Some Notes of Inequalities Under $C$-mixing Conditions and Their Applications to Variance Estimation

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Abstract

As a mixing condition including many interesting dynamic systems as special cases, $C$-mixing condition has drawn significant attention in recent years. This paper aims to do some contributions on the following points. First, we show a Bernstein-type inequality under $C$-mixing conditions. Compared with the pioneering work on this point, Hang and Steinwart (2017), our inequality is sharper under more general assumptions. Second, since the general definition of $C$-mixing condition is based on a covariance inequality whose upper bound relies on some given $C$-norm (see Definition 1), a natural difficulty arises when the $C$-norm is infinitely large. Under this circumstances, we show some inequalities bounding the variance of partial sums without requiring finite $C$-norms. Finally, up to our knowledge, there is few literature discussing central limit theorem under $C$-mixing conditions as general as that of Hang and Steinwart (2017). Thus, under Hang and Steinwart’s $C$-mixing conditions, we take one step forward on this point by deriving a central limit theorem with mild moment conditions. As for the applications, we apply the previously mentioned results to show Bahadur representation and asymptotic normality of weighted $M$-estimators.

1 Introduction

1.1 Literature Review

Concentration inequalities such as those of Hoeffding, Bernstein, McDiarmid or Talagrand are playing an increasingly important role in (non-)asymptotic analysis in many areas of statistics, particularly in nonparametric estimation and prediction problems. Good examples can be found in any of the following three textbooks: Devroye, Györfi, and Lugosi (2013), Devroye and Lugosi (2001) and Györfi, Kohler, Krzyzak, Walk, et al. (2002). All of the examples mentioned above, however, involve cases in which the process is independent or even iid – an assumption that is often violated in areas such as spatial data regression, time series forecasting, and text or speech recognition. Thus, deriving concentration inequalities for dependent data becomes an important task.

One of the most widely used settings of dependence is mixing, which can generally be understood as ”asymptotically independent or uncorrelated”. Among the several definitions of mixing (Doukhan, 1994), strong mixing is often regarded as a more general one that contains $\phi$- and $\rho$-mixing processes as special cases. For the case where indices lie in a (positive-)integer set, Bosq
obtained a very general exponential inequality adaptive to the arbitrary decay rate of a strong mixing coefficient. Modha and Masry (1996) derived a Bernstein-type inequality that holds for $m$-dependent and a strong mixing process simultaneously, but it seems that the sharpness of their inequality is even worse than that of Bosq, even when the mixing coefficient decay exponentially. Merlevède, Peligrad, and Rio (2009) discovered a Bernstein-type inequality for a geometric strong-mixing process, which is the same as the iid case, modulo a logarithm factor and some constants.

However, a strong-mixing condition may fail to include data collected from some dynamic systems. The first interesting example would be the real-valued $\tilde{\phi}$-mixing process proposed by Rio (1996). Subsequently, Dedecker and Prieur (2005) provided a definition equivalent to that of Rio (1996) and obtained a useful concentration inequality for arbitrary separately-Lipschitz-continuous function $f : \prod_{i=1}^{n} \mathcal{X}_i \to \mathbb{R}$. More specifically, they gave an exponential inequality as follows: for $\forall t > 0$,

$$
P \{ |f - E(f)| > t \} \leq \exp \left( -\frac{nt^2}{C} \right),$$

(1)

where $C$ is a constant associated only with the Lipschitz condition of $f$ and $E(f)$ is the expectation of $f$. Furthermore, Theorem 2 in Dedecker and Prieur (2005) demonstrates (1) for iterated random functions and Markov kernels. Even though their result is much more general compared with an exponential inequality (e.g. Bernstein-type inequality) for the partial sum, (1) becomes less sharper when the variance term $\sigma^2$ becomes sufficiently small. A simple but important example is the kernel smoother. Recall that a crucial step in deriving the uniform convergence rate is to investigate the following multiplier-type empirical process, (see Hansen (2008), Kristensen (2009), Li, Lu, and Linton (2012), Vogt (2012), etc.)

$$
\frac{1}{n} \sum_{i=1}^{n} \left( K \left( \frac{X_i - x}{h} \right) Y_i - E \left[ K \left( \frac{X_i - x}{h} \right) Y_i \right] \right),
$$

(2)

where $K(\cdot)$ represents some specific kernel function. It is obvious that $\text{Var} \left( K(X_i - x/h)Y_i \right) = O(h^D)$, provided that $X_i \in \mathbb{R}^D$, which would vanish as sample size $n$ tends to be infinitely large. More details on this will be given in Section 4.

Additionally, note that both Rio (1995)’s and Dedecker and Prieur (2004)’s definitions focus on measuring the dependence between a $\sigma$-algebra and a real-valued random variable. Olivier (2010) extended Rio (1995)’s idea by giving a definition that measures the dependence between sub-$\sigma$-algebra and an arbitrary random vector. Furthermore, a deviation inequality is shown based on this extended definition. But as has been pointed out by Hang and Steinwart (2017), Olivier’s extension turns out to be more restrictive than Rio (1995)’s original version, and the idea of $\tilde{\phi}$-mixing itself is still not general enough for many interesting dynamic systems. To achieve this goal, Maume-Deschamps (2006) proposed $C$-mixing coefficients. Motivated by this, Hang and Steinwart (2017) derived a Bernstein-type inequality for geometrically $C$-mixing process (see Definition 1 in Section 2). Modulo a logarithmic factor $\sim (\log n)^{2/\gamma} (\gamma > 0)$ and some constants, their result coincides with the standard Bernstein inequality for iid data. Furthermore, they derived an oracle inequality for generic regularized empirical risk minimization and applied this oracle inequality to support vector machines with Gaussian kernel. By using the newly derived Bernstein-type inequality,
Hang, Steinwart, Feng, and Suykens (2018) obtained $L^1$ and $L^\infty$ convergence rates of kernel density estimator for $C$-mixing-type dynamic systems. More recently, a key extension of Hang and Steinwart (2017) is Blanchard and Zadorozhnyi (2019). Blanchard and Zadorozhnyi obtained a deviation inequality for the summation of Banach-valued random variables satisfying $C$-mixing coefficients with both exponential and polynomial decay and under certain smoothness assumptions of the underlying Banach norm. Asymptotically, their result is as sharp as ours and hence sharper than that of Hang and Steinwart. However, Blanchard and Zadorozhnyi (2019) focused on the averaged sums of random variables that take values in a separable Banach space, whereas we concentrate on the arithmetic mean of centered empirical processes whose sample vector is allowed to be collected from any $Z$-valued $C$-mixing processes, where $Z$ could be any well-defined metric space. Generally speaking, both of our results can be regarded as updates of Hang and Steinwart’s result in different directions.

Except for the aforementioned Bernstein-type (or Exponential-type) inequalities, another set of important and useful inequalities are those dedicated to bounding the variance of partial sums under $C$-mixing conditions. This investigation is crucial for examining the asymptotic variance and MSE of estimators that can be generally regarded as an ”empirical mean”, like kernel smoothers. Numerous useful inequalities have been derived to bound the variance of the partial sum process under various mixing conditions. A straight but non-trivial example is the class of processes whose $\rho$-mixing (Doukhan (2012)) coefficient converges to 0. Since $\rho$-mixing is tightly connected with linear correlation coefficient, we can easily obtain that $\text{Var}(S_n) = O(n)$, where $\{S_n\}_{n \in \mathbb{N}}$ denotes the partial sum process. Note that only second moment condition is required here. It is also widely known that, if some higher order moment conditions exist, the $\rho$-mixing condition can be extended to $\alpha$-mixing condition which is much more general. Unfortunately, for generally defined $C$-mixing-type processes, up to our knowledge, there is a scarcity of relative literature.

1.2 Challenges and Contributions

According to the proof of the Bernstein-type inequality in Hang and Steinwart (2017), we suppose number ”2” in term “$(\log N)^{2}$” is more or less a technical compromise. Stationarity seems to be a necessary condition for the proof. However, Gray and Kieffer (1980) pointed out that some dynamic systems that coincide with the $C$-mixing condition are not stationary, but only asymptotically mean stationary, see e.g. Gray (2009) and Definition 2.2 in Steinwart, Hush, and Scovel (2009). Another key restriction of Hang and Steinwart (2017)’s work is the growth condition of the semi-norm associated with $C$-mixing. As will be shown below in Section 2, a particular kind of $C$-mixing process is often tightly connected with a specific choice of semi-norm, whereas for some applications, such as kernel-based smoothers, this semi-norm often grows with sample size. According to Theorem 3.2 in Hang and Steinwart’s paper (see also below in Section 3), when the function of a random variable has finite $L^\infty$-norm $\| \cdot \|_\infty$, their inequality would be valid only if the semi-norm is no larger than $O(N^2)$. Of course, there have been many important $C$-mixing processes suitable for this, but it would still be meaningful if we could relax this requirement significantly.

Furthermore, note that a very important example of the $C$-mixing process is $\phi$-mixing, which is also frequently used to model spatial data (e.g. (Jenish & Prucha, 2009)). A natural and necessary extension is to consider the situation in which the index set of processes lies in $\mathbb{N}^d$ for any $d \geq 2$. 

Hence non-isotropic sampling behavior (e.g. Hallin, Lu, and Tran (2004), Lu and Chen (2002)) needs to be considered. Indeed, sampling behavior is crucial to the degree of dependence. For example, consider a $C$-mixing process whose index set is $\mathbb{N}^d$ with some $d (d \geq 2)$, but for which we sample only the ”locations” along the x-axis. This means that the index set of our sampling process has a linear order that is similar to a time series. A reasonable conjecture in this case is that the decreasing rate of the probabilistic inequality of the difference between empirical and population means should be as sharp as that in the situation where $d = 1$. More generally, we hope that the sharpness of our inequality is only weakly affected by the number of ”directions” along which the quantity of sampling sites will diverge to $+\infty$ at different rates. In the rest of this paper, we will address this as the ”effective dimension”.

As for the inequalities that bound the variance of the partial sum of $C$-mixing processes, the crucial point lies in bounding the covariance between any two components. Although Definition 1 of this paper may seem to provide a direct inequality for the covariance, it is only non-trivial if the $C$-norm is finite. Unfortunately, this requirement is often too strong in many applications. One obvious example is as follows: suppose $\{Z_i\}_{i \in \mathbb{N}^+} := Z$ is a strictly stationary $C$-mixing process, and its $C$-mixing coefficient is defined with respect to the semi-norm $\| \cdot \|$. If $\| \cdot \|$ is not finite with respect to the identity mapping, then calculating $\text{Var}(\overline{Z}_n)$, where $\overline{Z}_n := n^{-1} \sum_{i=1}^{n} Z_i$ represents the empirical mean, would involve determining $|\text{Cov}(Z_i, Z_j)|$. According to Definition 1, obtaining a non-trivial upper bound directly becomes nearly impossible. This disadvantage would be lethal when investigating the mean squared error (MSE) of the kernel-based conditional mean estimator for fixed-design nonparametric regressions.

In short, under $C$-mixing conditions, this paper aims to establish a sharper Bernstein-type inequality and several new inequalities for the variance of partial sums that can overcome the barriers mentioned above. For Bernstein-type inequality, motivated by the tricks from Valenzuela-Domínguez, Krebs, and Frank (2017), J. T. Krebs (2018) and J. Krebs (2021), together with the definition of $C$-mixing, we use a more direct approach to bound the Laplace transformation of averaged partial sum process. This alternative approach fortunately allows us to bypass the technical compromise mentioned above and polish the logarithm factor to a sharper level. As shown by Theorem 1 in Section 3, the sharpness of our inequality is only affected by the ”effective dimension” up to a ”$\log N$”-level, which makes our result adaptive to non-isotropic sampling behavior. As for the requirement with regard to the growth condition of the semi-norm associated with the $C$-mixing coefficient, instead of being no larger than $O(N^2)$, our result only requires the the growth condition is at the order of $O(N^\alpha)$, for $\forall \alpha > 0$. We prove that, under very weak moment and $C$-mixing conditions, the variance of partial sums can be upper bounded by the variance of that of an iid process, up to a constant multiplier.

In our application, we focus on utilizing the newly derived inequalities to investigate the mean squared error (MSE) and uniform convergence rates of kernel smoothers. Firstly, we examine the MSE and uniform convergence rates of conditional estimators. Specifically, we demonstrate that the optimal MSE can be achieved by using a $C$-mixing coefficient with a weaker rate than the $\alpha$-mixing coefficient in Theorem 1 of Hansen (2008). Regarding the uniform convergence rate, we establish that, for geometric $C$-mixing processes, their optimality can be achieved by only requiring the unconditional moments of the output to be slightly larger than $2 + D/\alpha$, where $D$ and $\alpha$ represent
the input dimension and Hölder continuity, respectively. This aligns with the results obtained by Chen and Christensen (2015) for sieve estimators. Additionally, we revisit the MSE and uniform rates of kernel density estimators for \( C \)-mixing-type processes or dynamic systems, which can be seen as an extension and completion of the work by Hang et al. (2018).

The rest of this paper is organized as follows: In Section 2, we review the definition of the \( C \)-mixing process. Here, instead of borrowing Hang and Steinwart (2017)’s definition directly, we use its sufficient and necessary condition as the definition of the \( C \)-mixing condition. This gives us more flexibility when defining the case where \( d \geq 2 \). In Section 3, we demonstrate our newly discovered Bernstein-type inequality and compare it with previous research. In Section 4, inequalities for the variance of partial sums under \( C \)-mixing conditions will be discussed. Section 5 is dedicated to exhibit how we apply the inequalities to examine the theoretical guarantee of kernel smoothers under \( C \)-mixing conditions. The proof of theorems and propositions is delayed to the Appendix.

2 \( C \)-mixing

Suppose we have a \( Z \)-valued process \( Z = (Z_i)_{i \in \mathbb{N}^d} \) on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\), where \( \mathbb{N}^d \) denotes the non-negative \( d \)-dimensional integer-valued vectors. Hence, some spatial lattice processes are also taken into consideration. Based on this, we use \( L^p(\mathbb{P}) \) space of all real-valued and measurable functions, \( f : \Omega \rightarrow \mathbb{R} \), with finite \( L^p \)-norm with respect to measure \( \mathbb{P} \), which forms a complete and normed linear space (Banach space). Furthermore, for arbitrary \( \sigma \)-algebra of \( \mathcal{A} \), say \( \mathcal{A}' \), we use \( L^p(\mathcal{A}', \mathbb{P}) \) to denote the collection of all \( \mathcal{A}' \)-measurable functions with finite \( L^p \)-norm with respect to measure \( \mathbb{P} \). Moreover, for any Banach space \( E \), we use \( B_E \) to indicate the closed unit ball in \( E \).

Now we introduce the semi-norm term in \( C \)-mixing, which is tightly associated with the specific choice of dependence measure. Given a semi-norm \( || \cdot || \) on a vector space \( E \) with bounded measurable functions \( f : Z \rightarrow \mathbb{R} \), we define \( C \)-norm, denoted as \( || \cdot ||_C \), as follows,

\[
||f||_C = ||f||_\infty + ||f||
\]

Meanwhile, we denote the relative \( C \)-space by

\[
C(Z) := \{ f : Z \rightarrow \mathbb{R} ||f||_C < +\infty \}.
\]

For the image region, let us now assume that there exists a measurable space \((Z, \mathcal{B})\) and \( Z_i : \Omega \rightarrow Z \) for each \( i \in \mathbb{N}^d \). For any finite set \( U \subset \mathbb{N}^d \), we denote \( \sigma(U) \) as the \( \sigma \)-field generated by the vector \((Z_i)_{i \in U}\). We now give the definition of the \( C \)-mixing process \( Z \) in the following.

**Definition 1** Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space and \((Z, \mathcal{B})\) a measurable space, with \( Z \) defined as a \( Z \)-valued process on \( \Omega \) and norm \( || \cdot ||_C \) as (3) for a given semi-norm \( || \cdot || \). Then, for any finite subset \( U \subset \mathbb{N}^d \) and \( i \in \mathbb{N}^d \), we say process \( Z \) is a \( C \)-mixing process if the following inequality holds for all \( f \in L^1(\sigma(U), \mathbb{P}) \) and all \( g \in C(Z) \),

\[
|\text{Cov}(f, g)| \leq C(\rho(U, i))||f||_{L^1(\mathbb{P})}||g||_C,
\]
where $\rho(U, i) = \inf_{u \in U} \|u - i\|_\infty$ and $C(r)$ is a non-negative and non-increasing function on $\mathbb{R}^+$. More specifically, when $C(\rho(U, i)) = \nu - b\rho(U, i)\gamma$ for some $\nu, b, \gamma > 0$, we call this process a geometrical $C$-mixing process of order $(\nu, b, \gamma)$; when $C(\rho(U, i)) = b/\rho(U, i)^\gamma$, we call this process an algebraic $C$-mixing process of order $(b, \gamma)$.

Compared with the original definition of $C$-mixing coefficient in Hang and Steinwart (2017), our definition is actually its necessary and sufficient condition, which also describes the dependence between a sub-$\sigma$-algebra and a random variable. Our setting, however, can be more easily extended to a situation in which the dimension of the index set is large than 1, i.e., $d > 1$. When $d = 1$, if we let $U = \{0, 1, ..., k\}$ or $\{k + n, k + n + 1, ..., +\infty\}$ and $i = k + n$(or $k$), our definition immediately becomes a necessary and sufficient condition for the (time-reversed) $C$-mixing process which is given as Definition 2.5 in Hang and Steinwart. However, we do not mention the time-reverse setting here because, in the context of random field, this discussion becomes pointless due to the fact that there is no natural (linear) order in the index set $\mathbb{N}^d$. This being said, for the case in which $d = 1$, our result (Theorem 1) is also valid for the time-reverse $C$-mixing process. To check this, refer to the proof of this theorem in the Appendix.

Recall that the choice of semi-norm $\| \cdot \|$ is closely connected with the specific definition of the process. Due to (3) and (5), the choice of semi-norm significantly affects the upper bound of $\text{Cov}(f, g)$, which is crucial to our result. This naturally raises the question: "Which semi-norms are used frequently?" In Section 2 of Hang and Steinwart (2017)'s work, there has been some helpful conclusions in this regard and some reviews of the relevant literature. Therefore, we will concentrate here primarily on the following six examples of semi-norms, which will later serve as motivating examples for the assumptions of our technique with regard to the semi-norm.

**Example 1** For an arbitrary set $Z$, we set $\|f\| = 0$ for all real-valued and measurable function $f : Z \to \mathbb{R}$.

**Example 2** Let $Z \subset \mathbb{R}^D$ be an open subset. For a continuously differentiable function $f : Z \to \mathbb{R}$, we define

$$\|f\| := \sum_{i=1}^{d} \left\| \frac{\partial f}{\partial z_i} \right\|_\infty.$$  

**Example 3** Let $Z \subset \mathbb{R}^D$ and $C_b(Z)$ be the set of bounded continuous functions on $Z$. For each $f \in C_b(Z)$ and $0 < \alpha \leq 1$, let

$$\|f\| := \|f\|_\alpha := \sup_{z \neq z'} \frac{|f(z) - f(z')|}{\|z - z'\|_2^{\alpha}},$$  

where $\| \cdot \|_2$ denotes the Euclidean norm in $\mathbb{R}^D$.

**Example 4** Based on Example 3, by setting $\alpha = 1$, we immediately obtain a Lipschitz-norm, say $\text{Lip}(f)$, of $f$.

**Example 5** Let $Z \subset \mathbb{R}$ be an interval. We define the semi-norm $\| \cdot \|$ as the total variation of measurable function $f : Z \to \mathbb{R}$. Hence, for each $\|f\| < \infty$, $f \in BV(Z)$, and we write this semi-norm as $\|f\|_{BV}$, where $BV(Z)$ represents the bounded variation of the class on $Z$. 

6
Example 6 Let $Z \subset \mathbb{R}^D$ be a closed rectangle. Parallel to Example 5, where $\|f\|_{BV} = \int_Z |f'| \, d\mu$, here we set
\[
\|f\| := \int_Z \left| \frac{\partial f}{\partial z_1 \partial z_2 \ldots \partial z_D} \right| \, d\mu,
\]
with $\mu$ being the Lebesgue measure on $\mathbb{R}^D$.

