Extracting the Possible Intrinsic Relation between the Radiative Efficiency and Mass of QSOs: A Maximum Likelihood Method and its Application to the SDSS DR7 QSOs

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Abstract

Radiative efficiencies of QSOs and their distribution encode rich information on the evolution of both masses and spins of massive black holes (MBHs) across cosmic time. In this paper, we develop a maximum likelihood method to statistically extract the intrinsic relation between radiative efficiency ($\epsilon$) and mass ($M$) of QSOs from their distribution on the luminosity–(empirically estimated virial) mass plane. By using mock samples, we find that strong constraint can be put on the $\epsilon$–$M$ relation at redshift $z \lesssim 0.4$ from uniform QSO samples similar to those in the Sloan Digital Sky Survey (SDSS) and from QSO samples at $z \sim 0.6$ ($z \lesssim 1.0$) if the magnitude limit of the survey can be $\sim 1$–$2$ (or $2$–$3$) magnitude deeper. Applying this method to the SDSS DR7 QSOs with $z \lesssim 0.7$, we find $\epsilon \propto M^{0.0--1.1}$ ($\epsilon \propto M^{-1.0--0.0}$) correlation for QSOs with the masses obtained according to the $H\beta$ (or Mg II) empirical mass estimator. These contradictory results may be due to the unknown systematic errors in the two mass estimators, preventing an accurate constraint on the $\epsilon$–$M$ relation by using current available QSO samples. We find that both the estimates of MBH mass and Eddington ratio distribution functions can be affected by the $\epsilon$–$M$ relation, suggesting that the determination of this relation is important for understanding the accretion and growth history of MBHs. In the future, the intrinsic $\epsilon$–$M$ relation is expected to be strongly constrained by using QSO samples obtained from surveys deeper than the SDSS if the host galaxy contamination and systematic errors of the mass estimator(s) can be well modeled or removed.

Unified Astronomy Thesaurus concepts: Quasars (1319); Black hole physics (159); Supermassive black holes (1663); Accretion (14); Galaxy accretion disks (562)

1. Introduction

There is a consensus that massive black holes (MBHs; in the mass range $M \sim 10^5$–$10^{10}$ $M_\odot$) lie in the centers of most galaxies. The mass growth and spin evolution of these MBHs are mainly governed by their accretion histories (e.g., Soltan 1982; Krolik 1999; Yu & Tremaine 2002; Marconi et al. 2004; Yu & Lu 2004; King & Pringle 2006; Pogge et al. 2009; Volonteri et al. 2013; Zhang & Lu 2019). Most QSOs are probably accreting gaseous material via the standard thin disk (Novikov & Thorne 1973; Shakura & Sunyaev 1973), and the mass growth rate of their central MBHs and the energy radiated from them are determined by the accretion rate and radiative efficiency, with the latter one directly determined by the MBH spin. MBHs with different masses are likely residing in different types of galaxies and environments, for example, MBHs at the high-mass end mostly reside in giant ellipticals, while those at the low-mass end mostly reside in spiral bulges, and thus they may have different accretion histories. A correlation between radiative efficiency $\epsilon$ (or spin $a$) and mass of active MBHs may emerge because of environmental differences (e.g., coherent or chaotic accretion Dotti et al. 2013; Dubois et al. 2014; Reynolds 2014; Zhang & Lu 2019). Such a correlation, if it exists, may be a probe to the accretion histories of MBHs.

Currently, it is still not clear whether there is an intrinsic $\epsilon$–$M$ correlation or not. Davis & Laor (2011) directly estimated the radiative efficiencies for Palomar-Green (PG) QSOs by $\epsilon = L_{\text{bol}}/M_{\text{acc}} c^2$, where $L_{\text{bol}}$ is the bolometric luminosity, $M_{\text{acc}}$ the rate inferred from the standard thin accretion disk modeling, and $c$ the speed of light. They found a correlation between the radiative efficiencies and MBH masses of those PG QSOs. Wu et al. (2013) further estimated the radiative efficiencies for Sloan Digital Sky Survey (SDSS) QSOs and also found such a correlation, but they attributed it to an apparent correlation that is possibly induced by the sample flux limit and uncertainties in the MBH estimates (see also Raimundo et al. 2012). Therefore, it is important to model and remove those effects induced by the sample selection and measurement errors in the $\epsilon$–$M$ relation analysis in order to confirm whether there is an intrinsic correlation or not.

In the standard thin accretion disk model, the luminosity of a QSO at the optical band depends on the accretion rate, MBH mass, as well as spin. For systems with a fixed Eddington ratio $\lambda$ (e.g., 0.01–1), the mass accretion rate decreases with increasing radiative efficiency, and the optical-band luminosity is correspondingly affected. To estimate the optical-band luminosities accurately, the relativistic correction is important if the spin $a$ and correspondingly $\epsilon$ is large, and it is even substantial if $M > 10^4 M_\odot$ because the peak of spectral energy distribution shifts to the optical band and a larger $\epsilon$ corresponds to a higher peak. For these reasons, if there is an underlying intrinsic $\epsilon$–$M$ relation for a large sample of QSOs, their distribution on the luminosity–MBH (virial) mass plane may be significantly different from that of a sample without such an intrinsic relation.
In this paper, we develop a maximum likelihood method to extract the intrinsic \( c-M \) relation, if any, from the distribution of a large number of QSOs on the luminosity--MBH (virial) mass plane, by utilizing the relativistic thin accretion disk model (e.g., Novikov & Thorne 1973). We take into account both the selection effects and the uncertainties in the MBH mass estimates. We first test the validity of this method by using mock samples of QSOs with different settings on the magnitude/flux limit, and then apply it to the SDSS DR7 QSO sample to obtain constraint on the possible intrinsic \( c-M \) relation.

This paper is organized as follows. In Section 2, we develop an analytical approach to calculate the optical-band luminosity of individual QSOs by adopting the relativistic standard thin accretion disk model. Assuming such a model, we then introduce a maximum likelihood method in Section 3 for extracting the possible intrinsic \( c-M \) relation. We generate a number of mock QSO samples and use the Markov Chain Monte Carlo (MCMC) fitting technique to verify this method in Section 4. In Section 5, we apply this method to the SDSS DR7 QSO sample and obtain constraints on the \( c-M \) relation. We then introduce an analytical approach to calculate the optical-band luminosity with the ray-tracing technique. Details for the ray-tracing method are provided in a previous work (Shimura & Takahara 1995; Salvesen et al. 2013). Discussions and conclusions are given in Sections 6 and 7, respectively.

In this paper, we assume a flat Lambda cold dark matter cosmology with parameters \((h_0,\Omega_m,\Omega_{\Lambda}) = (0.7, 0.3, 0.7)\), where \( h_0 = H_0/100 \text{ km s}^{-1} \text{ Mpc}^{-1} \) with \( H_0 \) as the Hubble constant, \( \Omega_m \) and \( \Omega_{\Lambda} \) are the fractions of matter and cosmological constant in the local universe, respectively.

### 2. Relativistic Thin Accretion Disk Model: an Analytical Approach

In this section, we introduce an analytical model to estimate the optical luminosity radiated from a system that accretes material via the standard thin disk. This analytical method is calibrated by fitting to the numerical results obtained from the sophisticated relativistic accretion disk model implemented with the ray-tracing technique. Details for the ray-tracing method are provided in a previous work (Zhang et al. 2015). By adopting such a numerical method, we trace back each ray in a distant observer’s image plane to the accretion disk in the equatorial plane of the MBH. Below we show only the results obtained from the accretion disk models at the wavelength 2500 Å as it will be applied to the SDSS-like QSO surveys.

The optical luminosity of a QSO in the standard thin accretion disk model depends on the MBH mass \( M_\bullet \), accretion rate \( \dot{M}_{\text{acc}} \) (or Eddington ratio \( \lambda \)), efficiency \( (c, \text{ or the MBH spin } a) \), and the inclination angle \( i \) (defined as the angle between disk normal direction and line of sight). We adopt the ray-tracing method to obtain the observed luminosity at any frequency (\( \nu \)) by integrating over the surface of accretion disk

\[
L_\nu = 4\pi \int g^3 I_{E0}(r_e, \theta_e) d\alpha d\beta.
\]  

Here \( \alpha \) and \( \beta \) are the two impact parameters describing the position of a disk element at \((r_e, \theta_e)\) on the observer’s sky. \( g = E/E_{E0} \) is the relativistic correction factor (for more details see Cunningham 1975), where \( E_{E0} \) is the energy of a photon at the rest frame of the disk element and \( E \) is the energy of the photon received by the observer. \( I_{E0} \) is the intensity at the rest frame of the disk element that emits the photon, which is given by

\[
I_{E0} = f_{\text{col}}^{-4} F(\nu).
\]

Here \( f_{\text{col}} \) is the spectral hardening factor (Shimura & Takahara 1995; Salvesen et al. 2013), \( F(\nu) \) the intensity of a blackbody radiation at temperature \( T \), where

\[
T = f_{\text{col}} \left( \frac{3\epsilon_0^6}{8\pi G^2 \sigma} \right) \frac{M_{\text{acc}}^{1/4}}{M_\bullet^{1/2}} Q^{1/4}(x),
\]

with \( G \) the gravitational constant and \( \sigma \) the Thompson scattering cross section. The function \( Q(x) \), with \( x = r_g^2/2 \) (in unit of \( r_g = GM_\bullet/c^2 \)), can be found in Gierliński et al. (2001).

