New approach to summation of field-theoretical series in models with strong coupling

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Abstract

A new approach to summation of divergent field-theoretical series is suggested. It is based on the Borel transformation combined with a conformal mapping and does not imply the knowledge of the exact asymptotic parameters. The method is tested on functions expanded in their asymptotic power series and applied to estimating the ground state energy of simple quantum mechanical problems including anisotropic oscillators and calculating the critical exponents for certain conformal field models. It can be expected that the new approach to summation may be used to obtaining numerical estimates for important physical quantities represented by divergent series in two- and three-dimensional field models.
In this brief report we discuss the problem of summation of "bad" series arising in various fields of physics [1, 2]. By the strong coupling we mean the situation when the parameter of expansion does not belong to the range of convergence of the perturbative series. The typical situation is when the radius of convergence is just zero and the resulting series are asymptotic. On the other hand, these series can represent important physical quantities, and to extract reliable information from them, they should be processed by a proper resummation technique.

The main goal of the present work is to suggest a new approach to treating divergent series and to apply it to finding numerical estimates of the critical exponents for two- and three dimensional anisotropic field models. The resummation technique proposed is tested on simple model functions expanded in their asymptotic power series, estimating ground state energy of isotropic and cubic anharmonic oscillators, Yukawa potential, and critical exponents for conformal field theories.

At present, there are several resummation procedures such as simple Padé, Padé-Borel, and Padé-Borel-Leroy techniques, whose application, however, is limited to series with coefficients alternating in signs. For the more sophisticated method based on the Borel transformation combined with a conformal mapping, first proposed in Refs. [3, 4], this limitation is not crucial. This technique was then elaborated and systematically used for various phase transition problems [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] being now regarded to as a most universal procedure. But it requires the knowledge the exact asymptotic high-order behavior of the series. As a rule the coefficients of the series \( \sum_k f_k g^k \) behave at large \( k \) as \( k! k^\omega (-a_0)^k \). The numbers \( a_0 \) and \( b_0 \) characterize the main divergent part. Nowadays these parameters are found only for the simplest case of the \( O(N) \)-symmetric models [13, 16, 17], and calculating them for anisotropic models is a most difficult problem as yet unsolved. As an exception we mention the anisotropic quartic quantum oscillator which represents a one-dimensional \( \varphi^4 \) field theory with the cubic anisotropy. For the perturbation expansion of the ground state energy of this system the asymptotic parameters were found in Ref. [18]. Within the assumption of the weak anisotropy the large-order asymptotic behavior of the \( \beta \)-functions for the cubic model was deduced in Ref. [19] and then used for determination of the stability of the cubic fixed point in three dimensions [20].

Below, we make an attempt to overcome the outlined difficulties and suggest a new approach to summation of the divergent field-theoretical series, which is based on the standard technique Borel transformation combined with a conformal mapping [1, 2], but which does not involve the exact values of the asymptotic parameters. We start from the Borel-Leroy transformation modified with a conformal mapping in the form [7]

\[
F(g; a, b) = \sum_{k=0}^{\infty} A_k(\lambda) \int_0^\infty e^{-\frac{x}{ag}} \left( \frac{x}{ag} \right)^b d\left( \frac{x}{ag} \right) \frac{\omega^k(x)}{(1 - \omega^k(x))^{2\lambda}}.
\]

