Intractability of approximate multi-dimensional nonlinear optimization on independence systems

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Abstract

We consider optimization of nonlinear objective functions that balance $d$ linear criteria over $n$-element independence systems presented by linear-optimization oracles. For $d = 1$, we have previously shown that an $r$-best approximate solution can be found in polynomial time. Here, using an extended Erdős-Ko-Rado theorem of Frankl, we show that for $d = 2$, finding a $\rho n$-best solution requires exponential time.

1 Introduction

Given system $S \subseteq \{0, 1\}^n$, integer $d \times n$ matrix $W$, and function $f : \mathbb{Z}^d \to \mathbb{Z}$, consider the problem of minimizing the nonlinear composite function $f(Wx)$ over $S$, that is,

$$\min\{f(Wx) : x \in S\}.$$  

(1)

This problem can be interpreted as multi-criteria optimization, where row $W_i$ of $W$ gives a linear function $W_i x$ representing the value of feasible point $x \in S$ under criterion $i$, and the objective value $f(Wx) = f(W_1 x, \ldots, W_d x)$ is the balancing of these $d$ criteria.

Assume we can do linear optimization over $S$ to begin with, namely $S$ is presented by a linear-optimization oracle, which queried on $w \in \mathbb{Z}^n$, solves $\max\{wx : x \in S\}$. For restricted systems $S$, such as matroids and matroid intersections, or restricted functions $f$, such as concave functions, problem (1) can be solved in polynomial time [1, 2]. A comprehensive description of the state of the art on this area can be found in [3].

Here we continue our investigation from [4] of problem (1) where $S$ is an arbitrary independence system, that is, $S$ nonempty, and $x \leq y \in S$ with $x \in \{0, 1\}^n$ imply $x \in S$.

A feasible point $x^* \in S$ is called an $r$-best solution of problem (1) provided there are at most $r$ better objective function values attainable by other feasible points, that is,

$$|\{f(Wx) : f(Wx) < f(Wx^*), x \in S\}| \leq r.$$  

So it provides a suitable approximation to (1). In particular, a 0-best solution is optimal.

In [4], the case of $d = 1$ was considered, that is, the problem $\min\{f(wx) : x \in S\}$ with $w \in \mathbb{Z}^n$. It was shown that for any fixed positive integers $a_1, \ldots, a_p$ there is a polynomial time algorithm that, given any $w \in \{a_1, \ldots, a_p\}^n$, provides an $r(a_1, \ldots, a_p)$-best solution to the problem, where $r(a_1, \ldots, a_p)$ is a constant related to Frobenius numbers of some of the $a_i$. In particular, for any $p = 2$ integers, $r(a_1, a_2) = F(a)$ is the Frobenius number.
In this note we consider the problem in dimension $d = 2$. We restrict attention to $2 \times n$ matrices $W$ which are $\{0, 1\}$-valued. Then the image of $S$ under $W$ satisfies

$$WS := \{Wx : x \in S\} \subseteq \{0, 1, \ldots, n\}^2.$$  \hfill (2)

Therefore, the problem of computing the optimal objective function value of (1) is seemingly reducible to computing the image $WS$ by checking if $y \in WS$ for each of the $(n+1)^2$ points $y$ in the set on right-hand side of (2) and determining the minimum value of $f$ over $WS$. Unfortunately, this so called fiber problem, of checking if $y \in WS$, is computationally hard. In particular, already for $S$ the set of (indicators of) matchings in a bipartite graph, over which linear optimization is easy, this problem includes as a special case the notorious exact matching problem whose complexity is long open [6].

Here we show that there is a universal positive constant $\rho$ such that, already for $d = 2$, matrix $W$ each column of which is one of the two unit vectors in $\mathbb{Z}^2$, and very simple explicit function $f$ supported on $\{0, 1, \ldots, n\}^2$, there is no polynomial time algorithm that can produce even a $\rho n$-best solution of problem (1) for every independence system $S \subseteq \{0, 1\}^n$, let alone find a constant $r$-best or optimal solution. Our construction makes use of a beautiful extension of the classical Erdős-Ko-Rado theorem due to Frankl [3].

It is interesting whether our construction could be refined to shed some light on the exact matching and related open problems of [6], and whether other natural oracles for $S$ could lead to polynomial time solution of problem (1) in dimensions $d = 2$ and higher.

2 A $\rho n$-best solution cannot be found in polynomial time

**Theorem 2.1.** There exists a universal positive constant $\rho$ such that no polynomial time algorithm can compute a $\rho n$-best solution of the 2-dimensional nonlinear optimization problem

$$\min \{f(Wx) : x \in S\}$$

over every independence system $S \subseteq \{0, 1\}^n$ presented by a linear-optimization oracle, with $W$ an integer $2 \times n$ weight matrix each column of which is one of the unit vectors in $\mathbb{Z}^2$, and $f$ an explicit function supported on $\{0, 1, \ldots, n\}^2$.

