An Improved Analysis of (Variance-Reduced) Policy Gradient and Natural Policy Gradient Methods

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Abstract

In this paper, we revisit and improve the convergence of policy gradient (PG), natural PG (NPG) methods, and their variance-reduced variants, under general smooth policy parametrizations. More specifically, with the Fisher information matrix of the policy being positive definite: i) we show that a state-of-the-art variance-reduced PG method, which has only been shown to converge to stationary points, converges to the globally optimal value up to some inherent function approximation error due to policy parametrization; ii) we show that NPG enjoys a lower sample complexity; iii) we propose SRVR-NPG, which incorporates variance-reduction into the NPG update. Our improvements follow from an observation that the convergence of (variance-reduced) PG and NPG methods can improve each other: the stationary convergence analysis of PG can be applied to NPG as well, and the global convergence analysis of NPG can help to establish the global convergence of (variance-reduced) PG methods. Our analysis carefully integrates the advantages of these two lines of works. Thanks to this improvement, we have also made variance-reduction for NPG possible, with both global convergence and an efficient finite-sample complexity.

1 Introduction

Policy gradient (PG) methods, or more generally direct policy search methods, have long been recognized as one of the foundations of reinforcement learning (RL) [47]. Specifically, PG methods directly search for the optimal policy parameter that maximizes the long-term return in Markov decision processes (MDPs), following the policy gradient ascent direction [54, 50]. This search direction can be more efficient using a preconditioning matrix, e.g., using the natural PG direction [24]. These methods have achieved tremendous empirical successes recently, especially boosted by the power of (deep) neural networks for policy parametrization [43, 29, 34, 44]. These successes are primarily attributed to the fact that PG methods naturally incorporate function approximation for policy parametrization, in order to handle massive and even continuous state-action spaces.

In practice, the policy gradients are usually estimated via samples using Monte-Carlo rollouts and bootstrapping [54, 5]. Such stochastic PG methods notoriously suffer from very high variances, which not only destabilize but also slow down the convergence. Several conventional approaches have been advocated to reduce the variance of PG methods, e.g., by adding a baseline [50, 55], or by using function approximation for estimating the value function, namely, developing actor-critic algorithms [27, 59, 7]. More recently, motivated by the advances of variance-reduction techniques in stochastic optimization [23, 23, 22, 22], there have been surging interests in developing variance-reduced PG methods [62, 68, 57, 59, 63], which are shown to be faster.

In contrast to the empirical successes of PG methods, their theoretical convergence guarantees, especially non-asymptotic global convergence guarantees, have not been addressed satisfactorily.
Conventional approaches to reduce the high variance in PG.

Variance-Reduction (VR) for PG. with overparametrized neural networks [30], and for regularized MDPs [45]. Very recently, we establish the global convergence up to compatible function approximation errors due to policy approximation. The idea of variance reduction (VR) is first proposed to accelerate stochastic minimization. VR variance-reduced algorithm based on NPG, and establish its global convergence with an efficient stationary-point convergence (which might have arbitrarily bad performance: see (2.2)), in contrast applied to PG methods [22]. We highlight that all these sample complexity results are for first-order methods, and matches the sample complexity of in [59], and [40] applies the idea of SARAH and considers a more general setting with regularization. Finally, heavy-ball type of momentum has also been proposed for acceleration, and analyzed rigorously in [38]. Then, [57] improves the sample complexity of SVRG, and [59] proposes a new SRVR-PG method that uses recursively updated semi-stochastic algorithms such as SVRG [23, 2, 42], SAGA [12], SARAH [37], and Spider [15] achieve acceleration for acceleration, and analyzed rigorously in [38]. Then, [57] improves the sample complexity of SVRG, and [59] proposes a new SRVR-PG method that uses recursively updated semi-stochastic policy gradient, which leads to an improved sample complexity of \( O(\varepsilon^{-2}) \) over previous works.

MOTIVATED BY THESE ADVANCES AND THE QUESTIONS THAT REMAIN TO BE ANSWERED, WE AIM IN THIS PAPER TO IMPROVE THE CONVERGENCE OF PG AND NATURAL PG (NPG) METHODS, AND THEIR VARIANCE-REDUCED VARIANTS, UNDER GENERAL SMOOTH POLICY PARAMETRIZATIONS. OUR CONTRIBUTIONS ARE SUMMARIZED AS FOLLOWS.

**Contributions.** With a focus on the conventional Monte-Carlo-based PG methods, we propose a general framework for analyzing their global convergence. Our contribution is three-fold: first, we establish the global convergence up to compatible function approximation errors due to policy parametrization, for a variance-reduced PG method SRVR-PG [59]; second, we improve the global convergence of NPG methods established in [1], from \( O(\varepsilon^{-2}) \) to \( O(\varepsilon^{-3}) \); third, we propose a new variance-reduced algorithm based on NPG, and establish its global convergence with an efficient sample-complexity. These improvements are based on a framework that integrates the advantages of previous analyses on (variance reduced) PG and NPG, and rely on a (mild) assumption that the Fisher information matrix induced by the policy parametrization is positive definite (see Assumption 2.1). A comparison of previous results and our improvements is laid out in Table 1.

**Related Work.**

**Global Convergence of (Natural) PG.** Recently, there has been a surging research interest in investigating the global convergence of PG and NPG methods, which is beyond the folklore convergence to first-order stationary policies. In the special case with linear dynamics and quadratic reward, [16] shows that PG methods with random search converge to the globally optimal policy with linear rates. In [64], with a simple reward-reshaping, PG methods have been shown to converge to the second-order stationary-point policies. [6] shows that for finite-MDPs and several control tasks, the nonconvex RL objective has no suboptimal local minima. [52] prove that (natural) PG methods converge to the globally optimal value when overparametrized neural networks are used for function approximation. [1] provides a fairly general characterization of global convergence for these methods, and a basic sample complexity result for sample-based NPG updates. It is also worth noting that trust-region policy optimization (TRPO) [43], as a variant of NPG, also enjoys global convergence with overparametrized neural networks [30], and for regularized MDPs [45]. Very recently, for actor-critic algorithms, a series of non-asymptotic convergence results have also been established [60, 61, 56, 21], with global convergence guarantees when natural PG/PPO are used in the actor step.

**Variance-Reduction (VR) for PG.** Conventional approaches to reduce the high variance in PG methods include using (natural) actor-critic algorithms [27, 59, 7], and adding baselines [50, 55]. The idea of variance reduction (VR) is first proposed to accelerate stochastic minimization. VR algorithms such as SVRG [23, 2, 52], SAGA [12], SARAH [37], and Spider [15] achieve acceleration over SGD in both convex and nonconvex settings. SVRG is also accelerated by applying a positive definite preconditioner that captures the curvature of the objective [31]. Inspired by these successes in stochastic optimization, VR is also incorporated into PG methods [62], with empirical validations for acceleration, and analyzed rigorously in [33]. Then, [57] improves the sample complexity of SVRPG, and [59] proposes a new SRVR-PG method that uses recursively updated semi-stochastic policy gradient, which leads to an improved sample complexity of \( O(\varepsilon^{-1.5}) \) over previous works. More recently, [63] proposes a new STORM-PG method, which blends momentum in the update and matches the sample complexity of in [59], and [40] applies the idea of SARAH and considers a more general setting with regularization. Finally, heavy-ball type of momentum has also been applied to PG methods [22]. We highlight that all these sample complexity results are for first-order stationary-point convergence (which might have arbitrarily bad performance: see (2.2)), in contrast to (usually enjoys global optimality up to some compatible function approximation error due to policy parametrization), after a finite number of iterations/samples. These recent prominent guarantees are normally beyond the folklore first-order stationary-point convergence, as expected from a stochastic nonconvex optimization perspective of solving RL with PG methods. Special landscapes of the RL objective, though nonconvex, have enabled the convergence to even globally optimal values. On the other hand, none of the aforementioned variance-reduced PG methods [62, 38, 57, 59, 63] have been shown to enjoy these desired global convergence properties. It remains unclear whether these methods can converge to beyond first-order stationary policies.

1 That is, finding a parameter \( \theta \) such that \( \| \nabla J(\theta) \|^2 \leq \varepsilon \), where \( J \) is the expected return.
to the more desired global convergence guarantees (up to some function approximation errors that can be small) that we are interested in.

\[ NPG \quad \|NPG\| \quad \|TRPO\| \quad \|TRPO\| \]

\[ O(\varepsilon^{-4}) \quad O(T_{TD}\varepsilon^{-2}) \quad O(\varepsilon^{-8}) \quad O(\varepsilon^{-4}) \]

Table 1: Comparison of sample complexities of several methods to reach global optimality up to some compatible function approximation error (see (2.3)). Our results are listed in the second table (see App. A for their derivations). We compare the number of trajectories to reach \( \varepsilon \)-optimality in expectation, up to some inherent error due to the function approximation for policy parametrization (see (2.9)). \( \sigma^2 \) is an upper bound for the variance of gradient estimator (see Assumption 4.1), and \( W \) is an upper bound for the variance of importance weight (see Assumption 4.3).

2 Preliminaries

We first introduce some preliminaries regarding both the MDPs and policy gradient methods.

2.1 Markov Decision Processes

Consider a discounted Markov decision process defined by a tuple \((S, A, P, R, \gamma)\), where \(S\) and \(A\) denote the state and action spaces of the agent, \(P(s' \mid s, a) : S \times A \rightarrow \mathcal{P}(S)\) is the Markov kernel that determines the transition probability from \((s, a)\) to state \(s'\), \(\gamma \in (0, 1)\) is the discount factor, and \(r : S \times A \rightarrow [-R, R]\) is the reward function of \(s\) and \(a\).

At each time \(t\), the agent executes an action \(a_t \in A\) given the current state \(s_t \in S\), following a possibly stochastic policy \(\pi : S \rightarrow \mathcal{P}(A)\), i.e., \(a_t \sim \pi(\cdot \mid s_t)\). Then, given the state-action pair \((s_t, a_t)\), the agent observes a reward \(r_t = r(s_t, a_t)\). Thus, under any policy \(\pi\), one can define the state-action value function \(Q^\pi : S \times A \rightarrow \mathbb{R}\) as

\[
Q^\pi(s, a) := \mathbb{E}_{a_t, \sim \pi(\cdot \mid s_t), s_{t+1} \sim P(\cdot \mid s_t, a_t)} \left( \sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s, a_0 = a \right).
\]

One can also define the state-value function \(V^\pi : S \rightarrow \mathbb{R}\), and the advantage function \(A^\pi : S \times A \rightarrow \mathbb{R}\), under policy \(\pi\), as \(V^\pi(s) := \mathbb{E}_{a_t \sim \pi(\cdot \mid s)}[Q^\pi(s, a)]\) and \(A^\pi(s, a) := Q^\pi(s, a) - V^\pi(s)\), respectively. Suppose that the initial state \(s_0\) is drawn from some distribution \(\rho\). Then, the goal of the agent is to find the optimal policy that maximizes the expected discounted return, namely,

\[
\max_{\pi} J(\pi) := \mathbb{E}_{s_0 \sim \rho}[V^\pi(s_0)]. \tag{2.1}
\]

In practice, both the state and action spaces \(S\) and \(A\) can be very large. Thus, the policy \(\pi\) is usually parametrized as \(\pi_\theta\) for some parameter \(\theta \in \mathbb{R}^d\), using, for example, deep neural networks. As such, the goal of the agent is to maximize \(J(\pi_\theta)\) in the space of the parameter \(\theta\), which naturally induces an optimization problem. Such a problem is in general nonconvex [30, 45], making it challenging to find the globally optimal policy.

For notational convenience, let us denote \(J(\pi_\theta)\) by \(J(\theta)\). Many of the previous works focus on establishing stationary convergence of policy gradient methods. That is, finding \(\theta\) that satisfies

\[
\|\nabla J(\theta)\|^2 \leq \varepsilon. \tag{2.2}
\]

Obviously, such \(\theta\) may not lead to a large \(J(\theta)\). Instead, we are interested in finding \(\theta\) such that

\[
J^* - J(\theta) \leq O(\sqrt{\varepsilon_{\text{bias}}}) + \varepsilon, \tag{2.3}
\]

where \(J^* = \max_{\pi} J(\pi)\), and the \(O(\sqrt{\varepsilon_{\text{bias}}})\) term reflects the inherent error related to the possibly limited expressive power of the policy parametrization \(\pi_\theta\) (see Assumption 4.4 for the definition).
2.2 (Natural) Policy Gradient Methods

To solve the optimization problem (2.1), one standard way is via the policy gradient (PG) method \[50\]. Specifically, let \( \tau_i = \{s_0, a_0, s_1, \cdots \} \) denote the data of a sampled trajectory under policy \( \pi_\theta \). Then, a stochastic PG ascent update is given as

\[
\theta^{k+1} = \theta^k + \eta \cdot \frac{1}{N} \sum_{i=1}^{N} g(\tau_i \mid \theta^k),
\]

where \( \eta > 0 \) is a stepsize, \( N \) is the number of trajectories, and \( g(\tau_i \mid \theta^k) \) estimates \( \nabla J(\theta^k) \) using the trajectory \( \tau_i \). Common unbiased estimators of PG include REINFORCE \[54\], using the policy gradient theorem \[48\], and GPOMDP \[5\]. The commonly used GPOMDP estimator will be given by

\[
g(\tau_i \mid \theta) = \sum_{h=0}^{H-1} \left( \sum_{t=0}^{h} \nabla_\theta \log \pi_\theta(a_t^{i} \mid s_t^{i}) \right) \left( \gamma^h r(s_h^{i}, a_h^{i}) \right),
\]

where \( \nabla_\theta \log \pi_\theta(a_t^{i} \mid s_t^{i}) \) is the score function. If the expectation of this infinite sum exits, then (2.5) becomes an unbiased estimate of the policy gradient of the objective \( J(\theta) \) defined in (2.1). This unbiasedness is established in App. \[B\] for completeness.

In practice, a truncated version of GPOMDP is used to approximate the infinite sum in (2.5), as

\[
g(\tau_i^H \mid \theta) = \sum_{h=0}^{H-1} \left( \sum_{t=0}^{h} \nabla_\theta \log \pi_\theta(a_t^{i} \mid s_t^{i}) \right) \left( \gamma^h r(s_h^{i}, a_h^{i}) \right),
\]

where \( \tau_i^H = \{s_0, a_0, s_1, \cdots, s_{H-1}, a_{H-1}, s_H\} \) is a truncation of the full trajectory \( \tau_i \) of length \( H \). (2.6) is thus a biased stochastic estimate of \( \nabla J(\theta) \), with the bias being negligible for a large enough \( H \). For notational simplicity, we denote the \( H \)-horizon trajectory distribution induced by the initial state distribution \( \rho \) and policy \( \pi_\theta \) as \( p_H^\rho(\cdot \mid \theta) \), that is,

\[
p_H^\rho(\tau^H \mid \theta) = \rho(s_0) \prod_{h=0}^{H-1} \pi_\theta(a_h \mid s_h) P(s_{h+1} \mid a_h, s_h).
\]

Hereafter, unless otherwise stated, we refer to this \( H \)-horizon trajectory simply as trajectory, drawn from \( p_H^\rho(\cdot \mid \theta) \).

As a significant variant of PG, NPG \[24\] also incorporates a preconditioning matrix \( F_\rho(\theta) \), leading to the following update

\[
\theta^{k+1} = \theta^k + \eta \cdot F_\rho^\top(\theta^k) \nabla J(\theta^k),
\]

where \( F_\rho(\theta) = \mathbb{E}_{s \sim d^\pi_\theta} [F_s(\theta)] \), and \( F_s(\theta) = \frac{\nabla_\theta \log \pi_\theta(a \mid s) \nabla_\theta \log \pi_\theta(a \mid s) \top}{\nabla \log \pi_\theta(a \mid s) \top} \) is the Fisher information matrix of \( \pi_\theta(\cdot \mid s) \in \mathcal{P}(A) \), \( F_\rho^\top(\theta^k) \) is the Moore-Penrose pseudoinverse of \( F_\rho(\theta^k) \), and \( d^\pi_\rho(s) \in \mathcal{P}(S) \) is the state visitation measure induced by policy \( \pi_\theta \) and initial distribution \( \rho \), which is defined as

\[
d^\pi_\rho(s) := (1 - \gamma) \mathbb{E}_{s_0 \sim \rho} \sum_{t=0}^{\infty} \gamma^t P(s_t = s \mid s_0, \pi_\theta).
\]

The NPG update (2.7) can also be written as \[24\]

\[
\theta^{k+1} = \theta^k + \eta \cdot w^k, \quad \text{with} \quad w^k \in \arg\min_{w \in \mathbb{R}^d} L_{\nu_\rho^\pi}(w; \theta),
\]

where \( L_{\nu_\rho^\pi}(w; \theta) \) is the compatible function approximation error defined by

\[
L_{\nu_\rho^\pi}(w; \theta) = \mathbb{E}_{(s,a) \sim \nu_\rho^\pi} \left[ (A^\pi(s,a) - (1 - \gamma)w \top \nabla_\theta \log \pi_\theta(a \mid s))^2 \right].
\]

Here, \( \nu_\rho^\pi(s,a) = d^\pi_\rho(s) \pi(a \mid s) \) is the state-action visitation measure induced by \( \pi_\theta \) and initial state distribution \( \rho \), which can also be written as

\[
\nu_\rho^\pi(s,a) := (1 - \gamma) \mathbb{E}_{s_0 \sim \rho} \sum_{t=0}^{\infty} \gamma^t P(s_t = s, a_t = a | s_0, \pi_\theta).
\]
For convenience, we will denote $\nu^{\pi_\theta}$ by $\nu^\pi$ hereafter. In other words, the NPG update direction $w^k$ is given by the minimizer of a stochastic optimization problem. In practice, one obtains an approximate NPG update direction $w^k$ by SGD (see Procedure 1).

Regarding the NPG update (2.8), we make the following standing assumption on the Fisher information matrix induced by $\pi_\theta$ and $\rho$.

**Assumption 2.1.** For all $\theta \in \mathbb{R}^d$, the Fisher information matrix induced by policy $\pi_\theta$ and initial state distribution $\rho$ satisfies

$$F_\rho(\theta) = \mathbb{E}_{(s,a) \sim \nu^\pi} \left[ \nabla_\theta \log \pi_\theta(a \mid s) \nabla_\theta \log \pi_\theta(a \mid s)^\top \right] \succeq \mu_F \cdot I_d$$

for some constant $\mu_F > 0$.

