COHOMOLOGY CLASSES OF RANK VARIETIES AND A CONJECTURE OF LIU

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Abstract. To each finite subset of a discrete grid \( \mathbb{N} \times \mathbb{N} \) (a diagram), one can associate a subvariety of a complex Grassmannian (a diagram variety), and a representation of a symmetric group (a Specht module). Liu has conjectured that the cohomology class of a diagram variety is represented by the Frobenius characteristic of the corresponding Specht module. We give a counterexample to this conjecture.

However, we show that for the diagram variety of a permutation diagram, Liu’s conjectured cohomology class \( \sigma \) is at least an upper bound on the actual class \( \tau \), in the sense that \( \sigma - \tau \) is a nonnegative linear combination of Schubert classes. To do this, we consider a degeneration of Coskun’s rank varieties which contains the appropriate diagram variety as a component. Rank varieties are instances of Knutson–Lam–Speyer’s positroid varieties, whose cohomology classes are represented by affine Stanley symmetric functions. We show that the cohomology class of a rank variety is in fact represented by an ordinary Stanley symmetric function.

1. Introduction

A finite subset \( D \) of \( \mathbb{N} \times \mathbb{N} \) is called a diagram. Write \( [n] \) for \( \{1, 2, \ldots, n\} \). Given a diagram contained in \( [k] \times [n-k] \), define a subvariety \( X_D \) of the complex Grassmannian \( \text{Gr}(k,n) \) as the Zariski closure of

\[
\{ \text{rowspan}[A \mid I_k] : A \in M_{k,n-k}, A_{ij} = 0 \text{ if } (i,j) \in D \},
\]

where \( M_{k,n-k} \) is the set of \( k \times (n-k) \) complex matrices, and \( I_k \) is the \( k \times k \) identity matrix. This variety \( X_D \) is called a diagram variety. For example, if \( D = \{(1,1), (1,2), (2,2)\} \), \( k = 2 \), \( n = 4 \), then \( X_D \) is the closure of the set of 2-planes in \( \mathbb{C}^4 \) which are the rowspans of matrices of the form

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & * & 0 & 1
\end{bmatrix}.
\]

One can associate a complex representation \( S^D \) of the symmetric group \( S_{|D|} \) to a diagram \( D \), called the Specht module of \( D \). These generalize the usual irreducible Specht modules, which occur when \( D \) is the Young diagram of a partition; the definition for general diagrams is due to James and Peel [8].

Each of these objects, diagram variety and Specht module, naturally leads to a class in the cohomology ring \( H^*(\text{Gr}(k,n), \mathbb{Z}) \). For the diagram variety, we take the Chow ring class of \( X_D \) and use the natural isomorphism between \( H^*(\text{Gr}(k,n), \mathbb{Z}) \) and the Chow ring of \( \text{Gr}(k,n) \) to obtain a cohomology class \( [X_D] \in H^2_{|D|}(\text{Gr}(k,n), \mathbb{Z}) \).

As for the Specht module, let \( s_D \) be the Frobenius characteristic of \( S^D \), meaning \( s_D = \sum_{\lambda} a_{\lambda} s_{\lambda} \) if \( S^D \simeq \bigoplus_{\lambda} a_{\lambda} S^{\lambda} \), where \( s_{\lambda} \) is a Schur function. Here \( \lambda \) runs over partitions, and \( S^{\lambda} \) is an irreducible Specht module. There is a surjective ring homomorphism \( \phi \) from the ring of
symmetric functions to $H^*(\text{Gr}(k,n),\mathbb{Z})$, sending the Schur function $s_\lambda$ to the Schubert class $\sigma_\lambda = [X_\lambda]$, or to 0 if $\lambda \not\subseteq (k^{n-k})$. Hence we can consider the cohomology class $\phi(s_D)$.

**Conjecture** (Liu [13], Conjecture 2.5 below). For any diagram $D$, the cohomology classes $[X_D]$ and $\phi(s_D)$ are equal.

Liu proved Conjecture 2.5, or the weaker variant claiming equality of degrees, for various classes of diagrams [13]. However, it turns out that this conjecture fails in general, as we show in Section 2.

**Theorem.** If $D = \{(1,1),(2,2),(3,3),(4,4)\}$, $k = 4$, $n = 8$, then Conjecture 2.5 is false.

Work of Kraśkiewicz and Pragacz [10], and of Reiner and Shimozono [16], shows that for the diagram $D(w)$ of a permutation $w \in S_n$, $s_{D(w)}$ is the Stanley symmetric function $F_w$ (or $F_{w^{-1}}$, depending on conventions). Thus, if Conjecture 2.5 were to hold for $D(w)$, then $[X_{D(w)}] = \phi(F_w)$. Here $D(w)$ is the diagram with one cell $(i,w(j))$ for each inversion $i < j$, $w(i) > w(j)$ of $w$.

Extending work of Postnikov [15], Knutson, Lam, and Speyer [9] have defined a collection of subvarieties $\Pi_f$ of Grassmannians called *positroid varieties*, indexed by certain affine permutations $f$. These are exactly the varieties obtained by projecting a Richardson variety in the flag variety $\text{Fl}(n)$ to $\text{Gr}(k,n)$. They show that the positroid variety $\Pi_f$ has cohomology class $\phi(\tilde{F}_f)$, where $\tilde{F}_f$ is the affine Stanley symmetric function of $f$. Given an ordinary permutation $w \in S_n$, say $f_w$ is the bijection $\mathbb{Z} \to \mathbb{Z}$ with

$$f_w(i) = \begin{cases} i + n & \text{if } 1 \leq i \leq n \\ w(i) + 2n & \text{if } n \leq i \leq 2n \end{cases}$$

and $f(i + 2n) = f(i) + 2n$. It is easy to show that $\tilde{F}_{f_w} = F_w$. Bearing Conjecture 2.5 in mind, this suggests a relationship between $\Pi_{f_w}$ and $X_{D(w)}$.

In fact, Conjecture 2.5 can fail even when $D$ is a permutation diagram $D(w)$, which makes this reasoning dubious. Nevertheless, we will give a degeneration of $\Pi_{f_w}$ to a (possibly reducible) variety containing $X_{D(w)}$ as a component, which implies an upper bound on $[X_{D(w)}]$.

**Theorem** (Theorem 5.6). The cohomology class $\phi(F_w) - [X_{D(w)}]$ is a nonnegative integer combination of Schubert classes.

For $w \in S_n$, the varieties $\Pi_{f_w}$ are a special case of the *rank varieties* considered by Billey and Coskun [1]. Rank varieties are themselves a special class of positroid varieties, namely the varieties obtained by projecting a Richardson variety in a partial flag variety $\text{Fl}(k_1,\ldots,k_t)$ to $\text{Gr}(k_t,n)$. We work out necessary and sufficient conditions on $f$ for $\Pi_f$ to be a rank variety.

Coskun [3] gave a recursive rule for computing the cohomology class of a rank variety. We give a different formula for this class, in terms of ordinary Stanley symmetric functions.

**Theorem** (Theorem 4.1). If $X \subseteq \text{Gr}(k,n)$ is a rank variety, then $[X] = \phi(F_w)$ for some ordinary permutation $w$.

Example 4.2 below gives an example of a rank variety $X = \Pi_f$ where $\tilde{F}_f$ is not even Schur-positive, much less equal to any ordinary Stanley symmetric function $F_w$. Thus, Theorem 4.1 is not simply a trivial corollary of the result that $[X] = \phi(\tilde{F}_f)$.

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2. A COUNTEREXAMPLE TO LIU’S CONJECTURE

Definition 2.1. A diagram is a finite subset of \( \mathbb{N}^2 \).

