nu \in \mathcal{F}_{\ell} \). By [RW22, Remark 4.9], this implies that \( \nu \in W_{\ell \mu} \) and, hence, that \( u(\mu) = w(\mu - \lambda) + \lambda \) for some \( u \in W \). Rewriting,
\[
u \cdot \mu_\ell = w(\mu_\ell - \lambda_0) + \lambda_0.
\]
By [Jan03, Lemma II.7.7], this forces \( \mu_\ell = u \cdot \mu_\ell \), so that \( w(\mu - \lambda) = \mu - \lambda \), contradicting the assumption \( \nu \neq \mu \). Thus we reduce to \( \nu = \mu \), in which case we have \( 0 < \langle \alpha, \lambda \rangle < \ell \) and \( 0 \leq \langle \alpha, \mu \rangle \leq \ell \) for all \( \alpha \in \Phi_+ \); using (6.10), these inequalities imply \( (I_u^{\lambda,\mu})^\omega = 1 \) and thus that \( F = F \cap (I_\lambda \cdot L_\mu)^\omega = \{1\} \). □

We can thus combine the cartesian squares in (6.8) into one commutative diagram with \( q^\omega \).

Now let us perform the main calculation, showing that Smith restriction intertwines translation onto the wall and pushforward in Smith theory as follows.

\[
\begin{array}{ccc}
\mathcal{G}_f \oplus & \xrightarrow{\phi_\ell^\omega} & \mathcal{G}_f^\omega \\
\downarrow \eta & & \downarrow h_\ell^\omega \\
\mathcal{G}_f \oplus & \xleftarrow{w} & \mathcal{F}_{\ell}^s \\
\downarrow h_\ell & & \downarrow z^* \\
\mathcal{F}_{\ell} & \xrightarrow{z} & \mathcal{F}_{\ell}^s
\end{array}
\] (6.11)

Now, convolving the distinguished triangle (6.5) on the left by \( F \), we obtain
\[
F \star C \to F \star \mathcal{F}(\gamma) \to F \star \mathcal{K}_\gamma \xrightarrow{[1]}
\]
(6.12)
where \( F \star C \) belongs to the triangulated category generated by objects \( F \star \mathcal{F}(\zeta) \) for \( \zeta < \gamma \), by Lemma 6.2. It follows from the (equivariant) Iwahori–Whittaker version of the geometric Satake equivalence and [Jan03, Remark 7.7] that \( pr_\mu(F \star \mathcal{F}(\zeta)) = 0 \) in \( \text{Tilt}_{IW, gm} \), so that \( (h_\ell^s)^* \text{Sm}_{\ell}^s i_{\ell}^s (F \star \mathcal{F}(\zeta)) = 0 \). Accordingly, since \( (h_\ell^s)^* \text{Sm}_{\ell}^s i_{\ell}^s \) is a triangulated functor,
(h^*_\ell)^\ast_{\text{Sm}} i^\ast_\text{Sm} (\mathcal{F} \ast \mathcal{C}) = 0 and (6.12) yields a natural isomorphism

\[(h^*_\ell)^\ast_{\text{Sm}} i^\ast_\text{Sm} (\mathcal{F} \ast \mathcal{I}(\gamma)) \cong (h^*_\ell)^\ast_{\text{Sm}} i^\ast_\text{Sm} (\mathcal{F} \ast \mathcal{K}_\gamma).\]

Continuing from here, keeping in mind diagram (6.11),

\[(h^*_\ell)^\ast_{\text{Sm}} i^\ast_\text{Sm} (\mathcal{F} \ast \mathcal{K}) = (h^*_\ell)^\ast_{\text{Sm}} i^\ast_\text{Sm} (\phi_1)^\ast_\text{Sm}(\phi_1)_\ast^\ast_\text{Sm}\mathcal{F} = (h^*_\ell)^\ast_{\text{Sm}} (\phi_1)_\ast^\ast_\text{Sm}(\phi_1)_\ast^\ast_\text{Sm}\mathcal{F} = w^\ast_\text{Sm} \eta^\ast_\text{Sm}(\phi_1)_\ast^\ast_\text{Sm}\mathcal{F},\]

using smooth base change at the end. Observe \(i^\ast_\text{Sm} \mathcal{F} = (h^*_\ell)^\ast_{\text{Sm}} i^\ast_\text{Sm} \mathcal{F}\), so that