In fact, the following explicit statement holds. Let $l$ be any positive integer with $l \geq 2^{10}$, $k := 7l$, $m := 8l^2$, $n := 2m$, and $\rho := \frac{1}{17}$. Let $W$ be the $2 \times n$ matrix with first $m$ columns the unit vector $1_1$ and last $m$ columns the unit vector $1_2$. Define $f$ on $\mathbb{Z}^2$ explicitly by

$$f(y) = f(y_1, y_2) := \begin{cases} (y_1 - k) - l(y_2 - k) - 1 & \text{if } k + 1 \leq y_1, y_2 \leq k + l, \\ 0 & \text{otherwise}. \end{cases} \hfill (3)$$

Then at least $2^{\frac{1}{17} \sqrt{n}}$ queries to the oracle of $S$ are needed to compute a $\frac{1}{17} n$-best solution.

**Proof.** Let $l \geq 2^{10}$ be a positive integer, $k, m, n, \rho$ and $W$ as above, and $f$ as in (3) above. It is more convenient here to work with set systems over ground set $N := \{1, \ldots, n\}$ rather than sets of vectors in $\{0, 1\}^n$. As usual, vectors $x \in \{0, 1\}^n$ are in bijection with subsets $X \subseteq N$ with corresponding elements satisfying $X = \text{supp}(x)$ the support of $x$ and $x = 1_X$ the indicator of $X$. So we replace each $S \subseteq \{0, 1\}^n$ by the set system $S := \{X = \text{supp}(x) : x \in S\}$. Also, for $c \in \mathbb{Z}^n$ and $X \subseteq N$ we write $cX := c1_X$. Let

$$N_1 \supseteq N_2 = N$$

be the natural equipartition of the ground set defined by $N_1 := \{1, \ldots, m\}$ and $N_2 := \{m + 1, \ldots, 2m\}$. For each subset $X \subseteq N$ of the ground set we write

$$X_1 := X \cap N_1, X_2 := X \cap N_2,$$

with $X = X_1 \cup X_2$ the naturally induced partition of $X$. 
Moreover, the objective function values of the points in $S$ are equal to $(|X_1|, |X_2|)$. The image of a set system $S$ over $N$ is $WS := \{WX : X \in S\}$. We use several set systems over $N$, defined as follows. First, for each pair of integers $0 \leq y_1, y_2 \leq m$, let

$$S_{y_1,y_2} := \{X = X_1 \uplus X_2 : |X_1| = y_1, |X_2| = y_2\}.$$  

Next, let

$$S^* := \{X : (|X_1|, |X_2|) \leq (m, k) \text{ or } (|X_1|, |X_2|) \leq (k, m)\}.$$  

Then $S^*$ is an independence system whose image is given by

$$WS^* := WS \uplus \{(y_1, y_2) : (k + 1, k + 1) \leq (y_1, y_2) \leq (k + l, k + l)\}.$$  

Moreover, the objective function value of every $X \in S^*$, and hence in particular of every $\rho n$-best solution of the minimization problem over $S^*$, satisfies $f(WX) = 0$.

Next, for each $Y \in S_{k+l,k+l}$, let

$$S_Y := S^* \cup \{X : X \subseteq Y\}.$$  

Then $S_Y$ is also an independence system, with image

$$WS_Y := WS^* \uplus \{(y_1, y_2) : (k + 1, k + 1) \leq (y_1, y_2) \leq (k + l, k + l)\}.$$  

Moreover, the objective function values of the points in $S_Y \setminus S^*$, whose images lie in $WS_Y \setminus WS^*$, attain exactly all $l^2 = \frac{k}{l} n > pm$ values $-1, -2, \ldots, -l^2$, and so the value of every $\rho n$-best solution of the minimization problem over $S_Y$ satisfies $f(WX) \leq -1$.

For each vector $c \in \mathbb{Z}^n$ and each pair $1 \leq i_1, i_2 \leq l$, let

$$T_{i_1,i_2}(c) := \{Z \in S_{k+i_1,k+i_2} : cZ > \max \{cX : X \in S^*\}\}.$$  

Claim: For every $c \in \mathbb{Z}^n$ and every pair $1 \leq i_1, i_2 \leq l$, we have

$$|T_{i_1,i_2}(c)| \leq \binom{m}{l} \left( \frac{m}{k+l} \right).$$  

Proof of Claim: Consider any pair $U = U_1 \uplus U_2, V = V_1 \uplus V_2 \in T_{i_1,i_2}(c)$. We now show that either $|U_1 \cap V_1| \geq k + 1$ or $|U_2 \cap V_2| \geq k + 1$. Suppose, indirectly, this is not so. Put

