Hydrodynamic theory of rotating ultracold Bose–Einstein condensates in supersolid phase

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Abstract

Within the mean field Gross–Pitaevskii framework, ultracold atomic condensates with long-range interaction are predicted to have a supersolid-like ground state beyond a critical interaction strength. Such a mean field supersolid-like ground state has periodically modulated superfluid density which implies the coexistence of superfluid and crystalline order. An ultracold atomic system in such a mean field ground state can be subjected to an artificial gauge field created either through rotation or by introducing space dependent coupling among hyperfine states of the atoms using Raman lasers. Starting from this Gross–Pitaevskii energy functional that describes such systems at zero temperature, we construct a hydrodynamic theory to describe the low-energy long-wavelength excitations of a rotating supersolid of weakly interacting ultracold atoms in two spatial dimensions for a generic type of long-range interaction. We treat the supersolidity in such a system within the framework of the well known two-fluid approximation. Considering such a system in the fast rotation limit where a vortex lattice in superfluid coexists with the supersolid lattice, we analytically obtain the dispersion relations of collective excitations around this equilibrium state. The dispersion relation gives the modes of the rotating supersolid which can be experimentally measured within the current technology. We point out that this can clearly identify an ultracold atomic supersolid phase in an unambiguous way.

Keywords: ultra cold atomic condensate, supersolid phase, artificial gauge field and vortices, collective modes

(Some figures may appear in colour only in the online journal)

1. Introduction

The issue of observing supersolidity experimentally in solid $^4$He [1, 2] has now been settled conclusively by showing that there is no supersolidity in such a system [3]. As a result, the counter-intuitive co-existence of superfluidity and crystalline order [4–8] still remains an open question in spite of much progress in this direction [9]. In this context, an alternative possible route to observe supersolidity in a much more controllable and conspicuous way is via certain species of ultracold atomic condensate with long-range interaction.

These ultracold atom condensates with long-range interactions can have roton-instability in their excitation spectrum [10–12] and significant experimental [13–17] as well as theoretical [18–22] progress has taken place in realizing such systems. In recent experiments such roton-like mode softening has been demonstrated through cavity mediated long-range interaction in ultracold atomic BEC [23] and a self-organized supersolid phase has also been experimentally observed [24] in Dicke quantum phase transition where the long-range interaction is generated by a two-photon process in cavity.

In this paper, we show that one way of clearly identifying such an ultracold supersolid phase is to study its response to an artificial gauge field created through rotation or by other
means [25–27]. Study of the critical velocity of the nucleation of vortices in a rotating dipole-blockaded ultracold supersolid condensate [28] as well as supersolid vortex lattice phases in a fast rotating Rydberg-dressed Bose–Einstein condensate (BEC) within the Gross–Pitaevskii approach [29] was carried out recently. The same work [29] highlighted the important difference in the vortex lattice structure in such a supersolid-like ground state as compared to a similar vortex lattice structure in an ultracold atomic superfluid state. However, it is still not clear if within the standard time of flight measurement technique, one will be able to separately identify the vortex cores in vortex lattices from the superfluid density minimum in the supersolid lattices. A way out of this problem is to look for the collective excitation spectrum of such supersolid vortex lattices.

In an ultracold atomic ensemble with long-range interaction, a supersolid-like ground state implies a periodic modulation of the superfluid density when the relative strength of such interaction exceeds a critical value. This implies that the supersolid phase possesses phase coherence as well as periodic density distribution, which results in density modulated superfluid, where the density maxima or minima forms a lattice, referred to as supersolid lattice in this paper. It is worth noting that this is completely different from the density wave phase, which is an insulating phase with no phase coherence, but possesses a periodic or crystalline distribution of particles, with no superfluidity. Typically for weakly interacting bosons near absolute zero temperature, such an ultracold BEC is theoretically described within the framework of Gross–Pitaevskii equation for short range as well as for generic long-range interaction in mean field approximation. Within this framework a periodic modulation was discussed as early as 1957 by E Gross [30] and recently discussed in several contexts [9, 11, 20] that include ultracold atomic systems. Such a supersolid ground state is different from the Andreev–Lifshitz supersolid scenario [5] which is based on vacancies or interstitials with repulsive interactions, more appropriate for the solid $^4$He.

We study the effect of a sufficiently high artificial magnetic field on such a supersolid phase which results in the formation of a vortex lattice phase in such a system. As shown in recent literature [29] such vortices can arrange themselves either at the minima or maxima of the supersolid density periodic modulation. In more specific terms, the superfluid density is modulated in a periodic manner in the supersolid phase. When such a supersolid is rotated quickly, a vortex lattice is formed and there is modulation of the superfluid density due to the formation of vortices. Particularly at the core of such vortices the superfluid density goes to zero and there will be circulation around such a vortex core. This vortex core may coincide with the minima as well as the maxima of the superfluid density in the superfluid phase ([29]) under various conditions. For example in figure 1 we schematically described a situation where the vortex lattice co-exists with a supersolid lattice and the vortex core coincides with the superfluid density minimum in the supersolid phase.

![Figure 1. Schematic figure showing the co-existing vortex lattice and supersolid lattice in a rotating supersolid. The high density (dark) areas show the supersolid crystal which is hexagonal in shape, and the low density (light) areas show the vortex lattice, with arrow plots showing the winding of the single vortex.](image)

The high density (dark) areas in figure 1 show the supersolid crystal lattice (hexagonal), whereas the low density (light) areas show the vortex positions superposed with arrow plots to show the winding of the single vortex. Treating the small oscillation around such an equilibrium state in the low-energy long-wavelength limit, we construct a hydrodynamic theory of collective excitations of such a vortex lattice state in ultracold atomic supersolid. In particular, we calculate the dispersion of such low-energy long-wavelength collective excitations and explain how they demonstrate the supersolid behavior.

In this paper, in the framework of a hydrodynamic theory, we demonstrate that the collective excitation spectrum of the vortex lattice phase in a fast rotating ultracold atomic supersolid has important differences with the collective excitation of the Abrikosov vortex lattice phase in ultracold atomic superfluid. Many studies have been carried out for rapidly rotating BECs with their vortex lattices and their collective excitations [31–39]. The collective excitations have been studied within a hydrodynamic framework [36–38] where Tkachenko modes [31] have been studied and compared with similar theories developed earlier for superfluid helium [34, 35]. We argue that experimental detection of the same for a rotating supersolid can provide a conclusive test of supersolidity, and hence motivates the present work. The adaptation of a hydrodynamic theory in the present case implies writing the equations of motion in terms of density and phase, and describing the long wavelength behavior of the fluid with these equations. To validate such a hydrodynamic approximation, the variables used in these equations
are averaged over scales much larger than the inter-vortex spacing and the supersolid lattice spacing.

For low $q$ values, we show that our analytical results also qualitatively agree with the appearance of two distinct longitudinal modes for a supersolid, in a recent work by Saccani et al [40] derived from a microscopic model using a quantum Monte Carlo method. Our results show the appearance of longitudinal as well as transverse modes of a rotating supersolid by analytical means, and can be confirmed by numerical calculations qualitatively. A quantitative comparison of the results requires much more involved analytical calculations, including the mutual friction of the two co-existing lattices and also the different possible symmetry considerations of the vortex and the supersolid lattice, which is out of the context of the present work.

The rest of the paper is organized in the following way. In section 2 we derived the hydrodynamic Lagrangian from the Gross–Pitaevskii energy functional using a homogenization method. Section 3 shows the determination of hydrodynamic equations of motion, the calculation of dispersion relations and the corresponding sound modes for the rotating ultracold supersolid. We conclude by emphasizing the significance of the main finding of this work, namely the dispersion modes for the rotating supersolid system, and also point out the possibility of experimental verification. Further details on the calculations are provided in appendix A.

2. Effective hydrodynamic Lagrangian

We begin with a Gross–Pitaevskii mean field description of ultracold atomic BEC at $T = 0$ with suitable long-range interaction, rotated about the $z$-axis with a frequency $\Omega$ in two dimensions. It may be pointed out here that our equilibrium state is the one obtained in the limit of high rotation, the trap potential is almost cancelled by the centrifugal force [41] and the system can be very well approximated as a uniform two-dimensional system. The details of the derivation of such a Gross–Pitaevskii energy functional from the microscopic Hamiltonian of a typical ultracold system such as Rydberg-excited BEC is given in [20]. As already mentioned, at a sufficiently large interaction strength and fast enough rotational frequency $\Omega$, the ground state of the system is a vortex lattice phase of the supersolid, as shown by a recent numerical study [29]. We are interested in low energy excitations of such a system which has a wavelength much larger than the lattice parameters of the vortex lattice or the supersolid lattice.

The mean field Lagrangian for the rotating supersolid system is given as

$$L = \int \mathbf{dr} \left\{ \frac{i}{\hbar} \left[ \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right] - \mathcal{E}_R(\psi, \psi^*) \right\}. \quad (1)$$

Here $\mathcal{E}_R$ is the Gross–Pitaevskii energy functional, in the co-rotating frame [43, 44], related with the non-rotating energy functional $\mathcal{E}$ through the expression

$$\mathcal{E}_R = \mathcal{E} - \mathcal{\Omega} \cdot <(\mathbf{r} \times \mathbf{p})>. \quad \mathcal{E} \text{ is the usual Gross–Pitaevskii energy functional, given by}$$

$$\mathcal{E}(\psi, \psi^*) = \frac{\hbar^2}{2m} \big| \mathbf{V} \psi \big|^2 + \frac{1}{2} \int \mathbf{dr} \big| \psi(\mathbf{r}) \big|^2 U(\mathbf{r}) \big| \psi(\mathbf{r}) \big|^2. \quad (2)$$

In the usual zero temperature mean field description of an ultracold atomic superfluid [43], $\big| \psi(\mathbf{r}) \big|^2$ is identified with superfluid density and $\psi(\mathbf{r})$ as the superfluid order parameter. However in the present case for an ultracold atomic supersolid [45–47], one can extract a Landau two-fluid description from the same Gross–Pitaevskii energy functional, where the normal component of the two-fluid description corresponds to the solid part of the supersolid. We must mention at the outset that from now onwards, the superscript ‘ss’ stands for the supersolid lattice component, which plays the role of the normal component in the two-fluid description and super- script ‘uid’ stands for the vortex lattice component of the system, in our subsequent calculations.

