On Universal Constants of AdS Black Holes from Hawking-Page Phase Transition

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Abstract

We investigate the thermodynamic properties of the Hawking-Page phase transition of AdS black holes. We present evidence for the existence of two universal critical constants associated with the Hawking-Page (HP) and minimum black hole thermodynamical transition points. These constants are defined by $C_S = \frac{S_{HP} - S_{min}}{S_{min}}$ and $C_T = \frac{T_{HP} - T_{min}}{T_{min}}$ where $S_{min}(T_{HP})$ and $T_{min}(S_{HP})$ are the minimal (HP phase transition) entropy and temperature, respectively, below which no black hole can exist. For a large class of four dimensional non-rotating black holes, we find $C_S = 2$ and $C_T = \frac{2 - \sqrt{3}}{\sqrt{3}}$. For the rotating case, however, such universal ratios are slightly affected without losing the expected values. Taking small values of the involved rotating parameter, we recover the same constants. Higher dimensional models, with other universal constants, are also discussed in some details.

Keywords: AdS black holes, Universal behaviors, Hawking-Page phase transition.
1 Introduction

A classical stationary black hole (BH) solution in General Relativity is characterised by its mass $M$, angular momentum $J$ and charge $Q$ alone. In particular, its horizon area is a simple function of these three quantities. Identifying the horizon area as an entropy, the BHs satisfy a set of laws which are directly analogous to those of thermodynamics [1–3]. According to Hawking’s prediction [2,3], BHs emit thermal radiation at the semiclassical level which fixes the Bekenstein-Hawking area/entropy relation to be $S_{BH} \sim \frac{A}{4}$ (where $c = \hbar = G = 1$).

The thermodynamic interpretation of BH physics might be more than a mere analogy and could provide the existence of sublying degrees of freedom [4]. Any consistent theory of quantum gravity could address this challenge in some way or at least advance in this sense. More detailed studies of black holes could be found in many papers including [5,6]. The thermodynamic properties of anti de Sitter (AdS) black holes have been extensively studied, which includes the existence of a minimal Hawking temperature and the Hawking-Page phase transition [7,8]. The Hawking-Page phase transition between large stable black holes and thermal gas in the AdS space has been approached using different methods. For example, an analogy between phase structures of various AdS black holes and statistical models associated with Van der Waals like phase transitions has been suggested [9]. Interpreting the cosmological constant as a kind of thermodynamic pressure, and its conjugate variable as the thermodynamic volume, several non trivial results have been presented [10,11].

Thermodynamics of AdS black holes, in supergravity theories, have been also investigated by exploiting the AdS/CFT correspondence which provides an interplay between gravitational models in $d$-dimensional AdS geometries and $(d−1)$-dimensional conformal field theories living in the boundary of such AdS spaces. Using the physics of solitonic branes, different models in type IIB superstrings and M-theory have been studied by considering the cosmological constant in the bulk closely related to the number of colors associated with branes in question. The thermodynamic stability behavior of such AdS black holes, in higher dimensional known supergravity theories has been examined in this context [12,13].

More concretely, at the semiclassical level, it has been shown that AdS black holes can be in stable thermal equilibrium with radiation. Hawking and Page presented a theoretical evidence for the existence of certain phase transitions in the phase space of the (non-rotating uncharged) Schwarzschild-AdS black hole [14]. Later, a first order phase transition in the charged (non-rotating) Reissner–Nordstrom-AdS (RN-AdS) black hole space-time has been investigated [15–17]. Moreover, many efforts have been devoted to deal with various AdS black holes in different backgrounds including branes and Dark Energy (DE) [18–25].

A close inspection in thermodynamic behaviors of AdS black holes reveals that there exist two temperatures. The first one is the minimum temperature $T_{min}$ below which no black hole can survive. The second one is called Hawking-Page (HP) temperature $T_{HP}$ such that for $T < T_{HP}$ the thermal AdS is the preferred state while for $T > T_{HP}$ the black-hole is the preferred one: the one with the dominant contribution to the semiclassical partition.
function. This is the Hawking-Page phase transition [22].

