ON EXTENSION TO FOURIER TRANSFORMS

Vladimir Lebedev

Abstract. For \( n \)-point sets \( K \) in an LCA group \( \Gamma \) we obtain an estimate for the norm of “the best” extension operator from the space \( l^\infty(K) \) of bounded functions on \( K \) to the space \( A(\Gamma) \) of Fourier transforms. As a simple consequence our estimate implies that if \( K \) is an infinite closed subset of \( \Gamma \) then there does not exist a bounded linear extension operator from the space \( C_0(K) \) of continuous functions on \( K \) vanishing at infinity to \( A(\Gamma) \). The latter result generalizes a result by Graham who considered the case of compact subsets \( K \).

Key words and phrases. Fourier algebra, Helson sets, Sidon sets, extension operators.

2020 Mathematics Subject Classification. Primary 43A20, 43A25, 43A46, 47B38.

1. Introduction and statement of results

Let \( G \) be a locally compact abelian (LCA) group and \( \Gamma \) its dual. We consider the space \( A(\Gamma) \) of Fourier transforms, i.e., the space of all functions \( f \) of the form \( f = \hat{\lambda} \), where \( \lambda \in L^1(G) \). The norm on \( A(\Gamma) \) is defined by \( \|f\|_{A(\Gamma)} = \|\lambda\|_{L^1(G)} \). Here, \( L^1(G) \) is the space of all complex-valued functions on \( G \) integrable with respect to the Haar measure \( m_G \) on \( G \) and \( \|\lambda\|_{L^1(G)} = \int_G |\lambda(g)| \, dm_G(g) \). Note that \( \| \cdot \|_{A(\Gamma)} \) does not depend on the normalization of \( m_G \). Clearly, \( A(\Gamma) \) is a Banach space.

For a closed set \( K \subseteq \Gamma \) let \( C_0(K) \) stand for the space of all complex-valued continuous functions \( f \) on \( K \) vanishing at infinity. It is a Banach space with respect to the norm \( \|f\|_{C_0(K)} = \sup_{\gamma \in K} |f(\gamma)| \). We recall that a function \( f \) on \( K \) is said to vanish at infinity if for every \( \varepsilon > 0 \) the set \( \{ \gamma \in K : |f(\gamma)| > \varepsilon \} \) is contained in some compact set. If \( K \) is compact, then \( C_0(K) \) coincides with the space \( C(K) \) of continuous functions on \( K \). We invariably use the notation \( C_0(K) \) in both non-compact and compact cases. We have \( A(\Gamma) \subseteq C_0(\Gamma) \) and \( \|f\|_{C_0(\Gamma)} \leq \|f\|_{A(\Gamma)} \).
Closed sets \( K \subseteq \Gamma \) with the property that each function in \( C_0(K) \) is a restriction to \( K \) of some function in \( A(\Gamma) \), i.e., sets \( K \) satisfying
\[
A(\Gamma)|_K = C_0(K),
\] (1)
are of constant interest in harmonic analysis (we provide related references and a brief discussion on terminology at the end of the introduction). It is well-known that each infinite LCA group has a nontrivial, i.e., infinite, subset \( K \) satisfying (1) [5, Theorems 5.6.6 and 5.7.5]. Certainly, condition (1) just means that for every function \( f \in C_0(K) \) there exists an \( F \in A(\Gamma) \) which coincides with \( f \) on \( K \); one naturally calls \( F \) an extension of \( f \). At the same time, Graham [1] showed that if \( K \) is an infinite compact subset of \( \Gamma \) then this extension can never be realized by a bounded linear operator, namely he showed that if \( K \) is an infinite compact subset of \( \Gamma \), then there does not exist a bounded linear operator \( E : C(K) \to A(\Gamma) \) such that
\[
(Ef)|_K = f
\] (2)
for all \( f \in C(K) \).

It is worth noting that the proof of this result in [1] uses tensor algebras.

Generally, given linear spaces \( X(K) \) and \( Y(\Gamma) \) of functions on \( K \) and \( \Gamma \) respectively, we call a linear operator \( E : X(K) \to Y(\Gamma) \) an extension operator if (2) holds for all \( f \in X(K) \).

