Minimal Lagrangian tori in Kahler-Einstein manifolds

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Abstract

In this paper we use structure preserving torus actions on Kahler-Einstein manifolds to construct minimal Lagrangian submanifolds. Our main result is: Let $N^{2n}$ be a Kahler-Einstein manifold with positive scalar curvature with an effective $T^n$-action. Then precisely one regular orbit $L$ of the $T$-action is a minimal Lagrangian submanifold of $N$. Moreover there is an $(n-1)$-torus $T^{n-1} \subset T^n$ and a sequence of non-flat immersed minimal Lagrangian tori $L_k$, invariant under $T^{n-1}$ s.t. $L_k$ locally converge to $L$ (in particular the supremum of the sectional curvatures of $L_k$ and the distance between $L$ and $L_k$ go to 0 as $k \to \infty$).

1 Introduction

In this paper we will use torus actions on Kahler-Einstein manifolds with positive scalar curvature to construct minimal Lagrangian tori.

Let $N^{2n}$ be a Kahler-Einstein (K-E) manifold with positive scalar curvature and suppose we have a structure preserving $T^k$-action on $N$. We will look for $T$-invariant minimal Lagrangian submanifolds of $N$. If $k = n$ then we have shown in [4] that there is precisely one regular orbit of the $T$-action, which is a minimal Lagrangian submanifold of $N$. In this paper we study the case when $k = n - 1$ (a complexity one action). The main tool in our investigation will be a correspondence between minimal Lagrangian submanifolds of $N$ and certain Special Lagrangian submanifolds of $K(N)$- the total space of the canonical bundle of $N$.

The manifold $K(N)$ has a natural holomorphic volume form $\varphi$. Also since $N$ is K-E with positive scalar curvature, we have a (Calabi) metric $\omega_u$ on $K(N)$, which is a Ricci-flat Kahler metric (see Section 2.1). The form $\varphi$ is covariantly constant with respect to $\omega_u$, and we have Special Lagrangian (SLag) submanifolds $L' \subset K(N)$, defined by the conditions $\omega_u|_{L'} = 0$ and $\text{Im}\varphi|_{L'} = 0$ (see [4] and [5]). There is a radial vector field $Y$ on $K(N)$, whose flow is scaling of $K(N)$ by real numbers (see Section 2.1). Our main tool in studying minimal Lagrangian submanifolds on $N$ will be a correspondence between minimal La-
grangian submanifolds on $N$ and SLag submanifolds on $K(N)$, invariant under the flow of $Y$ (see Lemmas 1 and 2 in Section 2.1).

In Section 2.2 we study SLag submanifolds on $K(N)$ using a torus action on $N$. Suppose we have a $T^k$-action on $N$. This action of course induces a $T^k$-action on $K(N)$. We will see that there are canonical moment maps $\mu$ on $N$ and $\mu'$ on $K(N)$. Let $Z \subset N$ be the zero set of $\mu$ and $\pi : K(N) \rightarrow N$ be the projection. Then the zero set of $\mu'$ is $Z' = \pi^{-1}(Z)$. Suppose that $T$ acts freely on $Z'' = Z' - Z$. Then we have a symplectic reduction $Q = Z''/T$. We will see that $Q$ has a natural holomorphic volume form $\varphi'$ and a metric $\omega'$ and SLag submanifolds of $(Q,\varphi',\omega')$ lift to $T$-invariant SLag submanifolds of $K(N)$. Also the vector field $Y$ is tangent to $Z''$ and projects to a vector field $Y'$ on $Q$. Thus we reduced the problem of finding minimal Lagrangian submanifolds of $N$ to a problem of finding SLag submanifolds of $Q$, invariant under the flow of $Y'$.

In Section 2.3 we assume that $k = n - 1$. Let $X \subset Z''$ be the set of elements of $Z''$ of unit length in $K(N)$ and $S = X/T \subset Q$. We will see that there is a non-vanishing vector field $W$ on $S$ s.t. there is a correspondence between $Y'$-invariant SLag submanifolds of $Q$ and trajectories of the $W$-flow on $S$.

Next we would like to develop a criterion to see that $T^{n-1}$ acts freely on $Z''$. We also would like to understand periodic orbits of the vector field $W$ on $S$ (to construct immersed minimal Lagrangian tori on $N$). We can do it if we assume that $N$ is a toric K-E manifold (see Section 3.1). In this case we can prove the following Theorem:

**Theorem 1** Let $N^{2n}$ be a K-E manifold with positive scalar curvature with an effective $T^n$-action. Then precisely one regular orbit $L$ of the $T$-action is a minimal Lagrangian submanifold of $N$. Moreover there is an $(n-1)$-torus $T^{n-1} \subset T^n$ and a sequence of non-flat immersed $T^{n-1}$-invariant minimal Lagrangian tori $L_k \subset N$ s.t. $L_k$ locally converge to $L$ (in particular the supremum of sectional curvatures of $L_k$ and the distance between $L$ and $L_k$ go to 0 as $k \rightarrow \infty$).

Here by local convergence we mean the following: The distance between $L_k$ and $L$ goes to 0 as $k \rightarrow \infty$. Also for any point $l \in L$ we can choose a neighbourhood $U$ of $l$ in $N$ s.t. $L_k \cap U$ is a finite union $L_k^l$ of submanifolds of the form $L_k^l = exp(v_k^l)(L \cap U)$, where $v_k^l$ is a normal vector field to $L$ on $L \cap U$. Moreover any subsequence $v_k^l$ converges to 0 in a $C^\infty$ topology as $k \rightarrow \infty$.

This result is new even for $N = \mathbb{C}P^n$ for $n \geq 3$. For $n = 2$ examples of non-flat $S^1$-invariant immersed minimal Lagrangian tori in $\mathbb{C}P^2$ were constructed in [2] and [3] using harmonic maps.

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In this paper we use a number of results from our previous paper [1], including proofs for the completeness of exposition.

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2 Minimal Lagrangian submanifolds in complexity one K-E manifolds

2.1 A correspondence between minimal and Special Lagrangian submanifolds

Let $N^{2n}$ be a K-E manifold with positive scalar curvature. We begin by reviewing the geometry of $K(N)$ and the correspondence between minimal Lagrangian submanifolds of $N$ and certain Special Lagrangian submanifolds of $K(N)$.

Let $K(N)$ be the total space of the canonical bundle of $N$ and $\pi : K(N) \rightarrow N$ be the projection. There is a canonical $(n,0)$-form $\rho$ on $K(N)$ defined by $\rho(\alpha)(v_1, \ldots, v_n) = a(\alpha_1(v_1), \ldots, \alpha_n(v_n))$, $a \in K(N)$. The form $\varphi = d\rho$ is a holomorphic volume form on $K(N)$. If $z_1, \ldots, z_n$ are local coordinates on $N$ then $(z_1, \ldots, z_n, y = dz_1 \wedge \ldots \wedge dz_n)$ are coordinates on $K(N)$ and $\rho = ydz_1 \wedge \ldots \wedge dz_n$, $\varphi = dy \wedge dz_1 \wedge \ldots \wedge dz_n$.

