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Abstract

The role of the $so(2, 1)$ symmetry in General Relativity is analyzed. Cosmological solutions of Einstein field equations invariant with respect to a space-like Lie algebra $\mathcal{G}_r$, with $3 \leq r \leq 6$ and containing $so(2, 1)$ as a subalgebra, are also classified.

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Introduction

Gravitational models are usually classified in terms of their group of isometries, namely the number of Killing vectors, the values of the structure constants and the transitivity regions, i.e. the regions on which the isometry group acts transitively. The classification of all non-isomorphic isometry groups is a well known and solved issue in the context of Group Theory; for instance, there exist 9 non-isomorphic 3-dimensional groups, $G_3$, which yield the so-called Bianchi models. A simpler example is provided by the $G_2$ groups. In this case the corresponding Lie algebra $\mathcal{G}_2$ is described in terms of two Killing vectors $X, Y$ satisfying the commutation relation $[X, Y] = sY$ with $s = 0, 1$, this corresponding respectively to the Abelian and non-Abelian case.

Interesting gravitational fields are represented by metrics possessing a higher degree of symmetry. For an exhaustive review on the various classifications of such models available in the literature see for example [9]. Among them particular attention have attracted homogeneous or hypersurface-homogeneous models.
like the Friedmann-Robertson-Walker metric (FRW) which is $G_6$-symmetric, the Gödel metric which is $G_5$-symmetric or the Kantowski-Sachs metric, $G_4$-symmetric.

To understand the role of the symmetries in the classification let us give a few definitions.

The Lie algebra of all Killing vector fields of a given metric $g$ will be denoted by $\text{Kill}(g)$, while Killing algebra will denote any subalgebra $\mathcal{G}$ of $\text{Kill}(g)$. The group corresponding to a subalgebra $\mathcal{G}$ of $\text{Kill}(g)$ of a metric $g$ is called group of motions and denoted $G_r$, where $r$, the order of the group, is the number of generators. If $\mathcal{G} = \text{Kill}(g)$, then the corresponding group is called complete group of motions.

A metric manifold is said to be homogeneous if its group $G_r$ of motions acts transitively on it, that is the whole manifold is an orbit of $G_r$.

A metric manifold is said to be maximally symmetric if it has the maximum number of Killing vector fields, i.e. if it admits a complete group $G_{n(n+1)/2}$ of motions, where $n$ denotes the dimension of the manifold. Of course, a maximally symmetric manifold is homogeneous.

The isotropy or stability group of a point $p$ is a subgroup $H_p$ of $G_r$ leaving $p$ fixed. The manifold is said to be isotropic about $p$ if its isotropy group is $n$-dimensional. It can be easily shown that if the manifold is spacelike, then $H_p = SO(n)$.

A manifold isotropic about every point $p$ is said to be isotropic.

It is easy to see that:

- A maximally symmetric metric manifold has constant curvature. The converse is also true.

- A metric manifold of constant curvature is isotropic. The converse is also true.

Thus, a maximally symmetric manifold is isotropic and has a constant curvature.

A cosmological model or shortly a cosmology is a Lorentzian 4-dimensional differential manifold foliated by 3-dimensional submanifolds $S$ on which the restriction $g|_S$ of the metric $g$ is positive definite. These submanifolds $S$ will be also called leaves. A cosmology is said to be homogeneous and/or isotropic if the leaves $S$ are homogeneous and/or isotropic.

In this framework let us briefly review the cosmological models previously mentioned.
1 Some known cosmological models

It is well known that the FRW metric, which in spherical coordinates \((r \in [0, \infty[, \vartheta \in [0, \pi[, \varphi \in [0, 2\pi[)\) has the form

\[
g = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)\right],
\]

is a homogeneous and isotropic cosmology and the 3-dimensional leaves \(S\), defined by \(t = \text{const}\), have constant scalar curvature \(R\) proportional to \(k\) \((R = 6k/a^2)\). Indeed, \(\dim K\il(g|_S) = 6\) and the Killing vector fields \(X_i, P_i, \{i = 1, \ldots, 3\}\) of \(g\)

\[
X_i = -\epsilon_{ijk}x_j\partial_k, \quad P_i = (1 - kr^2/4)\partial_i + \frac{1}{2}kx_i(r\partial_r)
\]

close the Lie algebra

\[
[X_i, X_j] = \sum_k \epsilon_{ijk}X_k, \quad [P_i, P_j] = \sum_k \epsilon_{ijk}X_k, \quad [X_i, P_j] = \sum_k \epsilon_{ijk}Y_j.
\]

