Boson representations, 
non-standard quantum algebras 
and contractions

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Abstract

A Gelfan'd–Dyson mapping is used to generate a one-boson realization for 
the non-standard quantum deformation of $\mathfrak{sl}(2, \mathbb{R})$ which directly provides 
its infinite and finite dimensional irreducible representations. Tensor product 
decompositions are worked out for some examples. Relations between con-
traction methods and boson realizations are also explored in several contexts. 
So, a class of two-boson representations for the non-standard deformation of 
$\mathfrak{sl}(2, \mathbb{R})$ is introduced and contracted to the non-standard quantum (1+1) 
Poincaré representations. Likewise, a quantum extended Hopf $\mathfrak{sl}(2, \mathbb{R})$ alge-
bra is constructed and the Jordanian $q$-oscillator algebra representations are 
obtained from it by means of another contraction procedure.
1 Introduction

Boson realizations of many symmetry algebras and superalgebras are known to be useful in many problems of Condensed Matter [1] and Nuclear Physics [2]. Among the existent bosonization processes, we shall fix our attention in the so-called Gelfand–Dyson (GD) mapping of $sl(2, \mathbb{R})$ [3], initially introduced in spin systems. The aim of this paper is to show that deformed GD type realizations are the most appropriate tools in order to construct the representation theory of non-standard quantum $sl(2, \mathbb{R})$ and other non-standard quantum algebras linked to it by means of contraction limits. Therefore, we hope that the results reached in this paper can be directly applied to deformed shell models or coherent states methods where GD maps have been proven very successful.

Firstly, we recall that the standard deformation of $sl(2, \mathbb{R})$ [4, 5, 6] is associated to the (constant) solution $r = J_+ \wedge J_-$ of the modified classical Yang–Baxter equation (YBE). This quantum algebra has been fully developed and extensively applied (see [7]). However, there also exists a non-standard deformation linked to the solution $r = J_3 \wedge J_+$. This deformation (which was firstly introduced at a quantum group level [8, 9], and later as a quantum Hopf algebra [10]) has recently attracted much attention. For instance, it has been applied to build up higher dimensional non-standard quantum algebras [11] as well as the non-standard $q$-differential calculus [12, 13]. Its universal quantum $R$-matrix [14, 15] and its irreducible representations [16, 17, 18] have been also studied.

Furthermore, it is rather remarkable that there exists a close relationship between $U_zsl(2, \mathbb{R})$ and the non-standard quantum (1+1) Poincaré [19, 20] and oscillator algebras [21]. In particular, all of them have a similar Hopf subalgebra determined by the two generators involved in the classical $r$-matrix. As an important consequence, they have a formally identical universal $R$-matrix. We will show that most of these common features can be explained by a contraction scheme connecting all these non-standard quantum algebras.

As we shall see in section 2, the full representation theory of $sl(2, \mathbb{R})$ can be straightforwardly derived with the aid of the one-boson GD realization. Afterwards, we build the one-boson infinite dimensional representations for $U_zsl(2, \mathbb{R})$ by following the same approach. It turns out that their explicit form is somewhat more complicated than those of the standard deformation [22, 23] in the sense that they cannot be obtained by the mere substitutions of numbers by $q$-numbers. The corresponding finite dimensional representations are deduced in a very natural way obtaining closed expressions for their matrix elements in any dimension. It is remarkable that these results are in plain agreement with those derived in [14, 15] by using a recurrence method in another basis. The main advantage of the basis here used is the very simple form that the quantum universal $R$-matrix presents; this fact, combined with the representation theory just derived, is used to provide some explicit $R$-matrices.

Although these deformed representations may appear awkward to handle, we
know that they should be fully consistent with the deformed composition of representations given by the quantum coproduct. In section 3 we illustrate such a problem for some low dimensional representations showing that complete reducibility holds and preserves the same well known classical angular momentum decomposition rules in tensor product spaces. However, non-standard Clebsch-Gordan coefficients are shown to be essentially different to those of the standard deformation.

The remaining sections of the paper are devoted to the study of the relations between boson realizations and contractions in classical and deformed frameworks. In particular, two-boson GD representations for classical and quantum \(sl(2, \mathbb{R})\) are introduced in section 4. They are shown to be the most adequate objects to obtain the representations of the non-standard quantum Poincaré algebra by means of a contraction process. We also present in section 5 a suitable quantum deformation of the (pseudo)extended Lie algebra \(sl(2, \mathbb{R})\) together with its one-boson representations so that they give rise, also through a contraction procedure, to the representations of the non-standard quantum oscillator algebra (also quoted as the Jordanian \(q\)-oscillator). The extension here introduced contains some interesting features that will be discussed. In section 6 the non-standard \(sl(2, \mathbb{R})\), Poincaré and oscillator algebras are presented as quadratic structures by using a deformed boson algebra. Finally, some remarks end the paper.

2 One-boson \(U_z sl(2, \mathbb{R})\) representations

2.1 Classical one-boson representations

To start with we shall deal with the classical Lie algebra \(sl(2, \mathbb{R})\) whose generators \(\{J_3, J_+, J_-\}\) obey the commutation rules

\[
[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = J_3. \tag{2.1}
\]

This algebra is isomorphic to \(so(2, 1)\), the Lie algebra generating the group of motions of a \((1+1)\) De Sitter space with non-zero constant curvature, where \(J_3\) generates the boosts and \(J_+, J_-\) translations along the light-cone. An alternative physical interpretation for \(sl(2, \mathbb{R})\) is to consider it as the infinitesimal generators of the conformal group of a one-dimensional space; in this sense, \(J_3\) would generate dilations, \(J_+\) translations and \(J_-\) special conformal transformations. Obviously, these different interpretations come from different representations of the algebra on the spaces linked to the physical problem under consideration.

The irreducible representations of \(sl(2, \mathbb{R})\) are characterized by the eigenvalue of the quadratic Casimir element

\[
\mathcal{C} = \frac{1}{2}J_3^2 + J_+ J_- + J_- J_+. \tag{2.2}
\]

If the generators \(\{a_-, a_+\}\) close a boson algebra, i.e., \([a_-, a_+] = 1\), then, the realiza-
tion of \( sl(2, \mathbb{R}) \) given by

\[
J_+ = a_+, \quad J_3 = 2a_+a_- + \beta 1, \quad J_- = -a_+a_- - \beta a_-, \quad (2.3)
\]

where \( \beta \) is a free parameter, is known as the GD one-boson realization [3, 24]. Now we will see how the GD map (2.3) can be used in order to get easily any of the \( sl(2, \mathbb{R}) \approx su(1, 1) \) irreducible representation series [25].