Given a diagram \( D \) contained in \( [k] \times [n-k] \), define an open subset
\[
X_D^\circ = \{ \text{rowspan}(A | I_k) : A \in M_{k,n-k}, \text{if } (i,j) \in D \text{ then } A_{ij} = 0 \}
\]
of the complex Grassmannian \( \text{Gr}(k,n) \). For example, if \( D = \{(1,1),(1,2),(2,2),(2,3)\} \), \( k = 2 \), and \( n = 5 \), then
\[
X_D^\circ = \left\{ \text{rowspan} \begin{pmatrix} 0 & 0 & * & 1 & 0 \\ * & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.
\]

Definition 2.2. The diagram variety \( X_D \) of \( D \) is \( \overline{X_D^\circ} \), the closure being in the Zariski topology.

Notice that \( X_D^\circ \) is an open dense subset of \( X_D \) isomorphic to \( \mathbb{C}^{k(n-k)-|D|} \). In particular, it is irreducible, so \( X_D \) is also irreducible and has codimension \(|D|\).

We now describe a representation of \( S_m \) associated to each diagram \( D \). Say \(|D| = m \) for convenience. A bijective filling of \( D \) is a bijection \( T : D \to [m] \). The symmetric group \( S_m \) acts on bijective fillings of \( D \) by permuting entries. Fix a bijective filling \( T \) of \( D \). Let \( R(T) \) denote the group of permutations \( \sigma \in S_m \) for which \( i, \sigma(i) \) are always in the same row of \( T \).

Let \( C(T) \) be the analogous subgroup with “row” replaced by “column”.

Definition 2.3. The Specht module of \( D \) is the left ideal
\[
S^D = \mathbb{C}[S_m] \sum_{p \in R(T)} \sum_{q \in C(T)} \text{sgn}(q)qp
\]
of \( \mathbb{C}[S_m] \), viewed as an \( S_m \)-module.

The Specht modules associated to general diagrams were studied by James and Peel [8]. As \( D \) runs over (Young diagrams of) partitions of \( m \), the Specht modules provide a complete, irredundant set of complex irreducibles for \( S_m \); more about these classical Specht modules can be found in [19] or [6]. It is easy to show that the isomorphism type of \( S^D \) does not depend on the choice of \( T \), and that it is unaltered by permuting the rows or the columns of \( D \). If the rows and columns of \( D \) cannot be permuted to obtain a partition (equivalently, the rows of \( D \) are not totally ordered under inclusion), then \( S^D \) will not be irreducible. For example, if \( \lambda/\mu \) is a skew shape, then
\[
S^{\lambda/\mu} \simeq \bigoplus_\nu c_{\mu\nu}^\lambda S^\nu,
\]
where \( c_{\mu\nu}^\lambda \) is a Littlewood-Richardson coefficient.

In general it is an open problem to give a combinatorial rule for decomposing \( S^D \) into irreducibles. The widest class of diagrams for which such a rule is known are the percent-avoiding diagrams, studied by Reiner and Shimozono [18]; see also [12] and [17].

Given a diagram \( D \subset [k] \times [n-k] \), let \( D^\circ \) be the complement of \( D \) in \( [k] \times [n-k] \) rotated by \( 180^\circ \). For example, if \( \mu \subset \lambda \subset [k] \times [n-k] \) are partitions, then \( X_\lambda^\circ \cap X_\mu^\circ = X_{(\lambda/\mu)^\circ}^\circ \). This intersection is transverse on the dense open subset \( X_{(\lambda/\mu)^\circ}^\circ \) of \( X_{(\lambda/\mu)^\circ} \), and indeed one can show that \([X_{(\lambda/\mu)^\circ}] = \sum_\nu c_{\mu\nu}^\lambda \sigma_{\nu\nu}^\circ \) [13, Proposition 5.5.3].

Magyar has shown that Specht module decompositions behave as nicely as possible with respect to the box complement operation.

Theorem 2.4 (Magyar [14]). For any diagram \( D \) contained in \([k] \times [n-k] \), \( S^D \simeq \bigoplus_\lambda a_\lambda S^\lambda \) if and only if \( S^{D^\circ} \simeq \bigoplus_\lambda a_\lambda S^{\lambda^\circ} \).
In particular, \( S^{(\lambda/\mu)^\vee} \simeq \bigoplus_{\nu} c_{\lambda\mu}^{\nu} S^{\nu^\vee} \). Comparing this decomposition of \( S^{(\lambda/\mu)^\vee} \) to the expansion \( [X_{(\lambda/\mu)^\vee}] = \sum_{\nu} c_{\lambda\mu}^{\nu} \sigma_{\nu^\vee} \) discussed above suggests the next conjecture (and proves it when \( D = (\lambda/\mu)^\vee \)).

**Conjecture 2.5** (Liu [13]). For any diagram \( D \), the cohomology classes \( [X_D] \) and \( \phi(s_D) \) are equal.

Liu proved Conjecture 2.5 in the case above where \( D^\vee \) is a skew shape, or when it corresponds to a forest [13] in the sense that one can represent a diagram \( D \subset [k] \times [n-k] \) as the bipartite graph with white vertices \([k]\), black vertices \([n-k]\), and an edge between a white \( i \) and black \( j \) whenever \((i,j) \in D\). In [2], we proved Conjecture 2.5 when \( D^\vee \) is a permutation diagram and \( S^D \) is multiplicity-free.

One gets a weaker version of Conjecture 2.5 by comparing degrees. The degree of a codimension \( d \) subvariety \( X \) of \( \text{Gr}(k,n) \) is the integer \( \deg(X) \) defined by \( [X] \sigma_1^{k(n-k)-d} = \deg(X) \sigma_{(k,n-k)} \). Under the Plücker embedding, this gives the usual notion of the degree of a subvariety of projective space, namely the number of points in the intersection of \( X \) with a generic \( d \)-dimensional linear subspace. One can check using Pieri’s rule that \( \deg(\sigma_\lambda) = f^{\lambda^\vee} \), the number of standard Young tableaux of shape \( \lambda^\vee \). This is also \( \dim S^{\lambda^\vee} \). Since degree is additive on cohomology classes, Conjecture 2.5 predicts the following.

**Conjecture 2.6** (Liu). The degree of \( X_D \) is \( \dim S^{D^\vee} \).

Liu proved Conjecture 2.6 when \( D^\vee \) is a permutation diagram, and when \( D^\vee \) has the property that if \((i,j_1),(i,j_2) \in D \) and \( j_1 < j < j_2 \), then \((i,j) \in D \). In light of the assertion of Theorem 2.4 that taking complements in the decomposition of \( S^D \) gives the decomposition of \( S^{D^\vee} \), one may be tempted to gloss over the issue of \( D^\vee \) failing for the classes \([X_D]\), and Conjecture 2.5 can fail for \( D \) while holding for \( D^\vee \).

Suppose \( D = \{(1,1),(2,2),(3,3),(4,4)\} \), with \( k = 4 \) and \( n = 8 \). This is a skew diagram 4321/321. The Specht module \( S^D \) is simply the regular representation of \( S_4 \), with

\[
S^D \simeq S^{1111} \oplus 3S^{2111} \oplus 2S^{22} \oplus 3S^{31} \oplus S^4.
\]

Theorem 2.4 then says

\[
S^{D^\vee} \simeq S^{3333} \oplus 3S^{4332} \oplus 2S^{4422} \oplus 3S^{4431} \oplus S^{444},
\]

so \( \dim S^{D^\vee} = f^{3333} + 3f^{4332} + 2f^{4422} + 3f^{4431} + f^{444} = 24024 \).

On the other hand, an explicit calculation in Macaulay2 shows \( \deg X_D = 21384 \). Therefore Conjectures 2.6 and 2.5 both fail for \( D \).

The discrepancy in degrees is \( 24024 - 21384 = 2640 = f^{4422} \), which hints at how to see this discrepancy more explicitly. Given a \( k \)-subset \( I \) of \([n]\), write \( p_I \) for the corresponding Plücker coordinate on \( \text{Gr}(k,n) \), so \( p_I(A) \) is the minor of \( A \) in columns \( I \). Let \( Y \) be the scheme determined by the vanishing of the Plücker coordinates \( p_{1678}, p_{2578}, p_{3568}, p_{4567} \). These are exactly the Plücker coordinates which vanish on \( X_D \). One can check by computer that \( \text{codim} Y = 4 \), and so \( Y \) is a complete intersection. This implies that

\[
[Y] = \sigma_4^4 = \sigma_{1111} + 3\sigma_{211} + 2\sigma_{22} + 3\sigma_{31} + \sigma_4;
\]

see [5, Section 5.2.1].

