Stability Analysis of Reaction-Diffusion Equations with Double Diffusivity System

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ABSTRACT

Stability analysis for steady state solution of reaction-diffusion equations with double diffusivity discuss and arise in the solution of problems of flow of homogeneous liquids and heat conduction involving air-temperature \( u(x,t) \) and a grain-temperature \( v(x,t) \), the resulting of this analysis shows that the system is stable when:

\[ Dk^4 + a'_1 Lk^2 + b'_2 Lk^2 + b'_1 a'_1 L^2 - b'_2 a'_2 L^2 > 0 \]

1. Introduction

Reaction-diffusion give rise to texture synthesis based on the simulation of a process of local nonlinear interactions, which has been proposed as a model of biological pattern formation. A chemical mechanism that was first proposed by Alan \([10]\) (1952) to account for pattern formation in biological morphogenesis, he postulated that patterning is governed primarily by concurrently operating processes: diffusion of morphogens through the tissue and chemical reactions that produce and destroy morphogens at a rate that depends, among other things, on their concentrations. Such mechanisms are called reaction-diffusion (RD) systems \([11]\).

Antic and Hill (2000) \([1]\) are studied a mathematical model for heat transfer in grain store microclimates, this model is "double diffusivity" such that they are used the heat-balance integral method to transform the coupled partial differential equations to the coupled ordinary differential equations and solved it numerically by using the Fehlberg fourth-fifth order Range-kutta method.

Aggarwala and Nasim \([2]\) derived the solution of reaction-diffusion equations with double diffusivity by laplace technique and fourier transforms which appear to be simpler and more direct.

Chow Tanya \([4]\) are studied the derivation of similarity solutions for one-dimensional coupled systems of reaction-diffusion equations, these solutions are obtained by means of one-parameter group methods.
Molz [7] used a coupled system of a model for water transport through plant tissue and Rubinsein [9] is able to derive a coupled system in a one-dimensional case includes heat conduction in heterogeneous media.

In this paper, we will study the stability analysis of steady state solution of double diffusivity model.

2. Model of equations:

The one-dimensional case of reaction-diffusion equations with double-diffusivity is given by [2]

\[
\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} - a_1 u(x,t) + b_1 v(x,t) \tag{1a}
\]

\[
\frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + a_2 u(x,t) - b_2 v(x,t) \tag{1b}
\]

where \( u(x,t) \) and \( v(x,t) \) denote the air-temperature and a grain-temperature respectively, the self-diffusivities \( D_1, D_2, a_1, b_1, a_2 \) and \( b_2 \) are positive constants, with initial conditions:

\[ u(x,0) = u_0 \]

\[ v(x,0) = v_0 \]

where \( u_0 \) and \( v_0 \) are constant interior solutions of (1a)-(1b) and zero flux boundary conditions on \( u \) and \( v \) [2]:

\[
\frac{\partial u}{\partial x} = 0 \text{ at } x=0 \tag{2a}
\]

\[
\frac{\partial v}{\partial x} = 0 \text{ at } x=0 \tag{2b}
\]

when \( a_1 = a_2 \) and \( b_1 = b_2 \) this system represent mathematical model for heat transfer in grain store microclimates[1].

For dimensionless form, we introduce the following dimensionless quantities:

\[
x' = \frac{x}{L}, \quad \tau' = \frac{t D_2}{L^2}, \quad \frac{u}{U}, \quad \frac{v}{V}, \quad D = \frac{D_1}{D_2}, \quad a_i = \frac{a_i D_2}{L}, \quad b_i = \frac{b_i D_2}{L}, \quad b_2 = \frac{b_2 D_2}{L}
\]

substitute these non-dimensional quantities into equations (1a) , (1b) and the conditions (2a),(2b) we get:

\[
\frac{\partial u'}{\partial \tau'} = D \frac{\partial^2 u'}{\partial x'^2} - a_1' L u'(x', \tau') + b_1' L v'(x', \tau') \tag{3a}
\]

\[
\frac{\partial v'}{\partial \tau'} = \frac{\partial^2 v'}{\partial x'^2} + a_2' L u'(x', \tau') - b_2' L v'(x', \tau') \tag{3b}
\]

\[
\frac{\partial u'}{\partial x'} = 0 \text{ at } x=0 \tag{3c}
\]

\[
\frac{\partial v'}{\partial x'} = 0 \text{ at } x=0 \tag{3d}
\]

