ON THE ERDŐS-FALCONER DISTANCE PROBLEM FOR TWO
SETS OF DIFFERENT SIZE IN VECTOR SPACES OVER FINITE
FIELDS

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ABSTRACT. We consider a finite fields version of the Erdős-Falconer distance
problem for two different sets. In a certain range for the sizes of the two sets
we obtain results of the conjectured order of magnitude.

1. Introduction

Let \( E \subset \mathbb{R}^s \), and let
\[
\Delta(E) = \{ \| x - y \| : x, y \in E \}
\]
be the set of distances between elements in \( E \), where \( \| \cdot \| \) denotes the Euclidean
metric. Erdős’ distance conjecture [2] is that
\[
\# \Delta(E) \gg \varepsilon (\#E)^{s/2-\varepsilon}
\]
for \( s \geq 2 \) and finite \( E \). In a recent breakthrough paper by Guth and Katz [4], this
problem has been solved for \( s = 2 \), whereas it is still open for higher dimensions.
Later Falconer [3] considered a continuous version of Erdős’ distance problem, re-
placing \( \#E \) by the Hausdorff dimension of \( E \), and \( \#\Delta(E) \) by the Lebesgue measure
of \( \Delta(E) \). More recently, Iosevich and Rudnev [6] dealt with a finite fields version
of these problems. For a finite field \( \mathbb{F}_q \) and \( x \in \mathbb{F}_q^s \), let
\[
|x|^2 = \sum_{i=1}^{s} x_i^2.
\]
Note that this is a natural way of defining distance over finite fields, as for Euclidean
distance keeping the property of being invariant under orthogonal transformations,
whereas on the other hand \( |x|^2 = 0 \) no longer implies that \( x = 0 \), since for \( s \geq 3 \)
all quadratic forms over finite fields are isotropic.
In the following we will always assume that \( q \) is odd; in particular, \( q \geq 3 \). As
pointed out in the introduction of [6], the conjecture (1.1) no longer holds true over
finite fields irrespective of the size of \( E \). One example (see introduction of [4]) for
this phenomenon are sets \( E \) small enough to fall prey to certain number theoretic
properties of \( \mathbb{F}_q \): Let \( q \) be a prime such that \( q \equiv 1 \pmod{4} \), and let \( i \in \mathbb{F}_q \) be a
square root of \(-1\). For the set
\[
E = \{(x, ix) : x \in \mathbb{F}_q \}
\]
in \( \mathbb{F}_q^2 \) one then immediately verifies that \( \#E = q \), but \( \#\Delta(E) = 1 \). For sets of
large enough size, however, one should expect \( \Delta(E) \) to have order of magnitude
\( q \) many elements, or even be the set of all elements in \( \mathbb{F}_q \). In this context, one of

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Iosevich and Rudnev’s main results (see [6], Theorem 1.2) is that if $E \subset \mathbb{F}_q^s$ where $\#E \geq Cq^{s/2}$ for a sufficiently large constant $C$, then

$$\#\Delta(E) \gg \min\left\{ q, \frac{\#E}{q^{(s-1)/2}} \right\},$$

where

$$\Delta(E) = \{|x - y|^2 : x, y \in E\}.$$

In particular, if $\#E \gg q^{(s+1)/2}$, then $\#\Delta(E) \gg q$. For $s = 2$, the stronger result that $\#\Delta(E) \gg q$ if

$$\#E \gg q^{4/3}$$

has recently been established by Chapman, Erdogan, Hart, Iosevich and Koh (see [1], Theorem 2.2). Our focus in this paper is on a generalisation of this problem to the situation of distances between two different sets $E, F \subset \mathbb{F}_q^s$. Analogously to above, we define

$$\Delta(E, F) = \{|x - y|^2 : x \in E, y \in F\}.$$

It is straightforward to adapt Iosevich and Rudnev’s approach to show that if

$$\#(E)(\#F) \geq Cq^s$$

for a sufficiently large constant $C$, then

$$\#\Delta(E, F) \gg \min\left\{ q, \frac{(\#E)^{1/2}(\#F)^{1/2}}{q^{(s-1)/2}} \right\};$$

see also Theorem 2.1 in [10] for a similar result. In particular, if $(\#E)(\#F) \gg q^{s+1}$, then $\#\Delta(E, F) \gg q$. For $s = 2$, the stronger result that $\#\Delta(E, F) \gg q$ if

$$\#(E)(\#F) \gg q^{9/3}$$

has recently been proved by Koh and Shen ([8], Theorem 1.3), this way generalising (1.3), and they also put forward the following conjecture (see Conjecture 1.2 in [9]) generalising Conjecture 1.1 in [6] for even $s$.

