Enumerations of Universal Cycles for $k$-Permutations

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Abstract

Universal cycle for $k$-permutations is a cyclic arrangement in which each $k$-permutation appears exactly once as $k$ consecutive elements. Enumeration problem of universal cycles for $k$-permutations is discussed and one new enumerating method is proposed in this paper. Accurate enumerating formulae are provided when $k = 2, 3$.

Key words: $k$-permutation; Universal cycle; Eulerian tour; Laplacian matrix; Eigenvalue

1 Introduction

Given a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$. A $k$-permutation is an ordered arrangement of $k$ distinct elements in $[n]$, $1 \leq k \leq n$. Let $P_{n,k}$ be the set of all $k$-permutations of the $n$-set $[n]$. Obviously, $|P_{n,k}| = n!/(n-k)!$. Let $C = (x_1, x_2, \ldots, x_{|P_{n,k}|})$ be a cyclic arrangement (or periodic sequence), where each $x_i \in [n]$ for $1 \leq i \leq |P_{n,k}|$. If in $C$ each $k$-permutation appears exactly once as $k$ consecutive elements, then we say that $C$ is a universal cycle for $P_{n,k}$. For example, if $n = 4$ and $k = 2$, then $(12, 31, 43, 12)$ is a universal cycle for $P_{4,2}$, as follows.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 1 &  & \\
4 & 3 & \Rightarrow & P_{4,2} = \{12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43\} \\
1 & 2 & 4 & 2
\end{array}
\]

Universal cycles were introduced by Chung, Diaconis, and Graham \cite{3} as generalizations of de Bruijn cycles \cite{2}, which are binary sequences with period $2^n$ that contain every binary $n$-tuple. Universal cycles are connected with Gray codes deeply \cite{9, 11}. In this paper we consider the universal cycles for $k$-permutations. Jackson \cite{6} showed that the universal cycle for $k$-permutations always exists when $k < n$. There are lots of results about the construction of universal cycles for $k$-permutations, mainly for the case that $k = n - 1$ named shorthand permutations \cite{4, 5, 8, 10}. Another interesting problem is to compute the number of distinct universal cycles for $k$-permutations. This problem was formally presented in \cite{7}.

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Problem 1.1 (Problem 477 [7]) How many different universal cycles for $P_{n,k}$ exist?

Up to now, such enumeration problem has not been solved. When $k = 1$, the number of universal cycles for $P_{n,1}$ is obviously equal to $(n - 1)!$. But for $k \geq 2$, counting them is a little complicated. In this paper, we propose one new method to count them and accurate formulae for the number of universal cycles are provided when $k = 2$ and $3$. The following two theorems are main results of this paper.

Theorem 1.2 The number of universal cycles for $P_{n,2}$, $n \geq 3$, is equal to

$$n^{n-2}[(n-2)!]^n. \quad (1)$$

Theorem 1.3 The number of universal cycles for $P_{n,3}$, $n \geq 4$, is equal to

$$(n - 3)\frac{(n-1)(n-2)}{2}(n-2)^{n-1}(n-1)^{\frac{(n-1)(n-2)}{2}} - 2n^{n-2}[(n-3)!]^n(n-1). \quad (2)$$

The rest of this paper is presented as follows. In Section 2 we collect preliminary notions and known results. Related lemmas and detailed proofs of Theorems 1.2 and 1.3 are provided in Sections 3. We conclude this paper in the last section.

2 Preliminaries

Let us recall some definitions and concepts for digraphs. For a vertex $v$ in a digraph, its out-degree is the number of arcs with initial vertex $v$, and its in-degree is the number of arcs with final vertex $v$. A digraph is balanced if for each vertex, its in-degree and out-degree are the same. It is well-known that a digraph contains an Eulerian tour if and only if the digraph is connected and balanced (see, for example, [1, Theorem 1.7.2]).

