Abstract

We study chiral anomalies in $\mathcal{N} = (0, 1)$ and $(0, 2)$ two-dimensional minimal sigma models defined on the generic homogeneous spaces $G/H$. Such minimal theories contain only (left) chiral fermions and in certain cases are inconsistent because of ‘incurable’ anomalies. We explicitly calculate the anomalous fermionic effective action and show how to remedy it by adding a series of local counterterms. In this procedure, we derive a local anomaly matching condition, which is demonstrated to be equivalent to the well-known global topological constraint on $p_1(G/H)$, the first Pontryagin class. More importantly, we show that these local counterterms further modify and constrain ‘curable’ chiral models, some of which, for example, flow to the nontrivial infrared superconformal fixed point. Finally, we also observe an interesting relation between $\mathcal{N} = (0, 1)$ and $(0, 2)$ two-dimensional minimal sigma models and supersymmetric gauge theories.

Keywords: nonlinear sigma models, anomalies, gauge formulation, homogeneous spaces, isometry, holonomy, chiral fermions

1. Introduction and summary

Supersymmetric nonlinear sigma models in two dimensions present a useful theoretical laboratory which has deep connections with many other quantum theories, as well as with...
aspects of topology. In supersymmetric theories, the simplicity of the theory increases with the number of supercharges rather often. By simplicity we mean that the theory under consideration can have special properties allowing one to obtain exact results or uncover elegant mathematical structures. On the other hand, theories with less supersymmetry, presenting more difficulties for theoretical analysis, are sometimes closer to physical phenomena, and as such must be thoroughly studied. In this paper, we will focus on minimal supersymmetric models with $\mathcal{N} = (0, 1)$ or $\mathcal{N} = (0, 2)$ supersymmetry [1–7]. It has long been known that such models, generally speaking, exhibit a chiral fermion anomaly which imposes severe constraints on the topology of the target manifold [3, 8, 9]. For this reason, such minimal supersymmetric models have been explored to a lesser extent than nonchiral models. The guiding principle established [8] for the chiral $\mathcal{N} = (0, 1)$ or $\mathcal{N} = (0, 2)$ sigma models is the first Pontryagin class.

Our present work is motivated by the following considerations. Firstly, the global anomaly cancellation condition does not touch the local behavior of the theory. Even when one has a ‘good’ theory, which has no global anomaly [8], it does not automatically mean that one gets the well-defined theory for free. The ease of the global anomaly only implies that one is able to introduce ‘local counterterms’ to correctly integrated out chiral fermions and find the anomaly-free fermionic effective action. This can be shown simply by looking at fermionic loops in two-dimensional field theories, and the fermionic loops consist of two parts: the pole part and the local part. While the pole part is uniquely determined without any ambiguity using an optical theorem argument, the local part can only be fixed by considerations from elsewhere, such as gauge symmetries, etc. This is the part that we need to take care of, for which we need to add local counterterms. The ‘local counterterm’ here is not to be confused with the terms added to absorb various divergences in the process of renormalization, since the quantization of fermions in two dimensions is insensitive to RG flow. In fact, the roles played by what we call local counterterms are similar to that of the contact term in gauge theories, which is added to keep the transversality of certain polarization operators. Since the latter sometimes also refers to the Schwinger term, we refrain from using it here. Moreover, by explicitly curing such a theory (i.e. adding the appropriate local counterterms), one can exhibit many quantum aspects of the theory in a more understandable way, thus enabling one to initiate a discussion of the infrared (IR) behavior of the theory, which has not been carried out previously.

Secondly, many sigma models have more than one equivalent formulation: a nonlinear description based on the Riemannian metric that encodes geometric information, an embedding into a larger linear target space and then imposing extra gauge symmetries or constraints, or a hybrid lying between the above two formulations [7]. Although classically all these formulations are equivalent, at the quantum level one could have different considerations depending on the formulation. For example, in the nonlinear formulation it is easy to understand the global chiral fermion anomaly, while when using the gauge formulation, one will be focused on the gauge anomaly. Work has been done on these aspects [8–16], providing us with starting positions. The precise relation between the gauge anomaly and global anomaly for different formulations of the very same model has not been thoroughly discussed previously. In this paper, we study the chiral sigma models on homogeneous spaces, for which both nonlinear and gauge formulations are present. We reveal the relation between the different anomalies. Our result also provides us with a generalized context for the determinant line bundle consideration of the fermion anomaly. In nonhomogeneous spaces one cannot compare global and gauge anomalies on the nose, although an analogous structure has been

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4 A more mathematical interpretation of the ambiguity in the functional determinant of the (chiral) Dirac operators can be found in [8].
revealed in the case of the Kähler manifold. This paper generalizes and extends the results of [7].

Finally, we would like to emphasize the possible applications of our results in model building. Models with large supersymmetry may be viewed as being composed of theories with less supersymmetry. In this regard, an understanding of minimal supersymmetric models as the building blocks for all supersymmetric theories is of importance.

In practice, the usual situation is the opposite. For example, \( \mathcal{N} = 2, 2 \) theories are always better understood than \( \mathcal{N} = 0, 2 \) and were explored earlier. Softly broken to \( \mathcal{N} = 0, 2 \) theories (free from chiral anomalies), they are easier for explorations [17–27]. We hope that our work on the minimal models can give insights into the understanding of more complicated models. In the current paper we also discuss at length the IR behavior for many models and observe a new connection with the superconformal models [26, 28, 29].

To make the contents here more assessable to readers with various backgrounds, we summarize the type of anomalies one encounters in table 1. In this context, our analysis on isometry and holonomy anomalies implies an analog of anomalies in the nonlinear and gauge settings, which can also be seen at the end of section 2,

| Anomalies | Symmetries violated | Examples |
|-----------|---------------------|----------|
| (Local) gauge anomalies | Infinitesimal gauge transformations | Minimal \( \mathcal{N} = (0, 1) \) chiral gauge theories |
| Isometry anomalies | Infinitesimal isometry transformations | Minimal \( \mathcal{N} = (0, 2) \) \( \mathbb{CP}(N) \) models for \( N \geq 2 \) |
| Holonomy anomalies | Freedom in finding local charts of the target | See section 3.3 |
| Global anomalies (in nonlinear sigma models) | Homotopies in the moduli of bosonic fields | Minimal \( \mathcal{N} = (0, 2) \) \( \mathbb{CP}(N) \) models for \( N \geq 2 \) |

It is then a nontrivial fact that for homogeneous spaces such local anomaly cancellation conditions, as well as that for the global anomalies in nonlinear sigma models, are identical rationally, as we show in section 3.4. Some of the geometric intuition is given in section 4.2.

In our paper, we consider the bosonic and gauge fields to all be external. In the nonlinear sigma model setting, one does not lose anything, because the nonlinear bosonic fields encounter no anomaly. In the gauge picture, however, it could well be that the path integral of the gauge fields becomes singular due to the inconsistency between the renormalization scheme and the gauge constraints. To make the analysis clean, we do not consider dynamic gauge fields.

The paper is organized as follows. In section 2, we will first construct bosonic and \( \mathcal{N} = (0, 1) \) supersymmetric sigma models on homogeneous spaces by virtue of a hidden gauge formulation. The calculation of their isometry anomalies is given in section 2.3. As discussed in the previous work [7], we show that the isometry anomalies reflect the failure of bundle re-parameterization from the local section \( s \) to \( s' \) induced by the isometry transformations, where \( s, s': U_s \cap U_{s'} \subset M \to G \).

To offset these aforementioned anomalies, in section 3 we are led to consider more generic holonomy anomalies, of which isometry anomalies are a special class. We give
criteria, ensuring that the holonomy as well as isometry anomalies are removed, by adding well-defined local counterterms in section 3.1. With these criteria, and after adding the appropriate local counterterms, we discuss the low-energy behavior of the minimal $\mathcal{N} = (0, 1)$ sigma models in section 3.2. In section 3.3 several concrete examples are given to illustrate the idea. We review the appropriate tools that we had developed before. The topological origin of the anomalies and counterterms are discussed in section 3.4.

In section 4, we begin to relate the holonomy and isometry anomalies to topological anomalies in a general context. In section 4.1, a discussion of the isometry anomalies in the general Kähler sigma models is given. The isometry anomaly in the pure geometric formulation is related to the topological chiral fermion anomaly in terms of the determinant line bundle, parallel to the relation between the non-Abelian gauge anomaly and chiral anomaly in gauge theories. In section 4.2, we give the determinant line bundle description for the holonomy anomaly for sigma models over homogeneous spaces. This provides a unified picture conceptually showing that the holonomy (gauge) anomaly and the topological anomaly are due to the nontriviality of a single determinant line bundle over the space of the fields.

2. Isometry anomalies

We will formulate this section by following the logic line of our previous work \cite{7}, where we construct sigma models on $S^{2N-1}$ and gauge the U(1) factor to deduce the corresponding CP$^N$ models by the fibration:

$$U(1) \xrightarrow{i} S^{2N-1} \xrightarrow{\pi} \mathbb{C}P^{N-1}.$$ 

Similarly, for homogeneous spaces, we also have a canonical fibration:

$$H \xrightarrow{i} G \xrightarrow{\pi} M.$$ 

Therefore, we first construct sigma models on the group manifold $G$, and gauge the subgroup $H$ to obtain a sigma model on the homogeneous space $M$. Analogously to the CP$^N$ case, to define a sigma model on $M$, one needs to specify a local patch $U \subset M$ and a section $s : U \rightarrow G$. To discuss the isometry anomalies on model $M$, we will show that an isometric transformation $l_k : M \rightarrow M$ will induce a change of section $s$ to $s'$, and thus, an $H$-gauge transformation. For the chiral fermions nontrivially coupled to these $H$-gauges, isometry anomalies will be produced. We will calculate them at the end of this section. For simplicity, we assume that $G$ is a connected, compact, semi-simple Lie group and that $H$ is a closed Lie subgroup.

2.1. Sigma models on $M$ through gauge formulation

For sigma models on $M$, the construction can be traced back to the 1970s and the Callan–Coleman–Wess–Zumino (CCWZ) coset construction \cite{30}. In this subsection, we will review this construction in the so-called ‘hidden’ local gauge formulation, which will eventually be explained in the language of the principal bundle.

To have a sigma model on $M$, as mentioned at the beginning, one must first construct a sigma model on the group manifold $G$, and then ‘gauge’ it down to that of space $M$. We will soon see that such a construction is just a formulation with a ‘hidden’ right local $H$-gauge, see \cite{31}, in which the Nambu–Goldstone bosons take values in group $G$ instead of $M$, and the right local $H$-gauge helps eliminate the redundant degree of freedom. Each time that one chooses a fixed gauge, it is the equivalent of choosing a local section to ‘pull back’ the model defined on bundle $G$ to base space $M$. Thus, the language of the principal bundle is an ideal
mathematical framework for interpreting the model, as well as further anomalies if there are any.

Since $G$ is semi-simple, one can always use the Killing form $K : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$, which is negative definite, to define the metric $\gamma$ of $G$. We consider the Lie algebra $\mathfrak{g}$ in its fundamental representation\(^5\), and normalize the anti-Hermitian generators, $F_A$, as:

$$K(F_A, F_B) = \text{Tr}(F_A F_B) = -\delta_{AB}. \quad (2.1.1)$$

In most of the notes, we focus on the sigma models defined on a simple group $G$. For the bosonic sigma model on such a group, the action is given by

$$S_G = \frac{1}{2\lambda^2} \int_{\Sigma} d^2x \text{Tr}(\partial_\mu g^{-1}\partial_\mu g) = -\frac{1}{2\lambda^2} \int_{\Sigma} d^2x \text{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\mu g), \quad (2.1.2)$$

where $\Sigma$ is the two-dimensional spacetime manifold, $g = g(x)$ takes the value on the matrix group $G$, and $\lambda^2$ is a coupling constant\(^6\). It is seen in equation (2.1.2) that $g^{-1}\partial_\mu g$ is the Maurer–Cartan form $\theta_g \equiv g^{-1}dg$ pulling back to the cotangent space of spacetime $\Sigma$. $g^{-1}dg \in T^*G$ on $G$ defines the map:

$$\theta_g = L_{g^{-1}e} : T_eG \rightarrow T_eG = \mathfrak{g}, \quad (2.1.3)$$

where $L_{g^{-1}e}$ is the pushforward map induced by the left translation $L_{g^{-1}}$, and $T_eG$ is the tangent space of $G$ at the group identity $e$, and we thereby have the metric $\gamma$ defined as

$$\gamma(X_g, Y_g) \equiv -K(L_{g^{-1}e}X_g, L_{g^{-1}e}Y_g) = -L_{g^{-1}}^* K(X_g, Y_g), \quad (2.1.4)$$

where $X_g$ and $Y_g$ are two vector fields at point $g \in G$.

On a local chart $\{U, \phi^\alpha\}$ near the identity $e \in G$, we can use the exponential map to express $g(x)$ as\(^7\)

$$g(x) = \text{Exp}(\delta^\alpha_\beta \phi^\alpha(x) F_A), \quad \text{for } A, \alpha = 1, 2, \ldots, \text{dim } G,$$

where $\phi^\alpha(x)$ are Nambu–Goldstone bosons. Therefore, one can express $\theta_g$ and $\gamma$ in a more familiar way as

$$\theta(\phi) = \theta^A_\alpha(\phi) d\phi^\alpha F_A,$$

$$\gamma_{\alpha\beta}(\phi) = \delta_{AB} \partial^A_\alpha \theta^B_\beta, \quad (2.1.5)$$

where $\theta^A_\alpha$ is the vielbein to decompose $\gamma_{\alpha\beta}$. Notice that the vielbein one-form is left invariant and right equivariant,

$$L_{g_0}^* \theta = (g_0 g)^{-1} d(g_0 g) = \theta$$

$$R_{g_0}^* \theta = (g g_0)^{-1} d(g g_0) = g_0^{-1} \theta g_0, \quad \text{for } g_0 \in G. \quad (2.1.6)$$

The metric $\gamma$ defined above is consequently left and right invariant,

$$L_{g_0}^* \gamma = R_{g_0}^* \gamma = \gamma, \quad \text{for any } g_0 \in G.$$

Therefore, the action $S_G$ has isometries $G_L \times G_R$.

\(^5\) It is true that the Killing form is defined by means of an adjoint representation of $G$, but for semi-simple Lie algebra, one is free to rescale a constant for each simple factor and thus we can choose the fundamental representation as our benchmark.

\(^6\) For a semi-simple Lie group $G$, there are as many coupling constants $\lambda^2_i$ as the number of its simple factors $G_i$, and the Killing form $K$ is the direct sum of $K_i$ for each $G_i$.

\(^7\) We use Greek and capital letters to distinguish the indexes of the curved coordinates from that of the flat vector space $g$. 

5 4 0 0 1 0 2 5 4 0 1
Now we consider group $G$ as a principal bundle with the fiber $H$ and base space $M \equiv G/H$, 
$$
H \rightarrow i \rightarrow G \xrightarrow{\pi} M
$$
with the projection 
$$
\pi : G \rightarrow M \\
g \mapsto gH
$$
(2.1.7)
and the $H$-group action acting from the right on $G$ satisfies $\pi(gh) = \pi(g)$.

To define a sigma model on $M$, we first notice from formula (2.1.7) that the coset space is defined by an $H$-action from the right. This motivates us to gauge part of the right isometries $H \subset G_R$ of the sigma model on group $G$. Considering $g(x) \rightarrow g(x)h(x)$ for a right $h(x) \in H$ transformation, the Maurer–Cartan form changes to: 
$$
g^{-1}dg \rightarrow h^{-1}(g^{-1}dg)h + h^{-1}dh.
$$
(2.1.8)
To make it gauge invariant, we introduce the gauge fields 
$$
A(x) = A_i(x)dx^i H, 
$$
(2.1.9)
where $H_i \in \mathfrak{h}$ for $i = 1, 2, \ldots, \text{dim} \mathfrak{h}$, taking values on the Lie subalgebra $\mathfrak{h}$. This transforms to 
$$
A \rightarrow h^{-1}Ah + h^{-1}dh
$$
to remedy the additional $h^{-1}dh$ part of the gauge transformation of $g^{-1}dg$. Therefore 
$$
g^{-1}dg - A \rightarrow h^{-1}(g^{-1}dg - A)h
$$
is gauge covariant. The action on $M$ is thus given by 
$$
S_M = -\frac{1}{2L^2} \int \Sigma d^2x \text{Tr}[(g^{-1}\partial_\mu g - A_\mu)(g^{-1}\partial^\mu g - A^\mu)].
$$
(2.1.10)
After the appropriate gauge fixing, the action above will give the usual CCWZ coset construction. To see this, let us work out the action near the group identity $e$, where we will decompose the Maurer–Cartan form $\theta_\mathfrak{g} = g^{-1}dg$ locally—see equation (2.1.5)—along a vertical space $\mathfrak{h}$ and a horizontal space complimentary to $\mathfrak{h}$.