**Conjecture 1.** Let $s \geq 2$ be even and $(\#E)(\#F) \geq Cq^s$ for a sufficiently large constant $C$. Then $\#\Delta(E, F) \gg q$.

In this paper we establish the following result, which improves on (1.4) and (1.5) for sets $E, F$ of different size in a certain range for $(\#E)$ and $(\#F)$.

**Theorem 1.** Let $E, F \subset \mathbb{F}_q^s$ where $s \geq 2$. Further, let $\#E \leq \#F$ and $(\#E)(\#F) \geq (900 + \log q)q^s$. Then

$$\#\Delta(E, F) \gg \min\left\{ q, \frac{\#F}{q^{(s-1)/2} \log q} \right\}.$$

For $s = 2$ also the alternative lower bound

$$\#\Delta(E, F) \gg \min\left\{ q, \frac{(\#E)^{1/2}\#F}{q \log q} \right\}$$

holds true.

Note that (1.7) is superior to (1.6) for $s = 2$ if and only if $\#E \gg q$. Note also that Theorem 1 implies that if $(\#E)(\#F) \geq (900 + \log q)q^s$ and $\max\{\#E, \#F\} \geq q^{(s+1)/2} \log q$, then $\#\Delta(E, F) \gg q$. These conditions on $E$ and $F$ are for example satisfied if $\#E \geq q^{(s-1)/2}$ and $\#F \geq (900 + \log q)q^{(s+1)/2}$. Hence, apart from a factor $\log q$, Conjecture 1 holds true for a certain range of cardinalities of $E$ and $F$, both for even and odd dimension $s$. 
Our approach follows that of Iosevich and Rudnev, paying close attention to certain spherical averages of Fourier transforms.

2. Notation

Our notation is fairly standard. Let $\mathbb{C}$ be the field of complex numbers, and we write $\mathbb{F}_q$ for a fixed finite field having $q$ elements, where $q$ is odd, and we denote by $\mathbb{F}_q^*$ the non-zero elements of $\mathbb{F}_q$. Further, if $a \in \mathbb{F}_q^*$, we write $\overline{a}$ for the multiplicative inverse of $a$. Moreover, we write

$$e \left( \frac{j}{q} \right) \quad (1 \leq j \leq q)$$

for the additive characters of $\mathbb{F}_q$, the main character being that where $j = q$. If $q$ is a prime, then $e(j/q)$ is just

$$e \left( \frac{j}{q} \right) = e^{2\pi i \frac{j}{q}}$$

where $i^2 = -1$. If $f : \mathbb{F}_q^s \to \mathbb{C}$ is any function, then we denote by $\hat{f}$ its Fourier transform given by

$$\hat{f}(x) = q^{-s} \sum_{m \in \mathbb{F}_q^s} e \left( \frac{-mx}{q} \right) f(m),$$

where as usual $mx$ is the inner product

$$mx = \sum_{i=1}^s m_i x_i.$$

The function $f$ can be recovered from its Fourier transform $\hat{f}$ via the inversion formula

$$f(x) = \sum_{m \in \mathbb{F}_q^s} e \left( \frac{mx}{q} \right) \hat{f}(m).$$

The tool that underpins many arguments is Plancherel’s formula

$$\sum_{m \in \mathbb{F}_q^s} \left| \hat{f}(m) \right|^2 = q^{-s} \sum_{x \in \mathbb{F}_q^s} |f(x)|^2.$$

All these formulas are easy to verify, and proofs can be found in many textbooks on number theory or Fourier analysis. For a subset $E \subset \mathbb{F}_q^s$, we also write $E$ for its characteristic function, i.e.

$$E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise}, \end{cases}$$

and analogously for subsets $F \subset \mathbb{F}_q^*$. Moreover, let $S_r$ be the sphere

$$S_r = \{ x \in \mathbb{F}_q^s : |x|^2 = r \},$$

and as above we also write $S_r$ for the corresponding characteristic function. Moreover, for $E \subset \mathbb{F}_q^s$ and $r \in \mathbb{F}_q$, let $\sigma_E(r)$ be the spherical average

$$\sigma_E(r) = \sum_{a \in \mathbb{F}_q^s \mid |a|^2 = r} |E(a)|^2.$$
of the Fourier transform $\hat{E}(a)$ of $E$, and we define analogously $\sigma_F(r)$. Furthermore, we define
$$\sigma_{E,F}(r) = \sum_{m \in \mathbb{F}_q^*: |m|^2 = r} \overline{E(m)F(m)},$$
where as usual $\overline{\cdot}$ denotes complex conjugation. In particular, $\sigma_E(r) = \sigma_{E,E}(r)$. Our main tool for bounding $\#\Delta(E,F)$ below is the following upper bound on $\sigma_E \sigma_F$ on average. In the following, all implied $O$-constants depend at most on the dimension $s$.