(i) **Lower bounded representations.** When the operators \( a_-, a_+ \) act in the usual way on the number states Hilbert space spanned by \( \{ |m\rangle \}_{m=0}^{\infty} \), i.e.,

\[
a_+|m\rangle = \sqrt{m+1}|m+1\rangle, \quad a_-|m\rangle = \sqrt{m}|m-1\rangle, \quad (2.4)
\]

(2.3) leads to a lower bounded representation:

\[
J_+|m\rangle = \sqrt{m+1}|m+1\rangle, \quad J_3|m\rangle = (2m + \beta)|m\rangle, \quad J_-|m\rangle = -\sqrt{m}(m-1 + \beta)|m-1\rangle.
\]

The casimir eigenvalue being

\[
\mathcal{C} = \beta(\beta/2 - 1). \quad (2.6)
\]

For negative integer values of \( \beta \), hereafter denoted as \( \beta_- \in \mathbb{Z}^- \), the representation (2.3) is reducible leading to a finite dimensional irreducible quotient representation of dimension \( |\beta_- - 1| \). For instance, \( \beta_- = -1 \) (\( \mathcal{C} = 3/2 \)) provides the two-dimensional representations of \( sl(2, \mathbb{R}) \) by setting \( |2\rangle \equiv 0 \):

\[
J_+|0\rangle = |1\rangle \quad J_+|1\rangle = 0 \quad J_3|0\rangle = -|0\rangle \quad J_3|1\rangle = |1\rangle \quad J_-|0\rangle = 0 \quad J_-|1\rangle = |0\rangle.
\]

The numbers \( \langle m|X|m'\rangle \) where \( \langle m|m'\rangle = \delta_{m,m'} \) give the matrix elements of these representations; in the previous example, we have

\[
J_+ = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad J_- = \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad J_3 = \begin{pmatrix} -1 & \cdot \\ \cdot & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}.
\]

(2.7)

With this notation in mind, we shall schematically display the matrix form that the aforementioned representations take as a future reference with respect to the deformed algebra:

\[
J_+ = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot \\ \cdot & \sqrt{2} & 0 & \cdot \\ \cdot & \cdot & \sqrt{3} & 0 \\ \cdot & \cdot & \cdot & \sqrt{4} \\ \cdot & \cdot & \cdot & \cdot \\ \end{pmatrix}, \quad (2.9)
\]

\[
J_3 = \begin{pmatrix} \beta & \cdot & \cdot & \cdot \\ \cdot & 2 + \beta & \cdot & \cdot \\ \cdot & \cdot & 4 + \beta & \cdot \\ \cdot & \cdot & \cdot & 6 + \beta \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \end{pmatrix}, \quad (2.10)
\]
\[ J_- = - \begin{pmatrix} 0 & \beta & \cdot & \cdot & \cdot \\ . & 0 & \sqrt{2}(1 + \beta) & \cdot & \cdot \\ . & . & 0 & \sqrt{3}(2 + \beta) & \cdot \\ . & . & . & 0 & \sqrt{4}(3 + \beta) \\ . & . & . & . & \ddots \end{pmatrix}. \] (2.11)

(ii) Upper bounded representations. Quite similar upper bounded representations can be defined in the supplementary space \( \{|m\rangle\}_{m=-\infty}^{-1} \). However, in order to avoid the complex numbers in the accompanying square roots (2.15) inside this space, we shall redefine the basis vectors in the form
\[ |m\rangle \rightarrow \frac{1}{\sqrt{m}}|m\rangle, \]
so that the boson operators act as
\[ a_+|m\rangle = -(m+1)|m+1\rangle, \quad a_-|m\rangle = -|m-1\rangle, \] (2.12)
leading to the \( sl(2, \mathbb{R}) \) action
\[ J_+|m\rangle = -(m+1)|m+1\rangle, \quad J_3|m\rangle = (2m+\beta_+)|m\rangle, \quad J_-|m\rangle = (m-1+\beta_+)|m-1\rangle. \] (2.13)
The finite dimensional representations are now originated for \( \beta_+ - 2 \in \mathbb{Z}^+ \), with dimension \( \beta_+ - 1 \). However, note that in this case the action (2.13) allows for an invariant subspace, so that it is not necessary to make use of the quotient mechanism to reach irreducibility.

These representations are particularly well suited for describing the differential version of the GD map (2.3),
\[ J_+ = \partial_x, \quad J_3 = -2x\partial_x + \beta - 2, \quad J_- = -x^2\partial_x + (\beta - 2)x. \] (2.14)
The basis functions will be the positive integer powers \( \{x^n\}_{n=0}^{\infty} \), with the identification \( x^n \equiv |n-1\rangle \). In particular for the values of the label \( \beta \) given by \( \beta_+ - 2 \in \mathbb{Z}^+ \) the support space for the finite \( (\beta_+ - 1) \)-dimensional representations is generated by the monomials \( \{1, x, x^2, \ldots, x^{\beta_+ - 2}\} \). Note also that (2.14) reproduces for \( \beta = 2 \) the usual differential realization of the Lie algebra of the conformal group for the one-dimensional Euclidean space.

The finite dimensional representations obtained from the lower (labeled by \( \beta_- \)) or upper bounded ones (denoted by \( \beta_+ \)) are equivalent whenever \( \beta_+ - 2 = |\beta_-| = -\beta_- \). Indeed in this case the Casimir (2.6) is the same \( C_{\beta_-} = C_{\beta_+} \). Hereafter we shall introduce the notation \( |\beta_- - 1| = \beta_+ - 1 = 2j + 1 \), where \( j \) is a half positive integer that should be identified with the label of the integer \( (2j+1) \)-dimensional representations of \( sl(2, \mathbb{R}) \approx su(1,1) \) [24]. Using this notation, the Casimir (2.6) will turn into the more familiar expression \( C = 2j(j+1) \).

(iii) Non-bounded representations. When the operators \( a_-, a_+ \) act on the space spanned by \( \{|m\rangle\}_{m=-\infty}^{\infty} \) in the form
\[ a_+|m\rangle = \sqrt{m+\lambda+1}|m+1\rangle, \quad a_-|m\rangle = \sqrt{m+\lambda}|m-1\rangle, \] (2.15)
where $0 < Re(\lambda) < 1$ we get unbounded representations. These are less familiar because they cannot implement the hermiticity conditions $(a_-)^* = a_+$ inside a Hilbert space. However they can be substituted in (2.3), so that if $\beta - \lambda \notin \mathbb{Z}$ we get a family of $sl(2, \mathbb{R})$ representations with the same eigenvalue (2.6) which are not bounded either. Nevertheless we will not need to give more details because this paper shall be mainly concerned with the bounded cases.

The principal and complementary series of $sl(2, \mathbb{R}) \approx su(1, 1)$ representations can be derived from case (iii), while the discrete series come from the other (i) and (ii) cases. The implementation of the hermiticity conditions can be achieved by a wiseful definition of the inner product. This is a familiar problem of GD mappings which often is called ‘the unitarization process’ [2]. Anyway, we shall not address these questions here.

