On a Harmonic Univalent Subclass of Functions Involving a Generalized Linear Operator

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Abstract: In this paper, a subclass of complex-valued harmonic univalent functions defined by a generalized linear operator is introduced. Some interesting results such as coefficient bounds, compactness, and other properties of this class are obtained.

Keywords: harmonic univalent functions; generalized linear operator; differential operator; Salagean operator; coefficient bounds

1. Introduction

Let $H$ represent the continuous harmonic functions which are harmonic in the open unit disk $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$ and let $A$ be a subclass of $H$ which represents the functions which are analytic in $U$. A harmonic function in $U$ could be expressed as $f = h + \overline{g}$, where $h$ and $g$ are in $A$, $h$ is the analytic part of $f$, $g$ is the co-analytic part of $f$ and $|h'(z)| > |g'(z)|$ is a necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $U$ (see Clunie and Sheil-Smith [1]). Now we write,

$$ h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \quad (1) $$

Let $SH$ represents the functions of the form $f = h + \overline{g}$ which are harmonic and univalent in $U$, which normalized by the condition $f(0) = f_z(0) - 1 = 0$. The subclass $SH^0$ of $SH$ consists of all functions in $SH$ which have the additional property $f_z(0) = 0$. The class $SH$ was investigated by Clunie and Sheil-Smith [1]. Since then, many researchers have studied the class $SH$ and even investigated some subclasses of it. Also, we observe that the class $SH$ reduces to the class $S$ of normalized analytic univalent functions in $U$, if the co-analytic part of $f$ is equal to zero. For $f \in S$, the Salagean differential operator $D^m(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ was defined by Salagean [2]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [3] defined the modified Salagean operator of $f$ as

$$ D^m f(z) = D^m h(z) + (-1)^m D^m g(z), $$

where

$$ D^m h(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n, \quad D^m g(z) = \sum_{n=2}^{\infty} n^m b_n z^n. $$

Next, for functions $f \in A$, For $n \in \mathbb{N}_0$, $\beta \geq \gamma \geq 0$, Yalçın and Altunkaya [4] defined the differential operator of $I^m_{\beta,\gamma} f : SH^0 \rightarrow SH^0$. Now we define our differential operator:
Let the function \( f \) be defined by (1). Then \( f \) exists an analytic function and studied by Swamy [8], introduced and studied by Ramadan and Darus [6]. By taking different choices of \( \mu, \lambda, \delta, \tau \) and \( \varsigma \) we get \( I^m_{\delta, \mu, \lambda, \varsigma, \tau} f(z) \) was introduced and studied by Darus and Ibrahim [7]. \( I^m_{\mu, \lambda, \varsigma, \tau} f(z) \) was introduced and studied by Swamy [8]. \( I^m_{1-\mu, 0, 1, \delta} f(z) \) was introduced and studied by Al-Oboudi [9] and \( I^m_{0, 0, 1, \delta} f(z) \) was introduced and studied by Salagean [2].

If \( f \in A \), then when we take \( \mu = 1, \lambda = 0, \delta = 0, \tau = 1, \varsigma = 1 \) we obtain \( I^m_{0, \tau, \delta, \varsigma} f(z) \) was introduced and studied by Ramadan and Darus [6]. By taking different choices of \( \mu, \lambda, \delta, \tau \) and \( \varsigma \) we get \( I^m_{1-\mu, 0, \delta, \varsigma} f(z) \) was introduced and studied by Darus and Ibrahim [7]. \( I^m_{\mu, \lambda, \varsigma, \delta} f(z) \) was introduced and studied by Swamy [8]. \( I^m_{1-\mu, 0, 1, \delta} f(z) \) was introduced and studied by Al-Oboudi [9] and \( I^m_{0, 0, 1, \delta} f(z) \) was introduced and studied by Salagean [2].

