The weak Stokes problem associated with a flow through a profile cascade in $L^r$-framework

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Abstract
We study the weak steady Stokes problem, associated with a flow of a Newtonian incompressible fluid through a spatially periodic profile cascade, in the $L^r$-setup. The mathematical model used here is based on the reduction to one spatial period, represented by a bounded 2D domain $\Omega$. The corresponding Stokes problem is formulated using three types of boundary conditions: the conditions of periodicity on the “lower” and “upper” parts of the boundary, the Dirichlet boundary conditions on the “inflow” and on the profile and an artificial “do nothing”-type boundary condition on the “outflow.” Under appropriate assumptions on the given data, we prove the existence and uniqueness of a weak solution in $W^{1,r}(\Omega)$ and its continuous dependence on the data. We explain the sense in which the “do nothing” boundary condition on the “outflow” is satisfied.

KEYWORDS
artificial boundary condition, the Stokes problem, weak solution

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1 | INTRODUCTION

One spatial period: domain $\Omega$
Mathematical models of a flow through a three-dimensional turbine wheel often use the reduction to two space dimensions, where the flow is studied as a flow through an infinite planar profile cascade. In an appropriately chosen Cartesian coordinate system, the profiles in the cascade periodically repeat with the period $\tau$ in the $x_2$-direction. It can be naturally assumed that the flow is also $\tau$-periodic in variable $x_2$. This enables one to study the flow through the spatial period, which contains just one profile—see domain $\Omega$ and profile $P$ in Figure 1. This approach is used, e.g., in [7] and [17], where the authors present the numerical analysis of the models or corresponding numerical simulations, and in the papers [8] and [9] and [29]–[31] devoted to theoretical analysis of the mathematical models.

We assume that the fluid flows into the cascade through the straight line $\gamma_{\text{in}}$ (the $x_2$-axis, the inflow) and essentially leaves the cascade through the straight line $\gamma_{\text{out}}$, whose equation is $x_1 = d$ (the outflow). By “essentially” we mean that we do not exclude possible reverse flows on $\gamma_{\text{out}}$. The considered spatial period $\Omega$ is mainly determined by artificially chosen curves $\Gamma_0$ and $\Gamma_1 \equiv \Gamma_0 + r e_2$, which form the “lower” and “upper” parts of $\partial \Omega$ (the boundary of $\Omega$), respectively (see Figure 1). We denote by $e_2$ the unit vector in the $x_2$-direction. The parts of $\partial \Omega$, lying on the straight lines $\gamma_{\text{in}}$ and $\gamma_{\text{out}}$, are the line segments $\Gamma_{\text{in}} \equiv A_0 A_1$ and $\Gamma_{\text{out}} \equiv B_0 B_1$, respectively, of length $\tau$. The last part of $\partial \Omega$, i.e., the boundary of the profile $P$, is denoted by $\Gamma_P$. We assume that the domain $\Omega$ is Lipschitzian and the curves $\Gamma_0$ and $\Gamma_1$ are of the class $C^{\infty}$. 
The Stokes boundary-value problem on one spatial period

The flow of an incompressible Newtonian fluid is described by the Navier–Stokes equations. An important role in theoretical studies of these equations play the properties of solutions of the steady Stokes equation

\[-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f},\]  

(1.1)

which follows from the momentum equation in the Navier–Stokes system if one neglects the derivative with respect to time and the nonlinear “convective” term. Equation (1.1) is studied together with the equation of continuity (= condition of incompressibility)

\[\text{div } \mathbf{u} = 0.\]  

(1.2)

Equations (1.1) and (1.2) represent the so called steady Stokes system, or the steady Stokes equations. The unknowns are \(\mathbf{u} = (u_1, u_2)\) (the velocity) and \(p\) (the pressure). The positive constant \(\nu\) is the kinematic coefficient of viscosity and \(\mathbf{f}\) denotes the external body force. The density of the fluid can be without loss of generality supposed to be equal to one. The systems (1.1) and (1.2) are completed by appropriate boundary conditions on \(\partial \Omega\). One can naturally assume that the velocity profile on \(\Gamma_{in}\) is known, which leads to the inhomogeneous Dirichlet boundary condition

\[\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_{in}.\]  

(1.3)

Further, we consider the homogeneous Dirichlet boundary condition

\[\mathbf{u} = 0 \quad \text{on } \Gamma_{p},\]  

(1.4)

the condition of periodicity on \(\Gamma_0\) and \(\Gamma_1\)

\[\mathbf{u}(x_1, x_2 + \tau) = \mathbf{u}(x_1, x_2) \quad \text{for } \mathbf{x} \equiv (x_1, x_2) \in \Gamma_0\]  

(1.5)

and the artificial boundary condition

\[-\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p \mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_{out},\]  

(1.6)

where \(\mathbf{h}\) is a given vector function on \(\Gamma_{out}\) and \(\mathbf{n}\) denotes the unit outer normal vector, which is equal to \(\mathbf{e}_1 \equiv (1, 0)\) on \(\Gamma_{out}\). The boundary condition (1.6) (with \(\mathbf{h} = 0\)) is often called the “do nothing” condition, because it naturally follows from an appropriate weak formulation of the boundary-value problem, see [13] and [15].
On the results of this paper

The main results of the paper are theorems on properties of the corresponding weak Stokes operator $A_r$ (Theorem 3.1) and on the existence, uniqueness and continuous dependence on the data of a weak solution $u$ to the Stokes problem (1.1)–(1.6) in the $L^r$-setting (Theorem 3.3). Theorem 3.4 provides the existence of an associated pressure $p$ and explains the sense, in which $u$ and $p$ satisfy the boundary condition (1.6). These theorems (together with the definition of the weak solution and some auxiliary results) are listed, for readers’ convenience, in Section 3. The proofs are split to Sections 4–6.

Our results do not follow from previous papers on the Stokes problem (see the next paragraph), mainly because we consider three different types of boundary conditions on $\partial \Omega$, two of which “meet” in the “corner points” $A_0$, $A_1$, $B_0$, and $B_1$ of domain $\Omega$. Moreover, while the corresponding $L^2$-theory is relatively simple, the general $L^r$-case is much more difficult. An important tool, used in the proof of Theorem 3.1 in Section 4, is the $L^2$-maximum regularity property of the related Stokes operator, proven in [31]. The key inequality (4.6), which is also needed in the proof of Theorem 3.1, is proven in a separate section, see Section 5. The crucial estimate of the $W^{1,2}$-norm of the velocity in the neighborhood of $\Gamma_{\text{out}}$ is obtained, applying the results of Agmon et al. [2] on general elliptic systems. As an auxiliary result, we present Lemma 6.1 (in Section 6) on an appropriate extension of the velocity profile $g$ from $\Gamma_{\text{in}}$ to $\Omega$.

On some previous related results

The Stokes problem, in various domains and with various boundary conditions, has already been studied in many papers and books. As to weak solutions, the $L^2$-existential theory and the proof of uniqueness of an existing weak solution are relatively simple, because one works in a Hilbert space and can apply the Riesz theorem. The corresponding $L^r$-theory for a general $r \in (1, \infty)$ is much more difficult. Nevertheless, also in the $L^r$-setup, results on the existence and uniqueness of weak solutions of the Stokes problem in a smooth $N$-dimensional domain with Dirichlet’s boundary condition for the velocity can be found, e.g., in [6] (for $N = 3$), [10] and [34] ($N \geq 2$), with Navier’s boundary condition in [1] ($N = 3$) and with the Navier-type boundary condition in [3] ($N = 3$). Fundamental estimates have been basically obtained using the Stokes fundamental solution and the corresponding Green tensor in [6] and [10], respectively, also the inf–sup condition in [1] and [3]. The 2D case in a smooth domain has also been studied in [34] (with Dirichlet’s boundary condition, expressing the velocity using a stream function, applying the operator $\nabla \perp$ to the Stokes equation and using the results on the biharmonic boundary-value problem) and in [26] (the Neumann problem, applying the theory of hydrodynamical potentials).

The existence and uniqueness of a solution to the Stokes problem in an $N$-dimensional bounded Lipschitz domain with Dirichlet’s boundary condition for the velocity in the $L^r$-setup has been proven in [32] for $N \geq 3$ and $\frac{2}{N} < r < 3$ (if $N = 3$) or $2 \leq r \leq 2(N - 1)(N - 2)/(N(N - 3))$ (if $N \geq 4$) and in [11] (for $N \geq 2$) under the assumption that the Lipschitz constant of the boundary is sufficiently small. Planar Stokes problems in a bounded simply connected Lipschitz domain $\Omega$ with Navier’s or Navier-type boundary conditions, or with boundary conditions prescribing the tangential velocity and the pressure on $\partial \Omega$, have been studied in [27] and [28] using tools of complex analysis.

An artificial boundary condition on a part $\Gamma$ of the boundary has already been many times used in studies of the Navier–Stokes equations in channels or profile cascades. However, as the condition (1.6), in connection with the nonlinear term in the Navier–Stokes momentum equation, does not enable one to control the amount of kinetic energy in $\Omega$ in the case of a reverse flow on $\Gamma$, the condition (1.6) has usually been replaced by appropriate modifications, see e.g. [5], [8], [9], and [29–30]. The modifications are suggested so that one can derive an energy inequality, and consequently prove the existence of a weak solution. In papers [24] and [22], the authors use the boundary condition (1.6) on the outflow in connection with a flow in a channel, and they prove the existence of weak solutions of the Navier–Stokes equations for “small data”. Possible reverse flows on the “outflow” of a channel are controlled using additional conditions in [18], [19], and [20], where the Navier–Stokes equations are replaced by the Navier–Stokes variational inequalities. The existence of a solution in the general $L^r$-setting and the regularity up to the boundary of existing weak solutions (stationary or time-dependent, in the $L^2$ or $L^r$ settings) to the Navier–Stokes equations with the boundary condition (1.6) on a part of the boundary has not been studied in literature yet. This is mainly because one needs a detailed information on existence, uniqueness and regularity in the $L^r$-framework for the corresponding steady Stokes problem.

There are, to our knowledge, only two papers which provide information on the existence of a weak solution of the Navier–Stokes or only the Stokes problem with the boundary condition (1.6) on a part of the boundary, and its regularity: 1) the paper [23], where the authors studied a flow in a 2D channel $D$ of a special geometry, considering the homogeneous Dirichlet boundary condition on the walls and condition (1.6) on the outflow, and proved that the velocity is in $W^{2,\beta,2}(D)$ for certain $\beta \in (0, 1)$, provided that $f \in L^2(D)$ (see [23, Theorem 2.1]), and 2) the paper [31], where the existence of a weak
solution of the Stokes problem (1.1)–(1.6) in the $L^2$-setting and its regularity have been recently proven, under natural assumptions on $f$, $g$, and $h$.

