Sturm-Liouville problems with transfer condition Herglotz dependent on the eigenparameter – Hilbert space formulation

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Abstract. We consider a Sturm-Liouville equation \( \ell y := -y'' + qy = \lambda y \) on the intervals \((-a, 0)\) and \((0, b)\) with \(a, b > 0\) and \(q \in L^2(-a, b)\). We impose boundary conditions \(y(-a) \cos \alpha = y'(-a) \sin \alpha, \ y(b) \cos \beta = y'(b) \sin \beta\), where \(\alpha \in [0, \pi]\) and \(\beta \in (0, \pi]\), together with transmission conditions rationally-dependent on the eigenparameter via

\[
\begin{align*}
-\eta(0^+) \left( \lambda \eta - \xi - \sum_{i=1}^{N} \frac{b_i^2}{\lambda - c_i} \right) &= y'(0^+) - y'(0^-), \\
y'(0^-) \left( \lambda \kappa + \zeta - \sum_{j=1}^{M} \frac{a_j^2}{\lambda - d_j} \right) &= y(0^+) - y(0^-),
\end{align*}
\]

with \(b_i, a_j > 0\) for \(i = 1, \ldots, N\), and \(j = 1, \ldots, M\). Here we take \(\eta, \kappa \geq 0\) and \(N, M \in \mathbb{N}_0\). The geometric multiplicity of the eigenvalues is considered and the cases in which the multiplicity can be 2 are characterized. An example is given to illustrate the cases. A Hilbert space formulation of the above eigenvalue problem as a self-adjoint operator eigenvalue problem in \(L^2(-a, b) \ominus \mathbb{C}^{N^*} \oplus \mathbb{C}^{M^*}\), for suitable \(N^*, M^*\), is given. The Green’s function and the resolvent of the related Hilbert space operator are expressed explicitly.

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1. Introduction

There has been growing interest in spectral problems involving differential operators with discontinuity conditions. We refer to such conditions as transmission conditions (see [6, 11, 18]), although they appear under the guise of many names. These include point interactions in the physics literature, with important examples being the $\delta$ and $\delta'$ interactions from quantum mechanics (see for example [3, 7] and the references therein); interface conditions, [21]; as well as matching conditions on graphs, [20]. For an interesting exposition of transmission condition problems that arise naturally in applications we refer the reader to the book by A. N. Tikhonov and A. A. Samarskii, [15, Chapter II].

Direct and inverse problems for continuous Sturm-Liouville equations with eigenparameter dependent boundary conditions have been studied extensively (see [4, 5, 10, 14, 16] for a sample of the literature). Investigations into Sturm-Liouville equations with discontinuity conditions depending on the spectral parameter have been thus far limited to the affine case (see [2, 12, 17, 19]) and affine dependence of the square root of the eigenparameter, see [13]. In particular, transmission conditions of the form

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} c & 0 \\ h(\lambda) & c^{-1} \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix},$$

where $c \in \mathbb{R}^+$ and $h$ is affine in $\lambda$ were considered in [19], and $c = 1$ and $h(\lambda) = i\alpha \sqrt{\lambda}, \alpha > 0$ in [13]. Recently, the discontinuity condition

$$\begin{bmatrix} y_1(0^+) \\ y_2(0^+) \end{bmatrix} = \begin{bmatrix} c & 0 \\ h(\lambda) & c^{-1} \end{bmatrix} \begin{bmatrix} y_1(0^-) \\ y_2(0^-) \end{bmatrix},$$

with $c \in \mathbb{R}^+$ and $h$ a polynomial in $\lambda$ was considered in [8] for the Dirac operator

$$\left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \frac{dY}{dx} + \begin{bmatrix} p(x) & q(x) \\ q(x) & r(x) \end{bmatrix} Y = \lambda Y, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

with boundary conditions also polynomially dependent on the spectral parameter.

We consider a Sturm-Liouville equation

$$\ell y := -y'' + qy = \lambda y$$

on the intervals $(-a, 0)$ and $(0, b)$ with $y|_{(-a, 0)}$, $y'|_{(-a, 0)}$, $\ell y|_{(-a, 0)} \in L^2(-a, 0)$ and $y|_{(0,b)}$, $y'|_{(0,b)}$, $\ell y|_{(0,b)} \in L^2(0,b)$, where $a, b > 0$ and $q \in L^2(-a, b)$. We impose boundary conditions

$$y(-a) \cos \alpha = y'(-a) \sin \alpha,$$

$$y(b) \cos \beta = y'(b) \sin \beta,$$

where $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$, and transmission conditions

$$y(0^+) \mu(\lambda) = \Delta' y,$$

$$y'(0^-) \nu(\lambda) = \Delta y.$$
Here
\[ \Delta y = y_+(0) - y_-(0), \]
\[ \Delta' y = y'_+(0) - y'_-(0), \]
\[ \mu(\lambda) = -\left( \lambda \eta - \xi - \sum_{i=1}^{N} \frac{b_i^2}{\lambda - c_i} \right), \]
\[ \nu(\lambda) = \lambda \kappa + \zeta - \sum_{j=1}^{M} \frac{d_j^2}{\lambda - d_j}, \]
with \( \eta \geq 0, \kappa \geq 0, \)
\[ c_1 < c_2 < \cdots < c_N, \]
\[ d_1 < d_2 < \cdots < d_M, \]
and \( b_i, a_j > 0 \) for \( i = 1, \ldots, N, \) and \( j = 1, \ldots, M. \) We consider \( N, M \in \mathbb{N}_0, \)
with \( b_0 = a_0 = 0. \)

Further, we will write
\[ \frac{1}{\mu(\lambda)} = \sigma - \sum_{i=1}^{N'} \frac{\beta_i^2}{\lambda - \gamma_i}, \]
if \( \eta > 0, \)
\[ \frac{1}{\nu(\lambda)} = \tau + \sum_{j=1}^{M'} \frac{\alpha_j^2}{\lambda - \delta_j}, \]
if \( \kappa > 0. \)

Here \( N', M' \in \mathbb{N} \) and
\[ \gamma_1 < \gamma_2 < \cdots < \gamma_{N'}, \]
\[ \delta_1 < \delta_2 < \cdots < \delta_{M'}, \]
with \( \beta_i, \alpha_j > 0 \) for \( i = 1, \ldots, N', \) and \( j = 1, \ldots, M'. \)

To the best of our knowledge, this is the first time that spectral properties of \( (1.1) \) with boundary conditions \( (1.2)-(1.3) \) and transmission conditions dependent on the eigenparameter via general rational Nevanlinna-Herglotz functions (see \( (1.4)-(1.5) \)) have been studied.

Note that for \( \lambda \) a pole of \( \mu(\lambda) \) we have that condition \( (1.4) \) becomes \( y(0^+) = 0 \) which results in \( (1.5) \) becoming \( y'(0^-)\nu(\lambda) = -y(0^-), \) resulting in two separate eigenvalue problems on the intervals \( (-a, 0) \) and \( (0, b); \) if \( \lambda \) is a zero of \( \mu(\lambda) \) then \( (1.4) \) becomes \( \Delta'y = 0, \) i.e. \( y'(0^+) = y'(0^-). \) Similarly if \( \lambda \) is a pole of \( \nu(\lambda) \) then \( (1.5) \) at \( \lambda \) becomes \( y'(0^-) = 0 \) and \( (1.4) \) can be expressed as \( y(0^+)\mu(\lambda) = y'(0^+), \) again resulting in separate eigenvalue problems on the intervals \( (-a, 0) \) and \( (0, b); \) while if \( \lambda \) is a zero of \( \nu(\lambda) \) then \( (1.5) \) becomes \( \Delta y = 0, \) i.e. \( y(0^-) = y(0^+). \)

In Section 2 the geometric multiplicity of the eigenvalues of \( (1.1), (1.2)-(1.3) \) with \( (1.4)-(1.5) \) is considered and the cases in which the multiplicity can be 2 are characterized. An example is given to illustrate the cases. The Green’s function of \( (1.1), (1.2)-(1.3) \) with \( (1.4)-(1.5) \) is given in Section 3. A
Hilbert space formulation of (1.1), (1.2)-(1.3) with (1.4)-(1.5) as a operator eigenvalue problem in $L^2(-a,b) \oplus \mathbb{C}^N^* \oplus \mathbb{C}^M^*$, for suitable $N^*$ and $M^*$, is given in Section 4. In Section 5 this Hilbert space operator is shown to be self-adjoint. The related Hilbert space resolvent operator is constructed in Section 6.

