A QUASI-SOLUTION APPROACH TO BLASIUS SIMILARITY
EQUATION WITH GENERAL BOUNDARY CONDITIONS

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ABSTRACT. A recently developed method [2], [3], and [5] is used to find an analytic approximate solution with rigorous error bounds to the classical Blasius similarity equation with general boundary conditions. This provides detailed proofs for the results reported in [13].

1. INTRODUCTION

The classical similarity solution of Blasius to the boundary layer equation past a semi-infinite plate satisfies the two-point boundary value problem

\[ f'''(x) + f(x)f'' = 0 \quad \text{for} \quad x \in (0, \infty) \]

with no-slip boundary conditions:

\[ f(0) = 0, \quad f'(0) = 0, \quad \text{and} \quad \lim_{x \to +\infty} f'(x) = 1. \]

One may consider \[ f(0) = \tilde{\alpha}, \quad f'(0) = \tilde{\gamma}, \quad \text{and} \quad \lim_{x \to +\infty} f'(x) = 1. \]

In [6], using a transformation

\[ f(x) = a^{-1/2}F(a^{-1/2}x) \]

introduced by Töpfer [12], it is shown that the boundary value problem [1] and [2] can be written as the initial value problem

\[ F'''(x) + F(x)F''(x) = 0 \quad \text{for} \quad x \in (0, \infty) \]

with initial conditions

\[ F(0) = 0, \quad F'(0) = 0, \quad F''(0) = 1. \]

At infinity, \( \lim_{x \to \infty} F'(x) = a \in \mathbb{R}^+ \). Under the transformation, the generalized boundary conditions \[ \text{[3]} \]

become

\[ F(0) = a^{1/2} \tilde{\alpha} \equiv \alpha, \quad F'(0) = a\tilde{\gamma} \equiv \gamma, \quad F''(0) = 1. \]

The non-dimensionalized wall stress is given by

\[ f''(0) = a^{-3/2} \]

In this paper, the quasi-solution approach developed in [4] is adopted to find an approximate analytic solution to the problem [6] with generalized initial conditions [7] and to prove its rigorous error bounds.

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2. Representation of a quasi-solution and main results

For simplicity, we consider the initial value problem (5)–(7) with $\gamma = 0$ and $\alpha \in \mathcal{J} := [-\frac{1}{2}, \frac{1}{2}]$. (Throughout the rest of the paper, whenever (7) is referred, this condition will be assumed.) Through piecewise polynomial representations, other intervals in $\alpha$ can be examined in a similar fashion. Let

$$P(y; \beta) = \sum_{i=0}^{13} \frac{p_{i,j}}{(i+1)(i+2)(i+3)} \beta^j y^i$$

where $p_{i,j}$ is the $(i+1, j+1)$-entry of the following matrix

$$\begin{bmatrix}
935148 & -9845 & 274 & 241 & 422 & 308
493148 & -98042 & 40132715 & 11270972 & 16143111 & 28190517
15185 & -17096 & 36599 & 19441 & 6287 & -10649
-47267 & -473735 & 968864 & -3419898 & 892276 & 3570017
-203116 & -3042 & 15440 & 21239 & 114887 & 5024
65155 & 970153 & 235863 & 89658 & 372923 & 37953
-72804 & 239197 & 213995 & -110679 & 1323305 & 80021
-77453 & 179253 & 190583 & -28721 & 299224 & 35684
106800 & -112122 & 155285 & 525204 & -2092974 & 391166
45663 & 86717 & 19732 & 17519 & 49136 & 20741
587344 & 77473 & 304475 & 3049469 & 445437 & 56873
-30909 & 44072 & 10896 & -226885 & 2681 & 6814
3084825 & 1006071 & 171511 & -3723623 & 1097313 & 1207261
27614 & 9319 & 4286 & 24721 & 2915 & 3463
2254558 & 3595213 & -1689674 & 2081034 & 1011365 & -1249672
-15583 & 35581 & 4399 & 19944 & 3794 & 3794
1915077 & 3165632 & 5196992 & -3429722 & 3839299 & -2755673
2126 & 3527 & 3543 & -1237 & 2153 & 9363
-2862927 & 3706169 & 5245388 & 1764108 & 6522639 & 1116964
-19274 & -26247 & 1929 & -439 & 1666 & 833
281944 & 1574435 & 5003871 & 7633149 & 6098777 & -9281007
19 & 237 & 1621 & -1117 & 958 & 4506
-2906157 & 2704059 & 8285685 & 6455381 & 4186543 & 301817
-2481 & 3157 & 3899 & 1295 & 865 & 1914
2072736 & 1425478 & 3778762 & -980233 & 3100252 & -4063417
5813 & 4881 & 4529 & 486 & 1527 & 5821
-1051227 & 745495 & -1839247 & 1844827 & 2241089 & 3813801
19699 & 17357 & 13071 & 3276 & 6290 & 30274
\end{bmatrix}$$

**Definition 1.** For $\alpha \in \mathcal{J}$, define functions $a_0(\alpha)$, $b_0(\alpha)$, and $c_0(\alpha)$ by

$$a_0(\alpha) = \frac{3221}{1946} - \frac{797}{603} \alpha + \frac{176}{289} \alpha^2$$

$$b_0(\alpha) = -\frac{2763}{1765} - \frac{761}{284} \alpha - \frac{194}{237} \alpha^2$$

$$c_0(\alpha) = \frac{377}{1613} + \frac{174}{1357} \alpha + \frac{937}{6822} \alpha^2$$

and a subset $\mathcal{S}_\alpha$ of $\mathbb{R}^3$ by

$$\mathcal{S}_\alpha = \left\{ (a, b, c) \in \mathbb{R}^3 : \sqrt{(a-a_0(\alpha))^2 + \frac{1}{4}(b-b_0(\alpha))^2 + \frac{1}{4}(c-c_0(\alpha))^2} \leq \rho_0 \right\}$$

where $\rho_0 := 5 \times 10^{-4}$.

**Definition 2.** Given $a, b, c \in \mathbb{R}$ with $a > 0$, define

$$t(x; a, b) = \frac{a}{2} \left( x + \frac{a}{b} \right)$$

$$q_0(t; c) = 2e\sqrt{e^{-t}I_0(t) + c^2e^{-2t}(2J_0(t) - I_0(t) - I_0^2(t))}$$
in which
\begin{equation}
I_0(t) = 1 - \sqrt{\pi t} e^{t} \text{erfc}(\sqrt{t}) \quad \text{and} \quad J_0(t) = I_0(2t)
\end{equation}
where \text{erfc} denotes the complementary error function.