To do this we first write complex $\psi(\mathbf{r}) = n(\mathbf{r}) e^{i\phi(\mathbf{r})}$ in terms of the density $n$ and phase $\phi$. Then Lagrangian $L$ in the non-rotating case takes the form

$$L = \int \mathbf{dr} \left\{ - \left[ \hbar n \frac{\partial \phi}{\partial t} + \frac{\hbar^2}{2m} (\mathbf{V} n)^2 + \frac{1}{4m} (\nabla n)^2 \right] \right\} \mathbf{dr}$$

$$- \frac{1}{2} \int U(\mathbf{r} - \mathbf{r}') n(\mathbf{r}) n(\mathbf{r}') \mathbf{dr}' \right\}. \quad (3)$$

We also introduce $\mathbf{u}^u(\mathbf{r}, t)$ as the displacement field of the supersolid lattice. In an ordinary superfluid the average superfluid density $\bar{\rho} = \frac{1}{V} \int n(\mathbf{r}) \mathbf{dr}$ is constant, where $V$ is the total volume of the system. On the other hand, in a given classical crystalline solid, $\rho$ is defined by a fixed number of atoms per unit cell for a given set of lattice vectors such that the elastic deformation of lattice parameters obeys $\dot{\rho} = - \mathbf{V} \cdot \mathbf{u}^u$. In an ultracold atomic supersolid, quantum fluctuation leads to additional compression/dilation effects of the lattice ($\mathbf{u}^u(\mathbf{r}, t)$) which adds to the superfluid density. This is basically due to the change in displacement field $\mathbf{u}^u(\mathbf{r}, t)$ or equivalently by changing the density of superfluid component. Hence, this fact can be expressed through the following ansatz [47]

$$\rho = \bar{\rho} + \dot{\rho} \mathbf{V} \cdot \mathbf{u}^u + \delta \rho. \quad (4)$$

The above relation takes into account the fluctuations in the density of the supersolid. It is worth noting that $\rho$ is the total density, which is comprised of the superfluid density and the crystal density due to spatial modulations in density. We describe the lattice part of the supersolid as the normal component within the well known two-fluid description. $\delta \rho$ are the fluctuations around the steady state with density $\bar{\rho}$. For a usual prototype superfluid, $\rho$ is simply the superfluid density with $\delta \rho$ as the fluctuations around the steady state.

In the same way as in a crystalline solid where the presence of a lattice makes the effective mass of electron as a tensor, here also in the presence of a lattice-like normal component, the superfluid density will be a tensor-like quantity [48] in a supersolid. For a typical lattice structure such as the hcp and fcc lattice, it has been shown [49] that the
superfluid flow is same in all directions of the crystal and hence one can write the superfluid density tensor in an isotropic form. In the current work we also consider an isotropic supersolid so that the structure of this tensor is purely diagonal and is given by \( \rho^{ss}_k = \rho^{ss}(\rho) \delta_{ik} \) with all components having the same value.

When the system is rotated fast enough, a vortex lattice is formed that can also be characterized as a patterned modulation in the superfluid density and phase. To denote the fluctuations of the vortex lattice from its equilibrium position, we introduce the displacement field \( u^r(\mathbf{r}, t) \). This vortex lattice has an associated vortex crystal lattice effective mass. For the case of rotating superfluids, such an effective mass for the vortex lattice has been considered in the literature [34].

When such an effective mass is taken into account, it leads to an additional term in the kinetic energy of the system which will be proportional to the product of the mass density of the vortex lattice and the square of the velocity difference between the superfluid and the vortex lattice velocity. Additionally in the present case, it will also produce terms due to the relative motion between the vortex lattice and the normal component due to the supersolid lattice. This fact can also be appreciated by inspecting the expression of the Lagrangian (5) and \( \mathcal{E}_{bh}(\phi) \) in the subsequent derivation. In less technical language, by introduction of vortex lattice effective mass, the system will have mutual friction or relative motion between the different components.

To simplify further analysis, we ignore such relative motions that arise due to the effective mass of the vortex lattice, and take into account only the supersolid crystal effective mass. This approximation has also been explained and shown in detail mathematically in appendix A.

Thus the displacement field \( u^r(\mathbf{r}, t) \) as well as the average density \( \rho \) can be varied independently and hence the complex macroscopic wavefunction \( \psi(\mathbf{r}, t) \) is now a functional of three field variables \( \rho(\mathbf{r}, t), u^a(\mathbf{r}, t), \phi(\mathbf{r}, t) \). To construct a long-wavelength description of the system we use the homogenization technique [45-47] in which one separates the density and phase in fast and slow varying components, and the fast varying component is integrated out. This finally gives us the effective Lagrangian as

\[
L = \int d\mathbf{r} \left[ -\hbar \frac{\partial \phi}{\partial t} - \mathcal{E} \right]
\]

where

\[
\mathcal{E} = \mathcal{E}_{in}(\rho) + \mathcal{E}_{bh}(\phi) + \mathcal{E}^{ss}_{el}(\nabla \psi^{*}) + \mathcal{E}^{ss}_{el}(\nabla u^r)
\]

\[
\mathcal{E}_{in}(\rho) = \frac{\hbar^2}{2m} \left( \nabla \rho \right)^2 + \rho \int U(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}') d\mathbf{r}' = \mu \rho
\]

\[
\mathcal{E}_{bh}(\phi) = \frac{\hbar^2}{2m} \left( \nabla \phi \right)^2 + (\rho \delta_{ik} - \rho^{ss}_{ik}) \left( V\phi - \frac{m D u^a}{\hbar} \right)
\]

\[
\mathcal{E}_{ss}^{el}(\nabla u^r) = \frac{1}{2} \lambda_{iklm} \epsilon^i_{lm} \epsilon^j_{km};
\]

\[
\mathcal{E}_{el}(\nabla \psi^{*}) = \frac{1}{2} \lambda_{iklm} \epsilon^i_{lm} \epsilon^j_{km}.
\]

The details of the derivation are given in appendix A (section 1)\(^1\).

Let us briefly summarize the main approximations that we have made to arrive at the effective energy functional. As mentioned earlier, we consider a rapidly rotating condensate where the rotation takes place in the \( x-y \) plane about the \( z \)-axis with rotation frequency \( \Omega \) being very close to the two-dimensional trapping potential \( \omega_{\perp} \) [41]. Therefore, the effective trapping potential in the \( x-y \) plane is given by

\[
V_{\text{ext}} = \frac{2}{m} (\omega_{\perp}^2 - \Omega^2) r^2
\]

and for such a fast rotating condensate, it is set to zero for the rest of the calculation. In this regard we may point out that in experiments on rotating ultracold BECs, a rotational frequency \( \Omega \) as high as 0.99 part of the transverse trapping frequency \( \omega_{\perp} \) was achieved [42].

Also, the normal or crystal lattice component may have a different velocity than the superfluid component, with the velocity difference proportional to

\[
\left( V\phi - \frac{m D u^a}{\hbar} \right)
\]

where

\[
\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{h}{m} V\psi \cdot \nabla \phi,
\]

giving rise to a kinetic energy term corresponding to the mass density of supersolid lattice in \( \mathcal{E}_{bh}(\phi) \). Within the two-fluid description, \( \rho^{ss}_{ik} \) is the density of the superfluid part of the supersolid and \( (\rho \delta_{ik} - \rho^{ss}_{ik}) \) is the density of the normal (remaining lattice) part of the supersolid lattice, with \( \rho \) as the total density of the supersolid. Also, as stated earlier, we ignore the associated vortex lattice effective mass in the present set of calculations.

To include the elastic properties of the supersolid and vortex lattice we use free energy of the deformed crystal [50] such that the strain energy \( \epsilon_{ik} = \frac{1}{2} \left( \frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i} \right) \lambda_{iklm} \) is a tensor of rank four which relates the strains to the stresses and is called the elastic modulus tensor.

3. Hydrodynamic equations for an ultracold rotating supersolid

Extremization of the above Lagrangian gives the hydrodynamic equations for a rotating supersolid with an embedded vortex lattice and provides the theoretical framework of this paper. These equations are

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \frac{\hbar}{m} \nabla \phi \right) = 0
\]

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x_k} \left( (\rho - \rho^{ss}) \frac{m D u^a}{h} \right) = 0
\]

\[
\frac{m}{\hbar} \left( \frac{\partial \phi}{\partial t} + \Omega \times \mathbf{v}_L \right) = - \nabla P
\]

\[
2\Omega \rho \frac{\partial}{\partial t} \frac{\epsilon_{ss}^{ss}}{m} \left( \frac{\partial \phi}{\partial t} - \frac{\partial \phi^{ss}}{\partial t} \right) + \frac{m}{\hbar} \partial \phi \cdot (\Omega \times \mathbf{r}) \left( \frac{\partial \phi}{\partial t} - \frac{\partial \phi^{ss}}{\partial t} \right) = 0
\]

\(^1\) See appendix A for the derivation of effective Lagrangian for a rotating supersolid in section 1, and hydrodynamic equations of motion for rotating supersolid in section 2.
\[
\begin{align*}
\frac{m \partial}{\partial t} \left[ \rho - \rho^\alpha \right] (\dot{u}^\alpha - \frac{\hbar}{m} \partial \phi) \\
+ \hbar \frac{\partial}{\partial x_k} \left[ \rho - \rho^\alpha \right] (\dot{u}^\alpha - \frac{\hbar}{m} \partial \phi) \partial \phi \\
- \left[ \left( \lambda^\alpha + \mu^\alpha \right) \partial_{\alpha} u^\alpha + \mu^\alpha \nabla^2 u^\alpha \right] = 0.
\end{align*}
\]

Equations (10) and (11) correspond to the equations of motion for density and phase and imply conservation of mass and momentum respectively. In deriving them, higher order terms containing product of derivatives of different quantities like \( \frac{\partial}{\partial \rho}(\rho - \rho^\alpha) \left( V \partial \phi - \frac{m \partial \omega}{\hbar} \right)^2 \) and \( (V \cdot \nabla \omega)(\frac{\partial \omega}{\partial \rho})^2 \) are neglected. We perform an averaging over a vortex lattice cell to obtain \( \dot{\phi} = 2\Omega + V \times \nu \), as the averaged vorticity, with the time derivative \( v^\alpha \) giving the velocity of the vortex lattice \( \nu_L \) and \( \nu \), as the averaged superfluid velocity \( (\nu = \frac{\mu}{m} \nabla \phi) \). The pressure \( P^r = \rho \left( T + \frac{1}{2} U \left( |r - r'| \rho \right) \right) \), with \( T = \frac{\hbar}{m \rho} \nabla \sqrt{\rho} \). \( T \) shows the quantum mechanical nature as it contains \( \hbar^2 \) explicitly, and is hence named the quantum pressure term. As pointed out earlier, we assume throughout an isotropic supersolid lattice, such that the superfluid density tensor \( \rho^\alpha = \rho^\alpha \delta_{\alpha \beta} \).

