Localisation and Delocalisation for a Simple Quantum Wave Guide with Randomness

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Abstract. In this paper, we consider Schrödinger operators on $M \times \mathbb{Z}^{d_2}$, with $M = \{M_1, \ldots, M_2\}^{d_1}$ (‘quantum wave guides’) with a ‘$\Gamma$-trimmed’ random potential, namely a potential which vanishes outside a subset $\Gamma$ which is periodic with respect to a sub-lattice. We prove that (under appropriate assumptions) for strong disorder these operators have pure point spectrum outside the set $\Sigma_0 = \sigma(H_{0, \Gamma^c})$ where $H_{0, \Gamma^c}$ is the free (discrete) Laplacian on the complement $\Gamma^c$ of $\Gamma$. We also prove that the operators have some absolutely continuous spectrum in an energy region $\mathcal{E} \subset \Sigma_0$. Consequently, there is a mobility edge for such models. We also consider the case $-M_1 = M_2 = \infty$, i.e. $\Gamma$-trimmed operators on $\mathbb{Z}^d = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$. Again, we prove localisation outside $\Sigma_0$ by showing exponential decay of the Green function $G_{E+i\eta}(x, y)$ uniformly in $\eta > 0$. For all energies $E \in \mathcal{E}$ we prove that the Green’s function $G_{E+i\eta}$ is not (uniformly) in $\ell^1$ as $\eta$ approaches 0. This implies that neither the fractional moment method nor multi-scale analysis can be applied here.

1. Introduction

Quantum waveguides are quantum mechanical structures which are confined in certain spaces dimensions, but unconfined in others. The last decades showed a growing interest in these systems in the mathematical literature. The book [8] by Exner and Kovařík gives an overview on the state of art (as of 2015) as well as an extensive list of references on waveguides.

A quantum waveguide with the simplest geometry is given by a particle in a ($k$-dimensional) strip in $\mathbb{Z}^{k+m}$ or $\mathbb{R}^{k+m}$. Other examples are tubes or wires which are bended or twisted (see, for example, the discussion in Krejčiřík [21]). Of particular interest are waveguides with randomness either in the geometry of the system or in the potential energy.
In this paper, we consider waveguides with a simple geometry, namely on a strip in $\mathbb{Z}^d$, for example, on

$$\mathcal{X} = \{M_1 p, M_1 p + 1, \ldots, M_2 p - 1\}^{d_1} \times \mathbb{Z}^{d_2}, \quad d_1 + d_2 = d, p \geq 2, \quad (1)$$

with a random potential $V_\omega$. The potential we consider is ‘sporadic’ or ‘$\Gamma$-trimmed’, in the sense that $V_\omega(x) = 0$ for lattice points $x \notin \Gamma$. Here, $\Gamma$ is an $\mathbb{L}$-periodic subset of the strip for a sub lattice $\mathbb{L}$. In example (1), we may choose, for instance,

$$\Gamma = \{M_1 p, (M_1 + 1)p, \ldots, (M_2 - 1)p\}^{d_1} \times \mathbb{Z}^{d_2} \quad (2)$$

For $x \in \Gamma$, the potentials are independent and identically distributed. Random operators $H_\omega$ with such potentials are called ‘$\Gamma$-trimmed’.

Spectral theory for trimmed Anderson models (i.e. on $\mathbb{Z}^d$) was done in the PhD-thesis of Obermeit [23] and the papers of Rojas-Molina [24], Elgart-Klein [5], Elgart-Sodin [6] and Kirsch-Krishna [16]. The present paper was inspired by [6].

We show that models as in (1), (2) have a mobility edge (or rather mobility edges). The measure theoretical nature of the spectrum depends on the energy region. More precisely, denote by $H_{0,\Gamma^c}$ the free (discrete) Laplacian on the set $\Gamma^c$. Outside of the spectrum $\Sigma_0 := \sigma(H_{0,\Gamma^c})$ the operator $H_\omega$ has dense point spectrum for high enough disorder. On the other hand, we prove that $H_\omega$ has some absolutely continuous spectrum inside $\sigma(H_{0,\Gamma^c})$ regardless of the strength (or even existence) of the randomness.

The absolutely continuous spectrum comes from the existence of canonical extended states. More precisely, in an energy region inside $\Sigma_0$ we find periodic solutions of the free Schrödinger equation which vanish on the set $\Gamma$. Hence, these functions solve the Schrödinger equation with a random (in fact, with an arbitrary) potential on $\Gamma$ as well.

Kotani and Simon consider random operators on a strip [20] with $d_2 = 1$. They give abstract conditions for absolutely continuous spectrum in terms of Lyapunov exponents. The examples for absolutely continuous spectrum they give are deterministic potentials. Their method does not extend to $d_2 > 1$.

To prove pure point spectrum we employ the multiscale analysis (see, for example, Dreifus-Klein [27] or Disertori et.al. [4] and references given there). The critical ingredient in our case is a Wegner Estimate. To prove this estimate, we use in an essential way that we work outside the spectrum $\sigma(H_{0,\Gamma^c})$. In fact, the estimate blows up when we approach $\sigma(H_{0,\Gamma^c})$.

From the multiscale bounds not only pure point spectrum follows but also dynamical localisation (see Damanik–Stollmann [3] or [26]).

We also consider the case $-M_1 = M_2 = \infty$, in other words a $\Gamma$-trimmed potential on $\mathbb{Z}^d = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$. Again we use multiscale analysis outside $\Sigma_0$ for high enough disorder, which implies uniform exponential decay of the Green function and existence of pure point spectrum.
Inside a subset \( \mathcal{E} \) of \( \Sigma_0 \) we prove that the Green function \( G_{E+i\epsilon}(x,y) \) is not only not uniformly exponentially decaying, but that for \( x \notin \Gamma \) even

\[
\sup_{\epsilon \to 0} \sum_{y \in \mathbb{Z}^d} |G_{E+i\epsilon}(x,y)| = \infty.
\]

(3)

Under appropriate conditions, we even have \( \mathcal{E} = \Sigma_0 \).

Consequently, we have a ‘phase transition’ at \( \Sigma_0 = \sigma(H_0,\Gamma^c) \) which manifests itself in the behaviour of the Green function.

