STABILITY AND MOMENT ESTIMATES FOR THE STOCHASTIC SINGULAR \( \Phi \)-LAPLACE EQUATION

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Abstract. We study stability, long-time behavior and moment estimates for stochastic evolution equations with additive Wiener noise and with singular drift given by a divergence type quasilinear diffusion operator which may not necessarily exhibit a homogeneous diffusivity. Our results cover the singular stochastic \( p \)-Laplace equations and, more generally, singular stochastic \( \Phi \)-Laplace equations with zero Dirichlet boundary conditions. We obtain improved moment estimates and quantitative convergence rates of the ergodic semigroup to the unique invariant measure, classified in a systematic way according to the degree of local degeneracy of the potential at the origin. We obtain new concentration results for the invariant measure and establish maximal dissipativity of the associated Kolmogorov operator. In particular, we recover the results for the curve shortening flow in the plane by Es-Sarhir, von Renesse and Stannat, NoDEA 16(9), 2012, [26], and improve the results by Liu and Tölle, ECP 16, 2011, [41].

1. Introduction

The study of long-time behavior for Markovian dynamics associated to stochastic evolution equations has a long history. Its significance to mathematical physics is emphasized in the equations of Fokker and Planck, and Kolmogorov [34], which are discussed in detail in the monographs [12,48]. Extended to stochastic partial differential equations (SPDEs), the analysis of invariant probability distributions and Kolmogorov operators in infinite dimensions [17,18] provides improved insight into the averaged dynamics of PDEs perturbed by random noise with infinitely many frequency modes [19].

We are interested in stability and moment estimates for nonlinear SPDEs of the following type

\[
\partial_t X_t = \text{div} [\phi(\nabla X_t)] + B \partial_t W_t, \quad X_t|_{\partial \Omega} = 0, \quad t > 0, \quad X_0 = u_0,
\]

where \( u_0 \in L^2(\Omega) \) for a bounded, open and convex domain \( \Omega \subset \mathbb{R}^d \) with piecewise \( C^2 \)-boundary, \( d \geq 1 \). We require that the solution \( X_t, t > 0 \) vanishes on the boundary \( \partial \Omega \), that is, we impose zero Dirichlet boundary conditions. Here, \( B \) is a linear operator that is regularizing the spatial component and \( \{W_t\}_{t \geq 0} \) a cylindrical Wiener process on some filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) with values in some separable Hilbert space \( U \). The nonlinearity \( \phi : \mathbb{R}^d \to \mathbb{R}^d \) is monotone in the sense that

\[
\langle \phi(x) - \phi(y), x - y \rangle \geq 0, \quad x, y \in \mathbb{R}^d.
\]
Existence and uniqueness of solutions to SPDEs with nonlinear monotone drift were first systematically studied in [36,47].

In this work, we shall study the singular drift case, including examples where the Jacobian of \( \phi \) has a singularity in zero. We assume that \( \phi \) has at most linear growth, i.e. for some \( C > 0 \),

\[
|\phi(x)| \leq C(1 + |x|).
\]

We refer to [29,40,51] for a well-posedness theory for such equations. Examples of such \( \phi \) that satisfy our main Assumption 1.1 include, among others, the following ones.

**Stochastic singular \( p \)-Laplace.** For \( p \in (1,2) \), let \( \phi(x) = |x|^{p-2}x \) and yield the stochastic singular \( p \)-Laplace equation. It has been first addressed in [38]. The results have been extended in [28,29,31,40]. The multi-valued stochastic 1-Laplace equation, which is also known as the total variation flow is not within the scope of this work, instead, we refer to [7,28–31]. The \( L^1 \)-theory of the stochastic \( p \)-Laplace equation has been discussed in [52].

**Stochastic non-Newtonian fluids.** For \( p \in (1,2) \), let

\[
\phi(x) = 1 + \frac{|x|^2}{2} \left( \frac{p}{2} - 2 \right) x,
\]

which yields a simplified model for stochastic singular power law fluids without convection term. It exhibits a similar nonlinearity as in an equation modeling non-Newtonian fluids with stochastic perturbation in the shear-thinning case, see [8,13,53].

**Stochastic curve shortening flow.** In one spatial dimension \( d = 1 \), we may formally differentiate \( \phi(x) = \arctan(x) \) in order to obtain the so-called (additive noise) stochastic curve shortening flow

\[
\frac{\partial_t X_t}{1 + (\partial_x X_t)^2} + B \partial_t W_t, \quad t > 0, \quad X_0 = u_0,
\]

which is a formulation of the stochastic mean curvature flow in the plane, see [25,28]. It has been generalized toward two spatial dimensions in [32].

**Further examples.** Furthermore, we have the minimal surface flow \( \phi(x) = \frac{x}{\sqrt{1 + |x|^2}} \), which is related to the mean curvature flow, see [10,27], and the logarithmic diffusion \( \phi(x) = \log(1 + |x|) \frac{x}{|x|} \), as addressed e.g. in [29].

The main problem with these kinds of nonlinear diffusivities remains the lack of coercivity and dissipativity in the general situation. This may be achieved by Sobolev embeddings in certain cases [39], however, energy methods as e.g. stochastic variational inequalities [7,28,31,54] seem to deal with well-posedness issues better without loosing for instance the Markov property of the semigroup [30].

Other variational approaches to divergence type drift SPDEs can be found in [7,39,40,43,44]. The case of gradient type Stratonovich noise is addressed in [3,4,16,57].

**Long-time behavior.** Even in the case of analytically weak variational solutions, we may define a Feller semigroup \( \{P_t\}_{t \geq 0} \) associated to the dynamics of (1.1) via extension of

\[
P_t f(u) := \mathbb{E}[f(X^u_t)], \quad f \in \text{Lip}_b(L^2(O)), \quad t \geq 0,
\]

where \( \{X^u_t\}_{t \geq 0} \) denotes a solution to (1.1) with \( X_0 = u \), see [18]. We call a Borel probability measure \( \mu \) on \( L^2(O) \) invariant with respect to \( \{P_t\}_{t \geq 0} \) if

\[
\int_{L^2(O)} P_t f \ d\mu = \int_{L^2(O)} f \ d\mu
\]

for every \( t \geq 0 \) and every \( f \in C_b(L^2(O)) \). We refer to [56] for an overview article on this topic. Uniqueness of invariant measures for singular stochastic equations in low spatial dimensions was first proved with methods from [35] for the stochastic curve shortening flow in [25]. These
results were then extended to the stochastic singular $p$-Laplace equation in \[41\] and to more general equations in \[29,46\]. In particular, for additive noise, the decay behavior of deterministic solutions (that is, for $B \equiv 0$) plays an important role, see \[20,49,50\] for results of this type. We would like to point out that our result is novel even for the deterministic case. Second order estimates for the deterministic parabolic $p$-Laplace system have been obtained in \[14,15\].

Ergodicity of the semigroup and uniqueness of invariant measures for local and nonlocal singular $p$-Laplace equations in all spatial dimensions was proved in \[30\].

We are also interested in improved moment estimates for singular monotone drift SPDEs which yield more explicit convergence rates in the spirit of \[24,26,41\]. To this end, we may assume that the singularity is of a certain polynomial type, more precisely, we shall assume that $\phi$ has a radially symmetric potential, that is, $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is of the form

\[
\phi(z) = \begin{cases} 
\psi(|z|) \frac{z}{|z|}, & \text{if } z \neq 0, \\
0, & \text{if } z = 0,
\end{cases}
\]

where $\psi : \mathbb{R} \to \mathbb{R}$ satisfies the following Assumption 1.1.

\textbf{Assumption 1.1.} Suppose that $\psi \in W^{1,1}_{\text{loc}}(\mathbb{R})$ such that there exists $C > 0$ with

\begin{itemize}
  \item[(C1)] $\psi'(r) > 0$ for every $r > 0$,
  \item[(C2)] $\psi(r) = -\psi(-r)$ for every $r \in \mathbb{R}$,
  \item[(C3)] $\psi(r) \leq C(1 + |r|)$ for every $r \geq 0$,
  \item[(C4)] there exist $a \geq 0$, $b > 0$, $s \in (0, 2]$ with 
    \[ (a + b|r|^s)|\psi'(r)| \geq 1 \]
    for every $r \geq 0$,
  \item[(C5)] there exist $c > 0$ and $K \geq 0$ such that 
    \[ \psi(r) \cdot r \geq c|r| - K, \quad r \geq 0. \]
  \item[(C6)] $0 < s \leq \frac{4}{d} \wedge 2$.
\end{itemize}

We are ready to formulate our main result, see Theorems 4.3 and 5.1, which generalizes the results of \[24,26,29,41\] which assume the more restrictive assumption $s < \frac{4}{d+2} \wedge 1$ compared to (C6).

\textbf{Theorem 1.2.} Assume (C1)–(C6) and that\footnote{We denote the space of Hilbert-Schmidt operators from the separable Hilbert space $U$ to a separable Hilbert space $S$ by $L_2(U, S)$.} $B \in L_2(U, H^1_0(\mathcal{O}))$. Set

\[ s^* = (2 - s) \vee \frac{4 - s}{2 + s}. \]

(i) Then there exists a unique invariant Borel probability measure $\mu$ of $\{P_t\}_{t \geq 0}$ that is concentrated on the subset $W^{2,\alpha}_0(\mathcal{O}) \cap H^1_0(\mathcal{O}) \subset L^2(\mathcal{O})$ for any $1 < \alpha \leq d + 2$ and $\alpha = 1$ if $d = 1$ with

\[
\int_{L^2(\mathcal{O})} \|u\|_{W_0^{2,\alpha}(\mathcal{O})}^s \mu(du) + \int_{L^2(\mathcal{O})} \|u\|_{H^1_0(\mathcal{O})}^s \mu(du) < +\infty.
\]

(ii) Let $d_0 := 1 \vee \frac{d}{2}$ and

\[ 0 < \beta \leq \beta^* := 1 \wedge \frac{8 - 2s}{s(2 + s)d_0} \in \left[ \frac{1}{2d_0}, 1 \right]. \]
Then there exist non-negative constants $C_1, C_2$ and $C_3$ such that
\[
\limsup_{t \to +\infty} \left[ t^{-2} \frac{\|P_t f(u) - P_t f(v)\|}{\|u - v\|_{L^2(\Omega)}} \right] \leq C_1 |f|_\beta C_2 + C_3 \int_{L^2(\Omega)} \|u\|_{H^s_0(\Omega)} \mu(du)
\]
for every $u, v \in H_0^1(\Omega)$ and for every $f : L^2(\Omega) \to \mathbb{R}$ that is bounded and $\beta$-Hölder-continuous.\footnote{2 \(f : L^2(\Omega) \to \mathbb{R}\) is called $\beta$-Hölder-continuous if \(\sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|_{L^2(\Omega)}} =: |f|_\beta < +\infty.\)}

Finally, we shall address the maximal dissipativity of the associated Kolmogorov operator with infinitely many variables, see \([6, 11, 17, 23, 26, 45, 55, 56]\) for related works.

