SUBFACTORS AND HADAMARD MATRICES

WES CAMP AND REMUS NICOARA

Abstract. To any complex Hadamard matrix $H$ one associates a spin model commuting square, and therefore a hyperfinite subfactor. The standard invariant of this subfactor captures certain "group-like" symmetries of $H$. To gain some insight, we compute the first few relative commutants of such subfactors for Hadamard matrices of small dimensions. Also, we show that subfactors arising from Dita type matrices have intermediate subfactors, and thus their standard invariants have some extra structure besides the Jones projections.

1. Introduction

A complex Hadamard matrix is a matrix $H \in M_n(\mathbb{C})$ having all entries of absolute value 1 and all rows mutually orthogonal. Equivalently, $\frac{1}{\sqrt{n}}H$ is a unitary matrix with all entries of the same absolute value. For example, the Fourier matrix $F_n = (\omega^{ij})_{1 \leq i,j \leq n}, \omega = e^{2\pi i/n}$, is a Hadamard matrix.

In the recent years, complex Hadamard matrices have found applications in various topics of mathematics and physics, such as quantum information theory, error correcting codes, cyclic $n$-roots, spectral sets and Fuglede’s conjecture. A general classification of real or complex Hadamard matrices is not available. A catalogue of most known complex Hadamard matrices can be found in [TZ]. The complete classification is known for $n \leq 5$ ([H]) and for self-adjoint matrices of order 6 ([BeN]).

The connection between Hadamard matrices and von Neumann algebras arose from an observation of Popa ([Po2]): a unitary matrix $U$ is of the form $\frac{1}{\sqrt{n}}H$, $H$ Hadamard matrix, if and only if the algebra of $n \times n$ diagonal matrices $D_n$ is orthogonal onto $U D_n U^*$, with respect to the inner product given by the trace on $M_n(\mathbb{C})$. Equivalently, the square of inclusions:

$$\mathcal{E}(H) = \left( \begin{array}{c}
D_n \subset M_n(\mathbb{C}) \\
\bigcup \mathbb{C} \subset \bigcup U D_n U^*, \tau
\end{array} \right)$$

is a commuting square, in the sense of [Po1],[Po2]. Here $\tau$ denotes the trace on $M_n(\mathbb{C})$, normalized such that $\tau(1) = 1$. 
Such commuting squares are called spin models, the name coming from statistical mechanical considerations (see [JS]). By iterating Jones’ basic construction, one can construct a hyperfinite, index $n$ subfactor from $H$ (see for instance [JS]). The subfactor associated to $H$ can be used to capture some of the symmetries of $H$, and thus to classify $H$ to a certain extent (see [BHJ], [Jo2], [BN]).

Let $N \subset M$ be an inclusion of $II_1$ factors of finite index, and let $N \subset M \subset M_1 \subset M_2 \subset ...$ be the tower of factors constructed by iterating Jones’ basic construction (see [Jo1]), where $e_1, e_2, ...$ denote the Jones projections. The standard invariant $G_{N,M}$ is then defined as the trace preserving isomorphism class of the following sequence of commuting squares of inclusions of finite dimensional $*$-algebras:

$$C = N' \cap N < N' \cap M < N' \cap M_1 < N' \cap M_2 < ... \cup \cup \cup M' \cap M < M' \cap M_1 < M' \cap M_2 < ...$$

The Jones projections $e_1, e_2, ..., e_n$ are always contained in $N' \cap M_n$. If the index of the subfactor $N \subset M$ is at least 4, they generate the Temperley-Lieb algebra of order $n$, denoted $TL_n$. In a lot of situations the relative commutant $N' \cap M_n$ has some interesting extra structure, besides $TL_n$. For instance, the five non-equivalent real Hadamard matrices of order 16 yield different dimensions for the second relative commutant $N' \cap M_1$, and thus are classified by these dimensions ([BHJ]).

In this paper we investigate the relation between Hadamard matrices and their subfactors. We look at Hadamard matrices of small dimensions or of special types. The paper is organized as follows: in the first section we recall, in our present framework, several results of [Jo2], [JS] regarding computations of standard invariants for spin models.

In section 2 we study the subfactors associated to Hadamard matrices of Dita type. These are matrices that arise from a construction of [Di], which is a generalization of a construction of Haagerup ([H]). Most known parametric families of Hadamard matrices are of Dita type. We show that the associated subfactors have intermediate subfactors.

In the last section we present a list of computations of the second and third relative commutants $N' \cap M_1, N' \cap M_2$, for complex Hadamard matrices of small dimensions. We make several remarks and conjectures regarding the structure of the standard invariant. Most of the computations included were done using computers, with the help of the Mathematica and GAP softwares.

We would like to thank Teodor Banica, Kyle Beauchamp and Dietmar Bisch for fruitful discussions and correspondence. Wes Camp was supported in part
2. Subfactors associated to Hadamard matrices

Let $H$ be a complex $n \times n$ Hadamard matrix and let $U = \frac{1}{\sqrt{n}}H$. $U$ is a unitary matrix, with all entries of the same absolute value. One associates to $U$ the square of inclusions:

$$\mathcal{C}(U) = \left( \mathcal{D}_n \subset M_n(\mathbb{C}), \tau \right)$$

where $\mathcal{D}_n$ is the algebra of diagonal $n \times n$ matrices and $\tau$ is the trace on $M_n(\mathbb{C})$, normalized such that $\tau(1) = 1$.

Since $H$ is a Hadamard matrix, $\mathcal{C}(H)$ is a commuting square in the sense of \cite{Po1}, \cite{Po2}, i.e. $E_{\mathcal{D}_n}E_{U\mathcal{D}_nU^*} = E_{\mathcal{C}}$. The notation $E_A$ refers to the $\tau$-invariant conditional expectation from $M_n(\mathbb{C})$ onto the $*$-subalgebra $A$.

Recall that two complex Hadamard matrices are said to be equivalent if there exist unitary diagonal matrices $D_1, D_2$ and permutation matrices $P_1, P_2$ such that $H_2 = P_1D_1H_1D_2P_2$. It is easy to see that $H_1, H_2$ are equivalent if and only if $\mathcal{C}(H_1), \mathcal{C}(H_2)$ are isomorphic as commuting squares, i.e. conjugate by a unitary from $M_n(\mathbb{C})$.