Given a digraph $D$, its adjacency matrix is the $(0,1)$-matrix $A = (a_{i,j})$ where $a_{i,j} = 1$ if $v_iv_j$ is an arc of $D$, and $a_{i,j} = 0$ otherwise. Let $T$ be the diagonal matrix of the vertex out-degrees. The Laplacian matrix of $D$ is defined as $L = T - A$. The eigenvalues of $L$ are called the Laplacian eigenvalues of $D$. Obviously, the row sum of $L$ is zero, which implies that $0$ is an eigenvalue of $L$ with respect to the eigenvector $1$.

In order to count the number of distinct universal cycles for $P_{n,k}$, $k > 1$, we define a transition digraph. Let $D$ be a digraph with vertex set $P_{n,k-1}$. The arcs of $D$ satisfy the following rule: for any two vertices $i_1i_2\cdots i_{k-1}$ and $j_1j_2\cdots j_{k-1}$, there is an arc from $i_1i_2\cdots i_{k-1}$ to $j_1j_2\cdots j_{k-1}$ if and only if $i_s = j_{s-1}$ for $2 \leq s \leq k - 1$, and $i_1 \neq j_{k-1}$. Such a digraph is called the transition digraph of $P_{n,k}$. Let $uv$ be an arc in $D$ with initial vertex $u$ and final vertex $v$. If $u = i_1i_2\cdots i_{k-1}$, then $v = i_2i_3\cdots i_{k-1}i_k$, where $i_k \in [n]\{i_1, i_2, \ldots, i_{k-1}\}$, and so the arc $uv$ may be regarded as the $k$-permutation $i_1i_2\cdots i_{k-1}i_k$. On the other hand, any $k$-permutation $i_1i_2\cdots i_{k-1}i_k$ in $P_{n,k}$ is represented by an arc with initial vertex $i_1i_2\cdots i_{k-1}$ and final vertex $i_2i_3\cdots i_{k-1}i_k$. In [6], Jackson showed such transition digraph is balanced and connected. One can see that any Eulerian tour in this transition digraph corresponds to a universal cycle for $P_{n,k}$, which leads to the following proposition directly.
**Proposition 2.1** The number of distinct universal cycles for $P_{n,k}$ is equal to the number of Eulerian tours of its transition digraph.

This proposition implies that it is sufficient to consider the number of Eulerian tours in the transition digraph of $P_{n,k}$. In the following, we establish the formulae for the number of distinct universal cycles. Let $D$ be a connected balanced digraph, and let $\epsilon(D)$ denote the number of Eulerian tours of $D$. We use $d^+(v)$ to denote the out-degree of the vertex $v$.

**Lemma 2.2** ([12]) Let $D$ be a connected balanced digraph with vertex set $V$. If the Laplacian eigenvalues of $D$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|V|-1} > \mu_{|V|} = 0$, then

$$\epsilon(D) = \frac{1}{|V|} \prod_{v \in V} (d^+(v) - 1)!.$$  

According to Lemma 2.2, to count the number of universal cycles for $P_{n,k}$, it is enough to compute the corresponding Laplacian eigenvalues. In the following section, we will show how to determine them by a simple method.

### 3 Enumeration formulae for $k = 2$ and $3$

In this section we will count the number of universal cycles for $P_{n,2}$ and $P_{n,3}$, and the proofs of Theorems 1.2 and 1.3 are provided respectively.

Firstly we give the proof of Theorem 1.2 about the number of universal cycles for $P_{n,2}$.