Here \(\epsilon_{ijk}\) is the Levi-Civita tensor density and the relation between Cartesian and spherical coordinates is the obvious one. For positive \(k\) the six vectors span the Lie algebra of \(SO(4)\), for \(k = 0\) they span the Lie algebra of the \textit{semidirect product} \(SO(3) \times' \mathbb{R}^3\), whereas \(k\) negative corresponds to the proper Lorentz group \(SO(3;1)\). Since the maximally symmetric and isotropic leaves \(S\) are spacelike, the isotropy group is for all the three cases \(SO(3)\). In order to evidentiate the special role of \(SO(3)\) and because of some parallelism we may trace for \(so(2,1)\) invariant situations (cfr. section 3.4) it is useful to rewrite the three algebras respectively as \(so(3) \oplus so(3)\), \(so(3) \oplus' \mathbb{R}^3\), \(so(3) \oplus sb(2, C)\), where \(\oplus'\) denotes the \textit{semidirect sum} characterized by the coadjoint action of \(so(3)\) on \(\mathbb{R}^3\), and \(\oplus\) denotes a \textit{fully non-commutative sum} of two algebras which act on each other by coadjoint action (more details on this structure are given in section 3.4).

In the coordinates \(x \in [-\infty, \infty[, \vartheta \in [0, \pi[, \varphi \in [0, 2\pi[\), the \textit{Kantowski-Sachs} metric [4] reads

\[
g = dt^2 - a^2(t)dx^2 - b^2(t)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2).
\]

It has \(\dim K\il(g) = 4\), the above Killing fields \(X_i\), \textit{i.e.} the generators of rotations, and the Killing field \(P = \partial_x\), \textit{i.e.} the generator of the translations in the \(x\) direction; hence \(K\il(g)\) is a central extension of \(so(3)\). This yields a model which is still spatially homogeneous but not isotropic.

These two classes of metrics have been considered as possible cosmological solutions of Einstein field equations because none of the Killing fields is \textit{timelike}; \textit{i.e.}, no coordinate system exists in which the components of the metrics are independent on the \textit{time coordinate}. In this regard, when the Killing fields are
space-like the mentioned Bianchi models may be considered as well as possible cosmological models.

With respect to these considerations the Gödel metric [10]

\[ g = dx^2 + e^{2x} dy^2/2 + dz^2 - (e^x dy + cdt)^2, \]  

(1.5)

for which \( \text{dim} \, \text{Kill}(g) = 5 \) with Killing fields

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial y}, & X_2 &= \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, & X_3 &= y \frac{\partial}{\partial x} + (e^{-2x} - \frac{1}{2}y^2) \frac{\partial}{\partial y} - 2e^{-x} \frac{\partial}{\partial t}, \\
Y_1 &= \frac{\partial}{\partial z}, & Y_2 &= \frac{\partial}{\partial t}
\end{align*}
\]

is essentially different. It is a static solution whose isometry group acts transitively on the whole space-time which is then a homogeneous manifold. The Killing fields \( Y_i \) commute between them and the vector fields \( X_i \) close the \( so(2,1) \) Lie algebra, but do not have space-like orbits.

In this letter cosmological solutions of Einstein field equations invariant with respect to a spacelike Lie algebra \( G_r \), with \( 3 \leq r \leq 6 \) and containing \( so(2,1) \) as a subalgebra are classified.

This analysis was suggested by a previous work [14] where vacuum gravitational fields invariant for a non-Abelian 2-dimensional Killing algebra are exhaustively described as reported below.

Let \( g \) be a metric on the space-time and \( G_2 = \text{span}\{X, Y\} \) one of its Killing algebras

\[
X, Y \in G_2 \quad [X, Y] = sY, \quad s = 0, 1
\]  

(1.6)

The involutive distribution \( \mathcal{D} \) generated by \( X \) and \( Y \) is obviously integrable. It is 2-dimensional*. The orthogonal distribution is assumed to be integrable and transversal to \( \mathcal{G} \).

The class of metrics, so characterized, encompasses a wide variety of gravitational models. It suffices to mention that this class includes the Robinson-Bondi plane-waves, the cylindrical-wave solutions, the homogeneous cosmological models of Bianchi types I through VIII, the pseudo-Schwarzschild [14] and Kerr solutions, the Belinskii-Khalaktinov [5] general cosmological solution with a physical singularity on portions of the so-called long eras.

