Properties of a curve whose convex hull covers a given convex body

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Abstract
In this note, we prove the following inequality for the norm $N(K)$ of a convex body $K$ in $\mathbb{R}^n$, $n \geq 2$:

$$N(K) \leq \frac{\pi^{n-1}}{2 \Gamma \left( \frac{n+1}{2} \right)} \cdot \text{length}(\gamma) + \frac{\pi^{n-1}}{\Gamma \left( \frac{n}{2} \right)} \cdot \text{diam}(K),$$

where $\text{diam}(K)$ is the diameter of $K$, $\gamma$ is any curve in $\mathbb{R}^n$ whose convex hull covers $K$, and $\Gamma$ is the gamma function. If in addition $K$ has constant width $\Theta$, then we get the inequality

$$\text{length}(\gamma) \geq \frac{2(\pi - 1) \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n}{2} \right)} \cdot \Theta \geq 2(\pi - 1) \cdot \sqrt{\frac{n-1}{2\pi}} \cdot \Theta.$$

In addition, we pose several unsolved problems.

Keywords Bodies of constant width · Convex body · Convex hull · Curve · Diameter · Quermassintegral · Norm of a convex body

Mathematics Subject Classification 52A10 · 52A15 · 52A20 · 52A38 · 52A39 · 52A40 · 53A04

1 Introduction and main results
There are many unsolved problems related to the convex hull of a curve in Euclidean space, see e.g. Ghomi (2018) and Zalgaller (1997) for discussions of some of them.

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In this note, we consider some relationships between a convex body $K$ in Euclidean space and a curve $\gamma$, whose convex hull cover $K$. We refer to Burago et al. (2001) for standard results in the metric geometry and to Bonnesen and Fenchel (1987), Hadwiger (1957) and Martini et al. (2019) for classical results in the geometry of convex bodies.

We identify $n$-dimensional Euclidean space with $\mathbb{R}^n$ supplied with the standard Euclidean metric $d$, where $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. For any subset $A \subset \mathbb{R}^n$, $\text{co}(A)$ means the convex hull of $A$. For every points $B, C \in \mathbb{R}^n$, $[B, C]$ denotes the line segment between these points.

A convex body is any compact convex subset of $\mathbb{R}^n$ (convex bodies with empty interior are allowed). We shall denote by $\text{vol}(K)$, $\text{area}(K)$, $\text{bd}(K)$ and $\text{int}(K)$ respectively the volume, the surface area, the boundary, and the interior of a convex figure $K$. Note also that the diameter

$$\text{diam}(K) := \max\{d(x, y) \mid x, y \in K\}$$

of a convex body $K$ coincides with the maximal distance between two parallel support hyperplanes to $K$.

The symbol $B^n(x, \rho)$ denotes a closed ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $\rho \geq 0$. We consider also the unit ball $B^n := B^n(0, 1)$ and the unit sphere $S^{n-1} = \text{bd}(B^n)$. We use the symbols $\omega_n$ and $\sigma_{n-1}$ for the volume of the unit $B^n$ and for the surface area of the unit sphere $S^{n-1}$ respectively. Recall that $\sigma_{n-1} = n\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, where $\Gamma$ is the gamma function.

A curve $\gamma$ is the image of a continuous mapping $\phi : [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^n$. As usually, the length of $\gamma$ is defined as $\text{length}(\gamma) := \sup \left\{ \sum_{i=1}^{m} d(\phi(t_{i-1}), \phi(t_i)) \right\}$, where the supremum is taken over all finite increasing sequences $a = i_0 < i_1 < \cdots < i_{m-1} < i_m = b$ that lie in the interval $[a, b]$. A curve $\gamma$ is called rectifiable if length(\gamma) < \infty.

For $n = 2$, we use the term a convex figure for a convex body $K \subset \mathbb{R}^2$, $\text{area}(K)$ is the perimeter per$(K)$ of $K$ in this case. We call a curve $\gamma \subset \mathbb{R}^2$ convex if it is a closed connected subset of the boundary of the convex hull $\text{co}(\gamma)$ of $\gamma$.

