Sandpile probabilities on triangular and hexagonal lattices

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Received 11 August 2017
Accepted for publication 10 October 2017
Published 6 December 2017

Abstract
We consider the Abelian sandpile model on triangular and hexagonal lattices. We compute several height probabilities on the full plane and on half-planes, and discuss some properties of the universality of the model.

Keywords: Abelian sandpile model, uniform spanning trees, logarithmic conformal field theory

(Some figures may appear in colour only in the online journal)

1. Introduction
The Abelian sandpile model, introduced by Bak, Tang and Wiesenfeld [3], is one of the most studied complex system exhibiting self-organized criticality, meaning it naturally evolves towards and maintains itself in a critical state. Over the years, many analytical and numerical results have been obtained; the 2D isotropic model being particularly amenable to computations due to its Abelian property [8]. Among them are single-site height probabilities on the full lattice or half-lattice [17, 34], various joint probabilities of heights at isolated sites or between a certain type of clusters [15, 22, 23, 30, 31], the effects of boundary conditions and boundary probabilities [4, 11, 14, 16, 28, 37], and avalanche and toppling wave distributions [12, 13, 20, 35]. The model has also been studied, in the scaling limit, in a field-theoretical framework. Namely, it has been shown to be described by a logarithmic conformal field theory with central charge $c = -2$ [17, 22, 24, 25, 29] (for a recent review, see also [38]).

Most of the statistical results have been obtained for a sandpile model defined on the simplest 2D regular lattice, namely the square grid. However, other regular graphs may also be considered, such as the triangular and hexagonal (or honeycomb) lattices, to check the universality of the Abelian sandpile model. In particular, height probabilities and critical exponents have been investigated on such lattices in [10, 21, 27] using renormalization group transformations and numerical simulations. More recently, multisite height-one probabilities...
have been evaluated exactly on the hexagonal lattice [2]. All the results so far point towards a common set of critical exponents and scaling behavior for all lattices.

In this paper, we aim to provide further verifications of the universality of the Abelian sandpile model. To do so, we compute one-site height probabilities on the triangular and hexagonal graphs, for the full infinite lattice or half-lattices. We also give explicit expressions for two-site probabilities on the boundary of two types of hexagonal half-lattices.

The article is organized as follows. In section 2, we recall the definition of the Abelian sandpile model on a generic graph. In section 3, we discuss the relation between recurrent states of the model and uniform spanning forests with marked vertices called nodes. We give an overview of the techniques required to compute probabilities in the latter model, based on graphs equipped with a connection [9, 18, 19]. Using these results, we compute several explicit sandpile probabilities on the triangular and hexagonal lattices in sections 4 and 5. We compare our results with their analogues on the square lattice in section 6.

2. Abelian sandpile model

In this section we recall the definition of the Abelian sandpile model [3] on a generic undirected connected graph $G$. Let us denote by $G^*$ the extended graph obtained from $G$ by adding a vertex $s$ together with edges between $s$ and a collection of selected vertices $D$ of $G$. The vertex $s$ is called a sink or root, and its neighbors on $G^*$ (i.e. the vertices in $D$) are called dissipative or open sites ($D$ is also said to be wired to the sink). By contrast, the vertices in $G^* \setminus D$ are said to be closed. In all computations presented in this paper, we shall assume $G$ to be planar, such that $D$ is a subset of the vertices adjacent to the outer face of the graph.

With each vertex or site $i$ of $G$ we associate an integer $h_i \in \mathbb{N}^*$, which can be thought of as the number of sand grains piled up at $i$. We call the collection $C = \{h_i\}_{i \in G}$ a sandpile configuration on $G$. $C$ is said to be stable if $h_i \leq \text{deg}_{G^*}(i)$ for every site $i \in G$, where $\text{deg}_{G^*}(i)$ is the degree of $i$ in $G^*$. The toppling matrix $\Delta = \Delta_G$ of the Abelian sandpile model is defined as follows, for any $i, j \in G$:

$$
\Delta_{ij} = \begin{cases} 
\text{deg}_{G^*}(i) & \text{if } i = j, \\
-1 & \text{if } i \text{ and } j \text{ are neighbors on } G, \\
0 & \text{otherwise}.
\end{cases}
$$

Thus the toppling matrix is the discrete graph Laplacian with Dirichlet boundary conditions at $\{s\}$ (that is, the standard graph Laplacian of $G^*$ from which we remove the row and column indexed by $s$).

The sandpile model evolves in time through a two-step discrete algorithm, mapping a stable configuration $C_t$ at time $t$ onto another stable configuration $C_{t+1}$ at time $t+1$. First a grain of sand is added to a random site $i$, increasing its height $h_i$ by 1: $h_i^{\text{new}} = h_i^{\text{old}} + 1$. If $h_i^{\text{new}} \leq \text{deg}_{G^*}(i)$, the new configuration is stable and defines $C_{t+1}$. If rather $h_i^{\text{new}} > \text{deg}_{G^*}(i)$, $i$ topples, that is, the height at every site $j$ of $G$ is shifted by $-\Delta_{ij}$ (in particular $h_i^{\text{new}} \rightarrow h_i^{\text{new}} - \text{deg}_{G^*}(i)$). This relaxation procedure is repeated in the case where some of the shifted heights $h_i$ exceed $\text{deg}_{G^*}(i)$, until all sites of $G$ are stable. The resulting sandpile configuration defines $C_{t+1}$.

The sink $s$ plays a particular role, as it never topples (thus its height grows unboundedly over time). Its presence is required to ensure that only a finite number of topplings occur at each time step [8]. The model is said to be Abelian because the stable configuration obtained through relaxation does not depend on the order in which unstable sites topple.

The dynamics presented above defines a discrete Markov chain on the space of stable configurations. It has a unique invariant measure $\mathbb{P}_C$ given by
\begin{equation}
\Pr^*_G(C) = \begin{cases} 
\frac{1}{|\mathcal{R}|} & \text{if } C \text{ is recurrent,} \\
0 & \text{if } C \text{ is transient,}
\end{cases} \tag{2.2}
\end{equation}

where \( \mathcal{R} \) is the subset of recurrent configurations on \( G \), with \(|\mathcal{R}| = \det \Delta \) [8]. While \( \Pr^*_G \) is simple since it is uniform on \( \mathcal{R} \), it has a complicated support. Indeed, a stable configuration \( C \) is recurrent if and only if it contains no subconfigurations of a certain type, said to be forbidden [8]. Assessing whether a configuration is recurrent or transient therefore requires that the whole lattice be scanned.

An alternative, more useful, characterization of recurrent sandpile configurations has been provided by the so-called burning algorithm [24], which yields a bijection between such configurations on \( G \) and uniform spanning trees on \( G^* \). Most of the sandpile results have been obtained through this correspondence, in particular single-site and multisite height probabilities. Indeed, this algorithm allows one to trade a globally constrained model in terms of local degrees of freedom for an unconstrained one in terms of nonlocal degrees of freedom; the latter of which proving to be analytically more tractable. The calculations with standard graph-theoretical methods have however remained long and cumbersome. Recently, Kenyon and Wilson devised new techniques using graphs with connections [19], which greatly simplify spanning tree computations. In the next section, we recall the explicit equation relating height probabilities to certain classes of spanning trees, and give an overview of the methods developed in [19]. We refer the reader to our previous work [33] for a detailed exposition of these techniques applied specifically to the sandpile model on the square lattice.

3. Spanning tree probabilities

The authors of [24] defined a burning algorithm to establish a one-to-one correspondence between recurrent sandpile configurations on a graph \( G \) and rooted spanning trees on \( G^* = G \cup \{s\} \), that is, oriented spanning trees in which all edges are directed towards the root \( s \). A modified version of this algorithm was used in [34] to express one-site sandpile probabilities \( \Pr^*_G(i) \equiv \Pr^G(h_i=a) \) in terms of fractions of spanning trees with fixed topological properties:

\begin{equation}
\Pr^*_G(i) = \sum_{q=0}^{a-1} \frac{X^G_q(i)}{\deg_{G^*}(i)} - q = \Pr^*_G(i) + \frac{X^G_{a-1}(i)}{\deg_{G^*}(i) + 1 - a^*}, \quad (\Pr^*_G(0) \equiv 0), \tag{3.1}
\end{equation}

where \( X^G_q(i) \) denotes the fraction of rooted spanning trees on \( G^* \) such that \( i \) has \( q \) predecessors among its nearest neighbors (\( j \) is a predecessor of \( i \) on a rooted spanning tree if the unique path from \( j \) to the root \( s \) goes through \( i \)). A similar decomposition of spanning trees into subclasses can be utilized for multisite probabilities [16, 28, 33].

Among the fractions of spanning trees \( X^G_q(i) \), the case \( q = 0 \) is the easiest to compute (see [23] for the square lattice and [2] for the hexagonal lattice). Trees contributing to \( X^G_0(i) \) are such that \( i \) is a leaf, as it has no predecessor among its nearest neighbors. Such spanning trees on \( G^* \) are in \( \deg_{G^*}(i) \)-to-one correspondence with unconstrained spanning trees on a modified graph \( G^* \), obtained by cutting all but one adjacent edge to \( i \). We denote these removed edges by \( \{i,j\} \), with \( 1 \leq \ell \leq \deg_{G^*}(i) - 1 \). The factor \( \deg_{G^*}(i) \) in the correspondence comes from the fact that a leaf at \( i \) can be connected to any of its neighbors on \( G^* \), but to only one on \( \overline{G^*} \). Therefore

\begin{equation}
X^G_0(i) \det \Delta_G^* = \deg_{G^*}(i) \times \det \Delta_G. \tag{3.2}
\end{equation}
Note that the difference between the two Laplacians, $B = \Delta_{\sigma} - \Delta_{\mathcal{G}}$, is nonzero on a finite submatrix only: $B_{ij} = -\text{deg}_{\mathcal{G}^*}(i)+1$, $B_{j,i} = -1$, $B_{i,i} = B_{j,j} = 1$ and $B_{m,n} = 0$ otherwise. Its rank is equal to $r = \text{deg}_{\mathcal{G}^*}(i)-1$. The fraction of interest thus reads:
\[
X_0^G(i) = \text{deg}_{\mathcal{G}^*}(i) \times \det(\mathbb{1} + GB),
\]
where $G \equiv (\Delta_{\mathcal{G}})^{-1}$ is the Green function of the graph $\mathcal{G}^*$ with Dirichlet boundary conditions at $\{s\}$. Since the rank of $B$ is finite, one sees that the computation of $X_0^G(i)$ consists in evaluating the determinant of an $r \times r$ matrix, even if the graph $\mathcal{G}$ is infinite. Finding the explicit value of $X^G_0(i)$ on regular graphs is thus straightforward, as the Green functions for such graphs are well known (see e.g. [7] and the appendix of this paper for the triangular lattice).