Since the four Plücker coordinates cutting out \( Y \) vanish on \( X_D^\circ \), the diagram variety \( X_D \) is contained in \( Y \). However, \( Y \) has another component, namely the Schubert variety which is
the closure of
\[
\begin{pmatrix}
* & * & 1 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 1 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & * & * & 1 & 0 \\
* & * & 0 & 0 & * & * & 0 & 0 
\end{pmatrix}.
\]
This Schubert variety has degree \( \dim S^{(22)} = f^{4422} = 2640 \), which is \( \deg Y - \deg X_D \).

Therefore \[ [X_D] = [Y] - \sigma_{22} = \sigma_{1111} + 3\sigma_{211} + \sigma_{22} + 3\sigma_{31} + \sigma_4. \]

Although \( D = \{ (1, 1), (2, 2), (3, 3), (4, 4) \} \) is not itself a permutation diagram, this counterexample leads directly to one of the form \( XD(w) \). Take \( w = 21436587 \). Then \( D(w) = \{ (1, 1), (3, 3), (5, 5), (7, 7) \} \) can be obtained from \( D \) by permuting rows and columns, and viewing \( D \) in a larger rectangle. Neither of these operations on diagrams affects \( s_D \) or \( [X_D] \), identifying the latter with its pullback along an embedding of \( Gr(k,n) \) into \( Gr(k,n+1) \) or \( Gr(k+1,n+1) \). Thus Conjecture 2.5 can fail for permutation diagrams.

More counterexamples to Conjecture 2.5 can be easily manufactured from this one. For two diagrams \( D_1 \) and \( D_2 \) where \( D_1 \subseteq [a] \times [b] \), define \[ D_1 \cdot D_2 = D_1 \cup \{ (i + a, j + b) : (i, j) \in D_2 \}. \]

Graphically, \( D_1 \cdot D_2 \) is the diagram

\[ 
\begin{array}{c}
D_1 \\
\hline \\
\hline \\
D_2
\end{array}
\]

One can show that \([X_{D_1 \cdot D_2}] = [X_{D_1}][X_{D_2}]\) and similarly that \( s_{D_1 \cdot D_2} = s_{D_1}s_{D_2} \). Therefore if Conjecture 2.5 holds for \( D_1 \) but not \( D_2 \), then it will fail for \( D_1 \cdot D_2 \).

**Remark.** One might naturally wonder about the diagram

\( D' = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) \} \),

and whether Conjecture 2.5 fails for \( D' \). However, trying to repeat the analysis above runs into an immediate problem: the analogue of \( Y \), which is the scheme \( Z \) cut out by

\[ P_{1789(10)}, P_{2689(10)}, P_{3679(10)}, P_{4678(10)}, P_{56789} \]

no longer even has the same codimension as \( X_D \) (thanks to Ricky Liu for pointing this out).

Indeed, \( X_D \) has codimension 5 but \( Z \) contains the codimension 4 Schubert cell

\[ \begin{pmatrix}
* & * & * & 1 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & * & * & 1 & 0 & 0 \\
* & * & 0 & 0 & * & * & 0 & 1 & 0 \\
* & * & 0 & 0 & * & * & 0 & 0 & 1 
\end{pmatrix} \cdot \]

3. Positroid varieties and rank varieties

**Definition 3.1.** An affine permutation is a bijection \( f : \mathbb{Z} \to \mathbb{Z} \) such that

\[ f(i + n) = f(i) + n \]

for all \( i \) and some fixed \( n \). Write \( \tilde{S}_n \) for the set of affine permutations with a particular \( n \).
Note that the image of any set \( \{a, a+1, \ldots, a+n-1\} \) completely determines an affine permutation. Call such an image a window. We will write affine permutations in one-line notation as the image of \([n]: 14825 \) fixes 1, sends 3 to 8, 7 to 9, etc. Members of any window are all distinct modulo \( n \), so \( \sum_{i=1}^{n} f(i) \equiv n(n+1)/2 \pmod n \). Let \( av(f) \) be the integer \( \frac{1}{n} \sum_{i=1}^{n} f(i) - i \).

**Warning.** Affine permutations are usually required to satisfy \( av(f) = 0 \), which ours need not. However, for a fixed \( k \), affine permutations in \( \tilde{S}_n \) satisfying \( av(f) = k \) are in bijection with those satisfying \( av(f) = 0 \) by subtracting \( k \) from each entry in a window. When we refer to constructions on affine permutations that require a Coxeter group structure (e.g. length, reduced words, Stanley symmetric functions), we are implicitly using this isomorphism to transport that structure from the “usual” affine permutation group \( \{ f \in \tilde{S}_n : av(f) = 0 \} \).

The length \( \ell(f) \) of an affine permutation \( f \) is the number of inversions \( i < j, f(i) > f(j) \), provided that we regard any two inversions \( i < j \) and \( i + pn < j + pn \) as equivalent.

**Definition 3.2.** An affine permutation \( f \in \tilde{S}_n \) is bounded if \( i \leq f(i) \leq i + n \) for all \( i \). Let \( \text{Bound}(k, n) \) denote the set of bounded affine permutations in \( \tilde{S}_n \) with \( av(f) = k \).

Any affine permutation \( f \) has a permutation matrix, the \( \mathbb{Z} \times \mathbb{Z} \) matrix \( A \) with \( A_{i, f(i)} = 1 \) and all other entries 0. For any \( i, j \in \mathbb{Z} \), define
\[
|i, j|(f) = \{ p < i : f(p) > j \}.
\]
Thus, \( |i, j|(f) \) is the number of 1’s strictly northeast of \((i, j)\) in the permutation matrix of \( f \), in matrix coordinates.

Fix a basis \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \). We will abuse notation by writing \( \langle X \rangle \) both for the span of the vectors in \( X \), if \( X \subseteq \mathbb{C}^n \), and for the span of vectors \( e_i \) with \( i \in X \), if \( X \subseteq [n] \). If \( X \subseteq [n] \), let \( \text{Proj}_X : \mathbb{C}^n \to \langle X \rangle \) be the projection which fixes those basis vectors \( e_i \) with \( i \in X \) and sends the rest to 0. For integers \( i \leq j \), write \([i, j]\) for \( \{ i, i+1, \ldots, j \} \). We interpret indices of basis vectors modulo \( n \), so that \([i, j]\) \subseteq \( \mathbb{C}^n \) even if \( i, j \) fail to lie in \([1, n] \).

**Definition 3.3** ([9]). Given a bounded affine permutation \( f \in \text{Bound}(k, n) \), the positroid variety \( \Pi_f \subseteq \text{Gr}(k, n) \) is the closure of
\[
\{ V \in \text{Gr}(k, n) : \dim(\text{Proj}_{[i, j]} V) = k - |i, j|(f) \text{ for all } i \leq j \}.
\]

**Theorem 3.4** ([9], Theorem 5.9). The positroid variety \( \Pi_f \subseteq \text{Gr}(k, n) \) is irreducible of codimension \( \ell(f) \).

Knutson–Lam–Speyer also computed the cohomology class of \( \Pi_f \) in terms of affine Stanley symmetric functions. These are a class of symmetric functions indexed by affine permutations introduced by Lam in [11], and we now give a definition.

Let \( \tilde{S}_n^k \) be the set of affine permutations with \( av(f) = k \). Then \( \tilde{S}_n^0 \) is a Coxeter group with simple generators \( s_0, \ldots, s_{n-1} \), where \( s_i \) interchanges \( i + np \) and \( i + 1 + np \) for every \( p \). A reduced word for \( f \in \tilde{S}_n^0 \) is a word \( a_1 \cdots a_{\ell} \) in the alphabet \([0, n-1]\) with \( s_{a_1} \cdots s_{a_{\ell}} = f \) and such that \( \ell \) is minimal with this property. Let \( \text{Red}(f) \) denote the set of reduced words for \( f \).