2. Geometric multiplicity

**Lemma 2.1.** All eigenvalues of (1.1)-(1.5) not at poles of $\mu(\lambda)$ or $\nu(\lambda)$ are geometrically simple. In this case the transmission conditions (1.4)-(1.5) can be expressed as

$$
\begin{bmatrix}
y(0^+) \\
y'(0^+)
\end{bmatrix} = T
\begin{bmatrix}
y(0^-) \\
y'(0^-)
\end{bmatrix}, \quad \text{where } T =
\begin{bmatrix}
\frac{\nu(\lambda)}{\mu(\lambda)} & 1 + \mu(\lambda)\nu(\lambda) \\
1 & \mu(\lambda)
\end{bmatrix}.
$$

**Proof.** As $T$ is invertible, imposing (1.2) restricts the solution space of (1.1) to one dimension. □

**Theorem 2.2.** The maximum geometric multiplicity of an eigenvalue of (1.1)-(1.5) is 2 and such eigenvalues can only occur at poles of $\mu(\lambda)$ or $\nu(\lambda)$. An eigenvalue $\lambda$ has geometric multiplicity 2 if and only if:

I. $\lambda$ is a pole of $\mu$ and an eigenvalue of (1.1) on $(-a,0)$ with boundary conditions (1.2) and $\nu(\lambda)y'(0^-) + y(0^-) = 0$, and an eigenvalue of (1.1) on $(0,b)$ with boundary conditions $y(0^+) = 0$ and (1.3); or

II. $\lambda$ is a pole of $\nu$ and eigenvalue of (1.1) on $(-a,0)$ with boundary conditions (1.2) and $y'(0^-) = 0$, and an eigenvalue of (1.1) on $(0,b)$ with boundary conditions $y(0^+) = 0$ and (1.3); or

III. $\lambda$ is a pole of both $\mu$ and $\nu$ and an eigenvalue of (1.1) on $(-a,0)$ with boundary conditions (1.2) and $y'(0^-) = 0$, and an eigenvalue of (1.1) on $(0,b)$ with boundary conditions $y(0^+) = 0$ and (1.3).

**Proof.** The conclusion that these are only instances in which non-simple eigenvalues are possible follows from Lemma 2.1. That the maximum geometric multiplicity is 2 in the given circumstances follows from the maximum geometric multiplicity of the resulting eigenvalue problems on the intervals $(-a,0)$ and on $(0,b)$ being 1. If $\lambda$ is simultaneously an eigenvalue of the problems on the intervals $(-a,0)$ and on $(0,b)$ with say eigenfunctions $u$ on $(-a,0)$ and $v$ on $(0,b)$, then extending $u$ and $v$ by zero to $(-a,0) \cup (0,b)$ gives two linearly independent eigenfunctions for (1.1)-(1.5).

**Note** Let

$$\phi(\lambda) = \frac{\prod_{j=1}^{m}(s_j - \lambda)}{\prod_{k=1}^{m}(r_k - \lambda)}$$
where \( r_1 < s_1 < r_2 < s_2 < \cdots < s_{m-1} < r_m < s_m \) then 

\[
\varphi(\lambda) = 1 - \sum_{k=1}^{m} \frac{K_k}{\lambda - r_k}
\]

with 

\[
K_i = \frac{\prod_{j=1}^{m} (s_j - r_i)}{\prod_{k \neq i} (r_k - r_i)} > 0.
\]

If instead \( s_1 < r_1 < s_2 < \cdots < r_{m-1} < s_m < r_m \) then \( K_i < 0 \) for all \( i = 1, \ldots, m \).

**Theorem 2.3.** For any \( N, M \in \mathbb{N}_0 \) there are potentials \( q \in L^2(-\pi, \pi) \) and parameters \( c_1 < c_2 < \cdots < c_N, d_1 < d_2 < \cdots < d_M, \eta, \kappa \geq 0 \) and \( b_i, a_j > 0 \) for \( i = 1, \ldots, N \), and \( j = 1, \ldots, M \) such that (1.1)-(1.5) with \( a = b = \pi \) has precisely \( N + M \) double eigenvalues (the maximum number possible).

**Proof.** Assume that \( N \leq M \). We take boundary conditions \( y(\pm \pi) = 0 \) and set \( q(x) = 0 \) for \( x \in [0, \pi] \). Now \( 1^2/4, 3^2/4, \ldots, (2N-1)^2/4 \) are eigenvalues of (1.1) on \([0, \pi]\) with boundary conditions \( y(\pi) = 0 = y'(0^+) \), while \( 1^2, 2^2, \ldots, N^2 \) are eigenvalues of (1.1) on \([0, \pi]\) with boundary conditions \( y(\pi) = 0 = y(0^+) \). In particular \( \lambda = 1^2, 2^2, \ldots, N^2 \) are eigenvalues of (1.1) on \([0, \pi]\) with boundary conditions \( y(\pi) = 0 \) and \( \mu(\lambda)y(0^+) = y'(0^+) \) where 

\[
\mu(\lambda) = \prod_{k=1}^{N} \left( \frac{(k - \frac{1}{2})^2 - \lambda}{i^2 - \lambda} \right) = 1 + \sum_{i=1}^{N} \frac{b_i^2}{(\lambda - i^2)}
\]

where \( b_i > 0, i = 1, \ldots, N \).

Let \( d_j = (j-1/2)^2 \) for \( j = 1, \ldots, N \) and \( d_{N+1} < \cdots < d_M \) be eigenvalues of (1.1) on \([0, \pi]\) with boundary conditions \( y(\pi) = 0 \) and \( \mu(\lambda)y(0^+) = y'(0^+) \) with \( d_{N+1} > N^2 \). Define \( \delta_j = j^2 \), for \( j = 1, \ldots, N \), and \( \delta_j = (d_{j+1} + d_j)/2 \), \( j = N + 1, \ldots, M - 1 \) and \( \delta_M > d_M \). Let 

\[
\nu(\lambda) = \frac{\prod_{j=1}^{M} (\delta_j - \lambda)}{\prod_{k=1}^{M} (d_k - \lambda)} = 1 - \sum_{j=1}^{M} \frac{a_j^2}{(\lambda - d_j)}
\]

where \( a_j > 0, j = 1, \ldots, M \).

We now take \( q \) on \([-\pi, 0)\) to be an \( L^2 \) potential so that the eigenvalues of (1.1) on \([-\pi, 0)\) with boundary condition \( y(-\pi) = 0 \) and \( y(0^-) = 0 \) contains the set \( \{\delta_1, \ldots, \delta_M\} \) while the eigenvalues of (1.1) on \([-\pi, 0)\) with boundary condition \( y(-\pi) = 0 \) and \( y'(0^-) = 0 \) contains the set \( \{d_1, \ldots, d_M\} \). This is
possible via the Gelfand-Levitan theory of inverse spectral problems, see for example Marcenko \[9, \text{Theorem 3.4.3}\], since \(d_1 < \delta_1 < d_2 < \delta_2 < \cdots < \delta_{M-1} < d_M < \delta_M\).

Now \(d_1, \ldots, d_M\) are poles of \(\nu\), so for \(\lambda = d_i, i = 1, \ldots, M\), the transmission conditions become \(y'(0^-) = 0\) and \(\mu(\lambda)y(0^+) = y'(0^+)\). By construction of \(q\), we have that \(d_1, \ldots, d_M\) are eigenvalues of the Sturm-Liouville problem on \([-\pi, 0]\) with boundary conditions \(y(-\pi) = 0 = y(0^-)\). Also by construction of \(d_1, \ldots, d_M\) they are eigenvalues of the Sturm-Liouville problem on \([0, \pi]\) with boundary conditions \(\mu(\lambda)y(0^+) = y'(0^+), y(\pi) = 0\). Hence \(d_1, \ldots, d_M\) are double eigenvalues of the Sturm-Liouville problem on \([-\pi, \pi]\) with transmission condition at 0 and boundary conditions \(y(-\pi) = 0 = y(\pi)\).

Also \(\lambda = \delta_1, \ldots, \delta_N\) are poles of \(\mu\) so that at these values of \(\lambda\) the transmission conditions become \(y(0^+) = 0, \nu(\lambda)y'(0^-) = -y(0^-)\), but by construction \(\delta_1, \ldots, \delta_N\) are zeros of \(\nu\). So the transmission conditions become \(y(0^+) = 0 = y(0^-)\). By construction of \(q, \delta_1, \ldots, \delta_N\), are eigenvalues of the Sturm-Liouville problem on \([-\pi, 0]\) with boundary conditions \(y(-\pi) = 0 = y(0^-)\). However by choice of \(\delta_1, \ldots, \delta_N\), they are eigenvalues of the Sturm-Liouville problem on \([0, \pi]\) with boundary conditions \(y(0^+) = 0 = y(\pi)\).

Hence the problem has at least \(M + N\) double eigenvalues, which we know to be the maximum possible.

