Fully-dynamic risk-indifference pricing and no-good-deal bounds

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Abstract

The seller’s risk-indifference price evaluation is studied. We propose a dynamic risk-indifference pricing criteria derived from a fully-dynamic family of risk measures on the $L_p$-spaces for $p \in [1, \infty]$. The concept of fully-dynamic risk measures extends the one of dynamic risk measures by adding the actual possibility of changing the risk perspectives over time. The family is then characterised by a double time index. Our framework fits well the study of both short and long term investments. In this dynamic framework we analyse whether the risk-indifference pricing criterion actually provides a proper convex price system, for which time-consistency is guaranteed. It turns out that the analysis is quite delicate and necessitates an adequate setting. This entails the use of capacities and an extension of the whole price system to the Banach spaces derived by the capacity seminorms.

Furthermore, we consider the relationship of the fully-dynamic risk-indifference price with no-good-deal bounds. Recall that no-good-deal pricing guarantees that not only arbitrage opportunities are excluded, but also all deals that are “too good to be true”. We shall provide necessary and sufficient conditions on the fully-dynamic risk measures so that the corresponding risk-indifference prices satisfy the no-good-deal bounds. The use of no-good-deal bounds also provides a method to select the risk measures and then construct a proper fully-dynamic risk-indifference price system in the $L_2$-spaces.

Key-words: convex prices, risk-indifference prices, time-consistency, extension theorems, dynamic risk measures, no-good-deal.

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1 Introduction

Risk-indifference pricing has been proposed as an alternative to utility indifference as a pricing approach in incomplete markets, see e.g. [28]. The idea is to replace the performance of a portfolio described in terms of the agent’s utility function with the performance based on the risk exposure measured by convex risk measures. Indeed, instead of maximising the utility, which is often quite difficult for a trader to write down explicitly, the risk-indifference criterion minimises the risk exposure. The point of connection between the two approaches is represented by the entropic risk measure which corresponds to an exponential-type utility. The price of a financial claim is then defined as the amount such that the risk exposure remains the same whether the optimal trading strategy includes the claim or not.

This pricing approach has been introduced on a fixed time horizon $T$ characterised by the expiration time of the claim $X$. The corresponding price $x_{0T}(X)$ itself is evaluated at the present time represented by 0. In this context a large part of the literature has focused on the techniques of stochastic control necessary to be able to compute the price of $X$, typically a financial derivative. The different techniques depend on the choice of risk measure, on the market dynamics of the fundamental traded underlying assets, and on the available information flow. See e.g. [16, 20, 24]. Again depending on the underlying price dynamics and the information flow available, one can extend some of these results to obtain the price evaluation for any time $s \in [0, T]$.

On the other side, [23] presents a systematic study of the risk-indifference price system $(x_{sT}(X))_s$ for essentially bounded claims $X \in L_\infty(F_T)$ and it also includes a study of time-consistency. Similar to the present paper, the study is kept free of the particular choice of underlying price dynamics. In [23] the evaluation of each price $x_{sT}(X)$ is linked with one element $\rho_s$ from the dynamic convex risk measure $(\rho_s)_s$. The risk measure $\rho_s$ evaluates at time $s$ the risks of financial positions at $T$.

However, particularly thinking of long time horizons, it seems reasonable to allow for the possibility of modelling changes in the risk evaluation criteria. For this reason, in this paper, we consider a fully-dynamic convex risk measure, that is a family of risk measures indexed by two points in time $(\rho_{st})_{s,t}$. One can regard a dynamic risk measure $(\rho_s)_s$ as corresponding to the elements $\rho_{sT}$ in the fully-dynamic risk measure $(\rho_{st})_{s,t}$.

The notion of time-consistency for a fully dynamic risk measure was introduced in [6] where a complete characterization of time consistency is given. We introduce here the concept of weak time-consistency for the fully-dynamic risk measure $(\rho_{st})_{s,t}$. Both concepts of time-consistency and weak time-consistency are compared with the meaning of time-consistency for the dynamic risk measure $(\rho_s)_s$, see e.g. [1]. The delicate relationships among
these concepts are detailed in the paper.

We define the risk-indifference price system \( (x_{st})_{s,t} \) starting from the fully-dynamic convex risk measures \( (\rho_{st})_{s,t} \), which are not assumed a priori to be normalised.

The claims and all financial positions belong to an \( L_p \)-space for \( p \in [1, \infty) \).

We study first the properties of \( x_{st} \) for each \( s \leq t \) to verify under what circumstances \( x_{st} \) is a proper convex price operator, and then we study the time-consistency of the whole family \( (x_{st})_{s,t} \) to obtain a convex price system.

Specifically, let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P), \ t \in [0, T])\) be a complete filtered probability space with \( T \in (0, \infty) \). We assume that \( \mathcal{F}_0 \) is trivial. For any \( p \in [1, \infty] \) and a \( \sigma \)-algebra \( \mathcal{B} \) of events of \( \Omega \), we consider the \( L_p(\mathcal{B}) := L_p(\Omega, \mathcal{B}, P) \) of real valued random variables with the finite norms:

\[
\|X\|_p := \begin{cases} 
(E[|X|^p])^{1/p}, & p \in [1, \infty), \\
\text{esssup}|X|, & p = \infty.
\end{cases}
\]

We denote the expectation under \( P \) by \( E \). If other measures are used they will be specified. For any time \( t \in [0, T] \), we consider the convex subset:

\[
L_t \subseteq L_p(\mathcal{F}_t) : \ L_t \subseteq L_T.
\]

The prices are in general defined on the domain \( L_t \) in the set of purchasable assets. Note that, in general, \( L_t \subseteq L_p(\mathcal{F}_t) \) for some \( t \in [0, T] \). In view of the financial application, it is justified to require that:

A1) \( 1 \in L_t \),

A2) for the \( \sigma \)-algebra \( \mathcal{F}_t \) we have the property that \( 1_A X \in L_t \) for every \( A \in \mathcal{F}_t \) and every \( X \in L_t \).

To take care of the value of money over time, we consider a numéraire of reference \( R_t, t \in [0, T] \), used as discounting factor. This refers to a money market account. In this work we set \( R_t \equiv 1 \) to simplify the notation as we are dealing with discounted values. Hence we shall just write about prices, though effectively, we refer to discounted prices.

**Definition 1.1.** For any \( s, t \in [0, T] \) with \( s \leq t \), the operator

\[
x_{st} : L_t \rightarrow L_s
\]

is a (convex) price operator if it is:

- monotone, i.e. for any \( X', X'' \in L_t \),

\[
x_{st}(X') \geq x_{st}(X''), \quad X' \geq X''.
\]
• convex, i.e. for any $X', X'' \in L_t$ and $\lambda \in [0, 1]$,
\[
    x_{st}(\lambda X' + (1 - \lambda)X'') \leq \lambda x_{st}(X') + (1 - \lambda)x_{st}(X'')
\]

• it has the Fatou property, i.e. for any $X \in L_t$ and any dominated sequence $(X_n)_n$ in $L_t$ (i.e. there exists $Y \in L_p(F_t)$ such that $|X_n| \leq Y$ for all $n$) which is converging $P$-a.s. to $X$, we have:
\[
    \liminf_{n \to \infty} x_{st}(X_n) \geq x_{st}(X)
\]

• weak $\mathcal{F}_s$-homogeneous, i.e. for all $X \in L_t$
\[
    x_{st}(1_A X) = 1_A x_{st}(X), \quad A \in \mathcal{F}_s,
\]

• projection property
\[
    x_{st}(X) = X, \quad X \in L_p(\mathcal{F}_s) \cap L_t.
\]

In particular we have $x_{st}(0) = 0$, $x_{st}(1) = 1$, and $x_{tt}(X) = X$, $X \in L_t$.

We remark that, if $p \in [1, \infty)$ and the operator $x_{st}$ is monotone and linear (as in [2] and [13]), the weak $\mathcal{F}_s$-homogeneity is equivalent to $\mathcal{F}_s$-homogeneity, i.e. for all $X \in L_t$, $x_{st}(\xi X) = \xi x_{st}(X)$ for all $\xi \in L_p(\mathcal{F}_s)$ such that $\xi X \in L_t$.

If $p = \infty$ and the operator is linear and semi-continuous, then the same result holds (see [8]).

For the whole family of price operators we introduce the concepts of time-consistency in its different forms.

**Definition 1.2.** The family of convex price operators $x_{st} : L_t \rightarrow L_s$, $s, t \in [0, T]$: $s \leq t$, is

• time-consistent if, for all $r, s, t \in [0, T]$: $r \leq s \leq t$, we have that
\[
    x_{rt}(X) = x_{rs}(x_{st}(X)), \quad X \in L_t
\]

• weak time-consistent if, for all $t \in [0, T]$,
\[
    x_{st}(X) \geq x_{st}(Y), \quad \text{for } X, Y \in L_t,
\]

implies that
\[
    x_{rt}(X) \geq x_{rt}(Y), \quad \text{for all } r \leq s \leq t.
\]

Note that time-consistency implies weak time-consistency.

Finally, a price system is defined as follows:
Definition 1.3. A (convex) price system \((x_{st})_{s,t}\) on \((L_t)_t\) is the family of convex price operators \((1.2)\) if it is at least weak time-consistent.

In our study we find that for \(p \in [1, \infty)\) under mild assumptions, a risk-indifference price operator \(x_{st}\) is a mapping from \(L_p(\mathcal{F}_t)\) into the space of \(\mathcal{F}_s\)-measurable random variables, but the operators take values in \(L_p(\mathcal{F}_s)\) only on a restricted domain \(L_t := \text{Dom} x_{st} \subseteq L_p(\mathcal{F}_t)\).

On the contrary, if \(p = \infty\), then the operator \(x_{st}\) is well-defined on \(L_\infty(\mathcal{F}_t)\) with values in \(L_\infty(\mathcal{F}_s)\). The study of the Fatou property and of time-consistency for the family \((x_{st})_{s,t}\) is then far from trivial when \(p < \infty\). We tackle the issue by extending the operators \(x_{st}\) from \(\text{Dom} x_{st} \subseteq L_p(\mathcal{F}_t)\) to an appropriate Banach space \(L_t^c\). To achieve such an extension, which maintains the same structure of a risk-indifference operator, we have first to extend the risk measures themselves. For this we make use of the notions of sensitive and dominated risk measures. The domination is inspired by sandwich extension theorems as in \([9]\) and it is used in \([10]\) to obtain a representation of risk measures. The sensitivity is linked to the relevance as introduced in \([23]\) and it allows to write the representation of risk measures in terms of \(P\)-equivalent probability measures. See \([19]\). The extended risk-indifference operator is obtained directly from the extended risk measures. It is a mapping defined on \(L_t^c\), it takes values in \(L_s^c\), and it fulfills the requirements to be a convex price operators. Furthermore, the family of operators \((x_{st})_{s,t}\) on \((L_t^c)_t\) is weakly time-consistent and thus it is a convex price system. By this we can conclude that working with risk indifference pricing from a dynamic perspective is actually a delicate affair and this should be kept in mind when studying techniques of computation of the prices themselves.

We complete the study showing that, for all \(t\) and \(X \in L_t^c\), the processes \(x_{st}(X), s \in [0, t]\), admit càdlàg modification. This also ensures that stopping times can be considered.

Working with the Banach spaces \((L_t^c)_t\) is mathematically effective, though not straightforward. For this reason in the last part of the paper we propose to work with no-good-deal bounds on prices, so to obtain a risk-indifference price system \((x_{st})_{s,t}\) with domain \((L_2(\mathcal{F}_t))_t\). No-good-deal bounds have been introduced in a dynamic setting in \([8]\). Here we study first the representation of convex no-good-deal prices and then we combine this with the risk-indifference prices.

Our final result gives a characterisation of those risk measures for which the the risk-indifference prices are actually no-good-deal.
2 Risk-indifference pricing

In this section we are studying dynamic risk-indifference prices in the $L_p$ setting, $p \in [1, \infty]$. As we shall see, for the fixed times $s, t : s \leq t$, each risk-indifference price evaluation is substantially a convex price operator. However, we have to be cautious with the definition in order to guarantee the Fatou property and then the time-consistency of the whole price system. This section is divided in three parts. In the first we introduce and study properties of a family of risk measures $(\rho_{st})_{s,t}$ indexed by time to achieve a fully-dynamic structure. Then we study the corresponding risk-indifference price evaluations $(x_{st})_{s,t}$. Here we discuss the very definition of $x_{st}$ both in terms of domain and range. We shall see that while in the case $p = \infty$, $x_{st}$ is a well-defined convex price, for the case $p < \infty$ we have to introduce a new framework beyond the $L_p$-setting. It is only in this new framework that we can show that the risk-indifference evaluations $x_{st}$ have all the properties to be a convex price operator Definition 1.1. In particular we obtain the Fatou property and the dual representation of $x_{st}$. In the last parts of this section we study the time-consistency of the family $(x_{st})_{s,t}$ and the regularity of the trajectories.

2.1 Fully-dynamic risk measures, domination, sensitivity.

At first we introduce the notion of fully-dynamic risk measure. This object was already considered in [6] under the name of dynamic risk measure. Then we review the different concepts of time-consistency and we deal with the notions of domination and sensitivity.

For any two points of time $s \leq t$, the mapping $\rho_{st} : L_p(\mathcal{F}_t) \rightarrow L_p(\mathcal{F}_s)$ satisfies the following properties:

- **monotonicity**, i.e. $\rho_{st}(X') \leq \rho_{st}(X'')$, for any $X' \geq X''$ in $L_p(\mathcal{F}_t)$,

- **convexity**, i.e. for any $X', X'' \in L_p(\mathcal{F}_t)$ and $\lambda \in [0, 1]$, $\rho_{st}(\lambda X' + (1 - \lambda)X'') \leq \lambda \rho_{st}(X') + (1 - \lambda)\rho_{st}(X'')$,

- **$\mathcal{F}_s$-translation invariance**, i.e. for any $X \in L_p(\mathcal{F}_t)$, $\rho_{st}(X + f) = \rho_{st}(X) - f$, for all $f \in L_p(\mathcal{F}_s)$.

