Integrable Low Dimensional Models for Black Holes and Cosmologies from High Dimensional Theories

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Abstract

We describe a class of integrable models of 1+1 and 1-dimensional dilaton gravity coupled to scalar fields. The models can be derived from high dimensional supergravity theories by dimensional reductions. The equations of motion of these models reduce to systems of the Liouville equations endowed with energy and momentum constraints. We construct the general solution of the 1+1 dimensional problem in terms of chiral moduli fields and establish its simple reduction to static black holes (dimension 0+1), and cosmological models (dimension 1+0). We also discuss some general problems of dimensional reduction and relations between static and cosmological solutions.

1 Introduction

It is now long time since 1+1 and 0+1 (or, 1+0) dimensional dilaton gravity (DG) coupled to scalar matter fields proved to be a reliable model for high dimensional (HD) black holes (BHs), cosmological models (CMs), and branes (BRs).

The connection between high and low dimensions has been demonstrated in different contexts of gravity and string theory - symmetry reduction, compactification, holographic principle, AdS/CFT correspondence, duality (see e.g. [11] - [13]). For spherically symmetric configurations the description

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of static BHs, BRs and CMs even simplifies to 1-dimensional DG – matter models, often analytically integrable (see e.g. [7], [10] - [16]).

However, generally DG models coupled to matter are not integrable. Recently A.T. Filippov has proposed a general class of integrable 1+1 dimensional DG models that reduce to $N$ Liouville equations (a brief summary without proof is published in Refs. [15], [16]) and has found the general analytic solution of the constraints. These models are closely related to physically interesting solutions of higher dimensional supergravity (SG) theories describing the low energy limit of superstring theories.

A characteristic feature of the static solutions of the models derived from string theory is the existence of horizons with nontrivial scalar field distributions. In fact, it is well known that in the Einstein - Maxwell theories minimally coupled to scalar fields the spherical static horizons disappear if the scalar fields have a nontrivial space distribution (this is the ‘no-hair theorem’; we call ‘no horizon theorem’ its local version). However, the ‘no horizon’ theorem is not true for Einstein - Yang-Mills theories [17] as well as for solutions of HD supergravity theories, see e.g. [10] - [13].

In Section 2 we demonstrate that BHs, BRs and CMs derived from HD SG theories may be described in terms of 1+1 and 1-dimensional DG theories. In Sections 3, 4 we discuss a new class of integrable DG theories coupled to any number of scalar fields, construct the general solutions and study their main properties. In Section 5 we briefly outline possible applications of the integrable models and some unsolved problems. In the appendices we present a selection of known results as well as new ideas concerning dimensional reduction. We demonstrate that a too naive reduction may result in loosing interesting physical solutions. In particular, this gives a hint of explanation for the fact that dimensional reduction to black holes and cosmological models utilize very different approaches, as one may conclude by inspecting relevant textbooks and reviews.

We also give a detailed derivation of the effective potential for DG nonlinearly coupled to Abelian gauge fields. The models with nonlinear coupling of gauge fields to DG were not considered in the literature in full generality and the construction of the effective potential for them is, to the best of our knowledge, a new result. This result was mentioned in [15] and the proof was given in [18]. Here we present a more transparent and more general proof.

Note that in the lectures [18] the reader may find a substantial part of the material presented here from a different perspective. In fact, after completing [18] we have found new results concerning relations between BHs and cosmologies that significantly changed our understanding dimensional
reduction and the role of the low dimensional integrable models in theories of gravity and cosmology. Having this in mind, we decided to completely rewrite [18], preserving its formal structure but deleting introductory or outdated material while adding new original results (especially in Sections 3, 4). We also attempted to be as clear and rigorous as possible in formulating and interpreting our results as well as in discussing their relations to other approaches. For all these reasons, the present paper may be and should be read independently of the lectures [18].

2 From HD to (1+1) dimensional dilaton gravity

The HD theories which, under dimensional reductions, produce special examples of integrable theories come from the low energy limit of the theories described by 10-dimensional SGs. The bosonic part of the SG of type II (corresponding to the type II superstrings) is

\[ L^{(10)} = L^{(10)}_{NS-NS} + L^{(10)}_{RR} . \]  

Here it is sufficient to consider the first Lagrangian (the second one gives similar 1+1 dimensional theories). We have

\[ L^{(10)}_{NS-NS} = \sqrt{-g^{(10)} e^{-2\phi_s}} \left[ R^{(10)} + 4(\nabla \phi_s)^2 - \frac{1}{12} H_3^2 \right] , \]  

where \( \phi_s \) is the dilaton, related to the string coupling constant; \( H_3 = dB_2 \) is a 3-form; \( g^{(10)}_{\mu\nu} \) and \( R^{(10)} \) are the 10-dimensional metric and scalar curvature.

Among the many ways to reduce HD theories to low dimensions (LD) we only mention those that may lead to integrable theories. First, one may compactify a \( D \)-dimensional theory on a \( p \)-dimensional torus \( T^p \) and use the Kaluza - Klein - Mandel - Fock (KKMF) mechanism\(^1\). This introduces \( p \) Abelian gauge fields and at least \( p \) scalar fields. Antisymmetric tensor fields (\( n \)-forms), which may be present in the HD theory, will produce lower-rank forms and, eventually, other scalar fields. Thus we get a theory of gravity coupled to matter fields (scalars, Abelian gauge fields and, possibly, higher-rank forms) in a space of dimension \( d = D - p \). In the next step one reduces further its dimension by using some symmetry of the \( d \)-dimensional theory, most typically the spherical symmetry (the axial symmetry leads to much more complex low dimensional theories and will not be considered here). One thus arrives at a 1+1 DG theory coupled to scalar and gauge fields.

\(^1\)It is usually called the KK mechanism
The 1+1 dimensional theories so derived may describe spherically symmetric evolution of the BHs (collapse of matter) and of the universe (expansion of the universe). Usually the final step in the chain of dimensional reductions in CMs is somewhat different from that in BH physics since the CMs are normally obtained by reducing the $d$-dimensional theory directly to dimension 1+0 (see e.g. [19], [20]). Indeed, isotropy and homogeneity of the universe require that the whole space should have constant curvature $k = 0, \pm 1$ and all these cosmologies may be described by 1+0 dimensional DG theories. On the other hand, the spherical symmetry reduces HD gravity to 1+1 dimensional DG theories. In the case of the standard BHs coupled only to Abelian gauge fields these 1+1 DG theories automatically reduce to 1-dimensional DG. In this way only a very special cosmology may emerge. It is similar to the interior of the Schwarzschild black hole.

When there is scalar matter then, in general, there is no automatic reduction from dimension 1+1 to 1+0 or 0+1 and attempts to use a naive reduction by taking all geometric and matter fields depending on one variable only ($t$ or $r$, resp.) will result in losing important solutions. Moreover, in order to obtain the standard cosmological solutions from the 1+1 dimensional DG one has to modify the procedure of the dimensional reduction. The modifications that we discuss in a separate publication [21] give, as a bonus, some unusual cosmological and static solutions. We will not discuss these matters in detail, considering mainly naive reductions of the 1+1 DG.

Returning to the dimensional reduction of SG theories, we only note that one may use different sorts of dimensional reduction (KKMF, compactification on tori, etc.) but after several steps the resulting Lagrangian in dimension $d$ will typically depend on the curvature term $R^{(d)}$, a dilaton $\phi_d$, other scalar fields and Abelian forms (we do not consider reductions producing non-Abelian forms as they do not give integrable theories even in low dimensions). Thus, for our purposes the following expression for the effective Lagrangian $\mathcal{L}^{(d)}$ is sufficient:

$$\mathcal{L}^{(d)} = \sqrt{-g^{(d)}} e^{-2\phi_d} \left( R^{(d)} + 4(\nabla \phi_d)^2 - (\nabla \psi)^2 - X_0 - X_1 (\nabla \sigma)^2 - X_2 F_{2}^2 \right). \quad (3)$$

Here $F_2$ is a 2-form (an Abelian gauge field); the potentials $X_a$ are functions of $\phi_d$ and $\psi$. Actually, the Lagrangian should depend on several $F_2$-fields, several $\psi$-fields, and may depend on several $\sigma$-fields as well as on higher-rank forms\(^2\). However, after further reduction to dimension 1+1 only 2-

\(^2\)Typically, in dimensional reductions of supergravity by toroidal compactifications and the KKMF mechanism there is no $X_0$ terms but higher-rank forms usually appear. Nev-
forms and scalar fields will survive (the 2-forms can also be excluded by
writing an effective potential depending on electric or magnetic charges).

