Stability and Resolution Analysis of Topological Derivative Based Localization of Small Electromagnetic Inclusions

Abdul Wahab *

December 23, 2014

Abstract

The aim of this article is to elaborate and rigorously analyze a topological derivative based imaging framework for locating an electromagnetic inclusion of diminishing size from boundary measurements of the tangential component of scattered magnetic field at a fixed frequency. The inverse problem of inclusion detection is formulated as an optimization problem in terms of a filtered discrepancy functional and the topological derivative based imaging functional obtained therefrom. The sensitivity and resolution analysis of the imaging functional is rigorously performed. It is substantiated that the Rayleigh resolution limit is achieved. Further, the stability of the reconstruction with respect to measurement and medium noises is investigated and the signal-to-noise ratio is evaluated in terms of the imaginary part of free space fundamental magnetic solution.

AMS subject classifications 2000. Primary, 35L05, 35R30, 74B05; Secondary, 47A52, 65J20

Key words. Electromagnetic imaging; Topological derivative; Localization; Resolution analysis; Stability analysis; Medium noise; Measurement noise.

1 Introduction

The concept of derivatives with respect to geometry or topology has played a significant role in industrial and engineering optimization problems, especially for designing optimal shapes of various products subject to industrial constraints [33]. Soon after its emergence [22], the idea was embraced for imaging of diametrically small anomalies [18] and inverse scattering problems; see, for example, [2, 4, 14, 20, 21, 23, 24] and articles cited therein.

In topological derivative based imaging framework, a trial inclusion is created in the (inclusion-free) background medium at a search point, furnishing fitted data. Then a misfit functional is constructed using measurements and the fitted data. The search points that minimize the discrepancy between measured data and the fitted data are then sought. In order to find its minima, the misfit is expanded using the asymptotic expansions due to the perturbation of the wave-field in the presence of an inclusion versus its characteristic size. The leading order term in the expansion is then referred to as the topological derivative of the misfit, which synthesizes its sensitivity relative to the insertion of an inclusion at a given search location. Its maximum, which corresponds to the point at which the insertion of the inclusion maximally decreases the misfit, is therefore a potential candidate for the location of the true inclusion.

*Department of Mathematics, COMSATS Institute of Information Technology, 47040, Wah Cantt., Pakistan (wahab@ciitwah.edu.pk).
The topological derivatives have been used heuristically in the context of imaging and non-destructive testing lacking rigorous mathematical justifications, unlike in shape optimization wherein they attracted enormous interest from mathematical as well as numerical viewpoint. For the first time, the stability and resolution analysis of the topological derivative based imaging of small inclusions for the anti-plane elasticity was performed by Ammari et al. [5]. Therein, it is demystified that in order to get a stable and guaranteed localization with a good resolution, the use of a filtered discrepancy is indispensable whereas the filter needs to be defined in terms of a Neumann-Poincaré type boundary integral operator. The filtered topological derivative functional is proved to achieve Raleigh resolution limit. Moreover, it is elucidated that this topological sensitivity framework is stable and robust with respect to medium and measurement noises, and with limited view measurements. It performs far better than classical imaging frameworks including back-propagation technique, MUSIC-type imaging and Kirchhoff migration in worse imaging conditions.

The full elasticity case of topological sensitivity framework in a linear isotropic regime was rigorously explained by Ammari et al. [1]. The study surprisingly demystifies that the classical framework does not guarantee a localization of the inclusion even with a filtered discrepancy functional. Moreover, even if it is somehow able to locate the inclusion, the resolution of the functional degenerates thanks to nonlinear coupling between shear and pressure components at the boundary. In order to counter the coupling artifacts and to have a guaranteed localization of small inclusions, a modified imaging framework was proposed based on a weighted Helmholtz decomposition [3] applied to the initial guess furnished by filtered topological derivative functional. The modified framework is then proved to be stable with respect to medium and measurement noises. Furthermore, it achieves the Rayleigh resolution limit.

The aim in this article is to study a topological derivative based imaging framework for detecting diametrically small electromagnetic inclusions from single and multiple boundary measurements of the tangential component of scattered magnetic field over a fixed frequency. It is assumed that the magnetic field satisfies full three dimensional Maxwell equations and the inclusion is penetrable however homogeneous with electromagnetic parameters different from that of the background medium. The work is focused on the analysis of the detection capabilities of a filtered topological derivative based imaging functional wherein the filter is defined in terms of a boundary integral operator. Precisely, the aim of the article is three-fold: First to introduce a filtered topological derivative based imaging framework, then to perform sensitivity and resolution analysis of the algorithm and finally to investigate its stability with respect to measurement and medium noises. The potential applications envisioned by the imaging of electromagnetic inclusions of diminishing size can be found in non-destructive testing of small material impurities, medical diagnosis and therapeutic protocols, especially for detecting and curing cancers of vanishing size and for brain imaging. It is worthwhile precising that the problem of detecting small electromagnetic inclusions has been previously studied by using MUSIC-type algorithms [6], time reversal and phase conjugation techniques [34,36], reverse time migration [19], topological derivative based imaging [29], and asymptotic expansion techniques [8,9]. For the imaging of thin electromagnetic inclusions and cracks in a two dimensional setting, we refer the reader to [31,32] for instance. We will restrict ourselves only to the detection of the inclusion and will not discuss its morphology (shape, size and material properties) in this paper. In this regard, we refer for instance to the recent results by Asch and Mefire [12] and Bao et al. [13].

The rest of this article is organized in the following manner. In Section 2, we collect some notation and important results on electromagnetic Green’s functions, boundary layer potentials and polarization tensors. The inverse problem under taken in this study is then
mathematically formulated. In Section 3, a filtered quadratic misfit is defined and its topological
derivative is evaluated using asymptotic expansion of the scattered magnetic field with respect
to the characteristic size of the inclusion. The sensitivity and resolution analysis of the imaging
functional is performed in Section 4. Section 5 is dedicated to perform stability analysis of
the topological derivative based imaging with respect to measurement noise whereas Section 6
deals with its stability with respect to medium noise. Finally, a summary of the results obtained
herein is provided in Section 7.

2 Mathematical formulation

In this section, we introduce some notation and collect some basic results for electromagnetic
Green’s functions and layer potentials indispensable for this study. We also mathematically
formulate the inverse problem undertaken.

2.1 Notation

Let \( X \subset \mathbb{R}^3 \) be a smooth domain with simply connected boundary \( \Gamma \) and \( \nu \) denote the outward
unit normal vector on \( \Gamma \). We define the surface divergence of a complex valued vector field
\( \mathbf{u} \in C^k(\Gamma) \) by

\[
\text{div}_\Gamma \mathbf{u} = \text{div} \tilde{\mathbf{u}}_\Gamma - (\nabla \tilde{\mathbf{u}}_\Gamma) \cdot \nu,
\]

where \( k \in \mathbb{N} \) and the superposed \( \sim \) indicates a smooth extension to \( \mathbb{R}^3 \).

Let \( H^s(X) \) and \( H^s_{\text{loc}}(X) \) be the usual Sobolev spaces for \( s > 0 \). We will denote by \( H^s(\Gamma) \) the
trace space of \( H^{s+1/2}(X) \) and by \( H^{-s} \) the \( L^2 \)-dual space of \( H^s(\Gamma) \). Moreover, we define \( TH^s(\Gamma) \)
the tangential trace space of \( H^{s+1/2}(X) \) under the action of the trace operator \( \gamma_\nu [\mathbf{u}] = \nu \times \mathbf{u}|_\Gamma \)
and denote its \( L^2 \)-dual by \( TH^{-s}(\Gamma) \). We also define the Hilbert space

\[
TH^s_{\text{div}}(\Gamma) := \left\{ \mathbf{u} \in (TH^s(\Gamma))^3 \mid \text{div}_\Gamma \mathbf{u} \in (H^s(\Gamma))^3 \right\}.
\]

Similarly, the dual space of \( TH^s_{\text{div}}(\Gamma) \) is denoted by \( TH^{-s}_{\text{div}}(\Gamma) \).

Finally, we define the spaces \( H(X; \text{curl}) \) and \( H_{\text{loc}}(X; \text{curl}) \) by

\[
\begin{align*}
H(X; \text{curl}) & := \left\{ \mathbf{u} \in (H^s(X))^3 \mid \text{curl } \mathbf{u} \in L^2(X) \right\}, \\
H_{\text{loc}}(X; \text{curl}) & := \left\{ \mathbf{u} \in (H^s_{\text{loc}}(X))^3 \mid \text{curl } \mathbf{u} \in L^2_{\text{loc}}(X) \right\}.
\end{align*}
\]

Refer to [15, 16, 30] and references therein for further details.

For matrices \( \mathbf{A} = (a_{ij})_{i,j=1}^3 \) and \( \mathbf{B} = (b_{ij})_{i,j=1}^3 \), we define the contraction operator ‘ \cdot ‘ by

\[
\mathbf{A} : \mathbf{B} := \sum_{i,j=1}^3 a_{ij} b_{ij},
\]

and the Hilbert-Schmidt matrix norm \( \| \cdot \|_{HS} \) of a square matrix \( \mathbf{A} \) by

\[
\| \mathbf{A} \|_{HS} := \mathbf{A} : \mathbf{A}.
\]
2.2 Electromagnetic Green’s functions

We introduce the scalar function

\[ g(x, y) := \frac{e^{i \kappa |x-y|}}{4\pi |x-y|}, \quad x \neq y, \quad x, y \in \mathbb{R}^3, \] (2.5)

which is the outgoing fundamental solution of the Helmholtz operator \(-(\Delta + \kappa^2)\) in \(\mathbb{R}^3\). Using \(g\), we define the dyadic Green’s function \(\Gamma\) by

\[ \Gamma(x, y) := -\epsilon_0 \left[I_3 + \frac{1}{\kappa^2} \nabla_x \nabla^T_x\right] g(x, y), \] (2.6)

where \(I_3\) is the \(3 \times 3\) identity matrix. The function \(\Gamma(x, y)\) is the solution to

\[ \nabla_x \times \nabla_x \times \Gamma(x, y) - \kappa^2 \Gamma(x, y) = -\epsilon_0 \delta_y(x) I_3, \quad x, y \in \mathbb{R}^3, \] (2.7)

subject to the Silver-Müller condition

\[ \lim_{|x-y| \to \infty} |x-y| \left[ \nabla_x \times \Gamma(x, y) \times \frac{x-y}{|x-y|} - i \kappa \Gamma(x, y) \right] = 0. \] (2.8)

Here \(\delta_y(\cdot) = \delta_0(\cdot - y)\) is the Dirac mass at \(y\) and the operator \(\nabla \times\) acts on matrices column-wise, that is,

\[ \nabla \times [\Gamma p] := \nabla \times [\Gamma] p, \quad \text{for all constant vectors } p \in \mathbb{R}^3. \]

It is worthwhile precising that \(\Gamma\) possesses the following reciprocity properties in isotropic dielectric materials; see [6, 37],

\[ \Gamma(x, y) = \Gamma(y, x) \quad \text{and} \quad \nabla_x \times \Gamma(x, y) = [\nabla_y \times \Gamma(y, x)]^T. \] (2.9)

The following electromagnetic Helmholtz-Kirchhoff identities are the key ingredients to elucidate the localization capabilities of the imaging functional proposed in the next section.

**Lemma 2.1** (See [19]). Let \(B(0, r)\) be an open ball in \(\mathbb{R}^3\) with large radius \(r \to \infty\) and boundary \(\partial B(0, r)\). Then, for all \(x, y \in B(0, r)\), we have

\[ \int_{\partial B(0, r)} \left(\Gamma(x, z)\right)^T \Gamma(z, y) d\sigma(z) = -\frac{1}{\kappa \epsilon_0} \Im \{\Gamma(x, y)\} + Q(x, y), \] (2.10)

where \(Q = (q_{ij})_{i,j=1}^3\) is such that \(|q_{ij}(x, y)| + |\nabla_x q_{ij}(x, y)| \leq C r^{-1}\) uniformly for all \(x, y \in B(0, r)\). Here and throughout this paper \(d\sigma\) denotes the surface element.

**Lemma 2.2.** Let \(B(0, r)\) be an open ball in \(\mathbb{R}^3\) with large radius \(r \to \infty\) and boundary \(\partial B(0, r)\). Then, for all \(x, y \in B(0, r)\), we have

\[ \int_{\partial B(0, r)} \left(\nabla \times \Gamma(x, z)\right)^T \left(\Gamma(z, y) \times \nabla(z)\right) d\sigma(z) = -\frac{1}{\kappa \epsilon_0} \Im \{\Gamma(x, y)\} + \tilde{Q}(x, y), \] (2.11)

where \(\tilde{Q} = (\tilde{q}_{ij})_{i,j=1}^3\) is such that \(|\tilde{q}_{ij}(x, y)| + |\nabla_x \tilde{q}_{ij}(x, y)| \leq \tilde{C} r^{-1}\) uniformly for all \(x, y \in B(0, r)\).
Proof. The identity (2.11) can be proved trivially by mimicking the proof of Lemma 2.1 provided in [19, Lemma 3.2]. For the sake of completeness, we briefly sketch the proof here.

