Irreducible representations of quantum solvable algebras at roots of 1

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Abstract

We study the irreducible representations of quantum solvable algebras at roots of 1 which lie over a point of the variety of center. We characterize the quiver of fiber algebra and present the formulas on the dimension and the number of these representations in terms of Poisson structure of the variety of center.

1 Introduction.

Quantum algebras appears in papers on mathamatical physics as deformations of the algebra of regular functions $\mathbb{C}[G]$ on the Lie group and universal enveloping algebra $U(g)$. From algebraic point of view, quantizing $\mathbb{C}$-algebra $R$, we have got $\mathbb{C}$-algebra $R_q$ which is a free module over the ring of Laurent polynomials $\mathbb{C}[q,q^{-1}]$ and $R = R_q \mod (q - 1)$. If $R$ is a Hopf algebra, then it is natural to seek its quantizations in the class of Hopf algebras. The most familiar quantum algebras are quantum universal enveloping algebra $U_q(g)$ for semisimple Lie algebra $g$, its dual Hopf algebra $\mathbb{C}_q[G]$, algebra of Quantum matrices, Quantum Weyl algebra. One can extend the chain of examples considering the multiparatmeter versions of these algebras, quantum spaces of representations.

One sets up the problem of description of the space of primitive ideals. It is interesiting to construct some general theory in spirit of the orbit method and also to classify primitive ideals for specific quantum algebras. The problem reduces to specializations $R_\varepsilon = R_q \mod (q - \varepsilon)$ where $\varepsilon \in \mathbb{C}$. Two cases take place: $\varepsilon$ is a root of 1 and $\varepsilon$ is not a root of 1.

Up today the classification of primitive ideals is known for for $\mathbb{C}_q[G]$ and quantum universal enveloping algebra of maximal solvable (resp. nilpotent) subalgebra in $g$. The

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case of not a root of unity is studied in the book [J]. The papers [DC-K], [DCKP1,2], [DC-L], [DC-P1,2] are devoted to the case \( \varepsilon \) is a root of 1.

Here is the simplified plan of classification of primitive ideals for \( C_q[G] \). The classification is based on the description of symplectic leaves on \( G \) as orbits of dressing transformations [ST]. For any symplectic leaf \( \Omega \) one considers the ideal of functions vanishing on it. Its generators are some matrix elements of irreducible representations of the Lie group \( G \). One can construct quantum analog of this ideal as an ideal generated by the corresponding matrix elements of irreducible representations of \( U_q(\mathfrak{g}) \). The constructed ideal is primitive, if \( \varepsilon \) is not a root of 1. It helps to stratify primitive ideals, if \( \varepsilon \) is a root of 1.

The next example is algebra of Quantum matrices. Theses algebra is a bialgebra, but not a Hopf algebra. The above methods are not valid for it. For classification of prime winding-invariant ideals see [GLn1,2], [C2], [L]. One of the main goals is \( U_q(\mathfrak{g}) \). This problem is far from its solution [J], [DC-K].

Consideration of examples make possible to set up some conjectures. The next goal is to prove this conjectures in maximally weak assumptions imposed on \( R_q \). These assumptions must be easily checkable and the theory must cover the main examples.

This paper is devoted to the case of roots of 1. In what follows we suppose that \( \varepsilon \) is a primitive \( l \)th root of 1. In the above examples, \( R_\varepsilon \) is finite over its center. That is the algebra \( R_\varepsilon \) is an order. Notice that this property also holds for elliptic algebras [FO], some new quantum groups that appear in the framework of theory of special functions [IK], reflection algebras [BG1].

The problem of description of primitive ideals for orders is equivalent to problem of classification of irreducible representations. The restriction on the center of an irreducible representation \( \pi \) of \( R_\varepsilon \) is scalar \( \pi|_{Z_\varepsilon} = \chi \cdot \text{id} \) and it defines the character \( \chi \) (i.e. the point of the variety) of center \( Z_\varepsilon \). We set up the usual problem for orders: to classify all irreducible representations of \( R_\varepsilon \) lying over given point \( \chi \) of the variety of center.

There is one common feature of the above orders: the existence of the quantum adjoint action (see Section 2 and [DCKP2],[P3]). Acting on the center \( Z_\varepsilon \), the quantum adjoint action defines the Poisson bracket. The variety of center becomes a Poisson variety which splits into symplectic leaves. It is proposed that the problem of classification of irreducible representations can be solved in terms of geometrical and Poisson properties of the variety of center.

In the paper we study the quantum solvable algebras which are iterated skew polynomials extensions of \( K[q,q^{-1}] \). The examples of these algebras are the algebra of Quantum matrices (see 2.14), Quantum Weyl algebra, \( U_q(\mathfrak{b}) \) and \( U_q(\mathfrak{n}) \) (see 2.15) and their numerous subalgebras. The algebra \( C_q[G] \) is not solvable, but one can reduce it to some solvable algebra after the localization. For details in examples see [P2]. The main goal is the construction of quantum version of theory of Dixmier for \( U_q(\mathfrak{g}) \) where \( \mathfrak{g} \) is a solvable Lie algebra [D]. Here are some problems which stimulate general theory.

**Problem 1.** To prove that the symplectic leaves are algebraic (i.e Zariski-open in its Zariski closure);

**Problem 2.** To prove that the dimension of an irreducible representation over \( \chi \) is equal to \( l^2 \) where \( d \) is the dimension of symplectic leaf of \( \chi \). Conjectured in [DCP1,4.5],[DCP
Problem 3. To describe the quiver of the algebra \( R_{\varepsilon, \chi} := R_\varepsilon / m(\chi)R_\varepsilon \) where \( m(\chi) = \text{Ker}(\chi) \).

Problem 4. To find the formula for the number of irreducible representations over \( \chi \).

The solutions is known for \( \mathbb{C}_q[G] \) and \( U_q(b) \). The solution of Problem 1 for these algebras arises from the method of dressing transformations. The formulas on dimensions and the number of irreducible representations were obtained in [DC-P2]. The quivers were studied in [BG2].

In [P3] the Problems 1 and 2 were solved for rather great \( l \) (the point of good reduction of stratification process). The goal of this paper is to drop these undesirable restriction on \( l \) and to go forward in describing the quiver and determining the number of irreducible representations over \( \chi \).

The main definition of the paper is the definition of normal quantum solvable algebra (or NQS-algebra, see Definition 2.10). We require that this algebra obeys some Conditions CN1-CN2. We present two examples (Quantum matrices and \( U(n) \)). One can find the other examples in ([G], [P1-P2]). Our definition of admissible \( l \) (Definition 2.18) is easily checkable and necessary for solution of Problems 1-4.

We stratify the prime \( D \)-stable spectrum of NQS-algebra (see Theorem 3.2). It is proved that every prime \( D \)-stable ideal is completely prime (see Theorem 3.3). The Problems 1 and 2 are solved in Theorem 4.2.

One can correspond the quiver to any finite dimensional algebra \( A \) [Pie, 6.4]. The vertices of quiver are primitive idempotents \( e_1, \ldots, e_N \) such that their right ideals \( e_1A, \ldots, e_NA \) represent non isomorphic classes of principal indecomposable \( A \)-modulas. Two vertices \( e_i, e_j \) are linked with wedge \( (e_i, e_j) \) if \( e_iJe_j \neq 0 \) where \( J \) is the radical of \( A \). In the paper we prove (see Theorem 4.3) that any two vertices \( e_i, e_j \) of quiver of finite dimensional algebra \( R_{\varepsilon, \chi} \) is linked by wedges \( (e_i, e_j) \) and \( (e_j, e_i) \). In particular the quiver is connected.

In the last Section 5, we prove (Theorems 5.5,5.7) that the number of irreducible representations over \( \chi \) is equal to \( l^t \) where \( t \) is the dimension of some toric Lie subalgebra of the stabilizer \( g(\chi) \) of \( \chi \) (Definition 5.6).

We are very thankfull to J.Cauchon; he sent his new preprint [C1] to the author. The method of stratification of [C1] is used in this paper. We are very thankfull to C.De Concini, C.Procesi, K.Brown and I.Gordon for useful discussions.

## 2 Quantum solvable algebras and FA-elements

We begin with some general definitions and the properties of skew extensions which are used throughout this paper.

Let \( R_F \) be a domain and an algebra over a field \( F \).

**Definition 2.1.** We say that \( x \in R_F \) is an element of finite adjoint action (or \( x \) is a FA-element) if \( x \) is not a zero divisor and for every \( a \in R_F \) there exists a polynomial \( f_a(t) = c_0 t^N + c_1 t^{N-1} + \cdots + c_N \), \( c_0 \neq 0 \), \( c_N \neq 0 \) over \( F \) such that

\[
 c_0 x^N a + c_1 x^{N-1} ax + \cdots + c_N ax^N = 0. \tag{2.1}
\]
A FA-element $x$ generates a denominator set $S_x := \{x^n\}_{n \in \mathbb{N}}$ [P1, Proposition 3.3]. One can rewrite (2.1) in the form

$$f_a(\text{Ad}_x)a = 0 \quad (2.2)$$

where $\text{Ad}_x(a) = xax^{-1}$. If $x$ is a FA-element in $R$, then it is a FA-element in $RS_x^{-1}$. The following statements are easy to prove.

**Proposition 2.2.** Let $x, y \in \text{Fract}(R_F)$ be FA-elements in a domain $R_F$ and suppose that $xy = \gamma yx$ with some $\gamma \in F^\times$. Then $xy$ is also a FA-element.

**Proposition 2.3.** Suppose that the above domain $R_F$ is generated by $x_1, \ldots, x_n$ and $x \in \text{Fract}(R_F)$. Suppose that for every $j$ there exists a polynomial $f_j(t)$ obeying (2.1) with $a = x_j$. Then $x$ is a FA-element in $R_F$. If, in addition, $f_j(t)$ splits $f_j(t) = (t - \gamma_j^{(1)}) \cdots (t - \gamma_j^{(n_j)})$, then, for any $a \in R_F$, the polynomial $f_a(t)$ also splits with the roots in the semigroup generated by $\gamma_j^{(s)}$.

Let us have an endomorphism $\tau$ of $R_F$ ($\tau$ is identical on $F$) and a $\tau$-derivation $\delta$ of $R_F$ (i.e. $\delta(ab) = \delta(a)b + \tau(a)\delta(b)$ for all $a, b \in R_F$) which is zero on $F$. An Ore extension (skew extension) $T_F = R_F[x; \tau, \delta]$ of $R_F$ is generated by $x$ and $R_F$ with $xa = \tau(a)x + \delta(a)$ for all $a \in R_F$. Every element of $T$ can be uniquely presented in the form $\sum x^i r_i$ (or $\sum r_i x^i$) where $r_i \in R$.

**Proposition 2.4.** Let $R_F$ and $T_F = R_F[x; \tau, \delta]$ be as above with diagonalizable automorphism $\tau$. Suppose that $\tau \delta = \gamma \delta \tau$, $\gamma \neq 0$. The element $x$ is a FA-element in $T_F$ iff $\delta$ is locally nilpotent. Moreover, for $\tau$-eigenvector $a$, there exists a polynomial $f_a(t)$ of degree $N$ obeying (2.1) iff $\delta^N(a) = 0$.

**Proof.** Let $a$ be a $\tau$-eigenvector, i.e. $\tau(a) = \beta a$. There exists a polynomial $f(t)$ obeying (2.1). Then

$$0 = c_0 x^N a + c_1 x^{N-1} ax + \ldots + c_N ax^N = f(\beta)ax^N + \{ \text{terms of lower degree} \}.$$ 

It implies that $f(\beta) = 0$, $f(t) = f_1(t)(t - \beta)$ and $0 = f(\text{Ad}_x)a = f_1(\text{Ad}_x)(\text{Ad}_x - \beta)a = f_1(\text{Ad}_x)\delta(a)x^{-1}$. The element $\delta(a)$ is also a $\tau$-eigenvector. After $N$ steps we get $\delta^N(a) = 0$ where $N = \deg f(t)$. On the other hand, if $\delta^N(a) = 0$ and $\tau(a) = \beta a$, then the polynomial

$$f(t) := \prod_{i=1}^N (t - \beta \gamma_i)$$

obeys (2.1). $\square$

Let $K$ be an algebraic closed field of zero characteristic, $q$ be an indeterminate and $C$ be a localization $K[q, q^{-1}]$ over some finitely generated denominator set. Denote $\Gamma = \{q^k : k \in \mathbb{Z}\}$. Put $F = \text{Fract}(C) = K(q)$.

**Definition 2.5.** Let $R$ be a unital domain, an algebra over $C$ and a free $C$-module. Let $x$ be an element in $R$.

1) An element $x \in R$ is a FA-element if it is a FA-element in $R_F := R \otimes_C F$;
2) We say $x$ is a $F_A(q)$-element in $R$ if it is a FA-element in $R_F := R \otimes_C F$ and for any $a \in R$ one can choose the polynomial $f_a(t)$ obeying (2.1) such that it splits and all its roots belong to $\Gamma$.

