A Comparative Analysis of Fractional Space-Time Advection-Dispersion Equation via Semi-Analytical Methods

Noufe H. Aljahdaly, Rasool Shah, Muhammed Naeem, and Mohammad Asif Arefin

1Mathematics Department, Faculty of Sciences and Arts, King Abdulaziz University, Rabigh, Saudi Arabia
2Department of Mathematics, Abdul Wali Khan University Mardan 23200, Pakistan
3Dean of Joint First Year Umm Al-Qura University Makkah, P.O. Box 715, Saudi Arabia
4Jashore University of Science and Technology, Jashore 7408, Bangladesh

Correspondence should be addressed to Muhammed Naeem; mfaridoon@uqu.edu.sa and Mohammad Asif Arefin; asif.math@just.edu.bd

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The approximate solutions of the time fractional advection-dispersion equation are presented in this article. The nonlocal nature of solute movement and the nonuniformity of fluid flow velocity in the advection-dispersion process lead to the formation of a heterogeneous system, which can be modeled using a fractional advection-dispersion equation, which generalizes the classical advection-dispersion equation and replaces the time derivative with the fractional Caputo derivative. Researchers use a variety of numerical techniques to study such fractional models, but the nonlocality of the derivative having fractional order leads to high computation complexity and complex calculations, so the task is to find an efficient technique that requires less computation and provides greater accuracy when numerically solving such models. A innovative techniques, homotopy perturbation method and new iteration method, are used in connection with the Elzaki transform to solve the “fractional advection-dispersion equation” which provides the solution in the convergent series form. When the homotopy perturbation method is used with the Elzaki transform, fast convergent series solutions can be obtained with less computation. By solving some cases of time-fractional advection-dispersion equation with varied initial conditions with the help of new iterative transform method and homotopy perturbation transform method demonstrates the usefulness of the proposed methods.

1. Introduction

For the past 300 years, fractional calculus has been used to generalize the integration and differentiation of integer order to arbitrary order. Due to its nonlocal nature, fractional differential equations are well adapted to explain diverse phenomena in engineering and science, and the researchers’ growing interest in this field has led to solving real-world problems in type of fractional differential equations. In addition, fractional derivatives can be used for description in a variety of phenomena that have memory and hereditary properties by mathematical way [1–5]. Fractional order differential equations have been shown to be a valuable tool for revealing hidden characteristics in a variety of real-world processes, including physical sciences, signal processing, electromagnetics, earthquakes, traffic flow, and the study of viscoelastic material properties and many more processes [6–11]. The historical and nonlocal distributed effects are considered via fractional differential coefficients; an outstanding literature on this topic may be found in numerous monographs [12–15]. For this reason, many authors are attracted to knowing the properties of fractional differential equations and vast applications in modeling and engineering fields [16–19].

The ADE is used in the study of solute transport or Brownian motion of particles in a fluid that occurs when advection and particle dispersion occur at the same time [20, 21]. The fractional advection-dispersion equation better
represents the phenomenon of anomalous particle diffusion in the transport process; in anomalous diffusion, solute transport is faster or faster than the time’s inferred square root given by Baeumer et al. [22]. The equation is used to investigate groundwater pollution, smoke or dust pollution of the atmosphere, and the spread of chemical solutes and pollutant discharges [23]. As a result, FADE has caught the interest of numerous researchers. As a result, the researchers are interested in solving the FADE to determine the solute concentration at a specific time and location [24, 25]. Jaiswal et al. [26] discovered an analytical solution for one-dimensional ADE. Huang et al. [27] developed the fractional ADE, whereas Hikal and Abu Ibrahim [30] use the homotopy perturbation method for revisiting the time-fractional Rosenau-Hyman problem.

