Computational Complexity of Multi-Player Evolutionarily Stable Strategies*

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Abstract

In this paper we study the computational complexity of computing an evolutionary stable strategy (ESS) in multi-player symmetric games. For two-player games, deciding existence of an ESS is complete for $\Sigma^p_2$, the second level of the polynomial time hierarchy. We show that deciding existence of an ESS of a multi-player game is closely connected to the second level of the real polynomial time hierarchy. Namely, we show that the problem is hard for a complexity class we denote as $\exists^D \cdot \forall^R$ and is a member of $\exists^R \forall^R$, where the former class restrict the latter by having the existentially quantified variables be Boolean rather than real-valued. As a special case of our results it follows that deciding whether a given strategy is an ESS is complete for $\forall^R$.

A concept strongly related to ESS is that of a locally superior strategy (LSS). We extend our results about ESS and show that deciding existence of an LSS of a multiplayer game is likewise hard for $\exists^D \cdot \forall^R$ and a member of $\exists^R \forall^R$, and as a special case that deciding whether a given strategy is an LSS is complete for $\forall^R$.

1 Introduction

First introduced by Maynard Smith and Price in [MP73; May74], a central concept emerging from evolutionary game theory is that of an evolutionary stable strategy (ESS) in a symmetric two-player game in strategic form. Each pure strategy of the game is viewed as a type of possible individuals of a population. A mixed strategy of the game then corresponds to describing the proportion of each type of individual of the population, which as a simplifying assumption is considered to be infinite. The population is engaged in a pairwise conflict where two individuals are selected at random and receive payoffs depending on their respective types. The population is expected to evolve in a way where strategies that achieve a higher payoff than others will spread in the population. A strategy $\sigma$ is an ESS if it outperforms any “mutant” strategy $\tau \neq \sigma$ adopted by a small fraction of the population. Otherwise we say that $\sigma$ may be invaded. An ESS is in particular a symmetric Nash equilibrium (SNE), but, unlike a SNE, it is not guaranteed to exist.

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The Hawk-Dove game [MP73], presented with concrete payoffs in Fig. 1, is a classic example where an ESS may explain the proportion of the population tending to engage in aggressive behavior. The game has a unique SNE $\sigma$, where the players choose Hawk with probability $\frac{1}{2}$, and this is in fact an ESS. Note first that $u(\sigma, \sigma) = (-1) \left( \frac{1}{2} \right)^2 + 2 \left( \frac{1}{2} \right)^2 + 0 \left( \frac{1}{2} \right)^2 + 1 \left( \frac{1}{2} \right)^2 = \frac{1}{2}$. Consider now any strategy profile $\tau$ that chooses Hawk with probability $p$. Then $u(\tau, \sigma) = (-1 + 2)p/2 + (1 + 0)(1 - p)/2 = \frac{1}{2}$ as well. However, $u(\sigma, \tau) = \frac{1}{2} - 2p$ and $u(\tau, \tau) = 1 - 2p^2$, and thus $u(\sigma, \tau) - u(\tau, \tau) = 2(p - \frac{1}{2})^2$, which means that $\sigma$ outperforms $\tau$ if $p \neq \frac{1}{2}$.

While the two-player setting is the typical setting to study ESS, the concept may in a natural way be generalized to the setting of multi-player games, as established by Palm [Pal84] and Broom, Cannings, and Vickers [BCV97]. This allows one to model populations that engage in conflicts involving more than two individuals. Many of the two-player games typically studied in the context of ESS readily generalize to multi-player games, including the Hawk-Dove and Stag Hunt games (cf. [BR13]). For a naturally occurring example, Broom and Rychtář [BR13, Example 9.1] argue that the cooperative hunting method of carousel feeding by killer whales may be modeled as a multi-player Stag Hunt game.

The computational complexity of computing an ESS was first studied by Etessami and Lochbihler [EL07]. We shall denote the problem of deciding whether a given symmetric game in strategic form has an ESS as $\exists \text{ESS}$ and similarly the problem of deciding whether a given strategy is an ESS of the given game as IsESS. Previous work has been concerned only with two-player symmetric games in strategic form. Etessami and Lochbihler proved that $\exists \text{ESS}$ is hard both for NP and coNP and is contained in $\Sigma^p_2$. Nisan [Nis06] showed that $\exists \text{ESS}$ is hard for the class coDP, which is the class of unions of languages from NP and coNP. From both works it also follows that the problem IsESS is coNP-complete. Finally Conitzer [Con19] showed $\Sigma^p_2$-completeness for $\exists \text{ESS}$. The direct but important consequence of these results is that any algorithm for computing an ESS in a general game can be used to solve $\Sigma^p_2$-complete problems. For instance, we cannot expect to be able to compute an ESS in a simple way using a SAT solver.

One may observe that the above hardness results for two-player games also generalize to apply to $m$-player games, for any fixed $m \geq 3$. Note that, since a reduction showing $\Sigma^p_2$-hardness must produce an $m$-player symmetric game, this is not a trivial observation (in particular adding "dummy" players, each having a single strategy, to a nontrivial symmetric game would result in a non-symmetric game). One would however suspect that the problems $\exists \text{ESS}$ and IsESS become significantly harder for $m$-player games, when $m \geq 3$. Namely, starting with the work of Schaefer and Štefankovič [SS17], several works have shown that many natural decision problems concerning Nash equilibrium (NE) in 3-player strategic form games are $\exists \Re$-complete [Gar+18; BM16; BM17; Han19; BH19]. These results stand in contrast to the two-player setting, where the same decision problems are NP-complete [GZ89; CS08]. The class $\exists \Re$ is the complexity class that captures the decision problem for the existential the-
ory of the reals [ŚŚ17], or alternatively, is the constant-free Boolean part of the real analogue NP in the Blum-Shub-Smale model of computation [BC09]. Clearly we have NP ⊆ ∃R, and from the decision procedure for the existential theory of the reals by Canny [Can88] it follows that ∃R ⊆ PSPACE. We consider it likely that NP is a strict subset of ∃R, which would mean that the above mentioned decision problems concerning NE become strictly harder as the number of players increase beyond two.

We confirm that the problems ∃ESS and IsESS indeed are likely to become harder for multi-player games by proving hardness of the problems for discrete complexity classes defined in terms of real complexity classes that we consider likely to be stronger than Σp2 and NP. Our results are perhaps most easily stated in terms of the decision problem for the first order theory of the reals Th(R). Just like the class ∃R corresponds to the existential fragment Th∃(R) of Th(R), we can consider classes ∀R and ∃∀R corresponding to the universal fragment Th∀(R) and the existential-universal fragment Th∃∧∀(R) of Th(R), respectively. It is easy to see that the problem ∃ESS belongs to ∃∀R and that IsESS belongs to ∀R. We show that for 5-player games, the problem ∃ESS is hard for the subclass of ∃∀R where the block of universal quantifiers is restricted to range over Boolean variables. For the problem IsESS we completely characterize its complexity for 5-player games by proving that the problem is also hard for ∀R. Our hardness results thus imply that any algorithm for computing an ESS in a 5-player game can be used to solve quite general problems involving real polynomials. In particular it indicates that computing an ESS is significantly more difficult than deciding if a system of real polynomials has no solution, which is a basic problem complete for ∀R.

Our proof of hardness for ∃ESS combines ideas of the Πp2-completeness proof of the problem MINMAXCLIQUE by Ko and Lin [KL95], the reduction from the complement of MINMAXCLIQUE to ∃ESS for two-player games by Conitzer [Con19], and the direct translation of solutions of a polynomial system to strategies of a game by Hansen [Han19], in addition to new ideas.

A strongly related concept to an ESS is that of a locally superior strategy (LSS) which is equivalent to an ESS having a uniform invasion barrier [Pal84]. For the case of two-player games these concepts coincide [HSS79], but they differ for multi-player games [Mil08]. Analogously to the case of ESS we consider the two computational problems ∃LSS and IsLSS and prove the same results for these as for ∃ESS and IsESS.

We leave the problem of determining the precise computational complexity of ∃ESS and ∃LSS as an interesting open problem. The class ∃∀R is the natural real complexity class generalization of Πp2. Together with Σp2-completeness of ∃ESS for the setting of two-player games, this might lead one to expect that ∃ESS should be ∃∀R-hard for multi-player games. However, a basic property of the set of evolutionary stable strategies is that any ESS is an isolated point in the space of strategies [AMO19, Proposition 3], which means that the set of evolutionary stable strategies is always a discrete set. Expressing ∃ESS in Th∃∧∀(R), the universal quantifier range over all potential ESS and the existential quantifier over potential invading strategies. The fact that the set of ESS is a discrete set could possibly mean that the universal quantifier could be made discrete as well. We also note that we do not even know whether ∃ESS is hard for ∃R, which is clearly a prerequisite for ∃∀R-hardness.
2 Preliminaries

2.1 Strategic Form Games

We present here basic definitions concerning strategic form games, mainly to establish our notation. A finite $m$-player strategic form game $G$ is given by finite sets $S_1, \ldots, S_m$ of actions (pure strategies) together with utility functions $u_1, \ldots, u_m : S_1 \times \cdots \times S_m \to \mathbb{R}$. A choice of an action $a_i \in S_i$ for each player together form a pure strategy profile $a = (a_1, \ldots, a_m)$. Let $\Delta(S_i)$ denote the set of probability distributions on $S_i$. A (mixed) strategy for player $i$ is then an element $\pi_i \in \Delta(S_i)$. We may conveniently identify an action $a_i$ with the strategy that assigns probability 1 to $a_i$. A strategy $x_i$ for each player $i$ together form a strategy profile $x = (x_1, \ldots, x_m)$. For fixed $i$ we denote by $x_{-i}$ the partial strategy profile $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ for all players except player $i$, and if $x'_i \in \Delta(S_i)$ we denote by $(x'_i, x_{-i})$ the strategy profile $(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_m)$. The utility functions extend to strategy profiles by letting $u_i(x) = E_{\pi_{-i} \sim} u_i(a_1, \ldots, a_m)$. We shall also refer to $u_i(x)$ as the payoff of player $i$. A strategy profile $x$ is a Nash equilibrium (NE) if $u_i(x') \geq u_i(x)$ for all $i$ and all $x'_i \in \Delta(S_i)$. Every finite strategic form game $G$ has an NE [Nas51].