Fractions $X^G_{q>0}(i)$ are harder to evaluate because the predecessor property imposes nonlocal constraints on spanning trees. A closed form for one-point probabilities $P_n(i)$ on the square lattice was only obtained [5, 19, 32] 20 years after they were first written using an integral representation [34]. In particular, the authors of [19] developed a combinatorial method based on the so-called line bundle Laplacian introduced in [18] (see also [9]), which we have used to obtain the results presented in this article. We shall give an overview of their method and refer the reader to the original paper for more details, or to [33] for its specific application to the Abelian sandpile model.

The main tool of [19], the line bundle Laplacian $\Delta = \Delta_{\mathcal{G}}$, is an extension of the usual graph Laplacian $\Delta$. It is defined as follows, for Dirichlet boundary conditions at $\{s\}$ as in section 2,
\[
\Delta_{ij} = \begin{cases} 
\text{deg}_{\mathcal{G}^*}(i) & \text{if } i = j, \\
-\phi_{ij} & \text{if } i \text{ and } j \text{ are neighbors on } \mathcal{G}, \\
0 & \text{otherwise,}
\end{cases}
\]
for any vertices $i,j \in \mathcal{G}$ (thus excluding the sink $s$). The $\phi_{ij}$'s are nonzero complex numbers that satisfy the property $\phi_{ij} = \phi_{ji}^{-1}$. The quantity $\phi_{ij}$ is called the parallel transport from $i$ to $j$. If all parallel transports are trivial, $\Delta = \Delta$. This generalized\footnote{It should be noted that the definition (3.4) differs from that of a weighted graph Laplacian, since the $\phi_{ij}$'s do not contribute to the diagonal of $\Delta$.} Laplacian allows one to compute specific spanning forest events described hereafter.

Let $\mathcal{G}^*$ be a planar graph with a marked face $f$. We select a subset $\bar{N}$ of the vertices lying along the boundary of $f$, together with the sink $s$, and call them nodes. We label them from 1 to $n-1$ in counterclockwise order, with $n = s$. In [19] $\mathcal{G}^*$, together with $\bar{N}$, is called an annular-one surface graph. Let $\Delta$ be the line bundle Laplacian on $\mathcal{G}^*$ with trivial parallel transports everywhere except on the directed edges $(k, \ell)$ crossed by a zipper, that is, a path on the dual graph from $f$ to the outer face: $\phi_{k,\ell} = \phi_{\ell,k}^{-1} = z \in \mathbb{C}^*$. It follows that the Laplacian $\Delta(z)$ and its inverse $\mathcal{G}(z) = (\Delta(z))^{-1}$ depend on this parameter $z$, with $\Delta(1) = \Delta$ and $\mathcal{G}(1) = G$. The function $\mathcal{G}(z)$ is called the line bundle Green function of the graph $\mathcal{G}^*$. Up to relabeling, we can always assume that the edge adjacent to $f$ that the zipper crosses lies between nodes 1 and $n-1$, as depicted in figure 1.

Let $\sigma$ be a partition of the nodes $\bar{N}$ such that $n$ (the sink) is not in a singleton. We denote by $Z[\sigma]$ the number of spanning forests on $\mathcal{G}^*$ such that the nodes $\{1, \ldots, n\}$ lie on different trees according to $\sigma$ (see for instance figure 2); $Z \equiv Z[12 \cdots n]$ is thus the total number of spanning trees on $\mathcal{G}^*$. As we shall explain below, the fractions of spanning trees $X^G_0(i)$, and hence the one-site sandpile probabilities, can be expressed in terms of the $Z[\sigma]$'s. For partial pairings, that is, partitions of the form $\sigma = r_1s_1 \cdots |r_k|s_k|t_1| \cdots |t_\ell$, with $\bar{N} = R \cup S \cup T$ and $R, S, T$ disjoint, the all-minors matrix-tree theorem [6] states that
\[
\text{det}(\mathcal{G}(z)) = Z \prod_{i,j} \phi_{ij}^{-1}.
\]
\[
\det G_{R \cup T}^{S \cup T} = \sum_{\rho \in S_k} \varepsilon(\rho) Z[r_1 s_{\rho(1)}] \cdots [r_k s_{\rho(k)}] t_1 \cdots [t_\ell],
\]

where the sum is over all permutations of the symmetric group on \(k\) objects, and \(\varepsilon(\rho)\) is the signature of \(\rho\). The notation \(G_{R \cup T}^{S \cup T}\) refers to the restriction of the Green matrix to the rows (resp. columns) indexed by the nodes in \(R \cup T\) (resp. \(S \cup T\)), with the condition \(G_{i,\rho} = G_{n,j} = 1\) for any node \(1 \leq i \leq n-1\).

For annular-one surface graphs, there are \(\frac{1}{2} \binom{2k}{k}\) possible planar pairings of \(2k\) nodes (assuming the set \(T\) of isolated nodes is fixed, and does not contain the sink \(n\)), and as many ways to divide \(\mathcal{N} \setminus T\) into two subsets of order \(k\). However, the linear system for the \(Z[\sigma]'s\) provided by equation (3.5) is not invertible. A solution to this problem has been presented.
in [19]. It relies on a generalization of the all-minors matrix-tree theorem for graphs with a nontrivial connection, which relates minors of the line bundle Green function to partition functions of combinatorial objects called cycle-rooted groves. These are spanning subgraphs in which each component is either a tree with at least one node, or a cycle-rooted tree (i.e. a unicycle) containing no nodes. The partition function of all cycle-rooted groves in which the nodes are distributed in the tree components according to the partition $\sigma$ is denoted by $\mathcal{Z}[\sigma]$. The weight of a grove is given by the product of the weights of its components. The weight of a spanning tree is defined as the product of the weights of its edges, while that of a cycle-rooted tree includes an extra factor that vanishes for a trivial connection. For such a connection, a cycle-rooted grove is thus simply a spanning forest and $\mathcal{Z}[\sigma] = Z[\sigma]$.

The key property of this generalized version of equation (3.5) is that the linear system it provides becomes invertible on an annular-one surface graph equipped with a nontrivial connection. Thus the partition function $\mathcal{Z}[\sigma]$ for any partial pairing can be written as a linear combination of minors of the line bundle Green function $\mathscr{G}$. For a connection supported on a zipper with parallel transport $z \in \mathbb{C}^*$, the number of spanning forests $Z[\sigma]$ can then be obtained by taking the limit $z \to 1$ of $\mathcal{Z}[\sigma]$. The main result of [19] is the following:

$$\frac{Z[\sigma]}{Z} = \text{a function of } G, G' \text{ evaluated at the nodes } N \setminus \{n\},$$

(3.6)

where $G' = d\mathscr{G}/dz|_{z=1}$ is called the Green function derivative (a complicated but explicit combinatorial expression of $Z[\sigma]/Z$ for any partial pairing $\sigma$ has been given in [26]). This function can be expressed in terms of the standard Green function through the relation

$$G'_{ij} = \sum_{(k,l): \phi_{kl}=z} (G_{ik} G_{lj} - G_{ij} G_{kl}).$$

(3.7)

More generally, the number $Z[\pi]$ of spanning forests whose nodes are distributed in trees according to the generic partition $\pi$ can be computed as a linear combination of $Z[\sigma]$’s (provided that $n$ is not in a singleton), where $\sigma$ are partial pairings [19]. We shall give an example of such a reduction in the next section.

Before going any further, let us make two observations. First, a natural question is, why bother to introduce a connection on the graph and then take its trivial limit? Although the operation seems trivial, it is not, as the ratios $Z[\sigma]/Z$ are functions of the Green function derivative $G'$, which explicitly depends on the location of the zipper edges (see equation (3.7)). This dependence remains manifest even after taking the limit $z \to 1$, see (4.7) for an illustration. Second, requiring the zipper to start between the nodes 1 and $n-1$ on the boundary of $\mathcal{F}$ does not determine it uniquely. Indeed, many choices are admissible (see for instance figure 1), and lead to seemingly distinct expressions for the Green function derivative (3.7). It turns out that the functions $G'_{u,v}$ associated with different admissible zippers are closely related to one another: either they are identical, or they differ by $\pm G_{u,v}$ [19, 33]. In any case, they yield the same values for the ratios $Z[\sigma]/Z$.

As the standard Green functions of regular graphs are well known, most of the work consists in evaluating the Green function derivative. We shall discuss two methods for calculating $G'_{u,v}$, depending on whether its arguments $u, v$ are close to or far away from the marked face $f$. In the former case, a clever use [19] of the transformation properties of the derivative under translations and deformations of the zipper allows one to find the value $G'_{u,v}$ without actually computing the (possibly infinite) sum appearing in equation (3.7). If rather $u$ and/or $v$ are separated by a large distance $r$ from $f$, we shall give an asymptotic expression for $G'_{u,v}$ in terms of inverse powers of $r$. Since the discussion for both cases is rather technical, we shall reserve it for the appendix.

Let us now return to the main point of interest of this paper, namely sandpile probabilities. As recalled in equation (3.1), one can evaluate them by computing the fractions of spanning
trees on the extended graph $G^*$ such that the reference site $i$ has a fixed number of predecessors among its neighbors. The connection with spanning forests with nodes goes as follows. Let $T$ be a spanning tree on $G^*$ such that $i$ has $0 \leq q \leq \deg_{G^*}(i) - 1$ predecessors among its nearest neighbors, which we denote by $j_1, \ldots, j_q$. Let $F$ be the spanning forest obtained by removing all the edges of $T$ that contain $i$, as illustrated in figure 3. The vertex $i$, its neighbors and the sink are taken as the nodes, and are distributed in trees of $F$ according to a partition $\sigma$, with the following properties:

- $i$ is an isolated vertex;
- the neighbors of $i$ not in $\{j_1, \ldots, j_q\}$ lie on a single tree connected to the sink $s$;
- $j_1, \ldots, j_q$ lie on (possibly distinct) trees disconnected from the sink.

