On the Relative Strength of Split, Triangle and Quadrilateral Cuts

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Abstract

Integer programs defined by two equations with two free integer variables and nonnegative continuous variables have three types of nontrivial facets: split, triangle or quadrilateral inequalities. In this paper, we compare the strength of these three families of inequalities. In particular, we study how well each family approximates the integer hull. We show that, in a well defined sense, triangle inequalities provide a good approximation of the integer hull. The same statement holds for quadrilateral inequalities. On the other hand, the approximation produced by split inequalities may be arbitrarily bad.

1 Introduction

In this paper, we consider mixed integer linear programs with two equality constraints, two free integer variables and any number of nonnegative continuous variables. We assume that

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the two integer variables are expressed in terms of the remaining variables as follows.

\[
\begin{align*}
x &= f + \sum_{j=1}^{k} r^j s_j \\
x &\in \mathbb{Z}^2 \\
s &\in \mathbb{R}_+^k.
\end{align*}
\] (1)

This model was introduced by Andersen, Louveaux, Weismantel and Wolsey [1]. It is a natural relaxation of a general mixed integer linear program (MILP) and therefore it can be used to generate cutting planes for MILP. Currently, MILP solvers rely on cuts that can be generated from a single equation (such as Gomory mixed integer cuts [13], MIR cuts [16], lift-and-project cuts [3], lifted cover inequalities [8]). Model (1) has attracted attention recently as a way of generating new families of cuts from two equations instead of just a single one [1, 7, 10, 11, 14].

We assume \( f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2, k \geq 1, \) and \( r^j \in \mathbb{Q}^2 \setminus \{0\} \). So \( s = 0 \) is not a solution of (1). Let \( R_f(r^1, \ldots, r^k) \) be the convex hull of all vectors \( s \in \mathbb{R}_+^k \) such that \( f + \sum_{j=1}^{k} r^j s_j \) is integral. A classical theorem of Meyer [17] implies that \( R_f(r^1, \ldots, r^k) \) is a polyhedron. Andersen, Louveaux, Weismantel and Wolsey [1] showed that the facets of \( R_f(r^1, \ldots, r^k) \) are \( s \geq 0 \) (called trivial inequalities), split inequalities [6] and intersection cuts (Balas [2]) arising from triangles or quadrilaterals in \( \mathbb{R}^2 \). Borozan and Cornuèjols [5] investigated a relaxation of (1) where the vector \((s_1, \ldots, s_k) \in \mathbb{R}_+^k \) is extended to infinite dimensions by defining it for all directions \( r^j \in \mathbb{Q}^2 \) instead of just \( r^1, \ldots, r^k \). Namely let \( R_f \) be the convex hull of all infinite-dimensional vectors \( s \) with finite support that satisfy

\[
\begin{align*}
x &= f + \sum_{r \in \mathbb{Q}^2} r s_r \\
x &\in \mathbb{Z}^2 \\
s &\geq 0.
\end{align*}
\] (2)

Theorem 1.2 below shows that there is a one-to-one correspondence between minimal valid inequalities for \( R_f \) and maximal lattice-free convex sets that contain \( f \) in their interior. By lattice-free convex set we mean a convex set with no integral point in its interior. However integral points are allowed on the boundary. These maximal lattice-free convex sets are splits, triangles, and quadrilaterals as proved in the following theorem of Lovász [15].

**Theorem 1.1.** (Lovász [15]) In the plane, a maximal lattice-free convex set with nonempty interior is one of the following:

(i) A split \( c \leq ax_1 + bx_2 \leq c + 1 \) where \( a \) and \( b \) are coprime integers and \( c \) is an integer;

(ii) A triangle with an integral point in the interior of each of its edges;

(iii) A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges; Moreover, these four integral points are vertices of a parallelogram of area 1.

\( R_f(r^1, \ldots, r^k) \) is a polyhedron of blocking type, i.e. \( R_f(r^1, \ldots, r^k) \subseteq \mathbb{R}_+^k \) and if \( x \in R_f(r^1, \ldots, r^k) \), then \( y \geq x \) implies \( y \in R_f(r^1, \ldots, r^k) \). Similarly \( R_f \) is a convex set of blocking type. Any nontrivial valid linear inequality for \( R_f \) is of the form

\[
\sum_{r \in \mathbb{Q}^2} \psi(r) s_r \geq 1
\] (3)
where \( \psi : \mathbb{Q}^2 \to \mathbb{R} \). The nontrivial valid linear inequalities for \( R_f(r^1, \ldots, r^k) \) are the restrictions of (3) to \( r^1, \ldots, r^k \):

\[
\sum_{j=1}^{k} \psi(r^j)s_j \geq 1.
\]

(4)

A nontrivial valid linear inequality for \( R_f \) is minimal if there is no other nontrivial valid inequality \( \sum_{r \in \mathbb{Q}^2} \psi'(r)s_r \geq 1 \) such that \( \psi'(r) \leq \psi(r) \) for all \( r \in \mathbb{Q}^2 \).

**Theorem 1.2.** (Borozan and Cornuéjols [5]) Minimal nontrivial valid linear inequalities for \( R_f \) are associated with functions \( \psi \) that are nonnegative positively homogeneous piecewise linear and convex. Furthermore, the closure of the set

\[
B_\psi := \{ x \in \mathbb{Q}^2 : \psi(x - f) \leq 1 \}
\]

(5)

is a maximal lattice-free convex set containing \( f \) in its interior.

Conversely, any maximal lattice-free convex set \( B \) with \( f \) in its interior defines a function \( \psi_B : \mathbb{R}^2 \to \mathbb{R}_+ \) that can be used to generate a minimal nontrivial valid linear inequality. Indeed, define \( \psi_B(0) = 0 \) and \( \psi_B(x - f) = 1 \) for all points \( x \) on the boundary of \( B \). Then, the positive homogeneity of \( \psi_B \) implies the value of \( \psi_B(r) \) for any vector \( r \in \mathbb{R}^2 \setminus \{0\} \): If there is a positive scalar \( \lambda \) such that the point \( f + \lambda r \) is on the boundary of \( B \), we get that \( \psi_B(r) = 1/\lambda \). Otherwise, if there is no such \( \lambda \), \( r \) is an unbounded direction of \( B \) and \( \psi_B(r) = 0 \).

Note that the above construction of \( \psi_B \) is nothing but the derivation of the intersection cut as introduced by Balas [2].

Following Dey and Wolsey [10], the maximal lattice-free triangles can be partitioned into three types (see Figure 1):

- **Type 1 triangles**: triangles with integral vertices and exactly one integral point in the relative interior of each edge;
- **Type 2 triangles**: triangles with at least one fractional vertex \( v \), exactly one integral point in the relative interior of the two edges incident to \( v \) and at least two integral points on the third edge;
- **Type 3 triangles**: triangles with exactly three integral points on the boundary, one in the relative interior of each edge.

Figure 1 shows these three types of triangles as well as a maximal lattice-free quadrilateral and a split satisfying the properties of Theorem 1.1.

In this paper we will need conditions guaranteeing that a split, triangle or quadrilateral actually defines a facet of \( R_f(r^1, \ldots, r^k) \). Such conditions were obtained by Cornuéjols and Margot [7] and will be stated in Theorem 4.1.

### 1.1 Motivation

An unbounded maximal lattice-free convex set is called a split. It has two parallel edges whose direction is called the direction of the split. Split inequalities for (1) are valid inequalities
that can be derived by combining the two equations in (1) and by using the integrality of \( \pi_1 x_1 + \pi_2 x_2 \), where \( \pi \in \mathbb{Z}^2 \) is normal to the direction of the split. Similarly, for general MILPs, the equations can be combined into a single equality from which a split inequality is derived. Split inequalities are equivalent to Gomory mixed integer cuts \[18\]. Empirical evidence shows that split inequalities can be effective for strengthening the linear programming relaxation of MILPs \[4, 9\]. Interestingly, triangle and quadrilateral inequalities cannot be derived from a single equation \[1\]. They can only be derived from (1) without aggregating the two equations. Recent computational experiments by Espinoza \[11\] indicate that quadrilaterals also induce effective cutting planes in the context of solving general MILPs. In this paper, we consider the relative strength of split, triangle and quadrilateral inequalities from a theoretical point of view. We use an approach for measuring strength initiated by Goemans \[12\], based on the following definition and results.

Let \( Q \subseteq \mathbb{R}_+^n \setminus \{0\} \) be a polyhedron of the form \( Q = \{ x : a^i x \geq b_i \text{ for } i = 1, \ldots, m \} \) where \( a^i \geq 0 \) and \( b_i \geq 0 \) for \( i = 1, \ldots, m \) and let \( \alpha > 0 \) be a scalar. We define the polyhedron \( \alpha Q \) as \( \{ x : \alpha a^i x \geq b_i \text{ for } i = 1, \ldots, m \} \). Note that \( \alpha Q \) contains \( Q \) when \( \alpha \geq 1 \). It will be convenient to define \( \alpha Q \) to be \( \mathbb{R}_+^n \) when \( \alpha = +\infty \).

We need the following generalization of a theorem of Goemans \[12\].

**Theorem 1.3.** Suppose \( Q \subseteq \mathbb{R}_+^n \setminus \{0\} \) is defined as above. If convex set \( P \subseteq \mathbb{R}_+^n \) is a relaxation of \( Q \) (i.e. \( Q \subseteq P \)), then the smallest value of \( \alpha \geq 1 \) such that \( P \subseteq \alpha Q \) is

\[
\max_{i=1,\ldots,m} \left\{ \frac{b_i}{\inf\{a^i x : x \in P\}} : b_i > 0 \right\}.
\]

Here, we define \( \inf\{a^i x : x \in P\} \) to be \(+\infty\) if \( \inf\{a^i x : x \in P\} = 0 \).

In other words, the only directions that need to be considered to compute \( \alpha \) are those defined by the nontrivial facets of \( Q \). Goemans’ paper assumes that both \( P \) and \( Q \) are polyhedra, but one can easily verify that only the polyhedrality of \( Q \) is needed in the proof. We give the proof of Theorem 1.3 in Section 2 for completeness.

### 1.2 Results

Let the *split closure* \( S_f(r^1, \ldots, r^k) \) be the intersection of all split inequalities, let the *triangle closure* \( T_f(r^1, \ldots, r^k) \) be the intersection of all inequalities arising from maximal lattice-free
triangles, and let the quadrilateral closure $Q_f(r^1, \ldots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free quadrilaterals. In the remainder of the paper, to simplify notation, we refer to $R_f(r^1, \ldots, r^k)$, $S_f(r^1, \ldots, r^k)$, $T_f(r^1, \ldots, r^k)$ and $Q_f(r^1, \ldots, r^k)$ as $R^k_f$, $S^k_f$, $T^k_f$ and $Q^k_f$ respectively, whenever the rays $r^1, \ldots, r^k$ are obvious from the context.

Since all the facets of $R^k_f$ are induced by these three families of maximal lattice-free convex sets, we have

$$R^k_f = S^k_f \cap T^k_f \cap Q^k_f.$$  

It is known that the split closure is a polyhedron (Cook, Kannan and Schrijver [6]) but such a result is not known for the triangle closure and the quadrilateral closure. In this paper we show the following results.

**Theorem 1.4.** $T^k_f \subseteq S^k_f$ and $Q^k_f \subseteq S^k_f$.

This theorem may seem counter-intuitive because some split inequalities are facets of $R^k_f$. However, we show that any split inequality can be obtained as the limit of an infinite collection of triangle inequalities. What we will show is that, if a point is cut off by a split inequality, then it is also cut off by some triangle inequality. Consequently, the intersection of all triangle inequalities is contained in the split closure. The same is true of the quadrilateral closure.

