2- Dominator Coloring for Various Graphs in Graph Theory

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Abstract: Given a graph G, the dominator coloring problem seeks a proper coloring of G with the additional property that every vertex in the graph G dominates at least 2-color class. In this paper, as an extension of Dominator coloring such that various graph using 2-dominator coloring has been discussed.

Keywords: 2-Dominator Coloring, Barbell Graph, Star Graph, Banana Tree, Wheel Graph

1. Introduction

In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number \(\chi(G)\) of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If \(\chi(G) = k\), we say that G is k-chromatic.

A dominating set S is a subset of the vertices in a graph such that every vertex in the graph either belongs to S or has a neighbor in S. The domination number is the order of a minimum dominating set. Given a graph G and an integer k, finding a dominating set of order k is NP-complete on arbitrary graphs [2], [4].

Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. A graph has a dominator coloring if it has a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number \(\chi_d(G)\) is the minimum number of color classes in a dominator coloring of a graph G. A \(\chi_d(G)\) - coloring of G is any dominator coloring with \(\chi_d(G)\) colors. Our study of this was motivated by [9] and [7].

2. Various Kinds of Graphs in Graph Theory

Theorem 2.1
Let \(S_{1,n}\) be a star graph, \(C[S_{1,n}]\) be central graph of star graph, then \(\chi_d[\{C[S_{1,n}]\}] \leq n+2, n \geq 3\).

Proof:
Let \(U_0\) be the root vertex of G. Let \(\{v_0, v_1, v_2, ..., v_{n-1}\}\) be the remaining vertices of G. By definition of central graph \(C(G)\), subdivide the each edge exactly once and joining all the non-adjacent vertices of G. Let \(w_0, w_1, w_2, ..., w_{n-1}\) be the subdividing vertices of G. By definition of 2- dominator coloring, assign color \(C_0\) to the root vertices \(U_0\). Now color of all the pendent vertices of G by ‘n’ different colors. Introduce two new colors ‘\(C_1\)’, ‘\(C_2\)’ alternatively to the subdividing vertices of G, \(C_0\) will dominates ‘\(C_1\)’, ‘\(C_2\)’. Also all the pendent vertices of G dominates at least two color class. i.e. \(1+n+2 \leq n+3\). Therefore \(\chi_{d,2}[C[S_{1,n}] \leq n+3\), \(n \geq 3\).

Theorem 2.2
Let \(S_{1,n}\) be a star graph, then \(\chi_{d,2}(S_{1,n}) = 2, n \geq 2\).

Proof:
Let \(S_{1,n}\) be a star graph by 2 dominator coloring and from theorem 3.2 and 3.7[9], the assignment of colors to the vertices of G, May dominates itself and it forms a color class. Thus \(\chi_{d,2}(S_{1,n}) = 2\).

Theorem 2.3
Let \(G = B_0(K_n, K_n)\) where \(n = 3, 4, 5, ..\). Then \(\chi_{d,2}[B_0(K_n, K_n)] = n\).

Proof:
Let \(G = B_0(K_n, K_n)\) be a Barbell graph. Now divide the vertices of G be U and V. Let \(U = \{u_0, u_1, u_2, ..., u_{n-1}\}\) and \(V = \{v_0, v_1, v_2, ..., v_{n-1}\}\) respectively. By definition of Barbell graph, G is obtained by connecting two complete graphs \(K_n\) by \(a_n\) bridge \(e_1\). Using theorem 1.2 [2] and 3.1 [3],2-dominator coloring in any complete graph all the vertices must contain at least n colors. Let the color vertex \(u_0\) by \(c_0\) \(a_1\) by \(c_1, u_2\) by \(c_2, ..., u_{n-1}\) by \(c_{n-1}\). Similarly, color of the vertex \(v_0\) by \(c_0\) \(v_1\) by \(c_1\) \(v_2\) by \(c_2, ..., v_{n-1}\) by \(c_{n-1}\). Thus a bridge ‘\(e_1\)’ between U and V will receive the color ‘\(c_0 c_{n-1}\)’. Hence no two adjacent vertices have same color. Also every vertex of G dominates at least two color classes. i.e. \(\chi_{d,2}[B_0(K_n, K_n)] = n, n = 3, 4, 5, ..\).