In this paper we shall only consider symmetric games. The game $G$ is symmetric if all players have the same set $S$ of actions and where the utility function of a given player depends only on the action of that player (not the identity of the player) together with the multiset of actions of the other players. More precisely we say that $G$ is symmetric if there is a finite set $S$ such that $S_i = S$, for every $i \in [m]$, and such that for every permutation $\pi$ on $[m]$, every $i \in [m]$ and every $(a_1, \ldots, a_m) \in S^m$ it holds that $u_i(a_1, \ldots, a_m) = u_{\pi^{-1}(i)}(a_{\pi(1)}, \ldots, a_{\pi(m)})$. It follows that a symmetric game $G$ is fully specified by $S$ and $u$; for simplicity we let $u = u_1$. A strategy profile $x = (x_1, \ldots, x_m)$ is symmetric if $x_1 = \cdots = x_m$. If a symmetric strategy profile $x$ is an NE it is called a symmetric NE (SNE). Every finite strategic form symmetric game $G$ has a SNE [Nas51].

A single strategy $\sigma \in \Delta(S)$ defines the symmetric strategy profile $\sigma^m$. More generally, given $\sigma, \sigma_1, \ldots, \sigma_r \in \Delta(S)$ and $m_1, \ldots, m_r \geq 1$ with $m_1 + \cdots + m_r = m - 1$, we denote by $(\sigma; \sigma_1^{m_1}, \ldots, \sigma_r^{m_r})$ a strategy profile where player 1 is playing using strategy $\sigma$ and $m_i$ of the remaining players are playing using strategy $\sigma_i$, for $i = 1, \ldots, r$. By the assumptions of symmetry, the payoff $u(\sigma; \sigma_1^{m_1}, \ldots, \sigma_r^{m_r})$ is well defined.

2.2 Evolutionary Stable Strategies

Our main object of study is the notion of evolutionary stable strategies as defined by Maynard Smith and Price [MP73] for 2-player games and generalized to multi-player games by Palm [Pal84] and Broom, Cannins, and Vickers [BCV97]. We follow below the definition given by Broom et al.

**Definition 1.** Let $G$ be a symmetric game given by $S$ and $u$. Let $\sigma, \tau \in \Delta(S)$. We say that $\sigma$ is evolutionary stable (ES) against $\tau$ if there is $\varepsilon > 0$ such that for all $0 < \varepsilon < \varepsilon$ we have

$$u(\sigma; \tau^{m-1}) > u(\tau; \tau^{m-1}),$$

where $\tau_\varepsilon = \varepsilon \tau + (1 - \varepsilon)\sigma$ is the strategy that plays according to $\tau$ with probability $\varepsilon$ and according to $\sigma$ with probability $1 - \varepsilon$. We say that $\sigma$ is an evolutionary stable strategy (ESS) if $\sigma$ is ES against every $\tau \neq \sigma$. If $\sigma$ is not ES against $\tau$ we also say that $\tau$ invades $\sigma$. 

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While we are mainly interested in the computational complexity of discrete problems, it may simply define this class in terms of a restriction of the decision problem for the first-order theory of the reals, as explained in the next subsection. The reader may thus defer reading this subsection.

A concept strongly related to evolutionary stable strategies is that locally superior strategies.

**Definition 2.** A strategy $\sigma$ is a locally superior strategy (LSS) if there exists $\epsilon_\sigma > 0$ such that $u(\sigma; \tau^m) > u(\tau^m)$ for all $\tau$ satisfying $0 < \|\sigma - \tau\|_1 < \epsilon_\sigma$.

It is easy to see that $\sigma$ is locally superior if and only if $\sigma$ is an ESS with a uniform invasion barrier [Pal84]. Indeed, with $\tau_\epsilon$ as in Definition 1 we have $u(\tau^m) = \epsilon u(\tau; \tau^m_\epsilon) + (1 - \epsilon)u(\sigma; \tau^m_\epsilon)$, from which it follows that for $0 < \epsilon < 1$ we have $u(\sigma; \tau^m_\epsilon) > u(\tau; \tau^m_\epsilon)$ if and only if $u(\sigma; \tau^{m-1}_\epsilon) > u(\tau^{m-1}_\epsilon)$.

Since as stated, in the case of 2-player games any ESS has a uniform invasion barrier, it follows that the notions of ESS and LSS coincide for 2-player games.

**2.4 Real Computational Complexity**

While we are mainly interested in the computational complexity of discrete problems, it is useful to discuss a model of computation operating on real-valued input. We use this to define the complexity class $\mathbb{F}^D \cdot \forall \mathbb{R}$, used to formulate our main result. Alternatively we may simply define this class in terms of a restriction of the decision problem for the first-order theory of the reals, as explained in the next subsection. The reader may thus defer reading this subsection.

A standard model for studying computational complexity in the setting of reals is that of Blum-Shub-Smale (BSS) machines [BSS89]. A BSS machine takes a vector $x \in \mathbb{R}^n$ as an input and performs arithmetic operations and comparisons at unit cost. In addition the machine may be equipped with a finite set of real-valued machine constants. In this way a BSS machine accepts a real language $L \subseteq \mathbb{R}^n$, where $\mathbb{R}^n = \bigcup_{k \geq 0} \mathbb{R}^{nk}$. Imposing polynomial time bounds we obtain the complexity classes P$_\mathbb{R}$ and NP$_\mathbb{R}$ for deterministic and nondeterministic BSS machines, respectively, forming real-valued analogues of P and NP. Cucker [Cuc93] defined the real analogue PH$_\mathbb{R}$.
of the polynomial time hierarchy formed by the classes $\Sigma^R_k$ and $\Pi^R_k$, for $k \geq 1$. The class $\Sigma^R_{k+1}$ may be defined as real languages accepted by a nondeterministic oracle BSS machine in polynomial time using an oracle language from $\Sigma^R_k$ with $\Sigma^R_1 = \text{NP}_R$, and $\Pi^R_k$ is simply the class of complements of languages of $\Sigma^R_k$. For natural problems such as TSP or KNAPSACK with real-valued input the search space remains discrete. Goode [Goo94] introduced the notion of digital nondeterminism (cf. [CM96]) restricting nondeterministic guesses to the set $\{0,1\}$, which when imposing polynomial time bounds define the class DNP$_R$. One may also define a polynomial hierarchy based on digital nondeterminism giving rise to classes $\Sigma^R_k$ and $\Pi^R_k$, for $k \geq 1$.

Another convenient way to define the classes described above is by means of complexity class operators (cf. [Zac86; BS01]). Here we shall consider existential or universal quantifiers over either real-valued or Boolean variables whose number is bounded by a polynomial. For a real complexity class $\mathcal{C}$, define $\exists^R \mathcal{C}$ as the class of real languages $L$ for which there exists $L' \in \mathcal{C}$ and a polynomial $p$ such that $x \in L$ if and only if $\exists y \in [0,1]^p : (x,y) \in L'$. For a (real) complexity class $\mathcal{C}$, define $\exists^D \cdot \mathcal{C}$ as the class of real (or discrete) languages $L$ for which there exists $L' \in \mathcal{C}$ and a polynomial $p$ such that $x \in L$ if and only if $\exists y \in [0,1]^p : (x,y) \in L'$. Replacing existential quantifiers with universal quantifiers we analogously obtain definitions of classes $\forall^R \cdot \mathcal{C}$ and $\forall^D \cdot \mathcal{C}$. We now have that $\Sigma^R_{k+1} = \exists^R \cdot \Pi^R_k$, $\Sigma^D_{k+1} = \exists^D \cdot \Pi^R_k$, as well as $\Sigma^R_k \cap 1 = \exists^D \cdot \Pi^R_k$, for $k \geq 1$. We shall also consider mixing real and discrete operators. In such cases one may not always have an equivalent definition in terms of oracle machines. For instance, while $\exists^R \cdot \text{coNP}_R = \text{NP}_R$ we can only prove the inclusion $\exists^D \cdot \text{coNP}_R \subseteq \text{DNP}^R$ and in particular we do not know if $\text{NP}_R \subseteq \exists^D \cdot \text{coNP}_R$.

