ON THE ASYMPTOTICS OF MORSE NUMBERS OF FINITE COVERS OF MANIFOLDS

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ABSTRACT. Let $M$ be a closed connected manifold. We denote by $\mathcal{M}(M)$ the Morse number of $M$, that is, the minimal possible number of critical points of a Morse function $f$ on $M$. M. Gromov posed the following question: Let $N_k, k \in \mathbb{N}$ be a sequence of manifolds, such that each $N_k$ is an $a_k$-fold cover of $M$ where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\mathcal{M}(N_k)$ as $k \to \infty$?

In this paper we study the case $\pi_1(M) \cong \mathbb{Z}^n$, $\dim M \geq 6$. Let $\xi \in H^1(M, \mathbb{Z}), \xi \neq 0$. Let $M(\xi)$ be the infinite cyclic cover corresponding to $\xi$, with generating covering translation $t : M(\xi) \to M(\xi)$. Let $M(\xi, k)$ be the quotient $M(\xi)/t^k$. We prove that $\lim_{k \to \infty} \mathcal{M}(M(\xi, k))/k$ exists. For $\xi$ outside a subset $\mathcal{M} \subset H^1(M)$ which is the union of a finite family of hyperplanes, we obtain the asymptotics of $\mathcal{M}(M(\xi, k))$ as $k \to \infty$ in terms of homotopy invariants of $M$ related to the Novikov homology of $M$. It turns out that the limit above does not depend on $\xi$ (if $\xi \notin \mathcal{M}$). Similar results hold for the stable Morse numbers. Generalizations for the case of non-cyclic covers are obtained.

INTRODUCTION AND THE STATEMENT OF THE RESULT

Let $M$ be a closed connected smooth manifold. Denote by $\mathcal{M}(M)$ the Morse number of $M$, that is, the minimal possible number of critical points of a Morse function on $M$. In the case $\pi_1(M) = 0, \dim M \geq 6$, this number is easily computable in terms of homology of $M$ (see [14]). In the case of arbitrary fundamental group (even for $\dim M \geq 6$), the number $\mathcal{M}(M)$ is very difficult to compute: it depends on the simple homotopy type of $M$, the relevant algebraic constructions are rather complicated, and it is not easy to extract the needed numerical invariant (see [15], or [16], Ch. 7).

M. Gromov posed the following question:

Let $N_k, k \in \mathbb{N}$ be a sequence of manifolds, such that each $N_k$ is an $a_k$-fold cover of the manifold $M$ where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\mathcal{M}(N_k)$ as $k \to \infty$?

In the present article we study the problem for $\pi_1(M)$ free abelian and $\dim M \geq 6$. To formulate our results, we need some terminology from algebra. Denote $\mathbb{Z}[\mathbb{Z}^n]$ by $\Lambda$. Let

$$C_* = \{0 \leftarrow C_0 \overset{\partial_1}{\longrightarrow} C_1 \ldots \overset{\partial_k}{\longrightarrow} C_k \leftarrow 0\}$$

be a free finitely generated $\Lambda$-complex. Denote by $B_i(C_*)$ the rank of the module $H_i(C_*) \otimes_{\Lambda} \{\Lambda\}$ over the field of fractions $\{\Lambda\}$. Denote by $B(C_*)$ the sum of all $B_i(C_*)$. Consider now the homomorphism $\partial_{i+1} : C_{i+1} \to C_i$, and let $d = \text{rk} C_i$. Recall that the Fitting invariant $\mathcal{F}_i$ of the homomorphism $\partial_{i+1}$ (see e.g. [4], p.278) is the ideal of $\Lambda$ generated by the $(d-t) \times (d-t)$ subdeterminants of the matrix of $\partial_{i+1}$ (for $t \geq d$ one sets $\mathcal{F}_i = \Lambda$ by

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definition). We shall denote the sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_d$ of the Fitting invariants by $F(i)$.

Define the reduced Fitting sequence for $\partial i+1$ to be the sequence

$$FR(i) \quad \mathcal{F}_s \subset \ldots \subset \mathcal{F}_r$$

where $\mathcal{F}_s$, respectively $\mathcal{F}_r$, is the first, respectively the last, term of the Fitting sequence $F(i)$, not equal to 0, respectively to $\Lambda$. The sequence $F(i)$ is not a homotopy invariant of $\mathcal{C}_s$, but the sequence $FR(i)$ is (see e.g. [13], Ch.4, §2). We say that an ideal $J$ of $\Lambda$ is numerically prime if there is no number $l \in \mathbb{Z}, l \neq \pm 1$, such that every $R \in J$ is divisible by $l$, and we denote by $Q_i(\mathcal{C}_s)$ the number of ideals in the sequence $FR(i)$ which are not numerically prime. Denote by $Q(C_s)$ the sum of all $Q_i(\mathcal{C}_s)$.

A subgroup $G \subset \mathbb{Z}^m$ will be called an integral hyperplane if it is a direct summand of $\mathbb{Z}^m$ of rank $m - 1$.

Now let $M$ be a closed connected manifold, $\pi_1(M) \approx \mathbb{Z}^m, m \geq 1$. It is convenient to set $m = n + 1, n \geq 0$. For every non-zero $\xi \in H^1(M)$ there is a unique connected infinite cyclic covering $\mathcal{P}_\xi : M(\xi) \to M$ such that $\mathcal{P}_\xi(\xi) = 0$. Denote by $M(\xi, k) \to M$ the $k$-fold cyclic covering of $M$ obtained from $\mathcal{P}_\xi$. Let $\mathcal{C}_s(M)$ be the cellular chain complex of the universal cover $\widetilde{M}$. We shall abbreviate $B(C_s(M))$ to $B(M)$ and $Q(C_s(M))$ to $Q(M)$.

**Main Theorem.** Let $\dim M \geq 6, \pi_1(M) \approx \mathbb{Z}^{n+1}, n \geq 0$. Then:

1. For any non-zero $\xi \in H^1(M)$ the limit $\lim_{k \to \infty} M(M(\xi, k))/k$ exists.
2. There is a subset $\mathcal{M} \subset H^1(M)$ which is a finite union of integral hyperplanes in $H^1(M)$, and for every non-zero $\xi \notin \mathcal{M}$ there is a real number $a$ such that for every $k \in \mathbb{N}$ we have

$$k(B(M) + 2Q(M)) - a \leq M(M(\xi, k)) \leq k(B(M) + 2Q(M)) + a$$

**Remarks:**

1. A similar result holds for the stable Morse numbers of $M$, see §5.
2. The limit $\lim_{k \to \infty} M(M(\xi, k))/k$ will be denoted by $\mu(M, \xi)$. The second point of the Main Theorem implies that for a "generic" cohomology class $\xi$ we have $\mu(M, \xi) = B(M) + 2Q(M)$.
3. Denote by $\mathcal{M}_i(M)$ the minimal number of critical points of index $i$ of a Morse function on $M$. The methods of the present paper allow also to prove that (under the assumptions of the Main Theorem) the limit $\lim_{k \to \infty} M_i(M(\xi, k))/k$ exists, and that for all $\xi$, except those belonging to a finite union of integral hyperplanes, there is a real number $a$ such that for every natural $k$ we have

$$k(B_i(M) + Q_i(M) + Q_{i-1}(M)) - a \leq M_i(M(\xi, k)) \leq k(B_i(M) + Q_i(M) + Q_{i-1}(M)) + a$$

4. The numbers $B_s(M), Q_s(M)$ are closely related to the Novikov homology of $M$. Namely, $B_s(M)$, $Q_s(M)$ is equal to the Novikov Betti number $b_s(M, \xi)$ for every non-zero class $\xi \in H^1(M)$ (note that $B_s(M)$ is also equal to the $L^2$-Betti number $b_s^2(M)$). Further, for every non-zero $\xi \notin \mathcal{M}$ we have $Q_s(M) \leq q_s(M, \xi)$ where $q_s(M, \xi)$ is the Novikov torsion number (that follows from Remark 2.6 and Proposition 3.3 of the present paper).

