User Capacity of Pilot-Contaminated TDD Massive MIMO Systems

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Abstract—Pilot contamination has been regarded as a main limiting factor of time division duplexing (TDD) massive multiple-input–multiple-output (Massive MIMO) systems, as it will make the signal-to-interference-plus-noise ratio (SINR) saturated. However, how pilot contamination will limit the user capacity of downlink Massive MIMO, i.e., the maximum number of admissible users, has not been addressed. This paper provides an explicit expression of the Massive MIMO user capacity in the pilot-contaminated regime where the number of users is larger than the pilot sequence length. Furthermore, the scheme for achieving the user capacity, i.e., the uplink pilot training sequence and downlink power allocation, has been identified. By using this capacity-achieving scheme, the SINR requirement of each user can be satisfied and energy-efficient transmission is feasible in the large-antenna-size (LAS) regime. Comparison with two non-capacity-achieving schemes highlights the superiority of our proposed scheme in terms of achieving higher user capacity.

Index Terms—Massive MIMO, user capacity, pilot contamination, pilot-aided channel estimation, power allocation.

I. INTRODUCTION

Massive MIMO is regarded as an efficient and scalable approach for multicell multiuser MIMO implementation. By deploying base stations (BSs) with much more antennas than active user equipments (UEs), the asymptotic orthogonality among MIMO channels becomes valid and in turn it makes intra- and inter-cell interference more manageable. Hence, using simple linear precoder and detector can approach the optimal dirty-paper coding capacity [1]. Channel side information (CSI) at BSs plays an important role in the exploitation of channel orthogonality. In practice, TDD operation is assumed for CSI acquisition through uplink training. The advantage of uplink training is that the length of pilot training sequences is proportional to the number of active UEs rather than that of BS antennas. The training length is fundamentally limited by the channel coherence time, which can be short due to UEs of high mobility. It has been shown, nevertheless, in [2] that the effect of using short training sequences diminishes in the LAS regime. Specifically, among the poorly estimated channels, the asymptotic orthogonality can still hold.

However, a major problem with TDD Massive MIMO is that inevitably the same pilot sequences will be reused in multiple cells. The channels to UEs in different cells who share the same pilot sequence will be collectively learned by BSs. In other words, the desired channel learned by a BS is contaminated by undesired channels. Once this contaminated CSI is utilized for transmitting or receiving signals, intercell interference occurs immediately which limits the achievable SINR. This phenomenon, known as pilot contamination, can not be circumvented simply by increasing the BS antenna size [3]. Several attempts have been made to tackle this problem. In [4], a sophisticated precoding method is proposed to minimize intercell interference due to pilot contamination. A more direct approach to pilot decontamination, which promises to purify the polluted channel information, can be found in [5]. By harnessing second-order channel statistics, it can remove contamination from undesired channels which occupy different angle-of-arrival intervals from the desired channel. A recent study [6] claims that pilot contamination is due to inappropriate linear channel estimation. Hence, by using the proposed subspace-based estimation, unpolluted CSI is obtainable. One thing to note is that this method should be accompanied by suitable frequency reuse pattern and power control among UEs. Also, the asymptotic effectiveness of the last two methods in the LAS regime has been analytically presented.

The effect of pilot contamination is usually quantified as SINR saturation due to intercell interference. A number of studies have examined this saturation phenomenon and the corresponding uplink or downlink throughput [3], [7], [8]. The former two provide analysis in the LAS regime with a fixed number of active UEs, while in [8], it analyzes asymptotic SINRs with a fixed ratio of the BS antenna size to the active-UE number. All these studies lead to a similar conclusion that the SINR will saturate with an increasing antenna size, making system throughput limited.

So far, however, there has been little discussion about the user capacity of TDD Massive MIMO, which is the maximum number of UEs whose SINR requirements can be met for a given pilot sequence length. This term “user capacity” was originally coined for analyzing CDMA systems [9]. In this paper, we confine our discussion to the user capacity of single-cell downlink TDD Massive MIMO, where pilot contamination will occur once the number of UEs is greater than the pilot sequence length. Meanwhile, we consider a more general set of pilot sequences whose cross-correlations can range from $-1$ to $1$. In most studies of Massive MIMO, the cross-correlations are restricted to be $1$ or $0$. Our discussion will show that the user capacity can be characterized by a specific region within which there exists a capacity-achieving pilot sequence and

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power allocation such that the SINR requirements are satisfied and the user data can be energy-efficiently transmitted. This result is significant in the sense that within the specified region we have no worries about pilot contamination by using the proposed allocation scheme. Meanwhile, the derivation of this result only involves simple channel estimation and linear precoding. It means that this region can exist without relying on advanced estimation or precoding methods. Though this region is identified within a single cell, it sheds light on the possible existence of the similar region in the general multicell scenario.

