Highly anisotropic scaling limits

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Abstract

We consider a highly anisotropic $d = 2$ Ising spin model whose precise definition can be found at the beginning of Section 2. In this model the spins on a same horizontal line (layer) interact via a $d = 1$ Kac potential while the vertical interaction is between nearest neighbors, both interactions being ferromagnetic. The temperature is set equal to 1 which is the mean field critical value, so that the mean field limit for the Kac potential alone does not have a spontaneous magnetization. We compute the phase diagram of the full system in the Lebowitz-Penrose limit showing that due to the vertical interaction it has a spontaneous magnetization. The result is not covered by the Lebowitz-Penrose theory because our Kac potential has support on regions of positive codimension.

1 Introduction

This work focuses on the proof of the Lebowitz-Penrose limit for a highly anisotropic $d = 2$ Ising spin model which has been first studied in [2], its precise definition can be found at the beginning of Section 2. In this model the spins on a same horizontal line (layer) interact via a $d = 1$ Kac potential while the vertical interaction is between nearest neighbors, both interactions are ferromagnetic. The temperature is set equal to 1 which is the mean field critical value (without vertical interactions), so that the mean field limit for the Kac potential alone does not have spontaneous magnetization. However in [2] it is proved that even a small vertical interaction is sufficient to produce a phase transition at least for small values of the Kac scaling parameter $\gamma$. The idea in [2] is to study a model with fewer vertical interactions (those left have a chessboard structure): by the Ginibre inequalities if a spontaneous magnetization is present in the reduced model then it is also present in the true system as well. The advantage of working in the reduced system is that one can split the system into blocks of two layers, the vertical interaction is left only inside each block so that blocks do not interact vertically with each other; the horizontal interaction is unchanged. As a consequence in [2] it is shown that it is sufficient to carry out the Lebowitz-Penrose coarse graining procedure only for two-layer systems. It is then proved that this can be done, that there is a positive spontaneous magnetization in the limits volume to infinity and then $\gamma \to 0$ and that such a property remains valid also at

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finite small $\gamma > 0$. However the value of the spontaneous magnetization for the reduced system is certainly smaller than the real one because half of the vertical interactions has been dropped.

The problem of studying directly the original system and in particular to find its true spontaneous magnetization has been left open in [2], we attack it in this paper determining explicitly the limit phase diagram of the true system when first the volume goes to infinity and then $\gamma \to 0$. This is not covered by the Lebowitz-Penrose theory because our Kac potential is singular having support on regions of positive codimension. We hope in a successive paper to prove that there is a positive spontaneous magnetization also at $\gamma > 0$ which converges as $\gamma \to 0$ to the one found here.

This work is part of a more general project (which besides us involves several other colleagues) where we want to study systems with Kac potentials having support on regions of positive codimension plus short range interactions, both in equilibrium and non equilibrium. The description of the system is hybrid: referring to our Ising model we can make a coarse graining on each layer and introduce macroscopic variables but the interaction between layers is microscopic and it is described by an effective interaction to be determined. The purpose is to derive such an effective hamiltonian and find its ground states.

In this paper we compute the limit ground state energy but we hope in the future to study the excited states and derive the large deviations functional.

Similar structures are present in SOS models, for instance in the SOS interface models where the real valued spin variables $S_x$, $x \in \mathbb{Z}^d$, represent the position of an interface in $\mathbb{Z}^{d+1}$. Evidently the model is obtained by an anisotropic scaling limit for which the interface becomes sharp, the point $S_x$ on the “vertical” line through $x$, while the interaction among spins remains short range. We hope to establish such connections starting from models like the one considered here.

The appearance of a macroscopic description on the layers may also originate from a canonical constraint with or without the presence of a Kac potential. Considering the system in a finite box we may fix on each horizontal layer the total magnetization which gives rise to a multi-canonical ensemble. Indeed when we study the system with Kac potentials following the Lebowitz-Penrose procedure we coarse grain and get such multi-canonical ensembles. Our analysis will be based on a proof that equivalence of ensembles extends to such cases.

The multi-canonical constraint appears naturally in dynamical problems when we consider a Kawasaki dynamics on each layer so that the total magnetization on each layer is constant in time. The vertical interaction affects the rates of horizontal exchanges on the layers so that in the hydrodynamic limit the evolution is conjectured to be ruled by coupled diffusions. An interesting variant would be a weakly asymmetric simple exclusion on each layer with small interactions among layers which should be in the KPZ class of systems.

We refer to the introductions in [1]–[2] for more references and a list of open problems and conjectures, in particular the connection with quantum Ising models (via Feynman-Kac), phase transitions for the hard-rods Kac-Helfand model and the dependence on $\gamma$ of the critical value of the vertical interaction for a phase transition to occur.

We conclude the introduction by observing that highly anisotropic interactions are present in nature, the best example is the graphite where horizontal structures are rather free to slide one with respect to the other. However it may happen that even a small interaction among layers produces macroscopic effects. For instance for bilayer graphene samples
interacting via an interlayer coupling constant [5, 6, 7] the presence of a band gap in the energy spectrum, which is tunable by an external electric field, paves the way to a variety of applications in electronics [8]. Multilayer graphene samples have also gone, recently, under intense investigation [9, 10], which revealed the rise of exceptional thermal conduction properties for these materials as well as the possibility to control the thermodynamically stable crystalline structure of the material through an external voltage.

2 The model and the main result

As mentioned in the introduction one of our aims is the extension of the Lebowitz-Penrose theory to cases where the support of the Kac potential has a positive codimension. This is what we accomplish in this paper in the simple context of the \( d = 2 \) Ising model. Let \( \Lambda \) be a square in \( \mathbb{Z}^2 \), \( L \) its side, \((x,i)\) its points. Write \( \sigma \in \{-1,1\}^\Lambda \) for a spin configuration in \( \Lambda \), define \( \sigma(x,L+1) = \sigma(x,1) \), \( \sigma(x+L,i) = \sigma(x,i) \) and let

\[
H_{\gamma,L}^\text{per}(\sigma) = H_{\gamma,L}(\sigma) + H_{L}^\text{vert}(\sigma) + H_{\text{ext},L}(\sigma) \quad (2.1)
\]

\[
H_{\gamma,L}(\sigma) = \sum_{i=1}^{L} \{-\frac{1}{2} \sum_{x \neq y} \gamma(x,y) \sigma(x,i) \sigma(y,i)\}
\]

\[
H_{L}^\text{vert}(\sigma) = \sum_{x=1}^{L} \{-\lambda \sum_{i=1}^{L} \sigma(x,i) \sigma(x,i+1)\}
\]

\[
H_{\text{ext},L}(\sigma) = -\sum_{(x,i) \in \Lambda} \text{h}_{\text{ext}} \sigma(x,i)
\]

\( H_{\gamma,L}(\sigma) \) is the Kac hamiltonian, it has only horizontal interactions; \( H_{L}^\text{vert}(\sigma) \) is the hamiltonian of a nearest neighbor Ising model with only vertical interactions; \( H_{\text{ext},L}(\sigma) \) is the energy due to the external magnetic field \( \text{h}_{\text{ext}} \). We suppose that

\[
J_\gamma(x,y) = c_\gamma J(\gamma x, \gamma y) \quad (2.2)
\]

where \( J(r,r') \) is a smooth, symmetric probability kernel on \( \mathbb{R} \) which vanishes for \( |r-r'| \geq 1 \); \( c_\gamma \) is such that

\[
\sum_{y} J_\gamma(x,y) = 1 \quad (2.3)
\]

Since \( \int J(r,r') dr' = 1 \), \( c_\gamma \to 1 \) as \( \gamma \to 0 \). Let

\[
Z_{\gamma,\text{hext},L}^\text{per} = \sum_{\sigma \in \{-1,1\}^\Lambda} e^{-H_{\gamma,\text{hext},L}(\sigma)} \quad (2.4)
\]

be the partition function relative to the hamiltonian \( H_{\gamma,\text{hext},L}^\text{per}(\sigma) \). Call \( f_\lambda(m) \) the free energy density with magnetization density \( m \) relative to the hamiltonian \( H_{L}^\text{vert}(\sigma) \), since the horizontal interactions are absent \( f_\lambda(m) \) is equal to the free energy of the \( d = 1 \) Ising model with only nearest neighbor interactions of strength \( \lambda \).
Theorem 1 For $\lambda \geq 0$ small enough

$$\lim_{\gamma \to 0} \lim_{L \to \infty} \log \frac{Z_{\gamma, \text{ext}, L}}{|\Lambda|} = - \inf_{m \in [-1, 1]} \left\{ -h_{\text{ext}}m + \left[ -\frac{m^2}{2} + f_\lambda(m) \right] \right\}$$

(2.5)

After a few comments on Theorem 1 we give a heuristic derivation of (2.5) followed by a description of how proofs are organized in the various sections.

2.1 Remarks on Theorem 1

- (2.5) remains valid for general Van Hove regions and boundary conditions since the interaction has finite range for any fixed value of $\gamma > 0$. The restriction to small $\lambda$ is needed for cluster expansion, it is technical and could be presumably removed.

- The limit in (2.5) is the sum of the external magnetic field energy $-h_{\text{ext}}m$, the mean field energy $-m^2/2$ and the vertical free energy $f_\lambda(m)$: it reflects the analogous splitting of the Hamiltonian in (2.1).

- $\lim_{L \to \infty} \log Z_{\gamma, \text{ext}, L}^{\text{per}} |\Lambda| =: P_\gamma(h_{\text{ext}})$ is the pressure of the system with Hamiltonian $H^{\text{per}}_{\gamma, \text{ext}, L}$. By ferromagnetic inequalities $P_\gamma(h_{\text{ext}})$ is for any $\gamma > 0$ a convex function of $h_{\text{ext}}$ differentiable at any $h_{\text{ext}} \neq 0$; its derivative is the magnetization which is equal to the average spin for the unique DLR measure at the given values of $h_{\text{ext}}$ and $\gamma$. The limits (by subsequences) of $P_\gamma(h_{\text{ext}})$ as $\gamma \to 0$ are thus convex functions and Theorem 1 proves that the limit actually exists (without going to subsequences) and identifies its value.

- The limit of $P_\gamma(h_{\text{ext}})$ as $\gamma \to 0$ is the pressure $P(h_{\text{ext}})$ in the Lebowitz-Penrose limit when first $|\Lambda| \to \infty$ and then $\gamma \to 0$. (2.5) shows that $P(h_{\text{ext}})$ is the Legendre transform of the function $[-m^2/2 + f_\lambda(m)]$ and therefore the free energy $F_\lambda(m)$ defined as the Legendre transform of the pressure $P(h_{\text{ext}})$ is equal to the convex envelope:

$$F_\lambda(m) = CE \left\{ -\frac{m^2}{2} + f_\lambda(m) \right\}$$

(2.6)

- (2.6) is in agreement with the Lebowitz-Penrose result which states that the limit free energy density is the convex envelope of $-m^2/2$ plus the free energy density of the reference system (i.e. without the Kac interaction). The Lebowitz-Penrose analysis however applies if the Kac interaction is non degenerate being positive in two dimensional regions. Our theorem shows that this is not necessary.

- When $\lambda = 0$, $f_0(m) = -S(m)$ where $S(m)$ is the entropy of the free Ising model:

$$-S(m) = \frac{1+m}{2} \log \frac{1+m}{2} + \frac{1-m}{2} \log \frac{1-m}{2}$$

(2.7)

In this case $-m^2/2 + f_0(m)$ is strictly convex and coincides with $F_0(m)$. When $\lambda > 0$ we shall see that the function $-m^2/2 + f_\lambda(m)$ is no longer convex. In fact the Taylor expansion of $-S(m)$ gives

$$-S(m) = -\log 2 + \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{1}{2k+2} m^{2k+2}$$

(2.8)
and to leading orders in $\lambda$, $f_\lambda(m) = -S(m) - \lambda m^2$ so that $-\frac{m^2}{2} + f_\lambda(m)$ has a double well shape with minima at $\pm \sqrt{6\lambda}$ and $F_\lambda(m)$ is constant in the interval with endpoints $\pm \sqrt{6\lambda}$. The spontaneous magnetization is then $\sqrt{3\lambda}$ to be compared with the value $\sqrt{3\lambda}$ found in [2] for the system with reduced vertical interactions, as described in the introduction.

- The proof of Theorem 4 does not require the use of a non local free energy functional as the one introduced by Lebowitz-Penrose, but we have nonetheless established some basic ingredients for its derivation which will be used in a future work to study the large deviations.

2.2 Heuristic derivation of the mean field equation

Let $\langle \sigma(x,i) \rangle =: m$ be the average spin in an extremal, translation invariant DLR measure at $\gamma > 0$. Then

$$\langle \sigma(x,i) \rangle = \langle \tanh \{ \sum_y J_\gamma(x,y)\sigma(y,i) + \lambda[\sigma(x,i+1) + \sigma(x,i-1)] + h_{\text{ext}} \} \rangle$$

(2.9)

By the law of large numbers $\sum_y J_\gamma(x,y)(\sigma(y,i) - m) \to 0$ in the limit $\gamma \to 0$, recall that $\sum_y J_\gamma(x,y) = 1$. In such an approximation (2.9) becomes

$$\langle \sigma(x,i) \rangle = \langle \tanh \{ \lambda[\sigma(x,i+1) + \sigma(x,i-1)] + h_{\text{ext}} + m \} \rangle$$

(2.10)

This is the equation for the average spin in a $d=1$ Ising model with only nearest neighbor interactions of strength $\lambda$ and magnetic field $h_{\text{ext}} + m$. Then the average spin is equal to the thermodynamic magnetization $m$ which is related to the free energy $f_\lambda(m)$ by a variational principle which gives

$$0 = (h_{\text{ext}} + m) - f_\lambda'(m) = h_{\text{ext}} - \frac{d}{dm}\left(-\frac{m^2}{2} + f_\lambda(m)\right)$$

(2.11)

in agreement with (2.5)–(2.6).

2.3 Organization of the paper

The proof of (2.5) distinguishes large and small values of the magnetization and consequently of the magnetic field. Large magnetic fields are studied in Section 3 by using the Dobrushin high temperature theory based on the Vaserstein distance; the “small” values of the magnetic field are studied in the remaining sections. In Section 4 we give the scheme of proof of Theorem 4 which is based on the following steps (each step being discussed in a subsection). (1) a coarse graining procedure a la Lebowitz-Penrose to reduce to a $d=1$ system with only nearest neighbor interactions and without Kac potentials. The price is that we have a variational problem with multiple constraints as we have fixed the magnetization on each layer. (2) We then consider the analogous problem in the multi gran canonical ensemble where on each layer we have a magnetic field. The partition function of such a system is studied in details using cluster expansion under the assumption that $\lambda$ is sufficiently small. (3) We prove an extended equivalence of ensembles so that the
original variational problem with constraints given by the magnetization is replaced by a variational problem where one needs to optimize on the value of the auxiliary magnetic fields. (4) The proof proceeds by showing that the minimizer is made by magnetic fields equal to each other on each layer. (5) We then show that Theorem 1 follows.

In Section 3 we prove a combinatorial lemma which says that any monomial \( u_1^{n_1} \cdots u_k^{n_k} \) in the variables \( u_1, \ldots, u_k \), \( n_1 + \cdots + n_k = N \geq 2 \), can be written as a sum of one body monomials \( p_i u_i^{n_i} \), \( p_i \) positive numbers, plus a sum of terms proportional to gradients squared, \( \sum_{i<j} d_{i,j} (u_i - u_j)^2 \), the \( d_{i,j} \) polynomials of degree \( N - 2 \). This is the essential property needed to prove that the minimizers are homogeneous.