### 2.2 Quantum one-boson representations

The Hopf algebra $U_z sl(2, \mathbb{R})$ deforming the bialgebra generated by the classical $r$-matrix $r = zJ_3 \wedge J_+$ is characterized by the following coproduct, counit, antipode and commutation rules (see [15]):

$$
\Delta(J_+) = 1 \otimes J_+ + J_+ \otimes 1, \\
\Delta(J_3) = 1 \otimes J_3 + J_3 \otimes e^{2zJ_+}, \\
\Delta(J_-) = 1 \otimes J_- + J_- \otimes e^{2zJ_+},
$$

(2.16)

$$
\epsilon(X) = 0, \quad \text{for } X \in \{J_3, J_+, J_-\},
$$

(2.17)

$$
\gamma(J_+) = -J_+, \quad \gamma(J_3) = -J_3e^{-2zJ_+}, \quad \gamma(J_-) = -J_-e^{-2zJ_+},
$$

(2.18)

$$
[J_3, J_+] = \frac{e^{2zJ_+} - 1}{z}, \quad [J_3, J_-] = -2J_- + zJ_3^2, \quad [J_+, J_-] = J_3.
$$

(2.19)

The quantum Casimir is

$$
C_z = \frac{1}{2}J_3 e^{-2zJ_+} + \frac{1 - e^{-2zJ_+}}{2z} J_- + J_+ - \frac{1 - e^{-2zJ_+}}{2z} + e^{-2zJ_+} - 1,
$$

(2.20)

and the universal $R$-matrix takes the form

$$
R = \exp\{-zJ_+ \otimes J_3\} \exp\{zJ_3 \otimes J_+\}.
$$

(2.21)

A realization of $U_z sl(2, \mathbb{R})$ in terms of the boson algebra $[a_-, a_+] = 1$ reads

$$
J_+ = a_+, \quad J_3 = \frac{e^{2za_+} - 1}{z} a_- + \beta \frac{e^{2za_+} + 1}{2},
$$

(2.22)

$$
J_- = -\frac{e^{2za_+} - 1}{2z} a_- - \beta \frac{e^{2za_+} + 1}{2} a_- - z\beta^2 \frac{e^{2za_+} - 1}{8}.
$$

The Casimir for this realization takes the same expression (2.6) than in the classical case. The limit $z \to 0$ of (2.22) gives rise to the GD realization (2.3) for $sl(2, \mathbb{R})$ while the Casimir keeps the same eigenvalue (2.6) along the whole process.
The essential point is that GD like quantum formulas (2.22) allow us to compute easily closed expressions for any representation of this quantum algebra, (in this respect, compare to the derivation of such representations given in [17, 18]).

Lower bounded representations can be obtained from (2.4), by taking into account that
\[ e^{2za+}|m\rangle = |m\rangle + \sum_{k=1}^{\infty} \frac{(2z)^k}{k!} \sqrt{\frac{(m+k)!}{m!}} |m+k\rangle, \quad (2.23) \]
so that the action of (2.22) on the states \{ |m\rangle \}_{m=0}^{\infty} for any \( \beta \) is obtained:
\[ J_+|m\rangle = \sqrt{m+1} |m+1\rangle, \]
\[ J_3|m\rangle = (2m + \beta) |m\rangle + \sum_{k=1}^{\infty} \frac{(2z)^k}{k!} \sqrt{\frac{(m+k)!}{m!}} \left( \frac{2m}{k+1} + \beta \right) |m+k\rangle, \quad (2.24) \]
\[ J_-|m\rangle = -\sqrt{m} (m-1 + \beta) |m-1\rangle - \sum_{k=1}^{\infty} \frac{(2z)^k}{k!} \sqrt{\frac{(m+k)!}{m!}} \left\{ \frac{m}{\sqrt{m+k}} \left( \frac{m-1 + \beta}{k+1} + \frac{\beta}{2} \right) |m-1+k\rangle + \frac{z\beta^2}{8} |m+k\rangle \right\}. \]

The general matrix form for \( J_+ \) is the classical one (2.3) and for the two remaining generators we have
\[
J_3 = \begin{pmatrix}
\beta & 2 + \beta \\
\beta z & \sqrt{2}(2+\beta)z \\
\sqrt{2}\beta z^2 & \sqrt{2}(3+\beta)z \\
\frac{4}{\sqrt{3}}\beta z^3 & \frac{4}{\sqrt{3}}(4+\beta)z \\
\frac{1}{\sqrt{3}}\beta z^4 & \frac{1}{\sqrt{3}}(5+\beta)z \\
\end{pmatrix}
\]
\[
J_- = \begin{pmatrix}
0 & \beta z & \sqrt{2}(1+\beta) \\
\frac{1}{2}\beta z^2 & (\sqrt{2}+\frac{\beta}{2})z^2 & \sqrt{3}(2+\beta) \\
\frac{1}{2\sqrt{2}}\beta z^3 & (\sqrt{2}\beta + \frac{\beta}{2})z^2 & 3\beta(2+\beta) \\
\frac{1}{\sqrt{2}}\beta z^4 & (\sqrt{2}\beta + \sqrt{2}\beta^2)z^4 & (8+6\beta + \frac{8}{3}\beta^2)z^3 \\
\frac{1}{\sqrt{2}}\beta z^5 & (\sqrt{2}\beta + \sqrt{2}\beta^2)z^4 & (8+6\beta + \frac{8}{3}\beta^2)z^3 \\
\end{pmatrix}
\]

For \( \beta = \beta_- \in \mathbb{Z}^- \) this action directly provides, by means of a quotient space, the finite dimensional representations of dimension \( |\beta_- - 1| \) much in the same way as for the classical counterpart. Therefore, we will denote \( |\beta_- - 1| = 2|j_z + 1| \in \mathbb{Z}^+ \) being \( j_z = 0, 1/2, 1 \ldots \). Indeed it is clear that the expressions (2.24) contain a power series in \( z \) such whose first terms coincides with the non-deformed analogue shown in (2.3). We write down the 2, 3 and 4-dimensional matrix representations:

(a) \( \beta_- = -1, C_z = 3/2, j_z = 1/2 \).
\[
J_+ = \begin{pmatrix}
1 & \\
1 & \\
\end{pmatrix} \quad J_- = \begin{pmatrix}
\frac{1}{2} z^2 & 1 \\
\frac{1}{2} z & 1 \\
\end{pmatrix} \quad J_3 = \begin{pmatrix}
-1 & \\
-1 & \\
\end{pmatrix}
\]
(2.26)
(b) $\beta_- = -2$, $C_z = \frac{4}{9}$, $j_z = \frac{1}{3}$.

\[
J_+ = \begin{pmatrix}
1 & \cdots & \\
. & \sqrt{2} & . \\
. & \sqrt{3} & . \\
\end{pmatrix}
J_- = \begin{pmatrix}
-z^2 & 2z & \sqrt{2} \\
-\sqrt{2}z^3 & \sqrt{2}z^2 & 2z \\
\end{pmatrix}
J_3 = \begin{pmatrix}
-2z & . & . \\
-2\sqrt{2}z^2 & . & 2 \\
\end{pmatrix}
\] (2.27)

(c) $\beta_- = -3$, $C_z = \frac{15}{2}$, $j_z = \frac{3}{2}$.

\[
J_+ = \begin{pmatrix}
1 & \cdots & . \\
. & \sqrt{2} & . \\
. & \sqrt{3} & . \\
\end{pmatrix}
J_- = \begin{pmatrix}
-z^2 & 2z & \sqrt{2} \\
-\sqrt{2}z^3 & \sqrt{2}z^2 & 2z \\
\end{pmatrix}
J_3 = \begin{pmatrix}
-3z & -1 & . \\
-2\sqrt{2}z^2 & -\sqrt{2}z & 1 \\
-2\sqrt{6}z^3 & -5\sqrt{2}z^2 & \sqrt{3}z & 3 \\
\end{pmatrix}
\] (2.28)

