Ground-state fidelity of Luttinger liquids: A wave functional approach

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Abstract.
We use a wave functional approach to calculate the fidelity of ground states in the Luttinger liquid universality class of one-dimensional gapless quantum many-body systems. The ground-state wave functionals are discussed using both the Schrödinger (functional differential equation) formulation and a path integral formulation. The fidelity between Luttinger liquids with Luttinger parameters $K$ and $K'$ is found to decay exponentially with system size, and to obey the symmetry $F(K, K') = F(1/K, 1/K')$ as a consequence of a duality in the bosonization description of Luttinger liquids.

Keywords: bosonization, Luttinger liquids (theory), quantum phase transitions (theory).
1. Introduction

In recent years it has been realized that concepts from quantum information theory can be fruitfully applied to analyze and characterize aspects of the phase diagram of quantum many-body systems. In particular, the notion of entanglement has proven very powerful [1]. Quantum phase transitions (QPTs) [2] can be detected by studying ground state entanglement [1], and universal terms in the von Neumann entropy ("entanglement entropy") have been identified in classes of critical systems in both one [3, 4, 5] and two [6] dimensions, as well as in two-dimensional topologically ordered phases [7].

More recently it has also become clear that another useful quantity for studying QPTs is the fidelity between two ground states corresponding to different parameters in the Hamiltonian. Here the fidelity is simply (the modulus of) the overlap between the ground states. The basic idea [8] is that as one of the parameter sets is varied so that a quantum phase transition is crossed, one expects a sharp signature in the fidelity due to the qualitative difference between ground states in different phases. This idea has been elucidated and tested on various models, and generalized and extended in various directions [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

Most of the models which have been investigated from the fidelity point of view so far have been one-dimensional. One particularly important universality class in one dimension is the Luttinger (or Tomonaga-Luttinger) liquid universality class [34]. It includes all critical one-dimensional systems whose low-energy physics is described by a conformal field theory with central charge $c = 1$, regardless of the details of the microscopic Hamiltonian and whether it describes fermions, bosons, or spins [35, 36, 37]. The low-energy effective field theory for Luttinger liquids (LLs) is the Luttinger model [38]. Recently Yang [26] presented a calculation of the ground state fidelity of two LLs using the standard operator formalism.

In this paper we revisit the problem of the fidelity of LL ground states, using an alternative approach based on wave functionals. In this approach the overlap between two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is expressed as a functional (or path) integral $\langle \Psi_1 | \Psi_2 \rangle = \int D\phi \Psi_1^\dagger[\phi] \Psi_2[\phi]$, where $\Psi_1[\phi]$ and $\Psi_2[\phi]$ are wave functionals for the two states. We note that this expression resembles, and is the field-theoretical analogue of, the expression for the overlap of two states $|\psi_1\rangle$ and $|\psi_2\rangle$ in the Schrödinger (wave mechanics) formulation of quantum mechanics, which is given by an ordinary integral (for simplicity, consider a single particle in one dimension) $\langle \psi_1 | \psi_2 \rangle = \int dx \psi_1^\ast(x) \psi_2(x)$, where $\psi_1(x)$ and $\psi_2(x)$ are wave functions for the two states.

The ground state wave functional of the LL has been derived by Fradkin et al. [39] and Stone and Fisher [10] using path integral methods. The LL ground state wave functional has also been discussed in the context of the Schrödinger formulation of quantum field theory (the field-theoretical analogue of the Schrödinger formulation of quantum mechanics), in which wave functionals are obtained as solutions of functional...
differential equations. Fradkin et al. \cite{39} showed that the ground state wave functional they had obtained from their path integral formulation was indeed the lowest-energy eigen-functional of the Luttinger model Hamiltonian expressed as a (second-order) functional differential operator. Closely related Schrödinger-type derivations have been given in \cite{42} and \cite{43}.

In this paper we give an alternative derivation in which the LL ground state wave functional is obtained as the solution of a (first-order) functional differential equation that results from translating the relation $\hat{\beta}_q|\Psi_0\rangle = 0$ to the Schrödinger formulation. Here $|\Psi_0\rangle$ is the LL ground state and $\hat{\beta}_q$ is an arbitrary annihilation operator in the set of canonical boson operators which diagonalizes the Luttinger model Hamiltonian. We also present an alternative path integral derivation of the LL ground state wave functional.