We note that using similar methods to those of the above proof, it can be shown that any number of eigenvalues between 0 and \(M + N\) can be constructed to be double. Due to notational opacity we will only present a proof of the other extreme case, that of no double eigenvalues.

**Theorem 2.4.** For any \(N, M \in \mathbb{N}_0\) and potentials \(q = 0\) there are parameters \(c_1 < c_2 < \cdots < c_N, d_1 < d_2 < \cdots < d_M, \eta, \kappa \geq 0\) and \(b_i, a_j > 0\) for \(i = 1, \ldots, N, \) and \(j = 1, \ldots, M\) such that (1.7)-(1.9) with \(a = b = \pi, \alpha = 0, \beta = \pi\), has no double eigenvalues.

**Proof.** Let

\[
\mu(\lambda) = \frac{\prod_{j=1}^{N} (j^2 - \lambda)}{\prod_{k=1}^{N} \left( \left( k + \frac{1}{2} \right)^2 - \lambda \right)}
\]

and

\[
\nu(\lambda) = \frac{\prod_{j=1}^{M} \left( \left( \frac{j + 1}{2} \right)^2 - \lambda \right)}{\prod_{k=1}^{M} (k^2 - \lambda)}.
\]

The poles of \(\mu\) are \(3^2/4, 5^2/4, \ldots, (2N + 1)^2/4\) which are not eigenvalues of the Sturm-Liouville problem on \([0, \pi]\) with boundary conditions \(y(0^+) = 0 = y(\pi)\). The poles of \(\nu\) are at \(1^2, \ldots, M^2\) which are not eigenvalues of the
Theorem 3.1. For \( \psi \) not an eigenvalue of (1.1)-(1.5), the Green’s function of (1.1)-(1.5) is given by

\[
G(x,t;\lambda) = \begin{cases} 
\frac{u(x;\lambda)v(t;\lambda)}{\psi(\lambda)}, & \text{if } x < t \text{ and } x, t \in [-a,0) \cup (0,b], \\
\frac{u(t;\lambda)v(x;\lambda)}{\psi(\lambda)}, & \text{if } t < x \text{ and } x, t \in [-a,0) \cup (0,b], 
\end{cases}
\]  

in the sense that if \( h \in L^2(-a,b) \) then

\[
g(x;\lambda) = \int_{-a}^{b} G(x,t;\lambda)h(t)dt := \mathcal{G}_\lambda h
\]

is a solution of \( (\lambda - \ell)g = h \) on \( (-a,0) \) and \( (0,b) \), such that \( g \) obeys the boundary conditions (1.2)-(1.3) and the transmission conditions (1.4)-(1.5).
Theorem 3.2. For \( u \) so (1.4) and (1.5) are obeyed as these conditions are obeyed by \( v \) (1.5), the Green’s function of (1.1)-(1.5) is given by

\[
g(x) = u(x) \int_{x}^{b} v(t) h(t) \, dt + v(x) \int_{-a}^{x} u(t) h(t) \, dt,
\]

where for brevity we have suppressed the argument \( \lambda \). Differentiating \( g \) gives

\[
g'(x) = u'(x) \int_{x}^{b} v(t) h(t) \, dt + v'(x) \int_{-a}^{x} u(t) h(t) \, dt,
\]

and a further differentiation gives

\[
g''(x) = u''(x) \int_{x}^{b} v(t) h(t) \, dt + v''(x) \int_{-a}^{x} u(t) h(t) \, dt + h(x) \psi
\]

so (1.6) \( g = h \). Further from (3.7) and (3.8)

\[
\begin{pmatrix}
g(x) \\ g'(x)
\end{pmatrix}
\begin{pmatrix}
\psi \\
u(x)
\end{pmatrix}
= 
\begin{pmatrix}
u(x) \\ v'(x)
\end{pmatrix}
\int_{-a}^{x} u(t) h(t) \, dt,
\]

from which it follows that

\[
\begin{pmatrix}
\psi \\ u(-a)
\end{pmatrix}
= 
\begin{pmatrix}
u(-a) \\ u'(-a)
\end{pmatrix}
\int_{-a}^{b} v(t) h(t) \, dt,
\]

so (1.2) is obeyed as this condition is obeyed by \( u \), and

\[
\begin{pmatrix}
\psi \\ u'(b)
\end{pmatrix}
= 
\begin{pmatrix}
u(b) \\ v'(b)
\end{pmatrix}
\int_{-a}^{b} u(t) h(t) \, dt,
\]

so (1.3) is obeyed as this condition is obeyed by \( v \). Moreover,

\[
\begin{pmatrix}
\psi \\ u(0^\pm)
\end{pmatrix}
= 
\begin{pmatrix}
u(0^\pm) \\ v'(0^\pm)
\end{pmatrix}
\int_{0}^{b} v(t) h(t) \, dt + \begin{pmatrix}
v(0^\pm) \\ v'(0^\pm)
\end{pmatrix}
\int_{-a}^{0} u(t) h(t) \, dt,
\]

so (1.4) and (1.5) are obeyed as these conditions are obeyed by \( u \) and \( v \).

Proof. From the above definition of \( G \) and \( g \), we have

\[
g(x) = u(x) \int_{x}^{b} v(t) h(t) \, dt + v(x) \int_{-a}^{x} u(t) h(t) \, dt
\]

(3.7)

and for brevity we have suppressed the argument \( \lambda \). Differentiating \( g \) gives

\[
g'(x) = u'(x) \int_{x}^{b} v(t) h(t) \, dt + v'(x) \int_{-a}^{x} u(t) h(t) \, dt,
\]

(3.8)

and a further differentiation gives

\[
g''(x) = u''(x) \int_{x}^{b} v(t) h(t) \, dt + v''(x) \int_{-a}^{x} u(t) h(t) \, dt + h(x) \psi
\]

(3.9)

so (1.6) \( g = h \). Further from (3.7) and (3.8)

\[
\begin{pmatrix}
g(x) \\ g'(x)
\end{pmatrix}
\begin{pmatrix}
\psi \\
u(x)
\end{pmatrix}
= 
\begin{pmatrix}
u(x) \\ v'(x)
\end{pmatrix}
\int_{-a}^{x} u(t) h(t) \, dt,
\]

from which it follows that

\[
\begin{pmatrix}
\psi \\ u(-a)
\end{pmatrix}
= 
\begin{pmatrix}
u(-a) \\ u'(-a)
\end{pmatrix}
\int_{-a}^{b} v(t) h(t) \, dt,
\]

so (1.2) is obeyed as this condition is obeyed by \( u \), and

\[
\begin{pmatrix}
\psi \\ u'(b)
\end{pmatrix}
= 
\begin{pmatrix}
u(b) \\ v'(b)
\end{pmatrix}
\int_{-a}^{b} u(t) h(t) \, dt,
\]

so (1.3) is obeyed as this condition is obeyed by \( v \). Moreover,

\[
\begin{pmatrix}
\psi \\ u(0^\pm)
\end{pmatrix}
= 
\begin{pmatrix}
u(0^\pm) \\ v'(0^\pm)
\end{pmatrix}
\int_{0}^{b} v(t) h(t) \, dt + \begin{pmatrix}
v(0^\pm) \\ v'(0^\pm)
\end{pmatrix}
\int_{-a}^{0} u(t) h(t) \, dt,
\]

so (1.4) and (1.5) are obeyed as these conditions are obeyed by \( u \) and \( v \). □

Theorem 3.2. For \( \lambda \) a pole of \( \mu(\lambda) \) or \( \nu(\lambda) \) and not an eigenvalue of (1.1)-(1.3), the Green’s function of (1.1)-(1.3) is given by

\[
G(x, t; \lambda) =
\begin{cases}
  g^-(x, t; \lambda), & \text{if } x, t \in [-a, 0), \\
  g^+(x, t; \lambda), & \text{if } x, t \in (0, b], \\
  0, & \text{if } (x, t) \in (0, b) \times [-a, 0) \\
  & \text{or } (x, t) \in [-a, 0) \times (0, b],
\end{cases}
\]

(3.11)

in the sense that if \( h \in L^2(-a, b) \) then

\[
g(x; \lambda) = \int_{-a}^{b} G(x, t; \lambda) h(t) \, dt := \mathcal{G}_\lambda h
\]

(3.12)

is a solution of \( (\lambda - \ell) g = h \) on \((-a, 0)\) and \((0, b)\) such that \( g \) obeys the boundary conditions (1.2)-(1.3) and the transmission conditions (1.4)-(1.5).
Here $g^-(x,t;\lambda)$ is the Green’s function for the Sturm-Liouville equation (1.1) on $[-a,0]$ with boundary conditions (1.2) and

$$g'(0^-)\nu(\lambda) = -g(0^-), \quad \text{if } \lambda \text{ is a pole of } \mu \text{ but not of } \nu,$$