**Lemma 1.** In the notation from above, let $s \geq 2$. Then we have

$$\sum_{r \in \mathbb{F}_q^*} \sigma_E(r) \sigma_F(r) \ll \log q \left( q^{-2s-1}(\#E)(\#F) + q^{-\frac{s+1}{2}}(\#E)^2(\#F) \right).$$

For odd $s \geq 2$, also the bound

$$\sum_{r \in \mathbb{F}_q} \sigma_E(r) \sigma_F(r) \ll \log q \left( q^{-2s-1}(\#E)(\#F) + q^{-\frac{s+1}{2}}(\#E)^2(\#F) \right).$$

holds true, including the term $r = 0$. Moreover, for $s = 2$ we also have the alternative bound

$$\sum_{r \in \mathbb{F}_q^*} \sigma_E(r) \sigma_F(r) \ll (\log q)^2 q^{-5}(\#E)^{3/2}(\#F).$$

Note that (2.3) is superior to (2.1) for $s = 2$ if and only if $\#E \gg q$. Finally, for fixed $E,F \in \mathbb{F}_q^s$ and given $j \in \mathbb{F}_q^*$, we define

$$\nu(j) = \#\{(x,y) \in E \times F : |x - y|^2 = j\}.$$

### 3. Bounding the Fourier transform of a sphere

In this section we collect some useful bounds on the Fourier transform of a sphere in the finite fields setting.

**Lemma 2.** For $m \in \mathbb{F}_q^s$, let

$$\chi(m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$

Then

$$\hat{S}_r(m) = \frac{\chi(m)}{q} + q^{-\frac{s}{2}-1} c_q s \sum_{j \in \mathbb{F}_q^*} e \left( \frac{jr + |m|^2 j}{q} \right) \eta_q(j),$$

where the complex number $c_q$ depends only on $q$ and $s$, such that $|c_q| = 1$, and where $\eta_q$ denotes a quadratic multiplicative character of $\mathbb{F}_q^*$.

**Proof.** This is Lemma 4 in [5].

**Corollary 1.** Let $m \neq 0$. Then

$$|\hat{S}_r(m)| \leq q^{-s/2}.$$

Moreover, still assuming $m \neq 0$, for $r \neq 0$ or odd $s$, the stronger bound

$$\hat{S}_r(m) \ll q^{-\frac{s+1}{2}}$$
holds true. Further, for \( s \geq 2 \) and \( m = 0 \) we have the bound
\[
|\hat{S}_r(0)| \leq \frac{2}{q}.
\]
Finally,
\[
\hat{S}_0(m) = c_s(q^{-s/2} - q^{-s/2-1})
\]
for \( m \neq 0 \), \( |m|^2 = 0 \) and even \( s \), and
\[
\hat{S}_0(m) \ll q^{-s/2-1}
\]
for \( m \neq 0 \), \( |m|^2 \neq 0 \) and even \( s \).

Proof. The first and third bound follow immediately from Lemma 2 on trivially bounding the sum over \( j \). For the second one we make use of Weil’s seminal work (see for example Corollary 11.12 in [7]) to bound the resulting Kloosterman sum over \( j \) (even \( s \)), or use the elementary evaluation of the Salié sum (see for example Lemma 12.4 in [7]) to bound the relevant sum over \( j \) (odd \( s \)). The last two bounds follow on evaluating the summation over \( j \) after noting that the term \( \eta_s(j) \) vanishes for even \( s \). □