In agreement with Yang \cite{26} we find that the ground-state fidelity of LLs decays exponentially with system size, but we find that the rate of this exponential decay is a factor of two smaller than the prediction in \cite{26}. We stress, however, that this does not change Yang’s conclusion \cite{26} that followed from his application of his fidelity result to the spin-1/2 XXZ chain, namely that the \textit{fidelity susceptibility} (the second derivative of the fidelity \cite{8, 10, 11, 14, 18}) can signal the QPTs in the XXZ chain.\footnote{Yang’s result for the fidelity [Equation (10) in \cite{26}] has $\prod_{q\neq 0}$ instead of $\prod_{q>0}$ and is therefore the square of our result (3.4). We note that in Equation (8) for the ground state in \cite{26} the summation should have been over the positive wavevectors only (or, equivalently, a factor of 1/2 is missing in the exponent). When the correct expression for the ground state is used, we find, as expected, that the operator approach used in \cite{26} gives the same result (3.4) for the fidelity as the wave functional approach.} We also find that the ground-state fidelity of LLs obeys a certain symmetry which we show to be a consequence of a duality in the bosonization description of LLs.

This paper is organized as follows. In Sec. 2 we present our derivation of the LL ground state wave functional using the Schrödinger formulation. In Sec. 3 the ground-state fidelity is derived and its “duality symmetry” is explained. Some basic facts about LLs and their bosonization description, which form the backdrop for the discussion in the rest of the paper, are summarized in Appendix A. In Appendix B a path integral derivation of the LL ground state wave functional is presented. In this paper we follow the bosonization notation of \cite{36} rather closely.

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\footnote{For a nice introduction to the Schrödinger formulation of quantum field theory, including comparisons with the operator and path integral formulations, see \cite{41}.}

\footnote{The $S = 1/2$ XXZ spin chain is in the LL universality class for $-1 < \Delta \leq 1$, where $\Delta = J_z/J_{xy}$ is the exchange anisotropy ratio in the XXZ model. Yang used his expression for the ground-state fidelity of LLs to show that the fidelity susceptibility of the XXZ chain signals the QPTs at $\Delta = \pm 1$ by diverging at those two points. This conclusion is not affected by the different exponential decay rate of the fidelity found by us, which only changes the prefactor of the fidelity susceptibility, not its singular behaviour.}
2. Ground state wave functional of the Luttinger liquid: A derivation using the Schrödinger formulation

It is well-known that the ground state wavefunction \( \psi_0(x) = \langle x | \psi_0 \rangle \) of the harmonic oscillator can be found from the property \( \hat{a} | \psi_0 \rangle = 0 \), by expressing the bosonic annihilation operator \( \hat{a} \) in terms of the position and momentum operators \( \hat{x} \) and \( \hat{p} \), and going to the \( | x \rangle \) basis where these operators are represented as \( \hat{x} \rightarrow x \), \( \hat{p} \rightarrow -i \partial/\partial x \); this gives a first-order differential equation for \( \psi_0(x) \) which is easily solved. The derivation that follows is essentially the generalization of this procedure to the LL. The most important technical difference from the simple quantum mechanics problem is that now the argument of the wave “function” is a function, not a number, i.e. we are dealing with a wave functional, and consequently ordinary differentiation is replaced by functional differentiation.

We start by expanding the operators \( \hat{\phi}(x) \) and \( \hat{\theta}(x) \) in the Luttinger model Hamiltonian (A.1) as

\[
\hat{\phi}(x) = -\frac{i \pi}{L} \sum_{q \neq 0} \left( \frac{L|q|}{2\pi} \right)^{1/2} \frac{1}{q} e^{-iqx} (\hat{b}_q + \hat{b}_{-q}),
\]

\[
\hat{\theta}(x) = +\frac{i \pi}{L} \sum_{q \neq 0} \left( \frac{L|q|}{2\pi} \right)^{1/2} \frac{1}{|q|} e^{-iqx} (\hat{b}_q - \hat{b}_{-q}).
\]

The \( \hat{b} \)-operators obey canonical bosonic commutation relations \( [\hat{b}_q, \hat{b}_{q'}^\dagger] = \delta_{q,q'} \), and \( L \) is the length of the system. Next, we make a Bogoliubov transformation to another set of canonical boson operators \( \hat{\beta}_q \),

\[
\hat{b}_q = \cosh \xi \hat{\beta}_q - \sinh \xi \hat{\beta}_q^\dagger.
\]

The parameter \( \xi \) is chosen so that the off-diagonal terms in \( \hat{H} \) vanish. The ground state \( | \Psi_0 \rangle \) is the vacuum of the \( \hat{\beta} \)-bosons, i.e. \( \hat{\beta}_q | \Psi_0 \rangle = 0 \) for all \( q \neq 0 \), which implies

\[
\langle \hat{b}_q + \tanh \xi \hat{b}_q^\dagger | \Psi_0 \rangle = 0,
\]

where

\[
\tanh \xi = \frac{1 - K}{1 + K}.
\]

