Advantages of the Discrete Stochastic Arithmetic to Validate the Results of the Taylor Expansion Method to Solve the Generalized Abel’s Integral Equation

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Abstract: The aim of this paper is to apply the Taylor expansion method to solve the first and second kinds Volterra integral equations with Abel kernel. This study focuses on two main arithmetics: the FPA and the DSA. In order to apply the DSA, we use the CESTAC method and the CADNA library. Using this method, we can find the optimal step of the method, the optimal approximation, the optimal error, and some of numerical instabilities. They are the main novelties of the DSA in comparison with the FPA. The error analysis of the method is proved. Furthermore, the main theorem of the CESTAC method is presented. Using this theorem we can apply a new termination criterion instead of the traditional absolute error. Several examples are approximated based on the FPA and the DSA. The numerical results show the applications and advantages of the DSA than the FPA.

Keywords: Generalized Abel’s integral equation; Taylor expansion method; Discrete stochastic arithmetic; CESTAC method; CADNA library

1. Introduction

In 1823, the AIE was studied by Niels Abel for solving mathematical physics problems [1–3]. Moreover, the generalized form of AIE with finite interval has been presented by Zeilon [4]. The AIEs are singular form of the Volterra IEs. Singular integral equations are among the important and applicable kinds of integral equations which have been solved by many authors [5–8]. This problem has many applications in various areas such as simultaneous dual relations [9], stellar winds [10], water wave [11], spectroscopic data [12], and others [1,13,14].

In this paper, we consider the following first kind AIE:

\[
\int_a^x \frac{\mathcal{P}(t)}{(\phi(x) - \phi(t))^{\alpha}} dt = g(x), \quad 0 < \alpha < 1,
\]

and the second kind

\[
\mathcal{P}(x) = g(x) + \int_a^x \frac{\mathcal{P}(t)}{(\phi(x) - \phi(t))^{\alpha}} dt, \quad 0 < \alpha < 1,
\]

where \(a\) is a given real value, \(g(x)\) and \(\phi(x)\) are known functions, and \(\mathcal{P}(x)\) is an unknown function that \(\phi(t)\) is strictly monotonically increasing and differentiable function in some interval \(a < t < b\), and \(\phi'(t) \neq 0\) for every \(t\) in the interval.

AIEs (1) and (2) have some applications not only in different fields of physics such as optics, astrophysics, plasma, biophysics, nuclear, etc. [10,14], but also in the pure modeling of their related problems. Furthermore, many real-life applications of the Abel differential
equations can be found in [15–18]. Baker [19], Wazwaz [2,3], and Delves [20] have studied the numerical treatment of singular IEs. In recent years, many authors have solved the AIEs by using various methods [7,21–24]. The properties of the AIEs can be found in [3,8,25,26]. Existence of the solution of AIEs has studied in some researches [27–30]. Furthermore, smooth and non-smooth solutions cases and the regularity properties can be found in [31–35]. Many other topics of nonlinear integral equations with discontinuous symmetric kernels with application of group symmetry have remained.

One of the powerful and efficient methods to estimate the various problems is the collocation method [36]. This method is among expansion methods which can be used by combining with different basis functions [37,38]. Kanwal and Liu in [39] have presented the Taylor expansion to approximate the IEs. This method can be applied to solve the Volterra-Fredholm IEs [40–42], system of IEs [43], Volterra IEs [44], integro-differential equations [45,46], and others [39,47].

We know that generally the mentioned methods are based on the FPA. When we want to show the efficiency of the methods, the following conditions can be applied:

\[ |P(x) - P_m(x)| < \varepsilon \quad \text{or} \quad |P_m(x) - P_{m-1}(x)| < \varepsilon, \]  

(3)

where exact and approximate solutions denoted by \( P(x) \) and \( P_m(x) \), and \( \varepsilon \) is a small positive value. Focusing on condition (3), we need to know \( P(x) \) as the exact solution and also optimal \( \varepsilon \). But there is no exact solution in real life problems. Furthermore, the optimal \( \varepsilon \) can not be found in the FPA. Thus, for small and large values of \( \varepsilon \) we will have some difficulties to find the proper approximation. Thus, instead of applying the methods based on the FPA, we recommend the CESTAC method which is based on the DSA [48,49]. In this method, we apply the following condition:

\[ |P_m(x) - P_{m-1}(x)| = @.0, \]  

(4)

where \( P_m(x) \) and \( P_{m-1}(x) \) show consecutive approximations and @.0 denotes the informative zero [50,51] which can be produced only in the SA and using the CADNA library. In the CESTAC method, having the exact solution is not necessary. Furthermore, no need to have \( \varepsilon \) in the novel condition. The main difference between the DSA and the FPA is applying the CADNA library instead of other software [52–54]. We should write all CADNA codes by C, C++, FORTRAN, or ADA and we should run the codes on Linux operating system [55–58]. After that we will be able to identify the optimal iteration, solution, and error of the numerical algorithm. For more applications of this method and the library, we refer the reader to the papers in [59–62].