**Definition 2.6.** We say that two elements $a, b$ $q$-commute if $ab = q^k ba$ for some integer $k$. 

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Proposition 2.7 [C, Prop.2.1-2.3]. Let \( R \) be as in Definition 2.5 and \( T_F = R_F[x; \tau, \delta] \) be a skew extension where \( \tau \) is an automorphism, \( \delta \) is a locally nilpotent \( \tau \)-derivation and \( \tau \delta = q^s \delta \tau \) with \( s \neq 0 \). Denote

\[
\widehat{a} = \sum_{n=0}^{+\infty} \frac{(1 - q^s)^{-n}}{(n)_q!} a^n \tau^{-n}(a) x^{-n},
\]

(2.3)

where \((n)_q! = \frac{q^n - 1}{q - 1}\). Then

1) the set \( S_x = \{x^m\}_{m \in \mathbb{N}} \) is a denominator subset in \( T_F \),
2) the map \( a \mapsto \widehat{a} \) is an embedding of \( R \) is \( T_F S_x^{-1} \),
3) \( T_F S_x^{-1} = \hat{R}_F[x^{\pm 1}; \tau] \) where \( \hat{R}_F \) is the image of \( R \) under \( a \mapsto \widehat{a} \).

Throughout this paper \( \varepsilon \) is a primitive \( l^{th} \) root of 1 such that \( C \) admits specialisation by \( \varepsilon : C \to K \) with \( q \mapsto \varepsilon \). For any \( \varepsilon \) consider the specialisation \( R_\varepsilon \) of \( R \) over \( K \). In what follows we shall use two notations. If \( a \in R \), we put \( a_\varepsilon := a \mod (q - \varepsilon) \). For \( a \in R_\varepsilon \), we denote by \( a \in R \) an element of preimage of \( a \) under the map \( \pi_\varepsilon : R \to R_\varepsilon = R \mod (q - \varepsilon) \). For any algebra \( A \) of \( R \), we denote \( A_\varepsilon := (A + R(q - \varepsilon)) \mod (q - \varepsilon) = \pi_\varepsilon(A) \).

If \( u_\varepsilon = u \mod (q - \varepsilon) \) lies in the center \( Z_\varepsilon \) of \( R_\varepsilon \), then \( \mathcal{D}_u(a) = \frac{ua - au}{q - \varepsilon} \mod (q - \varepsilon) \) defines a derivation of \( R_\varepsilon \). We call \( \mathcal{D}_u \) the quantum adjoint action of \( u \) (see [DCKP1-2],[P3]). An ideal is stable with respect to the quantum adjoint action (call \( \mathcal{D} \)-stable) if it is stable with respect to all \( \mathcal{D}_u \). The formula \( \{a, b\} = \mathcal{D}_a(b), \) for \( a, b \in Z_\varepsilon \), defines the Poisson bracket on \( M = \text{Maxspec } Z_\varepsilon \).

Here are two versions of reduction of Proposition 2.7 modulo \( q - \varepsilon \).

Corollary 2.8. Let \( T, R, \tau, \delta, q^s \) be as in Proposition 2.7. Suppose that \( R \) is generated by the elements \( x_1, \ldots, x_n \) and \( \tau \) is a diagonal automorphism with eigenvalues in \( \Gamma \). Choose \( N \) such that \( \delta^N(x_i) = 0 \) for all \( 1 \leq i \leq n \). Suppose that \( l \) is relatively prime with \( s \) and \( l \geq N \). Then

1) \( T_\varepsilon S_x^{-1} \cong R_\varepsilon[x^{\pm 1}; \tau] \),
2) \( x^l_\varepsilon \) lies in the center \( Z(T_\varepsilon) \).

Proof. Denote by \( \mathfrak{N}_1 \) the denominator subset in \( C \) generated by \( q^{sn} - 1, 1 \leq n \leq d \). The elements \( x^{\pm 1}, \hat{x}_1, \ldots, \hat{x}_M \) generate \( \hat{T} := TS_x^{-1} \mathfrak{N}_1^{-1} \). We denote by \( \hat{R} \) the subalgebra generated by \( \hat{x}_1, \ldots, \hat{x}_M \) over \( C \mathfrak{N}_1^{-1} \). By Proposition 2.7, the map \( a \mapsto \widehat{a} \) provides an isomorphism of \( R \mathfrak{N}_1^{-1} \) onto \( \hat{R} \). We have \( \hat{T} = \hat{R}[x; \tau, \delta] \). After reduction modulo \( q - \varepsilon \) (we obtain 1).

Since \( x^l \hat{x}_j = q^{sn} \hat{x}_j x \) for some \( n_j \), then \( x^l_\varepsilon \) lies in the center \( Z(\hat{T}_\varepsilon) \). This proves 2). \( \Box \)

Corollary 2.9. Let \( T, R, \tau, \delta, q^s \) be as above and \( l \) be relatively prime to \( s \). Suppose that \( x_\varepsilon^l \in Z(T_\varepsilon) \). Then \( T_\varepsilon S_x^{-1} \cong R_\varepsilon[x^{\pm 1}; \tau] \).

Proof. Taking

\[
x^l a = \tau^l(a)x^l + \sum_{i=1}^{l-1} \binom{l}{i} q^{s_i} \delta^i(a)x^{l-i} + \delta^l(a)
\]

modulo \( q - \varepsilon \), we obtain \( x^l a = ax^l + \delta^l(a) \mod (q - \varepsilon) \) and \( \delta^l(a) \in (q - \varepsilon)R \) for any \( a \in R \). If \( n = lm + r, \ 0 \leq r < l \), then \( \delta^n(a) \in (q - \varepsilon)^mR \). On the other hand,
\((n)_q! = (q - \varepsilon)^m c(q)\) where \(c(\varepsilon) \neq 0\). Hence

\[
\frac{\delta^n(a)}{(n)_q!} \in Rc^{-1}(q).
\]

Consider the denominator subset \(\mathfrak{R}_x\) in \(C\) generated by \(q^n - 1\) where \(l\) does not divide \(n\) and \(\frac{d^{m-1}}{q^{\varepsilon}}\), \(m \in \mathbb{N}\). For any \(a \in R\) the element \(\hat{a}\) (see 2.3) lies in the localization of \(T\) over \(S_x\) and \(\mathfrak{R}_x\) and \(TS_x^{-1}\mathfrak{R}_x^{-1} = R\mathfrak{R}_x^{-1}[x; \tau]\). Taking modulo \(q - \varepsilon\), we get the claim. \(\square\)

Let \(S = (s_{ij})\) be a \(M \times M\) integer skew-symmetric matrix. Denote \(q_{ij} = q^{s_{ij}}\) and form the matrix \(Q = (q_{ij})\). Choose the subset, call distinguished subset, \(\mathfrak{t} := \{t_1, \ldots, t_m\}\) where \(1 \leq t_1 < \ldots < t_m \leq M\).

Definition 2.10. We say that \(R\) is a normal quantum solvable algebra (or a NQS-algebra) over \(C\), if \(R\) is generated by the elements \(x_i, 1 \leq i \leq M\) and \(x_j^{-1}, j \in \mathfrak{t}\) such that the monomials \(x_1^{t_1} \cdots x_M^{t_M}\) with \(t_j \in \mathbb{Z}\), \(j \in \mathfrak{t}\) and \(t_j \in \mathbb{N}\), \(1 \leq j \leq M\), \(j \notin \mathfrak{t}\) form a free \(C\)-basis, the algebra \(C\) lies in the center of \(R\) and the following relations hold

1) \(x_i x_j = q_{ij} x_j x_i\) for all \(i, j \in \mathfrak{t}\);  
2) for \(1 \leq i < j \leq M\),

\[
x_i x_j - q_{ij} x_j x_i + r_{ij} = 0
\]

where \(r_{ij}\) is a sum of monomials \(c x_{i+1}^{t_{i+1}} \cdots x_{j-1}^{t_{j-1}}\) with \(c \in C\). The definition of quantum solvable algebra is given in Remark 2.12.

The subalgebra \(Y_i\) generated by \(C\) and \(x_{i}^{\pm 1}\), \(i \notin \mathfrak{t}\), is an algebra of twisted Laurent polynomials. The subalgebras \(R_i\), generated by \(C\), \(x_j\), \(j \geq i\) and their inverses for the distinguished subscripts, form a chain \(R = R_1 \supset R_2 \supset \cdots \supset R_M\) (call it the right filtration). One can prove that each \(R_i\) is a skew extension of \(R_i+1\) [GL1,1.2]. This means that the map \(\tau_i : x_j \mapsto q_{ij} x_j, i < j\) is extended to an automorphism of \(R_{i+1}\) and the map \(\delta_i : x_j \mapsto r_{ij}\) is extended to a \(\tau_i\)-derivation of \(R_{i+1}\). All automorphisms \(\tau_i\) and \(\delta_i\) are identical on \(C\) and all \(\tau_i\)-derivations \(\delta_i\) are equal to zero on \(C\). The formula (2.4) yields, \(R_i = R_{i+1}[x_i; \tau_i, \delta_i]\) for \(i \notin \mathfrak{t}\) and \(R_i = R_{i+1}[x_i^{\pm 1}, \tau_i]\) for \(i \in \mathfrak{t}\). A NQS-algebra is a Noetherian domain [MC-R, 1.2.9], a \(C\)-algebra and a free \(C\)-module.

The NQS-algebra \(R\) has the other filtration (call it the left filtration)

\[
R'_1 \subset R'_2 \subset \cdots \subset R'_M = R
\]

with \(R'_i\) is generated by \(C\), \(x_1, \ldots, x_i\) and their inverses for distinguished subscripts. Again \(R'_i = R'_{i-1}[x_i; \tau'_i, \delta'_i]\) (resp. \(R'_i = R'_{i-1}[x_i^{\pm 1}, \tau'_i]\) for distinguished \(i\)) where \(\tau'_i\) (resp. \(\delta'_i\)) is the automorphism (resp. \(\tau'_i\)-derivation) of \(R_{i-1}\). We put \(\delta_i = \delta'_i = 0\) for distinguished \(i\).

Furthermore, for any \(1 \leq \alpha < \beta \leq M\) we denote by \(R_{[\alpha, \beta]}\) the subalgebra generated by \(C\), \(x_i\) and \(x_j^{\pm 1}\) such that \(\alpha \leq i, j \leq \beta\) and \(j \in \mathfrak{t}\).

Notice that \(R_{[\alpha, \beta]} = R_{[\alpha-1, \beta]}[x_{\alpha}; \tau_{\alpha}, \delta_{\alpha}]\), for \(\alpha \notin \mathfrak{t}\), and \(R_{[\alpha, \beta]} = R_{[\alpha, \beta-1]}[x_{\alpha}^{\pm 1}; \tau_{\alpha}]\), for \(\alpha \in \mathfrak{t}\). Similarly, \(R_{[\alpha, \beta]} = R_{[\alpha, \beta-1]}[x_{\beta}; \tau_{\beta}, \delta_{\beta}]\), for \(\beta \notin \mathfrak{t}\), and \(R_{[\alpha, \beta]} = R_{[\alpha, \beta-1]}[x_{\beta}^{\pm 1}; \tau_{\beta}]\), for \(\beta \in \mathfrak{t}\). We put the following conditions on a NQS-algebra.

Condition CN1. We require \(R\) be an iterated \(q\)-skew extension for the left and the right filtrations. This means that \(\tau_i \delta_i = q_i \delta_i \tau_i\), for some \(q_i = q^{s_i}, s_i \in \mathbb{Z}\), and
\( \tau'_i \delta'_i = q^i q'^i \), for some \( q'_i = q'^i, s'_i \in \mathbb{Z} \). We require that all \( s_i \neq 0 \) (resp. \( s'_i \neq 0 \)) if \( \delta_i \neq 0 \) (resp. \( \delta'_i \neq 0 \)). We call \( \{s_i\}, \{s'_i\} \) the systems of exponents \( R \).

**Condition CN2.** All \( \tau_i \) and \( \tau'_i \) are extended to diagonal automorphisms of \( R \) and generate the commuting diagonal subgroups \( H \) and \( H' \).

**Proposition 2.11.** Let \( R \) be a NQS-algebra over \( C \). Put \( n = M - m \). Let \( \bar{R}_i, i \notin \mathfrak{t} \) be a subalgebra generated by \( R_i \) and \( Y_i \). The chain \( \bar{R} = \bar{R}_1 \supset \bar{R}_2 \supset \cdots \supset \bar{R}_n \supset \bar{R}_{n+1} = Y \) is a chain of skew extensions \( \bar{R}_i \cong \bar{R}_{i+1}[x; \tau_i, \delta_i] \). If, in addition, \( R \) obeys Condition CN1, then \( \tau_i = q^i \), \( \delta_i = q'^i \) as in CN1.

