STABILITY OF A COLOCA TED FINITE VOLUME SCHEME FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We introduce a finite volume scheme for the two-dimensional incompressible Navier-Stokes equations. We use a triangular mesh. The unknowns for the velocity and pressure are both piecewise constant (colocated scheme). We use a projection (fractional-step) method to deal with the incompressibility constraint. We prove that the differential operators in the Navier-Stokes equations and their discrete counterparts share similar properties. In particular, we state an inf-sup (Babuška-Brezzi) condition. We infer from it the stability of the scheme.

Key Words. Incompressible fluids, Navier-Stokes equations, projection methods and finite volume.

1. Introduction

We consider the flow of an incompressible fluid in a open bounded set \( \Omega \subset \mathbb{R}^2 \) during the time interval \( [0,T] \). The velocity field \( u : \Omega \times [0,T] \to \mathbb{R}^2 \) and the pressure field \( p : \Omega \times [0,T] \to \mathbb{R} \) satisfy the Navier-Stokes equations

\[
\begin{align*}
\frac{d}{dt} u - \frac{1}{\text{Re}} \Delta u + (u \cdot \nabla) u + \nabla p &= f, \\
\text{div} \; u &= 0,
\end{align*}
\]

(1.1) (1.2)

with the boundary and initial condition

\[
\begin{align*}
u|_{\partial \Omega} &= 0, \\
u|_{t=0} &= u_0.
\end{align*}
\]

The terms \( \Delta u \) and \( (u \cdot \nabla) u \) are respectively associated with the physical phenomena of diffusion and convection. The Reynolds number \( \text{Re} \) measures the influence of convection in the flow. For equations (1.1)–(1.2), finite element and finite difference methods are well known and mathematical studies are available (see [10] for example). Numerous computations have also been conducted with finite volume schemes (e.g. [14] and [1]). However, in this case, few mathematical results are available. Let us cite EYMARD AND HERBIN [7] and EYMARD, LATCHÉ AND HERBIN [8]. In order to deal with the incompressibility constraint (1.2), these works use a penalization method. Another way is to use the projection methods which have been introduced by CHORIN [4] and TEMAM [15]. This is the case in FAURE [9]. In this work, however, the mesh is made of squares, so that the geometry of the problem is limited. Therefore, we introduce in what follows a finite volume scheme on triangular meshes for equations (1.1)–(1.2), using a projection method. An interesting feature of this scheme is that the unknowns for the velocity and
pressure are both piecewise constant (collocated scheme). It leads to an economic computer storage, and allows an easy generalization of the scheme to the 3D case. The layout of the article is the following. We first introduce (section 2) some notations and hypotheses on the mesh. We define (section 2.2) the spaces we use to approximate the velocity and pressure. We define also (section 2.3) the operators we use to approximate the differential operators in (1.1)–(1.2). Combining this with a projection method, we build the scheme in section 3. In order to provide a mathematical analysis for the scheme, we prove in section 4 that the differential operators for the convection term is positive, stable and consistent. The discrete operator for the divergence term is positive, stable and consistent. The discrete operator for the divergence satisfies an inf-sup (Babuška-Brezzi) condition. From these properties we deduce in section 5 the stability of the scheme.

We conclude with some notations. The spaces \((L^2, ||.|.||_2)\) and \((L^\infty, ||.|.||_\infty)\) are the usual Lebesgue spaces and we set \(L^2_0 = \{q \in L^2 : \int_{\Omega} q(x) \, dx = 0\}\). Their vectorial counterparts are \((L^2, ||.|.||_2)\) and \((L^\infty, ||.|.||_\infty)\) with \(L^2 = (L^2)^2\) and \(L^\infty = (L^\infty)^2\). For \(k \in \mathbb{N}^+\), \((H^k, ||.|.||_k)\) is the usual Sobolev space. Its vectorial counterpart is \((H^k, ||.|.||_k)\) with \(H^k = (H^k)^2\). For \(k = 1\), the functions of \(H^1\) with a null trace on the boundary form the space \(H^1_0\). Also, we set \(\nabla u = (\nabla u_1, \nabla u_2)^T\) if \(u = (u_1, u_2) \in H^1\).

If \(X \subset L^2\) is a Banach space, we define \(C(0,T;X)\) (resp. \(L^2(0,T;X)\)) as the set of the applications \(g : [0,T] \rightarrow X\) such that \(t \rightarrow |g(t)|\) is continuous (resp. square integrable). The norms \(||.|.|_{C(0,T;X)}\) and \(||.|.|_{L^2(0,T;X)}\) are defined respectively by \(\|g\|_{C(0,T;X)} = \sup_{t \in [0,T]} |g(t)|\) and \(\|g\|_{L^2(0,T;X)} = \left( \int_0^T |g(t)|^2 \, ds \right)^{1/2}\).

In all calculations, \(C\) is a generic positive constant, depending only on \(\Omega, u_0, f\).

2. Discrete setting

First, we introduce the spaces and the operators needed to build the scheme.

2.1. The mesh. Let \(T_h\) be a triangular mesh of \(\Omega: \overline{\Omega} = \bigcup_{K \in T_h} K\). For each triangle \(K \in T_h\), we denote by \(|K|\) its area and \(E_K\) the set of his edges. If \(\sigma \in E_K\), \(n_{K,\sigma}\) is the unit vector normal to \(\sigma\) pointing outward of \(K\). The set of edges of the mesh is \(E_h = \bigcup_{K \in T_h} E_K\). The length of an edge \(\sigma \in E_h\) is \(|\sigma|\) and its middle point \(x_\sigma\). The set of edges located inside \(\Omega\) (resp. on its boundary) is \(E^{int}_h\) (resp. \(E^{ext}_h\)): \(E_h^\sigma = E^{int}_h \cup E^{ext}_h\). If \(\sigma \in E^{int}_h\), \(K_\sigma\) and \(L_\sigma\) are the triangles sharing \(\sigma\) as an edge. If \(\sigma \in E^{ext}_h\), only the triangle \(K_\sigma\) inside \(\Omega\) is defined.

We denote by \(x_K\) the circumcenter of a triangle \(K\). We assume that the measure of all interior angles of the triangles of the mesh are below \(\frac{\pi}{7}\), so that \(x_K \in K\).

If \(\sigma \in E^{int}_h\) (resp. \(\sigma \in E^{ext}_h\)) we set \(d_\sigma = d(x_{K_\sigma}, x_{L_\sigma})\) (resp. \(d_\sigma = d(x_\sigma, x_{K_\sigma})\)). We define for all edge \(\sigma \in E_h\)

\[
\tau_\sigma = \frac{|\sigma|}{d_\sigma}
\]

The maximum circumradius of the triangles of the mesh is \(h\). We assume (\(\text{\cite{6}}\), p. 776) that there exists \(C > 0\) such that

\[
\forall \sigma \in E_h, \quad d(x_{K_\sigma}, \sigma) \geq C |\sigma| \quad \text{and} \quad |\sigma| \geq Ch.
\]

It implies that there exists \(C > 0\) such that

\[
\forall \sigma \in E_h, \quad \tau_\sigma \geq C.
\]
Lastly, if $K \in T_h$ we have (with $\sigma \in \mathcal{E}_K$ and $h_{K,\sigma}$ the matching altitude)
\begin{equation}
|K| = \frac{1}{2} |\sigma| h_{K,\sigma} \geq \frac{1}{2} |\sigma| d(x_K, x_\sigma) \geq C h^2. \tag{2.3}
\end{equation}

Lastly, if $K \in T_h$ and $L \in T_h$ are two triangles sharing the edge $\sigma \in \mathcal{E}_{h}^{int}$, we define
\[ \alpha_{K,L} = \frac{d(x_L, x_\sigma)}{d(x_K, x_\sigma)}. \]
Let us notice that $\alpha_{K,L} \in [0,1]$ and $\alpha_{K,L} + \alpha_{L,K} = 1$.

2.2. The discrete spaces. We first define
\[ P_0 = \{ q \in L^2 : \forall K \in T_h, \ q|_K \text{ is a constant} \}, \quad P_0 = (P_0)^2. \]

For the sake of concision, we set for all $q_h \in P_0$ (resp. $v_h \in P_0$) and all triangle $K \in T_h$: $q_K = q_h|_K$ (resp. $v_K = v_h|_K$). Although $P_0 \not\in \mathcal{H}^1$, we define the discrete equivalent of a $\mathcal{H}^1$ norm as follows. For all $v_h \in P_0$ we set
\begin{equation}
\|v_h\|_h = \left( \sum_{\sigma \in \mathcal{E}_{h}^{int}} \tau_\sigma |v_{L,\sigma} - v_{K,\sigma}|^2 + \sum_{\sigma \in \mathcal{E}_{h}^{ext}} \tau_\sigma |v_{K,\sigma}|^2 \right)^{1/2}, \tag{2.4}
\end{equation}

where $\tau_\sigma$ is given by (2.1). We have [6] a Poincaré-like inequality for $P_0$: there exists $C > 0$ such that for all $v_h \in P_0$
\begin{equation}
|v_h| \leq C \|v_h\|_h. \tag{2.5}
\end{equation}

We also have the following inverse inequality.

**Proposition 2.1.** There exists a constant $C > 0$ such that for all $v_h \in P_0$
\[ h \|v_h\|_h \leq C |v_h|. \]

**Proof.** According to (2.4)
\[ h^2 \|v_h\|_h^2 = \sum_{\sigma \in \mathcal{E}_{h}^{int}} h^2 \tau_\sigma |v_{L,\sigma} - v_{K,\sigma}|^2 + \sum_{\sigma \in \mathcal{E}_{h}^{ext}} h^2 \tau_\sigma |v_{K,\sigma}|^2. \]

We deduce from (2.2) and (2.3) that $h^2 \tau_\sigma \leq C |K_\sigma|$ and $h^2 \tau_\sigma \leq C |L_\sigma|$. Thus, since $|v_{L,\sigma} - v_{K,\sigma}|^2 \leq 2 (|v_{L,\sigma}|^2 + |v_{K,\sigma}|^2)$, we get
\[ h^2 \|v_h\|_h^2 \leq C \sum_{\sigma \in \mathcal{E}_{h}^{int}} (|K_\sigma||v_{K,\sigma}|^2 + |L_\sigma||v_{L,\sigma}|^2) + C \sum_{\sigma \in \mathcal{E}_{h}^{ext}} |K_\sigma||v_{K,\sigma}|^2. \]

Hence $h^2 \|v_h\|_h^2 \leq C \sum_{K \in T_h} |K| |v_K|^2 \leq C |v_h|^2$.

From the norm $\|\cdot\|_h$ we deduce a dual norm. For all $v_h \in P_0$ we set
\begin{equation}
\|v_h\|_{-1,h} = \sup_{\psi_h \in P_0} \frac{(v_h, \psi_h)}{\|\psi_h\|_h}. \tag{2.6}
\end{equation}

For all $u_h \in P_0$ and $v_h \in P_0$ we have $(u_h, v_h) \leq \|u_h\|_{-1,h} \|v_h\|_h$. Now we introduce some operators on $P_0$ and $P_0$. We define the projection operator $\Pi_{P_0} : L^2 \rightarrow P_0$ as follows. For all $w \in L^2$, $\Pi_{P_0} w \in P_0$ is given by
\begin{equation}
\forall K \in T_h, \quad (\Pi_{P_0} w)|_K = \frac{1}{|K|} \int_K w(x) \, dx. \tag{2.7}
\end{equation}
We easily check that for all $w \in L^2$ and $v_h \in P_0$ we have $(\Pi_{P_0} w, v_h) = (w, v_h)$. It implies that $\Pi_{P_0}$ is stable for the $L^2$ norm. We define also the interpolation operator $\Pi_{P_0} : H^2 \to P_0$. For all $q \in H^2$, $\Pi_{P_0} q \in P_0$ is given by

$$\forall K \in \mathcal{T}_h, \quad \Pi_{P_0} q|_K = q(x_K).$$

According to the Sobolev embedding theorem, $q \in H^2$ is a.e. equal to a continuous function. Therefore the definition above makes sense. We also set $\tilde{\Pi}_{P_0} = (\Pi_{P_0})^2$.

The operator $\tilde{\Pi}_{P_0}$ (resp. $\Pi_{P_0}$) is naturally stable for the $L^\infty$ (resp. $L^\infty$) norm. One also checks ([2] and [16]) that there exists $C > 0$ such that

$$|v - \Pi_{P_0} v| \leq C h \|v\|_1, \quad |q - \tilde{\Pi}_{P_0} q| \leq C h \|q\|_2$$

for all $v \in H^1$ and $q \in H^2$.

