Adiabatic charge pumping through a dot at the junction of \( N \) quantum wires

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(Dated: November 11, 2018)

We study adiabatic charge pumping through a quantum dot placed at the junction of \( N \) quantum wires. We explicitly map out the pattern of pumped charge as a function of the time-varying tunneling parameters coupling the wires to the dot and the phase between any two time varying parameters controlling the shape of the dot. We find that with \( N - 2 \) time-independent well-coupled leads, the maximum pumped charge in the remaining two leads is strongly suppressed with increasing \( N \), leading to the possibility of tuning of the pumped charge, by modulating the coupling of the \( N - 2 \) leads.

PACS numbers: 73.23Hk,72.10Bg,73.40Ei

An adiabatic charge pump\(^1\) was theoretically proposed by Thouless for an isolated system and later extended to open systems at finite temperature\(^2\).\(^3\).\(^4\). It is essentially a device that drives current through a system at zero bias by a time-dependent variation of two or more parameters. Although theoretically proposed about a decade ago, Switkes et al\(^5\) recently demonstrated that an adiabatic electron pump, which produces a dc voltage in response to a cyclic change in the confining potential of an open quantum dot was experimentally feasible. This has led to renewed interest\(^6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\) in the field.

In this paper, we study adiabatic charge pumping through a quantum dot placed at a junction of \( N \) quantum wires and which is well-coupled to the wires. We show that under a time dependent variation of two of the tunneling parameters to the dot, charge \( Q_{ij} = Q(i, \tau) - Q(j, \tau) \) can be pumped from wire \( i \) to wire \( j \). An important point to note is that unlike the case when \( N = 2 \), the transmission between any two wires for \( N \geq 3 \) never reaches unity anywhere in the parameter space, because of the possibility of leaking out through the other wires. Thus, analogously to the \( N = 2 \) case\(^14\), large pumping occurs when the ‘pumping contour’ (a closed curve in the \( J_i-J_j \) plane, where \( J_i \) and \( J_j \) denote the tunneling parameters between wire \( i, j \) and the dot) encloses transmission maxima peaks (when \( \sum_j T_{ij} = 1 \)) and cuts the resonance line at points where the transmission is small, thus ensuring that most of the resonant line is enclosed.

Consider a time-dependent scatterer connected to \( N \) single channel leads. In the adiabatic limit, at leading order, \( O(\omega^0) \), where \( \omega \) is the frequency), the incoming particle sees a static scatterer, and hence the scattering and the current can be computed using time-independent scattering theory (the Landauer-Buttiker formula). But as shown by Buttiker, Pretre and Thomas\(^3\) and Brouwer\(^4\), even at order \( O(\omega^1) \), the current and the charge pumped can be computed from the ‘frozen’ or ‘quasi-static’ scattering matrix\(^12\) data. For a simultaneous slow and small variation of two time-dependent parameters \( X_1 \) and \( X_2 \), the charge emitted from wire \( i \) is given by

\[
Q(i, \tau) = \frac{e}{2\pi} \int_0^T \left| r_{ii} \right| \left| \frac{d\theta_i}{dt} \right| + \sum_{j \neq i} |t_{ij}|^2 \left| \frac{d\psi_{ij}}{dt} \right| \tag{1}
\]

\[
= \frac{e}{2\pi} \int_0^T \left| \frac{d\theta_i}{dt} \right| + \sum_{j = 1}^{N} |t_{ij}|^2 \left| \frac{d\psi_{ij}}{dt} \right| \tag{2}
\]

where the first term is denoted as \( Q_{int} \) and the second as \( Q_{tran} \) and the \( S \)-matrix for an \( N \)-lead system is given by

\[
S = \begin{pmatrix}
|r_{11}|e^{i\theta_1} & |t_{12}|e^{i\psi_{12}} & |t_{13}|e^{i\psi_{13}} & \cdots \\
|t_{21}|e^{i\psi_{21}} & |r_{22}|e^{i\theta_2} & |t_{23}|e^{i\psi_{23}} & \cdots \\
|t_{31}|e^{i\psi_{31}} & |t_{32}|e^{i\psi_{32}} & |r_{33}|e^{i\theta_3} & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix} \tag{3}
\]

\( r_{ii}, t_{ij}, \theta_i \) and \( \psi_{ij} \) are time dependent in general. We will mainly use Eq.\(^2\) in our computations. Note that the first term on the RHS of Eq.\(^2\) is a total derivative and denotes the winding number around the \( R = 0 \) singularity. Thus \( Q_{int} \) is forced to be an integer. The second term is not quantised, however, and hence, the pumped charge is not an integer. But by choosing the contour appropriately, to cut the transmission resonance maxima when it is small, \( Q_{tran} \) can be made small, and the charge pumped is almost quantised\(^14\). As we shall see, this is harder to arrange when \( N > 2 \), because there is a possibility of transmission into the other wires. This explains why the value of the maximum pumped charge decreases as \( N \) increases, which is one of the main results of this report.

