FROM STATISTICAL PHYSICS TO HIGH ENERGY QCD

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I discuss recent progress in understanding the high–energy evolution in QCD, which points towards a remarkable correspondence with the reaction–diffusion problem of statistical physics.

Recently, there has been significant progress in our understanding of QCD at high energy, based on the observations that (i) the gluon number fluctuations play an important role in the evolution towards saturation and the unitarity limit \(^1\,^2\) and (ii) the QCD evolution in the presence of fluctuations and saturation is in the same universality class as a series of problems in statistical physics, the prototype of which being the ‘reaction–diffusion’ problem \(^2\,^4\). These observations have developed into a rich correspondence between high–energy QCD and modern problems in statistical physics, which relates topics of current research in both fields, and which has already allowed us to deduce some insightful results in QCD by properly translating the corresponding results from statistical physics \(^2\,^4\).

To put such theoretical developments into a specific physical context, let us consider \(\gamma^*–\)proton deep inelastic scattering (DIS) at high energy, or small Bjorken–\(x\). We shall view this process in a special frame in which most of the total energy is carried by the proton, whose wavefunction is therefore highly evolved, while the virtual photon has just enough energy to dissociate long before splitting into a quark–antiquark pair in a colorless state (a ‘color dipole’), which then scatters off the gluon distribution in the proton. The transverse size \(r\) of the dipole is controlled by the virtuality \(Q^2\) of \(\gamma^*\) (roughly, \(r^2 \sim 1/Q^2\)), so for \(Q^2 \gg \Lambda_{QCD}^2\) one can treat the dipole scattering in perturbation theory. But for sufficiently small \(x\), even such a small dipole can see a high–density gluonic system, and thus undergo strong scattering.

Specifically, the small–\(x\) gluons to which couple the projectile form a color glass condensate \(^5\) (CGC), i.e., a multigluonic state which is characterized by high quantum occupancy, of order \(1/\alpha_s\), for transverse momenta \(k_{\perp}\) below the saturation momentum \(Q_s(x)\), but which becomes rapidly dilute when increasing \(k_{\perp}\) above \(Q_s\). The saturation scale rises very fast with the energy, \(Q_s^2(x) \sim x^{-\lambda}\), and is the fundamental scale in QCD at high energy. In particular, the external dipole is strongly absorbed provided its size \(r\) is large on the scale set by \(1/Q_s\), whereas for \(r \ll 1/Q_s\) one rather has weak scattering, or ‘color transparency’.

In turn, the small–\(x\) gluons are produced through quantum evolution, i.e., through radiation from color sources (typically, other gluons) with larger values of \(x\), whose internal dynamics is ‘frozen’ by Lorentz time dilation. Let \(\tau = \ln 1/x\) denote the rapidity; it takes, roughly, a rapidity interval \(\Delta \tau \sim 1/\alpha_s\) to emit one small–\(x\) gluon; thus, in the high energy regime where \(\alpha_s\tau \gg 1\), the dipole meets with well developed gluon cascades, as illustrated in Fig. 1. Three types of processes can be distinguished in Fig. 1, which for more clarity are singled out in Fig. 2.

The first process, Fig. 2.a, represents one step in the standard BFKL evolution \(^5\); by iterating this step, one generates gluon ladders which are resummed in the solution to the BFKL
equation\textsuperscript{5}. However, by itself, the latter is well known to suffer from conceptual difficulties in the high energy limit: \(i\) The BFKL estimate for the dipole scattering amplitude \(T_\tau(r)\) grows exponentially with \(\tau\) (i.e., like a power of the energy), and thus eventually violates the unitarity bound \(T_\tau(r) \leq 1\). (The upper limit \(T_\tau = 1\) corresponds to the black disk limit, in which the dipole is totally absorbed by the target.) \(ii\) The BFKL ladder is not protected from deviations towards the non-perturbative domain at low transverse momenta \(k_\perp^2 < \sim \Lambda^2_{\text{QCD}}\) (‘infrared diffusion’). With increasing energy, the BFKL solution receives larger and larger contributions from such soft intermediate gluons, and thus becomes unreliable.

