New Examples of Locally Algebraically Integrable Bodies

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Abstract—Any compact body with regular boundary in $\mathbb{R}^N$ defines a two-valued function on the space of affine hyperplanes: the volumes of the two parts into which these hyperplanes cut the body. This function is never algebraic if $N$ is even and is very rarely algebraic if $N$ is odd: all known bodies defining algebraic volume functions are ellipsoids (and have been essentially found by Archimedes for $N=3$). We demonstrate a new series of locally algebraically integrable bodies with algebraic boundaries in spaces of arbitrary dimensions, that is, of bodies such that the corresponding volume functions coincide with algebraic ones in some open domains of the space of hyperplanes intersecting the body.

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1. INTRODUCTION

Notation. Let $\mathcal{P}$ denote the space of all affine hyperplanes in the Euclidean space $\mathbb{R}^N$. Any compact subset $W \subset \mathbb{R}^N$ with regular boundary $\partial W$ defines a two-valued function $V_W$ on $\mathcal{P}$, whose values at a hyperplane are the volumes of the two parts into which this hyperplane cuts the subset $W$. Let $\text{reg } \mathcal{P}$ denote the subset in $\mathcal{P}$ consisting of hyperplanes transversal to $\partial W$.

The function $V_W$ is $C^\infty$-smooth (analytic) in the set $\text{reg } \mathcal{P}$ if the hypersurface $\partial W$ is $C^\infty$-smooth (respectively, semialgebraic).

Definition (see [1], [2]). A compact subset $W \subset \mathbb{R}^N$ is algebraically integrable if the corresponding volume function $V_W$ coincides everywhere in $\mathcal{P}$ with some pair of values of an algebraic function (i.e., there is a nontrivial polynomial in $N+2$ variables vanishing on any set of numbers $(V, a_1, \ldots, a_N, b)$ such that $V$ is one of the volumes cut from the body $W$ by the hyperplane defined by the equation $a_1x_1 + \cdots + a_Nx_N = b$). A subset $W$ is locally algebraically integrable in some open domain of $\mathcal{P}$ if the volume function $V_W$ coincides with an algebraic one in this domain. Any connected component of the set $\text{reg } \mathcal{P}$, such that the local algebraicity property is satisfied in it is called an algebraic lacuna of $W$; cf. [3], [4]. The algebraic lacunas consisting of hyperplanes not intersecting the set $W$ are called trivial.

Archimedes [5] showed by an explicit calculation that the balls in $\mathbb{R}^3$ are algebraically integrable; this property can easily be proved also for all ellipsoids in odd-dimensional spaces. On the contrary, Newton [6, Lemma 28] proved that the convex bounded sets with smooth boundaries in $\mathbb{R}^2$ are never algebraically integrable. Arnold (see [7, Problems 1987-14, 1988-13, 1990-27]) conjectured that there are no examples of algebraically integrable bodies other than the ellipsoids in odd-dimensional spaces. This conjecture was proved in [8] for even-dimensional spaces. Several topological and geometrical obstructions to the integrability in the case of odd $N$ were found in [2] and [4], however, it is unclear whether they are strong enough to prohibit all cases other than the ellipsoids. Some of these obstructions
concern the local geometry of bodies and show that even local integrability (except in the case of trivial lacunas) is a rare phenomenon. Accordingly, the problem of finding the bodies allowing nontrivial lacunas arises.

In [2], the first (different from the odd-dimensional ellipsoids) examples of bodies with algebraic smooth boundaries in any $\mathbb{R}^N$, $N \geq 3$, having nontrivial lacunas were found: these are some tubular neighborhoods of standard two-dimensional spheres. In Sec. 2 below, we considerably expand this series, proving that some tubular neighborhoods of standard even-dimensional spheres $S^{2k} \subset \mathbb{R}^{2k+1} \subset \mathbb{R}^N$ with arbitrary $k < N/2$ also satisfy this local algebraicity condition.

**Remark 1.** The strongest geometric obstruction to local integrability (on which the guess of our examples is based) is as follows. Consider a connected component of the set $\text{reg} \mathcal{P}$ and the boundary point of this component corresponding to a hyperplane tangent to $\partial W$ and defined by a linear equation of the form $L(x) = c$, where the hyperplanes with the equation $L(x) = c + \varepsilon$ belong to our component for all sufficiently small $\varepsilon > 0$. Suppose that the second fundamental quadratic form of the hypersurface $\partial W$ is nondegenerate at this point. Let us choose the normal vector participating in the definition of this quadratic form in such a way that it is directed to the side where $L(x) > c$. Then, according to [2], our component cannot be a lacuna if the positive inertia index of such a quadratic form is odd for at least one such boundary point of this component.

