Structure and asymptotics for Motzkin numbers modulo primes using automata

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Abstract

We establish a lower bound of $2^{\frac{2}{p(p-1)}}$ for the asymptotic density of the Motzkin numbers divisible by a general prime number $p \geq 5$. We provide a criteria for when this asymptotic density is actually 1. We also provide a partial characterisation of those Motzkin numbers which are divisible by a prime $p \geq 5$. All results are obtained using the automata method of Rowland and Yassawi.

1 Introduction

The Motzkin numbers $M_n$ are defined by

$$M_n := \sum_{k \geq 0} \binom{n}{2k} C_k$$

where $C_k$ are the Catalan numbers.

There has been some work in recent years on analysing the Motzkin numbers $M_n$ modulo primes and prime powers. Deutsch and Sagan [3] provided a characterisation of Motzkin numbers divisible by 2, 4 and 5. They also provided a complete characterisation of the Motzkin numbers modulo 3 and showed that no Motzkin number is divisible by 8. Eu, Liu and Yeh [4] reproved some of these results and extended them to include criteria for when $M_n$ is congruent to $\{2, 4, 6\}$ mod 8. Krattenthaler and Müller [6] established identities for the Motzkin numbers modulo higher powers of 3 which include the modulo 3 result of [3] as a special case. Krattenthaler and Müller [5] have more recently extended this work to a full characterisation of $M_n \mod 8$ in terms of the binary expansion of $n$. The results in [6] and [5] are obtained by expressing the generating function of $M_n$ as a polynomial involving a special function. Rowland and Yassawi [7] investigated $M_n$ in the general setting of automatic sequences. The values of $M_n$ (as well as other sequences) modulo prime powers can be computed via automata. Rowland and Yassawi provided algorithms for creating
the relevant automata. They established results for \( M_n \) modulo small prime powers, including a full characterisation of \( M_n \) modulo 8 (modulo 5\(^2\) and 13\(^2\) are available from Rowland’s website). They also established that 0 is a forbidden residue for \( M_n \) modulo 8, 5\(^2\) and 13\(^2\). In theory the automata can be constructed for any prime power but computing power and memory quickly becomes a barrier. For example, the automata for \( M_n \) modulo 13\(^2\) has over 2000 states. Rowland and Yassawi also went on to describe a method for obtaining asymptotic densities of \( M_n \). We have previously [2] used Rowland and Yassawi’s work to analyse Motzkin numbers modulo specific primes up to 29. This current paper will deal with a general prime \( p \). It turns out that the behaviour of \( M_n \) modulo a general prime is similar to the behaviour modulo small primes.

We will use Rowland and Yassawi’s automata to establish a lower bound on the asymptotic density of the set of \( M_n \) divisible by a general prime \( p \geq 5 \). This lower bound is \( \frac{2}{p^2(p^2-1)} \). As shown in [2] the asymptotic density is actually 1 for some primes, e.g. \( p = 7, 17, 19 \). We will also make note of some structure results that appear from an examination of the relevant state diagrams of the automata. In particular \( M_n \equiv 0 \mod p \) when \( n \) takes certain forms depending on the prime \( p \). This generalises results that had already been shown to hold for particular small primes as mentioned in the previous paragraph. It is found that the behaviour of \( M_n \mod p \) depends to some extent on the value of \( p \mod 6 \) - this is either +1 or −1 of course.

Table 1 summarises the results that will be presented in subsequent sections. Firstly, we will explain some of the definitions we have used in this paper.

The asymptotic density of a subset \( S \) of \( \mathbb{N} \) is defined to be

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ n \in S : n \leq N \}
\]

if the limit exists, where \( \# S \) is the number of elements in a set \( S \). For a prime number \( p \geq 5 \) we mainly will be studying the asymptotic density of the set

\[
S_p(0) = \{ n \in \mathbb{N} : M_n \equiv 0 \mod p \}.
\]

(1)

However, for any \( x \in \mathbb{N} \) we define the set

\[
S_p(x) = \{ n \in \mathbb{N} : M_n \equiv x \mod p \}.
\]

For a number \( p \), we write the base \( p \) expansion of a number \( n \) as

\[
[n]_p = \langle n_r n_{r-1} ... n_1 n_0 \rangle
\]

where \( n_i \in [0, p - 1] \) and

\[
n = n_r p^r + n_{r-1} p^{r-1} + ... + n_1 p + n_0.
\]

2
Prime Density Values of \( n \) such that \( M_n \equiv 0 \mod p \)

| Prime \( p \) | Density | Values of \( n \) |
|----------------|----------|------------------|
| \( p \equiv 1 \mod 6 \) | \( \geq \frac{2}{p(p-1)} \) | \( n = (pi + 1)p^k - 2 \) for \( i \geq 0 \) and \( k \geq 1 \).  
\( n = (pi + p - 1)p^k - 1 \) for \( i \geq 0 \) and \( k \geq 1 \). |
| \( p \equiv -1 \mod 6 \) | \( \geq \frac{2}{p(p-1)} \) | \( n = (pi + 1)p^{2k} - 2 \) for \( i \geq 0 \) and \( k \geq 1 \).  
\( n = (pi + p - 2)p^{2k+1} - 2 \) for \( i \geq 0 \) and \( k \geq 0 \).  
\( n = (pi + 2)p^{2k+1} - 1 \) for \( i \geq 0 \) and \( k \geq 0 \).  
\( n = (pi + p - 1)p^{2k} - 1 \) for \( i \geq 0 \) and \( k \geq 1 \). |

Table 1: Table of results

Binomial coefficients are prominent in this paper. Here the binomial coefficient \( \binom{n}{m} \) is defined to be 0 when \( m > n \) or when either \( n \) or \( m \) is negative.