Assumption 2.1 essentially states that $F_\rho(\theta)$ behaves well as a pre-conditioner in the NPG update (2.5). This is a common (and minimal) requirement for the convergence of preconditioned algorithms in both convex and nonconvex settings in the optimization realm, for example, the quasi-Newton algorithms [8,17,19,46], and their stochastic variants [29,35,20,33,31]. In the RL realm, one common example of policy parametrizations that can satisfy this assumption is the Gaussian policy [34,13,38,59], where $\pi_\theta(\cdot \mid s) = \mathcal{N}(\mu_\theta(s), \Sigma)$ with mean parameterized linearly as $\mu_\theta(s) = \phi(s)^\top \theta$, where $\phi(s)$ denotes some feature matrix of proper dimensions, $\theta$ is the coefficient vector, and $\Sigma \succ 0$ is some fixed covariance matrix. In this case, the Fisher information matrix at each $s$ becomes $\phi(s)^\top \Sigma^{-1} \phi(s)^\top$, independent of $\theta$, and is uniformly lower bounded (positive definite sense) if $\phi(s)$ is full-row-rank, namely, the features expanded by $\theta$ are linearly independent, which is a common requirement for linear function approximation settings [51,53,59]. See App. B.2 for more detailed justifications, as well as discussions on more general policy parametrizations.

In the pioneering NPG work [24], $F(\theta)$ is directly assumed to be positive definite. So is in the follow-up works on natural actor-critic algorithms [39,7]. In fact, this way, $F(\theta)$ will define a valid Riemannian metric on the parameter space, which has been used for interpreting the desired convergence properties of natural gradient methods [4,32]. In a recent version of [11], a relevant assumption (specifically, Assumption 6.5, item 3) is made to establish the global convergence of NPG, in which it is assumed that $\lambda_{\min}(F(\theta))$ is not too small compared with the Fisher information matrix induced by a fixed comparator policy, this can be implied by our Assumption 2.1. To sum up, the positive definiteness on the Fisher preconditioning matrix is common and not very restrictive.

In Sec. 4, we shall see that under Assumption 2.1, the stationary convergence of NPG can be analyzed, and NPG enjoys a better sample complexity of $O(\varepsilon^{-3})$ in terms of its global convergence, compared with the existing sample complexity of $O(\varepsilon^{-2})$ in [11]. In addition, interestingly, PG and its variance-reduced version SRVR-PG also enjoy global convergence, although the Fisher information matrix does not appear explicitly in their updates.

### 3 Variance-Reduced Policy Gradient Methods

Recently, [39] proposes an algorithm called Stochastic Recursive Variance Reduced Policy Gradient (SRVR-PG, see Algorithm 2), which applies variance-reduction on PG. It achieves a sample complexity of $O(\varepsilon^{-1.5})$ to find an $\varepsilon$-stationary point, compared with the $O(\varepsilon^{-2})$ sample complexity of stochastic PG. However, it remains unclear whether SRVR-PG converges globally. In this work, we provide an affirmative answer to this question by showing that SRVR-PG has a sample complexity of $O(\varepsilon^{-3})$ to find an $\varepsilon$-optimal policy, up to some compatible function approximation error due to policy parametrization.

We also propose a new algorithm called SRVR-NPG to incorporate variance reduction into NPG, which is described in Algorithm 1. In Sec. 4 we provide a sample complexity for its global convergence, which is comparable to our improved NPG result.

In line 8 of Algorithm 1, $g_w(\tau^H_j | \theta^j_{t-1})$ is a weighted gradient estimator given by

$$g_w(\tau^H_j | \theta^j_{t-1}) = \sum_{h=0}^{H-1} w_{0:h} \nabla_\theta \log \pi_\theta(a_t \mid s_t) \left( \sum_{i=0}^h \nabla_\theta \log \pi_\theta(a_t \mid s_t) \right) \left( \gamma^h r(s_h, a_h) \right), \quad (3.1)$$
Assumption 4.1. The truncated GPOMDP estimator $\nu_w$ where the importance weight factor $w$ is produced by an unbiased estimator of $\nabla J^H (\theta^j_t)$.

This importance sampling makes $\nu_t^j$ an unbiased estimator of $\nabla J^H (\theta^j_t)$.

In lines 4 and 8 of Algorithm 1, $\nu_t^j$ is produced by SRVR-NPG-SGD (see Procedure 2), which applies SGD to solve the following subproblem:

$$w_t^j = \text{argmin}_w \left\{ \mathbb{E}_{(s,a) \sim \nu^j_t} \left[ \left( w^T \nabla_{\theta} \log \pi_{\theta} (a | s) \right)^2 \right] - 2 \langle w, u_t^j \rangle \right\},$$

where $\nu^j_t$ is the state-action visitation measure induced by $\pi_{\theta}$. The exact update direction given by (3.3) is $F_{\rho}^{-1}(\theta^j_t)u_t^j$, and as in NPG, $F_{\rho}^{-1}(\theta^j_t)$ also serves as a preconditioner.

4. Theoretical Results

Before presenting the global convergence results, we first introduce some standard assumptions.

Assumption 4.1. The truncated GPOMDP estimator $g(\tau^H | \theta)$ defined in (2.6) satisfies

$$\text{Var} \left( g(\tau^H | \theta) \right) = \mathbb{E}[||g(\tau^H | \theta) - \mathbb{E}[g(\tau^H | \theta)]||^2] \leq \sigma^2$$

for any $\theta$ and $\tau^H \sim p^H(\cdot | \theta)$.

Assumption 4.2.

1. $\|\nabla_{\theta} \log \pi_{\theta} (a | s)\| \leq G$ for any $\theta$ and $(s,a) \in \mathcal{S} \times \mathcal{A}$.

2. $\|\nabla_{\theta} \log \pi_{\theta_1} (a | s) - \nabla_{\theta} \log \pi_{\theta_2} (a | s)\| \leq M \|\theta_1 - \theta_2\|$ for any $\theta_1, \theta_2$ and $(s,a) \in \mathcal{S} \times \mathcal{A}$.

Assumption 4.3. For the importance weight $w_{0,h}(\tau^H | \theta_1, \theta_2)$ (3.2), there exists $W > 0$ such that

$$\text{Var}(w_{0,h}(\tau^H | \theta_1, \theta_2)) \leq W, \forall \theta_1, \theta_2 \in \mathbb{R}^d, \tau^H \sim p^H(\cdot | \theta_2).$$

Assumptions 4.1, 4.2 and 4.3 are standard in the analysis of PG methods and their variance reduced variants. These can be verified for simple policy parametrizations such as Gaussian policies; see [58, 41] for more justifications.

Following the Assumption 6.5 of [1], we assume that the policy parametrization $\pi_{\theta}$ achieves a good function approximation, as measured by the transferred compatible function approximation error.

1. Following [1], we apply SGD to make a fair comparison. One can also apply the SA algorithm and AC-SA algorithm [18].
Assumption 4.4. For any $\theta \in \mathbb{R}^d$, the transferred compatible function approximation error satisfies
\[
L_{\nu^*}(w_0^*; \theta) = \mathbb{E}_{(s,a) \sim \nu^*} \left[ (A^{\theta}(s,a) - (1 - \gamma)(w_0^*)^T \nabla_{\theta} \log \pi_{\theta}(a | s))^2 \right] \leq \epsilon_{\text{bias}},
\] (4.1)
where $\nu^*(s,a) = d_{\rho}^*(s) \cdot \pi^*(a | s)$ is the state-action distribution induced by an optimal policy $\pi^*$ that maximizes $J(\pi)$, and $w_0^* = \arg\min_{\nu \in \mathcal{B}} L_{\nu^*}(w; \theta)$ is the exact NPG update direction at $\theta$.

$\epsilon_{\text{bias}}$ reflects the error when approximating the advantage function from the score function, it measures the capacity of the parametrization $\pi_{\theta}$. When $\pi_{\theta}$ is the softmax parametrization, we have $\epsilon_{\text{bias}} = 0$ \cite{1}. When $\pi_{\theta}$ is a restricted parametrization, $\epsilon_{\text{bias}}$ is often positive as $\pi_{\theta}$ may not contain all stochastic policies. For rich neural parametrizations, $\epsilon_{\text{bias}}$ is very small \cite{52}.

4.1 A General Framework for Global Convergence

Inspired by the global convergence analysis of NPG in \cite{1}, we present a general framework that relates the global convergence rates of these algorithms to i) their stationary convergence rate on $J(\theta)$, and ii) the difference between their update directions and exact NPG update directions.

Proposition 4.5. Let $\{\theta^k\}_{k=1}^K$ be generated by a general update of the form
\[
\theta^{k+1} = \theta^k + \eta w^k, \quad k = 0, 1, \ldots, K - 1.
\]
Furthermore, let $w^k = F_{\rho}^{-1}(\theta^k)\nabla J(\theta^k)$ be the exact NPG update direction at $\theta^k$. Then, we have
\[
J(\pi^*) - \frac{1}{K} \sum_{k=0}^{K-1} J(\theta^k) \leq \frac{\sqrt{\epsilon_{\text{bias}}}}{1 - \gamma} + \frac{1}{\eta K} \mathbb{E}_{s \sim d^*_\rho} \left[ \text{KL} \left( \pi(\cdot | s) \| \pi_{\theta_{k+1}}(\cdot | s) \right) \right] + \frac{M\eta}{2K} \sum_{k=0}^{K-1} ||w^k||^2 + \frac{G}{K} \sum_{k=0}^{K-1} ||w^k - w^k||,
\] (4.2)
where $\pi^*$ is an optimal policy that maximizes $J(\pi)$.

The detailed proof of this global convergence framework can be found in \cite{1}. To obtain a high level idea, one first starts from the $M$-smoothness of the score function to get
\[
\mathbb{E}_{s \sim d^*_\rho} \left[ \text{KL} \left( \pi^*(\cdot | s) \| \pi_{\theta_{k+1}}(\cdot | s) \right) \right] - \text{KL} \left( \pi^*(\cdot | s) \| \pi_{\theta_{k}}(\cdot | s) \right) \]
\[
\geq \eta \mathbb{E}_{s \sim d^*_\rho} \left[ \nabla \log \pi_{\theta_k}(a | s) \cdot w^k \right] + \eta \mathbb{E}_{s \sim d^*_\rho} \left[ \nabla \log \pi_{\theta_k}(a | s) \cdot (w^k - w^k) \right] - \frac{M\eta^2}{2} ||w^k||^2.
\]
On the other hand, the renowned Performance Difference Lemma \cite{23} tells us that
\[
\mathbb{E}_{s \sim d^*_\rho} \mathbb{E}_{a \sim \pi^*(\cdot | s)} [A^{\theta_{k+1}}(s, a)] = (1 - \gamma) \left( J^* - J(\theta^k) \right).
\]
To connect the advantage term $\mathbb{E}_{s \sim d^*_\rho} \mathbb{E}_{a \sim \pi^*(\cdot | s)} [A^{\theta_{k+1}}(s, a)]$ with the inner product term $\mathbb{E}_{s \sim d^*_\rho} \mathbb{E}_{a \sim \pi^*(\cdot | s)} [\nabla \log \pi_{\theta_k}(a | s) \cdot w^k]$, we invoke Assumption 4.4
\[
\mathbb{E}_{s \sim d^*_\rho} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ (A^{\theta_{k}}(s, a) - (1 - \gamma)(w_0^*)^T \nabla_{\theta} \log \pi_{\theta}(a | s))^2 \right] \leq \epsilon_{\text{bias}}, \quad \text{for any } \theta \in \mathbb{R}^d.
\]
The final result follows from a telescoping sum on $k = 0, 1, \ldots, K - 1$.

Several remarks are in order. The first term on the right-hand side of (4.2) reflects the function approximation error due to the parametrization $\pi_{\theta}$, and the second term is of the form $O\left( \frac{1}{K} \right)$. The third term depends on the stationary convergence. With Assumption 2.1, it can be shown that $\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} ||w^k||^2 \rightarrow 0$ for both NPG and SRVR-NPG. The proof follows from an optimization perspective and is inspired by the stationary convergence analysis of stochastic PG (see App. E).

With Assumption 2.1, we can also show that the last term of (4.2) is small. Take stochastic PG as an example; then, we have $w^k = \frac{1}{N} \sum_{i=1}^{N} g(\tau^{H}(\theta^k))$, and
\[
\frac{1}{K} \sum_{k=0}^{K-1} ||w^k - w^k|| \leq \frac{1}{K} \sum_{k=0}^{K-1} ||w^k - \nabla J(\theta^k)|| + \frac{1}{K} \sum_{k=0}^{K-1} \left( 1 + \frac{1}{K} \right) ||\nabla J(\theta^k)||.
\]

\footnote{The stationary convergence of SRVR-PG has been established in \cite{59}.}
When \( H \) and \( N \) are large enough, \( w^k \) is a low-variance estimator of \( \nabla J^H(\theta^k) \), and \( \nabla J^H(\theta^k) \) is close to \( \nabla J(\theta^k) \), this makes the first term above small. The second term also goes to 0 as \( \theta^k \) approaches stationarity.

### 4.2 Global Convergence Results

By applying Proposition 4.5 on the PG, NPG, SRVR-PG, and SRVR-NPG updates and analyzing their stationary convergence, we obtain their global convergence rates. In the following, we only keep the dependences on \( \sigma^2 \) (the variance of the gradient estimator), \( W \) (variance of importance weight), \( 1/(1-\gamma) \) (the effective horizon) and \( \varepsilon \) (target accuracy). The specific choice of the parameters and sample complexities, as well as the proof, can be found in the appendix.

**Theorem 4.6.** In the stochastic PG (2.4) with the truncated GPOMDP estimator (2.6), take \( \eta = \frac{1}{4L_J} \), \( K = O\left(\frac{1}{(1-\gamma)^2\varepsilon^2}\right) \), \( N = O\left(\frac{\sigma^2}{\varepsilon^2}\right) \), and \( H = O\left(\log\left(\frac{1}{(1-\gamma)\varepsilon}\right)\right) \). Then, we have

\[
J(\pi^*) - \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[J(\theta^k)] \leq \frac{\sqrt{\text{bias}}}{1-\gamma} + \varepsilon.
\]

In total, stochastic PG samples \( O\left(\frac{\sigma^2}{(1-\gamma)^2\varepsilon^2}\right) \) trajectories.

**Remark 4.7.** \( L_J = \frac{MR}{(1-\gamma)^2} \) is the Lipschitz constant of \( \nabla J \), see Lemma B.1 for details.

**Remark 4.8.** Theorem 4.6 improves the result of [1] Thm. 6.11 from (impractical) full gradients to sample-based stochastic gradients.

**Theorem 4.9.** In the NPG update (2.8), let us apply \( O\left(\frac{1}{(1-\gamma)^2\varepsilon^2}\right) \) iterations of SGD as in Procedure 1 to obtain an update direction. In addition, take \( \eta = \frac{\mu^2}{4G^2L_J} \) and \( K = O\left(\frac{1}{(1-\gamma)^2\varepsilon}\right) \). Then,

\[
J^* - \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[J(\theta^k)] \leq \frac{\sqrt{\text{bias}}}{1-\gamma} + \varepsilon.
\]

In total, NPG samples \( O\left(\frac{1}{(1-\gamma)^2\varepsilon^2}\right) \) trajectories.

**Remark 4.10.** Compared with [1] Coro. 6.10, Theorem 4.9 improves the sample complexity of NPG by \( O(\varepsilon^{-1}) \). This is because our stationary convergence analysis on NPG allows for a constant stepsize \( \eta \), while [1] Coro. 6.10 applies a stepsize of \( \eta = O(1/K) \). It is worth noting that the \( O(\sqrt{\text{bias}}) \) term is the same as in [1], and we also apply the average SGD [4] to solve the NPG subproblem (2.8).

**Theorem 4.11.** In SRVR-PG (Algorithm 2), take \( \eta = \frac{1}{8L_J} \), \( S = O\left(\frac{1}{(1-\gamma)^2\varepsilon}\right) \), \( m = O\left(\frac{(1-\gamma)^{0.5}}{\varepsilon}\right) \), \( B = O\left(\frac{W}{(1-\gamma)^2\varepsilon}\right) \), \( N = O\left(\frac{\sigma^2}{\varepsilon}\right) \), and \( H = O\left(\log\left(\frac{1}{(1-\gamma)\varepsilon}\right)\right) \). Then, we have

\[
J^* - \frac{1}{S_M} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[J(\theta_{t+1}^s)] \leq \frac{\sqrt{\text{bias}}}{1-\gamma} + \varepsilon.
\]

In total, SRVR-PG samples \( O\left(\frac{W+\sigma^2}{(1-\gamma)^2\varepsilon^2}\right) \) trajectories.

**Remark 4.12.** Theorem 4.11 establishes the global convergence of SRVR-PG proposed in [59], where only stationary convergence is shown. Also, compared with stochastic PG, SRVR-PG enjoys a better sample complexity thanks to its faster stationary convergence.

**Theorem 4.13.** In SRVR-NPG (Algorithm 1), let us apply \( O\left(\frac{1}{(1-\gamma)^2\varepsilon}\right) \) iterations of SGD as in Procedure 2 to obtain an update direction. In addition, take \( \eta = \frac{\mu^2}{16L_J} \), \( S = O\left(\frac{1}{(1-\gamma)^2\varepsilon^2}\right) \), \( m = O\left(\frac{(1-\gamma)^{0.5}}{\varepsilon}\right) \), \( B = O\left(\frac{W}{(1-\gamma)^2\varepsilon}\right) \), \( N = O\left(\frac{\sigma^2}{\varepsilon}\right) \), and \( H = O\left(\log\left(\frac{1}{(1-\gamma)\varepsilon}\right)\right) \). Then,

\[
J^* - \frac{1}{S_M} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[J(\theta_{t+1}^s)] \leq \frac{\sqrt{\text{bias}}}{1-\gamma} + \varepsilon.
\]

In total, SRVR-NPG samples \( O\left(\frac{W+\sigma^2}{(1-\gamma)^2\varepsilon^2} + \frac{1}{(1-\gamma)^2\varepsilon^2}\right) \) trajectories.
Remark 4.14. Compared with SRVR-PG, our SRVR-NPG has a better dependence on \(W\) and \(\sigma^2\), which could be large in practice (especially \(W\)). The current sample complexity of SRVR-NPG is not better than our (improved) result of NPG since, in our analysis, the advantage of variance reduction is offset by the cost of solving the subproblems.

5 Numerical Experiments

In this section, we compare the numerical performances of stochastic PG, NPG, SRVR-PG, and SRVR-NPG. Specifically, we test on benchmark reinforcement learning environments Cartpole and Mountain Car. Our implementation is based on the implementation of SRVPG\(^1\) and SRVR-PG\(^2\), and can be found in the supplementary material.

For both tasks, we apply a Gaussian policy of the form \(\pi_\theta(a | s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\mu_\theta(s) - a)^2}{2\sigma^2}\right)\) where the mean \(\mu_\theta(s)\) is modeled by a neural network with Tanh as the activation function.

For the Cartpole problem, we apply a neural network of size \(32 \times 1\) and a horizon of \(H = 100\). In addition, each training algorithm uses 5000 trajectories in total. For the Mountain Car problem, we apply a neural network of size \(64 \times 1\) and take \(H = 1000\). 3000 trajectories are allowed for each algorithm. The numerical performance comparison, as well as the settings of algorithm-specific parameters, can be found in Figures 1 and 2. In App. O we provide more implementation details.

