Bounding the number of classes of a finite group in terms of a prime

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Abstract. Héthelyi and Külshammer showed that the number of conjugacy classes \( k(G) \) of any solvable finite group \( G \) whose order is divisible by the square of a prime \( p \) is at least \((49p + 1)/60 \). Here an asymptotic generalization of this result is established. It is proved that there exists a constant \( c > 0 \) such that, for any finite group \( G \) whose order is divisible by the square of a prime \( p \), we have \( k(G) \geq cp \).

1 Introduction

Let \( k(G) \) denote the number of conjugacy classes of a finite group \( G \). This is also the number of complex irreducible characters of \( G \). Bounding \( k(G) \) is a fundamental problem in group and representation theory.

Let \( G \) be a finite group and \( p \) a prime divisor of the order \( |G| \) of \( G \). In this paper, we discuss lower bounds for \( k(G) \) only in terms of \( p \).

Pyber observed that results of Brauer [1] imply that \( G \) contains at least \( 2\sqrt{p-1} \) conjugacy classes provided that \( p^2 \) does not divide \( |G| \). Building on works of Héthelyi and Külshammer [8], Malle [17], Keller [12], Héthelyi, Horváth, Keller and Maróti [7], it was shown in [19] that \( k(G) \geq 2\sqrt{p-1} \) for any finite group \( G \) and any prime \( p \) dividing \( |G| \), with equality if and only if \( \sqrt{p-1} \) is an integer, \( G = C_p \rtimes C_p \sqrt{p-1} \) and \( C_G(C_p) = C_p \).

The objective of the current paper is to provide a stronger lower bound for \( k(G) \) in the case where \( p^2 \) divides \( |G| \). Héthelyi and Külshammer [9] showed that, for any finite solvable group \( G \) and any prime \( p \) such that \( p^2 \) divides \( |G| \), the number

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of conjugacy classes of $G$ is at least $(49p + 1)/60$. This bound is sharp [9] for infinitely many primes $p$; however, it does not generalize [8] to arbitrary finite groups since there are infinitely many non-solvable groups $G$ and primes $p$ with $k(G) = 0.55p - 0.05$.

The main result of this paper is the following.

**Theorem 1.1.** There exists a constant $c > 0$ such that, for any finite group $G$ whose order is divisible by the square of a prime $p$, we have $k(G) \geq cp$.

Questions of Pyber and the papers [8,9] of Héthelyi and Kühlhammer motivated our result.

Let $B$ be a $p$-block of a finite group $G$, and let $D$ be a defect group of $B$. The number $k(B)$ of complex irreducible characters of $G$ associated to the block $B$ is a lower bound for $k(G)$. A recent result of Otokita [20, Corollary 4] states that $k(B) \geq (p^m + p - 2)/(p - 1)$, where $p^m$ denotes the exponent of the center of $D$.

Finally, note that Kovács and Leedham-Green [14] have constructed, for every odd prime $p$, a finite $p$-group $G$ of order $p^n$ with $k(G) = \frac{1}{2}(p^3 - p^2 + p + 1)$ (see also [21]).

## 2 Affine groups

The purpose of this section is to prove Proposition 2.2. For this, we need the following lemma. The base of the logarithms in this paper is always 2.

**Lemma 2.1.** Let $H$ be a finite group and $V$ a finite, faithful, completely reducible $H$-module over a finite field of characteristic $p$. Assume that $H$ has no composition factor isomorphic to an alternating group of degree larger than $(\log p)^3$ and has no composition factor isomorphic to a simple group of Lie type defined over a field of characteristic $p$. Put $p^n = |V|$. Then $H$ has an abelian subgroup of index at most $(c_1 \log p)^{(7n-1)}$ for some universal constant $c_1 > 1$.

Note that, once Lemma 2.1 is proved, it may be extended by a theorem of Chermak and Delgado [11, Theorem 1.41] as follows. Under the conditions of Lemma 2.1, the group $H$ contains a characteristic abelian subgroup of index at most $(c_1 \log p)^{14(n-1)}$ for some universal constant $c_1 > 1$.

**Proof of Lemma 2.1.** Assume first that $V$ is a primitive and irreducible $H$-module. We use the following structure result which is implicit in the proofs of [6] (see for example the proof of [6, Theorem 9.1]). Let $F$ be the largest field such that
$H$ embeds in $\Gamma L_F(V)$. Let $C$ be the subgroup of non-zero elements in $F$. We claim that $|H/(H \cap C)| \leq (c_1 \log p)^{7(n-1)}$ for some universal constant $c_1 > 1$. For this, we may assume that $C \leq H$.

Let $H_0$ be the centralizer of $C$ in $H$, and let $R$ be a normal subgroup of $H$ contained in $H_0$ minimal with respect to not being contained in $C$ (if such exists). There are two possibilities for $R$. It is of symplectic type and $|R/Z(R)| = r^{2a}$ for some prime $r$ and integer $a$ such that $r$ divides $|F| - 1$ or $R$ is a central product of $t$ isomorphic quasisimple groups.

Choose a maximal collection $J_1, \ldots, J_m$ of such non-cyclic normal subgroups in $H_0$ which pairwise commute (if such exist). Let $J$ be the central product of the subgroups $J_1, \ldots, J_m$. Then $H_0/(C \cdot \text{Sol}(J))$ embeds in the direct product of the automorphism groups of $J_i/Z(J_i)$, where Sol($J$) denotes the solvable radical of $J$. (Note that, in the proof of [6, Theorem 9.1], it was falsely asserted that $H_0/C$ embeds in the direct product of the automorphism groups; however, this did not affect the proof of [6, Theorem 9.1] nor [6, Theorem 10.1].)

Let $W$ be an irreducible constituent of $V$ for the normal subgroup $J$ of $H$ (provided that $J$ is non-trivial). Since $H$ is primitive on $V$, it follows that $J$ acts homogeneously on $V$ by Clifford’s theorem. Let $E = \text{End}_{FJ}(W)$. Now we have $W \cong U_1 \otimes \cdots \otimes U_m$, where $U_i$ is an absolutely irreducible $EJ_i$-module by [13, Lemma 5.5.5]. Notice that $E$ may be viewed as a subfield of $\text{End}_{FJ}(V)$, and since $J$ is normal in $H$, the multiplicative group of $E$ is normalized by $H$. Our choice of $F$ implies that $E = F$. If $J_i$ is of symplectic type with $J_i/Z(J_i)$ of order $r_i^{2a_j}$, then $\dim U_i = r_i^{a_j}$. If $J_i$ is a central product of $t$ isomorphic quasisimple groups $Q_{i,j}$ with $1 \leq j \leq t$, then $U_i \cong U_{i,1} \otimes \cdots \otimes U_{i,t}$, where $U_{i,j}$ is an absolutely irreducible (faithful) $FQ_{i,j}$-module for every $j$ with $1 \leq j \leq t$, by [13, Lemma 5.5.5 and Lemma 2.10.1].

