are twin primes. By using the identity \((2n - 1)n - (2n + 1)(n - 1) = 1\) with \(2n - 1 = p\) we find that \(x' = n = 1014\) and \(y' = n - 1 = 1013\) satisfy \(px' - qy' = 1\). Hence, \(x_0 = 1014\) and \(y_0 = 1013\), too. For \(t = 5\) the number \(y_0 + qt = 11159\) is prime. Also, for \(t = 10\) the number \(y_0 + pt = 21283\) is prime. Thus, we obtain
\[
(2n - 1)n - (2n + 1)(n - 1) = 1
\]
and
\[
px' - qy' = 1.
\]

\[
\det \begin{bmatrix} 2027 & 2029 \\ 11148 & 11159 \end{bmatrix} = 1, \quad \text{and} \quad \det \begin{bmatrix} 2027 & 2029 \\ 21283 & 21304 \end{bmatrix} = 1.
\]

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105.25 Families of curves orthogonal to the lines \(y = mx - 2m - m^3\)

Orthogonal pairs of family of curves represent a co-ordinate system. They may also represent an electrostatic system comprising of lines of force and equipotentials. Textbooks [1, 2] discuss the method and exercises of finding orthogonal family of curves to a given one-parameter family of curves. First, one sets up a first order ordinary differential equation (ODE) by eliminating the free parameter of the given family of curves, by differentiation. Next, in this ODE(\(c\)) one changes \(y\) to \(-\frac{1}{y}\) to set up a new ODE(\(c\)) to solve and get the required family of their orthogonal curves. Here \(c\) denotes the curve and \(o\) denotes the orthogonal curve/trajectory.

The orthogonal trajectories to the curves \(x^2 - y^2 = C_1\) is found by differentiating this equation and setting up the ODE(\(c\)) as \(y' = \frac{x}{y}\), the ODE(\(o\)) is obtained by changing \(y'\) to \(-1/y'\), we get ODE(\(o\)) as \(y' = -\frac{y}{x}\), solving this, we get \(xy = C_2\), where \(C_2\) is the constant/parameter of integration. The example just discussed represents a hyperbolic co-ordinate system and an electrostatic quadrupole as well. More involved orthogonal pairs of curves are discussed in the textbooks [1, 2].

Let us find orthogonal trajectories to the family of lines \(y = mx\), where \(m\) is a real parameter. Differentiating \(\frac{y}{x} = m\), we eliminate \(m\) and get the ODE(\(c\)) as \(y' = \frac{y}{x}\). Next we change \(y'\) to \(-1/y'\) to get ODE(\(o\)) as \(y' = -\frac{x}{y}\).
solving this we find orthogonal trajectories as \( x^2 + y^2 = r^2 \), where \( r \) is a real parameter. This pair of curves represent the polar co-ordinate system (when, \( m = \tan \theta \)) and the electric mono-pole (charge) where the lines are the lines of force and the circles are equi-potentials.

Similarly, we can find the orthogonal trajectories to the family of lines \( y = m(x + 1) \). The ODE(c) for this family is obtained by eliminating \( m \) by differentiation \( \frac{y}{x + 1} = m \), we get \((x + 1)y' - y = 0 \). By changing \( y' \) to \(-1/y' \), we get ODE(o) as \((x + 1) + yy' = 0 \), solving this we get \((x + 1)^2 + y^2 = r^2 \), where \( r \) is a real parameter. For any real value of \( m \) the lines are normal (orthogonal) to the circles.

We would like to remark that finding orthogonal trajectories to other one-parameter families of lines: \( y = mx + f(m) \), where \( f(m) \) becomes a more involved function of \( m \), remain elusive in the literature. The well known one-parameter family of lines \( y = mx \) are known to be normal to the simple parabola \( y^2 = 4x \) [3]. However, other curves which are orthogonal to the lines (1) are not known. In this Note, we propose to find them. Let us differentiate (1) with respect to \( x \), we get \( y' = m \), so the required ODE(c) is

\[ y = y'x - 2y' - (y')^3. \] (2)

The ODE(o) for the family of curves orthogonal to (1) can be obtained by replacing \( y' \) by \(-1/y' \).