**Note 3.** Let \( \alpha \in \mathcal{J} \) be arbitrary but fixed and let
\begin{align}
a_l & = a_0(\alpha) - \rho_0, \quad a_r = a_0(\alpha) + \rho_0, \\
b_l & = b_0(\alpha) - 2\rho_0, \quad b_r = b_0(\alpha) + 2\rho_0, \\
c_l & = c_0(\alpha) - 2\rho_0, \quad c_r = c_0(\alpha) + 2\rho_0.
\end{align}
Suppose \((a, b, c) \in S_\alpha\). Then it follows that \( a \in [a_l, a_r], \ b \in [b_l, b_r], \) and \( c \in [c_l, c_r] \). Since \( a_0(\alpha) \) and \( b_0(\alpha) \) are quadratic in \( \alpha \), simple calculations show that \( a \in [1.5, 1.75] \) and \( b \in [-1.75, -1.4] \), which implies that \( \frac{b}{a} \geq -1.17 \). In particular, the function \( t(x) := t(x; a, b) \) maps bijectively the interval \( x \in \left[ \frac{5}{2}, \infty \right) \) onto the interval \( t \in [t_m, \infty) \) where \( t_m := t(\frac{5}{2}) \). Moreover, since \( a_l > 0 \) and \( b_r < 0 \),
\begin{equation}
\frac{a_l}{2} \left( \frac{5}{2} + \frac{b_l}{a_l} \right)^2 < t_m < \frac{a_r}{2} \left( \frac{5}{2} + \frac{b_r}{a_r} \right)^2.
\end{equation}
Since \( \alpha \) was chosen arbitrary, we conclude that \( t_m \in (t_{m,l}, t_{m,r}) \) where
\begin{align}
t_{m,l} & = \inf_{\alpha \in \mathcal{J}} \left\{ \frac{a_l}{2} \left( \frac{5}{2} + \frac{b_l}{a_l} \right)^2 \right\} = 1.962257 \cdots \\
t_{m,r} & = \sup_{\alpha \in \mathcal{J}} \left\{ \frac{a_r}{2} \left( \frac{5}{2} + \frac{b_r}{a_r} \right)^2 \right\} = 2.043219 \cdots.
\end{align}
So, provided that \( a > 0 \), the domain \( t \in [T, \infty) \) where \( 1.96 \leq T \leq t_{m,l} \) corresponds to the domain \( x \in \left[ -\frac{b}{a} + \sqrt{\frac{2T}{a}}, \infty \right) \) which is guaranteed to include \( x \in \left[ \frac{5}{2}, \infty \right) \).

The theorem below provides an approximate analytic representation of solution \( F_\alpha \) to \([\mathcal{I} \times \mathcal{J}]\) and \([\mathcal{I} \times \{\gamma\}]\) with \( \alpha \in \mathcal{J} \) and \( \gamma = 0 \).

**Theorem 1.** Let \( \alpha \in \mathcal{J} \) and \( \gamma = 0 \). Then there exists a unique triple \((a, b, c) = (a(\alpha), b(\alpha), c(\alpha)) \in S_\alpha\) such that the function \( F_{0,\alpha} \) defined by
\begin{equation}
F_{0,\alpha}(x) = \begin{cases} 
\alpha + \frac{x^2}{2} + x^3 P \left( \frac{2}{5}x; \frac{25}{3}\alpha + \frac{1}{2} \right), & x \in \left[ 0, \frac{5}{2} \right] \\
ax + b + \sqrt{\frac{a}{2t(x)}} q_0(t(x); c), & x \in \left( \frac{5}{2}, \infty \right)
\end{cases}
\end{equation}
is a representation of the actual solution \( F_\alpha \) to the initial value problem \([\mathcal{I} \times \mathcal{J}]\) and \([\mathcal{I} \times \{\gamma\}]\) within small errors. More precisely, the error term \( E_{\alpha}(x) := F_\alpha(x) - F_{0,\alpha}(x) \) satisfies on \( \mathcal{I} := [0, \frac{5}{2}] \)
\begin{equation}
\|E_{\alpha}'\|_{\infty, \mathcal{I}} \leq 4.8916 \times 10^{-6}, \|E_{\alpha}''\|_{\infty, \mathcal{I}} \leq 3.7474 \times 10^{-6}, \|E_{\alpha}\|_{\infty, \mathcal{I}} \leq 7.9497 \times 10^{-6}
\end{equation}
and for \( x > \frac{5}{2} \)
\begin{equation}
|E_{\alpha}'(x)| \leq 5.4901 \times 10^{-4} t^{-1} e^{-3t}, \ |E_{\alpha}''(x)| \leq 9.8179 \times 10^{-5} t^{-3/2} e^{-3t} , \ |E_{\alpha}(x)| \leq 1.7558 \times 10^{-5} t^{-2} e^{-3t},
\end{equation}
where \( t = t(x; a, b) \).
The proof of Theorem \[1\] relies on the following three propositions.

**Proposition 2.** For each \(\alpha \in J\), let \(F_{1,\alpha}(x)\) be the solution of \([5]\) and \([7]\) on \(I\). Then the error term \(E_{\alpha}(x) = F_{1,\alpha}(x) - F_{0,\alpha}(x)\) verifies the equation

\[
(27) \quad L[E_{\alpha}] = E''_{\alpha} + F_{0,\alpha}E''_{\alpha} + F_{0,\alpha}''E_{\alpha} = -F_{0,\alpha} - F_{0,\alpha}'' - E_{\alpha}'',
\]

for \(x \in I\) and satisfies the bounds given in \([29]\).

**Proposition 3.** Let \(T \geq 1.96\). Given \((a, b, c)\) with \(a > 0\), \(|c| \leq \frac{1}{4}\), in the domain \(x \in [-\frac{b}{a} + \sqrt{\frac{2T}{a}}, \infty)\), which corresponds to the domain \(t = t(x; a, b) \in [T, \infty)\), there exists a unique solution to \([5]\) in the form

\[
(29) \quad F_2(x; a, b, c) = ax + b + \sqrt{\frac{a}{2t}}q(t; c)
\]

where the function \(q(t; c)\) satisfies the condition

\[
(30) \quad \lim_{t \to \infty} \frac{q(t; c)}{\sqrt{t}} = 0.
\]

Furthermore, the function \(E(t; c) = q(t; c) - q_0(t; c)\) satisfies the following bounds for \(t \in [T, \infty)\):

\[
(31) \quad |E(t; c)| \leq 1.6955 \times 10^{-4} e^{-3t} \frac{e^{-3t}}{9t^{3/2}}
\]

\[
(32) \quad \left| \sqrt{t}E''(t; c) - \frac{1}{2t}E(t; c) \right| \leq 1.6955 \times 10^{-4} \frac{e^{-3t}}{3t^{3/2}}
\]

\[
(33) \quad \left| \sqrt{t}E''(t; c) + \frac{1}{2t^{3/2}}E(t; c) \right| \leq 1.6955 \times 10^{-4} \frac{e^{-3t}}{t}
\]

**Proposition 4.** For each \(\alpha \in J\), there exists a unique triple \((a, b, c) \in S_\alpha\) so that the functions \(F_1\) and \(F_2\) in the previous two propositions and their first two derivatives agree at \(x = \frac{a}{2}\).

The proof of Theorem \[1\] follows from Propositions \[2\]-\[4\] as follows: Proposition \[2\] implies that for any \(\alpha \in J\), \(F_{1,\alpha}(x) = F_{0,\alpha}(x) + E_{\alpha}(x)\) satisfies \([5]\) and \([7]\) for \(x \in I\). Note that \(F_{0,\alpha}\) satisfies the initial conditions \(F_{0,\alpha}(0) = \alpha, F_{0,\alpha}'(0) = 0\), and \(F_{0,\alpha}''(0) = 1\).