We invert (2.1)-(2.2) to write the \( \hat{b} \)-bosons in terms of \( \hat{\phi}(x) \) and \( \partial_x \hat{\theta}(x) \),

\[
\hat{b}_q = -\frac{i \text{sgn}(q)}{\sqrt{2\pi L|q|}} \int dx e^{-iqx} \left[ |q| \hat{\phi}(x) + i \partial_x \hat{\theta}(x) \right],
\]

where \( \text{sgn}(q) \) is the sign of \( q \). Inserting (2.6) into (2.4), and using (2.5) and (A.2), we get

\[
\int dx e^{-iqx} \left[ |q| \hat{\phi}(x) + i\pi K \hat{\Pi}_\phi(x) \right] | \Psi_0 \rangle = 0.
\]

4 In these expansions we have neglected the \( q = 0 \) terms (“zero modes”) and also a factor involving a short-distance cut-off.
We now project this equation onto an eigenstate $|\phi\rangle$ of $\hat{\phi}(x)$ with eigenvalue $\phi(x)$. Defining the ground state wave functional in the $\{|\phi\rangle\}$ basis as

$$\Psi_0[\phi] \equiv \langle \phi | \Psi_0 \rangle$$

(2.8)

and making use of the Schrödinger representation of $\hat{\phi}(x)$ and $\hat{\Pi}_\phi(x)$ in this basis,

$$\langle \phi | \hat{\phi}(x) | \Psi_0 \rangle = \phi(x) \Psi_0[\phi],$$

(2.9)

$$\langle \phi | \hat{\Pi}_\phi(x) | \Psi_0 \rangle = -i \frac{\delta}{\delta \phi(x)} \Psi_0[\phi],$$

(2.10)

we transform (2.7) into the first-order functional differential equation

$$\int dx \, e^{-iqx} \left[q \phi(x) + \pi K \frac{\delta}{\delta \phi(x)}\right] \Psi_0[\phi] = 0.$$  

(2.11)

To solve this equation, we insert the Ansatz

$$\Psi_0[\phi] \propto \exp \left[-\frac{1}{2\pi K} \int \int dx \, dx' \, \phi(x) g(x-x') \phi(x')\right].$$

(2.12)

Here the coefficient matrix $g(x,x')$ in the quadratic form was taken to be symmetric without loss of generality, and also translationally invariant, i.e. $g(x,x') = g(x-x')$. Calculating the functional derivative in (2.11) and introducing the Fourier transforms \(\tilde{\phi}(q)\) and \(\tilde{g}(q)\) of $\phi(x)$ and $g(x)$, respectively, we find $||q - \tilde{g}(q)||\tilde{\phi}(q) = 0$, i.e.,

$$\tilde{g}(q) = |q|.$$  

(2.13)

From this result $g(x)$ can be found; however, since an explicit expression for $g(x)$ will not be needed in the following, we relegate a discussion of it to the path integral derivation of $\Psi_0[\phi]$ in Appendix B where it comes up naturally.

In section 3 the fidelity will be calculated from $\Psi_0[\phi]$. For the purpose of understanding a symmetry that the fidelity will be shown to possess, we will now briefly also discuss the ground-state wave functional in the basis in which the operator $\hat{\theta}(x)$ is diagonal. This wave functional, defined as $\bar{\Psi}_0[\theta] \equiv \langle \theta | \Psi_0 \rangle$ where $|\theta\rangle$ is an eigenstate of $\hat{\theta}(x)$ with eigenvalue $\theta(x)$, can e.g. be derived in a way that is completely analogous to the derivation of $\Psi_0[\phi]$ above. Expressing the $\hat{b}$-operators in terms of $\hat{\theta}(x)$ and $\partial_x \hat{\phi}(x)$, i.e. $\hat{b}_q = i(2\pi L|q|)^{-1/2} \int dx \, e^{-iqx} [q |\hat{\theta}(x) + i\partial_x \hat{\phi}(x)\rangle$, (2.14) leads to an equation that is identical in form to (2.7) but differs by the replacements $\phi \rightarrow \theta$ [which here amounts to $\hat{\phi}(x) \rightarrow \hat{\theta}(x)$ and $\hat{\Pi}_\phi(x) \rightarrow \hat{\Pi}_\theta(x)$] and $K \rightarrow 1/K$. It follows that $\bar{\Psi}_0[\theta]$ can be obtained from $\Psi_0[\phi]$ by making the same replacements. Thus

$$\bar{\Psi}_0[\theta] \propto \exp \left[-\frac{K}{2\pi} \int \int dx \, dx' \, \theta(x) g(x-x') \theta(x')\right].$$

(2.14)

The property that $K \leftrightarrow 1/K$ when $\phi \leftrightarrow \theta$ holds more generally \[35, 36\] and is referred to as a duality; $\phi(x)$ and $\theta(x)$ are often referred to as dual fields. Thus one can regard the wavefunctionals $\Psi_0[\phi]$ and $\bar{\Psi}_0[\theta]$ as dual representations of the LL ground state $|\Psi_0\rangle$.