Let $|F| = p^f$ and $d = \dim_F V$. The product of the orders of all abelian composition factors in any composition series of the factor group $H/C$ is less than $f \cdot d^{2 \log d + 3} \leq n^{2 \log n + 4}$ by [6, Theorem 10.1] and its proof. This is at most $(c_2 \log p)^{n-1}$ for some constant $c_2 > 2$. We may now assume that $J \neq 1$ and $n > 1$.

Let $b(X)$ denote the product of the orders of all non-abelian composition factors in any composition series of a finite group $X$. Since $|H/C| \leq (c_2 \log p)^{n-1} b(H)$, we proceed to bound $b(H)$.

Without loss of generality, assume that $J_1, \ldots, J_k$ are groups of symplectic type with $k \geq 0$ and $|J_i/Z(J_i)| = r_i^{2a_i}$ for some primes $r_i$ and integers $a_i$, and assume that $J_{k+1}, \ldots, J_m$ are groups not of symplectic type. For each $\ell$ with $k + 1 \leq \ell \leq m$, let $J_\ell$ be a central product of $t_\ell$ copies, say $Q_{\ell,1}, \ldots, Q_{\ell,t_\ell}$, of a quasisimple group $Q_\ell$. In this case, $U_\ell \cong U_{\ell,1} \otimes \cdots \otimes U_{\ell,t_\ell}$, where $U_{\ell,j}$ is an irreducible (faithful) $Q_{\ell,j}$-module for every $j$ with $1 \leq j \leq t_\ell$. Using this nota-
tion, we may write the following:

\[ n \geq d = \dim V \geq \dim W = \left( \prod_{i=1}^{k} \dim U_i \right) \cdot \left( \prod_{\ell=k+1}^{m} \dim U_{\ell} \right) \]

\[ = \left( \prod_{i=1}^{k} r_i^{a_i} \right) \cdot \left( \prod_{\ell=k+1}^{m} (\dim U_{\ell,1})^{t_{\ell}} \right) \geq \left( \prod_{i=1}^{k} r_i^{a_i} \right) \cdot 2^{\sum_{\ell=k+1}^{m} t_{\ell}}. \tag{2.1} \]

Since \( H/H_0 \) and \( C \cdot \text{Sol}(J) \) are solvable, \( b(H) = b(H_0/(C \cdot \text{Sol}(J))) \). Recall from the third paragraph of this proof that the group \( H_0/(C \cdot \text{Sol}(J)) \) embeds in the direct product of the automorphism groups of the \( J_i/Z(J_i) \). There exists a chain of subnormal subgroups

\[ H_0/(C \cdot \text{Sol}(J)) = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_m = \{ C \cdot \text{Sol}(J) \} \]

such that \( N_{i-1}/N_i \leq \text{Aut}(J_i/Z(J_i)) \) for every \( i \) with \( 1 \leq i \leq m \). These give

\[ b(H) \leq \left( \prod_{i=1}^{k} |N_{i-1}/N_i| \right) \cdot \left( \prod_{\ell=k+1}^{m} b(N_{\ell-1}/N_\ell) \right). \tag{2.2} \]

Since \( \prod_{i=1}^{k} t_i^{a_i} \leq n \) by (2.1), we have

\[ \prod_{i=1}^{k} |N_{i-1}/N_i| < \prod_{i=1}^{k} r_i^{4a_i^2} \leq \prod_{i=1}^{k} n^{4\log(r_i^{a_i})} \]

\[ \leq n^{4\sum_{i=1}^{k} \log(r_i^{a_i})} \leq n^{4\log n}. \tag{2.3} \]

We see by Schreier’s conjecture that, for every \( \ell \) with \( k + 1 \leq \ell \leq m \), we have \( b(N_{\ell-1}/N_\ell) \leq |T_\ell| \cdot b(Q_\ell)^{t_\ell}, \) where \( T_\ell \) is some permutation group of degree \( t_\ell \) having no composition factor isomorphic to an alternating group of degree larger than \( (\log p)^3 \). Now \( |T_\ell| \leq (2 \log p)^{3(t_\ell - 1)} \) by [18, Corollary 1.5]. Using the fact that \( \sum_{\ell=k+1}^{m} t_\ell \leq \log n \) (see (2.1)), we have

\[ \prod_{\ell=k+1}^{m} b(N_{\ell-1}/N_\ell) \leq (2 \log p)^{3(\log n - 1)} \cdot \left( \prod_{\ell=k+1}^{m} b(Q_\ell)^{t_\ell} \right). \tag{2.4} \]

It follows by (2.2), (2.3) and (2.4) that

\[ b(H) < (c_3 \log p)^{3(n-1)} \cdot \left( \prod_{\ell=k+1}^{m} b(Q_\ell)^{t_\ell} \right) \tag{2.5} \]

for some constant \( c_3 > 1 \).
Let $T$ be a quasisimple group with $T/Z(T)$ not isomorphic to an alternating group of degree larger than $(\log p)^3$ and not isomorphic to a simple group of Lie type defined over a field of characteristic $p$. Let $U$ be any finite, faithful $FT$-module over the finite field $F$ of order $p^f$. Put $|F|^s = |U|$. We claim that

$$b(T) = |T/Z(T)| < (c_4 \log p)^{3(s-1)}$$  \hspace{1cm}(2.6)$$

for some universal constant $c_4 > 1$. We use [15]. A consequence of [13, (5.3.2), Corollary 5.3.3 and Theorem 5.3.9] is that if $T/Z(T)$ is a simple group of Lie type in characteristic different from $p$, then $|T/Z(T)| < (c_4 \log p)^{3(s-1)}$ for some constant $c_4 > 1$. By choosing $c_4$ to be at least the maximum of the size of the Monster and the largest value of $r!$ for which $r! \geq r^{r-5}$, where $r$ is a positive integer, our bound on $|T/Z(T)|$ extends to the case when $T/Z(T)$ is a sporadic simple group or $T/Z(T)$ is an alternating group of degree $r$ with $r! \geq r^{r-5}$. If $T/Z(T)$ is an alternating group of degree $r \leq (\log p)^3$ such that $r! < r^{r-5}$, then

$$|T/Z(T)| < r! < r^{r-5} \leq (\log p)^{3(r-5)} \leq (\log p)^{3(s-1)},$$

where the last inequality follows from [13, (5.3.2), Corollary 5.3.3 and Proposition 5.3.7]. This proves our claim.

