Time Operators of Harmonic Oscillators and Their Representations I

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Abstract

A time operator \( \hat{T}_\varepsilon \) of the one-dimensional harmonic oscillator \( \hat{h}_\varepsilon = \frac{1}{2} (p^2 + \varepsilon q^2) \) is rigorously constructed. It is formally expressed as
\[
\hat{T}_\varepsilon = \frac{1}{2} \sqrt{\varepsilon} (\arctan \sqrt{\varepsilon} \hat{t} + \arctan \sqrt{\varepsilon} \hat{t}^* )
\]
with \( \hat{t} = \frac{1}{i} q \). It is shown that the canonical commutation relation \([\hat{h}_\varepsilon, \hat{T}_\varepsilon] = -i \mathbb{I}\) holds true on a dense domain in the sense of sesqui-linear forms, and the limit of \( \hat{T}_\varepsilon \) as \( \varepsilon \to 0 \) is shown. Finally a matrix representation of \( \hat{T}_\varepsilon \) and its analytic continuation are given.

1 Introduction

1.1 Time operators

In this paper, we are concerned with time operators of the one-dimensional harmonic oscillator. Time operators are defined in this paper as conjugate operators for self-adjoint operators.

We begin with defining time operators. Let \([A, B]\) be the commutator of linear operators \(A\) and \(B\) defined by
\[
[A, B] = AB - BA.
\]

Definition 1.1 Let \(A\) be self-adjoint and \(B\) symmetric on a Hilbert space \(\mathcal{H}\). If \(A\) and \(B\) satisfy the canonical commutation relation:
\[
[A, B] = -i \mathbb{I}
\]
on a non-zero subspace \(D_{A,B} \subset D(AB) \cap D(BA)\), then \(B\) is called a time operator of \(A\).

Although the term “time operator” may not be appropriate at some situations, we follow the conventions and use this terminology throughout this paper. We show several examples of time operators.

(Position-Momentum) Let \(p = \frac{1}{i} \frac{d}{dx}\) and \(q = M_x\) be the multiplication by \(x\). Both \(p\) and \(q\) are self-adjoint operators on
\[
\begin{align*}
D(p) &= \{ f \in L^2(\mathbb{R}) \mid k\hat{f}(k) \in L^2(\mathbb{R}) \}, \\
D(q) &= \{ f \in L^2(\mathbb{R}) \mid xf \in L^2(\mathbb{R}) \},
\end{align*}
\]

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respectively, where \( \hat{f} \) denotes the Fourier transform of \( f \in L^2(\mathbb{R}) \), which is given by 
\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx.
\]
Hence the canonical commutation relation
\[
[p, q] = -i \mathbb{I}
\] (1.1)
holds on \( D(pq) \cap D(qp) \). Thus \( q \) is a time operator of \( p \), and \( D(pq) \cap D(qp) \) is dense in \( L^2(\mathbb{R}) \).

(Energy-Time: continuous case) The operator \( \frac{1}{2}p^2 \) describes the energy of the one-dimensional free quantum particle with the unit mass and the spectrum of \( \frac{1}{2}p^2 \) is purely absolutely continuous, i.e., Spec(\( \frac{1}{2}p^2 \)) = \([0, \infty)\). Let
\[
\hat{T}_{AB} = \frac{1}{2} (p^{-1}q + qp^{-1}).
\] (1.2)
\( \hat{T}_{AB} \) is called the Aharonov-Bohm operator [1] or the time of arrival operator. See Section 1.2 for the detail. It holds that
\[
\left[ \frac{1}{2} p^2 , \hat{T}_{AB} \right] = -i \mathbb{I}
\] (1.3)
on a dense domain. Then \( \hat{T}_{AB} \) is a time operator of \( \frac{1}{2} p^2 \).

(Energy-Time: discrete case) Let \( \frac{1}{2}(p^2 + q^2) \) be the one-dimensional harmonic oscillator which is the main object in this paper. The spectrum of \( \frac{1}{2}(p^2 + q^2) \) is purely discrete, i.e., Spec(\( \frac{1}{2}(p^2 + q^2) \)) = \( \{n + \frac{1}{2}\}_{n=0}^{\infty} \). Let
\[
\hat{T} = \frac{1}{2} ( \arctan p^{-1}q + \arctan qp^{-1}).
\] (1.4)
\( \hat{T} \) is called the angle operator. The name comes from the fact below; in the classical phase space the angle \( \theta \) between \((p_c, q_c)\) and \( p_c \)-axis is described by \( \theta = \arctan(q_c/p_c) \). It can be seen that \( \hat{T} \) and \( \frac{1}{2}(p^2 + q^2) \) formally satisfy
\[
\left[ \frac{1}{2} (p^2 + q^2), \hat{T} \right] = -i \mathbb{I}
\] (1.5)
on some domain. Then \( \hat{T} \) is a time operator of \( \frac{1}{2}(p^2 + q^2) \). However, as far as we know, there are no rigorous definitions of both \( \arctan(p^{-1}q) \) and \( \arctan(qp^{-1}) \) so far. We discuss this in this paper. See Theorems 3.3 and 3.4.

(Number-Phase) Let \( a = q + ip \) and \( a^* = q - ip \). Then \( N = a^*a \) is called the number operator. It holds that \([a, a^*] = \mathbb{I}\). In physics literatures a symmetric operator \( \hat{\phi} \) satisfying
\[
[N, \hat{\phi}] = -i \mathbb{I}
\] (1.6)
is called the phase operator, and formally it is described as
\[
\hat{\phi} = \frac{i}{2}(\log a - \log a^*).
\] (1.7)
Thus \( \hat{\phi} \) is a time operator of \( N \). However, there are also no rigorous definitions of both \( \log a \) and \( \log a^* \). We discuss this in the second paper [26].
Four canonical commutation relations (1.1), (1.3), (1.11) and (1.6) have been studied historically in theoretical physics so far, but (1.11) and (1.6) are crucial to investigate from a mathematical point of view. In this paper we study (1.11), and in the second paper [26] we study (1.6) and relationships between (1.4) and (1.7).

In this paper we discuss two time operators of the one-dimensional harmonic oscillator. One is the angle operator and the other an operator defined by a positive operator valued measure (POVM). The purpose of this paper is to consider properties of both the angle operator and the operator defined by POVM, rigorously, and to show relationships between the angle operator and the Aharonov-Bohm operator. The later can be attributed to considering a limit of the family of angle operators \( \hat{T}_\varepsilon \) defined in (1.9) as \( \varepsilon \to 0 \).