Firstly, for a connected, compact and semi-simple Lie group $G$ with its closed subgroup $H$, the coset space $M$ is a reductive homogeneous space, i.e. the Lie algebra $\mathfrak{g}$ of $G$ can be decomposed as 
$$
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},
$$
(2.1.11)
where $\mathfrak{h}$ is the subalgebra corresponding to the subgroup $H$, and $\mathfrak{m}$ is a transverse subspace that is preserved by the adjoint action of $H$, i.e., 
$$
\text{ad}_g \mathfrak{m} = \mathfrak{m}
$$
(2.1.12)
In principle, the choice of subspace $\mathfrak{m}$ complimentary to $\mathfrak{h}$ is quite arbitrary. However, similar to the discussion of $\mathbb{C}P^{N-1}$ embedded into $S^{2N-1}$ [7], we can utilize the Killing form $K$, see equation (2.1.1), to define 
$$
\mathfrak{m} = \mathfrak{h}^\perp,
$$
so that the homogeneous space $M$ is a Riemann submersion of $G$, and the tangent space $T_oM$, with $o \equiv \pi(e)$, is identified with $\mathfrak{m}$. Under this decomposition, for $H_i \in \mathfrak{h}$ and $X_a \in \mathfrak{m}$, we have
\[
\text{Tr}(H_i H_j) = -\delta_{ij}, \quad \text{Tr}(X^a X^b) = -\delta_{ab}, \quad \text{and} \quad \text{Tr}(H_j X^a) = 0.
\]

(2.1.13)

Now \( \theta^g \) is decomposed as

\[
\theta^g(\phi) = e^g(\phi) + \omega^g(\phi) \equiv e^g_\phi X_a + \omega^g_\phi H_i,
\]

(2.1.14)

where \( e^g \) are called basic forms and \( \omega^g \) is the canonical connection for the bundle \( \pi : G \to M \).

Now we can use the gauge fields to eliminate the redundant degrees of freedom. For the CCWZ construction, the unitary gauge is chosen to remove all the Nambu–Goldstone bosons on \( \mathfrak{h} \), i.e.,

\[
g(\phi) = \text{Exp}(\delta^g_\alpha \phi^\alpha X_a),
\]

on a local chart \( \{ \phi^\alpha \in U \subset M \} \) near \( \phi \in M \). Such a choice, geometrically speaking, is the equivalent of specifying a local section \( s : U \subset M \to G \),

\[
s(\phi) \equiv g(\phi) = \text{Exp}(\delta^g_\alpha \phi^\alpha X_a).
\]

(2.1.15)

Therefore, one can use \( s^* : T^*G \to T^*M \), pulling back the basic forms \( e^g \) to \( M \),

\[
e^g = s^* e^g = e^g_{\phi^a} d\phi^a X_a,
\]

and thus define the vielbein one-form \( e^a \) on \( M \). Similarly, the canonical connection \( \omega^g \) is also a pullback:

\[
\omega^g = s^* \omega^g = \omega^g_{\phi^a} d\phi^a H_i
\]

as a connection one-form locally defined on \( M \).

After fixing the gauge by equation (2.1.15), we localized the Lagrangian on a local chart \( \{ \phi^\alpha \in U \subset M \} \):

\[
S_M = -\frac{1}{2\lambda^2} \int_{\Sigma} d^2x \text{Tr}(e^a_\phi X_a + \omega^i_\phi H_i - A^i_\mu H_i)^2,
\]

where

\[
e^a_\phi = e^a_{\phi^\alpha} \partial_\alpha \phi^\alpha, \quad \omega^i_\phi = \omega^i_{\alpha} \partial_\alpha \phi^\alpha
\]

are vielbeins and the connection one-form pulls back further to the spacetime \( \Sigma \) by the map \( \phi^\alpha : \Sigma \to U \subset M \).

Since the gauge fields \( A^i_\mu \) are classically nondynamical, one can solve and express them in terms of Goldstone fields \( \phi^\alpha \) by equations of motion, and we get

\[
A^i_\mu = \omega^i_{\alpha} \partial_\alpha \phi^\alpha.
\]

(2.1.16)

Getting this expression back to the action equation (2.1.10), we find the action \( S_M \) by the CCWZ construction,

\[
S_M = -\frac{1}{2\lambda^2} \int_{\Sigma} d^2x \text{Tr}(e^a_\alpha X_a e^b_\beta X_b) \partial_\mu \phi^\alpha \partial_\nu \phi^\beta
\]

\[
= \frac{1}{2\lambda^2} \int_{\Sigma} d^2x \delta_{\alpha\beta} e^a_\alpha e^b_\beta \partial_\mu \phi^\alpha \partial_\nu \phi^\beta \equiv \frac{1}{2\lambda^2} \int_{\Sigma} d^2x \gamma_{\alpha\beta} \partial_\mu \phi^\alpha \partial_\nu \phi^\beta.
\]

(2.1.17)

Note that

\[
\gamma_{\alpha\beta} = \delta_{\alpha\beta} e^a_\alpha e^b_\beta, \quad \text{for} \quad a, b, \alpha, \beta = 1, 2 \ldots \text{dim} \ m
\]

is the metric on \( M \) and \( e^a_\alpha \) is its vielbein, correspondingly.
2.2. $\mathcal{N}=(0,1)$ supersymmetric sigma model on $M$

In this subsection we will supersymmetrize the action of the sigma model on $M = G/H$, see equation (2.1.10). In two-dimensional spacetime, we have the Weyl–Majorana Grassmannian variable $\theta_R$ which helps form the smallest representation of the supersymmetry, i.e., the $(0, 1)$ supersymmetry. The superderivative in superspace is defined as

$$D_L = -i \frac{\partial}{\partial \theta_R} - \theta_R \partial_{LL}$$

satisfying

$$\{D_L, D_L\} = 2D_L^2 = 2i \partial_{LL}$$

where $\partial_{LL}$ denotes the partial derivative along the light-cone coordinate $x_L$, and $\partial_{RR}$ for that of $x_R$ in what follows. The integration over the Grassmannian variable $\theta_R$ is equal to the differentiation:

$$\int d\theta_R = \frac{\partial}{\partial \theta_R} = iD_L|_{\theta_R=0}.$$

An ordinary bosonic field $\phi$ will be promoted to its superversion $\Phi$, which consists of $\phi$ and a left-moving fermion $\psi_L$:

$$\Phi = \phi + i\theta_R \psi_L.$$ 

To supersymmetrize the action equation (2.1.10), besides the scalar superfield $g(\Phi)$, we also need the $(0, 1)$ supergauge multiplets $\{V_L, V_{RR}\}$ [2]. It is true that one can directly supersymmetrize the local form of the Lagrangian in equation (2.1.17), which is already localized on a certain patch of $M$, without introducing any auxiliary gauge fields. However, with the help of gauge fields, it is quite easy to track the information regarding the isometric transformations on the different local charts, and also facilitate a discussion of the holonomy anomalies in the next section.

The $(0, 1)$ supergauge potentials $\{V_L, V_{RR}\}$ are given as

$$V_L = \eta_L - \theta_R A_{LL},$$

$$V_{RR} = A_{RR} + i\theta_R \chi_R.$$  \hspace{1cm} (2.2.1)

Under the supergauge transformation

$$V_L \rightarrow \mathcal{H}^{-1} V_L \mathcal{H} + \mathcal{H}^{-1} D_L \mathcal{H},$$

$$V_{RR} \rightarrow \mathcal{H}^{-1} V_{RR} \mathcal{H} + \mathcal{H}^{-1} i\partial_{RR} \mathcal{H},$$  \hspace{1cm} (2.2.2)

where $\mathcal{H}$ is an arbitrary scalar superfield, one can remove field $\eta_L$ by choosing the Wess–Zumino gauge. After this choice of supergauge, the residual is the normal gauge transformation on the gauge field $A_{\mu} = (A_{LL}, A_{RR})$ and the Gaugino field $\chi_R$:

$$A_{\mu} \rightarrow h^{-1} A_{\mu} h + h^{-1} i\partial_{\mu} h,$$

$$\chi_R \rightarrow h^{-1} \chi_R h.$$ \hspace{1cm} (2.2.3)

where field $h$ is the bosonic component of superfield $\mathcal{H}$.

Now we have all the ingredients needed to supersymmetrize the Lagrangian equation (2.1.10). We promote the bosonic field $g(x)$ to a scalar superfield $\mathcal{G}(x, \theta_R)$, taking values on group $G$. The bosonic part of $\mathcal{G}$ is $g(x)$ while the fermionic part is defined as
\[ \psi_L = \psi_L^A F_A \equiv G^{-1} D_L \mathcal{G}_{|\theta_R=0}, \]  
\tag{2.2.4} \]

and thus,

\[ \mathcal{G} = g + i\theta_R g \psi_L^A F_A, \]

where \( F_A \) are the generators of the Lie algebra \( \mathfrak{g} \) in a fundamental representation as before. Under this definition, the fermionic action of \( S_M^{(0,1)} \) becomes canonical. The gauge fields \( A_{\mu} \) are also enhanced to \( \{ V_L, V_{RR} \} \), taking values on the Lie algebra \( \mathfrak{h} \).

The \( (0,1) \) supersymmetric action now written in the superspace is given as

\[ S_M^{(0,1)} = \frac{i}{2} \int_{\Sigma} d^2x \int d\theta_R \text{Tr} \{ (G^{-1} D_L \mathcal{G} - V_L) (G^{-1} \partial_{RR} \mathcal{G} - V_{RR}) \}. \]  
\tag{2.2.5} \]

Superfield \( \mathcal{G} \) admits an \( \mathcal{H} \) super-gauge transformation as designed,

\[ \mathcal{G} \rightarrow \mathcal{G} \mathcal{H}. \]

To obtain the action in components, we impose the Wess–Zumino gauge to remove \( \eta_L \),

\[ V_L = -\theta_R A_{LL}. \]

Integrating \( \theta_R \) out, we get

\[ S_M^{(0,1)} = -\frac{1}{2} \int_{\Sigma} d^2x \text{Tr} \{ (g^{-1} \partial_{LL} g - A_{LL}) (g^{-1} \partial_{RR} g - A_{RR}) \} - \frac{i}{2} \int_{\Sigma} d^2x \text{Tr} \{ \psi_L (\partial_{RR} + g^{-1} \partial_{RR} g + A_{RR}) \psi_L \} \]

\[ - \frac{i}{2} \int_{\Sigma} d^2x \text{Tr} (\chi_R \psi_L). \]  
\tag{2.2.6} \]

The action still has ordinary \( H \)-gauge invariance,

\[ g \rightarrow gh, \quad \psi_L \rightarrow h^{-1} \psi_L h; \]
\[ A_{\mu} \rightarrow h^{-1} A_{\mu} h + h^{-1} \partial_{\mu} h, \]
\[ \chi_R \rightarrow h^{-1} \chi_R h. \]  
\tag{2.2.7} \]

As before, we decompose \( g^{-1} \partial_{\mu} g \) and \( \psi_L \) along the horizontal and vertical directions,

\[ g^{-1} \partial_{\mu} g = \epsilon_{\mu}^i X_i + \omega_{\mu}^i H_i, \]
\[ \psi_L = G^{-1} D_L \mathcal{G}_{|\theta_R=0} = \psi_L^a X_a + \psi_L^i H_i. \]  
\tag{2.2.8} \]

Since \( A_{\mu} \) and \( \chi_R \) are nondynamical, we solve these constraints by varying \( A_{\mu} \) and \( \chi_R \), and have

\[ A_{RR}^i = \omega_{RR}^i, \]
\[ A_{LL}^i = \omega_{LL}^i + \frac{i}{2} C_{ab}^i \psi_L^a \psi_L^b, \]
\[ \psi_L^i = 0, \]  
\tag{2.2.9} \]

where we have used equation (2.1.13), the anti-symmetric property of \( \psi_L^a \), and the commutator relations,

\[ [H_i, H_j] = C_{ij}^k H_k, \quad [H_i, X_a] = C_{ia}^b X_b, \quad [X_a, X_b] = C_{ab}^c H_c. \]  
\tag{2.2.10} \]

From the first two formulas in equation (2.2.10) above, we see that under this decomposition, the Lie subalgebra \( \mathfrak{h} \) reducibly acts on \( \mathfrak{g} \), or say, the adjoint representation of \( \mathfrak{g} \) restricted to \( \mathfrak{h} \) is decomposed as
where \( \rho \) denotes the representation of \( \mathfrak{h} \) acting on subspace \( m \). We will soon see that this observation is very important for determining whether the anomalies produced by chiral fermions can be removed, and for us to write the most general action.

Substituting equation (2.2.9) back into the action (2.2.6), we have

\[
S_{M}^{(0,1)} = \frac{1}{2} \int_{\Sigma} d^{2}x \delta_{ab} e_{LL}^{a} e_{RR}^{b} \\
+ \frac{i}{2} \int_{\Sigma} d^{2}x \psi_{L}^{a} \left( \partial_{RR} \delta_{ac} + \omega_{RR}^{b} C_{abc} + \frac{1}{2} e_{RR}^{b} C_{abc} \right) \psi_{L}^{c}.
\]  

(2.2.12)

This is not the final result yet because we should assign the coupling constants \( \lambda_{a} \), and this is related to how the vielbeins \( e_{\mu}^{a} \) and fermion \( \psi_{L}^{a} \) transform under the gauge transformation. From equation (2.2.7) and (2.2.8), writing the transformations in components:

\[
e_{\mu}^{a} \rightarrow \rho (h^{-1})_{b}^{a} e_{\mu}^{b}, \quad \psi_{L}^{a} \rightarrow \rho (h^{-1})_{b}^{a} \psi_{L}^{b}, \\
\omega_{L}^{a,b} \rightarrow \rho (h^{-1}) a \rho(h) + \rho (h^{-1}) \partial_{\mu} \rho(h)_{\mu}^{a,b},
\]

(2.2.13)

where \( \rho \) denotes the \( H \)-isotropy representation on \( m \) corresponding to \( \rho \), i.e.,

\[
h^{-1} X_{a} h \equiv \rho (h^{-1})_{b}^{a} X_{b} \quad \text{for} \quad X_{a,b} \in m.
\]

Equation (2.2.13) implies that the tangent bundle \( TM \) is identified with the associated \( H \)-principal fiber bundle with the vector space \( m \),

\[
TM \simeq G \times_{\rho} m,
\]

(2.2.15)

on which the vielbeins \( e_{\mu}^{a} \) and fermions \( \psi_{L}^{a} \) are the basic form, and \( \omega_{L}^{a,b} \) is the connection in the \( \rho \) representation. Now if \( \rho \), or equivalently \( \rho \), is further reducible on \( m \),

\[
\rho = \bigoplus_{c} \rho_{c},
\]

we can assign a different coupling constant \( \lambda_{c} \) to each independent representation \( r_{a} \) of \( H \) on \( m \). Based on the argument above, we rescale the vielbein \( e_{\mu}^{a} \) and fermion \( \psi_{L}^{a} \) with respect to the representations they belong to,

\[
e_{\mu}^{a} \rightarrow \frac{1}{\lambda_{a}} e_{\mu}^{a}, \quad \psi_{L}^{a} \rightarrow \frac{1}{\lambda_{a}} \psi_{L}^{a},
\]

(2.2.16)

and the action changes to

\[
S_{M}^{(0,1)} = \frac{1}{2 \lambda_{a}^{2}} \int_{\Sigma} d^{2}x \delta_{ab} e_{LL}^{a} e_{RR}^{b} \\
+ \frac{i}{2 \lambda_{a}} \int_{\Sigma} d^{2}x \psi_{L}^{a} \left( \partial_{RR} \delta_{ac} + \omega_{RR}^{b} C_{abc} + \frac{1}{2} e_{RR}^{b} C_{abc} \right) \psi_{L}^{c},
\]

(2.2.17)

where we used the fact that the connection \( \omega_{RR}^{b} C_{abc} \) is block diagonal, and thus indexes \( a \) and \( c \) are forced in the same representation, say \( \lambda_{a} = \lambda_{c} \). Furthermore, the anticommutativity of fermions \( \psi_{L}^{a} \) requires us to antisymmetrize the indexes \( a \) and \( c \) of term \( \psi_{L}^{a} \).

8 Since we chose normalized and orthogonal bases \( \{X_{a}\} \), \( \rho \) is in fact an orthogonal real representation of \( H \) on \( m \), i.e., \( \rho (h^{-1}) = \rho (h^{-1})^{\ast} \), by which equation (2.2.13) can be verified.

9 If there exist right isometries after we gauge out \( H \subset G_{b} \), the number of coupling constants will be as many as the independent representation of the normalizer of \( H \). For more details, we refer readers to reference [32].