The proofs of all the above statements are reported in successive appendices.

3 Large magnetic fields

The heuristic argument presented in Subsection 2.2 is made rigorous for large magnetic fields in the following theorem.

**Theorem 2** For any \( \lambda > 0 \) let \( h_{\text{ext}} > 0 \) be so large that

\[
 r := \frac{1 + 2\lambda}{\cosh^2(h_{\text{ext}} - 1 - 2\lambda)} < \frac{1}{4} \quad (3.1)
\]

Then (i) for any \( \gamma > 0 \) there is a unique DLR measure (by ferromagnetic inequalities the statement actually holds for any \( h_{\text{ext}} \neq 0 \)); (ii) its magnetization \( m_{\gamma} \) (the average value of a spin) converges as \( \gamma \to 0 \), to the value \( m \) for which (2.11) holds; (iii) \( m \) is the unique minimizer of (2.5) and

\[
 \lim_{\gamma \to 0} \lim_{L \to \infty} \frac{\log Z_{\gamma,h_{\text{ext}},L}^{\text{per}}}{|\Lambda|} = - \left\{ - h_{\text{ext}} m + \left[- \frac{m^2}{2} + f_\lambda(m)\right] \right\} \quad (3.2)
\]

As we have already mentioned the proof of Theorem 2 is based on the techniques introduced by Dobrushin to prove his famous large temperature uniqueness theorem. In this way we get uniqueness of the DLR measures and exponential decay of correlations for any fixed \( \gamma > 0 \). We then use an interpolation procedure to derive the phase diagram of the system which was introduced in [3] to study the corrections in \( \gamma \) to the mean field limit and thus prove Theorem 2. The details are reported in Appendix A.

4 Theorem 1: scheme of proof

Theorem 1 is thus proved for large magnetic fields and the remaining part of the paper deals with the “bounded” magnetic fields. To be precise we suppose hereafter \( \lambda \in (0, 1) \), but further requests on the smallness of \( \lambda \) will be asked later on, and restrict to magnetic fields

\[
 h_{\text{ext}} \in [0, h^*], \quad h^* := \frac{3}{\cosh^2(h^* - 3)} = \frac{1}{4} \quad (4.1)
\]
as Theorem 2 covers the values $h_{ext} > h^*$. By default in the sequel $h_{ext} \in [0, h^*]$ (the analysis of negative magnetic follows by symmetry).

The first step is to use coarse graining as in Lebowitz-Penrose.

### 4.1 The Lebowitz-Penrose procedure

In this subsection we use the Lebowitz-Penrose procedure to reduce to a partition function where the Kac potential is absent. Let us first recall the Lebowitz-Penrose result and consider the partition function $Z_{\gamma, h_{ext}, L}^{\text{per}}$ with the same short range, vertical interaction as in our case (the "reference system" in the Lebowitz-Penrose terminology) but with a Kac potential which has support on a region of full dimension ($d = 2$). After coarse graining and exploiting (i) the smoothness of the Kac potential, (ii) the ferromagnetic nature of the Kac potential, Lebowitz-Penrose have proved that $Z_{\gamma, h_{ext}, L}^{\text{per}}$ has the same "asymptotics" as

$$Z_{\Delta}^{\text{max}} := \max_{m \in M_{\Delta}} e^{(h_{ext}m + m^2/2)|\Delta|} \sum_{\sigma \in \{-1, 1\}^\Delta} e^{-H_{\text{vert}}^{\ell}(\sigma)} 1_{\sum_{x \in \Delta} \sigma(x) = m|\Delta|}$$

where $\Delta$ is a square of side $\ell$, $\ell$ the integer part of $\gamma^{-1/2}$, and

$$M_{\Delta} = \{-1, -1 + \frac{2}{|\Delta|}, \ldots, 1 - \frac{2}{|\Delta|}, 1\}$$

the set of all possible values of the empirical spin magnetization in $\Delta$.

By same "asymptotics" we mean that

$$\lim_{\gamma \to 0} \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log Z_{\gamma, h_{ext}, L}^{\text{per}} = \lim_{|\Delta| \to \infty} \frac{1}{|\Delta|} \log Z_{\Delta}^{\text{max}}$$

The same procedure works in our case as well leading to Theorem 3 below whose proof is given in Appendix B.

**Theorem 3** Let $\Delta$ and $\ell$ be as above, $M_{\ell} = \{-1, -1 + \frac{2}{\ell}, \ldots, 1 - \frac{2}{\ell}, 1\}$, $m_{\Delta}(x, i), (x, i) \in \Delta$, a function with values in $M_{\ell}$ which depends only on $i$,

$$\phi_{\ell}(m_{\Delta}) = -\frac{1}{|\Delta|} \log \sum_{\sigma \in \{-1, 1\}^\Delta} e^{-H_{\text{vert}}^{\ell}(\sigma)} 1_{\sum_{x \in \Delta} \sigma(x) = m_{\Delta}(\cdot, i) \text{ for all } i}$$

Then there is $m_+ \in (0, 1)$ so that

$$\lim_{\gamma \to 0} \lim_{L \to \infty} \frac{1}{|\Lambda|} \log Z_{\gamma, h_{ext}, L}^{\text{per}} = \lim_{|\Delta| \to \infty} \frac{1}{|\Delta|} \log Z_{\Delta}^{\text{max}}$$

where

$$\log Z_{\Delta}^{\text{max}} := \max_{m_{\Delta} : |m_{\Delta}(x, i)| \leq m_+, (x, i) \in \Delta} \left\{ \frac{m(x, i)^2}{2} + h_{ext} m(x, i) - \phi_{\ell}(m_{\Delta}) \right\}$$
(4.2) and (4.4) are identical but the meaning of $Z_{\Delta}^{\text{max}}$ is different in the two cases. In (4.2) it is a max over a scalar $m$ of the canonical partition function with magnetization $m$. By classical results on the thermodynamic limit this is related to the free energy of the system and one gets a formula as on the right hand side of (2.5). Thus one has essentially finished once he gets (4.2), in our case instead (4.4) is just the beginning of the work. In fact the variational problem behind (4.5) involves a vector $m_{\Delta}$ in a space whose dimensions diverge in the thermodynamic limit. Moreover the relation between $\phi_{\ell}(m_{\Delta})$ and the $d = 1$ free energy $f_{\lambda}(m)$ which appears in (2.5) is not evident due to the multi-canonical constraint of fixing the magnetization on each layer.

The picture looks simpler if we replace the multi-canonical ensemble by a gran canonical ensemble with auxiliary magnetic fields on each layer: let then $\underline{h} = (h_1, \ldots, h_{\ell})$ and

$$Z_{\Delta, \underline{h}} = \sum_{\sigma \in \{-1,1\}^\Delta} e^{-H_{\text{vert}}^{\text{ext}}(\sigma) - \sum_{(x,i) \in \Delta} h_i \sigma(x,i)}$$

The goal is to rewrite $\phi_{\ell}(m_{\Delta})$ in terms $\log Z_{\Delta, \underline{h}}$ thus proving an extended version of the equivalence of ensembles theorem. The first step in this direction is to get a full understanding of $Z_{\Delta, \underline{h}}$ as provided by the cluster expansion.

### 4.2 Cluster expansion

We first observe that

$$\log Z_{\Delta, \underline{h}} = \ell \log Z_{\ell, \underline{h}}$$

$$Z_{\ell, \underline{h}} = \sum_{\sigma \in \{-1,1\}^{[1,\ell]}} e^{\sum_{i=1}^{\ell} \{\lambda \sigma(i) \sigma(i+1) + h_i \sigma(i)\}}, \quad \sigma(\ell + 1) = \sigma(1)$$

with $Z_{\ell, \underline{h}}$ the partition function of the $d = 1$ Ising model with nearest neighbor interactions of strength $\lambda$ and space dependent magnetic field $\underline{h}$. We define

$$Z_{\ell, \underline{h}}^* := Z_{\ell, \underline{h}} \left\{ \prod_{i=1}^{\ell} (e^{h_i} + e^{-h_i}) \right\}^{-1}$$

$$u_i := \tanh(h_i)$$

In Appendix C we shall suppose $\lambda$ small and use cluster expansion to prove:

**Theorem 4** For any $\lambda > 0$ small enough

$$\log Z_{\ell, \underline{h}}^* = \sum_{N(\cdot)} A_{N(\cdot)} u^{N(\cdot)}$$

where $N(\cdot) : [1, \ell] \to \mathbb{N}$ and

$$u^{N(\cdot)} = \prod_{i=1}^{\ell} u_i^{N(i)}$$

The coefficients $A_{N(\cdot)}$ satisfy the following bounds. Call

$$e^b := \lambda^{-5/12}, \quad |N(\cdot)| = \sum_x N(x), \quad \|N(\cdot)\| = \max\{|N(\cdot)|, R(N(\cdot))\}$$
where \( R(N(\cdot)) \) denotes the cardinality of the support of \( N(\cdot) \) (i.e. the smallest interval which contains the set \{i : N(i) > 0\}). Then for any \( i \in [1, \ell] \) and any positive integer \( M \)

\[
\sum_{N(\cdot):N(i)>0,\|N(\cdot)\| \geq M} |A_{N(\cdot)}| \leq e^{-bM}
\]

(4.13)

Moreover \( A_{N(\cdot)} = 0 \) if \( |N(\cdot)| \) is odd and there are coefficients \( \alpha_k, k > 0 \), and \( c \) so that

\[
\sum_{N(\cdot):\|N(\cdot)\|=2} A_{N(\cdot)}u^{N(\cdot)} = \sum_{i<j} \alpha_{j-i}u_iu_j \quad (4.14)
\]

\[
|\alpha_1 - \lambda| \leq c\lambda e^{-2b}, \quad |\alpha_{j-i}| \leq c\lambda|j-i|e^{|j-i|}
\]

(4.15)

### 4.3 Equivalence of ensembles

The magnetizations associated to \( Z_{\ell,\hbar} \), as defined in (4.7), are

\[
m_i = \frac{\partial}{\partial h_i} \log Z_{\ell,\hbar}
\]

(4.16)

which are thus expressed via \( \hbar \) in terms of \((u_1,\ldots,u_\ell)\). We write more explicitly (4.16) as

\[
m_i = u_i + \Psi_i(u), \quad \Psi_i(u) = (1 - u_i^2) \sum_{N(\cdot):N(i)>0} N(\cdot)A_{N(\cdot)}u^{N(\cdot)}
\]

(4.17)

with \( N^{(i)}(k) = N(k) \) for \( k \neq i \) and \( N^{(i)}(i) = N(i) - 1 \). In Appendix F we will prove that there is a one to one correspondence between \( u \) and \( \mathbf{m} \) so that we may write \( u \) as a function of \( \mathbf{m} \).

**Theorem 5** For any \( \lambda > 0 \) small enough the following holds. For any \( \mathbf{m} \) such that \( |m_i| \leq m_+ \) (\( m_+ \) as in (4.5)) there is a unique \( \hbar \) such that (4.16) holds for any \( i = 1,\ldots,\ell \) and there exists \( h_+ > 0 \) so that all the components of \( \hbar \) are bounded by \( h_+ \).

**Theorem 6** For any \( \lambda > 0 \) small enough the following holds. For any \( \mathbf{m}_\Delta = \mathbf{m} = (m_1,\ldots,m_\ell), \mathbf{m}_i \leq m_+, \ i = 1,\ldots,\ell \), call \( \hbar = (h_1,\ldots,h_\ell) \) the magnetic fields associated to \( \mathbf{m} \) via Theorem 5 then for any \( a \in \left(\frac{1}{\ell},1\right) \) there is \( c \) so that

\[
\left| \frac{1}{\ell^2} \log \{e^{-\ell \sum h_i m_i Z_{\Delta,\hbar}} + \phi_\ell(m_\Delta)\} \right| \leq c\ell^{a-1}
\]

(4.18)

where \( \phi_\ell(m_\Delta) \) is defined in (4.3) and \( Z_{\Delta,\hbar} \) in (4.6).

As a consequence:
Theorem 7 Let $Z_{\Delta}^{\max}$ be as in (4.5), then

$$\lim_{|\Delta| \to \infty} \frac{1}{|\Delta|} \log Z_{\Delta}^{\max} = \lim_{\ell \to \infty} \frac{1}{\ell} \max_{h | h_i \leq h} \sum_{i=1}^{\ell} \frac{m_i^2}{2} + (h_{\text{ext}} - h_i)m_i + \log Z_{\ell, h}$$ (4.19)

where $m_i$ is the function of $h$ defined in (4.16) – (4.17).

By (4.4) and (4.19) we get

$$\lim_{\gamma \to 0} \lim_{L \to \infty} \frac{1}{|A|} \log Z_{\gamma, h_{\text{ext}}, L}^{\text{ber}} = \lim_{\ell \to \infty} \frac{1}{\ell} \max_{h} \sum_{i=1}^{\ell} \frac{m_i^2}{2} + (h_{\text{ext}} - h_i)m_i + \log Z_{\ell, h}$$ (4.20)

4.4 The quadratic structure of the effective hamiltonian

The goal therefore is to study the ground state energy of the effective hamiltonian

$$H_{\ell, h}^{\text{eff}} = -\sum_{i=1}^{\ell} \left\{ \frac{m_i^2}{2} - h_i m_i + h_{\text{ext}} m_i \right\} - \log \left( e^{h_i} + e^{-h_i} \right) - \log Z_{\ell, h}^* + A_0$$ (4.21)

regarded as a function of $u = (u_1, \ldots, u_\ell)$. For convenience in (4.21) we have subtracted to log $Z_{\ell, h}^*$ (which is defined in (4.10)) the first term of the expansion (4.10) (with $N(\cdot) \equiv 0$), which is a constant.

By Theorem 3 we can restrict to the set of $u : |u_i| \leq u_+ = \tanh(h_+), i = 1, \ldots, \ell$ and in the sequel we will tacitly restrict to such a set. We will prove that the inf over $u$ of $H_{\ell, h}^{\text{eff}}$ is achieved by vectors $u$ with all components equal to each other which is maybe the most relevant/original result of this paper.