**Remark 2.** The local algebraic integrability condition is affine invariant; therefore, all bodies affine equivalent to those considered in Sec. 2 below also satisfy it.

**Remark 3.** Local algebraic integrability, in contrast to the global one, can occur for bodies with not semialgebraic boundaries. For instance, if $W \subset \mathbb{R}^n$ is an arbitrary bounded body with smooth boundary, then the cylindrical body $W \times [-1, 1] \subset \mathbb{R}^{n+1}$ with smoothed corners $\partial W \times \{-1\}$ defines an algebraic volume function on the space of hyperplanes in $\mathbb{R}^{n+1}$ which are sufficiently close to the hyperplane $\mathbb{R}^n \times \{0\}$. However, the case of semialgebraic boundaries is especially interesting, because, in this case, the integrability property can be interpreted in terms of the finiteness of certain orbits of the Picard–Lefschetz monodromy groups related to these hypersurfaces and simply not defined in the nonalgebraic case; see, e.g., [8].

**Remark 4.** Let $W$ be a noncompact subset with semialgebraic boundary in $\mathbb{R}^N$. It may happen that one of the two parts into which a hyperplane transversal to $\partial W$ cuts $W$ is bounded (and hence the same is true for all close hyperplanes in $\mathcal{P}$). Then we again obtain a volume function defined in a neighborhood of such a hyperplane and can investigate the algebraicity of the analytic continuation of this function. The first results in the arising theory also are due to Archimedes [5], who has calculated such volumes cut from the standard (rotation-invariant) two-component hyperboloid and elliptic paraboloid in $\mathbb{R}^3$ by the planes orthogonal to the axes of these bodies.

**Remark 5.** There is a deep analogy between our current problem and the theory of sharp wavefronts of hyperbolic operators studied in [3], [9], and many other works; see [4, p. 138]. In particular, the obstruction from Remark 1 corresponds to the Davydova–Borovikov condition of sharpness. Notice, however, that the condition of local algebraic integrability considered here is a priori much more restrictive than that of sharpness (i.e., $C^\infty$ or even holomorphic regularity) in Petrowsky’s theory.

## 2. TUBULAR NEIGHBORHOODS OF SPHERES

Let $N = n + m$, where $n = 2k + 1$ is odd and $m$ is an arbitrary natural number. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be the standard coordinates in Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Consider the $(n-1)$-dimensional sphere in $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ defined by the conditions

$$x_1^2 + \cdots + x_n^2 = 1, \quad y_1 = \cdots = y_m = 0.$$

Given a number $\varepsilon \in (0, 1)$, we define a body $W \subset \mathbb{R}^N$ as a neighborhood of this sphere whose boundary $\partial W$ satisfies the equation

$$(x_1^2 + \cdots + x_n^2 - 1)^2 + (y_1^2 + \cdots + y_m^2) = \varepsilon^2. \quad (1)$$
This body is invariant under the action of the group \(O(n) \times O(m)\) of independent orthogonal transformations of spaces \(\mathbb{R}^n\) and \(\mathbb{R}^m\), and hence the volume function \(V_W\) is constant on any orbit of the action of this group on the space \(P\) of affine hyperplanes in \(\mathbb{R}^{n+m}\). It is easy to see that any such orbit not consisting of hyperplanes parallel to or containing the coordinate subspace \(\mathbb{R}^n \equiv \mathbb{R}^n \times \{0\} \subset \mathbb{R}^N\) contains the hyperplane given by an equation of the form

\[
x_1 = ay_1 + b, \quad a \geq 0, \quad b \geq 0.
\]

Namely, the hyperplane defined by the equation

\[
\alpha_1 x_1 + \cdots + \alpha_n x_n + \gamma_1 y_1 + \cdots + \gamma_m y_m = \beta,
\]

where not all coefficients \(\alpha_j\) are equal to 0, can be reduced to the form (2) with

\[
a = \frac{\sqrt{\gamma_1^2 + \cdots + \gamma_m^2}}{\sqrt{\alpha_1^2 + \cdots + \alpha_n^2}}, \quad b = \frac{|\beta|}{\sqrt{\alpha_1^2 + \cdots + \alpha_n^2}}.
\]

The value \(b\) is here equal to the distance between the origin and the intersection of our hyperplane (3) with the coordinate subspace \(\mathbb{R}^n\) and \(a\) is the cotangent of the angle between this hyperplane and this subspace \(\mathbb{R}^n\).

