Financial investments by Portfolio optimization

Todor Stoilov 1,*, Krasimira Stoilova 1 and Miroslav Vladimirov 2

1 Institute of Information and Communication Technologies – Bulgarian Academy of Sciences, Sofia, Bulgaria
2 Varna University of Economics, Varna, Bulgaria
todor@hsi.iccs.bas.bg

Abstract. Application of the Portfolio theory to appreciation of the Bulgarian stock exchange following the most popular financial indices is presented. The data of these indices corresponds to the same watching periods an year before. Comparison of the optimization problem’s solutions and analysis of the financial activities are commented. Modification of the portfolio optimization problem is proposed for calculation facilitation.

1. The first section in your paper
The resource allocation is done by investing capital in financial assets (or goods) which give a return to the investor after a period of time. The theoretical and formalized background of the investment process is based on the Portfolio theory. For the investor the target is to maximize the return and to minimize the investment risk [1, 2, 3].

2. Description of the Portfolio Problem
The classical portfolio problem is analytically given by relation [1]

\[
\min_x \frac{1}{2} (1 - \sigma) x^T \text{cov}(.) x - \sigma E^T x \tag{1}
\]

where
\[
\text{cov}(.) \quad \text{is a symmetrical positively defined square matrix } n \times n \quad \text{which presents the covariance between returns of } n \quad \text{assets, participating in the portfolio. This matrix gives also numerical values of asset' risks;}
\]
\[
E \quad \text{is a } (n \times 1) \quad \text{vector of the average profits of the assets for the period of time } T = [1, 2, \ldots, N];
\]
\[
\sigma \quad \text{is a parameter of the investor’s preferences to undertake risk in the investment process, } 0 \leq \sigma \leq \infty,
\]
\[
1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \text{is a unity vector, } n \times 1;
\]
The constraint of the optimization problem is the equation \( x_1 + x_2 + \cdots + x_n = 1 \), which is a formalization of the assumption that the full amount of the investment resources will be used for the portfolio. The solutions \( x_i, i=1,n \) give the relative values of the investments, which are allocated for the assets \( i, i=1,n \).

The first component \( V_p = x^T \text{cov}(x)x \) of the goal function is the quantitative assessment of the portfolio risk. The second component \( E_p = E^T x \) is the quantitative value of the portfolio return. The goal function of problem (1) targets the minimization of the portfolio risk \( V_p \) and also the maximization of its return \( E_p \). The parameter \( \sigma \) formalizes the investor’s ability to undertake risk and it has numerical value in the range \([0, \infty]\). When \( \sigma = 0 \) the investor’s main goal is to decrease the risk of the investment, \( \min_{x^*}[x^T \text{cov}(x)] \). For \( \sigma = \infty \) the investor’s target is to obtain a maximal return from the investment \( \min_{x^*}[-\sigma E^T x] = \max_{x^*}[E^T x] \).

For different values of the parameter \( \sigma \), the portfolio problem (1) gives different solutions. The set of solutions for given set of \( \sigma \) graphically represents the “efficient frontier”, which is drown in the space “risk-return”, \( V_p(E_p) \). The MPT recommends that the investor has to choose such an investment, which portfolio is a point, lying on the “efficiency frontier”. An example of a graphical representation of the “efficient frontier” is given in figure 1.

![Efficiency frontier](image)

**Figure 1. Efficiency frontier of the portfolio optimization.**

The main characteristics of the Portfolio optimization problem are presented in [4]. The Portfolio theory was applied there for assessment of four indices of the Bulgarian stock exchange. In this paper we would like to see the changes of the same indices a year later. The data of their behaviour have been taken from the public data [5] from the same dates but a year later.

3. **Application of the Portfolio Problem for Appreciation of the Bulgarian Financial Market**

The Bulgarian stock exchange is presented of four indices: SOFIX, BGBX40, BGTR30 and BGREIT. Here we assume that each index represents particular security. According to the Portfolio theory we solve optimization problem - problem (1), which solutions show the preferable securities (here indices) for investments. Practically, each of the indices represents the behaviour of the included different Bulgarian companies (corresponding securities). For instance, the index SOFIX includes 16 Bulgarian companies (OEA, ALB, ATERA, BREF, CCB, CENHL, CHIM, EUBG, FIB, HVAR, MCH, MONBAT, SFAR, SKK, SKKW2, TRACE).