A function \( f : U \to C \) is subordinate to the function \( g : U \to C \) denoted by \( f(z) \prec g(z) \), if there exists an analytic function \( w : U \to U \) with \( w(0) = 0 \) such that
\[
f(z) = g(w(z)), (z \in U).
\]

Moreover, if the function \( g \) is univalent in \( U \), then we have (see [11,12]):
\[
f(z) \prec g(z) \text{ if and only if } f(0) = g(0), f(U) \subset g(U).
\]

Denote by \( SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) the subclass of \( SH^0 \) consisting of functions of the form (1) that satisfy the condition
\[
I^{m+1}_{\delta, \mu, \lambda, \varsigma, \tau} f(z) \leq \frac{1 + A \varsigma}{1 + B \varsigma}, -1 \leq A < B \leq 1
\]
where \( I^m_{\delta, \mu, \lambda, \varsigma, \tau} f(z) \) is defined by (4). For relevant and recent references related to this work, we refer the reader to [13–20].

In this paper we use the same techniques that have been used earlier by Dziołk [21] and Dzioł et al. [22], to investigate coefficient bound, distortion bounds, and some other properties for the class \( SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \).

### 2. Coefficient Bounds

In this section we find the coefficient bound for the class \( SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \).

**Theorem 1.** Let the function \( f(z) = h + g \) be defined by (1). Then \( f \in SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) if
\[
\sum_{n=2}^{\infty} (C_n |a_n| + D_n |b_n|) \leq B - A
\]
where
\[ C_n = \left( \frac{\mu + \lambda + (\delta - \zeta)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right)^m \left( \frac{(\delta - \zeta)(\lambda - \tau)(n - 1)[B + 1] - (\mu + \lambda)(B - A)}{\mu + \lambda} \right) \]  
(7)

and
\[ D_n = \left( \frac{\mu + \lambda + (\delta - \zeta)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right)^m \left( \frac{[A + B(2 + (\delta - \zeta)(\lambda - \tau)(n - 1))](\mu + \lambda)}{\mu + \lambda} \right). \]  
(8)

**Proof.** Let \( a_n \neq 0 \) or \( b_n \neq 0 \) for \( n \geq 2 \). Since \( C_n, D_n \geq n(B - A) \) by (6), we obtain
\[ |f'(z)| - |g'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} - \sum_{n=2}^{\infty} n|b_n||z|^{n-1} \]
\[ \geq 1 - |z| \sum_{n=2}^{\infty} (n|a_n| + n|b_n|) \]
\[ \geq 1 - \frac{|z|}{B - A} \sum_{n=2}^{\infty} (C_n|a_n| + D_n|b_n|) \]
\[ \geq 1 - |z| > 0. \]

Therefore, \( f \) is univalent in \( U \). To ensure the univalence condition, consider \( z_1, z_2 \in U \) so that \( z_1 \neq z_2 \). Then
\[ \left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| = \left| \sum_{m=1}^{n} z_1^{m-1} - z_2^{m-1} \right| \leq \sum_{m=1}^{n} |z_1|^{m-1} \left| \frac{z_1 - z_2}{n} \right| < n, \quad n \geq 2. \]

So, we have
\[ \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{n=2}^{\infty} b_n (z_1^n - z_2^n)}{z_1 - z_2 + \sum_{n=2}^{\infty} n|a_n| (z_1^n - z_2^n)} \right| \]
\[ > 1 - \frac{\sum_{n=2}^{\infty} n|a_n|}{\sum_{n=2}^{\infty} |z_1^n - z_2^n| |a_n|} \geq 1 - \frac{\sum_{n=2}^{\infty} |b_n|}{\sum_{n=2}^{\infty} |z_1^n - z_2^n| |a_n|} \geq 0, \]
which proves univalences.

