A SIMPLE SCHEME TO IMPLEMENT A NONLOCAL TURBULENT CONVECTION MODEL FOR CONVECTIVE OVERSHOOT MIXING

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ABSTRACT

Classical “ballistic” overshoot models show some contradictions and are not consistent with numerical simulations and asteroseismic studies. Asteroseismic studies imply that overshoot is a weak mixing process. A diffusion model is suitable to deal with it. The form of diffusion coefficient in a diffusion model is crucial. Because overshoot mixing is related to convective heat transport (i.e., entropy mixing), there should be some similarity between them. A recent overshoot mixing model shows consistency between composition mixing and entropy mixing in the overshoot region. A prerequisite to apply the model is to know the dissipation rate of turbulent kinetic energy. The dissipation rate can be worked out by solving turbulent convection models (TCMs). But it is difficult to apply TCMs because of some numerical problems and the enormous time cost. In order to find a convenient way, we have used the asymptotic solution and simplified the TCM to a single linear equation for turbulent kinetic energy. This linear model is easy to implement in calculations of stellar evolution with negligible extra time cost. We have tested the linear model in stellar evolution, and have found that it can well reproduce the turbulent kinetic energy profile of the full TCM, as well as the diffusion coefficient, abundance profile, and stellar evolutionary tracks. We have also studied the effects of different values of the model parameters and have found that the effect due to the modification of temperature gradient in the overshoot region is slight.

Key words: convection – stars: evolution – stars: interiors

1. INTRODUCTION

Convective motion beyond the boundary of local linear stability is called convective overshoot. The mixing caused by convective overshoot is a major uncertainty in current stellar evolutionary theory, since it deeply affects the stellar structure but there is still no robust and easy-to-use theory at present. The traditional treatment of overshoot is based on nonlocal mixing length theories, e.g., “ballistic” models (Maeder 1975; Bressan et al. 1981; Zahn 1991), which show an adiabatically stratified and completely mixed overshoot region with a typical length of about 0.2–0.4$H_p$, where $H_p = -dr/d
ln P$ is the local pressure scale height. Although nonlocal mixing length models are easy to implement in stellar evolution codes and are widely used, they have some contradictions and they do not have enough spatial resolution to accurately describe the overshoot process (Renzini 1987). A property of nonlocal mixing length models is that there is a jump in $\nabla$ (temperature gradient) from $\nabla_{ad}$ (the adiabatic temperature gradient) to $\nabla_R$ (the radiative temperature gradient) at the boundary of the overshoot region. For the Sun, the discontinuity of $\nabla$ predicted by nonlocal mixing length models leads to a characteristic oscillatory component in the frequencies of solar $p$-modes (Gough 1990). This has been used to estimate the length of the overshoot region below the solar convection zone, and an upper limit has been found as 0.05$H_p$ (Basu et al. 1994; Basu & Antia 1994; Monteiro et al. 1994; Roxburgh & Vorontsov 1994; Christensen-Dalsgaard et al. 1995; Basu 1997). That is too small compared with the prediction of nonlocal mixing length models. Christensen-Dalsgaard et al. (2011) have investigated the temperature gradient profile below the base of the solar convection zone and have found that, in order to improve the agreement between models and helioseismic constraints, we actually need a smooth profile of $\nabla$, which is outside the realm of nonlocal mixing length overshoot models. The helioseismic study may imply that the downward overshoot region below the base of the solar convection zone cannot be completely mixed. Overshoot mixes both entropy and composition (Zhang 2013), and if such mixing is efficient it leads to $dS/dr = 0$ and $dX/dr = 0$ in the overshoot region. Entropy and composition being constants results in $\nabla = \nabla_{ad}$, just like the case in the convection zone with efficient convective heat transport. In a recent asteroseismic study on a very slowly rotating slowly pulsating B star KIC 10526294 (Moravveji et al. 2015), it is found that assuming an exponential diffusion coefficient for the convective core overshooting is better than assuming a large constant diffusion coefficient in the whole overshoot region, which implies that the classical step-function overshoot model is not the best choice.

Convective overshoot is a nonlocal convection phenomenon and there exist, besides nonlocal mixing length models, also turbulent convection models (TCMs), which are based on statistical equilibrium equations of auto- and cross-correlations of velocity and temperature perturbations (e.g., Xiong 1981, 1985; Canuto 1997, 2011; Xiong et al. 1997; Canuto & Dubovikov 1998; Deng et al. 2006; Li & Yang 2007; Li 2012). Xiong’s (1981) TCM and Li & Yang’s (2007) TCM have been applied in solar structure models and have been found to provide the required smooth $\nabla$ profile (Zhang & Li 2012a; Zhang et al. 2012). The temperature gradient profile outside the Schwarzschild local convective boundary predicted by Xiong’s (1981) TCM or Li & Yang’s (2007) TCM is different from the prediction of nonlocal mixing length models. Zahn (1991) has proposed to use the words “penetration” and “overshoot” to distinguish those convective motions beyond the Schwarzschild boundary with high efficiency of convective heat transport (“penetration”) and with low efficiency (“overshoot”), and the efficiency of penetration convection is so high that the dominant region is nearly adiabatically stratified. This
TCM. This diffusion model predicts an exponentially decreasing diffusion coefficient $D$ in the overshoot region, and the characteristic length for mixing in that region is the same as the characteristic length in the convection zone. On the other hand, the form of the diffusion coefficient in overshoot mixing should be related to the convective heat transport in the overshoot region because the latter is actually caused by the entropy mixing. In an overshoot region with high Péclet number, the convective timescale is too short for flows to exchange their entropy or their composition. The similarity between composition mixing and entropy mixing implies that the form of both may be the same. Turbulent convection models show that the convective heat flux $u_r T_r$ in the overshoot region is (Xiong 1989; Deng et al. 1996; Li & Yang 2007; Li 2012; Zhang & Li 2012b)

$$\frac{u_r T_r}{\mathcal{S}} \propto -\frac{T}{\delta g} \varepsilon,$$

(1)

where $g$ is the gravitational acceleration, $T$ is temperature, $\delta = -(\partial \ln \rho/\partial \ln T)_\rho$ is the dimensionless expansion coefficient, and $\varepsilon$ is the dissipation rate of turbulent kinetic energy. Therefore the entropy flux $u_r \mathcal{S}$ of the overshoot region is

$$u_r \mathcal{S} \approx \frac{c_p}{T} u_r T_r \propto -\frac{T}{\delta g} \varepsilon \sim -\frac{\varepsilon}{N^2} \frac{\partial S}{\partial r},$$