## 2 NOTATION

**Notation**

Recall that $\Omega$ is a Lipschitzian domain in $\mathbb{R}^2$, sketched on Figure 1. Its boundary consists of the curves $\Gamma_{in}$, $\Gamma_{out}$, $\Gamma_0$, $\Gamma_1$, and $\Gamma_p$, described in Section 1. We denote by $n = (n_1, n_2)$ the outer normal vector field on $\partial \Omega$. We use $c$ as a generic constant, i.e., a constant whose values may change throughout the text.

- $\Gamma_{in}$, respectively $\Gamma_{out}$, denotes the open line segment without the end points $A_0$, $A_1$, respectively $B_0$, $B_1$. Similarly, $\Gamma_0$, respectively $\Gamma_1$, denotes the curve $\Gamma_0$, respectively $\Gamma_1$, without the end points $A_0$, $B_0$, respectively $A_1$, $B_1$.
- We denote vector functions and spaces of vector functions by boldface letters. If $\mathbf{v} \equiv (v_1, v_2)$ is a vector function then $\mathbf{v} \cdot \mathbf{n}$ is a tensor function with the entries $\partial v_i / \partial x_j \equiv \partial_j v_i$ on the positions $ij$ (for $i, j = 1, 2$). Spaces of second-order tensor functions are denoted by the superscript $2 \times 2$.
- We denote by $\| \cdot \|_r$ the norm in $L^r(\Omega)$ or in $L^r(\Omega)^{2 \times 2}$. Similarly, $\| \cdot \|_{s, r}$ is the norm in $W^{s, r}(\Omega)$ or in $W^{s, r}(\Omega)^{2 \times 2}$.
- $C_0^{\infty}(\Omega)$ denotes the linear space of all infinitely differentiable divergence-free vector functions in $\Omega$, whose support is disjoint with $\Gamma_0$ and that satisfy, together with all their derivatives (of all orders), the condition of periodicity (1.5). Note that each $\mathbf{w} \in C_0^{\infty}(\Omega)$ automatically satisfies the outflow condition $\int_{\Gamma_{out}} \mathbf{w} \cdot \mathbf{n} \, d\mathbf{l} = 0$.
- We denote by $C_{0, \sigma}(\Omega)$ the intersection $C_0^{\infty}(\Omega) \cap C_{0, \sigma}(\Omega)$, where $C_{0, \sigma}(\Omega)$ is the space of all infinitely differentiable vector functions in $\Omega$ with a compact support in $\Omega$.
- Let $1 < r < \infty$. $V_r^{1, r}(\Omega)$ denotes the closure of $C_0^{\infty}(\Omega)$ in $W_r^{1, r}(\Omega)$. The space $V_r^{1, r}(\Omega)$ can be characterized as a space of divergence-free vector functions from $W_r^{1, r}(\Omega)$, whose traces on $\Gamma_{in} \cup \Gamma_p$ are equal to zero and the traces on $\Gamma_0$ and $\Gamma_1$ satisfy the condition of periodicity (1.5). Note that the norm in $V_r^{1, r}(\Omega)$ is equivalent to $\| \nabla \|_r$.
- We denote by $r'$ the conjugate exponent to $r$, by $W^{-1, r'}(\Omega)$ the dual space to $W_0^{1, r}(\Omega)$ and by $W_0^{-1, r}(\Omega)$ the dual space to $W_0^{1, r}(\Omega)$. The corresponding norms are denoted by $\| \cdot \|_{W_0^{-1, r}}$ and $\| \cdot \|_{W_{-1, r}}$, respectively.
- $V_{\sigma}^{-1, r}(\Omega)$ denotes the dual space to $V_{\sigma}^{1, r'}(\Omega)$. The duality pairing between $V_{\sigma}^{-1, r}(\Omega)$ and $V_{\sigma}^{1, r'}(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle_{(V_{\sigma}^{-1, r}, V_{\sigma}^{1, r'})}$. The norm in $V_{\sigma}^{-1, r}(\Omega)$ is denoted by $\| \cdot \|_{V_{\sigma}^{-1, r}}$.
- Denote by $A_r$ the linear mapping of $V_r^{1, r}(\Omega)$ to $V_{\sigma}^{-1, r}(\Omega)$, defined by the equation

$$\langle A_r \mathbf{v}, \mathbf{w} \rangle_{(V^{1, r}_r, V^{-1, r}_{\sigma})} := (\nabla \mathbf{v}, \nabla \mathbf{w}) \quad \text{for} \ \mathbf{v} \in V^{1, r}_r(\Omega) \text{ and } \mathbf{w} \in V^{-1, r}_{\sigma}(\Omega), \quad (2.1)$$

where $(\nabla \mathbf{v}, \nabla \mathbf{w})$ represents the integral $\int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{w} \, d\mathbf{x}$. We call $A_r$ the weak Stokes operator. (The reason for this name is that the weak Stokes problem (1.1)–(1.6) can be transformed to the equation $\nabla A_r \mathbf{v} = \mathbf{f}$ for an appropriate element $\mathbf{f} \in V_r^{1, r}(\Omega)$, see Section 6.)
- $W_{1, r}^{1, r}(\Gamma_{in})$ denotes the space of periodic functions in $W_0^{1, r}(\gamma_{in})$ and $W_{per}^{1, r}(\Gamma_{in})$ denotes the space of functions from $W^{1, r}(\Gamma_{in})$, that can be extended from $\Gamma_{in}$ to $\gamma_{in}$ as functions in $W^{1, r}(\gamma_{in})$. (Recall that the straight line $\gamma_{in}$ and the line segment $\Gamma_{in}$ are sketched on Figure 1.)
- The space $W_{per}^{1, r}(\Gamma_{out})$ is defined by analogy with $W_{per}^{1, r}(\Gamma_{in})$. Let us denote by $W_{per}^{-1, r}(\Gamma_{out})$ the dual space to $W_{per}^{1, r}(\Gamma_{out})$ and by $\langle \cdot, \cdot \rangle_{(W_{per}^{1, r}(\Gamma_{in}), W_{per}^{-1, r}(\Gamma_{out}))}$ the duality pairing between $W_{per}^{1, r}(\Gamma_{in})$ and $W_{per}^{-1, r}(\Gamma_{out})$.

An alternative description of $W_{per}^{1, r}(\Gamma_{in})$ (which also holds for $W_{per}^{-1, r}(\Gamma_{out})$) is explained in the following remark.

**Remark 2.1.** Assume at first that $r > 2$. In this case, functions from $W_{per}^{1, r}(\Gamma_{in})$ have traces at the end points $A_0$ and $A_1$ of $\Gamma_{in}$. Denote $\tilde{W}_{per}^{1, r}(\Gamma_{in}) := \{ \mathbf{w} \in W^{1, r}(\Gamma_{in}) ; \mathbf{w}(A_0) = \mathbf{w}(A_1) \}$ in the sense of traces. Obviously, $W_{per}^{1, r}(\Gamma_{in}) \subset \tilde{W}_{per}^{1, r}(\Gamma_{in})$. Let us show that the opposite inclusion is also true. Thus, let $\mathbf{w} \in \tilde{W}_{per}^{1, r}(\Gamma_{in})$. Denote by the same symbol $\mathbf{w}$ the $r$-periodic extension from $\Gamma_{in}$ to $\gamma_{in}$. We can assume without loss of generality that $A_0 = (0, 0)$ and $A_1 = (0, \tau)$. Put
$A_2 := (0, 2\tau)$. In order to verify that $w \in W^{1−1/r, r}_{\text{per}}(\gamma_{\text{in}})$, it is sufficient to show that $w \in W^{1−1/r, r}(A_0, A_2)$. The norm of $w$ in $W^{1−1/r, r}(A_0 A_2)$ equals $\|w\|_{r; A_0 A_2} + \langle\langle w\rangle\rangle_{1−1/r, r; A_0 A_2}$ (see formulas (2) and (3) in [21], paragraph 8.3.2, p. 386), where

$$
\langle\langle w\rangle\rangle_{1−1/r, r; A_0 A_2} = \int_0^{2\tau} \int_0^{2\tau} \frac{|w(0, y) - w(0, z)|^r}{|y - z|^r} \, dy \, dz
$$

$$
= \left( \int_0^\tau \int_0^{2\tau} + \int_\tau^{2\tau} \int_0^{2\tau} + \int_\tau^{2\tau} \int_\tau^{2\tau} \right) \frac{|w(0, y) - w(0, z)|^r}{|y - z|^r} \, dy \, dz
$$

$$
\leq \langle\langle w\rangle\rangle_{1−1/r, r; A_0 A_1} + \langle\langle w\rangle\rangle_{1−1/r, r; A_1 A_2} + 2 \int_\tau^{2\tau} \frac{|w(0, y) - w(0, z)|^r}{|y - z|^r} \, dy \, dz.
$$

The first two terms on the right-hand side are equal due to the $\tau$-periodicity of $w(0, \cdot)$. The integral on the right-hand side is less than or equal to $c \langle\langle w\rangle\rangle_{1−1/r, r; A_0 A_1}$. This confirms that $w \in W^{1−1/r, r}(A_0, A_2)$, indeed.

In the critical case $r = 2$, one cannot characterize $W^{1/2, 2}_{\text{per}}(\Gamma_{\text{in}})$ as in the case $r > 2$, because the traces at the end points $A_0$ and $A_1$ generally do not exist. Moreover, although $W^{1/2, 2}_{\text{per}}(\Gamma_{\text{in}}) = W^{1/2, 2}(\Gamma_{\text{in}})$, see [25, Theorem II.1.1]), one cannot identify $W^{1/2, 2}_{\text{per}}(\Gamma_{\text{in}})$ with $W^{1/2, 2}(\Gamma_{\text{in}})$, because e.g., the linear function $g(0, y) := y$ for $0 < y < 1$ is in $W^{1/2, 2}(\Gamma_{\text{in}})$, but its $\tau$-periodic extension to $\gamma_{\text{in}}$ is not in $W^{1/2, 2}_{\text{per}}(\gamma_{\text{in}})$. Thus, one only has the inclusion $W^{1/2, 2}_{\text{per}}(\Gamma_{\text{in}}) \subset W^{1/2, 2}(\Gamma_{\text{in}})$.

