TOWARDS FUNCTORYLITY OF SPINOR L-FUNCTIONS

BERNHARD HEIM

Abstract. The object of this work is the spinor L-function of degree 3 and certain degeneration related to the functoriality principle. We study liftings of automorphic forms on the pair of symplectic groups (GSp(2), GSp(4)) to GSp(6). We prove cuspidality and demonstrate the compatibility with conjectures of Andrianov, Panchishkin, Deligne and Yoshida. This is done on a motivic and analytic level. We discuss an underlying torus and L-group homomorphism and put our results in the context of the Langlands program.

1. Introduction

The principle of functoriality describes deep conjectural relationships among automorphic representations on different groups [30]. In this paper we consider automorphic representations and liftings of the pair of groups GL(2), GSp(4) to GSp(6), where GSp(2n) denotes the symplectic group with similitudes of degree n. The lifting will be described in terms of automorphic representations $\pi \in \mathcal{A}(\text{GSp}(2n))_{k}$ with holomorphic discrete series at archimedean places (of weight $k$), and convolutions of their spinor L-functions. This leads to map

$$\mathcal{A}(\text{GL}(2)(\mathbb{A})_{k-2} \times \mathcal{A}(\text{GSp}(4)(\mathbb{A})_{k} \longrightarrow \mathcal{A}(\text{GSp}(6)(\mathbb{A})_{k}.$$}

Here $\mathbb{A}$ denotes the adèles over $\mathbb{Q}$. We prove a cuspidality criterion and show that all other formally possible weights can not occur. Moreover we prove that the lifting is compatible with conjectures of Andrianov [2], Yoshida [47], Panchishkin [35] and Deligne [14] relating the functional equation of the spinor L-function of degree 3 and the arithmetic nature of the critical values. Moreover the underlying L-function can be identified with the spinor L-function of

$$\text{GL}(2)(\mathbb{A}) \times \text{GSp}(4)(\mathbb{A}).$$}

An integral representaion, functional equation and special values results have been indepently obtained by Furusawa [15] and Heim [21], and Böcherer and Heim [9, 10]. The results and observations obtained in this paper give strong evidence that

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the Miyawaki conjecture [32], [22] of type II is essentially not only the lift of a pair of two elliptic modular forms, in contrast, it can be viewed as a lifting of a pair of an elliptic modular form and a Siegel modular form of degree 2. There is also promising numerical data due to Miyawaki [32] supporting this approach. Let $\lambda_p(F)$ be the $p$-th Hecke eigenvalue of Siegel cuspidal Hecke eigenform. By a tremendous effort Miyawaki succeeded in determine the second eigenvalue of the Hecke eigenform $F_{14}$ of degree 3 and weight 14: $\lambda_2(F_{14}) = -2^7 \cdot 2295$. In this interesting case our method gives the simple statement that

$$\lambda_2(F_{14}) = \tau(2) \cdot \lambda_2(G),$$

where $\tau(2) = -24$ is the second Fourier coefficient of the Ramanujan $\Delta$-function and $G$ the non-trivial Siegel cuspform of weight 14 and degree 2. Experts in the this field, as Ibukiyama, Poor and Yuen, are optimistic that such calculations could be done for higher weights in the near future. We also would like to mention the recent work of Chiera and Vankov [12], which is the first in a series of papers attacking this problem by starting with the weight of first occurrence $k = 12$, in which the conjecture of Miyawaki is now fully proven by Ikeda [26] and Heim [22].

Concerning numerical data, Langlands [29, p. 213] stated:

*The necessary equalities are too complicated to be merely coincidences, and we may assume with some confidence, ... .*

This was in reference to the paper of Kurokawa [31], where a statement of what is now called the Saito-Kurokawa lifting had been verified for small primes, broaching what has become an important research subject over the last 30 years. The underlying principle of our lifting can be considered as a higher dimensional analog of the $L$-group homomorphism:

$$\text{GL}(2)(\mathbb{C}) \times \text{GL}(2)(\mathbb{C}) \longrightarrow \text{GL}(4)(\mathbb{C})$$

given by Ramakrishnan [41], which goes beyond endoscopic considerations. To explain this will require some preparation. It will become apparent that the approach here is only the tip of an iceberg, suggesting optimism about further results in this direction. We note that not much is known concerning functoriality and lifting of automorphic representations without the assumption of a Whittaker model. At this point we would like to suggest the reader to consult the excellent overview article of Raghuram and Shahidi [40], and especially chapter 7, with the description of Harders fundamental work [19].
Let \( G/K \) be a reductive group over a number field \( K \) and \( \mathcal{A}(G) \) the space of automorphic representations \( \pi \) of \( G(\mathbb{A}_K) \), where \( \mathbb{A}_K \) are the adeles of \( K \). A general question addressed in the Langlands program [17] arises if one takes the convolution L-function of two automorphic representations, and asks whether there exists another automorphic representation with this L-function. Let \( G_1, G_2, G_3 \) be three reductive groups defined over the same number field \( K \). Let \( \rho_1, \rho_2 \) be finite dimensional representations of the \( L \)-groups of \( G_1 \) and \( G_2 \). In the sense of Langlands (cf. [18]) we attach to the data \((\pi, \rho)\) the Langlands L-function \( L(s, \pi, \rho) \). In this notation we can ask whether there exists a map

\[
\otimes \begin{cases}
\mathcal{A}(G_1) \times \mathcal{A}(G_2) & \longrightarrow \mathcal{A}(G_3), \\
(\pi^1, \pi^2) & \mapsto \pi^1 \boxtimes \pi^2,
\end{cases}
\]

with an automorphic representation \( \pi := \pi^1 \boxtimes \pi^2 \in \mathcal{A}(G_3) \) which has the property that the convolution L-function \( L(s, \pi^1 \boxtimes \pi^2, \rho_1 \boxtimes \rho_2) \) is equal to a L-function \( L(s, \pi^1 \boxtimes \pi^2, \rho_3) \), where \( \rho_3 \) is a suitable finite dimensional representation of the \( L \)-group of \( G_3 \). One is far away from having a general solution of this problem and it seems that even the first examples given in the literature demonstrate the depth of the issues. This is illustrated by the work of Ramakrishnan [41] and Kim and Shahidi [27]. Here we briefly recall results of Ramakrishnan. He constructed a map

\[
\mathcal{A}(GL_2) \times \mathcal{A}(GL_2) \longrightarrow \mathcal{A}(GL_4),
\]

where \( \rho_i, 1 \leq i \leq 3 \) is the standard representation. The existence of such a map had been expected for many years, but several difficult auxiliary results were needed. Since we are in a parallel situation, we mention some essential auxiliary results still needed in order to prove the Main Conjecture in [23]. We recall briefly some of the main ingredients in the proof. To quote Ramakrishnan [41]:

Our proof uses a mixture of converse theorems, base change and descent, and it also appeals to the local regularity properties of Eisenstein series and the scalar products of their truncation.