Examples 1–5 are directly inherited from Hang and Steinwart (2017), which contain many interesting stochastic processes and time discrete dynamical systems as special cases. For Example 3 and 5, due to the Propositions 2.7 and 3.8, as well as Corollary 4.11 and Theorem 1.15, in Viana (1997), given a dynamic system $(\Omega, \mathcal{A}, \mathbb{P}, T)$, where $T$ is a measure-preserving map$^1$, this system is a geometrically time-reversed $C$-mixing process with related spaces $BV(Z)$ or $C_{b,a}(Z)$, when $T$ is one of the following maps, smooth expanding maps on manifolds, piecewise expanding maps, uniformly hyperbolic attractors and nonuniformly hyperbolic uni-modal maps. Moreover, Maume-Deschamps (2006) proved that the system is also a time-reversed geometric $C$-mixing process if $T$ is Lasota-Yorke maps, uni-modal maps, piecewise expanding maps in higher dimension. Here, the related spaces are $BV(Z)$ or $Lip(Z)$, which coincides with Examples 4 and 5. For more literature ress on this topic, see page 715 in Hang and Steinwart (2017).

For Example 1, due to Davydov (1968), it directly includes a $\phi$-mixing ((Doukhan, 2012)) process as a special case. Now we concentrate on Example 6. Intuitively it is like a multivariate-version of Example 5. Actually, according to Rio (1996) and Dedecker and Prieur (2005), real-valued $\hat{\phi}$-mixing coincides with Example 5. But Rio (1996) only defines a coefficient measuring the dependence between a sub-$\sigma$-algebra and a real-valued random variable, while the Lemma 1 in Dedecker and Prieur provides us with an equivalent definition. Motivated by this lemma, we therefore propose an extension in/of (2), which measures the dependence between a sub-$\sigma$-algebra and a random vector. Our Definition 2 and Proposition 1 reveal that, this extension could be regarded as a $C$-mixing process by letting the associated space satisfy the condition shown in Example 6.

Definition 2 Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Given a sub-$\sigma$-algebra $\mathcal{A}' \subset \mathcal{A}$ and a random vector $X : \Omega \to \prod_{i=1}^q [a_i, b_i] \subset \mathbb{R}^q$, $\forall \ q \in \mathbb{N}^+$, where $a_i, b_i \in \mathbb{R}$. We define a mixing coefficient, called $\hat{\phi}_V$-mixing, as follows:
\[
\hat{\phi}_V(\mathcal{A}', X) = \sup_{t \in \mathbb{R}} \|F_{X|\mathcal{A}'}(t) - F_X(t)\|_\infty,
\]
where $F_{X|\mathcal{A}'}$ denotes the conditional distribution function and $F_X$ is the cumulative distribution function of vector $X$.

Similar to Olivier (2010), Definition 2 is also a "multi-dimensional version" of Rio (1996)’s $\hat{\phi}$-mixing coefficient. Unlike Olivier’s extension, however, (6) is a natural extension of the idea of Dedecker and Prieur (2005). More generally, (6) directly gives a measure between a sub-$\sigma$-algebra and an arbitrary, well-defined random vector. This flexibility enables us to easily extend the real-valued $\hat{\phi}$-mixing process to an arbitrary $q$-dimensional vector-valued process.

But how then do we connect (6) with (5)? For this, we demonstrate the following proposition.

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$^1T : \Omega \to \Omega$ is a measure-preserving map if it satisfies $\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{A}$.
Proposition 1 According to Definition 2, define $\Lambda$ as a set of real-valued measurable functions whose domain is $\prod_{i=1}^{q}[a_i, b_i]$, i.e., $f : \prod_{i=1}^{q}[a_i, b_i] \to \mathbb{R}$. For each $f \in \Lambda$, $\exists g \in C(\prod_{i=1}^{q}[a_i, b_i])$ such that, for every $x := (x_1, \ldots, x_q) \in \prod_{i=1}^{q}[a_i, b_i]$,

$$f(x) = \int_{\prod_{i=1}^{q}[a_i, x_i]} g(t) \, d\mu \text{ and } |f(x)| < +\infty,$$

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}^q$ and $C(\prod_{i=1}^{q}[a_i, b_i])$ indicates the collection of all continuous functions on $\prod_{i=1}^{q}[a_i, b_i]$. Then, by letting $\frac{\partial f}{\partial x_1 \partial x_2 \ldots \partial x_q}(x) = Df(x)$, $Y$ be an arbitrary $\mathcal{A}'$-measurable mapping, and $Y \in L^1(\mathcal{A}', \mathbb{P})$, we obtain that, for any $f \in \Lambda$,

$$\text{Cov}(f(X), Y) \leq \hat{\phi}_V(\mathcal{A}', X) ||Y||_{L^1(\mathbb{P})} ||g||_{L^1(\mu)} \text{ and } g(x) = Df(x),$$

(7)

where $\hat{\phi}_V(\mathcal{A}', X)$ is the $\hat{\phi}_V$-mixing coefficient introduced in Definition 2.

Proposition 1 can be regarded as a direct extension of Proposition 1 in Dedecker and Prieur (2005). Additionally, Proposition 1 indicates that the $\hat{\phi}_V$-mixing coefficient is tightly connected with the $\mathcal{C}$-mixing process, allowing us to propose the following definition of a vector-valued $\hat{\phi}$-mixing process.

Definition 3 Suppose $Z = \{Z_i\}_{i \in \mathbb{N}^d}$ is a Z-valued process, where $Z$ is compact in $\mathbb{R}^D$. We call it a vector-valued $\hat{\phi}$-mixing process if, given $i \in \mathbb{N}^d$ and for all $U \subset \mathbb{N}^d$, it holds that $|U| < \infty$ and

$$\hat{\phi}_V(\sigma(U), Z_i) \searrow 0 \text{ as } \rho \nearrow +\infty,$$

where $\rho = \rho(U, i)$ and $\rho(U, i)$ is defined in Definition 1.

Together with Proposition 1 and Definition 1, it is easy to see that the vector-valued $\hat{\phi}$-mixing process is also a $\mathcal{C}$-mixing process whose semi-norm is associated with Example 6. Therefore, parallel to Olivier (2010), Definition 3 here can be regarded as another multidimensional extension of the real-valued $\hat{\phi}$-mixing process defined by Rio (1996).

## 3 Bernstein-type Inequality

This section focuses on Bernstein-type inequalities under geometric and algebraic $\mathcal{C}$-mixing conditions. Note that, here, we only compare our Bernstein-type inequality with Hang and Steinwart (2017) and Blanchard and Zadorozhnyi (2019). As we mentioned in the abstract, Hang and Steinwart can be regarded as the pioneering work under geometric $\mathcal{C}$-mixing condition, and they have made insightful comparisons between their results and those of previous research (see Examples 3.2-3.6, Hang and Steinwart (2017)). According to our search of the literature, however, Blanchard and Zadorozhnyi (2019) is the first paper dedicated to deriving a Bernstein-type inequality under an algebraic $\mathcal{C}$-mixing condition. By comparing our results with these two important examples of previous research, we manage to demonstrate our contribution in a sufficiently clear manner. Additionally, we introduce the following assumptions of the semi-norm in (3), which is necessary for our proof of the theorems in Sections 3 and 4.
By denoting $\mu$ as the Lebesgue measure on Euclidean space, we assume, given a function $f \in C(Z)$, the following assumptions hold for the semi-norm associated with a given $C$-mixing condition.

**Assumption 1** $||e|| \leq ||e||_\infty ||f||$.

**Assumption 2** $||f|| = 0$, if $f$ is a constant function.

**Assumption 3** Suppose $Z = \mathcal{X} \times I \rightarrow \mathbb{R}$, where $\mathcal{X}$ is a subset of $\mathbb{R}^D$ and $I \subset \mathbb{R}$ is an interval such that $\mu(I) < +\infty$. Consider the function $fg$, where $f : \mathcal{X} \rightarrow \mathbb{R}$, $\ f \in C(\mathcal{X})$ and $g : I \rightarrow I$ is an identity mapping. Then we assume $||fg|| \leq (||f|| + ||f||_\infty)\mu(I)$, provided that $\mu(I) \geq 1$.

**Assumption 4** If $g$ is an identity mapping defined in Assumption 3, we assume $||g|| \leq \mu(I)$.

Assumption 1 is directly inherited from Hang and Steinwart (2017), whereas Assumption 2 is from Hang et al. (2018). Here, we only need to argue Assumptions 3 and 4. Recall that a $C$-mixing process is actually a general idea containing a large class of stochastic processes (dynamic systems) and that the specific definition of any particular one of these is closely connected with the choice of semi-norm. As we summarized before, Examples 1 to 6 already include many interesting and useful processes or dynamic systems as special cases. Hence, the only reason why we introduce Assumptions 3 and 4 is because these are satisfied by all of the examples (see more detail in Appendix). Lastly, note that, for each theorem, only part of these assumptions will be used and we will mark them clearly. More particularly, to obtain Theorem 1 below, we only need Assumption 1, which is the same as Theorem 3.1 in Hang and Steinwart (2017).

### 3.1 Geometric $C$-mixing

**Theorem 1** Suppose we collect $N$ observations, $(Z_i)_{i=1}^N$, from a $Z$-valued geometrical $C$-mixing process of order $(\nu, b, \gamma)$. For any given $h \in C(Z)$, we assume $E_{\nu}(h) = 0$, $E_{\nu}(h^2) \leq \sigma^2$, $||h||_\infty \leq A$ and $||h|| \leq B$, for some $\sigma$, $A$ and $B > 0$. Under Assumption 1, when $N \geq N_0 := \max\{N \in \mathbb{N}^+ : \frac{(N/m)^{\nu-1}A}{A+B} \geq 2^{d-1}\}$, for any $\omega > 1$, we have, for $\forall \ t > 0$

$$\mathbb{P}\left\{ \frac{1}{N} \sum_{i=1}^N h(Z_i) \geq t \right\} \leq 8 \exp \left( -\frac{Nt^2}{2(\frac{\omega}{7} \log N)^\frac{\nu}{\gamma}(\sigma^2 + tA)} \right). \quad (8)$$

More specifically, when $\nu = e$, we have, for $\forall \ N \geq N_0 = \max\{N \in \mathbb{N}^+, \frac{N^{\nu-1}A}{A+B} \geq 2^{d-1}\}$,

$$\mathbb{P}\left\{ \frac{1}{N} \sum_{i=1}^N h(Z_i) \geq t \right\} \leq 8 \exp \left( -\frac{Nt^2}{2(\frac{\omega}{7} \log N)^\frac{e}{\gamma}(\sigma^2 + tA)} \right). \quad (9)$$

Note that (8) and (9) are valid for any $\omega > 1$, and that the $\omega$ here is associated with the upper bound of the semi-norm $B$. This requires, provided that $A$ is a fixed positive number, that $B$ be, at most, as large as $O(N^\alpha)$, for some $\alpha > 0$. There have been some examples that satisfy this requirement. A typical one is kernel density estimator (e.g. Hang et al. (2018)), where we usually let $h(Z_i)$ as $K\left(\frac{Z_i - x}{b}\right)$, where $K(\cdot)$ is a Lipschitz kernel and $b$ denotes the bandwidth. Now we set the
data to be collected from a $C$-mixing process whose relative semi-norm is the Lipschitz norm as in Example 4. Then, we obviously have $\| K(\frac{\omega - z}{b}) \| = O(1/b)$, which means we only need to let $\omega = 2$. As we discussed in Section 1, another interesting characteristic of Theorem 1 is that the sharpness of inequalities is affected only by ”effective dimension” $d'$ up to a log $N$-level. This indicates that no matter how large the $d$ is, if $d' = 1$, the decreasing rate of (8) and (9) will still be as fast as a time series. Additionally, note that (9) is valid for a (time-reverse) $C$-mixing process as defined in Definition 2.5 in Hang and Steinwart (2017), as well. To check this, refer to the proof of Theorem 1 in the Appendix.

Now we compare Theorem 1 with the previous research. Because Hang and Steinwart (2017) thoroughly discussed the comparison between their result and other related literature and focused solely on the case where $d' = d = 1$ and $\nu = e$, we only need to compare inequality (9) with theirs. First, based on our notations, we brief their result below.

(Theorem 3.1, Hang and Steinwart (2017)) Suppose $(Z_i)_{i \geq 1}$ is a $Z$-valued stationary geometrically (time-reversed) $C$-mixing process at the order of $e$ and $d' = d = 1$. For $\forall h \in C(Z)$, we assume $E_P(h) = 0$, $E_P(h^2) \leq \sigma^2$, $\| h \|_{\infty} \leq A$ and $\| h \| \leq B$, for some $\sigma$, $A$ and $B > 0$. Then, under Assumption 1, for all $t > 0$ and all $N$ such that

$$N \geq N_0 := \max \left\{ \min \left\{ N \geq 3 : N^2 \geq \frac{808c(3B + A)}{A} \text{ and } \frac{N}{(\log N)^{\frac{3}{2}}} \geq 4 \right\}, e^{\frac{t}{\gamma}} \right\},$$

we have

$$P \left\{ \left| \frac{1}{N} \sum_{i=1}^{N} h(Z_i) \right| \geq t \right\} \leq 2 \exp \left( -\frac{Nt^2}{8(\log N)^{\frac{3}{2}}(\sigma^2 + \frac{tA^3}{3})} \right).$$

(10)

Compared with (10), our results, i.e., (8) and (9), can be regarded as two updates on the following directions,

1. (9) is ”$(\log N)^{\frac{1}{2}}$” sharper than (10). When $\gamma$ is very close to 0, the advantage of (9) is more obvious.

2. When $A$ is finite, (10) is only valid asymptotically if $B \leq O(N^2)$. Even if $B$ is finite, the requirement for the sample might still be harsh because $N_0$ might be a very large finite integer. However, both (8) and (9) only require $B = O(N^\alpha)$, for $\forall \alpha > 0$. Furthermore, we could make $N_0$ very small or even equal to 1 if we choose sufficiently large $\omega$. Note this would only enlarge the constant factor of the upper bound but never the sharpness. In other words, (8) and (9) deliver more ”freedom” for use by allowing for more general preconditions.

3. Another difference lies in the assumption about the stationarity of process. As shown by Theorem 1, except for the $C$-mixing dependent condition, there are no other restrictions to the stochastic process, which means a non-stationary $C$-mixing process is also included.

4. (8) broadens the horizon by regularly taking grid spatial data into consideration. Non-isotropic sampling behavior is considered and (8) indicates that the sharpness of inequality is affected
only by the “effective dimension” of the index set up to a logarithm level. This could be advantageous when the “effective dimension” is much smaller than the dimension of the index set.

3.2 Algebraic $C$-mixing

Compared with the geometric $C$-mixing process, algebraic decay is obviously a much weaker restriction of the dependence and involves many more interesting examples. Until now, Blanchard and Zadorozhnyi (2019) may have been the only concentration inequality available for the algebraic $C$-mixing process. The result we demonstrate in this section is numerically similar to theirs, but our preconditions and settings are rather different. A key difference is that Blanchard and Zadorozhnyi (2019) focused on investigating the Bernstein-type concentration phenomenon of the norm of Banach-valued partial sum process, whereas we concentrate on deriving a Bernstein-type inequality for a centered empirical process, “$(f(Z_i))_{i \in \mathbb{Z^+}^n}$”, where $f \in C(Z)$ and $E(f) = 0$. Here $(Z_i)_{i \in \mathbb{Z^+}^n}$ is a $Z$-valued $C$-mixing process and $Z$ could be any well-defined metric space. Therefore, generally speaking, compared with Blanchard and Zadorozhnyi (2019)’s work, our result lies in different categories.

**Theorem 2** Suppose, under Assumptions ?? and ??, we collect $N$ observations, $(Z_i)_{i=1}^N$, from a $Z$-valued algebraic $C$-mixing process of order $\gamma$, where $\gamma > \alpha d$ for $\alpha > 0$ and $d > 1$. For all $h \in C(Z)$, we assume $E_{\pi}(h) = 0$, $E_{\pi}(h^2) = \sigma^2$, $||h||_{\infty} \leq A$ and $||h|| \leq B$, for some $\sigma$, $A$ and $B > 0$. Furthermore, assume $B \leq N^\theta$, for some $\alpha > 0$. Under Assumption 1, for any $N \geq N_0 := \max\{N \in \mathbb{N}^+: (N/m)^\alpha > A\}$ and $t > 0$, we have

$$\mathbb{P}\left\{\left|\frac{1}{N} \sum_{i=1}^{N} h(Z_i)\right| \geq t\right\} \leq C_1 \exp\left(-\frac{N^\theta t^2}{2C_2(\sigma^2 + tA)}\right),$$

where $C_1 = m \exp\left(1 + \frac{2d^2 + 1}{A}\right)$, $C_2 = m^\theta b^{-\frac{1}{\gamma + d}}$ and $\theta = 1 - \frac{(\alpha + 1)d'}{\gamma + d}$. More specifically, when $B$ is a fixed and positive constant and $d = d' = 1$, for any $\gamma > 0$ and $N \in \mathbb{N}^+$, we have

$$\mathbb{P}\left\{\left|\frac{1}{N} \sum_{i=1}^{N} h(Z_i)\right| \geq t\right\} \leq C_3 \exp\left(-\frac{N^{\frac{\gamma}{2}\theta} t^2}{2C_4(\sigma^2 + tA)}\right).$$

Here, $C_3 = \exp(1 + 2(A + B)/A)$ and $C_4 = b^{-\frac{1}{\gamma + 1}}$.

By comparing (12) with Theorem 3.5 and Proposition 3.6 in Blanchard and Zadorozhnyi (2019), it is easy to see that even though these two results are not comparable to each other, they have the same sharpness. But there are still some differences. As we will show in Section 4, instead of the process $(Z_i)_{i \in \mathbb{N}^+}$, we are actually frequently confronted with the process $(f(Z_i))_{i \in \mathbb{N}}$, where $f$ is usually a real-valued measurable function with some smoothness. If we want to apply the results from Blanchard and Zadorozhnyi (2019), we need to ensure that $(f(Z_i))_{i \in \mathbb{N}}$ is still a $C$-mixing process and its $C$-mixing coefficient is smaller than $(Z_i)_{i \in \mathbb{N}}$’s, while our results are generically immune to this problem. Of course, Blanchard and Zadorozhnyi’s results only require the linear and quadratic functions have finite semi-norms. When considering the transformed process $\{f(Z_i)\}$, this amounts
to requiring that \( f \) and \( f^2 \) have finite \( C \)-norm. By contrast, our paper assumes that both \( f \) and \( \exp(f) \) have finite \( C \)-norm (Assumption 1), as did in Hang and Steinwart (2017).

4 Variance Inequalities

In this section, we will concentrate on delivering inequalities of the variance of partial sums under \( C \)-mixing conditions, provided that \( d \) introduced in Definition 1 is equal to 1. Meanwhile, considering that a fundamental requirement of the elements in space \( C(Z) \) is uniform boundedness, we here intend to derive the inequalities under much weaker requirements. Before we start to exhibit our results, the following mild but important assumption needs to be introduced.

**Assumption 5** We assume the following two conditions hold for the semi-norm \( || \cdot || \).

A1 Suppose \( f \) is a measurable mapping on \((Z, \mathcal{B})\) such that \( ||f|| < \infty \). Then, for any \( B \in \mathcal{B} \), we assume the restriction of \( f \) on it, denoted as \( f|_B \), satisfies \( ||f|_B|| \leq ||f|| \).

A2 When \( f \) is an identity mapping on \( Z \subset \mathbb{R} \), we assume, for any \( B \in \mathcal{B} \), \( ||f|_B|| \leq C_1 \text{Vol}(B) \), for some \( C_1 > 0 \). Here \( \text{Vol}(B) \) denotes the volume of set \( B \).

It’s easy to obtain that condition A1 in Assumption 5 is suitable for the semi-norms demonstrated in Examples 1–6. Condition A2 can be regarded as a natural corollary of the combination of Assumption ?? and condition A1, and it is suitable for Examples 1–6 as well. The argument of Assumption 5 is postponed to the Appendix.