The \( d \)-dimensional theory can be further reduced to dimension 1+1 by
spherical symmetry. Before and after doing so one may transform this La-
grangian by the Weyl conformal transformation, \( g_{\mu\nu} \Rightarrow \tilde{g}_{\mu\nu} \equiv \Omega^2 g_{\mu\nu} \), where
\( \Omega \) depends on the dilaton. Expressing \( R \) in terms of the new metric,

\[
R^{(d)} = \Omega^2 \left[ \tilde{R}^{(d)} + 2(d-1)\nabla^2 \ln \Omega - (d-1)(d-2)(\nabla \ln \Omega)^2 \right],
\]

one can easily find the new expression for the Lagrangian. For \( d > 2 \) the mul-
tiplier \( e^{-2\phi_d} \) can be eliminated by choosing an appropriate function \( \Omega(\phi_d) \)
and thus the Lagrangian can be written in the ‘Einstein frame’ (as distinct
from the ‘string frame’ expressions above). If \( d = 2 \) the dilaton multiplier
cannot be removed but, instead, one can remove the dilaton gradient term.

For the spherically symmetric solutions of the theory \(^3\) it is more con-
venient to remove the dilaton factor by a Weyl transformation and rewrite
the action \(^3\) in the Einstein frame,

\[
\mathcal{L}_E^{(d)} = \sqrt{-g^{(d)}} \left[ R^{(d)} - (\nabla \chi)^2 - (\nabla \psi)^2 - X_0 \epsilon^{a_0 \chi} - X_1 \epsilon^{a_1 \chi} (\nabla \sigma)^2 - X_2 \epsilon^{a_2} F_2^2 \right],
\]

where \( \chi \propto \phi_d \) and \( a_k \) are known constants depending on \( d \). Then we param-
eterize the spherically symmetric metric\(^3\) by the general 1+1 dimensional
metric \( g_{ij}(x^0, x^1) \) and the new dilaton \( \varphi(x^0, x^1) (\nu \equiv 1/n, n \equiv d-2) \),

\[
ds^2 = g_{ij} dx^i dx^j + \varphi^{2\nu} d\Omega_{(d-2)}^2(k),
\]

introduce appropriate spherical symmetry conditions for the fields, which
from now on will be functions of the variables \( x^0 \) and \( x^1 \) \((t \text{ and } r)\), and
integrate out the other (angular) variables from the action.

In concrete computations, it is often more convenient to use the diagonal
spherically symmetric metric

\[
ds^2 = e^{2\alpha} dr^2 + e^{2\beta} d\Omega_{(d-2)}^2(k) - e^{2\gamma(t)} dt^2,
\]

\(^3\) Actually, we here call spherical symmetry a somewhat more general symmetry. Below,
\( d\Omega_{(d-2)}(k) \) is the metric on the surface of the generalized \((d-2)\)-dimensional ‘unit sphere’
\( S^{(d-2)}(k) \) that is the flat space for \( k = 0 \), pseudo-sphere for negative curvature \( k = -1 \) and
the standard sphere for positive curvature \( k = 1 \). The cylindrical symmetry also reduces
the \( d \)-dimensional theory \(^3\) to a 1+1 dimensional dilaton gravity but the equations of
motion are more cumbersome than in the spherical case.

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where $\alpha, \beta, \gamma$ are functions of $r$ and $t$. It should be emphasized that one may pass to this metric only after deriving the equation of motion with the general metric (8). Otherwise one of the equations will be lost. Note that the 2-dimensional dilaton is essentially the metric coefficient in the HD theory: $\varphi(r, t) = e^{\beta(r, t)/\nu}$.

Applying, in addition, the Weyl transformation that removes the dilaton gradient term we may obtain the effective 1+1 dimensional action

$$L^{(2)} = \sqrt{-g} \left[ \varphi R + k n(n - 1) \varphi^{-\nu} - X_0 e^{a_0 \chi} \varphi^{\nu} - X_2 e^{a_2 \chi} \varphi^{2-\nu} F^2 - \varphi \left( (\nabla \chi)^2 + (\nabla \psi)^2 + 2 X_1 e^{a_1 \chi} (\nabla \sigma)^2 \right) \right]. \quad (8)$$

Here $R \equiv R^{(2)}$, $\varphi$ is the 2-dilaton field; the scalar fields $\psi$ may have different origins – former dilaton fields, KKMF scalar fields, reduced $p$-forms, etc. The potentials $X_m$ depend on the scalar fields $\chi$ and $\psi$, which from now on will be called scalar matter fields. Also the field $\sigma$ may be regarded as a matter field but it plays a special role that will be discussed later. Notice that the potentials $X_m$ are positive.

In general, these theories are not integrable, meaning that it is impossible to find the analytic form of the general solution of the equations of motion. The reason is that EMs are highly nonlinear. The well known exception is the theory in which $k = 0$, $X_0 = 0$, $F^2 = 0$ and $\sigma = 0$. Although formally the EMs are still nonlinear, using the light cone metric one can show that the scalar fields satisfy the linear Euler – Darboux equation, for the general solution of which one can write a rather complex integral representation [22]. These solutions describe plane waves of scalar matter coupled to gravity. However, the expression for the metric is very difficult to analyze.

As mentioned above, the CMs are usually obtained by a different dimensional reduction, where the metric is written in the form

$$ds^2 = -e^{2\gamma(t)} dt^2 + e^{2\beta(t)} d\Omega_{(d-1)}^2(k). \quad (9)$$

Also, in this case somewhat different reductions of the fields are of interest because terms generated by the higher rank forms are important. However, after reduction the higher rank forms give rise to scalars of the $\psi$ or $\sigma$ type.

More important is the fact that the 1-dimensional static and cosmological solutions are differently embedded into the 1+1 DG theories. For this reason, a relation between the standard cosmological models and black holes, and, more generally, between different 1-dimensional reductions of the 1+1 DG is by no means obvious. One problem is of course to check
that the solutions of the dimensionally reduced theories satisfy the original HD equations. The second problem is to classify and enumerate all possible reductions of both types. To clarify this matter one has to carefully review the whole procedure of dimensional reduction because: 1. total derivative terms which are usually omitted without discussion may give non trivial contributions to low dimensional Lagrangians and 2. gauge fixing looking quite innocent if you work in one and the same dimension may lead to a loss of information when you perform a dimensional reduction. The effects of a careless approach to these two problems may be twofold: 1. some LD solutions of the HD equations may be lost, and 2. some solutions of the LD equations (obtained by too naive reductions) may not satisfy the HD equations (these effects are demonstrated by simple examples in [21]).

3 Integrable 2-dimensional DG coupled to matter

The Lagrangian of the general 1+1 DG coupled to Abelian gauge fields $F_{ij}^{(a)}$ and to scalar fields $\psi_n$ is

$$L^{(2)} = \sqrt{-g} [U(\varphi)R(g) + V(\varphi) + W(\varphi)(\nabla \varphi)^2 + X(\varphi, \psi, F_{(a)}^2, ..., F_{(A)}^2) + Y(\varphi, \psi) + \sum_n Z_n(\varphi, \psi)(\nabla \psi_n)^2]. \quad (10)$$

Here $g_{ij}$ is a generic 1+1 metric with signature (-1,1) and $R$ is the Ricci curvature; $U(\varphi), V(\varphi), W(\varphi)$ are arbitrary functions of the dilaton field; $X, Y$ and $Z_n$ are arbitrary functions of the dilaton field and of $N-2$ scalar fields $\psi_n$ ($Z_n < 0$); $X$ also depends on $A$ Abelian gauge fields $F_{(a)ij} \equiv F_{ij}^{(a)}$, $F_{(a)}^2 \equiv g^{ij} g^{ij'} F_{ij}^{(a)} F_{ij'}^{(a)}$. Notice that in dimensionally reduced theories both the scalar fields and the Abelian gauge fields are non-minimally coupled to the dilaton (i.e. the corresponding potentials depend on the dilaton).

The equations of motion (EM) of the theory (10) can be solved for arbitrary potentials $U, V, W$ and $X$ if $Y \equiv 0$, $\partial_\psi X \equiv 0$, and $\nabla \psi_n \equiv 0$. Then the theory is equivalent to the pure dilaton gravity (the one for which the potential $X, Y, Z$ vanish) that is known to be an integrable topological theory (for the simplest explicit solution in case of $X$ linear in $F^2$ see e.g [7] and references therein as well as the recent review [23], where on may find further references).

The properties of the general DG (10) are much more complex. To simplify further consideration we solve the equations for $F_{ij}^{(a)}$ and construct
the effective action (see Appendix 6.3)

\[ L^{(2)}_{\text{eff}} = \sqrt{-g} \left[ \varphi R(g) + V_{\text{eff}}(\varphi, \psi) + \sum_n Z_n(\varphi, \psi) g^{ij} \partial_i \psi_n \partial_j \psi_n \right]. \quad (11) \]

Here the effective potential \( V_{\text{eff}} \) (below we omit the subscript) depends also on charges introduced by solving the equations for the Abelian fields. We have used in (11) a Weyl transformation to exclude the kinetic term for the dilaton and also choose the simplest, linear parameterization for \( U(\varphi) \).

