Critical exponents of the explosive percolation transition

R. A. da Costa,1 S. N. Dorogovtsev,1,2 A. V. Goltsev,1,2 and J. F. F. Mendes1

1Departamento de Física da Universidade de Aveiro & I3N,
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal
2A.F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia

In a new type of percolation phase transition, which was observed in a set of non-equilibrium models, each new connection between vertices is chosen from a number of possibilities by an Achlioptas-like algorithm. This causes preferential merging of small components and delays the emergence of the percolation cluster. First simulations led to a conclusion that a percolation cluster in this irreversible process is born discontinuously, by a discontinuous phase transition, which results in the term “explosive percolation transition”. We have shown that this transition is actually continuous (second-order) though with anomalously small critical exponent of the percolation cluster. Here we propose an efficient numerical method enabling us to find the critical exponents and other characteristics of this second order transition for a representative set of explosive percolation models with different number of choices. The method is based on sewing together the numerical solutions of evolution equations for the cluster size distribution and power-law asymptotics. For each of the models, with high precision, we obtain critical exponents and the critical point.

PACS numbers: 64.60.-i, 05.40.-a, 64.60.ah, 64.60.F-

I. INTRODUCTION

A phase transition in traditional percolation problems is well known to be continuous, i.e., the order parameter $S$ emerges continuously, without a jump at the critical point $\beta = 1$. Above the percolation threshold, a giant connected component (percolation cluster) is present in a system, while below that point all connected components (clusters) are finite. This transition is observed for percolation on lattices and on various networks [3,4]. For lattices, the percolation transition is of the second order with the $\beta$ exponent of the order parameter (the size of the percolation cluster) smaller or equal to 1 ($\beta = 1$ in the mean-field regime, i.e., at or above the upper critical dimension of a system). For highly heterogeneous, e.g., scale-free, networks, the exponent $\beta$ may be above 1, which corresponds to an order of this transition higher than second [5]. The simplest model of percolation (classical random graph) is formulated in the following way. Starting from a large number $N$ of isolated vertices, at each step we choose at random a pair of vertices and interconnect them. When the relative number of links $t = L/N$ in this graph exceeds the threshold $t_c = 1/2$, the graph has a percolation cluster containing a finite fraction $S$ of all vertices. This is an equilibrium transition since this process can be reversed. For the classical random graph model, in the asymptotic relation $S \propto (t-t_c)^\beta$ near the percolation threshold, the critical exponent $\beta$ is 1. In the neighborhood of the continuous phase transition, scaling behavior takes place. In particular, at the critical point, the cluster size distribution $n(s)$ (which is the probability that a finite cluster contains $s$ vertices), asymptotically, decays as a power law, $n(s) \sim s^{-\tau}$, where the critical exponent $\tau$ is 5/2 for the classical random graphs.

The common understanding of the percolation phase transition as continuous was shaken by the study [7] reporting a discontinuous percolation phase transition in models where each new edge is selected from several possibilities by a Metropolis-like local optimization algorithm (e.g., of two candidate connections, the edge joining two smallest clusters is chosen). The suggested discontinuity resulted in the new term, namely “explosive percolation”. This observation was confirmed in a number of subsequent works based on simulations, including Refs. [8–13]. In our work [14], we showed that the conclusions for the local optimization based models obtained from these simulations were incorrect, and the so-called explosive percolation transition is actually continuous for infinite systems. We explained that the exponent $\beta$ of this transition is surprisingly small, which makes the observation of the continuous transition in simulations of realistic size systems virtually impossible. The critical singularity with a small $\beta$ is perceived as a discontinuity for simulated systems. The continuity of the explosive percolation transition was afterwards confirmed by mathematicians [15] and was observed in Ref. [14] for other models.

