Fibonacci Graphs and their Expressions

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Abstract

The paper investigates relationship between algebraic expressions and graphs. We consider a digraph called a Fibonacci graph which gives a generic example of non-series-parallel graphs. Our intention in this paper is to simplify the expressions of Fibonacci graphs and eventually find their shortest representations. With that end in view, we describe the number of methods for generating Fibonacci graph expressions and carry out their comparative analysis.

Keywords: Fibonacci graph, series-parallel graph, two-terminal directed acyclic graph, reduction.

1. Introduction

A graph $G = (V,E)$ consists of a vertex set $V$ and an edge set $E$, where each edge corresponds to a pair $(v,w)$ of vertices. If the edges are ordered pairs of vertices (i.e., the pair $(v,w)$ is different from the pair $(w,v)$), then we call the graph directed or digraph; otherwise, we call it undirected. If $(v,w)$ is an edge in a digraph, we say that $(v,w)$ leaves vertex $v$ and enters vertex $w$. In a digraph,
the out-degree of a vertex is the number of edges leaving it, and the in-degree of a vertex is the number of edges entering it. A vertex in a digraph is a source if no edges enter it, and a sink if no edges leave it.

A path from vertex \( v_0 \) to vertex \( v_k \) in a graph \( G = (V, E) \) is a sequence of its vertices \([v_0, v_1, v_2, \ldots, v_{k-1}, v_k]\) such that \((v_{i-1}, v_i) \in E\) for \(1 \leq i \leq k\). \( G \) is an acyclic graph if there is no closed path \([v_0, v_1, v_2, \ldots, v_k, v_0]\) in \( G \). A two-terminal directed acyclic graph (st-dag) has only one source \( s \) and only one sink \( t \). In an st-dag, every vertex lies on some path from \( s \) to \( t \).

A graph \( G' = (V', E') \) is a subgraph of \( G = (V, E) \) if \( V' \subseteq V \) and \( E' \subseteq E \). A graph \( G \) is homeomorphic to a graph \( G' \) (a homeomorph of \( G' \)) if \( G \) can be obtained by subdividing edges of \( G' \) with new vertices.

We consider a labeled graph which has labels attached to its edges. Each path between the source and the sink (a sequential path) in an st-dag can be presented by a product of all edge labels of the path. We define the sum of edge label products corresponding to all possible sequential paths of an st-dag \( G \) as the canonical expression of \( G \). An algebraic expression is called an st-dag expression (a factoring of an st-dag in [1]) if it is algebraically equivalent to the canonical expression of an st-dag. An st-dag expression consists of terms (edge labels), the operators + (disjoint union) and \( \cdot \) (concatenation, also denoted by juxtaposition when no ambiguity arises), and parentheses.

We define the complexity of an algebraic expression in two ways. The complexity of an algebraic expression is (i) the total number of terms in the expression including all their appearances (the first complexity characteristic) or (ii) the number of plus operators in the expression (the second complexity characteristic). We will denote the first and the second complexity characteristic of an st-dag expression by \( T(n) \) and \( P(n) \), respectively, where \( n \) is the number of vertices in the graph (the size of the graph).

An equivalent expression with the minimum complexity is called an optimal representation of the algebraic expression.

A series-parallel graph is defined recursively as follows:
(i) A single edge \((u, v)\) is a series-parallel graph with source \( u \) and sink \( v \).
(ii) If \( G_1 \) and \( G_2 \) are series-parallel graphs, so is the graph obtained by either of the following operations:
   (a) Parallel composition: identify the source of \( G_1 \) with the source of \( G_2 \) and the sink of \( G_1 \) with the sink of \( G_2 \).
   (b) Series composition: identify the sink of \( G_1 \) with the source of \( G_2 \).

As shown in [1] and [8], a series-parallel graph expression has a representation
functions. A Boolean function is defined as read-once and consider the correspondence between series-parallel graphs and read-once \cite{1}, \cite{2}, \cite{3}, \cite{4}, \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{14}, and other works. Specifically, \cite{11}, \cite{12}, and \cite{15} consider the correspondence between series-parallel graphs and read-once functions. A Boolean function is defined as read-once if it may be computed by

![Figure 1.1: A series-parallel graph.](image)

in which each term appears only once. We proved in \cite{8} that this representation is an optimal representation of the series-parallel graph expression from the perspective of the first complexity characteristic. For example, the st-dag expression of the series-parallel graph presented in Figure 1.1 is \(ab\bar{d} + ab\bar{e} + a\bar{c}d + ace + f + f\bar{e}d\). Since it is a series-parallel graph, the expression can be reduced to \((a(b + c) + f)(d + e)\), where each term appears once.

The notion of a Fibonacci graph (FG) was introduced in \cite{6}. A Fibonacci graph has vertices \(\{1, 2, 3, \ldots, n\}\) and edges \n
\[\{(v, v + 1) \mid v = 1, 2, \ldots, n - 1\} \cup \{(v, v + 2) \mid v = 1, 2, \ldots, n - 2\}.\]

This graph is illustrated in Figure 1.2.

![Figure 1.2: A Fibonacci graph.](image)

As shown in \cite{3}, an st-dag is series-parallel if and only if it does not contain a subgraph homeomorphic to the forbidden subgraph positioned between vertices 1 and 4 of the Fibonacci graph shown in Figure 1.2. Thus, Fibonacci graphs are of interest as “through” non-series-parallel st-dags. Notice that Fibonacci graphs of size 2 or 3 are series-parallel.

Mutual relations between graphs and algebraic expressions are discussed in \cite{1}, \cite{2}, \cite{3}, \cite{4}, \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{15}, and other works. Specifically, \cite{11}, \cite{12}, and \cite{15} consider the correspondence between series-parallel graphs and read-once functions. A Boolean function is defined as read-once if it may be computed by
some formula in which no variable occurs more than once (read-once formula). On the other hand, a series-parallel graph expression can be reduced to the representation in which each term appears only once. Hence, such a representation of a series-parallel graph expression can be considered to be a read-once formula (Boolean operations are replaced by arithmetic ones).

An expression of a homeomorph of the forbidden subgraph belonging to any non-series-parallel st-dag has no representation in which each term appears once. For example, consider the subgraph positioned between vertices 1 and 4 of the Fibonacci graph shown in Figure 1.2. Possible optimal representations of its expression are \( a_1 (a_2 a_3 + b_2) + b_1 a_3 \) or \( (a_1 a_2 + b_1) a_3 + a_1 b_2 \). For this reason, an expression of a non-series-parallel st-dag can not be represented as a read-once formula. However, for arbitrary functions, which are not read-once, generating the optimum factored form is NP-complete [16]. Some heuristic algorithms developed in order to obtain good factored forms are described in [11], [5] and other works. Therefore, generating an optimal representation for a non-series-parallel st-dag expression is a highly complex problem.

The problem of factoring boolean functions into shorter, more compact formulae is one of the basic operations in algorithmic logic synthesis since compactification saves money. In logic synthesis, one standard measure of the complexity of a logic circuit is the number of terms. Computation time also depends on the number of terms. However, computation time is determined by the number of operations on terms as well. For this reason, the number of plus operators is another important characteristic of a logic circuit. Besides, the number of plus operators characterizes the number of computation levels in a logic circuit (its "branching out" degree).

Our intention in this paper is to simplify the expressions of Fibonacci graphs (we denote them by \( Ex(FG) \)) and eventually find their optimal representations. In [8] we presented a heuristic algorithm with that end in view and analyzed obtained expressions from the perspective of the first complexity characteristic. Here we describe the number of methods for generating Fibonacci graph expressions and carry out their comparative analysis from the perspective of both the first and the second complexity characteristics.

2. Simple Methods

This section considers three quite natural methods for generating expressions of Fibonacci graphs.
2.1. Sequential Paths Method

This method is based directly on the definition of an st-dag expression as the canonical expression of the st-dag.

**Theorem 2.1.** For an $n$-vertex $FG$:

1. The number of sequential paths $p(n)$ is defined recursively as follows:

\[
\begin{align*}
p(1) &= 1 \\
p(2) &= 1 \\
p(n) &= p(n-1) + p(n-2) \quad (n > 2).
\end{align*}
\]  

(2.1)

2. The total number of terms $T(n)$ in the expression $Ex(FG)$ derived by the sequential paths method is defined recursively as follows:

\[
\begin{align*}
T(1) &= 0 \\
T(2) &= 1 \\
T(n) &= T(n-1) + T(n-2) + p(n) \quad (n > 2).
\end{align*}
\]  

(2.2)

3. The number of plus operators $P(n)$ in the expression $Ex(FG)$ derived by the sequential paths method is defined recursively as follows:

\[
\begin{align*}
P(1) &= 0 \\
P(2) &= 0 \\
P(n) &= P(n-1) + P(n-2) + 1 \quad (n > 2).
\end{align*}
\]  

(2.3)

**Proof.**

1. Initial statements $p(1) = 1$ and $p(2) = 1$ follow clearly. All sequential paths in a Fibonacci graph (see Figure 1.2) subdivide into two groups. Paths of the first group start from the edge labeled $a_1$; paths of the second group start from the edge labeled $b_1$. Paths of the first group are all sequential paths of the $FG$ positioned between vertices 2 and $n$ and are supplemented by an edge labeled $a_1$. This graph includes $n - 1$ vertices, and, for this reason, the number of sequential paths in this graph, and by extension, in the first group, is equal to $p(n - 1)$. By analogy, the number of sequential paths in the second group is equal to $p(n - 2)$. Hence, the proof of the statement is complete.

