Q^2-evolution of parton densities at small x values and H1 and ZEUS experimental data.

A.V. Kotikov and B.G. Shaikhatdenov

Joint Institute for Nuclear Research, 141980 Dubna, Russia

Abstract. It is shown that in the leading twist approximation of the Wilson operator product expansion with “frozen” and analytic strong coupling constants, considering the Bessel-inspired behavior of the structure functions F_2 and the derivative ∂ ln F_2/∂ ln(1/x) at small x values, obtained for a flat initial condition in the DGLAP evolution equations, leads to a good agreement with the deep inelastic scattering H1 and ZEUS experimental data from HERA.

Keywords: structure functions, parton distribution functions

PACS: 12.38.-t, 12.38.Qk

INTRODUCTION

A reasonable agreement between HERA data [1]-[6] and the next-to-leading-order (NLO) approximation of perturbative Quantum Chromodynamics (QCD) has been observed for Q^2 ≥ 2 GeV^2 (see reviews in [7] and references therein), which gives us a reason to believe that perturbative QCD is capable of describing the evolution of the structure function (SF) F_2 and its derivatives down to very low Q^2 values, where all the strong interactions are conventionally considered to be soft processes.

A standard way to study the x behavior of quarks and gluons is to compare the data with the numerical solution to the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [8] by fitting the parameters of x-profile of partons at some initial Q^2 and the QCD energy scale Λ [9, 10]. However, for the purpose of analyzing exclusively the small-x region, there is an alternative to carry out a simpler analysis by using some of the existing analytical solutions to DGLAP equations in the small-x limit [11]–[14].

To improve the analysis at low Q^2 values, it is important to consider the well-known infrared modifications of the strong coupling constant. We will use its “frozen” and analytic versions (see, [15, 16] and references therein).

GENERALIZED DOUBLED ASYMPTOTIC SCALING APPROACH

At low-x values there is the simple analytical solution of DGLAP evolution [11]: the HERA small-x data can be interpreted in terms of the so-called doubled asymptotic scaling (DAS) phenomenon related to the asymptotic behavior of the DGLAP evolution discovered many years ago [17].

The original study of [11] was extended in [12, 13, 14] to include the finite parts of anomalous dimensions of Wilson operators 1. This has led to predictions [13, 14] of the small-x asymptotic form of parton distribution functions (PDFs) in the framework of the DGLAP dynamics starting at some Q^2 with the flat function

\[ f_a(Q^2_0) = A_a \quad \text{(hereafter } a = q, g), \]  

where f_a are the parton distributions multiplied by x and A_a are unknown parameters to be determined from the data.

We refer to the approach of [12, 13, 14] as generalized DAS approximation. In that approach the flat initial conditions in Eq. (1) determine the basic role of the singular parts of anomalous dimensions, as in the standard DAS case, while the contribution from finite parts of anomalous dimensions and from Wilson coefficients can be considered as corrections which are, however, important for better agreement with experimental data. In the present paper, similarly to [11]–[14], we neglect the contribution from the non-singlet quark component.

---

1 In the standard DAS approximation [17] only the singular parts of the anomalous dimensions were used.
The flat initial condition (1) corresponds to the case when parton density tend to some constant value at $x \to 0$ and at some initial value $Q_0^2$. The main ingredients of the results [13, 14], are:

- Both, the gluon and quark singlet densities are presented in terms of two components (“+” and “−”) which are obtained from the analytic $Q^2$-dependent expressions of the corresponding (“+” and “−”) PDF moments.
- The twist-two part of the “−” component is constant at small $x$ at any values of $Q^2$, whereas the one of the “+” component grows at $Q^2 \geq Q_0^2$ as

$$\sim e^\sigma, \quad \sigma = 2 \sqrt{\left| d_+ \right| s - \left( d_{++} + |d_+| \frac{B_1}{\beta_0} \right) p} \ln \left( \frac{1}{x} \right), \quad \rho = \frac{\sigma}{2 \ln(1/x)},$$

where $\sigma$ and $\rho$ are the generalized Ball–Forte variables,

$$s = \ln \left( \frac{a_s(Q^2)}{a_s(Q_0^2)} \right), \quad p = a_s(Q^2_0) - a_s(Q^2), \quad d_+ = -\frac{12}{\beta_0}, \quad d_{++} = \frac{412}{27 \beta_0}. \tag{3}$$