According to the usual, albeit rather qualitative, wisdom, the Hawking-Page transition indicates that theories must have a small number of states at lower energy, but a huge number of states at high energies with a sharp transition at $T_{HP}$. It turns out that such temperatures could be combined to reveal certain universal behaviors. Such an issue has been unveiled in a previous work [26] and similar ideas have been discussed in [27]. Moreover, universalities in AdS black holes have been approached from other angles [21]. This could open a new window for investigating universal behaviors which could be considered as tools for understanding physical models of black holes.

The aim of this work is to quantify in a precise way the jump in the number of states at the Hawking-Page transition by showing how some kind of universality behavior is at work. Precisely, we prove two universal critical constants associated with thermodynamic transitions of four dimensional AdS black holes. They are given by $C_S = \frac{S_{HP} - S_{min}}{S_{min}}$ and $C_T = \frac{T_{HP} - T_{min}}{T_{min}}$, where $S_{min}$ and $S_{HP}$ are the minimal and $HP$ phase entropies, respectively.

For a large class of non rotating black holes, we find $C_S = 2$ and $C_T = 2 - \frac{\sqrt{3}}{\sqrt{3}}$. For the rotating solution, however, such universal ratios are slightly affected without losing the expected values. For small values of the involved rotating parameter, we recover the same constants. Higher dimensional solutions, with other universal constants, are also examined in some details. In this work, we use dimensionless units in which one has $\hbar = G_4 = c = 1$.

This paper is structured as follows. In section 2, we give the strategy of this work. In section 3, we deal with the non-rotating AdS black hole by computing the associated critical ratios. In section 4, we investigate the rotating AdS black holes. Concluding discussions and open questions are presented in section 5.

## 2 AdS black hole universalities in the $(S, T)$ plane

Many constants arise in the formulation of fundamental physical theories. Such invariant quantities, called fundamental constants, are considered as scalar ones in any coordinate system. The dimensionless type of such universal constants has been of a particular interest. In this way, these constants are expressed as simple numbers with non-trivial meanings. Thus, theirs determinations are very important. With the help of experimental evaluations, one could predict wheather a theory is relevant or not. Besides, the universal constant values have been essential for a precise quantitative description of fundamental theories of the universe. In this paper, we investigate critical behaviors of AdS black holes by proposing some universal ratios and constants associated with the Hawking-Page phase transition from laws of thermodynamics by varying certain parameters [28]. The variation of such parameters have been considered from many reasons. For instance, this could provide a possible origin of such parameters from ‘more fundamental’ theories. Moreover, they could be supported by the need of keeping the scale law (the first degree homogeneity) of the ”internal energy” with
respect the extensive thermodynamic variables (the consistency of the first law of black hole thermodynamics with the Smarr relation) [23]. Before going ahead, we remind some black hole thermodynamics. In the simplest case, the Gibbs (or rather the Helmholtz) function is defined by

\[ G = M - TS, \]  

(2.1)

where \( M \) is the total "energy" or mass of the system, which obeys the "first law of black hole thermodynamics" and where the temperature is given by

\[ T = \frac{\partial M}{\partial S}. \]  

(2.2)

More generally, the Gibbs function is a function of an intensive variable the temperature and any other extensive magnitudes

\[ G = G(T, X_i) \]  

(2.3)

where \( X_i \) denote such extensive magnitude needed to describe the system.

In standard macroscopical thermodynamics, the third law says that the entropy of a system approaches a constant value, \( S_{\min} \) as its temperature approaches absolute zero. This constant value is absolute which cannot depend on any other parameters characterising the closed system. Moreover, at absolute zero the system must be in a state of minimum energy. In the simplest case, there is only an unique state (\( S_{\min} \equiv 0 \)). In other systems, however, there may remain some finite entropy, either because the system becomes locked in a non-minimal energy state, or, because the minimum energy state is not unique. This constant value, \( S_0 \) is usually called the "Residual Entropy" of the system, which occurs if a system can exist in many different states at lowest energies

\[ \lim_{T \to T_{\min}} S(T) = S_{\min}, \]  

(2.4)

\[ \lim_{T \to T_{\min}} \left( \frac{\partial S}{\partial X} \right)_T = 0. \]  

(2.5)

The condition (2.4) is equivalent to the constraint

\[ \frac{\partial T}{\partial S}_{S_{\min}} = \frac{\partial^2 M}{\partial S^2} = 0. \]  