1.2 Aharonov-Bohm operator, angle operators and POVM

Within the long-running discussion on time in quantum mechanics, Aharonov and Bohm introduced the operator \( \hat{T}_{AB} \) [1]. They considered the quantization of functions of the classical momentum \( P \) and the classical position \( Q \), which provides us with the time of arrival at a point \( y \) of a particle that at the instant \( t = 0 \) has position \( x \) and momentum \( P \), i.e.,

\[
\text{time of arrival} = \frac{x - y}{P}.
\]  

(1.8)

The classical expression (1.8) of the time of arrival suggests to define the symmetric time operator of the free Hamiltonian \( \frac{1}{2}p^2 \) in quantum mechanics by \( \hat{T}_{AB} \). It is established that \( \hat{T}_{AB} \) is densely defined and symmetric but no self-adjoint extensions exist, and it satisfies

\[
\left[ \frac{1}{2}p^2, \hat{T}_{AB} \right] = -i \mathbb{1}
\]

on a dense domain. See e.g., [29] and [4, Theorems 4.21 and 4.22].

In this paper we are concerned with time operators of the one-dimensional harmonic oscillator with parameter \( \varepsilon \):

\[
\frac{1}{2}(p^2 + \varepsilon q^2), \quad 0 < \varepsilon \leq 1.
\]

Operator \( \frac{1}{2}(p^2 + \varepsilon q^2) \) is self-adjoint on \( D(p^2) \cap D(q^2) \). We shall construct \( \hat{T}_\varepsilon \) such that

\[
\left[ \frac{1}{2}(p^2 + \varepsilon q^2), \hat{T}_\varepsilon \right] = -i \mathbb{1}.
\]

By a heuristic argument in Section 2.1, formally \( \hat{T}_\varepsilon \) is given by

\[
\hat{T}_\varepsilon = \frac{1}{2} \frac{1}{\sqrt{\varepsilon}} \left( \arctan(\sqrt{\varepsilon}p^{-1}q) + \arctan(\sqrt{\varepsilon}qp^{-1}) \right).
\]  

(1.9)

The reason to introduce the parameter \( \varepsilon \) is as follows. We are interested in the limit of \( \hat{T}_\varepsilon \) as \( \varepsilon \to 0 \). Since \( \frac{1}{2}(p^2 + \varepsilon q^2) \) converges to \( \frac{1}{2}p^2 \) as \( \varepsilon \to 0 \), one expects that \( \hat{T}_\varepsilon \) converges to \( \hat{T}_{AB} \) as \( \varepsilon \to 0 \), but the time operators of \( \frac{1}{2}(p^2 + \varepsilon q^2) \) are not unique. Then it is worth considering the limit.
One standard way of constructing time operators is to apply POVMs. The details are given in Section 6. Let \( P \) be a POVM on a measurable space \((\Omega, \mathcal{B})\) associated with a self-adjoint operator \( H \). We define
\[
T_{\text{POVM}} = \int_{\Omega} t dP_t,
\]
Then \( T_{\text{POVM}} \) is a time operator of \( H \). By using the POVM associated with \( \frac{1}{2}(p^2 + q^2) \) one can construct a time operator \( T_G \) of \( \frac{1}{2}(p^2 + q^2) \). See (6.4) and (6.5). It is interested in comparing the angle operator to \( T_G \), since time operators are not unique.

### 1.3 Momentum representation

In what follows we take the momentum representation instead of the position representation. Let \( F : L^2(\mathbb{R}_x) \to L^2(\mathbb{R}_k) \) be the Fourier transformation. Then
\[
F p F^{-1} = M_k, \quad F q F^{-1} = +i\frac{d}{dk},
\]
where \( M_k \) is the multiplication by \( k \) on \( L^2(\mathbb{R}_k) \):
\[
D(M_k) = \{ f \in L^2(\mathbb{R}_k) \mid kf(k) \in L^2(\mathbb{R}_k) \},
M_k f(k) = kf(k).
\]
Since we work in \( L^2(\mathbb{R}_k) \) for \( L^2(\mathbb{R}_x) \) in what follows because of taking the momentum representation, instead of notations \( L^2(\mathbb{R}_k) \), \( M_k \) and \( i\frac{d}{dk} \), we rewrite them as \( L^2(\mathbb{R}), q \) and \( -p \), respectively, unless confusions may arise. Thus \( [p, q] = -i \mathbb{I} \) also holds true in the momentum representation. The Aharonov-Bohm operator in the momentum representation is a time operator of \( \frac{1}{2} q^2 \). Since \( q \) is an injective self-adjoint operator, the inverse operator \( q^{-1} \) exists as a self-adjoint operator. In particular, the domain of \( q^{-1} \) is dense. Let
\[
t = q^{-1} p \quad (1.10)
\]
with the domain \( D(t) = \{ f \in D(p) \mid pf \in D(q^{-1}) \} \). The operator \( t \) is densely defined but not symmetric. In the momentum representation, the Aharonov-Bohm operator is given by
\[
T_{AB} = -\frac{1}{2}(t + t^*). \quad (1.10)
\]
The operator \( T_{AB} \) is a densely defined symmetric operator, and it follows that
\[
\left[ \frac{1}{2} q^2, T_{AB} \right] = -i \mathbb{I}
\]
on a dense domain.

Furthermore in the momentum representation, \( \hat{h}_\varepsilon \) is then transformed to
\[
\hat{h}_\varepsilon = \frac{1}{2}(\varepsilon p^2 + q^2).
\]
We shall construct an operator \( \hat{T}_\varepsilon \) such that
\[
[h_\varepsilon, T_\varepsilon] = -i \mathbb{I}. \quad (1.11)
\]
Throughout this paper from now on we investigate \( T_{AB} \) and \( T_\varepsilon \) instead of \( \hat{T}_{AB} \) and \( \hat{T}_\varepsilon \).
1.4 Heuristic derivation of angle operators

Heuristically we derive $T_\epsilon$ satisfying (1.11) in this section. Since $[h_\epsilon, t] = i(\mathbb{1} + \epsilon t^2)$, we observe

$$[h_\epsilon, f(t)] = i((\mathbb{1} + \epsilon t^2)f'(t)$$

for a function $f$ on $\mathbb{R}$. From this we may expect that

$$f'(t) = (\mathbb{1} + \epsilon t^2)^{-1}$$

and then

$$f(t) = -\frac{1}{\sqrt{\epsilon}}\arctan \sqrt{\epsilon} t.$$ Symmetrizing $f$, we see that

$$T_\epsilon = -\frac{1}{2}\frac{1}{\sqrt{\epsilon}}(\arctan \sqrt{\epsilon} t + \arctan \sqrt{\epsilon} t^*)$$