For every $\ell$ with $k + 1 \leq \ell \leq m$, define $s_\ell \geq 2$ by

$$|U_{\ell,1}| = |F|^{s_\ell}, \quad \text{that is,} \quad s_\ell = \dim U_{\ell,1}.$$  \hspace{1cm}(2.7)$$

Using (2.6) and (2.1), we find that

$$\prod_{\ell=k+1}^{m} b(Q_{\ell})^{t_\ell} < \prod_{\ell=k+1}^{m} (c_4 \log p)^{3(s_\ell-1)r_\ell} \leq \prod_{\ell=k+1}^{m} (c_4 \log p)^{3(s_\ell^t-1)}$$

$$\leq (c_4 \log p)^{3((\sum_{\ell=k+1}^{m} s_\ell^t)-1)}$$

$$\leq (c_4 \log p)^{3((\prod_{\ell=k+1}^{m} s_\ell^t)-1)}$$

$$\leq (c_4 \log p)^{3(n-1)}. \hspace{1cm}(2.7)$$

We have $b(H) < (c_3 c_4 \log p)^{6(n-1)}$ by (2.5) and (2.7). Thus

$$|H/C| \leq (c_2 \log p)^{n-1} b(H) < (c_2 c_3 c_4 \log p)^{7(n-1)}.$$  \hspace{1cm}$$

Finally, set $c_1 = c_2 c_3 c_4 > 2$.

This finishes the proof of the lemma in case $V$ is a primitive and irreducible $H$-module.

Let $H$ be a counterexample to the statement of the lemma with $\dim V$ minimal and with $c_1$ as before. Put $f(p) = (c_1 \log p)^7$. 

We claim that \( V \) must be an irreducible \( H \)-module. Assume that \( V = V_1 \oplus V_2 \), where \( V_1 \) and \( V_2 \) are non-trivial (completely reducible) \( H \)-modules. Let \( H_1 \) be the action of \( H \) on \( V_1 \), and let \( H_2 \) be the action of \( H \) on \( V_2 \). The groups \( H_1 \) and \( H_2 \) are factor groups of \( H \) and thus have no non-abelian composition factor which is not a composition factor of \( H \). The group \( H \) may be viewed as a subgroup of \( H_1 \times H_2 \). Since \( H \) is a counterexample with \( \dim V \) minimal, there exist an abelian subgroup \( A_1 \) in \( H_1 \) of index at most \( f(p)^m \) and an abelian subgroup \( A_2 \) in \( H_2 \) of index at most \( f(p)^{n-m-1} \), where \( p^m = |V_1| \). The group \( A = (A_1 \times A_2) \cap H \) is an abelian subgroup of \( H \). Moreover,

\[
|H : A| = |H(A_1 \times A_2)|/|A_1 \times A_2| \leq |H_1 \times H_2|/|A_1 \times A_2| \\
\leq f(p)^{n-2} < f(p)^{n-1}.
\]

This is a contradiction. Thus \( V \) is an irreducible \( H \)-module.

We claim that \( V \) cannot be an imprimitive \( H \)-module. Let \( V = V_1 + \cdots + V_t \) with \( t > 1 \) be an imprimitivity decomposition for \( V \) with each \( V_i \) a subspace in \( V \), and let \( N \) be the normal subgroup of \( H \) consisting of all elements leaving every \( V_i \) invariant. The group \( N \) acts completely reducibly on \( V \) and thus also on each \( V_i \) by Clifford’s theorem. For every \( i \) with \( 1 \leq i \leq t \), let \( H_i \) be the action of \( N \) on \( V_i \). The group \( H/N \) may be viewed as a permutation group of degree \( t \). In particular, \( H \) may be viewed as a subgroup of a full wreath product of the form \( W = (H_1 \times \cdots \times H_t) : \text{Sym}(t) \). Since \( H \) is a counterexample with \( \dim V \) minimal, there exists an abelian subgroup \( A_i \) in \( H_i \), for every \( i \) with \( 1 \leq i \leq t \), such that \( |H_i : A_i| \leq f(p)^{(n/t)-1} \). The group \( A_1 \times \cdots \times A_t \) is contained in \( W \). Thus \( A = (A_1 \times \cdots \times A_t) \cap N \) is an abelian subgroup in \( H \). As before,

\[
|N : A| = |N(A_1 \times \cdots \times A_t)|/|A_1 \times \cdots \times A_t| \\
\leq |\prod_{i=1}^{t} H_i|/|\prod_{i=1}^{t} A_i| \leq |\prod_{i=1}^{t} H_i : A_i| \leq \prod_{i=1}^{t} f(p)^{(n/t)-1} \\
= f(p)^{n-t}.
\]

The permutation group \( H/N \) of degree \( t \) has no composition factor isomorphic to an alternating group of degree larger than \( (\log p)^3 \). It follows that

\[
|H/N| \leq (2 \log p)^{3(t-1)} < f(p)^{t-1}
\]

by [18, Corollary 1.5]. We thus have \( |H : A| < f(p)^{n-t} f(p)^{t-1} = f(p)^{n-1} \) by (2.8) and (2.9). A contradiction.

This finishes the proof of the lemma. \( \Box \)
Let \( X \) be a finite group. Denote the number of orbits of \( \text{Aut}(X) \) on \( X \) by \( k^*(X) \). If \( X \) acts on a set \( Y \), then denote the number of orbits of \( X \) on \( Y \) by \( n(X, Y) \).

**Proposition 2.2.** There exists a universal constant \( c_5 > 0 \) such that if \( G \) is a finite group having an elementary abelian minimal normal subgroup \( V \) of \( p \)-rank at least 2 and \( |G/V| \) is not divisible by \( p^2 \), then \( k(G) \geq c_5 p \).

