Dirac Equation Redux by Direct Quantization of the 4-Momentum Vector

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Dirac Equation Redux by Direct Quantization of the 4-Momentum Vector

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Abstract. The Dirac Equation (DE) is a cornerstone of quantum physics. We prove that direct quantization of the 4-momentum vector $p$ with modulus equal to the rest energy $m$ ($c = 1$) yields a manifestly covariant Equation $\hat{p}\psi = m\psi$ with frame-free operator $\hat{p} = \hbar \nabla$. In coordinate representation, this is equivalent to DE with spacetime frame vectors $e^\mu$ replacing Dirac’s $\gamma^\mu$-matrices. Recall that standard DE is not manifestly covariant. Adding an independent Hermitian vector $e^5$ to the spacetime basis $\{e^\mu\}$ allows to accommodate the momentum operator in a real vector space with a complex structure arising alone from vectors and multivectors. The real vector space generated from the action of the Clifford or geometric product onto the quintet $\{e^\mu, e^5\}$ has dimension 32, the same as the equivalent real dimension for the space of Dirac matrices. $e^5$ proves defining for the C & CPT symmetries, distinction of axial vs. polar vectors, left and right handed rotors & spinors, etc. Therefore, we name it reflector and $\{e^\mu, e^5\}$ a basis for spacetime-reflection (STR).

The pentavector $I \equiv e^{05123}$ commutes with all elements of STR and depicts the pseudoscalar in STR. We develop the formalism by deriving all essential results from the novel STR DE: free field solutions, spin ½ magnetic angular momentum, conserved probability currents, symmetries and nonrelativistic approximation. In simple terms, we demonstrate how Dirac matrices are a redundant representation of spacetime-reflection frame vectors. In STR all properties of the Dirac electron / positron spring from the special relativistic 4-momentum and the postulate of quantization.

1. Introduction

The 93 years old Dirac Equation (DE) [1,2] is one of the most far-reaching equations in physics: it is Lorentz covariant, electron spin springs from the Equation and it predicted the first instance of antiparticle, the positron, discovered four years later by Anderson [3]. DE for a free electron is ($c = 1$) [4,5]:

$$(\gamma^\mu p_\mu - m)\psi = 0 \text{ (sum)}; \quad p_\mu = i\hbar \partial_\mu \equiv i\hbar \partial / \partial x^\mu; \quad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^\mu_\nu; \quad \mu, \nu = 0,1,2,3 \quad (1)$$
is the rest mass of the electron and \( \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the time-space signature. Dirac argued that \( \gamma^\mu \) have to be \( 4 \times 4 \) complex matrices “…describing some new degrees of freedom belonging to some internal motion in the electron” [1, 2]. This remains the standard position today [4, 5]. Dirac matrices and their generalizations are considered a fundamental representation to DE and to any domain of theoretical physics where spin 1/2 is relevant [4]. In the following we show that these extra degrees of freedom are redundant.

Based on the isomorphism of the Clifford algebras for Dirac \( \gamma^\mu \) matrices and spacetime vectors, (see Eq. (3) below) Hestenes [6] proposed a new version of DE in the formalism of his spacetime algebra STA [7]. There is no matrix and no imaginary number in STA DE. However, the spin has been put ‘by hand’ in the equation, thereby diminishing its predictive power and symmetry compared to standard DE. The real vector space of STA has dimension 16, which is half of the equivalent real dimension for the space of \( 4 \times 4 \) complex matrices. The Clifford algebra of STA is \( \mathcal{C}_\ell(1,3) \), the subscript (1,3) standing for the signature.

The Clifford algebras for spacetime frame vectors \( \{x^\mu\} \) and Dirac matrices \( \{\gamma^\mu\} \) being the same, the motivation for the present work has been to show that direct quantization of the 4-momentum vector \( p \) of modulus equal to the rest energy \( m \) \((c = 1)\), provides the simplest form of Dirac Equation: \( \hat{p}\psi = m\psi \). This is a realization of the “square root” for the relativistic invariant \( p^2 = m^2 \), with the spinor \( \psi \) ‘intermediating’ between a vectorial operator \( \hat{p} \) and a scalar \( m \). As shown below, ‘internal degrees of freedom’ of the electron follow from the Equation, extras, like matrices unneeded. This minimal approach together with the manifest covariance of the Equation mark immediate improvements over the standard DE, affirming that in order to derive all properties of the Dirac electron / positron we only need the special relativistic momentum 4-vector and the postulate of quantization. After a swift introduction to the Clifford algebra \( \mathcal{C}_\ell(1,3) \) over spacetime, we expand to \( \mathcal{C}_\ell(2,3) \) over spacetime-reflection (STR) in order to accommodate the momentum operator in a \textit{real} vector space. The STR formalism takes shape by proving how it works. Detailed derivation of some additional results in STR appear in the Appendices A-D.

This report comprises new material and changes relative to deposited preprints [8] with the same title. Apart from rearrangements and enhanced section structure, an expanded analysis of time-reversal symmetry and its geometric meaning appears in Section 4.3 and in App. D. A new Section (4.4) on Lorentz transformations of STR DE comprises significant improvements and corrections compared to previous versions; the covariant
transformation is the same as before, now standing on firmer grounds. Two tables have been added, Table 1
giving an overview of the notation (as suggested by one reviewer) and Table 2 summarizing the bilinears of
STR DE. In addition, a more symmetric form of the STR Dirac spinor as expressed by two Pauli spinors has
been introduced, which is though equivalent to the previous form. Following the suggestion of another
reviewer, the presentation of the STR spinor appear earlier in the report, see Eqs. (17-22) and Table 2. The
structure of the STR spinor is further elucidated by showing the STR free field solutions in Eqs. (23-29).

2. Clifford product in spacetime.

The Clifford or geometric product [6, 7] of two vectors \( u, v \) combines the symmetric Hamilton’s scalar
product with the antisymmetric Grassmann’s wedge product [8, 9]; in coordinate-free form it is:

\[
uv = u \cdot v + u \land v; \quad (uv)w = u(vw) = uvw
\]
\[
2u \cdot v = (uv + vu) = \{u, v\}; \quad 2u \land v = (uv - vu) = [u, v]
\]

The two parts relate naturally to the anticommutator \( \{, \} \) and commutator \( [,] \). The bivector \( u \land v = -v \land u \)
represents the oriented area encompassed by the two vectors. The geometric product (2) is linear and if not
zero, it is invertible. When normalized it renders rotors or boosts, the generators of Lorentz transformations.
Examples of both will appear in the following. The geometric product of orthonormal frame vectors is:

\[
e^\mu e^\nu \equiv e^{\mu\nu} = e^\mu \cdot e^\nu + e^\mu \land e^\nu = \eta^{\mu\nu} + e^\mu \land e^\nu; \quad \{e^\mu, e^\nu\} = 2\eta^{\mu\nu}; \quad u = u^\mu e^\mu; \quad \mu, \nu = 0,1,2,3
\]

(3)

Upright letters depict vectors; italics depict scalars. From (3) \( e^\mu \cdot e^\nu = \eta^{\mu\nu} \) defines the timespace signature.
Signature can ‘lift’ indices of frame vectors up and down, thus connecting the reciprocal bases. The
coincidence of Clifford algebras for \( e^\mu \) and Dirac matrices \( \gamma^\mu \) leads to STA - spacetime algebra [6, 7]. The
real vector space arising from the action of Clifford product onto spacetime vectors (2, 3) has dimension 16
with basis elements comprising: the real scalar unit, four vectors \( e^\mu \), six bivectors \( e^{\mu\nu} = e^\mu \land e^\nu (\mu \neq \nu) \),
four trivectors \( e^{\lambda\mu\nu} (\lambda \neq \mu \neq \nu \neq \lambda) \) and one tetravector \( e^{0123} \):

Basis of STA vector space: \( \{1, e^4, e^{4\mu}, e^{4\mu\nu}, e^{0123}; \lambda, \mu, \nu = 0,1,2,3\} \); generators: \( \{e^\mu\}, \quad \text{dim}16 \)

(4)

From (3) permutations of given orthogonal vectors can at most change the sign, not adding new basis
elements. Similarly to the geometric interpretation of basis bivectors as oriented area elements, trivectors
(tetra-vectors) in (4) are oriented 3-volume (resp. 4-volume) elements in spacetime, all unitless. Multivectors
in STA correspond to tensors, represented by matrices in the standard formalism. Multivectors formed by wedge product alone, known as blades, are anti-symmetric relative to the exchange of any two indices. \( e^{0123} \) squares to \(-1\) and defines the pseudo-scalar of STA. However, it commutes only with even grade elements of STA (scalars and bivectors) and anti-commutes with vectors and trivectors. We will define another real vector space in the following, therefore will not use here a specific symbol for \( e^{0123} \). The STA basis (4) is defined in terms of upper indices, but one can lift indices up and down with the help of the appropriate form of spacetime signature and define a reciprocal basis relative to (4). Another operation is that of reversing (\( \sim \)) the order of indices of a multivector, which corresponds to taking the transpose of a matrix. In STA the Hermite conjugate (\( \dagger \)) of a basis vector is defined by the parity transformation (see Eq. (35)) \( e^{\mu} = e^{0}e^{\mu}e^{0} \); it works without shift to the reciprocal basis. The Hermite conjugate of any elements \( A, B \) of STA is:

\[
A^{\dagger} \equiv e^{0}Ae^{0}, (A + B)^{\dagger} = A^{\dagger} + B^{\dagger}; \text{ e.g. } \left(a_{\mu\nu}e^{\mu\nu}\right)^{\dagger} = a_{\mu\nu}e^{0\nu\mu0} \text{ (sum over } \mu, \nu); a_{\mu\nu} \in \mathbb{R}
\]  

(5)