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differential equation having fractional order is given as [35–37]

\[ f^\sigma \psi(\varphi) = \frac{1}{\Gamma(\sigma)} \int_0^\varphi (\varphi - \mu)^{\sigma-1} \psi(\mu) d\mu, \quad \varphi > 0, \quad \sigma > 0. \]  

With basic properties:

\[ f^\sigma \psi' = \frac{\Gamma(j+1)}{\Gamma(j+\sigma+1)} \psi^{j+\sigma}, \]

\[ D^\sigma \psi' = \frac{\Gamma(j+1)}{\Gamma(j-\sigma+1)} \psi^{j-\sigma}. \]

2.2. Definition. The Abel–Riemann integration operator \( f^\sigma \) having fractional order is given as [35–37]

\[ c_{\mathcal{D}} D^\sigma \psi(\varphi) = \begin{cases} \frac{1}{\Gamma(j-\sigma)} \int_0^\varphi \frac{\psi(\mu)}{(\varphi - \mu)^{\alpha-j+1}} d\mu, & j-1 < \sigma < j, \\ \frac{d}{d\varphi} \psi(\varphi), & j = \sigma. \end{cases} \]

2.3. Definition. The fractional Caputo operator \( D^\sigma \) having order \( \sigma \) is calculated as [41–43]

\[ c_{\mathcal{D}} D^\sigma \psi(\varphi) = \frac{\Gamma(j+1)}{\Gamma(j-\sigma+1)} \psi^{j-\sigma}. \]

with the following properties:

\[ f^\sigma \psi^\sigma(\varphi) = g(\varphi) - \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^\varphi (\varphi - \mu)^{\alpha-k+1} \psi^k(\mu) d\mu, \quad \sigma > 0, \quad \varphi > 0, \quad j-1 < \sigma < j, \]

\[ D^\sigma f^\sigma g(\varphi) = g(\varphi). \]

2.4. Definition. The Elzaki transform of Caputo operator is calculated as [41, 42]

\[ E\left[D^\sigma \psi(\varphi)\right] = \mathcal{E}\left[\psi(\varphi)\right] - \sum_{k=0}^{\infty} \frac{1}{(\alpha-k)!} \mathcal{E}[g^{(k)}(0)], \quad \text{where} \quad j-1 < \sigma < j. \]

3. Idea of New Iterative Transform Method (NITM)

Let us consider the partial differential equation having fractional order in the form of

\[ D^\sigma \zeta(\mu, \rho) + N \zeta(\mu, \rho) + M \zeta(\mu, \rho) = h(\mu, \rho), \quad n \in \mathbb{N}, \quad n-1 < \sigma \leq n, \]

where \( j \in \mathbb{Z}^+, \sigma \in \mathbb{R}^+ \), and

\[ D^\sigma \psi(\varphi) = \begin{cases} \frac{d}{d\varphi} \psi(\varphi), & \sigma = j, \\ \frac{1}{\Gamma(j-\sigma)} \int_0^\varphi \psi(\mu) (\varphi - \mu)^{\sigma-1} d\mu, & j-1 < \sigma < j, \end{cases} \]
where perturbation parameter is denoted by $p$.

By substituting Equations (13), (14), and (15) in Equation (12), we get

$$
\sum_{m=0}^{\infty} \zeta_m(\mu, \rho) = E^{-1} \left[ \sigma^\rho \left( \sum_{k=0}^{m} s^{2-\sigma+k} u^{(k)}(\mu, 0) + s^\rho E[h(\mu, \rho)] \right) \right] - E^{-1} \left[ \sigma^\rho E[N[\zeta(\mu, \rho) + M\zeta(\mu, \rho)]] \right].
$$

By taking the Elzaki transform of Equation (19)

$$
E \left[ D^\rho E(\zeta(\mu, \rho) + N\zeta(\mu, \rho)] = E[h(\mu, \rho)], \quad \rho > 0, \quad 0 < \sigma \leq 1, \quad \zeta(\mu, 0) = g(\mu), \quad v \in \mathbb{R}.
$$

On taking Elzaki inverse transform, we have

$$
\zeta(\mu, \rho) = F(x, \rho) - E^{-1} \left[ \sigma^\rho E[M\zeta(\mu, \rho) + N\zeta(\mu, \rho)] \right],
$$

where

$$
(\mu, \rho) = E^{-1} \left[ \sigma^\rho \left( g(\mu) + s^\rho E[h(\mu, \rho)] \right) \right] = g(v) + E^{-1} \left[ \sigma^\rho E[h(\mu, \rho)] \right].
$$