In this paper, the Taylor expansion method is applied to find the numerical solution of generalized AIEs. The error analysis of the presented method is illustrated. Using the new method and the library, the numerical results are validated. Based on the obtained results, the optimal iteration, solution, and error are identified. Proving a theorem, we will be able to apply the new condition (4) instead of (3). Several numerical examples are solved and the numerical results are compared between the FPA and the DSA.

2. Taylor Expansion Method

In order to estimate the AIEs, the \( n \)-th order Taylor polynomials at \( x = z \) is introduced as follows:

\[ P_n(x) = \sum_{j=0}^{n} \frac{1}{j!} P^{(j)}(z)(x-z)^j, \quad a \leq x, z \leq b, \]  

(5)

where unknowns \( P^{(j)}(z), j = 0, 1, \ldots, n \) should be determined.

To approximate Equation (1), we rewrite it as

\[ \sum_{j=0}^{n} \frac{1}{j!} \int_{a}^{x} \frac{P^{(j)}(z)(t-z)^j}{[\phi(x) - \phi(t)]^a} \, dt = g(x), \quad 0 < a < 1 \quad \text{and} \quad a \leq x, z \leq b. \]  

(6)
By putting the following collocation points,

$$x_i = a + \left(\frac{b - a}{n}\right)i, \quad i = 0, 1, 2, ..., n,$$

(7)

into Equation (6) we get

$$\sum_{j=0}^{n} \frac{1}{\pi^2} \int_a^{x_j} \frac{(t-z)^j}{[\phi(x_i) - \phi(t)]^n} dt \mathcal{P}^{(j)}(z) = g(x_i), \quad 0 < \alpha < 1.$$  

(8)

Now, we can write Equation (8) in the form

$$AX = G,$$

(9)

where

$$A = \begin{bmatrix}
\frac{1}{\pi^2} \int_a^{x_0} \frac{(t-z)^0}{[\phi(x_0) - \phi(t)]^n} dt & \frac{1}{\pi^2} \int_a^{x_0} \frac{(t-z)^1}{[\phi(x_0) - \phi(t)]^n} dt & \cdots & \frac{1}{\pi^2} \int_a^{x_0} \frac{(t-z)^n}{[\phi(x_0) - \phi(t)]^n} dt \\
\frac{1}{\pi^2} \int_a^{x_1} \frac{(t-z)^0}{[\phi(x_1) - \phi(t)]^n} dt & \frac{1}{\pi^2} \int_a^{x_1} \frac{(t-z)^1}{[\phi(x_1) - \phi(t)]^n} dt & \cdots & \frac{1}{\pi^2} \int_a^{x_1} \frac{(t-z)^n}{[\phi(x_1) - \phi(t)]^n} dt \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\pi^2} \int_a^{x_n} \frac{(t-z)^0}{[\phi(x_n) - \phi(t)]^n} dt & \frac{1}{\pi^2} \int_a^{x_n} \frac{(t-z)^1}{[\phi(x_n) - \phi(t)]^n} dt & \cdots & \frac{1}{\pi^2} \int_a^{x_n} \frac{(t-z)^n}{[\phi(x_n) - \phi(t)]^n} dt
\end{bmatrix}^{(n+1)\times(n+1)}$$

$$X = \begin{bmatrix}
\mathcal{P}^{(0)}(z) \\
\mathcal{P}^{(1)}(z) \\
\vdots \\
\mathcal{P}^{(n)}(z)
\end{bmatrix}_{(n+1)\times 1}, \quad G = \begin{bmatrix}
g(x_0) \\
g(x_1) \\
\vdots \\
g(x_n)
\end{bmatrix}_{(n+1)\times 1}.$$  

Solving (9) and putting $\mathcal{P}^{(j)}(z), \ j = 0, 1, ..., n$ in Equation (5) we can find the solution of problem (1).