2. Initial statements $T(1) = 0$ and $T(2) = 1$ follow clearly. Consider the case of $n > 2$. As was mentioned above, each sequential path of an $n$-vertex $FG$ is a sequential path of an $n - 1$-vertex $FG$ or an $n - 2$-vertex $FG$ which is
supplemented by one edge. That is, each sequential path in an \( n - 1 \)-vertex \( FG \) and an \( n - 2 \)-vertex \( FG \) corresponds to an additional term in \( T(n) \). Hence,

\[
T(n) = T(n - 1) + p(n - 1) + T(n - 2) + p(n - 2) \\
= T(n - 1) + T(n - 2) + p(n).
\]

3. Initial statements \( P(1) = 0 \) and \( P(2) = 0 \) follow clearly. Consider the case of \( n > 2 \). Taking into consideration (2.1) and the obvious equality \( p(n) = P(n) + 1 \) for \( n \geq 1 \), formula (2.3) follows immediately.

**Remark 2.2.** The number of sequential paths \( p(n) \) in an \( n \)-vertex \( FG \) is equal to the Fibonacci number \( F_n \) \((F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2})\).

The following explicit formula for \( F_n \) and, consequently, for \( p(n) \) is obtained by the method for linear recurrence relations solving [14] (henceforth, the method [14]):

\[
p(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]. \tag{2.4}
\]

Using (2.1) and (2.2) the following recurrence for \( T(n) \) is derived:

\[
T(1) = 0 \\
T(2) = 1 \\
T(3) = 3 \\
T(4) = 7 \\
T(n) = 2T(n - 1) + T(n - 2) - 2T(n - 3) - T(n - 4) \quad (n > 4). \tag{2.5}
\]

**Corollary 2.3.** For an \( n \)-vertex \( FG \):

1. The total number of terms \( T(n) \) in the expression \( Ex(FG) \) derived by the sequential paths method is expressed explicitly as follows:

\[
T(n) = \frac{1}{5} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

2. The number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the sequential paths method is expressed explicitly as follows:

\[
P(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] - 1.
\]
**Proof.** 1. The proof is based on relation (5.1) and on the method [14].

2. The proof follows immediately from (2.4) and the above mentioned equality $p(n) = P(n) + 1$. ■

For $n = 9$, the corresponding algebraic expression is

\[
\begin{align*}
& a_1a_2a_3a_4a_5a_6b_7 + a_1a_2a_3a_4a_5b_6a_8 + a_1a_2a_3a_4b_5a_7a_8 + \\
& a_1a_2a_3a_4b_5b_7 + a_1a_2a_3b_4a_6a_7a_8 + a_1a_2a_3b_4a_6b_7 + a_1a_2a_3b_4b_6a_8 + \\
& a_1a_2b_3a_5a_7a_8 + a_1a_2b_3a_5a_6b_7 + a_1a_2b_3a_5b_6a_8 + a_1a_2b_3b_5a_7a_8 + \\
& a_1a_2b_3b_5b_7 + a_1b_2a_4a_5a_6a_7a_8 + a_1b_2a_4a_5a_6b_7 + a_1b_2a_4a_5b_6a_8 + \\
& a_1b_2a_4b_5a_7a_8 + a_1b_2a_4b_5b_7 + a_1b_2b_4a_6a_7a_8 + a_1b_2b_4a_6b_7 + \\
& a_1b_2b_4b_6a_8 + b_1a_3a_4a_5a_6a_7a_8 + b_1a_3a_4a_5a_6b_7 + b_1a_3a_4a_5b_6a_8 + \\
& b_1a_3a_4b_5a_7a_8 + b_1a_3a_4b_5b_7 + b_1a_3b_4a_6a_7a_8 + b_1a_3b_4a_6b_7 + \\
& b_1a_3b_4b_6a_8 + b_1b_2a_5a_6a_7a_8 + b_1b_2a_5a_6b_7 + b_1b_2a_5b_6a_8 + \\
& b_1b_3b_5a_7a_8 + b_1b_3b_5b_7.
\end{align*}
\]

It contains 34 products (that correspond to 34 sequential paths of the graph), 201 terms and 33 plus operators.

**2.1.1. Time and Space Expenses of the Method**

The expression $Ex(FG)$ will be implemented in this and in other methods by a linked list of the following characters: terms $a_i$ and $b_i$, parentheses "(" and ")", and a sign "+". Terms $a_i$ and $b_i$ conditionally considered as alone characters can be presented as character sequences consisting of characters "a" or "b" and digits of number $i$.

We propose the following recursive algorithm which realizes the sequential paths method:

\[
FG_{Sequential\_Paths}(i, j, n, Arr, Expr)
\]

1. if $i < n$

2. $Arr[j] \leftarrow a_i$

3. $FG_{Sequential\_Paths}(i + 1, j + 1, n, Arr, Expr)$

4. if $i < n - 1$
5. \( \text{Arr}[j] \leftarrow b_i \)

6. \( \text{FG}_\text{Sequential}_\text{Paths}(i + 2, j + 1, n, \text{Arr}, \text{Expr}) \)

7. \( \text{else} \)

8. \( \textbf{if not empty}(\text{Expr}) \)

9. \( \text{Insert}_\text{to}_\text{End}(" + ", \text{Expr}) \)

10. \( \text{Copy}_\text{to}_\text{End}(\text{Arr}, j, \text{Expr}) \)

Given integers \( i \) and \( j \) and an auxiliary array \( \text{Arr} \) of size \( n - 1 \), this procedure generates a linked list \( \text{Expr} \) which implements the expression of an \( n \)-vertex \( \text{FG} \) derived by the sequential paths method. Array \( \text{Arr} \) is used to accumulate a product of terms corresponding to a current sequential path. Integer \( i \) is a number of a given vertex which edges labeled \( a_i \) and \( b_i \) leave, and \( j \) is a subscript of an element in \( \text{Arr} \). The procedure is invoked with \( i = j = 1 \) and an empty list \( \text{Expr} \).

The procedure generates all admissible for our problem combinations of \( a_i \) \((i = 1, 2, ..., n - 1)\) and \( b_i \) \((i = 1, 2, ..., n - 2)\) After \( i \) reaches \( n \) (line 7) a current combination has been composed and content of the first \( j \) elements of \( \text{Arr} \) is copied at the end of \( \text{Expr} \) (line 10). If a derived product is not the first one in the expression, i.e., \( \text{Expr} \) is not an empty list, then a sign " + " is inserted before the product (lines 8-9).

The running time of this algorithm consists of two components.

In the general case of the recursion, the running time, as follows from lines 3 and 6 is \( t_1(n) = t_1(n - 1) + t_1(n - 2) + O(1) \), i.e., the time complexity increases as Fibonacci numbers (see (2.4)).

However, in the base case of the recursion, the time expenses are not constant but are proportional to the size of a part of the expression copied from \( \text{Arr} \) to \( \text{Expr} \). Therefore, this component of the algorithm’s running time is determined by the size of all the generated expression including all terms and plus operators. As follows from Theorem 2.1 and Corollary 2.3 the total number of terms is the most significant part of the expression’s size and, therefore, it states the complexity of the time \( t_2(n) \) for copying all parts of the expression from \( \text{Arr} \) to \( \text{Expr} \). That is, by Corollary 2.3 \( t_2(n) = \Theta \left( n \left( \frac{1 + \sqrt{5}}{2} \right)^n \right) \).

Thus, the total running time of the algorithm is

\[
t(n) = t_1(n) + t_2(n) = \Theta \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n \right) + \Theta \left( n \left( \frac{1 + \sqrt{5}}{2} \right)^n \right) = \Theta \left( n \left( \frac{1 + \sqrt{5}}{2} \right)^n \right).
\]
The algorithm uses only a $\Theta(n)$-size additional array and so, the amount of memory it requires is determined by the size of the derived expression and is also $\Theta(n \left(\frac{1+\sqrt{5}}{2}\right)^n)$.

