Maximum average degree of list-edge-critical graphs and Vizing’s conjecture

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Abstract

Vizing conjectured that $\chi'(G) \leq \Delta + 1$ for all graphs. For a graph $G$ and nonnegative integer $k$, we say $G$ is a $k$-list-edge-critical graph if $\chi'_\ell(G) > k$, but $\chi'_\ell(G - e) \leq k$ for all $e \in E(G)$. We use known results for list-edge-critical graphs to verify Vizing’s conjecture for $G$ with $\text{mad}(G) < \frac{\Delta + 3}{2}$ and $\Delta \leq 9$.

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1. Introduction

We consider only simple graphs in this paper. It will be convenient for us to define for a graph $G$, the vertex set $V_x = \{v \in V(G) \mid d(v) = x\}$ and the set $V_{[x,y]} = \{v \in V(G) \mid x \leq d(v) \leq y\}$. An \textit{edge-coloring} of $G$ is a function which maps one color to every edge of $G$ such that adjacent edges receive distinct colors. A $k$-edge-coloring of $G$ is an edge-coloring of $G$ which maps a total of $k$ colors to $E(G)$. The chromatic index $\chi'(G)$ is the minimum $k$ such that $G$ is $k$-edge-colorable. Vizing’s Theorem [10] gives us $\chi'(G) \leq \Delta + 1$ for all graphs $G$ where $\Delta$ is the maximum degree of $G$.

We are interested in a variation of edge-coloring called list-edge-coloring. A list-edge-coloring is an edge-coloring with the extra constraint that each edge can only be colored from a preassigned

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Conjecture 1 was proved in 1998 by Juvan, Mohar, Škrekovski [8]. Vizing [9] in 1976 and independently by Erdős, Rubin, and Taylor [5] in 1979. The Conjecture 1 relaxation of the LECC proposed by Vizing.

We say an edge-list-assignment of \( G \) is \( k \)-list-edge-colorable if \( G \) can be properly edge-colored with every edge \( e \) receiving a color from \( L(e) \). We say that \( G \) is \( k \)-list-edge-colorable if \( G \) is \( k \)-list-edge-colorable for all \( L \) such that \( |L(e)| \geq k \) for all \( e \in E(G) \). We note this concept is referred to as \( k \)-edge-choosable in other papers. The list-chromatic index, \( \chi'_l(G) \), is the minimum \( k \) such that \( G \) is \( k \)-list-edge-colorable. So, we want to achieve a list-edge-coloring for all list-assignments \( L \) with minimal list-size \( k \).

It is easy to see that \( \chi'_l(G) \geq \chi'(G) \geq \Delta \) for all graphs. The List-Edge Coloring Conjecture proposes that \( \chi'_l(G) = \chi'(G) \), but this has only been verified for a few special families of graphs, such as Galvin’s result for the family of bipartite graphs [6]. In this paper, we will focus on a relaxation of the LECC proposed by Vizing.

**Conjecture 1 (Vizing [9]).** If \( G \) is a graph, then \( \chi'_l(G) \leq \Delta + 1 \).

This conjecture has been verified for all graphs with \( \Delta \leq 4 \). The \( \Delta = 3 \) case was proved by Vizing [9] in 1976 and independently by Erdős, Rubin, and Taylor [5] in 1979. The \( \Delta = 4 \) case of Conjecture 1 was proved in 1998 by Juvan, Mohar, Škrekovski [8].

The average degree of a graph \( G \) is \( ad(G) = \frac{\sum \deg(v)}{|V(G)|} \). The maximum average degree of a graph \( G \) is \( mad(G) = \max\{ad(H) : H \subseteq G\} \). That is, \( mad(G) \) is the maximum of the set of average degrees of all subgraphs \( G \).