For X-ray binaries, the typical value for the hardening factor \( f_{\text{col}} \) is \( \sim 1.7 \) (Shimura & Takahara 1995). For QSOs, however, there are large uncertainties in the estimates of this parameter. For simplicity, we only consider the following two cases: (1) \( f_{\text{col}} = 1 \), corresponding to the most simple standard thin accretion disk model (Novikov & Thorne 1973) and (2) \( f_{\text{col}} = 1.7 \), considering the effect of Comptonization (Shimura & Takahara 1995).

We find that the resulting luminosity from the latter case is almost the same as that predicted by the TLUSTY model (Hubeny & Lanz 1995) for a system with the same physical parameters (the differences in predicted luminosities usually <0.04–0.2 dex).

The maximum likelihood method mentioned in Section 3 requires calculations of the (optical band) luminosities of a large number of QSOs under various parameter settings. For a practical point of view, it may be more efficient if a fast and accurate analytical approximation to Equation (1) can be obtained for such an implementation.

For most QSOs, the optical--UV band luminosities given by Equation (1) follow a simple scaling relation, i.e., \( L_\nu \propto \lambda^{1/3} \epsilon^{-2/3} M_\bullet^{4/3} \cos i \), where \( h\nu < kT \) and the relativistic effects are ignored. Therefore, we introduce an analytical form to approximate the optical-band luminosity for a wide range of conditions, i.e.,

\[
L_\nu = c_0 f_{\text{col}}^{-4/3} K \lambda^{1/3} \epsilon^{-2/3} M_\bullet^{4/3} \cos i.
\]

Here \( K \) is a correction factor by considering the relativistic effects and the bending of the blackbody radiation spectrum, \( c_{0.1} = \epsilon/0.1 \) and \( M_\bullet = M_\bullet/10^8 M_\odot \). \( c_0 \) is a constant and set to \( c_0 = 2.54 \times 10^{45} \text{ erg s}^{-1} \) in order to make \( K = 1 \) when \( M_\bullet = \lambda = c_{0.1} = f_{\text{col}} = \cos i = 1 \).

Using the analytical Equation (4) to fit the numerical results obtained from Equation (1) at wavelength 2500 Å, we find that \( K \) can be approximately described by

\[
K \approx \frac{u\eta}{\exp(wn)} - 1,
\]

where \( \eta = M_\bullet^{1/2} / \dot{M}_{\text{acc,2.2}}^{1/4} = \lambda^{1/4} M_\bullet^{1/4} \epsilon_{0.1}^{-1/4} \) and \( \dot{M}_{\text{acc,2.2}} = \dot{M}_{\text{acc}}/ (2.2 M_\odot \text{ yr}^{-1}) \). The value of \( u \) and \( w \) depend only on the spin parameter and \( f_{\text{col}} \). If assuming \( f_{\text{col}} = 1.7 \), we have

\[
u = 0.0355 r_{\text{ISCO}}/r_g + 0.131, \quad w = 0.0315 r_{\text{ISCO}}/r_g + 0.116.
\]

If assuming \( f_{\text{col}} = 1 \), we have

\[
u = 0.0655 r_{\text{ISCO}}/r_g + 0.242, \quad w = 0.0553 r_{\text{ISCO}}/r_g + 0.196.
\]
3. A Maximum Likelihood Method

In this section, we introduce a maximum likelihood method that can be used to extract the possible intrinsic $\epsilon - M$ relation. We start with a general formalism of the method in Section 3.1, and then consider more specific selection function(s) and probability distribution function (PDFs) of various QSO properties in Sections 3.2 and 3.3, respectively.

3.1. The Likelihood Function

We denote the joint PDF of QSOs as $\Phi(M_\epsilon, L_\nu, z)$, which describes the number density of QSOs with luminosity in the range from $L_\nu$ to $L_\nu + dL_\nu$, true MBH mass in the range from $M_\epsilon$ to $M_\epsilon + dM_\epsilon$, and redshift in the range from $z$ to $z + dz$. For most QSOs, $L_\nu$ and $z$ can be directly measured with high accuracy. However, the observational determined MBH mass $M_{\epsilon,\text{obs}}$ may substantially deviate from the true MBH mass $M_\epsilon$ and this deviation may be denoted by a PDF $P(M_{\epsilon,\text{obs}}|M_\epsilon)$. Therefore, the observational determined joint PDF of QSOs, $\Phi_\text{obs}(M_{\epsilon,\text{obs}}, L_\nu, z)$, is given by

$$\Phi_\text{obs}(M_{\epsilon,\text{obs}}, L_\nu, z) = \int \Phi(M_\epsilon, L_\nu, z) P(M_{\epsilon,\text{obs}}|M_\epsilon) dM_\epsilon.$$  (8)

Assuming that the selection of QSOs depends only on the flux/luminosity limit of a QSO survey, the selection function can then be denoted by $\Omega(L_\nu, z)$. Note that this assumption may be a good approximation for the SDSS QSO survey, however, it may be an oversimplification for QSO surveys in general. In order to extract the intrinsic joint PDF, we define a likelihood function $L$ as (see Marshall et al. 1983)

$$\ln L = \sum_{i=1}^{N} \ln \Phi_\text{obs}(M_{\epsilon,i}, L_{\nu,i}, z_i)$$

$$- \iint \Phi(M_\epsilon, L_\nu, z) \Omega(L_\nu, z) \frac{dV}{dz} dM_\epsilon dL_\nu,$$  (9)

where $N$ is the total number of QSOs in the observational sample. The first term in the right-hand side of Equation (9) sums over all the observed QSOs giving their individual masses $M_{\epsilon,i}$, luminosity $L_{\nu,i}$, and redshift $z_i$, and the second term integrates over $M_\epsilon$ and $L_\nu$ and $z$. It is practically convenient to use the joint distribution of the MBH and Eddington ratio, i.e., $\Phi(M_\epsilon, \lambda, z)$ rather than $\Phi(M_\epsilon, L_\nu, z)$ in modeling the intrinsic distribution of QSOs. According to Equation (4), the joint PDF $\Phi(M_\epsilon, L_\nu, z)$ can be also described as

$$\Phi(M_\epsilon, L_\nu, z) = \frac{\Phi(M_\epsilon, \lambda, z)}{\left| \frac{\partial L_\nu}{\partial \lambda} \right|},$$  (10)

We can then calculate Equation (9) according to the assumed joint distribution of $\Phi(M_\epsilon, \lambda, z)$ and the intrinsic $\epsilon - M$ relation.

3.2. Observational Bias and Selection Functions

For a given emission line, assuming that the true mass of the MBH $M_\epsilon$ can be estimated by an ideal empirical relation, e.g.,

$$\log M_\epsilon = \tilde{a} + \tilde{b} \log L_\nu + \tilde{c} \log \text{FWHM},$$  (11)

where $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ are constants, and FWHM is the full width at the half maximum of the adopted emission line. Due to various

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7 In some cases, $\Phi_\epsilon = 0$ as the input parameters of the model poorly describe the observed (or mock) samples. For example, the scatter of the observational determined MBH masses around the true MBH masses $\sigma_\text{M}$ given by Equation (14) below is set too small to explain the observed mass distribution of QSOs. For these cases we set $\Phi_\epsilon$ to be a tiny value to avoid odd results.

8 We assume that the mass estimator is a function of $L_\nu$, which is used for the selection in Equation (9). Usually, the mass estimator is a function of the continuum luminosity around the emission line $L_{\text{em}}$, which may be different from $L_\nu$ (at 2500 Å). Considering that the continuum spectrum of a QSO can be described by a power law, $L_\nu$ can always be obtained according to a given $L_{\text{em}}$ and the difference between these two luminosities is then a constant that can be absorbed into the parameter $\tilde{b}$. 