Let us recall a remarkable property of planar curves, that was obtained in Nikonorov and Nikonorova (2021).

**Theorem 1** (Nikonorov and Nikonorova 2021) For a given convex figure $K$ and for any planar curve $\gamma$ with the property $K \subset \text{co}(\gamma)$, the inequality

$$\text{length}(\gamma) \geq \text{per}(K) - \text{diam}(K) \quad (1)$$

holds. Moreover, this inequality becomes an equality if and only if $\gamma$ is a convex curve, $\text{bd}(K) = \gamma \cup [A, B]$, and $\text{diam}(K) = d(A, B)$, where $A$ and $B$ are the endpoints of $\gamma$.

Inequality (1) was suggested by A. Akopyan and V. Vysotsky, who proved it in the case when $\gamma$ is passing through all extreme points of $K$ (see Theorem 7 in Akopyan and Vysotsky 2017).

In this note, we obtain some generalization of Theorem 1 to the multidimensional case. Since length$(\gamma)$ and diam$(K)$ are, respectively, in some sense “one-dimensional”
characteristics of a curve \( \gamma \) and a convex body \( K \) in \( \mathbb{R}^n \), we should find some “one-dimensional” characteristic of the convex body \( K \subset \mathbb{R}^n \) instead of \( \text{per}(K) \), used in the two-dimensional case. Such a natural generalization of \( \text{per}(K) \) is the norm \( N(K) \) of convex body \( K \).

Let \( K \subset \mathbb{R}^n \) be a non-empty compact convex set. The support function \( h(K, \cdot) \) of \( K \) is defined by

\[
h(K, u) := \sup\{ (x, u) \mid x \in K \}
\]

for \( u \in \mathbb{R}^n \). The width function \( w(K, \cdot) \) of \( K \) is defined by

\[
w(K, u) := h(K, u) + h(K, -u), \quad u \in S^{n-1}.
\]

The mean value of the width function over \( S^{n-1} \) is called the mean width and denoted by \( w(\cdot) \), thus

\[
N(K) := \frac{1}{2} \int_{S^{n-1}} h(K, u) du = \frac{\sigma_{n-1}}{2} \cdot w(K).
\]  

See details e. g. in 6.1.7 of Hadwiger (1957).

For a convex body \( K \subset \mathbb{R}^n \) and for a number \( \varepsilon \geq 0 \), we consider the outer parallel body of \( K \) at distance \( \varepsilon \), which is defined as follows:

\[
K_{\varepsilon} := \{ x \in \mathbb{R}^n \mid d(K, x) \leq \varepsilon \} = \bigcup_{x \in K} B^n(x, \varepsilon).
\]

The Steiner formula says that its volume can be expressed as a polynomial of degree \( n \) in the parameter \( \varepsilon \):

\[
\text{vol}(K_{\varepsilon}) = \sum_{k=0}^{n} C_n^k \cdot W_k(K) \cdot \varepsilon^k, \quad C_n^k = \frac{n!}{k!(n-k)!}.
\]  

The functionals \( W_0, W_1, \ldots, W_{n-1}, W_n \) are called the quermassintegrals, see Bonnesen and Fenchel (1987) and Hadwiger (1957). Note that \( W_0(K) = \text{vol}(K) \), \( nW_1(K) = \text{area}(K) \), \( W_n = \omega_n \) and \( nW_{n-1} = N(K) \). In particular, we have \( N(K) = \text{area}(K) = \text{per}(K) \) for \( n = 2 \).

The main result of this note is the following theorem, that is proved in the next section.

