Elliptic flow in the Gaussian model of eccentricity fluctuations

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(Dated: February 1, 2008)

We discuss a specific model of elliptic flow fluctuations due to Gaussian fluctuations in the initial spatial $x$ and $y$ eccentricity components $\{\langle (\sigma_y^2 - \sigma_x^2)/(\sigma_x^2 + \sigma_y^2) \rangle, \langle 2\sigma_{xy}/(\sigma_x^2 + \sigma_y^2) \rangle\}$. We find that in this model $v_2\{4\}$, elliptic flow determined from 4-particle cumulants, exactly equals the average flow value in the reaction plane coordinate system, $\langle v_{RP} \rangle$, the relation which, in an approximate form, was found earlier by Bhalerao and Ollitrault in a more general analysis, but under the same assumption that $v_2$ is proportional to the initial system eccentricity. We further show that in the Gaussian model all higher order cumulants are equal to $v_2\{4\}$. Analysis of the distribution in the magnitude of the flow vector, the $Q$–distribution, reveals that it is totally defined by two parameters, $v_2\{2\}$, the flow from 2-particle cumulants, and $v_2\{4\}$, thus providing equivalent information compared to the method of cumulants. The flow obtained from the $Q$–distribution is again $v_2\{4\} = \langle v_{RP} \rangle$.

PACS numbers: 25.75.Ld, 25.75.-q

I. INTRODUCTION

Elliptic flow is an important observable in heavy ion collision experiments, which provides valuable information about the physics of the system evolution starting from very early times. Large elliptic flow values observed recently in experiments at RHIC 1 are often used as an evidence for early system thermalization and as an argument for the creation of a new form of matter, sQGP, the strongly interacting quark-gluon plasma. With high statistics data obtained in the last few years at RHIC the analysis of elliptic flow becomes dominated by systematic uncertainties, mostly by inability to separate the so-called non-flow correlations (azimuthal correlations not related to the orientation of the reaction plane) and the effects of flow fluctuations 2. Flow fluctuations can be due to different reasons: one that has attracted much attention recently is the fluctuations in initial eccentricity of the participant zone. Below we discuss only the flow fluctuations related to eccentricity fluctuations 3, 4, 5. In this paper we review the definitions of the different coordinate systems relevant to flow analysis. Then we discuss a particular model of eccentricity fluctuations. Within this model we show that by studying azimuthal correlations of produced particles at midrapidity it is in principle impossible to separate non-flow correlations from flow fluctuations effects as all observables contain the same combination of the two effects.

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We call the coordinate system defined by the impact parameter and the beam direction the reaction plane coordinate system, and use subscript $RP$ to denote quantities in this system (see Fig. 1). Then the orientation (azimuth) of the impact parameter vector in the laboratory frame is given by $\Psi_{RP}$. The principal axes of the participant zone will define the participant plane coordinate system with the corresponding angle $\Psi_{PP}$, and with the $x_{PP}$ axis pointing in the direction of the semi-minor axis of the participant zone. We use $PP$ subscript for quantities defined in this system.

The orientation of the flow vector $Q = \{Q_x, Q_y\} = \{\sum_i \cos \phi_i, \sum_i \sin \phi_i\}$, where the sum runs over all particles in some momentum window, defines the second harmonic event plane (see Fig. 2) with corresponding azimuth $\Psi_{EP}$, $Q_x = Q \cos 2\Psi_{EP}$, $Q_y = Q \sin 2\Psi_{EP}$. Although we use $Q$ in this paper, in practice one would use $q = Q/\sqrt{N}$ in order to minimize the effect of the multiplicity spread within a centrality bin \[2\]. For a given orientation of the participant plane, $\Psi_{PP}$, anisotropic flow develops along this participant plane.