2.2. Depth First Search (DFS) Method

An expression is derived by utilizing the well-known depth first search algorithm [2] and by using intermediate subexpressions which are accumulated in st-dag’s vertices. A subexpression which is accumulated in vertex $i$ of the st-dag corresponds to its subgraph which is positioned between vertices $i$ and $n$. The following recursive procedure is used:

1. The subexpression accumulated in vertex $n$ (see Figure 1.2) is equal to 1.
2. The subexpression accumulated in vertex $n-1$ is equal to $a_{n-1}$.
3. The subexpression accumulated in vertex $i$ ($i < n - 1$) is equal to $a_iE_{i+1} + b_iE_{i+2}$ where $E_{i+1}$ and $E_{i+2}$ are subexpressions accumulated in vertices $i+1$ and $i+2$, respectively.
4. The subexpression accumulated in vertex 1 is the resulting expression.

The special case of a subgraph consisting of a single vertex is considered in line 1 of the recursive procedure. It is clear that such a subgraph can be connected to other subgraphs only serially. For this reason, it is accepted that its subexpression is 1, so that, when it is multiplied by another subexpression, the final result is not influenced.

**Theorem 2.4.** For an $n$-vertex $FG$:

1. The total number of terms $T(n)$ in the expression $Ex(FG)$ derived by the DFS method is defined recursively as follows:

$$
T(1) = 0 \\
T(2) = 1 \\
T(n) = T(n-1) + T(n-2) + 2 \quad (n > 2).
$$  \hspace{1cm} (2.6)
2. The number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the DFS method is defined recursively as follows:

\[
P(1) = 0 \\
P(2) = 0 \\
P(n) = P(n-1) + P(n-2) + 1 \quad (n > 2).
\]

(2.7)

**Proof.** 1. Initial statements \( T(1) = 0 \) and \( T(2) = 1 \) follow clearly. The resulting expression \( Ex(FG) \) is equal to \( a_1E_2 + b_1E_3 \) where \( E_2 \) and \( E_3 \) are subexpressions accumulated in vertices 2 and 3, respectively (see Figure 1.2 and the DFS recursive procedure). \( E_2 \) is the symbolic expression of the \( FG \) which is positioned between vertices 2 and \( n \). This graph includes \( n-1 \) vertices and, for this reason, the total number of terms in \( E_2 \) is equal to \( T(n-1) \). By analogy, the total number of terms in \( E_3 \) is equal to \( T(n-2) \). Terms \( a_1 \) and \( b_1 \) are two additional terms in \( Ex(FG) \). Hence, the proof of the statement is complete.

2. This second proof is analogous to the first one. The expression \( a_1E_2 + b_1E_3 \) includes all plus operations of \( E_2 \) and \( E_3 \) and one additional plus operation. ■

As follows from Theorems 2.1 and 2.4, a method’s evaluation depends on the kind of complexity that has been chosen. If methods are compared by means of the second complexity characteristic, then sequential paths and DFS methods are equivalent. However, from the perspective of the first complexity characteristic, the DFS method is more efficient.

**Corollary 2.5.** For an \( n \)-vertex \( FG \):

1. The total number of terms \( T(n) \) in the expression \( Ex(FG) \) derived by the DFS method is expressed explicitly as follows:

\[
T(n) = \frac{1}{10} \left[ (5 + 3\sqrt{5}) \left( \frac{1 + \sqrt{5}}{2} \right)^n + (5 - 3\sqrt{5}) \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] - 2.
\]

2. The number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the DFS method is expressed explicitly as follows:

\[
P(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] - 1.
\]
Proof. 1. The proof uses the recurrence obtained in Theorem 2.4 and is based on the method [14].

2. The proof follows immediately from Corollary 2.3 and the equivalence of sequential paths and DFS methods from the perspective of the second complexity characteristic.

For \( n = 9 \), the corresponding algebraic expression is

\[
\begin{align*}
& a_1(a_2(a_3(a_4(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_5(a_7a_8 + b_7)) + b_4(a_6(a_7a_8 + b_7) + b_6a_8)) + b_3(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_2(a_4(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_5(a_7a_8 + b_7)) + b_1(a_3(a_4(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_0(a_7a_8 + b_7)) + b_4(a_6(a_7a_8 + b_7) + b_6a_8)) + b_3(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_2(a_4(a_5(a_6(a_7a_8 + b_7) + b_6a_8)) + b_1(a_3(a_4(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_0(a_7a_8 + b_7)) + b_1(a_3(a_4(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_0(a_7a_8 + b_7))))).
\end{align*}
\]

It contains 87 terms and 33 plus operators.

Hence, this algorithm optimizes prefix parts of all subexpressions. In principle, the DFS method can be applied by traversing the st-dag in the opposite direction. In such a case, suffix parts of subexpressions are optimized. Expression complexity characteristics will be the same.

2.2.1. Time and Space Expenses of the Method

We propose the following recursive algorithm which realizes the DFS method in accordance with the above procedure:

\[
FG_{DFS_{direct}}(i, n, Expr)
\]

1. if \( i < n - 1 \)

2. \( FG_{DFS_{direct}}(i + 1, n, Expr) \)

3. if \( i < n - 2 \)

4. Insert_to_Head ("\( ^1 \)", Expr)

5. Insert_to_End("\( ^n \)", Expr)

6. Insert_to_Head ("\( a_i \)", Expr)
The algorithm generates a linked list $Expr$ which implements an expression of a subgraph positioned between vertices $i$ and $n$. Parameter $i$ is substituted by 1 initially, for deriving the expression of an $n$-vertex $FG$.

Recursive calls in lines 2 and 7 of the algorithm generate expressions for subgraphs with sources $i+1$ and $i+2$, respectively. The first expression is presented as list $Expr$ and the second one is presented as an additional list $Expr2$. After corresponding insertions of terms $a_i$ and $b_i$, parentheses and a sign "+$", these lists are concatenated in $O(1)$ time into the unified list $Expr$ (line 12) by assigning the address of the first element in $Expr2$ to the pointer in the last element of $Expr$. In the base cases of the recursion, $Expr$ consists of the single term $a_{n-1}$ (line 15) or is an empty list (line 17).

Thus, the running time of the algorithm is $t(n) = t(n-1) + t(n-2) + O(1)$, i.e., its complexity increases as Fibonacci numbers and $t(n) = \Theta \left( \left( \frac{1+\sqrt{5}}{2} \right)^n \right)$.

The amount of memory that requires the algorithm is determined only by the size of the derived expression and is also $\Theta \left( \left( \frac{1+\sqrt{5}}{2} \right)^n \right)$.

The following algorithm realizes the DFS method applied in the opposite direction:

$$FG_{\text{DFS \_ opposite}}(i, n, Expr)$$
1. if $i < n - 1$

2. $FG_{DFS\_opposite}(i, n - 1, Expr)$

3. if $i < n - 2$

4. Insert\_to\_Head("\(\)", Expr)

5. Insert\_to\_End("\(\)", Expr)

6. Insert\_to\_End("\(a_{n-1} +\)", Expr)

7. $FG_{DFS\_opposite}(i, n - 2, Expr_2)$

8. if $i < n - 3$

9. Insert\_to\_Head("\(\)", Expr_2)

10. Insert\_to\_End("\(\)", Expr_2)

11. Insert\_to\_End("\(b_{n-2}\)", Expr_2)

12. Concatenate(Expr, Expr_2)

13. else

14. if $i = n - 1$

15. $Expr \leftarrow "a_i"$

16. else

17. $Expr \leftarrow NULL$

It is clear that time and space expenses of this algorithm are the same as of the previous one and are $\Theta \left(n \left(\frac{1+\sqrt{5}}{2}\right)^n\right)$. 
2.3. Depth Last Search (DLS) Method

While the DLS method is similar to the DFS method they differ in the fact that an st-dag expression in the DLS method is derived by using special subexpressions such as \( a_i a_{i+1} + b_i \) which are related to corresponding closed graph segments. The following recursive procedure is used:

1. The subexpression accumulated in vertex \( n \) is equal to 1.
2. The subexpression accumulated in vertex \( n - 1 \) is equal to \( a_{n-1} \).
3. The subexpression accumulated in vertex \( n - 2 \) is equal to \( a_{n-2} a_{n-1} + b_{n-2} \).
4. The subexpression accumulated in vertex \( i \) (\( i < n - 2 \)) is equal to \( (a_i a_{i+1} + b_i)E_{i+2} + a_i b_{i+1} E_{i+3} \) where \( E_{i+2} \) and \( E_{i+3} \) are subexpressions accumulated in vertices \( i + 2 \) and \( i + 3 \), respectively.
5. The subexpression accumulated in vertex 1 is the resulting expression.