To describe quantitatively the explosive percolation transition, in our work [14] we showed that the problem can be formulated as a specific aggregation process. The evolution equations for the explosive percolation problems resemble the Smoluchowski equation [20], which enables us to use traditional numerical algorithms [21]. The system of equations was conveniently organized in such a way that we could solve them one by one, sequentially. In this way we succeed to solve numerically $10^6$ evolution equations for the cluster size distribution, which corresponds to the range of cluster sizes $s \leq 10^6$. Nonetheless, these direct numerical calculations were so computationally demanding that proceeding in this way we could not further improve precision of our results including the critical exponents and amplitudes and the explosive percolation threshold position. In the present
work we present a new numerical approach to this problem. We demonstrate how to find characteristics of the explosive percolation transition with higher precision by implementing an efficient method, without solving a large array of evolution equations.

II. THE MODEL

Let us employ the version of the explosive percolation process first considered in Ref. [14]. This model belongs to the same class as the original so-called Achlioptas process simulated in Ref. [5] and in simulations produces the same seemingly discontinuous phase transition. Moreover, the exponent \( \beta \) in our model turns out to be even smaller than in the process from Ref. [7]. Our representative and actually elegant model naturally generalizes the classical random graph model of ordinary percolation (see Sec. I) and allows for analytical and numerical treatment. The process is defined in the following way. We start from an arbitrary initial configuration, for example, from a large number \( N \) of isolated vertices, and at each step, we select uniformly at random \( m \) vertices (\( m \) vertex sample) and choose that of them which is inside the smallest of the clusters to which these vertices belong. Then we again select \( m \) vertices and choose that of them belonging to the smallest cluster, and, finally, add an edge connecting the two vertices selected in this way, see Fig. 1. In other words, at each step, two sets of clusters are chosen with probability proportional to their sizes and two smallest clusters, taken from each of them belonging to the smallest cluster, and, finally, add an edge connecting the two vertices selected in this way, see Fig. 1. In other words, at each step, two sets of clusters are chosen with probability proportional to their sizes and two smallest clusters, taken from each of the sets, merge together. If \( m = 1 \), we recover the classical random graph model. Here we consider the cases of \( m = 2, 3, \) and 4.

This model can be treated as an aggregation process, in which clusters selected by our rules merge progressively. A complete description of this process is provided by the evolving distribution \( P(s, t) = s n(s)/(s) \), which is the probability that a uniformly randomly chosen vertex belongs to a cluster of size \( s \) at time \( t \). Here \( n(s) \) is the size distribution of clusters and \( ⟨s⟩ \) is the average size of clusters (including the percolation cluster). Let us introduce the probability \( Q(s, t) \) that a vertex chosen by our rules from an \( m \) vertex sample belongs to a cluster of size \( s \). This is the size distribution of merging clusters. Using formulas of probability theory [22], we express the distribution \( Q(s, t) \) in terms of the distribution \( P(s, t) \),

\[
Q(s, t) = \left[ \sum_{u=s}^{\infty} P(u, t) + S(t) \right]^m - \left[ \sum_{u=s+1}^{\infty} P(u, t) + S(t) \right]^m, \tag{1}
\]

which is the basic formula of extreme value statistics. Here \( \sum_{u=1}^{\infty} P(u) + S = 1 \) is the cumulative distribution. This is the probability that a uniformly randomly chosen vertex belongs to a cluster of size \( u \geq s \) including the giant component. Using the normalization condition \( \sum_{u=1}^{\infty} P(u) = 1 \), we obtain

\[
Q(s, t) = \left[ 1 - \sum_{u=1}^{s-1} P(u, t) \right]^m - \left[ 1 - \sum_{u=1}^{s} P(u, t) \right]^m. \tag{2}
\]

Note also the relation \( \sum_{u=1}^{\infty} Q(u) + S^m = 1 \). The evolution equations for the distributions \( P(s, t) \) and \( Q(s, t) \) describing this aggregation process in the infinite system (\( N \to \infty \)) have the form

\[
\frac{∂P(s, t)}{∂t} = s \sum_{u+v=s} Q(u, t)Q(v, t) - 2sQ(s, t). \tag{3}
\]