There is a canonical radial vector field $Y$ on $K(N)$, given at a point $m \in K(N)$ by the vector $m$ (viewed as a tangent vector to the linear fiber over $\pi(m)$). We have $iY \rho = 0$. Also the Lie derivative $L_Y \rho = \rho$. So $\rho = iY d\rho = iY \varphi$. So $L_Y \varphi = d(iY \varphi) = d\rho = \varphi$.

If $N$ is a Kähler-Einstein manifold with positive scalar curvature then $K(N)$ has a Ricci-flat Kähler metric on it (see [7], p.108). The metric is constructed as follows: The connection on $K(N)$ induces a horizontal distribution for the projection $\pi$, with a corresponding splitting of the tangent bundle of $K(N)$ into horizontal and vertical distributions. We can identify the horizontal space at each point $m \in K(N)$ with the tangent space to $N$ at $\pi(m)$. Let $r^2 : K(N) \rightarrow \mathbb{R}_+$ be the square of the length of an element in $K(N)$ and $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive function with a positive first derivative. We define the metric $\omega_u$ on $K(N)$ as follows: We put the horizontal and the vertical distributions to be orthogonal. On the horizontal distribution we define the metric to be $u(r^2)\pi^*(\omega)$ and on the vertical distribution we define it to be $t^{-1}u'(r^2)\omega^*$. Here $\omega$ is the Kähler-Einstein metric on $N$, $t$ is its scalar curvature and $\omega^*$ is the induced metric on the linear fibers of $\pi$. The Kähler-Einstein condition on $N$ ensures that the corresponding 2-form $\omega_u$ defining this metric on $K(N)$ is closed, i.e. the metric is Kähler. If we take $u(r^2) = (tr^2 + l)^{\frac{1}{t+1}}$ for some positive constant $l$ (see [7], p.109), then $\omega_u$ is complete and Ricci-flat (the Calabi metric). From now on we study $K(N)$ endowed with this metric $\omega_u$.

We begin with the following observation: Let $L$ be an oriented Lagrangian submanifold of $N$. For any point $l \in L$ there is a unique element $\kappa_l$ in the fiber of $K(N)$ over $l$ which restricts to the volume form on $L$. Various $\kappa_l$ give rise to a section $\kappa$ of $K(N)$ over $L$. Consider a submanifold $L^K \subset K(N)$ given by

$$L^K = \{(m|m = a\kappa_l \text{ for } l \in L, \ a \in \mathbb{R}\}$$

We have the following:

Lemma 1 $L$ is a minimal Lagrangian submanifold of $N$ iff $L^K$ is a Special Lagrangian submanifold of $K(N)$.
Here by a minimal submanifold we mean a submanifold, which is critical for the volume functional (i.e. the trace of the second fundamental form vanishes).

**Proof:** First we note that $L^K$ is Special, i.e. $\text{Im} \varphi |_{L^K} = 0$. Indeed one easily verifies that $\text{Im} \varphi |_{L^K} = 0$, hence $\text{Im} \varphi |_{L^K} = 0$.

We now prove that $L^K$ is Lagrangian with respect to $\omega_\lambda$ iff $L$ is minimal. Let $m$ be a point on $L^K - L$, $l = \pi(m)$ and $m = a \kappa l$. The tangent space of $L^K$ at $m$ is spanned by $\kappa l$ (viewed as a vertical vector in $T_m K(N)$) and vectors $(e + a \nabla_e \kappa)$. Here $e$ is any tangent vector to $L$ at $l$ (viewed as an element of the horizontal distribution of $T_m K(N)$) and $a \nabla_e \kappa$ lives in the vertical distribution of $T_m K(N)$. To compute $\nabla_e \kappa$ take an orthonormal frame $(v_j)$ of $T_l L$ and extend it to an orthonormal frame of $L$ in a neighbourhood $U$ of $l$ in $L$ s.t. $\nabla^L v_i = 0$ at $l$ (here $\nabla^L$ is the Levi-Civita connection of $L$). We get that

$$\nabla_e \kappa = \kappa \cdot \nabla_e \kappa (v_1, \ldots, v_n) = \kappa (e(v_1, \ldots, v_n)) - \Sigma \kappa (v_1, \ldots, \nabla_{v_j} v_j, \ldots, v_n)$$

Now $e(v_1, \ldots, v_n) = 0$. Also clearly

$$\kappa (v_1, \ldots, \nabla_{v_j} v_j, \ldots, v_n) = i < \nabla_{v_j} v_j, J v_j > = i < \nabla_{v_j} e, J v_j > = i < - e, J (\nabla_{v_j} v_j) >$$

Here $J$ is the complex structure on $N$. Thus we get that

$$a \nabla_e \kappa = -ia (Jh \cdot e) \kappa l$$

Here $h = \Sigma \nabla_{v_j} v_j$ is the trace of the second fundamental form of $L$. From this one easily deduces that $L^K$ is Lagrangian iff $h = 0$, i.e. $L$ is minimal. Q.E.D.

The manifold $L^K$ is invariant under the flow of the vector field $Y$ on $K(N)$ (which is just scaling of $K(N)$ by real numbers). Vice versa we have the following:

**Lemma 2** Let $L'$ be a Special Lagrangian submanifold of $K(N) - N$, invariant under the flow of $Y$. Then $L = \pi(L')$ is an (immersed) minimal Lagrangian submanifold of $N$.

**Proof:** Let $m \in L' - N$. Since $L'$ is Lagrangian and $Y$ is in the tangent space $T_m L'$ then the tangent space to $L'$ at $m$ clearly decomposes as

$$T_m L' = \text{span}(Y) \oplus T'$$

where $T'$ is in the horizontal distribution at $m$. The space $\pi_\lambda (T')$ can be viewed as a tangent space to $L$ at $l = \pi(m)$. Clearly this tangent space $T_l L$ is Lagrangian, i.e. $L$ is Lagrangian. Also $L'$ was Special and we have seen in the beginning of this section that $i_Y \varphi = \rho$. Thus $m$ (viewed as an $(n,0)$-form on $N$ at $l$) restricts to a real $n$-form on $T_l L$, i.e. $m \in L^K$. Hence locally $L'$ coincides with $L^K$. From Lemma 1 we deduce that $L$ is minimal. Q.E.D.

### 2.2 SLag submanifolds on $K(N)$ via symplectic reduction

In the previous section we showed how to find minimal Lagrangian submanifolds of $N$ from certain SLag submanifolds of $K(N)$. In this section we will see that
if we have a torus action on $N$ then we can find $T$-invariant SLag submanifolds of $K(N)$ from SLag submanifolds of a certain symplectic reduction of $K(N)$.