Let us now write down the lattice model for a quantum dot at the junction of \( N \) quantum wires. The wires are modeled as a one-dimensional chain of sites, with on-site energy zero and with nearest neighbour hopping amplitudes given by \(-J\). The dot is again modeled as a bunch of tight-binding sites with nearest neighbour hoppings given by \(-J_D\) and on-site energy \( \epsilon_0 \). Finally, the dot is coupled to the \( i^{th} \) lead by a hopping element \( J_i \). In the
work presented in this brief report, we have taken $J_D = J$ and $\epsilon = 0$ for simplicity, since taking $J_D \neq J$ and $\epsilon \neq 0$ makes no qualitative difference to the results\cite{18}. Thus, the Hamiltonian of the system is given by

$$H = H_{\text{wires}} + H_{\text{dot-wire}} + H_{\text{dot}}$$

$$= \sum_{i=1}^{N} \left[ -J \sum_{m=l}^{l+1} c_{im}^{\dagger} c_{im+1} - J_0 c_{10}^{\dagger} c_{11} + h.c. \right]$$

Here $L$ denotes the number of sites on each wire with wire index $i$ and $l$ denotes the number of sites on each wire representing the dot. $c_0$ denotes the dot point which is common to all the wires. Since our focus here is on charge pumping, we have not included spin in the above Hamiltonian.

For fixed values of $J$ and $J_j$, ($i = 1, \ldots, N$), the scattering problem is straightforward. Let us assume that the incoming wave is along wire 1, without loss of generality. Then the wave-function at the site $n$ in lead 1 is given by

$$c_{in} = \exp(-ikx_n) + r_{11} \exp(ikx_n); \quad (n \geq l + 1)$$

and the wave-function at site $n$ in all other leads $i = 2, \ldots, N$ are given by

$$c_{in} = t_{ij} \exp(ikx_n); \quad (n \geq l + 1)$$

where $x_n = na$ for the lattice spacing $a$. Note that the number of the sites, implies that incoming waves are denoted by $\exp(-ikx_n)$ whereas outgoing states are denoted by $\exp(ikx_n)$. The wave-function at the wire within the dot is given by

$$c_{in} = a_{jn} = P_j^{\dagger} \exp(-iqx_n) + P_j^{\dagger} \exp(iqx_n); \quad (n \leq l)$$

The wave-vectors $k$ and $q$ are related to the energy as

$$E_k = -2J \cos ka \quad \text{and} \quad E_k = \epsilon_0 - 2J_D \cos qa$$

as can be seen by solving the equations in the bulk of the wire and the dot. The set of boundary equations (for amplitudes at the boundary of the dot and the wire, i.e., sites $l$ and $l+1$ and at the junction point of the dot) can be solved to obtain $r_{11}$ and $t_{ij}$ in terms of $J, J_D$ and $J_i$, but it gets progressively more difficult for higher values of $N$ to get analytical results. However, the equations can be solved numerically and we can obtain the scattering $S$-matrix in the parameter space of the hopping amplitudes.

To obtain pumped charge, any two of the hopping parameters that couple the dot to the wire need to be adiabatically varied. Hence, in the lattice model, two of the hopping parameters that couple the dot to the wires are taken to oscillate with the frequency $\omega$, with a modulation parameter $P_i$ and a phase difference $2\phi$, i.e., we choose

$$J_i = J_{i0} + P_i \cos(\omega t + \phi)$$

$$J_{j\neq i} = J_{j0} + P_j \cos(\omega t - \phi)$$

where $j$ is one value not equal to $i$. In this report, we choose $P_i = P_j$. The case where $P_i \neq P_j$, as also the cases when more than two parameters are varied will be considered elsewhere\cite{18}.