If $1 < r < 2$ then $W^{1−1/r, r}_{\text{per}}(\Gamma_{\text{in}}) = W^{1−1/r, r}(\Gamma_{\text{in}})$, which follows from the density of $W^{1−1/r, r}(\Gamma_{\text{in}})$ in $W^{1/2, 2}(\Gamma_{\text{in}})$, the identity $W^{1/2, 2}(\Gamma_{\text{in}}) = W^{1/2, 2}_{\text{per}}(\Gamma_{\text{in}})$ and the density of $W^{1/2, 2}(\Gamma_{\text{in}})$ in $C_0^\infty(\Gamma_{\text{in}})$. This and similar estimates of $\langle\langle w\rangle\rangle_{1−1/r, r; A_0 A_2}$, as above, enable one to show that every function from $W^{1−1/r, r}(\Gamma_{\text{in}})$, periodically extended to $\gamma_{\text{in}}$ with the period $\tau$, is in $W^{1−1/r, r}_{\text{loc}}(\gamma_{\text{in}})$. Thus, one obtains the identity $W^{1−1/r, r}_{\text{per}}(\Gamma_{\text{in}}) = W^{1−1/r, r}(\Gamma_{\text{in}})$.

3 | THE MAIN RESULTS

Assume further on that the curve $\Gamma_\sigma$ (boundary of the profile $P$) is of the class $C^2$. (This assumption enables us to apply [31, Theorem 2] in Section 4.)

The next theorem is an analogue of results, known on the Stokes problem with the homogeneous Dirichlet or Navier or Navier-type boundary conditions on the whole boundary of domain $\Omega$, see [10], [1], and [3]. Recall that the theorem is non-trivial especially due to the variety of used boundary conditions and the fact that one cannot apply Riesz’ theorem in the general $L'$-setting in order to establish the existence and uniqueness of a solution of the equation $A_r v = f$ for $f \in V_{\sigma}^{1−1/r}(\Omega)$.

**Theorem 3.1** (on the weak Stokes operator $A_r$). The weak Stokes operator $A_r$ is a bounded, closed and injective operator from $V_{\sigma}^{1−1/r}(\Omega)$ to $V_{\sigma}^{1−1/r}(\Omega)$ with $D(A_r) = V_{\sigma}^{1−1/r}(\Omega)$ and $R(A_r) = V_{\sigma}^{1−1/r}(\Omega)$. The adjoint operator to $A_r$ is $A_r^* = A_r$.
The proof of Theorem 3.1 is given in Sections 4 and 5.

**Definition 3.2** (weak solution of the Stokes problem (1.1)–(1.6)). Let \( r \in (1, \infty) \). Let \( f \in W_{0}^{-1/r}(\Omega) \), \( g \in W_{\text{per}}^{1-1/r,r}(\Gamma_{\text{in}}) \) and \( h \in W_{\text{per}}^{-1/r,r}(\Gamma_{\text{out}}) \). Let \( F \in L'(\Omega)^{2 \times 2} \) be a tensor function, satisfying \( \text{div} \, F = f \) in the sense of distributions, provided by Lemma 4.1. A divergence-free function \( u \in W^{1/r}(\Omega) \), satisfying the conditions (1.3), (1.4), and (1.5) in the sense of traces on \( \Gamma_{\text{in}} \), \( \Gamma_{p} \), and \( \Gamma_{0} \), respectively, and the equation

\[
\nu \int_{\Omega} \nabla u : \nabla w \, dx = - \int_{\Omega} F : \nabla w \, dx - \langle h, w \rangle_{(W_{\text{per}}^{-1/r,r}(\Gamma_{\text{in}}), W_{\text{per}}^{1-1/r',r'}(\Gamma_{\text{out}}))}
\]

for all \( w \in V_{0}^{1/r}(\Omega) \), is said to be a weak solution to the Stokes problem (1.1)–(1.6). A distribution \( p \), such that \( u \) and \( p \) satisfy Equation (1.1) in the sense of distributions in \( \Omega \), is called an associated pressure.

Although the boundary condition (1.6) does not explicitly appear in Definition 3.2, Theorem 3.4 shows that condition (1.6) is in fact implicitly hidden in the definition.

The existence and uniqueness of a weak solution, under a little bit subtler assumptions on the given velocity profile \( g \) on \( \Gamma_{\text{in}} \) than in Definition 3.2, is provided by the following theorem.

**Theorem 3.3** (existence and uniqueness of a weak solution). Let \( 1 < r < \infty \), \( f \in W_{0}^{-1/r}(\Omega) \) and \( g \) be a given velocity profile on \( \Gamma_{\text{in}} \), such that \( g \in W_{\text{per}}^{s,r}(\Gamma_{\text{in}}) \), where \( s > 1/r \) if \( 1 < r \leq 2 \) and \( s = 1 - 1/r \) if \( r > 2 \). Let \( g \) satisfy the condition \( g(A_{0}) = g(A_{1}) \). Let \( h \) be an element of \( W_{\text{per}}^{-1/r,r}(\Gamma_{\text{out}}) \). Then the Stokes problem (1.1)–(1.6) has a unique weak solution \( u \) (in the sense of Definition 3.2). The solution satisfies the estimate

\[
\|u\|_{1,r} \leq c \left[ \|f\|_{W_{0}^{-1/r}} + \|g\|_{s,r,\Gamma_{\text{in}}} + \|h\|_{W_{\text{per}}^{-1/r,r}(\Gamma_{\text{out}})} \right],
\]

where \( c = c(\nu, \Omega, r) \).

The next theorem shows that if \( u \) is a weak solution then there exists \( p \in L'(\Omega) \) so that \( u \) and \( p \) satisfy a certain weak analogue of the condition (1.6) on \( \Gamma_{\text{out}} \).

**Theorem 3.4** (on an associated pressure). 1) Let \( r, f, F, g \) and \( h \) satisfy the assumptions from Definition 3.2. Let \( u \) be a weak solution of the Stokes problem (1.1)–(1.6). Then there exists an associated pressure \( p \in L'(\Omega) \), such that

\[
(-\nu \nabla u + p I - F) \cdot n = h
\]

holds as an equality in \( W_{\text{per}}^{-1/r,r}(\Gamma_{\text{out}}) \) and

\[
\|p\|_{r} \leq c \left[ \|f\|_{W_{0}^{-1/r}} + \|g\|_{s,r,\Gamma_{\text{in}}} + \|h\|_{W_{\text{per}}^{-1/r,r}(\Gamma_{\text{out}})} \right],
\]

where \( c = c(\nu, \Omega, r) \).

2) If, moreover, \( f \in L'(\Omega) \) then the tensor function \( F \) can be constructed so that it lies in \( W_{\text{per}}^{1/r}(\Omega)^{2 \times 2} \), satisfies the equation \( \text{div} \, F = f \) a.e. in \( \Omega \) and the condition \( F \cdot n = 0 \) a.e. on \( \Gamma_{\text{out}} \). In this case, (3.3) takes the form

\[
(-\nu \nabla u + p I) \cdot n = h,
\]

consistent with (1.6).

The proofs of Theorems 3.3 and 3.4 are given in Section 6.
PROOF OF THEOREM 3.1

Lemma 4.1. Let \( f \in W_0^{-1,r}(\Omega) \). Then there exists \( F \in L'(\Omega)^{2\times 2} \), satisfying \( \text{div} \ F = f \) in the sense of distributions in \( \Omega \) and

\[
\|F\|_r \leq c \|f\|_{W_0^{-1,r}},
\]

where \( c \) is independent of \( f \).

Proof. The proof is based on the results from paper [12] by Geissert et al. If the domain \( \Omega' \subset \mathbb{R}^3 \) is bounded and star-shaped with respect to some ball \( K \subset \Omega' \) and \( \omega \) is a function in \( C_0^\infty(K) \), such that \( \int_K \omega \, dx = 1 \), then it follows from Proposition 2.1 in [12] that there is a bounded linear operator \( \mathfrak{B} : W_0^{-1,r}(\Omega') \to L'(\Omega') \), such that \( \text{div} \mathfrak{B} f = f - \omega \int_{\Omega'} f \, dx \) for \( f \in L'(\Omega') \).

Applying an appropriate limit procedure, one can show that if \( f \in W_0^{-1,r}(\Omega') \) then

\[
\text{div} \mathfrak{B} f = f - \omega \langle f, 1 \rangle_{(W_0^{-1,r}(\Omega'), W^{1,r}(\Omega'))},
\]

holds in \( \Omega' \) in the sense of distributions. Then \( \text{div} \left[ \mathfrak{B} f + z \right] = f - \omega \langle f, 1 \rangle_{(W_0^{-1,r}(\Omega'), W^{1,r}(\Omega'))} + \text{div} \ z \) for \( z \in L'(\Omega') \). Let us choose \( z \) so that \( z = z_1 + z_2 \), where \( z_1 \) satisfies the equation

\[
\text{div} z_1 = (\omega - \bar{\omega}) \langle f, 1 \rangle_{(W_0^{-1,r}(\Omega'), W^{1,r}(\Omega'))},
\]

and

\[
z_2 = \frac{1}{3} x \bar{\omega} \langle f, 1 \rangle_{(W_0^{-1,r}(\Omega'), W^{1,r}(\Omega'))}.
\]

(We denote by \( \bar{\omega} \) the mean value of \( \omega \) in \( \Omega' \).) As the mean value of the right-hand side of (4.2) in \( \Omega' \) is zero, the equation (4.2) (for then unknown \( z_1 \)) is solvable in \( W_0^{1,r}(\Omega') \) due to [12, Theorem 2.5]. Moreover,

\[
\|z_1\|_{L^1;\Omega'} \leq c \|\omega - \bar{\omega}\|_{L^r;\Omega'} \|\langle f, 1 \rangle_{(W_0^{-1,r}(\Omega'), W^{1,r}(\Omega'))}\| \leq c \|f\|_{W_0^{-1,r}(\Omega)},
\]

where \( c \) depends only on \( \Omega', \omega \) and \( r \). Furthermore,

\[
\text{div} z_2 = \bar{\omega} \langle f, 1 \rangle_{(W_0^{-1,r}(\Omega'), W^{1,r}(\Omega'))} \quad \text{and} \quad \|z_2\|_{L^1;\Omega'} \leq c \|f\|_{W_0^{-1,r}(\Omega)}.
\]

Thus, the function \( \mathfrak{B} f + z \) satisfies the equation \( \text{div} \left[ \mathfrak{B} f + z \right] = f \) in the sense of distributions in \( \Omega' \) and \( \|\mathfrak{B} f + z\|_{L^1;\Omega'} \leq c \|f\|_{W_0^{-1,r}(\Omega)} \).