6 Concluding Remarks

In this work, we have introduced a framework for analyzing the global convergence of (natural) PG methods and their variance-reduced variants, under the assumption that the Fisher information matrix is positive definite. We have established the sample complexity for the global convergence of stochastic PG and its variance-reduced variant SRVR-PG, and improved the sample complexity of NPG. In addition, we have introduced SRVR-NPG, which incorporates variance-reduction into NPG, and enjoys both global convergence guarantee and an efficient sample complexity. Our improved analysis hinges on exploiting the advantages of previous analyses on (variance reduced) PG and NPG methods, which may be of independent interest, and can be used to design faster variance-reduced NPG methods in the future.

\(^1\)https://github.com/Dam930/rllab
\(^2\)https://github.com/xgfelicia/SRVRPG
**Broader Impact**

The results of this paper improve the performance of policy-gradient methods for reinforcement learning, as well as our understanding to the existing methods. Through reinforcement learning, our study will also benefit several research communities such as machine learning and robotics. We do not believe that the results in this work will cause any ethical issue, or put anyone at a disadvantage in our society.

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**References**

[1] Alekh Agarwal, Sham M Kakade, Jason D Lee, and Gaurav Mahajan. On the theory of policy gradient methods: Optimality, approximation, and distribution shift. *arXiv preprint arXiv:1908.00261v5*, 2019.

[2] Zeyuan Allen-Zhu and Elad Hazan. Variance reduction for faster non-convex optimization. In *International Conference on Machine Learning*, pages 699–707, 2016.

[3] Shun-Ichi Amari. Natural gradient works efficiently in learning. *Neural Computation*, 10(2):251–276, 1998.

[4] Francis Bach and Eric Moulines. Non-strongly-convex smooth stochastic approximation with convergence rate $o(1/n)$. In *Advances in Neural Information Processing Systems*, pages 773–781, 2013.

[5] Jonathan Baxter and Peter L Bartlett. Infinite-horizon policy-gradient estimation. *Journal of Artificial Intelligence Research*, 15:319–350, 2001.

[6] Jalaj Bhandari and Daniel Russo. Global optimality guarantees for policy gradient methods. *arXiv preprint arXiv:1906.01786*, 2019.

[7] Shalabh Bhatnagar, Richard S Sutton, Mohammad Ghavamzadeh, and Mark Lee. Natural actor-critic algorithms. *Automatica*, 45(11):2471–2482, 2009.

[8] Charles George Broyden. The convergence of a class of double-rank minimization algorithms I. general considerations. *IMA Journal of Applied Mathematics*, 6(1):76–90, 1970.

[9] Richard H Byrd, Samantha L Hansen, Jorge Nocedal, and Yoram Singer. A stochastic quasi-newton method for large-scale optimization. *SIAM Journal on Optimization*, 26(2):1008–1031, 2016.

[10] Corinna Cortes, Yishay Mansour, and Mehryar Mohri. Learning bounds for importance weighting. In *Advances in Neural Information Processing Systems*, pages 442–450, 2010.

[11] Anirban DasGupta. The exponential family and statistical applications. In *Probability for Statistics and Machine Learning*, pages 583–612. Springer, 2011.

[12] Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Advances in Neural Information Processing Systems*, pages 1646–1654, 2014.

[13] Yan Duan, Xi Chen, Rein Houthooft, John Schulman, and Pieter Abbeel. Benchmarking deep reinforcement learning for continuous control. In *International Conference on Machine Learning*, pages 1329–1338, 2016.

[14] John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of machine learning research*, 12(7), 2011.
[15] Cong Fang, Chris Junchi Li, Zhouchen Lin, and Tong Zhang. Spider: Near-optimal non-convex optimization via stochastic path-integrated differential estimator. In Advances in Neural Information Processing Systems, pages 689–699, 2018.

[16] Maryam Fazel, Rong Ge, Sham M Kakade, and Mehran Mesbahi. Global convergence of policy gradient methods for the linear quadratic regulator. In International Conference on Machine Learning, 2018.

[17] Roger Fletcher. A new approach to variable metric algorithms. The computer journal, 13(3):317–322, 1970.

[18] Saeed Ghadimi and Guanghui Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization i: A generic algorithmic framework. SIAM Journal on Optimization, 22(4):1469–1492, 2012.

[19] Donald Goldfarb. A family of variable-metric methods derived by variational means. Mathematics of computation, 24(109):23–26, 1970.

[20] Robert Gower, Donald Goldfarb, and Peter Richtárik. Stochastic block bfgs: Squeezing more curvature out of data. In International Conference on Machine Learning, pages 1869–1878, 2016.

[21] Mingyi Hong, Hoi-To Wai, Zhaoran Wang, and Zhuoran Yang. A two-timescale framework for bilevel optimization: Complexity analysis and application to actor-critic. arXiv preprint arXiv:2007.05170, 2020.

[22] Feihu Huang, Shangqian Gao, Jian Pei, and Heng Huang. Momentum-based policy gradient methods. In Thirty-seventh International Conference on Machine Learning (ICML 2020), 2020.

[23] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In Advances in Neural Information Processing Systems, pages 315–323, 2013.

[24] Sham M Kakade. A natural policy gradient. In Advances in Neural Information Processing Systems, pages 1531–1538, 2002.

[25] Sham M Kakade and John Langford. Approximately optimal approximate reinforcement learning. In International Conference on Machine Learning, volume 2, pages 267–274, 2002.

[26] Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.

[27] Vijay R Konda and John N Tsitsiklis. Actor-critic algorithms. In Advances in Neural Information Processing Systems, pages 1008–1014, 2000.

[28] Solomon Kullback. Information Theory and Statistics. Courier Corporation, 1997.

[29] Timothy P Lillicrap, Jonathan J Hunt, Alexander Pritzel, Nicolas Heess, Tom Erez, Yuval Tassa, David Silver, and Daan Wierstra. Continuous control with deep reinforcement learning. arXiv preprint arXiv:1509.02971, 2015.

[30] Boyi Liu, Qi Cai, Zhuoran Yang, and Zhaoran Wang. Neural trust region/proximal policy optimization attains globally optimal policy. In Advances in Neural Information Processing Systems, pages 10564–10575, 2019.

[31] Yanli Liu, Fei Feng, and Wotao Yin. Acceleration of svrg and katyusha x by inexact preconditioning. In International Conference on Machine Learning, pages 4003–4012, 2019.

[32] James Martens. New insights and perspectives on the natural gradient method. arXiv preprint arXiv:1412.1193, 2014.

[33] Francisco S Melo, Sean P Meyn, and M Isabel Ribeiro. An analysis of reinforcement learning with function approximation. In International Conference on Machine Learning, pages 664–671, 2008.

[34] Volodymyr Mnih, Adria Puigdomenech Badia, Mehdi Mirza, Alex Graves, Timothy Lillicrap, Tim Harley, David Silver, and Koray Kavukcuoglu. Asynchronous methods for deep reinforcement learning. In International Conference on Machine Learning, pages 1928–1937, 2016.

[35] Philipp Moritz, Robert Nishihara, and Michael Jordan. A linearly-convergent stochastic l-bfgs algorithm. In Artificial Intelligence and Statistics, pages 249–258, 2016.
[36] Arkadi Nemirovski, Anatoli Juditsky, Guanghui Lan, and Alexander Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4):1574–1609, 2009.

[37] Lam M Nguyen, Jie Liu, Katya Scheinberg, and Martin Takáč. Sarah: A novel method for machine learning problems using stochastic recursive gradient. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 2613–2621. JMLR. org, 2017.

[38] Matteo Papini, Damiano Binaghi, Giuseppe Canonaco, Matteo Pirotta, and Marcello Restelli. Stochastic variance-reduced policy gradient. In *Proceedings of the 35th International Conference on Machine Learning*, pages 4026–4035, 2018.

[39] Jan Peters and Stefan Schaal. Natural actor-critic. *Neurocomputing*, 71(7-9):1180–1190, 2008.

[40] Nhan H Pham, Lam M Nguyen, Dzung T Phan, Phuong Ha Nguyen, Marten van Dijk, and Quoc Tran-Dinh. A hybrid stochastic policy gradient algorithm for reinforcement learning. *arXiv preprint arXiv:2003.00430*, 2020.

[41] Matteo Pirotta, Marcello Restelli, and Luca Bascetta. Adaptive step-size for policy gradient methods. In *Advances in Neural Information Processing Systems*, pages 1394–1402, 2013.

[42] Sashank J Reddi, Ahmed Hefny, Suvrit Sra, Barnabás Póczos, and Alex Smola. Stochastic variance reduction for nonconvex optimization. In *International Conference on Machine Learning*, pages 314–323, 2016.

[43] John Schulman, Sergey Levine, Pieter Abbeel, Michael Jordan, and Philipp Moritz. Trust region policy optimization. In *International Conference on Machine Learning*, pages 1889–1897, 2015.

[44] John Schulman, Filip Wolski, Prafulla Dhariwal, Alec Radford, and Oleg Klimov. Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*, 2017.

[45] Lior Shani, Yonathan Efroni, and Shie Mannor. Adaptive trust region policy optimization: Global convergence and faster rates for regularized MDPs. *arXiv preprint arXiv:1909.02769*, 2019.

[46] David F Shanno. Conditioning of quasi-newton methods for function minimization. *Mathematics of computation*, 24(111):647–656, 1970.

[47] Richard S Sutton and Andrew G Barto. *Reinforcement Learning: An Introduction*. MIT press, 2018.

[48] Richard S Sutton, Andrew G Barto, and Ronald J Williams. Reinforcement learning is direct adaptive optimal control. *IEEE Control Systems Magazine*, 12(2):19–22, 1992.

[49] Richard S Sutton, Hamid Reza Maei, Doina Precup, Shalabh Bhatnagar, David Silver, Csaba Szepesvári, and Eric Wiewiora. Fast gradient-descent methods for temporal-difference learning with linear function approximation. In *International Conference on Machine Learning*, pages 993–1000, 2009.

[50] Richard S Sutton, David A McAllester, Satinder P Singh, and Yishay Mansour. Policy gradient methods for reinforcement learning with function approximation. In *Advances in Neural Information Processing Systems*, pages 1057–1063, 2000.

[51] John N Tsitsiklis and Benjamin Van Roy. Analysis of temporal-difference learning with function approximation. In *Advances in Neural Information Processing Systems*, pages 1075–1081, 1997.

[52] Lingxiao Wang, Qi Cai, Zhuoruan Yang, and Zhaoran Wang. Neural policy gradient methods: Global optimality and rates of convergence. *arXiv preprint arXiv:1909.01150*, 2019.

[53] Xiao Wang, Shiqian Ma, Donald Goldfarb, and Wei Liu. Stochastic quasi-newton methods for nonconvex stochastic optimization. *SIAM Journal on Optimization*, 27(2):927–956, 2017.

[54] Ronald J Williams. Simple statistical gradient-following algorithms for connectionist reinforcement learning. *Machine Learning*, 8(3-4):229–256, 1992.

[55] Cathy Wu, Aravind Rajeswaran, Yan Duan, Vikash Kumar, Alexandre M Bayen, Sham Kakade, Igor Mordatch, and Pieter Abbeel. Variance reduction for policy gradient with action-dependent factorized baselines. *arXiv preprint arXiv:1803.07246*, 2018.

[56] Yue Wu, Weitong Zhang, Pan Xu, and Quanquan Gu. A finite time analysis of two time-scale actor critic methods. *arXiv preprint arXiv:2005.01350*, 2020.
[57] Pan Xu, Felicia Gao, and Quanquan Gu. An improved convergence analysis of stochastic variance-reduced policy gradient. *arXiv preprint arXiv:1905.12615*, 2019.

[58] Pan Xu, Felicia Gao, and Quanquan Gu. Sample efficient policy gradient methods with recursive variance reduction. *arXiv preprint arXiv:1909.08610*, 2019.

[59] Pan Xu, Felicia Gao, and Quanquan Gu. Sample efficient policy gradient methods with recursive variance reduction. In *International Conference on Learning Representations*, 2020.

[60] Tengyu Xu, Zhe Wang, and Yingbin Liang. Improving sample complexity bounds for actor-critic algorithms. *arXiv preprint arXiv:2004.12956*, 2020.

[61] Tengyu Xu, Zhe Wang, and Yingbin Liang. Non-asymptotic convergence analysis of two time-scale (natural) actor-critic algorithms. *arXiv preprint arXiv:2005.03557*, 2020.

[62] Tianbing Xu, Qiang Liu, and Jian Peng. Stochastic variance reduction for policy gradient estimation. *arXiv preprint arXiv:1710.06034*, 2017.

[63] Huizhuo Yuan, Xiangru Lian, Ji Liu, and Yuren Zhou. Stochastic recursive momentum for policy gradient methods. *arXiv preprint arXiv:2003.04302*, 2020.

[64] Kaiqing Zhang, Alec Koppel, Hao Zhu, and Tamer Başar. Global convergence of policy gradient methods to (almost) locally optimal policies. *arXiv preprint arXiv:1906.08383*, 2019.
A Derivation of Previous Complexity Bounds

In this section, we briefly explain how to derive the sample complexities bounds in the first line of Table 1.

In the most recent version of [1], a complexity bound of $O(\varepsilon^{-6})$ can be obtained by taking $N = O(\varepsilon^{-4})$ and $N = O(\varepsilon^{-2})$ in its Corollary 6.2. Note this complexity bound can be improved to $O(\varepsilon^{-4})$ if a uniform upper bound for exact NPG update directions is applied. In this case, one can apply the convergence bound of SGD instead of Projected SGD for the NPG subproblem. In this paper, we establish an upper bound for $\|\nabla J(\theta)\|$ in Lemma B.1. Therefore the exact NPG update direction is also upper bounded thanks to Assumption 2.1.

For [52], the sample complexity bound of $O(T_{TD\varepsilon^{-2}})$ is achieved by its Theorem 4.13. To be specific, one takes $T = O(\varepsilon^{-2})$ and $T_{TD} = O(m)$ number of temporal difference updates at each iteration. Here, $m$ is width of the neural network.

Note that in the proof of its Corollary 4.14, we can choose $m = O(T^4)$ (instead of $O(T^6)$) to have a convergence bound of the form $O(\sqrt{\varepsilon} + \varepsilon$ (instead of $O(\varepsilon)$), which is similar to our $\frac{\sqrt{4 Class} + \varepsilon}{1 - \gamma}$ convergence bound.

For [30], by the Corollary 4.10 therein, one needs to take $K = O(\varepsilon^{-2})$ and $T = O(K^3) = O(\varepsilon^{-6})$, which results in a total sample complexity of $O(\varepsilon^{-8})$.

For [45], its Theorem 5 (item 1) gives a sample complexity of $\sum_{k=1}^{N} M_k = O(\varepsilon^{-4})$, where we have applied $N = O(\varepsilon^{-2})$ and $M_k = O(\varepsilon^{-2})$.

B Helper Lemmas

In this section, we lay out several results that will be useful in later analyses and proofs.

B.1 Properties of PG Estimator

First, for any $H > 1$, we define the $H$-horizon truncated versions of the return $J(\theta)$ as

$$J^H(\theta) := \mathbb{E}_{x_0 \sim \rho} \left( \sum_{t=0}^{H-1} \gamma^t r_t \right), \quad (B.1)$$

where the expectation is taken over the trajectories, starting from the state distribution $\rho$. Now we establish several properties of the GPOMDP policy gradient estimators and the return functions.

**Lemma B.1.** Recall the GPOMDP policy gradient estimate given in (2.5). The following properties hold:

- If the infinite-sum in (2.5) is well defined, $g(\tau_t | \theta)$ in (2.5) is an unbiased estimate of the PG $\nabla J(\theta)$. Similarly, the truncated GPOMDP estimate $g(\tau_t^H | \theta)$ given by (2.6) is an unbiased estimate of the PG $\nabla J^H(\theta)$.

- $J(\theta), J^H(\theta)$ are $L_J$-smooth, where $L_J = \frac{MR}{(1 - \gamma)^2} + \frac{2G^2R}{(1 - \gamma)^2}$. Furthermore, we have $\max \{\|\nabla J(\theta)\|, \|\nabla J^H(\theta)\|\} \leq \frac{GR}{1 - \gamma}.$

- We also have $\|\nabla J^H(\theta) - \nabla J(\theta)\| \leq GR \left( \frac{H+1}{1 - \gamma} + \frac{\gamma}{1 - \gamma^2} \right) \gamma^{H}$.

**Proof.** the unbiasedness of $g(\tau_t | \theta)$ follows directly from [5]. A similar decomposition can also be done for its truncated version $g(\tau_t^H | \theta)$.
The second argument follows directly from the Proposition 4.2 in [58].

For the third argument, one can calculate that
\[
\|g(\tau_i^H | \theta) - g(\tau_i | \theta)\| = \left\| \sum_{h=H}^{\infty} \left( \sum_{i=0}^{h} \nabla_{\theta} \log \pi_{\theta}(a_i^t | s_i^t) \right) (\gamma^h r(s_h^t, a_h^t)) \right\|
\]
\[
\leq \|GR \sum_{h=H}^{\infty} (h + 1) \gamma^h\|
\]
\[
= GR \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right) \gamma^H
\]

This rest of the proof follows from the unbiasedness of \(g(\tau_i | \theta)\) and \(g(\tau_i^H | \theta)\) for estimating \(\nabla J(\theta)\) and \(\nabla J^H(\theta)\), respectively. \(\square\)

### B.2 On the Positive Definiteness of \(F_{\rho}(\theta)\)

Now we remark that the positive definiteness on the Fisher information matrix induced by \(\pi_{\theta}\), as stated in Assumption 2.1, is not restricted. Assumption 2.1 essentially states that \(F(\theta)\) behaves well as a preconditioner in the NPG update (2.8). This is a common (and minimal) requirement for the convergence of preconditioned algorithms in both convex and nonconvex settings in the optimization realm [9][35][20][53][31].

In the RL realm, one common example of policy parametrizations that can satisfy this assumption is the Gaussian policy [54][13][38][59], where \(\pi_{\theta}(\cdot | s) = \mathcal{N}(\mu_{\theta}(s), \Sigma)\) with mean parametrized linearly as \(\mu_{\theta}(s) = \phi(s)^\top \theta\), where \(\phi(s)\) denotes some feature matrix of proper dimensions, \(\theta\) is the coefficient vector, and \(\Sigma > 0\) is some fixed covariance matrix. Suppose the action \(a \in A \subseteq \mathbb{R}^d\) and recall \(\theta \in \mathbb{R}^d\). Thus, \(\phi(s) \in \mathbb{R}^{d \times A}\). In this case, the Fisher information at each \(s\) becomes \(\phi(s)\Sigma^{-1}\phi(s)^\top\), independent of \(\theta\), and is positive definite if \(\phi(s)\) is full-row-rank. For the case \(d < A\), which is usually the case as a lower-dimensional (than \(a\)) parameter \(\theta\) is used, this can be achieved by designing the rows of \(\phi(s)\) to be linearly independent, a common requirement for linear function approximation settings [51][35][49].

