Relativistic Dynamics of Point Magnetic Moment

Johann Rafelski, Martin Formanek, and Andrew Steinmetz
Department of Physics, The University of Arizona, Tucson, Arizona, 85721, USA

Submitted: December 1, 2017 / Print date: January 18, 2018

Abstract. The covariant motion of a classical point particle with magnetic moment in the presence of (external) electromagnetic fields is revisited. We are interested in understanding Lorentz force extension involving point particle magnetic moment (Stern-Gerlach force) and how the spin precession dynamics is modified for consistency. We introduce spin as a classical particle property inherent to Poincaré symmetry of space-time. We propose a covariant formulation of the magnetic force based on a 'magnetic' 4-potential and show how the point particle magnetic moment relates to the Amperian (current loop) and Gilbertian (magnetic monopole) description. We show that covariant spin precession lacks a unique form and discuss connection to $g - 2$ anomaly. We consider variational action principle and find that a consistent extension of Lorentz force to include magnetic spin force is not straightforward. We look at non-covariant particle dynamics, and present a short introduction to dynamics of (neutral) particles hit by a laser pulse of arbitrary shape.

PACS. 21.10.Ky Electromagnetic moments 03.30.+p Special relativity 13.40.Em Electric and magnetic moments

1 Introduction

The (relativistic) dynamics of particle magnetic moment $\mu$, i.e. the proper time dynamics of spin $s^\mu(\tau)$, has not been fully described before. Our interest in this topic originates in a multitude of current research topics:

i) the ongoing effort to understand the magnetic moment anomaly of the muon [12];

ii) questions regarding how elementary magnetic dipoles (e.g. neutrons) interact with external fields [3,4];

iii) particle dynamics in ultra strong magnetic fields created in relativistic heavy ion collisions [5,6];

iv) magnetars, stellar objects with extreme $O(10^{11})$ T magnetic fields [7,8];

v) the exploration of particle dynamics in laser generated strong fields [9];

vi) neutron beam guidance and neutron storage rings [10];

vii) the finding of unusual quantum spin dynamics when gyromagnetic ratio $g \neq 2$ [11,12].

The results we present will further improve the understanding of plasma physics in presence of inhomogeneous magnetic fields, and improve formulation of radiation reaction forces, topics not further discussed in this presentation.

In the context of the electromagnetic (EM) Maxwell-Lorentz theory we learn in the classroom that

1. The magnetic moment $\mu$ has an interaction energy with a magnetic field $B$

$$E_m = -\mu \cdot B.$$  (1)

The corresponding Stern-Gerlach force $FSG$ has been written in two formats

$$FSG \equiv \begin{cases} \nabla(\mu \cdot B), & \text{Amperian Model}, \\ (\mu \cdot \nabla)B, & \text{Gilbertian Model}. \end{cases}$$  (2)

The name ‘Amperian’ relates to the loop current generating the force. The Gilbertian model invokes a magnetic dipole made of two magnetic monopoles. These two forces written here in rest frame of a particle are related [3,4]. We will show that a internal spin based magnetic dipole appears naturally; it does not need to be made of magnetic monopoles or current loops. We find that both force expressions in Eq. (2) are equivalent, this equivalence arises from covariant dynamics we develop and requires additional terms in particle rest frame complementing those shown in Eq. (2).

2. The torque $T$ that a magnetic field $B$ exerts on a magnetic dipole $\mu$ in a way that tends to align the dipole with the direction of a magnetic field $B$

$$T \equiv \frac{d\mathbf{s}}{dt} = \mu \times B = g\mu_B \frac{s}{\hbar/2} \times B, \quad \mu_B \equiv \frac{e\hbar}{2m}.$$  (3)

The magnetic moment is defined in general in terms of the product of Bohr magneton $\mu_B$ with the gyromagnetic ratio $g$, $|\mu| \equiv g\mu_B$. In Eq. (3) we used $|s| = h/2$ for a spin-1/2 particle, a more general expression will be introduced in subsection 3.1.1.

We used the same coefficient $\mu$ to characterize both the Stern-Gerlach force Eq. (2) and spin precession force.
Eq. (3). However, there is no compelling argument to do so and we will generalize this hypothesis — it is well known that Dirac quantum dynamics of spin-1/2 particles predicts both the magnitude $g = 2$ and identity of magnetic moments entering Eq. (2) and Eq. (3).

While the conservation of electrical charge is rooted in gauge invariance symmetry, the magnitude of electrical charge has remained a riddle; the situation is similar for the case of the magnetic moment $\mu$: spin properties are rooted in the Poincaré symmetry of space-time, however, the strength of spin interaction with magnetic field, Eq. (1) and Eq. (3), is arbitrary but unique for each type of (classical) particle. Introducing the gyromagnetic ratio $g$ we in fact create an additional conserved particle quality. This becomes clearer when we realize that the appearance of `$e$' does not mean that particles we study need to be electrically charged.

First principle considerations of point particle relativistic dynamics experience some difficulties in generating Eqs. (2, 3), as a rich literature on the subject shows we will cite only work that is directly relevant to our numerical study of spin effects and radiation reaction in a strong electromagnetic field.

For what follows it is important to know that the spin precession Eq. (3) is a result of spatial rotational invariance which leads to angular and spin coupling, and thus spin dynamics can be found without a new dynamical principle has been argued e.g. by Van Dam and Ruijgrok and Schwinger. Similar physics content is seen in the work of Skagerstam and Stern [15, 16], who considered the context of fiber bundle structure focusing on Thomas precession.

Covariant generalization of the spin precession Eq. (3) is often attributed to the 1959 work by Bergmann-Michel-Telegdi. However we are reminded [18, 19, 20] that this result was discovered already 33 years earlier by L.H. Thomas [23, 24] at the time when the story of the electron gyromagnetic ratio $g = 2$ was unfolding. Following Jackson [18] we call the corresponding equation TBMT.

J. Frenkel, who published [21, 22] at the same time with O. Stern-Gerlach extension of the Lorentz force. In the final part of this work section 5 we show some of the physical content of the theoretical framework. In subsection 5.1 we present a more detailed discussion of dynamical equations for the case of particle in motion with a given $\beta = v/c$ and $E, B$ fields in laboratory. In subsection 5.2 we study solution of the dynamical equations for the case of an EM light wave pulse hitting a neutral particle. We have obtained exact solutions of this problem, detail will follow under separate cover [27].

In section 6 is a brief summary of our findings.