Lemma 3. Let \( s \geq 2 \) and \( r \in \mathbb{F}_q^* \). Then
\[
\sigma_E(r) = q^{-s} \#E + q^{-s+1} (\#E)^2.
\]
This bound is also true for \( r = 0 \) and odd \( s \). Moreover, for \( s = 2 \), we also have the alternative bound
\[
\sigma_E(r) \ll q^{-3} (\#E)^{3/2}.
\]
Proof. The bound (3.1) for \( r \neq 0 \) is essentially Lemma 1.8 in [6], but in order to cover the case \( r = 0 \) and odd \( s \) as well let us give a complete proof. We have
\[
\sigma_E(r) = \sum_{m \in \mathbb{F}_q : |m|^2 = r} |E(m)|^2 \sum_{m \in \mathbb{F}_q} \overline{E(m)} \hat{E}(m) S_r(m)
\]
\[
= q^{-2s} \sum_{m \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q} E(x) e\left(\frac{mx}{q}\right) \sum_{y \in \mathbb{F}_q} E(y) e\left(-\frac{my}{q}\right) S_r(m)
\]
\[
= q^{-2s} \sum_{x, y \in \mathbb{F}_q} E(x) E(y) \sum_{m \in \mathbb{F}_q} e\left(\frac{mx - y}{q}\right) S_r(m)
\]
\[
= q^{-s} \sum_{x, y \in \mathbb{F}_q} E(x) E(y) \hat{S}_r(y - x)
\]
\[
\leq q^{-s} \#E \cdot |\hat{S}_r(0)| + q^{-s} (\#E)^2 \max_{m \in \mathbb{F}_q \setminus \{0\}} |\hat{S}_r(m)|.
\]
Corollary [1] now yields (3.1). The second bound (3.2) is Lemma 4.4 in [1]. □

4. PROOF OF LEMMA [1]

Clearly, by Plancherel’s formula,
\[
\sigma_F(r) \leq \sum_{a \in \mathbb{F}_q} |\hat{F}(a)|^2 = \frac{|F|}{q^s} \leq 1,
\]
and the same bound holds true for $\sigma_E(r)$. Hence, on writing
\[ T_i = \sum_{r \in \mathbb{F}_q^{*2i-1} \leq \sigma_F(r) \leq 2i} \sigma_E(r) \sigma_F(r) \]
for $i \in \mathbb{Z}$, by a dyadic intersection of the range of possible values of $\sigma_F$ we find that
\[ \sum_{r \in \mathbb{F}_q^*} \sigma_E(r) \sigma_F(r) \leq q^{-4s+1} + \sum_{-4s \leq i \leq 0} T_i \ll q^{-4s+1} + \log q \cdot \max_{-4s \leq i \leq 0} T_i. \]
We conclude that there exists a subset $M \subset \mathbb{F}_q^*$ such that
\[ \sum_{r \in M} \sigma_E(r) \sigma_F(r) \ll q^{-4s+1} + \log q \sum_{r \in M} \sigma_E(r) \sigma_F(r) \]
and
\[ A \leq \sigma_F(r) \leq 2A \]
for all $r \in M$, for a suitable positive constant $A$. By Cauchy-Schwarz,
\[ \sum_{r \in M} \sigma_E(r) \sigma_F(r) \leq \left( \sum_{r \in M} \sigma_E(r)^2 \right)^{1/2} \left( \sum_{r \in M} \sigma_F(r)^2 \right)^{1/2}. \]
Let us first bound $\sum_{r \in M} \sigma_E(r)^2$. Using Lemma 3 we obtain
\[ \sum_{r \in M} \sigma_E(r)^2 \leq \left( \max_{t \in \mathbb{F}_q^*} \sigma_E(t) \right)^2 \#M \ll (\#M) \left( q^{-2s-2}(\#F)^2 + q^{-3s-1}(\#E)^4 \right) \]
in general, and for $s = 2$ we also obtain the alternative bound
\[ \sum_{r \in M} \sigma_E(r)^2 \ll (\#M) q^{-6}(\#E)^3. \]
Next, let us bound $\sum_{r \in M} \sigma_F(r)^2$.

**Lemma 4.** We have
\[ \sum_{r \in \mathbb{F}_q} \sigma_F(r) = q^{-s} \#F. \]

**Proof.** Since
\[ \sum_{r \in \mathbb{F}_q} \sigma_F(r) = \sum_{a \in \mathbb{F}_q^*} |\hat{F}(a)|^2, \]
the result follows immediately from Plancherel’s formula
\[ \sum_{a \in \mathbb{F}_q^*} |\hat{F}(a)|^2 = q^{-s} \sum_{a \in \mathbb{F}_q^*} F(a)^2 = q^{-s} \#F. \]