**Proof.** We put \( \tau(a) = \tau_i(a) \) (resp. \( \delta(a) = \delta_i(a) \)), for \( a \in R_{i-1} \), and \( \tau(x_j) = q_{ij} x_j \) (resp. \( \delta(x_j) = 0 \)), for \( j < i, j \in \mathfrak{t} \). The direct calculations conclude the proof. □

**Remark 2.12.** A quantum solvable algebra is defined in [P1-P3] as an algebra generated \( R \) is generated by the elements \( x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \) with \( M = n+m \) such that the monomials \( x_1^{t_1} x_2^{t_2} x_{n+1}^{t_{n+1}} \cdots x_{n+m}^{t_{n+m}} \) with \( t_1, \ldots, t_n \in \mathbb{N} \) and \( t_{n+1}, \ldots, t_{n+m} \in \mathbb{Z} \) form a free \( C \)-basis and the relations hold: 1) \( x_i x_j = q_{ij} x_j x_i \), for all \( i \) and \( n + 1 \leq j \leq M \),

2) \( x_i x_j = q_{ij} x_j x_i + r_{ij}, 1 \leq i < j \leq n \) where \( r_{ij} \) is an element of the subalgebra \( R_{i+1} \) generated by \( x_{i+1}, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \). Proposition 2.9 claims that a NQS-algebra is a quantum solvable algebra. The Conditions CN1 and CN2 are comparable with more general Conditions Q1-Q4 of [P2] and Conditions 3.2-3.4 of [P3].

**Proposition 2.13.** Any FA-element in a quantum solvable algebra (in particular, is a NQS-algebra) \( R \) is a FA\(_q\)-element.

**Proof.** Let \( R \) be a quantum solvable algebra (see above Remark). For a monomial \( w = x_1^{t_1} \cdots x_M^{t_M} \), denote \( \deg(w) = (t_1, \ldots, t_M) \). Lexicographical order provides the filtration in \( R \). The algebra \( A_Q := \text{gr}(R) \) is generated by \( a_i = \text{gr}(x_i), 1 \leq i \leq M \) and \( a_j^{-1}, j \notin \mathfrak{t} \). The relations are \( a_i a_j = q_{ij} a_j a_i \). The algebra \( A_Q \) is the localization of algebra of twisted polynomials. As usual \( (\cdot, \cdot) \) denotes the standard scalar product in \( \mathbb{Z}^M \). For two monomials \( a, b \in A_Q \) with \( \deg(a) = m, \deg(b) = n \), we have \( ab = q^{(m+n)} ba \). For every \( u, v \in R \) with \( \deg(u) = m, \deg(v) = n \),

\[
uv = q^{(m+n)} vu + \{\text{terms of lower degree}\}.
\]

(2.5)

Let \( u, v \in R \) and the element \( u \) be a FA-element. Let \( f(t) \) be the corresponding polynomial obeying (2.1), for \( x := u \) and \( a := v \). Put \( \gamma := q^{(m+n)} \). By (2.5),

\[ 0 = c_0 u^N v + u^{N-1} v u + \ldots + c_N v u^N = f(\gamma) v u^N + \{\text{terms of lower degree}\}. \]

Hence, \( f(\gamma) = 0 \) and \( f(t) = (t - \gamma) f_1(t) \). The element \( v_1 := uv - \gamma vu \) is annihilated by \( f_1(Ad_u) \). The proof is concluded by induction on degree of polynomial \( f(t) \). □

Here are two the most familiar examples of NQS-algebras.

**Example 2.14.** Quantum matrices.

The algebra \( M_q(n, K) \) of regular functions on quantum matrices is generated over \( C := K[q, q^{-1}] \) and the entries of quantum matrix \( \{a_{ij}\}_{t, i=1}^n \) which obey the relations \( a_{ti} a_{sj} = q_{ij} a_{si} a_{ti} \) for \( i < j, t < s \) and \( a_{ti} a_{sj} = q a_{sj} a_{ti} \), for \( t < s, i = j \) and \( t = s, i < j \).

The algebra \( M_q(n, K) \) is a NQS-algebra with respect to generators \( x_{(i-1)n+j} = a_{ij} \). It obeys CN1 (see [G], [P2]) and CN2 (the map \( \tau_{ij} : a_{ij} \mapsto qa_{ij} \), i.e. the multiplication of \( i \)th row by \( q \), is an automorphism of \( R \)). One can obtain another examples...
Let $\varepsilon$ be a primitive $l$th root of unity. Consider the new system of generators $E_{\beta_1} = T_{i_1} \cdots T_{i_{s-1}} E_{i_s}$, for $1 \leq s \leq N$. There are the following relations on the $E_{\beta_1}$'s [LS]:

$$E_{\beta_1} E_{\beta_j} - q^{-(\beta_1, \beta_j)} E_{\beta_j} E_{\beta_1} = \sum_{m \in \mathbb{Z}^N_+} c_m E^m,$$

where $i < j$, $c_m \in K[q, q^{-1}]$ and $c_m \neq 0$ only when $m = (m_1, \ldots, m_N)$ is such that $m_s = 0$ for $s \leq i$ and $s \geq j$. The algebras $U_q(n)$, $U_q(b)$, and subalgebras $U_q^w(n)$ (see [C1],[DC-P1]) are NQS-algebras. They obey Conditions CN1 (see [G], [P2]) and CN2 (the map $\tau_\alpha : E_{\beta} \mapsto q^{(\alpha, \beta)} E_{\beta}$ is an automorphism of $R$).

**Proposition 2.16.** Let $R$ be a NQS-algebra obeying Condition CN1 with the systems of exponents $\{s_t\}$, $\{s_t'\}$.

1. All $x_\alpha$, $1 \leq \alpha \leq M$ are FA-elements in $R$. Choose $N_\alpha$ (see Proposition 2.4) such that $\delta_{N_\alpha}^\alpha (x_j) = 0$, $\alpha < j$.

2. For any $1 \leq \alpha \leq M$, $\alpha \notin \mathfrak{t}$ consider two denominator subsets $\mathfrak{t}_\alpha$ generated by $q^{\alpha j} - 1$, $q^{\alpha j} - 1$, $1 \leq t \leq \alpha$. The algebras $RS_{\alpha}^{-1} \mathfrak{t}_\alpha^{-1}$ is a NQS-algebra with distinguished subset $\mathbf{t} \cup \{\alpha\}$ over $C\mathfrak{t}_\alpha^{-1}$, with the same (as $R$) matrix $Q$ and systems of exponents.

**Proof.** The claim 1) is proved similarly [P1, Lemma 4.3]. To prove 2) we apply Proposition 2.5 for two extensions $R_\alpha = R_{\alpha+1}[x_\alpha; \tau_\alpha, \delta_\alpha]$ and $R'_\alpha = R'_{\alpha-1}[x_\alpha; \tau'_\alpha, \delta'_\alpha]$, and consider the new system of generators $\tilde{x}_1, \ldots, \tilde{x}_{\alpha-1}, x_\alpha^\pm, \tilde{x}_{\alpha+1}, \ldots, \tilde{x}_M$ of $RS_{\alpha}^{-1}$.

**Corollary 2.17.** Let $R$ be as in Proposition 2.16. Suppose that $l$ is relatively prime with $s_\alpha, s'_\alpha$, and $x'_{\alpha l}$ lies in the center of $R_{\varepsilon}$. If $R_{\varepsilon} S_{\alpha\varepsilon}^{-1}$ is isomorphic to the specialization of some NQS-algebra modulo $q - \varepsilon$.

**Proof.** Consider the multiplicatively closed subset $\mathfrak{t}_{\alpha l}$ generated by $q^{\alpha n l} - 1, q^{\alpha n l} - 1$ (where $1 \leq n < N_\alpha$ and $l$ does not divide $n$) and $q^{x_{\alpha l} m l - 1} - q^{x_{\alpha l} m l - 1}$ for $1 \leq m < N_\alpha$. Since $l$ is relatively prime with $s_\alpha$ and $s'_\alpha$, then polynomials of $\mathfrak{t}_{\alpha l}$ are not zero at $q = \varepsilon$. The element $x_{\alpha l}$ lies in the center of $R_{\varepsilon}$; by the proof of Corollary 2.9, $\tilde{x}_1, \ldots, \tilde{x}_{\alpha-1}, x_{\alpha l}^\pm, \tilde{x}_{\alpha+1}, \ldots, \tilde{x}_M$ lie in $RS_{\alpha l}^{-1} \mathfrak{t}_{\alpha l}^{-1}$. One can reduce the generators modulo $q - \varepsilon$ and get the system of generators $\tilde{x}_{1\varepsilon}, \ldots, \tilde{x}_{\alpha-1\varepsilon}, x_{\alpha l}^\pm, \tilde{x}_{\alpha+1\varepsilon}, \ldots, \tilde{x}_M\varepsilon$ of $R_{\varepsilon} S_{\alpha\varepsilon}^{-1}$.

For a NQS-algebra $R$, consider $N := N_R = \max \{N_\alpha\}$. For $1 \leq i_1 < \ldots < i_k < M$, $\mu := \{i_1, \ldots, i_k\} \supset \mathfrak{t}$ we denote by $S_{\mu}$ the submatrix $(s_{ij})$, $i, j \in \mu$ of $S$. 

**Definition 2.18.** We say that a positive integer $l$ (resp. a primitive $l$th root of unity $\varepsilon$) is admissible for a NQS-algebra $R$ if it obeys the conditions:

1) $l$ is relatively prime with all elementary divisors of all submatrices $S_{\mu}$, $\mu \supset \mathfrak{t}$;

2) $l$ is relatively prime with $s_i, s'_i$, $1 \leq i \leq M$;

3) $l \geq N$.

**Lemma 2.19.** Let $\varepsilon$ obeys the conditions 2) and 3) of Definition 2.18, and $R$ be a
NQS-algebra obeying Condition CN1. Then the elements \( \{ x_{i\varepsilon} \} \) lie in the center \( Z_\varepsilon \) of \( R_\varepsilon \).

**Proof.** Apply Corollary 2.8. \( \square \)

**Proposition 2.20.** Let \( R \) and \( \varepsilon \) be as in 2.19.

1) If \( x \) is a FA-element of \( R_F \) (resp. \( R_\varepsilon \)), then linear operator \( \text{Ad}_x \) is diagonalizable in \( R_F S_x^{-1} \) (resp. \( R_\varepsilon S_x^{-1} \)).

2) For any FA-element \( x \) in \( R \) the element \( x'_\varepsilon \) lies in \( Z_\varepsilon \).

**Proof.** Lemma 2.19 implies that \( R_\varepsilon \) is finite over its center. The set of roots of unity, that obey 2) and 3) of Definition 2.18, is infinite. The statement 1) is a corollary [P1, Cor.2.5, Proposition 3.4].

Let us prove 2). For the FA-element \( x \) and any \( a \) is \( R \) there exists a minimal polynomial \( f(t) \) that obeys (2.1). By Proposition 2.13, the roots of \( f(t) \) belong to \( \Gamma \). Suppose that \( q^{\alpha_1}, \ldots, q^{\alpha_N} \) are the roots of \( f(t) \). The element \( u = x^l \) are also a FA-element of \( R \). The operators \( \text{Ad}_x \) and \( \text{Ad}_x^l \) are simultaneously diagonalizable. The roots of corresponding polynomial \( f_x(t) \) for \( x^l \) are \( \lambda_i := q^{\alpha_id}, 1 \leq i \leq N \). It implies \( f_x(t) = (t - 1)^N \mod (q - \varepsilon) \) and \( (\text{Ad}_x^l - \text{id})a = 0 \). On the other hand, by 1), \( x_\varepsilon \) and \( x_\varepsilon^l \) are FA-elements in \( R_\varepsilon \). Hence, \( \text{Ad}_x^l \) is diagonalizable. It follows \( x_\varepsilon^l \in Z_\varepsilon \) and 2). \( \square 

**Definition 2.21.** \( R \) and \( \varepsilon \) as above. We say that an ideal is \( D_0 \)-stable if it is stable with respect to all derivations \( D_{x^l}, 1 \leq i \leq M \).

**Notation 2.22.** For any automorphism \( \tau \in H \) we denote by \( \theta \) the following diagonal derivation of \( R_\varepsilon \)

\[
\theta(a) = \frac{\tau^l - \text{id}}{q - \varepsilon} \mod (q - \varepsilon).
\]

Similar \( \theta' \) for \( \tau' \in H' \). By \( \Theta \) we denote the commutative subalgebra spanned by \( \theta_1, \ldots, \theta_M \). Similarly for \( \Theta' \).

### 3 Stratification of prime ideals

In this section, we stratify the prime spectrum of \( R \) and the prime \( D \)-stable spectrum of \( R_\varepsilon \) (Theorem 3.2). It is proved that any prime \( D \)-stable ideal of \( R_\varepsilon \) is completely prime (Theorem 3.3).

Throughout this section \( R \) is a NQS-algebra, obeying Conditions CN1 and CN2, and \( \varepsilon \) obeys the conditions 2) and 3) of Definition 2.18.

Consider the multiplicatively closed subset \( \mathfrak{K} = \prod_{\alpha} \mathfrak{K}_{\alpha,l} \) (see Corollary 2.17). The polynomials of \( \mathfrak{K} \) are not zero at \( q = \varepsilon \).