For \(\mu_{\theta}(s)\) being nonlinear functions of \(\theta\), e.g., neural networks, the positive definiteness can still be satisfied, if the Jacobian of \(\mu_{\theta}(s)\) at all \(\theta\) uniformly satisfies the aforementioned conditions of \(\phi(s)\) (the Jacobian in the linear case). In addition, beyond Gaussian policies, with the same conditions mentioned above on the feature \(\phi(s)\) or the Jacobian of \(\mu_{\theta}(s)\), Assumption 2.1 also holds more generally for any full-rank exponential family parametrization with mean parametrized by \(\mu_{\theta}(s)\), as the Fisher information matrix, in this case, is also positive definite, in replace of the covariance matrix \(\phi(s)\Sigma^{-1}\phi(s)\) in the Gaussian case [11].

Indeed, the Fisher information matrix is positive definite for any regular statistical model [28]. In the pioneering NPG work [24], \(F(\theta)\) is directly assumed to be positive definite. So is in the follow-up works on natural actor-critic algorithms [30][7]. In fact, this way, \(F_{\rho}(\theta)\) will define a valid Riemannian metric on the parameter space, which has been used for interpreting the desired convergence properties of natural gradient methods [3][32]. In sum, the positive definiteness on the Fisher preconditioning matrix is common and not restrictive.

### C SGD and Sampling Procedures

#### C.1 SGD for Solving the Subproblems of NPG and SRVR-NPG

Similar to the Algorithm 1 of [11], we also apply the averaged SGD algorithm as in [41] to solve the subproblems of NPG and SRVR-NPG.
Then, a stochastic gradient \( \widetilde{\nabla} l(w_t) \) is given by
\[
\widetilde{\nabla} l(w_t) = \left((w_t)^T \nabla \log \pi_{\theta_{t+1}}(a|s)\right) \nabla \log \pi_{\theta_{t+1}}(a|s) - w_t^{j+1}.
\] (C.4)

where \( (s, a) \sim \nu_{\pi_{\theta_{t+1}}} \) is obtained in a similar way as above. It is straightforward to verify that \( \widetilde{\nabla} l(w_t) \) is an unbiased estimate of \( \nabla l(w_t) \).
C.2 Sampling Procedures

Sampling \((s, a) \sim \mu^\pi_{s,a}\) and Obtaining \(\hat{A}^\pi_{s,a}(s, a)\) can be done in a standard way, for example, by apply Algorithm 3 of [1]. Both of them needs to sample \(\frac{1}{1-\gamma}\) state-action pairs in expectation.

D SRVR-PG Algorithm

The Stochastic Recursive Variance-Reduced PG (SRVR-PG) algorithm is introduced in [59], where a recursively updated semi-stochastic gradient \(u_t^{j+1}\) is applied as an update direction.

Algorithm 2 Stochastic Recursive Variance Reduced Policy Gradient (SRVR-PG)

Input: number of epochs \(S\), epoch size \(m\), stepsize \(\eta\), batch size \(N\), minibatch size \(B\), truncation horizon \(H\), initial parameter \(\theta_0^m = \theta_0 \in \mathbb{R}^d\).

1: for \(j \leftarrow 0, \ldots, S - 1\) do
2: \(\theta_0^{j+1} = \theta_0^m;\)
3: Sample \(N\) trajectories \(\{\tau^H_i\}_{i=1}^N\) from \(p^H(\cdot|\theta_0^{j+1});\)
4: \(u_0^{j+1} = \frac{1}{N} \sum_{i=1}^N g(\tau^H_i|\theta_0^{j+1});\)
5: \(\theta_1^{j+1} = \theta_0^{j+1} - \eta u_0^{j+1};\)
6: for \(t \leftarrow 1, \ldots, m - 1\) do
7: Sample \(B\) trajectories \(\{\tau^H_j\}_{j=1}^B\) from \(p^H(\cdot|\theta_t^{j+1});\)
8: \(u_t^{j+1} = u_t^{j+1} + \frac{1}{B} \sum_{j=1}^B \left( g(\tau^H_j|\theta_t^{j+1}) - g_w(\tau^H_j|\theta_t^{j+1}) \right);\)
9: \(\theta_{t+1}^{j+1} = \theta_t^{j+1} - \eta u_t^{j+1};\)
10: end for
11: end for
12: return \(\theta_{out}\) chosen uniformly from \(\{\theta\}_{j=1,\ldots,S;t=0,\ldots,m-1}.\)

Here, the gradient estimators \(g\) and \(g_w\) are defined in (2.6) and (3.1), respectively.

E Stationary Convergence

In this section, we proceed to establish the stationary convergence of stochastic PG, NPG, SRVR-PG, and SRVR-NPG from an optimization perspective.

The stationary convergence of stochastic PG follows from the analysis of SGD. For SRVR-PG, we adapt its analysis in [62].

For NPG and SRVR-NPG, the Fisher information matrix \(F(\theta)\) is applied as a preconditioner on top of PG and SRVR-PG, respectively. Regarding \(F(\theta)\), we know from Assumptions 2.1 and 4.2 that

\[
\mu_F I_d \preceq F(\theta) \preceq C^2 I_d \quad \text{for any} \quad \theta \in \mathbb{R}^d.
\]

Since \(\mu_F > 0\), we know that \(F(\theta)\) defines a nice metric around \(\theta\). Consequently, with the analysis of gradient methods in nonconvex optimization, one can show that NPG (SRVR-NPG) has a similar iteration complexity compared with PG (SRVR-PG), although at each iteration, a subproblem needs to be solved in order to obtain an approximate preconditioned update direction.

We next present the stationary convergence results, and prove them in the subsequent sections. These results are established for \(J(\theta)\), and we will apply the intermediate results in their proof to establish the global convergence on \(J(\theta)\) (up to function approximation errors due to policy parametrizations).

**Theorem E.1.** In the stochastic PG update (2.4), by choosing \(\eta = \frac{1}{4L_J}\), \(K = \frac{32L_J(\hat{J}^H(\theta_0))}{\varepsilon}\), and \(N = \frac{8\sigma^2}{\varepsilon^2}\), we have

\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J^H(\theta_k)\|^2] \leq \varepsilon.
\]
In total, stochastic PG samples $O\left(\frac{\sigma^2}{(1-\gamma)^2\varepsilon^2}\right)$ trajectories.

**Theorem E.2.** In the NPG update (2.8), let us apply $O\left(\frac{1}{(1-\gamma)^2}\right)$ iterations of SGD as in Procedure 1 to obtain an update direction $w^k$. In addition, let us take $\eta = \frac{\mu_F^2}{3\mu_F^2 L_j}$ and $K = \frac{32L_jG^4(J^* - J(\theta_0))}{\mu_F^2 \varepsilon}$. Then, we have

$$\frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla J(\theta^k)\|^2] \leq \varepsilon.$$ 

In total, NPG samples $O\left(\frac{1}{(1-\gamma)^2\varepsilon^2}\right)$ trajectories.

**Corollary E.3.** (Theorem 4.5 of [59]) In SRVR-PG (Algorithm 2), take $\eta = \frac{1}{4L_j}$, $N = \frac{12\sigma^2}{\varepsilon}$, $S = \frac{34M (J^* - J(\theta_0))}{(1-\gamma)^2 \varepsilon^2 \theta_0 s}$, $m = \frac{(1-\gamma)^2}{\varepsilon^2 \theta_0 s}$, and $B = \frac{72\eta G^2(2G^2 + M)(W+1)}{M(1-\gamma)^2}$. Then, we have

$$\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|\nabla J^H(\theta_{i+1})\|^2] \leq \varepsilon.$$ 

In total, SRVR-PG samples $O\left(\frac{W+2\sigma^2}{(1-\gamma)^2\varepsilon^2}\right)$ trajectories.

**Theorem E.4.** In SRVR-NPG (Algorithm 1), take $\eta = \frac{\mu_F^2}{4L_j}$, $S = \frac{24G^2(J^* - J(\theta_0))}{\eta^2 \mu_F^2 s}$, $m = \frac{1}{2\theta_0 s}$, $B = \left(\frac{n}{\mu_F} + \frac{\eta}{4G^2}\right) \frac{24G^2(2G^2 + M)(W+1)}{(1-\gamma)^2} \frac{1}{L_j \varepsilon^2 \theta_0 s}$, and $N = 3\left(\frac{8G^2}{\mu_F} + 2\right) \frac{\varepsilon^2}{\eta}$. In addition, assume that $\varepsilon$ is small enough such that

$$\varepsilon \leq \min\left\{ 3\left(\frac{8G^2}{\mu_F} + 2\right) \left(\frac{GR}{(1-\gamma)^2}\right)^2, 3\left(\frac{8G^2}{4} + \frac{8G^4}{4\mu_F}\right) \frac{2}{\mu_F} \left(\frac{GR}{(1-\gamma)^2}\right)^2, \right. \left(\frac{2}{3n\eta L_j} \left(\mu_F + \frac{\mu_F^2}{4G^2}\right)^4 \right)^2 \right\}.$$ 

Let us also apply $O\left(\frac{1}{(1-\gamma)^2\varepsilon^2}\right)$ iterations of SGD as in Procedure 2 to obtain an update direction $w^k$. Then, in order to have

$$\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|\nabla J^H(\theta_{i+1})\|^2] \leq \varepsilon,$$

SRVR-NPG samples $O\left(\frac{\sigma^2}{(1-\gamma)^2\varepsilon^2} + \frac{W}{(1-\gamma)^2\varepsilon^2} + \frac{1}{(1-\gamma)^2\varepsilon^2}\right)$ trajectories.

**F Proof of Theorem E.1**

**Proof.** Let $g^k = \frac{1}{N} \sum_{i=1}^{N} g(\tau_i^H|\theta^k)$. Then, we have

$$J^H(\theta^{k+1}) \geq J^H(\theta^k) + \langle \nabla J^H(\theta^k), \theta^{k+1} - \theta^k \rangle - \frac{L_j}{2} \|g^{k+1} - g^k\|^2$$

$$= J^H(\theta^k) + \eta \langle \nabla J^H(\theta^k), g^k \rangle - \frac{L_j \eta^2}{2} \|g^k\|^2$$

$$= J^H(\theta^k) + \eta \langle \nabla J^H(\theta^k), g^k - \nabla J^H(\theta^k) + \nabla J^H(\theta^k) \rangle - \frac{L_j \eta^2}{2} \|g^k - \nabla J^H(\theta^k) + \nabla J^H(\theta^k)\|^2$$

$$\geq J^H(\theta^k) + \eta \|\nabla J^H(\theta^k)\|^2 - \frac{\eta}{2} \|g^k - \nabla J^H(\theta^k)\|^2$$

$$- L_j \eta^2 \|g^k - \nabla J^H(\theta^k)\|^2 - L_j \eta^2 \|\nabla J^H(\theta^k)\|^2$$

$$= J^H(\theta^k) + \left(\frac{\eta}{2} - L_j \eta^2\right) \|\nabla J^H(\theta^k)\|^2 - \left(\frac{\eta}{2} + L_j \eta^2\right) \|g^k - \nabla J^H(\theta^k)\|^2.$$
where we have applied Lemma B.1 in the first inequality, and Cauchy-Schwartz in the second inequality.

Taking expectation on both sides and applying Lemma B.1 and Assumption 4.1 yields
\[ E[J^H(\theta^{k+1})] \geq E[J^H(\theta^k)] + \left( \frac{\eta}{2} - L_J \eta^2 \right) E[\|\nabla J^H(\theta^k)\|^2] - \left( \frac{\eta}{2} + L_J \eta^2 \right) \frac{\sigma^2}{N}. \]

Let us further telescope from \( k = 0 \) to \( K - 1 \) to obtain
\[ \frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla J^H(\theta^k)\|^2] \leq \frac{J^H(\theta_0)}{K} + \left( \frac{\eta}{2} + L_J \eta^2 \right) \frac{\sigma^2}{N}. \tag{F.2} \]

Taking \( \eta = \frac{1}{4L_J} \), \( K = \frac{32L_J (J^H(\theta_0))}{\epsilon} \), and \( N = \frac{6\sigma^2}{\epsilon} \) gives
\[ \frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla J^H(\theta^k)\|^2] \leq \epsilon. \]

Finally, by applying \( L_J = \frac{M R}{(1-\gamma)^2} + \frac{2G^2 R}{(1-\gamma)^2} \), we know that PG needs to sample \( KN = \mathcal{O}\left(\frac{\sigma^2}{(1-\gamma)^4 \epsilon^2}\right) \)
trajectories.

G Proof of Theorem E.2

Before proving Theorem E.2, let us first establish the sample complexity of SGD when applied to obtain an approximate NPG update direction \( w^k \).

**Proposition G.1.** In Procedure [1] take \( \alpha = \frac{1}{4\sigma^2} \) and let the objective be
\[ l(w) = \frac{1}{2(1-\gamma)^2} L_{\nu^}\theta(w; \theta^k) = \frac{1}{2} \mathbb{E}_{(s,a) \sim \nu^s_{\theta^k}} \left[ \frac{1}{1-\gamma} A_{\pi_{\theta^k}}(s, a) - w^T \nabla_\theta \log \pi_{\theta^k}(a | s) \right]^2. \]

Let \( w^*_k \) be the minimizer of \( l(w) \). Then, in order to achieve
\[ E[\|w_{out} - w^*_k\|^2] \leq \epsilon', \]
Procedure [1] requires sampling
\[ \frac{4 \left( \frac{2G^2 R + 4\mu_F R}{\mu_F(1-\gamma)^2} \sqrt{d} + \frac{G^2 R}{\mu_F(1-\gamma)^2} \right)^2}{\mu_F \epsilon^2} = \mathcal{O}\left( \frac{1}{(1-\gamma)^4 \epsilon^2} \right) \]
trajectories.

**Proof.** In this proof, we will suppress the superscript \( k \).

Let \( l^* = \min_{w \in \mathbb{R}^d} l(w) \) and \( w_* = \arg\min_{w \in \mathbb{R}^d} l(w) \).

By Theorem 1 of [4], we know that
\[ E[l(w_{out}) - l^*] \leq \frac{2(\xi \sqrt{d} + G E[\|w_*\|])^2}{T}, \]
where \( l^* \) is the minimum of \( l(w) \), and \( \xi \) is defined such that
\[ E[g_* (g^T_*) = \xi^2 \nabla^2_w l(w), \]
where \( g_* \) is a stochastic gradient of \( l(w) \) at \( w_* \).

Following the proof of Corollary 6.10 of [1](arXiv V2 version), we obtain an upper bound of \( \xi \) as follows.

By (C.2) we know that
\[ g_* = \left( w^T_{*} \nabla_\theta \log \pi_\theta(a | s) - \frac{1}{1-\gamma} A_{\pi_{\theta^k}}(s, a) \right) \nabla_\theta \log \pi_\theta(a | s), \]

where \( \pi_\theta(a | s) \) is the target distribution.
where \( s, a \sim \nu^\pi \) and \( a' \sim \nu^\pi(\cdot | s) \). Therefore, we can stipulate that

\[
\mathbb{E} \left[ \left( w^T \nabla_\theta \log \pi_\theta(a|s) - \frac{1}{1 - \gamma} \hat{A}^\pi(s, a) \right)^2 \right] \leq \xi^2
\]

From Lemma B.1 and Assumption 2.1 we have

\[
||w_*|| = ||F_\rho^{-1}(\theta)\nabla J(\theta)|| \leq \frac{GR}{\mu_F(1 - \gamma)^2}.
\]

Therefore, it suffices to have

\[
\mathbb{E} \left[ \left( w^T \nabla_\theta \log \pi_\theta(a|s) - \frac{1}{1 - \gamma} \hat{A}^\pi(s, a) \right)^2 \right] \leq \xi^2
\]

\[
\leq 2 \mathbb{E} \left[ \left( w^T \nabla_\theta \log \pi_\theta(a|s) \right)^2 \right] + \frac{2}{(1 - \gamma)^2} \mathbb{E} \left[ \hat{A}^\pi(s, a) \right] \leq \frac{8R^2}{(1 - \gamma)^2}
\]

We can safely take

\[
\xi = \frac{2G^2R + 4\mu_FR}{\mu_F(1 - \gamma)^2}
\]

This leads to

\[
\mathbb{E}[l(w_{out}) - l(w_*)] \leq \frac{2 \left( \frac{2G^2R + 4\mu_FR}{\mu_F(1 - \gamma)^2} \right)^2}{\sqrt{d} + \frac{G^2R}{\mu_F(1 - \gamma)^2}}.
\]

Since \( l(w) \) is \( \mu_F \)-strongly convex, in order to achieve \( \mathbb{E}[||w_{out} - w^k||^2] \leq \epsilon' \), let us set

\[
\mathbb{E}[l(w_{out}) - l(w_*)] \leq \frac{\mu_F}{2} \epsilon'.
\]

Then, we need

\[
T = 4 \left( \frac{2G^2R + 4\mu_FR}{\mu_F(1 - \gamma)^2} \right)^2 = \mathcal{O} \left( \frac{1}{(1 - \gamma)^2 \epsilon'} \right).
\]

Since each stochastic gradient of SGD has a cost of \( \frac{2}{1 - \gamma} \) (see App. C), this means to sample \( \mathcal{O} \left( \frac{1}{(1 - \gamma)^2 \epsilon'} \right) \) trajectories.

Now, we are ready to prove Theorem E.2

**Proof of Theorem E.2** We apply SGD to obtain a \( w^k \) such that

\[
\mathbb{E}\|w^k - F^{-1}(\theta^k)\nabla J(\theta^k)\|^2 \leq \frac{\mu_F^2 \epsilon}{32\eta^2G^4L^2_f} = \mathcal{O}(\epsilon), \tag{G.1}
\]

By Proposition G.1 we need to sample \( \mathcal{O} \left( \frac{1}{(1 - \gamma)^2} \right) \) trajectories.

From G.1 we have

\[
\mathbb{E}\|\theta^{k+1} - \hat{\theta}^{k+1}\|^2 = \eta^2\mathbb{E}\|w^k - F^{-1}(\theta^k)\nabla J(\theta^k)\|^2 \leq \frac{\mu_F^2 \epsilon}{32G^4L^2_f \left( \frac{2G^4}{\mu_F^2} + 1 \right)}, \tag{G.2}
\]
where $\theta_{k+1}^* = \theta^k + \eta F^{-1}(\theta^k)\nabla J(\theta^k)$.