We note that the degenerate $p$-Laplace, i.e. the case $p > 2$, may be treated more easily with the Krylov-Bogoliubov method \([18]\), as is done in the variational setup in \([5]\). Previous approaches include dimension-free Harnack inequalities \([33, 37, 42]\) and the more direct approach via irreducibility \([58]\). In the case of multiplicative noise, existence of invariant measures for monotone drift equations has been proved in \([21]\).

**Structure of the paper.** We start with collecting some preliminary results in Section 2 including a known well-posedness result for \((1.1)\) and results on invariant measures of Markovian Feller semigroups. We conclude Section 2 with a discussion of our assumptions and a collection of our main examples for $\phi$. In Section 3, as one the main results, we shall prove a stability result for the semigroup, see Proposition \([3, 3] 4.1\). In Section 4, we shall first record a result for the case of one spatial dimension, see Lemma \([1, 2](i)\). Next, we prove the improved moment estimates and a concentration result for the unique invariant measure, see Theorem \([5, 1](ii)\) which implies our main result Theorem \([5, 1](ii)\). In Section 5, the main result Theorem \([5, 1](ii)\) is proved and applied to the main examples, see Theorem \([5, 1](ii)\) and the subsequent examples. In Section 6, under an additional assumption, we prove the maximal dissipativity of the Kolmogorov operator associated to the semigroup, see Theorem \([6, 3]\).

2. Preliminary results

2.1. Existence and uniqueness of solutions. Let us recall the following conditions from \([29]\), simplified with regard to the time-dependence of the drift coefficients, which is not needed here. The results are adapted to the Gelfand triple \footnote{3 Let $H$ be a separable Hilbert space and let $V$ be another Hilbert space (with topological dual $V^*$) which is embedded densely and continuously into $H$. Then, $H$ is identified with its dual via the Riesz isometry. The triple $V \subset H \subset V^*$ is called Gelfand triple.}.

\[
V := H^1_0(\Omega) \subset H := L^2(\Omega) \subset V^* = H^{-1}(\Omega)
\]
that also encodes the zero Dirichlet boundary condition. Here, we use the shorthand notation $H^1_0(\Omega) = W^{1,2}_0(\Omega)$ and the notation $H^{-1}(\Omega)$ for the topological dual space of $H_0^1(\Omega)$.

Suppose that $A : V \to V^*$ satisfies the following conditions: There exists a constant $C > 0$ such that

(A1) The map $u \mapsto A(u)$ is maximal monotone \footnote{That is, the graph of $A$ is maximal in the class of monotone graphs, ordered by set-inclusion, see \([2]\) for details.}

(A2) For all $u \in V$:
\[
\|A(u)\|_{V^*} \leq C\|u\|_V.
\]

\[2f : L^2(\Omega) \to \mathbb{R}\] is called $\beta$-Hölder-continuous if \(\sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|_{L^2(\Omega)}} =: |f|_\beta < +\infty.\]
(A3) For all $u \in V$, and for all $n \in \mathbb{N}$:

$$2\langle A(u), T_n(u) \rangle_{V'} \geq -C\|u\|_{V'}^2,$$

such that $C$ is independent of $n$, where $T_n := n(\text{Id} - (\text{Id} - \frac{1}{n})^{-1})$ denotes the Yosida-approximation of the negative Dirichlet Laplace operator $-\Delta = -\sum_{i=1}^{d} \partial_x \partial_{x_i}$.

Let $U$ be a separable Hilbert space. For a separable Hilbert space $S$, denote the space of Hilbert-Schmidt operators from $U$ to $S$ by $L_2(U, S)$. Denote by $\{W_t\}_{t \geq 0}$ a cylindrical Wiener process in $U$ for a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ that satisfies the standard assumptions.

**Assumption 2.1.** Assume that

(B1) $B \in L_2(U, V)$.

**Definition 2.2.** We say that a continuous $\{\mathcal{F}_t\}_{t \geq 0}$-adapted stochastic process $X : [0, T] \times \Omega \rightarrow H$ is a solution to

$$dX_t + A(X_t) dt = B dW_t, \quad X_0 = u_0,$$

if $X \in L^2(\Omega; C([0, T]; H)) \cap L^2([0, T] \times \Omega; V)$ and solves the following integral equation in $V^*$

$$X_t = u_0 - \int_0^t A(X_\tau) \, d\tau + B W_t,$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$.

**Theorem 2.3.** Suppose that conditions (A1)–(A3), (B1) hold. Let $u_0 \in L^2(\Omega, F_0, \mathbb{P}; V)$. Then there exists a unique solution in the sense of the previous definition to the equation

$$dX_t + A(X_t) dt = B dW_t, \quad X_0 = u_0,$$

that satisfies

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|X_t\|_V^2 \right] < +\infty.$$

**Proof.** See [29, Theorem 4.4].

**Definition 2.4.** An $\{\mathcal{F}_t\}_{t \geq 0}$-adapted stochastic process $X \in L^2(\Omega; C([0, T]; H))$ is called a generalized solution to (2.2) with starting point $u_0 \in H$ if for all approximations $u^m_0 \in V$, $m \in \mathbb{N}$ with $\|u^m_0 - u_0\|_H \rightarrow 0$ as $m \rightarrow +\infty$ and all $B_m \in L_2(U, V)$ such that $B_m \rightarrow B \in L_2(U, H)$ strongly in $L_2(U, H)$ as $m \rightarrow +\infty$, we have that

$$X^m \rightarrow X \quad \text{strongly in} \quad L^2(\Omega; C([0, T]; H)) \quad \text{as} \quad m \rightarrow +\infty.$$

**Theorem 2.5.** Suppose that conditions (A1)–(A3) hold and that $B \in L_2(U, H)$. Let $u_0 \in L^2(\Omega, F_0, \mathbb{P}; H)$. Then there exists a unique generalized solution in the sense of the previous definition to the equation

$$dX_t + A(X_t) dt = B dW_t, \quad X_0 = u_0.$$

**Proof.** See [29, Theorem 4.6].

On a bounded, open and convex domain $O \subset \mathbb{R}^d$ with piecewise $C^2$-boundary, we shall consider the $\Phi$-Laplace SPDE

$$dX_t = \text{div} [\phi(\nabla X_t)] dt + B dW_t, \quad X_0 = u_0,$$

See [19] for the notion of a cylindrical Wiener process in Hilbert space.

Also referred to as limit solution.
with zero Dirichlet boundary conditions for \( t \in [0, T] \) and \( u_0 \in H := L^2(\Omega) \), where \( \phi \) is of the form \([14]\). We sometimes use the notation

\[
A(u) := - \text{div}[\phi(\nabla u)],
\]

which is made rigorous in the compact Gelfand triple \([2.1]\). Indeed,

\[
\langle A(u), v \rangle_V = \int_\Omega \langle \phi(\nabla u), \nabla v \rangle \, dx, \quad u, v \in V.
\]

**Lemma 2.6.** Suppose that \( \phi \) satisfies \((C1)\)–\((C3)\). Then \( A \) satisfies conditions \((A1)\)–\((A3)\).

**Proof.** Set

\[
\Psi(x) := \int_0^{|x|} \psi(r) \, dr,
\]

which is a radially symmetric, convex, continuous and convex function with \( \Psi(0) = 0 \) and at most quadratic growth. We have the representation

\[
A(u) = \nabla^* \circ \partial \left( \int_\Omega \Psi(\cdot) \, dx \right) \circ \nabla u, \quad u \in V,
\]

where \( \partial \) denotes the subgradient and \( \nabla^* = (-\Delta)^{-1} \circ \text{div} \) denotes the adjoint operator of \( \nabla : V = H^1_0(\Omega) \to L^2(\Omega; \mathbb{R}^d) \). We may apply the results from \([29\text{ Proposition 7.1 and Section 7.2.1}]\) to yield the claim. See, in particular, \([29\text{ Example 7.9}]\). We note that we use the convexity assumption on the boundary here, cf. \([29\text{ Proposition D.2 and 7\text{ Proposition 8.2}]\). □

Applying Theorem 2.5, we obtain the following existence and uniqueness result.

**Corollary 2.7.** Suppose that \( \phi \) satisfies \((C1)\)–\((C3)\), \((C5)\) and that \( B \) satisfies \((B1)\). Then there exists a unique generalized solution to \((2.5)\) in the sense of Definition \([2.4]\).

### 2.2. Feller semigroups and invariant measures

Following \([18]\), we set

\[
P_t f(u) := \mathbb{E}[f(X_t^u)], \quad f \in \text{Lip}_b(H), \quad t \geq 0,
\]

where \( \{X_t^u\}_{t \geq 0} \) denotes a (generalized) solution to \((2.5)\) with initial datum \( X_0 = u \). By the results in \([29\text{ Section 6.4}]\), the extension of \( \{P_t\}_{t \geq 0} \) to \( C_b(H) \) defines a Markovian transition semigroup. It can easily be seen that \( \{P_t\}_{t \geq 0} \) is stochastically continuous, that is,

\[
\lim_{t \to 0^+} P_t f(u) = f(u), \quad \text{for all } f \in \text{Lip}_b(H), \quad u \in H,
\]

cf. \([18\text{ Proposition 2.1.1}]\), and satisfies the Feller property, that is, for any \( f \in C_b(H) \) and \( t \geq 0 \) one has \( P_t f \in C_b(H) \). Denote its dual semigroup restricted to finite Borel measures by \( \{P_t^*\}_{t \geq 0} \). A probability measure \( \mu \) is called **invariant** for \( \{P_t\}_{t \geq 0} \) if \( P_t^* \mu = \mu \) for any \( t \geq 0 \).

We recall the following concepts defined e.g. in \([35]\).

**Definition 2.8.** We say that \( \{P_t\}_{t \geq 0} \) is **weak-*mean ergodic** if there exists a Borel probability measure \( \mu_* \) on \( \mathcal{B}(H) \), that is the Borel sets of \( H \), such that

\[
\text{w-} \lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t^* \nu \, dt = \mu_* \quad \text{for every Borel probability measure } \nu \text{ on } \mathcal{B}(H),
\]

where the limit is in the sense of weak convergence of probability measures.