2 Background on Motzkin numbers modulo primes

As mentioned in the introduction there have been results which characterise \( M_n \) modulo primes \( p \leq 29 \). We collect these below to allow comparison with the results for a general prime.

**Theorem 1.** (Theorem 5.5 of [4]). The \( n \)th Motzkin number \( M_n \) is even if and only if

\[
n = (4i + \epsilon)4^j + 1 - \delta \text{ for } i,j \in \mathbb{N}, \epsilon \in \{1,3\} \text{ and } \delta \in \{1,2\}.
\]

**Theorem 2.** (Corollary 4.10 of [3]). Let \( T(01) \) be the set of numbers which have a base-3 representation consisting of the digits 0 and 1 only. Then the Motzkin numbers satisfy

\[
M_n \equiv \begin{cases} 
-1 \mod 3 & \text{if } n \in 3T(01) - 1, \\
1 \mod 3 & \text{if } n \in 3T(01) \text{ or } n \in 3T(01) - 2, \\
0 \mod 3 & \text{otherwise}.
\end{cases}
\]

**Theorem 3.** (Theorem 5.4 of [3]). The Motzkin number \( M_n \) is divisible by 5 if and only if \( n \) is one of the following forms

\[
(5i + 1)5^{2j} - 2, \ (5i + 2)5^{2j-1} - 1, \ (5i + 3)5^{2j-1} - 2, \ (5i + 4)5^{2j} - 1 \text{ where } i,j \in \mathbb{N} \text{ and } j \geq 1.
\]
The above results and others have been used to establish asymptotic densities of the sets $S_q(0)$ for $q = 2, 4, 8$ and also for primes up to 29 - see [7], [1] and [2]. In particular the asymptotic density of $S_2(0)$ is $\frac{1}{3}$ ([7] example 3.12), the asymptotic density of $S_4(0)$ is $\frac{1}{6}$ ([7] example 3.14), the asymptotic density of $S_q(0)$ is 1 for $q \in \{3, 7, 17, 19\}$ [1] and [2], the asymptotic density of $S_q(0)$ is $\frac{2}{q(q-1)}$ for $q \in \{5, 11, 13, 23\}$ [1] and [2]. The asymptotic density of $S_{29}(0)$ satisfies $\frac{2}{29 \times 28} < S_{29}(0) < 1$. 

There are 2 questions which we will investigate in this article. Firstly, what is the asymptotic density of $S_p(0)$ for general primes $p \geq 5$? Secondly, what structural features are evident in the distribution of $M_n \mod p$? The investigation will proceed by constructing an automaton for a general prime following the instructions from [7] (Algorithm 1). The state diagram for the automaton provides an excellent tool for analysing the behaviour of $M_n \mod p$.

3 Three useful series and some modular identities involving binomial coefficients

There are 3 series that will appear regularly during the construction of the automaton. We will therefore devote this section to a discussion of these series which are interesting in their own right. We define the series as follows:

$$a_n := \sum_{k \geq 0} (-1)^k \binom{n-k}{k}$$

$$b_n := \sum_{k \geq 0} (-1)^k \binom{n-k}{k} k$$

$$c_n := \sum_{k \geq 0} (-1)^k \binom{n-k}{k} k^2.$$  \hspace{1cm} (4)

Theorem 4.

$$a_n = \cos\left(\frac{\pi n}{3}\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\pi n}{3}\right)$$

$$b_n = \frac{2n}{3} \cos\left(\frac{\pi n}{3}\right) - \frac{2}{3\sqrt{3}} \sin\left(\frac{\pi n}{3}\right)$$

and

$$c_n = \frac{n}{3} (n-1) \cos\left(\frac{\pi n}{3}\right) - \frac{1}{3\sqrt{3}} (n^2 + n - 2) \sin\left(\frac{\pi n}{3}\right)$$

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Proof. The three series each satisfy a linear difference equation. The solutions to these equations can be derived using standard methods. For $a_n$ we have $a_0 = 1$, $a_1 = 1$ and

$$a_{n+1} = \sum_{k \geq 0} (-1)^k \binom{n + 1 - k}{k}$$

$$= \sum_{k \geq 0} (-1)^k \binom{n - k}{k} + \sum_{k \geq 0} (-1)^k \binom{n - k}{k - 1}$$

$$= \sum_{k \geq 0} (-1)^k \binom{n - k}{k} - \sum_{k \geq 0} (-1)^k \binom{n - 1 - k}{k}$$

where we have used the binomial identity

$$\binom{s + 1}{r} = \binom{s}{r} + \binom{s}{r - 1}.$$  \hspace{1cm} (5)