It has been shown [14] that all the \( G_2 \) invariant solutions of the vacuum Einstein field equations are characterized either in terms of solutions of an algebraic equation (the tortoise equation) or in terms of solutions of a partial differential equation in the plane, depending on whether \( g(Y, Y) \neq 0 \) or \( g(Y, Y) = 0 \) respectively. In was also shown that in the first case there exists a third Killing field such that \( \text{Kill}(g) \), the complete Killing algebra is isomorphic to \( so(2,1) \) and the

*The fields \( X \) and \( Y \), leaving invariant a metric, i.e., a not degenerate \((0,2)\) tensor field, cannot be parallel.
Killing leaves, \textit{i.e.}, the transitivity regions, are 2-dimensional Riemann surfaces of constant curvature.

The $so(2,1)$ Lie algebra

\begin{equation}
[X_1, X_2] = X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2,
\end{equation}

in the pseudo-spherical coordinates $(r \in ]0, \infty[, \vartheta \in \mathbb{R}, \varphi \in [0, 2\pi])$, is spanned by

\begin{align*}
X_1 &= \sin \varphi \partial_\vartheta + \cos \varphi \coth \vartheta \partial_\varphi, \\
X_2 &= -\cos \varphi \partial_\vartheta + \sin \varphi \coth \vartheta \partial_\varphi, \\
X_3 &= \partial_\varphi.
\end{align*}

The most general $so(2,1)$ invariant metric can be written as

\begin{equation}
g = A(r,t)dt^2 - B(r,t)dr^2 - F(r,t)r^2(d\vartheta^2 + H(\vartheta)d\varphi^2)
\end{equation}

where $H(\vartheta)$ is one of the functions $\sinh^2 \vartheta$ or $-\cosh^2 \vartheta$.

With positive definite functions $A, B, F$ and $H(\vartheta) = \sinh^2 \vartheta$ the metric has Lorentzian signature and the 2-dimensional surfaces defined by $(r,t = \text{const})$ may be identified with one of the sheets of the two-sheeted space-like hyperboloid. They are also known as pseudo-spheres. In the other case, corresponding to $H(\vartheta) = -\cosh^2 \vartheta$, the 2-dimensional surfaces defined by $(r,t = \text{const})$ are one-sheeted time-like hyperboloids and will not be considered here.

The pseudo-sphere \cite{12} is a surface with constant negative Gaussian curvature $\mathcal{R} = -1/r^2$. It can be globally embedded in a 3-dimensional Minkowskian space. Let $y_1, y_2, y_3$ denote the coordinates in the Minkowskian space, where the separation from the origin is given by $y^2 = -y_1^2 + y_2^2 + y_3^2$. These coordinates are connected to the pseudo-spherical coordinates $(r, \vartheta, \varphi)$ by $y_1 = r \cosh \vartheta$, $y_2 = r \sinh \vartheta \cos \varphi$, $y_3 = r \sinh \vartheta \sin \varphi$. The equation $y^2 = -r^2$, \textit{i.e.} the locus of points equidistant from the origin, specifies a hyperboloid of two sheets intersecting the $y_1$ axis at the points $\pm r$ called poles in analogy with the sphere. Either sheet (say the upper sheet) models an infinite spacelike surface without a boundary; hence, the Minkowski metric becomes positive definite (Riemannian) upon it. This surface has constant Gaussian curvature $(\mathcal{R} = -1/r^2)$, and it is the only simply connected surface with this property. Other embeddings of the pseudo-sphere in the 3-dimensional Euclidean space are also available, for example it can be regarded as the 2-dimensional surface generated by the tractrix \cite{10}, but they are not global.

Next section will be devoted to the description of $so(2,1)$ invariant gravitational fields, both in the vacuum and in the matter, and of $so(2,1)$-invariant cosmologies \textit{i.e.} of solutions of Einstein field equations whose restriction $g|_S$ to the 3-dimensional leaves $S$ is $so(2,1)$ invariant.
2 The \(so(2,1)\) invariant gravitational fields

It has already been shown \cite{14} that locally there exists just one\(^1\) \(so(2,1)\) invariant solution of Einstein field equation in the vacuum. It is given by the metric (1.11) in which the components are given by

\[
A(r', t) = B(r', t)^{-1} = (1 - \frac{A}{r'}) \quad F(r', t) = 1,
\]

and \(r' = r(1 + A/4r)^2\). This is the static solution (pseudo-Schwarzschild) found in \cite{14} and also, in the context of warped solutions, in \cite{13}.