Hereafter we use the notation $a_s = a_s/(4\pi)$. The first two coefficients of the QCD $\beta$-function in the $\overline{\text{MS}}$-scheme are $\beta_0 = 11 - (2/3)f$ and $\beta_1 = 102 - (114/9)f$ with $f$ is being the number of active quark flavors.

Note here that the perturbative coupling constant $a_s(Q^2)$ is different at the leading-order (LO) and NLO approximations. Indeed, from the renormalization group equation we can obtain the following equations for the coupling constant

$$\frac{1}{a_s^{\text{LO}}(Q^2)} = \beta_0 \ln \left( \frac{Q^2}{\Lambda_{\text{LO}}} \right), \quad \frac{1}{a_s(Q^2)} = \frac{\beta_0}{\beta_0 + \beta_1} \ln \left( \frac{Q^2}{\Lambda^2} \right)$$

at the LO and NLO approximations, respectively. Usually at the NLO level $\overline{\text{MS}}$-scheme is used, so we apply $\Lambda = \Lambda_{\overline{\text{MS}}}$ below.

**PARTON DISTRIBUTIONS AND THE STRUCTURE FUNCTION $F_2$**

Here, for simplicity we consider only the LO approximation. The structure function $F_2$ and PDFs $f_a$ ($a = q, g$) have the form

$$F_2(x, Q^2) = e f_g(x, Q^2), \quad f_a(x, Q^2) = f_a^+(x, Q^2) + f_a^-(x, Q^2),$$

where $e = (\sum f_i^2)/f$ is an average charge squared.

The small-$x$ asymptotic expressions for parton densities $f_a^\pm$ look like

$$f_g^+(x, Q^2) = \left( A_g + \frac{4}{9} A_q \right) I_0(\sigma) e^{-\sigma s} + O(\rho), \quad f_q^+(x, Q^2) = \frac{f}{9} \frac{\rho I_1(\sigma)}{I_0(\sigma)} f_g^+(x, Q^2) + O(\rho),$$

$$f_g^-(x, Q^2) = -\frac{4}{9} A_g e^{-d_+ x} + O(x), \quad f_q^-(x, Q^2) = A_q e^{-d_-(1+s)} + O(x),$$

where $I_0 (\nu = 0, 1)$ are the modified Bessel functions and $\sigma$ and $\rho$ can be found in (2) when $p = 0$. The coefficient $d_+$ (see eq. (3)) and

$$\overline{d}_+ = 1 + \frac{20f}{27\beta_0}, \quad d_- = \frac{16f}{27\beta_0} \tag{7}$$

denote singular and regular parts of the anomalous dimensions $d_+(n)$ and $d_-(n)$, respectively, in the limit $n \to 1^+$. Here $n$ is a variable in the Mellin space.

---

2 Such an approach has been developed recently also for the fragmentation function, whose first moments (i.e., mean multiplicities of quarks and gluons) were analyzed. The results are in good agreement with the experimental data.
3 The NLO results can be found in [13, 14].
4 We denote the singular and regular parts of a given quantity $k(n)$ in the limit $n \to 1$ by $k/(n-1)$ and $\overline{k}$, respectively.
$Q^2 = \{1.5, 2.0, 2.5, 3.5, 5.0, 6.5, 8.5, 12, 15, 20, 25, 35, 60, 90, 120\} \text{GeV}^2$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$F_2(x, Q^2)$ as a function of $x$ for different $Q^2$ bins. The experimental points are from H1 [1] (open points) and ZEUS [2] (solid points) at $Q^2 \geq 1.5 \text{ GeV}^2$. The solid curve represents the NLO fit. The dashed curve (hardly distinguishable from the solid one) represents the LO fit.}
\end{figure}

**EFFECTIVE SLOPES**

Contrary to the approach in [11]-[14] various groups have been able to fit the available data using a hard input at small $x$: $x^{-\lambda}$, $\lambda > 0$ with different $\lambda$ values at low and high $Q^2$ (see [20]-[26]). Such results are well-known at low $Q^2$ values [21]. At large $Q^2$ values, for the modern HERA data it is also not very surprising, because imprssible to distinguish between the behavior based on a steep input parton parameterization, at quite large $Q^2$, and the steep form acquired after the dynamical evolution from a flat initial condition at quite low $Q^2$ values.