Alternatively, one could have defined Hermite conjugate by the combination of index lift and reversal, which changes a given basis vector to its reciprocal (co)vector. This is convenient when proving e.g. that the STA basis (4) is orthonormal, i.e. that each basis element has modulus 1 and the product of any pair of different elements has zero scalar part \( \langle \cdot \rangle_{0} \). E.g. for fixed indices (no sum over repeated indices, so that \( e_{2}e^{4} = 1 \)):

\[
\langle (e^{\lambda\mu})^{\dagger}e^{2\nu}\rangle_{0} = \langle e_{\mu\lambda}e^{2\nu}\rangle_{0} = \langle e_{\mu}e^{\nu}\rangle_{0} = \delta_{\mu}^{\nu}, \text{ or } \langle (e^{4})^{\dagger}e^{\mu\nu}\rangle_{0} = \langle e_{2}\eta^{\mu\nu}\rangle_{0} + \frac{1}{4}(\delta_{\mu}^{\nu}e^{\mu} - \delta_{\mu}^{\nu}e^{\nu})_{0} = 0
\]  

(6)

Now, the relativistic 4-momentum vector \( p \) of modulus \( m \) and different forms of its square, coordinate-free and coordinate-bound, are given in STA by \( (c = 1) \) (sum over repeated indices by default):

\[
p = e^{\mu}p_{\mu}; \text{ pp } = p \cdot p + p \wedge p = e^{\mu\nu}p_{\mu}p_{\nu} = \eta^{\mu\nu}p_{\mu}p_{\nu} = p \cdot p = m^{2}
\]  

(7)

A relativistic quantum vector Equation of first order in spacetime derivatives for a particle of mass \( m \) is then:

\[
\hat{p}\psi = m\psi \text{ with } \hat{p} = hi\nabla = hie^{\mu}\partial_{\mu}; \partial_{\mu} \equiv \partial/\partial x^{\mu} = \eta_{\mu\nu} \partial/\partial x^{\nu}; \text{ sum over repeated indices}
\]  

(8)

In most cases we will drop the hat and depict the operators \( \hat{p}, \hat{p}_{\mu} \) by \( p, p_{\mu} \). Due to the imaginary \( i \) entering with the canonical momentum operator, STA’s field of scalars has to expand from real to complex numbers. Then, in analogy to the definition of Dirac’s \( \gamma^{5} \) matrix, we can define \( e^{5} \) by (note that \( e^{0123} = e^{3210} \)):

\[
e^{5} = ie^{0123}; \text{ e}^{5}e^{5} = ie^{0123}ie^{0123} = 1 \Rightarrow (e^{5})^{\dagger} = -i(e^{0123})^{\dagger} = -ie^{0}e^{3210}e^{0} = e^{5}
\]  

(9)
i commutes with all the elements of STA in (4), therefore $e^5$ (like $\gamma^5$) is Hermitian, as shown. One can certainly build a formalism expanding the STA basis (4) on complex scalars, which as in (9) would mix the complex structures arising from scalars and vectors & multivectors. However, inspired from STA, we prefer taking an alternative path leading to a real vector space with the complex structure springing from vectors & multivectors alone.

3. The real vector space of spacetime-reflection, STR.

Using the same symbol as in (9) we assume a Hermitian vector $e^5$ to be a basis frame vector, on equal footing with $\{e^\mu\}$, and name the quintet $\{e^\mu, e^5\}$ an orthonormal vector basis for spacetime-reflection, STR; the reason for the name will become clear in the following. Let $X$ depict the real vector space generated by the action of the Clifford product onto $\{e^\mu, e^5\}$. An orthonormal basis for $X$ (dim 32) is:

$$X \text{ basis: } \{1, e^\mu, e^\mu_\lambda, e^{0123}, e^5, e^\lambda_\mu, e^{\lambda\mu5}, e^{01235}\}; \quad \lambda, \mu, \nu = 0,1,2,3; \quad \lambda \neq \mu \neq \nu \neq \lambda \quad (10)$$

STR vector product: $e^\tau e^\nu \equiv e^\tau e^\nu = e^\tau \cdot e^\nu + e^\tau \wedge e^\nu = \zeta^{\tau\nu} + e^\tau \wedge e^\nu; \quad \tau, \nu = 0,1,2,3,5; \quad \zeta^{\tau\nu} = e^\tau \cdot e^\nu = (+-+-+) \quad (11)$

The first 16 elements in (10) are the basis elements of STA in (4) [7]. The other 16 elements are obtained by multiplying with the frame vector $e^5$. Symbolically we could write $\text{STR} = \text{STA}(1 + e^5)$. 16 elements in (10) square to $-1$, allowing for a rich complex structure in $X$. Of these only the element of highest grade, the pentavector is both isotropic (i.e. not privileging any spacetime director) and commutes with all elements of $X$; it constitutes the (geometric) pseudoscalar of STR, depicted by (compare with (9)):

$$I \equiv e^{05123} = -e^{01235}; \quad I = I; \quad I^2 = -1; \quad I^\dagger = -e^{5\dagger}(e^{0123})^\dagger = -I; \quad Ie^\tau = e^\tau I; \quad \tau = 0,1,2,3,5 \quad (12)$$

A general element $y \in X$ is $y = e^\tau(a_\tau + Ib_\tau) + e^{\tau\nu}(c_{\tau\nu} + Id_{\tau\nu}); \tau, \nu = 0,1,2,3,5; \quad a_\tau, b_\tau, c_{\tau\nu}, d_{\tau\nu} \in \mathbb{R}$ (there is redundancy; summation over dummy indices is on). The first two terms stand for vectors and tetravectors, respectively of grade 1 and 4. The last two terms stand for scalars & pseudoscalars ($\tau = \nu$) and bivectors & trivectors ($\tau \neq \nu$), respectively of grade 0, 5, 2, 3. Table 1 summarizes the STR notation; the position vector is $x = x_\mu e^\mu$ with the unit of length attached to the scalar components. We adapt (5) to define the Hermite conjugate of STR elements in (10) taking care of $e^5$ factors, as we did for $I^\dagger$ in (12); for a pure multivector:
3.1. Subspaces of 3D relative vectors.

Two subspaces of relative vectors in $\mathbf{X}$ as well as their product will be of special interest ($j, k = 1, 2, 3$):

$\mathbf{X}: \{1(1), \mathbf{x}_j = e^{j0}(3), \mathbf{x}_{jk} = e^{k j}(3), \mathbf{x}_{123} = e^{0123}(1)\}; \text{ generators: } \{\mathbf{x}_j\}; \text{ dim8}$

$\mathbf{\Sigma}: \{1(1), \sigma_j = e^{j5} (3), \sigma_{jk} = \epsilon_{jkl} e^{l0}(3), \sigma_{123} = I(1)\}; \text{ generators: } \{\sigma_j\}; \text{ dim8}$

$\mathbf{\Xi}: \{1, e^5, x_j, \sigma_j, l x_j, l \sigma_j, e^5, l\}; \{\mathbf{X} \cup \mathbf{\Sigma}\} \subset \mathbf{\Xi}; \text{ generators: } \{e^5, \sigma_j\} \text{ or } \{e^5, x_j\}; \text{ dim16}$

Similarly to the standard case [4], a vector space isomorphic to (10) occurs from the direct product $\mathbf{X} = \mathbf{\Sigma} \otimes \mathbf{\Sigma}$ with scalars and the pseudo-scalar $I = \sigma_{123}$ freely passing through $\otimes$. In addition to providing an alternative route leading to all the results presented here, this representation marks a further argument in favor of expanding ST to STR.

The subspaces $\mathbf{X}, \mathbf{\Sigma}$ share their even grade members, i.e. the scalar 1 (grade 0) and the three bivectors $\mathbf{x}_{jk} = \sigma_{jk} = e^{k j} (j \neq k$ grade 2). We will present the parity transformation further down (Eq. (34-35)), but remind that it changes sign to 3D polar vectors (parity-odd) and leaves unchanged axial vectors (parity-even). As shown in Eq. (35) the vectors $\mathbf{x}_j$ behave as polar vectors (parity-odd), while $\sigma_j$ behave as axial vectors (parity-even). The subspace of axial vectors, spin and rotor generators $\mathbf{\Sigma}$ is isomorphic to the space of Pauli matrices [7]. The subspace of polar vectors and boost generators $\mathbf{X}$ is the same as the even subspace of Hestenes’ STA, in our notation having $\mathbf{x}_{123} = -I e^5$ as ‘local’ pseudoscalar [6,7,10]. The subspace $\mathbf{\Xi}$ in (14’) comprises the Lorentz group and is home to Dirac spinors, as explained further down. Notice that the vectors $\mathbf{x}_j, \sigma_j$ are Hermitian. In passing, the bivectors $e^{j5} = \sigma^j e^0$ are also Hermitian and parity-even, see (35).

3.2. Momentum quantization in STR – the STR DE.

Now we can write the momentum quantization Equation in STR (compare with (8)), obtaining the STR DE:

$p \psi = m \psi \quad \text{with} \quad p = h \nabla = h e^\mu \partial_\mu; \quad \partial_\mu = \partial / \partial x^\mu = \eta_{\mu \nu} \partial / \partial x^\nu; \quad \text{sum over repeated indices}$

$\nabla$ is both coordinate-free and shorthand for $e^\mu \partial_\mu$, therefore one can avoid the Feynman slash notation! $\psi$ must have four components, i.e. it is a 4-spinor because of the four spacetime dimensions leading to four pairwise orthogonal projectors, two relative to the time axis $e^0$ and two relative to one space axis in $\mathbf{\Sigma}$, per
convention $\sigma_3$ (see Eqs. (17-22) below). $p = \hbar \nabla$ is a tetravector operator in $X$ containing $e^\phi$. For the electron with charge $e$, in the presence of an electromagnetic 4-potential $\Lambda$, (15) generalizes to:

$$P\psi = m\psi \quad \text{where} \quad P \equiv p + e\Lambda = \hbar I \nabla + e\Lambda = e^\mu \{ \hbar \partial_\mu + eA_\mu \}$$

This is the STR DE minimally coupled to electromagnetic (EM) field. The first Equations in (15), (16) are frame-free, which underlines the manifest covariance of STR DE, as detailed further down by Equation (49).