The perturbation technique of parameter $p$ is given as

$$
\zeta(\mu, \rho) = \sum_{k=0}^{\infty} p^k \zeta_k(\mu, \rho),
$$

where perturbation parameter is denoted by $p$ and $p \in [0, 1]$. The nonlinear terms can be calculated as

$$
N\zeta(\mu, \rho) = \sum_{k=0}^{\infty} p^k H_k(\zeta_k).
$$

Thus, the iterative formula is given as

$$
\zeta_0(\mu, \rho) = E^{-1} \left[ \sigma^\rho \left( \sum_{k=0}^{m} s^{2-\sigma+k} u^{(k)}(\mu, 0) + s^\rho E[g(\mu, \rho)] \right) \right],
$$

$$
\zeta_1(\mu, \rho) = -E^{-1} \left[ \sigma^\rho E[N[\zeta_0(\mu, \rho) + M[\zeta_0(\mu, \rho)]]] \right],
$$

$$
\zeta_m(\mu, \rho) = -E^{-1} \left[ \sigma^\rho \left[ -N \left( \sum_{k=0}^{m} \zeta_k(\mu, \rho) \right) - M \left( \sum_{k=0}^{m} \zeta_k(\mu, \rho) \right) \right] \right].
$$

Lastly, Equations (8) and (9) give series form result for $m$-term as

$$
\zeta(\mu, \rho) \equiv \zeta_0(\mu, \rho) + \zeta_1(\mu, \rho) + \zeta_2(\mu, \rho) + \cdots + \zeta_m(\mu, \rho), \quad m = 1, 2, \cdots.
$$
where $H_n$ represents He’s polynomials in terms of $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n$, and can be expressed as

$$H_n(\zeta_0, \zeta_1, \ldots, \zeta_n) = \frac{1}{\sigma(n+1)} D^\frac{n}{\sigma} \left[ N \left( \sum_{k=0}^{\infty} p^k \zeta_k \right) \right]_{p=0}, \quad (25)$$

where $D^\frac{n}{\sigma} = \partial^\frac{n}{\sigma}/\partial p^\frac{n}{\sigma}$.

Substituting Equations (24) and (25) in Equation (21),

$$\sum_{k=0}^{\infty} p^k \zeta_k(\mu, \rho) = F(\mu, \rho) - \rho,$$

$$\times \left[ E^{-1} \left\{ s^E \left( M \sum_{k=0}^{\infty} p^k \zeta_k(\mu, \rho) + \sum_{k=0}^{\infty} p^k H_k(\zeta_k) \right) \right\} \right]. \quad (26)$$

On comparison of both sides coefficient of $p$, we get

$$p^0: \zeta_0(\mu, \rho) = F(\mu, \rho),$$

$$p^1: \zeta_1(\mu, \rho) = E^{-1} \left[ s^E \left( M \zeta_0(\mu, \rho) + H_0(\zeta) \right) \right],$$

$$p^2: \zeta_2(\mu, \rho) = E^{-1} \left[ s^E \left( M \zeta_1(\mu, \rho) + H_1(\zeta) \right) \right],$$

$$\vdots$$

$$p^k: \zeta_k(\mu, \rho) = E^{-1} \left[ s^E \left( M \zeta_{k-1}(\mu, \rho) + H_{k-1}(\zeta) \right) \right], k > 0, k \in \mathbb{N}. \quad (27)$$

The $\zeta_k(\mu, \rho)$ term can be calculated easily resulting convergent series. By taking $p \rightarrow 1,$

$$\zeta(\mu, \rho) = \lim_{M \rightarrow \infty} \sum_{k=1}^{M} \zeta_k(\mu, \rho). \quad (28)$$