### 1.1 Notation

For most of notation, see Ref. [28]. Here we note that we use SI unit system and the metric:

$\text{diag } g_{\mu \nu} = \{1, -1, -1, -1\}$, $p_{\mu}p^{\mu} = g_{\mu \nu}p^{\mu}p^{\nu} = \frac{E^2}{c^2} - p^2$

We further recognize the totally antisymmetric covariant pseudo tensor $\epsilon$:

$\epsilon_{\mu \nu \alpha \beta} = \sqrt{-g} \left\{ \begin{array}{ll} (-1)^{\text{perm}} & \text{if all indices are distinct} \\ 0 & \text{otherwise} \end{array} \right.$

where ‘perm’ is the signature of the permutation. It is important to remember when transiting to non-covariant notation in laboratory frame of reference that the analog contravariant pseudo-tensor due to odd number of space-like dimensions is negative for even permutations and positive for odd permutations. The Appendix B of Ref. [29] presents an introduction to $\epsilon$.

We will introduce an elementary magnetic dipole charge $d$ — the limitations of the alphabet force us to adopt the
characterized by eigenvalues of two Casimir operators \( (30,31) \), it has been established that
can solve for dynamical evolution of

\[
F^{\mu\nu}(E \rightarrow \mathbf{K}, B \rightarrow \mathbf{J}) = \mathcal{M}^{\mu\nu}.
\]

The generators \( \mathbf{J}, \mathbf{K} \) of space-time transformations are recognized by their commutation relations. They are used in a well known way to construct representations of the Lorentz group.

In terms of the generator tensor \( \mathcal{M}^{\mu\nu} \) the covariant definition of the particle spin (operator) vector is

\[
\bar{s}_\mu \equiv \frac{\bar{u}_\mu \bar{u}^{\nu}}{\sqrt{C_1}} = \frac{\bar{M}^{\mu\nu}}{c}, \quad \bar{u}^{\mu} \equiv \frac{c \bar{p}^{\mu}}{\sqrt{C_1^2/C_1}} = \frac{\bar{p}^{\mu}}{m}.
\]

According to Eq. (8), spin \( \bar{s}_\mu \) is a pseudo vector, as required for angular dynamics. The dimension of \( \bar{s}^{\mu} \) is the same as the dimension of the generator of space rotations \( \mathbf{J} \). We further find that \( \bar{s}^{\mu} \) is orthogonal to the 4-velocity (operator) \( \bar{u}^{\mu} \)

\[
c\bar{s} \cdot \bar{u} = \bar{u}^{\nu} \bar{M}^{\mu\nu} \bar{u}^{\mu} = 0,
\]

by virtue of the antisymmetry of \( \bar{M} \) evident in the definition Eq. (6). The definition of the particle spin (operator) is unique: no other space-like (space-like given the orthogonality \( \bar{s} \cdot \bar{u} = 0 \)) pseudo vector associated with the Poincaré group describing space-time symmetry transformations can be constructed.

We now transition to c-numbered quantities (dropping the bar): an observer ‘(0)’ co-moving with a particle measures the 4-momentum and 4-spin \( \bar{s}^{\mu} \)

\[
p'^{(0)}_\mu \equiv \{ \sqrt{C_1}, 0, 0, 0 \}, \quad s'^{(0)}_\mu \equiv \{ 0, 0, 0, \sqrt{C_2}/C_1 \},
\]

where according to convention \( \bar{s} \)-axis of the coordinate system points in direction of the intrinsic spin vector \( \mathbf{s} \). In the particle rest frame we see that

\[
0 = p'^{(0)}_\mu s'^{(0)}_\mu = p'^{\mu} s'_\mu = m(u^{\mu} s^\mu)|_{\text{any frame}}, \quad (11)
\]

consistent with operator equation Eq. (9); more generally, any space-like vector is normal to the time-like 4-velocity vector. For the magnitude of the spin vector we obtain

\[
-s^2 \equiv s'^{(0)}_\mu s'^{(0)}_\mu \equiv s'^{\mu} s'_{\mu} \vert_{\text{any frame}} = -\frac{C_2}{C_1}.
\]

We keep in mind that \( s^2 \) must always be a constant of motion in any frame of reference. Its value \( s \cdot s = -s^2 \) is always negative, appropriate for a space-like vector. Similary

\[
p'^{(0)}_\mu p'^{(0)}_\mu = p'^2 |_{\text{any frame}} = C_1 \equiv m^2 c^2,
\]

must be a constant of motion in any frame of reference and the value \( p^2 \) is positive; appropriate for a time-like vector.

As long as forces are small in the sense discussed in Ref. [23], we can act as if rules of relativity apply to both inertial and (weakly) accelerated frames of reference. This allows us to explore the action of forces on particles in their rest frame where Eq. (10) defines the state of a particle. By writing the force laws in covariant fashion we can solve for dynamical evolution of \( p^{\mu}(\tau) \), \( s^{\mu}(\tau) \) classical numbered variables.
3 Covariant dynamics

3.1 Generalized Lorentz force

3.1.1 Magnetic dipole potential and Amperian force

We have gone to great lengths in section 2 to argue for the existence of particle intrinsic spin. For all massive particles this implies the existence of a particle intrinsic magnetic dipole moment, without need for magnetic monopoles to exist or current loops. Spin naturally arises in the context of symmetries of Minkowski space-time, it is not a quantum property.

In view of above it is appropriate to study classical dynamics of particles that have both, an elementary electric charge $e$, and an elementary magnetic dipole charge $d$. The covariant dynamics beyond the Lorentz force needs to incorporate the Stern-Gerlach force. Thus the extension has to contain the elementary magnetic moment of a particle contributing to this force. To achieve a suitable generalization we introduce the magnetic potential

$$B_µ(x, s) \equiv F^µ_ν(x)s^ν d , \quad F^µ_ν = \frac{1}{2} e_µσ_αβ F^αβ . \quad (14)$$

We use dual pseudo tensor since $B_µ$ is a pseudo vector; the product in Eq. (14) results in a polar 4-vector $F$. We note that the magnetic dipole potential $B_µ$ by construction in terms of antisymmetric field pseudo tensor $F^µ_ν$ satisfies

$$\partial_µ B^µ = 0 , \quad s \cdot B = 0 \implies B \cdot \frac{ds}{dt} = -s \cdot \frac{dB}{dt} . \quad (15)$$

The additional potential energy of a particle at rest placed in this magnetic dipole potential is

$$U(0) \equiv B^0 c d = c F^0_ν(x)s^ν d = -|μ| \left| B \cdot \frac{s}{s} \right| \equiv -μ \cdot B . \quad (16)$$

This shows Eq. (14) describes the energy content seen in Eq. (11); all factors are appropriate.