This involves a converse theorem of Cogdell and Piatetski-Shapiro [13], base change results of Arthur and Clozel [6], results of Garrett [16] and Piateski-Shapiro and Rallis [36] towards the triple product L-function and finally applications of Arthur’s truncation [5] of non-cuspidal Eisenstein series to get boundness in vertical strips of the triple product L-functions. Recently the L-functions considered in this paper showed up several interesting papers. In dissertations of Agarwal [3] and Saha [43] the level
aspect and $p$-adic properties had been considered. Both have beautiful applications in mind. The first indicates applications towards functoriality à la Ramakrishnan and the second one goes towards the modularity approach of the Iwasawa conjecture related to ideas of Skinner and Wiles.

In two manuscripts Pitale and Schmidt \cite{38,39} consider the L-function of $\text{GSp}(4) \times \text{GL}(2)$ mainly for holomorphic modular forms and partly refine the results of Furusawa \cite{15} and Heim \cite{21}, and B"ocherer and Heim \cite{9,10} in the level aspect.

Hence it is obvious that considering the $\text{GSp}(4) \times \text{GL}(2)$ spinor L-function for the family of weights $k, k-2$, even in the highly assumed case of endoscopy (e.g. Saito-Kurokawa lifts) gives a significant new approach to several conjectures related to the spinor L-function of Siegel modular forms of degree 3.

Recently Miyawaki’s conjecture for $F_{12}$ had been proven \cite{22}. Applications are indicated by Dalla Piazza and van Geemen \cite{37}. These L-series and their relation to Galois representations are of great interest in mathematical physics, where they show up in the bosonic and superstring measures.

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\section{Summary of the main results}

The lifting we predict would be an example of Langlands' functoriality principle. Since our predicted lift goes beyond endoscopy, and since it satisfies all known motivic and analytic viewpoints and which is compatible with conjectures of Andrianov, Deligne, Yoshida, Panchiskin, we have to generalize the concept of $L$-group homomorphism.

The $L$-group of the group of projective symplectic similitudes of degree $n$ is the spin group $\text{Spin}(2n+1)(\mathbb{C})$. Let $r_n$ be complex representations of the $L$-groups $\text{Spin}(2n+1)(\mathbb{C})$. Langlands conjectures that for any $L$-group homomorphism

\begin{equation}
\Theta : \text{Spin}(3)(\mathbb{C}) \times \text{Spin}(5)(\mathbb{C}) \longrightarrow \text{Spin}(7)(\mathbb{C})
\end{equation}

there exists a map

\begin{equation}
\Theta^* : \mathcal{A}(\text{PGSp}(2)) \times \mathcal{A}(\text{PGSp}(4)) \longrightarrow \mathcal{A}(\text{PGSp}(6))
\end{equation}

such that the associated L-functions are equal, namely,

\begin{equation}
L(s, \Theta^*(\pi^1, \pi^2), r_3) = L(s, \pi^1 \otimes \pi^2, r_1 \otimes r_2).
\end{equation}
Let $\rho_n$ be the spin representation of $\text{Spin}(2n + 1)(\mathbb{C})$. Since we are interested in a reasonable generalization of the Miyawaki conjecture, and in a proof of Andrianov’s and Deligne’s conjecture for such liftings for the spinor L-function of degree 3, we restrict ourself to the space $\mathcal{A}(\text{PGSp}(2n))_k$ of all automorphic representations $\pi = \pi_F$, associated to Siegel modular forms $F$ of degree $n$ and weight $k$. Take $r_i = \rho_i$ (for $n = 1, 2, 3$). To study the spinor L-function of degree 3 fix a maximal torus $T(n)$ in $\text{Spin}(2n + 1)$ parametrized by $(a_0, a_1, \cdots, a_n)$ with $a_i \in \mathbb{C}^\times$. We take the torus homomorphism $\Theta : T(3) \times T(5) \rightarrow T(7)$, or lifted L-group homomorphism, given by

\begin{equation}
\Theta((a_0, a_1), (\beta_0, \beta_1, \beta_2)) = (a_0\beta_0, \beta_1, \beta_2, a_1).
\end{equation}

We are aware that this seems to be very bizarre, but since we cannot assume endoscopy in general (since we not only consider Saito-Kurokawa cusps), it seems that this is the only way to state such a lift in the framework of Langlands description of lifts in terms of Satake parameters.

Now suppose that the corresponding functorial map $\Theta^*(\pi_1, \pi_2)$ exists. Then put

\begin{equation}
\pi := \pi^1 \boxtimes \pi^2 := \Theta^*(\pi_1, \pi_2).
\end{equation}

\textit{A priori} it is not clear that these conjectural lifts are cuspidal. In this paper we show that they are. Let the corresponding cuspidal Siegel modular forms of degree 1, 2, 3 have weights $k_1, k_2, k_3 \in \mathbb{N}$. We prove that only the triple $k - 2, k, k$ can occur. This conclusion includes Miyawaki’s conjecture [32] of type II [23]. We also give a motivic interpretation of these lifts and deduce the same result. Now let $F = h \boxtimes G$ with suitable $h \in S_{k-2}, G \in S^2_k, F \in S^3_k$, where $S^n_k$ is the space of cuspidal Siegel modular forms of weight $k$ and degree $n$. Let $\lambda_p$ be the $p$-th Hecke eigenvalue. Then, for all primes $p$, the eigenvalues of the $h, G, F$ satisfy

\begin{equation}
\lambda_p(F) = \lambda_p(h) \cdot \lambda_p(G).
\end{equation}

We prove the conjecture of Andrianov and Panchishkin on the meromorphic continuation and functional equation of the spinor L-function of degree 3 for this type of cuspidal Hecke eigenform (see section 3.2 for details).

\textbf{Theorem [Conjecture of Andrianov and Panchishkin].} Let $k_1, k_2$ be positive even integers. Let $\pi^1 \in \mathcal{A}(\text{GL}(2))_{k_1}$ and $\pi^2 \in \mathcal{A}(\text{GSp}(4))_{k_2}$ be two automorphic representations. Let $\pi := \pi^1 \boxtimes \pi^2 \in \mathcal{A}(\text{GSp}(6))_k$ be modular. Then necessarily $k_1 = k - 2$ and $k_2 = k$. The L-function $L(s, \pi, \rho_3)$ has a meromorphic continuation
to the whole complex plane. Let the first Fourier-Jacobi coefficient of \( \pi^2 \) be non-trivial. Then the completed L-function

\[
\hat{L}(s, \pi, \rho_3) = L_\infty(s, \pi, \rho_3) L(s, \pi, \rho_3),
\]

has a functional equation under \( s \mapsto 3k - 5 - s \), where

\[
L_\infty(s, \pi) := \Gamma_C(s) \prod_{m=1}^{3} \Gamma_C(s - k + m).
\]

For \( \pi^2 \) attached to a Saito-Kurokawa lift the spinor L-function is entire. The special values of the lifts are apparently compatible with Deligne’s conjecture on the arithmetic of special values specialized to spinor L-function and to twisted spinor L-function by Yoshida and Panchishkin.