4.1. Example. Consider the time-fractional ADE

$$D^\nu_p \zeta(\mu, \rho) = \ell D^\ell_p \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho), \quad (29)$$

with initial condition

$$\zeta(\mu, 0) = e^{-\mu}, \quad (30)$$

where $\ell$ is the ratio of constant diffusivity and the drift velocity. The exact solution is

$$\zeta(\mu, \phi, \rho) = e^{(\ell + 1)\rho - \mu}. \quad (31)$$

By taking the Elzaki transform of Eq. (29), we get

$$E[\nu(\mu, \phi, \rho)] = s^E \left( e^{-\mu} \right) + s^E \left[ \ell D^\ell_p \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho) \right]. \quad (32)$$

On taking Elzaki inverse transform, we have

$$\nu(\mu, \phi, \rho) = e^{-\mu} + E^{-1} \left( s^E \left[ \ell D^\ell_p \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho) \right] \right). \quad (33)$$

Thus by using NITM, we have

$$\zeta_0(\mu, \rho) = e^{-\mu},$$

$$\zeta_1(\mu, \rho) = E^{-1} \left[ s^E \left( \ell D^\ell_p \zeta_0(\mu, \rho) - D_\mu \zeta_0(\mu, \rho) \right) \right],$$

$$\zeta_2(\mu, \rho) = E^{-1} \left[ s^E \left( \ell D^\ell_p \zeta_1(\mu, \rho) - D_\mu \zeta_1(\mu, \rho) \right) \right],$$

$$\vdots$$

$$\zeta_n(\mu, \rho) = E^{-1} \left[ s^E \left( \ell D^\ell_p \zeta_n(\mu, \rho) - D_\mu \zeta_n(\mu, \rho) \right) \right], \quad (34)$$

The series form solution is given as

$$\zeta(\mu, \rho) = \zeta_0(\mu, \rho) + \zeta_1(\mu, \rho) + \zeta_2(\mu, \rho) + \zeta_3(\mu, \rho) + \cdots + \zeta_n(\mu, \rho). \quad (35)$$

Thus, we have

$$\zeta(\mu, \rho) = e^{-\mu} \left\{ 1 + \frac{(\ell + 1)\rho^\sigma}{\Gamma(\sigma + 1)} + \frac{(\ell + 1)^2\rho^{2\sigma}}{\Gamma(2\sigma + 1)} + \cdots + \frac{(\ell + 1)^n\rho^{n\sigma}}{\Gamma(n\sigma + 1)} \right\}. \quad (36)$$

Now by using the HPTM, we have

$$\sum_{n=0}^{\infty} p^n w_n(\mu, \rho) = (e^{-\mu}) + \rho \left\{ E^{-1} \left( s^E \left[ \sum_{n=0}^{\infty} p^n H_n(w) \right] \right) \right\}. \quad (37)$$
By Comparing coefficient of $p$ on both sides, we get:

\begin{align*}
p^0 : w_0(\mu, \rho) &= e^{\mu},
p^1 : w_1(\mu, \rho) &= \left[ E^{-1}\{ s^\mu E(H_0(w)) \} \right] = e^{\mu} \frac{(\ell + 1)^{\rho^\sigma}}{\Gamma(\sigma + 1)},
p^2 : w_2(\mu, \rho) &= \left[ E^{-1}\{ s^\mu E(H_1(w)) \} \right] = e^{\mu} \frac{(\ell + 1)^2(\rho^\sigma)^2}{2\Gamma(2\sigma + 1)},
p^3 : w_3(\mu, \rho) &= \left[ E^{-1}\{ s^\mu E(H_2(w)) \} \right] = e^{\mu} \frac{(\ell + 1)^3(\rho^\sigma)^3}{3\Gamma(3\sigma + 1)},
\vdots
p^n : w_n(\mu, \rho) &= \left[ E^{-1}\{ s^\mu E(H_{n-1}(w)) \} \right] = e^{\mu} \frac{(\ell + 1)^n(\rho^\sigma)^n}{n\sigma + 1}.
\end{align*}

(38)

The solution in series form by means of HPM is given as

\[ \zeta(\mu, \rho) = \sum_{n=0}^{\infty} p^n w_n(\mu, \rho). \]

(39)