(2)

where $N^2$ describes the squared buoyancy frequency, $S$ is entropy, and $c_p$ is specific heat capacity at constant pressure. This expression shows that the diffusion coefficient for entropy mixing in an overshoot region with high Péclet number is $D_\varepsilon \propto \varepsilon/N^2$. In the convective mixing model of Zhang (2013), the diffusion coefficient is solved based on hydrodynamic equations and some closure assumptions. The solution is that the diffusion coefficient for convective mixing in the convection zone is of the form $D \propto k^2/\varepsilon \sim ul$ and the diffusion coefficient for convective overshoot mixing is of the form $D \propto \varepsilon/N^2$. This result is consistent with convective entropy mixing in both convection zone and overshoot region.

The prerequisite for applying Zhang’s (2013) convective mixing model is to know the dissipation rate of turbulent kinetic energy $\varepsilon$ in the overshoot region. At present, a practicable option is to use TCMs (e.g., Xiong 1981, 1985; Canuto 1997, 2011; Xiong et al. 1997; Canuto & Dubovikov 1998; Deng et al. 2006; Li & Yang 2007; Li 2012), which have been suggested to deal with convective overshoot by helioseismic studies (Christensen-Dalsgaard et al. 2011). Those TCMs are based on hydrodynamic equations and closure assumptions, and describe the evolution and distribution of averaged correlations of turbulent variables $(u_r/u'_r, u'_r T_r, T'_r, T'^r, \varepsilon, \cdots)$ in the stellar interior. However, TCMs are highly nonlinear equations, too complicated to apply in stellar evolution. Sometimes it is difficult to find a solution satisfying both TCM equations and the stellar structure equations because of problems in numerical calculation. Even for the converged stellar evolution models, the time cost is enormous (normal time cost multiplied by a factor of 50–100) (Zhang 2015). In order to apply the convective mixing model, it is necessary to simplify TCMs to stably and quickly solve the distribution of turbulent kinetic energy $\varepsilon$ in the stellar interior. In this paper, we introduce a simple scheme to implement Li & Yang’s (2007) TCM for convective overshoot mixing. The content of this paper is as follows: the overshoot mixing model is
introduced in Section 2, the TCM and its properties are introduced in Section 3, the details of the simple scheme are described in Section 4, the numerical results of this scheme are shown in Section 5, and Section 6 is a summary.

2. THE OVERSHOOT MIXING MODEL

In this paper, Zhang’s (2013) model of overshoot mixing is adopted. The model is derived from fluid dynamic equations and some assumptions. The model shows that the convective overshoot mixing in a region of high Péclet number can be treated as a diffusion process with the diffusion coefficient as follows:

\[ D = C_{OV} \frac{\varepsilon}{N_{turb}}. \]  

(3)

where \( N_{turb} \) is calculated as

\[ N_{turb}^2 = -\frac{\delta g}{H_p} \left[ \nabla - \nabla_{ad} \right] \]

\[ C_1 C_3 \sum_{i=1}^{I} \left( \frac{\partial \ln T}{\partial X_i} \right)_{p,H-X(X_i)} \frac{dX_i}{d \ln P} \]

(4)

where \( I \) is the number of independent elements, \( \varepsilon \) is the dissipation rate of turbulent kinetic energy, \( \nabla_{ad} \) is the adiabatic temperature gradient, \( \nabla \) is the real temperature gradient in the stellar interior, \( C_{OV}, C_1, \) and \( C_3 \) are model parameters, \( X_i \) is the mass fraction of the \( i \)th element, and the other symbols have their usual meanings. The parameter \( C_{OV} \) is a proportional factor that could be determined by calibrations of fitting observations, the parameter \( C_1 \) is used to model the turbulent abundance–abundance correlation \( X_i X_j \), and the parameter \( C_3 \) is used to model the dissipation of turbulent temperature–abundance correlation \( T X_i X_j \). \( N_{turb} \) is similar to the squared Brunt–Väisälä frequency \( N^2 \) since \( N_{turb}^2 = N^2 \) when \( C_1 C_3 = 1 \), which is assumed in Zhang (2013). However, according to Canuto & Dubovikov (1998) and Canuto (2011), \( C_1 = \sigma_i = 0.72 \) where \( \sigma_i \) is the turbulent Prandtl number.

The representation of the diffusion coefficient shows the image that the (radial) length scale for mixing \( l_{mix} = \sqrt{k}/N_{turb} \) and the lifetime \( \tau_{mix} = \tau = k/\varepsilon \) where \( k \) is the turbulent kinetic energy, since the diffusion coefficient is \( D \propto l_{mix}^2/\tau_{mix} \). The diffusion coefficient of matter mixing has the same form as convective heat transport (i.e., entropy mixing) in an overshoot region with high Péclet number (Zhang 2013), because the TCMs (e.g., Xiong 1989; Deng et al. 2006; Zhang & Li 2012b) show that the convective heat flux is proportional to the dissipation rate of turbulent kinetic energy in an overshoot region with high Péclet number. The physical reason is that convective heat transport in the region of high Péclet number is equivalent to entropy mixing, and entropy mixing is an accessory of matter mixing (Zhang 2013). The same form for matter mixing and for convective heat transport implies the consistency between TCMs and the overshoot mixing model.