Any spanning tree $T'$ possessing the same predecessorship relations between $i$ and its neighbors as $T$ will give rise to the same partition: $\sigma(T) = \sigma(T')$. Conversely, any forest on $G^*$ of the type $\sigma$ that meets these requirements can be extended into a spanning tree contributing to $X_{G^*}^q(i)$ by adding a finite number of edges between $i$ and its neighbors (note that the choice of these extra edges is not necessarily unique). We can thus use the result (3.6) to compute the fractions $X_{G^*}^q(i)$, provided we slightly modify the graph by removing a few edges, so that $i$ and its neighbors lie around a single face (to form an annular-one surface graph).

4. The triangular lattice

We first consider the Abelian sandpile model on the triangular lattice $G = \mathcal{L}_T$, and compute one-site probabilities using equation (3.1). We associate with each site $i \in G$ a height $h_i \in \{1, 2, 3, 4, 5, 6\}$ and coordinates $\mathbf{r} = (x, y) = x \mathbf{e}_1 + y \mathbf{e}_2$, where the unit vectors $\mathbf{e}_1, \mathbf{e}_2$ form an angle of $120^\circ$ (see figure 4). The six neighbors of the origin $(0, 0)$ are therefore located at $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 0)$, $(-1, -1)$ and $(0, -1)$. The Euclidean distance $r$ separating $\mathbf{r} = (x, y)$ from the origin $(0, 0)$ is given by $r^2 = x^2 + y^2 - xy$.

4.1. On the plane

The standard graph Laplacian $\Delta$ on $G$ with a sink $s$ is defined by equation (2.1), with each vertex having degree six on the lattice. Due to translation invariance, the standard Green function $G_{u,v} = (\Delta^{-1})_{u,v}$ only depends on the difference $v - u \equiv (x, y)$. Its Fourier representation reads
\[ G_{x,y} = G(y - u) \equiv G(x,y) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i\theta_1 + iy\theta_2} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \left( 6 - 2\cos \theta_1 - 2\cos \theta_2 - 2\cos(\theta_1 + \theta_2) \right) \]  
\hspace{2cm} (4.1)

Due to the symmetries of the lattice, the Green function satisfies 12 identities, which can be obtained by repeated applications of the two following relations,

\[ G(x,y) = G(x - y, x) = G(x - y, -y), \]  
\hspace{2cm} (4.2)

corresponding respectively to a counterclockwise rotation of 60° and a reflection with respect to the horizontal axis. Although the integral (4.1) diverges, the difference \( G(x,y) - G(0,0) \) is finite. Its values for short distances, as well as its asymptotic behavior, are collected in the appendix. To compute probabilities for predecessor diagrams, we use a line bundle Laplacian \( \Delta \) with a zipper attached to the face whose lower left corner is the origin (see the left panel of figure 5). We choose here a nontrivial parallel transport \( z \) on the edges of the form ((0, k), (1, k)) and ((0, k - 1), (1, k)) for \( k \leq 0 \).

The reference site \( i \) for the fractions \( X_q(i) \) is taken to be the origin without loss of generality. The site \( i \), its six neighbors and the sink form the subset of selected vertices called nodes (see section 3). To meet the requirement that all nodes but the sink lie along the boundary of a single face \( f \), we define a modified graph \( \hat{G} \) by cutting the edges between \( i \) and four of its neighbors, as depicted on the right panel of figure 5. We label \( i \) and its neighbors from 1 to 7 in counterclockwise order along the boundary of \( f \), starting at the right of the zipper.

The fraction \( X_0(i) \) is obtained using equation (3.3) and can be written in terms of the standard Green function (4.1) evaluated at \( i \) and its neighbors. Its explicit value immediately yields the height-one probability on the triangular lattice:

\[ P_1(i) = \frac{X_0(i)}{6} = -\frac{25}{648} - \frac{55}{72\sqrt{3}\pi} + \frac{7}{3\pi^2} + \frac{11\sqrt{3}}{\pi^3} - \frac{90}{\pi^4} + \frac{54\sqrt{3}}{\pi^5} \approx 0.054. \]  
\hspace{2cm} (4.3)

Next we compute the fraction of spanning trees on \( \hat{G}^* \) such that \( i \) has exactly one predecessor among its nearest neighbors (see figure 6), which we assume to be node 6 at (0, -1). By removing the edge between nodes 6 and 7, we can establish a bijection between the spanning trees with a unique predecessor of \( i \) to two-component spanning forests, due to the relative positions of the nodes 1, 6, 7 on \( \hat{G} \):
where the combinatorial factor takes into account the number of choices for the unique predecessor of \( i \) among its neighbors (6) and the number of ways to connect \( i \) to the tree with the sink (5). The second equality follows because, \( G \) being planar, the nodes 1 to 4 necessarily belong to the same component as 5, 7, 8. Considering the spanning trees in \( Z_{\{6|578\}} \), we can write

\[
Z_{\{6|78\}} = Z_{\{6|578\}} - Z_{\{56|78\}},
\]

since node 5 can either be with node 6 or with nodes 7 and 8, implying \( Z_{\{6|78\}} = Z_{\{6|578\}} + Z_{\{56|78\}} \).

The number of spanning forests of the types \( 6|78 \) and \( 56|78 \) can be expressed in terms of the usual Green function \( \mathcal{G} = \mathcal{G}^{|z|1} \) and the Green function derivative \( \mathcal{G}' = d\mathcal{G}/dz |_{z=1} \), where the bar is a reminder that these quantities are defined on the modified graph \( \mathcal{G} \). We can compute the number of forests of the first type directly with the all-minors matrix-tree theorem, as the sum on the right-hand side of equation (3.5) contains only one term:

\[
Z_{\{6|78\}} = Z \det \mathcal{G}^{\{6,8\}}_{\{6,7\}} = Z \mathcal{G}_{6,6,8,8} \mathcal{G}_{7,8} - \mathcal{G}_{6,6,8,8} \mathcal{G}_{7,6}) = Z \mathcal{G}_{6,6} - \mathcal{G}_{6,5},
\]

where we used \( \mathcal{G}_{ik} = 1 \) since node 8 is the sink, and \( \mathcal{G}_{ij} = \mathcal{G}_{ji} \) for an undirected graph.

The number of spanning forests of the type \( 56|78 \), on the other hand, is harder to compute. It requires the use of the Green function derivative \( \mathcal{G}' \) associated with the zipper. Using the
generalization of the all-minors matrix-tree theorem to graphs with a connection, one finds the following result (see section 5.4 in [19]):

\[
Z_{[56 \mid 78]} = \lim_{z \to 1} Z_{[56 \mid 78]} = \lim_{z \to 1} Z \times \frac{\det \mathcal{F}^{(6,8)}_{[5,7]} - \det \mathcal{F}^{(7,8)}_{[5,6]} - z^2 \det \mathcal{F}^{(5,8)}_{[6,7]}}{1 - z^2}
\]

\[
= Z \times \left( G_{5,6} - G_{7,6} + \frac{G_{6,7}}{2} \right)
\]

\[
= Z (G_{5,6} - G_{7,6} + G_{5,7} - G_{6,7}),
\]

where the antisymmetry \( G'_{ij} = -G'_{ji} \) has been used, see (3.7).

Both \( G, G' \) on the modified graph \( \mathcal{G}' \), as well as the ratio \( Z/Z \), are given in terms of \( G, G' \) on the original graph \( \mathcal{G} \) using the Woodbury formula, which we recall in the appendix along with selected values of \( G, G' \). The resulting analytical value of \( X_1(i) \) on the triangular lattice \( \mathcal{G} = \mathcal{L}_t \) reads:

\[
X_1(i) = \frac{485}{1296} + \frac{2395}{36} \frac{1}{\sqrt{3}} \frac{3}{2} - \frac{345}{2} \frac{1}{\pi^2} + \frac{2475}{\pi^4} - \frac{1620}{\pi^5} \approx 0.190,
\]

which, upon using equation (3.1), directly yields the height-two probability \( P_2(i) \), shown below.

Higher-height probabilities on the lattice are computed in the same way, although there are multiple inequivalent predecessor diagrams contributing to the fractions \( X_q(i) \) when \( q \geq 2 \) (see figure 7). The probability of each of these diagrams can be determined similarly to that

![Figure 7](image-url)
of the unique one contributing to $X_I(i)$. We find the following explicit expressions for $P_a(i)$ on $G = \mathcal{L}_\Gamma$:

\[ P_1(i) = -\frac{25}{648} - \frac{55}{72\sqrt{3}\pi} + \frac{7}{3\pi} + \frac{11\sqrt{3}}{3\pi} - \frac{90}{\pi^2} + \frac{54\sqrt{3}}{\pi^3} \approx 0.054, \quad (4.9a) \]

\[ P_2(i) = \frac{47}{1296} + \frac{301}{24\sqrt{3}\pi} - \frac{193}{6\pi^2} - \frac{29\sqrt{3}}{3\pi^3} + \frac{405}{\pi^4} - \frac{270\sqrt{3}}{\pi^5} \approx 0.092, \quad (4.9b) \]

\[ P_3(i) = \frac{3}{8} - \frac{5929}{144\sqrt{3}\pi} + \frac{1441}{12\pi^2} - \frac{9\sqrt{3}}{\pi^3} - \frac{720}{\pi^4} + \frac{540\sqrt{3}}{\pi^5} \approx 0.137, \quad (4.9c) \]

\[ P_4(i) = \frac{3427}{2592} + \frac{6515}{144\sqrt{3}\pi} - \frac{2125}{12\pi^2} + \frac{91\sqrt{3}}{3\pi^3} + \frac{630}{\pi^4} - \frac{540\sqrt{3}}{\pi^5} \approx 0.189, \quad (4.9d) \]

\[ P_5(i) = -\frac{2663}{1296} - \frac{71\sqrt{3}}{16\pi} + \frac{1331}{12\pi^2} - \frac{94\sqrt{3}}{3\pi^3} - \frac{270}{\pi^4} + \frac{270\sqrt{3}}{\pi^5} \approx 0.242, \quad (4.9e) \]

\[ P_6(i) = \frac{1175}{864} - \frac{365}{144\sqrt{3}\pi} - \frac{289}{12\pi^2} + \frac{30\sqrt{3}}{\pi^3} + \frac{45}{\pi^4} - \frac{54\sqrt{3}}{\pi^5} \approx 0.286, \quad (4.9f) \]

where we used $P_6(i) = 1 - \sum_{a=1}^5 P_a(i)$ to avoid direct computation of $X_5(i)$, which requires the evaluation of 22 separate predecessor diagrams.