These ‘small–\(x\) problems’ of the BFKL equation are both cured by the gluon recombination processes \((n \to 2)\) illustrated in Fig. 2.b which are important at high energy, when the gluon density in the target is large, and lead to gluon saturation and the formation of the CGC. Such processes are included in the Balitsky–JIMWLK equation\textsuperscript{5}, a non–linear, functional, generalization of the BFKL evolution which describes the approach towards gluon saturation in the target and preserves the unitarity bound in the evolution of the scattering amplitudes.

However, the Balitsky–JIMWLK equation misses\textsuperscript{4} the process in Fig. 2.c — the \(2 \to n\) gluon splitting — which describes the bremsstrahlung of additional small–\(x\) gluons in one step of the evolution. By itself, this process is important in the dilute regime, where it leads to the construction of higher–point gluon correlation functions from the dominant 2–point function. But once generated, the \(n\)–point functions with \(n > 2\) are rapidly amplified by the subsequent BFKL evolution (the faster the larger is \(n\)) and then they play an important role in the non–linear dynamics leading to saturation. Thus, such splitting processes are in fact important for the evolution towards high gluon density, as originally observed in numerical simulations\textsuperscript{6} of Mueller’s ‘color dipole picture’\textsuperscript{7}, and more recently reiterated in the analysis in Refs.\textsuperscript{1,2,4}.

Equations including both merging and splitting in the limit where the number of colors \(N_c\) is large have recently became available\textsuperscript{4} (see also Refs.\textsuperscript{8,9}), but their general solutions have not yet been investigated (except under some additional approximations\textsuperscript{4,10}). Still, as we shall
argue now, the \textit{asymptotic} behaviour of the corresponding solutions — where by ‘asymptotic’ we mean both the high–energy limit $\tau \to \infty$ and the weak coupling limit $\alpha_s \to 0$ — can be \textit{a priori} deduced from universality considerations, by exploiting the correspondence between high–energy QCD and the reaction–diffusion problem of statistical physics\textsuperscript{2}.

To that aim, it is convenient to rely on the event–by–event description\textsuperscript{2} of the scattering between the external dipole and the hadronic target (cf. Fig. 1) and to use the large–$N_c$ approximation to replace the gluons in the target wavefunction by color dipoles\textsuperscript{7}. Then, the dipole–target scattering amplitude corresponding to a given event can be estimated as

\[
T_r(r, b) \simeq \alpha_s^2 f_r(r, b),
\]

where $\alpha_s^2$ is the scattering amplitude between two dipoles with comparable sizes and nearby impact parameters, and $f_r(r, b)$ is the \textit{occupation number} for target dipoles with size $r$ at impact parameter $b$, and is a \textit{discrete} quantity: $f = 0, 1, 2, \ldots$. Thus, in a given event, the scattering amplitude is a multiple integer of $\alpha_s^2$.

In this dipole language, the $2 \rightarrow 4$ gluon splitting depicted in Fig. 2.c is tantamount to $1 \rightarrow 2$ dipole splitting, and generates \textit{fluctuations} in the dipole occupation number and hence in the scattering amplitude. Thus, the evolution of the amplitude $T_r(r, b)$ with increasing $\tau$ represents a \textit{stochastic process} characterized by an expectation value $\langle T(r, b) \rangle \simeq \alpha_s^2 \langle f(r, b) \rangle_{\tau}$, and also by fluctuations $\delta T \sim \alpha_s^2 \delta f \sim \sqrt{\alpha_s^2 T}$ (we have used the fact that $\delta f \sim \sqrt{T}$ for fluctuations in the particle number). Clearly, such fluctuations are relatively important (in the sense that $\delta T \gtrsim T$) only in the \textit{very} dilute regime where $\langle f \rangle \lesssim 1$, or $\langle T \rangle \lesssim \alpha_s^2$.