Motivated by Vizing and the List Edge Coloring Conjecture, Woodall conjectured [11] if \( G \) has \( mad(G) < \Delta - 1 \), then \( \chi'_l(G) = \Delta \). Together with Borodin and Kostochka, Woodall [2] was able to verify his conjecture when \( mad(G) < \sqrt{2\Delta} \).

We say that a graph \( G \) is \( k \)-list-edge-critical if \( \chi'_l(G) > k \), and \( \chi'_l(G - e) \leq k \) for all \( e \in E(G) \). By taking advantage of known results for list-edge-critical graphs, we relax Woodall’s conjecture by bounding \( \Delta(G) \leq 9 \) to verify Conjecture 1 when \( mad(G) \leq \frac{\Delta(G)+3}{2} \).

### 2. Main Result

In 1990, Borodin verified Conjecture 1 for planar graphs with \( \Delta \geq 9 \) (see [3]). This was improved to planar graphs with \( \Delta \geq 8 \) by Bonamy in 2015 (see [1]). In 2010, before Bonamy’s result, Cohen and Havet wrote a new proof of Borodin’s theorem which reduced the argument to about a single page (see [4]). Their new proof used the minimality of list-edge-critical graphs and a clever discharging argument. We state one of their lemmas below.

**Lemma 2.1 (Cohen & Havet [4]).** If \( G \) is \( (\Delta + 1) \)-list-edge-critical, then \( \deg(u) + \deg(v) \geq \Delta + 3 \).

Lemma 2.1, together with Borodin, Kostochka, Woodall’s generalization [2] of Galvin’s Theorem, were used to prove the following lemma. This lemma is listed as Lemma 9 in [7] and was used to achieve edge-precoloring results.

**Lemma 2.2 (Harrelson, McDonald, Puleo [7]).** Let \( a_0, a, b_0 \in \mathbb{N} \) such that \( a_0 > 2 \), \( b_0 > a \), and \( a + b_0 = \Delta + 3 \). If \( G \) is \( (\Delta + 1) \)-list-edge-critical, then...
For all values of $\Delta$ and $\alpha$ and their resulting inequalities from Lemma 2.2. Each table also presents the discharging step and $ad$.

Proof. Let $P$ and $\alpha$ be the sum of coefficients of $V_i$ from the first table. We will apply a discharging step and denote $\alpha'(v)$ as the final charge for $v \in V(G)$ after discharging. We will also use $\alpha'(P)$ and $\alpha'(G)$ to denote the final charges of $P$ and $G$, respectively, after the discharging step. To get a contradiction, we will prove $\alpha'(G) \geq m \cdot v(G)$ by showing $\alpha'(P) > 0$ and $\alpha'(v) \geq m$ for all $v \in V(G)$.

We note that this theorem is known for $\Delta \leq 4$ so we may assume $5 \leq \Delta \leq 9$. For each of these values of $\Delta$, we provide Tables 1 through 5. Each table provides a list of triples $(a_0, a, b_0)$ and their resulting inequalities from Lemma 2.2. Each table also presents the discharging step and verifies $\alpha'(v) \geq m$ for all $v \in V(G)$. We let $x_i$ be the sum of coefficients of $V_i$ from the first table. For all values of $\Delta$, we discharge in the following way: If $deg(v) = i < m$, then $v$ will give $x_i$ to $P$. If $deg(v) = i \geq m$, then $v$ will take $x_i$ from $P$.

For all values of $\Delta$, we verify $\alpha'(P) > 0$ by using only strict inequalities and noting the lesser size of every inequality only contains vertices with degree less than $m$ and the greater side every inequality contains vertices with degree greater than $m$. This means more charge is put into $P$ than is taken from $P$ due to how we defined $x_i$ in our discharging step.