In the case of \( m = 1 \), \( Q(s, t) \) in Eq. (3) should be replaced by the distribution \( P(s, t) \), and we arrive at well-known master equations for standard percolation, which can be solved explicitly [23]. This cannot be done for \( m > 1 \) because in this case the right-hand side of Eq. (3) is not bilinear due to relation (2) and cannot be treated by a generating function technique. Because of this nonlinearity, the case of \( m \geq 2 \) is far more difficult than of \( m = 1 \). So we will analyze Eq. (3) numerically, taking into account the relation (2) and a given initial distribution \( P(s, 0) \). In our work [14] we showed that the solution of this equation has a power-law asymptotics

\[
P(s, t_c) \cong f(0)s^{1-τ}, \tag{4}
\]

at the critical point, which indicates a continuous phase transition. Here \( f(0) \) is a critical amplitude which equals the value of the scaling function \( f(x) \) at \( x = 0 \). At \( t \neq t_c \), the size distribution of clusters decreases more rapidly than any power law. Near \( t_c \) we have \( P(s, t) = s^{1-τ}f(s(t-t_c^{1/σ})) \), where \( τ \) and \( σ \) are critical exponents. While the values of the critical exponents are independent on initial conditions, the form of the scaling function \( f(x) \), and so the critical amplitude \( f(0) \), depends on the initial distribution \( P(s, 0) \). Note that initial distributions should decay sufficiently rapidly, faster than a power law with exponent \(-2m/(2m-1)\), to produce a nonzero critical point, \( t_c > 0 \). The role of initial conditions will be considered elsewhere. Furthermore, the scaling function above the critical point differs from that below \( t_c \) [14].

![FIG. 1. Explosive percolation model rules. At each step, two samples of \( m \) vertices are selected at random. In each of the samples, a vertex belonging to the smallest (of \( m \)) cluster is chosen, and a new edges connecting these two vertices is added to the system.](image-url)
The dimensionality of our system is infinite, i.e., above the upper critical dimension, which guarantees an exact description in the framework of a mean-field approach. Exact Eqs. (2) and (3) provide this description. As it should be above the upper critical dimension, one can express any critical exponent in this problem in terms of one of them. So we need to find one critical exponent. Similarly to Ref. [14], we find the following relation between the critical exponents \( \tau \) and \( \beta \) for arbitrary \( m \):

\[
\beta = \frac{\tau - 2}{1 - (2m - 1)(\tau - 2)}(5)
\]

and the expression for the upper critical dimension

\[
d_{uc} = 2 + 4m\beta \tag{6}
\]

above which a mean-field description is exact. Here we exploited the fact that the upper critical dimension can be always expressed in terms of the mean-field theory critical exponents. Note that our models, in the present form, are defined in infinite dimensions, and, formally speaking, one cannot directly implement them in lower dimensions. Assume that one can introduce an explosive percolation model of the same universality class of critical behavior (i.e., with the same set of critical exponents) but defined on lattices. Then \( d_{uc} \) is the upper critical dimension of this model. Notice that there were considered a bunch of explosive percolation systems on finite-dimensional (mostly two-dimensional) lattices [10].

Let us obtain the critical exponent \( \tau \) and the critical amplitude \( f(0) \) for \( m = 2, 3, \) and \( 4 \), as well as the critical point (explosive percolation threshold) \( t_c \). We assume that at the initial moment all vertices are disconnected, that is \( P(s, 0) = \delta_{s,1} \), where \( \delta_{s,1} \) is the Kronecker symbol. This is our initial condition for the evolution equation.

Substituting Eq. (2) into Eq. (3) we find \( P(1, t) \). Substituting this result into the second equation and solving we obtain \( P(2, t) \), and so on. This procedure enables us to solve numerically the first \( s_{max} \) equations and find \( P(s \leq s_{max}, t) \) at any \( t \) with any desired precision.