The proof is outlined as follows. Assume that $\xi \in H^1(M)$ is indivisible. Let $f : M \to S^1$ be a Morse map, representing $\xi$, and let $V = f^{-1}(\lambda)$ be a regular level surface of $f$. We can assume that $V$ is connected and that $\pi_1(V) \to \pi_1(M)$ is an isomorphism onto $\text{Ker} \xi$. Cut $M$ along $V$, and obtain a cobordism $W$, such that the boundary $\partial W$ has two connected components $\partial_0 W$ and $\partial_1 W$, each diffeomorphic to $V$. The cyclic cover $M(\xi)$ is the union of a countable family of copies of $W$ glued successively to each other. The union $W_k$ of $k$ successive copies is a cobordism. Its boundary $\partial W_k$ has two connected components $\partial_0 W_k$
and $\partial_1 W_k$, each diffeomorphic to $V$ (see §4 for details). We show that $\mathcal{M}(\mathcal{M}(\xi, k))$ and $\mathcal{M}(W_k, \partial_0 W_k)$ have the same asymptotics as $k \to \infty$ (see §4). Further, $\mathcal{M}(W_k, \partial_0 W_k)$ is equal to the Morse number of the $\mathbb{Z}[\mathbb{Z}^n]$-complex $C_*(\widetilde{\mathcal{M}}(\xi), \partial)$, see §1 for definitions. It turns out that the asymptotic behaviour of this Morse number (as $k \to \infty$) depends only on the chain homotopy type of $C_*(\mathcal{M}(\xi))$ (moreover it depends only on the Novikov completion of this complex). The definition and the properties of the corresponding invariant of chain complexes are the subject of §§1 - 3 of the paper. These sections are purely algebraic. It follows from the author’s earlier result [3], that for $\xi$ outside a finite union of integral hyperplanes in $H^1(M)$, the Novikov-completed chain homotopy type of $C_*(\mathcal{M}(\xi))$ is easily computable. (This is the subject of the second half of §2 and of §3.) This leads to the effective computation of the asymptotics presented in the main theorem.

I am grateful to M. Gromov for a stimulating discussion on the subject. He suggested in particular, that asymptotically the numbers $\mathcal{M}(N)$ above should be related to Novikov numbers. He indicated also that the Morse number $\mathcal{M}(\mathcal{M}(\xi, k))$ should have the same asymptotics as the Morse number of the pair $(W_k, \partial_0 W_k)$ (see Prop. 4.1 of the present paper).

1. Morse numbers of chain complexes

In this section we define the notion of the Morse number for arbitrary chain complexes over $\mathbb{Z}[\mathbb{Z}^n]$ and we develop some basic properties of these numbers. We assume that the reader is familiar with §3 of [15] and with §1 of [1]. We denote $\mathbb{Z}[\mathbb{Z}^n]$ by $R$.

Terminological remark. Let $A_*, B_*$ be chain complexes. We shall denote the chain maps from $A_*$ to $B_*$ as follows: $f_* : A_* \to B_*$, so that $f_k$ is a homomorphism $A_k \to B_k$.

Definition 1.1. An $R$-complex is a chain complex $\{0 \leftarrow C_0 \leftarrow C_1 \ldots \leftarrow C_k \leftarrow 0\}$ of finitely generated $R$-modules. The length $l(C_*)$ of an $R$-complex $C_*$ is the maximal number $l$ such that $C_l \neq 0$. An $R$-complex $C_*$ is called a free $R$-complex (or simply $f$-complex) if every $C_i$ is a free finitely generated module over $R$.

Definition 1.2 [15]. Let $C_*$ be an $f$-complex over $R$. The minimal possible number of free generators of an $f$-complex $D_*$, having the same homotopy type as $C_*$, is called the Morse number of $C_*$ and denoted by $\mathcal{M}(C_*)$ (or by $\mathcal{M}_R(C_*)$, if we want to stress the base ring).

One of the consequences of the Quillen-Suslin theorem ([13], [17]) is that $R$ is an $s$-ring, that is, every projective $R$-module is free (see [3], Ch.5, §4). $R$ is also an IBN-ring, that is, the number of free generators of a free module is uniquely determined. Therefore, in the homotopy type of every $f$-complex over $R$ there exists a minimal chain complex, that is, a complex $D_*$ such that the number of free generators of $D_*$ in each dimension is minimal over all the free complexes in this homotopy type (see [15], Th.3.7).

Definition 1.3. Let $A_*$ be an $R$-complex. We call a free model of $A_*$ a free $R$-complex $A'_*$ together with a chain map $\alpha_* : A'_* \to A_*$ which is epimorphic and induces an isomorphism in homology. \footnote{Sometimes we shall say (by abuse of terminology) that the complex $A'_*$ itself is a free model of $A_*$.} If $\alpha_* : A'_* \to A_*$, $\beta_* : B'_* \to B_*$ are free models, and $f_* : A_* \to B_*$ is a chain map, then a chain map $F_* : A'_* \to B'_*$ is called covering of $f$ if $\beta_* F_* = f_* \alpha_*$. Similar terminology is accepted for chain homotopies.
Lemma 1.4.  Let $A_*$ be an $R$-complex. Then there is a free model of $A_*$ and

1. Every chain map $A_* \to B_*$ admits a covering with respect to any free models of $A_*$ and $B_*$. 
2. Let $h_* : A_* \to B_{*+1}$ be a chain homotopy from $f_*$ to $g_*$, and $F_*, G_*$ be coverings of $f_*$, respectively $g_*$, with respect to some free models of $A_*, B_*$. Then there is a chain homotopy $H_*$ from $F_*$ to $G_*$, covering $h_*$. 
3. Two free models of a complex $A_*$ are homotopy equivalent.