We say that the **weak law of large numbers** holds for \( \{P_t\}_{t \geq 0} \), for a function \( f \in \text{Lip}_b(H) \) and for a probability measure \( \nu \) on \( \mathcal{B}(H) \) if

\[
P_{\nu^*} \text{-} \lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t f(X_t) \, dt = \int_H f \, d\mu_*,
\]
where $\mu_*$ denotes the invariant measure of $\{P_t\}_{t \geq 0}$ and $\{X_t\}_{t \geq 0}$ denotes the Markov process related to $\{P_t\}_{t \geq 0}$ whose initial distribution is $\nu$ and whose path measure is $\mathbb{P}_\nu$, and where the convergence takes place in $\mathbb{P}_\nu$-probability.

As noted in [35, Remark 3], (2.8) implies uniqueness of the invariant measure.

**Theorem 2.9.** Suppose that $\phi$ satisfies $(C1)$–$(C5)$ and that

$$s < \frac{4}{d + 2} \wedge 1.$$  

Then there exists a unique invariant measure $\mu$ for the semigroup $\{P_t\}_{t \geq 0}$ such that $\{P_t\}_{t \geq 0}$ is weak-$*$-mean ergodic and the weak law of large numbers holds for any $f \in \text{Lip}_b(\mathcal{H})$ and any probability measure $\nu$.

**Proof.** We would like to use [29, Remark 5.5 and Proposition 7.1] to see that the hypotheses of [29, Theorem 5.6] are satisfied. For that we only have to verify that our equation admits a Lyapunov function with compact sublevel sets and that the solution to the deterministic counterpart of our equation (that is, $B \equiv 0$) vanishes for $t \to +\infty$.

By Lemma 2.10, we get for $u \in V$,

$$(2.10) \quad 2 \langle A(u), u \rangle_V = 2 \int_\Omega \langle \phi(\nabla u), \nabla u \rangle dx \geq 2 \int_\Omega |\nabla u|^{2-s} dx - 2K|\Omega| =: \Theta(u),$$

where we denote $|\Omega| := \int_\Omega dx$. By (2.9) and the Rellich-Kodrachov theorem, $W^{1,2-s}_0(\Omega) \subset L^2(\Omega)$ compactly, thus $\Theta$ is a Lyapunov function with compact sublevels. Let $(v_t)_{t \geq 0}$ be a solution to the deterministic equation,

$$dv_t + A(v_t) dt = 0, \quad t > 0, \quad v_0 = u.$$

It remains to prove that

$$\lim_{t \to +\infty} \|v_t\|_H = 0$$

for every initial datum $u \in H$. By the chain rule,

$$\frac{d}{dt} \|v_t\|_H^2 \leq -2C \left( \|v_t\|_H^2 \right)^{\frac{2-s}{s}} + 2K|\Omega|.$$

In analogy to [29, Proof of Remark 5.5], we see that $f(t) := \|v_t\|_H^2 e^{-2K|\Omega| t}$ is a subsolution to the ordinary differential equation

$$f'(t) = -2Cf(t)\frac{2-s}{s}.$$

Hence

$$\|v_t\|_H^2 \leq e^{2K|\Omega| t} \left( (\|u\|_H^s + Cst) \vee 0 \right)^{\frac{2}{s}} \to 0,$$

as $t \to +\infty$. Now, we may apply [29, Remark 5.5 and Proposition 7.1] and see that the hypotheses for [29, Theorem 5.6] are satisfied. Thus the claimed result follows.\[\square\]

The above result is improved in Theorems 4.3 and 3.3 below, as we just need to assume (C6) instead of (2.9). We refer to [0] for a new unified two-step approach to proving the existence of invariant measures.
2.3. Discussion of the assumptions. If \( s \in (0, 1] \), (C5) follows readily from the other assumptions.

**Lemma 2.10.** Assumptions (C1)–(C4), together with the assumption that (C4) holds with \( s \in (0, 1] \), imply that there exist constants \( c = c(a, b) > 0 \) and \( K = K(a, b, s, C) \geq 0 \) such that
\[
\psi(r) \cdot r \geq c|r|^{2-s} - K, \quad r \geq 0.
\]
In particular, then (C5) follows.

**Proof.** Let \( s \in (0, 1] \). Integrating \((a + b)|r|^s|\psi'(r)| \geq 1\) by parts over \([0, x] \), \( x \geq 0\) yields
\[
(a + b)|r|^s \psi(x) \geq x + bs \int_0^x \psi(r)|r|^{s-1} dr \geq x.
\]
Hence,
\[
a|x|\psi(x) + a\psi(x) + b|x|\psi(x) \geq a|x|^{1-s}\psi(x) + b|x|\psi(x) \geq |x|^{2-s}.
\]
Using (C3) and rearranging terms yields
\[
K(a, s, C) + \frac{1}{2}|x|^{2-s} + (a + b)|x|\psi(x) \geq |x|^{2-s},
\]
which proves the claim for \( s \in (0, 1] \). \( \square \)

In Section 3 we shall discuss the following examples.

**Example 2.11.** The following examples satisfy Assumptions (C1)–(C5).

- **Singular \( p \)-Laplace:** Let \( p \in (1, 2) \). Then \( \psi(r) = |r|^{p-2}r \) satisfies (C1)–(C5) with \( C = c = K = 1, a = 0, b = (p-1)^{-1}, s = 2 - p \).
- **Non-Newtonian fluids:** Let \( p \in (1, 2) \). Then \( \psi(r) = (1 + |r|^2)^{p/2}r \) satisfies (C1)–(C5) with \( C = K = c = 1, a = b = (p - 1)^{-1}, s = 2 - p \).
- **Log-diffusion:** Let \( \psi(r) = \log(1 + |r|) \operatorname{sgn}(r) \). Then \( \psi \) satisfies (C1)–(C5) with \( C = a = b = s = c = 1, K = 2 \).
- **Minimal surface flow:** Let \( \psi(r) = \frac{1}{\sqrt{1 + |r|^2}} \). Then \( \psi \) satisfies (C1)–(C5) with \( C = a = b = c = K = 1 \).
- **Curve shortening flow:** Let \( \psi(r) = \arctan(r) \). Then \( \psi \) satisfies (C1)–(C5) with \( C = a = b = c = K = 1, s = 2 \).

**Remark 2.12.** Note that for the first two examples, (0.9) translates to \( p > \frac{2d}{d+2} \) and \( d < \frac{2p}{2-p} \) respectively, which was assumed e.g. in [41]. However, for the first two examples, we just need to assume condition (C6) which is \( p \geq 2 - \frac{4}{d} \), and \( d \leq \frac{4}{2-p} \) respectively.

3. Stability

In this section, stability, that is, rates of convergence for large times for the solutions to 2.5 starting at two distinct initial data will be established. Unless otherwise stated, we assume that conditions (C1)–(C6) hold. On a bounded, open and convex domain \( \mathcal{O} \subset \mathbb{R}^d \) with piecewise \( C^2 \)-boundary, we shall consider the \( \Phi \)-Laplace SPDE 2.5 such that \( B \) satisfies (B1).

Let us first record a lemma for the situation that \( d = 1 \). Let \( \mathcal{O} = (0, L) \subset \mathbb{R} \), for some \( L > 0 \). Set \( I := \mathcal{O} = [0, L] \). Let \( \Delta = \partial_{xx} \) be the Dirichlet Laplace on \( \mathcal{O} \).

**Lemma 3.1.** Let \( u, v \in V = H_0^1(I) \). For the line segment \( \gamma : [0, 1] \to L^2(0, L), \; \lambda \mapsto \gamma(\lambda, u, v), \) where \( \gamma(\lambda, u, v)(x) := \partial_x u(x) + \lambda(\partial_x v(x) - \partial_x u(x)) \), we have that
\[
H^{-1}(A(u) - A(v), u - v)_{H^3} \geq \frac{\|u - v\|_H^2}{L} \int_0^1 \left( \int_0^L (\psi'(\gamma(\lambda, u, v)(x)))^{-1} dx \right)^{-1} d\lambda.
\]
Proof. First note that (C4) guarantees that the expression on the RHS of (3.1) is almost surely less or equal to zero as for $d\lambda$-a.e. $\lambda \in [0,1]$,

$$\int_0^L (\psi'(\gamma, u, v)(x))^{-1} dx \leq 2^{(s-1)\nu_0} \int_0^L (a + b|\partial_x u(x)|^s + b|\partial_x v(x)|^s) \, dx < +\infty.$$  

We find the identity

$$H^{-1}(A(u) - A(v), u - v)_{H^1} = \int_0^L (\phi(\partial_x u(x)) - \phi(\partial_x v(x))) (\partial_x u(x) - \partial_x v(x)) \, dx$$

$$= - \int_0^L \int_0^1 \frac{d}{d\lambda} \phi(\gamma(\lambda, u, v)(x))(\partial_x u(x) - \partial_x v(x)) \, d\lambda \, dx$$

$$= \int_0^L \int_0^1 \psi'(\gamma, u, v)(x)(\partial_x u(x) - \partial_x v(x))^2 \, d\lambda \, dx$$

Let $\tilde{u}, \tilde{v}$ denote continuous representatives of $u, v \in H^1_0(0, L)$. Note that $\tilde{u}(0) = \tilde{v}(0) = 0$. Using the fundamental theorem of calculus and Hölder inequality yields

$$\left|\tilde{u}(x) - \tilde{v}(x)\right|^2 = \left(\int_0^x (\partial_y u(y) - \partial_y v(y)) \, dy\right)^2$$

$$\leq \left(\int_0^L \psi'(\gamma, u, v)(x)(\partial_x u(x) - \partial_x v(x))^2 \, dx\right) \times \left(\int_0^L (\psi'(\gamma, u, v)(x))^{-1} \, dx\right).$$

Hence,

$$\frac{1}{L} \int_0^L |u(x) - v(x)| \, dx$$

$$\leq \left(\int_0^L \psi'(\gamma, u, v)(x)(\partial_x u(x) - \partial_x v(x))^2 \, dx\right) \times \left(\int_0^L (\psi'(\gamma, u, v)(x))^{-1} \, dx\right).$$

By dividing and integrating with respect to $\lambda$, we achieve

$$\frac{\|u - v\|_{H^1}^2}{L} \int_0^1 \left(\int_0^L (\psi'(\gamma, u, v)(x))^{-1} \, dx\right)^{-1} \, d\lambda$$

$$\leq \int_0^1 \int_0^L \psi'(\gamma, u, v)(x)(\partial_x u(x) - \partial_x v(x))^2 \, dx \, d\lambda.$$

Now, consider the general case that $d \geq 1$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded, open and convex domain with piecewise $C^2$-boundary. We denote $|\mathcal{O}| := \int_{\mathcal{O}} \, dx$. We define $\Psi : \mathbb{R}^d \to \mathbb{R}$ by

$$\Psi(x) := \int_0^{|x|} \psi(r) \, dr$$

(3.2)

We see that $\Psi$ is a radially symmetric convex function of at most quadratic growth, and its first and second derivatives are given by

$$D\Psi(x) = \phi(x) = \begin{cases} 0, & \text{if } x = 0, \\ \psi(|x|) \frac{x}{|x|}, & \text{if } x \neq 0, \end{cases}$$

and by

$$(D^2\Psi(x))_{i,j} = \psi(|x|) \frac{x_i x_j}{|x|^2} + \left(\psi(|x|) - \frac{\psi(|x|)}{|x|} |x| \right) \delta_{ij}.$$
with special case \((D^2\Psi(x))(1,1) = \psi'(|x|)\) for \(d = 1\).