Thus, we have

\[ \zeta(\mu, \rho) = e^{\mu} \left\{ 1 + \frac{(\ell + 1)^{\rho^\sigma}}{\Gamma(\sigma + 1)} + \frac{(\ell + 1)^2(\rho^\sigma)^2}{2\Gamma(2\sigma + 1)} + \cdots + \frac{(\ell + 1)^n(\rho^\sigma)^n}{n\sigma + 1} \right\}. \]

(40)

4.2. Example. Consider the time-fractional ADE

\[ D_0^\mu \zeta(\mu, \rho) = \ell D_0^\mu \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho), \]

(41)

with initial conditions

\[ \zeta(\mu, 0) = \mu^3 - \mu^2. \]

(42)

By taking the Elzaki transform of Equation (29), we get

\[ E[\nu(\mu, \phi, \rho)] = s^\mu (\mu^3 - \mu^2) + s^\mu E \left[ \ell D_0^\mu \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho) \right]. \]

(43)

On taking Elzaki inverse transform, we have

\[ \nu(\mu, \phi, \rho) = (\mu^3 - \mu^2) + E^{-1} \left[ s^\mu E \left[ \ell D_0^\mu \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho) \right] \right]. \]

(44)

Thus by using NITM, we have

\[ \zeta(\mu, \rho) = \mu^3 - \mu^2, \]

\[ \zeta_1(\mu, \rho) = \left. \left[ \ell D_0^\mu \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho) \right] \right|_{\zeta = \mu^3 - \mu^2} \]

\[ = \left\{ -3\mu^2 + 2\mu(1 + 3\xi) - 2\ell \right\} \frac{\rho^\sigma}{\Gamma(\sigma + 1)} , \]

\[ \zeta_2(\mu, \rho) = \left. \left[ \ell D_0^\mu \zeta_1(\mu, \rho) - D_\mu \zeta_1(\mu, \rho) \right] \right|_{\zeta_1 = \mu^3 - \mu^2} \]

\[ = \left\{ 6\mu - 2 + 12\ell \right\} \frac{\rho^\sigma}{\Gamma(3\sigma + 1)} , \]

\[ \vdots \]

The series form solution is given as

\[ \zeta(\mu, \rho) = \zeta_0(\mu, \rho) + \zeta_1(\mu, \rho) + \zeta_2(\mu, \rho) + \zeta_3(\mu, \rho) + \cdots \]

Thus, we have

\[ \zeta(\mu, \rho) = (\mu^3 - \mu^2) + \left\{ -3\mu^2 + 2\mu(1 + 3\xi) - 2\ell \right\} \frac{\rho^\sigma}{\Gamma(\sigma + 1)} , \]

\[ + \left\{ 6\mu - 2 + 12\ell \right\} \frac{\rho^\sigma}{\Gamma(3\sigma + 1)} + \cdots \]

(47)

Now by applying the HPTM, we have

\[ \sum_{n=0}^{\infty} p^n w_n(\mu, \rho) = (e^{\mu}) + p \left\{ E^{-1} \left[ s^\mu E \left[ \sum_{n=0}^{\infty} p^n H_n(w) \right] \right] \right\} \]

(48)
By comparing coefficient of \( p \) on both sides, we get
\[
p^0 : w_0(\mu, \rho) = \mu^3 - \mu^2,
\]
\[
p^1 : w_1(\mu, \rho) = \left[ E^{-1}\left( s^\rho E(H_0(w)) \right) \right]
= \left\{ -3\mu^2 + 2\mu(1 + 3\ell) - 2\ell \right\} \frac{\rho^\sigma}{\Gamma(\sigma + 1)},
\]
\[
p^2 : w_2(\mu, \rho) = \left[ E^{-1}\left( s^\rho E(H_1(w)) \right) \right]
= \left\{ 6\mu - 2 - 12\ell \right\} \frac{(\rho^\sigma)^2}{\Gamma(2\sigma + 1)},
\]
\[
p^3 : w_3(\mu, \rho) = \left[ E^{-1}\left( s^\rho E(H_2(w)) \right) \right]
= -6 \cdot \frac{(\rho^\sigma)^3}{\Gamma(3\sigma + 1)}. \tag{49}
\]

