On neighbour-dependent shifts preserving renewal process

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Abstract

Let $\mathcal{T} = \{0 = T_0 < T_1 < T_2 < \ldots\}$ denote a renewal process on the real line, so that the increments are i.i.d. strictly positive random variables $\tau_k = T_k - T_{k-1}$ with distribution $F$. Choose random points $X_k$ between each pair of neighbouring points $T_{k-1}, T_k$ as follows: $X_k = T_{k-1} + b_k(T_k - T_{k-1})$, where $b_k, k = 1, 2, \ldots$ are i.i.d. with a given distribution $G$ supported by $[0, 1]$. Now shift these points so that they start also from the origin, this defines an operator $\Phi_G$ such that $\mathcal{T}' = \Phi_G(\mathcal{T}) = \{0 = T'_0 < T'_1 < T'_2 < \ldots\}$ with $T'_{k} = X_{k+1} - X_1$, $k \in \mathbb{N}$.

Generally speaking, the resulting process $\mathcal{T}'$ is no longer a renewal process. However, apart from degenerate cases when $\tau_k$'s and $b_k$'s are constants, $\Phi_G(\mathcal{T})$ is a distributional copy of $\mathcal{T}$ if and only if $\tau_k$'s are Gamma-distributed and $b_k$'s are Beta-distributed. In particular, a homogeneous Poisson process is invariant with respect to (and only to) $B(a, 1-a)$-based shifts for some $0 \leq a < 1$.

Keywords: Gamma-process, renewal process, neighbour-dependent shifts, fixed point, Poisson process, Beta distribution

1 Introduction

It is a well-known property of a Poisson process that whenever we shift its points by i.i.d. random variables, we get again a Poisson process. In particular, the result of such a transformation of a homogeneous Poisson process of the real line is again a homogeneous Poisson process of the same rate.

The fact that a random independent shifts preserve the Poisson process distribution reflects ‘independence’ of its points. However, transformations which depend on two or more neighbouring points destroy this independence and hence the Poisson process. For instance, the mid-points of the consecutive segments in a homogeneous Poisson process on the line do not follow Poisson process

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distribution, as it can be easily checked. Every second iteration of this transformation by which the points move to the centre of their Voronoi cells is known as the adjustment process and it is used to model behaviour of repulsing particles or animals, see, e.g., [3, Chap. 7.3.2] and the references therein. Actually, the variance of the inter-point distances in 1D adjustment model vanishes, so iterations of the adjustment procedure converge (in a suitable sense) to a regular array of points, see [4].

There are, however, transformations depending on neighbouring points which preserve Poisson process. A notable example of this kind is a one-server queue with i.i.d. service times which takes the process as an input and transforms it into the process of the departure times from this queue. Burke [1] and Daley [2] show that the output process from $GI|M|1$ queue is renewal if and only if the input process is Poisson. In this case the output process is also Poisson with the same rate.

Although the departure time in a one-server queue may depend on any number of consecutive points of the input process, in this paper we fix a class of transformations which depend on pairs of consecutive points only calling them neighbour-dependent shifts. The shifts to the mid-points of the consecutive segments providing an example, we ask ourselves if it possible to come with a non-trivial neighbour-dependent shifts which preserve the Poisson process? Below we answer positively to this question, the segments must be divided randomly in Beta-distributed proportions by the new points. Moreover, we show that such Beta-based shifts are the only non-trivial ones which preserve a renewal process and that a renewal process preserved by neighbour-dependent shifts is necessarily a Gamma process.