**Proof.** Since \( k(G) \geq k(G/V) + n(G, V) - 1 \) by Clifford’s theorem, it is sufficient to show that \( k(G/V) + n(G, V) \geq c_6 p \) for some universal constant \( c_6 > 0 \). For this latter claim, we may assume that \( G/V \) acts faithfully on \( V \), that is, \( V \) is a faithful and irreducible \( H := G/V \)-module. This is because

\[
k(G/V) \geq k(G/C_G(V)) \quad \text{and} \quad n(G, V) = n(G/C_G(V), V).
\]

We may assume that \( p \) is sufficiently large.

Every non-abelian (simple) composition factor of \( H \) (provided that it exists) has order coprime to \( p \), except possibly one which has order divisible by \( p \) (but not by \( p^2 \)). There are the following possibilities for a non-abelian composition factor \( S \) of \( H \): (i) \( S \) is an alternating group; (ii) \( S \) is a simple group of Lie type in characteristic different from \( p \); (iii) \( S \cong \text{PSL}(2, p) \); (iv) \( S \) is a sporadic simple group.

Suppose that such a composition factor \( S \) exists. We have \( k(H) \geq k^*(S) \) by [21, Lemma 2.5]. Since \( k^*(\text{PSL}(2, p)) \geq (p - 1)/4 \), by considering diagonal matrices in \( \text{SL}(2, p) \), we may exclude case (iii) by assuming that \( p \) is sufficiently large. Let \( S \) be an alternating group of degree \( r \geq 5 \). Since \( |\text{Out}(S)| \leq 4 \), we have \( k^*(S) \geq k(S)/4 \). Since \( S \) is a normal subgroup of index 2 in the symmetric group of degree \( r \), we have \( k(S) \geq \pi(r)/2 \) where \( \pi(r) \) denotes the number of partitions of \( r \). We thus find that \( k^*(S) \geq c_7 \sqrt{r} \) for some constant \( c_7 > 1 \). If \( r > (\log p)^3 \), then \( k^*(S) > p \) for sufficiently large \( p \). Thus we assume that every alternating composition factor of \( H \) has degree at most \( (\log p)^3 \).

The group \( H \) contains an abelian subgroup \( A \) with

\[
|H : A| < |V|^\Theta(1)
\]

as \( p \to \infty \), by Lemma 2.1. Furthermore, \( k(H) \geq k(A)/|H : A| = |A|/|H : A| \) by [4, p. 502] and \( n(G, V) \geq |V|/|H| \). These give

\[
k(H) + n(G, V) \geq \frac{|A|}{|H : A|} + \frac{|V|}{|H : A|} = \frac{|A|}{|H : A|} + \frac{|V|/|A|}{|H : A|} > \frac{|A| + (|V|/|A|)}{|V|^\Theta(1)} \quad \text{as} \quad p \to \infty.
\]
Since the real function \( g(x) = x + (|V|/x) \) takes its minimum in the interval \([1, |V|]\) when \( x = \sqrt{|V|} \), we find that \( k(H) + n(G, V) > 2 \cdot |V|^{(1/2) - o(1)} > p \) for sufficiently large \( p \), unless \( |V| = p^2 \).

Let \( |V| = p^2 \). Note that, in [9, p. 661 and 662], it is shown that if \( G \) is solvable, we have

\[
k(G/V) + n(G, V) - 1 \geq \frac{49p + 1}{60}.
\]

Thus we may assume that \( G \) is non-solvable. In this case, \( H/Z(H) \) is either \( \text{Alt}(5) \) or \( \text{Sym}(5) \) (given that case (iii) above cannot occur) by [2, Section XII.260] or [10, Hauptsatz II.8.27]. Also, \( |Z(H)| < p \) since \( H \) is non-solvable by assumption. Thus there exists a constant \( c_8 > 0 \) such that \( k(G) \geq n(G, V) \geq |V|/|H| > c_8 p \).

This finishes the proof of the proposition. \( \square \)

3 Finite simple groups

In this section, we prove Propositions 3.7 and 3.8. We first prove a few preliminary lemmas.

**Lemma 3.1.** Let \( p, q \in \mathbb{N}^+ \setminus \{1\} \) such that \( p \mid q^i + (-1)^a \) and \( p \mid q^j + (-1)^b \) for some \( i, j \in \mathbb{N}^+ \) and some \( a, b \in \{0, 1\} \). If \( (i, a) \neq (j, b) \), then

\[
p \leq q^{\min\{i, j, |i-j|\}} + 1.
\]

**Proof.** Without loss of generality, we may assume that \( j \leq i \). By our assumptions, since \( p \leq q^j + 1 \), it is sufficient to show that \( p \leq q^{i-j} + 1 \). We have

\[
p \mid (q^i + (-1)^a) - (q^j + (-1)^b) = q^i - q^j + (-1)^a - (-1)^b.
\]

Assume first that \( i \neq j \) and \( a = b \). Then \( p \mid q^i - q^j = q^j(q^{i-j} - 1) \), and so \( p \leq q^{i-j} - 1 \). If \( a \neq b \), then

\[
p \mid (q^i + (-1)^a) + (q^j + (-1)^b) = q^i + q^j = q^j(q^{i-j} + 1),
\]

and so \( p \leq q^{i-j} + 1 \). \( \square \)

**Lemma 3.2.** Suppose that \( P(x) \) is a polynomial admitting a factorization of the form \( P^+(x)P^-(x) \), where

\[
P^+(x) = \prod_{i \in S^+} (x^i + 1)^{k^+_i} \quad \text{and} \quad P^-(x) = \prod_{i \in S^-} (x^i - 1)^{k^-_i}
\]

for some sets of integers \( S^+ \) and \( S^- \) and positive integers \( k^+_i \) and \( k^-_i \). Set

\[
m = \max\{S^+ \cup S^-\}.
\]