(2.6)

Moreover, it is recalled that

\[ \frac{\partial G}{\partial S} = \frac{\partial U}{\partial S} - \frac{\partial T}{\partial S} S - T = -\frac{\partial T}{\partial S} S. \]  

(2.7)

If we take \( S = S_{\min} \ (\partial T/\partial S = 0) \), then in this case one has \( \frac{\partial G}{\partial S} = 0 \). As illustrated in Fig.1, the free energy reaches a maximum at this point

\[ G(S_{\min}) = G_{\max}. \]  

(2.8)
In the present work, we will compute the minimal or the residual Entropy for a large class of AdS black hole systems and relate it to the Hawking-Page entropy $S_{HP} = S(T_{HP})$ constrained by

$$G(T_{HP}, X_{fix}) = 0. \quad (2.9)$$

It should be noted that the entropy in general and the particular values $S_{HP}$ and $S_{min}$ depend on any other parameters. It has been understood that both are computed at the same value of them. We demonstrate here some universal relations and constants associated with the $HP$ phase transition.

As it was indicated in the introduction, the Hawking-Page transition indicates that quantum theories of gravity must have a small number of states at lower energy, but a huge number of states at higher energies with a sharp transition at $T_{HP}$. The aim of this work is to quantify in a precise way the jump in the number of states at the Hawking-Page transition by showing how some kind of universal behavior is at work.

The main aim, through this work, is to investigate two universal constants at the Hawking-Page transition point. Precisely, they are given by

$$C_S = \frac{S_{HP} - S_{min}}{S_{min}} \quad (2.10)$$

$$C_T = \frac{T_{HP} - T_{min}}{T_{min}}. \quad (2.11)$$

In particular, we will show that the numerical values, for non-rotating black holes, are

$$C_S = 2 \quad (2.12)$$

$$C_T = \frac{2 - \sqrt{3}}{\sqrt{3}}. \quad (2.13)$$

Equivalently, they can be given also by

$$S_{HP} = 3S_{min}, \quad (2.14)$$

$$T_{HP} = \frac{2\sqrt{3}}{3}T_{min}. \quad (2.15)$$
being independent of any parameter. For rotating AdS black holes, however, these ratios depend on other thermodynamic parameters. They can be put like

\[ S_{HP} = 3S_{min} + \ldots, \]
\[ T_{HP} = \frac{2\sqrt{3}}{3} T_{min} + \ldots \]

(2.16)

(2.17)

In what follows, we illustrate such results by giving explicit models with AdS backgrounds. We expect that these explored universal constants could be exploited to unveil more physical data associated with thermodynamic aspects of AdS black holes in four dimensions.

Let us remark that the systematic study of other possible universal relations involving other parameters than \( T \) and \( S \) is beyond the scope of this work.

However, let us simply remark here that we could calculate other constants associated with other parameters including angular momentum \( J \) or the charge \( Q \) of the black hole. In the extended phase space \((P,V)\), for instance, it has been observed that there exists an universal constant involving the volume which will be given in the conclusion part. However, no universal constants corresponding to the pressure can be elaborated because such phase transitions can occur at all pressure values. This also holds for transitions of the charged and rotating black holes which can happen at generic values of the electric potential and the angular velocity.

3 Universal constants of non-rotating AdS-black holes

In this section, we investigate non rotating AdS black holes in four dimensions. Concretely, we first consider a Schwarzschild solution. Then, we deal with a charged black hole.

3.1 Schwarzschild AdS black holes

We first consider the simplest case of a non charged AdS-black hole defined by the following action

\[ I = \frac{1}{16\pi} \int_M d^4x\sqrt{-g}(R - 2\Lambda) \]

(3.18)

where \( \Lambda \) is the cosmological constant. According to [22], this theory involves a black hole solution given by the following line element

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \]

(3.19)

where \( d\Omega^2 \) is a 2-dimensional unit sphere. \( f(r) = 1 - \frac{m}{r} + \frac{r^2}{L^2} \) denotes the metric function where \( L \) is a fixed constant, considered as the AdS radius \( L \) being related to a negative cosmological constant, via \( \Lambda = -3/L^2 \). Here, we consider two cases associated with such a constant.

