Three-dimensional quantum gravity according to ST modular bootstrap

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ABSTRACT: We combine the large-c ST modular bootstrap equations with the Cardy formula for the asymptotic growth of the density of states to prove that any 2d unitary, compact, conformal field theory (CFT) with no higher spin conserved currents leads to conflicting inequalities whenever the entire spectrum of non-trivial primaries lies above the BTZ threshold. As a consequence, the holographic dual of 3d pure gravity, if it exists, cannot be a 2d CFT. Consistent solutions of ST bootstrap equations require additional primaries lying below the BTZ threshold. The lowest non-trivial primary necessarily has odd spin.
1. Introduction

One of the oldest precursors of the AdS/CFT correspondence [1] is the discovery by Brown and Henneaux [2] that any consistent quantum theory of gravity on $AdS_3$ is a two-dimensional conformal field theory (CFT) with central charge $c = \frac{3}{2G\sqrt{-\Lambda}}$ with $G$ being Newton’s constant and $\Lambda$ the cosmological constant. This CFT lives on the boundary at spatial infinity and according to holographic duality encodes all the properties of the quantum theory of gravity in the bulk. In particular the $AdS_3$ vacuum corresponds to the CFT identity module and the black hole solutions discovered by Banados Teitelboim and Zanelli (BTZ) [3, 4] correspond to non-trivial primaries.

In pure quantum gravity, owing to the absence of local degrees of freedom in the bulk, these primaries, including the identity which describes its edge modes, would form the expected complete spectrum of the corresponding CFT.

In the following we will be interested in a semi-classical description, which requires that the cosmological constant is small in Planck units, or equivalently that the central charge $c$ is large [5]. In this limit the BTZ black holes have a horizon of positive length $A$ and a corresponding Bekenstein-Hawking entropy $A/4G$ which nicely coincides [6] with the Cardy formula [7] for the asymptotic growth of the number of states in any unitary CFT. Thus, according to holographic duality, the spectrum of heavy states of any consistent 3d quantum gravity is universal and corresponds to heavy black holes [8].

BTZ black holes are locally equivalent to the $AdS_3$ vacuum, however they are inequivalent globally since they can be obtained from the true vacuum by discrete identifications of points. As a consequence they cannot be smoothly deformed into the vacuum. We therefore expect a gap in the mass spectrum of pure gravity as also predicted from the requirement of locality in the bulk theory [9]. Actually BTZ black holes can exist only above a mass threshold which corresponds in the large-$c$ limit to a primary of scaling dimensions $\Delta_{BTZ} = \frac{c}{2} + O(c^0)$. Prior experience with holographic duality would suggest that pure quantum gravity admits a quantum completion only if one could find a large-$c$ unitary theory in which the scaling dimensions $\Delta$ of all non-trivial primaries obey the condition...
$\Delta > \Delta_{BTZ}$. If however such a CFT does not exist, as we will argue in this paper, we can only conclude that the holographic dual of $AdS_3$ pure gravity, if it exists, is not a $2d$ CFT.

We consider, as usual, Euclidean $AdS_3$ whose conformal boundary geometry is a torus. The ensuing modular invariance under $PSL[2,\mathbb{Z}]$ and the holomorphy of the partition function of the CFT living on the boundary are sources of important information on the properties and consistency of the bulk theory.

What is the meaning of modular invariance on the gravity side? In a path-integral approach to pure quantum gravity one has to sum over different $AdS_3$ geometries with fixed asymptotic boundary conditions, i.e. Euclidean BTZ black holes. On the CFT side this sum amounts to a regularized sum over modular group images (i.e. a Poincaré sum) of the Virasoro character of the identity [10–14]. The quantity $Z_{MWK}$ obtained this way, known as the Maloney, Witten and Keller partition function, is finite and modular invariant by construction, however suffers of some unphysical features. In particular it does not posses a discrete spectrum and some states have a negative norm, both in regime of large spin $j$ [15] and at finite $j$ [16]. Different solutions have been proposed to cure this lack of unitarity, by adding some other kind of matter to the BTZ spectrum [13–15]. All these scenarios point toward the non-existence of pure quantum gravity as a consistent quantum theory.