3. Stability analysis:

Assume that the value of the concentrations \( u \) and \( v \) has the following from [5]:

\[
u'(x', \tau') = u_1'(x') + u_2'(x', \tau') \]

\[
v'(x', \tau') = v_1'(x') + v_2'(x', \tau') \tag{4}
\]
where $u_1(x')$ and $v_1(x')$, denote the steady state case, $u_2(x',t')$ and $v_2(x',t')$ denote the disturbance case.

If we substituted (4) in equations (3a)-(3d), we get the following systems:

**The steady state system:**

\[
D \frac{d^2 u'_1}{dx'^2} - a'_1 L u'_1(x') + b'_1 L v'_1(x') = 0 \quad \text{(5a)}
\]

\[
\frac{d^2 v'_1}{dx'^2} + a'_2 L u'_1(x') - b'_2 L v'_1(x') = 0 \quad \text{(5b)}
\]

with the boundary conditions:

\[
\frac{du'_1}{dx'} = 0 \quad \text{at} \quad x=0 \quad \text{(5c)}
\]

\[
\frac{dv'_1}{dx'} = 0 \quad \text{at} \quad x=0 \quad \text{(5d)}
\]

**The second system:**

\[
\frac{\partial u'_2}{\partial t'} = D \frac{\partial^2 u'_2}{\partial x'^2} - a'_1 L u'_2(x',t') + b'_1 L v'_2(x',t') \quad \text{(6a)}
\]

\[
\frac{\partial v'_2}{\partial t'} = \frac{\partial^2 v'_2}{\partial x'^2} + a'_2 u'_2(x',t') - b'_2 L v'_2(x',t') \quad \text{(6b)}
\]

with the related boundary conditions:

\[
\frac{\partial u'_2}{\partial x'} = 0 \quad \text{at} \quad x=0 \quad \text{(3c)}
\]

\[
\frac{\partial v'_2}{\partial x'} = 0 \quad \text{at} \quad x=0 \quad \text{(3d)}
\]

**4. Solution of the steady state case:**

To find the solution to (5a)-(5b) with the boundary conditions (5c)-(5d), we shall use the technique which Benson, D. and sherratt, J. [3] are used for the solution the linearized model of coupled ordinary differential equations as follows:

\[
\frac{d^2 u'_1}{dx'^2} - \frac{a'_1 L}{D} u'_1(x') + \frac{b'_1 L}{D} v'_1(x') = 0 \quad \text{(7a)}
\]

\[
S \frac{d^2 v'_1}{dx'^2} + S a'_2 L u'_1(x') - S b'_2 L v'_1(x') = 0 \quad \text{(7b)}
\]

adding (7a) to (7b) gives:

\[
\left( \frac{d^2 u'_1}{dx'^2} + S \frac{d^2 v'_1}{dx'^2} \right) + \left( S a'_2 L - \frac{a'_1 L}{D} \right) u'_1 + \left( \frac{b'_1 L}{D} - S b'_2 L \right) v'_1 = 0
\]

\[
\Rightarrow \left( \frac{d^2 (u'_1 + Sv'_1)}{dx'^2} \right) + \left( S a'_2 L - \frac{a'_1 L}{D} \right) \left( \frac{b'_1 L}{D} - S b'_2 L \right) v'_1 = 0 \quad \text{(8)}
\]

we choose (S) such that:
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\[ S = \left( \frac{b'_L - Sb'_L}{D} \right) \]

\[ \left( Sa'_L - \frac{a'_L}{D} \right) \]

which is a quadratic equation for (S) and thus:

\[ Da'_L S^2 + (Db'_L - a'_L) S - b'_L = 0 \]

\[ S = \frac{-(Db'_L - a'_L) \pm \sqrt{(Db'_L - a'_L)^2 + 4Da'_L^2b'_L}}{2DLa'_L} \]

which given:

\[ S_1 = \frac{-(Db'_L - a'_L) - \sqrt{(Db'_L - a'_L)^2 + 4Da'_L^2b'_L}}{2DLa'_L} \]

\[ S_2 = \frac{-(Db'_L - a'_L) + \sqrt{(Db'_L - a'_L)^2 + 4Da'_L^2b'_L}}{2DLa'_L} \]