**Case 1: Fixed cosmological constant**
Using a normalised entropy $S \equiv \frac{S_{BH}}{\pi} = \frac{4}{4\pi} = r_h^2$ with $r_h$ the radius of the black hole horizon and taking $L = 1$, the black hole mass is

$$M = \frac{\sqrt{S}(1 + S)}{2}.$$  \hspace{1cm} (3.20)

The temperature has the following form

$$T = \frac{\partial M}{\partial S} = \frac{1 + 3S}{4\sqrt{S}}.$$  \hspace{1cm} (3.21)

The variation of the temperature gives the minimal entropy and the minimal temperature $(S_{\text{min}}, T_{\text{min}})$. A simple calculation shows that

$$(S_{\text{min}}, T_{\text{min}}) = \left(\frac{1}{3}, \frac{\sqrt{3}}{2}\right).$$  \hspace{1cm} (3.22)

For the Hawking-Page phase transition, one should use the Gibbs free energy given by the following formula

$$G = M - T \cdot S = \frac{1}{4}\sqrt{S}(1 - S).$$ \hspace{1cm} (3.23)

It is easy to find

$$(S_{HP}, T_{HP}) = (1, 1).$$ \hspace{1cm} (3.24)

Combining Eq (3.22) and (3.24), we obtain the desired universal critical ratios

$$\frac{S_{HP}}{S_{\text{min}}} = 3, \quad \frac{T_{HP}}{T_{\text{min}}} = \frac{2}{\sqrt{3}}.$$ \hspace{1cm} (3.25)

**Case 2: Thermodynamical cosmological constant**

Treating the cosmological constant as a thermodynamical variable playing the role of a pressure $\Lambda = \frac{-3}{L^2} = -8\pi P$, the first law of BH thermodynamics becomes

$$dM = TdS + VdP,$$ \hspace{1cm} (3.26)

where $V$ is the thermodynamic volume conjugated to the pressure $P$ ($V = \partial M/\partial P$) \[29\]. In this case, the masse takes the following form

$$M = \frac{\sqrt{S}(3 + 8\pi SP)}{6}.$$ \hspace{1cm} (3.27)

The temperature and the volume are given by

$$T = \frac{1 + 8P\pi S}{4\sqrt{S}}$$ \hspace{1cm} (3.28)

$$V = \frac{4}{3}\pi S^3.$$ \hspace{1cm} (3.29)

An easy computation reveals that the minimal temperature (at fixed $P$) and the associated entropy are

$$(S_{\text{min}}, T_{\text{min}}) = \left(\frac{1}{8\pi P}, \sqrt{2\pi P}\right).$$ \hspace{1cm} (3.30)
For this model, the canonical Gibbs free energy reads as

\[ G = M - T S = \frac{\sqrt{S}}{12} (3 - 8\pi PS). \] (3.31)

Using the normalised entropy, we find the critical point in the \((S, T)\) space

\[ (S_{HP}, T_{HP}) = \left( \frac{3}{8\pi P}, \sqrt{\frac{8\pi P}{3}} \right). \] (3.32)

Combining Eq (3.30) and (3.32), we find \( \frac{S_{HP}}{S_{min}} = 3 \) and \( \frac{T_{HP}}{T_{min}} = \frac{2}{\sqrt{3}} \).

### 3.2 Reissner–Nordstrom-AdS (RN-AdS) black holes

This four dimensional AdS black hole is defined by the metric function

\[ f(r) = 1 - \frac{m}{r} + \frac{r^2}{L^2} + \frac{Q^2}{r^2}, \] (3.33)

where \( Q \) is the charge [22]. Using a entropy reduced expression and putting \( L = 1 \), the black hole mass reads as

\[ M = \sqrt{S} \left( S + (1 + \Phi^2) \right), \] (3.34)

where \( \Phi = \frac{Q}{r} \) is the electric potential conjugated to the charge which can be defined as the difference between the electric potential at the event horizon and the boundary \( r \to \infty \). For simplicity reasons, we will consider only a thermodynamic phase space corresponding to the variables \((S, Q)\) by fixing \( L \). It is worth to note that the extension to a more general case, where the latter is also variable, could be possible as in the non-charged case. Using the associated first law of thermodynamics, the expression of the temperature can be written as

\[ T = \frac{3S + (1 - \Phi^2)}{4\sqrt{S}}. \] (3.35)