III. THE APPROACH

The method we use in this paper is based on the fact that at the critical point, the cluster size distribution \( P(s, t_c) \) has the power-law asymptotics [1], whose parameters, \( \tau \) and \( f(0) \) we do not know yet. The idea is to sew together the numerical solution \( P(s, t) \) and a power-law function at some cluster size \( s_0 \leq s_{max} \) and then analyze the variation of the result with \( s_0 \). Let us assume first that we know precisely the value of the critical point \( t_c \), which is actually not the case. Then, after finding numerically \( P(s, t_c) \) for all \( s \leq s_0 \) and sewing it together with a power law at \( s_0 \), we easily obtain exponent \( \tau \) and critical amplitude \( f(0) \). For that, one uses two conditions: (i) \( P(s_0, t_c) = f(0)s_0^{\tau-1} \) and (ii) the normalization condition, namely,

\[
\sum_{s=1}^{\infty} P(s, t_c) = \sum_{s<s_0} P(s, t_c) + f(0) \sum_{s \geq s_0} s^{1-\tau} = 1. \tag{7}
\]

This condition can be conveniently rewritten in the following form:

\[
1 = \sum_{s<s_0} P(s, t_c) + P(s_0, t_c)s_0^{1-\tau} \left[ \zeta(\tau-1) - \sum_{s<s_0} s^{1-\tau} \right]. \tag{8}
\]

where \( \zeta(x) = \sum_{s=1}^{\infty} s^{-x} \) is the Riemann zeta function. With known \( t_c \), we would immediately find critical exponent \( \tau \) from this equation, leading to the precise value of \( \tau \) in the limit \( s_0 \to \infty \). The value of \( t_c \), however, is not known in advance. Nonetheless, we can formally perform this procedure at any \( t \), inserting the numerical solution of the evolution equations for \( s \leq s_0 \) into the following equations:

\[
P(s_0, t) = f(0)s_0^{1-\tau}, \tag{9}
\]

\[
1 = \sum_{s<s_0} P(s, t) + P(s_0, t)s_0^{1-\tau} \left[ \zeta(\tau-1) - \sum_{s<s_0} s^{1-\tau} \right]. \tag{10}
\]

From which we can find \( \tau \) and \( f(0) \) vs. \( s_0 \) for any \( t \). Clearly, the exact value of \( \tau \) is given by the last equation only for \( t = t_c \) (when \( s_0 \to \infty \)). However, we can still use it to find the set of points \( \tau(s_0) \) corresponding to some time \( t \neq t_c \). The idea is to analyze how the solutions \( \tau(s_0, t) \) of Eq. (10) vary with \( s_0 \) for a set of \( t \) chosen from a neighborhood of the supposed critical point. Below and above the critical point, these solutions behave quite differently as \( s_0 \) approaches infinity, and so it will be easy to identify \( t_c \). This difference is due to the fact that above \( t_c \), Eq. (10) neglects the giant component, and the real sum for finite components \( \sum_{s=1}^{\infty} P(s, t > t_c) \) actually equals \( 1 - S < 1 \).
for the case of the critical point (respectively, solid and dashed curves) particular values of to obtain the precise values apart from the critical point, the distribution $P$ manifests itself in the divergence of the sum $\sum_{s=s_0}^{\infty} s^{1-\tau}$ on its upper limit at $\tau = 2$.

If $t = t_c$, the curve $\tau(s_0, t)$ leads to the precise $\tau$ as $1/s_0 \to 0$. Otherwise, the curves run away from that point, which is the behavior demonstrated by the solid and dashed curves in Fig. 2. To find the precise values of $t_c$, $\tau$, and $f(0)$ we inspected a set of $t$. For each of these $t$, we find the minimum of $\tau(s_0)$, see Fig. 2. Namely, we find the value of this minimum $\tau^* = \min_{s_0} \tau(s_0, t)$ and the value $s^*$ of $s_0$ at which it takes place. We also find the value $f^*(0)$ corresponding to this minimum, $f^*(0) = P(s^*, t)s^{\tau^* - 1}$. In particular, these results provide the dependence of the position of the minima $s^*$ on $t$. It turns out however that instead of considering the function $s^*(t)$, it is more convenient to analyze the transformed inverse function $t$ vs. $1/s^2$. This gives a function $t(1/s^2)$, see Fig. 3(a). In the limit $s^* \to \infty$, the values $t(s^*)$, $\tau^*$, and $f^*(0)$ tend to the exact values of the critical point $t_c$, the critical exponent $\tau$, and the amplitude $f(0)$, respectively. This figure demonstrates that the curve $t$ vs. $1/s^2$ approaches $t_c$ almost linearly. Similarly, we plot $\tau^*$ and $f^*(0)$ vs. $1/s^*$, see Figs. 3(b) and 3(c), respectively, where each point on the plots corresponds to a different value of $t$. These figures demonstrate that $\tau^*$ and $f(0)^*$ approach $\tau$ and, respectively, $f(0)$ almost linearly with $1/s^*$. This enables us to make extrapolations to $s^* \to \infty$ (the maximum number of equations which we used, $s_{\max}$, was $10^5$) and obtain $t_c$, $\tau$, and $f(0)$ with very high precision.