The orientation of the participant plane can also be characterized by the eccentricity vector with coordinates

$$\varepsilon = \{\varepsilon_x, \varepsilon_y\} = \left\{ \langle \frac{\sigma_y^2 - \sigma_x^2}{\sigma_x^2 + \sigma_y^2} \rangle_{part}, \langle \frac{2\sigma_{xy}}{\sigma_x^2 + \sigma_y^2} \rangle_{part} \right\}, \tag{1}$$

where $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$, $\sigma_y^2 = \langle y^2 \rangle - \langle y \rangle^2$, and $\sigma_{xy} = \langle xy \rangle - \langle y \rangle \langle x \rangle$, and the average is taken over the coordinates of the participants in a given event \[3, 4\]. The eccentricity vector direction is given by $\Psi_{PP} = \text{atan}2(\varepsilon_y, \varepsilon_x)$, and its magnitude, $\varepsilon_{part} = \sqrt{\varepsilon_x^2 + \varepsilon_y^2} \equiv \varepsilon_{PP}$, is called the participant eccentricity (see Figs. 3, 4) in contrast with the reaction plane (or standard) eccentricity $\varepsilon_x \equiv \varepsilon_{RP}$ with its mean value defined to be

$$\langle \varepsilon_x \rangle = \langle \varepsilon_{RP} \rangle \equiv \bar{\varepsilon}. \tag{2}$$

This mean value is approximately $\varepsilon_{opt}$, the optical eccentricity determined by the optical Glauber model \[6\].
III. GAUSSIAN MODEL FOR ECCENTRICITY FLUCTUATIONS

In events with fixed $\varepsilon$, both in magnitude and orientation, the flow vector on average points along $\varepsilon$, but with the magnitude and orientation of the flow vector fluctuating due to finite multiplicity of particles used in its definition. As can be seen from simulations using the MC Glauber model [3, 4, 5] in Fig. 5, the distributions in $\varepsilon_x$ and $\varepsilon_y$ are well approximated by a Gaussian form with widths approximately equal in the two directions. There exists some deviation from a Gaussian form in peripheral collisions, but even there the deviations are small, so we proceed with the Gaussian ansatz. We denote the equal widths in $\varepsilon_x$ and $\varepsilon_y$ by $\sigma_{\varepsilon}$. The distribution in the magnitude of the eccentricity, $\varepsilon_{\text{part}}$, can be obtained by integration over angle of the vector $\varepsilon$ as a two-dimensional Gaussian (see, for example, the derivation in [7]), and is given by

$$
\frac{dn}{d\varepsilon_{\text{part}}} \frac{\varepsilon_{\text{part}}}{\sigma_{\varepsilon}^2}I_0 \left( \frac{\varepsilon_{\text{part}} \langle \varepsilon_{RP} \rangle}{\sigma_{\varepsilon}^2} \right) \exp \left( -\frac{\varepsilon_{\text{part}}^2 + \langle \varepsilon_{RP} \rangle^2}{2\sigma_{\varepsilon}^2} \right) \equiv \text{BG}(\varepsilon_{\text{part}}; \langle \varepsilon_{RP} \rangle, \sigma_{\varepsilon}),
$$

where we have introduced a short hand notation $\text{BG}(x; \bar{x}, \sigma)$ for the “Bessel-Gaussian” distribution with one variable argument and two constant parameters (see Fig. 6). Note that in $\text{BG}(\varepsilon_{\text{part}}; \langle \varepsilon_{RP} \rangle, \sigma_{\varepsilon})$, $\varepsilon_{\text{part}}$ is an eccentricity as given in PP but $\langle \varepsilon_{RP} \rangle$ and $\sigma_{\varepsilon}$ describe the 2-D Gaussian distribution in the RP system. The distribution is normalized to unity. For later use we provide a few moments of the distribution $\text{BG}(x; \bar{x}, \sigma)$, where $x$ is a generic variable (not the $x$-axis):

$$
\langle x \rangle = \frac{1}{2\sigma} \exp \left( -\frac{x^2}{4\sigma^2} \right) \sqrt{\frac{\pi}{2}} \left[ (2\sigma^2 + x^2)I_0 \left( \frac{x^2}{4\sigma^2} \right) + x^2 I_1 \left( \frac{x^2}{4\sigma^2} \right) \right],
$$

$$
\langle x^2 \rangle = \bar{x}^2 + 2\sigma^2,
$$

$$
\langle x^4 \rangle = \bar{x}^4 + 8\bar{x}^2 \sigma^2 + 8\sigma^4,
$$

$$
\langle x^6 \rangle = \bar{x}^6 + 18\bar{x}^4 \sigma^2 + 72\bar{x}^2 \sigma^4 + 48\sigma^6.
$$