Moreover,

- If $p = \infty$, then we assume that continuity from below holds with convergence point-wise, i.e. for any $X \in L_\infty(\mathcal{F}_t)$ and any sequence $(X_n)_n$ such that $X_n \uparrow X$ P-a.s., then $\rho_{st}(X_n) \downarrow \rho_{st}(X)$ P-a.s.

If $p < \infty$, we recall that the risk measure $\rho_{st}$ is always continuous from below both in the $L_p$-convergence and P-a.s. (see e.g. [4], [17]).
We stress that we do not assume a priori that the risk measures $\rho_{st}$ are normalised.

We observe that the risk measure $\rho_{st}$ satisfies weak $\mathcal{F}_s$-homogeneity, i.e.

$$1_A \rho_{st}(X) = 1_A \rho_{st}(1_A X), \quad X \in L_p(\mathcal{F}_t), \quad A \in \mathcal{F}_s. \quad (2.1)$$

Indeed for $p = \infty$, monotonicity and translation invariance imply (2.1) for all $X \in L_\infty(\mathcal{F}_t)$. See [11] Proposition 3.3 and [23] Section 3. In the case $p < \infty$, the weak $\mathcal{F}_s$-homogeneity for all $X \in L_p(\mathcal{F}_t)$ is then guaranteed by the continuity from below.

For any $s \leq t$ the risk measure above admits the following representation:

$$\rho_{st}(X) = \operatorname{essmax}_{Q \in P: \mathcal{Q}_s = P|\mathcal{F}_s} \left( E_Q[-X|\mathcal{F}_s] - \alpha_{st}(Q) \right),$$

$$= \operatorname{essmax}_{Q \in P: E_P[\alpha_{st}(Q)] < \infty} \left( E_Q[-X|\mathcal{F}_s] - \alpha_{st}(Q) \right), \quad X \in L_p(\mathcal{F}_t), \quad (2.2)$$

where $\alpha_{st}$ is the minimal penalty, i.e.

$$\alpha_{st}(Q) := \operatorname{esssup}_{X \in L_p(\mathcal{F}_t)} \left( E_Q[-X|\mathcal{F}_s] - \rho_{st}(X) \right). \quad (2.3)$$

The extreme value in representation (2.2) is achieved thanks to the continuity from below. See [5] for this result written in the case $p = \infty$. The case $p < \infty$ is proved in a similar fashion. See also [14] for the a representation result with esssup.

**Lemma 2.1.** Given the representation (2.2), let $Q$ be a probability measure such that $E(\alpha_{st}(Q)) < \infty$. Then we have

$$\alpha_{st}(Q) := \operatorname{esssup}_{X \in L_p(\mathcal{F}_t)} \left( E_Q(-X|\mathcal{F}_s) - \rho_{st}(X) \right)$$

$$= \operatorname{esssup}_{X \in L_p(\mathcal{F}_t)} \left( E_Q(-X|\mathcal{F}_s) - \rho_{st}(X) \right). \quad (2.4)$$

**Proof.** The first equation is just the definition of the minimal penalty $\alpha_{st}(Q)$. It is well known that $\alpha_{st}(Q) = \operatorname{esssup}_{X \in \mathcal{A}_{st}^P} E_Q(-X|\mathcal{F}_s)$ where $\mathcal{A}_{st}^P$ is the acceptance set for $\rho_{st}$, i.e., $\mathcal{A}_{st}^P := \{ X \in L_p(\mathcal{F}_t) | \rho_{st}(X) \leq 0 \}$. See [14]. Let $X \in \mathcal{A}_{st}^P$. For all $n > 0$, let $X_n = \sup(X, -n)$. Every $X_n$ belongs to $\mathcal{A}_{st}^P \subseteq L_p(\mathcal{F}_t)$. The sequence $-X_n$ is increasing to $-X$, it follows from the monotone convergence theorem that $E_Q(-X_n|\mathcal{F}_s)$ is increasing to $E_Q(-X|\mathcal{F}_s)$. Then the continuity from above of $\rho_{st}$ (which is a consequence of the representation of $\rho_{st}$) allows to restrict the evaluation of $\operatorname{esssup}_{X \in L_p(\mathcal{F}_t)} [E_Q(-X|\mathcal{F}_s) - \rho_{st}(X)]$ to the elements $X$ bounded from below. So, let $X \in L_p(\mathcal{F}_t)$ be bounded from below by $C$. Then $X$ is the increasing limit of the sequence of bounded random variables $Y_n = \inf(X, n)$. 

$$7$$
Moreover, $|Y_n| \leq \sup(|C|,X)$. The hypothesis $E(\alpha_{st}(Q)) < \infty$ implies that $Q \ll P$ with a density in $L_q(\mathcal{F}_t)$, where $q$ is the conjugate of $p$. It follows from the dominated convergence theorem for conditional expectations that $E_Q(-X|\mathcal{F}_s) = \lim_{n \to \infty} E_Q(-Y_n|\mathcal{F}_s)$. On the other hand, $\rho_{st}$ is continuous from below, thus $\rho_{st}(X) = \lim_{n \to \infty} \rho_{st}(Y_n)$. This proves that

$$E_Q(-X|\mathcal{F}_s) - \rho_{st}(X) \leq \text{esssup}_{X \in L_\infty(\mathcal{F}_t)} [E_Q(-X|\mathcal{F}_s) - \rho_{st}(X)].$$

The result follows easily.

As announced, in this paper we consider a family of convex risk measures $(\rho_{st})_{s,t}$ that is fully-dynamic. This means that we consider a family indexed by two points in time. In this way we allow for the possibility of modelling changes in the rules of the evaluation along with time. This is particularly reasonable whenever the time horizon $T$ is large.

**Definition 2.2.** A fully-dynamic risk measure on $L_p$ is

- time-consistent if for all $r, s, t \in [0, T] : r \leq s \leq t$, we have
  $$\rho_{rt}(X) = \rho_{rs}(-\rho_{st}(X)), \quad X \in L_p(\mathcal{F}_t),$$

- weak time-consistent if for all $r, s, t \in [0, T] : r \leq s \leq t$, for all $X, Y \in L_p(\mathcal{F}_t)$, we have
  $$\rho_{st}(X) = \rho_{st}(Y) \implies \rho_{rt}(X) = \rho_{rt}(Y).$$

**Proposition 2.3.** Let $\rho_{st} : L_p(\mathcal{F}_t) \rightarrow L_p(\mathcal{F}_s), s, t \in [0, T] : s \leq t$. The following assertions are equivalent:

i) The fully-dynamic risk measure $(\rho_{st})_{s,t}$ is time-consistent.

ii) The fully-dynamic risk measure $(\rho_{st})_{s,t}$ is weak time-consistent and

$$\rho_{rt}(Y) = \rho_{rs}(Y - \rho_{st}(0)), \quad 0 \leq r \leq s \leq t, \quad Y \in L_p(\mathcal{F}_s). \quad (2.5)$$

**Proof.** To see that (i) implies (ii), it is enough to use the translation invariance property of $\rho_{st}$.

Conversely, assume that (ii) is satisfied. Let $Z \in L_p(\mathcal{F}_t)$ and define $Y := \rho_{st}(0) - \rho_{st}(Z)$. From the translation invariance property of $\rho_{st}$ follows that $\rho_{st}(Y) = \rho_{st}(Z)$. Thus, from weak time-consistency, $\rho_{rt}(Y) = \rho_{rt}(Z)$. Applying (2.5) to $Y$, we get that $\rho_{rt}(Z) = \rho_{rs}(-\rho_{st}(Z))$. □

In the case when the risk measures $(\rho_{st})_{s,t}$ are normalised, the above result has an easy interpretation.
Corollary 2.4. Assume that the fully-dynamic risk measure \((\rho_{st})_{s,t}\) is normalised i.e. \(\rho_{st}(0) = 0\), for all \(s \leq t\). Then the fully-dynamic risk measure \((\rho_{st})_{s,t}\) is time-consistent if and only if it is weak time-consistent and satisfies the restriction property, i.e. for all \(0 \leq r \leq s \leq t\),

\[
\rho_{rt}(Y) = \rho_{rs}(Y), \quad Y \in L_p(\mathcal{F}_s). \tag{2.6}
\]

Remark 2.5. In most of the literature a so-called dynamic risk measure \((\rho_s)_s\) is characterised by only one time index. See e.g. [1], [23]. In this case the risk measure refers to the evaluation of positions at a fixed time horizon and this corresponds to \(\rho_s := \rho_{sT}\). For this dynamic risk measure, time-consistency is given as

\[
\rho_t(X) = \rho_t(Y) \implies \rho_s(X) = \rho_s(Y), \quad s \leq t, \ X, Y \in L_p(\mathcal{F}_T).
\]

From the corollary above, we can see that if the fully-dynamic risk measure \((\rho_{st})_{s,t}\) is normalised, there is a one-to-one correspondence between the time-consistent \((\rho_s)_s\) and the time-consistent \((\rho_{st})_{s,t}\). In fact,

\[
\rho_{st} = \rho_{sT} = \rho_s, \quad t \geq s.
\]

We stress that the result is not true if \(\rho_{st}\) is not normalised.

Time-consistent dynamic and fully-dynamic risk measures can be constructed by BSDE methods, see e.g. [25], [27], [26], and also from the cocycle characterisation of the time-consistency, see [6] and [7].

To summarise, in this work we consider a time-consistent fully-dynamic risk measure \((\rho_{st})_{s,t}\) not necessarily normalised. Also we do not assume the restriction property (2.6).

Hereafter we consider the concept of domination for a risk measure, first introduced in [10]. This is going to be crucial for defining the correct framework to study risk-indifference price operators.

Definition 2.6. Fix \(p < \infty\). Let \(\rho : L_p(\mathcal{F}_T) \to \mathbb{R}\) be a convex risk measure continuous from below. The risk measure \(\rho\) is dominated if there exists a sublinear (or coherent) risk measure \(\tilde{\rho} : L_p(\mathcal{F}_T) \to \mathbb{R}\) such that

\[
\rho(X) - \rho(0) \leq \tilde{\rho}(X), \quad X \in L_p(\mathcal{F}_T).
\]

In [10] the property of domination is characterised and it is proved that it guarantees a representation result. The following proposition summarises these findings.

Proposition 2.7. Fix \(p \in [1, \infty)\). Let \(\rho : L_p(\mathcal{F}_T) \to \mathbb{R}\) be a convex risk measure. The following statements are equivalent:

1. The risk measure \(\rho\) is dominated.
2. There exist \( K > 0 \) and \( C \in \mathbb{R} \) such that \( \rho(X) \leq K \|X\|_p + C \), for all \( X \in L_p(\mathcal{F}_T) \).

3. The risk measure \( \rho \) admits representation

\[
\rho(X) = \max_{Q \in \mathcal{B}^K} (E_Q(-X) - \alpha(Q)), \quad X \in L_p(\mathcal{F}_t),
\]

where \( \mathcal{B}^K \) is the set of probability measures on \( \mathcal{F}_T \):

\[
\mathcal{B}^K := \{ Q \ll P : \frac{dQ}{dP} \in L_q(\mathcal{F}_T), \|\frac{dQ}{dP}\|_q \leq K \}, \quad q = p(p - 1)^{-1}.
\]

Proof. The result is directly retrieved from Proposition 3.1 of [10] by applying the normalised convex risk measure \( \rho(X) - \rho(0) \) with the capacity \( c(X) = \|X\|_p \).

The above representation is known in the case of sublinear (or coherent) risk measures, see [17] and [22]. Also a representation result for finite convex risk measures on \( L_p \) (\( p < \infty \)) is given in [22] Theorem 2.11. This proof is unfortunately based on a wrong statement (see Proposition 2.10 in [22]).

Note that, if a convex risk measure \( \rho \) is dominated, then we have the sandwich:

\[-\bar{\rho}(-X) \leq \rho(X) - \rho(0) \leq \bar{\rho}(X), \quad X \in L_p(\mathcal{F}_T),\]

where \(-\bar{\rho}(-X)\) is superlinear and \( \bar{\rho} \) is sublinear.

The sandwich relationship above provides a motivation itself for the use of the property of domination. This is in view of the link with the extension theorem of time-consistent convex operators satisfying a sandwich condition as studied in [9]. Indeed if we consider a time-consistent family of operators \((\rho_{st})_{s,t}\) defined on the vector subspaces \((L_t)\), with \( L_t \subseteq L_p(\mathcal{F}_t) \) and if each \( \rho_{st} \) satisfies a sandwich condition and the Fatou property, then we can extend this family \((\rho_{st})_{s,t}\) to the whole \((L_p(\mathcal{F}_t))_t\).

Now we consider the sensitivity of a risk measure. This concept yields a representation of the risk measure in terms of probability measures equivalent to \( P \). See e.g. Section 3 in [23].

**Definition 2.8.** Let \( \rho : L_p(\mathcal{F}_T) \to \mathbb{R} \) be a convex risk measure.

- The risk measure \( \rho \) is strong sensitive or relevant (to \( P \)), if

\[ \rho(1_B) < \rho(0), \]

for all \( B \in \mathcal{F}_T \) such that \( P(B) > 0 \).

- The risk measure \( \rho \) is sensitive (to \( P \)) if there exists a probability measure \( \tilde{Q} \sim P \) such that \( \alpha(\tilde{Q}) < \infty \), where \( \alpha \) is the minimal penalty associated to \( \rho \).
Remark 2.9. The property of relevance implies sensitivity. This follows from Lemma 3.4 in [23] applied to \( \Phi(X) = -\rho(X) \), restricted to \( L_\infty(F_T) \).