For the Anderson model (with full randomness), it is expected that in higher dimension there is a mobility edge, namely a transition from pure point spectrum to absolutely continuous spectrum depending on the energy range and the strength of the disorder. All that is rigorously known (on \( \mathbb{Z}^d \)) is the existence of pure point spectrum (see, for example, Aizenman–Warzel [2] or Kirsch [15]). However, on the Bethe tree Klein [17] proved the existence of absolutely continuous spectrum (see also Klein–Sadel [19], Aizenman–Warzel [1], Froese et al. [9]).

There are random Schrödinger operators with decaying randomness for which a mobility edge is known to exist (see Krishna [22], Kirsch et.al. [10,13], and Jaksic-Last [11]). These models are not ergodic. However, the models we consider here are either ergodic in \( \mathbb{Z}^{d_1} \)-direction (for the strip) or even ergodic with respect to a \( d \)-dimensional sub-lattice.

The systems we consider are about the simplest wave guides possible. We expect that the localisation results we have in this paper can be extended to more complicated wave guide systems using essentially the same technique. We also expect the delocalisation results to hold more generally, assuming there are big enough regions without (random) potential. However, such results would presumably require refined methods.

2. Setup

We consider quantum systems (wave guides) on \( \mathcal{X} \subset \mathbb{Z}^d = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \).

For \( p = (p_1, \ldots, p_d) \in (\mathbb{N} \setminus \{1\})^d \) (periods) set (unit cell)

\[
C_0 = \{ x \in \mathbb{Z}^d \mid 0 \leq x_\nu \leq p_\nu - 1 \text{ for all } \nu \}
\]

(4)

By \( e_\nu \) we denote the standard basis of \( \mathbb{Z}^d \). The lattice \( \mathbb{L} \) and the subset \( \mathbb{L}_{M_1,M_2} \) are defined by

\[
\mathbb{L} := \left\{ \sum_{\nu=1}^{d} i_\nu p_\nu e_\nu \mid i_\nu \in \mathbb{Z} \right\}
\]

(5)

and

\[
\mathbb{L}_{M_1,M_2} := \left\{ \sum_{\nu=1}^{d} i_\nu p_\nu e_\nu \mid M_1 \leq i_\nu < M_2 \text{ for } \nu \leq d_1 \right\},
\]

(6)

with \( M_1, M_2 \in \mathbb{Z}, M_1 < M_2 \). In this paper, we always assume that \( d_1, d_2 > 0 \).

Then, we define ‘cubes’ \( \mathcal{X}_{M_1,M_2} \) by ‘periodising’ \( C_0 \):

\[
\mathcal{X}_{M_1,M_2} := C_0 + \mathbb{L}_{M_1,M_2}
\]

(7)
and
\[ X_\infty := C_0 + \mathbb{L}. \tag{8} \]
Informally, we consider \( X_\infty \) as \( X_{M_1, M_2} \) with \( M_1 = -\infty, M_2 = \infty \).

Thus, \( X_{M_1, M_2} \) is a strip (‘waveguide’) of restricted width in \( d_1 \) directions and unconfined in \( d_2 \) directions, \( X_\infty \) is (for the moment) just a complicated expression for \( \mathbb{Z}^d \).

Sometimes we omit the indices \( M_1, M_2 \) and \( \infty \) if they are clear from the context or if they are irrelevant. For simplicity of arguments and, in fact, without loss of generality we assume that \( M_2 - M_1 \) is even.

The discrete Laplacian \( H_0 \) on \( X_\infty = \mathbb{Z}^d \) is given by:
\[ H_0 u(n) := \sum_{\nu=1}^{d} \left( u(n + e_\nu) + u(n - e_\nu) \right) \tag{9} \]
When we restrict \( H_0 \) to subsets of \( \mathbb{Z}^d \) we have to impose boundary conditions.

In the following, we will most of the time work with either ‘simple’ boundary conditions or with ‘periodic’ boundary conditions.

**Definition 2.1.** If \( \Lambda \) is a subset of \( \mathbb{Z}^d \), then the operator \( H_{0, \Lambda} \) on \( \ell^2(\Lambda) \) given by
\[ H_{0, \Lambda} u(n) = \sum_{\nu=1}^{d} \left( \chi_\Lambda(n + e_\nu) u(n + e_\nu) + \chi_\Lambda(n - e_\nu) u(n - e_\nu) \right) \tag{10} \]
is called the Laplacian on \( \Lambda \) with *simple boundary conditions*.

Here
\[ \chi_\Lambda(n) = \begin{cases} 1, & \text{for } n \in \Lambda; \\ 0, & \text{otherwise}. \end{cases} \tag{11} \]

**Definition 2.2.** Suppose the box \( \Lambda \subset \mathbb{Z}^d \) is given by
\[ \Lambda = \{ x \in \mathbb{Z}^d \mid q_\nu \leq x_\nu \leq p_\nu \text{ for } \nu = 1, \ldots, d' \} \tag{12} \]
for some \( d' \leq d \), then we call the operator \( H_0^\Lambda \) defined by
\[ H_0^\Lambda u(x) = \sum_{\nu=1}^{d} \left( u(N^+_{\nu} x) + u(N^-_{\nu} x) \right) \tag{13} \]
where
\[ N^+_{\nu} x = \begin{cases} x + e_\nu, & \text{if } x + e_\nu \in \Lambda; \\ x - (p_\nu - q_\nu)e_\nu, & \text{if } x + e_\nu \notin \Lambda. \end{cases} \tag{14} \]
\[ N^-_{\nu} x = \begin{cases} x - e_\nu, & \text{if } x - e_\nu \in \Lambda; \\ x + (p_\nu - q_\nu)e_\nu, & \text{if } x - e_\nu \notin \Lambda. \end{cases} \tag{15} \]
the Laplacian on \( \Lambda \) with periodic boundary conditions.

For the operator \( H_0 \) on \( X_{M_1, M_2} \), we impose periodic boundary conditions.

Now, we define the set \( \Gamma \) of ‘active sites’, i.e. the sites where the potential \( V_\omega \) may be nonzero. The active sites inside \( C_0 \) are denoted by \( \Gamma_0 \), with
\[ \emptyset \neq \Gamma_0 \subsetneq C_0 \tag{16} \]
and
\[
\Gamma := \Gamma_0 + L_{M_1 M_2}
\]
where we include again the case \(-M_1 = M_2 = \infty\).