Using the Gram matrix, one obtains that the eigenvalues of \(D^2\Psi(x)\) are \(\psi'(|x|)\) and \(\frac{\psi(|x|)}{|x|}\) with respective multiplicities 1 and \(d - 1\). In particular,

\[
\langle D^2\Psi(x)h, h \rangle \geq \psi'(|x|) \wedge \left( \frac{\psi(|x|)}{|x|} \right) |h|^2 \quad \text{for every} \quad h \in \mathbb{R}^d.
\]

As above, for \(u, v \in V\), the operator \(A(u)\) can be defined via

\[
\langle V, \langle A(u), v \rangle \rangle_V = \int_\Omega \langle \phi(\nabla u), \nabla v \rangle \, dx.
\]

For \(u \in C^2(\mathcal{O})\) we have the representation

\[
A(u) = -\frac{\psi(|\nabla u|)}{|\nabla u|} \Delta u - \left( \psi'(\|\nabla u\|) - \frac{\psi(|\nabla u|)}{|\nabla u|} \right) \sum_{i,j=1}^d \frac{\partial_i \partial_j u \partial_i u \partial_j u}{|\nabla u|^2}.
\]

Also, on a formal level,

\[
\langle A(u), v \rangle_V = -\sum_{i=1}^d \int_\mathcal{O} \partial_{x_i} \left( \text{div}(\phi(\nabla u)) \right) \partial_{x_i} v \, dx
\]

\[
= \int_\mathcal{O} \text{div}(\phi(\nabla u)) \Delta v \, dx
\]

\[
= \sum_{i=1}^d \int_\mathcal{O} \langle D^2\Psi(\nabla u) \nabla \partial_{x_i} u, \nabla \partial_{x_i} v \rangle \, dx.
\]

**Lemma 3.2.** Let \(d \geq 2\). For \(u, v \in H^1_h(\Omega)\), define \(\gamma : [0,1] \to L^2(\mathcal{O}; \mathbb{R}^d), \lambda \mapsto \gamma(\lambda, u, v)\) via \(\gamma(\lambda, u, v)(x) = \nabla u(x) + \lambda(\nabla v(x) - \nabla u(x))\). Set \(\Lambda_{\min}(x) := \psi'(|x|) \wedge \frac{\psi(|x|)}{|x|}\). Then there exists \(C = C(\mathcal{O}, d) > 0\),

\[
H^{-1}(A(u) - A(v), u - v)_{H^1_h} \geq C \|u - v\|_{L^2}^2 \int_0^1 \left( \int_\mathcal{O} \Lambda_{\min}(\gamma(\lambda, u, v)(x))^{-\frac{2}{d}} \, dx \right)^{-\frac{1}{2}} d\lambda.
\]

**Proof.** As in the assertion, \(\Lambda_{\min}(x) := \psi'(|x|) \wedge \frac{\psi(|x|)}{|x|}\). First note that (C4) and (C6) guarantee that the expression on the RHS is well-defined as for \(d\lambda\)-a.e. \(\lambda \in [0,1]\),

\[
\int_\mathcal{O} \Lambda_{\min}(\gamma(\lambda, u, v)(x))^{-\frac{2}{d}} \, dx \leq \int_\mathcal{O} \left( a + b|\nabla u(x)|^{\frac{4}{d}} + b|\nabla v(x)|^{\frac{4}{d}} \right) \, dx < +\infty.
\]

We find the identity

\[
H^{-1}(A(u) - A(v), u - v)_{H^1_h} = \int_\mathcal{O} \langle \psi(\nabla u(x)) - \psi(\nabla v(x)), \nabla u(x) - \nabla v(x) \rangle \, dx
\]

\[
= -\int_\mathcal{O} \int_0^1 \frac{d}{d\lambda} \langle \psi(\gamma(\lambda, u, v)(x))(\nabla u(x) - \nabla v(x)) \rangle \, d\lambda \, dx
\]

\[
= \int_\mathcal{O} \int_0^1 \langle D^2\Psi(\gamma(\lambda, u, v)(x))(\nabla u(x) - \nabla v(x)), \nabla u(x) - \nabla v(x) \rangle \, d\lambda \, dx,
\]
where \( \Psi \) is as in (3.2). From the embedding of \( W_{0}^{1,\frac{d}{2}}(O) \) into \( L^{2}(O) \) and Hölder’s inequality, we derive

\[
\|u - v\|_{L^{2}} \leq C \|u - v\|_{W_{0}^{1,\frac{d}{2}}} \leq C \left( \int_{O} \left( D^{2}\Psi(\gamma(\lambda, u, v)(x))((\nabla u(x) - \nabla v(x)), \nabla u(x) - \nabla v(x))\right)^{\frac{d}{2}} \Lambda_{\min}(\gamma(\lambda, u, v)(x))^{-\frac{d}{2}} dx \right) \frac{d+2}{d}
\]

\[
\leq C \left( \int_{O} \left( D^{2}\Psi(\gamma(\lambda, u, v)(x))((\nabla u(x) - \nabla v(x)), \nabla u(x) - \nabla v(x))\right) dx \right)^{\frac{d}{2}} \times \left( \int_{O} \Lambda_{\min}(\gamma(\lambda, u, v)(x))^{-\frac{d}{2}} dx \right)^{\frac{1}{2}},
\]

where \( C > 0 \) may change from line to line. Integration with respect to \( \lambda \) yields

\[
\|u - v\|_{L^{2}}^{2} \int_{0}^{1} \left( \int_{O} \Lambda_{\min}(\gamma(\lambda, u, v)(x))^{-\frac{d}{2}} dx \right) d\lambda \leq C \int_{0}^{1} \int_{O} \left( D^{2}\Psi(\gamma(\lambda))(\nabla u(x) - \nabla v(x)), \nabla u(x) - \nabla v(x) \right) dx d\lambda
\]

Rearranging terms yields the result.

\[ \square \]

**Proposition 3.3.** Suppose that \( d \geq 1 \) with \( O \subset \mathbb{R}^{d} \) bounded, open and convex. Let \( u \) and \( v \) be solutions to (1.1) with \( \phi \) as in (1.4) and let \( u_{0}, v_{0} \in V \). Let \( d_{0} := 1 + \frac{d}{2} \). For \( 0 \leq \alpha \leq 1 \), \( l \geq 1 \) and \( s \in (0, 2] \) as in (C4), we have that

\[
\|u_{t} - v_{t}\|_{H}^{2a} \leq t^{-\alpha l}C(l, \alpha, O, d) \left( \frac{1}{l} \int_{0}^{l} (|a(O)|^{\alpha d_{0}} + b^{\alpha d_{0}}\|u_{t}\|_{V}^{\alpha d_{0}} + b^{\alpha d_{0}}\|v_{t}\|_{V}^{\alpha d_{0}}) \right)^{\frac{\alpha}{\alpha l}} \|u_{0} - v_{0}\|_{H}^{2a}.
\]

**Proof.** Starting from Lemmas 3.1 and 3.2 where \( \gamma(\lambda, u, v)(x) = \nabla u(x) + \lambda(\nabla v(x) - \nabla u(x)) \),

\[
\frac{1}{2} \frac{d}{dt} \|u_{t} - v_{t}\|_{H}^{2} \leq -C\|u_{t} - v_{t}\|_{H}^{2} \int_{0}^{1} \left( \int_{O} (\psi'(\gamma(\lambda, u_{t}, v_{t})(x)))^{-d_{0}} dx \right)^{-\frac{1}{d_{0}}} d\lambda,
\]

where in the \( d = 1 \) case, \( C = L^{-1} \). Gronwall’s Lemma yields

\[
\|u_{t} - v_{t}\|_{H}^{2} \leq \|u_{0} - v_{0}\|_{H}^{2} \exp \left( -2C \int_{0}^{t} \int_{O} (\psi'(\gamma(\lambda, u_{\tau}, v_{\tau})(x)))^{-d_{0}} dx \right)^{-\frac{1}{d_{0}}} d\tau d\tau.
\]

For \( l \geq 1 \), \( x \geq 0 \), the elementary estimate \( \log(\frac{x}{t}) \leq \frac{x}{t} - 1 \) implies

\[
e^{x} \geq c_{l}x^{l}
\]
Lemma 4.1. Let that we will utilize later. Let \(c\) with \(s\) as in Assumption \((C4)\), there exists constants \(C(1)\)–\(C6)\) and \((B1)\).

\[
\|u_0 - v_0\|_{\mathcal{H}}^{2\alpha} \geq C^2 \left(\int_0^t 2C \int_0^1 \left(\int_\Omega \left(\psi^\prime(\gamma(\lambda, u_\tau, v_\tau))(x)\right)^{-\alpha^0} dx\right)^{\frac{1}{\alpha}} d\lambda d\tau\right)^{\frac{1}{\alpha}} \|u_t - v_t\|_{\mathcal{H}}^{2\alpha}.
\]

\[
geq C^2 \left(\int_0^t \int_0^1 \left(\int_\Omega \left(\psi^\prime(\gamma(\lambda, u_\tau, v_\tau))(x)\right)^{-\alpha^0} dx\right)^{\frac{1}{\alpha}} d\tau\right)^{\frac{1}{\alpha}} \|u_t - v_t\|_{\mathcal{H}}^{2\alpha}.
\]

\[\geq C^2 (2C)^{\alpha t} \left(\int_0^t \int_0^1 \left(\int_\Omega \left(\psi^\prime(\gamma(\lambda, u_\tau, v_\tau))(x)\right)^{-\alpha^0} dx\right)^{\frac{1}{\alpha}} d\tau\right)^{\frac{1}{\alpha}} \|u_t - v_t\|_{\mathcal{H}}^{2\alpha}.
\]

\[
\geq C^2 (2C)^{\alpha t} \left(\int_0^t \int_0^1 \left(\int_\Omega \left(\psi^\prime(\gamma(\lambda, u_\tau, v_\tau))(x)\right)^{-\alpha^0} dx\right)^{\frac{1}{\alpha}} d\tau\right)^{\frac{1}{\alpha}} \|u_t - v_t\|_{\mathcal{H}}^{2\alpha}.
\]

\[\geq C^2 (2C)^{\alpha t} \left(\int_0^t \int_0^1 \left(\int_\Omega \left(\psi^\prime(\gamma(\lambda, u_\tau, v_\tau))(x)\right)^{-\alpha^0} dx\right)^{\frac{1}{\alpha}} d\tau\right)^{\frac{1}{\alpha}} \|u_t - v_t\|_{\mathcal{H}}^{2\alpha}.
\]

\[\square
\]

4. Moment estimates

Consider equation \((2.5)\) with the same assumptions as stated in the beginning of Section 3 in other words, we assume \((C1)\)–\((C6)\) and \((B1)\).