We recall from [19, Lemma 3.1], that for all constant vectors \( p, q \in \mathbb{R}^3 \) and \( x, y \in B(0, r) \)

\[
2\langle p, \Im m \{ \Gamma(x, z) \} \rangle q = \epsilon_0 \int_{\partial B(0, r)} \left( \frac{\Gamma(x, z) p \cdot \nu(z)}{\sqrt{(\kappa^2 - 1)(\kappa^2 + 3)}} \times \nabla_z \times \frac{1}{\kappa^2} \Gamma(z, y) q \right) d\sigma(z),
\]

\[
= \epsilon_0 \int_{\partial B(0, r)} \left( \left[ \frac{\Gamma(x, z) p \cdot \nu(z)}{\sqrt{(\kappa^2 - 1)(\kappa^2 + 3)}} \right] \cdot \nabla_z \times \frac{1}{\kappa^2} \Gamma(z, y) q \right) d\sigma(z),
\]

Moreover, in the far field where \( r \to \infty \), we have

\[
\Gamma(x, y) p = O(r^{-1}),
\]

\[
\frac{\partial}{\partial x_j} \Gamma(x, y) p = O(r^{-1}),
\]

\[
\nabla_x \times \Gamma(x, y) p + i\kappa \Gamma(x, y) p \times \nu(x) = O(r^{-2}),
\]

\[
\frac{\partial}{\partial x_j} \left( \nabla_x \times \Gamma(x, y) p + i\kappa \Gamma(x, y) p \times \nu(x) \right) = O(r^{-2}).
\]

By virtue of the estimates (2.13) and (2.15), the expression (2.12) renders

\[
\int_{\partial B(0, r)} \left( \frac{\Gamma(x, z) \times \nu(z)}{\kappa^2} \right)^T \left( \frac{\Gamma(z, y) \times \nu(z)}{\kappa^2} \right) d\sigma(z) = -\frac{1}{\kappa^2} \Im m \{ \Gamma(x, y) \} + O(r^{-1}).
\]

The above relation also shows that \( |\tilde{q}_j(x, y)| = O(r^{-1}) \). The estimate for \( |\nabla_x \tilde{q}_j(x, y)| \) can be proved analogously using (2.15) and (2.16). This completes the proof.

2.3 Layer potentials

We define the scalar single layer potential \( S^\epsilon \) associated with domain \( X \) of a scalar field \( \phi \in H^{s-1/2}(\Gamma) \) by

\[
S^\epsilon(\phi)(x) := \int_\Gamma g(x, y) \phi(y) d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma.
\]

The vector single layer potential is defined likewise and still represented by \( S^\epsilon \) by abuse of notation. Using \( S^\epsilon \), we define the electric single layer potential by

\[
S^E[j](x) := S^\epsilon[j](x) + \frac{1}{\kappa^2} \nabla S^\epsilon[\text{div} j](x) = -\frac{1}{\epsilon_0} \int_\Gamma \Gamma(y, x) j(y) d\sigma(y),
\]

for all \( j \in TH^{-1/2}_d(\Gamma) \).

For \( \psi \in TH^{-1/2}_d(\Gamma) \), we define the magnetic dipole operator \( \mathcal{P}^\epsilon \) by

\[
\mathcal{P}^\epsilon[\psi](x) := \int_\Gamma \nabla_x \times \left( g(x, y) \psi(y) \right) d\sigma(y) \times \nu(x), \quad x \in \Gamma,
\]

and the operator \( \mathcal{P}^\epsilon_\ast \) by

\[
\mathcal{P}^\epsilon_\ast[\psi](x) := \mathcal{P}^\epsilon[\psi \times \nu](x) \times \nu(x).
\]

Following results hold.
Lemma 2.3 (See [17, 30]). The operator $\mathcal{P}^\kappa$ is continuous mapping from $TH^{-1/2}_{\text{div}}(\Gamma)$ to itself. The operator $\frac{1}{2}I - \mathcal{P}^\kappa$ is Fredholm of index zero from $TH^{-1/2}_{\text{div}}(\Gamma)$ to itself. If $\Re\{\kappa\} > 0$, this operator is an isomorphism. Moreover, for all $\psi \in TH^{-1/2}_{\text{div}}(\Gamma)$

$$\mathcal{P}^\kappa[\psi] \times \nu = -\mathcal{P}^\kappa[\psi \times \nu].$$

(2.22)

Lemma 2.4 (See [26, 28]). The electric single layer potential $S^\kappa_E$ is continuous from $TH^{-1/2}_{\text{div}}(\Gamma)$ to $H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ and for all $j \in TH^{-1/2}_{\text{div}}(\Gamma)$, we have

$$(\nabla \times \nabla \times -\kappa^2 I) S^\kappa_E[j](x) = 0,$$

(2.23)

where $I : TH^{-1/2}_{\text{div}}(\Gamma) \to TH^{-1/2}_{\text{div}}(\Gamma)$ is the identity operator. Moreover, $S^\kappa_E$ satisfies the Silver-Müller condition.

Lemma 2.5 (See [17, 22, 30]). For all $j \in TH^{-1/2}_{\text{div}}(\Gamma)$, the traces $(S^\kappa_E[j] \times \nu)^\pm$ and $((\nabla \times S^\kappa_E[j]) \times \nu)^\pm$ are well defined and

$$\left( S^\kappa_E[j] \times \nu \right)^+ - \left( S^\kappa_E[j] \times \nu \right)^- = 0,$$

(2.24)

$$\left( (\nabla \times S^\kappa_E[j]) \times \nu \right)^+ - \left( (\nabla \times S^\kappa_E[j]) \times \nu \right)^- = -j.$$  

(2.25)

Moreover,

$$\left( (\nabla \times S^\kappa_E[j]) \times \nu \right)^\pm = \left( \frac{1}{2}I + \mathcal{P}^\kappa \right)[j].$$

(2.26)

Here superscripts $+$ and $-$ indicate the limiting values at $\Gamma$ from outside $X$ and from inside $X$ respectively.

Lemma 2.6 (See [27]). For all $j \in TH^{-1/2}_{\text{div}}(\Gamma)$

$$\nabla \cdot S^\kappa[j](x) = S^\kappa[\text{div}_\Gamma j](x), \quad x \in \mathbb{R}^3 \setminus \Gamma.$$  

(2.27)

2.4 Polarization tensor

Let us define the piece-wise constant function $\gamma$ by

$$\gamma(x) := \begin{cases} 
\gamma_0, & x \in \mathbb{R}^3 \setminus X, \\
\gamma_X, & x \in X,
\end{cases}$$

(2.28)

where $\gamma_0, \gamma_X \in \mathbb{C}$ such that $\Re\{\gamma_0\}, \Re\{\gamma_X\} > 0$, and let $v_i$ be the scalar potential defined as the solution to the transmission problem

$$\begin{cases}
\Delta v_i = 0, & (\mathbb{R}^3 \setminus X) \cup \mathbb{X}, \\
v_i^+ - v_i^- = 0, & \Gamma, \\
\frac{\gamma_0}{\gamma_1} \left( \frac{\partial v_i}{\partial n} \right)^+ - \left( \frac{\partial v_i}{\partial n} \right)^- = 0, & \Gamma, \\
v_i(x) - x_i \to 0, & |x| \to \infty.
\end{cases}$$

(2.29)

We define the polarization tensor $M_X(k) := (m_{ij})_{i,j=1}^3$, associated with the domain $X$ depending on the contrast $k := \gamma_0/\gamma_X$, by

$$m_{ij}(k) := \frac{1}{k} \int_X \frac{\partial v_i}{\partial x_j} \, dx.$$  

(2.30)
Lemma 2.7 (See [8]). The tensor $M_X(k)$ is real symmetric positive definite if $k \in \mathbb{R}_+$. Moreover, when $X$ is a ball

$$M_X(k) = \frac{3}{2k + 1} |X| I_3.$$  \hfill (2.31)

2.5 Problem formulation

Let $D = \rho B + z_D$ be a small three-dimensional bounded inclusion with a smooth and simply connected boundary $\partial D$, permittivity $\epsilon_1 > 0$ and permeability $\mu_1 > 0$, where $B$ is a regular enough bounded domain in $\mathbb{R}^3$ representing the volume of the inclusion, $z_D$ is the vector position of its center and $\rho > 0$ is the scale factor. The inclusion $D$ is compactly supported in the bounded open background domain $\Omega \subset \mathbb{R}^3$ with a smooth and simply connected boundary $\partial \Omega$. Let $\epsilon_0 > 0$ and $\mu_0 > 0$ be the permittivity and permeability of the background $\Omega$ without inclusion $D$, letting $\kappa := \omega \sqrt{\epsilon_0 \mu_0}$ and $c = 1/\sqrt{\epsilon_0 \mu_0}$ to be the background wave-number and speed of light, respectively, where $\omega > 0$ is the frequency pulsation. We define the piecewise constant functions $\mu_\rho$ and $\epsilon_\rho$ by

$$\mu_\rho(x) := \begin{cases} \mu_1, & x \in D, \\ \mu_0, & x \in \mathbb{R}^3 \setminus \overline{D} \end{cases}, \quad \text{and} \quad \epsilon_\rho(x) := \begin{cases} \epsilon_1, & x \in D, \\ \epsilon_0, & x \in \mathbb{R}^3 \setminus \overline{D}. \end{cases} \hfill (2.32)$$

Furthermore, suppose that the inclusion $D$ is of diminishing characteristic size and is separated apart from $\partial \Omega$, that is, there exists a constant $d_0 > 0$ such that

$$\inf_{x \in D} \text{dist}(x, \partial \Omega) \geq d_0 > 0 \quad \text{and} \quad \rho \kappa \ll 1. \hfill (2.33)$$

Let $H_\rho \in H_{\text{loc}}(\text{curl}, \Omega)$ denote the time-harmonic magnetic field in $\Omega$ in the presence of inclusion $D$, that is, the solution to

$$\begin{cases} \nabla \times (\epsilon_0^{-1} \nabla \times H_\rho) - \omega^2 \mu_0 H_\rho = 0, & \Omega \setminus \overline{D}, \\ \nabla \times (\epsilon_1^{-1} \nabla \times H_\rho) - \omega^2 \mu_1 H_\rho = 0, & D, \\ (H_\rho \times \nu)^+ - (H_\rho \times \nu)^- = 0, & \partial D, \\ \epsilon_0^{-1}(\nabla \times H_\rho)^+ \times \nu - \epsilon_1^{-1}(\nabla \times H_\rho)^- \times \nu = 0, & \partial D, \\ \mu_0 H_\rho^+ \cdot \nu - \mu_1 H_\rho^- \cdot \nu = 0, & \partial D, \\ \epsilon_0^{-1}(\nabla \times H_\rho) \times \nu = h, & \partial \Omega. \end{cases} \hfill (2.34)$$

We also define the background magnetic field $H_0 \in H_{\text{loc}}(\text{curl}, \Omega)$, that is in the absence of any inclusion inside $\Omega$, as the solution to

$$\begin{cases} \nabla \times (\epsilon_0^{-1} \nabla \times H_0) - \omega^2 \mu_0 H_0 = 0, & \Omega, \\ \epsilon_0^{-1}(\nabla \times H_0) \times \nu = h, & \partial \Omega. \end{cases} \hfill (2.35)$$

In this paper, we are interested in the following problem.

**Inverse problem**

Given the measurements $H_\rho \times \nu$ for all $x \in \partial \Omega$, find the position $z_D$ of the inclusion $D$ using a filtered topological derivative based imaging framework.
An identical problem has been studied by Masmoudi et al. [29] using topological derivative-based sensitivity framework by invoking an adjoint field. The aim here is to design and analyze the performance of topological derivative-based detection framework applied to a filtered quadratic misfit. Moreover, the approach adopted herein is based on the asymptotic expansion of the scattered magnetic field with respect to the size of the inclusion.

3 Topological derivative based imaging framework

Let \( z_0 \in \Omega \) be a search point. We nucleate an inclusion \( D_\delta = \delta B_\delta + z_0 \) inside the background \( \Omega \) with permittivity \( \varepsilon_\delta \) and permeability \( \mu_\delta \) defined by

\[
\mu_\delta(x) := \begin{cases} 
\mu_2, & x \in D_\delta, \\
\mu_0, & x \in \mathbb{R}^3 \setminus D,
\end{cases} \quad \text{and} \quad \varepsilon_\delta(x) := \begin{cases} 
\varepsilon_2, & x \in D_\delta, \\
\varepsilon_0, & x \in \mathbb{R}^3 \setminus D,
\end{cases}
\]  

(3.1)

where \( \varepsilon_2, \mu_2 > 0 \). Let \( H_\delta \) be the magnetic field in the presence of the inclusion \( D_\delta \) in \( \Omega \) satisfying a transmission problem analogous to that in (2.33). We collect \( H_\delta \times \nu(x) \) for all \( x \in \partial \Omega \).

Define the discrepancy functional

\[
\mathcal{H}_f[H_0](z_0) := \frac{1}{2} \int_{\partial \Omega} \left| \left( \frac{1}{2} \mathcal{I} - \mathcal{P}^\kappa \right) \left( \left( H_\rho - H_\delta \right) \times \nu \right) (x) \right|^2 dx. \tag{3.2}
\]

By construction, the search point \( z_0 \) relative to which the field \( H_\delta \) minimizes the functional \( \mathcal{H}_f[H_0] \) is a potential candidate for \( z_D \). In order to study the optimization problem (3.2), we define the topological derivative of misfit \( \mathcal{H}_f \) as follows.

**Definition 3.1** (Topological derivative). For any \( z_0 \in \Omega \) and incident field \( H_0 \), the topological derivative (imaging functional) of the misfit \( \mathcal{H}_f[H_0] \) is defined by

\[
\partial T \mathcal{H}_f[H_0](z_0) := -\frac{\partial \mathcal{H}_f[H_0](z_0)}{\partial (\delta^3)} \bigg|_{(\delta^3) = 0}. \tag{3.3}
\]

The following asymptotic expansion of the scattered magnetic field due to the presence of inclusion \( D_\rho \) versus scale factor \( \rho \), is the key ingredient to evaluate the topological derivative \( \partial T \mathcal{H}_f[H_0] \).

**Theorem 3.2** (See [10]). For all \( x \in \partial \Omega \), and \( D_\rho = \rho B_\rho + z_D \) satisfying condition (2.33)

\[
\left( \frac{1}{2} \mathcal{I} - \mathcal{P}^\kappa \right) \left( \left( H_\rho - H_0 \right) \times \nu \right)(x) = \rho^3 \omega^2 \frac{\mu_0}{\mu_1} (\mu_0 - \mu_1) \left[ \Gamma(z_D, x) \times \nu(x) \right] M_{B_\rho} \left( \frac{\mu_0}{\mu_1} \right) H_0(z_D) \]

\[
+ \rho^3 \left( \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_0} \right) \left[ \left( \nabla z_D \times \Gamma(x, z_D) \right)^T \times \nu(x) \right] M_{B_\rho} \left( \frac{\varepsilon_0}{\varepsilon_1} \right) \nabla \times H_0(z_D) + O(\rho^4), \tag{3.4}
\]

where the operator \( \nabla \times \) and \( \nu \times \) act column-wise on matrices. The term \( O(\rho^4) \) is bounded by \( C\rho^4 \) uniformly on \( x \), where constant \( C \) is independent of \( z_D \).