The explicit format of this new force is obtained when we use Eq. (14) to define a new antisymmetric tensor

$$G^µν = \partial^µ B^ν - \partial^ν B^µ = s_α [∂^µ F^αν - ∂^ν F^αµ] . \quad (17)$$

Equation (17) allows us to add to the Lorentz force

$$m\dot{u}^µ = H^µν u_ν , \quad H^µν = e F^µν + G^µν d . \quad (18)$$

In the $G$-tensor we note appearance in the force of the derivative of EM fields, required if we are to see the Amperian model variant of the Stern-Gerlach force Eq. (2) as a part of generalized Lorentz force.

The Amperian-Stern-Gerlach (ASG) force 4-vector is obtained multiplying with $u_ν d$ the $G$-tensor Eq. (17). Thus the total 4-force a particle of charge $e$ and magnetic dipole charge $d$ experiences is

$$F^µ_{ASG} = e F^µν u_ν - u_ν \partial F^νµ d s_α \left( u_ν \cdot F^α s d \right) . \quad (19)$$

In the particle rest frame we have

$$u^µ| RF = \{ e, 0 \} , \quad cs^ν d| RF = \{ 0, μ \} . \quad (20)$$

We can use Eq. (20) to read-off from Eq. (18) the particle rest frame force to be

$$F^µ_{ASG}| RF = \left\{ 0, e \mathcal{E} - \frac{1}{c^2} \mu \times \mathcal{E} + \nabla(μ \cdot B) \right\} , \quad (21)$$

where two contributions $∂(μ \cdot B)/∂t$ to $F^0$ cancel. Each of the three terms originates in one of the covariant terms in the sequence shown. The result is what one calls Amperian model originating in dipoles created by current loops. This is, however, not the last word in regard to the form of the force.

3.1.2 Gilbertian model Stern-Gerlach force

We restate the Stern-Gerlach-Lorentz force Eq. (18), showing the derivative terms explicitly,

$$\dot{m}u^µ = e F^µν u_ν + (∂^µ(u_ν \cdot F^α ν) - s_α u_ν \partial F^α µ ν) d . \quad (22)$$

Multiplying with $s^µ$ the last term vanishes due to antisymmetry of $F^α ν$ and we obtain

$$s_ν \ddot{u} = \frac{1}{m} e (e F - s_ν \partial F^ν d) \cdot u . \quad (23)$$

This equation suggests that we explore

$$e F^µν \to \tilde{F}^µν = e F^µν - s_ν \partial F^ν d \cdot u . \quad (24)$$

as the generalized Lorentz force replacing the usual field tensor $e F$ by $\tilde{F}$ in a somewhat simpler way compared to the original $H^µν$ Eq. (13) modification.

We demonstrate now that the field modification seen in Eq. (24) leads to a different and fully equivalent format of the force. We replace in the first term in Eq. (22) $F \to \tilde{F}$ and add the extra term from Eq. (24) to the two reminder terms. Changing the index naming these we can write symmetrically

$$\dot{m}u^µ = \tilde{F}^µν u_ν + s_α \left( ∂^µ F^α ν + ∂^ν F^α µ ν + ∂^ν F^α ν µ \right) u_β d . \quad (25)$$

The tensor appearing in the parentheses in the 2nd line of Eq. (25) is antisymmetric under any of the three exchanges of the indices. It is therefore proportional to the totally antisymmetric tensor $e_αβγ$ which must be contracted with some 4-vector $V_γ$ containing a gradient of the EM dual field tensor, there are two such available 4-vectors $∂^µ F^α γ$ which vanishes by virtue of Maxwell equations, and

$$V_γ = \frac{1}{2} e_γ κ η δ F^κ σ σ γ = \partial^µ F^α γ = μ_0 j_γ . \quad (26)$$

Thus we introduce the Gilbertian form of the 4-force

$$F^µ_{GSG} = \tilde{F}^µν u_ν - μ_0 j^γ c_γ α β ν s^β q^µ d . \quad (27)$$
Note that in our formulation the Amperian and the Gilbertian 4-forces are identical
\[ F_{\text{ASG}}^\mu = F_{\text{GSG}}^\mu , \]
they are just written differently.

In the rest frame of a particle, see Eq. (20) the Gilbertian force Eq. (27) is
\[ F_{\text{GSG}}^\mu |_{\text{RF}} = \{ 0, e E + (\mu \cdot \nabla) B + \mu_0 \mu \times j \} . \]

It is interesting to see the mechanism by which the two formats of the forces equal to each other in the particle rest frame. With
\[ \nabla (\mu \cdot B) - (\mu \cdot \nabla) B = \mu \times (\nabla \times B) , \]
we form the force difference between Eq. (21) and Eq. (25)
\[ [F_{\text{ASG}} - F_{\text{GSG}}]_{\text{RF}} = \mu \times \left( -\frac{1}{c^2} \frac{\partial E}{\partial t} + \nabla \times B - \mu_0 j \right) = 0 . \]
The terms in parenthesis cancel according to Maxwell equation confirming that both the Amperian and the Gilbertian forces are equal taking as an example the instantaneous rest frame. From now on we will use Gilbertian form of the force and in later examples we will focus on particle motion in vacuum, \( j^\mu = 0 \).

In this discussion of forces we kept the electrical charge \( e \) and the elementary magnetic moment ‘charge’ \( d \) Eq. (1) as independent qualities of a point particle. As noted in the introduction it is common to set \( |\mu| \equiv g \mu_B \), see below Eq. (3). Hence we can have both, charged particles without magnetic moment, or neutral particles with magnetic moment, aside from particles that have both charge and magnetic moment. For particles with both charge and magnetic moment we can write, using Gilbertian form of force
\[ m \dot{u}^\mu = F^{\mu \nu} u_\nu = e \left( F^{\mu \nu} - (1 + a) \frac{\lambda}{mc^2} \frac{s \cdot \partial}{[8]} F^{* \mu \nu} \right) u_\nu , \]
where \( a = (g - 2)/2 \) is the gyromagnetic ratio anomaly. The Compton wavelength \( \lambda = h/mc \) defines the scale at which the spatial field inhomogeneity is relevant; note that inhomogeneities of the field are boosted in size for a particle in motion, a situation which will become more explicit in section 3.1.3.

