Inverse Multiobjective Optimization Through Online Learning

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Abstract

We study the problem of learning the objective functions or constraints of a multi-objective decision making model, based on a set of sequentially arrived decisions. In particular, these decisions might not be exact and possibly carry measurement noise or are generated with the bounded rationality of decision makers. In this paper, we propose a general online learning framework to deal with this learning problem using inverse multiobjective optimization. More precisely, we develop two online learning algorithms with implicit update rules which can handle noisy data. Numerical results show that both algorithms can learn the parameters with great accuracy and are robust to noise.

1 Introduction

Understanding human participants’ preferences and desires is critical for an organization in designing and providing services or products. Nevertheless, as in most scenarios, we can only observe their decisions or behaviors, while cannot directly access their decision making schemes. Indeed, participants probably do not have exact information regarding their own decision making process [1]. To bridge the discrepancy, one idea has been proposed and received significant research attention, which is to infer or learn the missing information of the underlying decision models from observed data, assuming that human decision makers are making optimal decisions [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. This idea actually carries the data-driven concept and is more applicable as a large amount of data are generated and become readily available, especially those from digital devices and online transactions.

Inferring unknown parameters of optimization model from observed decisions is often cast into an inverse optimization problem. It seeks particular values for those parameters such that the difference between the actual observation and the expected solution to the optimization model (populated with those inferred values) is minimized. Although complicated, an inverse optimization model can often be simplified for computation through using KKT conditions or strong duality of the decision making model, provided that it is convex. Nowadays, extending from its initial form that only considers a single observation [2, 3, 4, 5, 6, 15], inverse optimization has been further developed and applied to handle many observations [1, 7, 8, 9, 16, 11]. Nevertheless, a particular challenge, which is almost unavoidable for any large data set, is that the data could be inconsistent due to measurement errors or decision makers’ sub-optimality. To address this challenge, the assumption on the observations’ optimality is weakened to integrate those noisy data, and KKT conditions or strong duality is relaxed to incorporate inexactness.

Different from the majority of existing literature, [17][18][19] take another perspective to explain the so called “data inconsistency”: decision makers are driven by multiple criteria, and different people have different preferences or weights over those criteria, which leads them to make a variety of responses or choices. Then, it can be anticipated that once we remove the variance caused by such multi-criteria decision making from data, their quality or consistency can be greatly improved. We

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note that this explanation matches a real situation where data are collected from multiple participants of different backgrounds or personalities. Indeed, it is not rare that the same customer, even when facing the same set of products, makes different purchases over time, reflecting that her preferences on multiple criteria might shift. Moreover, we would like to point out that for a service provider or a product supplier, it is more critical to correctly understand the whole customer population, their decision making criteria, and the distribution of their preferences, rather than to have a precise estimation on every single customer’s utility function. Actually the latter one is practically infeasible or unnecessary when the customer population is large.

In this paper, we aim to learn the constraints and a set of objective functions of a decision making problem with multiple objectives, instead of inferring parameters of a decision making problem with a single objective. In particular, we consider such learning problem in an online fashion, noting that in many practical scenarios observations are unveiled sequentially. Specifically, we study such learning problem as an inverse multiobjective optimization problem (IMOP) dealing with noisy data, develop online learning algorithms to derive parameters for each objective function and constraint, and finally output an estimation of the distribution of weights (which, together with those objective functions, define individuals’ utility functions) among human subjects.

1.1 Related work

Our work is most related to the subject of inverse multiobjective optimization. The goal is to find multiple objective functions or constraints that explains the observed efficient solutions well. This subject actually carries the data-driven concept and becomes more applicable as large amounts of data are generated and become readily available, especially those from digital devices and online transactions. There are several recent studies related to the presented research. One is in [6], which considers a single observation that is assumed to be an exact optimal solution. Then, given a set of well-defined linear functions, an inverse optimization is formulated to learn their weights. Another two are [17, 18], in which which propose the batch learning framework to infer the utility functions or constraints from multiple noisy decisions through inverse multiobjective optimization. These two work can be categorized as doing inverse multiobjective optimization in batch setting. In contrast, we do inverse multiobjective optimization in online setting, and the proposed online learning algorithms significantly accelerate the learning process with performance guarantees, allowing us to deal with more realistic and complex preference inference problems.

Also related to our work is the line of research conducted by [16] and [11], which develops online learning methods to infer the utility function or constraints from sequentially arrived observations. However, their approach is only possible to handle inverse optimization with single objective. More specifically, their methods apply to situations where observations are generated by decision making problems with only one objective function. Differently, our approach does not make the single-objective assumption and only requires the convexity of the underlying decision making problem with multiple objectives. Hence, we believe that our work generalize their methods and extends the applicability of online learning from solving inverse optimization problems to inverse multiobjective optimization problems.

1.2 Our contributions

To the best of authors’ knowledge, we propose the first general framework of online learning for inferring decision makers’ objective functions or constraints using inverse multiobjective optimization. This framework can learn the parameters of any convex decision making problem, and can explicitly handle noisy decisions. Moreover, we show that the online learning approach, which adopts an implicit update rule, has an $O(\sqrt{T})$ regret under suitable regularity conditions when using the ideal loss function. We finally illustrate the performance of two algorithms on both a multiobjective quadratic programming problem and a portfolio optimization problem. Results show that both algorithms can learn parameters with great accuracy and are robust to noise while the second algorithm significantly accelerate the learning process over the first one.
2 Problem setting

In this section, we review basic concepts in multiobjective decision making problem and introduce the framework for solving inverse multiobjective optimization problems in batch setting [17].

2.1 Decision making problem with multiple objectives

We consider a family of parametrized multiobjective decision making problems of the form

$$\min_{x \in \mathbb{R}^n} \left\{ f_1(x, \theta), f_2(x, \theta), \ldots, f_p(x, \theta) \right\}$$

$$\text{s.t. } x \in X(\theta)$$

where \( p \geq 2 \) and \( f_i(x, \theta) : \mathbb{R}^n \times \mathbb{R}^{n_o} \rightarrow \mathbb{R} \) for each \( l \in [p] \). Assume parameter \( \theta \in \Theta \subseteq \mathbb{R}^{n_o} \). We denote the vector of objective functions by \( f(x, \theta) = (f_1(x, \theta), f_2(x, \theta), \ldots, f_p(x, \theta))^T \). Assume \( X(\theta) = \{ x \in \mathbb{R}^n : g(x, \theta) \leq 0, x \in \mathbb{R}^n \} \), where \( g(x, \theta) = (g_1(x, \theta), \ldots, g_q(x, \theta))^T \) is another vector-valued function with \( g_k(x, \theta) : \mathbb{R}^n \times \mathbb{R}^{n_o} \rightarrow \mathbb{R} \) for each \( k \in [q] \).