**Example 1.5.** As an illustration, consider the simple example in Figure 2. The split inequality is $s_1 \geq 1$ and the split closure is given by $S^2_f = \{(s_1, s_2) \mid s_1 \geq 1, s_2 \geq 0\}$. All triangle inequalities are of the form $as_1 + bs_2 \geq 1$ with $a \geq 1$ and $b > 0$. The depicted triangle inequality is of the form $s_1 + 2s_2 \geq 1$ and by moving the corner $v$ closer to the boundary of the split, one can get a triangle inequality of the form $s_1 + bs_2 \geq 1$ with $b$ tending to 0, but remaining positive. Note that the split inequality can not be obtained as a positive combination of triangle inequalities, but any point cut by the split inequality is cut by one of the triangle inequalities. As a result, the triangle closure is $T^2_f = \{(s_1, s_2) \mid s_1 + bs_2 \geq 1, \text{ for all } b > 0, s_1 \geq 0, s_2 \geq 0\} = S^2_f$.}

\[\]
We further study the strength of the triangle closure and quadrilateral closure in the sense defined in Section 1.1. We first compute the strength of a single Type 1 triangle facet as $f$ varies in the interior of the triangle, relative to the entire split closure.

**Theorem 1.6.** Let $T$ be a Type 1 triangle as depicted in Figure 3. Let $f$ be in its interior and assume that the set of rays $\{r^1, \ldots, r^k\}$ contains rays pointing to the three corners of $T$. Let $\sum_{i=1}^{k} \psi(r^i)s_i \geq 1$ be the inequality generated by $T$. The value

$$\min \left\{ \sum_{i=1}^{k} \psi(r^i)s_i : s \in S_f^k \right\}$$

is a piecewise linear function of $f$ for which some level curves are depicted in Figure 3. This function varies from a minimum of $\frac{1}{2}$ in the center of $T$ to a maximum of $\frac{2}{3}$ at its corners.

![Figure 3: Illustration for Theorem 1.6](image)

Next we show that both the triangle closure and the quadrilateral closure are good approximations of the integer hull $R^k_f$ in the sense that

**Theorem 1.7.**

- $R^k_f \subseteq T^k_f \subseteq 2R^k_f$ and
- $R^k_f \subseteq Q^k_f \subseteq 2R^k_f$.

Finally we show that the split closure may not be a good approximation of the integer hull.

**Theorem 1.8.** For any $\alpha > 1$, there is a choice of $f, r^1, \ldots, r^k$ such that

$$S^k_f \not\subseteq \alpha R^k_f.$$  

These results provide additional support for the recent interest in cuts derived from two or more rows of an integer program [1, 5, 7, 10, 11, 14].
2 Proof of Theorem 1.3

Proof. Let
\[
\alpha = \max_{i=1,\ldots,m} \left\{ \frac{b_i}{\inf \{a^i x : x \in P \}} : b_i > 0 \right\}.
\]
We first show that \( P \subseteq \alpha Q \). This holds when \( \alpha = +\infty \) by definition of \( \alpha Q \). Therefore we may assume \( 1 \leq \alpha < +\infty \). Consider any point \( p \in P \). The inequalities of \( \alpha Q \) are of the form \( \alpha a^i x \geq b_i \) with \( a^i \geq 0 \) and \( b_i \geq 0 \). If \( b_i = 0 \), then since \( p \in P \subseteq \mathbb{R}^n_+ \), \( a^i p \geq 0 \) and hence this inequality is satisfied. If \( b_i > 0 \), then we know from the definition of \( \alpha \) that
\[
\frac{b_i}{\inf \{a^i x : x \in P \}} \leq \alpha.
\]
This implies
\[
b_i \leq \alpha \inf \{a^i x : x \in P \} \leq \alpha a^i p.
\]
Therefore, \( p \) satisfies this inequality.

We next show that for any \( 1 \leq \alpha' < \alpha \), \( P \not\subseteq \alpha' Q \). Say \( \alpha = \frac{b_j}{\inf \{a^j x : x \in P \}} \) (i.e. the maximum, possibly \( +\infty \), is reached for index \( j \)). Let \( \delta = \frac{b_j}{\alpha'} - \frac{b_j}{\alpha} \). We have \( \delta > 0 \). From the definition of \( \alpha \) we know that \( \inf \{a^j x : x \in P \} = \frac{b_j}{\alpha} \). Therefore, there exists \( p \in P \) such that \( a^j p < \frac{b_j}{\alpha} + \delta = \frac{b_j}{\alpha'} \). So \( \alpha' a^j p < b_j \) and hence \( p \not\in \alpha' Q \).

3 Split closure vs. triangle and quadrilateral closures

In this section, we present the proof of Theorem 1.4.

Proof. (Theorem 1.4). We show that if any point \( \bar{s} \) is cut off by a split inequality, then it is also cut off by some triangle inequality. This will prove the theorem.

Consider any split inequality \( \sum_{i=1}^k \psi_S(r^i) \bar{s}_i \geq 1 \) (see Figure 4) and denote by \( L_1 \) and \( L_2 \) its two boundary lines. Point \( f \) lies in some parallelogram of area 1 whose vertices \( y^1, y^2, y^3, \) and \( y^4 \) are lattice points on the boundary of the split.

Assume without loss of generality that \( y^1 \) and \( y^2 \) are on \( L_1 \). Consider the family \( \mathcal{T} \) of triangles whose edges are supported by \( L_2 \) and by two lines passing through \( y^1 \) and \( y^2 \) and whose interior contains the segment \( y^1 y^2 \). See Figure 4. Note that all triangles in \( \mathcal{T} \) are of Type 2. For \( T \in \mathcal{T} \) we will denote by \( \psi_T \) the minimal function associated with \( T \).

By assumption, \( \sum_{i=1}^k \psi_S(r^i) \bar{s}_i < 1 \). Let \( \epsilon = 1 - \sum_{i=1}^k \psi_S(r^i) \bar{s}_i \).

We now make the following simple observation. Given a finite set \( X \) of points that lie in the interior of the split \( S \), we can find a triangle \( T \in \mathcal{T} \) as defined above, such that all points in \( X \) are in the interior of \( T \). To see this, consider the convex hull \( \mathcal{C}(X) \) of \( X \). Since all points in \( X \) are in the interior of \( S \), so is \( \mathcal{C}(X) \). This implies that the tangent lines from \( y^1 \) and \( y^2 \) to \( \mathcal{C}(X) \) are not parallel to \( L_1 \). Two of these four tangent lines along with \( L_2 \) of \( S \) form a triangle in \( \mathcal{T} \) with \( X \) in its interior.

Let \( s_{\text{max}} = \max \{ \bar{s}_i : i = 1, \ldots, k \} \) and define \( \delta = \frac{\epsilon}{2 k s_{\text{max}}} > 0 \). For every ray \( r^i \) define \( c(r^i) = \psi_S(r^i) + \delta \). Therefore, by definition \( p^i = f + \frac{1}{c(r^i)} \cdot r^i \) is a point strictly in the interior
Figure 4: Approximating a split inequality with a triangle inequality. The triangle is formed by $L_2$ and the two dashed lines of $S$.

Using the observation stated above, there exists a triangle $T \in \mathcal{T}$ which contains all the points $p^i$. It follows that the coefficient $\psi_T(r^i)$ for any ray $r^i$ is less than or equal to $c(r^i)$.

We claim that for this triangle $T$ we have $\sum_{i=1}^{k} \psi_T(r^i) \bar{s}_i < 1$. Indeed,

$$\sum_{i=1}^{k} \psi_T(r^i) \bar{s}_i \leq \sum_{i=1}^{k} c(r^i) \bar{s}_i$$

$$= \sum_{i=1}^{k} (\psi_S(r^i) + \delta) \bar{s}_i = \sum_{i=1}^{k} \psi_S(r^i) \bar{s}_i + \sum_{i=1}^{k} \frac{\epsilon}{2 \cdot k \cdot s_{\text{max}}} \bar{s}_i$$

$$\leq \sum_{i=1}^{k} \psi_S(r^i) \bar{s}_i + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2} < 1$$

The first inequality follows from the definition of $c(r^i)$ and the last equality follows from the fact that $\sum_{i=1}^{k} \psi_S(r^i) \bar{s}_i = 1 - \epsilon$.

This shows that $T_{f}^k \subseteq S_{f}^k$. For the quadrilateral closure, we also use two lines passing through $y^3$ and $y^4$ on $L_2$ and argue similarly.

Note that even though there can be a zero coefficient in a split inequality for some ray $r$, in the proof above we exhibit a sequence of triangle inequalities with arbitrarily small coefficients for ray $r$. Any point cut off by the split inequality is also cut off by a cut in the sequence.
4 Tools

4.1 Conditions under which a maximal lattice-free convex set gives rise to a facet

Andersen, Louveaux, Weismantel and Wolsey [1] characterized the facets of $R^k_f$ as arising from splits, triangles and quadrilaterals. Cornuéjols and Margot [7] gave a converse. We give this characterization in Theorem 4.1. Roughly speaking, for a maximal lattice-free triangle or quadrilateral to give rise to a facet, it has to have its corner points on half-lines $f + \lambda r^j$ for some $j = 1, \ldots, k$ and $\lambda > 0$; or to satisfy a certain technical condition called the ray condition. Although the ray condition is not central to this paper (it is only used once in the proof of Theorem 7.2), we need to include it for technical completeness.

Let $B_\psi$ be a maximal lattice-free split, triangle or quadrilateral with $f$ in its interior. For any $j = 1, \ldots, k$ such that $\psi(r^j) > 0$, let $p^j$ be the intersection of the half-line $f + \lambda r^j$, $\lambda \geq 0$, with the boundary of $B_\psi$. The point $p^j$ is called the boundary point for $r^j$. Let $P$ be a set of boundary points. We say that a point $p \in P$ is active if it can have a positive coefficient in a convex combination of points in $P$ generating an integral point. Note that $p \in P$ is active if and only if $p$ is integral or there exists $q \in P$ such that the segment $pq$ contains an integral point in its interior. We say that an active point $p \in P$ is uniquely active if it has a positive coefficient in exactly one convex combination of points in $P$ generating an integral point.

Apply the following Reduction Algorithm:

0.) Let $P = \{p^1, \ldots, p^k\}$.

1.) While there exists $p \in P$ such that $p$ is active and $p$ is a convex combination of other points in $P$, remove $p$ from $P$. At the end of this step, $P$ contains at most two active points on each edge of $B_\psi$ and all points of $P$ are distinct.

2.) While there exists a uniquely active $p \in P$, remove $p$ from $P$.

3.) If $P$ contains exactly two active points $p$ and $q$ (and possibly inactive points), remove both $p$ and $q$ from $P$.

The ray condition holds for a triangle or a quadrilateral if $P = \emptyset$ at termination of the Reduction Algorithm.

The ray condition holds for a split if, at termination of the Reduction Algorithm, either $P = \emptyset$, or $P = \{p_1, q_1, p_2, q_2\}$ with $p_1, q_1$ on one of the boundary lines and $p_2, q_2$ on the other and both line segments $p_1q_1$ and $p_2q_2$ contain at least two integral points.