The values of the financial data are given in Table 1.

The volume of the financial data is in million leva. In order to have relative relations and to calculate the covariance matrix we transform Table 1 to relative units where the values are calculated according to the relation in percentage:

\[
[(\text{New value} - \text{Old value})/\text{Old value}] \times 100
\]

The transformed data are presented in Table 2.
Table 1. Data of the four Bulgarian indices

| Date       | SOFIX  | BGBX40 | BGTR30 | BGREIT |
|------------|--------|--------|--------|--------|
| 26.1.2019  | 573.65 | 113.3  | 487.28 | 120.45 |
| 29.12.2018 | 595.65 | 116.1  | 496.59 | 121.19 |
| 24.11.2018 | 588.93 | 114.9  | 488.74 | 117.35 |
| 29.10.2018 | 599.65 | 117.46 | 499.03 | 116.45 |
| 29.9.2018  | 624.39 | 121.93 | 520.34 | 117.79 |
| 25.8.2018  | 636.32 | 123.07 | 525.98 | 116.47 |
| 28.7.2018  | 632.31 | 122.52 | 523.56 | 116.44 |
| 30.6.2018  | 634.26 | 124.9  | 525.39 | 116.21 |
| 26.5.2018  | 642.09 | 127.44 | 531.11 | 114.77 |
| 28.4.2018  | 658.2  | 130.34 | 541.7  | 115.53 |
| 31.3.2018  | 649.17 | 128.54 | 536.29 | 114.93 |
| 24.2.2018  | 687.26 | 130.71 | 557.92 | 114.65 |
| 27.1.2018  | 709.62 | 137.91 | 567.67 | 116.09 |

Table 2. Relative data of the four Bulgarian indices in %

|        | SOFIX | BGBX40 | BGTR30 | BGREIT |
|--------|-------|--------|--------|--------|
| -3.69  | -2.41 | -1.87  | -0.61  |
| 1.14   | 1.04  | 1.61   | 3.27   |
| -1.79  | -2.18 | -2.06  | 0.77   |
| -3.96  | -3.67 | -4.10  | -1.44  |
| -1.87  | -0.93 | -1.07  | 1.13   |
| 0.63   | 0.45  | 0.46   | 0.03   |
| -0.31  | -1.91 | -0.35  | 0.20   |
| -1.22  | -1.99 | -1.08  | 1.25   |
| -2.45  | -2.22 | -1.95  | -0.66  |
| 1.39   | 1.40  | 1.01   | 0.52   |
| -5.54  | -1.66 | -3.88  | 0.24   |
| -3.15  | -5.22 | -1.72  | -1.24  |

Using data in Table 2 the covariance matrix is in the form:

\[
\text{Cov}(\cdot) = \begin{bmatrix}
    4.315599 & 2.760744 & 3.287769 & 1.404938 \\
    2.760744 & 3.305114 & 2.230184 & 1.483142 \\
    3.287769 & 2.230184 & 2.819019 & 1.213343 \\
    1.404938 & 1.483142 & 1.213343 & 1.415463 \\
\end{bmatrix}
\]
This matrix is obligatory for the evaluation of the goal function of the Portfolio Optimization Problem (1). For different values of the coefficient \( \sigma \) which represents the investor’s behaviour for taking risk in investments we solve the optimization problem (1). For different \( \sigma \) we obtain the solutions of the optimization problem, the portfolio risk, the portfolio return and the goal function \( F \), Table 3. The solutions of the optimization problem are the unknown variables \( x \) which here are the four indices of the Bulgarian Stock exchange. The columns SOFIX, BGBX40, BGTR30, BGREIT are \( x_1, x_2, x_3, x_4 \) and these solutions are the relative values of the investments, which are allocated for the SOFIX, BGBX40, BGTR30, BGREIT.

For \( \sigma=0 \) the investor should invest 0.89 units in BGREIT, 0.11 units in BGTR30 and zero in SOFIX and BGBX40.

According to the Portfolio optimization problem the best investment policy for \( \sigma=0.15 \) and more (here \( \sigma \) is changed to 3.5 but the tendency is the same for larger values) to invest all capital in BGREIT.