**Definition 2.1** (Efficiency). For fixed \( \theta \), a decision vector \( x^* \in X(\theta) \) is said to be efficient if there exists no other decision vector \( x \in X(\theta) \) such that \( f_i(x, \theta) \leq f_i(x^*, \theta) \) for all \( i \in [p] \), and \( f_k(x, \theta) < f_k(x^*, \theta) \) for at least one \( k \in [p] \).

In the study of multiobjective optimization, the set of all efficient solutions is denoted by \( X_E(\theta) \) and called the efficient set. The weighting method is commonly used to obtain an efficient solution through computing the problem of weighted sum (PWS) [20] as follows.

$$\min_{x \in X(\theta)} \{ w^T f(x, \theta) \}$$

where \( w = (w^1, \ldots, w^p)^T \). Without loss of generality, all possible weights are restricted to a simplex, which is denoted by \( \mathcal{W}_p = \{ w \in \mathbb{R}^p : 1^T w = 1 \} \). Next, we denote the set of optimal solutions for the (PWS) by

$$S(w, \theta) = \arg \min_{x} \{ w^T f(x, \theta) : x \in X(\theta) \}.$$

Let \( \mathcal{W}_p^+ = \{ w \in \mathcal{W}_p^+ : 1^T w = 1 \} \). Following from Theorem 3.1.2 of [21], we have:

**Proposition 2.1.** If \( x \in S(w, \theta) \) and \( w \in \mathcal{W}_p^+ \), then \( x \in X_E(\theta) \).

The next result from Theorem 3.1.4 of [21] states that all the efficient solutions can be found by the weighting method for convex [DMP]

**Proposition 2.2.** Assume that [DMP] is convex. If \( x \in X \) is an efficient solution, then there exists a weighting vector \( w \in \mathcal{W}_p \) such that \( x \) is an optimal solution of [PWS].

By Propositions 2.1, 2.2, we can summarize the relationship between \( S(w, \theta) \) and \( X_E(\theta) \) as follows.

**Corollary 2.2.1.** For convex [DMP]

$$\bigcup_{w \in \mathcal{W}_p^+} S(w, \theta) \subseteq X_E(\theta) \subseteq \bigcup_{w \in \mathcal{W}_p} S(w, \theta)$$

In the following, we make a few assumptions to simplify our understanding, which are actually mild and appear often in the literature.

**Assumption 1.** Set \( \Theta \) is a convex compact set. There exists \( D > 0 \) such that \( \| \theta \|_2 \leq D \) for all \( \theta \in \Theta \). In addition, for each \( \theta \in \Theta \), both \( f(x, \theta) \) and \( g(x, \theta) \) are convex in \( x \).

2.2 Inverse multiobjective optimization

Consider a learner who has access to decision makers’ decisions, but does not know their objective functions or constraints. In the inverse multiobjective optimization model, the learner aims to learn decision makers’ multiple objective functions from observed noisy decisions only, and no information regarding decision makers’ preferences over multiple objective functions is available. We denote \( y \) the observed noisy decision that might carry measurement error or is generated with a bounded rationality of the decision maker, i.e., being suboptimal. Throughout the paper we assume that \( y \) is a random variable distributed according to an unknown distribution \( \mathbb{P}_y \) supported on \( \mathcal{Y} \). As \( y \) is a noisy observation, we note that \( y \) does not necessarily belong to \( X(\theta) \), i.e., it might be either feasible or infeasible with respect to \( X(\theta) \).
2.2.1 Loss function and surrogate loss function

Ideally, the learner would aim to learn \( \theta \) by finding parameter values that minimizes the distance between the noisy decision and the predicted decision derived with those values. Without knowing decision makers’ preferences over multiple objective functions, however, the learner cannot predict a desired decision even when \( \theta \) is given. Hence, the traditional loss function of the distance between the observation and the prediction in the traditional online learning [22, 23] is not applicable. To address such a challenge, we begin with a discussion on the construction of an appropriate loss function for the inverse multiobjective optimization problem [17, 18].

Given a noisy decision \( y \) and a hypothesis \( \theta \), the following loss function can be defined as the minimum (squared) distance between \( y \) and the efficient set \( X_E(\theta) \):

\[
l(y, \theta) = \min_{x \in X_E(\theta)} \| y - x \|_2^2.
\]

For a general DMP, however, there might exist no explicit way to characterize the efficient set \( X_E(\theta) \). Hence, an approximation approach to practically describe this set can be adopted. Following from Corollary 2.2.1 and its following remarks, a sampling approach can be adopted to generate \( w_k \in \mathcal{W}_p \) for each \( k \in [K] \) and approximate \( X_E(\theta) \) as \( \bigcup_{k \in [K]} S(w_k, \theta) \). Then, the surrogate loss function is defined as

\[
l_K(y, \theta) = \min_{x \in \bigcup_{k \in [K]} S(w_k, \theta)} \| y - x \|_2^2.
\]

By using binary variables, this surrogate loss function can be converted into the Surrogate Loss Problem,

\[
l_K(y, \theta) = \min_{z_j \in \{0,1\}} \| y - \sum_{k \in [K]} z_k x_k \|_2^2
\]

\[\text{s.t. } \sum_{k \in [K]} z_k = 1, \quad x_k \in S(w_k, \theta)\]

Constraint \( \sum_{k \in [K]} z_k = 1 \) ensures that exactly one of efficient solutions will be chosen to measure the distance to \( y \). Hence, solving this optimization problem identifies some \( w_k \) with \( k \in [K] \) such that the corresponding efficient solution \( S(w_k, \theta) \) is closest to \( y \).

Remark 2.1. It is guaranteed that no efficient solution will be excluded if all weight vectors in \( \mathcal{W}_p \) are enumerated. As it is practically infeasible due to computational intractability, we can control the number of sampled weights \( K \) to balance the tradeoff between the approximation accuracy and computational efficacy. Certainly, if the computational power is strong, we would suggest to draw a large number of weights evenly in \( \mathcal{W}_p \) to avoid any bias. In practice, for general convex DMP, we evenly sample \( \{w_k\}_{k \in [K]} \) from \( \mathcal{W}_p^+ \) to ensure that \( S(w_k, \theta) \in X_E(\theta) \). If \( f(x, \theta) \) is known to be strictly convex, we can evenly sample \( \{w_k\}_{k \in [K]} \) from \( \mathcal{W}_p \) as \( S(w_k, \theta) \in X_E(\theta) \) by Proposition 2.1.

