Finding and classifying an infinite number of cases of the practically perfect phase transition in an Ising model in one dimension

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Very recently, the discovery of a marginal, or practically perfect, phase transition (MPT or PPPT) at finite temperature in the 2-leg ladder Ising model with trimer rungs was reported [1]. The MPT or PPPT stems from a new mathematical structure that has not appeared before in phase-transition problems. This opens the door to simulations and utilizations of the spontaneous phase-transition phenomena in one-dimensional systems, despite the proof that phase transitions do not exist in the one-dimensional Ising models with short-range interactions back to year 1924 [2]. Naturally, the urgent fundamental and practical question is how we, now guided by the new light on phase transitions, can find the next cases of MPT or PPPT in one-dimensional systems. Here, I present a generalization of the original idea to a new parents-children Ising model with ice-cream-cone rungs, and exactly prove that the model possesses the same mathematical structure and thus MPT or PPPT. Yet, the ice-cream-cone structure features internal degrees of freedom, making the MPT or PPPT cases countless and highly tunable—with interesting behaviors like phase reentrance, $T_c$ domes, pairing etc. These discoveries open the door wide to new interdisciplinary researches in studying, engineering, and utilizing the rich phenomenology of MPT or PPPT in various one-dimensional physical, biological, economical, industrial, and social systems, ranging from building the first-generation phase-transition-ready one-dimensional circuits to developing advanced theories that contain the new mathematical structure for frustration-driven phase transitions.
The Ising model is a basic mathematical model in statistical mechanics [3]. It has been widely used to describe collective phenomena in various physical, biological, economical, and social systems. The model consists of individuals that have one of two values (+1 or −1, e.g., open or close in neural networks, buy or sell in financial markets, yes or no in voting, magnetic moments of atomic spins pointing to the up or down direction, etc.). The individuals interact according to the simple rule that neighbors with like values are rewarded more than those with unlike values (Rule #1). Therefore, the society tends to form the order in which all the members have the same value. This tendency is however disturbed by heat, which favors the free choice of the values. On the other hand, if the opposite rule, namely neighbors with like values are rewarded less than those with unlike values (Rule #2), is adopted, the ordered state will have alternating values. Frustration happens when the alternation cannot be arranged. The central question of the Ising model is whether a spontaneous phase transition between the high-temperature disordered state and the low-temperature ordered state exists at a finite temperature. Low dimensionality and frustration are two well-known suppressors of phase transitions. Finding a transition in an Ising model in one dimension is very unusual, unless long-range interactions are used [4]. Surprisingly, frustration was recently found to drive a MPT or PPPT in a one-dimensional Ising model with short-range interactions [1]. It is urgent to know whether this is a very special case or the beginning of extensive studies of MPT or PPPT as implied by the wide applicability of the Ising model. It is reasonable to believe the latter, since the MPT or PPPT exposes a new mathematical structure of phase transition and the beauty of math is its generality. The purpose of this paper is to report a generalization of MPT or PPPT to an infinite number of cases and their classification. Examples in each classes with exotic properties such as phase reentrance and the $T_c$ doom will be shown and discussed.

I. THE ISING MODEL AS A SOCIAL SYSTEM

The generalized one-dimensional model is depicted in Fig. 1a. It is a ladder with advanced rungs: the spins on each rung form a polyhedron (or an ice cream cone to symbolize the far richer details the rung can have). It looks unnatural and not easy to imagine. However, once the model is mapped to a social system, everyone would agree that it makes sense. Suppose this model stands for a neighborhood on a long street. Every rung corresponds to a household. The two spins located on the legs are the two parents. In the case of a single-parent family, the eldest child is promoted
to the position of the other parent. The parents talk to the parents of the nearest neighbors. In the original model, every household has only one child, who makes essentially a triangle with his/her argumentative parents. Now it is clear that introducing a polyhedral rung means increasing the family size. Below, we hold the mapping convention: ladder=street, rung=household, outer spins=parents, and inner spins=children. And we may use them interchangeably.

Expressed in math, one of the infinite possible forms of the generalized 2-leg ladder model with ice-cream-cone rungs is the following Ising model

\[ H \sum_{i=1}^{N} \left[ H^{(i)}_{\text{parents}} + H^{(i)}_{\text{children}} + H^{(i)}_{\text{bias}} \right] \]

(where \( N \) is the total number of the households and we are interested in the large \( N \) limit) given by (c.f., Fig. 1a)

\[
H^{(i)}_{\text{parents}} = -J(\sigma_{i,1}\sigma_{i+1,1} + \sigma_{i,2}\sigma_{i+1,2}) - J'(\sigma_{i,1}\sigma_{i+1,1} + \sigma_{i,2}\sigma_{i+1,1}) - J_{12}\sigma_{i,1}\sigma_{i,2},
\]

\[
H^{(i)}_{\text{children}} = -\sum_{m=3}^{M+2} (J_{1m}\sigma_{i,1}\sigma_{i,m} + J_{2m}\sigma_{i,2}\sigma_{i,m}) - \sum_{m,m'} J_{mm'}\sigma_{i,m}\sigma_{i,m'},
\]

