Anisotropic spacetimes in chiral scalar field cosmology

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Abstract We study the behaviour and the evolution of the cosmological field equations in an homogeneous and anisotropic spacetime with two scalar fields coupled in the kinetic term. Specifically, the kinetic energy for the scalar field Lagrangian is that of the Chiral model and defines a two-dimensional maximally symmetric space with negative curvature. For the background space we assume the locally rotational spacetime which describes the Bianchi I, the Bianchi III and the Kantowski–Sachs anisotropic spaces. We work on the $H$-normalization and we investigate the stationary points and their stability. For the exponential potential we find a new exact solution which describes an anisotropic inflationary solution. The anisotropic inflation is always unstable, while future attractors are the scaling inflationary solution or the hyperbolic inflation. For scalar field potential different from the exponential, the de Sitter universe exists.

1 Introduction

Gravitational models with two or more scalar fields for the description of the matter part for the gravitational field equations have been widely studied in the literature during recent years [1–13]. Multi-scalar field cosmological models have been used as alternative mechanisms for the description of inflation [14] as also as unified dark energy models. Indeed, because of the additional degrees of freedom provided by the scalar field, the exit from the inflationary era is different from the single-scalar field theory. Specifically, it is not necessary the values for the scalar fields to be the same at the beginning of the inflation and at the end of the inflation. Hence, the curvature perturbations can be affected by the different number of e-folds [15,16]. On the other hand, multi-scalar fields provide non-adiabatic field perturbations which generate observable non-Gaussianities in the power spectrum [17–19]. As far as the late time universe is concerned, multi-scalar field models provide dark energy models which can cross the phantom divide line without the presence of ghosts [20], as also to describe the dark matter component of the universe [21].

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In this study, we focus upon the asymptotic dynamics for the field equations in a two-scalar field theory known as the Chiral model within an homogeneous and anisotropic background space [22–24,26]. The kinetic energy of the two scalar field lies on a two-dimensional maximally symmetric space of negative curvature, hyperbolic space. This multi-scalar field model provides the so-called hyperbolic inflation [23,25]. However, there are various applications of this model and in other areas of the cosmic evolution [26–35].

On the other hand, homogeneous and anisotropic are mainly expressed by the Bianchi class of spatially homogeneous spacetimes. Bianchi spacetimes have been mainly used for the discussion of anisotropies in the very early universe [36–38]. The presence of a cosmological constant in Bianchi spacetimes leads to isotropic universe as a future solution [39]. For the physical space in this study we assume the generic line element [40]

\[ ds^2 = -dr^2 + e^{2\alpha(t)} (e^{2\beta(t)} \, dx^2 + e^{-\beta(t)} (dy^2 + f^2(y) \, dz^2)) , \]

(1)

where the function \( f(y) \) has one of the following forms, \( f_A(y) = 1, f_B(y) = \sinh(\sqrt{|K|}y) \) and \( f_C(y) = \sin(\sqrt{|K|}y) \). The line element (1) corresponds to homogeneous locally rotational spacetimes (LRS) induced with four isometries. For \( f_A(y) \) the spacetime is that of Bianchi I, for \( f_B(y) \) is that of the Bianchi III metric while for \( f_C(y) \) the line element reduces to that of Kantowski-Sachs. These three different families of spacetimes reduce to the flat, closed and open Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes when the parameter, \( \beta(t) \), becomes constant. Indeed, the parameter, \( \beta(t) \), is the anisotropic parameter while \( \alpha(t) \) is the scale factor for the three-dimensional hypersurface. These spacetimes play an important role on the description of the very early universe and specifically during the pre-inflationary era [41–49].

In the following we investigate the asymptotic dynamics and the evolution for the field equations by investigating the stationary points for the field equations [50]. Every stationary point describes a specific era for the evolution of the field equations [51]. The analysis of the stability properties for the stationary points is essential in order to construct cosmological history [52,53]. Such an analysis provides important information for the viability of a given gravitational theory [54]. In addition this analysis provides important information about the initial condition problem. Such analysis has been widely studied in various gravitational models [55–57] while some studies in anisotropic universes can be found in [8,58–63]. The plan of the paper is as follows.

In Sect. 2 we present the field equations for the Chiral theory with anisotropic background space described by the anisotropic line element (1). Section 3 includes the new results of this analysis in which we study the general evolution and the asymptotic behaviour for the field equations for the Chiral theory for the potential function of the hyperbolic inflation [23]. In Sect. 4 we investigate the dynamics for a scalar field potential beyond the exponential. Finally, in Sect. 5 we summarize the results and we draw our conclusions.