**Theorem [Conjecture of Deligne, Yoshida, Panchishkin].** Let \( \tilde{\pi} = \pi' \boxtimes \pi'' \in \mathcal{A}(GSp(6))_k \) for \( \pi' \in \mathcal{A}(GL(2))_{k-2} \) and \( \pi'' \in \mathcal{A}(GSp(4))_k \) primitive. Let \( F_{\tilde{\pi}} \in S^3_k \) be normalized. Then the critical integers of the spinor L-function of \( F_{\tilde{\pi}} \) are exactly \( m = k, k + 1, \ldots, 2k - 5 \). Moreover, there exists a positive \( \Omega \in \mathbb{R} \) such that

\[
L(m, F_{\tilde{\pi}}, \rho_3) \frac{\pi_{4m^{2} - 3k + 6\Omega}}{\pi_{4m^{2} - 3k + 6\Omega}} \in E.
\]

It is surprising that the spinor L-function attached to a Siegel modular form \( F = h \boxtimes G \in S^3_k \) has a Rankin-Selberg integral representation with respect to an Eisenstein series of Siegel type of degree 5 [21], [9], [10].

### 3. Basic notation and facts

In this section we describe the relation between Siegel modular forms and automorphic representations. We use the approach of Langlands to systematically define the spinor, standard, and \( GL(2) \)-twisted spinor L-function.

Let \( k, n \) be positive integers and let \( S^n_k = S_k(\Gamma_n) \) be the space of cuspidal Siegel modular forms of weight \( k \) with respect to the Siegel modular group \( \Gamma_n = Sp(2n)(\mathbb{Z}) \). Let \( G_{2n} := GSp(2n) \) be the symplectic group with similitudes and let \( \overline{G}_{2n} := G/Z \), where \( Z \) is the center of \( G \). In the setting of Siegel modular forms let \( F \in S^m_k \) be a Hecke eigenform and \( \pi \) the associated automorphic representation of \( \overline{G}_{2n} \mathbb{A}_Q \) over \( Q \), where \( \mathbb{A}_Q \) is the adele ring of \( Q \). Then \( \pi \) factors over primes as \( \pi = \otimes'_v \pi_v \), where \( \pi_\infty \) is a holomorphic discrete series representation (or limit of discrete series) and, for finite \( v = p \), \( \pi_p \) is a spherical representation of \( Sp(2n)(\mathbb{Q}_p) \). Here \( p \) are the prime numbers.
in \( \mathbb{Q} \). The local representations are uniquely determined by the Satake parameters of \( F \):

\[
\mu_{0,p}^F, \mu_{1,p}^F, \ldots, \mu_{n,p}^F.
\]

These Satake parameters are unique up to the action of the Weyl group of \( G_{2n} \). There is a clear correspondence between the eigenvalues of the Hecke operators in the classical setting and the Satake parameters coming from the local representations. For example let \( \lambda_p(F) \) the eigenvalue of \( F \) with respect to the Hecke operator \( T_p^{(n)} \). Here \( T_p^{(n)} \) is the canonical generalization of the well-known Hecke operator \( T_p \) for elliptic modular forms on the upper half-space \( \mathbb{H} := \{ z = x + iy \mid y > 0 \} \). Then by standard normalization we have

\[
\lambda_p(F) = \mu_{0,p}^F \left( 1 + \sum_{r=1}^n \prod_{1 \leq i_1 \leq \ldots \leq i_r} \mu_{i_1,p}^F \cdot \ldots \cdot \mu_{i_r,p}^F \right).
\]

To fix notation, let \( H \) be any linear algebraic reductive group (split) over a number field \( K \). Then

\[
A(H) := \{ \pi \text{ cuspidal automorphic representation of } H(\mathbb{A}_K) \}.
\]

Let \( H \) be the projective group of symplectic similitudes and \( k \) a positive integer. Then we denote by \( A(H)_k \) the set of all representations \( \pi \) in \( A(H) \) such that there exists a \( F \in S^n_k \) with \( \pi = \pi_F \).

Now we can reformulate our lifting problem. This leads to the following necessary condition. Let

- \( \pi^1 \in A(PGSp(2)(\mathbb{A}))_{k_1} \)
- \( \pi^2 \in A(PGSp(4)(\mathbb{A}))_{k_2} \).

Then there has to exist a \( \pi^3 \in A(PGSp(6)(\mathbb{A}))_{k_3} \) such that for all finite primes \( p \):

\[
\lambda_p(F) = \lambda_p(h) \cdot \lambda_p(G).
\]

Here \( h, G, F \) are related to \( \pi^1, \pi^2, \pi^3 \) as above. Now we want to put this observation into the picture of the Langlands program to apply standard techniques and obtain possible generalizations.

3.1. Automorphic L-functions. Langlands’ set-up attaches automorphic L-functions to reductive algebraic groups in a systematic way and recovers the classical L-functions as the Hecke L-function, Rankin-Selberg L-function and Andrianov Spinor L-function. Let us recall the main ingredients. We fix the following data related to a linear algebraic group (split) over some number field \( K \).
• $G(\mathbb{A}_K)$ denote the restricted direct product $\prod_v G(K_v)$, here the $K_v$ are the local fields attached to the valuations $v$.
• $L^G$ the $L$-group of $G$, which is a well-defined subgroup of $GL(n)(\mathbb{C})$, $n$ suitable.
• $\rho$ a finite dimensional representation of the $L$-group of $G$.
• $\pi$ an automorphic representation of $G(\mathbb{A}_F)$, $\pi = \otimes_v \pi_v$.

For example let $G = PGSp(2n)$, then the $L$-group coincides in this special case with the universal covering $Spin(2n + 1)(\mathbb{C})$ of the orthogonal group $SO(2n + 1)(\mathbb{C})$. We denote by $\rho_n$ and $st_n$ the spin representation of dimension $2n$ and the standard imbedding into $GL(2n + 1)(\mathbb{C})$ of the $L$-group.