Proof. To prove the existence of a free model, we proceed by induction in the length of $A_*$. If $l(A_*) = 0$, then it follows from the fact that every finitely generated module over $R$ has a free finite resolution of finite length. To make the induction step, it suffices to construct a free model for a complex of the type $C_* = \{ 0 \leftarrow A_0 \overset{\partial_0}{\leftarrow} C_1 \overset{\partial_0}{\leftarrow} C_2 \ldots \overset{\partial_n}{\leftarrow} C_n \leftarrow 0 \}$, where $C_i$ are free finitely generated modules and $A_0$ is a finitely generated module. Let $B_* = \{ 0 \leftarrow A_0 \overset{\epsilon}{\leftarrow} E_0 \overset{\partial_0}{\leftarrow} E_1 \overset{\partial_0}{\leftarrow} \ldots \}$ be a finite free resolution of $A_0$. There is a chain map $\phi_* : C_* \to B_*$, such that $\phi_0 = \text{id}$. Define now an $R$-complex

$$F_* = \{ 0 \leftarrow E_0 \overset{D_1}{\leftarrow} C_1 \oplus E_1 \overset{D_2}{\leftarrow} C_2 \oplus E_2 \ldots \}$$

setting $D_1(c_1, e_1) = \phi_1(c_1) + d_1(e_1)$ and $D_i(c_i, e_i) = (\partial_i(c_i), d_i(e_i) + (-1)^i \phi_i(c_i))$ for $i \geq 2$.

Define further a map $\gamma_* : F_* \to C_*$ to be the projection $(x, y) \mapsto x$ when $* \geq 1$ and set $\gamma_0 = \epsilon$. It is easy to check that $F_*$ is indeed an $f$-complex, and that $\gamma_*$ is a free model. The points (1) and (2) of our lemma are proved by a standard homological algebra argument; (3) follows from (2).

Definition 1.5. The Morse number $\mathcal{M}(C_*)$ of a complex $C_*$ is the Morse number of any of its free models.

Proposition 1.6. Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of $R$-complexes. Then 1) $\mathcal{M}(B_*) \leq \mathcal{M}(A_*) + \mathcal{M}(C_*)$, and 2) $\mathcal{M}(A_*) \leq \mathcal{M}(C_*) + \mathcal{M}(B_*)$.

Proof. 1) The following lemma reduces the assertion to the case of free $R$-complexes.

Lemma 1.7. Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of $R$-complexes. Then there is a commutative diagram

$$
\begin{array}{cccccc}
0 & \leftarrow & A'_* & \leftarrow & B'_* & \leftarrow & C'_* & \leftarrow & 0 \\
\downarrow & \alpha_* & \downarrow & \beta_* & \downarrow & \gamma_* & \downarrow & \\
0 & \leftarrow & A_* & \leftarrow & B_* & \leftarrow & C_* & \leftarrow & 0
\end{array}
$$

where $\alpha_*, \beta_*, \gamma_*$ are free models.

Proof of the lemma. Let $g'_* : C'_* \to B'_*$ be a covering of $C_* \to B_*$ with respect to some free models $B'_*, C'_*$. We can assume that $g'_*$ is a monomorphism onto a direct summand (the proof repeats almost verbally the proof of Lemma 1.8 from [11] and will be omitted).

Now, setting $A'_* = B'_*/\text{Im} C'$, we obtain the first line of the commutative diagram above. □

For the case of free complexes the assertion follows from the next one.

Lemma 1.8. Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of free $R$-complexes. Then there is an exact sequence $0 \leftarrow A'_* \leftarrow B'_* \leftarrow C'_* \leftarrow 0$ of free $R$-complexes such that $A'_* \sim A_*$, $B'_* \sim B_*$, $C'_* \sim C_*$, and $A'_*, C'_*$ are minimal.
The proof of this lemma is an exercise in the theory of minimal complexes ([12], §4), and will be left to the reader. □

To prove 2) observe that there is an exact sequence $Z = \{0 \leftarrow \Sigma C_\ast \leftarrow D_\ast \leftarrow B_\ast \leftarrow 0\}$ where $D_\ast$ is the mapping cone of $j_\ast$, and $\Sigma C_\ast$ is the suspension of $C_\ast$. Now apply the point 1) to the sequence $Z$. □

In some cases the first inequality of the preceding proposition turns to equality. We shall say that a complex $C_\ast$ is concentrated in dimensions $[k, r]$ if $C_i = 0$ for $i < k$ and for $i > r$. We denote by $F(i, s)_\ast$ the chain complex $\{0 \leftarrow R^s \leftarrow 0\}$ concentrated in dimensions $[i, i]$.  

**Lemma 1.9.** 1) For every $f$-complex $C_\ast$ we have $M(C_\ast \oplus F(i, s)_\ast) = M(C_\ast) + s$. 2) Let $C_\ast, D_\ast$ be $f$-complexes, concentrated respectively in dimensions $[a, b]$, and $[b, c]$. Then $M(C_\ast \oplus D_\ast) = M(C_\ast) + M(D_\ast)$.

The proof of this lemma is easily obtained from V.V.Sharko’s criterion of minimality of chain complexes (see [12], Lemma 3.6). □

2. A numerical invariant of free chain $R((t))$-complexes

We denote $Z[Z^n]$ by $R$ (as in the previous section). Let us start with a free $R[[t]]$-complex $A_\ast$. For $k \in \mathbb{N}$ denote by $A[k]_\ast$ the free $R$-complex $A_\ast/t^k A_\ast$, and denote its Morse number by $\mu_k(A_\ast)$, so $\mu_k(A_\ast) = M_R(A[k]_\ast)$. Note that $\mu_k(A_\ast) + \mu_l(A_\ast) \geq \mu_{k+l}(A_\ast)$.

(Indeed, consider the short exact sequence $0 \leftarrow A[k]_\ast \leftarrow A[k + l]_\ast \leftarrow A[l]_\ast \leftarrow 0$ and apply Proposition 1.6.) Therefore the sequence $\{\mu_k/k\}_{k \in \mathbb{N}}$ has a limit (see [12], ex. 98) which will be denoted by $\sigma(A_\ast)$. It is clear that $\sigma(A_\ast)$ is a chain homotopy invariant of $A_\ast$.

Now we shall consider free complexes over the ring $R((t)) = \sigma^{-1} R[[t]]$ where $\sigma$ is the multiplicative set $\{t^l \mid l \in \mathbb{N}\}$. Let $C_\ast$ be such a complex. We say that a chain subcomplex $D_\ast \subset C_\ast$ is a basic subcomplex if 1) $D_\ast$ is a free $R[[t]]$-complex, and 2) $\sigma^{-1}D_\ast = C_\ast$. It is clear that each free complex $C_\ast$ over $R((t))$ has basic subcomplexes.

**Proposition 2.1.** Let $C_\ast$ be a free $R((t))$-complex. Then the number $\sigma(D_\ast)$ is the same for every basic subcomplex $D_\ast \subset C_\ast$.

