Random sequential adsorption on a dashed line

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Abstract

We study analytically and numerically a model of random sequential adsorption (RSA) of segments on a line, subject to some constraints suggested by two kinds of physical situations: (i) deposition of dimers on a lattice where the sites have a spatial extension; (ii) deposition of extended particles which must overlap one (or several) adsorbing sites on the substrate. Both systems involve discrete and continuous degrees of freedom, and, in one dimension, are equivalent to our model, which depends on one length parameter. When this parameter is varied, the model interpolates between a variety of known situations: monomers on a lattice, “car-parking” problem, dimers on a lattice. An analysis of the long-time behaviour of the coverage as a function of the parameter exhibits an anomalous $1/t^2$ approach to the jamming limit at the transition point between the fast exponential kinetics, characteristic of the lattice model, and the $1/t$ law of the continuous one.

LPTB 94-12
October 1994
PACS 05.70Ln, 68.10.Jy
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1 Introduction

The model of random sequential adsorption describes deposition processes in which desorption is negligible and surface diffusion very slow on the experimental time scale. Particles land successively and randomly on the surface; due to the geometrical exclusion effect, if an incoming particle overlaps a previously deposited one, it is rejected. This model applies to many physical situations such as adsorption of latex balls, proteins or chemisorption at low temperature\[1\]. The substrate is either a lattice or a continuous surface, depending on the size of the particles relative to the microscopic scale. Many versions of the model have been studied to adapt it to various physical situations, but all share a common universal behaviour for the long time approach to the asymptotic coverage, depending on the discrete or continuous nature of the substrate. As long as the minimum interval between two neighbouring particles on the substrate remains non-zero, which is the case on a lattice, one can show that the jamming limit is approached exponentially\[1,2\]. Conversely, on a continuum substrate, the kinetics follows a power-law decay whose exponent depends on the number of degrees of freedom per particle\[3-11\]. For instance, in the one dimensional case, where the model is exactly soluble, we have

\[ \theta_k(t) = \theta_k(\infty) - A(k)e^{-\phi t} \]

for the deposition of \(k\)-mers on a lattice with flux \(\phi\), whereas

\[ \theta(t) = \theta(\infty) - A/t \]

for the continuous “car-parking” problem\[12\].

In fact, at the mesoscopic scale, many physical situations involve both continuous and discrete degrees of freedom. For example, with the recent advances in nano-technologies, it is conceivable to realize a lattice of small gold droplets deposited on a silicon surface\[13\]. The adsorption on this substrate of elongated particles (polymers) whose length is of the order of the distance between two gold dots can be modelized by the adsorption of dimers\[14\]. However, since the size of the gold dots (typically some nanometers of radius) is not negligible with respect to the length of the particles, there will be a continuous range of positions for the dimer to be fixed on two neighbouring dots.

Alternatively, large particles like proteins or enzymes can be absorbed on a latticized substrate. Such a physical situation is described in ref.\[15\] and modelized in the following way: adsorbing sites are randomly or regularly disposed on a continuum substrate. Extended particles, represented by disks, land on the
surface but remain stuck only if they overlap one (or several) adsorbing site(s). Here again, the deposition process is driven by the discrete degree of freedom imposed by the location of the sites, and by a continuous one associated with the position of the adsorbed particle with respect to the site.

In this paper, we investigate a one-dimensional RSA model which involves both discrete and continuous degrees of freedom. We show that the two above-mentioned physical situations, when reduced to a one-dimensional substrate, are equivalent to our model. In section 2, we present the model and analyse its jamming limit using the result of a numerical simulation. Section 3 is devoted to the analysis of the long time behaviour of the coverage as derived from the master equations of the model. Our results and the conclusions are summarized in the section 4. The Appendices contain the technical details of the derivation of the master equations (Appendix A) and of their solution (Appendix B).

2 The model

2.1 Two equivalent one-dimensional models

Consider the first example depicted in the introduction. It can be considered as a generalisation of the lattice dimer model, in which the lattice sites have a non zero extension which we set to unity. In the deposition process, the dimer ends must stick on this discrete set of continuous intervals, the situation where a given interval contains both the end of one dimer and the origin of another being now allowed (see figure 1(a)). Moreover, the dimer length is no longer constrained to be equal to the lattice spacing, but is arbitrary within bounds compatible with these adsorbing rule, thereby introducing a new length scale. We denote by $a$ the edge-to-edge distance between two neighbouring sites. The dependence of the model on this parameter occurs only in a trivial redefinition of the effective flux of incoming particles. Therefore, the model is independent of $a$, provided we impose the condition that a deposited particle overlaps an inter-site interval. The length $\ell$ of the dimer lies in the interval $[a, a + 2]$. We set $\ell = a + r$, where $r \in [0, 2]$ will be the unique parameter of the model. In the following, we will refer to this model as model (I).