3 Online learning for IMOP

In our online learning setting, noisy decisions become available to the learner one by one. Hence, the learning algorithm produces a sequence of hypotheses \((\theta_1, \ldots, \theta_{T+1})\). Here, \( T \) is the total number of rounds, and \( \theta_1 \) is an arbitrary initial hypothesis and \( \theta_t \) for \( t > 1 \) is the hypothesis chosen after seeing the \((t-1)^{th}\) decision. Let \( l(y_t, \theta_t) \) denote the loss the learning algorithm suffers when it tries to predict \( y_t \) based on the previous observed decisions \( \{y_1, \ldots, y_{t-1}\} \). The goal of the learner is to minimize the regret, which is the cumulative loss \( \sum_{t=1}^{T} l(y_t, \theta_t) \) against the best possible loss when the whole batch of decisions are available. Formally, the regret is defined as

\[
R_T = \sum_{t=1}^{T} l(y_t, \theta_t) - \min_{\theta \in \Theta} \sum_{t=1}^{T} l(y_t, \theta).
\]

Unlike most online learning problems that assume the loss function to be smooth [22, 23], in this study, \( l(y, \theta) \) and \( l_K(y, \theta) \) are not necessarily smooth, due to the structures of \( X_E(\theta) \) and \( \bigcup_{k \in [K]} S(w_k, \theta) \). Thus, the popular gradient based online learning algorithm [24, 25] fails and our problem is significantly more difficult than most online learning problems. To address this challenge, two online learning algorithms are developed in the next subsection.
### 3.1 Online implicit updates

Once receiving the $t$th noisy decision $y_t$, the ideal way to update $\theta_{t+1}$ is by solving the following optimization problem using the ideal loss function:

$$
\theta_{t+1} = \arg \min_{\theta \in \Theta} \frac{1}{2} \| \theta - \theta_t \|^2 + \eta_t l(y_t, \theta) \tag{2}
$$

where $\eta_t$ is the learning rate in each round, and $l(y_t, \theta)$ is defined in the loss function.

As explained in previous section, $l(y_t, \theta)$ might not be computable due to the non-existence of closed form of the efficient set $X_E(\theta)$. Thus, we seek to approximate the update (2) by the following:

$$
\theta_{t+1} = \arg \min_{\theta \in \Theta} \frac{1}{2} \| \theta - \theta_t \|^2 + \eta_t l_K(y_t, \theta) \tag{3}
$$

where $\eta_t$ is the learning rate in each round, and $l_K(y_t, \theta)$ is defined in surrogate loss.

The update (3) approximates that in (2), and seeks to balance the tradeoff between "conservativeness" and "correctiveness", where the first term characterizes how conservative we are to maintain the current estimation, and the second term indicates how corrective we would like to modify with the new estimation. As no closed form exists for $\theta_{t+1}$ in general, this update method is an implicit approach.

To solve (3), we can replace $\theta_{t+1}$ by KKT conditions for each $k \in [K]$. See supplementary material Section A for details. Alternatively, solving (3) is equivalent to solving $K$ independent programs defined in the following and taking the one with the least optimal value (breaking ties arbitrarily).

$$
\min_{\theta \in \Theta} \frac{1}{2} \| \theta - \theta_t \|^2 + \eta_t \| y_t - x \|^2 \tag{4}
$$

s.t. $x \in S(w_k, \theta)$

Our application of the implicit update rule to learn the parameter of an DMP proceeds as outlined in Algorithm 1.

**Algorithm 1** Online Learning for IMOP

1. **Input:** noisy decisions $\{y_t\}_{t \in T}$, weights $\{w_k\}_{k \in K}$
2. **Initialize** $\theta_1 = 0$
3. **for** $t = 1$ to $T$ **do**
4. **receive** $y_t$
5. **suffer loss** $l_K(y_t, \theta_t)$
6. **if** $l_K(y_t, \theta_t) = 0$ **then**
7. $\theta_{t+1} \leftarrow \theta_t$
8. **else**
9. **set learning rate** $\eta_t \propto 1/\sqrt{t}$
10. **update** $\theta_{t+1}$ by solving (3) directly (or equivalently solving $K$ subproblems (4))
11. **end if**
12. **end for**

**Remark 3.1.** (i) When choosing (4) to update $\theta_{t+1}$, we can do parallel computing to implement the $K$ independent problems of (4), which would dramatically improve the computational efficiency. (ii) After the completion of Algorithm 1, we can allocate every $y_t$ to the $w_k$ that minimizes $l_K(y_t, \theta_{T+1})$, which provides an inference on the distribution of weights of component functions $f_i(x, \theta)$ over human subjects.

**Acceleration of Algorithm 1** Note that we update $\theta$ and the weight sample assigned to $y_t$ in (3) simultaneously, meaning that both $\theta$ and the weight sample index $k$ are variables when solving (3). In other words, one needs to solve $K$ subproblems (4) to get an optimal solution for (3). However, we note that the increment of $\theta$ by solving (3) is typically small for each update. Consequently, the weight sample assigned to $y_t$ using $\theta_{t+1}$ is roughly the same as using the previous guess of this parameter, i.e., $\theta_t$. Hence, it is reasonable to approximately solve (3) by first assigning a weight...
sample to \( y_t \) based on the previous updating result. Then, instead of computing \( K \) problems of (4), we simply compute a single one associated with the selected weight sample. Through such a procedure, we significantly ease the computational burden of solving (3). Our application of the accelerated implicit update rule proceeds as outlined in Algorithm 2.

**Algorithm 2 Online Learning with Acceleration**

1: **Input:** noisy decisions \( \{y_t\}_{t \in T} \), weights \( \{w_k\}_{k \in K} \)
2: **Initialize** \( \theta_1 = 0 \)
3: **for** \( t = 1 \) to \( T \) **do**
4: \( y_t \)
5: **suffer loss** \( l_K(y_t, \theta_t) \)
6: \( k^* = \arg \min_{k \in [K]} \|y_t - x_k\|^2_2 \), where \( x_k \in S(w_k, \theta_t) \) for each \( k \in [K] \)
7: **if** \( l_K(y_t, \theta_t) = 0 \) **then**
8: \( \theta_{t+1} \leftarrow \theta_t \)
9: **else**
10: **set learning rate** \( \eta_t \propto 1/\sqrt{t} \)
11: **update** \( \theta_{t+1} \) by solving (4) with \( k = k^* \)
12: **end if**
13: **end for**

**Mini-batches** One technique to enhance online learning is to consider multiple observations per update [26]. In online IMOP, this means that computing \( \theta_{t+1} \) using \( |N_t| > 1 \) noisy decisions:

\[
\theta_{t+1} = \arg \min_{\theta \in \Theta} \frac{1}{2} \|\theta - \theta_t\|^2_2 + \frac{\eta_t}{|N_t|} \sum_{t \in N_t} l_K(y_t, \theta) \tag{5}
\]

However, we should point out that applying Mini-batches might not be suitable here as the update (5) is drastically more difficult to compute even for \( |N_t| = 2 \) than the update (3) with a single observation.