Fix an integer \( i_1 \) which \( 1 \leq i_1 \leq M \). If \( i_1 \in \mathfrak{k} \), we put \( R^{(1)} := R \). If \( i_1 \notin \mathfrak{k} \), we consider the denominator subset \( S_1 \) generated by \( y_1 := x_{i_1} \). According to Proposition 2.16 and Corollary 2.17, \( R^{(1)} := RS_{1}^{-1}\mathfrak{K}^{-1} \) is a NQS-algebra over \( CR^{-1} \) with the same (as \( R \)) systems of exponents. The algebra \( R^{(1)} \) is generated by

\[
x_1, \ldots, x_{i_1-1}, x_{i_1}^{\pm 1}, x_{i_1+1}, \ldots, x_M
\]

where \( x_j := \hat{x}_j \). Recall that all generators \( q \)-commute with \( y_1 \) and are FA-elements in \( R^{(1)} \). It follows that, for all \( i \), the elements \( x_{i\varepsilon}^l \) lie in the center of \( R^{(1)}_\varepsilon = R_\varepsilon S_{1\varepsilon}^{-1} \) (see Proposition 2.20).
Let \( i_2 \) be any integer which \( i_1 < i_2 \leq n \). There exists a positive integer \( t \) such that \( y_2 := x'_{i_2}x''_{i_1} \in R \). Similarly to the first step of stratification process, we consider denominator subset \( S_2 \) generated by \( q \)-commuting elements \( y_1, y_2 \). As we saw the element \((x'_{i_2}x''_{i_1})^t \) lies in the center of \( R_\varepsilon(1) \). By Corollary 2.17, the algebra \( R^{(2)} := RS_2^{-1} \mathfrak{N}^{-1} = RS_1^{-1}S_{x'_{i_2}}^{-1} \mathfrak{N}^{-1} \) is a NQS-algebra with the generators

\[
x''_1, \ldots, x''_{i_1-1}, x''_1, x''_{i_1+1}, \ldots, x''_{i_2-1}, x''_{i_2}, x''_{i_2+1}, \ldots, x''_M. \tag{3.2}
\]

After \( k \) steps we get the denominator subset \( S := S_\mu, \mu := \{i_1, \ldots, i_k\} \) generated by the system of \( q \)-commuting elements \( y_1, \ldots, y_k \in R \) and \( \mathfrak{N} \). We call \( S \) as the standard denominator subset. The algebra \( \tilde{R} := R^{(k)} = RS_1^{-1} \) is a NQS-algebra over \( \mathcal{C} \mathfrak{N}^{-1} \) with the generators \( \tilde{x}_j := x_j^{(k)} \) and \( y_1^{1}, \ldots, y_k^{1} \). All generators are FA-elements in \( \tilde{R} \).

We denote by \( Y := Y_\mu \) the subalgebra, generated by \( y_1^{1}, \ldots, y_k^{1} \) (or \( x_1^{1}, \ldots, x_k^{1} \), \( (x''_i)^{\pm1}, \ldots, (x''_{i_k})^{\pm1} \)). The imposed relations of \( Y \) are \( y_iy_j = q^{ij}y_jy_i \). The integer matrix \( (t_{ij})_{i,j=1}^k \) is obtained by elementary transformations of submatrix \( S_\mu = (s_{ij})_{ij \in \mu} \) of \( S \). The algebra \( Y \) is an algebra of twisted Laurent polynomials.

By Proposition 2.11, we may treat \( \tilde{R} \) as an iterated \( q \)-skew extension

\[
\tilde{R} = \tilde{R}_1 \supset \tilde{R}_2 \supset \cdots \supset \tilde{R}_M \supset \tilde{R}_{M+1} =: Y
\tag{3.3}
\]

where \( M := M - k \) and \( \tilde{R}_i \cong \tilde{R}_{i+1}[x; \tilde{r}_i, \tilde{s}_i] \).

**Definition 3.1.**

1) We say that \( S := S_\mu \) is \( C \)-admissible if the ideal \( J := J_S \) of \( \tilde{R} \) generated by \( \tilde{x}_i \), \( i \in [1, M] - \mu \) has zero intersection with \( C \);
2) We say that \( S := S_\mu \) is \( \varepsilon \)-admissible if the ideal \( J := J_S \) of \( \tilde{R} \) generated by \( \tilde{x}_i, \varepsilon \), \( i \in [1, M] - \mu \) is proper.
3) We say that \( S := S_\mu \) is \((\varepsilon, \mathcal{D})\)-admissible if the \( \mathcal{D} \)-stable ideal (denote \( \mathcal{D} J \)) of \( \tilde{R}_\varepsilon \) generated by \( J \) (see 2)) is proper.

Notice that, in general, the ideal \( J \) (resp. \( J \)) may have nonzero intersection with \( Y \) (resp. \( Y_\varepsilon \)) and is not prime. For instance, it holds for the algebra \( R_f \) which is constructed by a polynomial \( f \) as follows. This algebra is generated by \( x_1, x_2, y_1^{1}, \ldots, y_k^{1} \) where the elements \( \{y_i\} \) lie in the center and \( x_1x_2 - qx_2x_1 = f(y_1, \ldots, y_k, q) \). The ideal \( J \), generated by \( x_1 \) and \( x_2 \), has nonzero intersection with \( Y \) and is not prime when the polynomial \( f \) is reducible. In the case \( f = f(q) \), the ideal \( J \) has nonzero intersection with \( C \).

**Theorem 3.2.** Let \( R \) be a NQS-algebra obeying Conditions CN1 and CN2, and \( \varepsilon \) be a specialisation of \( C \) obeying the conditions 2) and 3) of Definition 2.18.

1) For any \( \mathcal{I} \in \operatorname{Spec}(R), \mathcal{I} \cap C = 0 \), there exists a unique \( C \)-admissible standard denominator subset \( S := S_\mu \) such that \( \mathcal{I} \cap S = \emptyset \) and \( \mathcal{I} S^{-1} \supset J_S \).
2) Let \( R \) and \( \varepsilon \) be as above. For any prime \( \mathcal{D} \)-stable ideal \( I \) of \( R_\varepsilon \) there exists a unique \((\varepsilon, \mathcal{D})\)-admissible standard denominator subset \( S = S_\mu \) such that \( I \cap S_\varepsilon = \emptyset \) and \( IS_\varepsilon^{-1} \supset J_S \).

**Proof.** Let \( \mathcal{I} \in \operatorname{Spec}(R) \) and \( \mathcal{I} \cap C = 0 \). Suppose that \( x_{i_1}, \ldots, x_{i_{n-1}} \in \mathcal{I} \) and \( y_1 := x_{i_1} \notin \mathcal{I} \). All prime ideals of \( R \) are completely prime [GL2, Theorem 2.3]. (This is false for \( R_\varepsilon \)). Therefore, \( \mathcal{I} \cap \{y_1^t\}_{t \in \mathcal{N}} = \emptyset \). The ideal \( \mathcal{I} \) admits localization over \( S_1 \cdot \mathfrak{N} \)
The factor algebra ideal of an algebra of twisted Laurent polynomials is completely prime [P3, Corollary Z center to D obtain x IS follows that the ideal x IS with prime. Since the subset R y l, 1 ≤ l ≤ 2, we obtain 2).

Proof consider the left filtration I ε \subseteq \cdots \subseteq I_{ε} \subseteq \cdots \subseteq R_{ε} = R_{ε}. A prime (D, Θ')-stable ideal has prime intersections with all subalgebras R_{ε} [P3, Theorem 2.12]. Suppose that I contains x_{1,ε}, ..., x_{i-1,ε} and does not contain y_{i,ε} = x_{i,ε}. The ideal I_{Θ'} \cap R_{ε} is prime. Since

$$\frac{R_{ε}'}{I_{Θ'} \cap R_{ε}'} \cong K[x_{i,ε}],$$

the ideal I_{Θ'} \cap R_{ε} is completely prime. It follows that I_{Θ'} has empty intersection with the subset S_{1,ε} := \{y_{i,n}\}_{n \in \mathbb{N}}. Since y_{i,ε} is a Θ'-eigenvector, I has empty intersection with S_{1,ε}.

Since y_{i} is a FA-element in R, the element y_{i,ε} lies in the center of R_{ε} (see Proposition 2.20). By proof of Corollary 2.9, we can reduce x_{1,ε}, ..., x_{i-1,ε} modulo q - ε and obtain x'_{1,ε}, ..., x'_{i-1,ε} , ε \in IS_{1,ε}^{-1}.

On the second step suppose that x'_{i+1,ε}, ..., x'_{i+1,ε} \in I_{Θ'} S_{1,ε}^{-1} and x'_{i,ε} \notin I_{Θ'} S_{1,ε}^{-1}. The factor algebra

$$\frac{R_{ε}'}{I_{Θ'} \cap R_{ε}'} \cong \frac{K[x_{i,ε}]}{I_{Θ'} \cap K[x_{i,ε}]},$$

is a prime factor of the algebra generated by two q-commuting y_{i} and x'_{i,ε}. The image of x'_{i,ε} is either zero or regular [P3, Lemma 3.11]. Since x'_{i,ε} \notin I_{Θ'}, the image is regular. The ideal I_{Θ'} (and IS_{1,ε}^{-1}) has empty intersection with S_{2,ε} generated by y_{i,ε} and y_{2,ε} (see stratification process). We consider localization over S_{2,ε}. After k steps we obtain 2). □

We say that an ideal of Y_{ε} is D_{0}-stable, if it is stable with respect to all derivations D_{y_{i,j}}, 1 ≤ i ≤ k. By direct calculations [P3, Lemma 3.16],

$$D_{y_{i,j}} y_{i,ε} = t_{ij} l_{i,ε}^{-1} y_{i,ε} y_{j,ε},$$

$$\{y_{i,ε}, y_{j,ε}\} = t_{ij} l_{ij}^{2} \epsilon y_{i,ε} y_{j,ε}.$$  

Theorem 3.3. Let R, ε be as Theorem 3.2. Any prime D-stable ideal of R_{ε} is completely prime.

Proof. Let I be a prime D-stable ideal of R_{ε}. According the previous Theorem,

$$\frac{R_{ε} S_{ε}^{-1} Y_{ε}}{I S_{ε}^{-1} Y_{ε}} = Y_{ε}$$

(3.4)

where Y_{ε} is the algebra of twisted Laurent polynomials generated by y_{1,ε}, ..., y_{k,ε}. It follows that the ideal IS_{ε}^{-1} \cap Y_{ε} of Y_{ε} is prime. Since y_{i} is a FA-element, y_{i,ε} lies in the center Z_{ε} (see Proposition 2.20). The ideal I is D-stable, hence, it is stable with respect to D_{y_{i,j}} : R_{ε} → R_{ε}, 1 ≤ i ≤ k. The same is true for IS_{ε}^{-1} \cap Y_{ε}. Any prime D_{0}-stable ideal of an algebra of twisted Laurent polynomials is completely prime [P3, Corollary
The ideal $I S_{\varepsilon}^{-1} \cap Y_{\varepsilon}$ is completely prime and, therefore, $I$ is completely prime.

Till the end of this section we suppose that $\varepsilon$ is an admissible specialisation of $C$ (see Definition 2.18). One can choose the new generators (monomials) $h_1, \ldots, h_k$ of $Y$ with the following relations

$$h_1 h_2 = q^{m_1} h_2 h_1, \ldots, h_{2r-1} h_2 = q^{m_r} h_2 h_{2r-1}$$

(3.5)

where $m_1, \ldots, m_r$ are relatively prime with $l$ (see Definition 2.18) and $h_{2r+1}, \ldots, h_k$ generate the center of $G$. All elements $h_i$ are FA-elements in $\tilde{R}$. In what follows we suppose that the elements of $C$ and $z_1 := h_{2r+t+1}, \ldots, z_p := h_k$, $p = k - 2r - t$ generate the intersection $\mathfrak{Z} := Y \cap \tilde{Z}$ where $\tilde{Z} := \text{Center}(\tilde{R})$. Denote $u_1 := h_{2r+1}, \ldots, u_t := h_{2r+t}$. We have $\mathfrak{Z} = K[z_1^{\pm 1}, \ldots, z_p^{\pm 1}, q^{\pm 1}]$, $3_{\varepsilon} = K[z_1^{\pm 1}, \ldots, z_p^{\pm 1}]$ and

$$Z(Y) = K[u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1^{\pm 1}, \ldots, z_p^{\pm 1}, q^{\pm 1}],$$

$$Z(Y)_{\varepsilon} := Z(Y) \mod (q - \varepsilon) = K[u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1^{\pm 1}, \ldots, z_p^{\pm 1}],$$

$$Z(Y_{\varepsilon}) := \text{Center}(Y_{\varepsilon}) = K[u_1^{\pm 1}, \ldots, h_{2r}, u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1^{\pm 1}, \ldots, z_p^{\pm 1}].$$

The algebra $Z(Y)_{\varepsilon}$ coincides with subalgebra $Z(Y_{\varepsilon})^P$ which consists of the elements of $Z(Y_{\varepsilon})$ annihilated by all $D_{y_i}$. As above $\tilde{Z}_{\varepsilon} := \text{Center}(\tilde{R}_{\varepsilon})$. The intersection $\tilde{Z}_{\varepsilon} \cap Y_{\varepsilon}$ is a polynomial algebra

$$\tilde{Z}_{\varepsilon} \cap Y_{\varepsilon} = K[h_1^{\pm 1}, \ldots, h_{2r}^{\pm 1}, u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1^{\pm 1}, \ldots, z_p^{\pm 1}].$$

**Notations 3.4.**

1) $G$ is the subgroup in $\tilde{R}$ generated by $S$ (i.e. by $y_1, \ldots, y_k$),

2) $G'$ is its subgroup generated by $y_1', \ldots, y_k'$,

3) $W := \{a \in \tilde{R} : ay = ya \text{ for all } y \in Y\}$,

4) $W_{\varepsilon} := W \mod (q - \varepsilon)$.