\[
H^{(i)}_{\text{bias}} = -B \left( g_{1}\sigma_{i,1} + g_{2}\sigma_{i,2} + \sum_{m=3}^{M+2} g_{m}\sigma_{i,m} \right),
\]

where \( \sigma_{i,1} = \pm 1 \) and \( \sigma_{i,2} = \pm 1 \) denote the two parents on the \( i \)th household (rung) of the street. Starting from the index number 3, \( \sigma_{i,m} = \pm 1 \) stands for the children on the \( i \)th household. \( \sigma_{N+1,m} \equiv \sigma_{1,m} \) (i.e., the periodic boundary condition). \( M \) is the number of children per household, which is an arbitrary natural number. \( M = 1 \) was studied in details in Ref. [1]. \( J \) and \( J' \) are the interactions between parents of neighboring households. Inside one household, \( J_{12} \) is the interaction between the two parents, \( J_{1m} \) and \( J_{2m} \) the interaction between the children and their parents, \( J_{mm'} \) the interaction between child \( m \) and child \( m' \). To be complete, \( B \) is the external bias field, which will influence the individuals’ opinions or behaviors with a specific preference; it is irrelevant in this paper, as we are interested in spontaneous phase transitions at \( B = 0 \).

As proved exactly in the next section, we stress here that the forms of interactions among the children can be not only two-body—as \( J_{mm'} \) explicitly written in Eq. (1)—but also three-body, four-body, . . . , arbitrary-body interactions. Moreover, particularly for the field of quantum physics, quantum computing, quantum information, etc., the children spins can be quantum spins and the interactions among the children can be of quantum nature, too, with transverse components—because the commutator \([H^{(i)}_{\text{children}}, H] = 0\)—as long as the children-parents interactions are of the Ising type, i.e., the parents provide classical fields to their children. This abundance is exactly the reason that the rungs are called ice-cream-cone rungs, which include and are not limit to polyhedron rungs. The only two constraints on my proof are the following: (i) the parents are
classical spins and they interact with their children using only classical interactions like the ones explicitly written in Eq. (1), no matter the children are classical or quantum. (ii) The model has the minimum unit cell size ($L = 1$), meaning that the household dependences of $M$, member characters, and interactions are neglected. The important issues on such household dependences will be addressed in subsequent publications.

II. THE EXACT SOLUTIONS

The task is to compute the partition function $Z = \text{Tr} e^{-\beta H}$ for all possible combinations of the values $\sigma_{i,m}$ ($i = 1, 2, 3, \ldots, N$ and $m = 1, 2, 3, \ldots, M + 2$) in the large $N$ limit. Here $\beta = 1/(k_B T)$ with $T$ being the temperature and $k_B$ the Boltzmann constant. The thermodynamical properties are retrieved from the free energy per household $f(T) = \lim_{N \to \infty} -\frac{1}{N} k_B T \ln Z$, the entropy $S = -\frac{\partial f}{\partial T}$, and the specific heat $C_v = T \frac{\partial S}{\partial T}$. The order parameter for MPT or PPPT is the correlation function between the two parents: $C_{12}(0) = -\frac{\partial f}{\partial J_{12}}$ [1]. This model can be solved exactly by using the transfer matrix method [1, 3]. Once the largest eigenvalue $\lambda$ of the transfer matrix is obtained, the partition function $Z = \lim_{N \to \infty} \lambda N$ and the free energy per household $f(T) = -k_B T \ln \lambda$.

In the transfer matrix method, we deal with the following complex for the two neighboring households $i$ and $i + 1$:

$$e^{-\beta H_{\text{parents}}} - \frac{1}{2} \beta J_{12} (\sigma_{i,1} \sigma_{i,2} - \sigma_{i+1,1} \sigma_{i+1,2}) - \frac{1}{2} \beta H_{\text{children}} + H_{\text{children}}^{(i+1)}$$

(2)

The children’s values can be exactly integrated out, as they interact with the members of the same household only and interact via the Ising-type interaction with the parents of Ising type, who directly connect to the outside world, which yields the following $4 \times 4$ transfer matrix in the order of the two parents’ values $\sigma_2 / \sigma_1 = +, -, \pm, -\pm$:

$$\Lambda = \begin{pmatrix}
e^{2x+2x'+w} & + & + & + \\
+ & + & j & j \\
+ & + & + & j \\
e^{-2x-2x'+w} & - & - & - \\
- & - & j & j \\
e^{2x+2x'-w} & + & + & j \\
+ & + & - & j \\
e^{-2x-2x'+w} & - & - & j \\
- & - & - & j \\
e^{-2x+2x'+w} & + & + & j \\
+ & + & + & j \\
e^{-2x-2x'-w} & - & - & - \\
- & - & - & j \\
\end{pmatrix}$$

(3)
where \( j = i + 1 \), \( x = \beta J \), \( x' = \beta J' \), \( w = \beta J_{12} \), and the children’s contribution functions

\[
\left[ \begin{array}{c} \pm \end{array} \right] = \left[ \begin{array}{c} \sum_{\sigma_{1,3}, \ldots, \sigma_{i,m}, \ldots, \sigma_{i,M+2}} \left( e^{\beta H_{\text{children}}^{(i)}} \right)_{\sigma_{1,3}=\pm} \sigma_{i,1} = \pm \right]^{1/2}.
\tag{4}
\]