Let us start with recording a second order functional inequality for the one-dimensional case that we will utilize later. Let \(d = 1\). Let \(\Omega = (0, L) \subset \mathbb{R}\), for some \(L > 0\). Set \(I := \bar{\Omega} = [0, L] \). Let \(\Delta = \partial_{xx}\) be the Dirichlet Laplace on \(I\). Let us equip \(W_{0, 1}^1(I)\) with the norm \(\|u\|_{W_{0, 1}^1(I)} = \int_I |\partial_x u|^2 dx\).

Lemma 4.1 (compare with Lemma 2.2. in \([26]\)). For \(u \in C_0^\infty(I)\) and \(a \geq 0, b > 0\) and \(s \in (0, 2)\) as in Assumption \((C4)\), there exists constants \(C(L, a, s) > 0\) and \(C(L, b, s) > 0\) such that

\[
\left(\int_I |\partial_x u|^2 dx\right)^{\frac{s^*}{s}} \leq s^* \int_I \psi^\prime(\partial_x u) |\partial_x u|^2 dx + C(L, b, s)\|u\|_{W_{0, 1}^1(I)}^{1 + (2 - s)} + C(L, a, s),
\]

where

\[s^* := (2 - s) \sqrt{\frac{4 - s}{2 + s}}.\]

Proof. Let \(u \in C_0^\infty(I)\). Let \(a \in \left[\frac{1}{2}, 2 - \frac{s}{2}\right]\). By Hölder inequality, Young inequality and \((C4)\),

\[
\left(\int_I |\partial_x u| dx\right)^{\alpha} \leq \int_I \psi^\prime(\partial_x u) |\partial_x u|^2 dx \leq \left(\int_I \psi^\prime(\partial_x u) d\lambda\right)^{\frac{1}{2}} d\lambda \leq \frac{\alpha}{2} \int_I \psi^\prime(\partial_x u) |\partial_x u|^2 dx + \frac{2 - \alpha}{2} \left(\int_I (a + b(|\partial_x u|^s) dx\right)^{\frac{\alpha}{2 - \alpha}}
\]

\[
\leq \frac{\alpha}{2} \int_I \psi^\prime(\partial_x u) |\partial_x u|^2 dx + \frac{2 - \alpha}{2} (aL)^{\frac{\alpha}{2 - \alpha}} + b \frac{\alpha}{2 - \alpha} \left(\int_I |\partial_x u|^s dx\right)^{\frac{\alpha}{2 - \alpha}}.
\]
Furthermore, by Jensen’s inequality and the embedding $W^{1,1}_0(I) \hookrightarrow C(I)$ (note that $u(0) = 0$),
\[
L^\frac{s}{2} \left( \int_I |\partial_x u|^s dx \right)^{\frac{2}{s}} \leq \int_I |\partial_x u|^2 dx = -\int_I u \partial_{xx} u dx \leq \|u\|_\infty \int_I |\partial_x u| dx \leq \|\partial_x u\|_{L^1(I)} \int_I |\partial_x u| dx.
\]
Since $\alpha < 2 - \frac{s}{2}$, we can apply Young’s inequality again and get altogether,
\[
2 \left( \int_I |\partial_x u| dx \right)^\alpha \leq \alpha \int_I \psi'(\partial_x u) |\partial_x u|^2 dx + C(L, b, \alpha, s) \left( \int_I |\partial_x u|^s dx \right)^{\frac{2}{s}} + C(L, a, \alpha)
\]
\[
\leq \alpha \int_I \psi'(\partial_x u) |\partial_x u|^2 dx + C(L, b, \alpha, s) \left( \int_I |\partial_x u|^2 dx \right)^{\frac{2}{s}} + C(L, a, \alpha)
\]
\[
\leq \alpha \int_I \psi'(\partial_x u) |\partial_x u|^2 dx + \left( \int_I |\partial_x u| dx \right)^\alpha + C(L, b, \alpha, s) \|\partial_x u\|_{L^1(I)}^{\frac{2}{s}} + C(L, a, \alpha).
\]
Choosing $\alpha = s^* = (2 - s) \vee \frac{2}{2-s}$ yields the proof. □

Now, consider the general multivariate case $d \geq 1$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded, open and convex domain with piecewise $C^2$-boundary. To achieve the main result, we first show a pathwise regularity result for (2.5). This result combined with moment estimates for the finite dimensional Galerkin approximations of (2.5) yields the existence of a unique invariant measure for (2.5).

**Lemma 4.2.** For every $0 < s \leq 2$, and every $1 < \alpha \leq \frac{2-s}{d-s}$, there exists $C = C(d, s, \mathcal{O}) > 0$ such that for every $u \in C_0^\infty(\mathcal{O})$,
\[
\left( \int_{\mathcal{O}} \sum_{i=1}^d \sum_{j=1}^d |\partial_{x_i} \partial_{x_j} u|^\alpha dx \right)^{\frac{2}{\alpha}} \leq C \left( 1 + \sum_{i=1}^d \int_{\mathcal{O}} \langle D^2 \psi(\nabla u) \nabla \partial_{x_i} u, \nabla \partial_{x_i} u \rangle dx \right)^{\frac{2}{2-s}},
\]
where $s^*$ is as in (4.1) and $\Psi$ is defined as in (3.2).

**Proof.** Set $\Lambda_{\min}(x) := \psi'(|x|) \wedge \frac{\psi(|x|)}{|x|}$. Let $1 < \alpha \leq \frac{2-s}{d-s}$. First, by using H"older’s and Young’s inequalities, we get for $\varepsilon > 0$,
\[
\left( \int_{\mathcal{O}} \sum_{i=1}^d \sum_{j=1}^d |\partial_{x_i} \partial_{x_j} u|^\alpha dx \right)^{\frac{2}{\alpha}} \leq C \left( \int_{\mathcal{O}} \sum_{i=1}^d \langle D^2 \psi(\nabla u) \nabla \partial_{x_i} u, \nabla \partial_{x_i} u \rangle dx \right)^{\frac{2}{2-s}} \cdot \Lambda_{\min}^{-\frac{s}{2}}(\nabla u) dx \leq C \left( \int_{\mathcal{O}} \sum_{i=1}^d \langle D^2 \psi(\nabla u) \nabla \partial_{x_i} u, \nabla \partial_{x_i} u \rangle dx \right)^{\frac{2}{2-s}} \times \left( \int_{\mathcal{O}} \Lambda_{\min}^{-\frac{s}{2}}(\nabla u) dx \right)^{(2-s)(2-\alpha)} \frac{2}{2-s} \cdot \Lambda_{\min}^{-\frac{s}{2}}(\nabla u) dx \leq C \left[ \frac{2-s}{2} \cdot \varepsilon^{-\frac{2}{2-s}} \int_{\mathcal{O}} \sum_{i=1}^d \langle D^2 \psi(\nabla u) \nabla \partial_{x_i} u, \nabla \partial_{x_i} u \rangle dx + \frac{s}{2} \varepsilon^{\frac{2}{2-s}} \left( \int_{\mathcal{O}} \Lambda_{\min}^{-\frac{s}{2}}(\nabla u) dx \right)^{(2-s)(2-\alpha)} \right].
We note that $W^{2,\alpha}_0(O)$ is continuously embedded into $W^{1,\frac{2\alpha}{\alpha-2}}_0(O)$ for $1 < \alpha \leq d\frac{2\alpha}{\alpha-2}$. This embedding, together with (C4), yields
\[
\left(\int_{O} \Lambda_{\min}^{-\frac{\alpha}{\alpha-2}}(\nabla u) \, dx\right)^{\frac{\alpha-2(2-\alpha)}{\alpha}} \leq C \left(1 + \int_{O} |\nabla u|^{\frac{2\alpha}{\alpha-2}} \, dx\right)^{\frac{\alpha-2(2-\alpha)}{\alpha}} \leq C \left(1 + \|u\|_{W^{2,\frac{2\alpha}{\alpha-2}}(O)}\right)^{\frac{\alpha-2(2-\alpha)}{\alpha}}.
\]
Choosing $\varepsilon$ small enough completes the proof.

We shall prove the existence of the unique invariant measure for the ergodic semigroup and the main concentration result now.

**Theorem 4.3.** There exists a unique invariant Borel probability measure $\mu$ for the semigroup $\{P_t\}$ of equation (2.5) that is concentrated on the subset $W^{2,\alpha}_0(O) \cap W^{1,2}_0(O) \subset L^2(O)$ for any $1 < \alpha \leq d\frac{2\alpha}{\alpha-2}$ if $d \geq 2$ and $\alpha = 1$ if $d = 1$ with
\[
\int_{H} \|u\|_{W^{2,\alpha}_0(O)}^{s^*} \mu(du) + \int_{H} \|u\|_{V}^{s^*} \mu(du) + \sum_{i=1}^{d} \int_{H} \langle D^{2}\Psi(\nabla u) \partial_{x_i} \nabla u, \partial_{x_i} \nabla u \rangle_{H} \mu(du) < +\infty,
\]
where $s^*$ is as in (4.1) and $s \in (0, 2]$ is as in assumption (C4) and $\Psi$ is defined as in (3.2).