Assume that \( k^+_i = k^-_i = 1 \) for every index \( i \) strictly larger than \( m/2 \). If \( p \) is an odd prime such that \( p^2 \mid P(q) \) for some integer \( q \geq 2 \), then \( p \leq q^{m/2} + 1 \).
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\[ P^{-}(q) = \prod_{i=2}^{n+1} (q^i - 1) \]

| \( S, r \) | \( P^{-}(q) \) | \( P^{+}(q) \) |
|-------|-------------|-------------|
| \( A_n(q), n \) | \( \prod_{i=2}^{n+1} (q^i - 1) \) | 1 |
| \( B_n(q), C_n(q), n \) | \( \prod_{i=2}^{n+1} (q^i - 1) \) | \( \prod_{i=2}^{n+1} (q^i + 1) \) |
| \( D_n(q), n \) | \( \prod_{i=1}^{n+1} (q^i - 1) \) | \( \prod_{i=1}^{n+1} (q^i + 1) \) |
| \( 2D_n(q), n - 1 \) | \( \prod_{i=1}^{n} (q^i - 1) \) | \( \prod_{i=1}^{n} (q^i + 1) \) |
| \( G_2(q), 2 \) | \( (q - 1)(q^3 - 1) \) | \( (q + 1)(q^3 + 1) \) |
| \( F_4(q), 4 \) | \( (q - 1)(q^3 - 1)^2(q^4 - 1) \) | \( (q + 1)(q^3 + 1)^2(q^4 + 1) \cdot (q^6 + 1) \) |
| \( E_6(q), 6 \) | \( (q - 1)^2(q^3 - 1)^3(q^5 - 1) \) | \( (q + 1)^2(q^2 + 1)^3(q^3 + 1)^3 \cdot (q^4 + 1)(q^6 + 1) \) |
| \( E_7(q), 7 \) | \( (q - 1)^2(q^3 - 1)^2(q^5 - 1) \cdot (q^7 - 1)(q^9 - 1) \) | \( (q + 1)^2(q^2 + 1)(q^3 + 1)^2 \cdot (q^4 + 1)(q^5 + 1)(q^6 + 1) \cdot (q^7 + 1)(q^9 + 1) \) |
| \( E_8(q), 8 \) | \( (q - 1)^2(q^3 - 1)^2(q^5 - 1) \cdot (q^7 - 1)(q^9 - 1)(q^{15} - 1) \) | \( (q + 1)^2(q^2 + 1)(q^3 + 1)^2 \cdot (q^4 + 1)(q^5 + 1)(q^6 + 1)^2 \cdot (q^7 + 1)(q^9 + 1)(q^{10} + 1) \cdot (q^{12} + 1)(q^{15} + 1) \) |

Table 1

**Proof.** Let \( p \) be an odd prime such that \( p^2 \mid P(q) \) for some positive integer \( q \geq 2 \). Let \( i \in S^{+} \cup S^{-} \) be such that \( p \mid q^i + 1 \) or \( p \mid q^i - 1 \). If \( i \leq m/2 \), the assertion is clear, so assume that \( i > m/2 \). If \( p^2 \mid q^i + 1 \) or \( p^2 \mid q^i - 1 \), then

\[ p \leq \sqrt{q^i} \pm 1 < q^{i/2} + 1 \leq q^{m/2} + 1. \]

Otherwise, there exists \( j \in S^{+} \cup S^{-} \) distinct from \( i \) such that

\[ p \mid q^j + 1 \quad \text{or} \quad p \mid q^j - 1. \]

By Lemma 3.1, \( p \leq q^{\min\{i,j,|i-j|\} + 1} \). Observe that \( \min\{i,j,|i-j|\} \leq m/2 \).
For a proof of this observation, we may assume that \( i \leq j \leq m \), and so \( i \) or \( j - i \)

is at most \( m/2 \). \( \square \)

**Lemma 3.3.** Let \( S \) be a finite simple group of Lie type of Lie rank \( r \) defined over a field of size \( q \) as in Table 1. If \( p \) is an odd prime such that \( p \nmid q \) and \( p^2 \mid |S| \),

then \( p \leq q^{(r+1)/2} + 1 \) if \( r > 8 \) and \( p \leq q^{r/2} + 1 \) if \( r \leq 8 \).
\begin{align*}
S, r & \quad P(q) \\
\begin{array}{ll}
2B_2(q), 1 & (q - 1)(q - \sqrt{2q} + 1)(q + \sqrt{2q} + 1) \\
2G_2(q), 1 & (q - 1)(q + 1)(q - \sqrt{3q} + 1)(q + \sqrt{3q} + 1) \\
2F_4(q), 2 & (q - 1)^2(q + 1)(q - \sqrt{2q} + 1)(q + \sqrt{2q} + 1) \\
& (q^2 + 1)(q^3 - \sqrt{2q^3} + 1)(q^3 + \sqrt{2q^3} + 1)
\end{array}
\end{align*}

Table 2

\begin{align*}
S, r & \quad P(q) \\
\begin{array}{ll}
3D_4(q), 2 & (q - 1)^2(q + 1)(q^2 + q + 1)(q^3 + 1)(q^8 + q^4 + 1) \\
2E_6(q), 4 & (q^2 - 1)(q^3 - 1)^2(q^3 + 1)^2(q^4 - 1)(q^4 + 1) \\
& (q^5 + 1)(q^6 + 1)(q^9 + 1) \\
2A_n-1(q), [n/2] & \prod_{i=2, i\ even}^{n}(q^i - 1) \prod_{i=3, i\ odd}^{n}(q^i + 1)
\end{array}
\end{align*}

Table 3

**Proof.** Observe that $|S|$ divides $P^-(q)P^+(q)$ times a suitable power of $q$, so $p^2 \mid P^-(q)P^+(q)$. If $m$ is as in the statement of Lemma 3.2, then $p \leq q^{m/2} + 1$. According to Table 1, $m \leq r + 1$ if $r > 8$ and $m \leq 2r - 1$ if $r \leq 8$. The result follows.

**Lemma 3.4.** Let $S$ be a finite simple group of Lie type of Lie rank $r$ defined over a field of size $q$ as in Table 2. There exists a constant $c_{10}$ such that if $p$ is an odd prime with $p \nmid q$ and $p^2 \mid |S|$, then $p \leq c_{10} \cdot q^{r^{1/2}}$.

**Proof.** Let $P(q)$ be as in Table 2. Notice that $p^2 \mid P(q)$. If $p^2$ divides any of the three, four and eight factors in the factorizations of $P(q)$ in Table 2, in the respective three cases, then the statement holds. The statement also holds in case $S = 2F_4(q)$ when $p^2 \mid (q - 1)^2$. We may assume that there are two distinct factors $P_1(q)$ and $P_2(q)$ in the factorization of $P(q)$ given in Table 2 which are divisible by $p$. Hence $p \mid P_1(q) - P_2(q)$.