Let $T^k$ act on $N$. Then this action induces a $T^k$-action on $K(N)$. Let $\mathcal{T}$ be the Lie algebra of $T$, $v \in \mathcal{T}$, $X_v$ be the flow vector field on $N$ and $X'_v$ the flow vector field on $K(N)$. So $\pi_v(X'_v) = X_v$. Let $l \in N$ and $m \in K_i = \pi^{-1}(l)$. Let $R(m)$ be the vertical part of $X'_v$ at $m$. Since $R(m)$ is vertical, it can be viewed as an element of $K_i$. The correspondence $m \mapsto R(m)$ is a linear correspondence on $K_i$. Hence there is a complex number $\sigma_l(v)$ s.t. $R(m) = \sigma_l(v)m$. At a regular point $l$ of the $T$-action $\sigma_l(v)$ can also be found in a following way : Take any unit length element $\xi \in K_i$. Extend $\xi$ along the orbit of $X_v$ to be invariant under the flow of $X_v$. Then one easily computes that $\sigma_l(v) = \nabla_{X_v}\xi : \xi$. Since the flow of $X_v$ is given by holomorphic isometries, $\xi$ has unit length. Hence $\sigma_l(v)$ is purely imaginary. Also $\sigma_l(v)$ is linear in $v$ (because $R(m)$ is given by the vertical part of the differential of the $T$-action at $m$, and this differential is a linear map from $\mathcal{T}$ to $T_{m}K(N)$). Hence $i\sigma$ can be viewed as a map from $N$ to the dual Lie algebra $\mathcal{T}^*$. This map is $T$-invariant.

Let $t > 0$ be the scalar curvature of $N$.

**Lemma 3** The map $\mu = -it^{-1}\sigma$ is a moment map for the action.

**Proof:** Let $v \in \mathcal{T}$. We need to show that $d(-it^{-1}\sigma(v)) = iX_v \omega$. We will do it at a regular point $l$ of the action. Choose any unit length element $\xi$ of $K(N)$ over $l$. We can extend $\xi$ to be a local unit length section, invariant under the $X_v$-flow. $\xi$ defines a connection 1-form $\psi$, $\psi(u) = \nabla_u \xi : \xi$. $\psi$ is invariant under the $X_v$-flow and the K-E condition says that $i d \psi = i \omega$. So

$$0 = \mathcal{L}_{X_v} \psi = d(iX_v \psi) + iX_v d \psi = d \sigma(v) - it(iX_v, \omega)$$

So $\mu$ is a moment map. Q.E.D.

**Remark:** By the construction of $\mu$ we get that $\mu(v) = 0$ for some $v \in \mathcal{T}$ at a point $l \in N$ iff the vector field $X'_v$ is horizontal at $\pi^{-1}(l)$.

**Lemma 4** The map $\mu' = u\pi^{-1}(\mu)$ is a moment map for the $T$-action on $K(N)$.

**Proof:** Let $v \in \mathcal{T}$. We need to prove that $d\mu'(v) = iX_v \omega_u$.

We will study $\omega_u$ in more detail (see [9]). Let $m \in N$ be a regular point for the $T^n$-action and $\xi$ a unit length element of $K(N)$ over $m$. We can extend $\xi$ to be a local unit length section of $K(N)$, invariant under the flow of $X_v$. $\xi$ gives rise to a connection 1-form $\psi$ for the connection on $K(N)$ and the Einstein condition tells that $i d \psi = i \omega$. The section $\xi$ defines a complex coordinate $a$ on $K(N)$, which is invariant under the $X'_v$-flow. Also the form $b = da + a\pi^* \psi$ vanishes on the horizontal distribution (see [10], p. 108). We have $r^2 = a\pi$ and $u = u(r^2)$. Also the Kahler form $\omega_u$ on $K(N)$ is given by

$$\omega_u = u \pi^* \omega - it^{-1} u' b \wedge \bar{b}$$

One directly verifies that $\omega_u = d\eta$ for $\eta = it^{-1} u \pi^* \psi - it^{-1} \omega_u$. By our construction the flow of $X'_v$ leaves $\eta$ invariant. So

$$0 = \mathcal{L}_{X'_v} \eta = i X'_v d\eta + d(iX'_v \eta) = i X'_v \omega_u + d(it^{-1} u \psi(X_v)) = i X'_v \omega - d(\mu'(v))$$
Here we used the fact that $d\varphi(X'_v) = 0$ and $\psi(X_v) = \sigma(v)$. So $\mu'$ is a moment map and we are done. Q.E.D.

Let now $L'$ be a (connected) SLag submanifold of $K(N)$, invariant under the $T$-action and under the $Y$-flow. Since $L'$ is Lagrangian and $T$-invariant, the moment map $\mu'$ is constant on $L'$. But $\mu' = u\pi^{-1}(\mu)$ and $Y(\mu') = 2v^2u\pi^{-1}(\mu)$. So we have $\pi^{-1}(\mu) = 0$ on $L'$. Let $Z$ be the zero set of $\mu$. Then $L' \subset Z' = \pi^{-1}(Z) = \mu'^{-1}(0)$.

Let $Z'' = Z' - Z$. From now on we assume that $T$ acts freely on $Z''$ (we will demonstrate examples where this holds in Section 3). We have a symplectic reduction $N_{red} = Z/T$ and a (smooth) symplectic reduction $Q = Z''/T$, endowed with a Kähler metric $\omega'$.

Let $v_1, \ldots, v_k$ be a basis for $T$ and $X'_1, \ldots, X'_k$ be the corresponding flow vector fields on $K(N)$. Let $\varphi^* = i_{X'_1} \ldots i_{X'_k} \varphi$ be an $(n - k, 0)$-form on $K(N)$, obtained by contracting $\varphi$ by $X'_1, \ldots, X'_k$. Let $\rho^* = i_{X'_1} \ldots i_{X'_k} \rho$. We claim that

$$\varphi^* = (-1)^k d\rho^*$$

We prove this by induction on $k$. Namely let $\varphi^*_l = i_{X'_1} \ldots i_{X'_l} \varphi$ and $\rho^*_l = i_{X'_1} \ldots i_{X'_l} \rho$. We claim that $\varphi^*_l = (-1)^l d\rho^*_l$. For $l = 1$ we have that $\rho$ is $X'_1$-invariant. Hence

$$0 = L_{X'_1} \rho = d\rho^*_1 + \varphi^*_1$$

Now we use induction. The form $\rho^*_{l-1}$ is $X'_l$-invariant. Hence

$$0 = L_{X'_l} \rho^*_{l-1} = d\rho^*_l + (-1)^{l-1} \varphi^*_l$$

and we are done by induction.

Both $\varphi^*$ and $\rho^*$ are $T$-invariant. Let $\nu : Z \to N_{red}$ and $\nu' : Z'' \to Q$ be the quotient maps. One easily sees that there is a unique $(n - k, 0)$-form $\varphi'$ on $Q$ and a unique $(n - k - 1, 0)$-form $\rho'$ on $Q$ s.t.

$$\nu^*(\varphi') = \varphi^*, \quad \nu^*(\rho') = \rho^*, \quad \varphi' = (-1)^k d\rho'$$

We can define on $Q$ SLag submanifolds $L''$ by the conditions $\omega'|_{L''} = 0$, $\text{Im}\varphi'|_{L''} = 0$.