$$X := (U_1 \cap V_1) \uplus (U_2 \cap V_2),$$  

$$Y := (U_1 \cup V_1) \uplus (U_2 \cap V_2).$$  

Then $|U_1 \cap V_1| \leq k$ and $|U_2 \cup V_2| \leq m$, imply $X \in S^*$, and $|U_1 \cup V_1| \leq m$ and $|U_2 \cap V_2| \leq k + l$ imply $Y \in S^*$. We then obtain the following contradiction,

$$0 < cU - cX = c(U_1 \setminus V_1) - c(V_2 \setminus U_2) = cY - cV < 0.$$  

So indeed, for every pair $U = U_1 \uplus U_2, V = V_1 \uplus V_2 \in T_{i_1,i_2}(c) \subseteq S_{k+i_1,k+i_2}$, either $|U_1 \cap V_1| \geq k + 1$ or $|U_2 \cap V_2| \geq k + 1$. Therefore, we can now apply the extended Erdős-Ko-Rado theorem for direct products of Frankl [3] Theorem 2, which implies

$$\frac{|T_{i_1,i_2}(c)|}{|S_{k+i_1,k+i_2}|} \leq \max \left\{ \left( \frac{m}{k + i_1} \right), \left( \frac{m}{(k + i_2) - (k + 1)} \right) \right\}.$$  

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from which it is easy to conclude that, as claimed,

$$|T_1, i_2(c)| \leq \binom{m}{l} \binom{m}{k+l}.$$ 

We continue with the proof of our theorem. Since $k = 7l$, $m = 8l^2$ and $l \geq 2$ we get

$$\left( \frac{m}{k+l} \right) \left( \frac{m}{l} \right)^3 = \left( \frac{8l^2}{8l} \right) \left( \frac{8l^2}{l} \right)^3 = \left( \frac{4l^2}{8l} \right)^{8l} / (8l^2)^{3l} \geq (2^{-9})^{2l}.$$ 

Therefore

$$|S_{k+l, k+l}| = \left( \frac{m}{k+l} \right) \left( \frac{m}{l} \right)^3 \geq (2^{-9})^{2l} \left( \frac{m}{l} \right)^3 \left( \frac{m}{k+l} \right).$$

Consider any algorithm attempting to obtain a $\rho_n$-best solution to the nonlinear optimization problem over any system $S$, and let $c^1, \ldots, c^q \in \mathbb{Z}^n$ be the sequence of queries to the oracle of $S$ made by the algorithm. For each pair $1 \leq i_1, i_2 \leq l$ and each $Z \in T_{i_1, i_2}(c^p)$, the number of $Y \in S_{k+l, k+l}$ containing $Z$, and hence satisfying $Z \in S_Y$, is

$$\left( \frac{m - (k + i_1)}{l - i_1} \right) \left( \frac{m - (k + i_2)}{l - i_2} \right) \leq \left( \frac{m}{l} \right)^2.$$ 

So the number of $Y \in S_{k+l, k+l}$ containing some $Z$ which lies in some $T_{i_1, i_2}(c^p)$ is at most

$$\sum_{p=1}^{q} \sum_{i_1=1}^{l} \sum_{i_2=1}^{l} \binom{m}{l}^2 |T_{i_1, i_2}(c^p)| \leq q l^2 \binom{m}{l}^3 \binom{m}{k+l}.$$

Therefore, if the number of oracle queries satisfies $q < l^{-2} (2^{-9} l)^{2l}$, then there exists some $Y \in S_{k+l, k+l}$ which does not contain any $Z$ in any $T_{i_1, i_2}(c^p)$. This means that any $Z \in S_Y$ satisfies $c^p Z \leq \max\{c^p X : X \in S^*\}$. Hence, whether the linear-optimization oracle presents $S^*$ or $S_Y$, on each query $c^p$ it can reply with some $X^p \in S^*$ attaining $c^p X^p = \max\{c^p X : X \in S^*\} = \max\{c^p X : X \in S_Y\}.$

So the algorithm cannot tell whether the oracle presents $S^*$ or $S_Y$, whether the image is $WS^*$ or $WS_Y$, and whether the objective function value of every $\rho_n$-best solution is zero or negative, let alone compute any $\rho_n$-best solution. Therefore, with $l \geq 2^{10}$, every algorithm which can produce a $\rho_n$-best solution for the 2-dimensional nonlinear optimization problem (1) over every system $S$ must make at least an exponential number

$$q \geq l^{-2} (2^{-9} l)^{2l} \geq l^{-2} 2^{2l} > 2^l = 2^{\frac{1}{2} \sqrt{n}}$$

of queries to the oracle presenting $S$ and therefore cannot run in polynomial time. \qed
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