The more physically interesting gravitational field, generated by a distribution of matter described by an energy-momentum tensor \(T_{\mu\nu}\) and reducing in the vacuum to the previous one, is given by

\[
g = f(r)dt^2 - h(r)dr^2 - r^2(d\theta^2 + \sinh^2 \varphi d\varphi^2),
\]

where the \(so(2,1)\)-invariant positive functions \(f(r)\) and \(h(r)\) satisfy the equations

\[
8\pi T_{00} = h'(rh^2)^{-1} + \frac{1}{r^2}(1 - h^{-1})
\]

\[
8\pi T_{11} = f'(rhf)^{-1} - \frac{1}{r^2}(1 - h^{-1})
\]

\[
8\pi T_{22} = \frac{1}{2}f'(rhf)^{-1} - \frac{1}{2}h'(rh^2)^{-1} + \frac{1}{2}(fh)^{-1/2}[hf^{-1/2}f']^r,
\]

the apex denoting the derivation with respect to \(r\).

For a perfect fluid, with energy momentum tensor field of the type \(T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu)\), the above equations give

\[
h = \frac{1}{1 - 2m(r)/r},
\]

\[
\frac{d\psi}{dr} = \frac{4\pi Pr^3 + m}{r(r - 2m)}
\]

\[
\frac{dP}{dr} = -(P + \rho)\frac{4\pi Pr^3 + m}{r(r - 2m)}
\]

where \(\psi = \ln \sqrt{f}\) and \(m(r) = 4\pi \int_1^r \rho(r)r^2dr + C\), the constant \(C = h(1)^{-1} - 1\) being determined by the boundary conditions.

The quantity \(m(r)\), when integrated over the whole volume of the source, represents the total mass. Unlike the \(so(3)\) invariant compact case, the hyperbolic symmetry will give to the source an infinite extension. Then, a constant density is not allowed whereas each function of \(r\) decreasing faster than \(1/r^3\) will do.

\(^1\)This yields an extension of the Birkhoff theorem to the \(so(2,1)\) invariant case.
3 The $so(2,1)$ invariant cosmologies

In this section cosmologies invariant for a Lie algebra $G_r$ having $so(2,1)$ as a Lie subalgebra of space-like Killing fields will be analyzed.

3.1 $G_3 = SO(2,1)$ invariant cosmologies

The most general $so(2,1)$ invariant cosmology is given by the metric (1.11) with $A(r,t) = 1$:

$$g = dt^2 - B(r,t)dr^2 - F(r,t)r^2(d\theta^2 + \sinh^2 \varphi d\varphi^2)$$  \hspace{1cm} (3.14)

Cosmologies with maximal symmetry $so(2,1)$ are not homogeneous since the orbits of $so(2,1)$ are 2-dimensional. They belong to the class of Bianchi cosmologies [6]. An interesting point of view which deserves more investigation is to regard them as limiting cases of models with higher symmetries in the presence of sources which retain only the $so(2,1)$ symmetry. We will come back to this comment when discussing the $G_6$ models.

3.2 $G_4$ invariant cosmologies

An additional Killing field for the above metric has to commute with the $so(2,1)$ generators thus yielding a central extension of $so(2,1)$ and will have the general form

$$P = P^r(r,t) \frac{\partial}{\partial r} + P^t(r,t) \frac{\partial}{\partial t}$$  \hspace{1cm} (3.15)

The Killing equations

$$P^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu P^\alpha g_{\alpha\nu} + \partial_\nu P^\alpha g_{\mu\alpha} = 0$$  \hspace{1cm} (3.16)

are satisfied by

$$P = P(r) \frac{\partial}{\partial r}, \quad g = -dt^2 + \frac{B(t)}{P(r)^2}dr^2 + F(t)(d\theta^2 + \sinh^2 \varphi d\varphi^2),$$  \hspace{1cm} (3.17)

where $P(r)$ and $B(t)$ are arbitrary non vanishing positive functions and the time coordinate $t$ has been rescaled by $\sqrt{A(t)}$. By defining a new radial coordinate, this gravitational field may be reexpressed into the more suggestive form

$$g = -dt^2 + b(t)dr^2 + f(t)(d\theta^2 + \sinh^2 \varphi d\varphi^2).$$  \hspace{1cm} (3.18)

It may be obtained from the Kantowski-Sachs cosmological solution (1.4) replacing the $SO(3)$ orbits with those of $SO(2,1)$. It is spatially homogeneous as the leaves $S$ are orbits of $G_4$ but not isotropic, $SO(2,1)$ being only a global symmetry [4, 9].
3.3 $G_5$ invariant cosmologies

There are no $G_5$ invariant cosmologies because a 3-dimensional metric manifold cannot admit a complete group $G_5$ of motions [1].