Remark that, from Theorem 3.2, we also have for all \( x \in \partial \Omega \) and \( z_S \in \Omega \)

\[
\left( \frac{1}{2} \mathcal{I} - \mathcal{P}^\kappa \right) \left( H_\delta - H_0 \right) \times \nu \right)(x) = \delta^3 \omega^2 \frac{\mu_0}{\mu_2} (\mu_0 - \mu_2) \left[ \Gamma(z_S, x) \times \nu(x) \right] M_{B_\delta} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \]

\[
+ \delta^3 \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_0} \right) \left[ \left( \nabla z_S \times \Gamma(x, z_S) \right)^T \times \nu(x) \right] M_{B_\delta} \left( \frac{\varepsilon_0}{\varepsilon_2} \right) \nabla \times H_0(z_S) + O(\delta^4). \tag{3.5}
\]
Proof. Note that, using asymptotic expansion (3.5) we have for all $z_S \in \Omega$ where the back-propagator

$$\partial_I \mathcal{H}_f[H_0](z_S) = -\Re \left\{ \frac{\omega^2 \epsilon_0 \mu_0 (\mu_0 - \mu_2)}{\mu_2} U(z_S) \cdot \left[ M_{B_4} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \right] \right. $$

$$+ \epsilon_0 \left( \frac{1}{\epsilon_2^2} - \frac{1}{\epsilon_0^2} \right) \nabla_{z_S} \times U(z_S) \cdot \left[ M_{B_4} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla_{z_S} \times H_0(z_S) \right] \} , \quad (3.6)$$

where the back-propagator $U$ is defined by

$$U(z) := \mathcal{S}^\mu \left[ \nu \times \left( \frac{1}{2} I - \mathcal{P}^\mu \right) \left[ (H_\rho - H_0) \times \nu \right] \right] (z) . \quad (3.7)$$

We have made use of the fact that $f$ is given by

$$\partial H \left[ \left( \frac{1}{2} \nu \times \left( \frac{1}{2} I - \mathcal{P}^\mu \right) \left[ (H_\rho - H_0) \times \nu \right] \right] \right] (z) \} ,$$

and vectors $A$ and $W$ is defined by

$$W(z) := \left( \frac{1}{2} I - \mathcal{P}^\mu \right) \left[ (H_\rho - H_0) \times \nu \right] (z) . \quad (3.9)$$

We have made use of the fact that $\left( \frac{1}{2} I - \mathcal{P}^\mu \right) \left[ (H_\rho - H_0) \times \nu \right] = O(\rho^3)$ from Theorem 3.2.

Recall that for any matrix $A$, and vectors $u$, $v$ and $w$

$$[(A \times u) v] \cdot w = [(A \times u)^T w] \cdot v = [A^T (u \times w)] \cdot v.$$

Therefore, we have

$$\left[ \Gamma(z_S, z) \times \nu(z) \right] \left[ M_{B_4} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \right] \cdot W(z)$$

$$= \left[ (\Gamma(z_S, z))^T (\nu(z) \times W(z)) \right] \cdot \left[ M_{B_4} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \right] .$$
Moreover, from property (2.9) and since $\Gamma$ is symmetric, we have

$$\int_{\partial\Omega} \left( \nabla \times \nabla \times U(z) \right) \cdot \mathbf{W}(z) \, d\sigma(z),$$

$$= \left[ \int_{\partial\Omega} \left( \nabla \times \nabla \times U(z) \right) \cdot \mathbf{W}(z) \, d\sigma(z) \right],$$

$$= \left[ \int_{\partial\Omega} \left( \nabla \times \nabla \times U(z) \right) \cdot \mathbf{W}(z) \, d\sigma(z) \right],$$

$$= - \varepsilon_0 S_E^\delta \left[ \nabla \times \nabla \times U(z) \right] \cdot \mathbf{W}(z),$$

$$= -\varepsilon_0 U(z) \cdot \mathbf{W}(z).$$

Similarly,

$$\int_{\partial\Omega} \left( \nabla \times \nabla \times U(z) \right) \cdot \mathbf{W}(z) \, d\sigma(z),$$

$$= \left[ \int_{\partial\Omega} \left( \nabla \times \nabla \times U(z) \right) \cdot \mathbf{W}(z) \, d\sigma(z) \right],$$

$$= \left[ \int_{\partial\Omega} \left( \nabla \times \nabla \times U(z) \right) \cdot \mathbf{W}(z) \, d\sigma(z) \right],$$

$$= -\varepsilon_0 \nabla \times \nabla \times U(z) \cdot \mathbf{W}(z).$$

Therefore, by virtue of (3.10) and (3.11), expansion (3.8) renders

$$\mathcal{H}_f[H_0](z) = \frac{1}{2} \int_{\partial\Omega} \left( \frac{1}{2} I - \mathcal{P} \right) \left( \left[ \nabla \times \nabla \times U(z) \right] \cdot \mathbf{W}(z) \right) \, d\sigma(z),$$

$$= \varepsilon_0 \left[ \omega^2 \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \right] \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

$$= \varepsilon_0 \left( \mu_0 - \mu_2 \right) \left( \mu_0 - \mu_2 \right) \cdot \mathbf{W}(z),$$

Finally, the conclusion follows by tending $\varepsilon^3 \to 0$. □

To conclude this section, we precise that thanks to Lemmas 2.4 and 2.5, the back-propagator $U$ is the solution to boundary value problem

$$\begin{cases}
\nabla \times \nabla \times U(x) = \kappa^2 U(x), & x \in \Omega, \\
\nabla \times U(x) \times \nu(x) = \left( \frac{1}{2} I + \mathcal{P} \right) \left( \nu \times \left( \frac{1}{2} I - \mathcal{P} \right) \left[ \left( \nabla \times \nabla \times U(x) \right) \cdot \mathbf{W}(x) \right] \right), & x \in \partial\Omega.
\end{cases}$$

(3.13)
4 Sensitivity and resolution analysis

In this section, we demystify the reason why the topological derivative functional attain its maximum at the true location $z_D$ of the electromagnetic inclusion $D$.

4.1 Imaging with single incident field

In order to ascertain the localization and resolution of the imaging function $\partial_T H f [H_0]$, we entertain two special cases for simplicity. Precisely, we consider the permittivity contrast only case and the permeability contrast only case.

First, we consider the permeability contrast only case, and assume that $\epsilon_1 = \epsilon_0 = \epsilon_2$, thereby restricting $\partial_T H f [H_0]$ to

\[
\partial_T H f [H_0](z_S) = -\omega^2 \epsilon_0 \mu_0 (\mu_0 - \mu_2) \text{Re} \left\{ U(z_S) \cdot \left[ M_{B_1} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \right] \right\},
\]

After simple manipulations, we arrive at

\[
\partial_T H f [H_0](z_S) := -\frac{\mu_0}{\mu_2} (\mu_0 - \mu_2) \text{Re} \left\{ \left( \int_{\partial \Omega} \Gamma(z, z_S) (\nu(z) \times W(z)) \, d\sigma(z) \right) \cdot \left[ M_{B_1} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \right] \right\}.
\]

Note that

\[
W(z) = \left( \frac{1}{2} I - P^\ell \right) [(H_\rho - H_0) \times \nu] (z),
\]

\[
= \rho^2 \omega^2 \frac{\mu_0}{\mu_1} (\mu_0 - \mu_1) \left[ \overline{\Gamma(z_D, z)} \times \nu(z) \right] M_{B_1} \left( \frac{\mu_0}{\mu_1} \right) \overline{H_0(z_D)} + O(\rho^4).
\]

Therefore, on injecting back the expression for $W$, we obtain

\[
\partial_T H f [H_0](z_S) = \rho^3 C_\mu \omega^4 \text{Re} \left\{ \int_{\partial \Omega} \Gamma(z, z_S) \left[ \nu(z) \times \left( \overline{\Gamma(z_D, z)} \times \nu \right) \right] \cdot \left[ M_{B_1} \left( \frac{\mu_0}{\mu_1} \right) \overline{H_0(z_D)} \right] \, d\sigma(z) \right\} + O(\rho^4),
\]

where

\[
C_\mu := \frac{\mu_0^2}{\mu_1 \mu_2} (\mu_0 - \mu_1)(\mu_0 - \mu_2).
\] (4.1)

After simple manipulations, we arrive at

\[
\partial_T H f [H_0](z_S) := \rho^3 \omega^4 \text{Re} \left\{ C_\mu M_{B_1} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \cdot \overline{\mathcal{R}_1(z_S, z_D)} M_{B_1} \left( \frac{\mu_0}{\mu_1} \right) \overline{H_0(z_D)} \right\} + O(\rho^4),
\]

where

\[
\mathcal{R}_1(x, y) := \int_{\partial \Omega} \left( \overline{\Gamma(x, z) \times \nu(z)} \right)^T \left( \Gamma(y, z) \times \nu(z) \right) \, d\sigma(z).
\] (4.3)
Recall from Lemma 2.2 that for all \( x, y \in \Omega \) far from the boundary \( \partial \Omega \), which is indeed the case here by assumption 2.3, we have

\[
\mathcal{R}_1(x, y) \simeq -\frac{1}{\kappa} \Im \{ \Gamma(x, y) \} ,
\]

\[
= \frac{1}{\kappa} \left( I_3 - \frac{1}{\kappa^2} \nabla \nabla^T \right) \Im \{ g(x, y) \} ,
\]

\[
= \left( I_3 - \frac{1}{\kappa^2} \nabla \nabla^T \right) \frac{\sin \left( \kappa |x - y| \right)}{4\pi} .
\]  

(4.4)

Here \( \sin(t) = \sin(t)/t \) is the sinc function. Therefore, by substituting back the approximation of \( \mathcal{R}_1 \), we arrive at

\[
\partial_T \mathcal{H}_f[H_0](z_S) \simeq -\rho^3 \omega^3 \Re \left\{ \frac{C_\mu}{\epsilon_0 \sqrt{\epsilon_0 \mu_0}} M_{B_\mu} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_S) \right\} \Im \{ \Gamma(z_S, z_D) \} M_{B_\mu} \left( \frac{\mu_0}{\mu_2} \right) H_0(z_D) + O(\rho^4). \]

(4.5)

On the other hand, if we have only a permittivity contrast, that is, \( \mu_1 = \mu_0 = \mu_2 \), the topological derivative reduces to

\[
\partial_T \mathcal{H}_f[H_0](z_S)
\]

\[
= -\Re \left\{ \epsilon_0 \left( \frac{1}{\epsilon_2} - \frac{1}{\epsilon_0} \right) \nabla z_S \times U(z_S) \cdot \left[ M_{B_\mu} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla z_S \times H_0(z_S) \right] \right\} ,
\]

\[
= \Re \left\{ \left( \frac{1}{\epsilon_2} - \frac{1}{\epsilon_0} \right) \nabla z_S \times \left( \int_{\partial \Omega} \Gamma(z, z_S)(\nu(z) \times W(z)) d\sigma(z) \right) \cdot \left[ M_{B_\mu} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla z_S \times H_0(z_S) \right] \right\} .
\]

In this case the function \( W \) admits the expansion

\[
W(z) = \left( \frac{1}{2} I - \rho \kappa \right) \left[ (H_\rho - H_0) \times \nu \right](z),
\]

\[
= \rho^3 \left( \frac{1}{\epsilon_2} - \frac{1}{\epsilon_0} \right) \left[ (\nabla z_D \times \Gamma(z, z_D))^T \times \nu(z) \right] M_{B_\mu} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla \times H_0(z_D) + O(\rho^4) .
\]

Therefore, \( \partial_T \mathcal{H}_f[H_0](z_S) \) becomes

\[
\partial_T \mathcal{H}_f[H_0](z_S)
\]

\[
= \rho^3 C_\epsilon \Re \left\{ \int_{\partial \Omega} \nabla z_S \times \Gamma(z, z_S) \left[ \nu(z) \times \left( (\nabla z_D \times \Gamma(z, z_D))^T \times \nu(z) \right) \right] M_{B_\mu} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla z_S \times H_0(z_S) \right\} + O(\rho^4),
\]

\[
= \rho^3 C_\epsilon \Re \left\{ \int_{\partial \Omega} (\nabla z \times \Gamma(z, z_S))^T \left[ \nu(z) \times \left( (\nabla z \times \Gamma(z, z_D))^T \times \nu(z) \right) \right] M_{B_\mu} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla z_S \times H_0(z_S) \right\} + O(\rho^4),
\]
where for the latter identity, relations in (4.5) have been invoked and the constant \( C_\varepsilon \) is defined by

\[
C_\varepsilon := \left( \frac{1}{\epsilon_1} - \frac{1}{\epsilon_0} \right) \left( \frac{1}{\epsilon_2} - \frac{1}{\epsilon_0} \right).
\]

After straightforward operations, we get

\[
\partial T \mathcal{H}_f[H_0](z_S) = \rho^3 C_\varepsilon \Re \left\{ M_{B_3} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla_{z_S} \times H_0(z_S) : \mathcal{R}_2(z_S, z_D) M_{B_\rho} \left( \frac{\epsilon_0}{\epsilon_1} \right) \nabla_{z_D} \times H_0(z_D) \right\} + O(\rho^4),
\]

where

\[
\mathcal{R}_2(x, y) := \int_{\partial \Omega} \left( \nabla_z \times \Gamma(x, z) \times \nu(z) \right)^T \left( \nabla_z \times \Gamma(y, z) \times \nu(z) \right) d\sigma(z).
\]

Note that, by virtue of the assumption (2.33), and the Silver-Müller condition, for all \( x, y \in \Omega \)

\[
\mathcal{R}_2(x, y) \approx \kappa^2 \int_{\partial \Omega} \left( \Gamma(x, z) \right)^T \left( \Gamma(y, z) \right) d\sigma(z) \approx -\frac{\kappa}{\epsilon_0} \Im \left\{ \Gamma(x, y) \right\},
\]

where for the latter identity, Lemma 2.4 is invoked.