We denote by $\mathcal{C}^t(H)$ the commuting square obtained by flipping the upper left and lower right corners of $\mathcal{C}(H)$:

$$\mathcal{C}^t(H) = \left( U\mathcal{D}_nU^* \subset M_n(\mathbb{C}), \tau \right)$$

We have: $\mathcal{C}^t(H) = \text{Ad}(U)\mathcal{C}(H^*)$. Thus, $\mathcal{C}^t(H)$ and $\mathcal{C}(H)$ are isomorphic as commuting squares if and only if $H, H^*$ are equivalent as Hadamard matrices.

We now recall the construction of a subfactor from a commuting square. By iterating Jones' basic construction (\cite{Jo1}), one obtains from $\mathcal{C}^t(H)$ a tower of commuting squares of finite dimensional $*$-algebras:

$$\begin{align*}
UD_nU^* &\subset M_n(\mathbb{C}) \supset X_1 \supset X_2 \supset \ldots \\
\mathbb{C} &\subset \mathcal{D}_n \supset Y_1 \supset Y_2 \supset \ldots 
\end{align*}$$

(1)

together with the extension of the trace, which we will still denote by $\tau$, and Jones projections $g_{i+2} \in Y_i$, $i = 1, 2, \ldots$.
Let $M_H$ be the weak closure of $\cup_i X_i$, with respect to the trace $\tau$, and let $N_H$ be the weak closure of $\cup_i Y_i$. $N_H, M_H$ are hyperfinite $II_1$ factors, and the trace $\tau$ extends continuously to the trace of $M_H$, which we will still denote by $\tau$. It is well known that $N_H \subset M_H$ is a subfactor of index $n$, which we will call the subfactor associated to the Hadamard matrix $H$.

The standard invariant of $N_H \subset M_H$ can be expressed in terms of commutants of finite dimensional algebras, by using Ocneanu’s compactness argument (5.7 in [JS]). Consider the basic construction for the commuting square $\mathcal{C}(H)$:

\[
\begin{align*}
\mathcal{D}_n & \subset M_n(\mathbb{C}) \\
\cup & \cup \cup \cup \\
\mathbb{C} & \subset UD_n U^* \\
\mathcal{P}_1 & \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset \ldots \\
\mathcal{Q}_1 & \subset \mathcal{Q}_2 \subset \ldots \\
\mathcal{D}'_n \cap U D_n U^* & \subset \mathcal{D}'_n \cap \mathcal{Q}_1 \subset \mathcal{D}'_n \cap \mathcal{Q}_2 \subset \mathcal{D}'_n \cap \mathcal{Q}_3 \subset \ldots
\end{align*}
\]

Ocneanu’s compactness theorem asserts that the first row of the standard invariant of $N_H \subset M_H$ is the row of inclusions:

\[
\begin{align*}
\mathcal{D}'_n \cap U D_n U^* & \subset \mathcal{D}'_n \cap \mathcal{Q}_1 \subset \mathcal{D}'_n \cap \mathcal{Q}_2 \subset \mathcal{D}'_n \cap \mathcal{Q}_3 \subset \ldots
\end{align*}
\]

More precisely, if

\[
N_H \subset M_H \subset M_{H,1} \subset M_{H,2} \subset \ldots
\]

is the Jones tower obtained from iterating the basic construction for the inclusion $N_H \subset M_H$, then:

\[
\mathcal{D}'_n \cap \mathcal{Q}_i = N_H' \cap M_{H,i}, \text{ for all } i \geq 1.
\]

Thus, the problem of computing the standard invariant of the subfactor associated to $H$ is the same as the computation of $\mathcal{D}'_n \cap \mathcal{Q}_i$. However, such computations seem very hard, and even for small $i$ and for matrices $H$ of small dimensions they seem to require computer use. Jones ([Jo2]) provided a diagrammatic description of the relative commutants $\mathcal{D}'_n \cap \mathcal{Q}_i$ (see also [JS]), which we express below in the framework of this paper.

Let $\mathcal{P}_0 = M_n(\mathbb{C})$ and let $(e_{i,j})_{1 \leq i,j \leq n}$ be its canonical matrix units. Let

\[
e_2 = \frac{1}{n} \sum_{i,j=1}^{n} e_{i,j}.
\]

It is easy to check that $e_2$ is a projection. Moreover: $< \mathcal{D}_n, e_2 > = M_n(\mathbb{C})$ and $e_2 x e_2 = E_{\mathcal{D}_n} (x) e_2$ for all $x \in M_n(\mathbb{C})$. Thus, $e_2$ is realizing the basic construction

\[
\mathbb{C} \subset \mathcal{D}_n \subset M_n(\mathbb{C})
\]

Let $e_{k,l} \otimes e_{i,j}$ denote the $n^2 \times n^2$ matrix having only one non-zero entry, equal to 1, at the intersection of row $(i-1)n + k$ and column $(j-1)n + l$. Thus, $e_{k,l} \otimes e_{i,j}$ are matrix units of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. In what follows, we
Lemma 2.1. Let \( P_1 = M_n(\mathbb{C}) \otimes D_n, P_2 = M_n(\mathbb{C}) \otimes M_n(\mathbb{C}), e_3 = \sum_{i=1}^n e_i \otimes e_i \in P_1 \) and \( e_4 = I_n \otimes e_2 \in P_2 \). Then
\[
\mathcal{D}_n \subset M_n(\mathbb{C}) \subset P_1
\]
is a basic construction with Jones projection \( e_3 \) and
\[
M_n(\mathbb{C}) \subset P_1 \subset P_2
\]
is a basic construction with Jones projection \( e_4 \).