Equation (8) becomes:

\[ d^2(u'_j + S_jv'_j) + \left( Sa'_L - \frac{a'_L}{D} \right) (u'_j + S_jv'_j) = 0, \text{ for } j=1, \ldots (9) \]

The general solution of (9) is:

\[ u'_j + S_jv'_j = A_j \cos(\sqrt{\lambda}_j x) + B_j \sin(\sqrt{\lambda}_j x), \text{ for } j=1,2 \]

where \( \lambda_j = \left( Sa'_L - \frac{a'_L}{D} \right), \text{ for } j=1,2 \)

and the other solution is:

\[ \lambda_1 = \left( Sa'_L - \frac{a'_L}{D} \right) \]

\[ \lambda_1 = \frac{-(Db'_L + a'_L) - \sqrt{(Db'_L + a'_L)^2 + 4Da'_L^2b'_L}}{2D} \]

\[ \lambda_2 = \left( Sa'_L - \frac{a'_L}{D} \right) \]

\[ \lambda_2 = \frac{-(Db'_L + a'_L) + \sqrt{(Db'_L + a'_L)^2 + 4Da'_L^2b'_L}}{2D} \]

Then by applying zero flux boundary conditions (5c)-(5d), we get:

\[ u'_j(x') = S_j \cos(\sqrt{\lambda}_j x') - S_j \cos(\sqrt{\lambda}_j x') \]

\[ v'_j(x') = \cos(\sqrt{\lambda}_j x') - \cos(\sqrt{\lambda}_j x') \]

**5. Stability analysis (disturbance case):**

Assume that the value of \( u'_j(x', t') \) and \( v'_j(x', t') \), has the following form ([6],[8]):

\[ u'_j(x', t') = F_1 e^{\delta(x'-ct')} \]

\[ v'_j(x', t') = F_2 e^{\delta(x'-ct')} \]

\[ \ldots (11) \]

...
here \( c = c_1 + ic_2 \) is an eigenvalue represent the speed of the wave, the functions \( F_1 \) and \( F_2 \) are the constant amplitudes, \( k \) is the wave number. The flow is stable if the linearized equation correspond to eigenvalue \( c \) with negative part \( (c < 0) \) for presented configurations.

Now, substitute (11) in the equations (6a)-(6b), we get respectively:

\[-ik(c_1 + ic_2)F_1 = -Dk^2F_1 - a'_1LF_1 + b'_1LF_2\]
\[-ik(c_1 + ic_2)F_2 = -k^2F_2 + a'_2LF_1 - b'_2LF_2\]

by separate the real part and imaginary part, we get:

\[kc_1F_1 = -Dk^2F_1 - a'_1LF_1 + b'_1LF_2\]  
...(12a)

\[kc_1F_2 = -k^2F_2 + a'_2LF_1 - b'_2LF_2\]  
...(12b)

from (12a) we get:

\[(kc_1 + Dk^2 + a'_1)LF_1 = b'_1LF_2\]  
\[F_1 = \left(\frac{b'_1L}{kc_1 + Dk^2 + a'_1}\right)F_2\]  
...(13)

we substitute (13) in the equation (12b), we get:

\[kc_1F_2 = -k^2F_2 + \frac{a'_2b'_1L}{(kc_1 + Dk^2 + a'_1)L}F_2 - b'_2LF_2\]