The vector-operator $\nabla$ obeys to the chain rule $\nabla A = e^\mu \partial_\mu A_\nu = e^\mu \nu [\partial_\mu A_\nu] + A_\nu \partial_\mu$ and we need generalize the commutator in (2), as illustrated in Appendix A for $[x, p]$ (position-momentum operators). The generators of the Lorentz group appear naturally in $[x, p]$. Table 1 summarizes the notation in the STR formalism. In the remaining part of this Section we will define the Dirac form of the spinor $\psi$ in STR and further elucidate its structure by giving the free field solutions of STR DE.

| Multivector | Space $X$ (a) | Spacetime (b) | Subspace $X$ (c) | Subspace $\Sigma$ (c) |
|-------------|---------------|---------------|-----------------|----------------------|
| Scalars (d) | $e^\tau \cdot e^\nu = \xi^\tau\nu$ | $e^\mu \cdot e^\nu = \eta^\mu\nu$ | $x_j \cdot x_k = \delta_{jk}$ | $\sigma_j \cdot \sigma_k = \delta_{jk}$ |
| Basis vectors | $e^\tau$ | $e^\mu$ | $x_j = x^i = e^i$ | $\sigma_j = e^{j0} = -e^5 x^j$ |
| Vectors (e) | $x' = x e^\tau$ | $x = x_\mu e^\mu; p = p_\mu e^\mu; A = A_\mu e^\mu$ | $x = x_j e^j; E = E_j x^j$ | $(\sigma, B) \equiv B_j \sigma_j; s = \sigma_3^3$ |
| Basis bivectors | $e^\tau \wedge e^\nu$ | $e^\mu \wedge e^\nu$ | $x_j \wedge x_k = e^k \wedge e^j = \sigma_j \wedge \sigma_k = \epsilon \epsilon_{ijk} \sigma_i$ |
| Bivectors (f) | $e^{50}$ in $R_\omega = e^{-e^{50} \omega \tau}$ | $F = (V \wedge A)$ | $F = E + l(\sigma, B)$ |
| Trivectors | $l e^\tau \wedge e^\nu$ | $l e^{\mu5}$ | $x_{123} = -l e^5$ | $\sigma_{123} = l$ |
| Tetravectors | $l e^\tau, \bar{p} = \hbar \nabla 1 \hbar e^\mu \partial_\mu$ | $e^{0123} = -l e^5$ | - | - |
| Pseudoscalar | $l = e^{05123}$ | $l e^5 = -e^{0123}$ | $l e^5 = -x_{123}$ | $l = \sigma_{123}$ |

**Table 1. Summary of notation in STR.** Upright letters depict vectorial quantities, letters in italics depict real scalars.

(a) From the definition of the orthonormal basis for the full STR vector space $X$ in Equation (10);

(b) From Equations (3, 4);

(c) From Equation (14);

(d) The signature scalars for the different spaces in STR: $\xi^\tau\nu(+ -- +), \eta^\mu\nu(+ --),$ $\delta_{jk}(++ -)$;

(e) Examples of shown vectors: the 4-position vector $x = x_\mu e^\mu$, 4-momentum $p$, EM 4-potential $A$, relative position $x$, electrical field $E$, magnetic field as an axial vector $(\sigma, B)$ and the spin vector $s$ along $\sigma_3$;
The generalized rotor $R_{\omega}$ is relevant e.g. in passing from the Dirac to the Weyl basis (see Eq. (7A) in Appendix B). The invariant bivector Faraday $F$ is shown in two forms, see Eq. (30, 30'). In $(\nabla \wedge A)$ the action of the operator $\nabla$ is confined within the brackets.

Due to Lorentz covariance (Eq. (49) further down) the STR Dirac spinor must fulfill $\psi \in \Sigma$ with $\Sigma$ already presented in (14'). One realization of this requirement is to split $\psi$ into two independent parts $\varphi, \chi$ by the action of the two orthogonal projectors $(1 \pm e^\omega)$, with the additional condition that $\varphi, e^\omega \chi$ be Pauli spinors:

$$\psi = \frac{1}{2}[(1 + e^0)\psi + (1 - e^0)\psi] = \varphi + \chi; \quad \varphi \equiv \frac{1}{2}(1 + e^0)\psi; \quad \chi \equiv \frac{1}{2}(1 - e^0)\psi; \quad \varphi, e^\omega \chi \in \Sigma$$

(17)

Similarly to the treatment in the standard formalism [4], depending on the physical problem at hand, one may choose to explicate in STR a scalar + pseudoscalar (grade 0&5) factor in $\psi$. E.g., in the case of free field, one expands $\psi$ in plane waves of positive and negative energy and a constant spinor (with $p$ the 4-momentum vector and $s$ the spin degrees of freedom):

$$\psi_+ = e^{-lp \cdot \sigma_3/h}u(p, s) \quad \text{and} \quad \psi_- = e^{lp \cdot \sigma_3/h}v(p, s); \quad u(p, s), v(p, s) \quad \text{satisfy DE}$$

(18)

Or, when deriving the Pauli Equation as non-relativistic approximation to DE, following Feynman we split the Dirac spinor $\psi = \rho \psi_p$, isolating in $\rho$ the fast oscillating part and in $\psi_p$ the Pauli spinors proper:

$$\psi = \frac{1}{2}[\rho(1 + e^0)\psi_p + (1 - e^0)\psi_p] = e^{-lmt/h}(\varphi_p + \chi_p); \quad \varphi_p, e^\omega \chi_p \in \Sigma$$

(19)

$\varphi$ and $\chi$ not being limited to the slow particle regime are more general than $\varphi_p, \chi_p$. Each of the two Pauli spinors $\varphi$ and $e^\omega \chi$ splits into two independent components by the action of two orthogonal projectors in $\Sigma$.

We illustrate the projection of $\varphi$ relative to the $\sigma_3$ axis, chosen by convention as reference direction:

$$\varphi = \frac{1}{2}[(1 + \sigma_3)\varphi + (1 - \sigma_3)\varphi] \equiv \varphi_u + \sigma_1 \varphi_d;$$

$$\varphi_u \equiv \frac{1}{2}(1 + \sigma_3)\varphi; \quad \sigma_1 \varphi_d \equiv \frac{1}{2}(1 - \sigma_3)\varphi; \quad \varphi_u, \varphi_d \in \{a + i b; a, b \in \mathbb{R}\}$$

(20)

The two moduli $|\varphi_u|^2$ and $|\varphi_d|^2$ are proportional to the probabilities for spin up and spin down in the $\sigma_3$ direction, respectively. If normalized we will have $|\varphi_u|^2 + |\varphi_d|^2 = 1$. For spin depending on position $\varphi_u, \varphi_d$ are in general functions of position and the normalization condition above appears as an integral over the 3D space. For spin independent on position (spin-position decoupling, s-p), the spatial parts of $\varphi_u$ and $\varphi_d$ are equal to the common factor $\rho$ below (in (21) $R_\phi = e^{-l\sigma_2 \theta'/2} = \cos \frac{\theta}{2} - i \sigma_2 \sin \frac{\theta}{2}$):

$$\varphi_u + \sigma_1 \varphi_d = \rho R_\phi \quad \text{with} \quad \rho \cos \frac{\theta}{2} = \varphi_u; \quad -\rho \sigma_2 \sin \frac{\theta}{2} = \varphi_d$$

(21)
The $\sigma_1 \varphi_d$ can be traded with $-1\sigma_2$ by ‘extracting’ a $\sigma_3$ from the projector $(1 - \sigma_3)$ in (20). This renders the representations of spin up and spin down formally the same as in STA [7], however remember that $\sigma_k$ in STA are not parity-even (see also [8]). The same expressions as in (20, 21) are valid for the other STR Pauli spinor $e^5\chi$, generally with a different angle $\theta' \neq \theta$. $\rho_\varphi, \rho_\chi$ have grade 0&5 and may comprise normalization factors. The result of (17-21) is a Dirac spinor with four projectors $(1 \pm e^5\chi)(1 \pm e^5\chi)$:

$$\psi = \varphi + \chi; \quad \varphi, e^5\chi \in \Sigma; \quad \varphi = \varphi_u - 1\sigma_2\varphi_d = \rho_\varphi R_\varphi; \quad e^5\chi = \chi_u - 1\sigma_2\chi_d = \rho_\chi R_\chi, \quad (22)$$

We conclude Section (3.2) by showing how to resolve the STR DE in the case of free field, which offers a first illustration of the structure of the Dirac spinors in (22). From (18), $u(p, s) = \varphi + \chi$ and $v(p, s) = \varphi - \chi$ with $\varphi, e^5\chi \in \Sigma$

$$u(p, s) = \varphi_+ + \chi_+; \quad v(p, s) = \varphi_- + \chi_- \quad \text{with} \quad \varphi_\pm, e^5\chi_\pm \in \Sigma \quad (23)$$

We will show in Section (4.5) that the form of spinor (17) allows to write STR DE as two coupled equations:

$$\begin{cases} (E - m)\varphi - p\chi = 0 \\ (E + m)\chi - p\varphi = 0 \end{cases} \Rightarrow \begin{cases} (E - m)\varphi_+ - p\chi_+ = 0 \\ (E + m)\chi_+ - p\varphi_+ = 0 \end{cases} \quad \text{and} \quad \begin{cases} (E - m)\varphi_- - p\chi_- = 0 \\ (E + m)\chi_- - p\varphi_- = 0 \end{cases} \quad (24)$$

$E = p_0$ is the energy (scalar) and $p$ is the 3-momentum (vector in $X$, see (14)). The first coupled equations in (24) are formally the same as in the standard treatment, while the other two couples follow from (18, 23). The first $\varphi$-terms in the upper equations belong to $\Sigma$ and so do the second $\chi$-terms, as we can see by writing, e.g. $p\chi = pe^5e^5\chi = -p_k\sigma_k e^5\chi \equiv -(p, \sigma)(e^5\chi) \in \Sigma$. Similarly, left- multiplication by $e^5$ of the lower equations in (24) brings all their terms into $\Sigma$. After this consistency check, we can proceed with the free field solutions. For positive energy $E > 0$, the factor $E + m > 0$, therefore we can express $\chi_+$ from the lower equation in (24) as a function of $\varphi_+$ then substitute it into the upper equation:

$$\chi_+ = p\varphi_+/(E + m); \quad (E^2 - m^2 - p^2)\varphi_+ = 0 \quad (25)$$

For $\varphi_+ \neq 0$ the last Equation is just the relativistic invariant $E^2 - m^2 - p^2 = 0$. As shown above, we can write the first Equation in (25) as $e^5\chi_+ = -(p, \sigma)\varphi_+/(E + m)$. Now we can express $e^5\chi_+, \varphi_+$ in the last Equation by the corresponding probability amplitudes for spin up and spin down as in (22):

$$\chi_{+u} + \sigma_1\chi_{+d} = -(p, \sigma)(\varphi_{+u} + \sigma_1\varphi_{+d})/(E + m) \quad (26)$$
The last Equation is now easily resolved:

\[
\chi_{+u} = \left[-p_3 \varphi_{+u} - (p_1 - lp_2) \varphi_{+d}/(E + m)\right]/(E + m); \quad \chi_{+d} = \left[p_3 \varphi_{+d} - (p_1 - lp_2) \varphi_{+u}/(E + m)\right]/(E + m)
\]