2 Field equations

We assume the gravitational model in a Riemann manifold \( g_{\mu\nu}(x^\kappa) \) and Ricci scalar \( R(x^\kappa) \) with two-scalar fields minimally coupled to the gravity, which is described by the following Action Integral.

\[ S = \int \sqrt{-g} \, dx^4 \left( \frac{R}{2} + L_C (\phi, \nabla_\mu \phi, \psi, \nabla_\mu \psi) \right) . \]

(2)
Lagrangian \( L_C (\phi, \nabla \mu \phi, \psi, \nabla \mu \psi) \) is assumed to be that of the Chiral model, that is

\[
L_C (\phi, \nabla \mu \phi, \psi, \nabla \mu \psi) = -\frac{1}{2} g^{\mu \nu} (\nabla_{\mu} \phi \nabla_{\nu} \phi + e^{-2\kappa \phi} \nabla_{\mu} \psi \nabla_{\nu} \psi) + V (\phi). \tag{3}
\]

Consequently, the two scalar fields, \( \phi \) and \( \psi \), are defined on a two-dimensional space of constant and negative curvature, while their evolution is defined on the physical space with metric \( g_{\mu \nu} \). In the following we assume that \( \kappa \neq 0 \) and the scalar field potential is that of the hyperbolic inflation, that is, \( V (\phi) = V_0 e^{-\lambda \phi} \).

For the line element \( (1) \) we derive

\[
R (\alpha, \dot{\alpha}, \beta, \dot{\beta}) = 6\ddot{\alpha} + 12\dot{\alpha}^2 + \frac{3}{2} \dot{\beta}^2 - 2e^{\beta - 2\alpha} K \tag{4}
\]

and \( \sqrt{-g} = e^{3\alpha} \), where the overdot means total derivative with respect the independent parameter \( t \), i.e. \( \dot{\alpha} = \frac{da}{dt} \).

Hence, by substituting into \( (2) \) and assuming that the scalar fields inherit the symmetries of the background space, we obtain the following system of second-order differential equations \[64\]

\[
2\ddot{\alpha} + 3\dot{\alpha}^2 + \frac{3}{4} \dot{\beta}^2 + \frac{1}{2} (\dot{\phi}^2 + e^{-2\kappa \phi} \dot{\psi}^2) - V (\phi) - \frac{1}{3} e^{-2\alpha - \beta} K = 0, \tag{5}
\]

\[
\ddot{\beta} + 3\dot{\alpha} \dot{\beta} + \frac{2}{3} e^{-2\alpha - \beta} K = 0, \tag{6}
\]

\[
\ddot{\phi} + \kappa e^{-2\kappa \phi} \dot{\psi}^2 + 3\dot{\alpha} \dot{\phi} + V_{,\phi} = 0, \tag{7}
\]

\[
\ddot{\psi} - 2\kappa \phi \dot{\psi} + 3\dot{\alpha} \dot{\psi} = 0 \tag{8}
\]

and the constraint equation

\[
e^{3\alpha} \left( 3\dot{\alpha}^2 - \frac{3}{4} \dot{\beta}^2 - \frac{1}{2} (\dot{\phi}^2 + e^{-2\kappa \phi} \dot{\psi}^2) - V (\phi) \right) - e^{\alpha - \beta} K = 0. \tag{9}
\]

The parameter \( K \) denotes the spatial curvature of the three-dimensional hypersurface part for \( (1) \). Indeed, for Bianchi I space \( K = 0 \), for Bianchi III space is positive \( K > 0 \) while for the Kantowski-Sachs space, \( K < 0 \).

### 2.1 Dimensionless variables

In order to study the global evolution of the field equations we define the new set of variables

\[
\Sigma = \frac{\dot{\beta}}{2H}, \quad x = \frac{\dot{\phi}}{\sqrt{6}H}, \quad y^2 = \frac{V (\phi)}{3H^2}, \quad z = \frac{e^{-\kappa \phi} \dot{\psi}}{\sqrt{6}H}, \quad \omega_R = \frac{R^{(3)}}{3H^2}, \tag{10}
\]

where \( R^{(3)} = e^{\alpha - \beta} K \) and \( H (t) = \dot{\alpha} \) is the expansion rate.