We start by making explicit the leading terms in (4.21) for $\lambda$ small. To this end and recalling that $\log \left( e^{h_i} + e^{-h_i} \right) = h_i u_i + S(u_i)$, the entropy $S(u)$ being defined in (2.7) – (2.8), we write

$$\log(e^{h_i} + e^{-h_i}) = h_i u_i + \frac{u_i^2}{2} + T(u_i)$$

$$T(u) = -\log 2 + \sum_{k=1}^{\infty} \frac{1}{2k+1} \frac{1}{2k+2} u_i^{2k+2}$$ (4.22)

$$\log Z_{\ell, h}^* - A_0 = -\frac{\lambda}{2} \sum_{i=1}^{\ell} (u_{i+1} - u_i)^2 + \Theta$$ (4.23)

$$\xi_i := (h_i - u_i)(1 - u_i^2)$$ (4.24)

$$\Psi_i = \lambda (1 - u_i^2)(u_{i+1} + u_{i-1}) + \Phi_i$$ (4.25)

where $\Psi_i$ is defined in (4.17) and $\Theta$ and $\Phi_i$ are defined by the above equations. By some simple algebra, see Appendix G for details, we can rewrite $H_{\ell, h}^{\text{eff}}$ as:
Lemma 1 With the above notation:

\[
H_{\ell,h}^{\text{eff}} = \sum_{i=1}^{\ell} \left\{ T(u_i) - h_{\text{ext}} u_i - 2\lambda h_{\text{ext}} [u_i - u_i^3] + 2\lambda \xi_i u_i \right\} \\
+ \sum_{i=1}^{\ell} \left\{ \frac{\lambda}{2} (u_i - u_{i+1})^2 + (h_i - u_i) \Phi_i - h_{\text{ext}} \Phi_i - \frac{\Psi_i^2}{2} \right\} - \Theta \\
- \lambda \sum_{i=1}^{\ell} \left\{ h_{\text{ext}} (u_i + u_{i+1}) (u_{i+1} - u_i)^2 + (\xi_i - \xi_{i+1}) (u_i - u_{i+1}) \right\} 
\]

(4.26)

The terms with \(\Theta, \Psi_i^2\) and \(\Phi_i\) are “under control” in the following sense:

Theorem 8 Call

\[
H_{\ell,h}^{(1)} = \sum_{i=1}^{\ell} \left\{ (h_i - u_i) \Phi_i - h_{\text{ext}} \Phi_i - \frac{\Psi_i^2}{2} \right\} - \Theta 
\]

(4.27)

Then for any \(h_{\text{ext}} \in [0,h^*]\) there is a continuous function \(f^{(1)}(u)\) on \([-1,1]\) and functions \(b^{(1)}_{i,j}(u), i < j\), so that

\[
H_{\ell,h}^{(1)} = \sum_{i=1}^{\ell} f^{(1)}(u_i) + \sum_{1 \leq i < j \leq \ell} b^{(1)}_{i,j}(u)(u_i - u_j)^2
\]

(4.28)

with

\[
\sum_{1 \leq i < j \leq \ell} b^{(1)}_{i,j}(u)(u_i - u_j)^2 \geq -c\lambda^{1+\frac{c}{2}} \sum_{i=1}^{\ell} (u_i - u_{i+1})^2
\]

(4.29)

and \(c\) a positive constant.

The proof of Theorem 8 starts from (4.10) and it is based on a representation of the monomials \(u^N(\cdot)\) as sum of one body and gradients squared terms which is established in Section 5. After that we exploit the properties of the coefficients \(A_N(\cdot)\) stated in (4.13), (4.14) and (4.15). The computations are straightforward but lengthy, the details are reported in Appendix H.

Define

\[
\theta_i(u) := \frac{\xi_i - \xi_{i+1}}{u_i - u_{i+1}}
\]

(4.30)

when \(u_i \neq u_{i+1}\) and equal to \(d\xi/du(v)\) when \(u_i = u_{i+1} = v\). Then \(\theta_i(u)\) depends only on \(u_i\) and \(u_{i+1}\), and as a function of \(u_i\) and \(u_{i+1}\) is continuous, symmetric and bounded in \(|u_i| \leq R, |u_{i+1}| \leq R, R < 1\). It is essential in our proof that \(\theta_i(u) < \frac{1}{2}\). We checked numerically that this is “fortunately” true and indeed we found a rigorous proof, reported in Appendix H of an upper bound smaller than 1/2.
Proposition 1

\[ \theta_i(u) \leq \frac{3}{8} \quad (4.31) \]

It follows from (4.26) and (4.28) that

\[
H_{\ell,h}^{\text{eff}} = \sum_{i=1}^{\ell} f(u_i) + \lambda \sum_{i=1}^{\ell} \left\{ \frac{1}{2} - h_{\text{ext}}(u_i + u_{i+1}) - \theta_i(u) \right\} (u_i - u_{i+1})^2 \\
+ \sum_{i<j} \theta_i^{(1)}(u)(u_i - u_j)^2
\]

(4.32)

where

\[ f(u_i) = \{T(u_i) - h_{\text{ext}}u_i - 2\lambda h_{\text{ext}}[u_i - u_i^3] + 2\lambda \xi_i u_i\} + f^{(1)}(u_i) \]

(4.33)

Let \( u^* \) be the minimizer of \( f(\cdot) \), then by (4.32)

\[
\inf_{u} H_{\ell,h}^{\text{eff}} \leq \ell \inf_{u} f(u) =: \ell f(u^*)
\]

as the right hand side is the value obtained by choosing all \( u_i = u^* \).

Let \( h_0 := \frac{1}{4} \left[ \frac{1}{2} - \frac{3}{8} \right] \) (4.34)

then if \( h_{\text{ext}} \in [0, h_0] \) using (4.29) we get

\[
H_{\ell,h}^{\text{eff}} \geq \sum_{i=1}^{\ell} f(u_i) + \lambda \sum_{i=1}^{\ell} \left\{ \frac{1}{2} - \frac{3}{8}\right\} (u_i - u_{i+1})^2 - \sum_{i} c\lambda^1 + \frac{3}{8} (u_i - u_{i+1})^2 \\
\geq \sum_{i=1}^{\ell} f(u_i) + \lambda \sum_{i=1}^{\ell} \left\{ \frac{1}{2} - \frac{3}{8} - 2c\lambda^{1/2}\right\} (u_i - u_{i+1})^2
\]

(4.35)

Hence

\[
\inf_{u} H_{\ell,h}^{\text{eff}} \geq \ell f(u^*)
\]

for \( \lambda \) so small that \( 2c\lambda^{1/2} \leq \frac{1}{2} - \frac{3}{8} \), because by (4.35) we then get a lower bound by neglecting the sum of the terms with \( (u_i - u_{i+1})^2 \). We have thus proved:

**Theorem 9** Let \( h_{\text{ext}} \in [0, h_0] \) and \( \lambda \) be so small that \( 2c\lambda^{1/2} \leq \frac{1}{2} - \frac{3}{8} \), then the inf of \( H_{\ell,h}^{\text{eff}} \) is equal to the min of \( H_{\ell,h}^{\text{eff}} \) over homogeneous \( h \), namely \( h \) with all its components equal to each other.

Thus the inf of \( H_{\ell,h}^{\text{eff}} \) is achieved when all the components of \( h \) are equal to each other, in such a case we can compute explicitly the minimizer, see the next subsection. The result comes from the quadratic structure of the hamiltonian, (4.28)–(4.29), somehow reminiscent of the Ginzburg-Landau functional whose integrand has the form \( W(u) + |\nabla u|^2 \) and the gradient term forces the minimizer to be a constant.
The argument used to prove Theorem 9 does not extend to the complementary case when $h_{\text{ext}} \in [h_0, h^*]$ because $\left\{ \frac{1}{2} - h_{\text{ext}}(u_i + u_{i+1}) \right\}$ in (4.32) may become negative and we would then lose the positivity of the coefficients of the gradients squared. Nonetheless we can use the “strong convexity” of the one body term $T(u_i)$ in (4.26) when $u_i$ is away from 0 to prove:

**Theorem 10** Let $h_{\text{ext}} \in [h_0, h^*]$ and let $\lambda$ be small enough, then the inf of $H_{\text{eff}}^{\text{ext}}$ is equal to the min of $H_{\text{eff}}^{\text{hom}}$ over homogeneous $h$.

Theorem 10 is proved in Appendix J

### 4.5 Proof of Theorem 1

Using Theorem 9 and Theorem 10 we will next prove Theorem 1. We thus know that

$$\lim_{\gamma \to 0} \lim_{L \to \infty} \frac{1}{|\Lambda|} \log Z_{\gamma, h_{\text{ext}}, L}^{\text{per}} = \lim_{\ell \to \infty} \max_{m \in \mathcal{M}_\ell : |m| \leq m_+} \left[ \frac{m^2}{2} + h_{\text{ext}} m - \psi_{\lambda, \ell}(m) \right]$$

where, letting $\Delta = I \times I'$,

$$\psi_{\lambda, \ell}(m) = -\frac{1}{|\Delta|} \log \sum_{\sigma \in \{-1, 1\}^\Delta} 1_{\sum_{\sigma \neq \ell} \sigma(x,i) = \ell} m, \text{ for all } i \in I' e^{-H_{\ell, \text{vert}}(\sigma)}$$

Denote by

$$f_{\lambda, \ell}(m) = -\frac{1}{|\Delta|} \log \sum_{\sigma \in \{-1, 1\}^\Delta} 1_{\sum_{\sigma \neq \ell} \sigma(x,i) = |\Delta|} m e^{-H_{\ell, \text{vert}}(\sigma)}$$

the finite volume free energy of the system with only vertical interactions. Thus $\lim_{\ell \to \infty} f_{\lambda, \ell}(m) = f_{\lambda}(m)$. We obviously have $-\psi_{\lambda, \ell}(m) \leq -f_{\lambda, \ell}(m)$ and by classical results on the Ising model

$$-f_{\lambda, \ell}(m) \leq -f_{\lambda}(m) + \frac{c}{\ell}$$

so that

$$\limsup_{\gamma \to 0} \lim_{L \to \infty} \frac{1}{|\Lambda|} \log Z_{\gamma, h_{\text{ext}}, L}^{\text{per}} \leq \lim_{\ell \to \infty} \max_{m \in \mathcal{M}_\ell : |m| \leq m_+} \left[ \frac{m^2}{2} + h_{\text{ext}} m - f_{\lambda}(m) \right]$$

which proves that the left hand side of (2.5) is bounded by its right hand side.

We are thus left with the proof of the reverse inequality. Let

$$\hat{m} = \arg \min \left\{ -h_{\text{ext}} m - \frac{m^2}{2} + f_{\lambda}(m) \right\}$$

$$m^{(\ell)} = \max \left\{ m \in \mathcal{M}_\ell : m \leq \hat{m} \right\}$$

$$h^{(\ell)} : \frac{d}{dh} p_{\lambda, \ell}(h) \bigg|_{h=h^{(\ell)}} = m^{(\ell)}$$

where $p_{\lambda, \ell}(h) = \ell^{-1} \log Z_{\lambda, h, \ell}$ and $Z_{\lambda, h, \ell}$ is the partition function of the Ising model in $[1, \ell]$ with n.n. interaction of strength $\lambda$ and magnetic field $h$; $p_{\lambda}(h)$ is the corresponding pressure in the thermodynamic limit $\ell \to \infty$. It is well known that

$$\sup_h |p_{\lambda, \ell}(h) - p_{\lambda}(h)| \leq \frac{c}{\ell}$$
Then
\[
\left\{ h_{\text{ext}} m^{(\ell)} + \frac{(m^{(\ell)})^2}{2} - h^{(\ell)} m^{(\ell)} + \frac{1}{|A|} \log \left( \sum_{\sigma} 1_{\sigma(x,i)=\ell m^{(\ell)}} e^{-H_{\text{pert}}^{\sigma}(\sigma) + \sum h^{(\ell)} \sigma(x,i)} \right) \right\}
\] (4.44)
is a lower bound in the asymptotic sense for \( \frac{1}{|A|} \log Z_{\gamma, h_{\text{ext}}, L}^{\text{per}} \). By the equivalence of ensembles, see Theorem 6 in Subsection F.2, the lower bound becomes
\[
\left\{ h_{\text{ext}} m^{(\ell)} + \frac{(m^{(\ell)})^2}{2} - h^{(\ell)} m^{(\ell)} + p_{\lambda,\ell}(h^{(\ell)}) \right\}
\] (4.45)
where by (4.43) we can also replace \( p_{\lambda,\ell}(h^{(\ell)}) \) by \( p_{\lambda}(h^{(\ell)}) \). By compactness \( h^{(\ell)} \) converges by subsequences and if \( \ell_k \) is a convergent subsequence there is \( h \) so that \( h^{(\ell_k)} \to h \) and consequently
\[
\lim_k p'_{\lambda,\ell}(h^{(\ell_k)}) = p'_{\lambda}(h)
\] (4.46)
In fact in general any limit point of \( p'_{\lambda,\ell}(h^{(\ell_k)}) \) lies in the interval \([\frac{d}{dh} p_{\lambda}(h), \frac{d}{dh} p_{\lambda}(h)]\) of its left and right derivatives, but since we are considering a \( d = 1 \) system such derivatives are equal to each other. Moreover by the choice of \( h^{(\ell)} \)
\[
p_{\lambda,\ell}(h^{(\ell)}) = m^{(\ell)} \to \tilde{m}
\] (4.47)
Hence if \( h^{(\ell_k)} \to h \) then
\[
p'_{\lambda}(h) = \tilde{m}
\] (4.48)
and since there is a unique \( \tilde{h} \) such that \( p'_{\lambda}(\tilde{h}) = \tilde{m} \) it follows that for any convergent subsequence \( h^{(\ell_k)} \to \tilde{h} \) and therefore \( h^{(\ell)} \to \tilde{h} \). Thus the expression (4.45) converges to
\[
\left\{ h_{\text{ext}} \tilde{m} + \frac{(\tilde{m})^2}{2} - \tilde{h} \tilde{m} + p_{\lambda}(\tilde{h}) \right\}
\] (4.49)
which concludes the proof because \(-\tilde{h} \tilde{m} + p_{\lambda}(\tilde{h}) = f_{\lambda}(\tilde{m})\).

5 Monomials are sum of gradients

In this section we prove a combinatorial lemma, Theorem 11 below, which is the key ingredient in the proof of the gradient structure of the Hamiltonian. The whole section is self contained and can be read independently of the rest of the paper.

Theorem 11 Let \( u = (u_1, \ldots, u_k) \in \mathbb{R}^k, \overline{u} = (n_1, \ldots, n_k) \in \mathbb{N}_+^k \) and
\[
M_{\overline{u}}(u) = u_1^{n_1} \cdots u_k^{n_k}, \quad \sum_{i=1}^k n_i =: N
\]
a monomial of degree \( N \) in the \( k \) real variables \( u_1, \ldots, u_k \). Then for any \( N \geq 2 \)
\[
M_{\overline{u}}(u) = \sum_{i=1}^k p_i u_i^N + \sum_{1 \leq i < j \leq k} d_{i,j}(\overline{u})(u_i - u_j)^2
\] (5.1)
where \((p_1, \ldots, p_k)\) is a probability vector, its component \(p_i\) depending on \(n\); \(d_{i,j}(u)\) are polynomials of degree \(N-2\) with negative coefficients which depend on \(u\) and there is a constant \(c\) so that for any positive \(U \leq 1\)

\[
\sup_{|u_i| \leq U, i = 1, \ldots, k} |d_{i,j}(u)| \leq cU^{N-2}N^3
\]

**Proof** Call \(N_j = n_1 + \cdots + n_j, j = 1, \ldots, k\), so that \(N_k = N\). We will prove the theorem with

\[
d_{i,j}(u) = \sum_{m=1}^{N_j-1} c_{i,j;m}(u_i^{m-1}u_j^{N_j-m-1}u_{j+1}^{n_j+1} \cdots u_k^{n_k})
\]

with coefficients \(c_{i,j;m}, i < j, 1 \leq m \leq N_j - 1\), which depend on \(n_1, \ldots, n_j\) and satisfy the bound

\[
|c_{i,j;m}| \leq CN^2
\]

(5.4)

(5.2) follows from (5.3)–(5.4) which also show that the representation (5.1) of \(M_\alpha(u)\) is not unique as we can commute the factors \(u_{i,n_i}\) in the monomial \(M_\alpha(u)\) without changing its value.