(3.13)

$$g'(0^-) = 0, \quad \text{if } \lambda \text{ is a pole of } \mu \text{ and of } \nu,$$

(3.14)

and $g^+(x,t;\lambda)$ is the Green’s function for the Sturm-Liouville equation (1.1) on $[0,b]$ with boundary conditions (1.3) and

$$g(0^+)\mu(\lambda) = g'(0^+), \quad \text{if } \lambda \text{ is a pole of } \nu \text{ but not of } \mu,$$

(3.15)

$$g(0^+) = 0, \quad \text{if } \lambda \text{ is a pole of } \mu \text{ and of } \nu,$$

(3.16)

Proof. If $\lambda$ is not an eigenvalue of either (1.1) on $[-a,0]$ with (1.2) and (3.13)-(3.14), or (1.1) on $[0,b]$ with (1.3) and (3.16)-(3.15), then $g(x;\lambda)$ obeys (1.2)-(1.3) and (1.4)-(1.5), and $(\lambda - \ell)g = h$ a.e. on $(-a,0) \cup (0,b)$. It thus remains only to show that if $\lambda$ is not an eigenvalue of (1.1)-(1.5) then $\lambda$ is not an eigenvalue of either (1.1) on $[-a,0]$, (1.2) and (3.13)-(3.14) or (1.1) on $[0,b]$, (1.3) and (3.16)-(3.15).

If $\lambda$ is an eigenvalue of (1.1) on $[-a,0]$, (1.2) and (3.13)-(3.14), then let $w$ denote an eigenfunction for this eigenvalue. This eigenfunction, $w$, can then be extended to $(-a,0) \cup (0,b)$ by setting to 0 on $(0,b)$. It is now apparent that $w$ is an eigenfunction of (1.1)-(1.5), making $\lambda$ an eigenvalue of (1.1)-(1.5).

Similarly, if $\lambda$ is an eigenvalue of (1.1) on $[0,b]$, (1.3) and (3.16)-(3.15) with eigenfunction say $w$, then $w$, can then be extended to $(-a,0) \cup (0,b)$ by setting to 0 on $(-a,0)$. It is now apparent that $w$ is an eigenfunction of (1.1)-(1.5) making $\lambda$ an eigenvalue of (1.1)-(1.5). \qed

4. Hilbert space formulation

We now formulate (1.1) with boundary conditions (1.2)-(1.3) and transmission conditions (1.4)-(1.5) as an operator eigenvalue problem in a Hilbert space $\mathcal{H}$. For $\eta, \kappa > 0$ we set $\mathcal{H} = L^2(-a,b) \bigoplus \mathbb{C}^N \bigoplus \mathbb{C}^{M'}$ and

$$LY := \begin{bmatrix} \ell & 0 & 0 \\ \beta \Delta' & [\gamma_i] & 0 \\ \alpha \Delta & 0 & [\delta_j] \end{bmatrix} \begin{bmatrix} y \\ y^1 \\ y^2 \end{bmatrix}$$

(4.1)

with domain

$$\mathcal{D}(L) = \left\{ Y \begin{bmatrix} y_1 \\ y^1 \\ y_2 \\ y^2 \end{bmatrix}, \begin{bmatrix} [\gamma_i] := \text{diag}(\gamma_1, \ldots, \gamma_N) \end{bmatrix}, \beta := (\beta_i), [\delta_j] := \text{diag}(\delta_1, \ldots, \delta_{M'}) \begin{bmatrix} \gamma_i \end{bmatrix} := \text{diag}(\gamma_1, \ldots, \gamma_N), \beta := (\beta_i), y^1 := (y^1_i), [\delta_j] := \text{diag}(\delta_1, \ldots, \delta_{M'}) \right\},$$

where $Y := \begin{bmatrix} y \\ y^1 \\ y^2 \end{bmatrix}$, $[\gamma_i] := \text{diag}(\gamma_1, \ldots, \gamma_N)$, $\beta := (\beta_i)$, $y^1 := (y^1_i)$, $[\delta_j] := \text{diag}(\delta_1, \ldots, \delta_{M'})$, $\alpha := (\alpha_j)$ and $y^2 := (y^2_j)$. For later reference, we take the
inner product on $\mathcal{H}$ as
\[
\langle Y, Z \rangle := (y, z) + \langle y^1, z^1 \rangle + \langle y^2, z^2 \rangle,
\]
where $(y, z) = \int_a^b y z dx$ and $\langle \cdot, \cdot \rangle$ denotes Euclidean inner product.

Replacement of $\beta, \gamma, \sigma, y(0^+), \Delta' y$ and $N'$ by $b, c, -\xi, -\Delta y, y(0^+)$ and $N$ in the specification of $\mathcal{H}, L$ and $\mathcal{D}(L)$ above and the results of this section, below, yields the case of $\eta = 0$, while replacement of $\alpha, \delta, \tau, y(0^-), \Delta y$ and $M'$ by $a, d, -\zeta, -\Delta y, y(0^-)$ and $M$ yields the case of $\kappa = 0$.

**Theorem 4.1.** The eigenvalue problems $LY = \lambda Y$ and $(1.1)$ with boundary conditions $(1.2)$-$(1.3)$ and transmission conditions $(1.4)$-$(1.5)$ are equivalent in the sense that $\lambda$ is an eigenvalue of $LY = \lambda Y$ with eigenvector $Y$ if and only if $\lambda$ is an eigenvalue, with eigenfunction $y$, of $(1.1)$ with boundary conditions $(1.2)$-$(1.3)$ and transmission conditions $(1.4)$-$(1.5)$. Here, for $\eta, \kappa > 0$,
\[
y^1 = (\lambda I - [\gamma i])^{-1} \beta \Delta' y \tag{4.2}
\]
if $\lambda \neq \gamma_i$ for all $i = 1, \ldots, N'$ while if $\lambda = \gamma_i$ for some $I \in \{1, \ldots, N'\}$ then
\[
y^1 = -\frac{y(0^+)}{\beta_i} e^I, \tag{4.3}
\]
with $e^I$ the vector in $\mathbb{R}^{N'}$ with all entries 0 except the $I^{th}$ which is 1, and
\[
y^2 = (\lambda I - [\delta j])^{-1} \alpha \Delta y, \tag{4.4}
\]
if $\lambda \neq \delta_j$ for all $j = 1, \ldots, M'$, while if $\lambda = \delta_j$ for some $J \in \{1, \ldots, M'\}$ then
\[
y^2 = \frac{y'(0^-)}{\alpha_j} e^J, \tag{4.5}
\]
with $e^J$ the vector in $\mathbb{R}^{M'}$ with all entries 0 except the $J^{th}$ which is 1. The geometric multiplicity of $\lambda$ as an eigenvalue of $L$ is the same as the geometric multiplicity of $\lambda$ as an eigenvalue of $(1.4)$-$(1.5)$.

**Proof.** If $LY = \lambda Y$, then $\ell y = \lambda y$, where $y|_{(-a,0)}$, $y'|_{(-a,0)}$, $\ell y|_{(-a,0)} \in L^2(-a,0)$ and $y|_{(0,b)}$, $y'|_{(0,b)}$, $\ell y|_{(0,b)} \in L^2(0,b)$. Moreover $y$ obeys $(1.2)$ and $(1.3)$. By definition of $LY$, $\gamma_i y_i^1 + \beta_i \Delta' y = \lambda y_i^1$ for all $i$, the domain conditions give $-y(0^+) + \sigma \Delta' y - \langle y^1, \beta \rangle = 0$. Hence if $\lambda \neq \gamma_i$ for all $i$, then $y_i^1 = \frac{\beta_i}{\lambda - \gamma_i} \Delta' y$ which with the domain condition gives
\[
y(0^+) = \left[ \sigma - \sum_{i=1}^{N'} \frac{\beta_i^2}{\lambda - \gamma_i} \right] \Delta' y.
\]
If $\lambda = \gamma_i$ then $\gamma_i y_i^1 + \beta_i \Delta' y = \lambda y_i^1$ gives $\Delta' y = 0$ and $y_i^1 = 0$ for all $i \neq I$, which together with the domain condition gives $y_I^1 = -\frac{y(0^+)}{\beta_i}$. Hence $y$ obeys $(1.4)$.