**Notation.** Let \(v_p\) be the volume of the unit ball in Euclidean space \(\mathbb{R}^p\) (so that the \((p - 1)\)-dimensional area of the unit sphere is equal to \(pv_p\)). Let \(L(a, b)\) denote the hyperplane defined by (2), and let \(V(a, b)\) equal the greater value of the volume function \(V_W\) at this hyperplane \(L(a, b)\) (that is, the volume of the intersection of \(W\) with the component of \(\mathbb{R}^N \setminus L(a, b)\) containing the origin). Let \(C(\varepsilon)\) be the volume of the entire body \(W\).

The body \(W\) is symmetric; therefore, both values of \(V_W\) at any hyperplane containing the origin are equal to \(C(\varepsilon)/2\). In particular, \(V(a, 0) \equiv C(\varepsilon)/2\).

**Theorem.** For any \(\varepsilon \in (0, 1)\), the partial derivative \(\partial V(a, b)/\partial b\) is equal to a polynomial of degree \(k - 1\) in the variables \(a^2\) and \(b^2\) on the region of \(\mathbb{R}^2_+\) where \(a\) and \(b\) are sufficiently small with respect to \(1 - \varepsilon\).

**Corollary.** For any \(\varepsilon \in (0, 1)\), the volume function \(V_W\) defined by the body \(W\) with boundary (1) in a neighborhood of the set of hyperplanes containing the coordinate subspace \(\mathbb{R}^m\) is defined by a two-valued function of the parameters \(\alpha_i, \gamma_i\), and \(\beta\) of these hyperplanes (see (3)) which has the form

\[
\frac{C(\varepsilon)}{2} \pm P(a(\alpha, \gamma), b(\alpha, \beta)),
\]

where the functions \(a(\alpha, \gamma)\) and \(b(\alpha, \beta)\) are given in (4) and \(P\) is a polynomial of degree \(2k - 1\) containing only monomials of even degrees in \(a\) and odd degrees in \(b\).

In particular, the function \(V_W\) is defined in this neighborhood by an algebraic dilation-invariant function of the variables \(\alpha_i, \gamma_i\), and \(\beta\) of homogeneous degree 0.

**Proof of Theorem.** The partial derivative of the volume function \(V(a, b)\) with respect to \(b\) is obviously equal to the \((N - 1)\)-dimensional volume of the orthogonal projection of the hypersurface \(L(a, b) \cap W\) onto the coordinate subspace \(\{x_1 = 0\}\) in \(\mathbb{R}^N\). Let us calculate this volume.

For any \(y \in \mathbb{R}^m\) with \(|y| \leq \varepsilon\), consider the corresponding affine \(n\)-dimensional plane \(\mathbb{R}^n_y \equiv \mathbb{R}^n \times \{y\} \subset \mathbb{R}^N \times \mathbb{R}^m\).

The intersection of this plane and the body \(W\) is the spherical layer defined by the condition

\[
(x_1^2 + \cdots + x_n^2 - 1)^2 \leq \varepsilon^2 - |y|^2,
\]

i.e., the set of points in \(\mathbb{R}^n_y\) whose distances from the origin \((0, y)\) of this plane belong to the interval

\[
[\sqrt{1 - \varepsilon^2 - |y|^2}, \sqrt{1 + \varepsilon^2 - |y|^2}].
\]
If the constants $a$ and $b$ in (2) are sufficiently small, then the intersection of any such layer with the affine plane (2) is a spherical layer in some affine subspace of dimension $n - 1 \equiv 2k$ in $\mathbb{R}^n$; the latter layer is the difference of two $(n - 1)$-dimensional balls with radii equal to

$$\sqrt{1 - (ay_1 + b)^2 + \varepsilon^2 - |y|^2} \quad \text{and} \quad \sqrt{1 - (ay_1 + b)^2 - \varepsilon^2 - |y|^2}.$$  