**4.2. On the upper half-plane**

In addition to the infinite triangular lattice, we compute one-site probabilities on the semi-infinite lattice with a horizontal boundary, i.e. $\mathcal{G} = \{(x,y) \in \mathcal{L}_\Gamma | y > 0\}$. The usual boundary conditions on the upper half-plane are either uniformly open or uniformly closed ($\Delta_{ij} = 6$ or $\Delta_{ij} = 4$ resp. for boundary sites). For the latter, we have not been able to define a suitable reflection in order to use the image method. Therefore, we only discuss the case of an open boundary, for which a simple reflection through the line $y = 0$ works and yields the following Green function,

\[ G_{(x_1,y_1),(x_2,y_2)}^{op} = G_{(x_1,y_1),(x_2,y_2)} - G_{(x_1,y_1),(x_2,-y_2,y_2)}, \quad (4.10) \]

where $G$ is the standard Green function on the full triangular lattice (4.1). We choose the reference site $i$ to be located at $(0,p)$ with $p \gg 1$, and take the zipper to be the path on the dual graph crossing the edges of the form $((0,k), (-1,k))$ and $((0,k+1), (-1,k))$ for $k \geq p$ (see figure 5). For such a zipper, the Green function derivative $G'_{op}$ on the upper half-plane reads:

\[ G'_{op}^{(x_1,y_1),(x_2,y_2)} = \sum_{k=0}^{\infty} \left[ G'_{op}^{(x_1,y_1),(-1,p+k+1),(x_2,y_2)} - G'_{op}^{(x_1,y_1),(0,p+k),(x_2,y_2)} - G'_{op}^{(x_1,y_1),(0,p+k+1),(x_2,y_2)} + G'_{op}^{(x_1,y_1),(-1,p+k),(x_2,y_2)} \right]. \quad (4.11) \]

Using equation (4.10), we can write $G'_{op}$ in terms of $G'$ on the full lattice (with respect to the zipper depicted in figure 5) as follows:
For pairs of vertices \((x_1, y_1)\) close to the head of the zipper (i.e. of the form \((a_i, p+b_i)\) with \(a_i, b_i = o(1)\)), the first term of equation (4.12) can be computed exactly since it is independent of \(p\); the three remaining terms are evaluated as power series in \(1/p\) (see the appendix for more details).

Similarly to full-plane computations, the reference site \(i\) located at \((0, p)\) and its six neighbors are chosen as nodes (note that we have relabeled the nodes with respect to figure 5, so that the new zipper is once again located between nodes 1 and 7). We define a modified graph \(\mathcal{G}'\) by cutting the edges 7-2, 7-3, 7-4 and 7-5, so that nodes 1 to 7 lie along the boundary of a single face on \(\mathcal{G}\). Since the upper half-plane is not invariant under rotations of 60°, there are in total roughly four times as many distinct diagrams as on the full plane (the left-right symmetry is still preserved). The correspondence between predecessor diagrams and spanning forests with a fixed node partition \(\sigma\) holds on the upper half-plane as well, with \(Z[\sigma]/Z\) given as a function of \(G^{\sigma}, G'^{\sigma}\) instead of \(G, G'\). The final results for one-site probabilities at \(i = (0, p)\) on the upper half-plane take the form

\[
\sigma^{\text{op}}_a(r) \equiv \mathbb{P}^{\text{op}}_a(r) = \frac{1}{p^2} (c_a + d_a \log r) + \ldots, \tag{4.13}
\]

where \(\mathbb{P}_a\) are the one-site probabilities on the full plane (4.9) and \(r = \sqrt{3}p/2\) is the Euclidean distance between \(i = (0, p)\) and the symmetry axis for the image method, \(y = 0\). The coefficients \(c_a, d_a\) are given in table 1, from which we see that, like on the square half-lattice, all one-point height probabilities have a logarithmic term except for the height 1.

### 5. The hexagonal lattice

The second regular graph we consider is the hexagonal (or honeycomb) lattice \(\mathcal{L}_H\). In contrast with the triangular case, there are two types of vertices on this lattice, which we call \(A\) and \(B\) (see figure 9). Each vertex of type \(A\) has three neighbors of type \(B\), and vice versa. We choose
Table 1. Numerical coefficients of one-site probabilities on the triangular upper half-plane.

| $a$  | $c_a$                                                                 | $d_a$                              |
|------|-----------------------------------------------------------------------|------------------------------------|
| 1    | $-\frac{25}{144\sqrt{3}} - \frac{5}{28\pi} + \frac{33\sqrt{3}}{8\pi} - \frac{99}{8\pi} + \frac{22\sqrt{3}}{28\pi}$ | 0                                  |
| 2    | $\left( -\frac{25}{48\pi} + \frac{45\sqrt{3}}{16\pi} - \frac{27}{4\pi} + \frac{27\sqrt{3}}{16\pi} \right) \left( \gamma + \frac{1}{3} \log 48 \right)$ | $-\frac{25}{48\pi} + \frac{45\sqrt{3}}{16\pi} - \frac{27}{4\pi} + \frac{27\sqrt{3}}{16\pi}$ |
|      | $+ \frac{149}{96\sqrt{3}\pi} - \frac{41}{192\pi} - \frac{1321\sqrt{3}}{64\pi^2} + \frac{513}{4\pi^2} = \frac{1161\sqrt{3}}{16\pi^2}$   |                                    |
| 3    | $\frac{18}{6\pi} - \frac{209\sqrt{3}}{16\pi^2} + \frac{225}{4\pi} - \frac{27\sqrt{3}}{5\pi} \left( \gamma + \frac{1}{3} \log 48 \right)$ | $\frac{19}{6\pi} - \frac{209\sqrt{3}}{16\pi^2} + \frac{225}{4\pi} - \frac{27\sqrt{3}}{5\pi}$ |
|      | $- \frac{353}{72\sqrt{3}\pi} + \frac{275}{48\pi} + \frac{2491\sqrt{3}}{64\pi^2} - \frac{4275}{16\pi^2} + \frac{621\sqrt{3}}{4\pi^2}$         |                                    |
| 4    | $\left( -\frac{469}{64\pi} + \frac{341\sqrt{3}}{16\pi^2} - \frac{351}{8\pi} + \frac{81\sqrt{3}}{8\pi} \right) \left( \gamma + \frac{1}{3} \log 48 \right)$ | $-\frac{469}{64\pi} + \frac{341\sqrt{3}}{16\pi^2} - \frac{351}{8\pi} + \frac{81\sqrt{3}}{8\pi}$ |
|      | $+ \frac{1019}{144\sqrt{3}\pi} - \frac{5425}{384\pi^2} - \frac{2127\sqrt{3}}{64\pi^4} + \frac{4473}{16\pi^4} - \frac{1323\sqrt{3}}{8\pi^4}$ |                                    |
| 5    | $\frac{167}{48\pi^3} + \frac{235\sqrt{3}}{16\pi^2} + \frac{243}{4\pi} - \frac{27\sqrt{3}}{16\pi} \left( \gamma + \frac{1}{3} \log 48 \right)$ | $\frac{167}{48\pi^3} - \frac{235\sqrt{3}}{16\pi^2} + \frac{243}{4\pi} - \frac{27\sqrt{3}}{16\pi}$ |
|      | $- \frac{671}{144\sqrt{3}\pi} + \frac{2227}{192\pi^2} + \frac{751\sqrt{3}}{64\pi^4} - \frac{2349}{16\pi^4} + \frac{351\sqrt{3}}{4\pi^4}$       |                                    |
| 6    | $\left( -\frac{119}{96\pi} + \frac{29\sqrt{3}}{8\pi} - \frac{63}{4\pi} + \frac{27\sqrt{3}}{4\pi} \right) \left( \gamma + \frac{1}{3} \log 48 \right)$ | $-\frac{119}{96\pi} + \frac{29\sqrt{3}}{8\pi} - \frac{63}{4\pi} + \frac{27\sqrt{3}}{4\pi}$ |
|      | $- \frac{319}{288\sqrt{3}\pi} + \frac{369}{72\pi} - \frac{29\sqrt{3}}{16\pi} + \frac{495}{16\pi} - \frac{297\sqrt{3}}{16\pi}$               |                                    |

Figure 9. Left: unit cells marked by rectangles on the hexagonal lattice. Each of them contains two types of vertices, A and B (resp. in blue and in red). Right: coordinate system on the hexagonal lattice, where the positions of A vertices are expressed in terms of the unit vectors $\mathbf{e}_1, \mathbf{e}_2$.

The origin of the lattice to be of type A, and pick a coordinate system $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ where the position of each unit cell is specified by the coordinates $(x, y) \in \mathbb{Z}^2$ of its A vertex. Each individual vertex of $L_H$ is therefore referred to by the complete set of coordinates $(x, y; \alpha)$, with $\alpha = A, B$. Alternatively we can use polar coordinates $(r, \varphi; \alpha)$ with the angle $\varphi$ measured counterclockwise from the x-axis, which are related to $(x, y; \alpha)$ through $x = r \cos \varphi + \frac{\sqrt{3}}{2} \sin \varphi$, $y = \frac{\sqrt{3}}{2} \sin \varphi$.