(1.12)

may be a time operator of $h_\epsilon$. Operator $T_\epsilon$ appears in many literatures e.g., [40]. In a literature, however it is stated that $\arctan t$ is bounded because of $|\arctan x| \leq \pi/2$ for $x \in \mathbb{R}$. It is however incorrect. Since $t$ is unbounded and not symmetric, one can not apply the functional calculus for self-adjoint operators to define both $\arctan \sqrt{\epsilon} t$ and $\arctan \sqrt{\epsilon} t^*$. Then it is not trivial to define $\arctan \sqrt{\epsilon} t^#$ for $t^# = t, t^*$. In this paper we define them by the Taylor series:

$$\arctan \sqrt{\epsilon} t^# = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (\sqrt{\epsilon} t^#)^{2n+1}.$$ (1.13)

It is emphasised that $\arctan x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n + 1)$ is valid for $|x| < 1$. Moreover it is not straightforward to specify a dense domain $D$ such that $D \subset \bigcap_{n=0}^{\infty} (D(t^n) \cap D((t^*)^n))$ and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (\sqrt{\epsilon} t^#)^{2n+1} f \in L^2(\mathbb{R}), \ f \in D.$$

We give the exact definitions of $\arctan \sqrt{\epsilon} t^#$ in (3.3) and (3.4) in Section 3.1.

1.5 Literatures

We introduce several literatures on angle operators and phase operators. Literatures listed below are mainly taken from physics. The canonical commutation relation and the uncertainty relation are found by W. Heisenberg [25] at 1927. In 1945 the uncertain relation of time and energy is deduced from the principle of quantum mechanics in [28].

In 60’s Aharonov and Bohm [1] discuss the time operator for the case of a free quantum particle. The word “time operator” appears in [40] where a phase operator and an angle operator are studied. Also see [35, 15]. The mathematically rigorous study of $pq - qp = -i\mathbb{1}$ is given by [17]. In [36] time operators are studied on the set of distributions to avoid the difficulty of the domains of time operators. In [27] a relationship between angle operators and phase operators is studied.

In 80’s it is shown in [16] that no dense domains of canonical commutation relations exist under some conditions. Also in [37, 38] canonical commutation relations are studied from an operator theoretic point of view. In the series of papers [12, 8, 9, 10] the algebraic properties of time operators of harmonic oscillators and related operators are studied, but they are far away from firm mathematics. In [7] an extension of a Hilbert space is considered to define a self-adjoint phase operator. In [22, 23] $T_{AB}$ is defined on the Schwartz space and $p^{-1}$ is given by $\langle p^{-1} f | g \rangle = \int f^2 g(t) dt$. This is different from $p^{-1}$ defined in terms of the spectral measure of $p$. In [31] the phase operator is discussed, but from only physical point of view.
In 90’s in [13, 14] time operators of Hamiltonians with purely discrete spectrum are studied. In particular [13, (27b)] gives a time operator of a harmonic oscillator. In [33] a relationship between angle operators and phase operators is studied. We are inspired by papers [34, 39] where the difficulty of the definition of the angle operator is pointed out.

In the 21st century, in [29] time operators of Hamiltonians with purely absolutely continuous spectrum are given through the weak Weyl relations. On the other hand, in the series of papers [19, 18, 20, 21, 11], time operators of Hamiltonians with purely discrete spectrum are studied. See also [6, 3]. In [41] the existence of conjugate operators of Hamiltonians possessing purely point spectrum is established in an abstract setting. In [5] the hierarchy of time operators are introduced, and a time operator of the hydrogen atom which has both continuous spectrum and discrete spectrum, is given.

2 Main results

2.1 Hierarchy of time operators

The following hierarchy of classes of time operators is introduced in [5]:

\[ \text{Ultra-strong time} \subset \text{Strong time} \subset \text{Time} \subset \text{Weak time} \subset \text{Ultra-weak time} \]

We explain this hierarchy below. Let \( A \) and \( B \) be self-adjoint. If the Weyl relation

\[ e^{-isB} e^{-itA} = e^{its} e^{-itA} e^{-isB}, \quad s, t \in \mathbb{R} \]  

holds true, then \([A, B] = -i \mathbb{I}\) also holds true, which is deduced by taking derivatives \( d/ds \) and \( d/dt \) on both sides of (2.1). Then \( B \) is called the ultra-strong time operator of \( A \). It is known that the position operator \( q \) and the momentum operator \( p \) satisfy the Weyl relation. Furthermore by the von Neumann uniqueness theorem [42], if the Weyl relation (2.1) holds, and both \( e^{-itA} \) and \( e^{-isB} \) are irreducible, then \( A \) (resp. \( B \)) is unitarily equivalent to \( p \) (resp. \( q \)).

A pair \( \{A, B\} \) consisting of a self-adjoint operator \( A \) and a symmetric operator \( B \) is called the weak Weyl representation if \( e^{-itA} D(B) \subset D(B) \) for all \( t \in \mathbb{R} \) and

\[ Be^{-itA} \psi = e^{-itA}(B + t)\psi, \quad t \in \mathbb{R} \]  

holds true for all \( \psi \in D(B) \). If \( \{A, B\} \) is a weak Weyl representation, then the canonical commutation relation \([A, B] = -i \mathbb{I}\) also holds true. Then \( B \) is called the strong time operator of \( A \). It is shown in [29] that \( T_{AB} \) is a strong time operator of \( \frac{1}{2} q^2 \). The crucial fact is that if \( \{A, B\} \) is a weak Weyl representation, then the spectrum of \( A \) is purely absolutely continuous. In particular there are no eigenvalues of \( A \). We refer to see [29, 2, 4, 32] for the comprehensive study of weak Weyl representations.

One difficulty of the investigation of time operators is to specify their domains. The understanding of time in quantum mechanics as an observable has been hampered by the early recognition that no observable associated with time can exist. It was stated by W. Pauli in [30, p.63, footnote 2], but Pauli may forget domain arguments. Let \( He = Ec \). Using \([H, T] = -i \mathbb{I}\) formally, one obtains that \( He^{\varepsilon T} e = (E + \varepsilon) e^{\varepsilon T} e \) for all \( \varepsilon \in \mathbb{R} \). Hence \( e^{\varepsilon T} e \) is an eigenvector of \( H \) with eigenvalue \( E + \varepsilon \). Then each point in \( \mathbb{R} \) is an eigenvalue of \( H \), but this obviously contradicts the separability of \( L^2(\mathbb{R}) \). It should be noted, however, that this argument is very formal. In particular, no attention is paid to the domain of operators involved. E.g., \( e^{\varepsilon T} e \)
is not necessarily in the domain of $H$ and, if $e^{ieT}e \notin D(H)$, then $He^{ieT}e = (E + \varepsilon)e^{ieT}e$ is meaningless. See e.g., Theorem 3.14 in [18]. Furthermore at least either $e \notin D(T)$ or $Te \notin D(H)$ holds true.