Proposition \[3\] implies that \(F_2(x; a, b, c) = ax + b + \sqrt{\frac{a}{2t}}q_0(t; c) + E(t; c)\), where \(t = t(x; a, b)\), satisfies \([5]\) in the domain of \(x\) that includes \([\frac{a}{2}, \infty)\) when \((a, b, c) \in S_\alpha\). Proposition \[4\] ensures that both \(F_1\) and \(F_2\) solve the same ODE \([7]\). Furthermore, identifying \(F_{0,\alpha}(x)\) and \(E_{\alpha}(x)\) from Theorem \[1\] for \(x \in (\frac{a}{2}, \infty)\) with \(ax + b + \sqrt{\frac{a}{2t}}q_0(t; c)\) and \(\sqrt{\frac{a}{2t}}E(t; c)\) respectively, and relating \(x\)-derivatives to \(t\)-derivatives via \(t = t(x; a, b)\), the error bounds \([29]\) follows from Proposition \[3\].

The proofs of Propositions \[2\]-\[4\] are presented in the following sections.

### 3. Solution in the finite domain \(I = [0, \frac{5}{2}]\) and Proof of Proposition \[2\]

The quasi-solution \(F_0\) in the compact set \(I\) is obtained by fitting numerical solutions of \([5]\) on \(I\) with high accuracy satisfying various initial conditions. More
Based on how rapidly \( R \) by \( R(x;\alpha) \) for degree 3 and 3 respectively, its derivatives as well as the residual. this subsection, two methods are used to estimate sizes of the quasi-solution and the residual.

\[ F'''(x) + F(x)F''(x) = 0, \quad x \in \mathcal{I} \]
\[ F(0) = \alpha_k, \quad F'(0) = 0, \quad F'''(0) = 1 \]

with absolute errors of the order \( 10^{-16} \). Since numerical differentiation is ill-conditioned, we project the third derivative of the numerical solutions, rather than the solutions themselves, onto the subspace spanned by first several Chebychev polynomials to obtain the set of \( N \) approximate third derivatives written in the form

\[ P_k(x) = \sum_{n=0}^{M} c_{k,n}x^n, \quad k = 0, \ldots, N. \]

We then fit the coefficients \( c_{k,n} \) against \( (\alpha_k)^N \) by degree 5 polynomial \( c_n(\alpha) \) for all \( n = 0, \ldots, M \) and write

\[ P(x;\alpha) = \sum_{n=0}^{M} c_n(\alpha)x^n. \]

This is how the polynomial \( P \) is obtained.

We seek to control the error term \( E_\alpha(x) \) on the interval \( \mathcal{I} \) uniformly in \( \alpha \in \mathcal{J} \) by first estimating the size of the residual

\[ R_\alpha(x) = F'''_{\alpha}(x) + F_0(x)F''_\alpha(x). \]

**Notation 4.** In the following analysis, we will use two notations, for instance, \( R_\alpha(x) \) and \( R(x;\alpha) \) interchangeably. To be more precise, in places where we view the parameter \( \alpha \) as another variable, we regard \( R_\alpha(x) \) as a function \( R(x;\alpha) \) of two variables \( x \) and \( \alpha \) on \( \mathcal{I} \times \mathcal{J} \). In such a case, the derivatives with respect to \( x \) will be denoted by prime notations, e.g., \( \partial_x R(x;\alpha) = R'(x;\alpha) \); the derivatives with respect to \( \alpha \) will be denoted using the partial derivative symbols, e.g., \( \partial_\alpha R(x;\alpha) \).

We then invert the principal part of \( \mathcal{L}[E] \) in Proposition 2 by using initial conditions to obtain a nonlinear integral equation. The smallness of \( R \) and careful bounds on the resolvents allow us to use a contractive mapping argument to draw the desired conclusion.

### 3.1. Estimating sizes of the quasi-solution and the residual on \( \mathcal{I} \).

In this subsection, two methods are used to estimate sizes of the quasi-solution and its derivatives as well as the residual.

#### 3.1.1. Estimation using local Taylor series expansion.

Since \( F_0(x;\alpha) \), now viewed as \( F_0(x;\alpha) \), is a polynomial of \( x \) and \( \alpha \) of degree 16 and 5 respectively, \( R \) is a polynomial of \( x \) and \( \alpha \) of degree 30 and 10 respectively:

\[ R(x;\alpha) = \sum_{m=0}^{30} \sum_{n=0}^{10} c_{m,n}x^m. \]

Based on how rapidly \( R(x;\alpha) \) changes in \( \mathcal{I} \) and in \( \mathcal{J} \), we choose \( \{x_k\}_{k=0}^{15} \) \( \in \mathcal{I} \) given by

\[ \{0, 0.0625, 0.125, 0.25, 0.375, 0.5, 0.75, 1.0, 1.25, 1.4, 1.5, 1.75, 2, 2.25, 2.4, 2.5 \}. \]
and \( \{\alpha_l\}_{l=0}^5 \in \mathcal{J} \) given by
\[
\{-0.06, -0.05, -0.02, 0.02, 0.05, 0.06\}.
\]
We will show that \( R \) is small in the sense that the norm given by
\[
\|R\|_{\infty, \mathcal{I} \times \mathcal{J}} := \sup \{ |R(x; \alpha)| : x \in \mathcal{I}, \alpha \in \mathcal{J} \}
\]
is small.

On each subregion \([x_{k-1}, x_k] \times [\alpha_{l-1}, \alpha_l] \) in \( \mathcal{I} \times \mathcal{J} \), re-expand \( R \) in the scaled variables \( \tilde{x}_k \) and \( \tilde{\alpha}_l \) where
\[
\begin{align*}
\alpha &= \frac{\alpha_l + \alpha_{l-1}}{2} + \frac{\alpha_l - \alpha_{l-1}}{2} \tilde{\alpha}_l \quad \text{and} \quad x = \frac{x_k + x_{k-1}}{2} + \frac{x_k - x_{k-1}}{2} \tilde{x}_k.
\end{align*}
\]
Note that both \( \tilde{\alpha}_l \) and \( \tilde{x}_k \) are in \([-1, 1]\). So we have
\[
R(x; \alpha) = \sum_{m=0}^{30} \sum_{n=0}^{10} c_{m,n}^{(k,l)} \tilde{x}_k^m \tilde{\alpha}_l^n
\]
\[
= \sum_{m=0}^{3} c_{m,0}^{(k,l)} \tilde{x}_k^m + \sum_{n=1}^{3} c_{0,n}^{(k,l)} \tilde{\alpha}_l^n + \sum_{m,n} c_{m,n}^{(k,l)} \tilde{x}_k^m \tilde{\alpha}_l^n
\]
where the last sum is a double summation over all indices left out from the first two. Observe that the first two terms are single-variable cubic polynomials in \( \tilde{x}_k \) and \( \tilde{\alpha}_l \) respectively. So we can determine the maximum \( M_{k,l} \) and the minimum \( m_{k,l} \) of their sum in \( \tilde{x}_k \in [-1, 1] \) and \( \tilde{\alpha}_l \in [-1, 1] \) using calculus. The remaining term in (42) is bounded by its \( l^1 \)-norm:
\[
E_{k,l} := \sum_{m,n} |c_{m,n}^{(k,l)}|.
\]
It follows that on \([x_{k-1}, x_k] \times [\alpha_{l-1}, \alpha_l]\),
\[
m_{k,l} - E_{k,l} \leq R(x; \alpha) \leq M_{k,l} + E_{k,l}.
\]
The maximum and minimum over an arbitrary union of subregions are found by taking the minimum of \( m_{k,l} - E_{k,l} \) and the maximum of \( M_{k,l} + E_{k,l} \) over the appropriate indices \( k \) and \( l \). Note that, though elementary and tedious, these computations are executed easily and exactly with the aid of a computer algebra system since they only involve operations with rational numbers.