**Theorem 2.6.** For an \( n \)-vertex \( FG \):

1. The total number of terms \( T(n) \) in the expression \( Ex(FG) \) derived by the DLS method is defined recursively as follows:

\[
T(1) = 0 \\
T(2) = 1 \\
T(3) = 3 \\
T(n) = T(n - 2) + T(n - 3) + 5 \quad (n > 3).
\]

2. The number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the DLS method is defined recursively as follows:

\[
P(1) = 0 \\
P(2) = 0 \\
P(3) = 1 \\
P(n) = P(n - 2) + P(n - 3) + 2 \quad (n > 3).
\]

**Proof.** 1. Initial statements \( T(1) = 0, T(2) = 1, \) and \( T(3) = 3 \) follow clearly. The resulting expression \( Ex(FG) \) is equal to \( (a_1 a_2 + b_1)E_3 + a_1 b_2 E_4 \), where \( E_3 \) and \( E_4 \) are subexpressions accumulated in the vertices 3 and 4, respectively.
Figure 1.2 and the DLS recursive procedure). $E_3$ is the symbolic expression of the $FG$ which is positioned between vertices 3 and $n$. This graph includes $n - 2$ vertices, and for this reason, the total number of terms in $E_3$ is equal to $T(n - 2)$. By analogy, the total number of terms in $E_4$ is equal to $T(n - 3)$. Terms $a_1$, $a_2$, $b_1$, $a_1$, and $b_2$ are five additional terms in $Ex(FG)$. Hence, the proof of the statement is complete.

2. This second proof is analogous to the first one. The expression ($a_1 a_2 + b_1)E_3 + a_1 b_2 E_4$ includes all plus operations of $E_3$ and $E_4$ and two additional plus operations.

As follows from Theorems 2.1, 2.4, and 2.6, the DLS method is more efficient than sequential paths and DFS methods from the perspective of both complexity characteristics.

**Corollary 2.7.** For an $n$-vertex $FG$:

1. The total number of terms $T(n)$ in the expression $Ex(FG)$ derived by the DLS method is expressed explicitly as follows:

$$T(n) \approx 3.4912 (1.3247)^n + (-0.245 46 - 0.0449 7i) (-0.662 36 + 0.562 28i)^n + (-0.245 46 + 0.0449 7i) (-0.662 36 - 0.562 28i)^n - 5$$

or

$$T(n) \approx 3.4912 (1.3247)^n + (-1)^{n+1} (0.868 84)^n [0.49109 \cos (0.703 86n) + 0.088942 \sin (0.703 86n)] - 5.$$  

2. The number of plus operators $P(n)$ in the expression $Ex(FG)$ derived by the DLS method is expressed explicitly as follows:

$$P(n) \approx 1.2672 (1.3247)^n + (-0.133 62 - 0.128 28i) (-0.662 36 + 0.562 28i)^n + (-0.133 62 + 0.128 28i) (-0.662 36 - 0.562 28i)^n - 2$$

or

$$P(n) \approx 1.2672 (1.3247)^n + (-1)^{n+1} (0.868 84)^n [0.26724 \cos (0.703 86n) + 0.25655 \sin (0.703 86n)] - 2.$$  

The proof of Corollary 2.7 uses the recurrences obtained in Theorem 2.6 and is based on the method [14].
For $n = 9$, the corresponding algebraic expression is
\[(a_1a_2 + b_1)((a_3a_4 + b_3)((a_5a_6 + b_5)(a_7a_8 + b_7) + a_5b_6a_8) +
a_3b_4((a_6a_7 + b_6)a_8 + a_5b_7)) + a_1b_2((a_4a_5 + b_4)((a_6a_7 + b_6)a_8 + a_6b_7) +
a_4b_5(a_7a_8 + b_7)).\]
It contains 39 terms and 14 plus operators.

Like the DFS method, the DLS method can be employed by traversing the $FG$ in the opposite direction.

### 2.3.1. Time and Space Expenses of the Method

The algorithms which realize the method applied in both the direct and the opposite directions are similar to the above algorithms for the DFS method (section 2.2.1). Complexities of running time and memory required by these algorithms are defined as in the previous methods by the same recurrences as the expression size and, by Corollary 2.7, are about $\Theta((1.3247)^n)$.

### 3. Reduction Method

This method is based on the idea of reduction. A **series reduction** at vertex $v$ is possible when $a = (u, v)$ is the unique edge entering $v$, and $b = (v, w)$ is the unique edge leaving $v$: then $a$ and $b$ are replaced by $(u, w)$. A **parallel reduction** at vertices $v, w$ replaces two or more edges $a_1, \ldots, a_k$ joining $v$ to $w$ by a single edge $(v, w)$. A **node reduction** at $v$ can occur when $v$ has in-degree or out-degree 1 (a node reduction is a generalization of a series reduction). Suppose $v$ has in-degree 1, and let $a = (u, v)$ be the edge entering $v$. Let $b_1 = (v, w_1), \ldots, b_k = (v, w_k)$ be the edges leaving $v$. Replace $\{a, b_1, \ldots, b_k\}$ by $\{g_1, \ldots, g_k\}$, where $g_i = (u, w_i)$. We call such a reduction a **fork reduction**. The case where $v$ has out-degree 1 is symmetric: here $a = (v, w), b_i = (u_i, v)$, and $g_i = (u_i, w_i)$. We call such a reduction a **joint reduction**.

The algorithm for generating an st-dag expression from a sequence of series, parallel, and node reductions is proposed in [1]. From a sequence of series, parallel, and node reductions, reducing an arbitrary st-dag $G$ to a single edge, an expression of this st-dag can be obtained as follows. We denote, for the sake of brevity, the label of every edge $e$ before every reduction as $e$. Then, the new label for the edge resulting from a series reduction of $a$ and $b$ is $ab$. For a parallel reduction, the new
edge is labeled $a_1 + \ldots + a_k$. The new label for each edge resulting from a node reduction is $ab_i$ for a fork reduction or $b_ia$ for a joint reduction. Node reductions are used until series and parallel reductions are possible. Ultimately, the single edge to which $G$ is reduced has a label giving an expression of $G$.

In relation to a Fibonacci graph that has more than 3 vertices, this algorithm is transformed to the following special procedure.

1. A fork reduction is done at the second vertex (from the left) (see Figure 1.2) or a joint reduction is done at the last but one vertex (from the left).

2. A parallel reduction is done at the first and the second vertices (from the left) in the case of preceding fork reduction or at the last but one and the last vertices (from the left) in the case of preceding joint reduction.

3. If the resulting reduced $FG$ contains more than three vertices, then we return to step 1 of this algorithm. Otherwise, it is a series-parallel graph which is reduced to a single edge by a series reduction at the second vertex and a parallel reduction at the source and the sink. The single edge is labeled by the resulting expression.

The example of the reduction process in relation to a 6-vertex $FG$ is shown in Figure 3.1.

It is clear (see Figure 3.1) that if the $FG$ contains $n$ vertices, then the number of applied node reductions is equal to $y = n - 3$. Thus, the reduction method applied to an $n$-vertex $FG$ includes $2^{n-3}$ possible reduction processes that are due to different numbers and execution orders of fork and joint reductions. In the case of $n \leq 3$ the $n$-vertex $FG$ is a series-parallel graph and node reductions are not done. Our intention is to find reduction processes which lead to expression representation $Ex(FG)$ with a minimum complexity. We propose the following algorithm:

1. $y \leftarrow n - 3$

2. if $y \mod 2 = 0$

3. $fork\_count \leftarrow y/2 \quad joint\_count \leftarrow y/2$

4. else

5. $z \leftarrow \text{rand}(2)$
Parallel reduction at vertices 4 and 6

Fork reduction at vertex 2
Parallel reduction at vertices 1 and 3

Fork reduction at vertex 3
Parallel reduction at vertices 1 and 4

Series reduction at vertex 4
Parallel reduction at vertices 1 and 6

Figure 3.1: The example of a reduction process on a Fibonacci graph.
6. \textbf{if } z = 1 \\
7. \quad \textit{fork\_count} \leftarrow (y - 1)/2 \quad \textit{joint\_count} \leftarrow (y + 1)/2 \\
8. \textbf{else} \\
9. \quad \textit{fork\_count} \leftarrow (y + 1)/2 \quad \textit{joint\_count} \leftarrow (y - 1)/2 \\
10. \textbf{while } \textit{fork\_count} > 0 \textbf{ and } \textit{joint\_count} > 0 \\
11. \quad z \leftarrow \text{rand}(2) \\
12. \quad \textbf{if } z = 1 \\
13. \quad \quad \textbf{apply fork and parallel reductions} \\
14. \quad \quad \textit{fork\_count} \leftarrow \textit{fork\_count} - 1 \\
15. \quad \quad \textbf{else} \\
16. \quad \quad \textbf{apply joint and parallel reductions} \\
17. \quad \quad \textit{joint\_count} \leftarrow \textit{joint\_count} - 1 \\
18. \textbf{while } \textit{fork\_count} > 0 \\
19. \quad \quad \textbf{apply fork and parallel reductions} \\
20. \quad \quad \textit{fork\_count} \leftarrow \textit{fork\_count} - 1 \\
21. \textbf{while } \textit{joint\_count} > 0 \\
22. \quad \quad \textbf{apply joint and parallel reductions} \\
23. \quad \quad \textit{joint\_count} \leftarrow \textit{joint\_count} - 1 \\

Lines 2 and 3 of the algorithm determine the numbers of fork and joint reductions to be applied for even \( y \) (even \( n \)). For odd \( y \) (odd \( n \)) there are two possible values for the numbers of fork and joint reductions. These values are determined in lines 5-9 by function \text{rand}(2) which generates randomly 1 or 2. The \textbf{while} loop in lines 10-17 repeatedly applies a pair of node and parallel reductions. A kind of a node reduction (fork or joint) is also determined by function \text{rand}(2). After all possible joint or all possible fork reductions are done, the remaining fork (lines 18-20) or remaining joint (lines 21-23) reductions, respectively, are applied.

