Stick models of projective configurations

Taneli Luotoniemi

Department of Mathematics and Systems Analysis, Aalto University, Helsinki, Finland

ABSTRACT
Although projective geometry is an elegant and enlightening domain of spatial thinking and doing, it remains largely unknown to the general audience. This shortcoming can be mended with the aid of figures consisting of points, lines, and planes, that illustrate various projective phenomena. In practice, these configurations can be assembled physically from sticks tied together at their crossings. As an example, I discuss a set of five configurations and some of the projective topics connected to them. The activity of building the stick models offers an instructive, simple, and sculpturally engaging approach to projective geometry.

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CONTACT
Taneli Luotoniemi
Department of Mathematics and Systems Analysis, Aalto University, P.O. Box 11100 (Otakaari 1B), FI-00076 AALTO, Finland

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Background

Projective geometry is an elegant and enlightening domain of spatial thinking and doing. As a branch of mathematics, it is fairly unknown but intuitively accessible to a general audience. Projective geometry grew out of the linear perspective studies of Renaissance painters, although essentially projective results can be found already from Apollonius of Perga (c. 200 BC) and Pappus of Alexandria (c. 300AD).

For the purposes of the discussion below, projective geometry primarily differs from the more traditional Euclidean geometry in that it lacks the notion of parallelism. In Euclidean geometry two points in a plane will always determine a unique line, but two lines in a plane (or two planes in space), will determine a point (or a line) only if they are not parallel. Projective geometry does away with this blemish by adopting the principle that in a plane any pair of lines, and in space any pair of planes, will intersect each other. A remarkable result of this stance is a perfect symmetry between points and lines in a projective plane, and points and planes in projective space. This correspondence, which extends not only over the elements themselves, but over any theorems involving them, is called projective duality.

Unlike school geometry, projective geometry also does not involve taking measurements such as lengths, angles, areas, or volumes, and these concepts do not consequently belong to its vocabulary. Instead, projective geometry studies those characteristics of shapes and figures that remain unchanged regardless of the chosen point of view, such as the concurrency of lines and the collinearity of points. One might think that such a restricted toolbox would not produce any interesting results, but surprisingly enough, projective geometry offers a wealth of profound theorems and structures. The rich landscape of spatial phenomena unfolding through the projective approach offers many insights also to Euclidean geometry and to spatial practices of visual arts. The mathematical phenomena traditionally treated in visual arts, such as linear perspective and regular solids, are given fresh, engaging, and elegant descriptions in the context of projective geometry.

Configurations

Many phenomena of projective geometry can be explored through configurations: regular arrangements of a finite number of points, lines, and planes. Here, a projective configuration is regular if each of its elements that share a common dimensionality belongs to the same number of other elements of any other dimensionality: each point has the same number of lines or planes going through it, each line has the same number points along it or planes going through it, and each plane has the same number points or lines lying on it.

Four arbitrarily positioned lines in a projective plane, taken together with their intersections, gives a simple example (Figure 1). Each line of this configuration, called the complete quadrilateral, belongs to three points, and each of its six points belongs to two lines. Taking a projective dual of this configuration gives us another configuration, where the roles of the points and lines are reversed (Figure 2). This plane-dual of the complete quadrilateral is called the complete quadrangle, and each of its four points belongs to three lines, whereas its six lines belong to two points each.

Four arbitrarily positioned planes in projective space, taken together with their intersections, serves as a three-dimensional example. This configuration is called the complete
tetrahedron (Figure 3). The planes intersect each other in six lines and four points. Each of its points has three lines and three planes going through it, and each of its lines has two points along it, while being also an intersection of two planes. Each plane has three points and three lines lying on it. As the points and planes play identical roles in the structure, it is a self-dual figure in projective space.

To navigate the projective environments the configurations inhabit, it is important to understand their topology. Unlike their Euclidean counterparts, which extend indefinitely towards infinity, projective lines, planes, and spaces close back onto themselves. This characteristic of projective structures makes it challenging to portray them in our

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**Figure 1.** The complete quadrilateral.