The proof of (5.1) generalizes the equality

\[
uu = \frac{u^2}{2} + \frac{v^2}{2} - \frac{1}{2}(u-v)^2
\]

In fact we use the above identity to rewrite the factor \(u_1u_2\) in \(M_\alpha \equiv M_\alpha(u)\) getting

\[
M_\alpha = \frac{1}{2}M_{\alpha+\epsilon_1-\epsilon_2} + \frac{1}{2}M_{\alpha-\epsilon_1+\epsilon_2} - \frac{1}{2}M_{\alpha-\epsilon_1-\epsilon_2}(u_1-u_2)^2
\]

(5.5)

where

\[
\epsilon_1 = (1, 0, \ldots, 0), \quad \epsilon_2 = (0, 1, 0, \ldots, 0)
\]

We thus have

\[
\frac{1}{2}M_{\alpha+\epsilon_1-\epsilon_2} + \frac{1}{2}M_{\alpha-\epsilon_1+\epsilon_2} - M_\alpha = \frac{1}{2}M_{\alpha-\epsilon_1-\epsilon_2}(u_1-u_2)^2
\]

which reminds of the discrete version of the equation \(\Delta u = f\) that we will solve by iteration. There is a nice probabilistic interpretation under which the terms \(\frac{1}{2}M_{\alpha+\epsilon_1-\epsilon_2}\) and \(\frac{1}{2}M_{\alpha-\epsilon_1+\epsilon_2}\) will be interpreted by saying that \(n_1 \to n_1 \pm 1\) with probability \(1/2\), see the process \(\mu(t)\) defined below. With this in mind we introduce a Markov chain \(\xi(t), t \in \mathbb{N}, \xi(t) \in \Omega\), where

\[
\Omega = \bigcup_{i=1}^{k-1} \Omega_i, \quad \Omega_i = \{(i,x) : 1 \leq x \leq N_{i+1} - 1\}, \quad i < k - 1
\]

\[
\Omega_{k-1} = \{(k-1,x) : 0 \leq x \leq N_k\}
\]

(5.6)

The transition probabilities \(P(\cdot, \cdot)\) are all set equal to 0 except

\[
P((i,x),(i,y)) = \frac{1}{2}, \quad |x-y| = 1;
\]

\[
P((i,1),(i+1,N_{i+1})) = P((i,N_{i+1}-1),(i+1,N_{i+1})) = \frac{1}{2}
\]

\[
P((k-1,0),(k-1,0)) = P((k-1,N_k),(k-1,N_k)) = 1
\]

(5.7)
The first line in (5.7) describes the motion on the components $\Omega_i$ of $\Omega$: the second one the jump from $\Omega_i$ to $\Omega_{i+1}$ (the reverse jump having 0 probability) while the last line says that the endpoints of $\Omega_{k-1}$ are “traps”, namely once the chain reaches those points it is stuck there forever.

We start the chain at time 0 from

$$\xi(0) = (1, n_1)$$

(5.8)

We call $\tau_i, i = 0, \ldots, k - 2$, the first time $t$ when $\xi(t) \in \Omega_{i+1}$ ($\tau_0 = 0$) and define for $i \geq 1$, $\sigma_i = \pm$ if $\xi(\tau_i - 1) = N_{i+1}$, respectively $\xi(\tau_i - 1) = 1$. For $\tau_i \leq t < \tau_{i+1}$ $\xi(t)$ is a simple symmetric random walk, thus, by classical estimates, there are constants $b > 0, c > 0$ so that

$$P[\tau_{i+1} - \tau_i > s] \leq ce^{-bN_{i+2}s^2}$$

(5.9)

To establish a connection with $M_\omega(\bar{u})$ and the iterates of (5.5) we introduce new processes $\bar{u}(t)$, $f(t)$ and $a(t)$ which are all “adapted to the canonical filtration $\mathcal{F}_t$”, calling $\theta(t)$ adapted to the canonical filtration $\mathcal{F}_t$ if $\theta(t)$ is determined by $\{\xi(s), s \leq t\}$.

Let $\bar{u}(t) = (n_1(t), \ldots, n_k(t))$ be defined as follows. When $t < \tau_1$ we set

$$\bar{u}(t) = (\xi(t), N_2 - \xi(t), n_3, \ldots, n_k)$$

(5.10)

For $t = \tau_1$:

$$\bar{u}(\tau_1) = \begin{cases} (N_2, 0, n_3, \ldots, n_k) & \text{if } \sigma_1 = + \\ (0, N_2, n_3, \ldots, n_k) & \text{if } \sigma_1 = - \end{cases}$$

(5.11)

In the interval $\tau_1 \leq t < \tau_2$

$$\bar{u}(t) = \begin{cases} (\xi(t), 0, N_3 - \xi(t), n_4, \ldots, n_k) & \text{if } \sigma_1 = + \\ (0, \xi(t), N_3 - \xi(t), n_4, \ldots, n_k) & \text{if } \sigma_1 = - \end{cases}$$

(5.12)

By iteration the definition is extended to all $t \in \mathbb{N}$. The process $\bar{u}(t)$ is indeed quite simple: fix $2 \leq i \leq k$, then $n_i(t) = n_i(0)$ for $t \leq \tau_{i-2}$ after that it performs a simple symmetric random walk with absorption at 0. In the time interval $\tau_{i-2} \leq t \leq \tau_{i-1}$ all $n_j(t) = 0$ with $j < i$ except one, whose label is denoted $\ell_i$, which jumps with opposite sign as $n_i(t)$. For $j > i$, $n_j(t) = n_j(0)$ in $\tau_{i-1} \leq t \leq \tau_i$.

The process $f(t)$ is defined as

$$f(t) = \prod_{i=1}^k u_i^{n_i(t)}$$

(5.13)

while $a(t)$ is defined by setting

$$a(t) = \frac{1}{2} \left( u_{\ell_i}^{n_{\ell_i}(t)-1} u_i^{N_i-n_{\ell_i}(t)-1} \prod_{j>i} u_j^{n_j} \right) \left( u_{\ell_i} - u_i \right)^2, \quad \tau_{i-2} \leq t < \tau_{i-1}$$

(5.14)

We are going to prove that $f(0) = M_\omega(\bar{u}) = u_1^{n_1} \cdots u_k^{n_k}$ is equal to

$$f(0) = E[f(t)] - \sum_{s=-1}^{t-1} E[a(s)], \quad a(-1) = 0, \quad t \geq 0$$

(5.15)
(5.15) will be proved by showing that

\[ m(t) := f(t) - \sum_{s=-1}^{t-1} a(s) \]

is a \( \mathcal{F}_t \)-martingale namely that

\[ E[m(t+1) | \mathcal{F}_t] = 0, \quad E[f(t+1) - f(t) | \mathcal{F}_t] = a(t) \]

Since we are conditioning on \( \mathcal{F}_t \) we know the process till time \( t \), suppose that \( \tau_{i-2} \leq t < \tau_{i-1} \), call \( n(t) = (n'_1, \ldots, n'_{i+1}, n_{i+2}, \ldots, n_k) \), so that all \( n'_j = 0 \) with \( j < i \) except \( \ell_i \), while \( n'_j = n_j(0) = n_j \) for \( j > i \). Then, by (5.13),

\[ f(t) = u_{\ell_i} n'_i \prod_{j=i+1}^k u_j^{n_j} \]

and

\[ E[f(t+1) | \mathcal{F}_t] = \frac{1}{2} \left\{ u_{\ell_i}^{n'_i+1} u_i^{n'_i-1} + u_{\ell_i}^{n'_i-1} u_i^{n'_i+1} \right\} \prod_{j=i+1}^k u_j^{n_j} \]

so that \( E[f(t+1) | \mathcal{F}_t] - f(t) \) is equal to

\[ \frac{1}{2} \left\{ u_{\ell_i}^2 + u_i^2 - 2u_{\ell_i} u_i \right\} \prod_{j=i+1}^k u_j^{n_j} = \frac{1}{2} \left( u_{\ell_i} - u_i \right)^2 \prod_{j=i+1}^k u_j^{n_j} \]

which is equal to \( a(t) \). Thus \( E[f(t+1) - a(t) | \mathcal{F}_t] = 0 \) and therefore \( m(t) \) is a martingale. Since \( P[\tau_{k-1} < \infty] = 1 \) we can take the limit as \( t \to \infty \) in (5.15) which yields (5.1) with

\[ p_i = \frac{1}{2} P[\sigma_j +, j \geq 1], \quad p_i = \frac{1}{2} P[\sigma_{i-1} = -, \sigma_j +, j \geq i], \quad i > 1 \]

\[ c_{i,j;m} = -\frac{1}{2} P[\sigma_{i-1} = -, \sigma_n = +, i \leq n \leq j - 1] \times \sum_{t \geq 0} P_{N_{j-1}}[x^0(t) = m, x^0(s) \in [1, N_j - 1], s \leq t] \]

where \( x^0(t) \) is a simple symmetric random walk which starts from \( N_{j-1} \).

---

A Proof of Theorem 2

We preliminary observe that for any \( h_{\text{ext}} > 0 \) there is \( m \) so that \( h_{\text{ext}} + m = f'_\lambda^i(m) \): in fact \( h_{\text{ext}} + m - f'_\lambda^i(m) \) is positive at \( m = 0 \) and negative as \( m \to 1 \) with \( f'_\lambda^i(m) \) continuous.
If there are several $m$ for which the equality holds we arbitrarily fix one of them that we denote by $m_{h^{ext}}$, we shall see a posteriori that there is uniqueness. To compute the left hand side of (3.2) we introduce an interpolating hamiltonian. For $t \in [0,1]$ we set:

$$H_{t,\gamma,L}(\sigma) = tH^{\text{per}}_{\gamma,h^{ext},L}(\sigma) + (1-t)H^0_L(\sigma)$$

(A.1)

$$H^0_L(\sigma) = H^{\text{vert}}_L(\sigma) + H_{h^{ext},L} - \sum_{(x,i) \in \Lambda} m_{h^{ext}} \sigma(x,i)$$

Denote by $Z^0_L$ the partition function with hamiltonian $H^0_L$, by $P_{t,\gamma,L}$ the Gibbs measure with hamiltonian $H_{t,\gamma,L}$ and by $E_{t,\gamma,L}$ its expectation, then

$$\log Z^0_{\gamma,h^{ext},L} - \log Z^0_L = \int_0^1 E_{t,\gamma,L}[H^0_L - H^{\text{per}}_{\gamma,h^{ext},L}]dt$$

(A.2)

The thermodynamic limit of $\log Z^0_L/|\Lambda|$ is the pressure of the $d = 1$ Ising model with only vertical interactions and magnetic field $h^{ext} + m_{h^{ext}}$, thus, by the choice of $m_{h^{ext}}$:

$$\lim_{L \to \infty} \frac{\log Z^0_L}{|\Lambda|} = (h^{ext} + m_{h^{ext}})m_{h^{ext}} - f_\lambda(m_{h^{ext}})$$

(A.3)

To compute the left hand side of (3.2) we need to control the expectation on the right hand side of (A.2) that we will do by exploiting the assumptions on $h^{ext}$ which imply the validity of the Dobrushin uniqueness criterion as we are going to show. The criterion involves the Vaserstein distance of the conditional probabilities $P_{t,\gamma,L}[\sigma(x,i) | \{\sigma(y,j)\}]$ of a spin $\sigma(x,i)$ under different values of the conditioning spins $\{\sigma(y,j), (y,j) \neq (x,i)\}$. In the case of Ising spins such Vaserstein distance is simply equal to the absolute value of the difference of the conditional expectations and the criterion requires that for any pair of spin configurations outside $(x,i)$

$$|E_{t,\gamma,L}[\sigma(x,i) | \{\sigma(y,j)\}] - E_{t,\gamma,L}[\sigma(x,i) | \{\sigma'(y,j)\}]| \\ \leq \sum_{y,j} r_{\gamma,L}(x,i;y,j)|\sigma'(y,j) - \sigma(y,j)|, \quad \sum_{y,j} r(x,i;y,j) \leq r < 1$$

(A.4)

Since

$$E_{t,\gamma,L}[\sigma(x,i) | \{\sigma(y,j)\}] = \tanh \left\{ t \sum_{y \neq x} J_{\gamma,L}(x,y)\sigma(y,i) + (1-t)m_{h^{ext}} \right\} + \lambda[\sigma(x,i+1) + \sigma(x,i-1)] + h^{ext}$$

where

$$J_{\gamma,L}(x,y) = \cosh^{-2}(h^{ext} - 1 - 2\lambda) \left( J_{\gamma,L}(x,y) \mathbf{1}_{j=i} + \lambda \mathbf{1}_{x=y, j=i \pm 1 \mod L} \right)$$

(A.5)

By the Dobrushin uniqueness theorem there is a unique DLR measure $P_{t,\gamma}$ which is the weak limit of $P_{t,\gamma,L}$ as $L \to \infty$. We denote by $m_{t,\gamma,L}$ and $m_{t,\gamma}$ the average of a spin under $P_{t,\gamma,L}$ and $P_{t,\gamma}$. We call $\nu_0^\gamma$ and $\nu^\gamma$ the measures $P_{t,\gamma,L}$ and $P_{t,\gamma}$ when $t = 0$, thus $\nu_0^\gamma$ is the Gibbs measure for the Ising system in $\Lambda$ with hamiltonian $H^{\text{vert}}$ and magnetic field $h^{ext} + m_{h^{ext}}$, $\nu^0$ denoting its thermodynamic limit. We then have

$$\lim_{L \to \infty} m_{t,\gamma,L} = m_{t,\gamma}, \quad \lim_{L \to \infty} m_{0,\gamma,L} = m_{h^{ext}}$$

(A.6)
It also follows from the Dobrushin theory that under $P_{t,\gamma,L}$ the spins are weakly correlated: let $z \neq x$ then

$$|E_{t,\gamma,L}[(\sigma(x,i) - m_{t,\gamma,L}) | \sigma(z,i)]| \leq 2 \sum_{n \geq 0}^* \sum_{y_1,j_1,...,y_n,j_n} r_{t,\gamma,L}(y_1,j_1; y_n,j_n) \cdots r_{t,\gamma,L}(y_n-1,j_n-1; y_n,j_n)$$

(A.7)

where the *sum means that all the pairs $(y_k,j_k), k = 1,..,n$ must differ from $(z,i)$. Thus there is a constant $c$ so that

$$|E_{t,\gamma,L}[(\sigma(x,i) - m_{t,\gamma,L}) | \sigma(z,i)]| \leq c$$

(A.8)

and also (after using Chebitchev)

$$E_{t,\gamma,L} \left[ \left| \sum_y J_{t,\gamma,L}(x,y)(\sigma(y,i) - m_{t,\gamma,L}) \right| \right] \leq c$$

(A.9)

We can also use the Dobrushin technique to estimate the Vaserstein distance between $P_{t,\gamma,L}$ and $\nu_L^0$. The key bound is again the Vaserstein distance between single spin conditional expectations. We have

$$E_{t,\gamma,L}[\sigma(x,i) | \{\sigma(y,j)\}] - E_{\nu_L^0}[\sigma(x,i) | \{\sigma'(y,j)\}]$$

$$= \tanh \left\{ t \sum_{y \neq x} J_{t,\gamma,L}(x,y)\sigma(y,i) + (1-t)m_{\text{ext}} + \lambda[\sigma(x,i + 1) + \sigma(x,i - 1)] + h_{\text{ext}} \right\}$$

$$- \tanh \left\{ + \lambda[\sigma'(x,i + 1) + \sigma'(x,i - 1)] + h_{\text{ext}} + m_{\text{ext}} \right\}$$