**Proof.** Let $D_\ast, F_\ast$ be basic subcomplexes. The Noetherian property of $R[[t]]$ and the condition (2) in the definition of a basic subcomplex imply immediately that there is $s \in \mathbb{N}$ such that $t^s F_\ast \subset D_\ast$. Since $\sigma(D_\ast) = \sigma(t^s D_\ast)$ we can assume that $t^s F_\ast \subset D_\ast \subset F_\ast$. Now for every $l \in \mathbb{N}$ we obtain two exact sequences of finitely generated chain complexes over $R$.

\begin{align*}
(1) & \quad 0 \leftarrow F_\ast/D_\ast \leftarrow F_\ast/t^l D_\ast \leftarrow D_\ast/t^l D_\ast \leftarrow 0 \\
(2) & \quad 0 \leftarrow F_\ast/t^l D_\ast \leftarrow F_\ast/t^{l+s} F_\ast \leftarrow t^l D_\ast/t^{l+s} F_\ast \leftarrow 0
\end{align*}

Applying Prop. 1.6 we deduce from (1) and (2) that $\mu_{l+s}(F_\ast) \leq C + \mu_l(D_\ast)$ where $C$ does not depend on $l$. This implies easily that $\sigma(F_\ast) \leq \sigma(D_\ast)$; by symmetry we obtain $\sigma(F_\ast) = \sigma(D_\ast)$. □

Now we can define an invariant of $R((t))$-complexes. Namely, if $C_\ast$ is a free $R((t))$-complex, we set $s(C_\ast) = \sigma(D_\ast)$ where $D_\ast$ is any basic subcomplex of $C_\ast$. The number $s(C_\ast)$ depends only on the homotopy type of the $R((t))$-complex $C_\ast$. Indeed, a version of the Cockcroft-Swan theorem ([11], Prop. 1.7) shows that it is sufficient to check that $\sigma(C_\ast)$ does not change when we add to $C_\ast$ a complex of the form $\{0 \leftarrow R((t)) \leftarrow \sigma R((t)) \leftarrow 0\}$. But this is obvious.
For some free $R((t))$-complexes the asymptotic properties of the Morse numbers are still better. We shall say that a sequence $a_k$ of real numbers is *asymptotically linear* if $\exists C, \alpha, \forall k : ak - C \leq a_k \leq ak + C$. We shall say that a free $R((t))$-complex $C_*$ is of *asymptotically linear growth* (abbreviation: *aslg*) if for some basic subcomplex $D_* \subset C_*$ the sequence $\mu_k(D_*)$ is asymptotically linear. Similarly to the proof of Proposition 2.1, one can show that in an *aslg*-complex *every* basic subcomplex $D'_*$ has an asymptotically linear sequence $\mu_k(D'_*)$. Note also that the property of being *aslg* is homotopy invariant. We do not know if every $R((t))$-complex is *aslg*, but we shall prove that every complex of a certain class appearing in our geometrical setting is *aslg*, and we shall calculate its $s$-invariant. We need some definitions. A *monomial* of $R$ is an element of the form $ag$ where $a \in \mathbf{Z}$, and $g \in \mathbf{Z}^n$. Let $Z = z_k t^k + \ldots + z_l t^l \in R[t, t^{-1}]$ where $l, k \in \mathbf{Z}, k \leq l$, and $z_k, z_l \neq 0$. We shall say that $Z$ is:

- **monic** if $z_k = \pm g, g \in \mathbf{Z}^n$ (Our terminology differs here from the standard one.)
- **numerically prime** if it is not divisible by an integer not equal to $\pm 1$.
- **special** if each $z_i$ is a monomial in $R$.

We denote $R((t))$ by $\mathcal{L}$.

**Definition 2.2.** Let $C_*$ be a complex over $\mathcal{L}$. We shall say that $C_*$ is of *principal type* if for every $i$ an isomorphism

$$H_i(C_*) \approx (\bigoplus_{j=1}^{b_i} \mathcal{L}) \oplus (\bigoplus_{s=1}^{q_i} \mathcal{L}/a_s^{(i)} \mathcal{L})$$

is fixed, and for every $i, s$: 1) $a_s^{(i)} \in R[t, t^{-1}]$ and $a_s^{(i)}$ is special, non-zero and not monic 2) $a_s^{(i)} \mid a_{s+1}^{(i)}$.

For a complex $C_*$ of principal type we denote by $\nu_i$ the number of those polynomials $a_s^{(i)}$ which are not numerically prime.

**Theorem 2.3.** Let $C_*$ be a free $\mathcal{L}$-complex of principal type. Then $C_*$ is of asymptotically linear growth, and $s(C_*) = \sum_i b_i + 2 \sum_i \nu_i$.

**Proof.** We can assume that all the elements $a_s^{(i)}$ in $\mathcal{L}$ are of the form $z_0 + \ldots + z_k t^k$ where $z_0 \in \mathbf{Z}, z_0 \neq 0$. Denote by $\mathcal{F}(i)_*$ the free $\mathcal{L}$-complex $\{0 \leftarrow \mathcal{L}^{b_i} \leftarrow 0\}$ concentrated in dimensions $[i, i]$. For $\rho \in \mathcal{L}$ and $i \in \mathbf{N}$, denote by $\tau(\rho, i)_*$ the free complex $\{0 \leftarrow \mathcal{L} \leftarrow \mathcal{L} \leftarrow 0\}$ concentrated in dimensions $[i, i + 1]$. Note that if $\rho \in R[[t]]$ then $\tau(\rho, i)_*$ has a standard basic subcomplex $\{0 \leftarrow R[[t]] \leftarrow \mathcal{L} \leftarrow R[[t]] \leftarrow 0\}$ which will be denoted by $\tau'(\rho, i)_*$.

For a given $i$ denote by $\pi$ (resp. by $\nu$) the set of all $s$ such that $a_s^{(i)}$ is numerically prime (resp. not numerically prime). Set

$$\mathcal{T}\mathcal{P}(i)_* = \bigoplus_{s \in \pi} \tau(a_s^{(i)}, i)_* \quad \text{and} \quad \mathcal{T}\mathcal{N}(i)_* = \bigoplus_{s \in \nu} \tau(a_s^{(i)}, i)_* \quad \text{and} \quad \mathcal{T}(i)_* = \mathcal{T}\mathcal{P}(i)_* \oplus \mathcal{T}\mathcal{N}(i)_*.$$
Lemma 2.4. For every $i, k$ we have: (1) $\mu_k(\mathcal{T}N'(i)_*) = 2k_n$, (2) $\mu_k(\mathcal{T}'(i)_*) \geq 2k_n$. (3) For every $i$ the sequence $\{\mu_k(\mathcal{T}P'(i)_*)\}_{k \in \mathbb{N}}$ is bounded.

Proof. 1) Fix some $i$. The condition 2) from the definition implies that there is a prime number $p$ such that every polynomial $a_s^{(i)}$ which is not numerically prime is divisible by $p$. Abbreviate $\mathcal{T}N'(i)_*$ to $L_i$; the inequality $\mu_k(L_i) \leq 2k_n$ is immediate. To prove the inverse inequality consider an $R$-module $H$ not less than $\dim \rho F$. Let $\mu$ be divisible by $p$, and $\rho \geq 1$. Then $\mu_k$ implies our assertion.

Denote by $(4)$ the subdeterminant of the form $\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & 1 \end{pmatrix}$.