2 Model Description

Let $\mathcal{T} = \{0 = T_0 < T_1 < T_2 < \ldots\}$ be a realisation of a point process on $\mathbb{R}_+$ without multiple points. In this paper it will be a renewal process on the real line, so that the increments are i.i.d. non-degenerate positive random variables, $\tau_k = T_k - T_{k-1}$ drawn from a distribution $F$. Given a distribution $G$ supported by $[0, 1)$, define a (stochastic) shift operator $\Phi_G$ as follows: take a family $\{b_k, \ k \in \mathbb{N}\}$ of i.i.d. random variables with distribution $G$ and set $X_k = T_{k-1} + b_k(T_k - T_{k-1})$. Now shift the whole sequence $X_1 < X_2 < \ldots$ by $-X_1$, to obtain the result of the operator: $\Phi_G(\mathcal{T}) = \mathcal{T}' = \{0 = T'_0 < T'_1 < \ldots\}$, where $T'_k = X_k - X_1, \ k = 1, 2, \ldots$. It can also be expressed in terms of the increments:

\[
\begin{align*}
\tau'_k &= T'_k - T'_{k-1} = (1 - b_k) \tau_k + b_{k+1} \tau_{k+1}, \\
T'_0 &= 0, \ T'_{k+1} = T'_k + \tau'_k, \quad k \in \mathbb{N}.
\end{align*}
\]

Obviously, when $G$ is concentrated on $0$ or on $1$, the corresponding operator $\Phi_G$ preserves any renewal process. So we exclude these trivial cases from our consideration.
3 Main results

The main result is the characterisation of the class of fixed points of the shift operator $\Phi_G$. Throughout this paper, $\Gamma(\alpha, \gamma)$ denotes the Gamma distribution with shape parameter $\alpha$ and rate parameter $\gamma$, its density is given by

$$f_\Gamma(x) = \frac{\gamma^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\gamma x}, \ x > 0,$$

and $B(\alpha, \beta)$ is the Beta distribution with density

$$f_B(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, x \in (0, 1).$$

The degenerate distribution concentrated at point $x$ is denoted by $\delta_x$.

**Theorem 1.** Let $\mathcal{T}$ be a renewal process with increment distribution $F$ and $\Phi_G$ be a neighbour-dependent shift operator. Then $\Phi_G(\mathcal{T}) \overset{D}{=} \mathcal{T}$ if and only if one of the following alternatives is true:

1. $F = \Gamma(\alpha, \gamma)$ and $G = B(r\alpha, (1-r)\alpha)$ for some constants $\alpha > 0$, $\gamma > 0$ and $r \in (0, 1)$,

2. $F = \delta_s$ for some $s \in (0, \infty)$ and $G = \delta_b$ for some $b \in (0, 1)$.

**Proof.** The ‘if’ part for non-degenerated case follows from shape versus size independence for Gamma-distributed random vectors. The ‘only if’ part is proven via a coupling with a Gamma process with independent increments in continuous time.

**Necessity.** Considering two consecutive segments in $\mathcal{T}$, we note that $\Phi_G(\mathcal{T}) \overset{D}{=} \mathcal{T}$ implies the following condition:

If $X, Y, Z$ are independent random variables distributed with $F$, and $a, b, c$ are independent distributed with $G$, then the random variables

$$(1-a)X + bY, (1-b)Y + cZ$$

are also independent, both distributed with $F$.

In terms of characteristic functions,

$$\phi_{(1-a)X+bY,(1-b)Y+cZ}(x_1, x_2) = \mathbb{E}\exp\{ix_1((1-a)X + bY) + ix_2((1-b)Y + cZ)\}$$

$$= \mathbb{E}\exp\{ix_1(1-a)X\} \mathbb{E}\exp\{ix_2cZ\} \mathbb{E}\exp\{ix_1bY + ix_2(1-b)Y\}$$

$$= \phi_{(1-a)X}(x_1)\phi_cZ(x_2)\phi_{bY,(1-b)Y}(x_1, x_2).$$

On the other hand, the independence of $(1-a)X + bY$ and $(1-b)Y + cZ$ implies

$$\phi_{(1-a)X+bY,(1-b)Y+cZ}(x_1, x_2) = \phi_{(1-a)X+bY}(x_1)\phi_{(1-b)Y+cZ}(x_2)$$

$$= \phi_{(1-a)X}(x_1)\phi_{bY}(x_1)\phi_{(1-b)Y}(x_2)\phi_cZ(x_2),$$
yielding\\ \phi_{(1-b)Y,bY}(x_1,x_2) = \phi_{(1-b)Y}(x_1)\phi_{bY}(x_2),$

so the random variables $\eta_1 = bY$ and $\eta_2 = (1-b)Y$ are independent.