**Theorem 4.1.** (Cornuéjols and Margot [7]) The facets of $R^k_f$ are

(i) split inequalities where the unbounded direction of $B_\psi$ is $r^j$ for some $j = 1, \ldots, k$ and the line $f + \lambda r^j$ contains no integral point; or where $B_\psi$ satisfies the ray condition,

(ii) triangle inequalities where the triangle $B_\psi$ has its corner points on three half-lines $f + \lambda r^j$ for some $j = 1, \ldots, k$ and $\lambda > 0$; or where the triangle $B_\psi$ satisfies the ray condition,
(iii) quadrilateral inequalities where the corners of \( B_\psi \) are on four half-lines \( f + \lambda r^j \) for some \( j = 1, \ldots, k \) and \( \lambda > 0 \), and \( B_\psi \) satisfies a certain ratio condition (the ratio condition will not be needed in this paper; the interested reader is referred to [7] for details).

Note that the same facet may arise from different convex sets. For example quadrilaterals for which the ray condition holds define facets, but there is always also a triangle that defines the same facet, which is the reason why there is no mention of the ray condition in (iii) of Theorem 4.1.

### 4.2 Reducing the number of rays in the analysis

The following technical theorem will be used in the proofs of Theorems 1.6, 1.7, and 1.8, where we will be applying Theorem 4.1.

**Theorem 4.2.** Let \( B_1, \ldots, B_m \) be lattice-free convex sets with \( f \) in the interior of \( B_p \), \( p = 1, \ldots, m \). Let \( R_c \subseteq \{1, \ldots, k\} \) be a subset of the ray indices such that for every ray \( r^j \) with \( j \notin R_c \), \( r^j \) is the convex combination of some two rays in \( R_c \). Define

\[
z_1 = \min \left( \sum_{i=1}^{k} s_i \right) \sum_{i=1}^{k} \psi_{B_p}(r^i)s_i \geq 1 \text{ for } p = 1, \ldots, m \]

\[
s \geq 0
\]

and

\[
z_c = \min \left( \sum_{i \in R_c} s_i \right) \sum_{i \in R_c} \psi_{B_p}(r^i)s_i \geq 1 \text{ for } p = 1, \ldots, m \]

\[
s \geq 0 .
\]

Then \( z_1 = z_c \).

**Proof.** Assume there exists \( j \notin R_c \) and \( r^1, r^2 \) are the rays in \( R_c \) such that \( r^j = \lambda r^1 + (1-\lambda)r^2 \) for some \( 0 < \lambda < 1 \). Let \( K = \{1, \ldots, k\} - j \) and define

\[
z_2 = \min \left( \sum_{i \in K} s_i \right) \sum_{i \in K} \psi_{B_p}(r^i)s_i \geq 1 \text{ for } p = 1, \ldots, m
\]

\[
s \geq 0 .
\]

We first show that \( z_1 = z_2 \). Applying the same reasoning repeatedly to all the indices not in \( R_c \) yields the proof of the theorem.

Any optimal solution for the LP defining \( z_2 \) yields a feasible solution for the one defining \( z_1 \) by setting \( s_j = 0 \), implying \( z_1 \leq z_2 \). It remains to show that \( z_1 \geq z_2 \).
Consider any point \( \hat{s} \) satisfying \( \sum_{i=1}^{k} \psi_{B_p}(r^i)\hat{s}_i \geq 1 \) for every \( p \in \{1, \ldots, m\} \). Consider the following values \( \bar{s} \) for the variables corresponding to the indices \( t \in K \).

\[
\bar{s}_t = \begin{cases} 
\hat{s}_t & \text{if } t \not\in \{1, 2, j\} \\
\hat{s}_1 + \lambda \hat{s}_j & \text{if } t = 1 \\
\hat{s}_2 + (1 - \lambda) \hat{s}_j & \text{if } t = 2
\end{cases}
\]

One can check that \( \sum_{i \in K} \bar{s}_i = \hat{s}_j + \sum_{i \in K} \hat{s}_i \).

By Theorem 1.2, \( \psi_{B_p} \) is convex, thus \( \psi_{B_p}(r^j) \leq \lambda \psi_{B_p}(r^1) + (1 - \lambda) \psi_{B_p}(r^2) \) for \( p = 1, \ldots, m \). It follows that \( \sum_{i \in K} \psi_{B_p}(r^i)\bar{s}_i \geq \psi_{B_p}(r^j)\hat{s}_j + \sum_{i \in K} \psi_{B_p}(r^i)\hat{s}_i = \sum_{i=1}^{k} \psi_{B_p}(r^i)\hat{s}_i \geq 1 \) for \( p = 1, \ldots, m \). Hence \( \bar{s} \) satisfies all the inequalities restricted to indices in \( K \) and has the same objective value as \( \hat{s} \). It follows that \( z_1 \geq z_2 \).

5 Proof sketch for Theorems 1.6 and 1.7

In this section, we give a brief outline of the proofs of Theorems 1.6 and 1.7. A complete proof will be given in Sections 6 and 7 respectively.

In Theorem 1.6, we need to analyze the optimization problem

\[
\min \left\{ \sum_{i=1}^{k} \psi(r^i)s_i : s \in S^k \right\} ,
\]  

where \( \psi \) is the minimal function derived from the Type 1 triangle.

For Theorem 1.7, recall that all nontrivial facet defining inequalities for \( R^k \) are of the form \( a^ts \geq 1 \) with \( a^t \geq 0 \). Therefore, Theorem 1.2 shows that to prove Theorem 1.7 we need to consider all nontrivial facet defining inequalities and optimize in the direction \( a^t \) over the triangle closure \( T^k \) and the quadrilateral closure \( Q^k \). This task is made easier since all the nontrivial facets of \( R^k \) are characterized in Theorem 4.1. Moreover, Theorem 1.4 shows that every split inequality \( \sum_{i=1}^{k} \psi(r^i)s_i \geq 1 \) is valid for \( T^k \). Therefore, if we optimize over \( T^k \) in the direction \( \sum_{i=1}^{k} \psi(r^i)s_i \), we get a value of at least one. Theorem 1.4 shows that this also holds for \( Q^k \). Thus, we can ignore the facets defined by split inequalities.

Formally, consider a maximal lattice-free triangle or quadrilateral \( B \) with associated minimal function \( \psi \) that gives rise to a facet \( \sum_{j=1}^{k} \psi(r^j)s_j \geq 1 \) of \( R^k \). We want to investigate the following optimization problems:

\[
\inf \left\{ \sum_{j=1}^{k} \psi(r^j)s_j : s \in T^k \right\} \tag{7}
\]

and

\[
\inf \left\{ \sum_{j=1}^{k} \psi(r^j)s_j : s \in Q^k \right\} \tag{8}
\]
We first observe that, without loss of generality, we can make the following simplifying assumptions for problems (6), (7) and (8).

**Assumption 5.1.** Consider the objective function \( \psi \) in problems (6), (7) and (8). For every \( j \) such that \( \psi(r^j) > 0 \), the ray \( r^j \) is such that the point \( f + r^j \) is on the boundary of the lattice-free set \( B \) generating \( \psi \).

Indeed, this amounts to scaling the coefficient for the ray \( r^j \) by a constant factor in every inequality derived from all maximal lattice-free sets, including \( B \). Therefore, this corresponds to a simultaneous scaling of variable \( s_j \) and corresponding coefficients in problems (6), (7) and (8). This does not change the optimal values of these problems. Moreover, Cornuéjols and Margot [7] show that the equations of all edges of triangles of Type 1, 2, or 3, of quadrilaterals and the direction of all splits generating facets of \( R_f \) are rational. This implies that the scaling factor for ray \( r^j \) is a rational number and that the scaled ray is rational too.

As a consequence, we can assume that the objective function of problems (6), (7) or (8) is \( \sum_{j=1}^k s_j \).

Remark 5.2. If \( \{r^1, \ldots, r^k\} \) contains the corner rays of the convex set defining the objective functions of (6), (7) or (8), then Assumption 5.1 implies that the hypotheses of Theorem 4.2 are satisfied. Therefore, when analyzing (6), (7) or (8), we can assume that \( \{r^1, \ldots, r^k\} \) is exactly the set of corner rays.

### 6 Type 1 triangle and the split closure

In this section, we present the proof of Theorem 1.6.

Consider any Type 1 triangle \( T \) with integral vertices \( x^j \), for \( j = 1, 2, 3 \), and one integral point \( y^j \) for \( j = 1, 2, 3 \) in the interior of each edge. We want to study the optimization problem (6). Recall that Remark 5.2 says that we only need to consider the case with three corner rays \( r^1, r^2 \) and \( r^3 \).

We compute the exact value of

\[
zs_{\text{PLIT}} = \min \sum_{j=1}^3 s_j \quad \sum_{j=1}^3 \psi(r^j)s_j \geq 1 \quad \text{for all splits } B_\psi
\]

(9)

Observe that, using a unimodular transformation, \( T \) can be made to have one horizontal edge \( x^1x^2 \) and one vertical edge \( x^1x^3 \), as shown in Figure 3. Without loss of generality, we place the origin at point \( x^1 \).

We distinguish two cases depending on the position of \( f \) in the interior of triangle \( T \): \( f \) is in the inner triangle \( T_I \) with vertices \( y^1 = (1, 1) \), \( y^2 = (0, 1) \) and \( y^3 = (1, 0) \); and \( f \in \text{int}(T) \setminus T_I \). We show that \( zs_{\text{PLIT}} = \frac{1}{2} \) when \( f \) is in the inner triangle \( T_I \) and that
$z_{\text{SPLIT}}$ increases linearly from $\frac{1}{2}$ when $f$ is at the boundary of $T_I$ to $\frac{2}{3}$ at the corners of the triangle $T$ when $f \in \text{int}(T) \backslash T_I$. See the right part of Figure 3 for some level curves of $z_{\text{SPLIT}}$ as a function of the position of $f$ in $T$. By a symmetry argument, it is sufficient to consider the inner triangle $T_I$ and the corner triangle $T_C$ defined by $f_1 + f_2 \leq 1$, $f_1, f_2 \geq 0$.

**Theorem 6.1.** Let $T$ be a Type 1 triangle with integral vertices, say $(0,0)$, $(0,2)$ and $(2,0)$. Then

(i) $z_{\text{SPLIT}} = \frac{1}{2}$ when $f$ is interior to the triangle with vertices $(1,0)$, $(0,1)$ and $(1,1)$.

(ii) $z_{\text{SPLIT}} = 1 - \frac{1}{3 - f_1 - f_2}$ when $f = (f_1, f_2)$ is interior to the corner triangle $f_1 + f_2 \leq 1$, $f_1, f_2 \geq 0$. The value of $z_{\text{SPLIT}}$ when $f$ is in the other corner triangles follows by symmetry.

To prove this theorem, we show that the split closure is completely defined by only three split inequalities. In other words, all other split inequalities are dominated by these three split inequalities.

Define $S_1$ as the convex set $1 \leq x_1 + x_2 \leq 2$, $S_2$ as the convex set $0 \leq x_1 \leq 1$ and $S_3$ as the convex set $0 \leq x_2 \leq 1$. Define Split 1 (resp. Split 2, Split 3) to be the inequality obtained from $S_1$ (resp. $S_2$, $S_3$).

Let $S$ be a split $c \leq ax_1 + ba_2 \leq c + 1$ with $f$ in the interior of $S$. The shores of $S$ are the two half-planes $ax_1 + ba_2 \leq c$ and $ax_1 + ba_2 \geq c + 1$.

**Remark 6.2.** Let $A$, $B$, and $C$ be three points on a line, with $B$ between $A$ and $C$ and let $S$ be a split. If $A$ and $C$ are not in the interior of $S$ but $B$ is, then $A$ and $C$ are on opposite shores of $S$.