Proof. To show that \( \mathcal{D}_n \subset M_n(\mathbb{C}) \subset P_1 \) is a basic construction it is enough to check that \( \langle M_n(\mathbb{C}), e_3 \rangle = P_1 \) and \( e_3 \) is implementing \( E_{M_n(\mathbb{C})} \). First part is clear, since \( e_k \otimes e_i = e_k \otimes e_i \) are a basis for \( P_1 = M_n(\mathbb{C}) \otimes D_n \). To check that \( e_3 \) implements the conditional expectation, let \( X = (x_{i,j}) \in M_n(\mathbb{C}) \). We have:
\[
e_3(X \otimes I_n)e_3 = \sum_{i,j=1}^n (D_i \otimes D_j)(X \otimes I_n)(D_j \otimes D_j)
= \sum_{i=1}^n D_i X D_i \otimes D_i
= \sum_{i=1}^n (D_i X D_i \otimes I_n)e_3
= E_{\mathcal{D}_n \otimes I_n}(X)e_3
\]

Since \( \mathbb{C} \subset \mathcal{D}_n \subset M_n(\mathbb{C}) \) is a basic construction, after tensoring to the left by \( M_n(\mathbb{C}) \) it follows that \( M_n(\mathbb{C}) \subset P_1 \subset P_2 \) is a basic construction, with \( e_4 = I_n \otimes e_2 \).

Proposition 2.1. The algebras \( P_1, P_2, P_3, \ldots \) constructed in (2) are given by
\[
P_{2k} = \otimes_{i=1}^{k+1} M_n(\mathbb{C}), \quad P_{2k+1} = P_{2k} \otimes \mathcal{D}_n
\]
with the Jones projections
\[
e_{2k+2} = \otimes_{i=1}^k I_n \otimes e_2, \quad e_{2k+3} = \otimes_{i=1}^k I_n \otimes e_3
\]

Proof. Follows from the previous lemma, by tensoring successively by \( M_n(\mathbb{C}) \).
Proposition 2.2. Let $H$ be a complex $n \times n$ Hadamard matrix, let $U = \frac{1}{\sqrt{n}} H$, and
\[ D_U = \sum_{i,j=1}^{n} \bar{u}_{i,j} e_{j,j} \otimes e_{i,i}, \quad U_1 = U D_U. \]
Then the algebras $Q_1, Q_2, Q_3, \ldots$ constructed in (2) are given by
\[ Q_k = U_k P_{k-1} U_k^*, \quad k \geq 1 \]
where $U_k \in \mathcal{P}_k$ are the unitary elements:
\[ U_{2k+1} = \prod_{i=0}^{k} (\otimes^i I_n \otimes U_1 \otimes^{k-i} I_n), \quad U_{2k} = U_{2k-1} (\otimes^k I_n \otimes U), \quad k \geq 1. \]
Proof. The unitary $U_1$ satisfies:
\[ (\text{Ad} U_1)(D_n) = (\text{Ad} U)(D_n) \]
since $U^* U_1 = D_U \in D_n$. Moreover, we have:
\begin{align*}
(\text{Ad} U_1)(e_2) &= (\text{Ad} U) \text{Ad} \left( \sum_{i,j=1}^{n} \bar{u}_{i,j} e_{j,j} \otimes e_{i,i} \right) \left( \frac{1}{n} \sum_{k,l=1}^{n} e_{k,l} \right) \\
&= (\text{Ad} U) \left( \sum_{i,k,l=1}^{n} \bar{u}_{i,k} u_{i,l} e_{k,l} \otimes e_{i,i} \right) \\
&= (\text{Ad} U)(\text{Ad} U^*(e_3)) \\
&= e_3
\end{align*}
(4)
It follows that $\text{Ad} U_1$ takes the basic construction $\mathbb{C} \subset D_n \subset M_n(\mathbb{C})$ onto the inclusion $\mathbb{C} \subset U D_n U^* \subset U_1 M_n(\mathbb{C}) U_1^*$. Thus this is also a basic construction, which shows that $Q_1 = U_1 M_n(\mathbb{C}) U_1^*$. Moreover, it follows that each $\text{Ad} U_i$ takes the basic construction $\mathcal{P}_{i-1} \subset \mathcal{P}_i \subset \mathcal{P}_{i+1}$ onto $Q_i \subset Q_{i+1} \subset Q_{i+2}$, which ends the proof.
\[
\square
\]

The first relative commutant $\mathcal{D}_n' \cap U D_n U^*$ is equal to $\mathbb{C}$, since the commuting square condition implies $\mathcal{D}_n \cap U D_n U^* = \mathbb{C}$. Thus the subfactor $N_H \subset M_H$ is irreducible. In the following proposition we describe the higher relative commutants of the subfactor $N_H \subset M_H$ as the commutants of some matrices $P_i, i \geq 1$.

Proposition 2.3. With the previous notations, let $P_i$ denote the projection $U_i e_{i+3} U_i^* \in \mathcal{P}_{i+1}, i \geq 1$. Then we have the following formula for the $(i+1)$-th relative commutant:
\[ \mathcal{D}_n' \cap Q_i = P_i' \cap \mathcal{D}_n' \cap P_i. \]
Proof. We have:

\[
\mathcal{D}'_n \cap Q_i = \mathcal{D}'_n \cap \text{Ad}_i(\mathcal{P}_{i-1}) \\
= \mathcal{D}'_n \cap \text{Ad}_i(e'_{i+3} \cap \mathcal{P}_i) \\
= \mathcal{D}'_n \cap P'_i \cap \text{Ad}_i(\mathcal{P}_i) \\
= \mathcal{D}'_n \cap P'_i \cap \mathcal{P}_i 
\]

We used the fact that \(\mathcal{P}_{i-1} \subset \mathcal{P}_i \subset \mathcal{P}_{i+1}\) is a basic construction, and thus \(e'_{i+3} \cap \mathcal{P}_i = \mathcal{P}_{i-1}\). □

Remark 2.1. The \(n^2 \times n^2\) matrix \(P_1 = U_1 e_4 U_1^*\) can be written as

\[
P_1 = \sum_{a,b,c,d=1}^{n} p_{a,b}^{c,d} e_{a,b} \otimes e_{c,d}, \quad \text{where} \quad p_{a,b}^{c,d} = \sum_{i=1}^{n} u_{a,i} u_{b,i} u_{c,i} u_{d,i}.\]

This matrix is used in the theory of Hadamard matrices and it is called the profile of \(H\). It is a result of Jones ([Jo2]) that the matrices \(P_{2i+1}, i \geq 1\), depend only on \(P_1\). Indeed, one can check that

\[
P_{2i+1} = \sum_{k_1,l_1, \ldots, k_i,l_i=1}^{n} p_{a,b}^{k_1,l_1 \ldots, k_i,l_i} p_{k_1,l_1 \ldots, k_i,l_i}^{c,d} e_{a,b} \otimes e_{c,d}.\]

Thus, all higher relative commutants of even orders are determined by \(P_1\).