The Gödel model [9] is a $G_5$ invariant solution of the vacuum Einstein equations with a $SO(2, 1)$ subgroup of symmetries, but the $SO(2, 1)$-orbits are not space-like so that they do not fit our analysis.

3.4 $G_6$ invariant cosmologies

In this case, the leaves $S$ are maximally symmetric and then isotropic and of constant curvature. Hence we search for three additional Killing fields $P_i$ which, together with the $so(2, 1)$ generators, span the Lie algebra of any possible $G_6$ with space-like 3-dimensional orbits. Because of the isotropy it must contain $SO(3)$ as a subgroup. On the other side, maximally symmetric manifolds are uniquely specified by the numbers of eigenvalues of the metric that are positive or negative and by the sign of the scalar curvature [3, 7]; it follows that the $G_6$ group we are looking for is the proper Lorentz group $SO(3, 1)$ and the corresponding invariant cosmology has to be isomorphic to the FRW cosmology (1.1) with negative spatial curvature, no matter what cumbersome coordinate system is adopted. Nonetheless, it is always interesting to discover yet an alternative form of such a metric starting from a symmetry different from $SO(3)$ which is the standard one. Together with a family of indefinite metrics, this alternative form may be obtained as follows. The most general $G_6$ including $so(2, 1)$ as a subalgebra may be given the Lie algebra structure [8]

$$[X_i, X_j] = c_{ijk} X_k \quad (3.19)$$
$$[X_i, P_j] = \epsilon_1 c_{ijk} P_k + \epsilon_2 f_{ijk} X_k \quad (3.20)$$
$$[P_i, P_j] = \epsilon_3 c_{ijk} X_k + \epsilon_4 f_{ijk} P_k \quad (3.21)$$

where $c_{ijk}$ are the structure constants of $so(2, 1)$, $f_{ijk}$ are some unknown structure constants and $\epsilon_i = 0, \pm 1$, all of them constrained by the Jacobi identity. Up to isomorphisms we have the following cases.

i) $\epsilon_1 = 1, \ \epsilon_2 = \epsilon_4 = 0, \ \epsilon_3 = 0, \pm 1$. This corresponds to search $G_6$ in the form of a principal fibre bundle having $SO(2, 1)$ as structure group;

ii) $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0, \ \epsilon_4 = 1$. This is the direct (hence commutative) sum which can be considered for any of the 9 Bianchi types: $so(2, 1) \oplus G_3$.

Imposing the Killing equation to the generators of $G_3$ we find no solutions of this form, if $so(2, 1)$ is realized as in (1.10);

iii) $\epsilon_1 = \epsilon_4 = 1, \ \epsilon_3 = 0$. This is the sum of two 3-dimensional Lie algebras not commuting between them, one of them being $so(2, 1)$. The case $\epsilon_2 = 0$ corresponds to a semidirect sum, whereas $\epsilon_2 = 1$ yields a fully non-commutative sum of Lie algebras. In this case $so(2, 1)$ is said to be a Lie
bialgebra [11]. The compatibility condition for two Lie algebras to be given such a structure is

\[ c_{ijk} f_{krs} + c_{irk} f_{jks} - f_{irk} c_{jks} - c_{jrk} f_{iks} + f_{jrk} c_{kis} = 0. \] (3.22)

There are only two solutions for this equation, one with \( f_{ijk} \) all vanishing which yields the semidirect sum \( so(2,1) \oplus' R^3 \) (this is the only semidirect sum compatible with the \( so(2,1) \) structure), the other one with \( f_{ijk} \) given by the structure constants of \( sb(2,C) \), the Lie algebra of type \( V \) in the Bianchi classification. They both correspond to an indefinite metric; the first one is rediscovered below in the context of case (i).