The vector field $Y$ is tangent to $Z''$ and $T$-invariant, hence it projects to a vector field $Y'$ on $Q$. We had $i_{Y'} \varphi = \rho$ on $K(N)$. Hence we also have $i_{Y'} \varphi' = \rho'$ on $Q$. We obviously have the following:

**Lemma 5** Let $L''$ be a SLag submanifold of $Q$, invariant under $Y'$. Then $L' = \nu'^{-1}(L'')$ is a SLag submanifold of $K(N)$, invariant under $T^k$ and under the $Y$-flow.

The proof of the lemma is obvious.

### 2.3 Complexity one actions and periodic orbits

In the previous section we have shown that one can reduce the problem of finding $T$ and $Y$-invariant SLag submanifolds of $K(N)$ to finding $Y'$-invariant
SLag submanifolds of $Q$. In this section we will assume that $k = n - 1$. Let $X \subset Z''$ be the set of elements in $Z''$ of unit length and $S = X/T \subset Q$. We will show that there is a vector field $W$ on $S$ s.t. there is correspondence between $Y'$-invariant SLag submanifolds of $Q$ and the trajectories of the $W$-flow on $S$.

As we saw the tangent bundle of $K(N)$ decomposes as a direct sum $V \oplus H$ of the vertical and the horizontal distributions. Let $U$ be the image of the Lie algebra of $T$ under the differential of the action on $K(N)$. At points of $Z''$ $U$ is an $(n-1)$-dimensional vector space, and it is contained in the horizontal distribution $H$ (since on $\pi(Z'')$ the moment map $\mu$ vanishes). Also the Kahler form $\omega_\alpha$ restricts to 0 on $U$. Let $U^c$ be the complexification of $U$ in the tangent bundle to $K(N)$. Then $U^c$ can be viewed as a complex $(n-1)$-dimensional vector bundle over $Z''$. Let $H'$ be the orthogonal complement of $U^c$ in $H$. Then the tangent bundle of $Z''$ is a direct sum $V \oplus H' \oplus U$. Also the quotient of $V \oplus H'$ under the $T$-action can be identified with the tangent bundle to the symplectic reduction $Q = Z''/T$. Since $H'$ and $V$ are $T$-invariant the tangent bundle to $Q$ splits as a direct sum of 2 complex line bundles: $TQ = V \oplus H'$. Also $V$ and $H'$ are orthogonal both with respect to the symplectic form $\omega$ and the Riemannian metric on $Q$.

There is a natural circle action on $X$ (given by the multiplication by complex numbers of absolute value 1 on $K(N)$). This action is $T$-invariant, hence it induces a circle action on $S = X/T$. Let $F$ be the vector field generating this action on $S$. Then $F = J(Y')$ (here $J$ is the complex structure on $Q$). Also both $Y'$ and $F$ are in the vertical distribution $V$ along $S$ and the tangent bundle $TS$ of $S$ splits as a direct sum $TS = H' \oplus \text{span}(F)$.

Let $\gamma$ be some path in $S$ and let $\gamma^Q$ be the orbit of $\gamma$ under the $Y'$-flow in $Q$. We wish to understand when $\gamma^Q$ is a SLag submanifold of $Q$. Let $W$ be a tangent vector to $\gamma$. Clearly for $\gamma^Q$ to be Lagrangian we need $W$ to live in the horizontal distribution $H'$. The form $\rho' = i_{Y'} \varphi'$ is a (non-zero) $(1,0)$-form on $H'$. Hence the form $\text{Im}\rho'$ has a 1-dimensional kernel in $H'$. Clearly for $\gamma^Q$ to be Special we need $W$ to belong to this kernel. We can normalize $W$ s.t. $\text{Re}\rho'(W) = 1$. Those conditions give rise to a non-vanishing horizontal vector field $W$ on $S$. Let $\gamma$ be a trajectory of $W$ on $S$ and consider $\gamma^Q \subset Q$. The forms $\omega'$ and $\varphi'$ vanish on $\gamma^Q$ along $\gamma$. Also the $Y'$-flow preserves the horizontal distribution and $L_{Y'} \rho' = \rho'$. From this we easily deduce that $\gamma^Q$ is a $Y'$-invariant SLag submanifold of $Q$. From the above discussion we get the following:

**Lemma 6** Let $\gamma$ be a trajectory of $W$ on $S$. Then $L_\gamma = \pi(\nu^{-1}(\gamma^Q))$ is an immersed minimal Lagrangian submanifold of $N$. If $\gamma$ is periodic then $L_\gamma$ is an immersed minimal Lagrangian torus.

There is one general relation among trajectories of $W$, which will later be important: Consider the circle action on $K(N)$ as before. The $(n,0)$-form $\rho$ is equivariant with respect to this action, i.e. if $\lambda \in S^1$ then $\lambda^* (\rho) = \lambda \rho$. Since $\varphi = d\rho$ we get that $\varphi$ is also equivariant with respect to this action. Thus we also deduce that $\rho'$ and $\varphi'$ are equivariant with respect to the circle action on $Q$. Also this action preserves the horizontal distribution $H'$ on $S$. Consider an element $-1 \in S^1$. Then $-1^* (\rho') = -\rho'$. From this we deduce that the $-1$-action

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on S reverses the vector field W, i.e. \(-1_* (W) = -W\). Thus the \(-1\)-action sends W-trajectories to W-trajectories, but it reverses their directions.

3 Toric K-E manifolds

In Section 2.3 we saw that if we have a \(T^{n-1}\)-action on N, then one can construct minimal Lagrangian submanifolds of N from trajectories of the vector field W on S. In order to do this we needed T to act freely on \(Z''\). In this section we will show a class of examples where this holds. We will also investigate periodic orbits of W on S (to construct immersed minimal Lagrangian tori).

Let N be toric, i.e. we have an effective structure-preserving \(T^n\)-action on N. For recent results on toric K-E manifolds we refer the reader to [8] and [2]. We will use various \((n-1)\)-dimensional sub-tori of T to construct invariant minimal Lagrangian submanifolds. But first we will see that there is a unique minimal Lagrangian torus, invariant under the whole of T.

Suppose \(L\) is a regular orbit of the \(T\)-action, which is a minimal submanifold. Then \(L^K\) is a SLag submanifold of \(K(N)\). The moment map \(\mu'\) of \(K(N)\) is constant on \(L^K\). As we have seen in Section 2.2, we must have \(\mu = 0\) on \(L\) i.e. \(L \subseteq \mu^{-1}(0)\). By Atiyah’s result [3], \(\mu^{-1}(0)\) is connected, hence \(L = \mu^{-1}(0)\). So if a regular orbit, which is a minimal submanifold, exists, it must coincide with \(\mu^{-1}(0)\). Next we prove that such an orbit does exist.