### 3.2 Spin motion

#### 3.2.1 Conventional TBMT

For particles with \( m \neq 0 \) differentiating Eq. (11) with respect to proper time we find
\[ \dot{u} \cdot s + u \cdot \dot{s} = 0 , \]
where we introduced proper time derivative \( \dot{s}^\mu = ds^\mu/d\tau \).

Schwinger observed [14] that given Eq. (31) one can use covariant form of the dynamical Lorentz force equations for \( dw^\mu/d\tau \) to obtain
\[ u_\mu \left( \frac{ds^\mu}{d\tau} - \frac{e}{m} F^{\mu \nu} s_\nu \right) = 0 . \]
Here \( F^{\mu \nu} \) is the usual EM field tensor. Equation (32) has the general TBMT solution
\[ \frac{ds^\mu}{d\tau} = \frac{e}{m} F^{\mu \nu} s_\nu + \tilde{\alpha} e (u \cdot F^* s) , \]
with the notation \( u \cdot F \cdot s \equiv u_\mu F^{\mu \nu} s_\nu \).

In Eq. (33) \( \tilde{\alpha} \) is an arbitrary constant considering that the additional term multiplied with \( u^\mu \) vanishes. On the other hand we can read off the magnetic moment entering Eq. (3): the last term is higher order in \( 1/e^2 \). Hence in the rest frame of the particle we see that \( 2(1 + \tilde{\alpha}) = g \) i.e. Eq. (33) reproduces Eq. (2) with the magnetic moment coefficient when \( \tilde{\alpha} = \alpha \). Therefore, as introduced, \( \tilde{\alpha} = \alpha \) is the \( g \neq 2 \) anomaly. However, in Eq. (33) we could for example use \( \tilde{\alpha} = (g^2 - 4)/8 = a^2/2 \), which classical limit of quantum dynamics in certain specific conditions implies [12]. In this case \( \tilde{\alpha} \rightarrow a \) up to higher order corrections. This means that measurement of \( \tilde{\alpha} \) as performed in experiments [12] depends on derivation of the relation of \( \tilde{\alpha} \) with \( a \) obtained from quantum theory. These remarks apply even before we study gradient in field corrections.

#### 3.2.2 Gradient corrections to TBMT

The arguments by Schwinger, see Eqs. (31,32,33), are ideally positioned to obtain in a consistent way generalization of the TBMT equations including the gradient of fields terms required for consistency. We use Eq. (24) in Eq. (33) to obtain
\[ \frac{ds^\mu}{d\tau} = \frac{1 + \tilde{\alpha}}{m} \left( e F^{\mu \nu} - s \cdot \partial F^{s \mu \nu} d \right) s_\nu + \frac{\tilde{\alpha}}{mc^2} (s \cdot e F \cdot s - s \cdot \partial s \cdot F^{*} u \cdot d) u^\mu . \]

The dominant gradient of field correction arises for an elementary particle from the 2nd term in the first line in Eq. (34), considering the coefficient of the second line \( a = a_2/2\pi + \ldots = 1.2 \times 10^{-3} \). One should remember that given the precision of the measurement [12] of \( \tilde{\alpha} \) which is driven by the first term in the second line in Eq. (34) we cannot in general neglect the new 2nd term in first line in Eq. (34) even if the characteristic length defining the gradient magnitude is the Compton wavelength \( \lambda \), see Eq. (33).

#### 3.2.3 Non-uniqueness of gradient corrections to TBMT

It is not self evident that the form Eq. (34) is unique. To see that a family of possible extensions TBMT arises we recall the tensor Eq. (18) \( H^{\mu \nu} \) made of the two potentials \( A^\mu \) and \( B^\mu \). We now consider the spin dynamics in terms
of the two field tensors, $F$ and $G$ replacing the usual EM-tensor $F^{\mu\nu}$ in the Schwinger solution, Eq. (33). In other words, we explore the dynamics according to

$$\frac{d s^\mu}{d \tau} = \frac{1}{m} F^{\mu\nu} s_\nu + \tilde{a} \vec{c} \left( F^{\mu\nu} s_\nu - \frac{e}{c^2} \left( u \cdot F \cdot s \right) \right) \quad (35)$$

$$+ G^{\mu\nu} s_\nu \frac{d}{m} + \left( G^{\mu\nu} s_\nu - \frac{e}{c^2} \left( u \cdot G \cdot s \right) \right) \frac{d b}{m}.$$

Two different constants $\tilde{a}$ and $\tilde{b}$ are introduced now since the two terms shown involving $F$ and $G$ tensors could be included in Schwinger solution independently with different constants. Intuition demands that $\tilde{a} = \tilde{b}$. However, aside from algebraic simplicity we do not find any compelling argument for this assumption.

We return now to the definition of the $G$ tensor Eq. (17) to obtain

$$G^{\mu\nu} s_\nu = (s_\nu s_\alpha \partial^\mu F^{\alpha\nu} - s \cdot \partial F^{\mu\alpha} s_\alpha) \quad (36)$$

$$= -s \cdot \partial F^{\mu\nu} s_\nu.$$

The first term in the first line vanishes by antisymmetry of $F^\mu$. Also we have

$$u \cdot G \cdot s = -s \cdot \partial u \cdot F^\mu \cdot s \quad (37)$$

Using Eq. (36) and Eq. (37) we can combine in Eq. (35) the first two terms in both lines, and the last terms in both lines to obtain

$$\frac{d s^\mu}{d \tau} = \frac{1 + \tilde{a}}{m} \left( e F^{\mu\nu} - 1 + \tilde{b} \frac{e}{c} s \cdot \partial F^{\mu\nu} d \right) s_\nu$$

$$- \frac{\tilde{a} e}{m c^2} \left( u \cdot \left( e F - \tilde{b} \frac{e}{c} s \cdot \partial F^\mu d \right) \cdot s \right). \quad (38)$$

This equation agrees with Eq. (34) only when $\tilde{a} = \tilde{b}$. However, this requirement is neither mathematically nor physically necessary. For example using Eq. (26) we easily check $s \cdot \tilde{a} + u \cdot \tilde{b} = 0$ without any assumptions about $\tilde{a}, \tilde{b}$.

As Eq. (35) shows the physical difference between factors $\tilde{a}$ and $\tilde{b}$ is related to the nature of the interaction: the ‘magnetic’ tensor $G$ is related to $\tilde{b}$ only. Thus for a neutral particle $e \to 0$ we see in Eq. (35) that the torque depends only on $\tilde{b}$. Conversely, when the effect of magnetic potential is negligible Eq. (35) becomes the textbook spin dynamics that depends on $\tilde{a}$ alone.