Let \( \mathcal{I}_k \) be defined by
\[
\mathcal{I}_1 = [0, 1.25], \quad \mathcal{I}_2 = [1.25, 1.4], \quad \mathcal{I}_3 = [1.4, 2], \quad \mathcal{I}_4 = [2, 2.5].
\]
Using the method outlined above, we obtain estimates of the size of \( R \) on subregions \( \mathcal{I}_k \times \mathcal{J} \):
\[
\begin{align*}
-4.9058 \times 10^{-7} &\leq R(x; \alpha) \leq 5.1794 \times 10^{-7} \quad \text{on } \mathcal{I}_1 \times \mathcal{J}, \\
-8.4748 \times 10^{-8} &\leq R(x; \alpha) \leq 7.5413 \times 10^{-7} \quad \text{on } \mathcal{I}_2 \times \mathcal{J}, \\
1.1011 \times 10^{-7} &\leq R(x; \alpha) \leq 1.3040 \times 10^{-6} \quad \text{on } \mathcal{I}_3 \times \mathcal{J}, \\
4.9134 \times 10^{-7} &\leq R(x; \alpha) \leq 2.9344 \times 10^{-6} \quad \text{on } \mathcal{I}_4 \times \mathcal{J}.
\end{align*}
\]
This implies that \( \| R \|_{\infty, I \times J} \leq 2.9344 \times 10^{-6} \). The same method is used to estimate the size of \( F_0 \) on the subregions:

\[
\begin{align*}
(50) & \quad -0.0601 \leq F_0(x; \alpha) \leq 0.8004 \quad \text{on } I_1 \times J, \\
(51) & \quad 0.7157 \leq F_0(x; \alpha) \leq 0.9753 \quad \text{on } I_2 \times J, \\
(52) & \quad 0.9039 \leq F_0(x; \alpha) \leq 1.7938 \quad \text{on } I_3 \times J, \\
(53) & \quad 1.7819 \leq F_0(x; \alpha) \leq 2.6220 \quad \text{on } I_4 \times J.
\end{align*}
\]

The size of \( F_0' \) is similarly estimated:

\[
\begin{align*}
(54) & \quad -0.0001 \leq F_0'(x; \alpha) \leq 1.1990 \quad \text{on } I_1 \times J, \\
(55) & \quad 1.1179 \leq F_0'(x; \alpha) \leq 1.3091 \quad \text{on } I_2 \times J, \\
(56) & \quad 1.2134 \leq F_0'(x; \alpha) \leq 1.6066 \quad \text{on } I_3 \times J, \\
(57) & \quad 1.4660 \leq F_0'(x; \alpha) \leq 1.7036 \quad \text{on } I_4 \times J.
\end{align*}
\]

Lastly, the estimates of \( F_0'' \) are given:

\[
\begin{align*}
(58) & \quad 0.6770 \leq F_0''(x; \alpha) \leq 1.0144 \quad \text{on } I_1 \times J, \\
(59) & \quad 0.5927 \leq F_0''(x; \alpha) \leq 0.7778 \quad \text{on } I_2 \times J, \\
(60) & \quad 0.2599 \leq F_0''(x; \alpha) \leq 0.6890 \quad \text{on } I_3 \times J, \\
(61) & \quad 0.0881 \leq F_0''(x; \alpha) \leq 0.3099 \quad \text{on } I_4 \times J.
\end{align*}
\]

**Remark 5.** For any given \( x \in I, F_0, F_0', F_0'' \) behave “almost linearly” in \( \alpha \in J \). So we can still obtain fairly reasonable estimates without subdividing \( J \) as presented above.

### 3.1.2. Alternate method using Chebyshev polynomials.

Alternatively, we can find a bound on \( \| R \|_{\infty, I \times J} \) by projecting it onto the orthogonal space of Chebyshev polynomials. To be more precise, take the Chebyshev expansions of the monomials \( x^m \) and \( \alpha^n \) on \( I \) and \( J \) respectively and write

\[
(62) \quad x^m = \sum_{i=0}^{m} p_{m,i} T_i(\tilde{x}) \quad \text{and} \quad \alpha^n = \sum_{j=0}^{n} q_{n,j} T_j(\tilde{\alpha}),
\]

where \( T_k \) is the Chebyshev polynomial of the first kind with degree \( k \), \( \tilde{x} = \frac{x}{\bar{x}} - 1 \), and \( \tilde{\alpha} = \frac{50\alpha}{\bar{x}} \). On substitution, we can rewrite \( R \) as

\[
(63) \quad R(x; \alpha) = \sum_{i=0}^{30} \sum_{j=0}^{10} r_{i,j} T_j(\tilde{\alpha}) T_i(\tilde{x}),
\]

where

\[
(64) \quad r_{i,j} = \sum_{m=i}^{30} \sum_{n=j}^{10} c_{m,n} p_{m,i} q_{n,j}.
\]

Since \( |T_k(y)| = |\cos(k \cos^{-1} y)| \leq 1 \) for \( y \in [-1, 1] \), it immediately follows that

\[
(65) \quad \| R \|_{\infty, I \times J} \leq \sum_{i=0}^{30} \sum_{j=0}^{10} |r_{i,j}|.
\]
Using a computer algebra system, we obtain that \( \|R\|_{\infty, \mathcal{I} \times \mathcal{J}} \leq 3.5551 \times 10^{-6} \). Projecting \( R \) to Chebyshev polynomials in each of the subregions \( \mathcal{I}_k \times \mathcal{J} \) yields somewhat better bounds:

\[
\begin{align*}
(66) & \quad \|R\|_{\infty, \mathcal{I}_1 \times \mathcal{J}} \leq 5.3776 \times 10^{-7}, \quad \|R\|_{\infty, \mathcal{I}_2 \times \mathcal{J}} \leq 9.6144 \times 10^{-7}, \\
(67) & \quad \|R\|_{\infty, \mathcal{I}_3 \times \mathcal{J}} \leq 1.5004 \times 10^{-6}, \quad \|R\|_{\infty, \mathcal{I}_4 \times \mathcal{J}} \leq 2.9505 \times 10^{-6}.
\end{align*}
\]

Observe that these bounds are not as sharp as the ones obtained in (65–68). Yet, this method is simpler and more easily adapted.

### 3.2. Properties of some functions used in the subsequent sections

In this subsection, we will show that, for any fixed \( \alpha \in \mathcal{J} \), each of the functions

\[
G_1 := 2F_0'' - 2F_0, \quad G_2 := F_0'' - 2F_0, \quad G_3 := F_0'' - 2F_0 + 1,
\]

has a unique zero in the interval \( \mathcal{I} \).