To study time operators of self-adjoint operators possessing eigenvalues it is useful to introduce weak time operators and ultra-weak time operators to avoid the domain argument mentioned just above.

A symmetric operator $B$ is called the weak time operator of a self-adjoint operator $A$ if there exists a non-zero subspace $D \subset D(B) \cap D(A)$ on which the weak canonical commutation relation holds:

$$\langle A\phi, B\psi \rangle - \langle B\phi, A\psi \rangle = -i\langle \phi, \psi \rangle, \quad \phi, \psi \in D. \tag{2.3}$$

We can furthermore generalize weak time operators, which is the so-called ultra-weak time operator. Ultra-weak time operators are the key tool in this paper.

**Definition 2.1 (ultra-weak time operator [5])** Let $H$ be a self-adjoint operator on $\mathcal{H}$ and $D_1$ and $D_2$ be non-zero subspaces of $\mathcal{H}$. A sesqui-linear form

$$t : D_1 \times D_2 \rightarrow \mathbb{C}, \quad D_1 \times D_2 \ni (\phi, \psi) \mapsto t[\phi, \psi] \in \mathbb{C}$$

with the domain $D(t) = D_1 \times D_2$ is called the ultra-weak time operator of $H$ if there exist non-zero subspaces $D$ and $E$ of $D_1 \cap D_2$ such that the following (1)–(3) hold:

1. $E \subset D(H) \cap D$.
2. $t[\phi, \psi]^* = t[\psi, \phi]$ for all $\phi, \psi \in D$.
3. $HE \subset D_1$ and, for all $\psi, \phi \in E$,

$$t[H\phi, \psi] - t[H\psi, \phi]^* = -i\langle \phi, \psi \rangle. \tag{2.4}$$

We call $D$ the symmetric domain and $E$ the ultra-weak CCR-domain of $t$.

### 2.2 New insights and main results

In this paper we are concerned with time operators of Hamiltonians with purely discrete spectrum. We in particular focus on the one-dimensional harmonic oscillator. As far as we know, there has been huge amount of investigations of time operators of the one-dimensional harmonic oscillator, there has been however few rigorous mathematical investigations. Not even the domains of time operators have been established. In addition, although the existence of several time operators of harmonic oscillators is known, we have never found any classification of them. Not only are there no rigorous results, but unfortunately there are also misleading results. No matter how much the algebraic relations of $p^{-1}q$ and $qp^{-1}$ are derived, we have the impression so that they are far from mathematics. It then appears to be an ad hoc study and we aspire to a more systematic research. Then the new insights of this paper are to investigate (1)-(6) below: (1) definition of $T_\varepsilon$, (2) representation of $T_\varepsilon$, (3) extension of $T_\varepsilon$, (4) analytic continuation of representation of $T_\varepsilon$, (5) limit of $T_\varepsilon$ as $\varepsilon \to 0$, (6) comparison of $T_\varepsilon$ and $T_G$.

The problems discussed and solved in this paper are listed below:
1. **Definition of arctan \( t^\# \).** We give the firm mathematical definitions of both \( \arctan t \) and \( \arctan t^\# \), and discuss the domains of \( \arctan t \) and \( \arctan t^\# \). We also show that both are unbounded operators in Theorems 3.3 and 3.4.

2. **Representation of arctan \( t^\# \).** Let \( S^\#_\varepsilon = -\frac{1}{\sqrt{\varepsilon}} \arctan \sqrt{\varepsilon} t^\# \) be given by (1.13). We introduce three subspaces:

\[
\mathcal{M}_{\varepsilon,0} = \text{LH}\left\{ x^{2n}e^{-\alpha x^2/(2\sqrt{\varepsilon})} \bigg| \alpha \in (0, 1), n \in \mathbb{N} \right\},
\]

\[
\mathcal{M}_{\varepsilon,1} = \text{LH}\left\{ x^{2n+1}e^{-\alpha x^2/(2\sqrt{\varepsilon})} \bigg| \alpha \in (0, 1), n \in \mathbb{N} \right\},
\]

\[
\mathcal{R}_\varepsilon = \text{LH}\left\{ x^n e^{-x^2/(2\sqrt{\varepsilon})} \bigg| n \in \mathbb{N} \right\}.
\]

Here and in what follows LH\{...\} means the linear hull of \{...\} and \( \mathbb{N} = \{0, 1, 2, \ldots, \} \).

- \( \alpha \in (0, 1) \). We describe the action of \( S^\#_\varepsilon \) on \( \mathcal{M}_{\varepsilon,\#} \) in Lemmas 3.6 and 3.7:

\[
S^\varepsilon x^{2n}e^{-\alpha x^2/(2\sqrt{\varepsilon})} = -\frac{i}{2\sqrt{\varepsilon}} \left( \left( x^2 - 2\sqrt{\varepsilon} \frac{d}{d\alpha} \right) \log \frac{1 + \alpha}{1 - \alpha} \right) e^{-\alpha x^2/(2\sqrt{\varepsilon})},
\]

\[
S^\varepsilon x^{2n+1}e^{-\alpha x^2/(2\sqrt{\varepsilon})} = -\frac{i}{2\sqrt{\varepsilon}} \left( \left( x^2 - 2\sqrt{\varepsilon} \frac{d}{d\alpha} \right) \log \frac{1 + \alpha}{1 - \alpha} \right) xe^{-\alpha x^2/(2\sqrt{\varepsilon})}.
\]

- \( \alpha = 1 \). We show in Theorem 3.10 that \( D(S^\#_\varepsilon) \cap \mathcal{R}_\varepsilon = \{0\} \), and it is also shown that any eigenvector of \( h_\varepsilon \) does not belong to \( D(S^\#_\varepsilon) \). As a result we see that \( (f, S^\#_\varepsilon g) \) diverges to infinity for any \( f, g \in \mathcal{R}_\varepsilon \).