Therefore, we conclude that

\[
\partial T \mathcal{H}_f[H_0](z_S) \approx -\rho^3 \omega \sqrt{\mu_0 C_\varepsilon} \Re \left\{ M_{B_3} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla_{z_S} \times H_0(z_S) : \Im \left\{ \Gamma(z_S, z_D) \right\} M_{B_\rho} \left( \frac{\epsilon_0}{\epsilon_1} \right) \nabla_{z_D} \times H_0(z_D) \right\} + O(\rho^4).
\]

Remark that, for both permittivity contrast and permeability contrast cases,

\[
\partial T \mathcal{H}_f[H_0](z_S) \propto f(z_S, z_D) \text{sinc}(\kappa z_S - z_D)), \quad \text{for a smooth function } f.
\]

Therefore, it has a sharp peak when \( z_S \to z_D \) with a focal spot size of half a wavelength of the incident wave. Therefore, the resolution of the imaging functional \( \partial T \mathcal{H}_f[H_0](z_S) \) achieves the Rayleigh resolution limit. It is emphasized that the localization and resolution is essential and does not depend on the choice of permittivity or permeability contrast only.

### 4.2 Imaging with multiple incident fields

Let \( \theta_1, \theta_2, \cdots, \theta_n \) be \( n \)-equidistributed directions on the unit sphere and let

\[
H^j_0(x) = \theta^\perp_j e^{i\kappa \theta^\perp_j x}, \quad j \in \{1, 2, \cdots, n\},
\]

be the incident magnetic fields where \( \theta^\perp_j \) is the polarization direction such that \( \theta_j \cdot \theta^\perp_j = 0 \). The incident fields \( H^j_0 \) are the solutions to the Maxwell equations

\[
\begin{aligned}
\nabla \times \frac{1}{\epsilon_0} \nabla \times H^j_0 - \omega^2 \mu_0 H^j_0 &= 0, & \Omega, \\
\mathbf{h}^j(x) = \frac{1}{\epsilon_0} \nabla \times \left( \theta^\perp_j e^{i\kappa \theta^\perp_j x} \right) \times \nu(x), & \partial \Omega.
\end{aligned}
\]
We recall from [5] that for \( n \) sufficiently large
\[
\frac{1}{n} \sum_{j=1}^{n} e^{i\kappa \theta_j^T (x-y)} \simeq \frac{4\pi}{\kappa} \Im \{ \mathcal{J} \}.
\]  
(4.13)

Since \( \theta_j^\perp \cdot \theta_j = 0 \), we therefore have \( \theta_j^\perp (\theta_j^\perp)^T = (I_3 - \theta_j \theta_j^T) \) so that
\[
\frac{1}{n} \sum_{j=1}^{n} e^{i\kappa \theta_j^T (x-y)} \theta_j^\perp (\theta_j^\perp)^T = \frac{1}{n} \sum_{j=1}^{n} (I_3 - \theta_j \theta_j^T) e^{i\kappa \theta_j^T (x-y)},
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} \left( I_3 - \frac{1}{\kappa^2} \nabla_x \nabla_x^T \right) e^{i\kappa \theta_j^T (x-y)},
\]
\[
= \left( I_3 - \frac{1}{\kappa^2} \nabla_x \nabla_x^T \right) \frac{1}{n} \sum_{j=1}^{n} e^{i\kappa \theta_j^T (x-y)},
\]
\[
\simeq \frac{4\pi}{\kappa} \left( I_3 - \frac{1}{\kappa^2} \nabla_x \nabla_x^T \right) \Im \{ \mathcal{J} \},
\]
\[
= - \frac{4\pi}{\kappa} \Im \{ \mathcal{J} \}.
\]  
(4.14)

Moreover
\[
\frac{1}{n} \sum_{j=1}^{n} e^{i\kappa \theta_j^T (x-y)} (\theta_j \times \theta_j^\perp) (\theta_j \times \theta_j^\perp)^T \simeq - \frac{4\pi}{\kappa \epsilon_0} \Im \{ \mathcal{G} \},
\]  
(4.15)

indeed since \( \theta_j \times \theta_j^\perp \cdot \theta_j = 0 \) and consequently \( (\theta_j \times \theta_j^\perp) (\theta_j \times \theta_j^\perp)^T = (I_3 - \theta_j \theta_j^T) \).

Let us define the topological derivative for multiple incident fields by the superposition of the individual topological derivatives as
\[
\partial_T \mathcal{H}_f(z_S) := \frac{1}{n} \sum_{j=1}^{n} \partial_T \mathcal{H}_f[H_0](z_S),
\]  
(4.16)

where the factor \( 1/n \) is used for normalization.

The following result holds

**Theorem 4.1.** Let \( z_S \in \Omega, D = \rho B_\rho + z_D \) satisfy the condition (2.33) and \( n \in \mathbb{N} \) be sufficiently large. Then,

1. for \( \epsilon_0 = \epsilon_1 = \epsilon_2 \)
\[
\partial_T \mathcal{H}_f(z_S) \simeq \rho^3 \omega^2 \frac{4\pi C_J}{\epsilon_0 \mu_0 \mu_1} \Re \left\{ \Im \{ \mathcal{G}(z_S, z_D) \} \mathbf{M}_{B_\rho} \left( \frac{\mu_0}{\mu_1} \right) : \right. 
\]
\[
\left. \Im \{ \mathcal{G}(z_S, z_D) \} \mathbf{M}^T_{B_\rho} \left( \frac{\mu_0}{\mu_2} \right) \right\} + O(\rho^4),
\]  
(4.17)
2. for $\mu_0 = \mu_1 = \mu_2$

$$\partial_T \mathcal{H}_f(z_S) \simeq \rho^3 \omega^2 \frac{4\pi \mu_0 C_s}{\epsilon_0} \Re \left\{ \Im \left\{ \mathbf{\Gamma}(z_S, z_D) \right\} \mathbf{M}_{B^p} \left( \frac{\epsilon_0}{\epsilon_1} \right) : \right\}$$

$$\Im \left\{ \mathbf{\Gamma}(z_S, z_D) \right\} \mathbf{M}_{B^s} \left( \frac{\epsilon_0}{\epsilon_2} \right) \right\} + O(\rho^4), \quad (4.18)$$

where the constants $C_s$ and $C_e$ are defined by 4.11 and 4.16 respectively.

Proof. When $\epsilon_0 = \epsilon_1 = \epsilon_2$, for all $z_S \in \Omega$

$$\partial_T \mathcal{H}_f(z_S) = \frac{1}{n} \sum_{j=1}^{n} \partial_T \mathcal{H}_f \left( \mathbf{H}_{0}^j \right)(z_S),$$

$$\simeq - \rho^3 \omega^3 \frac{C_s}{\epsilon_0 \sqrt{\epsilon_0 \mu_0}} \frac{1}{n} \sum_{j=1}^{n} \Re \left\{ \mathbf{M}_{B^s} \left( \frac{\mu_0}{\mu_2} \right) \mathbf{H}_{0}^j(z_S) \cdot \Im \left\{ \mathbf{\Gamma}(z_S, z_D) \right\} \mathbf{M}_{B^p} \left( \frac{\mu_0}{\mu_1} \right) \mathbf{H}_{0}^j(z_D) \right\} + O(\rho^4),$$

$$= - \rho^3 \omega^3 \frac{C_s}{\epsilon_0 \sqrt{\epsilon_0 \mu_0}} \frac{1}{n} \sum_{j=1}^{n} \Re \left\{ \Im \left\{ \mathbf{\Gamma}(z_S, z_D) \right\} \mathbf{M}_{B^p} \left( \frac{\mu_0}{\mu_1} \right) : \left[ \frac{1}{n} \sum_{j=1}^{n} \theta_j^+ \left( \theta_j^+ \right)^T e^{i\kappa \theta_j^T(z_S - z_D)} \right] \mathbf{M}_{B^s} \left( \frac{\mu_0}{\mu_2} \right) \right\}$$

$$+ O(\rho^4).$$

Here we have made use of the fact that $\theta_j \cdot \mathbf{A} \theta_j = \theta_j^T \theta_j : \mathbf{A}$. Finally, 4.17 follows immediately by virtue of 4.13.

In order to prove the second identity, we proceed in the similar fashion. For only permittivity contrast case

$$\partial_T \mathcal{H}_f(z_S) = \frac{1}{n} \sum_{j=1}^{n} \partial_T \mathcal{H}_f \left( \mathbf{H}_{0}^j \right)(z_S),$$

$$\simeq - \rho^3 \omega^3 \frac{\sqrt{\mu_0 C_s}}{\sqrt{\epsilon_0}} \frac{1}{n} \sum_{j=1}^{n} \Re \left\{ \mathbf{M}_{B^s} \left( \frac{\epsilon_0}{\epsilon_2} \right) \nabla z_S \times \mathbf{H}_{0}^j(z_S).$$

$$\Im \left\{ \mathbf{\Gamma}(z_S, z_D) \right\} \mathbf{M}_{B^p} \left( \frac{\epsilon_0}{\epsilon_1} \right) \nabla z_D \times \mathbf{H}_{0}^j(z_D) \right\} + O(\rho^4),$$

$$\simeq - \rho^3 \omega^3 \frac{\sqrt{\mu_0 C_s}}{\sqrt{\epsilon_0}} \frac{1}{n} \sum_{j=1}^{n} \Re \left\{ \mathbf{M}_{B^s} \left( \frac{\epsilon_0}{\epsilon_2} \right) \left( \theta_j \times \theta_j^+ \right) \right\} \cdot$$

$$\Im \left\{ \mathbf{\Gamma}(z_S, z_D) \right\} \mathbf{M}_{B^p} \left( \frac{\epsilon_0}{\epsilon_1} \right) \left( \theta_j \times \theta_j^+ \right) e^{i\kappa \theta_j^T(z_S - z_D)} \right\} + O(\rho^4),$$
On further simplification, we arrive at
\[ \partial T H_f(z_S) \simeq -\rho^3 \omega^2 \tilde{C}_\mu \Im m \left\{ \Gamma(z_S, z_D) \right\} M_{B_\rho} \left( \frac{\epsilon_0}{\epsilon_1} \right) : \left[ \frac{1}{n} \sum_{j=1}^{n} (\theta_j \times \theta_j^\perp) (\theta_j \times \theta_j^\perp)^T e^{i \epsilon_j (z_S - z_D)} \right] M_{B_\delta} \left( \frac{\epsilon_0}{\epsilon_2} \right) + O(\rho^4), \]

The proof is completed by invoking approximation (4.15). □

As an immediate consequence of Theorem 4.1 and Lemma 2.7, the following result can be readily proved.

**Corollary 4.2.** Let \( z_S \in \Omega, D = \rho B_\rho + z_D \) be an open sphere in \( \mathbb{R}^3 \) such that condition (2.33) holds and \( n \in \mathbb{N} \) be sufficiently large. Then,

1. for \( \epsilon_0 = \epsilon_1 = \epsilon_2 \)
\[ \partial T H_f(z_S) \simeq \rho^3 \omega^2 \tilde{C}_\mu \Im m \left\{ \Gamma(z_S, z_D) \right\}^2_{HS} + O(\rho^4). \] (4.19)

2. for \( \mu_0 = \mu_1 = \mu_2 \)
\[ \partial T H_f(z_S) \simeq \rho^3 \omega^2 \tilde{C}_\epsilon \Im m \left\{ \Gamma(z_S, z_D) \right\}^2_{HS} + O(\rho^4). \] (4.20)

The constants \( \tilde{C}_\mu \) and \( \tilde{C}_\epsilon \) are defined by
\[ \tilde{C}_\mu := \frac{36\pi C_\mu \mu_1 \mu_2}{\epsilon_0^3 \mu_0 (2\mu_0 + \mu_1)(2\mu_0 + \mu_2)} |B_\rho| |B_\delta|, \quad \tilde{C}_\epsilon := \frac{36\pi C_\epsilon \mu_0 \epsilon_1 \epsilon_2}{\epsilon_0 (2\epsilon_0 + \epsilon_1)(2\epsilon_0 + \epsilon_2)} |B_\rho| |B_\delta|. \] (4.21)

In rest of this paper, we analyze the stability of the multi-incidence imaging functional (4.16) with respect to medium and measurement noises.

### 5 Statistical stability with respect to measurement noise

The aim here is to substantiate that the imaging functional proposed in Section 4.2 is stable with respect to additive measurement noise. For brevity, the simplest model of the measurement noise is entertained. Precisely, it is assumed that the accurate value of magnetic field at the boundary is corrupted by a mean-zero circular Gaussian noise \( \eta_{\text{noise}} \) : \( \partial \Omega \to \mathbb{C}^3 \), with covariance \( \sigma_{\text{noise}}^2 \), that is,
\[ H_\rho(z) := H_{\rho,\text{true}}(z) + \eta_{\text{noise}}(z), \quad z \in \partial \Omega, \]
where \( H_\rho \) is the corrupted value of the magnetic field at the boundary.

**Nota Bene.** In the sequel, \( E \) denotes the expectation with respect to the statistics of the noise. In this section, a superposed \( \text{true} \) indicates the true value of a quantity, that is, the value without noise corruption. For ease of presentation, in rest of this paper we assume that the permittivity \( \epsilon_0 \) and permeability \( \mu_0 \) are scaled to 1, that is, \( \epsilon_0 = 1 = \mu_0 \) without loss of generality.

We assume that \( \eta_{\text{noise}} \) satisfies following four properties.
1. The measurement noises at different locations on the boundary are uncorrelated.
2. The different components of the measurement noise are uncorrelated.
3. The real and imaginary parts of the measurement noise are uncorrelated.
4. The measurement noises corresponding to two different incident waves are uncorrelated.