Notations: \(\mathbb{R}\): real number, \(\mathbb{Z}\): integers, \(\|\cdot\|_p\): p-norm, \((\cdot)^T\): transpose, \((\cdot)^H\): Hermitian transpose, \(\otimes\): Kronecker product, \(\oplus\): Hadamard product, \(\mathbf{I}_N\): \(N \times N\) identity matrix, \(\mathcal{CN}\): complex normal distribution, \(\mathbb{E}[\cdot]\): expectation, \(\text{tr}(\cdot)\): trace, \(\text{diag}(\cdots)\): diagonal matrix, \(\succ\), \(\succeq\): vector inequalities, \(0\): zero vector, \(\text{Null}(\cdot)\): null space, \(\text{card}(\cdot)\): cardinality.

II. TDD System Model

Consider a single-cell network where a BS equipped with \(M\) antennas serves \(K\) single-antenna UEs, assuming TDD operation. The BS acquires downlink CSI through uplink pilot training. The acquired CSI will be utilized to form linear precoding matrix for downlink spatial multiplexing.

A. Uplink Training

During the uplink training phase, each UE transmits its own pilot sequence \(s_i \in \mathbb{R}^{\tau \times 1}\), where \(i\) is the UE index, \(\|s_i\|_2 = 1\), and \(s_i^H s_j = \rho_{ij}\), which is the correlation between different training sequences. The pilot data over the block-fading channel, synchronously received at the BS, can be expressed as

\[
y_{(\tau M \times 1)} = \sum_{i=1}^{K} S_i h_i + z, \tag{1}
\]

where \(S_i\) is the \(\tau M \times M\) matrix given by \(S_i = s_i \otimes I_{M}\), \(z\) is the additive Gaussian noise distributed as \(\mathcal{CN}(0, \sigma_z^2 I_{\tau M})\), and \(h_i\) for \(1 \leq i \leq K\) are identically and independently distributed (i.i.d.) channel vectors with distribution \(\mathcal{CN}(0, I_M)\). The pilot length is implicitly assumed to be less than one channel block. This channel model is commonly assumed in Massive MIMO systems [3], [4], [7]. A simple single-user least-squares estimate of the \(i\)th channel vector is given by

\[
\hat{h}_i = P_{s,i} Y = h_i + \sum_{j \neq i} \rho_{ij} h_j + S_i^T z, \tag{2}
\]

where \(P_{s,i} = S_i^T\). This estimate indicates how the desire channel information is polluted by undesired channels in the pilot-contaminated regime \((K > \tau)\), where \(\rho_{ij}\) may not be zero for some \(j \neq i\).

B. Downlink Transmission

Exploiting estimated CSI at the BS, maximum ratio transmission (MRT) precoded data are formed and simultaneously transmitted to UEs. The received signal at the \(i\)th UE is

\[
r_i = h_i^H \left( \sum_{j=1}^{K} t_j x_j \right) + w_i, \tag{3}
\]

where \(t_i \triangleq h_i/\|h_i\|_2\) is a MRT precoding vector, \(x_i\) denotes uncorrelated zero-mean data with power \(\mathbb{E}[x_i^H x_i] = P_i\), and \(w_i\) is the zero-mean noise with variance \(\sigma_n^2\). In the LAS regime \((M \gg K)\), the following asymptotic results can be applied [7]

\[
\lim_{M \to \infty} \frac{1}{M} h_i^H h_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. 
\end{cases}
\]