By means of the general formula for the $R$-matrix (2.21) we can get explicit solutions of the quantum Yang-Baxter equation by substituting the representations just found. The computations are considerably simplified due to the factorized form (2.21). As an example we write down the $4 \times 4$ and $9 \times 9$ $R$-matrices corresponding to the above 2 and 3-dimensional representations, respectively:

\[
R = \begin{pmatrix}
1 & . & . . & . \\
-z & 1 & . & . \\
z & . & 1 & . \\
z^2 & -z & z & 1 \\
\end{pmatrix}
\] (2.29)

\[
R = \begin{pmatrix}
1 & . & . & . & . & . \\
-2z & 1 & . & . & . & . \\
2\sqrt{2}z^2 & -2\sqrt{2}z & 1 & . & . & . \\
2z & . & 1 & . & . & . \\
. & . & . & 1 & . & . \\
. & 2\sqrt{2}z^2 & -2z & 1 & . & . \\
2\sqrt{2}z^2 & . & 2\sqrt{2}z & . & 1 & . \\
. & . & 2\sqrt{2}z^2 & 2z & 1 & . \\
. & -4z^3 & 2\sqrt{2}z^2 & 4z^3 & -2\sqrt{2}z & 2\sqrt{2}z & 1 \\
\end{pmatrix}
\] (2.30)

Upper bounded representations inside polynomial spaces $\{x^n\}_{n=0}^{+\infty}$ are supplied by a differential (difference) realization by means of the operator

\[
D_z \equiv (e^{2z\partial_x} - 1)/2z.
\] (2.31)

The action of $D_z$ is just a discrete derivative: $D_z\phi(x) = (\phi(x + 2z) - \phi(x))/2z$. Thus, we have

\[
J_+ = \partial_x, \quad J_3 = 2D_zx + z\beta D_z + \beta,
\]
\[ J_- = -D_z x^2 - z \beta D_z x - \frac{z^2 \beta^2}{4} D_z - \beta x. \]

(2.32)

From these expressions, we see that non-standard quantum deformations are related to a discrete difference calculus quite different to that of standard \( U_q sl(2, \mathbb{R}) \). In analogy to the classical case, the finite dimensional representations originated from (2.32) for \( \beta + 2 \in \mathbb{Z}^+ \) and \( \beta + 1 = 2 j_z + 1 \in \mathbb{Z}^+ \) is also supported by \( \langle 1, x, \ldots x^{\beta + 2} \rangle \). These representations, denoted by \( j_z \), will be put to work in some examples along the next section.

3 Tensor product representations and decomposition rules

Given a pair of representations for the \( U_z sl(2, \mathbb{R}) \) algebra acting on the vector spaces \( \mathcal{H}_1, \mathcal{H}_2 \) the coproduct (2.16) originates a new representation in the tensor product space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). Although the initial representations may be irreducible the final coproduct representation will in general be reducible. We shall see that, for some particular finite dimensional cases worked out below, the coproduct representation is completely reduced into irreducible components following the same well known rules valid for the classical \( sl(2, \mathbb{R}) \) integer representations. Using the conventional notation [26], this decomposition can be written as

\[ j_z \otimes j'_z = |j_z + j'_z| \oplus |j_z + j'_z - 1| \oplus \ldots \oplus |j_z - j'_z|, \]

(3.1)

where \( j_z \) and \( j'_z \) are positive half-integers corresponding to the quantum representations of dimension \( 2j_z + 1 \) or \( 2j'_z + 1 \), respectively. However, the vector basis of the irreducible support subspaces expressed in terms of the original basis (i.e., the Clebsch-Gordan coefficients) become quite different to those of the \( sl(2, \mathbb{R}) \) Lie algebra due to extra terms containing powers of the deformation parameter \( z \).

We shall examine these features in detail for two simple examples by making use of the differential realizations given in (2.32) that are particularly easy to handle for computations.

- \( 1/2 \otimes 1/2 \) representations.

The representation \( j = 1/2 \) for the Lie algebra \( sl(2, \mathbb{R}) \) is realized in the polynomial vector space spanned by \( \{ |-1\rangle = 1, | -2\rangle = x \} \). For \( j = 1 \) the basis is chosen in the form \( \{ |-1\rangle = 1, | -2\rangle = x, | -3\rangle = x^2 \} \); we shall use the variable \( y \) for the second space in the tensor product. The coproduct representation in this case is defined by \( \Delta_{\text{clas}}(X) = 1 \otimes U_{1/2}(X) + U_{1/2}(X) \otimes 1 \), where \( X \) is any of the algebra generators and \( U_{1/2} \) is for the \( j = 1/2 \)-spin representation. The decomposition \( 1/2 \otimes 1/2 = 1 \oplus 0 \) has support spaces whose basis are

\[ \mathcal{H}^{\text{clas}}_1 = \langle E_{-1}^{\text{clas}} \equiv 1, E_{-2}^{\text{clas}} \equiv \frac{1}{2}(x + y), E_{-3}^{\text{clas}} \equiv xy \rangle, \]

\[ \mathcal{H}^{\text{clas}}_0 = \langle U_{-1}^{\text{clas}} \equiv \frac{1}{2}(x - y) \rangle. \]

(3.2)
Obviously, the triplet generating $\mathcal{H}^{\text{clas}}_1$ is symmetric under the permutation map $\sigma(a \otimes b) = b \otimes a$, while the singlet underlying $\mathcal{H}^{\text{clas}}_0$ is antisymmetric.

Now, for the non-standard $U_zsl(2, \mathbb{R})$ the coproduct representation is defined according to (2.16), and we consider the value $\beta_+ = 3$ that corresponds to the case $j_z = 1/2$ (2.20). The reduction $1/2 \otimes 1/2 = 1 \oplus 0$ keeps on correct too. Here, the representation $j_z = 1$ is obtained with $\beta_+ = 4$ (2.27), while 0 is of course for the trivial representation. Explicitly, the invariant vector subspaces $\mathcal{H}_1 = \langle E_{-1}, E_{-2}, E_{-3} \rangle$ and $\mathcal{H}_0 = \langle U_1 \rangle$ are as follows in terms of the basis (3.2):

$$
\begin{align*}
E_{-1} & = E^{\text{clas}}_{-1} \\
E_{-2} & = E^{\text{clas}}_{-2} \\
E_{-3} & = E^{\text{clas}}_{-3} + \frac{3z^2}{4} E^{\text{clas}}_{-1} + z U^{\text{clas}}_{-1} \\
U_{-1} & = U^{\text{clas}}_{-1} + \frac{z}{2} E^{\text{clas}}_{-1}.
\end{align*}
$$

Note that symmetry in the basis of $\mathcal{H}_1$ and antisymmetry in $\mathcal{H}_0$ do not hold unless we assume that the permutation map $\sigma$ transforms the deformation parameter $z$ into $-z$. On the other hand, it can be easily proven that, for all $z$, the transformation (3.3) is always well-defined. Therefore, roots of unity seem to be not privileged for the non-standard deformation.

- $1 \otimes 1/2$ representations.