Then, under aforementioned assumptions, we have

\[
E \left[ \eta_{\text{noise}}(y) \eta_{\text{noise}}(y')^T \right] = \sigma_{\text{noise}}^2 \delta_y(y') I_3, \tag{5.2}
\]

and

\[
E \left[ \eta_{\text{noise}}^j(y) \eta_{\text{noise}}^l(y')^T \right] = \sigma_{\text{noise}}^2 \delta_{jl} \delta_y(y') I_3, \tag{5.3}
\]

where superposed \( j \) and \( l \) indicate respectively the \( j \)-th and \( l \)-th measurements and \( \delta_{jl} \) is the Kronecker’s delta function which assumes the value 1 when \( j = l \) and zero otherwise.

The imaging functional \( \partial_T \mathcal{H}_f(z) \) is mainly affected by the additive noise during the back-propagation step thanks to the construction of back-propagator in terms of the measurements at the boundary. In the presence of measurement noise, for all \( z \in \Omega \) the back-propagator takes on the form

\[
U(z) = -\int_{\partial \Omega} \Gamma(x, z) \nu(x) \times \left( \frac{1}{2} I - \mathcal{P}^\kappa \right) \left[ (H^\text{true} - H_0 + \eta_{\text{noise}}) \times \nu \right] (x) d\sigma(x),
\]

\[
= U^\text{true}(z) + U^\text{noise}(z), \tag{5.4}
\]

where \( U^\text{true} \) corresponds to the back-propagation of the noise-free data whereas \( U^\text{noise} \) corresponds to noise back-propagation and is given by

\[
U^\text{noise}(z) := -\int_{\partial \Omega} \Gamma(x, z) \nu(x) \times \left( \frac{1}{2} I - \mathcal{P}^\kappa \right) [\eta_{\text{noise}} \times \nu] (x) d\sigma(x). \tag{5.5}
\]

Let us now discuss the statistics of \( U^\text{noise}(z) \). We have the following lemma.

**Lemma 5.1.** The random field \( U^\text{noise}(z), z \in \Omega \), is a mean zero Gaussian field with covariance

\[
E \left[ U^\text{noise}(z) U^\text{noise}(z')^T \right] = -\frac{\sigma_{\text{noise}}^2}{4\kappa} 3m \left\{ \Gamma(z, z') \right\}, \quad z, z' \in \Omega. \tag{5.6}
\]

**Proof.** First of all note that, since \( \eta_{\text{noise}} \) is a mean-zero circular Gaussian random process, \( U^\text{noise}(z) \) is also a mean-zero circular Gaussian random process thanks to linearity. Moreover, its covariance can be calculated for all \( z, z' \in \Omega \) as

\[
E \left[ U^\text{noise}(z) U^\text{noise}(z')^T \right] := E \left[ \int_{\partial \Omega} \Gamma(x, z) \nu(x) \times \left( \frac{1}{2} I - \mathcal{P}^\kappa \right) [\eta_{\text{noise}} \times \nu] (x) d\sigma(x) \right. 
\]

\[
= \sum_{i=1}^4 E_i(z, z'), \tag{5.7}
\]

\[
\left. \left( \int_{\partial \Omega} \Gamma(x', z') \nu(x') \times \left( \frac{1}{2} I - \mathcal{P}^\kappa \right) [\eta_{\text{noise}} \times \nu] (x') d\sigma(x') \right)^T \right].
\]
where

\[
E_1(z, z') := \frac{1}{4} \int_{\partial \Omega} (\nabla(x, z) \times \nu(x))^T \left( \eta_{\text{noise}}(x) \times \nu(x) \right) d\sigma(x)
\]

\[
E_2(z, z') := -\frac{1}{2} \int_{\partial \Omega} (\nabla(x, z) \times \nu(x))^T \left( \eta_{\text{noise}}(x) \times \nu(x) \right) d\sigma(x)
\]

\[
E_3(z, z') := -\frac{1}{2} \int_{\partial \Omega} (\nabla(x, z) \times \nu(x))^T P^\kappa [\eta_{\text{noise}} \times \nu] (x) d\sigma(x)
\]

and

\[
E_4(z, z') := \int_{\partial \Omega} (\nabla(x, z) \times \nu(x))^T P^\kappa [\eta_{\text{noise}} \times \nu] (x) d\sigma(x)
\]

Let us now analyze each term individually. Note that

\[
E_1(z, z') := \frac{1}{4} \int_{\partial \Omega} \left( (\nabla(x, z) \times \nu(x))^T \left( \eta_{\text{noise}}(x) \right) \right) d\sigma(x)
\]

\[
E_2(z, z') := \frac{1}{2} \int_{\partial \Omega} \left( (\nabla(x, z) \times \nu(x))^T \eta_{\text{noise}}(x) \right) d\sigma(x)
\]

where in order to obtain the latter identity, expression (5.2) has been invoked. Assuming, \( z, z' \in \Omega \) far from \( \partial \Omega \) and utilizing the Helmholtz-Kirchhoff identities, we obtain

\[
E_1(z, z') \simeq \frac{\sigma_{\text{noise}}^2}{4} \int_{\partial \Omega} (\nabla(x, z))^T (\nabla(x, z')) d\sigma(x) \simeq -\frac{\sigma_{\text{noise}}^2}{4 \kappa} \Im \{ \nabla(z, z') \}. \quad (5.13)
\]
Now, remark that,

\[
(\nabla_x \times (\phi(x)I_3))^T \mathbf{p} = -\nabla_x \times (\phi(x)p),
\]

for any constant vector \( p \) and smooth function \( \phi \). Therefore,

\[
\nabla_x \times (g(y, x)\eta_{\text{noise}}(y) \times \nu(y)) = -[\nabla_x \times (g(y, x)I_3)]^T (\eta_{\text{noise}}(y) \times \nu(y)),
\]

\[
= -[\nabla_y \times \Gamma(y, x)]^T (\eta_{\text{noise}}(y) \times \nu(y)),
\]

\[
= (\nabla_y \times \Gamma(y, x)) \times \nu(y) \eta_{\text{noise}}(y).
\]

Consequently,

\[
\mathcal{P}^\kappa \eta_{\text{noise}} \times \nu(x) = \int_{\partial \Omega} [(\nabla_y \times \Gamma(y, x)) \times \nu(y)]^T \eta_{\text{noise}}(y) d\sigma(y) \times \nu(x).
\]

By virtue of (5.14), we have

\[
E_2(z, z') := -\frac{1}{2} \mathbb{E} \left[ \iint_{(\partial \Omega)^2} (\Gamma(x, z) \times \nu(x)) \times \nu(x) (\eta_{\text{noise}}(x) \times \nu(x))^T \Gamma(x', z') \times \nu(x') \eta_{\text{noise}}(x') d\sigma(x) d\sigma(x') \right],
\]

\[
= -\frac{1}{2} \iiint_{(\partial \Omega)^3} [(\Gamma(x, z) \times \nu(x)) \times \nu(x)]^T \mathbb{E} \left[ (\eta_{\text{noise}}(x) \times \nu(x))^T \eta_{\text{noise}}(y) \right] \times \nu(y) \times \nu(x') \times \nu(x') \times \nu(x) d\sigma(y) d\sigma(x) d\sigma(x'),
\]

\[
= -\frac{i \kappa \sigma_{\text{noise}}^2}{2} \iint_{(\partial \Omega)^2} \Gamma(y, z) |\Gamma(y, x')|^T \Gamma(x', z') d\sigma(y) d\sigma(x) d\sigma(x').
\]

A simple application of Helmholtz-Kirchhoff identity then yields

\[
E_2(z, z') \simeq \frac{\sigma_{\text{noise}}^2}{2 \kappa^2} \int_{\partial \Omega} \left\{ \iota \kappa \Gamma(y, z) \right\} \Im \left\{ \kappa \Gamma(y, z') \right\} d\sigma(y).
\]

Similarly, third term \( E_3(z, z') \) can be evaluated and appears to be

\[
E_3(z, z') \simeq \frac{\sigma_{\text{noise}}^2}{2 \kappa^2} \int_{\partial \Omega} \Im \left\{ \kappa \Gamma(y, z) \right\} \left\{ \iota \kappa \Gamma(y, z') \right\} d\sigma(y).
\]
In order to explicitly calculate $E_4(z, z')$, we observe by invoking (5.14) that

$$
E \left[ P^\kappa [\eta_{\text{noise}} \times \nu](x) P^\kappa [\eta_{\text{noise}} \times \nu](x')^T \right] = E \left[ \int_{(\partial \Omega)^2} \left( [\nabla_y \times \Gamma(y, x)] \times \nu(x) \right)^T \eta_{\text{noise}}(y) \times \nu(x) \right. \\
\left. \times \left( [\nabla_y \times \Gamma(y', x')] \times \nu(x') \right)^T \eta_{\text{noise}}(y') \times \nu(x') \right] \, d\sigma(y) d\sigma(y'),
$$

$$
= \int_{(\partial \Omega)^2} \left( [\nabla_y \times \Gamma(y, x)] \times \nu(x) \right)^T \eta_{\text{noise}}(y) \times \nu(x) \right] \, d\sigma(y) d\sigma(y'),
$$

$$
= \sigma_{\text{noise}}^2 \int_{\Omega^3} \left( [\nabla_y \times \Gamma(y, x)] \times \nu(x) \right)^T \eta_{\text{noise}}(y) \times \nu(x) \right] \, d\sigma(y) d\sigma(y'),
$$

Therefore,

$$
E_4(z, z') = \int_{(\partial \Omega)^2} \left( \Gamma(x, z) \times \nu(x) \right)^T E \left[ P^\kappa [\eta_{\text{noise}} \times \nu](x) P^\kappa [\eta_{\text{noise}} \times \nu](x')^T \right] \, d\sigma(x) d\sigma(x'),
$$

$$
\simeq \sigma_{\text{noise}}^2 \kappa^2 \int_{\Omega^3} \left( \Gamma(x, z) \times \nu(x) \right)^T \eta_{\text{noise}}(y) \times \nu(x) \right] \, d\sigma(y) d\sigma(y'),
$$

Adding all the contributions $E_i$ ($i = 1, \cdots, 4$), we obtain the covariance of $U_{\text{noise}}$

$$
E \left[ U_{\text{noise}}(z) U_{\text{noise}}(z')^T \right] = -\frac{\sigma_{\text{noise}}^2}{4\kappa} \Im m \left\{ \Gamma(z, z') \right\} + \frac{\sigma_{\text{noise}}^2}{2\kappa^2} \int_{\Omega^2} \left\{ i\kappa \Gamma(y, z) \right\} \Im m \left\{ \kappa \Gamma(y, z') \right\} d\sigma(y) + \frac{\sigma_{\text{noise}}^2}{2\kappa^2} \int_{\Omega^2} \Im m \left\{ \kappa \Gamma(y, z) \right\} \Im m \left\{ i\kappa \Gamma(y, z') \right\} d\sigma(y) + \frac{\sigma_{\text{noise}}^2}{\kappa^2} \int_{\Omega^2} \Im m \left\{ \kappa \Gamma(y, z) \right\} \Im m \left\{ \kappa \Gamma(y, z') \right\} d\sigma(y).
$$

Finally, recall that $z, z'$ are assumed far away from the boundary and we have the estimate

$$
\nabla_y \times \Gamma(y, z) \times \nu(y) = i\kappa \Gamma(y, z) + O(|y - z|^{-1}).
$$

Therefore, using the Helmholtz-Kirchhoff identities (taking their imaginary parts), we arrive at

$$
E \left[ U_{\text{noise}}(z) U_{\text{noise}}(z')^T \right] = -\frac{\sigma_{\text{noise}}^2}{4\kappa} \Im m \left\{ \Gamma(z, z') \right\}.
$$
The following remark is in order.

**Remark 5.2.** Lemma 5.1 indicates that $U^{\text{noise}}$ is a speckle pattern, that is, a random cloud of hot spots having typical diameters of the order of wavelength and amplitudes of the order of $\sigma_{\text{noise}}/(2\sqrt{n})$.

We are now ready to perform the stability analysis of the imaging functional $\partial_T \mathcal{H}_f$. For simplicity, we restrict ourselves to permittivity contrast only case and permeability contrast only case.

### 5.1 Stability analysis in permeability contrast case

Recall that the imaging functional in this case reduces to

$$
\partial_T \mathcal{H}_f (z) = -\kappa^2 \frac{1 - \mu_2}{\mu_2} \frac{1}{n} \sum_{j=1}^{n} \Re \left\{ (U^{\text{true},j}(z) + U^{\text{noise},j}(z)) \cdot \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) H_0^j (z) \right] \right\},
$$

where superposed $j$ indicates the fields associated with incident wave $H_0^j$.

It is readily noted that the first term in the above expression with $U^{\text{true},j}$ renders the image obtained in the case without noise as discussed in Section 4.2. The second term is therefore a corruption in the image due to the measurement noise. The covariance of the corrupted image is given by

$$
\text{Cov} \left( \partial_T \mathcal{H}_f (z), \partial_T \mathcal{H}_f (z') \right) = \frac{\alpha^2}{n^2} \sum_{j,l=1}^{n} \mathbb{E} \left\{ \Re \left\{ U^{\text{noise},j}(z) \cdot \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) H_0^j (z) \right] \right\} \Re \left\{ U^{\text{noise},l}(z') \cdot \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) H_0^l (z') \right] \right\} \right\},
$$

$$
= \frac{\alpha^2}{2n^2} \sum_{j=1}^{n} \mathbb{E} \left\{ \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) H_0^j (z) \right] \cdot \mathbb{E} \left\{ U^{\text{noise},j}(z) U^{\text{noise},j}(z')^T \right\} \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) H_0^j (z') \right] \right\},
$$

where

$$
a_{\mu} := \kappa^2 \left( \frac{1 - \mu_2}{\mu_2} \right),
$$

and we have made use of the assumption that $U^{\text{noise},j}$ and $U^{\text{noise},l}$ are uncorrelated.