Note that the parameter $\sigma$ is not the variance of this distribution; the latter would be given by

$$
\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2,
$$

with $\langle x^2 \rangle$ and $\langle x \rangle$ given above. Also, from Eqs. (4) and (5) it can be shown that

$$
2\langle x^2 \rangle^2 - \langle x^4 \rangle = \bar{x}^4
$$

FIG. 3: Definition of $\varepsilon_{\text{part}}$.  
FIG. 4: Flow vector distribution in events with fixed $\varepsilon$. 



\[3\]
In very central collision, the non-zero eccentricity of the overlap region is defined mostly by fluctuations, \( \langle \varepsilon_{\text{part}} \rangle \gg \langle \varepsilon_{\text{RP}} \rangle \). This limit corresponds to \( \bar{x} \ll \sigma \) in Eqs. (4–8). One finds in this limit \( \langle x \rangle = \sigma \sqrt{\pi/2} \) and \( \sigma_x/\langle x \rangle = \sqrt{4/\pi - 1} \), the relation first derived in [8].

Figure 6 shows the distribution in \( \varepsilon_{\text{part}} \) from the MC Glauber calculation, together with the fit to the BG form. The quality of the fit is good, and the extracted fit parameters shown in Table II agree well with those extracted directly from the distributions of Fig. 5 bottom for \( \varepsilon_x \) and \( \varepsilon_y \).
TABLE I: Comparison of a Gaussian distribution of \( \varepsilon \) in the RP system with the Bessel-Gaussian fit in the PP system for mid-central collision.

| \( \varepsilon \) | \( \sigma_\varepsilon \) |
|-----------------|-----------------|
| \( G, \varepsilon_x \), (Fig. [5]) | 0.1384 ± 0.0001 | 0.0935 ± 0.0001 |
| \( G, \varepsilon_y \), (Fig. [5]) | 0.0000 ± 0.0001 | 0.0923 ± 0.0001 |
| BG (Fig. [6]) | 0.1344 ± 0.0002 | 0.0957 ± 0.0001 |

IV. FLOW FLUCTUATIONS IN A GAUSSIAN MODEL OF ECCENTRICITY FLUCTUATIONS

We start our consideration by deriving the flow vector distribution. One can approach this problem starting from two different coordinate systems: the participant coordinate system or the reaction plane one (see Fig. [4]). In the PP-system the \( y \) coordinate of the flow vector is not affected by flow (and/or flow fluctuations), only the \( x \) component is, which might be taken as a simplification. On the other hand the fluctuations in participant eccentricity (and correspondingly, in flow) have the BG form, which is more difficult to take into account analytically. Somewhat easier (though, obviously, equivalent) is to perform the analysis of the \( Q \)-distribution in the reaction plane system. In the RP-system both components of the flow vector are affected by eccentricity fluctuations, but the fluctuations are of Gaussian form, with the same widths in the \( x \) and \( y \) directions. Assume that on average, flow is proportional to eccentricity with proportionality coefficient \( \kappa \):

\[
\nu_2 = \kappa \varepsilon_{\text{part}}.
\]

For events with fixed \( \varepsilon = \{ \varepsilon_x, \varepsilon_y \} \) this leads to \( \langle Q_x \rangle_\varepsilon = N \kappa \varepsilon_x, \langle Q_y \rangle_\varepsilon = N \kappa \varepsilon_y \). For the overall distribution one finds that the flow vector is a two-dimensional Gaussian distribution with \( \langle Q_x \rangle = N \kappa \langle \varepsilon_{\text{RP}} \rangle, \langle Q_y \rangle = 0 \), and widths in the
two directions given (see \cite{9, 10}) by

\[
\sigma_{Qy}^2 = \left\langle \left( \sum_i \sin 2\phi_i \right)^2 \right\rangle = \frac{1}{2} N [1 - \langle \cos(4\phi_i) \rangle + (N - 1)(2\kappa^2 \sigma_x^2 + \delta)], \tag{12}
\]

\[
\sigma_{Qx}^2 = \left\langle \left( \sum_i \cos 2\phi_i \right)^2 \right\rangle - (N\kappa \langle \varepsilon_x \rangle)^2 = \frac{1}{2} N [1 + \langle \cos(4\phi_i) \rangle - 2\kappa^2 \langle \varepsilon_x \rangle^2 + (N - 1)(2\kappa^2 \sigma_x^2 + \delta)], \tag{13}
\]