Equation (12) is obtained by putting the force \( f(t,v^\alpha = -\rho \omega \times (\nu_L - \nu)) \) acting per unit volume of the fluid moving with velocity \( \nu \), equal to the variation of the elastic response due to vortex displacements given by \( f_{\nu_L} = -\frac{\partial}{\partial \rho} \frac{\partial \rho}{\partial \phi} = \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi} \right) \right) \). Equation (13) gives the elastic response of the isotropic supersolid crystal lattice. Here \( \lambda^\alpha, \mu^\alpha \) are respectively second Lamé coefficient of the supersolid and vortex lattice, and \( K^{ss, v} \) and \( \mu^{ss, v} \) are the respective compressibility and shear modulus of these lattices [50].

The non-deformed steady state of a supersolid with embedded vortex lattice is characterized by \( u^\alpha = 0, \dot{u}^\alpha = 0 \) (and therefore \( \rho = 0 \)), and an average density \( \bar{\rho} \). Equations (10)–(13) are now aligned around such a steady state in terms of small perturbations \( \delta \rho, \delta \phi, \nabla \cdot \nabla \omega = \delta J^{ss} \), and \( \omega^\alpha \). Here, \( \delta J^{ss} \) is the elastic compressibility of the supersolid lattice, and equation (4) shows the relation between \( \delta \rho \) and \( \delta J^{ss} \). The resulting equations describe the low energy collective excitations of a rotating supersolid and are given by

\[
\frac{\partial \delta \rho}{\partial t} + \rho^\alpha \frac{\hbar}{m} \nabla^2 \delta \phi + (\rho - \rho^\alpha) \frac{\hbar}{m} \partial \phi = 0 \tag{14}
\]

\[
\rho^\alpha \left( \frac{\partial \delta \phi}{\partial t} + 2 \Omega \times \nu_L \right) = -\frac{\hbar}{m} \nabla \delta \rho \tag{15}
\]

\[
\rho \bar{2} \Omega \left[ \nabla \times \left( \nu_L - \nu \right) \right] = \left( \lambda^\alpha + \mu^\alpha \right) \nabla \left( \nabla \cdot \nabla \omega \right) + \mu^\alpha \nabla^2 \nabla \cdot \nabla \omega \tag{16}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\hbar}{m} \nabla \delta \phi \right) = \left( \lambda^\alpha + \mu^\alpha \right) \nabla \left( \nabla \cdot \nabla \omega \right) \frac{\hbar}{m} \left( \rho - \rho^\alpha \right) \nabla^2 \delta J^{ss}. \tag{17}
\]

In (15), \( c_{sm} \) is the modified sound velocity, namely
\[
c_{sm} = c_s^2 \left( \frac{\rho^\alpha}{\rho} \right),
\]
where \( c_s \) is the usual sound velocity that connects the pressure fluctuation \( \delta P \) to density through \( \delta P = m c_s^2 \delta \rho \). \( \nu_L \) and \( \nu \) in (15) and (16) are the vortex lattice velocity, and the averaged superfluid velocity respectively. Equation (17) has been obtained after taking the divergence of (13) and then performing the linearization, with

\[
\delta J^{ss} = \nabla \cdot \omega^\alpha
\]
as the elastic compressibility of the supersolid lattice.

Apart from the above set of equations, there is another equation that describes the decoupled shear waves for the rotating supersolid system. It is obtained by taking the curl of (13) after expanding in terms of small fluctuations, which gives

\[
m \left( \rho - \rho^\alpha \right) \frac{\partial^2}{\partial t^2} \sigma - \mu^\alpha \nabla^2 \sigma = 0 \tag{19}
\]

where

\[
\sigma = \nabla \times \omega^\alpha. \tag{20}
\]

The equation (19) gives the shear mode velocity which depends on the supersolid density, namely

\[
\nu^\alpha_{hear} = \sqrt{\frac{\rho^\alpha}{m \left( \rho - \rho^\alpha \right)}}. \tag{21}
\]

It is worth noting that the shear mode for the supersolid is obtained by taking the curl of the equation for elastic response of the supersolid lattice, and the divergence of the same equation is used to calculate the longitudinal modes of the supersolid lattice.

### 3.1. Low-energy long-wavelength modes from hydrodynamic equations

In the rapid rotation limit, after setting the reduced trapping potential to zero, we expand the small fluctuations as

\[
\delta \rho = \delta \rho(q) \exp(iq \cdot r - i\omega t) \quad \delta \phi = \delta \phi(q) \exp(iq \cdot r - i\omega t) \quad \delta J^{ss} = \delta J^{ss}(q) \exp(iq \cdot r - i\omega t) \tag{22}
\]

Here, \( q, r \) and \( q \) are the two-dimensional position vector in the plane normal to the rotation axis and the two-dimensional wave vector. We decompose superfluid velocity \( \nu \) and the vortex lattice velocity \( \nu_L \) in the longitudinal and transverse component in the \( q \) plane. Subsequent algebra in the Fourier space expresses the longitudinal and transverse components of the superfluid velocity \( \{\nu_L, \nu \} \) in terms of the longitudinal and transverse component of the vortex lattice velocity \( \{\nu_{Lq}, \nu_q\} \).

In terms of the longitudinal and transverse components of various velocities, we obtain the following equations for determining the dispersion relation:

\[
\iiota \nu_{Lq} = \nu_{Lq} \left( \frac{c_s^2 q^2}{2 \Omega^2} + 2 \Omega \right) = 0 \tag{23}
\]
which in the long-wavelength limit finally leads to the dispersion

\[
\omega^2 - \omega^2 \left( 4\Omega^2 + \omega^2 \right) + \omega^2 \left( \omega^2 + c_{st}^2 q^2 \right) = 0
\]

This gives the following dispersion equation

\[
\omega^2 - \omega^2 \left( \frac{c_{sm}^2 q^2}{2\Omega} + c_{ss}^2 q^2 + \left( \frac{c_{sm}^2 q^2}{2\Omega} + 2\Omega \right) \left( \frac{c_{sm}^2 q^2}{2\Omega} + 2\Omega \right) \right) + \omega^2 \left[ \left( \frac{c_{sm}^2 q^2}{2\Omega} + 2\Omega \right) \left( \frac{c_{sm}^2 q^2}{2\Omega} + 2\Omega \right) \right] c_{sm}^2 q^2
\]

where the velocities of longitudinal and shear modes are given by \( c_{l}^2 = \frac{\lambda + 2\mu}{mp} \) and \( c_{s}^2 = \frac{\lambda}{mp} \), and \( c_{km} = \sqrt{\frac{\lambda^2 + 4\mu^2}{m(\rho - \rho^*)}} \).

Equation (29) describes the dispersion of a rotating supersolid and is one of the main results in this work.

### 3.2. Limiting behavior of the collective modes: recovering the non-rotating supersolid and rotating superfluid

We first show that the dispersion relation (29) reproduces the correct limiting behavior. The dispersion relation [11, 45, 46] of a non-rotating supersolid can be obtained from (29) by setting the condition that for \( \Omega = 0 \), there is no vortex lattice.

This gives \( \omega^2 = \frac{\lambda^2}{2} \left( c_{sm}^2 + c_{ss}^2 \right) q^2 \left[ 1 \pm \sqrt{1 - \frac{4\mu^2}{(c_{sm}^2 + c_{ss}^2)^2}} \right] \) along with a decoupled shear mode through (19), reproducing the known result for modes for a non-rotating supersolid as calculated in [11, 45, 46]. It is worth pointing out that for low \( q \) values, our analytical results also qualitatively agree with the appearance of two distinct longitudinal modes for a supersolid, in a recent work by Saccani et al [40] derived from a microscopic model using a quantum Monte Carlo method. However, the analytical approach also gives us the third mode which is absent in the quantum Monte Carlo calculations [40]. We plot these modes in figure 2(a).

Results for a rotating superfluid with a vortex lattice can also be obtained from (29) where \( \rho \rightarrow \rho^* \) and \( c_{sm} \rightarrow c_{l} \) in the absence of any normal component. Under these circumstances the following things happen. Firstly, the modified elastic wave speed due to the presence of the normal component drops out of the description. Secondly, the modified second sound velocity

\[
c_{sm}^2 = c_{l}^2 \left( \rho^*/\rho \right)
\]

becomes the second sound velocity \( c_{sm} = c_{l} \).

To see how this limit correctly reproduces the result for a rotating superfluid, we separate out in the equation the terms that depend on \( c_{sm}^2 \) by writing it as

\[
\omega^2 - \omega^2 \left( 4\Omega^2 + c_{sm}^2 q^2 + \left( c_{l}^2 + c_{sm}^2 q^2 \right) \right) + \omega^2 \left( \left( 4\Omega^2 + c_{sm}^2 q^2 + c_{l}^2 q^2 \right) c_{sm}^2 q^2 + c_{l}^2 q^2 c_{sm}^2 q^2 \right) - \left( c_{sm}^2 q^2 \right) \left( c_{sm}^2 q^2 \right) = 0
\]

which simplifies to

\[
\omega^2 - \omega^2 \left( 4\Omega^2 + c_{sm}^2 q^2 \right) + \omega^2 \left( \left( 4\Omega^2 + c_{sm}^2 q^2 \right) c_{sm}^2 q^2 + c_{l}^2 q^2 c_{sm}^2 q^2 \right) - \left( c_{sm}^2 q^2 \right) \left( c_{sm}^2 q^2 \right) = 0
\]
Since \(\tilde{\Omega} = \lambda \mu \rho \rho c \ km \ m \) (\(\bar{s} s s s\)), we now multiply both sides of the equation by \(\rho \rho \bar{s} s s\) and take the limit \(\rho \rho \bar{s} s s\), which makes the left hand side of (31) zero. We also set \(C_{sm} = c_s\). This yields

\[
\omega^4 - \omega^2 \left(4 \Omega^2 + c_s^2 q^2 \left(c_{vl}^2 + c_{vs}^2\right) q^2\right) + c_s^2 q^2 c_s^2 q^2 = 0.
\]

(32)