One can even avoid extrapolation procedure, which may occur difficult at $m = 4$ and higher. The problem is that for higher $m$ the curves $\tau(s_0)$ oscillate (see Fig. 4), because the distribution $P(s, t)$ oscillates in the range of low $s$. The reason for these oscillations is the following. If $m$ tends to infinity, then according to our rules, two smallest clusters in the system merge at each step together. Consequently, single vertices initially merge together into the clusters of size 2, then these clusters merge into the clusters of 4 vertices, and so on. This results in the peaks of the distribution at $s = 2, 4, 8, \ldots$, which are seen already at $m = 4$. Fortunately, the amplitude of the oscillations in Fig. 4 decreases with decreasing $1/s_0$. This enables us to investigate the run away of the curves from the precise value of $\tau$ at small $1/s_0$. As $t$ approaches $t_c$ from below or above, the run away in direction of infinity or 2, respectively, occurs at smaller and smaller values of $1/s_0$. It is easy to identify an interval where $t_c$ should lie. The lower bound of this interval is the biggest value of $t$ for which the curve $\tau(1/s_0)$ still shows a trend to infinity at the smallest $1/s_0$, i.e., $1/s_{\max}$. The higher bound of the interval is the smallest value of $t$ that corresponds to a curve still demonstrating a trend to 2 at $1/s_0 = 1/s_{\max}$. Corresponding intervals for $\tau$ and $f(0)$ are then obtained by using Eqs. (9) and (10). We adjust

---

**FIG. 3.** Functions $t(s^*)$ (a) vs. $1/s^2$, and $\tau(s^*)$ (b), and $f^*(0, s^*)$ (c) vs. $1/s^*$ (see the text for the definitions of these functions) in the case of $m = 2$. In the limit $s^* \to \infty$, the values $t(s^*)$, $\tau^*$, and $f^*(0)$ approach the exact values of the critical point $t_c$, the critical exponent $\tau$, and the amplitude $f(0)$, respectively. These curves are extrapolated to $1/s^* \to 0$ to obtain the precise values $t_c$, $\tau$, and $f(0)$.


FIG. 4. Variation of \( \tau \), calculated from Eq. (10), with \( 1/s_0 \) in the case of \( m = 4 \). The solid and dashed curves correspond to two values of \( t \), namely, \( t > t_c \) and \( t < t_c \), respectively.

Another approach for a model of this class was used in Ref. [24] to estimate the percolation threshold position imposing the strong assumption that the cluster size distribution \( P(s, t) \propto s^{-1} e^{-cs} \), where \( c \) is time-dependent, and \( c(t_c) = 0 \). In this way, after solving \( 10^5 \) evolution equations, they achieved the same precision of \( t_c \) (or even worse) which our method provides with only 10 equations. One should emphasize that the actual scaling form of the cluster size distribution essentially deviates from a simple exponential, see Ref. [14].