Let \(\Gamma_H\) denote the graph of vertices \(\{1,2,\ldots,n\} \times \{1,2,\ldots,n\}\), in which the distinct vertices \((a,c)\) and \((b,d)\) are connected if and only if \(p_{a,b}^{c,d} \neq 0\). The second relative commutant can be easily described in terms of \(\Gamma_H\). We recall this in the following Proposition, which is a reformulation of a result in [Jo2] (see also [JS]).

**Proposition 2.4.** The second relative commutant of the subfactor \(N_H \subset M_H\) is abelian, its minimal projections are in bijection with the connected components of \(\Gamma_H\), and their traces are proportional to the sizes of the connected components.

Proof. Let \(\sum_{i,j=1}^{n} \lambda_i^j e_{i,i} \otimes e_{j,j},\lambda_i^j \in \{0,1\}\), be a projection in the second relative commutant \(P'_1 \cap (\mathcal{D}_n \otimes \mathcal{D}_n)\). We have:

\[
(\sum_{a,b,c,d=1}^{n} p_{a,b}^{c,d} e_{a,b} \otimes e_{c,d})(\sum_{i,j=1}^{n} \lambda_i^j e_{i,i} \otimes e_{j,j}) = (\sum_{i,j=1}^{n} \lambda_i^j e_{i,i} \otimes e_{j,j})(\sum_{a,b,c,d=1}^{n} p_{a,b}^{c,d} e_{a,b} \otimes e_{c,d})
\]

Equivalently:

\[
\sum_{a,c,i,j=1}^{n} \lambda_i^j p_{a,b}^{c,d} e_{a,b} \otimes e_{c,d} = \sum_{b,d,i,j=1}^{n} \lambda_i^j p_{i,b}^{j,d} e_{i,b} \otimes e_{j,d}.\]
By relabeling and identifying the set of indices, it follows:
\[
(\lambda^c_a - \lambda^i_j) p^c,j_{a,i} = 0.
\]
Thus, if the vertices \((a, c)\) and \((i, j)\) are connected then \(\lambda^c_a = \lambda^i_j\). This ends the proof. \(\square\)

3. Matrices of Dita Type

In this section we investigate the standard invariant of subfactors associated to a particular class of Hadamard matrices, obtained by a construction of P.Dita ([Di]), which is a generalization of an idea of U.Haagerup ([H]). These matrices have a lot of symmetries, and we show that for such matrices the second relative commutant has some extra structure besides the Jones projection.

Let \(n\) be non-prime, \(n = kl\) with \(k, l \geq 2\). Let \(A = (a_{i,j}) \in M_k(\mathbb{C})\) and \(B_1, ..., B_k \in M_m(\mathbb{C})\) be complex Hadamard matrices. It is possible to construct an \(n \times n\) Hadamard matrix from \(A, B_1, ..., B_k\) by using an idea of [Di] (see also [H], [Pe]). This construction is a generalization of the tensor product of two Hadamard matrices:

\[
H = \begin{pmatrix}
    a_{1,1}B_1 & a_{1,2}B_2 & \cdots & a_{1,k}B_k \\
    a_{2,1}B_1 & a_{2,2}B_2 & \cdots & a_{2,k}B_k \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{k,1}B_1 & a_{k,2}B_2 & \cdots & a_{k,k}B_k
\end{pmatrix}
\] (6)

Let \((f_{i,j})_{1 \leq i,j \leq k}\) be the matrix units of \(M_k(\mathbb{C})\). We identify \(M_n(\mathbb{C})\) with the tensor product \(M_m(\mathbb{C}) \otimes M_k(\mathbb{C})\), with the same conventions as before. Thus:

\[
H = \sum_{i,j=1}^{k} a_{i,j}B_j \otimes f_{i,j}
\]

One can use construct multi-parametric families of non-equivalent Hadamard matrices, by replacing \(B_1, ..., B_k\) by \(B_1D_1, ..., B_kD_k\), where \(D_1, ..., D_k\) are diagonal unitaries. Some of the families of Hadamard matrices of small orders considered in the next section arise from this construction.

Recall that the second relative commutant always contains the Jones projection \(e_3 = \sum e_{ii} \otimes e_{ii}\). In the next proposition we show that the second relative commutant of a Dita type subfactor contains another projection \(f \geq e_3\), so it has dimension at least 3.

**Proposition 3.1.** Let \(H = (a_{i,j}B_j)_{1 \leq i,j \leq k} \in M_n(\mathbb{C})\) be a Dita type matrix, where \(A = (a_{i,j})_{1 \leq i,j \leq k} \in M_k(\mathbb{C})\) and \(B_1, ..., B_k \in M_m(\mathbb{C})\) are complex
Hadamard matrices, \(n = mk\). Then the second relative commutant of the subfactor associated to \(H\) contains the projection:

\[
f = \sum_{1 \leq i, j \leq n, i \equiv j (\text{mod } m)} e_{i,i} \otimes e_{j,j} \in M_n(\mathbb{C}).
\]

**Proof.** For \(1 \leq i \leq n\), let \(i_0 = (i - 1)(\text{mod } m) + 1\) and \(i_1 = \frac{i - i_0}{m} + 1\). We will use similar notations for \(1 \leq j \leq n\). Thus, the \((i, j)\) entry of \(H\) is:

\[
h_{i,j} = a_{i_1,j_1} b_{i_0,j_0}^{j_1}
\]

where \(b_{r,s}^t\) is the \((r, s)\) entry of \(B_t\), for all \(1 \leq t \leq k\), \(1 \leq r, s \leq m\).