We introduce the finite element spaces

$$P^d_1 = \{ v \in L^2 ; \forall K \in \mathcal{T}_h, \ v|_K \text{ is affine} \},$$

$$P^{nc}_1 = \{ v_h \in P_1^d ; \forall \sigma \in \mathcal{E}_h^{int}, v_h|_{\partial \sigma} = v_h|_{x_{\sigma}} = v_h|_{L_{\sigma}}(x_{\sigma}) \},$$

$$P^c_1 = \{ v_h \in (P_1^d)^2 ; v_h \text{ is continuous and } v_h|_{\partial \Omega} = 0 \}.$$

We have $P^c_1 \subset H^1_0$. We define the projection operator $\Pi_{P^c_1} : H^1_0 \to P^c_1$. For all $v = (v_1, v_2) \in H^1_0$, $\Pi_{P^c_1} v = (v_1, v_2) \in P^c_1$ is given by

$$\forall \phi_h = (\phi^i_h, \phi^j_h) \in P^c_1, \quad \sum_{i=1}^2 (\nabla v^i_h, \nabla \phi^i_h) = \sum_{i=1}^2 (\nabla v^i, \nabla \phi^i_h).$$

The operator $\Pi_{P^c_1}$ is stable for the $H^1$ norm and ([2] p. 110) there exists $C > 0$ such that for all $v \in H^1$

$$|v - \Pi_{P^c_1} v| \leq C h \|v\|_1. \quad (2.9)$$

Let us address now the space $P^{nc}_1$. If $q_h \in P^{nc}_1$, we have usually $\nabla q_h \not\in L^2$. Thus we define the operator $\nabla h : P^{nc}_1 \to P_0$ by setting for all $q_h \in P^{nc}_1$ and all $K \in \mathcal{T}_h$,

$$\nabla h q_h|_K = \frac{1}{|K|} \int_K \nabla q_h \, dx.$$

The associated norm is given by

$$\|q_h\|_{1,h} = \left( |q_h|^2 + |\nabla h q_h|^2 \right)^{1/2}.$$

We also have a Poincaré inequality: there exists $C > 0$ such that for all $q_h \in P^{nc}_1 \cap L^2_0$ (2.10)

$$|q_h| \leq C |\nabla h q_h|.$$

We define the projection operator $\Pi_{P^{nc}_1}$. For all $q_h \in P^{nc}_1$, $\Pi_{P^{nc}_1} q_h$ is given by

$$\forall \phi \in L^2, \quad (\Pi_{P^{nc}_1} q_h, \phi) = (q_h, \phi). \quad (2.11)$$

We have the following result.

**Proposition 2.2.** If $q_h \in P_0$, $\Pi_{P^{nc}_1} q_h$ is given by

$$\forall \sigma \in \mathcal{E}_h^{int}, \quad (\Pi_{P^{nc}_1} q_h)|_{x_{\sigma}} = \frac{|K_{\sigma}|}{|K_{\sigma}| + |L_{\sigma}|} q_{K_{\sigma}} + \frac{|L_{\sigma}|}{|K_{\sigma}| + |L_{\sigma}|} q_{L_{\sigma}},$$

$$\forall \sigma \in \mathcal{E}_h^{ext}, \quad (\Pi_{P^{nc}_1} q_h)|_{x_{\sigma}} = q_{K_{\sigma}}.$$
One checks \cite{3} that there exists $C > 0$ such that:
\begin{equation}
q_h|_{\partial K_{\sigma}} = 0 \quad \forall \sigma \in K_{\sigma}, \quad \|q_h|_{\partial K_{\sigma}}\|_{L^\infty(\partial K_{\sigma})} \leq C h \|v\|_{H^1(\Omega)}
\end{equation}

Let us notice that $\psi_{\sigma}$ vanishes outside $K_{\sigma} \cup L_{\sigma}$ if $\sigma \in E_h^{ext}$ and outside $K_{\sigma}$ if $\sigma \in E_h^{int}$. Using a quadrature formula we get
\begin{equation}
\begin{aligned}
(\Pi_{P_1^r} q_h, \psi_{\sigma}) &= \left( \frac{|K_{\sigma}|}{3} + \frac{|L_{\sigma}|}{3} \right) (\Pi_{P_1^r} q_h)(x_{\sigma}) \\
(q_h, \psi_{\sigma}) &= q_{K_{\sigma}} \frac{|K_{\sigma}|}{3} + q_{L_{\sigma}} \frac{|L_{\sigma}|}{3}.
\end{aligned}
\end{equation}

For an edge $\sigma \in E_h^{ext}$ we have $(\Pi_{P_1^r} q_h, \psi_{\sigma}) = \frac{|K_{\sigma}|}{3} (\Pi_{P_1^r} q_h)(x_{\sigma})$ and $(q_h, \psi_{\sigma}) = q_{K_{\sigma}} \frac{|K_{\sigma}|}{3}$. By plugging these equations into (2.11) with $\phi = \psi_{\sigma}$, we get the result. 

We finally introduce the Raviart-Thomas spaces
\begin{align*}
RT_0^d &= \{v_h \in P_1^d : \forall \sigma \in E_h, \quad v_h|_{K} \cdot n_{K,\sigma} \text{ is a constant, and } v_h \cdot n|_{\partial K} = 0 \}, \\
RT_0 &= \{v_h \in RT_0^d : \forall K \in T_h, \quad \forall \sigma \in E_h, \quad v_h|_{K} \cdot n_{K,\sigma} = v_h|_{L_{\sigma}} \cdot n_{K,\sigma} \}.
\end{align*}

For all $v_h \in RT_0$, $K \in T_h$ and $\sigma \in E_h$ we set $(v_h \cdot n_{K,\sigma})_\sigma = v_h|_{K} \cdot n_{K,\sigma}$. We define the operator $\Pi_{RT_0} : H^1 \to RT_0$. For all $v \in H^1$, $\Pi_{RT_0} v \in RT_0$ is given by
\begin{equation}
\forall K \in T_h, \quad \forall \sigma \in E_h, \quad (\Pi_{RT_0} v \cdot n_{K,\sigma})_\sigma = \frac{1}{|\sigma|} \int_{\sigma} v \, d\sigma.
\end{equation}

One checks \cite{3} that there exists $C > 0$ such that for all $v \in H^1$,
\begin{equation}
|v - \Pi_{RT_0} v| \leq C h \|v\|_{H^1}.
\end{equation}

The following result will be useful.

**Proposition 2.3.** For all $v \in H^1$ such that $\text{div} \ v = 0$, we have $\Pi_{RT_0} v \in P_0$.

**Proof.** Let $v_h = \Pi_{RT_0} v$ and $K \in T_h$. According to \cite{3} there exists $a_K \in \mathbb{R}^2$ and $b_K \in \mathbb{R}$ such that: $\forall x \in K$, $v_h(x) = a_K + b_K \cdot x$. Thus $\text{div} \ v_h|_K = 2 b_K$. On the other hand, according to the divergence formula and (2.12)
\begin{equation}
0 = \int_{K} \text{div} \ v \, dx = \int_{\partial K} v \cdot n \, d\gamma = \int_{K} v_h \cdot n \, d\gamma = \int_{K} \text{div} \ v_h \, dx.
\end{equation}

Hence $b_K = 0$ and we get: $\forall x \in K$, $v_h(x) = a_K$.

2.3. The discrete operators. The equations (1.1) - (1.2) use the differential operators gradient, divergence and laplacian. Using the spaces of section 2.2 we define their discrete counterparts. The discrete gradient $\nabla_h : P_0 \to P_0$ is built using a linear interpolation on the edges of the mesh (see \cite{16} for details). This kind of construction has also be considered in \cite{5}. We set for all $q_h \in P_0$ and all $K \in T_h$
\begin{equation}
\nabla_h q_h|_K = \frac{1}{|K|} \sum_{\sigma \in E_h \cap E_h^{int}} |\sigma| \left( \alpha_{K_{\sigma}, L_{\sigma}} q_{K_{\sigma}} + \alpha_{L_{\sigma}, K_{\sigma}} q_{L_{\sigma}} \right) n_{K,\sigma}
\end{equation}

We have the following result \cite{16}.

**Proposition 2.4.** If $q_h \in L^2$ is such that $\nabla_h q_h = 0$, then $q_h = 0$. 

\[ \]
The discrete divergence operator $\text{div}_h : P_0 \to P_0$ is built so that it is adjoint to the operator $\nabla_h$ (proposition 1.1 below). We set for all $q_h \in P_0$ and all $K \in T_h$

$$\text{div}_h \mathbf{v}_h|_K = \frac{1}{|K|} \sum_{\sigma \in E_K \cap E_{h|n}^t} |\sigma| \left( \alpha_{L,\sigma} K, \mathbf{v}_{K,\sigma} + \alpha_{K,\sigma} L, \mathbf{v}_{L,\sigma} \right) \cdot \mathbf{n}_K,\sigma.$$  

The first discrete laplacian $\Delta_h : P_0 \to P_0$ ensures that the incompressibility constraint (1.2) is satisfied in a discrete sense (proposition 3.1). We set for all $q_h \in P_0$

$$\Delta_h q_h = \text{div}_h(\nabla_h q_h).$$

The second discrete laplacian $\tilde{\Delta}_h : P_0 \to P_0$ is the usual operator in finite volume schemes [6]. We set for all $\mathbf{v}_h \in P_0$ and all $K \in T_h$

$$\tilde{\Delta}_h \mathbf{v}_h|_K = \frac{1}{|K|} \sum_{\sigma \in E_K \cap E_{h|n}^t} \tau_\sigma (\mathbf{v}_{L,\sigma} - \mathbf{v}_{K,\sigma}) - \frac{1}{|K|} \sum_{\sigma \in E_K \cap E_{h|n}^t} \tau_\sigma \mathbf{v}_{K,\sigma}.$$  

In order to approximate the convection term $(u \cdot \nabla)u$ in (1.1) we define a bilinear form $\tilde{b}_h : P_0 \times P_0 \to P_0$ using the well-known upwind scheme (6 p. 766). For all $u_h \in P_0$, $v_h \in P_0$, and all $K \in T_h$ we have

$$\tilde{b}_h(u_h, v_h)|_K = \frac{1}{|K|} \sum_{\sigma \in E_K \cap E_{h|n}^t} |\sigma| \left( (u_{\sigma} \cdot \mathbf{n}_{K,\sigma})^+ u_K + (u_{\sigma} \cdot \mathbf{n}_{K,\sigma})^- v_{L,\sigma} \right).$$

We have set $u_{\sigma} = \alpha_{L,\sigma} K, u_{K,\sigma} + \alpha_{K,\sigma} L, u_{L,\sigma}$ and $a^+ = \max(a, 0)$, $a^- = \min(a, 0)$ for all $a \in \mathbb{R}$. Lastly, we define the trilinear form $b_h : P_0 \times P_0 \times P_0 \to \mathbb{R}^2$ as follows. For all $u_h \in P_0$, $v_h \in P_0$, $w_h \in P_0$, we set

$$b_h(u_h, v_h, w_h) = \sum_{K \in T_h} |K| \mathbf{w}_K \cdot \tilde{b}_h(u_h, v_h)|_K.$$  

3. The scheme

We have defined in section 2 the discretization in space. We now have to define a discretization in time, and treat the incompressibility constraint (1.2). We use a projection method to this end. This kind of method has been introduced by CHORIN [4] and TEMAM [15]. The basic idea is the following. The time interval $[0, T]$ is split with a time step $k$: $[0, T] = \bigcup_{n=0}^{N} [t_n, t_{n+1}]$ with $N \in \mathbb{N}^*$ and $t_n = n k$ for all $n \in \{0, \ldots, N\}$. For all $m \in \{2, \ldots, N\}$, we compute (see equation (3.2) below) a first velocity field $\mathbf{u}^m_h \simeq u(t_m)$ using only equation (1.1). We use a second-order BDF scheme for the discretization in time. We then project $\mathbf{u}^m_h$ (see equation (3.3) below) over a subspace of $P_0$. We get a a pressure field $p^m_h \simeq p(t_m)$ and a second velocity field $\mathbf{u}^{m+1}_h \simeq u(t_{m+1})$, which fulfills the incompressibility constraint (1.2) in a discrete sense. The algorithm goes as follows.