In order to apply the overshoot mixing model in stellar evolution, one must know the dissipation rate of turbulent kinetic energy, i.e., \( \varepsilon \), in the overshoot region. At present, we can calculate the dissipation rate of turbulent kinetic energy in the overshoot region by using TCMs.

3. LI & YANG’S (2007) NONLOCAL TCM AND ITS PROPERTIES

The TCM adopted in this paper was developed by Li & Yang (2007):

\[ \frac{\partial}{\partial m} \left[ \left( \frac{dm}{dr} \right)^2 \left( 2 C_{e1} k_r \tau \right) \frac{\partial k_r}{\partial m} \right] = \frac{1}{3} \tau^{-1} \frac{\delta g}{T} u_r T', \]

(5)

\[ + C_k \tau^{-1} \left( k_r - \frac{k}{3} \right). \]

\[ \frac{\partial}{\partial m} \left[ \left( \frac{dm}{dr} \right)^2 \left( 2 C_{e1} k_r \tau \right) \frac{\partial k_r}{\partial m} \right] = \frac{1}{3} \tau^{-1} \frac{\delta g}{T} u_r T', \]

(6)

\[ \frac{\partial}{\partial m} \left[ \left( \frac{dm}{dr} \right)^2 \left( 4 C_{e1} k_r \tau \right) \frac{\partial u_r T'}{\partial m} \right] = -\frac{\delta g}{T} T'' T', \]

(7)

\[ - 2 k_r \frac{\rho}{R_p} (\nabla - \nabla_{ad}) + C_1 (1 + P_e^{-1}) \tau^{-1} u_r T', \]

\[ + C_e (1 + P_e^{-1}) \tau^{-1} T'' T'. \]

In the above equations, the meanings of the symbols are as follows: \( k_r = u_r u_r'/2 \) is the radial kinetic energy, \( u_r T' \) is the convective heat flux, \( T'' T' \) is the temperature variance, \( \tau = k/\varepsilon \) is the turbulent dissipation timescale with the turbulent dissipation \( \varepsilon = k^{3/2}/l \) in which \( l = \alpha H_p \), \( P_e = l\sqrt{k}/D_R \) is the Péclet number in which the radiative diffusion coefficient \( D_R = \lambda_0/\rho c \) and the thermal conduction coefficient \( \lambda = 4aT^3/(3cp) \), \( C_e, C_1, C_2, C_{e1}, C_{e2}, \alpha \) are model parameters, and the other symbols have their usual meanings. The parameter \( C_e \) in this model is related to the overshoot mixing model by \( C_{OV} = C_e - C_e \) (Zhang 2013).

This TCM has been investigated theoretically by Zhang & Li (2012b). Now we recall the main results.

In the convection zone with high Péclet number, turbulence is nearly in local equilibrium, thus the localized model (ignoring the diffusion terms on the left-hand side of the equations of the nonlocal model) is reasonable to describe the turbulent convection (Li & Yang 2001). The approximate solution of the localized model in a convection zone of high Péclet number shows that the temperature gradient is very close to the adiabatic temperature gradient.

In the overshoot region, the diffusion of turbulent kinetic energy is necessary since the turbulent energy in the convective overshoot region is supported by nonlocal convective transport. By ignoring the diffusions of \( u_r T' \) and \( T'' T' \) (i.e., setting \( C_{e1} = C_{e2} = 0 \)), it has been found that the asymptotic solution in the overshoot region with \( P_e \gg 1 \) is

\[ k = k_C \left( \frac{p}{P_e} \right), \]

(9)

\[ u_r T' = \text{Max}\left\{ -\frac{T}{H_p} D_R (\nabla_{ad} - \nabla_R), -2C_e \omega \frac{T}{\delta g} \right\}, \]

(10)

\[ T'' T' = 2 \frac{T}{H_p} \frac{T}{\delta g} (\nabla_{ad} - \nabla) \omega k, \]

(11)

where \( k_C \) is the Schwarzschild convective boundary where \( \nabla_R = \nabla_{ad}, \omega = k_r/k \) is the degree of anisotropy, and
\[ \theta = d \ln k/d \ln P \] is the exponentially decreasing index of turbulent kinetic energy in the overshoot region.

The exponentially decreasing index of turbulent kinetic energy in the overshoot region \( \theta \) is determined by

\[ \theta = \pm \frac{1}{\alpha} \sqrt{\frac{1 + 2C_e \omega_o}{3C_e \omega_o}}, \tag{12} \]

where the sign depends on the direction of overshoot: positive for upward and negative for downward.

The value of \( k_e \) can be estimated by using the “the maximum of diffusion” method as

\[ k_e^2 \approx \frac{1}{e} \left[ k_{B,local}^2 \right] \approx \frac{1}{e} \left[ \delta g D_R (\nabla R - \nabla_{ad}) \right]_{B}, \tag{13} \]

where location \( B \) is a point in the convection zone with the distance to the convective boundary being

\[ |r_B - r_c| = \frac{4C_e \omega_c}{3} l \tag{14} \]

where \( r_c \) is the radius at the convective boundary.

Some typical values of the degree of anisotropy \( \omega \) in some cases are as follows: \( \omega_{CZ} \) the degree of anisotropy in the convection zone:

\[ \omega_{CZ} = \frac{2}{3C_k} + \frac{1}{3}, \tag{15} \]

\( \omega_c \) the degree of anisotropy at the convective boundary:

\[ \omega_c \approx \frac{1}{2} \left( \omega_{CZ} + \frac{1}{3} \right), \tag{16} \]

and \( \omega_o \) the asymptotic equilibrium value of the degree of anisotropy in the overshoot region satisfies the following equation:

\[ 2C_e \omega_o^2 - (C_k - 1 + 2C_e) \omega_o + \frac{1}{3} (C_k - 1) = 0. \tag{17} \]

The parameter \( C_k \) should be larger than 1, and the smaller root of \( \omega_o \) is the physical root (see the Appendix).