**Example 2.3.** In the following examples \(X\) may be either \(X_{M_1 M_2}\) or \(X_\infty = \mathbb{Z}^d\)

1. For some \(\nu \leq d_1\)
   \[
   \Gamma = \{ x \in X \mid x_\nu = 0 \}
   \] (18)

2. \(\Gamma = \{ x \in X \mid x_1 = 0 \text{ or } x_2 = 0 \text{ or } \ldots \text{ or } x_{d_1} = 0 \}\)

3. For some results, we can deal with the following less restrictive model
   \[
   \emptyset \neq \Gamma \subset \{ x \in X \mid x_1 = 0 \text{ or } x_2 = 0 \text{ or } \ldots \text{ or } x_{d_1} = 0 \}\]
   (19)

In this article, we investigate spectral properties of operators \(H\) on \(X\) of the form:
\[
Hu(n) = H_0 u(n) + V(n) u(n)
\] (20)
where the potential \(V\) is supported by \(\Gamma\), i.e. \(V(n) = 0\) for \(n \notin \Gamma\).

Most of the time we suppose that \(V\) is a random potential with independent, identically distributed random variables \(V_\omega(\gamma), \gamma \in \Gamma\), but some of our results are independent of such an assumption.

### 3. Results

We prove localisation under fairly weak assumptions on \(\Gamma\).

**Theorem 3.1.** Suppose that \(\emptyset \neq \Gamma \neq X\). Assume that the random variables \(V_\omega(n), n \in \Gamma\) are independent with a common distribution \(P_0\) which has a bounded density \(\rho\) (with respect to Lebesgue measure) with compact support.

If \(I \subset \{ E \mid \text{dist}(E, \sigma(H_0, \Gamma_c)) \geq \gamma \}\), then \(H_\omega\) has pure point spectrum inside \(I\) with exponentially decaying eigenfunctions if \(\|\rho\|_\infty\) is small enough, i.e. if \(\|\rho\|_\infty \leq c_\gamma\).

**Remark 3.2.** Observe that \(\|\rho\|_\infty\) small means high disorder.

The multiscale analysis gives a form of localisation which is more than merely pure point spectrum, namely dynamical localisation. Dynamical localisation appears in various forms (for a detailed discussion see, for example, [18]), we use it in the following form which we take from [26].

**Definition 3.3.** For a self-adjoint operator \(H\) and an interval \(I\) denote by \(P_I(H)\) the spectral projection for \(H\) on \(I\).

We say that the random operator \(H\) satisfies dynamical localisation in the energy interval \(I\) if for any \(p > 0\)
\[
\mathbb{E} \left[ \sup_{t>0} \| |x|^p e^{-iHt} P_I(H)\varphi \| \right] < \infty
\] (21)
for any compactly supported \(\varphi\).
Corollary 3.4. Under the assumptions of Theorem 3.1, there is dynamical localisation in $I$.

That dynamical localisation follows from multiscale analysis is proved in [3], see also [26].

The following result shows that Theorem 3.1 is not an empty statement.

Proposition 3.5. Under the assumptions of Theorem 3.1, there is an $\eta > 0$ such that

$$\left[\inf \Sigma, \inf \Sigma + \eta\right] \cap \sigma(H_0,\Gamma^c) = \emptyset \quad (22)$$

and

$$\left[\sup \Sigma - \eta, \sup \Sigma\right] \cap \sigma(H_0,\Gamma^c) = \emptyset \quad (23)$$

Theorem 3.1 and Corollary 3.4 are proved in Sect. 5. The proof of Proposition 3.5 is given in Sect. 4. These results reprove and extend previous results in [5,6,16,23,24].

Now, we turn to a class of examples for which we can prove the existence of absolutely continuous spectrum.

Definition 3.6. We call $\Gamma$ as in (16) and (17) a single layer set if $\Gamma \subset G \times \mathbb{Z}^{d_2}$ with

$$\emptyset \neq G \subset G_0 = \{x \in \mathcal{X} \mid x_1 = 0 \text{ or } x_2 = 0 \text{ or } \ldots \text{ or } x_{d_1} = 0\} \quad (24)$$

We call $\Gamma$ a strict single layer set if $\Gamma = G_0 \times \mathbb{Z}^{d_2}$.

Definition 3.7. We set

$$\mathcal{L} := \bigotimes_{\nu=1}^{d_1} \{1,2,\ldots,p_{\nu} - 1\} \quad (25)$$

and for $L := (\ell_1,\ldots,\ell_{d_1}) \in \mathcal{L}$ we set

$$e_L := 2 \left(\sum_{\nu=1}^{d_1} \cos \left(\frac{\pi \ell_{\nu}}{p_{\nu}}\right)\right) \quad (26)$$

$$\mathcal{E} := \{e_L \mid L \in \mathcal{L}\} + [-2d_2,2d_2] \quad (27)$$

Proposition 3.8. Consider the operator $H_0$ on $\mathcal{X}_{M_1,M_2}$, a set $\Gamma \subset \mathcal{X}_{M_1,M_2}$ and the operator $H_0,\Gamma^c$, the restriction of $H_0$ to $\Gamma^c$ (with simple boundary conditions).

1. If $\Gamma$ is a single layer set, then $\mathcal{E} \subset \Sigma_0 := \sigma(H_0,\Gamma^c)$.
2. If $\Gamma$ is a strict single layer set, then $\mathcal{E} = \Sigma_0$

Theorem 3.9. Consider $H_0$ on $\mathcal{X}_{M_1,M_2}$ with $M_1, M_2$ finite, $M_2 - M_1$ even and with periodic boundary conditions. Assume that $\Gamma$ is a single layer set.

If $W$ is an arbitrary potential vanishing outside $\Gamma$, then

$$\mathcal{E} \subset \sigma_{ac}(H_0 + W). \quad (28)$$

In particular, if $\Gamma$ is a strict single layer set, then

$$\Sigma_0 \subset \sigma_{ac}(H_0 + W). \quad (29)$$
Theorem 3.9 applies in particular to Γ-trimmed random potential as in Theorem 3.1. Thus, for such a random potential there is an energy region with pure point spectrum and a region with absolutely continuous spectrum. Consequently, there exists a mobility edge.