We define this algorithm as the \textit{optimal reduction method}. 

Theorem 3.1. The minimum complexity representation (for terms and for plus operators) among all possible expression representations \(Ex(FG)\) derived by the reduction method for an \(n\)-vertex \(FG\) \((n > 3)\) is achieved by the optimal reduction method.

Proof. Actually, as follows from lines 2-9 of the optimal decomposition method, we should prove that the minimum complexity representation (for terms and for plus operators) is achieved if and only if (i) the number of applied fork reductions is equal to the number of applied joint reductions for odd \(n\); (ii) the numbers of fork reductions and joint reductions are distinguished by one for even \(n\).

The initial number of terms on the graph edges is equal to the number of edges. The initial number of plus operators is equal to 0. Each pair of node and parallel reductions leads to an increase in the total number of terms and plus operators on the graph edges. This follows from the fact that each node reduction leads to duplicate copies of corresponding terms and subexpressions and to new plus operators. We should find the numbers of terms and plus operators on the single edge to which the \(FG\) is reduced and analyze how these numbers increase in comparison to with their initial values. The following basic points are used.

1. The increment value of the total number of terms and plus operators on the left side of the \(FG\) increases as the number of fork reductions increases; the increment value on the right side of the \(FG\) increases as the number of joint reductions increases.

   Indeed, the current increment depends only on the duplicate subexpression that labels the appointed edge. This edge is positioned between the first and the second vertices in the case of a fork reduction, or between the last but one and the last vertices in the case of a joint reduction. The increment of the first complexity characteristic is equal to the total number of terms in the duplicate subexpression. The increment of the second complexity characteristic is equal to the number of plus operators in this subexpression with an additional plus operator added. In the next step of the reduction algorithm, the above mentioned edge is the result of parallel reduction at two edges. One of these edges is labeled by the subexpression including the duplicate subexpression of the previous step (see Figure 3.1). Hence, the size of the duplicate subexpression for the next reduction increases and, therefore, the increment value increases.

2. The above mentioned increment values from the left side and from the right side of the \(FG\) are independent, i.e., the increment value after a current fork
reduction does not depend on the number of already applied joint reductions and vice versa.

We prove the stronger statement which asserts that the above mentioned duplicate subexpressions determining the increments are independent. In every reduction step, we conditionally denote the edge labels of the reduced $FG$ as the edge labels of the initial $FG$ shown in Figure 1.2. In such a case, in each reduction step, edges leaving the source of the reduced $FG$ are labeled $a_1$ and $b_1$, respectively, and edges entering the sink of the reduced $FG$ are labeled $a_{n-1}$ and $b_{n-2}$, respectively. Hence, $a_1$ is a current duplicate subexpression before a fork reduction and $a_{n-1}$ is a current duplicate subexpression before a joint reduction. The new (after a fork and parallel reduction) $a_1$ depends only on the old (before a fork reduction) $a_1$, $a_2$, and $b_1$. The new (after a joint and parallel reduction) $a_{n-1}$ depends only on the old (before a joint reduction) $a_{n-1}$, $a_{n-2}$, and $b_{n-2}$. Labels $a_2$ and $a_{n-2}$ are initial terms always. They do not change and depend on nothing. The new $b_1$ depends only on the old $a_1$ and $b_2$. The new $b_{n-2}$ depends only on the old $a_{n-1}$ and $b_{n-3}$. Labels $b_2$ and $b_{n-3}$ are also initial terms always, and depend on nothing. For this reason, the new $a_1$ in no step depends on the old $a_{n-1}$ and $b_{n-2}$ and the new $a_{n-1}$ in no step depends on the old $a_1$ and $b_1$. On the other hand, the $Ex(FG)$ accumulation takes place only in $a_1$ and $b_1$, and in $a_{n-1}$ and $b_{n-2}$. The pair of the fork and parallel reductions influences only the new $a_1$ and $b_1$, and the pair of the joint and parallel reductions influences only the new $a_{n-1}$ and $b_{n-2}$. All the above holds for all reduction steps including the last one, when $a_1$ and $a_{n-1}$ draw closer to one another. Figure 3.2 where the reduction method is applied to a 5-vertex $FG$, illustrates this phenomenon. Two possible algorithms in which either the fork reduction comes first (and the joint reduction follows) or the joint reduction comes first (and the fork reduction follows) are illustrated. As shown, both algorithms lead to the same result. Labels on edges $(1,3)$ and $(3,5)$ of the resulting 3-edge st-dag have no common terms. That is, $a_1$ does not depend on joint reductions and $a_{n-1}$ does not depend on fork reductions. Therefore, duplicate subexpressions from the left side and from the right side are independent, and, thus, corresponding increment values are independent as well.

3. The increment value related to the $i$-th fork reduction in the $FG$ is equal to the increment value related to the $i$-th joint reduction in the $FG$.

This follows from the symmetrical structure of an $FG$ and from independence of fork and joint reductions.

As noted above, if the $FG$ contains $n$ vertices, then the number of applied node reductions is equal to $y = n - 3$. Since in each step, two kinds of node
Figure 3.2: Two reduction algorithms on a 5-vertex Fibonacci graph leading to the same result.
reductions are possible, initially, the potential number of possible node reductions is equal to $2y$ ($y$ fork and $y$ joint reductions). Hence, we can present the reduction procedure as follows. There are two equal stacks $S_1$ and $S_2$ (Figure 3.3). Each of them contains $y$ elements. The size of an element in each stack increases from top to bottom. The elements of the same level in $S_1$ and $S_2$ are of equal size. Here a $\text{Pop}$ operation on $S_1$ corresponds to a fork reduction and a $\text{Pop}$ operation on $S_2$ corresponds to a joint reduction. The size of a stack element corresponds to an increment value. We should put out $y$ elements from two stacks. It is clear that for even $y$ (odd $n$) the total size of pulled out elements will be minimum if and only if a $\text{Pop}$ operation is done $y/2$ times on each stack. For odd $y$ (even $n$), in order to ensure the minimum total size, a $\text{Pop}$ operation should be done $(y − 1)/2$ times on $S_1$ and $(y − 1)/2 + 1$ times on $S_2$, or vice versa. Thus, the proof of the theorem is complete.

**Proposition 3.2.** The expression representation $Ex(FG)$ derived by the reduction method does not depend on the execution order of fork and joint reductions and depends only on their number.

**Proof.** The proof is similar to point 2 in the proof of Theorem 3.1. As noted, the $Ex(FG)$ accumulation takes place only in $a_1$ and $b_1$, and in $a_{n−1}$ and $b_{n−2}$, i.e., on edges leaving the source and entering the sink of the current reduced $FG$. Accumulation processes on these pairs of edges are independent. As shown in Figure 3.2, in the last reduction steps, when the source and the sink draw closer

Figure 3.3: Two equal stacks. Sizes of their elements correspond to increment values.
to one another, edge labels of the resulting 3-edge st-dag do not depend also on the order of the execution of reductions.

**Remark 3.3.** The number of possible reduction processes included by the optimal reduction method applied to an \( n \)-vertex \( FG \) is equal to

\[
\binom{n-3}{(n-3)/2} = \frac{(n-3)!}{(((n-3)/2)!)^2}
\]

for odd \( n \) and

\[
\binom{n-3}{(n-4)/2} = \frac{(n-3)!}{((n-4)/2)!(n-2)/2!}
\]

for even \( n \).

The different reduction processes are due to different execution orders of fork and joint reductions.

It may be noted that each pair of a fork and a parallel reduction corresponds to two parallel recursion steps of the DFS method, each of which is executed by traversing the \( FG \) in opposite directions. These steps are equivalent to an ordinary \( Ex(FG) \) accumulation step on two edges, leaving the source of the current reduced \( FG \). By analogy, each pair of a joint and a parallel reduction corresponds to two parallel recursion steps of the direct DFS method. They are equivalent to an ordinary \( Ex(FG) \) accumulation step on two edges, entering the sink of the current reduced \( FG \). Hence, the reduction process in an \( FG \) can be conditionally presented as four parallel DFS processes on four subgraphs of this \( FG \). The corresponding four subexpressions are linked in the final step of the reduction procedure (see Figure 3.1). The resulting expression is constructed from the following three elements: four subexpressions which are obtained by applying the DFS method to four corresponding subgraphs of the \( FG \); one additional term \( b_l \) (\( l \) determines the place where the \( FG \) is decomposed into subgraphs, and depends on the number of fork and joint reductions); and one additional plus operation.