Theorem 2.4
Let \(G = P_2 \square P_n\), \(n \geq 2\), then \(\chi_{d,2}(G) = 4\).

Proof:
Let G = \(P_2 \square P_n\) be a ladder graph, \(n \geq 2\) (i.e. two rails or paths connected by rungs). Now color the vertices in path \(P_1\) by two different colors \(c_0, c_1\) alternatively. Similarly color. The vertices in path \(P_2\) by another new two different colors.
Let \( G = B_{m,n} \), \( m,n \geq 2 \) be a banana tree then \( \chi_{d,2}(B_{m,n}) = 2 \).

**Proof:**

Let \( B_{m,n} \) be a banana tree of order \( m,n \geq 2 \) i.e. \((m,n)-\) banana tree is a graph created by connecting a leaf of each of \( m \) copies of \( n \)-star graph with a pendant vertex and it is different from all stars. Thus from theorem 3.2 and 3.7 [9], \( \chi_{d,2}(B_{m,n}) = 2 \).

**Theorem 2.6**

If \( G = F_{n,k} \), \( n,k \geq 2 \), then \( \chi_{d,2}(F_{n,k}) = 2 \).

**Proof:**

Let \( G = F_{n,k} \), \( n,k \geq 2 \), be a fire cracker graph i.e \( F_{n,k} \) is a graph connecting \( n,k \)- states by an leaf. By definition of 2 dominator coloring and by use of theorem 3.2 and 3.7 [9], every vertex may dominates its own class. Thus \( \chi_{d,2}(F_{n,k}) = 2 \).

**Theorem 2.7**

If \( G = D_{n}^{(m)} \), \( n \) be the number of vertex \( m \) be the copies of complete graph, \( n,m \geq 3 \), then \( \chi_{d,2}(D_{n}^{(m)}) = n \).

**Proof:**

Let \( G = D_{n}^{(m)} \) be a wind mill graph and it is obtained of connecting \( m \) copies of \( K_n \) with the common vertex. By the definition of 2- dominator coloring from theorem 1.2 and 3.1[3] \( D_{n}^{(m)} \) needs at least \( n \) colors and no two adjacent vertices will receive same color. Thus \( \chi_{d,2}(D_{n}^{(m)}) = n, n \geq 3,4,5, \ldots \).

**Theorem 2.8**

If \( G = G_n \) be any gear graph then \( \chi_{d,2}(G_n) = 4, n = 3,4,5,7, \ldots \).

**Proof:**

Let \( G = G_n \) be a gear graph or bipartite wheel graph. Let \( u_i \) be the root vertex of \( G \). Let \( v_1,v_2, \ldots, v_n \) be the main vertex of \( G \). Let \( w_1, w_2, \ldots, w_n \) be the sub vertices of \( G \). Procedure for 2- dominator coloring of \( G \):

Case 1: For \( n = 3 \), \( \chi_{d,2}(G_3) = 4 \).

Let a root vertex \( u \) is assigned by the color \( c_0 \). Now assign color to vertex \( v_0 \) by \( c_0 \) and \( v_1 \) by \( c_1 \). Similarly \( w_0 \) is assigned by \( c_0 \), \( w_1 \) by \( c_1 \) and \( w_2 \) by \( c_2 \) respectively. Therefore every vertex must dominate at least two color class. Hence \( \chi_{d,2}(G) = 4 \).

Case 2: For \( n = 4,5,6, \ldots \), \( \chi_{d,2}(G_3) = 4 \).