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observational uncertainties, the empirical mass estimator may be not accurately obtained, so that it is scattered around, or biased from the true ones (e.g., due to stochastic dispersions in luminosity, see Shen 2013). As a result, the observed mass \( M_{\text{obs}} \) of QSOs given by the empirical mass estimator for QSOs now becomes

\[
\log M_{\text{obs}} = a + b \log L_\nu + c \log \text{FWHM} + s, \tag{12}
\]

where \( s \) is a random deviation following a Gaussian distribution with a zero mean and a scatter of \( \sigma_s \).

In this paper, the adopted mass estimator is based on either the H\( \beta \) or Mg\( \text{II} \) line, thus, following Shen et al. (2011) we set \( c = c = 2 \), assuming that both of these two emission lines are coming from the same regions of the broad line region (BLR; see Onken & Kollmeier 2008 for details.) Under such assumption, we have ignored the complexities of any systematic errors dependent on the FWHM.\(^9\) Then the observed MBH masses will be biased from the true ones by the systematic that is not well probed by the empirical estimator, i.e.,

\[
\log M_{\text{obs}} = \log M + (a - \tilde{a}) \log L_\nu + (b - \tilde{b}) \log L_\nu + s,
\]

where \( b_{\text{obs}} = a - \tilde{a} + \beta_{\text{obs}} \log L_\nu \) describes the possible systematic error in the MBH mass estimator with a dependence on the QSO luminosity showing by the parameter \( \beta_{\text{obs}} \) (similar to the parameter \( \beta \) in Shen 2013 and Shen & Kelly 2012), and \( L_\nu \) is a characteristic luminosity.

To cover such complexities, we assume that the PDF of the observational determined MBH masses \( M_{\text{obs}} \) around the true MBH mass \( M \) is described by a Gaussian function

\[
P(M_{\text{obs}} | M) = \frac{1}{\sqrt{2\pi} \sigma_M} \exp \left\{ -\frac{(\log (M_{\text{obs}} / M) + b_M + \beta_L \log (L_\nu / L_\nu))^2}{2\sigma_M^2} \right\}, \tag{14}
\]

Here we set \( \log (L_\nu / \text{erg s}^{-1}) = 44.67 \), which is the QSO luminosity when \( M = 10^8 M_\odot \), \( \log \lambda = -0.5 \), and \( \epsilon = 0.1 \).

We also assume that the selection function \( \Omega(L_{\text{opt}}, z) \) is simply given by

\[
\Omega(L_\nu, z) = \begin{cases} 
\Omega_0, & L_\nu \geq L_{\text{lim}}(z), \\
0, & L_\nu < L_{\text{lim}}(z),
\end{cases}
\tag{15}
\]

for a flux limit QSO survey. Here \( \Omega_0 \) is the fraction of sky coverage of the survey, and \( L_{\text{lim}}(z) \) is the luminosity limit determined by the flux limit at wavelength 2500 Å, which is larger than the luminosity limit \( L_{\text{lim}}(z) \). The fraction of QSOs with mass measurements in these spectroscopically targeted samples is close to unity, thus, it can be safely ignored (see discussions in Schulze et al. 2015). The \( L_{\text{lim}} \) at redshift \( z \) for a given flux limit \( L_{\text{lim}} \) of the SDSS can be found in Richards et al. (2006).

\[\text{3.3. Distribution Functions of the Intrinsic QSO Properties}\]

We assume that the intrinsic PDF of \( M \) and \( \lambda \) for QSOs within a sufficiently small redshift bin \((z \pm dz)\) can be approximated as

\[
\Phi(M, \lambda, z) = \Psi(M, \lambda) \Phi(z), \tag{16}
\]

where \( \Phi(z) \) represents the redshift evolution of the normalization, and \( \Psi(M, \lambda) \) represents the joint distribution of MBH mass and Eddington ratio. For simplicity, we assume

\[
\Phi(z) = 10^{\gamma(z-z_0)}, \tag{17}
\]

where \( \gamma \) and \( z_0 \) are two free parameters. Below we fix \( z_0 = 0.6 \) simply because it is close to the middle of the redshift range that is considered in this paper.

We further set

\[
\Psi(M, \lambda) = \Phi_M(M) \Phi_\lambda(\lambda | M), \tag{18}
\]

where \( \Phi_M(M) \) is described by a Schechter-like function\(^{10}\)

\[
\Phi_M(M) = \frac{\Psi_s}{\log_{10} e} \left( \frac{M}{M_s} \right)^{\alpha + 1} \exp \left\{ -\left[ \frac{M}{M_s} \right]^{\psi} \right\}, \tag{19}
\]

where \( \Psi_s, M_s, \alpha_M, \) and \( \beta_M \) are all free parameters.

The conditional Eddington ratio distribution for QSOs with the same \( M \) is assumed to be described by

\[
\Phi_\lambda(\lambda | M) = \frac{1}{\log_{10} e} \left( \frac{\lambda}{\lambda_s(M)} \right)^{\alpha + 1} \exp \left\{ -\left[ \frac{\lambda}{\lambda_s(M)} \right] \right\}, \tag{20}
\]

or alternatively a Gaussian function

\[
\Phi_\lambda(\lambda | M) = \frac{1}{\log_{10} e \sqrt{2\pi} \alpha_\lambda} \exp \left\{ -\frac{(\log \lambda - \log \lambda_s(M))^2}{2\sigma_\lambda^2} \right\}, \tag{21}
\]

where

\[
\log \lambda_s(M) = \log \lambda_0 + k_\lambda (\log M - 8), \tag{22}
\]

\( \lambda_0, k_\lambda, \) and \( \alpha_\lambda \) are all free parameters.

We assume that the intrinsic \( \epsilon-M \) relation is simply described by

\[
\epsilon = \begin{cases} 
0.038, & \text{if } \epsilon c_i M^{\alpha_\epsilon} > 0.038, \\
0.42, & \text{if } c_i M^{\alpha_\epsilon} < 0.038,
\end{cases} \tag{23}
\]

where \( c_i \) and \( \alpha_\epsilon \) are two free parameters, \( M_b = M / 10^8 M_\odot \).

The MBH spin is in the range from \(-1\) to \(1\), and correspondingly, \( \epsilon \) is confined in the range of \(0.038-0.42\) in the standard thin disk accretion model. In the model, the inclination angle \( i \) for each QSO is fixed at \( \arccos 0.8 \).

As can be seen from the above settings, the maximum likelihood model contains a total of 13 free parameters, i.e.,

\[\text{9} \text{ However, we notice that observations do suggest that } c \text{ may be biased and not exactly 2 (e.g., Wang et al. 2009). For the possible systematic errors of the FWHM and its impact on our results, see discussions in Section 5.3.}\]

\[\text{10} \text{ We have also try an alternative double power-law model:}\]

\[
\Psi_M(M) = \frac{\Psi_s}{\log_{10} e} \left( \frac{M}{M_s} \right)^{\alpha + 1} \left[ 1 + \left( \frac{M}{M_s} \right)^{\frac{\beta_M}{\alpha_M}} \right],
\]

However, our following calculations show that there is no clear difference between the results obtained by adopting these two different distribution forms. In the rest of the paper we discuss only the results from the Schechter-like function.
We generate a number of intrinsic mock QSOs by the Monte Carlo method. For simplicity, we also assume \( f_{\text{col}} = 1.7 \) and \( \cos i = 0.8 \), if not otherwise stated. The radiative efficiency \( \epsilon \) of each mock QSO is given by Equation (23), by assuming \( c_r = 0.1 \) and \( \alpha_r \) adopts a value in between \(-1\) and \(1\). Then the optical-band luminosity \( L_v \), at 2500 Å of each mock QSO is estimated according to Equation (4).

We find that the constraint on the intrinsic \( \epsilon-M \) relation weakly depends on the sample size, or correspondingly, \( \Omega_0 \) for a given magnitude limit. Therefore, we simply assume that \( \Omega_0 = 1 \). We assume that a QSO at redshift \( z \) is “observable” if its \( L_v \geq L_{\text{lim}}(z) \), and then the “observed” mass of this mock QSO is set according to Equation (14) by assuming \( \sigma_M = 0.3 \), \( \beta_M = 0 \), and \( b_M = 0 \). Finally, we obtain a sample of \( N \) mock QSOs with “observational” properties of \( M_{\text{obs}}, i, \Omega_0, i \), and \( \zeta_r \), \( i = 1, \cdots, N \). According to such a mock sample, we can then use the maximum likelihood in Equation (9) to recover the input parameters and check how well they can be reproduced.