**Proposition 2** Suppose \((Z_i)_{i \in \mathbb{N}} := Z \) is a \( Z \)-valued process defined on a probability space \((\Omega, \mathcal{A}, \mu)\). Its \( C \)-mixing coefficient is upper bounded by a function \( C(t) \) satisfying \( \lim_{t \to +\infty} C(t) \leq 0 \) and we assume there exists some \( \delta \in (0, 1) \) such that \( \sum_{m=1}^{+\infty} m^{-\delta} C(m) < +\infty \). Provided that the semi-norm associated with this \( C \)-mixing condition is \( || \cdot || \) and that it is submitted to Assumptions 2 and 5, let \( h \in \mathcal{H} \), where \( \mathcal{H} = \{ f \text{ is measurable : } Z \to \mathbb{R} \mid ||f|| < K_n \} \) with \( K_n < +\infty \) for each \( n \). Moreover, we also assume that, for some \( s > 1 \), condition C1 (or C2) holds, where conditions C1 and C2 are as follows.

C1 \( \max_i \max_{j > i} E[|h(Z_j)|^s |Z_i] < K' < +\infty \).

C2 \( \max_i E|h(Z_i)|^{2s} < K'' < +\infty \).

Then, under Assumptions 5, if Condition C1 holds,

\[
\text{Var}(h(Z_i)) \leq \sum_{i=1}^{n} \text{Var}(h(Z_i)) + (2K'M + 8C_2) \sum_{i=1}^{n} E|h(Z_i)| + \triangle_n;
\]

and if Condition C2 holds,

\[
\text{Var}(h(Z_i)) \leq \sum_{i=1}^{n} \text{Var}(h(Z_i)) + 8C_2 \sum_{i=1}^{n} E|h(Z_i)| + 2MK'' \sum_{i=1}^{n} \sqrt{\text{Var}(h(Z_i))} + \triangle_n,
\]

12
where $\Delta_n = 2\sum_{i=1}^{n} \sum_{m=1}^{K_n} (E|h(Z_i)h(Z_{i+m})| + 7E|h(Z_i)|E|h(Z_{i+m})|), M = \sum_{m=1}^{+\infty} m^{-(1+\delta)},$ and $C_2 = \sum_{m=1}^{+\infty} m^{1+\delta} C(m)$.

The volume of term $\Delta_n$ is governed by moment conditions of function $h$ and growth conditions of $K_n$ simultaneously. This term appears because we involve the techniques used in the proof of Theorem 1 in Hansen (2008) to finish the proof of Proposition 2. Please note that the class $\mathcal{C}$ is much more general than $\mathcal{C}(Z)$ since its only requirement is the bound of the semi-norm with respect to each given $n$. This is quite useful in investigating the theoretical properties of kernel smoothers under $\mathcal{C}$-mixing conditions. As we will discuss in Section 5, the kernel density and conditional mean estimators serve as perfect examples, in which the semi-norm of a kernel function typically diverges as the sample size grows. However, this is still not sufficiently strong to cover the case of kernel smoothers for fixed-design nonparametric regression. To examine the mean squared error (MSE), we have to consider the term "$\sum_{i=1}^{n} w_i \xi_i$", where $\{\xi_i\}$ represents the error terms submitted to $\mathcal{C}$-mixing conditions. It is important to note that we usually only impose certain moment conditions on the error terms, and their support is often unbounded, such as in the case of a Gaussian process. Under this circumstance, by considering $w_i \xi_i$ as a mapping on $\xi_i$, the total variation semi-norm of this mapping is obviously infinitely large, which indicates that the covariance inequality delivered by Definition 1 is useless in understanding the MSE of $\sum_{i=1}^{n} w_i \xi_i$ under $\phi$-mixing condition. Hence, it is reasonable to consider the following scenario: Suppose $Z$ is a real-valued $\mathcal{C}$-mixing process with unbounded support, and $h(z) = az$, where $a \in \mathbb{R}$. In this case, neither the aforementioned class $\mathcal{H}$ (see Proposition 2) nor the space $\mathcal{C}(Z)$ contains an element like this. To tackle this problem, we deliver the following proposition.

**Proposition 3** Suppose $(Z_i)_{i \in \mathbb{N}} := Z$ is a $Z$-valued process defined on a probability space $(\Omega, \mathcal{A}, \mu)$, $Z \subset \mathbb{R}$. Its $\mathcal{C}$-mixing coefficient is upper bounded by a function $C(t)$ introduced in Proposition 2. Let $\{a_{in}\}$ be an array real numbers satisfying that $b_n < \min_{1 \leq i \leq n} |a_{in}| \leq \max_{1 \leq i \leq n} |a_{in}| < B_n$, for some $b_n, B_n > 0$ such that $\lim_n B_n/b_n < +\infty$. Moreover, we assume that, for some $s > 1$, condition $D1$ (or $D2$) holds, where conditions $D1$ and $D2$ are as follows.

**D1** $\max_{i, \max j > i} E[|Z_j|^s | Z_i] < B < +\infty$, for some $A > 0$.

**D2** $\max_i E[Z_i|Z_i]^{2s} < A' < +\infty$, for some $A' > 0$.

Then, under Assumptions 2 and 5, we obtain the following results.

Under condition $D1$,

$$\text{Var}(\sum_{i=1}^{n} a_{in} Z_i) \leq C^* \sum_{i=1}^{n} a_{in}^2 \text{Var}(Z_i) + (2^{s+2}(AB/b)M + 16C_2(4C_1 + 1)(B/b)) \sum_{i=1}^{n} a_{in}^2 E|Z_i|.$$ 

Under condition $D2$,

$$\text{Var}(\sum_{i=1}^{n} a_{in} Z_i) \leq C^* \sum_{i=1}^{n} a_{in}^2 \text{Var}(Z_i) + 2^{2s+1}(\sqrt{A'B/b}M + 16C_2(4C_1 + 1)(B/b)) \sum_{i=1}^{n} a_{in}^2 E|Z_i|.$$ 

Here constants $M$ and $C_2$ are introduced in Proposition 2 and $C_1$ is introduced in Assumption 5.
Remark 1 Please note that the parameter $\delta$ in Propositions 2 and 3 is allowed to be arbitrarily close to 0. Hence only some weak restrictions are delivered to the function $C(m)$. For example, when $C(m) = bm^{-\gamma}$, the restriction $\sum_{m=1}^{+\infty} m^{1+\delta/s-1} C(m) < +\infty$ is satisfied for any $\gamma > \frac{s}{s-1}$.

Regarding condition D1, a simple yet non-trivial example is Gaussian processes, where the correlation coefficient between any two components approaches zero as the difference between the indices of these two components diverges. Under this circumstance, the Gaussian process can be considered as an $\alpha$-mixing process, which includes the $\phi$-mixing condition as a special case. Since the only restriction for $s$ is "$s > 1$", Condition D2 actually requires the moment conditions of the process $Z$ to be only slightly stronger than the existence of the second moment. Section 5 shows that the kernel-based conditional mean estimator for fixed design regressions can achieve the minimax MSE under Condition D2 and a very weak requirement of the $C$-mixing coefficient. These moment and mixing conditions are also weaker than those required in obtaining the MSE of kernel smoothers under $\alpha$-mixing conditions (e.g., Theorem 1 in Hansen (2008)).

5 Central Limit Theorem Under $C$-Mixing Conditions

In this section, we focus on deriving a central limit theorem for the generally defined $C$-mixing-type processes.

6 Applications

In Section 5.1, we will explore the applications of the inequalities discussed in Sections 3 and 4 to examine the deviation inequalities of empirical processes with product-type structures. These results will be particularly useful in establishing the uniform convergence rates of kernel-based estimators in both fixed-design and random-design scenarios. Building upon the findings in Sections 3, 4 and 5.1, Section 5.2 will focus on investigating the $L_\infty$-norm rate and mean squared error of kernel density estimators when the density function is only point-wise Hölder-controllable. Section 5.3 will be dedicated to showing the mean squared error and uniform convergence rates of kernel-based conditional mean estimators nonparametric regressions. Notably, even under relatively weak moment and $C$-mixing conditions, it can be demonstrated that the mean square errors of these kernel smoothers achieve minimax optimality for Hölder-controllable classes. Overall, the goal of these sections is to showcase the practical implications of the inequalities discussed earlier and provide a theoretical foundation for the use of kernel-based estimators in various settings.

6.1 Mathematical Preparation

First, we skip the discussion about the measurability of the supreme of the empirical process. After all, according to our applications in Sections 5.2 and 5.3, the index set of the empirical process would be a separable space. Then, without alternative notation, we always admit the following assumption in Section 5.1.
Assumption 6 Suppose \((X_i, Y_i)_{i \geq 1}\) is a geometrically (time-reverse) \(C\)-mixing process at the order of \((e, b, \gamma)\) and \(d' = d = 1\). For each \(i\), we assume \((X_i, Y_i)\) shares the same distribution as random vector \((X, Y)\), where \(X \in X \subset \mathbb{R}^D\) and \(Y \in \mathbb{R}\). Assume \(\sup_{x \in X} E[Y^2 | X = x] \leq \sigma^2\) for some \(\sigma^2 > 0\).

It is widely known that, to discuss the uniform convergence rate of local smoother, a key element is the tail probability of the supremum of the centered multiplier-type empirical process. That is to say, given a real-valued function class \(\mathcal{F}\), our object in this section is to obtain a probabilistic upper bound for the following event,

\[
\sup_{f \in \mathcal{F}} |P_N(fW) - P(fW)| > t, \ \forall \ t > 0, 
\]

(13)

\[
P_N(fW) = \frac{1}{N} \sum_{i=1}^{N} f(X_i)W_i, \ \ \ P(fW) = \frac{1}{N} \sum_{i=1}^{N} E[f(X_i)W_i], \ \ W_i = Y_i 1[|Y_i| \leq L_N],
\]

where \(\{L_N\}_{N \in \mathbb{N}}\) is a non-decreasing non-negative number sequence.

For this purpose, under iid or independence assumptions, there has been much outstanding previous research. Except for the results shown in two standard textbooks, Wellner and van der Varrt (2013), Ledoux and Talagrand (1991), a more recent result is from Mendelson (2016), in which a high probability upper bound for (13) was obtained based on generic chaining techniques and a variation of the ”Talagrand-\(\gamma\) functional”. Parallel to deriving a probabilistic upper bound of (13), taking the expectation of (13) is also an important task. Proposition 3.1 of Giné, Latala, and Zinn (2000) demonstrated an arbitrary \(q\)-th moment \(\(q \geq 1\)\) inequality for (13). For the only concern of first moment, Han and Wellner (2019) discovered a general upper bound under very weak assumptions about the structure of the function class. Then they applied this result to obtain the convergence rate of the least square estimators of the nonparametric regression with a heavy tailed error term. Furthermore, Han (2022) derived a multiplier inequality for a higher-order multiplier empirical process, which is of huge importance in many areas, such as the central limit theorem for multiplier and bootstrap empirical processes or general theory for bootstrap M-estimators based on U-statistics.

However, all of the previous results rely heavily on the assumption of independent observations. For dependent data, according to our search of the literature, the number of directly related previous results is surprisingly small. For the \(C\)-mixing process, up to our best knowledge, there may be no literature available. Fortunately, the proofs of Theorem 3.3 from Van de Geer (1990) and Theorem 1 from Mammen, Rothe, and Schienle (2012) allow us to obtain a probabilistic upper bound of (13) based purely on a direct combination of typical chaining argument and (9).

Proposition 4 Under Assumptions 1-4 and 6, let \(W_i = Y_i 1[|Y_i| \leq L_N]\), as defined in (13). Consider a given subset \(\mathcal{X} \subset C(\mathbb{X})\) such that, for all \(f \in \mathcal{X}\), \(|f|_{\infty} \leq A\), \(|f| \leq B\) and \(E|f(X)|^2 \leq \sigma_f^2\), for some \(A, B, \sigma_f > 0\). For all \(N\) and \(t\), if all of the following four conditions \(C_1-C_4\) hold:

\(C_1.\) \(tA L_N \leq \frac{\sigma_2^* \sigma_f^2}{10}\), \(L_N \geq 1\),

\(C_2.\)

\[
\sqrt{\frac{N}{(\log N)^{\frac{3}{2}}}} \geq \frac{15 \sigma_2 \sigma_f^2}{t^{\frac{1}{2}} (\frac{b}{a})^{\frac{1}{2}}},
\]
C3. \( \exists \omega > 1, \text{s.t.} \)
\[ N^\omega - 1 \geq \left( 2 + \frac{32BL_N\sigma_F}{5At} \right) 2^{d'-1}, \]

C4.
\[ \sqrt{\frac{N}{(\log N)^{\gamma}}} t \geq \frac{60\sqrt{10}\sigma\sqrt{\sigma_F}}{(\frac{b}{\omega})^{\frac{1}{2\gamma}}} \int_{\frac{1}{2}N^{\frac{1}{4}}}^{(\sigma_F)^{\frac{1}{2}}} \sqrt{\log N(u^2, F, || \cdot ||_\infty)} du, \]

then we obtain,
\[ \mathbb{P} \left\{ \sup_{f \in F} |P_N(fW) - P(fW)| \geq t \right\} \leq 88 \exp \left( -\frac{(b/\omega)^{\frac{1}{2}}Nt^2}{2250(\log N)^{\frac{1}{4}}\sigma^2\sigma^2_F} \right), \quad (14) \]

where \( P_N(fW) \) and \( P(fW) \) are defined in (13). Here \( \mathcal{N}(\tau, F, || \cdot ||_\infty) \) denotes the covering number of the set \( F \) on the radius of \( \tau \) with respect to the metric \( || \cdot ||_\infty \). Because C1 is an assumption, only variance information is contained by the exponential upper bound. If the variance term, \( \sigma_F \), is asymptotically strictly larger than 0 or even remains as a fixed constant, C1 is no longer needed because a Hoeffding-type inequality is sharp enough to obtain the (optimal) rates (see Chapter 19, Györfi et al. (2002)). Additionally, note that we often replace \( t \) with a convergence rate. Hence, this assumption becomes trivial when \( L_N, \sigma \) and \( \sigma_F \) are fixed constants. C2 is a pure technique restriction for obtaining the non-asymptotic upper bound. C3 can be considered a natural variation of \( N_0 \) in Theorem 1 and the parameter \( \omega \) here depends on \( B \). Lastly, C4 is a mild requirement for covering number?? because the range of the integral is a bounded interval for every given \( N \). A natural consequence of Proposition 4 is the following corollary.

**Corollary 1** Under the same conditions as Proposition 6??, for any \( N, t > 0 \) that satisfy the four conditions shown below:

D1.
\[ tA \leq \frac{\sigma^2_F}{10}, \]

D2.
\[ \sqrt{\frac{N}{(\log N)^{\gamma}}} t \geq \frac{15\sigma_F}{(\frac{b}{\omega})^{\frac{1}{2\gamma}}} \]

D3. \( \exists \omega > 1, \text{s.t.} \)
\[ N^\omega - 1 \geq \left( \frac{3}{2} + \frac{16B\sigma_F}{5At} \right) 2^{d'-1} \]

D4.
\[ \sqrt{\frac{N}{(\log N)^{\gamma}}} t \geq \frac{60\sqrt{10}\sigma\sqrt{\sigma_F}}{(\frac{b}{\omega})^{\frac{1}{2\gamma}}} \int_{t^{\gamma/2}}^{(\sigma_F)^{\frac{1}{2}}} \sqrt{\log N(u^2, F, || \cdot ||_\infty)} du \]
we have,

$$\mathbb{P}\left\{ \sup_{f \in \mathcal{F}} |P_N(f) - P(f)| \geq t \right\} \leq 88 \exp \left( -\frac{(b/\omega)^{\frac{1}{2}} N t^2}{2250 (\log N)^{\frac{3}{2}} \sigma_f^2} \right),$$

(15)

Apparently, by using a Bernstein-type inequality (9), both (14) and (15) manage to reflect the variance information, which offers more sharpness when the variance becomes sufficiently small as $N \to +\infty$.

6.2 Kernel Density Estimation

Within this subsection, we investigate the uniform convergence rate of a kernel density estimator for the $C$-mixing process. The commonly considered density estimation problem is often stated as follows: Suppose we collect $X_1, ..., X_N$ as a sample drawn from an unknown distribution $P$ on $\mathbb{R}^D$ with density $f$. Thus, the goal of density estimation is to estimate this underlying function $f$ based on observations. Indeed, density estimation is always a fundamental and crucial topic in many branches of statistics and machine learning, such as mode regression, classification and clustering. One of the most widely used estimators is the kernel density estimator, which was initiated by Parzen (1962) and Rosenblatt (1956). According to (14), Chapter 4, Wand and Jones (1994), its most general form is

$$f_n(x) = (N|H|^{\frac{1}{2}})^{-1} \sum_{i=1}^{N} K \left( H^{-\frac{1}{2}} (X_i - x) \right),$$

(16)

where $H$ is a symmetric positive definite $D \times D$ matrix, satisfying $N^{-1}|H|^{-\frac{1}{2}}$ and all entries of $H$ approach zero as $N \to +\infty$. In this paper, to simplify the complexity of the calculation, we only consider the case where $H = b^2 I_D$, with $I_D$ denoting the $D$-dimensional identity matrix. Thus, we can rewrite (17) as

$$f_n(x) = \frac{1}{Nb^D} \sum_{i=1}^{N} K \left( \frac{X_i - x}{b} \right),$$

(17)

where, given the vector $X_i - x \in \mathbb{R}^D$, $\frac{X_i - x}{b} := (b1_D)^{-1}(X_i - x)$ for $i = 1, \ldots, n$. For the $C$-mixing process, by defining (17) as a spherical kernel, Hang et al. (2018) investigated the universal consistency of the kernel density estimator under the $L_1$-norm and derived its convergence rate under the $L_1$ and $L_\infty$ norms. Here we aim to update this result on the convergence rate of (17) under the $L_\infty$ norm within the assumptions that are somewhat more general. Based on our inequalities in Section 3, we could update the convergence rate under both $L_1$ and $L_\infty$ norms, but we concentrate here on the uniform convergence only.

It is widely known that an important precondition in nonparametric curve estimation is the smoothness of curve, like Hölder and Sobolev conditions. But as shown in Examples 2 and 3 of Hang and Steinwart (2017), for some dynamic systems, density functions often satisfy only weaker continuity conditions, such as a pointwise $\alpha$-Hölder controllable function. To better demonstrate our result, we first introduce the definition of a ”pointwise $\alpha$-Hölder controllable function”.

**Definition 4** (pointwise $\alpha$-Hölder controllable) A function $f : \mathbb{R}^D \to \mathbb{R}$ is called a pointwise $\alpha$-
Hölder controllable, if almost surely all \( x \in \mathbb{R}^D \), there exist two functions \( c(x) \geq 0 \) and \( r(x) > 0 \) such that for all \( x' \in \mathbb{R}^D \) with \( ||x'|| < r(x) \), we have

\[
|f(x + x') - f(x)| \leq c(x)||x'||^\alpha, \quad \alpha \in (0, 1].
\]

Moreover, \( f \) is called a uniformly \( \alpha \)-Hölder controllable function, if

\[
r_0 := \text{ess inf}_{x \in \Omega} r(x) > 0,
\]

where the function ess is \( \text{ess} \). Thus, the commonly defined \( \alpha \)-Hölder continuous (e.g., Chapter 1, Tsybakov (2004)) is a special case that is pointwise \( \alpha \)-Hölder controllable, with \( r(x) \) and \( c(x) \) setting as some universal positive constants \( c \) and \( r \), respectively.

As for the convolution kernel function used here, we assume the following conditions are satisfied,

**K1.** \( K(u) \) is a non-negative, uniformly upper bounded and Lipschitz continuous function of vector \( u \).

**K2.** \( \kappa_\alpha := \int_{\mathbb{R}^D} ||u||^\alpha K(u)du < +\infty \) and \( \int_{\mathbb{R}^D} K(u)du = 1 \).

**K3.** \( \text{supp}(K(u)) \subseteq B_M^D \), for some \( M > 0 \), where \( B_M^D \) denotes a \( D \)-dimensional ball centered at the original point at radius of \( M \).