On the other hand, \( f \in SH^p(\delta, \mu, \lambda, \zeta, \tau, m, A, B) \) if and only if there exists a function \( w \) with \( w(0) = 0 \), and \( |w(z)| < 1(z \in U) \) such that
\[ \frac{I_{\delta, \mu, \lambda, \zeta, \tau}^{m+1} f(z)}{I_{\delta, \mu, \lambda, \zeta, \tau}^m f(z)} < 1 + Az \]
\[ B \frac{J_{\delta, \mu, \lambda, \zeta, \tau}^{m+1} f(z) - J_{\delta, \mu, \lambda, \zeta, \tau}^m f(z)}{A I_{\delta, \mu, \lambda, \zeta, \tau}^{m+1} f(z) - A I_{\delta, \mu, \lambda, \zeta, \tau}^m f(z)} < 1, \quad (z \in U). \]  
(9)
The above inequality (9) holds, since for \(|z| = r (0 < r < 1)\) we obtain

\[
\left| \sum_{n=2}^{\infty} \left( \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} \right)^m \frac{(\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} a_n z^n \right| + (-1)^m \left| \sum_{n=2}^{\infty} \left( \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} \right)^m \frac{2(\mu + \lambda) + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} b_n z^n \right|
\]

\[
- (B - A) z + \sum_{n=2}^{\infty} \left( \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} \right)^m \left( B \frac{2(\mu + \lambda) + (\delta - \varsigma)(\lambda - \tau)(1-n)}{\mu + \lambda} + A \right) b_n z^n
\]

\[
\leq r \left( \sum_{n=2}^{\infty} (C_n |a_n| + D_n |b_n|) r^{n-1} - (B - A) \right) < 0.
\]

Therefore, \( f \in SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \), and so the proof is completed.

Next we show that the condition (6) is also necessary for the functions \( f \in H \) to be in the class \( SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) = T^m \cap SH^0(b, \mu, \lambda, \varsigma, \tau, m, A, B) \) where \( T^m \) is the class of functions \( f = h + \bar{g} \in SH^0 \) so that

\[
f = h + \bar{g} = z - \sum_{n=2}^{\infty} a_n z^n + (-1)^m \sum_{n=2}^{\infty} b_n \bar{z}^n (z \in U).
\]

\( \Box \)

**Theorem 2.** Let \( f = h + \bar{g} \) be defined by (10). Then \( f \in SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) if and only if the condition (6) holds.

**Proof.** For this proof, we let the fractions \( \frac{(\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} = L \) and \( \frac{2(\mu + \lambda) + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} = K \). The first part “if statement” follows from Theorem 1. Conversely, we suppose that \( f \in SH^0_T(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \), then by (9) we have

\[
\left| \sum_{n=2}^{\infty} \left( \frac{L^m (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} |a_n| \bar{z}^n + (K)^m \frac{2(\mu + \lambda) + (\delta - \varsigma)(\lambda - \tau)(n-1)}{\mu + \lambda} |b_n| \bar{z}^n \right) \right| < 1.
\]

\[
(B - A) z - \sum_{n=2}^{\infty} \left[ \left( L^m (BL - A) |a_n| \bar{z}^n + (K)^m (BK + A) |b_n| \bar{z}^n \right) \right] < 1.
\]
Theorem 3. The class $SH^0$ is convex and compact subset of $SH$.

Proof. Let $f_1 \in SH^0_1(\delta, \mu, \lambda, \zeta, \tau, m, A, B)$, where

$$f_1(z) = z - \sum_{n=2}^{\infty} [a_{1,n}] z^n + (-1)^m \sum_{n=2}^{\infty} [b_{2,n}] z^n (z \in U, t \in \mathbb{N}).$$