First we shall supply the basis of $\mathcal{H}_1 \otimes 1/2 = \mathcal{H}^{\text{clas}}_{3/2} \oplus \mathcal{H}^{\text{clas}}_{1/2}$ for the reduction $1 \otimes 1/2 = 3/2 \oplus 1/2$ in the Lie algebra context. For the $j = 3/2$ representation we use the basis \{ $| -1 \rangle = 1, | -2 \rangle = x, | -3 \rangle = x^2, | -4 \rangle = x^3$ \}, and the invariant subspaces in the classical tensor product are spanned by

$$
\begin{align*}
\mathcal{H}^{\text{clas}}_{3/2} & = \langle E^{\text{clas}}_{-1} \equiv 1, E^{\text{clas}}_{-2} \equiv \frac{1}{3}(y + 2x), E^{\text{clas}}_{-3} \equiv \frac{1}{3}(2xy + x^2), E^{\text{clas}}_{-4} \equiv x^2 y \rangle, \\
\mathcal{H}^{\text{clas}}_{1/2} & = \langle U^{\text{clas}}_{-1} \equiv \frac{1}{2}(y - x), U^{\text{clas}}_{-2} \equiv \frac{1}{2}xy - x^2 \rangle
\end{align*}
$$

With respect to the deformed quantum algebra $U_zsl(2, \mathbb{R})$ it can be checked directly that its coproduct leads to the same direct sum reduction $1 \otimes 1/2 = 3/2 \oplus 1/2$ on the same polynomial vector space $\mathcal{H}_1 \otimes 1/2$ but with new invariant subspaces $\mathcal{H}_{3/2} = \langle E_{-1}, E_{-2}, E_{-3}, E_{-4} \rangle$ and $\mathcal{H}_{1/2} = \langle U_{-1}, U_{-2} \rangle$ given by the following deformed change of basis:

$$
\begin{align*}
E_{-1} & = E^{\text{clas}}_{-1} \\
E_{-2} & = E^{\text{clas}}_{-2} \\
E_{-3} & = E^{\text{clas}}_{-3} + \frac{3z^2}{4} E^{\text{clas}}_{-1} - \frac{2z}{3} U^{\text{clas}}_{-1} \\
E_{-4} & = E^{\text{clas}}_{-4} + \frac{9z^2}{4} E^{\text{clas}}_{-2} - \frac{9z^3}{4} E^{\text{clas}}_{-1} - 2z U^{\text{clas}}_{-2} - \frac{z^2}{3} E^{\text{clas}}_{-1} \\
U_{-1} & = U^{\text{clas}}_{-1} - \frac{z}{2} E^{\text{clas}}_{-2}
\end{align*}
$$
\[
U_{-2} = U_{-2}^{\text{clas}} - \frac{3z}{8}E_{-3}^{\text{clas}} + \frac{3z^2}{8}E_{-2}^{\text{clas}}.
\]

As expected, the limit \( z \to 0 \) provides the classical partners of the reduction process.

At this point it is worth mentioning that representations of the standard deformation of \( sl(2, \mathbb{R}) \) are strongly different from their non-standard counterparts. On one hand, such standard representations can be essentially constructed by substituting some matrix elements of the classical matrices by the corresponding \( q \)-numbers \(^{27}\) and, consequently, the same holds for the Clebsch-Gordan coefficients. This straightforward method is no longer valid for the non-standard case, where \( q \)-numbers do not work and, moreover, some new non-vanishing Clebsch-Gordan coefficients have to be added with respect to the classical theory. Since \( q \)-numbers are directly related to the peculiarities of roots of unity, the loss of such properties in the non-standard case seems quite natural.

4 Two-boson \( U_zsl(2, \mathbb{R}) \) representations and their contraction to Poincaré

As it has been shown above the one-boson representations of \( sl(2, \mathbb{R}) \) are closely linked with its interpretation as a one-dimensional conformal algebra. In contrast, a description in terms of two-boson algebras is physically related to its role as a \((1+1)\)-dimensional kinematical algebra. This fact allows us to perform a contraction in order to reach the \((1+1)\) Poincaré algebra representations; such a process cannot be applied onto the one-boson representations of section 3.

4.1 Classical two-boson representations

Let us again begin with a discussion for \( sl(2, \mathbb{R}) \) since it will give us a natural reference. We consider two independent boson algebras

\[
[a_-, a_+] = 1, \quad [b_-, b_+] = 1.
\]

A two-boson representation of \( sl(2, \mathbb{R}) \) is the following

\[
J_+ = a_+, \quad J_3 = 2a_+a_- - 2b_+b_-, \quad J_- = -a_+a_-^2 + 2b_+b_-a_+ + ab_+.
\]

There are, of course a high arbitrariness for many other expressions realizing the same \( sl(2, \mathbb{R}) \) algebra, but the simple choice shown in (4.2) will be enough for our purposes. The only track left here of such wide range of possibilities is the free parameter \( \alpha \). This parameter should not be seen as an irreducible representation label as it was the case of \( \beta \), for instance, with respect to (2.3). We will call formulas (4.2) the two-boson GD realization. They should be compared to the more common Jordan–Schwinger realization \(^{28}\).
A differential realization is obtained by taking
\[ a_- = -x^+, \quad a_+ = \frac{\partial}{\partial x^+} \equiv \partial_+ , \quad b_- = -x^- , \quad b_+ = \frac{\partial}{\partial x^-} \equiv \partial_-, \quad (4.3) \]
where, \( x^+ = t + x \) and \( x^- = t - x \) can be identified as light-cone coordinates. Then, we get
\[ J_+ = \partial_+ , \quad J_3 = -2x^+ \partial_+ + 2x^- \partial_- , \]
\[ J_- = -(x^+)^2 \partial_+ + 2x^+ x^- \partial_- + \alpha \partial_-. \quad (4.4) \]
The expression for the Casimir (2.2) acting on the wavefunction space \( \phi(x^+, x^-) \) is the second order operator
\[ C = 2(x^-)^2 \partial_-^2 + 4x^- \partial_- + 2\alpha \partial_+^2 . \quad (4.5) \]
By means of the following Inöni–Wigner contraction, to be applied in (2.1),
\[ P_+ = \varepsilon J_+ , \quad P_- = \varepsilon J_- , \quad K = \frac{1}{2} J_3 , \quad (4.6) \]
the boost \( K \) and the light-cone translations \( P_{\pm} \) of the (1+1) Poincaré algebra \( \mathcal{P}(1+1) \) are generated in the limit \( \varepsilon \to 0 \), since the comutation rules between these new generators are
\[ [K, P_+] = P_+ , \quad [K, P_-] = -P_- , \quad [P_+, P_-] = 0 . \quad (4.7) \]
In order to contract the representation (1.2) we consider (4.6) together with [29]:
\[ a_- \to \varepsilon^{-1} a_- , \quad a_+ \to \varepsilon a_+ , \quad b_- \to b_- , \quad b_+ \to b_+ , \quad \alpha \to \varepsilon \alpha , \quad (4.8) \]
and the limit \( \varepsilon \to 0 \) provides the two-boson \( \mathcal{P}(1+1) \) representations
\[ P_+ = a_+ , \quad K = a_+ a_- - b_+ b_- , \quad P_- = \alpha b_+ . \quad (4.9) \]