**Theorem 2** For a given convex body \( K \subset \mathbb{R}^n \) and for any curve \( \gamma \subset \mathbb{R}^n \) with the property \( K \subset \text{co}(\gamma) \), where \( n \geq 2 \), the inequality

\[
N(K) = nW_{n-1}(K) \leq \frac{\pi^{n-1}}{2 \Gamma \left( \frac{n+1}{2} \right)} \cdot \text{length}(\gamma) + \frac{\pi^{n-1}}{\Gamma \left( \frac{n}{2} \right)} \cdot \text{diam}(K)
\]  

holds.
Remark 1  Inequality (4) is equivalent to (1) for \( n = 2 \) and does not hold in general for \( n = 1 \). For \( n = 3 \), (4) has the form \( N(K) \leq \frac{\pi}{2} \) length(\( \gamma \)) + 2 diam(\( K \)).

It should be noted that equality in (4) does not hold if \( n \geq 3 \) and diam(\( K \)) > 0 (i. e. \( K \) is not an one-point set), see Remark 3.

By virtue of the assertion of Theorem 2, the following problem seems to be interesting.

**Problem 1**  For a given \( n \), find all possible constants \( A(n) \) and \( B(n) \) such that

\[
N(K) \leq A(n) \cdot \text{length}(\gamma) + B(n) \cdot \text{diam}(K)
\]

for any convex body \( K \subset \mathbb{R}^n \) and for any curve \( \gamma \subset \mathbb{R}^n \) with the property \( K \subset \text{co}(\gamma) \).

Now we additionally suppose that in the conditions of Theorem 2 the convex body \( K \) has constant width \( \Theta \), i. e., its width function is constant over the unit sphere [for the so defined bodies of constant width see the monograph (Martini et al. 2019)]. Then, by the definition of the mean width \( w(K) \) we have \( w(K) = \Theta \) (since \( w(K, u) \equiv \Theta \) for all \( u \in \mathbb{S}^{n-1} \)), and by the definition of norm \( N(K) \) (see (2)) we have \( N(K) = \frac{\sigma_{n-1}}{2} \cdot w(K) = \frac{\sigma_{n-1}}{2} \cdot \Theta \). Obviously we have also \( \text{diam}(K) = \Theta \). Hence, in our case, (4) has the following form:

\[
\frac{\sigma_{n-1}}{2} \cdot \Theta = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot \Theta \leq \frac{\pi^{n-1}}{2\Gamma\left(\frac{n+1}{2}\right)} \cdot \text{length}(\gamma) + \frac{\pi^{n-1}}{\Gamma\left(\frac{n}{2}\right)} \cdot \Theta
\]

or

\[
\frac{\pi - 1}{\Gamma\left(\frac{n}{2}\right)} \cdot \Theta \leq \frac{\sqrt{\pi}}{2\Gamma\left(\frac{n+1}{2}\right)} \cdot \text{length}(\gamma).
\]

Therefore, we get the following

**Theorem 3**  For a given convex body \( K \subset \mathbb{R}^n \) of constant width \( \Theta \) and for any curve \( \gamma \subset \mathbb{R}^n \) with the property \( K \subset \text{co}(\gamma) \), where \( n \geq 2 \), the inequality

\[
\text{length}(\gamma) \geq \frac{2(\pi - 1)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right)} \cdot \Theta
\]

(5)

holds.

**Remark 2**  Inequality (5) has the form \( \text{length}(\gamma) \geq (\pi - 1) \cdot \Theta \) for \( n = 2 \) and the form \( \text{length}(\gamma) \geq 4 \left(1 - \frac{1}{\pi}\right) \cdot \Theta \) for \( n = 3 \), but (5) does not hold in general for \( n = 1 \).