To study discrete problems we define the Boolean part of a real language $L \subseteq \mathbb{R}^*$ as $\text{BP}(L) = L \cap \{0,1\}^\ast$ and of real complexity classes $\mathcal{C}$ as $\text{BP}(\mathcal{C}) = \{ \text{BP}(L) \mid L \in \mathcal{C} \}$. The Boolean part of a real complexity class is thus a discrete complexity class and may be compared with other discrete complexity classes defined for instance using Turing machines. Furthermore, since we are interested in uniform discrete complexity we shall disallow machine constants. Indeed, a single real number may encode an infinite sequence of discrete advice strings, which for instance implies that $\text{P/poly} \subseteq \text{BP}(\text{P})$. For a class $\mathcal{C}$ defined above we denote by $\mathcal{C}^0$ the analogously defined class without machine constants. Several classes given by Boolean parts of constant-free real complexity are defined specifically in the literature. Most prominently is the class $\text{BP}(\text{NP}_R^0)$ which also captures the complexity of the existential theory of the reals. It has been named $\exists^R$ by Schaefer and Štefankovič [SS17] as well as NPR by Bürgisser and Cucker [BC09]; we shall use the former notation $\exists^R$. We further let $\forall^R = \text{BP}(\text{coNP}_R^0)$ as well as $\exists^R \forall^R = \text{BP}(\exists^R \forall^R)$, $\exists^R \forall^R = \text{BP}(\Pi^R_1)$, $\forall^R \exists^R = \text{BP}(\Pi^R_2)$, $\forall^R \exists^R$ and $\forall^R \Pi^R_1$. We shall in particular be interested in the class $\exists^D \cdot \forall^R$. Clearly, from the definitions above we have that this class contains both the familiar classes $\forall^R$ and $\Sigma^D_2$ and is itself contained in $\forall^R$. In fact $\exists^D \cdot \forall^R$ contains the class $(\Sigma^D_2)_{\text{PosSLP}}$, where PosSLP is the problem of deciding whether an integer given by a division free arithmetic circuit is positive, as introduced by Allender et al. [All+09]. This follows since $\text{P}^{\text{PosSLP}} = \text{BP}(\text{P}^0_R)$ [All+09, Proposition 1.1], and thus

$$
(\Sigma^D_2)^{\text{PosSLP}} = \exists^D \cdot \forall^D \cdot \text{PosSLP} = \exists^D \cdot \forall^D \cdot \text{BP}(\text{P}^0_R) \\
\subseteq \exists^D \cdot \text{BP}(\forall^R \cdot \text{P}^0_R) = \exists^D \cdot \text{BP}(\text{coNP}^0_R) = \exists^D \cdot \forall^R.
$$
2.5 The First-Order Theory of the Reals

The discrete complexity classes BP($\Sigma^0_k$) and BP($\Pi^0_k$) may alternatively be characterized using the decision problem for the first-order theory of the reals. We denote by $\text{Th}(\mathbb{R})$ the set of all true first-order sentences over the reals. We shall consider the restriction to sentences in prenex normal form

$$(Q_1 x_1 \in \mathbb{R}^{m_1}) \cdots (Q_k x_k \in \mathbb{R}^{m_k}) \varphi(x_1, \ldots, x_k),$$

where $\varphi$ is a quantifier free Boolean formula of equalities and inequalities of polynomials with integer coefficients, where each $Q_i$ is one of the quantifiers $\exists$ or $\forall$, typically alternating, and gives rise to $k$ blocks of quantified variables. The restriction of $\text{Th}(\mathbb{R})$ to formulas in prenex normal form with $k$ being a fixed constant and also $Q_1 = \exists$ is complete for BP($\Sigma^0_k$); when instead $Q_1 = \forall$ it is complete for BP($\Pi^0_k$). In particular, the existential theory of the reals $\text{Th}_\exists(\mathbb{R})$, where $k = 1$ and $Q_1 = \exists$, is complete for $\exists \mathbb{R}$. Similarly $\text{Th}_{\forall \exists}(\mathbb{R})$ where $k = 2$ and $Q_1 = \forall$ is complete for $\forall \exists \mathbb{R}$; when we furthermore restrict the first quantifier block to Boolean variables the problem becomes complete for $\exists \mathbb{R} \forall \mathbb{R}$.

2.6 Real Polynomials with Discrete Quantification

In this section we shall prove that the following problem, $\forall \mathbb{D} \text{HOM4FEAS}(\Delta)$, is complete for the complexity class $\forall \mathbb{D}, \exists \mathbb{R}$. In Section 3 and Section 4 we use the complement of this problem to prove our main results of $\exists \mathbb{D} \forall \mathbb{R}$-hardness of $\exists \text{ESS}$ and of $\exists \text{LSS}$.

Denote by $\Delta^n \subseteq \mathbb{R}^{n+1}$ the $n$-simplex $\{ x \in \mathbb{R}^{n+1} | x \geq 0 \land \sum_{i=1}^{n+1} x_i = 1 \}$ and similarly by $\Delta^n_c \subseteq \mathbb{R}^n$ the corner $n$-simplex $\{ x \in \mathbb{R}^n | x \geq 0 \land \sum_{i=1}^n x_i \leq 1 \}$.

**Definition 3** ($\forall \mathbb{D} \text{HOM4FEAS}(\Delta)$). For the problem $\forall \mathbb{D} \text{HOM4FEAS}(\Delta)$ we are given as input rational coefficients $a_{i,\alpha}$, where $i \in \{0, \ldots, n\}$ and $\alpha \in [m]^4$, forming the polynomial

$$F(y; z) = F_0(z) + \sum_{i=1}^n y_i F_i(z),$$

where

$$F_i(z) = \sum_{\alpha \in [m]^4} a_{i,\alpha} \prod_{j=1}^4 z_{a_j}, \quad \text{for } i = 0, \ldots, n.$$

We are to decide whether for all $y \in \{0,1\}^n$ there exists $z \in \Delta^{n-1}$ such that $F(y; z) = 0$.

Analogously to the fact that a matrix representing a quadratic form may be assumed to be symmetric, we may assume that the polynomials of Definition 3 are symmetrized.

**Definition 4.** For $\alpha \in [m]^4$ and a permutation $\pi$ on $[4]$, define $\pi \cdot \alpha \in [m]^4$ by $(\pi \cdot \alpha)_i = \alpha_{\pi(i)}$. We say that a homogenous polynomial $G$ given in the form

$$G(z) = \sum_{\alpha \in [m]^4} b_{\alpha} \prod_{j=1}^4 z_{a_j},$$

is symmetrized if $b_{\alpha} = b_{\pi \cdot \alpha}$ for all $\alpha$ and $\pi$.

**Lemma 2.** Any homogeneous polynomial $G$ in the form of Equation (3) is, as a function, equal to a symmetrized homogeneous polynomial $H$ in the same form.
Proof. Let \( c_\alpha = \frac{1}{2^n} \sum_x b_{\alpha} \) for all \( \alpha \), and define \( H \) by

\[
H(z) = \sum_{\alpha \in [m]^4} c_\alpha \prod_{j=1}^4 z_{\alpha_j}.
\]

Then \( H \) is clearly symmetrized and it holds that \( G(z) = H(z) \) for all \( z \).

We next turn to the proof of \( \forall^D \cdot \exists^R \)-hardness of \( \forall^D \text{HOM4FEAS}(\Delta) \). The proof is mainly a combination of existing ideas and proofs, and the reader may thus defer reading it.

**Theorem 1.** The problem \( \forall^D \text{HOM4FEAS}(\Delta) \) is complete for \( \forall^D \cdot \exists^R \), and remains \( \forall^D \cdot \exists^R \)-hard even with the promise that for all \( y \in \{0,1\}^n \) and \( z \in \mathbb{R}^m \) it holds that \( F(y,z) \geq 0 \).

**Proof.** We shall prove hardness of \( \forall^D \text{HOM4FEAS}(\Delta) \) by describing a general reduction from a language \( L \) in \( \forall^D \cdot \exists^R \) in several steps making use of reductions that proves several problems involving real polynomials \( \exists^R \)-hard. Consider first the standard complete problem QUAD for \( \exists^R \) which is that of deciding if a system of multivariate quadratic polynomials have a common root [Blu+98; SŠ17]. The general reduction from a language \( L \) in \( \exists^R \) to QUAD works by treating the input \( x \) as variables and computes, based only on \( |x| \) and not the actual value of \( x \), a system of quadratic polynomials \( q_i(x,y), i = 1, \ldots, \ell \), where \( y \in \mathbb{R}^{p(|x|)} \) for some polynomial \( p \). The system has the property that for all \( x \) it holds that \( x \in L \) if and only if there exists \( y \) such that \( q_i(x,y) = 0 \), for all \( i \).

Suppose now that \( L \in \forall^D \cdot \exists^R \). Then there is \( L' \) in \( \exists^R \) and a polynomial \( p \) such that \( x \in L \) if and only if \( \forall y \in \{0,1\}^{p(|x|)} : \langle x, y \rangle \in L' \). On input \( x \) we may apply the reduction from \( L' \) to QUAD and in this way obtain a system of quadratic equations \( q_i(x,y,z), i = 1, \ldots, \ell \) where \( z \in \mathbb{R}^{p(|x|)} \), for some polynomial \( p_1 \), such that \( \langle x, y \rangle \in L' \) if and only if there exists \( z \) \( \in \mathbb{R}^{p(|x|)} \) such that \( q_i(x,y,z) = 0 \) for all \( i \). At this point we may just treat \( x \) as fixed constants, and we view the system as polynomials in variables \( (y,z) \), suppressing the dependence on \( x \) in the notation. Define \( n = p(|x|) \). We next introduce additional existentially quantified variables \( w \in \mathbb{R}^m \), substitute \( w_i \) for \( y_i \) in all polynomials, and then add new polynomials \( w_0 - y_i \), for \( i \in [n] \). Renaming polynomials and bundling the existentially quantified variables we now have a system of quadratic polynomials \( q_i(y,z), i \in [\ell_2] \) where \( z \in \mathbb{R}^{m_2} \) and \( m_2 \leq p_2(|x|) \) for some polynomial \( p_2 \), such that \( x \in L \) if and only if

\[
\forall y \in \{0,1\}^n \exists z \in \mathbb{R}^{m_2} \forall i \in [\ell_2] : q_i(y,z) = 0,
\]

and where each polynomial \( q_i \) depends on at most 1 coordinate of \( y \).