The proof of Theorem 3.9 is contained in Sect. 6.

Unfortunately, the proof of Theorem 3.9 does not work for the case \(-M_1 = M_2 = \infty\), i.e. for \(X = \mathbb{Z}^d\). However, for this case we can at least show, that both the fractional moment method and the multiscale analysis cannot work. In fact, the spectral values in \(E\) belong to ‘extended states’ in an informal sense.

Let \(\Gamma \subset X_\infty\) be a one layer set and \(V_\omega\) be a random potential on \(\Gamma\) satisfying the assumptions of Theorem 3.1. Denote by \(G_{E+i\zeta}^{V_\omega}(x,y)\) the Green function (i.e. the kernel of the resolvent \((H_\omega - E - i\zeta)^{-1}\) with \(\zeta > 0\)).

Then, we show

**Theorem 3.10.** We assume \(X = \mathbb{Z}^d\) and \(\Gamma\) is a single layer set.

1. If \(E \notin \sigma(H_0,\Gamma^c)\), then for high enough disorder
   \[
   \limsup_{\zeta \searrow 0} \|G_{E+i\zeta}^{V_\omega}(x,y)\| \leq C e^{-m|x-y|} \tag{30}
   \]
   \(\mathbb{P}\)-almost surely.

2. If \(E \in E\), then
   \[
   \limsup_{\zeta \searrow 0} \sum_{y \in \mathbb{Z}^d} |G_{E+i\zeta}^{V_\omega}(x,y)| = \infty \tag{31}
   \]
   for all \(x \notin \Gamma\) and all \(\omega\).

Part 2 of Theorem 3.10 is actually a deterministic result, and it holds for any potential vanishing outside \(\Gamma\).

We prove this theorem in Sect. 7.

### 4. The Random Operator and Its Spectrum

In this section, we consider operators with random potential. To emphasise this, we write

\[
H_\omega := H_0 + V_\omega. \tag{32}
\]

**Hypothesis 4.1.** We suppose that the potentials \(V_\omega(\gamma), \gamma \in \Gamma\) are i.i.d. with a common distribution \(P_0\). We assume that the support \(S\) of \(P_0\) is compact.

The assumption that \(S\) is compact can be relaxed considerably but we do not bother to do so.

We denote the corresponding probability space by \((\Omega, \mathcal{F}, \mathbb{P})\) which can and will be taken to be \(\left(S^\Gamma, \bigotimes_{\gamma \in \Gamma} \mathcal{B}(S), \bigotimes_{\gamma \in \Gamma} P_0\right)\).

We write the lattice \(L\) as \(L = L_1 \times L_2\) with \(L_1 \subset \mathbb{Z}^{d_1}\) and \(L_2 \subset \mathbb{Z}^{d_2}\) and points \(x\) in \(X\) as \(x = (x_1, x_2)\) with \(x_i \in \mathbb{Z}^{d_i}\).
We define ‘shift’ operators $T_j, j \in \mathbb{L} := \mathbb{L}_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by
\begin{equation}
T_j \omega(x_1, x_2) := \omega(x_1, x_2 - j)
\end{equation}
(33)

It is easy to see that the shift $T_j$ is measure preserving, i.e. $\mathbb{P}(T_j^{-1} A) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$.

The following result tells us that the family $\{T_j\}_{j \in \mathbb{L}'}$ is ergodic:

**Proposition 4.2.** If $A \in \mathcal{F}$ is invariant under $\{T_j\}_{j \in \mathbb{L}'}$, i.e. $T_j^{-1} A = A$ for all $j \in \mathbb{L}'$, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

This result can be found in [7], for example.

Define for $j \in \mathbb{L}'$ the shift operator
\begin{equation}
U_j u(x_1, x_2) := u(x_1, x_2 - j),
\end{equation}
(34)

for $(x_1, x_2) \in \mathcal{X}$.

The operators $U_j$ are unitary on $\ell^2(\mathcal{X})$; moreover, the operators $H_\omega$ are ergodic in the sense
\begin{equation}
H_{T_j \omega} = U_j H_\omega U_j^*.
\end{equation}
(35)

with ergodic $T_j$ by Proposition 4.2

It follows (see, for example, [14]):

**Proposition 4.3.**
1. The spectrum $\sigma(H_\omega)$ is non-random (almost surely).
2. The same is true for the measure theoretic parts of the spectrum (the absolutely continuous part $\sigma_{ac}(H_\omega)$, the singular continuous part, etc.).
3. There is (almost surely) no discrete spectrum.

**Definition 4.4.** We denote by $\Sigma$ the almost sure spectrum of $H_\omega$, i.e.
$\Sigma = \sigma(H_\omega)$ $\mathbb{P}$-almost surely.

We now investigate the spectrum (as a set).

**Definition 4.5.** A function $W : \mathcal{X} \in \mathbb{R}$ is called an admissible potential (with respect to $P_0$) if
\begin{align*}
W(x) &\in \mathcal{S} = \text{supp } P_0 & \text{if } x \in \Gamma, \\
W(x) &\equiv 0 & \text{otherwise}.
\end{align*}

We denote the set of admissible potentials by $\mathcal{A}$.

**Remark 4.6.** Taking
\begin{equation}
(\Omega, \mathcal{F}, \mathbb{P}) = \left( (\mathcal{S})^\Gamma, \bigotimes_{\gamma \in \Gamma} \mathcal{B}(\mathcal{S}), \bigotimes_{\gamma \in \Gamma} P_0 \right)
\end{equation}
(36)

there is a one-to-one correspondence $\tau$ between $\Omega$ and the set $\mathcal{A}$ of admissible potentials, namely $\tau(\omega)(n) = \sum_{\gamma \in \Gamma} \omega_\gamma \delta_\gamma n$.

We may therefore identify $\Omega$ and $\mathcal{A}$.