As it has been mentioned above and shown in [13, 14, 27], the behavior of parton densities and $F_2$ given in the Bessel-like form by generalized DAS approach can mimic a power law shape over a limited region of $x$ and $Q^2$

\begin{align*}
  f_a(x, Q^2) &\sim x^{-\lambda_{eff}^a(x, Q^2)} \quad \text{and} \quad F_2(x, Q^2) \sim x^{-\lambda_{eff}^F(x, Q^2)}. \\
\end{align*}

The effective slopes $\lambda_{eff}^a(x, Q^2)$ and $\lambda_{eff}^F(x, Q^2)$ have the form:

\begin{align*}
  \lambda_{eff}^a(x, Q^2) &= \frac{f_a^+(x, Q^2)}{f_a^-(x, Q^2)} \rho \frac{I_1(\sigma)}{I_0(\sigma)}, \quad \lambda_{eff}^F(x, Q^2) = \lambda_{eff}^F(x, Q^2) = \frac{f_q^+(x, Q^2)}{f_q^-(x, Q^2)} \rho \frac{I_1(\sigma)}{I_0(\sigma)}. \quad (8)
\end{align*}

The effective slopes $\lambda_{eff}^a$ and $\lambda_{eff}^F$ depend on the magnitudes $A_a$ of the initial PDFs and also on the chosen input values of $Q_0^2$ and $\Lambda$. To compare with the experimental data it is necessary the exact expressions (8), but for qualitative
FIGURE 2. $x$ dependence of $F_2(x, Q^2)$ in bins of $Q^2$. The experimental data from H1 (open points) and ZEUS (solid points) are compared with the NLO fits for $Q^2 \geq 0.5 \text{ GeV}^2$ implemented with the canonical (solid lines), frozen (dot-dashed lines), and analytic (dashed lines) versions of the strong-coupling constant. For comparison, also the results obtained in Ref. [14] through a fit based on the renormalon model of higher-twist terms are shown (dotted lines).

analysis it is better to use an approximation. At quite large values of $Q^2$, where the "−" component is negligible, the dependence on the initial PDFs disappears, having in this case for the asymptotic behavior the following expressions:

$$\lambda_{\text{eff},x}(x, Q^2) = \rho \frac{I_1(\sigma)}{I_0(\sigma)} \approx \rho - \frac{1}{4 \ln(1/x)}$$

where the symbol $\approx$ marks the approximation obtained in the expansion of the modified Bessel functions.

**“FROZEN” AND ANALYTIC COUPLING CONSTANTS**

In order to improve an agreement at low $Q^2$ values, the QCD coupland is modified in the infrared region. We considered [16] two modifications that effectively increase the argument of the coupling constant at low $Q^2$ values (see [34]).

In the first case, which is more phenomenological, we introduce freezing of the coupling constant by changing its argument $Q^2 \rightarrow Q^2 + M_\rho^2$, where $M_\rho$ is the $\rho$-meson mass (see [28]). Thus, in the above formulæ we have to carry out the following replacement:

$$a_s(Q^2) \rightarrow a_{\text{fr}}(Q^2) \equiv a_s(Q^2 + M_\rho^2)$$

The second possibility follows the Shirkov–Solovtsov idea [29] concerning the analyticity of the coupling constant that leads to additional power dependence of the latter. Then, in the above formulæ the coupling constant $a_s(Q^2)$
TABLE 1. The result of the LO and NLO fits to H1 and ZEUS data for different low $Q^2$ cuts. In the fits $f$ is fixed to 4 flavors.