Using the expression (4.11) for $H_0^0$, Lemma 5.1 and the approximation (4.11), we obtain

$$
\text{Cov} \left( \partial_T \mathcal{H}_f (z), \partial_T \mathcal{H}_f (z') \right)
= -\frac{\alpha^2 \sigma_{\text{noise}}^2}{8n^2 \kappa} \sum_{j=1}^{n} \Re \left\{ \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) H_0^j (z) \right] \cdot \Im \left\{ \Gamma(z,z') \right\} \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) H_0^j (z') \right] \right\},
$$

$$
= -\frac{\alpha^2 \sigma_{\text{noise}}^2}{8n^2 \kappa} \sum_{j=1}^{n} \Re \left\{ \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) \theta_j^T \right] \cdot \Im \left\{ \Gamma(z,z') \right\} \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) \theta_j^T e^{i \kappa \theta_j^T (z-z')} \right] \right\},
$$

$$
= -\frac{\alpha^2 \sigma_{\text{noise}}^2}{8n \kappa} \Re \left\{ \Im \left\{ \Gamma(z,z') \right\} \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) \theta_j^T \right] \cdot \left[ \sum_{j=1}^{n} \theta_j^T \theta_j^T e^{i \kappa \theta_j^T (z-z')} \right] \left[ M_{B_3}^T \left( \frac{1}{\mu_2} \right) \right] \right\},
$$

$$
\approx \frac{\alpha^2 \sigma_{\text{noise}}^2}{2n \kappa^2} \Re \left\{ \Im \left\{ \Gamma(z,z') \right\} \left[ M_{B_3} \left( \frac{1}{\mu_2} \right) \theta_j^T \right] \cdot \Im \left\{ \Gamma(z,z') \right\} \left[ M_{B_3}^T \left( \frac{1}{\mu_2} \right) \theta_j^T \right] \right\}.
$$
For analysis sake assume that $B_\delta$ is a ball in $\mathbb{R}^3$, so that Lemma 2.7 yields
\[
\text{Cov}(\partial_T \mathcal{H}_f(z), \partial_T \mathcal{H}_f(z')) \simeq \frac{\tilde{a}_\mu^2 \sigma^2_{\text{noise}}}{2n} \|\Im \{\Gamma(z, z')\}\|_{HS}^2.
\] (5.21)

where
\[
\tilde{a}_\mu = \frac{3 \mu_2 \sigma_{\text{noise}}}{(2 + \mu_2)\kappa^2}.
\]

Remark 5.3. Note that the perturbation in image due to measurement noise is of order $\sigma^2_{\text{noise}}/\sqrt{2n}$ and the typical shape of hot spots in the perturbation is identical with that of the main peak of functional $\partial_T \mathcal{H}_f$ related to accurate data. The main peak of $\partial_T \mathcal{H}_f$ is slightly affected in a stable manner due to the perturbations as well. Moreover, since the typical size of the perturbation is inversely proportional to $\sqrt{2n}$, the use of multiple incident fields further enhances the stability of the imaging framework based on $\partial_T \mathcal{H}_f$.

It follows immediately from (5.21) that the variance of functional $\partial_T \mathcal{H}_f$ at a search point $z_S$ is given by
\[
\text{Var}(\partial_T \mathcal{H}_f(z_S)) \simeq \frac{\tilde{a}_\mu^2 \sigma^2_{\text{noise}}}{2n} \|\Im \{\Gamma(z, z)\}\|_{HS}^2.
\] (5.22)

Therefore, the signal-to-noise ratio (SNR), defined by
\[
\text{SNR} := \frac{E[\partial_T \mathcal{H}_f(z_D)]}{\sqrt{\text{Var}[\partial_T \mathcal{H}_f(z_D)]}},
\] (5.23)

can be approximated by virtue of (5.22) and Corollary 4.2 as
\[
\text{SNR} \simeq \sqrt{2n\rho^2\omega^2\tilde{C}_\mu} \|\Im \{\Gamma(z_D, z_D)\}\|_{HS}.
\] (5.24)

5.2 Stability analysis in permittivity contrast case

In permittivity contrast case, assuming $B_\delta$ a ball in $\mathbb{R}^3$, we have
\[
\partial_T \mathcal{H}_f(z) = -\frac{a_\epsilon}{n\kappa^2} \sum_{j=1}^{n} \Re \left\{ \nabla_z \times (U_{\text{true},j}(z) + U_{\text{noise},j}(z)) \cdot [\nabla_{z'} \times H_0^j(z')] \right\}.
\]
where the constant $a_\epsilon$ is defined by
\[
a_\epsilon := \frac{3\kappa^2(1 - \epsilon_2)}{(2 + \epsilon_2)}.
\]

Observe again that the first term corresponds to the true image in the absence of the noise as in the permeability contrast case, whereas the covariance of the corrupted image here is now
given by

\[ \text{Cov}(\partial_T \mathcal{H}_f(z), \partial_T \mathcal{H}_f(z')) = \frac{a^2}{n^2 \kappa^2} \sum_{j,l=1}^{n} \text{Re} \left\{ \text{Re} \left[ \nabla_z \times U^{\text{noise},j}(z) \cdot \nabla_z \times H'_0(z) \right] \right\} \text{Re} \left\{ \nabla_{z'} \times U^{\text{noise},l}(z') \cdot \nabla_{z'} \times H'_0(z') \right\}, \]

\[ = \frac{a^2}{2n^2 \kappa^2} \sum_{j,l=1}^{n} \text{Re} \left\{ \text{Re} \left[ \nabla_z \times U^{\text{noise},j}(z) \cdot \nabla_z \times H'_0(z) \right] \right\} \text{Re} \left\{ \nabla_{z'} \times U^{\text{noise},l}(z') \cdot \nabla_{z'} \times H'_0(z') \right\}, \]

\[ = \frac{a^2}{2n^2 \kappa^2} \sum_{j=1}^{n} \text{Re} \left\{ \text{Re} \left[ \nabla_z \times U^{\text{noise},j}(z) \cdot \nabla_z \times H'_0(z) \right] \right\} \text{Re} \left\{ \nabla_{z'} \times U^{\text{noise},j}(z') \cdot \nabla_{z'} \times H'_0(z') \right\}, \]

Using the arguments as in Lemma (5.1), it can be easily proved that

\[ \mathbb{E} \left[ \nabla_z \times U^{\text{noise},j}(z) \nabla_{z'} \times U^{\text{noise},j}(z') \right] \simeq -\frac{\sigma^2_{\text{noise}} K}{4} \Im \left\{ \Gamma(z, z') \right\}. \]

Therefore,

\[ \text{Cov}(\partial_T \mathcal{H}_f(z), \partial_T \mathcal{H}_f(z')) \simeq \frac{a^2 \sigma_{\text{noise}}^2}{8n^2 \kappa^2} \sum_{j=1}^{n} \text{Re} \left\{ \left[ \nabla_z \times H'_0(z) \right] \cdot \Im \left\{ \Gamma(z, z') \right\} \right\}, \]

\[ = -\frac{a^2 \sigma_{\text{noise}}^2}{8n^2 \kappa} \sum_{j=1}^{n} \text{Re} \left\{ \left[ \nabla_z \times H'_0(z) \right] \right\} \cdot \Im \left\{ \nabla_{z'} \times H'_0(z') \right\}, \]

\[ = -\frac{a^2 \sigma_{\text{noise}}^2}{8n^2 \kappa} \text{Re} \left\{ \Im \left\{ \Gamma(z, z') \right\} \cdot \sum_{j=1}^{n} \left( \theta_j \times \theta_j^+ \right) \left( \theta_j \times \theta_j^+ \right)^T e^{i \kappa \theta_j^T(z-z')} \right\}. \]

Using approximation (4.15), we deduce that

\[ \text{Cov}(\partial_T \mathcal{H}_f(z), \partial_T \mathcal{H}_f(z')) \simeq \frac{a^2 \rho^2 \sigma_{\text{noise}}^2}{2n \kappa^2} \left\| \Im \left\{ \Gamma(z, z') \right\} \right\|_{HS}^2. \]

Consequently, the variance of functional \( \partial_T \mathcal{H}_f \) at a search point \( z_S \) is given by

\[ \text{Var}(\partial_T \mathcal{H}_f(z_S)) \simeq \frac{a^2 \rho^2 \sigma_{\text{noise}}^2}{2n \kappa^2} \left\| \Im \left\{ \Gamma(z_S, z_S) \right\} \right\|_{HS}^2. \]  

(5.25)

The signal-to-noise ratio in this case can be given by

\[ \text{SNR} := \frac{\mathbb{E} \left[ \partial_T \mathcal{H}_f(z_D) \right]}{\sqrt{\text{Var}(\partial_T \mathcal{H}_f(z_D))}} \simeq \frac{\rho^3 \kappa^3 C \sqrt{2n}}{\alpha \sqrt{n} \rho \sigma_{\text{noise}}} \left\| \Im \left\{ \Gamma(z_D, z_D) \right\} \right\|_{HS}, \]

(5.26)

by virtue of Corollary 4.2. Note that the Remark 5.3 still holds in this case and functional \( \partial_T \mathcal{H}_f \) remains stable in this case.
6 Statistical stability with respect to medium noise

In this section, we aim to investigate the statistical stability of the imaging functional $\partial_T \mathcal{H}_f$ with respect to medium noise. For simplicity, we assume that only one of the permittivity and permeability parameters fluctuates around the background value at a time. Moreover, it is assumed that the background electromagnetic parameters are normalized, that is, $\mu_0 = 1$ and $\epsilon_0 = 1$.

6.1 Fluctuations in permeability

Let the permeability of $\Omega$, denoted by $\mu(x)$ throughout in this section, be fluctuating around the background permeability such that

$$\mu(x) = 1 + \gamma(x),$$

(6.1)

where $\gamma(x)$ represents a random fluctuation such that the typical size of $\gamma$, denoted by $\sigma_\gamma$, is small enough so that the born approximation is valid. We emphasize that $\gamma$ is a real-valued function.

Nota Bene. Throughout this subsection, we term the homogeneous medium with parameter ($\epsilon_0 = 1, \mu_0 = 1$) as the reference medium, and the random medium without inclusion as the background medium still denoted by $\Omega$ by abuse of notation. Further, superposed 0 indicates a field in the reference medium and any field without superscript is related to the random medium with or without inclusion henceforth.

Let $G^0$ and $G$ be the reference and background dyadic Green’s functions with Neumann type boundary conditions, that is, the solutions to

$$\begin{aligned}
\nabla_x \times \nabla_x \times G^0(x, y) - \omega^2 G^0(x, y) &= -\delta_y(x)I_3, \quad \Omega, \\
(\nabla_x \times G^0(x, y)) \times \nu(x) &= 0, \quad \partial \Omega,
\end{aligned}$$

(6.2)

and

$$\begin{aligned}
\nabla_x \times \nabla_x \times G(x, y) - \mu(x) \omega^2 G(x, y) &= -\delta_y(x)I_3, \quad \Omega, \\
(\nabla_x \times G(x, y)) \times \nu(x) &= 0, \quad \partial \Omega.
\end{aligned}$$

(6.3)

The following born approximation is valid thanks to the smallness assumption on the size of $\gamma$,

$$G(x, y) = G^0(x, y) + \omega^2 \int_\Omega G^0(x, z)\gamma(z)G^0(z, y)dz + o(\sigma_\gamma).$$

(6.4)

As a consequence, we also have

$$H_0(x) = H^0_0(x) + \omega^2 \int_\Omega G^0(x, z)\gamma(z)H^0_0(z)dz + o(\sigma_\gamma).$$

(6.5)

The back-propagator $U$ is now constructed as follows,

$$U(z) = -\int_{\partial \Omega} I^0(x, z)\nu(x) \times \left( \frac{1}{2} I^0 - P^\kappa,0 \right) \left[ (H_\rho - H^0_0) \times \nu \right](x) d\sigma(x).$$

(6.6)

Remark that the back-propagation step uses reference fundamental solution and the reference magnetic solution since the background solutions are unknown. We express $H_\rho - H^0_0$ as the
sum of two terms $H_0 - H_0^0$ and $H_0 - H_0^0$. Subsequently, we invoke Lemma 3.2 and born approximations (6.4) and (6.5). Therefore,

$$U(z) = -\int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) [(H_0 - H_0^0) \times \nu](x)d\sigma(x)$$

$$= -\int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) [(H_0^0 - H_0^0) \times \nu](x)d\sigma(x),$$

$$- \kappa^2 \int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) \left[ \int_{\Omega} G^0(\cdot, y) \gamma(y)(H_0^0 - H_0^0)(y)dy \times \nu(\cdot) \right](x)d\sigma(x)$$

$$- \kappa^2 \int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) \left[ \int_{\Omega} G^0(\cdot, y) \gamma(y)H_0^0(y)dy \times \nu(\cdot) \right](x)d\sigma(x)$$

Note that the first term on the right hand side, that is,

$$-\int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) [(H_0^0 - H_0^0) \times \nu](x)d\sigma(x),$$

is exactly the reference back-propagator defined in (6.7). Therefore, we will denote this term by $U^{\text{true}}(z)$. From [11, Theorem 2.1], we have $(H_0^0 - H_0^0) = O(\rho^3)$. Consequently, the second term

$$- \kappa^2 \int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) \left[ \int_{\Omega} G^0(\cdot, y) \gamma(y)(H_0^0 - H_0^0)(y)dy \times \nu(\cdot) \right](x)d\sigma(x)$$

is of the order $O(\sigma^3 \rho^3)$ and is neglected henceforth. For the last term, we note that

$$\int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) \left[ \int_{\Omega} G^0(\cdot, y) \gamma(y)H_0^0(y)dy \times \nu(\cdot) \right](x)d\sigma(x),$$

$$\int_{\partial\Omega} \Gamma_0^0(x, z) \nu(x) \times \left( \frac{1}{2} I - P^{\kappa, 0} \right) \left[ \int_{\Omega} G^0(\cdot, y) \gamma(y)H_0^0(y)dy \times \nu(\cdot) \right](x)d\sigma(x),$$

$$= \int_{\partial\Omega} (\Gamma_0^0(x, z) \times \nu(x))^T \left( \frac{1}{2} I + P_{\kappa, 0} \right) \left[ \int_{\Omega} G^0(\cdot, y) \gamma(y)H_0^0(y)dy \right](x) \times \nu(x)d\sigma(x),$$

$$= \int_{\partial\Omega} (\Gamma_0^0(x, z) \times \nu(x))^T \int_{\Omega} \Gamma_{\partial\Omega}(x, y) \gamma(y)H_0^0(y) \times \nu(x)dyd\sigma(x),$$

$$= \int_{\partial\Omega} \gamma(y) \int_{\partial\Omega} (\Gamma_0^0(x, z) \times \nu(x))^T \Gamma_{\partial\Omega}(x, y) \times \nu(x)d\sigma(x) \int_{\Omega} H_0^0(y)dy.$$

In order to obtain the latter identity, we have made use of the following result that can be proved by similar arguments as in [11, Theorem 2.28].