The elements of $G$ are FA-elements on $\tilde{R}$. It follows that, for any $y \in G$, the linear operator $\text{Ad}_y$ is diagonalizable over $C\mathfrak{N}^{-1}$, (Proposition 2.20). Since the generators of $G$ are $q$-commuting elements, then $\{\text{Ad}_y : y \in G\}$ is the commutative subgroup of $\text{Aut}(\tilde{R})$. It follows that $\{\text{Ad}_y\}$ are simultaneously diagonalizable.

The map $\Delta_{y^i} := y_{\varepsilon}^{-i} D_{y^i} : \tilde{R}_{\varepsilon} \to \tilde{R}_{\varepsilon}$ is a diagonalizable derivation. Moreover, if $\text{Ad}_y v = q^\alpha v$, then $\Delta_{y^i}(v_{\varepsilon}) = \alpha v_{\varepsilon}$ where $\alpha := q^l_{\varepsilon} - 1$. If $D_{y^i}(v_{\varepsilon}) = 0$ for any $y \in G$, then $v_{\varepsilon} \in W_{\varepsilon}$.

The derivation $\Delta_{y^i}$ preserve the center $\tilde{Z}_{\varepsilon}$ and diagonalizable in it.

**Lemma 3.5.** Let $v \in \tilde{R}$ and $v_{\varepsilon} \in \tilde{Z}_{\varepsilon}$. Then

1) $D_v(Y_{\varepsilon})$ is contained in the ideal $\langle v_{\varepsilon} \rangle$ generated by $v_{\varepsilon}$.

2) If $v_{\varepsilon} \in W_{\varepsilon} \cap \tilde{Z}_{\varepsilon}$, then $D_v(Y_{\varepsilon}) = 0$.

**Proof.** One can assume, that $v_{\varepsilon}$ (resp. $v$) is $\Delta_{y^i}$-eigenvector (resp. $\text{Ad}_y$-eigenvector).

For $\text{Ad}_y v = q^\alpha v$, we have $\Delta_{y^i} v_{\varepsilon} = \alpha v_{\varepsilon}$ and $\alpha y_{\varepsilon}^i v_{\varepsilon} = D_{y^i} v_{\varepsilon}$. On the other hand,

$$D_{y^i} v_{\varepsilon} = \{y_{\varepsilon}^i, v_{\varepsilon} \} = -\{v_{\varepsilon}, y_{\varepsilon}^i \} = -D_v(y_{\varepsilon}^i) = -q^{l_{\varepsilon} - 1} D_v(y_{\varepsilon}).$$

It follows

$$D_v(y_{\varepsilon}) = -\alpha l_{\varepsilon}^{-1} y_{\varepsilon} v_{\varepsilon}.$$

(3.6)
Formula (3.6) yields 1).

To prove claim 2), we decompose $v$ into the sum $v = v_0 + v_1 + \ldots + v_n$ of $\text{Ad}_G$ eigenvectors. Suppose that $v_0 \in W$ (i.e $\text{Ad}_G v = v$ for all $y \in G$) and $\text{Ad} v_i = q^{\alpha_i} v_i$, $\alpha_i \neq 0$ for $1 \leq i \leq n$. Since $v_\epsilon \in W_\epsilon$, then $v_\epsilon = 0$ for $1 \leq i \leq n$. Using (3.6), we have

$$D_v(y_\epsilon) = D_{v_0}(y_\epsilon) + D_{v_1}(y_\epsilon) + \ldots + D_{v_n}(y_\epsilon) = D_{v_0}(y_\epsilon) = 0.$$  

This proves 2). \( \square \)

**Proposition 3.6.** Let $S := S_\mu$ be $(\epsilon, D)$-admissible and $D J$ denotes the lowest $D$-stable ideal which contains $J := J_S$ (see Definition 3.1). Then

$$\tilde{Z}_\epsilon = D J \cap \tilde{Z}_\epsilon + K[h_{1\epsilon}^{\pm l}, \ldots, h_{2t,\epsilon}^{\pm l}](W_\epsilon \cap \tilde{Z}_\epsilon). \quad (3.7)$$

**Proof.** Let $v_\epsilon \in \tilde{Z}_\epsilon$ be a common eigenvector for $\Delta_{G^l}$. If $v_\epsilon \notin D J \cap \tilde{Z}_\epsilon$, then $v_\epsilon = j_0 \epsilon + y_0 \epsilon$ where $j_0 \epsilon \in D J$, $y_0 \epsilon$ is a nonzero element of $Z(Y_\epsilon)$ and $j_0 \epsilon$, $y_0 \epsilon$ are $\Delta_{G^l}$-eigenvectors with the common eigenvalue. One can present $y_0 \epsilon$ in the form $y_0 \epsilon = h_\epsilon y'_0 \epsilon$ where $h_\epsilon$ is some monomial:

$$h_\epsilon := h_{1\epsilon}^{m_1 l} \ldots h_{2t, \epsilon}^{m_{2t} l}$$

with $m_1, \ldots, m_{2t} \in \mathbb{Z}$, $y'_0 \epsilon \in Y_\epsilon$ and $\Delta_{G^l} y'_0 \epsilon = 0$. Then the element $v'_\epsilon := h_\epsilon^{-1} v_\epsilon$ obeys $\Delta_{G^l} v'_\epsilon = 0$. Whence $v'_\epsilon \in W_\epsilon$. Notice that $D J \cap Y_\epsilon$ is a $D_Y$-stable ideal in the algebra of twisted Laurent polynomials $Y_\epsilon$. Hence [P3, Lemma 3.17], $D J \cap Y_\epsilon$ is generated by its intersection with $Z(Y)_\epsilon$.

**Proposition 3.7** 1) If $m$ is a maximal ideal of $Z(Y)_\epsilon$ which lies over $D J \cap Z(Y)_\epsilon$, then $L_m := D J + m Y_\epsilon$ is $D$-stable ideal in $\tilde{R}_\epsilon$;

2) if $\mathcal{M}$ is a maximal ideal of $W_\epsilon \cap \tilde{Z}_\epsilon$ over $D J \cap W_\epsilon \cap \tilde{Z}_\epsilon$, then

$$L_{\mathcal{M}} := D J \cap \tilde{Z}_\epsilon + K[h_{1\epsilon}^{\pm l}, \ldots, h_{2t, \epsilon}^{\pm l}] \mathcal{M}$$

is a Poisson ideal of $\tilde{Z}_\epsilon$.

**Proof.** By the formula $\tilde{R}_\epsilon = D J + Y_\epsilon$ (resp. (3.7)), $L_m$ (resp. $L_{\mathcal{M}}$) is a two-sided ideal in $\tilde{R}_\epsilon$ (resp. $\tilde{Z}_\epsilon$).

Let $v_\epsilon \in \tilde{Z}_\epsilon$ (resp. $v \in \tilde{R}$) be a common $\Delta_{G^l}$-eigenvector (resp. $\text{Ad}_G$). We are going to prove that both ideals $L_m$ and $L_{\mathcal{M}}$ are $D_v$-stable.

If $v_\epsilon \in D J \cap \tilde{Z}_\epsilon$ or $v_\epsilon \in W_\epsilon \cap \tilde{Z}_\epsilon$ the statement is a corollary of Lemma 3.5. For $v_\epsilon \in K[h_{1\epsilon}^{\pm l}, \ldots, h_{2t, \epsilon}^{\pm l}]$, the derivation $D_v$ is zero in $Z(Y)_\epsilon$ and $W_\epsilon$. Both $m$ and $\mathcal{M}$ are annihilated by $D_v$. The ideals $L_m$ and $L_{\mathcal{M}}$ are $D_v$-stable. \( \square \)

### 4 Irreducible representations

Let $R$ be an algebra and a free $C$-module. One can consider specialisation $R_\epsilon$ of $R$. As above $Z_\epsilon$ is a center of $R_\epsilon$. This algebra has a Poisson structure via quantum adjoint action (see Section 2).

Let $\chi$ be a central character $\chi : Z_\epsilon \to K$ and $m(\chi)$ the corresponding maximal ideal. We treat $\chi$ as a point of variety $\mathcal{M} := \text{Maxspec}(Z_\epsilon)$. We consider stratification of $\mathcal{M}$ [BG1]: $\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_m = \emptyset$ where $\mathcal{M}_{i+1} = (\mathcal{M}_i)_{\text{sing}}$. All $\mathcal{M}_i$,
are Poisson varieties. In the case $K = \mathbb{C}$, the smooth locuses $\mathcal{M}_i^0 := \mathcal{M}_i - \mathcal{M}_{i+1}$ are complex analytic Poisson varieties. Each symplectic leaf is a disjoint union of symplectic leaves. For $\chi$ we denote by $\Omega_\chi$ the corresponding symplectic leaf.

Let $m(\chi, \mathcal{D})$ be the greatest Poisson (i.e. $\mathcal{D}$-stable) ideal in $m(\chi)$. One can treat the algebra $\mathcal{F} := Z_\varepsilon/m(\chi, \mathcal{D})$ as the algebra of regular functions on Zariski closure $\mathcal{M}_\chi$ of $\Omega_\chi$. Denote by $R_{\varepsilon, \chi}$ the finite dimensional subalgebra $R_\varepsilon/m(\chi)R_\varepsilon$.

**Lemma 4.1** [P3, Lemma 5.1]. Let $K = \mathbb{C}$. Let $f$ be a nonzero element of $\mathcal{F}$. There exists $\chi' \in \Omega_\chi$ such that $f(\chi') \neq 0$ and the algebra $R_{\varepsilon, \chi'}$ is isomorphic to $R_{\varepsilon, \chi}$.

**Theorem 4.2.** Let $K = \mathbb{C}$, $R$ be a NQS-algebra obeying Conditions CN1 and CN2, and $\varepsilon$ is an admissible specialisation of $C$. Let $\pi$ be an irreducible representation with central charakter $\chi$. Then

1) $\dim(\pi) = l\dim(\Omega_\chi)$,
2) $\Omega_\chi$ is algebraic (i.e. it is Zariski open in its Zariski closure),
3) the algebras $R_{\varepsilon, \chi}$ and $R_{\varepsilon, \chi'}$ are isomorphic for any $\chi', \chi'' \in \Omega_\chi$.

**Proof.** For an irreducible representation $\pi$ with the central charakter $\chi$, we consider its kernel $I(\pi)$ in $R_\varepsilon$. This ideal is prime and the greatest $\mathcal{D}$-stable ideal $I(\pi, \mathcal{D})$ in $I(\pi)$ is completely prime (see Theorem 3.3). The ideal $m(\chi, \mathcal{D})$ coincides with $I(\pi, \mathcal{D}) \cap Z_\varepsilon$.

By Theorem 3.2, there exists $(\varepsilon, \mathcal{D})$-admissible denominator set $S := S_\mu$ with empty intersection with $I(\pi, \mathcal{D})$. The ideal $I(\pi, \mathcal{D})$ admits localization $\tilde{I}(\pi, \mathcal{D}) := I(\pi, \mathcal{D})S_\varepsilon^{-1}$ and $\tilde{I}(\pi, \mathcal{D}) \supset \mathcal{D}J$ (see Section 3). The subset $S_\lambda := \{h^i : h \in S\}$ is a denominator subset consisting of $q$-commuting FA-elements and $\tilde{R} := RS_\varepsilon^{-1} = RS_\lambda^{-1}$. The subset $S_{\lambda \varepsilon}$ belongs to the center $Z_\varepsilon$; it is a denominator subset in $R_\varepsilon$ and $\tilde{R}_\varepsilon := R_\varepsilon S_{\lambda \varepsilon}^{-1} = R_\varepsilon S_{\lambda \varepsilon}^{-1}$. The ideal $m(\chi, \mathcal{D})$ has empty intersection with $S_{\lambda \varepsilon}$. We denote $\tilde{Z}_\varepsilon = Z_\varepsilon S_{\lambda \varepsilon}^{-1}$ and $\tilde{m}(\pi, \mathcal{D}) := m(\chi, \mathcal{D})S_{\lambda \varepsilon}^{-1} = \tilde{I}(\pi, \mathcal{D}) \cap \tilde{Z}_\varepsilon \supset \mathcal{D}J \cap \tilde{Z}_\varepsilon$. By Lemma 4.1, we may require $\chi(y) \neq 0$ for any $y \in S_\lambda$. Since $\pi$ is an irreducible representation, $\pi(y^i) = \chi(y^i) \cdot \text{id}$. The ideal $I(\pi)$ admits localization over $S_{\lambda \varepsilon}$ and $\bar{\pi}$ is an irreducible representation of $\tilde{R}_{\varepsilon}$.