10 \( \tau \) and \( \kappa \) are respectively the torsion and contorsion of the homogeneous space \( M \), see also [32].
\[ \tau_{abc} \equiv -\frac{1}{2} \frac{\lambda_a}{\lambda_b \lambda_c} \nabla_{abc} \rightarrow \kappa_{abc} \equiv \frac{1}{2} \left( \frac{\lambda_b}{\lambda_a \lambda_c} - \frac{\lambda_c}{\lambda_a \lambda_b} \right) C_{abc}. \]

We finally have
\[ S_M^{(0,1)} = \frac{1}{2 \lambda_a^2} \int \tilde{\sigma} \partial_\mu \tilde{\epsilon}_a^\mu \tilde{\epsilon}_R^\nu \]
\[ + \frac{i}{2 \lambda_a^2} \int \tilde{\sigma} \partial_\mu \tilde{\epsilon}_a^\nu (\partial_R \kappa_{ac} + \omega^d_R \kappa_{abc} - \epsilon^d_R \kappa_{abc}) \tilde{\psi}_L^d. \tag{2.2.18} \]
\[ = \frac{1}{2 \lambda_a^2} \int \tilde{\sigma} \left[ \delta_{ab} \epsilon^a_R e^b_R + i \tilde{\psi}_L^a (\partial_R \kappa_{ac} + \tilde{\omega}^e_R \kappa_{abc}) \tilde{\psi}_L^e \right], \]
where the \( \kappa \) term is absorbed into connection \( \omega \) to define:
\[ \tilde{\omega}_{ac} \equiv \omega_{ac} - \kappa_{ac}, \]
as the Levi–Civita connection of the homogeneous space \( M \). Since field \( \kappa_{ab} \) is tensorial, under the \( H \)-gauge transformation, we still have
\[ \epsilon_a^\mu \rightarrow \rho(h^{-1}) \epsilon_a^\mu, \quad \tilde{\psi}_L^a \rightarrow \rho(h^{-1}) \tilde{\psi}_L^a, \]
\[ \tilde{\omega}_{ab} \rightarrow (\rho(h^{-1}) \tilde{\omega}_a \rho(h) + \rho(h^{-1}) \tilde{\partial}_b \rho(h)) \tilde{\psi}_L^a. \tag{2.2.19} \]

### 2.3. Isometry anomalies of the sigma model on \( M \)

In this subsection, we will disclose the relation between the isometries and \( H \)-gauge transformations, see equation (2.2.13), and then calculate the isometry anomalies of the action equation (2.2.18). For brevity, in what follows, also including the next section, we will only label one instead of two \( R \) or \( L \) as the subscription of all quantities when it leads to no confusion.

Now let us consider the isometries of the action. We start from the fibration:
\[ H \overset{i}{\rightarrow} G \xrightarrow{\pi} M, \]
with all (left) isometries \( l_k : M \rightarrow M \) induced from left translations\(^{11}\) \( L_k \):
\[ L_k : g(x) \rightarrow k g(x), \quad \text{for } k \in G, \tag{2.3.1} \]
and we have the following commuting diagram:
\[
\begin{array}{ccc}
G & \xrightarrow{l_k} & G \\
\pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{l_k} & M
\end{array}
\]
with \( \pi \circ L_k = l_k \circ \pi. \tag{2.3.2} \)

It is easily seen that these left translations keep action equation (2.2.6) invariant trivially since \( k \in G \) is a constant group element.

When investigating the isometric transformation \( l_k \) on \( M \), we are required to choose a local trivialization, or say, a local section \( s : U \subset M \rightarrow G \). Physically speaking, we fix a

\(^{11}\) As mentioned, there may also be right isometries on \( M \) induced by the right translation on \( G \) if the normalizer of \( H \) is larger than \( H \) itself. There are also corresponding right isometry anomalies, but discussion of them is similar to that of the left. More details can be found in [16].
gauge—for example the CCWZ coset construction where the unitary gauge is chosen (see equation (2.1.15))—and localize the action $S_{M}^{0,1}$ on $U_{\phi}$ by the coordinates $\{\phi^{\alpha}\} \in U_{\phi} \subset M$. More explicitly, we have

$$g = s(\phi). \quad (2.3.3)$$

Therefore, vielbeins $e_{\mu}^{a}$, connection $\omega_{\mu}^{ab}$ as well as the fermions $\psi_{L}^{a}$ are pulled back to $U_{\phi} \subset M$ and expressed by equations (2.3.3) and (2.2.8) as

$$s^{a}(e_{\mu}^{a}) = e_{\mu}^{a} = e_{\mu}^{a} \partial_{\mu} \phi^{\alpha} = -\text{Tr} \left( X^{a} s^{-1} \frac{\partial s}{\partial \phi^{\alpha}} \right) \partial_{\mu} \phi^{\alpha},$$

$$s^{a}(\omega_{\mu}^{ab}) = \omega_{\mu}^{ab} \partial_{\mu} \phi^{\alpha} = -\text{Tr} \left( H^{a} s^{-1} \frac{\partial s}{\partial \phi^{\alpha}} \right) \partial_{\mu} \phi^{\alpha} \mathcal{C}_{ib},$$

$$s^{a}(\psi_{L}^{a}) = -\text{Tr} \left( X^{a} s^{-1} \frac{\partial s}{\partial \phi^{\alpha}} \right) D_{L} \Phi^{a} |_{\phi = 0} = e_{\mu}^{a} \psi_{L}^{a}. \quad (2.3.4)$$

From now on, we will not label $s^{a}$ to distinguish these quantities as forms on the bundle $G \times_{\phi} m$ or locally pulled back to $U_{\phi} \subset M$. This should not lead to any confusion in context. Thanks to the gauge fixing, the action localized on $U_{\phi}$ is given as

$$S_{U_{\phi}}^{0,1}[\phi, \psi_{L}] = \frac{1}{2\lambda_{a}} \int_{\Sigma} d^{2}x \ e_{\mu}^{a} e_{\nu}^{b} e_{\lambda}^{c} \partial_{\mu} \phi^{\alpha} \partial_{\nu} \phi^{\beta} \partial_{\lambda} \phi^{\gamma}$$

$$+ \frac{i}{2\lambda_{a}} \int_{\Sigma} d^{2}x \ \psi_{L}^{a} \left( \partial_{R} \bar{\phi}_{ac} + \partial_{R} \phi^{a} \bar{\phi}_{ac} \right) \psi_{L}^{c}. \quad (2.3.5)$$

This action should be invariant under the isometric transformation

$$l_{k} : \phi \mapsto l_{k}(\phi). \quad (2.3.6)$$

We will show that vielbeins, connections and fermions are transformed under $l_{k}$ as a special type of $H$-gauge transformation, see equation (2.2.13). Then the invariance of action (2.3.5) is guaranteed.

To see this, one can directly calculate their Lie derivatives with respect to isometries $l_{k}$ (see [16] for example). Here instead we interpret this issue in the language of a fiber bundle, which we presented and explained in great detail for the case of $\text{CP}^{N-1}$ in [7]. For a given section $s$, or a fixed gauge, we map the local patch $U_{\phi}$ to $G$ by

$$s(\phi) = g \in G.$$

A left translation $L_{k}$ acting on $s(\phi)$ not only induces an isometric transformation $l_{k}$ on chart $\{\phi^{\alpha}\}$, but also changes the fixed gauge. When we consider the isometric transformations of quantities $e_{\mu}^{a}$, $\omega_{\mu}^{ab}$ and $\psi_{L}^{a}$ under the original fixed gauge, we are required to accompany them by an $H$-gauge transformation $h(\phi, k)$ to compensate for the change:

$$L_{k} s(\phi) h(\phi, k) = s(l_{k}(\phi)), \quad \text{for} \ k \in G. \quad (2.3.7)$$

Or in other words, the composition of $L_{k}^{-1} \circ s \circ l_{k}$ defines another section $s'$, see the commuting diagram:
Sections $s'$ and $s$ are related by the $H$-gauge transformation $h(\phi, k)$, i.e., equation (2.3.7),

$$ s'(\phi) = s(\phi) h(\phi, k). $$

Now, after the isometric transformation $l_k$, the vielbeins, connections and fermions are pulled back to $U_s$ by $s^*$ and are related to those pulled back by $s'$ as

$$ e'_\mu \rightarrow e^{\prime \alpha}_\mu = \rho(h^{-1}_{\alpha, \beta})_b^a e^b_\mu, \quad \psi^a_L \rightarrow \psi^{\prime a}_L = \rho(h^{-1}_{a, b})_b^a \psi^b_L, $$

$$ \omega^a_{\mu b} \rightarrow \omega^{\prime a}_{\mu b} = (\rho(h^{-1}_{a, b})_b^a) \omega^b_T + \rho(h^{-1}_{a, b})_b^a \partial_\mu \rho(h_{\alpha, \beta})_b^a, \quad (2.3.8) $$

where $h_{\alpha, \beta} \equiv h(\phi, k)$ for short. Infinitesimally, one can expand

$$ l_k \simeq 1 + \epsilon^A K_A(\phi), \quad L_k^{-1} \simeq 1 - \epsilon^A F_A, \quad \text{and} \quad h(\phi, k) \simeq 1 + \alpha^i(\phi, \epsilon) H_i, $$

and get them back to equation (2.3.7) to explicitly solve $K_A$, the Killing field for isometries $l_k$, and $\alpha^i$. However, it is unnecessary to know their explicit expression. We only need to know, infinitesimally,

$$ \delta_\epsilon e^\alpha_\mu = -\vartheta(\alpha)_b^a e^a_\mu, \quad \delta_\epsilon \psi^a_L = -\vartheta(\alpha)_b^a \psi^b_L, $$

$$ \delta_\epsilon \omega^{a}_{\mu b} = \partial_\mu \vartheta(\alpha)_b^a + [\omega^a_T, \vartheta(\alpha)_b^a], \quad (2.3.9) $$

where

$$ \vartheta(\alpha)_b^a \equiv \alpha^i \vartheta(H_i)_b^a = \alpha^i C^a_{ib}. $$

One can further show that contortion $\kappa^a_{ib}$ transforms tensorially,

$$ \delta_\epsilon \kappa^a_{ib} = [\kappa^a_{ib}, \vartheta(\alpha)_b^a] \quad \text{and thus} \quad \delta_\epsilon \omega^a_{\mu b} = \partial_\mu \vartheta(\alpha)_b^a + [\omega^a_T, \vartheta(\alpha)_b^a]. $$

Now, for isometry anomalies, we use action $S^{(0,1)}_U[\phi, \psi_L]$ to calculate the effective action. Similar to the discussion in [7], anomalies are only produced from fermionic integration effective action. We thereby integrate out the fermionic part of the action equation (2.3.5) and have

$$ i \mathcal{W}^s_f[\bar{\omega}_R] = \frac{i}{16\pi} \int_{\Sigma} d^2x \text{Tr} \left( \bar{\omega}_R \partial_\mu \partial_\nu \omega^\lambda_R \right) + \mathcal{O}(\bar{\omega}_R^3), \quad (2.3.10) $$

where the superscript $s$ denotes that our perturbative calculation is performed on the local chart $U_s$. Varying $\mathcal{W}^s_f$, we produce the isometry anomalies $\mathcal{I}_s$,

$$ \mathcal{I}_s = \delta_s \mathcal{W}^s_f = -\frac{1}{8\pi} \int_{\Sigma} d^2x \text{Tr}(\alpha \partial_\mu \bar{\omega}_R). \quad (2.3.11) $$

To conclude, in this chapter we have calculated the isometry anomalies of generic $(0, 1)$ supersymmetric sigma models defined on manifold $M = G/H$. To perform a perturbative calculation, we need to specify a local chart $U_k$ on $M$ to define the model and thus a section $s$ from $U_s$ to $G$. After integrating out the fermions, we find the effective action $\mathcal{W}_f$, which is also defined on the local patch $U_s$. However, in many cases the effective action does not bear the isometries $l_k$ for the $k \in G$ it had before, and thus produces isometry anomalies. We established a correspondence between the isometry anomalies and some specific $H$-gauge anomalies when considering the definition of the effective action $\mathcal{W}_f$ on the intersection of
two local patches \(U_i \cap U_j\), where the local patch \(U_i = l_i^{-1}(U_i)\) is induced by isometries. This observation actually inevitably leads us not only to consider anomalies localized on a specified coordinate or local chart, but also to evaluate whether the effective action \(\mathcal{W}_f\) can be consistently defined on different local patches and their intersections. If so, one is able to transit \(\mathcal{W}_f\) from patch to patch without producing any \(H\)-gauge anomalies, or holonomy anomalies. Since isometry anomalies are a specific type of \(H\)-gauge anomaly, they will vanish in this situation for certain. Otherwise, when a model suffers from holonomy anomalies, it is not even possible to globally define the quantum theory mechanically, and it thus makes no sense to consider its isometry anomalies. Therefore, in the next chapter, we will focus on holonomy anomalies and the criteria by which they vanish or can be canceled by counterterms.

3. Holonomy anomalies

In the last section, we argued that to define sigma models on Homogeneous spaces \(M = G/H\), prior to considering the isometry anomalies it is necessary for the theory to be independent of the choice of sections \(s : U_i \subset M \to G\), or free from holonomy anomalies. The holonomy anomalies will arise when we change from one section \(s\) to another \(s'\), or physically speaking, from a fixed gauge to another. Therefore, they correspond to an arbitrary \(H\)-gauge transformation, see equation (2.2.7) and equation (2.2.13), while isometry transformations are a special type of \(H\)-gauge transformation, see equation (2.3.8). Therefore, once the holonomy anomalies are removed, the isometry anomalies will automatically vanish as well. We will thus focus on the holonomy anomalies and their cancellation condition.

3.1. Anomaly matching condition

From equation (2.3.11), we know that \(\alpha\) and \(\omega_R\) take values in the \(\varrho\) representation of the \(\mathfrak{h}\) Lie subalgebra. On the other hand, to offset anomalies, we have bosonic fields \(g\) and \(A_\mu\) as ingredients to construct the possible local counterterms \(\mathcal{W}_{\varrho, c.t}[g, A]\). Therefore, the possible counterterms are restricted on in the fundamental representation \(F\) of \(g\) restricted on \(\mathfrak{h}\). Choosing the fundamental representation \(F\) is merely a convention, since we are free to redefine our coupling constants corresponding to other different representations.

Roughly speaking, the counterterm we can introduce is an analog of the gauged WZW term, which is well known to produce gauge anomalies when the gauge fields \(A_\mu\) taking values in \(\mathfrak{h}\) are not in an ‘anomaly free’ representation [33]. Our strategy is to use the anomalies produced by the gauged WZW term to match that from the chiral fermions. This method has been applied to construct worldsheet models for \((0, 2)\) string vacua since the 1990s—see the work of Berglund et al [34], where they introduced both left/right-handed fermions in different representations and the gauged WZW term, and arranged them to cancel out the anomalies among each other. Here, in the minimal \((0, 1)\) and \((0, 2)\) models, we instead restrict ourselves to only left-handed fermions in the isotropic representation \(\varrho\). Therefore, the anomalies between the gauged WZW term and the chiral fermions are not guaranteed to be canceled out, but only conditionally to be so. The condition to match these two anomalies, as we will show later, imposes a topological constraint on the target space \(G/H\), say the vanishing of the first Pontryagin class. From now on, to distinguish the representations \(\varrho\) and \(F\), we will use \(\text{Tr}_\varrho\) and \(\text{Tr}_F\) to label the representation under which we take the trace.

First, we will explore more regarding the structure of the effective fermionic action \(\mathcal{W}_f\). In what follows, we will not fix ourselves to any specific gauge, and will not solve for the
gauge field $A$ in terms of $g$ as equation (2.2.9), because it will help better track the information regarding the gauge transformations on $\mathcal{W}_f$ and, more importantly, give us an explicit expression of the fermionic effective action. We thus use equation (2.2.6) and rewrite the fermionic part as

$$S_f = -\frac{i}{2} \int d^2x \{ \text{Tr}_F \psi_L (\partial_R \psi_L + [A_R, \psi_L]) + \text{Tr}_F \psi_L (g^{-1} \partial_R g - A_R) \psi_L \}$$

(3.1.1)

with

$$\psi_L = \psi_L^a X^a, \quad A_R = A_R^i H_i.$$

The two parts of the above equation are classically separately gauge invariant. However, the second term $g^{-1} \partial_R g - A_R$ coupling to the fermions, transforms tensorially under an $H$-gauge transformation,

$$g^{-1} \partial_R g - A_R \rightarrow h^{-1} (g^{-1} \partial_R g - A_R) h,$$

while in the first term, the chiral fermions $\psi_L$ couple to the gauge fields $A_R$ and will produce genuine anomalies. If we can find counterterms to offset the anomalies from the first term in equation (3.1.1), the anomalies from the second one can also be removed by an analog of the Bardeen-like counterterm in two dimensions. Let us see how it works.