\[w^\ast_\text{Sm} \eta^\ast_\text{Sm}(\phi_1)_\ast^\ast_\text{Sm}\mathcal{F} = w^\ast_\text{Sm} (\phi_1)_\ast^\ast_\text{Sm}(h^*_\ell)^\ast_{\text{Sm}} i^\ast_\text{Sm}\mathcal{F})\]
\[= w^\ast_\text{Sm} (\phi_1)_\ast^\ast_\text{Sm}(h^*_\ell)^\ast_{\text{Sm}} (i^\ast_\text{Sm}\mathcal{F})\]
\[= w^\ast_\text{Sm} \eta^\ast_\text{Sm}(\phi_1)_\ast^\ast_\text{Sm}(i^\ast_\text{Sm}\mathcal{F}) = (z^\ast)^\ast_\text{Sm}(i^\ast_\text{Sm}\mathcal{F}) = (q^\ast)^\ast_\text{Sm}(i^\ast_\text{Sm}\mathcal{F}),\]

where we have applied another base change and used the identities \(z^\ast z^\ast = 1, z^\ast = q^\ast z\). Since the foregoing identifications are natural in \(\mathcal{F}\),

\[(q^\ast)^\ast_{\text{Sm}} : \text{Sm}^\lambda_{\text{IW}} \rightarrow \text{Sm}^\mu_{\text{IW}}\]

is the translation functor corresponding to \(T^\ast\). Formal properties of adjunctions now imply that \((q^\ast)^\ast_{\text{Sm}}\) corresponds to \(T^\ast\) and thus that \((q^\ast)^\ast_{\text{Sm}}(q^\ast)^\ast_{\text{Sm}}\) corresponds to \(\theta^\ast\). (This can also be proven directly by arguments analogous to those above.)

We have now constructed an action of \(\mathcal{H}\) on the tilting category \(\text{Tilt}(\text{Rep}_0(G))\) by wall-crossing functors. As explained in [RW18, Remark 5.1.2(1)], this induces a similar action on \(\text{Rep}_0(G)\) and completes the proof of our main result.

**Theorem 6.6.** There is a monoidal right action of \(\mathcal{H}\) on \(\text{Rep}_0(G)\), such that \(B_s(n)\) acts by the wall-crossing functor \(\theta^\ast_s\) for all \(s \in S, n \in \mathbb{Z}\). That is, there exists a monoidal functor \(a : \mathcal{H} \rightarrow \text{End}(\text{Rep}_0(G))\) with \(a(B_s(n)) = \theta^\ast_s\).

The following is an important corollary, already derived by Riche and Williamson, yielding the character formula for tilting modules referenced in the introduction. See [RW18, §5.2] and [RW18, §1.4] for proofs and additional discussion.

**Corollary 6.7.** The action of \(\mathcal{H}\) on the object \(T^\ast(\lambda_0) \in \text{Tilt}(\text{Rep}_0(G))\) descends to an additive functor \(\Psi : \mathcal{H}^\prime \rightarrow \text{Tilt}(\text{Rep}_0(G))\) which realises \(\mathcal{H}^\prime\) as a graded enhancement of \(\text{Tilt}(\text{Rep}_0(G))\): that is, \(\Psi\) induces an equivalence between the de-grading of \(\mathcal{H}^\prime\) and \(\text{Tilt}(\text{Rep}_0(G))\).

**Remark 6.8.** (1) To be precise, the discussion in [RW18] is in terms of the Hecke category \(\mathcal{H}^\prime\) for the standard realisation rather than the Hecke category \(\mathcal{H}\) for the loop realisation (recall §4.2). However, the proofs and consequences for representation theory are the same.

(2) We have established Theorem 6.6 and deduced Corollary 6.7 over the finite field \(k\), but they hold over any extension field \(k^\prime\) of \(k\). Indeed, this can be seen by changing the base field for the category of tilting modules and the associated Hecke category action: for \(\dagger \in \{\emptyset, 0\}\), the canonical functor

\[k^\prime \otimes_k \text{Tilt} (\text{Rep}_1(G)) \rightarrow \text{Tilt} (\text{Rep}_1(G_{k^\prime}))\]

is additive, fully faithful, and induces a bijection on indecomposable objects, hence is an equivalence (see, e.g., [Jan03, §E.22]). Compatibility with field extensions will also apply to our more complete Theorem 6.13 below.
6.4 Analysis of morphisms