These results can be carried over to the whole domain \( \Omega \), applying the same arguments as in [12], pp. 116–117, because \( \Omega \) can be expressed as a finite union of star-shaped domains. Thus, we can formulate the proposition: there exists a bounded linear operator \( \widetilde{\mathfrak{B}} : W_0^{-1,r}(\Omega) \to L'(\Omega), \) such that \( \text{div} \widetilde{\mathfrak{B}} f = f \) in the sense of distributions in \( \Omega \) for \( f \in W_0^{-1,r}(\Omega) \).

Now, it is just a technical step to extend this proposition from \( f \in W_0^{-1,r}(\Omega) \) to \( f \in W_0^{-1,r}(\Omega) \). This completes the proof. \( \square \)

Remark 4.2. As there is not a complete coincidence on the definition of the divergence of a tensor field in literature, note that if \( F = (F_{ij})(i, j = 1, 2) \) then \( \text{div} F \) in Lemma 4.1 (and also further on throughout the paper) denotes the vector \( \partial_j F_{ij} \) \((i = 1, 2)\). In accordance with this notation, \( \text{div} \mathbf{v} \mathbf{v} \) is the vector with the entries \( \partial_j (\partial_j v_i) = \Delta v_i \) \((i = 1, 2)\).

One can also assume that \( f \in W_0^{-1,r}(\Omega) \) (instead of \( f \in W_0^{-1,r}(\Omega) \)) in Lemma 4.1. However, in this case, one cannot apply [12] in order to obtain the tensorial function \( F \in L'((\Omega)^{2\times 2}) \) with the properties stated in the lemma. Nevertheless, the existence of \( \mathfrak{B} f \), satisfying the equation \( \text{div} F = f \) (in the sense of distributions) and the estimate \( \|F\|_r \leq c \|f\|_{W_0^{-1,r}} \) can be proven in this case, too, just appropriately modifying the proof of Lemma II.1.6.1 in [33], which concerns the case \( r = 2 \).

Proof of Theorem 3.1. The case \( r = 2 \) is proven in [31]. Thus, assume that \( r \neq 2 \). We split the proof to several parts.
(a) Denote by $\| A_r \|_{V_{1,r}^{1/r} \to V_{-1,r}^{1/r}}$ the norm of operator $A_r$. Since

$$\| A_r \|_{V_{1,r}^{1/r} \to V_{-1,r}^{1/r}} = \sup_{v \in V_{1,r}^{1/r}(\Omega), v \neq 0} \frac{\| A_r v \|_{V_{-1,r}^{1/r}}}{\| v \|_{1,r}} \leq \sup_{v \in V_{1,r}^{1/r}(\Omega), v \neq 0} \frac{\| A_r v \|_{V_{-1,r}^{1/r}}}{\| v \|_{1,r}} \| v \|_{1,r} \leq c,$$

the operator $A_r$ is bounded. The identity $D(A_r) = V_{1,r}^{1/r}(\Omega)$ follows from the definition of $A_r$. Operator $A_r$ is closed, as a bounded linear operator, defined on the whole space $V_{1,r}^{1/r}(\Omega)$.

(b) In this part, we consider $F \in V_{-1,2}^{-1}(\Omega)$ of a special form and deal with the equation $A_2 v = F$. Concretely, we assume that $F \in W^{1,2}(\Omega)$ satisfies the condition

$$F(x_1, x_2 + \tau) = F(x_1, x_2) \quad \text{for a.a. } (x_1, x_2) \in \Gamma_0 \quad (4.3)$$

and $F \in V_{-1,2}^{-1}(\Omega)$ is defined by the formula

$$\langle F, w \rangle_{(V_{-1,2}^{-1}, V_{1,2}^{1})} := - \int_{\Omega} F : \nabla w \, dx \quad \text{for all } w \in V_{1,2}^{1}(\Omega). \quad (4.4)$$

Then, due to [31, Lemma 1 and Theorem 2], the equation $A_2 v = F$ has a unique solution $v \in V_{-1,2}^{-1}(\Omega) \cap W^{2,2}(\Omega)$, there exists $p \in W^{1,2}(\Omega)$ (an associated pressure), such that $-\nabla v + p \mathbb{I} - F \in W^{1,2}(\Omega)^2 \times 2$, and $v, p$ satisfy the equation

$$-\Delta v + \nabla p - \text{div} F \equiv \text{div} (-\nabla v + p \mathbb{I} - F) = 0 \quad (4.5)$$

a.e. in $\Omega$. If $r > 2$ then there exists $c_1 > 0$, independent of $v$ and $F$, such that

$$\| v \|_{1,r} \leq c_1 \| F \|_r. \quad (4.6)$$

This estimate is the key part of the proof of Theorem 3.1. As the proof of (4.6) is relatively long, the used technique differs from other sections and we do not want to disturb the logical sequence of arguments, we postpone the derivation of (4.6) to a separate section (Section 5).

(c) In this part, we assume that $r > 2$, $F \in L^r(\Omega)^2 \times 2$ and the functional $F \in V_{-1,r}^{-1}(\Omega)$ is defined, by analogy with (4.4), by the formula

$$\langle F, w \rangle_{(V_{-1,r}^{-1}, V_{1,r}^{1})} := - \int_{\Omega} F : \nabla w \, dx \quad \text{for all } w \in V_{1,r}^{1}(\Omega). \quad (4.7)$$

We prove the solvability of the equation $A_r v = F$.

There exists a sequence $\{F_n\}$ in $W^{1,2}(\Omega)^2 \times 2$, satisfying (4.3), such that $F_n \to F$ in $L^r(\Omega)^2 \times 2$. Let $\{F_n\}$ be a sequence of functionals from $V_{-1,r}^{-1}(\Omega)$, related to $F_n$ through formula (4.7). The functionals $F$ and $F_n$ satisfy

$$\| F_n - F \|_{V_{-1,r}^{-1}} = \sup_{w \in C_0^\infty(\Omega), w \neq 0} \left| \int_{\Omega} (F_n - F) : \nabla w \, dx \right| \leq c \| F_n - F \|_r. \quad (4.8)$$

The restriction of $F_n$, which is a bounded linear functional on $V_{1,r}^{1}(\Omega)$, to $V_{1,r}^{1}(\Omega)$ can be considered to be an element of $V_{-1,2}(\Omega)$, related to $F_n$ through formula (4.4). Thus, due to part (b) of this proof, there exists a sequence $\{v_n\}$ in $V_{-1,2}(\Omega) \cap W^{1,2}(\Omega)$, such that $A_r v_n = F_n$ and

$$\| v_n \|_{1,r} \leq c_1 \| F_n \|_r. \quad (4.9)$$
As the space \( V^{1,r}_\sigma(\Omega) \) is reflexive, there exists a subsequence (which we again denote by \( \{v_n\} \)), such that \( v_n \rightharpoonup v \) in \( V^{1,r}_\sigma(\Omega) \). The functions \( v_n \) satisfy
\[
(\nabla v_n, \nabla w) = (F_n, w)_{(V^{-1,2}_\sigma, V^{1,2}_\sigma)} \quad \text{for all } w \in V^{1,2}_\sigma(\Omega).
\]

If \( w \in V^{1,r'}_\sigma(\Omega) \) then the right-hand side coincides with \( (F_n, w)_{(V^{-1,2}_\sigma, V^{1,2}_\sigma)} \). The space \( V^{1,2}_\sigma(\Omega) \) is dense in \( V^{1,r'}_\sigma(\Omega) \), because \( V^{1,2}_\sigma(\Omega) \) contains \( \mathcal{C}_\infty^\sigma(\Omega) \) and \( \mathcal{C}_\infty^\sigma(\Omega) \) is dense in \( V^{1,r'}_\sigma(\Omega) \). Thus, we also have
\[
(\nabla v_n, \nabla w) = (F_n, w)_{(V^{-1,2}_\sigma, V^{1,2}_\sigma)} \quad \text{for all } w \in V^{1,r'}_\sigma(\Omega).
\]

The limit transition for \( n \to \infty \) yields
\[
(\nabla v, \nabla w) = (F, w)_{(V^{-1,2}_\sigma, V^{1,2}_\sigma)} \quad \text{for all } w \in V^{1,r'}_\sigma(\Omega),
\]
which means that \( A_r v = F \). It also follows from the limit transition that the function \( v \) satisfies inequality (4.6).

(d) Here, we show that each functional \( f \in V_{-1,r}^{-1}(\Omega) \) (for any \( r \in (1, \infty) \)) can be expressed in the form (4.7) for some appropriate \( F \in L^r(\Omega)^{2 \times 2} \). This, together with part (c), shows that if \( r > 2 \) and \( f \in V_{-1,r}^{-1}(\Omega) \), then the equation \( A_r v = f \) is solvable. In other words, \( R(A_r) = V_{-1,r}^{-1}(\Omega) \).

Thus, let \( f \in V_{-1,r}^{-1}(\Omega) \). As \( V_{1,r'}^{-1}(\Omega) \) is a closed subspace of \( W_{1,r'}^{-1}(\Omega) \), the functional \( f \) can be extended (by the Hahn–Banach theorem) to a bounded linear functional \( f^* \in W_{-1,r}^{-1}(\Omega) \), such that \( \|f^*\|_{W_{-1,r}^{-1}} = \|f\|_{V_{-1,r}^{-1}} \). There exists (by Lemma 4.1) \( F \in L^r(\Omega)^{2 \times 2} \), such that \( \text{div} F = f^* \) in the sense of distributions in \( \Omega \) and \( \|F\|_r \leq c \|f^*\|_{W_{-1,r}^{-1}} \equiv \|f\|_{V_{-1,r}^{-1}} \).