The above solution is for the TCM in the overshoot region with \( P_e \gg 1 \), and the diffusions of \( u_e T \) and \( TT' \) are ignored. In the low \( P_e \) overshoot region, it is mathematically required that turbulent variables must be cut off in a short distance. Therefore it is reasonable to ignore the overshoot in low \( P_e \) region. The diffusion of \( u_e T \) can be ignored since the diffusion is much less than the local terms. The diffusion of \( TT' \) smoothes the profile of \( u_e T \) and \( TT' \) near the convective boundary but basically does not affect the profile of \( k \). For those reasons, we can use the above approximate/asymptotic solution instead of the numerical solution of the TCM.

4. LINEAR MODEL OF NONLOCAL TCM FOR OVERSHEET MIXING

By solving Li & Yang’s (2007) nonlocal TCM to obtain the dissipation rate of turbulent kinetic energy \( \varepsilon = k^{3/2}/l \) where \( l = \alpha H_P \), one can apply the overshoot mixing model. However, it is difficult to apply such a nonlocal TCM in stellar evolution (e.g., Zhang 2015): the time cost is enormous and numerical instability cannot be totally resolved yet. It is necessary to find a simple approach in order to work out the dissipation rate conveniently. The asymptotic solution is simple to use but the estimate of \( k_e \) cannot be used for a thin convection zone or a small convective core, and the accuracy of the estimate is not high enough. We need to find a better way.

The equation for turbulent kinetic energy in diffusion equilibrium, Equation (6), is equivalent to the following equation:

\[ \frac{\partial}{\partial m} \left[ \left( \frac{dm}{dr} \right)^2 \left( \frac{4}{3} \omega e C_k \right) \frac{\partial k^2}{\partial m} \right] = \varepsilon - \frac{\delta g}{T} u_e T'. \tag{18} \]

As mentioned above, the temperature gradient in a convection zone of high Péclet number is near the adiabatic temperature gradient, and the convective heat flux in an overshoot region of high Péclet number satisfies Equation (10). Therefore the convective heat flux in a region of high Péclet number (whether in the convection zone or the overshoot region) satisfies Equation (10):

\[ u_e T' = \text{Max} \left\{ - \frac{T}{H_P} D_R (\nabla_{ad} - \nabla_R), -2C_e \omega T \frac{\varepsilon}{\delta g} \right\}. \tag{19} \]

The point of intersection at which \( -(T/H_P)D_R (\nabla_{ad} - \nabla_R) = -2C_e \omega (T/\delta g) \varepsilon \) is located in the overshoot region, with the distance to the convective boundary being (Zhang & Li 2012b)

\[ l_{ad} \approx \varphi H_P, \quad \varphi = \frac{\alpha}{3} \frac{4C_e \omega}{2C_e \omega_c} + 1. \tag{20} \]

Therefore the convective heat flux in a region of high Péclet number can also be written as follows. For the case of \( \nabla_{ad} > \nabla_R \) and \( |\ln(P_{ad}/P)| > \varphi \),

\[ u_e T' = -2C_e \omega T \frac{\varepsilon}{\delta g}, \tag{21} \]

where \( P_C \) is the pressure of the closest convective boundary, and for other cases,

\[ u_e T' = - \frac{T}{H_P} D_R (\nabla_{ad} - \nabla_R). \tag{22} \]

Taking the representation of the convective heat flux given by Equations (21) and (22) into (18), and noting that \( \varepsilon = k^{3/2}/l \), we get a linear equation in \( k^{3/2} \).

Another variable needed to be determined is the degree of anisotropy \( \omega \). In the convection zone, \( \omega \) changes from \( \omega_{CZ} \) to \( \omega_o \) in the region near the convective boundary with the diffusion of \( k \) dominating. In the overshoot region, numerical calculations show that \( \omega \) changes from \( \omega_{CZ} \) to \( \omega_o \) near the convective boundary in a typical length of about \( 1H_k \) where \( H_k = |dr/d \ln k| = H_P/|\theta| \) is the scale height of turbulent kinetic energy. Thus, we estimate \( \omega \) by using linear interpolation as follows. For the case of \( \nabla_R \gg \nabla_{ad} \),

\[ \omega = \text{Min}(1, \chi) \omega_{CZ} + \text{Max}(0, 1 - \chi) \omega_o, \quad \chi = \frac{1}{\alpha} \sqrt{\frac{3}{4C_e \omega_C}} \left| \ln \frac{P}{P_C} \right|, \tag{23} \]

and for the case of \( \nabla_R < \nabla_{ad} \),

\[ \omega = \text{Min}(1, \beta) \omega_o + \text{Max}(0, 1 - \beta) \omega_{CZ}, \quad \beta = \frac{1}{\theta} \frac{1}{\theta} \left| \ln \frac{P}{P_C} \right|, \tag{24} \]
where $P_C$ is the pressure of the closest convective boundary.

A linear model of turbulent kinetic energy in diffusion equilibrium with $P_C \gg 1$ comprises Equations (18), (21), (22), (23), and (24). In order to solve the linear model, we need to set two boundary conditions. A reasonable set of boundary conditions are zero flux at the stellar center and the stellar surface:

$$\frac{\partial k}{\partial m} \bigg|_{m=0} = \frac{\partial k}{\partial m} \bigg|_{m=M} = 0. \quad (25)$$

The problem is that, in general, the Péclet number is low in a thin envelope of a star. This leads to some mistakes when the linear model with the assumption $P_C \gg 1$ is used. In thin convective envelope(s) below the stellar surface, the significant radiative heat exchange leads to a low Péclet number so that the temperature gradient is higher than the adiabatic temperature gradient. In this case the convective heat flux is smaller than the adiabatic heat flux (e.g., Equation (22)) so the kinetic energy should be smaller than the value determined by the linear model. However, the difference should exist only in the layer of low Péclet number extending several typical diffusion length scales $\sim l$. Therefore using those boundary conditions should not lead to mistakes for the overshooting of the convective core and downward overshooting of the thick convective envelope. If the final solution of turbulent kinetic energy shows some regions with $P_C = \sqrt{\epsilon_k / \varepsilon} < 1$ in the overshoot region, we suggest to reset zero turbulent kinetic energy in those regions, because the TCM shows that the turbulent variables cut off quickly in a region of low Péclet number, as mentioned in Section 3.