Remark 2.6. Let $L = \mathbb{Z}[t, t^{-1}], \hat{L} = \mathbb{Z}((t))$. The homomorphism $\epsilon : \mathbb{Z} \rightarrow \{1\}$ extends to ring homomorphisms $\epsilon : R[t, t^{-1}] \rightarrow L$ and $\hat{\epsilon} : \mathcal{L} \rightarrow \hat{L}$. Therefore for every $\mathcal{L}$-complex $C_* we can form an $\hat{L}$-complex $\hat{\mathcal{C}_*} = C_* \otimes \hat{L}$. Assume that $C_*$ is of principal type. Using the homotopy equivalence $C_* \sim \oplus (\mathcal{F}(i)_* \oplus \mathcal{T}(i)_*)$ from the proof of Theorem 2.3, it is easy to see that $\hat{\mathcal{C}_*}$ is also of principal type and

$$H_i(\hat{\mathcal{C}_*}) \approx \left( \bigoplus_{j=1}^{b_i} \hat{L} \right) \oplus \left( \bigoplus_{s=1}^{q_i} \hat{L}/\alpha_s^{(i)} \hat{L} \right)$$
with $\alpha_s^{(i)} = e(a_s^{(i)})$. Since $a_s^{(i)}$ are special and not monic, $b_i$ and $q_i$ are equal respectively to the rank and to the torsion number of $H_1(\mathcal{C}_s)$ over the principal ring $L$. It is easy to see that the above decomposition satisfies Definition 2.2, therefore $\mathcal{C}_s$ is astlg. Further, $a_s^{(i)}$ is numerically prime if and only if $\alpha_s^{(i)}$ is, and this implies $s(C_s) = s(\mathcal{C}_s)$.

3. A numerical invariant $S(C_\ast, \xi)$

In this section $\Lambda = \mathbb{Z} [\mathbb{Z}^{n+1}]$, $C_\ast$ is a free $\Lambda$-complex, and $\xi : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is a non-zero homomorphism. We define a numerical invariant $S(C_\ast, \xi)$. For the cohomology classes $\xi$ outside a finite union of integer hyperplanes we calculate $S(C_\ast, \xi)$ in terms of the reduced Fitting sequences of the boundary operators of $C_\ast$ (the mentioned finite union of integer hyperplanes depends on $C_\ast$). An element $z \in \Lambda$ is called $\xi$-monic if $z = \pm g + z_0$ where $g \in \mathbb{Z}^{n+1}$ and $\text{supp } z_0 \subset \{ h \in \mathbb{Z}^{n+1} | \xi(h) < \xi(g) \}$. An element $z$ is called $\xi$-special if any two different elements $a, b \in \text{supp } z$ satisfy $\xi(a) \neq \xi(b)$. We denote by $S_\xi$ the multiplicative subset of all $\xi$-monic polynomials, and we denote by $\Lambda(\xi)$ the localization $S_\xi^{-1}\Lambda$.

**Definition 3.1.** A subset $X \subset \mathbb{Z}^k$ will be called small if it is a finite union of integer hyperplanes.

**Theorem 3.2 (\cite{4}, Th. 0.1).** There is a small subset $\mathcal{R} \subset \text{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z})$ such that for every $\xi \notin \mathcal{R}$ and every $p$ we have:

$$S_\xi^{-1} H_p(C_\ast) \approx \left( \bigoplus_{i=1}^{b_p(C_\ast, \xi)} \Lambda(\xi) \right) \oplus \left( \bigoplus_{j=1}^{a_p(C_\ast, \xi)} \Lambda(\xi)/a_j^{(p)} \Lambda(\xi) \right)$$

where $a_j^{(p)} \in \Lambda$ are non-zero and not $\xi$-monic elements of $\Lambda$ (depending on $\xi$), and $a_j^{(p)} \mid a_j^{(p+1)}$. \hfill \square

**Sketch of the proof of Theorem 3.2.** We shall recall here the basic idea of the proof of 3.2 following \cite{8} and \cite{4}, see \cite{4} for the full proof. Let $\xi : \mathbb{Z}^{n+1} \to \mathbb{R}$ be a non-zero homomorphism. Similarly to the above, we define the notion of $\xi$-monic polynomial, and we introduce the ring $\Lambda(\xi) = S_\xi^{-1}\Lambda$. (We take here the occasion to note that for the first time the localization technique was applied to Novikov rings and Novikov inequalities in the paper \cite{4} of M.Farber. In this paper M.Farber considers the ring $S_\xi^{-1}\Lambda$, where $\Lambda = \mathbb{Z} [\mathbb{Z}]$, and $\xi$ is the inclusion of $\mathbb{Z}$ to $\mathbb{R}$.) Recall next the definition of the Novikov ring $\Lambda_\xi$ (see, e.g. \cite{11}, p. 326). Denote by $\hat{\Lambda}$ the abelian group of all the linear combinations of the form $\lambda = \sum_{g \in \mathbb{Z}^{n+1}} n_g g$ where $n_g \in \mathbb{Z}$ and the sum may be infinite. Let $\Lambda_\xi^-$ be the subset of $\hat{\Lambda}$ consisting of $\lambda \in \hat{\Lambda}$ such that for every $c \in \mathbb{R}$ the set supp $\lambda \cap \xi^{-1}(\{c, \infty\})$ is finite. This subset is called Novikov ring (it is not difficult to see that $\Lambda_\xi^-$ has a natural ring structure).

Proceeding to Theorem 3.2, recall that Theorem 1.4 of \cite{8} asserts that if $\xi$ is injective then $\Lambda(\xi)$ is euclidean. (The proof is based on a theorem by J.Cl.Sikorav, which asserts that if $\xi$ is injective then $\Lambda_\xi^-$ is euclidean, see \cite{8}, Th. 1.1.)

Therefore we obtain the decomposition (\cite{4}) for any monomorphism $\xi$. This implies that (\cite{4}) is true for every homomorphism $\eta$ belonging to an open conical set containing $\xi$ (see \cite{4}, the beginning of §7). Since the monomorphisms are dense in $\text{Hom}(\mathbb{Z}^{n+1}, \mathbb{R})$, we obtain
the decomposition \( [\xi] \) for every \( \xi \) belonging to some open and dense conical subset \( U \) in \( \text{Hom}(\mathbb{Z}^{n+1}, \mathbb{R}) \).

Analyzing further the algebraic structure of the rings \( \Lambda_\xi, \Lambda(\xi) \) (it is done in \( [3] \)), one can prove that \( U \) can be chosen in such a way that the complement \( \text{Hom}(\mathbb{Z}^{n+1}, \mathbb{R}) \setminus U \) is a finite union \( \mathfrak{M} = \bigcup_i L_i \) of hyperplanes \( L_i \). Moreover, each \( L_i \) is of the form \( l_i \otimes \mathbb{R} \) where \( l_i \in \text{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z}) \) is an integer hyperplane. That proves Theorem 3.2. (See \( [3] \) for more information about the numbers \( b_p(C_*, \xi), q_p(C_*, \xi) \).)

