A Unified Model of Dark Energy Based on
the Mandelstam–Tamm Uncertainty Relation

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Abstract

It is commonly recognized now that Dark Energy (Lambda-term) is of crucial importance both at the early (inflationary) stage of cosmological evolution and at the present time. However, little is known about its nature and origin till now. In particular, it is still unclear if Lambda-term is a new fundamental constant or represents just an effective contribution from the underlying field theory. Here, we show that a quite promising and universal approach to this problem might be based on the Mandelstam–Tamm uncertainty relation of quantum mechanics. As a result, we get the effective Lambda-term that is important throughout the entire history of the Universe. Besides, such an approach requires a substantial reconsideration of some other cosmological parameters, e.g., the age of the Universe.

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I. INTRODUCTION

As is known, the \( \Lambda \)-term (or cosmological constant) was introduced by Einstein immediately during development of General Relativity, but it was thought for a long time to be insignificant in cosmology. The situation changed in the early 1980’s, when the inflationary models were suggested to resolve the problem of homogeneity of the early Universe, the absence of singularity, etc. \cite{1,2}. The crucial ingredient of such models was the stage of exponential expansion caused by the dominance of the effective \( \Lambda \)-term, resulting from some kind of the underlying theory of elementary particles (such as supersymmetry, Higgs fields, etc.).

About two decades later, the redshift distribution of supernovae Ia as well as the analysis of CMB fluctuations enforced us to conclude that the modern evolution of the Universe is also governed by the \( \Lambda \)-term, but with much less magnitude than during inflation. Such \( \Lambda \)-term was pictorially called the Dark Energy.

So, the current point of view \cite{3} is that the \( \Lambda \)-term plays a decisive role both in the very early and present-day Universe. However, its values at these stages are absolutely different and, therefore, it is unclear if such \( \Lambda \)-terms are produced by the same physical mechanism or they are absolutely different physical entities? It is the aim of the present paper to demonstrate that a quite universal explanation of the time-dependent \( \Lambda \)-term, which is significant at different stages of cosmological evolution, can be given by the quantum-mechanical uncertainty relation in the Mandelstam–Tamm form.

II. THEORETICAL MODEL

The standard Heisenberg uncertainty relation

\[
\Delta x \Delta p \geq \frac{1}{2} \hbar
\]  \hspace{1cm} (1)

for variances of the coordinate and momentum (or any other pair of non-commuting quantities) can be obtained by the straightforward averaging of the corresponding quantum operators.

The situation is much more subtle for the relation between the energy and time

\[
\Delta E \Delta t \geq \frac{1}{2} \hbar,
\]  \hspace{1cm} (2)
because the time is an independent parameter in the quantum-mechanical equations rather than the operator acting in the Hilbert space of quantum states. In fact, the inequality (2) was written by Heisenberg from the dimensionality arguments already in the early days of quantum mechanics. However, its first rigorous proof in the framework of a particular model was given only about two decades later [4] and, therefore, it is usually called in the modern literature the Mandelstam–Tamm (rather than Heisenberg) uncertainty relation. A comprehensive review of the recent works on various interpretations of inequality (2) can be found in paper [5].

Apart from their primary application to the measurement problems, the uncertainty relations can be efficiently used for estimating the characteristic parameters of the quantum systems. (The most well-known example is the evaluation of typical size and energy of the ground-state hydrogen atom, provided that the equality sign is employed.) So, the basic idea of our subsequent consideration will be to use the Mandelstam–Tamm uncertainty relation for getting the quantum-mechanical vacuum energy, associated with Λ-term.

Returning to the cosmological model, we start with the usual Robertson–Walker metric:

$$ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right],$$

where $c$ is the speed of light, $R$ is the scale factor of the Universe; $r$, $\theta$, and $\varphi$ are the dimensionless coordinates; $k = 1, 0, \text{ and } -1$ for the closed, flat, and open three-dimensional space.

For this metric, the General Relativity equations are reduced to the Friedmann equation [6]:

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3c^2} \rho - k c^2 \frac{1}{R^2} + \frac{c^2}{3} \Lambda,$$

where $H$ is the Hubble parameter, $G$ is the gravitational constant, $\rho$ is the energy density of matter in the Universe, and dot denotes differentiation with respect to time.