If $\Delta = 9$, then we consider the ordered triples in the form of $(a_0, a, b_0)$ and the system of inequalities resulting from Lemma 2.2 as displayed in Table 1. We note that the final charge of $P$ is positive since adding all inequalities together yields:

$$x_3V_3 + x_4V_4 + x_5V_5 < x_7V_7 + x_8V_8 + x_9V_9.$$ 

The final charges from Table 1 gives

$$\alpha'(G) = \alpha'(P) + \sum_{v \in V(G)} \alpha'(v) > m \cdot v(G)$$

This is a contradiction for $\Delta = 9$. We proceed through the remaining values of $\Delta$ using the same argument. We present a table for each value of $\Delta$. Each table displays inequalities resulting from Lemma 2.2 and each table displays the discharging step to verify $\alpha'(v) > m$ and $\alpha'(P) > 0$. Note that, for $\Delta = 8$, we multiply the first inequality by $1/2$.

This completes the proof of Theorem 2.1. 

$$2 \sum_{i=a_0}^{a} |V_i| < \sum_{j=b_0}^{\Delta} (a + j - \Delta - 2)|V_j|.$$
Table 1. Inequalities and final charges for $\Delta = 9$.

**Lemma 2.2 inequalities for $\Delta = 9$**

| $(a_0, a, b_0)$ | Inequality | $\alpha(v) = i$ | $x_i$ | $\alpha'(v)$ |
|----------------|------------|-----------------|-------|--------------|
| (3,5,7)        | $V_3 + V_4 + V_5 < \frac{1}{2} V_7 + V_8 + \frac{3}{2} V_9$ | 3     | 3     | 6            |
| (3,4,8)        | $V_3 + V_4 < \frac{1}{2} V_8 + V_9$ | 4     | 2     | 6            |
| (3,3,9)        | $V_3 < \frac{1}{2} V_9$ | 5     | 1     | 6            |
|                |            | 6     | 0     | 6            |
|                |            | 7     | $\frac{1}{2}$ | $\frac{13}{7}$ |
|                |            | 8     | $\frac{3}{2}$ | $\frac{13}{7}$ |
|                |            | 9     | $\frac{6}{2}$ | 6            |

**Discharging for $\Delta = 9, m = 6$**

Table 2. Inequalities and final charges for $\Delta = 8$.

**Lemma 2.2 inequalities for $\Delta = 8$**

| $(a_0, a, b_0)$ | Inequality | $\alpha(v) = i$ | $x_i$ | $\alpha'(v)$ |
|----------------|------------|-----------------|-------|--------------|
| (3,5,6)        | $\frac{1}{2}[V_3 + V_4 + V_5] < \frac{1}{2} \left[ \frac{1}{2} V_6 + \frac{2}{3} V_7 + \frac{3}{2} V_8 \right]$ | 3     | $\frac{5}{2}$ | $\frac{11}{2}$ |
| (3,4,7)        | $V_3 + V_4 < \frac{1}{2} V_7 + \frac{2}{3} V_8$ | 4     | $\frac{3}{2}$ | $\frac{11}{2}$ |
|                |            | 5     | $\frac{1}{2}$ | $\frac{11}{2}$ |
|                |            | 6     | $\frac{1}{4}$ | $\frac{23}{4}$ |
|                |            | 7     | 1     | 6            |
|                |            | 8     | $\frac{9}{4}$ | $\frac{23}{4}$ |

**Discharging for $\Delta = 8, m = \frac{11}{2}$**

Table 3. Inequalities and final charges for $\Delta = 7$.

**Lemma 2.2 inequalities for $\Delta = 7$**

| $(a_0, a, b_0)$ | Inequality | $\alpha(v) = i$ | $x_i$ | $\alpha'(v)$ |
|----------------|------------|-----------------|-------|--------------|
| (3,4,6)        | $V_3 + V_4 < \frac{1}{2} V_6 + \frac{2}{3} V_7$ | 3     | 2     | 5            |
| (3,3,7)        | $V_3 < \frac{1}{2} V_7$ | 4     | 1     | 5            |
|                |            | 5     | 0     | 5            |
|                |            | 6     | $\frac{1}{2}$ | $\frac{11}{2}$ |
|                |            | 7     | $\frac{3}{2}$ | $\frac{11}{2}$ |
Table 4. Inequalities and final charges for $\Delta = 6$.