With these notations, the projection \(f\) can be written as

\[
f = \sum_{i,j=1}^{n} \lambda_{i,j}^1 e_{i,i} \otimes e_{j,j}
\]

where \(\lambda_{i,j}^1 = 1\) if \(i_0 = j_0\) and \(\lambda_{i,j}^1 = 0\) for all other \(i, j\).

According to Proposition 2.4, showing that \(f\) is in the second relative commutant is equivalent to showing that \(p_{i,c}^{j,d} = 0\) whenever \(c_0 \neq d_0\). Using the formula for the entries of \(P_1\) and the fact that \(i_0 = j_0\) we obtain:

\[
p_{i,c}^{j,d} = \sum_{x=1}^{n} u_{i,x} \bar{u}_{c,x} \bar{u}_{j,x} u_{d,x}
\]

\[
= \frac{1}{n^2} \sum_{x=1}^{n} h_{i,x} \bar{h}_{c,x} \bar{h}_{j,x} h_{d,x}
\]

\[
= \frac{1}{n^2} \sum_{x=1}^{n} a_{i_1,x_1} b_{i_0,x_0}^{x_1} \bar{a}_{c_1,x_1} \bar{b}_{c_0,x_0}^{x_1} \bar{a}_{j_1,x_1} \bar{b}_{j_0,x_0}^{x_1} a_{d_1,x_1} b_{d_0,x_0}^{x_1}
\]

\[
= \frac{1}{n^2} \sum_{x=1}^{n} a_{i_1,x_1} \bar{a}_{c_1,x_1} \bar{b}_{c_0,x_0}^{x_1} \bar{a}_{j_1,x_1} a_{d_1,x_1} b_{d_0,x_0}^{x_1}
\]

\[
= \frac{1}{n^2} \sum_{x=1}^{n} \left( a_{i_1,x_1} \bar{a}_{c_1,x_1} \bar{a}_{j_1,x_1} a_{d_1,x_1} \left( \sum_{x_0=1}^{m} \bar{b}_{c_0,x_0}^{x_1} b_{d_0,x_0}^{x_1} \right) \right)
\]

\[
= \frac{1}{n^2} \sum_{x=1}^{n} \left( a_{i_1,x_1} \bar{a}_{c_1,x_1} \bar{a}_{j_1,x_1} a_{d_1,x_1} \delta_{c_0}^{d_0} \right)
\]

\[
= 0
\]

whenever \(c_0 \neq d_0\). \(\square\)
We show that in fact the subfactor $N_H \subset M_H$ associated to the Dita matrix $H$ has an intermediate subfactor $N_H \subset R_H \subset M_H$, and the projection $f$ is the Bisch projection (in the sense of [Bi]) corresponding to $R_H$.

**Proposition 3.2.** Let $H = \sum_{1 \leq i, j \leq k} a_{i,j} B_j \otimes f_{i,j} \in M_n(\mathbb{C})$ be a Dita type matrix, where $A = (a_{i,j})_{1 \leq i, j \leq k} \in M_k(\mathbb{C})$ and $B_1, ..., B_k \in M_m(\mathbb{C})$ are complex Hadamard matrices, $n = mk$. Then:

(a). The commuting square $\mathcal{C}(H)$ can be decomposed into two adjacent symmetric commuting squares:

\[
\begin{align*}
\mathcal{D}_m \otimes D_k & \subset M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) \\
\cup & \\
\mathcal{D}_m \otimes I_k & \subset U(M_m(\mathbb{C}) \otimes D_k)U^* \\
\cup & \\
\mathbb{C} & \subset U(D_m \otimes D_k)U^*
\end{align*}
\]

(b). The commuting square $\mathcal{C}^t(H)$ can be decomposed into two adjacent symmetric commuting squares:

\[
\begin{align*}
U(D_m \otimes D_k)U^* & \subset M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) \\
\cup & \\
U(I_m \otimes D_k)U^* & \subset D_m \otimes M_k(\mathbb{C}) \\
\cup & \\
\mathbb{C} & \subset D_m \otimes D_k
\end{align*}
\]
Proof. (a). We first show that $D_m \otimes I_k \subset U(M_m(\mathbb{C}) \otimes D_k)U^*$. Equivalently, we check that $U^*(D_m \otimes I_k)U \subset (M_m(\mathbb{C}) \otimes D_k)$. Indeed, for $D \in D_m$ we have:

\[
U^*(D \otimes I_k)U = \frac{1}{n} \left( \sum_{1 \leq i,j \leq k} \bar{a}_{i,j} B_j^* \otimes f_{i,j} \right) (D \otimes I_k) \left( \sum_{1 \leq i,j \leq k} a_{i,j} B_j \otimes f_{i,j} \right)
\]

\[
= \frac{1}{n} \sum_{1 \leq i,j,j' \leq k} \bar{a}_{i,j} a_{i,j'} B_j^* B_j' \otimes f_{j,j'}
\]

\[
= \frac{1}{n} \sum_{1 \leq j,j' \leq k} (\sum_{i=1}^k \delta_{i,j} B_j^* B_j' \otimes f_{j,j'})
\]

\[
= \frac{1}{n} \sum_{1 \leq j \leq k} B_j^* B_j \otimes f_{j,j} \in (M_m(\mathbb{C}) \otimes D_k)
\]

The lower square of inclusions is clearly a commuting square, since $\mathcal{C}(H)$ is a commuting square. We check that

\[
D_m \otimes D_k \subset M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) \cup U \cup (M_m(\mathbb{C}) \otimes D_k) U^*
\]

is a commuting square. For $X \in M_m(\mathbb{C})$ and $D \in D_k$ we have:

\[
U(X \otimes D)U^* = \frac{1}{n} \left( \sum_{1 \leq i,j \leq k} a_{i,j} B_j \otimes f_{i,j} \right) (X \otimes D) \left( \sum_{1 \leq i',j' \leq k} \bar{a}_{i',j'} B_{j'}^* \otimes f_{i',j'} \right)
\]

\[
= \frac{1}{n} \sum_{1 \leq i,j,j' \leq k} \bar{a}_{i',j} a_{i,j} B_j B_{j'}^* \otimes D_{j,j} f_{i,j'}
\]
Hence:

\[
E_{D_n}(U(X \otimes D)U^*) = E_{D_n}\left(\frac{1}{n} \sum_{1 \leq i,i',j \leq k} \bar{a}_{i',j} a_{i,j} B_j X B_j^* \otimes D_{j,j} f_{i,i'}\right)
\]

\[
= \frac{1}{n} \sum_{1 \leq i,i',j \leq k} E_{D_m}(\bar{a}_{i',j} a_{i,j} B_j X B_j^*) \otimes D_{j,j} f_{i,i'}
\]

\[
= \frac{1}{n} \sum_{1 \leq i,j \leq k} D_{j,j} E_{D_m}(B_j X B_j^*) \otimes f_{i,i} \in D_m \otimes I_k
\]