**Theorem 4.7.**
1. If $W$ is an admissible potential, then $\sigma(H_0 + W) \subset \Sigma$.
2. We have
\begin{equation}
\Sigma = \bigcup_{W \in \mathcal{A}} \sigma(H_0 + W).
\end{equation}
(37)
Proof. 1. For $E \in \sigma(H_0 + W)$, there exists a Weyl sequence of functions $\varphi_n$ with compact (hence finite) support; more precisely we may suppose that $\|\varphi_n\| = 1$ and
\[
\|(H_0 + W - E) \varphi_n\| < \frac{1}{n}
\] (38)
Set $S_n = \text{supp} \varphi_n$ which is a finite set. By the Borel–Cantelli lemma, there is a vector $j_n \in L'$ such that
\[
\sup_{k \in S_n} |W(k) - V_\omega(k + j_n)| < \frac{1}{n}
\] (39)
With $\psi_n(x) = \varphi_n(x + j_n)$, we therefore get
\[
\|(H_\omega - E) \psi_n\| < \frac{2}{n},
\] (40)
thus $E \in \sigma(H_\omega)$.
2. Since the set $A$ has probability one (in the sense of Remark 4.6)
\[
\Sigma \subset \bigcup_{W \in A} \sigma(H_0 + W).
\] (41)
This together with 1. proves the theorem. □

Set $V_a = a \chi_\Gamma$ then $V_a$ is an admissible potential if $a \in \text{supp} \ P_0$. We define $H_a = H_0 + V_a$ and set $E_{\min}(a) = \inf \sigma(H_a)$ and $E_{\max}(a) = \sup \sigma(H_a)$.

Theorem 4.8. If $\text{supp} \ P_0 = [a, b]$, then
\[
\Sigma = [E_{\min}(a), E_{\max}(b)].
\] (42)

Proof. Since $V_x$ are admissible for all $x \in [a, b]$ and due to continuity, we have $\Sigma \supset [E_{\min}(a), E_{\max}(b)]$.

Suppose now, that $W$ is an admissible potential then by monotonicity
\[
H_a \leq H_0 + W \leq H_b.
\]
So, $\sigma(H_0 + W) \subset [\inf \sigma(H_a), \sup \sigma(H_b)]$. □

We will have a closer look at the operators $H_a$. Let us denote by $H_{0, \Gamma}$ and $H_{0, \Gamma^c}$ the operator $H_0$ restricted to $\ell^2(\Gamma)$ and $\ell^2(\Gamma^c)$, respectively, with simple boundary conditions.

Proposition 4.9. For any $a \in \mathbb{R}$
\[
\inf \sigma(H_a) \leq \inf \sigma(H_{0, \Gamma^c})
\] (43)
and $\sup \sigma(H_a) \geq \sup \sigma(H_{0, \Gamma^c})$. (44)

Moreover, if $a > 0$, then
\[
\sup \sigma(H_a) > \sup \sigma(H_{0, \Gamma^c})
\] (45)

Remark 4.10. For $a < 0$ we have $\inf \sigma(H_a) < \inf \sigma(H_{0, \Gamma^c})$.  

Proof. Take $\varphi \in \ell^2(\Gamma^c)$ with $\|\varphi\|_{\ell^2(\Gamma^c)} = 1$ and define $\tilde{\varphi} \in \ell^2(\mathcal{X})$ by
\[
\tilde{\varphi}(n) = \begin{cases} 
\varphi(n), & \text{for } n \in \Gamma^c; \\
0, & \text{otherwise.}
\end{cases}
\] (46)
Then,
\[
\langle \varphi, H_0, \Gamma \varphi \rangle_{\ell^2(\Gamma^c)} = \langle \tilde{\varphi}, H_a \tilde{\varphi} \rangle_{\ell^2(\mathcal{X})}. \tag{47}
\]
From (47), the equalities (43) and (44) follow by the Min-Max-Principle.

To prove the strict inequality (45), we show that $\sup H_a$ is strictly increasing with $a$.

The operator $H_a$ is periodic; therefore, $E_a = \sup \sigma(H_a)$ is given by the sup $h_a$ where $h_a$ is the operator $H_a$ restricted to the periodic cell $C_0$ with periodic boundary conditions. The corresponding eigenfunction $\psi_a$ is strictly positive.

By the Hellmann–Feynman theorem
\[
\frac{d}{da} E_a = \sum_{j \in C_0 \cap \Gamma} |\psi_a(j)|^2 > 0, \tag{48}
\]
hence $E_a < E_b$ if $a < b$. This proves 45. The inequality in Remark 4.10 is proved in a similar way. □

Now, we consider the case that $\Gamma = G \times \mathbb{Z}^{d_2}$. In this case, the operator $H_a$ separates in the following way.

Definition 4.11. The space $\mathcal{X}$ splits in a part $\mathcal{X} = Z \subset Z^{d_1} \times Z^{d_2}$, namely
\[
\mathcal{X}_\infty = Z^{d_1} \times Z^{d_2}, \tag{49}
\]
\[
\mathcal{X}_{M_1,M_2} = Z \times Z^{d_2}. \tag{50}
\]

We denote the Laplacian on $\ell^2(Z)$ (possibly with periodic boundary conditions) by $H_0^{(1)}$ and the Laplacian on $\ell^2(Z^{d_2})$ by $H_0^{(2)}$. We also set $H_a^{(1)} = H_0^{(1)} + a \chi_G$.

Then
\[
H_a = (H_a^{(1)} \otimes 1_{Z^{d_2}}) \oplus (1_Z \otimes H_0^{(2)}). \tag{51}
\]
Consequently
\[
\sigma(H_a) = \sigma(H_a^{(1)}) + [-2d_2, 2d_2]. \tag{52}
\]
This proves the following Corollary:

Corollary 4.12. If $\Gamma = G \times \mathbb{Z}^{d_2}$ and $\sup P_0 = [a, b]$ then
\[
\Sigma = [\inf \sigma(H_a^{(1)}), \sup \sigma(H_b^{(1)})] + [-2d_2, 2d_2]. \tag{53}
\]

The above consideration also allows us to prove Proposition 3.8.

Proof. (Proposition 3.8) Part 1. follows by computation. Part 2. follows from (52) □
5. Localisation

In this section, we prove pure point spectrum with exponentially decaying eigenfunction for energies $E \notin \sigma(H_{0,\Gamma^c})$ for high enough disorder.