![Figure 5: Illustration for the proof of Lemma 6.3](image)

**Lemma 6.3.** If $f$ is in the interior of the triangle $T_I$ with vertices $(0,1)$, $(1,0)$ and $(1,1)$, then the split closure is defined by Split 1, Split 2 and Split 3.

**Proof.** Let $S$ be a split defining an inequality that is not dominated by either of Split 1, Split 2, or Split 3. For $i = 1, 2, 3$, let $z^i$ (resp. $w^i$) be the boundary point for $r^i$ on the boundary of $T_I$ (resp. $S$) (see Figure 5). (Note that since $x^i = f + r^i$ is integer, these points exist.) Observe that if $z^1$ is not in the interior of $S$, then the inequality obtained from $S$ is
dominated by Split 1, since the three boundary points \( w_1, w_2, w_3 \) are closer to \( f \) than the corresponding three boundary points \( z^1, x^2, x^3 \) for the three rays on the boundary of \( S_1 \). A similar observation holds for \( z^2 \) and Split 2 and for \( z^3 \) and Split 3, yielding that \( z^2 \) and \( z^3 \) are also in the interior of \( S \).

Since the points \( y^1, y^2 \) and \( y^3 \) are integer, they are not in the interior of \( S \). Applying Remark 6.2 to \( y^1, z^3, y^2 \), we must have that \( y^1 \) and \( y^2 \) are on opposite shores of \( S \). Now, \( y^3 \) is in one of the two shores of \( S \). Assume without loss of generality that it is on the same shore as \( y^1 \). Applying Remark 6.2 to \( y^1, z^2, y^3 \), we have that \( y^1 \) and \( y^3 \) are on opposite shores of \( S \), a contradiction.

**Lemma 6.4.** If \( f \) is in the interior of the triangle \( T_I \) with vertices \( (0,1), (1,0) \) and \( (1,1) \), then \( z_{\text{SPLIT}} = \frac{1}{2} \).

**Proof.** By Lemma 6.3, \( z_{\text{SPLIT}} \) is given by

\[
\begin{align*}
z_{\text{SPLIT}} &= \min \sum_{j=1}^{3} s_j \\
&\quad \text{subject to } \sum_{j=1}^{3} \psi_i(r^j)s_j \geq 1 \quad \text{for } i = 1, 2, 3 \\
&\quad \text{where } \psi_i(r^j) \text{ is the coefficient of } s_j \text{ in Split } i, \text{ for } i = 1, 2, 3. 
\end{align*}
\]

where \( \psi_i(r^j) \) is the coefficient of \( s_j \) in Split \( i \), for \( i = 1, 2, 3 \). Let \( f = (f_1, f_2) \). The coefficient of \( s^j \) in the split inequality can be computed from the boundary point for \( r^j \) with the corresponding split. For example, the boundary points for \( r^2 \) and \( r^3 \) with \( S_1 \) are the integer points \( x^2 \) and \( x^3 \). This implies that \( \psi_1(r^2) = \psi_1(r^3) = 1 \). On the other hand, the boundary point for \( r^1 \) is the point \( t = \left( \frac{f_1}{f_1 + f_2}, \frac{f_2}{f_1 + f_2} \right) \). The length of \( r^1 \) divided by the length of the segment \( ft \) determines the coefficient \( \psi_1(r^1) \) of \( s_1 \) (This follows from the homogeneity of \( \psi_1 \) and the fact that \( \psi_1(t - f) = 1 \) since \( t \) is on the boundary of \( S_1 \)). We get \( \psi_1(r^1) = \frac{f_1 + f_2}{f_1 + f_2 - 1} \).

Repeating this for \( S_2 \) and \( S_3 \), we get that \( z_{\text{SPLIT}} \) is the optimal value of the following linear program.

\[
\begin{align*}
z_{\text{SPLIT}} &= \min \quad s_1 + s_2 + s_3 \\
&\quad \text{subject to } \frac{f_1 + f_2}{f_1 + f_2 - 1}s_1 + s_2 + s_3 \geq 1 \\
&\quad \text{subject to } s_1 + \frac{2-f_1}{1-f_1}s_2 + s_3 \geq 1 \\
&\quad \text{subject to } s_1 + s_2 + \frac{2-f_2}{1-f_2}s_3 \geq 1 \\
&\quad s \geq 0.
\end{align*}
\]

Its optimal solution \( s^* \) is \( s_1^* = \frac{f_1 + f_2 - 1}{2}, \ s_2^* = \frac{1-f_1}{2}, \ s_3^* = \frac{1-f_2}{2} \) with value \( z_{\text{SPLIT}} = s_1^* + s_2^* + s_3^* = \frac{1}{2} \). Indeed, note that all three inequalities in (11) are satisfied at equality and that the dual of (11) is feasible (for example, \( (0,0,0) \) is a solution). Therefore the complementary slackness conditions hold for \( s^* \) with any feasible solution of the dual.

Now we prove the second part of the theorem, when \( f \) is interior to the corner triangle with vertices \( (0,0), (1,0) \) and \( (0,1) \) or an inner point on the segment \( y^2 y^3 \).
Lemma 6.5. If \( f \) is in the interior of the triangle with vertices \((0,0), (0,1)\) and \((1,0)\), or an inner point on the segment joining \((0,1)\) to \((1,0)\), then the split closure is defined by Split 2 and Split 3.

Proof. Let \( S \) be a split defining a split inequality that is not dominated by either of Split 2, or Split 3. Let \( z^2 \) be the intersection point of \( r^2 \) with \( y^1y^3 \) and let \( z^3 \) be the intersection point of \( r^3 \) with \( y^1y^2 \). For \( i = 1, 2, 3 \), let \( w^i \) be the intersection point of \( r^i \) with either \( L_1 \) or \( L_2 \). (Note that since \( x^i \) is integer, \( r^i \) has to intersect one of the two lines.) Observe that if \( z^2 \) is not in the interior of \( S \), then the inequality obtained from \( S \) is dominated by Split 2, since the three intersection points \( w^1, w^2, w^3 \) are closer to \( f \) than the corresponding three intersection points \( x^1, z^2, x^3 \) for \( S_2 \). A similar observation holds for \( z^3 \) and \( S_3 \), yielding that \( z^3 \) is also in the interior of \( S \).

Since the points \( y^1, y^2 \) and \( y^3 \) are integer, they are not in the interior of \( S \). Applying Remark 6.2 to \( y^1, z^3, y^2 \), we have that \( y^1 \) and \( y^2 \) are on opposite shores of \( S \). Applying Remark 6.2 to \( y^1, z^2, y^3 \), we have that \( y^1 \) and \( y^3 \) are on opposite shores of \( S \). It follows that \( y^2 \) and \( y^3 \) are on the same shore \( W \) of \( S \) and thus the whole segment \( y^2y^3 \) is in \( W \). This is a contradiction with the fact that both \( f \) and \( z^3 \) are in the interior of \( S \), as the two segments \( y^2y^3 \) and \( fz^3 \) intersect. \( \square \)

Lemma 6.6. If \( f = (f_1, f_2) \) is in the interior of the triangle with vertices \((0,0), (0,1)\) and \((1,0)\), or an inner point on the segment joining \((0,1)\) to \((1,0)\), then \( z_{\text{SPLIT}} = 1 - \frac{1}{3-f_1-f_2} \).

Proof. The optimal solution of the LP

\[
\begin{align*}
    z_{\text{SPLIT}} = \min & \quad s_1 + s_2 + s_3 \\
    \text{subject to} & \quad s_1 + \frac{2-f_1}{1-f_1} s_2 + s_3 \geq 1 \\
    & \quad s_1 + s_2 + \frac{2-f_2}{1-f_2} s_3 \geq 1 \\
    & \quad s \geq 0.
\end{align*}
\]

is \( s_1 = 0, s_2 = \frac{1-f_1}{3-f_1-f_2}, s_3 = \frac{1-f_2}{3-f_1-f_2} \). \( \square \)

This completes the proof of Theorem 6.1. This theorem in conjunction with Theorem 1.3 implies that including all Type 1 triangle facets can improve upon the split closure only by a factor of 2.

Corollary 6.7. Let \( \mathcal{F} \) be the family of all facet defining inequalities arising from Type 1 triangles. Define

\[
\bar{S}_f = S_f^k \cap \left\{ \sum_{i=1}^k \psi(r^i)s_i \geq 1 : \psi \in \mathcal{F} \right\}.
\]

Then \( \bar{S}_f \subseteq S_f^k \subseteq 2\bar{S}_f \).

7 Integer hull vs. triangle and quadrilateral closures

In this section we present the proof of Theorem 1.7. We show that the triangle closure \( T_f^k \) and the quadrilateral closure \( Q_f^k \) both approximate the integer hull \( R_f^k \) to within a factor of
two. As outlined in Section 5, we can show this by taking a facet of $R^k_f$, and optimizing in that direction over $T^k_f$ or $Q^k_f$. As noted in that section, we need to analyze the optimization problems (7) and (8), where the objective function comes from a maximal lattice-free triangle or quadrilateral.

7.1 Approximating the integer hull by the triangle closure

We only need to consider facets of $R^k_f$ derived from quadrilaterals to obtain the objective function of problem (7). We prove the following result.

**Theorem 7.1.** Let $Q$ be a maximal lattice-free quadrilateral with corresponding minimal function $\psi$ and generating a facet $\sum_{i=1}^k \psi(r^i)s_i \geq 1$ of $R^k_f$. Then

$$\inf \left\{ \sum_{i=1}^k \psi(r^i)s_i : s \in T^k_f \right\} \geq \frac{1}{2}.$$

**Proof.** The theorem holds if the facet defining inequality can also be obtained as a triangle inequality. Therefore by Theorem 4.1 we may assume that rays $r^1, \ldots, r^4$ are the corners rays of $Q$ (see Figure 6). We remind the reader of Remark 5.2 showing that we can assume that $k = 4$ and that the four rays are exactly the corner rays of $Q$.

By a unimodular transformation, we may further assume that the four integral points on the boundary of $Q$ are $(0,0), (1,0), (1,1), (0,1)$. Moreover, by symmetry, we may assume that the fractional point $f$ satisfies $f_1 \leq \frac{1}{2}$ and $f_2 \leq \frac{1}{2}$ as rotating this region about $(\frac{1}{2}, \frac{1}{2})$ by multiples of $\frac{\pi}{2}$ covers the entire quadrilateral. Note that $f_1 < 0$ and $f_2 < 0$ are possible.

We relax Problem (7) by keeping only two of the triangle inequalities, defined by triangles $T_1$ and $T_2$. $T_1$ has corner $f + r^4$ and edges supported by the two edges of $Q$ incident with that corner and by the line $x = 1$. $T_2$ has corner $f + r^1$ and edges supported by the two edges of $Q$ incident with that corner and by the line $y = 1$. The two triangles are depicted in dashed lines in Figure 6.