In deriving Eqs. (3) and (4), the detailed form of \( H_{\text{children}}^{(i)} \) is not needed. It works for arbitrary forms of interactions in \( H_{\text{children}}^{(i)} \) and for both classical and quantum children, because the commutator \([H_{\text{children}}^{(i)}, H] = 0\). For the systems with mixed quantum particles and classical Ising spins [5], Eq. (4) means that one first obtains the \( 2^M \) eigenvalues (energy levels) of the quantum Hamiltonian \( H_{\text{children}}^{(i)} \) for one of the four \( \sigma_{i,1} = +, +, -; - \) combinations, say +, and thermally populates those energy levels to get +. Then move on to work out for the other three combinations one by one.

Then, using the spin up-down symmetry in the absence of an external bias field, i.e., \( \left[ \begin{array}{c} - \end{array} \right] = \left[ \begin{array}{c} + \end{array} \right] \) and \( \left[ \begin{array}{c} + \end{array} \right] = \left[ \begin{array}{c} - \end{array} \right] \), as well as the condition of the minimal unit cell (i.e., no household dependence; \( \left[ \begin{array}{c} \pm \end{array} \right] = \left[ \begin{array}{c} \pm \end{array} \right] \)), the transfer matrix is rewritten as

\[
\Lambda = \begin{pmatrix}
a & z & z & u \\
z & b & v & z \\
z & v & b & z \\
u & z & z & a
\end{pmatrix}
\tag{5}
\]

where \( a = e^{2x+2x'+w} \left[ \begin{array}{c} + \end{array} \right]^{2}, z = \left[ \begin{array}{c} + \end{array} \right], u = e^{-2x-2x'+w} \left[ \begin{array}{c} + \end{array} \right]^{2}, b = e^{2x-2x'-w} \left[ \begin{array}{c} + \end{array} \right]^{2}, v = e^{-2x+2x'-w} \left[ \begin{array}{c} + \end{array} \right]^{2}. \)

It is highly symmetric, it can be block diagonalized by the parity-symmetry operations \( U \) for the two spin-reversed pairs \((+, -)\) and \((-+, +)\),

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}
\tag{6}
\]

and the result is

\[
U^T \Lambda U = \begin{pmatrix}
a - u & 0 & 0 & 0 \\
0 & b - v & 0 & 0 \\
0 & 0 & b + v & 2z \\
0 & 0 & 2z & a + u
\end{pmatrix}
\tag{7}
\]
where \( a + u = 2 \cosh(2x + 2x')e^{w/2} \), \( b + v = 2 \cosh(2x - 2x')e^{-w/2} \), \( a - u = 2 \sinh(2x + 2x')e^{w/2} \), \( b - v = 2 \sinh(2x - 2x')e^{-w/2} \), and \( z = \frac{1}{2} \). The eigensystem problem is reduced to a quadratic equation for the even-parity states, which can be easily solved. Finally, the eigenvalues are \( a - u, b - v, \) and

\[
\lambda_{\pm} = \frac{a + u + b + v}{2} \pm \sqrt{\left(\frac{a + u - b - v}{2}\right)^2 + 4z^2}.
\]

(8)

Let us discuss two scenarios: \( x' = 0 \) and \( x' \neq 0 \).

A. \( J' = 0 \)

An elegant form of the largest eigenvalue of the transfer matrix is given by

\[
\lambda = \lambda_+ = \Upsilon_+ \left[ \cosh(2\beta J) + \sqrt{1 + (\Upsilon_- / \Upsilon_+)^2 \sinh^2(2\beta J)} \right],
\]

(9)

where

\[
\Upsilon_\pm = e^{\beta J_{12} \frac{1}{2}} \pm e^{-\beta J_{12} \frac{1}{2}}.
\]

(10)

\( \Upsilon_\pm \) do not dependent on \( J \) explicitly, while \( \lambda \) depends explicitly on the intra-household interactions solely via \( \Upsilon_\pm \). From now on, \( \Upsilon_\pm \), called the frustration function in Ref. \([1]\), will be referred to as the rainbow functions. The reason for this name upgrade will be made clear later on. The unconventional order parameter is the family member correlation functions \([1]\):

\[
C_{12}(0) = \langle \sigma_{i,1}\sigma_{i,2} \rangle_T = -\frac{\partial f(T)}{\partial J_{12}} = \frac{(\Upsilon_- / \Upsilon_+) \cosh(2\beta J)}{[1 + (\Upsilon_- / \Upsilon_+)^2 \sinh^2(2\beta J)]^{3/2}}.
\]

(11)

where \( \langle \cdots \rangle_T \) denotes the thermodynamical average. We arrive at the same mathematical structures of Eqs. (9), (10), and (11) as before \([1]\)—and with the children’s contribution to the rainbow function being generalized to (4).

