Double-negative flexural acoustic metamaterial

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Abstract
We present an acoustic metamaterial for flexural waves with both negative density and elastic modulus. By means of an analytical lumped-element approach, we introduce a combination of resonant elements on the surface of a plate. Following on from experimental demonstrations and theoretical work, negative density is achieved by introducing elements such as stubs that involve normal-force interactions, whereas negative modulus is achieved by introducing new types of resonant element that involve lateral forces and rotational inertia. Our approach therefore opens the way for the realization of double-negative acoustic meta-plates to control flexural waves. Applications include cloaking and filtering.

Keywords: acoustics, metamaterial, flexural wave, lumped-element model, double-negative material, elastic modulus, density

1. Introduction
Metamaterials based on locally vibrationally resonant structures can exhibit negative effective constitutive parameters controlling wave propagation [1]. For applications it is important that the parameters determining the propagation of the waves with wavelengths significantly exceeding the distance between the local resonators can be made negative, and that this does not require spatial periodicity of the resonant structure. In acoustics there exist proposals for the
design of one-, two- and three-dimensional (3D) metamaterials over particular frequency ranges for which both negative density and negative modulus arise simultaneously for bulk acoustic waves [2–10]; their design exploits overlapping resonances of resonant structures [2–9] or the use of space coiling [10]. Such ‘double-negativity’ in an acoustic system has been experimentally achieved for a variety of systems, notably, for example, for the propagation of pressure waves in air-filled tubes [4]. In such systems, negative density was synthesized from resonances in an array of thin membranes [11], whereas negative bulk modulus was synthesized from resonances in an array of Helmholtz resonators [12] or in side-holes in the tube walls [4]. Another example is a 3D array of hollow spheres [8]. There is, however, still scope for new double-negative systems involving other types of acoustic wave.

In particular, consider the example of flexural waves. Modifying the propagation of Lamb waves in plates or flexural waves in membranes in a controllable and tunable way is a topic of active investigation. (For a review and extended list of references see [13].) It was shown theoretically that phononic properties—acoustic properties arising from periodicity as opposed to metamaterial-related properties—such as forbidden bands or bands exhibiting opposite signs of the phase and group velocities can be introduced by regular spatial structuring of plates or membranes; for example, this can be done by the use of perforations in the case of plates [14–16]. Perforated holes can be filled with a material different from that of the plate [14] or left empty [15, 16].

It has in addition been predicted that the interaction of Lamb waves with local resonators in the metamaterial context can induce a complete band gap [17–19]. It was also demonstrated numerically that filling perforated holes in plates by a soft material [17] or by depositing stubs or rods on the surface of plates [18, 19] can significantly modify the Lamb-wave (plate-mode) dispersion relations through hybridization with the local resonances. Analogous effects are also theoretically predicted in the case of partial perforation of plates, e.g., by the use of holes covered by thin resonant membranes [20]. Most of the theoretical predictions for such local resonators have been confirmed with measurements. In some experiments [21–24] the local resonators are stubs or rods deposited on the surface, whereas in others [25] they are cantilevers of special form perforated in plates. Theoretical predictions and interpretation of the experimental observations for such flexural (Lamb-wave) metamaterials are mostly based on numerical modelling [17–22, 24–26]. Analytical approaches for the evaluation of the dispersion relations for flexural waves in plates periodically loaded by a single type of simple oscillator were only proposed quite recently, and were applied to the study of the interplay between local resonances and the Bragg scattering of waves and to the interaction of resonant- and Bragg-forbidden gaps [27, 28].

A survey of published theoretical studies for flexural waves in the presence of local resonances [16, 19, 20, 22, 26–28] indicates that there exist no theoretical predictions for frequency bands in which the phase and group velocities have opposite sign, corresponding to a double-negative acoustic meta-plate. Neither is there any indication of the presence of this type of band in experiments on flexural-wave metamaterials [22, 25]. Proposed and tested resonant metamaterial structures either provide resonances influencing a single constitutive parameter of the flexural waves or provide non-overlapping resonances, precluding double-negativity. Here we develop a simple analytical lumped-element theory to guide flexural-wave metamaterial design. We show that the vibrations of resonant structures that exhibit normal forces on the plate surface [22, 27, 28] influence only the effective density. We also propose rotationally-resonant structures that can directly influence the effective flexural rigidity. In addition we show how, by analogy with 3D solid-state acoustic metamaterials [3] or pressure waves in air-filled tubes [4], that combining these two types of resonant structure leads to the realization of a
double-negative meta-plate that admits the propagation of flexural waves with opposite phase and group velocities.