Consider now the second example of the introduction where the adsorbing sites are regularly disposed and the radius of the particles is smaller than the inter-site distance. In one dimension, the substrate is a regular lattice, the particle a segment of length $r < 2$ lattice units, and the adsorbing rule, aside from the RSA ones, is that the segment must cover a site (see figure 1(b)). We
call this model, model (II).

If we constrain the deposited particle to overlap only one adsorbing site, it is obvious that model (II) is equivalent to the previous one taken in the limit $a = 0$, where the inter-site interval, reduced to a point, is identified with the adsorbing site of model (II). In the following, we will consider the one dimensional model from this point of view and from now on everything will refer to model (II).

### 2.2 The jamming limit

The constraint that a deposited particle must overlap one and only one adsorbing site just modifies the incoming flux $\phi$ by a multiplicative factor of $r$ for $r < 1$ and $2 - r$ for $r > 1$. We define the occupancy rate $\theta$ as the fraction of occupied sites. The covering of the whole substrate is $r\theta$.

The jamming limit for this quantity, obtained from a numerical simulation, is presented in figure 2, as a function of $r$. Let us first analyse this curve qualitatively. Clearly, for $r \leq 1/2$, the adsorption of a particle on the site $i$, regardless of its position on the site, cannot prevent the deposition on neighbouring sites. Therefore, each site will be occupied independently of its neighbours, and the model is completely equivalent to the deposition of monomers on a lattice; we expect the asymptotic limit $\theta = 1$ to be reached exponentially fast.

Consider now for the site $i$ the most defavourable situation in the case $r < 1$, (fig. 3(a)), where the site $i - 1$ is occupied by the extreme left edge of a particle and the site $i + 1$ by the extreme right edge of another one. In this situation the interval for a deposition at site $i$ is minimum and has extension $2(1 - r)$. The deposition is allowed if $2(1 - r) > r \iff r < 2/3$. Therefore, for $r < 2/3$, we still expect a coverage of all the sites, which is effectively observed in figure 2 where $\theta(t = \infty) = 1$ up to $r = 2/3$. Furthermore, since in this $r$-interval the smallest target is of non-zero extension, the kinetics must remain lattice-like and the jamming limit is again approached exponentially fast.

Conversely consider now the most favourable situation in the case $r > 1$ (fig. 3(b)), where the particle deposited on the site $i - 1$ has its extreme left edge close to site $i - 2$. The remaining space to adsorb a particle on site $i$, has extension $3 - r$. If this interval is less than $r$ ($\Rightarrow r > 3/2$), deposition is impossible, and the particle at site $i - 1$ effectively occupies two sites. In this situation, for $r > 3/2$, the model is completely equivalent to the lattice dimer model \[16\] and one expects $\theta(t = \infty) = \frac{1 - e^{-2}}{2} = 0.43233...$ (see figure 2).

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\[1\]This result is more clearly seen in the framework of model (I)
Between these two extreme cases, for $2/3 \leq r \leq 3/2$, the asymptotic occupancy rate $\theta(t = \infty)$ decreases continuously. Furthermore, since the target intervals for a particle deposition can be arbitrarily small, one expects a power law dynamics, characteristic of a *continuous* model. The case $r = 1$ is special: the particle length matches exactly the inter-site distance and the constraint for a landing particle to overlap one and only one adsorbing site is automatically satisfied; the discrete nature of the substrate no longer plays any role and we recover the “car-parking” problem\[12\] with jamming limit $\theta = 0.747597$.

3 The long time behaviour of the coverage

3.1 The master equations

We have derived the master equations for the time evolution of the probability of finding at time $t$ a gap of given length. All the technical details are given in appendix A. The result is that the coverage $\theta(t)$ can be expressed in terms of a reduced probability $P(x,t)$:

$$\theta(t) = \int_0^t dt' e^{rt'} \int_0^r dx P(x,t') P(r-x,t') \quad \text{for } r \in [0, 1]$$

$$= \int_0^t dt' e^{(2-r)t'} \int_{r-1}^1 dx P(x,t') P(r-x,t') \quad \text{for } r \in [1, 2]$$

where $P(x,t)$ is solution of the integral equation

$$-\frac{\partial P(x,t)}{\partial t} = xP(x,t) + \int_x^r P(x',t)dx' + u(x+r-1) e^{-rt} \int_0^{x+r-1} P(x',t)d\xi$$

subject to the initial condition $P(x,0) = 1$. In eq.\[3\], $u(t)$ is the Heaviside step function and in the last integral of eq.\[4\], $b(x) = \text{Inf} \{x + r - 1, 1\}$.