Let us analyze case (i) in more detail. From the condition (3.20), with \( X_i \in so(2,1) \) the fields \( P_i \) have the following form

\[
\begin{align*}
P_1 &= b(r) \left[ \sinh \vartheta \cos \varphi \partial_r - \frac{\cos \varphi \cosh \vartheta}{r} \partial_\varphi + \frac{\sin \varphi}{r \sinh \vartheta} \partial_\vartheta \right] \\
P_2 &= b(r) \left[ \sinh \vartheta \sin \varphi \partial_r - \frac{\sin \varphi \cosh \vartheta}{r} \partial_\varphi - \frac{\cos \varphi}{r \sinh \vartheta} \partial_\vartheta \right] \\
P_3 &= b(r) \left[ \cosh \vartheta \partial_r - \frac{\sinh \vartheta}{r} \partial_\vartheta \right]
\end{align*}
\] (3.23)

The condition (3.21) fixes the \( r \) dependence to be

\[ b(r) = (-\epsilon_3 r^2 + C)^{1/2}, \] (3.24)

with \( C \) an arbitrary negative constant. Moreover the Killing equations for the metric (3.14) restricted to the leaves \( S \), give

\[ B(r) = \frac{B}{(-\epsilon r^2 + C)}; \quad F(r) = -\frac{B}{C}; \] (3.25)

with \( B \) a positive arbitrary constant.

Thus, the restricted metric takes the form

\[ g|_S = -\frac{1}{(1 + \epsilon_3 r^2/B)} dr^2 + r^2 (d\vartheta^2 + \sinh^2 \vartheta d\varphi^2) \] (3.26)

where the coordinate \( r \) has been rescaled by the factor of \( \sqrt{-B/C} \).

According to the value of \( \epsilon_3 \) three different situations occur. It can be easily checked that for \( \epsilon_3 = 0 \), \( Kil(g) \) is the semidirect sum \( so(2,1) \oplus' R^3 \); for \( \epsilon_3 = -1 \), \( Kil(g) \) is the Lorentz Lie algebra \( so(3,1) \), while for \( \epsilon_3 = 1 \), \( Kil(g) \) is the direct sum \( so(2,1) \oplus so(2,1) \), that is to say \( so(2,2) \). The submanifolds \( S \) have constant scalar curvature given by

\[ \mathcal{R}(S) = 6 \frac{\epsilon}{Ba^2} . \] (3.27)
Then the metric of the space-time is given by
\[ g = dt^2 - a^2(t)\left[\frac{1}{1 + \epsilon_3 r^2/B} dr^2 + r^2(\theta^2 + \sinh^2 \varphi d\varphi^2)\right], \]
\[ (3.28) \]

For \( \epsilon_3 = -1 \) and \( r^2 > B \) it has Lorentzian signature and the leaves \( S \) are space-like submanifolds with negative scalar curvature. The above metric is locally diffeomorphic to the FRW metric (1.1) with negative \( k \), but the coordinates are adapted to a different subgroup of the complete symmetry group \( SO(3,1) \).

Indeed, as it has already been mentioned, the vector fields \( X_i \in so(2,1) \) given by (1.10) together with the vector fields \( P_i \) (3.23) with \( \epsilon_3 = -1 \), span the Lie algebra of the proper Lorentz group. This algebra can be put into a more conventional form by redefining
\[ Y_1 = \sqrt{B} P_1, \quad Y_2 = \sqrt{B} P_2, \quad Y_3 = -X_3, \]
\[ Q_1 = X_1, \quad Q_2 = X_2, \quad Q_3 = \sqrt{B} P_3 \]
\[ (3.29) \]

Thus, we have found yet an alternative form of the FRW metric, for the case of negative scalar curvature, in terms of the symmetry subgroup \( SO(2,1) \). The leaves \( S \) are foliated by pseudospheres, \( i.e \) space-like orbits of \( SO(2,1) \) which are non-compact and of constant negative curvature. This result is not surprising because, as already recalled, two maximally symmetric manifolds with the same inertial indices and the same sign of the scalar curvature have to be diffeomorphic. On the other hand the Lorentz group has both \( SO(3) \) and \( SO(2,1) \) as subgroups, so that we can choose to adapt our system of coordinates to either one or the other.

Going back to case (i), the remaining choices \( \epsilon_3 = 1,0 \) respectively correspond to positive and null scalar curvature of the 3-dimensional orbits, but the associated metrics are of indefinite signature, that is such orbits are not space-like. The expected conclusion is that there is only one possible \( G_6 \)-invariant cosmology admitting \( SO(2,1) \) as a subgroup and this is the well known Lorentz group for the negative curvature FRW metric.

The interest in the indefinite metrics we have found could be for example in studying the symmetry breaking, from any of the reported six-dimensional symmetry groups, down to \( SO(2,1) \) through an appropriately chosen source.

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