**Figure 2.** The complete quadrangle.
locally-Euclidean space. Although projective geometry itself treats points and other elements impartially, our visualizations bounded in Euclidean space are heavily point-biased. As planar illustrations are usually contained in a topological disk (and three-dimensional models inhabit a topological ball), they portray points ‘intact’, i.e. a line through a point can be continuously rotated around it in a cyclic fashion. But because any conventional visualization will depict only a neighbourhood of a point, the continuous shapes of projective lines and planes are destroyed, and full cyclic translation of a point around them is not possible.

To give topologically correct interpretations to such bounded illustrations, we can imagine gluing together the opposite extremities of their containers (Weeks, 2020, pp. 57–58). Figure 4 provides an example portrayal of a projective line, divided into one-dimensional regions, segments, by two points. To convince ourselves that also the seemingly disjoint blue segment is projectively intact, we can imagine connecting the two ends of the visible portion of the line. It is now evident that two points partition the projective line into two segments, of which only the pink one gets portrayed intact in the centre of the illustration, and the blue one is broken into two parts.

The disk-shaped container of Figure 5 can be interpreted similarly: if the antipodal points of its circular boundary are glued together, it becomes evident that all the regions are legitimate triangles of a projective plane, bounded by the three lines just like the triangular region in the middle. The three lines partition the projective plane into four triangular regions altogether, of which only the pink one gets portrayed intact in the illustration.
Finally in Figure 6, we see how three-dimensional regions in projective space, bounded by four planes, might appear in a sculptural model. To appreciate their tetrahedral shapes intact, we imagine the three-dimensional solid ball where the model is contained, and glue together the antipodal points of its spherical boundary. All in all, four planes partition the projective space into eight tetrahedral regions. In a finite model built in our everyday Euclidean space, only one such volume can be portrayed intact at a time, and the other tetrahedra wrap around the projective space and include portions at seemingly opposite sides of the configuration.

**Stick models**

Although the configurations reside more naturally in projective space, they can be modelled in our locally-Euclidean physical space by finite sticks tied together at their crossings. Each stick portrays an intersection of two or more planes, and the vertices where the sticks meet portray the points of the configuration. A length of material can be left sticking outwards from each vertex to emphasize how the planes and the lines close back onto themselves.
During a series of online lectures and hands-on workshops, I introduced the students of our interdisciplinary Aalto Math&Arts minor program (Aalto University, n.d.) to the basics of projective geometry. Before participating in the workshop on-site, the students from arts, science, and engineering backgrounds investigated different projective configurations and the phenomena connected to them at home by assembling miniature models from bamboo sticks. I instructed these sessions via video teleconferencing software, and used a separate webcam to show detailed views of miniature models on my desk. I encouraged the students to build the configurations first by themselves through trial and error, but offered also building instructions in the form of videos (Luotoniemi, 2020b, 2020c, 2020d, 2020e).

During the final five-day outdoor workshop, the students painted and built five designs from 2.4 m wooden sticks and tie wire. The configurations portrayed were the complete pentahedron, the complete hexahedron, the complete hexachoron, Reye’s configuration, and Schlafli’s double-six.

**Complete pentahedron**

Perhaps the most famous projective statement, called Desargues’ theorem, relates a pair of triangles in the plane. It states that two triangles are *in perspective with respect to a point* if and only if they are also *in perspective with respect to a line*. This means that, if the lines joining the corresponding vertices of the two triangles are concurrent (at the centre of perspectivity), the intersection points of the corresponding sides are collinear (along the axis of perspectivity). By observing the situation in three-dimensional projective space, the triangles can be thought of as cross-sections of a three-sided cone with an apex at the centre of perspectivity. The axis of perspectivity is then revealed to be simply the line of intersection of the planes of the triangles (Robbin, 2006, p. 53).