\[(k^2c_1^2 + Dk^3c_1 + a'_1Lk + k^3 + b'_2Lk)c_2 + (k^4D + a'_1Lk^2 - a'_2L^2b'_1 + b'_2Lk + b'_2LDk^2 + b'_2L^2a'_1)F_2 = 0\]

such that \( F_2 \neq 0 \), then we get:

\[k^2c_1^2 + (Dk^3 + a'_1Lk + k^3 + b'_2Lk)c_2 + (k^4D + a'_1Lk^2 - a'_2L^2b'_1 + b'_2Lk^2 + b'_2LDk^2 + b'_2L^2a'_1) = 0\]  
when \( a'_1 = a'_2 \) and \( b'_1 = b'_2 \) [1] we get:

\[c_2 = -\left(\frac{(Dk^3 + a'_1Lk + k^3 + b'_2Lk)}{2k^2}\right)\]

\[\text{then:}\]

\[c_2 = -\left(\frac{(Dk^3 + a'_1Lk + k^3 + b'_2Lk)}{2k}\right)\]

\[\text{then:}\]

\[c_2 = \frac{(Dk^3 + a'_1Lk + k^3 + b'_2Lk)}{2k}\]

\[\text{since } a'_1 > 0, b'_2 > 0, D > 0, L > 0 \text{ and } k > 0 \text{ then:}\]

\[(Dk^3 + a'_1Lk + k^3 + b'_2Lk) > \left[Dk^3 + a'_1Lk + k^3 + b'_2Lk\right]^2 - 4k^2(D + a'_1Lk^2 + b'_2LDk^2)\]

this system is stable always.

If \( a'_1 \neq a'_2 \) and \( b'_1 \neq b'_2 \) [2] we get:

\[c_2 = -\left(\frac{(Dk^3 + a'_1Lk + k^3 + b'_2Lk)}{2k}\right)\]

\[\text{then:}\]

\[c_2 = \frac{(Dk^3 + a'_1Lk + k^3 + b'_2Lk)}{2k}\]
\[ c_2 = -\frac{\left( Dk^3 + a'_1Lk + k^2 + b'_2L \right)}{2k^2} \] 
\[ m\sqrt{\left( - Dk^3 - a'_1Lk + k^3 + b'_2L \right)^2 + 4k^2L^2a'_2b'_1} \] 
\[ \frac{2k^2}{2k} \] 
\[ \text{either:} \] 
\[ c_2 = -\frac{\left( Dk^3 + a'_1L + k^2 + b'_2L \right)}{2k} \] 
\[ \frac{\sqrt{\left( - Dk^3 - a'_1L + k^3 + b'_2L \right)^2 + 4L^2a'_2b'_1}}{2k} \] 

such that \( L > 0, D > 0, k > 0, a'_1 > 0, a'_2 > 0, b'_1 > 0, b'_2 > 0 \) and \( c_2 < 0 \) always, thus the system is stable.

or:
\[ c_2 = -\frac{\left( Dk^3 + a'_1L + k^2 + b'_2L \right)}{2k} \] 
\[ \frac{\sqrt{\left( - Dk^3 - a'_1L + k^3 + b'_2L \right)^2 + 4L^2a'_2b'_1}}{2k} \] 

and the system is stable when:
\[ (Dk^3 + a'_1L + k^2 + b'_2L)^2 > \left( - Dk^3 - a'_1L + k^3 + b'_2L \right)^2 + 4L^2a'_2b'_1 \]
and unstable otherwise.

The neutral stability curve, when \( (c_2 = 0) \) in equation (16) is:
\[ (Dk^3 + a'_1L + k^2 + b'_2L)^2 = \left( - Dk^3 - a'_1L + k^3 + b'_2L \right)^2 + 4L^2a'_2b'_1 \] 
\[ \Rightarrow Dk^4 + a'_1Lk^2 + b'_2LDk^2 + a'_1b'_2L^2 - a'_2b'_1L^2 = 0 \] 
\[ \Rightarrow Dk^4 + (a'_1L + b'_2LD)k^2 + (a'_1b'_2L^2 - a'_2b'_1L^2) = 0 \] 
\[ k^2 = \frac{-\left(a'_1L + b'_2LD\right)}{2D} \sqrt{\left[a'_1L + b'_2LD\right]^2 - 4D\left[a'_1b'_2L^2 - a'_2b'_1L^2\right]} \] 
\[ k^2 \] 