Moreover, we select the new independent variable to be \( dt = d\tau, \tau = \alpha \). Thus, in the new variables the field Eqs. \( (5)-(8) \) read

\[
\Sigma' = -y^2 (1 + \Sigma) + (2\Sigma - 1) (x^2 + z^2 + \Sigma^2 - 1), \tag{12}
\]

\[
x' = 2x^3 + \frac{\sqrt{6}}{2} (y^2 \lambda - 2z^2 \kappa) - x (y^2 - 2 (z^2 + \Sigma^2 - 1)), \tag{13}
\]
\[ y' = \frac{1}{2} y \left( 2 (1 - y^2) + 4 (x^2 + z^2 + \Sigma^2) - \sqrt{6} x \lambda \right) \]  
\[ (14) \]

and

\[ z' = z \left( \sqrt{6} \kappa + 2 (x^2 + z^2 + \Sigma^2 - 1) - y^2 \right), \]  
\[ (15) \]

where \( V(\phi) = V_0 e^{-\lambda \phi}, \lambda = \frac{V_{\phi}^2}{V}, \) and \( \Sigma' = \frac{d\Sigma}{d\tau}. \) Furthermore, constraint equation (9) reduces to the following algebraic equation

\[ \omega_R = 1 - \left( \Sigma^2 + x^2 + y^2 + z^2 \right). \]  
\[ (16) \]

By definition, the parameter \( y \) is positive, while the field equations remain invariant under the discrete transformation \( z \rightarrow -z. \) Hence we select to work with \( z > 0. \)

Moreover, we define the deceleration parameter \( q = -1 - \frac{\ddot{a}}{a^2}, \) which with the use of the dimensionless variables is

\[ q (\Sigma, x, y, z) = 2 \left( x^2 + z^2 + \Sigma^2 \right) - y^2. \]  
\[ (17) \]

3 Asymptotic dynamics

We continue our analysis with the study of the dynamics provided by dynamical system (12)–(16). Specifically, we determine the stationary points and we investigate their stability. Every stationary point corresponds to a specific era in the evolution of cosmological history. We summarize the stationary points, \( P = (\Sigma(P), x(P), y(P), z(P)), \) in three categories, (A) stationary points of General Relativity; (B) stationary points of quintessence and (C) stationary points with two scalar fields. The first family of stationary points describes exact solutions without any matter source. Thus, the exact solutions described by these points are those of General Relativity in the vacuum \((x, y, z) = (0, 0, 0).\) For the family (B) of points, only the scalar field \( \phi \) contributes in the cosmological solution, that is \( z = 0, \) \((x, y) \neq (0, 0),\) while for the third family of points, both the scalar fields contribute, i.e. \( \dot{\phi} \dot{\psi} \neq 0. \) It is important to mention that the stability properties of the points on the families (A) and (B) depend upon the existence of the second field, that is, of the dynamical variable \( z. \) Thus we should perform a detailed analysis on the stability conditions. Moreover, stationary points with \( \Sigma = 0, \) correspond to isotropic background space, while stationary points with \( \eta = 0, \) indicate that the exact solution is a static solution. Furthermore, the background space in an asymptotic solution is that of Bianchi I or spatially flat FLRW metric when \( \omega_R = 0, \) of Bianchi III or closed FLRW metric when \( \omega_R > 0, \) or Kantowski-Sachs or open FLRW universe when \( \omega_R < 0. \)

We determine the stationary points for dynamical system (12)–(16) for values of the dynamical variables in the finite regime.

3.1 Stationary points of family A

The stationary points which belong to the family A are

\[ A_1^\pm = (\pm 1, 0, 0, 0) , \quad A_2 = \left( \frac{1}{2}, 0, 0, 0 \right). \]

For each of the stationary points we calculate \( \omega_R (A_1^\pm) = 0 \) and \( \omega_R (A_2^\pm) = \frac{3}{4}. \)

We continue with the discussion of the physical properties for the asymptotic solutions at the stationary points while we investigate the stability properties of the points.
Points $A_1^\pm$ describe anisotropic spacetime with zero spatial curvature, that is, the asymptotic solution at the points correspond to Kasner universes. The eigenvalues of the linearized system near the asymptotic points are $e_1 \left(A_1^+\right) = 6$, $e_2 \left(A_1^+\right) = 3$, $e_3 \left(A_1^+\right) = 0$ and $e_4 \left(A_1^+\right) = 0; e_1 \left(A_1^-\right) = 3$, $e_2 \left(A_1^-\right) = 2$, $e_3 \left(A_1^-\right) = 0$ and $e_4 \left(A_1^-\right) = 0$. Thus, points $A_1^\pm$ are sources and the asymptotic solutions are always unstable.

Point $A_2$ describes an anisotropic exact solution with nonzero spatial curvature. The eigenvalues are $e_1 \left(A_3^-\right) = \frac{3}{2} , e_2 \left(A_3^-\right) = -\frac{3}{2} , e_3 \left(A_3^-\right) = -\frac{3}{2}$ and $e_4 \left(A_3^-\right) = -\frac{3}{2}$ from which we conclude that the vacuum anisotropic solution is unstable, while point $A_2$ is a saddle point for the dynamical system.

