Smooth Kernel Estimation of a Circular Density Function: A Connection to Orthogonal Polynomials on the Unit Circle

Yogendra P. Chaubey
Smooth Kernel Estimation of a Circular Density Function: A Connection to Orthogonal Polynomials on the Unit Circle

Yogendra P. Chaubey

Department of Mathematics and Statistics, Concordia University, Montréal, QC H3G 1M8
e-mail: yogen.chaubey@concordia.ca

Abstract: In this note we provide a simple approximation theory motivation for the circular kernel density estimation and further explore the usefulness of the wrapped Cauchy kernel in this context. It is seen that the wrapped Cauchy kernel appears as a natural candidate in connection to orthogonal series density estimation on a unit circle. This adds further weight to the considerable role of the wrapped Cauchy in circular statistics.

AMS 2000 subject classifications: Primary 62G07; secondary 62G20.

Keywords and phrases: Circular kernel density estimator, Orthogonal series density.

1. Introduction

Consider an absolutely continuous (with respect to the Lebesgue measure) circular density \( f(\theta), \theta \in [-\pi, \pi], \) i.e \( f(\theta) \) is \( 2\pi \)-periodic,

\[
f(\theta) \geq 0 \text{ for } \theta \in \mathbb{R} \text{ and } \int_{-\pi}^{\pi} f(\theta)d\theta = 1.
\] (1.1)

In the literature on modeling circular data, starting from the classical text of Mardia (1972), there appear many standard texts such as Fisher (1993), Jammalamadaka and SenGupta (2001) and Mardia and Jupp (1972) that cover parametric models along with many inference problems. More recently various alternatives to these classical parametric models, exhibiting asymmetry and multimodality have been investigated with respect to their mathematical properties and goodness of fit to some real data; see Abe and Pewsey
In cases where multimodal and/or asymmetric models may be appropriate, semiparametric or nonparametric modelling may be considered more appropriate. Fernández-Durán (2004) and Mooney et al. (2003) considered semi-parametric analysis based on mixture of circular normal and von Mises distributions and Hall et al. (1987), Bai et al. (1988), Fisher (1989), Taylor (2008) and Klemelä (2000) have considered nonparametric approaches.

Given a random sample \((\theta_1, \ldots, \theta_N)\) from the density \((1.1)\), the circular kernel density estimator is given by

\[
\hat{f}(\theta; h) = \frac{1}{N} \sum_{j=1}^{N} k_h(\theta - \theta_j) \tag{1.2}
\]

where \(k_h(\theta - \phi)\) is a circular kernel density function that is concentrated around \(\phi\) as \(h \to h_0\) for some known \(h_0\). As motivated in Taylor (2008), a natural choice for the kernel function is one of the commonly used circular probability densities, such as the wrapped normal distribution, or the von Mises distribution. Taylor (2008) investigated the use of von Mises kernel, in which case the density estimator is given by

\[
\hat{f}_{VM}(\theta; \nu) = \frac{1}{N(2\pi)I_0(\nu)} \sum_{j=1}^{N} \exp\{\nu(\theta - \theta_j)\} \tag{1.3}
\]

where \(I_0(\nu)\) is the Bessel function of order \(r\) and \(\nu\) is the concentration parameter. Di Marzio et al. (2009) considered the use of circular kernels to circular regression while extending the use of von Mises kernels to more general circular kernels. In the present note I demonstrate that the wrapped Cauchy kernel presents itself as the kernel of choice by considering an estimation problem on the unit circle. We also show that this approach leads to orthogonal series density estimation, however no truncation of the series is required. It may be noted that the wrapped Cauchy distribution with location parameter \(\mu\) and concentration parameter \(\rho\) is given by

\[
f_{WC}(\theta; \mu, \rho) = \frac{1}{2\pi} \frac{1}{1 + \rho^2 - 2\rho \cos(\theta - \mu)}, -\pi \leq \theta < \pi, \tag{1.4}
\]

that becomes degenerate at \(\theta = \mu\) as \(\rho \to 1\). The estimator of \(f(\theta)\) based on
the above kernel is given by
\[ \hat{f}_{WC}(\theta; \rho) = \frac{1}{N} \sum_{j=1}^{N} f_{WC}(\theta; \theta_j, \rho). \] (1.5)