In order to approximate the second kind form of AIE (2), by putting Equation (5) into Equation (2) we have

$$\sum_{j=0}^{n} \frac{1}{\pi^2} \int_a^{x} \frac{\mathcal{P}^{(j)}(z)(x-z)^j}{[\phi(x) - \phi(t)]^n} dt = g(x), \quad 0 < \alpha < 1.$$  

(10)

Then, by putting collocation points (7) into Equation (10), the following equation is obtained:

$$\sum_{j=0}^{n} \frac{1}{\pi^2} \left[ (x_i - z)^j - \int_a^{x_i} \frac{(t-z)^j}{[\phi(x_i) - \phi(t)]^n} dt \right] \mathcal{P}^{(j)}(z) = g(x_i), \quad 0 < \alpha < 1.$$  

(11)
Finally, we rewrite Equation (11) in the matrix form \((B - A)X = G\) where matrices 
\(A, X, G\) were presented and

\[
B = \begin{bmatrix}
\frac{1}{n!}(x_0 - z)^0 & \frac{1}{n!}(x_0 - z)^1 & \cdots & \frac{1}{n!}(x_0 - z)^n \\
\frac{1}{n!}(x_1 - z)^0 & \frac{1}{n!}(x_1 - z)^1 & \cdots & \frac{1}{n!}(x_1 - z)^n \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n!}(x_n - z)^0 & \frac{1}{n!}(x_n - z)^1 & \cdots & \frac{1}{n!}(x_n - z)^n
\end{bmatrix}_{(n+1)\times(n+1)}.
\]

Solving the system and substituting into Equation (5), we can find the solution of
Equation (2) approximately.

**Theorem 1.** Assume that \(P(x)\) is an exact solution, \(P_n(x)\) is an approximate solution of Equations (1)
and (2), \(S_n(x) = \sum_{i=0}^{n} \frac{\hat{P}(i)(x-z)^i}{i!}\) is the \(n\)-th order Taylor polynomial at \(x = z\) and \(\hat{P}(i)\) is the
Taylor coefficient of the exact solution. Then,

\[
\|P(x) - P_n(x)\|_\infty \leq M \frac{\hat{P}(n+1)(\xi)}{(n+1)!} + C \max_{0 \leq i \leq n} |e_i(z)|, \quad \text{for some } \xi \in [a, b],
\]

where \(M = \max_{a \leq x \leq b} |(x-z)^{n+1}|, \quad C = \|\ell\|_\infty = \max_{a \leq x \leq b} \{|l_0(c)|, |l_1(c)|, \ldots, |l_n(c)|\},\)
\(e_i(z) = \hat{P}(i)(z) - P(i)(z)\).

**Proof.** Assume that \(R_n(x)\) is the reminder term of \(n\)-th order Taylor polynomial \(S_n\) at
\(x = z\) which is given by

\[
R_n(x) = P(x) - S_n(x) = \frac{\hat{P}(n+1)(\xi)}{(n+1)!} (x-z)^{n+1}, \quad \text{for some } \xi \in [a, b].
\]

Thus,

\[
|R_n(x)| = |P(x) - S_n(x)| \leq \frac{\hat{P}(n+1)(\xi)}{(n+1)!} \max_{a \leq x \leq b} |(x-z)^{n+1}| = \frac{M}{(n+1)!} \hat{P}(n+1)(\xi), \quad (12)
\]

for some \(\xi \in [a, b]\).

Now, one can easily write that

\[
\|P(x) - P_n(x)\|_\infty \leq \|P(x) - S_n(x)\|_\infty + \|S_n(x) - P_n(x)\|_\infty \]

\[
\leq \|R_n(x)\|_\infty + \|S_n(x) - P_n(x)\|_\infty. \quad (13)
\]

Moreover, we get

\[
|S_n(x) - P_n(x)| = \sum_{i=0}^{n} \left| \frac{\hat{P}(i)(z) - P(i)(z)}{i!} (x-z)^i \right| \leq |E\ell| \leq \|E\|_\infty \|\ell\|_\infty \leq C\|E\|_\infty, \quad (14)
\]

where \(E = (e_0(z), e_1(z), \ldots, e_l(z), \ldots, e_n(z)), \quad \ell = (\ell_0(z), \ell_1(z), \ldots, \ell_l(z), \ldots, \ell_n(z))\),
and

\[
e_i(z) = \hat{P}(i)(z) - P(i)(z), \quad \ell_i(z) = \frac{(x-z)^i}{i!}.
\]
Substituting Equations (12) and (14) into Equation (13), we get
\[ \| P(x) - P_n(x) \|_\infty \leq \frac{M}{(n+1)!} \tilde{P}^{(n+1)}(\xi) + C \max_{0 \leq i \leq n} |e_i(z)|, \text{ for some } \xi \in [a,b]. \]