Building a three-dimensional model of the Desargues’ configuration amounts to introducing a fifth plane into the complete tetrahedron. This configuration may be called the *complete pentahedron* (Figure 7), as it is composed of five arbitrarily positioned planes of projective 3-space. It is remarkable that the planes will always intersect each other in the same pattern of ten lines (with three points along each) and ten points (with three lines through each). The planes partition the space into five tetrahedral regions, and ten triangular prisms. Two tetrahedra and two prisms appear intact in the stick model, and each of the other polyhedral regions is split in two. The colours of the model illuminate one pair of perspective triangles having corresponding sides painted with blue, pink, and yellow. White rods joining the corresponding vertices are concurrent at the centre of perspectivity, and the axis of perspectivity is another white rod.

As each element having the same dimensionality plays an identical role in the configuration, there are ten instances of Desargues’ theorem present in it. This means we can find a pair of triangles for each point (or each line) of the configuration, that is in perspective with respect to that point (or that line). Figure 8 shows another pair of such triangles, and their centre of perspectivity.

As the logo I had designed for our Aalto Math&Arts program (Aalto University, n.d.) features this configuration, I made an animation (Luotoniemi, 2020a) to illustrate how the students can locate planes, lines, points, polyhedral regions, and instances of the Desargues’ theorem present in the structure. The complete pentahedron has also a connection to hyperspatial geometry, as it can be interpreted as a four-dimensional solid.
Figure 7. The complete pentahedron.

called the expanded pentachoron, radially projected down to our 3-space from its centre (Luotoniemi, 2018).

**Complete hexahedron**

The ‘infinity’ that splits some volumes of the complete pentahedron in two, can be modelled by adding a sixth plane to the configuration. The reason a stick model of the complete pentahedron always has the same visible layout, is that six arbitrarily positioned planes will intersect each other in projective space in the same configuration of fifteen lines (each belonging to four points and two planes) and twenty points (with three lines and three planes through each). We may call this configuration the complete hexahedron. The six planes partition the space into two cuboids, twelve triangular prisms, six tetrahedra, and six wedge-shaped polyhedra composed of two pentagons, two quadrilaterals, and two triangles each (Locher-Ernst, 1940/2003, pp. 159–169).

As a seventh plane could be positioned in several distinct fashions with respect to the six planes of the configuration, there exist different visible layouts for a stick model of the complete hexahedron. The physical model in Figure 9 portrays a layout having a three-fold rotational symmetry around its vertical axis (Figure 10). Note that the three-fold symmetry leaves the colours orange and green unchanged, whereas pink, blue, and white are interchanged cyclically. In this layout three tetrahedra, six prisms, and one cuboid (Figure 11)
appear intact. To find the rest of the polyhedra with the students, we followed the edges sticking outward and imagined tracing them around their projective lines to the opposite side of the configuration. The model is painted with five colours (blue, pink, white, green, and orange) so that each colour appears exactly once along each plane.

The theorem embodied by the configuration states that two quadrilaterals are in perspective from a line (i.e. the four intersection points of the corresponding sides are collinear) if and only if the six lines determined by their corresponding vertices form a quadrangle – a natural consequence of Desargues’ theorem. Again, by observing the situation in three-dimensional stick figure, the quadrilaterals can be thought of as cross-sections of a complete tetrahedron, the axis of perspectivity being the intersection line of their planes.

**Complete hexachoron**

The plane-dual of the quadrilateral theorem – illustrated above by the complete hexahedron, can be stated as: two quadrangles are in perspective from a point if and only if the meets of their corresponding edges are the vertices of a quadrilateral. In three-dimensional projective space this fact becomes a generalization of the Desargues’ theorem, stating that two tetrahedra are in perspective from a point if and only if they are also in perspective from a plane (i.e. the intersection points of the corresponding edges are coplanar) (Veblen & Young, 1946, pp. 43–44).
Figure 9. The complete hexahedron.