(10)

The lower commuting square is symmetric, since the product of the dimensions of its upper left and lower right corners equals the dimension of its upper right corner. This also implies that the upper commuting square is symmetric, since \(\mathcal{C}(H)\) is symmetric.

(b). The proof is similar to the proof of part (a). \(\square\)

**Corollary 3.1.** The subfactors associated to Dita matrices have intermediate subfactors.

**Proof.** By iterating the basic construction for the decomposition of \(\mathcal{C}(H)\) in commuting squares, we obtain the towers of algebras:

\[
U(D_m \otimes D_k)U^* \subset M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) \subset \mathcal{X}_1 \subset \mathcal{X}_2 \subset \ldots
\]

\[
U(I_m \otimes D_k)U^* \subset D_m \otimes M_k(\mathbb{C}) \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots
\]

\[
\mathbb{C} \subset D_m \otimes D_k \subset \mathcal{Y}_1 \subset \mathcal{Y}_2 \subset \ldots
\]

where \(\mathcal{R}_i = \langle \mathcal{R}_{i-1}, e_{i+2} \rangle \subset \mathcal{X}_i\). Let \(R_H\) be the weak closure of \(\cup_i \mathcal{R}_i\). We have \(N_H \subset R_H \subset M_H\) and \(R_H\) is a II_1 factor since the subfactor \(N_H \subset M_H\) is irreducible. \(\square\)

**Remark 3.1.** It is immediate to check that the projection \(f \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})\) from Proposition 3.1 implements the conditional expectation from \(M_n(\mathbb{C}) \otimes I_n = M_n(\mathbb{C})\) onto \(D_m \otimes M_k(\mathbb{C})\). It follows that \(f\) is the Bisch projection for the intermediate subfactor \(N_H \subset R_H \subset M_H\).
4. Matrices of small order

In this section we compute the second relative commutants of the subfactors associated to Hadamard matrices of small dimensions. For some of the matrices considered we also specify the dimension of the third relative commutant. Most computations included were done with the help of computers, using GAP and Mathematica.

It is well known in subfactor theory that the dimension of the second relative commutant $D' \cap Q_1$ is at most $n$, with equality if and only if $H$ is equivalent to a tensor product of Fourier matrices. In this case the subfactor $N_H \subset M_H$ is well understood, being a cross-product subfactor. For this reason, we exclude from our analysis tensor products of Fourier matrices.

Some of the matrices we present are parameterized and they yield continuous families of complex Hadamard matrices. In such cases, the strategy for computing the second relative commutant will be to determine which entries of the profile matrix $P_1$ depend on the parameters, and for what values of the parameters are these entries 0. According to Proposition 2.4, the second relative commutant will not change as long as the 0 entries of $P_1$ do not change. Thus, to compute the second relative commutant for any other value of the parameters, it is enough to compute it for some random value.

We will describe the second relative commutant by specifying its minimal projections. Each such projection $p$ corresponds to a subset $S \subset \{1, 2, ..., n^2\}$: $p$ is the $n^2 \times n^2$ diagonal matrix having 1 on diagonal positions $i \in S$ and 0 on all other positions. Since the Jones projection $e_3$ is always in the second relative commutant, one of the subsets of our partitions will always be $\{1, n + 2, 2n + 3, ..., kn + k + 1, ..., n^2\}$.

Complex Hadamard matrices of dimension 4. There exists, up to equivalence, only one family of complex Hadamard matrices of dimension 4:

$$F_4(a) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & a & -1 & -a \\
1 & -1 & 1 & -1 \\
1 & -a & -1 & a
\end{pmatrix}, \quad |a| = 1$$

The entries of $P_1$ that depend on the parameter $a$ are $\frac{1}{8} + \frac{a^2}{8}$, $\frac{1}{8} - \frac{a^2}{8}$, $\frac{1}{8} + \frac{1}{8a^2}$, $\frac{1}{8} - \frac{1}{8a^2}$. Thus, the second relative commutant is the same for all values of $a$ that are not roots of these equations.

The roots $a = 1, a = -1$ yield matrices that are tensor products of $2 \times 2$ Fourier matrices. Thus the dimension of the second relative commutant
is 4, and its minimal projections are given by the partition \{1, 6, 11, 16\}, {2, 5, 12, 15}, {3, 8, 9, 14}, {4, 7, 10, 13}.

The roots \(a = i, a = -i\) yield the 4 \times 4 Fourier matrix, thus the minimal projections are \{1, 6, 11, 16\}, \{2, 7, 12, 13\}, \{3, 8, 9, 14\}, \{4, 5, 10, 15\}.

Any other values of \(a, |a| = 1\), yield relative commutants of dimension 3: \{1, 6, 11, 16\}, \{2, 4, 5, 7, 10, 12, 13, 15\}, \{3, 8, 9, 14\}. This is not surprising, since this matrix is of Dita type (see Proposition 3.1).