**K4.** For any given \( N > 0 \), \( \exists \kappa > 0 \), such that, for function \( K(\cdot - x/b) \in \mathcal{C}(X) \), we have

\[
\sup_{x \in \mathbb{R}^D} \||K(\cdot - x/b)|| \leq N^\kappa.
\]

Compared with (i)-(iv) of Assumption B in Hang et al. (2018), Assumptions K1 and K3 are nearly the same as (i), (iii). K2 is slightly different from (ii) because, except for the spherical kernel, we take some other widely used multivariate kernel functions into consideration, such as the product kernel. K3 is introduced to bound the deterministic errors of kernel-based estimators when the density function is only pointwisely \( \alpha \)-Hölder controllable. For K4, it actually requires that the semi-norm of kernel function grows asymptotically no more than \( O(N^\delta) \), for some \( \delta > 0 \). More specifically, this assumption easily holds for all kernels where the semi-norm is Lipschitz norm like Example 4. Since \( \sup_{x \in \mathbb{R}^D} \||K(x - \cdot/b)|| \leq O(b^{-1}) \), we only need to set \( \omega = 2 \). Thus, \( \omega \) is a user-defined parameter which is determined based on the choice of the semi-norm. Compared with (iv), our K4 is less restrictive.

**Theorem 3** Suppose \((X_i)_{i \geq 1}\) is a strict-stationary \( X \)-valued geometrically (time-reverse) \( C \)-mixing process at the order of \((e, b, \gamma)\) and \( d' = d = 1 \). \( X \subset \mathbb{R}^D \) and is bounded?? Assume Assumptions 1 and 2 are satisfied.

(i) Let \( K \) be a kernel function that satisfies K1, K2 and K4. Assume the density function \( f \) is \( \alpha \)-Hölder continuous with \( \alpha \in (0, 1] \).

(ii) Let \( K \) be a kernel function that satisfies K1, K3 and K4. Assume the density function \( f \) is pointwise \( \alpha \)-Hölder controllable with \( \alpha \in (0, 1] \). Suppose that, given \( x \in X \), \( r(x) < +\infty \). Define \( X^{+bM} = \{ x \in \mathbb{R}^D : \inf_{x' \in X} ||x - x'||_2 \leq bM \} \), \( X^*_b = \{ x \in X : r(x) > bM \} \). Finally, assume function \( c(x) \) is bounded on \( X^*_b \cap X^{+bM} \).
Then, for both cases (i) and (ii), by setting $b_n = \left( \frac{\log N}{N} \right)^{\gamma/2}$ we have

$$
||f_N - f||_\infty = O_P \left( \frac{\log N}{N} \right)^{\frac{\gamma}{2}\alpha + D}.
$$

(18)

More specifically, when $\gamma > 1$, by setting $b_n = \left( \frac{\log N}{N} \right)^{\frac{1}{2}\alpha + D}$, we can attain Stone (1982)’s optimal uniform convergence rate:

$$
||f_N - f||_\infty = O_P \left( \frac{\log N}{N} \right)^{\alpha^2 + D}.
$$

(19)

For case (ii) in Theorem 3, set $X^*_b$ is introduced to avoid the $x$ whose function $r(x)$ does not exist. This is because the function $r(x)$ is a function only associated with $x$ but independent of sample size $N$. However, bandwidth $b$ is governed by $N$, satisfying $\lim_{N \to \infty} b = 0$. Hence, for any $x \notin X^*_b$, we have $0 \leq r(x) = \lim_{N \to \infty} r(x) \leq \lim_{N \to \infty} bM = 0$. Thus, $r(x)$ does not exist for any $x \notin X^*$, which implies that there is no necessity to discuss a pointwise $\alpha$-Hölder controllable. Therefore, for case (ii), (18) and (19) are valid only over set $X^*_b$. Moreover, we only require the support set $X$ to be a bounded set. No compactness is mentioned here. This is because in some typical examples of measure-preserving dynamical systems, such as Examples 4 and 5 in Hang et al. (2018), the support set of density function $f$ is an open interval and $f$ is not continuous at the boundary points.

Now we focus on showing the mean squared error of the density estimator (17).

**Theorem 4** Suppose $(X_i)_{i \geq 1}$ is a strictly stationary $\mathbb{R}^D$-valued algebraic (time-reverse) $C$-mixing process at the order of $(b, \gamma)$ and $d' = d = 1$. The marginal density of this process, denoted as $f$, is uniformly $\alpha$-Hölder controllable and the function $c(x)$ defined in Definition 4 is almost surely finite on $X$ with respect to $\lambda$, where $\lambda$ denotes the Lebesgue measure. Based on the semi-norm $|| \cdot ||$ associated with this $C$-mixing condition, we assume $||K(u/b)|| \leq b^{-1}$. Then, based on conditions $K1$ to $K3$ and Assumptions 2 and 5, by choosing $b = N^{-\frac{1}{2\alpha + D}},$ we have, for any $\gamma > 1$,

$$
MSE(f_N) = O(N^{-\frac{2\alpha}{2\alpha + D}}),
$$

which is almost surely with respect to $\lambda$. When compared to condition $K4$, the assumption $||K(u/b)|| \leq b^{-1}$ is borrowed from Hang et al. (2018) and actually imposes a stronger requirement on the semi-norm $|| \cdot ||$ of the selected kernel function. Fortunately, it can be easily demonstrated that this assumption aligns well with all the significant examples of semi-norms mentioned in Section 2 (see Examples 1-6). Furthermore, it is important to note that the only constraint on $\gamma$ is that it must be greater than 1. This requirement is also the minimum condition for achieving optimal MSE of kernel density estimators under algebraic $\alpha$-mixing conditions.

### 6.3 Conditional Mean Estimation

In Section 5.3, we will focus on investigating the uniform and MSE convergence rates of Nadayara-Watson-type estimators for the conditional mean function of the following model.

$$
Y_i = m(X_i) + \epsilon_i, E[\epsilon_i | X_i] = 0,
$$

(20)
where \((X_i, Y_i)\)'s are identical to random vector \((X, Y)\). The estimator of mean function \(m(\cdot)\) can be expressed as follow,

\[
\hat{m}(x) = \frac{\sum_{i=1}^{N} K_{ih}(x_i) Y_i}{\sum_{i=1}^{N} K_{ih}(x)} = K \left( \frac{X_i - x}{h} \right).
\] (21)

Obtaining the uniform convergence rates of kernel-based estimators like (21) is always a crucial task since "uniform convergence" ensures the consistency of the "worst case". There have been many important previous contributions on this topic. Hansen (2008) did an outstanding pioneering work on this topic by showing the uniform rates for many kernel-based estimators with (un-)bounded support under algebraic \(\alpha\)-mixing conditions. Kristensen (2009) extended this work to heterogeneously dependent data. More recently, Vogt (2012) investigated the uniform convergence rate of (21) for locally stationary \(\alpha\)-mixing process whose mixing coefficient decays algebraically. More particularly, a technique commonly used in all three aforementioned research is the combination of truncation method and conditional moments slightly larger than two. That is to say,

\[
\sup_x E[|Y|^\rho |X = x] < +\infty, \text{ for some } \rho > 2.
\] (22)

However, this may lead harsh restriction on the mixing coefficient. Meanwhile, none of these three papers mentioned the attainability of optimal uniform rate.

Parallel to kernel-based estimators, Chen and Christensen (2015) carried out groundbreaking work in demonstrating the uniform convergence rates of sieve estimators for \(\beta\)-mixing processes. They also meticulously discussed the attainability of optimality. In contrast to the aforementioned research on kernel smoothers, Chen and Christensen focused on showing that the optimal uniform convergence rates of sieve estimators could be achieved by only requiring the existence of unconditional moments, i.e., \(E|Y|^\rho < +\infty\). However, in this case, \(\rho\) is required to be strictly larger than \(2 + D/\alpha\), where \(D\) and \(\alpha\) denote the dimension of the input and the degree of Hölder-smoothness, respectively. Of course, slightly stricter assumptions on moment conditions result in much weaker restrictions on the \(\beta\)-mixing coefficient.

In this paper, we focus on examining the uniform convergence rate of (21) by following the line of Chen and Christensen (2015).

**Theorem 5** Except for Assumptions K1–K4, 1–4 and 6, we assume the mean function \(m(x)\) is \(\alpha\)-Hölder continuous and \(E|Y|^\rho < +\infty\), for some \(\rho > 2 + \frac{D}{\alpha}\). We also assume the marginal density of \(X_1\), denoted as \(f_X\), is uniformly lower bounded by some positive constant. Then, provided that \(h \text{ or } b?? = ((\log N)^{\frac{\gamma + 1}{\gamma}} / N)^{-\frac{\alpha}{2\alpha + D}}\), we have

\[
\sup_x |\hat{m}(x) - m(x)| = O \left( \frac{(\log N)^{\frac{\gamma + 1}{\gamma}}}{N} \right)^{-\frac{\alpha}{2\alpha + D}}.
\] (23)

More particularly, if \(\gamma > 1\), we have

\[
\sup_x |\hat{m}(x) - m(x)| = O \left( \frac{(\log N)}{N} \right)^{\frac{\alpha}{2\alpha + D}}.
\]
Even though we manage to obtain the optimal uniform convergence rate of estimator (21) as sharp as that of sieve estimators in Chen and Christensen (2015), this is a result based on assuming the decay of \( C \)-mixing coefficient is at an exponential level. (13) and (14) in Theorem 2 are not sufficiently sharp to allow us to obtain (23). Meanwhile, if we can enhance the smoothing condition of \( \alpha \)-Hölder controllable class to \( \alpha \)-Hölder class with higher-order differentiability, like \( \alpha > D \), the unconditional moment conditions could be much weaker.

There has also been numerous literature dedicated to understanding the asymptotic MSE of estimator (20) under various mixing conditions. But, as we mentioned in Section 1.2, Definition 1 would only deliver a non-trivial covariance inequality when the semi-norm is finite. This is too restrictive to be applied to derive the upper bound of MSE. Please recall that, to show the asymptotic variance of kernel-based conditional mean(or variance) estimators, one of the most important goals is to deliver an upper bound of

\[
\text{Var} \left( \frac{1}{Nh^D} \sum_{i=1}^{N} K \left( \frac{X_i - x}{h} \right) Y_i \right).
\]

Mostly, except for some mild moment conditions, we would not impose boundedness to the \( Y_i \)'s. When the semi-norm associated with the underlying \( C \)-mixing condition is total variation, like Example 5, the semi-norm of function \( h(x, y) = K(x - u/h)y \) is equal to \( +\infty \). Under this circumstance, even our Propositions 2 and 3 cannot be applied directly. Hence, we introduce the following Proposition 5 to tackle this problem.

**Proposition 5** Denote \( Z_i = (X_i, \xi_i) \) and suppose \( Z := (Z_i)_{i \in \mathbb{N}} \) is an \( \mathbb{R}^D \)-valued strictly stationary and algebraic (time-reverse) \( C \)-mixing process of order \((b, \gamma)\) and \( d' = d = 1 \). The following two moment conditions are met.

\[
\text{C1} \quad \max_i \sup_x E[|\xi_i|^\rho |X_i = x] < +\infty, \text{ for some } \rho > 2.
\]

\[
\text{C2} \quad \max_{i,j} \sup_x E[|\xi_i \xi_j| |X_i = x] < +\infty.
\]

Moreover, based on the semi-norm \( \| \cdot \| \) associated with this \( C \)-mixing condition, we assume \( \|K(u/h)\| \leq h^{-1} \). Then, under conditions \( \mathbf{K1} \) to \( \mathbf{K3} \) and Assumptions 2, 3 and 5, for all \( \gamma > \frac{2\rho}{\rho - 2} \), we have

\[
\text{Var} \left( \frac{1}{Nh^D} \sum_{i=1}^{N} K \left( \frac{X_i - x}{h} \right) \xi_i \right) = \mathcal{O}(N^{-\frac{2\alpha}{\rho + \alpha}}), \quad (24)
\]

provided that \( h = \mathcal{O}(N^{-\frac{1}{2\rho + \alpha}}) \) and \( \alpha \in (0, 1] \).

Generally speaking, Proposition 5 can be regarded as an extension of Theorem 1 in Hansen (2008) to \( C \)-mixing processes. The conditions C1 and C2 in Proposition 5 are directly borrowed from there. Furthermore, (24) only requires the parameter \( \gamma \) to be strictly larger than \( \frac{2\rho}{\rho - 2} \), while Hansen’s Theorem 1 needs \( \gamma > \frac{2\rho - 2}{\rho - 2} \). Considering that \( \frac{2\rho - 2}{\rho - 2} = \frac{\rho}{\rho - 2} + 1 \), the restriction of our results on the rate mixing coefficient?? is significantly weaker. As a direct application of Proposition 5, we have the following Theorem 6, which exhibits the MSE of (21).
Theorem 6 Denote $Z_i = (X_i, Y_i)$ and suppose $Z := (Z_i)_{i \in \mathbb{N}}$ is an $\mathbb{R}^D$-valued strictly stationary and algebraic (time-reverse) $C$-mixing process of order $(b, \gamma)$ and $d' = d = 1$. The following moment conditions are met, $\max_i \sup_x E[|Y_i|^\rho|X_i = x] < +\infty$ and $\max_{i,j} \sup_x E[|Y_iY_j||X_i = x] < +\infty$, for some $\rho > 2$. Moreover, we assume $\|K(u/h)\| \leq h^{-1}$ and $\inf f_X(x) > 0$, where $\| \cdot \|$ is the semi-norm associated with the $C$-mixing condition and $f_X$ is the marginal density of $X_1$. Assume $m(x) = \alpha$-Hölder continuous. Then, under conditions $K_1$ to $K_3$ and Assumptions 2, 3 and 5, by setting $h = N^{-\frac{1}{2\alpha + D}}$, we have, for all $\gamma > \frac{\rho}{\rho - 2}$,

$$MSE(\hat{m}(x)) = O(N^{-\frac{2\alpha}{\alpha + D}}).$$

7 Comments and Conclusions

In this paper, we obtain the Bernstein-type inequalities for geometric and algebraic $C$-mixing type processes, which can be regarded as an update and extension of Hang and Steinwart (2017)’s pioneering work on this topic. More specifically, for geometric $C$-mixing type processes, we manage to obtain a sharper inequality under more general preconditions, like weaker restrictions to the growth condition of the semi-norm, arbitrary non-stationarity and non-isotropic sampling when index set lies in $\mathbb{Z}^d$. However, for algebraic $C$-mixing type processes, effective sample size of our Bernstein-type inequality is only $O(N^\theta)$, for some $\theta \in (0, 1)$. This will reject us to apply it colored to investigating the attainability of the optimality of kernel smoothing for algebraic $C$-mixing type processes. A possible approach to bypass this difficulty is to construct some useful coupling lemmas for $C$-mixing coefficient, like Bradley’s lemma (Lemma 1.1 in Bosq (2012)) and Barbee’s lemma (Theorem 2.1 in Dehling and Philipp (2002)). Actually, Dedecker and Prieur (2004) discovered such kind of result for real-valued $\tau$-mixing process. Hence it is very meaningful to develop coupling for more general $C$-mixing type processes.

Another significant contribution of this paper is the derivation of inequalities that bound the variance of partial sums under $C$-mixing conditions. We are able to obtain relatively precise inequalities even under very weak $C$-mixing and moment conditions. Unlike the Bernstein-type inequality for algebraic $C$-mixing processes mentioned earlier, our variance inequalities easily allow us to achieve the optimal mean squared error for various kernel-based estimators even when the $C$-mixing coefficient decays algebraically. However, it is unfortunate that we still need to assume that the order of conditional moments is slightly greater than two. Therefore, a meaningful direction for future research would be to explore whether it is possible to relax this restriction on moment conditions.

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Supplementary

Argument of Assumption 3 and 4

Assumption 4 is obviously satisfied under the situations like Examples 1-5. Hence we only argue Assumption 3 is also accepted by all examples except for Example 5. Note Example 5 focus on real-valued process which does not match the condition of Assumption 3. Since $g$ is an identity mapping, we have $f(x)g(y) = f(x)y := h(z)$, for $\forall z = (x, y) \in X \times I$.

Example 1 is obvious.

Example 2.

$$\sum_{i=1}^{D} \left\| \frac{\partial h}{\partial z_i} \right\|_\infty = \sum_{i=1}^{D-1} \left\| \frac{\partial h}{\partial x_i} \right\|_\infty + \left\| f \right\|_\infty \leq (\left\| f \right\| + \left\| f \right\|_\infty) \mu(I)$$

Example 3.

$$|f(x)|_\alpha = \sup_{(x,y) \neq (x',y')} \frac{|f(x)y - f(x')y'|}{\left\| (x, y) - (x', y') \right\|^\alpha_2}$$

$$\leq \sup_{(x,y) \neq (x',y')} \frac{|f(x)y - f(x')y'|}{\left\| (x, y) - (x', y') \right\|^\alpha_2} + \sup_{(x,y) \neq (x',y')} \frac{|f(x)y' - f(x')y'|}{\left\| (x, y) - (x', y') \right\|^\alpha_2}$$

$$\leq \sup_{y \neq y'} \frac{|f(x)y - f(x')y'|}{\left\| y - y' \right\|^\alpha_2} + \sup_{x \neq x'} \frac{|f(x)y' - f(x')y'|}{\left\| x - x' \right\|^\alpha_2}$$

$$\leq \left\| f \right\|_\infty \mu(I) + |f|_\alpha \mu(I),$$

where the second last inequality is due to the monotonicity of function $(\cdot)^{\frac{\alpha}{2}}$ and the last inequality is because $\alpha \leq 1$.

Example 4 is nothing else but a special case of Example 3, thus we omit it here.

Example 6. Note that

$$\left\| f(x) \right\| = \int_X \left| \frac{\partial f}{\partial x_1 \ldots \partial x_{D-1}} \right| \, d\mu = \left\| f \right\| < (\left\| f \right\| + \left\| f \right\|_\infty) \mu(I).$$

**Lemmas**

**Lemma 1** Except for Assumptions K1-K4, 1-4 and 6, we also assume that $h = (\log N/N)^{\frac{1}{2}}$ and $E|Y|^\rho < +\infty$, for some $\alpha \in (0, 1]$ and $\rho > 2 + \frac{D}{\alpha}$. Then, we have

$$\sup_{x \in X} \left| \frac{1}{N} \sum_{i=1}^{N} \left( K_h \left( \frac{X_i - x}{h} \right) Y_i - E \left[ K_h \left( \frac{X_i - x}{h} \right) Y_i \right] \right) \right| = O_p \left( \left( \frac{\log N}{N} \right)^{\frac{\alpha}{2(D+\rho)}} \right),$$

where $K_h(u/h) = h^{-D}K(u/h)$.

**Proof of Lemma 1**

The proof is divided into two parts. Within the "part A", we prove the theorem under the condition that $E|Y|^\rho < +\infty$, for all $\rho > 2 + D/\alpha$. "Part B" is dedicated to focusing on finishing the
proof provided that $\sup_x E|Y|^{\rho}|X = x| < +\infty$, for all $\rho > 2$.