If \( V_{\text{eff}} = V(\varphi) \) and \( Z_n \equiv 0 \) the theory (11) is the pure dilaton gravity that is integrable with arbitrary potential \( V(\varphi) \). If \( Z_n \neq 0 \) the theory may be integrable with very special potentials \( V_{\text{eff}}(\varphi, \psi) \) and \( Z_n(\varphi, \psi) \). Roughly speaking, the two dimensional models may be integrable if either the potentials \( Z_n \) can be transformed to constants or the potential \( V_{\text{eff}} \) is zero. In the first case \( V_{\text{eff}} \) should have very special form what is described below. In the second case, for the theory to be explicitly analytically integrable the potentials \( Z_n \) must be very special functions\(^5\). The best studied examples of the integrable models belonging to the first class (constant \( Z_n \)) are the CGHS model \( (V_{\text{eff}} = g_0) \) and the JT model \( (V_{\text{eff}} = g_1 \varphi) \). A generalization \( (V_{\text{eff}} = g_+ e^{g \varphi} + g_- e^{-g \varphi}) \) was proposed in \([7]\).

Here we will mainly discuss a much more general class of integrable 1+1 DG models with minimal coupling to scalar fields defined by the Lagrangian (11) with the following potentials:

\[ Z_n = -1; \quad |f| V_{\text{eff}} = \sum_{n=1}^N 2 g_n e^{q_n}. \quad (12) \]

Here \( f \) is the light cone metric, \( ds^2 = -4f(u,v) \, du \, dv \), and

\[ q_n \equiv F + a_n \varphi + \sum_{m=3}^N \psi_m a_{mn} \equiv \sum_{m=1}^N \psi_m a_{mn}, \quad (13) \]

where \( \psi_1 + \psi_2 \equiv \ln |f| \equiv F \) (\( F \equiv \epsilon e^F \), \( \epsilon = \pm 1 \)), \( \psi_1 - \psi_2 \equiv \varphi \) and thus \( a_{1n} = 1 + a_n \), \( a_{2n} = 1 - a_n \). By varying the Lagrangian (11) in \( N - 2 \) scalar

\(^4\)If \( U'(\varphi) \) has zeroes, this parameterization, as well as the more popular exponential one, \( U = e^{\lambda \varphi} \), is valid only locally, i.e. between two consecutive zeroes.

\(^5\)The theories proposed in \([24]\) are explicitly integrable. The theories with \( Z_n = -\varphi \) are essentially equivalent to the one implicitly solved in \([22]\). String theory may generate a more general scalar coupling in \( L^{(2)}_{\text{eff}} \) related to \( \sigma \)-models. Such theories were considered in many papers in connection to studies of the cosmological singularity (see \([25]\), where a discussion of the main results and further references may be found.)
fields, dilaton and in $g_{ij}$ and then passing to the light cone metric we find $N$ equations of motion for $N$ functions $\psi_n$,

$$\epsilon_n \partial_u \partial_v \psi_n = \sum_{m=1}^{N} \varepsilon g_m \epsilon^{g_{mn}} a_{mn}; \quad \epsilon_1 = -1, \quad \epsilon_n = +1, \text{ if } n \geq 2,$$

(14)

as well as two constraints,

$$C_i \equiv f \partial_i (\partial_i \varphi / f) + \sum_{n=3}^{N} (\partial_i \psi_n)^2 = 0, \quad i = (u, v).$$

(15)

For arbitrary coefficients $a_{mn}$ these EMs are not integrable. However, as proposed in [15] - [16], the equations (14) are integrable and the constraints (15) can be solved if the $N$-component vectors $v_n \equiv (a_{mn})$ are pseudo-orthogonal. Then (14) reduce to $N$ independent, explicitly integrable Liouville equations for $q_n$,

$$\partial_u \partial_v q_n - \tilde{g}_n \epsilon^{q_n} = 0,$$

(16)

where $\tilde{g}_n = \varepsilon \lambda_n g_n$, $\lambda_n = \sum \epsilon_m a_{mn}^2$, and $\varepsilon \equiv f / |f|$ (note that the equations for $q_n$ depend on $\epsilon_n$ only implicitly, through the normalization factor $\lambda_n$).6

The expression for the original fields in terms of the Liouville fields $q_n$ may be found by using the orthogonality relations for $a_{mn}$ ($m \neq n$) and the definition of $\lambda_n \equiv \gamma_n^{-1}$ (for $m = n$) combined in the equation:

$$\sum_{k=1}^{N} \epsilon_k a_{km} a_{kn} = \lambda_n \delta_{mn} \equiv \gamma_n^{-1} \delta_{mn} .$$

(17)

More convenient is to use the following sum rules that follow from (17):

$$\sum \gamma_n = 0 ; \quad \sum \gamma_n a_n = -\frac{1}{2} ; \quad \sum \gamma_n a_n^2 = 0 ; \quad \sum a_{mn} \gamma_n = 0 , \quad m \geq 3 .$$

(18)

Now, using (17) - (18) we can invert the definition (13) and get

$$\psi_n = \varepsilon_n \sum a_{nm} \gamma_m q_m, \quad F = -2 \sum \gamma_n a_n q_n , \quad \varphi = -2 \sum \gamma_n q_n .$$

(19)

Also, using the orthogonality relations (17) it can be proven that one and only one norm $\gamma_n$ is negative. We thus choose $\gamma_1 < 0$ while other $\gamma_n$ are positive. In physically motivated models the parameters $a_{mn}$, $\gamma_n$ and $g_n$ may satisfy some further relations. For example, the signs of $\gamma_n$ and $g_n$

6Here we suppose that $\lambda_n \neq 0$ and $g_n \neq 0$. Otherwise the solution of the constraints should be modified in a fairly obvious way. We also denote $\gamma_n \equiv \lambda_n^{-1}$. 

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may be correlated so that \( g_n/\gamma_n < 0 \). However, such relations do not follow from the orthogonality conditions and we ignore them in our discussion\(^7\).

The most important fact is that the constraints can be explicitly solved. First, we write the solutions of the Liouville equations \(^{10}\) in the form suggested by the conformal symmetry properties of the Liouville equation \(^{26}\),

\[
e^{-q_n/2} = a_n(u)b_n(v) - \frac{1}{2}g_n\bar{a}_n(u)\bar{b}_n(v) \equiv X_n(u,v),
\]

where the chiral fields \( a_n(u), b_n(v), \bar{a}_n(u) \) and \( \bar{b}_n(v) \) satisfy the equations (do not mix \( a_n(u) \) with \( a_n \) used above)

\[
a_n(u)a'_n(u) - a'_n(u)a_n(u) = 1, \quad b_n(v)b'_n(v) - b'_n(v)b_n(v) = 1.
\]

Using \(^{21}\) we can express \( \bar{a} \) and \( \bar{b} \) in terms of \( a \) and \( b \) and thus write \( X_n \) as

\[
X_n(u,v) = a_n(u)b_n(v) \left[ 1 - \frac{1}{2}g_n \int \frac{du}{a_n^2(u)} \int \frac{dv}{b_n^2(v)} \right].
\]

It is not so straightforward but not very difficult to rewrite the constraints \(^{15}\) in the form\(^8\)

\[
C_u \equiv 4 \sum_{n=1}^{N} \gamma_n \frac{a_n''(u)}{a_n(u)} = 0, \quad C_v \equiv 4 \sum_{n=1}^{N} \gamma_n \frac{b_n''(v)}{b_n(v)} = 0.
\]

Using the relation \( \sum \gamma_n = 0 \) we can explicitly solve these constraints. First, it can be shown that the constraints are equivalent to the equations

\[
a_n(u) = \frac{\mu_n(u) - \sum \gamma_n \mu_n'(u)}{2 \sum \gamma_n \mu_n(u)}, \quad b_n(v) = \frac{\nu_n(v) - \sum \gamma_n \nu_n'(v)}{2 \sum \gamma_n \nu_n(v)},
\]

where \( \mu_n(u) \) and \( \nu_n(v) \) are arbitrary functions satisfying the constraints

\[
\sum \gamma_n \mu_n^2(u) = 0, \quad \sum \gamma_n \nu_n^2(v) = 0.
\]

Now, by integrating the first order differential equations \(^{24}\) for \( a_n(u) \) and \( b_n(v) \) we find the general solution of the \( N \)-Liouville DG in terms of the chiral moduli fields \( \mu_n(u) \) and \( \nu_n(v) \) satisfying \(^{25}\):

\[
a_n(u) = \left| \sum \gamma_m \mu_m \right|^{-\frac{1}{2}} \exp \int du \mu_n(u), \quad b_n(v) = \left| \sum \gamma_m \nu_m \right|^{-\frac{1}{2}} \exp \int dv \nu_n(v).
\]

\(^7\)In fact, the coupling constants have nothing to do with integrability and may be arbitrarily chosen.