5.1. On the plane

The standard graph Laplacian on $\mathcal{G} = L_H$ is defined by
\[ \Delta_{u,v} = \begin{cases} 3 & \text{if } u = v, \\ -1 & \text{if } u \text{ and } v \text{ are neighbors,} \\ 0 & \text{otherwise.} \end{cases} \] (5.1)

A more appropriate way of writing \( \Delta \) to compute its inverse \( G \) is obtained by decomposing the lattice into unit cells [39]. Let \( a(\vec{r}_1, \vec{r}_2) \equiv a(\vec{r}_2 - \vec{r}_1) \) be the \( 2 \times 2 \) adjacency matrix for the vertices of the unit cells located at \( \vec{r}_1 \) and \( \vec{r}_2 \), that is,

\[ a_{\alpha_1, \alpha_2}(\vec{r}_2 - \vec{r}_1) = \begin{cases} 1 & \text{if the vertex } \alpha_1 \text{ of the cell } \vec{r}_1 \text{ is a neighbor of the vertex } \alpha_2 \text{ of the cell } \vec{r}_2, \\ 0 & \text{otherwise.} \end{cases} \] (5.2)

The only nonzero matrices \( a(\vec{r}) \) are therefore

\[ a(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a(1,0) = a(1,1) = a(-1,0) = a(-1,-1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \] (5.3)

The \( 2 \times 2 \) block entry of the Laplacian indexed by \( \vec{r}_1, \vec{r}_2 \) (with \( \alpha_1, \alpha_2 = A, B \)) can then be written as follows,

\[ \Delta_{(\vec{r}_1, \alpha_1), (\vec{r}_2, \alpha_2)} = \begin{pmatrix} 3 I_2 - a(0,0) \otimes \delta_{\vec{r}_1, \vec{r}_2} - a(1,0) \otimes \delta_{\vec{r}_1, \vec{r}_2-z} - a(1,1) \otimes \delta_{\vec{r}_1, \vec{r}_2-\vec{r}_1} \\ \delta_{\vec{r}_2, \vec{r}_1} - a(-1,0) \otimes \delta_{\vec{r}_2, \vec{r}_1+\vec{r}_1} - a(-1,-1) \otimes \delta_{\vec{r}_2, \vec{r}_1+\vec{r}_1+\vec{r}_1} \end{pmatrix}. \] (5.4)

Its inverse \( G \) depends only on the difference \( \vec{r} \equiv \vec{r}_2 - \vec{r}_1 = (x, y) \), and is given [2] by

\[ G_{\alpha_1, \alpha_2}(x,y) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{e^{i(x\theta_1 + y\theta_2)}}{6 - 2\cos\theta_1 - 2\cos\theta_2 - 2\cos(\theta_1 + \theta_2)} \times \left( \frac{3}{1 + e^{i\theta_1} + e^{-i(\theta_1 + \theta_2)}} \right). \] (5.5)

It readily follows that \( G_{AA}(x,y) = G_{BB}(x,y) \), \( G_{AB}(-x,-y) = G_{BA}(x,y) \) and

\[ G_{AB}(x,y) = \frac{1}{3} \left\{ G_{AA}(x,y) + G_{AA}(x+1,y) + G_{AA}(x+1,y+1) \right\}, \] (5.6)
Moreover, we observe that $G_{AA}$ is directly related to the Green function on the triangular lattice (4.1):

$$G_{AA}(x, y) = \frac{1}{3} \{ G_{AA}(x, y) + G_{AA}(x - 1, y) + G_{AA}(x - 1, y - 1) \}. \tag{5.7}$$

5.2. On upper half-planes

In addition to the lattice $\mathcal{L}_H$, we consider half-lattices with two kinds of boundaries: one parallel to the $x$ axis, and a horizontal one (see figure 11). For both half-planes, which, following [2], we refer to as principal and horizontal, respectively, we choose the reference site $i = (0, 0; A)$ with $p \gg 1$. As for the full lattice, we select $i$, its three neighbors and the sink as nodes on a modified graph, here obtained by cutting the edge between $i$ and its neighbor $(-1, p; B)$. For simplicity, the zipper is taken as a path on the dual graph starting on a face adjacent to $i$ and extending up to infinity. The edges with a nontrivial parallel transport $z \in \mathbb{C}^*$ are the following: $((0, k; B), (0, k; A))$ for $k \gg p + 1$. The same arguments as in section 3 are
used to write the fractions of spanning trees $X_p(i)$ in terms of the Green function $G$ of the half-lattice of interest and the Green function derivative $G'$ associated with the zipper.

Let us first look at the principal half-plane, whose boundary consists in vertices of the form $(x, y=1; A)$. The corresponding Green function can be written in terms of the full-plane Green function, for either closed or open boundary conditions, using the image method. For the closed boundary, each site is mirrored through the reflection axis $y = 2/3$, so the Green function reads [2]

$$G^{cl}_{(x_1, y_1; \alpha_1), (x_2, y_2; A)} = G_{(x_1, y_1; \alpha_1), (x_2, y_2; A)} + G_{(x_1, y_1; \alpha_1), (x_2 - y_2 - 1, y_2; B)},$$

(5.10)

Proceeding as in (4.12), the Green function derivative with respect to the zipper represented in figure 10 can be written in terms of that on the full lattice. For instance, if $\alpha_1 = \alpha_2 = A$, we find the relation

$$G'^{cl}_{(x_1, y_1; A), (x_2, y_2; A)} = G'_{(-x_1, p + 1 - y_1; A), (-x_2, p + 1 - y_2; B)} - G'_{(-x_1, p + 1 - y_1; B), (y_2 - x_2, p + y_2; A)} - G'_{(y_1 - x_1, p + y_1; A), (-x_2, p + 1 - y_2; B)} + G'_{(y_1 - x_1, p + y_1; B), (y_2 - x_2, p + y_2; A)}.$$  

(5.11)

The open case is subtler, as the mirror image of a vertex of type $A$ with respect to the natural reflection line $y = 1/3$ does not belong to the lattice. However, one can use the Poisson equation to get a suitable expression for the Green function [2]:

$$G^{op}_{(x_1, y_1; \alpha_1), (x_2, y_2; A)} = G_{(x_1, y_1; \alpha_1), (x_2, y_2; A)} - \frac{1}{3} \left[ G_{(x_1, y_1; \alpha_1), (x_2 - y_2 - 1, y_2; B)} + G_{(x_1, y_1; \alpha_1), (x_2 - y_2 - 1, y_2; B)} + G_{(x_1, y_1; \alpha_1), (x_2, y_2 - 1, y_2; B)} \right],$$

(5.12)

$$G^{op}_{(x_1, y_1; \alpha_1), (x_2, y_2; A)} = G_{(x_1, y_1; \alpha_1), (x_2, y_2; B)} - G_{(x_1, y_1; \alpha_1), (x_2, y_2 - 1, y_2; B)}.$$  

Here the Green function derivatives on the half-lattice and on the full lattice are connected through

$$G'^{op}_{(x_1, y_1; B), (x_2, y_2; B)} = G'_{(-x_1, p + 1 - y_1; A), (-x_2, p + 1 - y_2; A)} + G'_{(-x_1, p + 1 - y_1; B), (y_2 - x_2, p + y_2; A)} + G'_{(y_1 - x_1, p + y_1; A), (-x_2, p + 1 - y_2; A)} + G'_{(y_1 - x_1, p + y_1; B), (y_2 - x_2, p + y_2; A)}$$

(5.13)

for $\alpha_1 = \alpha_2 = B$. Similar relations hold for other values of $\alpha_1, \alpha_2$.

We use these formulas together with the integral representation of the Green function on the full lattice (5.5) to compute $G^{cl}$ and $G^{cl, op}$ as series expansions in $1/p$ (this is discussed in the appendix for the triangular lattice; the treatment of the hexagonal lattice is very similar). We find that one-site probabilities on the principal half-plane take the form

$$\sigma^\text{princ}_a(r) \equiv \mathbb{P}^\text{princ}_a(r) - P_a = \frac{1}{r^2} (c^\text{princ}_a + d^\text{princ}_a \log r) + O(r^{-3} \log r),$$

(5.14)

where $\mathbb{P}_a$ denotes the one-site probability on the full lattice, and $r$ is the Euclidean distance between the reference site $i = (0, p; A)$ and the reflection axis used in the image method, namely

$$r = \sqrt{3} \frac{p}{2} - \frac{1}{\sqrt{3}} \quad \text{(closed b.c.)}, \quad r = \sqrt{3} \frac{p}{2} - \frac{1}{2\sqrt{3}} \quad \text{(open b.c.)}.$$

(5.15)
for both types of boundary conditions.

\[ \log \frac{1}{1 + \log 1} \]

Table 3. Coefficients for one-site probabilities on the hexagonal half-lattice with a horizontal open boundary.

| \(a\) | \(c_{a}^{\text{princ}, \text{cl}}\) | \(d_{a}^{\text{princ}, \text{cl}}\) | \(c_{a}^{\text{princ}, \text{op}}\) | \(d_{a}^{\text{princ}, \text{op}}\) |
|------|----------------|----------------|----------------|----------------|
| 1    | \(-\frac{1}{16\sqrt{3}}\) | 0              | \(-\frac{1}{16\sqrt{3}}\) | 0              |
| 2    | \(-\frac{3}{16\sqrt{3}}(\gamma + \frac{1}{2} \log 48) + \frac{15}{64\pi^2}\) | \(-\frac{3}{16\sqrt{3}}(\gamma + \frac{1}{2} \log 48) - \frac{9}{64\pi^2}\) | \(-\frac{3}{16\sqrt{3}}\) | \(-\frac{3}{16\sqrt{3}}\) |
| 3    | \(-\frac{3}{64\sqrt{3}}(\gamma + \frac{1}{2} \log 48) - \frac{3(4 - 4\sqrt{3})}{192\pi^2}\) | \(-\frac{3}{64\sqrt{3}}(\gamma + \frac{1}{2} \log 48) + \frac{27 - 4\sqrt{3}}{192\pi^2}\) | \(-\frac{3}{64\sqrt{3}}\) | \(-\frac{3}{64\sqrt{3}}\) |

The numerical values of the coefficients \(c_{a}^{\text{princ}}, d_{a}^{\text{princ}}\) for both types of boundary conditions are collected in Table 2. We note the distinctive change of sign between the two boundary conditions in the most dominant terms (rational for the height-one, logarithmic for the higher heights), as encountered on the square half-lattice [17].

| \(a\) | \(c_{a}^{\text{hor}, \text{cl}}\) | \(d_{a}^{\text{hor}, \text{cl}}\) |
|------|----------------|----------------|
| 1    | \(-\frac{1}{16\sqrt{3}}\) | 0              |
| 2    | \(-\frac{1}{16\sqrt{3}}(\gamma + \frac{1}{2} \log 48) - \frac{9}{64\pi^2}\) | \(-\frac{3}{16\sqrt{3}}\) |
| 3    | \(-\frac{3}{64\sqrt{3}}(\gamma + \frac{1}{2} \log 48) + \frac{27 - 4\sqrt{3}}{192\pi^2}\) | \(-\frac{3}{64\sqrt{3}}\) |

Table 3. Coefficients for one-site probabilities on the hexagonal half-lattice with a principal boundary.