3. CESTAC Method-CADNA Library

For solving the mathematical or engineering problems, researchers apply condition (3) to consider the precision of mathematical methods. Thus, we have to know the exact solution and also the proper \( \epsilon \). Without knowing the solution and also choosing large or small \( \epsilon \), the accurate results cannot be produced and this is big fault of mathematical methods based on the FPA. Because of these problems, the CESTAC method will be considered and the CADNA library will be used. Assume that \( Y^* \) is a representable value of \( y^* \in \mathbb{R} \) which is generated by computer using the binary FPA as
\[ Y^* = y^* - \mu 2^E - \beta \psi, \]  
where the mantissa bits, the binary exponent of the result, the sign and the missing segment of the mantissa have been specified by \( \beta, E, \mu \) and \( 2^{-\beta} \psi \), respectively [53–56]. For finding the results with single to double accuracies, we can apply \( \beta = 24, 53 \). If \( \psi \) be a stochastic variable, then we will have uniformly distribution on \([-1, 1]\). \( \mu \) and \( \sigma \) show the average and standard deviation values which can be produced by making perturbation on the last mantissa bit of \( Y^* \). By doing this process for \( \Phi = \{ Y^*_1, Y^*_2, ..., Y^*_k \} \), we have \( \tilde{Y}^* = \frac{\sum_{i=1}^{k} Y^*_i}{k} \) and \( \sigma^2 = \frac{\sum_{i=1}^{k} (Y^*_i - \tilde{Y}^*)^2}{k-1} \). Now we can find the NCSDs as follows:
\[ C_{\tilde{Y}^*, Y^*} = \log_{10} \frac{\sqrt{E|\tilde{Y}^*|}}{\tau_0 \sigma}, \]
where \( \tau_0 \) is the value of \( T \) distribution as the confidence interval is \( 1 - \delta \), with \( k - 1 \) freedom degree [52]. The algorithm will be stopped when \( \tilde{Y}^* = 0 \) or \( C_{\tilde{Y}^*, Y^*} \leq 0 \). Furthermore, this is important to know that this process will be done using the CADNA codes [49,52,53]. Applying the method and the library, we can find some advantages and highlights as listed below.

- Having the exact solution in the CESTAC method is not necessary.
- In this method, there is no need to have \( \epsilon \).
- The method depends on two consecutive approximations.
- The stopping condition of the method depends on the informational zero sign @.0 and it can be generated by the library.
- In the new method, we do not need to produce extra iterations.
- Using the new procedure, we can find the optimal iteration, approximation, and error of the method.
- Different kinds of numerical instabilities can be identified.
- We have to apply the LINUX operating system.
- In the library, we have to write all codes applying C, C++, FORTRAN, or ADA codes.

**Definition 1** ([49]). For two real numbers \( \omega_1 \) and \( \omega_2 \), the NCSDs can be defined as
\[
\left\{ \begin{array}{c}
C_{\omega_1, \omega_2} = \log_{10} \left| \frac{\omega_1 + \omega_2}{2(\omega_1 - \omega_2)} \right| = \log_{10} \left| \frac{\omega_1}{\omega_1 - \omega_2} - \frac{1}{2} \right|, \quad \omega_1 \neq \omega_2, \\
C_{\omega_1, \omega_1} = +\infty.
\end{array} \right.
\]
Theorem 2. Let \( \mathcal{P}(t) \) and \( \mathcal{P}_n(t) \) be the exact or approximate solutions of the singular integral Equations (1) and (2). Then,

\[
\mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}_{n+1}(t)} \simeq \mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}(t)},
\]

where \( \mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}(t)} \) denotes the NCSDs of \( \mathcal{P}_n(t) \), \( \mathcal{P}(t) \) and \( \mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}_{n+1}(t)} \) show the NCSDs of two consecutive iterations \( \mathcal{P}_n(t) \), \( \mathcal{P}_{n+1}(t) \).

Proof. Applying Definition 1 and using Theorem 1 we can write

\[
\mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}_{n+1}(t)} = \log_{10} \left| \frac{\mathcal{P}_n(t)}{\mathcal{P}_n(t) - \mathcal{P}_{n+1}(t)} - \frac{1}{2} \right|
\]

\[
= \log_{10} \left| \frac{\mathcal{P}_n(t)}{\mathcal{P}_n(t) - \mathcal{P}_{n+1}(t)} \right| + \log_{10} \left| 1 - \frac{1}{2\mathcal{P}_n(t)} \mathcal{P}_n(t) - \mathcal{P}_{n+1}(t) \right|
\]

\[
= \log_{10} \left| \frac{\mathcal{P}_n(t)}{\mathcal{P}_n(t) - \mathcal{P}_{n+1}(t)} \right| + \mathcal{O}(\mathcal{P}_n(t) - \mathcal{P}_{n+1}(t))
\]

\[
= \log_{10} \left| \frac{\mathcal{P}_n(t)}{(\mathcal{P}_n(t) - \mathcal{P}(t)) \left[ 1 - \frac{\mathcal{P}_{n+1}(t) - \mathcal{P}(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} \right]} \right| + \mathcal{O}(\mathcal{E}_n) + \mathcal{O}(\mathcal{E}_{n+1})
\]

\[
= \log_{10} \left| \frac{\mathcal{P}_n(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} \right| - \log_{10} \left| 1 - \frac{\mathcal{P}_{n+1}(t) - \mathcal{P}(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} \right| + \mathcal{O} \left( \frac{1}{(n+1)!} \right).
\]