Recall $\tilde{R}_{\varepsilon} := \mathcal{D}J + Y_\varepsilon$ (Section 3). One can treat $\tilde{R}$ as a free left (and right) $Y$-module. We form the free basis which consists of monomials (in the lexicographical order) of $\{\tilde{x}_i\}$. Denote by $\rho_\varepsilon$ the natural projection $\rho : \tilde{R} \to Y$. The projection $\rho$ is a morphism of left (and right) $Y$-modules. Similarly, $\rho_\varepsilon : \tilde{R}_\varepsilon \to Y_\varepsilon$ is a morphism of $Y_\varepsilon$-modules and $\rho_\varepsilon \mathcal{D}_Y(a) = \mathcal{D}_Y \rho_\varepsilon(a)$, for any $y \in G$ and $a \in \tilde{R}_\varepsilon$. It follows that $\rho_\varepsilon(W_\varepsilon) = Z(Y)_\varepsilon$.

The representation $\pi$ passes through $\rho_\varepsilon$ and determined by

$$\nu_\alpha, 1 \leq \alpha \leq 2r; \quad \lambda_\beta, 1 \leq \beta \leq t; \quad \xi_\gamma, 1 \leq \gamma \leq p$$

where $\pi(h^i_{\alpha \varepsilon}) = \nu_\alpha \cdot \text{id}$, $\pi(u_{\beta \varepsilon}) = \lambda_\beta \cdot \text{id}$ and $\pi(z_{\gamma \varepsilon}) = \xi_\gamma \cdot \text{id}$.

The ideal $\tilde{I}(\pi) := I(\pi)S_{\lambda \varepsilon}^{-1}$ is the maximal ideal of $Z(Y_\varepsilon)$ generated by all $h^i_{\alpha \varepsilon} - \nu_\alpha$, $u_{\beta \varepsilon} - \lambda_\beta$, $z_{\gamma} - \xi_\gamma$. We obtain

$$\dim(\pi) = l^{2r}. \quad (4.1)$$

Denote by $\lambda$ the character of $Z(Y)^\varepsilon$ determined by $\pi$. The ideal

$$m_\pi := \text{Ker}(\lambda) = \sum_{1 \leq \beta \leq t} Y_\varepsilon(u_{\beta \varepsilon} - \lambda_\beta) + \sum_{1 \leq \gamma \leq p} Y_\varepsilon(z_{\gamma} - \xi_\gamma) \quad (4.2)$$

is a maximal $\mathcal{D}_Y$-stable ideal in $Y_\varepsilon$.
Character $\lambda$ obeys the condition $\lambda|_{D' \cap Z(V)} = 0$. We have

$$\bar{I}(\pi, D) \subset D J + Y_{\pi} m_{\pi} \subset \bar{I}(\pi).$$

By Proposition 3.17, the middle ideal is $D$-stable. It implies

$$\bar{I}(\pi, D) = D J + Y_{\pi} m_{\pi}. \quad (4.3)$$

Similarly, by (3.7), the central character $\chi$ also passes through $\rho_\varepsilon$ and determined by $\nu_\alpha = \chi(h^l_{\alpha e})$ and $\chi|_{W_\varepsilon \cap Z_\varepsilon}$.

We consider

$$M_\chi := \text{Ker} \chi|_{W_\varepsilon \cap Z_\varepsilon}$$

and obtain

$$\bar{m}(\pi, D) = D J \bigcap Z_\varepsilon + K[h^l_{1\varepsilon}, \ldots, h^l_{2r \varepsilon}] M_\chi. \quad (4.4)$$

Comparing (3.7) and (4.4), we see that the algebra $\bar{Z}_\varepsilon/\bar{m}(\pi, D)$ is isomorphic (as a Poisson algebra) to $\mathbb{C}[h^l_{1\varepsilon}, \ldots, h^l_{2r \varepsilon}]$ with the Poisson bracket

$$\{h^l_{1\varepsilon}, h^l_{2\varepsilon}\} = m_1 l^2 \varepsilon^{-1} h^l_{1\varepsilon} h^l_{1\varepsilon}, \ldots, \{h^l_{2r-1\varepsilon}, h^l_{2r \varepsilon}\} = m_{r} l^2 \varepsilon^{-1} h^l_{2r-1\varepsilon} h^l_{2r \varepsilon}.$$  

The maximal spectrum of above Poisson algebra has a single symplectic leaf. It follows that the symplectic leaf $\Omega_\chi$ contains the subset $O$ which is Zariski-open in the Zariski closure $M_\chi := \overline{\Omega}_\chi$. It follows dim $\Omega_\chi = \dim O = 2r$ and, by (4.3), dim($\pi$) = $l^2 \dim(\Omega_\chi)$. This proves 1).

2) For any $\chi' \in M_\chi - \Omega_\chi$, we see

$$\dim \Omega_\chi' \leq \dim (M_\chi - O) < \dim O = \dim \Omega_\chi.$$  

Then $M_\chi - \Omega_\chi = \{\chi' \in M_\chi : \dim \Omega_\chi' < 2r\}$. On the other hand, the subset $M_{< 2r} := \{\chi' \in M : \dim \Omega_\chi' < 2r\}$ is Zariski closed in $M$. The subset $M_\chi - \Omega_\chi$ coincides with $M_{< 2r} \bigcap M_\chi$ and is Zariski-closed. Hence $\Omega_\chi$ is Zariski-open in $M_\chi$.

3) Consider the equivalence relation on $\Omega_\chi$ as follows $\chi \cong \chi'$ iff $R_\varepsilon/m(\chi) R_\varepsilon \cong R_\varepsilon/m(\chi') R_\varepsilon$. Any equivalence class $[\chi] = \{\chi' : \chi' \cong \chi\}$ is an open subset of $\Omega_\chi$ in the topology of complex smooth manifold [P3, Lemma 5.1]. The manifold $\Omega_\chi$ is connected. This proves claim 3). □

**Theorem 4.3.** Let $K$ be an algebraic closed field of zero characteristic. Let $R$ and $\varepsilon$ be as in Theorem 4.2. Any two vertices $e_i, e_j$, $i \neq j$ of the quiver of algebra $R_\varepsilon, \chi$ are linked by the wedges $(e_i, e_j)$ and $(e_j, e_i)$. In particular the quiver is connected.

**Proof.**

**Step 1.** Let us prove that all irreducible representations over a common central character $\chi$ can be passed through suitable localization $\tilde{R}_\varepsilon$ such that $\pi(D J) = 0$ for any $\pi$ over $\chi$.

For any irreducible representation $\pi$ there exists $(\varepsilon, D)$-admissible standard denominator subset $S := S_\mu$ such that $I(\pi, D) \bigcap S_\varepsilon = \emptyset$ and $\bar{I}(\pi, D) \supset D J$. We may assume that $I(\pi) \bigcap S_\varepsilon = \emptyset$ (see Proof of Theorem 4.2). As above $\chi$ is the central character of $\pi$. The ideal $\bar{m}(\pi, D)$ admits localization over $S_\varepsilon$ and $\bar{I}(\pi) \supset R_\varepsilon \bar{m}(\chi) \supset R_\varepsilon \bar{m}(\chi, D)$.  

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For another irreducible representation $π'$ over $χ$, we also have $I(π') ⊇ R_εm(χ) ⊇ R_εm(π, D)$. For the greatest $D$-stable ideal $I(π', D)$ in $I(π')$, we obtain $I(π') ⊇ I(π', D) ⊇ R_εm(χ, D)$ and $I(π', D) ∩ Z_ε = m(χ, D)$. It implies that $I(π', D) ∩ S_ε = ∅$ and $I(π', D) ⊃ \{t(\tilde{r}_k), \tilde{r}_k \in [1, M - μ]\}$ (see Section 3). That is $I(π', D)$ contains $ΔJ$ and admits localization over $S_ε$. Ideal $I(π')$ also admits localization over $S_ε$. This proves the claim of Step 1.

Step 2. According to Step 1, any irreducible representation $π$ over $χ$ is a representation of $R_ε$ and its kernel contains $ΔJ$. Then $ΔJ \mod m(χ)$ is contained in the radical of $R_ε/m(χ)R_ε$.

By (3.7), $π$ lies over $χ$ iff $π(h_{ie}^j) = χ(h_{ie}^j) \cdot id$ and

$$π|_{W_ε} = χ|_{W_ε} \cdot id.$$ (4.5)

Denote $Z(Y_ε)' := ρ_ε(W_ε \cap \tilde{Z}_ε)$. Since $ρ_ε(W_ε) = Z(Y_ε)$, then $Z(Y_ε)'$ is a subalgebra of $Z(Y_ε)$.

The character $χ$ defines the character $χ'$ on $Z(Y_ε)'$ such that $χ'ρ_ε(w_ε) = χ(w_ε)$ for any $w_ε \in W_ε \cap \tilde{Z}_ε$. In particular, $χ'(z_{ie}) = χ(z_{ie})$, and $χ'(u_{ie}^j) = χ(u_{ie}^j)$.

According to the proof of Theorem 4.2, there exists 1-1 correspondence between irreducible representations over $χ$ and characters $λ$ of $Z(Y_ε)$ such that

$$λ|_{Z(Y_ε)'} = χ'|Z(Y_ε)'.$$ (4.6)

We will say that such $λ$ is comparable with $χ$. In particular, $λ(z_{ie}) = χ'(z_{ie})$, and $λ_{ie} = λ(u_{ie}^j) = χ'(u_{ie}^j) =: λ_i$.

Two characters $λ, λ'$ over $χ$ differ $λ_i' = ε_iλ_i$, $1 ≤ i ≤ r$ where $ε_1, \ldots, ε_r$ are $l$th roots of unity. Denote

$$ε_λ = l^{-r} \prod_{i=1}^{r}((λ_{-1}^{-1}u_{ie})^{l-1} + (λ_{-1}^{-1}u_{ie})^{l^2} + \cdots + 1).$$

The elements $\{ε_λ\}$ obey $ε_λ^2 = ε_λ$. If $λ$ is comparable (resp. non-comparable) with $χ_i$ then $ε_λ$ is a primitive idempotent corresponding $π$ (resp. is a zero element of $R_ε/\tilde{m}(χ)R_ε$).

By choice of $u_1, \ldots, u_t$ (see (3.5) and below), there exist $v_1, \ldots, v_t$ such that $v_iu_j = q^{a_{ij}}u_jv_i$ where $d := \det(n_{ij})_{ij=1}^r$ and $d$ is relatively prime with $l$ (see Definition 3.1).

For any system $(ε_i, \ldots, ε_r)$ of $t$th roots of unity, there exists $v \in R_ε$ such that

$$vu_i = ε_iu_iv.$$ (4.7)

Let us prove that one can choose $v \notin \tilde{m}(χ)R_ε$. Since the Ad-action of the subgroup $U_ε$ generated by $u_{ie}$, $1 ≤ i ≤ r$ is diagonalizable, one can decompose $\tilde{R}_ε = \tilde{m}(χ)R_ε \oplus V$ where $V$ is some finite dimensional $AdU_ε$-stable subspace. Consider the completion $\hat{R}_ε$ (resp. $\hat{Z}_ε$) of $R_ε$ (resp. $\tilde{Z}_ε$) in the $\tilde{m}(χ)$-adic topology. We have decomposition $\hat{R}_ε = \hat{Z}_ε \otimes R_ε \cong \hat{Z}_ε \oplus V$. The $AdU_ε$-action is identical in $\hat{Z}_ε$. One can choose $v \in V$.

Put $\hat{v} := v \mod \tilde{m}(χ)\tilde{R}_ε$. We have proved that $\hat{v} \neq 0$. For $(ε_1, \ldots, ε_r) \neq (1, \ldots, 1)$, the element $v$ lies in $ΔJ$. The formula (4.7) implies that for different primitive idempotents $ε_λ, ε_λ'$ of $R_ε/\tilde{m}(χ)\tilde{R}_ε$ there exits an nonzero element $\hat{v}$ of the radical such
that
\[ \hat{v} e_\lambda = e_{\lambda'} \hat{v}. \]

The idempotents \( \lambda \) and \( \lambda' \) are linked by the wedge (as vertices of the quiver of algebra \( R_{\varepsilon,\lambda} \)) [Pie, 6.4]. \( \square \)

5 On number of irreducible representations

The goal of this section is to prove the statements on the number of irreducible representations over the common central character.

We begin with the proof of the formula (5.1) for some ideal in iterated skew polynomial extension. The property (5.1) is well known for commutative rings [AM, Corollary 10.18]. Notice that, in general, (5.1) is false for noncommutative iterated extensions (for instance, take \( R = U(\mathfrak{g}) \) for two-dimensional Lie algebra \([x,y] = y\) and \( I = \langle x, y \rangle \)).

**Lemma 5.1** Let we have an iterated \( q \)-skew extension \( \mathcal{R} = \mathcal{R}_1 \supset \ldots \supset \mathcal{R}_n = \mathcal{Y} \) of the algebra \( \mathcal{Y} \) over the field \( \mathcal{F} \); \( \mathcal{R}_i = \mathcal{R}_i + \mathcal{R}_i \cdot x_i \cdot \tau_i \cdot \delta_i \) where \( \tau_i \) is a diagonal automorphism of \( \mathcal{R}_i + \mathcal{R}_i \cdot x_i \cdot \tau_i \cdot \delta_i \) with \( q_i \in \mathcal{F}^* \). We impose the following requirements.