Following Theorem 6.6, it remains to examine the actions of generating morphisms of $\mathcal{H}$. We aim to show there exist counit–unit pairs $(\varepsilon, \eta): T_s \dashv T^s$ and $(\psi, \varphi): T^s \dashv T_s$ such that

$$a\left(\begin{array}{c} x \\ y \end{array}\right) = \varepsilon : \theta_s \to \text{id}, \quad a\left(\begin{array}{c} x \\ y \end{array}\right) = \varphi : \text{id} \to \theta_s,$$

$$a\left(\begin{array}{c} x \\ y \end{array}\right) = T_s \eta T^s : \theta_s \to \theta_s, \quad a\left(\begin{array}{c} x \\ y \end{array}\right) = T^s \psi T_s : \theta_s \to \theta_s.$$  

(6.13) (6.14)

To do this, we make an argument via total cohomology. Recall the ring $R = R_k$ from §4.2 and note that

$\begin{align*}
R &\cong H^\bullet_{\tilde{I} \times \mathbb{G}_m}(\mathbb{F}_l, k), & R^s &\cong H^\bullet_{\tilde{I}^s \times \mathbb{G}_m}(\mathbb{F}_l^s, k),
\end{align*}$

(6.15)

induced by the two actions of $\tilde{I} \times \mathbb{G}_m$ on $L\tilde{G}$, respectively, the left action of $\tilde{I} \times \mathbb{G}_m$ and right action of $\tilde{I}^s \times \mathbb{G}_m$ on $LG$; for the latter, we use that

$$H^\bullet_{\tilde{I} \times \mathbb{G}_m}(\mathbb{F}_l, k) \cong H^\bullet_{\tilde{I}^s \times \mathbb{G}_m}((\tilde{I} \times \mathbb{G}_m) \setminus (L\tilde{G} \times \mathbb{G}_m), k).$$

Composing the total cohomology funtor $H^\bullet_{\tilde{I} \times \mathbb{G}_m}(\mathbb{F}_l, -)$, respectively, $H^\bullet_{\tilde{I}^s \times \mathbb{G}_m}(\mathbb{F}_l^s, -)$, with the restrictions of scalars induced by (6.15), we obtain functors

$$H^\bullet_{\tilde{I} \times \mathbb{G}_m}(\mathbb{F}_l, k) \cong H^\bullet_{\tilde{I}^s \times \mathbb{G}_m}((\tilde{I} \times \mathbb{G}_m) \setminus (L\tilde{G} \times \mathbb{G}_m), k).$$

These functors have many favourable properties, including intertwining convolution products with tensor products of graded bimodules and providing realisations of Bott–Samelson bimodules:

$$\mathbb{H}(\mathcal{E}(s)) = R \otimes_{R^s} R(1), \quad s \in S;$$

we refer the reader to [Ach21, §7.6], [RW18, §10.5], and [BY13, §3.2] for discussion in similar contexts. Here $\mathbb{H}$ and $\mathbb{H}^s$ are also faithful, by [MR18, Remark 3.19].

Next, we make an easily proven ‘base change’ observation on the following commutative diagram of commutative graded rings.

$$\begin{array}{c}
B & \xrightarrow{g} & B_0 \\
\downarrow f & & \downarrow f_0 \\
A & \xrightarrow{\alpha} & A_0
\end{array}$$

(6.16)

**Lemma 6.9.** The maps in (6.16) induce functors of restriction of scalars between the associated categories of graded right modules, such that the induced square of categories commutes up to natural isomorphism. There is also a morphism of graded right $A$-modules (natural in the $B_0$-module $M$),

$$M_B \otimes_B A \to (M \otimes_{B_0} A_0)_A, \quad m \otimes a \mapsto m \otimes \alpha(a).$$

(6.17)
Proposition 6.10. The functors $\mathbb{H}$ and $\mathbb{H}^s$ fit into the following two commutative squares (superimposed onto one diagram).

$$\text{Parity}_{\mathcal{I} \times \mathbb{G}_m} (\widetilde{\mathcal{F}}l, \mathbb{K}) \xrightarrow{H} (R, R)\text{-bimod} \mathbb{Z}$$

$$\text{Parity}_{\mathcal{I} \times \mathbb{G}_m} (\widetilde{\mathcal{F}}^s, \mathbb{K}) \xrightarrow{H^s} (R, R^s)\text{-bimod} \mathbb{Z}$$

The vertical dashes represent arrows which either both go up or both go down. The left arrows are $(q^s)_*$ and $(q^s)^*$, while the right arrows are restriction $(-) \otimes_R R^s$ and induction $(-) \otimes_R R$. 