In Section 2, we provide a simple approximation theory argument behind
the nonparametric density estimator of the type introduced in (1.2) and (1.3).
In Section 3, first we present some basic results from the literature on
orthogonal polynomials on the unit circle and then introduce the strategy of
estimating \( f(\theta) \) by estimating an expectation of a specific complex function,
that in turn produces the non-parametric circular kernel density estimator
in (1.5). The next section shows that the circular kernel density estimator is
equivalent to the orthogonal series estimation in a limiting sense. This equiv-
alence establishes a kind of qualitative superiority of the kernel estimator
over the orthogonal series estimator that requires the series to be truncated,
however the kernel estimator does not have such a restriction.

2. Motivation for the Circular Kernel Density Estimator

The starting point of the nonparametric density estimation is the theorem
given below from approximation theory (see Mhaskar and Pai (2000)). Before
giving the theorem we will need the following definition:

Definition 2.1. Let \( \{K_n\} \subset C^{*} \) where \( C^{*} \) denotes the set of periodic analytic
functions with a period 2\( \pi \). We say that \( \{K_n\} \) is an approximate identity if

A. \( K_n(\theta) \geq 0 \ \forall \ \theta \in [-\pi, \pi] \);
B. \( \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) = 1 \);
C. \( \lim_{n \to \infty} \max_{|\theta| \geq \delta} K_n(\theta) = 0 \) for every \( \delta > 0 \).

The definition above is motivated from the following theorem which is simi-
lar to the one used in the theory of linear kernel estimation (see Prakasa Rao
(1983)).

Theorem 2.1. Let \( f \in C^{*} \), \( \{K_n\} \) be approximate identity and for \( n = 1, 2, \ldots \),
set
\[ f^{*}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\eta) K_n(\eta - \theta) d\eta. \] (2.1)
Then we have
\[ \lim_{n \to \infty} \sup_{x \in [-\pi, \pi]} |f^{*}(x) - f(x)| = 0. \] (2.2)
Note that taking the sequence of concentration coefficients \( \rho \equiv \rho_n \) such that \( \rho_n \to 1 \), the density function of the Wrapped Cauchy will satisfy the conditions in the definition in place of \( 2\pi K_n \). The integral in the above theorem. In general \( 2\pi K_n \) may be replaced by a sequence of periodic densities on \([-\pi, \pi]\], that converge to a degenerate distribution at \( \theta = 0 \).

For a given random sample of \( \theta_1, \ldots, \theta_N \) from the circular density \( f \), the Monte-Carlo estimate of \( f^* \) is given by

\[
\hat{f}(\theta) = \frac{1}{(2\pi)^N} \sum_{j=1}^{N} K_n(\theta_j - \theta),
\]

the suffix \( n \) for the kernel \( K \) may be a function of the sample size \( N \). The kernel given by the wrapped Cauchy density satisfies the assumptions in the above theorem that provides the estimator proposed in (1.5).

3. Some Preliminary Results from Complex Analysis

Let \( D \) be the open unit disk, \( \{z \mid |z| < 1\} \), in \( \mathbb{Z} \) and let \( \mu \) be a continuous measure defined on the boundary \( \partial D \), i.e. the circle \( \{z \mid |z| = 1\} \). The point \( z \in D \) will be represented by \( z = re^{i\theta} \) for \( r \in [0, 1), \theta \in [0, 2\pi) \) and \( i = \sqrt{-1} \).