We consider \( G_3 \) first. Based on the calculations above, this function is positive on \( \mathcal{I}_1 \). In addition, using the bounds of \( R \), we see that its derivative \( F_0'' - 2F_0 = -F_0F_0'' - 2F_0 + R \) is negative on \( \mathcal{I}_2 \cup \mathcal{I}_3 \cup \mathcal{I}_4 \). Thus, \( G_3 \) has at most one root in \( \mathcal{I} \). Now, the function \( G_3(1.25; \alpha) \) is a polynomial in \( \alpha \) of order 5. Applying the method given in Subsection 3.1 we obtain that

\[
(69) \quad 0.0781 \leq G_3(1.25; \alpha) \leq 0.3463 \quad \text{whereas} \quad -0.3564 \leq G_3(1.4; \alpha) \leq -0.1190,
\]

for all \( \alpha \in \mathcal{J} \), which means that for any given \( \alpha \in \mathcal{J} \), the values of \( G_3 \) at \( x = 1.25 \) and \( x = 1.4 \) have the opposite signs. So by the intermediate value theorem, there exists a unique zero of \( G_3 \) between the two numbers. Similarly, we can show that there is a unique zero of \( G_1 \) in \( \mathcal{I} \) between \( x = 1.15 \) and \( x = 1.3 \) and that \( G_2 \) has its only zero in \( \mathcal{I} \) between \( x = 0.85 \) and \( x = 1.05 \).

### 3.3. The error estimation using the energy method

Let \( \alpha \in \mathcal{J} \) be fixed and consider the linear (generally) inhomogeneous equation

\[
(70) \quad \mathcal{L}[\phi](x) := \phi''(x) + F_0'(x; \alpha)\phi''(x) + F_0'(x; \alpha)\phi(x) = r(x)
\]

over an arbitrary subinterval \([x_1, x_r] \subset \mathcal{I}\), with known initial conditions \( \phi(x_1), \phi'(x_1), \) and \( \phi''(x_1) \). The solution to this equation is given by the standard variation of parameter formula:

\[
(71) \quad \phi(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_1)\Phi_{j, \alpha}(x) + \sum_{j=1}^{3} \Phi_{j, \alpha}(x) \int_{x_1}^{x} \Psi_{j, \alpha}(t)r(t) \, dt
\]

where \( \{\Phi_{j, \alpha}\}_{j=1}^{3} \) form a fundamental set of solutions to \( \mathcal{L}_\alpha[\phi] = 0 \) and \( \{\Psi_{j, \alpha}\}_{j=1}^{3} \) are elements of the inverse of the fundamental matrix constructed from the \( \Phi_{j, \alpha} \) and their derivatives. (In what follows, for the sake of notational simplicity, we will suppress the \( \alpha \)-subscript but remember that these fundamental solutions depend on \( \alpha \).) Since we seek to find the bounds on \( \|\phi\|_{\infty} = \|\phi\|_{\infty, [x_1, x_r]} \), the precise expressions are unimportant. Rather, we proceed by differentiating \( \phi \) twice using properties of \( \Phi_{j} \) and \( \Psi_{j} \) to obtain

\[
(72) \quad \phi''(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_1)\Phi_{j}''(x) + \sum_{j=1}^{3} \Phi_{j}''(x) \int_{x_1}^{x} \Psi_{j}(t)r(t) \, dt.
\]

\(^{(1)}\)In particular, \( \sum_{j=1}^{3} \Phi_{j}(x)\Psi_{j}(x) = 0, \sum_{j=1}^{3} \Phi_{j}'(x)\Psi_{j}(x) = 0 \)
Rewrite (72) by abstractly replacing the second term by an operator $\mathcal{G}$:

$$\phi''(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x) \Phi_j''(x) + \mathcal{G}[r](x). \tag{73}$$

From general properties of fundamental matrix and its inverse for the linear ODEs with polynomial coefficients, $\mathcal{G}$ is a bounded linear operator on $C([x_l, x_r])$; denote its norm by $M_\alpha$,

$$M_\alpha = \|\mathcal{G}\|. \tag{74}$$

Then, on the interval $[x_l, x_r]$, we have

$$\|\phi''\|_\infty \leq \sum_{j=1}^{\infty} M_{j,\alpha} \phi^{(j-1)}(x_l) + M_\alpha |r|_\infty \quad \text{where} \quad M_{j,\alpha} = \sup_{x \in [x_l, x_r]} |\Phi_j''(x)|. \tag{75}$$

To determine bounds on $M_{j,\alpha}$ and $M_\alpha$, we use the “energy method”: take the original ODE

$$\phi''' + F_0 \phi'' + F_0'' \phi = r, \tag{76}$$

multiply it by $2\phi''$, and then integrate from $x_l$ to $x$ using the known initial conditions to obtain

$$\begin{align*}
(\phi''(x))^2 &= (\phi''(x_l))^2 \\
&\quad - \int_{x_l}^{x} \left\{ 2F_0(y; \alpha) (\phi''(y))^2 + 2F_0''(y; \alpha) \phi''(y) \phi(y) - 2\phi''(y) r(y) \right\} dy.
\end{align*} \tag{77}$$

Note that we can express $\phi(x)$ in terms of $\phi''(x)$ by using integration by parts along with the known $\phi(x_l)$ and $\phi'(x_l)$:

$$\ddot{\phi}(x) := \phi(x) - \phi(x_l) - (x-x_l) \phi'(x_l) = \int_{x_l}^{x} (x-y) \phi''(y) dy. \tag{78}$$

Using (78), the equation (77) is now written as

$$\begin{align*}
(\phi''(x))^2 &= (\phi''(x_l))^2 \\
&\quad - \int_{x_l}^{x} \left\{ 2F_0(y; \alpha) (\phi''(y))^2 + 2F_0''(y; \alpha) \phi''(y) \phi(y) - 2\phi''(y) r(y) \right\} dy.
\end{align*} \tag{79}$$

Since the ODE of our interest is linear, we may consider separately the following cases to determine the bounds of $M_j$ and $M$:

- $r = 0$, $\phi(x_l) = 1$, $\phi'(x_l) = 0$, $\phi''(x_l) = 0$;
- $r = 0$, $\phi(x_l) = 0$, $\phi'(x_l) = 1$, $\phi''(x_l) = 0$;
- $r = 0$, $\phi(x_l) = 0$, $\phi'(x_l) = 0$, $\phi''(x_l) = 1$;
- $r \neq 0$, $\phi(x_l) = 0$, $\phi'(x_l) = 0$, $\phi''(x_l) = 0$. 