First, for all $m \in \{0, \ldots, N\}$, we set $f^m_h = \Pi_{P_0} f(t_m)$. Since the operator $\Pi_{P_0}$ is stable for the $L^2$-norm we get

$$|f^m_h| = |\Pi_{P_0} f(t_m)| \leq |f(t_m)| \leq \|f\|_{C(0,T;L^2)}.$$  

We start with the initial values

$$u^0_h \in P_0 \cap RT_0, \quad u^1_h \in P_0 \cap RT_0 \quad p^1_h \in P_0 \cap L^2_0.$$  

For all $n \in \{1, \ldots, N\}$, $(\mathbf{u}^{n+1}_h, p^{n+1}_h)$ is deduced from $(\mathbf{u}^n_h, p^n_h, u^n_h)$ as follows.

- $\mathbf{u}^{n+1}_h \in P_0$ is given by

$$3 \mathbf{u}^{n+1}_h - 4 \mathbf{u}^n_h + \mathbf{u}^{n-1}_h = \frac{1}{2k} \Delta_h \mathbf{u}^{n+1}_h + \tilde{b}_h(2 \mathbf{u}^n_h - \mathbf{u}^{n-1}_h, \mathbf{u}^{n+1}_h) + \nabla_h p^n_h = f^{n+1}_h,$$  

while the first discrete laplacian ensures that the incompressibility constraint (1.2) is satisfied in a discrete sense (proposition 3.1). We set for all $q_h \in P_0$
We show that the operator (3.3) and compare with (3.4).

(4.1)

There exists a constant $C > 0$ such that for all $v \in H^2$ and all $u \in H^2 \cap H^1_0$ satisfying $\text{div} \, u = 0$ 

$$
\|\Pi_{P_0} \tilde{b}(u, v) - \tilde{b}_h(\Pi_{RT_0} u, \Pi_{P_0} v)\|_{-1, h} \leq C h \|u\|_2 \|v\|_1.
$$

\text{Proof.} Let $u_h = \Pi_{RT_0} u$ and $v_h = \Pi_{P_0} v$. According to proposition 2.3 we have $u_h \in P_0$. Let $K \in \mathcal{T}_h$. According to the divergence formula and (2.7) we have 

$$
\Pi_{P_0} \tilde{b}(u, v)|_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_h} \int_{\sigma} v (u \cdot n) \, d\sigma.
$$

4. Properties of the discrete operators

We prove that the differential operators in (1.1)–(1.2) and the operators defined in section 2.3 share similar properties.

4.1. Properties of the discrete convective term. We define $\tilde{b} : H^1 \times H^1 \to L^2$.

For all $u \in H^1$ and $v = (v_1, v_2) \in H^1$ we set

$$
\tilde{b}(u, v) = (\text{div}(v_1 u), \text{div}(v_2 u)).
$$

We show that the operator $\tilde{b}_h$ is a consistent approximation of $\tilde{b}$.

Proposition 4.1. There exists a constant $C > 0$ such that for all $v \in H^2$ and all $u \in H^2 \cap H^1_0$ satisfying $\text{div} \, u = 0$ 

$$
\|\Pi_{P_0} \tilde{b}(u, v) - \tilde{b}_h(\Pi_{RT_0} u, \Pi_{P_0} v)\|_{-1, h} \leq C h \|u\|_2 \|v\|_1.
$$

\text{Proof.} Let $u_h = \Pi_{RT_0} u$ and $v_h = \Pi_{P_0} v$. According to proposition 2.3 we have $u_h \in P_0$. Let $K \in \mathcal{T}_h$. According to the divergence formula and (2.7) we have 

$$
\Pi_{P_0} \tilde{b}(u, v)|_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_h \cap \mathcal{E}_h^{int}} \int_{\sigma} v (u \cdot n) \, d\sigma.
$$
On the other hand, let us rewrite $\tilde{b}_h(u_h, v_h)$. Let $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}$. Setting

$$v_{K,L,\sigma} = \begin{cases} 
    v_K & \text{if } (u_h \cdot n_{K,\sigma})_\sigma \geq 0 \\
    v_L & \text{if } (u_h \cdot n_{K,\sigma})_\sigma < 0
\end{cases}$$

one checks that $v_K (u_\sigma \cdot n_{K,\sigma})^+ + v_L (u_\sigma \cdot n_{K,\sigma})^- = v_{K,L,\sigma} (u_\sigma \cdot n_{K,\sigma})$. By definition $u_\sigma \cdot n_{K,\sigma} = \alpha_{\sigma, K} (u_K \cdot n_{K,\sigma}) + \alpha_{K,L,\sigma} (u_L \cdot n_{K,\sigma})$; since $u_h \in \mathbb{R}^2_0$ we get $u_\sigma \cdot n_{K,\sigma} = (\alpha_{\sigma, K} + \alpha_{K,L,\sigma}) (u_K \cdot n_{K,\sigma}) = (u_K \cdot n_{K,\sigma}) = (u_h \cdot n_{K,\sigma})_\sigma$. Using at last (2.17), we deduce from (2.17)

$$\tilde{b}_h(u_h, v_h)|_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} \int_{\sigma} v_{K,L,\sigma} (u \cdot n_{K,\sigma}) \, d\sigma.$$ 

Thus

$$(\Pi_p b(u,v) - \tilde{b}_h(u_h, v_h))|_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} \int_{\sigma} (v - v_{K,L,\sigma}) (u \cdot n) \, d\sigma.$$ 

Let $\psi_h \in P_0$. We have

$$(\Pi_p b(u,v) - \tilde{b}_h(u_h, v_h), \psi_h) = \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} \int_{\sigma} (v - v_{K,L,\sigma}) (u \cdot n) \, d\sigma = \sum_{\sigma \in \mathcal{E}_h^{int}} (\psi_{K,\sigma} - \psi_{L,\sigma}) \int_{\sigma} (v - v_{K,L,\sigma}) (u \cdot n) \, d\sigma.$$ 

Let $\sigma \in \mathcal{E}_h^{int}$. We want to estimate the integral over $\sigma$. Since we work in a two-dimensional domain, we have the Sobolev injection $H^2 \subset L^\infty$. Thus

$$\left| \int_{\sigma} (v - v_{K,L,\sigma}) (u \cdot n) \, d\sigma \right| \leq \|u\|_{L^\infty} \int_{\sigma} |v - v_{K,L,\sigma}| \, d\sigma \leq C \|u\|_2 \int_{\sigma} |v - v_{K,L,\sigma}| \, d\sigma.$$ 

Let us first assume that $v \in C^1$. We set

$$x_{K,L,\sigma} = \begin{cases} 
    x_K & \text{if } (u_h \cdot n_{K,\sigma})_\sigma \geq 0 \\
    x_L & \text{if } (u_h \cdot n_{K,\sigma})_\sigma < 0
\end{cases}.$$ 

If $x \in \sigma$, we have the following Taylor expansion

$$v(x) - v_{K,L,\sigma} = v(x) - v(x_{K,L,\sigma}) = \int_0^1 \nabla v \left( t x + (1-t) x_{K,L,\sigma} \right) (x - x_{K,L,\sigma}) \, dt.$$ 

We have $|x - x_{K,L,\sigma}| \leq h$. Thus, integrating over $\sigma$ and using the Cauchy-Schwarz inequality, we get

$$\left( \int_{\sigma} |v - v_{K,L,\sigma}| \, d\sigma \right)^2 \leq \left( \int_{\sigma} \left( \frac{1}{h} \int_0^1 \left| \nabla v \left( t x + (1-t) x_{K,L,\sigma} \right) \right|^2 \, dt \right)^{1/2} \right)^2.$$ 

We then use the change of variable $(t,x) \rightarrow y = t x + (1-t) x_{K,L,\sigma}$. Let $D_{\sigma}$ be the quadrilateral domain given by the endpoints of $\sigma$, $x_{K,\sigma}$ and $x_{L,\sigma}$. The domain $[0,1] \times \sigma$ becomes $D_{K,L,\sigma}$ with

$$D_{K,L,\sigma} = \begin{cases} 
    D_{\sigma} \cap K_{\sigma} & \text{if } (u_h \cdot n_{K,\sigma})_\sigma \geq 0 \\
    D_{\sigma} \cap L_{\sigma} & \text{if } (u_h \cdot n_{K,\sigma})_\sigma < 0
\end{cases}.$$ 

For all $t \in [0,1]$ we have $h \sqrt{t} \leq h t \leq C d(x_{K,L,\sigma}, \sigma) t$ thanks to the hypothesis on the mesh. We check easily that $d(x_{K,L,\sigma}, \sigma) t \, dt \, d\sigma = dy$. Thus we get

$$\int_{\sigma} |v - v_{K,L,\sigma}| \, d\sigma \leq C h \left( \int_{D_{K,L,\sigma}} |\nabla v(y)|^2 \, dy \right)^{1/2}.$$
Since \((C^1)^2\) is dense in \(H^2\), this estimate still holds for \(v \in H^2\). Plugging this estimate into (4.12) and using the Cauchy-Schwarz inequality we get

\[
\left| \left( \Pi_h \tilde{b}(u,v) - \tilde{b}_h(\Pi_{RT_0}u, \Pi_h v), \psi_h \right) \right| \leq C h \|u\|_{H^2} \left( \sum_{\sigma \in E_h^{int}} |\psi_{L\sigma} - \psi_{K\sigma}|^2 \right)^{1/2} \left( \sum_{\sigma \in E_h^{int}} \int_{D_{K\sigma \cap \sigma}} |\nabla v(y)|^2 \, dy \right)^{1/2}
\]

so that

\[
\left| \left( \Pi_h \tilde{b}(u,v) - \tilde{b}_h(\Pi_{RT_0}u, \Pi_h v), \psi_h \right) \right| \leq C h \|u\|_{H^2} \| \psi_{h} \|_{1,A} \|v\|_1.
\]

Using then definition (2.9), we get the result.  

Let us consider now the operator \(b_h\). Let \(u \in H^1\) and \(v \in L^\infty \cap H^1\) with \(\|u\| \geq 0\). Integrating by parts we deduce from (4.1): \(\int_{\Omega} v \cdot \tilde{b}(u,v) \, dx = \int_{\Omega} \frac{|v|^2}{2} \, div \, u \, dx \geq 0\). The discrete operator \(b_h\) shares a similar property.

**Proposition 4.2.** Let \(u_h \in P_0\) such that \(\text{div} \, u_h \geq 0\). For all \(v_h \in P_0\) we have

\[
b_h(u_h, v_h, v_h) \geq 0.
\]

**Proof.** Remember that for all edges \(\sigma \in E_h^{int}\), two triangles \(K_\sigma\) et \(L_\sigma\) share \(\sigma\) as an edge. We denote by \(K_\sigma\) the one such that \(u_\sigma \cdot n_{K_\sigma,\sigma} \geq 0\). Using the algebraic identity \(2 \, a \cdot (a - b) = a^2 - b^2 + (a - b)^2\) we deduce from (2.18)

\[
2 \, b_h(u_h, v_h, v_h) = 2 \sum_{\sigma \in E_h^{int}} |\sigma| \, (v_{K\sigma} - v_{L\sigma}) \cdot (u_\sigma \cdot n_{K_\sigma,\sigma}) = \sum_{\sigma \in E_h^{int}} |\sigma| \, \left( |v_{K\sigma}|^2 - |v_{L\sigma}|^2 + |v_{K\sigma} - v_{L\sigma}|^2 \right) (u_\sigma \cdot n_{K_\sigma,\sigma})
\]

so that

\[
2 \, b_h(u_h, v_h, v_h) \geq \sum_{\sigma \in E_h^{int}} |\sigma| \, \left( |v_{K\sigma}|^2 - |v_{L\sigma}|^2 \right) (u_\sigma \cdot n_{K_\sigma,\sigma}).
\]

This sum can be written as a sum over the triangles of the mesh. We get

\[
2 \, b_h(u_h, v_h, v_h) \geq \sum_{K \in T_h} |K| \, |v_K|^2 \sum_{\sigma \in E_h^{int} \cap K} |\sigma| \, (u_\sigma \cdot n_{K_\sigma,\sigma}).
\]

Using finally definition (2.19) we get

\[
2 \, b_h(u_h, v_h, v_h) \geq \sum_{K \in T_h} |K| \, |v_K|^2 \, (\text{div}_h u_h)|_K \geq 0.
\]

The following result states that the operator \(b_h\) is stable for suitable norms.