With the plane wave prefactor in (18) the general solution (non normalized) takes the form:

\[
E > 0: \psi_+ = e^{-1 \chi/h}\left\{\varphi_u; \varphi_d; \left[-p_3 \varphi_u - (p_1 - lp_2) \varphi_d/(E + m); [p_3 \varphi_d - (p_1 + lp_2) \varphi_u]/(E + m)\right]\right\}
\]

which in the case of spin up \(\varphi_u = 1\) (respectively spin down \(\varphi_d = 1\)) yields:

\[
\text{Spin up: } \psi_{+1} = e^{-1 \chi/h}\left\{1; 0; - p_3/(E + m); -(p_1 + lp_2)/(E + m)\right\}
\]

\[
\text{Spin down: } \psi_{+1} = e^{-1 \chi/h}\left\{0; 1; - (p_1 - lp_2)/(E + m); p_3/(E + m)\right\}
\]

One finds in a similar fashion the ‘negative energy’ solutions from the third pair of equations in (24), in this case remembering that \(-E + m > 0\). We just show the final result here:

\[
E < 0: \psi_- = e^{1 \chi/h}\left\{[p_3 \chi_u + (p_1 - lp_2) \chi_d]/(-E + m); [-p_3 \chi_d + (p_1 + lp_2) \chi_u]/(-E + m); \chi_u; \chi_d\right\}
\]

4. STR at work

With the above preliminaries in place, it is instructive to start by showing how in the presence of external EM field the spin magnetic moment of the electron arises from spacetime-reflection.

4.1. The square of the STR DE.

\[
P\psi = m\psi \Rightarrow PP\psi = m^2\psi \Rightarrow [(hI\nabla + eA)(hI\nabla + eA) - m^2]\psi = \left(\eta^\mu_\nu (hI\partial_\mu + eA_\mu)(hI\partial_\nu + eA_\nu) - m^2 + ehI(\nabla \wedge A + A \wedge \nabla)\right)\psi = \left[KG + ehI[(\nabla \wedge A)]\psi = \left[KG + ehI[-(\partial_0 A) - le^5 (\nabla \times A)]\right]\psi = [KG + ehI(E - le^5 B)]\psi \equiv [KG + ehI(E + I(\sigma, B))]\psi \equiv (KG + ehIF)\psi = 0
\]

Brackets in e.g. \((\nabla \wedge A)\) or \((\nabla A_0)\) confine the action of the operator;

\[
A = A_j x_j \text{ (vector potential); } E = E_j x_j \text{ (electric field, polar 3D vector); } (\sigma, B) \equiv \sigma_j B_j = -e^5 B \text{ (magnetic field, axial 3D vector); } F \equiv E + I(\sigma, B) = \nabla \wedge A + A \wedge \nabla = (\nabla \wedge A) \text{ (Faraday)}
\]

\(KG\) stands for the Klein-Gordon term; it comprises grade 0, 5 components, including operators. The \(e^5\)-independent term \((\nabla \wedge A) = E + I(\sigma, B)\) is the Faraday \(F\), depicting the relativistic invariant EM field strength experienced by the electron, as marked by the prefactor \(eh\). \(F\) is a tensor in the standard formalism [4, 5]; in STR and STA it is a 4D bivector as clearly seen in (30) [7, 11]. The term \(ehIF\) distinguishes the
squared DE from the KG Equation. It represents the ‘internal degrees of freedom’ of the electron – the spin, interacting with the EM field. Indeed, in the nonrelativistic regime, Equation (30) (or the STR Pauli Equation (54)) yields Equation (31) below, as shown in Appendix C (orbital and spin angular momentum vectors: \( \mathbf{L} \equiv \mathbf{r} \times \mathbf{p}; \mathbf{S} = \hbar \sigma / 2 \). \( \mathbf{S} \) is the symbolic Pauli’s spin operator in STR – meaningful only as part of a product, e.g. \( \mathbf{B} \cdot 2 \mathbf{S} \equiv \hbar (\sigma, \mathbf{B}) = \hbar \mathbf{B} \sigma \)).

\[
\begin{align*}
(P \cdot P - m^2 + e\hbar \mathbf{F})\psi &= 0 \\
&\quad \text{by the nonrelativistic approximation,}
\end{align*}
\]

Equation (31) above, as shown in Appendix C (orbital and spin angular momentum vectors: \( \mathbf{L} \equiv \mathbf{r} \times \mathbf{p}; \mathbf{S} = \hbar \sigma / 2 \)). \( \mathbf{S} \) is the symbolic Pauli’s spin operator in STR – meaningful only as part of a product, e.g. \( \mathbf{B} \cdot 2 \mathbf{S} \equiv \hbar (\sigma, \mathbf{B}) = \hbar \mathbf{B} \sigma \)).

This is the famous prediction from DE that the unit of spin angular momentum interacts twice as strongly with the magnetic field as the unit of orbital angular momentum. In other words, according to (31) the gyromagnetic ratio, i.e. the ratio between magnetic moment and angular momentum is twice as great for the electron spin as it is for the orbital motion. The derivation of (30-31) proves that spin springs from 4-momentum in spacetime-reflection, without any preconceived internal degrees of freedom. The electron spin gyromagnetic ratio from (31) is a factor of \(~1.00116\) smaller than the experimental figure, the gap arising from QED corrections, beyond the scope of DE [4, 5].

4.2. Conserved currents [2, 4] of STR DE.

From (13, 14) \( \mathbf{x}^\dagger_j = \mathbf{x}_j \), which simplifies the derivation of the Hermite conjugate for DE and the Dirac conjugate \( \overline{\psi} \) for the spinor \( \psi \):

\[
(P - m)\psi = \left[ e^\mu (\hbar \partial_\mu + eA_\mu) - m \right] \psi = \left[ (\hbar \partial_0 + eA_0) + \mathbf{x}_j (-\hbar \partial_j + eA_j) - me^0 \right] e^0 \psi = 0
\]

\[
\psi^\dagger e^0 \left[ (\hbar \partial_0 + eA_0) + \mathbf{x}_j (-\hbar \partial_j + eA_j) - me^0 \right] \equiv \overline{\psi} \left[ e^\mu (\hbar \partial_\mu + eA_\mu) - m \right] e^0 = 0
\]

After Hermite conjugation \( \dagger \), \( \partial_\mu \) act to the left. Left-multiply the DE by \( \overline{\psi} = \psi^\dagger e^0 \) and right-multiply the last equation in (32) by \( \psi^\dagger e^0 \psi \); then subtract it from the first, obtaining the conserved probability current:

\[
\left( \partial_\mu \overline{\psi} \psi^\mu + \overline{\psi} e^\mu (\partial_\mu \psi) = \partial_\mu (\overline{\psi} e^\mu \psi) = 0 \right. \text{ with probability density } \langle \overline{\psi} e^0 \psi \rangle = \langle \psi^\dagger \psi \rangle \geq 0
\]

Adapting the form of \( \varphi, \chi \) from (21), in the spin-position decoupling regime, one can go further with the calculation in the Pauli basis, eventually deriving expressions for the components \( \langle \overline{\psi} e^i \psi \rangle \) consisting of only the probability amplitudes \( \varphi_\alpha, \varphi_d, \chi_\alpha, \chi_d \) (not shown here). Table 2 summarizes the bilinears of STR DE.
Table 2. Dirac bilinears in standard and STR formalisms. Expanded forms of the STR Dirac bilinears appear in the last column, in terms of the Pauli spinors (21, 22). The angled brackets $\langle \cdot \rangle$ in STR yield the expectation value.

(a) The angle $\beta$ is standard in the STA literature. In STR it is defined by the relative moduli of the two Pauli spinors, where $\rho^2 = \rho^+\rho = \langle \bar{\psi}e^0\psi \rangle = \langle \psi^+ \psi \rangle = \rho_\theta^2 + \rho_\phi^2 + \rho_\chi^2$. All $\rho$-s are sums of a scalar and a pseudoscalar! See also footnote (d) below.

(b) The standard antisymmetric traceless tensor is defined by the commutator of Dirac matrices $\sigma^{\mu\nu} \equiv \frac{1}{2} [\gamma^\mu, \gamma^\nu]$;

(c) $\varepsilon \equiv (\delta^{00} - \delta^{00})$ and $\delta \equiv \delta^{00}$ $\delta^{00}$;

(d) As anticipated in Eq. (14), $\sigma_j$ are axial, therefore they appear naturally here. For $m = 0$, the axial currents $\bar{\psi}e^5\psi$ are conserved.

(e) In STR $\langle \bar{\psi}e^5\psi \rangle = \langle \phi^+e^5\chi - \chi^+e^5\phi \rangle = \langle R_{R_5}^{\dagger}R_5 \rangle_\theta (\rho_\theta^4\rho_\phi - \rho_\phi^4\rho_\theta) - i \sin \beta$, which changes sign under Hermite conjugation, as a pseudoscalar should. The corresponding expression in STA is in our notation $\rho^2 \sin \beta$, which is a scalar and hints to a simplistic definition of $\psi$ in STA. $\langle R_{R_5}^{\dagger}R_5 \rangle_\theta$ extracts the scalar part of $R_{R_5}^{\dagger}R_5$.