It should be noted that the equality does not hold in inequality (5) if \( n \geq 2 \) and diam(\( K \)) > 0 (i. e. \( K \) is not a one-point set), see Remark 4.
Now we will apply Gautschi’s inequality (see Gautschi 1959)

\[ x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+1)^{1-s}, \]

where 0 < s < 1 and x > 0, to inequality (5). Taking \( x = (n-1)/2 \) and \( s = 1/2 \), we get

\[ \sqrt{\frac{n-1}{2}} \leq \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \leq \sqrt{n+1} \]

for all \( n \geq 1 \). (6)

This implies the following corollary of Theorem 3:

**Corollary 1** For a given convex body \( K \subset \mathbb{R}^n \) of constant width \( \Theta \) and for any curve \( \gamma \subset \mathbb{R}^n \) with the property \( K \subset \text{co}(\gamma) \), the inequality

\[ \text{length}(\gamma) \geq 2(\pi - 1) \cdot \sqrt{\frac{n-1}{2\pi}} \cdot \Theta \]

(7)

holds.

Note also, that (6) implies the asymptotic value

\[ \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\Gamma\left(\frac{n}{2}\right)}} \rightarrow \frac{1}{\sqrt{2}} \]

as \( n \rightarrow \infty \).

The following problem is naturally induced by Theorem 3.

**Problem 2** For a given \( n \), find the maximal constant \( C(n) \) such that

\[ \text{length}(\gamma) \geq C(n) \cdot \Theta \]

for any convex body \( K \subset \mathbb{R}^n \) of constant width \( \Theta \) and for any curve \( \gamma \subset \mathbb{R}^n \) with the property \( K \subset \text{co}(\gamma) \).

### 2 Proof of Theorem 2 and additional remarks

Let us recall some useful information of the Grassmann manifold \( G_{n,k} \). It could be considered as a set of \( k \)-dimensional unoriented linear subspaces in \( \mathbb{R}^n \) with a natural manifold structure. Note that the group \( O(n) \) of all orthogonal transformation of \( \mathbb{R}^n \) act transitively on \( G_{n,k} \), hence, \( G_{n,k} \) is a compact homogeneous space (in fact, we have \( G_{n,k} = O(n)/O(k) \times O(n-k) \)). The space \( G_{n,k} \) admits \( O(n) \)-invariant measure, which is unique, up to a positive multiple. For our goals, it is useful to consider the measure \( \mu_{n,k} \) on \( G_{n,k} \) that is introduced in (Santaló 2004, II.12.4) (although we use different notations here). It should be noted that

\[ \int_{G_{n,k}} 1 \, d\mu_{n,k} = \mu_{n,k}(G_{n,k}) = \frac{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_{n-k}}{\sigma_{k-1}\sigma_{k-2}\ldots\sigma_1\sigma_0} =: C_{n,k}. \]
In particular, the measure \( \mu_{n,k} := (C_{n,k})^{-1} \cdot \mu_{n,k} \) is such that \( \mu_{n,k}(G_{n,k}) = 1 \) and hence coincides with the Haar measure on \( G_{n,k} \).

Let us fix a convex body \( A \) in \( \mathbb{R}^n \) and an integer \( k, 1 \leq k \leq n - 1 \). Now, let us consider some \( P \in G_{n,k} \) and the orthogonal projection \( A' \) of \( A \) to \( P \). We will use notation \( W_j' \) for the quermassintegrals in \( P \). If \( j \) is such that \( 0 \leq j \leq k \leq n - 1 \), then we have the following equality (see Santaló 2004, (13.31)):

\[
\int_{G_{n,k}} W_j'(A') \, d\mu_{n,k} = \frac{n \sigma_{n-2} \cdots \sigma_{n-k}}{k \sigma_{k-2} \cdots \sigma_0} \cdot W_{n-k+j}(A)
\]

\[
= \frac{n}{k} \frac{\sigma_{k-1}}{\sigma_{n-1}} \cdot \mu_{n,k}(G_{n,k}) \cdot W_{n-k+j}(A).
\] (8)

This equality could be rewritten for the Haar measure \( \mu_{n,k} \) as follows:

\[
\int_{G_{n,k}} W_j'(A') \, d\mu_{n,k} = \frac{n}{k} \frac{\sigma_{k-1}}{\sigma_{n-1}} \cdot W_{n-k+j}(A).
\] (9)