For the next step we use that QUAD remains \( \exists^R \)-hard when asking for a solution in the unit ball [Sch10], or analogously in the corner simplex [Han19]. Applying the reduction of [Han19, Proposition 2] we first rewrite each variable \( z_t \) as a difference \( z_t = z_t^+ - z_t^- \) of two non-negative real variables \( z_t^+ \) and \( z_t^- \) and then introduce additional existentially quantified variables \( w_0, \ldots, w_t \) for suitable \( t = O(\log \tau + m_2) \), where \( \tau \) is the maximum bitlength of the coefficients of the given system. Then polynomials are added that together implement \( t \) steps of repeated squaring of \( \frac{1}{2} \), i.e. we add polynomials \( w_t - \frac{1}{2} \), and \( w_{j-1} - w_j^2 \), for \( j \in [t] \), which means that any solution must then have \( w_0 = 2^{-\tau/2} \).
In the polynomial system we now substitute each occurrence of \(z_i\) by \((z_i^+ - z_i^-)/w_0\) and afterwards multiply by \(w_0^2\) in every polynomial where a substitution occurred, in order to clear \(w_0\) from the denominators. For suitable \(t\) this implies that if for fixed \(y\), the given system of polynomials has a solution \(z \in \mathbb{R}^{m_2}\), then the transformed system has a solution \((z^+, z^-, w)\) in \(\Delta_{c}^{2m_2+t+1}\). Note that, since the variables \(y_i\) are not divided by \(w_0\), the polynomials are no longer of degree at most 2 after multiplication with \(w_0^2\). However, they remain of degree at most 2 in the variables \((z^+, z^-, w)\).

Again, renaming polynomials and bundling the existentially quantified variables we now have a system of polynomials \(q_i(y, z), i \in [\ell]\), where \(z \in \mathbb{R}^{m_3}\) and \(m_3 \leq p_3(|x|)\) for some polynomial \(p_3\), such that \(x \in L\) if and only if

\[
\forall y \in \{0, 1\}^n \exists z \in \Delta_{c}^{m_3} \forall i \in [\ell] : q_i(y, z) = 0
\]

and where each polynomial \(q_i\) depends on at most 1 coordinate of \(y\) and is of degree at most 2 in the variables \(z\).

The next step simply consists of homogenizing the polynomials in the existentially quantified variables. For this we simply introduce a slack variable \(z_{m_3+1} = 1 - \sum_{i=1}^{m_3} z_i\) and homogenize by multiplying terms by \(\sum_{i=1}^{m_3} z_i\) or \(\sum_{i=1}^{m_3+1} z_i z_j\) as needed. Letting \(q_i^0\) be the homogenization of \(q_i\), we now have that \(x \in L\) if and only if

\[
\forall y \in \{0, 1\}^n \exists z \in \Delta^{m_3} \forall i \in [\ell] : q_i^0(y, z) = 0
\]

and where each polynomial \(q_i^0\) depends on at most 1 coordinate of \(y\) and are homogeneous of degree 2 in the variables \(z\).

For the final step we reuse the idea of the reduction from QUAD to 4FEAS, which merely takes the sum of the squares of every given polynomial. Thus we let

\[
F(y, z) = \sum_{i=1}^{\ell} (q_i^0(y, z))^2
\]

We note that \((q_i^0(y, z))^2 \geq 0\) for all \(y\) and \(z\) and is homogeneous of degree 4 in the variables \(z\). Further, since \(y_j^2 = y_j\) for any \(y_j \in \{0, 1\}\) we may replace all occurrences of \(y_j^2\) by \(y_j\) thereby obtaining an equivalent polynomial (when \(y \in \{0, 1\}^n\)) of the form of Definition 3. We have that for every fixed \(y \in \{0, 1\}^n\) and all \(z \in \mathbb{R}^m\) that \(F(y, z) = 0\) if and only if \(q_i(y, z) = 0\) for all \(i\). Thus \(x \in L\) if and only if

\[
\forall y \in \{0, 1\}^n \exists z \in \Delta^{m} F(y, z) = 0
\]

which completes the proof of hardness. Let us also note that the definition of \(F\) guarantees that \(F(y, z) \geq 0\) for all \(y \in \{0, 1\}^n\) and \(z \in \mathbb{R}^m\). Since on the other hand clearly \(\forall^D \text{HOM4FEAS}(\Delta) \in \forall^D \mathcal{R}\), the result follows.

As a special case, (when there are no universally quantified variables) the proof gives a reduction from the \(\exists \mathcal{R}\)-complete problem QUAD to the problem HOM4FEAS(\(\Delta\)), where we are given as input a homogeneous degree 4 polynomial \(F(z)\) in \(m\) variables with rational coefficients and are to decide whether there exists \(z \in \Delta^{m-1}\) such that \(F(z) = 0\). Also, we clearly have that HOM4FEAS(\(\Delta\)) is a member of \(\exists \mathcal{R}\) and therefore have the following result.

**Theorem 2.** The problem HOM4FEAS(\(\Delta\)) is complete for \(\exists \mathcal{R}\), and remains \(\exists \mathcal{R}\)-hard even when assuming that for all \(z \in \mathbb{R}^m\) it holds that \(F(z) \geq 0\).
3 Complexity of ESS

In this section we shall prove our results for deciding existence of an ESS. In the proof we will re-use a trick used by Conitzer [Con19] for the case of 2-player games, where by duplicating a subset of the actions of a game we ensure that no ESS can be supported by any of the duplicated actions, as shown in the following lemma. Here, by duplicating an action we mean that the utilities assigned to any pure strategy profile involving the duplicated action is defined to be equal to the utility for the pure strategy profile obtained by replacing occurrences of the duplicated action by the original action. The precise property is as follows.

Lemma 3. Let $\mathcal{G}$ be an $m$-player symmetric game given by $S$ and $u$. Suppose that $s, s' \in S$ are such that for all strategies $\tau$ we have $u(s; \tau^{m-1}) = u(s'; \tau^{m-1})$. Then $s$ can not be in the support of an ESS $\sigma$.

Proof. Suppose $\sigma$ is a strategy with $s \in \text{Supp}(\sigma)$. Let $\sigma'$ be obtained from $\sigma$ by moving the probability mass of $s$ to $s'$. From our assumption we then have $u(\sigma; \tau^{m-1}) = u(\sigma'; \tau^{m-1})$ for all $\tau$. In particular we have $u(\sigma; \sigma_\epsilon^{m-1}) = u(\sigma'; \sigma_\epsilon^{m-1})$, for all $\epsilon > 0$, where $\sigma_\epsilon$ is given by $\sigma_\epsilon = \epsilon \sigma' + (1 - \epsilon) \sigma$. This means that $\sigma'$ invades $\sigma$ and $\sigma$ is therefore not an ESS. 

We now state and prove our first main result.

Theorem 3. $\exists \text{ESS}$ is $\exists \text{D} \cdot \forall \mathbb{R}$-hard for 5-player games.

Proof. We prove our result by giving a reduction from the complement of the problem $\exists \text{D} \text{HOM4FEAS}(\Delta)$ to $\exists \text{ESS}$. It follows from Theorem 1 that the former problem is complete for $\exists \text{D} \cdot \forall \mathbb{R}$. Thus let $a_{i, \alpha}$ be given rational coefficients, with $i = 0, \ldots, n$ and $\alpha \in [m]^4$, forming the polynomials $F_i(y, z)$ and $F_i(z)$, for $i = 0, \ldots, n$ as in Definition 3. We may assume that for all $y \in \{0, 1\}^n$ and all $z \in \mathbb{R}^m$ it holds that $F_i(y, z) \geq 0$. We may also without loss of generality assume that each $F_i$ is symmetrized by Lemma 2. This will ensure that the game defined below is well defined and symmetric.

We next define a 5-player game $\mathcal{G}$ based on $F$. The strategy set is naturally divided in three parts $S = S_1 \cup S_2 \cup S_3$. These are defined as follows.

$S_1 = \{(i, \alpha, b) \mid i \in \{0, \ldots, n\}, \alpha \in [m]^4, b \in \{0, 1\}\}$

$S_2 = \{\gamma\}$

$S_3 = \{1, \ldots, m\}$

An action $(i, \alpha, b)$ of $S_1$ thus identifies a term of $F_i$ together with $b \in \{0, 1\}$, which is supposed to be equal to $y_i$. When convenient we may describe the actions of $S_1$ by pairs $(t, b)$, where $t = (i, \alpha)$ for some $i$ and $\alpha$. The single action $\gamma$ is used for rewarding inconsistencies in the choices of $b$ among strategies of $S_1$. Finally, a probability distribution on $S_1$ will define an input $z$. Let $M = (n + 1)m^4$ be the total number of terms of $F$. Thus $|S_1| = 2M$.