As \( f^* = f \) on \( V_{1,r'}^{-1}(\Omega) \), we have
\[
\langle \langle \text{div} F, w \rangle \rangle = \langle \langle f, w \rangle \rangle \quad \forall w \in C_0^{\infty}(\Omega),
\]
where \( \langle \langle \cdot, \cdot \rangle \rangle \) denotes the action of a distribution on a function from \( C_0^{\infty}(\Omega) \). Since the left-hand side is equal to
\[
-\langle \langle F, \nabla w \rangle \rangle = -\int_\Omega F : \nabla w \, dx,
\]
we obtain
\[
\langle f, w \rangle_{(V_{-1,r}^{-1}, V_{1,r'}^{-1})} = \langle \langle f, w \rangle \rangle = -\int_\Omega F : \nabla w \, dx \quad \text{for all } w \in C_0^{\infty}(\Omega).
\]

Both the left-hand and right-hand sides can be continuously extended so that they equal each other for all \( w \in V_{1,r'}^{-1}(\Omega) \). This means that \( f = F \), where \( F \) is related to \( F \) through formula (4.7). It follows from part (c) that there exists \( v \in V_{1,r'}^{-1}(\Omega) \), such that \( A_r v = f \). There exists \( c > 0 \), independent of \( f \) and \( v \), such that the solution satisfies the estimate
\[
\|v\|_{1,r} \leq c \|f\|_{V_{-1,r}^{-1}}. \quad (4.10)
\]

(e) In this part, we derive an information on the adjoint operator \( A_r^* \) to \( A_r \) for any \( r \in (1, \infty) \). The adjoint operator acts from \( V_{-1,r}^{-1}(\Omega)^{**} = V_{1,r'}^{1,r}(\Omega) \) to \( V_{1,r}^{-1}(\Omega) \). However, as \( V_{1,r'}^{1,r}(\Omega) \) is reflexive, \( V_{1,r'}^{-1}(\Omega)^{**} \) can be identified with \( V_{1,r'}^{-1}(\Omega) \). Thus, \( A_r^* \) is an operator from \( V_{1,r'}^{-1}(\Omega) \) to \( V_{-1,r'}^{-1}(\Omega) \). The domain of \( A_r^* \) is, by definition, the set of all \( w \in V_{1,r'}^{-1}(\Omega) \), such that \( \langle A_r z, w \rangle_{(V_{-1,r}^{-1}, V_{1,r'}^{-1})} \) is, in dependence on \( z \), a bounded linear functional on \( V_{1,r}^{-1}(\Omega) \). This functional is exactly \( A_r^* w \) and it satisfies
\[
\langle A_r^* w, z \rangle_{(V_{-1,r}^{-1}, V_{1,r'}^{-1})} = \langle A_r z, w \rangle_{(V_{-1,r}^{-1}, V_{1,r'}^{-1})}.
\]
By definition of $A_r$, the right-hand side equals $(\nabla z, \nabla w)$. This can be also written as $(\nabla w, \nabla z)$ for each fixed $w \in V^{1,r'}_{\sigma}(\Omega)$. From this, we deduce that $D(A^*_r) = V^{1,r'}_{\sigma}(\Omega)$ and $A^*_r = A_r$.

f) Here, we assume that $1 < r < \infty$ and prove the uniqueness of the solution $v$ of the equation $A_r v = f$ for any $f \in V^{-1,r}_{\sigma}(\Omega)$.

Assume at first that $r > 2$. Then, as $V^{1,r}_{\sigma}(\Omega) \subset V^{1,r'}_{\sigma}(\Omega)$, one can use Equation (2.1) with $w = v$ and deduce that $A^*_r v = 0$ implies $v = 0$, which means that the solution of the equation $A_r v = f$ is unique.

Assume now that $1 < r < 2$. Then the null space of $A_r$ and the range of $A_r'$ satisfy the identity $N(A_r) = R(A_r')^\perp$, see [16, p. 168]. However, as $R(A_r')$ is the whole space $V^{-1,r'}_{\sigma}(\Omega)$ (because $r' > 2$ and due to parts (c) and (d) of this proof), the space of annihilators $R(A_r')^\perp$ is trivial. Thus, we obtain the identity $N(A_r) = \{0\}$. This implies the uniqueness of the solution $v$ and the injectivity of operator $A_r$.

(g) Finally, we assume that $1 < r < 2$ and prove the existence of a solution $v$ of the equation $A_r v = f$ for any $f \in V^{-1,r}_{\sigma}(\Omega)$.

Since $R(A_r) = N(A_r') = \{0\}$, $R(A_r)$ is dense in $V^{-1,r}_{\sigma}(\Omega)$, which follows from the inequality $r' > 2$, parts (c) and (d) of this proof and the closed graph theorem. Consequently, due to [16, p. 169], the operator $(A_r)^{-1} \equiv A_r^{-1}$ is bounded from $V^{-1,r}_{\sigma}(\Omega)$ to $V^{1,r}_{\sigma}(\Omega)$. Hence, $A_r$ maps a closed set in $V^{1,r}_{\sigma}(\Omega)$ onto a closed set in $V^{-1,r}_{\sigma}(\Omega)$. It means that $R(A_r)$ is closed in $V^{-1,r}_{\sigma}(\Omega)$. Since it is also dense, we obtain $R(A_r) = V^{-1,r}_{\sigma}(\Omega)$. This completes the proof of Theorem 3.1.

5 | PROOF OF THE INEQUALITY (4.6)

Recall that in part (b) of the proof of Theorem 3.1, we assume that $F \in W^{1,2}(\Omega)^{2 \times 2}$ satisfies (4.3) and the functions $v \in V^{1,2}_{\sigma}(\Omega) \cap W^{2,2}(\Omega)$ and $p \in W^{1,2}(\Omega)$ satisfy Equation (4.5) a.e. in $\Omega$, the boundary condition

$$v = 0$$

on $\Gamma_{in} \cup \Gamma_p$ and the condition of periodicity (1.5) on $\Gamma_0$. It follows from [31, Theorem 2] that $v$ and $p$ also satisfy the conditions of periodicity

$$\frac{\partial v}{\partial n}(x_1, x_2 + \tau) = - \frac{\partial v}{\partial n}(x_1, x_2) \quad \text{for a.a. } x \equiv (x_1, x_2) \in \Gamma_0,$$

$$p(x_1, x_2 + \tau) = p(x_1, x_2) \quad \text{for a.a. } x \equiv (x_1, x_2) \in \Gamma_0$$

and the boundary condition

$$(-\nu \nabla v + p \mathbb{I} - F) \cdot n \equiv -\nu \frac{\partial v}{\partial n} + p n - F \cdot n = 0 \quad \text{a.e. on } \Gamma_{out}.$$  

Extending $v$, $p$, and $F$ $\tau$-periodically in the $x_2$-direction, we deduce that the extended functions (which we again denote by $v$, $p$, and $F$) satisfy Equation (4.5) a.e. in

$$\Omega_{ext} := \Omega \cup \Gamma^0_{in} \cup \Gamma^0_{out} \cup \{x = (x_1, x_2) \in \mathbb{R}^2; \ 0 < x_1 < d, \ x \pm \tau e_2 \in \Omega\}.$$ 

The extended functions satisfy $v \in W^{2,2}(\Omega_{ext})$, $p \in W^{1,2}(\Omega_{ext})$ and $F \in W^{1,2}(\Omega_{ext})$. Note that

$$\partial \Omega_{ext} = (\Gamma_{in})_{ext} \cup (\Gamma_{out})_{ext} \cup (\Gamma_0 - \tau e_2) \cup (\Gamma_1 + \tau e_2) \cup (\Gamma_p)_{ext},$$

where $(\Gamma_{in})_{ext}$ is the part of the boundary of $\Omega_{ext}$ on $\gamma_{in}$, $(\Gamma_{out})_{ext}$ is the part of the boundary of $\Omega_{ext}$ on $\gamma_{out}$ and

$$(\Gamma_p)_{ext} := \Gamma_p \cup (\Gamma_p + \tau e_2) \cup (\Gamma_p - \tau e_2).$$

We assume that $r > 2$ throughout this section and we split the derivation of the estimate (4.6) into two parts, in which we obtain the desired estimate in the interior of $\Omega_{ext}$ plus a neighborhood of $\Gamma_{in} \cup \Gamma_p$ (see Lemma 5.1) and in a neighborhood of $\Gamma_{out}$ (see Lemma 5.2).
Lemma 5.1. Let $\Omega'$ be a sub-domain of $\Omega_{\text{ext}}$, such that the distance between $\Omega'$ and any of the curves $\Gamma_0 - e_2$, $\Gamma_1 + e_2$, $\Gamma_p - e_2$, $\Gamma_p + e_2$, and $(\Gamma_{\text{out}})_{\text{ext}}$ is positive. Then $v$ satisfies the estimate
\[ \|v\|_{1,r;\Omega'} \leq c \|F\|_r, \quad (5.5) \]
where $c = c(\nu, \Omega, \Omega')$.

Proof. We may assume, without loss of generality, that $\Gamma_p \subset \partial \Omega'$. There exists $\rho > 0$ so small that $U_{\rho}(\Omega') := \{ x = (x_1, x_2) \in \mathbb{R}^2 \setminus P; \, 0 < x_1 < d, \dist(x, \Omega') < \rho \}$ is a subset of $\Omega_{\text{ext}}$. (Recall that $P$ is a compact set, see Figure 1.) There exists an infinitely differentiable function $\eta$ in $\Omega_{\text{ext}}$, such that $\eta = 1$ in $\Omega'$, $\text{supp} \eta \subset U_{\rho}(\Omega')$ and $\Omega'' := \text{supp} \eta$ is of the class $C^2$. Denote $\tilde{v} := \eta v$ and $\tilde{p} := \eta p$. The functions $\tilde{v}, \tilde{p}$ represent a strong solution of the problem
\[ -\nu \Delta \tilde{v} + \nabla \tilde{p} = \tilde{f} \quad \text{in} \ \Omega'', \quad (5.6) \]
\[ \div \tilde{v} = \tilde{h} \quad \text{in} \ \Omega'', \quad (5.7) \]
\[ \tilde{v} = 0 \quad \text{on} \ \partial \Omega'', \quad (5.8) \]
where
\[ \tilde{f} := \eta \div F - 2\nu \nabla \eta \cdot \nabla v - \nu (\Delta \eta)v - (\nabla \eta)p \quad \text{and} \quad \tilde{h} := \nabla \eta \cdot v. \]

Applying Proposition I.2.3 from [34], p. 35, we obtain the estimate
\[ \|\tilde{v}\|_{1,r;\Omega''} \leq c (\|\tilde{f}\|_{W^{-1,r}} + \|\tilde{h}\|_r). \quad (5.9) \]
Since $L^2(\Omega) \hookrightarrow W^{-1,r}(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L'(\Omega)$, we can estimate the terms on the right-hand side as follows:
\[ \|\tilde{f}\|_{W^{-1,r}} \leq c \|F\|_r + c \|v\|_r + c \|p\|_{-1,r} \leq c \|F\|_r + c \|v\|_{1,2} + c \|p\|_2, \]
\[ \|\tilde{h}\|_r \leq c \|v\|_r \leq c \|v\|_{1,2}. \]
Due to [31, Lemma 1 and estimate (2.5)], we have $\|v\|_{1,2} + \|p\|_2 \leq c \|F\|_{W^{-1,r}} \leq c \|F\|_r$. Substituting these estimates to (5.9), we obtain: $\|\tilde{v}\|_{1,r;\Omega''} \leq c \|F\|_r$. Since $\Omega' \subset \Omega''$ and $\eta = 1$ on $\Omega'$, we obtain (5.5). \[ \square \]

Lemma 5.2. Let $\Omega'$ be a sub-domain of $\Omega_{\text{ext}}$, such that $\Omega'$ has a positive distance from any of the sets $\Gamma_0 - e_2$, $\Gamma_1 + e_2$, $(\Gamma_{\text{in}})_{\text{ext}}$, and $(\Gamma_p)_{\text{ext}}$. Then $v$ satisfies the estimate
\[ \|v\|_{1,r;\Omega'} \leq c \|F\|_r, \quad (5.10) \]
where $c = c(\nu, \Omega, \Omega')$.