A different line of thought, first advocated by Hellerman [17], only assumes a discrete spectrum and modular invariance of the partition function, regarded as a smooth function of the modular parameter $\tau$ of the torus. The core of this method, known as modular bootstrap, relies on the observation that there are special values of $\tau$ where the partition function is smooth only if its derivatives fulfill specific linear constraints. The resulting infinite set of equations are known as modular bootstrap equations. The special values are the points of the upper half-plane $H_+$ which are left invariant under the action of some non-trivial subgroup of $PSL(2,\mathbb{Z})$. Its fundamental domain accommodates three points of this kind, namely the $\mathbb{Z}_2$ elliptic point at $\tau = i$, left invariant by the modular inversion $S : \tau \rightarrow -1/\tau$; the parabolic point, or cusp, at $\tau = i\infty$, stabilized by the modular translation $T : \tau \rightarrow \tau + 1$; and finally the $\mathbb{Z}_3$ elliptic point at $\tau = \exp(2i\pi/3)$, stabilized by $ST : \tau \rightarrow -\frac{1}{\tau+1}$.

As first pointed out by Hellerman, the equations associated with $\tau = i$ ($S$ bootstrap) yield an upper bound for the allowed scaling dimension $\Delta_1$ of the first non-trivial primary of any unitary CFT with $c > 1$. Hellerman’s upper bound reads $\Delta_1 < c/6 + 0.4737$. It has since been improved numerically as well as analytically [18–21]. To date, the strongest upper bound obtained with $S$ modular bootstrap by extrapolating large-$c$ data, computed with the linear programming method [22], is $\Delta_1 < c/9.08$ [20]. This is far weaker than the sought-after $\Delta < \Delta_{BTZ} = \frac{c}{12} + O(c^0)$, necessary to question pure quantum gravity.

Analyticity at the cusp $\tau = i\infty$, in the large spin limit and for $c > 1$, yields the upper bound $\Delta - |j| < \frac{c-1}{12}$ on the twist gap [19,15].

$ST$ modular bootstrap [23–25] introduces a new scale in the spectrum of primaries of any CFT. It turns out that in the bootstrap equations the scaling dimensions $\Delta_A$ of any non-trivial primary $A$ always appears in the combination $\Delta_A - \Delta_+ + \Delta = \Delta_{BTZ} + O(c^0)$ (1.1)
In the large-c limit the ST modular bootstrap equations simplify dramatically. Excluding higher spin conserved currents they simply read [24]

\[ \sum_A (-1)^{j_A} N_A e^{-\sqrt{3}\pi \Delta_A} \left( \frac{\Delta_A - \Delta_+}{\Delta_+} \right)^n = \gamma^2 (-1)^{n+1} + O(1/\Delta_+) \quad (n = 1, 2, \ldots) \]  

where the sum is over the non-trivial primaries, \( N_A \) is their multiplicity, \( j_A \) their spin and \( v = 1 + e^{-\sqrt{3}\pi} \).

In this paper we will describe some important consequences of these equations. We assume their convergence and a discrete spectrum (i.e. a compact CFT) which we organize in an ascending order

\[ \Delta_1 < \Delta_2 < \Delta_3 < \ldots \]

A salient observation is that the series in (1.2) are endowed with an intrinsic cut-off, namely \( N_A \) is bounded by the Cardy formula for large enough \( \Delta_A \) [8]. It follows that the terms of the series with \( \Delta_A > 4\Delta_+ = \frac{c}{3} + O(c^0) \) decay exponentially with \( c \), giving rise to a further, substantial simplification of (1.2).

By exploiting these new equations we obtain a simple expression for the multiplicity of the light spectrum in the large-c limit. In particular, it turns out that \( \Delta_+ \), which appears to be the CFT counterpart of the BTZ threshold, behaves as a forbidden level. To rephrase it in an apparently different way, a primary with \( \Delta_A = \Delta_+ \) decouples from the other states, as (1.2) clearly shows.