Let us now turn to the horizontal half-plane (drawn on the right panel of Figure 11), whose boundary sites \((x, y; \alpha)\) satisfy the equality \(x = 2y - 2\). For an open boundary, the image method allows one to write the Green function as [2]

\[ G_{(x_1,y_1,\alpha_1),(x_2,y_2,\alpha_2)}^{\text{op}} = G_{(x_1,y_1,\alpha_1),(x_2,y_2,\alpha_2)} - G_{(x_1,y_1,\alpha_1),(x_2,y_2+1,\alpha_2)}. \]  

(5.16)

With respect to the zipper pictured in Figure 11, the Green function derivative reads:

\[ G_{(x_1,y_1,\alpha_1),(x_2,y_2,\alpha_2)}^{\text{op}} = G_{(x_1,y_1+1,\alpha_1),(x_2,y_2,\alpha_2)} - G_{(x_1,y_1,\alpha_1),(x_2,y_2+1,\alpha_2)} - G_{(x_1,y_1,\alpha_1),(x_2,y_2,\alpha_2)} + G_{(x_1,y_1+1,\alpha_1),(x_2,y_2+1,\alpha_2)}. \]  

(5.17)

where \(G\) is the derivative on the full lattice with a zipper going down to infinity (see Figure 10), and \(\alpha = B\) (resp. \(A\)) if \(\alpha = A\) (resp. \(B\)). For a closed boundary, we have not been able to find a suitable reflection axis for the image method so we only give results for the open half-plane.

Let us define \(r\) as the distance between the reference site \(i = (0, p; A)\) and the reflection axis used in the image method \(y = (x + 1)/2\), i.e. \(r = p - 1/2\). Up to third-order terms, one-site probabilities on the open horizontal half-plane are given by

\[ \sigma_{a}^{\text{hor, op}}(r) \equiv \mathbb{P}_{a}^{\text{hor, op}}(r) - \mathbb{P}_{a} = \frac{1}{r^2} (c_{a}^{\text{hor, op}} + d_{a}^{\text{hor, op}} \log r) + O(r^{-3} \log r), \]  

(5.18)

with coefficients \(c_{a}^{\text{hor, op}}, d_{a}^{\text{hor, op}}\) given in Table 3. As expected, one-site probabilities on the principal and horizontal half-planes with open boundary conditions coincide at order \(1/r^2\), namely \(c_{a}^{\text{hor, op}} = c_{a}^{\text{princ}, \text{op}}\) and \(d_{a}^{\text{hor, op}} = d_{a}^{\text{princ}, \text{op}}\).

5.3. On boundaries

In this subsection, we consider probabilities involving vertices located on the boundary of the two hexagonal upper half-planes described in the previous subsection. Due to the small
number of admissible heights (two on a closed boundary, three on an open one), the calculations are quite straightforward, as they do not require the use of a nontrivial connection on the graph. Indeed, as explained in section 3, height-one probabilities can be computed in terms of the standard Green function only. The same is true for height-three probabilities on an open boundary, see below, yielding height-two probabilities by subtraction.

We shall compute one-site and two-site probabilities on the boundary of the two half-lattices described above. The simplest case is the principal boundary with closed boundary conditions, since the height of a closed boundary site takes on the values 1 or 2. Therefore height-two probabilities can be obtained by subtraction from height-one probabilities, which can be computed via defect matrices. This has been done in [2] with the following results,

\[
\begin{align*}
\mathbb{P}^{\text{princ,cl}}_1 &= \frac{\sqrt{3}}{\pi} - \frac{1}{3}, \quad \mathbb{P}^{\text{princ,cl}}_2 = \frac{4}{3} - \frac{\sqrt{3}}{\pi}, \\
\mathbb{P}^{\text{princ,cl}}_{1,1}(i,j) - \mathbb{P}^{\text{princ,cl}}_1(i) &= -\frac{3}{16\pi^2x^2} + \ldots,
\end{align*}
\]

where \( x = |x_2 - x_1| \) denotes the Euclidean distance between sites \( i = (x_1, 1; A) \) and \( j = (x_2, 1; A) \).

For open boundary conditions on the principal half-plane, boundary heights \( h \) take their value in \( \{1, 2, 3\} \). The height-three probability at site \( i \) can be evaluated as follows [28]: define a new Laplacian \( \tilde{\Delta} \) such that \( \tilde{\Delta}_{ij} = \Delta_{ij} - 1 \), with \( \tilde{\Delta} \) and \( \Delta \) coinciding everywhere else. The burning algorithm [24] gives a bijection between recurrent configurations with \( h_i = 3 \) and spanning trees that use the edge between \( i \) and the sink \( s \). As \( \det \Delta \) counts precisely the number of spanning trees that do not use that particular edge, it follows that

\[
\mathbb{P}^3(i) = 1 - \frac{\det \tilde{\Delta}}{\det \Delta} = (\Delta^{-1})_{ii}.
\]

The remaining height probability \( \mathbb{P}^2(i) \) can be obtained from the relation \( \sum_{i=1}^3 \mathbb{P}^i(i) = 1 \). With the appropriate Green function (5.12), we find the following one-site and two-site probabilities for the open boundary conditions\(^2\):

\[
\begin{align*}
\mathbb{P}^{\text{princ,op}}_1 &= \frac{1}{36} + \frac{4}{\sqrt{3}\pi} - \frac{9}{\pi^2}, \quad \mathbb{P}^{\text{princ,op}}_2 = -\frac{7}{36} - \frac{2}{\sqrt{3}\pi} + \frac{9}{\pi^2}, \quad \mathbb{P}^{\text{princ,op}}_3 = \frac{8}{9} - \frac{2}{\sqrt{3}\pi}, \\
\mathbb{P}^{\text{princ,op}}_{a,b}(i,j) - \mathbb{P}^{\text{princ,op}}_a(i) &= -\frac{\alpha_a^{\text{princ,op}}\alpha_b^{\text{princ,op}}}{4\pi^2} + \ldots, \quad a, b = 1, 2, 3,
\end{align*}
\]

with \( \alpha_1^{\text{princ,op}} = \frac{11}{2\sqrt{3}\pi} - \frac{9}{\pi^2}, \alpha_2^{\text{princ,op}} = -\frac{7}{2\sqrt{3}\pi} + \frac{9}{\pi^2}, \alpha_3^{\text{princ,op}} = -\frac{2}{\pi^2} \) (up to a global sign).

On the horizontal half-plane with open boundary conditions, we obtain similar results using the Green function (5.16):

\[
\begin{align*}
\mathbb{P}^{\text{hor,op}}_1(i) &= -\frac{37}{36} + \frac{8}{\sqrt{3}\pi} - \frac{3}{\pi^2}, \quad \mathbb{P}^{\text{hor,op}}_2(i) = \frac{55}{36} - \frac{8}{\sqrt{3}\pi} + \frac{3}{\pi^2}, \quad \mathbb{P}^{\text{hor,op}}_3(i) = \frac{1}{2}, \\
\mathbb{P}^{\text{hor,op}}_{a,b}(i,j) - \mathbb{P}^{\text{hor,op}}_a(i) &= -\frac{\alpha_a^{\text{hor,op}}\alpha_b^{\text{hor,op}}}{4\pi^4} + \ldots, \quad a, b = 1, 2, 3,
\end{align*}
\]

\(^2\) Only height-one probabilities were given in [2].
with $\alpha_{1,\text{hor,op}} = -\frac{1}{\sqrt{3}\pi} + \frac{1}{\sqrt{2}}$, $\alpha_{2,\text{hor,op}} = -\frac{5}{6\sqrt{3}\pi} - \frac{1}{\sqrt{2}}$, $\alpha_{3,\text{hor,op}} = -\frac{1}{\sqrt{3}\pi}$ (again up to a global sign).

More generally, multisite boundary probabilities on hexagonal half-lattices could easily be computed using equations (3.3) and (5.21). For the triangular half-plane with an open boundary, the situation is more complicated, as there are three nontrivial probabilities to evaluate separately, namely 2, 3 and 4 (heights 5 and 6 can be handled in the same way as height 3 on the hexagonal lattice).

6. Discussion

On the square lattice, joint height probabilities at sites separated by large distances have been considered on the full plane and in the bulk of the upper half-plane. The probabilities involving only unit heights are given by rational functions of the distances [4, 22, 23]. In contrast, probabilities including at least one height strictly larger than one contain logarithmic terms [17, 31]. In light of these results, the variables $h_a(i) \equiv \delta h_a(i), a - P_a$ in the bulk of the lattice (far from the boundaries) are expected to converge to conformal fields $h_a(z, \bar{z})$ of weight $(1,1)$ in that limit, where $P_a$ is the one-site probability on the full infinite lattice, $a = 1, 2, 3, 4$. The field $h_1(z, \bar{z})$ has been identified as a primary field $\phi$, degenerate at level two, and the other fields $h_{a>1}(z, \bar{z})$ as (appropriately normalized) logarithmic partners $\psi$ of $\phi$, in a rank 2 Jordan cell [17, 29]. More precisely, if $\psi$ refers to a fixed logarithmic partner of $\phi$, the height fields have been expressed as linear combinations $h_a(z, \bar{z}) = \alpha_a \psi(z, \bar{z}) + \beta_a \phi(z, \bar{z})$, with $\alpha_1 = 0$.

Our results for the subtracted one-point probabilities $\sigma_a(r)$ on the triangular and hexagonal half-lattices, given respectively by equations (4.13), (5.14) and (5.18) are perfectly consistent with this picture.

The logarithmic conformal field theory in play has further been described by the identification of the operator $\mu$ implementing the change of boundary condition, from open to closed or vice versa. This chiral field has been found to be primary, of weight $-1/8$ [36]. It follows that the correlators $\langle h_a(z, \bar{z}) \rangle_{\text{op}}$ and $\langle h_a(z, \bar{z}) \rangle_{\text{cl}}$ corresponding to the scaling limit of $\sigma_a^{\text{op}}(r)$ and $\sigma_a^{\text{cl}}(r)$ must be related by the following equations [17]:

$$c_a^{\text{op}} = -c_a^{\text{cl}} - \frac{1}{2}d_a^{\text{cl}}, \quad d_a^{\text{op}} = -d_a^{\text{cl}}.$$  (6.1)

Again, we find these relations to be in full agreement with the coefficients in table 2 on the principal hexagonal half-plane. Unfortunately, we cannot make the comparison for the other two half-lattices considered in this paper, namely the triangular half-plane and the horizontal hexagonal half-plane, as we have not been able to compute the Green function for a closed boundary.

In contrast to their bulk analogues, all subtracted probabilities between boundary sites of the square half-lattice are given by rational functions. The boundary height fields have thus been identified as nonlogarithmic fields, which have weight 2 [11, 16, 28]. Moreover, for open boundary conditions, the observation that height correlations on the square lattice factorize signal the fact that all boundary height fields are proportional to a single conformal field (this assumption was shown to be false for a closed boundary). Equations (5.23) and (5.25) indeed confirm this assertion for both hexagonal half-planes with an open boundary. One notices that the factorization property also holds for the closed hexagonal half-plane; however, it is merely due to the fact the two height fields are required to sum to zero.
Acknowledgments

This work is supported by the Belgian Interuniversity Attraction Poles Program P7/18 through the network DYGEST (Dynamics, Geometry and Statistical Physics). PR is Senior Research Associate of the Belgian Fonds National de la Recherche Scientifique (FNRS).