| $Q^2 \geq 1.5\text{GeV}^2$ | $\Lambda_x$ | $\Lambda_y$ | $Q_0^2$ [GeV$^2$] | $\chi^2/n.o.p.$ |
|-----------------------------|-------------|-------------|-------------------|-----------------|
| LO                          | 0.784±0.016 | 0.801±0.019 | 0.304±0.003       | 754/609         |
| LO&an.                      | 0.932±0.017 | 0.707±0.020 | 0.339±0.003       | 632/609         |
| LO&fr.                      | 1.022±0.018 | 0.650±0.020 | 0.356±0.003       | 547/609         |
| NLO                         | -0.200±0.011| 0.903±0.021 | 0.495±0.006       | 798/609         |
| NLO&an.                     | 0.310±0.013 | 0.640±0.022 | 0.702±0.008       | 655/609         |
| NLO&fr.                     | 0.180±0.012 | 0.780±0.022 | 0.681±0.007       | 669/609         |

| $Q^2 \geq 0.5\text{GeV}^2$ | $\Lambda_x$ | $\Lambda_y$ | $Q_0^2$ [GeV$^2$] | $\chi^2/n.o.p.$ |
|-----------------------------|-------------|-------------|-------------------|-----------------|
| LO                          | 0.641±0.010 | 0.937±0.012 | 0.295±0.003       | 1090/662        |
| LO&an.                      | 0.846±0.010 | 0.771±0.013 | 0.328±0.003       | 803/662         |
| LO&fr.                      | 1.127±0.011 | 0.534±0.015 | 0.358±0.003       | 679/662         |
| NLO                         | -0.192±0.006| 1.087±0.012 | 0.478±0.006       | 1229/662        |
| NLO&an.                     | 0.281±0.008 | 0.634±0.016 | 0.680±0.007       | 633/662         |
| NLO&fr.                     | 0.205±0.007 | 0.650±0.016 | 0.589±0.006       | 670/662         |

should be replaced as follows:

$$a_{\text{an}}^{LO}(Q^2) = a_s^{LO}(Q^2) - \frac{1}{\Lambda^2 - \Lambda^2_0} \frac{\Lambda^2_0}{Q^2}, \quad a_{\text{an}}^{LO}(Q^2) = a_s(Q^2) - \frac{1}{2b_0} \frac{\Lambda^2}{Q^2 - \Lambda^2} + \ldots,$$

in the LO and NLO approximations, respectively. Here the the symbol \ldots stands for the terms that provide negligible contributions when $Q^2 \geq 1$ GeV [29]. Note that the perturbative coupling constant $a_s(Q^2)$ is different in the LO and NLO approximations (see eq. (4) above).

**COMPARISON WITH EXPERIMENTAL DATA**

Using the results of previous section we have analyzed [13, 14, 16] HERA data for $F_2$ and the slope $\partial \ln F_2 / \partial \ln(1/x)$ at small $x$ from the H1 and ZEUS Collaborations [1]-[6]. In order to keep the analysis as simple as possible, we fix $f = 4$ and $\alpha_s(M_Z^2) = 0.1166$ (i.e., $\Lambda^4 = 284$ MeV) in agreement with the more recent ZEUS results [2].

As it is possible to see in Fig. 1 (see also [13, 14]), the twist-two approximation is reasonable at $Q^2 \geq 2$ GeV$^2$. Moreover, the results of fits in [14] have an important property: they are very similar in LO and NLO approximations of perturbation theory. The similarity is related to the fact that the small-$x$ asymptotics of the NLO corrections are usually large and negative (see, for example, $\alpha_4$-corrections [30, 31] to Balitsky–Fadin–Kuraev–Lipatov (BFKL) kernel [32]). Then, the LO form $\sim \alpha_s(Q^2)$ for some observable and the NLO one $\sim \alpha_s(Q^2)(1 - K \alpha_s(Q^2))$ with a large value of $K$ are similar, because $\Lambda \gg \Lambda_0$ and, thus, $\alpha_s(Q^2)$ at LO is considerably smaller than $\alpha_s(Q^2)$ at NLO for HERA $Q^2$ values.