**IV. CRITICAL EXPONENTS AND AMPLITUDES**

The results of the application of this numerical method to the models with \( m = 2, 3, \) and \( 4 \) are presented in Table I. For comparison, in the first column of the table, we show the exact values for the ordinary percolation problem (\( m = 1 \)). The values of the exponent \( \beta \) and the upper critical dimension \( d_{uc} \) are found by using relations (5) and (6). In the case of \( m = 2 \), the values presented in the table agree with our results [14], although the precision of the numbers obtained in the present work is much higher despite here we solved 10 times less evolution equations than in Ref. [14]. Furthermore, the results in the table for the models with \( 2 \leq m \leq 4 \) agree with those obtained from equations for scaling functions (we will consider this alternative method elsewhere). As \( m \) increases, the difference \( 1-t_c \) decreases, and the exponent \( \beta \) of the giant component size also decreases. The critical amplitude \( f(0) \) is close to \( P(1,t_c) \), especially when \( m \leq 3 \). This closeness indicates that the deviations from a power-law asymptotics in these problems are small even at low values of \( s \). Note that \( f(0) > P(1,t_c) \) for classical percolation, while the opposite is true for the explosive percolation transition. The values of \( \beta \) are remarkably small. In particular, in the case of \( m = 4 \), \( \beta \) is close to 1/400. This produces an extremely “sharp” transition whose continuity is virtually unobservable even in astronomically large though finite systems. One should note that our way to vary exponents in these non-universal systems by changing \( m \) is not unique. For example, explosive percolation models introduced in Ref. [25, 26] showed a a decay of \( \beta \), controlled by a different, specially introduced model parameter.

**V. DISCUSSIONS AND CONCLUSIONS**

Our results show a rapid decrease of the critical exponent \( \beta \) values with increasing \( m \). For \( m = 4 \), exponent \( \beta \) is about 20 times smaller than already its tiny value for \( m = 2 \). This indicates that exploration of this transition by means of numerical simulations at higher \( m \) is hardly possible. Table I demonstrates that the upper critical dimension \( d_{uc} \) quickly approaches 2 with increasing \( m \). So the explosive percolation transition in models of this class placed on two dimensional lattices is very close to its upper critical dimension. This suggests that the critical characteristics of explosive percolation on two-dimensional systems should be close to what was obtained in this paper.

We emphasize that our results were found for models in which evolution is determined by purely local optimization rules. It means that each new interconnection uses only a finite amount of information. In particular, to establish a new link, we do not need to know which of clusters is the biggest in the system. (Indeed, to find the largest cluster, one has to know about all of them.) It is the local optimization rule that leads to continuity of the explosive percolation transition in these models. In more exotic models employing various global optimization algorithms and their variations, discontinuities may occur [27, 32].

In summary, we have proposed an effective numerical method enabling us to find characteristics of explosive percolation transitions with high precision. We obtained the critical points, critical exponents and critical amplitudes in a set of representative models. The fact that critical exponents are model dependent demonstrates the non-universality of critical phenomena for this phase transition. Our results confirm the conclusion that explosive percolation transitions are continuous, with a power-law form of the cluster size distribution at the critical point. Based on our observations, we suggest that in a wide range of models of this kind the explosive percolation transition is continuous, including in particular, the models considered in Refs. [5, 11, 16, 34, 37].

Our approach provides a useful tool for a quantitative description of a new class of critical phenomena
TABLE I. Characteristics of the standard percolation \((m = 1)\) and explosive percolation \((m = 2, 3, 4)\) transitions.

| \(m\) | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| \(t_c\) | 1/2 | 0.923207509297(2) | 0.9817953173509(2) | 0.99497356260563(2) |
| \(\beta\) | 1 | 0.05557108(1) | 0.010428725(1) | 0.0024806707(2) |
| \(\tau\) | 5/2 | 2.04763045(1) | 2.009911883(1) | 2.0024383299(1) |
| \(d_{uc}\) | 6 | 2.4445686(1) | 2.12514470(2) | 2.03969731(3) |
| \(f(0)\) | 1/\(\sqrt{2}\pi \approx 0.3989\) | 0.04619071(1) | 0.009831398(1) | 0.0024320386(1) |
| \(P(1, t_c)\) | \(1/e \approx 0.3678\) | 0.0485992895546(4) | 0.01172146480245(2) | 0.003334067143133(1) |

in non-equilibrium systems and irreversible processes. Moreover, the applicability of this numerical method is not limited to explosive percolation models. Since the method relies only on generic scaling properties, it is suitable to a wide range of continuous phase transitions in non-equilibrium systems.