**Lemma 6.1.** For all $x \in \partial\Omega$ and $y \in \Omega$, we have

$$\left( \frac{1}{2} + P_{\kappa, 0} \right) G^0(x, y) = \Gamma_0^0(x, y).$$

Finally, combining all the three observations, we conclude that

$$U(z) = U^{\text{true}}(z) + U^{\text{noise}}(z) + O(\sigma_\ast \rho^3) + o(\sigma_\ast).$$
where we define $\mathbf{U}^{\text{noise}}$ by

$$
\mathbf{U}^{\text{noise}}(z) := -\kappa^2 \int_\Omega \gamma(y) \left[ \int_{\partial\Omega} \left( \mathbf{G}(x, z) \times \mathbf{v}(x) \right)^T \mathbf{F}(x, y) \times \mathbf{v}(x) d\sigma(x) \right] \mathbf{H}^0_0(y) dy.
$$

(6.9)

The expression (6.9) clearly shows that the back-propagator in the random medium is the sum of the reference back-propagator and the error term thanks to the medium noise. The reference back-propagator $\mathbf{U}^{\text{true}}$ produces exactly the same results as without medium noise. The back-propagator $\mathbf{U}^{\text{noise}}$ generates a speckle field corrupting the reconstructed image. In rest of this subsection, we consider the case of only permittivity contrast and only permeability contrast for simplicity in order to analyze the speckle field generated by $\mathbf{U}^{\text{noise}}$. Moreover the situation when there are multiple incident fields of the form (4.11) is taken into account.

### 6.1.1 Speckle field analysis for permeability contrast

Before any further discussion, remark that thanks to Helmholtz-Kirchhoff identity in Lemma 2.2

$$
\mathbf{U}^{\text{noise}}(z) \simeq \kappa \int_\Omega \gamma(y) \Im \left\{ \mathbf{G}(y, z) \right\} \mathbf{H}^0_0(y) dy.
$$

(6.10)

Let us now compute the covariance of the speckle field generated by the back-propagation of $\mathbf{U}^{\text{noise}}$ as

$$
\text{Cov} \left( \partial_T \mathcal{H}_f(z), \partial_T \mathcal{H}_f(z') \right) = \left( \frac{1}{\mu^2} (1 - \mu_2) \right)^2 \frac{1}{n^2} \sum_{j,l=1}^{n} \mathbb{E} \left[ \Re \left\{ \mathbf{U}^{\text{noise}}(z) \cdot \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) \mathbf{H}^{0,j}_0(z) \right\} \Re \left\{ \mathbf{U}^{\text{noise}}(z') \cdot \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) \mathbf{H}^{0,j}_0(z') \right\} \right].
$$

(6.11)

for all $z, z' \in \Omega$, where $\mathbf{H}^{0,j}_0$ are the incident fields of the form (4.11).

The expression $\Re \left\{ \mathbf{U}^{\text{noise}}(z) \cdot \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) \mathbf{H}^{0,j}_0(z) \right\}$ can be approximated explicitly using (6.10), (6.11) and (6.14). Indeed,

$$
\frac{1}{n} \sum_{j=1}^{n} \mathbf{U}^{\text{noise}}(z) \cdot \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) \mathbf{H}^{0,j}_0(z)
$$

$$
\simeq \kappa \frac{1}{n} \sum_{j=1}^{n} \int_\Omega \gamma(y) \Im \left\{ \mathbf{G}(y, z) \right\} \mathbf{H}^{0,j}_0(y) dy \cdot \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) \mathbf{H}^{0,j}_0(z),
$$

$$
= \kappa \frac{1}{n} \sum_{j=1}^{n} \int_\Omega \gamma(y) \Im \left\{ \mathbf{G}(y, z) \right\} : \mathbf{H}^{0,j}_0(y) \mathbf{H}^{0,j}_0(z)^T \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) dy,
$$

$$
= \kappa \int_\Omega \gamma(y) \Im \left\{ \mathbf{G}(y, z) \right\} : \left[ \frac{1}{n} \sum_{j=1}^{n} \hat{\mathbf{g}}_j \cdot \hat{\mathbf{g}}_j^T e^{i \kappa \mathbf{d}_j^T (\mathbf{z} - \mathbf{y})} \right] \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) dy,
$$

$$
\simeq -4\pi \int_\Omega \gamma(y) \Im \left\{ \mathbf{G}(y, z) \right\} : \Im \left\{ \mathbf{G}(y, z) \right\} \mathbf{M}_{B_3} \left( \frac{1}{\mu^2} \right) dy.
$$

26
Assume for the sake of analysis, that $B_δ$ is a ball in $\mathbb{R}^3$. Then, thanks to Lemma 2.7 we have

$$\frac{1}{n} \sum_{j=1}^{n} \mathbf{U}_{\text{noise}}(z) \cdot \left[ \mathbf{M}_{B_δ} \left( \frac{1}{\mu^2} \right) \mathbf{H}^0_\gamma(z) \right] \simeq -\frac{12\mu_2\pi|B_δ|}{\epsilon_0^2(2 + \mu_2)} \int_\Omega \gamma(y) \left\| \Im \left\{ \Gamma^0(y, z) \right\} \right\|_{HS}^2 dy.$$  

(6.12)

Consequently, the covariance of the speckle field can be approximated by

$$\text{Cov}\left( \partial_T H_f(z), \partial_T H_f(z') \right) = b_\mu^2 \int_{\Omega \times \Omega} C_\gamma(y, y') \left\| \Im \left\{ \Gamma^0(y, z) \right\} \right\|_{HS}^2 \left\| \Im \left\{ \Gamma^0(y', z') \right\} \right\|_{HS}^2 dy dy',$$  

(6.13)

where $b_\mu$ and $C_\gamma(y, y')$ are defined by

$$b_\mu := -12\pi \frac{(1 - \mu_2^2)|B_δ|}{2 + \mu_2} \quad \text{and} \quad C_\gamma(y, y') := \mathbb{E} \left[ \gamma(y) \gamma(y') \right].$$  

(6.14)

The function $C_\gamma(y, y')$ is the two-point correlation function of the fluctuations in permeability.

The following remark is in order.

**Remark 6.2.** The expression in (6.12) elucidates that the speckle field in the image is essentially the medium noise smoothed by an integral kernel of the form $\|\Im \{\Gamma^0\}\|_{HS}^2$. Similarly, (6.13) demystifies that the correlation structure of the speckle field is essentially that of the medium noise smoothed by the same kernel. Since the typical width of $\Im \{\Gamma^0\}$ is about half the wavelength, the correlation length of the speckle field is roughly the maximum between the correlation length of medium noise and the wavelength. Moreover, unlike measurement noise case discussed in the previous section, the factor $\sqrt{n}$ disappeared. Therefore, the functional $\partial_T H_f$ is moderately stable with respect to medium noise.

### 6.1.2 Speckle field analysis for permittivity contrast

Let us now compute the covariance of the speckle field generated by the back-propagation of $\mathbf{U}_{\text{noise}}$ in the case of permittivity contrast. For all $z, z' \in \Omega$

$$\text{Cov}\left( \partial_T H_f(z), \partial_T H_f(z') \right) = \left( \frac{3\epsilon_2|B_δ|}{2 + \epsilon_2} \left( \frac{1}{\epsilon_2} - 1 \right) \right)^2 \frac{1}{n^2} \sum_{j,m=1}^{n}$$

$$\mathbb{E} \left[ \Re \left\{ \nabla_z \times \mathbf{U}_{\text{noise}}(z) \cdot \nabla_z \times \mathbf{H}^0_{\gamma,j}(z) \right\} \Re \left\{ \nabla_{z'} \times \mathbf{U}_{\text{noise}}(z') \cdot \nabla_{z'} \times \mathbf{H}^0_{\gamma,l}(z') \right\} \right].$$  

(6.15)
Again $H_0^{0,j}$ are the incident fields of the form (4.11) and we have assumed that $B_\delta$ is a ball in $\mathbb{R}^3$ for analysis sake. In this case, we have

$$
\frac{1}{n} \sum_{j=1}^{n} \nabla_z \times U_{\text{noise}}(z) \cdot \nabla_z \times H_0^{0,j}(z)
$$

\[ \simeq k \frac{1}{n} \sum_{j=1}^{n} \frac{1}{n} \int_{\Omega} \gamma(y) \nabla_z \times \Im \{ \Gamma^0(y, z) \} : H_0^{0,j}(y) \cdot \nabla_z \times H_0^{0,j}(z) dy,
\]

\[ = k \frac{1}{n} \sum_{j=1}^{n} \frac{1}{n} \int_{\Omega} \gamma(y) \nabla_z \times \Im \{ \Gamma^0(y, z) \} : \left( \nabla_z \times H_0^{0,j}(z) \right) \overline{H_0^{0,j}(y)}^T dy,
\]

\[ = k \int_{\Omega} \gamma(y) \nabla_z \times \Im \{ \Gamma^0(y, z) \} : \nabla_z \times \left( \nabla_z \times H_0^{0,j}(z) \right) \overline{H_0^{0,j}(y)}^T dy,
\]

\[ \simeq -4\pi \int_{\Omega} \gamma(y) \nabla_z \times \Im \{ \Gamma^0(y, z) \} : \nabla_z \times \Im \{ \Gamma^0(y, z) \} dy,
\]

Therefore, the covariance turns out to be

$$
\text{Cov} \left( \partial_r \mathcal{H}_f(z), \partial_r \mathcal{H}_f(z') \right) = \overline{b}_\mu^2 \int_{\Omega \times \Omega} C_\gamma(y, y') \left\| \nabla_z \times \Im \{ \Gamma^0(y, z) \} \right\|_{HS}^2 \left\| \nabla_{z'} \times \Im \{ \Gamma^0(y', z') \} \right\|_{HS}^2 dy dy',
$$

(6.16)

where the constant $\overline{b}_\mu$ is defined by

$$
\overline{b}_\mu := -12\pi \frac{\epsilon_2 |B_\delta|}{(2 + \epsilon_2)} \left( \frac{1}{\epsilon_2} - 1 \right).
$$

(6.17)

The conclusions drown in Remark 5.2 still hold in this case and the imaging functional is moderately stable.

### 6.2 Fluctuations in permittivity

Let us now investigate the stability of the imaging framework with respect to medium noise when the permittivity, hereafter denoted by $\epsilon$, is fluctuating randomly around the reference permittivity. We assume that the fluctuating background permittivity is such that

$$
\frac{1}{\epsilon(x)} := 1 + \alpha(x),
$$

(6.18)

where the reference permittivity is 1 and $\alpha$ is a random fluctuation. It is again assumed that the fluctuation is weak so that the Born approximation is appropriate. We will make use of the same conventions as in Section 6.1 for reference and background media, and fields.
The equation for the magnetic field with fluctuating permittivity is then given by
\[ \nabla \times \nabla \times H_\rho(x) - \omega^2 H_\rho(x) = -\nabla \times \alpha(x) \nabla \times H_\rho(x). \tag{6.19} \]

Since, the Born approximation is appropriate thanks to assumption of weak fluctuations, we have \( H_\rho \approx H_\rho^0 - H_\rho^1 \), where \( H_\rho^0 \) solves the reference problem and \( H_\rho^1 \) solves
\[ \nabla \times \nabla \times H_\rho^1(x) - \omega^2 H_\rho^1(x) = -\nabla \times \alpha(x) \nabla \times H_\rho^0(x). \tag{6.20} \]

Consequently, we have
\[ H_\rho^1(x) = \int_\Omega G^0(x, y) \left( \nabla_y \times \alpha(y) \nabla \times H_\rho^0(y) \right) dy, \tag{6.21} \]

where \( G^0(x, y) \) is given by (6.2).