The Efficiency frontier for the presented data is shown in figure 2.

| sigma | SOFIX | BGBX40 | BGTR30 | BGREIT | Risk   | Return | \( F \)       |
|-------|-------|--------|--------|--------|--------|--------|--------------|
| 0     | 0     | 0      | 0      | 0.111805 | 0.888195 | 1.392865 | 0.139757 | 0.696433       |
| 0.001 | 0     | 0      | 0      | 0.110939 | 0.889061 | 1.392867 | 0.141111 | 0.696292       |
| 0.01  | 0     | 0      | 0      | 0.103149 | 0.896851 | 1.393001 | 0.153303 | 0.694967       |
| 0.02  | 0     | 0      | 0      | 0.094492 | 0.905508 | 1.393407 | 0.166849 | 0.693567       |
| 0.03  | 0     | 0      | 0      | 0.085836 | 0.914164 | 1.394084 | 0.180395 | 0.69163        |
| 0.05  | 0     | 0      | 0      | 0.068523 | 0.931477 | 1.396252 | 0.207487 | 0.687752       |
| 0.1   | 0     | 0      | 0      | 0.025242 | 0.974758 | 1.406411 | 0.275218 | 0.675684       |
| 0.15  | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.660524       |
| 0.2   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.644788       |
| 0.25  | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.629052       |
| 0.3   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.613316       |
| 0.35  | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.59758        |
| 0.4   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.581844       |
| 0.45  | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.566108       |
| 0.5   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.550372       |
| 0.6   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.5189        |
| 0.8   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.455957       |
| 1     | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.393013       |
| 1.5   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.235654       |
| 2     | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | 0.078294       |
| 2.5   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | -0.07907       |
| 3     | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | -0.23642       |
| 3.5   | 0     | 0      | 0      | 1       | 1.415463 | 0.314719 | -0.39378       |
Figure 2. The efficiency frontier of the current stock exchange data.

The variations of BGTR30 and BGREIT according to $\sigma$ are given in figure 3 and figure 4. The other two indices – SOFIX and BGBX40 have solutions zero that means that the investments are not recommended.

Figure 3. Variation of BGTR30 according to $\sigma$.

Figure 4. Variation of BGREIT according to $\sigma$.

The variations of Risk and Return to sigma are given in figure 5 and figure 6.
The results from the application of the Portfolio theory to real data of Bulgarian stock exchange show the following. The variations of problem’s solutions are in changing $\sigma$ from 0 to 0.15. When the coefficient of investor’s preferences of taking risk $\sigma$ is greater than 0.15 the problem’s solutions, the Risk and Return obtain constant values.

4. Modification of the Portfolio optimization problem
The solution of the portfolio optimization problem (1) gives the optimal values of $x_i$ that represent the best investment activities. Each solution of this problem gives one point of the Efficiency Frontier of the plane Return/Risk. By giving different values of $\sigma$, $0 \leq \sigma \leq \infty$ the Portfolio optimization problem is solved and accordingly one point of the Efficiency Frontier is received. The variations of $\sigma$ are determined by the person, making decision. We propose more universal approach for decision making by normalization of the Portfolio optimization problem by proposing new variable $\psi$, $0 \leq \psi \leq 1$. The Portfolio optimization problem (1) has the following modification

$$\min_x \frac{1}{2} (1-\psi)x^T \text{cov}(\cdot)x - \psi E^T x$$

$$x^T 1 = 1,$$

$$0 \leq \psi \leq 1$$

Problem (2) is solved for different values of $\psi$, $0 \leq \psi \leq 1$. 

Figure 5. Variation of Risk according to $\sigma$.

Figure 6. Variation of Return according to $\sigma$. 
The results are shown in Table 4.

The efficiency frontier of the initial (1) and modified (2) portfolio optimization problem is the same for both cases according to figures 2 and 7. The similarity can be seen also from the solutions, presented in Table 3 and Table 4. This equivalence can be used for changing $\psi$ in smaller interval, $0 \leq \psi \leq 1$ instead of changing $\sigma$ in bigger interval, $0 \leq \sigma \leq \infty$.