By Lemma 3.1 and Assumption 4.2 we have

$$J(\theta^{k+1}) \geq J(\theta^k) + \langle \nabla J(\theta^k), \theta^k_{k+1} - \theta^k \rangle + \langle \nabla J(\theta^k), \theta^k_{k+1} - \theta^k \rangle - \frac{L_J}{2} \| \theta^k_{k+1} - \theta^k \|^2$$

$$= J(\theta^k) + \eta \nabla J(\theta^k) F^{-1}(\eta \nabla J(\theta^k))$$

$$+ \langle \nabla J(\theta^k), \theta^k_{k+1} - \theta^k \rangle - \frac{L_J}{2} \| \theta^k_{k+1} - \theta^k \|^2$$

$$\geq J(\theta^k) + \eta \frac{G^2}{G^2} \| \nabla J(\theta^k) \|^2 + \langle \nabla J(\theta^k), \theta^k_{k+1} - \theta^k \rangle - \frac{L_J}{2} \| \theta^k_{k+1} - \theta^k \|^2.$$ 

Therefore,

$$J(\theta^{k+1}) \geq J(\theta^k) + \frac{\eta L_J}{G^2} \| \nabla J(\theta^k) \|^2 - \frac{G^2}{2\eta} \| \theta^k_{k+1} - \theta^k \|^2 - \frac{L_J}{2} \| \theta^k_{k+1} - \theta^k \|^2$$

$$\geq J(\theta^k) + \frac{\eta}{2G^2} \| \nabla J(\theta^k) \|^2 - \left( \frac{G^2}{2\eta} + L_J \right) \| \theta^k_{k+1} - \theta^k \|^2 - \frac{L_J}{2} \| \theta^k_{k+1} - \theta^k \|^2$$

$$\geq J(\theta^k) + \left( \frac{\eta}{2G^2} - \frac{L_J \eta^2}{\mu_F^2} \right) \| \nabla J(\theta^k) \|^2 - \left( \frac{G^2}{2\eta} + L_J \right) \| \theta^k_{k+1} - \theta^k \|^2$$

where we have applied Cauchy-Schwartz in the first and second inequalities, and $\theta^k_{k+1} = \theta^k + \eta F^{-1}(\theta^k)\nabla J(\theta^k)$ in the last step.

Taking full expectation on both sides yields

$$\mathbb{E}[J(\theta^{k+1})] \geq \mathbb{E}[J(\theta^k)] + \left( \frac{\eta}{2G^2} - \frac{L_J \eta^2}{\mu_F^2} \right) \mathbb{E}[\| \nabla J(\theta^k) \|^2] - \left( \frac{G^2}{2\eta} + L_J \right) \mathbb{E}[\| \theta^k_{k+1} - \theta^k \|^2]$$

$$\geq \mathbb{E}[J(\theta^k)] + \left( \frac{\eta}{2G^2} - \frac{L_J \eta^2}{\mu_F^2} \right) \mathbb{E}[\| \nabla J(\theta^k) \|^2] - \left( \frac{G^2}{2\eta} + L_J \right) \frac{\mu_F^2 \varepsilon}{32G^4 L_J^2 \left( \frac{2G^4}{\mu_F^2} + 1 \right)}$$

where we have applied (G.2) in the second inequality.

Telescoping the above inequality from $k = 0$ to $k = K - 1$ gives

$$\frac{J^* - J(\theta_0)}{K} \geq \left( \frac{\eta}{2G^2} - \frac{L_J \eta^2}{\mu_F^2} \right) \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| \nabla J(\theta^k) \|^2] - \left( \frac{G^2}{2\eta} + L_J \right) \frac{\mu_F^2 \varepsilon}{32G^4 L_J^2 \left( \frac{2G^4}{\mu_F^2} + 1 \right)}$$

Finally, by taking $\eta = \frac{\mu_F^2}{4G^2 L_J}$ and $K = \frac{32L_J G^4 (J^* - J(\theta_0))}{\mu_F^2 \varepsilon}$, we arrive at

$$\frac{J^* - J(\theta_0)}{K} \leq \frac{J^* - J(\theta_0)}{K} + \frac{\mu_F^2 \varepsilon}{32G^4 L_J^2 \left( \frac{2G^4}{\mu_F^2} + 1 \right) K} = \varepsilon.$$ 

Recall that at each iteration of NPG, we apply SGD as in Procedure 1 to reach (G.1). By Proposition G.1 we know that in total, NPG requires to sample

$$\frac{32L_J G^4 (J^* - J(\theta_0))}{\mu_F^2 \varepsilon} \cdot \frac{4 \left( \frac{G^2 R + 2\mu_F R}{\mu_F (1-\gamma)^2} \sqrt{\delta} + \frac{G^2 R}{\mu_F (1-\gamma)^2} \right)^2}{\mu_F^2 \varepsilon} = \mathcal{O} \left( \frac{1}{(1 - \gamma)^6 \varepsilon^2} \right)$$

trajectories.
\textbf{H Proof of Theorem E.3}

\textit{Proof.} By Theorem 4.5 of \cite{59}, we know that if $\eta = \frac{1}{3W}$ and

$$B = \frac{3\eta C_{\gamma} m}{L J} = \frac{72\eta G^2 (2G^2 + M)(W + 1)\gamma m}{M(1 - \gamma)^3},$$

then

$$\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E} \|\nabla J^H (\theta_t^{i+1})\|^2 \leq \frac{8(J* - J^H(\theta_0))}{\eta Sm} + \frac{6\sigma^2}{N}.$$ 

Therefore, taking $N = \frac{12\sigma^2}{\varepsilon}$ and $Sm = \frac{64MR(J* - J^H(\theta_0))}{(1 - \gamma)^2 \varepsilon}$ yields

$$\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E} \|\nabla J^H (\theta_t^{i+1})\|^2 \leq \varepsilon.$$ 

Let us take $S = \frac{64MR(J* - J^H(\theta_0))}{(1 - \gamma)^2 \varepsilon \sigma^5}$ and $m = \frac{(1 - \gamma)^5}{\sigma^5}$ Then, the number of trajectories required by SRVR-PG is

$$S(N + mB) = S \frac{12\sigma^2}{\varepsilon} + \frac{64MR(J* - J^H(\theta_0))B}{(1 - \gamma)^2 \varepsilon} = S \frac{12\sigma^2}{\varepsilon} + \frac{64MR(J* - J^H(\theta_0))(72\eta G^2 (2G^2 + M)(W + 1)\gamma m}{(1 - \gamma)^2 \varepsilon} = \mathcal{O} \left( \frac{\sigma^2}{(1 - \gamma)^2 \varepsilon \sigma^5} + \frac{W}{(1 - \gamma)^2 \varepsilon \sigma^5} \right) = \mathcal{O} \left( \frac{W + \sigma^2}{(1 - \gamma)^2 \varepsilon^{1.5}} \right).$$

Therefore, SRVR-PG needs to sample $\mathcal{O} \left( \frac{W + \sigma^2}{(1 - \gamma)^2 \varepsilon^{1.5}} \right)$ trajectories. \hfill $\square$

\textbf{I Proof of Theorem E.4}

In order to prove Theorem E.4, we need the following technical results.

\textbf{Lemma I.1} (Equation B.10 of \cite{57}). We have

$$\mathbb{E} \|\nabla J^H (\theta_t^{i+1}) - u_t^{i+1}\|^2 \leq \frac{C_{\alpha}}{B} \sum_{t=1}^{t} \mathbb{E} \|\theta_t^{i+1} - \theta_t^{i+1}\|^2 + \frac{\sigma^2}{N},$$

where

$$C_{\alpha} = \frac{24RG^2 (2G^2 + M)(W + 1)\gamma}{(1 - \gamma)^5}.$$

\textit{Proof.} This lemma is adapted from the Equation B.10 of \cite{57}, where SRVR-PG is analyzed. It is also true for our SRVR-NPG since the update rule of $u_t^{i+1}$ is the same for both algorithms. \hfill $\square$

\textbf{Proposition 1.2.} In SRVR-NPG, apply SGD as in Procedure 2 to solve the subproblems. Take $\alpha = \frac{1}{\lambda G^2}$ and let the objective be

$$l(w) = \frac{1}{2} \left( \mathbb{E}_{(s, a) \sim \nu_{\theta_t}^{T}} [w^{T} \nabla \log \pi_{\theta_t}^{a+1}(a | s)]^2 - 2\langle \eta w, u_t^{i+1} \rangle \right).$$

Let $w_{t, *}^{i+1} = F^{-1}(\theta_t^{i+1}) \nabla J^H (\theta_t^{i+1})$ be the minimizer of $l(w)$. Assume in addition that

$$\frac{\sigma^2}{N} \leq \left( \frac{GR}{(1 - \gamma)^2} \right)^2, \quad \varepsilon' \leq \frac{2}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2, \quad \frac{C_{\alpha} m \sigma^2}{B} \leq \frac{1}{3} \mu_F^2.$$
Then, in order to achieve

\[ \mathbb{E}[\|w_t^{j+1} - w_{j,*}^{j+1}\|^2] \leq \varepsilon' \]

for each \( s = 0, 1, ..., S - 1 \) and \( t = 0, 1, ..., m - 1 \), Procedure 2 requires sampling

\[
4 \left( \frac{2 \mu_F^2 G R^2 \theta^2 + 2 G R}{\mu_F^2 (1-\gamma)^2} \sqrt{d} + \frac{2 G R^2}{\mu_F (1-\gamma)^2} \right)^2 \leq \mathcal{O} \left( \frac{1}{(1-\gamma)^2 \varepsilon'} \right)
\]

trajectories.

**Proof of Proposition I.2** Recall that we are applying SGD as in Procedure 2 to solve the SRVR-NPG subproblem (3.3).

Let us focus on \( t = 0 \), where \( u_0^{j+1} = \frac{1}{N} \sum_{i=1}^{N} g(\pi_i \theta_0^{j+1}) \). As a result, \( \mathbb{E}[u_0^{j+1}] = \nabla J^H(\theta_0^{j+1}) \) and

\[
\text{Var}(u_0^{j+1}) \leq \frac{\sigma^2}{N}. \quad \text{Therefore,}
\]

\[
\mathbb{E}[\|u_0^{j+1}\|^2] = \mathbb{E}[\|F^{-1}(\theta_0^{j+1}) u_0^{j+1}\|^2] 
\leq \frac{1}{\mu_F^2} \mathbb{E}[\|u_0^{j+1}\|] \leq \frac{1}{\mu_F^2} \left( \frac{G R}{(1-\gamma)^2} \right)^2 + \frac{1}{\mu_F} \frac{\sigma^2}{N} \leq \frac{4}{\mu_F} \left( \frac{G R}{(1-\gamma)^2} \right)^2.
\]

Recall from (C.4) that a stochastic gradient \( \nabla l(w_t) \) is given by

\[
\nabla l(w) = \left( w^T \nabla \log \pi_{\theta_{j+1}}(a|s) \right) \nabla \log \pi_{\theta_{j+1}}(a|s) - u_0^{j+1}.
\]

Here, \( s, a \sim \nu_{\pi_{\theta_{j+1}}} \).

Therefore, By Theorem 1 of [4], we know that in order to reach \( \mathbb{E}[\|w_0^{j+1} - (w_0^{j+1})_{\pi_{\theta_{j+1}}}\|^2] \leq \varepsilon' \), we need

\[
\mathbb{E}[l(w_{\text{out}}) - l^*] \leq \frac{2(\xi \sqrt{d} + G \|w_0^{j+1}\|)}{T} \leq \frac{\mu_F}{2} \varepsilon',
\]

where \( l^* \) is the minimum of \( l(w) \), and \( \xi \) is defined such that the stochastic gradient \( g_* \) at the solution \( w_0^{j+1} \) satisfies

\[
\mathbb{E}[g_*(g_*)^T] \leq \xi^2 \nabla w l(w).
\]

Similar as Proposition G.1, we know that \( \xi \) can be chosen by

\[
\xi^2 = \left( \frac{2 \mu_F G R \theta^2 + 2 G R}{\mu_F (1-\gamma)^2} \right)^2.
\]

As a result, the number of iterations, \( T \), should be

\[
T = 4 \left( \frac{2 \mu_F^2 G R^2 \theta^2 + 2 G R^2}{\mu_F^2 (1-\gamma)^2} \sqrt{d} + \frac{2 G R^2}{\mu_F (1-\gamma)^2} \right)^2 \leq \mathcal{O} \left( \frac{1}{(1-\gamma)^2 \varepsilon'} \right).
\]

Since each stochastic gradient of \( l(w) \) only needs to sample a state-action pair, this is equivalent to sampling \( \mathcal{O} \left( \frac{1}{(1-\gamma)^2 \varepsilon'} \right) \) trajectories.
Now, let us turn to \( t \geq 1 \). \( u_{t+1}^{j+1} \) is an unbiased estimate of \( \nabla J(\theta_t^{j+1}) \), and its variance is bounded as in Lemma [1]. Therefore,

\[
E[\|u_{t+1}^{j+1}\|] = E[\|F^{-1}(\theta_t^{j+1})u_{t+1}^{j+1}\|] \leq \frac{1}{\mu_F} E[\|u_{t+1}^{j+1}\|]
\]

\[
= \frac{1}{\mu_F} E[\|\nabla J^H(\theta_t^{j+1})\|] + \frac{1}{\mu_F} E[\|u_{t+1}^{j+1} - \nabla J^H(\theta_t^{j+1})\|]
\]

\[
\leq \frac{1}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{1}{\mu_F} \left( \frac{C_\gamma}{B} \sum_{i=1}^{t} E[\|\theta_t^{j+1} - \theta_{t-1}^{j+1}\|] + \frac{\sigma^2}{N} \right)
\]

\[
= \frac{1}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{1}{\mu_F} \left( \frac{C_\gamma}{B} \sum_{i=1}^{t} \eta^2 E[\|w_{t-1}^{j+1}\|] + \frac{\sigma^2}{N} \right)
\]

\[
\leq \frac{1}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{1}{\mu_F} \left( \frac{C_\gamma}{B} \sum_{i=1}^{t} 2\eta^2 \left( E[\|w_{t-1}^{j+1}\|] + E[\|w_{t-1}^{j+1} - w_{t-1}^{j+1}\|] + \frac{\sigma^2}{N} \right) \right)
\]

(1.1)

Now, we are ready to prove the desired results by induction.

Assume that for all \( t' < t \), we have

\[
E[\|u_{t'}^{j+1}\|] \leq \frac{4}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2,
\]

and we have applied

\[
T = \frac{4 \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{2GR}{\mu_F} \sqrt{d} + \frac{2G^2}{\mu_F (1 - \gamma)^2}}{\mu_F \varepsilon'} = O \left( \frac{1}{(1 - \gamma)^4 \varepsilon'} \right)
\]

iterations of SGD as in Procedure [2].

Similar to the case of \( t = 0 \), we know that this yields

\[
E[\|u_{t'}^{j+1} - u_{t'}^{j+1}\|] \leq \varepsilon'.
\]

Then, by (1.1), we have

\[
E[\|u_{t+1}^{j+1}\|] \leq \frac{1}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{1}{\mu_F} \left( \frac{C_\gamma}{B} \sum_{i=1}^{t} 2\eta^2 \left( E[\|w_{t-1}^{j+1}\|] + E[\|w_{t-1}^{j+1} - w_{t-1}^{j+1}\|] + \frac{\sigma^2}{N} \right) \right)
\]

\[
\leq \frac{1}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{1}{\mu_F} \left( \frac{C_\gamma}{B} \sum_{i=1}^{t} \eta^2 \left( \frac{4}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \varepsilon' \right) + \frac{\sigma^2}{N} \right)
\]

\[
= \frac{1}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{1}{\mu_F} \left( \frac{C_\gamma}{B} \sum_{i=1}^{t} \eta^2 \frac{4}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2 \right)
\]

\[
+ \frac{1}{\mu_F} \left( \frac{C_\gamma}{B} \sum_{i=1}^{t} \eta^2 \varepsilon' \right) + \frac{1}{\mu_F} \frac{\sigma^2}{N}
\]

\[
\leq \frac{4}{\mu_F} \left( \frac{GR}{(1 - \gamma)^2} \right)^2.
\]

As a result, we can apply

\[
T = \frac{4 \left( \frac{GR}{(1 - \gamma)^2} \right)^2 + \frac{2GR}{\mu_F} \sqrt{d} + \frac{2G^2}{\mu_F (1 - \gamma)^2}}{\mu_F \varepsilon'} = O \left( \frac{1}{(1 - \gamma)^4 \varepsilon'} \right)
\]

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iterations of SGD as in Procedure 2 so that
\[ \mathbb{E}[\|w_t^{j+1} - w_{t,*,t}^{j+1}\|_2^2] \leq \varepsilon'. \]

Since each stochastic gradient of \( l(w) \) has a cost of \( \frac{1}{1-\gamma} \) (see App. C), this is equivalent to sample
\[
4 \left( \frac{2 \beta_{GR} \frac{1}{1-\gamma} G^2 + \eta GR}{\sqrt{\mu_F}} \right)^2 \frac{1}{(1-\gamma)^2} \mu_F \varepsilon' = O \left( \frac{1}{(1-\gamma)^4 \varepsilon'} \right)
\]
trajectories.

We are now ready to prove Theorem E.4.

**Proof of Theorem E.4.** Line 9 of Algorithm 1 reads
\[ w_t^{j+1} \approx \arg\min_w \left\{ \mathbb{E}_{(s,a) \sim \nu_t^{j+1}}[w^T \nabla \log \pi_{\theta_t^{j+1}}(s,a)] - 2\eta w, u_t^{j+1} \right\}. \]

And we want to apply SGD as in Procedure 2 to obtain a \( w_t^{j+1} \) that satisfies
\[ \mathbb{E}[\|w_t^{j+1} - F^{-1}(\theta_t^{j+1}) u_t^{j+1}\|_2^2] \leq \frac{\varepsilon}{3 \left( \frac{8 G^2 \mu_F}{4} + \frac{8 G^4}{4} \right)}. \] (I.2)

Recall that the parameters \( S, m, B \) and \( N \) are chosen as
\[
S = \frac{24 G^2 (J^* - J^H(\theta_0))}{\eta \epsilon^{0.5}},
\]
\[ m = \frac{1}{\epsilon^{0.5}}, \]
\[ B = \left( \frac{\eta}{\mu_F} + \frac{\eta}{4 G^2} \right) \frac{4 C \gamma m}{L_J \epsilon^{0.25}}, \]
\[ N = 3 \left( \frac{8 G^2}{\mu_F} + 2 \right) \frac{\sigma^2}{\varepsilon}. \]

Since
\[ \varepsilon \leq \min \left\{ 3 \left( \frac{8 G^2}{\mu_F} + 2 \right) \left( \frac{GR}{(1-\gamma)^2} \right)^2, \frac{8 G^2}{4} + \frac{8 G^4}{4 \mu_F}, \frac{2}{3} \left( \frac{GR}{(1-\gamma)^2} \right)^2, \frac{2}{3} \eta L_J \left( \frac{\mu_F}{4 G^2} \right)^4 \right\}, \]
the requirements of Proposition I.2 are satisfied:
\[ \frac{\sigma^2}{N} = \frac{\varepsilon}{3 \left( \frac{8 G^2}{\mu_F} + 2 \right) \left( \frac{GR}{(1-\gamma)^2} \right)^2} \leq \frac{GR}{(1-\gamma)^2}, \]
\[ \varepsilon' = \frac{\varepsilon}{3 \left( \frac{8 G^2}{4 \mu_F} + \frac{8 G^4}{4} \right) \left( \frac{GR}{(1-\gamma)^2} \right)^2} \leq \frac{2}{\mu_F} \left( \frac{GR}{(1-\gamma)^2} \right)^2, \]
\[ \frac{C \gamma m}{B} \frac{2 \eta^2 \epsilon^{0.25}}{L_J} \leq \frac{1}{3} \mu_F. \]

By applying Proposition I.2 we know that in order to have (I.2), one needs to sample
\[
4 \left( \frac{2 \beta_{GR} \frac{1}{1-\gamma} G^2 + \sqrt{\eta G^2 (1-\gamma)}}{\sqrt{\mu_F}} \sqrt{d} + \frac{\sqrt{2 G^2 R}}{\mu_F (1-\gamma)^2} \right)^2 \mu_F \varepsilon' = O \left( \frac{1}{(1-\gamma)^4 \varepsilon'} \right)
\]
where $\theta_{t+1,*} = \theta_t^1 + \eta F^{-1}(\theta_t^1)u_t^1$. 