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5 One can be led to this Ansatz e.g. by comparing (2.11) to the differential equation $(x+x_0^2 \frac{d^2}{dx^2})\psi_0(x) = 0$ obtained for the simple harmonic oscillator problem discussed at the beginning of this section, which has the solution $\psi_0(x) \propto \exp[-(x/x_0)^2/2]$. 
3. Fidelity between Luttinger liquid ground states

In this section we discuss the ground state fidelity between two LLs with Luttinger parameters $K$ and $K'$\footnote{The Luttinger velocities $u$ and $u'$ do not come into consideration here since the LL ground states are independent of these velocities.}. Denoting the two (normalized) ground states by $|\Psi_{0,K}\rangle$ and $|\Psi_{0,K'}\rangle$, the fidelity between them is defined as the modulus of their overlap,

$$F(K, K') = |\langle \Psi_{0,K} | \Psi_{0,K'} \rangle|.$$  \hfill (3.1)

We begin by considering, as in the previous section, a continuum field theory description of a system of length $L$ with periodic boundary conditions. Using the resolution of the identity in the form $I = \int \prod_x d\phi(x) |\phi\rangle \langle \phi|$, the ground state overlap can be expressed as a path integral involving the ground state wave functionals in the $\{ |\phi\rangle \}$ basis,

$$\langle \Psi_{0,K} | \Psi_{0,K'} \rangle = \int \prod_x d\phi(x) \Psi_{0,K}^* \phi \Psi_{0,K'} \phi$$

$$= \mathcal{N}_K \mathcal{N}_{K'} \int \prod_x d\phi(x) \exp \left[ -\frac{1}{2\pi} \left( \frac{1}{K} + \frac{1}{K'} \right) \int \int dx dx' \phi(x)g(x-x')\phi(x') \right].$$  \hfill (3.2)

Here $\mathcal{N}_K$ is the normalization factor in $\Psi_{0,K} |\phi\rangle$. The quadratic form is diagonalized by introducing the Fourier transform $\phi(x) = (1/L) \sum_q \tilde{\phi}(q)e^{iqx}$ and similarly for $g(x)$; the $q$'s are discrete (due to the finite size $L$) and unbounded (due to the continuum nature of the theory). This gives

$$\langle \Psi_{0,K} | \Psi_{0,K'} \rangle = \mathcal{N}_K \mathcal{N}_{K'} J \int d\tilde{\phi}(0) \int \prod_{q>0} d\tilde{\phi}(q)$$

$$\times \exp \left[ -\frac{1}{\pi} \left( \frac{1}{K} + \frac{1}{K'} \right) \frac{1}{L} \sum_{q>0} \tilde{g}(q) |\tilde{\phi}(q)|^2 \right] = \mathcal{N}_K \mathcal{N}_{K'} J \Omega_0 \prod_{q>0} \frac{2\pi i}{\pi \left( \frac{1}{K} + \frac{1}{K'} \right) L \tilde{g}(q)}.$$  \hfill (3.3)

Here $J$ is the Jacobian for the change of integration variables from $\phi(x)$ to $\tilde{\phi}(q)$ in the path integral measure, and $\Omega_0 = \int d\tilde{\phi}(0)$ is a (divergent) quantity which arises because $\tilde{g}(0) = 0$. Setting $K = K'$ in (3.3) and using $\langle \Psi_{0,K} | \Psi_{0,K} \rangle = 1$ we find

$$\mathcal{N}_K = \left( J \Omega_0 \prod_{q>0} \frac{i\pi^2 KL}{\tilde{g}(q)} \right)^{-1/2}.$$  \hfill (3.3)

Inserting this in (3.3), most quantities cancel out, resulting in the following expression for the ground state fidelity of two LLs \footnote{The Luttinger velocities $u$ and $u'$ do not come into consideration here since the LL ground states are independent of these velocities.} (see also footnote 2):

$$F(K, K') = \prod_{q>0} \frac{2}{\sqrt{\frac{K}{K'} + \sqrt{\frac{K'}{K}}}}.$$  \hfill (3.4)