These equations can be exactly solved in a few special cases, corresponding to $r \leq 1/2$, $r = 1$ and $r \geq 3/2$, which are presented in the next section. For the generic case $1/2 < r < 3/2$ ($r \neq 1$), we have devised an iterative construction.
of the general solution, which is developed in appendix B and which leads to the following result:

Due to the last term in eqs (3,4), \( P(x,t) \) appears to be piecewise defined with respect to \( x \), in successive intervals of length \((1-r)\) within the range \((0, r)\) for \( r \leq 1 \), or of length \((r-1)\) within the range \((r-1, 2-r)\) for \( r \geq 1 \). The number of intervals thus depends on the value of \( r \) and is given by (see appendix A):

\[
k = \left[ \frac{r}{1-r} \right] + 1 \quad \text{for} \quad 1/2 < r < 1 \quad \quad k = \left[ \frac{2-r}{r-1} \right] + 1 \quad \text{for} \quad 1 < r < 3/2 \quad (5)
\]

where \([X]\) denotes the integer part of \( X \).

The function \( P_\ell(x,t) \) which coincides with \( P(x,t) \) in the \( \ell \)th interval is expressed in terms of a unique \( x \)-independent function \( q(t) \)

\[
P_\ell(x,t) = q(t) + \int_0^t K_\ell(x,t|t') q(t') \, dt' \quad 2 \leq \ell \leq k \quad (6)
\]

where the kernels \( K_\ell(x,t|t') \) are constructed recursively. The function \( q(t) \) is itself solution of a linear integral equation

\[
q(t) = u_0(t) \left\{ 1 + \int_0^t \rho_k(t,t') q(t') \, dt' \right\} \quad (7)
\]

where \( u_0(t) \) is defined by

\[
1/2 < r < 1 \quad \quad u_0(t) = e^{-rt}
\]

\[
1 < r < 3/2 \quad \quad u_0(t) = \exp \left[ e^{-(2-r)t} - (2-r)t - 1 \right]
\]

and the kernels \( \rho_k \) are expressed in terms of \( K_\ell \).

Even in the simplest case, \( k = 2 \), we are unable to find analytic solutions of equation (7). However, from these expressions we can extract both approximate solutions and the exact asymptotic behaviour for the dynamics of the model, which allow us to understand the various regimes which interpolate between the special cases depicted in section 3.2.

This is the object of the following sections.

### 3.2 Special cases

There are 3 special cases where the rate equations (3,4) reduce to a simple form which is exactly solvable.

- \( 0 \leq r \leq 1/2 \). The number \( k \) of intervals is equal to one (see eq.(5)). Equation (4) yields directly

\[
P(x,t) = e^{-rt}
\]

\[
\theta(t) = 1 - e^{-rt}
\]
which is, as expected, the exact result for the deposition of monomers on a lattice, up to a factor $r$ in the flux corresponding to the target area for each deposition.

- $r = 1$. The two rate equations become identical and reduce to the known car-parking equation under the ansatz $P(x, t) = e^{-xt} q(t)$.

- $3/2 \leq r \leq 2$. There is again only one interval. From the rate equation we get directly

$$P(x, t) \equiv u_0(t) = \exp \left[ e^{-(2-r)t} - (2-r)t - 1 \right]$$

and

$$\theta(t) = (2-r) \int_0^t dt' e^{(2-r)t'} u_0^2(t') = \frac{1}{2} \left[ 1 - \exp[-2 + 2e^{-(2-r)t}] \right]$$

which is exactly the lattice dimer solution, up to a factor $1/2$ due to our definition of $\theta$.

### 3.3 The case $1/2 < r \leq 2/3$

According to eq.(5), this is the interval (for $r < 1$) corresponding to the lowest non trivial value $k = 2$. There are only two reduced probabilities, $q(t)$ and $P_2(x, t)$. The function $\rho_2(t, t')$ of eq.(7) is given by

$$\rho_2(t, t') = \frac{e^{(2r-1)(t-t')} - 1}{t - t'} - (2r - 1)$$

and $K_2(x, t|t')$ of eq.(6) by

$$K_2(x, t|t') = e^{-rt'} \frac{e^{-x(t-t')} - e^{-(1-r)(t-t')}}{t - t'}$$

(see the appendix). After some elementary algebra based on the integral representation eq.(3), one gets from eq.(11) a very simple expression for $\theta$:

$$\theta = 1 - q^2(t) e^{rt}$$

(8)

We obtain the exact asymptotic behaviour of the coverage in the following way.