Lemma 7 Let \((M^{2n}, \omega)\) be a compact symplectic manifold and g some Riemannian metric on M. Suppose that we have an effective Hamiltonian n-torus action on M, which preserves g. Then there is a regular orbit of the action, which is a minimal submanifold with respect to g. In fact this orbit maximizes volume among the orbits.

Proof: We have a moment map \(\mu\) and smooth orbits are levels set of the moment map. For a regular orbit \(L\) to be a minimal submanifold, it is obviously necessary to be a critical point for the volume functional on the orbits. We note that it is also sufficient. Indeed let \(v\) be any element of the Lie algebra \(T\) of the torus \(T^n\). Then \(\mu(v)\) is \(T^n\)-invariant, and so is the gradient \(\nabla \mu(v)\). Also this gradient is orthogonal to the orbits. Consider now this gradient flow. It commutes with the \(T^n\)-action, hence it sends orbits to orbits. Since \(L\) is critical for the volume functional on the orbits, we get from the first variation formula \(\int_L h \cdot \nabla \mu(v) = 0\). Here \(h\) is a trace of the second fundamental form of \(L\). But both \(h\) and \(\nabla \mu(v)\) are \(T^n\)-invariant, hence we are integrating a constant. So \(h \cdot \nabla \mu(v) = 0\) pointwise. Now \(v\) was arbitrary, hence \(h = 0\).

We want to find a regular orbit, which is maximum point for the volume functional on the orbits. First we need to prove that the volume functional is continuous on the space of orbits. Let \(L'\) be a regular orbit for the torus action. Then the differential of the moment map is surjective along \(L'\). From this one easily deduces that orbits of the action near \(L'\) coincide with level sets of the moment map. So obviously the volume functional is continuous near \(L'\). Next
we prove that the volume functional is continuous near the singular orbits. This follows from the following easy Lemma:

**Lemma 8** Let $L$ be an orbit with a positive dimensional stabilizer $T' \subset T$ and $x \in L$. Then for any $\epsilon > 0$ there is a neighbourhood $U$ of $x$ s.t. any orbit passing through $U$ has volume $< \epsilon$.

**Proof:** Take a (unit) vector $e_1$ in the Lie Algebra of $T'$. Then the corresponding flow vector field $X_1$ vanishes along $L$. Extend $e_1$ to an o.n. basis $e_2, \ldots, e_n$ of $T$. The flow vector fields $X_i$ will have uniformly bounded lengths. We choose a neighbourhood $U$ of $x$ in which $X_1$ has sufficiently small length and it is clear that volumes of the orbits through $U$ will be sufficiently small. Q.E.D.

So the volume functional is continuous on the space of orbits and we can find an orbit $L$, which maximizes volume among the orbits. Obviously $L$ must be a regular orbit (since singular orbits have zero volume). As we have seen, $L$ is a minimal submanifold of $N$ and we are done. Q.E.D.

Let now $T'' \subset T^n$ be some $(n-1)$-torus in $T$ and let $\mu''$ be the canonical moment map for the $T''$-action on $N$ as in Section 2.2. Then $\mu''$ is just the restriction of $\mu$ to the dual Lie algebra of $T''$. In order to apply the constructions of Section 2.3 we want $T''$ to act freely on $Z''$. The following lemma guarantees the existence of such $T''$:

**Lemma 9** Let $N^{2n}$ be a K-E manifold with an effective $T^n$-action as above. Then there is an $(n-1)$-torus $T'' \subset T$ s.t.

i) The differential of the $T''$-action on $N$ is injective along $Z$ and $T$ acts freely on $Z''$.

ii) There is an element $v$ in the Lie algebra of $T''$ s.t. the flow vector field $X_v$ doesn't have a constant length along $Z$.

**Remark:** Condition ii) in the lemma will be used to show that certain minimal Lagrangian tori we shall construct have Killing fields of non-constant length, hence they are not flat.

**Proof:** Let $T'' \subset T^n$ be some $(n-1)$-torus. First we prove that if the differential of the $T''$-action on $N$ is injective along $Z$, then the $T$-action on $Z''$ is free. Suppose not. Then there is a point $l \in Z''$ and an element $1 \neq t \in T$ s.t $t \cdot l = l$. In that case $t$ also preserves the points on the $T''$-orbit through $l$. The tangent space $P$ to this orbit at $l$ is in the horizontal distribution at $l$ (since we are at the zero set of the moment map $\mu'$). Also $\omega_u|_P = 0$. So the differential $dt$ of the $t$-action at $l$ acts trivially on the complexification $P^c$ of $P$. Also $dt$ acts trivially on the vertical distribution $V(l)$ at $l$. The vector space $P^c \oplus V(l)$ is a complex vector space of dimension $n$ and $dt$ acts trivially on it. Also $dt$ preserves the holomorphic volume form $\varphi$ at $l$. Hence $dt$ is trivial at $l$. Hence $t$ acts trivially on $K(N)$ and on $N$, but the $T$-action on $N$ was effective—a contradiction.

Next we wish to understand for which $(n-1)$-tori $T'' \subset T$ the differential of the $T''$-action is injective along the zero set $Z$ of the canonical moment map of $T''$. Let $T^*$ be the dual Lie algebra of $T$ and let $\Lambda \subset T^*$ be the weight lattice.
of $T$. Any element $0 \neq v \in \Lambda$ defines an $(n-1)$-torus $T_v \subset T$ s.t. $v$ vanishes on the Lie algebra of $T_v$. Let $\mu$ be the canonical moment map of $T$ and $\mu_v$ be the canonical moment map of $T_v$. Then $\mu_v$ is just the restriction of $\mu$ to the dual Lie algebra of $T$. It is therefore clear that $\mu_v$ vanishes at a point $n \in N$ iff $\mu(n)$ is proportional to $v$. Since $N$ is a toric variety, the moment polytope is convex and has no faces in the interior. Since $0$ is in the interior of the moment polytope, it is clear that $Z = \mu^{-1}[t_1v, t_2v]$ with $t_1 < 0 < t_2$. For any $t_1 < t < t_2$ the value $tv$ is in the interior of the moment polytope, while $t_1v$ and $t_2v$ are not.