The $(n - 1)$-dimensional Euclidean volume of this layer is equal to

$$v_{2k} \left( \left( 1 - (ay_1 + b)^2 + \varepsilon^2 - |y|^2 \right) - \left( 1 - (ay_1 + b)^2 - \varepsilon^2 - |y|^2 \right) \right)^k. \quad (5)$$

We must integrate these volumes over the ball in $\mathbb{R}^m$ consisting of all points $y$ with $|y| \leq \varepsilon$. Let us fiber this ball into the family of $(m - 1)$-dimensional balls on any of which the coordinate $y_1$ is constant. Then, we integrate over $y_1 \in [-\varepsilon, \varepsilon]$ the integrals of the values (5) over the corresponding $(m - 1)$-dimensional balls of this family; i.e., in the case $m > 1$, we take the integral

$$\int_{-\varepsilon}^{\varepsilon} \int_0^{\varepsilon^2 - y_1^2} (m - 1)v_{m-1}\rho^{m-2} v_{2k} \left( \left( 1 - (ay_1 + b)^2 + \varepsilon^2 - (y_1^2 + \rho^2) \right)^k - \left( 1 - (ay_1 + b)^2 - \varepsilon^2 - (y_1^2 + \rho^2) \right)^k \right) d\rho dy_1. \quad (6)$$

The substitution $y_1 = \varepsilon \sin \varphi$, $\rho = \varepsilon \cos \varphi \sin \vartheta$ turns this integral into

$$(m - 1)v_{m-1}v_{2k} \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} (\varepsilon \cos \varphi \sin \vartheta)^{m-2} \left( \left( 1 - (a\varepsilon \sin \varphi + b)^2 + \varepsilon \cos \varphi \cos \vartheta \right)^k - \left( 1 - (a\varepsilon \sin \varphi + b)^2 - \varepsilon \cos \varphi \cos \vartheta \right)^k \right) d\varphi d\vartheta d(\varepsilon \sin \varphi)$$

$$\equiv (m - 1)v_{m-1}v_{2k} \varepsilon^m \int_{-\pi/2}^{\pi/2} \cos^m \varphi \int_0^{\pi/2} \left( \left( 1 - (a\varepsilon \sin \varphi + b)^2 + \varepsilon \cos \varphi \cos \vartheta \right)^k - \left( 1 - (a\varepsilon \sin \varphi + b)^2 - \varepsilon \cos \varphi \cos \vartheta \right)^k \right) d\varphi \sin^{m-2} \vartheta \cos \vartheta d\vartheta d\varphi.$$  

The function under the inner integral sign in the last expression is a polynomial in $\sin \vartheta$ and $\cos \vartheta$ whose coefficients are some polynomials in the variables $\varepsilon$, $\cos \varphi$, $b$, and $a \sin \varphi$. Therefore, the values of these integrals corresponding to varying values of $\varphi$ also form a polynomial in the latter variables. Any monomial of the latter polynomial containing an odd power of $a$ also contains the same odd power of $\sin \varphi$ and hence vanishes in the final integration over $\varphi \in [-\pi/2, \pi/2]$. The result of this integration is a polynomial in $a$ and $b$ (for any fixed $\varepsilon$) containing only even powers of $a$. But the sum of powers of $a$ and $b$ in any monomial of (5) is also even; hence the same is true for the final polynomial, which therefore contains only even powers of $b$, too.

In the remaining case $m = 1$, the function (6) is equal to the integral

$$v_{2k} \int_{-\varepsilon}^{\varepsilon} \left( 1 - (ay + b)^2 + \varepsilon^2 - y^2 \right)^k - \left( 1 - (ay + b)^2 - \varepsilon^2 - y^2 \right)^k dy$$

$$= v_{2k} \varepsilon \int_{-\pi/2}^{\pi/2} \left( 1 - (a\varepsilon \sin \varphi + b)^2 + \varepsilon \cos \varphi \right)^k - \left( 1 - (a\varepsilon \sin \varphi + b)^2 - \varepsilon \cos \varphi \right)^k \varepsilon \cos \varphi d\varphi$$

$$= 2v_{2k} \sum_{j=1}^{k/2} \left( \begin{array}{c} k \varepsilon^j \int_{-\pi/2}^{\pi/2} \left( 1 - (a\varepsilon \sin \varphi + b)^2 \right)^{k-2j+1} \cos^{2j} \varphi d\varphi, \end{array} \right.$$  

also satisfying the assertion of our theorem for the same reasons as above.  

\[\square\]

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