Figure 2 shows the distributions of mock QSOs on the \( L_v-M_{\text{obs}} \) plane resulting from the same \( \Phi(M, \lambda, z) \) but two different \( \epsilon-M \) relations, one with \( \alpha_r = 1 \) (blue contours) and the other with \( \alpha_r = -1 \) (red contours). These two distributions are quite different, suggesting that the apparent \( L_v-M \) joint distribution depend not only on \( \Phi(M, \lambda, z) \) but also on the intrinsic \( \epsilon-M \) correlation, if any. Due to such differences, the intrinsic \( \epsilon-M \) relation can be extracted from the 2D distribution of QSOs on the luminosity versus the observed MBH mass plane.

4. Mock Sample Test of the Method

The distribution of QSOs from a survey on the \( L_v-M_{\text{obs}} \) plane only shows those QSOs above the magnitude limit of the survey. Therefore, the constraints on the \( \epsilon-M \) relation obtained from such a distribution may strongly depend on the flux/magnitude limit of the survey.

To investigate how the constraints on the model parameters depend on the flux limit, we perform a number of Monte Carlo Markov Chain (MCMC) calculations. We consider three cases, i.e., \( \alpha_r \) is assumed to be \(-0.5, 0, \) and \(0.5\), respectively. For each of these cases, we first generate mock QSO samples at the redshift \( 0.5 < z < 0.7 \)\(^{11} \) and the \( i \)-band magnitude limit is set to be \( \lim_i = 19.1 \) mag (similar to the SDSS QSO survey), 20, 21, or 22 mag. Then we use the MCMC method to recover the input values of those model parameters by adopting the likelihood function given by Equation (9). The model considered here contains 11 parameters (with \( P_M = 0 \) and \( \beta_M = 0 \)), or a model with 12 parameters by only fixing \( \beta_M = 0 \), so that we can see the effects of constant mass bias \( b_M \) by comparing these two models.

To show more clearly the constraints of the intrinsic \( \epsilon-M \) relation, we obtain the two-dimensional (2D) confidence level for the reconstructed \( \epsilon-M \) relation according to the accepted values of \( \alpha_r \) and \( c_r \) in each chain of the MCMC fittings. The results for the case \( \alpha_r = 0.5 \) are shown in Figure 3. We can see that the input \( \epsilon-M \) relation can be well recovered when the flux limit is sufficiently low, i.e., \( \lim_i \geq 20-21 \) mag. Note that the best-fit values of \( \alpha_r \) and \( c_r \) slightly deviate from the input ones when \( \lim_i = 22 \), which is simply due to the random noises.

\(^{11}\) It can be similarly done for any other redshift bins.
introduced by the Monte Carlo method in generating the mock QSO samples.

When \( i_{\text{lim}} \lesssim 19 \) mag, there is a strong degeneracy between \( \alpha_c \) and \( c_e \) (top panel of Figure 3) and the \( \epsilon-M_* \) relation cannot be well constrained (bottom panel of Figure 3). Such a degeneracy is expected as low-mass QSOs cannot be observed, and the observed sample covers only a small range of \( M_* \). If these faint QSOs can be included in the sample, as illustrated in Figure 3 when \( i_{\text{lim}} \gtrsim 20-21 \) mag, the degeneracy will then be almost disappeared.

Figure 4 shows the constraints on \( \epsilon-M_* \) relation for cases with different input values of \( \alpha_c \) by using either the 11- or 12-parameter model. We can see that, in most cases, the constraints on both \( \alpha_c \) and \( c_e \) are significantly improved if the magnitude limit of the QSO survey increases to \( \gtrsim 20-21 \) mag. Note here that the value of \( c_e \) is always poorly constrained (see bottom panel of Figure 4) if there is no intrinsic \( \epsilon-M_* \) relation (i.e., \( \alpha_c \sim 0 \)) and if the \( b_M \) is included in the MCMC fitting. As \( b_M \) reflects the systematic errors in the mass estimates, \( c_e \) is strongly degenerate with \( b_M \) when \( \alpha_c \sim 0 \). However, such a degeneracy can be broken when \( \alpha_c \) is significantly different from 0.

We find that the constraints on the intrinsic \( \epsilon-M_* \) relation depend also on the initial value of \( \alpha_c \). If \( \alpha_c \) < 0, the QSOs in the mock samples are statistically brighter than those if \( \alpha_c \) > 0 (see Figure 2), thus the constraints are slightly stronger (see Figure 4). For other parameters in the model, we also find that they can be reproduced with considerable accuracy.

Similar constraints on the intrinsic \( \epsilon-M_* \) relation can be also obtained for mock samples at other redshift bins if assuming that the intrinsic distribution function \( \Phi(M_*, L_\nu, z) \) depends weakly on the redshift. According to the relation between the luminosity and the \( i \)-band magnitude limit given by Richards et al. (2006), \( i_{\text{lim}} = 20 \sim 21 \) mag at \( z \sim 0.6 \) corresponds to \( L_\nu > 10^{44.5-10^{44}} \) erg s\(^{-1}\) and this luminosity range corresponds to \( i_{\text{lim}} \sim 19-20 \) mag at \( z = 0.4 \). Therefore, it is possible to get a reasonably constraints by using the SDSS QSO sample at redshift \( z \lesssim 0.4 \) as its magnitude limit is 19.1 mag. Similarly, if a QSO survey can have the magnitude limit of \( i_{\text{lim}} = 22 \) mag, then it would provide strong constraints on the \( \epsilon-M_* \) relation at redshift \( z \lesssim 1.5 \).

4.3 Scattering of \( \cos i \) and \( \epsilon \)

For the mock simulations in the previous section, we have set \( \cos i = 0.8 \) and assume that the \( \epsilon-M_* \) relation is explicitly given by Equation (23). In reality, however, it is likely that \( \cos i \) is randomly distributed over a range of values, and the \( \epsilon-M_* \) relation may have some scatters. To check whether the maximum likelihood method can still work well under these circumstances, we generate a group of mock samples with settings similar to those in Section 4.1 except \( \cos i \) is uniformly distributed between (0.6, 1.0) and a Gaussian scatter of 0.1 dex on the \( \epsilon \) given by Equation (23). We generate mock QSO samples at \( 0.3 < z < 0.5 \) and set \( i_{\text{lim}} = 19.1 \) mag to mimic the case of SDSS DR7 QSOs. We then use the MCMC fitting technique to obtain constraints on the \( \epsilon-M_* \) relation.

Figure 5 shows the results of the constraints on the intrinsic relation. The green dots show the exemplified value of \( \epsilon \) for the “observed” sample. The blue solid line represents the best-fit value and the region enclosed by the gray shadows shows the \( 2-\sigma \) constraints. It can be seen that the \( \epsilon-M_* \) relation can still be well reproduced, even if there are some scatters in the radiative efficiencies. As demonstrated in Section 4.2, the constraints on the intrinsic correlations is stronger for the samples with \( \alpha_c < 0 \) than those with \( \alpha_c > 0 \).

If the scatter is large, then Equation (23) is not so meaningful and the reconstructed value of \( \alpha_c \) will then likely be zero, which may only reflect an effective intrinsic relation. We find that the other model parameters, defining the form of \( \Phi(M_*, \lambda, z) \), can also be well reconstructed within the 3\( \sigma \) confidence level. Therefore, we conclude that the maximum likelihood method introduced in this paper can work well even if there is a large scatter in the \( \epsilon-M_* \) relation. Note here that there might be a scatter for the intrinsic \( \epsilon-M_* \) relation, if any, which is not considered above. In principle, one may add more model parameters to include such a possibility for the \( \epsilon-M_* \) relation when considering the likelihood function. However, the calculations will be more complicated and
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4.4. Effect of Host Galaxy Contamination

The contamination from host galaxies makes some of those QSOs with intrinsic luminosities below the magnitude limit become observable, and thus leads to a modification of the distribution of those QSOs on the observed $M_e-L_{QSO}$ plane. Therefore, the recovered intrinsic $\epsilon-M_e$ relation may be biased if the host galaxy contamination is not well modeled for the QSO sample(s).