On the other hand, we have

\[
J^H(\theta_{t+1}^j) \geq J^H(\theta_t^1) + \langle \nabla J^H(\theta_t^1), \theta_{t+1}^j - \theta_t^1 \rangle - \frac{L_J}{2} \| \theta_{t+1}^j - \theta_t^1 \|^2 \\
= J^H(\theta_t^1) + \langle \nabla J^H(\theta_t^1) - u_t^1, \theta_{t+1}^j - \theta_t^1 \rangle + \langle u_t^1, \theta_{t+1}^j - \theta_t^1 \rangle - \frac{L_J}{2} \| \theta_{t+1}^j - \theta_t^1 \|^2 \\
\geq J^H(\theta_t^1) - \frac{\eta}{\mu_F} \| \nabla J^H(\theta_t^1) - u_t^1 \|^2 - \frac{\mu_F}{4\eta} \| \theta_{t+1}^j - \theta_t^1 \|^2 + \langle u_t^1, \theta_{t+1}^j - \theta_t^1 \rangle - \frac{L_J}{2} \| \theta_{t+1}^j - \theta_t^1 \|^2
\]

where we have applied Lemma B.1 in the first inequality, and Cauchy-Schwartz in the second one. Rearranging gives

\[
J^H(\theta_{t+1}^j) \geq J^H(\theta_t^1) - \frac{\eta}{\mu_F} \| \nabla J^H(\theta_t^1) - u_t^1 \|^2 - \frac{\mu_F}{4\eta} \| \theta_{t+1}^j - \theta_t^1 \|^2 \\
\frac{1}{2} \langle u_{t+1}^j, \theta_{t+1,*}^j - \theta_t^1 \rangle + \frac{1}{2} \langle u_t^1, \theta_{t+1}^j - \theta_t^1 \rangle + \frac{1}{2} \langle \frac{1}{\eta} F(\theta_{t+1}^j)(\theta_{t+1,*}^j - \theta_t^1), \theta_{t+1}^j - \theta_t^1 \rangle \\
- \frac{L_J}{2} \| \theta_{t+1}^j - \theta_t^1 \|^2 = J^H(\theta_t^1) - \frac{\eta}{\mu_F} \| \nabla J^H(\theta_t^1) - u_t^1 \|^2 \\
- \frac{\mu_F}{4\eta} \| \theta_{t+1}^j - \theta_t^1 \|^2 + \frac{1}{2} \langle u_{t+1}^j, \eta F^{-1}(\theta_t^1)u_t^1 \rangle + \frac{1}{2} \langle u_t^1, \theta_{t+1}^j - \theta_t^1 \rangle + \frac{1}{2} \langle \frac{1}{\eta} F(\theta_{t+1}^j)(\theta_{t+1,*}^j - \theta_t^1), \theta_{t+1}^j - \theta_t^1 \rangle \\
- \frac{L_J}{2} \| \theta_{t+1}^j - \theta_t^1 \|^2.
\]
Applying $F^{-1}(\theta) \geq \frac{1}{c_F} I$ on the first inner product, and Cauchy-Schwartz on the second inner product term leads to
\[
J^H(\theta_{t+1}^{i+1}) \geq J^H(\theta_t^{i+1}) - \frac{\eta}{\mu_F} \| \nabla J^H(\theta_t^{i+1}) - u_t^{i+1} \|^2 \\
- \frac{\mu_F}{4\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 + \frac{\eta}{2G^2} \| u_t^{i+1} \|^2 \\
- \frac{\eta}{4G^2} \| u_t^{i+1} \|^2 - \frac{G^2}{4\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 \\
+ \frac{1}{\eta} \left( F(\theta_t^{i+1})(\theta_t^{i+1} - \theta_t^{i+1}), \theta_t^{i+1} - \theta_t^{i+1} \right) - \frac{L_F}{2} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 \\
= J^H(\theta_t^{i+1}) - \frac{\eta}{\mu_F} \| \nabla J^H(\theta_t^{i+1}) - u_t^{i+1} \|^2 \\
- \left( \frac{\mu_F}{4\eta} + \frac{L_F}{2} \right) \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 + \frac{\eta}{4G^2} \| u_t^{i+1} \|^2 \\
+ \frac{1}{\eta} \left( F(\theta_t^{i+1})(\theta_t^{i+1} - \theta_t^{i+1}), \theta_t^{i+1} - \theta_t^{i+1} \right) \\
+ \frac{1}{\eta} \left( F(\theta_t^{i+1})(\theta_t^{i+1} - \theta_t^{i+1}), \theta_t^{i+1} - \theta_t^{i+1} \right) \\
- \frac{G^2}{4\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2.
\]

Applying $\| \nabla J^H(\theta_t^{i+1}) \|^2 \leq 2 \| \nabla J^H(\theta_t^{i+1}) - u_t^{i+1} \|^2 + 2 \| u_t^{i+1} \|^2$ and $F(\theta_t^{i+1}) \geq \mu_F I_d$ yields
\[
J^H(\theta_t^{i+1}) \geq J^H(\theta_t^{i+1}) - \left( \frac{\eta}{\mu_F} + \frac{\eta}{4G^2} \right) \| \nabla J^H(\theta_t^{i+1}) - u_t^{i+1} \|^2 \\
- \left( \frac{\mu_F}{4\eta} + \frac{L_F}{2} \right) \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 + \frac{\eta}{8G^2} \| \nabla J^H(\theta_t^{i+1}) \|^2 \\
+ \frac{\mu_F}{2\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 + \frac{1}{\eta} \left( F(\theta_t^{i+1})(\theta_t^{i+1} - \theta_t^{i+1}), \theta_t^{i+1} - \theta_t^{i+1} \right) \\
- \frac{G^2}{4\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 \\
\geq J^H(\theta_t^{i+1}) - \left( \frac{\eta}{\mu_F} + \frac{\eta}{4G^2} \right) \| \nabla J^H(\theta_t^{i+1}) - u_t^{i+1} \|^2 \\
- \left( \frac{\mu_F}{4\eta} + \frac{L_F}{2} \right) \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 + \frac{\eta}{8G^2} \| \nabla J^H(\theta_t^{i+1}) \|^2 \\
+ \frac{\mu_F}{2\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 - \frac{\mu_F^2}{2\mu_F \eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 - \frac{\mu_F}{8\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2 \\
- \frac{G^2}{4\eta} \| \theta_t^{i+1} - \theta_t^{i+1} \|^2.
\]

(I.4)
From Lemma [1], we further know that

\[
E[J^H(\theta^i_{t+1})] \geq E[J^H(\theta^i_{t+1})] - \left(\frac{\eta}{\mu_F} + \frac{\eta}{4G^2}\right) \left(\frac{C_\gamma}{B}\sum_{t=0}^{m-1} E[\|\theta^i_{t+1} - \theta^i_{t+1}\|^2 + \sigma^2] \right)
\]

\[
+ \left(\frac{\mu_F}{8\eta} - \frac{L_J}{2}\right) E[\|\theta^i_{t+1} - \theta^i_{t+1}\|^2]
\]

\[
+ \frac{\eta}{8G^2} E[\|\nabla J^H(\theta^i_{t+1})\|^2] - (\frac{\mu_F}{4\eta} + \frac{G^2}{4\eta^2}) E[\|\theta^i_{t+1,*} - \theta^i_{t+1}\|^2].
\]

Telescoping for \(s = 0, 1, ..., S - 1\) and \(t = 0, 1, ..., m - 1\) and dividing by \(Sm\) gives

\[
\frac{8G^2}{\eta} \left(\frac{\mu_F}{8\eta} - \frac{L_J}{2} - \left(\frac{\eta}{\mu_F} + \frac{\eta}{4G^2}\right) \frac{C_\gamma m}{B}\right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|\theta^i_{t+1} - \theta^i_{t+1}\|^2]
\]

\[
+ \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|\nabla J^H(\theta^i_{t+1})\|^2]
\]

\[
\leq \frac{8G^2}{\eta} \frac{J^* - J^H(\theta_0)}{Sm} + \left(\frac{8G^2}{\mu_F} + 2\right) \frac{\sigma^2}{N} + \left(\frac{8G^2}{4\eta^2} + \frac{8G^4}{4\eta^4}\right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|\theta^i_{t+1,*} - \theta^i_{t+1}\|^2].
\]

(1.5)

Let us first show that the first term on the left hand side of (1.5) is non-negative. In fact, from \(\eta = \frac{\mu_F}{8L_J}\) and \(B = \left(\frac{\eta}{\mu_F} + \frac{\eta}{4G^2}\right) \frac{4C_\gamma m}{L_J e^{0.5}}\) we have

\[
\frac{8G^2}{\eta} \left(\frac{\mu_F}{8\eta} - \frac{L_J}{2} - \left(\frac{\eta}{\mu_F} + \frac{\eta}{4G^2}\right) \frac{C_\gamma m}{B}\right) \geq \frac{16G^2 L_J^2}{\mu_F} > 0.
\]

Therefore, in order to have

\[
\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|\nabla J^H(\theta^i_{t+1})\|^2] \leq \varepsilon,
\]

we can set all the three terms on the right hand side of (1.5) to be \(\frac{\varepsilon}{3}\), which gives

\[
S_m = \frac{24G^2(J^{H,*} - J^H(\theta_0))}{\eta \varepsilon},
\]

\[
N = 3 \left(\frac{8G^2}{\mu_F} + 2\right) \frac{\sigma^2}{\varepsilon}
\]

\[
E[\|\theta^i_{t+1} - \theta^i_{t+1}\|^2] \leq \frac{8G^2 \mu_F}{4\eta^2} + \frac{8G^4}{4\eta^4} \cdot \frac{\varepsilon}{3},
\]

where the last requirement is satisfied according to (1.3).

For the parameters \(S, m, B, \) and \(N\), we have

\[
S = \frac{24G^2(J^{H,*} - J^H(\theta_0))}{\eta \varepsilon^{0.5}} = O\left(\frac{1}{(1 - \gamma)^2 \varepsilon^{0.5}}\right),
\]

\[
m = \frac{1}{\varepsilon^{0.5}},
\]

\[
B = \left(\frac{\eta}{\mu_F} + \frac{\eta}{4G^2}\right) \frac{4C_\gamma m}{L_J e^{0.25}} = O\left(\frac{W}{(1 - \gamma)^{0.25}}\right),
\]

\[
N = 3 \left(\frac{8G^2}{\mu_F} + 2\right) \frac{\sigma^2}{\varepsilon} = O\left(\frac{\sigma^2}{\varepsilon}\right),
\]

where we have applied the definition of \(C_\gamma\) in Lemma [1] in the third equality.
Therefore in total, the number of trajectories required by SRVR-NPG to reach ε-stationarity is

\[
S \left( N + mB + (m+1) \frac{4 \left( \frac{2G^2 R}{\mu_d(1-\gamma)^2} + \frac{2}{1-\gamma} \sqrt{d} + \frac{2G^2 R}{\mu_d(1-\gamma)^2} \right)^2}{\mu_d^3 \left( \frac{2G^2 R}{\mu_d(1-\gamma)^2} + \frac{2}{1-\gamma} \right)} \right) \\
= O \left( \frac{\sigma^2}{(1-\gamma)^2 \varepsilon^{1.5}} + \frac{W}{(1-\gamma)^3 \varepsilon^{1.75}} + \frac{1}{(1-\gamma)^2 \varepsilon^2} \right).
\]

\[ \square \]

### J Proof of Proposition 4.5

In this section, we proceed to prove Proposition 4.5, which establishes a general global convergence result on policy gradient methods of the form \( \theta^{k+1} = \theta^k + \eta w^k \).

**Proof.** First, by the \( M \)-smoothness of score function (see Assumption 4.2), we know that

\[
\begin{align*}
\mathbb{E}_{s \sim d^*_\theta} \left[ \text{KL} \left( \pi^* (\cdot | s) \| \pi_{\theta^k} (\cdot | s) \right) - \text{KL} \left( \pi^* (\cdot | s) \| \pi_{\theta^{k+1}} (\cdot | s) \right) \right] \\
= \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \log \frac{\pi_{\theta^{k+1}} (a | s)}{\pi_{\theta^k} (a | s)} \right] \\
\geq \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot (\theta^{k+1} - \theta^k) \right] - \frac{M \eta^2}{2} \| w^k \|^2 \\
= \eta \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot w^k \right] - \frac{M \eta^2}{2} \| w^k \|^2 \\
+ \eta \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot (w^k - w^*_*) \right] - \frac{M \eta^2}{2} \| w^k \|^2
\end{align*}
\]

where we have applied \( \text{KL} (p||q) = \mathbb{E}_{s \sim p} [ - \log \frac{q(s)}{p(s)} ] \) in the first step.

On the other hand, by the performance difference lemma [25] we know that

\[
\mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ A^{\pi^*_\theta} (s, a) \right] = (1 - \gamma) \left( J^* - J(\theta^k) \right).
\]

Therefore,

\[
\begin{align*}
\mathbb{E}_{s \sim d^*_\theta} \left[ \text{KL} \left( \pi^* (\cdot | s) \| \pi_{\theta^k} (\cdot | s) \right) - \text{KL} \left( \pi^* (\cdot | s) \| \pi_{\theta^{k+1}} (\cdot | s) \right) \right] \\
\geq \eta \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot w^k \right] \\
+ \eta \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot (w^k - w^*_*) \right] - \frac{M \eta^2}{2} \| w^k \|^2 \\
= \eta \left( J^* - J(\theta^k) \right) + \eta \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot w^k - \frac{1}{1 - \gamma} A^{\pi^*_{\theta^k}} (s, a) \right] \\
+ \eta \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot (w^k - w^*_*) \right] - \frac{M \eta^2}{2} \| w^k \|^2 \\
= \eta \left( J^* - J(\theta^k) \right) + \eta \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot (1 - \gamma) w^*_* - A^{\pi^*_{\theta^k}} (s, a) \right] \\
+ \eta \mathbb{E}_{s \sim d^*_\theta} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ \nabla_{\theta^k} \log \pi_{\theta^k} (a | s) \cdot (w^k - w^*_*) \right] - \frac{M \eta^2}{2} \| w^k \|^2
\end{align*}
\]

Therefore,
Now, let us apply Jensen’s inequality and Assumption 4.2 to obtain
\[
E_{s \sim d^\pi} \left[ KL(\pi^*(\cdot \mid s) \parallel \pi_{\theta_k} (\cdot \mid s)) - KL(\pi^*(\cdot \mid s) \parallel \pi_{\theta_{k+1}} (\cdot \mid s)) \right] \\
\geq \eta \left( J^* - J(\theta_k) \right) - \frac{1}{1 - \gamma} \sqrt{E_{s \sim d^\pi} \left[ E_{a \sim \pi^*(\cdot \mid s)} \left[ (\nabla_{\theta} \log \pi_{\theta_k}(a \mid s) \cdot (1 - \gamma) w^k_s - A_{\theta_k}(s, a))^2 \right] \right]} - \eta G \| w^k - w^*_k \| - \frac{M \eta^2}{2} \| w^k \|^2
\]
Combining this with Assumption 4.4 yields
\[
E_{s \sim d^\pi} \left[ KL(\pi^*(\cdot \mid s) \parallel \pi_{\theta_k} (\cdot \mid s)) - KL(\pi^*(\cdot \mid s) \parallel \pi_{\theta_{k+1}} (\cdot \mid s)) \right] \\
\geq \eta \left( J^* - J(\theta_k) \right) - \eta \sqrt{\frac{1}{(1 - \gamma)^2} \epsilon_{bias} - \eta G \| w^k - w^*_k \| - \frac{M \eta^2}{2} \| w^k \|^2}.
\]

Finally, let us telescope the above inequality from \( k = 0 \) to \( K - 1 \), and divide by \( K \), which gives
\[
J(\pi^*) - \frac{1}{K} \sum_{k=0}^{K-1} J(\theta_k) \leq \frac{\sqrt{\epsilon_{bias}}}{1 - \gamma} + \frac{1}{\eta K} E_{s \sim d^\pi} \left[ KL(\pi^*(\cdot \mid s) \parallel \pi_{\theta^0} (\cdot \mid s)) \right] \\
+ \frac{G}{K} \sum_{k=0}^{K-1} \| w^k - w^*_k \| + \frac{M \eta}{2K} \sum_{k=0}^{K-1} \| w^k \|^2.
\]

On the right hand side of (J.2), the first term reflects the function approximation error due to the possibly imperfect policy parametrization. The second term vanishes as \( K \to \infty \).

By looking at the third and fourth term, we know that for an update of the form \( \theta_{k+1} = \theta_k + \eta w^k \), its global convergence rate depends crucially on i) the difference between its update directions \( w^k \) and the exact NPG update direction \( w^*_k \), and ii) its stationary convergence rate.

In the rest of this paper, we shall see that for stochastic PG, NPG, SRVR-PG, and SRVR-NPG, both the third and fourth terms of (J.2) go to 0 as \( K \to \infty \), whose speed lead to different global convergence rates for different algorithms. In order to achieve this, we will apply some intermediate results in the previous proof of stationary convergence.

### K Proof of Theorem 4.6

Let us take \( w^k \) as the update direction of PG and apply Proposition 4.5. To this end, we need to upper bound \( \frac{1}{K} \sum_{k=0}^{K-1} \| w^k - w^*_k \| \), \( \frac{1}{K} \sum_{k=0}^{K-1} \| w^k \|^2 \), and \( \frac{1}{K} E_{s \sim d^\pi} \left[ KL(\pi^*(\cdot \mid s) \parallel \pi_{\theta^0} (\cdot \mid s)) \right] \), where \( w^k = F^{-1}(\theta^k) \nabla J(\theta^k) \) is the exact NPG update direction at \( \theta^k \).

- Bounding \( \frac{1}{K} \sum_{k=0}^{K-1} \| w^k - w^*_k \| \).