### 4.2 Quantum two-boson representations: the Poincaré algebra

As far as the quantum non-standard algebra \( U_z sl(2, \mathbb{R}) \) is concerned, the corresponding two-boson version takes the form
\[ J_+ = a_+ , \quad J_3 = \frac{e^{2a_+} - 1}{z} a_- - 2b_+ b_- , \]
\[ J_- = -\frac{e^{2a_+} - 1}{2z} a_-^2 + 2b_+ b_- a_- + \alpha b_+ + 2z(b_+ b_- + b_+^2 b_-^2) . \quad (4.10) \]
Algebraically \( U_z sl(2, \mathbb{R}) \) can be contracted to a non-standard \( U_\varepsilon \mathcal{P}(1+1) \) algebra by defining the generators as in (4.6) and at the same time setting
\[ z \to \varepsilon^{-1} z , \quad (4.11) \]
so that we get the Hopf algebra

\[
\Delta(P_+) = 1 \otimes P_+ + P_+ \otimes 1, \\
\Delta(K) = 1 \otimes K + K \otimes e^{2zP_+}, \\
\Delta(P_-) = 1 \otimes P_- + P_- \otimes e^{2zP_+},
\]

\[
\epsilon(X) = 0, \quad \text{for } X \in \{K, P_+, P_-\}, \\
\gamma(P_+) = -P_+, \quad \gamma(K) = -Ke^{-2zP_+}, \quad \gamma(P_-) = -P_-e^{-2zP_+}, \\
[K, P_] = \frac{e^{2zP_+} - 1}{2z}, \quad [K, P_-] = -P_-, \quad [P_+, P_-] = 0.
\]

The quantum Casimir is found by contracting (2.20) as the \( \lim_{\varepsilon \to 0} (\varepsilon^2 C_z) \) and the \( R \)-matrix comes directly from the contraction of (2.21):

\[
C_z = \frac{1 - e^{-2zP_+}}{z} P_- \\
R = \exp\{-2zP_+ \otimes K\} \exp\{2zK \otimes P_+\}.
\]

The corresponding classical \( r \)-matrix is \( r = 2zK \wedge P_+ \). These results are in full concordance with those obtained in [20] by a \( T \)-matrix approach.

After applying the transformations (4.6), (4.8) and (4.11) on (4.10), the contracted two-boson representation becomes

\[
P_+ = a_+, \quad K = \frac{e^{2za_+} - 1}{2z} a_- - b_+ b_-, \quad P_- = \alpha b_+.
\]

5 Extended \( U_zsl(2, \mathbb{R}) \) and its contraction to the oscillator algebra

The infinite dimensional representations for the non-standard quantum oscillator algebra [21] can be deduced by performing a contraction on a (non-standard) quantum deformation \( U_zsl(2, \mathbb{R}) \) of the pseudo-extended \( sl(2, \mathbb{R}) \) Lie algebra. In this section we develop such a process at both classical and quantum levels.

5.1 Classical level

It is well-known that a trivial central extension of the Lie algebra \( sl(2, \mathbb{R}) \) leads through a careful contraction to a non-trivial extension of the \( (1+1) \) Poincaré algebra corresponding to a constant non-null background field [30]; this contracted extended algebra is isomorphic to the oscillator \( \mathfrak{h}_4 \) Lie algebra. Let us review here such properties for the sake of comprehension and unification of notation. The trivial extension, designed by \( sl(2, \mathbb{R}) \), obeys to the commutation rules

\[
[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = J_3 - I, \quad [I, \cdot] = 0.
\]

13
where $I$ is the central extension generator. The second order Casimir is now

$$
C = \frac{1}{2} J_3^2 - J_3 I + J_+ J_- + J_- J_+.
$$

(5.2)

The one-boson realization for (5.1) is

$$
J_+ = a_+, \quad J_3 = 2a_+ a_- + \beta 1, \quad J_- = -a_+ a_- - \beta a_- + \delta a_-, \quad I = \delta 1,
$$

(5.3)

where $\delta$ and $\beta$ are free parameters related with the eigenvalue of the Casimir by

$$
C = \beta (\beta/2 - 1) + \delta (1 - \beta).
$$

(5.4)

An Inönü-Wigner contraction can be applied by defining the new generators

$$
A_+ = \varepsilon J_+, \quad A_- = \varepsilon J_-, \quad N = J_3/2, \quad M = \varepsilon^2 I,
$$

(5.5)

so that in the limit $\varepsilon \to 0$ we reach the oscillator $h_4$ Lie algebra,

$$
[N, A_+] = A_+, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = M, \quad [M, \cdot] = 0.
$$

(5.6)

The corresponding second order Casimir is obtained as $\lim_{\varepsilon \to 0} (-\varepsilon^2 C)$:

$$
C = 2NM - A_+ A_- - A_- A_+.
$$

(5.7)

The additional replacements

$$
a_- \to \varepsilon^{-1} a_-, \quad a_+ \to \varepsilon a_+, \quad \beta \to \beta/2, \quad \delta \to \varepsilon^2 \delta,
$$

(5.8)

provide the one-boson $h_4$ realization:

$$
N = a_+ a_- + \beta, \quad A_+ = a_+, \quad A_- = \delta a_-, \quad M = \delta 1.
$$

(5.9)

Hence the eigenvalue of (5.7) is $C = \delta (2\beta - 1)$.

As for the infinite dimensional irreducible representations of $h_4$ in number state spaces, they are trivially derived from those of the boson algebra (2.4) and (2.15). Note that (5.9) clarifies the difference between considering a boson algebra and the harmonic oscillator algebra $h_4$.

### 5.2 Quantum level: the non-standard oscillator

In the quantum context following closely the classical approach first we must define an appropriate extension of $U_z \mathfrak{sl}(2, \mathbb{R})$ by the addition of a new central generator $I$. In this way we will call the extended non-standard quantum algebra of $\mathfrak{sl}(2, \mathbb{R})$ to the Hopf algebra denoted $U_z \mathfrak{sl}(2, \mathbb{R})$ and given by

$$
\Delta(J_+) = 1 \otimes J_+ + J_+ \otimes 1,
$$
\[
\Delta(J_3) = 1 \otimes J_3 + J_3 \otimes e^{2zJ_+},
\]
\[
\Delta(J_-) = 1 \otimes J_- + J_- \otimes e^{2zJ_+} + zJ_3 \otimes I \ e^{2zJ_+},
\]
\[
\Delta(I) = 1 \otimes I + I \otimes 1,
\]
\[
\epsilon(X) = 0, \quad \text{for } X \in \{J_3, J_+, J_-, I\},
\]
\[
\gamma(J_+) = -J_+ + 1, \quad \gamma(I) = -I, \quad \gamma(J_3) = -J_3 e^{-2zJ_+},
\]
\[
\gamma(J_-) = -J_- e^{-2zJ_+} + zJ_3 I \ e^{-2zJ_+},
\]
\[
[J_3, J_+] = \frac{e^{2zJ_+} - 1}{z}, \quad [J_3, J_-] = -2J_- + zJ_3^2,
\]
\[
[J_+, J_-] = J_3 - I \ e^{2zJ_+}, \quad [I, \cdot] = 0.
\]

The (coboundary) Lie bialgebra underlying this Hopf algebra is again generated by \(r = zJ_3 \wedge J_+\). Note that the new generator \(I\) remains central and primitive; there is another quantum Casimir given by
\[
C_z = \frac{1}{2} J_3 e^{-2zJ_+} J_3 - J_3 I + \frac{1 - e^{-2zJ_+}}{2z} J_- + J_- \frac{1 - e^{-2zJ_+}}{2z} + e^{-2zJ_+} - 1.
\]

The Hopf subalgebra generated by \(J_3\) and \(J_+\) is the same as in the non-extended case, hence the universal \(R\)-matrix [21] is obviously a solution of the quantum YBE for \(U_{z\mathfrak{sl}}(2, \mathbb{R})\). Furthermore, cumbersome computations show this \(R\)-matrix verifies \(\mathcal{R}\Delta(J_-)\mathcal{R}^{-1} = \sigma \circ \Delta(J_-)\) in \(U_{z\mathfrak{sl}}(2, \mathbb{R})\) (the proof for \(I\) is trivial), hence we conclude that element is also a universal \(R\)-matrix for the whole \(U_{z\mathfrak{sl}}(2, \mathbb{R})\).