2. Lumped-element theory of flexural metamaterials

2.1. Flexural wave equation

A classical derivation of the equation for unidirectional plane flexural wave propagation [29] in the $x$ direction combines the equation for the vertical displacement $u_z(x, t)$ of a plate

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial Q_z}{\partial x} + q,$$  

(1)

where $\rho$ is the plate density, $h$ is its thickness and $q(x, t)$ is the normal force applied per unit surface area of the plate surface (see figure 1), with the relation for the shear force $Q_z(x, t)$ per unit length along the lateral direction $y$ and the spatial derivatives of the displacement field $u_z(x, t)$. Here $z$ is the coordinate pointing directly downwards into the plate. An expression for $Q_z$ in equation (1) can be obtained by first considering the formula relating the normal strain $\frac{\partial u_z}{\partial x}$ to the curvature of the plate:

$$\frac{\partial u_z}{\partial x} = -z \frac{\partial^2 u_z}{\partial x^2}.$$  

(2)

This allows one to evaluate the normal stress

$$\sigma_z = \frac{E}{1 - \nu^2} \frac{\partial u_z}{\partial x} = -z \frac{E}{1 - \nu^2} \frac{\partial^2 u_z}{\partial x^2}$$

and the bending moment per unit length along $y$ acting on a mechanical element of infinitesimal surface area $dx\,dy$, as shown in figure 1:

$$M_x = \int_{-h/2}^{h/2} z \sigma_z \, dz = -D \frac{\partial^2 u_z}{\partial x^2}.$$  

(3)
Here the flexural (i.e., bending) modulus of the plate is defined by \( D = (h^3/12)E/(1 - \nu^2) \), where \( E \) is the Young modulus and \( \nu \) is the Poisson’s ratio. By substituting equation (2) in the relation between the shear force and bending moment (both per unit length along the \( y \) axis—see figure 1)

\[
Q_x = \frac{\partial M_x}{\partial x},
\]

the relation

\[
Q_x = -D \frac{\partial^3 u_z}{\partial x^3}
\]

can be obtained. Combination of equations (1) and (5) leads to the closed-form equation for flexural waves:

\[
\rho \frac{\partial^2 u_z}{\partial t^2} = -D \frac{\partial^4 u_z}{\partial x^4} + q.
\]

The so-called equation for flexural waves, equation (6), is known to provide an excellent theoretical description of the propagation of zeroth-order anti-symmetric Lamb modes of vanishingly small amplitude, i.e., linear waves, when the wave number does not exceed the reciprocal of the plate thickness [29].

2.2. Effective density of a flexural metamaterial

A flexural metamaterial, i.e. an acoustic meta-plate, can be constructed from an array of resonant elements attached to a plate. For a flexural wave of wavelength \( \lambda \), the distribution of normal force arising from the distribution of resonant elements is continuous to a good approximation provided that the distances between adjacent elements on the plate surface are significantly smaller than \( \lambda \). This is the condition for the term ‘metamaterial’ to apply. If we model a resonant element as a lumped-element oscillator composed of a bead of mass \( m_\perp \) free to move in the \( z \) direction on a spring of stiffness \( \eta_\perp \) attached to the top surface of the plate [30], as shown by the example of the regular lattice in figure 2(a) of the element shown in figure 3(a), then

\[
\eta_\perp = -\frac{q}{\partial x} \left( \frac{\partial u_z}{\partial t} \right) (x, t)
\]

where \( u_z \) is the spatially-averaged time-dependent vertical displacement distribution of the masses from their equilibrium positions, induced by the motion \( u_z(x, t) \) of the upper surface of the plate, and \( n \) is the number of oscillators per unit area. (In the example of the regular array of figure 2(a), \( n = 1/a_x a_y \), where \( a_x \) and \( a_y \) are the sides of the unit cell.) Combining equation (7) with the equation of the oscillator motion, \( m_\perp \partial^2 u_\perp /\partial t^2 = \eta_\perp [u_z(x, t) - u_\perp] \), the normal surface force can be evaluated in the frequency domain as \( \tilde{q}(x, \omega) = m_\perp [\omega^2 / (\omega_\perp^2 - \omega^2)] \tilde{u}_z(x, \omega) \), where \( \omega \) is the angular frequency and \( \omega_\perp = \sqrt{\eta_\perp / m_\perp} \), indicating resonant enhancement of the oscillator influence on plate motion at \( \omega = \omega_\perp \). Combination of the temporal Fourier transform \( \tilde{q}(x, \omega) \) with the frequency-domain version of equation (6) leads to the dispersion relation for +\( x \)-propagating flexural waves with displacement \( u_z(x, t) \propto \exp (i\omega t - ikx) \), where \( k \) is the wave number,