Similarly $\delta_j y_j^2 + \alpha_j \Delta y = \lambda y_j^2$ for all $j$, giving $y_j^2 = \frac{\alpha_j}{\lambda - \delta_j} \Delta y$ if $\lambda \neq \delta_j$ for all $j$, and $\Delta y = 0$ if for some $J, \lambda = \delta_j$ and $y_j^2 = 0$ for all $j \neq J$. The domain
condition \( y'(0^-) - \tau \Delta y - \langle y^2, \alpha \rangle = 0 \) in the case of \( \lambda \neq \delta_j \) for all \( j \) now gives that

\[
y'(0^-) = \left[ \tau + \sum_{j=1}^{M'} \frac{\alpha_j^2}{\lambda - \delta_j} \right] \Delta y, \tag{4.6}
\]

while for \( \lambda = \delta_j \) the domain condition forces \( y_j^2 = \frac{y'(0^-)}{\alpha_j} \) from which \((1.5)\) follows.

Hence, the eigenvalues of \( L \) are eigenvalues of \((1.1)\) with boundary conditions \((1.2)-(1.3)\) and transmission conditions \((1.4)-(1.5)\), with eigenfunction \( y = [Y]_0 \) (i.e. the functional component of \( Y \)).

For the converse, let \( \lambda \) be an eigenvalue, with eigenfunction \( y \), of \((1.1)\) with boundary conditions \((1.2)-(1.3)\) and transmission conditions \((1.4)-(1.5)\). Here \( \ell y = \lambda y \) on \((-a,0) \cup (0,b)\), with \( y|_{(-a,0)}, y'|_{(-a,0)}, \ell y|_{(-a,0)} \in L^2(-a,0) \) and \( y|_{(0,b)}, y'|_{(0,b)}, \ell y|_{(0,b)} \in L^2(0,b) \). Define \( Y \) as given in \((1.2)-(1.5)\).

For \( \lambda \neq \gamma_i \) for all \( i \), from the form given for \( y^1 \),

\[
[LY]_1 = [\beta \Delta', \gamma_i], \quad [0] Y = \beta \Delta'y + [\gamma_i] y^1 = \lambda y^1,
\]

and since \( y \) obeys \((1.4)\),

\[
\langle y^1, \beta \rangle = \sum_{i=1}^{N'} \frac{\beta_i^2}{\lambda - \gamma_i} \Delta'y = -y(0^+) + \sigma \Delta'y,
\]

so the domain condition for \( y^1 \) is obeyed.

For \( \lambda = \gamma_I \), for some \( I \in \{1, \ldots, N'\} \), we have \( \Delta'y = 0 \) so

\[
[LY]_1 = [\beta \Delta', \gamma_i], \quad [0] Y = [\gamma_i] y^1 = \lambda y^1,
\]

since \( y_i^1 = 0 \) for \( i \neq I \). Also, since \( y_i^1 = -y(0^+)/\beta_I \),

\[
\langle y^1, \beta \rangle = -y(0^+) = -y(0^+) + \sigma \Delta'y,
\]

and the domain condition relating to \( y^1 \) is obeyed.

If \( \lambda \neq \delta_j \) for all \( j = 1, \ldots, M' \), from \((4.4)\),

\[
[LY]_2 = [\alpha \Delta, 0, [\delta_j]], Y = \alpha \Delta y + [\delta_j] y^2 = \lambda y^2,
\]

while \((1.5), (1.11)\) and \((4.4)\) combined give that the domain condition

\[
y'(0^-) - \tau \Delta y - \langle y^2, \alpha \rangle = y'(0^-) - \tau \Delta y - \sum_{j=1}^{M'} \frac{\alpha_j^2}{\lambda - \delta_j} \Delta y = 0
\]

is satisfied.

For \( \lambda = \delta_j \), for some \( J \in \{1, \ldots, M'\} \), from \((1.5)\) and \((1.11)\) we have \( \Delta y = 0 \) which, together with \((4.5)\), gives

\[
[LY]_2 = [\alpha \Delta, 0, [\delta_j]], Y = [\delta_j] y^2 = \lambda y^2,
\]

while \( \Delta y = 0 \) and \((4.5)\) give that the domain condition

\[
y'(0^-) - \tau \Delta y - \langle y^2, \alpha \rangle = 0
\]

is satisfied.
Next we consider the correspondence of geometric multiplicities. If \( \lambda \) is an eigenvalue of (1.1)-(1.5) with eigenfunctions \( y^{[1]}, \ldots, y^{[k]} \) which are linearly independent then the vectors \( Y^{[1]}, \ldots, Y^{[k]} \) as given by (4.2)-(4.5) are linearly independent eigenvectors of \( L \) for the eigenvalue \( \lambda \). Hence the geometric multiplicity of \( \lambda \) as an eigenvalue of \( L \) is at least as large as the geometric multiplicity of \( \lambda \) as an eigenvalue (1.1)-(1.5).

If \( Y^{[1]}, \ldots, Y^{[k]} \) are linearly independent eigenvectors of \( L \) for the eigenvalue \( \lambda \) then, from the first part of this theorem, the functional components \( y^{[1]} = [Y^{[1]}]_0, \ldots, y^{[k]} = [Y^{[k]}]_0 \) are eigenvectors of (1.1)-(1.5) for the eigenvalue \( \lambda \). It remains only to prove their linear independence. If there are \( \rho_1, \ldots, \rho_k \), not all zero, with

\[
0 = \sum_{n=1}^{k} \rho_n y^{[n]},
\]

then from (4.2)-(4.5)

\[
0 = \sum_{n=1}^{k} \rho_n Y^{[n]},
\]

contradicting the linear independence of \( Y^{[1]}, \ldots, Y^{[k]} \). Hence \( y^{[1]}, \ldots, y^{[k]} \) are linearly independent and the geometric multiplicity of \( \lambda \) as an eigenvalue of \( L \) coincides with its geometric multiplicity as an eigenvalue of (1.1)-(1.5). □

5. Self-adjointness

In this section we show that \( L \) is a self-adjoint (densely defined) operator in \( \mathcal{H} \).

**Theorem 5.1.** The operator \( L \) is self-adjoint in \( \mathcal{H} \).

**Proof.** We present the proof for the case of \( \eta, \kappa > 0 \), the proofs for the other cases being similar.

We begin by showing that \( D(L) \) is dense in \( \mathcal{H} \). Let \( A, B, C, D \in \mathbb{R} \),

\[
F = \begin{bmatrix} f \\ f^1 \\ f^2 \end{bmatrix} \in \mathcal{H} \quad \text{and} \quad W = \begin{bmatrix} w \\ f^1 \\ f^2 \end{bmatrix}
\]

(5.1)

where \( w \) is \( C^\infty \) on \([-a, 0) \) and \((0, b] \) with \( w(-a) = w'(a) = 0 = w(b) = w'(b) \) so that \( w|_{[-a,0]} \) has an extension to a function \( C^\infty \) on \([-a, 0] \), \( w|_{(0,b]} \) has an extension to a function \( C^\infty \) on \([0, b] \) and

\[
w(0^-) = (\sigma C - A - 1) \langle f^1, \beta \rangle + (\sigma D - B) \langle f^2, \alpha \rangle,
\]

\[
w(0^+) = (\sigma C - 1) \langle f^1, \beta \rangle + \sigma D \langle f^2, \alpha \rangle,
\]

\[
w'(0^-) = \tau A \langle f^1, \beta \rangle + (\tau B + 1) \langle f^2, \alpha \rangle,
\]

\[
w'(0^+) = (\tau A + C) \langle f^1, \beta \rangle + (\tau B + D + 1) \langle f^2, \alpha \rangle,
\]

where \( \sigma, \tau \) are constants to be chosen.
then $W \in \mathcal{D}(L)$ and
\begin{align}
\Delta w &= A \langle f^1, \beta \rangle + B \langle f^2, \alpha \rangle, \\
\Delta' w &= C \langle f^1, \beta \rangle + D \langle f^2, \alpha \rangle.
\end{align}

As $q \in L^2(-a, b)$ it follows that $(C_0^\infty(-a, 0) \bigoplus C_0^\infty(0, b)) \bigoplus \{0\} \bigoplus \{0\} \subset \mathcal{D}(L)$. Here, $C_0^\infty(-a, 0) \bigoplus C_0^\infty(0, b)$ is dense in $L^2(-a, b)$ so there is a sequence $\{g_n\} \subset C_0^\infty(-a, 0) \bigoplus C_0^\infty(0, b)$ with $g_n \to f - w$ in norm. Here, $G_n := [g_n \ 0]^T \in \mathcal{D}(L)$ and thus $W + G_n \in \mathcal{D}(L)$. Now, $W + G_n \to F$ in norm as $n \to \infty$ giving that $\mathcal{D}(L)$ is dense in $\mathcal{H}$.