Suppose the line $\text{span}(v)$ doesn’t intersect any of the $(n-2)$-faces of the moment polytope. This means that any point in $Z$ has either a trivial or a 1-dimensional stabilizer in $T$. We claim that the differential of the $T''$-action is injective along $Z$. Suppose not. Then there is a point $n \in Z$ and a vector $0 \neq w$ in the Lie algebra of $T''$ s.t. the flow vector field $X_w$ vanishes at $n$. Since $n \in Z$ the flow vector field $X'_w$ of $w$ on $K(N)$ is horizontal along $\pi^{-1}(n) \subset K(N)$, hence it vanishes along $\pi^{-1}(n)$. Let $g = \exp(tw)$ for some $t \in \mathbb{R}$. Then the $g$-action on $\pi^{-1}(n)$ is trivial. But this means that the differential of the $g$-action on the tangent space $T_nN$ has Jacobian 1. Also $g$ acts trivially on the orbit $L'$ of the $T$-action through $n$. The tangent space $T_nL'$ of $L'$ at $n$ is $(n-1)$-dimensional and $\omega$ restricts to 0 on it. Hence it’s complexification $T_nL'$ is a complex $(n-1)$-dimensional space and $dg$ acts trivially on it. Also $dg$ has Jacobian 1. Hence $dg$ is trivial, hence $g$ acts trivially-a contradiction.

A generic line in the projective space $PT^*$ doesn’t intersect the $(n-2)$-faces of the moment polytope of $\mu$. Also the set of lines passing through points of $\Lambda$ is dense in $PT^*$. So we can easily find $v \in \Lambda$ so that i) holds for $T_v$. In order to ensure that ii) holds, consider a point $b$ in the $(n-2)$-face of the moment polytope. The orbit $\mu^{-1}(b)$ has a stabilizer of dimension at least 2. Hence we can find a vector $0 \neq w \in T$ s.t. $b(w) = 0$ and the flow vector field $X_w$ vanishes along $\mu^{-1}(b)$. We can find a sequence of elements $v_k \in \Lambda$ s.t. the lines $(v_k) = \text{span}(v_k)$ do not intersect the $(n-2)$-faces of the moment polytope and $(v_k)$ converge to the line $(b) = \text{span}(b)$ in $PT^*$. We can also find a sequence of vectors $w_k \in T$ s.t. $v_k(w_k) = 0$ and $w_k \rightarrow w$.

Each $v_k$ defines an $(n-1)$-torus $T_k \subset T$. Let $\mu_k$ be the canonical moment map of $T_k$, and $Z_k$ be the zero set of $\mu_k$. We can find points $n_k$ on $Z_k$ s.t. $n_k$ converge to a point $n \in \mu^{-1}(b)$. Let $X_k$ be the flow vector field of $w_k$. Then the length of $X_k$ at points $n_k$ goes to 0 as $k \rightarrow \infty$. On the other hand the torus $L = \mu^{-1}(0)$ is contained in all of $Z_k$. Moreover the lengths of $X_k$ along $L$ are a-priori bounded from below. So we deduce that for $k$ large enough the torus $T''_k = T_k$ satisfies the conditions i) and ii) of the lemma. Q.E.D.

From now on we pick a sub-torus $T'' \subset T$ satisfying the conditions of Lemma 9. We can use the results of Section 2.3 to deduce that one can construct minimal Lagrangian submanifolds of $N$ from the trajectories of the vector field $W$ on $S$. From Lemma 6 we deduce that in order to obtain immersed minimal Lagrangian tori we need the orbits to be periodic. A first step in finding such orbits will be the following observation: The circle $R = T/T''$ acts freely on $Q$ and on $S$. Let $w \neq 0$ be some element in the Lie algebra of $R$. We have the flow vectors field $A_w$ for the $w$-action on $Q$ and the vector fields $A_w$ and $W$ commute. We also have
a $(1,0)$-form $\rho'$ and a holomorphic $(2,0)$-form $\varphi'$ on $Q$ with $\varphi' = (-1)^{n-1}d\rho'$. The flow of $A_w$ preserves $\rho'$ and $\varphi'$. A key point in finding periodic trajectories of $W$ is the fact that there is a function on $S$ constant along the trajectories:

**Lemma 10** Let $h = \rho'(A_w)$ and $f = \text{Re}(h)$. Then $f$ is constant along the trajectories of $W$.

**Proof:** We have

$$0 = \mathcal{L}_{A_w} \rho' = d(i_{A_w} \rho') + i_{A_w} dp' = dh + (-1)^{n-1}i_{A_w} \varphi'$$

So $dh = (-1)^{n}i_{A_w} \varphi'$. So $dh(W) = (-1)^{n} \varphi'(A_w, W)$. The vector field $A_w$ is in the tangent bundle to $S$, hence we can decompose it into $A_w = A_w^H + \lambda F$. Here $A_w^H$ is the horizontal part of $A_w$ (i.e. the part in the distribution $H'$). $F$ is the generator of the $S^1$-action on $S$ and $\lambda \in \mathbb{R}$ (see Section 2.3). $W$ is horizontal and $H'$ is a 1-dimensional complex vector bundle. The form $\varphi'$ is a $(2,0)$-form on $Q$. Hence $\varphi'(A_w^H, W) = 0$. Also $F = JY'$. Hence $\varphi'(F, W) = i\varphi'(Y', W)$. By the construction of $W$ we had that $\varphi'(Y', W)$ is real. From all this we deduce that $dh(W)$ is purely imaginary, hence $df(W) = 0$, i.e. $f$ is constant along the trajectories of $W$. Q.E.D.

From the previous lemma we deduce that the trajectories of $W$ live on level sets of the function $f$. We need to understand those level sets in more detail.

We had our symplectic reductions $N_{\text{red}} = Z/T''$ and $Q$ and we have a natural projection $\pi' : Q \to N_{\text{red}}$. Let $v$ be an element of the weight lattice $\Lambda$ of $T^*$ defining the torus $T''$. As we have seen $Z$ is equal to $\mu^{-1}[t_1v, t_2v]$ for $t_1 < 0 < t_2$. $T''$ acts freely on $Z_0 = \mu^{-1}(t_1v, t_2v)$ and we have $N_0 = Z_0/T'' \subset N_{\text{red}}$, which is the smooth part of $N_{\text{red}}$. We also have 2 points $a_1 = \mu^{-1}(t_1v)/T'' \in N_{\text{red}}$ and $N_{\text{red}}$ is a disjoint union of $a_1, a_2$ and $N_0$. We have $S_0 = \pi'^{-1}(N_0) \cap S$, and $S_0$ is a fiber bundle over $N_0$ with fibers being the orbits of the $S^1$-action on $S$ (see Section 2.3). This action is free on $S_0$. Also each $K_i = \pi'^{-1}(a_i)$ is an orbit of the $S^1$-action on $S$, but this action on each $K_i$ might have a finite stabilizer.

We have seen in Section 2.3 that the form $\rho'$ is equivariant with respect to the $S^1$-action on $S$. The flow vector field $A_w$ is invariant under $S^1$-action. Hence the function $h = \rho'(A_w)$ is $S^1$-equivariant.

On $Z_0$ we had an oriented Lagrangian distribution $D$, given by the image of $T$ under the differential of the action on $N$. This distribution gives rise to a unit length section $\kappa$ of $K(N)$ over $Z_0$ as in Lemma 1. This section is $T$-invariant, hence it gives rise to an $R$-invariant section $\kappa'$ of $S_0$ over $N_0$. By definition $\kappa$ restricts to a positive real $n$-form on the distribution $D$. From this we deduce that $h = \rho'(A_w)$ is real and positive along $\kappa'$.