1) \( \mathcal{Y} \) is a free module over its center and a Noetherian domain;
2) any ideal of \( \mathcal{Y} \) is generated by its intersection with the center;
3) \( \delta_i(\mathcal{Y}) = 0 \); 
4) any \( \delta_i \) is locally nilpotent in \( \mathcal{R}_i + \mathcal{R}_i \cdot x_i \cdot \tau_i \cdot \delta_i \).

Let \( \mathcal{I} \) be an ideal of \( \mathcal{R} \) which contains \( x_1, \ldots, x_n \). Then
\[ \bigcap_{m=1}^{\infty} \mathcal{I}^m = 0. \] (5.1)

**Proof.** We shall prove by induction on \( n \). If \( n = 1 \), then \( \mathcal{R} = \mathcal{Y} \). Let \( \{f_\alpha\} \) be the free basis of \( \mathcal{Y} \) over its center \( Z(\mathcal{Y}) \). The ideal \( \mathcal{I} \) is generated by its intersection with \( Z(\mathcal{Y}) \). Then \( \mathcal{I} = \{ \sum c_\alpha f_\alpha : c_\alpha \in \mathcal{I} \cap Z(\mathcal{Y}) \} \). Hence \( \mathcal{I}^m \subset \{ \sum c_\alpha f_\alpha : c_\alpha \in (\mathcal{I} \cap Z(\mathcal{Y}))^m \} \).

The property (5.1) is true for \( \mathcal{I} \cap Z(\mathcal{Y}) \); it is true for \( \mathcal{I} \).

Suppose that (5.1) is true for extensions of the length \( \leq n \). Let us prove for an extension of the length \( n + 1 \). Let \( \mathcal{R} \) be the iterated extension of \( \mathcal{Y} \) that obeys the requirements of the Lemma
\[ \mathcal{R} = \mathcal{R}_s [x; \tau, \delta] \supset \mathcal{R}_s \supset \mathcal{R}_{s+1} \supset \ldots \supset \mathcal{R}_n = \mathcal{Y}. \]

By the induction hypothesis, (5.1) is true for the ideal \( \mathcal{I}_* = \mathcal{I} \cap \mathcal{R}_s \) of \( \mathcal{R}_s \).

Since \( x \in \mathcal{I}_* \), then \( \delta(\mathcal{R}_s) \subset \mathcal{I}_* \). Any element of \( \mathcal{I} \) has the form \( r_0 + x r_1 + x^2 r_2 + \ldots \) where \( r_0 \in \mathcal{I}_* \) and \( r_i \in \mathcal{R}_s \), \( i \geq 1 \). Therefore, \( \mathcal{I}^m \) is the span of \( x^k b_k \),
\[ b_k = \delta^{\alpha_1}(r_1) \cdots \delta^{\alpha_n}(r_n) j_1^{\beta_1} \cdots j_t^{\beta_t} \] (5.2)
where \( k, \alpha_i, \beta_i, n, t \) are nonnegative integers, \( r_i \in \mathcal{R}_s \), \( j_i \in \mathcal{I}_* \) and
\[ k + \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_t \geq m. \] (5.3)
Suppose that \( a \in \bigcap_{m=1}^{\infty} \mathcal{I}^m \) and \( a \neq 0 \). Then \( a = x^k b_k + x^{k+1} b_{k+1} + \ldots, \ b_k \neq 0 \). For any \( m \) one can present \( b_k \) in the form (5.2) where \( \alpha_i, \beta_i, t, n \) depends on the choice of \( m \) and (5.3) holds.

On the other hand, since \( \bigcap \mathcal{I}_s^m = 0 \), there exists \( m_0 \) such that

\[
b_k \in \mathcal{I}_{s}^{m_0} \quad \text{and} \quad b_k \notin \mathcal{I}_{s}^{m_0+1}. \tag{5.4}\]

The condition (5.4) yields \( \beta_1 + \ldots + \beta_i \leq m_0 \) and the number of nonzero \( \alpha_i \) is also \( \leq m_0 \).

Recall that \( \delta \) is locally nilpotent \( \tau \)-derivation; there exists \( N \) such that \( \delta^N (x_i) = 0 \) for all \( i \). It implies \( \delta^{\alpha_i N} (R_i) \subseteq \mathcal{I}_s^\alpha \) for all \( n \). In particular, \( \delta^{(m_0+1)N} (R_i) \subseteq \mathcal{I}_s^{m_0+1} \). Since \( b_k \notin \mathcal{I}_s^{m_0+1} \), then \( \alpha_i < (m_0 + 1)N \) for any \( i \).

We conclude that the left side of inequality (5.3) is restricted as \( m \) tends to infinity. This leads to a contradiction. The ideals \( \mathcal{I}_s^m \) have zero intersection. □.

Let \( S := S_\mu \) be the standard denominator subset (see Section 3). Recall that after localisation of a NQS-algebra, we obtain an iterated \( q \)-skew extension \( \bar{R} := \bar{R}_1 \supset \cdots \supset \bar{R}_3 \supset \bar{Y} \) where \( M := M - k \) and \( \bar{R}_i \cong \bar{R}_{i+1}[x; \bar{t}_i, \bar{\delta}_i] \) (see (3.3)). As above we denote by \( \bar{J} = \bar{J}_S \) the lowest ideal of \( \bar{R} \) which contains \( \bar{x}_i \) for all \( i \). Let \( \{ Q_1, \ldots, Q_m \} \) be the set of all minimal prime ideals over \( \bar{J} \). Denote

\[
X_1 := X_{1S} := \{ Q_i : Q_i \cap C = 0 \},
\]

\[
X_2 := X_{2S} := \{ Q_i : Q_i \cap C \neq 0 \}.
\]

**Proposition 5.2.** Suppose that \( X_1 \neq \emptyset \) and \( Q \in X_1 \).

1) The ideal \( Q \cap Y \) is generated by \( Q \cap 3 \).
2) \( Q = \bar{J} + \bar{R}(Q \cap Y) \).

**Proof.** The second statement is the easy corollary of the first. Our goal is to prove statement 1).

First notice that any prime ideal of \( X_1 \) is completely prime. [GL2, Theorem 3.2]. We will prove the statement by induction on \( M \). The statement is obviously true for \( M = 0 \).

We assume that the statement is true for an algebra of length \( \bar{M} \). Our aim is to prove the statement for \( \bar{R} \) of the length \( \bar{M} + 1 \). Let \( \bar{R}_s \) be the subalgebra generated by \( Y \) and all \( \{ \bar{x}_i \} \) apart from the first, \( \bar{R} = \bar{R}_s[\bar{x}; \bar{t}, \bar{\delta}], \bar{t}\bar{\delta} = q^s \bar{t}\bar{\delta} \) with \( s \neq 0 \) and \( \bar{J}_s \) be the ideal \( \bar{R} \) which contains \( \{ \bar{x}_i \} \).

Since completely prime ideal \( Q \cap \bar{R}_s \) of \( \bar{R}_s \) contains \( \bar{J}_s \), then there exists some minimal prime ideal \( Q_s \) of \( \bar{R}_s \) such that \( Q \cap \bar{R}_s \supset Q_s \supset \bar{J}_s \).

Since \( 0 = Q \cap C \supset Q \cap \bar{J}_s \), then \( Q_s \cap C = 0 \). Whence \( Q_s \) obeys the requirements of Proposition. In particular, \( Q_s \) is a completely prime ideal of \( \bar{R}_s \). The ideal \( q := \bar{R}(Q_s \cap 3) \) of \( 3 \) is completely prime. We retain the former notations \( \bar{R}, Y, Q \) for \( \bar{R}_s \).

By this agreement, \( Q_s \cap Y = 0 \). We obtain the natural projection \( \pi_S : \bar{R}_s \to Y \) with the kernal \( Q_s \). We denote by \( B \) the denominator subset \( Y - \{ 0 \} \). The algebra \( R_s := \bar{R}_s B^{-1} \) is an iterated \( q \)-skew polynomial extension of \( \mathcal{Y} := Y B^{-1} \). The ideal \( J_s := \mathcal{Q} B^{-1} \) of \( R_s \) obeys the requirements of Lemma 5.1. One can choose the subset \( \Psi \subset \{ \bar{x}_i \in \bar{R}_s \} \) which form \( \mathcal{Y} \)-basis of \( J_s \) over \( \mathcal{I}_s^2 \). By (5.1), the set \( \Psi^m \) which consists...
of products of arbitrary $m$ elements of $\Psi$, generate $\mathcal{J}_s$ over $\mathcal{J}_s^{m+1}$. This implies that, if an element of $Z(Y)$ commute with all elements of $\Psi$, then it lies in the center of $\tilde{R}$.

We put $\tilde{\tau}(u_i) = q^{n_i1}u_i$ and denote by $\mathcal{J}_s$ the intersection of the center of $\tilde{R}$ with $Y$. The subalgebra $\mathcal{J}_s$ is contained in $Z(Y)$. The following case take place.

Case 1. $\mathcal{J}_s = \mathcal{J}$. There exist elements $\Phi := \{v_1, \ldots, v_t\} \subset \Psi$ such that $u_iv_j = q^{n_{ij}}v_iu_j$, $d := \det(n_{ij})_{ij=1}^t \neq 0$. We put $\Phi_s := \Phi$, if there is no $v_{j_0} \notin \Phi$ such that $u_jv_{j_0} = q^{n_{j_0}}v_{j_0}u_j$ for all $i$. If the above $j_0$ exists, it is unique and we put $\Phi_s := \Phi - \{v_{j_0}\}$.

Case 2. $\mathcal{J}_s \neq \mathcal{J}$. One can suppose that $\mathcal{J}_s$ is generated by $\mathcal{J}$ and $u_i$. Remark $\alpha_t \neq 0$ (otherwise $u_i \in \mathcal{J}$). There exist elements $\Phi_s = \{v_1, \ldots, v_{t-1}\} \subset \Psi$ such that all $v_i$ commute with $u_i$, then $u_iv_j = q^{n_{ij}}v_iu_j$, $d' := \det(n_{ij})_{ij=1}^{t-1} \neq 0$. In the Case 2, we put $\Phi_s := \Phi_s \cup \{\tilde{x}\}$.

Step 1. We are going to prove that $\tilde{\delta}(v_j) \in Q_s$ for any $v_j \in \Phi_s$. That is $b_j := \pi_S(\tilde{\delta}(v_j)) = 0$ (for $v_j \in \Phi_s$).

Since $u_iv_j = q^{n_{ij}}v_iu_j$, then, using $\tilde{\delta}(u_i) = 0$, we obtain $q^{n_{ij}}\bar{u}_j\tilde{\delta}(v_j) - q^{n_{ij}}\tilde{\delta}(v_j)u_i = 0$. The element $u_i$ lies in the center $Z(Y)$ and is invertible. We have $(q^{n_{ij}} - q^{n_{ij}})b_j = 0$. Recall $v_j \in \Phi_s$; there exists $i_0$ such that $q^{n_{i_0}} \neq q^{n_{j_0}}$. It implies $b_j = 0$. This concludes Step 1.

Step 2. Recall that $\tilde{\tau}$ (but not any $\tilde{\tau}_i$) is an automorphism of $\tilde{R}$. Then the ideal $Q$ (and $Q \cap Y$) is $\tilde{\tau}$-stable. As for $\tilde{\tau}_i$, $1 \leq i$, this map is the automorphism of $\tilde{R}_{i+1}$ (but not of $\tilde{R}$). We are going to prove that the ideal $Q \cap Y$ is $\tilde{\tau}$-stable for $v_i \in \Phi_s$. It suffices to verify that, for any generator $a \in \{\tilde{x}_i\}$ of $\mathcal{J}_s$, the element $b := \pi_S(\tilde{\delta}(a))$ is $\tilde{\tau}$-eigenvector.

Each $a$ is an $\text{Ad}_{C}$-eigenvector, then $b$ is also $\text{Ad}_{C}$-eigenvector with the same eigenvalue. Multiplying $a$ (and $b$) by suitable monomial $h_1^{m_1}\cdots h_{2r}^{m_{2r}}$, we may assume that $\text{Ad}_{\tilde{\tau}}b = b$. That is $b \in Z(Y)$.

Each generator $a$ is a FA-element (indeed FA$_q$-element) of $\tilde{R}$. Take $v_i \in \Phi_s$. Since all generators are $\tilde{\tau}$-eigenvector, $\tilde{\tau}(v_i) = q^{\beta_i}v_i$. There exists a polynomial $f(t) = c_0b^N + c_1t^{N-1} + \cdots + c_N$ such that $f(t) := c_0 \prod_{m=1}^N (t - q^m)$ such that
$$c_0a^Nv_i + c_1a^{N-1}v_i + \cdots + c_Nv_i = 0. \quad (5.5)$$

We act by $\delta$ $N$ times on (5.5) and obtain
$$c_0^\delta(a)Nv_i + c_1^\delta(a)N-1v_i + \cdots + c_N^\delta(a)N = 0 \mod (Q_s^2)$$
where $c_j := q^{\beta_j}c_j$. Then
$$c_0^\delta b^Nv_i + c_1^\delta b^{N-1}v_i + \cdots + c_N^\delta b^N = 0 \mod (Q_s^2),$$
$$c_0^\delta b^Nv_i + c_1^\delta b^{N-1}\tilde{\tau}_i(b) + \cdots + c_N^\delta \tilde{\tau}_i(b)^N = 0 \mod (Q_s^2).$$
Each $v_i$ is an element of $\mathcal{J}$-basis of $\mathcal{J}$ over $\mathcal{J}^2$. Therefore
$$c_0^\delta b^N + c_1^\delta b^{N-1}\tilde{\tau}_i(b) + \cdots + c_N^\delta \tilde{\tau}_i(b)^N = 0,$$
$$\prod_{m=1}^N (b - q^m + \beta_i\tilde{\tau}_i(b)) = 0.$$
Since \( Y \) is a domain, then \( b \) is \( \tilde{\tau}_t \)-eigenvector. The ideal \( Q \cap Y \) is \( \tilde{\tau}_t \)-stable.