\[
k^2 = \pm \frac{\omega}{\sqrt{D}} \sqrt{\frac{\rho h + nm_\perp}{1 - \omega^2 / \omega_\perp^2}},
\]

which indicates that the normal loading of the plate by the oscillators modifies the effective density of the plate. The solution in equation (8) predicts a gap opening in the frequency band from \( \omega_\perp \) up to \( \sqrt{1 + (nm_\perp / \rho h) \omega_\perp^2} \), i.e., above the resonance frequency \( \omega_\perp \). Thus the width of
the band forbidden for propagation, induced by the negative effective density

\[ \rho_{\text{eff}} = \rho + \frac{nm_\perp}{h} \frac{1}{1 - \omega^2/\omega_\perp^2} \]

of the meta-plate, is controlled by the ratio per unit surface area of the mass of the resonators to that of the plate. To avoid phononic effects from a regular lattice, the arrangement of the resonators on the plate corresponding to the lumped elements can be chosen to be random.

2.3. Effective modulus of a flexural metamaterial

To modify the flexural modulus, the resonant elements should directly participate in and influence the bending motion of the meta-plate. Consider the horizontal lumped element, shown by the example of the regular lattice in figure 2(b) of the element shown in figure 3(b),
consisting of a bead of mass $m_i$ and two $x$-aligned collinear springs, the latter attached to points A and B of the plate. This introduces additional rigidity to in-plane out-of-phase motion of points A and B, such motion being characteristic of flexure (bending). The resistance to bending is proportional to the difference in the displacements of these points, i.e., to the elongation

$$u_x^B - u_x^A = u_x \left( x + \frac{L_x}{2}, z = -\frac{h}{2}, t \right) - u_x \left( x - \frac{L_x}{2}, z = -\frac{h}{2}, t \right)$$

$$\approx \left[ \frac{\partial u_x}{\partial x} \left( x, z = -\frac{h}{2}, t \right) \right] L_x$$

of the two springs each of stiffness $\eta_i$. In equation (9), $L_x$ is the length from A to B of the lumped element along the $x$-axis, which is much smaller than $\lambda$ (since the separation of the elements is). The interaction of the flexural waves with this element is non-resonant, because symmetry dictates that equal and opposite displacements of the points A and B do not induce any motion of the mass (see arrows in figure 3(b)). To produce a resonant change in modulus we shall therefore require a somewhat different mechanical element.
A new type of lumped element that can resonate under bending loads is shown by the example of the regular lattice in figure 2(c) of the element shown in figure 3(c), consisting of a spherical bead of mass \(m\) and radius \(R\) that is connected at points A and B a distance \(L_x\) apart in the \(x\) direction to two \(x\)-aligned non-collinear springs of separation \(L_y = 2R\). Similar resonant elements have previously been proposed in mass–spring models to achieve negative modulus for bulk acoustic waves [31]. As in the case for the resonant element of figure 3(b), the mass \(m\) does not displace along the \(x\)-axis when \(u_x = -u_x^A\). However, it can rotate about the \(z\)-axis with corresponding moment of inertia \(I_z\). The forces applied to the points A and B are \(F_{A,B} = -\eta_0[u_x^{A,B}(x, t) \pm R\phi]\), where \(\phi\) is the rotation angle (as defined in figure 3(c)). Combining \(F_{A,B}\) with the equation of the oscillator motion, \(\dot{\phi} + \eta_0\dot{\phi} = \eta_0[\omega^2/(\omega_\|^2 - \omega^2)][\ddot{u}_z^B(x, \omega) - \ddot{u}_z^A(x, \omega)]\), where \(\omega_\| = \sqrt{2R^2\eta_0/I_z}\), indicating resonant enhancement of the influence of the oscillator on plate motion at \(\omega = \omega_\|\). Because the length \(L_x\) of the element along the \(x\) axis is much smaller than \(\lambda\), in the analysis of the plate motion \(L_x\) can be considered as an infinitesimal and treated similarly to \(d\) in the derivation of equation (6). The Fourier component \(\Delta M_x(x, \omega)\) of the induced bending moment per unit length along the \(y\) axis acting on the mechanical element of length \(L_x\) is then equal to \(-\eta_0[(1/2)\tilde{\lambda} + \tilde{F}_A - \tilde{F}_B]\).