If we take the equilibrium state as a hexagonal isotropic lattice of vortices following the standard literature [34–36, 38] and assume that the shear mode velocity is much smaller than other mode velocities, the corresponding mode frequencies are given as (figure 2(b)) \(\omega_f^2 = \left(c_{vl}^2 q^2 + 4 \Omega^2 + \frac{4 (C_1 + C_2)}{m \rho}\right)\) and \(\omega_f^2 = \frac{2 C_2}{m \rho} \left(c_s^2 q^2 + 4 \Omega^2 + \frac{4 (C_1 + C_2)}{m \rho}\right)\). This agrees with the earlier results [36, 38] where \(C_1\) and \(C_2\) are given as

\[
\frac{c_s^2}{2 \Omega} = \frac{1}{2 \Omega} \frac{4 C_1 + 2 C_2}{m \rho} ; \quad \frac{c^2_{sl}}{2 \Omega} = \frac{1}{2 \Omega} \frac{2 C_2}{m \rho}.
\]

3.3. Results and discussion

We shall now analytically determine and analyze the roots of the dispersion equation (29) for a rotating supersolid. Even though the general nature of solutions of such cubic (in terms of \(\omega^2\)) equations (29) are quite involved, the above dispersion relation is simplified when the velocity associated with the shear mode of the vortex lattice is smaller compared to the other mode velocities. This criterion is generally met for the rotating ultracold atomic superfluid [36, 37] and therefore it is

---

**Figure 2.** Dispersion roots for (a) non-rotating supersolid, (b) rotating superfluid and (c) rotating supersolid, \(\omega\) as a function of wave vector \(q\), scaled according to the respective cases. It is worth noting that the parameters for the elastic wave velocity \(c_e\) and the superfluid velocity \(c_s\) are taken from [40] with \(f_s = 0.3\). Here, \(f_s\) is the amount of superfluid fraction in the supersolid, and the value of the vortex lattice velocity \(c_{vl}\) and \(c_{vs}\) has been taken from [36].
reasonable to assume a similar condition for the ultracold atomic supersolid as well.

In the above three modes we consider a condition reads as
\[
(c_i^2 q_i^2)/c_m^2 \ll (4\Omega^2 + c_m^2 q_i^2 + c_i^2 q_i^2)^2 \approx (4\Omega^2 + c_i^2 q_i^2)^2.
\]

This means that the fast term in (29) can be neglected to obtain a quadratic equation of the form \( x^2 = B'x + C' = 0 \). Here \( x = \omega_i^2 \), \( B' = (4\Omega^2 + c_i^2 q_i^2 + (c_i^2 + c_m^2 + c_v^2 + c_{x}^2 q_i^2)^2 \) and \( C' = (4\Omega^2 + c_i^2 q_i^2 + (c_i^2 + c_m^2)q_i^2) c_m^2 q_i^2 + (c_i^2 q_i^2)^2 \). In the limit of high rotation frequency and low \( q, C' < B' \) and the roots can be approximated as \( B' \) and \( C' \).

Consequently we get two mode frequencies, namely
\[
\omega_1^2 \approx 4\Omega^2 + c_i^2 q_i^2 + c_m^2 q_i^2 + (c_i^2 + c_m^2)q_i^2 \quad (33)
\]
\[
\omega_2^2 \approx \left( 4\Omega^2 + c_i^2 q_i^2 + (c_i^2 + c_m^2 + c_v^2 + c_{x}^2 q_i^2)^2 \right) (c_m^2 q_i^2 + c_i^2 q_i^2 + c_i^2 q_i^2)^2 \quad (34)
\]

There is a decoupled shear mode which also exists along with the above two modes, which is given by
\[
\omega_3^2 = \frac{\mu^\mu}{m(\rho - \rho^\mu)} q_i^2. \quad (35)
\]

The above three modes provide us with the bulk excitation spectrum of the rotating supersolid within this hydrodynamic approximation and form one of the main findings of the work. The first mode (33) is the inertial mode of the rotating supersolid, which for \( \Omega \ll c_m q_i \), behaves as a sound wave, while for \( \Omega >> c_m q_i \), the frequency of the mode begins essentially at \( 2\Omega \) i.e. it is a gapped mode for rotating supersolid for \( \Omega \gg c_m q_i \). Corresponding inertial modes for rotating superfluids have been calculated and observed experimentally \[31\], with \( c_i \) as the sound speed. In the second mode, as can be seen from (34), all the three velocities, the supersolid lattice velocity, the superfluid velocity and the vortex lattice velocity are coupled. This mode is unique for the case of a rotating supersolid, and can be used to identify and study the co-existing properties of the supersolid lattice and the vortex lattice. The third mode (35) is decoupled from the other two modes, and it arises due to the existence of supersolid lattice structure, and gives a signature of supersolidity in the system. This mode also appears for the supersolid phase without any rotation, and stays unaffected by the vortex lattice structure co-existing with the supersolid lattice in the present case.

The existence of the decoupled shear mode in non-rotating (figure 2(a)) as well as rotating supersolid (figure 2(c)) characterizes the signature of periodic crystalline order embedded within the superfluid in the super solid phase (be it non-rotating or rotating). However, one can note the change in the mode frequencies and their dispersion relations from figure 2(a) and (c). This figure shows the change in the mode frequencies between the rotating super solid with the counterpart non-rotating super solid and rotating superfluid.

The appearance of complex modes due to the interplay of the vortex lattice and the super solid lattice is evident from the equations (33), (34) and (35). All three modes have been plotted in the figure 2(c). A more detailed analysis of these modes as symmetric and anti-symmetric combinations of individual modes of the vortex lattice and the supersolid lattice is given in the next section.

### 3.4. Symmetric and antisymmetric combination of modes of a rotating ultracold supersolid

To understand the significance of the mode frequencies, we can rewrite the first collective mode frequency (33) \( \omega_1^2 = \omega_1^{\mu} + \omega_1^{\alpha} \) with \( \omega_1^{\mu} = [4\Omega^2 + (c_i^2 + c_m^2)q_i^2] \), \( \omega_1^{\alpha} = (c_m^2 + c_i^2)q_i^2 \). This is a symmetric combination of the square of modes corresponding to the vortex lattice and the supersolid lattice. In the limit of fast rotation and small wave vector, the second mode (34) can be written as
\[
\omega_2^2 = \frac{c_m^2 q_i^2}{\omega_1^2} \left[ \omega_1^2 - \omega_1^{\alpha} + \left( c_m^2 + c_i^2 \right) q_i^2 \right]. \quad (36)
\]

To understand this mode, in the limit of fast rotation and small \( q \) behavior, we set the simplifying assumption \( (c_m^2 + c_i^2 q_i^2)q_i^2 \ll 4\Omega^2 \) and hence the left hand side term can be dropped. This sets \( \omega_2^2 = \frac{c_m^2 q_i^2}{\omega_1^2} (\omega_1^{\mu} - \omega_1^{\alpha}) \). The same limit also ensures that the preceeding expression is always positive and will not lead to any instability. This expression gives an antisymmetric coupling between the square of the modes of the vortex and the supersolid lattice. For a more realistic situation one can readily calculate the modification of this expression by including neglected terms. The square of the normal mode frequencies \( \omega_1^{\mu} \) and \( \omega_1^{\alpha} \) can therefore be interpreted as a symmetric and antisymmetric combination of the square of the modes corresponding to the vortex lattice and the supersolid lattice. Since we do not have the expression of the corresponding eigenvectors, it is not possible to comment conclusively on the type of coupling between the oscillation of the vortex lattice and supersolid lattice in the real space corresponding to such modes. Nevertheless, the occurrence of such modes indeed signifies a coupled motion of the supersolid and vortex lattice. This coupled motion took place for the lowest order hydrodynamic Lagrangian (5) where there is no direct coupling between two lattice displacement fields, \( u^\alpha \) and \( u^\mu \).

Thus the general nature of our results within hydrodynamic approximations suggests its applicability to minimally provide signatures for the supersolidity in cold atomic systems. We hope this can be tested with more detailed numerical investigations with specific microscopic models in future. The same experimental techniques \[31, 33\] used to study the collective oscillation of vortex lattices in rapidly rotating superfluid may be implemented here. Namely one needs to perturb the system to induce a deformation in the co-existing supersolid and vortex lattice for the case of a rotating
supersolid. The oscillations of these lattices under these perturbations can be observed using the TOF expansion technique and the information about the modes can be extracted and compared with the well known Tkachenko modes for rotating superfluid. This will possibly provide a route for the confirmation of supersolidity in such ultracold atomic condensates.

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Appendix A. Derivation of the effective Lagrangian for a rotating supersolid

A.1. Homogenization technique for long-wave effective Lagrangian

Here we derive the coupled equations for the three fields \( \rho (r, t), u(\mathbf{r}, t) \) which is a function of \( u^s(\mathbf{r}, t) \) and \( u^v(\mathbf{r}, t) \) and \( \phi (r, t) \), following the method called homogenization. This technique splits the long-wave behavior of various parameters and the short range periodic dependence on the lattice parameters [45].

We use the ansatz for density and phase as

\[
n(r, t) = \rho_0 (r - u(\mathbf{r}, t) \dot{\phi}(\mathbf{r}, t)) + \ddot{\rho}(r, t), \rho, \tau + \ldots \tag{A.1}
\]

\[
\Phi(r, t) = \phi(r, t) + \dot{\phi}(r - u(\mathbf{r}, t), \rho, \tau, t) + \ldots \tag{A.2}
\]

Here, the displacement of the vortex lattice and the supersolid lattice enters the modulated density as \( \rho_0 (r - u(\mathbf{r}, t) \dot{\phi}(\mathbf{r}, t)) \rho(r, t) \). Also, \( \phi, u(u^s, u^v) \) and \( \rho \) are slowly varying fields and \( \dot{\phi} \) and \( \ddot{\rho} \) are small and fast varying periodic functions.