The dimension of the third relative commutant is 10, and the dimension of the fourth relative commutant is 35 unless \(a\) is a primitive root of order 8 of unity, in which case the dimension is 36. Based on this evidence, we conjecture that the principal graph of the subfactor associated to \(F_4(a)\) is part of an affine 2-parameter family of Dita matrices:

\[
\begin{align*}
1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 \\
1 & \quad a e^{\frac{2\pi}{3}} & \quad b e^{\frac{2\pi}{3}} & \quad -1 & \quad \frac{a}{e^{\frac{2\pi}{3}}} & \quad \frac{b}{e^{\frac{2\pi}{3}}} \\
1 & \quad e^{\frac{2\pi}{3}} & \quad e^{-\frac{2\pi}{3}} & \quad 1 & \quad e^{\frac{2\pi}{3}} & \quad e^{-\frac{2\pi}{3}} \\
1 & \quad -a & \quad b & \quad -1 & \quad a & \quad -b \\
1 & \quad e^{-\frac{2\pi}{3}} & \quad e^{\frac{2\pi}{3}} & \quad 1 & \quad e^{-\frac{2\pi}{3}} & \quad e^{\frac{2\pi}{3}} \\
1 & \quad \frac{a}{e^{\frac{2\pi}{3}}} & \quad \frac{b}{e^{\frac{2\pi}{3}}} & \quad -1 & \quad a e^{\frac{2\pi}{3}} & \quad b e^{\frac{2\pi}{3}}
\end{align*}
\]

The entries of \(P_1\) that depend on \(a, b\) are: \(2 + a^{-2} + b^{-2}\), \(2 + \frac{2(-1)^{\frac{3}{2}}}{a^2} - \frac{2(-1)^{\frac{3}{2}}}{b^2}\)\), \(2 - \frac{2(-1)^{\frac{3}{2}}}{a^2} + \frac{2(-1)^{\frac{3}{2}}}{b^2}\), \(2 + 2(-1)^{\frac{3}{4}}a^2 - 2(-1)^{\frac{3}{4}}b^2\), \(2 - 2(-1)^{\frac{3}{4}}a^2 + 2(-1)^{\frac{3}{4}}b^2\).

Making one of these entries 0 yields the following possibilities: \(a = -\frac{1}{2} - \frac{i}{2} \sqrt{3}, b = -\frac{1}{2} + \frac{i}{2} \sqrt{3}\) or \(a = -\frac{1}{2} + \frac{i}{2} \sqrt{3}, b = -\frac{1}{2} - \frac{i}{2} \sqrt{3}\) or \(a = \frac{1}{2} + \frac{i}{2} \sqrt{3}, b = \frac{1}{2} - \frac{i}{2} \sqrt{3}\) or \(a = -\frac{1}{2} - \frac{i}{2} \sqrt{3}, b = \frac{1}{2} - \frac{i}{2} \sqrt{3}\) or \(a = -\frac{1}{2} + \frac{i}{2} \sqrt{3}, b = \frac{1}{2} + \frac{i}{2} \sqrt{3}\) or \(a = \frac{1}{2} - \frac{i}{2} \sqrt{3}, b = -\frac{1}{2} - \frac{i}{2} \sqrt{3}\) or \(a = \frac{1}{2} + \frac{i}{2} \sqrt{3}, b = -\frac{1}{2} + \frac{i}{2} \sqrt{3}\) or \(a = -1, b = -1\) or \(a = 1, b = 1\) or \(a = -1, b = 1\) or \(a = 1, b = -1\).

In each of these cases the matrix \(F_6(a, b)\) is a tensor product of Fourier matrices.

For all other pairs \((a, b)\) satisfying \(|a| = |b| = 1\), the second relative commutant has dimension 4: \{1, 8, 15, 22, 29, 36\}, \{2, 4, 6, 7, 9, 11, 14, 16, 18, 19, 21, 23, 26, 28, 30, 31, 33, 35\}, \{3, 10, 17, 24, 25, 32\}, \{5, 12, 13, 20, 27, 34\}.
The following family of self-adjoint, non-affine, complex Hadamard matrices was obtained in [BeN], one of the motivations being the search for Hadamard matrices of small dimensions that might yield subfactors with no extra structure in their relative commutants, besides the Jones projections.

\[
BN_6(\theta) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -y & -\bar{x} & y & 1 \\
1 & x & -1 & t & -t & -x \\
1 & -\bar{y} & \bar{t} & -1 & \bar{y} & -\bar{t} \\
1 & -x & -\bar{t} & y & 1 & \bar{z} \\
1 & \bar{y} & -\bar{x} & -t & z & 1
\end{pmatrix}
\]

where \( \theta \in [-\pi, -\arccos(\sqrt{-\frac{1+\sqrt{3}}{2}})] \cup [\arccos(\sqrt{-\frac{1+\sqrt{3}}{2}}), \pi] \) and the variables \( x, y, z, t \) are given by:

\[
y = \exp(i\theta), \quad z = \frac{1 + 2y - y^2}{y(-1 + 2y + y^2)}
\]

\[
x = \frac{1 + 2y + y^2 - \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{1 + 2y - y^2}
\]

\[
t = \frac{1 + 2y + y^2 - \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{-1 + 2y + y^2}
\]

The entries of \( BN_6 \) do not depend linearly on the parameters, thus this is not a Dita-type family. The corresponding subfactors have the second relative commutant generated by the Jones projection. We conjecture that \( BN_6(\theta) \) give supertransitive subfactors, i.e. all the relative commutants of higher orders are generated by the Jones projections.

There are other interesting complex Hadamard matrices of order 6, such as the one found by Tao in connection to Fuglede’s conjecture ([T]), or the Haagerup matrix ([H],TZ). We computed the second relative commutant for these matrices, and it is generated by the Jones projection.

**Complex Hadamard matrices of dimension 7.** The following one-parameter family was found in [Pe], providing a counterexample to a conjecture of Popa regarding the finiteness of the number of complex Hadamard matrices of prime dimension.
by the Jones projection, for all \(|a| = 1\). For \(a = 1\) we also computed the third relative commutant, and it is just the Temperley-Lieb algebra \(TL_2\). We conjecture that \(P_7(a)\) yield subfactors with no extra structure in their higher order relative commutants, besides the Jones projections.