**Proof of Theorem 1.2.** Let $D$ be the transition digraph of $P_{n,2}$. The vertex set of $D$ is $[n]$. For any two distinct vertices $i$ and $j$, by the definition of transition digraph, $ij$ is an arc of $D$. Note also that the out-degree of each vertex equals $n - 1$. Thus, the Laplacian matrix of $D$ is

$$L = \begin{bmatrix}
 n - 1 & -1 & -1 & \cdots & -1 \\
 -1 & n - 1 & -1 & \cdots & -1 \\
 -1 & -1 & n - 1 & \cdots & -1 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 -1 & -1 & -1 & \cdots & n - 1
\end{bmatrix} = nI - J,$$

where $J$ is the $n \times n$ matrix of all ones, and $I$ is the $n \times n$ identity matrix. It is well known that the eigenvalues of $J$ are $n$ (once) and $0$ ($n - 1$ times) [12]. A simple calculation shows that the eigenvalues of this Laplacian matrix are

$$\lambda_1, \lambda_2, \ldots, \lambda_n, 0, \underbrace{n, \ldots, n}_{n-1}.$$

According to Lemma 2.2, we obtain that

$$\epsilon(D) = \frac{1}{n} n^{n-1} [(n - 2)!]^n = n^{n-2} [(n - 2)!]^n.$$
Thus, it follows from Proposition 2.1 that there are \( n^{n-2} [(n-2)!]^n \) distinct universal cycles for \( P_{n,2} \).

Next we consider the universal cycles for \( P_{n,3} \). Let \( D \) be the transition digraph of \( P_{n,3} \) with adjacency matrix \( A \). One can see that \( D \) is balanced, and the out-degree (in-degree) of any vertex equals \( n - 2 \). The following lemma presents the property of the adjacency matrix \( A \).

**Lemma 3.1** Let \( D \) be the transition digraph of \( P_{n,3} \) with adjacency matrix \( A \). Then

\[
A^4 + A^3 + (n - 3)A^2 - A - (n - 2)I = (n - 2)(n - 3)J,
\]

where \( J \) is the \( n(n-1) \times n(n-1) \) matrix of all ones, and \( I \) is the \( n(n-1) \times n(n-1) \) identity matrix.

**Proof.** Let \( \tau_\ell(u, v) \) denote the number of walks from \( u \) to \( v \) in \( D \) of length \( \ell \geq 1 \). In the following, we determine \( \tau_\ell(u, v) \) by distinguishing seven cases for any ordered pair \( (u, v) \).

**Case 1:** \( u = ab \) and \( v = ab \), where \( a, b \in [n] \).

Obviously, \( \tau_1(ab, ab) = \tau_2(ab, ab) = 0 \). The 3-walk from \( ab \) to \( ab \) may be indicated as

\[
ab \rightarrow bx \rightarrow xa \rightarrow ab
\]

where \( x \in [n]\{a, b\} \). Thus, \( \tau_3(ab, ab) = n - 2 \). Similarly, the 4-walk from \( ab \) to \( ab \) is presented as

\[
ab \rightarrow bx \rightarrow xy \rightarrow ya \rightarrow ab
\]

where \( x, y \in [n]\{a, b\} \) and \( x \neq y \). Then \( \tau_4(ab, ab) = (n - 2)(n - 3) \).

**Case 2:** \( u = ab \) and \( v = ba \), where \( a, b \in [n] \).

It is easy to see that there is not walk from \( ab \) to \( ba \) of length \( \ell \leq 3 \), and so \( \tau_1(ab, ba) = \tau_2(ab, ba) = \tau_3(ab, ba) = 0 \). The 4-walk from \( ab \) to \( ba \) is expressed as

\[
ab \rightarrow bx \rightarrow xy \rightarrow ya \rightarrow ba.
\]

Thus, \( x, y \in [n]\{a, b\} \) and \( x \neq y \), which yielding \( \tau_4(ab, ba) = (n - 2)(n - 3) \).