Here we use MCMC simulations to investigate how the constraints of the intrinsic correlations are affected by the uncorrected host galaxy contamination. For the demonstration purpose, here we consider two different models for host contamination. The first one is from Shen et al. (2011), in which the ratio of host to QSO luminosity at a given QSO luminosity is given by

$$ \frac{L_{host}}{L_{QSO}} = 0.5621 - 0.8971x + 0.4104x^2 - 0.05620x^3, \quad (25) $$

where $x = \log(L_{QSO}/\text{erg s}^{-1}) - 44$, and $x + 44 < 45.03$. Another one is derived from Schulze et al. (2015; see their Figure 2):

$$ \frac{L_{host}}{L_{QSO}} = 0.7158 - 0.2941x + 0.03812x^2 - 0.01021x^3 \quad (26) $$

and $x + 44 < 46.67$. The upper limits of $x$ for the above two equation are obtained by setting the host to QSO luminosity ratio to zero. For QSOs with luminosities larger than the upper limit, the host contamination is negligible. Figure 6 shows the ratio of host to QSO luminosity given by these two models.

We generate a group of samples similar to those in Section 4.1 but at lower redshift $0.3 < z < 0.5$ (such that the host galaxy contamination becomes more important). Initially we set $\alpha_e = -0.5$ (or $\alpha_e = 0.5$). We then add the luminosity of galaxies into the samples according to Equation (25) or (26). For each group of samples we use MCMC simulations (of which the host galaxy contamination is not included) to investigate whether the $\epsilon-M_e$ correlations can be correctly recovered.

The results are as follows. We found that if the host galaxy contamination is similar to that given by Shen et al. (2011), the recovered $\alpha_e$ will be biased to higher values. For example, the recovered $\alpha_e$ is $0.2$ (or $0.7$) if the intrinsic $\alpha_e = -0.5$ (or $0.5$). However, if the contamination is similar to that given by Schulze et al. (2015), the effect of host galaxy contamination is not important. Both the intrinsic $\epsilon-M_e$ relation and other model parameters can be well recovered.

These results suggest that whether the intrinsic $\epsilon-M_e$ relation can be well recovered or not depends on the exact form of the host galaxy contamination. For example, the host galaxy contamination from Shen et al. (2011) is more severe than those in Schulze et al. (2015) for faint QSOs (see Figure 6). The $M_e-L_{QSO}$ distribution of the observed samples are more significantly modified, resulting in a strong bias of the intrinsic correlations. The above results suggests that, for a comprehensive study, it is important to investigate the effects of host contamination.

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Footnotes:

12 The scatterings of $\epsilon$ can be described by a probability function $\Phi(\epsilon|M_e)$ similar to that shown in Equations (20) and (21), then the mass–luminosity joint distribution is given by

$$ \Phi(M_e, L_e) = \frac{\partial}{\partial L_e} \int^{\infty}_0 \Phi(M_e, L_e')dL_e' $$

$$ = \frac{\partial}{\partial L_e} \left[ \int_{L_e(\lambda, \epsilon)}^{\infty} \Phi(M_e, \lambda) \Phi(\epsilon|M_e) d\lambda d\epsilon \right] $$

where $L_e(\lambda, \epsilon)$ is given by Equation (4). Then the likelihood function of Equation (9) can be accordingly modified (by using the integrals of the above equation) and calculated.

13 Note that this correction is for the $5100$ Å luminosity. However, we adopt it here only to illustrate the importance of the host galaxy contamination. The correction in Shen et al. (2011) is a function of the total luminosity (see Equation (27)), which is not initially known for the mock QSOs, thus, we need to convert it as a function of intrinsic QSO luminosity by refitting to the original equation.
galaxy contamination in the recovery of \( \epsilon-M \) correlations, especially for low redshift QSO samples.

5. Application to the SDSS DR7 QSO Samples

In this section, we apply our maximum likelihood method to the SDSS QSOs samples to extract the underlying intrinsic correlations. We apply our method to fit the QSOs in each redshift bin by varies different assumptions on the model (see Table 1). The details are in the following sections.

5.1. SDSS QSOs Samples

In this section, we briefly describe the SDSS QSO samples (Shen et al. 2011). In the catalog of SDSS QSOs (Data Release 7 (DR7)), 104,746 QSOs have i-band magnitude \( M_i < -22 \), at least one broad emission line broader than 1000 km s\(^{-1}\) and the estimations of central MBH masses. About half of these SDSS QSOs (total 57,959) QSO at 0.3 < \( z \) < 5 were selected to form a homogeneous, statistical sample, which is primarily a flux limited sample with i-band magnitude \( m_i < 19.1 \) at \( z \leq 2.9 \), \( m_i < 20.2 \) at \( z > 2.9 \) and an additional bright limit of \( m_i \geq 15 \). We select the QSOs with 0.34 < \( z \) < 0.9 in this sample (total number 15,356) to form our SDSS QSO samples. Among them, those QSOs with mass estimation errors >0.5 dex were removed from the QSO sample. The total number of these QSOs is only \( \sim 3\% \) of the whole sample, thus the removal of them only slightly affects the overall completeness.

We have shown in Section 4.2 that the SDSS-like QSO samples beyond \( z \sim 0.6-0.8 \) are unlikely set useful constraints on the intrinsic correlations, thus, we do not include the QSO samples with \( z > 0.9 \). Also, we notice that the completeness of the QSO samples at \( z \leq 0.9 \) are almost unity (Richards et al. 2006), which greatly simplifies our maximum likelihood method. The \( L_{2500} \) luminosity at 2500 Å is converted from \( M_i(z = 2) \), the K-corrected i-band absolute magnitude (Richards et al. 2006).

We divide those QSOs with \( z < 0.9 \) into three redshift bins, i.e., 0.34 < \( z \) < 0.5, 0.5 < \( z \) < 0.7, and 0.7 < \( z \) < 0.9. For QSOs with 0.34 < \( z \) < 0.7, their masses can be obtained by utilizing the H/\( \beta \) estimator, while for those with 0.5 < \( z \) < 0.9, their masses can be obtained by utilizing the Mg II estimator. For the redshift bin 0.5 < \( z \) < 0.7, QSO masses can be obtained by using either the H/\( \beta \) or Mg II estimator. Therefore, a comparison between the results obtained by utilizing the H/\( \beta \) estimator and those by the Mg II estimator in this bin may provide an estimation of the effects induced by possible systematical errors in different mass estimators. The total numbers of QSOs using the H/\( \beta \) estimator are 3420 and 4025 in the bins 0.34 < \( z \) < 0.5 and 0.5 < \( z \) < 0.7, respectively, while those using the Mg II estimator are 4039 and 3872 in the bins 0.5 < \( z \) < 0.7 and 0.7 < \( z \) < 0.9, respectively. Figure 9 shows the 2D number distribution of these QSOs in the \( L_{2500} - \log M_{\text{obs}} \) plane.

5.2. Models

The MCMC fitting results of the SDSS QSOs samples may depend on the details of the modeling. For example, the choices of essential model parameters, forms of BHMF, correction of the host galaxy contamination, and \( f_{\text{col}} \), may all affect the fitting results. To explore their possible impacts, here we adopt eight
different models, with the underlying assumptions of each model listed in Table 1.

All models ignore the host galaxy contamination, expect models M12S2S and M12US, of which we have assumed that the host galaxy correction is given by Shen et al. (2011) and Schulze et al. (2015), respectively. According to Shen et al. (2011), this ratio is given by

\[
\frac{L_{\text{host}}}{L_{\text{QSO}}} = 0.8052 - 1.5502x + 0.9121x^2 - 0.1577x^3, \quad (27)
\]

where  \(x = \log(L_{\nu}/\text{erg s}^{-1}) - 44\), and  \(x + 44 < 45.03\); while according to Schulze et al. (2015, see their Figure 2), it is given by

\[
\frac{L_{\text{host}}}{L_{\text{QSO}}} = 0.7917 - 0.3405x + 0.05689x^2 - 0.01461x^3, \quad (28)
\]

where  \(x + 44 < 46.67\).

The above two fitting formulas are valid only when \(\log(L_{\nu}/\text{erg s}^{-1}) \gtrsim 44\) (Shen et al. 2011) and \(\log(L_{\nu}/\text{erg s}^{-1}) \gtrsim 43.8\), respectively. By adopting these corrections, we can recover the intrinsic QSO luminosity, and use the MCMC model to extract the intrinsic correlations from these samples.

5.3. The Intrinsic \(\epsilon-M\) Relation

The intrinsic \(\epsilon-M\) relation is assumed to be a power law with slope \(\alpha_\epsilon\) and normalization \(c_\epsilon\) at \(M_\epsilon = 10^8 M_\odot\) (see Equation (23)). The best fits of these two parameters obtained from different models are summarized in Table 2 and shown also in Figure 7. Since the constraints on \(c_\epsilon\) are generally weak, the confidence level of the intrinsic relation in the \(\epsilon-M_\epsilon\) plane can be more clearly seen and is shown in Figure 8.