**Proposition 4.3.** There exists a constant \(C > 0\) such that for all \(v_h \in P_0\), \(w_h \in P_0\), \(u_h \in P_0\) satisfying \(\text{div} \, u_h = 0\)

\[
|b_h(u_h, v_h, v_h)| \leq C \|u_h\| \|v_h\| \|v_h\|_h.
\]

**Proof.** For all triangle \(K \in T_h\) and all edge \(\sigma \in E_h \cap E_h^{int}\), we have

\[
(u_\sigma \cdot n_{K_\sigma,\sigma})^+ v_K + (u_\sigma \cdot n_{K_\sigma,\sigma})^- v_{L\sigma} = (u_\sigma \cdot n_{K_\sigma,\sigma}) v_K - |(u_\sigma \cdot n_{K_\sigma,\sigma})| (v_{L\sigma} - v_K).
\]

This way, we deduce from (4.17) \(b_h(u_h, v_h, w_h) = S_1 + S_2\) with

\[
S_1 = \sum_{K \in T_h} v_K \cdot w_K \sum_{\sigma \in E_h^{int} \cap K} |\sigma| \, (u_\sigma \cdot n_{K_\sigma,\sigma}),
\]

\[
S_2 = - \sum_{K \in T_h} w_K \cdot v_K \sum_{\sigma \in E_h^{int} \cap K} |\sigma| \, |u_\sigma \cdot n_{K_\sigma,\sigma}| (v_{L\sigma} - v_K).
\]
By writing the sum over the edges as a sum over the triangles we get

$$S_2 = - \sum_{\sigma \in \mathcal{E}_h^{int}} |\sigma| (|u_\sigma \cdot n_{K,\sigma}| (v_{L,\sigma} - v_K) \cdot (w_{L,\sigma} - w_K)).$$

Using the Cauchy-Schwarz inequality we get

$$|S_2| \leq h \|u_h\|_\infty \left( \sum_{\sigma \in \mathcal{E}_h^{int}} |v_{L,\sigma} - v_K|^2 \right)^{1/2} \left( \sum_{\sigma \in \mathcal{E}_h^{int}} |w_{L,\sigma} - w_K|^2 \right)^{1/2}.$$ 

Since $u_h \in P_0$ we have the inverse inequality $h \|u_h\|_\infty \leq C \|u_h\|$. Using (2.2) and (2.4) we have

$$\sum_{\sigma \in \mathcal{E}_h^{int}} |v_{L,\sigma} - v_K|^2 \leq C \sum_{\sigma \in \mathcal{E}_h^{int}} \tau_\sigma |v_{L,\sigma} - v_K|^2 \leq C \|v_h\|^2$$

and $\sum_{\sigma \in \mathcal{E}_h^{ext}} |w_{L,\sigma} - w_K|^2 \leq C \|w_h\|^2$. Therefore $|S_2| \leq C \|u_h\| \|v_h\| \|w_h\|$. On the other hand we deduce from definition (2.15)

$$S_1 = \sum_{K \in T_h} |K| (v_K \cdot w_K) (\nabla_h u_h)|_K = 0.$$ 

By combining the estimates for $S_1$ and $S_2$ we get the result.

4.2. Properties of the discrete gradient.

**Proposition 4.4.** There exists a constant $C > 0$ such that for all $q_h \in P_0$:

$$h \|\nabla q_h\| \leq C |q_h|.$$

**Proof.** Using (2.14) and the Minkowski inequality, we have for all triangle $K \in T_h$

$$|K| \|\nabla q_h\|_K^2 \leq \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} \frac{6 |\sigma|^2}{|K|} (q_{K}^2 + q_{L,\sigma}^2) + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{ext}} \frac{6 |\sigma|^2}{|K|} q_{K}^2.$$

Let us sum over $K \in T_h$. Since $|\sigma| \leq h$, using (2.3), we get

$$|\nabla q_h|^2 \leq \frac{C}{h^2} \left( \sum_{K \in T_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} (|K| q_{K}^2 + |L_\sigma| q_{L,\sigma}^2) + \sum_{K \in T_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{ext}} |K| q_{K}^2 \right).$$

Thus $h^2 |\nabla q_h|^2 \leq C \sum_{K \in T_h} |K| q_{K}^2 \leq C |q_h|^2$.  

We now prove that $\nabla_h$ is a consistent approximation of the gradient.

**Proposition 4.5.** There exists a constant $C > 0$ such that for all $q \in H^2$

$$\|\Pi_{P_h}(\nabla q) - \nabla_h(\Pi_{P_h} q)\| \leq C h \|q\|_2.$$

**Proof.** Let $K \in T_h$. Using the gradient formula and definition (2.14) we get

$$|K| \left( \Pi_{P_h}(\nabla q) - \nabla_h(\Pi_{P_h} q) \right)|_K = \int_K \nabla q \, dx - |K| \nabla_h(\Pi_{P_h} q)|_K = \sum_{\sigma \in \mathcal{E}_K} I_{K,\sigma}^\sigma,$$

where we have set for all edge $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}$

$$I_{K,\sigma}^\sigma = \int_\sigma \left( q - \left( \alpha_{K,L,\sigma} q(x_K) + \alpha_{L,\sigma,K} q(x_{L,\sigma}) \right) \right) n_{K,\sigma} \, d\sigma.$$
and for all edge σ ∈ E_K ∩ E_e: \( I_K^\sigma = \int_\sigma (q - q(x_K)) n_{K,\sigma} \, d\sigma \). Squaring and using (2.3) we get
\[
|K| \left| \left( \Pi_0 \nabla q - \nabla_h(\Pi P_0 q) \right) \right|_K^2 \leq \frac{3}{|K|} \sum_{\sigma \in \mathcal{E}_K} |I_K^\sigma|^2 \leq C \frac{h^2}{|K|} \sum_{\sigma \in \mathcal{E}_K} |I_K^\sigma|^2.
\]
Summing over the triangles \( K \in T_h \) we get
\[
(4.3) \quad \left| \Pi_0 \nabla q - \nabla_h(\Pi P_0 q) \right|^2 \leq C \frac{h^2}{|K|} \sum_{\sigma \in \mathcal{E}_K} |I_K^\sigma|^2.
\]
We must estimate the integral terms \( I_K^\sigma \). Let \( K \in T_h \). Let us first assume that \( q \in C^2(\bar{\Omega}) \). Let \( \sigma \in \mathcal{E}_K \cap \mathcal{E}_h \). For \( x \in \sigma \) we have the following Taylor expansions
\[
q(x_K) = q(x) + \nabla q(x) \cdot (x_K - x) + \int_0^1 \nabla q(x_K + (1-t)x_K)(x_K - x) \cdot (x_K - x) \, dt,
\]
\[
q(x_{L,\sigma}) = q(x) + \nabla q(x) \cdot (x_{L,\sigma} - x) + \int_0^1 \nabla q(x_{L,\sigma} + (1-t)x)(x_{L,\sigma} - x) \cdot (x_{L,\sigma} - x) \, dt,
\]
\[
\nabla q(x) = \nabla q(x_K) - \int_0^1 \nabla \nabla q(t x_K + (1-t)x)(x_K - x) \, dt.
\]
Plugging the last expansion into the two others and integrating over \( \sigma \) we get
\[
(4.4) \quad \int_\sigma (q(x_K) - q) \, d\sigma = |\sigma| \nabla q(x_K) \cdot (x_K - x) - A^\sigma_T + B^\sigma_T,
\]
\[
(4.5) \quad \int_\sigma (q(x_{L,\sigma}) - q) \, d\sigma = |\sigma| \nabla q(x_K) \cdot (x_{L,\sigma} - x) - A^\sigma_L + B^\sigma_L.
\]
We have set for \( T \in \{K,\sigma\} \)
\[
(4.6) \quad A^\sigma_T = \int_\sigma \int_0^1 \nabla \nabla q(t x_T + (1-t)x)(x_T - x) \, dt \, d\sigma,
\]
\[
(4.7) \quad B^\sigma_T = \int_\sigma \int_0^1 \nabla \nabla q(t x_T + (1-t)x)(x_T - x) \cdot (x_T - x) \, dt \, d\sigma.
\]
One can bound these terms as in the proof of proposition. We get
\[
(4.8) \quad |A^\sigma_T|^2 \leq C h^2 \int_{D_\sigma} |\nabla \nabla q(\gamma)|^2 \, dy, \quad |B^\sigma_T|^2 \leq C h^4 \int_{D_\sigma} |\nabla \nabla q(\gamma)|^2 \, dy.
\]
Now, let us multiply (1.4) by \(-\alpha_{K,\sigma,\lambda} n_{K,\sigma}\), (1.5) by \(-\alpha_{L,\sigma,\lambda} n_{K,\sigma}\) and sum the equalities. Since \( \alpha_{L,\sigma} + \alpha_{L,\sigma} = 1 \) we have
\[
-\alpha_{L,\sigma} \int_\sigma (q(x_K) - q) n_{K,\sigma} \, d\sigma - \alpha_{L,\sigma} \int_\sigma (q(x_{L,\sigma}) - q) n_{K,\sigma} \, d\sigma
\]
\[
= \int_\sigma (q - (\alpha_{K,\sigma,\lambda} q(x_K) + \alpha_{L,\sigma,\lambda} q(x_{L,\sigma}))) n_{K,\sigma} \, d\sigma = I^\sigma_T.
\]
On the other hand
\[
-\alpha_{K,\sigma,\lambda} (x_K - x_\sigma) \cdot n_{K,\sigma} - \alpha_{L,\sigma,\lambda} (x_{L,\sigma} - x_\sigma) \cdot n_{K,\sigma} = -\alpha_{K,\sigma,\lambda} \alpha_{L,\sigma,\lambda} (d_\sigma - d_\sigma) = 0.
\]
Therefore we get \( I^\sigma_T = -\alpha_{L,\sigma,\lambda} (A^\sigma_T + B^\sigma_T) n_{K,\sigma} - \alpha_{K,\sigma,\lambda} (A^\sigma_L + B^\sigma_L) n_{K,\sigma} \). Using estimates (4.8) we obtain
\[
|I^\sigma_K|^2 \leq C h^4 \int_{D_\sigma} (|\nabla \nabla q(\gamma)|^2 + |\nabla \nabla q(\gamma)|^2) \, dy.
\]
We now consider the case $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}$. For $x \in \sigma$ we have

$$q(x_K) = q(x) + \nabla q(x) \cdot (x_K - x) + \int_0^1 H(q)((tx_K + (1 - t)x)(x_K - x) \cdot (x_K - x) dt.$$  

Multiplying by $n_{K,\sigma}$ and integrating over $\sigma$, we get $-I_K^\sigma = J_K^\sigma n_{K,\sigma} + B_K^\sigma n_{K,\sigma}$ with

$$J_K^\sigma = \int_\sigma \nabla q(x) \cdot (x_K - x) \, dx.$$  

Since $|x_K - x| \leq h$ if $x \in \sigma$, using a trace theorem, we have

$$|J_K^\sigma| \leq C h^2 \|\nabla q\|_{L^\infty(\sigma)} \leq C h^2 \left( \int_{D_\sigma} (|\nabla q(y)|^2 + |\nabla \nabla q(y)|^2) \, dy \right)^{1/2}.$$  

By combining this estimate with (4.3), we get

$$|J_K^\sigma|^2 \leq 2 |J^\sigma|^2 + 2 |B_K^\sigma|^2 \leq C h^4 \int_{D_\sigma} |H(q)(y)|^2 \, dy + C h^4 \int_{D_\sigma} (|\nabla q(y)|^2 + |\nabla \nabla q(y)|^2) \, dy.$$  

The space $C^2(\Omega)$ is dense in $H^2$. Therefore the bounds for $I_K$ still hold for $q \in H^2$. Plugging these bounds into (4.3) we get the result. \hfill \blacksquare

4.3. Properties of the discrete divergence. The operators divergence and gradient are adjoint: if $q \in H^1$ and $v \in H^1$ with $v \cdot n_{\partial \Omega} = 0$, we get $(v, \nabla q) = -(q, \text{div} v)$ by integrating by parts. For $\nabla_h$ and $\text{div}_h$ we state

**Proposition 4.6.** If $v_h \in P_0$ and $q_h \in P_0$ we have: $(v_h, \nabla_h q_h) = -(q_h, \text{div}_h v_h)$.  

**Proof.** Using (2.14) one checks that $(v_h, \nabla_h q_h) = \sum_{K \in T_h} q_K \left( S_1 + S_2 + S_3 \right)$ with

$$S_1 = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| \alpha_{K,L_\sigma} v_K \cdot n_{K,\sigma}, \quad S_2 = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| \alpha_{K,L_\sigma} v_{L_\sigma} \cdot n_{L_\sigma,\sigma},$$

and

$$S_3 = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| v_K \cdot n_{K,\sigma}.$$  