**Case 3:** \( u = ab \) and \( v = ac \), where \( a, b, c \in [n] \).

There is not walk from \( ab \) to \( ac \) of length \( \ell \leq 2 \), hence \( \tau_1(ab, ac) = \tau_2(ab, ac) = 0 \). The 3-walk from \( ab \) to \( ac \) may be presented as

\[
ab \rightarrow bx \rightarrow xa \rightarrow ac
\]

where \( x \) belongs to \( [n]\{a, b, c\} \). Then \( \tau_3(ab, ac) = n - 3 \). The 4-walk from \( ab \) to \( ac \) is

\[
ab \rightarrow bx \rightarrow xy \rightarrow ya \rightarrow ac.
\]

Obviously, \( x \in [n]\{a, b\} \), \( y \in [n]\{a, b, c\} \) and \( x \neq y \). Thus, \( \tau_4(ab, ac) = n - 3 + (n - 3)(n - 4) = (n - 3)^2 \).
Case 4: \( u = ab \) and \( v = ca \), where \( a, b, c \in [n] \).

In this case, \( uv \) is not an arc in \( D \), thus \( \tau_1(ab, ca) = 0 \). Moreover, \( ab \rightarrow bc \rightarrow ca \) is the unique 2-walk from \( ab \) to \( ca \), and so \( \tau_2(ab, ca) = 1 \). Since the 3-walk from \( ab \) to \( ca \) is

\[
ab \rightarrow bx \rightarrow xc \rightarrow ca,
\]

it means that \( x \in [n] \setminus \{a, b, c\} \). Therefore, \( \tau_3(ab, ca) = n - 3 \). Let

\[
ab \rightarrow bx \rightarrow xy \rightarrow yc \rightarrow ca
\]

be the 4-walk from \( ab \) to \( ca \). It follows that \( x, y \in [n] \setminus \{a, b, c\} \) and \( x \neq y \), hence \( \tau_4(ab, ca) = (n - 3)(n - 4) \).

Case 5: \( u = ab \) and \( v = bc \), where \( a, b, c \in [n] \).

Clearly, \( uv \) is an arc in \( D \), that is, \( \tau_1(ab, bc) = 1 \). Since there is no walk from \( ab \) to \( bc \) of length 2 or 3, we have \( \tau_2(ab, bc) = \tau_3(ab, bc) = 0 \). The 4-walk from \( ab \) to \( bc \) is presented as

\[
ab \rightarrow bx \rightarrow xy \rightarrow yb \rightarrow bc.
\]

It follows that \( x \in [n] \setminus \{a, b\} \), \( y \in [n] \setminus \{b, c\} \) and \( x \neq y \). Therefore, \( \tau_4(ab, bc) = (n - 3)^2 + (n - 2) \).

Case 6: \( u = ab \) and \( v = cb \), where \( a, b, c \in [n] \).

It is easy to see that the length of any walk from \( ab \) to \( cb \) is at least 3. Suppose that

\[
ab \rightarrow bx \rightarrow xc \rightarrow cb
\]

is a 3-walk from \( ab \) to \( cb \). Thus, \( x \in [n] \setminus \{a, b, c\} \), and so \( \tau_3(ab, cb) = n - 3 \). Let

\[
ab \rightarrow bx \rightarrow xy \rightarrow yc \rightarrow cb
\]

be a 4-walk from \( ab \) to \( cb \). It follows that \( x \in [n] \setminus \{a, b, c\} \), \( y \in [n] \setminus \{b, c\} \) and \( x \neq y \), which implies that \( \tau_4(ab, cb) = n - 3 + (n - 3)(n - 4) = (n - 3)^2 \).

Case 7: \( u = ab \) and \( v = cd \), where \( a, b, c, d \in [n] \).

Since \( uv \) is not an arc in \( D \), \( \tau_1(ab, cd) = 0 \). Note that \( ab \rightarrow bc \rightarrow cd \) is the unique 2-walk from \( ab \) to \( cd \). Hence \( \tau_2(ab, cd) = 1 \). For any \( x \in [n] \setminus \{a, b, c, d\} \), \( ab \rightarrow bx \rightarrow xc \rightarrow cd \) forms a 3-walk, thus \( \tau_3(ab, cd) = n - 4 \). The 4-walk from \( ab \) to \( cd \) is presented as

\[
ab \rightarrow bx \rightarrow xy \rightarrow yc \rightarrow cd,
\]

in which \( x \in [n] \setminus \{a, b, c\} \), \( y \in [n] \setminus \{b, c, d\} \) and \( x \neq y \). It follows that \( \tau_4(ab, cd) = n - 3 + (n - 4)^2 \).