**Proof.** For a complete orthonormal system $\{e_k\}_{k \in \mathbb{N}}$ of $H = L^2(O)$, and for $H_m := \text{span}\{e_k : 1 \leq k \leq m\}$, we consider the (finite dimensional) approximating stochastic differential equation in $H$
\[
u_{t}^{m} dt = P_m B \, dW_t,
\]
\[u_{0}^{m} = P_m u_{0},\]
where $A$ is as in (2.6). Note that here, we cannot use the standard Galerkin approximation as in [25], for there is no guarantee that the nonlinear operator $A$ maps $H_m$ into $V = H^{1}_0(O)$. One option would be to add an extra smoothing step for the operator $A$ as e.g. in [31, Proof of Theorem 3.1]. Instead, we follow the strategy from [29, Section 6.3.1] to find solutions $u_{n}^{m}$ converging to $u$ in $L^2(\Omega, C([0,T];H))$ with the aim to consider the corresponding invariant measures $\nu_{n}$. For $n \in \mathbb{N}$, let $J_n := (\text{Id} - \frac{\Delta}{n})^{-1}$ and $T_n := -\Delta J_n = n(\text{Id} - J_n)$, respectively, be the resolvent and the smooth Yosida approximation of the nonnegative definite differential operator $-\Delta = -\sum_{i=1}^{d} \partial_{x_i} \partial_{x_i}$, respectively. Let $\langle u, v \rangle_n := \langle u, T_n v \rangle_H$. The induced norms $\|\cdot\|_n$ are equivalent to the $H$-norm and converge monotonically as follows
\[
\|u\|_n \uparrow \|u\|_V, \quad u \in V, \quad +\infty, \quad \text{otherwise}.
\]
Applying Itô’s formula to $\|u_{n}^{m}(t)\|_n^2$, we get
\[
\|u_{n}^{m}\|_n^2 = \|u_{0}^{m}\|_n^2 - 2 \int_{0}^{t} \langle A(u_{n}^{m}), T_n(u_{n}^{m}) \rangle \, d\tau + \int_{0}^{t} \|P_m B\|_{L^2(U,H_m)}^2 \, d\tau + M_{t}^{n}
\]
where $t \rightarrow M_{t}^{n} = 2 \int_{0}^{t} \langle u_{n}^{m}, P_m B \, dW_{\tau} \rangle_n$ is a local martingale. From [7, Proposition 8.2] and (C3), applied to [29, Proposition 7.1], it follows that
\[
\langle A(u_{n}^{m}), T_n(u_{n}^{m}) \rangle_H \geq 0 \quad d\tau\text{-a.e.}
\]
We note that the convexity assumption of for the domain $O$ enters here, as it is needed to apply [7, Proposition 8.2]. By taking expectation, together with Fatou’s lemma, employing the bounds $\|u_{n}^{m}\|_n \leq \|u_{0}\|_V$ and $\|P_m B\|_{L^2(U,H_m)}^2 \leq \|B\|_{L^2(U,V)}^2$, it follows from (4.3) for every $t \geq 1$
that

\[
\frac{1}{t} \mathbb{E} \left( \int_0^t \sum_{i=1}^d \langle D^2 \Psi(\nabla u^m_{\tau}), \nabla \partial_z u^m_{\tau}, \nabla \partial_x u^m_{\tau} \rangle \, dx \, d\tau \right) \leq \frac{1}{2} \left( \mathbb{E} \|u_0\|_V^2 + \|B\|_{L_2(U,V)}^2 \right) =: K_1,
\]

independently of \(m\). On the other hand, the application of Itô’s formula to \(t \mapsto \|u^m_t\|_H^2\) yields

\[
\|u^m_t\|_H^2 = \|u^m_0\|_H^2 - 2 \int_0^t \langle A(u^m), u^m \rangle_H \, dt + \int_0^t \|P_m B\|_{L_2(U,H,m)}^2 \, dt + N^m_t
\]

where \(t \mapsto N^m_t = 2 \int_0^t \langle u^m, P_m B dW^r \rangle_H + \langle A(u^m), u^m \rangle = \int_0^t (\phi(\nabla u^m), \nabla u^m) \, dx\). Note that by (C5) and Lemma 2.10, we get

\[
\psi(r) \cdot r \geq c|r|^{1/2-s} - K,
\]

for suitable constants \(c > 0\) and \(K \geq 0\). This combined with the same arguments as before entails that for every \(t \geq 1\)

\[
\frac{1}{t} \mathbb{E} \left( \int_0^t \int_O |\nabla u^m_{\tau}|^{1/2-s} \, dx \, d\tau \right) \leq \frac{1}{2c} \left( \|u_0\|_H^2 + \|B\|_{L_2(U,H)} + K \right) =: K_2
\]

for a positive constant \(K_2 > 0\).

Let us treat the case \(d = 1\) separately first, so let us assume that \(O = (0, L)\). By the above estimates, it is easy to prove that the family of probability measures \(\left\{ \frac{1}{t} \int_0^t P^* \delta_{u^m} \, d\tau \right\}_{t \geq 1}\) (where \(\delta_{u^m}\) denotes the Dirac measure in \(u_0\)) is tight in the finite dimensional space \(H_m\), and we thus find that by the Krylov-Bogoliubov existence theorem [18, Theorem 3.1.1] that there exists an invariant measure \(\nu_m\) for the Markov semigroup of (4.2) such that for arbitrary \(N > 0\)

\[
\int_I \int_I (|\partial_x u|^1 \, dx \wedge N) \, \nu_m(du) + \int_I \int_I (\psi'(\partial_x u) |\partial_x u|^2 \, dx \wedge N) \, \nu_m(du) \leq K_3,
\]

where \(K_3 = K_1 + K_2\). An application of the monotone convergence lemma yields

\[
\sup_{m \geq 1} \int_I \int_I (|\partial_x u|^1 \, dx \nu(du)) + \sup_{m \geq 1} \int_I \int_I (\psi'(\partial_x u) |\partial_x u|^2 \, dx \nu(du)) \leq K_3.
\]

Combining (4.6) with Lemma 4.1, we obtain

\[
\sup_{m \geq 1} \int_I \left( \int_0^L |\partial_x u| \, dx \right)^s \nu(du) < +\infty.
\]

Due to the compactness of the embedding \(W^{2,1}_0(0, L) \subset W^{1,1}_0(0, L)\), we get that the sequence of measures \(\{\nu_m\}_{m \geq 1}\) is tight w.r.t. the \(W^{1,1}\) topology. Let \(\nu\) denote the limit of a weakly convergent subsequence in the sense of weak convergence of probability measures. For a function \(u \in L^1_{loc}(B)\), we define its total variation on an open subset \(B \subset \mathbb{R}^d\) by

\[
[Du](B) := \sup \left\{ \int_B u \text{div} v \, dx : v \in C^1(\overline{B}, \mathbb{R}^d), \|v\|_\infty \leq 1 \right\}.
\]

Recall that \(u \in L^1(B)\) belongs to the space \(BV(B)\) of functions of bounded variation if \([Du](B) < +\infty\). Note that

\[
u \mapsto \left\{ \begin{array}{ll}
[Du](\mathbb{R}), & \text{if } \partial_x u \in BV(\mathbb{R}) \cap L^1(0, L), \ u \in W^{1,1}_0(0, L), \\
+\infty, & \text{if } \partial_x u \in L^1(0, L) \setminus BV(\mathbb{R}), \ u \in W^{1,1}_0(0, L),
\end{array} \right.
\]
is the lower semi-continuous envelope of
\[ u \mapsto \int_0^L |\partial_{xx} u| \, dx, \quad u \in W^{2,1}_0(0, L), \]
in \( W^{1,1}_0(0, L) \). The lower semi-continuity of \( u \mapsto \|D\partial_x u\|_{\mathbb{R}} \) w.r.t. the strong \( W^{1,1}_0(0, L) \)-topology, together with \( 4.8 \) yields by \[ 1 \) Proposition 1.62 (a)],
\[ \int_H (\|D\partial_x u\|_{\mathbb{R}})^* \nu(du) < +\infty. \]
However, by the tightness, it can be seen by standard arguments that the support of \( \nu \) is contained in \( W^{2,1}_0(0, L) \) and thus
\[ \int_H \left( \int_0^L \partial_{xx} u \, dx \right)^* \nu(du) < +\infty. \]
From the boundedness of the embedding of \( W^{2,1}_0(0, L) \) into \( W^{1,2}_0(0, L) \) it follows that
\[ \int_H \|u\|_V^* \nu(du) < +\infty. \]
Hence the support of \( \nu \) is included in \( W^{2,1}_0(0, L) \cap H^1_0(0, L) \).

Let us suppose \( d \geq 1 \) now. By the above estimates, it is easy to prove that the family of probability measures \( \left\{ \frac{1}{t} \int_0^t P^{\tau_\delta} \, d\tau \right\}_{t \geq 1} \) is tight in the finite dimensional space \( H_m \) and thus by the Krylov-Bogoliubov existence theorem [18, Theorem 3.1.1], as the bound \( 4.5 \) is independent of \( t \), there exists an invariant measure \( \nu_m \) for \( 4.2 \). To show uniqueness, note that for any Lipschitz function \( F \) and \( \nu_m \)-almost every \( u_0, v_0 \) in \( H_m \), Proposition 3.3 yields (for \( \alpha = \frac{1}{2} \) and \( l = d_0 = 1 \vee \frac{d}{2} \)) that
\[
\left| \mathbb{E} \left( \frac{1}{t} \int_0^t F(u_\tau) \, d\tau \right) - \mathbb{E} \left( \frac{1}{t} \int_0^t F(v_\tau) \, d\tau \right) \right| \\
\leq \text{Lip}(F) \mathbb{E} \left( \frac{1}{t} \int_0^t \|u_\tau - v_\tau\|_H \, d\tau \right) \\
\leq t^{-\frac{d}{2}} C \left( 1 + \frac{1}{t} \mathbb{E} \left( \int_0^t \|u_\tau\|_{V^\alpha}^\alpha + \|v_\tau\|_{V^\alpha}^\alpha \, d\tau \right) \right) \|u_0 - v_0\|_H \\
\leq t^{-\frac{d}{2}} C \left( 1 + \|u_0\|_{V^\alpha}^\alpha + \|v_0\|_{V^\alpha}^\alpha \right) \|u_0 - v_0\|_H \rightarrow 0,
\]
as \( t \to +\infty \), where \( C > 0 \) may change from line to line. The latter implies uniqueness of the invariant measure \( \nu_m \) of the ergodic semigroup of \( 4.2 \) by standard arguments, see e.g. [41] Section 2.1, proof of Theorem 1.2 (iii)].