We can normalize $w$ s.t. $v(w) = 1$. We have a function $\tau = \mu(w)$ on $N_{\text{red}}$, and the image of this function is the interval $[t_1, t_2]$. For each $t_1 \leq t \leq t_2$ the level set $\tau^{-1}(t)$ is an orbit of the $R$-action on $N_{\text{red}}$. Let $L' = \tau^{-1}(0)$, $L_+ = \kappa'(L') \subset S$ and $L_- = (-1) \cdot L_+$. Each $L_{\pm}$ is an orbit of the $R$-action on $S$. Also at points of $L_{\pm}$ the vector field $A_w$ is horizontal (since $\mu(w) = 0$) and $\rho'(A_w)$ is real. The vector field $W$ also satisfies those properties, hence $W$ is proportional to $A_w$ along $L_{\pm}$. So we see that $L_{\pm}$ are trajectories $W$ (of course
the minimal Lagrangian torus of $N$ coming from these trajectories is the torus $L = \mu^{-1}(0)$. We have the following:

**Lemma 11** The differential $df$ of $f$ is non-vanishing on $S - (L_- \cup L_+)$. 

**Proof:** We have seen in the proof of Lemma 10 that $dh = (-1)^w i_{A_w} \varphi'$. On $S \cap \pi^{-1}(N_{red} - L')$ the vertical part of the vector field $A_w$ doesn’t vanish. Hence the form $i_{A_w} \varphi'$ restricts as a non-vanishing $(1,0)$-form on the horizontal distribution $H'$. From this it is clear that $df|_{H'} \neq 0$.

On $S \cap \pi^{-1}(L') - (L_- \cup L_+)$ $h$ is not real. Also $h$ is equivariant with respect to the $S^1$-action. Let $F$ be the vector field generating the $S^1$-action as before. Then the derivative of $f = Reh$ is non-zero in the direction of $F$. Q.E.D.

$f$ attains a constant value $f_+$ along $L_+$ and a value $f_- = -f_+$ along $L_- = (-1)^i \cdot L_+$. Since $S$ is compact and connected, it is clear from Lemma 11 that $f_+$ is the absolute maximum of $f$, attained only at $L_+$, and $f_-$ is the absolute minimum of $f$, attained only at $L_-$. Also for any $s \in (f_-, f_+)$, the level set $\Sigma_s = f^{-1}(s)$ is smooth.

We will also need the fact that $|f|_{K_i} = 0$. To prove that we note that along $K_i A_w$ is vertical. Indeed the action of $\exp(tw)$ on $a_i$ is trivial for any $t \in \mathbb{R}$. Hence the action of $\exp(tw)$ on $S$ preserves the fiber $K_i = \pi^{-1}(a_i)$, so the vector field $A_w$ is tangent to $K_i$, i.e. vertical along $K_i$. But from this we deduce that $h = \rho'(A_w) = 0$ at $K_i$, and so $f = 0$ at $K_i$.

Let now $\Phi = f^{-1}(f_-, f_+)$. Take any point $m \in \Phi$ and consider the level set $\Sigma_m$ of $f$ passing through $m$. Let $\Sigma_m'$ be the connected component of $\Sigma_m$ containing $m$. The vector field $W$ is tangent to $\Sigma_m'$. We have a free $R$-action on $\Sigma_m'$, and this action preserves the vector field $W$. Also $A_w$ is transversal to $W$ at all points of $\Sigma_m'$, and $W$ contains the vector field $A_w$ is not horizontal, so it can’t be proportional to $W$. If $m' \in S \cap \pi^{-1}(N_{red} - L')$, then $h = \rho'(A_w)$ is not real, while $\rho'(W)$ is real. So again $A_w$ and $W$ can’t be proportional. Thus we get that the quotient of $\Sigma_m'$ by the $R$-action is a circle and $W$ projects to a non-vanishing vector field on it. From this we deduce that the $W$-trajectory starting at $m$ will intersect the $R$-orbit of $m$. Suppose it intersects this orbit for the first time at a point $\xi(m)m, \xi(m) \in R$. This gives rise to a well-defined function $\xi : \Phi \mapsto R$. Clearly $\xi$ is continuous, $R$-invariant and constant along the trajectories of $W$. Also we have seen in Section 2.3 that the $-1$-action on $S$ sends $W$-trajectories to $W$-trajectories in the reverse direction. From this we easily deduce that 

$$\xi(-1 \cdot m) = \xi(m)^{-1}$$

Obviously the trajectory through $m$ is periodic if $\xi(m)$ is a root of unity in $R$. Let $R'$ be the set of roots of unity in $R$. Since $\xi$ is continuous, $\xi^{-1}(R')$ will be everywhere dense in $\Phi$ unless $\xi$ assumes a constant value not in $R'$ on some open subset of $\Phi$. The next lemma shows that it is impossible:

**Lemma 12** Suppose that $\xi$ is constant on some open set $U \subset \Phi$. Then $\xi$ is equal to a constant $g$ on the whole of $\Phi$ and $g^2 = 1$. 

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**Proof:** Let \( S_+ = f^{-1}(0, f_+) \), \( S_- = f^{-1}(f_-, 0) \). Thus \( S_- = -1 \cdot S_+ \). Suppose that \( \xi \) is constant on some open set \( U \in \Phi \). Then \( \xi \) is constant on some open ball \( U' \) either in \( S_+ \) or in \( S_- \). We can assume w.l.o.g. that \( U' \subset S_+ \). We note that \( S_+ \) is connected. In fact \( S_+ \) is given by

\[
S_+ = (\kappa(n)e^{i\theta}|n \in N_0, -\pi/2 < \theta < \pi/2) - (L_+ \cup L_-)
\]

First we prove that \( \xi \) is a constant \( g \) on \( S_+ \). Let \( A_w^H \) be the horizontal part of the vector field \( A_w \). Since \( S_+ \subset \pi^{-1}(N_0) \) we deduce that \( A_w^H \) doesn’t vanish on \( S_+ \). We also note that \( A_w^H \) cannot be proportional to \( JW \). Indeed suppose that \( A_w^H = \lambda JW \) for some \( \lambda \in \mathbb{R} \) at some point \( m \in S_+ \). But then

\[
h(m) = \rho'(A_w) = \rho'(A_w^H) = i\lambda\rho'(W)
\]

So \( h(m) \) is purely imaginary, hence \( f(m) = 0 \) - a contradiction. Since both \( A_w^H \) and \( W \) lie in \( H' \), which is a complex 1-dimensional distribution, we deduce that we can find a function \( b : S_+ \to \mathbb{R} \) s.t. the vector fields \( W' = A_w^H + bJA_w^H \) and \( W \) are proportional. Hence the trajectories of \( W' \) and \( W \) coincide. We will use \( W' \) instead of \( W \) to prove that \( \xi \) is constant on \( S_+ \). Suppose that for a point \( m \in S_+ \) it takes time \( t(m) \) for the \( W' \)-flow to hit the \( R \)-orbit of \( m \). We have the following:

**Lemma 13** \( \xi(m) = \exp(t(m)w) \)

**Proof:** Let \( \gamma \) be the trajectory of \( W' \) through \( m \) and \( \gamma' = \pi'(\gamma) \) be the corresponding path in \( N_0 \). We have an \( R \)-action on \( N_0 \), and the corresponding flow vector field \( B_w \) for the \( w \)-flow on \( N_0 \). We obviously have \( \pi'(A_w^H) = B_w \). Hence the tangent field to \( \gamma' \) is \( B_w + bJB_w \).