Since $\alpha_{K,L_\sigma} + \alpha_{L_\sigma,K} = 1$ we have

$$S_1 = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| (1 - \alpha_{L_\sigma,K}) v_K \cdot n_{K,\sigma} = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| v_K \cdot n_{K,\sigma} - \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| \alpha_{L_\sigma,K} v_K \cdot n_{K,\sigma}.$$  

Since $n_{L_\sigma,\sigma} = -n_{K,\sigma}$, we also have

$$S_2 = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| \alpha_{L_\sigma,K} v_{L_\sigma} \cdot n_{L_\sigma,\sigma} = - \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| \alpha_{K,L_\sigma} v_{L_\sigma} \cdot n_{K,\sigma}.$$  

Therefore

$$(v_h, \nabla_h q_h) = - \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{int}} |\sigma| \left( \alpha_{L,K_\sigma} v_K + \alpha_{K,L_\sigma} v_{L_\sigma} \right) \cdot n_{K,L_\sigma} + \sum_{\sigma \in \mathcal{E}_K} |\sigma| v_K \cdot n_{K,L_\sigma}.$$  

Using definition (2.13) we get

$$(v_h, \nabla_h q_h) = - \sum_{K \in T_h} |K| \text{div}_h v_h |_K + \sum_{K \in T_h} \sum_{\sigma \in \mathcal{E}_K} |\sigma| v_K \cdot n_{K,L_\sigma}.$$  

Since $\sum_{\sigma \in \mathcal{E}_K} |\sigma| v_K \cdot n_{K,L_\sigma} = v_K \cdot \sum_{\sigma \in \mathcal{E}_K} |\sigma| n_{K,L_\sigma} = 0$ we obtain finally

$$(v_h, \nabla_h q_h) = - \sum_{K \in T_h} q_K |K| \text{div}_h v_h |_K = -(q_h, \text{div}_h v_h).$$  

\hfill \blacksquare
The divergence operator and the spaces \( L_0^2, H_0^1 \) satisfy the following property, called \( \text{inf-sup} \) (or Babuška-Brezzi) condition (see [10] for example). There exists a constant \( C > 0 \) such that

\[
\inf_{q \in L_0^2 \setminus \{0\}} \sup_{v \in H_0^1 \setminus \{0\}} \frac{(q, \text{div} v)}{\|v\|_1} \geq C.
\]

We will now prove that the operator \( \text{div}_h \) and the spaces \( P_0 \cap L_0^2, P_0 \) satisfy an analogous property. The proof is based on the following lemma.

**Lemma 4.1.** We assume that the mesh is uniform (i.e. the triangles of the mesh are equilateral). Then we have for all \( q_h \in P_0 \)

\[
\nabla q_h = \tilde{\nabla} q_h(\Pi_{P_0} q_h).
\]

**Proof.** Since the mesh is uniform we have: \( \forall \sigma \in \mathcal{E}_h^{\text{int}}, \alpha_{K, \sigma} = \frac{1}{2} \). Let \( K \in T_h \). Using definition (2.14) and the gradient formula we get

\[
\int_K \left( \nabla q_h - \tilde{\nabla} q_h(\Pi_{P_0} q_h) \right) d\mathbf{x} = \sum_{\sigma \in \mathcal{E}_h^{\text{int}}} \frac{|\sigma|}{2} (q_{K, \sigma} + q_{L, \sigma}) n_{K, \sigma}
\]

\[
+ \sum_{\sigma \in \mathcal{E}_h^{\text{ext}}} |\sigma| q_{K, \sigma} n_{K, \sigma} - \sum_{\sigma \in \mathcal{E}_h^{\text{ext}}} \int_{\sigma} (\Pi_{P_0} q_h) n_{K, \sigma} d\sigma.
\]

Since \( q_h \in P_0 \) we deduce from proposition 2.2

\[
\int_{\sigma} \Pi_{P_0} q_h d\sigma = |\sigma| (\Pi_{P_0} q_h)(\mathbf{x}_\sigma) = \left\{ \begin{array}{ll} \frac{|\sigma|}{2} (q_{K, \sigma} + q_{L, \sigma}) & \text{if } \sigma \in \mathcal{E}_h^{\text{int}}, \\ |\sigma| q_{K, \sigma} & \text{if } \sigma \in \mathcal{E}_h^{\text{ext}}. \end{array} \right.
\]

Plugging this into the equation above, we get \( \nabla q_h|_K = \tilde{\nabla} q_h(\Pi_{P_0} q_h)|_K \)

**Lemma 4.2.** We assume that the mesh is uniform. There exists a constant \( C > 0 \) such that

\[
\forall q_h \in P_0 \cap L_0^2, \quad \sup_{v_h \in P_0 \setminus \{0\}} \frac{(q_h, \text{div}_h v_h)}{\|v_h\|_h} \geq C h \|\Pi_{P_0} q_h\|_{1, h}.
\]

**Proof.** If \( q_h = 0 \) the result is trivial. Let \( q_h \in P_0 \cap L_0^2 \setminus \{0\} \). Let \( v_h = \nabla q_h \in P_0 \setminus \{0\} \). Using proposition 4.0 we have

\[
-(q_h, \text{div}_h v_h) = (v_h, \nabla q_h) = |\nabla q_h|_h^2 = |\nabla q_h|_h |v_h|.
\]

Let \( \chi_\Omega \) be the characteristic function of \( \Omega \). Putting \( \psi = \chi_\Omega \) in (2.11) we get \( \Pi_{P_0} q_h \in L_0^2 \). So according to (2.10) and (4.1) we have

\[
|\nabla q_h| = |\tilde{\nabla} q_h(\Pi_{P_0} q_h)| \geq C \|\Pi_{P_0} q_h\|_{1, h}.
\]

On the other hand, according to proposition 2.1

\[
|v_h| \geq C h \|v_h\|_h.
\]

Therefore

\[
-(q_h, \text{div}_h v_h) \geq C h \|\Pi_{P_0} q_h\|_{1, h} \|v_h\|_h.
\]

**Proposition 4.7.** We assume that the mesh is uniform. There exists a constant \( C > 0 \) such that for all \( q_h \in P_0 \cap L_0^2 \)

\[
\sup_{v_h \in P_0 \setminus \{0\}} \frac{(q_h, \text{div}_h v_h)}{\|v_h\|_h} \geq C \|\Pi_{P_0} q_h\|.
\]
Proof. If $q_h = 0$ the result is clear. Let $q_h \in P_0 \cap L_0^2 \backslash \{0\}$. According to (4.9) there exists $v \in H_0^1$ such that

$$\text{div } v = -\Pi_{p_0^c} q_h \quad \text{and} \quad \|v\|_1 \leq C \|\Pi_{p_0^c} q_h\|.$$

(4.10)

We set $v_h = \Pi_{p_1^c} v$. We want to estimate $-(q_h, \text{div}_h(\Pi_{p_0^c} v_h))$. Since $\text{div} q_h \in P_0$ we deduce from proposition 4.6

$$-(q_h, \text{div}_h(\Pi_{p_0^c} v_h)) = (\Pi_{p_0^c} v_h, \text{div}_h q_h) = (v_h, \text{div}_h q_h).$$

Splitting the last term we get

$$-(q_h, \text{div}_h(\Pi_{p_0^c} v_h)) = (v, \text{div}_h q_h) - (v - v_h, \text{div}_h q_h).$$

One hand, integrating by parts, we get

$$\langle v, \text{div}_h q_h \rangle = -\langle (\Pi_{p_1^c} q_h, \text{div} v) + \sum_{K \in T_h} \sum_{\sigma \in \partial K} \int_{\sigma} (\Pi_{p_1^c} q_h) (v \cdot n_{K,\sigma}) \rangle d\sigma.$$

According to (4.10) we have $-(\Pi_{p_1^c} q_h, \text{div} v) = \|\Pi_{p_1^c} q_h\|^2$. Moreover

$$\sum_{K \in T_h} \sum_{\sigma \in \partial K} \int_{\sigma} (\Pi_{p_1^c} q_h) (v \cdot n_{K,\sigma}) d\sigma = \sum_{\sigma \in \partial K} \int_{\sigma} (\Pi_{p_1^c} q_h) (v \cdot n_{K,\sigma}) d\sigma$$

since $v\vert_{\partial \Omega} = 0$. Using [2] p.269 and (4.10) we have

$$\left| \sum_{\sigma \in \partial K} \int_{\sigma} (\Pi_{p_1^c} q_h) (v \cdot n_{K,\sigma}) d\sigma \right| \leq C h \|v\|_1 \|\Pi_{p_1^c} q_h\|_{1,h} \leq C h \|\Pi_{p_1^c} q_h\| \|\Pi_{p_1^c} q_h\|_{1,h}.$$

So we get

$$\langle v, \text{div}_h q_h \rangle \geq \|\Pi_{p_1^c} q_h\| - C h \|\Pi_{p_1^c} q_h\|_{1,h} \|\Pi_{p_1^c} q_h\|.$$ 

(4.12)

On the other hand, using lemma 4.1 and the Cauchy-Schwarz inequality

$$|\langle v - v_h, \text{div}_h q_h \rangle| = |\langle v - v_h, \text{div}_h(\Pi_{p_1^c} q_h) \rangle| \leq |v - v_h| \|\text{div}_h(\Pi_{p_1^c} q_h)\|.$$ 

Using (2.3) and (4.10) we get

$$|v - v_h| = |v - \Pi_{p_1^c} v| \leq C h \|v\|_1 \leq C h \|\Pi_{p_1^c} q_h\|.$$

Thus

$$\langle v - v_h, \text{div}_h q_h \rangle \leq C h \|\Pi_{p_1^c} q_h\| \|\Pi_{p_1^c} q_h\|_{1,h} \leq C h \|\Pi_{p_1^c} q_h\| \|\Pi_{p_1^c} q_h\|_{1,h}.$$

Let us plug this estimate and (4.12) into (4.11). We get

$$-(q_h, \text{div}_h(\Pi_{p_0^c} v_h)) \geq \|\Pi_{p_1^c} q_h\| - C h \|\Pi_{p_1^c} q_h\|_{1,h} \|\Pi_{p_1^c} q_h\|.$$ 

We now introduce the norm $\|\cdot\|_1$. We have $v_h = \Pi_{p_1^c} v \in P_1^p \subset H^1$. Thus, using [6] p. 776, we get $\|\Pi_{p_0^c} v_h\| \leq C \|v_h\|$. Since $\Pi_{p_1^c}$ is stable for the $H^1$ norm, we deduce from (4.10)

$$\|v_h\|_1 = \|\Pi_{p_1^c} v\|_1 \leq \|v\|_1 \leq C \|\Pi_{p_1^c} q_h\|.$$ 

Therefore $\|\Pi_{p_0^c} v_h\| \leq C \|\Pi_{p_1^c} q_h\|$. Using this inequality in (4.12) we obtain that there exists constants $C_1 > 0$ and $C_2 > 0$ such that

$$-(q_h, \text{div}_h(\Pi_{p_0^c} v_h)) \geq (C_1 \|\Pi_{p_1^c} q_h\| - C_2 h \|\Pi_{p_1^c} q_h\|_{1,h}) \|\Pi_{p_0^c} v_h\|.$$ 

We deduce from this

$$\sup_{v_h \in P_0 \setminus \{0\}} \frac{- (q_h, \text{div}_h v_h)}{\|v_h\|} \geq C_1 \|\Pi_{p_1^c} q_h\| - C_2 h \|\Pi_{p_1^c} q_h\|_{1,h}.$$
Let us first assume that
\[ \forall t \geq 0, \quad \max \left( C t, C_1 |\Pi_{P_1} q_h| - C_2 t \right) \geq \frac{C C_1}{C + C_2} |\Pi_{P_1} q_h| \]
we get the result. \( \blacksquare \)

4.4. Properties of the discrete laplacian. We first prove the coercivity of the discrete laplacian.

**Proposition 4.8.** For all \( u_h \in P_0 \) and \( v_h \in P_0 \) we have
\[ -(\Delta_h u_h, u_h) = \| u_h \|_h^2 - (\Delta_h u_h, v_h) \leq \| u_h \|_h \| v_h \|_h. \]

**Proof.** Using definition (2.3) and writing the sum over the triangles as a sum over the edges, we have
\[ -(\Delta_h u_h, v_h) = -\sum_{K \in T_h} v_K \cdot \left( \sum_{\sigma \in K \cap E_{h}^{int}} \tau_{\sigma} (u_{L_\sigma} - u_K) - \sum_{\sigma \in K \cap E_{h}^{ext}} \tau_{\sigma} u_K \right) \]
\[ = \sum_{\sigma \in E_{h}^{int}} \tau_{\sigma} (v_{L_\sigma} - v_K) \cdot (u_{L_\sigma} - u_K) + \sum_{K \in T_h} \sum_{\sigma \in K \cap E_{h}^{ext}} \tau_{\sigma} u_K \cdot v_K. \]

We get the first half of the result by taking \( v_h = u_h \). On the other hand, using the Cauchy-Schwarz inequality and the algebraic identity \( a b + c d \leq \sqrt{a^2 + c^2} \sqrt{b^2 + d^2} \), we get the second half. \( \blacksquare \)

If \( v \in H^2 \), we have \( |\Delta v| \leq \| v \|_2 \). The operator \( \Delta_h \) shares a similar property.