In other words, performing some resummation procedure (such as Grunberg’s effective-charge method [33]), one can see that the results up to NLO approximation may be represented as $\sim \alpha_s(Q_{\text{eff}}^2)$, where $Q_{\text{eff}}^2 \gg Q^2$. Indeed, from different studies [34, 35, 36], it is well known that at small-$x$ values the effective argument of the coupling constant is higher than $Q^2$.

At smaller $Q^2$, some modification of the twist-two approximation should be considered. In Ref. [14] we have added the higher twist corrections. For renormalon model of higher twists, we have found a good agreement with experimental data at essentially lower $Q^2$ values: $Q^2 \geq 0.5$ GeV$^2$ (see Figs. 2 and 3 in [14]), but we have added 4 additional parameters: amplitudes of twist-4 and twist-6 corrections to quark and gluon densities.

---

5 It seems that it is a property of any processes in which gluons, but not quarks play a basic role.

6 The equality of $\alpha_s(M_Z^2)$ at LO and NLO approximations, where $M_Z$ is the Z-boson mass, relates $\Lambda$ and $\Lambda_0$: $\Lambda^4 = 284$ MeV (as in [2]) corresponds to $\Lambda_0 = 112$ MeV (see [14]).
FIGURE 3. $Q^2$ dependence of $\lambda_{F_2}^{\text{eff}}(x, Q^2)$ for an average small-$x$ value of $x = 10^{-3}$. The experimental data from H1 (open points) and ZEUS (solid points) are compared with the NLO fits for $Q^2 \geq 0.5$ GeV$^2$ implemented with the canonical (solid line), frozen (dot-dashed line), and analytic (dashed line) versions of the strong-coupling constant. The linear rise of $\lambda_{F_2}^{\text{eff}}(x, Q^2)$ with $\ln Q^2$ is indicated by the straight dashed line. For comparison, also the results obtained in the phenomenological models by Capella et al. [25] (dash-dash-dotted line) and by Donnachie and Landshoff [38] (dot-dot-dashed line) are shown.

To improve the agreement at small $Q^2$ values without additional parameters, we modified [16] the QCD coupling constant. We considered two modifications: analytic and frozen coupling constants, which effectively increase the argument of the coupling constant at small $Q^2$ values (in agreement with [34, 35, 36]). Figure 2 and Table 1 show a strong improvement of the agreement with experimental data for $F_2$ (almost 2 times!). Similar results can be seen also in Fig. 3 for the experimental data for $\lambda_{F_2}^{\text{eff}}(x, Q^2)$ at $x \sim 10^{-3}$, which represents an average of the $x$-values of HERA experimental data. Note that the “frozen” and analytic coupling constants $\alpha_{\text{fr}}(Q^2)$ and $\alpha_{\text{an}}(Q^2)$, lead to very close results (see also [37, 15]).

Indeed, the fits for $F_2(x, Q^2)$ in [14] yielded $Q_0^2 \approx 0.5$–0.8 GeV$^2$. So, initially we had $\lambda_{F_2}^{\text{eff}}(x, Q_0^2) = 0$, as suggested by Eq. (1). The replacements of Eqs. (10) and (11) modify the value of $\lambda_{F_2}^{\text{eff}}(x, Q_0^2)$. For the “frozen” and analytic coupling constants $\alpha_{\text{fr}}(Q^2)$ and $\alpha_{\text{an}}(Q^2)$, the value of $\lambda_{F_2}^{\text{eff}}(x, Q_0^2)$ is nonzero and the slopes are quite close to the experimental data at $Q^2 \approx 0.5$ GeV$^2$. Nevertheless, for $Q^2 \leq 0.5$ GeV$^2$, there is still some disagreement with the data for the slope $\lambda_{F_2}^{\text{eff}}(x, Q^2)$, which needs additional investigation. Note that at $Q^2 \geq 0.5$ GeV$^2$ our results for $\lambda_{F_2}^{\text{eff}}(x, Q^2)$ are even better the results of phenomenological models [25, 38].