In fact, we can ask for a more explicit structure on the anomalous part of $\mathcal{W}_f$ in two-dimensional spacetime due to Polyakov and Wiegmann [35]. In two dimensions, one can parameterize the gauge fields as

$$A_R = h^{-1} \partial_R \tilde{h} \quad \text{and} \quad A_L = \tilde{h}^{-1} \partial_L \tilde{h}^\prime,$$

(3.1.2)

where fields $\tilde{h}(x)$ and $\tilde{h}^\prime(x)$ are elements in $H$ and under the gauge transformation

$$\tilde{h} \rightarrow h \tilde{h}, \quad \text{and} \quad \tilde{h}^\prime \rightarrow h^\prime \tilde{h}.$$

Notice that since $\tilde{h} = \tilde{h}^\prime$, $A_{\mu}$ is not a flat connection. One can solve $\tilde{h}$ and $\tilde{h}^\prime$ in terms of the Wilson lines of $A_R$ and $A_L$, although the expression is surely nonlocal,

$$\tilde{h}(x) = -P \text{e}^{-\int_{C_{\mu}} d\tilde{\epsilon} \cdot A_{\mu}}, \quad \text{and} \quad \tilde{h}^\prime(x) = -P \text{e}^{-\int_{C_{\mu}} d\tilde{\epsilon} \cdot A_{\mu}},$$

where $C_{\mu}$ is a path from a certain fixed point to $x$, and $P$ denotes the path-ordered integral. With the help of $\tilde{h}$, one can explicitly write down the anomalous part of $\mathcal{W}_f$. Let us first rewrite term $g^{-1} \partial_R g - A_R$ as

$$g^{-1} \partial_R g - A_R = g^{-1} \partial_R g - \tilde{h}^{-1} \partial_R \tilde{h} = g^{-1} \partial_R (g \tilde{h}^{-1}) (g \tilde{h}^{-1})^{-1} g.$$

(3.1.3)

Clearly, $g \tilde{h}^{-1}$ is gauge invariant. Actually, if we redefine the fermion $\psi_L$ as

$$\psi_L = \tilde{h}^{-1} \zeta_L \tilde{h} \quad \text{or in components} \quad \psi_L^a = \rho (\tilde{h}^{-1})^a b \zeta_L^b,$$

(3.1.4)

the action $S_f$ changes to

$$S'_f = -\frac{i}{2} \int d^2x \text{Tr}_F [\zeta_L \partial_R \zeta_L + \zeta_L (g \tilde{h}^{-1})^{-1} \partial_R (g \tilde{h}^{-1}) \zeta_L].$$

(3.1.5)

Notice that both $\zeta_L$ and $g \tilde{h}^{-1}$ are gauge invariant. After integrating out $\zeta_L$, the effective fermionic action is guaranteed to be gauge invariant as well,

$$\mathcal{W}_f^\prime (g \tilde{h}^{-1}) = -i \log \int D \zeta_L \text{e}^{iS'_f}.$$

(3.1.6)

Therefore, we can interpret that the anomaly equation (2.3.11) is raised in a functional determinant when we change fermionic measure,
\[ \int \mathcal{D}\psi_L = \int \mathcal{D}\alpha^{-1} \left[ \frac{\delta\psi_L}{\delta \xi_L} \right] \mathcal{D}\xi_L, \]

and we will calculate the determinant above. The method we will use is mainly based on \([35]\).

The determinant is an integrated version of anomaly equation (2.3.11). Now, since we keep the gauge fields explicitly, the anomaly equation becomes:

\[
A_{\alpha} = -\frac{1}{8\pi} \int_{\Sigma} d^2x \text{Tr}_c \alpha \partial_L \left( \frac{1}{2} A_R + \frac{1}{2} \omega_R - \kappa \right)
\]

\[
= -\frac{1}{8\pi} \int_{\Sigma} d^2x \text{Tr}_c \alpha \partial_L A_R - \frac{1}{16\pi} \int_{\Sigma} d^2x \text{Tr}_c \alpha \partial_L (\omega_R - A_R)
\]

(3.1.7)

where, in the second equality, we use

\[
\text{Tr}_{c,F_H, X_b} = 0,
\]

see equation (2.1.13). The first term in anomaly equation (3.1.7) corresponds to the first part of action equation (3.1.1):

\[ S^1_0 = -\frac{i}{2\lambda_0} \int_{\Sigma} d^2x \text{Tr}_{c L} \psi_L (\partial_R \psi_L + [A_R, \psi_L]) = \frac{i}{2\lambda_0} \int_{\Sigma} d^2x \text{Tr}_{c L} \psi_L (\partial_R \psi_L - \frac{1}{2} C_{a b} C_{i b} \psi_L), \]

where we write the action in components and rescale the fermions to make the coupling constants explicit. For \(A_R\) parameterized as \(\tilde{h}^{-1} \partial_R \tilde{h}\), we now aim to find an effective action \(\mathcal{W}^1_f[\tilde{h}]\) which corresponds to \(S^1_0\) and satisfies

\[ \delta_{\alpha} \mathcal{W}_f^1[\tilde{h}] = A^1_{\alpha} = -\frac{1}{8\pi} \int_{\Sigma} d^2x \text{Tr}_c (\tilde{h}^{-1} \partial_R \tilde{h}) \partial_L (\tilde{h}^{-1} \partial_R \tilde{h}), \]

where we also put \(\alpha = \tilde{h}^{-1} \partial_R \tilde{h}\). Due to Polyakov and Weigmann, the effective action can be solved as

\[ \mathcal{W}_f^1 = \mathcal{W}_{PW}[\tilde{h}] \equiv \frac{1}{16\pi} \int_{\Sigma} d^2x \text{Tr}_c (\tilde{h}^{-1} \partial_R \tilde{h}) (\tilde{h}^{-1} \partial_R \tilde{h}) - \frac{1}{24\pi} \int_{\Sigma} \text{Tr}_c (\tilde{h}^{-1} \partial_R \tilde{h})^3, \]

(3.1.8)

where, in the second term, \(\tilde{h} = \tilde{h}(x, t)\) has been extended\(^{12}\) to bulk \(B\) bounded by \(\Sigma\). It is well known that the second term is multi-valued and can be rewritten as a local form on spacetime \(\Sigma\); thus, we still have a local theory defined on \(\Sigma\) rather than the bulk \(B\).

Besides this part, there is also the second term left in the anomaly equation (3.1.7),

\[ A^2_{\alpha} = -\frac{1}{16\pi} \int_{\Sigma} d^2x \text{Tr}_c \alpha \partial_L (\omega_R - A_R). \]

We have argued that \(g^{-1} \partial_R g - A_R\) as well as \(\omega_R - A_R\) transform tensorially and thus do not produce anomalies themselves, unless they are coupled to gauge fields as probes. Therefore, we can easily verify that a Bardeen-like counterterm

\[ \mathcal{W}_f^2[\tilde{h}, \omega_R - A_R] = \frac{1}{16\pi} \int_{\Sigma} d^2x \text{Tr}_c (\omega_R - A_R) (\tilde{h}^{-1} \partial_R \tilde{h}), \]

(3.1.9)

satisfies

\[ \delta_{\alpha} \mathcal{W}_f^2[\tilde{h}, \omega_R - A_R] = A^2_{\alpha}, \]

and, thus, is the second part of the anomalous effective action.

\(^{12}\) The extensions of \(\tilde{h}(x, t)\), and also of \(g(x, t)\) later, are always assumed to exist. We will present situations when \(\pi_2(\tilde{H})\) or \(\pi_2(G)\) are nontrivial in our future work on global anomalies.
Overall, we explicitly solve the anomalous part of the effective action $\mathcal{W}_f$, and the whole effective action $\mathcal{W}_f$ is given as
\[
\mathcal{W}_f = \mathcal{W}_f^1[\hat{h}] + \mathcal{W}_f^2[\hat{h}, \omega_R - A_R] + \mathcal{W}_f^3[g\hat{h}^{-1}]
\]
\[= -\frac{1}{24\pi} \int_{\hat{h}(\mathbf{R})} \text{Tr}_f(\hat{h}^{-1}d\hat{h})^3 + \frac{1}{16\pi} \int_{\Sigma} d^2x \text{Tr}_f \omega_R(\hat{h}^{-1}\partial_i \hat{h}) + \mathcal{W}_f^3[g\hat{h}^{-1}].
\] (3.1.10)

Now, based on the anomalous effective action above, we seek the conditions and counterterms $\mathcal{W}_{c.t.}\{g, A_R\}$. The key hint from equation (3.1.10) is that we need an analog of term $\text{Tr}_f(\hat{h}^{-1}d\hat{h})^3$. It should first have the same gauge transformation rule as $\hat{h}^{-1}d\hat{h}$, and secondly be able to pull back to spacetime $\Sigma$ to define our theory in two dimensions. However, the only ingredient we have with which to satisfy the two conditions is
\[
\mathcal{W}_{c.t.} \sim \text{Tr}_f(g^{-1}dg)^3.
\]

An infinitesimal $H$-gauge transformation, see equation (2.2.7), is given as:
\[
\delta_g(g^{-1}dg) = d\alpha^i H_i + [g^{-1}dg, \alpha^i H_i], \quad \text{for } \delta_g g = g\alpha^i H_i,
\]
where we explicitly display $\alpha$ above taking the values in $F(\mathfrak{h})$. Therefore, we have:
\[
\delta_g \text{Tr}_f(g^{-1}dg)^3 \sim \text{Tr} \alpha d(g^{-1}dg) \sim \alpha^i d\omega^i \text{Tr} H_i H_j.
\]

Since we have already normalized the generators $H_i$ in equation (2.1.13), as
\[
\text{Tr}_f H_i H_j = -\delta_{ij}, \quad \text{for any } H_{i,j} \in \mathfrak{h},
\]
the anomaly matching condition, under our conventions, is
\[
\text{Tr}_f H_i H_j = c \text{ Tr}_f H_i H_j = -c \delta_{ij}, \quad \text{for any } H_{i,j} \in \mathfrak{h},
\] (3.1.11)
with some constant $c$. So long as the anomaly matching condition is satisfied, we can construct the counterterms $\mathcal{W}_{c.t.}$ as
\[
\mathcal{W}_{c.t.} = \frac{c}{24\pi} \int_{\hat{h}(\mathbf{R})} \text{Tr}_f(g^{-1}dg)^3 - \frac{c}{16\pi} \int_{\Sigma} d^2x \text{Tr}_f A_R(g^{-1}\partial_i g).
\] (3.1.12)

One can verify that when equation (3.1.11) is met,
\[
\delta_g \mathcal{W}_{c.t.} + A_0 = 0.
\]

At last, combining equation (3.1.12) and equation (3.1.10), we would expect the modified fermionic action $\mathcal{W}_f^{\text{eff}}$ to be gauge invariant,
\[
\mathcal{W}_f^{\text{eff}} = \mathcal{W}_f + \mathcal{W}_{c.t.} = \frac{c}{24\pi} \int_{\hat{h}^{-1}(\mathbf{R})} \text{Tr}_f[(g\hat{h}^{-1})^{-1}d(g\hat{h}^{-1})]^3 + \mathcal{W}_f^3[g\hat{h}^{-1}].
\] (3.1.13)

### 3.2. Comments on counterterms

So far, we have derived the anomaly matching condition equation (3.1.11), based on which the gauge invariant effective action, equation (3.1.13), is constructed above. There are some interesting results and comments that we want to make.

#### 3.2.1. Anomaly matching condition

The anomaly matching condition is a group theoretical result. In principle, if we understand how a subgroup $H$ is embedded into $G$, we can determine, by equation (3.1.11), whether a minimal $(0, 1)$ supersymmetric sigma model can
be well defined. Actually, the statement is topological, as we will show in subsection 3.4 that 

\[ \text{equation (3.1.11)} \]

will be satisfied if and only if the first Pontryagin form of \( M \) vanishes, i.e., \( p_1(M) = 0 \).

3.2.2. \( \mathcal{W}_{\text{eff}} \) incorporated with (0, 1) supersymmetry. Until now, besides requiring the (0, 1) supersymmetry on model building, we have not fully considered the role that supersymmetry may play in the game. The counterterm \( \mathcal{V}_{\ell \xi} \) that we added is apparently nonsupersymmetric, but is required to define our theory. Now we want to proceed one step further and find the gauge invariant fermionic action \( \mathcal{W}_{\text{eff}} \). For brevity, we use \( \varphi \equiv g \hat{h}^{-1} \) as the gauge invariant field, and \( \mathcal{W}_{\text{eff}} \) is rewritten as

\[ \mathcal{W}_{\text{eff}}[\varphi] = \frac{c}{24\pi} \int_{\varphi(B)} \text{Tr}_F(\varphi^{-1}d\varphi)^3 + \mathcal{W}_f[\varphi]. \]

The second term is due to a path integral over the fermion \( \zeta_L \), see equations (3.1.1) and (3.1.6),

\[ e^{i\mathcal{V}_{\ell \xi}[\varphi]} = \int D\zeta_L \exp \int \Sigma d^2x \left( \frac{i}{2\lambda^2} \right) \text{Tr}_F(\zeta_L \partial_R \zeta_L + \zeta_L \varphi^{-1} \partial_R \varphi \zeta_L), \]

which has its supersymmetric counterpart \( \mathcal{S}_\ell \), see equation (2.1.10). On the other hand, the first term, as a combination of anomalous and anomaly counterterms,

\[ \mathcal{V}_{\ell \xi} \equiv \frac{c}{24\pi} \int_{\varphi(B)} \text{Tr}_F(\varphi^{-1}d\varphi)^3, \]

has no superpartner. Therefore, we will supersymmetrize this term. Actually, the \( \mathcal{N} = (1, 1) \) supersymmetrization of \( \mathcal{V}_{\ell \xi} \) is well known in literature going back to the 1980s, see [36] and [37] for example. Here, we proceed similarly to construct an \( \mathcal{N} = (0, 1) \) supersymmetric action from \( \mathcal{V}_{\ell \xi} \). Since field \( \varphi \) is now gauge invariant, its (0, 1) superpartner is also gauge invariant, and thus must be \( \zeta_L \). The supersymmetrization of \( \mathcal{V}_{\ell \xi} \) can be formally performed in the (0, 1) superspace as equation (2.2.5):

\[ \mathcal{W}_{\ell \xi} = \frac{c}{16\pi} \int d^2x \int d\theta_R \text{Tr}_F(\Psi^{-1} \partial_R |\Psi| D_L \Psi, \Psi^{-1} \partial_R |\Psi|) \]

\[ = \frac{c}{24\pi} \int_{\varphi(B)} \text{Tr}_F(\varphi^{-1}d\varphi)^3 - \frac{ic}{16\pi} \int \Sigma d^2x \text{Tr}_F(\zeta_L \varphi^{-1} \partial_R \varphi \zeta_L), \]

where we define the superfield \( \Psi \), see equation (2.2.4):

\[ |\Psi|_{\theta_R = 0} \equiv \varphi \quad \text{and} \quad |\Psi^{-1}D_L \Psi|_{\theta_R = 0} \equiv \zeta_L. \]

As mentioned, for now all fields are gauge invariant, and one should not worry about anomalies for the fermionic part \( \mathcal{V}_{\ell \xi} \). Overall, we have a supersymmetric effective action:

\[ S^{(0,1)} = S_0 + S_f + \mathcal{W}_{\ell \xi}. \]

3.2.3. Renormalization flow and superconformal fixed point in the IR region. Now we want to investigate the nonperturbative behavior of the modified theories in the deep infrared region. It is interesting to realize that the modified theory contains the supersymmetric ‘WZW’ term with the gauge invariant variable \( \varphi = g \hat{h}^{-1} \). We are trying to argue that in an ad hoc gauge:
(a) for $M$ as a symmetric space, the ‘WZW’ action vanishes and the theory is equivalent to a bosonic sigma model with the left fermions decoupled, and therefore, supersymmetry should be broken in the IR region;
(b) for $M$ as a nonsymmetric homogeneous space with the nontrivial third cohomology $H^3(M) \neq 0$, the ‘WZW’ term corresponds to an element in $H^3(M)$, and the theory would flow to a (super)conformal fixed point in the IR region.

To illustrate part (a), we fix the gauge on variable $\varphi$, so that

$$e^{\varphi}^{-1} d\varphi \in \Omega^1(M) \otimes \mathfrak{m}, \quad \text{or say,} \quad e^{\varphi}^{-1} d\varphi = e^a X_a,$$

where $e^a$ will be shown as vielbein 1-forms on $M$ soon. This gauge can always be chosen, although $\varphi$ cannot be expressed in terms of an exponential map. This is because if we notice $g = \varphi \tilde{h}$, $\varphi$ is exactly a coset representative for $M = G/H$, and thus $e^{-\varphi} d\varphi$ is a 1-form on $T^*M$.