So $a_n$ satisfies the difference equation

$$a_{n+1} - a_n + a_{n-1} = 0$$  \hspace{1cm} (6)

The solution of this difference equation is given in the statement of the theorem. The initial values of $b_n$ are $b_0 = 0$, $b_1 = 0$ and, using the identity (5) again, we can show $b_n$ satisfies the non-homogeneous difference equation

$$b_{n+1} - b_n + b_{n-1} = -a_{n-1}.$$  \hspace{1cm} (7)

Finally, $c_0 = c_1 = 0$ and $c_n$ satisfies the non-homogeneous difference equation

$$c_{n+1} - c_n + c_{n-1} = -a_{n-1} - 2b_{n-1}.$$  \hspace{1cm} (8)

The results can be extended to the series

$$\sum_{k \geq 0} (-1)^k \binom{n - k}{k} k^m.$$

where $m$ is arbitrary but we only need the equations for $a_n$, $b_n$ and $c_n$. We list here some of the properties of the series which will be needed later. Firstly, $a_n$ is a periodic sequence with period 6. Starting with $n = 0$ the sequence $a_n$ is 1, 1, 0, −1, −1, 0, 1, 1, ... For any $n \in \mathbb{N}$,

$$a_{n-1} = \begin{cases} 1 & \text{if } n \equiv 1 \mod 6; \\ -1 & \text{if } n \equiv -1 \mod 6; \\ \end{cases}$$
\[ a_n = \begin{cases} 
1 & \text{if } n \equiv 1 \mod 6; \\
0 & \text{if } n \equiv -1 \mod 6; 
\end{cases} \]
\[ b_n = \begin{cases} 
\frac{n-1}{3} & \text{if } n \equiv 1 \mod 6; \\
\frac{n+1}{3} & \text{if } n \equiv -1 \mod 6; 
\end{cases} \]
and
\[ c_n = \begin{cases} 
-\frac{(n-1)}{3} & \text{if } n \equiv 1 \mod 6; \\
\frac{n^2-1}{3} & \text{if } n \equiv -1 \mod 6. 
\end{cases} \]

The above identities for \( a_n \) show that if \( p \) is prime (so \( p \equiv \pm 1 \mod 6 \))

\[ a_{p-1} - 2a_p + 1 = 0 \]  \hspace{1cm} (9)

and

\[ b_p + c_p = \begin{cases} 
0 & \text{if } p \equiv 1 \mod 6; \\
\frac{p(p+1)}{3} & \text{if } p \equiv -1 \mod 6. 
\end{cases} \]  \hspace{1cm} (10)

We will also need the following identities

\[ a_p - b_p - 1 + \frac{1}{2}b_{p+1} + \frac{1}{2}c_{p+1} - b_{p+2} - c_{p+2} = \begin{cases} 
\frac{p(p+5)}{6} & \text{if } p \equiv 1 \mod 6; \\
\frac{p(p+1)}{6} - 1 & \text{if } p \equiv -1 \mod 6. 
\end{cases} \]  \hspace{1cm} (11)

\[ -a_{p-1} + a_p + b_p - 2b_{p+1} - 2a_{p+1} + 1 = \begin{cases} 
p + 2 & \text{if } p \equiv 1 \mod 6; \\
-p - 1 & \text{if } p \equiv -1 \mod 6. 
\end{cases} \]  \hspace{1cm} (12)

There are also a few modular identities that will be useful in simplifying some of the equations that will appear later. Firstly, for \( k, l \in \mathbb{N} \),

\[ \binom{p-1}{l} - k \equiv (-1)^l \binom{k+l}{k} \mod p. \]  \hspace{1cm} (13)

In particular,

\[ \binom{p-1}{l} \equiv (-1)^l \mod p \]
\[
\binom{p-2}{l} \equiv (-1)^l (l + 1) \mod p
\]
\[
\binom{p-3}{l} \equiv (-1)^l \left(\frac{l + 2}{2}\right) = \frac{1}{2} (-1)^l (l + 2)(l + 1) \mod p.
\]

4 Background on automata for \(M_n \mod p\)

Rowland and Yassawi showed in [7] that the behaviour of sequences such as \(M_n \mod p\) can be studied by the use of finite state automata. The automaton has a finite number of states and rules for transitioning from one state to another. In the form described in [7] each state \(s\) is represented by a polynomial in 2 variables \(x\) and \(y\). Each state has a value obtained by evaluating the polynomial at \(x = 0\) and \(y = 0\). All calculations are made modulo \(p\). For the Motzkin case the initial state \(s_1\) is represented by the polynomial

\[
R(x, y) = y(1 - xy - 2x^2y^2 - 2x^2y^3).
\] (14)