The \( R \)-action on \( N_0 \) is Hamiltonian with the moment map \( \tau = \mu(w) \). Also the \( B_w \)-flow on \( N_0 \) commutes with the complex structure \( J \) on \( N_0 \). Hence the vector fields \( B_w \) and \( JB_w \) commute. Let \( P_1 = \gamma'(0) \) and \( P_2 = \gamma'(t(m)) \). Then \( P_2 = \xi(m)P_1 \). We need to prove that \( P_2 = \exp(t(m)w)P_1 \) and since the \( R \)-action on \( N_0 \) is free we would be done.

Let \( \exp(xJB_w) \) be the time \( x \) flow of \( B_w \). Note that the \( JB_w \)-flow is not complete. In fact we have

\[
JB_w(\tau) = \omega_{red}(JB_w, B_w) = |B_w|^2 > 0
\]

So \( \tau \) increases on the \( JB_w \)-trajectories. Let \( c(r) = \int_{(0, r]} b(t)dt \) for \( 0 \leq r \leq t(m) \). Consider a path \( \gamma''(r) = \exp(c(r)(JB_w))\exp(rw)(P_1) \) (note that we flow \( P_1 \) with respect to \( rB_w \) first). Then \( \gamma''(r) \) is defined for small values of \( r \). Also the tangent vector to \( \gamma'' \) is \( B_w + b(r)JB_w \). So \( \gamma'' \) coincides with \( \gamma' \) whenever it is defined.

Suppose on \( \gamma' \) \( \tau \) ranges between \( s_1 \) and \( s_2 \). Then \( t_1 < s_1 \) and \( s_2 < t_2 \). Pick any \( r \) for which \( \gamma''(r) \) is defined and consider the path \( \exp(tJB_w)\exp(rw)(P_1) \) for \( t \) ranging between 0 and \( c(r) \). The function \( \tau \) is increasing along the path, and on the endpoints it’s values are between \( s_1 \) and \( s_2 \). Hence this path lives.
in the compact set $A = \tau^{-1}[s_1, s_2]$ in $N_0$. From this one can easily deduce that $\gamma''(r)$ is well defined for all $0 \leq r \leq s$ and coincides with $\gamma'(r)$. In particular $P_2 = \exp(c(t(m))JB_w)\exp(t(m)w)(P_1)$. Now

$$\tau(P_2) = \tau(P_1) = \tau(\exp(t(m)w)P_1)$$

and $\tau$ increases on the trajectories of $JB_w$. So we get that $c(t(m)) = 0$, i.e. $P_2 = \exp(t(m)w)P_1$. Q.E.D.

Now we can prove that $\xi$ is constant on $S_+$. Since $\xi$ is constant on $U'$ we get that $t(m)$ is a constant $t$ on $U'$. Let $\phi_t$ be the time $t$ flow of $W'$ on $S_+$. Consider the map $\chi = \exp(-tw)\cdot \phi_t : S_+ \rightarrow S_+$. $S_+$ is a connected real analytic manifold and $\chi$ is a real analytic map. Also $\chi$ is the identity map on $U'$. So we deduce that $\chi$ is the identity map. So $\phi_t$ is the multiplication by $g = \exp(tw)$ on $S_+$. From this we easily deduce that $\xi = g$ on $S_+$.

So $\xi$ assumes a constant value $g$ on $S_+$, and hence it assumes a constant value $g^{-1}$ on $S_- = -1 \cdot S_+$. Let $\Delta = f^{-1}(0)$. Then $\Delta$ is the common boundary of $S_+$ and $S_-$ in $\Phi$. Since $\xi$ is continuous, we must have $g = g^{-1}$, i.e. $g^2 = 1$. Q.E.D.

From this we get an immediate corollary

**Corollary 1** The set $\xi^{-1}(R')$ is everywhere dense in $\Phi$.

We are now ready to state and prove our main result:

**Theorem 1** Let $N^{2n}$ be a K-E manifold with positive scalar curvature with an effective $T^n$-action. Then precisely one regular orbit of the action is a minimal Lagrangian submanifold of $N$. Moreover there is an $(n - 1)$-torus $T^{n-1} \subset T^n$ and a sequence of non-flat $T^{n-1}$-invariant immersed minimal Lagrangian tori $L_k \subset N$ s.t. $L_k$ locally converge to $L$ (in particular the supremum of sectional curvatures of $L_k$ and the distance between $L$ and $L_k$ goes to 0 as $k \rightarrow \infty$).

**Proof:** Choose a torus $T'' = T^{n-1}$ which satisfies the conditions of Lemma 9. By Corollary 1 we can choose a sequence of points $m_k \in \xi^{-1}(R')$ s.t. $m_k$ converge to a point $m$ in $L_+$. The $W$-trajectories $\gamma_k$ through $m_k$ are periodic and live on level sets $\Sigma_k$ of $f$ with $\Sigma_k$ converging to $L_+$. From this we easily see that $\gamma_k$ locally converge to the trajectory $L_+$. One easily deduces that the immersed minimal Lagrangian tori $L_k$ which $\gamma_k$ define as in Lemma 6 locally converge to the minimal, $T$-invariant orbit $L$.

Finally we prove that $L_k$ are not flat. From Lemma 9 we get a vector $v$ in the Lie algebra of $T''$ s.t. the flow vector field $X_v$ of $v$ doesn’t have a constant length on $Z$. Now the vector field $X_v$ along $L_k$ is a Killing vector field of $L_k$. So to prove that $L_k$ is not flat it is enough to prove that $|X_v|^2$ is non-constant on $L_k$.

The function $|X_v|^2$ is $T$-invariant on $Z$. Thus it can be viewed as an $R$-invariant function on $N_{red}$, i.e. it can be viewed as a function of $\tau = \mu(w)$ on $N_{red}$. Also $|X_v|^2$ is a real analytic function on $N_0 = \tau^{-1}(t_1, t_2)$. Since $|X_v|^2$ is non-constant, it is nowhere a locally constant function of $\tau$. Since $\gamma_k$ are different from $L_k$, we easily deduce that $\tau(\gamma_k)$ are non-trivial intervals in $(t_1, t_2)$. Hence $|X_v|^2$ is non-constant on $L_k$ and we are done. Q.E.D.
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