Combining the above relations with equation (9), one can write this additional bending moment in the form

\[
\Delta M_x(x, \omega) = -(h/2)(L_x/L_y)\eta_0[I\ddot{u}_x(x, \omega) / (\omega_\|^2 - \omega^2)]\partial \ddot{u}_x(x, z = -h/2, \omega)/\partial x.
\]

When this solution for the moment is substituted in the spectral version of equation (4) and additionally differentiated with respect to \(x\), we obtain in the frequency domain the additional element-induced force acting in the \(z\) direction per unit surface area beneath the lumped element in terms of \(u_z\) by the use of \(\partial \ddot{u}_z / \partial x = -(h/2)\partial^2 \ddot{u}_z / \partial x^2\). One further differentiation with respect to \(x\) then yields

\[
\Delta Q_x(x, \omega) / \partial x = (h/2)^2(L_x/L_y)\eta_0[\omega^2/(\omega_\|^2 - \omega^2)]\partial^4 \ddot{u}_z(x, \omega) / \partial x^4.
\]

Multiplying this result by the surface area \(L_xL_y\) of the mechanical element, one obtains the additional vertical force that acts on the surface of the plate when loaded by a single rotationally-resonant lumped element. The induced force per unit surface area of the plate due to the distribution of these elements can be obtained by an additional multiplication by the number of oscillators per unit area \(n\). This modified expression for \(\Delta Q_x(x, \omega) / \partial x\) should be substituted in the spectral version of equations (1) and (6)

\[
-\omega^2\rho \ddot{u}_z = \partial \ddot{Q}_z / \partial x + \Delta(\partial \ddot{Q}_z / \partial x) + \bar{q} = -D\partial^4 \ddot{u}_z / \partial x^4 + \Delta(\partial \ddot{Q}_z / \partial x) + \bar{q},
\]

where \(\bar{q} = 0\) in the case of the absence of vertical mechanical elements.

This results in the following dispersion relation for flexural waves in the meta-plate in the case when a distribution of rotationally-resonant elements is alone present:
The form of this dispersion relation indicates that the proposed lateral loading of the plate by rotationally-resonant oscillators modifies the effective flexural modulus of the plate. The solution in equation (10) predicts a gap opening in the frequency band from \( \omega_\parallel \sqrt{1 + n (h/2)^2 L_x^2 \eta_\parallel /D} \) up to \( \omega_\parallel \), i.e., below the resonance frequency \( \omega_\parallel \). Thus the width of the forbidden band for wave propagation, induced by the negative effective modulus

\[ D_{\text{eff}} = D - n (h/2)^2 L_x^2 \eta_\parallel \frac{\omega^2 / \omega_\parallel^2}{1 - \omega^2 / \omega_\parallel^2}, \]

is controlled by the ratio of the characteristic effective stiffness of the attached element to that of the plate, i.e. \( n L_x^2 \eta_\parallel \) and \( hE/(3(1 - \nu^2)) \), respectively. Simultaneous fabrication of distributions of the two types of the above-mentioned resonant elements on the plate surface, i.e., vertical and rotationally-resonant element types, therefore provides a double-negative acoustic meta-plate in the frequency range of the overlapping band gaps, provided that the condition for overlap is satisfied.

2.4. Double-negative flexural metamaterial

By combining the distributions of the vertically and horizontally resonant mechanical elements of figures 3(a) and (c), one may independently choose the effective density and modulus. The dispersion relation can be written in the following dimensionless form for propagation along the \( x \) axis:

\[ q^2 = \pm \frac{\sqrt{1 + p_\perp / \Omega_\perp}}{\sqrt{1 - p_\parallel / \Omega_\parallel}} \Omega_\perp. \] (11)