ACKNOWLEDGMENTS

This work was partially supported by the FCT project PTDC/MAT/114515/2009 and FET IP Project MULTIPLEX 317532.

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1994).
[2] D. Stauffer, Scaling theory of percolation clusters, Phys. Rep. 54, 1 (1979).
[3] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Critical phenomena in complex networks, Rev. Mod. Phys. 80, 1275 (2008).
[4] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, New York, 2003).
[5] S. N. Dorogovtsev and J. F. F. Mendes, Evolution of networks, Adv. Phys. 51, 1079 (2002).
[6] S. N. Dorogovtsev, *Lectures on Complex Networks* (Oxford University Press, Oxford, 2010).
[7] D. Achlioptas, R. M. D’Souza, and J. Spencer, Explosive percolation in random networks, Science 323, 1453 (2009).
[8] Y. S. Cho, J. S. Kim, J. Park, B. Kahng, and D. Kim, Percolation transitions in scale-free networks under Achlioptas process, Phys. Rev. Lett. 103, 135702 (2009).
[9] F. Radicchi and S. Fortunato, Explosive percolation in scale-free networks, Phys. Rev. Lett. 103, 168701 (2009).
[10] R. M. Ziff, Explosive growth in biased dynamic percolation on two-dimensional regular lattice networks, Phys. Rev. Lett. 103, 045701 (2009).
[11] Y. S. Cho, B. Kahng, and D. Kim, Cluster aggregation model for discontinuous percolation transition, Phys. Rev. E 81, 030103 (R) (2010).
[12] R. M. Ziff, Scaling behavior of explosive percolation on the square lattice, Phys. Rev. E 82, 051105 (2010).
[13] N. A. M. Araújo, J. S. Andrade Jr., R. M. Ziff, and H. J. Herrmann, Tricritical point in explosive percolation, Phys. Rev. Lett. 106, 095703 (2011).
[14] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Explosive percolation transition is accidentally continuous, Phys. Rev. Lett. 105, 255701 (2010).
[15] O. Riordan and L. Warnke, Achlioptas process phase transitions are continuous, Science 333, 322 (2011).
[16] J. Nagler, A. Levina, and M. Timme, Impact of single links in competitive percolation, Nature Phys. 7, 265 (2011).
[17] P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, and M. Paczuski, Explosive percolation is continuous, but with unusual finite size behavior, Phys. Rev. Lett. 106, 225701 (2011).
[18] H. K. Lee, B. J. Kim, and H. Park, Continuity of the explosive percolation transition, Phys. Rev. E 84, 020101 (2011).
[19] J. H. Qian, D. D. Han, and Y. G. Ma, Criticality and Continuity of Explosive Site Percolation in Random Networks, Europhys. Lett., 100, 48006 (2012).
[20] M. v. Smoluchowski, Über Brownsche Molekularbewegung unter Einwirkung äußerer Kräfte und den Zusammenhang mit der verallgemeinerten Diffusionsgleichung, Ann. Phys. 353, 1103 (1915).
[21] F. Leyvraz and H. R. Tschudi, Singularities in the kinetics of coagulation processes, J. Phys. A: Math. Gen. 14, 3389 (1981).
[22] W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, New York, 1968), Vol.1.
[23] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, 2010).
[24] V. S. Vijayaraghavan, P.-A. Noël, A. Waagen, and H. J. Herrmann, Tricritical point in explosive percolation on two-dimensional regular lattice networks, Phys. Rev. Lett. 106, 095703 (2011).
[25] P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, and M. Paczuski, Explosive percolation is continuous, but with unusual finite size behavior, Phys. Rev. Lett. 106, 225701 (2011).
[26] H. K. Lee, B. J. Kim, and H. Park, Continuity of the explosive percolation transition, Phys. Rev. E 84, 020101 (2011).