5. NUMERICAL RESULTS

We use the stellar evolution code YNEV (Zhang 2015) to test the linear model in overshoot mixing. We use two approaches to calculate the stellar evolutionary models to compare with each other. The first approach is to implement the full nonlocal TCM (e.g., Equations (5)–(8)) and the overshoot mixing model Equation (3) in stellar evolution (see Zhang 2015, Section 3.2), denoted as “full TCM” approach. Mixing length theory (MLT) is replaced by the TCM. The convective heat flux determining the temperature gradient and the dissipation rate $\varepsilon$ determining the diffusion coefficient of mixing in the overshoot region are calculated by using the TCM. The second approach is a standard stellar model (using MLT to determine the convective heat flux) with an extra diffusion overshoot mixing (Equation (3)) with the dissipation rate $\varepsilon$ determined by using the linear model, denoted as “linear model” approach.

In the adopted stellar evolution code, the convection zone is artificially fully mixed at first and then we solve the overshoot mixing. This may lead to a problem in using Equation (4). Because the artificial mixing may lead to a discontinuity at the convective boundary, $N_{\text{turb}}^2$ may not exist at the boundary in that case if using Equation (4) to calculate it. In order to avoid this problem resulting from the artificial mixing, we use the following formula to calculate $N_{\text{turb}}^2$:

$$N_{\text{turb}}^2 = -\frac{\delta g}{H_P} \left[ \nabla - \nabla_{ad} \right]$$

$$- \psi C_i C_A \sum_{i=1}^{l} \left( \frac{\partial T}{\partial X_i} \right)_{P_r, X_0 - X_i} \frac{dX_i}{d\ln P}$$

where the parameter $\psi$ is defined as

$$\psi = \text{Min} \left[ 1, \text{Max} \left( 0, \frac{\nabla_R - \nabla_{ad} - d}{d} \right) \right]$$

and $d$ is a small value (we set $d = 0.002$) to determine the depth of the swap region between the convective boundary and the location $\psi = 1$. Using that formula to calculate $N_{\text{turb}}^2$ is reasonable, because the swap region is small and it should be efficiently mixed due to the high diffusion coefficient. In solving the diffusion equation of mixing,

$$\frac{\partial X_i}{\partial t} = \frac{\partial}{\partial m} \left( \left( \frac{dm}{dr} \right)^2 D \frac{\partial X_i}{\partial m} \right),$$

the diffusion coefficient $D$ is calculated by using Equation (3) in the overshoot region and its upper limit is set to be $D = 10^{10}$ (enough for ensure full mixing); the boundary conditions are zero-flux conditions at the stellar center and the stellar surface.

The OPAL equation-of-state tables EoS2005 (Rogers & Nayfonov 2002) are used to calculate the thermodynamic functions. The Rosseland mean opacities in high- and low-temperature regions are interpolated from the OPAL tables (Iglesias & Rogers 1996) and the low-temperature tables (Ferguson et al. 2005), respectively. The rates of all nuclear reactions are based on Angulo et al. (1999) and Caughlan & Fowler (1988) and enhanced by a factor due to weak electron screening (Salpeter 1954). The composition in heavy elements is set as the GN93 (Grevesse & Noels 1993) or AGSS09 (Asplund et al. 2009) solar composition. Except for the overshoot mixing, non-standard physical processes (settling, mass loss, rotation, and so on) are not taken into account.

In our tests, the basic values of parameters are as follows. The convection parameters are $a_{\text{MLT}} = 1.75$ for MLT and $a = 0.8$ for the TCM. They are typical values for solar calibrations. Solar calibrations show that $a_{\text{MLT}} = (2.1 - 2.2)a$. Other TCM parameters are $C_t = C_t = 0$, $C_t = 7.5$, $C_e = 0.2$, $C_s = 0.08$, and $C_k = 2.5$. TCM parameters $C_t$, $C_e$, $C_s$, and $C_k$ are based on the reproduction of the temperature gradient below the solar convection zone to match the required helioseismic profile (Zhang & Li 2012a). The parameters in the overshoot mixing model are $C_{OV} = 10^{-3}$ based on calibrations on some observations (Zhang 2013, Meng & Zhang 2014) and $C_1 = 0.72$ (Canuto & Dubovikov 1998; Canuto 2011). In order to test the effects of different values of parameters, the parameters are varied in large ranges around their basic values.

5.1. Validating the Linear Model

We calculate stellar evolutionary models in the mass range $1.5 M_\odot - 10 M_\odot$ from zero-age main sequence to the asymptotic giant branch phase (or red giant branch (RGB) phase for the 1.5$M_\odot$ star). We then compare the model properties (e.g.,
turbulent kinetic energy, diffusion coefficient, abundance in stellar interior, and the stellar evolution tracks) between two approaches to calculating the stellar evolutionary models: the “full TCM” and the “linear model” approaches. Parameters of the TCM and the overshoot model are set as their basic values.

Turbulent kinetic energy $k$ is a direct indicator to check whether the linear model is a good approximation to the full TCM. Figures 1 and 2 show the profiles of turbulent rms speed $\sqrt{k}$ obtained by using the linear model and full TCM for $4M_\odot$ and $7M_\odot$ stellar models of different composition and state. The stellar models in Figures 1(a) and 2(a) are in the main-sequence stage with the hydrogen abundance in the center $X_C = 0.4$. The turbulent rms speed profiles obtained by using the linear model and full TCM are almost identical, showing that the linear model is a reasonable simplification of the full TCM. In the linear model, we cut off $k$ at about $\lg T = 6.85$ because the Péclet number is smaller than unity in the overshoot layer with $\lg T < 6.85$. The TCM (dashed line) shows that $k$ decreases quickly to zero in the overshoot region with $Pe \ll 1$. The stellar models in Figures 1(b) and 2(b) are at the top of the RGB where the central helium burning just begins. There is a thick convective envelope and a convective helium-burning core in each stellar model. The linear model well reproduces the result of the full TCM except in the surface layer. The difference is because $Pe \ll 1$ in the surface layer so the assumption of the linear model is not valid. However, as mentioned in Section 4, the difference exists only in the layer of low Péclet number that extends for several $l$ and it does not affect the mixing diffusion coefficient in the overshoot regions.