Notice that a scalar state with \( \Delta = \Delta_+ = \Delta_{BTZ} + O(c^0) \) corresponds on the gravity side to a massless BTZ black hole. It has a vanishing near horizon, hence a vanishing Bekenstein-Hawking entropy [3, 4]. As a consequence its degeneracy is expected to be of order of 1. In the MWK partition function \( Z_{MWK} \) it has a degeneracy of -6, one of the unphysical features of \( Z_{MWK} \) [14]. ST bootstrap avoids the issue by setting \( N_{\Delta_+} = 0 \).

We will obtain a general expression for the multiplicity \( N_i \) in terms of the scaling dimensions of all the primary operators \( O_j \) with \( \Delta_j \leq 4\Delta_+ = c/3 + O(c^0) \), namely

\[ N_i = \gamma^2 (-1)^{j_i} e^{\sqrt{3}\pi \Delta_i} \Delta_+ \left( \frac{\prod_{j \neq i} \Delta_j}{(\Delta_i - \Delta_+)^n} + O(1/\Delta_+) \right), \quad \Delta_j \leq 4\Delta_+ \]  

There are both positive and negative factors in the denominator of (1.3). Requiring \( N_i \geq 0 \) entails the following curious selection rule

\[ \frac{1 + \text{sign}(\Delta_i - \Delta_+)}{2} + i + j_i = \text{even integer} \].

This gives information on the parity of the spins in the ordered sequence of primaries. For instance, if the lightest primary has odd spin, then (1.4) implies \( \Delta_1 < \Delta_+ \); if the subsequent primary has even spin then \( \Delta_2 < \Delta_+ \) and so on. The possible solutions of the selection rule are encoded in the number \( \kappa \) of primaries whose dimensions lie below the threshold. The parity of spins alternates (then \( i + j_i \) is even for \( i \leq \kappa \)) always beginning with an odd spin.

If \( \kappa = 0 \) then \( j_1 \) is even, hence all primaries lie above the threshold \( \Delta_+ \). This solution is the only one compatible with the expected spectrum of pure quantum gravity. We will
show that such a solution is contradictory. The same conclusion was already reached in [24] under some restrictive conditions. Here we prove it in its full generality. More precisely we will demonstrate that the large-\(c\) ST modular bootstrap equations, when applied to any unitary, compact CFT with no higher spin conserved currents, generate conflicting inequalities whenever it is assumed that the entire spectrum of non-trivial primaries lies above the BTZ threshold. In other terms such a CFT does not exist.

This paper is organized as follows. In the next section we combine the ST bootstrap equations at large \(c\) with the Cardy formula, discuss the convergence properties of these equations, map the problem of existence of a compact CFT with a prescribed spectrum into a simple algebraic problem and prove our main theorem. In section 3 we obtain an expression for the multiplicity of primaries in terms of the light spectrum of the theory. Finally in section 4 we draw some conclusions.

2. Conflicting inequalities

In this section we will prove that in the large-\(c\) limit there are no consistent unitary compact CFTs with no higher spin conserved currents in which the spectrum of all non-trivial primaries lies above the BTZ threshold of \(c/12\).

Our starting point is the set of ST modular bootstrap equations (1.2) that we rewrite here with some further details

\[
\sum_{i=1}^{\infty} (-1)^{j_i} N_i e^{-\sqrt{3}\pi\Delta_i} \left( \frac{\Delta_i - \Delta_+}{\Delta_+} \right)^n = v^2 (-1)^{n+1} + O(1/\Delta_+), \quad (n = 1, 2, \ldots). \tag{2.1}
\]

\(j_i\) is the spin of the primary \(O_i\), however only its parity matters. If \(\Delta_i\) is a degenerate level housing primaries of different spins, we have \((-1)^{j_i} N_i = N_i^e - N_i^o\) where \(N_i^e (N_i^o)\) is the sum of the multiplicities of the primaries with even (odd) spins. The scaling dimensions of the primary operators are arranged in an ascending order \(\Delta_1 < \Delta_2 < \ldots\). We expand in \(1/\Delta_+\) instead of \(1/c\) because in the former case we obtain simpler expressions. We are working in the semiclassical limit \(\Delta_+ \to \infty\) and \(\Delta_i/\Delta_+\) constant, thus only primaries with \(\Delta_i \sim c\) contribute.