Thus, Problem (7) can be relaxed to the LP

$$\min \quad s_1 + s_2 + s_3 + s_4$$

$$\sum_{i=1}^4 \psi_{T_1}(r^i)s_i \geq 1 \quad \text{(Triangle } T_1)$$

$$\sum_{i=1}^4 \psi_{T_2}(r^i)s_i \geq 1 \quad \text{(Triangle } T_2)$$

Let $(\alpha, \beta) = f + r^2$ and $(\gamma, \delta) = f + r^3$. Computing the coefficients $\psi_{T_1}(r^2)$ and $\psi_{T_2}(r^3)$, LP (13) becomes
Figure 6: Approximating a quadrilateral inequality with triangle inequalities

\[ \begin{align*}
\min & \quad s_1 + s_2 + s_3 + s_4 \\
& \quad s_1 + \frac{\alpha - f_1}{1 - f_1} s_2 + s_3 + s_4 \geq 1 \quad (T_1) \\
& \quad s_1 + s_2 + \frac{\delta - f_2}{1 - f_2} s_3 + s_4 \geq 1 \quad (T_2) \\
& \quad s \in \mathbb{R}_+^4.
\end{align*} \]

Using the equation of the edge of \( Q \) connecting \( f + r^2 \) and \( f + r^3 \), we can find bounds on \( \psi_{T_1}(r^2) \) and \( \psi_{T_2}(r^3) \). The edge has equation \( x_1 \frac{1}{t} + \frac{t-1}{t} x_2 = 1 \) for some \( 1 < t < \infty \). Therefore \( \alpha \leq t \) and \( \delta \leq \frac{t}{t-1} \). Using these two inequalities together with \( f_1 \leq \frac{1}{2} \) and \( f_2 \leq \frac{1}{2} \), we get

\[
\frac{\alpha - f_1}{1 - f_1} = \frac{\alpha - 1}{1 - f_1} + 1 \leq 2(t - 1) + 1 = 2t - 1 \quad \text{and} \quad \frac{\delta - f_2}{1 - f_2} \leq 2 \frac{t}{t - 1} - 1.
\]

Using these bounds, we obtain the relaxation of LP \((14)\)

\[ \begin{align*}
\min & \quad s_1 + s_2 + s_3 + s_4 \\
& \quad s_1 + (2t - 1) s_2 + s_3 + s_4 \geq 1 \quad (T_1) \\
& \quad s_1 + s_2 + \left( \frac{2}{t - 1} - 1 \right) s_3 + s_4 \geq 1 \quad (T_2) \\
& \quad s \in \mathbb{R}_+^4.
\end{align*} \]

Set \( \lambda = 2t - 1 \) and \( \mu = 2 \frac{t}{t-1} - 1 \). Then \( t > 1 \) implies \( \lambda > 1 \) and \( \mu > 1 \). The optimal solution of the above LP is \( s_1 = s_4 = 0, s_2 = \frac{\mu - 1}{\lambda \mu - 1} \) and \( s_3 = \frac{\lambda - 1}{\lambda \mu - 1} \) with value
\[ s_1 + s_2 + s_3 + s_4 = \frac{\lambda + \mu - 2}{\lambda \mu - 1} = \frac{t^2 - 2t + 2}{t^2}. \]

To find the minimum of this expression for \( t > 1 \), we set its derivative to 0, and get the solution \( t = 2 \). Thus the minimum value of \( s_1 + s_2 + s_3 + s_4 \) is equal to \( \frac{1}{2} \).

### 7.2 Approximating the integer hull by the quadrilateral closure

In this section, we study Problem (8). We can approximate the facets derived from Type 1 and Type 2 triangles using quadrilaterals in a similar manner as the splits were approximated by triangles and quadrilaterals. See Figure 7. We again define the set \( X \) of points which lie strictly inside the Type 1 or Type 2 triangle, similar to the proof of Theorem 1.4. Then we can find quadrilaterals as shown in Figure 7 that contain the set \( X \). The proof goes through in exactly the same manner.

However triangles of Type 3 pose a problem. They cannot be approximated to any desired precision by a sequence of quadrilaterals.

![Figure 7: Approximating a triangle inequality with a quadrilateral inequality](image)

In this section, we work under Assumption 5.1. By Theorem 4.1 a Type 3 triangle \( T \) defines a facet if and only if either the ray condition holds, or all three corner rays are present. First we consider the case where the ray condition holds.

**Theorem 7.2.** Let \( T \) be a triangle of Type 3 with corresponding minimal function \( \psi_T \) defining a facet of \( \mathbb{R}^n \). If the ray condition holds for \( T \) then Problem (8) has optimal value 1.

**Proof.** We first prove that if the ray condition holds the points \( p^j = f + r^j \) are integral points on the boundary of \( T \), for \( j = 1, \ldots, k \).

For \( i = 1, 2, 3, \) let \( P_i \) be the set of points left at the end of Step \( i \) of the Reduction Algorithm given in Section 4. Each \( p^j \in P_i \) with \( p^j \) integral is uniquely active and is removed.
during Step 2 of the Reduction Algorithm. Hence, all points in $P_2$ are non-integral. Observe that Step 3 can only remove boundary points $p$ and $q$ when the segment $pq$ contains at least two integral points in its relative interior. Therefore this step does not remove anything in a Type 3 triangle and $P_3 = P_2$.

Since the ray condition holds, we have $P_3 = P_2 = \emptyset$ and $P_1$ contains only integral points. But then $P_1 = P$, showing that all boundary points at the beginning of the Reduction Algorithm are integral.

It is then straightforward to construct a maximal lattice-free quadrilateral $Q$ with $p^j$, $j = 1, \ldots, k$ on its boundary, and containing $f$ in its interior. It follows that the value of Problem (8) is equal to 1.

We now consider the case where $T$ is a Type 3 triangle with the three corner rays present. In this case, we can approximate the facet obtained from $T$ to within a factor of two by using inequalities derived from triangles of Type 2. Define another relaxation $\bar{T}_f^k$ as the convex set defined by the intersection of the inequalities derived only from Type 1 and Type 2 triangles. By definition, $T_f^k \subseteq \bar{T}_f^k$. From the discussion at the beginning of this section, we also know $Q_f^k \subseteq \bar{T}_f^k$. Hence (8) can be relaxed to

$$\inf \left\{ \sum_{i=1}^{k} \psi(r^i)s_i : s \in \bar{T}_f^k \right\}. \quad (16)$$

**Theorem 7.3.** Let $T$ be a triangle of Type 3 with corresponding minimal function $\psi$ and generating a facet $\sum_{i=1}^{k} \psi(r^i)s_i \geq 1$ of $R_f^k$. Then,

$$\inf \left\{ \sum_{i=1}^{k} \psi(r^i)s_i : s \in \bar{T}_f^k \right\} \geq \frac{1}{2}. \quad (16)$$

This theorem implies directly the following corollary.

**Corollary 7.4.** $Q_f^k \subseteq 2R_f^k$.

**Proof of Theorem 7.3.** We first make an affine transformation to simplify computations. Let $y^1, y^2, y^3$ be the three lattice points on the sides of $T$. We choose the transformation such that the two following properties are satisfied.

(i) The standard lattice is mapped to the lattice generated by the vectors $v^1 = (1,0)$ and $v^2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, i.e. all points of the form $z_1v^1 + z_2v^2$, where $z_1, z_2$ are integers.

(ii) $y^1, y^2, y^3$ are respectively mapped to $(0,0), (1,0), (0,1)$ in the above lattice.

We use the basis $v^1, v^2$ for $\mathbb{R}^2$ for all calculations and equations in the remainder of the proof. See Figure 8.

With this transformation, we can get a simple characterization for the Type 3 triangles. See Figure 8 for an example. We make the following claim about the relative orientations of the three sides of the Type 3 triangle.
Lemma 7.5. Let $\mathcal{F}$ be the family of triangles formed by three lines given by:

Line 1: $-\frac{x_1}{t_1} + x_2 = 1$ with $0 < t_1 < \infty$;

Line 2: $t_2 x_1 + x_2 = 0$ with $0 < t_2 < 1$;

Line 3: $x_1 + \frac{x_2}{t_3} = 1$ with $1 < t_3 < \infty$.

Any Type 3 triangle is either a triangle from $\mathcal{F}$ or a reflection of a triangle from $\mathcal{F}$ about the line $x_1 = x_2$.

Proof. Take any Type 3 triangle $T$. Consider the edge passing through $(1,0)$. Since it cannot go through the interior of the equilateral triangle $(0,0), (0,1), (1,0)$, there are only two choices for its orientation: a) It can go through the segment $(0,1), (1,1)$, or b) It can go through the segment $(0,0), (1,-1)$.

In the first case its equation is that of Line 3. This now forces the edge of $T$ passing through $(0,0)$ to have the equation for Line 2. This is because the only other possibility for this edge would be for the line to pass through the segment $(-1,1), (0,1)$. But then the lattice point $(1,-1)$ is included in the interior of the triangle. Similarly, the third edge must now have Line 1’s equation, because $(-1,1)$ needs to be excluded from the interior.

Case b) can be mapped to Case a) by a reflection about the line $x_1 = x_2$.

Remark 7.6. We can choose any values for $t_1, t_2, t_3$ independently in the prescribed ranges, and we get a lattice free triangle. This observation shows that the family $\mathcal{F}$ defined above is exactly the family of all Type 3 triangles modulo an affine transformation.

We now show how to bound Problem (16) and hence prove Theorem 1.8.

Consider any Type 3 triangle $T$. It is sufficient to consider the case where the lines supporting the edges of $T$ have equations as stated in the Lemma 7.5.

We consider two cases for the position of the fractional point $f = (f_1, f_2)$. 

Figure 8: Approximating a Type 3 triangle inequality with Type 2 triangle inequalities. The Type 3 triangle is in solid lines. The basis vectors are $v^1 = (1,0)$ and $v^2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.
(i) \( f_1 \leq \frac{1}{2}, f_2 \leq \frac{1}{2} \);

(ii) \( f_1 \leq 0, f_1 + f_2 \leq \frac{1}{2} \).

The union of the two regions described above when rotated by \( 2\pi/3 \) and \( 4\pi/3 \) about the point \((\frac{1}{3}, \frac{1}{3})\) (the centroid of the triangle formed by \((0,0),(0,1)\) and \((1,0))\), cover all of \(\mathbb{R}^2\). Hence they cover all of \(T\) and by rotational symmetry, investigating these two cases is enough.

For the first case, we relax Problem [16] by using only two inequalities from \(\bar{T}_k\). These are derived from Type 2 triangles \(T_1\) and \(T_2\) (see Figure 8), which are defined as follows. \(T_1\) has Line 1 and Line 2 supporting two of its edges and \(x_1 = 1\) supporting the third one (with more than one integral point). \(T_2\) has Line 2 and Line 3 supporting two of its edges and \(x_2 = 1\) supporting the third one (with more than one integral point). Let \(\psi_{T_1}\) and \(\psi_{T_2}\) be the corresponding minimal functions derived from \(T_1\) and \(T_2\).

The following LP is a relaxation of Problem [17].

\[
\min \sum_{i=1}^{k} s_i \\
\sum_{i=1}^{k} \psi_{T_1}(r^i)s_i \geq 1 \quad \text{(Triangle } T_1) \\
\sum_{i=1}^{k} \psi_{T_2}(r^i)s_i \geq 1 \quad \text{(Triangle } T_2) \\
s \in \mathbb{R}^k_+.
\]

Theorem 4.2 and Remark 5.2 imply that LP [17] is equivalent to

\[
\min s_1 + s_2 + s_3 \\
\psi_{T_1}(r^1)s_1 + \psi_{T_1}(r^2)s_2 + \psi_{T_1}(r^3)s_3 \geq 1 \quad \text{(Triangle } T_1) \\
\psi_{T_2}(r^1)s_1 + \psi_{T_2}(r^2)s_2 + \psi_{T_2}(r^3)s_3 \geq 1 \quad \text{(Triangle } T_2) \\
s \in \mathbb{R}^3_+.
\]

where \(r^1\), \(r^2\) and \(r^3\) are the three corner rays (see Figure 8).

We now show that this LP has an optimal value of at least \(\frac{1}{2}\).