The profiles of the diffusion coefficient of convective mixing for main-sequence stellar models with different mass are shown in Figure 3. The result of the linear model is almost identical to the result of the full TCM. We have not calculated the diffusion coefficient in the convective core but set it to be $10^{10}$, which is large enough to ensure complete mixing. According to the convective mixing model (Zhang 2013), the diffusion coefficient in the convective core is $\sim \sqrt[k]{k}$ which is much larger than $10^{10}$ and leads to a fully mixed core. The convective mixing model shows that the form of diffusion coefficient changes from $D \sim \sqrt{k}$ in the convective core to $D \approx C_{OV} \varepsilon / N_{\text{turb}}$ in the overshoot region (Zhang 2013). It is shown in the figure that the diffusion coefficient decreases quickly near the convective boundary and exponentially in the overshoot region. The latter is because $\varepsilon$ decreases exponentially and $N_{\text{turb}}$ changes much more slowly than $\varepsilon$ in most of the overshoot region. The diffusion coefficient in the overshoot region is not high enough to ensure complete mixing. This leads to a smooth profile of the
abundance in the stellar interior, as shown in Figure 4. The classical overshoot mixing model, which extends the fully mixing region from the convective boundary by a distance, cannot produce such a smooth profile of abundance. Figure 4 shows the hydrogen abundance profile in the stellar interior for different masses in the main sequence. The nearly identical profiles show that the linear model can be used to replace the full TCM in modeling overshoot mixing.

The stellar evolutionary tracks with the linear model and full TCM for different masses are shown in Figures 5 and 6. The tracks with the linear model are almost identical to those with the full TCM in all stages for different stellar masses. This shows that the linear model results in the same strength of overshoot mixing as the full TCM. In the RGB, there is a small difference in temperature (about 50 K) between stellar models with the linear model and full TCM. This is caused by the difference in turbulent heat transport efficiency between the TCM and MLT (adopted in the stellar models using the linear model) in the super-adiabatic convection zone near the surface. In the solar case, in order to generate a similar turbulent heat transport efficiency in the super-adiabatic convection zone, we require \( \alpha_{\text{MLT}} \approx 2.1 \) as mentioned above. However, this ratio may change a little for a different stellar mass or different evolutionary stage. This effect can be ignored for intermediate-mass main-sequence stars because the thin convective envelope is dominated by the radiative heat transport and the convective core is almost adiabatically stratified, regardless of which convection theory (MLT or TCM) is adopted.

**5.2. Effects of the Parameters**

The parameters involved in the linear model of nonlocal TCM are \( C_{\alpha}, C_{\alpha}, C_b \), and \( \alpha \), and the parameters involved in the overshoot mixing model are \( C_{\text{OV}} \) and \( C_1 \). We have tested the effects of those parameters on \( 2M_\odot \) (\( X = 0.715, Z = 0.014, \) AGSS09 composition) stellar models. Figure 7 shows the evolutionary tracks in the main sequence with different values of parameters (for the parameters that are not specific, they are set as their basic values: \( C_{\alpha} = 0.2, C_\alpha = 0.08, C_b = 2.5, \alpha = 0.8, C_1 = 0.72, \) and \( C_{\text{OV}} = 10^{-3} \)). It is found that the overshoot mixing is enhanced when \( C_\alpha \) or \( C_{\text{OV}} \) becomes larger or \( C_1 \) or \( C_b \) becomes smaller. Figure 7(d) shows that the tracks are insensitive to the parameter \( C_b \). In Figures 7(a) and (e), the track for \( C_\alpha = 0.08 \) is same one as for \( \alpha = 0.8 \), the track for \( C_\alpha = 0.02 \) is almost identical to the track for \( \alpha = 0.4 \), and the track for \( C_\alpha = 0.32 \) is almost identical to the track for \( \alpha = 1.6 \). This seems to imply that changing \( C_\alpha \) by a factor \( a \) is almost equivalent to changing \( \alpha \) by the factor \( \sqrt{a} \).
Those properties can be explained as follows. Since the linear model and the overshoot mixing model affect the stellar evolutionary tracks via the diffusion coefficient, let us analyze the dependence of the diffusion coefficient on parameters. As shown in Equation (3), the diffusion coefficient is proportional to the turbulent dissipation rate, which is determined by the linear model. By using Equations (9), (12), and (13), we find

\[
\ln \varepsilon = \ln \frac{k^2}{l} \approx \ln \left\{ \frac{[\delta g D _{R}(\nabla R - \nabla _{ad})]_B}{e_{H_P}} \left( \frac{P}{P_C} \right)^{\frac{3}{2}} \right\} \\
\approx \text{const.} + f(\alpha^2 C_\alpha) + \frac{3}{2} \theta(\alpha^2 C_\alpha, C_e, C_k) \ln \left( \frac{P}{P_C} \right) \\
= \text{const.} + f(\alpha^2 C_\alpha) + \frac{3}{2} \sqrt{\frac{1 + 2C_\epsilon \omega_0}{3\alpha^2 C_\epsilon \omega_0}} \ln P, \tag{29}
\]

where the function of the \( \alpha^2 C_\epsilon \) term \( f(\alpha^2 C_\epsilon) \) represents the location of the point B depending on \( \alpha^2 C_\epsilon \) (see Equation (14)).