Now we calculate the gradients and time derivatives of various expressions which will be further used in the calculations.

\[
(\nabla n)_i = (\delta_i - \partial_i u^s - \partial_i u^v) \partial_i \rho_0 + \frac{\partial^2_0 \rho_0}{\partial \rho} \frac{\partial}{\partial x_i}, \tag{A.3}
\]

Next,

\[
\partial_t \phi = \partial_t \phi - \partial_i u^s \partial_i \phi - \partial_i u^v \partial_i \phi + \dot{\phi} + \frac{\partial \dot{\phi}}{\partial \rho} \tag{A.4}
\]

\[
(\nabla \phi)_i = (\nabla \phi)_i + (\delta_i - \partial_i u^s - \partial_i u^v) \partial_i \phi + \frac{\partial \dot{\phi}}{\partial \rho} (\nabla \rho)_i. \tag{A.5}
\]

Now keeping the relevant contributions for the long-wave description and calculating

\[
n \partial_t \Phi = (\rho_0 + \ddot{\rho}) \left( \partial_t \phi - \partial_i u^s \partial_i \phi - \partial_i u^v \partial_i \phi + \dot{\phi} + \frac{\partial \dot{\phi}}{\partial \rho} \right) \tag{A.6}
\]

(h.o.t) stands for higher order terms through out the calculations.

\[
(\nabla n)_i^2 = \left( (\delta_i - \partial_i u^s - \partial_i u^v) \partial_i \rho_0 + \frac{\partial^2_0 \rho_0}{\partial \rho} \frac{\partial}{\partial x_i} \right)^2 + \left( (\delta_i - \partial_i u^s - \partial_i u^v) \partial_i \dot{\rho} + \frac{\partial \dot{\rho}}{\partial \rho} \frac{\partial}{\partial x_i} \right)^2 + 2 \left( (\delta_i - \partial_i u^s - \partial_i u^v) \partial_i \rho_0 + \frac{\partial^2_0 \rho_0}{\partial \rho} \frac{\partial}{\partial x_i} \right) \times \left( (\delta_i - \partial_i u^s - \partial_i u^v) \partial_i \dot{\rho} + \frac{\partial \dot{\rho}}{\partial \rho} \frac{\partial}{\partial x_i} \right) \tag{A.7}
\]

We calculate this quantity (A.7) term by term as follows:

**Term 1:**

\[
\left( \delta_i - \partial_i (u^{s^2} + u^{v^2}) \right) \partial_i \rho_0 + \frac{\partial^2_0 \rho_0}{\partial \rho} \frac{\partial}{\partial x_i} \tag{A.8}
\]

\[
\left( \delta_i + \partial_i (u^{s^2} + u^{v^2}) \right)^2 - 2 \delta_{ii} \partial_i (u^{s^2} + u^{v^2}) \frac{\partial \rho_0}{\partial x_i} \frac{\partial \rho_0}{\partial x_k} \tag{A.9}
\]

\[
(\delta_{ii} - 2 \partial_{ii} u^{s^2} - 2 \partial_{ii} u^{v^2} + \partial_{ii} u^{s^2} + \partial_{ii} u^{v^2} + \partial_{ii} u^{s^2} + \partial_{ii} u^{v^2}) \frac{\partial \rho_0}{\partial x_i} \frac{\partial \rho_0}{\partial x_k} \tag{A.10}
\]

\[
\left( \delta_{ii} + 2 e^{s^2}_i + 2 e^{v^2}_i \right) \frac{\partial \rho_0}{\partial x_i} \frac{\partial \rho_0}{\partial x_k} + h.o.t \tag{A.11}
\]

where

\[
e^{s^2}_i = \frac{1}{2} \left( \partial_{ii} u^{s^2} + \partial_{ii} u^{s^2} \right) + \frac{1}{2} \partial_{ii} u^{s^2} \partial_{ii} u^{s^2} \tag{A.12}
\]

is the strain tensor for the supersolid lattice and,

\[
e^{v^2}_i = \frac{1}{2} \left( \partial_{ii} u^{v^2} + \partial_{ii} u^{v^2} \right) + \frac{1}{2} \partial_{ii} u^{v^2} \partial_{ii} u^{v^2} \tag{A.13}
\]

is the strain tensor for the vortex lattice [50].

**Term 2**

\[
\left( \delta_{ii} - \partial_{ii} u^{s^2} + \partial_{ii} u^{v^2} \right)^2 \frac{\partial \rho_0}{\partial x_i} \frac{\partial \rho_0}{\partial x_k} \tag{A.14}
\]

\[
+ \left( \partial \rho \frac{\partial \rho}{\partial x_i} \right) \left( \partial \rho \frac{\partial \rho}{\partial x_k} \right) + 2 \left( \delta_{ii} - \partial_{ii} u^{s^2} + \partial_{ii} u^{v^2} \right) \frac{\partial \rho \partial \rho}{\partial x_i} \frac{\partial \rho \partial \rho}{\partial x_k} \tag{A.15}
\]

\[
= \delta_{ii} \left( \frac{\partial \rho}{\partial x_i} \right)^2 + h.o.t \tag{A.16}
\]

\[
= (\partial \rho)^2 + h.o.t. \tag{A.17}
\]

As mentioned earlier, in order to keep the relevant terms for long-wavelength description, terms which are quadratic in
fast varying variable $\dot{\rho}$ multiplied by other derivatives are ignored.

**Term 3**

\[
2 \left( (\delta_{ik} - \partial_t u^k + \partial_t u^k) \partial_t \rho_0 + \frac{\partial \rho_0}{\partial \rho} \frac{\partial \rho}{\partial x_i} \right) \times \left( (\delta_{ik} - \partial_t u^k + \partial_t u^k) \partial_t \dot{\rho} + \frac{\partial \rho_0}{\partial \dot{\rho}} \frac{\partial \dot{\rho}}{\partial x_i} \right)
\]
\[
= 2(\delta_{ik} - \partial_t u^k + \partial_t u^k) \frac{\partial \rho_0}{\partial x_i} \frac{\partial \rho}{\partial x_k} + h.o.t
\]
\[
\simeq 2(\delta_{ik} + 2e^i_k + 2e^k_i) \partial_t \rho \partial_t \dot{\rho} + h.o.t. \quad (A.12)
\]

Substituting equations (A.8), (A.11), (A.12) into equation (A.7), we get

\[
(Vn)^2 = (\delta_{ik} + 2e^i_k + 2e^k_i) \partial_t \rho \partial_t \rho_0 + 2(\delta_{ik} + 2e^i_k + 2e^k_i) \partial_t \rho \partial_t \dot{\rho}
\]
\[
+ (\partial_t \dot{\rho})^2 + h.o.t. \quad (A.13)
\]

Next we calculate

\[
(V\Phi)^2 = \left[ (V\rho)^2 + (\delta_{ik} - (\partial_t u^k + \partial_t u^k)) \frac{\partial \Phi}{\partial x_i} + \frac{\partial \Phi}{\partial \rho} (V\rho) \right]^2
\]
\[
= (\partial_t \Phi)^2 + (\Phi)^2
\]
\[
+ 2(\delta_{ik} - (\partial_t u^k + \partial_t u^k)) \partial_t \Phi \partial_t \dot{\Phi} + h.o.t. \quad (A.14)
\]

The higher order terms are the terms quadratic in fast varying variable $\dot{\Phi}$ multiplied by other derivatives, which we again neglect in the long-wavelength description. The next term is

\[
n V\Phi = (\rho_0 + \dot{\rho})
\]
\[
\times \left[ (V\Phi) + (\delta_{ik} - (\partial_t u^k + \partial_t u^k)) \frac{\partial \Phi}{\partial x_i} + \frac{\partial \Phi}{\partial \rho} (V\rho) \right]
\]
\[
= \rho_0 V\Phi + \rho_0 (\delta_{ik} - (\partial_t u^k + \partial_t u^k)) \frac{\partial \Phi}{\partial x_i}
\]
\[
+ \rho_0 \frac{\partial \Phi}{\partial \rho} (V\rho) + h.o.t. \quad (A.15)
\]

Before going into the calculation for the non-local interaction term, we calculate the first and second term of Lagrangian (3) and label their contribution to the corresponding energy part of the Lagrangian, which is explained later.

In the first term $L_1$ we use equation (A.6) and get the following expression

\[
L_1 = -\int \frac{\hbar}{2m} \frac{\partial \Phi}{\partial t} dr
\]
\[
= -\int \frac{\hbar}{2m} \partial_t \rho \partial_t \Phi \left[ \rho_0 - \int_{L_q} \rho_0 \partial_t u^i \partial_t \Phi dr + \int_{L_q} \rho_0 \partial_t u^i \partial_t \Phi dr \right]
\]
\[
+ \int_{L_q} \rho_0 \partial_t u^i \partial_t \Phi \left[ \int_{L_q} \partial_t \rho \partial_t \Phi dr \right] dr.
\]
\[
\text{neglected} \quad (A.16)
\]

The second term of Lagrangian (3) is calculated using (A.14) as

\[
L_2 = -\int \frac{\hbar^2}{2m} (V\Phi)^2 dr
\]
\[
= -\frac{\hbar^2}{2m} \int (V\Phi)^2 \rho_0 (r) dr
\]
\[
+ \frac{\hbar^2}{m} \int (\partial_t u^i + \partial_t u^i) \partial_t \rho \partial_t \rho_0 (r) dr
\]
\[
- \frac{\hbar^2}{m} \int V\Phi \cdot \nu \rho_0 (r) dr
\]
\[
- \frac{\hbar^2}{2m} \int (V\Phi)^2 \rho_0 (r) dr. \quad (A.17)
\]

Considering the non-local term now, given by

\[
N(\rho (r), \rho (r')) = \frac{1}{2} \int U(|r - r'|) \rho (r - u^m (r) - u^m (r')) \rho (r - u^m (r') - u^m (r')) dr dr'. \quad (A.18)
\]

**STEPS**

(1) Using the change of variables, $R = r - u (r)$ and, $R' = r' - u (r')$, we can determine

\[
dR = d \left( r - \left( u^m (r) + u^m (r) \right) \right)
\]

with

\[
(dR)_i = d \left( r - \left( u^m (r) + u^m (r) \right) \right)_i = \left[ dx_i - \frac{\partial \left( u_i^m + u_i^m \right)}{\partial x_k} dx_k \right]
\]

which implies

\[
|dR| = |dR| \cdot |dR|
\]
\[
= \left( \delta_{ik} - 2(\partial_t u_i^m + \partial_t u_i^m) + (\partial_t u_i^m + \partial_t u_i^m) \right) dx_i dx_k
\]
\[
= \left( \delta_{ik} + 2e^i_k + 2e^k_i \right) dx_i dx_k \quad (A.19)
\]

with strain tensors $e^i_k$ and $e^k_i$ defined in equations (A.9) and (A.8). Similarly,

\[
|dR| = \left| \delta_{ik} + 2e^i_k + 2e^k_i \right| dx_i' dx_k'. \quad (A.20)
\]

(2) Any integral with argument $r - u (r)$ may be transformed to

\[
\int Q (r - u (r)) dr = \int Q (R) dr
\]
\[
= \int Q (R) \sqrt{det (\delta_{ik} + 2e^i_k + 2e^k_i)} \quad (A.21)
\]
\[ \approx \int Q(r) (1 - e_{ik} - e_{ik}^\prime) \, dr. \] (A.22)

The step (A.21) in equation (A.22) is obtained by using equation (A.19).