Proof. Since $\div v = 0$, there exists $\varphi \in W^{3,2}(\Omega_{\text{ext}})$, such that $v \equiv (v_1, v_2) = \nabla^2 \varphi \equiv (-\partial_2 \varphi, \partial_1 \varphi)$. As $v = 0$ on $(\Gamma_{\text{in}})_{\text{ext}}$, the function $\varphi$ satisfies $\nabla^2 \varphi = 0$ on $(\Gamma_{\text{in}})_{\text{ext}}$. Since $\varphi$ is determined uniquely up to an additive constant, we can choose $\varphi$ so that
\[ \varphi = 0 \quad \text{and} \quad \nabla \varphi = 0 \quad \text{on} \ (\Gamma_{\text{in}})_{\text{ext}}. \quad (5.11) \]
Put
\[ \mathbb{Z} := -\nu \nabla v + pI - F. \]
Thus, if we denote by \( Z_{ij} \) \((i, j = 1, 2)\) the entries of \( Z \) and by \( F_{ij} \) \((i, j = 1, 2)\) the entries of \( F \), we can write this formula in the form
\[
\begin{pmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{pmatrix}
= -\nu
\begin{pmatrix}
\partial_1 v_1 & \partial_2 v_1 \\
\partial_1 v_2 & \partial_2 v_2
\end{pmatrix}
+ \begin{pmatrix}
p & 0 \\
0 & p
\end{pmatrix}
- \begin{pmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{pmatrix}
\]
\[
= -\nu
\begin{pmatrix}
-\partial_2 \varphi & -\partial_2 \varphi \\
\partial_1 \varphi & \partial_1 \varphi
\end{pmatrix}
+ \begin{pmatrix}
p & 0 \\
0 & p
\end{pmatrix}
- \begin{pmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{pmatrix}.
\]

Equation (4.5) says that \( \text{div} Z = 0 \), which means that \( \partial_j Z_{ij} = 0 \) for \( i = 1, 2 \). Hence there exist functions \( \psi_1, \psi_2 \in W^{2,2}(\Omega_{\text{ext}}) \), such that \( Z_{11} = -\partial_2 \psi_1, Z_{12} = \partial_1 \psi_1, Z_{21} = -\partial_2 \psi_2, Z_{22} = \partial_1 \psi_2 \). Thus, we obtain the equation
\[
\begin{pmatrix}
-\partial_2 \psi_1 & \partial_1 \psi_1 \\
-\partial_2 \psi_2 & \partial_1 \psi_2
\end{pmatrix}
= -\nu
\begin{pmatrix}
-\partial_2 \varphi & -\partial_2 \varphi \\
\partial_1 \varphi & \partial_1 \varphi
\end{pmatrix}
+ \begin{pmatrix}
p & 0 \\
0 & p
\end{pmatrix}
- \begin{pmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{pmatrix}.
\]

This tensorial equation can also be considered to be a system of four equations for four unknowns: \( \varphi, p, \psi_1 \) and \( \psi_2 \). Since \( \mathbf{n} = \mathbf{e}_1 \) on \( \Gamma_{\text{out}} \), the boundary condition (5.4) yields \( Z_{11} = Z_{12} = 0 \) on \( \Gamma_{\text{out}} \). This means that \( \partial_2 \psi_1 = \partial_2 \psi_2 = 0 \) on \( \Gamma_{\text{out}} \), which implies that \( \psi_1 \) and \( \psi_2 \) are constant on \( \Gamma_{\text{out}} \). Let us denote the constants by \( k_1 \) and \( k_2 \). Thus, we obtain the boundary conditions
\[
\psi_1 = k_1, \quad \psi_2 = k_2 \quad \text{on} \ (\Gamma_{\text{out}})_{\text{ext}}.
\]

Let us concretely choose \( k_1 \) and \( k_2 \) so that
\[
\int_\Omega \psi_1 \, dx = \int_\Omega \psi_2 \, dx = 0.
\]

Later on in this proof, we shall need estimates of \( \| \psi_1 \|_r \) and \( \| \psi_2 \|_r \). Let us begin with \( \| \psi_1 \|_r \):
\[
\| \psi_1 \|_r = \sup_{w \in L^{r'}(\Omega); w \neq 0} \left| \int_\Omega \psi_1 w \, dx \right| \| w \|^{-1}_{r'}.
\]

The function \( w \in L^{r'}(\Omega) \) can be written in the form \( w = \overline{w} + w' \), where \( \overline{w} = \int_\Omega w \, dx \) and \( \int_\Omega w' \, dx = 0 \). Then, obviously, \( \| w' \|_{r'} \leq c \| w \|_{r'} \), where \( c = c(\Omega) \). Moreover, there exists \( z \in W^{1,r}_0(\Omega) \), such that \( \text{div} \, z = w' \) and \( \| z \|_{1,r} \leq c \| w' \|_{r'} \), see [10, Theorem III.3.3]. Hence,
\[
\| \psi_1 \|_r \leq c \sup_{w \in L^{r'}(\Omega); w \neq 0} \left| \int_\Omega \psi_1 w \, dx + \int_\Omega \psi_1 w' \, dx \right| \| w' \|^{-1}_{r'}.
\]

As \( W^{1,r}_0(\Omega) \subset L^2(\Omega) \), we also have \( L^2(\Omega) \subset W^{-1,r}(\Omega) \). Hence \( \| p \|_{W^{-1,r}} \leq c \| p \|_2 \). Furthermore, as \( W^{1,2}(\Omega) \subset L'(\Omega) \), we also have \( \| v \|_{r} \leq c \| v \|_{1,2} \). Thus, (5.15) yields
\[
\| \psi_1 \|_r \leq c \left( \| v \|_{1,2} + \| p \|_2 + \| F \|_r \right).
\]

Due to [31, Lemma 1 and estimate (2.5)], the right-hand side is less than or equal to \( c \left( \| F \|_{W^{1,2}_r} + \| F \|_r \right) \leq c \left( \| F \|_2 + \| F \|_r \right) \), which is less than or equal to \( c \| F \|_r \). Substituting this to (5.16) and taking into account that \( \psi_2 \) can be estimated in the
same way, we finally obtain
\[ \| \psi_1 \|_r + \| \psi_2 \|_r \leq c \| F \|_r. \]  
(5.17)

In order to obtain the desired estimate (4.6), we will apply Theorem 10.2 from the paper [2] by Agmon et al. This theorem, however, concerns a boundary value problem in a half-space, which means a half-plane in the case of a problem in 2D. In order to transform the problem (5.12), (5.13) to a problem in the half-plane \( x_1 < d \), we apply the cut-off function technique; there exists \( \rho > 0 \) so small that
\[ U_\rho(\Omega') := \{ x = (x_1, x_2) \in \mathbb{R}^2; \ x_1 < d, \ \text{dist}(x, \Omega') < \rho \} \]
is a subset of \( \Omega_{ext} \). Let \( \eta \) be an infinitely differentiable function in \( \Omega_{ext} \), such that \( \eta = 1 \) in \( \Omega' \) and \( \eta = 0 \) in \( \Omega_{ext} \setminus U_\rho(\Omega') \). Multiplying (5.12) by \( \eta \), and denoting \( \tilde{\varphi} := \eta \varphi, \tilde{\psi}_1 := \eta \psi_1, \tilde{\psi}_2 := \eta \psi_2, \) and \( \tilde{p} := p \), we get
\[ \left( -\partial_2 \tilde{\psi}_1, \partial_1 \tilde{\psi}_1 \right) + \nu \left( -\partial_2 \tilde{\varphi}, \partial_1 \tilde{\varphi} \right) - \left( \tilde{p}, 0 \right) \]
\[ = -\left( \eta F_{11}, \eta F_{12} \right) + \left( -\eta \partial_2 \eta \psi_1, (\partial_1 \eta) \psi_1 \right) + \left( -\eta \partial_2 \eta \psi_2, (\partial_1 \eta) \psi_2 \right) + \left( -\eta \partial_2 \eta \varphi, -(\partial_2 \eta) \varphi \right) + \frac{2(\eta \partial_2 \eta) \psi_1}{2(\partial_1 \eta)(\partial_2 \varphi)} - 2(\partial_2 \eta)(\partial_2 \varphi). \]
\[ (5.18) \]

Since \( \eta \) is supported in the closure of \( U_\rho(\Omega') \), we can treat (5.18) as a system of four equations in the half-plane \( x_1 < d \) for the unknowns \( \tilde{\varphi}, \tilde{\psi}_1, \tilde{\psi}_2, \) and \( \tilde{p} \) with the boundary conditions
\[ \tilde{\psi}_1 = \eta k_1, \quad \tilde{\psi}_2 = \eta k_2 \quad \text{on } \gamma_{out}. \]
\[ (5.19) \]

In order to simplify Equation (5.18) and to use a notation, consistent with [2], we eliminate \( \tilde{p} \) by subtracting the two corresponding equations and denote \( u_1 := \varphi, u_2 := \psi_1, \) and \( u_3 := \psi_2 \). Then the problem (5.18) and (5.19) reduces to the system of three equations
\[ 2 \nu \partial_1 \partial_2 u_1 + \partial_2 u_2 + \partial_1 u_3 = -\eta (F_{22} - F_{11}) + (\partial_1 \eta) \psi_2 + (\partial_2 \eta) \psi_1 + 2(\partial_1 \eta) \varphi \]
\[ + 2(\partial_1 \eta)(\partial_2 \varphi) + 2(\partial_2 \eta)(\partial_1 \varphi), \]
\[ (5.20) \]
\[ \nu \partial_2 u_1 - \partial_1 u_3 = -\eta F_{21} - (\partial_2 \eta) \psi_2 + (\partial_1 \eta) \varphi + 2(\partial_1 \eta)(\partial_1 \varphi), \]
\[ (5.21) \]
\[ \nu \partial_2 u_1 - \partial_1 u_2 = -\eta F_{12} - (\partial_1 \eta) \psi_1 + (\partial_2 \eta) \varphi + 2(\partial_2 \eta)(\partial_2 \varphi) \]
\[ (5.22) \]

with the boundary conditions
\[ u_2 = \eta k_1, \quad u_3 = \eta k_2 \quad \text{on } \gamma_{out}. \]
\[ (5.23) \]

Obviously, the leading differential operator in (5.20)–(5.22) is
\[ \mathcal{L}(\partial_1, \partial_2) := \begin{pmatrix} 2 \nu \partial_1 \partial_2, & \partial_2, & \partial_1, \\ \nu \partial_1^2, & 0, & -\partial_2, \\ \nu \partial_2^2, & -\partial_1, & 0 \end{pmatrix}. \]

The determinant of \( \mathcal{L}(\xi_1, \xi_2) \) equals \( -\nu (\xi_1^2 + \xi_2^2)^2 \), which is a polynomial of degree \( 2m = 4 \). With this information, one can verify that the system (5.20)–(5.22) is “uniformly elliptic” of order 4 in the sense of Agmon–Dougls–Nirenberg, which means that it satisfies the conditions (1.1)–(1.7) from [2], pp. 38, 39, and the so called “supplementary condition,” see [2, p. 39]. Moreover, the boundary conditions (5.23), the number of which is \( m = 2 \), satisfy the condition (2.2) from [2, p. 42].
and have all required properties of the so called “complementing boundary conditions,” formulated in [2], p. 42 and 43. The verification is elementary, however technical, hence we do not provide the details here.