The mathematical structure of conventional phase transitions is the non-analyticity of the system’s thermodynamic free energy \( f(T) \) where \( T \) denotes temperature. A \( k \)-th-order phase transition means that the \( k \)-th derivative of \( f(T) \) starts to be discontinuous at the transition. At a glance, Eq. (9) and thus \( f(T) \) are analytic. This is a direct consequence of the fact that the transfer matrix, made up from Boltzmann factors, i.e., exponentials, is always strictly positive, irreducible, and analytic \([4]\). However, Eq. (9) has a novel mathematical structure for two features: Firstly, as the
difference between two positive quantities, the rainbow function $\Upsilon$ can change sign when the following condition is satisfied

$$
\pm^2 \neq \mp^2. 
$$

(12)

For example, for the likely situation of $\pm^2 > \mp^2$ (i.e., the children contribute more when the parents hold the same values than when the parents argue), a negative $J_{12}$ decreases the impact of $\pm^2$ while increasing the impact of $\mp^2$. Hence, the sign of $\Upsilon$ can change for a suitable combination of the intra-household interactions with $J_{12} < 0$. This also explain why MPT and PPPT was not found in the ordinary 2-leg ladder without the children. In that case, $\pm^2 = \mp^2 = 1$, violate the condition of Eq. (12) and thus $\Upsilon = 2 \sinh 2\beta J_{12}$, which mathematically changes sign only at $\beta = 0$, i.e., $T = \infty$. This could never happen.

Secondly, $(\Upsilon_\mp/\Upsilon_\pm)^2$ in Eq. (9) has a prefactor of $\sinh^2(2\beta J)$, which is exponentially large near $T_c$, the temperature at which $\Upsilon$ changes sign. So, if Eq. (9) is approximated by neglecting 1 inside $\sqrt{\cdots}$ as it is exponentially smaller than the other terms in the $\sqrt{\cdots}$,

$$
\lambda \simeq \Upsilon_\pm \cosh(2\beta J) + |\Upsilon_\mp| \sinh(2\beta |J|),
$$

(13)

which becomes non-analytic. This mimicking of $|\Upsilon_\mp|$ can also be regarded as a virtual level crossing at $T_c$ between $\lambda$ and the other eigenvalues of the transfer matrix, which was shown to be important for realizing phase transitions in one dimension [4]. The difference between Eq. (9) and Eq. (13) takes place in a region of $(T_c - \delta T, T_c + \delta T)$, where $\delta T$ can be estimated by $|\Upsilon_\mp/\Upsilon_\pm| \sinh(2\beta |J|) = 1$ near $T_c$. $\delta T$ is proportional to $\sinh(2\beta |J|)^{-1}$ and exponentially approaches to zero (the transition is much sharper and much sharper) as $T_c$ decreases. Ref. [1] gives an example of $\delta T \approx 0.4 \times 10^{-30}$ K for $T_c \sim 9$ K. However, for the mathematically strict $\delta T = 0$ transition, $T_c = 0$ as well [1]. Therefore, this kind of super-sharp transitions is named marginal phase transition or practically perfect phase transition (MPT or PPPT). These two features must be satisfied simultaneously. That $\Upsilon$ changes sign at a finite temperature $T^*$ but without an exponentially large prefactor at $T^*$ is a phase crossover.

B. $J' \neq 0$

Suppose $|J'| < |J|$ without loss of generality; otherwise, we just exchange $J$ and $J'$ in all the equations presented above. In general, Eq. (8) tells us that the simulated non-analyticity of $|\Upsilon_\mp|$
takes place at \( a + u - b - v = 0 \). When near \( T_c \), the condition of
\[
e^{- \frac{2|J_1 \pm J'|}{\beta k_B T}} \ll 1,
\]
is satisfied, all the formulae for \( J' = 0 \) are accurately retained with only one modification:
\[
\Upsilon_\pm = e^{\beta |J_{12} + 2J'\text{sgn}(J)|} \pm e^{-\beta |J_{12} + 2J'\text{sgn}(J)|}.
\]
Thus, the effect of \( J' \) is to make the substitution: \( J_{12} \rightarrow J_{12} + 2J'\text{sgn}(J) \). The condition of Eq. (14) holds when \(|J'|\) is appreciably different from \(|J|\), which is generally true. This has two significant implications: One, MPT or PPPT generally take places for \( J' \neq 0 \). This enlarges the parameter space of the model and stabilize the MPT or PPPT against perturbations. The other is that the straightforward effect of \( J' \) on \( J_{12} \) provides a simple way to tune the parameters. In the following, we will focus on the presentation on \( J' = 0 \) and keep in mind \( J_{12} \rightarrow J_{12} + 2J'\text{sgn}(J) \) in general. The unusual cases where the condition of Eq. (14) does not hold will be addressed elsewhere.

### III. CLASSIFICATION

Since the above proof implies that we have obtained an infinite number of the MPT or PPPT cases, one of the first worthy scientific activities is to classify them into distinct categories. I hereby propose a two-class classification. One is called the *regular class* for the cases satisfying
\[
J_{12} < 0 \quad \text{and} \quad \begin{vmatrix} + \end{vmatrix}^2 > \begin{vmatrix} - \end{vmatrix}^2,
\]
which means that the children contribute more when the parents hold the same values than when the parents don’t. The most regular systems have \( J_{1m} = J_{2m} \) for all the children \( m = 3, M + 2 \), which means that the system has the mirror symmetry in the parents-children interactions. In these systems, Eq. (16) becomes apparent because the two parents having opposite values in \( - \) zero out the \( J_{1m} \) and \( J_{2m} \) part of the children’s contribution to the total energy.