Proposition 2.10. Fix \( p < \infty \). Let \( \rho : L_p(F_T) \rightarrow \mathbb{R} \) be a convex risk measure continuous from below and sensitive, then the convex risk measure \( \rho \) admits representation

\[
\rho(X) = \sup_{Q \in \mathcal{Q}} \left( E_Q(X) - \alpha(Q) \right), \quad X \in L_p(F_T),
\]

(2.8)

where

\[
\mathcal{Q} := \{ Q \sim P : \alpha(Q) < \infty \}.
\]

Moreover, if \( \rho \) is dominated, then

\[
\mathcal{Q} \subseteq \mathcal{B}^K
\]

where \( K \) is the constant in the domination property and \( q = p(p - 1)^{-1} \).

Proof. The representation is a direct consequence of Theorem 3.1 in [23]. Let \( Q \in \mathcal{Q} \), then the domination implies that \( E_Q(X) \leq \alpha(Q) + K \|X\|_p + C \). Applying this to \( \lambda X \) for all \( \lambda > 0 \) we have

\[
E_Q[\lambda X] \leq \frac{\alpha(Q) + C}{\lambda} + K \|X\|_p.
\]

By taking \( \lambda \rightarrow \infty \) we conclude.

2.2 A risk-indifference price operator \( x_{st} \).

Risk-indifferent pricing was introduced in a static set-up as an alternative pricing technique to utility-indifference pricing in incomplete markets. Instead of considering the agents’ attitude to a financial investment in terms of utility functions, the risk-indifference approach uses risk measures. The point of connection between these two approaches is given by the fact that utility-indifference with exponential utility function corresponds substantially to a risk-indifference pricing with entropic risk measure.

Hereafter we consider the whole family of risk-indifference prices \( (x_{st})_{s,t} \) generated by a time-consistent fully-dynamic risk measure \( (\rho_{st})_{s,t} \) on \( L_p(F_t) \), \( 1 \leq p \leq \infty \). Risk-indifference pricing is also studied in [23] for the case of dynamic risk measures \( (\rho_t) \) as in Remark 2.5 and in the case \( p = \infty \) only. Our first goal is to identify the conditions under which a risk-indifferent price evaluation \( x_{st} \), associated to \( \rho_{st} \), satisfies the properties for being a convex price operator.
First of all fix $s, t \in [0, T]: s \leq t$. From a qualitative perspective, the risk-
indifference (seller’s discounted) price $x_{st}(X)$ at time $s$ for any (discounted)
financial position $X$, at time $t$, is given by the equation:

$$
\text{essinf}_{\theta \in \Theta_{st}} \rho_{st}(x_{st}(X) + Y_{st}(\theta) - X) = \text{essinf}_{\theta \in \Theta_{st}} \rho_{st}(y_s + Y_{st}(\theta)) \quad P-a.s., \quad (2.9)
$$

where the $\mathcal{F}_s$-measurable $y_s$ represents the money market account and, to-
gether with the price $x_{st}(X)$, is the initial capital at $s$. The $\mathcal{F}_t$-measurable
$Y_{st}(\theta)$ represents the value of an admissible portfolio $\theta$ on the time horizon
$(s, t)$ and the set $\Theta_{st}$ represents the admissible portfolios. Under suitable
integrability conditions, the equation above can be rewritten as

$$
x_{st}(X) = \text{essinf}_{\theta \in \Theta_{st}} \rho_{st}(Y_{st}(\theta) - X) - \text{essinf}_{\theta \in \Theta_{st}} \rho_{st}(Y_{st}(\theta)) \quad (2.10)
$$

by the translation invariance of $\rho_{st}$. Here below we are more specific about
portfolios and value processes, so to achieve a general definition of risk-
indifference price.

The market is characterised by a number of underlying assets, whose (dis-
counted) price is given by $(\Pi_t)_{t \in [0, T]}$ which is a $\mathcal{F}$-adapted locally bounded
semimartingale in $\mathbb{R}^d$.

**Definition 2.11.** Denote $\Xi$ the set of all $\mathcal{F}$-predictable processes $(\theta_t)_{t \in [0, T]}$
with values in $\mathbb{R}^d$, integrable with respect to $(\Pi_t)_{t \in [0, T]}$, and such that for all
$0 \leq s \leq T$, the integral process $(Y_{st}(\theta))_{t \in [0, T]}$ is $\mathcal{F}$-adapted and bounded from
below.

**Definition 2.12.** The set of admissible strategies on $[0, T]$ is constituted
by a subset $\Theta := \Theta_{0T} \subseteq \Xi$ satisfying the stability property: for any $A \in \mathcal{F}_s$
and any $\theta^{(1)}, \theta^{(2)}, \theta^{(3)} \in \Theta$ the strategy $\theta = (\theta_t)_t$ given by

$$
\theta_t := \begin{cases} 
\theta_t^{(1)}, & t \in (0, s] \\
1_A \theta_t^{(2)} + 1_A^c \theta_t^{(3)}, & t \in (s, T]
\end{cases}
$$

belongs to $\Theta$. Moreover we consider $0 \in \Theta$, $Y_{st}(0) = 0$.
For any $s \leq t$, the set $\Theta_{st}$ of admissible strategies on $(s, t]$ is constituted by
all strategies $\theta_1(s, t]$ with $\theta \in \Theta$.

Clearly if $\Theta = \Xi$, then the stability property is naturally satisfied. Our
choice to consider $\Theta \subseteq \Xi$ allows for a framework where it is possible to
consider exogenous constrains on the applicable strategies.

**Definition 2.13.** For $p \in [1, \infty]$, the sets of feasible claims $(C^p_{st})_{s,t}$ are
defined by

$$
C^p_{st} := \{ g \in L_p(\mathcal{F}_t) : \exists \theta \in \Theta \text{ such that } g \leq Y_{st}(\theta) \}.
$$
Note that $0 \in C^p_{st}$.

For the risk-indifference price to be well defined, we introduce the following technical assumption, standing for this paper.

Assumption:

$$\essinf_{g \in C^p_{st}} \rho_{st}(g) > -\infty \quad P - a.s. \quad (2.11)$$

Motivated by the above considerations we give the following definition.

**Definition 2.14.** Let $p \in [1, \infty]$. For any $s, t \in [0, T] : s \leq t$, the operator

$$x_{st}(X) := \essinf_{g \in C^p_{st}} \rho_{st}(g - X) - \essinf_{g \in C^p_{st}} \rho_{st}(g) \quad (2.12)$$

is well-defined $P$-a.s. for all $X \in L_p(F_t)$ as an $F_s$-measurable random variable. We call $x_{st}(X)$ the risk-indifference price of $X$ given by $\rho_{st}$ when the operator $x_{st}$ is considered on

$$\Dom x_{st} := \{ X \in L_p(F_t) : x_{st}(X) \in L_p(F_s) \}.$$

**Remark 2.15.** Due to the stability property of $\Theta$, the set $\{ \rho_{st}(g - X), g \in C^p_{st} \}$ has the lattice property for all $X \in L_p(F_t)$.

**Lemma 2.16.** Let $p \in [1, \infty]$.

1. If $\essinf_{g \in C^p_{st}} \rho_{st}(g)$ belongs to $L_p(F_t)$, then $\Dom x_{st} = L_p(F_t)$.

2. For all $s \leq t$, we have that, for all $X \in L_\infty(F_t)$, $x_{st}(X)$ belongs to $L_\infty(F_s)$. In particular, $L_\infty(F_t) \subseteq \Dom x_{st}$.

**Proof.** For all $X \geq 0$, we have

$$0 \leq x_{st}(X) = \essinf_{g \in C^p_{st}} \rho_{st}(g - X) - \essinf_{g \in C^p_{st}} \rho_{st}(g) \leq \rho_{st}(-X) - \essinf_{g \in C^p_{st}} \rho_{st}(g) \in L_p(F_s).$$

From the monotonicity, the convexity of $x_{st}$, and $x_{st}(0) = 0$, we obtain that, for all $X \in L_p(F_t)$,

$$-x_{st}(|X|) \leq -x_{st}(-X) \leq x_{st}(X) \leq x_{st}(|X|).$$

Then $|x_{st}(X)| \leq x_{st}(|X|)$ and $x_{st}(X) \in L_p(F_s)$. By this we have proved 1.

Now, let $X \in L_\infty(F_t)$, then there are two real numbers $N, M$ such that $N \leq X \leq M$. It follows easily from the translation invariance property that for all $s, N \leq x_{st}(X) \leq M$. This yields 2. \qed
In particular we stress that, for any \( p \in [1, \infty] \), the operator \( x_{st} \) in (2.12) restricted to \( L_\infty(\mathcal{F}_t) \) is always well-defined with values in \( L_\infty(\mathcal{F}_s) \).

We give now an alternative formula for \( x_{st} \).

**Lemma 2.17.** For all \( s \leq t \), for all \( X \in L^p(\mathcal{F}_t) \), we have

\[
\inf_{g \in C_{st}^p} \rho_{st}(g - X) = \inf_{g \in C_{st}^\infty} \rho_{st}(g - X),
\]

where

\[
C_{st}^\infty := \{ g \in L_\infty(\mathcal{F}_t) : \exists \theta \in \Theta \text{ such that } g \leq Y_{st}(\theta) \}.
\]

Then, for all \( X \in L^p(\mathcal{F}_t) \),

\[
x_{st}(X) = \inf_{g \in C_{st}^\infty} \rho_{st}(g - X) - \inf_{g \in C_{st}^\infty} \rho_{st}(g). \tag{2.13}
\]

**Proof.** Since \( C_{st}^\infty \subset C_{st}^p \), then we have that, for all \( X \in L_p(\mathcal{F}_t) \),

\[
\inf_{g \in C_{st}^p} \rho_{st}(g - X) \leq \inf_{g \in C_{st}^\infty} \rho_{st}(g - X).
\]

Let \( g_0 \in C_{st}^p \). Let \( \theta \in \Theta \) such that \( g_0 \leq Y_{st}(\theta) \). From Definition 2.11 we have that there is \( C > 0 \) and \( Y_{st}(\theta) \geq -C \). It follows that \( g' = \sup\{g_0, -C\} \) satisfies \( g_0 \leq g' \leq Y_{st}(\theta) \) and \( |g'| \leq \sup(C, |g_0|) \). Thus \( g' \in C_{st}^p \) and \( \rho_{st}(g_0 - X) \geq \rho_{st}(g' - X) \).

The random variable \( g' \) is bounded from below and thus it is the increasing limit of the sequence \( g'_n = \inf\{g', n\} \). Observe that \( g'_n \in L_\infty(\mathcal{F}_t) \), \( g'_n \leq Y_{st}(\theta) \).

The continuity from below of \( \rho_{st} \) yields \( \rho_{st}(g' - X) = \lim_{n \to \infty} \rho_{st}(g'_n - X) \).

This gives

\[
\rho_{st}(g_0 - X) \geq \inf_{g \in C_{st}^\infty} \rho_{st}(g - X).
\]

The proof is complete. \( \square \)

Hereafter we study the properties of the risk-indifference price operator \( x_{st} \) for fixed \( s \leq t \).

**Proposition 2.18.** The operator \( x_{st}(X) \), \( X \in L_p(\mathcal{F}_t) \), is monotone, convex, it has the projection property, it is weak \( \mathcal{F}_s \)-homogeneous, and it is continuous from above.

**Proof.** The monotonicity and convexity of \( x_{st} \) follow from the corresponding properties of \( \rho_{st} \). The projection property of \( x_{st} \) follows from the \( \mathcal{F}_s \)-translation invariance of \( \rho_{st} \).
The weak $F_s$-homogeneity is justified as follows. Let $A \in F_s$, then

$$1_A x_{st}(X) = 1_A [\inf_{g \in C^p_{st}} \rho_{st}(g - X) - \inf_{g \in C^p_{st}} \rho_{st}(g)]$$

$$= \inf_{g \in C^p_{st}} 1_A \rho_{st}(g - X) - \inf_{g \in C^p_{st}} 1_A \rho_{st}(g)$$

$$= \inf_{g \in C^p_{st}} 1_A \rho_{st}(g - 1_A X) - \inf_{g \in C^p_{st}} 1_A \rho_{st}(g)$$

$$= 1_A x_{st}(1_A X),$$

where the third inequality comes from the weak $F_s$-homogeneity of $\rho_{st}$ as follows:

$$1_A \rho_{st}(g - X) = 1_A \rho_{st}(1_A (g - X)) = 1_A \rho_{st}(1_A (g - 1_A X)) = 1_A \rho_{st}(g - 1_A X).$$

Finally, recall that for every $0 \leq s \leq t \leq T$, the risk measure $\rho_{st}$ is continuous from below. Let $(X_n)_n$ in $L^p(F_t)$ be a sequence decreasing to $X \in L^p(F_t)$. Then we have that $\rho_{st}(g - X)$ is the decreasing limit of $\rho_{st}(g - X_n)$, for $n \to \infty$. Hence,

$$x_{st}(X) = \inf_{n} \inf_{g \in C^p_{st}} \rho_{st}(g - X_n) - \inf_{g \in C^p_{st}} \rho_{st}(g)$$

$$= \inf_{n} \left[ \inf_{g \in C^p_{st}} \rho_{st}(g - X_n) - \inf_{g \in C^p_{st}} \rho_{st}(g) \right]$$

$$= \inf_{n} x_{st}(X_n).$$

The monotonicity of $x_{st}$ implies that $x_{st}(X)$ is the decreasing limit of $x_{st}(X_n)$. The continuity from above of $x_{st}$ is then proved. \qed

In the last part of this subsection we study the Fatou property for the risk-indifference price operator $x_{st}$. For this we shall distinguish the two cases when $p = \infty$ and $p \in [1, \infty)$. We have to recall that

$$L^\infty(F_t) \subseteq \text{Dom} x_{st} \subseteq L^p(F_t), \quad p \geq 1,$$

from Lemma 2.16 item 2.