A natural question related to the result of Graham is whether a similar statement holds in the case of non-compact closed sets \( K \). For instance, let \( G \) and \( \Gamma \) be the circle group \( \mathbb{T} \) and the group of integers \( \mathbb{Z} \), respectively. Consider the subset \( K = \{2^j, j = 0, 1, 2, \ldots \} \) of \( \mathbb{Z} \). Being a Sidon set, \( K \) has the property that for each sequence \( \{c_j, j = 0, 1, 2, \ldots \} \) convergent to zero there exists a function \( g \in L^1(\mathbb{T}) \) with \( \hat{g}(2^j) = c_j, j = 0, 1, 2, \ldots \). Thus, condition (1) holds. Does there exist a bounded linear extension operator from \( C_0(K) \) to \( A(\mathbb{Z}) \)?

The results of this note are as follows. For a set \( K \) let \( l^\infty(K) \) denote the (Banach) space of all complex-valued bounded functions \( f \) on \( K \) with \( \|f\|_{l^\infty(K)} = \sup_{\gamma \in K} |f(\gamma)| \). Theorem 1 below is of a quantitative character; it shows, in particular, that if \( K \) is an \( n \)-point subset of \( \Gamma \), then the norm of any extension operator from \( l^\infty(K) \) to \( A(\Gamma) \) is inevitably large as soon as \( n \) is large. As a simple and natural consequence this implies Theorem 2 that establishes non-existence of a bounded linear extension operator in the general case of closed sets \( K \subseteq \Gamma \), not necessarily compact ones (thus, in particular,
the answer to the above question about the set \( \{2^j, j = 0, 1, 2, \ldots \} \subseteq \mathbb{Z} \) is negative).

Given a finite set \( K \subseteq \Gamma \), define
\[
\alpha_\Gamma(K) \overset{\text{def}}{=} \inf\{\|\mathcal{E}\| : \mathcal{E} \text{ is a linear extension operator from } l^\infty(K) \text{ to } A(\Gamma)\}.
\]

**Theorem 1.** Let \( \Gamma \) be an LCA group. Let \( K \) be an \( n \)-point subset of \( \Gamma \). Then \( \sqrt{n/2} \leq \alpha_\Gamma(K) \leq \sqrt{n} \).

**Theorem 2.** Let \( \Gamma \) be an LCA group. Let \( K \) be an infinite closed subset of \( \Gamma \). Then, there does not exist a bounded linear extension operator from \( C_0(K) \) to \( A(\Gamma) \).

For basic results on sets satisfying (1) we refer the reader to [2–5]. We note that some authors ([2] and [3]) use the term “Helson set” for closed sets \( K \) with (1), in which case Sidon sets are just Helson subsets of a discrete group, while authors of earlier works use the term “Helson set” referring only to compact sets which is the case with the work [1] by Graham (as well as with the book [5]). To avoid confusion, in this note we refrain from using any terminology related to condition (1).

## 2. Proofs of the Theorems

We write the Fourier transform \( \hat{\lambda} \) of \( \lambda \in L^1(G) \) as
\[
\hat{\lambda}(\gamma) = \int_G (-g, \gamma) \lambda(g) dm_G(g), \quad \gamma \in \Gamma,
\]
where \((g, \gamma)\) stands for the value of the character \( \gamma \in \Gamma \) at the point \( g \in G \). We actually use \( dx \) instead of \( dm_G(x) \) in integrals. By \( \bar{z} \) we denote the complex conjugate of a number \( z \).

**Proof of Theorem 1.** We begin by proving the lower bound. The idea of the proof is to consider all functions on \( K \) which take values \( \pm 1 \) and average the norms of their images under an extension operator.

Let \( r_j, j = 1, 2, \ldots, \) be the Rademacher functions on the interval \([0, 1]\):
\[
r_j(\theta) = \text{sgn} \sin(2^j \pi \theta), \quad \theta \in [0, 1]
\]
(here sgn \( a = a/|a| \) for \( a \neq 0 \) and sgn 0 = 0). According to the well-known Khinchin inequality, for an arbitrary \( n \) and arbitrary complex numbers \( a_1, a_2, \ldots, a_n \) we have

\[
    c \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \leq \int_{[0,1]} \left| \sum_{j=1}^{n} a_j r_j(\theta) \right| d\theta,
\]

where \( c > 0 \) does not depend on \( n \) and \( a_1, a_2, \ldots, a_n \). It was shown by Szarek [6, Theorem 1, Remark 3] that (3) holds with \( c = 1/\sqrt{2} \).