Such asymptotic orthogonality has been experimentally verified in realistic propagation environments [1]. With this in mind, the received signal \(r_i\) can be approximated by

\[
r_i \approx \sum_{j=1}^{K} \rho_{ij} x_j + \sqrt{\frac{P_i}{M}} \sum_{j=1}^{K} \rho_{ij}^2 + w_i, \tag{4}
\]

due to

\[
\frac{\|h_i\|_2}{M} \approx \sqrt{\frac{1}{M} \left( \sum_{j=1}^{K} \rho_{ij}^2 \right)}, \tag{5}
\]

and some second-order results, i.e., \(\lim_{M \to \infty} \frac{h_i^H S_i^T s_j^H / M}{\|s_j\|_2} = 0\) and \(\lim_{M \to \infty} s_j^H S_i^T s_j^T / M^2 = 0\). The corresponding SINR is given by

\[
\text{SINR}_i \approx \frac{P_i}{\sum_{j \neq i} \rho_{ij}^2 P_j}, \tag{6}
\]

where \(D = \text{diag}(P_1, \ldots, P_K)\), \(S = [s_1, s_2, \ldots, s_K]\), and the fact \(\lim_{M \to \infty} \sigma_z^2 / M = 0\) is applied. This SINR expression tells that the downlink transmission operates in the interference-limited regime because of using a large number of antennas. Moreover, the interference part, \(\sum_{j \neq i} \rho_{ij}^2 P_j\), can not be simultaneously eliminated for every user in the pilot-contaminated regime as non-orthogonal pilot sequences have to be used.

III. User Capacity

A group of UEs is said to be admissible in the specified TDD Massive MIMO system if there exists a feasible pilot sequence matrix \(S\) and a power allocation vector \(p = [P_1, \ldots, P_K]^T \succeq 0\) such that the SINR requirements, \(\text{SINR}_i \geq \gamma_i\) for \(1 \leq i \leq K\), can be jointly satisfied. A pilot sequence matrix \(S\) is feasible if \(S \in \mathcal{S} = \{[s_1, s_2, \ldots, s_K]\}, s_i \in \mathbb{R}^{\tau \times 1}\|s_i\|_2 = 1\}$. 


In the following discussion, we will treat the approximation in \((7)\) as the exact SINR expression, and focus on the pilot-contaminated regime. The proposition below gives the upper bound of the maximum number of admissible UEs.

**Proposition 1.** If \(K\) UEs are admissible in the TDD Massive MIMO system, then

\[
K \leq \left[ \tau \left( \sum_{i=1}^{K} \frac{1 + \frac{1}{\gamma_i}}{\gamma_i} \right) \right]^{1/2},
\]

Proof: Making use of \((7)\), we have

\[
\sum_{i=1}^{K} \frac{1 + \text{SINR}_i}{\text{SINR}_i} = \sum_{i=1}^{K} \frac{1}{P_i} \text{tr} \left( s_i^T S D S^T S_i \right),
\]

\[
= \text{tr} \left( D^{-1} S^T S D S^T S \right),
\]

\[
= \text{tr} \left( D^{-1/2} G_s D G_s D^{-1/2} \right),
\]

where

\[
G_s \triangleq S^T S = 
\begin{bmatrix}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1K} \\
\rho_{12} & 1 & \rho_{23} & \cdots & \rho_{2K} \\
\rho_{13} & \rho_{23} & 1 & \cdots & \rho_{3K} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{1K} & \rho_{2K} & \rho_{3K} & \cdots & 1
\end{bmatrix}.
\]

Also, we can expand the trace in \((9)\) and obtain its lower bound as below

\[
\text{tr} \left( D^{-1/2} G_s D G_s D^{-1/2} \right) = K + \sum_{i=1}^{K} \sum_{j>i=1}^{K} \left( \frac{P_j}{P_i} + \frac{P_i}{P_j} \right) \rho_{ij}^2,
\]

\[
\geq K + \sum_{i=1}^{K} \sum_{j>i=1}^{K} 2\rho_{ij}^2,
\]

\[
= \text{tr} \left( G_s G_s \right),
\]

where the inequality is due to \((P_j/P_i + P_i/P_j) \geq 2\). The Gram matrix \(G_s\) has an eigendecomposition \(UD_G U^T\), where \(U\) is an unitary matrix and \(D_G = \text{diag}(d_1, \cdots, d_K)\) with \(d_1 \sim d_T > 0, d_{T+1} \sim d_K = 0\), and \(\sum_{i=1}^{\tau} d_i = K\). Then, we have

\[
\text{tr} \left( G_s G_s \right) = \text{tr} \left( UD_G U^T \right),
\]

\[
= \sum_{i=1}^{\tau} d_i^2,
\]

\[
\geq \frac{1}{\tau} \left( \sum_{i=1}^{\tau} d_i \right)^2 = \frac{K^2}{\tau}.
\]