A reduced word \( a = a_1 \cdots a_{\ell} \) is cyclically decreasing if all the \( a_i \) are distinct, and if whenever \( j \) and \( j + 1 \) appear in \( a \) (modulo \( n \)), \( j + 1 \) precedes \( j \). An affine permutation is cyclically decreasing if it has a cyclically decreasing reduced word. For a partition \( \lambda \), let \( m_\lambda \) be the monomial symmetric function indexed by \( \lambda \).

**Definition 3.5.** The affine Stanley symmetric function of \( f \in \tilde{S}_n^0 \) is
\[
\tilde{F}_f = \sum_{\lambda \vdash \ell(f)} c_{f, \lambda} m_\lambda,
\]
where \( c_{f,\lambda} \) is the number of factorizations \( f^1 \cdots f^{\ell(\lambda)} = f \) where \( \ell(f) = \sum_i \ell(f_i) \), each \( f_i \) is cyclically decreasing, and \( \ell(f_i) = \lambda_i \).

As mentioned above, subtracting \( k \) from each entry of a window for \( f \in \tilde{S}_n^k \) gives an isomorphism \( \tilde{S}_n^k \to \tilde{S}_n^0 \), which we use to extend the definition of \( \tilde{F}_f \) to all of \( \tilde{S}_n \).

**Theorem 3.6** ([9], Theorem 7.1). For \( f \in \text{Bound}(k, n) \), the cohomology class \([\Pi_f] \) is \( \phi(\tilde{F}_f) \).

The ordinary Stanley symmetric functions indexed by members of \( S_n \), introduced by Stanley in [20], are a special case of affine Stanley symmetric functions. To be precise, we can view \( w \in S_n \) as the affine permutation in \( \tilde{S}_n^0 \) sending \( i + pm \) to \( w(i) + pm \) for \( 1 \leq i \leq n \). Then the Stanley symmetric function \( F_w \) of \( w \) is \( \tilde{F}_w \). This is Proposition 5 in [11], but we will simply take it as a definition of \( F_w \). One should be aware, however, that the \( F_w \) defined in [20] is our \( F_{w^{-1}} \).

Now we discuss a subset of positroid varieties whose cohomology classes will turn out, in Section 4, to be represented by ordinary Stanley symmetric functions.

**Definition 3.7** ([1]). A rank set is a finite set of intervals \( M = \{[a_1, b_1], \ldots, [a_k, b_k]\} \) with \( a_i \leq b_i \) positive integers, where all \( a_i \) are distinct and all \( b_i \) are distinct. If \( S \) is a set of positive integers, let \( S(M) \) denote the set of intervals \( S' \in M \) such that \( S' \subseteq S \).

Suppose \( M \) is a rank set with \( b \leq n \) for all \( a, b \in M \), and \#(\( M \)) = \( k \). Coskun [3] defines a closed subvariety \( \Sigma_M \) of \( \text{Gr}(k, n) \) as the closure of the locus

\[
\{ V \in \text{Gr}(k, n) : \dim(V \cap (S \cap T)) = \#(S \cap T)(M) \text{ for } S, T \in M \}. 
\]

Here \( \#(S \cap T)(M) \) is the number of intervals in \( M \) contained in \( S \cap T \), as defined above. \( \Sigma_M \) is called a rank variety.

**Theorem 3.8** ([3], Lemma 3.29). The rank variety \( \Sigma_M \) is irreducible of dimension \( \sum_{S \in M}(\#S - \#S(M)) \).

The variety \( \Sigma_M \) has a useful interpretation in coordinates.

**Lemma 3.9.** Let \( U_M \) be the locus of \( k \)-planes with a basis \( \{v_S : S \in M\} \), indexed by the intervals in \( M \), such that the coefficient of \( e_i \) in \( v_S \) is nonzero if and only if \( i \in S \). Then \( \overline{U}_M = \Sigma_M \).

**Example 3.10.** If \( M = \{[1, 2], [3, 4], [2, 5]\} \) and \( n = 5 \), then

\[
U_M = \left\{ \begin{array}{c}
\text{rowspan} \\
00000
\end{array}
\right\} : \text{every } * \text{ nonzero} = \left\{ \begin{array}{c}
\text{rowspan} \\
00000
\end{array}
\right\}.
\]

Being defined by rank conditions on intersections with interval subspaces, rank varieties should be special instances of positroid varieties. Say \( M = \{[a_1, b_1], \ldots, [a_k, b_k]\} \) is a rank set with \( b_1 < \cdots < b_k \leq n \). Define

\[
\{c_1 < \cdots < c_{n-k}\} = [n] \setminus \{a_1, \ldots, a_k\} \text{ and } \\
\{d_1 < \cdots < d_{n-k}\} = [n] \setminus \{b_1, \ldots, b_k\}.
\]

Let \( f_M \) be the affine permutation mapping \( b_i \) to \( a_i + n \) and \( d_i \) to \( c_i \). Then \( f_M \) is bounded because \( a_i \leq b_i \), which implies \( d_i \leq c_i \). This provides a bijection between rank sets for \( \text{Gr}(k, n) \) and members \( f \) of \( \text{Bound}(k, n) \) such that the subsequence of \( f(1) \cdots f(n) \) with entries in \( [n] \) is increasing.

**Example 3.11.** Take \( M = \{[1, 1], [3, 4], [2, 5]\} \) and \( n = 5 \) as above. Then \( b_1 = 1, b_2 = 4, b_3 = 5 \) and \( a_1 = 1, a_2 = 3, a_3 = 2, \) so \( d_1 = 2, d_2 = 3 \) while \( c_1 = 4, c_2 = 5 \). Hence \( f_M = 64587 \).
Theorem 3.12. The rank variety $\Sigma_M$ is the positroid variety $\Pi_{f_M}$.

Proof. First we show that $\Pi_{f_M} \subseteq \Sigma_M$. To do this we check that for any interval $[r, s] \subseteq [n]$,
\[ \#[r, s](M) = \#[s + 1, n + r - 1](f_M). \]

The following are equivalent:
- $[a_p, b_p] \in [r, s](M)$,
- $a_p \leq s$ and $b_p \geq r$,
- $b_p < s + 1$ and $f_M(b_p) > n + r - 1$,
- $b_p \in [s + 1, n + r - 1](M)$.

This shows that $\#[r, s](M)$ is the number of elements of $[s + 1, n + r - 1](M)$ of the form $b_p$. But in fact every $q \in [s + 1, n + r - 1](M)$ is some $b_p$, because $f_M(q) > n + r - 1 \geq n$ and $1 \leq q \leq s \leq n$ imply $q \in \{b_1, \ldots, b_p\}$.

Now say $V$ is in the open subset of $\Pi_{f_M}$ where
\[ \dim \text{Proj}_{s+1, n+r-1}(M) = k - \#[s + 1, n + r - 1](f_M) \]
for all $r \leq s$. If $[r, s] = [a_i, b_i] \cap [a_j, b_j]$ for some $i$ and $j$, then
\[
\dim(V \cap ([r, s])) = k - \dim \text{Proj}_{s+1, n+r-1}(M) V = \#[s + 1, n + r - 1](f_M) = \#[s, b_p](M).
\]
That is, $V$ is in $\Sigma_M$.

Both $\Pi_{f_M}$ and $\Sigma_M$ are irreducible, so equality will follow if we show they have the same codimension $\ell(f_M)$. By Theorem 3.8,
\[
\text{codim } \Sigma_M = k(n - k) - \sum_{S \in M} (\#S - \#S(M))
\]
\[ = k(n - k) - \sum_{i=1}^{k} (b_i - a_i) + \sum_{S \in M} (\#S(M) - 1) \]
\[ = k(n - k) - \sum_{i=1}^{n} i + \sum_{i=1}^{n-k} d_i + \sum_{i=1}^{n-k} f_M(d_i) + \sum_{S \in M} (\#S(M) - 1) \]
\[ = \sum_{i=1}^{n-k} (k + d_i - f_M(d_i)) + \sum_{S \in M} (\#S(M) - 1). \]

Inversions of $f_M$ come in three types:
1. $b_i < b_j$ with $f_M(b_i) > f_M(b_j)$,
2. $b_j < d_i$ (with $f_M(b_j) > f_M(d_i)$ automatically), and
3. $d_i < b_j < d_i + n$ with $f_M(b_j) > f_M(d_i) + n$.