**Part A**

We use truncation method to finish the proof. Firstly, given a positive number sequence $\{L_N\}$ such that $\lim N L_N^{-2} = 0$, we rewrite

$$Y_i = Y_i 1[|Y_i| \leq L_N] + Y_i 1[|Y_i| > L_N] := W_iN + V_iN.$$ 

Thus,

$$\sup_{x \in \mathcal{X}} \left| \frac{1}{N} \sum_{i=1}^{N} \left( K_h \left( \frac{X_i - x}{h} \right) Y_i - E \left[ K_h \left( \frac{X_i - x}{h} \right) Y_i \right] \right) \right|$$

$$\leq \sup_{x \in \mathcal{X}} \left| \frac{1}{N} \sum_{i=1}^{N} \left( K_h \left( \frac{X_i - x}{h} \right) W_iN - E \left[ K_h \left( \frac{X_i - x}{h} \right) W_iN \right] \right) \right|$$

$$+ \sup_{x \in \mathcal{X}} \left| \frac{1}{N} \sum_{i=1}^{N} K_h \left( \frac{X_i - x}{h} \right) |W_iN| \right| + \sup_{x \in \mathcal{X}} \left| \frac{1}{N} \sum_{i=1}^{N} E \left[ K_h \left( \frac{X_i - x}{h} \right) V_iN \right] \right|$$

$$:= S_{1N} + S_{2N} + S_{3N}.$$ 

Note that, for any given $N$ and $a_N \searrow 0$,

$$\{S_{2N} > a_N\} \subset \{ \exists i = 1, 2, \ldots, N : |Y_i| > L_N \} = \bigcup_{i=1}^{N}\{|Y_i| > L_N\}.$$ 

Thus,

$$\mathbb{P}(S_{2N} > a_N) \leq \sum_{i=1}^{N} \mathbb{P}(|Y_i| > L_N) \leq N \max_i E|Y_i|^{\rho}/L_N^{\rho}.$$ 

This indicates $S_{2N} = O_P(a_N)$. To investigate the convergence rate of $S_{1N}$, we need to apply Proposition 4. Define

$$K_h(\mathcal{X}) = \left\{ K_h \left( \frac{\cdot - x}{h} \right) : \mathcal{X} \to \mathbb{R}; x \in \mathcal{X} \right\},$$

Then, let $a_N = \sqrt{\left(1 + \delta \right) \frac{\log N}{N k h^D}} = \mathcal{O} \left( \frac{(\log N)^{\gamma+1}}{N} \right)^{\frac{\alpha}{2\alpha + D}}$, for some $\delta > 0$. Then

$$\mathbb{P} (S_{1N} > a_N) = \mathbb{P} \left( \sup_{K_h \in K_h(\mathcal{X})} |P_N(K_h W) - P(K_h W)| > a_N \right),$$

where

$$P_N(K_h W) = \frac{1}{N} \sum_{i=1}^{N} K_h \left( \frac{X_i - x}{h} \right) W_iN,$$

$$P(K_h W) = \frac{1}{N} \sum_{i=1}^{N} E \left[ K_h \left( \frac{X_i - x}{h} \right) W_iN \right].$$
In order to apply Proposition 4, we need to ensure preconditions C1-C4 are satisfied for sufficiently large $N$. Note that for any $K_h \in K_h(\mathcal{X})$, $\|K_h\|_\infty \leq K_h^{-D}$, $\text{Var}(K_h) \leq Vh^{-D}$ and $\|K_h\| \leq N^\kappa h^{-D}$, for some $K$, $V$ and $\kappa > 0$. Thus, constants $A$, $B$ and $\sigma_F$ mentioned in Proposition 6 here are equal to $K_h^{-D}$, $N^\kappa h^{-D}$ and $Vh^{-D}$ respectively. Note the only requirement of $L_N$ is $\lim_{N \to \infty} N^{-\kappa} = 0$. Then, due to Assumption 6 and condition C1, setting $L_N = \frac{a^2 V}{10K_N a_N}$ is a suitable choice, since $Na_N = o(1)$ due to some direct calculation. Thus, condition C1 in Proposition 4 is satisfied. Recall the parameter $\omega$ in condition C2 is user-defined. Thus, we set

$$\omega^* := \inf \{ \omega > 0 : 15\sigma \sqrt{V}/(b \omega) \leq (\log 2)^{-\frac{1}{2}} \}$$

Then, C2 is satisfied for any $\omega > \omega^*$ and $N \geq 2$. For C3, we only need to note, under condition $h = O(N^{-\theta})$, we have

$$\frac{\sup_{K_h \in K_h(\mathcal{X})} \|K_h\|_\infty \sup_{K_h \in K_h(\mathcal{X})} \sqrt{\text{Var}(K_h)} L_N}{\sup_{K_h \in K_h(\mathcal{X})} \|K_h\|_\infty} \leq \frac{N^\kappa L_N}{a_N h^{D}/2} \leq \frac{N^\kappa + 1 h^{D}/2}{(\log N)^{\frac{7}{3} + \gamma}}.$$

Hence, by letting $\omega > (2 + \kappa \lor \omega^*)$, we instantly get

$$\frac{N^\kappa + 1 h^{D}/2}{(\log N)^{\frac{7}{3} + \gamma}} \leq N^{\omega^{-1}},$$

which ensures C3 can be satisfied. At last, we check C4. Define $T : x \to K_h(\cdot - x/h)$ as a mapping from Banach space $(\mathcal{X}, \| \cdot \|_2)$ to Banach space $(K_h(\mathcal{X}), \| \cdot \|_\infty)$. Note that, for any $x_1 \neq x_2 \in \mathcal{X}$, we have

$$\|T(x_1) - T(x_2)\|_\infty = \left\| K_h \left( \frac{x - x_1}{h} \right) - K_h \left( \frac{x - x_2}{h} \right) \right\|_\infty \leq \frac{\|x_1 - x_2\|_2}{h^{D+1}} \leq \frac{\text{diam}(\mathcal{X})}{2h^{D+1}}.$$

Thus mapping $T$ is a Lipschitz continuous mapping with Lipschitz modulo as $\frac{\text{diam}(\mathcal{X})}{2h^{D+1}}$, where $\text{diam}(\mathcal{X}) < +\infty$ since $\mathcal{X}$ is compact. According to Lemma 19, 21 and Proposition 13 in Hang et al. (2018), we know there exists some constant $c > 0$, such that, for any $\tau > 0$,

$$N(\tau, K(\mathcal{X}), \| \cdot \|_\infty) \leq \left( \frac{c}{\tau h^{D+1}} \right)^D.$$

Therefore,

$$\sqrt{\sigma_F} \int_{\frac{1}{2} a_N^{1/4} h^{-D}}^{\frac{1}{2} a_N^{1/4} h^{-D}} \sqrt{\log N(u^2, K, \| \cdot \|_\infty)} du \lesssim \left( \frac{1}{h^{D}} \right)^{\frac{1}{4}} \int_{a_N^{1/4}}^{\frac{1}{2} h^{-D}} \sqrt{D \log \frac{1}{a_N h^{D+1}}} du \lesssim \left( \frac{1}{h^{D}} \right)^{\frac{1}{4}} \sqrt{\frac{1}{a_N h^{D+1}}} \left( h^{-\frac{D}{4}} - a_N^{1/2} \right) \lesssim \left( \frac{1}{h^{D}} \right)^{\frac{1}{4}} \sqrt{\frac{N a_N h^{0.25D-1}}{(\log N)^{\frac{7}{3} + \gamma}}},$$

\(*\)
On the other hand,
\[ \sqrt{\frac{N}{(\log N)^{\gamma}}} t = \sqrt{(1 + \delta) \frac{\log N}{h^D}}. \]  

Obviously, by comparing (*) with (\(\star\star\)), we know there exists sufficiently large \(N\) such that C4 is satisfied. Thus, by applying Proposition 4, we manage to show that
\[ S_{1N} = \mathcal{O}_P \left( \sqrt{\frac{(\log N)^{\frac{\gamma+1}{\gamma}}}{Nh^D}} \right). \]

As for \(S_{3N}\),
\[ S_{3N} \leq \sup_{x \in A} \frac{1}{NL_N} \sum_{i=1}^{N} E \left[ K \left( \frac{X_i - x}{h} \right) |Y_i|^2 \right] \lesssim L_N^{-1} = \mathcal{O}(a_N) = \mathcal{O} \left( \sqrt{\frac{(\log N)^{\frac{\gamma+1}{\gamma}}}{Nh^D}} \right). \]

Proof of Theorem 1

**Step 1**(Simplify the Argument)

According to Assumption 5 and 6, it’s obvious that we can uniquely equip each scalar index ”i” with a vector ”\(i\)” and \(i = (i_1, \ldots, i_d) \in \mathbb{Z}^{d+}\). Then, we can rewrite the partial sums as follow,
\[ \sum_{i=1}^{N} h(Z_i) = \sum_{i} h(Z_i) = \sum_{1 \leq k \leq d', i_k = 1}^{n_0} \sum_{d-d'+1 \leq k \leq d} \sum_{i_k = 1}^{n_k} h(Z_{(i_1, \ldots, i_d)}), \]
which implies
\[ \mathbb{P} \left( \left| \frac{1}{N} \sum_{i=1}^{N} h(Z_i) \right| > t \right) \]
\[ = \mathbb{P} \left( \left| \frac{1}{N} \sum_{1 \leq k \leq d'} \sum_{i_k = 1}^{n_0} \sum_{d-d'+1 \leq k \leq d} \sum_{i_k = 1}^{n_k} h(Z_{(i_1, \ldots, i_d)}) \right| > t \right) \]
\[ \leq \mathbb{P} \left( \left| \sum_{1 \leq k \leq d'} \sum_{i_k = 1}^{n_0} \frac{1}{N} \sum_{d-d'+1 \leq k \leq d} \sum_{i_k = 1}^{n_k} h(Z_{(i_1, \ldots, i_d)}) \right| > t \right) \]
\[ \leq \sum_{1 \leq k \leq d'} \sum_{i_k = 1}^{n_0} \mathbb{P} \left( \left| \frac{1}{N} \sum_{d-d'+1 \leq k \leq d} \sum_{i_k = 1}^{n_k} h(Z_{(i_1, \ldots, i_d)}) \right| > \frac{t}{m} \right). \]

Since \(n_0\) is a fixed number, we only need to consider the situation that \(d = d'\). Therefore, from now on, we can assume that, we have sample \(\{Z_i\}_{i=1}^{\tilde{N}}\) and according to Assumption 5, \(\tilde{N} = \prod_{k=d-d'+1}^{d} n_k\).
Now we tend to derive an upper bound for

\[ \mathbb{P} \left( \left| \frac{1}{N} \sum_{i=1}^{N} h(Z_i) \right| > \frac{t}{m} \right) = \mathbb{P} \left( \left| \frac{1}{N} \sum_{i=1}^{N} h(Z_i) \right| > t \right). \]

Due to Chernoff’s approach, for \( \forall \lambda > 0 \), the right-side tail has upper bound

\[ \mathbb{P} \left( \frac{1}{N} \sum_{i=1}^{N} h(Z_i) > t \right) \leq m \mathbb{P} \left( \frac{1}{N} \sum_{i=1}^{N} h(Z_i) > t \right) \leq m e^{-\lambda t E \left[ \frac{\lambda}{N} \sum_{i=1}^{N} h(Z_i) \right]} \].

**Step 2** (Blocking)

Again, similar to Step 1, we know there exists an one-on-one mapping between scalar \( i \) and vector \( i = (i_1, \ldots, i_{d'}) \), \( i = 1, \ldots, \hat{N} \). For \( 1 \leq k \leq d' \), set \( P_N \in \mathbb{N}^+ \), \([n_k/P_N] = L_N^k \), \( r_k = n_k - L_N^k P_N \), where \([a]\) stands for the integer part of \( a \in \mathbb{R} \). Along the \( k \)-th direction, define

\[ I(j_k) = \begin{cases} \{j_k, j_k + P_N, \ldots, j_k + L_N^k P_N\} & 1 \leq j_k \leq r_k \\ \{j_k, j_k + P_N, \ldots, j_k + (L_N^k - 1)P_N\} & r_k < j_k \leq P_N. \end{cases} \tag{26} \]

Let \( I_j = I(j_1, \ldots, j_{d'}) = \prod_{k=1}^{d'} I(j_k) \). Then the projection of the indices in block \( I_j \) on the \( k \)-th axis is \( I(j_k) \). Moreover, there are \( P_N^{d'} \)-many different \( j \)'s. Equally, for each \( j = (j_1, \ldots, j_{d'}), 1 \leq j_k \leq P_N, 1 \leq k \leq d' \). Meanwhile, due to (26), given \( j \), for any \( i, i' \in I_j \) and \( i \neq i' \), we have

\[ \rho(i, i') \geq P_N. \tag{27} \]

Since (27) is the only we rely on later, we can replace vector \( j \) and \( i \) with scalar \( j \) and \( i \), which would significantly simplify our notation. Hence, we have

\[ \frac{1}{N} \sum_{i=1}^{\hat{N}} h(Z_i) = \frac{1}{N} \sum_{j=1}^{P_N^{d'}} \sum_{i \in I_j} h(Z_i) = \frac{P_N^{d'}}{N} \sum_{j=1}^{P_N^{d'}} \sum_{i \in I_j} \frac{h(Z_i)}{|I_j|} = \sum_{j=1}^{P_N^{d'}} p_j \sum_{i \in I_j} \frac{h(Z_i)}{|I_j|}, \tag{28} \]

where \(|A|\) denotes the cardinality of set \( A \). Obviously, according to the construction of block, for each \( \hat{N} = \frac{N}{m} \), we have

\[ \frac{\hat{N}}{P_N^{d'}} \leq |I_j| \leq \left( \frac{2}{P_N} \right)^{d'} \hat{N} \text{ and } \sum_{j=1}^{P_N^{d'}} p_j = 1. \tag{29} \]
Thus, based on (28) and (29), together with Jensen inequality, we obtain

$$E \left[ \exp \left( \frac{\lambda}{N} \sum_{i=1}^{N} h(Z_i) \right) \right] = E \left[ \exp \left( \sum_{j=1}^{p_f} p_j \sum_{i=1}^{I_j} \frac{h(Z_i)}{|I_j|} \right) \right] \leq \sum_{j=1}^{p_f} p_j E \left[ \exp \left( \lambda \frac{\sum_{i=1}^{I_j} h(Z_i)}{|I_j|} \right) \right].$$

**Step 3 ($E_j$)**

For any fixed $j$, define

$$g(z) = \exp \left( \frac{\lambda h(z)}{|I_j|} \right), \quad S_l = \exp \left( \lambda \sum_{i=1}^{l} \frac{h(Z_i)}{|I_j|} \right), \quad l = 1, 2, \ldots, |I_j|.$$

Since $||h||_{\infty} \leq A$, for any $l = 1, 2, \ldots, |I_j|$, $||S_l||_{\infty} \leq \exp(\lambda A/|I_j|)$ and $S_l$ is measurable with respect to the $\sigma$-algebra generated by $Z_i, i = 1, \ldots, l$, denoted as $\sigma(Z_i, i = 1, \ldots, l)$. Thus, for $\forall \lambda > 0$, $S_l \in L^1(\sigma(Z_i, i = 1, \ldots, l), P)$. Additionally, based on Assumption 1 and $||h|| \leq B$,

$$||g(z)|| \leq e^{\lambda A} \left( \frac{\lambda h(z)}{|I_j|} \right) \leq e^{\lambda A} \frac{\lambda B}{|I_j|} < +\infty,$$

which implies, for $\forall \lambda > 0$, we have $g(z) \in C(Z)$. Therefore, according to Definition 1, we have, for any $l = 0, \ldots, |I_j| - 1$

$$E[S_{|I_j| - l}] = E \left[ \exp \left( \lambda \sum_{i=1}^{l} \frac{h(Z_i)}{|I_j|} \right) \exp \left( \frac{\lambda h(Z_{|I_j| - l})}{|I_j|} \right) \right] \leq \text{Cov} \left( \exp \left( \lambda \sum_{i=1}^{l} \frac{h(Z_i)}{|I_j|} \right), \exp \left( \frac{\lambda h(Z_{|I_j| - l})}{|I_j|} \right) \right)$$

$$+ E \left[ \exp \left( \lambda \sum_{i=1}^{l} \frac{h(Z_i)}{|I_j|} \right) \right] E \left[ \exp \left( \frac{\lambda h(Z_{|I_j| - l})}{|I_j|} \right) \right].$$

For $I_1$, according to (27) and Definition 1,

$$I_1 \leq \nu^{-h_{P_N}} \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_{c} E \left[ \exp \left( \lambda \sum_{i=1}^{l} \frac{h(Z_i)}{|I_j|} \right) \right].$$

Then

$$I_1 + I_2 \leq \left( \nu^{-h_{P_N}} \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_{c} + E \left[ \exp \left( \frac{\lambda h(Z_{|I_j| - l})}{|I_j|} \right) \right] \right) E[S_{|I_j| - l - 1}].$$

(30)
By running this argument recursively, we obtain

$$E_j \leq \prod_{l=2}^{|I_j|} \left( e^{-hP_l^2} \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_c + E \left[ \exp \left( \frac{\lambda h(Z_i)}{|I_j|} \right) \right] \right) E[S_1]. \quad (31)$$

According to (3) and Assumption 1, by setting $\lambda < \frac{|I_j|}{A}$

$$\left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_c = \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_\infty + \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_c \leq \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_\infty \left( 1 + \frac{||h(z)||}{A} \right) \leq e \left( \frac{A+B^2}{A} \right). \quad (32)$$

On the other hand, since $Eh = 0$, $Eh^2 \leq \sigma^2$, $||h||_\infty \leq A$, for any $l$, we have

$$\exp \left( \frac{\lambda h(Z_i)}{|I_j|} \right) \leq 1 + \frac{\lambda h(Z_i)}{|I_j|} + \frac{1}{2} \left( \frac{\lambda h(Z_i)}{|I_j|} \right)^2 + \sum_{q=3}^{\infty} \frac{1}{q!} \left( \frac{\lambda h(Z_i)}{|I_j|} \right)^q$$

$$\leq 1 + \frac{\lambda h(Z_i)}{|I_j|} + \frac{1}{2} \left( \frac{\lambda h(Z_i)}{|I_j|} \right)^2 \left( 1 + \sum_{q=3}^{\infty} \left( \frac{\lambda ||h(z)||_\infty}{|I_j|} \right)^{q-2} \right)$$

$$= 1 + \frac{\lambda h(Z_i)}{|I_j|} + \frac{1}{2} \left( \frac{\lambda h(Z_i)}{|I_j|} \right)^2 \left( 1 - \frac{A}{|I_j|} \right)$$

$$= 1 + \frac{\lambda h(Z_i)}{|I_j|} + \frac{\lambda^2 h^2(Z_i)}{2|I_j|(|I_j| - \lambda A)};$$

which implies

$$E \left[ \exp \left( \frac{\lambda h(Z_i)}{|I_j|} \right) \right] \leq 1 + \frac{\lambda^2 \sigma^2}{2|I_j|(|I_j| - \lambda A)} \leq \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j|(|I_j| - \lambda A)} \right). \quad (33)$$
By applying (33) and (32) to (31) and letting $P_N = \left( \frac{1}{\lambda} \log_N \hat{N} \right)^{\frac{1}{\lambda}}$, we thus have

$$
E_j \leq \left( \frac{\nu}{\nu - bP_N} e \left( \frac{A + B}{A} \right) + \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j| |I_j| - \lambda A} \right) \right)^{|I_j|} \|S_1\|_\infty
$$

(34)

\[
\begin{align*}
&\leq e \left( \frac{\nu}{\nu - bP_N} e \left( \frac{A + B}{A} \right) + 1 \right)^{|I_j|} \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j| - \lambda A} \right) \\
&\leq e \exp \left( e \left( \frac{A + B}{A} \right) - bP_N |I_j| \right) \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j| - \lambda A} \right) \\
&= e \exp \left( \frac{1}{2^{d-1}} \left( \frac{A + B}{A(N/m)^{2\omega-1}} \right) \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j| - \lambda A} \right) \right) \\
&\leq 4 \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j| - \lambda A} \right),
\end{align*}
\]

where the fourth inequality uses the fact that $\log_N |I_j| \leq \log_N \hat{N}$ while the last inequality is based on the condition that $N \geq N_0 := \min \{ N \in \mathbb{N}, \left( \frac{A + B}{A(N/m)^{2\omega-1}} \right) \leq 2^{d-1} \}$ and $e < 2$. Note that, due to (34), the upper bound of $E_j$ is independent from $j$, which yields

$$
E \left[ \exp \left( \frac{\lambda}{N} \sum_{i=1}^{\hat{N}} h(Z_i) \right) \right] \leq 4 \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j| - \lambda A} \right).
$$

(35)

Hence $\frac{1}{N} \sum_{i=1}^{\hat{N}} h(Z_i)$ is a sub-Gamma random variable and

$$
\mathbb{P} \left( \left| \frac{1}{N} \sum_{i=1}^{\hat{N}} h(Z_i) \right| > t \right) < 8 \exp \left( \frac{\lambda^2 \sigma^2}{2|I_j| - \lambda A} - \lambda t \right).
$$

(36)

Let $\lambda = \frac{|I_j| \sigma^2 + tA}{t \sigma^2 + tA}$. It’s easy to specify that $\lambda < \frac{|I_j|}{A}$ and according to some simple algebra, we obtain

$$
\mathbb{P} \left( \left| \frac{1}{N} \sum_{i=1}^{\hat{N}} h(Z_i) \right| > t \right) < 8 \exp \left( \frac{|I_j| \sigma^2}{2|I_j| - \lambda A} \right).
$$

(37)

Together with the fact that

$$
|I_j| \geq \frac{\hat{N}}{P_N^{d'}} = \frac{N}{m(\frac{1}{\lambda} \log_N \hat{N})^{d'}} \geq \frac{N}{m(\frac{1}{\lambda} \log_N N)^{d'}},
$$

we finish the proof of Theorem 5.