Let \(S_n^{(k)}\) be the partial sum of the first \(k\) terms of the \(n^{\text{th}}\) equation. The assumed convergence of these series means that for every \(\varepsilon > 0\) there exists an integer \(m_n\) such that for all \(k \geq m_n\) it follows that \(|S_n^{(k)} - S_n| < \varepsilon\), \(S_n = v^2 (-1)^{n+1} + O(1/\Delta_+)\) being the sum of the series. If \(\varepsilon\) is chosen small enough, then all the \(S_n^{(k)}\)'s with \(k > m_n\) have the same sign as \(S_n\). In this way, we get the following exact inequalities

\[
S_{n=2\ell-1}^{(k)} > 0, \quad S_{n=2\ell}^{(k)} < 0, \quad \forall \ k \geq m_n. \tag{2.2}
\]

By exploiting the Cardy formula for the asymptotic growth of states we can now estimate \(m_n\) and show that it is independent of \(n\) for any finite \(n\).

The Cardy formula for the microcanonical entropy \(S(\Delta) \sim \log \rho(\Delta)\) at large \(\Delta\) entails the spectral density \(\rho(\Delta)\), which for a discrete spectrum is a sum of delta functions

\[
\rho(\Delta) = \sum_i N_i \delta(\Delta - \Delta_i), \tag{2.3}
\]
thus a precise definition of \( S(\Delta) \) requires averaging over an interval, i.e.

\[
S(\Delta) = \log \int_{\Delta-\delta}^{\Delta+\delta} \rho(\Delta') d\Delta',
\]

where \( \delta \) is chosen large enough to include energy levels. In the \( c \to \infty \) limit, with \( \Delta/c \) fixed and \( \Delta > c/6 \), it has been shown, assuming the sparseness of the spectrum [8], that [26]

\[
S(\Delta) = 2\pi \sqrt{\frac{c}{3} \left( \Delta + \frac{c}{12} \right)} - \frac{1}{2} \log c + O(c^0),
\]

with \( \delta \sim c^\alpha \), \( 0 \leq \alpha < 1 \) and \( \delta > \sqrt{\frac{3}{\pi}} \).

Actually we only need the leading behaviour of this quantity. Using the threshold \( \Delta+ \) instead of \( c \) it reads

\[
S(\Delta) = 4\pi \sqrt{(\Delta - \Delta+) \Delta^+}. \quad (2.5)
\]

For \( \Delta_i \) large enough, the terms of the series (2.1) behave as

\[
(-1)^i N_i e^{-\sqrt{3} \pi \Delta_i} \left( \frac{\Delta_i - \Delta^+}{\Delta^+} \right)^n \sim (-1)^i e^{S(\Delta_i) - \sqrt{3} \pi \Delta_i} \left( \frac{\Delta_i - \Delta^+}{\Delta^+} \right)^n, \quad (2.6)
\]

therefore all the terms with \( \Delta_i > 4 \Delta^+ = \frac{c}{3} + O(c^0) \) exponentially decrease with \( \Delta_i \), so they can be all absorbed in the asymptotic symbol \( O(1/\Delta^+) \) of (2.1).

The most important consequence is that the series (2.1) are dominated by the first \( m \) terms, with \( \Delta_m \sim c/3 \), thus we can replace all the \( m_n \)'s with \( m \) in (2.2).

The partial sum \( S^{(m)}_n \) is composed of \( p \) even-spin terms and \( q \) odd-spin terms \(^2\) with \( p + q = m \). To simplify the notation we define

\[
S^{(m)}_n = A_n - B_n
\]

with

\[
A_n = \sum_{i=1}^p w_i a^n_i, \quad B_n = \sum_{j=1}^q z_j b^n_j, \quad w_i \equiv N_i e^{-\sqrt{3} \pi \Delta_i} > 0, \quad z_j \equiv N_j e^{-\sqrt{3} \pi \Delta_j} > 0,
\]

\[
a_1 < a_2 < \cdots < a_p, \quad b_1 < b_2 < \cdots < b_q, \quad (2.7)
\]

where \( a_i = \frac{\Delta_i - \Delta^+}{\Delta^+} \) and \( b_j = \frac{\Delta_j - \Delta^+}{\Delta^+} \). We recast (a subset of) the inequalities (2.2) in the form

\[
A_{2\ell-1} > B_{2\ell-1}, \quad A_{2\ell} < B_{2\ell}, \quad \ell = 1, 2, \ldots \quad (2.8)
\]