**Lemma 2.2 inequalities for $\Delta = 6$**

| $(a_0, a, b_0)$ | Inequality | $\alpha(v) = i$ | $x_i$ | $\alpha'(v)$ |
|-----------------|------------|-----------------|-------|--------------|
| $(3,4,5)$       | $V_3 + V_4 < \frac{1}{2}V_5 + \frac{3}{2}V_6$ | 3     | 2     | 5            |
| $(3,3,6)$       | $V_3 < \frac{1}{2}V_6$                   | 4     | 1     | 5            |
|                 |                                        | 5     | $\frac{1}{2}$ | $\frac{9}{2}$ |
|                 |                                        | 6     | $\frac{3}{2}$ | $\frac{9}{2}$ |

**Discharging for $\Delta = 6$, $m = \frac{9}{2}$**

Table 5. Inequalities and final charges for $\Delta = 5$.

**Lemma 2.2 inequalities for $\Delta = 5$**

| $(a_0, a, b_0)$ | Inequality | $\alpha(v) = i$ | $x_i$ | $\alpha'(v)$ |
|-----------------|------------|-----------------|-------|--------------|
| $(3,3,5)$       | $V_3 < \frac{1}{2}V_5$                   | 3     | 1     | 4            |
|                 |                                        | 4     | 0     | 4            |
|                 |                                        | 5     | $\frac{1}{2}$ | $\frac{9}{2}$ |

3. Conclusion

The application of Lemma 2.2 can be improved for some values of $\Delta(G)$ presented in Theorem 2.1 to yield slightly greater values of $\text{mad}(G)$. We can also apply Lemma 2.2 to any value of $\Delta(G)$, but this will lower the bound on $\text{mad}(G)$. Specifically, we can find optimum values of $\text{mad}(G)$ given $\Delta(G)$ for graphs of higher max-degree by “reverse-engineering” the inequalities of Lemma 2.2 as shown in the following example for $\Delta(G) = 10$.

**Example 1.** Finding an optimal $\text{mad}(G)$ for $\Delta(G) = 10$.

**Proof.** Let $\text{mad}(G) < m$ for some $m$, let $\alpha(P) = 0$, and let $\alpha(v) = d(v)$ for all $v \in V(G)$. We wish to determine the largest number $m$ such that $\alpha'(P) > 0$ and $\alpha'(v) \geq m$ for all $v \in V(G)$. We begin by presenting a table of triples and their resulting inequalities from Lemma 2.2; however, we multiply each inequality by an arbitrary constant.

Table 6. **Lemma 2.2 inequalities for $\Delta = 10$**

| $(a_0, a, b_0)$ | Inequality | $c_1(V_3 + V_4 + V_5 + V_6 < \frac{1}{2}V_7 + \frac{3}{2}V_8 + \frac{3}{2}V_9 + \frac{1}{2}V_{10})$ |
|-----------------|------------|--------------------------------------------------|
| $(3,6,10)$      | $c_2(V_3 + V_4 + V_5 < \frac{1}{2}V_6 + \frac{3}{2}V_9 + \frac{3}{2}V_{10})$ |                                |
| $(3,5,10)$      | $c_3(V_3 + V_4 < \frac{1}{2}V_9 + \frac{1}{2}V_{10})$ |                                |
| $(3,3,10)$      | $c_4(V_3 < \frac{1}{2}V_{10})$ |                                |
As in Theorem 2.1, we let $x_i$ be the sum of coefficients of $V_i$ from this table. We will let “high-degree” vertices give charge to $P$ while “low-degree” vertices take charge from $P$ in the rules that follow. If $\deg(v) = i \geq \lceil \frac{1}{2} \Delta + 2 \rceil$, then $v$ gives $x_i$ to $P$. If $\deg(v) = i \leq \lfloor \frac{1}{2} \Delta + 1 \rfloor$, then $v$ takes $x_i$ from $P$. This yields the list of final charges displayed in Table 6. We set each final charge greater than or equal to $m$.