**Time-Reverse $C$-mixing**

We only consider time-reversed $C$-mixing process when $d' = d = 1$. Under this circumstance,
since \( m = 1 \), \( N = \hat{N} \), similar to Step 1 above, we only need to bound the Laplace transformation of \( \frac{1}{N} \sum_{i=1}^{N} h(Z_i) \). Then, by following Step 2, we introduce the following block

\[
I_j = \begin{cases} 
\{j, j + P_N, \ldots, j + L_N P_N \} & 1 \leq j \leq r \\
\{j, j + P_N, \ldots, j + (L_N - 1) P_N \} & r < j \leq P_N,
\end{cases}
\]

(38)

where \( P_N \in \mathbb{N}^+ \), \( L_N = [N/P_N] \) and \( r = N - P_N L_N \). Thus, within each block \( I_j \), the minimum distance between any two different indices is \( P_N \). Based on this, we manage to rewrite the averaged partial sum as follow

\[
\frac{1}{N} \sum_{i=1}^{N} h(Z_i) = \sum_{j=1}^{P_N} p_j \sum_{i \in I_j} h(Z_i),
\]

where \( p_j = \frac{|I_j|}{N} \) and \( \sum_{j=1}^{P_N} p_j = 1 \) for every \( N \). Then, similar to previous procedure, we tend to bound

\[
E \left[ \exp \left( \sum_{i \in I_j} \frac{h(Z_i)}{|I_j|} \right) \right]
\]

uniformly over \( j \). Then, according to the definition of time-reversed \( C \)-mixing (see Definition 2.5, Hang and Steinwart (2017)), by denoting \( I_j = \{j_l, l = 1, \ldots, |I_j|\} \), where \( j_{l+1} - j_l = P_N \), we have

\[
E \left[ \exp \left( \sum_{i \in I_j} \frac{h(Z_i)}{|I_j|} \right) \right] = E \left[ \exp \left( \sum_{l=1}^{|I_j|} \frac{h(Z_{j_l})}{|I_j|} \right) \right] = E \left[ \exp \left( \frac{h(Z_{j_1})}{|I_j|} \right) \exp \left( \sum_{l=2}^{|I_j|} \frac{h(Z_{j_l})}{|I_j|} \right) \right]
\]

\[
\leq \left| \text{Cov} \left( \exp \left( \frac{h(Z_{j_1})}{|I_j|} \right), \exp \left( \sum_{l=2}^{|I_j|} \frac{h(Z_{j_l})}{|I_j|} \right) \right) \right| + E \left[ \exp \left( \frac{h(Z_{j_1})}{|I_j|} \right) \right] E \left[ \exp \left( \sum_{l=2}^{|I_j|} \frac{h(Z_{j_l})}{|I_j|} \right) \right].
\]

Now by following the same argument used before we can obtain result (9) for time-reversed \( C \)-mixing.

**Proof of Theorem 2**

Considering the proof of Theorem 2 is nearly the same as the proof of Theorem 1, here we only highlight the differences. Firstly, ”Step 1” and ”Step 2” are totally unchanged. By inheriting all the notations used in ”Step 3”, according to the assumption of the decay of dependence, we can instantly obtain that, for any fixed \( j \) and \( l = 1, \ldots, |I_j| \),

\[
E[S_{|I_j|-l}] \leq \left( b P_N^{-\gamma} \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_{\mathcal{C}} + E \left[ \exp \left( \frac{\lambda h(Z_{|I_j|-l})}{|I_j|} \right) \right] \right) E[S_{|I_j|-l-1}].
\]

Then, by doing this recursively, we obtain,

\[
E_j \leq \prod_{l=2}^{|I_j|} \left( b P_N^{-\gamma} \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_{\mathcal{C}} + E \left[ \exp \left( \frac{\lambda h(Z_{|I_j|-l})}{|I_j|} \right) \right] \right) E[S_1].
\]
Note that, by assuming $\lambda < \frac{|I_j|}{A}$,

$$\left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_\infty \leq \left\| \exp \left( \frac{\lambda h(z)}{|I_j|} \right) \right\|_\infty \left( \left\| \frac{\lambda h(z)}{|I_j|} \right\| + 1 \right) \leq \exp \left( \frac{\lambda A}{|I_j|} \right) \frac{A + B}{A} \leq e^{A + B}.$$ 

Therefore, by letting $P_N = (\hat{N})^{\frac{\alpha + 1}{\gamma + \theta} b^{\theta + \sigma}}$,

$$E_j \leq \left( bP_N^{-\gamma} e \left( \frac{A + B}{A} \right) + \exp \left( \frac{\lambda^2 \sigma^2}{2(|I_j|(|I_j| - \lambda A))} \right) \right)^{|I_j|} \|S_1\|_\infty \leq e^{\left( bP_N^{-\gamma} e \left( \frac{A + B}{A} \right) + 1 \right)^{|I_j|} \|S_1\|_\infty} \exp \left( \frac{\lambda^2 \sigma^2}{2(|I_j| - \lambda A)} \right) \leq e^{\left( bP_N^{-\gamma} e \left( \frac{A + B}{A} \right) \hat{N} \right)^{|I_j|} \|S_1\|_\infty} \exp \left( \frac{\lambda^2 \sigma^2}{2(|I_j| - \lambda A)} \right) \leq e^{\left( 2^\theta bP_N^{-\gamma} e \left( \frac{A + B}{A} \right) \hat{N} \right)^{|I_j|} \|S_1\|_\infty} \exp \left( \frac{\lambda^2 \sigma^2}{2(|I_j| - \lambda A)} \right) \leq e^{\left( 2^\theta e \left( 1 + \frac{N^\alpha}{A} \right) \hat{N} \right)^{|I_j|} \|S_1\|_\infty} \exp \left( \frac{\lambda^2 \sigma^2}{2(|I_j| - \lambda A)} \right).$$

Then, when $(N/m)^\alpha > A$, we obtain

$$E_j \leq \exp \left( 1 + \frac{2^\theta e \left( 1 + \frac{N^\alpha}{A} \right) \hat{N} \right)^{|I_j|} \|S_1\|_\infty} \exp \left( \frac{\lambda^2 \sigma^2}{2(|I_j| - \lambda A)} \right).$$

Then, by following the same procedure demonstrated in the proof of Theorem 1, we can finish the proof. As for the case that $B$ is a fixed positive number, we only need to set $P_N = (\hat{N})^{\frac{1}{\gamma + \theta} b^{\theta + \sigma}}$ and repeat the calculation above.

**Proof of Proposition 1**

We firstly prove the covariance inequality. The proof is divided into two steps.

**Step 1**

We aim to prove the following equation at first.

$$\text{Cov}(Y, f(X)) = \int_{\mathbb{R}^q} \text{Cov}(Y, 1[X \leq t])g(t)dt, \ t \in \mathbb{R}^q,$$
where \([X \leq t] = \prod_{k=1}^{n}(-\infty < X_k \leq t_k]\). Note that, by setting \(X^*\) as a copy of \(X\) but independent from \(Y\), together with the definition of set \(\Lambda\), we have

\[
\text{Cov}(Y, f(X)) = E[Y(f(X) - f(X^*))] = E\left(Y \int_{\mathbb{R}^q} (1[X \leq t] - 1[X^* \leq t])g(t)dt\right).
\]

In order to apply Fubini Theorem, we need to ensure

\[
E\left(|Y| \int_{\mathbb{R}^q} |(1[X \leq t] - 1[X^* \leq t])g(t)|dt\right) < +\infty.
\]

Actually, due to the fact that \(Y\) is \(\mathcal{A}'\)-measurable, we have

\[
E\left(|Y| \int_{\mathbb{R}^q} |(1[X \leq t] - 1[X^* \leq t])g(t)|dt\right) \\
\leq E\left(|Y| \int_{\mathbb{R}^q} |1[X^* \leq t]|g(t)|dt\right) + E\left(|Y| \int_{\mathbb{R}^q} |1[X \leq t]|g(t)|dt\right) \\
= E\left(|Y|E\left(\int_{\mathbb{R}^q} |1[X^* \leq t]|g(t)|dt \mid \mathcal{A}'\right)\right) + E\left(|Y|E\left(\int_{\mathbb{R}^q} |1[X \leq t]|g(t)|dt \mid \mathcal{A}'\right)\right) \\
:= E_1 + E_2.
\]

For \(E_1\), please note

\[
E\left(\int_{\mathbb{R}^q} |1[X^* \leq t]|g(t)|dt \mid \mathcal{A}'\right) = E\left(\int_{\prod_{i=1}^{n}[a_i, b_i]} |1[X^* \leq t]|g(t)|dt \mid \mathcal{A}'\right) \\
= \int_{\prod_{i=1}^{n}[a_i, b_i]} P_{X^*}(X^* \leq t \mid \mathcal{A}') |g(t)|dt \leq \int_{\prod_{i=1}^{n}[a_i, b_i]} |g(t)|dt < +\infty.
\]

The second equality above is due to the fact that function \(|1[x^* \leq t]|g(t)|\) is uniformly bounded on its support, \((\prod_{i=1}^{n}[a_i, b_i])^2\). We thus obtain

\[
E_1 \leq E|Y| \int_{\prod_{i=1}^{n}[a_i, b_i]} |g(t)|dt < +\infty.
\]

Note \(X\) is identical to \(X^*\), the argument above works the same on \(E_2\). Hence, Fubini Theorem can be applied, which yields

\[
\text{Cov}(Y, f(X)) = \int_{\mathbb{R}^q} E(Y(1[X \leq t] - 1[X^* \leq t]))g(t)dt = \int_{\mathbb{R}^q} [E(Y(1[X \leq t])) - E(1[X^* \leq t])]|g(t)|dt.
\]
Similarly, according to the fact that $X^*$ is independent of $Y$ but identical to $X$, we have

\[
E(Y(1|X \leq t)) - E(1|X^* \leq t)) - E(Y(1|X \leq t)) + E(Y[E(1|X \leq t)] - E(1|X^* \leq t)) - E(Y(1|X \leq t)) + E(Y[E(1|X \leq t)] - E(1|X^* \leq t)) = Cov(Y, 1|X \leq t)) - Cov(Y, 1|X^* \leq t)) = Cov(Y, 1|X \leq t)) = 0.
\]

Thus,

\[
Cov(Y, f(X)) = \int_{R^q} Cov(Y, 1|X \leq t))|g(t)|dt.
\]

**Step 2**

\[
\therefore |Cov(Y, f(X))| \leq \int_{R^q} |Cov(Y, 1|X \leq t))||g(t)|dt
\]

\[
\leq E(\int_{R^q} |F_X(Av(t) - F_X(t))|g(t)|dt)
\]

\[
= E(\int_{R^q} \sup_{t} |F_X(Av(t) - F_X(t))|g(t)|dt)
\]

\[
\leq \phi_v(A', X)|E|Y| \int_{\prod_{i=1}^{q} [a_i, b_i]} |g(t)|dt.
\]

Now we start to prove that $g(x) = Df(x)$. Since for $\forall x \in \prod_{i=1}^{q} [a_i, b_i]$, $|f(x)| < +\infty$ and $\mu$ is the Lebesgue measure, we have

\[
\int_{\prod_{i=1}^{q} [a_i, x_i]} |g(t)|\mu(dt_1 \ldots dt_q) < +\infty
\]

for $\forall x \in \prod_{i=1}^{q} [a_i, b_i]$. Then for any $k = 1, \ldots, q$, fix $t_k = t_0$, $g(t_1, \ldots, t_0, \ldots, t_q)$ is also Lebesgue integrable. Furthermore, almost surely on $[a_k, b_k]$, we have

\[
\int_{a_k}^{b_k} \cdots \int_{a_{k-1}}^{b_{k-1}} \int_{a_{k+1}}^{b_{k+1}} \cdots \int_{a_1}^{b_1} |g(t_1, \ldots, t_0, \ldots, t_q)|\mu(dt_1 \ldots dt_{k-1}dt_{k+1} \ldots dt_q) < +\infty.
\]
Therefore, for \( \forall x_1 \in (a_1, b_1) \), we have

\[
\Delta := \left| \frac{f(x_1 + h, x_2, \ldots, x_q) - f(x_1, \ldots, x_q)}{h} - \int_{a_q}^{b_q} \cdots \int_{a_2}^{b_2} |g(x_1, t_2, \ldots, t_q)| \mu(dt_2 \ldots dt_q) \right|
\]

\[
= \frac{1}{h} \int_{a_q}^{b_q} \cdots \int_{a_2}^{b_2} |g(t_1, t_2, \ldots, t_q)| \mu(dt_1 \ldots dt_q) - \frac{1}{h} \int_{a_q}^{b_q} \cdots \int_{a_1}^{x_1+h} |g(x_1, t_2, \ldots, t_q)| \mu(dt_1 \ldots dt_q)
\]

\[
\leq \frac{1}{h} \int_{a_q}^{b_q} \cdots \int_{a_1}^{x_1+h} |g(t_1, \ldots, t_q) - g(x_1, t_2, \ldots, t_q)| \mu(dt_1 \ldots dt_q).
\]

Note \( g \) is continuous on \( C[\prod_{i=1}^{q} [a_i, b_i]] \) hence uniformly continuous. Thus, for \( \forall \epsilon > 0 \), \( \exists \) sufficiently small \( h > 0 \), such that, for \( \forall t_1 \in [x_1, x_1 + h] \), we have

\[
|g(t_1, \ldots, t_q) - g(x_1, t_2, \ldots, t_q)| < \epsilon \frac{1}{\prod_{i=1}^{q} (b_i - a_i)}.
\]

This implies \( \Delta \leq \epsilon \). Thus \( \frac{\partial f}{\partial x_1} = \int_{a_q}^{b_q} \cdots \int_{a_2}^{b_2} g(x_1, t_2, \ldots, t_q) \mu(dt_2 \ldots dt_q) \). Then, by repeating this procedure \( q - 1 \) times, we finish the proof.

**Proof of Proposition 2**

By denoting \( \overline{h}_i := h(Z_i) - E(h(Z_i)) \), it suffice to bound \( \text{Var}(\sum_{i=1}^{n} \overline{h}_i) \). Based on some simple algebra, we have

\[
\text{Var}(\sum_{i=1}^{n} \overline{h}_i) \leq \sum_{i=1}^{n} \text{Var}(\overline{h}_i) + 2 \sum_{i<j} \text{Cov}(\overline{h}_i, \overline{h}_j)
\]

\[
\leq \sum_{i=1}^{n} \text{Var}(\overline{h}_i) + 2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} \text{Cov}(\overline{h}_i, \overline{h}_{i+m})
\]

\[
\leq \sum_{i=1}^{n} |\overline{h}_i|^2 + 2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} |E[\overline{h}_i | \overline{h}_{i+m} | | \overline{h}_{i+m} | \leq L_m]| + 2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} |E[\overline{h}_i | \overline{h}_{i+m} | | \overline{h}_{i+m} | > L_m]|
\]

Here \( L_m = m^{\frac{s+1}{s-1}} \). Please note that both conditions C1 and C2 indicate that \( \max_i E[|\overline{h}_i|^s] < +\infty \), for some \( s > 1 \). Thus,

\[
2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} |E[\overline{h}_i | \overline{h}_{i+m} | | \overline{h}_{i+m} | > L_m]| \leq 2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} L_m^{-(s-1)} E[|\overline{h}_i | \overline{h}_{i+m} |]
\]

Under condition C1, we have

\[
E[|\overline{h}_i | \overline{h}_{i+m} |] = E[|\overline{h}_i | E[|\overline{h}_{i+1} |^s | Z_i]] < K\text{ ||}\overline{h}_i\text{ ||}_1.
\]

When condition C2 holds, by using Cauchy-Schwartz inequality, we obtain

\[
E[|\overline{h}_i | \overline{h}_{i+m} |] \leq ||\overline{h}_i||_2 ||\overline{h}_{i+m}||_2 \leq K''||\overline{h}_i||_2.
\]
Thus,

\[
2 \sum_{i=1}^{\infty} \sum_{m=1}^{n} |E[\tilde{h}_i \tilde{h}_{i+m} | \tilde{h}_{i+m} > L_m]| \leq 2(K') \sum_{i=1}^{n} ||\tilde{h}_i||_1 \left( \sum_{m=1}^{+\infty} m^{-(1+\delta)} \right), \text{ if condition C1 holds.}
\]

\[
2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} |E[\tilde{h}_i \tilde{h}_{i+m} | \tilde{h}_{i+m} > L_m]| \leq 2MK'' \sum_{i=1}^{n} ||\tilde{h}_i||_2, \text{ if condition C2 holds.}
\]

Note that, for each \( m \in \mathbb{N}^+ \), let \( \tilde{h}_{i+m} = g_{i+m}(Z_{i+m}) \). Then \( g_{i+m}(\cdot) \) can be represented as a subtraction between the following two components. The first component is the restriction of \( h \) on \( [E[h(Z_{i+m})] - L_m, E[h(Z_{i+m})] + L_m] \), which is a measurable subset of \( Z \). The second component is a constant \( E[h(Z_{i+m})]1[\tilde{h}_{i+m} \leq L_m] \). That is to say,

\[
g_{i+m}(\cdot) = h(\cdot)1[E[h(Z_{i+m})] - L_m, E[h(Z_{i+m})] + L_m] - E[h(Z_{i+m})]1[E[h(Z_{i+m})] - L_m, E[h(Z_{i+m})] + L_m] = g_{i+m}^1(\cdot) - g_{i+m}^2(\cdot).
\]

According Assumption 3 and basic triangle inequality of semi-norm, we immediately know \( ||g_{i+m}|| \leq ||g_{i+m}^1|| \leq ||h|| \leq K_n \), which indicates \( g \in C(Z) \) and

\[
||g_{i+m}||_{C} \leq L_m + K_n \leq 2m^{\frac{1+\delta}{\gamma}}, \text{ when } L_m \geq K_n;
\]

\[
||g_{i+m}||_{C} \leq L_m + K \leq 2K_n, \text{ when } L_m < K_n.
\]

Then, by defining \( m_n := \max\{m : L_m < K_n\} \), we have \( m_n = \lceil K_n^{\frac{1-\delta}{\gamma}} \rceil \). Therefore, according to the definition of \( C \)-mixing, we have

\[
2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} |E[\tilde{h}_i g_{i+m} | \tilde{h}_{i+m} \leq L_m]| = 2 \sum_{i=1}^{n} \sum_{m=1}^{m_n} |E[\tilde{h}_i g_{i+m}||\tilde{h}_{i+m}||g_{i+m}|_{C}.
\]

Note that

\[
|E[\tilde{h}_i g_{i+m}]| \leq E[|\tilde{h}_i g_{i+m}|] \leq E[\tilde{h}_i]E|h_{i+m}| \leq E|h_i|E|h_{i+m}| + 7E|h_i|E|h_{i+m}|.
\]

Then

\[
2 \sum_{i=1}^{n} \sum_{m=1}^{+\infty} |E[\tilde{h}_i \tilde{h}_{i+m} | \tilde{h}_{i+m} \leq L_m]| \leq 2 \sum_{i=1}^{n} \sum_{m=1}^{m_n} (E|h_i h_{i+m}| + 7E|h_i|E|h_{i+m}|) + 2 \sum_{i=1}^{n} \sum_{m=m_n}^{+\infty} C(m)||\tilde{h}_i||_1||g_{i+m}||_{C}
\]

\[
\leq 2 \sum_{i=1}^{n} \sum_{m=1}^{m_n} (E|h_i h_{i+m}| + 7E|h_i|E|h_{i+m}|) + 8 \sum_{i=1}^{n} ||h_i||_1 \left( \sum_{m=m_n}^{+\infty} m^{\frac{1+\delta}{\gamma}}C(m) \right) := \Delta_n
\]

:= C_2 < +\infty
\]