As

\[
\sum_{i=1}^{K} \frac{1 + \text{SINR}_i}{\text{SINR}_i} \leq \sum_{i=1}^{K} \frac{1 + \gamma_i}{\gamma_i},
\]

the desired inequality follows. \(\square\)

It is clear from this proposition that the number of admissible UEs is fundamentally limited once the length of pilot sequences and the SINR requirements are given. To offer another explanation, let's define the normalized mean-square error seen by the \(i\)th UE as \(\text{MSE}_i = 1/\text{SINR}_i\). A lower bound on the sum of \(\text{MSE}_i\) is given by

\[
\sum_{i=1}^{K} \text{MSE}_i = \text{tr} \left( \left( D^{-1} S^T S D S^T S \right) - K \right),
\]

\[
\geq \frac{K^2}{\tau} - K.
\]

Appealing to \((13)\) and \((14)\) leads to the same result as Proposition 1, which links the bound on the admissible UEs to the bound on the sum of mean-square errors.

Proposition 1 provides an upper bound for the user capacity. The next question to ask is whether \(K\) UEs are admissible if the inequality \((8)\) is satisfied, i.e., the achievability issue. In other words, once the UE number is less than or equal to the upper bound, we wonder if there exists a set of \(S \in \mathcal{S}\) and \(p > 0\), fulfilling the SINR requirements. The following section will show that the answer to this question is positive.

**IV. CAPACITY-ACHIEVING PILOT SEQUENCE AND POWER ALLOCATION**

Validating the converse of Proposition 1 requires to identify \(S\) and \(p\) with which the given SINR requirements are met. The constraints, \(\text{SINR}_i \geq \gamma_i\) for \(1 \leq i \leq K\), can be recast as

\[
A = \begin{bmatrix}
\frac{1}{\gamma_1} & -\frac{\rho_{12}}{\gamma_1} & \cdots & -\frac{\rho_{1K}}{\gamma_1} \\
-\frac{\rho_{21}}{\gamma_2} & 1 & \cdots & -\frac{\rho_{2K}}{\gamma_2} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\rho_{K1}}{\gamma_K} & -\frac{\rho_{K2}}{\gamma_K} & \cdots & 1
\end{bmatrix},
\]

If there exists \(\mathbf{p} > 0\) in the null space of \(A\), any linearly scaled \(\alpha \mathbf{p}\) for \(\alpha > 0\) is still a valid solution of the problem. In this case, the total transmission power can be made fairly small. The result is due to downlink Massive MIMO being in the interference-limited regime. The specific definition of a valid set of \(S\) and \(p\) is given below.

**Definition 2.** The set of a pilot sequence matrix \(S \in \mathcal{S}\) and a power allocation vector \(p > 0\) is said to be valid if

\[
\mathbf{p} \in \text{Null} \left( \mathbf{T} - G_s^T \circ G_s \right),
\]

where \(\mathbf{T} = \text{diag} \left( \frac{1}{\gamma_1}, \cdots, 1 + \frac{1}{\gamma_k} \right)\), \(\hat{\gamma}_i \geq \gamma_i\) for \(1 \leq i \leq K\), and \(\text{rank} \left( \text{Null} \left( \mathbf{T} - G_s^T \circ G_s \right) \right) > 1\).

In this definition, \(\mathbf{A} = \mathbf{T} - G_s^T \circ G_s\) when \(\hat{\gamma}_i = \gamma_i\). Also, if the achieved SINR is \(\hat{\gamma}_i\) higher than the required \(\gamma_i\), we regard the corresponding \(S\) and \(p\) as valid. The following proposition will specify a region where a valid pilot sequence and power allocation can exist.

**Proposition 3.** If

\[
\sum_{i=1}^{K} \left( \frac{\gamma_i}{1 + \gamma_i} \right) \leq \tau,
\]

...
then $K \leq \left( \frac{1}{\tau} \sum_{i=1}^{K} 1 + \frac{1}{\gamma_i} \right)^{1/2}$ and there exists a valid pilot sequence and power allocation.