The solution in series form by means of HPM is given as
\[
\zeta(\mu, \rho) = \sum_{n=0}^{\infty} p^n w_n(\mu, \rho). \tag{50}
\]

Thus, we have
\[
\zeta(\mu, \rho) = (\mu^3 - \mu^2) + \left\{ -3\mu^2 + 2\mu(1 + 3\ell) - 2\ell \right\} \frac{\rho^\sigma}{\Gamma(\sigma + 1)}
+ \left\{ 6\mu - 2 - 12\ell \right\} \frac{(\rho^\sigma)^2}{\Gamma(2\sigma + 1)} - 6 \cdot \frac{(\rho^\sigma)^3}{\Gamma(3\sigma + 1)} + \ldots. \tag{51}
\]

4.3. Example. Consider the time-fractional ADE
\[
D^\rho_{\rho} \zeta(\mu, \rho) = \ell D^2_{\mu} \zeta(\mu, \rho) - D_{\mu} \zeta(\mu, \rho), \tag{52}
\]
with initial conditions
\[
\zeta(\mu, 0) = \cos (\mu). \tag{53}
\]

By taking the Elzaki transform of Equation (29), we get
\[
E[\nu(\mu, \phi, \rho)] = s^2(\cos (\mu)) + s^\rho E\left[ \ell D^2_{\mu} \zeta(\mu, \rho) - D_{\mu} \zeta(\mu, \rho) \right]. \tag{54}
\]

On taking Elzaki inverse transform, we have
\[
\nu(\mu, \phi, \rho) = \cos (\mu) + E^{-1}\left( s^\rho E\left[ \ell D^2_{\mu} \zeta(\mu, \rho) - D_{\mu} \zeta(\mu, \rho) \right] \right). \tag{55}
\]

Thus by using NITM, we have
\[
\zeta_0(\mu, \rho) = \cos (\mu),
\zeta_1(\mu, \rho) = E^{-1}\left[ s^\rho E\left\{ \ell D^2_{\mu} \zeta_0(\mu, \rho) - D_{\mu} \zeta_0(\mu, \rho) \right\} \right]
= (\sin (\mu) - \ell \cos (\mu)) \frac{\rho^\sigma}{\Gamma(\sigma + 1)},
\zeta_2(\mu, \rho) = E^{-1}\left[ s^\rho E\left\{ \ell D^2_{\mu} \zeta_1(\mu, \rho) - D_{\mu} \zeta_1(\mu, \rho) \right\} \right]
= (\cos (\mu) - \ell \sin (\mu) + \ell^2 \cos (\mu)) \frac{(\rho^\sigma)^2}{\Gamma(2\sigma + 1)},
\zeta_3(\mu, \rho) = E^{-1}\left[ s^\rho E\left\{ \ell D^2_{\mu} \zeta_2(\mu, \rho) - D_{\mu} \zeta_2(\mu, \rho) \right\} \right]
= (-\sin (\mu) + \ell^2 \cos (\mu) + 3\ell^3 \sin (\mu) - \ell^3 \cos (\mu)) \frac{(\rho^\sigma)^3}{\Gamma(3\sigma + 1)} + \ldots. \tag{56}
\]

The series form solution is given as
\[
\zeta(\mu, \rho) = \zeta_0(\mu, \rho) + \zeta_1(\mu, \rho) + \zeta_2(\mu, \rho) + \zeta_3(\mu, \rho) + \ldots. \tag{57}
\]

Thus, we have
\[
\zeta(\mu, \rho) = \cos (\mu) + (\sin (\mu) - \ell \cos (\mu)) \frac{\rho^\sigma}{\Gamma(\sigma + 1)}
+ (\cos (\mu) - \ell \sin (\mu) + \ell^2 \cos (\mu)) \frac{(\rho^\sigma)^2}{\Gamma(2\sigma + 1)}
+ (-\sin (\mu) + \ell^2 \cos (\mu) + 3\ell^3 \sin (\mu) - \ell^3 \cos (\mu)) \frac{(\rho^\sigma)^3}{\Gamma(3\sigma + 1)} + \ldots. \tag{58}
\]