The charge pumped out from wire $i$, $Q(i, \tau)$ is then computed from the time dependence of the $S$-matrix parameters $r_{ii}$, which, in turn, are known in terms of the hopping parameters, using Eq\cite{2}. Finally, we can obtain the charge pumped between different wires as

$$Q_{ij} = Q(i, \tau) - Q(j, \tau) ,$$

Note that $Q(i, \tau)$ is computed with the wave incident in wire $i$, whereas $Q(j, \tau)$ is computed with the wave incident in wire $j$. The results for the pumped charges for the different cases are given below.

The parameters that affect the value of the pumped charge are the pumping amplitude $P$ and the phase $\phi$. We have varied $\phi$ over its full range. Within the adiabatic approximation, the pumped charge is independent of $\omega$ and so we have chosen a specific value of $\omega$.

We have solved the set of boundary equations for $N = 2, 3$ and 4 wires. For $N = 2$, we have chosen $4N = 4$, and both the hopping amplitudes coupling the dot to the wire are made time-dependent using Eq\cite{6} and we obtain a single pumped charge $Q = \Delta Q_{12} - \Delta Q_{21}$. For $N = 3$, any two of the three hopping amplitudes coupling the dot (with $6N = 4$ sites) to the wire can be made time-dependent. Let us call them $J_1$ and $J_2$. We then obtain the pumped charge $Q = \Delta Q_{12} - \Delta Q_{21}$. (Similarly, by making the other hopping amplitudes time dependent, we can obtain $Q' = \Delta Q_{23} - \Delta Q_{32}$ and $Q'' = \Delta Q_{31} - \Delta Q_{13}$). The pumped charge $Q$ is computed when the third hopping amplitude $J_3 = 1.1$ which is of the same order as the mean values of $J_{1,2}$ within each cycle.

We have plotted the pumped charge $Q$ as a function of the pumping strength $P_{1,2} = P$ and the phase $\phi$ in the contour plots of Fig\cite{1} for $N = 2$ and 3. We have chosen $\phi$ to go from $0 \rightarrow \pi$, since Eq\cite{10} shows that the difference in the phase between the tunneling parameters on the two leads connected to the dot, is given by $2\phi$, and we have chosen a maximum of 10 for $P$. In Fig\cite{2} we have plotted the different classes of contours in the $P - \phi$ plane for $N = 2$ and 3. We find that the contours can be classified into 3 categories. Those which (A) do not enclose either of the two $|T| = 1$ points, (B) includes one of them more number of times than the other and finally (not shown for $N = 3$) (C) includes both of them equal number of times. We see (as was shown in Ref.\cite{14}) that the pumped charge is large only in the case (B) with the sign of the pumped charge depending on the sense in which the contour encloses the $|T| = 1$ point. We note that the pattern of pumped charge remains the same for $N = 2$ and 3, since the form of the variation of the parameters remains the same. But the magnitude of the pumped charge decreases for $N = 3$. This is due to the "leak" of the wavefunction in the third direction, so that the transmission never reaches unity in the 1-2 direction.
FIG. 1: Contour plot of the pumped charge for $N = 2$ and 3 wires going from left to right. The $x$- and $y$- axes are $P \cos \phi$ and $P \sin \phi$. The colour scale is on the right of each plot. Note that the pattern remains the same in both; for $N = 2$, the pumped charge reaches a maximum of $\sim \pm 2$, whereas for $N = 3$, it reaches a maximum of $\pm 1.11$.

FIG. 2: Typical pumping contours as a function of the pumping parameters $X_1 = J_{12}$ and $X_2 = J_{23}$. For $N = 2$, the full pumping contours are shown for three classes corresponding to the regions $A,B$ and $C$ in Fig. 1(a). For $N = 3$, only a small part of contours of classes $A$ and $C$ corresponding to regions $A$ and $C$ in Fig. 1(b) is shown, along with the resonance points $R = 0$. Note that contour $A$ does not enclose the resonance points and contour $C$ encloses both once.

FIG. 3: The different contributions to the pumped charge $Q$ for $N = 2$ (a) and $N = 3$ (b). The full red line is $Q_{\text{int}}$, the blue dash-dot line is the total pumped charge $Q$ and the green (and magenta for $N = 3$) dotted lines are $Q_{\text{tran}}$. Note that large pumped charge is correlated with $Q_{\text{int}} \neq 0$. In Fig. 3(b), only a small region of $P$ is shown, because the width of $P$ for which pumped charge is large is smaller. The numerical values chosen to obtain $Q$ are $J_D = J = 1.0, J_{10} = J_{20} = 2.0, \phi = 0.05\pi k a = 0.001\pi, \omega = 1.0$.