### 3.2 Analysis of convergence

Note that the proposed online learning algorithms are generally applicable to learn the parameter of any convex DMP. In this section, we show that the average regret converges at a rate of \( O(1/\sqrt{T}) \) under certain regularity conditions based on the ideal loss function \( l(y, \theta) \). Namely, we consider the regret bound when using the ideally implicit update rule (2). Next, we introduce a few assumptions that are regular in literature [1, 7, 9, 8, 17, 11].

**Assumption 3.1.** \textbf{(a)} \( X(\theta) \) is closed, and has a nonempty relatively interior. \( X(\theta) \) is also bounded. Namely, there exists \( B > 0 \) such that \( \|x\|_2 \leq B \) for all \( x \in X(\theta) \). The support \( \mathcal{Y} \) of the noisy decisions \( y \) is contained within a ball of radius \( R \) almost surely, where \( R < \infty \). In other words, \( \mathbb{P}(\|y\|_2 \leq R) = 1 \).

\textbf{(b)} Each function in \( f \) is strongly convex on \( \mathbb{R}^n \), that is for each \( l \in [p] \), \( \exists \lambda_l > 0 \) for all \( x, y \in \mathbb{R}^n \)

\[
\left( \nabla f_l(y, \theta_l) - \nabla f_l(x, \theta_l) \right)^T (y - x) \geq \lambda_l \|x - y\|^2_2.
\]

Regarding Assumption 3.1(a), assuming that the feasible region is closed and bounded is very common in inverse optimization. The finite support of the observations is needed since we do not hope outliers have too many impacts in our learning. Let \( \lambda = \min_{l \in [p]} \{\lambda_l\} \). It follows that \( w^T f(x, \theta) \) is strongly convex with parameter \( \lambda \) for \( w \in \mathcal{V}_p \). Therefore, Assumption 3.1(b) ensures that \( S(w, \theta) \) is a single-valued set for each \( w \).

The performance of the algorithm also depends on how the change of \( \theta \) affects the objective values. For \( \forall w \in \mathcal{V}_p, \theta_1 \in \Theta, \theta_2 \in \Theta \), we consider the following function

\[
h(x, w, \theta_1, \theta_2) = w^T f(x, \theta_1) - w^T f(x, \theta_2).
\]
Assumption 3.2. \( \exists \kappa > 0, \forall w \in \mathcal{W}_p, h(\cdot, w, \theta_1, \theta_2) \) is \( \kappa \)-Lipschitz continuous on \( \mathcal{Y} \). That is, \( \forall x, y \in \mathcal{Y} \),
\[
|h(x, w, \theta_1, \theta_2) - h(y, w, \theta_1, \theta_2)| \leq \kappa \|\theta_1 - \theta_2\|_2 \|x - y\|_2.
\]

Basically, this assumption says that the objective functions will not change much when either the parameter \( \theta \) or the variable \( x \) is perturbed. It actually holds in many common situations, including the multiobjective linear program and multiobjective quadratic program.

From now on, given any \( y \in \mathcal{Y}, \forall \theta \in \Theta \), we denote \( x(\theta) \) the efficient point in \( X_E(\theta) \) that is closest to \( y \). Namely, \( l(y, \theta) = \|y - x(\theta)\|_2^2 \).

Lemma 3.1. Under Assumptions 3.1-3.2, the loss function \( l(y, \theta) \) is uniformly \( \frac{4(B + R)\kappa}{\lambda} \)-Lipschitz continuous in \( \theta \). That is, \( \forall y \in \mathcal{Y}, \forall \theta_1, \theta_2 \in \Theta \), we have
\[
|l(y, \theta_1) - l(y, \theta_2)| \leq \frac{4(B + R)\kappa}{\lambda} \|\theta_1 - \theta_2\|_2.
\]

The key point in proving Lemma 3.1 is the observation that the perturbation of \( S(w, \theta) \) due to \( \theta \) is bounded by the perturbation of \( \theta \) by applying Proposition 6.1 in [27]. Details of the proof are given in supplementary material.

Assumption 3.3. For the DMP, \( \forall y \in \mathcal{Y}, \forall \theta_1, \theta_2 \in \Theta, \forall \alpha, \beta \geq 0 \) s.t. \( \alpha + \beta = 1 \), we have either of the following:

(a) if \( x_1 \in X_E(\theta_1) \), and \( x_2 \in X_E(\theta_2) \), then
\[
\alpha x_1 + \beta x_2 \in X_E(\alpha \theta_1 + \beta \theta_2).
\]

(b) Moreover,
\[
\|\alpha x(\theta_1) + \beta x(\theta_2) - x(\alpha \theta_1 + \beta \theta_2)\|_2 \\
\leq \alpha \beta \|x(\theta_1) - x(\theta_2)\|_2/(2(B + R)).
\]

The definition of \( x(\theta_1), x(\theta_2) \) and \( x(\alpha \theta_1 + \beta \theta_2) \) is given before Lemma 3.1. This assumption requires that the convex combination of \( x_1 \in X_E(\theta_1) \), and \( x_2 \in X_E(\theta_2) \) belongs to \( X_E(\alpha \theta_1 + \beta \theta_2) \). Or there exists an efficient point in \( X_E(\alpha \theta_1 + \beta \theta_2) \) that is close to the convex combination of \( x(\theta_1) \) and \( x(\theta_2) \). Examples are given in supplementary material.

Let \( \theta^* \) be an optimal inference to \( \min_{\theta \in \Theta} \sum_{t \in [T]} l(y_t, \theta), \) i.e., an inference derived with the whole batch of observations available. Then, the following theorem asserts that under the above assumptions, the regret \( R_T = \sum_{t \in [T]} (l(y_t, \theta^*) - l(y_t, \theta)) \) of the online learning algorithm is of \( O(\sqrt{T}) \).

Theorem 3.2. Suppose Assumptions 3.1-3.3 hold. Then, choosing \( \eta_t = \frac{DL}{2\sqrt{2(B + R)\kappa}} \), we have
\[
R_T \leq \frac{4\sqrt{2(B + R)D\kappa}}{\lambda} \sqrt{T}.
\]

We establish the above regret bound by extending Theorem 3.2 in [25]. Our extension involves several critical and complicated analyses for the structure of the optimal solution set \( S(w, \theta) \) as well as the loss function, which is essential to our theoretical understanding. Moreover, we relax the requirement of smoothness of loss function to Lipschitz continuity through a similar argument in Lemma 1 of [28] and [29].