The following proposition relates the above numbers and the elements \( a_j^{(p)} \) to the Fitting invariants of the boundary operators of \( C_* \). We need some definitions. Let \( A : F_1 \to F_2 \) be a homomorphism of free finitely generated \( \Lambda \)-modules. Let \( J_0 \subset \ldots \subset J_r \) be the reduced Fitting sequence of \( A \), let \( \rho_i \in \Lambda \) be the g.c.d. of the elements of \( J_i \), and denote \( \rho_i/\rho_{i+1} \) by \( \zeta_i(A) \). Let \( \xi : \mathbb{Z}^{n+1} \to \mathbb{Z} \) be a non-zero homomorphism. Denote by \( k(A, \xi) \) the number of those \( \rho_i \) which are not \( \xi \)-monic. Set \( R_j(A, \xi) = \zeta_{k-j}(A, \xi) \) where \( k = k(A, \xi) \). Now let \( C_* = \{ 0 \leftarrow \ldots \leftarrow C_{i-1} \leftarrow C_i \leftarrow \ldots \} \) be a free \( \Lambda \)-complex.

**Proposition 3.3.** Assume that for \( \xi \in \text{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z}) \) and every \( p \) the decomposition \( [\xi] \) holds. Then: 1) \( b_p(C_*, \xi) = B_p(C_*) \). 2) \( q_p(C_*, \xi) = k(\partial_{p+1}, \xi) \). 3) For every \( p, s \) the elements \( a_s^{(p)} \) and \( R_s(\partial_{p+1}, \xi) \) are equal up to multiplication by a \( \xi \)-monic element. 4) \( Q_p(C_*) \) equals to the number of not numerically prime \( a_s^{(p)} \).

**Proof.** Recall that the reduced Fitting sequences are homotopy invariants of \( C_* \). This implies that \( k(\partial, \xi) \) and \( \pi(\partial, \xi) \) are homotopy invariants of \( C_* \) for fixed \( \xi \). 1) is obvious. Further, let \( 0 \leq p \leq n \) and let \( J_0 \subset \ldots \subset J_r \) be the reduced Fitting sequence for \( \partial_{p+1} : C_{p+1} \to C_p \). Then the reduced Fitting sequence \( FR(p) \) of the localized complex is a part of the sequence \( S^{-1}_\xi J_0 \subset \ldots \subset S^{-1}_\xi J_r \), and the g.c.d. of \( S^{-1}_\xi J_i \) is still \( \rho_i \). Therefore the sequence \( FR(p) \) has \( k(\partial_{p+1}, \xi) \) terms. Using the principal model for \( C_* \), it is easy to prove that \( FR(p) \) equals to the sequence of principal ideals \( (a_1^{(p)}, \ldots, a_N^{(p)}), (a_1^{(p)} \ldots a_1^{(p)}) \), \ldots, \lowercase{(a_1^{(p)})} \) where \( N = q_p(C_*, \xi) \). 2), 3) and 4) follow easily. \( \square \)

Let \( \xi = \overline{\xi} \cdot \xi \) where \( \overline{\xi} : \mathbb{Z}^{n+1} \to \mathbb{Z} \) is an epimorphism. Choose an isomorphism \( \text{Ker} \xi \approx \mathbb{Z}^n \), and an element \( t \in \mathbb{Z}^{n+1} \) such that \( \overline{\xi}(t) = -1 \). We obtain a decomposition \( \mathbb{Z}^{n+1} = \overline{\xi} \mathbb{Z} \oplus \mathbb{Z} \) and an isomorphism \( I(\xi) : \Lambda \approx R[t, t^{-1}] \). Consider the free \( \mathcal{L} \)-complex \( \tilde{C}_*(\xi) = C_* \otimes \Lambda \), where \( C_* \) is an \( R[t, t^{-1}] \) module via the isomorphism \( I(\xi)^{-1} \). Set \( S(C_*, \xi) = s(\tilde{C}_*(\xi)) \) (it is easy to check that \( S(C_*, \xi) \) depends indeed only on \( \xi \) and \( C_* \)).

**Theorem 3.4.** There is a small subset \( \mathfrak{M} \subset \text{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z}) \) such that for every \( \xi \notin \mathfrak{M} \) the complex \( \tilde{C}_*(\xi) \) is of asymptotically linear growth and \( S(C_*, \xi) = B(C_*) + 2Q(C_*) \).

**Proof.** The \( I(\xi) \)-image of a \( \xi \)-monic polynomial is obviously invertible in \( \mathcal{L} \), therefore the homomorphism \( \Lambda \to R[t, t^{-1}] \to R((t)) \) factors through \( \Lambda(\xi) \). Therefore, for every \( \xi \) outside a small subset \( \mathfrak{M} \) the formula (3) holds with \( C_* = \tilde{C}_*(\xi) \). The complex \( \tilde{C}_*(\xi) \) is not necessarily of principal type, since the polynomials \( a_s^{(i)} \) in the decomposition (3) are not necessarily special. But Proposition 3.3 implies that the elements \( a_s^{(i)} \) in the decomposition (3) can be chosen between the elements of the finite set \( \{ \zeta_j(\partial_{i+1}) \} \). Therefore, adding to \( \mathfrak{M} \) some integer hyperplanes if necessary, we can assume that all \( a_s^{(i)} \) are special. Now our theorem follows from 2.3. \( \square \)
4. Proof of the main theorem

Let $M$ be a closed connected manifold and $\xi \in H^1(M, \mathbb{Z})$ be an indivisible cohomology class. Denote by $\mathcal{P}_\xi : M(\xi) \to M$ the infinite cyclic covering such that $\mathcal{P}_\xi^* (\xi) = 0$. Choose a generator $t \in \mathbb{Z} \cong \pi_1(M)/\ker \xi$ of the structure group of $\mathcal{P}_\xi$ such that $\xi(t) = -1$. Let $f : M \to S^1$ be a Morse map representing $\xi$, and let $V$ be its regular level surface, say, $V = f^{-1}(\lambda)$. Then $f$ lifts to a Morse function $F : M(\xi) \to \mathbb{R}$ and $V$ lifts to $F^{-1}(\lambda') \subset M(\xi)$. Denote by $V^c$ the subset $F^{-1}([-\infty, \lambda'])$. For $k \geq 1$ denote by $W_k$ the cobordism $F^{-1}([\lambda' - k, \lambda'])$, $\partial W_k \approx V \cup t^k V$, and denote by $\alpha(k, V)$ its Morse number, that is, the minimal number of critical points of a Morse function on the cobordism $W_k$. Note that $\alpha(k + n, V) \leq \alpha(k, V) + \alpha(n, V)$. Therefore the sequence $\alpha(k, V)/k$ has a limit as $k \to \infty$. Denote this limit by $\alpha(V)$. It is easy to see that $\alpha(V)$ depends only on $M$ and $\xi$, so we denote it by $\alpha(M, \xi)$. An elementary construction, using the gluing of the upper part of $\partial W_k$ to the lower part $t^k V$, allows to obtain the inequality $\mathcal{M}(M, \xi, k) \leq \alpha(k, V) + 2\mathcal{M}(V)$. In particular, if $\xi$ is represented by a fibration over $S^1$, the sequence $\mathcal{M}(M(\xi), k)$ is bounded.