The other category is called the *exotic class* for the cases satisfying
\[
J_{12} > 0 \quad \text{and} \quad \begin{vmatrix} + \end{vmatrix}^2 < \begin{vmatrix} - \end{vmatrix}^2,
\]
which generally lacks the mirror symmetry. Therefore, even though the two parents have opposite values in \( - \), the children’s contribution to the total energy will not be zeroed out by the unequal \( J_{1m} \) and \( J_{2m} \).
Be alert that every regular system has two trivial exotic system companies: They are connected by the $J_{1m} \rightarrow -J_{1m}$ or $J_{2m} \rightarrow -J_{2m}$ transformation for all the children simultaneously. Thus, the first nontrivial exotic system appears for $M = 2$, which breaks the mirror symmetry and retain the inversion symmetry. The existence of nontrivial exotic systems justify the generalization of the studies of MPT or PPPT to increased family sizes. It also demonstrates the children’s power in flipping the argumentative parents to the cooperative parents and increasing the diversity and colorfulness of MPT or PPPT. Therefore, the name of $\Upsilon_\pm$ is upgraded to the rainbow functions from the frustration functions used in Ref. [1].

For the regular systems, the parents’ direct interaction $J_{12} < 0$ must be hold for the MPT or PPPT to occur. The MPT or PPPT is generally characterized by the order parameter $C_{12}(0) \simeq +1$ and $-1$ below $T_c - \delta T$ and above $T_c + \delta T$, respectively (exactly zero at $T_c$). A typical phase diagram is shown in Fig. 1b. Although the parents tend to have unlike values, since their interactions with the children satisfy $\frac{\sum^+}{2} > \frac{\sum^-}{2}$ in the low-temperature phase where energy contributions matter the most, the parents are in unison. As the system is heated up, they become less care about the family’s energy need and go on to take unlike values, leaving their children in strong frustration. This leads to a large gain in the entropy’s contribution to the free energy $f(T)$. Thus, the MPT or PPPT is an entropy-driven first-order transition with a large latent heat, a waterfall behavior of the entropy, and a super-sharp peak in heat capacity at $T_c$ [1].

The situation is opposite for the exotic systems, where the parents’ direct interaction $J_{12} > 0$ must be hold for the MPT or PPPT to occur. A typical phase diagram is shown in Fig. 1c. Although the parents tend to have like values, since their interactions with the children satisfy $\frac{\sum^+}{2} < \frac{\sum^-}{2}$ in the low-temperature phase where energy contributions matter the most, the parents are in disagreement. As the system is heated up, they become less care about the family’s energy need and go on to take like values, being effectively detached from their children. This leads to a large gain in the entropy’s contribution to the free energy $f(T)$. Thus, the MPT or PPPT is also an entropy-driven first-order transition with a large latent heat, as will exemplified with a $M = 2$ case below.

**IV. EXAMPLES AND NOVEL RESULTS**

We proceed with a few examples to reveal the rich phenomena of this generalized model, including phase reentrance and the $T_c$ dome. We will focus on the Ising model of Eq. (1).
A. The regular class with the mirror symmetry

We begin with the most regular cases where \( \beta J_{1m} = \beta J_{2m} = y \) and \( \beta J_{mm'} = g \), i.e., all the children have the same interactions with both of their parents and the interactions between the kids are all the same. In terms of the rung geometry, they form triangles, squares (or tetrahedra), trigonal bipyramids, octahedra for \( M = 1, 2, 3, 4 \), respectively (Fig. 1a excluding the rightmost one). The results for \( M = 1, 2, 3, 4 \) are listed in Table I. \( w < 0 \) must hold for the transition to take place, as discussed in the last section. \( T_c \geq 0 \) sets the other general constraint for the model parameters on achieving MPT or PPPT. To have insight into how the interactions between the children affect the MPT or PPPT, the results are divided into three regions: \( g > 0, g = 0, \) and \( g < 0 \). They will be discussed in passing. Note that for shorthand notation, when we describe that \( x, y, w, \) or \( g \) is strong or weak, e.g., weak \( w \) (though defined as \( w = \beta J_{12} \)) means weak \( J_{12} \), not weak \( \beta J_{12} \), which is very large near \( T_c \).