Let $\pi \in A(G)$ with $\pi = \otimes_p \pi_v$, then for almost all $v$ the local representations $\pi_v$ are spherical and unramified and correspond to the conjugacy class of a semi-simple element $t^\pi_v$ in the $L$-group. Let $S$ be a finite set, which contains all places at infinity and all finite $v$, which are ramified. Then the Langlands $L$-function attached to this data is defined by

$$(3.4) \quad L^{(S)}(s, \pi, r) := \prod_{v \notin S} \det (1 - r(t^\pi_v)(N v)^{−s})^{−1}.$$ 

Here $N v$ is the order of the residue field related to $v$. If we work over $\mathbb{Q}$ and $S = \{\infty\}$ we skip the $S$. This $L$-function is holomorphic with respect to $s \in \mathbb{C}$ for $\text{Re}(s)$ large enough. Langlands conjectured that the $L$-function can be completed such that one gets a meromorphic function on the whole $s$-plane with a functional equation.

**Definition.** Let $n, k \in \mathbb{N}$ and $\pi \in A(GSp(2n))_k$. Then the spinor and standard $L$-functions are given by

$$(3.5) \quad L(s, \pi, \rho_n) := \prod_p (\prod_{k=0}^n \prod_{1 \leq i_1 < \ldots < i_k \leq n} (1 - \mu_0 \mu_{i_1} \ldots \mu_{i_k} p^{-s}))^{−1}.$$ 

$$(3.6) \quad L(s, \pi, st_n) := \prod_p \left( (1 - p^{-s})^{\prod_{j=1}^n (1 - \mu_j p^{-s})} (1 - \mu_j^{-1} p^{-s}) \right)^{−1}.$$ 

Let $\pi = \pi_\infty \otimes_p \pi_p \in A(GSp(2n))$. Then $\pi$ satisfies the Ramanujan-Petersson conjecture at finite places if for all primes $p$ and local finite representations $\pi_p$ the Satake parameters $\mu_0, \mu_1, \ldots, \mu_n$ satisfy

$$(3.7) \quad |\mu_1| = |\mu_2| = \ldots = |\mu_n| = 1.$$ 

**Two remarks:**

a) The spinor and standard $L$-function converges absolutely and locally uniformly for real part of $s$ large enough. For example without any assumption one can choose $\text{Re}(s) > n + 1$ in the case of the standard $L$-function.
b) Let $\pi \in \mathcal{A}(\text{GSp}(2n))_k$ satisfy the Ramanujan-Petersson conjecture, then the Euler products converge already for

\[(3.8) \quad \text{Re}(s) > \left( nk - n(n + 1)/2 \right)/2 \text{ and } \text{Re}(s) > 1.\]

It is well known that the Ramanujan-Petersson conjecture is fullfilled in the case of elliptic cusp forms. In the case $n = 2$ the Saito-Kurokawa lifts (CAP-representations) contradict the conjecture. Nevertheless for all other representations $\pi \in \mathcal{A}(\text{GSp}(4))_k$ the Ramanujan-Petersson conjecture is satisfied [45], [46]. Let $c_1 := (k + 1)/2$, $c_2 := k - 1/2$, and $c_3 := (3k)/2 - 2$. Then we have the sharp bounds $\text{Re}(s) > c_n$ for the spinor L-functions in the cases $n = 1, 2, 3$, if the conjecture is satisfied.

3.2. Conjectures of Andrianov and Deligne. Let $\pi \in \mathcal{A}(\text{GSp}(6))_k$. Then there are several open conjectures on the analytic and arithmetic properties of the spinor L-function attached to $\pi$.

**Functional equation** [Conjecture of Andrianov [2], [35]]

The completed spinor L-function $\hat{L}(s, \pi, \rho_3)$ attached to an automorphic representation $\pi \in \mathcal{A}(\text{GSp}(6))_k$ has a meromorphic continuation to the whole complex plane and satisfies the functional equation

\[
\text{s} \rightarrow 3k - 5 - s.
\]

The completion is given by

\[(3.9) \quad \hat{L}(s, \pi, \rho_3) := L_\infty(s, \pi) L(s, \pi, \rho_3),\]

where $\Gamma_C(s) := 2(2\pi)^{-s}\Gamma(s)$ and

\[(3.10) \quad L_\infty(s, \pi) := \Gamma_C(s) \prod_{m=1}^{3} \Gamma_C(s - k + m).\]

**Arithmetic of critical values** [Conjecture of Deligne [14], Panchiskin [35], Yoshida [17]]

Let $\mathcal{M}$ be a (hypothetical) motive attached to $\pi \in \mathcal{A}(\text{GSp}(2n))_k$ with $L$-function $L(s, \pi, \rho_n)$. Hence $\mathcal{M}$ is a motive over $\mathbb{Q}$ with coefficients in an algebraic number field.
Let $R := E \otimes \mathbb{Q} \subset E \subset R$. Then the L-function $L(s, \mathcal{M})$ of the motive takes values in $R$. Let $m \in \mathbb{Z}$ be a critical value, then Deligne’s conjecture predicts that

$$L(m, \mathcal{M}) = \frac{1}{(2\pi i)^{d^{\pm} m c^{\pm}(\mathcal{M})}} \in E.$$  

Here $\pm$ has the same sign as $(-1)^m$ and $c^{+}(\mathcal{M}), c^{-}(\mathcal{M})$ are Deligne’s periods. The natural integers $d^{+}$ and $d^{-}$ are the dimension of the $+$ and $-$ eigenspace of the Betti realization of $\mathcal{M}$. Let $n = 3$ then Panchiskin [35] has determined explicit the interval of the involved critical values. They are given by the positive integers $k, k + 1, \ldots, 2k - 5$. Since we prove that the period does not depend on the parity of the special values we omit the discussion on the meaning of this fact in this paper. We only want to note that this somehow seems to reflect the degeneration of the lifted Siegel modular form of degree 3 on the level of periods. Of course one could speculate and ask if this already determine lifts in the case of Siegel modular forms of degree 3.

4. Candidates for the lifting - first properties

**Definition.** Let $h \in S_{k_1}$ and $G \in S_{k_2}$ be Hecke eigenforms. Let $\pi^1$ and $\pi^2$ be the associated automorphic representations and let $r_1$ and $r_2$ be two complex representations of dimension $m_1$ and $m_2$ of the corresponding $L$-groups. Then

$$L(s, \pi^1 \otimes \pi^2, r_1 \otimes r_2) := \prod_p \left( 1^{m_1 + m_2} - r_1(t_{p}^{\pi^1}) \otimes r_2(t_{p}^{\pi^2}) \mu_p^{-s} \right)^{-1}.$$  

If there is any automorphic representation $\pi^3$ (and complex representation $r_3$ of the related $L$-group) such that

$$L(s, \pi^3, r_3) = L(s, \pi^1 \otimes \pi^2, r_1 \otimes r_2),$$

then we put $\pi^3 = \pi^1 \otimes \pi^2$ to indicate the modularity of the tensor product.