New states are constructed by applying the Cartier operator \(\Lambda_{d,d}\) to the polynomials

\[
s_i * Q^{p-1}(x, y)
\]

for \(d \in \{0, 1, \ldots, p-1\}\) where \(\{s_i\}\) are the already calculated states and the polynomial \(Q\) is defined by

\[
Q( x, y ) = x^2 y^3 + 2x^2 y^2 + x^2 y + xy + x - 1 = x^2 y(y + 1)^2 + x(y + 1) - 1.
\] (15)

The Cartier operator is a linear map on polynomials defined by

\[
\Lambda_{d_1,d_2} \left( \sum_{m,n \geq 0} a_{m,n} x^m y^n \right) = \sum_{m,n \geq 0} a_{pm+d_1,pm+d_2} x^m y^n.
\] (16)

Since the Cartier operator maintains or reduces the degree of the polynomial and there are only finitely many polynomials modulo \(p\) of each degree, all states of the automaton are obtained within a known finite time. It will be seen later that the automaton has at most \(p + 6\) states. If

\[
\Lambda_{d,d}(s * Q^{p-1}) = t
\]

for states (i.e. polynomials) \(s\) and \(t\) then the transition from state \(s\) to state \(t\) under the input \(d\) is part of the automaton.
To calculate $M_n \mod p$, $n$ is first represented in base $p$. The base $p$ digits of $n$ are fed into the automaton starting with the least significant digit. The automaton starts at the initial state $s_1$ and transitions to a new state as each digit is fed into it. The value of the final state after all $n$’s digits have been used is equal to $M_n \mod p$. Refer to [7] for more details.

In the remainder of this article we will provide details of the automata for a general prime $p \geq 5$. We will provide the polynomials and values for the states and the relevant transitions between states. States are listed as $s_1$, $s_2$, ... . Transitions, when provided, will be in the form $(s, j) \rightarrow t$ which means that if the automaton is in state $s$ and receives digit $j$ then it will move to state $t$. We will call a state $s$ a **loop state** if all transitions from $s$ go to $s$ itself, i.e. $(s, j) \rightarrow s$ for all choices of $j$.

States and transitions are represented visually in the form of a directed graph. For example, figure 1 represents an automaton which moves from state $s_1$ to state $s_2$ when it receives the digit 3. It also moves from state $s_2$ to state $s_2$ (i.e. loops) if it is in state $s_2$ and receives a digit 4.

### 5 Preliminary calculations

Before we start constructing the automata it will be convenient to first precompute $\Lambda_{d,d}(s(x, y) * Q(x, y)^{p-1})$ for some simple choices of the polynomial $s$. The relevant results are contained in tables 2 and 3. When reading the table note that $\binom{n}{m} = 0$ for $m < 0$. We will go through a few of the calculations from tables 2 and 3.

Firstly, the polynomial $Q^{p-1}$ can be written as

$$Q^{p-1}(x, y) = \left( x^2 y(y + 1)^2 + x(y + 1) - 1 \right)^{p-1}$$

$$= \sum_{k \geq 0} \binom{p-1}{k} x^{2k} y^k (y + 1)^{2k} \left( x(y + 1) - 1 \right)^{p-1-k}$$

$$= \sum_{k, l \geq 0} \binom{p-1}{k} x^{2k} y^k (y + 1)^{2k} \binom{p-1-k}{l} x^l (y + 1)^l (-1)^{p-1-k-l}$$
| State $s$ | $\Lambda_{d,d}(s \ast Q^{p-1})$ |
|-----------|----------------------------------|
| 1         | $1$ for $0 \leq d \leq 1$  
$\sum_{k \geq 0} \binom{p-1-k}{d-2k} \binom{d}{k} (-1)^d$ for $2 \leq d \leq p - 3$  
$b_p$ for $d = p - 2$  
$a_{p-1}$ for $d = p - 1$ |
| $y$       | $0$ for $d = 0$  
$1$ for $d = 1$  
$\sum_{k \geq 0} \binom{p-1-k}{d-2k} \binom{d}{k+1} (-1)^d$ for $2 \leq d \leq p - 3$  
$- \sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-2}{k+1} + xy + xy^2$  
$= -a_p - b_p + 1 + xy + xy^2$ for $d = p - 2$  
$\sum_{k \geq 0} \binom{p-1-k}{k+1} (p-1)$ = $-a_{p-1}$ for $d = p - 1$ |
| $x^2y^2$  | $- \sum_{k \geq 0} \binom{p-1}{k+1} \binom{p-2}{k} xy = b_p xy$ for $d = 0$  
$\sum_{k \geq 0} \binom{p-1-k}{k} \binom{p-1}{k} xy = a_{p-1} xy$ for $d = 1$  
$\sum_{k \geq 0} \binom{p-1-k}{d-2k-2k} \binom{d-2}{k} (-1)^d$ for $2 \leq d \leq p - 1$ |
| $xy^2$    | $\sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-1}{k} xy = -a_{p-1} xy$ for $d = 0$  
$0$ for $d = 1$  
$\sum_{k \geq 0} \binom{p-1-k}{d-1-2k} \binom{d-1}{k+1} (-1)^{d+1}$ for $2 \leq d \leq p - 3$  
$- \frac{1}{2} (c_{p+1} + b_{p+1})$ for $d = p - 2$  
$- \sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-2}{k+1} + xy + xy^2$  
$= -a_p - b_p + 1 + xy + xy^2$ for $d = p - 1$ |