The variation of the temperature with respect to the entropy gives

\[ (S_{min}, T_{min}) = \left( \frac{(1 - \Phi^2)}{3}, \frac{\sqrt{3}}{2} (1 - \Phi^2)^{1/2} \right). \] (3.36)

To get the remaining quantities corresponding to the Hawking-Page phase transition, one can exploit the following grand canonical Gibbs free energy

\[ G = \sqrt{S} \left[ (1 - \Phi^2) - S \right]. \] (3.37)

Imposing \( G(T_{HP}) = 0 \), we find

\[ (S_{HP}, T_{HP}) = \left( (1 - \Phi^2), (1 - \Phi^2)^{1/2} \right). \] (3.38)

Using Eq.(3.36) and Eq.(3.38), we can determine the expressions of the universal critical ratios. After a simple examination of such the involved relations, we find \( C_S = 2 \) and
$C_T = \frac{2-\sqrt{3}}{\sqrt{3}}$. It is interesting to precise that one can also consider other non-rotating AdS black holes by implementing other external parameters including non-trivial backgrounds. We expect that such models provide similar universal critical constants.

Having discussed explicit models for non-rotating AdS models, we move to consider a rotating model. We will show that the latter exhibits identical universal ratios by considering small values of the involved parameters.

4 Rotating AdS black holes and universalities

Kerr black holes are characterised by their mass and angular momentum. The latter provides non trivial behaviors, even for optical aspects. Following [30, 31], the line element of a four dimensional Kerr-AdS metric reads as

$$ds^2 = \frac{\Sigma^2}{\Delta_r} dr^2 + \frac{\Sigma^2}{\Delta_\theta} d\theta^2 + \frac{\Delta r \sin^2 \theta}{\Sigma^2} \left( a \frac{dt}{\Xi} - (r^2 - a^2) \frac{d\phi}{\Xi} \right)^2 - \frac{\Delta r}{\Sigma^2} \left( \frac{dt}{\Xi} - a \sin^2 \theta \frac{d\phi}{\Xi} \right)^2. \quad (4.39)$$

The involved terms are

$$\Delta_r = r^2 - 2rn + a^2 + \frac{r^2}{L^2} (r^2 + a^2), \quad \Delta_\theta = 1 - \frac{a^2}{L^2} \cos^2 \theta, \quad \Xi = 1 - \frac{a^2}{L^2}, \quad \Sigma^2 = r^2 + a^2 \cos^2 \theta. \quad (4.40)$$

Using the normalized entropy \(S \equiv \frac{S_{BH}}{\pi}\) and the scaled quantities

$$\Omega = \frac{J}{MS} (1 + S), \quad (4.41)$$

$$\frac{J}{S} = \frac{M\Omega}{1 + S}, \quad (4.42)$$

$$a = \frac{J}{M} = S\Omega (1 + S)^{-1}. \quad (4.43)$$

where $\Omega$ is the difference between the angular velocities at the event horizon ($\Omega_h$) and at infinity ($\Omega_\infty$), one obtains the mass which is given by

$$M^2 = \frac{S (1 + S)^2}{4 (1 - \frac{\Omega^2 S^2}{1 + S})}. \quad (4.44)$$

Similarly, the Hawking temperature is found to be

$$T = \sqrt{\frac{S (1 + S)}{(1 + S - S\Omega^2)} \left[ \frac{1 - 2S(\Omega^2 - 2) - 3S^2(\Omega^2 - 1)}{4S(1 + S)} \right]}. \quad (4.45)$$