Here \( \Omega = \omega / \omega_\parallel \) the angular frequency is normalized with respect to \( \omega_\parallel \), \( q = k / \sqrt{\rho h/D \omega_\parallel} \) is the flexural wave number normalized with respect to that of flexural waves at frequency \( \omega_\parallel \) in the unloaded plate, and \( \Omega_0 = \omega_\parallel / \omega_\perp \) is the ratio of the resonance frequencies of the horizontal and vertical lumped elements. Also, \( p_\perp = nm / (\rho h) \) and \( p_\parallel = n^2 h^2 L_x^2 \eta_\parallel / (4D) \). Equation (11) admits four solutions, two for each direction. For \( +x \) propagation only, one solution corresponds to a mode with a wave number proportional to \( (1 + i) \), which is evanescent in the bare plate (i.e. in the case when \( p_\parallel = p_\perp = 0 \)) and cannot be made propagating in the meta-plate. However, the second solution corresponds to a mode which is propagating in the bare plate but can made evanescent in some frequency regions in the case of the meta-plate. Moreover, for the case of double-negativity the meta-plate can be constructed to achieve opposite signs of the phase and group velocities in this second mode. Below we shall analyse and illustrate the dispersion relations for this second mode.

The dimensionless parameters \( p_\perp \) and \( p_\parallel \) respectively control the widths of the negative-density and negative-modulus frequency bands for the evanescent modes. The former band starts from \( \Omega = 1 \) and ends at \( \sqrt{(1 + p_\perp)} \), whereas the latter starts from \( \Omega = \Omega_0 / \sqrt{(1 + p_\parallel)} \) and ends at \( \Omega_0 \). A double-negative propagating region thus exists in the frequency band.
under the condition that the right-hand side in the inequality of equation (12) is larger than the left-hand side. The conditions for double negativity, i.e., for overlap of the negative-density and negative-modulus bands, are \( \Omega_0 > 1 \) and \( \Omega_0 / \sqrt{1 + p_{||}} < \sqrt{1 + p_{\perp}} \). Under our approximation of lossless materials, waves propagating under these conditions are unattenuated. The double-negative frequency band, expressed by equation (12), is, in the absence of degeneracies, surrounded by two bands (characterized by single-negativity) forbidden for flexural-wave propagation, whereas at lower and higher frequencies there are double-positive, i.e. propagating, bands.

Figure 4(a) shows the normalized dispersion relation in the case when the conditions for double-negativity hold; a double-negative propagating band is enclosed by the negative-density forbidden band from below and by the negative-modulus forbidden band from above. The parameters chosen in figure 4(a) are \( \Omega_0 = 3 \), \( p_{||} = 3 \) and \( p_{\perp} = 3 \), and \( \Omega_0 \). Figures 4(b) and (c) show the corresponding variations in normalized effective density and modulus as a function of \( \Omega \).

Figure 4(d) shows the normalized dispersion relation in the complementary case when the conditions for double-negativity do not hold; a double-positive propagating band is enclosed by the negative-density forbidden band from below and by the negative-modulus forbidden band from above. The parameters chosen in figure 4(d) are \( \Omega_0 = 4 \), \( p_{||} = 2 \) and \( p_{\perp} = 2 \). Figures 4(e) and (f) show the corresponding variations in normalized effective density and
modulus as a function of $\Omega$. Other cases where degeneracies cause one or more of these bands to vanish also exist.

3. Design of the resonant elements

The simplest resonant vertical element for achieving an effective negative density, which admits the proposed lumped-element description, consists of a sandwich of two vertical rods of equal cross section, $S_1 = S_2 = S$, made of different materials, as shown in figure 3(d) for the example of a square cross section. The influence of this type of resonant element on the propagation of Lamb modes in plates has previously been studied by numerical simulation [19]. If the material (1), contacting the plate, is much softer than the plate material, then the solution for the lowest resonant frequency, under the conditions $\omega \leq H_{1i}$ and $\omega \leq \sqrt{S_i}/c_i$, can be approximated by