Denoting \( y' \) as \( p \), we get,

\[ yp^3 = p^2(2 - x) + 1 \] (3)

differentiating with respect to \( y \), we get,

\[ 3p^2 \frac{dp}{dy} + p^3 = -2px \frac{dp}{dy} - p^2 \frac{dx}{dy} + 4p \frac{dp}{dy} \] (4)

Substituting \( x \) from (4) and \( \frac{dx}{dy} = \frac{1}{p} \) in (5), we get an ODE in only two variables, \( y \) and \( p \):

\[ (p^3 + p) dy + \left( yp^2 + \frac{2}{p} \right) dp = 0. \] (5)

Comparing (6) with \( M \frac{dy}{dx} + N \frac{dp}{dy} = 0 \), \( \frac{\partial M}{\partial p} = 3p^2 + 1 \), \( \frac{\partial N}{\partial y} = p^2 \). Since the differential equation (6) is not exact [1, 2], it can be made so by multiplying with the integrating factor \( \mu(p) \), where \( \mu(p) \) [1, 2] is

\[ \mu(p) = \exp \left( \frac{\partial N}{\partial y} - \frac{\partial M}{\partial p} \right) M \right) dp = \exp \left( \frac{-2p^2 - 1}{p^3 + p} \right) dp = \frac{1}{p \sqrt{1 + p^2}}. \] (7)
Multiplying (6) by \( \mu(p) \) in (7), we get the exact ODE,

\[
\sqrt{1 + p^2} \, dy + \left( py + \frac{2}{p^2} \right) \frac{1}{\sqrt{1 + p^2}} \, dp = 0.
\]  

(8)

The solution of (8) can be written as

\[
\int \sqrt{1 + p^2} \, dy \, \text{[treat } p \text{ as constant]} + \int \frac{2}{p^2} \frac{1}{\sqrt{1 + p^2}} \, dp \, \text{[terms of } N \text{ not containing } y \text{]} = C.
\]

(9)

Integrating (9), we get

\[
y\sqrt{1 + p^2} - \frac{2\sqrt{1 + p^2}}{p} = C.
\]

(10)

Thus,

\[
y = \frac{2}{p} + \frac{C}{\sqrt{1 + p^2}}
\]

(11)

where \( C \) is a constant of integration. Inserting (11) in (4), we get

\[
x = \frac{1}{p^2} - \frac{Cp}{\sqrt{1 + p^2}}.
\]

(12)

Equations (11) and (12) are the parametric solution of (3), where eventually \( p \) acts merely as a real parameter. By choosing \( t = 1/p \), we get more convenient parametrisation of family of curves which are orthogonal to the family of lines in (1). We write,

\[
x = t^2 - \frac{C}{\sqrt{1 + t^2}}, \quad y = 2t + \frac{Ct}{\sqrt{1 + t^2}}.
\]

(13)

Where the most popular curve, parabola \( (y^2 = 4x) \) emerges when \( C = 0 \). Eventually, any member of the family of lines (1) is orthogonal to any of the curves (13) only at one point. The line may cut the curve non-orthogonally at some other point.

Now the question is what the non-parabolic family of curves would look like. In Figure 1, we give a graphical depiction of the parabolic \( (C = 0) \) and the non-parabolic \( (C \neq 0) \) curves which are always orthogonal to the lines (1) for any real value of \( m \).

Finally, we conclude that the family of orthogonal curve(s) (13) to the family of lines (1) is the parabola \( y^2 = 4x \) when \( (C = 0) \) else for \( C \neq 0 \) they are new and non-parabolic even if they look like a parabola. However, surprisingly, for large negative values of \( C \) they are not even parabola-like, see the innermost curve in Figure 1 for \( C = -4 \). Eventually, for \( C \neq 0 \) the orthogonal curves (trajectories) (13) are new, non-standard and non-parabolic. It may be remarked that a parabola is always represented as a (rational function of \( x, y \)) quadratic: \( (y - m_1x - c_1)^2 = A(y - m_2x - c_2) \),
where \( m_1 \neq m_2 \). If we eliminate \( t \) in (13), for \( C \neq 0 \) we do not get a rational relation between \( x \) and \( y \).

**Figure 1:** Depicting 7 members of the family of curves (13) orthogonal to the family of lines (1) \((m = 1, 2, -3)\). Only the dashed curve is a parabola, the others are all non-parabolic.

**References**
1. G. F. Simmons and S. G. Krantz, *Differential Equations* (McGraw-Hill, New York) (2007).
2. M. L. Krasnov, A. I. Kiselyov, G. I. Makarenko, *A Book of Problems in Ordinary Differential Equations* (MIR, Moscow) (1978).
3. S. L. Loney, *Elements of Co-ordinate geometry*, Part 1.

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**105.26 Areas above and below a curve**

In [1] the author considers a strictly increasing, continuous, function \( f(x) \) for \( x \geq 0 \) with \( f(0) = 0 \). The rectangle with base \([0, x]\) and height \( f(x) \) is divided into two parts by the curve \( y = f(x) \), and the areas of the parts in the rectangle above, and below, the curve are denoted by \( A(x) \) and \( B(x) \), respectively. The author then asks for an explanation based on ‘dynamic geometry’ of the fact that if \( f(x) = x^n \) then \( A(x)/B(x) = n \). In this Note we shed light on this topic by discussing the case of a general function \( f \) and, in particular, we ask whether, given a positive function \( \psi \),