**Proof:** The Cauchy–Schwarz inequality gives
\[
\sum_{i=1}^{K} \left( 1 + \frac{1}{\gamma_i} \right) \geq \frac{K^2}{\sum_{i=1}^{K} \left( \frac{\gamma_i}{1 + \gamma_i} \right)},
\]
which is equivalent to (8) and proves the first part of the statement. Before going on to the second part, a definition and a lemma to be utilized later are provided.

**Definition 4.** Given $x, y \in \mathbb{R}^N$, $x$ majorizes $y$ if
\[
\sum_{k=1}^{n} x[k] \geq \sum_{k=1}^{n} y[k], \text{ for } n = 1, \cdots, N,
\]
where $x[k]$ and $y[k]$ are respectively the elements of $x$ and $y$ in decreasing order.

**Lemma 5.** (Theorem 9.2.2) Given $x, y \in \mathbb{R}^N$, if $x$ majorizes $y$ and $\sum_{k=1}^{K} x[k] = \sum_{k=1}^{N} y[k]$, then there exists a real symmetric matrix $H$ with diagonal elements $y[k]$ and eigenvalues $x[k]$.

First consider the case of $\sum_{i=1}^{K} \left( \frac{\gamma_i}{1 + \gamma_i} \right) = \tau$.

Given that the $1 \times K$ vector of eigenvalues $e = [\lambda_1, \cdots, \lambda_K, 0, \cdots, 0]^T$ majorizes $p$ and $\sum_{i=1}^{K} \lambda_i = \sum_{i=1}^{K} P_i$, because of Lemma 5, there exists a real symmetric matrix $H = QAQ^T$, where the vector of diagonal entries of $H$ is equal to $p$, $A = \text{diag} (\lambda_1, \cdots, \lambda_K, 0, \cdots, 0)$, and the orthogonal matrix $Q$ can be presented as
\[
\begin{bmatrix}
V_{K\times \tau} & \hat{V}_{K \times (K-\tau)}
\end{bmatrix}.
\]
The approach to constructing $Q$ as well as $H$ is provided in [11, Sec. IV-A]. Define that
\[
S \triangleq \Sigma^{1/2}V^T D^{-1/2},
\]
where $\Sigma = \text{diag} (\lambda_1, \cdots, \lambda_K)$. Then, $S \in \mathcal{S}$ is true as the diagonal entries of $S^T S = D^{-1/2}H D^{-1/2}$ are equal to 1. Moreover, we have
\[
\text{SDS}^T = \Sigma.
\]
Let’s specify that
\[
\lambda_1 = \cdots = \lambda_\tau = \frac{\sum_{i=1}^{K} P_i}{\tau},
\]
and
\[
P_i = c \frac{\gamma_i}{1 + \gamma_i}, \text{ for some } c > 0.
\]
It can be verified that $e$ majorizes $p$ since for $1 \leq i \leq \tau$,
\[
\lambda_i = \frac{c \sum_{k=1}^{K} \gamma_k}{\tau}, \quad = c, \quad > \max \{P_k, \text{ for } 1 \leq k \leq K\},
\]
where the second equality is due to the case under consideration.

Next we will check if the SINR requirements are satisfied by using such pilot sequence and power allocation. Making use of (18), we have
\[
\text{SINR}_i = \frac{P_i}{\text{tr} (s_i^T \Sigma s_i) - P_i},
\]
\[
= \frac{c}{c - c \frac{\gamma_i}{1 + \gamma_i}},
\]
\[
= \gamma_i, \quad \forall i = 1, \cdots, K.
\]
Based on this result, it can be easily shown that $p \in \text{Null} \left( T - G^T \circ G \right)$ where the diagonal matrix $T = \text{diag} (1 + \frac{1}{\gamma_1}, \cdots, 1 + \frac{1}{\gamma_K})$.

Now we turn to the case of $\sum_{i=1}^{K} \left( \gamma_i / (1 + \gamma_i) \right) < \tau$. As $f(x) = x/(1+x)$ is monotonically increasing for $x > 0$, there exists a set $\{\tilde{\gamma}_i \geq \gamma_i \text{ for } 1 \leq i \leq K\}$ such that $\sum_{i=1}^{K} \left( \tilde{\gamma}_i / (1 + \tilde{\gamma}_i) \right) = \tau$. At the same time, $K \leq \left[ \tau \left( \sum_{i=1}^{K} 1 + \frac{1}{\gamma_i} \right) \right]^{1/2}$ holds. By exploiting the previous result, we can find a valid set of $S \in \mathcal{S}$ and $p > 0$ for which $T = \text{diag} (1 + \frac{1}{\tilde{\gamma}_1}, \cdots, 1 + \frac{1}{\tilde{\gamma}_K})$.