(i) \( g > 0 \), i.e., all the children tend to have the same value. For sufficiently strong \( g \) (no need

| \( M \) | \( g > 0 \) | \( g = 0 \) | \( g < 0 \) |
|-----|-----|-----|-----|
| 1   | \( 2 \cosh(2y) \) | \( 2 \cosh(2y) \) | \( 2 \) |
| 2   | \( 2e^g \cosh(4y) + 2e^{-g} \) | \( 2e^g + 2e^{-g} \) | \( 2 \) |
| 3   | \( 2e^{3g} \cosh(6y) + 6e^{-g} \cosh(2y) \) | \( 2e^{3g} + 6e^{-g} \) | \( 12 \) |
| 4   | \( 2e^{6g} \cosh(8y) + 8 \cosh(4y) + 6e^{-2g} \) | \( 2e^{6g} + 8 + 6e^{-2g} \) | \( 2 \) |
| 2*  | \( 2e^g \cosh(2y_1 + 2y_2) + 2e^{-g} \) | \( 2e^g + 2e^{-g} \cosh(2y_1 - 2y_2) \) | \( 4 \) |

\( \alpha = \frac{w + M|y|}{|x|} \) in this nontrivial exotic system
to be very large; here we show $g > 0.3J$), $\uparrow \downarrow$ and $\downarrow \uparrow$ will pick their values from the dominant first term listed in the table. The $g$ terms will be canceled out, yielding a constant $T_c$, which is proportional to the frustration parameter $\alpha = (w + M|y|)/|x|$. This means that the children are in unison; they act like one child with $y$ the interaction with the parents strengthened to as $M$ times as large as that in the $M = 1$ case. The transition occurs at $T_c = 2\alpha/\ln 2$ with the half-width $\delta T \approx 0.5, 0.2 \times 10^{-2}, 0.1 \times 10^{-9}$, and $0.4 \times 10^{-30}$ K for $\alpha = 0.1, 0.05, 0.025$, and $0.01$, respectively, for a typical $|J| = 300$ K (room temperature). When $g$ is positive but weak, $T_c$ will show a strong $g$ dependence. With the other model parameters fixed, $T_c$ drops by a factor of $M$ from the strong $g > 0$ case to the $g = 0$ case (Fig. 2b: the transition from the red regime to the purple one).

(ii) $g = 0$, i.e., the children are neutral about influencing or being influenced by the other kids. $T_c$ is rescaled with $\alpha = (w + M|y|)/(M|x|)$. This means an increasing in the range of the model parameters in the case of weak $g > 0$. For example, to achieve $\alpha = 0.05$, $M|y|$ and $|w|$ differ by 0.05 for strong $g > 0$, and by 0.05$M$ for weak $g > 0$. Even with the same $T_c$, the transition becomes sharper for weak $g$ by a factor of $1/M$ than for strong $g > 0$ cases.

(iii) $g < 0$, i.e., the children want to have unlike values from each other. Now, the solutions are highly $M$-dependent because with $g < 0$, frustration emerges if alternating arrangement of the values cannot be satisfied simultaneous. For weak $g < 0$, $\uparrow \downarrow$ still picks the first item while $\downarrow \uparrow$ picks the last item listed in the table. $\alpha$ is estimated to be $\alpha = \frac{(g + w + M|y|)}{|x|} \ln 2 \ln 2$. Thus, weak $g < 0$ reduces $T_c$ until the phenomenon of MPT or PPPT disappears. So, if changed from the regime which has MPT or PPPT, (i.e., $0 < \frac{w + M|y|}{|x|} < 0.15$), small $|g|/|x| \sim (w + M|y|)/|x|$ can switch off the transition (Fig. 2b: around $g = 0$).

Moreover, starting from $M = 3$, there is a strong $g < 0$ solution for the MPT and PPPT. For example, for $M = 3$, both $\uparrow \downarrow$ and $\downarrow \uparrow$ can pick the last items in the table, which requires the strong $g < -|y| + \frac{1}{3} \ln 3$ and $\alpha = (w + |y|)/|x| > 0$ to achieve a constant $T_c$ for strong $g < 0$ (Fig. 2b: the transition from the red regime to the purple one). The changing of $M|y|$ to $|y|$ in $\alpha$ and $T_c$ means that when the children are strongly frustrated among themselves (strong $g < 0$), they no longer contribute to the transition in unison but individually. Correspondingly, for the transition to take place for strong $g < 0$, the parents have to reduce the size of their disagreement parameter $w < 0$ from the level of about $-M|y|$ to $-|y|$.

So, the systems need to balance the degree of frustration in order to achieve MPT or PPPT. This means the intriguing reentrance behavior of MPT or PPPT. As shown in Figs. 2, how the
parents adopt to their children’s frustration can dramatically change the landscape of the phase diagram. In Fig. 2, the parents interact with $w = \alpha|x| - |y|$ for all $g$, and the transition happens only at strong $g < 0$. In Fig. 2b, the parents interact with $w = \alpha|x| - M|y|$ for all $g$, and the transition happens only at weak $g < 0$ as a continuation of the $g > 0$ solution. In Fig. 2c, the parents adopt a step function for adjusting $w$ to the above change in $g$. This creates two regimes with MPT or PPPT separated by a regime without the transition near $g = 0$. In Fig. 2d, the parents adopt an approximately linear relationship in between. Most interestingly, a doom-like shape of the low-temperature phase appears. The $T_c$ doom is a hallmark of the phase diagrams of many strongly correlated systems. It is remarkable that we have generate something similar in this exactly solvable model with two argumentative parents and three frustrated children, although the peak are is a crossover. It is worth further studies to characterize this behavior and to reveal the $T_c$ dome in other more complicated cases with larger $M$.