For our goal, the most important case is \( j = k - 1 \):

\[
\int_{G_{n,k}} W_{k-1}'(A') \, d\mu_{n,k} = \frac{n}{k} \frac{\sigma_{k-1}}{\sigma_{n-1}} \cdot W_{n-1}(A).
\] (10)

Now, take a line segment \( L \) in \( \mathbb{R}^n \) as the body \( A \) in (10). Let us put \( l = \text{length}(L) \) and \( l' = \text{length}(L') \). Using (3), we easily get

\[
\text{vol}(L_\varepsilon) = \omega_{n-1} \cdot l \cdot \varepsilon^{n-1} + \omega_n \cdot \varepsilon^n \quad \text{and} \quad \text{vol}(L'_\varepsilon) = \omega_{k-1} \cdot l' \cdot \varepsilon^{k-1} + \omega_k \cdot \varepsilon^k.
\]

Consequently, \( W_{n-1}(L) = \frac{\omega_{n-1}}{n} l = \frac{\sigma_{n-2}}{(n-1)n} l \) and \( W_{k-1}'(L') = \frac{\omega_{k-1}}{k} l' = \frac{\sigma_{k-2}}{(k-1)k} l \). Therefore, (10) implies

\[
\int_{G_{n,k}} l' \, d\mu_{n,k} = \frac{(k-1) \sigma_{n-2} \sigma_{k-1}}{(n-1) \sigma_{k-2} \sigma_{n-1}} \cdot l = \frac{\sigma_n \sigma_{k-1}}{\sigma_k \sigma_{n-1}} \cdot l.
\] (11)

The last equality is due to the relation

\[
\sigma_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)} = \frac{4\pi^{(m+1)/2}}{(m-1) \Gamma((m-1)/2)} = \frac{2\pi}{m-1} \sigma_{m-2}
\]

for every natural \( m \). Since (11) is fulfilled for any line segment \( L \) in \( \mathbb{R}^n \), then we get the equality

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where $\gamma$ is any broken line in $\mathbb{R}^n$ and $\gamma'$ is its orthogonal projection to the subspace $P \in G_{n,k}$. Using the passage to the limit, we find that formula (12) is valid for any rectifiable curve $\gamma$ in $\mathbb{R}^n$. This result is also well known, see e. g. (Alexandrov and Reshetnyak 1989, Theorem 4.8.1)

It should be noted that the assertion of Theorem 2 is trivial if the curve $\gamma$ is not rectifiable (in this case $\text{length}(\gamma) = \infty$). Now we consider any convex body $K$ in $\mathbb{R}^n$ and any rectifiable curve $\gamma$ in $\mathbb{R}^n$ such that $K \subset \text{co}(\gamma)$. For a two-dimensional linear subspace $P$ of $\mathbb{R}^n$, we denote by $K'$ and $\gamma'$ the orthogonal projections of $K$ and $\gamma$ to $P \in G_{n,2}$ respectively. Then, as easy to see, we get $K' \subset \text{co}(\gamma') \subset P$ and we can apply Theorem 1 in this situation. We get the following inequality:

$$\text{per}(K') \leq \text{length}(\gamma') + \text{diam}(K')$$

Now, let us integrate this inequality by $G_{n,2}$ with respect to the Haar measure $\mu_{n,2}$, using the equality $\text{per}(K') = 2W_1(K')$:

$$2 \int_{G_{n,2}} W_1(K') d \mu_{n,2} \leq \int_{G_{n,2}} \text{length}(\gamma') d \mu_{n,2} + \int_{G_{n,2}} \text{diam}(K') d \mu_{n,2}.$$ 

Due to (10) and (12) for $k = 2$, it is equivalent to the inequality

$$\frac{\sigma_1}{\sigma_{n-1}} \cdot N(K) = n \frac{\sigma_1}{\sigma_{n-1}} \cdot W_{n-1}(K) \leq \frac{\sigma_n \sigma_1}{\sigma_2 \sigma_{n-1}} \cdot \text{length}(\gamma) + \int_{G_{n,2}} \text{diam}(K') d \mu_{n,2}.$$ 