Remark 3.2. The above regret bound applies for ideal case where the loss function \( l(y, \theta) \) is used for the online learning. Regret bound for using the surrogate loss is currently under investigation as it requires more complicated analyses about the structure of \( \bigcup_{k \in [K]} S(w_k, \theta) \) and the corresponding \( l_K(y, \theta) \). Nonetheless, we numerically demonstrate that \( l_K(y, \theta) \), the approximation to \( l(y, \theta) \), indeed works well in learning the parameters of DMP under various environments.

4 Experiments

In this section, we will provide a multiobjective quadratic program (MQP) and a portfolio optimization problem to illustrate the performance of the proposed online learning Algorithms [1] and [2]. The mixed integer second order conic programs, which are derived from using KKT conditions in (3), are solved by Gurobi. All the algorithms are programmed with Julia [30]. The experiments have been run on an Intel(R) Xeon(R) E5-1620 processor that has a 3.60GHz CPU with 32 GB RAM.
Consider the following multiobjective quadratic optimization problem.

\[
\min_{x \in \mathbb{R}^2} \begin{cases} 
    f_1(x) = \frac{1}{2} x^T Q_1 x + c_1^T x \\
    f_2(x) = \frac{1}{2} x^T Q_2 x + c_2^T x 
\end{cases} \\
\text{s.t. } Ax \leq b,
\]

where the parameters of the two objective functions are

\[Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -6 \\ -5 \end{bmatrix},\]

and the parameters for the feasible region are

\[A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.\]

We suppose there are \( T \) decision makers. In each round, the learner would receive one noisy decision. Her goal is to learn the objective functions or restrictions of these decision makers. The noisy decision in each round \( t \in [T] \) is generated as follows. In round \( t \), we suppose that the decision maker derives an efficient solution \( x_t \) by solving PWS with weight \( w_t \), which is uniformly chosen from \( W_2 \). Next, the learner receives the noisy decision \( y_t \), which has been corrupted by noise that has a jointly uniform distribution with support \([−0.5, 0.5]^2\). Namely, \( y_t = x_t + \epsilon_t \), where each element of \( \epsilon_t \sim U(−0.5, 0.5) \).

### 4.1 Learning the Objectives Functions

In the first set of experiment, the learner seeks to learn \( c_1 \) and \( c_2 \) given the noisy decisions that arrive sequentially in \( T \) rounds. We assume that \( c_1 \) is within range \([1, 0]^2\), \( c_2 \) is within range \([-6, -1]^2\). \( T = 1000 \) rounds of noisy decisions are generated, and \( K = 41 \) weights from \( W_2 \) are evenly sampled. The learning rate is set to \( \eta_T = 5/\sqrt{T} \). Then, we implement Algorithms 1 and 2. At each round \( t \), we solve (4) using parallel computing with 6 workers.

To illustrate the performance of the algorithms in a statistical way, we run 100 repetitions of the experiments. Figure 1a shows the total estimation errors of \( c_1 \) and \( c_2 \) in each round over the 100 repetitions for the two algorithms. We also plot the average estimation error of the 100 repetitions. As can be seen in this figure, convergence for both algorithms is pretty fast. Also, estimation errors over rounds for different repetitions concentrate around the average, indicating that our algorithm is pretty robust to noise. The estimation error in the last round is not zero because we use a finite \( K \) to approximate the efficient set. We see in Figure 1b that Algorithm 2 is much faster than Algorithm 1, especially when \( K \) is large. To further illustrate the performance of algorithms, we randomly pick one repetition. The estimated efficient set after \( T = 1000 \) rounds is indicated by red line. The real efficient set is shown by the yellow line.

4.1.1 Learning the Objective Functions

In the first set of experiment, the learner seeks to learn \( c_1 \) and \( c_2 \) given the noisy decisions that arrive sequentially in \( T \) rounds. We assume that \( c_1 \) is within range \([1, 0]^2\), \( c_2 \) is within range \([-6, -1]^2\). \( T = 1000 \) rounds of noisy decisions are generated, and \( K = 41 \) weights from \( W_2 \) are evenly sampled. The learning rate is set to \( \eta_T = 5/\sqrt{T} \). Then, we implement Algorithms 1 and 2. At each round \( t \), we solve (4) using parallel computing with 6 workers.

To illustrate the performance of the algorithms in a statistical way, we run 100 repetitions of the experiments. Figure 1a shows the total estimation errors of \( c_1 \) and \( c_2 \) in each round over the 100 repetitions for the two algorithms. We also plot the average estimation error of the 100 repetitions. As can be seen in this figure, convergence for both algorithms is pretty fast. Also, estimation errors over rounds for different repetitions concentrate around the average, indicating that our algorithm is pretty robust to noise. The estimation error in the last round is not zero because we use a finite \( K \) to approximate the efficient set. We see in Figure 1b that Algorithm 2 is much faster than Algorithm 1, especially when \( K \) is large. To further illustrate the performance of algorithms, we randomly pick one repetition. The estimated efficient set after \( T = 1000 \) rounds is indicated by red line. The real efficient set is shown by the yellow line. (d) Each bar represents the proportion of the 1000 decision makers that has the corresponding weight for \( f_1(x) \).
one repetition using Algorithm \[1\] and plot the estimated efficient set in Figure 1c. We can see clearly that the estimated efficient set almost coincides with the real efficient set.

We also plot our prediction of the distribution for the preferences of \( f_1(x) \) and \( f_2(x) \) among the 1000 decision makers. Since there are only two objective functions, it is sufficient to draw the distribution of the weight for \( f_1(x) \) (given that the sum of weights of \( f_1(x) \) and \( f_2(x) \) equals to 1). As shown in Figure 1d, except in two endpoint areas, the distribution follows roughly uniformly distribution, which matches our uniformly sampled weights. Indeed, comparing Figures 1c and 1d, we would like to point out that a bound effect probably occurs in these two endpoint areas. As can be seen, although different weights are imposed on component functions, the noiseless optimal solutions, as well as observed decisions, do likely to merge together due to the limited feasible space in those areas. We believe that it reflects an essential challenge in learning multiple objective functions in practice and definitely deserves a further study.

### 4.1.2 Learning the Right-hand Side

In the second set of experiment, the learner seeks to learn \( b \) given the noisy decisions that arrive sequentially in \( T \) rounds. We assume that \( b \) is within \([-10, 10]^2\). \( T = 1000 \) rounds of noisy decisions are generated. \( K = 81 \) weights from \( \mathbb{W}_2 \) are evenly sampled. The learning rate is set to \( \eta_t = 5/\sqrt{t} \). Then, we apply Algorithms 1 and 2.