We start with the observation that by (4.2), we have
\[ \sum_{r \in M} \sigma_F(r)^2 \leq 4 \cdot \#M \cdot A^2. \]
Next, by Lemma 4

\[(4.7) \quad q^{-2s}(\#F)^2 = \left(\sum_{r \in \mathbb{F}_q} \sigma_F(r)\right)^2 = \sum_{m,n \in \mathbb{F}_q} \sigma_F(m)\sigma_F(n).\]

Moreover, by \[(4.2),\]

\[(4.8) \quad \sum_{m,n \in \mathbb{F}_q} \sigma_F(m)\sigma_F(n) \geq \sum_{m,n \in \mathbb{M}} \sigma_F(m)\sigma_F(n) \gg (\#M)^2 A^2.\]

By \[(4.6), (4.7),\] and \[(4.8)\] we obtain

\[\sum_{r \in \mathbb{M}} \sigma_F(r)^2 \ll \#M \cdot A^2 \ll (\#M)^{-1} \sum_{m,n \in \mathbb{M}} \sigma_F(m)\sigma_F(n)\]

\[(4.9) \quad \ll (\#M)^{-1} q^{-2s}(\#F)^2.\]

Summarising \[(4.1), (4.3), (4.4)\] and \[(4.9)\], and noting that

\[q^{-4s+1} \ll (\log q)q^{-2s-1}(\#E)/(\#F)\]

since \(\#E, \#F \geq 1\), we obtain

\[\sum_{r \in \mathbb{F}_q} \sigma_E(r)\sigma_F(r) \ll (\log q)q^{-2s-1}(\#E)/(\#F) + q^{-2s+1}((\#E)^2/(\#F)).\]

In case of odd \(s\), Lemma 3 also applies for \(r = 0\), so in the argument above we can replace \(\mathbb{F}_q^*\) by \(\mathbb{F}_q\), this way arriving at \[(2.2).\] Further, using \[(4.5)\] instead of \[(4.4)\], for \(s = 2\) we also obtain

\[\sum_{r \in \mathbb{F}_q} \sigma_E(r)\sigma_F(r) \ll (\log q)q^{-5}(\#E)^{3/2}/(\#F).\]

This completes the proof of Lemma 1. \(\square\)

5. PREPARATIONS FOR THE PROOF OF THEOREM 1

Before we embark on the proof of Theorem 1, we first need to collect some useful lemmata.

**Lemma 5.** Let \(j \in \mathbb{F}_q\). Then

\[\nu(j) = q^{2s} \sum_{m \in \mathbb{F}_q} \hat{S}_j(m)\overline{E(m)\hat{F}(m)}.\]

**Proof.** We have

\[\nu(j) = \sum_{x,y \in \mathbb{F}_q} E(x)F(y)S_j(x - y)\]

\[= \sum_{x,y \in \mathbb{F}_q} E(x)F(y) \sum_{m \in \mathbb{F}_q} e\left(\frac{(x - y)m}{q}\right)\hat{S}_j(m)\]

\[= \sum_{m \in \mathbb{F}_q} \hat{S}_j(m) \left(\sum_{x \in \mathbb{F}_q} E(x)e\left(\frac{xm}{q}\right)\right) \left(\sum_{y \in \mathbb{F}_q} F(y)e\left(-\frac{ym}{q}\right)\right)\]

\[= q^{2s} \sum_{m \in \mathbb{F}_q} \hat{S}_j(m)\overline{E(m)\hat{F}(m)}.\]
Lemma 6. Let $s \geq 2$ and $(\#E)(\#F) \geq 900q^s$. Then

$$\nu(0) \leq \frac{21}{30}(\#E)(\#F).$$

Proof. Since

$$\hat{E}(0) = q^{-s}\#E$$

and

$$\hat{F}(0) = q^{-s}\#F,$$

Lemma 5 yields

$$\nu(0) = (\#E)(\#F)\hat{S}_0(0) + \delta,$$

where

$$\delta = q^{2s} \sum_{m \in \mathbb{F}_q^* : m \neq 0} \hat{S}_0(m)\hat{E}(m)\hat{F}(m).$$

By Corollary 1, it follows that

$$\nu(0) \leq \frac{2(\#E)(\#F)}{q} + |\delta|.$$  

Moreover, Corollary 1 gives

$$|\hat{S}_0(m)| \leq q^{-s/2}$$

for $m \neq 0$. Hence, by Cauchy-Schwarz and Plancherel’s formula,

$$|\delta| \leq q^{2s} \left( \sum_{m \in \mathbb{F}_q^*} |\hat{E}(m)|^2 \right)^{1/2} \left( \sum_{m \in \mathbb{F}_q^*} |\hat{F}(m)|^2 \right)^{1/2} \leq q^{s/2}(\#E)^{1/2}(\#F)^{1/2}.$$  