**Proposition 2.19.** Let $p = \infty$. The risk-indifference price operator $x_{st}$ admits the following representation

$$x_{st}(X) = \sup_{Q \ll P, Q|F_s = P} (E_Q(X|F_s) - \gamma_{st}(Q)), \quad X \in L^\infty(F_t),$$

(2.15)

where $\gamma_{st}(Q)$ is the minimal penalty:

$$\gamma_{st}(Q) = \sup_{X \in L^\infty(F_t)} (E_Q(X|F_s) - x_{st}(X)).$$

In particular $x_{st}$ has the Fatou property and is continuous on $L^\infty(F_t)$. \hfill 15
Proof. The representation follows from the continuity from above, see Proposition 2.18. Indeed we refer to the dual representation for conditional risk measures as in [14] or in [5]. This dual representation is written for the conditional risk measure $x_{st}(-X)$. The second assertion follows from the representation (2.15) itself and [14].

Remark 2.20. If the operator $x_{st}$ was considered on a Fréchet lattice $L_t \supseteq L_{\infty}(\mathcal{F}_t)$, then one could obtain the Fatou property by applying the extension of the Namioka-Klee theorem, see [7], to the functional $E[x_{st}(X)]$, $X \in L_t$. We have to remark that $\text{Dom} \ x_{st}$ is not a Fréchet lattice, in fact it is not complete and also in general we have $\Theta \subseteq \Xi$ hence it may not even be a vector space.

As Remark 2.20 shows the study of the Fatou property (and also time-consistency as we shall see later) is delicate and the major issues are related to the domain of the operators involved. This is again noticed in e.g. [15] in the context of forward utilities.

In view of the remark above, we here propose a different approach to study the Fatou property for $p \in [1, \infty)$. For all $s, t \in [0, T] : s \leq t$, we shall introduce an extension of the operator $x_{st}$ by constructing an adequate extension of $\rho_{st}$. For this, when $p \in [1, \infty)$, we assume that $\rho_{0T}$ is dominated and sensitive.

We introduce the seminorm

$$c(X) := \sup_{Q \in \mathcal{Q}} E_Q(|X|),$$

for all $\mathcal{F}_T$ measurable random variables $X$ and

$$\mathcal{Q} := \{Q \sim P : \alpha_{0T}(Q) < \infty\}$$

where $\alpha_{0T}$ is the minimal penalty of $\rho_{0T}$ (see Proposition 2.10). We observe that

$$c(X) = 0 \iff X = 0 \ P-a.s.$$ 

This follows directly from $Q \sim P$ in the definition of $\mathcal{Q}$. Then $c$ induces a relationship of equivalence among random variables.

Definition 2.21. For all $t$, we define $L^c_t$ the completion, with respect to the seminorm $c$, of the set of essentially bounded $\mathcal{F}_t$-measurable random variables. We define $L^c_t := L^c_t / \sim$.

The space $L^c_t$ is a Banach space with norm $c$.

Lemma 2.22. For all $t$, the following relationship holds for all $Q \in \mathcal{Q}$:

$$L_p(\mathcal{F}_t, P) \subseteq L^c_t \subseteq L_1(\mathcal{F}_t, Q).$$
Proof. The relationship is directly proved from Proposition 2.10 in fact $\mathbb{E}_Q(|X|) \leq c(X) \leq K\|X\|_p$, for all $Q \in \mathcal{Q}$. \hfill \Box

In the sequel, we extend the risk measure $\rho_{st}$ to obtain the map:

$$\tilde{\rho}_{st} : L^c_t \rightarrow L^c_s.$$ 

With this we can extend the corresponding risk-indifference price $x_{st}$ as a map:

$$x_{st} : L^c_t \rightarrow L^c_s.$$ 

It is with this operator that we can study the Fatou property. Also we shall see that the extensions above are instrumental in the study of time-consistency for the price system.

The construction of the extension of $\rho_{st}$ is engaging and it requires several steps. First of all we obtain the following lemmas.

**Lemma 2.23.** For fixed $t \in [0,T]$, consider a convex risk measure $\psi_{tT} : L^\infty(F_T) \rightarrow L^\infty(F_t)$ continuous from below. Let $\beta_{tT}$ be its minimal penalty. Assume that there exists a probability measure $Q \sim P$ such that $\mathbb{E}_Q(\beta_{tT}(Q)) < \infty$. Then the following representation holds:

$$\psi_{tT}(X) = \esssup_{R \sim P, \beta_{tT}(R) \in L^\infty(F_t)} (E_R(-X|F_t) - \beta_{tT}(R)), \quad X \in L^\infty(F_T).$$

(2.16)

**Proof.** The proof is organised in steps.

Step 1. Consider $\Phi_t(X) := -\psi_{tT}(X), X \in L^\infty(F_T)$, and $\alpha_t(R) := -\beta_{tT}(R), R \sim P$. Then the mapping $\Phi_t$ satisfies the property I) in Theorem 3.1 of [23]. This result presents a number of equivalent statements and then property II) is true. From the proof that II) implies I) (see Appendix in [23]) we have that for all $\varepsilon > 0$ there is a probability measure $\check{Q} \sim P$ such that $\alpha_t(\check{Q}) + \varepsilon \geq -\Phi_t(0)$. On the other hand, we have that

$$-\Phi_t(0) = \psi_{tT}(0) \geq -\beta_{tT}(\check{Q}) = \alpha_t(\check{Q}).$$

This proves that there exists $\check{Q} \sim P$ such that

$$\beta_{tT}(\check{Q}) = -\alpha_t(\check{Q}) \text{ belongs to } L^\infty(F_t).$$

If follows from the property of the minimal penalty that we can consider $\check{Q}|\mathcal{F}_t = P$.

Step 2. Let $X \in L^\infty(F_T)$. For all $R \sim P$ such that $R|\mathcal{F}_t = P$ let

$$A = \{ E_R(-X|F_t) - \beta_{tT}(R) > E_Q(-X|F_t) - \beta_{tT}(\check{Q}) \}. $$
Let \( \tilde{R} \) be defined by
\[
\frac{d\tilde{R}}{dP} = 1_A \frac{dR}{dP} + 1_A \frac{d\tilde{Q}}{dP}.
\]
Then we have that \( \tilde{R} \sim P \), \( \tilde{R}_{|\mathcal{F}_t} = P \), and \( \beta_{tT}(\tilde{R}) = 1_A \beta_{tT}(R) + 1_A \beta_{tT}(\tilde{Q}) \).
From the definition of the event \( A \), we can see that \( \beta_{tT}(R)1_A \leq \beta_{tT}(\tilde{Q})1_A + 2\|X\|_\infty 1_A \). On the other hand we have \( \beta_{tT}(R) \geq -\psi_{tT}(0) \). Thus we conclude that \( \beta_{tT}(\tilde{R}) \in L_\infty(\mathcal{F}_t) \) and \( E_{\tilde{Q}}(X|\mathcal{F}_t) - \beta_{tT}(\tilde{R}) \geq E_R(X|\mathcal{F}_t) - \beta_{tT}(R) \). By this we have obtained representation (2.16).

**Lemma 2.24.** Set \( p \in [1, \infty) \). Let \((\rho_{st})_{s,t}\) be a time-consistent fully-dynamic risk measure such that \( \rho_{0T} \) is dominated and weak sensitive. For all \( Q \in \mathcal{Q} \), for all \( s \in [0, T] \), we have that the minimal penalties \((\alpha_{st})_{s,t}\) satisfy the following:
\[
\alpha_{0s}(Q) \in \mathbb{R}, \text{ and } E_Q(\alpha_{sT}(Q)) \in \mathbb{R}.
\]

**Proof.** The minimal penalties \((\alpha_{st})_{s,t}\) satisfy the cocycle condition:
\[
\alpha_{0T}(Q) = \alpha_{0s}(Q) + E_Q(\alpha_{sT}(Q)), \quad Q \in \mathcal{Q}, \quad (2.17)
\]
see [6]. Furthermore, it follows from the definition of minimal penalty that
\[
\alpha_{0s}(Q) \geq -\rho_{0s}(0) \quad \alpha_{sT}(Q) \geq -\rho_{sT}(0).
\]
By hypothesis we have that \( \rho_{sT}(0) \in L_p(\mathcal{F}_s) \) and, from Proposition 2.10, we have that \( \frac{dQ}{dP} \in L_q(\mathcal{F}_T) \), with \( q = p(p - 1)^{-1} \). Thus \( E_Q(-\rho_{sT}(0)) \in \mathbb{R} \). The result follows from \( \alpha_{0T}(Q) < \infty \) and the cocycle condition (2.17).

We introduce the following sets of probability measures for all \( s \leq t \) on \((\Omega, \mathcal{F}_t)\):
\[
\mathcal{P}_{st} := \left\{ R \sim P : R_{|\mathcal{F}_s} = P \text{ and } \alpha_{st}(R) \in L_p(\mathcal{F}_s) \right\}
\]
and
\[
\tilde{\mathcal{P}}_{st} := \left\{ R \sim P : R_{|\mathcal{F}_s} = P \text{ and } \sup_{Q \in \mathcal{Q}} E_Q(\alpha_{st}(R)) < \infty \right\}
\]
We also remark immediately that
\[
\mathcal{P}_{st} \subseteq \tilde{\mathcal{P}}_{st} = \left\{ R \sim P : R_{|\mathcal{F}_s} = P \text{ and } \sup_{Q \in \mathcal{Q}} E_Q(|\alpha_{st}(R)|) < \infty \right\}. \quad (2.18)
\]
In fact, for \( R \in \tilde{\mathcal{P}}_{st} \), we have that \( \alpha_{st}(R) \geq -\rho_{st}(0) \). Then \( |\alpha_{st}(R)| \leq \alpha_{st}(R) + 2|\rho_{st}(0)| \). We remind that \( \rho_{st}(0) \in L_p(\mathcal{F}_t) \) and \( ||\frac{dQ}{dP}||_q \leq K \) for all \( Q \in \mathcal{Q} \), see Proposition 2.10. Then we conclude that both relations hold.
Proposition 2.25. Set $p \in [1, \infty)$. Let $(\rho_{st})_{s,t}$ be a time-consistent fully-dynamic risk measure such that $\rho_{0T}$ is dominated and sensitive. Then we have that

1. for all $t \in [0, T]$, the risk measure $\rho_{0t}$ is dominated and sensitive,
2. for all $t \in [0, T] : s \leq t$, the following representation holds:
   \[
   \rho_{st}(X) = \text{esssup}_{R \in \mathcal{P}_{st}} (E_R(-X|\mathcal{F}_s) - \alpha_{st}(R)), \quad X \in L_p(\mathcal{F}_t), \quad (2.19)
   \]
3. for all $Q \in \mathcal{Q}$ and $R \in \tilde{\mathcal{P}}_{st}$, there exists $h \in L_q(\mathcal{F}_t) : \|h\|_q \leq K$, with $q = p(p-1)^{-1}$, such that
   \[
   E_Q(E_R(X|\mathcal{F}_s)) = E(hX), \quad X \in L_p(\mathcal{F}_t),
   \]
4. for all $t \in [0, T] : s \leq t$, the following representation holds:
   \[
   \rho_{st}(X) = \text{esssup}_{R \in \mathcal{P}_{st}} (E_R(-X|\mathcal{F}_s) - \alpha_{st}(R)), \quad X \in L_p(\mathcal{F}_t). \quad (2.20)
   \]

Proof.

1. For all $Z \in L_p(\mathcal{F}_t)$ we have $\rho_{0T}(Z) = \rho_{0t}(-\rho_{0T}(Z)) = \rho_{0t}(Z - \rho_{0T}(0))$. By hypothesis, $\rho_{0T}(0) \in L_p(\mathcal{F}_t)$, thus for all $Y \in L_p(\mathcal{F}_t)$ we have $\rho_{0t}(Y) = \rho_{0T}(Y + \rho_{0T}(0))$. This shows that $\rho_{0t}$ is dominated. From Lemma 2.24, $\rho_{0t}$ is sensitive. This completes the proof of item 1.

2. and 3. The proofs of items 2 and 3 proceed together, first proving the result in item 2 for $X \in L_\infty(\mathcal{F}_t)$, then item 3, and finally, item 2 for $X \in L_p(\mathcal{F}_t)$. The argument is split in steps.

Step 1. Let $0 \leq s \leq t \leq T$. Define
   \[
   \tilde{\rho}_{st}(X) := \rho_{st}(X) - \rho_{st}(0), \quad X \in L_p(\mathcal{F}_t). \quad (2.21)
   \]
The translation invariance and the monotonicity of $\rho_{st}$ imply that
   \[
   \tilde{\rho}_{st} : L_\infty(\mathcal{F}_t) \longrightarrow L_\infty(\mathcal{F}_s).
   \]
The minimal penalty associated to the restriction of $\tilde{\rho}_{st}$ to $L_\infty(\mathcal{F}_t)$ is
   \[
   \tilde{\alpha}_{st}(Q) = \text{esssup}_{X \in L_\infty(\mathcal{F}_t)} (E_Q(-X|\mathcal{F}_s) - \tilde{\rho}_{st}(X)).
   \]
for all $Q \sim P$. Thus
   \[
   \tilde{\alpha}_{st}(Q) = \text{esssup}_{X \in L_\infty(\mathcal{F}_t)} (E_Q(-X|\mathcal{F}_s) - \rho_{st}(X)) + \rho_{st}(0).
   \]
Then Lemma 2.1 yields
\[ \tilde{\alpha}_{st}(Q) = \alpha_{st}(Q) + \rho_{st}(0). \] (2.22)

From (2.22) and Lemma 2.24 we obtain that for all \( Q \in \mathcal{Q} \), \( E_Q(\tilde{\alpha}_{st}(Q)) < \infty \) for all \( Q \) such that \( E(\alpha_{st}(Q)) < \infty \). From Lemma 2.23 we have that the risk measure \( \tilde{\rho}_{st} \) satisfies the representation (2.16). This, together with (2.21) and (2.22) and \( \rho_{st}(0) \in L_p(\mathcal{F}_s) \), proves that (2.19) is satisfied for all \( X \in L_{\infty}(\mathcal{F}_t) \).