For all the stationary points we derive positive value for the deceleration parameter, $q \left(A_1^+\right) = 2, q \left(A_2\right) = \frac{1}{2}$.

3.2 Stationary points of family B

The family B for the stationary points of dynamical system (12)–(16) is consist of the points,

$$
B_1^\pm = \left(\pm \sqrt{1-x^2}, x, 0, 0\right),
\quad B_2 = \left(0, \frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}}, 0\right),
\quad B_3 = \left(1, 1 - \frac{3}{2 (1 + \lambda^2)}, \frac{\sqrt{6} \lambda}{2 (1 + \lambda^2)}, \sqrt{6} \left(2 + \frac{\lambda^2}{2}\right), 0\right)
$$

with $\omega_R \left(B_1^\pm\right) = 0, \omega_R \left(B_2^\pm\right) = 0$, and $\omega_R \left(B_3^\pm\right) = \frac{3 (\frac{\lambda^2}{2} - 4)}{4 (1 + \lambda^2)}$.

Points $B_1^\pm$, exist when $x^2 \leq 1$, for $x^2 < 1$. They describe families of points where the asymptotic solutions are Kasner universes, respectively. Moreover, for $x^2 = 1$, the solution is that of spatially flat FLRW universe dominated by a stiff fluid. For the asymptotic solutions at the stationary points the decelerating parameter is derived to be $q = 2$. Hence, the points do not describe acceleration. The eigenvalues are derived to be $e_1 \left(B_1^\pm\right) = 2 \left(\pm \sqrt{1 - x^2}\right), e_2 \left(B_1^\pm\right) = \sqrt{6} \kappa x, e_3 \left(B_1^\pm\right) = \frac{1}{2} \left(6 - \sqrt{6} \kappa \lambda\right)$ and $e_4 \left(B_1^\pm\right) = 0$. Consequently, points $B_1^\pm$ are sources or saddle points. Specifically, points $B_2^\pm$ are sources when $\left\{\kappa < 0, -1 \leq x < 0, \lambda > \frac{\sqrt{6}}{\lambda}\right\}$ or $\left\{\kappa > 0, 0 < x < 1, \lambda < \frac{\sqrt{6}}{\lambda}\right\}$. Otherwise they are saddle points.

Point $B_2$ is real and physically accepted when $\lambda^2 \leq 6$ and describes the scaling solution for the quintessence field with the exponential scalar field potential in a spatially flat FLRW background space. The deceleration parameter is calculated to be $q \left(B_2\right) = \frac{\lambda^2 - 2}{2}$, which means that for $\lambda^2 < 2$ the asymptotic solution describes an accelerating universe. The eigenvalues of the linearized system are $e_1 \left(B_2\right) = \frac{\lambda^2 + 2 \sqrt{\lambda^2 - 6}}{2}, e_2 \left(B_2\right) = \frac{\lambda^2 - 6}{2}, e_3 \left(B_2\right) = \frac{\lambda^2 - 6}{2}, e_4 \left(B_2\right) = (\lambda^2 - 2)$. Hence, the asymptotic solution at the point $B_2$ is stable and the point $B_2$ is an attractor when $\left\{-\sqrt{2} < \lambda < 0, \kappa > 6 - \frac{\lambda^2}{2\kappa}\right\}$ or $\left\{0 < \lambda < \sqrt{2}, \kappa < 6 - \frac{\lambda^2}{2\kappa}\right\}$. The region where point $B_2$ is an attractor is presented in Fig. 1.

Point $B_3$ describes the exact solution with anisotropic spacetime, with positive spatial curvature when $\lambda^2 > 4$, or with negative spatial curvature when for $\lambda^2 < 4$, while for $\lambda^2 = 4$ it describes a Bianchi I universe. The deceleration parameter is $q \left(B_3\right) = \frac{\lambda^2 - 2}{2(\lambda^2 + 1)}$. Hence there is acceleration for $\lambda^2 < 2$. The eigenvalues of the linearized system are $e_1 \left(B_3\right) =
Fig. 1 Region plot in the space of the free parameters \((\lambda, \kappa)\) where the asymptotic solutions at points \(B_2, B_3\) and \(C_1\) are stable solutions and points are attractors, and when point \(C_2\) is physically accepted and describes anisotropic inflation.