Table 2: Table of values of $\Lambda_{d,d}(s \ast Q^{p-1})$
| State s | $\Lambda_{d,d}(s \ast Q^{p-1})$ |
|---------|----------------------------------|
| xy      | $a_{p-1}xy$ for $d = 0$ $\sum_{k \geq 0} \binom{p-1-k}{d-1-2k} \binom{p-2}{d-1-2k} \sum_{k \geq 0} \frac{1}{2}(c_{p+1} - b_{p+1})$ for $d = p - 2$ $b_p$ for $d = p - 1$ |
| $x^2y^3$ | $-\sum_{k \geq 0} \frac{1}{2}(c_{p+1} + b_{p+1})$ for $d = p - 1$ |
| $x^2y^4$ | $-\sum_{k \geq 0} \frac{1}{2}(c_{p+1} + b_{p+1})$ for $d = p - 1$ |

Table 3: Table of values of $\Lambda_{d,d}(s \ast Q^{p-1})$
\[
\sum_{k,l \geq 0} \left( \frac{p-1}{k} \right) \left( \frac{p-1}{l} \right) x^{2k+l} y^k (y+1)^{2k+l} x^l (-1)^{k+l}.
\]

Using the identity
\[
\left( \frac{p-1}{k} \right) \equiv (-1)^k \mod p.
\]
we then have
\[
Q^{p-1}(x, y) = \sum_{k,l,m \geq 0} \left( \frac{p-1}{l} \right) \left( \frac{2k+l}{m} \right) (-1)^l x^{2k+l} y^{k+m}.
\]  \hspace{1cm} (17)

We define \(a_{k,l,m}\) by
\[
a_{k,l,m} := \left( \frac{p-1}{l} \right) \left( \frac{2k+l}{m} \right) (-1)^l.
\]  \hspace{1cm} (18)

We then have
\[
Q^{p-1}(x, y) = \sum_{k,l,m \geq 0} a_{k,l,m} x^{2k+l} y^{k+m} = \sum_{i,j \geq 0} b_{i,j} x^i y^j
\]
where
\[
b_{i,j} = \sum_{k \geq 0} \left( \frac{p-1}{i-2k} \right) \left( \frac{i}{j-k} \right) (-1)^i.
\]  \hspace{1cm} (19)

We will next calculate the effect of the Cartier operator \(\Lambda_{d,d}\) on a general monomial \(x^r y^t\). For \(0 \leq d \leq p - 1\) we have
\[
\Lambda_{d,d}(x^r y^t Q^{p-1}) = \Lambda_{d,d}(\sum_{i,j \geq 0} b_{i,j} x^{i+r} y^{j+t}).
\]
\[
= \sum_{i,j \geq 0} c_{i,j} x^i y^j = \sum_{i \geq r, j \geq t} c_{pi+d, pj+d} x^i y^j
\]
where \(c_{i,j}\) is defined by \(c_{i,j} := b_{i-r, j-t}\). So
\[
\Lambda_{d,d}(x^r y^t Q^{p-1}) = \sum_{i \geq r, j \geq t} \left( \sum_{k \geq 0} \left( \frac{p-1}{p_i+d-r-2k} \right) \left( \frac{p_i+d-r}{p_j+d-t-k} \right) (-1)^{i+d+r} \right) x^i y^j.
\]  \hspace{1cm} (20)
The sum above is finite as the indices $i$ and $j$ satisfy the restrictions
\[ r \leq pi + d \leq 2(p - 1) + r \quad \text{and} \quad t \leq pj + d \leq 3(p - 1) + t. \]

We will first look at the monomial $1$, for which we have $r = t = 0$. For this choice of $r$ and $t$ we need to determine
\[ \binom{p - 1 - k}{pi + d - 2k} \binom{p - 1 - k}{pj + d - k} (-1)^{i+d}. \]

for all possible choices of $i$ and $j$. In this case it turns out all terms are $0 \mod p$ except for the $i = j = 0$ term. So, for $0 \leq d \leq p - 1$,
\[ \Lambda_{d,d}(1 * Q^{p-1}) = \sum_{k \geq 0} \binom{p - 1 - k}{d - 2k} \binom{d}{k} (-1)^d. \]

For $d = p - 2$ this sum reduces to $b_p$ as
\[
\sum_{k \geq 0} \binom{p - 1 - k}{p - 2 - 2k} \binom{p - 2}{k} (-1)^{p-2} = \sum_{k \geq 0} \binom{p - 1 - k}{k + 1} (-1)^{k+1}(k + 1) \\
= \sum_{k \geq 1} \binom{p - k}{k} (-1)^k k \\
= b_p.
\]

For $d = p - 1$ the sum is $a_{p-1}$ as
\[
\sum_{k \geq 0} \binom{p - 1 - k}{p - 1 - 2k} \binom{p - 1}{k} (-1)^{p-1} = \sum_{k \geq 0} \binom{p - 1 - k}{k} (-1)^k = a_{p-1}
\]

The remaining cases can be treated similarly giving the results stated in table 2 and table 3.