**Step 2.** Let \( Q \in \mathcal{Q} \) and \( R \in \tilde{\mathcal{P}}_{st} \). For all \( X \in L_p(\mathcal{F}_t) \) we have
\[ E_Q(E_R(-X|\mathcal{F}_s)) - \alpha_{0s}(Q) - E_Q(\alpha_{st}(R)) \leq \rho_{0s}(-\rho_{st}(X)) = \rho_{0t}(X) \]

From Step 1, \( \rho_{0t} \) is dominated: \( \rho_{0t}(X) \leq K \|X\|_p + C \). Moreover, \( \alpha_{0s}(Q) < \infty \) and \( \sup_{Q \in \mathcal{Q}} E_Q(\alpha_{st}(R)) < \infty \). Thus there is some \( \kappa \in \mathbb{R} \) such that
\[ E_Q(E_R(-X|\mathcal{F}_s)) \leq K \|X\|_p + \kappa, \quad X \in L_p(\mathcal{F}_t) \]

This proves item 3.

**Step 3.** Let \( Q \in \mathcal{Q} \). The functional \( \phi(X) := E_Q(\rho_{st}(X)), X \in L_p(\mathcal{F}_t) \), defines a finite risk measure on \( L_p(\mathcal{F}_t) \). It is thus continuous for the \( L_p \)-norm, see [17]. From the definition of the minimal penalty we have that
\[ \rho_{st}(X) \geq \text{esssup}_{R \in \mathcal{P}_{st}} (E_R(-X|\mathcal{F}_s) - \alpha_{st}(R)) \]

Furthermore, for any given \( X \), the set
\[ \left\{ E_R(-X|\mathcal{F}_s) - \alpha_{st}(R), \ R \in \mathcal{P}_{st} \right\} \]
is a lattice upward directed. Thus, in order to prove (2.19) for all \( X \in L_p(\mathcal{F}_t) \), it is enough to prove that
\[ \phi(X) := E_Q(\rho_{st}(X)) \]
\[ \phi(X) = \sup_{R \in \mathcal{P}_{st}} \left( E_Q(E_R(-X|\mathcal{F}_s) - E_Q(\alpha_{st}(R)), X \in L_p(\mathcal{F}_t) \right). \]

From Step 1, we already know that (2.23) is satisfied for all \( X \in L_{\infty}(\mathcal{F}_t) \). Observe that \( \phi \) is continuous in the \( L_p \)-norm. Moreover, the continuity of the right-hand side of (2.23) in the \( L_p \)-norm follows from Step 2 and the inclusion \( \mathcal{P}_{st} \subseteq \tilde{\mathcal{P}}_{st} \). This ends the proof of item 2.

4. The representation (2.20) follows directly from (2.19) and the observation that \( \mathcal{P}_{st} \subseteq \tilde{\mathcal{P}}_{st} \).

By this the proof is complete. □
Lemma 2.26.

1. For all $0 \leq r \leq s \leq t \leq T$, for all $R \in \tilde{P}_{st}$, there is $S \in \tilde{P}_{sT}$, such that $R$ is the restriction of $S$ to $\mathcal{F}_t$.

2. For all $0 \leq r \leq s \leq t \leq T$, $\tilde{P}_{rt} \subset \tilde{P}_{rs}$, which means that the restriction to $\mathcal{F}_s$ of an element of $\tilde{P}_{rt}$, belongs to $\tilde{P}_{rs}$.

Proof.

1. $R \in \tilde{P}_{st}$, let $\tilde{Q} \in P_{tT}$. Let $S$ be the probability measure equivalent with $P$ such that $dS = dR \frac{d\tilde{Q}}{dP} E(\frac{d\tilde{Q}}{dP}|\mathcal{F}_t)$. In particular the restriction of $S$ to $\mathcal{F}_t$ is equal to $R$. It follows from the definition of $S$ and the properties of the minimal penalty that $\alpha_{sT}(S) = \alpha_{st}(R) + E_R(\alpha_{tT}(\tilde{Q})|\mathcal{F}_s)$. Thus

$$\sup_{Q \in \mathcal{Q}} E_Q(\alpha_{sT}(S)) \leq \sup_{Q \in \mathcal{Q}} E_Q(\alpha_{st}(R)) + \sup_{Q \in \mathcal{Q}} E_Q[E_R(\alpha_{tT}(\tilde{Q})|\mathcal{F}_s)]. \quad (2.24)$$

From item 3 of Proposition 2.25, we have that, for every $Q \in \mathcal{Q}$, there is $h \in L^q(\mathcal{F}_t)$ with $||h||_q \leq K$ such that, for all $X$ in $L_p(\mathcal{F}_t)$, $E_Q[E_R(X|\mathcal{F}_s)] = E(hX) \leq K||X||_p$. By definition of $\tilde{P}_{st}$ and $P_{tT}$ and using (2.24), we have that $S \in \tilde{P}_{sT}$.

2. Let $R \in \tilde{P}_{rt}$. It follows from the definition of the minimal penalty that $\alpha_{st}(R) \geq -\rho_{st}(0)$. We deduce then from the cocycle condition that $\alpha_{rs}(R) \leq \alpha_{rt}(R) + E_R(\rho_{st}(0)|\mathcal{F}_s)$ where $\rho_{st}(0)$ belongs to $L^p(\mathcal{F}_s)$. Then, from Proposition 2.25 item 3, we obtain $\sup_{Q \in \mathcal{Q}} E_Q(\alpha_{rs}(R)) < \infty$. This gives the result.

The proof is complete. \qed

We will now extend $\rho_{st}$ to $L^c_i$ for every $0 \leq s \leq t \leq T$. We first prove the following result.

Proposition 2.27. Let $0 = s_0 < s_1 < \ldots < s_n = T$. For all $Q_i \in \tilde{P}_{s_is_{i+1}}$ with $i = 0, \ldots, n-1$, let $Q$ be the unique probability measure on $\mathcal{F}_T$ such that $E_Q(X) = E_{Q_i}(E_{Q_i}(\ldots E_{Q_{n-1}}(X|\mathcal{F}_{s_{n-1}})\ldots|F_{s_1}))$, $X \in L_\infty(\mathcal{F}_T)$. Then $Q$ belongs to $\mathcal{Q}$.
Proof. For \( n = 1 \) observe that \( \tilde{P}_{0T} = Q \). We prove the result by induction for \( n \geq 2 \).

Step 1. For \( n = 2 \), we have \( 0 = s_0 < s_1 < s_2 = T \) and \( E_Q(X) = E_{Q_0}(E_{Q_1}(X|\mathcal{F}_{s_1})) \), for any \( X \in L_\infty(\mathcal{F}_T) \). From the cocycle condition and the properties of the minimal penalty, it is
\[
\alpha_{0T}(Q) = \alpha_{0s_1}(Q_0) + E_{Q_0}(\alpha_{s_1T}(Q_1)).
\]
By assumption the probability measure \( Q_0 \in \tilde{P}_{0s_1} \). From item 1 of Lemma 2.26 we can see that \( Q_0 \) is the restriction to \( \mathcal{F}_{s_1} \) of an element of \( Q \). Then we easily see also that \( \alpha_{0T}(Q) < \infty \) from the definition of \( \tilde{P}_{s_1T} \). Thus \( Q \in Q \).

Step 2. We assume that the result holds for \( n \) and we prove it for \( n + 1 \). Let \( 0 = s_0 < s_1 < \ldots < s_n < s_{n+1} = T \). From the induction hypothesis \( Q_{n-1} \in \tilde{P}_{s_{n-1}s_n} \). From item 1 of Lemma 2.26 we obtain that \( Q_{n-1} \) is the restriction to \( \mathcal{F}_{s_n} \) of an element \( R_{n-1} \in \tilde{P}_{s_{n-1}s_n} \). Then by induction we can see that the probability measure \( R \) on \( \mathcal{F}_T \) defined by
\[
E_R(X) = E_{Q_0}(E_{Q_1}(\cdots E_{R_{n-1}}(X|\mathcal{F}_{s_{n-1}})\cdots|\mathcal{F}_{s_1}))
\]
for all \( X \in L_\infty(\mathcal{F}_T) \), belongs to \( Q \). From item 2 of Lemma 2.26 the restriction \( R \) of \( R \) to \( \mathcal{F}_{s_n} \) belongs to \( \tilde{P}_{0s_n} \) and
\[
E_Q(X) = E_{Q_0}(E_{Q_1}(\cdots E_{Q_n}(X|\mathcal{F}_{s_n})\cdots|\mathcal{F}_{s_1})) = E_R(E_{Q_n}(X|\mathcal{F}_{s_n})).
\]
The result follows then from Step 1. \( \square \)

**Theorem 2.28.** Set \( p \in [1, \infty) \). Let \( (\rho_{st})_{s,t} \) be a time-consistent fully-dynamic risk measure on \((L_p(\mathcal{F}_t))_{t} \) such that \( \rho_{0T} \) is dominated and sensitive.

1. For all \( s \leq t \), the risk measure \( \rho_{st} \) admits a unique extension \( \tilde{\rho}_{st} : L^c \to L^c \) such that
\[
c(|\rho_{st}(X) - \tilde{\rho}_{st}(Y)|) \leq c(|X - Y|), \quad X, Y \in L^c. \tag{2.25}
\]

2. The extension \( \tilde{\rho}_{st} \) admits the following representation
\[
\tilde{\rho}_{st}(X) = \operatorname{esssup}_{R \in \mathcal{P}_{st}}(E_R(-X|\mathcal{F}_s) - \alpha_{s,t}(R)), \quad X \in L^c. \tag{2.26}
\]

**Proof.**

1. Let \( X, Y \in L_\infty(\mathcal{F}_t) \). Consider \( A \in \mathcal{F}_s \) such that \( |\rho_{st}(X) - \rho_{st}(Y)| = \left|\rho_{st}(X) - \rho_{st}(Y)\right|1_A + \left|\rho_{st}(Y) - \rho_{st}(X)\right|1_{A^c} \). From (2.20) we get
\[
|\rho_{st}(X) - \rho_{st}(Y)| \leq \operatorname{esssup}_{R \in \mathcal{P}_{st}} E_R(|X - Y| |\mathcal{F}_s). \tag{2.27}
\]
For all \( X,Y \in L_\infty(\mathcal{F}_t) \), the set \( \{ E_R(|X - Y||\mathcal{F}_s), R \in \tilde{\mathcal{P}}_{st} \} \) satisfies the lattice property. It follows that, for all \( Q \in \mathcal{Q} \),

\[
E_Q\left( \text{esssup}_{R \in \tilde{\mathcal{P}}_{st}} E_R(|X - Y||\mathcal{F}_s) \right) = \sup_{R \in \tilde{\mathcal{P}}_{st}} E_Q\left( E_R(|X - Y||\mathcal{F}_s) \right).
\]

From equation (2.27), from item 1 of Lemma 2.26, and from Proposition 2.27, we obtain that

\[
c(|\rho_{st}(X) - \rho_{st}(Y)|) \leq c(|X - Y|), \quad X,Y \in L_\infty(\mathcal{F}_t). \tag{2.28}
\]

On the other hand, \( \rho_{st}(X) \in L_p(\mathcal{F}_s) \subset L_c^{cs} \), for all \( X \in L_\infty(\mathcal{F}_t) \). Then item 1 in the statement of the theorem follows from the density of \( L_\infty(\mathcal{F}_t) \) in \( L_c^{cs} \) and from equation (2.28).

2. The representation (2.26) is satisfied for all \( X \in L_\infty(\mathcal{F}_t) \), see item 4 of Proposition 2.25. Item 1 of the theorem provides the continuity of \( \tilde{\rho}_{st} \) for the \( c \) norm. Moreover, from Lemma 2.26 item 1 and Proposition 2.27, we have continuity for the \( c \) norm of the right-hand side of (2.26).

By continuity, we can then see that (2.26) is satisfied for all \( X \) in \( L_c^{cs} \).

This completes the proof. \( \square \)

Corollary 2.29. The family \((\tilde{\rho}_{st})_{s,t}\) is a time-consistent dynamic risk measure on \((L_c^{ct})_t\). Furthermore \( \tilde{\rho}_{0T} \) is dominated by \( \sup_{Q \in \mathcal{Q}} E_Q(-X) \).

Proof. The properties of monotonicity, translation invariance, and time-consistency for \((\tilde{\rho}_{st})_{s,t}\) follow from (2.20) and the corresponding properties for \((\rho_{st})_{s,t}\) on \((L_p(\mathcal{F}_t))_t\). The domination by \( \sup_{Q \in \mathcal{Q}} E_Q(-X) \) follows from the representation (2.20) of \( \tilde{\rho}_{0T} \) and the observation that \( \tilde{\mathcal{P}}_{0T} = \mathcal{Q} \). \( \square \)

Now that we have the extension \((\tilde{\rho}_{st})_{s,t}\) on \((L_c^{ct})_t\), we can extend the price system \((x_{st})_{s,t}\) on \((L_c^{ct})_t\).

Definition 2.30. Set \( p \in [1,\infty) \). Let \((\rho_{st})_{s,t}\) be a time-consistent, fully-dynamic risk measure on \((L_p(\mathcal{F}_t))_t\) such that \( \rho_{0T} \) is dominated and sensitive. Let \( s \leq t \). For all \( X \in L_c^{ct}_t \), define

\[
x_{st}(X) := \text{essinf}_{g \in C_{ct}^s} \tilde{\rho}_{st}(g - X) - \text{essinf}_{g \in C_{ct}^s} \tilde{\rho}_{st}(g),
\]

where \( \tilde{\rho}_{st} \) is the extension of \( \rho_{st} \) to \( L_c^{ct}_t \) defined in Theorem 2.28.

Proposition 2.31.

1. The operator \( x_{st} \) is well-defined on \( L_c^{ct}_t \) with values in \( L_c^{cs} \). It extends the operator in Definition 2.14 (see also (2.13)). Moreover,

\[
c(|x_{st}(X) - x_{st}(Y)|) \leq c(|X - Y|), \quad X,Y \in L_c^{ct}_t. \tag{2.29}
\]
2. The operator \( x_{\text{st}} \) is convex, monotone, and satisfies the projection property.