From the inequality $0 \leq q(t) \leq 1$ and equation (7), it follows that $q(t) < C \frac{e^{-(1-r)t}}{t}$ which implies that the functions

$$C_n(x, t) = \int_0^t e^{-(r-x)t'} q(t') t'^n dt'$$
have a positive, finite, \( t \to \infty \) limit for all \( n \geq 0 \) and for all \( x \in [0,r] \), that we denote \( C_n(x) \). Expanding the denominator of the kernel \( \rho_2(t,t') \) in power of \( \frac{t'}{t} \), we get from eq.(7) the asymptotic expansion of \( q(t) \) at large \( t \)

\[
q(t) = \frac{e^{-(1-r)t}}{t} \left[ C_0(1-r) + \frac{C_1(1-r)}{t} + O(\frac{1}{t^2}) \right] \tag{9}
\]

Inserting this result in eq.(8), we obtain for the coverage the asymptotic value \( \theta(t=\infty) = 1 \), as expected, with the approach:

\[
\theta(t) = 1 - C_0^2(1-r) \frac{e^{-2(3r-1)t}}{t^2} + \cdots \tag{10}
\]

This equation shows that the kinetics remains exponentially driven over the open interval \( 1/2 < r < 2/3 \) but becomes \( 1/t^2 \) at the end point \( r = 2/3 \).

### 3.4 The case \( 2/3 < r < 3/4 \)

We have explicitly studied this case, where three \( x \)-intervals (\( k = 3 \)) are involved, since we expect it to be typical of what happens over the remaining range \( r < 1 \).

Following the method of the previous section, we can derive for the reduced probabilities \( P_2 \) and \( P_3 \) the following large-time behaviour

\[
P_2(x,t) = C_0(x,t) \frac{e^{-xt}}{t} \left[ 1 + O(\frac{1}{t^2}) \right] \]

\[
P_3(x,t) = P_2(x,t) + O(\frac{e^{-(2r-1)t}}{t^2})
\]

whereas the asymptotic behaviour of \( q(t) \) is still given by eq.(9) in spite of its being defined from a different kernel \( \rho_3 \).

The time derivative of \( \theta \) may be expressed in terms of the reduced probabilities as follows

\[
\dot{\theta}(t) = e^{rt} \left\{ \int_{1-r}^{2r-1} P_2(x,t) P_2(r-x,t) \, dx + 2q(t) \int_{2r-1}^{2(1-r)} P_2(x,t) \, dx + 2q(t) \int_{2(1-r)}^{r} P_3(x,t) \, dx \right\}
\]

Inserting in this expression the previous asymptotic expansion, we get

\[
\dot{\theta}(t) = \frac{A}{t^2} + \frac{2B}{t^3} \tag{11}
\]

where \( A \) and \( B \) are constants depending only on \( r \). In particular

\[
A = \lim_{t \to \infty} \int_{1-r}^{2r-1} C_0(x,t) C_0(r-x,t) \, dx
\]
We see that, except at $r = 2/3$ where $A$ vanishes, the leading behaviour of $\theta$ is of the form

$$\theta(t) = \theta(\infty) - \frac{A}{t}$$

We expect this behaviour to hold over the whole range up to $r = 1$ where it is proved. We have checked this assumption numerically for $r = 0.70$ and $r = 0.9$.

For $r = 2/3$ we recover the result of the previous section

$$\theta(t) = \theta(\infty) - \frac{B}{t^2}$$

which confirms that this unusual behaviour occurs only at this special value of $r$.