Now in this gauge, by the property of symmetric space

$$[m, m] \subset \mathfrak{h},$$

and the orthogonality equation (2.1.13), one verifies that:

$$\text{Tr}_F(e^{\varphi}^{-1} d\varphi) = 0, \quad \text{and} \quad \text{Tr}_F((\zeta_L e^{\varphi}^{-1} \partial_R \varphi \zeta_L)^*) = 0,$$

for $\zeta_L = \zeta^a X_a$ as well. Therefore, the fermion $\zeta_L$ is totally decoupled and free. Now let us turn to the bosonic part, see equation (2.1.10). We rewrite the action in the light-cone coordinate as

$$S_M = -\frac{1}{2\lambda^2} \int d^2x \text{Tr}_F[(g^{-1} \partial_R g - A_R)(g^{-1} \partial_L g - A_L)].$$

By using equations (3.1.3) and (3.1.2), we further express the action in terms of $\varphi$, $\tilde{h}$ and $A_L$:

$$S_M = -\frac{1}{2\lambda^2} \int d^2x \text{Tr}_F[(e^{-\varphi} \partial_R \varphi)(e^{-\varphi} \partial_L \varphi) + (e^{-\varphi} \partial_R \varphi)(\partial_L \tilde{h} \tilde{h}^{-1}) - (e^{-\varphi} \partial_R \varphi)(\tilde{h} A_L \tilde{h}^{-1})].$$

The last two terms vanish because of orthogonality again. Therefore, we finally have

$$S_M^{(0,1)} = -\frac{1}{2\lambda^2} \int d^2x \text{Tr}_F[(e^{-\varphi} \partial_R \varphi)(e^{-\varphi} \partial_L \varphi) + i(\zeta_L \partial_R \zeta_L)].$$

(3.2.3)

It is well known that the bosonic theory is asymptotic free. In the deep infrared region, there is a mass gap generated, while the free fermion $\zeta_L$ is chiral, and thus there is no way to pair the mass term; the supersymmetry will thereby be broken.

When we use equation (3.1.2) to parametrize the gauge fields, there are functional determinants to take into account,

$$\int D\mathcal{A}_R D\mathcal{A}_L = \int (\text{Det}_R \nabla_R)(\text{Det}_L \nabla_L) D\tilde{h} D\tilde{h}',$$

where $\nabla_{R,L} \equiv \partial_{R,L} + [A_{R,L}]$. The product of the two determinants is gauge invariant, and gives an additional Polyakov–Wiegmann functional [38, 39],

$$(\text{Det}_R \nabla_R)(\text{Det}_L \nabla_L) = \exp(-i\lambda \mathcal{W}_{PW}[\tilde{h} \tilde{h}^{-1}]) \partial_R \partial_L,$$

where $c_H$ is the eigenvalue of the second Casimir operator for $\mathfrak{h}$ in its adjoint representation. Nevertheless, this additional term will not affect our argument above.

Now we are aiming to argue part (b) under the same gauge equation (3.2.2). Since for nonsymmetric homogeneous spaces, the Lie algebra structure constant $C_{abc}$ is nonzero, we will have a nonvanishing WZW term and a fermionic interaction, see equation (3.2.1). The WZW term

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is a closed and horizontal basic 3-form, which vanishes under action of the \( h \)-Lie derivative \( \mathcal{L}_h \). Therefore, it is an element in \( H^3(M) \), when \( H^3(M) = 0 \), see [40] and [41]. Combining this term with the original bosonic action, see equation (3.2.3), we have

\[
S_{M,b} = -\frac{1}{2\lambda^2} \int_{\Sigma} d^3x \text{Tr}_F (\varphi^{-1} \partial_{R} \varphi) (\varphi^{-1} \partial_{L} \varphi) + \frac{c}{24\pi} \int_{\varphi(I)} \text{Tr}_F (\varphi^{-1} d\varphi)^3.
\]

By the standard argument, we know that for\(^{13}\)

\[
\frac{\lambda^2}{8\pi} = 1
\]  
(3.2.4)

the bosonic theory will be conformally invariant. Now let us temporarily reside at this critical point and check the fermionic action. Combining equation (3.1.5) and the fermionic part of equation (3.2.1), we get

\[
S_{M,f} = S'_f - \frac{ic}{16\pi} \int_{\Sigma} d^3x \text{Tr}_F (\zeta_L \varphi^{-1} \partial_{R} \varphi \zeta_L)
\]

\[
= -\frac{ic}{16\pi} \int_{\Sigma} d^3x \text{Tr}_F (\zeta_L \partial_{R} \zeta_L + 2\zeta_L \varphi^{-1} \partial_{R} \varphi \zeta_L).
\]  
(3.2.5)

Similar to equation (3.1.4), we further rotate \( \zeta_L \) to define a new fermionic variable \( \xi_L \) satisfying

\[
\zeta_L \equiv \varphi^{-1} \xi_L \varphi.
\]

We obtain a free fermionic action on \( \xi_L \) as:

\[
S_{M,f} = -\frac{ic}{16\pi} \int_{\Sigma} d^3x \text{Tr}_F \xi_L \partial_{R} \xi_L.
\]

Such a redefinition for chiral fermions will certainly lead us to the Polyakov–Wiegmann functional as before, although our theory was gauge invariant as it was modified. Such an additional functional only seems to contribute a shift to level \( c \) of the conformal theory. To sum up, because of the existence of the WZW term, the theory will flow to a nontrivial infrared conformal fixed point where the fermionic fields are free, while due to conformal symmetries, there is no mass gap for the bosonic sector, and thus the \((0, 1)\) supersymmetry seems to hold.

3.3. Examples

In this subsection, we turn to the use of anomaly matching condition equation (3.1.11) to analyze some examples.

3.3.1. The simple Lie group \( G \). Our first examples are sigma models defined on simple Lie \( G \) groups. Although we construct sigma models on \( M = G/H \) by gauging an \( H \) subgroup of \( G \), see equation (2.2.5), the Lie group \( G \) itself is a symmetric space as well, i.e.,

\[
G \simeq G_L \times G_R / G_V.
\]

\(^{13}\) For simplicity, we only assume one coupling constant, say \( \lambda^2 \), even though \( \varphi \) may be reducible.
The Lie algebra of $GL \times GR$ is

$$\mathfrak{g}_L \oplus \mathfrak{g}_R, \text{ with that } \mathfrak{g}_L = \mathfrak{g}_R = \mathfrak{g}. $$

We label the generators $L_A \in \mathfrak{g}_L$ and $R_A \in \mathfrak{g}_R$, and their commutators are

$$[L_A, L_B] = C^C_{AB} L_C, \quad [R_A, R_B] = C^C_{AB} R_C, \quad [L_A, R_B] = 0.$$

The diagonal group $G_V$ acting on $GL \times GR$ gives its Lie subalgebra $H_A \in \mathfrak{g}_V$.

$$H_A = L_A + R_A.$$  

By using the Killing form with the normalization

$$\text{Tr}(L_A L_B) = -\delta_{AB}, \quad \text{Tr}(R_A R_B) = -\delta_{AB}, \quad \text{and } \text{Tr}(L_A R_B) = 0,$$

we find other generators belonging to $\mathfrak{m}$, complimentary to $\mathfrak{h} = \mathfrak{g}_V$,  

$$X_A = L_A - R_A,$$

and their commutator relationship given by

$$[H_A, H_B] = C^C_{AB} H_C, \quad [H_A, X_B] = C^C_{AB} X_C, \quad \text{and } [X_A, X_B] = C^C_{AB} H_C.$$

Therefore, $G \simeq GL \times GR/G_V$ is a symmetric space with the isotropy representation $\rho$,  

$$(\mathfrak{g}_L \oplus \mathfrak{g}_R)\big|_{\mathfrak{g}_V} = \mathfrak{g}_L \oplus \rho = \mathfrak{g}_V \oplus \mathfrak{ad} \mathfrak{g}_V,$$

and we see that

$$\text{Tr}_F(H_A H_B) = -T_G \delta_{AB} = \frac{T_G}{2} \text{Tr}_F(H_A H_B),$$

where $T_G$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$. By the anomaly matching condition, we know that the sigma model is well defined on the simple group manifold $G$.

Besides this, there is another reason for us to consider the sigma model on $G$. In the following, we want to argue that the ease of the holonomy anomalies, or say the independence of the theory from the choices of section $s$, is prior to that of the isometry anomalies. To illustrate this point, we first look at the action equation $(2.2.6)$, without the gauge and Gaugino fields $A_\mu$ and $\psi_R$.

$$S_G^{(0,1)} = -\frac{1}{2\lambda^2} \int_{\Sigma} \bar{J}^2 \text{Tr}(g^{-1} \partial_L g)(g^{-1} \partial_R g) + i \text{Tr}(\psi_L (\partial_R - g^{-1} \partial_R g) \psi_L). \quad (3.3.1)$$

For this action, in fact, we already fix a gauge, or say a section $s : U_i \subset G \rightarrow GL \times GR$. Near the identity of $GL \times GR$, one can assign the coordinates $\{\phi\} \in U_i$ and use the exponential map to write $s$ explicitly,

$$g = s(\phi) = e^{2 \phi L_A}. \quad (3.3.2)$$

From the above equation, we also know that the gauge fixing removes the degrees of freedom on $GR$. In the following we will show that under this gauge fixing there is no isometry anomaly.

We consider the isometries of the action equation $(3.3.1)$. One can either interpret these isometries as left isometries of $GL$ and right ones of $GR$, or as all left isometries acting on equation $(3.3.2)$. Isometries of $GR$, parameterized by $e^{\lambda R_s}$, acting on $s(\phi)$ from the left, break the fixed gauge,

$$e^{\lambda R_s} s(\phi) = e^{\lambda R_s + 2 \phi L_A}.$$
Therefore, we need to compensate for this with a $G_A$-gauge transformation $h = e^{-\epsilon^s h L}$, 
\[ e^{\epsilon^s h L} s(\phi) h = e^{2\phi^t L} e^{-\epsilon^s h L}, \]
where in the two equations above we used the fact that $L_A$ and $R_B$ commute, and are thus equivalent to a right $G_R$ group action. Since the isometries from $G_L$ need no gauge compensation, whereas the isometries from $G_R$ need a constant gauge compensation, say $h = e^{-\epsilon^s h L}$, neither $G_L$ nor $G_R$ isometries produce isometry anomalies in the choice of section $s$. One can also confirm this statement directly from the fermionic part of the action equation (3.3.1),

\[
\begin{align*}
\text{for } g &\rightarrow kg, \quad g^{-1} \partial_k g \rightarrow g^{-1} \partial_k g; \\
\text{for } g &\rightarrow k \bar{g}, \quad g^{-1} \partial_k g \rightarrow \bar{k}^{-1} (g^{-1} \partial_k g) \bar{k},
\end{align*}
\]
where $k, \bar{k} \in G_L, G_R$ are constant group elements. We get the same result that $g^{-1} \partial_k g$ is invariant under the left isometries, and tensorially transformed under the right ones. Hence, after integrating out the fermions from the action equation (3.3.1), the fermionic effective action will not produce isometry anomalies.

From the analysis above, it seems that the theory is well defined even without adding counterterms. However, in what follows, we will argue that introducing counterterms as equation (3.1.12) is a must. First we notice that there is a classically held discrete symmetry. In the bosonic part of the action equation (3.3.1), we realize that
\[ g \rightarrow g^{-1}, \quad S_{G,b} \rightarrow S_{G,b}. \]
On the other hand, the fermionic part is changed to
\[ S_{G,f} \rightarrow -\frac{i}{2\chi^2} \int_\Sigma d^3x \text{Tr} [\psi_L (\partial_R + g \partial_k g^{-1}) \psi_L]. \]
To get it back to $S_{G,f}$, one must rotate the chiral fermions $\psi_L$ simultaneously with $g$,
\[ g \rightarrow g^{-1}, \quad \psi_L \rightarrow g \psi_L g^{-1}. \]
Now, since the transformation of the chiral fermion is $x$-dependent, such a rotation will produce an integrated anomaly at a quantum level, which is a WZW-like term that breaks this symmetry explicitly. Adding a WZW-like counterterm as equation (3.1.12) serves exactly to offset this anomaly and keep the discrete symmetry above. Up to now it is still not necessary to introduce a counterterm, for this discrete symmetry is not an \textit{a priori} hypothesis in our theory. In fact, a four-dimensional sigma model describing Goldstone bosons does not apply to the symmetry $g \rightarrow g^{-1}$ individually, but it should be combined with the parity inversion on spacetime, see \cite{42}.

Nevertheless, in our case, the anomaly of this discrete symmetry is a signal of the nonequivalence of different choices of sections, or gauge fixings. To see this, let us recall the CCWZ coset construction on the group $G$ manifold, i.e., the unitary gauge equation (2.1.15),
\[ s'(\phi) = e^{\phi^s L_A} = e^{\phi^s L_A - \phi^s R_A}. \]
In this gauge, we describe our theory by writing its vielbeins and connection. From equation (2.1.14) and (2.3.5), we have the pullback Maurer–Cartan 1-form
\[ s'^{-1} ds' = e^{-\phi^s L_A} d e^{\phi^s L_A} + e^{\phi^s R_A} d e^{-\phi^s R_A} \]
because $L_A$ and $R_B$ commute. Furthermore, because $L_A$ and $R_B$ satisfy the same commutation rules, we will have the same functional form, $\theta(\phi)$ for example, for the two terms with arguments up to a minus sign, i.e.,
From this, we can read off the vielbein and connection 1-form in the unitary gauge,

\[ e^A(\phi) = \frac{1}{2} [\theta^A(\phi) - \theta^A(-\phi)], \quad \omega^A(\phi) = \frac{1}{2} [\theta^A(\phi) + \theta^A(-\phi)]. \]

Apparently, \( e^A(\phi) \) and \( \omega^A(\phi) \) are odd and even 1-forms separately. One can check that with the help of the parities of \( e^\prime \) and \( \omega^\prime \), the theory indeed has the discrete symmetry mentioned above, which in the coordinates \( \phi \) and fermions \( \psi_L \) is given as:

\[ \phi \rightarrow -\phi, \quad \psi_L \rightarrow -\psi_L. \]  

On the other hand, if we choose the section as equation (3.3.2), the Maurer–Cartan 1-form is

\[ s^{-1}ds = \theta^A(2\phi)L_A = \frac{1}{2} \theta^A(2\phi)H_A + \frac{1}{2} \theta^A(2\phi)X_A \equiv \omega^A(\phi)H_A + e^A(\phi)X_A. \]

In this gauge, the vielbeins and connection 1-form coincide \(^{14}\) with each other, but their parities are sacrificed.

Now we are in a situation in which we do not ask for the theory to have or deny the symmetry (3.3.4), but rather require it to be equivalently described in different choices of the sections, e.g. \( s \) or \( s^\prime \). We know that sections (3.3.2) and (3.3.3) are connected by an \( H \)-gauge transformation,

\[ s^\prime(\phi) = s(\phi)e^{-\phi\mu_h}. \]

Therefore, the theory equation (3.3.1) is required to be \( H \)-gauge invariant even if it has been shown to have vanishing isometry anomalies. Furthermore, with the counterterm (3.1.12) added, applying the result of section 3.2, we know that the \( \mathcal{N} = (0, 1) \) supersymmetric sigma model defined on the simple Lie group \( G \) is equivalent to its bosonic principal sigma model plus a free chiral fermion, which is also different from the one predicted by action equation (2.1.2).

### 3.3.2. Oriented real Grassmannian manifolds

Our second example is the oriented real Grassmannian manifold:

\[ M = \frac{\text{SO}(p + q)}{\text{SO}(p) \times \text{SO}(q)}. \]

We know that for \( p = 1 \) (or \( q = 1 \)), the manifold is just a sphere \( S^q \) with vanishing isometry anomalies [7]. Now we will consider the more general case by anomaly matching condition equation (3.1.11).