3. Choose \( Q_0 \in \mathcal{Q} \), then \( x_{\text{st}} \) admits the following dual representation:

\[
x_{\text{st}}(X) = \esssup_{R \in \mathcal{K}} (E_R(X|\mathcal{F}_s) - \gamma_{\text{st}}(R)), \quad X \in \mathcal{L}_c^\infty,
\]

where \( \mathcal{K} \) is a set of probability measures in the dual of \( \mathcal{L}_c^\infty \) compact for the weak* topology such that every \( R \in \mathcal{K} \) is absolutely continuous with respect to \( P \), such that the restriction of \( R \) to \( \mathcal{F}_s \) is equal to \( Q_0 \), and

\[
\gamma_{\text{st}}(R) = \esssup_{Y \in \mathcal{L}_c^\infty} (E_R(Y|\mathcal{F}_s) - x_{\text{st}}(Y)).
\]

Furthermore, for all \( X \in \mathcal{L}_c^\infty \), there exists \( Q_X \in \mathcal{K} \) such that

\[
x_{\text{st}}(X) = E_{Q_X}(X|\mathcal{F}_s) - \gamma_{\text{st}}(Q_X)).
\]

\[(2.31)\]

Proof.

1. From (2.26) in Theorem 2.28 we have that

\[
|x_{\text{st}}(X) - x_{\text{st}}(Y)| \leq \esssup_{R \in \mathcal{P}_{\text{st}}} E_R(||X - Y|||\mathcal{F}_s), \quad X, Y \in \mathcal{L}_c^\infty.
\]

Then equation (2.29) follows from Lemma 2.26 item 1 and Proposition 2.27.

Recall that \( x_{\text{st}}(X) \in L_\infty(\mathcal{F}_s) \subset \mathcal{L}_c^\infty \), for all \( X \in \mathcal{L}_\infty(\mathcal{F}_t) \). From the continuity of \( x_{\text{st}} \) with respect to the \( c \) norm we conclude that \( x_{\text{st}}(X) \in \mathcal{L}_c^\infty \), for all \( X \in \mathcal{L}_c^\infty \).

Finally, Definition 2.30 of \( x_{\text{st}} \) extends the one given in Definition 2.14 (see also (2.13)), because \( \tilde{\rho}_{\text{st}} \) is the extension of \( \rho_{\text{st}} \) to \( \mathcal{L}_c^\infty \).

2. Statement 2 follows directly from the properties of \( \tilde{\rho}_{\text{st}} \), see Corollary 2.29.

3. Let \( Q_0 \in \mathcal{Q} \). The map \( E_{Q_0}(x_{\text{st}}(X)), X \in \mathcal{L}_c^\infty \), is up to a minus sign a normalised convex risk measure on \( \mathcal{L}_c^\infty \) with values in \( \mathbb{R} \) majorized by \( \sup_{Q \in \mathcal{Q}} E_Q(X) \). We refer to Proposition 3.1 and Theorem 3.2 in [10] to prove the existence of a set \( \mathcal{K} \) of probability measures in the dual of \( \mathcal{L}_c^\infty \), compact for the weak* topology, such that

\[
E_{Q_0}(x_{\text{st}}(X)) = \sup_{R \in \mathcal{K}} (E_R(X) - \gamma(R)) = E_{Q_X}(X) - \gamma(Q_X)
\]

for some \( Q_X \in \mathcal{K} \) (depending on \( X \)). Thus the representations (2.30) and (2.31) follow from standard arguments, see e.g. [14].

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The proof is complete.

Now we are ready to discuss the Fatou property of a risk-indifferent evaluation, as given in Definition 2.30. This brings to the following natural definition.

**Definition 2.32.** For any \( s \leq t \), an operator \( x : L^c_t \rightarrow L^c_s \) has the Fatou property on \( L^c_t \) if for any sequence \( (X_n)_n \in L^c_t \), dominated in \( L^c_t \), and converging P-a.s. to \( X \in L^c_t \), we have

\[
x(X) \leq \liminf_{n \to \infty} x(X_n). \tag{2.32}
\]

Here above \( (X_n)_n \) dominated in \( L^c_t \) means that there is \( Y \in L^c_t \) such that \( |X_n| \leq Y \) P-a.s. for all \( n \).

**Proposition 2.33.** Set \( p \in [1, \infty) \). Let \( (p_{st})_{s,t} \) be a time-consistent, fully-dynamic risk measure on \( (L^p(F_t))_t \) such that \( p_{0T} \) is dominated and sensitive. For all \( s \leq t \), the operator \( x_{st} \) in Definition 2.30 has the Fatou property on \( L^c_t \).

**Proof.** Let \( (X_k)_k \in L^c_t \) be dominated in \( L^c_t \) by \( Y \) and converge P-a.s. to \( X \in L^c_t \). Set \( \tilde{X}_n := \inf_{k \geq n} X_k \). The sequence \( (\tilde{X}_n)_n \) is increasing, \( |\tilde{X}_n| \leq Y \), and \( X = \lim_{n \to \infty} \tilde{X}_n \) P-a.s. From Proposition 2.31 for all \( X \in L^c_t \), there is a probability measure \( Q_X \) in the dual of \( L^c_t \) such that \( x_{st}(X) = E_{Q_X}(X|F_s) - \gamma_{st}(Q_X) \). Then, by the dominated convergence theorem, we obtain that

\[
x_{st}(X) = \lim_{n \to \infty} \left( E_{Q_X}(\tilde{X}_n|F_s) - \gamma_{st}(Q_X) \right) \\
\leq \liminf_{n \to \infty} \left( E_{Q_X}(X_k|F_s) - \gamma_{st}(Q_X) \right). \tag{2.33}
\]

Observe that, for all \( k \), \( E_{Q_X}(X_k|F_s) - \gamma_{st}(Q_X) \leq x_{st}(X_k) \). Then from (2.33) we get that

\[
x_{st}(X) \leq \liminf_{n \to \infty} x_{st}(X_k) = \liminf_{n \to \infty} x_{st}(X_n).
\]

This proves the Fatou property. \( \square \)

### 2.3 Risk-indifference price system \( (x_{st})_{s,t} \)

At this stage we have studied the properties of \( x_{st} \) defined on \( L^c_t \) with values in \( L^c_s \). We now study the time-consistency of the family \( (x_{st})_{s,t} \).

**Lemma 2.34.** Let \( 0 \leq r \leq s \leq t \). Every \( g \in C^\infty_{rt} \) can be written \( g = g_1 + g_2 \) for some \( g_1 \in C^\infty_{rs} \) and \( g_2 \in C^\infty_{st} \).
Hence, we obtain that $g \leq Y_{rs}(\theta) = Y_{rs}(\theta) + Y_{st}(\theta)$. Let $M \in \mathbb{R}$ such that $M \leq g$, $M \leq Y_{rs}(\theta)$, $M \leq Y_{st}(\theta)$. Let

$$g_1 := \inf \left( Y_{rs}(\theta), ||g||_{\infty} - M \right)$$

We remark that $g_1 \in L_{\infty}(\mathcal{F}_s)$ and $g_1 \leq Y_{rs}(\theta)$. Let $g_2 := g - g_1$, clearly $g_2 \in L_{\infty}(\mathcal{F}_1)$. Then we can see that $g_2 = \sup \left( g - Y_{rs}(\theta), g - ||g||_{\infty} + M \right)$. Hence, we obtain that $g_2 \leq \sup(Y_{st}(\theta), M) = Y_{st}(\theta)$.

**Lemma 2.35.** Let $p \in [1, \infty]$. Let $(\rho_{st})_{s,t}$ be a time-consistent fully-dynamic risk measure on $(L_p(\mathcal{F}_t))_t$. Let $r \leq s \leq t$. Consider the operator $x_{st}(X)$, $X \in L_p(\mathcal{F}_t)$ defined $P$-a.s. as in (2.12). For any $X, Y \in L_p(\mathcal{F}_t)$, assume that $x_{st}(X) \geq x_{st}(Y)$ $P$-a.s. Then $x_{rt}(X) \geq x_{rt}(Y)$.

Observe that the operator in Lemma 2.35 would be a risk-indifference price if defined on $Dom \ x_{st}$, see Definition 2.13.

**Proof.** For the proof we need to deal with both $L_p$-convergence and $P$-a.s. convergence. For this reason we work with an extension of the risk measure $\rho_{st}$ from $L_p(\mathcal{F}_s)$ to the set

$$D_s := \{ \mathcal{F}_s \text{-measurable } X : \exists (X_n)_n \in L_p(\mathcal{F}_s) \text{ s.t. } X_n \uparrow X \ P-\text{a.s.} \}.$$ 

We define this extension as the monotone limit

$$\bar{\rho}_{st}(X) := \lim_{n \to \infty} \rho_{st}(X_n) \quad P-\text{a.s.}$$

First of all we show that the definition is well-posed. We consider two sequences $(X_n)_n$ and $(Y_n)_n$ in $L_p(\mathcal{F}_s)$ such that both $X_n \uparrow X$ and $Y_n \uparrow X$ $P$-a.s., where $X$ is $\mathcal{F}_s$-measurable. We denote $A := \lim_{n \to \infty} \rho_{st}(X_n)$ and $B := \lim_{n \to \infty} \rho_{st}(Y_n)$ $P$-a.s. Then for any constant $M$, we would have $X_n \wedge M \uparrow X \wedge M$ and also $Y_n \wedge M \uparrow X \wedge M$ in the $L_p$-convergence. Observe that $X \wedge M \in L_p(\mathcal{F}_s)$ and $\rho_{st}$ is continuous from below, thus we have

$$\inf_M \inf_n \rho_{st}(X_n \wedge M) = \inf_M \rho_{st}(X \wedge M) = \inf_M \rho_{st}(Y_n \wedge M).$$

Hence $A = \bar{\rho}_{st}(X) = B$. It is also clear that $\bar{\rho}_{st}$ is monotone and continuous $P$-a.s. from below on $D$.

We now proceed with the proof of the statement. Let $X, Y \in L_p(\mathcal{F}_t)$ such that $x_{st}(X) \geq x_{st}(Y)$. From (2.13) and assumption 2.11, we have that

$$\text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g - X) \geq \text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g - Y) \quad (2.34)$$

From Lemma 2.34 any $g \in C_{st}^\infty$ is the sum $g = g_1 + g_2$ of $g_1 \in C_{st}^\infty$ and $g_2 \in C_{st}^\infty$. First of all from (2.34) and the lattice property of $\{\rho_{st}(g - X), \ g \in C_{st}^\infty\}$
we observe that \( g_1 - \essinf_{g_2 \in C^{rs}_{rt}} \rho_{st}(g_2 - X) \) belongs to \( D_s \). Then we can apply the extension \( \bar{\rho}_{rs} \) of \( \rho_{rs} \) and from its monotonicity we obtain that

\[
\bar{\rho}_{rs}(g_1 - \essinf_{g_2 \in C^{rs}_{rt}} \rho_{st}(g_2 - X)) \geq \bar{\rho}_{rs}(g_1 - \essinf_{g_2 \in C^{rs}_{rt}} \rho_{st}(g_2 - Y)),
\]

(2.35)

for all \( g_1 \) in \( C^{rs}_{rt} \).

On the other hand the time-consistency of \( (\rho_{st})_{s,t} \) yields

\[
\essinf_{g \in C^{rs}_{rt}} \rho_{rt}(g - X) = \essinf_{g \in C^{rs}_{rt}} \rho_{rs}(g_1 - \rho_{st}(g_2 - X)).
\]

(2.36)

Now, let us consider a sequence \((h_n)\) such that \(-\essinf_{g \in C^{rs}_{rt}} \rho_{st}(g - X)\) is the increasing limit of \(-\rho_{st}(h_n - X)\). Then \(-\essinf_{g \in C^{rs}_{rt}} \rho_{st}(g - X)\) belongs to \( D_s \). Then we have

\[
\bar{\rho}_{rs}(g_1 - \essinf_{g_2 \in C^{rs}_{rt}} \rho_{st}(g_2 - X)) = \lim_{n \to \infty} \rho_{rs}(g_1 - \rho_{st}(h_n - X))
\]

From which

\[
\bar{\rho}_{rs}(g_1 - \essinf_{g_2 \in C^{rs}_{rt}} \rho_{st}(g_2 - X)) = \essinf_{g \in C^{rs}_{rt}} \rho_{rs}(g_1 - \rho_{st}(g_2 - X)).
\]

So from equations (2.36) and (2.35), we get

\[
\essinf_{g \in C^{rs}_{rt}} \rho_{rt}(g - X) = \essinf_{g \in C^{rs}_{rt}} (\bar{\rho}_{rs}(g_1 - \essinf_{g_2 \in C^{rs}_{rt}} \rho_{st}(g_2 - X))) \geq \essinf_{g \in C^{rs}_{rt}} \rho_{rt}(g - Y).
\]

The result follows.

**Proposition 2.36.** Fix \( p \in [1, \infty] \). Let \((\rho_{st})_{s,t}\) be a time-consistent fully-dynamic risk measure on \((L_p(F_t))_t\). Then

\[
x_{rt}(x_{st}(X)) = x_{rt}(X), \quad X \in L_\infty(F_t).
\]

(2.37)

In particular this results show that, if \( p = \infty \), the family \( x_{st} \) is time-consistent, while for \( p < \infty \), the relationship is true only for essentially bounded claims.

**Proof.** For all \( X \in L_\infty(F_t) \), the value \( x_{st}(X) \) belongs to \( L_\infty(F_s) \), see Lemma 2.16 item 2. Then, as in [1], \( x_{rt}(x_{st}(X)) = x_{rt}(X) \) follows from the weak time-consistency (Proposition 2.35) applied with \( X \) and \( Y = x_{st}(X) \).