The Λ-term can be related to the vacuum energy density $\rho_v$ [6]:

$$\Lambda = \frac{8\pi G \rho_v}{c^2}.$$  

Next, assuming that $\Delta E = \rho_v l_P^3$ is the vacuum energy in the Planck volume (where $l_P = \sqrt{\frac{G\hbar}{c^3}}$), and $\Delta t \equiv t$ is the total time of cosmological evolution, we can apply the Mandelstam–Tamm uncertainty relation (2) with the equality sign. This results in

$$\Lambda(t) = \frac{4\pi}{c l_P} \frac{1}{t}.$$  

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Finally, substituting expression (6) into (4), we get the basic equation of our cosmological model:

\[
H^2 \equiv \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3c^2} \rho - \frac{kc^2}{R^2} + \frac{4\pi}{3\tau} \frac{1}{t},
\]

(7)

where \( \tau = l_P/c = \sqrt{G\hbar/c^5} \) is the Planck time.

To reveal the most important features of this equation, let us consider the case when the Universe is spatially flat \((k=0)\), and the energy density of matter is ignorable \((\rho \approx 0)\). Then, formula (7) is simplified to

\[
H^2 \equiv \left( \frac{\dot{R}}{R} \right)^2 = \frac{4\pi}{3\tau} \frac{1}{t},
\]

(8)

which can be trivially integrated and results in

\[
R(t) = R^* \exp \left[ \sqrt{\frac{16\pi}{3}} \sqrt{\frac{t}{\tau}} \right],
\]

(9)

where the integration constant was chosen so that \( R(0) = R^* \), and we consider only the solution increasing with time.

Finally, let us note that according to formula (7)

\[
T = \frac{4\pi}{3} \frac{1}{\tau H_0} \frac{1}{H_0},
\]

(10)

where \( T \) is the age of the Universe, and \( H_0 \) is the present-day value of the Hubble parameter.

III. DISCUSSION AND CONCLUSIONS

1. The proposed cosmological model provides a natural explanation for the existence of Dark Energy (effective \( \Lambda \)-term) throughout the entire life of the Universe.

2. As distinct from the “standard” cosmology, where scale factor \( R(t) \) evolves either exponentially (when the Dark Energy is dominant) or by a power law (when ordinary matter dominates), in our model \( R(t) \) evolves by the same universal law (9), which is much slower than a pure exponential expansion but much faster than any power-like dependence.

3. While in the standard cosmology the Hubble parameter either remains constant with time (when evolution is determined by the \( \Lambda \)-term) or decays as \( \alpha/t \) (where \( \alpha = 1/2 \) when radiation dominates, and \( \alpha = 2/3 \) when a non-relativistic matter dominates),
our equation (8) predicts the inverse square-root dependence $H(t) \propto 1/\sqrt{t}$, i.e., again some intermediate case between the two extremal situations.

4. Taking into account that age of the Universe in the standard cosmology is, roughly speaking, inversely proportional to the present-day value of the Hubble parameter, $T^* \approx 1/H_0$, we see that our relation (10) predicts that $T \approx (T^*/\tau) T^*$. Since $T^* \approx 4\times10^{17}$ s, and $\tau = 5\times10^{-44}$ s, this age in our model will be increased by $(T^*/\tau) \approx 10^{61}$ times. In other words, the Universe becomes “quasi-perpetual”. However, this anomalous lifetime should not be a fatal failure of the model: Really, as far as we know, the most of problematic issues in the modern cosmology are caused just by the insufficient lifetime of the Universe (so that some astronomical objects do not have sufficient time to be formed). On the other hand, it is difficult to say without a detailed analysis if there will be some crucial obstacles in the case of the anomalously long lifetime.

5. It will be desirable, of course, to consider more general solutions of the equation (7), involving a few matter components. Besides, due to the substantially different temporal evolution of the Universe as compared to the “standard” model, the processes of nucleosynthesis, cosmological structure formation, etc. should be carefully recalculated.

6. At last, one of hot topics of cosmology in the recent years is a systematic discrepancy in the values of the present-day Hubble parameter $H_0$ derived by the various methods (namely, by the employment of supernovae Ia and Cepheids as the “standard candles”, on the one hand, and by the analysis of CMB spectrum, on the other hand) [7, 8]. Since in the second case the resulting value of $H_0$ substantially depends on the expansion history of the Universe $R(t)$, the most of the recent attempts to resolve the above-mentioned discrepancy were based on the empirical modifications of the equation of state of the Dark Energy [9–12]. In this sense, the substantially modified temporal dependence (9), naturally following from our model, might be an additional option for such attempts.

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