\(^8\)If some \( g_n \) vanish, the corresponding fields \( q_n \) should be moved to the right-hand side of the equations \(^{15}\). It is not difficult to explicitly solve these generalized constraints.
The moduli fields $\mu_n(u)$ and $\nu_n(v)$ are not independent due to the constraints (25). In addition, we may use coordinate transformation $u \to U(u)$ and $v \to V(v)$ to choose two gauge conditions. We will show in a moment that writing
\[ | \sum \gamma_n \mu_n(u) | \equiv U'(u), \quad | \sum \gamma_n \nu_n(v) | \equiv V'(v) \] (27)
is indeed equivalent to choosing $(U,V)$ as a new coordinate system. With this aim, we first write
\[ A_n \equiv \exp \int du \mu_n(u), \quad B_n \equiv \exp \int dv \nu_n(v). \] (28)
Using (26) and (27) we may rewrite eq.(21) in terms of $A_n(U)$, $B_n(V)$:
\[ Y_n(U,V) = A_n(U)B_n(V) \left[ 1 - \frac{1}{2} g_n \int \frac{du}{A_n^2(U)} \int \frac{dv}{B_n^2(V)} \right], \] (29)
where we have defined
\[ Y_n(U,V) \equiv [U'(u)V'(v)]^{\frac{1}{2}} X_n(u,v) \] (30)
Now, with the above definitions we may derive the metric $f$, the dilaton $\varphi$, and the scalar fields $\psi_m (m \geq 3)$:
\[ e^{\varphi} = \prod_{n=1}^{N} [Y_n(U,V)]^4 \gamma_n, \quad e^{\psi_m} = \prod_{n=1}^{N} [Y_n(U,V)]^{-2a_{mn} \gamma_n}. \] (32)
We see that $ds^2 = -4f(u,v) \equiv -4f(U,V)dUdV$ and thus everything is expressed in terms of the new coordinates $(U,V)$.

The constraints (25) for the moduli parameters can easily be solved. However, it may be more convenient and instructive to introduce new moduli that are unit $(N-1)$-vectors (recall that $\gamma_1 < 0$ and $\gamma_k > 0$ for $k \geq 2$):
\[ \hat{\xi}_k(u) \equiv \frac{\mu_k(u) \sqrt{\gamma_k}}{\mu_1(u) \sqrt{|\gamma_1|}}, \quad \hat{\eta}_k(v) \equiv \frac{\nu_k(v) \sqrt{\gamma_k}}{\nu_1(v) \sqrt{|\gamma_1|}}, \quad k = 2, ..., N. \] (33)
These vectors moving on the surface of the unit sphere $S^{(N-2)}$ determine the solution up to a choice of the coordinate system, which can be fixed by the above gauge conditions (27). They now look as follows:
\[ U'(u) = |\gamma_1 \mu_1(u)|(1 - \cos \theta_{\xi}(u)), \quad V'(v) = |\gamma_1 \nu_1(v)|(1 - \cos \theta_{\eta}(v)), \] (34)
\[
\cos \theta_\xi(u) = \sum_{k=2}^{N} \hat{\gamma}_k \hat{\xi}_k(u), \quad \cos \theta_\eta(v) = \sum_{k=2}^{N} \hat{\gamma}_k \hat{\eta}_k(v),
\]

(35)

where \( \hat{\gamma} \) is the constant unit \((N-1)\)-vector, \( \hat{\gamma}_k = (\gamma_k/|\gamma_1|)^{\frac{1}{2}} \).

We thus have the general solution of the 1+1 dimensional dilaton gravity coupled to any number of scalar fields. It is explicitly expressed in terms of a sufficient number of arbitrary chiral fields and thus may be regarded as a highly nontrivial generalization of the D’Alembert solution for massless scalar fields. Using this solution one may solve the Cauchy problem and study the evolution of cosmological or black hole type solutions, etc. The representation of the general solution in terms of the chiral fields \( a_n(u) \) and \( b_n(v) \) may give us a good starting point in attempts to quantize our \( N \)-Liouville DG. Even more useful may be the chiral moduli fields \( \mu_n(u) \) and \( \nu_n(v) \) (or \( \hat{\xi}_k(u) \) and \( \hat{\eta}_k(v) \)). In terms of these moduli fields the dimensional reduction of the solutions becomes very transparent and this may simplify the derivation and physical interpretation of the evolution of one dimensional solutions and suggest new approaches to quantization based on analogy with the simple 1-dimensional case.

4 Integrable 1-dimensional DG coupled to matter

The naive reduction from dimension 1+1 to 0+1 (1+0) in the light cone coordinates \((u,v)\) is very simple. Supposing that \( \varphi = \varphi(\tau), \psi_n = \psi_n(\tau) \), where \( \tau = a(u) + b(v) \), we find from the 1+1 dimensional EMs (see Appendix)

\[
f(u,v) = \mp h(\tau) a'(u) b'(v), \quad ds^2 = -4f(u,v) du dv = \pm 4h(\tau) da db.
\]

(36)

Defining the space and time coordinates \( r = a \pm b \) and \( t = a \mp b \) we find

\[
ds^2 = h(\tau)(dr^2 - dt^2), \quad \text{where} \quad \tau = r \text{ or } \tau = t.
\]

(37)

Thus the reduced solution may be static or cosmological\(^9\).

This is not the most general way for obtaining 0+1 or 1+0 dimensional theories from higher dimensional ones. Not all possible reductions can be derived by this naive dimensional reduction of the 1+1 gravity. For example, the cosmological solutions corresponding to the reductions \( \Box \) should be

\(^9\)Of course, in 2-dimensional theories this distinction is not important. However, when we know the higher dimensional theory from which our DG originated, we can reconstruct the higher dimensional metric and thus find the higher dimensional interpretation of our solutions.
derived by a more complex dimensional reduction of the 1+1 dimensional DG, which will be discussed in [21].

The 0+1 EMs are described by the Lagrangian \((h = \varepsilon e^F, \varepsilon = \pm 1)\) [7]:

\[
\mathcal{L}^{(1)} = -\frac{1}{l(\tau)} \left[ \dot{\varphi} \dot{F} + \sum_n Z_n(\varphi, \psi) \dot{\psi}_n^2 \right] + l(\tau) \varepsilon e^F V(\varphi, \psi),
\]

where \(l(\tau)\) is the Lagrange multiplier (related to the general metric \(g_{ij}\)).

The integrable 2-dimensional \(N\)-Liouville theories are also integrable in dimension 0+1. Moreover, as we can solve the Cauchy problem in dimension 1+1, we can study the evolution of the initial configurations to stable static solutions, e.g. BHs, which are special solutions of the 0+1 reduction. However, the reduced theories can be explicitly solved for much more general potentials \(Z_n\) and \(V\), including those of realistic HD theories of branes, black holes and cosmologies. In this case, the 1-dimensional model [9S] is embedded in the nonintegrable 1+1 dimensional field theory\(^{10}\). Nevertheless, even in this case, integrable 1+1 dimensional theories may be used for approximate, qualitative description of processes leading to creation of BH as well as for studies of inhomogeneous cosmologies etc.

Suppose that for \(N - 2\) scalar fields \(\psi_n\) (\(n = 3, ..., N\)) the ratios of the \(Z\)-potentials are constant so that we can write \(Z_n = -\zeta_n/\phi'(\varphi)\). Let all the potentials \(Z_n\) and \(V\) be independent of the scalar fields \(\psi_{N+k}\) with \(k = 1, ..., K\). Then we first remove the factor \(\phi'(\varphi)\) by defining the new Lagrange multiplier \(\bar{l} = l(\tau)\phi'(\varphi)\) and absorb the factors \(\zeta_n > 0\) in the corresponding scalar fields. So we introduce the new dynamical variable \(\phi\) instead of \(\varphi\). Now we can solve the equations for the \(\sigma\)-fields and construct the effective Lagrangian:

\[
\mathcal{L}^{(1)}_{\text{eff}} = -\frac{1}{\bar{l}} \left[ \dot{\phi} \dot{F} - \sum_{n=3}^N \dot{\psi}_n^2 \right] + \bar{l} \varepsilon e^F V_{\text{eff}}(\phi, \psi) + V_{\sigma}(\phi, \psi).
\]

Here \(V_{\text{eff}} = V/\phi'(\varphi)\) and \(V_{\sigma} = \sum_k C_k^2 (Z_{N+k} \phi'(\varphi))^{-1}\) is the effective potential derived by solving the EMs for the \(\sigma\)-fields \(\psi_{N+k}\) (of course, now \(\varphi\) should be expressed in terms of \(\phi\)). If the original potentials in eq. [9S] are such that \((Z_{N+k} \phi'(\varphi))^{-1}\) and \(V/\phi'\) can be expressed in terms of sums of exponentials of linear combinations of the fields \(\phi\) and \(\psi\), then there is a chance that the 1-dimensional theory can be reduced to the \(N\)-Liouville theory (or the Toda theory if \(V_{\sigma} \neq 0\)).