On the other hand,

\[
\mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}(t)} = \log_{10} \left| \frac{\mathcal{P}_n(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} - \frac{1}{2} \right|
\]

\[
= \log_{10} \left| \frac{\mathcal{P}_n(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} \right| + \mathcal{O}(\mathcal{P}_n(t) - \mathcal{P}(t))
\]

\[
= \log_{10} \left| \frac{\mathcal{P}_n(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} \right| + \mathcal{O} \left( \frac{1}{(n+1)!} \right).
\]

Now, we can apply Equations (18) and (19) and write

\[
\mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}_{n+1}(t)} = \mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}(t)} - \log_{10} \left| 1 - \frac{\mathcal{P}_{n+1}(t) - \mathcal{P}(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} \right| + \mathcal{O} \left( \frac{1}{(n+1)!} \right).
\]

For the second term of logarithm, we get \( \frac{\mathcal{P}_{n+1}(t) - \mathcal{P}(t)}{\mathcal{P}_n(t) - \mathcal{P}(t)} = \frac{\mathcal{O} \left( \frac{1}{n+2} \right)}{\mathcal{O} \left( \frac{1}{n+1} \right)} = \mathcal{O} \left( \frac{1}{n+2} \right) \). Thus, when \( n \to \infty \), then \( \mathcal{O} \left( \frac{1}{n+2} \right) \) and \( \mathcal{O} \left( \frac{1}{(n+1)!} \right) \) tend zero and

\[
\mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}_{n+1}(t)} \simeq \mathcal{C}_{\mathcal{P}_n(t), \mathcal{P}(t)}.
\]

According to Theorem 2, we have equality between the NCSDs of two successive iterations and the exact and approximate solutions. Thus, the condition (4) can be applied instead of (3).
4. Numerical Illustrations

In this section, several examples of the generalized AIEs are solved by using the presented method [2,3,7]. For all examples the numerical results based on the FPA and the DSA are obtained. Furthermore, the number of iterations for different values of \( \varepsilon \) using the FPA are presented. Using the CESTAC method, the optimal step of the Taylor expansion method, the optimal approximation and the optimal error are found. Moreover, some of numerical instabilities are reported by the CADNA library.

Example 1. The following AIE [2,3]

\[
\frac{2}{3} \pi x^3 = \int_0^x \frac{P(t)}{\sqrt{x^2 - t^2}} dt,
\]

is discussed where \( P(x) = \pi x^3 \). Applying the mentioned method, the approximate solution is obtained for \( n = 5 \) as follows:

\[
P_5(x) = 0.000211964 - 0.00380121 x + 0.0198716 x^2 + 3.09737 x^3 + 0.0441592 x^4 - 0.0162574 x^5.
\]

In Table 1, the results are obtained for \( \varepsilon = 10^{-5} \) and \( x = 0.5 \). The number of iterations for various \( \varepsilon \) and using the FPA are presented in Table 2. According to these results, it is obvious that because we do not know the optimal value of \( \varepsilon \), we cannot find that which iteration is suitable to stop the numerical procedure. For large values of \( \varepsilon \), the algorithm is stopped very soon and for small values of \( \varepsilon \) we will have extra iterations. The numerical results based on the CESTAC method and applying the CADNA library are demonstrated in Table 3. According to this table, the optimal step of the Taylor expansion method for solving this example is \( \text{n}_{\text{opt}} = 8 \), the optimal approximation is \( P_{\text{opt}}(x) = 0.392698 \) and the optimal error is \( 0.39 \times 10^{-5} \). The informatical zero sign @.0 shows that the NCSDs for both termination criteria (3) and (4) are almost equal. In general form, we do not need to add the absolute error to Table 1, and the third column is presented only to compare the results. As we know, in the numerical procedures some numerical instabilities can be done. In this example, 8 numerical instabilities include 1 unstable intrinsic function and 7 losses of accuracy due to cancellation, are reported by the CADNA library.