For the ($\mathbb{L}$-periodic) set $\Gamma$ of ‘active’ sites, we merely assume that $\Gamma \neq \emptyset$. We may also assume that $\Gamma \neq \mathbb{C}_0$, since otherwise $\Gamma$ is the whole space, a case which is known, of course.

In the following, we consider the case $X = X_\infty$, at the end of this section we comment on the case $X_{M_1 M_2}$.

We will use multiscale analysis (see [15] and references given there). During the proof we need to consider boxes $\Lambda$ which are unions of shifted $\mathbb{C}_0$. Recall that the unit cell $\mathbb{C}_0$ is defined by:

\[ \mathbb{C}_0 = \{ x \in \mathbb{Z}^d | 0 \leq x_{\nu} \leq p_{\nu} - 1 \text{ for all } \nu \} \]  \hspace{1cm} (54)

**Definition 5.1.** We call a set $\Lambda \subset \mathbb{Z}^d$ a $\mathbb{C}_0$-box, if

\[ \Lambda = \{ x \in \mathbb{Z}^d | L_{\nu}p_{\nu} \leq x_{\nu} \leq L'_{\nu}p_{\nu} - 1 \text{ for all } \nu \} \]  \hspace{1cm} (55)

One of the crucial ingredients of (most versions of) multiscale analysis is the Wegner–Estimate. To prove this, we need the following result. By $H_\Lambda^\omega$, we denote the operator $H_\omega = H_0 + V_\omega$ restricted to $\ell^2(\Lambda)$ with periodic boundary conditions.

**Proposition 5.2.** Suppose $\Lambda$ is a $\mathbb{C}_0$-cube and $E \notin \sigma(H_{0,\Gamma^c})$ and

\[ H_\omega^\Lambda \psi = E \psi \]  \hspace{1cm} (56)

then

\[ \| \psi \|_{\ell^2(\Lambda)} \leq \frac{C}{\operatorname{dist}(E, \sigma(H_{0,\Gamma^c}))} \| \psi \|_{\ell^2(\Lambda \cap \Gamma)} \]  \hspace{1cm} (57)

with a constant $C$ which is independent of $\Lambda$ and $E$.

**Proof.** Set $\Lambda_1 = \Lambda \cap \Gamma^c$ and $\Lambda_2 = \Lambda \cap \Gamma$. We write $\ell^2(\Lambda)$

\[ \ell^2(\Lambda) = \ell^2(\Lambda_1) \oplus \ell^2(\Lambda_2) \]  \hspace{1cm} (58)

Accordingly, we may write the operator $H_\omega$ in block matrix form:

\[ H_\omega^\Lambda = \begin{pmatrix} H_0^{\Lambda_1} & T \\ T^* & H_0^{\Lambda_2} + V_\omega \end{pmatrix} \]  \hspace{1cm} (59)

The operator $T : \ell^2(\Lambda_2) \to \ell^2(\Lambda_1)$ ‘restores’ the links between $\Lambda_2$ and $\Lambda_1$.

The eigenvalue equation (56) reads

\[ \begin{pmatrix} H_0^{\Lambda_1} & T \\ T^* & H_0^{\Lambda_2} + V_\omega \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]  \hspace{1cm} (60)

So, in particular

\[ (H_0^{\Lambda_1} - E) \psi_1 = -T \psi_2 \]  \hspace{1cm} (61)
Thus,
\[ \| \psi \|_{L^2(\Lambda_1)} = \| \psi_1 \| \]  \quad (62)
\[ \leq \| (H_{\Lambda_1}^0 - E)^{-1} \| \| T \| \| \psi_2 \| \]  \quad (63)

Since for given \( m \) we have \( |T(n, m)| = 1 \) for at most \( 2d \) point \( n \) and \( T(n, m) = 0 \) otherwise, we conclude \( \| T \| \leq 2d. \)

Since we consider periodic boundary conditions on \( \Lambda \), we have
\[ \sigma(H_{\Lambda_1}^0) \subset \sigma(H_{0, \Gamma^c}) \]  \quad (64)

as any eigenfunction on \( \Lambda \) can be periodically extended to an eigenfunction on \( \mathcal{X} \).

Now we turn to the Wegner estimate. By \( N(A, E) \) we denote the number of eigenvalues of the operator \( A \) up to \( E \).

**Theorem 5.3.** If \( \text{dist}(E, \sigma(H_{0, \Gamma^c})) \geq \gamma \) and \( 0 \leq \varepsilon \leq \frac{\gamma}{2} \), then
\[ \mathbb{E}(N(H_\omega(\Lambda), E + \varepsilon) - N(H_\omega(\Lambda), E - \varepsilon)) \leq \frac{C}{\gamma} \| \rho \|_{\infty} \varepsilon |\Lambda| \]  \quad (65)

where \( |\Lambda| \) denote the volume (number of points) of \( \Lambda \).

**Proof.** The proof is a combination of the proofs from [15] and [12]. We sketch the main ideas. We use the abbreviation \( H = H_\omega(\Lambda) \).

Let \( g \) be a monotone \( C^\infty \)-function with \( 0 \leq g(t) \leq 1 \), \( g(t) = 0 \) for \( t \leq -2\varepsilon \) and \( g(t) = 1 \) for \( t \geq 2\varepsilon \).

We obtain
\[ N(H, E + \varepsilon) - N(H, E + \varepsilon) \leq \text{tr} g(H - E + 2\varepsilon) - \text{tr} g(H - E - 2\varepsilon) \]  \quad (66)
\[ = \int_{E-2\varepsilon}^{E+2\varepsilon} \text{tr} g'(H - \lambda) \, d\lambda \]  \quad (67)

Let \( E_n \) denote the eigenvalues of \( H_\omega(\Lambda) \) labelled in increasing order. These eigenvalues depend on the values \( v_j := V_\omega(j), j \in \Lambda_2 \).