(3) Relative distance

\[ \Delta R = R - R' = \Delta r - \Delta (u'(r) + u''(r)) \] (A.23)

where \( \Delta r = r - r' \). The above equation (A.23) implies

\[ |\Delta R|^2 \approx |\Delta r|^2 + |\Delta u(r)|^2 = 2\Delta r \cdot \Delta u(r) = |\Delta r|^2 + 2e_{ik}^\prime \Delta x_i \Delta x_k + 2e_{ik}^\prime \Delta x_i \Delta x_k. \]

Thus,

\[ |\Delta r| \approx \Delta R - \frac{(e_{ik} + e_{ik}^\prime)\Delta x_i \Delta x_k}{\Delta R}. \] (A.24)

The final result of the non-local term given by (A.18) can thus be calculated as

\[ N(\rho(r), \rho(r')) = \frac{1}{2} \int \left( |\Delta R| - (e_{ik} + e_{ik}^\prime) \Delta x_i \Delta x_k/|\Delta R| + \ldots \right) \rho(\Delta R) \, dR \times \frac{\sqrt{\det(\delta_{ik} + 2e_{ik}^\prime + 2e_{ik})}}{\rho(\Delta R') \, dR'} \times \frac{\sqrt{\det(\delta_{ik} + 2e_{ik}^\prime + 2e_{ik}^\prime)}}{\rho(\rho') \, dR'} \]
\[ \approx \frac{1}{2} \int \left( (1 - (e_{ik} + e_{ik}^\prime)) - (e_{ik} + e_{ik}^\prime) \right) f_{\rho}(r - r') \rho(\rho(r') \, dR' + \ldots) \rho(\rho(r') \, dR' \). \] (A.25)

Here, \( f_{\rho}(r - r') = (x_i - x'_i)(x_k - x'_k) u(r - r') \). The first and second term of the Lagrangian have already been calculated. Here we determine the third and fourth terms by substituting the ansatz in (A.1) and (A.2) in the Lagrangian (3). The third term of the Lagrangian (3) is calculated using (A.13) and (A.22) as

\[ L_3 = -\frac{\hbar^2}{2m} \int \frac{1}{4n} (\nabla n^2) \, dr \]
\[ = -\frac{\hbar^2}{2m} \int \left[ (\delta_{ik} + 2e_{ik}^\prime + 2e_{ik}^\prime) \partial_{\rho_0} \partial_{\rho_0} \rho_0 + 2(\delta_{ik} + 2e_{ik}^\prime + 2e_{ik}^\prime) \partial_{\rho_0} \partial_{\rho_0} \partial_{\rho_0} + (\partial_{\rho_0} \rho_0)^2 + h.o.t \right] \times \left( \frac{1}{4\rho_0(r)} + \frac{\bar{\rho}(r)}{4\rho_0^2(r)} \right) (1 - e_{ik} - e_{ik}^\prime) \, dr \]
\[ = -\frac{\hbar^2}{8m} \int \frac{(V\rho_0)^2}{\rho_0} \, dr + \frac{\hbar^2}{8m} \int \frac{(V\rho_0)^2}{\rho_0} (e_{ik} + e_{ik}^\prime) \, dr \]
\[ - \frac{\hbar^2}{4m} \int \left( e_{ik} + e_{ik}^\prime \right) \frac{\partial \rho_0}{\partial \rho_0} \frac{\partial \rho_0}{\partial \rho_0} (1 - e_{ik} - e_{ik}^\prime) \, dr \]
\[ - \frac{\hbar^2}{4m} \int \frac{\partial \rho_0}{\partial \rho_0} \, dr \]
\[ - \frac{\hbar^2}{2m} \int \left( e_{ik} + e_{ik}^\prime \right) \frac{\partial \rho_0}{\partial \rho_0} \, dr \]
\[ - \frac{\hbar^2}{8m} \int \frac{(V\bar{\rho})^2}{\rho_0} \, dr + \frac{\hbar^2}{8m} \int \frac{(V\rho_0)^2}{\rho_0^2} \, dr + \ldots \] (A.26)

Now, the fourth term of the Lagrangian (3) is calculated using (A.25) as

\[ L_4 = -\frac{1}{2} \int U(|r - r'|) n(r)n(r') \, dr dr' \]
\[ = -\frac{1}{2} \int \left( 1 - (e_{ik} + e_{ik}^\prime) - (e_{ik} + e_{ik}^\prime) \right) U(|r - r'|) \]
\[ \times (\rho_0(r) \rho_0(r') \rho_0(r) \rho_0(r') + \bar{\rho}(r) \rho_0(r') \rho_0(r) \rho_0(r') + \bar{\rho}(r) \rho_0(r') \rho_0(r) \rho_0(r') + \bar{\rho}(r) \rho_0(r') \rho_0(r) \rho_0(r')) \, dr dr' \]
\[ = -\frac{1}{2} \int L_{\rho} U(|r - r'|) \rho_0(r) \rho_0(r') \, dr dr' \]
\[ - \frac{1}{2} \int L_{\rho} U(|r - r'|) \bar{\rho}(r) \bar{\rho}(r') \, dr dr' \]
\[ - \frac{1}{2} \int L_{\rho} U(|r - r'|) \rho_0(r) \rho_0(r') \, dr dr' \]
\[ \left[ -\frac{1}{2} \int \left[ (e_{ik} + e_{ik}^\prime) f_{\rho}(r - r') \right] \right] \]
\[ \times \left( \rho_0(r) \bar{\rho}(r') + \bar{\rho}(r) \rho_0(r') \right) \, dr dr' \]
\[ \left[ -\frac{1}{2} \int \left[ (e_{ik} + e_{ik}^\prime) f_{\rho}(r - r') \right] \right] \]
\[ \times \left( \rho_0(r) \bar{\rho}(r') + \bar{\rho}(r) \rho_0(r') \right) \, dr dr'. \] (A.27)

So, adding and collecting all the terms, we get following five kinds of terms

\[ L = L_{\rho} + L_{\phi} + L_u + L_{\phi} + L_{\beta}. \] (A.28)
(1) $L_ρ$ is the internal energy part, which only depends on $ρ_0(r)$ which is slowly varying, and is given by

$$L_ρ = -\frac{ℏ^2}{8m} \int (Vρ_0)^2 \rho_0 \,dr$$

$$-\int U(|r-r'|)ρ_0(r)ρ_0(r') \,dr'dr'$$ \hspace{1cm} (A.29)

(2) $L_u$ is the hydrodynamical part I, which mixes the slowly varying phase $ϕ(r, t)$ and the slowly varying density $ρ_0(r)$, and is given below

$$L_u = -\int \left( ℏ \partial_ϕ ϕ + \frac{ℏ^2}{2m}(Vϕ)^2 \right) ρ_0(r) \,dr.$$ \hspace{1cm} (A.30)

The term in the integral is the Lagrangian density and we obtain an average energy density that depends on parameter $ϕ$ only, shown below as

$$E(ϕ) = \frac{1}{V} \int \left( ℏ \partial_ϕ ϕ + \frac{ℏ^2}{2m}(Vϕ)^2 \right) ρ_0(r) \,dr \approx \left( ℏ \partial_ϕ ϕ + \frac{ℏ^2}{2m}(Vϕ)^2 \right) ρ_0(r).$$ \hspace{1cm} (A.31)

Thus, the equation (A.30) when averaged directly looks like

$$L_ϕ = -\int \left( ℏ \partial_ϕ ϕ + \frac{ℏ^2}{2m}(Vϕ)^2 \right) ρ_0(r) \,dr.$$ \hspace{1cm} (A.32)

(3) $L_u$ is the elastic part I, given by

$$L_u = -\frac{ℏ^2}{4m} \int \left( \frac{ℏ}{ρ_0} \right)^2 \rho_0 \,dr$$

$$+ \frac{ℏ^2}{8m} \int \left( \frac{ℏ}{ρ_0} \right)^2 \rho_0 \left( 1 - ε_σ^0 - ε_τ^0 \right) \,dr$$

$$+ \frac{1}{2} \int \left( \frac{ℏ}{ρ_0} \right)^2 \rho_0 \left( r - r' \right)$$

$$+ \left( ε_σ^0 + ε_τ^0 + ε_σ^0 + ε_τ^0 \right) \times \int U(|r-r'|) \rho_0(r) \rho_0(r') \,dr'dr'.$$ \hspace{1cm} (A.33)

It can be averaged directly. However, it involves both quadratic and linear terms. They can be grouped and simplified, and hence the elastic part I of the Lagrangian reduces to,

$$L_u = \int \left( \frac{ℏ}{ρ_0} \right)^2 \rho_0 \,dr + \int \left( \frac{ℏ}{ρ_0} \right)^2 \rho_0 \,dr$$ \hspace{1cm} (A.34)

where $ε_σ^{(2)}$ is the elastic constant entering through the quadratic term, and is given by

$$ε_σ^{(2)} = \frac{1}{V} \int \frac{ℏ^2}{2m} \partial_ρ \partial_ρ \rho_0 \,dr$$

and

$$μ = \frac{ℏ^2}{4m} \left( \frac{(Vρ_0)^2}{2ρ_0^2} - \frac{V^2ρ_0}{ρ_0} \right) + \int U(|r-r'|) \rho_0(r') \,dr'.$$ \hspace{1cm} (A.35)

The chemical potential defined in equation (A.35) is for the usual Gross–Pitaevskii equation with long-range interaction. $ρ_0$ is the ground state density in terms of which the chemical potential is defined.