\(^{10}\)Unfortunately, we do not know any realistic HD theory with non vanishing potentials \(Z_n = -\phi\) that can be reduced to an integrable theory in dimension 1+1.
It should be emphasized that potentials $V(\varphi, \psi)$, which are exponential (in $\varphi$) in the original 1+1 dimensional theory, may become not exponential in the effective 1-dimensional theory \(^{39}\), and vice versa. For example, this is evident in the realistic theories where $Z_n = -\varphi$ and thus $\phi(\varphi) = \ln \varphi$. This example shows why integrable 1-dimensional theories may be naturally embedded in nonintegrable 2-dimensional theories. Of course, we may embed the theory \(^{39}\) in the integrable 2-dimensional theory with the dilaton $\phi$, the potential $V_{\text{eff}}(\phi, \psi)$ and $Z_n = -1$ but the relation of such a theory to the original HD (and 2-dimensional) theory is not quite clear.

Now let us forget about these subtleties and consider the 1-dimensional Lagrangian \(^{38}\) obtained by dimensional reduction from the 2-dimensional integrable $N$-Liouville theory (with $Z_n = -1$). We thus have

$$L^{(1)} = \frac{1}{l}(-\dot{\psi}_1^2 + \sum_{n=2}^N \dot{\psi}_n^2) + l \sum_{n=1}^N 2\varepsilon g_n e^{\gamma_n} = \sum_{n=1}^N \left(\frac{1}{l} \gamma_n \tilde{g}_n^2 + 2l \varepsilon g_n e^{\gamma_n}\right),$$

(40)

where $g_n$ is defined by eq. \(^{13}\) and $a_{mn}$ satisfy pseudo orthogonality conditions \(^{17}\). Then EMs are reduced to $N$ independent Liouville equations whose solutions have to satisfy the energy constraint that can be derived by varying $L$ in $l(\tau)$. The solutions are expressed in terms of elementary exponentials (for simplicity, we write the solution in the gauge $l(\tau) \equiv 1$ but all the results are actually gauge invariant):

$$e^{-q_n} = \left[\frac{\tilde{g}_n}{2\mu_n^2}\right]^{\mu_n(\tau-\tau_n)} + e^{-\mu_n(\tau-\tau_n)} + 2\varepsilon_n,$$

(41)

where $\tilde{g}_n = \varepsilon \lambda_n g_n \equiv \varepsilon g_n / \gamma_n$, $\varepsilon_n \equiv -\tilde{g}_n / |\tilde{g}_n|$. The real parameters $\mu_n^2$ and $\tau_n$ are the integration constants. In what follows we analyze the solutions with $\mu_n^2 > 0$. In this case it is sufficient to take $\mu_n > 0$, as $g_n(\mu_n) = g_n(-\mu_n)$. The constraint is simply $\sum \gamma_n \mu_n^2 = 0$, and its solution is trivial. The space of the solutions is thus defined by the $(2N-2)$-dimensional moduli space (one of the $\tau_n$ may be fixed)\(^{11}\). The solutions with $\mu_n^2 > 0$ may have horizons. In fact, they have at most two horizons, and the space of the solutions with horizons has dimension $N-1$. There exist integrable models having

\(^{11}\)Of course, \(^{11}\) is a solution of the 1+1 dimensional equations that may be obtained from eqs. \(^{20}, 26\) by choosing constant moduli $\mu_n = \nu_n$ satisfying the constraint \(^{26}\). Then also the moduli $\xi_k$ and $\eta_k$ are constant and equal. If the vectors $\xi$ and $\eta$ are constant but not equal, the corresponding solution of the 1+1 dimensional theory given by \(^{27}\) - \(^{35}\) may be interpreted (for $\epsilon_n > 0$) in terms of ingoing and outgoing localized (‘soliton’-like) waves of scalar fields. We will discuss these interesting new solutions in a separate publication.
solutions with horizons and no singularities but their relation to the high dimensional world is at the moment not clear.

To prove these statements one should analyze the behavior of \( q_n \) for \( |\tau| \to \infty \) and for \( |\tau - \tau_n| \to 0 \) (if \( \varepsilon_n < 0 \)). The horizons appear when \( F \to -\infty \) while \( \varphi \) and \( \psi_n \) for \( n \geq 2 \) tend to finite limits. This is possible for \( |\tau| \to \infty \) if and only if \( \mu_n = -\mu \). When \( F \to F_0 \) and \( \varphi \to \infty \) while the scalar matter fields are finite we have the flat space limit, e.g. the asymptote of the BH. This is possible if and only if \( \mu_n = a_n\mu \). The singularities in general appear for \( |\tau - \tau_n| \to 0 \) if \( \varepsilon_n < 0 \). The scalar fields and the dilaton may be free of singularity if and only if all \( \tau_n \) are equal.

These statements may be verified if we first write the expression for \( q_n(\tau) \) in the form

\[
- q_n(\tau) = \mu_n|\tau - \tau_n| + \ln(|\tilde{g}_n|/2\mu_n^2) + \ln(1 + \varepsilon_n \exp(-\mu_n|\tau - \tau_n|))²
\] (42)

that directly follows from (41) for \( \mu_n > 0 \), and then look at the corresponding analytic expressions for \( F \), \( \varphi \) and \( \psi_n \) (for \( n \geq 3 \)) that can be derived by using (19). For \( |\tau| \to \infty \) we have the following asymptotic behaviour

\[
\psi_n = -\varepsilon_n \sum a_{nm}\gamma_m\mu_m|\tau - \tau_m| + \ln(2\tilde{g}_m/\mu_m²) + o(1) . \] (43)

In particular,

\[
F = 2 \sum a_n\gamma_n\mu_n|\tau - \tau_n| + \ln(2\tilde{g}_n/\mu_n²) + o(1) , \] (44)

\[
\varphi = 2 \sum \gamma_n\mu_n|\tau - \tau_n| + \ln(2\tilde{g}_m/\mu_m²) + o(1) \] (45)

The divergent parts of these functions for \( \tau \to \infty \) are simply

\[
\psi_n = -\varepsilon_n|\tau| \sum a_{nm}\gamma_m\mu_m , \quad F = 2|\tau| \sum a_n\gamma_n\mu_n , \quad \varphi = 2|\tau| \sum \gamma_n\mu_n . \] (46)

When \( \varepsilon_n < 0 \) and all \( \tau_n \) are equal (we may choose \( \tau_n = 0 \)) the divergent parts of these functions are \( (\varepsilon_1 = -1, \varepsilon_n = 1, n \geq 2) \)

\[
\psi_n = -\varepsilon_n \ln(\tau² \sum a_{nm}\gamma_m) , \quad F = 2 \ln(\tau² \sum a_n\gamma_n) , \quad \varphi = 2 \ln(\tau² \sum \gamma_n) . \] (47)

Using the sum rules (18) one can now prove the above statements. In particular, we see that \( \psi_{n \geq 3} \) and \( \varphi \) are finite for \( |\tau| \to \infty \) if \( \mu_m \equiv \mu \) and finite for \( |\tau| \to 0 \) if \( \tau_n \equiv \tau_0 \). On the other hand, \( F \) is singular in both these limits, its divergent parts being \( F = -\mu|\tau| \) and \( F = -\ln(\tau²) \), respectively. Obviously the horizons appear when \( |\tau| \to \infty \). Thus we have the structure of the horizons and of the singularity similar to that of the Reissner - Nordstrøm
BH\textsuperscript{12}. To get the black hole of the Schwarzschild type, i.e. with one horizon, one has to use a limiting procedure that will be discussed elsewhere.

We have seen that integrability of the theory and asymptotic behaviour of the solutions depend only on the $a_{mn}$. Geometric and physical features of the solutions may depend also on the coupling constants $g_n$. We illustrate this dependence by deriving the two dimensional curvature. To simplify our formulas we use the light cone coordinates ($u, v$) and the 'Weyl frame', in which there is no $(\nabla \varphi)^2$ term in $\mathcal{L}^{(2)}$ and thus no $\dot{\varphi}^2$ term in $\mathcal{L}^{(1)}$. Then the two dimensional Ricci curvature for the 1-dimensional solutions may be derived by using a very simple formula:

$$R = \frac{1}{f} \partial_u \partial_v \ln |f| = \frac{1}{h(\tau)} \partial_\tau^2 \ln |h(\tau)| \equiv \varepsilon e^{-F} \tilde{F}(\tau).$$  \hspace{1cm} (48)

Using this formula and the expressions for $F$ derived above one may easily prove the following statements. 1. If $\mu_n \equiv \mu$ the curvature $R$ is finite on the horizons, i.e. for $|\tau| \to \infty$. If in addition all moduli $\tau_n$ are equal, $R$ is given by the following very compact expression:

$$R \to -\varepsilon_0 \prod |\tilde{g}_n|^{-2\gamma_n a_n}, \quad |\tau| \to \infty,$$  \hspace{1cm} (49)

where it is supposed that all $\varepsilon_n$ are equal and $\varepsilon_n \equiv \varepsilon_0$. 2. If all $\varepsilon_n$ in eq.(41) are negative ($\varepsilon_n \equiv \varepsilon_0 = -1$) and $\tau_n \equiv \tau_0$, then the curvature $R$ approaches the finite value given in eq.(49) also for $|\tau - \tau_0| \to 0$. In this case $\mu_n$ may be arbitrary real constants. 3. If $\mu_n = a_n \mu$ we find that $R \to 0$ for $\tau \to \infty$.