To illustrate the performance of the two algorithms, we run 100 repetitions of the experiments. Figure 3a shows the estimation error of \( b \) in each round over the 100 repetitions for the two algorithms. We also plot the average estimation error of the 100 repetitions. As can be seen in the figure, convergence for both algorithms is pretty fast. In addition, we see in Figure 3b that Algorithm 2 is much faster than Algorithm 1.

### 4.2 Learning the expected returns in portfolio optimization

In this example, we consider various noisy decisions arising from different investors in a stock market. More precisely, we consider a portfolio selection problem, where investors need to determine the fraction of their wealth to invest in each security in order to maximize the total return and minimize the total risk. The portfolio selection process typically involves the cooperation between an investor and a portfolio analyst, where the analyst provides an efficient frontier on a certain set of securities to the investor and then the investor selects a portfolio according to her preference to the returns and risks. The classical Markovitz mean-variance portfolio selection \cite{31} in the following is
used by analysts.

$$\min \begin{pmatrix} f_1(x) = -r^T x \\ f_2(x) = x^T Q x \end{pmatrix} \quad \text{s.t.} \quad 0 \leq x_i \leq b_i, \quad \forall i \in [n], \quad \sum_{i=1}^{n} x_i = 1,$$

where $r \in \mathbb{R}_+^n$ is a vector of individual security expected returns, $Q \in \mathbb{R}^{n \times n}$ is the covariance matrix of securities returns, $x$ is a portfolio specifying the proportions of capital to be invested in the different securities, and $b_i$ is an upper bound on the proportion of security $i$, $\forall i \in [n]$.

**Dataset:** The dataset is derived from monthly total returns of 30 stocks from a blue-chip index which tracks the performance of top 30 stocks in the market when the total investment universe consists of thousands of assets. The true expected returns and true return covariance matrix for the first 8 securities are given in Appendix. Suppose a learner seeks to learn the expected return for the first five securities that an analyst uses.

The noisy decision is generated as follows. We set the upper bounds for the proportion of the 8 securities to $b_i = 1.0, \forall i \in [8]$. Then, we sample $T = 1000$ weights such that the first element of $w_1$, ranging from 0 to 1, follows a truncated normal distribution derived from a normal distribution with mean 0.5 and standard deviation 0.1. In what follows, we will not distinguish truncated normal distribution from normal distribution because their difference is negligible. These weights are then used to generate optimal portfolios on the efficient frontier that is plot in Figure 4a. Subsequently, each component of these portfolios is rounded to the nearest thousandth, which can be seen as measurement error. The learning rate is set to $\eta_t = 5/\sqrt{t}$. At each round $t$, we solve (4) using parallel computing.

| $K$ | 6     | 11    | 21    | 41    |
|-----|-------|-------|-------|-------|
| $\| \hat{r} - r_{true} \|_2$ | 0.1270 | 0.1270 | 0.0420 | 0.0091 |

In Table 1 we list the estimation error and estimated expected returns for different $K \in \{6, 11, 21, 41\}$. As is shown in the table, the estimation error becomes smaller when $K$ increases, indicating that we have a better approximation accuracy of the efficient set when using a larger $K$. We also plot the estimated efficient frontier using the estimated $\hat{r}$ for $K = 41$ in Figure 4a. We can see that the estimated efficient frontier is very close to the real one, showing that our algorithm works quite well in learning expected returns in portfolio optimization. We also plot our estimation on the distribution of the weight of $f_1(x)$ among the 1000 decision makers. As shown in Figure 4b, the distribution follows roughly normal distribution. The result of Chi-square goodness-of-fit test supports our hypotheses.


Figure 4: Learning the expected return of a Portfolio optimization problem over $T = 1000$ rounds with $K = 41$. (a) The red line indicates the real efficient frontier. The blue dots indicates the estimated efficient frontier using the estimated expected return for $T = 1000$ and $K = 41$. (b) Each bar represents the proportion of the 1000 decision makers that has the corresponding weight for $f_1(x)$.

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Before giving the reformulations, we first make some discussions about the surrogate loss functions.

\[
l_K(y, \theta) = \min_{z_k \in \{0,1\}} \| y - \sum_{k \in [K]} z_k x_k \|^2_2
\]

\[
= \min_{z_k \in \{0,1\}} \sum_{k \in [K]} \| y - z_k x_k \|^2_2 - (K-1)\|y\|^2_2
\]

where \(x_k \in S(w_k, \theta)\) and \(\sum_{k \in [K]} z_k = 1\).

Since \((K-1)\|y\|^2_2\) is a constant, we can safely drop it and use the following as the surrogate loss function when solving the optimization program in the implicit update,

\[
l_K(y, \theta) = \min_{z_k \in \{0,1\}} \sum_{k \in [K]} \| y - z_k x_k \|^2_2
\]

where \(x_k \in S(w_k, \theta)\) and \(\sum_{k \in [K]} z_k = 1\).
A.1 Single level reformulation of the IMOP for DMP

The single level reformulation for the Implicit update in the paper is given in the following

\[
\begin{align*}
\min_b & \quad \frac{1}{2} \| \theta - \theta_t \|^2 + \eta_t \sum_{k \in [K]} \| y_t - \vartheta_k \|^2 \\
\text{s.t.} & \quad \theta \in \Theta \\
& \quad g(x_k) \leq 0, \quad u_k \geq 0 \\
& \quad u_k^T g(x_k) = 0 \\
& \quad \nabla_{x_k} u_k^T f(x_k, \theta) + u_k \cdot \nabla_{x_k} g(x_k) = 0 \\
& \quad 0 \leq \vartheta_k \leq M_k z_k \\
& \quad x_k - M_k (1 - z_k) \leq \vartheta_k \leq x_k \\
& \quad \sum_{k \in [K]} z_k = 1 \\
& \quad x_k \in \mathbb{R}^n, \quad u_k \in \mathbb{R}^m, \quad t_k \in \{0, 1\}^m, \quad z_k \in \{0, 1\} \quad \forall k \in [K]
\end{align*}
\]

A.2 Single level reformulation for the Inverse Multiobjective Quadratic Problem

When the objective functions are quadratic and the feasible region is a polyhedron, the multiobjective optimization has the following form

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad \frac{1}{2} x^T Q_1 x + c_1^T x \\
& \quad \vdots \\
& \quad \frac{1}{2} x^T Q_p x + c_p^T x \\
\text{s.t.} & \quad A x \geq b
\end{align*}
\]

MQP

where \( Q_l \in \mathbb{S}^m_+ \) (the set of symmetric positive semidefinite matrices) for all \( l \in [p] \).