| Table 1. The numerical results of Example 1 based on the FPA \( x = 0.5 \) and \( \varepsilon = 10^{-5} \). |
|---|---|---|
| \( n \) | \( P_n(x) \) | \( |P_n(x) - P_{n-1}(x)| \) | \( |P(x) - P_n(x)| \) |
| 2 | 1.25081099999999989514 | 1.25081099999999989514 | 0.858119183012757564 |
| 3 | 0.75796625000000017458 | 0.49284474999999972056 | 0.3652671683012763508 |
| 4 | 0.3606010000000000486 | 0.3973652500000016972 | 0.032098016987241364 |
| 5 | 0.3932226875000000104 | 0.0326216874999999617 | 0.00052360580127586154 |
| 6 | 0.3927024152499999205 | 0.00052027225000000898 | 0.00000333355127585255 |

| Table 2. The number of iterations of Example 1 for different values of \( \varepsilon \) based on the FPA. |
|---|---|---|---|---|---|
| \( \varepsilon \) | Small Values | \( \varepsilon = 10^{-5} \) | \( \varepsilon = 10^{-3} \) | \( \varepsilon = 10^{-1} \) | \( \varepsilon = 0.5 \) |
| \( n \) | \( >> 6 \) | 6 | 5 | 4 | 3 |
|   | Large Values | \| | | | 2 |
Table 3. The numerical results of Example 1 based on the DSA and using the CESTAC method for \( x = 0.5 \).

| \( n \) | \( P_n(x) \) | \( |P_n(x) - P_{n-1}(x)| \) | \( |P(x) - P_n(x)| \) |
|------|-------------|-----------------|-----------------|
| 2    | 0.1250810 \( \times 10^1 \) | 0.1250810 \( \times 10^1 \) | 0.8581120 |
| 3    | 0.7579662   | 0.492844        | 0.3652672       |
| 4    | 0.3606009   | 0.397365        | 0.320981 \( \times 10^{-1} \) |
| 5    | 0.393222    | 0.32621 \( \times 10^{-1} \) | 0.5236 \( \times 10^{-3} \) |
| 6    | 0.3927023   | 0.5202 \( \times 10^{-3} \) | 0.33 \( \times 10^{-5} \) |
| 7    | 0.3926984   | 0.39 \( \times 10^{-5} \) | 0.5 \( \times 10^{-6} \) |
| 8    | 0.392698    | @.0             | @.0             |

Example 2. In this problem, we solve the AIE \([2]\):

\[
\frac{4}{3} \sin^3(x) = \int_0^x \frac{P(t)}{(\sin(x) - \sin(t))^2} dt, \tag{21}
\]

with \( P(x) = \cos x \). By using the presented scheme for \( n = 9 \) the approximate solution is obtained as follows

\[
P_9(x) = 0.9999999996930083 + 1.36821367568944 \times 10^{-8} x
-0.500001933589282 x^2 + 1.3397660985165968 \times 10^{-6} x^3
+0.041661365627426505 x^4 + 0.000012743412705181756 x^5
-0.0014077144612610363 x^6 + 0.000016437031283826046 x^7
+0.000017405771897191456 x^8 + 9.088496847739541 \times 10^{-7} x^9. \tag{22}
\]

Furthermore, the numerical results for \( \varepsilon = 10^{-7} \) and the number of iterations for different values of \( \varepsilon \) using the Taylor expansion method based on the FPA are presented in Tables 4 and 5. The numerical results of the CESTAC method and the CADNA library based on the DSA are shown in Table 6. Using this method, the optimal iteration is \( n_{opt} = 9 \) and the optimal approximation is \( P_{opt}(x) = 0.980066 \) which are obtained for \( x = 0.2 \). Thus, applying the DSA we do not need to make more iterations to show the accuracy of the method. The CADNA library shows that for solving this problem we have 1 unstable intrinsic function, 11 losses of accuracy due to cancellation and totally 12 numerical instabilities.

Table 4. The numerical results of Example 2 based on the FPA for \( x = 0.2 \) and \( \varepsilon = 10^{-7} \).

| \( n \) | \( P_n(x) \) | \( |P_n(x) - P_{n-1}(x)| \) | \( |P(x) - P_n(x)| \) |
|------|-------------|-----------------|-----------------|
| 2    | 0.66597839999999997040 | 0.66597839999999997040 | 0.31408819746170046905 |
| 3    | 0.98810932000000006870 | 0.322130920000000009829 | 0.00804272253829962924 |
| 4    | 0.9785839189999919122 | 0.09525401600000014957 | 0.00148267906000008720 |
| 5    | 0.98022671040000017183 | 0.001642842640000025271 | 0.000012743412705181756 |
| 6    | 0.980078957680008463 | 0.00014780336000008720 | 0.0000126021829964518 |
| 7    | 0.980063576780401008 | 0.00001539999296704745 | 0.00000303977466042937 |
| 8    | 0.98006638960010816763 | 0.00000283191306615755 | 0.000000207861927182 |
| 9    | 0.98006657764296079005 | 0.0000001880428526224 | 0.0000001981873964940 |