**Proposition 4.9.** There exists a constant \( C > 0 \) such that for all \( v \in H^2 \)
\[ |\Delta_h (\Pi_{P_0} v)| \leq C \| v \|_2. \]

**Proof.** Let \( v_h = \Pi_{P_0} v \). Let \( K \in T_h \). According to definition (2.10)
\[ (4.13) \quad \Delta_h v_h|_K = \frac{1}{|K|} \sum_{\sigma \in K \cap E_{h}^{int}} \tau_{\sigma} (v(x_{L_\sigma}) - v(x_K)) - \frac{1}{|K|} \sum_{\sigma \in K \cap E_{h}^{ext}} \tau_{\sigma} v(x_K). \]

Let us first assume that \( v = (v_1, v_2) \in (C_0^\infty)^2 \). Let \( i \in \{1, 2\} \). If \( \sigma \in E_K \cap E_{h}^{int} \) and \( x \in \sigma \) we have the Taylor expansions
\[ v_i(x_{L_\sigma}) = v_i(x) + \nabla v_i(x) \cdot (x_{L_\sigma} - x) + \int_0^1 \nabla^2 v_i(t x_{L_\sigma} + (1-t) x) (x_{L_\sigma} - x) (x_{L_\sigma} - x) \, dt, \]
\[ v_i(x_K) = v_i(x) + \nabla v_i(x) \cdot (x_K - x) + \int_0^1 \nabla^2 v_i(t x_K + (1-t) x) (x_K - x) (x_K - x) \, dt, \]
\[ \nabla v_i(x) = \nabla v_i(x_K) - \int_0^1 \nabla v_i(t x_K + (1-t) x) (x_K - x) \, dt. \]

The notation \( \nabla^2 v_i(x) \) refers to the hessian matrix of \( v_i \). Plugging the last expansion into the two others and integrating over \( \sigma \), we get
\[ \int_\sigma (v_i(x_{L_\sigma}) - v_i(x)) \, dx = \nabla v_i(x_K) \cdot (x_K - x) - A_{L_\sigma}^{i} + B_{L_\sigma}^{i}, \]
\[ \int_\sigma (v_i(x_K) - v_i(x)) \, dx = \nabla v_i(x_K) \cdot (x_K - x) - A_{K}^{i} + B_{K}^{i}. \]

The terms \( A_{L_\sigma}^{i} \) and \( B_{L_\sigma}^{i} \) are the same as in (4.6) and (4.7), with \( v_i \) instead of \( q \). We substract these equations. Since \( x_{L_\sigma} - x_K = d_{\sigma} n_{K,\sigma} \) we infer from (2.1)
\[ \tau_{\sigma} (v_i(x_{L_\sigma}) - v_i(x_K)) = \nabla v_i(x_K) \cdot n_{K,\sigma} + \frac{1}{d_{\sigma}} \left( -A_{L_\sigma}^{i} + B_{L_\sigma}^{i} + A_{K}^{i} - B_{K}^{i} \right). \]
Let us consider now the case $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{h,ext}$. If $x \in \sigma$ we have the Taylor expansions

$$v_i(x_K) = v_i(x) + \nabla v_i(x) \cdot (x_K - x) + \int_0^1 \mathbf{H}(v_i)(t x_K + (1-t) x)(x_K - x) \cdot (x_K - x) \, dt,$$

$$\nabla v_i(x) = \nabla v_i(x_K) - \int_0^1 \nabla \nabla v_i(t x_K + (1-t) x)(x_K - x) \, dt.$$ 

Since $v_i \in C_0^\infty$ we have $v_i(x) = 0$. We plug the last expansion into the other and integrate over $\sigma$. Since $x_K - x = -d_{\sigma} n_{K,\sigma}$ we deduce from (2.1)

$$-\tau_{\sigma} v_i(x_K) = \nabla v_i(x_K) \cdot n_{K,\sigma} + \frac{1}{d_{\sigma}} (A_{\sigma}^{\tau_{\sigma} i} - B_{\sigma}^{\tau_{\sigma} i}).$$

Thus we get

$$\frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{h,ext}} \tau_{\sigma} (v_i(x_{L,\sigma}) - v_i(x_K)) = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{h,ext}} \tau_{\sigma} v_i(x_K)$$

$$= \frac{1}{|K|} \nabla v_i(x_K) \cdot \sum_{\sigma \in \mathcal{E}_K} |\sigma| n_{K,\sigma} + \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} R_{\sigma}$$

where we have set for all edge $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{h,ext}$

$$R_{\sigma} = \frac{1}{d_{\sigma}} \left( -A_{L,\sigma}^{\tau_{\sigma} i} + B_{L,\sigma}^{\tau_{\sigma} i} + A_{K,\sigma}^{\tau_{\sigma} i} - B_{K,\sigma}^{\tau_{\sigma} i} \right)$$

and for all edge $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{h,ext}$: $R_{\sigma} = \frac{1}{d_{\sigma}} (A_{K,\sigma}^{\tau_{\sigma} i} - B_{K,\sigma}^{\tau_{\sigma} i})$. Since $\sum_{\sigma \in \mathcal{E}_K} |\sigma| n_{K,\sigma} = 0$, setting $R_{\sigma} = (R_{\sigma}^1, R_{\sigma}^2)$, we get

$$\frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{h,ext}} \tau_{\sigma} (v(x_{L,\sigma}) - v(x_K)) = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} \tau_{\sigma} v(x_K) = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} R_{\sigma}.$$ 

Since the space $(C_0^\infty)^2$ is dense in $H^2$, one checks that this equation still holds for $v \in H^2$. Using (4.13) we infer from it

$$\left| \Delta_h v_h \right|^2 = \sum_{K \in \mathcal{T}_h} |K| \left| \Delta_h v_h |_K \right|^2 \leq \sum_{K \in \mathcal{T}_h} \frac{3}{|K|} \sum_{\sigma \in \mathcal{E}_K} |R_{\sigma}|^2.$$

Using estimates (4.6) and (4.7) we obtain

$$\left| \Delta_h v_h \right|^2 \leq C \sum_{i=1}^2 \sum_{\sigma \in \mathcal{E}_h} \int_{D_{\sigma}} \left( |\nabla \nabla v_i|^2 + |\mathbf{H}(v_i)|^2 \right) \, dx \leq C \|v\|_2^2.$$ 

**5. Stability of the scheme**

We now use the results of section 4 to prove the stability of the scheme. We first show an estimate for the computed velocity (theorem 5.1). We then state a similar result for the increments in time (lemma 5.2). Using the inf-sup condition (proposition 5.1), we infer from it some estimates on the pressure (theorem 5.2).

**Lemma 5.1.** For all $m \in \{0, \ldots, N\}$ and $n \in \{0, \ldots, N\}$ we have

$$\langle u^n_m, \nabla_h p^n_h \rangle = 0,$$

$$|u^n_m|^2 - |\tilde{u}_h^n|^2 + |u^n_h - \tilde{u}_h^n|^2 = 0.$$

**Proof.** First, using propositions 3.1 and 1.6 we get

$$\langle u^n_m, \nabla_h p^n_h \rangle = \langle (u^n_m, \nabla_h (p^n_h - p^{n-1}_h) \rangle = 0.$$ 

Thus we deduce from (3.4)

$$2(u^n_m, u^n_h - \tilde{u}_h^n) = -\frac{4k}{3} (u^n_m, \nabla_h (p^n_h - p^{n-1}_h)) = 0.$$
Using the algebraic identity $2a(a - b) = a^2 - b^2 + (a - b)^2$ we get
\[ 2(u^m_h, u^m_h - \tilde{u}^m_h) = |u^m_h|^2 - |\tilde{u}^m_h|^2 + |u^m_h - \tilde{u}^m_h|^2 = 0. \]

We introduce the following hypothesis on the initial data.

**Hypothesis (H1)** There exists $C > 0$ such that $|u^0_h| + |u^1_h| + k|\nabla_h p^0_h| \leq C$.

Hypothesis (H1) is fulfilled if we set $u^0_h = \Pi_T u_0$ and we use a semi-implicit Euler scheme to compute $u^1_h$. We have the following result.

**Theorem 5.1.** We assume that the initial values of the scheme fulfill (H1). For all $m \in \{2, \ldots, N\}$ we have
\[ |u^m_h|^2 + k \sum_{n=2}^m \|\tilde{u}^n_h\|_h^2 \leq C. \]

**Proof.** Let $m \in \{2, \ldots, N\}$ and $n \in \{1, \ldots, m - 1\}$. Taking the scalar product of (3.2) with $4k\tilde{u}^{n+1}_h$ we get
\[ 4k \left( \tilde{u}^{n+1}_h, \frac{3\tilde{u}^{n+1}_h - 2u^{n+1}_h + u^n_h}{2} \right) - \frac{4k}{\text{Re}} (\tilde{\Delta}_h \tilde{u}^{n+1}_h, \tilde{u}^{n+1}_h) \]
\[ + 4k b_h(2u^n_h - u^{n-1}_h, \tilde{u}^{n+1}_h, \tilde{u}^{n+1}_h) + 4k (\nabla_h p^n_h, \tilde{u}^{n+1}_h) = 4k (f^{n+1}_h, \tilde{u}^{n+1}_h). \]

First of all, using lemma 5.1 we get as in (12)
\[ 4k \left( \tilde{u}^{n+1}_h, \frac{3\tilde{u}^{n+1}_h - 2u^{n+1}_h + u^n_h}{2} \right) \]
\[ = |u^{n+1}_h|^2 - |u^n_h|^2 + 6|u^{n+1}_h - u^n_h|^2 + 2u^{n+1}_h - u^n_h|^2 + 2u^{n+1}_h - u^n_h|^2 \]
\[ + |u^{n+1}_h - u^n_h|^2. \]

According to proposition 4.3 we have $-\frac{4k}{\text{Re}} (\tilde{\Delta}_h \tilde{u}^{n+1}_h, \tilde{u}^{n+1}_h) = \frac{4k}{\text{Re}} \|\tilde{u}^{n+1}_h\|_h^2$. Also, using lemma 5.1 and (3.4), we have
\[ 4k (\nabla_h p^n_h, \tilde{u}^{n+1}_h) = 4k (\nabla_h p^n_h, \tilde{u}^{n+1}_h - u^{n+1}_h) \]
\[ = \frac{4k^2}{3} (|\nabla p^{n+1}_h|^2 - |\nabla p^n_h|^2 - |\nabla p^{n+1}_h - \nabla p^n_h|^2). \]

Multiplying (3.4) by $4k\nabla_h (p^{n+1}_h - p^n_h)$ and using the Young inequality we get
\[ \frac{4k^2}{3} |\nabla (p^{n+1}_h - p^n_h)|^2 \leq 3 |u^{n+1}_h - \tilde{u}^{n+1}_h|^2. \]

According to proposition 4.3 we have $4k b_h(2u^n_h - u^{n-1}_h, \tilde{u}^{n+1}_h, \tilde{u}^{n+1}_h) \geq 0$. At last using the Cauchy-Schwarz inequality, (2.2) and (3.1) we have
\[ 4k (f^{n+1}_h, \tilde{u}^{n+1}_h) \leq 4k |f^{n+1}_h| \|\tilde{u}^{n+1}_h\| \leq C k \|f\|_{L^2(T)} \|\tilde{u}^{n+1}_h\|_h. \]

Using the Young inequality we get
\[ 4k (f^{n+1}_h, \tilde{u}^{n+1}_h) \leq 3 k \|\tilde{u}^{n+1}_h\|_h^2 + C k \|f\|_{L^2(T)}^2 \|\tilde{u}^{n+1}_h\|_h. \]

Let us plug these estimates into (5.1). We get
\[ |u^{n+1}_h|^2 - |u^n_h|^2 + 2|u^{n+1}_h - u^n_h|^2 + |u^{n+1}_h - 2u^n_h + u^{n-1}_h|^2 \]
\[ + 3|\tilde{u}^{n+1}_h - u^{n+1}_h|^2 + k \|\tilde{u}^{n+1}_h\|_h^2 + \frac{4k^2}{3} (|\nabla_h p^{n+1}_h|^2 - |\nabla_h p^n_h|^2) \leq C k. \]
Summing from \( n = 1 \) to \( m - 1 \) we have

\[
|u_h^m|^2 + |2u_h^m - u_h^{m-1}|^2 + 3 \sum_{n=1}^{m-1} |u_h^{n+1} - u_h^{n+1}|^2 + k \sum_{n=1}^{m-1} |u_h^{n+1}|^2 + \frac{4k^2}{3} |\nabla_h p_h^m|^2 \\
\leq C + 4 |u_h^1|^2 + |2u_h^1 - u_h^0|^2 + k^2 |\nabla_h p_h^1|^2.
\]

Using hypothesis (H1) we get the result.