Using the simple inequality $-2ab \leq a^2 + b^2$, the relation (78), and Gronwall’s inequality, it is shown (see [5] for details) that

\[
M_{1,\alpha} \leq ((F_0'(x;\alpha) - F_0'(x;\alpha))^{1/2} \exp \left[ \frac{1}{2} \int_{x_1}^{x_r} Q_1(y;\alpha) \, dy \right] ,
\]

\[
M_{2,\alpha} \leq \left( \int_{x_1}^{x_r} (y-x_1)^2F_0''(y;\alpha) \, dy \right)^{1/2} \exp \left[ \frac{1}{2} \int_{x_1}^{x_r} Q_1(y;\alpha) \, dy \right] ,
\]

\[
M_{3,\alpha} \leq \exp \left[ \frac{1}{2} \int_{x_1}^{x_r} Q_2(y;\alpha) \, dy \right] ,
\]

\[
M_{\alpha} \leq (x_r-x_1)^{1/2} \exp \left[ \frac{1}{2} \int_{x_1}^{x_r} Q(y;\alpha) \, dy \right] ,
\]

where

\[
Q_1(x;\alpha) = \begin{cases} 
\frac{(x-x_1)^4}{4}F_0''(x;\alpha) + G_1(x;\alpha) & \text{if } G_1(x;\alpha) > 0 \\
\frac{(x-x_1)^4}{4}F_0''(x;\alpha) & \text{if } G_1(x;\alpha) \leq 0,
\end{cases}
\]

\[
Q_2(x;\alpha) = \begin{cases} 
\frac{(x-x_1)^4}{4}F_0''(x;\alpha) + G_2(x;\alpha) & \text{if } G_2(x;\alpha) > 0 \\
\frac{(x-x_1)^4}{4}F_0''(x;\alpha) & \text{if } G_2(x;\alpha) \leq 0,
\end{cases}
\]

\[
Q(x;\alpha) = \begin{cases} 
\frac{(x-x_1)^4}{4}F_0''(x;\alpha) + G_3(x;\alpha) & \text{if } G_3(x;\alpha) > 0 \\
\frac{(x-x_1)^4}{4}F_0''(x;\alpha) & \text{if } G_3(x;\alpha) \leq 0.
\end{cases}
\]

(See Section 4.2 for the definition of $G_i$'s.)

Now, using the estimation method introduced in Subsection 3.4.1 we can show that the $\alpha$-derivatives of the following functions

\[
F_0'(x;\alpha) - F_0'(x;\alpha), \quad F_0''(x;\alpha), \quad G_1(x;\alpha), \quad G_2(x;\alpha), \quad G_3(x;\alpha),
\]

are all negative for any given $x \in \mathcal{I}$. This implies that these functions are decreasing in $\alpha$ on the interval $\mathcal{J}$ and thus they attain the maximal values at $\alpha = -\frac{1}{ab}$ for any $x \in \mathcal{I}$. This allows us to uniformly bound $M_{j,\alpha}$ and $M_{\alpha}$ by $M_{j}$ and $M$ respectively. The results are summarized in Table 1.

### 3.4. The existence of solution and the error estimates.

Using the results from the previous subsection, we can now not only show the existence and the uniqueness of the error $E_{\alpha}$ in the decomposition of the solution

\[
F_\alpha(x) = F_{0,\alpha}(x) + E_\alpha(x),
\]
but also show that it is small uniformly in $\alpha$ on an arbitrary subinterval $[x_l, x_r] \subset \mathcal{I}$. Suppose $E_\alpha(x_l), E'_\alpha(x_l), E''_\alpha(x_l)$ are known. Then on $[x_l, x_r]$, $E_\alpha$ satisfies

$$L_\alpha[E_\alpha] = -E_\alpha E''_\alpha - R_\alpha$$

where $R_\alpha = F''_\alpha + F_\alpha F'_\alpha$. As in (73), this equation is equivalent to the integral equation

$$E''_\alpha = \sum_{j=1}^{3} E^{(j-1)}_\alpha(x_l) \Phi''_\alpha(x) - G[R_\alpha](x) - G[E_\alpha E''_\alpha](x) =: N[E''_\alpha](x)$$

The following lemma which we directly quote from [5] shows that the equation (90) has a unique solution using a contractive mapping argument and provides an error estimate:

**Lemma 5.** Let $\alpha \in \mathcal{J}$ be fixed and assume that for some $\varepsilon > 0$ we have

$$M(|E_\alpha(x_l)| + (x_r - x_l)|E'_\alpha(x_0)|) (1 + \varepsilon) + \frac{1}{2}(x_r - x_l)^2 MB_0(1 + \varepsilon)^2 < \varepsilon,$$

$$M(|E_\alpha(x_l)| + (x_r - x_l)|E'_\alpha(x_0)|) + (x_r - x_l)^2 MB_0(1 + \varepsilon) < 1,$$

where

$$B_0 = M \|R_\alpha\|_{\infty,[x_l,x_r]} + \sum_{j=1}^{3} M_j \left| E^{(j)}_\alpha(x_l) \right|.$$

Then there exists a unique solution $E''_\alpha$ of (90) in a ball of radius $B_0(1 + \varepsilon)$ in the space $C([x_l, x_r])$ equipped with the sup-norm $\|\cdot\|_{\infty,[x_l,x_r]}$.

We refer readers to [5] for proof.

3.5. **End of proof of Proposition 2** Starting from $\mathcal{I}_1 = [0, 1.25]$ with the known initial conditions $E_\alpha(0) = E''_\alpha(0) = E''''_\alpha(0) = 0$, it is verified that the lemma applies to all the subintervals $\mathcal{I}_k$’s and yields small error bounds as shown in Table 2. Hence we conclude that $E_\alpha$ satisfies the equation (24) – (28) with the bounds given in Theorem 1.  

| $B_0$ | $\varepsilon$ | $\|E\|_{\infty,\mathcal{I}_j}$ | $\|E'\|_{\infty,\mathcal{I}_j}$ | $\|E''\|_{\infty,\mathcal{I}_j}$ |
|--------|----------------|---------------------|---------------------|---------------------|
| $\mathcal{I}_1$ | $1.6538 \times 10^{-6}$ | $5 \times 10^{-6}$ | $1.6538 \times 10^{-6}$ | $2.0673 \times 10^{-6}$ | $1.2921 \times 10^{-6}$ |
| $\mathcal{I}_2$ | $2.4371 \times 10^{-6}$ | $7 \times 10^{-6}$ | $2.4371 \times 10^{-6}$ | $3.6556 \times 10^{-6}$ | $1.6296 \times 10^{-6}$ |
| $\mathcal{I}_3$ | $4.3873 \times 10^{-6}$ | $3 \times 10^{-6}$ | $4.3873 \times 10^{-6}$ | $2.6324 \times 10^{-6}$ | $2.6386 \times 10^{-6}$ |
| $\mathcal{I}_4$ | $7.4947 \times 10^{-6}$ | $4 \times 10^{-6}$ | $7.4947 \times 10^{-6}$ | $3.7474 \times 10^{-6}$ | $4.8916 \times 10^{-6}$ |

4. **Solution in $t \geq T \geq 1.96$ for $a > 0$, $|c| < \frac{1}{4}$ and proof of Proposition 3**