We know from Jensen’s inequality and \( \left( \mathbb{E}[\| w^j_{t+1} - w^j_{t+1} \|^2] \right)^2 \leq \mathbb{E}[\| w^j_{t+1} - w^j_{t+1} \|^2] \) that
\[
\left( \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| w^k - w^*_k \|^2] \right)^2 \leq \frac{1}{K} \sum_{k=0}^{K-1} \left( \mathbb{E}[\| w^k - w^*_k \|^2] \right)^2 \\
\leq \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| w^k - w^*_k \|^2] \\
\leq \frac{2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| w^k - \nabla J(\theta^k) \|^2] + \frac{2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| \nabla J(\theta^k) - F^{-1}(\theta^k) \nabla J(\theta^k) \|^2] \\
\leq \frac{2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| w^k - \nabla J(\theta^k) \|^2] + \frac{2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| \nabla J(\theta^k) - F^{-1}(\theta^k) \nabla J(\theta^k) \|^2]
\]

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Since
\[ w^k = \frac{1}{N} \sum_{i=1}^{N} g(\tau^k_i | \theta^k), \]
we have from Lemma B.1 and Assumption 4.1 that
\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|w^k - \nabla J(\theta^k)\|^2] \\
\leq \frac{2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|w^k - \nabla J(\theta^k)\|^2] + \frac{2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J^H(\theta^k) - \nabla J(\theta^k)\|^2] \\
\leq 2 \sigma^2 + 2G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H}. \tag{K.2}
\]
Furthermore, Assumption 2.1 tells us that
\[
\mathbb{E}[\|\nabla J(\theta^k) - F^{-1}(\theta^k) \nabla J(\theta^k)\|^2] \\
\leq \left( 1 + \frac{1}{\mu_F} \right)^2 \mathbb{E}[\|\nabla J(\theta^k)\|^2] \\
\leq \left( 1 + \frac{1}{\mu_F} \right)^2 \left( 2\mathbb{E}[\|\nabla J^H(\theta^k)\|^2] + 2G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H} \right). \tag{K.3}
\]
Combining (K.2) and (K.3) with (K.1) gives
\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|w^k - w^k_*\|] \\
\leq \left( 2 \sigma^2 + 2G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H} \\
+ 2 \left( 1 + \frac{1}{\mu_F} \right)^2 \right) \left( 2\mathbb{E}[\|\nabla J^H(\theta^k)\|^2] + 2G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H} \right)^{0.5} \tag{K.4}
\]
And recall from (F.2) that
\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J^H(\theta^k)\|^2] \leq \frac{J^{H,*} - J^H(\theta_0)}{\frac{1}{\gamma} G^2 \sigma^2} + \left( \frac{\eta}{2} + L J \eta^2 \right) \frac{\sigma^2}{N}. \tag{K.2}
\]
Let us take \( \eta = \frac{1}{4L J} \). In addition, let \( H, N, \) and \( K \) satisfy
\[
\frac{1}{3} \left( \frac{\varepsilon}{3G} \right)^2 \geq \left( 2 + 4 \left( 1 + \frac{1}{\mu_F} \right)^2 \right) G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H} \\
N \geq \frac{2 + 12 \left( 1 + \frac{1}{\mu_F} \right)^2 \sigma^2}{\frac{1}{\gamma} \left( \frac{\varepsilon}{3G} \right)^2}, \tag{K.5}
\]
\[
K \geq \frac{\left( 1 + \frac{1}{\mu_F} \right)^2}{\frac{1}{3} \left( \frac{\varepsilon}{3G} \right)^2} 64 L J (J^{H,*} - J^H(\theta_0)) \tag{K.6}
\]
Then, we have
\[
\frac{G}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|w^k - w^k_*\|] \leq \frac{\varepsilon}{3}. \tag{K.6}
\]
• Bounding $\frac{1}{K} \sum_{k=0}^{K-1} \|w^k\|^2$.

We have from (K.2) and (F.2) that

$$\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|w^k\|^2] \leq \sigma^2 \frac{1}{N} + \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J^H (\theta^k)\|^2] \leq \sigma^2 \frac{1}{N} + \frac{j^H - J^H(\theta_0) + \left( \frac{\eta}{2} + L_J \eta^2 \right) \sigma^2}{\frac{1}{2} - L_J \eta^2}.$$

Taking $\eta = \frac{1}{\eta L_J}$ and

$$N \geq \frac{12 M \eta \sigma^2}{\varepsilon}, \quad K \geq \frac{48 L_J M \eta (J^{H,*} - J^H(\theta_0))}{\varepsilon}, \quad (K.7)$$

we arrive at

$$\frac{M \eta}{2K} \sum_{k=0}^{K-1} \mathbb{E}[\|w^k\|^2] \leq \frac{\varepsilon}{3}. \quad (K.8)$$

• Bounding $\frac{1}{K} \mathbb{E}_{s \sim d_{\pi^*}} \| KL (\pi^*(\cdot | s) || \pi_\theta (\cdot | s)) \|$.

By taking

$$K \geq \frac{3 \mathbb{E}_{s \sim d_{\pi^*}} \| KL (\pi^*(\cdot | s) || \pi_\theta (\cdot | s)) \|}{\eta \varepsilon} \quad (K.9)$$

we have

$$\frac{1}{\eta K} \mathbb{E}_{s \sim d_{\pi^*}} \| KL (\pi^*(\cdot | s) || \pi_\theta (\cdot | s)) \| \leq \frac{\varepsilon}{3}. \quad (K.10)$$

In summary, we require $N$ and $K$ to satisfy (K.5), (K.7), and (K.9), which leads to

$$N = \mathcal{O} \left( \frac{\sigma^2}{\varepsilon^2} \right), \quad K = \mathcal{O} \left( \frac{1}{(1 - \gamma)^2 \varepsilon^2} \right), \quad H = \mathcal{O} \left( \log((1 - \gamma)^{-1} \varepsilon^{-1}) \right).$$

By combining (K.6), (K.8), (K.10) and (J.2), we can conclude that

$$J(\pi^*) - \frac{1}{K} \sum_{k=0}^{K-1} J(\theta^k) \leq \frac{\sqrt{\varepsilon_{\text{bias}}}}{1 - \gamma} + \varepsilon.$$ 

In total, stochastic PG requires to sample $KN = \mathcal{O} \left( \frac{\sigma^2}{(1 - \gamma)^3 \varepsilon^3} \right)$ trajectories.

**L Proof of Theorem 4.9**

Let us take $w^k$ as the update direction of NPG and apply Proposition 4.5. To this end, we need to upper bound $\frac{1}{K} \sum_{k=0}^{K-1} \|w^k - w^k\|$, $\frac{1}{K} \sum_{k=0}^{K-1} \|w^k\|^2$, and $\frac{1}{K} \mathbb{E}_{s \sim d_{\pi^*}} \| KL (\pi^*(\cdot | s) || \pi_\theta (\cdot | s)) \|$, where $w^k = F^{-1}(\theta^k) \nabla J(\theta^k)$ is the exact NPG update direction at $\theta^k$.

Let us take $\eta = \frac{\mu^2}{4G^2L_J}$ and apply SGD as in Procedure 1, to obtain a $w^k$ that satisfies

$$\mathbb{E}[\|w^k - w^k\|^2] \leq \min \left\{ \frac{\varepsilon}{12M \eta} \left( \frac{1 \varepsilon}{G \mu^2} \right)^{2}, \frac{\varepsilon}{12M \eta^2} \frac{\mu^2}{2} \frac{2G^2 - \frac{L_J}{G}}{2\eta} + L_J \right\}. \quad (L.1)$$

From Proposition G.1 we know that this requires sampling $\mathcal{O} \left( \frac{1}{(1 - \gamma)^3 \varepsilon^3} \right)$ trajectories at each iteration.
• Bounding $\frac{1}{K} \sum_{k=0}^{K-1} \| w^k - w^k \|$. 
Recall that the update direction $w^k \approx w^k = F^{-1}(\theta^k)\nabla J(\theta^k)$ is obtained by solving the subproblem

$$w^k \approx \underset{w\in\mathbb{R}^d}{\text{argmin}} L_{\theta^k}(w; \theta^k) = \underset{w\in\mathbb{R}^d}{\text{argmin}} E_{(s,a)\sim \pi^k} [A^{\pi^k}(s,a) - (1 - \gamma)w^\top \nabla_{\theta} \log \pi^k(a|s)]^2.$$ 

By (L.1) and Jensen’s inequality, we can write

$$\left(\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[[\| w^k - w^k \|]]\right)^2 \leq \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[[\| w^k - w^k \|]] \leq \left(\frac{1}{G^3} \right)^2 \cdot (L.2)$$

On the other hand, by replacing (G.1) with (L.1), the stationary convergence of NPG stated in (G.3) becomes

$$\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[[\| \nabla J(\theta^k) \|]^2] \leq \frac{J^* - J(\theta_0)}{K} + \frac{\varepsilon}{12M\eta} \frac{\mu^2}{\eta^2}\left(\frac{2}{K} \sum_{k=0}^{K-1} \mathbb{E}[[\| w^k - w^k \|]]\right)^2 + \frac{\eta}{\mu^2} K \frac{\beta^2}{\mu^2} \frac{\eta}{\mu^2} \leq \frac{\mu^2}{12M\eta}.$$

(L.1) and the above inequality yields

$$\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[[\| w^k \|]^2] \leq \frac{2}{2M\eta} \sum_{k=0}^{K-1} \mathbb{E}[[\| w^k - w^k \|]^2] + \frac{1}{\mu^2} \sum_{k=0}^{K-1} \mathbb{E}[[\| \nabla J(\theta^k) \|]^2] \leq \frac{2\varepsilon}{12M\eta} + \frac{2}{\mu^2} \frac{\mu^2}{12M\eta} = \varepsilon \cdot \frac{3}{3M\eta} \cdot (L.4)$$

• Bounding $\frac{1}{\eta} \mathbb{E}_{s\sim d^\pi \theta^k} [\text{KL}(\pi^\ast(\cdot|s)||\pi_{\theta^k}(\cdot|s))]$. 
Let us also set

$$K \geq \frac{3\mathbb{E}_{s\sim d^\pi \theta^k} [\text{KL}(\pi^\ast(\cdot|s)||\pi_{\theta^k}(\cdot|s))]}{\eta \varepsilon}, \quad (L.5)$$

so that

$$\frac{1}{\eta K} \sum_{k=0}^{K-1} \mathbb{E}_{s\sim d^\pi \theta^k} [\text{KL}(\pi^\ast(\cdot|s)||\pi_{\theta^k}(\cdot|s))] \leq \frac{\varepsilon}{3}. \quad (L.6)$$

In summary, we require $K$ to satisfy (L.3) and (L.5), which leads to

$$K = O \left( \frac{1}{(1 - \gamma)^2\varepsilon} \right) \quad (L.7)$$

By combining (L.2), (L.4), (L.6) and (L.2), we can conclude that

$$J(\pi^\ast) - \frac{1}{K} \sum_{k=0}^{K-1} J(\theta^k) \leq \frac{\sqrt{\varepsilon_{\text{bias}}}}{1 - \gamma} + \varepsilon.$$

Since at each iteration, SGD needs to sample $O \left( \frac{1}{(1 - \gamma)^2\varepsilon} \right)$ trajectories so that (L.1) is satisfied, NPG requires to sample $O \left( \frac{1}{(1 - \gamma)^2\varepsilon} \right)$ trajectories in total.
M Proof of Theorem 4.11

Let us take $w_{t+1}^*$ as the update direction of SRVR-PG and apply Proposition 4.5. To this end, we need to upper bound $\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \| w_{t+1}^* - w_{t+1} \|$, $\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \| w_{t+1} \|^2$, and $\frac{1}{Sm} \mathbb{E}_{s \sim d_{π}^*} [\text{KL}(π^*(\cdot|s)||π_θ^*(\cdot|s))]$, where $w_{t+1}^* = F^{-1}(θ_{t+1}) \nabla J(θ_{t+1})$ is the exact NPG update direction at $θ_{t+1}^*$.

- Bounding $\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \| w_{t+1}^* - w_{t+1} \|$.
  
  Since $w_{t+1}^* = w_{t+1}$ and $w_{t+1} = F^{-1}(θ_{t+1}) \nabla J(θ_{t+1})$, we have from Lemmas I.1 and B.1 that
  
  $$
  \mathbb{E}[||w_{t+1}^* - w_{t+1}||^2] 
  \leq 2\mathbb{E}[||w_{t+1} - \nabla J(θ_{t+1})||^2] + 2\mathbb{E}[||\nabla J(θ_{t+1}) - F^{-1}(θ_{t+1}) \nabla J(θ_{t+1})||^2]
  \leq 2\mathbb{E}[||w_{t+1} - \nabla J(θ_{t+1})||^2] + 2(1 + \frac{1}{\mu_F})^2 \mathbb{E}[||\nabla J(θ_{t+1})||^2]
  + 2G^2R^2 \left( \frac{H + 1}{N} + \frac{\gamma}{(1 - \gamma)^2} \right) \gamma^{2H}
  \leq 2 \left( \frac{C_γ}{B} \sum_{t=1}^{m-1} \mathbb{E}[||θ_{t+1} - θ_{t+1}||^2] + \frac{σ^2}{N} \right) + 2(1 + \frac{1}{\mu_F})^2 \mathbb{E}[||\nabla J(θ_{t+1})||^2]
  + 2G^2R^2 \left( \frac{H + 1}{N} + \frac{\gamma}{(1 - \gamma)^2} \right) \gamma^{2H},
  $$

  where we have applied Lemma B.1 and Assumption 2.1 in the second inequality, and Lemma I.1 in the third one.

Telescoping this over $s = 0, 1, \ldots, S - 1$, $t = 0, 1, m - 1$ and dividing by $Sm$ gives

$$
\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[||w_{t+1}^* - w_{t+1}||^2]
\leq 2(1 + \frac{1}{\mu_F})^2 \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[||\nabla J(θ_{t+1})||^2]
+ 2 \left( \frac{C_γm}{B} \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[||θ_{t+1} - θ_{t+1}||^2] + \frac{σ^2}{N} \right)
+ 2G^2R^2 \left( \frac{H + 1}{N} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H}.
\quad \text{(M.1)}
$$

On the other hand, from Equation (B.14) of [59] we know that

$$
\left( \frac{2}{η^2} \frac{4L_J}{η} - \frac{12mC_γ}{B} \right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[||θ_{t+1} - θ_{t+1}||^2]
+ \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[||\nabla J^H(θ_{t+1})||^2]
\leq 8(\lambda H^* - J(θ_0)) \frac{6\sigma^2}{ηSm} + \frac{6\sigma^2}{N}. \quad \text{(M.2)}
$$
By the definition of $C_\gamma$ in Lemma 1, we have

$$B = \frac{3\eta C_\gamma m}{L_J} = \frac{72\eta RG(2G^2 + M)(W + 1)\gamma}{L_J(1 - \gamma)^5} m$$

(M.3)

Therefore, (M.2) becomes

$$\left(\frac{2}{\eta^2} - \frac{8L_J}{\eta}\right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||\theta_{t+1}^s - \theta_t^s||^2] + \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||\nabla J^H(\theta_{t+1}^s)||^2]$$

$$\leq \frac{8(J_{t,*}^H - J^H(\theta_0))}{\eta Sm} + \frac{6\sigma^2}{N},$$

where we have also applied

$$\left(\frac{2}{\eta^2} - \frac{8L_J}{\eta}\right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||\theta_{t+1}^s - \theta_t^s||^2] + \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||\nabla J^H(\theta_{t+1}^s)||^2]$$

(M.4)

Since $\eta = \frac{1}{8L_J}$, we further have

$$\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||\theta_{t+1}^s - \theta_t^s||^2] \leq \frac{J_{t,*}^H - J^H(\theta_0)}{L_J Sm} + \frac{6\sigma^2}{64L_J N},$$

(M.4)

Putting these inequalities back into (M.1) yields

$$\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||w_{t+1}^s - w_{t,*}^s||^2] \leq 2(1 + \frac{1}{\mu_F})^2 \left(\frac{64L_J(J_{t,*}^H - J^H(\theta_0))}{Sm} + \frac{6\sigma^2}{N}\right)$$

$$+ 2 \left(\frac{8L_J^2}{3} \left(\frac{J_{t,*}^H - J^H(\theta_0)}{L_J Sm} + \frac{6\sigma^2}{64L_J N}\right) + \sigma^2\right)$$

$$+ 2G^2R^2 \left(\frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2}\right) \gamma^{2H}.$$

Let us set

$$\frac{1}{3} \left(\frac{\varepsilon}{3G}\right)^2 \geq 2G^2R^2 \left(\frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2}\right) \gamma^{2H},$$

$$N \geq \frac{\left(12(1 + \frac{1}{\mu_F})^2 + 2.5\right)\sigma^2}{\frac{1}{3} \left(\frac{\varepsilon}{3G}\right)^2},$$

$$Sm \geq \frac{\left(128(1 + \frac{1}{\mu_F})^2 + \frac{16}{\varepsilon}\right) L_J(J_{t,*}^H - J^H(\theta_0))}{\frac{1}{3} \left(\frac{\varepsilon}{3G}\right)^2},$$

so that

$$\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||w_{t+1}^s - w_{t,*}^s||^2] \leq \left(\frac{1}{G^3}\right)^2$$

By Jensen’s inequality, we further have

$$\left(\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||w_{t+1}^s - w_{t,*}^s||]\right)^2 \leq \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \left(E[||w_{t+1}^s - w_{t,*}^s||]\right)^2$$

$$\leq \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[||w_{t+1}^s - w_{t,*}^s||^2] \leq \left(\frac{1}{G^3}\right)^2$$

(M.6)

where we have also applied

$$\left(E[||w_{t+1}^s - w_{t,*}^s||]\right)^2 \leq E[||w_{t+1}^s - w_{t,*}^s||^2].$$
To achieve this, we require

\[ \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[w_t^{t+1}]^2 \]

Since \( \mathbb{E}[w_t^{t+1}] = \mathbb{E}[u_t^{t+1}] = \nabla J^H(\theta_t^{t+1}) \), by Lemma I.1 we have

\[ \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[w_t^{t+1}]^2 \]

\[ \leq \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[\|\theta_t^{t+1} - \theta_{t-1}^{t+1}\|^2] + \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[\|\nabla J^H(\theta_t^{t+1})\|^2] \]

\[ \leq \frac{C.m}{B} \cdot \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[\|\theta_t^{t+1} - \theta_{t-1}^{t+1}\|^2] + \frac{\sigma^2}{N} + \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[\|\nabla J^H(\theta_t^{t+1})\|^2]. \]

By setting \( \eta = \frac{1}{8L_J} \) and applying (M.3) and (M.4), we further have

\[ \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[w_t^{t+1}]^2 \]

\[ \leq \frac{8L_J^2}{3} \left( \frac{J^H, - J^H(\theta_0)}{L_J Sm} + \frac{6\sigma^2}{64L_J^2 N} + \frac{\sigma^2}{N} + \frac{64L_J(J^H, - J^H(\theta_0))}{Sm} + \frac{6\sigma^2}{N} \right). \]

Therefore, we can set

\[ N \geq \frac{174M \sigma^2}{32L_J \varepsilon}, \quad (M.7) \]

\[ Sm \geq \frac{50M(J^H, - J^H(\theta_0))}{\varepsilon}, \quad (M.8) \]

so that

\[ \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[w_t^{t+1}]^2 \leq \frac{\varepsilon}{3M \eta}. \]

- Bounding \( \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[d_t^{s, d^*}] \) \( [\text{KL}(\pi^*\cdot | s) \| \pi_{\theta^0}\cdot | s)] \).