Just like Desargues’ theorem, which became almost self-evident when observed in a higher-dimensional space, the tetrahedra theorem can be thought of as a shadow from four-dimensional projective space. The pair of tetrahedra can be understood as two crosssections through a bundle of four hyperplanes, i.e. four three-dimensional slices of the hyperspace concurrent at the centre of perspectivity. The plane where the corresponding edges of the tetrahedra meet is then the plane of intersection for the hyperplanes of the tetrahedra.

We may call this configuration the complete hexachoron, as it consists of six arbitrarily positioned hyperplanes. The hyperplanes intersect each other in fifteen planes, twenty lines, and fifteen points. Each of its planes goes through six points and four lines, whereas each line goes through three points, and is an intersection of three planes. Each of the points of the configuration has four lines and six planes going through it. There are six instances of the complete pentahedron present in the configuration, corresponding to the six hyperplanes. As such it also presents a method of constructing the Desargues’ configuration by intersecting a complete five-point with a plane (Veblen & Young, 1946, pp. 41–42).

Finding the polytopes bounded by the hyperplanes requires understanding of not just projective geometry, but also of four-dimensional space. The configuration can be
interpreted as a five-dimensional solid called the expanded 5-simplex, radially projected from its centre. This polytope has three kinds of four-dimensional hyperregions – pentachora, tetrahedral prisms, and 3–3 duoprisms (Luotoniemi, 2018).

The pentachoron is a four-dimensional generalization of the tetrahedron, and tetrahedral prism corresponds similarly to the triangular prism, but the 3–3 duoprism does not have an analog in three dimensions. The three-dimensional boundary of the duoprism consists of two sets of three triangular prisms, interlocked together like links in a chain. As the exclusively hyperspatial duoprism deserves special attention, the colouring of the stick model (Figure 12) illuminates one of them as a perspective construction, with the sticks standing for its six facets coloured blue, pink, yellow, orange, green, and purple. The vanishing points where these lines meet lie on two horizon lines, illustrated by white sticks (Luotoniemi, 2019, pp. 114–115).

**Reye’s configuration**

When a four-dimensional solid called the 24-cell, bounded by twenty-four octahedra, is radially projected from its centre into our three-dimensional space, it yields a projective
figure called Reye’s configuration. This structure consists of twelve points, sixteen lines, and twelve planes. Each of its points has four lines and six planes through it, each plane goes through six points and four lines, and each of its lines is incident with three points and three planes (Hilbert & Cohn-Vossen, 1932/1990, pp. 154–156).

The regularity of the configuration allows a four-colouring of the lines (blue, orange, white, pink), where none of the colours touch each other, but each of them appears at every point and in every plane (Figure 13).

I was already familiar with modelling Reye’s configuration, as I had built the structure from four-meter-long bamboo poles for the 2019 Future Art Lab exhibition (ADE Future Lab, n.d.) at the West Bund Art Centre in Shanghai with my fellow Aalto Math&Arts teachers Laura Isoniemi and Kirsi Peltonen. The resulting sculpture, called Space Hug, also served as an invitation for the exhibition audience to participate in a hands-on activity, where they were asked to make their own configurations by joining small bamboo sticks with rubber bands. These miniature configurations (some of them built according to given sets of instructions, others following artistic freedom) were then placed on the floor next to the large-scale sculpture to make an interactive installation that kept growing during the exhibition (Virtanen, 2019).
Another interesting subject for a stick model portrayal is the Schläfli’s double-six configuration (Figure 14), consisting of two sets (yellow and purple in the model) of six skew lines in projective space. Although the lines belonging to the same set are mutually skew, each of line of one set intersects five lines of the other set, amounting to thirty points in total. The lines of the configuration determine a cubic surface, which has not just twelve but twenty-seven straight lines (Hilbert & Cohn-Vossen, 1932/1990, pp. 164–170).

**Discussion**

Students building and investigating the stick models described above faced several spatial challenges pertaining to conceptual differences between projective and Euclidean geometries of three and four dimensions, as well as to sculptural issues.