In particular, there are no inversions just among the entries $f_M(d_i)$.

For fixed $i$, inversions of type (1) correspond to elements of $[a_i, b_i](M) \setminus \{a_i, b_i\}$. Indeed, $[a_i, b_i] \subseteq [a_i, b_i]$ if and only if $b_j \leq b_i$ and
\[ f_M(b_j) - n \geq a_j \geq a_i = f_M(b_i) - n. \]
Hence the total number of inversions of this type is \( \sum_{S \in M} (\#S(M) - 1) \).

It remains to show that \( \sum_{i=1}^{n-k} (k + d_i - f_M(d_i)) \) counts the inversions of types (2) and (3). Alternatively, we can show that of the pairs $(b_j, d_i)$ for fixed $i$, exactly $f_M(d_i) - d_i$ of them
are not inversions. Let $A$ be the set of values $f_M(b_j) - n$ for $(b_j, d_i)$ a noninversion with $i$ fixed, i.e., for $b_j > d_i$ and $f_M(b_j) - n < f_M(d_i)$. Then

$$A = [1, f_M(d_i) - 1] \setminus \{(f_M(d_1), \ldots, f_M(d_{n-k})) \cup \{f_M(b_j) - n : b_j < d_i\}\}.$$ 

Say $g \in S_n$ is the ordinary permutation with $g(d_p) = f_M(d_p)$ and $g(b_p) = f_M(b_p) - n$. We have $f_M(d_1) < f_M(d_2) < \cdots < f_M(d_i)$, so

$$A = [1, f_M(d_i) - 1] \setminus \{(g(d_1), \ldots, g(d_{i-1})) \cup \{g(b_j) : b_j < d_i\}\} = [1, f_M(d_i) - 1] \setminus g([1, d_i - 1]).$$

Since $f_M$ is bounded, $b_j < d_i$ implies $g(b_j) < d_i \leq f_M(d_i)$. This together with $f_M(d_1) < f_M(d_2) < \cdots$ show that $g([1, d_i - 1]) \subseteq [1, f_M(d_i) - 1]$, so $A$ has size $f_M(d_i) - d_i$. □

4. COHOMOLOGY CLASSES OF RANK VARIETIES

Let $Λ$ be the ring of symmetric functions over $\mathbb{Z}$, and $\phi : Λ \rightarrow H^*(\text{Gr}(k, n), \mathbb{Z})$ the ring homomorphism sending the Schur function $s_λ$ to the Schubert class $σ_λ$, or to 0 if $λ$ is not contained in a $k \times (n - k)$ rectangle.

Coskun gives a recursive rule to calculate the cohomology class of a rank variety [3]. Since rank varieties are posisotropic varieties, Theorem 3.6 gives a more direct answer, namely that $[\Sigma_M] = φ(F_M)$. The goal of this section is to show that $[\Sigma_M]$ is actually represented by an ordinary Stanley symmetric function.

**Theorem 4.1.** For any rank variety $\Sigma_M \subseteq \text{Gr}(k, n)$, there is a permutation $w_M$ such that $[\Sigma_M] = φ(F_w)$.

We will not prove this theorem by showing that $F_M$ is an ordinary Stanley symmetric function, since this is not true in general, as the next example shows.

**Example 4.2.** Let $M = \{[1, 1], [3, 3]\}$ with $\Sigma_M \subseteq \text{Gr}(2, 4)$. Then $f_M = 5274$, but $F_{5274} = s_{22} + s_{31} - s_4$ is not equal to any $F_w$, since ordinary Stanley symmetric functions are Schur-positive. On the other hand, $F_{31524} = s_{22} + s_{31}$, and $φ(F_{31524}) = σ_{22} = φ(F_{5274})$.

Instead, our strategy will be to replace a rank set $M$ with a new rank set $M'$ in such a way that the truth of Theorem 4.1 for $Σ_{M'}$ implies it for $Σ_M$, and so that after enough replacements we end up with a rank variety $Σ_N = Π_f$ where $F_f$ is obviously an ordinary Stanley symmetric function.

Specifically, if $M$ is a rank set, define a new rank set

$$κ(M) = \{[a, b + 1] : [a, b] \in M\}.$$ 

We call the operation $κ$ stretching. Say that $M$ is stretched if whenever $S, T \in M$, we have $\min(S) < \max(T)$. For any $M$, there is an $m$ so that $κ^m(M)$ is stretched. Given a rank variety $Σ_M$ in $\text{Gr}(k, n)$, we always interpret $Σ_{κ^m(M)}$ as a subvariety of $\text{Gr}(k, n + 1)$. Let $τ$ be the affine permutation $τ(i) = i + 1$.

**Lemma 4.3.** Suppose $M$ is stretched and $Σ_M \subseteq \text{Gr}(k, n)$. Let $b$ be minimal such that $[a, b] \in M$ for some $a$, and let $y = f_M(b - 1)$. Then $τ^{-y+1}f_M^yτ^{b-2}$ restricted to $[n]$ is an ordinary permutation $w \in S_n$, and $[Σ_M] = φ(F_w)$.

**Proof.** We first check that $f_M$ maps $[b - 1, n + b - 2]$ into $[y, y + n - 1]$, which shows that $τ^{-y+1}f_M^yτ^{b-2}$ is an ordinary permutation on $[n]$. Suppose $i \in [b - 1, n + b - 2]$.

- If $n < i \leq n + b - 2$, then $y \leq n < i \leq f_M(i)$ since $f_M$ is bounded. On the other side, $f_M(i) < f_M(n + b - 1) = y + n$, because the entries $f_M(n+1), \ldots, f_M(n+b-1)$ come in increasing order by the choice of $b$. 


If $b - 1 \leq i \leq n$ and $f_M(i) > n$, then certainly $y \leq n \leq f_M(i)$. On the other side, $f_M(i) - n$ is the left endpoint of an interval in $M$, while $b$ is the right endpoint, so $f_M(i) - n < b$ since $M$ is stretched. Hence $f_M(i) \leq b - 1 + n \leq f_M(b - 1) + n = y + n$. But we cannot have $f_M(i) = y + n$ since $f_M^1(y + n) = b - 1 + n > n$ ($b = 1$ is impossible because $M$ is stretched), so $f_M(i) \leq y + n - 1$.

If $b - 1 \leq i \leq n$ and $f_M(i) \leq n$, then $f_M(i) \leq y + n - 1$ is clear. On the other side, the entries of $f_M$ lying in $[n]$ come in increasing order, so $y = f_M(b - 1) \leq f_M(i)$.

Let $w$ be the restriction of $\tau^{-y+1} f_M \tau^{b-2}$ to $[n]$. By definition,

$$\tilde{F}_{\tau^{-y+1} f_M \tau^{b-2}} = F_w.$$  

The isomorphisms $\tilde{S}_n^r \rightarrow S_n^0$ are given by left multiplying by $\tau^r$, so $\tilde{F}_f = \tilde{F}_f$ for any $f$. On the other hand, $\tau^{-1} s_i \tau = s_{i-1}$, and so conjugation by $\tau$ gives a bijection between the cyclically decreasing factorizations of $f$ and those of $\tau^{-1} f \tau$ which preserves the lengths of the factors. This means $\tilde{F}_{\tau^{-1} f \tau} = \tilde{F}_f$, so

$$\tilde{F}_{\tau f} = \tilde{F}_{\tau^r(\tau^{-1} f \tau)} = \tilde{F}_{\tau^{-1} f \tau} = \tilde{F}_f.$$

Now we are done by Theorem 3.6, since $\tilde{F}_{f_M} = \tilde{F}_{\tau^{-y+1} f_M \tau^{b-2}} = F_w$.