3. **Ultra-weak time operator.** We can see that \( [h_\varepsilon, S_\varepsilon] = -i \mathbb{I} \) on \( \mathcal{M}_{\varepsilon,0} \) and \( [h_\varepsilon, S_\varepsilon^\varepsilon] = -i \mathbb{I} \) on \( \mathcal{M}_{\varepsilon,1} \), but \( \mathcal{M}_{\varepsilon,0} \cap \mathcal{M}_{\varepsilon,1} = \{0\} \). Let \( \mathcal{M}_\varepsilon = \mathcal{M}_{\varepsilon,0} \oplus \mathcal{M}_{\varepsilon,1} \). We define the ultra-weak time operator \( t_\varepsilon \) associated with \( T_\varepsilon \). Note that \( t_\varepsilon \) is a sesqui-linear form: \( t_\varepsilon : \mathcal{M}_\varepsilon \times \mathcal{M}_\varepsilon \to \mathbb{C} \). We can see that \( [h_\varepsilon, t_\varepsilon] = -i \mathbb{I} \) holds true on \( \mathcal{M}_\varepsilon \) in the sense of sesqui-linear forms in Theorem 4.3.

4. **Matrix representations and analytic continuation** Let \( \varepsilon = 1 \) and set \( t_\varepsilon = 1 = t \). We give the explicit form of

\[
an_{nm} = \mathcal{I}[x^n e^{+ixz^2/2}, x^m e^{+ixz^2/2}].
\]

for \( z \in \mathbb{C} \) such that \( 0 < -iz < 1 \) in Theorem 4.4. This gives a matrix representation of \( t \). Since \( z = i \) is a singular point of arctan \( z \) and \( q^{-1}pe^{-x^2/2} = i e^{-x^2/2} \), in Section 3.3 we show that \( \mathcal{I}[x^n e^{+ixz^2/2}, x^m e^{+ixz^2/2}] \) diverges to infinity for \( z = i \). We discuss however the analytic continuation of the map \( z \mapsto \mathcal{I}[x^n e^{+ixz^2/2}, x^m e^{+ixz^2/2}] \) to \( \mathbb{H} \setminus \{Re \alpha \in [1, \infty)\} \) in Theorem 4.4. Here \( \mathbb{H} \) denotes the open upper half plane in \( \mathbb{C} \).

5. **Continuum limit.** We are interested in the limit of \( T_\varepsilon \) as \( \varepsilon \to 0 \) and deriving \( T_{AB} \) as the limit. By (1.12) we boldly hope that \( T_\varepsilon \to T_{AB} \) as \( \varepsilon \to 0 \). Thus this is one reason to study the time operator of the form (1.12) in this paper. We shall consider this problem in the sense of sesqui-linear forms. We are interested in the limit of \( t_\varepsilon \) as \( \varepsilon \to 0 \), and we expect that the limit is the sesqui-linear form \( t_{AB} \) associated with \( T_{AB} \), i.e.,

\[
limit_{\varepsilon \to 0} t_\varepsilon = t_{AB} \quad (2.8)
\]
(2.8) is called the continuum limit in this paper. We prove this in Theorem 5.1. (2.8) is formally rewritten as
\[
\lim_{\epsilon \to 0} -\frac{1}{2} \frac{1}{\sqrt{\epsilon}} (\arctan \sqrt{\epsilon t} + \arctan \sqrt{\epsilon t^*}) = -\frac{1}{2} (t + t^*).
\]

This convergence gives a connection between time operators of self-adjoint operators with purely absolutely continuous spectrum and those of self-adjoint operators with purely discrete spectrum.

6. POVM. We compare \( t \) to the ultra-weak time operator \( t_G \) associated with \( T_G \). The result is that \( t \neq t_G \), which is shown in Theorem 6.8.

3 Definitions and properties of \( S_\varepsilon \) and \( S_\varepsilon^* \)

3.1 Definitions of \( S_\varepsilon \) and \( S_\varepsilon^* \)

To consider time operators of \( h_\varepsilon \), it is useful to decompose \( h_\varepsilon \) with the even part and the odd part. Let
\[
L^2_0 = \{ f \in L^2(\mathbb{R}) \mid \forall x \in \mathbb{R} \ f(-x) = f(x) \},
\]
\[
L^2_1 = \{ f \in L^2(\mathbb{R}) \mid \forall x \in \mathbb{R} \ f(-x) = -f(x) \},
\]
i.e., \( L^2_0 \) denotes the set of all even \( L^2 \)-functions, and \( L^2_1 \) all odd \( L^2 \)-functions. We fix \( 0 < \varepsilon \leq 1 \). We set \( h_{\text{even}} = h_\varepsilon |_{L^2_0} \) and \( h_{\text{odd}} = h_\varepsilon |_{L^2_1} \). Since \( h_\varepsilon \) is reduced by \( L^2_0 \) and \( L^2_1 \), we have
\[
h_\varepsilon = h_{\text{even}} \oplus h_{\text{odd}}. \tag{3.1}
\]

The spectrum of \( h_\varepsilon \) is \( \text{Spec}(h_\varepsilon) = \{ \sqrt{\varepsilon(n + \frac{1}{2})} \}_{n=0}^\infty \) and the eigenvector \( e_n \) associated with the eigenvalue \( \sqrt{\varepsilon(n + \frac{1}{2})} \) with odd integer \( n \) is an odd function, and that with even integer \( n \) is an even function. They are given by \( e_n(x) = c_n H_n(x/\sqrt{\varepsilon}) e^{-x^2/(2\sqrt{\varepsilon})} \), where \( H_n(x) \) is the \( n \)-th Hermite polynomial given by
\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{k! (n-2k)!} \tag{3.2}
\]
and \( c_n \) is the normalized constant such that \( \| e_n \| = 1 \). \( H_n \) is odd for odd integer \( n \) and even for even integer \( n \). Set \( \{ c_n H_n(x/\sqrt{\varepsilon}) e^{-x^2/(2\sqrt{\varepsilon})} \mid n = 2m \} \) is a complete orthonormal system of \( L^2_0 \), and set \( \{ c_n H_n(x/\sqrt{\varepsilon}) e^{-x^2/(2\sqrt{\varepsilon})} \mid n = 2m + 1 \} \) is of \( L^2_1 \). We have \( \text{Spec}(h_{\text{even}}) = \{ \sqrt{\epsilon(2m + \frac{1}{2})} \}_{m=0}^\infty \) and \( \text{Spec}(h_{\text{odd}}) = \{ \sqrt{\epsilon(2m + \frac{1}{2})} \}_{m=0}^\infty \).