A standard result in complex analysis involves the Poisson representation that involves the real and complex Poisson kernels that are defined as

\[
P_r(\theta, \varphi) = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \varphi)} \quad (3.1)
\]

for \( \theta, \varphi \in [0, 2\pi) \) and \( r \in [0, 1) \) and by

\[
C(z, \omega) = \frac{\omega + z}{\omega - z} \quad (3.2)
\]

for \( \omega \in \partial D \) and \( z \in D \). The connection between these kernels is given by the fact that

\[
P_r(\theta, \varphi) = \text{Re } C(re^{i\theta}, e^{i\varphi}) = (2\pi) f_{\text{WC}}(\theta; \varphi; \rho). \quad (3.3)
\]

The Poisson representation says that if \( g \) is analytic in a neighborhood of \( \mathbb{D} \) with \( g(0) \) real, then for \( z \in D \),

\[
g(z) = \int \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \text{Re}(g(e^{i\theta})) \frac{d\theta}{2\pi} \quad (3.4)
\]
(see (Simon, 2005, p. 27)). This representation leads to the result (see (ii) in §5 of Simon (2005)) that for Lebesgue a.e. $\theta$,

$$\lim_{r \uparrow 1} F(re^{i\theta}) \equiv F(e^{i\theta})$$

(3.5)

exists and if $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$ with $d\mu_s$ singular, then

$$w(\theta) = \text{Re} F(e^{i\theta}),$$

(3.6)

where

$$F(z) = \int \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) d\mu(\theta).$$

(3.7)

Our strategy for smooth estimation is the fact that for $d\mu_s = 0$ we have

$$f(\theta) = \frac{1}{2\pi} \lim_{r \uparrow 1} \text{Re} F(re^{i\theta}),$$

(3.8)

where

$$F(z) = \int \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) f(\theta) d\theta.$$

(3.9)

We define the estimator of $f(\theta)$ motivated by considering an estimator of $F(z)$, the identity (3.6) and (3.8), i.e.

$$\hat{f}_r(\theta) = \frac{1}{2\pi} \text{Re} F_N(re^{i\theta})$$

(3.10)

where

$$F_N(re^{i\theta}) = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{e^{i\theta_j} + re^{i\theta}}{e^{i\theta_j} - re^{i\theta}} \right),$$

(3.11)

where $r$ has to be chosen appropriately. Recognize that

$$F_N(re^{i\theta}) = \frac{1}{N} \sum_{j=1}^{N} C(z, \omega_j),$$

(3.12)

where $\omega_j = e^{i\theta_j}$, then using (3.4), we have

$$\text{Re} F_N(re^{i\theta}) = \frac{1}{N} \sum_{j=1}^{N} P_r(\theta, \theta_j),$$

(3.13)
and therefore

\[
\hat{f}_r(\theta) = \frac{1}{(2\pi)N} \sum_{j=1}^{N} P_r(\theta, \theta_j)
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} f_{WC}(\theta; \theta_j, r),
\]

that is of the same form as in (1.5).

4. Orthogonal Series Estimation

We get the orthogonal expansion of \(F(z)\) with respect to the basis \(\{1, z, z^2, \ldots\}\) as

\[
F(z) = 1 + 2 \sum_{n=1}^{\infty} c_n z^n
\]

where

\[
c_n = \int e^{-i n \theta} f(\theta) d\theta,
\]

is the \(j^{th}\) trigonometric moment. The series is truncated at some term \(N^*\) so that the error is negligible. However, we show below that estimating the trigonometric moment \(c_n, n = 1, 2, \ldots\) as

\[
\hat{c}_n = \frac{1}{N} \sum_{j=1}^{N} e^{-i n \theta_j},
\]
the estimator of $F(z)$ is the same as given in the previous section. This can be shown by writing

$$
\hat{F}(z) = 1 + \frac{2}{N} \sum_{j=1}^{N} \left\{ \sum_{n=1}^{\infty} e^{-im\theta_j} z^n \right\} \\
= 1 + \frac{2}{N} \sum_{j=1}^{N} \left\{ \sum_{n=1}^{\infty} (\tilde{\omega}_j z) ^n \right\}; \omega_j = e^{im\theta_j} \\
= 1 + \frac{2}{N} \sum_{j=1}^{N} \left( \frac{\tilde{\omega}_j z}{1 - \tilde{\omega}_j z} \right) \\
= \frac{2}{N} \sum_{j=1}^{N} \left( \frac{1}{2} + \frac{\tilde{\omega}_j z}{1 - \tilde{\omega}_j z} \right) \\
= \frac{1}{N} \sum_{j=1}^{N} \left( 1 + \tilde{\omega}_j z \right) \\
= \frac{1}{N} \sum_{j=1}^{N} C(z, \omega_j),
$$

which is the same as $F_N(z)$ given in (3.12). This ensures that the orthogonal series estimator of the density coincides with the circular kernel estimator.