Let us first suppose $Y \sim F$ is degenerate. Then the only case where we can represent $Y$ as a sum of the two independent random variables $bY$ and $(1-b)Y$ is when both of them are degenerate, too. That means, the random variable $b \sim G$ must be degenerate, leading us to alternative (ii).

Now let us suppose $Y \sim F$ is non-degenerate. Then so are $\eta_1$ and $\eta_2$. In that case the random variables $b = \eta_1/\eta_2$ and $Y = \eta_1 + \eta_2$ can be understood as shape and size variables of the random vector $(\eta_1, \eta_2)$. Note that $b$ and $Y$ are independent by construction, i.e. the shape is independent of the size. Therefore, the only possibility for the joint distributions of $\eta_1, \eta_2$ is them being independent, Gamma-distributed with some positive shape parameters $a_1, a_2$ and a common rate $\gamma$, see [5]. Put $r = a_1/(a_1 + a_2)$ and $\alpha = a_1 + a_2$. Then $b$ becomes $B(r\alpha, (1-r)\alpha)$-distributed and $Y$ conforms to $\Gamma(\alpha,\gamma)$, arriving to alternative (i).

Sufficiency. Alternative (ii) trivially leads to the invariance.

For alternative (i), let positive $\alpha, \gamma$ and $0 < r < 1$ be fixed. Consider a Gamma process $Y(t), t \in [0, \infty)$ which is a strictly increasing Lévy process with Gamma increments, so that

- $Y(0) = 0$, and
- for any $n$, $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$, the vector $(Y(t_1) - Y(t_0), \ldots, Y(t_n) - Y(t_{n-1}))$ consists of $n$ independent Gamma-distributed random variables with a common rate parameter $\gamma$, and shape parameters $(t_1-t_0)\alpha, \ldots, (t_n - t_{n-1})\alpha$, respectively.

See [6] for an example of constructive definition of such $Y(t)$. Then, for $k = 0, 1, 2, \ldots$, put $T_k = Y(k)$ and $T_k' = Y(k+r) - Y(r)$, see Figure 1. By construction, the sequence $T = \{Y(0), Y(1), Y(2), \ldots\}$ is a renewal process with $\Gamma(\alpha,\gamma)$-increments. So is the sequence $T' = \{0, Y(1+r) - Y(r), Y(2+r) - Y(r), \ldots\}$.

Now, in order to prove (i), we need to show that the sequence of random variables

$\tau'_k := Y(k+r) - Y(k-1+r), \quad k = 1, 2, \ldots$

is a distributional copy of the increments of $\Phi_G(T)$ with an i.i.d. of Beta-distributed random variables $b_k$ independent of $\tau_k = Y(k) - Y(k-1), \quad k = 1, 2, \ldots$.

Since

$\tau'_k = Y(k) - Y(k-1+r) + Y(k+r) - Y(k)$

$= \frac{Y(k) - Y(k-1+r)}{Y(k) - Y(k-1)} (Y(k) - Y(k-1)) + \frac{Y(k+r) - Y(k)}{Y(k) + Y(k)} (Y(k+1) - Y(k)),$

define

$\quad b_k = \frac{Y(k-1+r) - Y(k-1)}{Y(k) - Y(k-1)}, \quad k = 1, 2, \ldots$
Notice that the denominator

$$Y(k) - Y(k - 1) = (Y(k) - Y(k - 1 + r)) + (Y(k - 1 + r) - Y(k - 1))$$

is the sum of two independent Gamma-distributed random variables, so $b_k$'s are independent for different $k$ and $B(r\alpha, (1 - r)\alpha)$-distributed. Moreover, by the shape vs. size independence property of the Gamma distribution (see [5]), the sequence $\{b_k\}_{k \geq 1}$ is independent of $\{\tau_k\}_{k \geq 1}$. □

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