4.3. Symmetries of STR DE.

Let us look now at the symmetries parity $P$, time reversal $T$ and charge conjugation $C$ for the STR DE and for the STR Dirac spinor $\psi$. In the following we will present three forms $T, T', T''$ for time-reversal symmetry in STR as well as discuss the relation between the three (see also Appendix D).

**Parity** $P$: $e^\mu \rightarrow e^0e^\mu e^0 = e^\mu_P$. As in the standard treatment left-multiply STR DE by $e^0$ (below $\hbar = 1$):

$P$: $(P - m)\psi = 0 \rightarrow e^0(P - m)\psi = (e^0Pe^0 - m)e^0\psi \equiv (P_p - m)\psi_p =$

$[e^0(1\partial_0 + eA_0) - e^l(1\partial_l + eA_l) - m]\psi_p = [e^\mu_P(1\partial_\mu + eA_\mu) - m]\psi_p \Rightarrow P_p = \eta_{\mu\nu}e^\nu(1\partial_\nu + eA_\nu)$;

$\psi_p = e^0\psi = \psi - \chi$  \hspace{1cm} (34)

It is clear from (34) that both 3-momentum and 3-position, therefore also the vector potential, change sign in STR DE under parity. Applying the parity transformation (34) to the 3D vectors from (14) we see that:
$x_j \rightarrow e^0 x_j e^0 = -x_j$ are odd; $\sigma_j \rightarrow e^0 \sigma_j e^0 = \sigma_j$ and $e^{i\xi} \rightarrow e^0 e^{i\xi} e^0 = e^0 \sigma_j e^0 = e^{i\xi}$ are even \hspace{1cm} (35)

The last equation in (34) shows that under parity the Pauli spinors $\varphi$ and $\chi$ are respectively even and odd.

**Time reversal** $\mathcal{T}$: $x_0 \rightarrow -x_0 \Leftrightarrow \partial_0 \rightarrow -\partial_0$. We write first DE in terms of the time-reversed quantities:

\[ \text{DE: } (e^0(p_\mu + eA_\mu) - m)\psi = [e^0(-p'_0 + eA_0) + e^i(p'_j - eA'_j) - m] \overline{\psi} = 0 \hspace{1cm} (36) \]

In order to render the energy term positive in the time-reversed system we need to invert the sign of $-p'_0$ in (36) without changing $eA_0$. This can be achieved in different ways, e.g. by Hermite conjugation & reversal ($\dagger$) ($\mathcal{T}$), by $K_3$- or $K_5$- conjugations (leading respectively to $\mathcal{T}'$, $\mathcal{T}''$) of DE as now shown. We first present the three transformations then discuss and compare them. The Hermite conjugate of DE was shown in (32).

\[ \mathcal{T}: \text{DE} \rightarrow e^0 (e^i(-hI\partial_\mu + eA_\mu) - m) \overline{\psi} = [e^0(1\partial'_0 + eA_0) + e^i(1\partial_j - eA_j) - m] e^0 \overline{\psi} = 0 \Rightarrow \]

\[ \Rightarrow \psi_T = e^0 \overline{\psi} = \overline{\psi}; \quad TIT^{-1} = I\dagger = -I; \quad T^2 = 1 \hspace{1cm} (37) \]

Alternatively, we introduce $K_\omega$-conjugation that inverts only the $e^{i\omega}$ (below we look at $\omega = 3$ or 5):

\[ K_\omega: \{ e^\tau \rightarrow K_\omega e^\tau K_\omega = (1 - 2\delta_{\omega\tau})e^\tau, \text{ or } e^{i\omega} \rightarrow -e^{i\omega}; \quad e^{i\pi \omega} \rightarrow e^{i\tau}; \quad \tau = 0,1,2,3,5 \} \hspace{1cm} (38) \]

First the case $\omega = 3$. We define $\mathcal{T}'$ by the product of a unitary transformation and $K_3$, $\mathcal{T}' = UK_3 = e^{i3}K_3$:

\[ \mathcal{T}': \text{DE} \rightarrow e^{i3}K_3 [e^0(1\partial_\mu + eA_\mu) - m] \overline{\psi} = [e^0(1\partial'_0 + eA_0) + e^i(1\partial_j - eA_j) - m] e^{i3}K_3 \overline{\psi} = 0 \Rightarrow \]

\[ \Rightarrow \psi_{T'} = e^{i3}K_3 \overline{\psi} = 1\sigma_3 K_3 \overline{\psi} \equiv 1\sigma_3 \psi^{K_3}; \quad T'IT'^{-1} = e^{i3}K_3IK_3 = -I; \quad T'^2 = -1 \hspace{1cm} (39) \]

For $\omega = 5$ in (38), $\mathcal{T}''$: $\text{DE} \rightarrow K_5 [e^0(1\partial_\mu + eA_\mu) - m] \overline{\psi} = [e^0(1\partial'_0 + eA_0) - e^i(1\partial_j - eA_j) - m] K_5 \overline{\psi} = 0 \Rightarrow \]

\[ \psi_T' = \overline{\psi}; \quad T''IT'^{-1} = K_5 IK_5 = -I; \quad T''^2 = 1 \hspace{1cm} (40) \]

Note that the final operators in (37) and (39) are identical, while that in (40) has opposite sign at the spatial part (see below). Let list in bullet form few salient properties of the transformations (37), (39), (40).

- All the three transformations invert the pseudoscalar $I$ and are antiunitary [12] in STR.
- The vector operators for 3-momentum and position transform opposite (according) to the expectation for $\mathcal{T}$, $\mathcal{T}'$ (resp. $\mathcal{T}''$), which is clear by inspection of the transformed operators. Similarly to the cases of $\mathcal{T}, \mathcal{T}'$ the standard $t$-reversed DE has the wrong sign in front of the 3-momentum.
In the absence of magnetic field ($A_j = 0$) the Dirac Hamiltonian remains unchanged under $\mathcal{T}, \mathcal{T}'$

\[
\mathcal{L}_0 \psi = H\psi = \left( e^{i0} \mathcal{L}_j - eA_0 + m e^0 \right) \psi \xrightarrow{\mathcal{T}, \mathcal{T}'} \mathcal{L}_0 \psi_{\mathcal{T}, \mathcal{T}'} = H \psi_{\mathcal{T}, \mathcal{T}'} \quad \text{(41)}
\]

The 3-momentum term in the same Hamiltonian changes sign under $\mathcal{T}''$ (as it should under $t$-reversal!):

\[
\mathcal{L}_0 \psi \xrightarrow{T''} \mathcal{L}_0 \psi_{\mathcal{T}''} = K_z \mathcal{H} K_z \psi_{\mathcal{T}''} = \left( -e^{i0} \mathcal{L}_j - eA_0 + m e^0 \right) \psi_{\mathcal{T}''}; \; e^K \psi_{\mathcal{T}''} = H \quad \text{(41')}\]

- The final DE operators in (37), (39) differ by a parity transformation compared to the proper time-reversal operator (40). This leads to vectors for 3-momentum and position bearing wrong sign in (37), (39) – the price to pay for preserving the Hamiltonian (41) above; see also the last equation in (41').

- The vector potential $\mathbf{A} = A_j \mathbf{x}_j$ does not invert under the proper $t$-reversal (40). An additional parity transformation is required, as in (37), (39) in order to invert $\mathbf{A}$. In the following, keeping with tradition we will identify $t$-reversal with (37), (39), which both invert $\mathbf{A}$ and preserve the Hamiltonian (41).

- The transformed operator parts being the same in (37) and (39) one expects $\psi_\mathcal{T}$ and $\psi_{\mathcal{T}'}$ to represent the same state. Indeed, the $\mathcal{T} = \mathcal{T}^{-1}$ transformation is equivalent to a reversal of the spatial directors alone $e^j \rightarrow -e^j$, leaving $e^0, e^5$ unchanged; we call it $K_j$-conjugation ($K_j = K_1 K_2 K_3$). The property $K_j e^K K_j = e^5$ distinguishes it from the parity transformation (34). $K_3, K_j$ preserve the Clifford algebra $\{e^\mu, e^\nu\} = 2\eta^{\mu\nu}$. Both reverse handedness of spatial coordinates, therefore the resulting bases from the two conjugations differ by a proper rotation, as illustrated in Figure 1. In particular, $\mathcal{T}, \mathcal{T}'$ both flip spin.

- Kramers’ degeneracy [13] for a system consisting of an odd number of spin $\frac{1}{2}$ particles in an electric field is more directly proved for the transformation $\mathcal{T}'$ in (39) with $T'^2 = -1$, just as in the standard case [4]. However, it is of course valid for $\mathcal{T}$ in (37), which as argued above, leads to the same state.

The discussion above illustrates the power of STR to elucidate the geometric meaning of $t$-reversal, a subject that is quite involved in the standard formalism [4]. Two remarks before closing the paragraph. First, the standard $t$-reversal transformation corresponds to $\omega = 2$ in (38) with $T_{xt} = e^{13} K_2$. Second, conjugations $K_j, K_1, K_2, K_3$ from (38) (see also (16-17A)) relate in 3D to inversion and three frame-plane reflections, the same improper rotations characterizing the four maximally entangled states of pairs of spin $\frac{1}{2}$ particles [14].

\textit{Charge conjugation} $\mathcal{C}: e \rightarrow -e$. Similarly to the different forms of $\mathcal{T}$-symmetry above, one can construct different forms of $\mathcal{C}$, e.g. $\mathcal{C}: e^{123} K_j$; $\mathcal{C}: e^5 K_5$; $\mathcal{C}: e^3 K_3$, etc. E.g. $\mathcal{C}$ corresponding to (37) is obtained by first $K_j$-conjugating STR DE (16), then left-multiplying by $e^{123}$. 

14
\[ C: (e^\mu P_\mu - m)\psi = 0 \rightarrow C(e^\mu P_\mu - m)\psi = e^{123}e^0 \left[-e^\mu (\hbar \partial_\mu - eA_\mu) - m\right]\bar{\psi} = \left[e^\mu (\partial_\mu - eA_\mu) - m\right]\psi_c = 0 \Rightarrow \psi_c \equiv e^{123}\psi^+ = e^{123}\psi^{Kj}; \quad CIC^{-1} = e^{123}K_jK_ie^{123} = -I; \quad CC = e^{123}K_je^{123}K_j = -1 \] (42)

With (16) describing the electron, (42) describes the positron. Notice the role of \( e^h \) in \( e^{123} = Ie^h \).