We now show that $L$ is symmetric. Let $F, G \in \mathcal{D}(L)$, then the functional components $f$ and $g$ of $F$ and $G$ respectively obey
\begin{align}
\langle \ell f, g \rangle - \langle f, \ell g \rangle &= (-f' \bar{g} + f \bar{g}'(0^-)) + (f' \bar{g} - f \bar{g}'(0^+)).
\end{align}

Moreover, the vector components satisfy
\begin{align}
\langle \beta \Delta' f + [\gamma_i]f^1, g^1 \rangle - \langle f^1, \beta \Delta' g + [\gamma_i]g^1 \rangle &= \langle \beta \Delta' f, g^1 \rangle - \langle f^1, \beta \Delta' g \rangle, \\
\langle \alpha \Delta f + [\delta_j]f^2, g^2 \rangle - \langle f^2, \alpha \Delta g + [\delta_j]g^2 \rangle &= \langle \alpha \Delta f, g^2 \rangle - \langle f^2, \alpha \Delta g \rangle,
\end{align}
where the domain conditions give
\begin{align}
\langle \beta \Delta' f, g^1 \rangle - \langle f^1, \beta \Delta' g \rangle &= \Delta'[f(-\bar{g}(0^+) + \sigma \Delta' \bar{g}) - \Delta \bar{g}[f(0^+) + \sigma \Delta f]], \\
\langle \alpha \Delta f, g^2 \rangle - \langle f^2, \alpha \Delta g \rangle &= \Delta f[\bar{g}'(0^-) - \tau \Delta \bar{g}] - \bar{g}[f'(0^-) - \tau \Delta f].
\end{align}

Hence
\begin{align}
\langle \beta \Delta' f + [\gamma_i]f^1, g^1 \rangle - \langle f^1, \beta \Delta' g + [\gamma_i]g^1 \rangle &= f(0^+) \Delta' \bar{g} - \bar{g}(0^+) \Delta' f, \\
\langle \alpha \Delta f + [\delta_j]f^2, g^2 \rangle - \langle f^2, \alpha \Delta g + [\delta_j]g^2 \rangle &= \bar{g}'(0^-) \Delta f - f'(0^-) \Delta \bar{g}.
\end{align}

Direct computation gives
\begin{align}
(f' \bar{g} - f \bar{g}')'(0^-) - (f' \bar{g} - f \bar{g}')'(0^+) &= f(0^+) \Delta' \bar{g} - \bar{g}(0^+) \Delta' f + \bar{g}'(0^-) \Delta f - f'(0^-) \Delta \bar{g},
\end{align}
thus $\langle LF, G \rangle - \langle F, LG \rangle = 0$ and so $L$ is symmetric, giving $\mathcal{D}(L) \subset \mathcal{D}(L^*)$.

To show that $L$ is self-adjoint it remains only to verify that $\mathcal{D}(L^*) \subset \mathcal{D}(L)$. Let $G \in \mathcal{D}(L^*)$ then $\langle LF, G \rangle = \langle F, L^* G \rangle$ for all $F \in \mathcal{D}(L)$, and the map $F \mapsto \langle F, L^* G \rangle$ defines a continuous linear functional on $\mathcal{H}$. Hence, the map $F \mapsto \langle LF, G \rangle$ is a continuous linear functional on $\mathcal{H}$ restricted to the dense subspace $\mathcal{D}(L)$. In particular, there is $k \geq 0$ so that for all $F \in (C_0^\infty(-a, 0) \bigoplus \{0\}) \bigoplus \{0\} \bigoplus \{0\}$ we have that
\begin{align}
\left| \int_{-a}^0 f'' \left( -\bar{g} + \int_{-a}^x \int_{-a}^t q \bar{g} \ d\tau \ dt \right) \ dx \right| \leq k \|f\|_2,
\end{align}
for all $f \in C_0^\infty(-a, 0)$. Hence, see [1] Chapters 1 & 2,
\begin{align}
g - \int_{-a}^x \int_{-a}^t q \bar{g} \ d\tau \ dt \in H^2(-a, 0).
\end{align}
We note here that \( qg \in L^1(-a,0) \), giving that \( \int_{-a}^x qg \, d\tau \, dt \in L^2(-a,0) \). Hence, \( g \in H^1(-a,0) \) and differentiating (5.5) gives
\[
g' - \int_{-a}^x qg \, d\tau \in H^1(-a,0).
\] (5.6)

Thus \( g'' \) exists as a weak derivative and is in \( L^1(-a,0) \). Applying the above in (5.4) gives
\[
\left| \int_{-a}^0 f (-\overline{\mathcal{g}}'' + q\overline{g}) \, dx \right| \leq k\|f\|_2,
\] (5.7)
and hence \( \ell^*g = \ell g \) exists in \( L^2(-a,0) \). Similarly, we obtain \( g, g', \ell^*g = \ell g \) exists in \( L^2(0,b) \). Thus \( g \in H^2(-a,0) \oplus H^2(0,b) \) with \( \ell^*g = \ell g \in L^2(-a,b) \).

In the light of the above, taking \( F = [f \ 0 \ 0]^T \) and varying \( f \) through \( C^\infty[-a,0) \oplus C^\infty(0,b) \) obeying (1.2), (1.3) and having \( f^{(m)}(\pm 0) = 0 \) for all \( m = 0,1,2,\ldots \), we obtain that \( g \) obeys (1.2) and (1.3).

Now let \( W \) be as in (5.1), then we have that
\[
\begin{align*}
\int_{-a}^b \ell w \overline{g} \, dx &+ \langle \beta \Delta' w + [\gamma_i]f^1, [G]_1 \rangle + \langle \alpha \Delta w + [\delta_j]f^2, [G]_2 \rangle \\
&= \langle LW, G \rangle \\
&= \langle W, L^*G \rangle \\
&= \int_{-a}^b w \ell g \, dx + \langle f^1, [L^*G]_1 \rangle + \langle f^2, [L^*G]_2 \rangle.
\end{align*}
\]

Applying integration by parts to the pair of integrals in the above expression we have
\[
\begin{align*}
(\ell' w)(0^+) - (\ell' w)(0^-) &+ \langle \beta \Delta' w + [\gamma_i]f^1, [G]_1 \rangle + \langle \alpha \Delta w + [\delta_j]f^2, [G]_2 \rangle \\
&= (\ell w')(0^+) - (\ell w')(0^-) + \langle f^1, [L^*G]_1 \rangle + \langle f^2, [L^*G]_2 \rangle.
\end{align*}
\]
Using (5.2), (5.3) and the domain conditions obeyed by \( W \) to simplify the above we obtain
\[
0 = \langle f^1, V_1 - (AS_2 + CS_1)\beta \rangle + \langle f^2, V_2 - (BS_2 + DS_1)\alpha \rangle,
\] (5.8)
where
\[
\begin{align*}
V_1 &= -\Delta' g \beta - [\gamma_i][G]_1 + [L^*G]_1, \\
V_2 &= -\Delta g \alpha - [\delta_j][G]_2 + [L^*G]_2, \\
S_1 &= g(0^+) - \sigma \Delta' g + \langle [G]_1, \beta \rangle, \\
S_2 &= -g'(0^-) + \tau \Delta g + \langle [G]_2, \alpha \rangle.
\end{align*}
\]

Varying \( f^1 \) and \( f^2 \) in (5.8) gives
\[
\begin{align*}
0 &= V_1 - (AS_2 + CS_1)\beta, \\
0 &= V_2 - (BS_2 + DS_1)\alpha.
\end{align*}
\] (5.9) (5.10)

However (5.9) and (5.10) hold for all \( A, B, C, D \in \mathbb{R} \), giving that \( S_1, S_2, V_1, V_2 = 0 \). In particular \( S_1 = 0 = S_2 \) give \( G \in D(L) \). \( \square \)
6. The resolvent operator

The block operator form for $L$ given in (4.1), together with the Green’s function for (1.1), (1.2)-(1.3) with (1.4)-(1.5) given in Theorems 3.1 and 3.2 along with the domain conditions of $L$ lead to the expressions for the resolvent of $L$ given in the following theorems. In the interest of readability the construction of the resolvent will be done in two parts.