For the QSO samples utilizing the H\(\beta\) mass estimator, we summarize our main results at 0.34 < \(z < 0.7\) as follows. As shown in Table 2, the most likely value of \(\alpha_\epsilon\) is 0.3 ~ 1.1 obtained from the QSO sample with redshift 0.34 < \(z < 0.5\) by using different models, and it is still positive but slightly smaller (0.0 ~ 0.8) from the QSO sample with 0.5 < \(z < 0.7\). The most likely values of \(c_\epsilon\) are 1.4 ~ 0.7 obtained from the QSO samples with 0.34 < \(z < 0.7\). Apparently, most models suggest a positive correlation between \(\epsilon\) and \(M_\epsilon\), i.e., the larger the MBH mass, the higher \(\epsilon\) (or spin). As also seen from Figure 8, the constraint on the \(\epsilon-M_\epsilon\) relation is quite tight when it is obtained from the QSO sample with 0.34 < \(z < 0.5\) (panels in the first and fourth columns), while it has larger uncertainty when obtained from the QSO sample with 0.5 < \(z < 0.7\). For all models, \(\epsilon\) is low at \(M_\epsilon < 10^7 M_\odot\) and becomes high with \(\epsilon \sim 0.4\) at \(M_\epsilon > 10^8 M_\odot\).

For the QSO samples utilizing the Mg\(\text{II}\) mass estimator, we summarize the main results at 0.5 < \(z < 0.7\) as follows. The best-fit values of \(\alpha_\epsilon\) is -1.0 ~ -0.0 and \(c_\epsilon\) is -1.0 ~ -0.6. Most of those models tend to result in a negative \(\epsilon-M_\epsilon\) correlation, though only four among the eight models have \(\alpha_\epsilon > 0\) at 2\(\sigma\) confidence level. The constraints on the \(\epsilon-M_\epsilon\) relation shown in Figure 8 clearly suggest that \(\epsilon\) is large when \(M_\epsilon \sim 10^7 M_\odot\) and small \(\epsilon\) when \(M_\epsilon \sim 10^8\cdots 10^{10} M_\odot\).

For most models, the obtained results are inconsistent (at the 2\(\sigma\) level) with those constraints obtained from the QSO sample in

### Table 1
Different Models and the Settings Considered in this Paper

| Name         | Fixed Parameters | Host Galaxy Correction | BHMF | ERDF |
|--------------|------------------|------------------------|------|------|
| M11N2S       | \(b_{\text{Mz}} = 0, b_{\text{z1}} = 0\) | None | 1.7  | Schchter |
| M12N2S       | \(b_{\text{Mz}} = 0, b_{\text{z1}} = 0\) | None | 1.7  | Schchter |
| M12N1S       | \(b_{\text{Mz}} = 0, b_{\text{z1}} = 0\) | None | 1.7  | Schchter |
| M12S2S       | \(b_{\text{Mz}} = 0, b_{\text{z1}} = 0\) | None | 1.7  | Schchter |
| M12US        | \(b_{\text{Mz}} = 0, b_{\text{z1}} = 0\) | None | 1.7  | Schchter |
| M11N2SG      | \(b_{\text{Mz}} = 0, b_{\text{z1}} = 0\) | None | 1.7  | Schchter |
| M13N2G       | \(b_{\text{Mz}} = 0, b_{\text{z1}} = 0\) | None | 1.7  | Schchter |

Note. Here \(b_{\text{Mz}}\) and \(b_{\text{z1}}\) are two parameters describing the luminosity-dependent bias of the QSO mass estimator (defined in Equation (13)). Host galaxy contamination correction is given by either Equation (27) (“Shen11”) or Equation (28) (“Schulze15”), respectively.

### Table 2
Constraints on \(\alpha_\epsilon\) and \(c_\epsilon\) Obtained from the Fitting of SDSS QSOs in Different Redshift Bins to Those Models Listed in Table 1

| Model     | \(z = 0.34-0.5(\text{H}\beta)\) | \(z = 0.5-0.7(\text{H}\beta)\) | \(z = 0.5-0.7(\text{Mg}\text{II})\) | \(z = 0.7-0.9(\text{Mg}\text{II})\) |
|-----------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \(\alpha_\epsilon\) | \(c_\epsilon\) | \(\alpha_\epsilon\) | \(c_\epsilon\) | \(\alpha_\epsilon\) | \(c_\epsilon\) | \(\alpha_\epsilon\) | \(c_\epsilon\) |
| M11N2S    | 0.75 ± 0.32 | -1.13 ± 0.29 | 0.54 ± 0.25 | -1.11 ± 0.25 | -0.50 ± 0.41 | -1.01 ± 0.59 | 0.63 ± 1.08 | -0.89 ± 0.49 |
| M12N2S    | 0.91 ± 0.41 | -1.39 ± 0.09 | 0.65 ± 0.15 | -1.32 ± 0.09 | -0.74 ± 0.54 | -0.66 ± 0.26 | 0.02 ± 0.64 | -0.95 ± 0.55 |
| M12N1S    | 1.13 ± 0.51 | -1.35 ± 0.11 | 0.55 ± 0.30 | -1.31 ± 0.11 | -0.36 ± 0.51 | -0.65 ± 0.25 | 0.13 ± 0.52 | -0.94 ± 0.54 |
| M12US     | 1.06 ± 0.21 | -1.39 ± 0.09 | 0.67 ± 0.21 | -1.38 ± 0.07 | -1.03 ± 0.67 | -0.67 ± 0.27 | 0.09 ± 0.72 | -0.93 ± 0.53 |
| M12US     | 0.87 ± 0.42 | -1.27 ± 0.18 | 0.77 ± 0.15 | -1.33 ± 0.14 | -0.68 ± 0.88 | -0.70 ± 0.30 | -0.08 ± 0.67 | -0.91 ± 0.51 |
| M13N2S    | 0.55 ± 0.36 | -1.03 ± 0.21 | 0.52 ± 0.25 | -1.26 ± 0.16 | -0.69 ± 0.98 | -0.73 ± 0.33 | -0.14 ± 0.60 | -1.00 ± 0.42 |
| M11N2SG   | 0.27 ± 0.14 | -0.72 ± 0.11 | 0.02 ± 0.26 | -0.62 ± 0.16 | -0.09 ± 1.07 | -0.82 ± 0.42 | -0.54 ± 1.34 | -0.91 ± 0.51 |
| M13N2G    | 0.53 ± 0.26 | -0.96 ± 0.46 | 0.32 ± 0.37 | -1.07 ± 0.67 | -0.87 ± 1.09 | -0.79 ± 0.39 | -0.33 ± 1.08 | -0.91 ± 0.51 |

Note. Here “H\beta” and “Mg II” indicate those QSO samples in which the black hole masses are obtained by using the H\beta and Mg II mass estimators (“Schulze15”), respectively. The subscript and superscript associated with each of the best-fit values make the 2\(\sigma\) uncertainty range.

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the masses of SDSS QSOs are obtained by adopting the H β bars associated with each symbol indicate the 2σ confidence level. The red-dashed lines show the reference values of $\alpha_c = 0$ and $c_\beta = -1.0$, respectively. Note that the masses of SDSS QSOs are obtained by adopting the H β mass estimator in the redshift bin $0.34 < z < 0.7$ (filled squares) and the Mg II mass estimator in redshift bin $0.5 < z < 1.5$ (filled circles). Note that the results obtained from different models at each redshift bin are slightly offset from each other in the horizontal direction, for the clarity of the figure.

We find that our above conclusions on the $\epsilon-M$ relation for SDSS QSOs are not affected much by including host galaxy contamination or not in the model. This is possibly because of the complexities of host galaxy contamination functions. For example, the two kinds of host galaxy contamination adopted may be either overestimated or underestimated. On the other hand, the host contamination may vary from galaxy to galaxy, however, the simple functional correction shown in Equations (27) and (28) may not be accurate enough.

We also find that our conclusions on $\epsilon-M$ correlation does not affected by changing $f_{\text{col}} = 1.7$ to $f_{\text{col}} = 1$, or adopting a different distribution function of $\Phi(M, \lambda)$ (e.g., using Equation (21) rather than Equation (20)) in the modeling. For all eight models listed in Table 1, we have also tried other variations of them, e.g., by changing the form of Equation (19) in model M13N2S to a double power-law model (see its footnote), or by making a host galaxy correction by Equation (27) in addition to model M13N2S, we do not see significant differences in the recovered correlations. These results suggest that the inconsistencies are not due to these complexities. Note that only a few cases were test here and the above conclusion may need further confirmation by using a more sophisticated model, etc. in the future.