Appendix. Green functions and derivatives

In this appendix, we first recall some exact values of the Green function of the triangular lattice for small distances, as well as its asymptotic expansion for large distances. Then we discuss the methods we used to compute the Green function derivative on the full plane, with respect to the zipper depicted in figure 5. We work out an example in details for the short-distance regime, to illustrate the general principle of computation, and provide some explanations and two typical values for the large-distance regime. Finally, for a pair of graphs \((G, \overline{G})\) related to one another by the addition or removal of a few edges, we recall the relation between their respective Laplacians and Green functions (this was needed in sections 4 and 5). The analogous developments for the hexagonal lattice will be skipped (indeed, remember that the Green functions of both lattices are related through equation (5.8)).

A.1. Values of Green functions

In the coordinate system of section 4, the Green function of the triangular lattice was given in integral form in (4.1),

\[
G_{(x_1, y_1), (x_2, y_2)} = G(x_1 - x_2, y_1 - y_2) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \frac{e^{i(x_1 - x_2)\theta_1 + i(y_1 - y_2)\theta_2}}{6 - 2\cos \theta_1 - 2\cos \theta_2 - 2\cos(\theta_1 + \theta_2)}.
\]  (A.1)

In table A1 we list the value of the Green function for small separations [1], where \(G_{0,0} \equiv G(0, 0)\) is the divergent part of the integral (A.1). The large-distance expansion of \(G(x, y)\) can be computed from the corresponding result on the hexagonal lattice [2] and is given by

\[
G(x, y) = G_{0,0} - \frac{1}{2\sqrt{3}\pi} \left( \log r + \frac{1}{2} \log 12 \right) + \frac{\cos 6\varphi}{60\sqrt{3}\pi r^4} + O(r^{-6}),
\]  (A.2)

where \(r = \sqrt{x^2 + y^2 - xy} \gg 1\) and \(\varphi\) is the angle between the horizontal axis and \((x, y)\), i.e. \(x = r \cos \varphi + \frac{2\sqrt{3}}{3} \sin \varphi\) and \(y = \frac{2\sqrt{3}}{3} \sin \varphi\). Here \(\gamma \simeq 0.57722\) is the Euler–Mascheroni constant.

A.2. Green function derivative for small distances

The definition (3.7) of the Green function derivative \(G'\) with respect to the zipper in figure 5 yields:

\[
G'_{(x_1, y_1), (x_2, y_2)} = \sum_{k=0}^{\infty} \left[ G_{(x_1, y_1), (1-k, y_2)} G_{(0-k, (x_2, y_2))} - G_{(x_1, y_1), (0-k, y_2)} G_{(1-k, (x_2, y_2))} 
+ G_{(x_1, y_1), (1-k, y_2)} G_{(0-k-1, (x_2, y_2))} - G_{(x_1, y_1), (0-k-1, y_2)} G_{(1-k, (x_2, y_2))} \right].
\]  (A.3)
For generic vertices \( u_i = (x_i, y_i) \), \( G' \) must be computed through an explicit summation over all zipper edges using the integral representation of the Green function (A.1). However if \( u_1, u_2 \) are close to the head of the zipper (i.e. close to the origin of the lattice), we can find the value \( G' \) using the following symmetries and transformation properties \([19, 33]\):

(a) \( G' \) is antisymmetric, \( G'_{u_1, u_2} = -G'_{u_2, u_1} \);

(b) \( G' \) only depends on the relative position of its arguments with respect to zipper edges;

(c) Deforming the zipper in the direction of its arrow on the dual lattice while keeping its endpoints fixed changes the value of \( G'_{u_1, u_2} \) by \(-G_{u_1, u_2}\) (resp. \(+G_{u_1, u_2}\)) if the zipper crosses \( u_1 \) (resp. \( u_2 \)) only, and \( G'_{u_1, u_2} \) remains unchanged if the zipper crosses both \( u_1, u_2 \) or neither of them:

\[
G'_{u_1, u_2} - G'_{\text{new}} = \begin{cases} 
-G_{u_1, u_2} & \text{if the zipper crosses } u_1 \text{ only,} \\
+G_{u_1, u_2} & \text{if the zipper crosses } u_2 \text{ only,} \\
0 & \text{otherwise.}
\end{cases}
\]  

(A.4)

The first two assertions come directly from equation (3.7) and the invariance under translations of the triangular lattice. The third one follows from the fact that \( G' \) satisfies a Poisson-like equation \([33]\):

\[
(\Delta G')_{u_1, u_2} = -(G' \Delta)_{u_2, u_1} = \sum_{(k, \ell) \phi_{k, \ell} = z} [-\delta_{u_1, k} G_{k, u_2} + \delta_{u_1, \ell} G_{k, u_2}].
\]  

(A.5)

Let us illustrate how these properties can be used to obtain an explicit expression for the Green function derivative \([19]\). Consider for instance the vertices \( u_1 = (0, 0) \) and \( u_2 = (3, 1) \) together with the zipper represented in the first panel of figure A1. We begin by moving the head of the zipper while keeping \( u_1, u_2 \) fixed, that is, we add the following edges to the sum in equation (A.3): \((1, 1)\)–\((1, 0)\), \((2, 1)\)–\((1, 0)\), \((2, 1)\)–\((2, 0)\). Then we deform and rotate the zipper counterclockwise so that it points upwards (see panels (c)–(d) in figure A1). Since in doing so we drag the zipper across \( u_2 \), the derivative picks up an additional term \(+G_{u_1, u_2}\). Finally, we
rotating the whole lattice by $180^\circ$ to the left, with both $u_1, u_2$ and the zipper included, and we shift it so that $u_2$ now lies at the original position of $u_1$ (and vice versa). The antisymmetry of $G_{a_1, a_2}$ allows one to recast $G_{(0,0),(3,1)}$ as the sum of a finite number of terms. One finds

$$G_{(0,0),(3,1)} = \frac{1}{2} \left\{ G_{(0,0),(1,1)} G_{(1,1),(3,1)} - G_{(0,0),(1,0)} G_{(1,1),(3,1)} + G_{(0,0),(2,1)} G_{(1,0),(3,1)} - G_{(0,0),(2,0)} G_{(1,1),(3,1)} - G_{(0,0),(3,1)}\right\}$$

$$= \frac{1}{2} \left\{ G(1,1)G(2,1) - G(1,0)G(2,0) + G(2,1)G(2,1) - G(1,0)G(1,0) + G(1,1)G(1,1) - G(2,0)G(1,0) - G(3,1)\right\}$$

$$= G_{0,0} \left( \frac{5}{3} - 4\sqrt{3} \pi \right) - \frac{107}{72} + \frac{8}{\sqrt{3}\pi} + \frac{3}{2\pi^2},$$

(A.6)

where the values of $G(x, y)$ given in table A1 have been used together with the symmetry relation $G(3, 1) = G(-2, 1)$. Proceeding this way, one finds the values of the Green function derivative evaluated at the origin and its six neighbors with respect to this particular zipper, as tabulated in table A2.

A.3. Green function derivative for large distances

On the upper half-lattice, the image method allows one to write $G^{\text{op}}$ in terms of $G'$ on the full triangular lattice, with respect to the zippers depicted in figures 5 and 8 for the full lattice and the half-lattice, respectively. In particular, for vertices $u_i$ of the form $(a_i, p + b_i)$ with $a_i, b_i = a(1)$, equation (4.12) yields

$$G_{(a_1, p + b_1), (a_2, p + b_2)}^{\text{op}} = G_{(-a_1, -b_1), (-a_2, -b_2)}^{\prime} - G_{(-a_1, -b_1), (p - a_2, 2p + b_2)}^{\prime} - G_{(p - a_1, 2b_1 + b_1), (-a_2, -b_2)}^{\prime} + G_{(p - a_1, b_2 + b_1), (p - a_2, 2b_2 + b_2)}^{\prime}.$$  

(A.7)

In section A.2 we have shown how to compute the first term on the right-hand side, using the transformation properties of the Green function derivative $G'$. The remaining terms can in principle be evaluated in the same fashion. However, it would require the addition of $O(p)$ extra edges to move the head of the zipper, and then evaluating the resulting sum for large $p$. Instead we found it more convenient to start from (3.7) and the integral representation (A.1) of the Green function. The calculation essentially follows the same steps as the corresponding one on the square lattice, made explicit in [33]. We therefore skip the details and give the final result in a case that is relevant to (A.7), namely

| $G(x, y) - G_{0,0}$ | $y = -2$ | $y = -1$ | $y = 0$ | $y = 1$ | $y = 2$ |
|---------------------|-----------|-----------|-----------|-----------|-----------|
| $x = -2$            | $-\frac{1}{2} + 2\sqrt{3}$ | $\frac{1}{2} - \sqrt{3}$ | $-\frac{1}{2} + 2\sqrt{3}$ | $\frac{1}{2} - \sqrt{3}$ | $-8 + 14\sqrt{3}$ |
| $x = -1$            | $\frac{1}{2} - \sqrt{3}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2} - \sqrt{3}$ | $\frac{1}{2} - \sqrt{3}$ |
| $x = 0$             | $-\frac{1}{2} + 2\sqrt{3}$ | $-\frac{1}{8}$ | $0$ | $-\frac{1}{8}$ | $-\frac{1}{8}$ |
| $x = 1$             | $\frac{1}{2} - \sqrt{3}$ | $\frac{1}{8}$ | $-\frac{1}{8}$ | $\frac{1}{8}$ | $-\frac{1}{8}$ |
| $x = 2$             | $-8 + 14\sqrt{3}$ | $\frac{1}{2} - \sqrt{3}$ | $-\frac{1}{2} + 2\sqrt{3}$ | $\frac{1}{2} - \sqrt{3}$ | $-\frac{1}{2} + 2\sqrt{3}$ |

Table A1. Values of the Green function of the triangular lattice for small distances.
Table A2. Values of the Green function derivative of the triangular lattice around the origin, with respect to the zipper shown in panel (a) of figure A1.