Let \( K = \{t_j, j = 1, 2, \ldots, n\} \), where \( t_j \)'s are all distinct. Let \( E \) be an arbitrary linear extension operator from \( l^\infty(K) \) to \( A(\Gamma) \).

For each \( j = 1, 2, \ldots, n \) let \( \psi_j \) be a function on \( K \) whose value at the point \( t_j \) equals 1 and whose values at all other points of \( K \) equal 0. Let \( \Psi_j = E \psi_j \). We have \( \Psi_j = \Phi_j \), where \( \Phi_j \in L^1(G) \).

For each \( \theta \in [0,1] \) let \( f_\theta \) be the function on \( K \) defined by \( f_\theta(t_j) = r_j(\theta), j = 1, 2, \ldots, n \). We have \( \|f_\theta\|_{l^\infty(K)} \leq 1 \) and

\[
    f_\theta = \sum_{j=1}^{n} r_j(\theta) \psi_j.
\]

So,

\[
    \left\| \sum_{j=1}^{n} r_j(\theta) \Psi_j \right\|_{A(\Gamma)} = \left\| \sum_{j=1}^{n} r_j(\theta) E \psi_j \right\|_{A(\Gamma)} = \|E f_\theta\|_{A(\Gamma)} \leq \|E\|.
\]

This yields

\[
    \left\| \sum_{j=1}^{n} r_j(\theta) \Phi_j \right\|_{L^1(G)} \leq \|E\|.
\]

Thus, for all \( \theta \in [0,1] \) we have

\[
    \int_{G} \left| \sum_{j=1}^{N} r_j(\theta) \Phi_j(x) \right| dx \leq \|E\|.
\]

By integrating this inequality with respect to \( \theta \in [0,1] \) and applying the Khinchin inequality with the Szarek constant, we obtain

\[
    \int_{G} \frac{1}{\sqrt{2}} \left( \sum_{j=1}^{n} |\Phi_j(x)|^2 \right)^{1/2} dx \leq \|E\|.
\]
At the same time for all $x$

$$\sum_{j=1}^{n} |\Phi_j(x)| \leq n^{1/2} \left( \sum_{j=1}^{n} |\Phi_j(x)|^2 \right)^{1/2},$$

so (4) implies

$$\frac{1}{\sqrt{2}} \int_G \sum_{j=1}^{n} |\Phi_j(x)| dx \leq n^{1/2} \|\mathcal{E}\|.$$ 

Thus,

$$\frac{1}{\sqrt{2}} \sum_{j=1}^{n} \|\Psi_j\|_{A(\Gamma)} \leq n^{1/2} \|\mathcal{E}\|. \tag{5}$$

It remains to note that, since for each $j$

$$\|\Psi_j\|_{A(\Gamma)} \geq \sup_{\gamma \in \Gamma} |\Psi_j(\gamma)| \geq \Psi_j(t_j) = \psi_j(t_j) = 1,$$

estimate (5) yields

$$\frac{1}{\sqrt{2}} n \leq n^{1/2} \|\mathcal{E}\|,$$

which completes the proof of the lower bound.

To prove the upper bound we need the lemma below. This lemma is a modification of the theorem on local units [5, Theorem 2.6.1].

**Lemma.** Let $V$ be a neighborhood of zero in $\Gamma$. Then there exists $\Delta \in A(\Gamma)$ such that

(i) $\Delta(t) = 0$ for all $t \notin V$;
(ii) $\Delta(0) = 1$;
(iii) $0 \leq \Delta(t) \leq 1$ for all $t \in \Gamma$;
(iv) $\Delta = \hat{\lambda}$, where $\lambda \in L^1(G)$, $\lambda(x) \geq 0$ for all $x \in G$, $\|\lambda\|_{L^1(G)} = 1$.