In general, it is all what we can say about the numbers $\alpha(k, V)$ and their relation to the asymptotics of the Morse numbers of cyclic covers. However, if the fundamental group of $M$ is free abelian and $\dim M \geq 6$, one can say much more.

**Proposition 4.1.** Let $M$ be a closed connected manifold with a free abelian fundamental group. Assume that $\dim M \geq 6$. Let $\xi \in H^1(M), \xi \neq 0$. Then the sequence $\alpha(k, V) - \mathcal{M}(M(\xi), k)$ is bounded.

**Proof.** Let $\pi_1(M) \approx \mathbb{Z}^{n+1}, n \geq 0$. An argument similar to the one used in (3, p.325) shows that one can choose $V$ above such that the embedding $V \hookrightarrow M$ induces an isomorphism $\pi_1(V) \to \ker \xi$ (such $V$ will be called admissible $\xi$-splittings, see [3], p. 371). In this case all the embeddings $V \subset W_k \subset M(\xi) \supset t^k V$ induce isomorphisms of $\pi_0$ and of $\pi_1$. Choose an element $T \in \mathbb{Z}^{n+1}$, such that $\xi(T) = -1$. Let $\tilde{M}(\xi, k) = \tilde{M}/T^k$, then there is a $\mathbb{Z}^n$-covering $\tilde{M}(\xi, k) \to M(\xi, k)$. Choose a triangulation of $M$ such that $V$ is a subcomplex of $\tilde{M}$; then we obtain a $t$-invariant triangulation of $\tilde{M}(\xi, k)$ and the corresponding triangulations of all the covers. There are two exact sequences of corresponding $\mathbb{Z}[\mathbb{Z}^n]$-complexes:

\begin{align*}
0 & \to C_* (\tilde{V}) \to C_* (\tilde{M}(\xi, k)) \to C_* (\tilde{M}(\xi, k), \tilde{V}) \to 0 \\
0 & \to C_* (\tilde{V}) \to C_* (\tilde{W}_k, t^k V) \to C_* (\tilde{M}(\xi, k), \tilde{V}) \to 0
\end{align*}

Proposition [4,4] implies that there is $C = C(V)$ such that for every $k > 0$ we have: $\mathcal{M}(C_* (\tilde{M}(\xi, k))) \geq \mathcal{M}(C_* (\tilde{W}_k, t^k V)) - C$. Since $\mathcal{M}(C_* (\tilde{W}_k, t^k V)) = \alpha(k, V)$ (see the proof of Corollary 6.3 in [13]), our Proposition is proved.

**Proof of the Main theorem.** The point 1) follows immediately from [14] (with $\alpha(M, \xi) = \lim_{k \to \infty} \mathcal{M}(M(\xi), k)/k$). To prove 2) note that Theorem 3.4 implies that for all $\xi \in H^1(M)$ outside a small subset $\mathcal{W} \subset H^1(M)$ the complex $(C_*(\tilde{M})) \wedge (\xi)$ is aslg, and $S(C_*(\tilde{M}), \xi) = B(M) + 2Q(M)$. Note further that for every admissible $\xi$-splitting $V$ the complex $D_* = C_*(\tilde{V}/\partial) \otimes R[[t]]$ is a basic subcomplex of $C_* (\tilde{M}) \otimes R((t))$, and that $\mu_k(D_*) = \mathcal{M}(C_* (\tilde{W}_k, t^k V)) = \alpha(k, V)$. Now just apply Proposition 4.1.

**Remark 4.2.** A similar argument, together with Remark 2.6, shows that for $\xi$ outside a small subset of $H^1(M)$ the sequence $(B(M) + 2Q(M))k - \mathcal{M}_Z(C_*(M(\xi)))$ is
bounded where $C_*(M(\xi, k))$ is the chain complex of $M(\xi, k)$, defined over $\mathbb{Z}$ (see 1.2 for the definition of $\mathcal{M}_*(\cdot)$).

5. Further results and conjectures

5.1. Stable Morse numbers. Let $M$ be a closed connected manifold. Recall that a stable Morse function on $M$ is a Morse function $f : M \times \mathbb{R}^N \to \mathbb{R}$ such that there is a compact $K \subset M \times \mathbb{R}^N$, and a non-degenerate quadratic form $Q$ of index 0 on $\mathbb{R}^N$ such that $f(x, y) = Q(y)$ outside $K$. Let $f : M \times \mathbb{R}^N \to \mathbb{R}$ be a stable Morse function. Denote by $\tilde{\mathcal{M}}_p(f)$ the number of critical points of $f$ of index $p + N/2$. The Morse-Pitcher inequalities hold: $\tilde{\mathcal{M}}_p(f) \geq b_p(M) + q_p(M) + q_{p-1}(M)$.

Denote by $\mathcal{M}_S(M)$ the minimal possible number of critical points of a stable Morse function on $M$; we have $\mathcal{M}_S(M) \leq \mathcal{M}(M)$.

**Theorem 5.1.** Let $\dim M \geq 6, \pi_1(M) \approx \mathbb{Z}^{n+1}, n \geq 0$. There is a subset $\mathcal{M} \subset H^1(M)$ which is a finite union of integral hyperplanes in $H^1(M)$, and for every $\xi \notin \mathcal{M}$ there is a real number $a$ such that for every $k \in \mathbb{N}$ we have

$$k(B(M) + 2Q(M)) - a \leq \mathcal{M}_S(M(\xi, k)) \leq k(B(M) + 2Q(M)) + a$$

For the proof just recall that (by Remark 4.2) for every $\xi$ outside a small subset of $H^1(M)$ we have $k(B(M) + 2Q(M)) \leq \mathcal{M}_S(C_*(M(\xi, k))) + C$. □

We refer to [4] for a systematic exposition of the theory of stable Morse functions and its applications to Lagrangian intersection theory.

5.2. Non generic cohomology classes $\xi \in H^1(M)$. Here we construct a manifold $M$ with $\pi_1(M) \approx \mathbb{Z}^2$ and $\dim M \geq 6$, and a class $\xi \in H^1(M)$ such that $\mu(M, \xi) \neq B(M) + 2Q(M)$. Let $N$ be a closed connected manifold with $\pi_1(N) \approx \mathbb{Z}, \dim N \geq 5$ and $B(N) \neq 0$. Set $M = N \times S^1$. Let $\lambda : \pi_1(M) \to \mathbb{Z}$, resp. $\xi : \pi_1(M) \to \mathbb{Z}$, be epimorphisms with $\ker \lambda = \pi_1(N)$, resp. $\ker \xi = \pi_1(S^1)$. Then $\lambda$ is represented by the fibration $N \times S^1 \to S^1$. Therefore there is an open cone $C \subset H^1(M, \mathbb{R})$ containing $\lambda$, such that every integral non-divisible $\lambda' \in C$ can be represented by a fibration, and so $\mu(M, \lambda') = B(M) + 2Q(M) = 0$.