Varying the temperature with respect to the normalised entropy, we find the minimal entropy

$$S_{min} = -\frac{2}{3} \left( \frac{1 - \Omega^2/2}{1 - \Omega^2} \right) + (\lambda_1 + \lambda_2), \quad (4.46)$$
where the terms $\lambda_1$ and $\lambda_2$ are given respectively by

$$
\lambda_1 = \frac{1}{3} \sqrt{\frac{(1 - \Omega^2 + \Omega^4)}{(1 - \Omega^2)^2}} + \frac{3 \times \Omega^{4/3}}{2^{2/3} (-1 + 4\Omega^2 - 6\Omega^4 + 4\Omega^6 - \Omega^8)^{1/3}},
$$

$$
\lambda_2 = \frac{2}{3} \left[ \frac{(1 - \Omega^2 + \Omega^4)}{2 (1 - \Omega^2)} - \frac{2^{1/3} \times \Omega^{1/3}}{8 (-1 + 4\Omega^2 - 6\Omega^4 + 4\Omega^6 - \Omega^8)^{1/3}} \right.
$$

$$
\left. + \frac{2^{2/3}}{6 (1 - \Omega^2)^{3/2}} \sqrt{\frac{4(1-\Omega^2-\Omega^4)}{9(1-\Omega^2)^4} + \frac{24/3\Omega^{1/3}}{3(-1+4\Omega^2-6\Omega^4+4\Omega^6-\Omega^8)^{1/3}}} \right]^{1/2}.
$$

Using the grand canonical Gibbs free energy

$$
G = \sqrt{\frac{S(1+S)}{1+S-S\Omega^2}} (1 - S^2 (1 - \Omega^2)),
$$

we get the Hawking-Page entropy

$$
S_{HP} = \frac{1}{\sqrt{1-\Omega^2}}.
$$

A close inspection shows that the calculation is quite complicated deserving appropriate simplifications. Replacing Eq.(4.46) in Eq.(4.45), we get the following formula

$$
T_{min} = \frac{\sqrt{3}}{4} \times \lambda_3 \left( \lambda_3 + 2\sqrt{1 - \Omega^2 + \Omega^4} \right) \times \frac{1}{\left[ (1 - 2\Omega^2 + \sqrt{1 - \Omega^2 + \Omega^4} + \lambda_3) (-2 + \sqrt{1 - \Omega^2 + \Omega^4} + \lambda_3) (1 + \sqrt{1 - \Omega^2 + \Omega^4} + \lambda_3) \right]^{1/2}},
$$

where the $\lambda_3$ term is

$$
\lambda_3 = \sqrt{2(1 - \Omega^2 + \Omega^4) + \frac{(2 - \Omega^2)(1 - \Omega^2 - 2\Omega^4)}{\sqrt{1 - \Omega^2 + \Omega^4}}}. \quad (4.53)
$$

The Hawking-Page temperature, associated with the entropy given in Eq.(4.51), reads as

$$
T_{HP} = \sqrt{\frac{1 - \Omega^2 (3 + \sqrt{1 - \Omega^2})}{4 (1 - \sqrt{1 - \Omega^2})}}.
$$

To obtain the desired critical ratios, several approximations are needed. For small values of $\Omega$, we obtain the following simplified forms

$$
C_S(\Omega) = \frac{S_{HP} - S_{min}}{S_{min}} = 2 + \frac{27\Omega^4}{8},
$$

$$
C_T(\Omega) = \frac{T_{HP} - T_{min}}{T_{min}} = \frac{2 - \sqrt{3}}{\sqrt{3}} + \frac{3\sqrt{3}\Omega^2}{4}.
$$

It is easy to recover the previous constants by taking the limit $\Omega$ goes to zero. At first sight, this rotation case seems strange. A close inspection, however, shows that this black
hole solution usually exhibits non-trivial behaviors compared to the non-rotating one. Such behaviors have been clearly observed in the investigation of optical aspects of black holes. Concretely, it has been shown that the shadows of non-rotating black holes processes a perfect circle. This geometry, however, has been distorted by implementing the spin rotation parameters $[32,33]$. Moreover, circular behaviors can be recovered by taking small values of such parameters. It is not surprising to have similar ideas for thermodynamic properties. It is worth noting that the angular velocity dependency in the ratio of Eq(4.46) by Eq(4.51) is just analytical. In fact, numerical computations shows that the ratio $S_{min}/S_{HP}$ is equal to the value $1/3$ for small values of the angular velocity. Since analytical expressions are needed, we have applied several approximations to obtain the ones given in Eq(4.55) and Eq(4.56).