Thus, we may consider
\[ \sum_{j \in \Lambda_2} \frac{\partial}{\partial v_j} \text{tr} g(H - \lambda) = \sum_n \sum_{j \in \Lambda_2} \frac{\partial}{\partial v_j} g(E_n - \lambda) \]  \quad (68)
\[ = \sum_n g'(E_n - \lambda) \sum_{j \in \Lambda_2} \frac{\partial E_n}{\partial v_j} \]  \quad (69)
\[ \geq \sum_n g'(E_n - \lambda) \sum_{j \in \Lambda_2} |\psi_n(j)|^2 \]  \quad (70)
\[ \geq C' \text{dist}(E, \sigma(H_{0, \Gamma^c})) \text{tr} g'(H - \lambda) \]  \quad (71)

where \( \psi_n \) is a normalised eigenfunction of \( H \) with eigenvalue \( E_n \). Above we used the Hellmann–Feynman theorem and, in the final step, Proposition 5.2.
Summarising we proved
\[
\mathbb{E}\left( N(H, E + \varepsilon) - N(H, E + \varepsilon) \right) \leq C' \text{dist}(E, \sigma(H_0\Gamma^c)) \sum_{j \in \Lambda_2} \int_{E-2\varepsilon}^{E+2\varepsilon} \mathbb{E}\left( \frac{\partial}{\partial v_j} \text{tr} g(H - \lambda) \right) d\lambda
\]  (72)

Suppose supp \( P_0 \subset [a, b] \) and denote by \( H(v_j = c) \) the operator \( H \) with \( V_j \) replaced by the value \( c \), then
\[
\int \frac{\partial}{\partial v_j} \text{tr} g(H - \lambda) \rho(v_j) dv_j \leq \| \rho \|_\infty \left( \text{tr} g(H(v_j = b) - \lambda) - \text{tr} g(H(V_j = a) - \lambda) \right) \leq \| \rho \|_\infty
\]  (73)

We used that changing the potential at one site \( j \) is a rank one perturbation and \( 0 \leq g(\lambda) \leq 1 \).

Performing the integrals over the \( v_k, k \neq j \) gives the desired result. \( \Box \)

Once we have the Wegner estimate, the multiscale analysis follows the usual path. We need an initial length scale estimate and the induction step over growing length scales.

The initial length scale estimate follows directly from the Wegner estimate Theorem 5.3. In fact, as long as we are away from the spectrum of \( H_0\Gamma^c \), we can make the right hand side of (65) as small as we like by taking \( \| \rho \|_\infty \) small. This corresponds to high disorder. Using a Combes—Thomas estimate, this allows us to prove the initial scale estimate. For details see Section 11.1 in [15].

The induction step follows the lines in [15] sections 9 and 10. The only difference being that we deal with periodic boundary conditions while [15] uses simple boundary conditions.

If \( X = X_{M_1, M_2} \), we start the induction with a cube of the form:
\[
\Lambda = \{ x \in X_{M_1, M_2} \mid L_{\nu \nu}^p \leq x_{\nu} \leq L_{\nu \nu}^p - 1 \text{ for } \nu = d_1 + 1, \ldots, d_1 + d_2 \} \]  (76)

Corollary 3.4 follows from the work [3] of Damanik and Stollmann.

6. Absolutely Continuous Spectrum

In this section we consider a special case of the above operators.

We start with the following observation:

**Proposition 6.1.** Assume \( \Gamma = G \times \mathbb{Z}^{d_2} \) and suppose the (otherwise arbitrary) potential \( W \) is concentrated on \( \Gamma \). If there exists a polynomially bounded solution \( \psi \) of
\[
H_0^{(1)} \psi = e \psi
\]  (77)

which vanishes on \( G \), then
\[
e + [-2d_2, 2d_2] \subset \sigma(H_0 + W)
\]  (78)
Remark 6.2. \( H_0^{(1)} \) and \( H_0^{(2)} \) were defined in Definition 4.11.

Proof. Any \( \eta \in [-2d_2, 2d_2] \) is of the form \( \eta = 2 \sum_{\nu=1}^{d_2} \cos(\kappa_\nu \nu) \) and \( \varphi(x) = \prod_{\nu=1}^{d_2} \sin(\kappa_\nu x_\nu) \) is a (bounded) function solving
\[
H_0^{(2)} \varphi = \eta \varphi. \tag{79}
\]
This can be verified by applying the addition theorem for the sinus. Consequently, \( \Psi(x, y) := \psi(x) \varphi(y) \) is a bounded solution to
\[
H_0 \Psi = (e + \eta) \Psi. \tag{80}
\]
Since \( \psi \) vanishes on \( G \), \( \Psi \) vanishes on \( \Gamma \), so
\[
(H_0 + W) \Psi = H_0 \Psi = (e + \eta) \Psi. \tag{81}
\]
Thus, \( e + \eta \) is a generalised eigenvalue of \( H_0 + W \). By Sch’nol’s theorem any generalised eigenvalue belongs to the spectrum (see [25], Section C4 or [15], Section 7.1).

We discuss a class of examples for which Proposition 6.1 applies.

We look at \( \mathcal{X} \) or at the strip \( \mathcal{X}_{M_1, M_2} \). In the latter case, we impose periodic boundary conditions and take \( M_2 - M_1 \) is even. This way we avoid to discuss various cases separately.

For \( L = (\ell_1, \ldots, \ell_{d_1}) \), \( \ell_\nu \in \{1, 2, \ldots, p_\nu - 1\} \) we set
\[
\Phi_L(x_1, \ldots, x_{d_1}) := \prod_{\nu=1}^{d_1} \sin\left(\frac{\pi \ell_\nu}{p_\nu} x_\nu\right) \tag{82}
\]

Lemma 6.3. Under condition (24), the function \( \Phi_L(x) \) is a solution to
\[
(H_0^{(1)} + W) \psi = 2 \left( \sum_{\nu=1}^{d_2} \cos\left(\frac{\pi \ell_\nu}{p_\nu} x_\nu\right) \right) \psi \tag{83}
\]
and
\[
\Phi_L(x) = 0 \quad \text{for } x \in G \tag{84}
\]

Proof. Again by applying addition theorems and the fact that \( \Phi_L \) vanishes on \( G_0 \), hence on \( G \), we see that (83) holds. Moreover, since \( M_2 - M_1 \) is even \( \Phi_L \) satisfies periodic boundary conditions.

Now, we are ready to prove Theorem 3.9.

Proof. (Theorem 3.9) Take \( E \in \mathcal{E} \), then \( E = e_L + \eta \) for some \( \eta \in [-2d_2, 2d_2] \). Denote by \( E_L \) the (one-dimensional) subspace of \( \ell^2(\mathbb{Z}) \) generated by the eigenfunction \( \Phi_L \).