**Proof of the Lemma.** In the ordinary way, assume that the Fourier transform is extended from $L^1 \cap L^2(G)$ to $L^2(G)$ so that it is a one-to-one mapping of $L^2(G)$ onto $L^2(\Gamma)$. Assume also that the Haar measure $m_\Gamma$ is normalized so that $\|\hat{\lambda}\|_{L^2(\Gamma)} = \|\lambda\|_{L^2(G)}$, $\lambda \in L^2(G)$. This ensures that $\hat{ab} = \hat{a} \ast \hat{b}$ for all $a, b \in L^2(G)$, where $\ast$ stands for convolution (see [5] for details). Given a set $D \subseteq \Gamma$, by $1_D$ we denote its indicator function:
$1_D(\gamma) = 1$ for $\gamma \in D$ and $1_D(\gamma) = 0$ for $\gamma \in \Gamma \setminus D$. One obtains the functions $\Delta$ and $\lambda$ as follows. Choose a neighborhood $I$ of zero in $\Gamma$ so that $I - I \subseteq V$, $0 < m_\Gamma(I) < \infty$. Define

$$\Delta = \frac{1}{m_\Gamma(I)} 1_I * 1_{-I}.$$ 

Clearly, $1_I = \hat{\xi}$, where $\xi$ is a certain function in $L^2(G)$, whence $1_{-I} = \hat{\xi}$. Let

$$\lambda = \frac{1}{m_\Gamma(I)} |\xi|^2.$$ 

We have $\lambda \in L^1(G)$ and $\Delta = \hat{\lambda}$. The properties (i) – (iv) are obvious. The lemma is proved.

To obtain the upper bound in Theorem 1 choose a neighborhood $V$ of zero in $\Gamma$ so small that

$$((K - K) \setminus \{0\}) \cap V = \emptyset$$

Let $\Delta \in A(\Gamma)$ and $\lambda \in L^1(G)$ be corresponding functions from the Lemma ($\Delta = \hat{\lambda}$). Note that (6) implies

$$\Delta(q - p) = 0 \quad \text{for} \quad p, q \in K, p \neq q.$$ 

Define an operator $\mathcal{E} : l^\infty(K) \to A(\Gamma)$ as follows. For a function $f$ on $K$

$$\mathcal{E}f(t) = \sum_{\gamma \in K} f(\gamma) \Delta(t - \gamma), \quad t \in \Gamma.$$ 

Clearly, $\mathcal{E}$ is a linear extension operator from $l^\infty(K)$ to $A(\Gamma)$. To estimate its norm consider an arbitrary function $f$ on $K$ with $\|f\|_{l^\infty(K)} = 1$. We have $\mathcal{E}f = \hat{S}$ where

$$S(x) = \sum_{\gamma \in K} f(\gamma)(x, \gamma) \lambda(x), \quad x \in G.$$ 

So,

$$\|\mathcal{E}f\|_{A(\Gamma)} = \|S\|_{L^1(G)} = \int_G \left| \sum_{\gamma \in K} f(\gamma)(x, \gamma) \lambda(x) \right| dx.$$  

---

6
Note that the measure $m$ given by $dm(x) = \lambda(x)dx$ is a probability measure on $G$, hence,

$$\int_G \left| \sum_{\gamma \in K} f(\gamma)(x, \gamma) \right| \lambda(x)dx \leq \left( \int_G \left| \sum_{\gamma \in K} f(\gamma)(x, \gamma) \right|^2 \lambda(x)dx \right)^{1/2}.$$  

Thus (see (9), (7))

$$\|Ef\|_{A(\Gamma)}^2 \leq \int_G \left| \sum_{\gamma \in K} f(\gamma)(x, \gamma) \right|^2 \lambda(x)dx$$

$$= \int_G \left( \sum_{p, q \in K} f(p)\overline{f(q)}(x, p - q) \right) \lambda(x)dx$$

$$= \sum_{p, q \in K} f(p)\overline{f(q)} \left( \int_G (x, p - q)\lambda(x)dx \right)$$

$$= \sum_{p, q \in K} f(p)\overline{f(q)} \Delta(q - p) = \sum_{\gamma \in K} |f(\gamma)|^2 \leq n.$$

This completes the proof of Theorem 1.