3.5 The case $4/3 \leq r < 3/2$

As in the previous section, this interval corresponds to the first non-trivial one where $k = 2$ in the case $r > 1$. The kernels $K_2$ and $\rho_2$ are defined by

$$K_2(x, t|t') = e^{-t'} e^{(r-1)t} \frac{e^{-x(t-t')} - e^{-(2-r)(t-t')}}{t-t'}$$

$$\rho_2(t, t') = e^{-2r t'} \int_0^{t-t'} e^{-(3-2r)t''} \left[ 1 + (3 - 2r) t'' \right] - 1 \frac{1 - e^{-(2-r)(t'+t'')}}{t'^2} dt''$$

The expression for $\theta$ in terms of the reduced probability $q(t)$ is obtained from eq.(2):

$$\theta = 1 - q^2(t) e^{2(2-r)t} - (2 - r) \int_0^t e^{2(2-r)t'} q^2(t') dt'$$

(12)

One observes that $\rho_2$ is negative, which implies the bound $0 < q(t) \leq u_0(t)$ and, from the mean-value theorem, the estimate

$$\rho_2(t, t') = e^{-(2-r)t'} G(t-t') \left\{ \frac{1 - e^{-(3-2r)(t-t')}}{t-t'} - (3 - 2r) \right\}$$

where $1 \leq \exp(1 - e^{-(2-r)t'}) \leq G(t-t') \leq \exp(1 - e^{-(2-r)t'}) < e = 2.718...$. Inserting this result in eq.(12) leads to the asymptotic time expansion of $q(t)$

$$q(t) = e^{-2r t} \left\{ A(r) + \frac{B(r)}{t} + \cdots \right\}$$

where the bracket reduces to a constant when $r$ goes to $3/2$. This shows from eq.(12) that the jamming limit $\theta(\infty)$ is reached as $1/t$. 

8
4 Summary and conclusions

We have studied a one-dimensional model of random sequential adsorption with discrete and continuous degrees of freedom, which depends on one length parameter $r$. When this parameter varies over its range $0 < r < 2$ the model goes through the following regimes :

- $0 < r \leq 1/2$
  monomer on a lattice; $\theta(t) = 1 - e^{-rt}$

- $1/2 \leq r < 2/3$
  total coverage; non trivial lattice dynamics : $\theta(t) = 1 - A e^{-(2-3r)t}/t^2$ for large $t$.

- $r = 2/3$
  total coverage; anomalous continuous dynamics : $\theta(t) = 1 - A/t^2$ for large $t$.

- $2/3 < r < 3/4$
  non-trivial asymptotic coverage; normal continuous dynamics : $\theta(t) = \theta(\infty) - A/t$ for large $t$.

- $r = 1$
  “car-parking” problem : $\theta(t) = 0.7476 \cdots - A/t$ for large $t$.

- $4/3 \leq r \leq 3/2$
  same as for $2/3 < r < 3/4$.

- $3/2 \leq r < 2$
  lattice dimer model; $\theta(t) = \frac{1}{2} \left[ 1 - \exp[-2 + 2e^{-(2-r)t}] \right]$

- $3/4 < r < 4/3$
  Although we have not investigated this interval analytically (except for $r = 1$), we expect the same regime as in the bordering intervals $2/3 < r < 3/4$ and $4/3 \leq r \leq 3/2$ to hold, that is a non trivial asymptotic coverage decreasing with $r$ and a $1/t$ normal continuous dynamics. We have checked this conjecture numerically.

The remarkable point is that the kinetics of the model exhibits three “phases” : for $0 \leq r < 2/3$ where it is lattice-like, for $2/3 < r < 3/2$ where it is continuous-like and for $3/2 \leq r \leq 2$ where it is lattice-like again. At the transition point $r = 2/3$ it becomes “anomalous” since the jamming limit is approached in $1/t^2$ in contrast with the general belief that the exponent $n$ of the power-law decay is equal to the inverse of the number of degrees of freedom per particle.
The regime on both sides of the transition is characterised by a same typical cross-over time $\tau = \frac{1}{|2-3r|}$, defined from the slope of the exponential in eq.(11) or from the ratio $B/A$ in eq.(11). This time is such that for $t \ll \tau$ the dynamics is dominated by a $1/t^2$ behaviour in the two “phases”; only for $t \gg \tau$ does the characteristic long time behaviour, exponential for $r < 2/3$, in $1/t$ for $r > 2/3$, emerge. Since $\tau \to \infty$ when $r \to 2/3$, this long time regime is squeezed at the transition point $r = 2/3$, leaving only the $1/t^2$ behaviour.

To find for $\theta(t)$ an expression in a closed form is a very difficult technical problem, comparable to the determination of the correlation function in the standard “car-parking” model. However, the properties of the kernel $\rho_k$ allow us to obtain an iterative solution of eq.(7) from which approximate expressions for the long time coverage can be obtained.

Finally, it may be interesting to investigate these kinds of models in a more realistic physical context, such as a two dimensional substrate, a disordered distribution of adsorbing sites or the possibility for a particle to overlap several sites. It is presumably difficult to get an analytical insight in such models, but their properties can be numerically analysed using as a guide the one dimensional results.