Let $S$ be $2B_2(q)$, where $q = 2^{2r+1}$. In this case, $|P_1(q) - P_2(q)| \leq 2 \sqrt{2q}$. Similarly, if $S$ is $2G_2(q)$, then $p \leq 2 \sqrt{3q}$.

Let $S$ be $2F_4(q)$, where $q = 2^{2r+1}$. Assume first that $P_1(q)$ and $P_2(q)$ have the same degree. In this case, $|P_1(q) - P_2(q)| \leq 2 \sqrt{2q^5}$. Otherwise, $p$ divides a factor of degree 1, and so $p \leq q + \sqrt{2q} + 1$. In any case, the result follows.
Lemma 3.5. Let $S$ and $r$ be as in Table 3. Then both $k(3D_4(q))$ and $k(2E_6(q))$ are at least $c_{11} \cdot q^{r+2}$ for some constant $c_{11} > 0$. We also have

$$k(2A_{n-1}(q)) \geq q^{2r-1} / \min\{2r + 1, q + 1\}.$$  

Proof. See [16] and [5, Corollary 3.11].

Lemma 3.6. Let $S$ be a finite simple group of Lie type of Lie rank $r$ defined over a field of size $q$ as in Table 3. There exists a constant $c_{12}$ such that if $p$ is an odd prime with $p \nmid q$ and $p^2 \mid |S|$, then $p \leq c_{12} \cdot q^r + 1$.

Proof. First let $S$ be $3D_4(q)$. If $p$ divides a factor of $P(q)$ as in Table 3 of degree at most 3, then the claim is clear. Otherwise,

$$p^2 \mid q^8 + q^4 + 1 = \frac{q^{12} - 1}{q^4 - 1} = \frac{(q^6 - 1)(q^6 + 1)}{q^4 - 1}.$$  

Since $p$ is an odd prime and $p^2 \mid (q^6 - 1)(q^6 + 1)$, we have $p^2 \leq q^6 + 1$. Next, let $S$ be $2E_6(q)$. If $p$ divides a factor of $P(q)$ in Table 3 of degree at most 5, then the claim follows. So we may assume that $p^2 \mid (q^6 + 1)(q^9 + 1)$. By Lemma 3.2, $p \leq q^{4.5} + 1$.

Finally, let $S$ be $2A_{n-1}(q)$. In this case, $p \leq q^{r+\frac{1}{2}} + 1$, by Lemma 3.2.

Let $M(S)$ denote the Schur multiplier of a non-abelian finite simple group $S$.

Proposition 3.7. There exists a constant $c_9 > 0$ such that $k^*(S) \geq c_9 p$ for any non-abelian finite simple group $S$ and any prime $p$ such that

$$p^2 \mid |S| \quad \text{or} \quad p \mid |\text{Out}(S)| \quad \text{or} \quad p \mid |M(S)|.$$  

Proof. We may assume that $S$ and $p$ are sufficiently large. In particular, we may ignore sporadic simple groups and small alternating groups, and we may assume that $p$ is odd.

Let $S$ be an alternating group $\text{Alt}(r)$. Since $p$ is odd, $p^2$ must divide $|S|$, and so $p \leq r$. Since there are $\lfloor r/3 \rfloor$ conjugacy classes of elements of order 3 in $\text{Sym}(r)$, we have $\lfloor r/3 \rfloor \leq k^*(\text{Alt}(r))$, for $r \geq 7$. The constant $c_9$ can be chosen such that $\lfloor r/3 \rfloor \geq c_9 r$.

Let $S$ be a finite simple group of Lie type of Lie rank $r$ defined over the field of size $q = \ell^f$ for some prime $\ell$ and positive integer $f$. By [17, p. 657], we have

$$k^*(S) \geq \frac{q^r}{|M(S)| \cdot |\text{Out}(S)|}.$$
Since both $|\text{Out}(S)|$ and $|\text{M}(S)|$ are at most $c_{13} \cdot \min\{r, q\} \cdot f$ for some constant $c_{13}$, we find that

$$k^*(S) \geq \frac{q^r}{(c_{13} \cdot \min\{r, q\} \cdot f)^2}. \quad (3.1)$$

It follows that if $p \mid q$, then $k^*(S) \geq c_{14} \cdot p$ for some constant $c_{14} > 0$. Thus assume that $p$ does not divide $q$. Notice that $f \leq \log q$.

Assume first that $p^2$ does not divide $|S|$. Then $p \leq c_{13} \cdot \min\{r, q\} \cdot \log q$. In order to establish the claim in this case, it is sufficient to find a constant $c_{15} > 0$ such that $q^r \geq c_{15} \cdot (c_{13} \cdot \min\{r, q\} \cdot \log q)^3$. For any fixed constant $c_{15}$, this is certainly true for sufficiently large $q$ or sufficiently large $r$. Thus we may assume that $p^2 \mid |S|$.

Assume first that $r$ is bounded. Let $S$ be as in Tables 1 or 2. By Lemmas 3.3 and 3.4, $p$ is at most $c_{16} \cdot q^{r-\frac{1}{2}}$ for some constant $c_{16}$. In this case, by (3.1), it is sufficient to find a constant $c_{17} > 0$ such that

$$q^r \geq c_{17} \cdot (c_{13} \cdot \min\{r, q\} \cdot \log q)^2 \cdot c_{16} \cdot q^{r-\frac{1}{2}}.$$ For any fixed $c_{17}$, this inequality holds apart from at most finitely many pairs $(r, q)$.

Next let $S$ be one of the first two groups in Table 3. In this case,

$$k^*(S) \geq \frac{c_{18} \cdot q^{r+2}}{(c_{13} \cdot \min\{r, q\} \cdot f)^2}$$

for some constant $c_{18} > 0$, by Lemma 3.5. Also, $p \leq c_{12} \cdot q^{r+1}$ by Lemma 3.6. Again, it is sufficient to find a constant $c_{19} > 0$ such that

$$c_{18} \cdot q^{r+2} \geq c_{19} \cdot (c_{13} \cdot \min\{r, q\} \cdot \log q)^2 \cdot c_{12} \cdot q^{r+1}.$$ But this is possible since $r$ is bounded.