Note, that one can also find the description and explanation of the conditions that enable one to call a considered system “uniformly elliptic” and considered boundary conditions to be “complementing” in [4, Appendix D].

Denote by $\mathbb{R}^2_{d-}$ the half-plane $x_1 < d$.

The inclusions $u_1 \in W^{3,2}(\mathbb{R}^2_{d-}) \hookrightarrow W^{2,2}(\mathbb{R}^2_{d-})$, $u_2, u_3 \in W^{2,2}(\mathbb{R}^2_{d-}) \hookrightarrow W^{1,r}(\mathbb{R}^2_{d-})$ and the verification of the aforementioned conditions from [2] enable us to apply Theorem 10.2 from [2], which yields the estimate

$$
\|u_1\|_{2,r;\mathbb{R}^2_{d-}} + \|u_2\|_{1,r;\mathbb{R}^2_{d-}} + \|u_3\|_{1,r;\mathbb{R}^2_{d-}} \leq \sum_{i=1}^3 \|\tilde{f}_i\|_{r;\mathbb{R}^2_{d-}} + c \left( \|\eta k_1\|_{1-1/r,\gamma_{out}} + \|\eta k_2\|_{1-1/r,\gamma_{out}} \right) 
$$

(5.24)

where $\tilde{f}_1$, $\tilde{f}_2$, and $\tilde{f}_3$ denote the right-hand sides of Equations (5.20), (5.21), and (5.22), respectively, and $c$ is independent of $u_i$, $\tilde{f}_i$ ($i = 1, 2, 3$) and $k_1, k_2$. The first term on the right-hand side of (5.24) can be estimated using the inequalities

$$
\sum_{i=1}^3 \|\tilde{f}_i\|_{r;\mathbb{R}^2_{d-}} \leq c \left( \|F\|_r + \|\psi_1\|_r + \|\psi_2\|_r + \|\varphi\|_1,r \right).
$$

(5.25)

The norms $\|\psi_1\|_r$ and $\|\psi_2\|_r$ can be estimated using (5.17). The norm $\|\varphi\|_1,r$ can be estimated as follows:

$$
\|\varphi\|_{1,r} \leq c \|v\|_r \leq c \|v\|_{1,2} \leq c \|F\|_{\mathcal{V}^{-1,2}} \leq c \|F\|_2 \leq c \|F\|_r.
$$

(5.26)

(Here, the first inequality holds due to the definition of $\varphi$ and (5.11), the second inequality follows from the continuous imbedding $W^{1,2}(\Omega) \hookrightarrow L^r(\Omega)$, the third inequality follows from [31, Lemma 1], the fourth inequality follows from the definition of $F$ and the last inequality holds, because $r > 2$.) The second term on the right-hand side of (5.24) can be estimated using these inequalities:

$$
|k_1| + |k_2| = |\Omega|^{-1} \left| \int_{\Omega} (\psi_1 - k_1) \, dx \right| + |\Omega|^{-1} \left| \int_{\Omega} (\psi_2 - k_2) \, dx \right|
\leq c \|\nabla (\psi_1 - k_1)\|_2 + c \|\nabla (\psi_2 - k_2)\|_2 \leq c \|Z\|_2 \leq c \left( \|\nabla v\|_2 + \|p\|_2 + \|F\|_2 \right)
\leq c \|F\|_{\mathcal{V}^{-1,2}} + c \|F\|_2 \leq c \|F\|_2 \leq c \|F\|_r.
$$

(5.27)

(The inequalities $\int_{\Omega} \psi_j \, dx \leq c \|\nabla (\psi_j - k_j)\|_2$ (for $j = 1, 2$) hold due to (5.13). The first inequality on the last line follows from Lemma 1 and estimate (2.5) in [31] and the last estimate follows from the condition $r > 2$.) Estimates (5.24)–(5.27) now yield

$$
\|\varphi\|_{2,r;\Omega'} \leq \|\tilde{\varphi}\|_{2,r;\mathbb{R}^2_{d-}} = \|u_1\|_{2,r;\mathbb{R}^2_{d-}} \leq c \|F\|_r
$$

Since $\|v\|_{1,r;\Omega'} \leq c \|\varphi\|_{2,r;\Omega'}$, we obtain inequality (5.10).

6 | PROOFS OF THEOREMS 3.3 and 3.4

In order to prove Theorem 3.3, we shall need the next lemma. It is a modification of a result, proven in [9, Section 3].
Lemma 6.1. Let $1 < r < \infty$ and $g \in W^{s,r}(\Gamma_{in})$, where $s > 1/r$ if $1 < r \leq 2$ and $s = 1 - 1/r$ if $r > 2$. Let $g$ satisfy the condition $g(A_0) = g(A_1)$. There exists a divergence-free extension $g_*$ of $g$ from $\Gamma_{in}$ to $\Omega$ and a constant $c_2 > 0$, independent of $g$, such that $g_* \in W^{1,r}(\Omega)$, $g_*=0$ on $\Gamma_p$ in the sense of traces,

a) $\|g_*\|_{1,r} \leq c_2 \|g\|_{s,r;\Gamma_{in}}$.

b) $g_*$ satisfies the condition of periodicity (1.5) in the sense of traces on $\Gamma_0 \cup \Gamma_1$.

c) $g_*= (\Phi/r)e_1$ in a neighborhood of $\Gamma_{out}$, where $\Phi = -\int_{\Gamma_{in}} g \cdot n \, dl$.

Principles of the proof. The assumptions on number $s$ guarantee that $rs > 1$ and it makes therefore sense to speak about the traces of the function $g$ at the end points $A_0$ and $A_1$ of $\Gamma_{in}$. This enables one to extend at first $g$ from $\Gamma_{in}$ to $\partial \Omega$ in the way, described in [8, subsection 3.2]. The extended function (which is again denoted by $g$) satisfies the condition of periodicity (1.5), the condition $g=0$ on $\Gamma_p$ and the equality $\int_{\partial \Omega} g \cdot n \, dl = 0$. The inequality $\|g\|_{1-1/r,r;\partial \Omega} \leq c \|g\|_{s,r;\Gamma_{in}}$ can be proven by analogy with [8, Lemma 1], expressing the norm of $g$ in $W^{s,r}(\Gamma_{in})$ by formulas (2) and (3) in [21], p. 386, and the norm of $g$ in $W^{1-1/r,r}(\partial \Omega)$ by formulas (II.4.8) and (II.4.9) in [10], p. 64.

The existence of a divergence-free extension $g_* \in W^{1,r}(\Omega)$ of function $g$ from $\partial \Omega$ to $\Omega$ follows from [10, Exercise III.3.5]. The extension satisfies the estimate $\|g_*\|_{1,r} \leq c \|g\|_{1-1/r,r;\partial \Omega} \leq c \|g\|_{s,r;\partial \Omega}$.

The function $g_*$ can be further modified in the way described in [9, Subsection 3.3], so that it finally has the property c), too. Note that $r=2$ in [9]. However, considering a general exponent $r \in (1, \infty)$ does not affect the used procedure and the resulting estimates.

Proof of Theorem 3.3. Note that if $g$ satisfies the assumptions of Lemma 6.1 then $g$ also lies in $W^{1-1/r,r}_{per}(\Gamma_{in})$.

Let $F$ be a tensor function in $L^r(\Omega)^{2\times2}$, provided by Lemma 4.1, and $F \in V_{\sigma}^{-1,r}(\Omega)$ be the functional in $V_{\sigma}^{-1,r}(\Omega)$, defined by the formula (4.7). The norm of $F$ in $V_{\sigma}^{-1,r}(\Omega)$ satisfies the estimate $\|F\|_{V_{\sigma}^{-1,r}} \leq c \|F\|_{L^r(\Omega)} \leq c \|F\|_{W^{1-1/r,r}}$. Furthermore, let $g_*$ be the function, given by Lemma 6.1. Define a functional $G \in V_{\sigma}^{-1,r}(\Omega)$ by the formula

$$\langle G, w \rangle_{V_{\sigma}^{-1,r},V_{\sigma}^{1,r'}} := -\int_{\Omega} \nabla g_* : \nabla w \, dx \quad \text{for all } w \in V_{\sigma}^{1,r'}(\Omega). \quad (6.1)$$

Then $G$ satisfies $\|G\|_{V_{\sigma}^{-1,r}} \leq c \|\nabla g_*\|_{L^r(\Omega)} \leq c \|g_*\|_{s,r;\Gamma_{in}}$. Finally, let $H$ be a functional in $V_{\sigma}^{-1,r}(\Omega)$, defined by the formula

$$\langle H, w \rangle_{V_{\sigma}^{-1,r},V_{\sigma}^{1,r'}} := -\langle h, w \rangle_{(W_{per}^{-1/r,r}(\Gamma_{in})),(W_{per}^{1/r',r}(\Gamma_{out}))} \quad \text{for all } w \in V_{\sigma}^{1,r'}(\Omega). \quad (6.2)$$

Obviously, $H$ satisfies the estimate $\|H\|_{V_{\sigma}^{-1,r}} \leq c \|h\|_{W^{1-1/r,r}(\Gamma_{out})}$.