[27] J. H. Qian, D. D. Han, and Y. G. Ma, Criticality and Continuity of Explosive Site Percolation in Random Networks, Europhys. Lett., 100, 48006 (2012).
[28] M. v. Smoluchowski, Über Brownsche Molekularbewegung unter Einwirkung äußerer Kräfte und den Zusammenhang mit der verallgemeinerten Diffusionsgleichung, Ann. Phys. 353, 1103 (1915).
[29] F. Leyvraz and H. R. Tschudi, Singularities in the kinetics of coagulation processes, J. Phys. A: Math. Gen. 14, 3389 (1981).
[30] W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, New York, 1968), Vol.1.
[31] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, 2010).
[32] V. S. Vijayaraghavan, P.-A. Noël, A. Waagen, and H. J. Herrmann, Tricritical point in explosive percolation on two-dimensional regular lattice networks, Phys. Rev. Lett. 106, 095703 (2011).
[33] P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, and M. Paczuski, Explosive percolation is continuous, but with unusual finite size behavior, Phys. Rev. Lett. 106, 225701 (2011).
[34] H. K. Lee, B. J. Kim, and H. Park, Continuity of the explosive percolation transition, Phys. Rev. E 84, 020101 (2011).
[35] J. H. Qian, D. D. Han, and Y. G. Ma, Criticality and Continuity of Explosive Site Percolation in Random Networks, Europhys. Lett., 100, 48006 (2012).
[36] M. v. Smoluchowski, Über Brownsche Molekularbewegung unter Einwirkung äußerer Kräfte und den Zusammenhang mit der verallgemeinerten Diffusionsgleichung, Ann. Phys. 353, 1103 (1915).
[37] F. Leyvraz and H. R. Tschudi, Singularities in the kinetics of coagulation processes, J. Phys. A: Math. Gen. 14, 3389 (1981).
[26] R. F. S. Andrade and H. J. Herrmann, Percolation model with continuously varying exponents, Phys. Rev. E 88, 042122 (2013).
[27] K. J. Schrenk, N. A. M. Araújo, and H. J. Herrmann, The Gaussian model of explosive percolation in three and higher dimensions, Phys. Rev. E 84, 041136 (2011).
[28] Y. S. Cho, S. Hwang, H. J. Herrmann, and B. Kahng, Avoiding a Spanning Cluster in Percolation Models, Science 339, 1185 (2013).
[29] N. A. M. Araújo and H. J. Herrmann, Explosive percolation via control of the largest cluster, Phys. Rev. Lett. 105, 035701 (2010).
[30] W. Chen, X. Cheng, Z. Zheng, N. N. Chung, R. M. D’Souza, and J. Nagler, Unstable supercritical discontinuous percolation transitions, Phys. Rev. E 88, 042152 (2013).
[31] W. Chen, J. Nagler, X. Cheng, X. Jin, H. Shen, Z. Zheng, and R. M. D’Souza, Phase transitions in supercritical explosive percolation, Phys. Rev. E 87, 052130 (2013).
[32] K. J. Schrenk, A. Felder, S. Deflorin, N. A. M. Arajo, R. M. D’Souza, and H. J. Herrmann, Bohman-Frieze-Wormald model on the lattice, yielding a discontinuous percolation transition, Phys. Rev. E 85, 031103 (2012).
[33] O. Riordan and L. Warnke, Achlioptas processes are not always self-averaging, Phys. Rev. E 86, 011129 (2012).
[34] R. M. D’Souza and M. Mitzenmacher, Local cluster aggregation models of explosive percolation, Phys. Rev. Lett. 104, 195702 (2010).
[35] S. S. Manna and A. Chatterjee, A new route to explosive percolation, Physica A 390, 177 (2011).
[36] E. J. Friedman and A. S. Landsberg, Construction and analysis of random networks with explosive percolation, Phys. Rev. Lett. 103, 255701 (2009).
[37] W. Choi, S.-H. Yook, and Y. Kim, Explosive site percolation with a product rule, Phys. Rev. E 84, 020102 (2011).