Proof. To show the squares commute, we work with the following extended diagram.

$$\text{Parity}_{\mathcal{I} \times \mathbb{G}_m} (\widetilde{\mathcal{F}}l, \mathbb{K}) \xrightarrow{H} H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}l, \mathbb{K})\text{-mod} \mathbb{Z} \xrightarrow{r} (R, R)\text{-bimod} \mathbb{Z}$$

$$\text{Parity}_{\mathcal{I} \times \mathbb{G}_m} (\widetilde{\mathcal{F}}^s, \mathbb{K}) \xrightarrow{H^s} H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}^s, \mathbb{K})\text{-mod} \mathbb{Z} \xrightarrow{r^s} (R, R^s)\text{-bimod} \mathbb{Z}$$

Here the functors $r$ and $r^s$ are restrictions of scalars along the maps (6.15), and the middle vertical arrows are restriction and induction. The left-most horizontal functors are known to be fully faithful; see again [MR18, Remark 3.19]. Now, there is a canonical isomorphism of graded $H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}^s, \mathbb{K})$-modules, $H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}^s, \mathbb{K}) \cong H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}l, \mathbb{K})$, where the latter is a right module via the map $H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}^s, \mathbb{K}) \to H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}l, \mathbb{K})$ induced by $q^s$. We also have an obvious morphism $(6.18)$.

Meanwhile, Lemma 6.9 proves that the square of restriction functors between graded module categories commutes; thus, our claim holds for the intertwining of $(q^s)_*$ and restriction by $H$ and $H^s$. The same lemma provides a natural transformation $\text{ind} \circ r^s \to r \circ \text{ind}$. Composing on the right with $H^*_{\mathcal{I} \times \mathbb{G}_m} (\mathcal{F}^s, -)$ and using (6.18), we obtain a morphism

$$\text{ind} \circ H^s \to H \circ (q^s)^*.$$ (6.19)

We conclude by showing this is an isomorphism for objects in $\text{Parity}_{\mathcal{I} \times \mathbb{G}_m} (\widetilde{\mathcal{F}}^s, \mathbb{K})$. Note first that, for $w \in W^s$,

$$\mathbb{H}((q^s)^*(q_s) E(w)) = \mathbb{H}(E(w) \star E(s)[-1]) = \mathbb{H}(E(w)) \otimes_R B_s(-1) = (\mathbb{H}(E(w)) \otimes_R R^s) \otimes_R R = \mathbb{H}^s((q^s)_* E(w)) \otimes_R R,$$

since $(q_s)_* E(w) \cong E(w) \star E(s)(1)$ and $\mathbb{H}, \mathbb{H}^s$ intertwine convolution and tensor products. Thus, (6.19) is an isomorphism for objects of the form $(q^s)_* E(w), w \in W^s$. Now, every indecomposable object $E^s(w)$ in the Krull–Schmidt category $\text{Parity}_{\mathcal{I} \times \mathbb{G}_m} (\widetilde{\mathcal{F}}^s, \mathbb{K})$ is a multiplicity-one factor of $(q^s)_* E(w)$, so our claim follows by induction on the length of $w$ and the fact that the direct summand of an isomorphism is an isomorphism.

Now let us consider the following morphisms in $\text{Parity}_{\mathcal{I} \times \mathbb{G}_m} (\widetilde{\mathcal{F}}l, \mathbb{K})$:

$$\Delta \left( \begin{array}{c} \uparrow \\ w_s \end{array} \right) = u_s : E(s) \to E(1)[1], \quad \Delta \left( \begin{array}{c} \downarrow \\ \ell_s \end{array} \right) = \ell_s : E(1) \to E(s)[1],$$

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\[ \Delta \left( \begin{array}{c} \ast \\ 1 \end{array} \right) = b_s : \mathcal{E}(s) \to \mathcal{E}(ss)[-1], \quad \Delta \left( \begin{array}{c} \ast \\ 2 \end{array} \right) = c_s : \mathcal{E}(ss) \to \mathcal{E}(s)[-1]. \]