We now want to estimate the computed pressure. From now on, we make the following hypothesis on the initial values of the scheme: there exists a constant \( C > 0 \) such that, using hypothesis (H2), there exists a solution \((u, p)\) to the equations (1.1)–(1.2) such that

\[
(u, p) \in C(0, T; H^2), \quad u(t) \in C(0, T; L^2), \quad \nabla p \in C(0, T; L^2).
\]

We introduce the following hypothesis on the initial values of the scheme: there exists a constant \( C > 0 \) such that

\[
(H2) \quad |u_h^0 - u_0| + \frac{1}{h} \|u_h^1 - u(t_1)\|_{\infty} + |p_h^1 - p(t_1)| \leq C h, \quad |u_h^1 - u_h^0| \leq C k.
\]

One checks easily that this hypothesis implies (H1). We have the following result.

**Lemma 5.2.** We assume that the initial values of the scheme fulfill (H2). Then there exists a constant \( C > 0 \) such that for all \( m \in \{1, \ldots, N\} \)

\[
\frac{1}{k} |\delta u_h^m| \leq C.
\]

**Proof.** We prove the result by induction. The result holds for \( m = 1 \) thanks to hypothesis (H2). Let us consider the case \( m = 2 \). We set \( u_h^1 = u_h^0 \). Let \( u_h^{-1} \in P_0 \) given by

\[
u_h^{-1} = 4u_h^0 - 3u_h^1 + \frac{2}{Re} \Delta_h u_h^1 - 2k \tilde{b}_h(u_h^0, u_h^1) - 2k \nabla_h p_h^1 - 2k f_h^1.
\]

We subtract this equation from equation (5.3) written for \( n = 1 \). Since

\[
\tilde{b}_h(2u_h^1 - u_h^0, \delta u_h^2) - \tilde{b}_h(u_h^0, \delta u_h^2) = \tilde{b}(2u_h^1 - 2u_h^0, \delta u_h^2) + \tilde{b}_h(u_h^0, \delta u_h^2),
\]

upon setting \( \delta u_h^0 = u_h^0 - u_h^{-1} \), we get

\[
\frac{3\delta u_h^2 - 4\delta u_h^1 + \delta u_h^0}{2k} - \frac{1}{Re} \Delta_h(\delta u_h^2) + \tilde{b}(2u_h^1 - 2u_h^0, \delta u_h^2) + \tilde{b}_h(u_h^0, \delta u_h^2) = \delta f_h^2.
\]

Taking the scalar product with \( 4k \delta u_h^2 \) we get

\[
2 \left( 3 \delta u_h^2 - 4 \delta u_h^1 + \delta u_h^0, \delta u_h^2 \right) - \frac{1}{Re} \left( \Delta_h(\delta u_h^2), \delta u_h^2 \right)
\]

\[
+ 4k b_h(u_h^0, \delta u_h^2, \delta u_h^2) + 4k b_h(2u_h^1 - 2u_h^0, \delta u_h^2, \delta u_h^2) = 4k (\delta f_h^2, \delta u_h^2).
\]

According to proposition 4.3 we have

\[
4k |b_h(2u_h^1 - 2u_h^0, \delta u_h^2, \delta u_h^2)| \leq C k |2u_h^1 - 2u_h^0| ||\delta u_h^2||_h \leq C k^2 ||\delta u_h^2||_h .
\]

From the Young inequality and theorem 5.1 we deduce

\[
4k |b_h(2u_h^1 - u_h^0, \delta u_h^2, \delta u_h^2 - \delta u_h^0)| \leq \frac{k}{Re} ||\delta u_h^2||_h^2 + C k^3 \|\delta u_h^2\|_h^2 \leq \frac{k}{Re} ||\delta u_h^2||_h^2 + C k^2.
\]
On the other hand
\[ \delta f_h^2 = f_h^2 - f_{ti}^1 = \Pi_{P_0} f(t_2) - \Pi_{P_0} f(t_1) = \Pi_{P_0} \left( \int_{t_1}^{t_2} f_i(s) \, ds \right). \]

Since \( \Pi_{P_0} \) is stable for the \( L^2 \) norm, using the Cauchy-Schwarz inequality, we get
\[ |\delta f_h^2| \leq \int_{t_1}^{t_2} |f_i(s)| \, ds \leq \sqrt{k} \left( \int_{t_1}^{t_2} |f_i(s)|^2 \, ds \right)^{1/2} \leq \sqrt{k} \| f_i \|_{L^2(0,T;L^2)}. \]

Thus
\[ 4k |\delta f_h^2, \delta u_h^2| \leq 4k |\delta f_h^2| \leq Ck^{3/2} |\delta u_h^2|. \]

So that, using (2.5) and the Young inequality
\[ 4k |\delta u_h^2|^2 \leq |\delta u_h^2|^2 + |2 \delta u_h^2 - \delta u_0^2|^2. \]
We know (5.2) for \( m = 1 \) that \( |\delta u_h^1|^2 \leq Ck^2 \). It remains to estimate the term \( |2 \delta u_h^1 - \delta u_0^1|^2 \). According to (5.3)
\[ 2 \delta u_h^1 - \delta u_0^1 = -\delta u_h^1 + 2 \frac{k}{Re} \Delta_h u_h^1 - 2k \tilde{B}(u_0^1, u_h^1) - 2k \nabla_h p_h - 2k f_h^1. \]

by taking the scalar product with \( 2 \delta u_h^1 - u_h^1 \) and using the Cauchy-Schwarz inequality we get
\[ |2 \delta u_h^1 - \delta u_h^1|^2 \leq 2k \left( \frac{|\delta u_h^1|}{2k} + \frac{1}{Re} |\Delta_h u_h^1| + |\nabla_h p_h| + |f_h^1| \right) 2 |\delta u_h^1 - \delta u_h^1| \]
\[ + 2k \left( b(u_0^1, u_h^1) \right) |2 \delta u_h^1 - \delta u_h^0| \]

Let us bound the terms between braces. First, we have
\[ \Delta_h u_h^1 = \Delta_h (u_h^1 - \Pi_{P_0} u(t_1)) + \Delta_h (\Pi_{P_0} u(t_1)). \]

On one hand, according to proposition 4.8
\[ |\Delta_h (u_h^1 - \Pi_{P_0} u(t_1))|^2 = \left( \Delta_h (u_h^1 - \Pi_{P_0} u(t_1)), \Delta_h (u_h^1 - \Pi_{P_0} u(t_1)) \right) \]
\[ \leq \| \Delta_h (u_h^1 - \Pi_{P_0} u(t_1)) \|_h \| u_h^1 - \Pi_{P_0} u(t_1) \|_h. \]

Applying proposition 2.2, we get
\[ \left| \Delta_h (u_h^1 - \Pi_{P_0} u(t_1)) \right|^2 \leq \frac{C}{h^2} |\Delta_h (u_h^1 - \Pi_{P_0} u(t_1))| \| u_h^1 - \Pi_{P_0} u(t_1) \|. \]

Using the embedding \( L^\infty \subset L^2 \) we have
\[ |u_h^1 - \Pi_{P_0} u(t_1)| = \| \Pi_{P_0} (u_h^1 - u(t_1)) \| \leq \| \Pi_{P_0} (u_h^1 - u(t_1)) \|_\infty; \]

since \( \Pi_{P_0} \) is stable for the \( L^\infty \) norm, we get using hypothesis (H2)
\[ |u_h^1 - \Pi_{P_0} u(t_1)| \leq \| u_h^1 - u(t_1) \|_\infty \leq C h^2. \]

Therefore \( |\Delta_h (u_h^1 - \Pi_{P_0} u(t_1))| \leq C \). And according to proposition 4.9
\[ \left| \Delta_h (\Pi_{P_0} u(t_1)) \right| \leq C \| u(t_1) \| \leq C \| u \|_{C(0,T;H^2)}. \]

Hence \( |\Delta_h u_h^1| \leq C \). Let us now bound the pressure term in (5.6). We have
\[ \nabla_h p_h = \nabla_h (p_h - \Pi_{P_0} p(t_1)) + \left( \nabla_h (\Pi_{P_0} p(t_1)) - \Pi_{P_0} \nabla p(t_1) \right) + \Pi_{P_0} \nabla p(t_1). \]
According to proposition 4.3 we have \( |\nabla_h (p_h^1 - \Pi_h p(t_1))| \leq C h |p_h^1 - \Pi_h p(t_1)| \). Using (22.8) we get
\[
|\nabla_h (p_h^1 - \Pi_h p(t_1))| \leq C \|p(t_1)\|_2 \leq C \|p\|_{C(0,T;H^2)}.
\]
Since \( P_0 \) is stable for the \( L^2 \) norm we have \( |\Pi_{P_0} \nabla p(t_1)| \leq |\nabla p(t_1)| \leq \|p\|_{C(0,T;H^1)} \). Using proposition 4.5 to treat last term we get \( |\nabla_h p_h^1| \leq C \). And according to (5.1) and (5.2) for \( m = 1 \) we have \( \frac{1}{\sqrt{h}} |f_0^1| \leq C \). We are left with the term \( \left| b_h(u_h^0, u_h^1, 2 \delta u_h^1 - \delta u_h^0) \right| \) in (5.0). We use the following splitting
\[
\tilde{b}_h(u_h^0, u_h^1) = \tilde{b}_h(u_h^0, \Pi_{PT_0} u_0, u_h^1) + \tilde{b}_h(\Pi_{PT_0} u_0, u_h^1, \Pi_{P_0} u(t_1)) + \tilde{b}_h(\Pi_{PT_0} u_0, \Pi_{P_0} u(t_1)).
\]
Let us take the scalar product with \( 2 \delta u_h^1 - \delta u_h^0 \). We get
\[
b_h(u_h^0, u_h^1, 2 \delta u_h^1 - \delta u_h^0) = B_1 + B_2 + B_3
\]
with
\[
B_1 = b_h(u_h^0, \Pi_{PT_0} u_0, u_h^1, 2 \delta u_h^1 - \delta u_h^0),
\]
\[
B_2 = b_h(\Pi_{PT_0} u_0, u_h^1, \Pi_{P_0} u(t_1), 2 \delta u_h^1 - \delta u_h^0),
\]
and
\[
B_3 = \left( \tilde{b}_h(\Pi_{PT_0} u_0, \Pi_{P_0} u(t_1)), 2 \delta u_h^1 - \delta u_h^0 \right).
\]
Applying propositions 4.3 and 4.4 we have
\[
|B_1| \leq C \frac{1}{h} \|u_h^0 - \Pi_{PT_0} u_0\| \|u_h^1\|_h \|2 \delta u_h^1 - \delta u_h^0\|.
\]
According to (22.8) and (22.9) we have have
\[
|u_h^0 - \Pi_{PT_0} u_0| = |\Pi_{P_0} u_0 - \Pi_{PT_0} u_0| \leq |\Pi_{P_0} u_0 - u_0| + |u_0 - \Pi_{PT_0} u_0| \leq C h \|u_0\|_1.
\]
According to proposition 4.5 and 4.7
\[
||u_h^1||^2_\delta = - (\Delta_h u_h^1, u_h^1) \leq |\Delta_h u_h^1| \|u_h^1\| \leq C \|\Delta_h u_h^1\| \|u_h^1\|_h:
\]
since \( |\Delta_h u_h^1| \) is bounded we get \( ||u_h^1||_h \leq C \). Hence \( |B_1| \leq C |2 \delta u_h^1 - \delta u_h^0| \). In a similar way, using propositions 4.4 and 4.7 we get
\[
|B_2| \leq C \frac{1}{h^2} \|\Pi_{PT_0} u_0\| \|u_h^1 - \Pi_{P_0} u(t_1)\| \|2 \delta u_h^1 - \delta u_h^0\|.
\]
We have \( ||\Pi_{PT_0} u_0|| \leq ||\Pi_{PT_0} u_0 - u_0|| + |u_0| \leq C h \|u_0\|_1 + |u_0| \leq C \|u_0\|_1 \). Using moreover (5) we get \( |B_2| \leq C |2 \delta u_h^1 - \delta u_h^0| \). Lastly using the following splitting
\[
\tilde{b}_h(\Pi_{PT_0} u_0, \Pi_{P_0} u(t_1)) = \left( \tilde{b}_h(\Pi_{PT_0} u_0, \Pi_{P_0} u(t_1)) - \Pi_{P_0} \tilde{b}(u_0, u(t_1)) \right) + \Pi_{P_0} \tilde{b}(u_0, u(t_1)),
\]
we have \( B_3 = B_{31} + B_{32} \) with
\[
B_{31} = \left( \tilde{b}_h(\Pi_{PT_0} u_0, \Pi_{P_0} u(t_1)) - \Pi_{P_0} \tilde{b}(u_0, u(t_1)), 2 \delta u_h^1 - \delta u_h^0 \right),
\]
\[
B_{32} = \left( \Pi_{P_0} \tilde{b}(u_0, u(t_1)), 2 \delta u_h^1 - \delta u_h^0 \right).
\]
We have
\[
|B_{31}| \leq \|\tilde{b}_h(\Pi_{PT_0} u_0, \Pi_{P_0} u(t_1)) - \Pi_{P_0} \tilde{b}(u_0, u(t_1))\|_{-1,h} \|2 \delta u_h^1 - \delta u_h^0\|_h.
\]
So that, using proposition 4.1 \(|B_{31}| \leq C h \|u_0\|_2 \|u(t_1)\|_2 \|2 \delta u_h^1 - \delta u_h^0\|_h\). Using proposition 2.1 we obtain
\(|B_{31}| \leq C \|u_0\|_2 \|u\|_{C(0,T;H^2)} \|2 \delta u_h^1 - \delta u_h^0\|_h\).