To make further contact with textbook physics we note that the coefficient of the first term in Eq. (35)

$$\frac{1 + \tilde{a}}{m} = 2 \left( 1 + \tilde{a} \right) \frac{e \hbar}{2 m \hbar} = \tilde{g} \mu_B \frac{1}{\hbar}, \quad \tilde{g} = 2 \left( 1 + \tilde{a} \right), \quad (39)$$

should reproduce in leading order the torque coefficient in Eq. (3) as is expected from study of quantum correspondence. However, quantum correspondence could mean $\tilde{a} = a + a^2/2$, which follows comparing exact solutions of the Dirac equation with spin precession for the case we explored [12] and which is not exactly the motion of a muon in storage ring. However, this means that in order to compare the measurement of magnetic moment of the muon carried out on macroscopic scale [12] with quantum computations requires a further step, the establishment of quantum correspondence at the level of precision at which the anomaly is measured.

### 4 Search for variational principle action

At the beginning of earlier discussions of a covariant extension to the Lorentz force describing the Stern-Gerlach force was always a well invented covariant action. However, the Lorentz force itself is not a consistent complement of the Maxwell equations. The existence of radiation means that an accelerated particle experiences radiation friction. The radiation-reaction force has not been incorporated into a variational principle [28,32]. Thus we should not expect that the Stern-Gerlach force must originate in a simple action.

We seek a path $x^\mu(\tau)$ in space-time that a particle will take considering an action that is a functional of the 4-velocity $u^\mu(\tau) = dx^\mu/d\tau$ and spin $s^\mu(\tau)$. Variational principle requires an action $I(u, x; s)$. When $I$ respects space-time symmetries the magnitudes of particle mass and spin are preserved in the presence of electromagnetic (EM) fields. We also need to assure that $u^2 = c^2$ which constrains the form of force and thus $I$ that is allowed. Moreover, we want to preserve gauge invariance of the resultant dynamics.

The component in the action that produced the LHS (inertia part) of the Lorentz force remains in discussion. To generate the Lorentz force one choice of action is

$$I_{Lx}(u, x) = - \int d\tau \int m c \sqrt{u^2} - e \int d\tau u(\tau) \cdot A(x(\tau)) \quad (40)$$

We note that reparametrization of $\tau \to k \tau$ considering $u = dx/d\tau$ has no effect on value of $I_{Lx}$.

Variation with respect to path lead to

$$\frac{d}{d\tau} \frac{mc u^\mu}{\sqrt{u^2}} = L^\mu_{Lx} = u^\nu \partial^\mu A^\nu - \frac{d e A^\mu}{d\tau}, \quad (41)$$

where the RHS produces upon differentiation of $e A^\mu(x(\tau))$ the usual Lorentz force

$$L^\mu_{Lx} = e (\partial^\mu A^\nu - \partial^\nu A^\mu) u_\nu = e F^{\mu\nu} u_\nu. \quad (42)$$

Multiplying Eq. (41) with $mc u^\mu/\sqrt{u^2}$ we establish by antisymmetry of the tensor $F^{\mu\nu}$ Eq. (42) that also the product with the LHS in Eq. (41) vanishes. This means that $(mc u^\mu/\sqrt{u^2})^2 = m^2 c^2 \equiv p^2 = \text{Const.}$. Henceforth

$$p^\mu \equiv \frac{mc u^\mu}{\sqrt{u^2}}. \quad (43)$$

There is a problem when we supplement in Eq. (40) the usual action $I_{Lx}$ by a term $I_0$ based on our prior consideration of $A^\mu \to A^\mu + B^\mu$, see subsection 5.1.1. The
problem one encounters is that the quantity $B^\mu$ contains additional dependence on $s^\mu(\tau)$ which adds another term to the force. Let us look at the situation explicitly

$$I(u, x; s) = I_{Lx} + I_m , \quad I_m \equiv -\int d\tau u \cdot B(u, x; s) \, d .$$

Here the dependence on $s^\mu(\tau)$ is akin to a parameter dependence; some additional consideration defines the behavior, in our case this is the TBMT equations.

Varying with respect to the path the modified action $\tilde{S}_{\text{TBMT}}$ equation Eq. (33) or as would be more appropriate based equation system. On the other hand we have presented before a formulation of spin dynamics which does not require a variational principle in the study the particle dynamics: as is we have obtained a dynamical equation system empirically. Our failing in the search for an underlying action is not critical. A precedent situation comes to mind here: the radiation emitted by accelerated charges introduces a ‘radiation friction’which must be studied \cite{32} without an available action, based on empirical knowledge about the energy loss arising for accelerated charges.

### 5 Experimental consequences

#### 5.1 Non covariant form of dynamical equations

In most physical cases we create a particle guiding field which is at rest in the laboratory. Particle motion occurs with respect to this prescribed field and thus in nearly all situations it is practical to study particle position $z^\mu(\tau)$ in the laboratory frame of reference. Employing the Lorentz-coordinate transformations from the particle rest frame to the laboratory frame we obtain

$$\frac{dz^\mu}{d\tau} \equiv u^\mu|_L = c\gamma\{1, \beta\} , \quad \beta \equiv \frac{dz}{dct} = \frac{v}{c} , \quad s^\mu|_L = \left\{ \gamma \beta \cdot s , \left( \frac{\gamma}{\gamma + 1} \gamma \beta \cdot s \right) \beta + s \right\} ,$$

where as usual $\gamma = 1/\sqrt{1 - \beta^2}$ and one often sees the spin written with $\gamma^2/(\gamma + 1) = (\gamma - 1)/\beta^2$.

One easily checks that Eq. (48) and Eq. (49) also satisfy Eq. (47): $\mu s^\mu = 0$. A classic result of TBMT reported in textbooks is that the longitudinal polarization $\beta \cdot s$ for $g \approx 2$ and $\beta \to 1$ is a constant of motion. This shows that for a relativistic particle the magnitude of both time-like and space-like components of the spin 4-vector Eq. (49) can be arbitrarily large, even if the magnitude of the 4-vector is bounded $s_\mu s^\mu = -s^2$. This behavior parallels the behavior of 4-velocity $u^\mu u_\mu = c^2$.