The first two of these give rise to natural transformations,
\[ \bar{\varepsilon} = (\ast) \ast b_s[-1] : (q^s)^*(q^s)_s \to \text{id}, \quad \bar{\varphi} = (\ast) \ast c_s : \text{id} \to (q^s)^*(q^s)_s[2]. \]

**Proposition 6.11.** The map \( \bar{\varepsilon} \) is a counit for \((q^s)^* \dashv (q^s)_s\) and the map \( \bar{\varphi} \) is a unit for \((q^s)_s \dashv (q^s)^*\).[2]

**Proof.** For \( \mathcal{F} \in \text{Parity}_{\mathcal{I} \times \mathcal{G}}(\mathcal{F}_l, k) \) and \( \mathcal{G} \in \text{Parity}_{\mathcal{I} \times \mathcal{G}}(\mathcal{F}_l, k) \), consider
\[ \bar{A}_{\mathcal{F}, \mathcal{G}} : \text{Hom}(\mathcal{F}, (q^s)_s \mathcal{G}) \to \text{Hom}((q^s)^* \mathcal{F}, \mathcal{G}), \quad f \mapsto \bar{\varepsilon} \circ (q^s)^*(f); \]
\[ \bar{B}_{\mathcal{G}, \mathcal{F}} : \text{Hom}((q^s)^* \mathcal{G}, \mathcal{F}) \to \text{Hom}(\mathcal{G}, (q^s)_s \mathcal{F}[2]), \quad f \mapsto (q^s)_s(f)[2] \circ \bar{\varphi}. \]

We wish to show these are isomorphisms. Because we know in each case that the source and target are finite dimensional and isomorphic, it will suffice for our purposes to prove injectivity for any \( \mathcal{F}, \mathcal{G} \).

Suppose therefore that \( \bar{A}_{\mathcal{F}, \mathcal{G}}(f) = 0 \). Then
\[ 0 = \mathcal{H}(\bar{A}_{\mathcal{F}, \mathcal{G}}(f)) = \mathcal{H}(\bar{\varepsilon}) \circ \mathcal{H}((q^s)^*(f)), \]
where \( \mathcal{H}(\bar{\varepsilon}) = \mathcal{H}(\mathcal{G} \otimes_R \mathcal{H}(u_s[-1]) = \mathcal{H}(\mathcal{G}) \otimes_R m_s \), for \( m_s : R \otimes_R R \to R \), \( g \otimes h \mapsto gh \), and \( \mathcal{H}((q^s)^*(f)) = \mathcal{H}^s(f) \otimes_R R : \mathcal{H}^s(\mathcal{F}) \otimes_R R \to (\mathcal{H}(\mathcal{G}) \otimes_R R_R^\ast) \otimes_R R \),

using Proposition 6.10. But then we can calculate explicitly
\[ 0 = (\mathcal{H}(\mathcal{G}) \otimes_R m_s)(\mathcal{H}^s(f) \otimes_R R_r) R_r(x \otimes 1) = (\mathcal{H}(\mathcal{G}) \otimes m_s)(\mathcal{H}^s(f)(x) \otimes 1) \]
for any \( x \in \mathcal{H}^s(\mathcal{F}) \).

Now, there exists \( \chi \in \mathcal{H}(\mathcal{G}) \) such that \( \mathcal{H}^s(f)(x) = \chi \otimes 1 \) and hence \( (\mathcal{H}(\mathcal{G}) \otimes_R m_s)(\mathcal{H}^s(f)(x) \otimes 1) = \chi \). Since \( x \) was arbitrary, we get \( \mathcal{H}^s(f) = 0 \), proving that \( f = 0 \) by faithfulness of \( \mathcal{H}^s \). Thus, \( \bar{A}_{\mathcal{F}, \mathcal{G}} \) is injective and, therefore, an isomorphism. A suitable unit for \( \bar{\varepsilon} \) is then given by \( \bar{A}^{-1}_{\mathcal{F}, (q^s)_s \mathcal{G}}(1_{(q^s)_s \mathcal{F}}) \).

On the other hand, suppose \( \bar{B}_{\mathcal{G}, \mathcal{F}}(f) = 0 \), so that
\[ 0 = \mathcal{H}(\bar{B}_{\mathcal{G}, \mathcal{F}}(f)) = \mathcal{H}((q^s)\ast(f))(2) \circ \mathcal{H}(\bar{\varphi}), \]
where \( \mathcal{H}(\bar{\varphi}) = \mathcal{H}(\mathcal{F} \otimes_R \mathcal{H}(\delta_s), \delta_s : R \to R \otimes R_\mathcal{F} R, 1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \).