Aknowlegements

We thank G. Tarjus for discussions and for pointing out to us that model (I) and model (II) are equivalent.
APPENDIX

A The master equations

In this appendix, we derive the master equations for the probability of finding at time $t$ an unoccupied interval (gap) of given length. Let $P_n(x, y, t)$ be the probability for finding a gap of length at least $x + y + n - 1$, where $n$ is the number of adsorbing sites in the gap and $x(y)$ the distance between the left (right) edge of the gap and the last left(right) site in the gap (see figure 4). $P_1(0, 0, t)$ is the probability for finding a gap containing at least one site and its complement, $1 - P_1(0, 0, t)$, defines the probability that a site is covered by a particle, which is equivalent to the occupancy rate:

$$\theta(t) = 1 - P_1(0, 0, t)$$

The rate equation for this quantity is (in these equations and in the subsequent rate equations we set the effective flux of particles to unity)

$$- \frac{\partial P_1(0, 0, t)}{\partial t} = \int_0^r P_1(x, r - x, t) \, dx \quad \text{for } r \in [0, 1] \quad (13)$$

$$= \int_{r - 1}^1 P_1(x, r - x, t) \, dx \quad \text{for } r \in [1, 2] \quad (14)$$

Therefore to determine $\theta$ one needs to know $P_1(x, y, t)$ only for $x$ and $y$ less than $r$ (case $r \leq 1$) or greater than $r - 1$ (case $r \geq 1$) and set $x + y = r$. The general rate equations are obtained as usual, by counting the different ways of destroying a gap:

case $r \leq 1 \quad x \in [0, r] \quad y \in [0, r] \quad n \geq 2$

$$- \frac{\partial P_n(x, y, t)}{\partial t} =$$

$$[x + y + (n - 2)r]P_n(x, y, t) + \int_x^r P_n(x', y, t) \, dx' + \int_y^r P_n(x, y', t) \, dy'$$

$$+ u(x + r - 1) \int_0^{x + r - 1} P_{n+1}(x', y, t) \, dx' + u(y + r - 1) \int_0^{y + r - 1} P_{n+1}(x, y', t) \, dy' \quad (15)$$

case $r \geq 1 \quad x \in [r - 1, 1] \quad y \in [r - 1, 1] \quad n \geq 2$

$$- \frac{\partial P_n(x, y, t)}{\partial t} = [\{(x - r + 1) + (y - r + 1) + (n - 2)(2 - r)\}]P_n(x, y, t)$$

$$+ \int_x^1 P_n(x', y, t) \, dx' + \int_y^1 P_n(x, y', t) \, dy'$$

$$+ \int_{r - 1}^{b(x)} P_{n+1}(x', y, t) \, dx' + \int_{r - 1}^{b(y)} P_{n+1}(x, y', t) \, dy' \quad (16)$$
where \( u(t) \) is the Heaviside step function and, in the last equation, \( b(x) = \text{Inf}\{x + r - 1, 1\} \). The probability \( P_n(x, y, t) \) is subject to the initial condition

\[
P_n(x, y, t = 0) = 1
\]

From these equations, we see that for \( n \geq 2 \), the \( n \) dependence can be factorised out:

\[
P_n(x, y, t) = e^{-(n-2)rt} P_2(x, y, t) \quad \text{for } r \in [0, 1] \tag{17}
\]

\[
P_n(x, y, t) = e^{-(n-2)(2-r)t} P_2(x, y, t) \quad \text{for } r \in [1, 2] \tag{18}
\]

The function \( P_2(x, y, t) \) must be symmetric in its spatial arguments. Moreover, for \( n \geq 2 \) the left and the right parts of the gap cannot be simultaneously affected by the deposition of one dimer, hence the \( x \) and \( y \) dependence are uncorrelated. We deduce from these considerations that:

\[
P_2(x, y, t) = P(x, t) P(y, t) \tag{19}
\]

reducing the problem to finding one unknown function, \( P(x, t) \), which we call in the following, a reduced probability.