Now we shall show that $\mu(M, \xi) \neq 0$. Note that $M(\xi, k) = N_k \times S^1$ where $N_k$ is the $k$-fold cyclic cover of $N$, and therefore $\mathcal{M}(M(\xi, k)) \geq \mathcal{M}_S(C_*(N_k \times S^1))$. We shall obtain a lower estimate for $\mathcal{M}_S(C_*(N_k \times S^1))$. Let $\xi_0 : \pi_1 N \to \mathbb{Z}$ be the restriction $\xi \mid \pi_1 N$. Let $\overrightarrow{\pi} \to N$ be the infinite cyclic covering and $V \subset N$ be an admissible $\xi_0$-splitting. Let $W_k$ be the corresponding cobordism in $\overrightarrow{\pi}$. Using exact sequences similar to the exact sequences (2) from 4, it is easy to prove that $\mathcal{M}_S(C_*(N_k \times S^1)) = \mathcal{M}_S(C_*(W_k \times S^1, t^kV \times S^1))$ is bounded. Let $X_k = W_k / t^kV, Y_k = (W_k / t^kV) \times S^1$. Then $\mathcal{M}_S(C_*(Y_k)) = \sum p(b_p(Y_k) + q_p(Y_k)+q_{p-1}(Y_k)).$ Since $H_*(X_k \times S^1) = H_*(X_k) \oplus H_*(S^1)$ we have $q_p(X_k \times S^1) \geq q_p(X_k)$, and $\mathcal{M}_S(C_*(Y_k)) \geq \mathcal{M}_S(C_*(X_k))$. Recall from Proposition 4.1 that the sequence $\mathcal{M}(N_k) - \mathcal{M}_S(C_*(X_k))$ is bounded. Therefore $\mathcal{M}_S(C_*(N_k \times S^1)) \geq \mathcal{M}(N_k) + C \geq kB(N) + C'$ (where $C$ and $C'$ do not depend on $k$), and, finally, $\mu(M, \xi) \geq B(N)$.

5.3. Non cyclic finite coverings. The Main Theorem of the present paper allows also to deal with some non cyclic finite coverings.

**Proposition 5.2.** Let $M$ be a closed connected manifold, $\dim M \geq 6$, and $\pi_1(M) \approx \mathbb{Z}^{n+1}, n \geq 0$. Let $M_k \to M$ be the finite covering corresponding to the subgroup $k\mathbb{Z}^{n+1}$.
Then
\[ \lim_{k \to \infty} \frac{\mathcal{M}(M_k)}{k^{n+1}} = B(M) + 2Q(M) \]

Proof. We shall give only the main idea of the proof. Let \( \xi : \pi_1(M) \to \mathbb{Z} \) be an epimorphism not belonging to the small set \( \mathfrak{M} \) of the Main Theorem. Then \( M_k \to M \) factors through \( M(\xi,k) \to M \), and therefore \( \mathcal{M}(M_k)/k^n \leq k(B(M) + 2Q(M)) + C \). To obtain the lower estimate, note that the \( \mathbb{Z}^n \)-covering \( \mathcal{M}(\xi,k) \to M(\xi,k) \) factors through \( \mathcal{M}(\xi,k) \to M_k \) which corresponds to the subgroup \( G_k = k\mathbb{Z}^n \subset \mathbb{Z}^n \). Therefore \( \mathcal{M}(M_k) \geq \mathcal{M}_{\mathbb{Z}[G_k]}(C_\ast(M(\xi,k))) \). To obtain the lower estimate for \( \mathcal{M}_{\mathbb{Z}[G_k]}(C_\ast(M(\xi,k))) \), use (7) and (8) to reduce the question to finding the corresponding lower estimate for \( \mathcal{M}_{\mathbb{Z}[G_k]}(C_\ast(\tilde{W}_k, t^kV)) \). Then proceed similarly to the proof of the Main Theorem. \( \square \)

It seems that a similar result must hold for more general systems of non-cyclic finite coverings. To discuss a more general setting we need some definitions. Let \( G \) be a group. A sequence of subgroups \( G = G_0 \supset G_1 \supset \ldots \) will be called a tower if for every \( i \) the index of \( G_i \) in \( G \) is finite. It will be called a nested tower if, moreover, \( \cap_n G_n = \{0\} \).

If \( M \) is a closed connected manifold with \( \pi_1(M) = G \) and \( \mathcal{G} = \{G_n\} \) is a tower of subgroups of \( \pi_1(M) \), then there is the corresponding tower of finite coverings \( M = M_0 \leftarrow \ldots \leftarrow M_n \ldots \) of \( M \). The sequence \( \mathcal{M}(M_k)/G/G_k \) is decreasing, therefore it has a limit which will be denoted by \( \mu(\mathcal{G}) \). Recall a theorem of W. L"uck \([3]\), saying that if \( \mathcal{G} \) is nested, then the limit of the sequence \( b_p(M_k)/G/G_k \) exists and is equal to \( b_p(\mu) \).

Problem. Is it true in general (at least for dim \( M \geq 6 \)) that \( \mu(\mathcal{G}) \) does not depend on the choice of the nested tower \( \mathcal{G} \)?

We believe that \( \mu(\mathcal{G}) \) does not depend on \( \mathcal{G} \) for the case of free abelian fundamental group. Here is a result in this direction. Let \( G = \mathbb{Z}^{n+1} \). For a tower \( \mathcal{G} = \{G_n\} \) denote \( \max_i m(G_i/G) \) by \( r(\mathcal{G}) \) (here \( m(H) \) stands for the minimal number of generators of \( H \)). Denote by \( \mathcal{G}^{[i]} \) the tower \( G_i \supset G_{i+1} \supset \ldots \). The sequence \( r(\mathcal{G}^{[i]}) \) is decreasing, denote its limit by \( \rho(\mathcal{G}) \); then \( \rho(\mathcal{G}) \leq \text{rk} \ G \).

**Proposition 5.3.** Let \( M \) be a closed connected manifold, \( \dim M \geq 6 \), \( \pi_1(M) \approx \mathbb{Z}^{n+1}, n \geq 0 \). Let \( \mathcal{G} = \{G_k\} \) be a tower, and let \( M_k \to M \) be the finite covering corresponding to \( G_k \subset \pi_1(M) \). Assume that \( \rho(\mathcal{G}) = n + 1 \). Then
\[ \lim_{k \to \infty} \frac{\mathcal{M}(M_k)}{G/G_k} = B(M) + 2Q(M) \]

Proof. It is not difficult to show that if \( \rho(\mathcal{G}) = n + 1 \), then for each \( k \) there are \( a_k, b_k \in \mathbb{N} \) such that \( a_k G \supset G_k \supset b_k G \) with \( a_k, b_k \to \infty \) as \( k \to \infty \). Now our Proposition follows from Proposition 5.2. \( \square \)

Conjecture. Equality (9) is true for every nested tower in \( \mathbb{Z}^{n+1} \).

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