Quasi-stationary mechanics of elastic continua with bending stiffness wrapping on a pulley system

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Abstract. In many engineering applications elastic continua such as ropes and belts often are subject to bending when they pass over pulleys / sheaves. In this paper the quasi-stationary mechanics of a cable-pulley system is studied. The cable is modelled as a moving Euler-Bernoulli beam. The distribution of tension is non-uniform along its span and due to the bending stiffness the contact points at the pulley-beam boundaries are not unknown. The system is described by a set of nonlinear ordinary differential equations with undetermined boundary conditions. The resulting nonlinear Boundary Value Problem (BVP) with unknown boundaries is solved by converting the problem into the ‘standard’ form defined over a fixed interval. Numerical results obtained for a range of typical configurations with relevant boundary conditions applied demonstrate that due to the effects of bending stiffness the angels of wrap are reduced and the span tensions are increased.

1. Introduction
Elastic continua such as ropes and belts are used in lifting and hoisting installations as suspension/power transmission members and mass compensation means. In these applications elastic continua wrap around rotating sheaves and pulleys. Their behaviour has been studied extensively. One approach is to model the elastic continua without taking into consideration the contact mechanics at the sheave/pulley boundary. The continua are then modelled without taking into consideration of thir bending stiffness as a string [1-6]. Alternatively, a straight beam model is used to accommodate the bending stiffness effects [7-10]. It is then assumed in the analysis that they approach the sheave/pulley along a straight tangent line. However, due to their finite bending stiffness a transient region exists in the vicinity of the contact point on the sheave/pulley. At the contact/boundary point their curvature matches the curvature of the sheave/pulley, and then as the sections move away from this point, the curvature changes continuously [11-13]. Bending of moving continua around the sheaves/pulleys reduces the wrap angles, tend to change the natural frequencies and results in a non-trivial steady-state configuration about which the system can vibrate [14].

In this paper the cable span is modelled as an Euler-Bernoulli beam. The distribution of tension is non-uniform along its span and due to the bending stiffness the contact points at the pulley-beam boundaries are not unknown. The system is described by a set of nonlinear ordinary differential equations with undetermined boundary conditions. The resulting nonlinear Boundary Value Problem
(BVP) with unknown moving boundaries is solved by converting the problem into the ‘standard’ form defined over a fixed interval.

2. Mathematical model
A quasi-stationary motion of a system comprising an axially moving elastic one-dimensional continuum with bending stiffness divided into $p = 1, 2, 3, \ldots, P$ sections of slowly-varying length constrained by discrete elements such as rigid-body masses and rotating pulleys can be described by the following nonlinear Boundary Value Problem (BVP) with unknown boundaries

$$
y'_p = f_p(s_p, y_p; \tau), \quad 0 \leq s_p \leq \hat{L}_p(\tau)
$$

$$
g_p \left[ y_p(0), y_p(\hat{L}_p) \right] = 0
$$

where $y_p$, $f_p$, $g_p$ have $n$ components, $s_p$ represents an Eulerian coordinate and $\tau = \varepsilon t$ is a slowing time [15]. The formulation of the problem is described in what follows.

2.1. Governing equations
The continuum is modelled as an Euler-Bernoulli beam in vertical configuration moving at speed $v$. A free-body diagram of a segment of length $ds$ of the beam section $p$ is shown in figure 1, where $s$ (after dropping the subscript $p$) represents the arc length and is used as an Eulerian coordinate measured along the deformed beam. The Euler-Bernoulli beam model requires that the bending moment $M$ and the shear force $Q$ are given as

$$
M = EI \kappa
$$

$$
Q = \frac{dM}{ds} = EI \frac{d\kappa}{ds}
$$

respectively, where $EI$ is the bending stiffness and $\kappa$ represents the curvature. The curvature is equal to the reciprocal of the radius of curvature $\rho$ can be expressed as the rate of change of the slope angle $\varphi$ as

$$
\kappa = \frac{1}{\rho} = \frac{d\varphi}{ds}
$$

The equilibrium of forces in the normal direction and the tangential direction, respectively, yield the following equations to determine the unknown curvature $\kappa(s; \tau)$, tension $T(s; \tau)$ and the slope angle $\varphi(s; \tau)$

$$
\kappa'' - \frac{1}{EI} (T - mv^2) \kappa + \frac{mg}{EI} \sin \varphi = 0
$$

$$
T' - mv \varphi' + EI \kappa \varphi' + mg \cos \varphi = 0
$$

where $g$ is the acceleration of gravity, $m$ is the mass per unit length of the beam, $(\quad)'$ denotes differentiation with respect to $s$ and the unknown quantities are defined over the interval $0 \leq s \leq \hat{L}(\tau)$, where the slowly varying boundary $\hat{L}(\tau)$ represents the total arc length of the beam section and is also unknown. Furthermore, the Cartesian coordinates of cable particles are expressed as
\[ \frac{\partial x}{\partial s} = \cos \varphi, \quad \frac{\partial y}{\partial s} = \sin \varphi \]  

Equations (5-8) can be brought into the form (1) by denoting

\[ y = \begin{bmatrix} \kappa, \kappa', T, L, \varphi, x, y \end{bmatrix}^T \]  

with the function \( f \) defined as the following vector

\[ f = \begin{bmatrix} \kappa', \frac{1}{EI} (T - mv^2) \kappa - mg \frac{mv}{EI} \sin \varphi, \quad mv' - EI \kappa \kappa' - mg \cos \varphi, \quad 0, \quad \kappa, \quad \cos \varphi, \quad \sin \varphi \end{bmatrix}^T \]