For the case of core overshoot, the positive value of \( \theta \) is adopted, and \( f' > 0 \) in the general case because local kinetic energy in the convective core decreases toward the Schwarzschild boundary. It should be noticed that the two parameters \( \alpha \) and \( C_\epsilon \) can be combined into one. This is the reason why changing \( C_\epsilon \) by a factor \( a \) is almost equivalent to changing \( \alpha \) by the factor \( \sqrt{a} \). It is not difficult to find that \( d\theta / d(\alpha^2 C_\epsilon) < 0 \) and \( d\theta / dC_\epsilon > 0 \). This means that the exponential index of the diffusion coefficient becomes smaller when \( \alpha^2 C_\epsilon \) becomes larger or \( C_\epsilon \) becomes smaller. A smaller index of the diffusion coefficient leads to a higher efficiency for the mixing. \( \alpha_0 \) affects \( \omega_0 \) only. For the testing range of \( \alpha_0 \), \( \omega_0 \) changes a little so the tracks are insensitive to \( \alpha_0 \). When \( C_\epsilon \) becomes larger, the weight of abundance gradient in \( N^\text{turb}_2 \) is larger so \( N^\text{turb}_2 \) increases and the diffusion coefficient decreases. The diffusion coefficient is proportional to the parameter \( C_{OV} \) so the overshoot mixing is enhanced as \( C_{OV} \) increases.

5.3. Effects of the Modification of Temperature Gradient in the Overshoot Region

The convective heat flux could modify the temperature gradient in the overshoot region. This affects the value of \( N^\text{turb}_2 \) and thus affects the diffusion coefficient of overshoot mixing. In the previous calculations of stellar models based on the “linear model,” this effect was not taken into account since the temperature gradient is calculated in the traditional way (MLT in convection zones and the radiative temperature gradient is adopted outside convection zones). In this subsection, we investigate the effects of taking into account the modification of temperature gradient in the overshoot region.

We have implemented the modification of temperature gradient in the overshoot region as follows: step 1, solving the linear model to work out/update the convective heat flux in the overshoot region (e.g., Equation (10)) before every iteration in solving the stellar structure equations; step 2, solving the stellar structure equations by one iteration with the updated convective heat flux in the overshoot region. The iterations stop when the required accuracy is achieved.

It is not difficult to understand that the negative convective heat flux in the overshoot region should enlarge the temperature gradient and make it closer to the adiabatic temperature gradient, thus reducing \( N^\text{turb}_2 \) and increasing the diffusion coefficient of mixing. It can be found in Equation (20) that enlarging the parameter \( \alpha^2 C_\epsilon \) or \( C_e \) enhances the modification of the temperature gradient. Here, we take a large value of \( C_e = 0.8 \) as an example. Other parameters are set as their basic values. We have tried \( C_e = 1.6 \) but we cannot get converged stellar models.

Based on the linear model, \( 2M_\odot \) stellar models with \( X = 0.715, \ Z = 0.014 \), and AGSS09 mixture have been calculated, with or without the modification of temperature gradient in the overshoot region. Figure 8 shows the temperature gradient near the convective core boundary for the stellar model with the modification at the moment \( X_\epsilon \approx 0.32 \). The modification of temperature gradient is significant in a range \( \sim 0.1H_P \) outside the convective core. Figure 9 shows the diffusion coefficient and hydrogen abundance profile of that stellar model, as well as the stellar model without the modification of temperature gradient. It can be found that the diffusion coefficient near the convective boundary has been enlarged as a result of the modified temperature gradient. However, the difference between
hydrogen abundance profiles of the two models is not significant. This is because the diffusion coefficient near the convective boundary is very high so the abundance is very close to the abundance in the core as shown in the figure. In this case, enlarging the diffusion coefficient has little effect. The inhomogeneous region is in the region of low diffusion coefficient, in which the convective heat flux can be ignored, so the diffusion coefficient is not affected. Therefore the hydrogen abundance profiles of the two models are very close to each other. Figure 10 shows the evolutionary tracks for the stellar models with and without the modification of temperature gradient in the overshoot region. It is also shown that, although the modification can enhance the overshoot mixing, the effect is not significant. The comparison is for a large value of
$C_r = 0.8$. If the basic value is adopted, the effect should be less important.

6. SUMMARY

Asteroseismic studies do not support the classical “ballistic” overshoot model and imply that convective overshoot is a weak mixing process. Zhang’s (2013) overshoot mixing model describes the overshoot as a weak mixing process and the diffusion coefficient shows consistency with the overshoot entropy mixing. However, we need to know the dissipation rate of turbulent kinetic energy in the overshoot region before applying that mixing model. A practicable option to work out the dissipation rate is to solve nonlocal TCMs, but this is difficult because of some numerical problems, and the time cost is hard to bear.

In this paper, we have simplified the full nonlocal TCM developed by Li & Yang (2007) to a linear model (e.g., Equations (18), (21), (22), (23), and (24)) in order to obtain the dissipation rate of turbulent kinetic energy that is required in the overshoot mixing model. The linear model is a single linear diffusion equation for turbulent kinetic energy. It is very easy to implement in a stellar evolution code. The time cost of solving the linear model is negligible in comparison with solving the full TCM. And there is no numerical difficulty in solving the linear model. We have tested the linear model in a stellar evolution code, and have found that it can well reproduce the turbulent kinetic energy profile of the full TCM, as well as the diffusion coefficient, abundance profile, and stellar evolutionary tracks. We have also studied the effects of different values of the model parameters and have found that the effect due to the modification of temperature gradient in the overshoot region is slight.