Already in the work of Ramakrishnan [41] it was a non-trivial task to show that the possible lifting is forced to be cuspidal. Now since our groups have higher rank one can expect that this is the same case here also. In contrast to the $GL_n$ case the Ramanujan-Petersson conjecture fails in general. Let $M^\nu_k$ be the space of Siegel modular forms of degree $n$, weight $k$ with respect to the Siegel modular group $\Gamma_n$.

**Theorem 4.1.** Let $\pi = \pi(F)$ be a holomorphic automorphic representation of $GSp(6)(\mathbb{A})$ associated to a Hecke eigenform $F$ in $M^3_k$. Assume that $\pi = \pi^1 \otimes \pi^2$ is the lifting of the two cuspidal automorphic representations $\pi^1 \in A(GL(2))_{k_1}$ and $\pi^2 \in A(GSp(4))_{k_2}$ of even weights $k_1, k_2$. Then $\pi$ is cuspidal.
In the case of the lifting of $\text{GL}(2) \times \text{GL}(2)$ to $\text{GL}(4)$ this was called by Ramakrishnan cuspidality criterion, and employed in [41], section 3.2.

Proof. Let $\pi^1 \in A(\text{GL}(2))_{k_1}$ and $\pi^2 \in A(\text{GSp}(4))_{k_2}$ be two automorphic representations with even weights $k_1, k_2$. Let us assume that an automorphic representation $\pi$ exists coming from some Siegel eigenform $F$ of degree 3 of weight $k_3$, such that

$$L(s, \pi, \rho_3) = L(s, \pi^1 \otimes \pi^2, \rho_1 \otimes \rho_2).$$

Let us recall a result of Chai and Faltings [[11], p. 107].

**Proposition 4.2** (Chai-Faltings). Let $F \in S_k^n$ be a Hecke eigenform. Let $k > n$. Then we have for every finite prime $p$, that there exists at least one element in the Weyl group orbit of the $p$-Satake parameters $\mu_0, \mu_1, \mu_2, \ldots, \mu_n$ such that

$$|\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n| = 1.$$  \hspace{1cm} (4.3)

Here we have the normalization $\mu_0^2 \mu_1 \ldots \mu_n = p^{nk-n(n+1)/2}$.

Let $\alpha_0, \alpha_1$ and $\beta_0, \beta_1, \beta_2$ be the Satake parameters of $\pi'_p$ and $\pi''_p$. Here we have choosen the normalization

$$\alpha_0^2 \alpha_1 = p^{k_1-1}, \quad \beta_0^2 \beta_1 \beta_2 = p^{2k_2-3}.$$  

If we assume that $\pi$ is not cuspidal, then it follows from the Darstellungssatz of Klingen and [20] that $F$ is a Siegel Eisenstein series or a Klingen Eisenstein series. The case $k = 4$ is treated as the case of Siegel type Eisenstein series, although we have Hecke summation. Let $\mu_0, \mu_1, \mu_2, \mu_3$ be the Satake parameter of $\pi_p$ with $\mu_0^2 \mu_1 \mu_2 \mu_3 = p^{3k-6}$. From the lifting assumption we can deduce that

$$\mu_0 = \alpha_0 \beta_0, \quad \mu_1 = \beta_1, \quad \mu_2 = \beta_2, \quad \text{and} \quad \mu_3 = \alpha_1.$$  \hspace{1cm} (4.4)

Hence at least one of the Satake parameter $\mu_1, \mu_2, \mu_3$ of $\pi_p$ has the absolute value one. We now proceed case by case.

a) Let first $F$ be a Siegel Eisenstein series. We can assume that the weight $k$ is larger then 3. It is well known that the Satake parameters $\mu_1, \mu_2, \mu_3$ have the values $p^{k-3}, p^{k-2}, p^{k-1}$. This is unique up to the action of the Weyl group of the symplectic group. Since none of the values has absolute value one we have a contradiction.

b) Next we assume that $F$ is a Klingen Eisenstein series attached to the cusp form $H \in S_k^n$. Here $H$ is also a Hecke eigenform. Let $\gamma_0, \gamma_1, \gamma_2$ be the $p$-th Satake parameter of $H$. Then it follows from a Theorem of Chai and Faltings, that these
parameters can be chosen in such a way, that $|\gamma_1 \gamma_2| = 1$. Then the Satake parameters of $F$ are given by $\mu_1 = \gamma_1$, $\mu_2 = \gamma_2$, $\mu_3 = p^{k-3}$. From (4.4) we know that (up to the action of the Weyl group) at least one of the Satake parameters $\mu_1, \mu_2, \mu_3$ has absolute values 1. But if this would be the case for the Klingen Eisenstein series attached to $H$, then it would follow together with the Theorem of Chai and Faltings that $|\mu_1| = |\mu_2|$. Hence $|\mu_1 \mu_2 \mu_3|$ is always equal to $p^{k-3}$ or $p^{3-k}$, hence a contradiction to the Theorem of Chai and Faltings in degree 3.

c) Now let $F$ be a Klingen Eisenstein series attached to a Hecke eigenform $f \in S_k$ with $p$-th Satake parameters $\gamma_0, \gamma_1$, where $|\gamma_1| = 1$. Then the Satake parameters $\mu_1, \mu_2, \mu_3$ of $F$ can be chosen by

$$\mu_1 = \gamma_1, \, \mu_2 = p^{k-2}, \, \mu_3 = p^{k-3}.$$ 

But this is a contradiction. \qed

There is another possibility to prove this result by analytic methods instead of our algebraic proof. This is done by applying the method given in [20] to the properties of the standard $L$-function.

5. Motivic viewpoint

In this section we consider the lifting problem from the motivic point of view. We determine the possible weights for the candidates and show that this is compatible with our previous results obtained in [23]. We examine the possibility of the decomposition of a motive $M$ over $\mathbb{Q}$ into the tensor product of certain motives $M_1$ and $M_2$, which are attached to automorphic forms:

$$M = M_1 \otimes_{\mathbb{Q}} M_2.$$ 

5.1. Motivic decompositions. In 1999 we first considered the opposite question, we started with two constructed motives $M_1, M_2$ and had been interested in critical values [14] of the motive $M_1 \otimes_{\mathbb{Q}} M_2$ and the explicit formula for Deligne’s conjecture on the arithmetic of the $L$-function evaluated at this point. This was a natural question after finding a new integral representation of the $GL(2)$-twisted Andrianov spinor $L$-function [21]. This motivated Yoshida to work out the formula and a general procedure which describes how invariants and periods of motives $M_1, M_2, \ldots, M_r$ behave by standard algebraic operations (see also Yoshida [47], section 4, page 1193). Finally main parts of Deligne’s conjecture to the $L$-function from above had been proven by Böcherer and Heim [9], [10]. Since $M_1$ and $M_2$ had been motives attached to automorphic forms it is an interesting problem to study the tensor product and ask
if the new motives is a motive of an automorphic form - modularity of motives. The question if this motivic approach of the lifting considered in this paper leads to the same results, which we obtain by concrete analytic methods, was raised by Harder during the Japanese-German conference held at the MPI Mathematik in Bonn in February 2008. We calculate the Hodge numbers to delete all forbidden lifts. Moreover we can predict with this approach the functional equation of the spinor L-function attached to a Siegel modular form of degree 3 and the arithmetic of the critical values.