**CPT.** We can apply the three symmetries (34), (37), (42) in series onto the STR DE (16) (notice \( e^h \) again!)

\[ CPT: (e^\mu (\partial_\mu + eA_\mu) - m)\psi = 0 \rightarrow e^{123}K_je^0K_j(e^\mu (\partial_\mu + eA_\mu) - m)\psi = Ie^5(e^\mu (\partial_\mu + eA_\mu) - m)\psi = \left(e^\mu (-\partial_\mu - eA_\mu) - m\right)e^5\psi = 0 \Rightarrow \psi_{CPT} = Ie^5\psi = -e^{0123}\psi \] (43)

**CPT** reverses time, 3-momentum and charge, leaving invariant the Dirac Lagrangian (see Appendix D). The discussion above makes it clear that \( CPT = CT'' \), with \( T'' \) the *proper* time-reversal (see (40) and Fig. 1).

Combinations of different forms of \( C, T \) in **CPT** conserve the Lagrangian and reveal some of its symmetries.

**Figure 1.** Effect of conjugations \( K_0, K_1, K_j, K_5 \) onto the STR directors. The transformed frames and operators are related as shown by the two-sided arrows. From (29-32), \( T = K_j; T' = I\sigma_3K_3; T'' = K_5 \); we add here \( T''' = e^{123}K_\theta \). All \( T, T', T'', T''' \) invert \( I \) and thereby time; of these only \( T'', T''' \) keep 3-position unchanged and invert 3-momentum. However, as shown above, the Hamiltonian (33) preserves form under \( t \)-reversal when DE transforms by \( T = K_j \), or \( T' = I\sigma_3K_3 \); we accept these by convention to stand for \( t \)-reversal.

### 4.4. Relativistic covariance of STR DE (15, 16) and change of basis in STR

We have already seen that the geometric product of two vectors comprises a scalar and a bivector, see (3), (11). The frame bivectors generate Lorentz boosts and rotations as shown below (a factor of 2 ensures consistency with the definition in (1A)):

\[ e^\mu e^\nu = \eta^{\mu\nu} + e^\mu \wedge e^\nu; \quad e^\mu \wedge e^\nu = \left\{ \begin{array}{ll}
2(\delta_{ij}\delta_{\nu0} - \delta_{i0}\delta_{\nu j}) & K_j = (\delta_{ij}\delta_{\nu0} - \delta_{i0}\delta_{\nu j})x_j \quad \text{Boost gnrs.} \\
2\varepsilon_{ijk}\mathbf{l}_i & -\varepsilon_{ijk}\mathbf{l}_i & \text{ for } \mu = j, \nu = k \quad \text{Rotor gnrs.}
\end{array} \right. \] (44)

Notice that \( I \) is both parity and Lorentz invariant. Lorentz (boost-rotor) transformations \( S, S^{-1} \) appear as exponentials of the boost and rotor generators \( K_j, J_j \) in (36) (\( J_j, K_j \) do not comprise \( e^5 \!)): \( S = e^{S_j\omega_j} \) with \( \left\{ \begin{array}{l}
S_j = K_j = x_j/2; \omega_j = \alpha_j \text{ (rapidity, hyp. angle) boosts} \\
S_j = J_j = -I\sigma_j/2; \omega_j = \theta_j \text{ (Euclidian angle) rotors}
\end{array} \right. \quad e^0S^+e^0 = \bar{S} = S^{-1} \) (45)
The last relations in (37) show that from the definition of \( S_j, \bar{S} = S^{-1} \). The boost-rotor \( S \) is non-Hermitian, which follows from the opposite behavior of boosts and rotors under Hermite conjugation:

\[
\text{Rotor: } (e^{-1\sigma_{j}/2})^\dagger = e^{1\sigma_{j}/2}; \quad \text{Boost: } (e^{x_{j}/2})^\dagger = e^{x_{j}/2} \quad (46)
\]

Frame vectors (respectively Dirac spinors) Lorentz-transform by two-sided (one-sided) \( S \)-operation:

\[
\mathcal{L}: \quad e^\mu \rightarrow e'^\mu = Se^\mu \bar{S}; \quad e^\mu \rightarrow e'^\mu = Se^\mu \bar{S} \quad \text{and} \quad \mathcal{L}: \quad \psi \rightarrow \psi' = S\psi; \quad \bar{\psi} = \psi^\dagger e^0 \rightarrow (S\psi)^\dagger e^0 = \bar{\psi}\bar{S} \quad (47)
\]

The operator \((p - m)\) is coordinate-free and relativistic-invariant, i.e. it remains unchanged as a whole under a Lorentz transformation \( \mathcal{L} \). This allows easy linking of momentum operator components in different Lorentz frames (similar relations as in (48) below apply for all 4-vectors, like \( x, p, A, \nabla \), etc.):

\[
\mathcal{L}: \quad p = e^\mu p_\mu \rightarrow p' = e'^\mu p'_\mu = Se^\mu \bar{S}p_\mu = p; \quad p' = e'^\mu \cdot p = Se^\mu \bar{S} \cdot p; \quad p_\mu = e_\mu \cdot p' = S'e_\mu \bar{S}' \cdot p' \quad (48)
\]

\( S' = e^{S_j/\omega_j} = e^{-x_{j}/2+1\sigma_{j}/2} \) (compare with (45)) are the Lorentz rotor-boost from the perspective of the primed frame. When it comes to DE, it transforms as a spinor and (48) helps convert to primed components:

\[
\mathcal{L}: \quad (p - m)\psi = 0 \rightarrow S(p - m)\psi = (Sp\bar{S} - m)S\psi = (e'^\mu p'_\mu - m)\psi' = [e'^\mu (e_\mu \cdot p') - m]\psi' = 0, \quad \text{or} \quad (49)
\]

\[
\mathcal{L}: \quad (p - m)\psi = 0 \rightarrow S(p' - m)\psi = (Sp'\bar{S} - m)S\psi = (Sp'\bar{S}p'_\mu - m)\psi' = (S^2 e'^\mu \bar{S}^2 p'_\mu - m)\psi' = 0 \quad (49')
\]

Note, \( Sp\bar{S} \neq p' \) (though \( Sp\bar{S}Sp\bar{S} = p^2 \)). From \( p = p' \), the two forms (49), (49') expressed in the same frame stand for the same transformation. One could be tempted to write down the alternative transformation:

\[
\mathcal{L}_{alt}: \quad (p - m)\psi = 0 \rightarrow (p' - m)\psi' = \bar{\psi}'(Sp\bar{S} - m)\psi' = \bar{\psi}'(Sp\bar{S}p'_\mu - m)\psi = S(e'^\mu p'_\mu - m)\psi = 0 \quad (49'')
\]

However, only the transformation (49-49'') keeps the Lagrangian \( \mathcal{L} \) (see (13A)) Lorentz-invariant (see (47)):

\[
\mathcal{L}: \quad \mathcal{L} = \bar{\psi}'(Sp\bar{S} - m)\psi' = \bar{\psi}'(Sp\bar{S}p'_\mu - m)\psi = \bar{\psi}'(Se'^\mu \bar{S}p'_\mu - m)\psi = \bar{\psi}(p - m)\psi = \mathcal{L} \quad (50)
\]

While the alternative transformation (49'') alters the Lagrangian:

\[
\mathcal{L}_{alt}: \quad \mathcal{L} = \bar{\psi}'(p' - m)\psi' = \bar{\psi}'(p' - m)\psi = \bar{\psi}'(e'^\mu p'_\mu - m)\psi \neq \mathcal{L} \quad (50')
\]

Equations (48-50') make the case for a Lorentz transformation of STR DE as a whole entity, instead of the piecewise transformation in (49'') with the operator transforming as a 4-vector, see (47-48). Ultimately this is due to the Lorentz invariance of \( p \) combined with the covariance of \( \psi \). The form (49') is convenient because it directly explicates \( p'_\mu = \partial / \partial x'^\mu \). From (47), the Lorentz transformation of three bilinears take the form:

\[
\mathcal{L}: \quad \{ \bar{\psi}\psi \rightarrow \bar{\psi}'\psi' = \bar{\psi}'\psi; \quad \bar{\psi}e^{5}\psi \rightarrow \bar{\psi}'e^{5}\psi' = \bar{\psi}'e^{5}\psi; \quad \bar{\psi}e^\mu \psi \rightarrow \bar{\psi}'e^\mu \psi' \text{ (from invariance of} \nabla \text{ in (33))} \} \quad (51)
\]
Relations (45-50) demonstrate how boosts and rotations actually take place in the physical spacetime. In the discussion above we did not mention any (linearized) infinitesimal transformations \[2, 4, 5\] to prove the covariance of STR DE and the Lorentz invariance of the STR Dirac Lagrangian. STR stands at an advantage point when infinitesimal transformations are demanded, e.g. in curved spacetime \[4\].

How do we change basis in STR? (47) hints to the answer: by two sided general rotor transformations, i.e. 
\[e^\tau \rightarrow e^{\tau'} = S e^\tau S^{-1}; \quad \tau = 0,1,2,3,5.\] S here generalizes \[S\] in (45). E.g. in the deep relativistic regime \(|p| \gg m\) one can neglect the mass term in DE and transform the STR Dirac basis adapted until now into the Weyl basis and Weyl spinors. The generalized rotor 
\[S = R_\omega = \rho e^{\omega/2}\] acting one-sided to spinors and two-sided to vectors, can swap \(e^0\) with \(e^i\) leaving \(e^j\) unchanged, \(R_\omega e^i R_\omega^* = e^j\), as recounted in Appendix B. The two/one-sided transformations of vectors / spinors appeared also in Eqs. (34-43) on symmetries of STR DE.