If only fork or only joint reductions are applied to a Fibonacci graph, then the reduction method gives the DFS method. Hence, the DFS method is a special (worst from the perspective of the complexity) case of the reduction method.

The optimal reduction method is equivalent to applying the DFS method to four subgraphs which are revealed by decomposing the \( FG \) in the middle. This
results in the following correlations between the reduction method and the DFS method:

\[ T_r(n) = T_{\text{DFS}} \left( \left\lceil \frac{n}{2} \right\rceil \right) + T_{\text{DFS}} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + T_{\text{DFS}} \left( \left\lceil \frac{n}{2} \right\rceil - 1 \right) + T_{\text{DFS}} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + 1 \quad (3.1) \]

\[ P_r(n) = P_{\text{DFS}} \left( \left\lceil \frac{n}{2} \right\rceil \right) + P_{\text{DFS}} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + P_{\text{DFS}} \left( \left\lceil \frac{n}{2} \right\rceil - 1 \right) + P_{\text{DFS}} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + 1. \quad (3.2) \]

Here \( T_r(n) \) and \( T_{\text{DFS}}(n) \) are the first complexity characteristics of \( \text{Ex}(FG) \) which is derived by the optimal reduction method and the DFS method, respectively, for an \( n \)-vertex \( FG \); \( P_r(n) \) and \( P_{\text{DFS}}(n) \) are the second complexity characteristics of \( \text{Ex}(FG) \) which is derived by the optimal reduction method and the DFS methods, respectively, for an \( n \)-vertex \( FG \).

**Theorem 3.4.** For an \( n \)-vertex \( FG \):

1. The total number of terms \( T(n) \) in the expression \( \text{Ex}(FG) \) derived by the optimal reduction method is defined recursively as follows:

   \[
   \begin{align*}
   T(1) &= 0 \\
   T(2) &= 1 \\
   T(3) &= 3 \\
   T(4) &= 6 \\
   T(5) &= 9 \\
   T(n) &= T(n - 2) + T(n - 4) + 7 \quad (n > 5). 
   \end{align*}
   \]

2. The number of plus operators \( P(n) \) in the expression \( \text{Ex}(FG) \) derived by the optimal reduction method is defined recursively as follows:

   \[
   \begin{align*}
   P(1) &= 0 \\
   P(2) &= 0 \\
   P(3) &= 1 \\
   P(4) &= 2 \\
   P(5) &= 3 \\
   P(n) &= P(n - 2) + P(n - 4) + 3 \quad (n > 5). 
   \end{align*}
   \]
Proof. 1. Initial statements $T(1) = 0$, $T(2) = 1$, $T(3) = 3$, $T(4) = 6$, $T(5) = 9$ can be checked. For $n > 5$, we consider odd and even $n$ and use formulae (2.3) and (3.1).

(i) Odd $n$

$$T_r(n) = 2T_{DFS}\left(\frac{n-1}{2} + 1\right) + 2T_{DFS}\left(\frac{n-1}{2}\right) + 1$$
$$= 2\left(T_{DFS}\left(\frac{n-1}{2}\right) + T_{DFS}\left(\frac{n-1}{2} - 1\right) + 2\right) +$$
$$2\left(T_{DFS}\left(\frac{n-1}{2} - 1\right) + T_{DFS}\left(\frac{n-1}{2} - 2\right) + 2\right) + 1$$
$$= 2T_{DFS}\left(\frac{n-1}{2}\right) + 2T_{DFS}\left(\frac{n-1}{2} - 1\right) +$$
$$2T_{DFS}\left(\frac{n-1}{2} - 1\right) + 2T_{DFS}\left(\frac{n-1}{2} - 2\right) + 9$$
$$= T_r(n - 2) - 1 + T_r(n - 4) - 1 + 9 = T_r(n - 2) + T_r(n - 4) + 7.$$

(ii) Even $n$

$$T_r(n) = T_{DFS}\left(\frac{n}{2} + 1\right) + 2T_{DFS}\left(\frac{n}{2}\right) + T_{DFS}\left(\frac{n}{2} - 1\right) + 1$$
$$= T_{DFS}\left(\frac{n}{2}\right) + T_{DFS}\left(\frac{n}{2} - 1\right) + 2 +$$
$$2\left(T_{DFS}\left(\frac{n}{2} - 1\right) + T_{DFS}\left(\frac{n}{2} - 2\right) + 2\right) +$$
$$T_{DFS}\left(\frac{n}{2} - 2\right) + T_{DFS}\left(\frac{n}{2} - 3\right) + 2 + 1$$
$$= T_r(n - 2) - 1 + T_r(n - 4) - 1 + 9 = T_r(n - 2) + T_r(n - 4) + 7.$$

2. The proof is analogous and is based on formulae (2.7) and (3.2). ■

As follows from Theorems 2.1, 2.4, 2.6 and 3.4, the optimal decomposition method is more efficient than all the methods presented in section 2 from the perspective of both complexity characteristics.

Corollary 3.5. For an $n$-vertex $FG$:

1. The total number of terms $T(n)$ in the expression $Ex(FG)$ derived by the
optimal reduction method is expressed explicitly as follows:

\[ T(1) = 0 \]

\[ T(n) \approx (4.8896 + (-1)^n \cdot 0.070089) \left( \sqrt{\frac{\sqrt{5} + 1}{2}} \right)^n + \]

\[ \left[ (1 + (-1)^n) 0.020163 + (1 + (-1)^{n+1}) 0.082996 i \right] \left( i \sqrt{\frac{\sqrt{5} - 1}{2}} \right)^n - 7 \]

(n > 1)

or

\[ T(1) = 0 \]

\[ T(n) \approx (4.8896 + (-1)^n \cdot 0.070089) \left( \sqrt{\frac{\sqrt{5} + 1}{2}} \right)^n + \]

\[ \left[ 0.040325 \cos \left( \frac{n\pi}{2} \right) - 0.16599 \sin \left( \frac{n\pi}{2} \right) \right] \left( i \sqrt{\frac{\sqrt{5} - 1}{2}} \right)^n - 7 \]

(n > 1).

2. The number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the optimal reduction method is expressed explicitly as follows:

\[ P(1) = 0 \]

\[ P(n) \approx (1.8677 + (-1)^n \cdot 0.026772) \left( \sqrt{\frac{\sqrt{5} + 1}{2}} \right)^n + \]

\[ \left[ (1 + (-1)^n) 0.052786 + (1 + (-1)^{n+1}) 0.21729i \right] \left( i \sqrt{\frac{\sqrt{5} - 1}{2}} \right)^n - 3 \]

(n > 1)
or

\[ P(1) = 0 \]

\[ P(n) \approx (1.8677 + (-1)^n \cdot 0.026772) \left( \sqrt{\frac{\sqrt{5} + 1}{2}} \right)^n + \]

\[ \left[ 0.10557 \cos \left( \frac{n\pi}{2} \right) - 0.43457 \sin \left( \frac{n\pi}{2} \right) \right] \left( \sqrt{\frac{\sqrt{5} - 1}{2}} \right)^n \]

\[ (n > 1). \]

The proof of Corollary 3.5 uses the recurrences obtained in Theorem 3.4 and is based on the method [14].

For \( n = 9 \), the algebraic expression derived by the optimal reduction method is

\[ (((a_1a_2 + b_1)a_3 + a_1b_2)a_4 + (a_1a_2 + b_1)b_3)(a_5(a_6(a_7a_8 + b_7) + b_6a_8) + b_5(a_7a_8 + b_7)) + ((a_1a_2 + b_1)a_3 + a_1b_2)b_4(a_6(a_7a_8 + b_7) + b_6a_8). \]

It contains 35 terms and 13 plus operators.