In the Grassmannian case, \( G = \text{SO}(p + q) \) and \( H = \text{SO}(p) \times \text{SO}(q) \) with standard embedding. We choose generators \( T_{AB} \) in the fundamental representation of the Lie algebra \( \mathfrak{g} = \mathfrak{so}(p + q) \) as:

\[ (T_{AB})_{CD} = -\delta_{AC}\delta_{BD} + \delta_{AD}\delta_{BC}, \quad \text{with} \ A, B, C, D = 1, 2, ..., p, p + 1, ..., p + q. \]

Their commutators are

\[ [T_{AB}, T_{CD}] = \delta_{AC}T_{BD} + \delta_{BD}T_{AC} - \delta_{AD}T_{BC} - \delta_{BC}T_{AD}. \]

\(^{14}\) It should be noticed that although they have the same form, they follow different transformation rules, see equation (2.2.13). This difference will not be detected by isometric transformation, for they only induce constant gauge transformation, as we showed.
and the normalized Killing form is
\[ \text{Tr}(T_{AB}T_{CD}) = -2(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \]

For the Lie subalgebra \( \mathfrak{h} = so(p) \oplus so(q) \equiv \mathfrak{h}_p \oplus \mathfrak{h}_q \) we label the generators as
\[ H_i \equiv T_{ij} \in so(p), \quad \text{for} \ i, j = 1, 2, \ldots, p, \]
\[ H_a \equiv T_{ab} \in so(p), \quad \text{for} \ a, b = p + 1, p + 2, \ldots, p + q, \]
where we use subscripts ‘i’ and ‘a’ to label two indices for brevity. The rest of the generators form the subspace \( m \) complimentary to \( \mathfrak{h} \), where we label them as
\[ X_{\alpha} \equiv T_{\alpha \beta} \quad \text{for} \ \alpha = 1, 2, \ldots, q, \quad \text{for} \ \beta = 1, 2, \ldots, p. \]

Now we will investigate the isotropy representation of \( \mathfrak{h} \) on \( m \). For \( \mathfrak{h} \otimes \mathfrak{h} = h \), we have the decomposition by equation (2.2.11),
\[ (ad \mathfrak{g})|_{\mathfrak{h}} = (ad \mathfrak{h}) \oplus \mathfrak{g} = (ad \mathfrak{h}_p \oplus ad \mathfrak{h}_q) \oplus \mathfrak{g}. \quad (3.3.5) \]

Actually, we need only care about \( \text{Tr}_p(H_i H_j), \text{Tr}_q(H_a H_b) \) and \( \text{Tr}_p(H_i H_b) \). From equation (3.3.5), we have the equality
\[ \text{Tr}_p = \text{Tr}_{ad \mathfrak{g}} - \text{Tr}_{ad \mathfrak{h}_p}, \quad (3.3.6) \]
while the latter two traces are easy to calculate by the commutation relationship and normalization above. After a short calculation,
\[ \text{Tr}_{ad \mathfrak{g}}(H_i H_j) = -2(p + q - 2)\delta_{ij}, \quad \text{Tr}_{ad \mathfrak{g}}(H_a H_b) = -2(p + q - 2)\delta_{ab}, \quad \text{Tr}_{ad \mathfrak{g}}(H_i H_b) = 0; \]
\[ \text{Tr}_{ad \mathfrak{g}}(H_a H_b) = -2(p - 2)\delta_{ij}, \quad \text{Tr}_{ad \mathfrak{g}}(H_a H_b) = -2(q - 2)\delta_{ab}, \quad \text{Tr}_{ad \mathfrak{g}}(H_i H_b) = 0. \]

Thus, we have
\[ \text{Tr}_p(H_i H_j) = -2q \delta_{ij}, \quad \text{Tr}_p(H_a H_b) = -2p \delta_{ab}, \quad \text{Tr}_p(H_i H_b) = 0. \]

Therefore, to meet the anomaly matching condition, we only have two cases in which minimal \( \mathcal{N} = (0, 1) \) supersymmetric sigma models exist.

Case 1: \( p = 1, M = S^q \) : \( \text{Tr}_p(H_a H_b) = \text{Tr}_p(H_a H_b) \).

Case 2: \( p = q, M = SO(2p)/(SO(p) \times SO(p)) \) : \( \text{Tr}_p(H_i H_j) = p \text{Tr}_p(H_i H_j) \).

The result of case 2 should have no further difficulty in being generalized to the case in which \( H \) contains more than two identical factors, \( H \simeq H_1 \times H_2 \times \cdots \times H_k \).

For the anomaly on the oriented real Grassmannian manifold \( M \), there is also another interesting observation that helps verify our anomaly matching condition. Instead of constructing sigma models on \( SO(p + q) \) followed by gauging its subgroup \( SO(p) \times SO(q) \), one can consider another fibration,
\[ SO(p) \xrightarrow{i} V_q(\mathbb{R}^{p+q}) \xrightarrow{\pi} M, \]
where \( V_q(\mathbb{R}^{p+q}) \simeq SO(p + q)/SO(q) \) is the Stiefel manifold which is the set of all the orthonormal \( q \)-frames in \( \mathbb{R}^{p+q} \). The sigma models built on the Stiefel manifold are always well defined, as we will show in our next example. Here let us just assume it and consider how a real Grassmannian sigma model can be constructed in the fibration above.

In [7] we introduced a dual formalism for the \( O(N) \) model. With a little modification, we can work out the \( \mathcal{N} = (0, 1) \) supersymmetric action on the Grassmannian manifold,
\[ S_M = \frac{1}{2g_0^2} \int d^2x \, \text{Tr} ((\nabla_R n)^T \nabla_L n + i\psi_L^T \nabla_R \psi_L), \]

\[ (n^T)^a_a n^a_b = \delta^a_b, \quad (n^T)^a_\alpha \psi^\alpha_{Lb} = 0, \quad \text{(3.3.7)} \]

where \( n^a_a \) and \( \psi^a_{La} (\alpha = 1, 2, \ldots, p + q), (a = 1, 2, \ldots, p) \) are real bosonic fields and their chiral fermion partners, and the covariant derivative \( \nabla \) is defined as

\[ (\nabla_{R,L} n)^a_a = \partial_{R,L} n^a_a - n^a_b \Lambda^b_{R,La}. \]

The model is obtained by gauging the color symmetries, i.e., those on indexes \('a, b ...'\), of the action on Stiefel manifolds. Thus, the action on the Stiefel manifold is obtained by removing gauge fields, and weakening the constraints on \( n \) and \( \psi_L \) as

\[ n^T \psi_L + \psi_L^T n = 0. \]

In a standard \((0, 1)\) superspace construction, one can introduce a super Lagrange multiplier \( \Lambda_{R}^a \)

\[ \Lambda_{R}^a = \lambda_{R}^a + \theta_{R} \sigma^a_{b}, \]

with the indexes \( a, b \) symmetrized. Thus, the superconstraint term is

\[ S_c = \int d^2x \, \int d\theta_R \text{Tr} \Lambda_R (N^T N - I), \]

where

\[ N = n + i\theta_R \psi_L \]

is the superfield version of fields \( n \) and \( \psi_L \) and \( I \) is the \( p \times p \) unit matrix. The sigma model on the Stiefel manifold is given by

\[ S_V = \int d^2x \, \int d\theta_R \left[ - \frac{1}{2g_0^2} \text{Tr} ((D_L N)^T \partial_{R R} N) + \text{Tr} \Lambda_R (N^T N - I) \right]. \]

Correspondingly, after gauging its ‘color’ symmetry \( \text{SO}(p) \), we obtain the sigma model on the Grassmannian manifold \( M \),

\[ S_M = \int d^2x \, \int d\theta_R \left[ - \frac{1}{2g_0^2} \text{Tr} ((D_L N - NV_L)^T (\partial_{R R} N - NV_{R R}) + \text{Tr} \Lambda_R (N^T N - I) \right], \]

where the supergauge multiplets \( \psi_{L, R} \) were introduced in equation \((2.2.2)\).

In fact, the above two sigma models can be obtained by considering the (classically) low energy limit of \( \mathcal{N} = (0, 1) \) two-dimensional gauge theories and Yukawa theories respectively. We build the Yukawa theories as

\[ S_Y = \int d^2x \, \int d\theta_R \left[ - \frac{1}{2g_0^2} \text{Tr} ((D_L N)^T \partial_{R R} N) + \text{Tr} \Lambda_R (N^T N - I) \right] 
\]

\[ + \int d^2x \, \int d\theta_R \left[ - \frac{1}{2g_0^2} \text{Tr} (\Lambda_R^a D_L \Lambda_R^a) \right]. \quad \text{(3.3.8)} \]

It is noticed that the coupling constant \( \lambda_0 \) has a mass dimension because \( \Lambda_R \) has mass a dimension 3/2. In the low energy limit, we put \( \lambda_0 \to \infty \), and obtain the action \( S_V \). In a sense, we can interpret the UV completion of the sigma model on the Stiefel manifold as a Yukawa theory, although the sigma model itself can be considered as a renormalizable theory in two-dimensions.
Similarly, let us find the UV completion of $S_M$ by gauging the Yukawa theory $S_Y$ and adding gauge sectors. Noticing that the Yukawa interaction $S_Y$ is gauge invariant, we have

$$S_{Y+V} = \int d^2x \int d\theta_R \left[ -\frac{1}{2g_0^2} \text{Tr} \left( (D_L N - NV_L)^T (\partial_{RR} N - NV_R) \right) + \text{Tr} \Lambda_R (N^T N - I) \right]$$

$$+ \int d^2x \int d\theta_R \left[ -\frac{1}{2\lambda_0^2} \text{Tr} (\Lambda_R^T (D_L \Lambda_R + [V_L, \Lambda_R])) \right]$$

$$+ \int d^2x \int d\theta_R \left[ -\frac{1}{4\epsilon_0^2} \text{Tr} \left( W_R (D_L W_R + [V_L, W_R]) \right) \right]$$

(3.3.9)

where $W_R$ is the field strength of the gauge potential $V_{L,RR}$,

$$W_R \equiv [D_L + V_L, \partial_{RR} + V_{RR}].$$

The couplings $\epsilon_0$ and $\lambda_0$ are of the same nonvanishing dimensionality, so in a low energy limit the last two terms fade away, and we obtain the sigma model $S_M$ on the Grassmannian $M$.

Now, due to the observation of ’t Hooft’s consistency condition, we should expect $S_{Y+V}$ and $S_M$ to produce the same anomalies or to be anomaly-free. Therefore, we focus ourselves on the gauge fields and bi-fermion interactions of the action $S_{Y+V}$ and calculate their anomalies. The relevant part of the Lagrangian is

$$L_{\text{G-A}} = \frac{i}{2g_0^2} (\psi_L^T)^a (\partial_R \psi_L - \psi_L A_R)^a + \frac{i}{2\lambda_0^2} \chi^k_R (\partial_L \chi^k_R + [A_L, \chi^k_R]) + \frac{i}{2\epsilon_0^2} \chi_{R,1} (\partial_L \chi^i_R + A_L^i \epsilon_{jk} \chi^k_R).$$

(3.3.10)

To see if there are gauge-anomalies produced, we need to consider a vector rotation and compare the gauge anomalies from the left and right fermions. For the right, since the gauge fields are in the fundamental representation, and we also need to sum up the flavors, say $\alpha$ indexes, we finally have:

$$A_R \sim (p + q) \text{Tr}_R (H_I H_J) = -2(p + q) \delta_{ij}.$$ 

On the other hand, the gauge fields interacting with Gauginos are in adjoint representation of $SO(p)$, and with $\lambda_R$ in the fundamental representation. We have:

$$A_L \sim \text{Tr}_{\text{adj}} (H_I H_J) + (p + 2) \text{Tr}_R (H_I H_J) = -4p \delta_{ij}.$$ 

Therefore, the gauge anomaly vanishes only when $p = q$, consistent with the result we obtained for the sigma model.

This observation regarding the correspondence of anomalies in two-dimensional gauge theories and sigma models could be useful for considering theories in the deep infrared region. We made some predictions in section 3.2. We expect to verify them in the anomaly-free gauge theories by considering the large $N$-expansion as well. We will present this work somewhere else in the future.

3.3.3. $(G \times U(1))/H$ for group $G$ semi-simple and $H$ simple. Our third example is inspired by equation (3.3.6), by which we will show that for homogeneous space $M = (G \times U(1))/H$ with the $G$ semi-simple and $H$ simple, the anomaly matching condition equation (3.1.11) will be satisfied. Therefore, there always exists a minimal $N = (0, 1)$ supersymmetric sigma model on them.
The proof of the above statement is quite transparent when both \( G \) and \( H \) are simple groups. Since \( G \) and \( H \) are simple, their Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \) will be simple, and thus contain no nontrivial ideals. Therefore, their adjoint representation \( \text{ad} \mathfrak{g} \) and \( \text{ad} \mathfrak{h} \) are irreducible respectively. By choosing an appropriate basis, generators \( H_i \in \mathfrak{h} \) will satisfy the following relations,

\[
\text{Tr}_{\text{ad}} \mathfrak{g}(H_i H_j) = -T_G \delta_{ij}, \quad \text{Tr}_{\text{ad}} \mathfrak{h}(H_i H_j) = -T_H \delta_{ij}; \quad \text{Tr}_{\text{ad}} \mathfrak{g}(H_i H_j) = -\delta_{ij}, \quad (3.3.11)
\]

where \( T_G \) and \( T_H \) are the dual Coxeter numbers of \( \mathfrak{g} \) and \( \mathfrak{h} \) respectively. Combining equation (2.2.11) and equation (3.3.6), we have

\[
\text{Tr}_G(H_i H_j) = -(T_G - T_H) \delta_{ij} = (T_G - T_H) \text{Tr}_H(H_i H_j). \quad (3.3.12)
\]

Now we can easily improve the above result. Because \( G \) is semi-simple, the only difference is that we have distinguished normalization for each of its simple factors. By assigning independent coupling constants to each simple factor \( G_a \subset G \), we can still normalize

\[
\text{Tr}_{\text{ad}} \mathfrak{g}(H_i H_j) = -\delta_{ij},
\]

as our convention. For the adjoint representation of \( \mathfrak{g} \), we have

\[
\text{ad} \mathfrak{g} = \oplus_i \text{ad} \mathfrak{g}_i.
\]

Therefore, the first relation in equation (3.3.11) turns out to be

\[
\text{Tr}_{\text{ad}} \mathfrak{g}(H_i H_j) = -\left( \sum \alpha \right) \delta_{ij}.
\]

On the other hand, since \( H \) is simple as before, the second relation of equation (3.3.11) holds. Therefore

\[
\text{Tr}_G(H_i H_j) = \left( \sum \alpha \right) - T_H \text{Tr}_H(H_i H_j).
\]

At last, the subgroup \( H \) contains no \( U(1) \) factors, so the above result is thus unchanged.

Now let us apply this result to some typical examples. We simply enumerate some classical homogeneous (symmetric) space, satisfying the above condition, on which a minimal \( N = (0, 1) \) supersymmetric sigma model can be constructed.

1. \( G \times U'(1) \) with \( G \) semi-simple: because the chiral fermions on \( U(1) \) are free;
2. Real, complex and symplectic Stiefel manifolds:
   \( \text{SO}(p + q)/\text{SO}(p), \ \text{SU}(p + q)/\text{SU}(p), \ \text{and} \ \text{Sp}(p + q)/\text{Sp}(p); \)
3. \( \text{SU}(n)/\text{SO}(n), \ \text{SU}(2n)/\text{Sp}(n), \ \text{and} \ \text{SO}(2n)/\text{Sp}(n); \) are all symmetric spaces.

From the argument above, we see that it is crucial for condition \( H \) to be simple. In fact, as we mentioned earlier, the anomaly matching condition is actually topological. In the next subsection, we will show (for the example when \( H \) is simple) that the first Pontryagin class \( p_1(M) \) will always vanish.

### 3.3.4. \( H \) containing \( U(1) \) factors

Before proceeding to give a topological (characteristic class) explanation of the anomaly matching condition equation (3.1.11), we want to consider another type of homogeneous space where the subgroup \( H \) in turn contains \( U(1) \) factors. We
are motivated by realizing that when $H$ contains $U(1)$ factors, the homogeneous spaces will have a complex structure, and thus $\mathcal{N} = (0, 1)$ supersymmetry will be enhanced to $\mathcal{N} = (0, 2)$. Unfortunately, however, we will soon see that many of these sigma models suffer from nonremovable anomalies, and thus do not exist.

First we want to clarify the method used in section 3.1 to derive the anomaly matching condition. Although the Polyakov–Wiegmann functional, see equation (3.1.10), was given in the context of non-Abelian gauge theories, this is also true when we have some Abelian $U(1)$ gauge fields. For these Abelian gauge fields, labeled by $B_{R}^{i}T_{i}$ for example, $T_{i} \subset \mathfrak{h}$ commutes with all other generators in $\mathfrak{h}$, and forms the nontrivial center of the Lie algebra $\mathfrak{h}$. Therefore, in the fermionic anomalous effective action equation (3.1.10), there are no WZW-like terms for them, but only the second one exists, i.e.,

$$\mathcal{W}_{\text{anom.}} = -\frac{1}{24\pi} \int_{\mathfrak{h}(\mathfrak{h})} \text{Tr}_{c}(\tilde{h}^{-1}d\tilde{h})^{3} + \frac{1}{16\pi} \int_{\Sigma} d^{2}x \text{Tr}_{c}\omega_{c}(\tilde{h}^{-1}\partial_{\bar{c}}\tilde{h} + \partial_{\bar{c}}u'T_{i}),$$

where $\tilde{h}$ as before parameterizes the non-Abelian gauge fields $A_{R}$, while $u'$ for $B_{R}^{i}$ satisfies

$$B_{R}^{i}T_{i} = \partial_{\bar{c}}u'T_{i}.$$ 

Meanwhile, the counterterm (3.1.12), which we are able to add, will also be transformed under Abelian gauge rotation. Therefore, the anomaly matching condition will be the same as before.

Nevertheless, because of the discrepancy above we will show in the following that the anomaly matching condition can never be fulfilled for $H = H' \times U(1)$ when $H'$ is semisimple. Therefore, lots of minimal $\mathcal{N} = (0, 2)$ supersymmetric sigma models, for example the complex Grassmannian manifolds $\text{U}(p + q)/(\text{U}(p) \times \text{U}(q))$ (except for $\mathbb{C}P^{1}$), are ruled out.