Note that \(\psi_{T_1}(r^2) = \psi_{T_1}(r^3) = 1\). \(\psi_{T_1}(r^1)\) needs to be computed. First we compute \(r^1\) and \(r^2\) in terms of \(t_1, t_2, t_3, f_1\) and \(f_2\).

The intersection of Line 2 and Line 3 is given by

\[
\left( \frac{t_3}{t_3-t_2}, \frac{-t_2t_3}{t_3-t_2} \right)
\]

and thus

\[
r^1 = \left( \frac{t_3}{t_3-t_2}, \frac{-t_2t_3}{t_3-t_2} \right) - (f_1, f_2).
\]

As \(\psi_{T_1}(r^1) = \frac{1}{\gamma}\) where \(\gamma\) is such that \((f_1, f_2) + \gamma r^1\) lies on the line \(x_1 = 1\), we get

\[
\psi_{T_1}(r^1) = \frac{t_1}{t_1-t_2} - \frac{f_1}{1-f_1}.
\]
Similarly, we only need $\psi T_2(r^2)$ as $\psi T_2(r^1) = \psi T_2(r^3) = 1$. The intersection of Line 1 and Line 3 is

$$\left( \frac{t_1(t_3 - 1)}{1 + t_1 t_3}, \frac{t_3(t_1 + 1)}{1 + t_1 t_3} \right)$$

and thus

$$r^2 = \left( \frac{t_1(t_3 - 1)}{1 + t_1 t_3}, \frac{t_3(t_1 + 1)}{1 + t_1 t_3} \right) - (f_1, f_2).$$

Computing the coefficient like before, we get

$$\psi T_2(r^2) = \frac{t_3(t_1 + 1)}{1 + t_1 t_3} - f_2.$$

Hence LP (18) becomes

$$\min \quad s_1 + s_2 + s_3$$

$$\begin{align*}
\frac{t_1}{t_1 - t_2} s_1 + s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_1) \\
\frac{t_3(t_1 + 1)}{1 + t_1 t_3} - f_2 \quad s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_2) \\
s & \in \mathbb{R}_+^3.
\end{align*}$$

As

$$\frac{t_3}{t_3 - t_2} = \frac{t_3 - 1}{1 - f_1} + 1,$$

the assumptions $f_1 \leq \frac{1}{2}$ and $t_2 < 1$ yield

$$\frac{t_3}{1 - f_1} + 1 \leq 2 \frac{t_3}{t_3 - 1} - 1.$$

Similarly,

$$\frac{t_3(t_1 + 1)}{1 + t_1 t_3} - f_2 \quad \frac{t_3(t_1 + 1)}{1 + t_1 t_3} - 1 + 1 \quad \frac{t_3(t_1 + 1)}{1 + t_1 t_3} - f_2 + 1,$$

and the assumption $f_2 \leq \frac{1}{2}$ gives

$$\frac{t_3(t_1 + 1)}{1 + t_1 t_3} - 1 + 2 \frac{t_3(t_1 + 1)}{1 + t_1 t_3} - 1 = 2 \frac{t_3 - 1}{1 + t_1 t_3} + 1.$$

Under the assumption $t_1 > 0$, we obtain $2 \frac{t_3 - 1}{1 + t_1 t_3} + 1 \leq 2t_3 - 1$.

To get a lower bound on (19), we thus can relax its constraints to

$$\min \quad s_1 + s_2 + s_3$$

$$\begin{align*}
(2 \frac{t_3}{t_3 - 1} - 1) s_1 + s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_1) \\
(2t_3 - 1) s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_2) \\
s & \in \mathbb{R}_+^3.
\end{align*}$$

(20)
Set $\lambda = 2t_3 - 1$ and $\mu = 2t_3 - 1 - 1$. Then $t_3 > 1$ implies $\lambda > 1$ and $\mu > 1$. The optimal solution of LP (20) is $s_1 = \frac{\lambda - 1}{\lambda \mu - 1}$, $s_2 = \frac{\mu - 1}{\lambda \mu - 1}$ and $s_3 = 0$ with value

$$s_1 + s_2 + s_3 = \frac{\lambda + \mu - 2}{\lambda \mu - 1} = \frac{t_3^2 - 2t_3 + 2}{t_3^2}.$$

To find the minimum over all $t_3 > 1$, we set the derivative to 0, which gives the solution $t_3 = 2$. Thus the minimum value of $s_1 + s_2 + s_3$ is $\frac{1}{2}$.

Next we consider $f_1 \leq 0$ and $f_1 + f_2 \leq \frac{1}{2}$, the shaded region in Figure 9. We relax Problem (16) using only two inequalities. We take $T_1$ as before, but $T_2$ is the triangle formed by the following three lines: Line 2 from Lemma 7.5 line parallel to Line 1 from Lemma 7.5 but passing through $(-1,1)$ and the line passing through $(1,0), (0,1)$.

Figure 9: Approximating a Type 3 triangle inequality with Type 2 triangles inequalities - Case 2

As in the previous case, we formulate the relaxation as an LP with constraints corresponding to $T_1$ and $T_2$. The only difference from LP (18) is the coefficient $\psi_{T_2}(r^2)$. This time the point $(f_1, f_2) + \gamma r^2$ lies on the line $x_1 + x_2 = 1$ (recall that $\psi_{T_2}(r^2) = \frac{1}{4}$). This gives us

$$\psi_{T_2}(r^2) = \frac{2t_1t_3 + t_1 - t_1}{1 + t_3} - f_1 - f_2.$$

We then have the following LP.

$$\begin{align*}
\min \quad & s_1 + s_2 + s_3 \\
\text{s.t.} \quad & \frac{t_3 - t_2}{t_3 - t_2} - f_1 \geq 1 \quad (\text{Triangle } T_1) \\
\quad & s_1 + \frac{2t_1t_3 + t_1 - t_1}{1 + t_3} - f_1 - f_2 \geq 1 \quad (\text{Triangle } T_2) \\
\quad & s_1 + \frac{2t_1t_3 + t_1 - t_1}{1 + t_3} - f_1 - f_2 \geq 1 \quad (\text{Triangle } T_2) \\
\quad & s \in \mathbb{R}_+^3.
\end{align*}$$

23
We simplify the coefficients as earlier:

$$\frac{t_3}{1-t_3} - f_1 = 1 + \frac{t_3}{1-t_3} - 1$$

and

$$\frac{2t_1t_3 + t_3 - t_1}{1 + t_1t_3} - f_1 - f_2 = 1 + \frac{2t_1t_3 + t_3 - t_1}{1 + t_1t_3} - 1.$$

Using the assumptions $f_1 \leq 0$, $f_1 + f_2 \leq \frac{1}{2}$, we get that

$$1 + \frac{t_3}{1-t_3} - 1 \leq \frac{t_3}{t_3-t_2}$$

and

$$1 + \frac{2t_1t_3 + t_3 - t_1}{1 + t_1t_3} - 1 \leq 2(\frac{2t_1t_3 + t_3 - t_1}{1 + t_1t_3}) - 1.$$

We also have the conditions $t_2 < 1$ and $t_1 > 0$. $t_2 < 1$ implies $\frac{t_3}{t_3-t_2} \leq \frac{t_3}{t_3-1}$. Moreover

$$2\left(\frac{2t_1t_3 + t_3 - t_1}{1 + t_1t_3}\right) - 1 = 2\left(2 + \frac{t_3 - t_1 - 2}{1 + t_1t_3}\right) - 1$$

decreases in value as $t_1$ increases. Its maximum value is less than the value for $t_1 = 0$, because of the condition $t_1 > 0$. It follows that $2(\frac{2t_1t_3 + t_3 - t_1}{1 + t_1t_3}) - 1 \leq 2t_3 - 1$. After putting these relaxations into the constraints of LP (21), we get

$$\begin{align*}
\min & \quad s_1 + s_2 + s_3 \\
\text{s.t.} & \quad \frac{t_3}{t_3-1}s_1 + s_2 + s_3 \geq 1 \quad (\text{Triangle } T_1) \\
& \quad s_1 + (2t_3 - 1)s_2 + s_3 \geq 1 \quad (\text{Triangle } T_2) \\
& \quad s \in \mathbb{R}^3_+.
\end{align*}$$

The optimal solution of LP (22) is

$$s_3 = 0, \quad s_1 = \frac{2(t_3 - 1)^2}{2t_3^2 - 2t_3 + 1}, \quad s_2 = \frac{1}{2t_3^2 - 2t_3 + 1}, \quad \text{and} \quad s_1 + s_2 + s_3 = \frac{2t_3^2 - 4t_3 + 3}{2t_3^2 - 2t_3 + 1}.$$

Under the condition $t_3 > 1$, the minimum value of $s_1 + s_2 + s_3$ is achieved for $t_3 = 1 + \sqrt{\frac{1}{2}}$, with value $\frac{1}{1 + \sqrt{\frac{1}{2}}} > \frac{1}{2}$.

8 Split closure vs. a single triangle or quadrilateral inequality

In this section, we prove Theorem 1.8.

This is done by showing that there exist examples of integer programs (1) where the optimal value for optimizing in the direction of a triangle (or quadrilateral) inequality over the split closure $S^k_j$ can be arbitrarily small. We give such examples for facets derived from triangles of Type 2 and Type 3, and from quadrilaterals. These examples have the property that the point $f$ lies in the relative interior of a segment joining two integral points at distance 1.

Furthermore, in these examples, the rays end on the boundary of the triangle or quadrilateral and hence the facet corresponding to it is of the form $\sum_{j=1}^k s_j \geq 1$. We show that the following LP has optimal value much less than 1.
\[ z_{\text{SPLIT}} = \min \sum_{j=1}^{k} s_j \]

\[ \sum_{j=1}^{k} \psi(r^j)s_j \geq 1 \quad \text{for all splits } B_\psi \]

(23)

Theorem 1.3 then implies Theorem 1.8. A key step in the proof is a method for constructing a polyhedron contained in the split closure (Lemma 8.3). The resulting LP implies an upper bound on \( z_{\text{SPLIT}} \). We then give a family of examples showing that this upper bound can be arbitrarily close to 0. We start the proof with an easy lemma.

### 8.1 An easy lemma

Refer to Figure 10 for an illustration of the following lemma.

**Lemma 8.1.** Let \( r^1 \) and \( r^2 \) be two rays that are not multiples of each others and let \( H_1 \) and \( H_2 \) be the half-lines generated by nonnegative multiples of \( r^1 \) and \( r^2 \) respectively. Let \( p := k_1 r^1 + k_2 r^2 \) with \( k_1, k_2 > 0 \). Let \( L_1, L_2, \) and \( L_3 \) be three distinct lines going through \( p \) such that each of the lines intersects both \( H_1 \) and \( H_2 \) at points other than the origin. Let \( d_{ij} \) be the distance from the origin to the intersection of line \( L_i \) with the half-line \( H_j \) for \( i = 1, 2, 3 \) and \( j = 1, 2 \). Assume that \( d_{11} < d_{21} < d_{31} \). Then there exists \( 0 < \lambda < 1 \) such that

\[
\frac{1}{d_{21}} = \lambda \frac{1}{d_{11}} + (1 - \lambda) \frac{1}{d_{31}} \quad \text{and} \quad \frac{1}{d_{22}} = \lambda \frac{1}{d_{12}} + (1 - \lambda) \frac{1}{d_{32}}.
\]

**Proof.** Let \( u^i \) be a unit vector in the direction of \( r^i \) for \( i = 1, 2 \). Using \( \{u^1, u^2\} \) as a base of \( \mathbb{R}^2 \), for \( i = 1, 2, 3 \), \( L_i \) has equation

\[
\frac{1}{d_{1i}} x_1 + \frac{1}{d_{2i}} x_2 = 1
\]
As \( L_2 \) is a convex combination of \( L_1 \) and \( L_3 \), there exists \( 0 < \lambda < 1 \) such that \( \lambda L_1 + (1-\lambda)L_3 = L_2 \). The result follows.