**Remark 2.37.** Notice that when \( p < \infty \) equation (2.37) cannot be proved in \( L_p \). Indeed it is not true in general that \( x_{st}(X) \) belongs to \( L_p(F_t) \) for all \( X \in L_p(F_t) \). This is the reason why we need to work with the extension of \( x_{st} \) to \( L^p_s \). Then we know that for every \( X \) in \( L^p_s \), \( x_{st}(X) \) belongs to \( L^p_s \subseteq L^p_t \).
Theorem 2.38. Set $p \in [1, \infty)$. Let $(\rho_{st})_{s,t}$ be a time-consistent fully-dynamic risk measure on $(L_p(\mathcal{F}_t))_t$, such that $\rho_{0T}$ is dominated and sensitive. Let $x_{st}$ be the risk-indifference price on $L_c^c$ defined as in Definition 2.37. Then $(x_{st})_{s,t}$ on $(L_c^c)_t$ is weak time-consistent. For every given time horizon $t \leq T$, the price system $(x_s)_s$, defined as $x_s(X) := x_{st}(X)$, $s \leq t$, is time-consistent on the whole $L_c^c$. Namely, for all $0 \leq r \leq s \leq t$, for all $X \in L_c^c$, we have $x_s(X) \in L_c^s$ and

$$x_r(X) = x_r(x_s(X)).$$

Proof. Fix $t \in [0, T]$. We obtain that

$$x_{rt}(x_{st}(X)) = x_{rt}(X), \forall X \in L_c^r,$$

by the density of $L_{\infty}(\mathcal{F}_t)$ in $L_c^c$ and the uniform continuity for the c norm of $x_{st}$ for all $s \leq t$ (Proposition 2.31 and 2.37). In turn this gives the time-consistency of the family $(x_s)_s$, where $s \leq t$.

In the last part of this section we prove the regulatory of the trajectories for the risk-indifferent price operators.

Theorem 2.39. Fix some $t \in [0, T]$. Assume that for some $Q \in \mathcal{Q}$, $\gamma_{0t}(Q) = 0$, where $\gamma_{0t}$ is the minimal penalty for $x_{0t}$. Then for all $X \in L_c^t$, the stochastic process $x_{st}(X)$, $0 \leq s \leq t$, admits a càdlàg modification.

Proof. Notice that $\gamma_{0t}(Q) = 0$ implies that

$$0 = \sup_{X \in L_{\infty}(\mathcal{F}_t)} (E_Q(x_{st}(X)) - x_{0t}(X)).$$

(2.38)

For all $r \leq s \leq t$, let $y_{rs}$ be the restriction of $x_{rt}$ to $L_c^r$. From Theorem 2.38, we deduce that the family $(y_{rs})_{0 \leq r \leq s \leq t}$ of operators on $(L_c^s)_s$ is time-consistent. Its restriction to $(L_{\infty}(\mathcal{F}_s))_s$ is up to a minus sign a time-consistent normalised dynamic risk measure on $(L_{\infty}(\mathcal{F}_s))_s$. Making use of (2.33), the proof of Lemma 3 and of Lemma 4 in Section 3.1 of [7], applied with deterministic times, we can prove that $x_{st}(X)$ is the limit of $x_{sn,t}(X)$ in $L_1(Q)$, for every decreasing sequence $(s_n)_n$ converging to $s$ and for every $X \in L_{\infty}(\mathcal{F}_t)$. We also get that $x_{rt}(X) \geq E_Q(x_{st}(X)|\mathcal{F}_r)$ for all $X \in L_{\infty}(\mathcal{F}_t)$. Moreover, by the density of $L_{\infty}(\mathcal{F}_t)$ in $L_c^r$ and the uniform equicontinuity of $x_{st}$ for the c norm (equation (2.29)), we can show that, for all $X \in L_c^t$, the value $x_{st}(X)$ is the limit of $x_{sn,t}(X)$ in $L_1(Q)$ and that $x_{st}(X)$ is a Q-supermartingale for all $X \in L_c^t$. Recall that the probability measure $Q \in \mathcal{Q}$ is equivalent with $P$, then the modification theorem (see Theorem 4 page 76 in [13]) proves that, for all $X \in L_c^t$, the process $x_{st}(X)$ admits a càdlàg modification. 

\[28\]
Remark 2.40. In view of the proposition above, our results are valid also with stopping times. Indeed, we could also have started our work with a fully dynamic risk measure indexed by stopping times \((\rho_{\sigma,\tau})_{0\leq \sigma \leq \tau \leq T}\) as in [7] and obtain the same results of the present paper replacing deterministic times by stopping times. We stress that our framework allows to give price evaluations to all American-type financial claims.

3 Risk-indifference prices in \(L_2\) and no-good-deal bounds

Good-deal bounds were suggested simultaneously by Cochrane and Saa Requejo [12] and Bernardo and Ledoit [3] with the idea of identifying those deals that are “too good to be true”. The concept was introduced in a static setting by fixing bounds on the Sharpe ratio. In [8] the relationship between bounds on the Sharpe ratio and no-good-deal pricing measures was detailed providing an equivalent definition of no-good-deal bounds expressed in terms of bounds on the Radon-Nykodim derivatives. Also it was possible to define the concept of dynamic no-good-deal bounds.

In this section, in view of the nature of these concepts, it is natural to work with \(p = 2\).

We shall use dynamic no-good-deal bounds to provide a construction of risk-indifference prices in \(L_2\). Indeed by the use of the bounds we can guarantee that, for all \(s \leq t\), the risk-indifference price \(x_{st}\) (satisfying these bounds) is a well defined operator from \(L_2(\mathcal{F}_t, P)\) to \(L_2(\mathcal{F}_s, Q)\). We shall related this approach with the results of Section 2. Moreover, our study provides a characterisation of the risk measures \((\rho_{st})_{s,t}\) so that the associated risk-indifferent prices are no-good-deal prices.

In the first part of this section we revise the fundamental concepts of no-good-deal bounds and provide some first results on the role of the bounds for the construction of convex operators in \(L_2\).

3.1 No-good-deal prices

Definition 3.1. A probability measure \(Q \sim P\) is a no-good-deal pricing measure if there are no good-deals of level \(\delta > 0\) under \(Q\), that is, the Sharpe ratio is bounded:

\[-\delta \leq \frac{E(X) - E_Q(X)}{\sqrt{\text{Var}(X)}} \leq \delta,\]  

for all \(X \in L_2(\mathcal{F}_T, P) \cap L_1(\mathcal{F}_T, Q)\). Equivalently, we can say that \(Q \sim P\) is a no-good-deal pricing measure if \(\frac{dQ}{dP} \in L_2(\mathcal{F}_T)\) satisfies

\[E\left[\left(\frac{dQ}{dP} - 1\right)^2\right] \leq \delta^2.\]
From a dynamic perspective, it is suitable to work with the following set of probability measures.

**Definition 3.2.** Let \( s \leq t \). Define the set \( \mathcal{Q}_{st} \) of probability measures on \( \mathcal{F}_t \) as

\[
\mathcal{Q}_{st} := \{ Q \ll P : Q|\mathcal{F}_s = P \text{ and } \frac{dQ}{dP} \in \mathcal{D}_{st} \},
\]

where

\[
\mathcal{D}_{st} := \{ 1 + h_{s,t} : h_{s,t} \in L_2(\mathcal{F}_t), E[h_{s,t}|\mathcal{F}_s] = 0, E[h_{s,t}^2|\mathcal{F}_s] \leq \delta_{st}^2 \}.
\]

Here the family of non-negative real numbers \( \delta_{st} \), \( s, t \in [0, T] \) : \( s \leq t \), satisfies the condition for all \( r \leq s \leq t \):

\[
(\delta_{rs}\delta_{st} + \delta_{rs} + \delta_{st}) = \delta_{rt} \quad \text{(3.3)}
\]

and \( \delta_{st} \to 0, \ t \downarrow s \).

Remark that we can connect the bounds on the Sharpe ratio (3.1) with the one here above by choosing \( \delta_{st} := \delta_{t-s} - 1 \), for some \( \delta > 1 \).

Then the following definition is given.

**Definition 3.3.** A probability measure \( Q \sim P \) is a dynamic no-good-deal pricing measure if

\[
E \left[ \left( \left( \frac{dQ}{dP} \right)_t \left( \frac{dQ}{dP} \right)_s^{-1} - 1 \right)^2 |\mathcal{F}_s \right] \leq \delta_{st}^2,
\]

for every \( s \leq t \) and constants \( \delta_{st} > 0 \) satisfying (3.3). Here \( \left( \frac{dQ}{dP} \right)_t := E [\frac{dQ}{dP} |\mathcal{F}_t] \).

Corresponding to these bounds on the Radon-Nykodim derivatives, we can characterise the no-good-deal bounds on prices.

**Definition 3.4.** The no-good-deal bounds on prices are the sub-linear and super-linear operators here below:

\[
M_{st}(X) := \text{esssup}_{Q \in \mathcal{Q}_{st}} E_Q [X|\mathcal{F}_s], \quad X \in L_2(\mathcal{F}_t),
\]

\[
m_{st}(X) := \text{essinf}_{Q \in \mathcal{Q}_{st}} E_Q [X|\mathcal{F}_s], \quad X \in L_2(\mathcal{F}_t),
\]

where \( \mathcal{Q}_{st} \) is given in Definition 3.2.

Clearly, \( m_{st}(X) = -M_{st}(-X) \), for \( X \in L_2(\mathcal{F}_t) \). The properties of these operators are studied in Proposition 5.8 in [8].

Hereafter we study the representation of a general convex price operator \( x_{st} : L_\infty(\mathcal{F}_t) \rightarrow L_\infty(\mathcal{F}_s) \) satisfying the no-good-deal bounds. This is a crucial result for the study of no-good-deal risk-indifference prices.
Proposition 3.5. Set $s \leq t$. Let $x_{st}: L_{\infty}(\mathcal{F}_t) \rightarrow L_{\infty}(\mathcal{F}_s)$ be a convex price operator Definition 1.1 satisfying the no-good-deal bound: for all $X \in L_{\infty}(\mathcal{F}_t) : X \geq 0$,
\[ x_{st}(X) \leq M_{st}(X). \tag{3.5} \]
Let $\gamma_{st}$ be the minimal penalty of $x_{st}$ on $L_{\infty}(\mathcal{F}_t)$. Then, for any probability measure $Q \ll P : Q|_{\mathcal{F}_s} = P$ and $E[\gamma_{st}(Q)] < \infty$, we have that $Q \in \Omega_{st}$ and the following representation holds
\[ x_{st}(X) = \text{esssup}_{Q \in \Omega_{st}} \left( E_Q(X|\mathcal{F}_s) - \gamma_{st}(Q) \right), \quad X \in L_{\infty}(\mathcal{F}_t). \tag{3.6} \]
Conversely, if (3.6) holds than (3.5) is satisfied.

Proof. From Proposition 1 in [7] we have that a convex price operator $x_{st}$ defined on $L_{\infty}(\mathcal{F}_t)$ admits representation in the form
\[ x_{st}(X) = \text{esssup}_{Q \ll P : Q|_{\mathcal{F}_s} = P \, \text{and} \, E[\gamma_{st}(Q)] < \infty} \left( E_Q(X|\mathcal{F}_s) - \gamma_{st}(Q) \right), \quad X \in L_{\infty}(\mathcal{F}_t). \tag{3.7} \]
Consider the operator
\[ y_{st}(X) := x_{st}(X) - E(X|\mathcal{F}_s), \quad X \in L_{\infty}(\mathcal{F}_t). \]
From (3.5) and Definition 3.2 we have that, for $X \in L_{\infty}(\mathcal{F}_t) : X \geq 0$
\[ y_{st}(X) \leq \text{esssup}_{h \in \mathcal{D}_{st}} E((h - 1)X|\mathcal{F}_s) \leq \delta_{st}(E(X^2|\mathcal{F}_s))^{1/2}. \tag{3.8} \]
Hence, from (3.7) and (3.8), we obtain
\[ \text{esssup}_{Q \ll P : Q|_{\mathcal{F}_s} = P \, \text{and} \, \frac{dQ}{dP} = k} \left( E((k - 1)X|\mathcal{F}_s) - \gamma_{st}(Q) \right) = y_{st}(X) \leq \delta_{st}(E(X^2|\mathcal{F}_s))^{1/2}, \]
for $X \in L_{\infty}(\mathcal{F}_t) : X \geq 0$. Consider any $Q \ll P$ such that $Q|_{\mathcal{F}_s} = P$, $\frac{dQ}{dP} = k$ and $E[\gamma_{st}(Q)] < \infty$. Then for every $\lambda > 0$ we have
\[ E((k - 1)\lambda X|\mathcal{F}_s) - \gamma_{st}(Q) \leq \lambda \delta_{st}(E(X^2|\mathcal{F}_s))^{1/2}. \]
Thus
\[ E((k - 1)X|\mathcal{F}_s) \leq \delta_{st}(E(X^2|\mathcal{F}_s))^{1/2} + \frac{1}{\lambda} \gamma_{st}(Q). \]
Taking $\lambda \rightarrow \infty$ we obtain that $E((k - 1)X|\mathcal{F}_s) \leq \delta_{st}(E(X^2|\mathcal{F}_s))^{1/2}$. Hence, setting $h := k - 1$, we conclude that $E(h^2|\mathcal{F}_s) \leq \delta_{st}^2$. That is $Q \in \Omega_{st}$. Then $x_{st}$ admits representation (3.6). The converse is immediate. \qed
We remark that from the representation (3.6) we can see that the bound (3.5) is satisfied for all $X \in L^\infty(F_t)$.