Let \( k, n \) be positive integers. We assume that \( k > n \), since we are mainly interested in \( n = 1, 2, 3 \). Let \( \pi \in \mathcal{A}(\mathrm{GSp}(2n))_k \) be an automorphic representation with holomorphic discrete series at infinity.

We denote by \( E_\pi \) the totally real number field generated by the Hecke eigenvalues of \( \pi_f := \bigotimes_p \pi_p \). Now it is well known that we can pick a fix vector \( F = F(\pi) \in S_k^\pi \) attached to the representation \( \pi \), such that the field generated by the Fourier coefficients is contained in \( E_\pi \). We denote such \( F \) normalized. In the case \( n = 1 \) this can easily obtained by choosing \( F \) to be primitive. For Siegel modular form there is no such explicit property available.

5.2. Determination of the possible Hodge types. Let \( \mathcal{M} \) be a motive over \( \mathbb{Q} \) with coefficients in a totally real number field \( E \). Let \( H_B(\mathcal{M}) \) be the Betti realization of \( \mathcal{M} \). Then \( H_B(\mathcal{M}) \) is a vector space over \( E \) and its finite dimension \( d \) is called the rank of the motive \( \mathcal{M} \). The Hodge decomposition is given by

\[
H_B(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}(\mathcal{M}).
\]

Then \( H^{p,q}(\mathcal{M}) \) are free \( R := E \otimes_{\mathbb{Q}} \mathbb{C} \) modules. If \( H^{p,q}(\mathcal{M}) = \{0\} \) whenever \( p + q \neq w \). Then we denote \( \mathcal{M} \) a pure motive and \( w \) the (pure) weight. In the following let \( \mathcal{M} = \mathcal{M}(F) \) be the motive attached to a cuspidal Siegel Hecke eigenform with spinor representation of the L-group. Let \( h \in S_k \) then the Hodge type of \( \mathcal{M}(h) \) is given by

\[
(0, k - 1) + (k - 1, 0).
\]

Let \( G \in S_l^2 \) then the Hodge type of \( \mathcal{M}(G) \) is given by

\[
(0, 2l - 3) + (l - 2, l - 1) + (l - 1, l - 2) + (2l - 3, 0).
\]
By employing the Künneth formula we obtain for the tensor product of $\mathcal{M}(h) \otimes \mathcal{M}(G)$ the Hodge type
\[(0, 2l + k - 4) + (l - 2, l + k - 2) + (l - 1, l + k - 3) + (k - 1, 2l - 3) + (k + l - 3, l - 1) + (2l - 3, k - 1) + (l + 2, l - 2) + (k + 2l - 4, 0).
\]
Assuming that this tensor product is isomorphic to the motive $\mathcal{M}(F)$ attached to $F \in S^3_K$, which has the Hodge type
\[(0, 3K - 6) + (K - 3, 2K - 3) + (K - 2, 2K - 4) + (K - 1, 2K - 5) + \text{symmetric terms},
\]
then we obtain $k = K - 2, l = K$. This gives the desired result.

Remark. We are very much indebted to Alexei Panchiskin, for sharing some of his ideas with us.

Remark. It’s also possible to compare the Hodge numbers in the setting of vector-valued modular forms (see [7], for the relevant Hodge decompositions).

6. Analytic properties of Spinor L-functions

In this section we prove the meromorphic continuation of the spinor L-function attached to the lifting $\pi = \pi' \boxtimes \pi''$. We complete the L-function at infinity and obtain the functional equation. This result is compatible with the conjecture of Andrianov and Panchiskin.

**Theorem 6.1.** Let $k_1, k_2$ be even positive integers. Let $\pi^1 \in A(GL(2))_{k_1}$ and $\pi^2 \in A(GSp(4))_{k_2}$ be two automorphic representations. Let $\pi := \pi^1 \boxtimes \pi^2 \in A(GSp(6))_k$ be modular. Then $k_1 = k - 2$ and $k_2 = k$. Let $\rho_3$ be the spinor representation of $\text{Spin}(7)$. Then the L-function $L(s, \pi, \rho_3)$ has a meromorphic continuation to the whole complex plane. Let the first Fourier-Jacobi coefficient of $\pi^2$ be non-trivial. Then the completed L-function

\[(6.1) \quad \hat{L}(s, \pi, \rho_3) = L_\infty(s, \pi, \rho_3) L(s, \pi, \rho_3),
\]
satisfies the functional equation $s \mapsto 3k - 5 - s$. Here
\[L_\infty(s, \pi) := \Gamma_C(s) \prod_{m=1}^{3} \Gamma_C(s - k + m).
\]
Proof. Let \( \alpha_0, \alpha_1 \) be the Satake parameters of \( \pi'_p \) and \( \beta_0, \beta_1, \beta_2 \) be the Satake parameters of \( \pi''_p \). They correspond to the semi-simple elements (conjugacy classes) \( t(\pi'_p) \) and \( t(\pi''_p) \) in the related L-groups. Let \( \rho_1 \) and \( \rho_2 \) be the spinor representations of \( SL(2)(\mathbb{C}) \) and \( Spin(5)(\mathbb{C}) \) then the spinor L-function of \( \pi \) is given by

\[
L(s, \pi, \rho_3) = \prod_p \left\{ \det \left( 1 - \rho_1 \left( t(\pi'_p) \right) \otimes \rho_2 \left( t(\pi''_p) \right) \right) p^{-s} \right\}^{-1}
\]

Then the local factors \( L_p(X, \pi_p, \rho_3) \) are:

\[
\begin{align*}
(1 - \alpha_0 \beta_0 X)(1 - \alpha_0 \beta_0 \beta_1 X)(1 - \alpha_0 \beta_0 \beta_2 X)(1 - \alpha_0 \beta_0 \beta_1 \beta_2 X) \\
(1 - \alpha_0 \alpha_1 \beta_0 X)(1 - \alpha_0 \alpha_1 \beta_0 \beta_1 X)(1 - \alpha_0 \alpha_1 \beta_0 \beta_2 X)(1 - \alpha_0 \alpha_1 \beta_0 \beta_1 \beta_2 X).
\end{align*}
\]

Since \( \pi \) is associated with a Siegel modular form with trivial central character one can employ the Langlands-Shahidi method to obtain the meromorphic continuation of \( L(s, \pi, \rho_3) \). This works since in this case the problem can be reduced to the theory of Euler products applied to a Levi subgroup of the exceptional group of type \( F_4 \). This has been demonstrated in [4]. To get the functional equation one has to use a different method. Surprisingly there also exists a Rankin-Selberg integral which involves the L-function \( L(s, \pi' \otimes \pi'', \rho_1 \otimes \rho_2) \) (if we assume the technical restriction \( \pi'' \) primitive) for all even weights \( k_1, k_2 \). This has been discovered in [21] and generalized in [9]. Once such an integral representation has been found one can apply advanced techniques to get the desired properties of the L-function.