Then for $0 \leq \psi \leq 1$, let $f_1, f_2 \in SH^0_1(\delta, \mu, \lambda, \zeta, \tau, m, A, B)$ be defined by (12). Then

$$\xi(z) = \psi f_1(z) + (1 - \psi) f_2(z)$$

$$= z - \sum_{n=2}^{\infty} (\psi [a_{1,n}] + (1 - \psi) [a_{2,n}]) z^n + (-1)^m \sum_{n=2}^{\infty} (\psi [b_{1,n}] + (1 - \psi) [b_{2,n}]) z^n$$

and

$$\sum_{n=2}^{\infty} \left[ C_n (\psi [a_{1,n}] + (1 - \psi) [a_{2,n}]) + D_n \left(\psi [b_{1,n}] + (1 - \psi) [b_{2,n}]\right)\right]$$

$$= \psi \sum_{n=2}^{\infty} \left[ C_n [a_{1,n}] + D_n [b_{1,n}]\right] + (1 - \psi) \sum_{n=2}^{\infty} \left[ C_n [a_{2,n}] + D_n [b_{2,n}]\right]$$

$$\leq \psi (B - A) + (1 - \psi) (B - A) = B - A.$$

Thus, the function $\xi = \psi f_1(z) + (1 - \psi) f_2(z)$ is in the class $SH^0_1(\delta, \mu, \lambda, \zeta, \tau, m, A, B)$. This implies that $SH^0_1(\delta, \mu, \lambda, \zeta, \tau, m, A, B)$ is convex.
For \( f_t \in \text{SH}^0_T(\delta, \mu, \lambda, \zeta, \tau, m, A, B), \ t \in \mathbb{N} \) and \( |z| \leq r \ (0 < r < 1) \), then we have

\[
|f_t(z)| \leq r + \sum_{n=2}^{\infty} \left( |a_{t,n}| + |b_{t,n}| \right) r^n \\
\leq r + \sum_{n=2}^{\infty} \left[ C_n |a_{t,n}| + D_n |b_{t,n}| \right] r^n \\
\leq r + (B - A) r^2.
\]

Therefore, \( \text{SH}^0_T(\delta, \mu, \lambda, \zeta, \tau, m, A, B) \) is uniformly bounded. Let

\[
f_t(z) = z - \sum_{n=2}^{\infty} |a_{t,n}| z^n + (-1)^m \sum_{n=2}^{\infty} |b_{t,n}| z^n (z \in U, \ t \in \mathbb{N}).
\]

also, let \( f = h + \overline{g} \) where \( h \) and \( g \) are given by (1). Then by Theorem 2 we get

\[
\sum_{n=2}^{\infty} \left[ C_n |a_{n}| + D_n |b_{n}| \right] \leq B - A.
\]

(13)

If we assume \( f_t \rightarrow f \), then we get that \( |a_{t,n}| \rightarrow |a_n| \) and \( |b_{t,n}| \rightarrow |b_n| \) as \( n \rightarrow +\infty \) \((t \in \mathbb{N})\). Let \( \{\rho_n\} \)
be the sequence of partial sums of the series \( \sum_{n=2}^{\infty} \left[ C_n |a_{t,n}| + D_n |b_{t,n}| \right] \). Then \( \{\rho_n\} \) is a non-decreasing sequence and by (13) it is bounded above by \( B - A \). Thus, it is convergent and

\[
\sum_{n=2}^{\infty} \left[ C_n |a_{t,n}| + D_n |b_{t,n}| \right] = \lim_{n \rightarrow +\infty} \rho_n \leq B - A.
\]

Therefore, \( f \in \text{SH}^0_T(\delta, \mu, \lambda, \zeta, \tau, m, A, B) \) and therefore the class \( \text{SH}^0_T(\delta, \mu, \lambda, \zeta, \tau, m, A, B) \) is closed. As a result, the class is closed, and the class \( \text{SH}^0_T(\delta, \mu, \lambda, \zeta, \tau, m, A, B) \) is also compact subset of \( \text{SH} \), which completes the proof. \( \square \)

Lemma 1 [23]. Let \( f = h + \overline{g} \) be so that \( h \) and \( g \) are given by (1). Furthermore, let

\[
\sum_{n=2}^{\infty} \left\{ \frac{n - \alpha}{1 - \alpha} |a_n| + \frac{n + \alpha}{1 - \alpha} |b_n| \right\} \leq 1 (z \in U)
\]

where \( 0 \leq \alpha < 1 \). Then \( f \) is harmonic, orientation preserving, univalent in \( U \) and \( f \) is starlike of order \( \alpha \).