As it stands, the rhs of this expression vanishes (for $K \neq K'$) even for a system of finite size $L$, because we haven’t yet taken into account the short-distance cut-off of the field theory. Thus it is necessary to regularize (3.4). In the present context this is most straightforwardly done by considering its logarithm which involves the sum $\sum_{q>0} 1$. We use a soft cut-off $\alpha$ to remove this divergence by inserting a factor $e^{-\alpha q}$ in
the sum. For $L \gg \alpha$ this then gives $F(K, K') = [\kappa(K, K')]^{L/2\pi\alpha}$, where we have defined $\kappa(K, K') = 2/\sqrt{K/K' + \sqrt{K'/K}}$. As $1/\alpha$ is a measure of the (effective) maximum wavevector $q_{\text{max}}$, and the distance between adjacent wavevectors is $\Delta q = 2\pi/L$, the exponent $L/2\pi\alpha$ appearing in the fidelity is $\sim q_{\text{max}}/\Delta q$ and thus just represents the effective number of wavevectors in the product in (3.4) after the regularization, as expected.

An alternative regularization approach involves discretizing the field theory, i.e., putting it on a one-dimensional lattice with lattice constant $a$, so that the number of sites is $N = L/a$. Here the lattice constant plays the role of a hard cut-off. The $q$’s then become restricted to the first Brillouin zone, i.e., $|q| < \pi/a \equiv q_{\text{max}}$, and it can be shown that (2.13) must be replaced by $\tilde{g}(q) = a^{-1}\sqrt{2(1 - \cos qa)}$ (which reduces to (2.13) in the limit $qa \to 0$). Calculating the fidelity, one again arrives at the result (3.4) where now the upper limit $\pi/a$ for the $q$’s is implicitly understood. For $N \gg 1$ this gives $F(K, K') = [\kappa(K, K')]^{N/2}$. The obtained exponent $(N/2)$ can be written $q_{\text{max}}/\Delta q$, consistent with what we found when using a soft cut-off. Thus for $K \neq K'$ the fidelity is seen to decay exponentially with system size.

From these results we can deduce the fidelity per site $d(K, K')$ (equivalently the fidelity per wavevector), defined as $\ln d(K, K') = N^{-1}\ln F(K, K')$. This gives

$$d(K, K') = \sqrt{\frac{2}{\sqrt{\frac{K}{\pi}} + \sqrt{\frac{K'}{\pi}}}}. \quad (3.5)$$

It can be argued (cf. [9]) that this is a more fundamental quantity than the fidelity itself, because unlike $F(K, K')$, $d(K, K')$ is independent of the short-distance cut-off and remains finite in the thermodynamic/continuum limit $N = L/a \to \infty$ where $F(K, K')$ vanishes.

For the exactly solvable Luttinger liquid field theory studied here, both regularization approaches considered above are viable for studying the fidelity. The first approach is closer in spirit to most of the literature on Luttinger liquids. On the other hand, the second approach, based on lattice regularization, is more suitable for numerical studies of the fidelity of more general continuum field theories that are not analytically solvable [44]. Again, the main quantity of interest is the fidelity per site, which can be numerically extracted from the fidelity of the discretized version of the field theory in the limit as the lattice constant is sent to zero. As such numerical studies could also address field theories describing gapped (massive) phases, they would open up the possibility of investigating quantum phase transitions in continuum field theories by studying the behaviour of the fidelity per site as a function of the coupling constants.

We conclude this section by discussing some aspects of the fidelity’s dependence on the Luttinger parameters. We first note that (3.4) satisfies $F(K, K) = 1$ and $F(K, K') = F(K', K)$. These two properties are however evident already in the definition (3.1) of the fidelity. A much more interesting property of (3.4) is the symmetry $F(K, K') = F(1/K, 1/K')$. \(\quad (3.6)\)
Ground-state fidelity of Luttinger liquids: A wave functional approach

As we will now show, this symmetry is a consequence of the duality of the LL discussed at the end of section 2. Expressing the overlap \( \langle \Psi_{0,K} | \Psi_{0,K'} \rangle \) in terms of the \( \{|\theta\rangle\} \) basis gives (cf. the first line in (3.2))

\[
\langle \Psi_{0,K} | \Psi_{0,K'} \rangle = \int \prod_x d\theta(x) \, \bar{\Psi}_{0,K}^{*}[\theta] \Psi_{0,K'}[\theta]. \tag{3.7}
\]

Inserting (2.14) one sees that the resulting expression differs from the second line of (3.2) only by the replacement \((K,K') \rightarrow (1/K,1/K')\) which thus must leave the overlap and hence \(F(K,K')\) invariant. Thus (3.6) follows.