An explanation of the constraint, $\sum_{i=1}^{K} \left( \gamma_i / (1 + \tilde{\gamma}_i) \right) \leq \tau$, is as follows. The UE with a high SINR requirement should be allocated with a pilot sequence which is orthogonal to others. Overall, only $\tau$ such assignments are allowed in the system. Another thing to note is that the given proof is constructive, within which the method of obtaining pilot sequences and allocating sequences and powers to UEs can be found.

A corollary which follows from Propositions 1 and 5 is provided below.

**Corollary 6.** Given the identical SINR requirement $\gamma_i$, $K$ UEs are admissible in the TDD Massive MIMO system if and only if
\[
K \leq \left( 1 + \frac{1}{\gamma_i} \right) \tau.
\]
1) The valid pilot sequence and power allocation used in Proposition 3 is also referred to as the capacity-achieving pilot sequence and power allocation. It means that any $K$ UEs having the SINR requirements within $\mathbf{R}_{\text{WBE}}$ can be admitted by using this allocation. In the next section, it will be shown that other non-capacity-achieving schemes can not guarantee this.

2) When using the capacity-achieving pilot sequence and power allocation, the converse of Proposition 3 can be shown to be true.

3) In the case of identical SINR requirements, the pilot sequences in use are called Welch bound equality (WBE) sequences (with properties: $\mathbf{S} \in \mathcal{S}$, $\mathbf{SS}^T = \frac{\tau}{\tau} \mathbf{I}_r$, and $\rho_{ij}^2 = (K-\tau)/(K-1)\tau$ for $i \neq j$) [12], [13].

4) Generally, the regime $K \leq \left[ \tau \left( \sum_{i=1}^K 1 + \frac{1}{\gamma_i} \right) \right]^{1/2}$ while $\sum_{i=1}^K (\gamma_i / 1 + \gamma_i) > \tau$, has not been characterized, in which the existence of a valid pilot sequence and power allocation is unknown. However, when all the SINR requirements are the same, $K \leq \left[ \tau \left( \sum_{i=1}^K 1 + \frac{1}{\gamma_i} \right) \right]^{1/2}$ implies $\sum_{i=1}^K (\gamma_i / 1 + \gamma_i) \leq \tau$. In this special case, the setting: $\sum_{i=1}^K (\gamma_i / 1 + \gamma_i) > \tau$, is not of interest.

V. COMPARISON WITH NON-CAPACITY-ACHIEVING SCHEMES

The superiority of the proposed capacity-achieving allocation over other existing schemes will be presented in this section. Let’s first define two pilot sequence allocation schemes which are independent of the SINR requirements. Meanwhile, in both schemes, the transmit power $P_i$ allocated to the $i$th UE is $c \gamma_i / 1 + \gamma_i$ for some $c > 0$.

1) WBE Scheme: Pilot sequences in use are the WBE sequences.

2) Finite Orthogonal Sequence (FOS) Scheme: Given a pilot sequence length $\tau$, only $\tau$ orthogonal pilot sequences will be repeatedly used in the pilot-contaminated regime. Assume that $K = qr + r$ where $q, r \in \mathbb{Z}$ and $0 \leq r < \tau$. Each pilot sequence $s_t$ is used by a collection $E_i$ of UEs. Let card $(E_i) = q + 1$ for $1 \leq i \leq r$, card $(E_i) = q$ for $r + 1 \leq i \leq \tau$, and $E_i \cap E_j = \emptyset$ for $i \neq j$.

The following lemmas will show the potential reduction of the user capacity when the WBE and FOS scheme are applied to the case of general SINR constraints.