The boundary conditions (2) are then be formulated according to the span configuration accordingly.

\[ 2.2. \text{The solution strategy} \]

In order to solve equations (1-2) the BVP with the unknown boundary is reformulated into ‘standard’ form [16] defined over a fixed interval. This is accomplished by introducing the following nondimensional variables

\[ \tilde{s} = \frac{s}{L}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}, \quad \tilde{\kappa} = \frac{\kappa}{L}, \quad \tilde{T} = \frac{TL^2}{EI}, \quad \tilde{v} = \frac{v}{c_0} \]  

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where \( c_0 = \sqrt{T_0/m} \) with \( T_0 \) representing the tension, assumed to be known, at the start point or at the end point of the solution interval. Substituting (11) into equations (6-7) yields

\[
\tilde{\kappa}'' - \left( \tilde{T} - \beta^2 \tilde{L}^2 \tilde{v}^2 \right) \tilde{\kappa} + \frac{mg}{EI} \tilde{L}^3 \sin \phi = 0 \tag{12}
\]

\[
\tilde{T}' - \beta^2 \tilde{L} \tilde{v} \tilde{v}' + \tilde{\kappa} \tilde{v}'' + \frac{mg}{EI} \tilde{L} \cos \phi = 0 \tag{13}
\]

where \( \beta = \sqrt{T_0/EI} \) and \((\cdot)'\) denotes now differentiation with respect to the nondimensional variable \( \tilde{s} \) defined over the fixed interval \( 0 \leq \tilde{s} \leq 1 \). Equations (9-10) assume then the following form

\[
y = \left[ \tilde{\kappa}, \tilde{\kappa}', \tilde{T}, \tilde{L}, \phi, \tilde{x}, \tilde{y} \right]^T \tag{14}
\]

\[
f = \left[ \tilde{\kappa}', \left( \tilde{T} - \beta^2 \tilde{L}^2 \tilde{v}^2 \right) \tilde{\kappa} - \frac{mg}{EI} \tilde{L} \sin \phi, \beta^2 \tilde{L} \tilde{v} \tilde{v}' + \tilde{\kappa} \tilde{v}'' - \frac{mg}{EI} \tilde{L} \cos \phi, 0, \tilde{\kappa}, \cos \phi, \sin \phi \right]^T \tag{15}
\]

respectively. In this formulation the problem can be solved by using a standard BVP solver.

3. Numerical example and results
The solution is demonstrated for span \( AB \) of the cable – pulley system shown in figure 2. It is assumed that no slip occurs at the cable – pulley interface. The following boundary conditions expressed in non-dimensional coordinates are applied

\[
\kappa(0) = \tilde{L}(0) \frac{R}{r}, \quad \kappa(\tilde{L}) = -\tilde{L}(1) \frac{R}{r}, \quad \tilde{T}(1) = \frac{\tilde{L}(1)}{EI} T_0, \quad x(0) = -\frac{R}{\tilde{L}(0)} \sin \phi(0), \quad y(0) = \frac{R}{\tilde{L}(0)} \cos \phi(0)
\]

\[
x(\tilde{L}) = \frac{1}{\tilde{L}(1)} \left[ \tilde{L} - r \sin \phi(1) \right], \quad y(\tilde{L}) = \frac{1}{\tilde{L}(1)} \left[ R + r \left( \cos \phi(1) - 1 \right) \right] \tag{16}
\]

where \( T_0 = \frac{Mg}{2 \cos \phi(\tilde{L})} \) is the tension at \( s = \tilde{L} \).

The BVP described by equations (1-2) and equations (14-15) with the boundary conditions (16) is solved over a time interval of 5 s using the parameters shown in Table 1. Sample results are presented in figures 3–5. Figure 3 shows the slope angle at the contact points at \( s = 0 \) and at \( s = \tilde{L} \), respectively. It is evident that the bending stiffness decreases the angle of wrap. At the boundary with larger curvature the influence of bending stiffness is more prominent. The curvature changes over the span length as demonstrated in figure 4. The distribution curves of tension along the span are shown in figure 5. It is evident that the tension is increased with bending stiffness.
Table 1. Fundamental parameters of the system

| Parameter | Value    | Unit  |
|-----------|----------|-------|
| $M$       | 100      | kg    |
| $m$       | 0.343; 2.15 | kg/m |
| $EI$      | 1.6; 15  | Nm$^2$|
| $V$       | 1.5      | m/s   |
| $L(0)$    | 2        | m     |
| $R$       | 0.75     | m     |
| $r$       | 0.25     | m     |

Figure 3. Slope angles at contact points (a) $s = 0$, (b) $s = \bar{L}$. 
Figure 4. Curvature $\kappa(s;\tau)$. Figure 5. Tensions $T(s;\tau)$.

4. Conclusions
Axially moving elastic continua such as belts, cables and ropes are subject to bending when being wrapped on rigid pulleys/sheaves. The continua can be represented as a moving Euler-Bernoulli beam. The quasi-stationary mechanics of the system is then described by a nonlinear BVP with unknown slowly varying boundaries at the contact points. The model accommodates the weight of the cable and centrifugal effects. The BVP can be solved by converting the problem into the ‘standard’ form defined over a fixed interval. The solution yields the span shape, curvature values, slope angles and the distribution of tension along the span. Due to the bending stiffness the wrap angles are decreased and the distribution of tension is nonuniform. The bending stiffness results in the tension being increased which affects useful life of the system.

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