The proof is quite straightforward. With no loss of generality, let us only consider $H = H' \times U(1)$ for both the $G$ and $H'$ simple Lie groups. From a previous result, equation (3.3.12), we see that for $H_{ij}' \subset \mathfrak{h}'$

$$\text{Tr}_{c}(H_{ij}'H_{ij}') = (T_{G} - T_{H'})\text{Tr}_{c}(H_{ij}H_{ij}),$$

for $[H_{ij}', T] = 0$ and makes no contribution to the above equation, where $T$ is the generator of $U(1)$. For the same reason,

$$\text{Tr}_{c}T^{2} = \text{Tr}_{\text{ad}}gT^{2} - \text{Tr}_{\text{ad}}\bar{g}T^{2} = \text{Tr}_{\text{ad}}gT^{2} = T_{G}\text{Tr}_{c}T^{2}.$$

The anomaly matching condition is thus not satisfied. This finishes our proof.

A corollary we can obtain is that for homogeneous spaces $M = G/T'$ with $T = U(1)$ being the torus group, the anomaly matching condition equation (3.1.11) is satisfied. Therefore, the minimal $\mathcal{N} = (0, 2)$ supersymmetric sigma models on $G/T'$ can be well defined.

### 3.4. Topological origin of anomaly cancellation

In this subsection, we will establish a relation between the (local) anomaly matching condition and the global topological property of the homogeneous spaces $M = G/H$. More concretely, we will show that the anomaly matching condition equation (3.1.11) will be satisfied if and only if the first Pontryagin class on $M$ vanishes, i.e., $p_{1}(M) = 0$, which thereby agrees with Moore–Nelson’s constraint in the case of homogeneous spaces [8].

The main argument is based on a proposition in [43], see prop. (3.2), and a main theorem due to Borel and Hirzebruch, see theorem 10.7 in [44]. Here, we only rephrase the result in terms of anomaly matching condition equation (3.1.11). The idea can be intuitively
interpreted by equation (3.3.6),
\[ \text{Tr}_g(H,H) = \text{Tr}_{\text{ad}}(H,H) - \text{Tr}_{\text{ad}}(H,H). \]
Since by rescaling the coupling constants we always require the equality
\[ \text{Tr}_{\text{ad}}(H,H) = c' \text{Tr}_g(H,H), \]
the anomaly matching condition is thus equivalent to
\[ \text{Tr}_{\text{ad}}(H,H) = c'' \text{Tr}_g(H,H) \sim \text{Tr}_{\text{ad}}(H,H), \tag{3.4.1} \]
where \( c' \) and \( c'' \) are some constants. Now that we have evaluated the traces in the \( h \) and \( g \)-adjoint representations, one can express them by means of group theoretical invariants, say the symmetric functions of roots for \( h \) and \( g \) respectively. These symmetric functions are directly related to the characteristic classes of \( M \). We will explain that when equation (3.4.1) is satisfied, the first Pontryagin class \( p_1(M) = 0 \).

In the following, \( G \) is assumed to be a compact connected Lie group, \( H \) a closed subgroup of \( G \), and \( S \) and \( T \) are maximal tori of \( G \) and \( H \) respectively, chosen properly so that \( S \subset T \). Let \( s \subset t \) be the corresponding Lie algebras of \( S \) and \( T \), say the Cartan algebra of \( H \) and \( G \). We further set \( \{\beta_1, \ldots, \beta_k\} \subset s^* \) and \( \{\alpha_1, \ldots, \alpha_l\} \subset t^* \) as positive roots with respect to \( H \) and \( G \), arrange them satisfying
\[ \beta_1 = \alpha_1|_s, \beta_2 = \alpha_2|_t, \ldots, \beta_k = \alpha_k|_s, \]
and define
\[ \gamma_1 \equiv \alpha_{x+1}|_s, \gamma_2 \equiv \alpha_{x+2}|_s, \ldots, \gamma_{x-k} \equiv \alpha_{x}|_s, \]
which are called the roots complimentary to \( H \). With the help of the roots, one can rewrite the traces in equation (3.4.1) on the generators \( S_i \subset S \subset H \) and \( T_j \subset T \subset G \) as
\[ \text{Tr}_{\text{ad}}(S,S) = \sum_{b=1}^{k} \beta_b(S_i)\beta_b(S_j), \quad \text{and} \quad \text{Tr}_{\text{ad}}(T,T) = \sum_{a=1}^{l} \alpha_a(T_j)\alpha_a(T_j). \]
Therefore, the trace operator can be expressed in terms of quadratic symmetric polynomials on \( \sum \alpha_a^2 \) or \( \sum \beta_b^2 \) on the Cartan algebra \( s \) and \( t \), respectively. Actually, it is sufficient to focus only on the Cartan algebra \( s \) and \( t \). This is because in our case, the Lie algebras \( h \) and \( g \) can always be regarded as a direct sum of several simple algebras and \( u(1) \) factors. For each simple factor with a proper basis (the Cartan–Weyl basis for example), the traces on the Cartan algebra and other generators can be normalized the same, but they cannot be made the same among different simple factors.

On the other hand, one can identify \( \{\gamma\} \), the set of roots complementary to \( H \), with \( H^1(S;Z) \), the first cohomology class of tori \( S \), since they are integral functionals in \( \text{Hom}(\pi_1(S),Z) = \text{Hom}(H_1(S;Z),Z) \). The \( H^1(S;Z) \) can be further identified with \( H^2(BS;Z) \) via transgression, where \( BS \) is the classifying space of the tori \( S \). Therefore, the complimentary roots \( \{\gamma\} \) will be considered as elements in \( H^2(BS;Z) \). In what follows, we will only work under the real cohomology, which will considerably simplify our argument. First, the inclusion map
\[ i : S \to H \]
induces an isomorphism \( i^* \) on the cohomology rings of \( BH \) and \( BS \),
\[ i^* : H^*(BH;R) \simeq H^*(BS;R)^W(S,H), \tag{3.4.2} \]
where \( H^*(BS;R)^W(S,H) \) denotes those elements invariant under the Weyl group \( W(S,H) \). Secondly, from the universal fibration:
we have the fibration by module $H$,

$$G \xrightarrow{\pi} EG \xrightarrow{i} BG,$$

This induces the exact cohomology class chain

$$H^\ast(BG; R) \xrightarrow{q^\ast} H^\ast(BH; R) \xrightarrow{j^\ast} H^\ast(G/H; R).$$

Since $T(G/H)$ is the vector bundle associated with the $H$-principle bundle, see equation (2.2.15), the total Pontryagin classes $p(G/H)$ are pullback elements from some universal elements $a \in H^\ast(BH; R)$,

$$p(G/H) = f^\ast(a).$$

Now, with the identification in equation (3.4.2), we can express elements $a \in H^\ast(BH; R)$ in terms of the symmetric functions of the complimentary roots $\gamma_c$ in $H^\ast(BS; R)^{W(S,H)}$:

$$a = \prod_{c=1}^{t-s} (1 + \gamma_c^2).$$

Specific to the first Pontryagin class $p_1(G/H)$, we have

$$p_1(G/H) = j^\ast \left( \sum_{c=1}^{t-s} \gamma_c^2 \right).$$

From the exact sequence, a vanishing $p_1$ is equivalent to

$$\sum_{c=1}^{t-s} \gamma_c^2 \in \text{Im} q^\ast.$$ 

At last, it is noticed that

$$\sum_{c=1}^{t-s} \gamma_c^2 = \sum_{a=1}^{t} \alpha_a^2 a_a - \sum_{b=1}^{t} \beta_b^2 b_b,$$

while similar to equation (3.4.2), we have an isomorphism on $BG$ and $BT$,

$$H^\ast(BG; R) \simeq H^\ast(BT; R)^{W(T,G)}.$$ 

Since $\sum \alpha_a^2$ is always in $H^\ast(BG; R)$, the condition $p_1(G/H) = 0$, or say $\sum \gamma_c^2 \in \text{Im} q^\ast$, is the equivalent to the requirement $\sum \beta_b^2 \in \text{Im} q^\ast$. It is just the anomaly matching condition (3.4.1).

4. The determinant line bundle of homogeneous space sigma models

The aim of this section is twofold. On the one hand, we would like to see in the nonlinear formulation, how much our understanding of gauge anomalies can benefit our understanding of anomalies in a pure geometric model. Isometries on Riemannian manifolds come in various cases, which some gauge formulation is far from reaching. Still, one would like to be able to understand, for example, the relation between chiral anomalies, isometry anomalies and topological anomalies. On the other hand, so far as sigma models on homogeneous spaces are concerned, we would like to see how the gauge-like holonomy anomalies could arise with a view toward the determinant line bundle of certain Dirac operators parameterized by the space of the bosonic field. The hope is to gain a full picture that touches each of the four corners:
local versus global, gauge versus nonlinearity. A context like this can be useful in exploring interesting mathematical structures that closely tie up to each corner.

4.1. A digression on the Kähler sigma model anomaly in Fujikawa’s method

Here we shall look at the issue of local anomalies in the geometric formulation. Isometries in our system form a subset of the diffeomorphism group of the target manifold, which is accomplished via field-redefinition alone. We would like to explore whether such symmetries remain in the quantized system, and what the anomaly implies. Since we shall not be dealing with an unphysical degree of freedom, this is similar to the case of the axial anomaly, and Fujikawa’s method can thus be generalized to our current situation.

We first clarify the types of manipulations we shall use in the discussion. Consider a vector field on X, which is locally given by $V = K^i(x) \delta_i$, where $x$'s are the local coordinates on X. There are two possible manipulations that can be induced by $V$, namely the field redefinition and the infinitesimal diffeomorphism. The former is via

$$\phi^i \to \phi^i + \epsilon K^i(\phi)$$

where $\phi \in C^\infty(\Sigma, X)$ is a bosonic field. Since this does not correspond to any symmetry in the action, this will generally change the interaction. However, the field redefinition is a valid manipulation in field theory which should not cause any observational phenomena. The reason for this is that one can always get a contribution from the Jacobian of the path integral measure to overcome the change. The diffeomorphism transformation, on the other hand, is the aforementioned field redefinition together with the induced tensorial transformation for all geometric quantities. For example, under such a transformation, the metric tensor transforms according to

$$g_{ij}(\phi) \to g_{ij}(\phi) \frac{\partial \phi^i}{\partial \phi^k + \epsilon K^k(\phi)} \frac{\partial \phi^j}{\partial \phi^l + \epsilon K^l(\phi)},$$

and $\partial \phi^i$ transforms as a tangent vector. This definitely preserves the Lagrangian at the classical level. But in field theory language, when one interprets $g_{ij}$ as a function of the field $\phi$, there will also be an accompanying transformation for the ‘coupling constants’ of $\phi$ in $g_{ij}$. To make sense of these, one can view the (infinitely many) constants as background fields damped at classical values. The path integral measure would need further justification. This, however, is not the case that we are interested in.

The isometry symmetries are a subset of both classes. Defined solely by the field redefinition, it satisfies the property that

$$g_{ij}(\phi) \frac{\partial \phi^i}{\partial \phi^k + \epsilon K^k(\phi)} \frac{\partial \phi^j}{\partial \phi^l + \epsilon K^l(\phi)} = g_{ij}(\phi + \epsilon K(\phi)) + O(\epsilon^2) \quad (4.1.1)$$

and hence preserves the action, viewed as a local functional over the space of the fields at the classical level. The same is true for the quantum bosonic model, and this is a pure consequence of the property of field redefinition. Indeed, we are forced to have that the Jacobian from the path integral measure cancels the anomalous effective action. But a perturbative calculation shows that the effective action respects the isometry, thus forcing the path integral measure to respect the same symmetry up to an overall factor. Indeed, one can see this explicitly by writing down the measure explicitly, where we have used a standard volume form $[D\phi]$ on X associated with the metric $g$,

$$[D\phi] = \sqrt{\det g_{ij}} \, d\phi^1 \wedge \cdots \wedge d\phi^n. \quad (4.1.2)$$
If our model is coupled to chiral fermions, the path integral measure might not preserve such symmetries, and if this is true, nor shall the effective action after integrating out the fermions. This is the anomaly that we are interested in.

In supersymmetric models with a target manifold $X$, the fermions take the value in the tangent bundle $TX$. To build the path integral measure, one has to contract the indexes on $TX$ using a standard volume form. Together with the contribution from the bosonic part, we have that

$$[D\psi] = \frac{1}{\sqrt{\det g_{ij}}} d\psi^1_L \cdots d\psi^n_L d\psi^1_R \cdots d\psi^n_R.$$  \hspace{1cm} (4.1.3)

Note that $\psi_R$ are decoupled from our system, and we write them down to show the comparison between our case and the nonchiral case. Also, to use Fujikawa’s method, it is important to have Dirac fermions. Now we perform the isometry transformation induced by the Killing vector field $K_A$, where the index $A$ labels the isometries:

$$\phi^i(x) \to \phi^i(y) + \epsilon^A \int d^2x \ K_A^i[\phi(x)]\delta(x-y),$$

$$\psi^i_L(x) \to \psi^i_L(y) + \epsilon^A \int d^2x \ \partial_j K_A^i[\phi(x)]\delta(x-y)\psi^j_L(y).$$  \hspace{1cm} (4.1.4)

Note that the transformation is linear with respect to the fermionic degrees of freedom. Indeed, we can learn from the case of chiral anomaly that, as far as only the local anomalies are concerned, it is really the phase factor of such a transformation that matters.

Let us suppose we have Weyl fermions. Also, from here to the end of this section, we shall assume the target manifold to be Kähler to get the most elegant result. Write the path integral measure explicitly as

$$[D\psi] = \frac{1}{\sqrt{\det G}} d\bar{\psi}^\beta d\psi^\alpha \cdots d\bar{\psi}^\beta d\psi^\alpha.$$  \hspace{1cm} (4.1.5)

where each $\psi$ has two components $\psi_L$ and $\psi_R$. The metric $G$ expanded in the basis of $P_L\psi, \psi_L P_R, \psi_R P_L, \bar{\psi}_R P_R$ is given by table 2. Hence

$$\det G = \det g_{ij}.$$  \hspace{1cm} (4.1.6)

Now under the transformation

$$\psi^i(x) \to \psi^i(y) + \epsilon^A \text{Re}(\partial_j K_A^i[\phi(y)]) P_L \psi^j(y) + i \epsilon^A \text{Im}(\partial_j K_A^i[\phi(y)]) P_L \bar{\psi}_R^j(y),$$

$$\bar{\psi}^j(x) \to \bar{\psi}^j(y) + \epsilon^A \text{Re}(\partial_j K_A^i[\phi(y)]) \bar{\psi}^j(y) P_R + i \epsilon^A \text{Im}(\partial_j K_A^i[\phi(y)]) \bar{\psi}^j(y) P_R$$

$$= \bar{\psi}^j(y) + \epsilon^A \text{Re}(\partial_j K_A^i[\phi(y)]) \bar{\psi}^j(y) P_R - i \epsilon^A \text{Im}(\partial_j K_A^i[\phi(y)]) \bar{\psi}^j(y) P_R.$$  \hspace{1cm} (4.1.7)

Recall that the Jacobian, as in the pure bosonic case, has only a nontrivial real part, which cancels the change of $d\psi$ and $d\bar{\psi}$ induced by $\text{Re}(\partial_j K_A^i[\phi(y)])$. However, the transformation induced by $\text{Im}(\partial_j K_A^i[\phi(y)])$ is anomalous. The situation here is precisely the same as in the
case of the chiral anomaly, and for the time being, we take the bosonic degrees of freedom to
be external, or classical.

In Fujikawa’s method, infinitesimal isometry transformation gives the following extra
factor for the fermion integral measure:

$$\delta_{\epsilon} \cdot (\det iD)[\phi, \tilde{\phi}] = \exp \left(-i\epsilon A \int d^2x \, \text{Tr}[\Im (\partial_\gamma K_A^\dagger [\phi(x)]) \gamma_5] \right), \quad (4.1.8)$$

where the trace is taken over the basis from the right eigenstates of the Dirac operator

$$iD_{ij}^A \equiv ig_{ij}^A \gamma^a + g_{ij}^A \gamma^a \gamma^b \phi^b \gamma^c \phi^c + i\partial_\gamma \delta_{ij}^A P_L + i\bar{\gamma}^a \partial_\gamma \delta_{ij}^A P_R + i\bar{\gamma}^a \partial_\gamma \delta_{ij}^A P_R \quad (4.1.9)$$

and its left eigenstates. Evaluation of equation (4.1.8) is in general hard, due to the non-
flatness of $g_{ij}$ and the bosonic degree of freedom. But for the result in 2D, as an analog to the gauge
theory case [11], we obtain that up to the lowest order in the external fields,

$$\delta_{\epsilon} \cdot i \varepsilon_{\alpha ij} = \frac{i\epsilon A}{4\pi} \int \Im (\partial_\gamma K_A^\dagger) R_{ij}^A d\phi^j \wedge d\tilde{\phi} \wedge d\phi \wedge d\tilde{\phi} \wedge + \text{higher order terms in } \Gamma_{ij}^A. \quad (4.1.10)$$

Indeed, we only need the leading term from $\bar{\partial} \Gamma_{ij}^A$, which is also, up to a sign, the leading
term of the curvature tensor $R_{ij}^A$. Note that there is a special feature in the non-Abelian
anomaly (or, correspondingly, the linear isometry anomaly): if one only cares about the
lowest order in the ‘gauge’ field $A$ (or, correspondingly, the Christoffels $\Gamma$), then the
contribution to the non-Abelian anomaly is, up to a constant factor, the same as the Abelian
anomaly [11, 45]. This shows why the anomaly diagram in the perturbative calculation looks
similar to the one involved in the axial anomaly. The constant factor, in $2n$ dimensional
spacetime, might depend on the kinematics of the $(n + 1)$-gon Feynman diagrams. But in the
2D case this is extremely simple. To determine the full structure of such an anomaly, one can
either do a thorough calculation of equation (4.1.8), or use an argument like the Wess–
Zumino consistency condition, as mentioned in [11].