where \( N \) is the number of particles, and \( \delta \) is the non-flow contribution defined by \( \langle uu^* \rangle = \langle \cos(2\phi_i - 2\phi_j) \rangle = v_2^2 + \delta \), with \( u \) being the single-particle unit (second harmonic) flow vector. Neglecting the contributions of the fourth harmonic flow and the \( \langle \kappa \langle \varepsilon_x \rangle \rangle^2 \) term, both less than or of the order of \( 10^{-3} \) – \( 10^{-4} \) compared to unity, one finds that the widths in both directions are the same:

\[
\sigma_{Qx}^2 = \sigma_{Qy}^2 = \frac{1}{2} N [1 + (N - 1)(2\kappa^2 \sigma_x^2 + \delta)], \tag{14}
\]

Note that \( \kappa \langle \varepsilon_{RP} \rangle = \langle v_{RP} \rangle \equiv \bar{v} \) gives the real flow as calculated with respect to the reaction plane and the standard deviation of \( v \) along the reaction plane axis is \( \kappa \sigma_x = \sigma_{v_{x}} \). The distribution in flow vector magnitude would be given then by

\[
dn/dQ = BG(Q; N\kappa \langle \varepsilon_{RP} \rangle, \sigma_{Qx}). \tag{15}
\]

Let us now calculate \( v_2 \) from 2-particle and four-particle cumulants \cite{2, 11}, \( v_2 \{2\} \) and \( v_2 \{4\} \), using the Gaussian ansatz for flow fluctuations.

\[
v_2 \{2\}^2 \equiv \langle \cos(2\phi_i - 2\phi_j) \rangle = \langle v_2^2 \rangle + \delta = \kappa^2 \langle \varepsilon_{part}^2 \rangle + \delta. \tag{16}
\]

Using Eq. \cite{2} this becomes

\[
v_2 \{2\}^2 = \kappa^2 \langle \varepsilon_{RP} \rangle^2 + 2\sigma_x^2 + \delta = \langle v_{RP} \rangle^2 + 2\sigma_{v_{x}}^2 + \delta. \tag{17}
\]

Similarly, for the fourth order cumulant result, using Eq. \cite{9},

\[
v_2 \{4\}^4 \equiv 2 \langle \cos(2\phi_i - 2\phi_j) \rangle^2 - \langle \cos(2\phi_i + 2\phi_j - 2\phi_k - 2\phi_m) \rangle = 2 \langle v_2^2 \rangle^2 - \langle v_4^2 \rangle - \bar{v}_2^4 = \bar{v}_2^4 = \langle v_{RP} \rangle^4. \tag{18}
\]

Note that in this approach (Gaussian ansatz) \( v_2 \{4\}^4 \) is always well defined as the cumulant does not change sign. In our model the relation \cite{18} is exact, but in an approximate form (and using a different treatment of the eccentricity fluctuations) it was derived earlier by Bhalerao and Ollitrault \cite{12}, who were the first to note that the fourth order cumulant flow measurements are mostly unaffected not only by non-flow effects but also by flow fluctuations.

Proceeding further, for the difference of the two cumulant results one obtains from Eqs. \cite{17} and \cite{18}

\[
v_2 \{2\}^2 - v_2 \{4\}^2 = 2\kappa^2 \sigma_x^2 + \delta = 2\sigma_{v_{x}}^2 + \delta, \tag{19}
\]

unfortunately the same parameter that defines the \( Q \) distribution width in Eq. \cite{14}. The last observation rules out (in the Gaussian ansatz) the possibility to measure both fluctuations and non-flow by combining information from \( Q \)-distributions and cumulants. Neither do higher order cumulants provide new information. Using Eq. \cite{10} one finds out that

\[
v_2 \{6\}^6 = \left( \langle v_2^6 \rangle - 9 \langle v_2^4 \rangle \langle v_2^2 \rangle + 12 \langle v_2^3 \rangle^2 \right) / 4 = \langle v_{RP} \rangle^6. \tag{20}
\]
One can show that in this model all higher order cumulants are given by the corresponding power of $\langle v_{RP} \rangle$. Another way to look at this is to apply Eqs. (9) and (10) directly to the $Q$ distribution Eq. (15). One finds that the combinations usually associated with flow cumulants [11], are given by corresponding powers of $N v_{RP}$, for example $2 \langle Q^2 \rangle - \langle Q^4 \rangle = (N v_{RP})^4$.