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Above all, we manage to show that
\[
\text{Var}(\overline{h}_i) \leq \sum_{i=1}^{n} \text{Var}(\overline{h}_i) + (2K'M + 8C_2) \sum_{i=1}^{n} E|h(Z_i)| + \triangle_n, \text{ if condition C1 holds.}
\]
\[
\text{Var}(\overline{h}_i) \leq \sum_{i=1}^{n} \text{Var}(\overline{h}_i) + 8C_2 \sum_{i=1}^{n} E|h(Z_i)| + 2MK'' \sum_{i=1}^{n} \sqrt{\text{Var}(h(Z_i))} + \triangle_n, \text{ if condition C2 holds.}
\]
Thus, we finish the proof. 5

**Proof of Proposition 3**

The proof of Proposition 3 is essentially the same as the proof of Proposition 2. Here we only highlight the differences. Similarly, we denote \( \overline{h}_i := Z_i - E[Z_i] \). Then, according to some simple algebra, we get
\[
\text{Var}\left(\sum_{i=1}^{n} a_{in}Z_i\right) = \text{Var}\left(\sum_{i=1}^{n} a_{in}\overline{h}_i\right) \leq \sum_{i=1}^{n} a_{in}^2 \text{Var}(\overline{h}_i) + 2 \sum_{i<j} \left|a_{in}a_{jn} \text{Cov}(\overline{h}_i, \overline{h}_j)\right|
\]
\[
\leq \sum_{i=1}^{n} a_{in}^2 \text{Var}(\overline{h}_i) + 2 \sum_{i=1}^{n} |a_{in}| \sum_{m=1}^{+\infty} |a_{i+m,n}| |E[\overline{h}_i\overline{h}_{i+m}1[|\overline{h}_{i+m}| \leq L_m]]|
\]
\[
+ 2 \sum_{i=1}^{n} |a_{in}| \sum_{m=1}^{+\infty} |a_{i+m,n}| |E[\overline{h}_i\overline{h}_{i+m}1[|\overline{h}_{i+m}| > L_m]]|
\]
\[
\leq \sum_{i=1}^{n} a_{in}^2 \text{Var}(\overline{h}_i) + 2(B/b) \sum_{i=1}^{n} a_{in}^2 \sum_{m=1}^{+\infty} |E[\overline{h}_i\overline{h}_{i+m}1[|\overline{h}_{i+m}| \leq L_m]]|
\]
\[
+ 2(B/b) \sum_{i=1}^{n} (a_{in})^2 \sum_{m=1}^{+\infty} |E[\overline{h}_i\overline{h}_{i+m}1[|\overline{h}_{i+m}| > L_m]]|
\]
where the last inequality is due to the uniform bound of array \( \{a_{in}\} \) and \( L_m = m^\frac{1+\epsilon}{4} \). Similar to the proof of Proposition 2, by letting \( g_{i+m} := g_{i+m}(Z_{i+m}) := \overline{h}_{i+m}1[|\overline{h}_{i+m}| \leq L_m] \), we can decompose the mapping \( g_{i+m}(\cdot) \) as follow.
\[
g_{i+m}(z) = z1[B] - E[Z_{i+m}]1[B] := g^1_{i+m}(z) - g^2_{i+m}(z),
\]
where \( B = [E[Z_{i+m}] - L_m, E[Z_{i+m}] + L_m] \).

Thus, due to Assumption 5, we have \( ||g_{i+m}|| \leq C_1 \text{Vol}(B) = 2C_1 L_m = 4C_1 m^\frac{1+\epsilon}{4} \) and \( ||g_{i+m}||_c \leq (4C_1 + 1)m^\frac{1+\epsilon}{4} \). Thus, by letting \( \overline{g}_{i+m} = g_{i+m} - E[g_{i+m}] \), we have
\[
||\overline{g}_{i+m}||_c \leq ||g_{i+m}||_c + \max_m E|g_{i+m}| \leq (4C_1 + 1)m^\frac{1+\epsilon}{4} + 2 \max_j E|Z_j| \leq 2(4C_1 + 1)m^\frac{1+\epsilon}{4}
\]
and
\[
|E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} \leq L_m]| = |E[\bar{\theta}_1 \bar{g}_{i+m}]| \leq |E[\bar{\theta}_1 \bar{g}_{i+m}]| + E[|\bar{\theta}_1|E|g_{i+m}|
\]
\[
\leq |E[\bar{\theta}_1 \bar{g}_{i+m}]| + 4E|Z_i|E|Z_{i+m}| \leq |\text{Cov}(\bar{\theta}_1, \bar{g}_{i+m})| + 4E|Z_i|E|Z_{i+m}|
\]
\[
\leq C(m)|\bar{\theta}_1||\bar{g}_{i+m}||c + 4(\max E|Z_i|E|Z_i| \leq 8(4C_1 + 1)m^{1+\delta}C(m)E|Z_i|.
\]

Therefore, similar to the proof of Proposition 2, by denoting \( C_2 = \sum_{m=1}^{\infty} m^{1+\delta}C(m) \), we obtain
\[
2(B/b) \sum_{i=1}^{n} a_{m}^2 \sum_{m=1}^{\infty} |E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} \leq L_m]| \leq 16C_2(4C_1 + 1)(B/b) \sum_{i=1}^{n} a_{m}^2 E|Z_i|.
\]

By repeating the steps taken in the proof of Proposition 2, we can also obtain that
\[
|E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} > L_m]| \leq L_m^{-s-1}E|\bar{\theta}_1 \bar{\theta}_{i+m}| \leq L_m^{-s-1}E|\bar{\theta}_1|E|\bar{\theta}_{i+m}|^s|Z_i|].
\]

Under condition D1, we have
\[
\max_{j \geq 1} E[\bar{\theta}_j^s |Z_i] \leq 2^{s-1} \max_{j \geq 1} E[|Z_j|^s |Z_i] + 2^{s-1} \max_{j \geq 1}(E|Z_j|)^s < 2^s A.
\]

Here the last inequality is because condition D1 indicates \( \max_i E|Z_i| \leq \max_i(E|Z_i|^s)^{1/s} < A^{1/s} \).
Thus \( \max_i(E|Z_i|)^s < A \). Then
\[
|E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} > L_m]| \leq 2^{s+1} L_m^{-(s-1)}AE|Z_i|. (+)
\]

Under condition D2, we have
\[
|E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} > L_m]| \leq L_m^{-(s-1)}E|\bar{\theta}_1 \bar{\theta}_{i+m}| \leq L_m^{-(s-1)}|\bar{\theta}_1||\bar{\theta}_{i+m}|_2 = L_m^{-(s-1)} \sqrt{\text{Var}(Z_i)}|\bar{\theta}_{i+m}|_2.
\]

Note that
\[
\max_i |\bar{\theta}_i|^2 \leq 2^{2s-1}(\max_i |Z_i|^s_2 + \max(E|Z_i|)^s) \leq 2^{2s} \max_i |Z_i|^s_2 \leq 2^{2s}(A')^{1/2}.
\]

This indicates
\[
|E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} > L_m]| \leq 2^{2s} \sqrt{A'} L_m^{-(s-1)} \sqrt{\text{Var}(Z_i)}. (++).
\]

Based on (+) and (++), by letting \( M = \sum_{m=1}^{\infty} m^{-(1+\delta)} \), we manage to obtain that
\[
2(B/b) \sum_{i=1}^{n} a_{m}^2 \sum_{i=1}^{\infty} |E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} > L_m]| \leq 2^{s+2}(AB/b)M \sum_{i=1}^{n} a_{m}^2 E|Z_i|, \text{ if condition D1 holds};
\]
\[
2(B/b) \sum_{i=1}^{n} a_{m}^2 \sum_{i=1}^{\infty} |E[\bar{\theta}_1 \bar{\theta}_{i+m} | \bar{\theta}_{i+m} > L_m]| \leq 2^{2s+1}(\sqrt{A'}B/b)M \sum_{i=1}^{n} a_{m}^2 \sqrt{\text{Var}(Z_i)}, \text{ if condition D2 holds}.
\]

Above all, we manage to prove that,
under condition D1,
\[ \text{Var}(\sum_{i=1}^{n} a_{in}Z_i) \leq C^* \sum_{i=1}^{n} a_{in}^2 \text{Var}(Z_i) + (2^{s+2}(AB/b)M + 16C_2(4C_1 + 1)(B/b)) \sum_{i=1}^{n} a_{in}^2 E|Z_i|; \]

under condition D2,
\[ \text{Var}(\sum_{i=1}^{n} a_{in}Z_i) \leq C^* \sum_{i=1}^{n} a_{in}^2 \text{Var}(Z_i) + 2^{2s+1}(\sqrt{A'B/b})M + 16C_2(4C_1 + 1)(B/b)) \sum_{i=1}^{n} a_{in}^2 E|Z_i|. \]

**Proof of Proposition 4**

**Step 1** (Chaining)

Let \( \{f_1, \ldots, f_{N_n}\} \) be the minimum \( \sigma_{\mathcal{F}}/2^{2s} \)-net of \( \mathcal{F} \) with respect to infinite norm \( || \cdot ||_\infty \). Then, for any \( f \in \mathcal{F} \), we can construct chaining

\[ f = f - f^S + \sum_{s=1}^{S} f^s - f^{s-1}, \]

where \( f^0 = 0 \) and \( S = \min\{s \in \mathbb{N} : \frac{2s}{2s} \leq \frac{1}{4L_N} \} \). Then, we have

\[ |P_N(fW) - P(fW)| \]
\[ \leq |P_N((f - f^S)W) - P((f - f^S)W)| + \sum_{s=1}^{S} |P_N((f^s - f^{s-1})W) - P((f^s - f^{s-1})W)| \]
\[ \leq 2L_N ||f - f^S||_\infty + \sum_{s=1}^{S} |P_N((f^s - f^{s-1})W) - P((f^s - f^{s-1})W)| \]
\[ \leq \frac{t}{2} + \sum_{s=1}^{S} |P_N((f^s - f^{s-1})W) - P((f^s - f^{s-1})W)|. \]

Let \( \{\eta_s, s = 1, \ldots, S\} \) be a set of positive real numbers such that \( \eta_s > 0 \) and \( \sum_{s=1}^{S} \eta_s \leq 1 \). Then

\[ \mathbb{P} \left( \sup_{f \in \mathcal{F}} |P_N(fW) - P(fW)| \geq t \right) \]
\[ \leq \sum_{s=1}^{S} \mathbb{P} \left( \exists f : |P_N((f^s - f^{s-1})W) - P((f^s - f^{s-1})W)| \geq \frac{\eta_s t}{2} \right) \]
\[ \leq \sum_{s=1}^{S} N_s N_{s-1} \max_{f \in \mathcal{F}} \mathbb{P} \left( |P_N((f^s - f^{s-1})W) - P((f^s - f^{s-1})W)| \geq \frac{\eta_s t}{2} \right). \]

**Step 2** (Application of Bernstein-type Inequality)

Firstly, define

\[ h^s(x, w) = (f^s(x) - f^{s-1}(x))w - E[(f^s(X) - f^{s-1}(X))W], \quad \forall (x, w) \in \mathcal{X} \times [-L_N, L_N]. \]

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Note that, due to stationarity $E_s$ is a constant only associated with "s". In order to apply Theorem 1, we need to specify the following preconditions, $h^s \in C(\mathcal{X} \times [-L_N, L_N])$, $||h^s||_{\infty}$, $||h^s||$, $\text{Var}(h^s)$ and existence of $N_0$ defined in Theorem 1.

For $||h^s|| (s \geq 2)$, since $E_s$ is a constant for any given $s$, due to the triangle inequality of seminorm and Assumption 2, we know $||h^s|| \leq ||(f^s(x) - f^{s-1}(x))w||$. According to the definition of $C$-space (see (3) and (4)), we know $\mathcal{C}(\mathcal{X})$ is a linear space. Hence, for any $s$, $f^s - f^{s-1} \in \mathcal{C}(\mathcal{X})$, since $f^s$, $f^{s-1} \in \mathcal{F} \subset \mathcal{C}(\mathcal{X})$. Then, according to Assumption 3, we can show

$$||f^s(x) - f^{s-1}(x))w|| \leq 2(||f^s - f^{s-1}||_{\infty} + ||f^s - f||_{\infty})L_N \leq 2(2B + ||f^s - f||_{\infty} + ||f^s - f||_{\infty})L_N \leq 2(2B + \frac{5\sigma_F}{2^{2s}})L_N = 4BL_N + \frac{10AL_N}{2^{2s}}.$$  

For $||h^s|| (s = 1)$, only need to note that

$$||h^1|| \leq ||f^1w|| \leq 2(||f^1|| + ||f^1||_{\infty})L_N \leq 2BL_N + 2AL_N < 4BL_N + \frac{10AL_N}{2^{2s}}.$$  

Thus, for any $s = 1, \ldots, S$, $||h^s|| \leq 4BL_N + \frac{10AL_N}{2^{2s}}$.

For $||h^s||_{\infty} (s \geq 2)$,

$$||h^s||_{\infty} \leq 2(||f^s - f^{s-1})w||_{\infty} \leq 2(||f^s - f||_{\infty} + ||f^{s-1} - f||)L_N \leq \frac{10AL_N}{2^{2s}}.$$  

When $s = 1$, based on Assumption 3, we have

$$||h^1||_{\infty} \leq 2||f^1||_{\infty}L_N = 2AL_N < \frac{10AL_N}{2^{2s}}.$$  

Thus, for any $s$, $||h^s||_{\infty} \leq \frac{10AL_N}{2^{2s}}$. A natural consequence is that, for any fixed $s$,

$$||h^s||_{C} = ||h||_{\infty} + ||h|| \leq \frac{20AL_N}{2^{2s}} + 4BL_N < +\infty.$$  

This implies that, for every given $N$, $h^s \in \mathcal{C}(\mathcal{X} \times I)$. As for the existence of $N_0$, recall that its definition actually requires, for some $\omega > 1$, $N_0 = \min\{N \in \mathbb{N}, N^{\omega-1} \geq \frac{||h^s||_{\infty} + ||h^s||}{9^{\omega-1}}\}$. Note

$$\frac{||h^s||_{\infty} + ||h^s||}{||h^s||_{\infty}} = 1 + \frac{4BL_N + \frac{10AL_N}{2^{2s}}}{\frac{10AL_N}{2^{2s}}} = 2 + \frac{2B}{S/2^s} \leq 2 + \frac{2B}{5A/2^{2s}}.$$  

where $S$ is defined in Step 1. According its definition, we have

$$\frac{5A}{2^{2s}} = \frac{5}{4} \frac{\sigma_F}{2^{2(8-1)}} \geq \frac{5}{4} \frac{A}{\sigma_F} \frac{t}{4L_N}.$$
Then
\[
\frac{\|h^s\|_\infty + \|h^s\|}{\|h^s\|_\infty} \leq 2 + \frac{32BLN\sigma_F}{5At}.
\]

By using condition C3, we know \(N_0\) is existed.

For \(\text{Var}(h^s)(s \geq 2)\), since for any \(s\), \(E[h^s] = 0\),
\[
\text{Var}(h^s) = E[h^s]^2 = E[(f^s(X) - f^{s-1}(X))^2Y^2] = E[(f^s(x) - f^{s-1}(x))^2E[Y^2|X = x]] \leq \sigma^2\|f^s - f^{s-1}\|_\infty^2
\]
\[
\leq \sigma^2 \left(\frac{\sigma_F}{2s} + \frac{\sigma_F}{2(s-1)}\right)^2 \leq \sigma^2 \frac{9\sigma^2\sigma_F^2}{2s^2}.
\]

When \(s = 1\), obviously \(\text{Var}(h^1) = E[(f^1(x))^2E[Y^2|X = x]] \leq \sigma^2\frac{\sigma_F^2}{2} < \frac{9\sigma^2\sigma_F^2}{2^2}\). Therefore, for any \(s\), \(\text{Var}(h^s) \leq \frac{9\sigma^2\sigma_F^2}{2s^2}\).

Based on all the argument above, we apply Theorem 1 and get
\[
P \left(\sup_{f \in \mathcal{F}} |P_N(fW) - P(fW)| \geq t\right)
\leq \sum_{s=1}^{N} N_sN_{s-1} \max_{f \in \mathcal{F}} P \left(|P_N((f^s - f^{s-1})W) - P((f^s - f^{s-1})W)| \geq \frac{\eta_s t}{2}\right)
\leq 8 \sum_{s=1}^{N} \exp \left\{ 2\log N - \frac{(b/\omega)^\frac{1}{2} N(\eta_s t/2)^2}{2(\log N)^\frac{1}{2} (b\sigma^2\sigma_F^2 + \frac{110ALN}{2s^2})} \right\}
\leq 8 \sum_{s=1}^{N} \exp \left\{ 2\log N - \frac{(b/\omega)^\frac{1}{2} N(\eta_s t)^2}{80(\log N)^\frac{1}{2} (\frac{\sigma^2\sigma_F^2}{2s^2})} \right\},
\]
where the last inequality is due to condition C1.

**Step 3**

Note the only restrictions to \(\eta_s\) are \(\eta_s > 0\) and \(\sum_{s=1}^{S} \eta_s \leq 1\). For each \(s\), let
\[
2\log N_s \leq \frac{(b/\omega)^\frac{1}{2} N(\eta_s t)^2}{720(\log N)^\frac{1}{2} (\frac{\sigma^2\sigma_F^2}{2s^2})},
\]
which yields
\[
\eta_s \geq \frac{12\sqrt{10\sigma}}{(b/\omega)^\frac{1}{2}} \sqrt{\frac{(\log N)^\frac{1}{2}}{N}} \frac{1}{t} \sqrt{\frac{\sigma_F}{\log N}} \sqrt{\frac{\log N(\sigma_F^2/2s, \mathcal{F}, \|\cdot\|_\infty)}{\eta_s}} := \bar{\eta}_s.
\]
Set $\eta_s = \max\{\frac{\sqrt{s}}{\sigma_F^2}, \bar{\eta}_s\}$. Note on one hand, we have

$$\sum_{s=1}^{S} \frac{\sqrt{s}}{5 \cdot 2^s} = \frac{1}{5} \sum_{s=1}^{+\infty} s \left(\frac{1}{2}\right)^s = \frac{4}{5}.$$ 

On the other hand,

$$\sum_{s=1}^{S} \frac{\eta_s}{(b/\omega)^{2s}} = \frac{12 \sqrt{10} \sigma F}{(b/\omega)^{2s}} \sqrt{\frac{(\log N)^{1/\gamma}}{N}} \sum_{s=1}^{S} \frac{\sigma F}{2s} \sqrt{\log N\left(\frac{\sigma F}{2s}, F, || \cdot ||_\infty\right)}$$

$$= \frac{12 \sqrt{10} \sigma F}{(b/\omega)^{2s}} \sqrt{\frac{(\log N)^{1/\gamma}}{N}} \sum_{s=1}^{S} \frac{\sigma F}{2s} \sqrt{\log N\left(\frac{\sigma F}{2s}, F, || \cdot ||_\infty\right)}.$$ 

Please remark

$$\frac{\sigma F}{2s} = \frac{\sigma F}{2s-1} - \frac{\sigma F}{2s} = \sqrt{\frac{2}{2s-1}} - \sqrt{\frac{2}{2s}}.$$ 

Then

$$\sum_{s=1}^{S} \eta_s \leq \frac{12 \sqrt{10} \sigma F}{(b/\omega)^{2s}} \sqrt{\frac{(\log N)^{1/\gamma}}{N}} \sum_{s=1}^{S} \left(\sqrt{\frac{\sigma F}{2s-1}} - \sqrt{\frac{\sigma F}{2s}}\right) \sqrt{\log N\left(\frac{\sigma F}{2s}, F, || \cdot ||_\infty\right)} := I_s.$$ 

According to the definition of Stieltjes integral and monotonicity of covering number, we can obtain

$$\sum_{s=1}^{S} I_s \leq \int_{\frac{\sigma F}{2s}}^{\frac{\sigma F}{2s-1}} \sqrt{\log N(\tau, F, || \cdot ||_\infty)} d\sqrt{\tau}.$$ 

Note for any $S \in \mathbb{N}$, since $\sigma_F > 0$ and function $\tau \to \sqrt{\tau}$ is continuous and differentiable on interval $[\frac{\sigma F}{2s}, \sigma_F]$. By denoting ”(S) $\int^\tau$” and ”(R) $\int^\tau$” as Stieltjes and Riemann integral respectively, we have

$$(S) \int_{\frac{\sigma F}{2s}}^{\frac{\sigma F}{2s-1}} \sqrt{\log N(\tau, F, || \cdot ||_\infty)} d\sqrt{\tau} = (R) \int_{\frac{\sigma F}{2s}}^{\frac{\sigma F}{2s-1}} \frac{1}{2\sqrt{\tau}} \sqrt{\log N(\tau, F, || \cdot ||_\infty)} d\tau.$$ 

According to change in variable for Riemann integral, i.e. $\sqrt{\tau} = u$,

$$(R) \int_{\frac{\sigma F}{2s}}^{\frac{\sigma F}{2s-1}} \frac{1}{2\sqrt{\tau}} \sqrt{\log N(\tau, F, || \cdot ||_\infty)} d\tau = (R) \int_{\frac{\sigma F}{2s-1/2}}^{\frac{\sigma F}{2s-1/2}} \sqrt{\log N(u^2, F, || \cdot ||_\infty)} du.$$ 

Therefore,

$$\sum_{s=1}^{S} \eta_s \leq \frac{12 \sqrt{10} \sigma F}{(b/\omega)^{2s}} \sqrt{\frac{(\log N)^{1/\gamma}}{N}} \sum_{s=1}^{S} \frac{1}{2s-1/2} \sqrt{\log N\left(u^2, F, || \cdot ||_\infty\right)}.$$
Under condition C4, we get $\sum_{s=1}^{S} \eta_s \leq \frac{4}{5}$. Hence,

$$\sum_{s=1}^{S} \eta_s \leq \sum_{s=1}^{S} \frac{\sqrt{s}}{2^s} + \sum_{s=1}^{S} \bar{\eta}_s \leq \frac{4}{5} + \frac{1}{5} = 1,$$

which coincides with the requirement of $\eta_s$.