The construction of quasi-solution $F_0$ for $x \in [\frac{1}{2}, \infty)$ relies on large $x$ asymptotics, which as it turns out, gives a desirable accurate solution in the entire interval. For the Blasius solution, it is known that any solution with $\lim_{x \to \infty} F'(x) = a > 0$ must have the representation

$$F(x) = ax + b + G(x)$$
where $G(x)$ is exponentially small in $x$ for large $x$. Indeed, through change of variable $t = t(x; a, b)$ given in Definition 2 and $G(t) = \sqrt{\frac{x}{2t}} q(t)$ with $q$ satisfying

\begin{equation}
\frac{d^3}{dt^3} q + \left( 1 + \frac{q}{2t} \right) \frac{d^2}{dt^2} q + \left( -\frac{1}{2t} + \frac{3}{4t^2} - \frac{q}{4t^2} \right) \frac{dq}{dt} + \left( \frac{1}{2t^2} - \frac{3}{4t^3} \right) q + \frac{q^2}{4t^3} = 0
\end{equation}

and from a general theory [11] it may be deduced that small solutions $q$ must have the convergent series representation

\begin{equation}
q(t) = \sum_{n=1}^{\infty} \xi^n Q_n(t), \text{ where } \xi = \frac{ce^{-t}}{\sqrt{t}}
\end{equation}

where the equations for $Q_n$ may be deduced by plugging in (95) into (96) and equating different powers of $\xi$. With appropriate matching at $\infty$, one obtains

\begin{equation}
Q_1(t) = 2tI_0(t) \quad \text{and} \quad Q_2(t) = -tI_0(t) - tI_0(t)^2 + 2tJ_0(t)
\end{equation}

where

\begin{equation}
I_0(t) = 1 - \sqrt{\pi}te^t \text{erfc}(\sqrt{t}) = \frac{1}{2} \int_0^\infty \frac{e^{-st}}{(1+s)^{3/2}} ds,
\end{equation}

\begin{equation}
J_0(t) = 1 - \sqrt{\pi}te^{2t} \text{erfc}(\sqrt{2t}) = \frac{1}{4} \int_0^\infty \frac{e^{-st}}{(1+s/2)^{3/2}} ds.
\end{equation}

The two term truncation of (96) proved adequate to determine an accurate quasi-solution in an $x$-domain that corresponds to $t \geq 1.96$ if $|c| \leq \frac{1}{7}$ to within the quoted accuracy. Note that the solution is only complete after determining $(a, b, c)$ through matching of $F_0$, $F_a'$, and $F_a''$ at $x = \frac{x}{2}$. Since $(a, b, c)$ only needs to be restricted to some small neighborhood of $(a_0(\alpha), b_0(\alpha), c_0(\alpha))$ to accomplish matching (see Proposition 4), the restriction $t \geq 1.96$ is seen to include $x \geq \frac{x}{2}$ as shown in Note 3. Furthermore, the restriction $|c| \leq \frac{1}{7}$ in Proposition 4 is appropriate for the quoted error estimates in $x \geq \frac{x}{2}$ in Theorem 1.

We decompose

\begin{equation}
q(t) = q_0(t) + \mathcal{E}(t),
\end{equation}

where

\begin{equation}
q_0(t) = \frac{ce^{-t}}{\sqrt{t}} Q_1(t) + \frac{c^2 e^{-2t}}{t} Q_2(t).
\end{equation}

**Note 6.** The functions $q$ and $q_0$ (as well as some others to be introduced later) are dependent on $c$, but for the simplicity of notation, it will be suppressed in the current section.

On substituting in (95), we obtain a nonlinear integral equation for $\mathcal{E}$:

\begin{equation}
\mathcal{E}'' + \left( 1 + \frac{q_0}{2t} \right) \mathcal{E}'' + \left( -\frac{1}{2t} + \frac{3}{4t^2} - \frac{q_0}{4t^2} \right) \mathcal{E}'
+ \left( \frac{1}{2t^2} - \frac{3}{4t^3} + \frac{q_0''}{2t} - \frac{q_0'}{4t^2} + \frac{q_0}{2t^3} \right) \mathcal{E}
= -\frac{\mathcal{E}}{2t} + \frac{\mathcal{E}''}{4t^2} - \frac{\mathcal{E}'}{4t^3} - R,
\end{equation}

\begin{small}
(2) Though the non-degeneracy condition stated in [11] does not hold, a small modification leads to the same result.
\end{small}
where the remainder $R = R(t)$ is given by

\[(103) \quad R = \frac{d^3}{dt^3}q_0 + \left(1 + \frac{q_0}{2t}\right) \frac{d^2}{dt^2}q_0 + \left(-\frac{1}{2t} + \frac{3}{4t^2} - \frac{q_0}{4t^2}\right) \frac{d}{dt}q_0 + \left(\frac{1}{2t^2} - \frac{3}{4t^3}\right)q_0 + \frac{q_0^2}{4t^3}.\]

Using the auxiliary function

\[(104) \quad h(t) = e^t \left(\sqrt{t}E''(t) - \frac{E'(t)}{2\sqrt{t}} + \frac{E(t)}{2t^{3/2}}\right)\]

which is related to $E$ by

\[(105) \quad E(t) = \sqrt{t} \int_{\infty}^{t} \frac{ds}{\sqrt{s}} \int_{\infty}^{s} e^{-\tau} \sqrt{\tau} h(\tau) d\tau,\]

the equation (102) is now written as

\[(106) \quad h' = -\frac{q_0 e^t}{2t} h + e^t B E - \frac{E}{2t} h - \sqrt{te^t} R,\]

where

\[(107) \quad B(t) = -\frac{q_0'(t)}{2t^{1/2}} + \frac{q_0(t)}{4t^{3/2}} - \frac{q_0(t)}{4t^{3/2}}.\]

This equation can be rewritten in an integral form

\[(108) \quad h(t) = h_0(t) - \int_{\infty}^{t} \frac{q_0(\tau)e^\tau}{2\tau} h(\tau) d\tau + \int_{\infty}^{t} e^\tau B(\tau) E(\tau) d\tau - \int_{\infty}^{t} \frac{E(\tau)}{2\tau} h(\tau) d\tau =: N[h](t),\]

where

\[(109) \quad h_0(t) = -\int_{\infty}^{t} \sqrt{te^t} R(\tau) d\tau.\]

A contractive mapping argument in a small ball inside the Banach space of $C([T, \infty))$ equipped with the weighted norm

\[(110) \quad \|h\| := \sup_{t \geq T} te^t |h(t)|\]

is possible by utilizing the smallness of the residual $R = R(t)$ as shown in the following proposition:

**Proposition 6.** For $|c| \leq \frac{1}{4}$, $c < 0.03$, and $T \geq 1.96$, there exists a unique solution to the integral equation (108) in a ball of radius $(1 + \varepsilon)\|h_0\|$, implying that $\|h\| \leq (1 + \varepsilon)\|h_0\| \leq 1.6955 \times 10^{-4}$.

This proposition is also found in Costin and Tanveer [5], the only difference being $T \geq 1.96$ here whereas $T \geq 1.99$ in their paper. The proof is omitted here. The error bounds given in Theorem 4 follows immediately from:
Lemma 7. For $a > 0$, $|c| \leq \frac{1}{2}$, and $t \geq T \geq 1.96$, the function $E$ satisfies the following bounds:

\begin{align}
\left| \sqrt{\frac{a}{2t}}E(t) \right| &\leq \sqrt{\frac{a}{9}t^{-2}} e^{-3t} \|h\|, \\
\left| \frac{d}{dx} \sqrt{\frac{a}{2t}}E(t) \right| &\leq \frac{a}{3} t^{-3/2} e^{-3t} \|h\|, \\
\left| \frac{d^2}{dx^2} \sqrt{\frac{a}{2t}}E(t) \right| &\leq \sqrt{2a^{3/2}} t^{-1} e^{-3t} \|h\|,
\end{align}

where $t = t(x; a, b) = \frac{a}{2} \left( x + \frac{b}{a} \right)^2$.