Since $(\#E)(\#F) \geq 900q^s$, we conclude that

$$|\delta| \leq \frac{(\#E)(\#F)}{30}.$$  

Therefore, since $q \geq 3$, we have

$$\nu(0) \leq 2\frac{(\#E)(\#F)}{q} + |\delta| \leq \frac{21}{30}(\#E)(\#F).$$

Lemma 7. We have

$$\sum_{j \in \mathbb{F}_q^*} \nu(j)^2 \leq \frac{(\#E)^2(\#F)^2}{q} + q^{s-1}(\#E)(\#F)$$

$$+ q^{3s} |\sigma_{E,F}(0)|^2 + q^{3s} \sum_{r \in \mathbb{F}_q^*} \sigma_E(r)\sigma_F(r)$$

$$\leq \frac{(\#E)^2(\#F)^2}{q} + q^{3s} \sum_{r \in \mathbb{F}_q^*} \sigma_E(r)\sigma_F(r) + q^{s-1}(\#E)(\#F).$$

□
Proof. By Lemma 5 and Lemma 2, we have

\[
\sum_{j \in \mathbb{F}_q} \nu(j)^2 = q^{4s} \sum_{j \in \mathbb{F}_q} \sum_{m,n \in \mathbb{F}_q} \hat{S}_j(m) \overline{S}_j(n) \overline{E(m) \hat{F}(m) \hat{E}(n) \hat{F}(n)}
\]

\[
= q^{4s} \sum_{m,n \in \mathbb{F}_q} \overline{E(m) \hat{F}(m) \hat{E}(n) \hat{F}(n)} \sum_{j \in \mathbb{F}_q} \times
\]

\[
\left( \frac{\chi(m)}{q} + q^{-s/2-1} c^s_q \sum_{k \in \mathbb{F}_q^*} e\left( \frac{kj + |m|^2 \overline{4k}}{q} \right) \eta_q^s(k) \right)
\]

\[
\times \left( \frac{\chi(n)}{q} + q^{-s/2-1} c^s_q \sum_{l \in \mathbb{F}_q^*} e\left( \frac{-lj - |n|^2 \overline{4l}}{q} \right) \eta_q^s(l) \right).
\]

We are now going to expand the product and interchange the order of summation of \( j \) and \( k, l \). Since

\[
\sum_{l \in \mathbb{F}_q^*} \sum_{j \in \mathbb{F}_q} e\left( \frac{-lj - |n|^2 \overline{4l}}{q} \right) \eta_q^s(l) = 0,
\]

the two cross terms turn out to be zero. Moreover,

\[
\hat{E}(0) = \overline{E(0)} = q^{-s} \#E
\]

and

\[
\hat{F}(0) = \overline{F(0)} = q^{-s} \#F.
\]

Therefore,

\[
\sum_{j \in \mathbb{F}_q} \nu(j)^2 = \frac{(#E)^2(#F)^2}{q} + q^{3s-2} \sum_{m,n \in \mathbb{F}_q} \overline{E(m) \hat{F}(m) \hat{E}(n) \hat{F}(n)}T(m,n),
\]

where

\[
T(m,n) = c^2_q c^s_q \sum_{j \in \mathbb{F}_q} \sum_{k \in \mathbb{F}_q^*} e\left( \frac{kj + |m|^2 \overline{4k}}{q} \right) \eta_q^s(k) \sum_{l \in \mathbb{F}_q^*} e\left( \frac{-lj - |n|^2 \overline{4l}}{q} \right) \eta_q^s(l)
\]

\[
= q \sum_{k \in \mathbb{F}_q^*} e\left( \frac{4k(|m|^2 - |n|^2)}{q} \right)
\]

\[
= q \left( \sum_{k \in \mathbb{F}_q^*} e\left( \frac{4k(|m|^2 - |n|^2)}{q} \right) - 1 \right)
\]

\[
= \begin{cases} 
q^2 - q & \text{if } |m|^2 = |n|^2 \\
-q & \text{if } |m|^2 \neq |n|^2.
\end{cases}
\]