Due to Theorem 3.1, the equation

$$\nu A_\sigma v = F + \nu G + H \quad (6.3)$$

has a unique solution $v \in V_{\sigma}^{1,r}(\Omega)$, which satisfies

$$\|\nabla v\|_{r} \leq c (\|F\|_{V_{\sigma}^{-1,r}} + \|G\|_{V_{\sigma}^{-1,r}} + \|H\|_{V_{\sigma}^{-1,r}}), \quad (6.4)$$

where $c = c(\nu, \Omega, r)$. Equation (6.3) implies that

$$\nu \int_{\Omega} \nabla v : \nabla w \, dx = -\int_{\Omega} F : w \, dx - \nu \int_{\Omega} \nabla g_* : \nabla w \, dx - \langle h, w \rangle_{(W_{per}^{-1/r,r}(\Gamma_{in})),(W_{per}^{1/r',r}(\Gamma_{out}))}$$

for all $w \in V_{\sigma}^{1,r'}(\Omega)$. Put $u := v + g_*$. One can now easily verify that $u$ has all properties, stated in Definition 3.2, which means that $u$ is a weak solution to the problem (1.1)–(1.6). The estimate (3.2) follows from (6.4) and from the estimates of the norms of the functionals $F, G$ and $H$ in $V_{\sigma}^{-1,r}(\Omega)$. It implies the uniqueness of the weak solution $u$.

Proof of Theorem 3.4. Suppose at first that the test function $w$, used in (3.1), is in $C^\infty_{\sigma,\partial \Omega}(\Omega)$. Then, since $w=0$ on $\Gamma_{out}$, (3.1) and the equation $f = \text{div} F$ imply that

$$\langle \nu A u + \text{div} F, w \rangle = 0.$$
As this holds for all \( \mathbf{w} \in C_\infty^0(\Omega) \), we can apply De Rham’s lemma (see [34, p. 14]) and deduce that there exists a distribution \( p_0 \) in \( \Omega \), such that the equation

\[ \nu \Delta \mathbf{u} + \text{div} \mathbf{F} = \nabla p_0 \]  

(6.5)

holds in \( \Omega \) in the sense of distributions. As both \( \nu \Delta \mathbf{u} \) and \( \text{div} \mathbf{F} \) can also be identified with elements of \( W^{-1,r}(\Omega) \), \( \nabla p_0 \) belongs to \( W^{-1,r}(\Omega) \), too. It follows from [10, Lemma IV.1.1] that \( p_0 \) is such that the equation

\[ \int_{\Omega} \nu \Delta \mathbf{u} : \nabla \mathbf{w} \, d\mathbf{x} \leq c \| \nu \Delta \mathbf{u} + \text{div} \mathbf{F} \|_{W^{-1,r}} \]  

(6.6)

where \( c \) depends only on \( r, \nu \), and \( \Omega \).

Since \( -\nu \nabla \mathbf{u} - p_0 \mathbf{I} - \mathbf{F} \in L^r(\Omega) \) and \( \text{div}(-\nu \nabla \mathbf{u} + p_0 \mathbf{I} - \mathbf{F}) = 0 \) in the sense of distributions, we can apply Theorem III.2.2 from [10] and deduce that \( (\nu \nabla \mathbf{u} + p_0 \mathbf{I} - \mathbf{F}) \cdot \mathbf{n} \in W^{-1/r,r}_{\text{per}}(\partial \Omega) \), that is it holds in the sense of traces.

Let \( \mathbf{w} \in C_\infty^2(\Omega) \). Equation (3.1) and the generalized Gauss identity (see [10, p. 160]) imply that

\[ 0 = \left\langle \text{div}[\nu \nabla \mathbf{u} + \mathbf{F} - p_0 \mathbf{I}], \mathbf{w} \right\rangle_{(W^{-1/r,r}_{\text{per}}(\partial \Omega), W^{1/r',r'}_{\text{per}}(\partial \Omega))} \]

\[ = \left\langle (\nu \nabla \mathbf{u} + p_0 \mathbf{I}) \cdot \mathbf{n}, \mathbf{w} \right\rangle_{(W^{-1/r,r}_{\text{per}}(\partial \Omega), W^{1/r',r'}_{\text{per}}(\partial \Omega))} + \left\langle \mathbf{h}, \mathbf{w} \right\rangle_{(W^{-1/r,r}_{\text{per}}(\partial \Omega), W^{1/r',r'}_{\text{per}}(\partial \Omega))} \]

\[ - \left\langle \mathbf{h}, \mathbf{w} \right\rangle_{(W^{-1/r,r}_{\text{per}}(\partial \Omega), W^{1/r',r'}_{\text{per}}(\partial \Omega))} - \int_{\Omega} [\nu \nabla \mathbf{u} + \mathbf{F}] : \nabla \mathbf{w} \, d\mathbf{x} \]

The set of traces of all functions from \( C_\infty(\Omega) \) on \( \Gamma_{\text{out}} \) is dense in the set of all functions \( \mathbf{w} \in W^{-1/r,r}_{\text{per}}(\partial \Omega) \), such that \( \int_{\Gamma_{\text{out}}} \mathbf{w} : \mathbf{n} \, d\mathbf{l} = 0 \). (This follows from the density of the space of all functions \( \mathbf{w} \in C_\infty(\Gamma_{\text{out}}) \), such that \( \int_{\Gamma_{\text{out}}} \mathbf{w} : \mathbf{n} \, d\mathbf{l} = 0 \), and from the possibility of extension of any function in \( C_\infty(\Gamma_{\text{out}}) \) to \( \Omega \) so that the extended function is in \( C_\infty(\Omega) \).) Hence there exists \( c_3 \in \mathbb{R} \) such that \( \mathbf{u} \) and \( p_0 \) satisfy

\[ (\nu \nabla \mathbf{u} + p_0 \mathbf{I}) \cdot \mathbf{n} + \mathbf{h} = c_3 \mathbf{n}, \]  

(6.7)

as an equality in \( W^{-1/r,r}_{\text{per}}(\Gamma_{\text{out}}) \). Put \( p := p_0 + c_3 \). Then \( \mathbf{u} \) and \( p \) satisfy Equation (1.1) in the sense of distributions in \( \Omega \) and the boundary condition (3.3) as an equality in \( W^{-1/r,r}_{\text{per}}(\Gamma_{\text{out}}) \). The inequality (3.4) follows from (3.2), (6.6), the formula

\[ p = p_0 + c_3 \]  

and from (6.2). This completes the proof of part 1) of the theorem. In order to prove part 2), we shall need the next lemma.

Denote by \( W^{1/r,r}_{\text{per}}(\Omega) \) the space of functions from \( W^{1/r}(\Omega) \), that satisfy in the sense of traces the condition of periodicity on \( \Gamma_0 \) and \( \Gamma_1 \), analogous to (1.5).

Lemma 6.2. Let \( \mathbf{f} \in L'(\Omega) \) be given. Then there exists \( \mathbf{F} \in W^{1/r,r}_{\text{per}}(\Omega)^{2 \times 2} \), such that \( \text{div} \mathbf{F} = \mathbf{f} \) a.e. in \( \Omega \), \( \mathbf{F} \cdot \mathbf{n} = 0 \) a.e. on \( \Gamma_{\text{out}} \) in the sense of traces and

\[ \| \mathbf{F} \|_{1,r} \leq c \| \mathbf{f} \|_r, \]  

(6.8)

where \( c = c(\Omega, r) \).

Proof. Denote \( \tilde{\Omega} := \Omega \cup P \). Then \( |\tilde{\Omega}| = \tau d \). Define \( \mathbf{f} := 0 \) in \( P \). Thus, \( \mathbf{f} \in L'(\tilde{\Omega}) \). Put \( \mathbf{k} := |\tilde{\Omega}|^{-1} \int_{\tilde{\Omega}} \mathbf{f} \, d\mathbf{x} \).

Let \( \zeta = \zeta(x_1) \) be a smooth real function in \([0, d]\), such that \( \zeta(0) = 1 \) and \( \zeta \) is supported in \([0, \delta]\), where \( \delta > 0 \) is so small that the profile \( P \) (see Figure 1) lies in the strip \( \delta < x_1 < d \). Since \( \int_{\tilde{\Omega}} \zeta'(x_1) \, dx = -\tau \), we have \( \int_{\tilde{\Omega}} (\mathbf{f} + \mathbf{k} \zeta'(x_1)) \, dx = 0 \). Thus, due to [10, Theorem III.3.3], there exists \( \mathbf{F}_0 \in W^{1/r,r}_{\text{per}}(\tilde{\Omega})^{2 \times 2} \), such that \( \text{div} \mathbf{F}_0 = \mathbf{f} + \mathbf{k} \zeta' \) a.e. in \( \tilde{\Omega} \) and

\[ \| \mathbf{F}_0 \|_{1,r,\tilde{\Omega}} \leq c \| \mathbf{f} + \mathbf{k} \zeta' \|_{r,\tilde{\Omega}} \leq c \| \mathbf{f} \|_r, \]
where \( c = c(\tau, d, \zeta) \). Since \( \mathbf{k} d \zeta'(x_1) = \text{div} [\mathbf{k} d \zeta(x_1) \otimes \mathbf{e}_1] \), \( F_0 \) satisfies
\[
\text{div} [F_0 - \mathbf{k} d \zeta(x_1) \otimes \mathbf{e}_1] = f
\]
a.e. in \( \Omega \). Put \( F := F_0 - \mathbf{k} d \zeta(x_1) \otimes \mathbf{e}_1 \). The tensor function \( F \) has all the properties, stated in the lemma.

The statements in part 2) of Theorem 3.4 now follow from Lemma 6.2. The proof of Theorem 3.4 is completed.

**Remark 6.3.** The identification of the right-hand side of (1.1) with \( \text{div} F \) enables us to deduce that \( u \) and \( p \) satisfy (3.3), and particularly also (3.5), which are equalities in \( W_{\text{per}}^{-1/r,r}(\Gamma_{\text{out}}) \).

If \( f \in W_{0}^{-1,2}(\Omega) \) and \( F \) is only in \( L^{r}(\Omega)^{2 \times 2} \), as in part 1) of Theorem 3.4, then \( (-\nu \nabla u + p I - F) \cdot \mathbf{n} \) cannot be written as a difference \( (-\nu \nabla u + p I - F) \cdot \mathbf{n} - F \cdot \mathbf{n} \), because neither \( (-\nu \nabla u + p I) \cdot \mathbf{n} \), nor \( F \cdot \mathbf{n} \) need not be in \( W_{\text{per}}^{-1/r,r}(\Gamma_{\text{out}}) \). Thus, the sole term \( F \cdot \mathbf{n} \) need not have a sense on \( \Gamma_{\text{out}} \) and it is therefore generally not possible to require \( F \) to satisfy \( F \cdot \mathbf{n} = 0 \) on \( \Gamma_{\text{out}} \). The situation is different if \( f \in L^{r}(\Omega) \), see part 2) of Theorem 3.4.

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