**Theorem 6.1.** For $\lambda$ not an eigenvalue of $L$, $h \in L^2(-a,b)$ and $\lambda \neq \gamma_i, \delta_j$ for all $i,j$, if $\eta > 0$ and $\kappa > 0$, we have that

$$
(\lambda - L)^{-1} \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{G}_\lambda h \\ (\lambda I - [\gamma_i])^{-1} \beta \Delta' \mathcal{G}_\lambda h \\ (\lambda I - [\delta_j])^{-1} \alpha \Delta \mathcal{G}_\lambda h \end{bmatrix} =: G_0 h =: \begin{bmatrix} f \\ f^1 \\ f^2 \end{bmatrix}, \quad (6.1)
$$

where for $\lambda = \gamma_i$ we replace $(\lambda I - [\gamma_i])^{-1} \beta \Delta' \mathcal{G}_\lambda h$ by $-\frac{1}{\beta_i} \mathcal{G}_\lambda h(0^+) e^I$, and for $\lambda = \delta_j$ we replace $(\lambda I - [\delta_j])^{-1} \alpha \Delta \mathcal{G}_\lambda h$ by $\frac{1}{\alpha_j} (\mathcal{G}_\lambda h)'(0^-) e^J$. For $\eta = 0$ replace $\beta \Delta' \mathcal{G}_\lambda h, \frac{1}{\alpha_j} (\mathcal{G}_\lambda h)'(0^-)$ and $\gamma_i$ by $b\mathcal{G}_\lambda h(0^+), \frac{1}{\alpha_j} \Delta' \mathcal{G}_\lambda h$ and $e_i$. For $\kappa = 0$, replace $\alpha \Delta \mathcal{G}_\lambda h, \frac{1}{\alpha_j} (\mathcal{G}_\lambda h)'(0^-)$ and $\delta_j$ by $a(\mathcal{G}_\lambda h)'(0^-), -\frac{1}{\alpha_j} \Delta \mathcal{G}_\lambda h$ and $d_j$.\[\]

**Proof.** We present the proof for $\eta, \kappa > 0$, the other cases are similar with the symbol replacements noted above.

We begin by showing that $G_0 h \in D(L)$. From the definition of $\mathcal{G}_\lambda h$ it follows that $f|_{(-a,0)}, f'|_{(-a,0)}, \ell f|_{(-a,0)}, f|_{(0,b)}, f'|_{(0,b)}, \ell f|_{(0,b)} \in L^2(-a,0), f|_{(0,b)} \in L^2(0,b)$, and that $f$ obeys (1.2) and (1.3). Moreover,

$$
f(0^+) = \mathcal{G}_\lambda h(0^+) = \left[ \sigma - \beta T (\lambda - [\gamma_i])^{-1} \beta \right] \Delta' \mathcal{G}_\lambda h = \sigma \Delta' f - \langle f^1, \beta \rangle, \quad (6.2)
$$

if $\lambda \neq \gamma_i$ for all $i$, and

$$
f'(0^-) = (\mathcal{G}_\lambda h)'(0^-) = \left[ \tau + \alpha T (\lambda - [\delta_j])^{-1} \alpha \right] \Delta \mathcal{G}_\lambda h = \tau \Delta f + \langle f^2, \alpha \rangle. \quad (6.3)
$$

if $\lambda \neq \delta_j$ for all $j$. Here (6.2) and (6.3) follow from the definition of the Green’s operator in Theorem 3.1 and Theorem 3.2. For $\lambda = \gamma_i$, it follows from the definition of the Green’s operator that $\Delta' \mathcal{G}_\lambda h = 0$. Thus setting $f^1 = -\frac{1}{\beta_i} \mathcal{G}_\lambda h(0^+) e^I$ we have that (6.2) is replaced by

$$
f(0^+) = - \langle f^1, \beta \rangle = \sigma \Delta' f - \langle f^1, \beta \rangle. \quad (6.4)
$$

For $\lambda = \delta_j$ we have that $\Delta \mathcal{G}_\lambda h = 0$ by definition of $\mathcal{G}_\lambda h$, and setting $f^2 = \frac{1}{\alpha_j} (\mathcal{G}_\lambda h)'(0^-) e^J$ replaces (6.3) by

$$
f'(0^-) = \langle f^2, \alpha \rangle = \tau \Delta f + \langle f^1, \alpha \rangle. \quad (6.5)
$$

Thus $G_0 h \in D(L)$. The formal verification that $(\lambda - L)G_0 h = [h, 0, 0]^T$ is straightforward. \[\]

**Theorem 6.2.** If $\lambda$ is not an eigenvalue of $L$ then we have that

$$
(\lambda - L)^{-1} \begin{bmatrix} 0 \\ h^1 \\ h^2 \end{bmatrix} = \begin{bmatrix} f \\ f^1 \\ f^2 \end{bmatrix}, \quad (6.6)
$$
where

\[ f = A\chi_{[-a,0)}u_- + B\chi_{(0,b]}v_+ . \quad (6.7) \]

Here

\[
\begin{bmatrix} A \\ B \end{bmatrix} = \Lambda \begin{bmatrix} h^1, p \\ h^2, q \end{bmatrix}
\]

with

\[
p = \begin{cases} (\lambda I - [c_i])^{-1}b, & \eta = 0, \\ \mu(\lambda)(\lambda I - [\gamma_i])^{-1}\beta, & \eta > 0, \end{cases}
\]

\[
q = \begin{cases} (\lambda I - [d_j])^{-1}a, & \kappa = 0, \\ \nu(\lambda)(\lambda I - [\delta_j])^{-1}\alpha, & \kappa > 0, \end{cases}
\]

\[ \Lambda = \frac{1}{D} \begin{bmatrix} -v_+(0^+) & -v_+(0^+) + \mu(\lambda)v_+(0^+) \\ -u_-(0^-) - \nu(\lambda)u_-(0^-) & -u_-(0^-) \end{bmatrix} \]

and \( D = u'_-(0^-)v_+(0^+) - (v_+(0^+) - \mu(\lambda)v_+(0^+))(u_-(0^-) + \nu(\lambda)u_-(0^-)) \).

Therefore \( f, f^1 \) and \( f^2 \) are given uniquely. At poles, \( f \) can be determined by residue calculations. Also,

\[
-\beta \Delta' f + (\lambda I - [\gamma_i])f^1 = h^1, \quad \eta > 0, \quad (6.11)
\]

\[
-bf(0^+) + (\lambda I - [c_i])f^1 = h^1, \quad \eta = 0, \quad (6.12)
\]

\[
-\alpha \Delta f + (\lambda I - [\delta_j])f^2 = h^2, \quad \kappa > 0, \quad (6.13)
\]

\[
-af'(0^-) + (\lambda I - [d_j])f^2 = h^2, \quad \kappa = 0. \quad (6.14)
\]

**Proof.** The general solution of \((\lambda - \ell)f = 0\) on \([-a, 0) \cup (0, b]\) obeying boundary conditions (1.2) and (1.3) is given by (6.7), where \( u_-, v_+ \) are as defined at the beginning of Section 3 and \( u_-, v_+ \neq 0 \) obey (6.1) and (6.2) respectively. By (6.6), the operator conditions (6.11)-(6.14) follow.

From the domain of the operator \( L \) we obtain the domain conditions

\[
-f(0^+) + \sigma \Delta' f - \langle f^1, \beta \rangle = 0, \quad \eta > 0, \quad (6.15)
\]

\[
\Delta' f - \xi f(0^+) - \langle f^1, b \rangle = 0, \quad \eta = 0, \quad (6.16)
\]

\[
f'(0^-) - \tau \Delta f - \langle f^2, \alpha \rangle = 0, \quad \kappa > 0, \quad (6.17)
\]

\[
-\Delta f + \zeta f'(0^-) - \langle f^2, a \rangle = 0, \quad \kappa = 0. \quad (6.18)
\]

\( \eta > 0 \): Suppose that \( \lambda \neq \gamma_i \) for all \( i \). Substituting \( f^1 \) from (6.11) into (6.15) results in

\[
-f(0^+) + \sigma \Delta' f - \langle (\lambda I - [\gamma_i])^{-1}(h^1 + \beta \Delta' f), \beta \rangle = 0.
\]

Using (1.10) we get

\[
-f(0^+) + \frac{1}{\mu(\lambda)} \Delta' f = \langle h^1, (\lambda I - [\gamma_i])^{-1}\beta \rangle. \quad (6.19)
\]
If \( \lambda = \gamma_1 \) then from (6.11) we have \( \Delta' f = -h_1^1 \). Also for \( i \neq I \), \( f_i^1 = h_i^1 + \beta_i^1 \Delta' f_i^1 \). Thus (6.15) gives

\[
-f(0^+) - \sigma \frac{h_I^1}{\beta_I} - \sum_{i \neq I} \frac{\beta_i \beta_I h_i^1 - \beta_i h_I^1}{\gamma_I - \gamma_i} = \beta_I f_I^1.
\] (6.20)

\( \kappa \geq 0 \): If \( \lambda \neq \delta_j \) for all \( j \), substituting \( f^2 \) from (6.13) into (6.17) results in

\[
f'(0^-) - \tau \Delta f - \langle (\lambda I - [\delta_j])^{-1} (h^2 + \alpha \Delta f), \alpha \rangle = 0,
\]

using (1.11) we get

\[
f'(0^-) - \frac{1}{\nu(\lambda)} \Delta f = \langle h^2, (\lambda I - [\delta_j])^{-1} \alpha \rangle.
\] (6.21)