We now give a precise relationship between the classes of $\Sigma_M$ and $\Sigma_{\kappa M}$. Given a map $\iota : \text{Gr}(k, n) \rightarrow \text{Gr}(k, n + 1)$, we get a pullback

$$\iota^* : H^p(\text{Gr}(k, n + 1), \mathbb{Z}) \rightarrow H^p(\text{Gr}(k, n), \mathbb{Z})$$

and a pushforward

$$\iota_* : H^p(\text{Gr}(k, n), \mathbb{Z}) \rightarrow H^{2k+p}(\text{Gr}(k, n + 1), \mathbb{Z}),$$

the latter obtained from the pushforward on homology via Poincaré duality. These two maps on cohomology are related by the projection formula

$$\iota_*(\beta \iota^*(\alpha)) = \alpha \iota_*(\beta).$$

More information about the cohomology class of a variety and the pullback and pushforward can be found in [6, Appendix B].

**Theorem 4.4.** Let $\iota : \text{Gr}(k, n) \hookrightarrow \text{Gr}(k, n + 1)$ be the inclusion induced by an inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$. Then $\iota^*[\Sigma_{\kappa M}] = [\Sigma_M]$.

Before proving Theorem 4.4, we explicitly describe an algorithm which, given a rank variety $\Sigma_M \subseteq \text{Gr}(k, n)$, produces a permutation $w_M$ such that $[\Sigma_M] = \phi(F_{w_M})$.

**Step 1.** Choose $m$ such that $\kappa^m M$ is stretched, the minimal (positive) such $m$ being

$$m = \max(0, 1 + \max(\min(S) - \max(T))).$$

**Step 2.** Find $b$ minimal such that $[a, b] \in \kappa^m M$, and set $y = f_{\kappa^m M}(b - 1)$.

**Step 3.** Define $w_M \in S_n$ by $w_M(i) = f_{\kappa^m M}(b - 2 + i) - y + 1$.

The correctness of this algorithm follows from Lemma 4.3 and Theorem 4.4; for more details, see the proof of Theorem 4.1 below.
Example 4.5. Let \( M = \{[1, 3], [3, 6], [4, 5]\} \), so \( \Sigma_M \subseteq \text{Gr}(3, 6) \) has a dense open subset consisting of rowspans of matrices with the form
\[
\begin{bmatrix}
* & * & 0 & 0 & 0 \\
0 & 0 & * & * & * \\
0 & 0 & 0 & * & *
\end{bmatrix}
\]
The minimal \( m \) such that \( \kappa^m M \) is stretched is \( m = 2 \), and \( \kappa^2 M = \{[1, 5], [3, 8], [4, 7]\} \). We have \( f_{\kappa^2 M} = 256798(12)(11) \). The minimal \( b \) with \([a, b] \in \kappa^2 M \) is \( b = 5 \), and then \( y = f_{\kappa^2 M}(b - 1) = 7 \).
Write out more entries of \( f_{\kappa^2 M} \), demarcating the window from 1 to \( n \) with vertical bars and denoting negative numbers with horizontal bars:
\[
f_{\kappa^2 M} = \cdots 63211043|256798(12)(11)|0(10)|13(14)|15(17)|16(18)|19(19) \cdots
\]
Shift the window so that its leftmost entry is in position \( b - 1 = 4 \):
\[
f_{\kappa^2 M} = \cdots 63211043256|798(12)(11)|0(10)|13(14)|15(17)|16(18)|19(19) \cdots
\]
The entries of the window now fill out exactly the interval \([7, 14] \). That is, \( \tau^{-6} f_{\kappa^2 M} \tau^3 \) restricted to \([8] \) is an ordinary permutation, namely \( w_M = 13265478 \).

Theorem 4.4 will be straightforward after the next two lemmas. Say \( M = \{[a_1, b_1], \ldots, [a_k, b_k]\} \) with \( b_1 < \cdots < b_k \). Define \( h : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \) by
\[
h(e_i) = \begin{cases} e_{b_j+1} - e_{b_{j+1}+1} & \text{if } i = b_j + 1 \text{ for } j < k \\ e_i & \text{otherwise} \end{cases}
\]
Let \( \iota : \text{Gr}(k, n) \hookrightarrow \text{Gr}(k, n + 1) \) be the inclusion induced by \( h \).

**Lemma 4.6.** With \( \iota \) as above, \( \Sigma_M \cap \iota(\Sigma_M) = \iota(\Sigma_M) \).

**Proof.** We must show that
\[
\dim(V \cap \langle S \cap T \rangle) \geq \#(S \cap T)(M)
\]
for all \( S, T \in M \) is equivalent to
\[
\dim(h(V) \cap \langle \kappa S \cap \kappa T \rangle) \geq \#(\kappa S \cap \kappa T)(\kappa M)
\]
for all \( S, T \in M \). In fact, it will turn out that for each \( i, j \), the left (resp. right) side of the first inequality is equal to the left (resp. right) side of the second.

It is not hard to check that \([a, b] \subseteq S \cap T \) (with \( a \leq b \)) if and only if \([a, b + 1] \subseteq \kappa S \cap \kappa T \), and so \( \#(S \cap T)(M) = \#(\kappa S \cap \kappa T)(\kappa M) \). Since \( h \) is injective,
\[
\dim(V \cap \langle S \cap T \rangle) = \dim(h(V) \cap h(\langle S \cap T \rangle)),
\]
so we will be done if we show that \( \text{im}(h) \cap \langle \kappa S \cap \kappa T \rangle = h(\langle S \cap T \rangle) \).

Suppose \( p \in S \cap T \), and write \( S = [a_i, b_i], T = [a_j, b_j] \). If \( p \neq b_q + 1 \) for any \( q < k \), then \( h(e_p) = e_p \in \langle \kappa S \cap \kappa T \rangle \), so assume \( p = b_q + 1 \). Then \( b_q + 1 \leq \min(b_i, b_j) \). Since \( b_1 < b_2 < \cdots \), this means \( b_{q+1} \leq \min(b_i, b_j) \). Therefore \( b_{q+1} + 1 \in \kappa S \cap \kappa T \), so \( h(e_{b_{q+1}}) = e_{b_{q+1}} - e_{b_{q+1}+1} \in \langle \kappa S \cap \kappa T \rangle \).

Hence
\[
h(\langle S \cap T \rangle) \subseteq \langle \kappa S \cap \kappa T \rangle \cap \text{im} h. \tag{2}
\]

Let \( \alpha \in (\mathbb{C}^{n+1})^* \) be the linear functional defined by \( \alpha(\sum_p e_p e_p) = e_{b_1+1} + \cdots + e_{b_k+1} \). Then \( \ker \alpha = \text{im} h \). In particular, \( e_{b_{q+1}} \notin \text{im} h \) since \( \alpha(e_{b_{q+1}}) = 1 \). If \( \kappa S \cap \kappa T \) is non-empty, then it contains \( b_1 + 1 \) or \( b_2 + 1 \), so \( \langle \kappa S \cap \kappa T \rangle \cap \text{im} h \) is properly contained in \( \langle \kappa S \cap \kappa T \rangle \). But then
\[
\dim(\langle \kappa S \cap \kappa T \rangle \cap \text{im} h) < \dim \langle \kappa S \cap \kappa T \rangle = 1 + \dim h(\langle S \cap T \rangle),
\]
so the containment (2) must be an equality. \( \square \)
Write \( p \) for the space of linear maps \( \mathbb{C}^n \to \mathbb{C}^n \) sending \( \langle e_1, \ldots, e_k \rangle \) into itself. The tangent space to \( \text{Gr}(k, n) \) at \( \langle e_1, \ldots, e_k \rangle \) is \( \mathfrak{gl}_n/p \cong \text{Hom}(\langle e_1, \ldots, e_k \rangle, \mathbb{C}^n/\langle e_1, \ldots, e_k \rangle) \). More generally, the tangent space to \( \text{Gr}(k, n) \) at \( V \) is \( \text{Hom}(V, \mathbb{C}^n/V) \). With this identification, the differential of the quotient map \( q : \text{GL}_n(\mathbb{C}) \to \text{Gr}(k, n) \) sending \( A \) to the span of its first \( k \) rows is

\[
dq_A(\phi) = \pi_q(A) \circ \phi \circ A^{-1}|_{q(A)},
\]

where \( \phi \in \mathfrak{gl}_n \) and \( \pi_V : \mathbb{C}^n \to \mathbb{C}^n/V \) is the quotient map. Because of our convention of writing members of \( \text{Gr}(k, n) \) as row spans, \( A^{-1} \) should be interpreted as the linear transformation sending \( e_i \) to the \( i \)th row of \( A^{-1} \).