\( D \quad k \quad \text{or} \quad D \quad k \)
\begin{tabular}{|c|c|c|c|c|}
\hline
D & k & D & k \\
\hline
0.0500 & 0.9554 & 0.0500 & 1.5962 \\
0.1000 & 0.9419 & 0.1000 & 1.5040 \\
0.1500 & 0.8880 & 0.1500 & 1.4342 \\
0.2000 & 0.8612 & 0.2000 & 1.3780 \\
0.2500 & 0.8376 & 0.2500 & 1.3312 \\
0.3000 & 0.8165 & 0.3000 & 1.2910 \\
0.3500 & 0.7974 & 0.3500 & 1.2559 \\
0.4000 & 0.7801 & 0.4000 & 1.2247 \\
0.4500 & 0.7641 & 0.4500 & 1.1968 \\
0.5000 & 0.7494 & 0.5000 & 1.1714 \\
0.5500 & 0.7357 & 0.5500 & 1.1483 \\
0.6000 & 0.7229 & 0.6000 & 1.1270 \\
0.6500 & 0.7109 & 0.6500 & 1.1073 \\
0.7000 & 0.6997 & 0.7000 & 1.0889 \\
0.7500 & 0.6891 & 0.7500 & 1.0718 \\
0.8000 & 0.6790 & 0.8000 & 1.0557 \\
\hline
\end{tabular}
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|      |      |      |      |
|------|------|------|------|
| 0.8500 | 0.6695 | 0.8500 | 1.0406 |
| 0.9000 | 0.6604 | 0.9000 | 1.0263 |
| 0.9500 | 0.6518 | 0.9500 | 1.0128 |
| 1.0000 | 0.6436 | 1.0000 | 1.0000 |

Table(2)

\[
\begin{array}{ll}
\alpha'_1 = 1, \beta'_2 = 1, L = 1 & \alpha'_1 = 1, \beta'_2 = 1, L = 1 \\
D = 1, \beta'_1 = 1 & D = 1, \beta'_1 = 1 \\
\hline
\alpha'_2 & k \\
1.0000 & 0.0000 \\
1.0500 & 0.1571 \\
1.1000 & 0.2209 \\
1.1500 & 0.2690 \\
1.2000 & 0.3089 \\
1.2500 & 0.3436 \\
1.3000 & 0.3744 \\
1.3500 & 0.4024 \\
1.4000 & 0.4280 \\
1.4500 & 0.4518 \\
1.5000 & 0.4741 \\
1.5500 & 0.4950 \\
1.6000 & 0.5147 \\
1.6500 & 0.5334 \\
1.7000 & 0.5512 \\
1.7500 & 0.5682 \\
1.8000 & 0.5845 \\
1.8500 & 0.6001 \\
1.9000 & 0.6151 \\
1.9500 & 0.6296 \\
2.0000 & 0.6436 \\
\end{array}
\]

Fig(1). The natural stability curve in (17) when
\[\alpha'_1 = 1, \alpha'_2 = 1, \beta'_2 = 1, \beta'_1 = 2, L = 1\]
Fig(2). The natural stability curve in (17) when $a_1' = 1, a_2' = 2, b_2' = 1, b_1' = 2, L = 1$

Fig(3). The natural stability curve in (17) when $a_1' = 1, D = 1, b_2' = 1, b_1' = 1, L = 1$

6-conclusion:

The main conclusion, and from the figures is that the system is stable if $Dk^4 + a_1' Lk^2 + b_2' Lk^2 D + b_2' a_1' L^2 - b_1' a_2' L^2 > 0$, however when the coefficients $a_2'$ and $b_1'$ are increase then the unstable region is increase as shown in figures (1), (2) and (3) and table (1) and (2).
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