At this point it is worth mentioning that another quantum deformation of the extended \(\mathfrak{sl}(2, \mathbb{R})\) has been recently proposed in [31] leading to a deformed oscillator algebra with classical commutation relations, but not keeping the aforementioned subalgebra.

The one-boson realization of \(U_{z\mathfrak{sl}}(2, \mathbb{R})\) turns out to be a slight modification of the non-extended case [22]. Besides the new generator \(I = \delta 1\), the only generator that changes is \(J_-\) that reads
\[
J_- = -\frac{e^{2za_+} - 1}{2z} a_- - \beta \frac{e^{2za_+} + 1}{2} a_- - z\beta^2 \frac{e^{2za_+} - 1}{8} + \delta e^{2za_+} a_- + \frac{\beta z}{2} \delta e^{2za_+}.
\]

Now we proceed to carry out the contraction from \(U_{z\mathfrak{sl}}(2, \mathbb{R})\) to the non-standard quantum oscillator algebra [21], denoted \(U_{z\mathfrak{h}_4}\). At the Hopf algebra level, we consider the new generators defined by (5.3) and also the transformation of the deformation parameter \(z\) (4.11). Thus, when \(\varepsilon \to 0\) we arrive at the Hopf structure of \(U_{z\mathfrak{h}_4}\) given by
\[
\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1, \quad \Delta(M) = 1 \otimes M + M \otimes 1,
\]
\[
\Delta(N) = 1 \otimes N + N \otimes e^{2zA_+},
\]
\[
\Delta(A_-) = 1 \otimes A_- + A_- \otimes e^{-2zA_+} + 2z N \otimes M e^{2zA_+},
\]

\[
\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1, \quad \Delta(M) = 1 \otimes M + M \otimes 1,
\]
\[
\Delta(N) = 1 \otimes N + N \otimes e^{2zA_+},
\]
\[
\Delta(A_-) = 1 \otimes A_- + A_- \otimes e^{-2zA_+} + 2z N \otimes M e^{2zA_+},
\]
\[ e(X) = 0, \quad X \in \{N, A_+, A_-, M\}, \quad (5.17) \]

\[
\begin{align*}
\gamma(A_+) &= -A_+, \\
\gamma(M) &= -M, \\
\gamma(N) &= -Ne^{-2zA_+}, \\
\gamma(A_-) &= -A_+e^{-2zA_+} + 2zNe^{-2zA_+},
\end{align*}
\]

satisfying the commutators

\[ [N, A_+] = \frac{e^{2zA_+} - 1}{2z}, \quad [N, A_-] = -A_- - A_+e^{2zA_+}, \quad [M, \cdot] = 0, \quad (5.19) \]

where the classical r-matrix is \( r = 2zN \wedge A_+ \). Besides the generator \( M \) there is another central operator which is directly obtained from (5.14) by means of the limit \( \lim_{\varepsilon \to 0} (-\varepsilon^2 C_z) \) giving rise to the expression

\[ C_z = 2NM + \frac{e^{-2zA_+} - 1}{2z} A_- + A_- \frac{e^{-2zA_+} - 1}{2z}. \quad (5.20) \]

Likewise, the corresponding universal \( R \)-matrix is found by contracting (2.21):

\[ R = \exp\{-2zA_+ \otimes N\} \exp\{2zN \otimes A_+\}. \quad (5.21) \]

The boson representation of \( U_z(h_4) \) can be found using the same routine taking into account (5.5), (4.11) plus the extra replacements (5.8). Thus we have

\[
\begin{align*}
A_+ &= a_+, \\
M &= \delta 1, \\
A_- &= \delta e^{2za_+}a_- + \delta \beta z e^{2za_+}, \\
N &= e^{2za_+} - 1 \left( \frac{a_- + \beta}{2} \right).
\end{align*}
\]

Hence the action on the states \( \{|m\rangle\}_{m=0}^\infty \) reads

\[
\begin{align*}
A_+|m\rangle &= \sqrt{m+1} |m+1\rangle, \\
M|m\rangle &= \delta |m\rangle, \\
A_-|m\rangle &= \delta \sqrt{m} |m-1\rangle + \delta \sum_{k=0}^{\infty} \frac{(2z)^{k+1}}{k!} \sqrt{\frac{(m+k)!}{m!}} \left( \frac{m + \beta}{2} \right) |m+k\rangle, \\
N|m\rangle &= (m + \beta) |m\rangle + \sum_{k=1}^{\infty} \frac{(2z)^k}{k!} \sqrt{\frac{(m+k)!}{m!}} \left( \frac{m + \beta}{2} \right) |m+k\rangle.
\end{align*}
\]

The explicit matrix form is

\[ A_- = \delta \begin{pmatrix}
\beta z & 1 & & & \\
2\beta z^2 & (2 + \beta)z & \sqrt{2} & & \\
2\sqrt{2}\beta z^3 & 2\sqrt{2}(1 + \beta)z^2 & (4 + \beta)z & \sqrt{3} & \\
\frac{8}{\sqrt{6}}\beta z^4 & 2\sqrt{6}(\frac{1}{2} + \beta)z^3 & 2\sqrt{3}(2 + \beta)z^2 & (6 + \beta)z & \sqrt{4} \\
\frac{8}{\sqrt{6}}\beta z^5 & 8\sqrt{2}(\frac{1}{2} + \beta)z^4 & 4\sqrt{3}(\frac{3}{2} + \beta)z^3 & 4(3 + \beta)z^2 & (8 + \beta)z \\
& & & & \\
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\end{pmatrix}, \quad (5.24)\]
$$N = \begin{pmatrix}
\beta & \cdot & \cdot & \cdot & \cdot \\
\beta z & 1 + \beta & \cdot & \cdot & \cdot \\
\sqrt{2}\beta z^2 & \sqrt{2}(1 + \beta)z & 2 + \beta & \cdot & \cdot \\
\frac{1}{\sqrt{6}}\beta z^3 & 2\sqrt{6}(\frac{1}{3} + \frac{1}{2}\beta)z^2 & \sqrt{3}(2 + \beta)z & 3 + \beta & \cdot \\
\frac{1}{\sqrt{6}}\beta z^4 & 4\sqrt{2}(\frac{1}{3} + \beta)z^3 & 2\sqrt{3}(\frac{4}{3} + \beta)z^2 & 2(3 + \beta)z & 4 + \beta \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}, \quad (5.25)$$

being (2.9) the matrix for $A_+$ and $\delta 1$ with $1$ the identity matrix for $M$.