Eq. (1) applies so long as the scattering is weak, $T \ll 1$, but by extrapolation it shows that the unitarity corrections are expected to be important when the dipole occupation factor becomes of order $1/\alpha_s^2$. Consider first the formal limit $\alpha_s^2 \to 0$, in which the maximal occupation number $N \sim 1/\alpha_s^2$ becomes arbitrarily large. Then one can neglect the particle number fluctuations and follow the evolution of the scattering amplitude in the \textit{mean field approximation} (MFA). This is described by the Balitsky–Kovchegov equation\textsuperscript{5}, a non–linear version of the BFKL equation which, as shown in Ref. 3, lies in the same universality class as the Fisher–Kolmogorov–Petrovsky–Piscounov (FKPP) equation (the MFA for the reaction–diffusion process and related phenomena in biology, chemistry, astrophysics, etc; see\textsuperscript{13} for recent reviews and more references). The FKPP equation reads, schematically,

\[
\partial_\tau T(\rho, \tau) = \partial_\rho^2 T(\rho, \tau) + T(\rho, \tau) - T^2(\rho, \tau),
\]

in notations appropriate for the dipole scattering problem: $T(\rho, \tau) \equiv \langle T(r) \rangle_{\tau}$ and $\rho \equiv \ln(r_0^2/r^2)$, with $r_0$ a scale introduced by the initial conditions at low energy. Note that weak scattering ($T \ll 1$) corresponds to small dipole sizes ($r \ll 1/Q_s$), and thus to large values of $\rho$. The three terms on the r.h.s. of Eq. (2) describe, respectively, diffusion, growth and recombination. The first two among them represent (an approximate version of) the BFKL dynamics, while the latter is the non–linear term which describes multiple scattering and thus ensures that the evolution is consistent with the unitarity bound $T \leq 1$.

Specifically, the solution $T_r(\rho)$ to Eq. (2) is a \textit{front} which interpolates between two fixed points : the saturation fixed point $T = 1$ at $\rho \to -\infty$ and the unstable fixed point $T = 0$ at $\rho \to \infty$ (see Fig. 3). The position of the front, which marks the transition between strong scattering ($T \sim 1$) and, respectively, weak scattering ($T \ll 1$), defines the \textit{saturation scale}: $\rho_s(\tau) \equiv \ln(r_0^2 Q_s^2(\tau))$. With increasing $\tau$, the front moves towards larger values of $\rho$, as illustrated in Fig. 3. Note that the dominant mechanism for propagation is the BFKL growth in the tail of the distribution at large $\rho$: the front is \textit{pulled} by the rapid growth of a small perturbation around the unstable state. In view of that, the \textit{velocity} of the front $\lambda \equiv dp_s/d\tau$ is fully determined by the linearized version of Eq. (2), which describes the dynamics in the tail. Specifically, by solving
Figure 3: Evolution of the continuum front of the Balitsky–Kovchegov equation with increasing $\tau$.

the BFKL equation one finds$^{3,11,12}$ that, for $\rho > \rho_s(\tau)$ and sufficiently large $\tau$,

$$T_\tau(\rho) \simeq e^{\omega_2 s_\tau} e^{-\gamma\rho} = e^{-\gamma(\rho-\rho_s(\tau))}, \quad \rho_s(\tau) \equiv c\alpha_s\tau,$$

(3)

where $\alpha_s = \alpha_s N_c/\pi$, $\gamma = 0.63..$, and $c \equiv \omega/\gamma = 4.88..$. From Eq. (3) one can immediately identify the velocity of the front in the MFA as $\lambda_0 = c\alpha_s$. Since $Q^2_s(\tau) \simeq Q^2_0 e^{\lambda_0 \tau}$, it is furthermore clear that $\lambda_0$ plays also the role of the saturation exponent (here, in the MFA).