Thus, the inconsistency between the constraints on $\epsilon-M$ relation obtained by using the H β estimator and that using the Mg II one may be partly due to the possible bias induced by using the FWHM in the same way for H β and Mg II in the estimator. It is commonly assumed that the Mg II and H β emitting regions are the same and have the same line width (e.g., Salviander et al. 2007; Onken & Kollmeier 2008). In the SDSS sample we adopted, they use a fix ratio of 2 for the FWHM in both the H β and Mg II estimators (Shen et al. 2011). However, observations suggest that it can be different from 2, as they may be from different line-emitting locations in the BLR (e.g., Wang et al. 2009). Currently, it is still unclear whether the estimates of $M_*$ based on Mg II are consistent with those based on H β. In the current work, we cannot include the FWHM in our modeling, thus, if there are some systematics in the FWHM, we cannot reduce them here.

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**5.4. The Bias Due to the Mass Estimators, Host Galaxy Contamination, and other Parameters in the Model**

If inconsistencies between the results from H β and Mg II mass estimator are due to the bias of luminosity, then including $\beta_L$ should make their fitting results consistent with each other. However, we do not see such consistencies within the 2σ confidence level for those models including $\beta_L$ (e.g., M13N2S, M12N2SG), but we see larger uncertainties in the recovered $\epsilon-M$ correlation. Such inconsistencies do not disappear by including a constant $b_M$ as a free parameter or do not appear not in the MCMC simulations, although $b_M$ is usually nonzero ($\pm 0.2 \sim 0.4$, see Figure 12) resulting from all these models.
5.5. Constraints of BHMF and ERDF

In addition to the constraint on the $\epsilon$–$M$ relation, even strong constraints on the BHMF and ERDF at redshift $0.34 < z < 0.9$ can be also obtained from our model. The marginalized intrinsic BHMF and ERDF of the QSOs in each redshift bin are given by

$$\Phi(M_\star) = \frac{\int \Phi(M_\star, \lambda) d\lambda \int \Phi_\ell(z) \frac{dV}{dz} dz}{\int \frac{dV}{dz} dz},$$

and

$$\Phi(\lambda) = \frac{\int \Phi(M_\star, \lambda) dM_\star \int \Phi_\ell(z) \frac{dV}{dz} dz}{\int \frac{dV}{dz} dz},$$

respectively. Note that the results of $\Phi(\lambda)$ and $\Phi_{\text{obs}}(\lambda)$ depend strongly on the lower limit of the integration of $M_\star$, especially if $\lambda$ is small. It may not be so meaningful to consider the ERDF for MBHs with mass substantially below $\log(M_{\text{MBH}}) \sim 7.8$, as the completeness of $M_\star$ drops rapidly below $10^8 M_\odot$ (see Figure 10). To exclude the parameter ranges where the QSO samples are highly incomplete, we set the lower limit of $M_\star$ for the integrations in the above Equations (30) and (32), below which the completeness of $M_\star$ drops rapidly below $10^8 M_\odot$. (see Figure 10).

Figure 10 shows the BHMFs at different redshifts obtained from all eight models listed in Table 1. As seen from this figure, the resulting BHMFs are slightly model dependent for QSOs at $z < 0.7$, and some of them are significantly different from those given by Shen & Kelly (2012) and Kelly & Shen (2013). This may suggest that including the $\epsilon$–$M$ relation and the systematic errors in the estimated MBH masses in the model can affect the determination of BHMFs (similarly the ERDFs, see below). We find that, the larger the slope $\alpha_\ell$ of the $\epsilon$–$M$ relation, the larger the difference between the resulting
BHMF and that given by Shen & Kelly (2012) and Kelly & Shen (2013). For example, the models M11N2SG and M13N2SG result in a small value of $a$, and the resulting BHMFs are very close to those given by Shen & Kelly (2012) and Kelly & Shen (2013). For other models, the difference are more apparent as the $\alpha$ of them is larger.

Figure 11 shows the ERDFs in different redshift bins obtained from our models. As can be seen from this figure, the resulting ERDFs are also slightly model dependent. In the redshift bin $0.3 < z < 0.5$, the ERDFs obtained from our models have a relatively shallower slope and do not drop as rapidly around $\lambda = 1$ as those in Shen & Kelly (2012) and Kelly & Shen (2013). This is partly because we set a boundary for the Eddington ratio at $\lambda = 1$, and partly because of the positive $\epsilon - M$ correlations. Similar results for ERDFs are also obtained from the QSO sample with $0.5 < z < 0.7$ utilizing the H$\beta$ mass estimator. For the QSO sample with $0.5 < z < 0.7$ utilizing the Mg II mass estimator, the resulting ERDFs drop rapidly near $\lambda = 1$, and negative (rather than positive) $\epsilon - M$ correlations are obtained from different models. These results again suggest that a nonzero $\epsilon - M$ correlation can affect the constraints on ERDF as well as BHMF and raise the importance of considering the $\epsilon - M$ relation.

In the redshift bin $0.5 < z < 0.7$, both the BHMFs and ERDFs resulting from the fittings by utilizing the H$\beta$ (lines in green) and Mg II (lines in blue) mass estimators have some discrepancies. Such discrepancies are likely due to the fact that the H$\beta$ and Mg II mass estimators are not well modeled. As all parameters describing the BHMF, ERDF, and the $\epsilon - M$ relation are set free in the MCMC fittings, and a biased modeling of the mass estimator will then result in a biased $\epsilon - M$ relation (as shown in Section 5.4), BHMF, and ERDF. In the future, we expect that such discrepancies will disappear if we have a better model of the mass estimators.

Figure 11 shows the ERDFs in different redshift bins obtained from our models. As can be seen from this figure, the resulting ERDFs are also slightly model dependent. In the redshift bin $0.3 < z < 0.5$, the ERDFs obtained from our models have a relatively shallower slope and do not drop as rapidly around $\lambda = 1$ as those in Shen & Kelly (2012) and Kelly & Shen (2013). This is partly because we set a boundary for the Eddington ratio at $\lambda = 1$, and partly because of the positive $\epsilon - M$ correlations. Similar results for ERDFs are also obtained from the QSO sample with $0.5 < z < 0.7$ utilizing the H$\beta$ mass estimator. For the QSO sample with $0.5 < z < 0.7$ utilizing the Mg II mass estimator, the resulting ERDFs drop rapidly near $\lambda = 1$, and negative (rather than positive) $\epsilon - M$ correlations are obtained from different models. These results again suggest that a nonzero $\epsilon - M$ correlation can affect the constraints on ERDF as well as BHMF and raise the importance of considering the $\epsilon - M$ relation.

In the redshift bin $0.5 < z < 0.7$, both the BHMFs and ERDFs resulting from the fittings by utilizing the H$\beta$ (lines in green) and Mg II (lines in blue) mass estimators have some discrepancies. Such discrepancies are likely due to the fact that the H$\beta$ and Mg II mass estimators are not well modeled. As all parameters describing the BHMF, ERDF, and the $\epsilon - M$ relation are set free in the MCMC fittings, and a biased modeling of the mass estimator will then result in a biased $\epsilon - M$ relation (as shown in Section 5.4), BHMF, and ERDF. In the future, we expect that such discrepancies will disappear if we have a better model of the mass estimators.

Note that the way to obtain $\lambda$ in our method is very different from those in Kelly & Shen (2013) and Schulze et al. (2015). The Eddington ratio $\lambda$ is derived directly from the optical luminosity and the mass of QSOs in Kelly & Shen (2013) and Schulze et al. (2015), i.e., $\lambda \propto L_{\nu} M_{\ast}^{-1}$ as they adopt the empirical bolometric correction to directly get the total luminosity. In their methods, the Eddington ratio can be
obtained for each sample object. In our method, \( \lambda \) of each object cannot be separately obtained from observations and \( \ell \) of each individual object is unknown. In addition, in our method, \( \ell \) is also different from those adopted in Kelly & Shen (2013) and Schulze et al. (2015), according to Equation (4).

We notice that Shen & Kelly (2012) and Kelly & Shen (2013) adopt different modeling between the MBH and luminosity, and their constraints on ERDF are significantly different from each other, especially at the high Eddington ratio end. In our model, the Eddington ratio is generally complete if \( \lambda > -1.5 \sim -1.0 \) for the QSOs at redshift \( 0.34 < z < 0.9 \). Note that this completeness is only for the samples in which the completeness of the black hole masses is larger than \( 10^{-3} \).