Now, it follows from ergodicity that for arbitrary \( L > 0 \)
\[ \int_H \sum_{i=1}^d \int_O \left( \langle D^2 \Psi(\nabla u) \nabla u_{x_i}, \nabla u_{x_i} \rangle \, dx \wedge L \right) \nu_m(du) \leq C. \]
The monotone convergence lemma yields
\[ \sup_{m \geq 1} \int_H \sum_{i=1}^d \int_O \left( \langle D^2 \Psi(\nabla u) \nabla u_{x_i}, \nabla u_{x_i} \rangle \, dx \right) \nu_m(du) \leq C. \]
Combining (4.11) with Lemma 4.2, we obtain

\begin{equation}
\sup_{m \geq 1} \int_{\mathcal{H}} \|u^*\|_{W^{2, (2+\frac{\alpha}{s})}_0}^s \nu_m(du) < +\infty.
\end{equation}

Keeping assumption (C6) in mind, the compactness of the embedding $W^{2, \frac{2\alpha}{s}}_0(\mathcal{O})$ into $V = W^{2, \frac{2\alpha}{s}}_0(\mathcal{O})$ for $s < \frac{\alpha}{2}$ implies that the sequence of measures $\{\nu_m\}_{m \geq 1}$ is tight in $V$. Thus, there exists an invariant measure $\mu$ such that

$$\int_{\mathcal{H}} \|u^*\|_{V}^s \mu(du) < +\infty.$$  

The moment estimate for the $W^{2, \alpha}(\mathcal{O})$-norm, where $\alpha$ is as in the statement of the theorem, follows by lower semi-continuity from (4.12) and Proposition 1.62 (a).

The uniqueness of $\mu$ can be proved by the same arguments as for $\nu_m$, see the computation combined with Section 2.1, proof of Theorem 1.2 (iii).\]

As a consequence, we obtain the proof of Theorem 1.2 (i).

**Corollary 4.4** (Stochastic $p$-Laplace). Suppose that $\psi(r) = |r|^{p-2}r$, $p \in (1, 2)$. Then, under assumptions (C6) and (B1) we get that the unique invariant measure $\mu$ is concentrated on $\{u^*\}_{W^{2, \frac{2\alpha}{s}}_0(\mathcal{O}) \cap H^1_0(\mathcal{O}) \subset L^2(\mathcal{O})$ and satisfies

$$\int_{L^2(\mathcal{O})} \|u\|^p_{W^{2, \frac{2\alpha}{s}}_0(\mathcal{O})} \mu(du) + \int_{L^2(\mathcal{O})} \|u\|^p_{H^1_0(\mathcal{O})} \mu(du) + \int_{L^2(\mathcal{O})} \int_{\mathcal{O}} |\nabla u|^{p-2} |\nabla^2 u|^2 \, dx \mu(du) < +\infty.$$  

**Proof.** Note that here, $s^* = p$. The result follows from the fact that for $\psi(r) = |r|^{p-2}r$, $\Psi(x) = \frac{1}{p}|x|^p$,

$$(D^2\Psi(x))_{(i,j)} = |x|^{p-2}\delta_{ij} - (2-p)|x|^{p-2}\frac{2x_i x_j}{|x|^2}.$$  

\[\square\]

5. Decay estimate and examples

Proposition 3.3 is applied to prove the following main decay estimate for the semigroup $\{P_t\}$.

**Theorem 5.1.** Let $\{P_t\}$ be the semigroup from (2.7) and let $\mu$ denote its unique invariant measure. Let $d_0 := 1 + \frac{\alpha}{2}$. Let $s^* = (2-s) + \frac{\alpha}{2+\alpha}$. Let $0 < \beta \leq \beta^*: = 1 \wedge \frac{8 - 2\alpha}{\alpha + 2\alpha + 3\alpha} \in \left[\frac{1}{2}, 1\right]$. Then, under the assumptions of Proposition 3.3, there exist non-negative constants $C_1, C_2$ and $C_3$ such that

$$\limsup_{t \to +\infty} \left[ t^{s^*} \frac{P_t f(u) - P_t f(v)}{\|u - v\|_{H}^{3}} \right] \leq C_1 |f|_{\beta} \left( C_2 + C_3 \int_{\mathcal{H}} \|u\|_{V}^s \mu(du) \right),$$

for every $u, v \in V$ and every $f : H \to \mathbb{R}$ that is bounded and $\beta$-Hölder-continuous, i.e.,

$$\sup_{u, v \in \mathcal{H}} \frac{|f(u) - f(v)|}{\|u - v\|_{H}^{3}} =: |f|_{\beta} < +\infty.$$
Note that dimensions, i.e. $1 + 1$.

Stochastic mean curvature flow. Following convergence rates for the main examples.

For Corollary 5.4.

As before, we have $5.3$. Therefore, we obtain the following.

5.1. Stochastic mean curvature flow. For the stochastic mean curvature flow equation in $1 + 1$ dimensions, i.e.

$$A(u) = - \frac{\partial_x u}{1 + (\partial_x u)^2} = - \frac{\partial}{\partial x} \arctan(\partial_x u), \quad \psi(r) = \arctan(r),$$

we have $s = 2$, $\beta^* = \frac{1}{2}$, so our result recovers the result in [26].

Corollary 5.2. For $f$ Hölder continuous with $\beta = \frac{1}{2}$, we have that

$$\limsup_{t \to +\infty} \left[ t^{\frac{1}{2}} \frac{|P_t f(u) - P_t f(v)|}{|u - v|^{\frac{3}{2}}_{L^2(0,L)}} \right] \leq C_1|f|_\frac{1}{2} \left( C_2 + C_3 \int_H |u|_{V}^p \mu(du) \right),$$

where the constants do not depend on $f$.

5.2. Singular $p$-Laplace. Let $d_0 := 1 \lor \frac{d}{2}$. Consider, for $p \in (1,2)$, the operator

$$A(u) = - \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad \psi(r) = |r|^{p-2} r.$$

With $s = 2 - p$, we get $\beta^* = \frac{8 - 2(2 - p)}{(2 - p)(4 - p)d_0} \land 1$, $s^* = p$ and therefore get the following result which improves the result in [11], see also Remark 2.12.

Corollary 5.3. For $f$ Hölder continuous with $0 < \beta \leq \beta^*$, we have that

$$\limsup_{t \to +\infty} \left[ t^{\frac{1}{2} - \frac{2}{p}} \frac{|P_t f(u) - P_t f(v)|}{|u - v|^{\frac{p}{p-2}}_{L^2(O)}} \right] \leq C_1|f|_\beta \left( C_2 + C_3 \int_H |u|_{V}^p \mu(du) \right),$$

where the constants do not depend on $f$.

5.3. Non-Newtonian Fluids. Let $d_0 := 1 \lor \frac{d}{2}$. Consider the following operator for $p \in (1,2)$,

$$A(u) = - \text{div} \left( (1 + |\nabla u|^2)^{-\frac{p}{2}} \nabla u \right), \quad \psi(r) = (1 + r^2)^{-\frac{p}{2}} r.$$

As before, we have $s = 2 - p$, and get $\beta^* = \frac{8 - 2(2 - p)}{(2 - p)(4 - p)d_0} \land 1$, $s^* = p$. Therefore, we obtain the following.

Corollary 5.4. For $f$ Hölder continuous with $0 < \beta \leq \beta^*$, we have that

$$\limsup_{t \to +\infty} \left[ t^{\frac{1}{2} - \frac{2}{p}} \frac{|P_t f(u) - P_t f(v)|}{|u - v|^{\frac{p}{p-2}}_{L^2(O)}} \right] \leq C_1|f|_\beta \left( C_2 + C_3 \int_H |u|_{V}^p \mu(du) \right),$$

where the constants do not depend on $f$. 

Proof. Use Proposition 3.3 and get for $\beta =: \alpha \leq \frac{1}{2} \land \frac{1}{\alpha d_0} = \frac{\beta^*}{2}, l := \frac{1}{\alpha d_0} \geq 1$, $|P_t f(u) - P_t f(v)| = |E[f(u) - f(v)]| \leq |f|_{2\alpha} E \left( \|u_t - v_t\|_{H^0}^2 \right)

\leq |f|_{2\alpha} C(l, \alpha, O, d) E \left[ \left( \frac{1}{t} \int_0^t ((u|O)|)^{\alpha d_0} + b^{\alpha d_0} \|u_t\|_{V^{\alpha d_0}} + b^{\alpha d_0} \|v_t\|_{V^{\alpha d_0}} \right) d\tau \right] \|u_0 - v_0\|_{H^0}^2

\leq |f|_{2\alpha} C(l, \alpha, O, d) \beta^{\frac{1}{2}} \alpha^{-\frac{1}{2}} (\beta^{l-1}) \mu_0 E \left[ \left( \frac{1}{t} \int_0^t ((u|O)|)^\frac{\beta^*}{2} + b^{\frac{\beta^*}{2}} \|u_t\|_{V^\frac{\beta^*}{2}} + b^{\frac{\beta^*}{2}} \|v_t\|_{V^\frac{\beta^*}{2}} \right) d\tau \right] \|u_0 - v_0\|_{H^0}^2.

Note that $E \left( \frac{1}{t} \int_0^t \|u_t\|_{V^\frac{\beta^*}{2}}^\beta \right)$ converges to $\int_H \|u\|_{V}^\beta \mu(du)$ due to the ergodicity of $\{P_t\}$. 

As a consequence, we obtain the proof of Theorem 1.2 (ii). As an application, we find the following convergence rates for the main examples.
6. Maximal dissipativity of the associated Kolmogorov operator

Set
\[ \kappa(r) = \psi'(|r|) \wedge \frac{\psi(|r|)}{|r|}, \]
i.e. \( \Lambda_{\text{min}}(x) = \kappa(|x|) \), where \( \Lambda_{\text{min}} \) is defined as in Lemma 3.2, and set
\[ D_0 := \left\{ u \in W^{1,1}_0(\mathcal{O}) : \sqrt{\kappa(|\nabla u|)} \partial_x \partial_x u \in L^2(\mathcal{O}), 1 \leq i, j \leq d \right\}. \]

Recall that Theorem 4.3 yields that the support of the invariant measure \( \mu \) is included in \( D_0 \), so that
\[ \| \sqrt{\kappa(|\nabla u|)} \partial_x \partial_x u \|_H^2 \]
is well-defined (and finite) \( \mu \)-a.e., and
\[ \sum_{i,j=1}^d \int_H \| \sqrt{\kappa(|\nabla u|)} \partial_x \partial_x u \|_H^2 \mu(du) < +\infty. \] 

The Kolmogorov operator associated with (1.1) on sufficiently smooth functions \( F \) is given by
\[ J_0 F(u) := \frac{1}{2} \text{Tr}_H \left( B B^* D^2 F(u) \right) - \langle A(u), DF(u) \rangle. \]

In order to realize \( J_0 \) as an operator in \( L^1(H, \mu) \) with domain \( D(J_0) := C_0^\infty(H) \), we need to impose conditions on \( A \) ensuring that \( \| A(u) \| \in H \). To this end we now pose the following additional assumption.