6 Constructing the automata for $M_n \mod p$

In this section we will describe the states and transitions of the automata for $M_n \mod p$. These are summarised in table 4 for the case $p \equiv 1 \mod 6$ and tables 5...
and 6 for the case $p \equiv -1 \mod p$. A 'c' appearing in the tables represents a constant state (constant polynomial). The value of c depends on d and p. For given $d : 0 \leq d \leq p - 1$ and state s the tables give the state equal to

$$\Lambda_{d,d}(s \ast Q^{p-1}).$$

The transition $(s,d) \rightarrow \Lambda_{d,d}(s \ast Q^{p-1})$ is then part of the automaton.

The behaviour of the automaton depends on the value of $p \mod 6$. When $p \equiv 1 \mod 6$ there are up to $p + 4$ states consisting of the p constant polynomials modulo p and the 4 polynomials

$$s_1 = -2x^2y^3(y+1) - xy^2 + y, \quad s_2 = x^2y^2(y+1) + xy, \quad -xy(y+1), \quad x^2 + xy^2 + 2.$$  

When $p \equiv -1 \mod 6$ there are up to $p + 6$ states consisting of the p constant polynomials modulo p and the 6 polynomials

$$s_1, \quad s_2, \quad -xy(y+1) - 1, \quad xy(y+1) - 1, \quad xy(y+1), \quad xy(y+1) + 2.$$

It is unclear whether all p constant polynomials always appear as states.
\[
d s_1 s_2 \quad 2x^2y^2(y + 1) + xy \quad 1 \quad -xy(y + 1) - 1
\]

| \(d\) | \(s_2\) | \(s_2\) | 1 | \(-xy(y + 1) - 1\) |
|-------|--------|--------|---|---------------------|
| \(d = 0\) | \(s_2\) | \(s_2\) | 1 | \(-1\) |
| \(d = 1\) | 1 | 1 | 1 | \(c\) |
| \(2 \leq d \leq p - 4\) | \(c\) | \(c\) | \(c\) | \(c\) |
| \(d = p - 3\) | \(c\) | \(c\) | \(c\) | 0 |
| \(d = p - 2\) | \(-xy(y + 1) - 1\) | \(c\) | \(c\) | \(c\) |
| \(d = p - 1\) | \(xy(y + 1) - 1\) | \(c\) | \(-1\) | \(-xy(y + 1)\) |

Table 5: Table of states and transitions for \(p \equiv -1 \mod 6\).

| \(d\) | \(xy(y + 1) - 1\) | \(-xy(y + 1)\) | \(xy(y + 1) + 2\) |
|-------|-----------------|----------------|------------------|
| \(d = 0\) | -1 | 0 | 2 |
| \(1 \leq d \leq p - 3\) | \(c\) | \(c\) | \(c\) |
| \(d = p - 2\) | \(c\) | \(c\) | 0 |
| \(d = p - 1\) | \(xy(y + 1) + 2\) | \(-xy(y + 1) - 1\) | \(xy(y + 1) - 1\) |

Table 6: Table of states and transitions for \(p \equiv -1 \mod 6\).
Figures 2, 3 and 4 provide an alternative pictorial summary of the automata for $M_n \mod p$.

The calculation of the states will rely on the data contained in table 2 and table 3. As mentioned earlier, the initial state $s_1$ for the automata is the polynomial defined in equation (14). The second state $s_2$ is then given by

$$s_2 = \Lambda_{0,0}(s_1 * Q(x,y)^{p-1})$$

$$= \Lambda_{0,0}(y(1 - xy - 2x^2y^2 - 2x^2y^3) * Q(x,y)^{p-1})$$

$$= a_{p-1}xy - 2((-a_p - b_p + 1)xy + x^2y^2 + x^2y^3) - 2((2a_p + b_p - 2)xy - 2x^2y^2 - 2x^2y^3)$$
Figure 3: Another part of the state diagram for $M_n \mod p$ when $p \equiv 1 \mod p$.

Figure 4: Partial state diagram for $M_n \mod p$ when $p \equiv -1 \mod p$. 
\[
= 2x^2y^2 + 2x^2y^3 + (a_{p-1} - 2a_p + 2)xy
= 2x^2y^2 + 2x^2y^3 + xy
\]

from equation (9).