**Step 4**

Together with the design of $\eta_s$ in Step 3, we have

$$P \left( \sup_{f \in F} |P_N(fW) - P(fW)| \geq t \right) \leq 8 \sum_{s=1}^{S} \exp \left\{ -\frac{(b/\omega)^{\frac{1}{2}} N t^2}{90 (\log N)^{\frac{1}{4}} (s/5 \cdot 2^s)^{\frac{1}{2}}} \right\} \leq 8 \sum_{s=1}^{+\infty} \exp \left\{ -\frac{(b/\omega)^{\frac{1}{2}} N t^2}{2250 (\log N)^{\frac{1}{4}} \sigma^2 \sigma_F^2} \right\} = 8 \frac{1}{1 - \exp \left\{ -\frac{(b/\omega)^{\frac{1}{2}} N t^2}{2250 (\log N)^{\frac{1}{4}} \sigma^2 \sigma_F^2} \right\}} \exp \left\{ -\frac{(b/\omega)^{\frac{1}{2}} N t^2}{2250 (\log N)^{\frac{1}{4}} \sigma^2 \sigma_F^2} \right\} \leq 88 \exp \left\{ -\frac{(b/\omega)^{\frac{1}{2}} N t^2}{2250 (\log N)^{\frac{1}{4}} \sigma^2 \sigma_F^2} \right\}$$

where the last inequality is due to condition C2.

**Proof of Theorem 3**

Similar to Hang et al. (2018)’s method, we tend to obtain the convergence rate by bounding the stochastic error and deterministic error respectively. i.e.

$$||f_N - f||_\infty \leq \underbrace{||f_N - E[f_N]||_\infty}_{\text{stochastic}} + \underbrace{||E[f_N] - f||_\infty}_{\text{deterministic}}.$$

For stochastic error, according to Lemma 1, by setting $Y_i = 1$, we immediately obtain

$$\sup_{x \in \mathcal{X}} \left| \frac{1}{N} \sum_{i=1}^{N} \left( K_h \left( \frac{X_i - x}{h} \right) - E \left[ K_h \left( \frac{X_i - x}{h} \right) \right] \right) \right| = O_p \left( \sqrt{\frac{(\log N)^{\frac{3}{2} + 1}}{Nh^D}} \right)$$

and

$$\sup_{x \in \mathcal{X}} \left| \frac{1}{N} \sum_{i=1}^{N} \left( K_h \left( \frac{X_i - x}{h} \right) - E \left[ K_h \left( \frac{X_i - x}{h} \right) \right] \right) \right| = O_p \left( \sqrt{\frac{\log N}{Nh^D}} \right).$$
Now we only need to bound the deterministic error. It can be proved that the deterministic error is $O(h^\alpha)$ by following exactly the same calculation procedure as (ii) of Proposition 10 and (iv) of Proposition 11 in Hang et al. (2018), which is omitted here as well. Then, together with the design of $h$, we finish the proof of Theorem 3.

**Proof of Proposition 5**

Define $W_i = \xi_i 1[|\xi_i| \leq L_N]$, $V_i = \xi_i 1[|\xi_i| > L_N]$ and $L_N = (Nh^D)^{\frac{1}{\alpha}}$,

$$Var\left(\frac{1}{Nh^D} \sum_{i=1}^{N} K\left(\frac{X_i - x}{h}\right) \xi_i\right) = Var\left(\frac{1}{Nh^D} \sum_{i=1}^{N} K\left(\frac{X_i - x}{h}\right) W_i\right) + Var\left(\frac{1}{Nh^D} \sum_{i=1}^{N} K\left(\frac{X_i - x}{h}\right) V_i\right)$$

$$+ 2Cov\left(\frac{1}{Nh^D} \sum_{i=1}^{N} K\left(\frac{X_i - x}{h}\right) W_i, \frac{1}{Nh^D} \sum_{i=1}^{N} K\left(\frac{X_i - x}{h}\right) V_i\right)$$

$$:= V_{1N} + V_{2N} + V_{3N}$$

For $V_{2N}$, it suffice to bound $E\left[\frac{1}{Nh^D} \sum_{i=1}^{N} K(X_i - x/h)V_i\right]^2 := E_{2N}$. According to the proof of Theorem 4, by denoting $\frac{1}{Nh^D} K(X_i - x/h) = K_i(x)$ and $E_{2N}' := E_{2N}(Nh^D)$, it's easy to know that, for any fixed $x$,

$$\lim_{N} \mathbb{P}\left(\sum_{i=1}^{N} K_i(x) - f(x) > 1\right) = 0.$$ 

Thus, for arbitrary $\{N_k : k \in \mathbb{N}\} \subset \{N : N \in \mathbb{N}\}$, there exists $\{N_{k_j} : j \in \mathbb{N}\}$ such that

$$\mathbb{P}(\lim_j \{\sum_{i=1}^{N_{k_j}} K_{iN_{k_j}}(x) - f(x) > 1\}) = 0.$$ 

Thus, $\mathbb{P}(\lim_j \{\sum_{i=1}^{N_{k_j}} K_{iN_{k_j}}(x) - f(x) > 1\}) = 0$ and

$$\mathbb{P}(\lim_j \{\sum_{i=1}^{N_{k_j}} K_{iN_{k_j}}(x) - f(x) \leq 1\}) = 1.$$ 

Together with the non-negativity of $\sum_{i=1}^{N} K_i(x)$, this indicates that there exists $\{N_{k_j'}\} \subset \{N_{k_j}\}$ such that

$$0 \leq \sum_{i=1}^{N_{k_j'}} K_{iN_{k_j'}}(x) \leq 1 + f(x),$$

almost surely for every $j'$.

Based on the index set $\{N_{k_j}\}$, by denoting $K_{iN_{k_j'}}(x) = (1 + f(x))^{-1} K_{iN_{k_j}}(x)$, we have $0 \leq \sum_{i=1}^{N} K_{iN_{k_j'}}(x) \leq 1$ almost surely. Hence, by letting $K_0 = 1 - \sum_{i=1}^{N} K_{iN_{k_j}}(x)$, we have

$$E_{2N_{k_j'}} = (N_{k_j'} h^D)E\left[\sum_{i=1}^{N} K_{iN_{k_j'}}(x)V_{iN_{k_j'}} + K_0 \cdot 0\right] \leq (N_{k_j'} h^D) \sum_{i=1}^{N} E[|K_{iN_{k_j'}}(x)(V_{iN_{k_j'}})|^2]$$

$$\leq (N_{k_j'} h^D)(L_{N_{k_j'}})^{-(\rho - 2)} \sum_{i=1}^{N} E[|K_{iN_{k_j'}}(x)|\xi_{iN_{k_j'}}]^\rho \leq (N_{k_j'} h^D)(L_{N_{k_j'}})^{-(\rho - 2)} C \sum_{i=1}^{N} E[|K_{iN_{k_j'}}(x)|]$$(

$$\lesssim C,$$ 

where $C = \max_x E[|\xi_{iN_{k_j'}}|^\rho|x_i = x] < +\infty$.

Until now, we manage to prove that, for any sub-sequence $\{E_{2N_k}\} \subset \{E_{2N}'\}$, there exists a sub-subsequence $\{E_{2N_{k_j'}}\}$ such that $\lim_{j'} E_{2N_{k_j'}} < +\infty$. Thus, $\lim_{N} E_{2N}' < +\infty$ and $E_{2N} \lesssim \frac{1}{Nh^\alpha}$. This
indicates $V_{2N} \lesssim \frac{1}{Nh^D}$.

For $V_{1N}$, note that $V_{1N} = \frac{1}{N^{hD}} Var(\sum_{i=1}^N K(\frac{X_i - x}{h})W_i)$. By letting $h(Z_i) = K(\frac{X_i - x}{h})W_i$ and $s = 0.5\rho$, we have max$_i E|h(Z_i)|^{2s} \lesssim h^D$. We can also obtain that max$_j E|h(Z_i)h(Z_j)| \lesssim h^{2D}$.

According to Assumptions 2, 3 and 5, we obtain that $\|h\| \lesssim (Nh^D)^{\frac{1}{\rho-2}} h^{-1} := K_N$. Meanwhile, for any $\delta > 0$, we have

$$\sum_{m=1}^{+\infty} m^{\frac{s+1}{\rho-2}} C(m) = b \sum_{m=1}^{+\infty} m^{- (\gamma - \frac{1+\delta}{\rho-2})}.$$ 

Note that $\gamma > \frac{\delta}{\rho-2} = \frac{s}{s-1}$. Thus, for any $0 < \delta < (0.5\rho - 1)\gamma - 0.5\rho = (s-1)\gamma - s$, we have $\gamma > \frac{1+\delta}{\rho-2} + 1$, which indicates $\gamma - \frac{1+\delta}{\rho-2} > 1$ holds strictly. Hence, we prove that, given $s = 0.5\rho$, when $\gamma > \frac{s}{s-1} = \frac{\rho}{\rho-2}$, $\sum_{m=1}^{+\infty} m^{\frac{s+1}{\rho-2}} C(m) = \sum_{m=1}^{+\infty} m^{- \gamma - \frac{1+\delta}{\rho-2}} < +\infty$ holds for any $\delta \in (0, (s-1)\gamma - s)$, where $s = 0.5\rho$. Now, by applying Proposition 2 (based on its condition C2), we obtain

$$Var(\sum_{i=1}^N h(Z_i)) \lesssim N \cdot K_N^{\frac{1+\delta}{\rho-2}} h^{2D} + \sum_{i=1}^N (E|h(Z_i)|^2 + E|h(Z_i)|) + h^D \sum_{i=1}^N (E|h(Z_i)|^2)^{1/2} \lesssim NK_N^{\frac{1+\delta}{\rho-2}} h^{2D} + Nh^D, \text{ where } s = 0.5\rho.$$ 

Now we focus on calculating $K_N^{\frac{1+\delta}{\rho-2}} h^{2D}$. Recall that $K_N = \mathcal{O}((Nh^D)^{\frac{1}{\rho-2}} h^{-1})$. According to some simple calculation, provided that $h = \mathcal{O}(N^{-\frac{\gamma}{\rho-2}})$, for any $\delta > 0$,

$$K_N^{\frac{1+\delta}{\rho-2}} h^{2D} = h^{D - \frac{2\alpha(s-1)}{(\rho-2)(\rho+2)} - \frac{\gamma-1}{\rho-2}} := h^{\psi(D, s, \delta, \alpha)}.$$ 

Note $2(s-1) = \rho - 2$. Thus,

$$\psi(D, s, \delta, \alpha) = D - \frac{\alpha + 0.5\rho - 1}{1 + \delta} := \psi(D, \rho, \delta, \alpha).$$ 

Please note $\delta$ could be arbitrary number belonging to $(0, (0.5\rho - 1)\gamma - 0.5\rho)$ and $\psi$ is continuous with respect to $\delta$ on $\mathbb{R}$. Thus, for any given $D, \rho, \alpha, \psi'$ could take value on any point of the following open interval,

$$\left(D - (0.5\rho - (1 - \alpha)), \frac{0.5\rho - (1 - \alpha)}{1 + (0.5\rho - 1)\gamma - 0.5\rho}\right).$$ 

Recall that $\gamma \in (\frac{\rho}{\rho-2}, +\infty)$. Hence, for every $\gamma > \frac{\rho}{\rho-2}$, there exists some $\epsilon_\gamma > 0$ such that $\gamma = \frac{\rho + \epsilon_\gamma}{\rho-2}$, $\epsilon_\gamma = (\rho - 2)\gamma - \rho$. Hence, the interval above can be rewritten as

$$\left(D - (0.5\rho - (1 - \alpha)), \frac{0.5\rho - (1 - \alpha)}{1 + \epsilon_\gamma}\right).$$ 

Now if we can show $D - \frac{0.5\rho - (1 - \alpha)}{1 + \epsilon_\gamma} > 0$, then we manage to show, for the given $D, \gamma, \alpha$, there exists a $\delta \in (0, (0.5\rho - 1)\gamma - 0.5\rho)$ such that $\psi' > 0$. Actually, this is easy to be true. Since $\alpha \in (0, 1]$, we have $2 + \epsilon_\gamma - \alpha > 1 + \epsilon_\gamma > 1$. By denoting $\rho^* := \max\{\rho > 2 : 0.5\rho < 2 + \epsilon_\gamma - \alpha\}$, we know $\rho^*$ exists and
\[ D - 0.5e^{-\{1-\alpha\}} > 0 \] holds for any \( 2 < \rho < \rho^* \). Please note that this is the only requirement for \( \rho \) within the whole proof of Proposition 5. Therefore, if the moment condition \( \max_i \sup_x E[|\xi_i|^\rho |X_i = x] < +\infty \) holds for \( \rho \leq \rho^* \), this arguments automatically holds. If it holds for \( \rho > \rho^* \), it means for every \( \rho' \leq \rho^* \) this moment condition still holds. Thus, we finish the proof based on \( \rho' \) instead of \( \rho \).

Until now, we manage to prove \( K_{s - 1} + \delta N h D = O(1) \). This indicates \( V_1N \sim Nh D \).

Finally, as for \( V_3N \), we can prove it is also of order \( 1Nh D \) by applying Cauchy-Schwartz inequality and the results above. Q.E.D

Proof of Theorem 4

Step 1 (Asymptotic Bias)

The calculation of the asymptotic bias is quite simple and direct. Note that, according condition K2, we have \( h^{-D} \int_{\mathbb{R}^D} K(\frac{z - x_0}{h})dz = 1 \). Thus, due to the stationarity, for any \( x_0 \in \mathcal{X} \),

\[
|E[f_N(x_0)] - f(x_0)| \leq h^{-D} \int_{\mathcal{X}} K(\frac{z - x_0}{h})|f(z) - f(x_0)|dz
\]

\[
\leq \int_{\mathcal{X}(x_0) \cap B_{hD}^\alpha} K(u)|f(x_0 + uh) - f(x_0)|du \leq \int_{B_{hD}^\alpha} K(u)|f(x_0 + uh) - f(x_0)|du.
\]

Here \( \mathcal{X}(x_0) = \{ x \in \mathcal{X} : h^{-1}(x - x_0) \} \) which is a linear transformation of set \( \mathcal{X} \). Since \( f \) is uniformly \( \alpha \)-Holder controllable (see Definition 4), for sufficiently small \( h \leq r_0 \), we have

\[
|f(x_0 + uh) - f(x_0)| \leq c(x)||u||^\alpha h^\alpha \leq c(x)(Mh)^\alpha.
\]

Since \( c(x) \) is almost surely finite, we have

\[
|E[f_N(x_0)] - f(x_0)|^2 \leq h^{2\alpha}, \text{a.s.} - [\lambda].
\]

Step 2 (Asymptotic Variance)

Here we majorly rely on Proposition 5. We take \( \xi_i = 1 \), for any \( i \). Thus, for any \( \rho > 2 \), we have \( \max \sup_x E[|\xi_i|^\rho |X_i = x] < +\infty \) and \( \max_{i,j} \sup_x E[|\xi_i|\xi_j|X_i = x] < +\infty \). Thus, for any \( \gamma > 1 \), we have sufficiently large \( \rho_\gamma \) such that \( \gamma > \frac{\rho_\gamma}{\rho_\gamma - 2} \). Then, we have

\[
\text{Var}(f_N(x_0)) \sim N^{-\frac{2\alpha}{2\alpha + D}}.
\]

Proof of Theorem 5

Obviously,

\[
m(x) = E[Y | X = x] = \frac{\int y f(x, y)dy}{f_X(x)} := \frac{a(x)}{f_X(x)}.
\]
Define \( \hat{a}(x) = \frac{1}{N} \sum_{i=1}^{N} K_h \left( \frac{x - x_i}{h} \right) Y_i \) and \( \hat{f}_X(x) = \frac{1}{N} \sum_{i=1}^{N} K_h \left( \frac{x - x_i}{h} \right) \). We thus have the following result.

\[
||\hat{m}(x) - m(x)||_\infty = \left\| \frac{\hat{a}(x)}{f_X(x)} - m(x) \right\|_\infty
= \left\| \left( \frac{\hat{a}(x)}{f_X(x)} - m(x) \right) \left( \frac{\hat{f}_X(x)}{f_X(x)} + 1 - \frac{\hat{f}_X(x)}{f_X(x)} \right) \right\|_\infty
\leq \left\| \frac{\hat{a}(x) - m(x)}{f_X(x)} \right\|_\infty + \left\| \frac{(\hat{m}(x) - m(x))(\hat{f}_X(x) - f_X(x))}{f_X(x)} \right\|_\infty
:= ||M_1(x)||_\infty + ||M_2(x)||_\infty.
\]

Due to the fact that \( \hat{m}(x) = \frac{\hat{a}(x)}{f_X(x)} \), we have

\[
|M_1(x)| = ||(\hat{m}(x) - m(x))\left( \frac{\hat{f}_X(x)}{f_X(x)} \right)||.
\]

While, based on Theorem 3, we can also obtain

\[
\left| \frac{\hat{f}_X(x)}{f_X(x)} - 1 \right| \leq \frac{|\hat{f}_X(x) - f_X(x)|}{\inf_{x \in X} f_X(x)} = o_p(1).
\]

Therefore, compared with \( M_1(x) \), \( M_2(x) \) is a higher order term and

\[
||\hat{m}(x) - m(x)||_\infty = O_P(||M_1(x)||_\infty).
\]

Furthermore,

\[
||M_1(x)||_\infty \leq ||\hat{a}(x) - E[\hat{a}(x)]||_\infty + ||E[\hat{a}(x)] - m(x)f_X(x)||_\infty
= ||M_{11}(x)||_\infty + ||M_{12}(x)||_\infty.
\]

By using Lemma 1, we immediately have

\[
||M_{11}(x)||_\infty = O_{a.s.} \left( \sqrt{\frac{\log N}{Nh^D}} \right) \text{ and } ||M_{11}(x)||_\infty = O_p \left( \sqrt{\frac{(\log N)^{\frac{D+1}{2}}}{Nh^D}} \right).
\] (39)

As for \( M_{12}(x) \), according to the smoothness of \( m \) and standard calculation of kernel smoothers, we can easily obtain \( \sup_x |M_{12}(x)| = O(h^a) \).

Q.E.D

7.1 Proof of Theorem 6

The proof of Theorem 6 is a direct combination of Proposition 5 and the calculation of term \( M_{12}(x) \) in the proof of Theorem 5. Thus, we omit the detail here.