We remind that to obtain in the laboratory frame the usual Lorentz force we use the 4-velocity with respect to the laboratory frame Eq. (48), with laboratory defined tensor $F$, i.e. with laboratory given $\mathcal{E}, \mathcal{B}$ EM-fields

$$\frac{d(mu^\mu)|_L}{d\tau} = (eF^{\mu
u}u_\nu)|_L = eF_{\mu\nu}|_L \, u_\nu|_L .$$

Sometimes it is of advantage to transform Eq. (50) to the particle rest frame. Such a transformation $L$ with $Lu|^\text{rest} = u_L$ when used on the left hand side in Eq. (50) produces proper time differentiation of the transformation operator, see also \cite{33}. Such transformation into a co-rotating frame of reference originates the Thomas precession term in particle rest frame for the torque equation. This term is naturally present in covariant formulation when we work in the laboratory reference frame.

For the full force Eq. (16) we thus have

$$\frac{d(mu^\mu)|_L}{d\tau} = eF^{\mu\nu}|_L \, u_\nu|_L$$

$$-d s_\alpha|_L \, (\partial^\alpha F^{\mu\nu})|_L \, u_\nu|_L .$$

We see that in laboratory frame of reference a covariant gradient of the fields is prescribed, i.e. that some apparatus prescribes the magnitude

$$Q^{\alpha\mu}|_L \equiv \partial^\alpha F^{\mu\nu}|_L ,$$
which allows for a moving particle with \( u^\mu \big|_L \) Eq. (18) and \( s^\mu \big|_L \) Eq. (19) to experience the Lorentz force \( F_{SG}^{\text{L}} \)
\[
F_{SG}^{\text{L}} \equiv -d s_{\alpha} Q^{\mu \nu} \big|_L u^\nu \big|_L .
\]
We have gone to extraordinary length in arguing Eq. (51) to make sure that the forthcoming finding of the Lorentz boost of field inhomogeneity is not questioned.

5.1.2 Magnetic potential in the laboratory frame

We evaluate in the laboratory frame the form of Eq. (14). The computation is particularly simple once we first recall the laboratory format of the Lorentz force \( F_{SG}^{\text{L}} \)
\[
F_{SG}^{\text{L}} = F^{\mu \nu}(x) u_\nu \big|_L = -s^\mu \big|_L (\mu^{\rho \nu} u^\nu) \big|_L
\]
where we used in 2nd line i) \( F^{\mu \nu}_{\text{SB}} \) follows from the usual exchange of \( \mathcal{E}/c \leftrightarrow \mathcal{B} \) and ii) flip \( \beta \to -\beta \) to account for contravariant and not covariant 4-velocity. In the 3rd line we used \( \gamma (\gamma / (\gamma + 1) - 1) = -\gamma / (\gamma + 1) \). Notable in Eq. (56) is the absence of the highest power \( \gamma^2 \) as all terms cancel, the result is linear in (large) \( \gamma \).

For the magnetic action potential energy of a particle in lab frame we obtain
\[
U \equiv B \cdot u \big|_L d = \gamma \left( K \beta \cdot \mu \beta \cdot B - \mu \cdot (B - \beta \times \mathcal{E}/c) \right),
\]
\[
K = \beta^2 \frac{\gamma}{\gamma + 1} = 1 - \sqrt{1 - \beta^2} = \begin{cases} \frac{\beta^2}{2}, & \text{for } \beta \to 0 \\ 1, & \text{for } \beta \to 1 \end{cases}
\]
Equation (57) extends the rest frame \( \beta = 0 \) Eq. (18) and represents covariant generalization of Eq. (14). In ultrarelativistic limit all terms in Eq. (57) have the same magnitude.

5.1.3 Field to particle energy transfer

We now consider the energy gain by a particle per unit of laboratory time, that is we study the 0th component of laboratory time, 
\[
\frac{dE}{dt} = c \frac{dt}{d\tau} \frac{d(mu^\mu)}{d\tau} = c \gamma^{-1} \bar{F}^{0\nu} \big|_L u^\nu \big|_L
\]
\[
= c \gamma \mathcal{E} \cdot v + c d s^\mu \big|_L (\partial_{\alpha} \mathcal{B}) \big|_L \cdot v
\]
\[
\frac{dE}{dt} = (c \mathcal{E} + (\mu \cdot \nabla) \mathcal{B}) \cdot v + \gamma \beta \cdot \mu \left( \frac{\partial \mathcal{B}}{c dt} + \frac{\gamma}{\gamma + 1} (\beta \cdot \nabla) \mathcal{B} \right) \cdot v.
\]
A further simplification is achieved considering
\[
\frac{\partial \mathcal{B}}{c dt} + (\beta \cdot \nabla) \mathcal{B} = \frac{\partial \mathcal{B}}{c dt} + \sum_{i=1}^{3} \frac{dx_i}{c dt} \frac{\partial \mathcal{B}}{\partial x_i} = \frac{d \mathcal{B}}{c dt},
\]
where the total derivative with respect to time accounts for both, the change in time of the laboratory given field \( \mathcal{B} \), and the change due to change of position in the field by the moving particle. We thus find two parts
\[
\frac{dE}{dt} = \mathbf{v} \cdot \left( c \mathcal{E} + (\mu \cdot \nabla) \mathcal{B} - K \beta \cdot \mu (\beta \cdot \nabla) \mathcal{B} \right)
\]
\[
+ \beta \cdot \frac{d \mathcal{B}}{dt} \gamma \beta \cdot \mu,
\]
where the 2nd line is of particular interest as it is proportional to \( \gamma \). Focusing our attention on this last term: we can use \( \beta = c \mathcal{E}/\mathcal{B} \) and \( \gamma \beta = p/mc \). Upon multiplication with \( E \) and remembering that \( c^2 p^p = E dE \) we obtain
\[
\frac{dE}{dt} = \mathbf{v} \cdot \left( c \mathcal{E} + (\mu \cdot \nabla) \mathcal{B} - K \beta \cdot \mu (\beta \cdot \nabla) \mathcal{B} \right)
\]
\[
+ \beta \cdot \frac{d \mathcal{B}}{dt} \gamma \beta \cdot \mu,
\]
which in qualitative terms implies an exponential response of particle momentum as it crosses a magnetic field
\[
|\mathbf{p}| \approx m c e^{\pm (|\mathcal{B}| - |\mathbf{B}_0|)\mu/mc^2}.
\]
However, even a magnetar magnetic field of up \( 10^{11} \text{T} \) will not suffice to impact electron momentum decisively in view of the smallness of the electron magnetic moment \( 5.810^{-11} \text{MeV}/\text{T} \). However, in ultrarelativistic heavy ion collisions at LHC a 10,000 stronger very non-homogeneous \( \mathcal{B} \)-fields arise.