Theorem 4. Let \( 0 \leq \alpha < 1, C_n \) and \( D_n \) be defined by (7) and (8). Then

\[
r^*_\alpha \left( \text{SH}^0_T(\delta, \mu, \lambda, \zeta, \tau, n, A, B) \right) = \inf_{n \geq 2} \left[ \frac{1 - \alpha}{B - A} \min \left\{ \frac{C_n}{n + \alpha}, \frac{D_n}{n + \alpha} \right\} \right]^{1/\alpha},
\]

(14)

where \( r^*_\alpha \) is the radius of starlikeness of order \( \alpha \).
Proof. Let \( f \in SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) be of the form (10). Then, for \(|z| = r < 1\), we get

\[
\begin{align*}
\frac{|l_{0,n} f(z) - (1+\alpha) f(z)|}{|l_{0,n} f(z) + (1+\alpha) f(z)|} &= \frac{-\alpha - \sum_{k=2}^{n+1} \alpha^k |\alpha|^{k-1} \sum_{k=n+1}^{n+1} \alpha^k |\alpha|^{k-1} |b_k|}{(2-\alpha)z \sum_{k=2}^{n+1} \alpha^k |\alpha|^{k-1} \sum_{k=n+1}^{n+1} \alpha^k |\alpha|^{k-1} |b_k|} \\
&\leq \alpha - \sum_{n=2}^{\infty} \left( (n+1-\alpha)|b_n| - (1-\alpha) \frac{n+\alpha}{1+\alpha} |b_n| \right) \sum_{k=n+1}^{\infty} \left( (n+1-\alpha)|b_k| - (1-\alpha) \frac{n+\alpha}{1+\alpha} |b_k| \right)^{n-1}.
\end{align*}
\]

By using Lemma 1, we observe that \( f \) is starlike of order \( \alpha \) in \( U_r \) if and only if

\[
\frac{|l_{0,n} f(z) - (1+\alpha) f(z)|}{|l_{0,n} f(z) + (1+\alpha) f(z)|} < 1, z \in U_r,
\]

or

\[
\sum_{n=2}^{\infty} \left( \frac{n-\alpha}{1-\alpha} |b_n| + \frac{n+\alpha}{1+\alpha} |b_n| \right) |b_n|^{n-1} \leq 1. \tag{15}
\]

Furthermore, by using Theorem 2, we get

\[
\sum_{n=2}^{\infty} \left( \frac{C_n}{1-\alpha} + \frac{D_n}{1+\alpha} |b_n| \right) |b_n|^{n-1} \leq 1.
\]

Condition (15) is true if

\[
\frac{n-\alpha}{1-\alpha} \rho^{n-1} \leq \frac{C_n}{B-A} \rho^{n-1}.
\]

This proves

\[
\frac{n+\alpha}{1+\alpha} \rho^{n-1} \leq \frac{D_n}{B-A} \rho^{n-1} \quad (n = 2, 3, \ldots).
\]

So, the function \( f \) is starlike of order \( \alpha \) in the disk \( U_{r_{\alpha}} \) where

\[
r_{\alpha} = \inf_{n \geq 2} \left( \frac{1-\alpha}{B-A} \min \left( \frac{C_n}{n+\alpha}, \frac{D_n}{n+\alpha} \right) \right)^{\frac{1}{n-1}},
\]

and the function

\[
f_n(z) = h_n(z) + g_n(z) = z - \frac{B-A}{C_n} z^n + (\frac{B-A}{D_n} z^n).
\]

So, the radius \( r_{\alpha} \) cannot be larger. Then we get (14). \( \square \)

4. Extreme Points

In this section we find the extreme points for the class \( SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \).