Now by applying the HPTM, we have
\[
\sum_{n=0}^{\infty} p^n w_n(\mu, \rho) = (e^{-\mu}) + p\left\{ E^{-1}\left( s^\rho E\left[ \sum_{n=0}^{\infty} p^n H_n(w) \right] \right) \right\}. \tag{59}
\]
By Comparing coefficients of \( p \) on both sides, we get:

\[
p^0 : w_0(\mu, \rho) = \cos(\mu),
\]

\[
p^1 : w_1(\mu, \rho) = \left[ E^{-1} \left\{ \mathcal{L}^\omega E(H_0(w)) \right\} \right]
= \left( \sin(\mu) - \ell \cos(\mu) \right) \frac{\rho^\sigma}{\Gamma(\sigma + 1)},
\]

\[
p^2 : w_2(\mu, \rho) = \left[ E^{-1} \left\{ \mathcal{L}^\omega E(H_1(w)) \right\} \right]
= \left( -\cos(\mu) - 2\ell \sin(\mu) + \ell^2 \cos(\mu) \right) \frac{(\rho^\omega)^2}{\Gamma(2\sigma + 1)},
\]

\[
p^3 : w_3(\mu, \rho) = \left[ E^{-1} \left\{ \mathcal{L}^\omega E(H_2(w)) \right\} \right]
= \left( -\sin(\mu) + 3\ell \cos(\mu) + 3\ell^2 \sin(\mu) - \ell^3 \cos(\mu) \right) \frac{(\rho^\omega)^3}{\Gamma(3\sigma + 1)}.\tag{60}
\]

The solution in series form by means of HPM is given as

\[
\zeta(\mu, \rho) = \sum_{n=0}^{\infty} p^n w_n(\mu, \rho). \tag{61}
\]

Thus, we have

\[
\zeta(\mu, \rho) = \cos(\mu) + \left( \sin(\mu) - \ell \cos(\mu) \right) \frac{\rho^\sigma}{\Gamma(\sigma + 1)}
+ \left( -\cos(\mu) - 2\ell \sin(\mu) + \ell^2 \cos(\mu) \right) \frac{(\rho^\sigma)^2}{\Gamma(2\sigma + 1)}
+ \left( -\sin(\mu) + 3\ell \cos(\mu) + 3\ell^2 \sin(\mu) - \ell^3 \cos(\mu) \right) \frac{(\rho^\sigma)^3}{\Gamma(3\sigma + 1)}. \tag{62}
\]

5. Results and Discussion

We implemented NITM and HPTM for finding the approximate solutions of time-fractional ADE. The analytical solution and exact solution are shown in Figures 1(a) and 1(b) at \( \sigma = 1 \), whereas Figures 1(c) and 1(d) show the absolute error and the solution at various fractional order. Figures 2 and 3 show the behavior of the proposed method solution at various fractional orders. Table 1 shows the comparison of the exact and suggested methods solution in addition with the absolute error at various fractional order. Finally, the figures and table show that the suggested techniques have higher degree of accuracy and rapid convergence towards the exact results.
Figure 2: Nature of the proposed method solutions at different fractional orders for problem 2.

Figure 3: Nature of the proposed method solutions at different fractional orders for problem 3.
6. Conclusion

The solutions for time-fractional ADE are successfully obtained using NITM and HPTM in this paper. The study reveals that the derivative having fractional order, as well as the location and time factors, has an impact on solute concentration. For varying values of the fractional parameter \( \sigma \), solutions are plotted with spatial and time coordinates for three cases. We compare actual and analytical results with the use of graphs and tables, which are in strong agreement with one another, to demonstrate the effectiveness of the proposed methods. Also, the results achieved by implementing the suggested approaches are compared at various fractional orders, confirming that the result comes closer to the exact solution as the value moves from fractional to integer order. The methods should be extended to solve space-time fractional ADE in two or three dimensions. As a result, the NITM and HPTM are effective methods in finding exact and approximate solutions for nonlinear differential equations arising in science and engineering.

Data Availability

The numerical data used to support the findings of this study are included within the article.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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