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Appendix A. Some basics of Luttinger liquids and their bosonization description

The effective low-energy Hamiltonian for a LL is given by the Luttinger model. Using the bosonization description \[35, 36\], the Luttinger model Hamiltonian can be written

\[
\hat{H} = \frac{u}{2\pi} \int dx \left[ K : (\partial_x \hat{\theta}(x))^2 : + \frac{1}{K} : (\partial_x \hat{\phi}(x))^2 : \right]. \tag{A.1}
\]

Here \(\hat{\phi}(x)\) and \(\hat{\theta}(x)\) are Hermitian fields and \(\ldots:\) represents normal-ordering. The operators

\[
\hat{\Pi}_{\phi}(x) \equiv \frac{1}{\pi} \partial_x \hat{\theta}(x),
\]

\[
\hat{\Pi}_{\theta}(x) \equiv \frac{1}{\pi} \partial_x \hat{\phi}(x),
\]

are the conjugate momenta of \(\hat{\phi}(x)\) and \(\hat{\theta}(x)\), respectively; i.e. the following canonical equal-time commutation relations hold:

\[
[\hat{\phi}(x), \hat{\Pi}_{\phi}(x')] = i\delta(x - x'), \tag{A.4}
\]

\[
[\hat{\theta}(x), \hat{\Pi}_{\theta}(x')] = i\delta(x - x'). \tag{A.5}
\]

The LL is characterized by the two parameters \(u\) and \(K\) \[34, 35, 36\]. The former is the velocity of the low-energy excitations (whose energy disperses linearly with the wavevector), while the latter, known as the Luttinger parameter, determines the exponents of the asymptotic power-law decays of the correlation functions of the LL. The dependence of \(K\) and \(u\) on the parameters in a lattice model in the LL universality class

\[ We note that in the alternative operator approach, which was used to calculate the fidelity in \[26\], the symmetry (3.6) can be understood as follows. If \(K \rightarrow 1/K\), (2.5) gives \(\tanh \xi \rightarrow -\tanh \xi\), i.e. \(\xi \rightarrow -\xi\). The fidelity only depends on \(\cosh(\xi - \xi')\) \[26\] and is therefore invariant under \((\xi, \xi') \rightarrow (-\xi, -\xi')\). \]
Ground-state fidelity of Luttinger liquids: A wave functional approach

may be determined numerically or, in some cases, analytically. Analytical expressions are usually limited to regions of parameter space which are within reach of approximate analytical treatments, unless the model in question is integrable, in which case exact analytical results for $K$ and $u$ valid in the entire LL regime may be available. A model of this latter type is the spin-1/2 XXZ chain, which is exactly solvable by the Bethe Ansatz [15]; the LL regime for this model is $-1 < \Delta \leq 1$ where $\Delta = J_z/J_{xy}$ is the exchange anisotropy. By comparing LL predictions with the exact solution one finds

$$K = \pi/[2(\pi - \arccos \Delta)]$$  
$$u = \pi J_{xy} \sqrt{1 - \Delta^2}/(2 \arccos \Delta).$$

Thus as $\Delta$ is reduced from 1, $K$ increases from $1/2$, passing through 1 at $\Delta = 0$ and diverging as $\Delta \to -1$.

Appendix B. A path integral derivation of the ground state wave functional

In this Appendix we present a path integral derivation of the LL ground state wave functional $\Psi_0[\phi]$. Like the path integral derivation of this quantity given in [10] (see also [17]), the derivation discussed here is based on a path integral representation of the matrix element $\langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle$ of the imaginary-time evolution operator of the Luttinger model in the limit $\Delta \tau \to \infty$. In contrast to [10] and [17], however, we do not set $\phi_i(x) = 0$, i.e. we take both $\phi_f(x)$ and $\phi_i(x)$ to be arbitrary. Furthermore, we determine the dependence of this matrix element on $\phi_f(x)$ and $\phi_i(x)$ for an arbitrary time interval $\Delta \tau$. Another difference from [10] is that we use Poisson’s integral formula instead of Green function methods to find the classical action.

Expanding $|\phi_i\rangle$ and $|\phi_f\rangle$ in terms of the complete set of eigenstates $\{|\Psi_n\rangle\}$ of $\hat{H}$ gives

$$\langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle = \sum_n \Psi_n[\phi_f] \Psi_n^*[\phi_i] e^{-E_n \Delta \tau},$$

(B.1)

where $\{E_n\}$ is the corresponding set of eigenvalues and $\Psi_n[\phi] \equiv \langle \phi | \Psi_n \rangle$. In the limit $\Delta \tau \to \infty$ the contribution from the excited states in the sum is completely suppressed compared to that of the ground state, giving

$$\Psi_0[\phi_f] \Psi_0^*[\phi_i] = \lim_{\Delta \tau \to \infty} \frac{\langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle}{e^{-E_0 \Delta \tau}},$$

(B.2)

where $E_0$ is the ground state energy. This relation will be used to deduce the ground state wave functional $\Psi_0[\phi]$.