The (closed) subspace \( \mathfrak{h}_L = E_L \otimes \ell^2(\mathbb{Z}^{d_2}) \) of \( \ell^2(\mathcal{X}) \) is invariant under the operator \( H_0 + W \) and restricted to \( \mathfrak{h}_L \) the operators \( H_0 + W \) and \( H_0 \) agree. Consequently, \( H_0 + W \) on \( \mathfrak{h}_L \) is unitarily equivalent to \( H_0^{(2)} + e_L \) on \( \ell^2(\mathbb{Z}^{d_2}) \), an operator with purely absolutely continuous spectrum.
7. Absence of Exponential Localisation

We start with a general observation. Let $W$ be an arbitrary potential and denote by $G_W(x,y)$ the Green’s function for $H := H_0 + W$, i.e. the kernel of the operator $(H_0 + W - z)^{-1}$.

**Theorem 7.1.** If for some $E \in \mathbb{R}$

$$H \psi = (H_0 + W) \psi = E \psi$$

for a bounded function $\psi$, then for all $x \in \mathbb{Z}^d$ with $\psi(x) \neq 0$

$$\liminf_{\zeta \searrow 0} \zeta \sum_{y \in \mathbb{Z}^d} |G_W^{E+i\zeta}(x,y)| > 0,$$

in particular

$$\sup_{\zeta \searrow 0} \sum_{y \in \mathbb{Z}^d} |G_W^{E+i\zeta}(x,y)| = \infty. \quad (87)$$

Observe that $|G_{E+i\zeta}(x,y)| \leq \frac{C}{\eta} e^{-c\eta|x-y|}$ by the Combes–Thomas estimate (see, for example, [15]). Thus, for any $\zeta > 0$

$$\sum_{y \in \mathbb{Z}^d} |G_W^{E+i\zeta}(x,y)| < \infty. \quad (88)$$

**Proof.** Take $\varepsilon > 0$ arbitrary and assume that $|\psi(x)| \leq A < \infty$.

Let

$$\Lambda_L = \{ n \in \mathbb{Z}^d \mid |n_\nu| \leq L \text{ for } \nu = 1, \ldots, d \}$$

and

$$\partial' \Lambda_L = \{ n \in \mathbb{Z}^d \mid |n_\nu - L| \leq 2 \text{ for some } \nu \}$$

Denote by $\chi_L$ the characteristic function of $\Lambda_L$ and set $\psi_L := \chi_L \psi$.

We compute

$$H \psi_L(x) = \chi_L(x)(H \psi)(x) + \sum_{|j|=1} \psi(x + j)(\chi_L(x + j) - \chi_L(x))$$

$$= E \psi_L(x) + \sum_{|j|=1} \psi(x + j)(\chi_L(x + j) - \chi_L(x)).$$

It follows that for $L$ big enough

$$\psi(x) = \sum_{y \in \mathbb{Z}^d} G_W^{E+i\zeta}(x,y) \left( \sum_{|e|=1} \psi(y + e)(\chi_L(y + e) - \chi_L(y)) - i\zeta \psi(y) \chi_L(y) \right)$$

Observe that $\chi_L(x + j) - \chi_L(x)$ (with $|j| = 1$) vanishes outside $\partial' \Lambda_L$. Consequently,

$$|\psi(x)| \leq 2d A \sum_{y \in \partial' \Lambda_L} |G_W^{E+i\zeta}(x,y)| + \zeta \sum_{y \in \mathbb{Z}^d} |G_W^{E+i\zeta}(x,y)| \quad (89)$$
Since \( \sum_{y \in \mathbb{Z}^d} |G_{E+i\zeta}^W(x,y)| < \infty \) for \( \zeta > 0 \), we can choose \( L \) (depending on \( \zeta > 0 \)) such that \( 2dA \sum_{y \in \partial' \Lambda_L} |G_{E+i\zeta}^W(x,y)| < \varepsilon \) Then,
\[
|\psi(x)| \leq \zeta \sum_{y \in \mathbb{Z}^d} |G_{E+i\zeta}^W(x,y)| + \varepsilon \quad (90)
\]
Now suppose that \( \liminf_{\zeta \searrow 0} \zeta \sum_{y \in \mathbb{Z}^d} |G_{E+i\zeta}^W(x,y)| = 0 \) Then, as \( \varepsilon \) was arbitrary (90) implies \( \psi(x) = 0 \) which is a contradiction. \( \square \)

We apply the above theorem to our model.

**Theorem 7.2.** Assume Condition (24) holds and let \( W \) be an admissible potential on \( \mathcal{X}_\infty \).

Then, for each \( x_0 \in \mathbb{Z}^d \) and for all \( E \in \mathcal{E} \)
\[
\sup_{x \in x_0 + C_0} \sup_{\zeta \searrow 0} \sum_{y \in \mathbb{Z}^d} |G_{E+i\zeta}^W(x,y)| = \infty \quad (91)
\]
**Proof.** Take \( E \in \mathcal{E} \), then \( E = e_L + \eta \) with
\[
\eta = 2 \sum_{k=1}^{d_2} \cos(\pi \kappa_k) \quad (92)
\]
for some \( L \) and some \( \kappa_k \). It follows that
\[
\psi(x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}) := \prod_{j=1}^{d_1} \sin\left(\frac{\pi \ell_j}{p_j} x_j\right) \prod_{k=1}^{d_2} e^{i\pi \kappa_k y_k} \quad (93)
\]
vanishes on \( \Gamma \) and is a solution to
\[
H_0 \psi = (H_0 + W) \psi = E \psi, \quad (94)
\]
with \( ||\psi||_{\infty} \leq 1 \). An application of Theorem 7.1 gives the result. \( \square \)

[15]

**Acknowledgements**

A substantial part of this research was done during a visit of the first named author (WK) at Ashoka University, Haryana, India. The warm hospitality at Ashoka University and financial support by this institution is gratefully acknowledged.

**Funding** Open Access funding enabled and organized by Projekt DEAL.
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Communicated by Claude-Alain Pillet.  
Received: March 9, 2021.  
Accepted: March 13, 2022.