Theorem 5. The extreme points of \( SH^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) are the functions \( f \) of the form (1) where \( h = h_k \) and \( g = g_k \) are of the form

\[
\begin{align*}
h_1(z) &= z, \\
h_n(z) &= z - \frac{B-A}{C_n} z^n, \\
g_n(z) &= (-1)^m \frac{B-A}{D_n} z^n, (z \in U, n \geq 2).
\end{align*}
\]
**Proof.** Suppose that \( g_n = \psi f_1 + (1 - \psi) f_2 \) where \( 0 < \psi < 1 \) and \( f_1, f_2 \in SH_0^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) are written in the form

\[
f_t(z) = z - \sum_{n=2}^{\infty} |a_{t,n}| z^n + (-1)^m \sum_{n=2}^{\infty} |b_{t,n}| z^n (z \in U, t \in \{1, 2\}).
\]

Then, by (16), we get

\[
|b_{1,n}| = |b_{2,n}| = \frac{B - A}{D_n},
\]

and \( a_{1,t} = a_{2,t} = 0 \) for \( t \in \{2, 3\ldots\} \) and \( b_{1,t} = b_{2,t} = 0 \) for \( t \in \{2, 3\ldots\} \setminus \{n\} \). It follows that \( g_n(z) = f_1(z) = f_2(z) \) and \( g_n \) are in the class of extreme points of the class \( SH_0^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \). We also can ensure that the functions \( h_n(z) \) are the extreme points of the class \( SH_0^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \).

Now, assume that a function \( f \) of the form (1) is in the class of the extreme points of the class \( SH_0^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) and \( f \) is not of the form (16). Then there exists \( k \in \{2, 3\ldots\} \) such that

\[
0 < |a_k| < \frac{B - A}{\left( \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right)^m \left\{ \frac{\lambda + B(2 + (\delta - \varsigma)(\lambda - \tau)(n - 1))}{\mu + \lambda} \right\}}{\eta},
\]

or

\[
0 < |b_k| < \frac{B - A}{\left( \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right)^m \left\{ \frac{\lambda + B(2 + (\delta - \varsigma)(\lambda - \tau)(n - 1))}{\mu + \lambda} \right\}}{\eta}.
\]

If

\[
0 < |a_k| < \frac{B - A}{\left( \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right)^m \left\{ \frac{\lambda + B(2 + (\delta - \varsigma)(\lambda - \tau)(n - 1))}{\mu + \lambda} \right\}},
\]

then putting

\[
\psi = \frac{\left\{ \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right\}^m \left\{ \frac{\lambda + B(2 + (\delta - \varsigma)(\lambda - \tau)(n - 1))}{\mu + \lambda} \right\}}{B - A},
\]

and

\[
\chi = \frac{f - \psi h_k}{1 - \psi},
\]

we have \( 0 < \psi < 1 \), \( h_k \neq \chi \). Therefore, \( f \) is not in the class of the extreme points of the class \( SH_0^0(\delta, \mu, \lambda, \eta, \varsigma, \tau, m, A, B) \). Similarly, if

\[
0 < |b_k| < \frac{B - A}{\left( \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right)^m \left\{ \frac{\lambda + B(2 + (\delta - \varsigma)(\lambda - \tau)(n - 1))}{\mu + \lambda} \right\}},
\]

then putting

\[
\psi = \frac{\left\{ \frac{\mu + \lambda + (\delta - \varsigma)(\lambda - \tau)(n - 1)}{\mu + \lambda} \right\}^m \left\{ \frac{\lambda + B(2 + (\delta - \varsigma)(\lambda - \tau)(n - 1))}{\mu + \lambda} \right\}}{B - A},
\]

and

\[
\chi = \frac{f - \psi g_k}{1 - \psi},
\]

we have \( 0 < \psi < 1 \), \( g_k \neq \chi \). It follows that \( f \) is not in the family of extreme points of the class \( SH_0^0(\delta, \mu, \lambda, \varsigma, \tau, m, A, B) \) and so the proof is completed. □
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