A path integral representation of $\langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle$ can be obtained by Trotter-decomposing the exponential and inserting resolutions of the identity in a standard way (see e.g. [18]). This gives

$$\langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle \propto \int_{\phi(x,\tau_f) = \phi_f(x)}^{\phi(x,\tau_i) = \phi_i(x)} D\phi(x, \tau) \exp \left(-S[\phi(x, \tau)]\right),$$

(B.3)

where the Euclidean action is

$$S[\phi(x, \tau)] = \frac{1}{2\pi K} \int dx \int_{\tau_i}^{\tau_f} d\tau \left(u(\partial_x \phi)^2 + \frac{1}{u}(\partial_\tau \phi)^2\right).$$

(B.4)
In these expressions $\tau_i$ and $\tau_f$ are the initial and final time, respectively, with $\Delta \tau = \tau_f - \tau_i$. Making the variable change $y = u\tau$, the action takes the more symmetric form $S[\phi(x, y)] = (2\pi K)^{-1} \int dx \int_{y_i}^{y_f} dy \left[ (\partial_x \phi)^2 + (\partial_y \phi)^2 \right]$ where $y_i = u\tau_i$, $y_f = u\tau_f$. We see that $\langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle$ is given by a path integral of $\exp(-S[\phi(x, y)])$ over real-valued functions $\phi(x, y)$ defined on the horizontal strip in the $xy$ plane bounded by $y = y_i$ and $y = y_f$, with boundary conditions $\phi(x, y_i) = \phi_i(x)$ and $\phi(x, y_f) = \phi_f(x)$ on the lower and upper edge of the strip, respectively.

Because the action is quadratic, the path integral can be calculated exactly. We expand $S$ around the classical action $S_{cl}$ corresponding to $\phi_{cl}(x, y)$, the solution of the classical equation of motion $\delta S/\delta \phi(x, y) = 0$ (which for the LL is the Laplace equation in two dimensions), subject to the boundary conditions. The path integral in (B.3) can then be written as a product of $\exp(-S_{cl})$ and a path integral over the deviations $\phi(x, y) - \phi_{cl}(x, y)$ from the classical solution. The boundary conditions at $y = y_i$ and $y = y_f$ imply that the entire dependence on $\phi_i(x)$ and $\phi_f(x)$ lies in $S_{cl}$, i.e.

$$\langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle \propto \exp \left( -S_{cl}^{(\Delta y)}[\phi_f, \phi_i] \right).$$

Here the superscript on $S_{cl}$ indicates its dependence on the width $\Delta y = y_f - y_i$ of the strip.

Now we integrate by parts in the classical action and invoke the boundary conditions in the $x$ and $y$ direction (for the $x$ direction we use periodic boundary conditions $\phi_{cl}(-L/2, y) = \phi_{cl}(L/2, y)$ and then send the system length $L$ to infinity) as well as the fact that $\phi_{cl}(x, y)$ obeys the Laplace equation. This gives

$$S_{cl}^{(\Delta y)}[\phi_f, \phi_i] = \frac{1}{2}\pi K \int_{-\infty}^{\infty} dx \left[ \phi_f(x)\partial_y \phi(x, y_f) - \phi_i(x)\partial_y \phi(x, y_i) \right].$$

Next the boundary-value problem for $\phi_{cl}(x, y)$ defined on the strip is mapped to a different geometry where it is more easily solved. Defining the complex coordinate $z = x + iy$, the conformal transformation

$$w = \exp \left( \frac{\pi (z - iy_i)}{\Delta y} \right)$$

maps the horizontal strip of width $\Delta y$ in the complex $z$ plane to the upper half of the complex $w$ plane. In particular, the lower (upper) edge of the strip in the $z$ plane is mapped to the positive (negative) real axis in the $w$ plane [8]. Defining $u = \Re(w)$, $v = \Im(w)$, and $\Phi(u, v) \equiv \phi_{cl}(x, y)$, we then have $\Phi(u < 0, 0) = \phi_f(x)$ and $\Phi(u > 0, 0) = \phi_i(x)$. Thus $\Phi(u, v)$ is known on the entire real axis $v = 0$. Furthermore, it satisfies the Laplace equation in the upper half plane $v > 0$. These properties imply that $\Phi(u, v)$ is given by the Poisson integral formula [9].

$$\Phi(u, v) = \frac{v}{\pi} \int_{-\infty}^{\infty} du' \frac{\Phi(u', 0)}{(u - u')^2 + v^2}. \tag{B.8}$$

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8 We thank Andrew Doherty for suggesting this mapping.