Note that the solution (41) is written in a rather unusual coordinate system. One may write a more standard representation remembering that the dilaton $\varphi$ is related to the coordinate $r$ (see (3)). This may be useful for a geometric analysis of some simple solutions (e.g. the Schwarzschild or the Reissner - Nordstrøm black holes) but in general the standard representation is rather inconvenient in analyzing the solutions of the $N$-Liouville theory.

5 Discussion and outlook

The explicitly analytically integrable models presented here may be of interest for different applications. Most obviously we may use them to construct first approximations to generally non integrable theories. Realistic theories

\textsuperscript{12}Let us emphasize that when we have scalar fields varying in space the horizons do not appear in the standard Einstein - Maxwell theory. Thus the derived black holes are not identical to the standard ones. These new BH were earlier obtained in string theory and supergravity by using other approaches (see e.g. reviews [11] - [13]).
describing BHs and CMs are usually not integrable. However, explicit general solutions of the integrable approximations may allow one to construct different sorts of perturbation theories.

For example, spherically symmetric static BHs non minimally coupled to scalar fields are described by the integrable 0+1 dimensional $N$-Liouville model. However, the corresponding 1+1 theory is not integrable because the scalar coupling potentials $Z_n$ are not constant (see eq. (8)). To obtain approximate analytic solutions of the 1+1 theory one may try to approximate $Z_n$ by properly chosen constants.

This approach may be combined with the recently proposed analytic perturbation theory allowing to find solutions close to horizons for the most general non integrable 0+1 DG theories [27]. Near the horizons we can also use the integrable 1+1 dimensional DG (with $Z_n = -1$) as a good approximation to a realistic theory (with $Z_n$ depending on $\varphi$).

In cosmological applications, the behaviour of the 1+1 dimensional solutions for $\varphi \to 0$ (i.e. near the singularity at $\varphi = 0$) is of great interest. Integrable 1+1 dimensional theories could give, at best, a rough qualitative approximation of the exact solutions near the singularity. A more quantitative approximation might be obtained by first asymptotically solving the exact theory in the vicinity of $\varphi = 0$ and then sewing the asymptotic solutions with those of the integrable theory. To realize such a program one needs a very simple and explicit analytic solutions of the integrable theory. Our simple model having the solutions represented in terms of the moduli $\hat{\xi}$ and $\hat{\eta}$ may give a good starting point for such a work. Of course, before applications to realistic cosmologies become possible, one should study in detail and completely classify and interpret the behaviour of the 1+1 dimensional solutions and their precise relation to the 1+0 dimensional reduction.

The reduction from dimension 1+1 both to dimension 1+0 and to dimension 0+1 is especially transparent in the moduli representation for the solutions of the 1+1 dimensional $N$-Liouville model. However, as we emphasized above, the whole procedure of the dimensional reduction should be reconsidered from a more general point of view. A more detailed motivation for reconsidering the usual procedures will be given in a forthcoming paper [21].
6 Appendix

6.1 Reduction of the Curvature

Let the block diagonal dimensional reduction be $ds^2 = g_{ij} dx^i dx^j + h_{mn} dx^m dx^n$, where the metric $g_{ij}$ depends only on the coordinates of the first subspace, $x^i$. The Ricci curvature scalar for this metric is then

$$R = R[g] + R[h] - \frac{2}{\sqrt{h}} \nabla^m \nabla_m \sqrt{h} + \frac{1}{4} g^{ij} \partial_i h^{mn} \partial_j h_{mn} +$$

$$+ \frac{1}{4} g^{ij} (h^{mn} \partial_i h_{mn}) (h^{pq} \partial_j h_{pq}) .$$

(50)

Using this expression, partial integrations, and the Weyl transformations one may easily derive the reductions presented in the main text. If the second subspace is a $(d-2)$-sphere $S^{(d-2)}(k)$ of radius $e^{\beta}$ then

$$R[h] = R[S^{(d-2)}(k)] = e^{-2\beta} k (d-2)(d-3) , \; k = \pm 1, 0 .$$

For the analysis of the geometric properties of different 4-dimensional cosmologies it may help to use the easily derivable formula for the scalar curvature of the 3-dimensional spherically symmetric subspace of the space with the metric (7) (with $d = 4$, $k = 1$ and fixed $t$)

$$R = R^{(3)} = 2 e^{-2\beta} - 2 e^{-2\alpha} (2\beta'' + 3\beta'^2 - 2\beta'\alpha')$$

(51)

To analyze the geometric properties of the 3-space in more detail one may use the expressions for its Ricci tensor

$$R_1^1 = -2 e^{-2\alpha} (\beta'' + \beta'^2 - \beta'\alpha'), \; R_2^2 = R_3^3 = \frac{1}{2} (R^{(3)} - R_1^1) .$$

(52)

To help the reader in keeping trace of relations between the dimensions $d$, 1+1, 1+0 and 0+1 we also write here a simple expression for the curvature in dimension 1+1. Given the metric in diagonal form as $ds^2 = -e^{2\gamma} dt^2 + e^{2\alpha} dr^2$, its Ricci scalar $R$ is

$$R = 2 e^{-2\gamma} (\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha} \dot{\gamma}) - 2 e^{-2\alpha} (\ddot{\gamma} + \dot{\gamma}^2 - \dot{\gamma} \dot{\alpha}).$$

(53)

Further, for any scalar field $\varphi$ the expression for $\nabla^2 \varphi$ is

$$\nabla^2 \varphi \equiv \nabla^m \nabla_m \varphi = -e^{-2\gamma} (\ddot{\varphi} + (\ddot{\alpha} - \dot{\gamma}) \dot{\varphi}) + e^{-2\alpha} (\ddot{\varphi}' + (\dot{\gamma}' - \dot{\alpha}') \varphi').$$

(54)

All these expressions simplify in the $(u,v)$ coordinates that can be obtained by taking $\gamma = \alpha$ and introducing the light cone coordinates (always
possible for the 1+1 metric having the Minkowski signature). Denoting $e^{2\gamma} = e^{2\alpha} = f$ we have $ds^2 = f(dr^2 - dt^2)$, $R = f^{-1}(\partial^2_t - \partial^2_r) \ln |f|$. The $(u,v)$ metric, which drastically simplifies the equations of motion and all computations, can be obtained e.g. writing $t = u + v$ and $r = u - v$:

$$ds^2 = -4 f(u,v) du \, dv.$$  

There is a residual symmetry in the $(u,v)$ coordinates, namely, $u \to a(u)$, $v \to b(v)$. Under this transformation the previous equation becomes

$$ds^2 = -4 f(a(u), b(v)) a'(u) b'(v) du \, dv = -4 f(a, b) da \, db. \quad (55)$$

Thus the metric in the coordinates $(a,b)$ is the same as in the $(u,v)$ coordinates. Also the curvature and equations of motion remain invariant.

This freedom is useful for many reasons. For example, suppose we have found a solution of the EMs for which the metric $f$ and the dilaton $\varphi$ depend only on $uv$. Then, choosing $a = \ln u$, $b = \ln v$, we may go to coordinates $(a,b)$ in which the metric function and the dilaton depend on $a + b$. More generally, the solutions of integrable models may usually be written in terms of massless free fields $\chi_n$ which are solutions of the D’Alembert equation and thus may be written as $\chi_n(u,v) = a_n(u) + b_n(v)$. If all $\chi_n$ are equal, i.e. $\chi_n = a(u) + b(v)$, the theory reduces to one dimension.

In the same way one may dimensionally reduce the general, non integrable models. We describe the simplest approach using the light cone coordinates. The more standard approach that uses $r$ and $t$ is more cumbersome but may be of use in interpreting the low dimensional solutions as solutions of higher dimensional theories. As emphasized in the main text, the described naive reduction does not allow to obtain some physically interesting one-dimensional solutions of higher dimensional gravity coupled to matter fields. It seems that a more general approach should use the dimensional reduction of the equations of motion, in other words, it should employ a general procedure for separation of the time and space variables.