Finally, the explicit infinite dimensional representations in the monomial basis $\{x^n\}$ as well as a differential difference realization in terms of the operator (2.31) can be readily obtained. The latter is:

$$A_+ = \partial_x, \quad M = \delta 1, \quad N = D_x x + z \beta D_x + \beta,$$
$$A_- = 2 \delta z D_x x + 2 z \delta \beta z^2 D_x + \delta x + \delta \beta z. \quad (5.26)$$

6 Quadratic algebras

It is sometimes convenient to choose a suitable basis for the quantum Hopf algebras in order to get quadratic commutators. In doing so the computation of commutators and representations at the algebra level is greatly facilitated, although this procedure can add some extra complications at the coalgebra level. We shall briefly consider here that point with the help of an auxiliary deformed boson algebra:

$$\alpha_+ = e^{2z\alpha_+} - \frac{1}{2z}, \quad \alpha_- = \alpha_- + \mu z, \quad (6.1)$$

satisfying

$$[\alpha_-, \alpha_+] = 1 + 2z\alpha_+, \quad (6.2)$$

where $\mu$ is a free parameter to be fixed for each case.

(a) Quadratic $U_z sl(2, \mathbb{R})$.

Let us change to the basis

$$J_+ = e^{2zJ_+} - \frac{1}{2z}, \quad J_3 = J_3, \quad J_- = J_- \quad (6.3)$$

Then, a (deformed) boson realization is written in the following form by taking $\mu = \beta/2$:

$$J_+ = \alpha_+, \quad J_3 = 2\alpha_+ \alpha_- + \beta 1, \quad J_- = -\alpha_+ \alpha_- - \beta \alpha_- + \frac{1}{2} \beta^2 z, \quad (6.4)$$

to be confronted with (2.3). The commutators are easily derived from (6.2):

$$[J_3, J_+] = 2J_+(1 + 2zJ_+),$$
$$[J_3, J_-] = -2J_- + zJ_3,$$
$$[J_+, J_-] = (J_3 - 2zJ_+)(1 + 2zJ_+). \quad (6.5)$$
(b) Quadratic non-standard Poincaré.

Here we can choose \( \mu = 0 \), i.e., \( \bar{\pi}_- = a_- \). Let us define the basis

\[
\begin{align*}
\bar{\mathcal{P}}_+ &= \frac{e^{2z\mathcal{P}_+} - 1}{2z}, & \bar{\mathcal{P}}_- &= \mathcal{P}_-, & \bar{\mathcal{K}} &= \mathcal{K}.
\end{align*}
\]

(6.6)

The boson realization is given in terms of \( \{\bar{\pi}_+, \pi_-\} \) and \( \{b_+, b_-\} \),

\[
\bar{\mathcal{P}}_+ = \pi_+, \quad \bar{\mathcal{P}}_- = ab_+, \quad \bar{\mathcal{K}} = \pi_+\pi_- - b_+b_-,
\]

cf. equations (4.9). Now the commutators are

\[
\begin{align*}
[\bar{\mathcal{K}}, \bar{\mathcal{P}}_+] &= \bar{\mathcal{P}}_+(1 + 2z\bar{\mathcal{P}}_+), & [\bar{\mathcal{K}}, \bar{\mathcal{P}}_-] &= \bar{\mathcal{P}}_-, & [ar{\mathcal{P}}_+, \bar{\mathcal{P}}_-] &= 0.
\end{align*}
\]

(6.8)

(c) Quadratic \( U_z h_4 \).

Now we take the basis generators in the form

\[
\begin{align*}
\bar{\mathcal{A}}_+ &= \frac{e^{2z\mathcal{A}_+} - 1}{2z}, & \bar{\mathcal{A}}_- &= e^{-2z\mathcal{A}_+} A_-, & \bar{\mathcal{N}} &= N, & \bar{\mathcal{M}} &= M.
\end{align*}
\]

(6.9)

The corresponding (deformed) boson realization for \( \mu = \beta \) reads

\[
\begin{align*}
\bar{\mathcal{A}}_+ &= \pi_+, & \bar{\mathcal{A}}_- &= \delta \pi_- , & \bar{\mathcal{N}} &= \pi_+\pi_- + \beta 1, & \bar{\mathcal{M}} &= \delta 1.
\end{align*}
\]

to be compared with (6.9). The commutation rules become

\[
\begin{align*}
[\bar{\mathcal{N}}, \bar{\mathcal{A}}_+] &= (1 + 2z\bar{\mathcal{A}}_+)\bar{\mathcal{A}}_+, & [\bar{\mathcal{N}}, \bar{\mathcal{A}}_-] &= -(1 + 2z\bar{\mathcal{A}}_+)\bar{\mathcal{A}}_-, \\
[\bar{\mathcal{A}}_-, \bar{\mathcal{A}}_+] &= (1 + 2z\bar{\mathcal{A}}_+)\bar{\mathcal{M}}, & [\bar{\mathcal{M}}, \cdot] &= 0.
\end{align*}
\]

(6.11)

7 Concluding remarks

Along this paper we have given a unified treatment for a class of non-standard algebras related to \( U_z \mathfrak{sl}(2, \mathbb{R}) \): the extended \( U_z \mathfrak{sl}(2, \mathbb{R}) \), Poincaré \( U_z \mathcal{P}(1 + 1) \), and oscillator \( U_z h_4 \) deformed algebras. All these algebras share the same Hopf subalgebra (in the \( U_z \mathfrak{sl}(2, \mathbb{R}) \) case is generated by \( \{J_3, J_+\} \)) which leads to a common universal \( R \)-matrix.

At the same time we have computed the representations of these non-standard algebras by means of boson operators. We have remarked when it was suitable the use of one or two-boson algebras to describe each type of representations. The general conclusion is that the functions involved generalize the GD map for the angular momentum, in contrast to the usual Jordan–Schwinger map, that turns out to be more adequate for standard deformations. We have obtained simple closed expressions and practical differential realizations for the \( U_z \mathfrak{sl}(2, \mathbb{R}) \) representations (see also in this respect [17]) which parallel the Lie algebraic classification. For instance, finite dimensional representations have been shown to be labelled by integers \( j_z \) and
we have proven how their coproduct representations decompose by following exactly the classical addition of angular momenta [26].

Contraction processes onto the (one or two) bosonic representations relating all these non-isomorphic quantum algebras have been found. A careful attention was paid to define the most adequate non-standard quantum deformation of the centrally pseudo-extended algebra \( \mathfrak{sl}(2, \mathbb{R}) \). Indeed, it has some original features with respect to other extensions already defined in the context of quantum deformations; for example the altered commutator was not by writing a combination of the primitive generators as usual. At the same time the coproduct of the initial generators was also changed with the help of the new primitive generator \( M \). This extension allows for a contraction to \( U_q \mathfrak{su}_4 \) keeping intact the Hopf subalgebra \( \{ J_3, J_+ \} \).

We would also like to stress that we have preferred to present the properties under study in parallel with the classical situation at the Lie algebra level. In this way we wanted to keep closer to the physical language as well as to appreciate the prime role played by GD like maps in non-standard deformations.

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