The next interesting state is \( \Lambda_{p-2,p-2}(s_1 \ast Q(x, y)^{p-1}) \). We have

\[
\Lambda_{p-2,p-2}(s_1 \ast Q(x, y)^{p-1}) = -a_p - b_p + 1 + xy + xy^2 + \frac{1}{2}(b_{p+1} + c_{p+1})
+ 2\sum_{k \geq 0} \left( \frac{p - 1 - k}{k + 3} \right) \left( \frac{p - 4}{k + 1} \right) + 2\sum_{k \geq 0} \left( \frac{p - 1 - k}{k + 3} \right) \left( \frac{p - 4}{k + 2} \right) - xy - y^2
= -xy(y + 1) - a_p - b_p + 1 + \frac{1}{2}(b_{p+1} + c_{p+1}) + 2\sum_{k \geq 0} \left( \frac{p - 1 - k}{k + 3} \right) \left( \frac{p - 3}{k + 2} \right).
\]

Now since

\[
\sum_{k \geq 0} \left( \frac{p - 1 - k}{k + 3} \right) \left( \frac{p - 3}{k + 2} \right) = \frac{1}{2} \sum_{k \geq 0} \left( \frac{p - 1 - k}{k + 3} \right)(-1)^k(k + 4)(k + 3)
= -\frac{1}{2} \sum_{k \geq 3} \left( \frac{p + 2 - k}{k} \right)(-1)^k k(k + 1)
= -\frac{1}{2} (c_{p+2} + (p + 1) - 4\left( \frac{p}{2} \right) + b_{p+2} + (p + 1) - 2\left( \frac{p}{2} \right))
\equiv -\frac{1}{2} (c_{p+2} + b_{p+2} + 2) \mod p
\]

we have

\[
\Lambda_{p-2,p-2}(s_1 \ast Q(x, y)^{p-1}) = -xy(y + 1) - a_p - b_p + 1 + \frac{1}{2}b_{p+1} + \frac{1}{2}c_{p+1} - b_{p+2} - c_{p+2}.
\]

So, using equation 11 and operating modulo \( p \),

\[
\Lambda_{p-2,p-2}(s_1 \ast Q(x, y)^{p-1}) = \begin{cases} 
-xy(y + 1) & \text{ if } p \equiv 1 \mod 6; \\
-xy(y + 1) - 1 & \text{ if } p \equiv -1 \mod 6.
\end{cases}
\]

The next state to appear is \( \Lambda_{p-1,p-1}(s_1 \ast Q(x, y)^{p-1}) \).

\[
\Lambda_{p-1,p-1}(s_1 \ast Q(x, y)^{p-1}) = -a_{p-1} - ( -a_p - b_p + 1 + xy + xy^2 + (c_{p+1} + b_{p+1})
\]

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\[-(c_{p+1} + 3b_{p+1} + 2a_{p+1} - 2) + 2(xy + xy^2)\]
\[= xy(y + 1) - a_{p-1} + a_p + b_p - 2b_{p+1} - 2a_{p+1} + 1\]
\[= \begin{cases} 
  xy(y + 1) + 2 & \text{if } p \equiv 1 \mod 6; \\
  xy(y + 1) - 1 & \text{if } p \equiv -1 \mod 6; 
\end{cases}\]

using equation (12). The transitions \(\Lambda_{d,d}(s_2 * Q(x, y)^{p-1})\) all produce constant states except for \(\Lambda_{0,0}(s_2 * Q(x, y)^{p-1})\).

\[
\Lambda_{0,0} \ (s_2 * Q(x, y)^{p-1}) = a_{p-1}xy + 2b_pxy + 2(-a_p - b_p + 1)xy + 2x^2y^2 + 2x^2y^3 \\
= (a_{p-1} - 2a_p + 2)xy + 2x^2y^2 + 2x^2y^3 \\
= s_2
\]

using equation (9). In order to complete the entries from tables 4, 5 and 6 it is enough to examine the transitions \(\Lambda_{d,d}(xy(y + 1) * Q(x, y)^{p-1})\) (noting that \(xy(y + 1)\) is not actually a state). A similar calculation to the one for \(\Lambda_{p-2,p-2}(s_1 * Q(x, y)^{p-1})\) can be used to show that

\[
\Lambda_{p-3,p-3}(xy(y + 1) * Q(x, y)^{p-1}) = \frac{1}{2}(b_{p+2} - c_{p+2}); \\
\Lambda_{p-3,p-3}(1 * Q(x, y)^{p-1}) = \frac{1}{2}(c_{p+1} - b_{p+1}).
\]

We then have

\[
\Lambda_{p-3,p-3} \ ((-xy(y + 1) - 1) * Q(x, y)^{p-1}) = \frac{1}{2}\left((c_{p+2} - c_{p+1}) - (b_{p+2} - b_{p+1})\right) \\
= \frac{1}{2}\left((-a_p - 2b_p - c_p) - (-a_p - b_p)\right) \\
= -\frac{1}{2}(c_p + b_p) \\
\equiv 0 \mod p
\]

using equation (10).