First, students had to adopt the projective mindset, from which some familiar Euclidean notions such as lengths, angles, and parallelism are omitted. Sticks that appear parallel in a model can be interpreted as meeting ‘at infinity’. A slight modification to their angles does not have consequences for the projective structure of the configuration, but will make
such a vertex appear in view. If the bindings of the vertices are loose enough, the entire configuration will adjust itself to accommodate this transformation. A particularly delightful aspect of the craft is that no preparatory measurements of the parts are needed, and students learn in practice that there is geometry also beyond the lengths and angles. I emphasized this characteristic by using fixed length sticks, so that apart from its colour, a stick could take any position in the configuration. Sticks of varying length could be used also, but then their placement should be planned more carefully.

Another striking property of projective space is that it closes back onto itself – it has finite volume, yet no boundary. To illustrate this, any outermost vertex of the model (or a group of vertices in the case of Schläfli’s double-six) can be transferred to the opposite side of the visible configuration by conceptually ‘sliding it through infinity’. In practice this corresponds to unfastening its binding, and slightly rotating the sticks going through it. After the sticks are slid through their remaining vertices to shift the excess lengths to the opposite side, they can be tied together again. As a result of the manipulation the visible layout of the configuration changes handedness, but stays the same in other aspects (excluding lengths and angles, naturally).
The significance of projective geometry arises from its power to contest and enrich our spatial thinking and doing. Especially the concept of projective dualism can challenge our atomistic world view, in which the point is assumed to be the indivisible primitive – the building block for other spatial forms. Instead, assigning the role of the indivisible primitive to the plane yields a ‘counterspace’, which has surprising cultural connections to e.g. theosophical symbolism (Levin, 2020). The account of these connotations is, however, outside the scope of this paper. What is safe to say is that dualizing the projective figures and theorems is a task that requires a rigorous and unassuming attitude. During the activities described above, the students experimented with the plane-oriented viewpoint by dualizing simple projective theorems and phenomena, and by building configurations composed of intersecting planes.

To appreciate the complete hexachoron as a collection of hyperplanes, and to understand how the other configurations can be interpreted as gnomonic projections of regular four-dimensional figures, the students needed to come in terms also with 4D geometry. Luckily this is also a topic that permits visual and engaging approaches, especially with the aid of crafted models (Luotoniemi, 2019).
Because the full-length sticks traversing the entire model necessarily have some physical thickness, there was the challenge to make them pass each other without having to bend along the way. Finding the correct weaving for the sticks so that they can be tied together snugly at the vertices presents an interesting design problem. A geometrically natural approach would be to classify all of the essentially different ways that a given number of sticks can meet at a crossing. If such a classification does not exist, I think it warrants an investigation. I experimented using the same arrangement at each crossing of a configuration, translating and rotating it from one vertex to the next following the symmetries of the shape. However, as the physical thickness of the sticks is a Euclidean property, it was always required to deviate from such principle at last and resort to some trial-and-error modifications.

Another challenge was to find articulate and pleasant proportions for the regions bounded by the sticks. It turned out to be a good idea to bind them somewhat loosely at first, making it easier to modify the configuration. A natural objective is to have the points along a line and lines through a point as equally spaced as possible. Another possible design goal is to find a maximally symmetric 3D layout for the visible configuration. Before declaring a configuration finished, one must check carefully that it is as clear and balanced as possible. Neighbourhoods where vertices are crowded together would evidently increase the difficulty of appreciating the structure of the configuration.

Additional care is required when representing a configuration in a two-dimensional illustration, such as a photograph or a drawing. As the proportions of the visible configuration can be modified without any damage to its projective structure, it is perfectly acceptable to deform the model freely for the sole purpose of making it appear favourably in the picture. Depending on one’s personal taste, this might amount to e.g. maximizing the planar symmetry, or minimizing the number of crossings (i.e. apparent intersections of lines that are not, however, points of the configuration).