5.6. Fitting Results of Other Parameters

The maximum likelihood method applied here can put constraints on not only the intrinsic \( \epsilon-M \), relation, BHMF, and ERDF, but also simultaneously the other parameters in the model, e.g., such as the parameters related to the mass estimators. Figure 12 shows that our best-fit mass scatter \( \sigma_M \) for QSOs with \( 0.3 < z < 1.5 \) is around \( \sim 0.3 \) dex, consistent with the value 0.3–0.4 dex that is commonly used in the literature (e.g., McGill et al. 2008; Kelly et al. 2010; Schulze et al. 2015).

According to the models with 13 parameters, we find that the constant mass bias \( b_M \) and the possible luminosity-dependent systematic errors introduced by \( \beta_L \) in the H\( \beta \) and Mg\( II \) estimators may be different. For the QSO sample with \( 0.34 < z < 0.7 \), all models result in \( b_M \sim 0.1-0.4, \beta_L = -0.25 \sim -0.15 \) by using the H\( \beta \) estimator, while for the QSO sample with \( 0.5 < z < 0.7 \), they result in \( b_M \sim -0.1 \) to 0.2, \( \beta_L \sim 0.02-0.2 \) by using the Mg\( II \) estimator. These results suggest that both the constant mass bias and the luminosity-dependent systematic error in the Mg\( II \) estimator may be different from those in the H\( \beta \) one.

6. Discussion

Due to the discrepancies between the results obtained by using the Mg\( II \) and H\( \beta \) mass estimators, it is difficult to make solid conclusions regarding the \( \epsilon-M \) relation, the BHMF, and ERDF of the SDSS QSOs. If both mass estimators are well modeled, we expect that these results should be consistent with each other. If not, both the \( \epsilon-M \) relation and the BHMF or ERDF would depend on the correctness of the mass estimators, and they will be biased if there are any remaining systematics. Such unknown systematics could possibly be addressed in the future by extending our models to include the FWHM, velocity dispersion, or other details of the mass estimator(s) (similar to some previous works, e.g., Kelly et al. 2009). On the other
hand, it should be straightforward to expand our model likelihood function (Equation (9)) to fit QSOs samples with masses from two different mass estimators with equal weight. However, the accuracy of such a model is unknown, as currently we do not know whether the $\dot{\mathcal{M}}$–$M$ relations obtained by using the Mg II and H$\beta$ mass estimators are biased to negative or positive values. Only in the cases where the Mg II and H$\beta$ mass estimators are biased in opposite ways are we able to get more accurate results. Thus, this problem may be solved if the uncertainties in the Mg II and H$\beta$ mass estimators can be well determined or a better mass estimator can be provided, in which case more robust and consistent constraints on the $\dot{\mathcal{M}}$–$M$ relation can be obtained.

The $\dot{\mathcal{M}}$–$M$ relation of QSOs has also been investigated and discussed in some previous works but in a different way from the maximum likelihood method introduced in this paper. In these works, the radiation efficiency $\epsilon$ of individual QSOs were mostly estimated according to the QSO bolometric luminosities and the inferred accretion rates (e.g., Davis & Laor 2011; Raimundo et al. 2012; Wu et al. 2013; Schulze et al. 2017; Trakhtenbrot et al. 2017; Shankar et al. 2019). However, the selection effects of the flux limits of QSO surveys must be carefully considered, as they may lead to an apparent correlation between $\epsilon$ and $M$, even if there is no intrinsic one (e.g., see Raimundo et al. 2012; Wu et al. 2013). Furthermore, the model adopted in these works is not physical and self-consistent, as the averaged bolometric correction is used for those QSOs with the same luminosity at a given band but different masses and accretion rates. According to the accretion disk model, the bolometric corrections for QSOs should be dependent on the masses, spin, and accretion rate (see further discussion in Netzer & Trakhtenbrot 2014). As a comparison, our work is a statistical method, in which we do not focus on individual QSOs but the statistical distributions of QSO samples on the mass–luminosity plane. In our method, we have also taken account of the complexities due to the selection functions and the biases in the mass estimators, which is self-consistent as demonstrated by using the mock data.

The $\dot{\mathcal{M}}$–$M$ correlation may be also inferred from the (cosmological) evolution of black hole spins and mean radiative efficiencies (e.g., Li et al. 2015; Zhang & Lu 2019). The cosmological growth of the QSO masses and total luminosity radiated from QSOs are directly linked by the (mean) radiative efficiency, and thus the radiative efficiency (or spin) can be constrained according to the cosmological evolution of black hole mass density and the total radiated energy inferred from the QSO luminosity functions (see the So\'l\'tan argument in So\'l\'tan 1982 and the extended one in Yu & Lu 2004; the possible evidence for cosmological evolution and mass dependence of radiative efficiency in Wang et al. 2009; Li et al. 2015). However, the results on the $\dot{\mathcal{M}}$–$M$ relation possibly obtained by using the above method suffer from the large uncertainties in the empirical relations between black hole mass and galaxy properties that are adopted to estimate black hole mass densities and the uncertainties in the bolometric corrections and the luminosity functions used to estimate the QSO total energy densities. Considering also that our current model does not include the detailed physics to trace the

![Figure 11](image-url)
The intrinsic relation between radiative efficiency ($\epsilon$) and mass of MBHs ($M_{\bullet}$) in QSOs, if any, is important for understanding the MBH/QSO cosmological evolutions. In this paper, we develop a maximum likelihood method to extract the intrinsic $\epsilon-M_{\bullet}$ relation from the distribution of QSOs in the optical luminosity–MBH mass plane. We adopt a simple relativistic thin accretion disk model to estimate the radiation of QSOs in the optical-UV band.

We first use mock QSO samples to demonstrate that the intrinsic relation, if any, can be extracted from a SDSS DR7-like QSO sample with $z \sim 0.4$–0.6. For QSOs at higher redshift with $0.6 \lesssim z \lesssim 0.9$, the intrinsic relation can be robustly extracted only if a QSO survey can go about $\sim 1$–2 magnitude ($i_{\text{mag}} \approx 20$–21 mag) fainter than the current SDSS survey. If an SDSS-like survey has a flux limit down to $i_{\text{mag}} = 22$ mag, then it is possible to identify the intrinsic $\epsilon-M_{\bullet}$ relation for a QSO sample at redshift up to $z \sim 1.5$.

We find that such an intrinsic relation can still be extracted even if it has a scatter. Under some circumstances, if the samples are contaminated by the host galaxy and it is not corrected, the recovered $\epsilon-M_{\bullet}$ relation may be significantly biased from the intrinsic ones.

We apply our method to the SDSS QSOs in the redshift range $0.34 < z < 0.9$ and adopt a number of models to perform MCMC simulations and get constraints on the intrinsic $\epsilon-M_{\bullet}$ relation. We find signs of positive $\epsilon-M_{\bullet}$ correlations ($\epsilon \propto M_{\bullet}^{-0.5-1.1}$) for QSOs at $0.34 < z < 0.7$ utilizing the Hβ mass estimator. However, for QSOs at $0.5 < z < 0.7$ utilizing the Mg II mass estimator, we find that the correlations become negative ($\epsilon \propto M_{\bullet}^{-1.0-0}$). This inconsistency is likely due to the unknown different systematic errors in these two mass estimators, but not due to the host galaxy contamination.

We also derive constraints on the BHMF and ERDF of QSOs. Although our results are broadly consistent with some previous studies, e.g., Shen & Kelly (2012) and Kelly & Shen (2013), we find that the $\epsilon-M_{\bullet}$ relation, if it does exist, can affect the determination of BHMF and ERDF significantly; This may suggest that to obtain precise constraints on both the BHMF and ERDF, it is necessary to include the $\epsilon-M_{\bullet}$ relation in the modeling.

We conclude here that the maximum likelihood method introduced in this paper can set robust constraints on the intrinsic $\epsilon-M_{\bullet}$ relation, if any. However, if there are some unknown systematic errors in the mass estimators that cannot be well modeled, the recovered $\epsilon-M_{\bullet}$ relation may be strongly biased from the true one. We expect that the understanding of the systematics on the MBH mass estimator(s) will be improved a lot and the QSO surveys will go much deeper than the SDSS QSO survey adopted here, which will enable revealing the intrinsic $\epsilon-M_{\bullet}$ relation and better determinations of the BHMF and the ERDF of QSOs. In addition, it is possible to solve this problem if we can improve our model to consider the complexities of the FWHM of the mass estimators.

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