The rate equation for the function \( P_1(x, y, t) \) is different from eqs(15,16) and depends on the value of \( x + y \) with respect to \( r \). However, the arguments leading to the factorisation of the \( x \) and \( y \) dependences and of the \( n \) dependence remain valid for \( x + y \geq r \) which is precisely the region of interest, leading to:

\[
P_1(x, y, t) = e^{rt} P(x, t) P(y, t) \quad x + y \geq r
\]

From this factorisation property and the rate equations(13,14), we can express the occupancy rate in terms of the function \( P \), leading to eqs.(1,2) :

\[
\theta(t) = \int_0^t dt' e^{rt'} \int_0^x dx P(x, t') P(r-x, t') \quad \text{for } r \in [0, 1]
\]

\[
\theta(t) = \int_0^t dt' e^{(2-r)t'} \int_{r-1}^1 dx P(x, t') P(r-x, t') \quad \text{for } r \in [1, 2]
\]

From eqs.(15,16) and the factorisation properties of \( P_n(x, y, t) \), eqs.(17,18,19), we deduce the rate equations for the function \( P \), eqs.(3,4) of section 3.1 :

\[
\text{case } r \leq 1 \quad x \in [0, r]
\]

\[
-\frac{\partial P(x, t)}{\partial t} = xP(x, t) + \int_x^r P(x', t)dx' + u(x + r - 1) e^{-rt} \int_0^{x+r-1} P(x', t)dx' \tag{20}
\]

\[
\text{case } r \geq 1 \quad x \in [r-1, 1]
\]

\[
-\frac{\partial P(x, t)}{\partial t} = (x - r + 1)P(x, t) + \int_x^1 P(x', t)dx' + e^{-(2-r)t} \int_{r-1}^{b(x)} P(x', t)dx' \tag{21}
\]
Due to the last term in eqs. (20,21), \( P(x,t) \) appears to be piecewise defined with respect to \( x \), in successive intervals of length \((1-r)\) within the range \((0,r)\) for \( r \leq 1 \), or of length \((r-1)\) within the range \((r-1,2-r)\) for \( r \geq 1 \). To be more explicit, let us define the set of intervals \( I_\ell \) in the following way:

For \( 0 \leq r < 1 \) define \( k \geq 1 \) such that \( \frac{k-1}{k} < r \leq \frac{k}{k+1} \); then:

\[
I_\ell = [(\ell - 1)(1-r), \ell(1-r)] \quad 1 \leq \ell \leq k-1 \\
I_k = [(k-1)(1-r), r]
\]

which are such that \( \bigcup_{\ell=1}^{k} I_\ell = [0,r] \).

For \( 1 < r \leq 2 \) define \( k \geq 1 \) such that \( \frac{k+2}{k+1} < r \leq \frac{k+1}{k} \); then:

\[
I_1 = [(2-r), 1] \quad \text{only for } k \geq 2 \\
I_2 = [(k-1)(r-1), (2-r)] \quad \text{only for } k \geq 3 \\
I_\ell = [(k-\ell+1)(r-1),(k-\ell+2)(r-1)] \quad 3 \leq \ell \leq k
\]

which are such that \( \bigcup_{\ell=1}^{k} I_\ell = [r-1,1] \).

The number \( k \) of such intervals is directly expressed in terms of \( r \):

\[
k = \left\lceil \frac{r}{1-r} \right\rceil + 1 \quad \text{for } 0 \leq r < 1 \\
k = \left\lceil \frac{2-r}{r-1} \right\rceil + 1 \quad \text{for } 1 < r \leq 2
\]

where \( \lceil X \rceil \) means integer part of \( X \). In the following, for a given value of \( r \), we will denote by \( P_\ell(x,t), \ell = 1,...,k \) the restriction of \( P(x,t) \) to the interval \( I_\ell \).

**B** Integral equations for the generic case \( 2/3 < r < 3/2, \ r \neq 1 \)

In this appendix, we derive the representation of eqs. (11,12) from eqs (20,21) and we give the expression of the kernels \( K_\ell \) and \( \rho_k \) and of the function \( u_0 \).

We first remark that the reduced probability \( P_1(x,t) \) is in fact independent of \( x : P_1(x,t) \equiv q(t) \). Consider the left bordering site of the gap, in the case \( r \leq 1 \). There is a vacant space of length at least \( 1-r \) from its right edge independent on the occupation of its left neighbour; this means that \( P(x,t) \) is independent of \( x \) for \( x \in [0,1-r] \). The same property holds, in the case \( r \geq 1 \), for \( x \in [2-r,1] \).
We start with the case $r < 1$. For $1 - r \leq x \leq r$, we derive eq.(20) with respect to $x$ to get

$$-\frac{\partial^2 P(x, t)}{\partial x \partial t} = x \frac{\partial P(x, t)}{\partial x} + e^{-rt} P(x + r - 1, t)$$

Then integrating with respect to $t$, using the initial condition $\frac{\partial P(x, 0)}{\partial x} = 0$, we obtain

$$\frac{\partial P(x, t)}{\partial x} = -\int_0^t e^{-r'x - x(t-t')} P(x + r - 1, t') \, dt'$$