Table 7. Final charges for Example 1.

| $V_i$ | Final Charge $\geq m$ | Name |
|-------|-----------------------|------|
| $V_3$ | $3 + c_1 + c_2 + c_3 + c_4 \geq m$ | A    |
| $V_4$ | $4 + c_1 + c_2 + c_3 \geq m$ | B    |
| $V_5$ | $5 + c_1 + c_2 \geq m$ | C    |
| $V_6$ | $6 + c_1 \geq m$ | D    |
| $V_{10}$ | $10 - \frac{4}{2} c_1 - \frac{3}{2} c_2 - \frac{2}{2} c_3 - \frac{1}{2} c_4 \geq m$ | E    |

Increasing the constants $c_1, c_2, c_3, c_4$ increases the final charge of our “low-degree” vertices, but decreases the final charge of our “high-degree” vertices. We need all final charges to be greater than or equal to $m$ so we must chose $m$ carefully. While all vertices in $V_{[7,10]}$ give charge away, the vertices in $V_{10}$ give the most, meaning inequality $E$ has the strictest bound on $m$. With this in mind, we can find an optimal bound for $m$ by adding inequalities in the following way:

$$2E + A + B + C + D \implies 38 + 0x_1 + 0x_2 + 0x_3 \geq 6m \implies \frac{19}{3} \geq m$$

We can now use this bound and the inequalities of the “low-degree” vertices from Table 6 to solve for $c_1, c_2, c_3, c_4$.

$$D \quad V_6 : 6 + c_1 \geq \frac{19}{3} \implies c_1 = \frac{1}{3}$$

$$C \quad V_5 : 5 + c_1 + c_2 \geq \frac{19}{3} \implies c_2 = 1$$

$$B \quad V_4 : 4 + c_1 + c_2 + c_3 \geq \frac{19}{3} \implies c_3 = 1$$

$$A \quad V_3 : 3 + c_1 + c_2 + c_3 + c_4 \geq \frac{19}{3} \implies c_4 = 1$$

We have shown that $\alpha'(v) \geq \frac{19}{3}$ for our “low-degree” vertices in $V_{[3,6]}$. We only need to verify the values of $c_1, c_2, c_3, c_4$, and $m$ give us appropriate inequalities for the “high-degree” vertices.
V_7 : 7 - \frac{1}{2} c_1 > \frac{19}{3}
V_8 : 8 - \frac{2}{2} c_1 - \frac{1}{2} c_2 > \frac{19}{3}
V_9 : 9 - \frac{3}{2} c_1 - \frac{2}{2} c_2 - \frac{1}{2} c_3 > \frac{19}{3}
V_{10} : 10 - \frac{4}{2} c_1 - \frac{3}{2} c_2 - \frac{2}{2} c_3 - \frac{1}{2} c_4 = \frac{19}{3}

So \( m = \frac{19}{3} \) is a feasible bound for \( \text{mad}(G) \) when \( \Delta(G) = 10 \). This means if a graph \( H \) has \( \Delta(H) \leq 10 \) and \( \text{mad}(H) < \frac{19}{3} \), then \( \chi'_\ell(H) \leq \Delta + 1 \).

Lemma 2.2 can be thought of as a generalization Cohen and Havet’s argument in [4]. Both of these results use forbidden structures to force good counts of low and high degree vertices by relying on Galvin’s Theorem [6]. In this sense, good counts are achieved from knowing the list-edge-colorability of bipartite graphs. We are interested in how the list-edge-colorability of other simple families of graphs could be used to develop counts to verify Vizing’s Conjecture or even the LECC for a wider range of graphs than is currently known.

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