Proof. Note that by the definition of the weighted norm $\| \cdot \|$ given in (110),

\begin{equation}
\int_t^\infty s^{-1/2} e^{-\tau} h(s) \, ds \leq \frac{1}{3} \tau^{-3/2} e^{-\tau} \|h\|.
\end{equation}

Using (115) and the above inequality,

\begin{equation}
|E(t)| = \left| \int_{t}^{\infty} s^{-1/2} \int_{s}^{\infty} \tau^{-1/2} e^{-\tau} h(\tau) \, d\tau \, ds \right| \leq \frac{1}{9} t^{-3/2} e^{-3t} \|h\|
\end{equation}

and from this the first inequality follows immediately. To see the second statement, we note from (110) that

\begin{equation}
\frac{d}{dx} \sqrt{\frac{a}{2t}}E(t) = a \left( E'(t) - \frac{1}{2} E(t) \right) = a \int_{t}^{\infty} \tau^{-1/2} e^{-\tau} h(\tau) \, d\tau,
\end{equation}

and use the inequality (114). The last one follows from checking that

\begin{equation}
\frac{d^2}{dx^2} \sqrt{\frac{a}{2t}}E(t) = \sqrt{2a^{3/2}} e^{-1} h(t).
\end{equation}

and using the definition of the norm.

This leads to the proof of Proposition 3

5. Matching of solutions and proof of Proposition 4

Let $\alpha \in J$ and $(a, b, c) \in S_\alpha$. In order for the two representations of the solution, (SS) and (H), to coincide at $x = \frac{a}{2}$, we match them and their first two derivatives at the point. Let $t_m = t(\frac{a}{2}; a, b)$. Then by (111) and (105), we get

\begin{align}
a &= F'_\alpha(\frac{5}{2}) - a \left( q_0(t_m; c) - \frac{q_0(t_m; c)}{2t_m} \right) - a \int_{t_m}^{\infty} e^{-\tau} \sqrt{\tau} h(\tau; c) \, d\tau =: N_1(a, b, c) \\
b &= F_\alpha(\frac{5}{2}) - \frac{5}{2} N_1(a, b, c) - \sqrt{\frac{a}{2t_m}} q_0(t_m; c) \\
&\quad - \sqrt{\frac{a}{2}} \int_{t_m}^{\infty} \tau^{-1/2} \int_{t_m}^{\infty} s^{-1/2} e^{-\tau} h(s; c) \, ds := N_2(a, b, c) \\
c &= \frac{1}{\sqrt{2a^{3/2}}} \left[ V(t_m; c) + \frac{1}{c} h(t_m; c) \right]^{-1} e^{t_m} F''_\alpha(\frac{5}{2}) =: N_3(a, b, c)
\end{align}
where
\[
V(t; c) = -\frac{2}{c}te^t B(t; c).
\]

**Definition 7.** We define \( A = (a, \frac{1}{2}b, \frac{1}{2}c) \) and
\[
N[A] = (N_1(a, b, c), \frac{1}{4}N_2(a, b, c), \frac{1}{2}N_3(a, b, c)).
\]
For each \( \alpha \in \mathcal{J} \), define
\[
A_{\alpha} = (a_0(\alpha), \frac{1}{2}b_0(\alpha), \frac{1}{2}c_0(\alpha)).
\]
Define also
\[
S_{A, \alpha} = \{ \| A - A_{\alpha, \alpha} \|_2 \leq \rho_0 = 5 \times 10^{-4} \}
\]
where \( \| \cdot \|_2 \) is the Euclidean norm and let
\[
J = \frac{\partial N}{\partial A} = \begin{pmatrix}
\frac{1}{2}b_0 N_1 & 2b_0 N_2 & 2b_0 N_3 \\
\frac{1}{2}b_0 N_2 & \frac{1}{2}b_0 N_1 & 2b_0 N_2 \\
\frac{1}{2}b_0 N_3 & \frac{1}{2}b_0 N_2 & \frac{1}{2}b_0 N_3
\end{pmatrix}
\]
be the Jacobian. Let \( \| J \|_2 \) denote the \( l^2 \) (Euclidean) norm of the Jacobian matrix:
\[
\| J \|_2^2 = (\partial_a N_1)^2 + 4(\partial_b N_1)^2 + 4(\partial_c N_1)^2
+ \frac{1}{4}(\partial_a N_2)^2 + (\partial_b N_2)^2 + (\partial_c N_2)^2 + \frac{1}{4}(\partial_a N_3)^2 + (\partial_b N_3)^2 + (\partial_c N_3)^2.
\]

**Note 8.** \( A \in S_{A, \alpha} \) implies that \( (a, b, c) \in S_\alpha \). The system of equations (118)–(120) is now succinctly written as
\[
A = N[A].
\]

**Lemma 8.** Let \( \alpha \in \mathcal{J} \). Suppose that there exists some \( \beta \in (0, 1) \) satisfying
\[
\| A_{\alpha, \alpha} - N[A_{\alpha, \alpha}] \|_2 \leq (1 - \beta)\rho_0,
\]
\[
\sup_{A \in S_{A, \alpha}} \| J \|_2 \leq \beta.
\]
Then the equation \( A = N[A] \) has a unique solution in \( S_{A, \alpha} \).

**Proof.** Fix an \( \alpha \in \mathcal{J} \) and let \( A \in S_{A, \alpha} \). By the mean-value theorem,
\[
\| N[A] - N[A_{\alpha, \alpha}] \|_2 \leq \| N[A] - N[A_{\alpha, \alpha}] \|_2 + \| N[A_{\alpha, \alpha}] - A_{\alpha, \alpha} \|_2 \\
\leq \| J \|_2 \rho_0 + \rho_0(1 - \beta) \leq \rho_0.
\]
Moreover, if \( A_1, A_2 \in S_{A, \alpha} \),
\[
\| N[A_1] - N[A_2] \|_2 \leq \| J \|_2 \| A_1 - A_2 \|_2 \leq \beta \| A_1 - A_2 \|_2
\]
This implies that the map \( N \) maps the ball \( S_{A, \alpha} \) back to itself and is contractive there. Hence, by the Banach space fixed point theorem, the conclusion follows.

The Proposition 3 follows from Lemma 3 once we show that the conditions (128) and (129) are satisfied for any \( \alpha \in \mathcal{J} \). Following the procedures outlined in [5], it is not difficult to show that \( \beta \leq 0.8381 \) and that \( \| A_{\alpha, \alpha} - N[A_{\alpha, \alpha}] \|_2 \leq 4.1443 \times 10^{-5} \leq (1 - \beta)\rho_0 \) for any \( \alpha \in \mathcal{J} \). This completes the proof of Proposition 3.
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