(4) $L_ϕ$ is the hydrodynamical part II, given as

$$L_ϕ = ℏ \int \partial_ρ \left( \partial_ρ ϕ + ϕ + \partial_ρ ϕ \right)$$

$$+ \frac{ℏ}{m} \partial_ρ \left( \partial_ρ ϕ + ϕ \right)$$

$$- \frac{ℏ}{m} Vϕ \cdot \vec{V}ϕ - \frac{ℏ}{2m} \left( Vϕ \right)^2 \,dr.$$ \hspace{1cm} (A.36)

Now, the above equation (A.36) can be rewritten as

$$L_ϕ = -\frac{ℏ^2}{2m} \int \left( 2\partial_ρ A' \cdot \vec{V}ϕ + 2\partial_ρ A' \cdot \vec{V}ϕ + \partial_ρ \left( Vϕ \right)^2 \right) \,dr$$

with

$$A' = \left( \vec{V}ϕ - \left( \vec{V}ϕ \cdot \vec{V} \right) u^σ - \frac{m}{ℏ} \partial_ρ \vec{u}^σ \right)$$ \hspace{1cm} (A.37)

and,

$$A' = -\left( \vec{V}ϕ - \left( \vec{V}ϕ \cdot \vec{V} \right) u^σ - \frac{m}{ℏ} \partial_ρ \vec{u}^σ \right).$$ \hspace{1cm} (A.38)

The Euler–Lagrange condition for this part of Lagrangian $L_ϕ$ is

$$A' \cdot \vec{V}ϕ + A' \cdot \vec{V}ϕ + \vec{V} \left( ρ_0 \vec{V}ϕ \right) = 0.$$ \hspace{1cm} (A.39)

Solving this equation for $ϕ$, we get $ϕ = K_i(A'_i + A'_i)$ with $K_i$ a periodic function [45] which satisfies $Vϕ_i + V \cdot (ρ_0 K_i) = 0$. The above contribution to the Lagrangian can be written in simplified form as [45]

$$L_ϕ = \frac{ℏ^2}{2m} \int \rho_0 A'_i A'_i \,dr$$

with $ρ_0$ the tensor which for symmetric crystal structures is $ρ_0 = ρ_0$, defined as

$$ρ_0 = \frac{1}{V} \int \rho_0(r) \left( VK_i \cdot VK_i \right) \,dr.$$
where the terms which are quadratic in the gradients of \( \vec{\rho} \) are the relevant terms because the terms linear in \( \vec{\rho} \) disappear and the action is at minimum when \( n = \rho_0(\vec{r}) \) (see equation (A.1)).

Also, the line (A.40) in the above equation is equal to \( -\mu \int \vec{\rho}(\vec{r}) \) dr. Thus, keeping only relevant terms as below:

\[
\begin{align*}
\frac{\hbar^2}{4m} \lambda \cdot \left( \frac{\vec{V}\vec{\rho}}{\rho_0} \right) - \int U(\vec{r} - \vec{r'}) |\vec{\rho}(\vec{r'})|\, d\vec{r'}
\end{align*}
\]

\[
= \mu(\epsilon_{ik} + \epsilon_{ik}) + \frac{\hbar^2}{4m} \left( \frac{\partial_\rho \partial_\rho \rho_0}{\rho_0^2} - 2 \frac{\partial_\rho \rho_0}{\rho_0} \right) + \epsilon_{ik} \int \left( f_{ikl}(\vec{r} - \vec{r'}) + 2\delta_{ik} U(\vec{r} - \vec{r'}) \right) \rho_0(\vec{r'})\, d\vec{r'}.
\]

The solution of the above equation is the periodic function \( E_{ik}(\vec{r}) \) [45, 51] and of the form \( \vec{\rho} = \epsilon_{ik} E_{ik}(\vec{r}) \) [47]. Putting in expression (A.42) and adding the expression (A.34) we get

\[
L_u + L_\rho = \frac{1}{2} \int \left( \lambda_{iklm} \epsilon_{ik} \epsilon_{lm} + \lambda_{iklm} \epsilon_{ik} \epsilon_{lm} \right) \, d\vec{r'}
\]

where \( \lambda_{iklm} \) is given by

\[
\lambda_{iklm} = -\frac{1}{V} \int \frac{\hbar^2}{2m} \frac{\partial_\rho \partial_\rho \rho_0 \delta_{lm}}{\rho_0} \, d\vec{r}
\]

\[
- \frac{1}{V} \int \mu(\delta_{ik} E_{ik}'(\vec{r}) + \delta_{lm} E_{lm}'(\vec{r})) \, d\vec{r}
\]

\[
- \frac{1}{V} \int \left( \frac{\hbar^2}{4m} \frac{1}{\rho_0} (VE_{ik}' \cdot VE_{lm}') \right) \, d\vec{r}
\]

\[
+ \int U(\vec{r} - \vec{r'}) E_{ik}(\vec{r}) E_{lm}(\vec{r'}) \, d\vec{r}'.
\]

(44)

and, \( \lambda_{iklm} = \frac{1}{V} \int \frac{\hbar^2}{2m} \frac{\partial_\rho \partial_\rho \rho_0 \delta_{lm}}{\rho_0} \, d\vec{r}
\]

\[
- \frac{1}{V} \int \mu(\delta_{ik} E_{ik}'(\vec{r}) + \delta_{lm} E_{lm}'(\vec{r})) \, d\vec{r}
\]

\[
- \frac{1}{V} \int \left( \frac{\hbar^2}{4m} \frac{1}{\rho_0} (VE_{ik}' \cdot VE_{lm}') \right) \, d\vec{r}
\]

\[
+ \int U(\vec{r} - \vec{r'}) E_{ik}(\vec{r}) E_{lm}(\vec{r'}) \, d\vec{r}'.
\]

The expression \( \frac{1}{2} \lambda_{iklm} \epsilon_{ik} \epsilon_{lm} \) is the expression for the elastic energy density of a solid, and \( \lambda_{iklm} \) is the elastic modulus tensor [50].

Hence, we can write the effective Lagrangian for the long-wave perturbations of displacement of both lattices, of average density and of the phase as the sum of various contributions mentioned above.

\[
L_{\text{eff}} = \int \left[ -\hbar^2 \frac{\partial^2 \phi}{\partial t^2} - \frac{\hbar^2}{2m} \rho (\nabla \phi)^2 - \rho \phi^2 - \rho \phi^2 \right. \]

\[
\left. \times \left( V \phi - \frac{m}{\hbar} Du^{\phi} \right) \right] \cdot \left( V \phi - \frac{m}{\hbar} Du^{\phi} \right) \, d\vec{r}
\]

\[
- \mathcal{E}(\rho) = \frac{1}{2} \lambda_{iklm} \epsilon_{ik} \epsilon_{lm} - \frac{1}{2} \lambda_{iklm} \epsilon_{ik} \epsilon_{lm} - m \rho \nabla \cdot (\Omega \times \mathbf{r})
\]

(46)

where

\[
\frac{Du^{\phi}}{Dt} = \frac{\partial u^{\phi}}{\partial t} + \frac{\hbar}{m} \nabla \phi \cdot \nabla u^{\phi}.
\]

(47)

The above Lagrangian can also be written as

\[
L_{\text{eff}} = \int \left[ -\hbar^2 \frac{\partial^2 \phi}{\partial t^2} - \mathcal{E} \right]
\]

where

\[
\mathcal{E} = \mathcal{E}_w(\rho) + \mathcal{E}_{\phi}(\phi) + \mathcal{E}_{uu}(\nabla u^{\phi}) + \mathcal{E}_{uu}(\nabla u^{\phi})
\]

(48)

with

\[
\mathcal{E}_w(\rho) = \frac{\hbar^2}{2m} \frac{(V\phi)^2}{4\rho} + \rho \int U(\vec{r} - \vec{r'}) |\phi(\vec{r'})|\, d\vec{r'} - \mu \rho \phi
\]

(49)

\[
\mathcal{E}_{\phi}(\phi) = \frac{\hbar^2}{2m} \rho (\nabla \phi)^2 - \rho \phi^2 - \rho \phi^2 \times \left( \frac{V \phi}{\hbar} - \frac{m}{\hbar} Du^{\phi} \right)
\]

\[
\left( \frac{V \phi}{\hbar} - \frac{m}{\hbar} Du^{\phi} \right)
\]

(50)

\[
\mathcal{E}_{uu}(\nabla u^{\phi}) = \frac{1}{2} \lambda_{iklm} \epsilon_{ik} \epsilon_{lm}
\]

(51)

\[
\mathcal{E}_{uu}(\nabla u^{\phi}) = \frac{1}{2} \lambda_{iklm} \epsilon_{ik} \epsilon_{lm}
\]

(52)
equations (A.51) and (A.52) have no contribution and similarly, the second and third terms in equation (A.50) vanish along with the long-range interaction term in equation (A.49), hence recovering the Lagrangian for cold superfluids.

It can be clearly seen from the above equation (A.46) that the crystal lattice may have a different velocity than the superfluid component, with the velocity difference proportional to \( \frac{\partial u^{\text{ss}}}{\partial \mathbf{r}} \), with \( u^{\text{ss}} \) as the displacement field of the crystal lattice due to density modulations in supersolid. Hence the third term in equation (A.46) gives the product of the mass density of the supersolid lattice and the square of the supersolid lattice velocity.

Here \( \rho^{\text{ss}}_{ik} \) is the superfluid density tensor \([48]\), which is in general a symmetric matrix. In our further calculations, we express that the superfluid density is a function of local number density \( \rho \) and for isotropic symmetry of lattice, it is given by \( \rho^{\text{ss}}_{ik} = \rho^{\text{ss}} (\rho) \delta_{ik} \).

It may be noted from the structure of the proposed Lagrangian that we do not take into account coupling of the two lattices with displacement fields \( u^{\text{ss}}, u \) in the lowest order expansion and thus the elastic deformations of the two lattices do not interact with each other directly.

### A.2. Hydrodynamic equations of motion for a rotating supersolid

Here we provide the detailed derivations of the hydrodynamic equation of a rotating supersolid that appears in the main paper through the extremization of the hydrodynamic Lagrangian. The dynamical equations are derived by variation of action \( S = \int L \, dt \) taken as a functional of \( \rho, \phi, u^{\text{ss}}, \mathbf{v} \). This yields a set of coupled partial differential equations for those fields. The action to be extremized is \( S = \int L \, dt \), which gives the condition

\[
\delta \int L \, dt = 0
\]

where \( L = L(\rho, \phi, \mathbf{V} u^{\text{ss}}, \mathbf{V} \mathbf{v}) \), which implies

\[
\frac{\partial L}{\partial \rho} d \rho + \frac{\partial L}{\partial \phi} d \phi + \frac{\partial L}{\partial (\mathbf{V} u^{\text{ss}})} d (\mathbf{V} u^{\text{ss}}) + \frac{\partial L}{\partial (\mathbf{V} \mathbf{v})} d (\mathbf{V} \mathbf{v}) = 0. \tag{A.53}
\]

We calculate

\[
\frac{\partial L}{\partial \rho} = -\hbar \frac{\partial \phi}{\partial t} - \frac{\mathbf{V}^2 \rho}{2 \rho} - \frac{(\mathbf{V} \phi)^2}{4 \rho^2} \frac{\hbar^2}{2 m} - \int U (|\mathbf{r} - \mathbf{r}'|) \rho (\mathbf{r}') \, d\mathbf{r}' - \frac{\hbar^2}{2 m} (\mathbf{V} \phi)^2 + m \mathbf{v} \cdot (\Omega \times \mathbf{r}) = 0.
\]

Taking gradient on both sides,

\[
\frac{\partial \mathbf{v}_i}{\partial t} + \nabla \left( \frac{1}{2} \mathbf{v}_i^2 \right) = -\nabla \left[ T + V + g \rho + \int U (|\mathbf{r} - \mathbf{r}'|) \rho (\mathbf{r}') \, d\mathbf{r}' \right. - \left. \mathbf{v} \cdot (\Omega \times \mathbf{r}) \right] \tag{A.54}
\]

where \( \mathbf{v} \) is the superfluid velocity defined as \( \mathbf{v} = \frac{\hbar}{m} \mathbf{V} \phi \).