\[
-\frac{3(2-2\lambda+\lambda^2)}{2(1+\lambda^2)}, \quad e_2(B_3) = -\frac{3(2+3\lambda^2+\lambda^4)}{2(1+\lambda^2)}, \quad e_3(B_3) = -\frac{3\left(2+3\lambda^2+\lambda^4+\sqrt{((1+\lambda)^2(2+\lambda^2)(7\lambda^2-18))}\right)}{4(1+\lambda^2)^2},
\]

and \(e_4(B_3) = -\frac{3\left(2+3\lambda^2+\lambda^4+\sqrt{((1+\lambda)^2(2+\lambda^2)(7\lambda^2-18))}\right)}{4(1+\lambda^2)^2}\). In Fig. 1 we present the region in the space \((\lambda, \kappa)\) in which the point \(B_3\) is an attractor.
3.3 Stationary points of family C

The third family of stationary points for dynamical (12)–(16) includes the points

\[ C_1 = \left( 0, \frac{\sqrt{6}}{2\kappa + \lambda}, \frac{\sqrt{2\kappa}}{2\kappa + \lambda}, \frac{\sqrt{\lambda^2 + 2\kappa \lambda - 6}}{2\kappa + \lambda} \right), \]

\[ C_2 = \left( 2 - \frac{6\kappa}{2\kappa + \lambda}, \frac{\sqrt{6}}{2\kappa + \lambda}, \frac{\sqrt{6\kappa (2\kappa - \lambda)}}{2\kappa + \lambda}, \frac{\sqrt{6\kappa \lambda - 3(\lambda^2 + 2)}}{2\kappa + \lambda} \right). \]

The stationary point, \( C_1 \), exists when \( 2\kappa + \lambda \neq 0 \) and \( \{ \lambda \leq -\sqrt{6}, \kappa < 0 \} \), \( \{ -\sqrt{6} < \lambda < 0, \kappa < \frac{6 - \lambda^2}{2\lambda} \} \), \( \{ 0 < \lambda < \sqrt{6}, \kappa > \frac{6 - \lambda^2}{2\lambda} \} \), while the spatial curvature for the background space is zero, that is, \( \omega_R (C_1) = 0 \). Hence the background space at the stationary point is that of spatially flat FLRW universe. The point \( C_1 \) describes hyperbolic inflation [23]. The deceleration parameter is written as \( q (C_1) = 2 - \frac{6\kappa}{2\kappa + \lambda} \). Hence, the asymptotic solution at \( C_1 \) describes inflationary isotropic inflation when \( \{ \lambda \leq -\sqrt{2}, \kappa < \lambda \} \), \( \{ -\sqrt{2} < \lambda < 0, \kappa < \frac{6 - \lambda^2}{2\lambda} \} \), \( \{ 0 < \lambda < \sqrt{2}, \kappa > \frac{6 - \lambda^2}{2\lambda} \} \) and \( \{ \lambda \geq \sqrt{2}, \kappa > \lambda \} \). In Fig. 1 we present the region in the space \((\lambda, \kappa)\) in which the point \( C_1 \) is an attractor. Note that, when \( C_1 \) is an attractor, the asymptotic solution describes an inflationary universe, i.e. \( q (C_1) < 0 \).

The point \( C_2 \) describes anisotropic solutions with \( \omega_R (C_2) = -\frac{12\kappa (\kappa - \lambda)}{(2\kappa + \lambda)^2} \). The points are real and physical acceptable when \( 2\kappa \lambda > (2 + \lambda^2)^2 \). The deceleration parameter is derived to be \( q (C_2) = 2 - \frac{6\kappa}{2\kappa + \lambda} \). Consequently, when the point is real and physically accepted, it describes accelerated anisotropic inflationary solution in a Kantowski-Sachs background space. We derive the eigenvalues of the linearized system and we find that the four eigenvalues do not have real parts with negative values, for the same values of the parameters, \( \lambda \) and \( \kappa \). Moreover, they do not have real parts with positive values for the same values of the parameters, \( \lambda \) and \( \kappa \). We conclude that the asymptotic anisotropic solution is unstable and point \( C_2 \) is a saddle point. In Fig. 1 we present the region in the two-dimensional space \((\lambda, \kappa)\) where the point is real and physical acceptable.

We summarize the results of this analysis in Table 1. We present the stationary points, their physical properties as also we summarize their stability conditions.