Finally, assume that $r$ is unbounded. Let $S$ be as in Table 1. By Lemma 3.3, $p$ is at most $q^{(r+1)/2} + 1$. Since there exists a constant $c_{20} > 0$ such that

$$q^r \geq c_{20} \cdot (c_{13} \cdot \min\{r, q\} \cdot \log q)^2 \cdot (q^{(r+1)/2} + 1),$$

the lemma follows by (3.1). The only remaining case is $S = 2A_{n-1}(q)$. Here

$$k^*(S) \geq \frac{q^{2r-1}}{\min\{2r + 1, q + 1\} \cdot (c_{13} \cdot \min\{r, q\} \cdot f)^2}$$

by Lemma 3.5. Also, $p \leq c_{12} \cdot q^{r+1}$ by Lemma 3.6. Again, there exists a constant $c_{21} > 0$ such that

$$q^{2r-1} \geq c_{21} \cdot \min\{2r + 1, q + 1\} \cdot (c_{13} \cdot \min\{r, q\} \cdot \log q)^2 \cdot c_{12} \cdot q^{r+1}.$$ The proof is complete with $c_9$ the minimum of $c_{14}$, $c_{15}$, $c_{17}$, $c_{19}$, $c_{20}$ and $c_{21}$. □
Proposition 3.8. There exists a universal positive constant $c_{22}$ such that, for every non-abelian finite simple group $S$ and every prime $p$ dividing $|S|$, the inequalities $k(S) \geq c_{22} \cdot |\text{Out}(S)| \cdot \sqrt{p}$ and $k^*(S) \geq k(S)/|\text{Out}(S)|$ hold.

Proof. The second inequality follows from [21, Lemma 2.6].

Since $c_{22}$ is allowed to be chosen small enough, it may be assumed that $S$ is different from a sporadic group, different from $\text{Alt}(5)$, $\text{Alt}(6)$, and different from $\text{PSL}(2, 16)$, $\text{PSL}(2, 32)$ and $^{2}B_2(32)$.

Let $S$ be an alternating group $\text{Alt}(r)$ with $r \geq 7$. We have

$$k(S) \geq k^*(\text{Alt}(r)) \geq c_9 r \geq c_9 p$$

from the proof of Proposition 3.7. The claimed inequality holds if $c_{22}$ is chosen to be at most $c_9/2$.

Let $S$ be a finite simple group of Lie type of Lie rank $r$ defined over a field of size $q$. Malle in [17, p. 657] showed that $k(S) \geq q^r/|M(S)|$ and

$$\frac{q^r}{|M(S)|} \geq |\text{Out}(S)| \cdot 2 \cdot \sqrt{p - 1} \geq |\text{Out}(S)| \cdot \sqrt{p}$$

for all $S$ except for $\text{PSL}(2, 16)$, $\text{PSL}(2, 32)$ and $^{2}B_2(32)$. \hfill \Box

4 Proof of Theorem 1.1

Let $G$ be a counterexample to Theorem 1.1 with $c = \min\{c_5, c_9, c_{22}, c_{22}/2, 1/2\}$ and $|G|$ minimal.

Lemma 4.1. Let $N$ be a non-trivial normal subgroup of $G$. Then $p$ divides $|N|$, and $p^2$ does not divide $|G/N|$.

Proof. The number of complex irreducible characters of $G$, which is equal to $k(G)$, is at least $k(G/N)$, the number of complex irreducible characters of $G$ with $N$ in their kernel. If $|N|$ is not divisible by $p$, then $|G/N|$ is divisible by $p^2$, and so $k(G/N) \geq cp$ since $|G/N| < |G|$. \hfill \Box

Let $M = \text{soc}(G)$ be the socle of $G$ which is defined to be the product of all minimal normal subgroups of $G$. This group $M$ is a direct product of some of the minimal normal subgroups of $G$ by [3, Theorem 4.3A(ii)]. By Lemma 4.1, we may write $M$ in the form $M_1 \times M_2$, where $M_1$ is a (possibly trivial) elementary abelian $p$-group and $M_2$ is a (possibly trivial) direct product of non-abelian finite simple groups.
Lemma 4.2. The group $M_1$ is trivial or is cyclic of order $p$.

Proof. Assume that $p^2$ divides $|M_1|$. By Lemma 4.1 and Proposition 2.2, we may assume that every abelian minimal normal subgroup of $G$ is cyclic of order $p$. Furthermore, by the minimality of $G$, we may assume that $M = M_1 = C_p \times C_p$. Indeed, since $M = C_p \times \cdots \times C_p \times M_2$, a factor group of $G$ will have order divisible by $p^2$ unless $M = C_p \times C_p$.

We claim that $k(G) \geq k(G/M) + n(G, M) - 1 \geq p - 1 \geq cp$. For this, let $C = C_G(M)$ and $H = G/C$. Since $H$ acts faithfully on $M$, it is an abelian group of exponent dividing $p - 1$. Let $H_1$ be the kernel of the action of $H$ on the first direct factor $C_p$ of $M$. Then, since $H$ is abelian, $k(G/M) \geq k(H) = |H|$, and we get

$$k(G/M) + n(G, M) - 1 \geq |H| + n(H/H_1, C_p) \cdot n(H_1, C_p) - 1.$$ 

Observe that $n(H/H_1, C_p) = 1 + \frac{p-1}{|H/H_1|}$ and $n(H_1, C_p) = 1 + \frac{p-1}{|H_1|}$. Thus

$$|H| + n(H/H_1, C_p) \cdot n(H_1, C_p) - 1 > |H| + \frac{(p-1)^2}{|H|} \geq p - 1. \quad \Box$$

Lemma 4.3. The group $G$ cannot contain a normal subgroup which is a direct product of $t \geq 2$ copies of a non-abelian finite simple group.