What is the validity of the mean field approximation? We have earlier argued that the gluon splitting processes (cf. Fig. 2.c) responsible for dipole number fluctuations should play an important role in the dilute regime. This is further supported by the above considerations on the pulled nature of the front: Since the propagation of the front is driven by the dynamics in its tail where the fluctuations are a priori important, the front properties should be strongly sensitive to fluctuations. This is indeed known to be the case for the corresponding problem in statistical physics$^{13,14}$, as it can be understood from the following, qualitative argument:

Consider a particular realization of the stochastic evolution of the target, and the corresponding scattering amplitude which is discrete (in steps of $\alpha_s^2$). Because of discreteness, the microscopic front looks like a histogram and thus is necessarily compact: for any $\tau$, there is only a finite number of bins in $\rho$ ahead of $\rho_s$ where $T_\tau$ is non–zero (see Fig. 4). This property has important consequences for the propagation of the front. In the empty bins on the right of the tip of the front, the local, BFKL–like, growth is not possible anymore (as this would require a seed). Thus, the only way for the front to progress there is via diffusion, i.e., via radiation from the occupied bins at $\rho < \rho_{tip}$ (compare in that respect Figs. 3 and 4). But since diffusion is less effective than the local growth, we expect the velocity of the microscopic front (i.e., the saturation exponent) to be reduced as compared to the respective prediction of the MFA.

To obtain an estimate for this effect, we shall rely again on the universality of the asymptotic ($\tau \to \infty$ and $N \equiv 1/\alpha_s^2 \gg 1$) behaviour$^2$. Namely, from the experience with the reaction–diffusion process and related problems in statistical physics$^{13,14}$, one knows indeed that the dominant behaviour for large evolution ‘time’ and large (but finite) occupancy $N \gg 1$ is independent of the details of the microscopic dynamics, and thus is the same for all the processes whose mean field limit ($N \to \infty$) is governed by the FKPP equation (2). In particular, the dominant contribution to the correction $\lambda_N - \lambda_0$ to the front velocity is known to be universal, and can be obtained through the following, intuitive, argument, due to Brunet and Derrida$^{14}$:

For a given microscopic front and $N \gg 1$, the MFA should work reasonably well everywhere except in the vicinity of the tip of the front, where the occupation number $J$ becomes of order one (corresponding to $T \sim \alpha_s^2$ in the QCD problem) and thus the linear growth term becomes ineffective. Accordingly, Brunet and Derrida suggested a modified version of the FKPP equation
(2) in which the ‘BFKL–like’ growth term is switched off when $T < \alpha_s^2$:
\[
\partial_\tau T(\rho, \tau) = \partial_\rho^2 T + \Theta(T - \alpha_s^2) T(1 - T).
\]  
By solving this equation in the linear regime, they have obtained the first correction to the front velocity as compared to the MFA (in notations adapted to QCD; see Ref. 2 for details):
\[
\lambda \approx \bar{\alpha}_s \left[ c - \frac{\kappa}{\ln^2(1/\alpha_s^2)} + \mathcal{O}(1/\ln^3 \alpha_s^2) \right],
\]  
where the numbers $c \approx 4.88$ and $\kappa \approx 150$ are fully determined by the linear (BFKL) equation. In QCD, the same result has been first obtained through a different but related argument by Mueller and Shoshi 1. Note the extremely slow convergence of this result towards its mean field limit: the corrective term vanishes only logarithmically with decreasing $1/N = \alpha_s^2$, rather than the power–like suppression usually found for the effects of fluctuations. This is related to the high sensitivity of the pulled fronts to fluctuations, as alluded to above. Such a slow convergence, together with the relatively large value of the numerical factor $\kappa$, make that the above estimate for $\lambda$, although exact for asymptotically small $\alpha_s^2$, is in fact useless for practical applications. To understand the subasymptotic corrections and, more generally, the behaviour of the saturation momentum and of the scattering amplitudes for realistic values of $\tau$ and $\alpha_s$, one needs to solve the actual evolution equations of QCD 4, 8, 9, a program which is currently under way 10.

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