Hence, for any \( x \in \mathcal{H}(\mathcal{G}) \),
\[ 0 = (\mathcal{H}^s(f) \otimes_R R_r(2))(\mathcal{H}(\mathcal{G} \otimes_R \mathcal{H}(\delta_s))(2x \otimes 1) = (\mathcal{H}^s(f) \otimes_R R_r(2))(x \otimes (\alpha_s \otimes 1 + 1 \otimes \alpha_s)) \]
\[ = \mathcal{H}^s(f)(x \otimes \alpha_s) \otimes 1 + \mathcal{H}^s(f)(x \otimes 1) \otimes \alpha_s. \]

However, in \( \mathcal{H}^s(\mathcal{F}) \otimes_R R_r(2), \) an equation \( a \otimes 1 = b \otimes \alpha_s \) implies \( a = b = 0 \), so \( \mathcal{H}^s(f)(x \otimes \alpha_s) = \mathcal{H}^s(f)(x \otimes 1) = 0 \) and thus \( \mathcal{H}^s(f) = 0 \), i.e. \( f = 0 \), as desired. \( \square \)

As we see from the preceding proof,
\[ \bar{\eta}_{\mathcal{F}} = \bar{A}_{\mathcal{F}, (q^s)_s \mathcal{F}}^{-1}(1_{(q^s)_s \mathcal{F}}), \quad \bar{\psi}_{\mathcal{F}} = \bar{B}_{(q^s)_s \mathcal{F}[2], \mathcal{F}}^{-1}(1_{(q^s)_s \mathcal{F}[2]} \mathcal{F}) \]
are the unit and counit for \( \bar{\varepsilon} \) and \( \bar{\varphi} \), respectively. We then wish to compare \((q^s)^* \bar{\eta}(q^s)_s \) with \((\ast) \ast b_s[-1] \) and \((q^s)^* \bar{\psi}(q^s)_s \) with \((\ast) \ast c_s \).

**Proposition 6.12.** There are identifications of natural transformations,
\[(q^s)^* \bar{\eta}(q^s)_s = (\ast) \ast b_s[-1], \quad (q^s)^* \bar{\psi}(q^s)_s = (\ast) \ast c_s. \]
Hecke category actions via Smith–Treumann theory

Proof. Note that we have natural transformations

\[ \epsilon = (-) \otimes_R m_s : (-) \otimes_R B_s(-1) \to \text{id}, \quad \phi = (-) \otimes_R \delta_s : \text{id} \to (-) \otimes_R B_s(1), \]

which are such that \( \epsilon \mathbb{H} = \mathbb{H} \mathcal{F} \) and \( \phi \mathbb{H} = \mathbb{H} \mathcal{F} \), as well as maps of Hom spaces,

\[
A_{M,N} : \text{Hom}(M, N \otimes_R R^s) \to \text{Hom}(M \otimes_R R, N), \quad f \mapsto \epsilon_N \circ (f \otimes_R R) \\
B_{N,M} : \text{Hom}(N \otimes_R R^s, M) \to \text{Hom}(N, M \otimes_R R(2)), \quad f \mapsto (f \otimes_R R(2)) \circ \phi_N,
\]

which are adjunction isomorphisms for all graded \((R, R^s)\)-bimodules \(M\) and graded \((R, R)\)-bimodules \(N\). Indeed, inverses to \(A_{M,N}\) and \(B_{N,M}\) can be obtained using the projection and injection maps for the \((R^s, R^s)\)-bimodule decomposition [EMTW20, §4]

\[
R \simeq R^s \oplus R^s(-2), \quad x \mapsto \left( \partial_s \left( x \frac{\alpha_s}{2} \right), \partial_s(x) \right),
\]

where \(\partial_s : R \to R^s(-2)\) is the Demazure operator, \(\partial_s(x) = (x - s(x))/\alpha_s\). Now

\[
A_{\mathbb{H}^+(\mathcal{F}), \mathbb{H}(b(q^*) \cdot \mathcal{F})}(\mathbb{H}^s(\tilde{\eta} \cdot \mathcal{F})) = \epsilon_{\mathbb{H}(b(q^*) \cdot \mathcal{F})} \circ (\mathbb{H}^s(\tilde{\eta} \cdot \mathcal{F}) \otimes_R R) \\
= \mathbb{H}(\tilde{\eta} \cdot \mathcal{F}) \circ (b(q^*) \cdot \mathcal{F}) = 1,
\]