| \(G_{u_i, u_j}^\prime\) | (0, 0) | (1, 0) | (1, 1) | (0, 1) |
|------------------------|--------|--------|--------|--------|
| (0, 0)                 | 0      | \(-\frac{1}{6}G_{0,0} + \frac{1}{2}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{2}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{2}\) |
| (1, 0)                 | \(\frac{2}{3}G_{0,0} - \frac{7}{12}\) | 0      | \(\frac{1}{6}G_{0,0} - \frac{3}{4}\) | \(\frac{1}{6}G_{0,0} + \frac{1}{3}\) |
| (1, 1)                 | \(\frac{1}{3}G_{0,0} - \frac{3}{2}\) | 0      | \(\frac{1}{3}G_{0,0} + \frac{3}{2}\) | 0 |
| (0, 1)                 | \(\frac{1}{6}G_{0,0} - \frac{1}{3}\) | 0      | \(\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) |
| (−1, 0)                | \(G_{0,0} \left(1 - \frac{1}{2}G_{0,0}\right) - \frac{1}{2}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) |
| (−1, −1)               | \(G_{0,0} \left(1 - \frac{1}{2}G_{0,0}\right) + \frac{1}{2}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) |
| (0, −1)                | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) | \(-\frac{1}{6}G_{0,0} + \frac{1}{3}\) |

\[G_{0,0}^{(0,0), (p, 2p+1)} = G_{0,0} \left(-\frac{1}{3} + \frac{1}{2\sqrt{3\pi p}} - \frac{1}{6\sqrt{3\pi p^2}}\right) + (\log p + \gamma + \log 6) \times \left(\frac{1}{6\sqrt{3\pi}} - \frac{1}{24\pi^2 p} + \frac{1}{48\pi^2 p^2}\right) + \frac{1}{12\sqrt{3\pi}} \frac{9 + 2\sqrt{3\pi}}{432\pi^2 p^2} + O(p^{-3}\log p).\]  

(A.8)

This expansion allows one to compute the asymptotic expansion for the second and third terms of equation (A.7), that is, for \(G_{u_i, u_j}^\prime\) when only one of its arguments is close to the head of the zipper. The fourth term however corresponds to a derivative \(G_{u_i, u_j}^\prime\) where both \(u_i = ( p + a, 2p + b )\) are far away from the zipper. In that case, the arguments of the Green functions appearing on the right-hand side of equation (A.3) are large. We can thus use the asymptotic expansion (A.2) before summing over \(k\) to get the power expansion of \(G_{u_i, u_j}^\prime\) through the Euler–Maclaurin formula. For example, for \(a_1 = b_1 = 0, a_2 = b_2 = -1\), the Green function derivative reads:

\[G_{(p, 2p), (p-1, 2p-1)}^{(0,0)} = G_{0,0} \left(\frac{1}{4\sqrt{3\pi p}} + \frac{7}{24\sqrt{3\pi p^2}}\right) + (\log p + \gamma + \log 6) \left(-\frac{1}{24\pi^2 p} - \frac{7}{144\pi^2 p^2}\right) - \frac{1}{24\pi^2 p} + \frac{1}{32\pi^2 p^2} + O(p^{-3}\log p).\]  

(A.9)
A.4. Modified graphs

The spanning tree calculations presented in sections 4 and 5 require that the nodes (excluding the sink) be located around a single face. This can be achieved by defining a modified graph $\tilde{G}$ obtained by cutting multiple (nonzipper) edges between the reference site $i$ and its neighbors, as in figures 5 and 8. The line bundle Laplacian $\Delta(z)$ on $\tilde{G}$ is therefore a local symmetric modification of the Laplacian $\Delta(z)$ on the original graph $G$, which may be written as $\Delta = \Delta - UU^t$. Here the perturbation $U^tU$ has finite rank $r$, so $U$ can be chosen as a $r \times |G|$ matrix. The Woodbury formula allows one to write $\tilde{G} = \Delta^{-1}$ and $\det \Delta$ in terms of $G = \Delta^{-1}$ and $\det \Delta$.

$$\tilde{G} = G + GU^t (I - UGU^t)^{-1} UGU = G + GU^t (I - UGU^t)^{-1} UGU, \quad \text{(A.10)}$$

$$\det \Delta = \det \Delta \times \det (I - UGU^t). \quad \text{(A.11)}$$

In particular, if $G$ is obtained by cutting a single nonzipper edge $\{s,t\}$, the Woodbury formula reduces to the Sherman–Morrison formula. The modified Green function and its derivative then read

$$G_{u,v} = G_{u,v} - \frac{(G_{u,s} - G_{u,s})(G_{s,v} - G_{s,v})}{G_{s,t} + G_{t,s} - G_{t,s} - 1}. \quad \text{(A.12)}$$

$$G_{u,v}' = G_{u,v}' - \frac{(G_{u,s} - G_{u,s})(G_{s,v} - G_{s,v}) + (G_{u,s} - G_{u,t})(G_{t,v} - G_{t,v})}{G_{s,t} + G_{t,s} - G_{t,s} - 1}. \quad \text{(A.13)}$$

Both formulas can be used iteratively in case several edges are removed from the original graph, as an alternative to the more general Woodbury formula.

References

[1] Atkinson D and van Steenwijk F J 1999 Infinite resistive lattices Am. J. Phys. 67 486–92
[2] Azimi-Tafreshi N, Dashi-Naserabadi H, Moghimi-Araghi S and Ruelle P 2010 The Abelian sandpile model on the honeycomb lattice J. Stat. Mech. P02004
[3] Bak P, Tang C and Wiesenfeld K 1987 Self-organized criticality—an explanation of 1/f noise Phys. Rev. Lett. 59 381–4
[4] Brankov J G, Ivashkevich E V and Priezzhev V B 1993 Boundary effects in a 2D Abelian sandpile J. Phys. A 13 1729–40
[5] Caracciolo S and Sportiello A 2012 Exact integration of height probabilities in the Abelian Sandpile model J. Stat. Mech. P09013
[6] Chen W-K 1976 Applied Graph Theory: Graphs and Electrical Networks (Amsterdam: North-Holland)
[7] Cserti J 2000 Application of the lattice Green’s function for calculating the resistance of an infinite network of resistors Am. J. Phys. 68 896–906
[8] Dhar D 1990 Self-organized critical state of sandpile automaton models Phys. Rev. Lett. 64 1613–6
[9] Forman R 1993 Determinants of Laplacians on graphs Topology 32 35–46
[10] Hu C-K and Lin C-Y 2003 Universality in critical exponents for toppling waves of the BTW sandpile model on 2D lattices Physica A 318 92–100
[11] Ivashkevich E V 1994 Boundary height correlations in a 2D Abelian sandpile J. Phys. A: Math. Gen. 27 3643
[12] Ivashkevich E V, Kitareev D V and Priezzhev V B 1994 Critical exponents for boundary avalanches in 2D Abelian sandpile J. Phys. A: Math. Gen. 27 L585
[13] Ivashkevich E V, Kitareev D V and Priezzhev V B 1994 Waves of topplings in an Abelian sandpile Physica A 209 347–60
[14] Jeng M 2004 Boundary conditions and defect lines in the Abelian sandpile model Phys. Rev. E 69 051302
[15] Jeng M 2005 Conformal field theory correlations in the Abelian sandpile model Phys. Rev. E 71 036153
[16] Jeng M, Piroux G and Ruelle P 2006 Height variables, boundary correlations, and dissipative defects in the Abelian sandpile model Phys. Rev. E 71 036153
[17] Jeng M, Piroux G and Ruelle P 2006 Height variables in the Abelian sandpile model: scaling fields and correlations J. Stat. Mech. P10015
[18] Kenyon R 2011 Spanning forests and the vector bundle Laplacian Ann. Probab. 39 1983–2017
[19] Kenyon R W and Wilson D B 2015 Spanning trees of graphs on surfaces and the intensity of loop-erased random walk on planar graphs J. Am. Math. Soc. 28 985–1030
[20] Kitaev D V and Priezzhev V B 1998 Expansion and contraction of avalanches in the 2D Abelian sandpile Phys. Rev. E 58 2883–8
[21] Lin C-Y and Hu C-K 2002 Renormalization-group approach to an Abelian sandpile model on planar lattices Phys. Rev. E 66 021307
[22] Majumdar S N and Dhar D 1991 Height correlations in the Abelian sandpile model J. Phys. A: Math. Gen. 24 L357–62
[23] Majumdar S N and Dhar D 1992 Equivalence between the Abelian sandpile model and the $q \to 0$ limit of the Potts model Physica A 185 129–45
[24] Moghimi-Araghi S, Rajabpour M A and Rouhani S 2005 Abelian sandpile model: a conformal field theory point of view Nucl. Phys. B 718 362–70
[25] Panova G and Wilson D B 2017 Pfaffian formulas for spanning tree probabilities Comb. Probab. Compd. 26 118–37
[26] Papoyan V V and Povolotsky A 1996 Renormalization group study of sandpile on the triangular lattice Physica A 246 241–52
[27] Piroux G and Ruelle P 2005 Boundary height fields in the Abelian sandpile model J. Phys. A: Math. Gen. 38 1451
[28] Poghosyan V S, Grigorev S Y, Priezzhev V B and Ruelle P 2008 Pair correlations in sandpile model: a check of conformal field theory J. Stat. Mech. P07025
[29] Poghosyan V S, Grigorev S Y, Priezzhev V B and Ruelle P 2011 Return probability for the loop-erased random walk and mean height in the Abelian sandpile model: a proof J. Stat. Mech. P10004
[30] Poncelet A and Ruelle P 2013 Logarithmic conformal invariance in the Abelian sandpile model J. Phys. A: Math. Theor. 46 494014
[31] Poncelet A and Ruelle P 2018 015002
[32] Priezzhev V B 1994 Structure of 2D sandpile I. Height probabilities J. Stat. Phys. 74 955–79
[33] Priezzhev V B, Kitarev D V and Ivashkevich E V 1996 Formation of avalanches and critical exponents in an Abelian sandpile model Phys. Rev. Lett. 76 2093–6
[34] Ruelle P 2002 A $c = -2$ boundary changing operator for the Abelian sandpile model Phys. Lett. B 539 172–7
[35] Ruelle P 2007 Wind on the boundary for the Abelian sandpile model J. Stat. Mech. P09013
[36] Ruelle P 2013 Logarithmic conformal invariance in the Abelian sandpile model J. Phys. A: Math. Gen. 33 3881–902
[37] Shrock R and Wu F Y 2000 Spanning trees on graphs and lattices in $d$ dimensions J. Phys. A: Math. Gen. 33 881–902