**Conclusion**

The activity provided an enlightening, simple, and embodied method to study mathematically challenging phenomena. Stick configurations facilitate intuitive, playfully experimental, and imaginative approaches to at least two visual art topics: Desargues’ theorem is arguably one of the most important statements involving linear perspective, and by organizing of the elements of the models, students observe the often poetically expressed logic of sculptural composition in a pragmatic, matter-of-fact fashion. Models can be built in reasonable time by students during workshops, and they present also architectural possibilities, e.g. as designs for playground structures. Finding instances of theorems and polyhedral regions in the figures is an instructive activity, although participants without previous experience of projective geometry or regular solids require detailed instruction and support. With my students we always first observed some examples collectively before individual investigations. I gave the solutions in the form of black-and-white photographs, where I had illuminated the relevant parts of miniature models with colours. In art studies it is usually not relevant to measure the absorption of ideas quantitatively by, e.g. exams, so the emergence of pleasant structures and complimentary feedback from the participants must do for evidence of success.
All in all, the stick configurations illuminate new connections between arts and mathematics, and introduce projective geometry to new audiences. As the configurations afford an intuitive approach to this field of modern geometry, they advance on their part the democratization and popularization of pure mathematics.

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ORCID

Taneli Luotoniemi http://orcid.org/0000-0002-2534-1157

References

Aalto University. (n.d.). Aalto Math&Arts. http://matharts.aalto.fi/.
ADE Future Lab. (n.d.). FutureLab. http://www.ade-futurelab.com/.
Hilbert, D., & Cohn-Vossen, S. (1990). Geometry and the imagination. Chelsea Publishing Company. (Original work published 1932).
Levin, M. (2020). Patterns in nested platonic solids. Journal of Mathematics and the Arts, 14(1–2), 94–96. https://doi.org/10.1080/17513472.2020.1734518
Locher-Ernst, L. (2003). Space and counterspace – An introduction to modern geometry. AWSNA Publications. (Original work published 1940).
Luotoniemi, T. (2018). Desargues configuration as a gnomonic projection. Bridges Conference Proceedings (pp. 559–562) Stockholm, Sweden, Jul. 25–29, 2018. http://archive.bridgesmathart.org/2018/bridges2018-559.html
Luotoniemi, T. (2019). Hyperspatial interlace – Grasping four-dimensional geometry through crafted models [Doctoral dissertation, Aalto University]. Aaltodoc publication archive. http://urn.fi/URN:ISBN:978-952-60-8480-0
Luotoniemi, T. (2020a, January 3). Desargues configuration [Video]. YouTube. https://youtu.be/X1BuiRdpR00
Luotoniemi, T. (2020b, January 27). Building a stick model of the Desargues configuration [Video]. YouTube. https://youtu.be/lOxkYVcRbQc
Luotoniemi, T. (2020c, January 27). Building a stick model of the six-plane configuration [Video]. YouTube. https://youtu.be/9lkI3_1CZ2M
Luotoniemi, T. (2020d, January 28). Building a stick model of a tetrahedral configuration [Video]. YouTube. https://youtu.be/fby53U_n4o8
Luotoniemi, T. (2020e, January 28). Building a stick model of the Reye configuration (slightly blurry) [Video]. YouTube. https://youtu.be/ZVtHbLF2G9M
Robbin, T. (2006). Shadows of reality the fourth dimension in relativity, cubism, and modern thought. Yale University Press.
Veblen, O., & Young, J. (1946). Projective geometry. Vol I. Blaisdell Publishing.
Virtanen, J. P. (2019, December 9). News: Aalto Math&Arts in Shanghai Future Lab exhibition. Aalto University. Retrieved June 27, 2021, from https://www.aalto.fi/en/news/aalto-math-arts-in-shanghai-future-lab-exhibition.
Weeks, J. R. (2020). The shape of space. Chapman & Hall/CRC.