Since the operator \( t = q^{-1}p \) maps \( D(q^{-1}p) \cap L^2_0 \) to \( L^2_0 \), we define the operator \( S_\varepsilon \) on \( L^2_0 \) by
\[
D(S_\varepsilon) = \left\{ f \in L^2_0 \cap \bigcap_{n=0}^\infty D(t^{2n+1}) \mid \lim_{N \to \infty} \sum_{n=0}^N \frac{(-1)^n}{2n+1} (\sqrt{\epsilon} t)^{2n+1} f \text{ exists} \right\}, \tag{3.3}
\]
\[
S_\varepsilon = -\frac{1}{\sqrt{\varepsilon}} \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon} t)^{2n+1}. \tag{3.4}
\]
Formally we write $S_\varepsilon$ as

$$S_\varepsilon = -\frac{1}{\sqrt{\varepsilon}} \arctan \sqrt{\varepsilon} t. \tag{3.5}$$

We give a remark on the expression (3.5). The use of formal notation (3.5) is often useful since it enhances our imagination, but it can be seriously misleading. We need to be careful to use (3.5).

To study the properties of $t$ and $S_\varepsilon$, we introduce two spaces below:

$$\mathcal{L}_0 = \text{LH}\{e^{-\alpha x^2/(2\sqrt{\varepsilon})} \mid \alpha \in (0, 1)\},$$

$$\mathcal{L}_1 = \text{LH}\{xe^{-\alpha x^2/(2\sqrt{\varepsilon})} \mid \alpha \in (0, 1)\}.$$

**Lemma 3.1** The subspace $\mathcal{L}_0 + \mathcal{L}_1$ is dense in $L^2(\mathbb{R})$, and $\mathcal{L}_0 \perp \mathcal{L}_1$.

**Proof:** It is immediate to see that $\mathcal{L}_0 \perp \mathcal{L}_1$. By the definition of $\mathcal{L}_0$ we obtain that

$$\frac{e^{-\alpha x^2/(2\sqrt{\varepsilon})} - e^{-\alpha' x^2/(2\sqrt{\varepsilon})}}{\alpha - \alpha'} \in \mathcal{L}_0$$

for $\alpha, \alpha' \in (0, 1)$. In particular

$$x^2e^{-\alpha x^2/(2\sqrt{\varepsilon})} = -2\sqrt{\varepsilon} \lim_{\alpha' \to \alpha} \frac{e^{-\alpha x^2/(2\sqrt{\varepsilon})} - e^{-\alpha' x^2/(2\sqrt{\varepsilon})}}{\alpha - \alpha'} \in \mathcal{L}_0,$$

where $\overline{\mathcal{L}_0}$ is the closure of $\mathcal{L}_0$ in $L^2(\mathbb{R})$. Similarly we can see that $x^2ne^{-\alpha x^2/(2\sqrt{\varepsilon})} \in \overline{\mathcal{L}_0}$ for $n \in \mathbb{N}$. In the same way we can also see that $x^{2n+1}e^{-\alpha x^2/(2\sqrt{\varepsilon})} \in \overline{\mathcal{L}_1}$ for $n \in \mathbb{N}$. Since $\text{LH}\{x^ne^{-\alpha x^2/(2\sqrt{\varepsilon})} \mid n \in \mathbb{N}\}$ is dense, the lemma follows. \qed

**Lemma 3.2** It follows that $\mathcal{L}_0 \subset D(S_\varepsilon)$.

**Proof:** Let $\alpha \in (0, 1)$ and $f(x) = e^{-\alpha x^2/(2\sqrt{\varepsilon})} \in \mathcal{L}_0$. Then $\sqrt{\varepsilon}tf = \alpha if$, i.e., $f$ is an eigenvector of $\sqrt{\varepsilon}t$ with eigenvalue $\alpha i$. Thus $(\sqrt{\varepsilon}t)^{2n+1}f = (-1)^n i \alpha^{2n+1}f$. Hence

$$\left\| \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon}t)^{2n+1}f \right\| \leq \|f\| \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{2n+1} < \infty.$$

Then the lemma is proven. \qed

Let $\text{Spec}_p(A)$ be the set of all eigenvalues of $A$.

**Theorem 3.3** It follows that $i(0, \infty) \subset \text{Spec}_p(t)$ and $-i(0, \infty) \subset \text{Spec}_p(S_\varepsilon)$. In particular $t$ and $S_\varepsilon$ are unbounded.

**Proof:** For $f(x) = e^{-\alpha x^2/(2\sqrt{\varepsilon})}$ with $\alpha \in (0, \infty)$ it follows that $\sqrt{\varepsilon}tf = \alpha if$. Then the first statement follows. Let $\alpha \in (0, 1)$. We have

$$S_\varepsilon f = -\frac{1}{\sqrt{\varepsilon}} \sum_{n=0}^{\infty} \frac{(-1)^n(i\alpha)^{2n+1}}{2n+1}f = -\frac{i}{\sqrt{\varepsilon}2} \log \frac{1 + \alpha}{1 - \alpha} f.$$
Since \( \{ \log\left((1 + \alpha)/(1 - \alpha)\right) \mid \alpha \in (0, 1) \} = (0, \infty) \), the second statement is proven. \( \square \)

Let us study the formal adjoint of \( S_\varepsilon \), but the discussion is parallel to that of \( S_\varepsilon \). Set

\[
D(S_\varepsilon^*) = \left\{ f \in L^2 \cap \bigcap_{n=0}^{\infty} D((t^*)^{2n+1}) \mid \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon t^*})^{2n+1} f \text{ exists} \right\},
\]

\[
S_\varepsilon^* = -\frac{1}{\sqrt{\varepsilon}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon t^*})^{2n+1}.
\]

Formally we also write

\[
S_\varepsilon^* = -\frac{1}{\sqrt{\varepsilon}} \arctan \sqrt{\varepsilon t^*}.
\]

In a similar way to Lemma 3.2 and Theorem 3.3, we can show the theorem below.

**Theorem 3.4** It follows that \( S_1 \subset D(S_\varepsilon^*) \), \( i(0, \infty) \subset \text{Spec}_p(t^*) \) and \( -i(0, \infty) \subset \text{Spec}_p(S_\varepsilon^*) \). In particular \( t^* \) and \( S_\varepsilon^* \) are unbounded.

### 3.2 Representations of \( S_\varepsilon \) and \( S_\varepsilon^* \) for \( \alpha \in (0, 1) \)

To define a time operator of \( h_\varepsilon \), we prepare several technical lemmas on the properties of \( S_\varepsilon \# \).