We shall duplicate all actions of $S_2 \cup S_3$ and let duplicates behave exactly the same regarding the utility function defined below. By Lemma 3 it then follows that any ESS $\sigma$ of $\mathcal{G}$ must have $\text{Supp}(\sigma) \subseteq S_1$. For simplicity we describe the utilities of $\mathcal{G}$ without the duplicated actions.
When all players are playing an action of $S_1$ we define

$$u((t_1, b_1), \ldots, (t_5, b_5)) = \begin{cases} 
2 & \text{if } t_1 \notin \{t_2, \ldots, t_5\} \\
1 & \text{if } t_1 \in \{t_2, \ldots, t_5\} \text{ and } t_1 = t_j \Rightarrow b_1 = b_j \\
0 & \text{otherwise}
\end{cases} \quad (5)$$

Before defining the remaining utilities, we consider the payoff of strategies that play uniformly on the set of terms and according to a fixed assignment $y$. Define the number $T$ by

$$T = 1 + \left( 1 - \frac{1}{M} \right)^4 . \quad (6)$$

**Lemma 4.** Let $y \in \{0, 1\}^n$, let $y_0 \in \{0, 1\}$ be arbitrary, and define $\sigma_i$ to be the strategy that plays $(i, \alpha, y_i)$ with probability $\frac{1}{M}$ for all $\alpha$, and the remaining strategies with probability 0. Then $u(\sigma^5_5) = T$.

**Proof.** Note that $u(\sigma^5_5) - 1$ is precisely the probability of the intersection of the events $t_1 \neq t_j$, where $j = 2, \ldots, 5$, and $t_j$ is the term chosen by player $j$. For fixed $t_1$, these events are independent and each occurs with probability $1 - \frac{1}{M}$. We thus have

$$u(\sigma^5_5) = 1 + \Pr \left\{ \bigwedge_{j=2}^{4} t_1 \neq t_j \right\} = 1 + \left( 1 - \frac{1}{M} \right)^4 = T .$$

We will construct the game $\mathcal{G}$ in such a way that any ESS $\sigma$ will have $u(\sigma^5) = T$. Making use of Lemma 4, we now define utilities when at least one player is playing the action $\gamma$. In case at least two players are playing $\gamma$, these players receive utility 0 while the remaining players receive utility $T$. In case exactly one player is playing $\gamma$, the player receives utility $T + 1$ in case there are two players that play actions $(i, \alpha, b)$ and $(i, \alpha', b')$ with $b \neq b'$; otherwise the player receives utility $T$. In either case, when exactly one player is playing $\gamma$, the remaining players receive utility $T$.

We finally define utilities when one player is playing an action from $S_1$ and the remaining four players are playing an action from $S_3$. Suppose for simplicity of notation that player $j$ is playing action $\beta_j \in S_3$, for $j = 1, \ldots, 4$, while player 5 is playing action $(i, \alpha, b)$. We let player 5 receive utility $T$. Suppose that there exists a permutation $\pi$ on $[4]$ such that $\tilde{\beta} = \pi \cdot \alpha$. Then, let $K_\alpha = \{ \pi \cdot \alpha : \pi \in \text{Sym}([4]) \}$ (i.e. the size of the orbit $\text{Sym}([4]) \cdot \alpha$ with respect to the defined group action for the group of permutations on $[4]$). If either (i) $i = 0$, or (ii) $i > 0$, and $b = 1$, the first four players receive utility $T - \frac{M}{K_\alpha} a_{i, \alpha}$; otherwise they receive utility $T$. The first four players also receive utility $T$ in case $\tilde{\beta} \neq \pi \cdot \alpha$ for all $\pi$.

The above definition is well defined, since we assumed that each $F_j$ is symmetrized. We observe the following relationship between $\mathcal{G}$ and $F$.

**Lemma 5.** Let $y \in \{0, 1\}^n$, let $\sigma_i$ be defined as in Lemma 4, and let $z \in \Delta^{n-1} = \Delta(S_3)$. Then $u(z; \varepsilon^5, \sigma_5) = T - F(y, z)$. 

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Proof. Using the above definitions we have

\[
T - u(z; \gamma^3, \sigma_j) = \sum_{i=0}^{n} \sum_{\alpha \in [m]^4} \sum_{\beta \in [n]^4} \left( T - u(\beta_1, \beta_2, \beta_3, (i, \alpha, y_i)) \right) \frac{1}{M} \prod_{j=1}^{4} z_{\beta_j}
\]

\[
= \sum_{\alpha \in [m]^4} \sum_{\beta \in \text{Sym}(4)} \frac{M}{K\alpha} \left( a_{0, \alpha} + \sum_{i=1}^{n} y_i a_i, \alpha \right) \prod_{j=1}^{4} z_{\beta_j}
\]

\[
= \sum_{\alpha \in [m]^4} \left( a_{0, \alpha} + \sum_{i=1}^{n} y_i a_i, \alpha \right) \prod_{j=1}^{4} z_{\beta_j}
\]

\[
= F(y, z)
\]

At this point we have only partially specified the utilities of the game \( \mathcal{G} \); we simply let all remaining unspecified utilities equal \( T \), thereby completing the definition of \( \mathcal{G} \).

We are now ready to prove that \( \mathcal{G} \) has an ESS if and only if there exists \( y \in \{0,1\}^n \) such that \( F(y, z) > 0 \) for all \( z \in \Delta^{m-1} \). Suppose first that \( y \in \{0,1\}^n \) exists such that \( F(y, z) > 0 \) for all \( z \in \Delta^{m-1} \). We define \( \sigma = \sigma_y \) as in Lemma 4 and show that any \( \tau \neq \sigma \) satisfies the conditions of Lemma 1 thereby proving that \( \sigma \) is an ESS of \( \mathcal{G} \).

Suppose that \( \tau \neq \sigma \) invades \( \sigma \). Consider first playing \( \tau \) against \( \sigma^4 \). From the proof of Lemma 4 it follows that playing a strategy of the form \((i, \alpha, b)\) against \( \sigma^4 \) gives payoff \( T \) if \( b = y_i \) and otherwise payoff strictly below \( T \). The strategies of \( S_2 \cup S_3 \) all give payoff \( T \) against \( \sigma^4 \). It follows that to invade \( \sigma \), \( \tau \) can only play strategies from \( S_1 \) contained in \( \text{Supp}(\sigma) \). Let us write \( \tau = \delta_1 \tau_1 + \delta_2 \tau_2 + \delta_3 \tau_3 \) as a convex combination of strategies \( \tau_j \) with \( \text{Supp}(\tau_j) \subseteq S_j \) for \( j = 1, 2, 3 \). We shall consider playing \( \tau \) against \((\sigma, \sigma^4)\) and argue that \( \tau_1 = \sigma \) if \( \delta_1 > 0 \) and that \( \delta_2 = 0 \). Note first that if a strategy of \( S_2 \) is played, all players receive utility \( T \), so we may focus on the case when all players play using strategies from \( S_1 \cup S_2 \). Suppose that \( \delta_1 > 0 \) and for a term \( t = (i, \alpha) \) of \( F \) let \( p_t = \Pr_{\gamma \in \mathcal{G}([t, y_i])} \). We now have

\[
u(t_1; \tau_1, \sigma^3) = 1 + \sum_t p_t (1 - p_t) \left( 1 - \frac{1}{M} \right)^3 = 1 + \left( 1 - \frac{1}{M} \right)^3 \sum_t p_t (1 - p_t)
\]

and by Chebyshev’s sum inequality it follows that

\[
\sum_t p_t (1 - p_t) \leq \frac{1}{M} \left( \sum_t p_t \right) \left( \sum_t (1 - p_t) \right) = 1 - \frac{1}{M},
\]

and that equality holds if and only if \( p_t = \frac{1}{M} \) for all \( t \). Observe also that

\[
u(\sigma; \tau_1, \sigma^3) = 1 + \sum_t \frac{1}{M} (1 - p_t) \left( 1 - \frac{1}{M} \right)^3 = 1 + \frac{1}{M} (M - 1) \left( 1 - \frac{1}{M} \right)^3 = T
\]

Thus if \( \tau_1 \neq \sigma \), it follows that \( u(t_1; \tau_1, \sigma^3) < u(\sigma; \tau_1, \sigma^3) \). Now, since \( \text{Supp}(\tau_1) \subseteq \text{Supp}(\sigma) \) when \( \delta_1 > 0 \), playing \( \gamma \) can give utility at most \( T \), but also gives utility 0 in case another player plays \( \gamma \) as well.

Combining these observations it follows that unless \( \delta_2 = 0 \) and that \( \tau_1 = \sigma \) when \( \delta_1 > 0 \) we have \( u(\sigma; \tau, \sigma^3) > u(\sigma; \tau^1, \sigma^3) \). Thus we may now assume that this is the case, i.e., that \( \tau = \delta_1 \sigma + \delta_2 \tau_2 + \delta_3 \tau_3 \). From the definition of \( \mathcal{G} \) we now have that \( u(\tau; \tau^1, \sigma^{4-j}) \leq... \)
when replacing all utilities of 2 by 1. Similarly $\sigma_u$ proof.