(A.10)

thus, calling $A := \cosh^{-2}(h_{\text{ext}} - 1 - 2\lambda)$, we can bound the absolute value of the left hand side of (A.10) by:

$$\sum_{j=\pm 1} r_{t,\gamma,L}(x,i;x,j)\sigma(x,j) - \sigma'(x,j) + At|\sum_y J_{t,\gamma,L}(x,y)\sigma(y,i) - m_{\text{ext}}|$$

After adding and subtracting $m_{t,\gamma,L}$ to each $\sigma(y,i)$ and recalling that $\sum_y J_{t,\gamma,L}(x,y) = 1$, we use the Dobrushin analysis to claim that there exists a joint representation $P_{t,\gamma,L}$ of $P_{t,\gamma,L}$ and $\nu_L^0$ such that

$$\mathcal{E}_{t,\gamma,L}[|\sigma(x,i) - \sigma'(x,i)|] \leq \sum_{j=\pm 1} r_{t,\gamma,L}(x,i;x,j)\mathcal{E}_{t,\gamma,L}[|\sigma(x,j) - \sigma'(x,j)|]$$

$$+ At \left( \mathcal{E}_{t,\gamma,L}[|\sum_y J_{t,\gamma,L}(x,y)\sigma(y,i) - m_{t,\gamma,L}|] + |m_{t,\gamma,L} - m_{\text{ext}}| \right)$$

(A.11)

Since $\sum_y J_{t,\gamma,L}(x,y)\sigma(y,i) - m_{t,\gamma,L})$ does not depends on $\sigma'$ we can replace the $\mathcal{E}_{t,\gamma,L}$ expectation by the $E_{t,\gamma,L}$ expectation and after using (A.9) we get by iteration

$$\mathcal{E}_{t,\gamma,L}[|\sigma(x,i) - \sigma'(x,i)|] \leq \frac{At}{1-r} \left( c\gamma + |m_{t,\gamma,L} - m_{\text{ext}}| \right)$$

(A.12)

with $r$ as in (A.11). Since $|m_{t,\gamma,L} - m_{0,\gamma,L}| \leq \mathcal{E}_{t,\gamma,L}[|\sigma(x,i) - \sigma'(x,i)|]$, (A.12) yields

$$|m_{t,\gamma,L} - m_{0,\gamma,L}| \leq \frac{At}{1-r} \left( c\gamma + |m_{t,\gamma,L} - m_{0,\gamma,L}| + |m_{0,\gamma,L} - m_{\text{ext}}| \right)$$
By (3.1) \( \frac{\Delta r}{1 - r} \leq \frac{r}{1 - r} < \frac{1}{3} \), so that
\[
\frac{2}{3} m_{t,\gamma,L} - m_{0,\gamma,L} \leq \frac{1}{3} \left( c\gamma + |m_{0,\gamma,L} - m_{\text{ext}}| \right) \\
|m_{t,\gamma,L} - m_{\text{ext}}| \leq |m_{\text{ext}} - m_{0,\gamma,L}| + (c\gamma + |m_{0,\gamma,L} - m_{\text{ext}}|)
\]
(A.13)

Thus \( m_{t,\gamma,L} \to m_{\text{ext}} \) as first \( L \to \infty \) and then \( \gamma \to 0 \). This holds for all \( t \) and in particular for \( t = 1 \) hence properties (i) and (ii) are proved. Moreover, since \( m_{\gamma} \equiv m_{1,\gamma} \) converges as \( \gamma \to 0 \) to \( m_{\text{ext}} \) the latter is uniquely determined, as a consequence the equation \( h_{\text{ext}} + m = f_\lambda'(m) \) has a unique solution \( m_{\text{ext}} \) which is the limit of \( m_\gamma \) as \( \gamma \to 0 \). To prove (iii) we go back to (A.2) and observe that
\[
H_{L,\gamma,\text{ext},L}^0 - H_{\gamma,\text{ext},L}^\text{per} = \sum_{(x,i) \in \Lambda} \sigma(x,i) \left( \frac{1}{2} \sum_{y \neq x} J_{\gamma,L}(x,y) \sigma(y,i) - m_{\text{ext}} \right)
\]

Therefore
\[
|E_{t,\gamma,L}[H_{L,\gamma,\text{ext},L}^0 - H_{\gamma,\text{ext},L}^\text{per}] - \sum_{(x,i) \in \Lambda} E_{t,\gamma,L}[\sigma(x,i)] \left( \frac{m_{t,\gamma,L}}{2} - m_{\text{ext}} \right)| \\
\leq \sum_{(x,i) \in \Lambda} \frac{1}{2} E_{t,\gamma,L}[\left| \sum_{y \neq x} J_{\gamma,L}(x,y) (\sigma(y,i) - m_{t,\gamma,L}) \right|] \leq |\Lambda| c\gamma
\]

(A.2) and (A.3) then yield (3.2) because \( m_{t,\gamma,L} \to m_{\text{ext}} \) as \( L \to \infty \) and then \( \gamma \to 0 \). This is the same as taking the inf over all \( m \) because we have already seen that \( h_{\text{ext}} + m = f_\lambda'(m) \) has a unique solution.

### B Proof of Theorem 3

Following Lebowitz and Penrose we do coarse graining on a scale \( \ell, \ell \) the integer part of \( \gamma^{-1/2} \). Without loss of generality we restrict \( L \) in (2.4) to be an integer multiple of \( \ell \). We then split each horizontal line in \( \Lambda \) into \( L/\ell \) consecutive intervals of length \( \ell \) and call \( I \) the collection of all such intervals in \( \Lambda \). Thus
\[
\mathcal{M}_\ell = \{-1, -1 + \frac{2}{\ell}, \ldots, 1 - \frac{2}{\ell}, 1\}
\]
is the set of all possible values of the empirical spin magnetization in an interval \( I \in \mathcal{I} \). We denote by \( \mathcal{M} \) the set of all functions \( m = \{m(x,i), (x,i) \in \Lambda\} \) on \( \Lambda \) with values in \( \mathcal{M}_\ell \) which are constant on each one of the intervals \( I \) of \( \mathcal{I} \). Due to the smoothness assumption on the Kac potential there is \( c \) so that for all \( \sigma, \gamma \) and \( L \)
\[
\left| \sum_{(x,i) \in \Lambda} \frac{1}{2} J_{\gamma,L}(x,y) (\sigma(x,i) - m(x,i|\sigma)) \right| \leq c\gamma^{1/2} \Lambda \tag{B.1}
\]
where, denoting by \( I_{x,i} \) the interval in \( \mathcal{I} \) which contains \( (x,i) \),
\[
m(x,i|\sigma) = \frac{1}{\ell} \sum_{y \neq (y,i) \in I_{x,i}} \sigma(y,i) \tag{B.2}
\]
Thus \( m(x, i | \sigma) \) does not change when \((x, i)\) varies in an interval of \( \mathcal{I} \) and therefore \( m = \{ m(x, i | \sigma), (x, i) \in \mathcal{I} \} \in \mathcal{M} \). Then the partition function
\[
Z_{\gamma, L} := \sum_{m \in \mathcal{M}} e^{\frac{1}{2} \sum_{x, y} J_{\gamma, L}(x, y)m(x, i)m(y, i) + \sum_{x, i} h_{\text{ext}} m(x, i)} \sum_{\sigma} 1_{m_i(\sigma) = m} e^{-H_L^{\text{vert}}(\sigma)}
\]
has the same asymptotics as \( Z_{\gamma, L, \text{ext}, \ell} \) in the sense that
\[
\lim_{\gamma \to 0} \lim_{L \to \infty} \frac{1}{|\Lambda|} \log Z_{\gamma, L} - \log Z_{\gamma, L, \text{ext}, \ell} = 0 \quad (B.4)
\]
We next change the vertical interaction \( H_L^{\text{vert}}(\sigma) \) by replacing
\[-\lambda \sigma(x, n \ell) \sigma(x, n \ell + 1) \to -\lambda \sigma(x, n \ell) \sigma(x, (n - 1) \ell + 1)\]
and call \( H_L^{\text{vert}}(\sigma) \) the new vertical energy. We then split each vertical column into intervals of length \( \ell \), calling \( I' \) such intervals and \( \Delta \) the squares \( I \times I' \). Let \( \Delta = I \times I' \), \( m_\Delta \) the restriction of \( m \) to \( \Delta \), so that \( m_\Delta(x, i), \sigma(x, i) \in \mathcal{I} \) is only a function of \( i \) with values in \( \mathcal{M}_\ell \). Recalling the definition \( [1.3] \) of \( \phi_\ell(m_\Delta) \) we have that \( Z_{\gamma, L} \) has the same asymptotics as
\[
Z_{\gamma, L, \ell} := \sum_{m \in \mathcal{M}} e^{\frac{1}{2} \sum_{x, y} J_{\gamma, L}(x, y)m(x, i)m(y, i) + \sum_{x, i} (h_{\text{ext}} m(x, i) - \phi_\ell(m_{\Delta, i}))}
\]
where \( \Delta_{x,i} \) denotes the square \( \Delta \) which contains \((x, i)\).

The cardinality of \( \mathcal{M} \) is \( \ell^{|\Lambda|/\ell} \), hence \( Z_{\gamma, L, \ell} \) has the same asymptotics as
\[
Z_{\gamma, L, \ell}^{\text{max}} := \max_{m \in \mathcal{M}} e^{\frac{1}{2} \sum_{x, y} J_{\gamma, L}(x, y)m(x, i)m(y, i) + \sum_{x, i} (h_{\text{ext}} m(x, i) - \phi_\ell(m_{\Delta, i}))}
\]
Recalling the definition \( [1.5] \) of \( Z^{\text{max}}_\Delta \), we are going to show that
\[
\frac{1}{|\Lambda|} \log Z_{\gamma, L, \ell}^{\text{max}} = \frac{1}{|\Delta|} \log Z^{\text{max}}_\Delta
\]
To prove \( (B.7) \) we write
\[
m(x, i)m(y, i) = \frac{1}{2} \left( m(x, i)^2 + m(y, i)^2 \right) - \frac{1}{2} \left( m(x, i) - m(y, i) \right)^2
\]
and use that \( \sum_{y} J_{\gamma, L}(x, y) = 1 \). In this way the exponent in the right hand side of \( (B.6) \) becomes a sum over all the squares \( \Delta \) of terms which depend on \( m_\Delta \) plus an interaction given by
\[-\sum_{i,x,y} J_{\gamma, L}(x, y) \frac{1}{2} \left( m(x, i) - m(y, i) \right)^2\]
Due to the minus sign the maximizer is obtained when all \( m_\Delta \) are equal to each other and to the maximizer in \( (1.5) \). To complete the proof of \( (B.7) \) we still need to prove the bound on the magnetization:

**Proposition 2** There are \( \lambda_0 > 0 \) and \( m_+ < 1 \) so that for any \( \lambda \leq \lambda_0 \) the maximum in \( (B.6) \) is achieved on configurations \( m_\Delta \) such that for all \((x, i) \in \Delta, |m_\Delta(x, i)| \leq m_+ \).
Proof Given $h > 0$ let $S(m)$ be the entropy defined in (2.7) and let $m_h$ be such that

$$- [S'(m_h) + m_h] = h$$

(B.8)

Call $m^*$ the value of $m_h$ at $h^*$, $h^*$ as in (4.1) and choose $m_+ > m^*$. Fix any horizontal line $i$ in $\Delta$, take a magnetization $m_i$ such that $m_i \geq m_+$, it is then sufficient to prove that for all $\sigma(x, i + 1) + \sigma(x, i - 1) =: h_i(x),$

$$e^{-\ell U(m_i)} \sum_\sigma \sum_\sigma 1 e^{\lambda \sum_x \sigma(x)h_i(x)} \leq e^{-\ell U(m^*)} \sum_\sigma \sum_\sigma 1 e^{\lambda \sum_x \sigma(x)h_i(x)}$$

(B.9)

where $U(m) = -\frac{m^2}{2} - h_{ext}$. Since $|h_i| \leq 2$, this is implied (for $\ell$ large enough) by

$$- U(m_i) + S(m_i) + 4\lambda < - U(m^*) + S(m^*)$$

(B.10)

Since $m_i > m^*$ and $h_{ext} \leq h^*$, (B.10) is implied by

$$\frac{m_i^2}{2} + h^*m_i + S(m_i) + 4\lambda < \frac{(m^*)^2}{2} + h^*m^* + S(m^*)$$

(B.11)

The function $m^2 + S(m) + h^*m$ is strictly concave in a neighborhood of $m^*$ where it reaches its maximum, hence (recalling that $m_i \geq m_+ > m^*$

$$\left(\frac{(m^*)^2}{2} + h^*m^* + S(m^*)\right) - \left(\frac{m_i^2}{2} + h^*m_i + S(m_i)\right)$$

is strictly positive and (B.9) follows for $\lambda$ small enough.

C Cluster expansion

In this appendix we will study the partition function $Z_{\ell,h}^*$ defined in (4.8) using cluster expansion.

C.1 Reduction to a gas of polymers

We shall first prove in Proposition 3 below that $Z_{\ell,h}^*$ can be written as the partition function of a gas of polymers $\Gamma$. The definition of polymers and the main notation of this section are given below.

- $\Gamma = (C, S, X)$ denotes a polymer, $C$ its spatial support, $X$ and $S$ its specifications. $C$ is a collection of pairs of consecutive points ($L$ and $1$ being consecutive points), and calling connected two pairs if they have a common point, then $C$ is connected. We write $i \in C$ or sometimes $i \in \Gamma$ if $i$ is in one of the pairs of $C$. Each pair in $C$ is either a $X$-pair or a $S$-pair, $S$ and $X^*$ are the collection of all the $S$ and respectively all the $X$ pairs. $X$ is the set of all points $i$ which belong to one and only one of the $X$-pairs.
• \(|C|\) is the number of pairs in \(C\), \(|S|\) the number of pairs in \(S\) and \(|X|\) the number of points in \(X\). It follows directly from its definition that \(|X|\) is even.

• \(\Gamma\) and \(\Gamma'\) are compatible, \(\Gamma \sim \Gamma'\), if the spatial supports of \(\Gamma\) and \(\Gamma'\) do not have any point in common.

• \(w(\Gamma)\) is the weight of the polymer \(\Gamma\). If \(C\) consists of all the possible pairs (so that \(|C| = \ell\)) and \(S = \emptyset\) then we set

\[
w(\Gamma) = \sinh(\lambda)^\ell \quad \text{(C.1)}
\]

Otherwise:

\[
w(C, S, X) = \sinh(\lambda)^{|C|} \left( \frac{[\cosh(\lambda) - 1]}{\sinh(\lambda)} \right)^{|S|} \prod_{x \in X} u_x, \quad u_x = \tanh(h_x) \quad \text{(C.2)}
\]

Each \(X\)-pair in \(\Gamma\) contributes to the weight of \(\Gamma\) by a factor \(\sinh(\lambda)\) while each \(S\)-pair contributes with a factor \([\cosh(\lambda) - 1]\), as it readily follows from (C.2). The dependence of the weight on \(h_i\) is through the terms \(u_i, \ i \in X\).