We can now reformulate the claim at the beginning of this section, namely that there are no unitary compact CFTs without conserved higher spin currents, in which the entire

\(^1\)Note that (2.4) refers to the asymptotic multiplicity of states, while \( N_i \) refers to a proper subset of them, that is the primaries with spins of a given parity, hence \( N_i \leq e^{S(\Delta_i)} \). Notice also that the Cardy formula has been generalized to the case of high spin values, useful in the light cone limit [27, 28]. In the following we do not need this generalization.

\(^2\)In the next section we will discover that \( p \sim q \) (see Fig.1), in contrast with the assumption \( q = 0 \) of [23].
The spectrum of non-trivial primary operators lies above the BTZ threshold, as a simple algebraic Theorem: If the quantities $a_i$ ($i = 1, 2, \ldots, p$) and $b_j$ ($j = 1, 2, \ldots, q$) are all positive, they cannot fulfill the first $p + q + 1$ inequalities (2.8).

The detailed proof is fairly easy and goes as follows. We start from the following pivotal identity

$$
\sum_{j=0}^{p} \lambda_j A_{p-j+k} = 0, \ (k = 1, 2, \ldots), \quad (2.9)
$$

which can be rewritten as

$$
\sum_{i=1}^{p} w_i \left( \sum_{j=0}^{p} \lambda_j a_i^{p-j} \right) = 0. \quad (2.10)
$$

The expression enclosed in parentheses clearly shows that the $\lambda_j$'s can be chosen to be the coefficients of the polynomial

$$
x^k \prod_{j=1}^{p} (x - a_j) \equiv x^k \left( \sum_{j=0}^{p} \lambda_j x^{p-j} \right), \quad (2.11)
$$

with $\lambda_0 = 1$, $\lambda_1 = -\sum_{i=1}^{p} a_i$, $\lambda_2 = \sum_{i \neq j} a_i a_j$, and so on. Since all the $a_i$'s are positive, the $\lambda_j$'s have alternating signs, therefore the inequalities (2.8), when applied to (2.9), immediately yield

$$
B_{p+k} + \lambda_1 B_{p+k-1} + \ldots \lambda_p B_k \left\{ \begin{array}{l}
< 0 \text{ if } p + k \text{ odd} \\
> 0 \text{ if } p + k \text{ even}.
\end{array} \right. \quad (2.12)
$$

We take $k$ in the interval $k = 1, 2, \ldots q + 1$ and rewrite the left-hand-side (LHS) of these inequalities as

$$
\sum_{j=1}^{q} y_j b_j^k \text{ with } y_j = z_j \prod_{i=1}^{p} (b_j - a_i). \quad (2.13)
$$

Note that if all the $y_j$'s had the same sign, then, as $k$ varies, the sign of the LHS would always remain the same, contrary to the alternating sign required by (2.12). We then split the $q$ variables $b_j$, in accordance with the signs of the $y_j$'s, into two subsets $a_{i'}$, $(i' = 1, \ldots, p')$ and $b_{j'}$, $(j' = 1, \ldots, q')$ with $q = p' + q'$. In this way we can recast (2.12) as $q + 1$ inequalities of the form (2.8), but with a reduced number, $q$, of variables. We repeat this process starting with new identities of the type (2.9) and further reducing the number $q$ of variables through inequalities of type (2.12) to $q'' = q - p'$ variables fulfilling $q'' + 1$ inequalities, and so on. The iteration process terminates when we are left with a single variable $b$ and the two conflicting inequalities $b > 0$ and $b^2 < 0$. We conclude that the set of the first $p + q + 1$ inequalities (2.8) does not admit any solution as long as all the variables $a_i$ and $b_j$ are positive, QED.