Hence

\[
(5.1) \quad \sum_{j \in \mathbb{F}_q} \nu(j)^2 - \frac{(#E)^2(#F)^2}{q} \leq U + |V|
\]
Another application of Cauchy-Schwarz shows that

\[ U = q^{3s} \sum_{m, n \in \mathbb{F}_q} \overline{E(m)} \hat{F}(m) \hat{E}(n) \overline{F(n)} = q^{3s} \sum_{r \in \mathbb{F}_q} |\sigma_{E,F}(r)|^2 \]

and

\[ V = q^{3s-1} \sum_{m, n \in \mathbb{F}_q} \overline{E(m)} F(m) \hat{E}(n) \overline{F(n)}. \]

By Cauchy-Schwarz’ inequality,

\[ |\sigma_{E,F}(r)|^2 \leq \left( \sum_{m \in \mathbb{F}_q : |m|^2 = r} |\hat{E}(m)|^2 \right) \left( \sum_{m \in \mathbb{F}_q : |m|^2 = r} |\hat{F}(m)|^2 \right) = \sigma_E(r) \sigma_F(r). \]

Thus

(5.2) \[ U \leq q^{3s} |\sigma_{E,F}(0)|^2 + q^{3s} \sum_{r \in \mathbb{F}_q} \sigma_E(r) \sigma_F(r) \leq q^{3s} \sum_{r \in \mathbb{F}_q} \sigma_E(r) \sigma_F(r). \]

Another application of Cauchy-Schwarz shows that

\[ \left| \sum_{m, n \in \mathbb{F}_q} \overline{E(m)} \hat{F}(m) \hat{E}(n) \overline{F(n)} \right| \leq \left( \sum_{m \in \mathbb{F}_q} |\hat{E}(m)| \right)^2 \left( \sum_{m \in \mathbb{F}_q} |\hat{F}(m)| \right)^2 \leq \sum_{m \in \mathbb{F}_q} |\hat{E}(m)|^2 \sum_{m \in \mathbb{F}_q} |\hat{F}(m)|^2. \]

Hence, by Plancherel’s formula,

(5.3) \[ |V| \leq q^{s-1} (\#E)(\#F). \]

The result now follows from (5.1), (5.2) and (5.3). □

Lemma 8. Let \( s \geq 2 \) be even, \( \#E \leq \#F \) and \( (\#E)(\#F) \geq 900q^s \). Then we have

\[ |\sigma_{E,F}(0)|^2 = q^{-3s} \nu(0)^2 + O \left( q^{-3s-1} (\#E)^2 (\#F)^2 \right). \]

Proof. As in the proof of Lemma 3

\[
\sigma_{E,F}(0) = \sum_{m \in \mathbb{F}_q^* : |m|^2 = 0} \overline{E(m)} \hat{F}(m) = \sum_{m \in \mathbb{F}_q^*} \overline{E(m)} \hat{F}(m) S_0(m) \\
= q^{-2s} \sum_{m \in \mathbb{F}_q^*} \sum_{x \in \mathbb{F}_q^*} E(x) e \left( \frac{mx}{q} \right) \sum_{y \in \mathbb{F}_q^*} F(y) e \left( -\frac{my}{q} \right) S_0(m) \\
= q^{-2s} \sum_{x,y \in \mathbb{F}_q^*} E(x) F(y) \sum_{m \in \mathbb{F}_q^*} e \left( \frac{m(x-y)}{q} \right) S_0(m) \\
= q^{-s} \sum_{x,y \in \mathbb{F}_q^*} E(x) F(y) \hat{S}_0(y-x).
\]
By Corollary 1 and Cauchy-Schwarz’ inequality we obtain
\[ \sigma_E(0) = q^{-s}c_q^s \sum_{x, y \in \mathbb{P}_q^2 : x \neq y, |x - y|^2 = 0} E(x)F(y) \left( q^{-s/2} - q^{-s/2-1} \right) + O \left( q^{-s} \sum_{x \in \mathbb{P}_q^2} E(x)F(x)q^{-1} \right) + O \left( q^{-s} \sum_{x, y \in \mathbb{P}_q^2 : x \neq y, |x - y|^2 \neq 0} E(x)F(y)q^{-s/2-1} \right) = q^{-\frac{3}{2}s}c_q^s (\nu(0) + O(#E)) + O \left( q^{-s-1} \sum_{x \in \mathbb{P}_q^2} E(x)F(x) \right) + O \left( q^{-\frac{3}{2}s-1} \sum_{x, y \in \mathbb{P}_q^2 : x \neq y} E(x)F(y) \right) = q^{-\frac{3}{2}s}c_q^s \nu(0) + O \left( q^{-\frac{3}{2}s} #E \right) + O \left( q^{-s-1}(\#E)^{1/2}/(\#F)^{1/2} \right) + O \left( q^{-s-1}(\#E)/\#F \right) = q^{-\frac{3}{2}s}c_q^s \nu(0) + O \left( q^{-\frac{3}{2}s-1}(\#E)(\#F) \right). \]