**Corollary 8.2.** In the situation of Lemma 8.1 let \( L_4 \) be a line parallel to \( r^1 \) going through \( p \). Let \( d_{42} \) be the distance between the origin and the intersection of \( H_2 \) with \( L_4 \). Then there exists \( 0 < \lambda < 1 \) such that

\[
\frac{d_{21}}{d_{11}} = \lambda \frac{d_{11}}{d_{12}} \quad \text{and} \quad \frac{d_{22}}{d_{12}} = \lambda \frac{d_{12}}{d_{42}} + (1-\lambda) \frac{1}{d_{42}}.
\]

**Proof.** Similar to the proof of Lemma 8.1.

---

8.2 A polyhedron contained in the split closure

Our examples for proving Theorem 1.8 have the property that the point \( f \) lies in the relative interior of a segment joining two integral points \( y^1, y^2 \) at distance 1.

To obtain an upper bound on the value \( z_{\text{SPLIT}} \) of the split closure, we define some inequalities which dominate the split closure \( \mathcal{F} \). A *pseudo-split* is the convex set between two distinct parallel lines passing through \( y^1 \) and \( y^2 \) respectively. The direction of the lines, called *direction* of the pseudo-split, is a parameter. Figure 11 illustrates three pseudo-splits in the directions of three rays. The *pseudo-split inequality* is derived from a pseudo-split exactly in the same way as from any maximal lattice-free convex set. Note that pseudo-splits are in general not lattice-free and hence do not generate valid inequalities for \( R_f^k \). However, we can dominate any split inequality cutting \( f \) by an inequality derived from these convex sets. Indeed, consider any split \( S \) containing the fractional point \( f \) in its interior. Since \( f \) lies
on the segment \( y^1y^2 \), both boundary lines of \( S \) pass through the segment \( y^1y^2 \). The pseudo-split with direction identical to the direction of \( S \) generates an inequality that dominates the split inequality derived from \( S \), as the coefficient for any ray is smaller in the pseudo-split inequality.

The next lemma states that we can dominate the split closure by using only the inequalities generated by the pseudo-splits with direction parallel to the rays \( r^1, \ldots, r^k \) under mild assumptions on the rays.

**Lemma 8.3.** Assume that none of the rays \( r^1, \ldots, r^k \) has a zero first component and that \( f = (0, f_2) \) with \( 0 < f_2 < 1 \). Let \( y^1 = (0,1) \) and \( y^2 = (0,0) \), these two points being used to construct pseudo-splits. Let \( S_1, \ldots, S_k \) be the pseudo-splits in the directions of rays \( r^1, \ldots, r^k \) and denote the corresponding minimal functions by \( \psi_{S_1}, \ldots, \psi_{S_k} \). Let \( S \) be any split with \( f \) in its interior and let \( S' \) be the corresponding pseudo-split. Then the inequality \( \sum_{j=1}^k \psi_{S}(r^j)s_j \geq 1 \) corresponding to \( S' \) is dominated by a convex combination of the inequalities \( \sum_{j=1}^k \psi_{S_i}(r^j)s_j \geq 1, i = 1, \ldots, k \). Therefore, the split inequality corresponding to \( S \) is dominated by a convex combination of the inequalities corresponding to \( \psi_{S_1}, \ldots, \psi_{S_k} \).

**Proof.** As a convention, the direction of a pseudo-split forms an angle with the \( x_1 \)-axis in the range of \( ] - \frac{\pi}{2}, \frac{\pi}{2} [ \). Without loss of generality, assume that the slope of the directions of the pseudo-splits corresponding to the rays \( r^1, \ldots, r^k \) are monotonically non increasing. We can assume that the direction of \( S' \) is different than the direction of any of the rays in \( \{ r^1, \ldots, r^k \} \) as otherwise the result trivially holds.

![Figure 12: Bounding the split closure with a finite number of pseudo-splits](image)

First note that, if \( S' \) has a direction with slope greater than the slope of \( r^1 \), then the inequality generated by \( S' \) is dominated by the one generated by \( S_1 \). Indeed, any ray \( r^j \) having a slope smaller than \( r^1 \) has its boundary point for \( S' \) closer to \( f \) than the one for \( S_1 \).
It follows that $\psi_{S'}(r^j) \geq \psi_{S_1}(r^j)$. See Figure[12]. A similar reasoning holds for the case where $S'$ has a direction with slope smaller than the slope of $r^k$.

Thus we only have to consider the case where the slope of the direction of $S'$ is strictly between the slopes of the directions of $S_i$ and $S_{i+1}$, for some $1 \leq i \leq k - 1$. We claim the following.

**Observation 8.4.** There exists a $0 < \lambda < 1$ such that $\psi_{S'}(r) = \lambda \psi_{S_i}(r) + (1 - \lambda)\psi_{S_{i+1}}(r)$ for every ray $r \in \{r^1, \ldots, r^k\}$.

**Proof.** For each pseudo-split $S \in \{S', S_i, S_{i+1}\}$, we denote by $L^S_i$ its boundary line passing through $(0, 1)$ and by $L^S_j$ its boundary line passing through $(0, 0)$.

Consider first any ray $r^j$ with $j < i$ and let $L^r_j$ be the half-line $f + \mu r^j$, $\mu \geq 0$. We have that $L^r_j$ has a slope greater than the slope of the direction of $S_i$ and thus $L^r_j$ intersects the boundaries of $S', S_i$ and $S_{i+1}$ on $L^S_i$, $L^S_{i+1}$ and $L^S_j$. By Lemma S.1 there exists a $0 < \lambda_1 < 1$ such that, for all $r \in \{r^1, \ldots, r^{i-1}\}$

$$\psi_{S'}(r) = \lambda_1 \psi_{S_i}(r) + (1 - \lambda_1)\psi_{S_{i+1}}(r). \quad (24)$$

By Corollary S.2, equation (24) also holds for $r = r^i$.

Using a similar reasoning for the rays $\{r^{i+1}, \ldots, r^k\}$ and the boundary lines $L^S_i$, $L^S_{i+1}$ and $L^S_j$, there exists a $0 < \lambda_2 < 1$ such that, for all $r \in \{r^{i+1}, \ldots, r^k\}$

$$\psi_{S'}(r) = \lambda_2 \psi_{S_i}(r) + (1 - \lambda_2)\psi_{S_{i+1}}(r). \quad (25)$$

It remains to show that $\lambda_1 = \lambda_2$. Consider any line $L$ through $f$ that is not collinear with $r^i$ or $r^{i+1}$ and not going through $y^1$ and $y^2$. For each $S \in \{S', S_i, S_{i+1}\}$, let $z^S_i$ (resp. $z^S_j$) be the intersection of $L$ with $L^S_i$ (resp. $L^S_j$) and let $d^S_i$ (resp. $d^S_j$) be the distance from $f$ to $z^S_i$ (resp. $z^S_j$). See Figure[12]. By Lemma S.1

$$\frac{1}{d^S_i} = \lambda_1 \frac{1}{d^S_i} + (1 - \lambda_1) \frac{1}{d^S_{i+1}} \quad \text{and} \quad \frac{1}{d^S_j} = \lambda_2 \frac{1}{d^S_j} + (1 - \lambda_2) \frac{1}{d^S_{j+1}}. \quad (26)$$

The length of segments $f y^1, f z^S_i y^2$ and $f z^S_j y^2$ are respectively $1 - f_2, d^S_i, f_2$ and $d^S_j$. Observe that the triangles $f z^S_i y^1$ and $f z^S_j y^2$ are homothetic with homothetic ratio $t = \frac{1 - f_2}{f_2}$. It follows that $d^S_i = t \cdot d^S_j$ in (26) yields $\lambda_1 = \lambda_2$. \qed

This observation proves the lemma. \qed

Using the above lemma, we can bound the split closure for three rays. We assume that none of the rays has a zero first component and that the three rays generate $\mathbb{R}^2$. Without loss of generality, we make the following assumptions. The rays are $r^1 = \mu_1 (-1, t_1)$, $r^2 = \mu_2 (1, t_2)$ and $r^3 = \mu_3 (-1, t_3)$, where $t_i$’s are rational numbers in the range $]-\infty, \infty[$, with $t_1 > t_3$ and $\mu_i$’s are scaling factors with $\mu_i > 0$. Any configuration of three rays satisfying the above assumptions either fits this description or is a reflection of it about the segment $(0, 0), (0, 1)$. In addition, we must have $-t_1 < t_2 < -t_3$. See Figure[14] for an illustration.
**Theorem 8.5.** Assume that \( f = (0, f_2) \) with \( 0 < f_2 < 1 \). Consider rays \( r^1 = \mu_1(-1, t_1) \), \( r^2 = \mu_2(1, t_2) \) and \( r^3 = \mu_3(-1, t_3) \), where \( t_i \)'s are rational numbers with \( -t_1 < t_2 < -t_3 \) and \( \mu_i > 0 \). Then

\[
z_{\text{SPLIT}} \leq \frac{1}{t_1 - t_3} \left( \frac{1 - f_2}{\mu_1} + \frac{f_2}{\mu_3} \right).
\]

**Proof.** Let \( y^1 = (0,1) \) and \( y^2 = (0,0) \), these two points being used to construct pseudo-splits. By Lemma 8.3 we know that the three pseudo-splits \( S_1, S_2, S_3 \) corresponding to the directions of \( r^1, r^2, r^3 \) dominate the entire split closure. More formally, the following LP is a strengthening of (23) in this example of three rays.

\[
\begin{align*}
\min & \quad s_1 + s_2 + s_3 \\
& \quad \psi_{S_1}(r^1)s_1 + \psi_{S_1}(r^2)s_2 + \psi_{S_1}(r^3)s_3 \geq 1 \\
& \quad \psi_{S_2}(r^1)s_1 + \psi_{S_2}(r^2)s_2 + \psi_{S_2}(r^3)s_3 \geq 1 \\
& \quad \psi_{S_3}(r^1)s_1 + \psi_{S_3}(r^2)s_2 + \psi_{S_3}(r^3)s_3 \geq 1 \\
& \quad s \in \mathbb{R}^3_+.
\end{align*}
\]

(27)

It is fairly straightforward to compute the coefficients in the above inequalities. We give the calculations for \( S_1 \); the coefficients for the other two follow along similar lines.

\( \psi_{S_1}(r^1) \) is 0, since \( r^1 \) is parallel to the direction of \( S_1 \).

Consider \( r^2 \) and let its boundary point \( p \) for \( S_1 \) be \( (0, f_2) + \gamma \mu_2(1, t_2) \), for some \( \gamma \geq 0 \). Then \( \psi_{S_1}(r^2) \) is \( \frac{1}{\gamma} \). To compute \( \gamma \), we observe that \( p \) is on boundary 1 of \( S_1 \), by assumption of \( t_2 > -t_1 \). Hence, the slope of the line connecting \( p \) and \( (0,1) \) is \( -t_1 \). Therefore,

\[
\frac{f_2 + \gamma \mu_2 t_2 - 1}{0 + \gamma \mu_2} = -t_1
\]

which yields \( \gamma = \frac{1-f_2}{\mu_2(t_1 + t_2)} \). Hence \( \psi_{S_1}(r^2) = \frac{\mu_2(t_1 + t_2)}{1-f_2} \).