**Lemma 3.5** Suppose that \( \alpha \in (0, 1) \). Then, for all \( m \in \mathbb{N} \), \( x^{2m}e^{-\alpha x^2/(2\sqrt{\varepsilon})} \in D(S_\varepsilon) \) and

\[
S_\varepsilon x^{2m}e^{-\alpha x^2/(2\sqrt{\varepsilon})} = -\frac{i}{2\sqrt{\varepsilon}} \sum_{k=0}^{m} \binom{m}{k} (-2\sqrt{\varepsilon})^k \left( \log \frac{1 + \alpha}{1 - \alpha} \right)^{(k)} x^{2m-2k}e^{-\alpha x^2/(2\sqrt{\varepsilon})}.
\]

Here \( F^{(k)} = \frac{d^k F}{d\alpha^k} \).

**Proof:** Let \( f(x) = e^{-\alpha x^2/(2\sqrt{\varepsilon})} \). We can compute \( (\sqrt{\varepsilon} t)^n x^{2m}f(x) \) directly as

\[
(\sqrt{\varepsilon} t)^n x^{2m}f(x) = \sum_{k=0}^{m} \binom{m}{k} k! 2^k x^{2m-2k}(\alpha i)^{n-k}(-i\sqrt{\varepsilon})^k f(x).
\]

From this, we see that

\[
\sum_{n=0}^{M} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon} t)^{2n+1} x^{2m}f(x)
\]

\[
= \sum_{n=0}^{M} \sum_{k=0}^{m} \frac{(2n+1)}{k} \binom{m}{k} k! 2^k x^{2m-2k}(\alpha i)^{2n+1-k}(-i\sqrt{\varepsilon})^k f(x)
\]

\[
= i \sum_{n=0}^{M} \frac{e^{2n+1}}{2n+1} x^{2m} - 2m\sqrt{\varepsilon} \alpha x^{2m-2} + \sum_{k=2}^{m} \binom{m}{k} (-2\sqrt{\varepsilon})^k (\alpha^n)^{(k-1)} x^{2m-2k} f(x).
\]
Hence we can see that as $M \to \infty$,

$$\sum_{n=0}^{M} \frac{(-1)^n}{2n+1} f^{2n+1}_n f(x) \to i \left( \frac{1}{2} \log \frac{1 + \alpha}{1 - \alpha} x^{2m} - \frac{2m\sqrt{\varepsilon}}{1 - \alpha^2} x^{2m-2} + \sum_{k=2}^{m} \binom{m}{k} (-2\sqrt{\varepsilon})^k \left( \frac{1}{1 - \alpha^2} \right)^{(k-1)} x^{2m-2k} \right) f(x)$$

$$= \frac{i}{2} \sum_{k=0}^{m} \binom{m}{k} (-2\sqrt{\varepsilon})^k \left( \log \frac{1 + \alpha}{1 - \alpha} \right)^{(k)} x^{2m-2k} f(x).$$

Then the lemma follows.

Let us set

$$\xi_{\alpha i, \varepsilon} = e^{-\alpha x^2/(2\sqrt{\varepsilon})}, \quad t_{\alpha i, \varepsilon} = 2 \left( \frac{x^2}{2} - \sqrt{\varepsilon} \frac{d}{d\alpha} \right).$$

**Lemma 3.6** Let $\alpha \in (0, 1)$ and $f_{2m} = x^{2m} \xi_{\alpha i, \varepsilon}$ for $m \in \mathbb{N}$. Then

$$S_\varepsilon f_{2m} = -\frac{i}{2\sqrt{\varepsilon}} \left( t_{\alpha i, \varepsilon}^m \log \frac{1 + \alpha}{1 - \alpha} \right) \xi_{\alpha i, \varepsilon}.$$ \hfill (3.6)

In particular, the matrix representation of $S_\varepsilon$ is given by

$$(f_{2n}, S_\varepsilon f_{2m}) = -\frac{i}{2\sqrt{\varepsilon}} \left( \xi_{\alpha i, \varepsilon}, x^{2n} \left( t_{\alpha i, \varepsilon}^m \log \frac{1 + \alpha}{1 - \alpha} \right) \xi_{\alpha i, \varepsilon} \right).$$ \hfill (3.7)

**Proof:** By Lemma 3.5, we have

$$S_\varepsilon f_{2m} = -\frac{i}{2\sqrt{\varepsilon}} \sum_{k=0}^{m} \binom{m}{k} (-2\sqrt{\varepsilon})^k \left( \log \frac{1 + \alpha}{1 - \alpha} \right)^{(k)} x^{2m-2k} \xi_{\alpha i, \varepsilon}$$

$$= -\frac{i}{2\sqrt{\varepsilon}} \left( t_{\alpha i, \varepsilon}^m \log \frac{1 + \alpha}{1 - \alpha} \right) \xi_{\alpha i, \varepsilon}.$$

Thus the lemma is proven. \hfill \qed

In the same way as in the proof of Lemma 3.5, we can show that, for all $m \in \mathbb{N}$,

$$x^{2m+1} e^{-ax^2/(2\sqrt{\varepsilon})} \in D(S_\varepsilon^*)$$

and we have the representation of $S^*$ as in Lemma 3.6.

**Lemma 3.7** Suppose that $\alpha \in (0, 1)$. Set $f_{2m+1} = x^{2m+1} \xi_{\alpha i, \varepsilon}$ for $m \in \mathbb{N}$. Then

$$S^*_\varepsilon f_{2m+1} = -\frac{i}{2\sqrt{\varepsilon}} \left( t_{\alpha i, \varepsilon}^m \log \frac{1 + \alpha}{1 - \alpha} \right) x \xi_{\alpha i, \varepsilon}.$$ \hfill (3.8)

In particular, the matrix representation of $S_\varepsilon$ is given by

$$(f_{2n+1}, S^*_\varepsilon f_{2m+1}) = -\frac{i}{2\sqrt{\varepsilon}} \left( \xi_{\alpha i, \varepsilon}, x^{2n+2} \left( t_{\alpha i, \varepsilon}^m \log \frac{1 + \alpha}{1 - \alpha} \right) \xi_{\alpha i, \varepsilon} \right).$$ \hfill (3.9)
Proof: The proof is similar to that of Lemma 3.6. 