**Proposition 3** Let \(\Gamma\) and \(w(\Gamma)\) be as above, then

\[
Z^*_{\ell, h} = \sum \prod_{\Gamma \in \Gamma} w(\Gamma) \quad \text{(C.3)}
\]

where the sum is over all collections \(\Gamma = \Gamma_1, \ldots, \Gamma_n\) of mutually compatible polymers.

**Proof** We write

\[
Z^*_{\ell, h} = \sum_{\sigma} \left\{ \prod_i e^{h_{\sigma_i}} \left\{ \prod_i \left[ e^{\lambda \sigma_i \sigma_{i+1}} - 1 + 1 \right] \right\} \right\}
\]

By expanding the last product we get a sum of terms each one being characterized by the pairs \((i, i+1)\) with \(e^{\lambda \sigma_i \sigma_{i+1}} - 1\). We fix one of these terms and perform the sum over \(\sigma\). We call cluster a maximal connected set of pairs with \([e^{\lambda \sigma_i \sigma_{i+1}} - 1]\), this will be the spatial support of a polymer. The sum over \(\sigma\) factorizes over the clusters. After writing

\[
e^{\lambda \sigma_i \sigma_{i+1}} - 1 = \sinh(\lambda) \sigma_i \sigma_{i+1} + [\cosh(\lambda) - 1]
\]

we call \((i, i+1)\) a \(X\)-pair if it has the term \(\sinh(\lambda) \sigma_i \sigma_{i+1}\) and a \(S\)-pair if it has the term \([\cosh(\lambda) - 1]\). Notice that if \(i\) belongs to two \(X\)-pairs then we have a product of two \(\sigma_i\) which is equal to 1. Thus the sum over the spins in a cluster \(C\) becomes a sum over \(w(\Gamma)\) with the spatial support of \(\Gamma\) equal to \(C\). In this way we get (C.3).

We shall also consider the partition function

\[
Z'_{\ell} = \sum \prod_{\Gamma \in \Gamma} w_1(\Gamma) \quad \text{(C.4)}
\]

where \(w_1(\Gamma)\) is obtained from \(w(\Gamma)\) by putting \(u_i \equiv 1\).
C.2 The K-P condition

The K-P condition for cluster expansion requires that after introducing a weight $|\Gamma|$ then for any $\Gamma$

$$\sum_{\Gamma' \neq \Gamma} |w(\Gamma')|e^{\Gamma'|} \leq |\Gamma|$$

**Proposition 4** For $\lambda$ small enough we have that

$$\sum_{\Gamma' \neq \Gamma} |w(\Gamma')|e^{\Gamma'| (1 + b)} \leq |\Gamma|, \quad b > 0 \quad (C.5)$$

with

$$|\Gamma| = |C(\Gamma)| + 1, \quad e^b := \lambda^{-5/12} \quad (C.6)$$

having called $C(\Gamma)$ the spatial support of $\Gamma$.

**Proof** We are first going to prove that for $\lambda$ small enough

$$\sum_{\Gamma' : C(\Gamma') \ni i} w_1(\Gamma')e^{(1 + b)|\Gamma'|} \leq 1 \quad (C.7)$$

Fix $C$ and consider all $\Gamma$ with spatial support $C$, i.e. $C(\Gamma) = C$, so that $|\Gamma| = |C| + 1 =: n$. Then

$$\sum_{\Gamma : C(\Gamma) = C} w_1(\Gamma)e^{(1 + b)|\Gamma|} \leq e^{(1 + b)[\sinh(\lambda)e^{(1 + b)}]^{n-1}} \left(1 + \frac{[\cosh(\lambda) - 1]}{\sinh(\lambda)}\right)^{n-1} \quad (C.8)$$

Therefore the left hand side of (C.7) is bounded by

$$\sum_{n \geq 2} ne^{(1 + b)[\sinh(\lambda)e^{(1 + b)}]^{n-1}} \left(1 + \frac{[\cosh(\lambda) - 1]}{\sinh(\lambda)}\right)^{n-1}$$

which vanishes when $\lambda \to 0$, because by (C.6) $\lambda e^{2b}$ vanishes as $\lambda \to 0$. Hence (C.7) holds for $\lambda$ small enough.

To prove (C.5) we first write

$$\sum_{\Gamma' \neq \Gamma} |w(\Gamma')|e^{(1 + b)|\Gamma'|} \leq \sum_{\Gamma' \neq \Gamma} w_1(\Gamma')e^{(1 + b)|\Gamma'|} \quad (C.9)$$

and then use (C.7) to get

$$\sum_{\Gamma' \neq \Gamma} w_1(\Gamma')e^{(1 + b)|\Gamma'|} \leq \sum_{i \in C(\Gamma')} \sum_{\Gamma' : C(\Gamma') \ni i} w_1(\Gamma')e^{(1 + b)|\Gamma'|} \leq |\Gamma|$$
C.3 The basic theorem of cluster expansion

The theory of cluster expansion states that if the K-P condition is satisfied then the log of the partition function can be written as an absolutely convergent series over “clusters” of polymers. To define the clusters it is convenient to regard the space \( \{\Gamma\} \) of all polymers as a graph where two polymers are connected if they are incompatible, as defined in Subsection C.1. Then a cluster is a connected set in \( \{\Gamma\} \) whose elements may also have multiplicity larger than 1. We thus introduce functions \( I : \{\Gamma\} \to \mathbb{N} \) such that \( \{\Gamma : I(\Gamma) > 0\} \) is a non-empty connected set which is the cluster defined above, \( I(\Gamma) \) being the multiplicity of appearance of \( \Gamma \) in the cluster. With such notation the theory says that

\[
\log Z^*_{L,h} = \sum W^I, \quad W^I := a_I \prod_{\Gamma} w(\Gamma)^{I(\Gamma)} \quad (C.10)
\]

where the sums in (C.10)–(C.11) are absolutely convergent. The coefficients \( a_I \) are combinatorial (signed) factors, in particular \( a_I = 1 \) if \( I \) is supported by a single \( \Gamma \). We will not need the explicit expression of the \( a_I \) and only use the bound provided by Theorem 12 below. We use the notation:

\[
|I|_1 = \sum_{\Gamma} I(\Gamma), \quad ||I|| = \sum_{\Gamma} |\Gamma| I(\Gamma) \quad (C.12)
\]

**Theorem 12 (Cluster expansion)** Let \( \lambda \) be so small that the K-P condition (C.5) holds. Let \( \Gamma \) be a polymer and \( I \) a subset in \( \{I\} \) such that \( I(\Gamma) \geq 1 \) for all \( I \in I \) (\( I \) could be the whole \( \{I\} \)). Then

\[
\sum_{I \in I} |W^I_1| e^{||I||} \leq w_1(\Gamma)e^{(1+b)|\Gamma|} \sup_{I \in I} e^{-b||I||} \quad (C.13)
\]

Observe that the absolute convergence of the sum in (C.10)–(C.11) is implied by (C.13) with \( I = \{I : I(\Gamma) \geq 1\} \) as it becomes

\[
\sum_{I : I(\Gamma) \geq 1} |W^I_1| e^{||I||} \leq w_1(\Gamma)e^{||\Gamma||} \quad (C.14)
\]

because \( \inf_{I \in I} e^{-b||I||} = e^{-b|\Gamma|} \) as the inf is realized by \( I^* \) which has \( I^*(\Gamma) = 1 \) and \( I^*(\Gamma') = 0 \) for all \( \Gamma' \neq \Gamma \). (C.14) proves that the sum in (C.11) and hence the sum in (C.10) are both absolutely convergent.

D Proof of Theorem 4

In this section we will prove Theorem 4 as a direct consequence of Theorem 12.
D.1 Proof of \((4.10)\)

We start from \((C.10)\) and observe that

\[ W^I := a_I \prod_{\Gamma} w(\Gamma)^{I(\Gamma)} = \{a_I \prod_{\Gamma} w_1(\Gamma)^{I(\Gamma)} \} \{ \prod_{\Gamma} (u_{X(\Gamma)})^{I(\Gamma)} \} \]

The last factor is equal to \(u^N(\cdot)\) where \(N(\cdot)\) is determined by \(I:\)

\[ N(x) = \sum_{\Gamma} I(\Gamma) \mathbb{1}_{x \in X(\Gamma)} \tag{D.15} \]

hence \((4.10)\). Recalling \((4.12)\) we observe that \(|N(\cdot)|\) is even because the cardinality of each \(X(\Gamma)\) is even.

D.2 The term with \(|N(\cdot)| = 0\)

The term with \(|N(\cdot)| = 0\) is a constant \(A_0\) (i.e. it does not depends on \(u\)) and it will not play any meaningful role. It is bounded as follows:

**Lemma 2** There is a constant \(c\) (independent of \(u\) and \(\ell\)) such that

\[ |A_0| \leq c \lambda^2 \ell \]

**Proof** By \((C.14)\)

\[
|A_0| \leq \sum_{i=1}^{\ell} \sum_{C \supseteq i} \sum_{I: I(C,C,\emptyset) > 0} |W^I| \\
\leq \sum_{i=1}^{\ell} \sum_{C \supseteq i} \left[ \cosh(\lambda) - 1 \right]^{|C|} e^{|C| + 1} \leq \ell c \lambda^2
\]

D.3 Proof of \((4.15)\)

We have

\[ \alpha_{j-i} u_i u_j = \sum_{\Gamma=(C,S,X)} 1_{X\{i,j\}} \sum_{I: I(\Gamma) = 1; I(\Gamma') = 0} W^I \]

Thus by \((C.13)\)

\[
|\alpha_{j-i}| \leq \sum_{\Gamma=(C,S,X)} 1_{X\{i,j\}} w_1(\Gamma) e^{|\Gamma|} \\
\leq \sum_{\Gamma=(C,S,X)} 1_{X\{i,j\}} (\sinh(\lambda))^{|i-j|} (\cosh(\lambda) - 1)^{|C|-|i-j|} e^{|C| + 1}
\]

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which is bounded by

\[
|\alpha_{j-i}| \leq \sum_{n \geq 0, m \geq 0} (\sinh(\lambda))^{i-j} (\cosh(\lambda) - 1)^{n+m} e^{i-j+n+m+1}
\]

We have thus proved the second inequality in (4.15).

To prove the first one we call \( \Gamma^* = (C^*, S^*, X^*) \) where \( C^* = (i, i+1) \), \( S^* = \emptyset \), \( X^* = \{i, i+1\} \) and write

\[
\alpha_{i,i+1} u_i u_{i+1} = \sinh(\lambda) u_i u_{i+1} + \sum_{I; |I|>1; I(\Gamma^*)=1; I(\Gamma')=0 \text{ if } \Gamma' \neq \Gamma^* \text{ and } X' \neq \emptyset} W^I
\]

\[
+ \sum_{\Gamma=(C,S,X), \Gamma \neq \Gamma^*} 1_{X=\{i,j\}} \sum_{I: I(\Gamma)=1; I(\Gamma')=0 \text{ if } X' \neq \emptyset \text{ and } \Gamma' \neq \Gamma} W^I \quad (D.16)
\]

If \( I \) is the second term on the right hand side then \(|I| \geq 2+2\) so that this term is bounded by

\[
|w(\Gamma^*)| e^{|\Gamma^*|} e^{-2b} \leq \sinh(\lambda) e^2 e^{-2b}
\]

Proceeding as in the proof of the second inequality in (4.15) we can bound the last term on the right hand side of (D.16) by

\[
\leq \sum_{n \geq 0, m \geq 0, n+m > 0} (\sinh(\lambda))(\cosh(\lambda) - 1)^{n+m} e^{2+n+m+1}
\]

which proves the first inequality in (4.15).

D.4 Proof of (4.13)

If \( I \) determines \( N(\cdot) \) then for all \( j \)

\[
N(j) \leq \sum_{\Gamma; C(\Gamma) \ni j} I(\Gamma) \quad (D.17)
\]

hence

\[
R(N(\cdot)) \leq \sum_{\Gamma; I(\Gamma) > 0} |\Gamma|, \quad |N(\cdot)| = \sum_j N(j) \leq \sum_j \sum_{\Gamma; C(\Gamma) \ni j} |I(\Gamma)| \quad (D.18)
\]

Thus

\[
||I|| \geq ||N(\cdot)|| \quad (D.19)
\]

so that the left hand side of (4.13) is bounded by:

\[
\sum_{\Gamma \ni i} \sum_{I; I(\Gamma) > 0, ||I|| \geq M} |W^I| \leq \sum_{\Gamma \ni i} w_1(\Gamma)e^{|\Gamma|(1+b)} e^{-bM} \quad (D.20)
\]

having used (C.13), (4.13) then follows from (C.7).
E  A priori bounds

We will extensively use the bounds in this section which are corollaries of Theorem 4.

Corollary 1  There are constants $c_k$, $k \geq 0$, so that for any $i \in \{1, \ldots, \ell\}$, $k \geq 0$ and $M \geq 4$,

$$\sum_{\|N(\cdot)\| \leq M, |N(\cdot)| > 0} |N(\cdot)|^k |A_{N(\cdot)}| \leq c_k M^k e^{-bM} = c_k \lambda^{5/3} e^{-b(M-4)} \quad (E.1)$$

Proof
It follows from Theorem 4, see (4.13).

Corollary 2  There are constants $c'_k$, $k \geq 1$, so that for any $\ell$ and $i \in [1, \ell]$

$$\sum_{i_1, \ldots, i_{k-1}} \frac{\partial^{k-1}}{\partial u_{i_1} \cdots \partial u_{i_{k-1}}} \frac{\partial}{\partial u_i} \log Z^*_\ell \leq c'_k \lambda \quad (E.2)$$

for any $\lambda$ as small as required in Theorem 4. Moreover

$$\Psi_i(u) = 0 \text{ if } |u_i| = 1 \quad (E.3)$$

Proof  We write $\log Z^*_\ell = K_1 + K_2$ where $K_1$ is obtained by restricting the sum on the right hand side of (4.10) to $|N(\cdot)| \leq 2$, $K_2$ is the sum of the remaining terms. By (4.14)–(4.15) we easily check that $K_1$ satisfies the bound in (E.2). We bound

$$\sum_{i_1, \ldots, i_{k-1}} \frac{\partial^{k-1}}{\partial u_{i_1} \cdots \partial u_{i_{k-1}}} \frac{\partial}{\partial u_i} K_2$$

by

$$\sum_{M > 2} \sum_{\|N(\cdot)\| = M, N(\cdot) > 0} |N(\cdot)|^k R(N(\cdot))^k$$

(E.2) then follows from (E.1). (E.3) follows directly from the definition of $\Psi_i(u)$.

Corollary 3  Recalling (4.14) and writing $\alpha = \sum_{j>i} \alpha_{j-i}$,

$$\sum_{i<j} \alpha_{j-i} u_i u_j = \alpha \sum_i u_i^2 - \frac{1}{2} \sum_{j>i} \alpha_{j-i} (u_i - u_j)^2 \quad (E.4)$$
We write \( \|v\| \) for the sup norm of the vector \( v \): \( \|v\| := \max_{i=1,\ldots,l} |v_i| \).