Recall that the \((u, v)\)-entry of the matrix \( A^r \) is equal to \( \tau_r(u, v) \). According to the above cases, we determine the entries of matrices \( A, A^2, A^3 \) and \( A^4 \) respectively, as shown in Table 1. Assume that the matrix \( A \) satisfies the following equation

\[
\alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \alpha_3 A^3 + \alpha_4 A^4 = J.
\]
The solution to this system of equations is given by

\[
\begin{align*}
\alpha_0 + \alpha_3(n - 2) + \alpha_4(n - 2)(n - 3) &= 1, \\
\alpha_4(n - 2)(n - 3) &= 1, \\
\alpha_3(n - 3) + \alpha_4(n - 3)^2 &= 1, \\
\alpha_2 + \alpha_3(n - 3) + \alpha_4(n - 3)(n - 4) &= 1, \\
\alpha_1 + \alpha_4((n - 3)^2 + (n - 2)) &= 1, \\
\alpha_2 + \alpha_3(n - 4) + \alpha_4((n - 3) + (n - 4)^2) &= 1.
\end{align*}
\]

The solution to this system of equations is given by

\[
\begin{align*}
\alpha_0 &= \frac{1}{n - 3}, \\
\alpha_1 &= \frac{1}{(n - 2)(n - 3)}, \\
\alpha_2 &= \frac{1}{n - 2}, \\
\alpha_3 &= \frac{1}{(n - 2)(n - 3)}, \\
\alpha_4 &= \frac{1}{(n - 2)(n - 3)}.
\end{align*}
\]

Thus, it follows that

\[-\frac{1}{n - 3}I - \frac{1}{(n - 2)(n - 3)}A + \frac{1}{n - 2}A^2 + \frac{1}{(n - 2)(n - 3)}A^3 + \frac{1}{(n - 2)(n - 3)}A^4 = J,
\]

that is,

\[-(n - 2)I - A + (n - 3)A^2 + A^3 + A^4 = (n - 2)(n - 3)J,
\]

and Equation (5) has been deduced.

**Lemma 3.2** Let D be the transition digraph of \(P_{n,3}\) with adjacency matrix A. Then the eigenvalues of A are

\[
n - 2, \frac{1}{(n - 3)(n - 2)}p, \ldots, \frac{1}{n - 3}p, \ldots, q, \ldots, \frac{1}{(n - 3)(n - 2)}q,
\]

where p and q are the roots of \(\lambda^2 + \lambda + n - 2 = 0\).
Proof. Here we define a function \( \phi(x) = x^4 + x^3 + (n - 3)x^2 - x - (n - 2) \). If the eigenvalues of \( A \) are 
\[ \lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_{n(n-1)}, \]
then the eigenvalues of \( A^4 + A^3 + (n - 3)A^2 - A - (n - 2)I \) are
\[ \phi(\lambda_1), \phi(\lambda_2), \ldots, \phi(\lambda_i), \ldots, \phi(\lambda_{n(n-1)}), \]
which will be the eigenvalues of \((n - 2)(n - 3)J\)
\[ n(n-1)(n-2)(n-3), 0, 0, \ldots, 0, \]
\[ \frac{n(n-1)-1}{n} \]
according to Lemma 3.1.