**Complex Hadamard matrices of dimension 8.** The following 5-parameter family of Hadamard matrices contains the Fourier matrix and is of Dita type:

\[
P_7(a) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & a e^{\frac{i}{4} \pi} & \frac{a}{e^{\frac{i}{4} \pi}} & e^{\frac{i}{4} \pi} & -1 & -1 & e^{\frac{i}{4} \pi} \\
1 & \frac{a}{e^{\frac{i}{4} \pi}} & a e^{\frac{i}{4} \pi} & -1 & e^{\frac{i}{4} \pi} & -1 & e^{\frac{i}{4} \pi} \\
1 & e^{\frac{i}{4} \pi} & -1 & e^{\frac{i}{4} \pi} & \frac{1}{a e^{\frac{i}{4} \pi}} & e^{\frac{i}{4} \pi} & -1 \\
1 & -1 & e^{\frac{i}{4} \pi} & \frac{1}{a e^{\frac{i}{4} \pi}} & a e^{\frac{i}{4} \pi} & e^{\frac{i}{4} \pi} & -1 \\
1 & -1 & -1 & e^{\frac{i}{4} \pi} & e^{\frac{i}{4} \pi} & e^{\frac{-2i}{3} \pi} & -1 & e^{\frac{i}{4} \pi} & e^{\frac{-2i}{3} \pi} \\
1 & e^{\frac{i}{3} \pi} & e^{\frac{i}{3} \pi} & -1 & -1 & e^{\frac{i}{3} \pi} & e^{\frac{-2i}{3} \pi} & e^{\frac{i}{3} \pi} & e^{\frac{-2i}{3} \pi}
\end{pmatrix}
\]

The second relative commutant of the associated subfactors is generated by the Jones projection, for all \(|a| = 1\). For \(a = 1\) we also computed the third relative commutant, and it is just the Temperley-Lieb algebra \(TL_2\). We conjecture that \(P_7(a)\) yield subfactors with no extra structure in their higher order relative commutants, besides the Jones projections.

The list of possible values of \(a, b, c, d, z\) that yield 0 entries for \(P_1\) is very long and we do not include it here. Outside these values, the second relative commutant has dimension 4 and it is given by \(\{1, 10, 19, 28, 37, 46, 55, 64\}\), \(\{2, 4, 6, 8, 9, 11, 13, 15, 18, 20, 22, 24, 25, 27, 29, 31, 34, 36, 38, 40, 41, 43, 45, 47, 50, 52, 54, 56, 57, 59, 61, 63\}\), \(\{3, 7, 12, 16, 17, 21, 26, 30, 35, 39, 44, 48, 49, 53, 58, 62\}\), \(\{5, 14, 23, 32, 33, 42, 51, 60\}\).

We analysed several other complex Hadamard matrices besides those included in this paper, such as those found by [MRS], [Sz]. We tried to cover
most known examples of complex Hadamard matrices of dimensions 2, 3, ..., 11. We draw some conclusions:

(1) Matrices of Dita type yield subfactors with intermediate subfactors, and thus the second relative commutant has some extra structure besides the Jones projection. We note that parametric families of Dita type exist for every \( n \) non-prime, and they contain the Fourier matrix \( F_n \).

(2) All non-Dita, non-Fourier matrices we tested have the second relative commutant generated by the Jones projection. The third relative commutant is also generated by the first two Jones projections for all cases we could compute. It remains an open problem whether there exist such complex Hadamard matrices that admit symmetries of higher order.

References

[BHJ] R. Bacher, P. de la Harpe, V. Jones, *Carres commutatifs et invariants de structures combinatoires*, Comptes rendus Acad. Sci. Math., 1995, vol. 320, no9, 1049-1054

[Bi] D. Bisch, *A note on intermediate subfactors*, Pacific J. Math. 163 (1994), no. 2, 201-216.

[BaN] T. Banica and R. Nicoara, *Quantum groups and Hadamard matrices*, preprint, math.OA/0610529

[BeN] K. Beauchamp and R. Nicoara, *Orthogonal maximal abelian *-subalgebras of the 6 \times 6 matrices*, preprint, math.OA/0609076

[Di] P. Dita, *Some results on the parametrization of complex Hadamard matrices*, J. Phys. A, 37 (2004) no. 20, 5355-5374

[GHJ] F. M. Goodman, P. de la Harpe and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Math. Sciences Res. Inst. Publ. Springer Verlag 1989

[H] U. Haagerup, *Orthogonal maximal abelian *-subalgebras of the \( n \times n \) matrices and cyclic \( n \)-roots*, Operator Algebras and Quantum Field Theory (ed. S. Doplicher et al.), International Press (1997).296-322

[HJ] P. de la Harpe and V. F. R. Jones, *Paires de sous-algebres semi-simples et graphes fortement reguliers*, C.A. Acad. Sci. Paris 311, serie I (1990), 147-150

[Jo1] V. F. R. Jones, *Index for subfactors*, Invent. Math 72 (1983), 1–25.

[Jo2] V. F. R. Jones, *Planar Algebras I*, math.QA/9909027

[JS] V. F. R. Jones and V. S. Sunder, *Introduction to subfactors*, Cambridge University press, (1997).

[MRS] M. Matolcsi, J. Reffy, F. Szollosi, *Constructions of complex Hadamard matrices via tiling abelian groups*, preprint quant-ph/0607073.

[Ni] R. Nicoara, *A finiteness result for commuting squares of matrix algebras*, J. Operator Theory 55 (2006), no. 2, 295–310.

[Pe] M. Petrescu, *Existence of continuous families of complex Hadamard matrices of certain prime dimensions and related results*, PhD thesis, University of California Los Angeles, 1997.

[Po1] S. Popa, *Classification of subfactors : the reduction to commuting squares*, Invent. Math., 101(1990),19-43
[Po2] S. Popa, *Orthogonal pairs of \( \ast \)-subalgebras in finite von Neumann algebras*, J. Operator Theory 9, 253-268 (1983)

[Sz] F. Szöllősi, *Parametrizing Complex Hadamard Matrices*, preprint, math.CO/0610297

[TZ] W. Tadej and K. Zyczkowski, *A concise guide to complex Hadamard matrices* Open Systems & Infor. Dyn., 13(2006), 133-177.

[T] T. Tao, *Fuglede’s conjecture is false in 5 and higher dimensions*, Math. Res. Letters 11 (2004), 251

W.C.: Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA
E-mail address: wes.camp@vanderbilt.edu

R.N.: Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA
E-mail address: remus.nicoara@vanderbilt.edu