Now consider \( r^3 \). As before, let its boundary point \( p' \) for \( S_1 \) be \( (0, f_2) + \gamma' \mu_3(-1, t_3) \), for some \( \gamma' \geq 0 \). This time note that the ray intersects boundary 2 (by the assumption \( t_3 < t_1 \)). Equating slopes, we get

\[
\frac{f_2 + \gamma' \mu_3 t_3}{0 - \gamma' \mu_3} = -t_1
\]

which yields \( \gamma' = \frac{f_2}{\mu_3(t_1 - t_3)} \). Hence \( \psi_{S_1}(r^3) = \frac{\mu_3(t_1 - t_3)}{f_2} \).

So we have that the inequality corresponding to \( S_1 \) is

\[
0 \cdot s_1 + \frac{\mu_2(t_1 + t_2)}{1-f_2} s_2 + \frac{\mu_3(t_1 - t_3)}{f_2} s_3 \geq 1.
\]

By very similar calculations, we can get the inequalities corresponding to \( \psi_{S_2} \) and \( \psi_{S_3} \). LP (27) becomes
\[ \begin{align*}
\text{min } & \ s_1 + s_2 + s_3 \\
0 \cdot s_1 + \frac{\mu_2(t_1 + t_2)}{1 - f_2} s_2 + \frac{\mu_3(t_1 - t_3)}{f_2} s_3 & \geq 1 \\
\frac{\mu_1(t_1 + t_2)}{1 - f_2} s_1 + 0 \cdot s_2 + \frac{\mu_3(-t_3 - t_2)}{f_2} s_3 & \geq 1 \\
\frac{\mu_1(t_1 - t_3)}{1 - f_2} s_1 + \frac{\mu_2(-t_3 - t_2)}{f_2} s_2 + 0 \cdot s_3 & \geq 1 \\
s & \in \mathbb{R}_+^3.
\end{align*} \tag{28} \]

As a sanity check, note that the assumption \(-t_1 < t_2 < -t_3\) implies that all the coefficients are nonnegative.

The following solution is feasible for LP (28):
\[ s_1 = \frac{1 - f_2}{\mu_1(t_1 - t_3)}, \quad s_2 = 0, \quad s_3 = \frac{f_2}{\mu_3(t_1 - t_3)} \quad \text{and} \quad s_1 + s_2 + s_3 = \frac{1}{t_1 - t_3} \left(\frac{1 - f_2}{\mu_1} + \frac{f_2}{\mu_3}\right). \]

Since the above LP was a strengthening of (23), we obtain
\[ z_{\text{SPLIT}} \leq s_1 + s_2 + s_3 = \frac{1}{t_1 - t_3} \left(\frac{1 - f_2}{\mu_1} + \frac{f_2}{\mu_3}\right). \]

If the rays are such that \(\mu_1 = \mu_3 = 1\), then the above expression is \(\frac{1}{t_1 - t_3}\). This implies that in this case if we have rays such that \((t_1 - t_3)\) tends to infinity, then \(z_{\text{SPLIT}}\) tends to 0.

### 8.3 Type 2 triangles that do much better than the split closure

In Section 8.2, we showed that we can bound the value of the split closure under mild conditions on \(f\) and the rays. In particular, we showed that as \(t_1 - t_3\) increases in value, the split closure does arbitrarily bad. In this section, we consider an infinite family of Type 2 triangles with rays pointing to its corners which satisfy these conditions.

Consider the same situation as in Section 8.2 and consider the Type 2 triangle \(T\) with the following three edges. The line parallel to the \(x_2\)-axis and passing through \((-1,0)\) supports one of the edges, and the other two edges are supported by lines passing through \((0,1)\) and \((0,0)\) respectively. See left part of Figure 13. Note that in this example, the rays are of the form \(r^1 = (-1,t_1), r^2 = \mu(1,t_2), r^3 = (-1,t_3)\). In the notation of Section 8.2, \(\mu_1 = \mu_3 = 1\).

**Theorem 8.6.** Given any \(\alpha > 1\), there exists a Type 2 triangle \(T\) as shown in Figure 13 such that for any point \(f\) in the relative interior of the segment joining \((0,0)\) to \((0,1)\), LP (23) has value \(z_{\text{SPLIT}} \leq \frac{1}{\alpha}\).

**Proof.** Let \(M = \lfloor \alpha \rfloor\). When the fractional point \(f\) is on the segment connecting \((0,0)\) and \((0,1)\), consider the triangle \(T\) with \(M\) integral points in the interior of the vertical edge (the triangle on the left in Figure 13). This implies \(t_1 - t_3 \geq M\). Therefore, from the result of Section 8.2, \(\mu_1 = \mu_3 = 1\) implies that \(z_{\text{SPLIT}} \leq \frac{1}{t_1 - t_3} \leq \frac{1}{\alpha}\). 

In this example, for any large constant \(\alpha\), optimizing over the split closure in the direction of the facet defined by these Type 2 triangles yields at most \(\frac{1}{\alpha}\). This implies Theorem 1.8.
8.4 More bad examples

The examples of Section 8.3 can be modified in various ways while keeping the property that the split closure is arbitrarily bad. The proofs are similar to that of Theorem 1.8.

8.4.1 Type 2 triangles when \( f \) is not on the segment joining \((0,0)\) to \((0,1)\)

The example of Section 8.3 can be generalized to the case where \( f \) is not on the segment connecting the points \((0,0)\) and \((0,1)\) as follows. Let \( T \) be a Type 2 triangle as shown on the right part of Figure 13. Let \( \Delta \) be the triangle with vertices \((0,0), (0,1)\) and the vertex \( x_2 \) of \( T \) with positive first coordinate. When the fractional point \( f \) is in the interior of triangle \( \Delta \), and triangle \( T \) has \( 2M \) integral points on its vertical edge, one can show that \( z_{\text{SPLIT}} \leq \frac{1}{M} \).

However, such bad examples cannot be constructed for any position of point \( f \) in the triangle \( T \). In particular, define the triangle \( \Delta' \) obtained from \( \Delta \) by a homothetic transformation with center \( x_2 \) and factor 2 (so one vertex of \( \Delta' \) is \( x_2 \) and points \((0,0)\) and \((0,1)\) become the middle points of the two edges of \( \Delta' \) with endpoint \( x_2 \)). When \( f \) is an interior point of \( T \) outside \( \Delta' \), it is easy to see that the split inequality obtained from the split parallel to the \( x_2 \)-axis \(-1 \leq x_1 \leq 0\) approximates the triangle inequality defined by \( T \) to within a factor at most 2. Indeed the linear program is

![Figure 13: Facets from Type 2 triangles with large gap versus the split closure](image-url)
\[ \min \ s_1 + s_2 + s_3 \]
\[ s_1 + \frac{f_1 - u}{f_1} s_2 + s_3 \geq 1 \]
\( s \in \mathbb{R}^3_+ \), \hspace{1cm} (29)

where \( u \) is the first coordinate of \( x^2 \). The optimal solution is \( s_1 = 0, s_2 = \frac{f_1}{f_1 - u}, s_3 = 0 \). Thus \( s_1 + s_2 + s_3 = \frac{f_1}{f_1 - u} \geq \frac{1}{2} \) since \( f_1 \leq -u \) for any \( f \in T \setminus \Delta' \). This implies that the split inequality approximates the triangle inequality by a factor at most 2 when \( f \) is outside \( \Delta' \).

\[ \bullet \]
\[ \bullet \]
\[ f \]
\[ r_1 \]
\[ r_2 \]
\[ r_3 \]

Figure 14: Facets from Type 3 triangles and quadrilaterals on which the split closure does poorly

8.4.2 Triangles of Type 3 and quadrilaterals

We now show how to modify the construction of Section 8.3 to get examples of Type 3 triangles and quadrilaterals that do arbitrarily better than the split closure.

To get a Type 3 triangle, we tilt the vertical edge of the triangle in Figure 13 around its integral point with minimum \( x_2 \)-value. See Figure 14. The same bound on \( z_{\text{SPLIT}} \) is then achieved.

Similarly, quadrilaterals can be constructed by breaking the vertical edge in Figure 13 into two edges of the quadrilateral. See Figure 14. By very similar arguments as in the previous section, we can show that \( z_{\text{SPLIT}} \) tends to 0.

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References

[1] K. Andersen, Q. Louveaux, R. Weismantel and L. A. Wolsey, Cutting Planes from Two Rows of a Simplex Tableau, *Proceedings of IPCO XII*, Ithaca, New York (June 2007), Lecture Notes in Computer Science 4513, 1–15.

[2] E. Balas, Intersection Cuts - A New Type of Cutting Planes for Integer Programming, *Operations Research* 19 (1971) 19–39.

[3] E. Balas, S. Ceria and G. Cornuéjols, A Lift-and-project Cutting Plane Algorithm for Mixed 0-1 Programs, *Mathematical Programming* 58 (1993) 295–324.

[4] E. Balas and A. Saxena, Optimizing over the Split Closure, *Mathematical Programming A* 113 (2008) 219–240.

[5] V. Borozan and G. Cornuéjols, Minimal Valid Inequalities for Integer Constraints, to appear in *Mathematics of Operations Research*.

[6] W. Cook, R. Kannan and A. Schrijver, Chvátal Closures for Mixed Integer Programming Problems, *Mathematical Programming* 47 (1990) 155–174.

[7] G. Cornuéjols and F. Margot, On the Facets of Mixed Integer Programs with Two Integer Variables and Two Constraints, to appear in *Mathematical Programming*.

[8] H. Crowder, E.L. Johnson and M. Padberg, Solving Large-Scale Zero-One Linear Programming Problems, *Operations Research* 31 (1983) 803–834.

[9] S. Dash, O. Günlük and A. Lodi, On the MIR Closure of Polyhedra, *Proceedings of IPCO XII*, Ithaca, New York (June 2007), Lecture Notes in Computer Science 4513, 337–351.

[10] S.S. Dey and L.A. Wolsey, Lifting Integer Variables in Minimal Inequalities Corresponding to Lattice-Free Triangles, *IPCO 2008*, Bertinoro, Italy (May 2008), Lecture Notes in Computer Science 5035, 463–475.

[11] D. Espinoza, Computing with multi-row Gomory cuts, *IPCO 2008*, Bertinoro, Italy (May 2008), Lecture Notes in Computer Science 5035, 214–224.

[12] M.X. Goemans, Worst-case Comparison of Valid Inequalities for the TSP, *Mathematical Programming* 69 (1995) 335–349.

[13] R.E. Gomory, An Algorithm for Integer Solutions to Linear Programs, *Recent Advances in Mathematical Programming*, R.L. Graves and P. Wolfe eds., McGraw-Hill, New York (1963) 269–302.

[14] R.E. Gomory, Thoughts about Integer Programming, 50th Anniversary Symposium of OR, University of Montreal, January 2007, and Corner Polyhedra and Two-Equation Cutting Planes, George Nemhauser Symposium, Atlanta, July 2007.
[15] L. Lovász, Geometry of Numbers and Integer Programming, *Mathematical Programming: Recent Developments and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989) 177–201.

[16] H. Marchand and L.A. Wolsey, Aggregation and Mixed Integer Rounding to Solve MIPs, *Operations Research* 49 (2001) 363–371.

[17] R.R. Meyer, On the Existence of Optimal Solutions to Integer and Mixed-Integer Programming Problems, *Mathematical Programming* 7 (1974) 223–235.

[18] G.L. Nemhauser and L.A. Wolsey, A Recursive Procedure to Generate All Cuts for 0-1 Mixed Integer Programs, *Mathematical Programming* 46 (1990) 379–390.