5 Conclusions and discussions

Various numerical constants appear in the formulation of fundamental physical theories. The dimensionless ones have been of a particular interest. The determination of such scalar constants is very important. With the help of experimental evaluations, one could predict whether a theory is relevant or not. Besides, the universal constant values have been essential for a precise quantitative description of fundamental theories of the universe. In this paper, we have investigated certain universal constants arising in four dimensional AdS black holes involving two primordial critical points. The first critical point corresponds to the Hawking Page phase transition described by the entropy $S_{HP}$, while the second one is associated with the bound under which no black hole can survive characterized by the entropy $S_{min}$. These critical values produce $T_{HP}$ and $T_{min}$, respectively. Precisely, we have shown that the ratios $S_{HP}/S_{min}$ and $T_{HP}/T_{min}$ can be considered as universal numbers predicted by a large class of AdS black hole in four dimensions. For four dimensional non-rotating AdS black holes, we have found two universal constants $C_S = S_{HP}/S_{min} = 2$ and $C_T = T_{HP}/T_{min} = \frac{2 - \sqrt{3}}{\sqrt{3}}$. For the rotating solution, however, such universal ratios are slightly affected without losing the expected values. For very small values of the involved rotating parameter, we have recovered the same critical constants.

The present approach could be adaptable for other black holes. In particular, a close examination shows that this analysis might be extended to non trivial theories of gravity with AdS geometries supported by branes. A concrete model derived from the compactification of M-theory on the sphere $\mathbb{S}^7$ in the presence of $N$ coincident M2-branes is described by the line element $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2h_{ij}dx^i dx^j + L^2 d\Omega_7^2$ where $f(r) = 1 - \frac{m}{r} + \left(\frac{r}{L}\right)^2$. It is noted that $h_{ij} dx^i dx^j$ is the line element of a 2-sphere and $d\Omega_7^2$ is the 7-dimensional sphere with the radius $L$ depending on the brane colors $N^{3/2}$ $[26]$. Using the reduced entropy $S = \frac{S_{BH}}{\pi^{3/2}}$ and $L^9 = \frac{\pi^{N^{3/2}}}{2^{7/2}}$, one obtains

$$M = \left[ S^{\frac{3}{2}} N^{\frac{7}{12}} + 96\sqrt{2} \times S^3 N^{-\frac{11}{12}} \right].$$

(5.57)
Using the first law of thermodynamics, one finds the Hawking temperature

\[ T = \frac{\sqrt{3\pi^{1/6}}}{48 2^{7/12} \ell_p} \left[ S^{-\frac{1}{2}} N^{\frac{7}{12}} + 3 \times 96\sqrt{2} S^{\frac{3}{2}} N^{-\frac{1}{12}} \right]. \quad (5.58) \]

The minimal value of the entropy and the corresponding minimal temperature are

\[ (S_{\text{min}}, T_{\text{min}}) = \left( \frac{N^{3/2}}{3 \cdot 2^{11/2}}, \frac{3^{1/2} \pi^{1/6}}{256 N^{1/6} \ell_p} \right). \quad (5.59) \]

Using the grand canonical Gibbs free energy given by

\[ G = \frac{\sqrt{3\pi^{1/6}}}{48 2^{7/12} \ell_p} \left[ S^{\frac{3}{2}} N^{\frac{7}{12}} - 96\sqrt{2} \times S^{\frac{3}{2}} N^{-\frac{1}{12}} \right], \quad (5.60) \]

we get the following critical point

\[ (S_{HP}, T_{HP}) = \left( \frac{N^{3/2}}{3 \cdot 2^{11/2}}, \frac{2^{1/6} \pi^{1/6}}{N^{1/6} \ell_p} \right). \quad (5.61) \]

Combining Eq.(5.59) and Eq.(5.61), we provide the universal critical ratios of the non rotating case. According to [26], these constants can be also obtained with DE contributions.