5.2 Neutral particle hit by a light pulse

5.2.1 Properties of equations

The dynamical equations developed here have a considerably more complex form compared to the Lorentz force and TBMT spin precession in constant fields. We need field gradients in the Stern-Gerlach force, and in the related correction in the TBMT equations. Since the new physics appears only in the presence of a particle magnetic moment, we simplify by considering neutral particles. We now show that the external field described by a light wave (pulse) lends itself to an analytical solution effort. This context could be of practical relevance in the study of laser interaction with magnetic atoms, molecules, the neutron and maybe neutrinos.

For \( e = 0 \) our equations Eq. (26) and Eq. (33) read
\[
\dot{u}^\mu = -s - \partial F^{\mu \nu} u_\nu \frac{d^\mu}{m},
\]
\[
\dot{s}^\mu = -s - \partial F^{\mu \nu} s_\nu \frac{1 + \bar{b}}{m} d + w^\nu u \cdot (s \cdot \partial) E^{\nu} \cdot s \frac{b \cdot d}{mc^2}.
\]
The external light wave field is a pulse with
\[ A^\mu = \epsilon^\mu f(\xi) , \quad \xi = k \cdot x , \quad k \cdot \epsilon = 0. \] (66)
The derivative of the dual EM tensor for linear fixed in space pulse polarization \( \epsilon^\mu \) is
\[ (s \cdot \partial) F^{+\mu
u} = (k \cdot s) \epsilon^{\mu\alpha\beta} k_\alpha \epsilon_\beta f''(\xi) , \] (67)
prime \( ' \) indicates derivative with respect to the phase \( \xi \).
Notice that if we contract Eq. (67) with \( k_\mu \) or \( \epsilon_\mu \) we get zero because Levi-Civita tensor \( \epsilon^{\mu\alpha\beta} \) is totally antisymmetric. Therefore contracting Eq. (67) with either \( k_\mu \) or \( \epsilon_\mu \) we find
\[ 0 = k \cdot \dot{u} \rightarrow k \cdot u = k \cdot u(0) , \quad u^\mu(0) = u^\mu(\tau_0) \] (68)
\[ = \epsilon \cdot \dot{u} \rightarrow \epsilon \cdot u = \epsilon \cdot u(0) . \] (69)
We further note that the argument of the light pulse Eq. (66) satisfies
\[ \xi = k \cdot x \rightarrow \dot{\xi} = k \cdot \dot{x} = k \cdot u = k \cdot u(0) . \] (70)
where we used Eq. (68). Thus we conclude that the particle follows the pulse such that
\[ \xi = k \cdot x = \tau k \cdot u(0) + \xi_0 , \quad \xi_0 = k \cdot x(0) . \] (71)
The two conservation laws Eq. (68) and Eq. (69) along with Eq. (70) make the light pulse an interesting example amenable to an analytical solution.

We now evaluate several invariants in the laboratory frame seeking understanding of their relevance. A particle moving in the laboratory frame in consideration of Eq. (15) experiences in its rest frame a plane wave with the Doppler shifted frequency
\[ k \cdot u(0) = \gamma_0 (1 - n \cdot \beta_0) \omega \] (72)
which is unbounded as it grows with particle laboratory Lorentz-\( \gamma_0 \). However, \( k \cdot s \), the projection of spin onto plane wave 4-momentum \( k^\mu \) is bounded. To see this we recall the constraint Eq. (11) which in the laboratory frame reads
\[ S_0^0 - \beta \cdot S_L = 0 . \] (73)
We thus obtain
\[ k \cdot s(\tau) = k \cdot s(\tau)|_{L} = |k| (S_0^0 - n \cdot S_L) = |k| (\beta - n) \cdot S_L , \] (74)
where we used Eq. (72) in last equality. Since \( \beta \) and \( n = k/|k| \) are unit-magnitude vectors we find
\[ (k \cdot s(\tau))^2 \leq 4 k^2 S_L^2 . \] (75)
The magnitude of the spin vector in the lab frame is constrained by Eq. (12)
\[ - s^2 = S_0^0 - S_L^0 = (\beta \cdot S_L)^2 - S_L^2 = - \sin^2 \theta S_L^2 , \] (76)
where we again used Eq. (73). Combining Eq. (75) and Eq. (76) we see that except when particle is moving exactly in direction of \( S_L (\sin^2 \theta = 0) \), the magnitude of \((k \cdot s(\tau))^2 \) is bounded.

5.2.2 Invariant acceleration and spin precession

Even without knowing the explicit form for \( u^\mu(\tau) \), \( s^\mu(\tau) \) we were able to obtain \[ 27 \] the invariant acceleration
\[ \dot{u}^2(\tau) = - \left( \frac{d}{d \tau} f''(\xi(\tau)) \cdot k \cdot s(\tau) \cdot k \cdot u(0) \right)^2 . \] (77)
This result follows using the usual trick of taking a further (proper) time derivative of Eq. (64) (multiplied by a suitable factor) and on RHS eliminating \( \dot{u} \) by using Eq. (54). Multiplying the result with \( u_\mu \) and eliminating \( u \cdot \dot{u} \) using the 2nd differential of \( u^2 = c^2 \) produces Eq. (77).

We see in Eq. (77) that the magnitude of the 4-force created by a light pulse and acting on an ultrarelativistic particle is dependent on square of the product of the 2nd derivative of pulse function with respect to \( \xi \), \( f''(\xi) \), with the Doppler shifted frequency Eq. (72). The value Eq. (77) is negative since acceleration is a space-like vector.

As we discussed below Eq. (76) the spin precession factor \( k \cdot s \) seen in Eq. (74) is bounded. We were able to obtain a soluble formulation of the spin precession dynamics described by the dimensionless variable
\[ y = k \cdot s(\tau) \frac{\ddot{b} d}{mcC_1} \] (78)
which satisfies the differential equation
\[ \left( \frac{d y(s)}{ds} \right)^2 = y^2 (1 - y^2) \quad s = (f'(\xi(\tau)) - f'(\xi(0))) C_1 \] (79)
performing suitable manipulations of dynamical equations prior to solving for \( u^\mu(\tau) \), \( s^\mu(\tau) \). We are seeking bounded periodic solutions of nonlinear Eq. (79) no matter how large the constant \( C_1 \) determined by the initial conditions
\[ C_1 \equiv \frac{\ddot{b} d}{mc} k \cdot s(0) C_2 , \quad C_2 \geq 1 , \] (80)
\[ C_2 \equiv \sqrt{\frac{|(k \cdot u)^2| |s^2| - |(k \cdot u)(\epsilon \cdot s) - (\epsilon \cdot u)(k \cdot s)|^2 |}{c^2(k \cdot u)^2}} \] (81)
\( C_2 \) contains the initial particle Lorentz-\( \gamma_0 \) factor. One can see several possible solutions of interest of Eq. (79); for example \( y = \sin(\phi(s)) \) satisfies all constraints. It leads to the pendulum type differential equation and we recognize that high intensity light pulses can flip particle spin. However, there are other relevant solutions, \( e.g. \ y \propto 1/ \cosh z \).