9 Poisson’s integral formula is easily derived using Cauchy’s integral formula. See, e.g., the derivation of Equation (4.2.13) in [49].
To calculate (B.6) we need \( \partial_y \phi(x, y_a) = (\pi/\Delta y) u(x, y_a) \partial_v \Phi(u(x, y_a), 0) \) where \( a = i, f \). From (B.8) we have \( \partial_v \Phi(u, 0) = - \int_{-\infty}^{\infty} du' g(u-u') \Phi(u', 0) \), where

\[
g(u) \equiv - \lim_{v \to 0} \partial_v \delta_v(u).
\]  

(B.9)

Here \( \delta_v(u) \) is a Lorentzian centered at \( u = 0 \) whose width is determined by \( v \),

\[
\delta_v(u) = \frac{v}{\pi u^2 + v^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \, e^{-|q|v} e^{iqu}.
\]  

(B.10)

Note the scaling relation \( g(bx) = b^{-2} g(x) \). Using these results we find

\[
S_{cl}^{(\Delta y)}[\phi_f, \phi_i] = \frac{1}{2\pi K} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \left\{ \sum_{a=i,f} \phi_a(x) g(f_-(x-x', \Delta y)) \phi_a(x') + 2 \phi_f(x) g(f_+(x-x', \Delta y)) \phi_i(x') \right\}
\]

(B.11)

where

\[
f_-(x, \Delta y) = \left( \frac{\pi}{2\Delta y} \right)^{-1} \sinh \left( \frac{\pi x}{2\Delta y} \right), \quad f_+(x, \Delta y) = \left( \frac{\pi}{2\Delta y} \right)^{-1} \cosh \left( \frac{\pi x}{2\Delta y} \right).
\]  

(B.12)

For a generic value of \( \Delta y = u\Delta \tau \) the second term in (B.11) couples \( \phi_i \) and \( \phi_i \), so that, in agreement with (B.1), \( \langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle \) cannot then be written as a product of two factors with one depending only on \( \phi_f \) and the other on \( \phi_i \). According to (B.1) - (B.2) such a factorization should however occur in the limit \( \Delta y \to \infty \). This indeed follows from (B.11); using that \( f_+(x, \Delta y) \to \infty \) and \( f_-(x, \Delta y) \to x \) in this limit, we see that \( g(f_+(x-x', \Delta y)) \to g(\infty) = 0 \) [from \( g(x \neq 0) = -1/\pi x^2 \)] and \( g(f_-(x-x', \Delta y)) \to g(x-x') \). Thus for \( \Delta y \to \infty \) the second term in (B.11) vanishes and the classical action reduces to

\[
S_{cl}^{(\infty)}[\phi_f, \phi_i] = \frac{1}{2\pi K} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \sum_{a=i,f} \phi_a(x) g(x-x') \phi_a(x').
\]  

(B.13)

It now follows from (B.2), (B.5) and (B.13) that \( \Psi_0[\phi] \) is given by (2.12), provided that the function \( g(x) \) defined in (B.9) is identical to \( g(x) \) in (2.12). As our notation suggests, this is indeed the case, as is easily seen from the Fourier representation of the Lorentzian given in (B.10), which implies \( \hat{g}(q) = -\lim_{v \to 0} \partial_v e^{-|q|v} = |q| \), in agreement with (2.13). Thus this path integral derivation gives exactly the same result for \( \Psi_0[\phi] \) as the very different derivation using the Schrödinger formulation in Sec. 2.

Finally, we note that an analogous derivation of \( \psi_0[\theta] = \langle \theta | \psi_0 \rangle \) can be given by considering a path integral representation of \( \langle \theta_f | e^{-\hat{H} \Delta \tau} | \theta_i \rangle \). The only difference from the derivation for \( \Psi_0[\phi] \) above is that the Euclidean action for the \( \theta \)-field is given by

\[
S[\theta(x, \tau)] = \frac{K}{2\pi} \int dx \int_{\tau_1}^{\tau_f} d\tau \left( \frac{1}{u} (\partial_x \theta)^2 + \frac{1}{u} (\partial_\tau \theta)^2 \right).
\]  

(B.14)

This action displays the same duality as discussed at the end of Sec. 2, i.e. it can be obtained from the corresponding action (B.4) for \( \langle \phi_f | e^{-\hat{H} \Delta \tau} | \phi_i \rangle \) by making the replacements \( \phi \to \theta \) and \( K \to 1/K \). Thus the result (2.11) for \( \psi_0[\theta] \) follows.
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