V. FITTING $Q$-DISTRIBUTIONS

As can be seen by comparing Eqs. (14) and (19), $v^2\{2\}$ and $v^2\{4\}$ completely define the form of the $Q$-distribution, and can be used as an alternative set of parameters compared to that in Eq. (15). If one tries to fit the $Q$-distribution with a functional form determined by three parameters, e.g. $\langle v \rangle$, $\sigma_v$, and $\delta$, these parameters should satisfy the values of $v^2\{2\}$ and $v^2\{4\}$ (which provides only two equations), and all three can not be determined.

There can be different functional forms used to describe flow fluctuations along the PP axis. Most often used are the Gaussian form $G(v; \langle v \rangle, \sigma_v)$ and the Bessel-Gaussian $BG(v; v_0, \sigma)$ discussed above. Both of them have two parameters, which, as we know can not be determined separately, so they must be correlated.

Assuming the $BG(v; v_0, \sigma)$ form for flow fluctuations to fit the $Q$-distribution, which would correspond to a two-dimensional Gaussian distribution in the reaction plane coordinate system, one would find from Eqs. (17) and (18) that the parameters are correlated according to

$$v^2\{2\}^2 = const = v_0^2 + 2\sigma^2 + \delta$$
$$v^2\{4\} = const = v_0.$$  \hspace{1cm} (21)

The mean and the variance of the $v$ distribution would be given by Eqs. (4) and (8), but since $\sigma$ can not be determined independent of $\delta$, $\langle v \rangle$ is also undetermined.

If one uses the Gaussian form for flow fluctuations in the $PP$-system, one would find that the parameters are correlated according to

$$v^2\{2\}^2 = const = \langle v \rangle^2 + \sigma_v^2 + \delta$$
$$v^2\{4\}^2 = const = \sqrt{\langle v \rangle^4 - 2\langle v \rangle^2 \sigma_v^2} \approx \langle v \rangle^2 - \sigma_v^2,$$ \hspace{1cm} (23)

or equivalently

$$v^2\{2\}^2 - v^2\{4\}^2 = const \approx 2\sigma_v^2 + \delta.$$ \hspace{1cm} (24)

The above two equations are derived in the approximation of $\sigma_v \ll \langle v \rangle$ but for Gaussian fluctuations in $v$ the exact formula can be used. Again, as $\sigma$ can not be determined independently, $\langle v \rangle$ is also undetermined.

VI. SUMMARY

We find that in the Gaussian ansatz, fitting $Q$-distributions does not bring any more information than that provided by cumulants. It is not surprising - if the distribution is defined just by two parameters one can not get more than $v^2\{2\}$ and $v^2\{4\}$ already provided. Note that under this ansatz all the higher order cumulant $v$ values are the same.
The origin of the “problem” can be traced to the Gaussian ansatz. It is known that for a Gaussian distribution all the cumulants higher than rank two are zero. The latter means that if the collective fluctuations are of the Gaussian type one can never prove that the fluctuations exist by any type of correlation analysis using only particles under consideration (no external information). A similar problem was observed earlier in a temperature fluctuation study of many-particle transverse momentum correlations [14]. Unfortunately, deviations from a Gaussian distribution might be too small to observe. Such deviations would show up in the bad quality of the $Q$-distribution fits based on the Gaussian ansatz, or in a small differences between higher order cumulant $v$ values.

The fact that all higher order cumulants are the same and determined by the value of flow in the reaction plane (not the participant plane), and that that fitting of $Q$-distribution yields the same value, explains the consistency between $v_2\{4\}$ and $v_2\{\text{ZDCSMD}\}$ [13], which is calculated with ZDC-SMD as event plane and is supposed to be sensitive to $v_2$ in the reaction plane, as well as the consistency between $v_2\{4\}$ and $v_2\{Q-dist\}$ [2].

Ref. [15] used a model of flow fluctuations in which flow fluctuates only in the impact parameter direction. The use of the detectors which measure spectator neutrons, advocated in [15], is justified for that model, but would yield zero results for the case of fluctuations discussed in this paper.

The authors thank P. Sorensen for fruitful discussion. This work was supported in part by the HENP Divisions of the Office of Science of the U.S. Department of Energy.

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