The coefficients \( A_k(\lambda) \) are determined from the equality \( B(x(\omega)) = \frac{A(\lambda, \omega)}{(1 - \omega)^{2\lambda}} \) where
\[ \omega = \frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1} \] and the Borel transform \( B(x) \) is the analytical continuation of the series 
\[ \sum_k \frac{f_k}{a_1(b+k+1)} x^k \] absolutely convergent in the unit circle, \( f_k \) being the coefficients of the original series. Parameter \( \lambda \) is chosen from the condition of the most rapid convergence of series (1), that is from minimizing the quantity 
\[ |1 - \frac{F_L(g; a, b)}{F_{L-1}(g; a, b)}|, \]
where \( L \) is the step of truncation and \( F_L(g; a, b) \) is the \( L \)-partial sum for \( F(g; a, b) \). In the regular scheme [4, 7, 8] parameters \( a \) and \( b \) are related to the exact asymptotic values \( a_0 \) and \( b_0 \). Since, in practice we deal with a piece of the series only, where the asymptotic regime might not be established, we vary parameters \( a \) and \( b \) in a neighbourhood of their exact values [21]. Our principle observation is that the result of processing \( F_L(g; a, b) \) exhibit very weak dependence on the transformation parameters \( a \) and \( b \) varying in a wide range. The dependence becomes weaker with the growth of the approximation order and the smaller is the parameter of expansion \( g \) the better this property holds. We put the stability of the result of processing with respect to variation of \( a \) and \( b \) into the foundation of our technique to summation of divergent series. Such an approach allows us to apply the transformation (1) even if the exact asymptotic behaviour of the series being processed is unknown.

The detailed analysis of the formulated resummation scheme applied to the model functions

\[ F(g) = \int_{-\infty}^{+\infty} e^{-x^2 - gx^4} dx \sim \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(2k + \frac{1}{2})}{k!} g^k, \]

\[ E(g) = \int_{0}^{\infty} e^{-x} (x \partial_x)^b_0 \frac{1}{1 + gx} dx \sim \sum_{k=0}^{\infty} (-1)^k k! b_0^k g^k \]

was performed in Ref. [22]. The convergence of the process for the function \( E(g) \), \( g = 1 \) is presented in Fig.1. Let us demonstrate our summation approach by estimating the ground state energy \( E(g) \) for several simple quantum mechanical systems. Consider first the isotropic anharmonic oscillator [23] with the Hamiltonian \( H = x^2 + gx^4 \). We observe the same stability of the result of processing \( E(g) \) with respect to \( a \) and \( b \) as for the model functions. The ground state energy estimates for \( g = 1 \) depending on the approximation order are listed in Table I.

| L | \( E(1) \) |
|---|---|
| 8 | 1.392376 |
| 9 | 1.392357 |
| 10 | 1.392344 |
| 11 | 1.392349 |
| 12 | 1.392351 |

For \( L = 8 \) our estimate is closer by one order to the exact value than the number \( 1.391655 \pm 0.004562 \) found in Ref. [24] on the basis of Wynn’s \( \epsilon \)-algorithm.

The ground state energy \( E(g) \) of the cubic anharmonic oscillator with the Hamiltonian

\[ H = \frac{1}{2}(x^2 + y^2) + \frac{g}{4}[x^4 + 2(1 - \delta)x^2y^2 + y^4] \]
depending of the anisotropy parameter $\delta$ and the value of the coupling constant $g$ was estimated in Ref. [18]. Those calculations were based on the knowlege of the exact values of the asymptotic parameters. The ground state energy estimated on the basis of our approach proved to be very close to the values of Ref. [18]. The results given by two different methods are listed in Table II. The stability of the result of processing $E(g)$ ($g/4 = 0.1$, $\delta = -2.5$) with respect to the variation of the parameters $a$ and $b$ is shown in Fig.2.

We have also studied the ground state energy $E(g)$ of the Yukawa potential

$$V_g(x) = -\frac{1}{x} \exp(-gx).$$

The dependence $E(g)$ presented in Fig.3 is in agreement with the exact results [24, 25]. The exact critical value of $g$ when the bound state disappears is $g_c = 1.190612...$. Using Winn’s $\epsilon$-algorithm [24] gives $g_c = 1.1836$. The best estimate for $g_c$ in the frame of our approach yields $g_c = 1.191$. In Fig.4 we demonstrate the behavior of curves $E(b)$ for $g = 1.11$ close to $g_c$ where the domain of stability of the result of processing begins to dissipate (c.f. with Fig.2 for the cubic anharmonic oscillator where the parameter of expansion $g$ is small).