The determination of the smoothing constant may be handled based on the cross validation method outlined in Taylor (2008).

**Remark:** Note that the simplification used in the above formulae does not work for $r = 1$. Even though, the limiting form of (4.1) is used to define an orthogonal series estimator as given by

$$
\hat{f}_S(\theta) = \frac{1}{2\pi} + \frac{1}{\pi N} \sum_{j=1}^{N} \sum_{n=1}^{n^*} \cos n(\theta - \theta_j),
$$

where $n^*$ is chosen according to some criterion, for example to minimize the integrated squared error. Thus the above discussion presents two contrasting situations: in one we have to determine the number of terms in the series and in the other number of terms in the series is allowed to be infinite, however, we choose to evaluate $\text{Re} F(re^{i\theta})$ for some $r$ close to 1 as an approximation to $\text{Re} F(e^{i\theta})$. 
References

Abe, T. and Pewsey, A. (2011). Symmetric Circular Models Through Duplication and Cosine Perturbation. *Computational Statistics & Data Analysis*, **55**(12), 3271–3282.

Bai, Z.D., Rao, C.R., Zhao, L.C. (1988). Kernel Estimators of Density Function of Directional Data. *Journal of Multivariate Analysis*, **27**(1), 24–39.

Di Marzio, M., Panzera, A., Taylor, C.C. (2009). Local Polynomial Regression for Circular Predictors. *Statistics & Probability Letters*, **79**(19), 2066–2075.

Fernández-Durán, J.J. (2004). *Circular distributions based on nonnegative trigonometric sums*, *Biometrics*, **60**, 499–503.

Fisher, N.I. (1989). Smoothing a sample of circular data, *J. Structural Geology*, **11**, 775–778.

Fisher, N.I. (1993). *Statistical Analysis of Circular Data*. Cambridge University Press, Cambridge.

Hall P, Watson GP, Cabrera J (1987). Kernel Density Estimation for Spherical Data. *Biometrika*, **74**(4), 751–762.

Jammalamadaka, S. R., SenGupta, A. (2001). *Topics in Circular Statistics*. World Scientific, Singapore.

Jones, M.C. and Pewsey, A (2012). Inverse Batschelet Distributions for Circular Data. *Biometrics*, **68**(1), 183–193.

Kato, S. and Jones, M.C. (2010). A family of distributions on the circle with links to, and applications arising from, Möbius transformation. *J. Amer. Statist. Assoc.*, **105**, 249-262.

Kato, S. and Jones, M. C. (2015). A tractable and interpretable four-parameter family of unimodal distributions on the circle. *Biometrika*, **102**(9), 181–190.

Klemelä, J. (2000). Estimation of Densities and Derivatives of Densities with Directional Data. *Journal of Multivariate Analysis*, **73**(1), 18–40.

Mardia, K.V. (1972). *Statistics of Directional Data*. Academic Press, New York.

Mardia, K.V. and Jupp, P. E. (2000). *Directional Statistics*. John Wiley & Sons, New York, NY, USA.

Mhaskar, H.N. and Pai, D.V. (2000). *Fundamentals of approximation Theory*. Narosa Publishing House, New Delhi, India.

Minh, D. and Farnum, N. (2003). Using bilinear transformations to induce probability distributions. *Commun. Stat.-Theory Meth.*, **32**, 1–9.