Lemma 7. The general SINR requirements $\gamma_i$ are satisfied by using the WBE scheme if and only if

$$\sum_{k \in E_i} \left( \frac{\gamma_k}{1 + \gamma_k} \right) \leq 1, \text{ for } 1 \leq i \leq \tau,$$  

(23)

and

$$\sum_{i=1}^K \left( \frac{\gamma_i}{1 + \gamma_i} \right) \leq \tau.$$  

(24)

\textbf{Proof:} Similar to the proof of Lemma 7.

To verify the results in Proposition 3 and in Lemmas 7 and 8 we consider a pilot-contaminated Massive MIMO system with $K = 6$ and $\tau = 3$. By fixing certain SINR requirements $\{\gamma_4 = \gamma_5 = \gamma_6 = 1\}$, we look into admissible regions of the remaining SINR requirements given by

$$R_{\text{GWBE}} = \left\{ \gamma_{1-3} \in \mathbb{R}^+ | \sum_{i=1}^3 \gamma_i/1 + \gamma_i \leq 3/2 \right\},$$  

(25)

$$R_{\text{WBE}} = R_{\text{GWBE}} \cap \left\{ \gamma_{1-3} \in \mathbb{R}^+ \mid \sum_{j=1}^3 \gamma_i/1 + \gamma_i \leq (7/2 - 4\gamma_i/1 + \gamma_i), \text{ for } 1 \leq i \leq 3 \right\},$$  

(26)

and

$$R_{\text{FOS}} = R_{\text{GWBE}} \cap \left\{ \gamma_{1-3} \in \mathbb{R}^+ \mid \gamma_i \leq 1, \text{ for } 1 \leq i \leq 3 \right\},$$  

(27)

for the GWBE, WBE, and FOS schemes. Note that it is implicitly assumed that $E_1 = \{\text{UE}_1, \text{UE}_4\}$, $E_2 = \{\text{UE}_2, \text{UE}_5\}$, and $E_3 = \{\text{UE}_3, \text{UE}_6\}$ for the FOS scheme. The upper boundaries of these regions in the positive orthant are plotted in Fig. 1.

For the GWBE scheme, an extra restriction $\gamma_3 = \min \{ \gamma_3, 5 \}$ is placed as the admissible $\gamma_3$ can go to infinity. It can be observed that the boundary surface of $R_{\text{GWBE}}$ lies well above those of $R_{\text{WBE}}$ and $R_{\text{FOS}}$. This implies that $R_{\text{GWBE}}$ contains more admissible points than $R_{\text{WBE}}$ and $R_{\text{FOS}}$, so more general SINR constraints $\gamma_{1-3}$ can be met in the GWBE scheme.

To explore the effects of having different numbers of UEs, Fig. 2 plots the achievable SINR versus the number of UEs given the fixed $\tau = 3$. For the FOS scheme, the grouping...
among UEs for any given $K$ is assumed to be optimal in the sense of maximizing the achievable SINR. It can be observed that increasing $K$, making pilot contamination more serious, leads to decreasing achievable SINRs for all three schemes. Our proposed GWBE scheme, however, attains relatively higher SINRs compared with the WBE and FOS schemes. Interestingly, the WBE scheme does not always outperform the FOS scheme for $K < 7$, but does so for $K \geq 7$. This highlights that the GWBE scheme exhibits a consistent superiority over the FOS scheme compared with the WBE scheme.

By specifying the SINR-requirement pattern of $K = 3l$ UEs, how many UEs are admissible for a given pilot length is depicted in Fig. 3. It can be observed that the number of admissible UEs scales almost linearly with the pilot length whatever scheme is adopted. This linear relationship directly demonstrates how the user capacity is limited by the pilot length. Also shown in the same figure, the GWBE scheme, without doubt, substantially outperforms the other two schemes in terms of admitting more UEs. In addition, two non-capacity-achieving schemes exhibit comparable user capacities especially at short pilot lengths.

VI. CONCLUSIONS

This paper has investigated the user capacity of downlink TDD Massive MIMO systems in the pilot-contaminated regime. The necessary condition for admitting a group of UEs with general SINR requirements has been provided. It shows an intrinsic capacity upper bound due to the limited length of pilot sequences. Meanwhile, the capacity-achieving pilot sequence and power allocation, which can achieve the identified user capacity and satisfy the SINR requirements, has been proposed and compared with the non-capacity-achieving WBE and FOS schemes. The results of this study indicate that the capacity-achieving allocation is necessary for the purpose of enhancing the user capacity.

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