Since $\sigma_1 = 2\pi$ and $\sigma_2 = 4\pi$, we get

$$N(K) \leq \frac{\sigma_n}{4\pi} \cdot \text{length}(\gamma) + \frac{\sigma_{n-1}}{2\pi} \int_{G_{n,2}} \text{diam}(K') d \mu_{n,2}. \quad (13)$$

Obviously, we have $\text{diam}(K') \leq \text{diam}(K)$ for every $P \in G_{n,2}$, and hence we get

$$N(K) \leq \frac{\sigma_n}{4\pi} \cdot \text{length}(\gamma) + \frac{\sigma_{n-1}}{2\pi} \text{diam}(K). \quad (14)$$

Since $\sigma_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$ and $\sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, we obtain (4). Thus, we have proved Theorem 2.
Remark 3 It should be noted that the equality does not hold in inequality (4) if $n \geq 3$ and $\text{diam}(K) > 0$ (i.e. $K$ is not a one-point set). Let suppose the contrary, i.e. there are a convex body $K$ and a curve $\gamma$ in $\mathbb{R}^n$ for some $n \geq 3$ such that $K \subset \text{co}(\gamma)$ and $N(K) = \frac{\pi^{\frac{n+1}{2}}}{2^\frac{n}{2} \Gamma\left(\frac{n+1}{2}\right)} \cdot \text{length}(\gamma) + \frac{\pi^{\frac{n-1}{2}}}{2^\frac{n}{2} \Gamma\left(\frac{n-1}{2}\right)} \cdot \text{diam}(K)$. Now, as we can see in the above proof, the equalities $\text{per}(K') = \text{length}(\gamma') + \text{diam}(K')$ and $\text{diam}(K') = \text{diam}(K)$ hold, where $K'$ and $\gamma'$ are respectively the orthogonal projections of $K$ and $\gamma$ to any two-dimensional linear subspace $P \in G_{n,2}$. On the other hand, from Theorem 1 we get that $\gamma'$ is a convex curve, $\text{bd}(K') = \gamma' \cup [A', B']$, and $\text{diam}(K') = d(A', B')$, where $A'$ and $B'$ are the endpoints of $\gamma'$. But the latter is impossible. Indeed, $A'$ and $B'$ are the orthogonal projections of the endpoints of the curve $\gamma$ (recall that we consider the orthogonal projection from $\mathbb{R}^n$ onto $P$), hence there are two-dimensional subspaces $P \in G_{n,2}$ with the property $B' = A'$. For any such subspace we have $0 = d(A', B') = \text{diam}(K') < \text{diam}(K)$, that is impossible by our assumptions.

Remark 4 The arguments in Remark 3 show that the equality does not hold in inequality (5) if $n \geq 3$ and $\text{diam}(K) > 0$ (i.e. $K$ is not a one-point set). Moreover, the same we have also for $n = 2$. Indeed, let us suppose that there are a planar convex figure of constant width $\Theta$ and a planar curve $\gamma$ such that $K \subset \text{co}(\gamma)$ and $\text{length}(\gamma) = (\pi - 1) \cdot \Theta$. It is clear that $\text{diam}(K) = \Theta$ and $\text{per}(K) = N(K) = \pi \Theta$ (see (2)). On the other hand, from Theorem 1 we get that $\gamma$ is a convex curve, $\text{bd}(K) = \gamma \cup [A, B]$, and $\text{diam}(K) = d(A, B)$, where $A$ and $B$ are the endpoints of $\gamma$. We know that the boundary of any planar figure of constant width contains no line segment (i.e. a figure of constant width is strictly convex), see e.g. Theorem 3.1.1 and the discussion after its statement in Martini et al. (2019). Hence, $B = A$ and $\text{diam}(K) = d(A, B) = 0$.

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