Mooney A., Helms P.J. and Jolliffe, I.T. (2003). Fitting mixtures of von
Mises distributions: a case study involving sudden infant death syndrome, *Comput. Stat. Data Anal.*, 41, 505–513.
Prakasa Rao, B.L.S. (1983). *Non Parametric Functional Estimation*. Academic Press, Orlando, Florida.
Simon, B. (2005). *Orthogonal Polynomials on the Unit Circle*, Part 1: Classical Theory. American Mathematical Society, Providence, Rhode Island.
Shimizu, K. and Iida, K. (2002). Pearson type vii distributions on spheres. *Commun. Stat.–Theory Meth.*, 31, 513–526.
Taylor, C.C. (2008). Automatic Bandwidth Selection for Circular Density Estimation. *Computational Statistics & Data Analysis*, 52(7), 3493–3500.
List of Recent Technical Reports

85. H. Brito–Santana, R. Rodríguez–Ramos, R. Guinovart–Díaz, J. Bravo–Castillero and F.J. Sabina, *Variational Bounds for Multiphase Transversely Isotropic Composites*, August 2006

86. José Garrido and Jun Zhou, *Credibility Theory for Generalized Linear and Mixed Models*, December 2006

87. Daniel Dufresne, José Garrido and Manuel Morales, *Fourier Inversion Formulas in Option Pricing and Insurance*, December 2006

88. Xiaowen Zhou, *A Superprocess Involving Both Branching and Coalescing*, December 2006

89. Yogendra P. Chaubey, Arusharka Sen and Pranab K. Sen, *A New Smooth Density Estimator for Non-Negative Random Variables*, January 2007

90. Md. Sharif Mozumder and José Garrido, *On the Relation between the Lévy Measure and the Jump Function of a Lévy Process*, October 2007

91. Arusharka Sen and Winfried Stute, *A Bi-Variate Kaplan-Meier Estimator via an Integral Equation*, October 2007

92. C. Sangüesa, *Uniform Error Bounds in Continuous Approximations of Nonnegative Random Variables Using Laplace Transforms*, January 2008

93. Yogendra P. Chaubey, Naâmane Laib and Arusharka Sen, *A Smooth Estimator of Regression Function for Non-negative Dependent Random Variables*, March 2008

94. Alejandro Balbás, Beatriz Balbás and Antonio Heras, *Optimal Reinsurance with General Risk Functions*, March 2008

95. Alejandro Balbás, Raquel Balbás and José Garrido, *Extending Pricing Rules with General Risk Functions*, April 2008

96. Yogendra P. Chaubey and Pranab K. Sen, *On the Selection of the Smoothing Parameter in Poisson Smoothing of Histogram Estimator: Computational Aspects*, December 2008
97. Runhuan Feng and José Garrido, *Actuarial Applications of Epidemiological Models*, December 2008

98. Alejandro Balbás, Beatriz Balbás and Raquel Balbás, *CAPM and APT Like Models with Risk Measures*, June 2009

99. Jun Zhou and José Garrido, *A Loss Reserving Method Based on Generalized Linear Models*, June 2009

100. Yogendra P. Chaubey and Isha Dewan, *Smooth Estimation of Survival and Density Functions for a Stationary Associated Process Using Poisson Weights*, September 2009

101. Krishna K. Saha and Debaraj Sen, *Improved Confidence Interval for the Dispersion Parameter in Count Data Using Profile Likelihood*, September 2009

102. Araceli Reyes, *Difficulties Teaching Mathematics with the ACE Cycle*, September 2009

103. Mansi Khurana, Yogendra P. Chaubey and Shalini Chandra, *Jackknifing the Ridge Regression Estimator: A Revisit*, February 2012.

104. Yogendra P. Chaubey and Rui Zhang, *Survival Distributions with Bathtub Shaped Hazard: A New Distribution Family*, November 2013.

105. Yogendra P. Chaubey, *Smooth Kernel Estimation of a Circular Density Function: A Connection to Orthogonal Polynomials on the Unit Circle*, January 2016.

Copies of technical reports can be requested from:

Dr. Wei Sun  
Department of Mathematics and Statistics  
Concordia University  
1455 de Maisonneuve Blvd. West,  
Montreal, QC, H3G 1M8 CANADA