**Lemma 4.7.** The intersection \( \Sigma_{k,n} \cap \iota(\text{Gr}(k, n)) \) is generically transverse.

**Proof.** Let \( \alpha \) be as in the proof of Lemma 4.6. The tangent space to \( \iota(\text{Gr}(k, n)) = \text{Gr}(k, \ker \alpha) \) at \( V \) is \( \text{Hom}(V, (\ker \alpha)/V) \). As \( V \subseteq \ker \alpha \), the functional \( \alpha \) descends to \( \tilde{\alpha} \in (\mathbb{C}^{n+1}/V)^* \), so we can also write the tangent space \( T_V \text{Gr}(k, \ker \alpha) \) as \( \{ \phi \in \text{Hom}(V, \mathbb{C}^{n+1}/V) : \tilde{\alpha} \circ \phi = 0 \} \).

Say the first \( k \) rows of \( A \in \text{GL}_{n+1}(\mathbb{C}) \) are \( v_1, \ldots, v_k \). Define a spanning set \( \{ \phi_{ij} : i \leq k \} \) for \( T_{q(A)} \text{Gr}(k, n+1) \) by

\[
\phi_{ij}(v_p) = \begin{cases} e_j + V & \text{if } p = i \\ 0 & \text{otherwise} \end{cases}.
\]

Define

\[
Z = \{ A \in \text{GL}_{n+1}(\mathbb{C}) : A_{ij} = 0 \text{ if } i \leq k \text{ and } j \notin [a_i, b_i + 1] \}.
\]

Then \( q(Z) \) contains a dense open subset \( U \) of \( \Sigma_{k,n} \), where \( q : \text{GL}_{n+1}(\mathbb{C}) \to \text{Gr}(k, n+1) \) sends \( A \) to the span of its first \( k \) rows.

Say \( \psi_{ij} \in T_A Z \) is the map sending \( e_i \) to \( e_j \) and all other \( e_p \)'s to 0, for \( i \leq k \) and \( j \in [a_i, b_i+1] \). Using equation (3), \( dq_A(\psi_{ij}) = \phi_{ij} \). Therefore if \( V \in U \subseteq \Sigma_{k,n} \), then the tangent space \( T_V \Sigma_{k,n} \) contains \( \phi_{ij} \) whenever \( j \in [a_i, b_i + 1] \).

If \( j \) is not equal to any \( b_p + 1 \), then \( \tilde{\alpha} \phi_{ij} = 0 \), so \( \phi_{ij} \in T_V \text{Gr}(k, \ker \alpha) \). If \( j = b_p + 1 \), write

\[
\phi_{i,b_p+1} = (\phi_{i,b_p+1} - \phi_{i,b_i+1}) + \phi_{i,b_i+1}.
\]

The first summand is in \( T_V \text{Gr}(k, \ker \alpha) \), and the second is in \( T_V \Sigma_{k,n} \). Thus \( T_V \text{Gr}(k, \ker \alpha) + T_V \Sigma_{k,n} = T_V \text{Gr}(k, n+1) \), and so \( \Sigma_{k,n} \) and \( \text{Gr}(k, \ker \alpha) \) intersect transversely on \( U \).

**Proof of Theorem 4.4.** Since \( \iota \) maps distinct Schubert varieties in \( \text{Gr}(k, n) \) onto distinct Schubert varieties in \( \text{Gr}(k, n+1) \), \( \iota_* \) is injective. Therefore it suffices to show that

\[
\iota_* \iota^* [\Sigma_{k,n}] = \iota_* [\Sigma_M].
\]

The right side here is \( [\iota(\Sigma_M)] \) since \( \iota \) is an embedding. By the projection formula, the left side is \( [\Sigma_{k,n}] [\iota(\text{Gr}(k, n))] \). Lemmas 4.6 and 4.7 below show that for a suitable choice of \( \iota_* \), \( \Sigma_{k,n} \) and \( \iota(\text{Gr}(k, n)) \) intersect generically transversely in \( \iota(\Sigma_M) \), so we are done.

**Proof of Theorem 4.1.** Having fixed \( k \), write \( \phi_n \) for the map \( \phi : \Lambda \to H^*(\text{Gr}(k, n), \mathbb{Z}) \), and likewise \( \iota_n \) for the inclusion \( \text{Gr}(k, n) \hookrightarrow \text{Gr}(k, n+1) \). The pullback \( \iota_n^* : H^*(\text{Gr}(k, n+1), \mathbb{Z}) \to H^*(\text{Gr}(k, n), \mathbb{Z}) \) sends \( \sigma_{\lambda} \) to \( \sigma_{\lambda} \) if \( \lambda_1 \leq n \), and to 0 otherwise. Thus the diagram

\[
\begin{array}{ccc}
H^*(\text{Gr}(k, n), \mathbb{Z}) & \xrightarrow{\phi_n} & H^*(\text{Gr}(k, n+1), \mathbb{Z}) \\
\downarrow \phi_n & & \downarrow \phi_{n+1} \\
\Lambda & \xrightarrow{\iota_n} & \Lambda
\end{array}
\]

commutes.
Say $\Sigma_M \subseteq \text{Gr}(k, n)$, and take $m$ large enough that $\kappa^m M$ is stretched. Then Lemma 4.3 shows that $[\Sigma_{\kappa^m M}] = \phi_{n+m}(F_w)$. By Theorem 4.4,

$$[\Sigma_M] = i_n^{*} \cdots i_{n+m-1}^{*} [\Sigma_{\kappa^m M}] = i_n^{*} \cdots i_{n+m-1}^{*} \phi_{n+m}(F_w) = \phi_n(F_w).$$

\[ \square \]

5. DEGENERATIONS OF RANK VARIETIES

Let $\phi_{t,i \to j}$ be the linear transformation sending $e_i$ to $te_i + (1-t)e_j$. For $t \neq 0$, the varieties $\phi_{t,i \to j} \Sigma_M$ are all isomorphic, so they form a flat family [4, Proposition III-56]. The flat limit $\lim_{t \to 0} \phi_{t,i \to j} \Sigma_M$ then exists as a scheme [7, Proposition 9.8]. The key fact for us is that $\Sigma_M$ and $\lim_{t \to 0} \phi_{t,i \to j} \Sigma_M$ have the same Chow ring class, hence the same cohomology class. In this section we will show that for an appropriate choice of $M$, $\lim_{t \to 0} \phi_{t,i \to j} \Sigma_M$ contains the diagram variety $X_{D(w)}$ as an irreducible component. Other authors have used these degenerations or similar ones to calculate cohomology classes or K-theory classes of subvarieties of Grassmannian, including Coskun [3] and Vakil [21].

Define an operator $C_{i \to j}$ on matrices of a fixed size as follows:

$$(C_{i \to j} A)_{pq} = \begin{cases} A_{pi} & \text{if } q = j \text{ and } A_{pj} = 0 \\ 0 & \text{if } q = i \text{ and } A_{pj} = 0 \\ A_{pq} & \text{otherwise.} \end{cases}$$

For example,

$$C_{1 \to 2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & -7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 3 & -7 \\ 0 & 0 \end{bmatrix}.$$  

Sometimes we also apply $C_{i \to j}$ to $k$-planes, or sets of $k$-planes. Strictly speaking this is ill-defined, since it can happen that rowspan $A = \text{rowspan } A'$ but rowspan $C_{i \to j} A \neq \text{rowspan } C_{i \to j} A'$, so we only use this notation when the $k$-planes are represented by specific matrices.