Upon solution of Eq. (79) \( k \cdot s(\tau) \) is known, given Eq. (71) we also know the dependence of Eq. (67) on proper time. Hence Eq. (64) can be solved for \( u^\mu \) and Eq. (63) can be solved for \( s^\mu \) resulting in an analytical solution of the dynamics of a neutral magnetic dipole moment in the field of a light pulse of arbitrary shape. The full description of the dynamics exceeds in length this presentation and will follow \[ 27 \].
6 Conclusions
The Stern-Gerlach covariant extension of the Lorentz force has seen considerable interest as there are many immediate applications listed in first paragraph. Here we have:
1) introduced in Eq. (10) the covariant classical 4-spin vector $s^a$ in a way expected in the context of Poincare symmetry of space-time;
2) presented a unique linear in fields form of the covariant magnetic moment potential, Eq. (14), which leads to a natural generalization of the Lorentz force;
3) shown that the resultant Amperian, Eq. (19), and Gibeletian, Eq. (26), forms of the magnetic moment force are equivalent;
4) extended the TBMT torque dynamics, Eq. (35), making these consistent with the modifications of the Lorentz force;
5) demonstrated the need to connect the magnetic moment magnitude entering the Stern-Gerlach force with the one seen in the context of torque dynamics, subsection 5.2;
6) shown that variational principle based dynamics has systemic failings when both position and spin are addressed within present day conceptual framework, see section 4;
7) reduced the covariant dynamical equations to laboratory frame of reference uncovering important features governing the coupled dynamics, see section 5.3;
8) obtained work done by variations of magnetic field in space-time on a particle, Eq. (61);
9) shown salient features of solutions of neutral particles with non-zero magnetic moment hit by a laser pulse, see section 5.1;

References
1. J. Grange et al. [Muon g-2 Collaboration], “Muon (g-2) Technical Design Report,” FERMILAB-FN-0992-E, 666 pages, arXiv:1501.06858 [physics.ins-det].
2. G. W. Bennett et al. [Muon g-2 Collaboration], “Final Report of the Muon ES21 Anomalous Magnetic Moment Measurement at BNL,” Phys. Rev. D 73 (2006) 072003 doi:10.1103/PhysRevD.73.072003.
3. Kirk T. McDonald, "Forces on Magnetic Dipoles," Joseph Henry Laboratories, Princeton University, Princeton, USA; Retrieved November 2017; document dated October 26, 2014; updated February 17, 2017.
4. F. Mezei “La nouvelle vague in polarized neutron scattering,” Physica B+C 137 (1986) 295 doi:10.1016/0378-4363(86)90335-9.
5. X. G. Huang, “Electromagnetic fields and anomalous transports in heavy-ion collisions — A pedagogical review,” Rept. Prog. Phys. 79 (2016) no.7, 076302 doi:10.1088/0034-4885/79/7/076302 [arXiv:1509.04073 [nucl-th]].
6. M. Greif, C. Greiner and Z. Xu, “Magnetic field influence on the early time dynamics of heavy-ion collisions,” Phys. Rev. C 96 (2017) 014903 doi:10.1103/PhysRevC.96.014903 [arXiv:1704.06505 [hep-ph]].
7. R. Turolla, S. Zane and A. Watts, “Magnetars: the physics behind observations. A review,” Rept. Prog. Phys. 78 (2015) no.11, 116901 doi:10.1088/0034-4885/78/11/116901 [arXiv:1507.02924 [astro-ph.HE]].
8. A. Sedrakian, X. G. Huang, M. Sinha and J. W. Clark, “From microphysics to dynamics of magnetars,” J. Phys. Conf. Ser. 861 (2017) no.1, 012025 doi:10.1088/1742-6596/861/1/012025 [arXiv:1701.00895 [astro-ph.HE]].
9. M. Wen, C. H. Keitel and H. Banke, “Spin-one-half particles in strong electromagnetic fields: Spin effects and radiation reaction,” Phys. Rev. A 95 (2017) 042102 doi:10.1103/PhysRevA.95.042102 [arXiv:1610.08951 [physics.plasm-ph]].
27. M. Formanek, et. al, in preparation.
28. J. Rafelski, *Relativity Matters: From Einstein’s EMC2 to Laser Particle Acceleration and Quark-Gluon Plasma*, XXV+468 pages, ISBN: 978-3-319-51230-3, Springer (Heidelberg, New York 2017) doi:10.1007/978-3-319-51231-0
29. R. M. Wald, “General Relativity,” XIII+491 pages, ISBN: 0-226-87033-2, Chicago U. Press (Chicago 1984) doi:10.7208/chicago/9780226870373.001.0001
30. Walter Greiner and Johann Rafelski, *Spezielle Relativitäts-theorie: Ein Lehr- und Übungsbuch für Anfangssemester* (translated: Special Theory of Relativity: A Text and exercise book for undergraduates); appeared as Volume 3a of Walter Greiner Series in Theoretical Physics; H. Deutsch (Frankfurt 1984, 1989, 1992) ISBN3-87144-711-0; 1989: ISBN 3-97171-1063-4; and 1992: ISBN 3-8171-1205-X.
31. For contemporary account see T. Ohlsson, “Relativistic quantum physics: From advanced quantum mechanics to introductory quantum field theory,” 297p.ISBN: 9781139210720 (eBook), 9780521767262 (Print) (Cambridge University Press 2011)
32. Y. Hadad, L. Labun, J. Rafelski, N. Elkinda, C. Klier and H. Ruhl, “Effects of Radiation-Reaction in Relativistic Laser Acceleration,” Phys. Rev. D 82 (2010) 096012 doi:10.1103/PhysRevD.82.096012 [arXiv:1005.3980 [hep-ph]].
33. A. E. Lobanov and O. S. Pavlova, “Solutions of the classical equation of motion for a spin in electromagnetic field,” Theoretical and Mathematical Phys. 121 (1999) 1601 doi:10.1007/BF02557213 translated from Russian: 121 (1999) 509