4 Beyond the exponential potential

We proceed with our analysis by considering a potential function different from the exponential potential. In particular, we assume the existence of a cosmological constant term, such that the scalar field potential is

\[ V (\phi) = V_0 \left( e^{-\sigma \phi} - \Lambda \right). \]

For this potential function parameter \( \lambda = \frac{V_0}{V} \) is not a constant, but it depends upon time variable. Indeed, \( \lambda = \frac{\sigma e^{-\sigma \phi}}{e^{-\sigma \phi} - \Lambda} \), such that \( \phi = -\frac{1}{\sigma} \ln \left( \frac{\lambda \Lambda}{\lambda - \sigma} \right) \). Consequently, the derivative of \( \lambda \) is different from zero, that is,

\[ \lambda' = \sqrt{6} x \lambda (\sigma - \lambda). \]
Table 1 Stationary points and their stability for the anisotropic Chiral model

| Point          | (Σ, x, y, z) | $\omega_R$ | q | Stability       |
|---------------|-------------|------------|---|---------------|
| $A_1^{\pm}$   | $\pm(1, 0, 0, 0)$ | 0          | 2 | Source        |
| $A_2$         | $1 \pm (0, 0, 0)$ | $\frac{3}{4}$ | $\frac{1}{2}$ | Saddle        |
| $B_1^{\pm}$   | $\pm\sqrt{1 - x^2}, x, 0, 0$ | 0          | 2 | Source/Saddle |
| $B_2$         | $0, \frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}}, 0$ | 0          | $\frac{\lambda^2 - 2}{2}$ | Attractor Fig. 1 |
| $B_3$         | $\left(2 \left(1 - \frac{3}{2(1 + \lambda^2)}\right), \frac{\sqrt{6x \lambda}}{2(1 + \lambda^2)}, \frac{\sqrt{6(2 + \lambda^2)}}{2(1 + \lambda^2)}, 0\right)$ | $\frac{3(\lambda^2 - 4)}{4(1 + \lambda^2)^2}$ | $\frac{\lambda^2 - 2}{2(\lambda^2 + 1)}$ | Attractor Fig. 1 |
| $C_1$         | $0, \frac{\sqrt{6x \lambda}}{2(1 + \lambda^2)}, \frac{\sqrt{6x^2 \lambda}}{2(1 + \lambda^2)}, \frac{\sqrt{6(2 + \lambda^2)}}{2(1 + \lambda^2)}$ | 0          | $2 - \frac{6x \lambda}{2x + \lambda}$ | Attractor Fig. 1 |
| $C_2$         | $\left(2 - \frac{6x \lambda}{2x + \lambda}, \frac{\sqrt{6x \lambda}}{2(1 + \lambda^2)}, \frac{\sqrt{6x^2 \lambda}}{2(1 + \lambda^2)}, \frac{\sqrt{6(2 + \lambda^2)}}{2(1 + \lambda^2)}\right)$ | $-\frac{12x(\lambda - \lambda)}{(2x + \lambda)^2}$ | $2 - \frac{6x \lambda}{2x + \lambda}$ | Saddle |

Therefore, for a non-exponential scalar field potential, the gravitational field equations have one extra dimension, i.e. Eq. (19). Because of the existence of Eq. (19) new stationary points follow, but the stability properties of the previous points may change.

As above, we summarize the stationary points $P = (\Sigma (P), x (P), y (P), z (P), \lambda (P))$ on three families of points, families $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. The stationary points are categorized according to the contribution of the scalar field in the cosmological fluid as above.

4.1 Stationary points of the family $\tilde{A}$

The family $\tilde{A}$ is defined by the stationary points

$$\tilde{A}_1^{\pm} = (\pm 1, 0, 0, 0, \lambda), \quad \tilde{A}_2 = \left(\frac{1}{2}, 0, 0, 0, \lambda\right), \quad \lambda \text{ arbitrary}. \quad (20)$$

The physical properties of the asymptotic solutions at the stationary points are similar to those of the exponential potential. The eigenvalues for the five-dimensional linearized system are derived to be $e_1 (\tilde{A}_1^{+}) = 6$, $e_2 (\tilde{A}_1^{+}) = 3$, $e_3 (\tilde{A}_1^{+}) = 0$, $e_4 (\tilde{A}_1^{+}) = 0$, $e_5 (\tilde{A}_1^{+}) = 0$; $e_1 (\tilde{A}_1^{-}) = 3$, $e_2 (\tilde{A}_1^{-}) = 2$, $e_3 (\tilde{A}_1^{-}) = 0$, $e_4 (\tilde{A}_1^{-}) = 0$, $e_5 (\tilde{A}_1^{-}) = 0$; $e_1 (\tilde{A}_2) = \frac{3}{2}$, $e_2 (\tilde{A}_2) = -\frac{3}{2}$, $e_3 (\tilde{A}_2) = -\frac{3}{2}$, $e_4 (\tilde{A}_2) = -\frac{3}{2}$, $e_5 (\tilde{A}_2) = 0$. We observe that the stability properties do not change for the stationary points. Hence, the points $\tilde{A}_1^{\pm}$ are sources, while the point $\tilde{A}_2$ is a saddle point.