We also have

\[
\Lambda_{p-1,p-1} \ ((xy(y + 1)) * Q(x, y)^{p-1})
\]
\[ b_p + (-a_p - b_p + 1) + xy + xy^2 \]
\[ = 1 - a_p + xy(y + 1) \]
\[ = \begin{cases} 
xy(y + 1) & \text{if } p \equiv 1 \mod 6; \\
xy(y + 1) + 1 & \text{if } p \equiv -1 \mod 6.
\end{cases} \]

7 Conclusions

The values of \( n \) mentioned in table 1 for which \( M_n \equiv 0 \mod p \) can be immediately derived from an inspection of tables 4, 5 and 6 and the associated state diagrams in figures 2, 3 and 4. A lower bound for the asymptotic density of the set \( S_p(0) \) can then be derived from the following result from [1]

**Theorem 5.** Let

\[ S(q, r, s, t) = \{(qi + r)q^{sj+t} : i, j \in \mathbb{N}\} \]

and

\[ S'(q, r, s, t) = \{(qi + r)q^{sj+t} : i, j \in \mathbb{N}, j \geq 1\} \]

for integers \( q, r, s, t \in \mathbb{Z} \) with \( q, s > 0, t \geq 0 \) and \( 0 \leq r < q \). Then the asymptotic density of the set \( S \) is \( (q^{t+1-s}(q^s - 1))^{-1} \). The asymptotic density of the set \( S' \) is \( (q^{t+1}(q^s - 1))^{-1} \).

From above we know that if \( p \equiv 1 \mod p \) then \( M_n \equiv 0 \mod p \) when \( n \) is in the forms

\[ n = (pi + 1)p^k - 2 \text{ for } i \geq 0 \text{ and } k \geq 1. \]
\[ n = (pi + p - 1)p^k - 1 \text{ for } i \geq 0 \text{ and } k \geq 1. \]

Each of the 2 forms has asymptotic density \( \frac{1}{p(p-1)} \). Therefore, when \( p \equiv 1 \mod 6 \) the asymptotic density of \( S_p(0) \) is \( \frac{2}{p(p-1)} \).

For \( p \equiv -1 \mod 6 \) there are 4 forms of numbers to consider. These are

\[ n = (pi + 1)p^{2k} - 2 \text{ for } i \geq 0 \text{ and } k \geq 1. \]
\[ n = (pi + p - 2)p^{2k+1} - 2 \text{ for } i \geq 0 \text{ and } k \geq 0. \]
\[ n = (pi + 2)p^{2k+1} - 1 \text{ for } i \geq 0 \text{ and } k \geq 0. \]
\[ n = (pi + p - 1)p^{2k} - 1 \text{ for } i \geq 0 \text{ and } k \geq 1. \]
The first and fourth forms have asymptotic density \( (p(p^2 - 1))^{-1} \). The second and third forms have asymptotic density \( (p^2 - 1)^{-1} \). Therefore, when \( p \equiv -1 \mod 6 \) the asymptotic density of \( S_p(0) \) is again \( \geq \frac{2}{p(p-1)} \).

Tables 4, 5 and 6 and the state diagrams in Figures 2, 3 and 4 can be used to determine which numbers \( n \) have \( M_n \equiv x \mod p \) for other values of \( x \). For example, figure 3 shows that if \( p \equiv 1 \mod 6 \) then \( M_n \equiv 1 \mod p \) when the base \( p \) representation of \( n \) contains only 0’s and 1’s. Figure 2 shows that if \( p \equiv 1 \mod 6 \) then \( M_n \equiv 2 \mod p \) when \( n = p^k - 1 \) for some \( k \in \mathbb{N} \).

As mentioned in the introduction results on forbidden residues of \( M_n \mod p^k \) have been proved for some primes \( p \) and some \( k \geq 2 \). In order to whether there are forbidden residues \( \mod p \) itself the constant states of the automata would need to be examined. Since

\[
\Lambda_{d,d}(1 \ast Q(x,y)^{p-1}) = c(p,d) := \sum_{k \geq 0} \binom{p^k - 1 - k}{d - 2k} \binom{d}{k} (-1)^d
\]

in order to show there are no forbidden residues \( \mod p \) it is sufficient to show that the set

\[
\{ c(p,d) : 0 \leq d \leq p - 1 \}
\]

generates \( \left( \frac{\mathbb{Z}}{p \mathbb{Z}} \right)^\times \).

As shown in [2] the asymptotic density of \( S_p(0) \) is actually 1 for some primes, e.g. \( p = 7, 17, 19 \). In these cases, there is a \( d : 2 \leq d \leq p - 2 \) such that

\[
\Lambda_{d,d}(1 \ast Q(x,y)^{p-1}) = 0.
\]

It then follows that

\[
\Lambda_{d,d}(c \ast Q(x,y)^{p-1}) = 0
\]

for all constants \( c \). As a result, any \( n \) which has a base-\( p \) representation containing 2 or more digits \( d \) satisfies \( M_n \equiv 0 \mod p \). The asymptotic density of this set is 1. It would therefore be of interest to determine for which \( p \) and \( d \)

\[
\sum_{k \geq 0} \binom{p - 1 - k}{d - 2k} \binom{d}{k} \equiv 0 \mod p.
\]

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