In this case the functional equation of an Eisenstein series of Siegel type of degree 5 leads to the meromorphic continuation and functional equation of the L-function \( L(s, \pi, \rho_3) \) we are interested in. Let \( h \in S_{k_1} \) and \( G \in S_{k_2} \) be Hecke eigenforms with \( (0 < k_1 \leq k_2) \) even integers. For technical reasons we assume that the first Fourier-Jacobi coefficient of \( G \) is non-trivial. Then the function

\[
\mathcal{L}(s) := 2^{-3} (2\pi)^{4-2k_2-k_1} \Gamma_C(s) \Gamma_C(s-k_2+1) \Gamma_C(s-k_2+2) \Gamma_C(s-k_1+1) \prod_p L_p \left( p^{-s}, \pi_h \otimes \pi_G, \rho_1 \otimes \rho_2 \right)
\]

has a meromorphic continuation to the whole complex plane and satisfies the functional equation

\[
(6.3) \quad s \mapsto k_1 + 2k_2 - 3 - s.
\]
Let \( k := k_1 + 2 = k_2 = k_3 \). Then

\[
(6.4) \quad \Gamma_C(s) \prod_{m=1}^{3} \Gamma_C(s - k + m) L(s, \pi' \times \pi'', \rho_1 \otimes \rho_2).
\]

has the functional equation \( s \mapsto 3k - 5 - s \). Hence the theorem is proven. \( \square \)

7. On Deligne’s conjecture

Let \( F_\pi \in S_n^k \) be a Hecke eigenform attached to \( \pi \in \mathcal{A}(GSp(2n)(\mathbb{A})_k) \). If \( k > 2n \) be even then \( F_\pi \) can be normalized such that the Fourier coefficients of \( F_\pi \) are contained in the totally real number field \( E \) generated by eigenvalues. Here we are interested the cases \( n = 1, 2, 3 \). For \( n = 1 \) this is obtained by choosing the eigenform to be primitive, i.e. the first Fourier coefficient \( a_g(1) = 1 \). For Siegel modular forms there is no such choice. Hence we make the assumption that the first Fourier coefficient in degree 2 is non-trivial and denote such forms also primitive.

Let further \( \| F_\pi \| \) denote the norm of \( F_\pi \), related to the Petersson scalar product.

If for \( \pi \in \mathcal{A}(GSp(4))_k \) any primitive \( F \) exists we denote \( \pi \) primitive. Finally using all the observations and results of this paper we obtain our main result:

**Theorem 7.1.** Let \( \tilde{\pi} = \pi' \boxtimes \pi'' \in \mathcal{A}(GSp(6))_k \) for \( \pi' \in \mathcal{A}(GL(2))_{k-2} \) and \( \pi'' \in \mathcal{A}(GSp(4))_k \) primitive. Let \( F_\tilde{\pi} \in S_k^3 \) be normalized. Then the critical integers of the spinor \( L \)-function of \( F_\pi \) are exactly \( m = k, k + 1, \ldots, 2k - 5 \). Moreover there exists a positive \( \Omega \in \mathbb{R} \) such that

\[
(7.1) \quad \frac{L(m, F_\tilde{\pi}, \rho_3)}{\pi^{4m-3k+6} \Omega^2} \in E.
\]

**Proof.** For the readers convenience we recall the basic steps. All the ingredients have already been prepared in this paper. Let \( F_\tilde{\pi} \) be given. We have proven the the full Andrianov conjecture for \( L(s, \tilde{\pi}, \rho_3) \). Hence we have a meromorphic continuation on the whole complex plane. After completion we get a functional equation. From the explicit form of the \( \Gamma \)-factors we can deduce the critical values in the sense of Deligne and obtain exactly \( m = k, k + 1, \ldots, 2k - 5 \). With the functoriality property determined in this paper the lifting is directly related to another \( L \)-function. It can be identified with the so-called \( GL(2) \)- twisted spinor \( L \)-function. And hence properties of this \( L \)-function can be transferred to \( L(s, \tilde{\pi}, \rho_3) \). Let

\[
\Omega := \| h_{\pi'} \|^2 \cdot \| G_{\pi''} \|^2.
\]
Here $h_{\pi'}$ and $G_{\pi''}$ are primitive forms as described above. Then finally form the arithmetic results given in [10] we obtain the desired result of the theorem. □

Identifying the period $\Omega$ with the period given in Deligne’s conjecture $e^{\pm}(\mathcal{M})$ of the attached motive leads to a proof of this conjecture for the lifts considered in this paper. This would imply that the the dimension of the $+$ and $-$ eigenspace of the Betti realization of $\mathcal{M}$ are equal. Finally we would like to remark that Miyawaki’s conjecture of TypII [32], [23] extended to special values result is contained in this theorem. Moreover the assumption to be primitive can be removed.