Assign color \( C_0 \) to the vertex \( u_0 \). Now color \( C_1,C_2 \) alternatively to the vertices \( v_0,v_1,\ldots,v_{n-1} \). Hence a root vertex \( u_0 \) colored by \( C_0 \) will dominates the color classes \( C_1 \) and \( C_2 \) respectively. The vertices \( w_0,w_1,\ldots, w_{n-1} \) is assigned by the color \( C_3 \) such that \( C_3 \) will dominates the color classes \( C_1 \) and \( C_2 \) in \( G \). By definition of 2-Dominator coloring the vertices of \( G \) must dominates at least two color classes. Hence \( \chi_{d,2}(G_n) = 4, n = 4,5,6, \ldots \).

**Theorem 2.9**

If \( G = W_n \) be a wheel graph then \( \chi_{d,2}(W_n) = \{3,n = 6,8,10, \ldots \} \).

**Proof:**

Let \( G = W_n \) be a wheel graph. Let \( u_1 \) be the root vertex of \( G \). Let \( v_1,v_2, \ldots, v_n \) be the main vertex of \( G \).

2- Dominator coloring of \( G \) as follows.

Case 1: \( \chi_{d,2}(W_n) = 3, n = 6,8,10, \ldots \)

First color \( c_0 \) to the root vertex \( u_1 \). Assign color ‘\( C_{1}, C_{2} \)’ alternatively to the remaining vertices of \( G \). i.e \( v_1,v_2 \) receive color \( C_{1}, C_{2} \). \( v_3,v_4 \) receive color \( C_{2}, C_{1} \). \( v_5,v_6 \) receive \( C_{1}, C_{2} \) and so on. By coloring it forms a clique. Hence every vertex must dominate at least two color classes. Thus \( \chi_{d,2}(W_n) = 3, n = 6,8,10, \ldots \).

Case 2: \( \chi_{d,2}(W_n) = 4, n = 7,9,11, \ldots \)

Assign color \( C_0 \) to the root vertex \( u_1 \). Now the colors \( C_1,C_2,C_3 \) will be assigned alternatively to the remaining vertices of \( G \). i.e \( v_1,v_2 \) by \( C_1,C_2 \), \( v_3,v_4 \) by \( C_2,C_3 \), \( v_5,v_6 \) by \( C_3,C_1 \) and \( v_7,v_8 \) by \( C_1,C_2 \) and so on. By coloring it forms a clique while assigning the color in the above alternative sequence, at last there may or may not remains some vertices of \( G \), it depends up on ‘\( n \)’ suppose the vertices of \( G \) remains, to color the remaining vertices of \( G \) using the above same three colors \( c_1,c_2,c_3 \) colored in any order, but there should not any two adjacent vertices receive the same color. Hence every vertex must dominate at least two color classes. Thus \( \chi_{d,2}(W_n) = 4, n = 7,9,11, \ldots \).

**Theorem 2.10:**

For any prism graph \( Y_{2,n}, \chi_{d,2}(Y_{2,n}) \leq n, n = 3,4,5, \ldots \).

**Proof:**

Let \( G = Y_{2,n} \) be a prism graph. Let \( G = U \cup V \). Let \( U = \{u_0,u_1,u_2, \ldots, u_{n-1}\} \) be the inner vertices of \( G \). \( V = \{v_0, v_1,v_2, \ldots, v_{n-1}\} \) be the outer vertices of \( G \) respectively. Let \( E(U) = \{u_i,u_{i+1}; 0 \leq i \leq n-2\} \) and \( E(V) = \{v_i,v_{i+1}; 0 \leq i \leq n-2\} \). Thus for any \( Y_{2,n}, E(G)=3n \). Here \( x=1 \) be the edge joining the vertex of \( U \) and \( V \).