Now we calculate the explicit form of equation (4.1.10) using the second method. Recall
that the Abelian anomaly for a nonlinear sigma model over a Kähler target manifold in 4D is
given by

$$\text{Tr} \left[ R^2 \right] = (R_{ij}^m R_{kl}^m + R_{ij}^m R_{kl}^m) d\phi^i \wedge d\tilde{\phi}^k \wedge d\phi^k \wedge d\tilde{\phi}^i. \quad (4.1.11)$$

This combination is invariant with respect to isometry transformation, and lifts up to a
cohomology class. So, locally there is a 3-form $\omega^3$ such that $d\omega^3 = \text{Tr} \left[ R^2 \right]$. Note that making
use of Kähler geometry, the bootstrapping procedure is similar to that of non-Abelian gauge
theories [10], if we consider the following relation:

$$R_{ij}^m d\phi^i \wedge d\tilde{\phi}^j \equiv -d\Gamma^m_n - \Gamma^m_{ba} \wedge \Gamma^b_{an}, \quad \Gamma^m_n \equiv g^{ma} \partial_b g_{an} d\phi^i. \quad (4.1.12)$$

The isometry transformation is induced by the Killing vector field on the target manifold,
which gives

$$\delta_{\epsilon} = dK_A + K_A d, \quad (4.1.13)$$

where the vector field is of the form

$$K_A = K_A^i \partial_i + K_A^j \bar{\partial}_j. \quad (4.1.14)$$

Using the fact that the Kähler metric is compatible with the Killing vector fields, we get that

$$\delta_{\epsilon} \Gamma^m_n = -\partial (\partial_m K_A^a) - \partial_n K_A^a \Gamma^m_a + \partial_n K_A^m \Gamma^a_a \equiv -d (\partial_m K_A^a) - [\Gamma, \partial_n K_A^a]. \quad (4.1.15)$$
Finally we have that $\delta_s \omega_3^1 = d\omega_3^1[K_A]$, which bears the form

$$\omega_3^1 \propto (\partial_m K_A^n d\Gamma_n^m - \partial_n K_A^m d\Gamma^m_n). \quad (4.1.16)$$

This, at the first order level, coincides with equation (4.1.10).

### 4.2. The determinant line bundle analysis

In this section, we analyze the origin of holonomy and the global anomalies in the models on homogeneous spaces using the setting of the determinant line bundle. Recall that in the process of canceling the isometry variation, the counterterm is indeed predicted by the trivialization of a 4-form, which represents the first Pontryagin class. We first want to clarify here what we mean by the first Pontryagin class when we have a complex vector bundle over a smooth manifold. Indeed, the argument of [8] gave the anomaly in terms of a second real Chern character, which by definition is defined on the real vector bundles by taking the complexification first, and then applying the complex Chern character. In this way, one can verify that for real vector bundles, the second real Chern character precisely gives the $p_1$ of the bundle, and in the case of a complex vector bundle, this gives $2 \text{ch}_2$ of the complex bundle.

Using the Chern–Weil construction, choosing a connection $\Theta$ over the bundle $E$, one clearly sees that the 4-form representing the obstruction is

$$\int_{Y \times \Sigma} \text{ev}^* \text{tr} \left[ \frac{1}{2\pi^2} R^{\Theta} \right]^2,$$

where $Y$ is an arbitrary 2-cycle in the space of the bosonic field $C^\infty(\Sigma, X)$ and $\text{ev}$ is the evaluation map

$$\text{ev} : \Sigma \times C^\infty(\Sigma, X) \to X.$$

We want to trivialize the expression, and one of the sufficient conditions is that $\text{ch}_2$ vanishes before we pull it back. This Chern–Weil form of $\text{ch}_2$ can always be locally trivialized by the Chern–Simons transgression 3-form $CS(\Theta)$ on $X$. Moreover, if $\text{ch}_2$ is trivial, the Chern–Simons form is globally defined on $X$. Then there is a guarantee that the isometry variation of $CS(\Theta)$ is trivialized by a 2-form, which is able to compensate for the anomalous transformation of the functional determinant,

$$\delta_s CS(\Theta) = d(\omega_2), \quad \omega_2 = \text{tr}(\alpha d\Theta) \sim \delta_s \Gamma_{\text{eff}}.$$

So the counterterm in this case is given by $CS(\Theta)$. If, furthermore, the Chern–Weil form turns out to be trivial, then $CS(\Theta)$ is a closed form, representing a cohomology class in $H^3(X; \mathbb{Q})$. Then the counterterm to be added is genuinely 2-dimensional, which is determined by

$$\delta_s CS(\Theta) = d\Omega_2, \quad \delta_s CS(\Theta) = d(\delta_s \Omega_2) = d\text{tr}(\alpha d\Theta). \quad (4.2.1)$$

Next we explain why the holonomy anomaly, which arises genuinely from a gauge description, can be viewed as the nontriviality of a certain determinant line bundle; the latter has been discussed extensively by [8] and [9].

Starting with the bosonic field $g \in C^\infty(\Sigma, G)$ of the theory, we have that

$$\text{ev} : C^\infty(\Sigma, G) \times \Sigma \to G, \quad (g, x) \mapsto g(x),$$
and at the level of differential forms, we also have a pushforward map
\[ e_g : \Omega^q(C^\infty(\Sigma, G) \times \Sigma) \rightarrow \Omega^q(C^\infty(\Sigma, G)) \]
induced by integration along \( \Sigma \). The classical action of the theory should be viewed as a \( \dim \Sigma \)-form on \( C^\infty(\Sigma, G) \times \Sigma \) pushed down to \( \Omega^q(C^\infty(\Sigma, G)) \), and hence is a function of the field. Path integral quantization amounts to say that there is also a certain pushforward map by integrating along \( C^\infty(\Sigma, G) \). As we do not have the applicable mathematical tools, we shall just consider it as given by the canonical quantization.

To build a coset model using the chiral gauge method, we introduce a gauge field \( A \) coming from a connection in \( \text{Conn}(\text{ad}P) \) for an adjoint \( \mathfrak{h} \)-bundle of \( H \rightarrow P \rightarrow \Sigma \), and promote the bosonic field \( g \) to a smooth section \( \mathcal{G} S \). When the bundle \( P \) has a global section, \( g \) can be viewed as a \( G \)-valued smooth map from \( \Sigma \). In the following analysis, we shall use a local trivialization of \( P \) to write \( g \) as a smooth map \( \mathcal{G} S \) while keeping in mind the nontrivial gluing of \( g \) across the open covers of \( \Sigma \).

The infinitesimal dimensional topological group \( C^\infty(\Sigma, H) \) acts on the space of fields:
\[ C^\infty(\Sigma, H) \times \Gamma(\Sigma, P \times_H G) \rightarrow \Gamma(\Sigma, P \times_H G) \] \( : (h, g) \mapsto gh, \)
and
\[ C^\infty(\Sigma, H) \times \text{Conn}(\text{ad}P) \rightarrow \text{Conn}(\text{ad}P) : (h, A) \mapsto h^{-1}Ah + h^{-1}dh. \]
The action is a functional over the space of fields, which is invariant with respect to gauge transformation, and is thus a functional over the orbit space of the diagonal action of the gauge transformation, which we call the reduced space of field
\[ \Gamma(\Sigma, P \times_H G) \times C^\infty(\Sigma, H) \times \text{Conn}(\text{ad}P). \]
Note that the gauge group acts on the bosonic field freely, so the quotient space can be taken as the honest orbit space without invoking the ghost degree of freedom.

Now the gauge fixing is a local functional \( f \) over the unreduced space of field whose critical locus intersects the \( C^\infty(\Sigma, H) \)-orbits transversely. By solving out the gauge fixing condition, one picks out a unique element in \( \Gamma(\Sigma, P \times_H G) \) for each orbit, and correspondingly the action functional will be restricted to \( \text{Crit}(f) \times \text{Conn}(\text{ad}P) \), which models the reduced space of the bosonic fields. In the gauged formalism of bosonic homogeneous space sigma models, one fixes the gauge by asking the connection to be a particular one pulled back via \( g \) from the principal \( H \)-bundle \( \pi : G \rightarrow G/H \). Note that \( G \rightarrow G/H \) is a Riemannian submersion, and this hence gives a sub-bundle \( \pi^*T(G/H) \subset TG \). Now the Maurer–Cartan form splits into a spin connection on \( G/H \) and the vielbein 1-form
\[ g^{-1}dg = \omega H + e^\psi \mathcal{X}_D. \]

We now describe the fermions coming from the \((0, 1)\) supersymmetry. These are, from the target side, sections of the vector bundles associated with the principal \( H \)-bundle via the isotropic representation \( \phi \) where \( \text{ad} g = \text{ad} \mathfrak{h} \oplus \phi \) as before.

Let \( S_L, S_R \) denote the bundles associated with \( \text{Spin}(\Sigma) \) with the half-spin representation, and we have that
\[ \psi \in \Gamma(\Sigma, S_L \otimes g^* G \times_{\phi} \mathfrak{m}), \quad g \in \Gamma(\Sigma, P \times_H G); \]
\[ D_{RR} : \Gamma(\Sigma, S_L \otimes g^* G \times_{\phi} \mathfrak{m}) \rightarrow \Gamma(\Sigma, S_R \otimes g^* G \times_{\phi} \mathfrak{m}). \]

There is also a linear gauge group action on the fermions induced from the isotropic \( H \)-actions on \( \phi \), and due to the pulling back of \( g \in \Gamma(\Sigma, P \times_H G) \), the gauge connection \( A \) is coupled to \( \psi \). The Dirac operator we need to consider comes from a Dirac operator on the
pulled-back bundle of $TG$
\[ D^{\rho \gamma}_{RR} = \partial_{RR} + g^{-1} \partial_{RR} g + A_{RR}, \]
whose component in the isotropy representation is
\[ (D^{\rho \gamma}_{RR})^{ab} = \partial_{RR} b^{ab} + \frac{1}{2} C^{iab}_{ij} (A_{RR}^i + \omega_{RR}^i) + \frac{1}{2} C^{iab}_{ij} e_{RR}^i. \] (4.2.3)

The operator is parameterized by $\Gamma(\Sigma, P \times_H G) \times \text{Conn}(\text{ad}_bP)$ and is gauge covariant. If there is no chiral fermion anomaly, taking its functional determinant should result in a gauge invariant expression, and thus descend to a functional over $\Gamma(\Sigma, P \times_H G) \times \text{Conn}(\text{ad}_bP)$. The fermionic anomaly is present because of the fact that the fermionic effective action might be a section of nontrivial complex line bundle over the space of fields for two reasons. Firstly, it is possible that the effective action is already a line bundle over the unreduced total space $\Gamma(\Sigma, P \times_H G) \times \text{Conn}(\text{ad}_bP)$, even before we check the gauge invariance; and secondly, it is possible that the nontriviality of the anomaly comes from the failure of a descent condition at the quantum level.

Repeating the analysis in [8], one knows that the line bundle is characterized by its first Chern class, which, upon integrating over a two-cycle in the space, gives the Chern number. In this way, one reduces the task of understanding the infinite dimensional space of field $\Gamma(\Sigma, P \times_H G) \times \text{Conn}(\text{ad}_bP)$ to an arbitrary 2-dimensional 2-cycle in it.

We need to be more specific about the choice of 2-cycles. It is hard to lift up a 2-cycle from the base to the larger space precisely because of the interaction between the 2-cycle in the base and the gauge group. But here we have some convenient choices because of the special form of the Dirac operator. Note that the connection-dependence of the Dirac operator decouples into two parts
\[ A_1 = \frac{1}{2} (\omega_{RR} - A_{RR}), \quad A_2 = A_{RR} \mid_{\rho} + \frac{1}{2} e_{RR}, \]
the former being covariant with respect to the gauge transformation, while the latter is not. In fact, $A_1$ is the difference of two connections on the very same bundle $P \to \Sigma$. This is based on two facts: (1) $\omega$ is a principal $H$-connection on $G \to G/H$; and (2) a section of the associated bundle $P \times_H G \to \Sigma$ can be used to pull the connection back to $P \to \Sigma$. To understand how the connection can be pulled back, it is enough to see that the sections pull back via $\text{ev} \circ A_{\Sigma} \circ G$, which is obvious. Along this line, one can view an element in $\Gamma(\Sigma, S_G \otimes g^*G \times \rho \text{ ad}_bP)$ as one in $\Gamma(\Sigma, S_G \otimes P \times_{\rho \text{ ad}_bP})$. A characteristic computation at a rational cohomology level would not depend on $A_1$. Now the analysis from the determinant line bundle says that the anomaly is given by
\[ \int_{\Sigma \times \Sigma} \left( Y \times \Sigma \right) \cdot \text{ev}^* \text{ch}(F^A_2). \]

The space $Y$ is a 2-cycle in the space of the bosonic fields. Now we should be able to understand conceptually why the global and holonomy anomaly cancellation conditions are the same. In the current setting, if we take $Y$ to be a 2-sphere in $\Gamma(\Sigma, P \times_H G)$, which intersects the gauge orbits transversely, then this expression gives rise to the known $p_1$ anomaly condition. If we take $Y$ to be a 2-sphere suspended from the gauge orbit $\Sigma$, and use $A_{RR}$ as a representative for $A_2$, this gives the cancellation condition on non-Abelian chiral gauge anomalies, as shown in section 3.1.
5. Conclusion

In this paper, we systematically study the anomalies in minimal $\mathcal{N} = (0, 1)$ and $(0, 2)$ supersymmetric sigma models on homogeneous spaces. The investigation starts from our previous observation \cite{7} on isometry/gauge anomaly correspondence for the sigma models realized in nonlinear/linear gauged formalisms respectively. This leads us to consider more general holonomy anomalies and how to remove them.

Following Polyakov and Wiegmann, we systematically explore the anomalous fermion effective action and obtain its explicit form. Later, in the procedure of mending the anomalous action, we derive an anomaly matching condition as the criteria for sieving out ill-defined models. This condition is equivalent to the global topological constraint of $p_1(G/H)$ thoroughly discussed by Moore and Nelson \cite{8}. More importantly and surprisingly, we demonstrate that these local counterterms will further modify and constrain the behavior of the ‘curable’ theories in the deep IR region. Supersymmetry will be broken in some theories, whereas some others flow to nontrivial infrared superconformal fixed points.

In addition to the general discussion above, we also analyzed various concrete examples, applying the anomaly matching condition to different types of $G$ and $H$. We find that most survived minimal models are $\mathcal{N} = (0, 1)$ supersymmetric, while $\mathcal{N} = (0, 2)$ minimal models, due to their nontrivial center in $H$, are typically topologically obstructed.

We also reveal an interesting correspondence between two-dimensional $\mathcal{N} = (0, 1)$ minimal sigma models and gauge theories, analogous to 't Hooft’s anomaly matching observation in the four-dimensional case. Finally, we discussed the isometry/holonomy anomalies and the anomaly matching condition from the standpoint of the determinant line bundle. We obtained a more general expression on the anomaly equation with the help of a more powerful mathematical tool operative in field configuration spaces.

Because of the simplicity of the fermion sector in the minimal models, we should expect these models to either be destroyed or strongly constrained by anomalies. This expectation is more or less substantiated in this paper: our refined treatment of the anomalies and their remedies displays very interesting features of the minimal $\mathcal{N} = (0, 2)$ and $(0, 1)$ sigma models. Our subsequent work will continue along these lines. It should be interesting to work out some solid examples to further verify our results on the low-energy behavior of the minimal sigma models. Good candidates include models on $G/T'$ (not necessarily maximal tori), since the complex structures on them will enhance the supersymmetry to $\mathcal{N} = (0, 2)$, which makes them particularly easy to handle.

On the other hand, it is also noteworthy that the $\mathcal{N} = (0, 1)$ minimal sigma model on $SO(2p)/SO(p) \times SO(p)$ corresponds to a $\mathcal{N} = (0, 1)$ two-dimensional gauge theory with the gauge group $SO(p)$. It is thus interesting to ask whether or not every curable minimal model will have its own corresponding gauge theory, and how to find them. Investigating these gauge theories may also shed light on the minimal sigma models, and vice versa. We expect to answer some of these questions in subsequent work.

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