Let us set

\[ Sm \geq \frac{3}{\eta} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[d_t^{s, d^*}] \] \( [\text{KL}(\pi^*\cdot | s) \| \pi_{\theta^0}\cdot | s)] \) \( \eta \varepsilon \)

so that

\[ \frac{1}{\eta Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[d_t^{s, d^*}] \) \( [\text{KL}(\pi^*\cdot | s) \| \pi_{\theta^0}\cdot | s)] \) \( \leq \frac{\varepsilon}{3}. \]

By combining (M.6), (M.8), (M.10) and (J.2), we can conclude that

\[ J(\pi^*) - \frac{1}{K} \sum_{k=0}^{K-1} J(\theta^k) \leq \frac{\sqrt{\varepsilon \text{bias}}}{1 - \gamma} + \varepsilon. \]

To achieve this, we require \( Sm \) and \( N \) to satisfy (M.5), (M.7), and (M.9), which leads to

\[ Sm = \mathcal{O}\left(\frac{1}{(1 - \gamma)^2 \varepsilon^2}\right), \quad N = \mathcal{O}\left(\frac{\sigma^2}{\varepsilon^2}\right), \quad H = \mathcal{O}\left(\log\left(\frac{1}{(1 - \gamma)\varepsilon}\right)\right). \]

By (M.3), we know that \( B = \mathcal{O}(W(1 - \gamma)^{-1} m) \).
Therefore, by taking $S = \mathcal{O}\left(\frac{1}{(1-\gamma)^2}\right)$ and $m = \mathcal{O}\left(\frac{1}{(1-\gamma)^3}\right)$, the sample complexity of SRVR-PG is

$$S(N + mB) = \mathcal{O}\left(\frac{\sigma^2}{(1-\gamma)^2} + \frac{W}{(1-\gamma)^3}\right) = \mathcal{O}\left(\frac{W + \sigma^2}{(1-\gamma)^3}\right).$$

\section{Proof of Theorem 4.13}

Let us take $w_t^{j+1}$ as the update direction of SRVR-NPG and apply Proposition 4.5. To this end, we need to upper bound $\frac{1}{\sigma m} \sum_{s=0}^{S-1} \sum_{l=0}^{m-1} \|w_t^{j+1} - w_l^{j+1}\|$, $\frac{1}{\sigma m} \sum_{s=0}^{S-1} \sum_{l=0}^{m-1} \|w_t^{j+1}\|^2$, and $\frac{1}{\sigma m} \text{KL}(\pi^*(\cdot | s) || \pi^0(\cdot | s))$, where $w_t^{j+1} = F^{-1}(\theta_t^{j+1}) \nabla J(\theta_t^{j+1})$ is the exact NPG update direction at $\theta_t^{j+1}$.

Let us take $\eta = \frac{\mu_F}{4G^{3/2}}$ and apply SGD as in Procedure 2 to obtain a $w_t^{j+1}$ that satisfies

$$\mathbb{E}[\|w_t^{j+1} - F^{-1}(\theta_t^{j+1})u_t^{j+1}\|^2] \leq \min \left\{ \frac{1}{2} \left( \frac{1-\gamma}{\gamma} \right)^2, \frac{4\mu_F^2}{\mu_F G^2 + \sigma^2}, \frac{64\mu_F^2 L_F^2}{\mu_F G^2 + \sigma^2}, \frac{1}{9M\eta} \right\}. \tag{N.1}$$

In order to apply Proposition 4.2 let assume the following so that its assumptions are satisfied:

$$\min \left\{ \frac{1}{2} \left( \frac{1-\gamma}{\gamma} \right)^2, \frac{4\mu_F^2}{\mu_F G^2 + \sigma^2}, \frac{64\mu_F^2 L_F^2}{\mu_F G^2 + \sigma^2}, \frac{1}{9M\eta} \right\} \leq \frac{2}{\mu_F^2} \left( \frac{GR}{(1-\gamma)^2} \right)^2, \tag{N.2}$$

$$\frac{C_s m}{B} 2\eta^2 \leq \frac{1}{3} \mu_F^2.$$

At the end of this proof, we will see that these assumptions are indeed satisfied for small $\varepsilon$.

From Proposition 4.2 we know that this requires sampling $\mathcal{O}\left(\frac{1}{(1-\gamma)^2}\right)$ trajectories at each iteration.

- **Bounding $\frac{1}{\sigma m} \sum_{s=0}^{S-1} \sum_{l=0}^{m-1} \|w_t^{j+1} - w_t^{j+1}\|$**:

  First of all, we have

  $$\mathbb{E}[\|w_t^{j+1} - w_l^{j+1}\|^2] \leq 2\mathbb{E}[\|w_t^{j+1} - F^{-1}(\theta_t^{j+1})u_t^{j+1}\|^2] + 2\mathbb{E}[\|F^{-1}(\theta_t^{j+1})u_t^{j+1} - F^{-1}(\theta_t^{j+1})\nabla J(\theta_t^{j+1})\|^2]$$

  $$\leq 2\mathbb{E}[\|w_t^{j+1} - F^{-1}(\theta_t^{j+1})u_t^{j+1}\|^2] + 2\frac{1}{\mu_F} \mathbb{E}[\|u_t^{j+1} - \nabla J(\theta_t^{j+1})\|^2]$$

  $$\leq 2\mathbb{E}[\|w_t^{j+1} - F^{-1}(\theta_t^{j+1})u_t^{j+1}\|^2] + 2\frac{1}{\mu_F} \left( \frac{C_s}{B} \sum_{i=1}^{t} \mathbb{E}[\|\theta_t^{j+1} - \theta_t^{j+1}\|^2] + \frac{\sigma^2}{N} \right)$$

  $$+ 4G^2 R^2 \left( \frac{H + 1}{1-\gamma} + \frac{\gamma}{(1-\gamma)^2} \right)^2 \gamma^{2H}$$

  $$\leq 2\mathbb{E}[\|w_t^{j+1} - F^{-1}(\theta_t^{j+1})u_t^{j+1}\|^2] + 2\frac{1}{\mu_F} \left( \frac{C_s}{B} \sum_{i=0}^{m-1} \mathbb{E}[\|\theta_t^{j+1} - \theta_t^{j+1}\|^2] + \frac{\sigma^2}{N} \right)$$

  $$+ 4G^2 R^2 \left( \frac{H + 1}{1-\gamma} + \frac{\gamma}{(1-\gamma)^2} \right)^2 \gamma^{2H},$$

  where we have applied Assumption 2.1 in the second inequality, and Lemmas 1.1 and B.1 in the third one.

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Telescoping this over \( s = 0, 1, \ldots, S - 1, t = 0, 1, m - 1 \) and dividing by \( Sm \) gives

\[
\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[|w_{i,t+1} - w_{i,t+1}^*|^2] \\
\leq 2 + \frac{1}{N} \left( \frac{8G^2}{\mu_F} + \frac{\eta}{4G^2} \right)^2 + 4 \frac{1}{\mu_F} \left( \frac{C_G m}{B} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[|\theta_{t+1}^j - \theta_{t+1}^j|^2] + \frac{\sigma^2}{N} \right) \\
+ 4G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^2 H.
\]

(N.3)

On the other hand, from (1.5) we know that

\[
\frac{8G^2}{\eta} \left( \frac{\mu_F}{8 \eta} \frac{L_J}{2} - \left( \frac{\eta}{\mu_F} + \frac{\eta}{4G^2} \right) C_G m \right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[|\theta_{t+1}^j - \theta_{t+1}^j|^2] \\
+ \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[|\nabla J^H(\theta_{t+1}^j)|^2] \\
\leq \frac{8G^2}{\eta} \left( J^{H,*} - J^H(\theta_0) \right) \\
+ \left( \frac{8G^2}{\mu_F} + 2 \right) \frac{\sigma^2}{N} + \left( \frac{8G^2 \mu_F}{4} + \frac{8G^4}{4} \right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E\|F^{-1}(\theta_{t+1}^j)w_{t+1}^j - w_{t+1}^j\|^2.
\]

(N.4)

Let us set

\[
B \geq \left( \frac{\eta}{\mu_F} + \frac{\eta}{4G^2} \right) \frac{2C_G m}{L_J} = \left( \frac{\eta}{\mu_F} + \frac{\eta}{4G^2} \right) \frac{48RG^2(2G^2 + M)(W + 1)\gamma}{L_J(1 - \gamma)^5} m. \quad \text{(N.5)}
\]

Since \( \eta = \frac{\mu_F}{16L_J} \), (N.4) becomes

\[
\frac{8G^2 L_J}{\eta} \left( \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[|\theta_{t+1}^j - \theta_{t+1}^j|^2] + \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[|\nabla J^H(\theta_{t+1}^j)|^2] \right) \\
\leq \frac{8G^2}{\eta} \left( \frac{J^{H,*} - J^H(\theta_0)}{Sm} \right) + \left( \frac{8G^2}{\mu_F} + 2 \right) \frac{\sigma^2}{N} \\
+ \left( \frac{8G^2 \mu_F}{4} + \frac{8G^4}{4} \right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E\|F^{-1}(\theta_{t+1}^j)w_{t+1}^j - w_{t+1}^j\|^2,
\]

from which we have

\[
\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[|\theta_{t+1}^j - \theta_{t+1}^j|^2] \\
\leq \frac{J^{H,*} - J^H(\theta_0)}{L_J Sm} + \left( \frac{8G^2}{\mu_F} + 2 \right) \frac{\mu_F}{128G^2 L_J^2} \frac{\sigma^2}{N} \\
+ \left( \frac{8G^2 \mu_F}{4} + \frac{8G^4}{4} \right) \frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E\|F^{-1}(\theta_{t+1}^j)w_{t+1}^j - w_{t+1}^j\|^2. \quad \text{(N.6)}
\]
Putting these inequalities back into (N.3) and applying (N.1) yields

\[
\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|w_{t+1}^j - w_{t*}^{j+1}\|^2] 
\leq 2 + \frac{1}{2} \left( \frac{(1/\varepsilon)^2}{\mu_F^2} \right)^2 \frac{G^2 \mu_F + G^4}{\mu_F^2} + \frac{1}{2} \left( \frac{(1/\varepsilon)^2}{\mu_F^2} \right)^2 \frac{4G^2 \mu_F + G^4}{\mu_F^2} 
+ 4G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H} 
\]

\[
\leq 2 + \frac{1}{2} \left( \frac{(1/\varepsilon)^2}{\mu_F^2} \right)^2 \frac{G^2 \mu_F + G^4}{\mu_F^2} + \frac{1}{2} \left( \frac{(1/\varepsilon)^2}{\mu_F^2} \right)^2 \frac{4G^2 \mu_F + G^4}{\mu_F^2} 
+ 2 \frac{C_{\gamma m} J^H J^H - J^H(\theta_0)}{L_J Sm} + 4G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H}. 
\]

From (N.5) we know that

\[
\frac{C_{\gamma m} J^H J^H - J^H(\theta_0)}{L_J Sm} \leq \frac{32L_J^2}{4 + \mu_F^2}, 
\]

which gives us

\[
\frac{1}{Sm} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E[\|w_{t+1}^j - w_{t*}^{j+1}\|^2] 
\leq \left( 2 + \frac{G^2 \mu_F + G^4}{\mu_F^2} \right) \frac{\mu_F}{4G^2 + \mu_F} + \frac{1}{2} \left( \frac{(1/\varepsilon)^2}{\mu_F^2} \right)^2 \frac{4G^2 \mu_F + G^4}{\mu_F^2} 
+ 2 \frac{C_{\gamma m} J^H J^H - J^H(\theta_0)}{L_J Sm} + 4G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H} 
\]

\[
= \frac{1}{4} \left( \frac{1}{G^3} \right)^2 + 2 \frac{C_{\gamma m} J^H J^H - J^H(\theta_0)}{L_J Sm} + 3 \frac{\sigma^2}{\mu_F^2 N} + 4G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H}, 
\]

where we have applied (N.1) in the first equality.
Let us set
\[ N \geq \frac{108G^2 \sigma^2}{\mu_F^2 \varepsilon^2}, \]
\[ B \geq \frac{72C_n m}{\mu_F L_j^2 \varepsilon}, \]
\[ S_m \geq \frac{L_j (J_i, \ast - J_i(\theta_0)) G^2}{\mu_F \varepsilon}, \] (N.7)
\[ \frac{1}{4} \left( \frac{\varepsilon}{3G} \right)^2 \geq 4G^2 R^2 \left( \frac{H + 1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)^2 \gamma^{2H}, \]
so that
\[ \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \mathbb{E}[\|w^{j+1}_t - w^{j+1}_{t, \ast}\|^2] \leq \left( \frac{1}{G^3} \right)^2 \]
By Jensen’s inequality, we further have
\[ \left( \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \mathbb{E}[\|w^{j+1}_t - w^{j+1}_{t, \ast}\|] \right)^2 \leq \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \left( \mathbb{E}[\|w^{j+1}_t - w^{j+1}_{t, \ast}\|] \right)^2 \]
\[ \leq \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \mathbb{E}[\|w^{j+1}_t - w^{j+1}_{t, \ast}\|^2] \leq \left( \frac{1}{G^3} \right)^2, \] (N.8)
where we have also applied \( \left( \mathbb{E}[\|w^{j+1}_t - w^{j+1}_{t, \ast}\|] \right)^2 \leq \mathbb{E}[\|w^{j+1}_t - w^{j+1}_{t, \ast}\|^2] \).

- Bounding \( \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \|w^{j+1}_t\|^2 \).
  By (N.6) we have
\[ \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \mathbb{E}[\|w^{j+1}_t\|^2] \]
\[ = \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \mathbb{E}[\|\theta^{j+1}_t - \theta^{j+1}_t\|^2] \]
\[ \leq \frac{J_i - J(\theta_0)}{L_j \eta^2 S_m} + \left( \frac{8G^2}{\mu_F} + 2 \right) \frac{\mu_F}{128G^2 \eta^2 L_j^2} \frac{\sigma^2}{N} \]
\[ + \left( \frac{8G^4}{4} \right) \frac{\mu_F}{128G^2 \eta^2 L_j^2} \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \mathbb{E}[\|F^{-1}(\theta^{j+1}_t)w^{j+1}_t - w^{j+1}_t\|^2] \]
\[ \leq \frac{J_i - J(\theta_0)}{L_j \eta^2 S_m} + \left( \frac{8G^2}{\mu_F} + 2 \right) \frac{\mu_F}{128G^2 \eta^2 L_j^2} \frac{\sigma^2}{N} + \frac{\varepsilon}{9M \eta}, \]
where we have applied (N.6) in the first inequality, and (N.1) in the last step.

We can set
\[ N \geq \frac{9M \mu_F (8G^2 + 2) \sigma^2}{128G^2 \eta L_j^2 \varepsilon}, \] (N.9)
\[ S_m \geq \frac{9M (J_i - J(\theta_0))}{L_j \eta \varepsilon}, \]
so that
\[ \frac{1}{S_m} \sum_{s=0}^{S-1} \sum_{m=0}^{m-1} \mathbb{E}[\|w^{j+1}_t\|^2] \leq \frac{\varepsilon}{3M \eta}. \] (N.10)
• Bounding $\frac{1}{Sm} \mathbb{E}_{s \sim d^*_\pi} [\text{KL}(\pi^*(\cdot \mid s) || \pi_{\theta^0}(\cdot \mid s))]$.

Let us set

$$Sm \geq \frac{3 \mathbb{E}_{s \sim d^*_\pi} [\text{KL}(\pi^*(\cdot \mid s) || \pi_{\theta^0}(\cdot \mid s))]}{\eta \varepsilon}$$

(N.11)

so that

$$\frac{1}{\eta Sm} \mathbb{E}_{s \sim d^*_\pi} [\text{KL}(\pi^*(\cdot \mid s) || \pi_{\theta^0}(\cdot \mid s))] \leq \frac{\varepsilon}{3}.$$  (N.12)

By combining (N.8), (N.10), (N.12) and (J.2), we can conclude that

$$J(\pi^*) - \frac{1}{K} \sum_{k=0}^{K-1} J(\theta^k) \leq \frac{\sqrt{\varepsilon \text{bias}}}{1 - \gamma} + \varepsilon.$$  

To achieve this, we require $Sm$, $B$, and $N$ to satisfy (N.5), (N.7), (N.9), and (N.11), which leads to

$$Sm = O\left(\frac{1}{(1 - \gamma)^2 \varepsilon^2}\right), \quad N = O\left(\frac{\sigma^2}{\varepsilon^2}\right),$$

$$B = O\left(\frac{W}{(1 - \gamma) \varepsilon m}\right), \quad H = O\left(\log\left(\frac{1}{(1 - \gamma) \varepsilon}\right)\right).$$

By Proposition I.2, we know that in order to achieve (N.1), SGD requires sampling $O\left(\frac{1}{(1 - \gamma)^4 \varepsilon^2}\right)$ trajectories per iteration.

Therefore, by taking $S = O\left(\frac{1}{(1 - \gamma)^2 \varepsilon^2 m}\right)$ and $m = O\left(\frac{1}{(1 - \gamma)^{0.5} \varepsilon^2}\right)$, the amount of trajectories required by SRVR-NPG is

$$S\left(N + mB + (1 + m)O\left(\frac{1}{(1 - \gamma)^4 \varepsilon^2}\right)\right)$$

$$= O\left(\frac{\sigma^2}{(1 - \gamma)^2 \varepsilon^2} + \frac{W}{(1 - \gamma)^2 \varepsilon^2 m} + \frac{1}{(1 - \gamma)^{0.5} \varepsilon^3}\right).$$

It is straightforward to verify that the requirements listed in (N.2) are also satisfied as long as $\varepsilon$ is small enough.

O Implementation Details

In this section, we provide additional details on the implementation of PG, NPG, SRVR-PG and SRVR-NPG.

1. For NPG, we use the default implementation provided by rllab\(^1\), which actually implements the trust region policy optimization (TRPO) algorithm [43]. For cartpole, we sample 200 trajectories at each iteration to solve the subproblem of TRPO. For mountain car, we sample 120 trajectories at each iteration.

2. We found that the naive implementation of PG and SRVR-PG typically do not work for our tests. For example, PG and SRVR-PG often give an average reward around $-90$ for the mountain-car test, despite of our best efforts.

3. As in [38] and [59], we found that it is necessary to apply Adagrad [14] or Adam [26] type of averaging to improve their performances.

4. In our experiments, we apply Adagrad type of averaging for PG and SRVR-PG, which results in much better performances. As for SRVR-NPG, we apply Adam type of averaging, which gives an approximation of the Fisher information matrix at each iteration (see section 11.2 of [32]). We leave the implementation of a better approximation of the Fisher information matrix to the future work.

\(^1\)https://github.com/rll/rllab

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