and similarly \(B_{\mathbb{H}(b(q^*) \cdot \mathcal{F}), \mathbb{H}(\mathcal{F})}(\mathbb{H}^s(\bar{\eta} \cdot \mathcal{F})) = 1\). On the other hand, there are natural transformations of graded \((R, R^s)\)-bimodule endofunctors,

\[
\zeta_M : M \to M \otimes_R R^s, \quad m \mapsto m \otimes 1, \\
\omega_M : M \otimes_R R(2) \to M, \quad m \otimes r \mapsto m \partial_s(r),
\]

where \(\partial_s : R(2) \to R^s\) is the Demazure operator associated to \(s\) (see [EW16, §3.3]). By direct calculation, these satisfy

\[
A_{M,M \otimes_R R}(\zeta_M) = \epsilon_{M \otimes_R R} \circ (\zeta_M \otimes_R R) = 1, \\
B_{M \otimes_R R, M}(\omega_M) = (\omega_M \otimes_R R(2)) \circ \phi_{M \otimes_R R} = 1.
\]

Since \(A\) and \(B\) are isomorphisms, we deduce \(\mathbb{H}^s(\tilde{\eta} \cdot \mathcal{F}) = \zeta_{\mathbb{H}^s(\mathcal{F})}\) and \(\mathbb{H}^s(\bar{\eta} \cdot \mathcal{F}) = \omega_{\mathbb{H}^s(\mathcal{F})}\). Hence

\[
\mathbb{H}(b(q^*) \cdot \mathcal{F}) \otimes_R R = \mathbb{H}(\tilde{\eta} \cdot \mathcal{F}) \otimes_R R = \mathbb{H}(\bar{\eta} \cdot \mathcal{F}) \oplus_R R
\]

using the fact [RW18, §10.5.4] that \(\mathbb{H}(b_1) : B_s \to B_{ss}(-1)\) is given by \(f \otimes g \mapsto f \otimes 1 \otimes g\); meanwhile,

\[
\mathbb{H}(b(q^*) \cdot \mathcal{F}) \otimes_R R = \mathbb{H}(\bar{\eta} \cdot \mathcal{F}) \oplus_R R
\]

using that \(\mathbb{H}(c_s) : B_{ss} \to B_s(-1)\) is given by \(f \otimes g \otimes h \mapsto f(\partial_s g) \otimes h\). Faithfulness of \(\mathbb{H}\) then yields the result. \(\square\)

After successive passage to the quotient \(\text{Parity}_{\text{IW}, \text{sm}}(\tilde{\mathcal{H}}, \mathbb{K})\), to the de-grading \(\text{Sm}_{\text{IW}}\), and to the equivalent right \(\mathcal{H}\)-module category \(\text{Tilt}_0(\mathcal{G})\), the relations given in Propositions 6.11

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and 6.12 for the pairs
\((\tilde{\varepsilon}_s, \tilde{\eta}_s) : (q^s)^* \dashv (q^s)_s, \quad (\tilde{\psi}_s, \tilde{\varphi}_s) : (q^s)^* \dashv (q^s)_s[2]\)

manifest precisely as (6.13) and (6.14) for the pairs given by their correspondents under the various functors,
\((\varepsilon_s, \eta_s) : T_s \dashv T^s, \quad (\psi_s, \varphi_s) : T^s \dashv T_s.\)

This allows us to state a more complete version of Theorem 6.6.

**Theorem 6.13.** There is a monoidal right action of \(H^0\) on \(\text{Rep}_0(G)\) such that, for all \(s \in S\) and for explicit counit–unit pairs \((\varepsilon_s, \eta_s) : T_s \dashv T^s\) and \((\psi_s, \varphi_s) : T^s \dashv T_s\), the following properties hold:

(i) \(B_s(n)\) acts by the wall-crossing functor \(\theta_s\) for all \(n \in \mathbb{Z}\);
(ii) the upper and lower dots act by \(\varepsilon_s\) and \(\varphi_s\), respectively;
(iii) the trivalent vertices act by \(T_s\eta_sT^s\) and \(T^s\psi_sT^s\).

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**Conflicts of Interest**

None.

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