References

[1] A. N. Andrianov: Shimura’s conjecture for Siegel modular group of genus 3. Soviet Math. Dokl. 8 (1967), 1474-1477.
[2] A. N. Andrianov: Euler products corresponding to Siegel modular forms of genus 2. Russian Math. Surveys 29 (1974), 45-116.
[3] M. Agarwal: $p$-adic L-functions for $GSp(4) \times GL(2)$. Dissertation at the University of Michigan (2007), advisor: C. Skinner.
[4] M. Asgari, R. Schmidt: Siegel modular forms and representations. manuscr. math. 104 (2001) 173-200.
[5] J. Arthur: A trace formula for reductive groups II. Applications of a truncation operator. Compos. Math. 40 (1980), 87-121.
[6] J. Arthur, L. Clozel: Simple Algebras, Base Change and the Advanced Theory of the Trace Formula. Ann. Math. Studies 120 (1989), Princeton.
[7] J. Bergström, G. van der Geer: The Euler characteristic of local systems on the moduli of curves and abelian varieties of genus three. [arXiv:0705.0293v1 [math.AG] 2.5.2007]
[8] S. Böcherer: Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe. J. reine angew. Math. 362 (1985), 146-168.
[9] S. Böcherer, B. Heim: L-functions on $GSp_2 \times GL_2$ of mixed weights. Math. Zeitschrift 235 (2000), 11-51.
[10] S. Böcherer, B. Heim: Critical values of L-functions on $GSp_2 \times GL_2$. Math. Zeitschrift 254 (2006), 485-503.
[11] Ch.-L. Chai, G. Faltings: Degeneration of Abelian Varieties. 22. Ergebnisse der Mathematik, Berlin,Heidelberg, New York: (1990).
[12] F. Chiera, K. Vankov: On special values of spinor L-functions of Siegel cusp eigenforms of genus 3. [arXiv:0805.2114v1 [math.NT] 14.5.2008]
[13] J. Cogdell, I. Piatetski-Shapiro: A converse theorem for $GL(4)$. Math. Research Letters 3 no.1 (1996), 67-71.
[14] P. Deligne: Valeurs de fonctions L et periodes d’integrales. Proc. Sympos. Pure Math. 55 (1979), 313-346.
[15] M. Furusawa: On L-functions for $GSp(4) \times GL(2)$ and their special values. J. Reine Angew. Math. 438 (1993), 187-218.
[16] P. Garrett: *Decomposition of Eisenstein series: triple product L-functions.* Ann. Math. **125** (1987), 209-235.
[17] S. Gelbart: *An elementary introduction to the Langlands program.* Bulletin of the AMS **10**, no.2 (1984), 177-219.
[18] S. Gelbart, F. Shahidi: *Analytic properties of automorphic L-functions.* Perspectives in Mathematics (1988).
[19] G. Harder: *General aspects in the theory of modular symbols.* Seminaire de Theorie des Nombres, Delange-Pisot-Poitou, Paris 1981-82, Prog. Math. Birkäuser, Boston (1983), 73-88.
[20] M. Harris: *The Rationality of Holomorphic Eisenstein series.* Invent. math. **63** (1981), 305-310.
[21] B. Heim: *Pullbacks of Eisenstein series, Hecke-Jacobi theory and automorphic L-functions.* In: Automorphic Forms, Automorphic Representations and Arithmetic. Proceedings of Symposia of Pure Mathematics **66**, part 2 (1999).
[22] B. Heim: *Miyawaki’s $F_{12}$ spinor L-function conjecture.* arXiv:0712.1286v1 [math.NT] 8.12.2007
[23] B. Heim: *On the Modularity of the GL$_2$-twisted Spinor L-function.* MPIM Bonn Preprint No. 2008-66
[24] T. Ikeda: *On the functional equations of triple L-functions.* J. Math. Kyoto Univ. **29** (1989), 175-219.
[25] T. Ikeda: *On the lifting of elliptic cusp forms to Siegel cusp forms of degree $2n$.* Ann. of Math. **154** no. 3 (2001), 641-681.
[26] T. Ikeda: *Pullback of the lifting of elliptic cusp forms and Miyawakis conjecture.* Duke Math. Journal, **131** no. 3 (2006), 469-497.
[27] H. Kim, F. Shahidi: *Functorial products for GL$_2$, GL$_3$ and symmetric cube for GL$_2$.* Ann. math. **155** (2002), 837-893.
[28] R. Langlands: *Problems in the theory of automorphic forms.* LNM **170** (1970) 18-61.
[29] R. Langlands: *Automorphic representations, Shimura varieties and motives. Ein Märchen.* Proc. Symp. Pure Math. **33** part 2 (1979), 205-246.
[30] R. Langlands: *Where Stands Functoriality Today.* Proc. Symp. Pure Math. **61** (1997).
[31] N. Kurokawa: *Examples of eigenvalues of Hecke operators on Siegel cuspforms of degree two.* Inventiones Math. **49** (1978), 149-165.
[32] I. Miyawaki: *Numerical examples of Siegel cusp forms of degree 3 and their zeta functions.* Mem. Fac. Sci. Kyushu Univ., **46** Ser. A. (1992), 307-339.
[33] S. Mizumoto: *Poles and residues of standard L-functions attached to Siegel modular forms.* Math. Ann. **289** (1991), 589-612.
[34] A. Panchishkin: *Admissible Non-Archimedean standard zeta functions of Siegel modular forms.* Proc. of the Joint AMS Summer Conference on Motives, Seattle **2** (1994), 251-292.
[35] A. Panchishkin: *L-functions of Siegel modular forms, motives and p-adic constructions.* Expose pour la Conference: Formes de Jacobi et Applications. 7-11 mai 2007 (CIRM, Luminy).
[36] I. Piatetski-Shapiro, S. Rallis: *Rankin triple L-functions.* Comp. Math. **64** (1987), 31-115.
[37] F. Dalla Piazza, B. van Geemen: Siegel modular forms and finite symplectic groups. arXiv:0804.3769v2v1 [math.AG] 5.05.2008

[38] A. Pitale, R. Schmidt: Integral Representation for L-functions for $GSp(4) \times GL(2)$. arXiv:0807.3522v1 [math.NT] 22.07.2008

[39] A. Pitale, R. Schmidt: L-functions for $GSp(4) \times GL(2)$ in the case of high $GL(2)$ conductor. arXiv:0808.1438v1 [math.NT] 11.08.2008

[40] A. Raghuram, F. Shahidi: Functoriality and Special Values of L-Functions. Eisenstein Series and Applications Series: Progress in Mathematics, Vol. 258 Gan, Wee Teck; Kudla, Stephen S.; Tschinkel, Yuri (Eds.) (2008), 271-293.

[41] D. Ramakrishnan: Modularity of the Rankin-Selberg L-series, and multiplicity one for $SL(2)$. Ann. math. 152 (2000), 45-111.

[42] D. Ramakrishnan, F. Shahidi: Siegel modular forms of genus 2 attached to elliptic curves. Math. Res. Lett. 14(2) (2007), 315-332.

[43] A. Saha: L-functions for holomorphic forms on $GSp(4) \times GL(2)$ and their special values. Dissertation at the Caltech Institute of Technology 2008, California. Advisor: D. Ramakrishnan.

[44] G. Shimura: On modular forms of half-integral weight. Ann. Math. 97 (1973), 440-481.

[45] R. Weissauer: The Ramanujan Conjecture of Genus two. (1994) University Heidelberg

[46] R. Weissauer: Four dimensional Galois representation. Asterisque 302 (2005).

[47] H. Yoshida: Motives and Siegel modular forms. American Journal of Mathematics 123 (2001), 1171-1197.

Max-Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: heim@mpim-bonn.mpg.de