4.5. STR DE in terms of Pauli spinors and its nonrelativistic limit – the STR Pauli Equation.

Now we turn to the form (17) of the STR Dirac spinor as a sum of the spinors \(\varphi\) and \(\chi\) and write down the two equations obtained by the sum and difference of the Dirac and of its parity-transformed equations:

\[
\begin{align*}
(P_0 e^0 + P_j e^j - m)(\varphi + \chi) &= 0 \\
(P_0 e^0 - P_j e^j - m)(\varphi - \chi) &= 0 \\
\Rightarrow \quad (P_0 e^0 - m)\varphi + P_j e^j \chi &= 0 \\
(P_0 e^0 - m)\chi + P_j e^j \varphi &= 0 \\
&\Rightarrow \quad (P_0 - m)\varphi - P \chi = 0 \\
&\quad (P_0 + m)\chi - P \varphi = 0
\end{align*}
\]

In the last step we use \(P_0 e^0 \varphi = P_0 \varphi; P_0 e^0 \chi = -P_0 \chi\), in accordance with the definition of \(\varphi, \chi\) in (17) and parity in (34). The STR Pauli Equation is the lowest order non-relativistic approximation to STR DE (52) (see also Appendix C). As anticipated in (19), following Feynman, we isolate the fast oscillating part of \(\psi\) as a common factor \(\rho = \rho(t)\) to \(\varphi\) and \(\chi\) in (52), leaving behind the nonrelativistic Pauli spinors proper \(\varphi_p, \chi_p\):

\[
\begin{align*}
(P_0 - m)\rho \varphi_p - P \rho \chi_p &= 0 \quad \rho = e^{-\frac{\text{int}}{\hbar}} \\
(P_0 + m)\rho \chi_p - P \rho \varphi_p &= 0 \\
\Rightarrow \quad (\hbar \partial_t + e A_0) \varphi_p - P \chi_p &= 0 \\
(\hbar \partial_t + e A_0 + 2m) \chi_p - P \varphi_p &= 0
\end{align*}
\]

For \(|(\hbar \partial_t + e A_0)\chi| \ll 2m|\chi|\) (nonrelativistic regime) the lower of the two coupled equations approximates in lowest order to: \(\chi_p \approx P \varphi_p / 2m\). I.e. for slow electrons \(|\chi_p| \ll |\varphi_p|\). Substituting into the upper equation one obtains the Pauli Hamiltonian \(H_p\) (below: \(PP = P \cdot P + P \wedge P = P \cdot P + he(\sigma, B)\), where: \(P \cdot P = (p + eA) \cdot (p + eA) = (-h(1)\nabla + eA) \cdot (-h(1)\nabla + eA) \equiv P^2\) is a grade 0, 5 operator):

\[
\hbar \partial_t \varphi_p = H_p \varphi_p = \left[ \frac{P^2}{2m} + e A_0 + \frac{he}{2m} (\sigma, B) \right] \varphi_p
\]
This is the STR Pauli Equation (PE), identical in form to the standard PE [15], but here without matrices and with a complex structure surging from the real vector space (10)! The term \((\hbar e/2m)(\sigma, B)\), which we met earlier in Eq. (31) marks the additional potential energy due to the spin magnetic moment of a slow electron. It distinguishes STR PE from the STR Schrödinger Equation [16], which is obtained from (54) by removing it (no spin) and by freezing the spinor \(\varphi\) let say to spin up. Of course, Eq. (31) can also be derived from (54).

**Final remarks and conclusions.**

The couple of Equations (52) is equivalent to DE in (16). By substituting \(\varphi, \chi\) from (21-22) into (52) one can write down four coupled first order partial differential equations in terms of \(\varphi_u, \varphi_d, \chi_u, \chi_d\) and with the \(e^\mu\) absent; these are equivalent to STR DE.

In conclusion, alike STA, STR promotes a geometric view of physics, where vectors and their Clifford combinations, not the scalar components set the complex structure. The definition of STR DE \(p = \hbar \mathcal{V} = \hbar e^\mu \partial_\mu\) shows its clear physical and geometric meaning as minimal path to quantization of the classical 4-momentum vector with modulus \(m\). Its demonstrated working hints to the expectation that all the formal machinery developed in nine+ decades to handle DE and its generalizations, adapt easily to the STR formalism. Spacetime-reflection stands on clear physical grounds in contrast to the fictional ‘internal degrees of freedom’ represented by \(\gamma\)-matrices; therefore, it becomes relevant to question the role of the same matrices in areas of modern physics assuming them as a fundamental representation. With its inborn distinction between (polar) boost and (axial) rotation/spin vectors, the STR formalism holds potential to spread to other areas of physics. Finally, by developing Dirac’s ideas, we recognize reflection / handedness as fundamental, side by side with space & time.

**Appendices**

**Appendix A. Commutation relations of boost and rotor generators and left and right rotors in STR**

We first calculate the commutator \([x, p]\) (below \(x = x_\mu e^\mu = x^\mu e_\mu; p = \hbar \mathcal{V} = \hbar e^\mu \partial_\mu; \mathcal{V} = e^{j0} \partial_{x_j} = x_j \partial/\partial x_j; \mathbf{L} = x \times p; (\sigma, \mathbf{L}) = \sigma_j x_j L_j x_j = L_j \sigma_j):\)

\[
[x, p] = \hbar (x \nabla - \nabla x) = \hbar (x \cdot \nabla - \nabla \cdot x + x \wedge \nabla - \nabla \wedge x) = \hbar (-(\nabla \cdot x) + 2x \cdot \nabla) = \hbar [-\partial_\mu(x^\mu) + 2e^{j0}(x_j \partial_0 - x_0 \partial_j) + 2e^{jk}(x_j \partial_k - x_k \partial_j)] = \hbar [-4 + 2x_j (x_0 \partial_{x_j} + x_j \partial_0) + 2e_{k\ell}l \sigma_l (x_j \partial_{x_k} - x_k \partial_{x_j})]
\]
\[ i\hbar [-4 + 2(x_0 \nabla + x \partial_0) + 2I(\sigma, (x \times \nabla))] = -4i\hbar + 2(x p_0 - x_0 p) - 2I(\sigma, L) = 4i\hbar [-1 + K]
\]
\[
K_j \left( x_j \partial_t + t \partial x_j \right) - \epsilon_{jkl} I_l \left( x_j \partial x_k - x_k \partial x_j \right) = 4i\hbar [\hbar K + J] \tag{1A}
\]

The commutator (1A) unites dimensionality (cumulative ‘uncertainty relation’), boost \( K \) and rotation \( J \) – the generators of proper Lorentz transformations, with \( K_j \equiv e^{j0}/2 \equiv x_j/2 \), \( J_l \equiv \epsilon_{jkl} e^{jk}/2 = -\epsilon_{jkl} \sigma_{jk}/2 = -J_l \sigma_l/2 \). \( K_j, J_l \) appear as directors for the Killing vectors, \( x_j \partial_t + t \partial x_j \) and \( x_j \partial x_k - x_k \partial x_j \) being components of the Killing vector in spacetime. Translation and rotation symmetries are the basic symmetries of spacetime. Notice the well-known similarity with the electric and magnetic fields from the potentials in (30).

The commutators of boost and rotor generators \( K_j, J_j \) from (1A) are:
\[
[J_j, J_k] = -\frac{1}{2} \sigma_{jk} = -\frac{1}{2} \epsilon_{jkl} l \sigma_l = \epsilon_{jkl} I_l: [K_j, K_k] = \frac{1}{2} x_{jk} = -\epsilon_{jkl} I_l: [J_j, K_k] = -\frac{1}{2} \hbar^5 \sigma_{jk} = \frac{1}{2} \epsilon_{jkl} x_l = 
\]
\[
\epsilon_{jkl} K_l \tag{2A}
\]

By the definition of \( J_j, K_j \) it is clear that they do not comprise \( e^5 \). The rotor-boost space can split into right and left handed disjoint subspaces, \( S_{+j} \) and \( S_{-j} \), illustrating the double coverage of the group \( SO(1,3) \) by the spinor group \( SU(2) \), i.e. \( SO(1,3) = SU(2) \otimes SU(2) \) (where = stands for isomorphic to):
\[
[S_{\pm j} \equiv 1/2 (J_j \pm iK_j)] = \frac{1}{2} J_j (1 \pm e^5) \Rightarrow \left\{ [S_{\pm j}, S_{\pm k}] = \frac{1}{4} \epsilon_{jkl} I_l (1 \pm e^5)^2 = \frac{1}{2} \epsilon_{jkl} I_l (1 \pm e^5) = \epsilon_{jkl} S_{\pm l}; [S_{+j}, S_{-k}] = 0 \right\}; \ 	ext{the two projectors} \ (1 \pm e^5) \text{ are orthogonal. Note the entrance of} e^5 \text{ into} S_{\pm j} \tag{3A}
\]

**Appendix B. Weyl spinors in STR.**

STR renders explicit the similarity between the chiral representation of rotors in (3A) and Right and Left handed Weyl spinors that are orthonormal \((1 \pm e^5)/2\) projections of Dirac spinors:
\[
\psi = \psi_L + \psi_R = \frac{1}{2} (1 - e^5) \psi + \frac{1}{2} (1 + e^5) \psi \tag{4A}
\]

Let see where this form is useful. The Dirac spinor and basis are preferred to describe slow electrons. For fast electrons, \( p \gg m \), we can neglect the mass term; then DE in momentum representation takes the form:
\[
(p - m) \psi(p) \rightarrow \overset{\text{pom}}{\overset{\text{pom}}{\rightarrow}} \text{p} \psi(p) \approx 0 \tag{5A}
\]

If \( \psi(p) \) is a solution to the STR DE at the r.h.s. then \( e^5 \psi(p) \) will also be a solution, \( e^5 \) anticommuting with \( p \). Now, in analogy to the Dirac spinor in (17) we write \( \psi(p) \) as:
\[
\psi(p) = \frac{1}{2} \left[ (1 + e^5) \psi(p) + (1 - e^5) \psi(p) \right] \equiv \psi_R(p) + \psi_L(p) \tag{6A}
\]
This form of the spinor is convenient to describe fast electrons or massless fermions.