In the context of this report, it is interesting to study the convergence of the RG series of certain conformal field models, for which the exact results are known, using our summation procedure. In two dimensions, summation of $\epsilon$-series is a difficult problem because the parameter of expansion is large ($\epsilon = 2$). However, application of our approach gives, as a whole, relatively good estimates. So, using five-loop $\epsilon$-expansions for $O(N)$-symmetric model [26] and setting $N = 2 \cos(\frac{m}{2})$, for the Ising model ($m = 3$) we obtain $\nu = 0.925$, $\eta = 0.220$, $\gamma = 1.650$ while the exact theory predicts $\nu = 1$, $\eta = 0.25$, $\gamma = 1.75$. Obviously, the difference between our results and the exact values does not exceed 8%. For the model describing polymers ($m = 2$), our estimates are even better: $\nu = 0.747$ (exact 0.75), $\eta = 0.190$ (exact 0.208), and $\gamma = 1.352$ (exact 1.350). Above numbers are also in accordance with the results of Refs. [8, 11], where a substantially different method was employed.

The analysis of the simple models fulfilled above enables one to apply the summation approach introduced to finding numerical estimates of important physical quantities in a number of real models. For example, it was used for calculation of critical exponents for the $N$-vector field models describing magnetic phase transitions in cubic and tetragonal crystals [22, 27]. It can be expected that the developed technique may be useful in such areas as QCD and QED as well.

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TABLE II. Estimates of the ground state energy of the cubic anharmonic oscillator for various anisotropy parameters $\delta$, coupling constant $g$, and approximation order $L$.

Results by Kleinert et al., $g/4 = 0.1$

| $L \setminus \delta$ | -2.5 | -1.5 | -0.5 | 0.5  | 1.5  |
|----------------------|------|------|------|------|------|
| 7                    | 1.217107 | 1.192033 | 1.164803 | 1.134735 | 1.100604 |
| 9                    | 1.217107 | 1.192034 | 1.164810 | 1.134736 | 1.100604 |
| 11                   | 1.217107 | 1.192035 | 1.164810 | 1.134739 | 1.100604 |

Our estimates, $g/4 = 0.1$

| $L \setminus \delta$ | -2.5 | -1.5 | -0.5 | 0.5  | 1.5  |
|----------------------|------|------|------|------|------|
| 12                   | 1.21705  | 1.19203 | 1.16480 | 1.134730 | 1.00600  |

relative error $< 0.001\%$
relative deviation from the results of Ref.18 $< 0.001\%$
$0 \leq b \leq 60 \quad 0.5 \leq a \leq 1.5$

Results by Kleinert et al., $g/4 = 1.0$

| $L \setminus \delta$ | -2.5 | -1.5 | -0.5 | 0.5  | 1.5  |
|----------------------|------|------|------|------|------|
| 7                    | 1.941172  | 1.862806 | 1.773888 | 1.669172 | 1.535454 |
| 9                    | 1.941172  | 1.862815 | 1.773909 | 1.669188 | 1.535425 |
| 11                   | 1.941180  | 1.862823 | 1.773924 | 1.669199 | 1.535418 |

Our estimates, $g/4 = 1.0$

| $L \setminus \delta$ | -2.5 | -1.5 | -0.5 | 0.5  | 1.5  |
|----------------------|------|------|------|------|------|
| 12                   | 1.9411  | 1.8627 | 1.7731 | 1.6691 | 1.5363 |

relative error $< 0.06\%$
relative deviation from the results of Ref.18 $< 0.05\%$
$0 \leq b \leq 60 \quad 0.5 \leq a \leq 1.5$

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Fig. 1: Convergence of the estimates for the function $E(g)$, $g=1$.

Fig. 2: Resulting curves for the ground state energy $E(g)$ of the cubic anharmonic oscillator from the 12-th approximation order.
Fig. 3: Dependence of the ground state energy $E(g)$ for the Yukawa potential from the 8-th approximation order.

Fig. 4: Resulting curves for the Yukawa potential ground state energy for $g=1.11$. 