At the next step we considered [39] the combined H1&ZEUS data for $F_2$ [3]. As can be seen from Fig. 4 and Table 2, the twist-two approximation is reasonable for $Q^2 \geq 2$ GeV$^2$. At lower $Q^2$ we observe that the fits in the cases with “frozen” and analytic strong coupling constants are very similar (see also [37, 16, 15]) and describe the data in the low $Q^2$ region significantly better than the standard fit. Nevertheless, for $Q^2 \leq 1.5$ GeV$^2$ there is still some disagreement with the data, which needs to be additionally studied. In particular, the BFKL resummation [32] may be important...
FIGURE 4. $x$ dependence of $F_2(x, Q^2)$ in bins of $Q^2$. The combined experimental data from H1 and ZEUS Collaborations [3] are compared with the NLO fits for $Q^2 \geq 0.5 \text{ GeV}^2$ implemented with the standard (solid lines), frozen (dot-dashed lines), and analytic (dashed lines) versions of the strong coupling constant.

here [40]. It can be added in the generalized DAS approach according to the discussion in Ref. [41].

CONCLUSIONS

We have shown the $Q^2$-dependence of the structure functions $F_2$ and the slope $\lambda_{F_2}^{\text{eff}} = \partial \ln F_2 / \partial \ln (1/x)$ at small-$x$ values in the framework of perturbative QCD. Our twist-two results are in a very good agreement with precise HERA data for $Q^2 \geq 2 \text{ GeV}^2$, where perturbative theory is applicable. Using the “frozen” and analytic coupling constants
| $Q^2$ ≥ 5 GeV$^2$ | $A_0$ | $A_1$ | $Q_0^2$ [GeV$^2$] | $\chi^2$/ndf |
|------------------|------|------|-----------------|-------------|
| LO               | 0.623±0.055 | 1.204±0.093 | 0.437±0.022 | 1.00 |
| LO&an.           | 0.796±0.059 | 1.105±0.095 | 0.494±0.024 | 0.85 |
| LO&fr.           | 0.782±0.058 | 1.110±0.094 | 0.485±0.024 | 0.82 |
| NLO              | -0.252±0.041 | 1.335±0.100 | 0.700±0.044 | 1.05 |
| NLO&an.          | 0.102±0.046 | 1.029±0.106 | 1.017±0.060 | 0.74 |
| NLO&fr.          | -0.132±0.043 | 1.219±0.102 | 0.793±0.049 | 0.86 |

$\alpha_f(Q^2)$ and $\alpha_{an}(Q^2)$ improves an agreement with the recent HERA data [4, 5, 6] for the slope $A_{F_2}^{\text{eff}}(x, Q^2)$ for small $Q^2$ values, $Q^2 ≥ 0.5$ GeV$^2$.

As the next step, we are going to adopt the Grunberg approach [33] together with the “frozen” and analytic modifications of the strong coupling constant for analyse of the combined H1&ZEUS data for $F_2$ [3]. The similar study has been done recently [42] for experimental data of the Bjorken sum rule.

**ACKNOWLEDGMENTS**

This work was supported by RFBR grant 13-02-01060-a. A.V.K. thanks the Organizing Committee of II Russian-Spanish Congress for invitation and support.

**REFERENCES**

1. H1 Collab. (C. Adloff et al.), Nucl. Phys. B 497 (1997) 3; Eur. Phys. J. C 21 (2001) 33.
2. ZEUS Collab. (S. Chekanov et al.), Eur. Phys. J. C 21 (2001) 443.
3. H1 and ZEUS Collab. (F. D. Aaron et al.), JHEP 1001 (2010) 109.
4. H1 and ZEUS Collab. (B. Surrow), Phenomenological studies of inclusive e p scattering at low momentum transfer $Q^2$, hep-ph/0201025.
5. H1 Collab. (C. Adloff et al.), Phys. Lett. B 520 (2001) 183.