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APPENDIX

ANALYSIS OF THE ROOT OF THE QUADRATIC EQUATION FOR $\omega_D$

The asymptotic equilibrium value of the degree of anisotropy in the overshoot region $\omega_D$ satisfies the following equation:

$$2C_r \omega_D^2 - (C_k - 1 + 2C_r) \omega_D + \frac{1}{3} (C_k - 1) = 0.$$  \hfill (30)

For convenience, we define $c = (C_k - 1)/(2C_r)$, thus the equation above can be written as

$$F(\omega_D) \equiv \omega_D^2 - (1 + c) \omega_D + \frac{c}{3} = 0.$$  \hfill (31)

Since the discriminant is always positive, let $\omega_1$ and $\omega_2$ be the two roots of the quadratic equation and $\omega_1 < \omega_2$. According to Vieta’s theorem, we have

$$\omega_1\omega_2 = \frac{c}{3},$$  \hfill (32)

$$\omega_1 + \omega_2 = 1 + c.$$  \hfill (33)

The values of function $F$ at 0, 1/3, and 1 are

$$F(0) = \frac{c}{3}, \quad F\left(\frac{1}{3}\right) = \frac{2}{9}, \quad F(1) = -\frac{2c}{3}.$$ \hfill (34)

1. Case A: $C_k \leq 1$. In this case, $c \leq 0$ so $\omega_1 \leq 0 \leq \omega_2$. Because $F(1/3)F(1) \leq 0$, we find $\omega_2 \geq 1/3$. A physically acceptable root should be $0 < \omega_D < 1/3$ because the work of buoyancy on the radial turbulent kinetic energy is negative in the overshoot region, so there is no acceptable root when $C_k \leq 1$.

2. Case B: $C_k > 1$. In this case, $c > 0$, $\omega_2 > (\omega_1 + \omega_2)/2 = (1 + c)/2 > 1/2$, so $\omega_2$ is not acceptable. Because $F(0)F(1/3) < 0$, we find $0 < \omega_1 < 1/3$, which is a physically acceptable root.

Finally, $C_k$ must be larger than 1 and the physically acceptable root is the small one.

REFERENCES

Angulo, C., Arnould, M., Rayet, M., et al. 1999, NuPhA, 656, 3
Asplund, M., Grevesse, N., Sauval, A. J., & Scott, P. 2009, ARA&A, 47, 481
Basu, S. 1997, ApJ, 482, 827
Basu, S., & Antia, H. M. 1994, MNRAS, 269, 1137
Basu, S., Antia, H. M., & Narasimha, D. 1994, MNRAS, 267, 209
Bressan, A. G., Chiosi, C., & Bertelli, G. 1981, A&A, 102, 25
Brummell, N. H., Clune, T. L., & Toomre, J. 2002, ApJ, 570, 825
Canuto, V. M. 1997, ApJ, 482, 827
Canuto, V. M. 2011, A&A, 528, 76
Canuto, V. M., & Dubovikov, M. 1998, ApJ, 493, 834
Caughlan, G. R., & Fowler, W. A. 1988, A&DNDT, 40, 283
Christensen-Dalsgaard, J., Monteiro, M. J. P. F. G., Rempel, M., & Thompson, M. J. 2011, MNRAS, 414, 1158
Christensen-Dalsgaard, J., Monteiro, M. J. P. F. G., & Thompson, M. J. 1995, MNRAS, 276, 243
Deng, L., Bressan, A., & Chiosi, C. 1996, A&A, 313, 145
Deng, L., Xiong, D. R., & Chan, K. L. 2006, ApJ, 643, 426
Ferguson, J. W., Alexander, D. R., Allard, F., et al. 2005, ApJ, 623, 585
Gough, D. O. 1990, in Progress of Seismology of the Sun and Stars, ed. Y. Osaki, & H. Shibahashi (Berlin: Springer), 283
Grevesse, N., & Noels, A. 1993, in Origin and Evolution of the Elements, ed. N. Prantzos, E. Vangioni-Flam, & M. Casse (Cambridge: Cambridge Univ. Press), 15
Herwig, F. 2000, A&A, 360, 952
Iglesias, C. A., & Rogers, F. J. 1996, ApJ, 464, 943
Li, Y. 2012, ApJ, 756, 37
Li, Y., & Yang, J. Y. 2001, ChJAA, 1, 66
Li, Y., & Yang, J. Y. 2007, MNRAS, 375, 388
Maeder, A. 1975, A&A, 40, 303
Meakin, C. A., & Arnett, D. 2007, ApJ, 667, 448
Meng, Y., & Zhang, Q. S. 2014, ApJ, 787, 127
Monteiro, M. J. P. F. G., Christensen-Dalsgaard, J., & Thompson, M. J. 1994, A&A, 283, 247
Moravveji, E., Aerts, C., Pápics, P. I., Triana, S. A., & Vandoren, B. 2015, A&A, 580, 27
Petrovay, K., & Marik, M. 1995, in ASP Conf. Ser. 76, Proc. GONG94: Helio- and Asteroseismology from Earth and Space, ed. R. K. Ulrich, E. J. Rhodes, & W. Däppen (San Francisco, CA: ASP), 216
Renzini, A. 1987, A&A, 188, 49
Rogers, F. J., & Nayfonov, A. 2002, ApJ, 576, 1064
Roxburgh, I. W., & Vorontsov, S. V. 1994, MNRAS, 268, 880
Salpeter, E. E. 1954, AuJPh, 7, 373
Singh, H. P., Roxburgh, I. W., & Chan, K. L. 1995, A&A, 295, 703
Ventura, P., Zeppieri, A., Mazzitelli, I., & D’Antona, F. 1998, A&A, 344, 953
Xiong, D. R. 1981, SciSn, 24, 1406
Xiong, D. R. 1985, A&A, 150, 133
Xiong, D. R. 1989, A&A, 213, 176
Xiong, D. R., Cheng, Q. L., & Deng, L. 1997, ApJS, 108, 529
Xiong, D. R., & Deng, L. 2001, MNRAS, 327, 1137
Zahn, J. P. 1991, A&A, 252, 179
Zhang, C., Deng, L., Xiong, D., & Christensen-Dalsgaard, J. 2012, ApJL, 759, L14
Zhang, Q. S. 2013, ApJS, 205, 18
Zhang, Q. S. 2015, RAA, 15, 549
Zhang, Q. S., & Li, Y. 2012a, ApJ, 746, 50
Zhang, Q. S., & Li, Y. 2012b, ApJ, 750, 11