Proof. Let $N$ be a normal subgroup of $G$ which is a direct product of $t \geq 2$ copies of a non-abelian finite simple group $S$. The prime $p$ divides $|N|$ and therefore $|S|$ by Lemma 4.1. Since $t \geq 2$, we have $p^2 \mid |N|$. On the other hand, by $C_G(N) \cap N = 1$ and by the minimality of $G$, we may assume that $C_G(N) = 1$. We then have

$$N \leq G \leq \text{Aut}(S) \cdot \text{Sym}(t).$$

Let $s = k^*(S)$. Choose a representative conjugacy class of $S$ from every $\text{Aut}(S)$-orbit on $S$. Let these be $C_1, \ldots, C_s$. Put $C = C_{i_1} \cdots C_{i_t}$, where, for each $j$ between 1 and $t$, the integer $i_j$ is between 1 and $s$. Note that $C$ is a conjugacy class of $N$ which can be uniquely labelled by a non-negative integer vector $(r_1, \ldots, r_s)$, where $r_i$ ($1 \leq i \leq s$) is the number of $j$ such that $i_j = i$, and hence it is contained in a unique conjugacy class of $G$. Note that the conjugation action of $\text{Aut}(S) \cdot \text{Sym}(t)$ on $N$ can only fuse $N$-classes which carry the same $(r_1, \ldots, r_s)$ label. Hence we have a family of conjugacy classes of $G$ which are uniquely labelled by these vectors. The set of all such vectors is the set of all non-negative integer solutions to the equation $x_1 + \cdots + x_s = t$. Therefore,

$$k(G) \geq \binom{t + s - 1}{t} \geq \binom{s + 1}{2} = s(s + 1)/2.$$ 

Since $s \geq c_{22} \cdot \sqrt{p}$ by Proposition 3.8, we have $k(G) > (c_{22}^2/2) \cdot p. \quad \Box$
The group $M_1$ is $C_p$ or is trivial and $M_2$ is trivial or is a direct product of pairwise non-isomorphic non-abelian finite simple groups, by Lemmas 4.2 and 4.3.

Lemma 4.4. If $M_2 \neq 1$, then $M_2$ is simple.

Proof. Assume that the group $M_2$ is non-trivial and not simple. The minimality of $G$ and Lemma 4.1 imply that $M_2 = S \times F$, where $S$ and $F$ are non-isomorphic non-abelian finite simple groups both of order divisible by $p$. There are at least
$$k^*(S) \cdot k^*(F) \geq (c_{22} \cdot \sqrt{p})^2 = c_{22}^2 \cdot p$$
conjugacy classes of $G$ contained in $M_2$ by Proposition 3.8. This is a contradiction. \hfill \Box

Lemma 4.5. The group $G$ cannot be almost simple.

Proof. This follows from Proposition 3.7. \hfill \Box

Lemma 4.6. We must have $M_2 = 1$.

Proof. Assume for a contradiction that $M_2 \neq 1$. Then $M_2$ is a non-abelian finite simple group $S$ by Lemma 4.4. Consider the normal subgroup $R = C_G(S) \times S$ of $G$. The group $G/R$ can be considered as a subgroup of $\text{Out}(S)$. Since $C_G(S)$ is normal in $G$, it is either trivial or $p$ divides $|C_G(S)|$ by Lemma 4.1. The first possibility cannot occur by Lemma 4.5. Thus $p$ must divide $|C_G(S)|$. On the other hand, $p^2$ cannot divide $|C_G(S)|$ by the minimality of $G$. By a result of Brauer [1], $k(C_G(S)) \geq 2\sqrt{p-1}$. By Proposition 3.8, it then follows that
$$k(G) \geq \frac{k(R)}{|\text{Out}(S)|} = \frac{k(C_G(S)) \cdot k(S)}{|\text{Out}(S)|} \geq 2\sqrt{p-1} \cdot c_{22} \cdot \sqrt{p} > c_{22} \cdot p.$$ \hfill \Box

Observe that $M = M_1 = C_p$ by Lemmas 4.2 and 4.6. Put $C = C_G(M)$. Then $|G/C|$ divides $p-1$. Consider a maximal chain of normal subgroups of $G$ from $C$ to 1 containing $M$. Let $K_1$ be the smallest group in this chain with the property that $p^2$ divides $|K_1|$. Let $K_2$ be the next smaller neighbor of $K_1$ in this chain. The group $M$ is contained in the center of $K_2$, but $|K_2/M|$ is not divisible by $p$. By the Schur–Zassenhaus theorem, $K_2 = M \times K$ for a $p'$-subgroup $K$ of $K_2$. Since $K$ is characteristic in $K_2$ and $K_2$ is normal in $G$, the group $K$ is normal in $G$. This occurs only if $K = 1$ by Lemma 4.1. By the maximality of the chain of normal subgroups of $G$, the group $K_1/M$ is a direct product of isomorphic simple groups $T$. Since $p^2$ divides $|K_1|$, the prime $p$ must divide $|T|$. By the minimality of $G$, the factor group $K_1/M$ is isomorphic to $T$. 


Lemma 4.7. The group $T$ cannot be $C_p$.

Proof. Assume that $T$ is cyclic of order $p$. Then we have that $|K_1| = p^2$. Since $k(G) \geq k(G/M)$ and $G$ is a minimal counterexample, we see that $|G/M|$ is not divisible by $p^2$ but divisible by $p$. Thus $G/K_1$ is a $p'$-group. By the Schur–Zassenhaus theorem, there is a $p'$-subgroup $H$ of $G$ such that $G = HK_1$. The group $C_H(K_1)$ is centralized by $K_1$, and it is the kernel of the action of $H$ on the normal subgroup $K_1$ of $G$. Thus $C_H(K_1)$ is normalized by $HK_1 = G$. Since $H$ is a $p'$-group, $C_H(K_1)$ must be trivial by Lemma 4.1. We conclude that $H$ may be considered as an automorphism group of $K_1$. Since $|K_1| = p^2$, the group $H$ and so $G$ must be solvable. The claim follows by [8].

The group $T$ must be a non-abelian simple group by Lemma 4.7 and the fact that $p$ divides $|T|$; see the paragraph before Lemma 4.7. By Lemma 4.6, $K_1$ is thus perfect and therefore a quasisimple group.

Notice that $k(G)$ is at least $k^*(K_1)$. We claim that $k^*(K_1) \geq k^*(T)$. Let $T_1$ and $T_2$ be two distinct $\text{Aut}(T)$-orbits in $T$. Consider $\phi^{-1}(T_1)$ and $\phi^{-1}(T_2)$, where $\phi$ is the natural projection from $K_1$ to $T$. Notice that these two sets are disjoint and $\text{Aut}(K_1)$-invariant. This proves the claim.

The following lemma completes the proof of Theorem 1.1.

Lemma 4.8. Let $T$ be a non-abelian finite simple group. Let $p$ be a prime divisor of $|T|$ such that $p$ divides the size of the Schur multiplier of $T$. Then $k^*(T) \geq c_9 p$.

Proof. This follows from Proposition 3.7.

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