**Lemma 5.1.** Say $F$ is any subset of $[k] \times [n-k]$ and $U$ is the set of $k$-planes $\text{rowspan}(A)$ where $A_{ij} = 0$ whenever $(i, j) \in F$. If $\text{rowspan } A \in U$ and $C_{i \to j} A$ has rank $k$, then $\text{rowspan } C_{i \to j} A \in \lim_{t \to 0} \phi_{t,i \to j} U$.

**Proof.** Given $i, j$, define a matrix $A(t)$ by

$$A(t)_{pq} = \begin{cases} t^{-1}a_{pq} & \text{if } q = i \text{ and } a_{pj} \neq 0 \\ a_{pq} - (t^{-1} - 1)a_{pi} & \text{if } q = j \text{ and } a_{pj} \neq 0 \\ a_{pq} & \text{otherwise.} \end{cases}$$

Then $\text{rowspan } A(t) \in U$ for $t \neq 0$, and $\lim_{t \to 0} \phi_{t,i \to j} \text{rowspan } A(t) = \text{rowspan } C_{i \to j} A$. \[ \square \]

Rank varieties and diagram varieties both have dense open subsets to which Lemma 5.1 applies. We will apply this lemma to rank varieties, but it has an interesting interpretation for diagram varieties as well. Define $C_{i \to j}$ on diagrams as on matrices (e.g. by viewing diagrams as 0,1-matrices). Lemma 5.1 shows that $X_{C_{i \to j} D} \subseteq \lim_{t \to 0} \phi_{t,i \to j} X_D$, which implies that $[X_D] - [X_{C_{i \to j} D}]$ is a nonnegative linear combination of Schubert classes. On the other hand, James and Peel [8] showed that for the diagram Specht modules $S^D$, there is always an $S^D[E]$-equivariant injection $S^C_{i \to j} D \hookrightarrow S^D$. Equivalently, $s_D - s_{C_{i \to j} D}$ is a nonnegative linear
combination of Schur functions. A more powerful version of this connection is important in Liu’s proofs of several cases of Conjectures 2.5 and 2.6 in [13].

Given a permutation \( w \in S_n \), define a rank set \( M(w) = \{ [w(i), i + n] : 1 \leq i \leq n \} \), so \( \Sigma_{M(w)} \subseteq \text{Gr}(n, 2n) \). Then

\[
f_{M(w)} = (n + 1) \cdots (2n)(w(1) + 2n) \cdots (w(n) + 2n) = (w \times 12 \cdots n)^{−n}.
\]

Here, for \( w \in S_n \) and \( v \in S_m \), \( w \times v \) is the permutation in \( S_{n+m} \) sending \( i \) to \( w(i) \) if \( i \leq n \) and to \( v(i - n) + n \) otherwise. Thus \( \tilde{F}_{M(w)} = \tilde{F}_{w \times 12 \cdots n} = F_{w \times 12 \cdots n} = F_w \). This shows that \( \Sigma_{M(w)} = \phi(F_w) \).

We also associate a diagram to each permutation.

**Definition 5.2.** The diagram of \( w \in S_n \) is

\[
D(w) = \{ (i, w(j)) \in [n] \times [n] : i < j, w(i) > w(j) \}.
\]

**Example 5.3.** If \( w = 24153 \), then

\[
\begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

Here we are using \( \circ \) for points of \([n] \times [n]\) in the diagram and \( \cdot \) for points not in it. We also use matrix coordinates, meaning that \((1, 1)\) is at the upper left.

Define

\[
C_w = C_{n+1 \to w(1)} \circ C_{n+2 \to w(2)} \circ \cdots \circ C_{2n \to w(n)}
\]

and

\[
\phi_{t, w} = \phi_{t, n+1 \to w(1)} \circ \phi_{t, n+2 \to w(2)} \circ \cdots \circ \phi_{t, 2n \to w(n)}.
\]

We will show that \( \lim_{t \to 0} \phi_{t, w} \Sigma_{M(w)} \) contains the diagram variety \( X_{D(w)} \) as an irreducible component. First we give an explicit example of this degeneration.

**Example 5.4.** Take \( n = 5 \) and \( w = 24153 \), so \( M(w) = \{ [2, 6], [4, 7], [1, 8], [5, 9], [3, 10] \} \). Then \( \Sigma_{M(w)} \) contains

\[
\begin{bmatrix}
0 & * & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * & * \\
\end{bmatrix}
\]

Here \( C_w = C_{6 \to 2}C_{7 \to 4}C_{8 \to 1}C_{9 \to 5}C_{10 \to 3} \), and, for example,

\[
C_{7 \to 4} \Sigma_{M(w)} \supseteq \begin{bmatrix}
0 & * & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * & * \\
\end{bmatrix}
\]

In total,

\[
C_w \Sigma_{M(w)} \supseteq \begin{bmatrix}
0 & * & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * \\
\end{bmatrix}
\]
Notice that the 0’s forced in columns 1 through 5 of these matrices form the diagram $D(24153)$. Columns 6 through 10 do not quite form an identity matrix, but at least on the open subset where this submatrix is invertible, we can clear out the underlined entries below the diagonal. Crucially, whenever $(i, j + n)$ needs to be cleared, row $j$ of $D(24153)$ contains row $i$, which means this clearing out does not affect the pattern of 0’s and $*$’s in columns 1 through 5. Hence,

$$C_w \Sigma_{M(w)} \supseteq \left\{ \text{rowspan} \begin{pmatrix} 0 & * & * & * & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 1 & 0 & 0 \\ * & 0 & * & * & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} = X_{D(24153)}^\circ.$$

By Lemma 5.1, this shows that $X_{D(24153)} \subseteq \lim_{t \to 0} \phi_t M \Sigma_{M(w)}$.

**Theorem 5.5.** The flat limit $\lim_{t \to 0} \phi_t M \Sigma_{M(w)}$ contains $X_{D(w)}$ as an irreducible component.

**Proof.** Let $DM(w)$ be the diagram $\{(i, j) : 1 \leq i \leq n$ and $w(i) \leq j \leq i + n\}$. As in Example 5.4, it is enough to show that $C_w DM(w)$ has the following two properties:

(a) If $j \leq n$, then $(i, j) \in C_w DM(w)$ if and only if $(i, j) \notin D(w)$.

(b) If $j > n$, then $(i, j) \in C_w DM(w)$ implies row $j - n$ of $D(w)$ contains row $i$ of $D(w)$.

Say $j \leq n$. Then $(i, j) \in C_w DM(w)$ if and only if $j \geq w(i)$ or $w^{-1}(j) + n \leq i + n$, if and only if $(i, j) \notin D(w)$.

Say $j > n$. Then $(i, j) \in C_w DM(w)$ if and only if $j \leq i + n$ and $w(i) \leq w(j - n)$. That is, $w$ has an inversion in positions $j - n < i$ (or they are the same position). It is easy to check that this implies row $j - n$ of $D(w)$ contains row $i$.

Since $\lim_{t \to 0} \phi_t M \Sigma_{M(w)} = [\Sigma_{M(w)}]$, an immediate corollary is an upper bound on $[X_{D(w)}]$.

**Theorem 5.6.** $\phi(F_w) - X_{D(w)}$ is a nonnegative combination of Schubert classes.

However, this difference of classes can be nonzero. Indeed, the counterexample $D = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ to Conjecture 2.5 discussed in Section 2 provides an example. Take $w = 21436587$. Then $D(w) = \{(1, 1), (3, 3), (5, 5), (7, 7)\}$ can be obtained from $D$ by permuting rows and columns, and viewing $D$ in a larger rectangle. Neither of these operations on diagrams affects $s_D$ or $[X_D]$, identifying the latter with its pullback along an embedding of $Gr(k, n)$ into $Gr(k, n + 1)$ or $Gr(k + 1, n + 1)$. Thus Conjecture 2.5 also fails for permutation diagrams.

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