Suppose now on the other hand that $\sigma$ is an ESS of $\mathcal{G}$. First, since we duplicated the actions of $S_2 \cup S_3$, it follows from Lemma 3 that $\text{Supp}(\sigma) \subseteq S_1$. We next show that for all terms $t$, if $\text{Pr}_{\sigma}([t, b]) > 0$, then unless $\text{Pr}_{\sigma}([t, 1 - b]) = 0$, $\sigma$ can be invaded. Suppose that $t$ is a term of $F$, let $p_0 = \text{Pr}_{\sigma}([t, 0])$ and $p_1 = \text{Pr}_{\sigma}([t, 1])$, and suppose that $p_0 > 0$ and $p_1 > 0$. Suppose without loss of generality that $p_0 \geq p_1$. Note now that

$$u((t, 0); \sigma^4) - u((t, 1); \sigma^4) = (1 - p_1)^4 - (1 - p_0)^4 \geq 0,$$

which can be seen by noting that that the left hand side of the equality does not change when replacing all utilities of 2 by 1. Similarly

$$u((t, 0); (t, 0), \sigma^3) - u((t, 1); (t, 1), \sigma^3) = (1 - p_1)^3 - (1 - p_0)^3 \geq 0.$$

Define the strategy $\sigma'$ from $\sigma$ by playing the strategy $(t, 0)$ with probability $p = p_0 + p_1$, the strategy $(t, 1)$ with probability 0, and otherwise according to $\sigma$. Then

$$u(\sigma'; \sigma^4) - u(\sigma^5) = (p_0 + p_1)u((t, 0); \sigma^4) - p_0u((t, 0); \sigma^4) - p_1u((t, 1); \sigma^4) = p_1(u((t, 0); \sigma^4) - u((t, 0); \sigma^4)) \geq 0.$$

By definition, $u((t, 0); (t, 1), \sigma_3) = u((t, 1); (t, 0), \sigma_3) = 0$, and we thus have

$$u(\sigma'; \sigma', \sigma^3) - u(\sigma; \sigma', \sigma^3) = (p_0 + p_1)^2u((t, 0); (t, 0), \sigma^3) - p_0(p_0 + p_1)u((t, 0); (t, 0), \sigma^3) = (p_0p_1 + p_1^2)u((t, 0); (t, 1), \sigma^3) > 0.$$

which means that $\sigma'$ invades $\sigma$. Since $\sigma$ is an ESS, this means that for each term $t$ there is $b_t \in \{0, 1\}$ such that $\sigma$ plays $(t, 1 - b_t)$ with probability 0. Let $p_t = \text{Pr}_{\sigma}([t, b_t])$ for all $t$. Defining the function $h : \mathbb{R} \rightarrow \mathbb{R}$ by $h(p) = (1 - p)^4$ we now have

$$u(\sigma^2) = 1 + \sum_i p_i h(p_i).$$

Suppose there exists terms $t$ and $t'$ such that $p_t < p_{t'}$. Since $h$ is strictly decreasing on $[0, 1]$ we then then have $h(p_t) > h(p_{t'})$, and therefore $p_t h(p_t) + p_{t'} h(p_{t'}) < p_t h(p_t) + p_{t'} h(p_{t'})$. Define $\sigma'$ to play $t$ with probability $p_t$, $t'$ with probability $p_{t'}$, and otherwise according to $\sigma$. We then have

$$u(\sigma') - u(\sigma') = p_t(h(p_t) - h(p_{t'})) + p_{t'}(h(p_{t'}) - h(p_t)) < 0,$$

which means that $\sigma'$ invades $\sigma$. Since $\sigma$ is an ESS this means that $p_i = 1/3$ for all $i$. From the proof of Lemma 4 it then follows that $u(\sigma^3) = T$.

Suppose now that there exists $i \in [n]$ and $\alpha, \alpha'$ such that $b_{i, \alpha} \neq b_{i, \alpha'}$. But then $u(\gamma; \sigma^4) > T = u(\sigma^4)$, which means that $\gamma$ invades $\sigma$. Since $\sigma$ is an ESS there must exist $y \in \{0, 1\}^v$ (and some $y' \in \{0, 1\}$) such that $\sigma = \sigma_y$, using the notation of Lemma 4.

Finally, let $z \in \Delta^{n-1} = \Delta(S_3)$. By definition of $u$ we have $u(z, z', \sigma^3) = T = u(\sigma; z', \sigma^3')$, for all $j \in \{0, 1, 2\}$. Next $u(z, z', \sigma) = T - F(y, z)$ while we have $u(\sigma; z', \sigma) = T$. For $\sigma$ to be ES against $z$ we must thus have $F(y, z) > 0$, and this concludes the proof. □
The best upper bound on the complexity of $\exists \text{ESS}$ we know is membership of $\forall \exists R$ which easily follows from either Definition 1 or from the characterization of Lemma 1. For the simpler problem $\text{IsESS}$ of determining whether a given strategy is an ESS we can fully characterize its complexity.

**Theorem 4.** $\text{IsESS}$ is $\forall R$-complete for 5-player games.

**Proof.** Clearly $\text{IsESS}$ belongs to $\forall R$ by the characterization of Lemma 1. To show $\forall R$-hardness we reduce from the complement of the problem $\text{HOM4FEAS}(\Delta)$ to $\text{IsESS}$. It follows from Theorem 2 that the former problem is complete for $\forall R$. From $F$ we construct the game $G$ as in the proof of Theorem 3 letting $n = 0$. We let $\sigma$ be the uniform distribution on the set of actions $(0, \alpha, 0)$, where $\alpha \in [m]^4$. It then follows from the proof of Theorem 3 that $\sigma$ is an ESS of $G$ if and only if $F(z) > 0$ for all $z \in \Delta^{m-1}$. Since we may assume that $F(z) \geq 0$ for all $z \in \mathbb{R}^m$ this completes the proof.

4 Complexity of LSS

In this section we extend our results for deciding existence of an ESS to that of deciding existence of a LSS. The results are obtained by reusing the reduction from the complement of $\forall D \text{HOM4FEAS}(\Delta)$ to $\exists \text{ESS}$ as a reduction to $\exists \text{LSS}$. Since any LSS is also an ESS it will suffice to prove that if the game constructed has an ESS it also has a LSS. While the proofs of this section thus subsumes parts of the proof of Theorem 3 (and Theorem 4), we presented those separately, since the proofs the proofs of this section are more involved.

We state our second main result below.

**Theorem 5.** $\exists \text{LSS}$ is $\exists D \cdot \forall R$-hard for 5-player games.

Like for the problem $\exists \text{ESS}$ the best upper bound on the complexity of $\exists \text{LSS}$ we know is membership of $\forall \exists R$ which follows directly from Definition 2.

The proof of Theorem 5 gives as a special case a reduction from the complement of $\forall D \text{HOM4FEAS}(\Delta)$ to $\exists \text{LSS}$, analogously to the proof of Theorem 4, thereby showing that $\exists \text{LSS}$ is $\forall R$-hard. On the other hand, proving $\forall R$-membership of $\exists \text{LSS}$ is not as simple as for $\exists \text{ESS}$. This is because Definition 2 defining that $\sigma$ is LSS involves a leading existential quantifier in front of the universal quantification over other strategies $\tau$, and we have no alternative definition without such a leading existential quantifier, unlike the case of ESS where this was given by Lemma 1. The existential quantifier is however just used for expressing universal quantification over sufficiently close strategies $\tau$ to $\sigma$, and it follow by a general result of Bürgisser and Cucker [BC09, Theorem 9.2] that this can be done in $\forall R$.

**Lemma 6** (Bürgisser and Cucker). The problem of deciding if a sentence of the form

$$\exists \epsilon > 0 \forall \epsilon \in (0, \epsilon_0) \forall x : \varphi(\epsilon, x)$$

is true, for a given a quantifier-free formula over the reals $\varphi(\epsilon, x)$ belongs to $\forall R$.

With this in hand, membership of $\exists \text{LSS}$ in $\forall R$ is straightforward and combined with the result of Theorem 5, we obtain the following.

**Theorem 6.** $\exists \text{LSS}$ is $\forall R$-complete for 5-player games.
We are to prove that there exists such that \( F(y, z) \geq 0 \) for all \( y \in \{0, 1\}^n \) and \( z \in \mathbb{R}^m \), and that each \( F_i \) is symmetrized.

As explained above, to complete the proof of Theorem 5 we just need to show that if there exist \( y \in \{0, 1\}^n \) exists such that \( F(y, z) > 0 \) for all \( z \in \Delta^{m-1} \), then the game \( G \) defined in the proof of Theorem 3 has an LSS. We thus assume that \( y \in \{0, 1\}^n \) exists such that \( F(y, z) > 0 \) for all \( z \in \Delta^{m-1} \), and define \( \sigma = \sigma_j \) as in Lemma 4, with \( y_0 \in \{0, 1\} \) arbitrarily chosen.

For \( \varepsilon > 0 \) to be specified later, consider a strategy \( \tau \neq \sigma \) such that \( ||\sigma - \tau||_\infty < \varepsilon \).

We are to prove that \( u(\sigma; \tau_1^4) > u(\tau_5^4) \). Let us write \( \tau = \delta_1 \tau_1 + \delta_2 \tau_2 + \delta_3 \tau_3 \) as a convex combination of strategies \( \tau_j \) with \( \text{Supp}(\tau_j) \subseteq S_j \), for \( j = 1, 2, 3 \). Since \( \text{Supp}(\sigma) \subseteq S_i \), we have

\[
||\sigma - \tau||_\infty = \max (||\sigma - \delta_1 \tau_1||_\infty, ||\delta_2 \tau_2||_\infty, ||\delta_3 \tau_3||_\infty) \leq \varepsilon .
\]

Since \( |S_2| = 2 \) and \(|S_3| = 2m \) (recall that the actions of \( S_2 \cup S_3 \) are duplicated), it follows that \( ||\tau_2||_\infty \geq \frac{1}{2} \) and \( ||\tau_3||_\infty \geq \frac{1}{2m} \). Combining this with the inequalities \( ||\delta_2 \tau_2||_\infty \leq \varepsilon \) and \( ||\delta_3 \tau_3||_\infty \leq \varepsilon \), it follows that \( \delta_2 \leq 2\varepsilon \) and \( \delta_3 \leq 2m\varepsilon \). Since also \( ||\sigma - \delta_1 \tau_1||_\infty \leq \varepsilon \) we now have

\[
||\sigma - \tau||_\infty \leq ||\sigma - \delta_1 \tau_1||_\infty + (1 - \delta_1) \leq \varepsilon + \delta_2 + \delta_3 \leq (2m + 3)\varepsilon \leq 4m\varepsilon , \tag{7}
\]

assuming, without loss of generality, \( m \geq 2 \) for the last inequality.