### Table 4. Solutions of the modified Portfolio optimization problem.

| $\psi$ | SOFIX | BGBX40 | BGTR30 | BGREIT | Risk | Return |
|--------|--------|--------|--------|--------|------|--------|
| 0      | 0      | 0      | 0.107216 | 0.892784 | 1.392903 | 0.146938 |
| 0.01   | 0      | 0      | 0.107216 | 0.892784 | 1.392903 | 0.146938 |
| 0.05   | 0      | 0      | 0.066245 | 0.933755 | 1.396618 | 0.211052 |
| 0.1    | 0      | 0      | 0.015624 | 0.984376 | 1.409589 | 0.290269 |
| 0.15   | 0      | 0      | 0.015624 | 0.984376 | 1.409589 | 0.290269 |
| 0.2    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.25   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.3    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.35   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.4    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.45   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.5    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.55   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.6    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.65   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.7    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.75   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.8    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.85   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.9    | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 0.95   | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |
| 1      | 0      | 0      | 0      | 1      | 1.415463 | 0.314719 |

![Efficiency Frontier](image)

**Figure 7.** Efficiency Frontier of the modified Portfolio optimization problem.
5. Comparison of Portfolio problem’s solutions

In [4] the Portfolio theory was applied for making decision of financial investments of the same financial indices like in this research. The data here of the four indices is taken at the same period like in [4]. We want to know how one year period reflects to the problem’s solutions or how the financial investments are changed. Here are presented the previous graphical results from [4].

The optimal solutions of the portfolio optimization problem from the both groups of data are quite different. The first solution $x_1$ is about SOFIX. The current results show that it is not recommended the investments in SOFIX because the solutions are zero (see Table 3). A year before this index is preferable for investments for $\sigma > 2$ (see figure 8).

The second solution $x_2$ is about BGBX40. For both periods this index is not suitable for investments according to the portfolio optimization problem (see Table 3 for current results and figure 9 about previous results).

The third solution $x_3$ is about BGTR30. This index is preferable for both periods however for different changes of $\sigma$. For the current results BGTR30 is suitable for investments for $0 \leq \sigma \leq 0.15$ (see figure 3). The previous variation of BGTR30 shows that preferable investments are for $1 \leq \sigma \leq 2.5$ (see figure 10).

The fourth solution $x_4$ is about BGREIT. This is the preferable index about current solution for $\sigma \geq 0.15$ (see figure 4). In the previous calculations BGREIT has decreasing character for $0 \leq \sigma \leq 2$ and for $\sigma \geq 2$ it is not recommended for investments, fig.11.

The efficiency frontier has similar variation for both cases. The variations of Risk according to $\sigma$ and the variations of Return according to $\sigma$ are similar for the both experiments.

![Figure 8. Optimal solutions changes SOFIX /sigma [4].](image)

![Figure 9. Optimal solutions changes BGBX40/sigma [4].](image)
Nevertheless the above similarities, the received calculations show the differences about the preferable financial investments. It means that one year period is quite different for making decisions for financial investments at Bulgarian stock exchange. One year is too long period for using the previous results of the same indices and for better results the Portfolio optimization problem has to be more frequently solved for determining the best investments.

6. Conclusions
The portfolio optimization is applied for optimal resource investments. Four typical financial indices of the Bulgarian stock exchange are analysed and compared with previous results. The financial market in our country is very dynamic and the obtained results an year before are not valid today. The authors’ recommendation is to apply more frequently the portfolio optimization for better financial investments.

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References
[1] W. Sharpe 2010 Adaptive asset allocation polices. *J. Financial Analysts, 66*, 5, pp.45-49 (2010)
[2] P. Magiera, A. Karbowski 2001 Dynamic portfolio optimization with expected value-variance criteria. *Preprints of the 9th IFAC Symposium on LSS'01, Bucharest, Romania*, pp. 308-313,
[3] J. Campbell, G. Chacko, J. Rodriguez, L. Viceira 2002 Strategic asset allocation in a continuous-time VAR model, Harvard Institute of Economic Research, Harvard University Cambridge, Massachusetts, pp.1-21
[4] K. Stoilova, T. Stoilov, M. Vladimirov 2018 Resource allocation by portfolio optimization. *Journal of the Technical University - Sofia Plovdiv branch, Bulgaria, “Fundamental Sciences and Applications”* 24 pp.19-24
[5] http://infostock.bg