Multiplying with \( \sigma_{E,F}(0) \) and noting that \( \nu(0) = O((\#E)(\#F)) \) by Lemma 6 then yields the result. \[ \square \]

**Lemma 9.** Let \( s \geq 2 \), \( \#E \leq \#F \) and \( (\#E)(\#F) \geq (\log q + 900)q^s \). Then
\[
\sum_{r \in \mathbb{P}_q^*} \nu(r)^2 \ll \frac{(\#E)^2(\#F)^2}{q} + (\log q)\frac{q^{-s}}{}(\#E)^2(\#F).
\]

For \( s = 2 \), we also have the alternative bound
\[
\sum_{r \in \mathbb{P}_q^*} \nu(r)^2 \ll \frac{(\#E)^2(\#F)^2}{q} + (\log q)(\#E)^{3/2}(\#F).
\]

**Proof.** By Lemma 6 and Lemma 9 for odd \( s \geq 2 \) we obtain
\[
\sum_{r \in \mathbb{P}_q^*} \nu(r)^2 \ll \frac{(\#E)^2(\#F)^2}{q} + q^{s-1}(\#E)(\#F) + (\log q)(q^{s-1}(\#E)(\#F) + q^{s-1}(\#E)^2(\#F)).
\]
Note that
\[(\log q)q^{s-1}(\#E)(\#F) \ll \frac{(\#E)^2(\#F)^2}{q}\]
since \((\#E)(\#F) \gg (\log q)q^s\), whence
\[
\sum_{r \in \mathbb{F}_q} \nu(r)^2 \ll \frac{(\#E)^2(\#F)^2}{q} + (\log q)q^{\frac{s-1}{2}}(\#E)^2(\#F).
\]
Since \(\nu(0)^2 \geq 0\), this is even stronger than the claim (5.4). For even \(s \geq 2\), Lemma 7 and Lemma 8 yield
\[
\sum_{r \in \mathbb{F}_q} \nu(r)^2 \leq \frac{(\#E)^2(\#F)^2}{q} + q^{-1}(\#E)(\#F)
+ \nu(0)^2 + O(q^{-1}(\#E)^2(\#F)^2) + q^{3s} \sum_{r \in \mathbb{F}_q^*} \sigma_E(r)\sigma_F(r).
\]
As above, subtracting \(\nu(0)^2\) and applying Lemma 11 then gives
\[
\sum_{r \in \mathbb{F}_q^*} \nu(r)^2 \ll \frac{(\#E)^2(\#F)^2}{q} + (\log q)q^{\frac{s-1}{2}}(\#E)^2(\#F).
\]
To obtain the alternative bound for \(s = 2\), we just use the alternative bound in Lemma 11 and keep the rest of the proof the same. \(\square\)

6. PROOF OF THEOREM 1

We follow the argument leading to formula (2.6) in [6]. By definition (2.4) of \(\nu(j)\), clearly
\[
\sum_{j \in \mathbb{F}_q} \nu(j) = (\#E)(\#F).
\]
Hence, by Lemma 6
\[
\left(\sum_{j \in \mathbb{F}_q} \nu(j)\right)^2 - 2\nu(0)^2 \geq \frac{1}{50}(\#E)^2(\#F)^2.
\]
Moreover, by Cauchy-Schwarz,
\[
\left(\sum_{j \in \mathbb{F}_q} \nu(j)\right)^2 \leq 2\nu(0)^2 + 2 \left(\sum_{j \in \mathbb{F}_q} \nu(j)\right)^2
\leq 2\nu(0)^2 + 2 \left(\sum_{j \in \mathbb{F}_q^*} \nu(j)\right)^2 \cdot \left(\sum_{j \in \mathbb{F}_q^*; \nu(j) > 0} 1\right)
\leq 2\nu(0)^2 + 2\#\Delta(E, F) \cdot \sum_{j \in \mathbb{F}_q^*} \nu(j)^2.
\]
Thus
\[
\#\Delta(E, F) \gg \frac{(\#E)^2(\#F)^2}{\sum_{j \in \mathbb{F}_q^*} \nu(j)^2}.
\]
The conclusion now follows immediately from Lemma 9.
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