In Fig. 3, we have separately shown the contributions of the the winding number term $Q_{\text{int}}$ and the transmission term $Q_{\text{tran}}$ see Eq. (2) for $N = 2$ and 3. These graphs clearly show that large pumped charge is always correlated with the winding number term being non-zero. Fig. 3(b) also shows that the range of values of $P$ for which the pumped charge is large is narrower for $N = 3$. This can be correlated with the coming together of the resonance lines as the coupling to the third wire is increased. This is shown explicitly in Fig. 4(a), and also in Fig. 2(b), where it is clear that it is hard to find contours which enclose one resonance but not the other. Thus, the parameter range $(P_c, \phi)$, for which the contours enclose one maxima and not the other is quite small, and so the region where pumped charge is large is small.

In Fig. 4(b), we have shown the plots of the pumped charge when $J_3 \ll J_{1,2}$ and $J_3 \gg J_{1,2}$. In either case, it can be checked that the plots are very similar to the $N = 2$ case studied earlier [14]. This is not surprising because both for $J_3$ very small and $J_3$ very large, the leak to the third wire is very small. But for intermediate $J_3$, we see that not only is the maximum pumped charge smaller, it occurs for a narrower range of parameters as expected.

What happens when 4 wires are attached to the dot and two of the hopping amplitudes coupling the wire to the dot are made time-dependent? Can we identify trends as the number of wires $N$ increases? To check this, we obtained the numerical results for $N = 4$ wires and we find that the magnitude of pumped charge decreases, with regions of large pumped charge getting thinner as we increase $N$. In Fig. 5, we plot the maximum pumped charge as a function of the number of wires well-coupled
FIG. 4: (a) Contour plot of transmission maxima \((T_{12} + T_{13})\) for three values of \(J_3\). Lines \(A\), \(B\) and \(C\) denote resonance lines for \(J_3 = 0.1\), 0.9 and 1.1 respectively. The crosses denote the \(R = 0\) points. Note that in the lower left quadrant, lines \(A\) and \(B\) are indistinguishable and the resonance point on \(A\) and \(B\) is indistinguishable and denoted by a single cross. (b) Pumped charge \(Q\) as a function of the pumping amplitude \(P\) for three values of \(J_3 = 0.1, 1.1\) and \(J_3 = 10.0\). The other parameters are the same as in Fig. 4.

Therefore, we conclude that it is harder to tune for ‘almost quantised’ pumped charge for pumping through a dot at a junction of \(N\) wires except for \(N = 2\). However, the magnitude of pumped charge is not vanishingly small for \(N = 3, 4\). There is a suppression of the maximum pumped charge from \(\sim 2\) for \(N = 2\) to about \(1.2\) for \(N = 3\) and to 0.45 for \(N = 4\). This can be easily explained as a ‘leak’ of the electrons into the wires that are not participating in the pumping process. Hence, it is easy to see that no pumped charge is expected in the large \(N\) limit.

In conclusion, we have explicitly obtained the pumped charge as a function of the various parameters such as pumping amplitude, phase \(\phi\), pumping frequency, etc, for pumping through a quantum dot placed at the junction of \(N\) wires where \(N = 2, 3\) and 4. We note that the maximum pumped charge reduces in magnitude as the number of wires is increased because some of the charge can leak out through the other wires which are not involved in the pumping process. The pumped charge is also seen to be large only when the pumping contour encloses the transmission maxima (not necessarily a \(|T| = 1\) point) appropriately and cuts the resonance line at a point where the transmission is low.

More general cases which are possible for \(N \geq 3\), which involve time dependent variation of all the tunneling parameters coupling the wire to the dot can also be studied\(^{[18]}\). One might expect that unless the time dependences are carefully adjusted the pumped charge would be quite low. However, they have not been explicitly considered here. The extensions to higher \(\omega\), where the adiabatic approximation breaks down will also be presented elsewhere.

ACKNOWLEDGEMENTS

We acknowledge use of the Beowulf cluster at the Harish-chandra research institute in our computations.

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