Finally, we integrate with respect to $x$, taking into account the boundary condition $P(x = 1 - r, t) = q(t)$, to get

$$P(x, t) = q(t) - e^{-(1-r)t} \int_0^t dt' e^{-(2r-1)t'} \int_0^{x+r-1} dx' e^{x'(t-t')} P(x', t') \quad (22)$$

By using repeatedly eq.(22) as $x$ increases, one obtains equation (6):

$$P_{\ell}(x, t) = q(t) + \int_0^t K_{\ell}(x, t|t') \, q(t') \, dt' \quad 2 \leq \ell \leq k$$

For $x \in I_2$ then $x' \in I_1$ where $P(x', t') = q(t')$ and eq.(22) gives eq.(3) with $\ell = 2$, and the following expression for the kernel $K_2$

$$K_2(x, t|t') = e^{-rt'} \frac{e^{-x(t-t')} - e^{-(1-r)(t-t')}}{t - t'} \quad r < 1 \quad (23)$$

When $x \in I_3$, by including the previous result in eq.(22), we get eq.(1) for $\ell = 3$ with the explicit form of the kernel $K_3$, and so on.

Analogous manipulations can be performed on eq.(21) in the case $r > 1$ for $x \leq 2 - r$, where the boundary condition is now $P(x = 2 - r, t) = q(t)$. The equation equivalent to eq.(22) is

$$P(x, t) = q(t) + e^{2(r-1)t} \int_0^t dt' e^{-rt'} \int_1^{x+r-1} dx' e^{x'(t-t')} P(x', t') \quad (24)$$

which, by repeated application as $x$ decreases, gives the representation of eq.(1).

For $x \in I_2$, one gets the kernel $K_2$

$$K_2(x, t|t') = e^{-r'+(r-1)t} \frac{e^{-x(t-t')} - e^{-(2-r)(t-t')}}{t - t'} \quad r > 1 \quad (25)$$

Assuming that, for a given value of $r$ all the $K_\ell$ are known, one can define a kernel $\sigma_k(t, t')$

$$\sigma_k(t, t') = \sum_{\ell=2}^{k} \int_{I_\ell} dx \, K_\ell(x, t|t')$$
and derive an integral equation for \( q(t) \). Considering eqs. (20, 21) where \( x \) is fixed to the value \( x = 1 - r \) (case \( r < 1 \)) or \( x = 2 - r \) (case \( r > 1 \)) and using eq. (5) for \( P_t(x, t) \), one gets respectively:

\[
\begin{align*}
  r < 1 & \quad -\frac{dq}{dt}(t) = rq(t) + \int_0^t q(t') \sigma_k(t, t') \, dt' \\
  r > 1 & \quad -\frac{dq}{dt}(t) = (2 - r)[1 + e^{-(2-r)t}]q(t) + e^{-(2-r)t} \int_0^t q(t') \sigma_k(t, t') \, dt' 
\end{align*}
\]

By integrating with respect to \( t \) with the initial condition \( q(0) = 1 \), we get the equation (7)

\[
q(t) = u_0(t) \left\{ 1 + \int_0^t \rho_k(t, t') q(t') \, dt' \right\}
\]

with

\[
\begin{align*}
  r < 1 \quad u_0(t) &= e^{-rt} \quad , \quad \rho_k(t, t') = \int_t^{t'} e^{r(t'' - t')} \sigma_k(t'', t') \, dt'' \\
  r > 1 \quad u_0(t) &= \exp \left[ e^{-(2-r)t} - (2-r)t - 1 \right] \quad , \quad \rho_k(t, t') = \int_t^{t'} \exp \left[ 1 - e^{-(2-r)t''} \right] \sigma_k(t'', t') \, dt'' 
\end{align*}
\]
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Figure captions

**Figure 1** (a) description and parametrization of model (I) : deposition of dimers on extended adsorbing sites. (b) description and parametrization of model (II) : deposition of segments on localised adsorbing sites.

**Figure 2** The jamming limit of the model as a function of the length $r$ of the adsorbed segments. The values of $r$ separating the different regimes are indicated.

**Figure 3** Two extreme situations for the deposition in model (II).

**Figure 4** The parametrisation of the gap.
Figure 1: Diagram illustrating two models of adsorption. (a) Model (I) with a dimer and adsorbing sites. (b) Model (II) showing a particle and adsorbing sites.
Figure 2
Figure 3
Figure 4

Gap of length $x+y+n-1$