### 6.2 Reduction of the Equations

The equations of motion for consistently reduced gravity theories are equivalent to the Einstein equations and there is no loss of information. Having the complete set of the equation one may use any convenient gauge (coordinate system). Let us write the EMs in the light cone $(u,v)$ coordinates. To
simplify the formulas we keep only one scalar field and thus use, instead of (11), the following effective Lagrangian
\[
L^{(2)} = \sqrt{-g} \left[ \varphi R + V(\varphi, \psi) + \sum Z_n(\varphi, \psi)(\nabla \psi_n)^2 \right].
\] (56)

By first varying this Lagrangian in generic coordinates and then going to the light cone ones we obtain the equations of motion
\[
\partial_u \partial_v \varphi + f V(\varphi, \psi) = 0,
\] (57)
\[
f \partial_i \left( \frac{\partial_i \varphi}{f} \right) = \sum Z_n(\partial_i \psi_n)^2 \quad (i = u, v)
\] (58)
\[
\partial_u (Z_n \partial_u \psi_n) + \partial_u (Z_n \partial_i \psi_n) + f V_{\psi_n}(\varphi, \psi) = \sum Z_{m, \psi_n} \partial_u \psi_m \partial_v \psi_m,
\] (59)
\[
\partial_u \partial_v \ln |f| + f V_{\varphi}(\varphi, \psi) = \sum Z_{n, \varphi} \partial_u \psi_n \partial_v \psi_n,
\] (60)
where \(V_{\varphi} = \partial_\varphi V, V_{\psi_n} = \partial_{\psi_n} V, Z_{n, \varphi} = \partial_\varphi Z_n, \) and \(Z_{m, \psi_n} = \partial_{\psi_n} Z_m.\) These equations are not independent. Actually, (60) follows from (57) − (58).

Alternatively, if (57), (58), (60) are satisfied, one of the equations (59) is also satisfied.

The most important equations are the constraints (58). A general formulation of the (naive) dimensional reduction in the \((u,v)\) coordinates that is valid both for static and cosmological solutions is suggested by the following simple observation. Consider the solutions with constant scalar field \(\psi \equiv \psi_0\) (the ‘vacuum’ solution). This solution exists if \(V_{\psi_0}(\varphi, \psi_0) = 0,\) see eq. (59). The constraints (58) can now be solved because their right-hand sides are identically zero. It is a simple exercise to prove that there exist chiral fields \(a(u)\) and \(b(v)\) such that \(\varphi(u, v) \equiv \varphi(\tau)\) and \(f(u, v) \equiv \varphi'(\tau) a'(u) b'(v)\) (the primes denote derivatives with respect to the corresponding argument). Using this result it is easy to prove that eq. (57) has the integral \(\varphi' + N(\varphi) = M,\) where \(N(\varphi)\) is defined by the equation \(N'(\varphi) = V(\varphi, \psi_0)\) and \(M\) is the integral of motion which for the BH solutions is proportional to the mass of the BH. The horizon, defined as a zero of the metric \(h(\tau) = M - N(\varphi),\) exists because the equation \(M = N(\varphi)\) has at least one solution in some interval of values of \(M.\) The EMs in the case considered are actually dimensionally reduced. Their solutions can be interpreted as BHs (Schwarzschild, Reissner Nordstrom and other known BH solutions in any dimension) or as CMs.

Taking into account the lesson of the scalar vacuum solutions, we introduce a more general dimensional reduction by supposing that the scalar
fields and the dilaton depend on one free field \( \tau = a(u) + b(v) \) (after dimensional reduction it is interpreted either as the space or the time coordinate). Then from eq. (57) it follows that the metric should have the form

\[
f(u,v) = \varepsilon h(a + b) a'(u) b'(v),
\]

where \( \varepsilon \) is introduced in order to have the same type of metric for the 0+1 and 1+0 cases. Defining \( \bar{\tau} \equiv a - b \), we have

\[
ds^2 = -4f(u,v) du dv = -4\varepsilon h da db = -\varepsilon h (d\tau^2 - d\bar{\tau}^2),
\]

and thus both reduced metrics may be written as by choosing \( \tau = r \), \( \varepsilon = -1 \) or \( \tau = t \) and \( \varepsilon = +1 \).

The reduced EMs for the dilaton and the scalar field

\[
d\tau - \varepsilon h \psi = 0, \quad 2\partial_\tau (Z \partial_\tau \psi) + \varepsilon h \psi = Z \psi (\partial_\tau \psi)^2,
\]

depend on \( \varepsilon \) while the constraints are the same for both reductions and give just one reduced constraint, \( \partial^2_\tau \varphi - \partial_\tau \varphi \partial_\tau \ln |h| = Z (\partial_\tau \psi)^2 \), equivalent (in the standard terminology) to the energy constraint.

Thus we have the rule for the reduction of the EMs: using the equations in the light cone gauge, derive the equations for \( \varphi(\tau), \psi(\tau), \varepsilon h(\tau) \) and then take \( \tau = r \) and \( \varepsilon = -1 \) or \( \tau = t \) and \( \varepsilon = +1 \). Using this rule we may write down reduced equations without calculations. First take the gauge fixed Lagrangian in the \((u, v)\) metric and using the residual covariance with respect to the transformation \( u \to a(u), v \to b(v) \) transform it to the new coordinates \((a, b)\),

\[
L^{(2)}_{g.f.} = \varphi \partial_a \partial_b F + fV - Z \partial_u \psi \partial_v \psi \to \varphi \partial_a \partial_b F + \varepsilon hV - Z \partial_a \psi \partial_b \psi
\]

(recall that \( F = \ln |f| \)). Then, using the reduction rule, we obtain

\[
L^{(1)}_{g.f.} = \varphi \bar{F} - Z \dot{\psi}^2 + \varepsilon hV,
\]

where the dot denotes \( \tau \)-differentiation. To get rid of the second derivative, we neglect the total derivative in the Lagrangian and replace \( \varphi \bar{F} \) by \( \dot{\varphi} \bar{F} \). This Lagrangian gives the correct gauge fixed equations of motion. To restore the lost constraint (the gauge fixed Lagrangian does not give the constraints) we recall that the constraint is just \( H = 0 \), where \( H \) is the Hamiltonian correspondent to the Lagrangian. It is evident that

\footnote{To simplify notation and formulas we consider here only the case of one scalar field. Generalizing to any number of fields is obvious.}
\[ H = -\dot{\phi} \dot{F} - Z \dot{\psi} - \varepsilon hV. \] Now it is easy to guess that the correct Lagrangian giving the equations of motion and the constraint \( H = 0 \) is simply

\[ L^{(1)} = -\frac{1}{l(\tau)} \left[ \dot{\phi} \dot{F} + Z \dot{\psi} \right] + l(\tau) \varepsilon hV. \quad (62) \]

In order to obtain from here the 0+1 theory we simply take \( \tau = r \) and \( \varepsilon = -1 \). The 1+0 theory can be written taking \( \tau = t \) and \( \varepsilon = +1 \).

Finally, let us write an example of cosmological reduction directly from a higher dimensional theory. We take the \( d \)-dimensional metric [9], suppose that the scalar functions depend on one variable \( t \), see e.g. [11]. Then using eq. (50) with the 1-dimensional metric \( g_{ij} \) and the \((d - 1)\)-dimensional metric \( h_{mn} \) we can find for example the reduced action for the Lagrangian [5]. We write here only the reduced curvature part (the derivation of the other terms is obvious):

\[ S = \int d^d x \sqrt{-g} \sqrt{h} R^{(d)} = (d - 1)(d - 2) \int dt e^\gamma e^{\alpha(d-1)} \left[ k e^{-2\alpha} - e^{-2\gamma} \dot{\alpha}^2 \right]. \quad (63) \]

It is not difficult to check that the naive cosmological reduction of the 1+1 theory does not give this expression for the cosmological action (see [21]). The geometric explanation of this fact is given by eq. (52), from which it is clear that for \( \beta' = 0 \) we have \( R^1_1 \neq R^2_2 \) and thus the 3-space is not isotropic while for all three cosmologies described by this action it must be isotropic. To obtain these standard cosmologies one has to use a different dimensional reduction from 1+1 to 1+0 dimension in which both the metric and the dilaton may depend on \( r \) and \( t \). Indeed, according to [52], the isotropy condition \( R^1_1 = R^2_2 \) requires that

\[ e^{2(\beta - \alpha)} (\beta' - \alpha') = -1 \quad (64) \]

This may be satisfied if we suppose that \( \alpha = \alpha_0(t) + \alpha_1(r) \) and \( \beta = \beta_0(t) + \beta_1(r) \). Then it follows that \( \alpha_0(t) - \beta_0(t) = \text{const} \) and, choosing the gauge in which \( \alpha_1(r) = \text{const} \), one can show that the solutions, up to a gauge choice, may be written as \( e^{\beta_1} = \sinh(kr)/\sqrt{k} \), where \( k = 0, \pm 1 \). This gives the standard cosmological solutions from the dimensional reduction of the 1+1 dimensional dilaton gravity. The detailed derivation of the reduced Lagrangian and HD equations of motion will be presented in [21].
6.3 Nonlinear coupling of gauge fields