When trying to learn \( \{c_l\}_{l \in [p]} \), the single level reformulation for the Implicit update in the paper is given in the following

\[
\begin{align*}
\min_{c_l} & \quad \frac{1}{2} \sum_{l \in [p]} \| c_l - c_l^f \|^2 + \eta_t \sum_{k \in [K]} \| y_t - \vartheta_k \|^2 \\
\text{s.t.} & \quad c_l \in \tilde{C}_l \\
& \quad A x_k \geq b, \quad u_k \geq 0 \\
& \quad u_k \leq M t_k \\
& \quad A x_k - b \leq M (1 - t_k) \\
& \quad (w^1_k Q_1 + \cdots + w^p_k Q_p)x_k + w^1_k c_1 + \cdots + w^p_k c_p - A^T u_k = 0 \\
& \quad 0 \leq \vartheta_k \leq M_k z_k \\
& \quad x_k - M_k (1 - z_k) \leq \vartheta_k \leq x_k \\
& \quad \sum_{k \in [K]} z_k = 1 \\
& \quad x_k \in \mathbb{R}^n, \quad u_k \in \mathbb{R}^m, \quad t_k \in \{0, 1\}^m, \quad z_k \in \{0, 1\} \quad \forall l \in [p] \forall k \in [K]
\end{align*}
\]

where \( c^f_l \) is the estimation of \( c_l \) at the \( t \)th round, and \( \tilde{C}_l \) is a convex set for each \( l \in [p] \).

We have a similar single level reformulation when learning the Right-hand side \( b \). Clearly, this is a Mixed Integer Second Order Cone program (MISOCOCP) when learning either \( c_l \) or \( b \).
B Omitted Proofs

B.1 Strongly Convex of $w^T f(x, \theta)$ as stated under Assumption 3.1

Proof. By the definition of $\lambda$, 
\[
\left( \nabla w^T f(y, \theta) - \nabla w^T f(x, \theta) \right)^T (y - x) = \left( \nabla \sum_{i=1}^{p} w_i f_i(y, \theta) - \nabla \sum_{i=1}^{p} w_i f_i(x, \theta) \right)^T (y - x)
\]
\[
= \sum_{i=1}^{p} w_i \left( \nabla f_i(y, \theta_i) - \nabla f_i(x, \theta_i) \right)^T (y - x)
\]
\[
\geq \sum_{i=1}^{p} w_i \lambda_i \|x - y\|_2^2 \geq \eta \|x - y\|_2^2 \sum_{i=1}^{p} w_i
\]
\[
= \lambda \|x - y\|_2^2
\]

Thus, $w^T f(x, \theta)$ is strongly convex for $x \in \mathbb{R}^n$. \(\square\)

B.2 Proof of Lemma 3.1

Proof. By Assumption 3.1(b), we know that $S(w, \theta)$ is a single-valued set for each $w \in \mathcal{W}_p$. Thus, $\forall y \in Y, \forall \theta_1, \theta_2 \in \Theta, \exists w^1, w^2 \in \mathcal{W}_p$, s.t. 
\[
x(\theta_1) = S(w^1, \theta_1), \ x(\theta_2) = S(w^2, \theta_2)
\]

Without loss of generality, let $l_K(y, \theta_1) \geq l_K(y, \theta_2)$. Then, 
\[
|l_K(y, \theta_1) - l_K(y, \theta_2)| = l_K(y, \theta_1) - l_K(y, \theta_2)
\]
\[
= \|y - x(\theta_1)\|_2^2 - \|y - x(\theta_2)\|_2^2
\]
\[
= \|y - S(w^1, \theta_1)\|_2^2 - \|y - S(w^2, \theta_2)\|_2^2
\]
\[
\leq \|y - S(w^2, \theta_1)\|_2^2 - \|y - S(w^2, \theta_2)\|_2^2
\]
\[
= \langle S(w^2, \theta_2) - S(w^1, \theta_1), 2y - S(w^2, \theta_1) - S(w^2, \theta_2) \rangle
\]
\[
\leq 2(B + R)\|S(w^2, \theta_2) - S(w^2, \theta_1)\|_2
\]

The last inequality is due to Cauchy-Schwartz inequality and the Assumptions 3.1(a), that is 
\[
\|2y - S(w^2, \theta_1) - S(w^2, \theta_2)\|_2 \leq 2(B + R) \tag{7}
\]

Next, we will apply Proposition 6.1 in [27] to bound $\|S(w^2, \theta_2) - S(w^2, \theta_1)\|_2$.

Under Assumptions 3.1, 3.2, the conditions of Proposition 6.1 in [27] are satisfied. Therefore, 
\[
\|S(w^2, \theta_2) - S(w^2, \theta_1)\|_2 \leq \frac{2\kappa}{\lambda} \theta_1 - \theta_2 \|_2 \tag{8}
\]

Plugging (7) and (8) in (6) yields the claim. \(\square\)

B.3 Proof of Theorem 3.2

Proof. we will use Theorem 3.2 in [25] to prove our theorem.

Let $G_t(\theta) = \frac{1}{2} \|\theta - \theta_t\|_2^2 + \eta l(y_t, \theta)$. 

We will now show the loss function is convex. The first step is to show that if Assumption 3.3 holds, then the loss function $l(y, \theta)$ is convex in $\theta$.

First, suppose Assumption 3.3(a) hold. Then, 
\[
\alpha l(y, \theta_1) + \beta l(y, \theta_2) - l(y, \alpha \theta_1 + \beta \theta_2)
\]
\[
= \alpha \|y - x(\theta_1)\|_2^2 + \beta \|y - x(\theta_2)\|_2^2 - \|y - x(\alpha \theta_1 + \beta \theta_2)\|_2^2
\]
\[
\geq \alpha \|y - x(\theta_1)\|_2^2 + \beta \|y - x(\theta_2)\|_2^2 - \|y - \alpha x(\theta_1) - \beta x(\theta_2)\|_2^2 \tag{By Assumption 3.3(a)}
\]
\[
= \alpha \beta \|x(\theta_1) - x(\theta_2)\|_2^2
\]
\[
\geq 0
\]