Let color the vertices of \( U \) be \( C_0=\{0,2,4,6,8,\ldots\} \). \( M \) different colors. Note \( M=\left\lceil\frac{1}{2}(2x+1)\right\rceil \), \( x=1 \) be the edge joining the vertices of \( U \) colored by another different colors \( 1,2,3 \), \( n \) in clockwise direction respectively. Similarly the vertices of \( V \) can be colored by the above said same \( M \) colors, but it starts from the vertices \( v_2 \), it may not be end at the vertex \( v_0 \) i.e., \( v_2,v_4,v_6, \ldots \) Assign and So on. Also the remaining vertices of \( V \) colored by above said same coloring using same concept. Therefore by coloring, there exists a clique between the vertices \( U \) and \( V \) of \( G \). Thus \( G \) dominates at least two color classes. Hence \( G \) admits \( \chi_{d,2}(Y_{2,n}) \leq n, n = 3,4,5, \ldots \).
Theorem 2.11
For any graph $G$, $G=W_{3,n}$ then $\chi_{d,2}(G) \leq n, n \geq 3$.

Proof:
Let $W_n$ be a web graph.
Let $G=UU_1V \cup Wb$ the vertex set.
Let $U=\{u_0, u_1, \ldots, u_{n-1}\}$ and $V=\{v_0, v_1, \ldots, v_{n-1}\}$ and $W=\{w_0, w_1, \ldots, w_{n-1}\}$ respectively.
Let $E(U)=\{u_iu_{i+1}: 0 \leq i \leq n-2\}$ and $E(V)=\{v_iv_{i+1}: 0 \leq j \leq n-2\}$ be the edge set of $U$ and $V$.
Let $E(t)=\{u_kv_k: 0 \leq K \leq n-1\}$ be the edge between the vertex set $UV$ and $VW$ respectively. Thus for any web graph $W_{3,n}$, $E(G)=3n+n$. Consider $x=1$ be the edge joining the vertex $UV$ and $VW$.

Now the procedure for $2$-dominator coloring of $G$ follows:
Let color the vertex $G \in U \cup V \cup W$ by $C=\{0,2x+1,2(2x+1), \ldots, (M-1)(2x+1)\}$ by $M$ different colors in clockwise direction. Note $M=\left\lceil \frac{n}{2(x+1)} \right\rceil \times 1$. Also the remaining vertices of $G \in U \cup V \cup W$ colored by $n-M$ different color $1, 2, \ldots, n-M$ in clockwise direction. To color the vertices of $U$, starts from the vertex $u_0$ using $C=\{0,2x+1,2(2x+1), \ldots, (M-1)(2x+1)\}$ and it may or may not end at the vertex $u_{n-1}$ and the remaining vertices of $G$ colored by $n-M$ different colors. Next color the vertices of $V$ starts from vertex $v_0$ using above said the $M$ colors $C=\{0,2x+1,2(2x+1), \ldots, (M-1)(2x+1)\}$ and it may or may or may not ends at the vertex $v_{n-1}$ then the remaining vertices of $V$ colored by the same said $M$ different colors in same previous manner. Similarly to color the remaining vertices of $W$ needs those same $n-M$ colors using same procedure as above. Likewise coloring there exist a cliques in $G$.

Hence every vertex of $G$ dominates at least two color classes. Thus for any $W_n$, $\chi_{d,2}(W_{3,n}) \leq M + n - m \leq n, n \geq 3$.

Theorem 2.12
Let $G$ be a group with generator set $S$ and let $\phi$ be a color-preserving permutation on $V(D(G, S))$. Then $\phi$ is a color-preserving automorphism of $D(G, S)$ if and only if $\phi(gh)=\phi(g)\phi(h)$.

Proof:
To show that $\phi$ is color-preserving, we need to show that if $gh=1=s$, then $\phi(gh)=s$. Suppose $gh=1=s$. Then $\phi(gg^{-1})=\phi(g)g^{-1}=s$.

3. Conclusion

In this paper, we found out some result of various graphs like: Central Graph of Star Graph $C[S_{1,n}]$, Star Graph: $S_{1,n}$, Barbell Graph $B(K_4, K_3)$, Ladder Graph, banana tree $B_{m,n}$, fire cracks graph: $F_{n,k}$, $n, k \geq 2$, wind Mill Graph $D_{n,m}$, Gear Graph $G_{n,m}$, Wheel Graph $W_n$, Helm Graph $H_n$, Cayley Graph using $2$-dominator coloring. This will be initiative study of extension of $2$-dominator coloring.

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