A close inspection shows that the two universal constants depend on the space-time dimension. To see that, we consider a concrete example of a \( d \) dimensional charged AdS black hole given by the following metric

\[ f(r) = 1 - \frac{m}{r^{d-3}} + \frac{q^2}{r^{2d-6}} + \frac{\ell^2}{r^2}, \quad (5.62) \]

where \( q \) is a quantity related to the charge \( Q = \sqrt{2(d-2)(d-3)} \left( \frac{q}{8\pi} \right) \). The electrostatic potential conjugated to such a quantity is \( \phi = \frac{\sqrt{2(d-2)} q}{r^2}. \) The calculations show that the temperature and the Gibbs free energy for this charged non-rotating solution are given by

\[ T = \frac{1}{4\pi} \left[ \frac{\omega_{d-2}}{4S} \right]^\frac{1}{d-2} \left( (d-3) \left( 1 - \frac{2(d-3)}{d-2} \phi^2 \right) + (d-1) \left[ \frac{4S}{\omega_{d-2}} \right]^\frac{2}{d-2} \right), \quad (5.63) \]

\[ G = \frac{1}{\pi} \left[ \frac{S^{d-3}}{4d-1 \omega_{d-2}} \right]^\frac{1}{d-2} \left( \omega_{d-2}^\frac{2}{d-2} \left( 1 - \frac{2(d-3)}{d-2} \phi^2 \right) - (4S)^\frac{2}{d-2} \right). \quad (5.64) \]

Using these expressions, we obtain

\[ S_{\text{min}} = \frac{\omega_{d-2}}{4} \left( \frac{d-3}{(d-2)(d-1)} \right)^\frac{d-2}{d-2} \left( (d-2) - 2(d-3) \phi^2 \right)^\frac{d-2}{d-2}, \quad (5.65) \]

\[ T_{\text{min}} = \frac{\sqrt{(d-3)(d-1)}}{2\pi \sqrt{(d-2)}} \sqrt{(d-2) - 2(d-3) \phi^2}, \quad (5.66) \]

\[ S_{HP} = \frac{\omega_{d-2}}{4(d-2)^{\frac{d-2}{d-1}}} \left( (d-2) - 2(d-3) \phi^2 \right)^{\frac{d-2}{d-1}}, \quad (5.67) \]

\[ T_{HP} = \frac{\sqrt{(d-2)}}{2\pi} \sqrt{(d-2) - 2(d-3) \phi^2}. \quad (5.68) \]
Computing the universal constants for such a model, we get

\[
C_S(d) = \frac{(d - 3)^{\frac{d}{2}} - (d - 1)^{\frac{d}{2}}}{(d - 1)^{\frac{d}{2}}},
\]

(5.69)

\[
C_T(d) = \frac{(d - 2) \sqrt{(d - 1)(d - 3)} - (d - 1)(d - 3)}{(d - 1)(d - 3)},
\]

(5.70)

For \(d = 4\), we recover the values given in section 3.

At this point, we remark that other universal constants related to the angular momentum \(J\) or the charge \(Q\) of the black hole can be shown to appear in presence of a Hawking-Page transition. In the extended phase space \((P, V)\), indeed, there exists an universal constant by considering the volume \(V\), conjugated variable to \(P\), the Cosmological constant “pressure”. In the \(d\)-dimensional Schwarzschild AdS black hole, for instance, calculations provide the following relations

\[
V = \frac{(4\mathcal{S})^{d-1}}{(d - 1)\omega_{d-2}},
\]

(5.71)

\[
S_{\text{min}} = \frac{\omega_{d-2}}{4^{d-1}} \times \left(\frac{(d - 3)(d - 2)}{\pi P}\right)^{\frac{d-2}{2}},
\]

(5.72)

\[
S_{\text{HP}} = \frac{\omega_{d-2}}{4^{d-1}} \times \left(\frac{(d - 2)(d - 1)}{\pi P}\right)^{\frac{d-2}{2}},
\]

(5.73)

where \(\omega_{d-2}\) is the volume of the unit \((d - 2)\)-sphere. Combining these relations, we get a new universal constant, related to the cosmological constant

\[
\frac{V_{\text{HP}}}{V_{\text{min}}} = \left(\frac{d - 1}{d - 3}\right)^{\frac{d-1}{2}},
\]

(5.74)

which depends only on the dimension \(d\). Obviously, due to Eq.(5.71), this ratio is closely related to the \(C_S\) or \(C_T\) quantities.

This work comes up with certain open questions. One of them is to understand such universal ratio behaviors for higher dimensional AdS black holes using shadow analysis. The associated physics need deeper reflections and investigations. Another question concerns the variation of external parameters associated with non-trivial backgrounds on which black hole solutions will be built. We leave these questions for future works.

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