**Assumption 6.1.** Suppose that

(C7) there exists a constant \( M > 0 \) such that for every \( r \in \mathbb{R} \),
\[ \left| \psi'(|r|) - \frac{\psi(|r|)}{|r|} \right| + \frac{\psi(|r|)}{|r|} \leq M \sqrt{\kappa(r)}. \]

**Example 6.2.** We shall collect the following examples for (C7) and leave their verification to the reader.

(i) For \( p \in (1, 2) \), there exists as constant \( C(p) > 0 \) such that \( \psi(r) = (1 + |r|^2)^{\frac{p-2}{2}} r \) satisfies (C7) with \( M = C(p) \).

(ii) \( \psi(r) = \log(1 + |r|) \text{sgn}(r) \) satisfies (C7) with \( M = 2 \).

(iii) \( \psi(r) = \arctan(r) \) satisfies (C7) with \( M = 4 \).

**Remark 6.3.** We would like to point out that for the singular \( p \)-Laplace \( \psi(r) = |r|^{p-2} r \), (C7) reduces to
\[ \sqrt{p-1} |r|^{p-2} \leq |r|^{\frac{p-2}{2}} \quad \text{for every } r \in \mathbb{R}. \]

Clearly, the existence of such \( M > 0 \) cannot be proved for \( p \neq 2 \). Neither does the minimal surface flow satisfy (C7).

An immediate implication of (C7) is that for \( u \in D_0 \) due to (3.4)
\[ |A(u)| \leq \left| \psi'(|\nabla u|) \right| \left( \sum_{i,j=1}^d \left( \partial_x \partial_x u \right)^2 \right)^{\frac{1}{2}} + \frac{\psi(|\nabla u|)}{|\nabla u|} |\Delta u| \]
\[ \leq M \sqrt{\kappa(|\nabla u|)} \left( \sum_{i,j=1}^d \left( \partial_x \partial_x u \right)^2 \right)^{\frac{1}{2}} \]
for some finite constant. Moreover, (6.1) now implies that \( \|A(u)\| \in L^2(H, \mu) \), so that for \( F \in C_b^2(H) \) it follows that \( (A(u), DF(u)) \in L^2(H, \mu) \), hence \( J_0F(u) \in L^1(H, \mu) \), too. From Itô’s formula for applied to \( F(u(t)) \) with regular initial condition, it follows that \( \mu \) is infinitesimally invariant for \( J_0 \). Therefore, since

\[
J_0F^2 = 2FJ_0F + (BB^*DF, DF), \quad F \in D(J_0),
\]

also,

\[
\int_H FJ_0Fd\mu = \frac{1}{2} \int_H \|B^*DF\|^2 d\mu.
\]

Thus \( J_0 \) is dissipative in the Hilbert space \( L^2(H, \mu) \). By similar arguments as in [22] one can prove that \( J_0 \) is also dissipative in \( L^1(H, \mu) \). As a consequence, it is closable and its closure \( J := \overline{J_0} \) with the domain \( D(J) \) is dissipative.

**Theorem 6.4** (compare with Theorem 4.1 in [26]). The operator \((J, D(J))\) generates a \( C_0 \)-semigroup of contractions on \( L^1(H, \mu) \).

**Proof.** We shall prove that for \( \lambda > 0 \), range\((\lambda - J)\) is dense in \( L^1(H, \mu) \). To this aim, for \( \alpha > 0 \), consider the Yosida approximation of \( A \) defined by

\[
A_\alpha(u) = A(J\alpha(u)), \quad \text{where} \quad J\alpha(u) = (\text{Id} + \alpha A)^{-1}(u).
\]

For the sequence \( A_\alpha \) we have the following:

(i) For any \( \alpha > 0 \), \( A_\alpha \) is dissipative and Lipschitz continuous,

(ii) \( ||A\alpha(u)||_H \leq ||A(u)||_H \) for any \( u \in D(A) \).

Note that the function \( A_\alpha \) is not differentiable in general. Therefore we consider a \( C^1 \)-approximation. For \( \alpha, \beta > 0 \) we set

\[
A_{\alpha,\beta}(u) := \int_H e^{\beta \Delta}A\alpha(e^{\beta \Delta}u + v)\mathcal{N}_{0,\sigma_\alpha}(dv)
\]

where \( \mathcal{N}_{0,\sigma_\alpha} \) is the Gaussian measure on \( H \) with mean \( 0 \) and covariance operator \( \sigma_\alpha := \int_0^\beta e^{2\tau \Delta} d\tau \), and \( e^{\tau \Delta}, \tau > 0 \), denotes the semigroup generated by the Dirichlet Laplace operator on \( L^2(\Omega) \).

Then, \( A_{\alpha,\beta} \) is dissipative since

\[
\langle A_{\alpha,\beta}(u) - A_{\alpha,\beta}(v), u - v \rangle = \int_H \langle A_\alpha(e^{\beta \Delta}u + w) - A_\alpha(e^{\beta \Delta}v + w), e^{\beta \Delta}(u - v) \rangle \mathcal{N}_{0,\sigma_\alpha}(dw) \geq 0.
\]

We would like to use the Cameron–Martin theorem to show that \( A_{\alpha,\beta} \) is differentiable. To this end we first need to check that \( e^{\beta \Delta}u \in \text{range}(\sigma_\beta^{1/2}) \). According to Proposition B.1 in [10] it suffices to prove that there exists a constant \( M \) such that \( ||e^{\beta \Delta}v||^2_H \leq M||\sigma_\beta^{1/2}v||^2_H \) for all \( v \in H \). But this follows from the fact that

\[
||e^{\beta \Delta}v||^2_H = \frac{1}{\beta} \int_0^\beta ||e^{(\beta - \tau)\Delta}e^{\tau \Delta}v||^2_H d\tau \leq \frac{1}{\beta} \int_0^\beta ||e^{\tau \Delta}v||^2_H d\tau
\]

\[
= \frac{1}{\beta} \int_0^\beta \langle e^{2\tau \Delta}v, v \rangle d\tau = \frac{1}{\beta} ||\sigma_\beta v||^2_H = \frac{1}{\beta} ||\sigma_\beta^{1/2}v||^2_H.
\]

An application of the Cameron–Martin theorem yields

\[
\langle DA_{\alpha,\beta}(u), h \rangle = \int_H e^{\beta \Delta}A\alpha(e^{\beta \Delta}u + v)\langle \sigma_\beta^{-1}e^{\beta \Delta}h, v \rangle \mathcal{N}_{0,\sigma_\alpha}(dv).
\]

and by the Cameron–Martin formula it is \( C^\infty \) differentiable. Moreover, as \( \alpha, \beta \to 0, A_{\alpha,\beta} \to A \) pointwise. Let us introduce the following approximating equation

\[
du_{\alpha,\beta}(t) = -A_{\alpha,\beta}(u_{\alpha,\beta}(t)) dt + B dW_t, \quad t > 0, \quad u_{\alpha,\beta}(0) = u.
\]
Since $A_{\alpha,\beta}$ is globally Lipschitz, equation (6.2) has a unique strong solution $(u_{\alpha,\beta}(t))_{t \geq 0}$. Moreover, by the regularity of $A_{\alpha,\beta}$ the process $(u_{\alpha,\beta}(t))_{t \geq 0}$ is differentiable on $H$. For any $h \in H$ we set $\eta_t(u) := Du_{\alpha,\beta}(t, u) \cdot h$. It holds that

$$
\frac{d}{dt} \eta_t(u) = -DA_{\alpha,\beta}(u_{\alpha,\beta}(t, u)) \cdot \eta_t(u), \quad \eta_t(0, u) = h.
$$

From the dissipativity of $A_{\alpha,\beta}$, we have that

$$
\langle DA_{\alpha,\beta}(z) h, h \rangle \geq 0, \quad h \in H, z \in D(A).
$$

Hence by multiplying both sides of (6.3) with $\eta_t(u)$ and integrating with respect to $t$, we get

$$
\|\eta_t(u)\|^2 \leq \|h\|^2.
$$

Now, for $\lambda > 0$ and $F \in C_b^0(H)$ consider the following elliptic equation

$$
(\lambda - J_{A_{\alpha,\beta}}) \varphi_{\alpha,\beta} = F, \quad \lambda > 0,
$$

where $J_{A_{\alpha,\beta}}$ is the Kolmogorov operator corresponding to the stochastic differential equation (6.2). It is well-known that this equation has a solution $\varphi_{\alpha,\beta} \in C_b^0(H)$ and can be written in the form $\varphi_{\alpha,\beta} = R(\lambda, J_{A_{\alpha,\beta}})F$, where

$$
R(\lambda, J_{A_{\alpha,\beta}})F(u) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}(F(u_{\alpha,\beta})(t, u)) \, dt
$$

is the pseudo resolvent associated with $J_{A_{\alpha,\beta}}$. Thus we have

$$
\|\lambda \varphi_{\alpha,\beta}\|_\infty \leq \|F\|_\infty.
$$

We have, moreover, for all $h \in H$,

$$
D \varphi_{\alpha,\beta}(u) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}(DF(u_{\alpha,\beta})(t, u)((Du_{\alpha,\beta})(t, u)h)) \, dt.
$$

Consequently, using (6.4), it follows that

$$
\|D \varphi_{\alpha,\beta}\| \leq \frac{1}{\lambda} \|DF\|_\infty.
$$

From (6.5), we have

$$
\lambda \varphi_{\alpha,\beta}(u) = \frac{1}{2} \text{Tr}_H \left( BB^* D^2 \varphi(u) \right) + \langle A(u), D\varphi_{\alpha,\beta}(u) \rangle = F(u) + \langle A(u) - A_{\alpha,\beta}(u), D\varphi_{\alpha,\beta}(u) \rangle, \quad \lambda > 0, \ u \in D(A).
$$

Using gradient bound (6.6), we deduce that for $\alpha, \beta \to 0$,

$$
\int_H \|\langle A(u) - A_{\alpha,\beta}(u), D\varphi_{\alpha,\beta}(u) \rangle\| \mu(du) \leq \frac{1}{\lambda} \|DF\|_\infty \|A_{\alpha,\beta} - A\|_{L^2(H, \mu)}.
$$

By Lebesgue’s theorem $\|A_{\alpha,\beta} - A\|_{L^2(H, \mu)}$ converges to 0 as $\alpha, \beta \to 0$. Therefore we deduce that for $\alpha, \beta \to 0$

$$
\lambda \varphi_{\alpha,\beta}(u) \to \frac{1}{2} \text{Tr}_H \left( BB^* D^2 \varphi(u) \right) + \langle A(u), D\varphi_{\alpha,\beta}(u) \rangle \to F
$$

strongly in $L^1(H, \mu)$. This implies that

$$
C^2_b(H) \subset \overline{(\lambda - J_0)(D(J_0))}.
$$

Since $C^2_b(H)$ is dense in $L^1(H, \mu)$, the proof is complete.
Statements and Declarations

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