Let us now bound \(B_{32}\). Using the Cauchy-Schwarz inequality and the stability of \(\Pi p_h\) for the \(L^2\) norm, we have
\(|B_{32}| \leq \left|\Pi p_h \tilde{b}(u_0, u(t_1))\right| \|2 \delta u_h^1 - \delta u_h^0\|_h \leq \left|\tilde{b}(u_0, u(t_1))\right| \|2 \delta u_h^1 - \delta u_h^0\|_h|.

Integrating by parts, we deduce from (4.1)
\[\left|\tilde{b}(u_0, u(t_1))\right| \leq \sum_{i=1}^{2} |u_0 \cdot \nabla u_i(t_1)| \leq |u_0| \|u(t_1)\|_2 \leq C |u_0| \|u\|_{C(0,T;H^2)}\]

Thus \(|B_{32}| \leq C \|2 \delta u_h^1 - \delta u_h^0\|_h\). By gathering the estimates for \(B_1, B_2, B_3\) we get
\[|b_h(u_h^0, u_h^1, 2 \delta u_h^1 - \delta u_h^0)| \leq C.\]

Thus we have bounded the right-hand side in (5.10). We infer from it
\[\|2 \delta u_h^1 - \delta u_h^0\|_h \leq C k.\]

Plugging this estimate into (5.3) and using (5.2) for \(m = 1\), we get (5.2) for \(m = 2\). Let \(m \in \{3, \ldots, N - 1\}\). We assume that the induction hypothesis is satisfied up to rank \(n = m - 1\). Let us subtract equation (3.2) with the same for \(n - 1\). Since the operator \(\tilde{b}_h\) is bilinear we get
\[\frac{3 \delta u_h^{n+1} - 4 \delta u_h^n + \delta u_h^{n-1}}{2k} - \frac{1}{Re} \Delta_h(\delta u_h^{n+1}) + \tilde{b}_h(2 \delta u_h^n - \delta u_h^{n-1}, \tilde{u}_h^{n+1}) + \tilde{b}_h(2 u_h^n - u_h^{n-1}, \delta u_h^{n+1}) + \nabla_h(\delta p_h^n) = \delta f_h^{n+1}.\]

Let us take the scalar product with \(4k \delta \tilde{u}_h^{n+1}\). We get
\[\left(\frac{3 \delta u_h^{n+1} - 4 \delta u_h^n + \delta u_h^{n-1}}{2k}, 4k \delta \tilde{u}_h^{n+1}\right) - \frac{4k}{Re} \left(\Delta_h(\delta u_h^{n+1}), \delta \tilde{u}_h^{n+1}\right) + 4k b_h(2 \delta u_h^n - \delta u_h^{n-1}, \tilde{u}_h^{n+1}, \delta \tilde{u}_h^{n+1}) + 4k b_h(2 u_h^n - u_h^{n-1}, \delta u_h^{n+1}, \tilde{u}_h^{n+1}) + 4k (\nabla_h(\delta p_h^n), \delta u_h^{n+1}) = 4k (\delta f_h^{n+1}, \delta \tilde{u}_h^{n+1}).\]

According to proposition 4.3 we have
\[|4k b_h(2 \delta u_h^n - \delta u_h^{n-1}, \tilde{u}_h^{n+1}, \delta \tilde{u}_h^{n+1})| \leq C k |2 \delta u_h^n - \delta u_h^{n-1}| \|\tilde{u}_h^{n+1}\|_h \|\delta \tilde{u}_h^{n+1}\|_h.\]

Using the induction hypothesis we get
\[|4k b_h(2 \delta u_h^n - \delta u_h^{n-1}, \tilde{u}_h^{n+1}, \delta \tilde{u}_h^{n+1})| \leq C k^2 \|\tilde{u}_h^{n+1}\|_h \|\delta \tilde{u}_h^{n+1}\|_h.\]

Using the Young inequality and (5.1) we infer that
\[|4k b_h(2 \delta u_h^n - \delta u_h^{n-1}, \tilde{u}_h^{n+1}, \delta \tilde{u}_h^{n+1})| \leq \frac{k}{Re} \|\delta \tilde{u}_h^{n+1}\|_h^2 + C k^2.\]

The other terms are treated like the case \(m = 2\). We finally obtain (5.2).

**Theorem 5.2.** We assume that the initial values of the scheme fulfill (H2). There exists a constant \(C > 0\) such that for all \(m \in \{2, \ldots, N\}\)
\[k \sum_{n=2}^{m} ||P_{T^n}p_h^n||^2 \leq C.\]
Proof. Let \( m \in \{2, \ldots, N\} \). We set \( n = m - 1 \). Using the inf-sup condition (4.7) and proposition 4.6, we get that there exists \( v_h \in P_0 \backslash \{0\} \) such that
\[
C \|v_h\|_h \|\Pi_{P_m^r} p_h^{n+1}\| \leq -\langle p_h^{n+1}, \text{div}_h v_h \rangle = \langle \nabla_h p_h^{n+1}, v_h \rangle.
\]
Plugging (5.1) into (5.2) we have
\[
\nabla_h p_h^{n+1} = \frac{3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}}{2k} + \frac{1}{Re} \Delta_h \hat{u}_h^{n+1} - \bar{b}_h (2 u_h^n - u_h^{n-1}, \hat{u}_h^{n+1}) + f_h^{n+1}.
\]
so that
\[
\left( \nabla_h p_h^{n+1}, v_h \right) = \left( \frac{3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}}{2k}, v_h \right) + \frac{1}{Re} \left( \Delta_h \hat{u}_h^{n+1}, v_h \right) - b_h (2 u_h^n - u_h^{n-1}, \hat{u}_h^{n+1}, v_h) + (f_h^{n+1}, v_h).
\]
Using the Cauchy-Schwarz inequality, (2.5) and (3.1) we have
\[
\left| \left( \frac{3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}}{2k}, v_h \right) \right| \leq C \left| \frac{3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}}{2k} \right| \|v_h\|_h
\]
and
\[
(f_h^{n+1}, v_h) \leq |f_h^{n+1}| \|v_h\| \leq C |v_h| \leq C \|v_h\|_h.
\]
Thanks to proposition 4.3 and theorem 5.1 we have
\[
|b_h (2 u_h^n - u_h^{n-1}, \hat{u}_h^{n+1}, v_h) | \leq (2 |u_h^n| + |u_h^{n-1}|) \|\hat{u}_h^{n+1}\|_h \|v_h\|_h \leq C \|\hat{u}_h^{n+1}\|_h \|v_h\|_h.
\]
And according to proposition 4.8 we have\( (\Delta_h \hat{u}_h^{n+1}, v_h) \leq \|\hat{u}_h^{n+1}\|_h \|v_h\|_h \). Thus
\[
(\nabla_h p_h^{n+1}, v_h) \leq C + C \left( \frac{3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}}{2k} + \|\hat{u}_h^{n+1}\|_h \right) \|v_h\|_h.
\]
Comparing with (5.7) we get
\[
\|\Pi_{P_m^r} p_h^{n+1}\| \leq C + C \left( \frac{3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}}{2k} + \|\hat{u}_h^{n+1}\|_h \right).
\]
Squaring and summing from \( n = 1 \) to \( m - 1 \) we obtain
\[
k \sum_{n=2}^m |\Pi_{P_m^r} p_h^n|_h^2 \leq C + C k \sum_{n=1}^{m-1} \frac{3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}}{2k} + C k \sum_{n=1}^{m-1} \|\hat{u}_h^{n+1}\|_h^2.
\]
The last term on the right-hand side is bounded, thanks to theorem 5.1. And since
\[
3 u_h^{n+1} - 4 u_h^n + u_h^{n-1} = 3(u_h^{n+1} - u_h^n) - (u_h^n - u_h^{n-1}) = 3 \delta u_h^{n+1} - \delta u_h^n
\]
we deduce from lemma 5.2
\[
k \sum_{n=1}^{m-1} \frac{|3 u_h^{n+1} - 4 u_h^n + u_h^{n-1}|^2}{4k^2} \leq C k \sum_{n=1}^{m-1} \frac{\|\delta u_h^n\|_h^2}{k^2} \leq C.
\]

References

[1] S. Boivin, F. Cayre, J. M. Herard, A finite volume method to solve the Navier-Stokes equations for incompressible flows on unstructured meshes, Int. J. Therm. Sci., 39 (2000) 806-825.
[2] S. C. Brenner, L. R. Scott, The Mathematical Theory of Finite Element Methods, Springer, 2002.
[3] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.
[4] J. Chorin, On the convergence of discrete approximations to the Navier-Stokes equations, Math. Comp. 23 (1969) 341-353.
[5] R. Eymard, T. Gallouët, R. Herbin, A cell-centered finite-volume approximation for anisotropic diffusion operators on unstructured meshes in any space dimension, IMA J. Numer. Anal. 26 (2006) 326-353.
[6] R. Eymard, T. Gallouët and R. Herbin, Finite volume methods. In Handbook of Numerical Analysis, P.G. Ciarlet and J.L. Lions eds, North-Holland, 2000.
[7] R. Eymard and R. Herbin, A staggered finite volume scheme on general meshes for the Navier-Stokes equations in two space dimensions, Int. J. Finite Volumes (2005).
[8] R. Eymard, J. C. Latché and R. Herbin, Convergence analysis of a colocated finite volume scheme for the incompressible Navier-Stokes equations on general 2D or 3D meshes, preprint LATP (2004).
[9] S. Faure, Stability of a colocated finite volume scheme for the Navier-Stokes equations, Num. Methods Partial Differential Equations 21(2) (2005) 242-271.
[10] V. Girault and P. A. Raviart, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Springer-Verlag, 1986.
[11] J.L. Guermond, Some implementations of projection methods for Navier-Stokes equations, M2AN 30(5) (1996) 637-667.
[12] J. L. Guermond, Un résultat de convergence l’ordre deux en temps pour l’approximation des équations de Navier-Stokes par une technique de projection, M2AN 33(1) (1999) 169-189.
[13] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal., 19(26) (1982) 275-311.
[14] D. Kim and H. Choi, A second-order time-accurate finite volume method for unsteady incompressible flow on hybrid unstructured grids, J. Comput. Phys. 162 (2000) 411-428.
[15] R. Temam, Sur l’approximation de la solution des équations de Navier-Stokes par la méthode de pas fractionnaires II, Arch. Ration. Mech. Anal. 33 (1969) 377-385.
[16] S. Zimmermann, Étude et implémentation de méthodes de volumes finis pour les fluides incompressibles, PhD, Blaise Pascal University, 2006.

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