**Proof of Theorem 2.** First note that if $K \subseteq \Gamma$ is a finite set, then there exists a linear extension operator from $l^\infty(K)$ to $C_0(\Gamma)$ whose norm is equal to 1. Indeed, the operator defined by (8) has this property. Let now $K$ be an infinite closed subset of $\Gamma$. For each $n = 1, 2, \ldots$ chose an $n$-point subset $K_n$ of $K$. Let $\Sigma_n : l^\infty(K_n) \to C_0(\Gamma)$ be a linear extension operator with norm equal to 1. Let $\mathcal{R} : C_0(\Gamma) \to C_0(K)$ stand for the operator of restriction to $K$, i.e., the operator that takes $f \in C_0(\Gamma)$ to its restriction $f|_K$. Assuming that there exists a bounded linear extension operator $\mathcal{E} : C_0(K) \to A(\Gamma)$, consider the operator $\mathcal{E}\mathcal{R}\Sigma_n$. This operator is a linear extension operator from $l^\infty(K_n)$ to $A(\Gamma)$ and its norm is at most $\|\mathcal{E}\|$. This contradicts Theorem 1 if $n$ is large enough.

**Remark.** Let $M(G)$ be the space of all complex regular bounded measures on $G$. Consider the space $B(\Gamma)$ of Fourier–Stieltjes transforms, i.e, the space of all functions $f$ of the form $f = \widehat{\mu}$, where $\mu \in M(G)$ ([5, Section 1.3]). The norm on $B(\Gamma)$ is defined by $\|f\|_{B(\Gamma)} = \|\mu\|_{M(G)}$. Clearly, $B(\Gamma)$ is a Banach space and $A(\Gamma) \subseteq B(\Gamma)$ (these spaces coincide when $\Gamma$ is compact). Theorem 2 can be strengthened by replacing $A(\Gamma)$ with $B(\Gamma)$. To
see this, we define for a finite set $K \subseteq \Gamma$

$$\beta_{\Gamma}(K) \overset{\text{def}}{=} \inf\{\|E\| : E \text{ is a linear extension operator from } l^\infty(K) \text{ to } B(\Gamma)\}.$$  

Note that

$$\alpha_{\Gamma}(K) = \beta_{\Gamma}(K). \quad (10)$$

Indeed, relation $\alpha_{\Gamma}(K) \geq \beta_{\Gamma}(K)$ is obvious. To verify that $\alpha_{\Gamma}(K) \leq \beta_{\Gamma}(K)$, we let $\delta > 0$ and consider a function $\chi_\delta \in A(\Gamma)$ such that $\chi_\delta = 1$ on $K$ and $\|\chi_\delta\|_{A(\Gamma)} \leq 1 + \delta$ (see [3, Proposition A.5.1]). Assuming that $E : C_0(K) \to B(\Gamma)$ is an extension operator, we let $E_\delta f = \chi_\delta E f$ for $f \in C_0(K)$. Note that if $a \in A(\Gamma), b \in B(\Gamma)$, then $ab \in A(\Gamma)$ and $\|ab\|_{A(\Gamma)} \leq \|a\|_{A(\Gamma)} \|b\|_{B(\Gamma)}$. So, $E_\delta$ is an extension operator from $C_0(K)$ to $A(\Gamma)$ and $\|E_\delta\|_{C_0(K) \to A(\Gamma)} \leq (1 + \delta) \|E\|_{C_0(K) \to B(\Gamma)}$. Thus, we obtain (10). It remains to proceed as in the proof of Theorem 2 with obvious modifications.

References

1. C. C. Graham, “Helson sets and simultaneous extension to Fourier transforms”, Studia Math. 43 (1972), 57–59.

2. C. C. Graham, K. E. Hare, Interpolation and Sidon Sets for Compact Groups, CMS Books in Mathematics, Springer, Boston, MA, 2013.

3. C. C. Graham, O. C. McGehee, Essays in commutative harmonic analysis, Grundlehren der mathematischen Wissenschaften, vol. 238, Springer, Berlin, 1979.

4. J.-P. Kahane, Série de Fourier absolument convergentes, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

5. W. Rudin, Fourier analysis on groups, Interscience Publishers, New York–London, 1962.
6. S. J. Szarek, “On the best constants in the Khinchin inequality”, Studia Math. 58 (1976), 197–208.

School of Applied Mathematics
National Research University Higher School of Economics (HSE University)
34 Tallinskaya St.
Moscow, 123458 Russia

e-mail: lebedevhome@gmail.com