### F.1 Proof of Theorem 5

Existence. By (E.2) we can use the implicit function theorem to claim existence of a small enough time \( T > 0 \) such that the equation

\[
m = u(t) + t\Psi(u(t)) \tag{F.1}
\]

has a solution \( u(t), t \in [0,T], \) such that: \( u(0) = m, u(t) \) is differentiable and \( \|u(t)\| < 1, \) recall that \( \|m\| < 1. \)

If \( \lambda \) is small enough (E.2) with \( k=1 \) yields

\[
\max_i \sup_{\|v\| \leq \lambda} \sum_j j|\frac{\partial}{\partial v_j} \Psi_i(u)| =: r < 1 \tag{F.2}
\]

so that the matrix \( 1 + t\nabla \Psi(u(t)), (\nabla \Psi)_{i,j} = \frac{\partial}{\partial v_j} \Psi_i, \) is invertible for \( t \leq \min\{T,1\} \) and therefore for \( t \leq \min\{T,1\} \)

\[
\dot{u}(t) = f(u(t),t) := -\left(1 + t\nabla \Psi(u(t))\right)^{-1}\Psi(u(t)), \quad u(0) = m \tag{F.3}
\]

By (F.2–E.2) \( f(u,t) \) is bounded and differentiable for \( t \leq 1 \) and \( \|u\| \leq 1, \) thus we can extend \( u(t) \) till \( \min\{1,\tau\} \) where \( \tau \) is the largest time \( \leq 1 \) such that \( \|u(t)\| \leq 1 \) for \( t \leq \tau. \)

Thus for \( t \leq \tau \) (F.1) has a solution \( u(t) \) which we claim to satisfy \( \|u(t)\| < 1. \) To prove the claim we suppose by contradiction that there is a time \( t \leq \tau \) and \( i \) so that \( |u_i(t)| = 1. \) By (F.1), \( m_i = u_i + t\Psi_i(u) = u_i \) (having used (E.3)). We have thus reached a contradiction because \( \|m\| < 1. \) Thus the claim is proved and as a consequence \( \tau = 1 \) and therefore we have a solution of (F.1) for all \( t \leq 1 \) with.

Uniqueness. Suppose there are two solutions \( u \) and \( v. \) Then

\[ u - v = \Psi(v) - \Psi(u) \]

Define \( u(s) = su + (1-s)v, s \in [0,1], \) then

\[
\|u - v\| \leq \int_0^1 \|\nabla \Psi(u(s))(u - v)\| \, ds
\]

Since \( \|u(s)\| < 1 \) by (E.2) \( \|\nabla \Psi(u(s))(u - v)\| \leq r\|u - v\|, \) so that \( \|u - v\| \leq r\|u - v\| \) and therefore \( u = v. \)

Boundedness. Calling \( u = u(t) \) when \( t = 1, \) by (E.1) and (E.2)

\[
\|u\| \leq \|m\| + \|\Psi(u)\| \leq \|m\| + c_1 \lambda \tag{F.4}
\]

so that if \( \|m\| \leq m_+ \) then for \( \lambda \) small enough \( \|u\| < 1 \) and therefore there exists \( h_+ \) such that \( \|h\| \leq h_+. \)
F.2 Proof of Theorem 6

Since
\[-φℓ(m) = \frac{1}{ℓ^2} \log \left\{ e^{-ℓ \sum_i h_i m_i} \sum_{σ \in \{-1,1\}^Δ} 1_{m(σ)=m} e^{-\sum_{x,i} (-λσ(x,i)σΔ(x,i+1)-h_i σ(x,i))} \right\} \quad \text{(F.5)}\]
we have for free
\[\frac{1}{ℓ^2} \log \left\{ e^{-ℓ \sum_i h_i m_i} Zγ,Δ,h \right\} ≥ -φℓ(m) \quad \text{(F.6)}\]
and we are thus left with the proof of a lower bound for \(-φℓ(m)\).

Call \(I_i = \{(x,i) : x ≤ ℓ - ℓ^{a'}\}\), let \(a' ∈ (\frac{1}{2}, a)\) and
\[B_i = \{σ(·, i) : \left| \sum_{(x,i) ∈ I_i} [σ(x,i) - m_i] \right| ≤ ℓ^{a'} \} \quad \text{(F.7)}\]

Let \(μ\) be the Gibbs probability for the system with vertical interactions and magnetic fields \(h\). We look for a lower bound for
\[μ \left[ \bigcap_i B_i \right] \cap \left\{ m(\cdot | σ) = m \right\} \]

By the central limit theorem
\[μ \left[ B_i^c \right] ≤ e^{-bℓ^{2a' - 1}}, \quad b > 0 \quad \text{(F.8)}\]

because the spins in \(I_i\) are i.i.d. with mean \(m_i\). Moreover
\[μ \left[ \left\{ m(\cdot | σ) = m \right\} | \left\{ \bigcap_i B_i \right\} \right] ≥ e^{-4λℓ^{1+a}} 2^{-ℓ^{1+a}} \quad \text{(F.9)}\]

because, given \(\bigcap_i B_i\), there is at least one configuration in the complement of \(I_i\) on each horizontal line. Thus
\[μ \left[ \bigcap_i B_i \right] \cap \left\{ m(\cdot | σ) = m \right\} \geq (1 - ℓe^{-bℓ^{2a' - 1}}) e^{-4λℓ^{1+a}} 2^{-ℓ^{1+a}} \]

hence
\[-φℓ(m) ≥ \frac{1}{ℓ^2} \log \left\{ e^{-ℓ \sum_i h_i m_i} Zγ,Δ,h \right\} - \frac{1}{ℓ^2} \log \left\{ (1 - ℓe^{-bℓ^{2a' - 1}}) e^{-4λℓ^{1+a}} 2^{-ℓ^{1+a}} \right\} \]

which together with (F.6) proves (4.18).

G Proof of Lemma 1

We first write
\[H_{eff}^{ℓ,h} = \sum_{i=1}^ℓ \left\{ -\frac{u_i^2}{2} - (h_{ext} - h_i) u_i - \log(e^{h_i} + e^{-h_i}) \right\} + \sum_{i=1}^ℓ \left\{ (h_i - u_i - h_{ext}) \Psi_i - \frac{Ψ_i^2}{2} \right\} - \log Z_{eff}^{ℓ,h} + A_\Φ \quad \text{(G.1)}\]
We have \( \log(e^{h_i} + e^{-h_i}) = h_i u_i + S(u_i) \), the entropy \( S(u) \) being defined in (2.7)–(2.8). Thus

\[
H_{\text{eff}} = \sum_{i=1}^{\ell} \left\{ (T(u_i) - h_{\text{ext}} u_i + (h_i - u_i - h_{\text{ext}}) \Psi_i - \frac{\Psi_i^2}{2} \right\} - \log Z_{\ell,h}^* + A_0 \tag{G.2}
\]

The term with \( h_{\text{ext}} \Psi_i \) in (G.2) becomes

\[
- h_{\text{ext}} \sum_i \Phi_i + \lambda h_{\text{ext}} \sum_i \left( u_i^2 u_{i+1} + u_i u_{i+1}^2 \right) - 2 \lambda h_{\text{ext}} \sum_i u_i
\]

which can be written as

\[
- h_{\text{ext}} \sum_i \Phi_i + 2 \lambda u_i + \lambda h_{\text{ext}} \sum_i \left( 2u_i^3 - (u_i + u_{i+1})(u_{i+1} - u_i)^2 \right) \tag{G.3}
\]

After an analogous procedure for the term with \((h_i - u_i)\Psi_i\) we get (4.26).

**H. Proof of Theorem 8**

We say that a function \( F(u) \) is “sum of one body and gradients squared terms” if

\[
F(u) = \sum_{i=1}^{\ell} f(u_i) + \sum_{1 \leq i < j \leq \ell} b_{i,j}(u)(u_i - u_j)^2
\]

for some functions \( f(u) \) and \( b_{i,j}(u) \). Thus (4.28) claims that \( H_{\ell,h}^{(1)} \) is “sum of one body and gradients squared terms”. We say in short that the “gradients squared terms are bounded as desired” if

\[
\sum_{1 \leq i < j \leq \ell} |b_{i,j}(u)|(u_i - u_j)^2 \leq c \lambda^{1+\frac{2}{3}} \sum_i (u_i - u_{i+1})^2
\]

Hence (4.29) will follow by showing that the gradients squared terms of \( H_{\ell,h}^{(1)} \) are bounded as desired.

We will examine separately the various terms which contribute to \( H^{(1)} \) and prove that each one of them is sum of one body and gradients squared terms and that the latter are bounded as desired.

**H.1 The \( \Theta \) term**

By (4.23)

\[
\Theta = \sum_{N(\cdot) \neq 0} A_{N(\cdot)} u^{N(\cdot)} + \frac{\lambda}{2} \sum_{i=1}^{\ell} (u_{i+1} - u_i)^2
\]

Call \( \Theta^{(2)} \) the above expression when we restrict the sum to \( N(\cdot) : |N(\cdot)| = 2 \) and call \( \Theta^{(>2)} = \Theta - \Theta^{(2)} \). Thus \( \Theta^{(>2)} \) is equal to the sum of \( A_{N(\cdot)} \) over \( N(\cdot) : |N(\cdot)| > 2 \), i.e.
$|N(\cdot)| \geq 4$, recall in fact from Theorem 4 that $A_{N(\cdot)} = 0$ if $N(\cdot)$ is odd. We start from

$$\Theta^{(2)} = \alpha \sum u_i^2 - \frac{1}{2} \sum (\alpha_1 - \lambda) (u_{i+1} - u_i)^2 - \frac{1}{2} \sum_{i<j, j-i>1} \alpha_{j-i} (u_j - u_i)^2 \quad (H.1)$$

Thus $\Theta^{(2)}$ is sum of one body and gradients squared terms. To prove that the latter are bounded as desired we write

$$(u_j - u_i)^2 \leq (j - i) \sum_{k=i}^{j-1} (u_{k+1} - u_k)^2 \quad (H.2)$$

and call $n = k - i \geq 0$, $m = j - k \geq 1$. We then use (4.15) to bound the sum of the terms with the gradients by

$$\sum_k (u_{k+1} - u_k)^2 \left\{ c \lambda e^{-2b} + \sum_{n \geq 0, m \geq 1, n+m > 1} (m+n) c \lambda^{m+n} e^{m+n} \right\} \quad (H.3)$$

which is the desired bound because $\frac{2}{3} \leq \frac{5}{6}$.

We rewrite $\Theta^{(>2)}$ using (5.1) for each one of the factors $u^{N(\cdot)}$. Thus given $N(\cdot)$ we call $i_1 < i_2 < \cdots < i_k$ the sites where $N(\cdot) > 0$ and call $\bar{n} = (N(i_1), \ldots, N(i_k))$. We then apply (5.1) with $u_1 = u_{i_1}, \ldots, u_k = u_{i_k}$ so that $p_i$ and $d_{i,j}$ in (5.1) become functions of $\bar{u}$ and $N(\cdot)$. We then get

$$\Theta^{(>2)} = \sum_{N(\cdot) : |N(\cdot)| \geq 4} A_{N(\cdot)} \left\{ \sum_{i:N(i)>0} p_i u_i^{N(\cdot)} + \sum_{j>i:N(j)>0, N(i)>0} d_{i,j} (u_i - u_j)^2 \right\} \quad (H.4)$$

which is sum of one body and gradients squared terms. To get the desired bound on the latter we use (1.2) and (5.2) to get

$$\sum_k (u_k - u_{k+1})^2 \sum_{i,j \geq i} \sum_{N(\cdot) : N(\cdot) \geq 4, N(i)>0, N(j)>0} c |N(\cdot)|^5 |A_{N(\cdot)}|$$

Since both $N(i) > 0$, $N(j) > 0$ then $j - i \leq R(N(\cdot))$ and given $R(N(\cdot)) \geq k - i$ there are at most $R(N(\cdot))$ possible values of $j$. Therefore the above expression is bounded by

$$\sum_k (u_k - u_{k+1})^2 \sum_{i \leq k N(\cdot) : N(\cdot) \geq 4, N(i)>0, R(N(\cdot)) \geq k-i} \|N(\cdot)\|^5 |A_{N(\cdot)}|$$

We upper bound the above if we extend the sum over $N(\cdot)$ such that

$$|N(\cdot)| \geq 4, N(i) > 0, \|N(\cdot)\| \geq \gamma_k - i, \gamma_k - i := \max \{4, k-i\}$$

We then apply (1.1) with $k = 5$ to get

$$\sum_k (u_k - u_{k+1})^2 \sum_{i \leq k} c_5 \gamma_k^5 e^{-b \gamma_k} = e^{-b} \sum_k (u_k - u_{k+1})^2 \left\{ \sum_{i \leq k} c_5 \gamma_k^5 e^{-b(\gamma_k - i)} \right\}$$

The curly bracket is bounded by

$$4^5 5 + \sum_{n \geq 1} (n+4)^5 e^{-bn} \leq c$$

Thus also $\Theta^{(>2)}$ is bounded as desired.
H.2 The term $h_{\text{ext}} \sum_i \Phi_i$

By (4.25)

$$\Phi_i = (1 - u_i^2)(\alpha_1 - \lambda)(u_{i+1} + u_{i-1}) + \sum_{j>i+1} \alpha_{j-i} u_j + \sum_{N(i):|N(i)| \geq 4} N(i)A_{N(i)} u^{N(i) - \epsilon_i}$$  \hspace{1cm} (H.5)

where $\epsilon_i(j) = 0$ if $j \neq i$ and $= 1$ if $j = i$.

Call $g_i := (1 - u_i^2)(\alpha_1 - \lambda)$ then the first term contributes to $\sum_i \Phi_i$ by

$$\sum_i (2g_i u_i - (g_i - g_{i+1})(u_i - u_{i+1})) = \sum_i 2g_i u_i + (\alpha_1 - \lambda) \sum_i (u_i + u_{i+1})(u_i - u_{i+1})^2$$

which is sum of one body and gradients squared terms. By (H.2) the coefficients of the gradients squared are bounded in absolute value by $2c\lambda e^{-2b}$ which is the desired bound because $2/3 \leq 5/6$.

By an analogous argument and writing $g_i' := (1 - u_i^2)$, the contribution of the second term in (H.5) is

$$\sum_{i<j} \alpha_{j-i} (2g_i' u_i - (g_i' - g_j')(u_i - u_j)) = \sum_{i<j} \alpha_{j-i} (2g_i' u_i + (u_i + u_j)(u_i - u_j)^2)$$

which is sum of one body and gradients squared terms. We bound the latter using (H.2) and the second inequality in (H.15) to get

$$\sum_k (u_{k+1} - u_k)^2 \left\{ \sum_{i \leq k < j, j - i > 2} 2c(\epsilon\lambda)^k \right\}$$

which is the desired bound because the curly bracket is bounded by $c^2\lambda^2$.