Following the analysis in Section 6.1, it can be noticed that the back-propagator, again defined in terms of the reference fundamental solution and associated reference solution, consists of two terms, one leading to the true image whereas the second giving rise to a speckle field corrupting the image thanks to permittivity fluctuations. This noise back-propagating term has the expression
\[ U_{\text{noise}}(z) = \int_{\partial \Omega} \Im \left\{ \Gamma^0(y, z) \right\} \nabla_y \times \alpha(y) \nabla \times H_\rho^0(y) \cdot d\sigma(x), \tag{6.23} \]

where we have assumed that domain \( B_\delta \) is a ball and the constant \( s_\mu \) is defined by
\[ s_\mu := 3 \rho^3 \omega^2 |B_\delta| \left( \frac{1 - \mu_2}{2 + \mu_2} \right). \]

Since \( \Im \left\{ \Gamma^0(y, z) \right\} \) is symmetric, we have
\[ \partial_T^{\text{noise}} \mathcal{H}_f(z) \simeq -\frac{s_\mu}{kn} \sum_{j=1}^{n} \Re \left\{ \int_\Omega \Im \left\{ \Gamma^0(y, z) \right\} \nabla_y \times \alpha(y) \nabla \times H_\rho^0(y) \cdot H_\rho^{0,j}(y) \left( y \right) dy \right\}. \]
Further, on assuming that \( \alpha(x) = 0 \) for all \( x \) in the neighborhood of boundary \( \partial\Omega \) and using the Green’s theorem, the above expression simplifies to

\[
\partial_T^{\text{noise}} H_f(z) \simeq -\frac{s\mu}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} H^{0,j}(z) \cdot \alpha(y) \nabla_y \times \overline{H}_0^{0,j}(y) \, dy \right\}.
\]

Therefore, we get

\[
\partial_T^{\text{noise}} H_f(z) \\
\simeq -\frac{s\mu}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} \cdot \alpha(y) \left( \nabla_y \times \overline{H}_0^{0,j}(y) \right) \left( H_0^{0,j}(z) \right)^T \, dy \right\},
\]

\[
= -\frac{s\mu}{\kappa} \Re \left\{ \int_{\Omega} \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} \cdot \alpha(y) \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} \, dy \right\},
\]

\[
= -\frac{4\pi s\mu}{\kappa^2} \Re \left\{ \int_{\Omega} \alpha(y) \left\| \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} \right\|_{HS}^2 \, dy \right\}.
\]

Consequently, the covariance of the speckle field turns out to be

\[
\text{Cov} \left( \partial_T^{\text{noise}} H_f(z), \partial_T^{\text{noise}} H_f(z') \right) \\
= \frac{16\pi^2 s^2}{\kappa^4} \int_{\Omega \times \Omega} C_\alpha(y, y') \left\| \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} \right\|_{HS}^2 \left\| \nabla_{y'} \times \Im \left\{ \Gamma^0(y', z') \right\} \right\|_{HS}^2 \, dy \, dy',
\]

where \( C_\alpha(y, y') := \mathbb{E}[\alpha(y)\alpha(y')] \) is the two point correlation of fluctuation \( \alpha \). The expression \[6.24]\ is very similar to that studied in \[6.10\]. As already pointed out in Remark \[6.2\] the speckle field is indeed the medium noise smoothed with an integral kernel whose width is of the order of wavelength.

### 6.2.2 Speckle field analysis for permittivity contrast

For the permittivity contrast only case, the speckle field generated by functional \( \partial_T H_f \) at a search point \( z \in \Omega \) is given by

\[
\partial_T^{\text{noise}} H_f(z) := \frac{1}{n} \sum_{j=1}^{n} \Re \left\{ \nabla_z \times U^{\text{noise}, j}(z) \cdot \nabla_z \times H_0^{0,j}(z) \right\},
\]

\[
= -\frac{s\epsilon}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \nabla_z \times \Im \left\{ \Gamma^0(y, z) \right\} \nabla_y \times \alpha(y) \nabla_y \times \overline{H}_0^{0,j}(y) \cdot \nabla_z \times \overline{H}_0^{0,j}(z) \, dy \right\},
\]

\[
= -\frac{s\epsilon}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \left( \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} \right)^T \nabla_y \times \alpha(y) \nabla_y \times \overline{H}_0^{0,j}(y) \cdot \nabla_z \times \overline{H}_0^{0,j}(z) \, dy \right\},
\]

\[
= -\frac{s\epsilon}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \nabla_y \times \Im \left\{ \Gamma^0(y, z) \right\} \nabla_z \times \overline{H}_0^{0,j}(z) \cdot \nabla_y \times \alpha(y) \nabla_y \times \overline{H}_0^{0,j}(y) \, dy \right\},
\]

30
where \( s_\varepsilon := (3 - 3\epsilon_2)/(2 + \epsilon_2) \). Using the hypothesis that \( \alpha \) is zero near boundary and the Green’s theorem, we simplify the above expression to

\[
\partial_T^{\text{noise}} H_f(z) \simeq -\frac{s_\varepsilon}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \nabla_{y} \times \nabla_{y} \Im \{ \Gamma^0(y, z) \} \nabla_{z} \times H^0_{ij}(z) \cdot \alpha(y) \nabla_{y} \times \overline{H^0_{ij}(y)} dy \right\},
\]

\[
\simeq -\frac{s_\varepsilon}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \Im \{ \Gamma^0(y, z) \} \nabla_{z} \times H^0_{ij}(z) \cdot \alpha(y) \nabla_{y} \times \overline{H^0_{ij}(y)} dy \right\},
\]

\[
\simeq -\frac{s_\varepsilon}{\kappa n} \sum_{j=1}^{n} \Re \left\{ \int_{\Omega} \Im \{ \Gamma^0(y, z) \} \cdot \alpha(y) \nabla_{y} \times \overline{H^0_{ij}(y)} \left( \nabla_{z} \times H^0_{ij}(z) \right)^T dy \right\},
\]

\[
\simeq -s_\varepsilon \kappa^3 \Re \left\{ \int_{\Omega} \Im \{ \Gamma^0(y, z) \} \cdot \alpha(y) \left[ \frac{1}{n} \sum_{j=1}^{n} (\theta_j \times \theta_j^\perp)(\theta_j \times \theta_j^\perp)^T e^{i\kappa \gamma_j(z-y)} \right] dy \right\}.
\]

Finally, invoking approximation (6.15), we arrive at

\[
\partial_T^{\text{noise}} H_f(z) \simeq 4\pi s_\varepsilon \kappa^2 \Re \left\{ \int_{\Omega} \alpha(y) \Im \{ \Gamma^0(y, z) \} : \Im \{ \Gamma^0(y, z) \} dy \right\},
\]

\[
= 4\pi s_\varepsilon \kappa^2 \int_{\Omega} \alpha(y) \| \Im \{ \Gamma^0(y, z) \} \|^2_{HS} dy,
\]

and as a consequence,

\[
\text{Cov} \left( \partial_T^{\text{noise}} H_f(z), \partial_T^{\text{noise}} H_f(z') \right) \simeq 16\pi^2 s_\varepsilon^2 \kappa^4 \int_{\Omega \times \Omega} C_\mu(y, y') \| \Im \{ \Gamma^0(y, z) \} \|^2_{HS} \| \Im \{ \Gamma^0(y', z') \} \|^2_{HS} dyd'y'.
\]

7 Conclusions

In this paper, we investigated a topological derivative based electromagnetic inclusion detection algorithm using the measurements of the tangential components of scattered magnetic field, considering a full Maxwell equations setting. It is elucidated that the topological derivative based imaging functional behaves like the square of the imaginary part of a free space fundamental magnetic solution and attains its maximum at the true location of the inclusion with Rayleigh resolution limit. The detection algorithm is proved to be very stable with respect to measurement noise and moderately stable with respect to medium noise. Moreover, it is indicated that multiple incident waves significantly enhance the stability of the functional. Albeit, the case of a single inclusion is discussed herein, the results extend to the case of multiple inclusions with a common characteristic size evidently.

In a forthcoming work, we intend to numerically investigate the results of this study. The problem of locating electromagnetic inclusions from far field scattering amplitude at single as well as multiple frequencies will also be the subject of a forthcoming investigation.
References

[1] H. Ammari, E. Bretin, J. Garnier, W. Jing, H. Kang, and A. Wahab, Localization, stability, and resolution of topological derivative based imaging functionals in elasticity, SIAM Journal on Imaging Sciences, 6(4):(2013), pp. 2174–2212.

[2] H. Ammari, E. Bretin, J. Garnier, H. Kang, H. Lee, and A. Wahab, Mathematical Methods in Elasticity Imaging, Princeton Series in Applied Mathematics, Princeton University Press, New Jersey, USA, 2015, ISBN: 978-0-69116531-8.

[3] H. Ammari, E. Bretin, J. Garnier, and A. Wahab, Time reversal algorithms in visco-elastic media, European Journal of Applied Mathematics, 24(4):(2013), pp. 565–600.

[4] H. Ammari, J. Garnier, W. Jing, H. Kang, M. Lim, K. Solna, and H. Wang, Mathematical and Statistical Methods for Multistatic Imaging, Lecture Notes in Mathematics, Vol. 2098, Springer, 2014.

[5] H. Ammari, J. Garnier, V. Jugnon, and H. Kang, Stability and resolution analysis for a topological derivative based imaging functional, SIAM Journal on Control and Optimization, 50(1):(2012), pp. 48–76.

[6] H. Ammari, E. Iakovleva, D. Lesselier, and G. Perrusson, MUSIC-type electromagnetic imaging of a collection of small three-dimensional inclusions, SIAM Journal on Scientific Computing, 29:(2007), pp. 674–709.

[7] H. Ammari, and H. Kang, Polarization and Moment Tensors: With Applications to Inverse Problems and Effective Medium Theory, Appl. Math. Sci. 162, Springer-Verlag, New York, 2007.

[8] H. Ammari, and H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements, Lecture Notes in Mathematics, Vol. 1846, Springer-Verlag, Berlin, 2004.

[9] H. Ammari, and H. Kang, A new method for reconstructing electromagnetic inhomogeneities of small volume, Inverse Problems, 19:(2003), pp. 63–71.

[10] H. Ammari, M. Vogelius, and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations, J. Math. Pur. Appl., 80(8):(2001), pp. 769–814.

[11] H. Ammari, and D. Volkov, The leading-order term in the asymptotic expansion of the scattering amplitude of a collection of finite number of dielectric inhomogeneities of small diameter, International Journal for Multiscale Computational Engineering, 3(3):(2005), pp. 149-160.

[12] M. Asch, and S. M. Mefire, Numerical localization of electromagnetic imperfections from a perturbation formula in three dimensions, Journal of Computational Mathematics, 26(2): 2008, pp. 149–195.

[13] G. Bao, J. Lin, and S. M. Mefire, Numerical reconstruction of electromagnetic inclusions in three dimensions, SIAM Journal on Imaging Science, 7(1): (2014), pp. 558–577.

[14] M. Bonnet and B. B. Guzina, Sounding of finite solid bodies by way of topological derivative, Internat. J. Numer. Methods Engrg., 61:(2004), pp. 2344–2373.
[15] A. Buffa, and P. Ciarlet Jr., On traces for functional spaces related to Maxwell’s equations. I. An integration by parts formula in Lipschitz polyhedra, *Math. Methods Appl. Sci.*, 24:(2001), pp. 9–30.

[16] A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab, Boundary element methods for Maxwell transmission problems in Lipschitz domain, *Numer. Math.*, 95:(2003), pp. 459–485.

[17] D. Colton, and R. Kress, *Integral Equation Methods in Scattering Theory*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1983.

[18] J. Céa, S. Garreau, P. Guillaume, and M. Masmoudi, The shape and topological optimization connection, *Comput. Methods Appl. Mech. Engrg.*, 188:(2001), pp. 703–726.

[19] J. Chen, Z. Chen, and G. Huang, Reverse time migration for extended obstacles: electromagnetic waves, *Inverse Problems*, 29:(2013), 085006.

[20] N. Dominguez, and V. Gibiat, Non-destructive imaging using the time domain topological energy method, *Ultrasonics*, 50:(2010), pp. 172–179.

[21] N. Dominguez, V. Gibiat, and Y. Esquerrea, Time domain topological gradient and time reversal analogy: An inverse method for ultrasonic target detection, *Wave Motion*, 42:(2005), pp. 31–52.

[22] A. Eschenauer, V. V. Kobelev, and A. Schumacher, Bubble method for topology and shape optimization of structures, *Struct. Optim.*, 8:(1994), pp. 42–51.

[23] G. R. Feijóo, A new method in inverse scattering based on the topological derivative, *Inverse Problems*, 20:(2004), pp. 1819–1840.

[24] M. Hintermüller, and A. Laurain, Electrical impedance tomography: From topology to shape, *Control Cybernet.*, 37:(2008), pp. 913–933.

[25] R. Hiptmair, Symmetric coupling for eddy current problems, *SIAM J. Numer. Anal.*, 40:(2002), pp. 41–65.

[26] G. C. Hsiao, and W. L. Wendland, *Boundary Integral Equations*, Applied Mathematical Sciences, Vol. 164, Springer-Verlag, Berlin, 2008.

[27] R. McCamy, and E. Stephan, Solution procedures for three dimensional eddy-current problems, *J. Math. Anal. Appl.*, 101:(1984), pp. 348–379.

[28] P. A. Martin, and P. Ola, Boundary integral equations for the scattering of electromagnetic waves by a homogeneous dielectric obstacle, *Proc. R. Soc. Edinb. A*, 123:(1993), pp. 185–208.

[29] M. Masmoudi, J. Pommier, and B. Samet, The topological asymptotic expansion for the Maxwell equations and some applications, *Inverse Problems*, 21:(2005), pp. 547–564.

[30] J. C. Nédélec, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Applied Mathematical Sciences, Vol. 144, Springer-Verlag, New York, 2001.
[31] W.-K. Park, Topological derivative strategy for one-step iteration imaging of arbitrary shaped thin, curve-like electromagnetic inclusions, *Journal of Computational Physics*, 231(4): (2012), pp. 1426–1439.

[32] W.-K. Park, Multi-frequency topological derivative for approximate shape acquisition of curve-like thin electromagnetic inhomogeneities, *Journal of Mathematical Analysis and Applications*, 404(2):(2013), pp. 501–518.

[33] J. Sokolowski, and A. Żochowski, On the topological derivative in shape optimization, *SIAM J. Control Optim.*, 37:(1999), pp. 1251–1272.

[34] S. Gdoura, A. Wahab, and D. Lesselier, Electromagnetic time reversal and scattering by a small dielectric inclusion, *Journal of Physics: Conference Series*, 386:(2012), 012010.

[35] A. Wahab, A. Rasheed, T. Hayat, R. Nawaz, Electromagnetic time reversal algorithms and source localization in lossy dielectric media, *Communications in Theoretical Physics*, 62(6):(2014), pp. 779–789.

[36] A. Wahab, A. Rasheed, R. Nawaz, and S. Anjum, Localization of extended current source with finite frequencies, *Comptes Rendus Mathématique*, 352:(2014), pp. 917–921.

[37] K. Wapenaar, General representations for wavefields modeling and inversion in geophysics, *Geophysics*, 75(5):(2007), pp. SM5–SM17.