(9)
Second, suppose Assumption 3.3(b) hold. Then,
\[
\alpha l(y, \theta_t) + \beta l(y, \theta_2) - l(y, \alpha \theta_t + \beta \theta_2)
\]
\[
= \alpha \|y - x(\theta_t)\|^2 + \beta \|y - x(\theta_2)\|^2 - \|y - x(\alpha \theta_t + \beta \theta_2)\|^2
\]
\[
= \alpha \|y - x(\theta_1)\|^2 + \beta \|y - x(\theta_2)\|^2 - \|y - \alpha x(\theta_t) - \beta x(\theta_2)\|^2
\]
\[
+ \|y - \alpha x(\theta_1) - \beta x(\theta_2)\|^2 - \|y - x(\alpha \theta_t + \beta \theta_2)\|^2
\]
\[
= \alpha \beta \|x(\theta_1) - x(\theta_2)\|^2 + \|y - \alpha x(\theta_t) - \beta x(\theta_2)\|^2 - \|y - x(\alpha \theta_t + \beta \theta_2)\|^2
\]
\[
= \alpha \beta \|x(\theta_1) - x(\theta_2)\|^2 - \langle x(\alpha \theta_t + \beta \theta_2) - x(\alpha x(\theta_1) - \beta x(\theta_2))\rangle
\]
\[
\geq \alpha \beta \|x(\theta_1) - x(\theta_2)\|^2 - \|\alpha x(\theta_t) + \beta x(\theta_2) - x(\alpha \theta_t + \beta \theta_2)\|^2
\]
\[
= \alpha \beta \|x(\theta_t) - x(\theta_2)\|^2 \tag{10}
\]

The last inequality is by Cauchy-Schwarz inequality. Note that
\[
\alpha \beta \|x(\theta_1) - x(\theta_2)\|^2 + 2\|y - x(\alpha \theta_t + \beta \theta_2) - x(\alpha \theta_t + \beta \theta_2)\|^2\]
\[
\leq 2(B + R)\|\alpha x(\theta_1) + \beta x(\theta_2) - x(\alpha \theta_t + \beta \theta_2)\|^2 \tag{11}
\]
\[
\leq \alpha \beta \|x(\theta_1) - x(\theta_2)\|^2 \tag{By Assumption 3.3(b)}
\]

Plugging (11) in (10) yields the result.

Using Theorem 3.2. in [25], for \( \alpha_t \leq \frac{G_t(\theta_{t+1})}{G_t(\theta_t)} \), we have
\[
R_T \leq \sum_{t=1}^{T} \frac{1}{\eta_t} (1 - \alpha_t) \eta_t l(y_t, \theta_t)
\]
\[
+ \frac{1}{2\eta_t} \left(\|\theta_t - \theta^*\|^2 - \|\theta_{t+1} - \theta^*\|^2\right) \tag{12}
\]

Notice that
\[
G_t(\theta_t) - G_t(\theta_{t+1})
\]
\[
= \eta_t (l(y_t, \theta_t) - l(y_t, \theta_{t+1})) - \frac{1}{2} \|\theta_t - \theta_{t+1}\|^2
\]
\[
\leq \frac{4(B + R)\alpha}{\lambda} \|\theta_t - \theta_{t+1}\|^2 - \frac{1}{2} \|\theta_t - \theta_{t+1}\|^2 \tag{13}
\]

The first inequality follows by applying Lemma 3.1

Let \( \alpha_t = \frac{R_t(\theta_{t+1})}{R_t(\theta_t)} \). Using (13), we have
\[
(1 - \alpha_t) \eta_t l(y_t, \theta_t) = (1 - \alpha_t) G_t(\theta_t)
\]
\[
= G_t(\theta_t) - G_t(\theta_{t+1})
\]
\[
\leq \frac{4(B + R)^2 \kappa^2 \eta_t}{\lambda^2} \tag{14}
\]

Plug (14) in (12), and note the telescoping sum,
\[
R_T \leq \sum_{t=1}^{T} \frac{8(B + R)^2 \kappa^2 \eta_t}{\lambda^2}
\]
\[
+ \sum_{t=1}^{T} \frac{1}{2\eta_t} \left(\|\theta_t - \theta^*\|^2 - \|\theta_{t+1} - \theta^*\|^2\right)
\]

Setting \( \eta_t = \frac{D^2}{2(B + R)^2 \kappa \sqrt{T}} \), we can simplify the second summation to \( \frac{D^2(B + R)^2 \kappa}{\lambda \sqrt{T}} \) since the sum telescopes and \( \theta_1 = 0, \|\theta^*\|^2 \leq D \). The first sum simplifies using \( \sum_{t=1}^{T} \frac{1}{\sqrt{T}} \leq 2\sqrt{T} - 1 \) to obtain the result
\[
R_T \leq \frac{4\sqrt{2}(B + R)D\kappa}{\lambda \sqrt{T}}.
\]
C Omitted Examples

C.1 Examples for which Assumption 3.3 holds

Consider for example the following quadratic program

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & & x^T x - 2\theta_1^T x \\
\text{s.t.} & & 0 \leq x \leq 10
\end{align*}
\]

One can check that Assumption 3.3 (a) is indeed satisfied. For example, let \( n = 1 \). Then, W.L.O.G, let \( \theta_1 \leq \theta_2 \). Then, \( X_E(\theta) = [\theta_1, \theta_2] \). Consider two parameters that \( \theta^1 = (\theta^1_1, \theta^1_2), \theta^2 = (\theta^2_1, \theta^2_2) \in [0, 10]^2 \). For all \( \alpha \in [0, 1] \),

\[
X_E(\alpha \theta^1 + (1 - \alpha) \theta^2) = [\alpha \theta^1_1 + (1 - \alpha) \theta^2_1, \alpha \theta^1_2 + (1 - \alpha) \theta^2_2]
\]

Although tedious, one can check that one can check that Assumption 3.3 (a) is indeed satisfied.

D Data for the Portfolio optimization problem

| Security | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| Expected Return | 0.1791 | 0.1143 | 0.1357 | 0.0837 | 0.1653 | 0.1808 | 0.0352 | 0.0368 |

| Security | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----------|----|----|----|----|----|----|----|----|
| 1        | 0.1641 | 0.0299 | 0.0478 | 0.0491 | 0.058 | 0.0871 | 0.0603 | 0.0492 |
| 2        | 0.0299 | 0.0720 | 0.0511 | 0.0287 | 0.0527 | 0.0297 | 0.0291 | 0.0326 |
| 3        | 0.0478 | 0.0511 | 0.0794 | 0.0498 | 0.0664 | 0.0479 | 0.0395 | 0.0523 |
| 4        | 0.0491 | 0.0287 | 0.0498 | 0.1148 | 0.0336 | 0.0503 | 0.0326 | 0.0447 |
| 5        | 0.0580 | 0.0527 | 0.664 | 0.0336 | 0.1073 | 0.0483 | 0.0402 | 0.0533 |
| 6        | 0.0871 | 0.0297 | 0.0479 | 0.0503 | 0.0483 | 0.1134 | 0.0591 | 0.0387 |
| 7        | 0.0603 | 0.0291 | 0.0395 | 0.0326 | 0.0402 | 0.0591 | 0.0704 | 0.0244 |
| 8        | 0.0492 | 0.0326 | 0.0523 | 0.0447 | 0.0533 | 0.0387 | 0.0244 | 0.1028 |