Another intuitive proof that does not make use of the iteration process relies on the observation that by choosing the coefficients $w_i$'s small enough we could invert the sign of the inequalities (2.8), while the inequalities (2.12), which directly follow from (2.8), cannot change sign, as they do not depend on the $w_i$'s.
The question of whether this theorem can be extended to cases in which some $a_i$’s or some $b_j$’s are negative then arises. The answer is simple: if some $a_i$’s are negative, that is if there are even-spin primaries lying below the BTZ threshold, then, provided that all the $b_j$’s are positive, the theorem holds true because deleting the negative $a_i$’s in the inequalities (2.8) reinforces them. On the contrary if some $b_j$’s are negative the theorem is no longer true and the $ST$ bootstrap equations admit infinitely many solutions, as next section explains.

3. Allowed spectra

In this section we attempt to extract useful information on the allowed spectra of a unitary, compact CFT at large $c$ from the $ST$ bootstrap equations (1.2).

By summing each of these equations with the subsequent one, they can be recast in the form of a single inhomogeneous equation and an infinite set of homogeneous ones

$$
\begin{align*}
\sum_{i=1}^{\infty} w_i a_i - \sum_{j=1}^{\infty} z_j b_j &= v^2 \\
\sum_{i=1}^{\infty} w_i a_i^n (1 + a_i) - \sum_{j=1}^{\infty} z_j b_j^n (1 + b_j) &= 0, \quad (n = 1, 2, \ldots),
\end{align*}
$$

(3.1)

where we used the shorthand notation introduced in the previous section. These equations have the same formal structure of the bootstrap equations associated with a boundary CFT.

More specifically the even-spin terms play the role of surface operators and the odd-spin ones the role of bulk operators in the special surface transition of the critical $3d$ Ising model. This system has been studied both with the extremal functional method [29] as well as with the method of determinants [30,31]. The latter aims to obtain an approximate estimate of the low lying spectrum of the theory by looking for common zeros of the minors resulting from a suitable truncation of the above equations. In the present case it is easy to check that these minors have only trivial zeros, i.e. those giving $\Delta_i = \Delta_j$ for $i \neq j$. Similarly we do not expect that the extremal functional method used in [29] yields a solution in the present case.

In conclusion we are unable to extract a direct information on the allowed spectrum of the theory under study. We can however find a precise relation between the low-lying spectrum of the primary operators and their multiplicity. Actually in the previous section we found that only the primaries with dimensions $\Delta_i \leq 4 \Delta_+$ can contribute to the $ST$ bootstrap equations at order $O(1/\Delta_+)$, hence we rewrite (2.1) as

$$
\begin{align*}
\sum_{i=1}^{p} w_i a_i^n - \sum_{j=1}^{q} z_j b_j^n &= v^2 (-1)^{n+1} + O(1/\Delta_+), \quad p + q = m, \quad \Delta_m \sim 4\Delta_+, \quad n = 1, 2, \ldots
\end{align*}
$$

(3.2)

In (2.8) we were only interested to the sign of the partial sum of the first $m$ terms. Here we exploit a more detailed information on the value of this sum. We take into account the

\[\text{footnote text}\]

\[\text{footnote text}\]
first $m$ equations and regard the $w_i$’s and the $z_j$’s as the unknowns. Note that the matrix of the coefficients of this linear system is a Vandermonde matrix, thus the solution can be written in a particularly simple way, namely

\begin{align*}
w_i &= \frac{v^2}{a_i} \frac{\prod_{k \neq i} (a_k + 1)}{\prod_{j=1}^{q} (b_j + 1)} + O(1/\Delta_+), \\
z_j &= -\frac{v^2}{b_j} \frac{\prod_{k \neq j} (b_k + 1)}{\prod_{i=1}^{p} (a_i + 1)} + O(1/\Delta_+). \tag{3.3}
\end{align*}

Unitary CFTs require $w_i > 0$ and $z_j > 0$. This leads to some information on the sequence of spins and scaling dimensions of the primaries. Remember that we set the $a_i$’s and the $b_j$’s in ascending order, therefore the sign of $(a_r - a_s)$ or $(b_r - b_s)$ is the same as $r - s$, while the sign of $(a_r - b_s)$ is unknown for the moment. Assume for instance that the lowest primary has even spin, that is $\min(a_1, a_2, \ldots, b_1, b_2, \ldots) = a_1$. Then the constraint $w_1 > 0$ entails $a_1 \equiv \frac{\Delta_1 - \Delta_+}{\Delta_+} > 0$, hence the entire set of primaries lies above the BTZ threshold, a spectrum that has been excluded in the previous section.