Next,

\[
\frac{\partial L}{\partial \phi} = -\hbar \frac{\partial \phi}{\partial t} - \hbar \frac{\partial \mathbf{v}}{\partial \phi} \left[ \mathcal{E}_{\text{ph}} (\phi) \right] = 0. \tag{A.55}
\]

We determine the second term in above equation separately,

\[
\frac{\partial}{\partial \phi} \left[ \mathcal{E}_{\text{ph}} (\phi) \right] = \frac{\partial}{\partial \phi} \left[ \frac{\hbar^2}{m} \mathbf{V} \phi \right] - \frac{\partial}{\partial \phi} \left[ \left( \rho \delta_{ik} - \rho^{\text{ss}}_{ik} \right) \left( \frac{m}{\hbar} \mathbf{u} \right) \right]
\]

\[
\times \left( \mathbf{V} \phi - \frac{m}{\hbar} \mathbf{u} \right) \left( \mathbf{V} \phi - \frac{m}{\hbar} \mathbf{u} \right) \right]
\]

\[
- \phi \frac{\partial}{\partial \phi} \left[ \rho \cdot (\Omega \times \mathbf{r}) \right] \]

\[
\frac{1}{2} \rho \frac{\partial}{\partial \phi} \left[ \rho \cdot (\Omega \times \mathbf{r}) \right]. \tag{A.56}
\]

Putting the above expression for \( \frac{\partial}{\partial \phi} \left[ \mathcal{E}_{\text{ph}} (\phi) \right] \) in equation (A.55), we get

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \left[ \frac{\hbar}{m} \mathbf{V} \phi \right]
\]

\[
+ \frac{\partial}{\partial x_i} \left[ \left( \rho - \rho^{\text{ss}} \right) \left( \delta_{ik} - \partial_{ik} u^{\text{ss}} \right) \left( \partial_{ik} \frac{\hbar}{m} \phi \right) \right]
\]

\[
- \frac{\partial}{\partial x_i} \left[ \rho \cdot (\Omega \times \mathbf{r}) \right] = 0.
\]

Equations (A.54) and (A.57) are the modified Euler and continuity equation for a condensate rotating at an angular frequency \( \Omega \). The energies in the laboratory and rotating frame are related by \( E_R = E \cdot \mathbf{r} \times \mathbf{p} \). The transformation to a rotating frame of reference introduces the \( \mathbf{v} \cdot \Omega \times \mathbf{r} \) term and \( \rho \cdot \Omega \times \mathbf{r} \) term in equation (A.54) and (A.57) respectively [52]. Here, \( \mathbf{v} \) is the superfluid velocity in the lab (inertial) frame of reference. In the rotating frame of reference, these equations along with the equations of elastic response of the system due to the supersolid lattice and the
vortex lattice are given as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \frac{h}{m} \nabla \phi \right) + \frac{\partial}{\partial x_k} \left[ (\rho - \rho^s)(\delta_{ik} - \partial_k u_i^s) \left( \dot{u}_i^s - \frac{h}{m} \partial_i \phi \right) \right] = 0 \tag{A.58}
\]

and,

\[
m \left( \frac{\partial \phi}{\partial t} + 2 \Omega \times \mathbf{v}_L \right) = -\frac{-\nabla P'}{\rho}. \tag{A.59}
\]

Here, \( \rho^s_{ik} \) is the superfluid density tensor [48] which assumes the form \( \rho^s_{ik} = \rho^{ss} \delta_{ik} \) for isotropic symmetry of the system and,

\[
P' = \rho \left( T + \int U(\rho - \rho') \rho(\rho') d\mathbf{r}' \right). \tag{A.60}
\]

In the equation (A.59), we have kept only the linearized terms and the nonlinear terms with higher orders of derivatives have been dropped. The neglected terms in equation (A.59) are given below.

\[
\frac{\partial}{\partial \rho} \left( \rho - \rho^s \right) \left( \nabla \phi - \frac{m}{h} \frac{D \mathbf{u}^s}{Dt} \right)^2 = \left( \frac{\rho^2 - 1}{\rho} - \nabla \cdot \mathbf{u}^s \right) \left( \nabla \phi - \frac{m}{h} \frac{D \mathbf{u}^s}{Dt} \right)^2 = \left( \frac{\rho^2 - 1}{\rho} - \nabla \cdot \mathbf{u}^s \right) \left( \nabla \phi \right)^2 + \left( \frac{m}{h} \right)^2 \left( \frac{D \mathbf{u}^s}{Dt} \right)^2 - 2 \frac{m}{h} \nabla \phi \cdot \left( \frac{D \mathbf{u}^s}{Dt} \right).
\]

When averaged over a vortex lattice cell, equation (A.59) can be written as

\[
m \left( \frac{\partial \phi}{\partial t} + \hat{\omega} \times \mathbf{v}_L \right) = -\frac{-\nabla P'}{\rho} \tag{A.61}
\]

with \( \mathbf{v}_L \) as the averaged velocity and \( \hat{\omega} = 2 \Omega + \nabla \times \mathbf{v}_L \) as the averaged vorticity [38]. The velocity of the vortex is given by \( \mathbf{v}_L \) and it is equal to the time derivative of the displacement vector of the vortex lattice \( \mathbf{u}' \).

The force \( \mathbf{f} \) acting per unit volume of the fluid moving with velocity \( \mathbf{v}_L \) is

\[
\mathbf{f} = \frac{m}{h} \frac{D \mathbf{u}^s}{Dt} - \rho \hat{\omega} \times (\mathbf{v}_L - \mathbf{v}_L) \tag{A.62}
\]

and it should be connected with a variation of energy due to vortex displacements. Thus,

\[
\mathbf{f} = \frac{\partial E^s_{\mathbf{v}}}{\partial t} = -\frac{\partial}{\partial x_i} \left( \frac{\partial E^s_{\mathbf{v}}}{\partial x_i} \right) = -\left( \lambda^s + \mu_s^s \right) \nabla \left( \nabla \cdot \mathbf{u}^s \right) + \mu_s^s \nabla^2 \mathbf{u}^s. \tag{A.63}
\]

Hence using equations (A.62) and (A.63) we get

\[
\rho \{ \hat{\varepsilon} \times (\mathbf{v}_L - \mathbf{v}_L) \} = \left( \lambda^s + \mu_s^s \right) \frac{\nabla \cdot \mathbf{u}^s}{m} + \mu_s^s \frac{\nabla^2 \mathbf{u}^s}{m}. \tag{A.64}
\]

where \( \lambda^s = K^s - \frac{2}{3} \mu^s \) is the Lamé coefficient, and \( K^s \) and \( \mu^s \) are the compressibility and shear modulus of the vortex lattice. Equation (12) is the equation of motion of the system due to the elastic response of the vortex lattice.

Next we determine the equation of motion due to the elastic response of the supersolid crystal lattice by considering the fourth term in (A.53). Finally,

\[
\frac{\partial \mathcal{L}}{\partial \mathbf{u}_i} = m \frac{\partial}{\partial t} \left[ (\rho \delta_{ik} - \rho_i^s) \left( \dot{u}_i^s - \frac{h}{m} \partial_i \phi \right) \right] + \frac{h}{m} \frac{\partial}{\partial x_i} \left[ (\rho \delta_{ik} - \rho_i^s) \left( \dot{u}_i^s - \frac{h}{m} \partial_i \phi \right) \right] \partial_s \phi + \frac{\partial}{\partial x_i} \left( \lambda_{\alpha \beta s}^s e_{s \beta}^s \right) = 0. \tag{A.65}
\]

Thus,

\[
m \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial \mathbf{u}^s} \right] = m \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial \mathbf{u}^s} \right] + \frac{h}{m} \frac{\partial}{\partial x_i} \left[ \frac{\partial}{\partial \mathbf{u}^s} \right] \partial_s \phi + \frac{\partial}{\partial x_i} \left( \lambda_{\alpha \beta s}^s e_{s \beta}^s \right) = 0. \tag{A.66}
\]

Here too, when the lattice is assumed to be isotropic then the above equation (A.66) can be written as

\[
m \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial \mathbf{u}^s} \right] = m \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial \mathbf{u}^s} \right] + \frac{h}{m} \frac{\partial}{\partial x_i} \left[ \frac{\partial}{\partial \mathbf{u}^s} \right] \partial_s \phi + \left( \lambda^s + \mu_s^s \right) \partial_s \mathbf{u}^s + \mu_s^s \nabla^2 \mathbf{u}^s = 0. \tag{A.67}
\]

where \( \lambda_s = K^s - \frac{2}{3} \mu_s^s \) is the second Lamé coefficient, and \( K^s \) and \( \mu_s^s \) are the compressibility and shear modulus of the solid [50].

Thus, equations (A.58), (A.61), (A.64) and (A.67) are the equations of motion for a rotating supersolid with elastic properties of both vortex lattice and supersolid crystal lattice taken into account.
This set of four equations forms the hydrodynamic equations of motion for a rotating supersolid system.

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\[ \frac{\partial}{\partial t} \left( \rho \rho' \right) \left( \hat{u}_i' - \frac{\hbar}{m} \partial \phi \right) + \hbar \frac{\partial}{\partial \lambda_k} \left( \rho \rho' \right) \left( \hat{u}_i' - \frac{\hbar}{m} \partial \phi \right) \partial \psi_k \frac{\partial \psi_k}{\partial \lambda_k} - \left( \lambda_\alpha + \mu_\alpha \right) \hat{u}_i \hat{u}_i' + \mu_\alpha V^2 \hat{u}_i^{ss} = 0. \] (A.71)

\[ \rho \rho' \left( \hat{u}_i' - \frac{\hbar}{m} \partial \phi \right) \]

\[ \hat{u}_i' - \frac{\hbar}{m} \partial \phi \]

\[ \partial \psi_k \frac{\partial \psi_k}{\partial \lambda_k} \]

\[ \left( \lambda_\alpha + \mu_\alpha \right) \hat{u}_i \hat{u}_i' \]

\[ \mu_\alpha V^2 \hat{u}_i^{ss} = 0. \]