### 3.1. Time and Space Expenses of the Method

As noted above, the optimal reduction method is equivalent to applying the DFS method to four subgraphs which are revealed by decomposing an \( n \)-vertex \( FG \) in the middle. The DFS method has to be employed twice in the direct and twice in the opposite direction. For this reason, we will implement the optimal reduction method using the algorithms presented in section 2.2.1 as its base.

\[ FG\_Reduction\_Optimal(n, Expr) \]

1. if \( n > 2 \)
2. \( FG\_DFS\_opposite(1, \left\lceil \frac{n}{2} \right\rceil, Expr) \)
3. if \( \left\lceil \frac{n}{2} \right\rceil > 2 \)
4. \( \text{Insert\_to\_Head} (\text{""}, Expr) \)
5. \texttt{Insert\_to\_End(","}, \textit{Expr})
6. \texttt{FG\_DFS\_direct(}[\frac{n}{2}], n, \textit{Expr2})
7. \texttt{if } \left\lfloor \frac{n}{2} \right\rfloor > 1
8. \texttt{Insert\_to\_Head(","}, \textit{Expr2})
9. \texttt{Insert\_to\_End(","}, \textit{Expr2})
10. \texttt{Concatenate(Expr, Expr2)}
11. \texttt{Concatenate(Expr, Expr2)}
12. \texttt{FG\_DFS\_opposite(1, } \left\lceil \frac{n}{2} \right\rceil - 1, \textit{Expr2})
13. \texttt{if } \left\lfloor \frac{n}{2} \right\rfloor > 3
14. \texttt{Insert\_to\_Head(","}, \textit{Expr2})
15. \texttt{Insert\_to\_End(","}, \textit{Expr2})
16. \texttt{Concatenate(Expr, Expr2)}
17. \texttt{Insert\_to\_End("b\left\lceil \frac{n}{2} \right\rceil - 1", Expr)}
18. \texttt{FG\_DFS\_direct(}[\frac{n}{2}] + 1, n, \textit{Expr2})
19. \texttt{if } \left\lfloor \frac{n}{2} \right\rfloor > 2
20. \texttt{Insert\_to\_Head(","}, \textit{Expr2})
21. \texttt{Insert\_to\_End(","}, \textit{Expr2})
22. \texttt{Concatenate(Expr, Expr2)}
23. \texttt{else}
24. \texttt{if } n = 2
25. \texttt{Expr \leftarrow "a_1"}
26. \texttt{else}
Expr ← NULL

For even $n$, the number of joint reductions implemented by this algorithm will be greater than the number of fork reductions by one. Hence, there exists the second version of the algorithm with another choice of the middle of the graph for even $n$ which implements the greater number of fork reductions. Both realizations give the same expression for odd $n$.

Running times of procedures for the DFS method applied to an $\frac{n}{2}$-vertex $FG$ (lines 2 6 12 18) are, as follows from section 2.2.1, $\Theta \left( \left( \frac{1+\sqrt{5}}{2} \right)^{n/2} \right)$. Other operations of the algorithm are performed in $O(1)$ time. Thus, the running time of the algorithm is $\Theta \left( \left( \frac{1+\sqrt{5}}{2} \right)^{n/2} \right)$.

The amount of memory that requires the algorithm is determined only by the size of the derived expression and, by Corollary 3.5, is also $\Theta \left( \left( \frac{1+\sqrt{5}}{2} \right)^{n/2} \right)$.

4. Decomposition Method

This method is based on revealing subgraphs in the initial graph. The resulting expression is produced by a special composition of subexpressions describing these subgraphs.

Consider the $n$-vertex $FG$ presented in Figure 1.2. Denote by $E(p, q)$ a subexpression related to its subgraph (which is an $FG$ as well) having a source $p$ ($1 \leq p \leq n$) and a sink $q$ ($1 \leq q \leq n$, $q \geq p$). If $q - p \geq 2$, then we choose any decomposition vertex $i$ ($p + 1 \leq i \leq q - 1$) in a subgraph, and, in effect, split it at this vertex (Figure 4.1). Otherwise, we assign final values to $E(p, q)$. As follows from the $FG$ structure, any path from vertex $p$ to vertex $q$ passes through vertex $i$ or avoids it via edge $b_{i-1}$. Therefore, $E(p, q)$ can be generated by the following recursive procedure (decomposition procedure):

1. case $q = p$ : $E(p, q) \leftarrow 1$
2. case $q = p + 1$ : $E(p, q) \leftarrow a_p$
3. case $q \geq p + 2$ : $\text{choice}(p, q, i)$
4. $E(p, q) \leftarrow E(p, i)E(i, q) + E(p, i - 1)b_{i-1}E(i + 1, q)$
Figure 4.1: Decomposition of a Fibonacci subgraph at vertex $i$.

Lines 1 and 2 contain conditions of exit from the recursion. The special case when a subgraph consists of a single vertex is considered in line 1. It is clear that such a subgraph can be connected to other subgraphs only serially. For this reason, it is accepted that its subexpression is 1, so that when it is multiplied by another subexpression, the final result is not influenced. Line 2 describes a subgraph consisting of a single edge. The corresponding subexpression consists of a single term equal to the edge label. The general case is processed in lines 3 and 4. The procedure, choice($p, q, i$), in line 3 chooses an arbitrary decomposition vertex $i$ on the interval $(p, q)$ so that $p < i < q$. A current subgraph is decomposed into four new subgraphs in line 4. Subgraphs described by subexpressions $E(p, i)$ and $E(i, q)$ include all paths from vertex $p$ to vertex $q$ passing through vertex $i$. Subgraphs described by subexpressions $E(p, i - 1)$ and $E(i + 1, q)$ include all paths from vertex $p$ to vertex $q$ passing through edge $b_{i-1}$.

$E(1, n)$ is the expression of the initial $n$-vertex $FG$ ($Ex(FG)$). Hence, the decomposition procedure is initially invoked by substituting parameters 1 and $n$ instead of $p$ and $q$, respectively.

In [8] we proved the following theorem that determines an optimal location of the decomposition vertex $i$ in an arbitrary interval $(p, q)$ of a Fibonacci graph from the perspective of the first complexity characteristic.

**Theorem 4.1.** The representation with a minimum total number of terms among all possible representations of $Ex(FG)$ derived by the decomposition method is achieved if and only if in each recursive step $i$ is equal to $\frac{q+p}{2}$ for odd $q - p + 1$ and to $\frac{q+p-1}{2}$ or $\frac{q+p+1}{2}$ for even $q - p + 1$, i.e., when $i$ is a middle vertex of the interval $(p, q)$. Such a decomposition method is called optimal.

The following theorem for the second complexity characteristic is proven in [7].

**Theorem 4.2.** The representation with a minimum number of plus operators among all possible representations of $Ex(FG)$ derived by the decomposition method can be achieved by the optimal decomposition method.
It can be easily shown that for an \( n \)-vertex \( FG \):

1. The total number of terms \( T(n) \) in the expression \( Ex(FG) \) derived by the optimal decomposition method is defined recursively as follows:

\[
T(1) = 0 \\
T(2) = 1 \\
T(n) = T(\left\lceil \frac{n}{2} \right\rceil) + T(\left\lfloor \frac{n}{2} \right\rfloor + 1) + T(\left\lceil \frac{n}{2} \right\rceil - 1) + T(\left\lfloor \frac{n}{2} \right\rfloor) + 1 \quad (n > 2).
\]

2. The number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the optimal decomposition method is defined recursively as follows:

\[
P(1) = 0 \\
P(2) = 0 \\
P(n) = P(\left\lceil \frac{n}{2} \right\rceil) + P(\left\lfloor \frac{n}{2} \right\rfloor + 1) + P(\left\lceil \frac{n}{2} \right\rceil - 1) + P(\left\lfloor \frac{n}{2} \right\rfloor) + 1 \quad (n > 2).
\]

For large \( n \)

\[
T(n) \approx 4T(\left\lceil \frac{n}{2} \right\rceil) + 1.
\]

By the master theorem \cite{2} recurrences like

\[
T(n) = \alpha T(\left\lceil \frac{n}{\beta} \right\rceil) + f(n),
\]

where \( \alpha \geq 1 \) and \( \beta > 1 \) are constants, and \( \left\lfloor \frac{n}{\beta} \right\rfloor \) is interpreted as either \( \left\lfloor \frac{n}{\beta} \right\rfloor \) or \( \left\lceil \frac{n}{\beta} \right\rceil \),

can be bounded asymptotically as follows:

\[
T(n) = \Theta \left( n^{\log_{\beta} \alpha} \right)
\]

if \( f(n) = O \left( n^{\log_{\beta} \alpha - \epsilon} \right) \) for some constant \( \epsilon > 0 \).

Therefore, \( T(n) \) and \( P(n) \) are \( \Theta \left( n^2 \right) \).

For \( n = 9 \), the possible algebraic expression derived by the optimal decomposition method is

\[
((a_1a_2 + b_1)(a_3a_4 + b_3) + a_1b_2a_4)((a_5a_6 + b_5)(a_7a_8 + b_7) + a_5b_6a_8) + \\
(a_1(a_2a_3 + b_2) + b_1a_3)b_4(a_6(a_7a_8 + b_7) + b_6a_8).
\]

It contains 31 terms and 11 plus operators.
We conjecture that the optimal decomposition method provides an optimal representation (for both our complexity characteristics) of an algebraic expression related to a Fibonacci graph.