$$\omega_\perp \approx \sqrt{\left(E_i/H_i\right)/\rho_2 H_2}. \quad (13)$$

Here $H_i$, $S_i$, $\rho_i$ and $c_i = \sqrt{E_i/\rho_i}$ denote the heights, cross-sectional areas, densities and longitudinal sound velocities of the materials ($i = 1, 2$). The solution in equation (13) indicates that the lower, soft rod plays the role of a lumped spring with a stiffness $\eta_\perp$ equal to $(E_1/H_1)S$, whereas the upper rod plays the role of a lumped mass $m_\perp$ equal to $\rho_2 H_2 S$. The low-frequency edge of the forbidden gap described by equation (13) diminishes in the case of a higher-density upper rod and also on increasing the heights of either rod. For example, using gold (of density $\rho_2 = 19,300 \text{ kg m}^{-3}$) with $H_2 = 50 \text{ nm}$ for the dense element and polymethyl methacrylate (of Young’s modulus $E_1 = 6.3 \text{ GPa}$ from [32]) with $H_1 = 460 \text{ nm}$ for the soft element, the resonance frequency $f_\perp = \omega_\perp/2\pi$ of the vertical element is estimated to be equal to 0.6 GHz. An additional decrease of the gap frequency can be achieved by reducing the ratio $S_1/S_2$. For example, for $S_1/S_2 = 1/2$ one finds $f_\perp = 0.42 \text{ GHz}$. Such near-GHz frequencies are very important in signal filtering applications. Vertical GHz-order resonance elements can also be realized in practice with attached spheres [33], as demonstrated for the case of acoustic metamaterials based on Rayleigh surface acoustic waves [34]. Our proposed design should be more versatile, having more degrees of freedom than a single sphere.

It is instructive to also make estimates for the lowest resonances of the horizontal rotationally-resonant element shown in figure 3(f), corresponding to the lumped element of figure 3(c), assuming that the materials are the same as those chosen for the vertical lumped element. These horizontal resonant elements could be installed in grooves prepared in the plate surface or attached to humps deposited on the surface. For illustration we have chosen a simple design based on rectangular-cross-section rods. In the formula for the resonant angular frequency $\omega_\parallel = \sqrt{2R^2 \eta_\parallel / I_\parallel}$, we approximate the bead diameter $2R$ by $h_\parallel/2$, $\eta_\parallel$ by the lumped element rigidity $(E_i/H_i)(h_\parallel/2)h_\perp$ and $I_\parallel$ by the moment of inertia of a rectangular body $\rho_2 H_2 h_\parallel h_\perp (H_2^2 + h_\parallel^2)/12$ about its centre of mass and the $z$ axis (where $m_\parallel = \rho_2 H_2 h_\parallel h_\perp$ is the lumped mass). This yields

$$\omega_\parallel \approx \sqrt{\left[E_i/H_i\right]/\rho_2 H_2} \left[3/4\right] \left[1 + H_2^2 / h_\parallel^2\right]^{-1}, \quad (14)$$

demonstrating that for $h_\parallel \geq H_2$ the forbidden gap arising from the negative effective flexural modulus can be chosen to end at an angular frequency relatively close to $\omega_\perp$ if, for example, one
uses the same values for $H_i$ in both types of resonant element. Here we shall take the example of $h_0 = 2H_2 = 100$ nm, with $H_1, H_2, E_1$ and $\rho_2$ entering equation (14) assumed to be the same as the values suggested above for the vertical element. The resonance frequency $f_{\parallel} = \omega/2\pi$ of the horizontal element is then equal to 0.46 GHz. Thus the cases $f_{\parallel} < f_\perp$ or $f_{\parallel} > f_\perp$ can be realized when $S_1/S_2 = 1$ or $1/2$, respectively, for the vertical elements. These formulas for $\alpha_\perp$ and $\omega_{\parallel}$ clearly indicate how different parameters quantitatively influence their relative values, and how they can be modified to achieve overlap of the band gaps and satisfy the condition for double-negativity. However, precise design of the proposed double-negative acoustic meta-plate would require numerical evaluation of the lowest resonance frequency of the rotationally-resonant mechanical elements.

4. Conclusions

By means of a lumped-element analytical theory, we demonstrate that a combination of resonant mechanical elements attached to a plate can be used to produce an flexural acoustic metamaterial with arbitrary density and elastic modulus, and in particular negative values. Negative density, previously demonstrated in experiment and theory [19, 22, 27, 28], is achieved by normal-force interactions in stub-like structures. Negative modulus is achieved by lateral-force interactions obtained thanks to the introduction of rotationally-resonant mechanical elements. We have also shown through analytical calculations how the relevant acoustic frequency band for double-negative behaviour can be chosen through the resonant-element geometry. The realization of double-negative acoustic meta-plates will allow the replacement of flexural-wave phononic crystals in wave-focusing applications [15] and will provide new perspectives for guiding and cloaking [35–37] of flexural waves, with potential applications in ultrasonic imaging and filtering for communication systems.

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