**Step 3.** Any ideal of \( Y \) is generated by its intersection with the center; \( Q \cap Y \) is generated by \( Q \cap Z(Y) \). We have proved that \( Q \cap Y \) is \( \tilde{\tau}_t \)-stable for \( v_i \in \Phi \), that is \( Q \cap Y \) is generated by \( \tilde{\tau}_t \)-eigenvectors (for \( v_i \in \Phi \)). All these eigenvectors have the form \( u_i^m \cdot u_t^m \cdot z \) where \( z \in \mathfrak{J} \). The elements \( u_t \) are invertible; the ideal \( Q \cap Y \) is generated by \( Q \cap Y \). \( \square \).

**Notation 5.3.** For any \( S := S_{\mu} \) and any \( Q \in X_{2S} \), we consider the finite subset \( E_{S,Q} \subset K \) which consists of elements \( \varepsilon \in K \) such that \( Q \cap C_{\varepsilon} = 0 \). We denote

\[
E_S := \bigcup_{Q \in X_{2S}} E_{S,Q},
\]

\[
E := \bigcup_S E_S.
\]

Notice that the sets \( E_S \) and \( E \) is finite.

We consider specialisation of NQS-algebra \( R_\varepsilon \) at admissible root of unity. Let \( J := J_S \) and \( \mathcal{P}J \) be the ideals of \( R_\varepsilon \) that were defined in Definition 3.1. Let \( P \) be some minimal prime ideal of \( \tilde{R}_\varepsilon \) over \( J \).

**Proposition 5.4.** \( R, \varepsilon, P \) as above. Suppose that \( \varepsilon \notin E_S \). Then

1) \( P \cap Y_\varepsilon \) is generated by \( P \cap \mathfrak{J}_\varepsilon \), \( P = \mathcal{P}J + \tilde{R}_\varepsilon (P \cap \mathfrak{J}_\varepsilon) \);

2) \( P \) is a \( \mathcal{D} \)-stable ideal.

**Proof.** The ideal \( \mathcal{P} := \pi_\varepsilon^{-1}(P) \) is prime and \( \mathcal{P} \cap C = (q - \varepsilon)C \). By definition of \( P \), the ideal \( \mathcal{P} \) contains \( \mathcal{J} \). Then \( \mathcal{P} \) contains some minimal prime ideal \( Q \) over \( \mathcal{J} \). If \( Q \in X_{2S} \), then \( (q - \varepsilon)C = \mathcal{P} \cap C \supset Q \cap C \). Whence \( Q \cap C \) is zero at \( q = \varepsilon \). This leads to contradiction.

Hence \( Q \in X_{1S} \). According to Proposition 5.2, \( Q \cap Y \) is generated by \( Q \cap \mathfrak{J} \). Then \( Q = \mathcal{J} + \tilde{R}(Q \cap \mathfrak{J}) \). Specialising modulo \( q - \varepsilon \), we obtain \( Q_\varepsilon = J + \tilde{R}_\varepsilon q \) where \( q := Q_\varepsilon \cap \mathfrak{J}_\varepsilon \). We have \( P \supset Q_\varepsilon \supset J \). The ideal \( Q_\varepsilon \) is \( \mathcal{D} \)-stable [P3, Lemma 3.12]. It implies \( Q_\varepsilon \supset \mathcal{P}J, Q_\varepsilon = \mathcal{P}J + \tilde{R}_\varepsilon q \) and \( \mathcal{P}J \cap Y_\varepsilon \subset Y_\varepsilon q \).

Recall that the ideal \( P \) is prime; \( P \cap \mathfrak{J}_\varepsilon \) is a prime ideal of \( \mathfrak{J}_\varepsilon \). There exists minimal prime ideal \( \mathfrak{p} \) of \( \mathfrak{J}_\varepsilon \) such that \( P \cap \mathfrak{J}_\varepsilon \supset \mathfrak{p} \supset q \). We have \( P \supset \tilde{R}_\varepsilon \mathfrak{p} \). Since \( P \supset Q_\varepsilon \supset \mathcal{P}J \), then \( P \supset \mathcal{P}J + \tilde{R}_\varepsilon \mathfrak{p} \supset J \). The middle ideal is prime, then \( P = \mathcal{P}J + \tilde{R}_\varepsilon \mathfrak{p} \). Similarly to Proposition 3.7, \( P \) is a \( \mathcal{D} \)-stable ideal. \( \square \)

**Theorem 5.5.** Let \( R \) be a NQS-algebra obeying Conditions CN1, CN2 and \( \varepsilon \) be an admissible root of 1. Suppose, in addition, that \( \varepsilon \notin E \). Then the number of irreducible representations over central character \( \chi \) equals to \( l^t \) for some nonnegative integer \( t \). (To explicit the geometrical sense of \( t \), see Theorem 5.7).

**Proof.** As in the Section 4 we may assume that \( \chi(S_{\mu}) \neq 0 \). The ideals \( I(\pi) \) and \( I(\pi, \mathcal{D}) \) admit localization on \( S_{\mu} \). After localization we obtain the ideals \( \tilde{I}(\pi) \) and \( \tilde{I}(\pi, \mathcal{D}) \) of \( \tilde{R}_\varepsilon \) (see the proof of Theorem 4.2) and \( \tilde{I}(\pi) \supset \tilde{I}(\pi, \mathcal{D}) \supset \mathcal{P}J \supset J \). For some minimal prime ideal \( P \) (see Proposition 5.4):

\[
\tilde{I}(\pi) \supset \tilde{I}(\pi, \mathcal{D}) \supset P = \mathcal{P}J + \tilde{R}_\varepsilon \mathfrak{p} \supset J.
\]

As above we put \( \chi(z_{j_\varepsilon}) = \xi_j \). For maximal ideal \( \Xi := \sum_{i=1}^\nu \mathfrak{J}_\varepsilon (z_{j_\varepsilon} - \xi_j) \) of \( \mathfrak{J}_\varepsilon \), we see \( \Xi \supset \mathfrak{p} \supset \mathcal{P}J \cap \mathfrak{J}_\varepsilon \supset J \cap \mathfrak{J}_\varepsilon \).
Denote $\tilde{R}_\xi := \frac{R_\xi}{\xi R}, Y_\xi := \frac{Y_\xi}{\xi Y}$, $\chi_{Y_\xi} \cong K[h_{i_1}^{\pm 1}, \ldots, h_{2\varepsilon}^{\pm 1}, u_{1\varepsilon}^{\pm 1}, \ldots, u_{\varepsilon}^{\pm 1}]$ and $J_\xi := (D + \tilde{R}_\xi \Xi) \mod \Xi = (J + \tilde{R}_\xi \Xi) \mod \Xi$. Notice that

$$\tilde{R}_\xi = J_\xi \oplus Y_\xi. \quad (5.6)$$

The subset $B := Y_\xi - \{0\}$ is a denominator subset in $\tilde{R}_\xi$. After localization we get the algebras $\mathfrak{K}_\xi := \tilde{R}_\xi B^{-1}$, $\mathfrak{Y}_\xi := Y_\xi B^{-1}$ and the ideal $\mathfrak{J}_\xi := J_\xi B^{-1}$ which obeys the requirements of Lemma 5.1. Therefore, $\bigcap_m \mathfrak{J}_\xi^m = 0$. We retain the notations for generators $\tilde{x}_{i\varepsilon}, h_{i\varepsilon}, u_{i\varepsilon}$ of $\tilde{R}_\xi$ for there images in $\mathfrak{K}_\xi$. Choose the basis $\Psi := \{x_{i\varepsilon}\}$ of $\mathfrak{Y}_\xi$-linear space $\mathfrak{J}_\xi$ over $\mathfrak{J}_\xi^2$. This basis generate $\mathfrak{J}_\xi$ in the following sense: for any element $a$ of $\mathfrak{J}$ and any $m \in \mathbb{N}$ there exists an expression $b$ of the elements $\Psi$ with coefficients of $\mathfrak{Y}_\xi$ such that $a = b$ mod $\mathfrak{J}_\xi^m$. A monomial of $\mathfrak{Y}_\xi$ lies in the center of $\mathfrak{K}_\xi$ if it commutes with all elements of $\Psi$.

The central character $\chi$ defines character $\chi'$ of the subalgebra $Z(Y_\xi)'$ (see Section 4) which is contained in $Z(Y_\xi)$. Recall (see the proof of Proposion 4.4) that there exists $1$-$1$ correspondence between the irreducible representations which lie over $\chi$ and characters $\lambda$ of $Z(Y_\xi)$ such that $\lambda|_{Z(Y_\xi)'} = \chi'|_{Z(Y_\xi)'}$. After factorization over $\Xi$ we obtain the similar statement for the subalgebra $Z(Y_\xi)' = Z(Y_\xi)'\Xi$.

Let us show that

$$Z(Y_\xi)' = K[u_{1\varepsilon}^{\pm 1}, \ldots, u_{\varepsilon}^{\pm 1}]. \quad (5.7)$$

Suppose that $y_0 \in Z(Y_\xi)$. Then there exists an element $w \in (W_\xi \cap \tilde{Z}_\xi) \mod \Xi$ such that $w = j_0 + y_0$ where $j_0 \in J_\xi$ and $y_0 \in Y_\xi$.

Let $a = x_{i\varepsilon} \in \Psi$. Since $aw = wa$ and $aj_0 - j_0 a$ mod $\Xi$ is $\mathfrak{J}_\xi^2$, then $aj_0 - j_0 a = y_0 a - a y_0 = (y_0 - \tilde{\tau}(y_0)) a$ modulo $\tilde{R}_\xi \Xi$ lies in $\mathfrak{J}_\xi^2$. By definition of $\Psi$, $y_0 = \tilde{\tau}(y_0)$. Thus, $\text{Ad}_{y_0}$ is identical on $\Psi$ and, therefore, $y_0 \in Z(R_\xi)$. This proves (5.7). According to (3.7),

$$Z(R_\xi) = (J_\xi \cap Z(R_\xi)) \oplus K[h_{i\varepsilon}^{\pm 1}, \ldots, h_{2\varepsilon}^{\pm 1}, u_{1\varepsilon}^{\pm 1}, \ldots, u_{\varepsilon}^{\pm 1}]. \quad (5.8)$$

The number of irreducible representations over $\chi$ equales to $l'$.

**Definition 5.6.** For a point $\chi \in \mathcal{M}$, we denote by $G(\chi)$ the Poisson subalgebra $\{a \in m(\chi) : \{a, m(\chi)\} \subset m(\chi)\}$ in $Z_\chi$. One can see $G(\chi) \supset m(\chi)^2$. The finite dimensional Lie algebra $\mathfrak{g}(\chi) := G(\chi)/m(\chi)^2$ is called the stabilizer of $\chi$ [KM, 1.1].

**Theorem 5.7.** $R, \varepsilon, t$ as in 5.5. For any central character $\chi$ the stabilizer $\mathfrak{g}(\chi)$ is a semidirect sum $\mathfrak{g}(\chi) := j + t$ where $j$ is an ideal and $t$ is the toric subalgebra of dimension $t$.

**Proof.** We may suppose that the central character $\chi$ and all $\pi$ over $\chi$ admit localization over $S_\varepsilon$ for some standard denominator subset $S = S_\mu$. After specialisation on $S_\varepsilon$ and, without loss of generality, we factorize the algebra $\tilde{R}_\xi$ modulo the ideal $\Xi$ (see Theorem 5.5).

The cotangent subspace $T^*_\chi(M)$ is a span of the images under $D : \tilde{Z}_\xi \to m(\chi)/m(\chi)^2$ of the elements $h_{i\varepsilon}^{\pm 1}, u_{i\varepsilon}^{\pm 1}$, and $j_\varepsilon \in J_\xi \cap \tilde{Z}_\xi$. We put $j := D(J_\xi \cap Z(R_\xi))$ and $t := D(K[u_{1\varepsilon}^{\pm 1}, \ldots, u_{\varepsilon}^{\pm 1}])$. The subalgebra $\mathfrak{g}(\chi)$ decomposes into the direct sum of linear subspaces $\mathfrak{g}(\chi) := j + t$. 

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Since the ideal \( J_\chi \) is \( D \)-stable (see Proposition 5.4), the subspace \( j \) is an ideal of \( g(\chi) \). Since \( \text{Ad}_{u_i} \) is diagonalizable, the subalgebra \( t \) is toric. □

**Conjecture 5.8.** The number of irreducible representations over \( \chi \) is equal to \( \text{rank}(\chi) \).

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