B. The exotic class with the inversion symmetry

Here we use the simplest $M = 2$ nontrivial exotic diamond rung (the rightmost one in Fig. 1) to exemplify the essential difference from the regular class. The exotic diamond rung has inversion symmetry with $\beta J_{13} = \beta J_{24} = y_1$ and $\beta J_{14} = \beta J_{23} = y_2$ but breaks the mirror symmetry with $y_1 \neq y_2$. Suppose $y_1 > |y_2|$ without loss of generality. The results are listed in the last line of Table I and discussed in passing.

(i) $g > 0$ (the right half of Fig. 3a). Strong $g > 0$ in fact sends the system to the regular class, because the solution for MPT or PPPT simply views the system as the one with the \textit{averaged structure} with $y_{1m} = y_{2m} = (y_1 + y_2)/2 = y$ that recovers the mirror symmetry. Thus, $w < 0$ and $\alpha = \frac{(w + M|y|)}{|x|}$. Therefore, whether a system should be classified into the regular or exotic class cannot be simply judged by the mirror symmetry between $y_{1m}$ and $y_{2m}$, which is a necessary condition for the exotic class, but not adequate.

This observation has a significant impact on one of the model’s two constraints, namely minimal unit cell (i.e., no household dependence of $M$ and interactions.) While the changes in the interaction parameters can be treated continuously, the change in $M$ (and thus the structure of the interactions) is discrete and its impact can be dramatic. However, there exists at least one hopeful in the parameter space to MPT and PPPT that views the system as if it has the averaged structure.

(ii) $g < 0$ (the left half of Fig. 3a). What is remarkably new is that for strong $g < 0$, both $\uparrow^2$
and pick the last terms as listed in the table and the latter is larger now. This means that \( w \) must be positive for the transition to happen and the resulting \( \alpha = \frac{(|y_1 - y_2| - w)}{|x|} \). At a glance, this twist seems as good as the aforementioned cases. However, it brings a great benefit. In the regular cases, \( \alpha = \frac{(M|y| - |w|)}{|x|} \). And we know that the MPT or PPPT requires strong frustration, i.e., small \( 0 < \alpha < 0.15 \). This means that \( |w| \) should be close to \( M|y| \). Due to the geometry, the distance for \( |w| \) (which is the distance between the two parents or the width of the ladder) is considerably longer than that for \( |y| \) (which is the distance between a child to his/her parents). Thus, \( |w| \sim M|y| \) may not be easily found in natural settings. Now, in the exotic diamond rung for strong \( g < 0 \), \( \alpha = \frac{(|y_1 - y_2| - |w|)}{|x|} \), which means \( |w| \sim |y_1 - y_2| \). That is, \( |w| \) is compared with the difference between two like values. This solves the problem.

As shown in Figs. 3a and 3b, how the parents adopt to their children’s frustration interestingly changes the landscape of the phase diagram. In Fig. 3a, the parents adopt a step function for adjusting \( w \) to the aforementioned change in \( g \). This creates two oppositely colored (regular vs exotic) regimes with MPT or PPPT separated by a regime without the transition. In Fig. 3b, the parents adopt an approximately linear relationship in between. Two hump shapes of the low-temperature phase appear. Compared with the doom in Fig. 2d, the low-temperature phases live completely under the other phase, yielding the \( T_c \) dooms.

In the exotic system \( (w > 0) \), the parents tend to have like values. However, the parents appear to have unlike values in the ground state, because the parents have different favorite child \( (y_1 > y_2) \) to follow and the children have unlike values \( (g < 0) \). Above the transition temperature, the parents have the like values. This is not something obvious to understand. In the regular system, the parents have unlike values in the higher temperature phase, and their children feel frustrated, leading to huge gain in entropy. For the exotic system, we verify that the MPT or PPPT is also entropy driven, as shown in Fig. 3c—the waterfall behavior of the entropy for strong \( g \).

To get more insights, we examine all the same-family correlation functions

\[
C_{mm'}(0) = \langle \sigma_{i,m} \sigma_{i,m'} \rangle_T = -\frac{\partial f(T)}{\partial J_{mm'}}. \tag{18}
\]

As shown in Fig. 3d, the parents’ correlation function \( C_{12}(0) \) changes sign at \( T_c \), but the children’s \( C_{34} \) remains to be strongly negative. The correlations between a parent and all the children change to zero. That is, in the higher temperature phase, both the parent pair and the children pair enjoy their respective direct interactions and the two pairs are effectively decoupled, leading to the gain of \( \ln 2 \) in entropy per household from this pairing dynamics.
V. IMPLICATIONS

The MPT or PPPT stems from the new mathematical structure [Eqs. (9) and (10)] that has not appear before in phase-transition problems. The present finding of an infinite number of nontrivial one-dimensional systems that contain MPT or PPPT reflects the principle that the beauty of mathematical abstraction is its generality. This makes the studies and utilization of the model attractive projects for scientific discovery and technological applications. A few thoughts on the immediate impacts of the present studies and Ref. [1]—without any substantial modification—follow.