4.2 Stationary points of family $\tilde{B}$

Family $\tilde{B}$ is composed of the following points

$$\tilde{B}_1^{\pm} = \left(\pm\sqrt{1 - x^2}, x, 0, 0, 0, \sigma\right), \quad \tilde{B}_2 = \left(0, \frac{\sigma}{\sqrt{6}}, \sqrt{1 - \frac{\sigma^2}{6}}, 0, 0, \sigma\right),$$

$$\tilde{B}_3 = \left(\frac{1}{2} \left(1 - \frac{3}{2(1 + \sigma^2)}\right), \frac{\sqrt{6} \sigma}{2(1 + \sigma^2)}, \frac{\sqrt{6(2 + \sigma^2)}}{2(1 + \sigma^2)}, 0, 0, \sigma\right).$$
The stationary points, $\tilde{B}_1^\pm$, $\tilde{B}_2$ and $\tilde{B}_3$, have the same physical properties with the corresponding points of family $B$. Thus we study only their stability properties.

The eigenvalues for the linearized system around $\tilde{B}_1^\pm$ are $e_1 (\tilde{B}_1^\pm) = 2 \left( 2 + \sqrt{1 - x^2} \right)$, $e_2 (\tilde{B}_1^\pm) = \sqrt{6} x \kappa$, $e_3 (\tilde{B}_1^\pm) = - \sqrt{6} x \sigma$, $e_4 (\tilde{B}_1^\pm) = \frac{1}{2} \left( 6 - \sqrt{6} x \sigma \right)$, $e_5 (\tilde{B}_1^\pm) = 0$. Therefore, the family of points $\tilde{B}_1^\pm$ are saddle points. Moreover, the eigenvalues around $\tilde{B}_2$ are $e_1 (\tilde{B}_2) = \frac{\sigma^2 + 2 \sigma - 6}{2}$, $e_2 (\tilde{B}_2) = \frac{\sigma^2 - 6}{2}$, $e_3 (\tilde{B}_2) = - \sqrt{6} x \sigma$, $e_4 (\tilde{B}_2) = 2^2 - 2$, $e_5 (\tilde{B}_2) = - \sigma^2$, that is, point $\tilde{B}_2$ has similar stability properties with point $B_2$, as presented in Fig. 1. Moreover, for point $\tilde{B}_3$ we calculate the eigenvalues $e_1 (\tilde{B}_3) = - \frac{3(2 - 2 \kappa \sigma + \sigma^2)}{2(1 + \sigma^2)}$, $e_2 (\tilde{B}_3) = - \frac{3(2 + 3 \sigma^2 + \sigma^4)}{2(1 + \sigma^2)}$, $e_3 (\tilde{B}_3) = - \frac{3(2 + 3 \sigma^2 + \sigma^4 + \sqrt{(1 + \sigma^2)(2 + \sigma^2)(7 \sigma^2 - 18)})}{4(1 + \sigma^2)^2}$, $e_4 (\tilde{B}_3) = - \frac{3(2 + 3 \sigma^2 + \sigma^4 + \sqrt{(1 + \sigma^2)(2 + \sigma^2)(7 \sigma^2 - 18)})}{4(1 + \sigma^2)^2}$ and $e_5 (\tilde{B}_3) = - \frac{3 \sigma^2}{1 + \sigma^2}$. The eigenvalue $e_5 (\tilde{B}_3)$ is always negative, while the rest are similar to those of point $B_3$. Thus, point $\tilde{B}_3$ is an attractor in the region presented in Fig. 1, where $\lambda = \sigma$.

The stationary points $\tilde{B}_4^\pm$, $\tilde{B}_5$, $\tilde{B}_6$ correspond to asymptotic solutions for which the scalar field potential plays the role of the cosmological constant, that is $V, \phi = 0$, and $V (\phi) = const$. Points $\tilde{B}_4^\pm$ have the same physical properties with $\tilde{B}_1^\pm$, while the eigenvalues of the linearized system are the same, for $\sigma = 0$.

The point $B_5$ describes an anisotropic inflationary exact solution in a Kantowski-Sachs spacetime, $\omega_R (\tilde{B}_5) = -3$, $q (\tilde{B}_5) = -1$. The eigenvalues are $e_1 (\tilde{B}_5) = -6$, $e_2 (\tilde{B}_5) = -3$, $e_3 (\tilde{B}_5) = -3$, $e_4 (\tilde{B}_5) = 3$, $e_5 (\tilde{B}_5) = 0$, which means that $\tilde{B}_5$ is a saddle point.