Recall that the out-degree of any vertex in \( D \) is equal to \( n - 2 \), that is, the row sum of \( A \) is \( n - 2 \). Let \( 1 \) be a column vector of all ones. It is obvious that \( A1 = (n - 2)1 \), which implies that \( n - 2 \) is an eigenvalue of \( A \). Moreover, since \( \phi(n - 2) = n(n - 1)(n - 2)(n - 3) \), the multiplicity of \( n - 2 \) as an eigenvalue of \( A \) is 1. Therefore, the remaining eigenvalues of \( A \) are the roots of
\[ \lambda^4 + \lambda^3 + (n - 3)\lambda^2 - \lambda - (n - 2) = (\lambda - 1)(\lambda + 1)(\lambda^2 + \lambda + n - 2) = 0. \]
So we can deduce that the characteristic polynomial of \( A \) is
\[ |\lambda I - A| = (\lambda - n + 2)(\lambda - 1)s_1(\lambda + 1)s_2(\lambda^2 + \lambda + n - 2)s_3, \]
where \( s_1, s_2, s_3 \geq 0 \) and \( s_1 + s_2 + 2s_3 = n(n-1) - 1 \). In other words, the eigenvalues of \( A \) are
\[ n-2, 1, \ldots, 1, -1, \ldots, -1, p, \ldots, p, q, \ldots, q, \]
where \( p \) and \( q \) are the roots of \( \lambda^2 + \lambda + n - 2 = 0 \). From Table 1, we obtain that the traces of \( A, A^2 \) and \( A^3 \) are 0, 0 and \( n(n-1)(n-2) \), respectively. Using the relationship between eigenvalues and trace of a matrix, we obtain
\[
\begin{aligned}
&n-2 + s_1 - s_2 + s_3(p + q) = 0, \\
&(n-2)^2 + s_1 + s_2 + s_3(p^2 + q^2) = 0, \\
&(n-2)^3 + s_1 - s_2 + s_3(p^3 + q^3) = n(n-1)(n-2).
\end{aligned}
\]
(6)

Since \( p \) and \( q \) are the roots of \( \lambda^2 + \lambda + n - 2 = 0 \), it follows that
\[
\begin{aligned}
p + q &= -1, \\
p^2 + q^2 &= (p + q)^2 - 2pq = -2n + 5, \\
p^3 + q^3 &= (p + q)(p^2 + q^2 - pq) = 3n - 7.
\end{aligned}
\]
(7)
Combining (6) and (7), we obtain that \( s_1 = (n - 1)(n - 2)/2, s_2 = n(n - 3)/2 \) and \( s_3 = n - 1 \), and the result follows.
Now, we provide the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Let $D$ be the transition digraph of $P_{n,3}$ with adjacency matrix $A$ and Laplacian matrix $L$. Since the out-degree of any vertex in $D$ is $(n-2)$, we have $L = (n-2)I - A$. Thus, it follows from Lemma 3.2 that the eigenvalues of $L$ are

$$0, n-3, \ldots, n-3, n-1, \ldots, n-1, n-2 - p, n-2 - q, \ldots, n-2 - q,$$

where $p$ and $q$ are the roots of $\lambda^2 + \lambda + n - 2 = 0$. Since $pq = n-2$ and $p + q = -1$, we have

$$(n-2-p)(n-2-q) = (n-2)^2 - (p+q)(n-2) + pq = n(n-2).$$

According to Lemma 2.2, we obtain that

$$\epsilon(D) = \frac{1}{n(n-1)}(n-3)^{(n-1)(n-2)}(n-1)^{n(n-3)}[n(n-2)]^{n-1}[(n-3)!]^{n(n-1)}
= (n-3)^{(n-1)(n-2)}(n-2)^{n-1}(n-1)^{(n-1)(n-2)}2^{-2}n^{n-2}[(n-3)!]^{n(n-1)}.$$}

The number of distinct universal cycles for $P_{n,3}$ can be deduced directly based on Proposition 2.1.

\[ \square \]

4 Conclusions

By discussing the entries of powers of adjacency matrix $A$, we construct a polynomial of $A$, and then the eigenvalues of $A$ can be provided. According to this new method, enumerating results of universal cycles for $P_{n,2}$ and $P_{n,3}$ are proposed precisely. This new method can be applied to general case by analyzing the entries of $A^i$.

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