Table 5. The number of iterations for different values of \( \varepsilon \) based on the FPA.

| \( \varepsilon \) | Small Values | \( \varepsilon = 10^{-7} \) | \( \varepsilon = 10^{-5} \) | \( \varepsilon = 10^{-3} \) | \( \varepsilon = 10^{-1} \) | Large Values |
|--------------|---------------|-----------------|-----------------|-----------------|-----------------|--------------|
| \( n \)      | \( n_{opt} = 9 \) | 9               | 7               | 5               | 3               | 2            |
Table 6. The numerical results of Example 2 based on the DSA and using the CESTAC method for \( x = 0.2 \).

| \( n \) | \( P_n(x) \) | \( |P_n(x) - P_{n-1}(x)| \) | \( |P(x) - P_n(x)| \) |
|-----|---------|----------------|----------------|
| 2   | 0.6659784 | 0.6659784       | 0.3140881      |
| 3   | 0.9881093 | 0.3221309       | 0.80427 \times 10^{-2} |
| 4   | 0.9785838 | 0.95254 \times 10^{-2} | 0.1482 \times 10^{-2} |
| 5   | 0.9802267 | 0.1642 \times 10^{-2} | 0.160 \times 10^{-3} |
| 6   | 0.980078  | 0.147 \times 10^{-3} | 0.1 \times 10^{-4} |
| 7   | 0.9800636 | 0.15 \times 10^{-4} | 0.2 \times 10^{-5} |
| 8   | 0.9800672 | 0.3 \times 10^{-5} | 0.6 \times 10^{-6} |
| 9   | 0.980066  | \@.0            | \@.0           |

Example 3. The AIE [2]

\[
\frac{6}{5}(e^x - 1)^\frac{5}{3} = \int_0^x \frac{P(t)}{(e^x - e^t)^2} dt,
\]

(23)
is considered with exact solution \( P(x) = e^x \). By using presented method, the following estimation is obtained for \( n = 9 \) as

\[
P_9(x) = 1.000000012857142 + 0.9999998135066253 x + 0.5000012303151726 x^2 + 0.04168066961287148 x^3 + 0.00003578465217356253 x^9.
\]

(24)

Tables 7 and 8 show the numerical results and the number of iterations for \( x = 0.4 \) based on the FPA. The numerical results of the CESTAC method based on the DSA are presented in Table 9 for \( x = 0.4 \). The CADNA library is applied to find the results. According to the CADNA library results there are 6 numerical instabilities include 6 losses of accuracy due to cancellation.

Table 7. The numerical results of Example 3 based on the FPA \( x = 0.4 \) and \( \varepsilon = 10^{-2} \).

| \( n \) | \( P_n(x) \) | \( |P_n(x) - P_{n-1}(x)| \) | \( |P(x) - P_n(x)| \) |
|-----|---------|----------------|----------------|
| 2   | 1.68538919999999992072 | 0.168538919999999992072 | 0.19356450235872957322 |
| 3   | 1.5257732800000012054 | 0.15961591999999980018 | 0.0339485823876775 |
| 4   | 1.489556159999992630 | 0.1609325 \times 10^{-4} | 0.028690816412704120 |

Table 8. The number of iterations of Example 3 for different values of \( \varepsilon \) based on the FPA.

| \( \varepsilon \) | Small Values | \( \varepsilon = 10^{-9} \) | \( \varepsilon = 10^{-7} \) | \( \varepsilon = 10^{-5} \) | \( \varepsilon = 10^{-2} \) | Large Values |
|-----|------------|----------------|----------------|----------------|----------------|------------|
| \( n \) | \( \gg \gg 11 \) | 11 | 8 | 6 | 4 | 4 |
Example 4. Let us consider the second kind AIE [63]
\[
\mathcal{P}(x) = x^\frac{7}{2} + \frac{35}{128} \pi x^4 - \int_0^x \frac{\mathcal{P}(t)}{(x-t)^{\frac{1}{2}}} dt, \quad 0 \leq x, t \leq 1,
\]
with non-smooth solution \( \mathcal{P}(x) = x^3 \sqrt{x} \). Table 10 shows the numerical results of the FPA for \( \varepsilon = 10^{-4} \) and \( x = 0.3 \). The number of iteration of the Taylor expansion method based on the FPA are presented in Table 11. In Table 12, the numerical results of this problem for \( x = 0.3 \) are obtained applying the CESTAC method and CADNA library. In this method, not only we do not have the disadvantages of the FPA but also we will be able to find the optimal error, optimal step, optimal approximation and some of numerical instabilities. According to Table 12, the optimal step of the Taylor expansion method is \( n_{\text{opt}} = 10 \), the optimal approximation is \( 0.1478846 \times 10^{-1} \) and the optimal error is \( 0.3 \times 10^{-7} \). Furthermore, we have 8 numerical instabilities include 8 losses of accuracy due to cancellation.