Firstly, they will stimulate exploration of the model for possibilities in functionalities and their optimized performance within the huge capacity of the model. We can change $M$ and the arrangement of the children spins. We can deal with two-body, three-body, four-body, . . . , arbitrary-body interactions among the children spins. We can study in the classical-children fashion or the quantum-children fashion. Don’t forget we can tune $J'$. And their combinations. I would like to mention one specific mission, namely engineering the first-generation phase-transition-ready one-dimensional circuits. An Ising spin is nothing but a two-level object, or a classical bit, which can be readily simulated with circuits. Even quantum bits (qubits) have been recently built with circuits [6]. The phase-transition-ready one-dimensional circuits will be useful in temperature-sensitive applications.

Fundamentally, they will motivate reexamination of the existing advanced theories that deal with strong frustration to figure out what are needed to be done to achieve the new mathematical structure presented here. This will not only provide refreshed insights into old problems [7], but more importantly also guide new theoretical development.

In addition, the studies of the MPT or PPPT have educational benefits. One-dimensional systems are ubiquitous and critically important in the universe and the human knowledge domain. They range from nanotubes to circuit wires and from DNA to superstring. Moreover, they have dominated the exact-math-based microscopic-level teaching and learning in classrooms—with one notorious annoying exception, i.e., the nonexistence of conventional phase transitions in the one-dimensional Ising models with short-range interactions. The practically perfect phase transition in the 2-leg Ising model (especially for the simple $M = 1$ case) may be used in classrooms as a remedy.

Last but not least, they will find interdisciplinary applications. For example, the current exact results on the dual presentation of the electron spin and social-science problem will help encour-
age the hope that a quarreling society (i.e., $J_{12} < 0$ and/or $J < 0$) can be in principle phase transitioned into social harmony (i.e., $C_{12}(0) = +1$). This is the beauty and power of the Ising model, which genuinely connects issues in various physical, biological, and social systems. The present discovery and classification of an infinite number of nontrivial cases of the practically perfect phase transition in the Ising model in one dimension are anticipated to add a new long-lasting excitement to this almost one-hundred-year-old interdisciplinary research domain.

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FIG. 1. The model. (a) The 2-leg ladder with (from left to right) ice-cream-cone, triangle, diamond, trigonal-bipyramid, octahedron, and exotic diamond rungs. The outer (yellow) and inner (orange) balls stand for the parent and children spins, respectively. The bonds represent the interactions $J(x)$, $J'$ (dashed lines), $J_{12}(w)$, $J_{1m}/J_{2m}$ ($y$ or $y_1, y_2$), and $J_{mm'}(g)$: the letters in the parentheses are shorthand notations for their values multiplied by the inverse temperature $\beta = 1/T$. Unlike the other regular rungs, which have the mirror symmetry to link the two parents, the deformed diamond rung (rightmost) has the inversion symmetry but no mirror symmetry. The phase diagrams of the model with the deformed diamond rungs in terms of the order parameter $C_{12}(0)$ as a function of the temperature $T$ and the frustration parameter (b) $\alpha = (|y_1 + y_2| + w)/|x|$ for the regular case $g = 3$ and (c) $\alpha = (|y_1 - y_2| - w)/|x|$ for the exotic case $g = -3$. Red stands for the $+1$ region (where the parents have like values), purple stands for the $-1$ region (where the parents have unlike values). The sharp transitions between them with $T_c = 2\alpha|J|/k_B \ln 2$ take place for strong frustration $0 < \alpha < 0.1$. 
FIG. 2. **Phase diagrams of the ladder with regular trigonal bipyramid rungs** \((M = 3)\) in terms of the order parameter \(C_{12}(0)\) as a function of \(T\) the temperature and \(g\) the interaction between the children. Red stands for the \(+1\) region (where the parents have like values), purple stands for the \(-1\) region (where the parents have unlike values). The parents’ interaction \(w\) is (a) \(\alpha |x| - |y|\), (b) \(\alpha |x| - M|y|\) for all \(g\), (c) a step function of \(g\) and (d) approximately a linear function of \(g\) between \(\alpha |x| - |y|\) and \(\alpha |x| - M|y|\). The last one shows a doom-like shape of the low-temperature phase near \(g = 0\). Here \(x, w, y_1, y_2, g\) are defined in Fig. [1].
FIG. 3. **Phase diagram of the ladder with the deformed diamond rungs** in terms of the order parameter $C_{12}(0)$ as a function of $T$ the temperature and $g$ the interaction between the children. Red stands for the +1 region (where the parents have like values), purple stands for the −1 region (where the parents have unlike values). The parents’ interaction $w$ is (a) a step function of $g$ and (b) approximately a linear function of $g$ between $\alpha |x| - |y_1 + y_2|$ at $g = 3$ and $|y_1 - y_2| - \alpha |x|$ at $g = -3$. The latter shows two hump-like shaped distinct low-temperature phases. (c) The waterfall behavior of the entropy per rung as a function of temperature for $\alpha = 0.05$ (i.e., $w = 0.45$, $y_1 = 1.55$, $y_2 = 1$) and three $g$’s. $|x|$ is the energy unit. (d) The same-family correlation functions $C_{mm'}(0)$ for the same set of parameters. $x$, $w$, $y_1$, $y_2$, $g$ are defined in Fig.[1].