It will be useful to introduce notation for the probabilities of the strategy \( \tau_i \).

**Definition 5.** For a term of the form \( t = (\alpha, i) \), define \( b_i = y_i \). Next, let \( p_{i,0} = \text{Pr}_{\tau_i}[(t, b_i)] \), \( p_{i,1} = \text{Pr}_{\tau_i}[(t, 1 - b_i)] \), and \( p_i = p_{i,0} + p_{i,1} \).

We also introduce notation for the set of actions in \( S_1 \) that are inconsistent with \( y \).

**Definition 6.** \( B = \{(t, b) \in S_1 \mid b \neq b_i \} \).

In order to prove that \( u(\sigma; \tau_4^4) > u(\tau_5^4) \) we shall analyze \( u(\tau_j; \tau_4^4) \), for \( j \in \{1, 2, 3\} \), separately, and compare to \( u(\sigma; \tau_4^4) \). Note that \( u(\sigma; \tau_4^4) = \delta_1^4 u(\sigma; \tau_4^4) + (1 - \delta_1^4) T \).

### 4.1 Comparison of \( u(\sigma; \tau_4^4) \) to \( u(\tau_1^4; \tau_4^4) \)

Note first that \( u(\tau_1^4; \tau_4^4) = \delta_1^4 u(\tau_1^4; \tau_4^4) + (1 - \delta_1^4) T \), so it suffices to consider \( u(\tau_1^4; \tau_4^4) \).

**Definition 7.** For a fixed term \( t \) and for \( (t_j, b_j) \) chosen according to \( \tau_i \) for \( j = 2, \ldots, 5 \), we define events \( A_t, B_{t,0}, B_{t,1} \) as follows. \( A_t \) denotes the set of outcomes where \( t \notin \{t_2, \ldots, t_5\} \). \( B_{t,0} \) denotes the set of outcomes where \( t \in \{t_2, \ldots, t_5\} \), and \( b_j = b_i \) whenever \( t_j = t \). Finally, \( B_{t,1} \) denotes the set of outcomes where \( t \in \{t_2, \ldots, t_5\} \), and \( b_j = 1 - b_i \) whenever \( t_j = t \).

It is straightforward to compute the probability of these events.

**Lemma 7.** For a fixed \( t \) we have \( \text{Pr}_{\tau_1^4}[A_t] = (1 - p_i)^4 \), \( \text{Pr}_{\tau_1^4}[B_{t,0}] = (1 - p_{i,0})^4 - (1 - p_i)^4 \), and \( \text{Pr}_{\tau_1^4}[B_{t,1}] = (1 - p_{i,0})^4 - (1 - p_i)^4 \).

Using these we can express the payoffs of the two strategies \( \sigma \) and \( \tau_1 \) against \( \tau_4^4 \) in terms of the probabilities \( p_{i,0} \) and \( p_{i,1} \).
Lemma 8. The payoffs \( u(\sigma; \tau_4^i) \) and \( u(\tau_5^i) \) satisfy the following equations.

\[
\begin{align*}
    u(\sigma; \tau_4^i) &= \sum_{t} \frac{1}{M} \left( (1 - p_t)^4 + (1 - p_{t,0})^4 \right) \\
    u(\tau_5^i) &= \sum_{t} p_t (1 - p_t)^4 + p_{t,0} (1 - p_{t,1})^4 + p_{t,1} (1 - p_{t,0})^4
\end{align*}
\]

Proof. By the definition of \( u \) we have

\[
\begin{align*}
    u(\sigma; \tau_4^i) &= \sum_{t} \frac{1}{M} \left( 2 \Pr[A_t] + \Pr[B_{t,0}] \right) \\
    u(\tau_5^i) &= \sum_{t} 2p_t \Pr[A_t] + p_{t,0} \Pr[B_{t,0}] + p_{t,1} \Pr[B_{t,1}]
\end{align*}
\]

The statement then follows from Lemma 7.

Lemma 9. We have the inequality

\[
\sum_{t} \frac{1}{M} (1 - p_t)^4 \geq \left( 1 - \frac{1}{M} \right)^4,
\]

and when \( p_t \leq \frac{2}{3} \) for all \( t \) we furthermore have

\[
\left( 1 - \frac{1}{M} \right)^4 \geq \sum_{t} p_t (1 - p_t)^4.
\]

Both inequalities holds with equality if and only if \( p_t = \frac{1}{M} \) for all \( t \).

Proof. The function \((1 - x)^4\) is strictly convex and thus Jensen’s inequality gives

\[
\left( 1 - \frac{1}{M} \right)^4 = \left( 1 - \frac{\sum_t p_t}{M} \right)^4 \leq \frac{1}{M} \sum_t (1 - p_t)^4,
\]

with equality if and only if \( p_t = \frac{1}{M} \) for all \( t \), resulting in the first of the stated inequalities. Next, since \( \frac{d^2}{dx^2} x(1 - x)^4 = 4(1 - x)^2(5x - 2) \), the function \( x(1 - x)^4 \) is strictly concave in the interval \([0, \frac{2}{3}]\) and thus Jensen’s inequality gives

\[
\frac{1}{M} \sum_t p_t (1 - p_t)^4 \leq \left( \frac{\sum_t p_t}{M} \right) \left( 1 - \frac{\sum_t p_t}{M} \right) \leq \frac{1}{M} \left( 1 - \frac{1}{M} \right)^4,
\]

with equality if and only if \( p_t = \frac{1}{M} \) for all \( t \), resulting in the second of the stated inequalities.

Next we relate the remaining terms in the two summations in Lemma 8.

Lemma 10. Suppose that \( p_{t,0} \geq \frac{4}{5M} \) for all \( t \). Then

\[
\sum_{t} \frac{1}{M} (1 - p_{t,1})^4 \geq \sum_{t} p_{t,0}(1 - p_{t,1})^4 + p_{t,1} (1 - p_{t,0})^4 + \frac{1}{M} p_{t,1}.
\]
Furthermore, it can be verified that for all \( 0 \leq x \leq \frac{1}{6} \) it holds that \( 1 - 4x \leq (1 - x)^4 \leq 1 - 3x \). Using this we get

\[
\sum_{\tau} \frac{1}{M} (1 - p_{\tau,1})^4 - p_{\tau,0}(1-p_{\tau,1})^4 - p_{\tau,1}(1-p_{\tau,0})^4 \\
\geq \sum_{\tau} \frac{1}{M} (1 - 4p_{\tau,1}) - p_{\tau,0}(1-3p_{\tau,1}) - p_{\tau,1}(1-3p_{\tau,0}) \\
= \sum_{\tau} p_{\tau,1} \left( 6p_{\tau,0} - \frac{4}{M} \right) \\
\geq \sum_{\tau} \frac{1}{M} p_{\tau,1} .
\]

Combining these we obtain the following.

**Proposition 1.** Assume that \( \|\sigma - \tau_1\|_\infty \leq \frac{1}{M} \). Then

\[
u(\sigma; \tau_1^4) \geq \nu(\tau_1^4) + \frac{1}{M \tau_1} \Pr[B] .
\]

Furthermore, \( u(\sigma; \tau_1^4) = u(\tau_1^4) \) if and only if \( \sigma = \tau_1 \).

**Proof.** The inequality follows by the equations for \( u(\sigma; \tau_1^4) \) and \( u(\tau_1^4) \) given in Lemma 8 together with the inequalities of Lemma 9 and Lemma 10. If \( u(\sigma; \tau_1^4) = u(\tau_1^4) \) it follows that \( \Pr_{\tau_1}[B] = 0 \) in which case Lemma 9 expresses the inequality \( u(\sigma; \tau_1^4) \geq u(\tau_1^4) \) and states that it holds with equality if and only if \( \sigma = \tau_1 \). \( \square \)

This finally allows us to compare playing the strategies \( \sigma \) and \( \tau_1 \) against \( \tau^4 \).

**Corollary 1.** Assume that \( \|\sigma - \tau_1\|_\infty \leq \frac{1}{M} \). Then

\[
u(\sigma; \tau^4) \geq \nu(\tau_1; \tau^4) + \frac{1}{M \tau_1} \Pr[B] .
\]

Furthermore, \( u(\sigma; \tau^4) = u(\tau_1; \tau^4) \) if and only if \( \sigma = \tau_1 \).

**Proof.** The inequality follows from Proposition 1 together with the observations that

\[
u(\sigma; \tau^4) = \delta^4 u(\sigma; \tau^4) + (1 - \delta^4) \Pr[B] \quad \text{and} \quad \nu(\tau_1; \tau^4) = \delta^4 u(\tau_1; \tau^4) + (1 - \delta^4) \Pr[B] .
\]

4.2 **Comparison of** \( u(\sigma; \tau^4) \) **to** \( u(\tau_2; \tau^4) \) **and** \( u(\tau_3; \tau^4) \).