### Table 10. The numerical results of Example 4 based on the FPA for \( x = 0.3 \) and \( \varepsilon = 10^{-4} \).

| \( n \) | \( \mathcal{P}_n(x) \) | \( |\mathcal{P}_n(x) - \mathcal{P}_{n-1}(x)| \) | \( |\mathcal{P}(x) - \mathcal{P}_n(x)| \) |
|---|---|---|---|
| 2 | 0.0438779999999999340 | 0.0438779999999999340 | 0.029089490368492881 |
| 3 | -0.024190800000000129 | 0.0680688000000001263 | 0.03897930960731507688 |
| 4 | 0.01437464999999993678 | 0.03856544999999994561 | 0.0004138596219953012607 |
| 5 | 0.01462970429999993147 | 0.0002505042999999468 | 0.00015880530731513139 |
| 6 | 0.0147636053190009453 | 0.00013390101900016306 | 0.00002490428831496833 |

### Table 11. The number of iterations of Example 4 for different values of \( \varepsilon \) based on the FPA.

| \( \varepsilon \) | Small Values | \( \varepsilon = 10^{-7} \) | \( \varepsilon = 10^{-4} \) | \( \varepsilon = 10^{-1} \) | \( \varepsilon = 0.5 \) | Large Values |
|---|---|---|---|---|---|---|
| \( n \) | \( > > 8 \) | 8 | 6 | 2 | 2 | 2 |

### Table 12. The numerical results of Example 4 based on the DSA and using the CESTAC method for \( x = 0.3 \).

| \( n \) | \( \mathcal{P}_n(x) \) | \( |\mathcal{P}_n(x) - \mathcal{P}_{n-1}(x)| \) | \( |\mathcal{P}(x) - \mathcal{P}_n(x)| \) |
|---|---|---|---|
| 2 | 0.1478799 \times 10^{-1} | 0.1478799 \times 10^{-1} | 0.290894 \times 10^{-1} |
| 3 | -0.421907 \times 10^{-1} | 0.680687 \times 10^{-1} | 0.389793 \times 10^{-1} |
| 4 | 0.143746 \times 10^{-1} | 0.385654 \times 10^{-1} | 0.4138 \times 10^{-3} |
| 5 | 0.146297 \times 10^{-1} | 0.25505 \times 10^{-3} | 0.1588 \times 10^{-3} |
| 6 | 0.147636 \times 10^{-1} | 0.13390 \times 10^{-3} | 0.290 \times 10^{-4} |
| 7 | 0.147857 \times 10^{-1} | 0.2213 \times 10^{-4} | 0.276 \times 10^{-5} |
| 8 | 0.147884 \times 10^{-1} | 0.268 \times 10^{-5} | 0.7 \times 10^{-7} |
| 9 | 0.147884 \times 10^{-1} | 0.3 \times 10^{-7} | 0.3 \times 10^{-7} |
| 10 | 0.1478846 \times 10^{-1} | 0.0 | 0.45 \times 10^{-7} |

5. Conclusions

In this paper, the Taylor expansion method has been applied to estimate the generalized form of the first and second kinds AIEs. The error analysis of the method has been illustrated. Furthermore, we have used the CESTAC method and the CADNA library, which are based on the DSA, to validate the obtained numerical results. Instead of applying the traditional absolute error we have used the new termination criterion to show the abilities of the CESTAC method. Moreover, we have compared the results between both arithmetics the FPA and the DSA. Proving a theorem we can apply the novel stopping condition instead of the previous one. Based on the mentioned theorem we showed the equality between two sides of relation (17). Several examples of the first and second kinds AIEs have been solved and using the mentioned technique the optimal iteration, approximation, error and some of instabilities have been found.
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Abbreviations
The following abbreviations are used in this paper:

AIE Abel Integral Equation
FPA Floating Point Arithmetic
DSA Discrete Stochastic Arithmetic
CESTAC Controle et Estimation Stochastique des Arrondis de Calculs
CADNA Control of Accuracy and Debugging for Numerical Applications
NCSDs Number of Common Significant Digits

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