If the lowest state is an odd-spin primary instead, then $z_1 > 0$ clearly implies $\Delta_1 < \Delta_+$. Moreover, if $\Delta_2 < \Delta_+$, then $\Delta_2$ is even, while if $\Delta_2 > \Delta_+$, $\Delta_2$ is odd. Continuing in this way it is easy to check that the general rule fixing the parity of the spin of the $n$th primary is

\[
\frac{1 + \text{sign}(\Delta_n - \Delta_+)}{2} + n + j_n = \text{even integer}. \tag{3.4}
\]

It follows that the allowed spectra are characterized by the number $\kappa$ of primaries lying below the threshold $\Delta_+$. The parity of the spin alternates both below and above $\Delta_+$, like in typical Regge trajectories. The lowest state always has odd spin, and the two closest states above and below $\Delta_+$ have spins with the same parity (see Fig. 1). Replacing $a_i, b_j, w_i, z_j$ in (3.3) with their definitions we get Eq.(1.3) discussed in the Introduction.

The states lying below the BTZ threshold $\Delta_+$ cannot correspond to BTZ black holes, of course. The apparent exponential growth of their multiplicity suggests that they could be dual to black holes of a gravity theory coupled to some kind of matter.
4. Conclusions

In this paper we studied some consequences of ST modular bootstrap, that is the set of equations arising from the requirement of analyticity at the $\mathbb{Z}_3$ elliptic point of the partition function of a compact CFT formulated on a torus. We combined these equations in the large-$c$ limit with the Cardy formula describing the asymptotic exponential growth of the spectral density, in this way obtaining useful information on primaries with scaling dimensions $\Delta \leq c/3$. In section 2 we provided a detailed proof of a simple algebraic theorem that has important consequences for the allowed spectra of compact CFTs. In particular, it implies that at large $c$ any unitary, compact CFT with no higher spin conserved currents, in which the entire spectrum of non-trivial primaries lies above the BTZ threshold leads to conflicting inequalities. This implies that the holographic dual of pure quantum gravity in $AdS_3$, if it exists, cannot be a unitary compact CFT.

There is another known example of a gravity theory whose dual is not an ordinary quantum mechanical system on the asymptotic boundary of space-time: Jakiw-Teitelboim (JT) dilaton gravity in two dimensions [33,34]. It has been recently understood that the JT model is dual to a matrix model, a statistical ensemble of quantum mechanical systems [35]. It has been very recently suggested that if pure $AdS_3$ gravity has a holographic dual, it could be an ensemble which generalizes matrix theory [36]. Conversely, by averaging over a suitably defined moduli space of free 2d CFTs, bulk duals have been obtained which resemble exotic theories of 3d gravity endowed with a large number of abelian fields [37–39].

Our algebraic theorem can be evaded by assuming a larger spectrum with additional primaries lying below the BTZ threshold. We found infinitely many consistent solutions of ST modular bootstrap equations, characterized by the number $\kappa$ of these new primaries. The lowest non-trivial primary necessarily has odd spin and the spin parity of any other primary is uniquely fixed by a simple selection rule described in (1.4). We also obtained a general expression for the multiplicity of primaries in terms of the scaling dimensions of all primaries with $\Delta \leq c/3$ (see Eq.(1.3)).

In a sense, our findings resemble those obtained with the MWK method [12, 13] used to compute the torus partition function of pure gravity by summing over saddle points of the gravitational path integral. Among the other unphysical features, the resulting density of states has some negative values. This is in some way the analogue of our conflicting identities. In both cases the cure is to add new matter to the pure gravity [13–15]. In our case the additional primaries lying below the BTZ threshold, in the semiclassical limit $c \to \infty$, $\Delta \sim c$, have an exponentially large multiplicity, nevertheless they cannot correspond to BTZ black holes; it would be important to understand the nature of these objects on the gravitational side.

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