To write the contribution to $\sum_i \Phi_i$ of the last term in (H.5) we introduce the following notation. Given $N(\cdot) : N(i) > 0$ we call $N'(\cdot) = N(\cdot) - \epsilon_i$ and $N''(\cdot) = N(\cdot) + \epsilon_i$. Let then $i_1 < i_2 < \cdots < i_k$ the sites $j$ where $N'(j) > 0$, $n = (N'(i_1), \ldots, N'(i_k))$ and denote by $p_j^-, d_{j,j'}^-$ the corresponding coefficients in (5.1). Similarly let $i'_1 < i'_2 < \cdots < i'_k$ the sites $j$ where $N''(j) > 0$, $n = (N''(i_1), \ldots, N''(i_k))$ and denote by $p_j^+, d_{j,j'}^+$ the corresponding coefficients in (5.1). Then the contribution to $\sum_i \Phi_i$ of the last term in (H.5) can be written as

$$\sum_{N(\cdot):|N(\cdot)| \geq 4} A_{N(\cdot)} \sum_{i, N(i) > 0} N(i) \left( \sum_{j: N'(j) > 0} [p_j^- u_j^{N(\cdot) - 1} - \sum_{j: N''(j) > 0} p_j^+ u_j^{N(\cdot) + 1}] \right)$$

$$+ \sum_{j < j': N'(j) > 0, N'(j') > 0} d_{j,j'}^-(u_j - u_{j'})^2 - \sum_{j < j': N''(j) > 0, N''(j') > 0} d_{j,j'}^+(u_j - u_{j'})^2$$  \hspace{1cm} (H.6)

which is sum of one body and gradients squared terms. To bound the latter we examine the terms with $d^-$, those with $d^+$ are analogous and their analysis is omitted. For the $d^-$ terms we get the bound:

$$\sum_{N(\cdot):|N(\cdot)| \geq 4} |A_{N(\cdot)}| \sum_{i, N(i) > 0} N(i) \sum_{j < j': N'(j) > 0, N'(j') > 0} c|N(\cdot)|^3(u_j - u_{j'})^2$$

$$\leq \sum_{N(\cdot):|N(\cdot)| \geq 4} |A_{N(\cdot)}| \sum_{j < j': N'(j) > 0, N'(j') > 0} c|N(\cdot)|^4(u_j - u_{j'})^2$$

which has an analogous structure as the gradient term in (H.4). Its analysis is similar and thus omitted. We have thus proved that $h_{\text{ext}} \sum_i \Phi_i$ has the desired structure.
H.3 The term \( \sum_i \Psi_i^2 \)

We introduce the following notation: given \( i, N(\cdot), N'(\cdot), \sigma, \sigma', \sigma \in \{-1, 1\}, \sigma' \in \{-1, 1\}, N(i) > 0, N'(i) > 0 \), we call

\[
\tilde{N}(\cdot) = N(\cdot) + N'(\cdot), \quad K \equiv K_{i, \tilde{N}(\cdot), \sigma, \sigma'} := \tilde{N}(\cdot) + (\sigma + \sigma')e_i
\]

Then \( \sum_i \Psi_i^2 \) is equal to

\[
\sum_i \sum_{N(\cdot), N'(\cdot), \sigma, \sigma'} N(i)N'(i)A_{N(\cdot)}A_{N'(\cdot)}(-1)^{\frac{\sigma + \sigma'}{2} + 1} \left( \sum_{j:K(j) > 0} p_j(K)u_j^{|K|} \right) + \sum_{j < j':K(j) > 0, K(j') > 0} d_{j,j'}(K)(u_{j'} - u_j)^2
\]

(II.7)

which is sum of one body and gradient squared terms. Let

\[
C_{j,j'} := \sum_i \sum_{N(\cdot), N'(\cdot), \sigma, \sigma'} N(i)N'(i)|A_{N(\cdot)}||A_{N'(\cdot)}| \sum_{j < j':K(j) > 0, K(j') > 0} |d_{j,j'}(K)|
\]

then the gradient squared terms are bounded by \( \sum_{j < j'} C_{j,j'}(u_{j'} - u_j)^2 \). We have

\[
C_{j,j'} \leq 4 \sum_i \sum_{N(\cdot), N'(\cdot)} N(i)N'(i)|A_{N(\cdot)}||A_{N'(\cdot)}| \sum_{j < j':\tilde{N}(j) > 0, \tilde{N}(j') > 0} c(|N(\cdot)| + |N'(\cdot)| + 2)^3
\]

because 4 is the cardinality of \( (\sigma, \sigma') \). Moreover

\[
C_{j,j'} \leq 4c \sum_i \sum_{N(\cdot), N'(\cdot)} |A_{N(\cdot)}||A_{N'(\cdot)}|(2|N(\cdot)|)^4(2|N'(\cdot)|)^4
\]

By the symmetry between \( N(\cdot) \) and \( N'(\cdot) \) we get with an extra factor 2:

\[
C_{j,j'} \leq 8c^4 \sum_i \sum_{N(\cdot), N'(\cdot)} |A_{N(\cdot)}||A_{N'(\cdot)}||N(\cdot)|^4N'(\cdot)|^4
\]

Moreover either \( R(N(\cdot)) \geq (j' - j)/2 \), or \( R(N'(\cdot)) \geq (j' - j)/2 \) or both, hence

\[
C_{j,j'} \leq 8c^4 \left( \sum_{N(\cdot):N(j) > 0, R(N(\cdot)) \geq \frac{j' - j}{4}} |A_{N(\cdot)}||N(\cdot)|^4 \sum_{i:N(i) > 0, N'(\cdot):N'(i) > 0} N'(\cdot)|^4 \right. \\
+ \left. \sum_{N(\cdot):N(j) > 0} |A_{N(\cdot)}||N(\cdot)|^4 \sum_{i:N(i) > 0, N'(\cdot):N'(i) > 0, R(N'(\cdot)) \geq \frac{j' - j}{4}} N'(\cdot)|^4 \right)
\]

By (II.1)

\[
C_{j,j'} \leq 8c^4 \left( \sum_{N(\cdot):N(j) > 0, R(N(\cdot)) \geq \frac{j' - j}{4}} |A_{N(\cdot)}||N(\cdot)|^4|N(\cdot)|c_4e^{-2b} \right. \\
+ \left. \sum_{N(\cdot):N(j) > 0} |A_{N(\cdot)}||N(\cdot)|^4|N(\cdot)|c_4e^{-b \max\{2, \frac{j' - j}{4}\}} \right)
\]
Using again (15.1)
\[ C_{j,j'} \leq 8c_4^4 2c_4 e^{-2b} c_5 e^{-b \max\{2, \frac{L'-1}{2}\}} =: c' e^{-2b} e^{-b \max\{2, \frac{L'-1}{2}\}} \]
Hence
\[ \sum_{j<j'} C_{j,j'}(u_{j'} - u_j)^2 \leq \sum_{k} (u_{k+1} - u_k)^2 \sum_{j,j': j<k<j'} (j' - j)c' e^{-2b} e^{-b \max\{2, \frac{L'-1}{2}\}} \]
The last sum is bounded proportionally to \( e^{-4b} \) (details are omitted) which gives the desired bound.

**H.4 The term \( \sum_i \xi_i \Phi_i \)**

Recalling (4.27) and (4.25) the contribution to \( H^{(1)}_{\ell, A} \) due to \( \sum_i \xi_i \Phi_i \) is
\[ \sum_{i=1}^{\ell}(h_i - u_i)(1 - u_i^2}\{ \sum_{j>i+1} \alpha_{j-i} u_j + \sum_{N(\cdot): N(i)>0, |N(\cdot)| \geq 4} N(i) A_{N(\cdot)} u^{N(\cdot)} \} \quad (H.8) \]
We have
\[ (h - u)(1 - u^2) = \frac{u^3}{3} - 2 \sum_{k=2}^{\infty} \frac{1}{4k^2 - 1} u^{2k+1} =: \sum_{k=1}^{\infty} \kappa_k u^{2k+1} \quad (H.9) \]
with \( |\kappa_k| < 1 \); since \( |u| \leq u_+ < 1 \) the series converges exponentially. We start from the terms with \( \alpha_{j-i} \):
\[ \sum_{i=1}^{\ell} \sum_{j>i+1} \sum_{k \geq 1} \kappa_k u_+^{2k+1} u_j = \sum_{i=1}^{\ell} \sum_{j>i+1} \sum_{k \geq 1} \kappa_k \{(p_i u_+^{2k+2} + p_j u_+^{2k+2}) + d(u_i - u_j)^2\} \]
where \( (p_i, p_j) \) is the probability vector introduced in Theorem (11) and \( d \) the corresponding coefficient. They depend on the pair \( (2k + 1, 1) \) and \( |d| \leq c k^6 u_+^{2k} \). This is sum of one body and squared gradients terms and we are left with bounding the latter. We have the bound
\[ \sum_{i=1}^{\ell} \sum_{j>i+1} |\alpha_{j-i}| \sum_{k \geq 1} c k^6 u_+^{2k} (u_i - u_j)^2 \leq \sum_{i=1}^{\ell} \sum_{j>i+1} |\alpha_{j-i}| c' (u_i - u_j)^2 \]
which satisfies the desired bound as proved in Subsection (H.1).

We next study the last term on the right hand side of (H.8). Proceeding as before we check that it is sum of one body and gradients squared terms and next prove that the gradients are bounded as desired. We first bound them by
\[ \sum_{i,j<j', N(\cdot): N(i)>0, N(j)>0, |N(\cdot)| \geq 4} N(i) A_{N(\cdot)} |c N(\cdot)|^3 u_+^{2k} (u_{j'} - u_j)^2 \]
We have \( (2k + |N(\cdot)|)^3 \leq (2k)^3 |N(\cdot)|^3 \) so that we get the bound
\[ \sum_{i,j<j', N(\cdot): N(i)>0, N(j)>0, |N(\cdot)| \geq 4} N(i) A_{N(\cdot)} |c' N(\cdot)|^3 (u_{j'} - u_j)^2 \]
with
\[ c' := \sum_{k \geq 1} (2k)^3 u_{2k} \]

We can perform the sum over \( i \) to get
\[ \sum_{j < j'} N(j) N(j') \sum_{N(j') > 0, |N(\cdot)| \geq 4} |A_{N(j)}| c' |N(\cdot)|^4 (u_{j'} - u_j)^2 \]

We are thus reduced to the case considered in Subsection H.1, we omit the details.

I Proof of Proposition 1

Recalling that \( \xi(u) := (h(u) - u)(1 - u^2) \), we have, supposing \( u' > u' \),
\[ \xi(u') - \xi(u) = \int_u^{u'} \frac{d\xi}{du} du \leq a(u_i - u_j), \quad (I.1) \]

with \( a = \max_{|u| < 1} \frac{d\xi}{du} \). Thus \( \theta_i(u) \leq a \) and by (I.9)
\[ a = \max_{|u| < 1} \left( u^2 - 2u \sum_{k=1}^{\infty} \frac{u^{2k+1}}{2k+1} \right) < \max_{|u| < 1} \left( u^2 - \frac{2}{3} u^4 \right) = \frac{3}{8} \]

having retained only the term with \( k = 1 \).

J Proof of Theorem 10

We shall use in the proof that in \( H_{\ell, h}^{\text{eff}} \) all terms but \( (T(u) - h_{\text{ext}} u) \), cf. (4.22), are proportional to \( \lambda \).

Calling \( \tilde{u} \) the minimizer of \( (T(u) - h_{\text{ext}} u) \):

- It will follow from Lemma 4 that the minimizer \( u^* \) of \( H_{\ell, h}^{\text{eff}} \) has components \( u_i^* \) such that \( |u_i^* - \tilde{u}| < \lambda^{1/4} \) (for all \( \lambda \) small enough), and that the minimizer \( v \) of \( f(u) \), \( f(u) \) the one body term defined in (4.28), is such that \( |v - \tilde{u}| < \lambda^{1/4} \);
- Since the gradient of \( H_{\ell, h}^{\text{eff}} \) vanishes at \( v = (v_i = v, \ i = 1, \ldots, \ell) \), cf. (4.28), \( v \) is a critical point of \( H_{\ell, h}^{\text{eff}} \);
- \( T(u) \) is a convex function and its second derivative \( T''(u) \) is a strictly increasing, positive function of \( u \in (0, 1) \) which diverges as \( u \to 1 \), as it follows from (4.22). Then the matrix \( \frac{\partial^2}{\partial u_i \partial u_j} H_{\ell, h}^{\text{eff}} \) is positive definite in the ball \( u : |u_i - \tilde{u}| < \lambda^{1/4} \), cf. Proposition 5.
As a consequence, the minimizer of $H_{\ell,h}^{\text{eff}}$ in the ball coincides with $v$ and since $\mathbf{u}^*$ is in the ball it coincides with $\mathbf{u}$, thus proving that all the components of $\mathbf{u}^*$ are equal to each other. We are thus left with the proof of Lemma 4 and Proposition 5. We need a preliminary lemma.

**Lemma 3** For any $h_{\text{ext}} \in [h_0,h^*]$ there is a unique $\tilde{u}$ such that

$$\left. \frac{d}{du}(T(u) - h_{\text{ext}}u) \right|_{u=\tilde{u}} = 0 \quad (J.1)$$

and there is $c_{h_0} > 0$ so that

$$\inf_{h_{\text{ext}} \in [h_0,h^*]} \left. \frac{d^2}{du^2} T(u) \right|_{u=\tilde{u}} \geq c_{h_0} \quad (J.2)$$

**Proof** The proof follows from the fact that the second derivative of $T(u)$ is positive away from 0 and in $(0,1)$ increases to $\infty$ as $u \to 1$.

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Fix all $u_j, j \neq i$ and call $F(u_i)$ the energy $H_{\ell,h}^{\text{eff}}(u)$ as a function of $u_i$. Then

**Lemma 4** There is $c'_{h_0} > 0$ so that for all $\lambda$ small enough the following holds. Let $h_{\text{ext}} \in [h_0,h^*]$ and $\tilde{u}$ as in Lemma 3 then

$$\inf_{u_i:|u_i-\tilde{u}| \geq \lambda^{1/4}} F(u_i) \geq F(\tilde{u}) + c'_{h_0} \lambda^{1/2} \quad (J.3)$$

**Proof** By (J.2)

$$\inf_{u_i:|u_i-\tilde{u}| \geq \lambda^{1/4}} |\{T(u) - h_{\text{ext}}u\} - \{T(\tilde{u}) - h_{\text{ext}}\tilde{u}\}| \geq \frac{c_{h_0}}{2} \lambda^{1/2}$$

We are going to show that the variation of all the other terms in (G.2) are bounded proportionally to $\lambda$ and this will then complete the proof of the lemma. We have

$$| (h_i - u_i)(1 - u_i^2) | \leq c, \quad (1 - u_i^2)^{-1} |\Psi_i| \leq c\lambda$$

(the first inequality by (H.9), the last inequality by (E.2)).

Call $G(u_i)$ the value of $\log Z_{\ell,h}^*$ when $\text{tanh}(h_i) = u_i$ and the other $h_j$ are fixed, then

$$|G(u_i) - G(u_j)| = \left| \sum_{N(\cdot):N(\cdot)>0} A_N(u) u_i^{N(\cdot)}(u_i - u_j) \right| \leq c\lambda |u_i - u_j|$$

where, to derive the last inequality, we have used Theorem 4.

•

As a corollary of the above lemmas
Lemma 5 For λ small enough the inf of $H_{\ell,h}^{\text{eff}}$ is achieved in the ball $u : \max\{|u_i - \tilde{u}| \leq \lambda^{1/4}, i = 1, \ldots, \ell\}$.

Proposition 5 For λ small enough the matrix $\frac{\partial^2}{\partial u_i \partial u_j} H_{\ell,h}^{\text{eff}}$ is strictly positive in the ball $u : \max\{|u_i - u_{\text{ext}}| \leq \lambda^{1/4}, i = 1, \ldots, \ell\}$.

Proof From Lemma 3 and Corollary 2 one obtains

$$\frac{\partial^2}{\partial u_i^2} H_{\ell,h}^{\text{eff}} \geq c_{h_0} - \lambda c_1, \text{for } i = 1, 2, \ldots, L$$

For any $i$,

$$\sum_{j \neq i} |\frac{\partial^2}{\partial u_i \partial u_j} H_{\ell,h}^{\text{eff}}| \leq c_2 \lambda$$

from [13] and Corollary 2.

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