As shown in [7], the optimal decomposition method is not always the only one that provides an expression for a Fibonacci graph with a minimum number of plus operators. There exist special values of $n$ when an $n$-vertex Fibonacci graph has several expressions with the same minimum number of plus operators (among expressions derived by the decomposition method). These special values are grouped as follows:

$$7, 13 \div 15, 25 \div 31, 49 \div 63, 97 \div 127, 193 \div 255, \ldots$$

In the general view, they can be presented in the following way:

$$n_{first_\nu} \leq n_{sp_\nu} \leq n_{last_\nu},$$

$$n_{first_1} = n_{last_1} = 7,$$

$$n_{first_\nu} = 2n_{first_{\nu-1}} - 1,$$

$$n_{last_\nu} = 2n_{last_{\nu-1}} + 1.$$  

Here $\nu$ is a number of a group of special numbers; $n_{sp_\nu}$ is a special number of the $\nu$-th group; $n_{first_\nu}$ and $n_{last_\nu}$ are the first value and the last value, respectively, in the $\nu$-th group. For all these values of $n$, not only the values of $i$ which are mentioned in Theorem 4.1, provide a minimum number of plus operators in $Ex(FG)$.

It can be shown that if $i = 3$ or $i = n - 2$ in every recursive step (the same value in each step) then the decomposition method turns out to be the DLS method. If $i$ is an arbitrary number in the first recursive step, and in all subsequent steps $i = 2$ or $i = n - 1$ (the same value in each step) then the decomposition method may be interpreted as the reduction method. Specifically, if $i$ in the first step is the same as $i$ in all following steps, then the decomposition method coincides with the DFS method.

4.1. Time and Space Expenses of the Method

We propose the following recursive algorithm which realizes the optimal decomposition method in accordance with the above procedure and Theorem 4.1:

$$FG_{Decomposition_{Optimal}}(p, q, Expr)$$

1. if $q - p > 1$
2. \(FG_{\text{Decomposition\_Optimal}}(p, \floor{\frac{q+p}{2}}, \text{Expr})\)

3. if \(\floor{\frac{q+p}{2}} - p > 1\)

4. \(\text{Insert\_to\_Head}(\"\), \text{Expr}\)

5. \(\text{Insert\_to\_End}(\")\", \text{Expr}\)

6. \(FG_{\text{Decomposition\_Optimal}}(\floor{\frac{q+p}{2}}, q, \text{Expr2})\)

7. if \(q - \floor{\frac{q+p}{2}} > 1\)

8. \(\text{Insert\_to\_Head}(\"\), \text{Expr2}\)

9. \(\text{Insert\_to\_End}(\")\", \text{Expr2}\)

10. \(\text{Concatenate}(\text{Expr}, \text{Expr2})\)

11. \(\text{Insert\_to\_End}(\" + \")\", \text{Expr})\)

12. \(FG_{\text{Decomposition\_Optimal}}(p, \floor{\frac{q+p}{2}} - 1, \text{Expr2})\)

13. if \(\floor{\frac{q+p}{2}} - p > 2\)

14. \(\text{Insert\_to\_Head}(\"\), \text{Expr2}\)

15. \(\text{Insert\_to\_End}(\")\", \text{Expr2}\)

16. \(\text{Concatenate}(\text{Expr}, \text{Expr2})\)

17. \(\text{Insert\_to\_End}(\"b\floor{\frac{q+p}{2}} - 1\")\", \text{Expr})\)

18. \(FG_{\text{Decomposition\_Optimal}}(\floor{\frac{q+p}{2}} + 1, q, \text{Expr2})\)

19. if \(q - \floor{\frac{q+p}{2}} > 2\)

20. \(\text{Insert\_to\_Head}(\"\), \text{Expr2}\)

21. \(\text{Insert\_to\_End}(\")\", \text{Expr2}\)

22. \(\text{Concatenate}(\text{Expr}, \text{Expr2})\)

23. else
24. if $q - p = 1$
25. \[ \text{Expr} \leftarrow "a_p" \]
26. else
27. \[ \text{Expr} \leftarrow \text{NULL} \]

The algorithm is initially invoked by substituting 1 and $n$ instead of $p$ and $q$, respectively, for deriving the expression of an $n$-vertex $FG$. Four recursive calls (lines 2, 3, 12, 13) include a middle of the interval $(p, q)$ in their parameter list ($\lfloor \frac{q + p}{2} \rfloor$ is a number of the decomposition vertex while its another possible number is $\lceil \frac{q + p}{2} \rceil$). Hence, the running time of the algorithm is $t(n) \approx 4t\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(1) = \Theta(n^2)$.

The amount of memory that requires the algorithm is determined only by the size of the derived expression and, thus, is also $\Theta(n^2)$.

5. Generalized Decomposition (GD) Method

As follows from the previous section, the decomposition method is based on splitting an $FG$ in each recursive step into two parts via decomposition vertex $i$ and edge $b_{i-1}$. The GD method entails splitting an $FG$ in each recursive step into an arbitrary number of parts (we will denote this number by $m$) via decomposition vertices $i_1, i_2, \ldots, i_{m-1}$ and edges $b_{i_{1-1}}, b_{i_{2-1}}, \ldots, b_{i_{m-1-1}}$, respectively. An example for $m = 3$ is illustrated in Figure 5.1

![Figure 5.1: Decomposition of a Fibonacci subgraph at vertices $i_1$ and $i_2$.](image)

In all cases when $m > 2$, the decomposition procedure used in the previous section is transformed to the more complex form. Specifically, for $m = 3$, the general line of the new decomposition procedure, corresponding to line 4 of the
decomposition procedure with $m = 2$ is presented as:

$$E(p, q) \leftarrow E(p, i_1)E(i_1, i_2)E(i_2, q) +
E(p, i_1 - 1)b_{i_1 - 1}E(i_1 + 1, i_2)E(i_2, q) +
E(p, i_1)E(i_1, i_2 - 1)b_{i_2 - 1}E(i_2 + 1, q) +
E(p, i_1 - 1)b_{i_1 - 1}E(i_1 + 1, i_2 - 1)b_{i_2 - 1}E(i_2 + 1, q).$$

The sum above consists of four parts, with each part including three subexpressions corresponding to the three parts of a split subgraph. Hence, a current subgraph is decomposed into twelve new subgraphs.

Suppose that an $FG$ is split into approximately equal parts in each recursive step (distances between decomposition vertices are equal or approximately equal). It will be the uniform GD method.

The following theorem is proven in [9].

**Theorem 5.1.** For an $n$-vertex $FG$, both the total number of terms $T(n)$ and the number of plus operators $P(n)$ in the expression $Ex(FG)$ derived by the uniform GD method (the $FG$ is split into $m$ parts) are $O\left(n^{1+\log_m 2^{n-1}}\right)$.

As follows from Theorem 5.1, $T(n)$ and $P(n)$ reach the minimum complexity among $2 \leq m \leq n-1$ when $m = 2$. Substituting 2 for $m$ gives $O(n^2)$ (we have the optimal decomposition method in this case). Further, the complexity increases with the increase in $m$. For example, we have $O(n^{1+\log_3 4})$ for $m = 3$, $O(n^{2.5})$ for $m = 4$, etc. In the extreme case, when $m = n - 1$, all inner vertices (from 2 to $n - 1$) of an $n$-vertex $FG$ are decomposition vertices. The single recursive step is executed in this case, and all revealed subgraphs are individual edges (labeled $a$ with index) connected by additional edges (labeled $b$ with index). That is, in this instance, the uniform GD method is reduced to the sequential paths method. Substituting $n - 1$ for $m$ gives

$$O\left(n^{1+\log_{n-1} 2^{n-2}}\right) > O\left(n^{1+\log_n 2^{n-2}}\right) = O\left(2^{n-2n}\right).$$

These results do not contradict our conjecture that the optimal decomposition method provides an optimal representation. At least, it is the best one among uniform GD methods (asymptotically).
6. Conclusions

Various algorithms generating an algebraic expression for a Fibonacci graph were proposed. Complexities of representations derived by sequential paths, DFS, DLS, and reduction methods increase exponentially as the number of the graph’s vertices increases. The generalized decomposition (GD) method has algorithms generating representations with polynomial complexity. Specifically, the decomposition method provides an algorithm for constructing the expression with $O(n^2)$ complexity. The methods we considered are closely related one to another. The GD method encompasses the widest class of algorithms, and among them, all algorithms of the decomposition method. One of them, the optimal decomposition method, is assumed to be the best, from the perspective of complexity. The decomposition algorithms class comprises as subclasses reduction algorithms, two DLS algorithms, etc. The subclass of reduction algorithms includes the optimal reduction method, two DFS algorithms, etc. The DFS method is the worst among reduction algorithms and is assumed to be the worst among decomposition algorithms. The sequential paths method is a special case of the GD method, but it does not belong to the class of decomposition algorithms. It generates an expression with the maximum complexity. The different methods relationship is illustrated in Figure 6.1.

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Figure 6.1: Relationships among methods of an algebraic expression generation for a Fibonacci graph.

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