Finally, points $\tilde{B}_6$ describe the de Sitter solution in a spatially flat FLRW spacetime, $\omega_R (\tilde{B}_6) = 0$, $q (\tilde{B}_6) = -1$. The eigenvalues of the linearized system are $e_1 (\tilde{B}_6) = -3$, $e_2 (\tilde{B}_6) = -3$, $e_3 (\tilde{B}_6) = -3$, $e_4 (\tilde{B}_6) = -2$, $e_5 (\tilde{B}_6) = 0$. In Fig. 2 we discuss the stability properties for the stationary point $\tilde{B}_6$. We observe that the point is in general a saddle space, but it has a stable manifold in the subspace $\{ \Sigma, x, y, z \}$ for $\lambda = 0$. The exact form for the stable manifold can be derived with the application of the centre manifold theorem. We select to omit such presentation because it does not contribute in the physical discussion for the anisotropic model.

4.3 Stationary points of family $\tilde{C}$

The stationary points with $xz \neq 0$ are

$$\tilde{C}_1 = \left( 0, \frac{\sqrt{6}}{2 \kappa + \lambda}, \sqrt{\frac{2 \kappa}{2 \kappa + \lambda}}, \sqrt{\frac{\lambda^2 + 2 \kappa \lambda - 6}{2 \kappa + \lambda}}, \sigma \right)$$

and

$$\tilde{C}_2 = \left( 2 - \frac{6 \kappa}{2 \kappa + \lambda}, \frac{\sqrt{6}}{2 \kappa + \lambda}, \sqrt{\frac{6 \kappa (2 \kappa - \lambda)}{2 \kappa + \lambda}}, \sqrt{\frac{6 \kappa \lambda - 3 (\lambda^2 + 2)}{2 \kappa + \lambda}}, \sigma \right).$$
Fig. 2 Phase-space portraits for the dynamical system around the stationary point \( B_6 \). Left figures are for \( \sigma = +1 \), while right figures are for \( \sigma = -1 \). Figures of the first row are in the plane \( \{ x, \lambda \} \), where we observe that \( B_6 \) is a saddle point. However, because the dynamical system at the stationary point has four eigenvalues with negative real parts, there is a stable submanifold when \( \lambda = 0 \), as it can be seen from the figures of the second row.

Thus the physical properties of the solutions are similar to those of points \( C_1 \) and \( C_2^\pm \), respectively. Moreover, the stability properties are the same as above. Indeed, point \( C_1 \) is an attractor as presented in Fig. 1 while point \( C_2 \) is always a saddle point.

Finally, there are no (real valued) stationary points for \( \lambda = 0 \).

We conclude that the consideration of a different potential function different from the exponential, provides new stationary points only in family \( B \), that is, of the quintessence case, in which the second scalar field does not contribute in the cosmological fluid, \( z = 0 \). The latter can be easily seen and, if we consider an arbitrary potential function \( V(\phi) \), where no new stationary points in the family \( C \) follow.

Finally, the new physical solutions are anisotropic inflationary solutions in a Kantowski-Sachs spacetime with cosmological constant and the de Sitter universe.
5 Conclusions

We performed a detailed analysis on the dynamics for the Chiral cosmological theory in an anisotropic background space. The Chiral model belongs to the multi-scalar field theories, in which the energy-momentum tensor of the theory is consisted of two interacting scalar fields minimally coupled to the gravity. The two scalar fields interact in the kinetic part, such that the scalar fields to lie on the hyperbolic plane.

For this model, and for the generic LRS background space which describes the Bianchi I, the Bianchi III and the Kantowski-Sachs spacetimes we wrote the field equations by using dimensionless variables in the $H$-normalization approach. Because of the large number of the dependent variables, we selected to work on the $H$-normalization instead of other dimensionless variables. We determined the stationary points of the field equations and we investigated their stability properties. Every stationary point corresponds to a specific exact solution for the field equations which describe a specific asymptotic behaviour during the cosmological evolution.

The stationary points have been categorized in three families. Points of family $A$ describe the limit of General Relativity without matter source, points of family $B$ correspond to the stationary points with a quintessence matter source, while points of the third family, namely family $C$, describe exact solutions where the two fields contribute. The points of the third family are of special interests because they can describe isotropic and anisotropic inflationary solutions with two scalar fields. In particular we recovered the isotropic inflationary model known hyperbolic inflation [23], while the anisotropic inflationary solution can be seen as the analogue of hyperbolic inflation. Because the anisotropic hyperbolic inflationary solution is always unstable, point $C_2$ is a saddle point, we can say that the anisotropic inflationary solution played role in the very early universe such is in the beginning of the inflation.

This work contributes to the anisotropic inflationary models. In a future study we plan to investigate further the effects of the Chiral model in anisotropic background spaces.

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