Let in the following \( \tilde{\tau}_1 \) denote the strategy obtained from \( \tau \) by conditioning on the outcome belonging to \( S_1 \setminus B \). We first consider playing \( \sigma \) against \( \tilde{\tau}_1 \).

**Lemma 11.** \( u(\sigma; \tilde{\tau}_1^4) \geq T \).

**Proof.** Similarly to Definition 5, for a term \( \tau \) we let \( \hat{\rho}_{\tau,0} = \Pr_{\tau_1}[(t, b_t)], \hat{\rho}_{\tau,1} = \Pr_{\tau_1}[(t, 1 - b_t)], \) and \( \hat{\rho} = \hat{\rho}_{\tau,0} + \hat{\rho}_{\tau,1} \). Clearly \( \hat{\rho}_{\tau,1} = 0 \) and \( \hat{\rho} = \hat{\rho}_{\tau,0} \). Using \( \tilde{\tau}_1 \) in place of \( \tau_1 \) in Lemma 8 and Lemma 9 we then have

\[
u(\sigma; \tilde{\tau}_1^4) = \sum_{\tau} \frac{1}{M} ((1-\hat{\rho})^4 + (1-\hat{\rho}_{\tau,1})^4) \\
= 1 + \sum_{\tau} \frac{1}{M} (1-\hat{\rho})^4 \geq 1 + \left( 1 - \frac{1}{M} \right)^4 = T .
\]

\( \square \)
We first compare playing the strategies $\sigma$ and $\tau_2$ against $\tau^4$.

**Proposition 2.** We have

$$u(\sigma; \tau^4) \geq u(\tau_2; \tau^4) - 12\delta_1 \Pr_{\tau_1}[B] + \delta_2.$$

**Proof.** We divide the outcomes of $\tau^4$ into events. Let $C_1$ be the event that the four players play an action in $S_1 \setminus B$, i.e. $C_1 = (S_1 \setminus B)^4$. Let $C_2$ be the event that at least one of the four players play an action in $S_2$. Let $C_3$ be the event that none of the four players play an action in $S_2$ and that at least one player is playing an action of $S_3$. Finally let $C_4$ be the event consisting of all remaining outcomes.

Conditioned on the event $C_1$, the strategy $\sigma$ receive payoff at least $T$ against $\tau^4$ by Lemma 11, whereas $\tau_2$ receives payoff $T$. When $C_2$ occurs, which happens with probability at least $\delta_2$, the strategy $\sigma$ receives payoff $T$ while the strategy $\tau_2$ receives payoff 0. When $C_3$ occurs, both strategies $\sigma$ and $\tau_2$ receive payoff $T$. We finally consider the case of $C_4$ occurring. Note that for this to happen, at least one of the four players need to play an action of $B$, which in turn happens with probability at most $4\delta_1 \Pr_{\tau_1}[B]$. By definition of $u$, the strategy $\tau_2$ receives at most payoff $T + 1$ while $\sigma$ receives at least 0 in any event and in particular in the event $C_4$. Combining these observations we obtain

$$u(\sigma; \tau^4) - u(\tau_2; \tau^4) \geq \delta_2 T - 4\delta_1 \Pr_{\tau_1}[B](T+1),$$

from which the stated inequality follows by using also that $1 \leq T \leq 2$. \hfill $\square$

We next compare playing the strategies $\sigma$ and $\tau_3$ against $\tau^4$.

**Proposition 3.** We have

$$u(\sigma; \tau^4) \geq u(\tau_3; \tau^4) + 4\delta_1\delta_2 \left(T - u(\tau_3; \tau_3)\right) - 8\delta_1 \Pr_{\tau_1}[B].$$

**Proof.** We first consider playing $\sigma$ against $\tau^4$. In the event that the four players play an action in $S_1 \setminus B$, the strategy $\sigma$ receive payoff at least $T$ by Lemma 11. When at least one player plays an action outside $S_1$, the strategy $\sigma$ receives $T$. It follows that for $\sigma$ to receive payoff less than $T$ at least one of the four players need to play an action of $B$ which happens with probability at most $4\delta_1 \Pr_{\tau_1}[B]$. We thus have

$$u(\sigma; \tau^4) \geq (1 - 4\delta_1 \Pr_{\tau_1}[B])T = T - 4\delta_1 \Pr_{\tau_1}[B]T.$$

We next consider playing $\tau_3$ against $\tau^4$. The strategy $\tau_3$ receives payoff $T$ unless exactly one player plays an action of $S_1$ and the remaining three players play an action of $S_3$. We thus have

$$u(\tau_3; \tau^4) = 4\delta_1\delta_2 u(\tau_3; \tau_3) + (1 - 4\delta_1\delta_2)T = T - 4\delta_1\delta_2(T - u(\tau_3; \tau_3)).$$

By combining these, and also using $T \leq 2$, the stated inequality follows. \hfill $\square$

### 4.3 Comparison of $u(\sigma; \tau^4)$ to $u(\tau^5)$.

We now combine the analysis of the previous subsections and make use of the assumption about $F(\gamma; z) > 0$ for all $z \in \Delta^m$. We make use of the latter assumption in connection with the relationship between $\mathcal{G}$ and $F$ given by Lemma 5.
Lemma 12. There exists \( \varepsilon > 0 \) such that when \( \tau_1 \in \Delta(S_1) \) is such that \( \|\sigma - \tau_1\|_\infty \leq \varepsilon \), we have \( u(\tau_1; \tau_2; \tau_1) < T \) for all \( \tau_3 \in \Delta(S_3) \).

Proof. By assumption we have \( F(y, z) > 0 \) for all \( z \in \Delta^{m-1} \). Since \( F \) is continuous and \( \Delta^{m-1} \) compact this implies that

\[
\min_{z \in \Delta^{m-1}} F(y, z) > 0.
\]

Next, define the function \( G : \Delta(S_1) \rightarrow \mathbb{R} \) by

\[
G(\tau_1) = \max_{\tau_2 \in \Delta(S_2)} u(\tau_1; \tau_2; \tau_1)
\]

Lemma 5 and Equation (10) together implies that \( G(\sigma) < T \). Since the function \( u \) is continuous and again using compactness of \( \Delta^{m-1} \) it follows that \( G \) is continuous as well. This means there exists \( \varepsilon > 0 \) such that \( G(\tau_1) < T \) whenever \( \|\sigma - \tau_1\|_\infty < \varepsilon \). \( \Box \)

We now define \( \varepsilon = \frac{1}{4m} \min \left( \varepsilon_y, \frac{1}{11M} \right) \). This means by Equation (7) that in particular \( \|\sigma - \tau_1\|_\infty \leq \frac{1}{4m} \), satisfying the condition of Corollary 1, and \( \|\sigma - \tau_1\|_\infty \leq \varepsilon_y \), satisfying the condition of Lemma 12.

Combining the inequalities of Corollary 1, Proposition 2, and Proposition 3, and using that \( \tau = \delta_1 \tau_1 + \delta_2 \tau_2 + \delta_3 \tau_3 \) results in the following inequality.

\[
u(\sigma; \tau^*) \geq u(\tau^*) + \left( \frac{\delta_1^2}{M} - 12 \delta_1 \delta_2 - 8 \delta_1 \delta_3 \right) \Pr[B] \tau_1^3 + 4 \delta_1^2 \left( T - u(\tau_1; \tau_2; \tau_1) \right)
\]

(11)

The definition of \( \varepsilon \) also gives the inequality \( \varepsilon \leq \frac{1}{44mM} \). Note that \( M(12\delta_2 + 8\delta_3) \leq M(24\varepsilon + 16m\varepsilon) \leq 32mM\varepsilon \). On the other hand \( \delta_1^2 \geq (1 - \delta_2 - \delta_3)^4 \geq (1 - 3m\varepsilon)^4 > 1 - 12m\varepsilon \geq 1 - 12mM\varepsilon \). This gives that \( \left( \frac{\delta_1^2}{M} - 12 \delta_1 \delta_2 - 8 \delta_1 \delta_3 \right) > \frac{\delta_1}{M}(1 - 44mM\varepsilon) \geq 0 \).

From this and Lemma 12 it follows that \( u(\sigma; \tau^*) \geq u(\tau^*) \). Furthermore, an equality, \( u(\sigma; \tau^*) = u(\tau^*) \) implies that \( \delta_2 = 0 \) and \( \delta_3 = 0 \). This means that \( \tau = \tau_1 \), and Corollary 1 then implies that in fact \( \sigma = \tau \). This concludes the proof of Theorem 5.

5 Conclusion

We have shown the problems \( \exists \text{ESS} \) and \( \exists \text{LSS} \) to be hard for \( \exists \text{P}, \forall \text{R} \) and members of \( \exists \forall \text{R} \). The main open problem is to characterize the precise complexity of \( \exists \text{ESS} \) and \( \exists \text{LSS} \), perhaps by improving the upper bounds. Another point is that our hardness proofs construct 5-player games, whereas the recent and related \( \exists \text{R} \)-completeness results for decision problems about NE in multi-player games holds already for 3-player games. This leads to the question about the complexity of \( \exists \text{ESS} \) and IsESS as well as \( \exists \text{LS} \) and IsLSS in 3-player and 4-player games. The reason that we end up with 5-player games is that we construct a degree 4 polynomial in the reduction, rather than (a system of) degree 2 polynomials as used in the related \( \exists \text{R} \)-completeness results. In both cases a number of players equal to the degree is used to simulate evaluation of a monomial and a last player is used to select the monomial. For our proof we critically use that the degree 4 polynomial involved in the reduction may be assumed to be non-negative.
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