If \( \lambda = \delta_j \) then from (6.13) we have \( \Delta f = -h_j^2 / \alpha_j \). Also for \( j \neq J \), \( f_j^2 = h_j^2 + \alpha_j \Delta f_j^2 \). Thus (6.17) gives

\[
f'(0^-) + \tau \frac{h_j^2}{\alpha_j} - \sum_{j \neq J} \frac{\alpha_j \alpha_j h_j^2 - \alpha_j h_j^2}{\delta_j - \delta_j} = \alpha_J f_J^2.
\] (6.22)

\( \eta = 0 \): For \( \lambda \neq c_i \) for all \( i \), combining (6.12) with (6.16) and using (1.6) gives

\[
\Delta' f - \mu(\lambda) f(0^+) = \langle h^1, (\lambda I - [c_i])^{-1} b \rangle.
\] (6.23)

For \( \lambda = c_I \) we obtain \( f(0^+) = -h_i^1 / \beta_i \) and

\[
\Delta' f + \frac{h_i^1}{\beta_i} - \sum_{i \neq I} \frac{b_i b_i h_i^1 - b_i h_i^1}{c_i - c_i} = b_I f_I^1.
\] (6.24)

\( \kappa = 0 \): For \( \lambda \neq d_j \) for all \( j \), combining (6.14) with (6.18) and using (1.7) we obtain

\[
-\Delta f + \nu(\lambda)f'(0^-) = \langle h^2, (\lambda I - [d_j])^{-1} a \rangle.
\] (6.25)

When \( \lambda = d_J \) we get \( f'(0^-) = -h_j^2 / \alpha_j \) and

\[
-\Delta f - \frac{h_j^2}{\alpha_j} - \sum_{j \neq J} \frac{a_j a_j h_j^2 - a_j h_j^2}{d_J - d_J} = a_J f_J^2.
\] (6.26)

Now using the domain conditions (6.19), (6.23), (6.21) and (6.25) we get the following equations for \( A \) and \( B \):

\[
\eta > 0 : \quad -\frac{u_-(0^-)}{\mu(\lambda)} A + \left( \frac{v_+(0^+)}{\mu(\lambda)} - v_+(0^+) \right) B = \langle h^1, (\lambda I - [\gamma_i])^{-1} \beta \rangle,
\]

\[
\eta = 0 : \quad -u_-(0^-) A + (v_+(0^+) - \mu(\lambda)v_+(0^+)) B = \langle h^1, (\lambda I - [c_i])^{-1} b \rangle,
\]

\[
\kappa > 0 : \quad \left( u_-(0^-) + \frac{u_-(0^-)}{\nu(\lambda)} \right) A - \frac{v_+(0^+)}{\nu(\lambda)} B = \langle h^2, (\lambda I - [\delta_j])^{-1} \alpha \rangle,
\]

\[
\kappa = 0 : \quad (u_-(0^-) + \nu(\lambda)u_-(0^-)) A - v_+(0^+) B = \langle h^2, (\lambda I - [d_j])^{-1} a \rangle,
\]

which can be written in the matrix form

\[
\begin{bmatrix}
-u_-(0^-) & v_+(0^+) - \mu(\lambda)v_+(0^+)
\end{bmatrix}

[ A ] = \left[ \begin{bmatrix}
\langle h^1, p \rangle \\
\langle h^2, q \rangle
\end{bmatrix}
\right],
\]
hence giving (6.8), where \( p, q \) are given by (6.9) and (6.10).

Note that \( D \neq 0 \) since if \( D = 0 \), then \( \begin{bmatrix} A \\ B \end{bmatrix} \) could be taken as a non-zero vector in the null space of the matrix

\[
\begin{bmatrix}
-u'_-(0^-) & v'_-(0^+) - \mu(\lambda)v_+(0^+)
\\
-u_-(0^-) + \nu(\lambda)u'_-(0^-) & -v_+(0^+)
\end{bmatrix}
\]

and the resulting function \( f \) would be an eigenfunction of (1.1)-(1.5), which contradicts the assumption that \( \lambda \) is not an eigenvalue of \( L \). Hence \( D \neq 0 \) and \( A \) and \( B \) are as given in (6.8).

Thus \( f \) has been uniquely determined, provided that \( \lambda \neq \gamma_i, c_i \) for all \( i \) and \( \lambda \neq \delta_j, d_j \) for all \( j \). We provide an example at the end to demonstrate how residue calculations are used to determine \( A \) and \( B \) from (6.8) at poles.

Now \( f^1_i \) and \( f^2_j \) are uniquely determined by (6.11), (6.12), (6.13) and (6.14) for \( \lambda \neq \gamma_i (\eta > 0), \lambda \neq c_i (\eta = 0), \lambda \neq \delta_j (\kappa > 0) \) and \( \lambda \neq d_j (\kappa = 0) \) respectively.

**Example.** We consider the case where \( \mu(\lambda) \) has a pole at \( \lambda = c_I \) and \( \nu(\lambda) \in \mathbb{C} \). For \( \lambda \neq c_I \) we can rewrite equation (6.8) as

\[
\begin{bmatrix} A \\ B \end{bmatrix} = \frac{\mu(\lambda)}{D} \begin{bmatrix}
-v_+(0^+) & \left(\frac{-v'_-(0^+)}{\nu(\lambda)} + \frac{v_+(0^+)}{\mu(\lambda)}\right)
\\
-u_-(0^-) - \nu(\lambda)u'_-(0^-) & -\left(\frac{-u'_-(0^-)}{\nu(\lambda)}\right)
\end{bmatrix} \begin{bmatrix} \langle h^1, p \rangle \\ \mu(\lambda) \langle h^2, q \rangle \end{bmatrix}.
\]

Here, \( \frac{D}{\mu(\lambda)} \rightarrow (u_-(0^-) + \nu(\lambda)u'_-(0^-))v_+(0^+) \) as \( \lambda \rightarrow c_I \). Note that this limit is non-zero else (1.4) and (1.5) would imply that either \( \chi_{[-\alpha, 0]}u_- \) or \( \chi_{(0, b)}v_+ \) are eigenfunctions, contradicting the assumption that \( \lambda = c_I \) is not an eigenvalue. Moreover, by (1.6) we obtain for the case of \( \eta = 0 \) that

\[
(\lambda - c_I)\mu(\lambda) = \xi \prod_{i=1}^N (\lambda - c_i) + \sum_{i=1}^N b_i^2 \prod_{k \neq i} (\lambda - c_k) \prod_{i \neq I} (\lambda - c_i) \rightarrow \frac{1}{b_I^2} \text{ as } \lambda \rightarrow c_I.
\]

Hence, as \( \lambda \rightarrow c_I \),

\[
\frac{\langle h_1, p \rangle}{\mu(\lambda)} \rightarrow \begin{cases} 
\frac{b_I^1}{b_I^2}, & \eta = 0, \\
\langle h_1, (c_I - \gamma_i)^{-1} \beta \rangle, & \eta > 0.
\end{cases}
\]

If, in addition, we have that \( \nu(c_I) = 0 \) then \( c_I = \delta_j \) and from (1.11) we obtain, for the case of \( \kappa > 0 \), that

\[
(\lambda - \delta_j)\frac{1}{\nu(\lambda)} = \frac{\tau \prod_{j=1}^{M'} (\lambda - \delta_j) + \sum_{j=1}^{M'} \alpha_j^2 \prod_{k \neq j} (\lambda - \delta_k)}{\prod_{j \neq I} (\lambda - \delta_j)} \rightarrow \alpha_j^2 \text{ as } \lambda \rightarrow \delta_j.
\]
So, as $\lambda \to \delta_J$, 
\[
\frac{\langle \mathbf{h}^2, \mathbf{q} \rangle}{\mu(\lambda)} \to \begin{cases} 
\langle \mathbf{h}^2, (\delta_J I - [d_j])^{-1} \mathbf{b} \rangle, & \kappa = 0 \\
\frac{h_j^2}{\alpha_j}, & \kappa > 0.
\end{cases}
\]

Thus $A$ and $B$ can be obtained uniquely. In particular, for $\eta = 0$ and $\kappa > 0$ with $\lambda = c_I = \delta_J$ we obtain 
\[
A = \frac{1}{u_-(0^-)} \left[ -\frac{h_l^1}{b_I} + \frac{h_j^2}{\alpha_j} \right], \quad B = -\frac{1}{v_+(0^+)} \frac{h_l^1}{b_I},
\]
which agrees with the conditions $f(0^+) = -\frac{h_l^1}{b_I} (\eta = 0, \lambda = c_I)$ and $\Delta f = -\frac{h_j^2}{\alpha_j} (\kappa > 0, \lambda = \delta_J)$ obtained in the proof of Theorem 6.2.

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