Suppose that in place of the standard Abelian gauge field term, \( X(\varphi, \psi)F^2 \), the Lagrangian contains a more general coupling of \( A \) gauge fields \( F^{(a)}_{ij} = \partial_i A^{(a)}_j - \partial_j A^{(a)}_i \) to dilaton and scalar fields, \( X(\varphi, \psi; F^2) \), where \( F^2 \equiv g^{ik}g^{jl}F_{ij}^{(a)}F_{kl}^{(a)} \) and \( a = 1, ..., A \). Without loss of generality we may write the two dimensional Lagrangian as

\[
L^{(2)} = \sqrt{-g} \left[ \varphi R + X(\varphi, \psi; F^2) + \sum Z_n(\varphi, \psi)(\nabla \psi_n)^2 \right].
\]

(65)

As in the case of the Lagrangian (56) the equations of motion are obtained by varying the Lagrangian (in generic coordinates) with respect to all variables, including now the gauge fields, and then going to the light cone coordinates (36). Variations of \( \psi_n \) and of \( \varphi \) give the equations (59) and (60) with \( V \) replaced by \( X \). The equations (57) and (58) are obtained by varying the metric. The two equations (58) correspond to variations of the diagonal part \( g_{ii} \). It is not difficult to see that the additional term produced by the \( F^2 \) dependence of \( X \) vanishes when we pass to the light cone metric. Indeed, this contribution is proportional to the sum of the terms \( \left( \partial X / \partial F^2 \right) \delta F^2 / \delta g_{ii} \) and this vanishes in the light cone metric because \( \delta F^2 / \delta g^{ii} = 2g^{jk}F_{ij}F_{ik} \equiv 0 \) if \( g_{jj} = 0 \). In other words, the constraint equations (58) are insensitive to the \( F^2 \) dependence of \( X \) in the \((u, v)\) coordinates. In contrast, the equation (61) has additional terms even in the light cone coordinates:

\[
\partial_u \partial_v \varphi + f \left[ X(\varphi, \psi; F^2) + f \sum \frac{\partial X}{\partial F^2} \cdot \frac{\partial F^2}{\partial f} \right] = 0.
\]

(66)

Finally, we have the equations of motion for the gauge fields,

\[
\partial_i \left( \sqrt{-g} \frac{\partial X}{\partial \left( \partial_i A^{(a)}_j \right)} \right) = 0,
\]

(67)

where \( \partial X / \partial \left( \partial_i A^{(a)}_j \right) = 4F^{ij}_{(a)} \partial X / \partial F^2_{(a)} \). In dimension 1+1 the equations (67) reduce to the conservation laws

\[
\sqrt{-g} F^{ij}_{(a)} \frac{\partial X}{\partial F^2_{(a)}} = \varepsilon^{ij} \lambda Q_a
\]

(68)

\footnote{In what follows we denote the set of all \( F^2_{(a)} \) in the potential \( X \) simply by \( F^2 \), i.e. \( F^2 = \{ F^2_{(1)}, ..., F^2_{(A)} \} \) and thus \( X = X(\varphi, \psi; F^2) \).}
where $\varepsilon^{ij} = -\varepsilon^{ji}$, $\varepsilon^{01} = 1$, $\lambda$ is a constant to be defined later and $Q_a$ are conserved charges. From this it is easy to obtain the equations for $F^2_{(a)}$ (recall that $2g = \varepsilon^{ij}\varepsilon^{lk}g_{il}g_{jk}$):

$$F^2_{(a)} = -2\lambda^2 Q_a^2 \left( \frac{\partial X}{\partial F^2_{(a)}} \right)^{-2}.$$  
(69)

This allows us (in principle) to write $F^2_{(a)}$ (and $F^i_{(a)}$) in terms of $\varphi, \psi, Q$ where $Q$ is the set of all the charges, $Q = \{Q_1, ..., Q_A\}$. Let us denote the solution as $\bar{F}^2_{(a)}(\varphi, \psi; Q)$ (or simply $\bar{F}^2_{(a)}$). Now we can write $\bar{F}^i_{(a)}$ in terms of $\bar{F}^2_{(a)}(\varphi, \psi; Q)$. Eq. (69) gives

$$\frac{\partial \bar{X}}{\partial F^2_{(a)}} \equiv \frac{\partial X}{\partial F^2_{(a)}} (F^2 \Rightarrow \bar{F}^2) = \epsilon_a \sqrt{\frac{2}{-\bar{F}^2_{(a)}}} \lambda Q_a.$$  
(70)

where $\bar{X} \equiv X(\varphi, \psi; \bar{F}^2)$, $\epsilon_a = \text{sign}[\partial X/\partial F^2_{(a)}]$ and $\sqrt{-\bar{F}^2_{(a)}} > 0$. Then from the equations (69) and (70) we get

$$\bar{F}^i_{(a)} = \frac{\epsilon^{ij} \lambda Q_a}{\sqrt{-g}} \left( \frac{\partial \bar{X}}{\partial F^2_{(a)}} \right)^{-1} = \epsilon^{ij} \epsilon \sqrt{\frac{2}{\bar{F}^2_{(a)}}/2g}.$$  
(71)

Now we can exclude the gauge fields from the equations of motion and find an effective potential $X_{\text{eff}}$ depending only on $\varphi, \psi$ and $Q$. To do this we go to the $(u, v)$ coordinates and recall that after computing the constraints, which are insensitive to the $F^2$ dependence, one may derive other equations using the reduced (gauge fixed) Lagrangian

$$L^{(2)}_{g.f.} = \varphi \partial_u \partial_v \ln |f| + f X(\varphi, \psi; F^2) - \sum Z_n \partial_u \psi_n \partial_v \psi_n.$$  
(72)

It is not difficult to guess that if we exclude the dependence on $F^2$ from the expression in the square brackets in eq. (69) we will get the desired effective potential $X_{\text{eff}}$ depending on $\varphi, \psi$ and $Q$. To write the explicit expression for it we first note that (we don’t set yet $F^2 = \bar{F}^2$)

$$X + f \sum \frac{\partial X}{\partial F^2_{(a)}} \cdot \frac{\partial F^2_{(a)}}{\partial f} = X - \sum 2F^2_{(a)} \frac{\partial X}{\partial F^2_{(a)}}$$  
(73)

\footnote{For small values of $F^2_{(a)}$ we usually have $\epsilon_a < 0.$}
Replacing $F^2$ by $\bar{F}^2$ and using in (73) the relations (70) and (71) we obtain the expression for the effective potential (we now choose $2\sqrt{2}\lambda = 1$)

$$X_{\text{eff}}(\varphi, \psi, \bar{F}^2) \equiv X_{\text{eff}}(\varphi, \psi, Q) \equiv X(\varphi, \psi; \bar{F}^2) + \sum Q_a \epsilon_a \sqrt{-\bar{F}^2(a)}$$

(74) that should replace $X$ in the Lagrangian (72). If we consider $X_{\text{eff}}$ as a function of $\bar{F}^2$ we may reproduce the expressions for $\bar{F}^2(a)(\varphi, \psi, Q)$. Indeed, varying the effective Lagrangian

$$L^{(2)}_{\text{eff}} = \varphi \partial_u \partial_v \ln |f| + f X_{\text{eff}}(\varphi, \psi; \bar{F}^2) - \sum Z_n \partial_u \psi_n \partial_v \psi_n.$$  

(75) with respect to $\bar{F}^2$ we find the condition

$$\frac{\delta L^{(2)}_{\text{eff}}}{\delta \bar{F}^2(a)} = \frac{\partial X}{\partial \bar{F}^2(a)} - \frac{1}{2} \epsilon Q(a)/\sqrt{-\bar{F}^2(a)} = 0$$

(76) that gives for $\bar{F}^2(a)$ the expression defined by (69) (with $\lambda^2 = 1/8$).

If these conditions are satisfied, the total derivative of $X_{\text{eff}}$ in $\varphi$ coincides with the partial derivative of $X$ (with fixed $F^2$). Indeed, we have

$$\frac{dX_{\text{eff}}}{d\varphi} = \frac{\partial X}{\partial \varphi} + \sum \left[ \frac{\partial X}{\partial \bar{F}^2(a)} - \frac{1}{2} \epsilon Q(a)/\sqrt{-\bar{F}^2(a)} \right] \frac{\partial \bar{F}^2(a)}{\partial \varphi} = \frac{\partial X}{\partial \varphi}$$

due to (70). Analogously, $dX_{\text{eff}}/d\psi = \partial X/\partial \psi$. This means that the equations (59), (60) obtained from the effective Lagrangian (75) by varying $\psi, \varphi$ coincide with the corresponding equations obtained from the original Lagrangian $L^{(2)}$. The constraints (58) are obviously the same and variation of $\varphi$ gives the correct equation (65). It follows that we may forget about the fields $F_{ij}$, $A_i$ (that can be derived from eq. (71) if needed) and work with the effective theory for the fields $f, \varphi, \psi$ simply replacing the original Lagrangian (72) with the effective Lagrangian.

It is not difficult to apply this construction to known Lagrangians of the Dirac - Born - Infeld type as well as to find new integrable models with nonlinear coupling of Abelian gauge fields to gravity.

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