The swap $e^0 \leftrightarrow e^5$ constitutes the passage from the Dirac ($e^0$ in the projectors) to Weyl basis ($e^5$ in the projectors). As mentioned in the main text in the discussion following Eq. (46) the change of basis in STR is achieved by two-sided generalized rotor transformations. Let illustrate it here in relation to the passage from Dirac to Weyl basis. It takes the form:

$$\{e^5\} \rightarrow \{e^5\}: e^{\tau} = R_{\omega} e^\tau R_{\omega}^{-1} = e^{e^{50}/2} e^{\tau} e^{-e^{50}/2} = e^{\tau} = e^{j}; \; \tau = 0, 1, 2, 3, 5; \; j = 1, 2, 3 \; (7A)$$

Within a sign (not relevant for a basis element) we have realized by (8A) the mentioned transformation of basis. Notice that $(e^{50})^2 = -1$, which makes exponentiation in (7A) meaningful.

**Appendix C. Derivation of Eq. (31).**

For convenience, let us rewrite the squared DE (30) here:

$$[\text{KG} + e\hbar I (E + I(\sigma, B))]\psi = 0 \; (8A)$$

If $\psi$ is a solution to DE then it is also a solution to its square, Eq. (8A). Now we follow the same strategy as earlier and try to write (8A) completely in the $\Sigma$ subspace with the help of the two Pauli spinors:

$$\psi = (1 + e^0)\varphi + (1 - e^0)\chi \; (8A) \implies (\text{KG} + e\hbar I (E + I(\sigma, B)))((1 + e^0)\varphi + (1 - e^0)\chi) =$$

$$\{[\text{KG} - e\hbar(\sigma, B)]\varphi + e\hbar I E\chi = 0 \}$$
$$\{[\text{KG} - e\hbar(\sigma, B)]\chi + e\hbar I E\varphi = 0 \} \; (9A)$$

The coupled equations (9A) are symmetric relative to the exchange $\varphi \leftrightarrow \chi$. We take the nonrelativistic limit of (9A). Under the derivation of the Pauli Equation we saw that in this limit $|\chi| \ll |\varphi|$. This is expected, as by writing DE in the rest frame of the electron in the momentum representation one finds $\chi = 0$. Therefore, in this regime we can take $\psi \approx \rho \varphi$ (this can be done also after removing the fast oscillations, which we do after (10A)) and the two equations in (9A) decouple:

$$\{[\text{KG} - e\hbar(\sigma, B)]\rho \varphi = 0 \}$$
$$\{ e\hbar I(\sigma, E)\rho \varphi = 0 \} \; (10A)$$

In the following we will look only at the top equation. As done earlier we isolate the fast oscillations in $\rho = e^{-i\omega t}/\hbar$, so that second derivative $\partial_x^2 \varphi$ of $\varphi$ from the KG term can be ignored after differentiating $\rho$ and removing the exponential. We also assume a weak constant EM field and in this way can also ignore the $A^2$ terms. These simplifications will make treatment easier allowing to focus on few essential properties of DE.
\[ (P \cdot P - m^2 - e\hbar(\sigma, B))\rho \varphi = \left( -\hbar^2 \partial_x^2 - p^2 + e^2 (A^2 - A_x^2) + e\hbar \left( 2A_0 \partial_0 + (\partial_0 A_0) - 2A_j \partial_j - (\partial_j A_j) \right) \right) - m^2 - e\hbar(\sigma, B) \exp(-imt/\hbar) \approx \exp(-imt/\hbar) \left( -\hbar^2 \left( -2 \frac{\text{i} m \partial \varphi}{\hbar} - \frac{m^2}{\hbar^2} + \frac{\partial^2 \varphi}{\partial x^2} \right) - p^2 + e\hbar \left( -2A \frac{\text{i} m}{\hbar} + (\partial_t A_0) - 2A_j \partial_j - (\partial_j A_j) \right) - m^2 - e\hbar(\sigma, B) \right) \varphi \Rightarrow \left[ I \hbar \partial_\tau - \frac{p^2}{2m} + eA_0 - \frac{e\hbar}{2m} \left[ 2A_j \partial_j + (\partial_j A_j) \right] \right] \varphi \approx 0 \] 

The last equation can be derived from the Pauli Equation (54) in the main text, as well. Now, we know that the Schrödinger Equation [16] can handle magnetic orbital momentum, which must be the term within square brackets in (11A). Let us render it explicit by trying to express \( A_j \) as a function of the components of the magnetic field \( B_k \). Start with \( B = \nabla \times A \Rightarrow B_j = (\partial_x A_y - \partial_y A_x) \) with indices varying cyclically \( \text{mod}3 \), e.g. \( B_2 = \partial_3 A_1 - \partial_1 A_3 \). The brackets in e.g. \( (\partial_j A_j) \) mean that the derivative operates only to that term. Taking \( A_j = (B_{j+1} x_{j+2} - B_{j+2} x_{j+1})/2 \), or \( 2A = B \times x \) satisfies the equation for \( B \), remembering that the \( B_j \) are constant. With this choice the term \( (\partial_j A_j) = 0 \) in (11A) and \( 2\hbar I \partial_j = 2A \cdot \) 

\[ p = B \times x \cdot p = B \cdot x \times p \equiv B \cdot L, \] 

where \( L = x \times p \) is the orbital angular momentum. The Pauli spin vector being \( S = \hbar \sigma / 2 \), we put it all together into the last equation in (11A) and obtain Eq. (31) from the main text: 

\[ \left[ I \hbar \partial_\tau - \frac{p^2}{2m} + eA_0 - \frac{e\hbar}{2m} B \cdot (L + 2S) \right] \varphi = 0 \] 

(12A)

Appendix D. Lagrangian of STR DE and orthogonality of direct vs. time-reversed STR Dirac spinors.

The Lagrangian in STR is (the last equality shows its invariance under CPT transformation): 

\[ \mathcal{L} = \bar{\psi} (p - m) \psi = \bar{\psi} (I \hbar \nabla - m) \psi = \bar{\psi} (Ihe^\mu \partial_\mu - m) \psi = \bar{\psi} e^5 (-Ihe^\mu \partial_\mu - m) e^5 \psi = \mathcal{L}_{CPT} \] 

(13A)

As in the standard case \( \bar{\psi} \) and \( \psi \) are independent and one can recover STR DE from the field equations most directly by differentiating \( \mathcal{L} \) in (13A) with respect to \( \partial_\mu \bar{\psi} \) and \( \bar{\psi} \) obtaining (\( \nabla \) in (14A) acts to the right): 

\[ \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}} - \frac{\delta \mathcal{L}}{\delta \bar{\psi}} = 0 \Rightarrow \frac{\delta \mathcal{L}}{\delta \bar{\psi}} = (I \hbar \nabla - m) \psi = 0 \] 

(14A)

The conservation of probability currents follows from the gauge symmetry of the Lagrangian and Noether’s theorem. Gauge symmetry: \( \{ \psi \rightarrow e^{i \theta} \psi; \mathcal{L} \rightarrow \mathcal{L} \} \Rightarrow \partial_\mu \bar{\psi} e^\mu \psi = 0 \) 

(15A)

**Orthogonality** of Dirac spinor \( \psi = (1 + e^0) \psi + (1 - e^0) \chi \equiv (1 \pm e^0) \psi_{\pm} \); \( \psi_{\pm} = \rho_{\psi_{\pm}} R_{\psi_{\pm}} (1 + \sigma_3) \) vs. time-reversed spinors (for ease of selection of terms I explicate the four projectors – not strictly needed).
\[ \mathcal{T}(\omega) : \psi \rightarrow T(\omega) \psi = \begin{cases} K_{\pm} \psi = e^{0} K_{\pm} \psi = e^{0} (1 \pm e^{0}) \psi_{\pm} \pm e^{0} \psi_{\mp}, & \text{for } \omega = 5 \\ I_{\pm} K_{\pm} \psi = I_{\pm} (1 \pm e^{0}) \psi_{\pm}, & \text{for } \omega = 1 \\ I_{\pm} K_{\pm} \psi = I_{\pm} (1 \pm e^{0}) \psi_{\pm}, & \text{for } \omega = 2 \\ I_{\pm} K_{\pm} \psi = I_{\pm} (1 \pm e^{0}) \psi_{\pm}, & \text{for } \omega = 3 \end{cases} \] (16A)

Now we can prove the orthogonality relations (below for \( \mathbf{I} \mathbf{\sigma} = e^{0}, I \mathbf{\sigma} \), \( \mathbf{I} \mathbf{\sigma} \), \( \mathbf{I} \mathbf{\sigma} \) for \( \omega = 5, 1, 2, 3 \), respectively):

\[ \langle \psi T(\omega) \psi \rangle = \psi_{\pm} (1 \pm e^{0}) e^{0} I_{\pm} \mathbf{\sigma}_{\omega} (1 \pm e^{0}) \psi_{\pm}^{K_{\omega}} = \langle \varphi_{\pm}^{+} I_{\pm} \mathbf{\sigma}_{\omega} \varphi_{\pm}^{K_{\omega}} \pm \chi_{\pm}^{+} I_{\pm} \mathbf{\sigma}_{\omega} \chi_{\omega}^{K_{\omega}} \rangle; \quad \varphi_{\pm}^{+} I_{\pm} \mathbf{\sigma}_{\omega} \varphi_{\pm}^{K_{\omega}} = (1 + \mathbf{\sigma}_{3}) \rho_{\varphi}^{R_{\varphi}}(1 - \mathbf{\sigma}_{3}) = 0, \text{ which completes the proof.} \] (17A)

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**Figures**

*Figure 1.* Effect of conjugations $K_0, K_3, K_f, K_5$ onto the STR directors. The transformed frames and operators are related as shown by the two-sided arrows. From $(29-32)$, $T = K_f; T' = l_03 K_3; T'' = K_5$; we add here $T''' = e^{123} K_0$. All $T, T', T'', T'''$ invert $I$ and thereby time; of these only $T'', T'''$ keep 3-position unchanged and invert 3-momentum. However, as shown above, the Hamiltonian $(33)$ preserves form under $t$-reversal when DE transforms by $T = K_f$, or $T' = l_03 K_3$; we accept these by convention to stand for $t$-reversal.

The Figure has the legend attached.