**Corollary 3.6.** Set $s \leq t$. Let $x_{st} : L^\infty(F_t) \to L^\infty(F_s)$ be a convex price operator satisfying the no-good-deal bound (3.5). Then $x_{st}$ is continuous in the $L^2$-norm and admits a unique extension

$$x_{st} : L^2(F_t) \to L^2(F_s).$$

This extension admits the representation

$$x_{st}(X) = \text{esssup}_{Q \in \mathcal{Q}_{st}} \left( E_Q(X|F_s) - \gamma_{st}(Q) \right), \quad X \in L^2(F_t), \quad (3.9)$$

and satisfies the no-good-deal bounds for all $X \in L^2(F_t)$:

$$m_{st}(X) \leq x_{st}(X) \leq M_{st}(X). \quad (3.10)$$

**Proof.** We can see that $x_{st}$ is continuous in the $L^2$-norm from (3.8). Hence the operator can be uniquely extended as a mapping from $L^2(F_t)$ to $L^2(F_s)$ by continuity. Moreover, the right-hand side of representation (3.6) is continuous in the $L^2$-norm, hence also extendable by continuity as a mapping from $L^2(F_t)$ to $L^2(F_s)$. Thus representation (3.9) holds. Moreover, for all $X \in L^2(F_t)$, $m_{st}(X) \leq -x_{st}(-X) \leq x_{st}(X) \leq M_{st}(X)$.

### 3.2 Risk-indifference prices with no-good-deal bounds

In this section we aim at finding necessary and sufficient conditions on the fully-dynamic risk measures, such that the associated risk-indifference price system satisfies the no-good-deal bounds.

We recall that the risk-indifference price system $(x_{st})_{s,t}$ on $(L^\infty(F_t))_t$ is a convex price system according to Definition 1.3.

In view of the setting $p = 2$, we consider $(\rho_{st})_{s,t}$ to be a fully-dynamic risk measure on $(L^2(F_t))_t$ and $(x_{st})_{s,t}$ to be the family of operators defined as in (2.12), from Definition 2.14. For $s \leq t$, we define

$$\tilde{\rho}_{st}(X) := \rho_{st}(X) - \rho_{st}(0), \quad X \in L^2(F_t).$$

Then the restriction of $\tilde{\rho}_{st}$ to $L^\infty(F_t)$ takes values in $L^\infty(F_s)$ and the risk-indifference price $\tilde{x}_{st}$ associated to $\tilde{\rho}_{st}(X)$ coincides with $x_{st}$. In the following $\tilde{\alpha}_{st}$ denotes the minimal penalty associated to $\tilde{\rho}_{st}$ on $L^\infty(F_t)$.

**Theorem 3.7.** Let $(\rho_{st})_{s,t}$ be a fully-dynamic risk measure on $(L^2(F_t))_t$ with $\tilde{\rho}_{st}$ and $\tilde{\alpha}_{st}$ as before. The following two groups of assertions $A$ and $B$ are equivalent for all $s \leq t$:
A1. $E(\text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g)) \in \mathbb{R}$.

A2. $x_{st}(X) = \text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g - X) - \text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g)$ satisfies the no-good-deal bounds on $L_\infty(\mathcal{F}_t)$.

B1. For any probability measure $R \ll P$ on $\mathcal{F}_t$, $R|_{\mathcal{F}_s} = P$, such that $E(\bar{\alpha}_{st}(R)) < \infty$. Then either

- $R \in \mathcal{Q}_{st} \cap I_{st}$
- or $R \in I_{st}^c$

with

$$I_{st} = \left\{ R \ll P, R|_{\mathcal{F}_s} = P : E[\text{esssup}_{g \in C_{st}^\infty} E_R(g|\mathcal{F}_s)] < \infty \right\}.$$

B2. The set $\mathcal{Q}_{st} \cap I_{st}$ is non empty and there exists $R_0 \in \mathcal{Q}_{st} \cap I_{st}$ such that $E(\bar{\alpha}_{st}(R_0)) < \infty$.

Proof. First we assume that A holds and we prove assertions B. B1) Recall that $x_{st}(X) = \bar{x}_{st}(X)$, $X \in L_\infty(\mathcal{F}_t)$. Denote $\gamma_{st}$ the minimal penalty for the restriction of $x_{st}$ to $L_\infty(\mathcal{F}_t)$. Observe that for $R \ll P$, $R|_{\mathcal{F}_s} = P$:

$$\gamma_{st}(R) = \text{esssup}_{X \in L_\infty(\mathcal{F}_t)} [E_R(X|\mathcal{F}_s) - x_{st}(X)]$$

$$= \text{esssup}_{X \in L_\infty(\mathcal{F}_t)} [E_R(X|\mathcal{F}_s) - \text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g - X) + \text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g)]$$

$$= \text{esssup}_{X \in L_\infty(\mathcal{F}_t)} \text{esssup}_{g \in C_{st}^\infty} [E_R(X - g|\mathcal{F}_s) - \tilde{\rho}_{st}(g - X) + E_R(g|\mathcal{F}_s)]$$

$$+ \text{essinf}_{g \in C_{st}^\infty} \tilde{\rho}_{st}(g)$$

$$= \text{esssup}_{Y \in L_\infty(\mathcal{F}_t)} [E_R(-Y|\mathcal{F}_s) - \tilde{\rho}_{st}(Y)] + \text{esssup}_{g \in C_{st}^\infty} E_R(g|\mathcal{F}_s) + \text{essinf}_{g \in C_{st}^\infty} \tilde{\rho}_{st}(g)$$

Hence,

$$\gamma_{st}(R) = \bar{\alpha}_{st}(R) + \text{esssup}_{g \in C_{st}^\infty} E_R(g|\mathcal{F}_s) + \text{essinf}_{g \in C_{st}^\infty} \tilde{\rho}_{st}(g) \quad (3.11)$$

From (3.11) and (A1), taking $R \ll P$ on $\mathcal{F}_t$, $R|_{\mathcal{F}_s} = P$ with $E(\bar{\alpha}_{st}(R)) < \infty$, we have that

$$E(\gamma_{st}(R)) < \infty \iff E[\text{esssup}_{g \in C_{st}^\infty} E_R(g|\mathcal{F}_s)] < \infty. \quad (3.12)$$

So either $E[\text{esssup}_{g \in C_{st}^\infty} E_R(g|\mathcal{F}_s)] = \infty$, which means that $R \in I_{st}^c$, or $E[\text{esssup}_{g \in C_{st}^\infty} E_R(g|\mathcal{F}_s)] < \infty$, and $R \in I_{st}$. In this case, from (3.12), assumption A2 and Proposition 3.3 we conclude that $R \in \mathcal{Q}_{st}$. 
B2) From the representation \textcolor{red}{[3.7]}, there exists a probability measure $R_0$ such that $E(\gamma_{st}(R_0)) < \infty$. The statement follows from the same arguments as above.

Now the converse, assume that B holds and we prove assertions A.

A1) Take $R_0 \in \Omega_{st} \cap I_{st}$ such that $E(\tilde{\gamma}_{st}(R_0)) < \infty$. Then

$$E[\text{essinf}_{g \in C_{st}^\infty} \tilde{\rho}_{st}(g)] \geq E[\text{essinf}_{g \in C_{st}^\infty} E R_0 (-g | F_s)] - E(\tilde{\alpha}_{st}(R_0))$$

$$= -E[\text{esssup}_{g \in C_{st}^\infty} E R_0 (g | F_s)] - E(\tilde{\alpha}_{st}(R_0)) > -\infty$$

A2) Consider $R \ll P$ on $\mathcal{F}_t$, $R | F_s = P$, such that $E(\gamma_{st}(R)) < \infty$. Note that $\tilde{\alpha}_{st}(R) \geq -\tilde{\rho}_{st}(0) = 0$, and $\text{esssup}_{g \in C_{st}^\infty} E R(g | F_s) \geq 0$. From \textcolor{red}{[3.11]}, taking expectation, we can see that $E(\tilde{\alpha}_{st}(R)) < \infty$ and $E[\text{esssup}_{g \in C_{st}^\infty} E R(g | F_s)] < \infty$. From assumption B1, we have $R \in \Omega_{st}$. Then we have

$$x_{st}(X) = \text{esssup}_{R \in \Omega_{st}} [E R (X | F_s) - \gamma_{st}(R)], \quad X \in L_\infty(\mathcal{F}_t)$$

(3.13)

and $x_{st}$ satisfies the no-good-deal bounds on $L_\infty(\mathcal{F}_t)$. \hfill \Box

Moreover, we have the following result.

**Corollary 3.8.** Let $(\rho_{st})_{s,t}$ be a fully-dynamic risk measure on $(L_2(\mathcal{F}_t))_t$ such that $E(\text{essinf}_{g \in C_{st}^\infty} \rho_{st}(g)) \in \mathbb{R}$. Assume that the operator $x_{st}(X), \quad X \in L_\infty(\mathcal{F}_t)$, as in \textcolor{red}{[2.12]} satisfies the no-good-deal bounds on $L_\infty(\mathcal{F}_t)$. Then $x_{st}$ has a unique continuous extension

$$x^\text{ngd}_{st} : L_2(\mathcal{F}_t) \rightarrow L_2(\mathcal{F}_s)$$

satisfying the no-good-deal bounds on $L_2(\mathcal{F}_t)$.

Furthermore, this extension admits the representation

$$x^\text{ngd}_{st}(X) = \text{esssup}_{Q \in \Omega_{st} \cap I_{st}} \left( E Q (X | F_s) - \gamma_{st}(Q) \right), \quad X \in L_2(\mathcal{F}_t).$$

Proof. The extension follows directly from Corollary \textcolor{red}{3.6}. The representation follows from \textcolor{red}{[3.9]} in Corollary \textcolor{red}{3.6} and the arguments of the proof of Theorem \textcolor{red}{3.7}. \hfill \Box

We remark that we do not know whether the extension $x^\text{ngd}_{st}$ is also a risk-indifferent price. We recall that in Section 2 we had assumed that $\rho_{0T}$ is dominated and with this, starting from $x_{st}$ on $L_\infty(\mathcal{F}_t)$ we obtained a risk-indifferent extension:

$$x^{\text{rI}}_{st} : L_\infty^\xi \rightarrow L_\infty^\xi$$

where $L_2(\mathcal{F}_t) \subseteq L_\infty^\xi$. See Definition \textcolor{red}{2.30} and Theorem \textcolor{red}{2.28}. Now we study the relations between these two extensions.
Theorem 3.9. Let \((\rho_{st})_{s,t}\) be a fully-dynamic convex risk measure such that hypotheses B in Theorem 3.7 are satisfied. Assume \(\rho_{0T}\) is dominated (for a constant \(K > 0\)) and sensitive. Then

\[
x^{ri}_{st}(X) = x^{ndg}_{st}(X) \quad P - a.s., \quad X \in L_2(\mathcal{F}_t),
\]

and \(x^{ri}_{st}(X) \in L_2(\mathcal{F}_s), \ X \in L_2(\mathcal{F}_t)\).

Moreover, the following representation holds:

\[
\rho_{st}(X) = \esssup_{R \in (\mathcal{Q}_{st} \cap B^K_{st} \cup I_{st}) \cup (B^K_{st} \cap I_{st}^c)} \left( E_R(-X|\mathcal{F}_s) - \alpha_{st}(R) \right), \quad (3.14)
\]

where

\[
B^K_{s,t} := \left\{ R \ll P : R|_{\mathcal{F}_s} = P, \sup_{Q \in \mathcal{Q}} E_Q \left( E_R(X|\mathcal{F}_s) \right) \leq K \|X\|_2, \forall X \in L_2(\mathcal{F}_t) \right\},
\]

and

\[
x^{ri}_{st}(X) = \esssup_{R \in (\mathcal{Q}_{st} \cap I_{st})} \left( E_R(X|\mathcal{F}_s) - \tilde{\alpha}_{st}(R) \right). \quad (3.15)
\]

Proof. From Proposition 2.31 item 1 and the definition of the semi-norm \(c\), we have that

\[
c(|x^{ri}_{st}(X) - x^{ri}_{st}(Y)|) \leq c(|X - Y|) \leq K \|X - Y\|_2,
\]

for all \(X, Y \in L_2(\mathcal{F}_t)\). Then \(x^{ri}_{st}\) is continuous for the \(L_2\)-norm and we can conclude \(x^{ri}_{st}(X) = x^{ndg}_{st}(X) \quad P - a.s. \) and then also \(x^{ri}_{st}(X) \in L_2(\mathcal{F}_s)\), for all \(X \in L_2(\mathcal{F}_t)\).

The representation

\[
\rho_{st}(X) = \esssup_{R \in \mathcal{P}_{st}} \left( E_R(-X|\mathcal{F}_s) - \alpha_{st}(R) \right), \quad X \in L_2(\mathcal{F}_t),
\]

is derived from Proposition 2.23 item 2. We observe that \(\mathcal{Q} = \tilde{\mathcal{P}}_{0T}\) and that for any \(Q \in \mathcal{Q}\) we have \(Q|_{\mathcal{F}_s} \in \tilde{\mathcal{P}}_{0s}\), from Lemma 2.26 item 2. Moreover, from item 1 in Lemma 2.26 we have that any \(R \in \mathcal{P}_{st}\) is \(R = \tilde{R}|_{\mathcal{F}_s}\), i.e. the restriction to \(\mathcal{F}_t\) of some \(\tilde{R} \in \tilde{\mathcal{P}}_{st}\).

Now take any \(R \in \mathcal{P}_{st} \subseteq \tilde{\mathcal{P}}_{st}\). From Proposition 2.27 we have that, for any \(Q \in \mathcal{Q}\), there exists \(S \in \mathcal{Q}\) such that

\[
E_Q(E_R(X|\mathcal{F}_s)) = E_S(X) \leq K \|X\|_2, \quad X \in L_2(\mathcal{F}_t),
\]

where the last inequality is justified by Proposition 2.10. Then \(R \in B^K_{st}\). Moreover, we observe that the definition of minimal penalty implies that \(\tilde{\alpha}_{st}(R) \leq \alpha_{st}(R) + \rho_{st}(0)\) and then \(E(\tilde{\alpha}_{st}(R)) < \infty\) comes from the definition of \(\mathcal{P}_{st}\). This allows to use property B1 from Theorem 3.7 and we can conclude that \(R \in ((\mathcal{Q}_{st} \cap I_{st}) \cup I_{st}^c) \cap B^K_{st}\). Hence representation (3.14) follows. \(\square\)
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