We have a brief comment on the derivation of representations of $S_\varepsilon$ and $S^*_\varepsilon$. These representations can be formally derived from a generating-function-type argument. Note that
\[ \arctan(\alpha i) = \frac{i}{2} \log \frac{1 + \alpha}{1 - \alpha} = i \int \frac{1}{1 - \alpha^2} \mathrm{d}\alpha. \]
Since $-\sqrt{\varepsilon} S_\varepsilon \xi_{\alpha i} = \arctan(\alpha i) \xi_{\alpha i, \varepsilon}$, taking the derivative on $\alpha$ on both sides we obtain that
\[ -\sqrt{\varepsilon} S_\varepsilon \left( -\frac{x^2}{2 \sqrt{\varepsilon}} \right) \xi_{\alpha i, \varepsilon} = \left( \frac{d}{d\alpha} \arctan(\alpha i) \right) \xi_{\alpha i, \varepsilon} + \arctan(\alpha i) \left( -\frac{x^2}{2 \sqrt{\varepsilon}} \right) \xi_{\alpha i, \varepsilon}. \]
Then we have
\[ S_\varepsilon x^2 \xi_{\alpha i, \varepsilon} = -\frac{2}{\sqrt{\varepsilon}} \left\{ \left( \frac{x^2}{2} - \sqrt{\varepsilon} \frac{d}{d\alpha} \right) \arctan(\alpha i) \right\} \xi_{\alpha i, \varepsilon}. \]
Repeating this we can see Lemma 3.6. The following statement is immediate.

**Lemma 3.8** Suppose that $\alpha \in (0, 1)$. Let $\rho$ be a polynomial. Then
\[
S_\varepsilon \rho(x^2) \xi_{\alpha i, \varepsilon} = -\frac{i}{2 \sqrt{\varepsilon}} \left( \rho(t_{\alpha i, \varepsilon}) \log \frac{1 + \alpha}{1 - \alpha} \right) \xi_{\alpha i, \varepsilon}, \tag{3.10}
\]
\[
S^*_\varepsilon \rho(x^2) x \xi_{\alpha i, \varepsilon} = -\frac{i}{2 \sqrt{\varepsilon}} \left( \rho(t_{\alpha i, \varepsilon}) \log \frac{1 + \alpha}{1 - \alpha} \right) x \xi_{\alpha i, \varepsilon}. \tag{3.11}
\]

Although Lemma 3.8 may be extended to general functions $\rho$, we do not discuss it here, but give an example.

**Example 3.9** Let $\rho(x^2) = e^{+\beta x^2/(2\sqrt{\varepsilon})}$ with $\alpha > \beta$. Then
\[
\rho(t_{\alpha i}) = e^{\beta t_{\alpha i}/(2\sqrt{\varepsilon})} = e^{\beta x^2/(2\sqrt{\varepsilon})} e^{-\beta x^2/(2\varepsilon)}. \]
Thus
\[
e^{\beta t_{\alpha i}/(2\sqrt{\varepsilon})} \log \frac{1 + \alpha}{1 - \alpha} = e^{\beta x^2/(2\sqrt{\varepsilon})} \log \frac{1 + (\alpha - \beta)}{1 - (\alpha - \beta)},
\]
and hence
\[
S_\varepsilon e^{-(\alpha - \beta) x^2/(2\sqrt{\varepsilon})} = -\frac{i}{2 \sqrt{\varepsilon}} \left( e^{\beta t_{\alpha i}/(2\sqrt{\varepsilon})} \log \frac{1 + \alpha}{1 - \alpha} \right) e^{-\alpha x^2/(2\sqrt{\varepsilon})} \\
= -\frac{i}{2 \sqrt{\varepsilon}} \log \frac{1 + (\alpha - \beta)}{1 - (\alpha - \beta)} e^{-(\alpha - \beta) x^2/(2\sqrt{\varepsilon})}.
\]

### 3.3 Representation of $S_\varepsilon$ and $S^*_\varepsilon$ for $\alpha = 1$

In this section, we discuss the case of $\alpha = 1$. We see that, for $\alpha \in (0, 1)$,
\[
S_\varepsilon e^{-\alpha x^2/(2\sqrt{\varepsilon})} = -\frac{i}{2 \sqrt{\varepsilon}} \log \frac{1 + \alpha}{1 - \alpha} e^{-\alpha x^2/(2\sqrt{\varepsilon})}.
\]
We also see that
\[
S_\varepsilon x^2 e^{-\alpha x^2/(2\sqrt{\varepsilon})} = - \left( \frac{i}{2 \sqrt{\varepsilon}} \log \frac{1 + \alpha}{1 - \alpha} x^2 - i \frac{2}{1 - \alpha^2} \right) e^{-\alpha x^2/(2\sqrt{\varepsilon})}.
\]
Hence the coefficient of the leading term in the curly bracket diverges to infinity as $\alpha \to 1$. Then it is expected that $e^{-x^2/(2\sqrt{\varepsilon})}, x^2 e^{-x^2/(2\sqrt{\varepsilon})} \notin \mathcal{D}(S_\varepsilon)$. Let $\mathcal{K}_\varepsilon$ be (2.7).
Theorem 3.10  It follows that \( \mathfrak{r}_\varepsilon \cap D(S_\varepsilon^\#) = \{0\} \).

Proof: We consider only \( S_\varepsilon \). Let \( \rho_1(x) \) be a polynomial with \( \rho_1(x) \neq \rho_1(-x) \) for any \( x \in \mathbb{R} \). We see that \( x^{2m+1}e^{-x^2/(2\varepsilon^2)} \notin D((\sqrt{\varepsilon}t)^{m+1}) \) and hence \( \rho_1(x)e^{-x^2/(2\sqrt{\varepsilon})} \notin D(S_\varepsilon) \).

Let \( \rho_0(x) \) be a monic polynomial of degree \( 2m \) such that \( \rho_0(x) = \rho_0(-x) \) for any \( x \in \mathbb{R} \). Set \( \xi_{i,\varepsilon} = e^{-x^2/(2\sqrt{\varepsilon})} \). We have

\[
\sum_{n=0}^{M} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon}t)^{2n+1} \rho_0(x) \xi_{i,\varepsilon} = i \sum_{n=0}^{M} \left( \frac{x^{2m}}{2n+1} - \frac{\sqrt{\varepsilon}\rho_0(x)}{x} \right) + \sum_{k=2}^{M} \frac{(-1)^k}{k!} 2n(2n-1) \cdots (2n-k+2) \left( (\sqrt{\varepsilon}t)^{k} \rho_0(x) \right) \xi_{i,\varepsilon}.
\]

(3.12)

Let \( e_{2n} \) be the eigenvector of \( h_\varepsilon \) with eigenvalue \( 2n \). We have

\[
\left\{ \left( (\sqrt{\varepsilon}t)^{k} \rho_0(x) \right) \xi_{i,\varepsilon} \bigg| 1 \leq k \leq m \right\} \subset LH\{e_0,e_2,\ldots,e_{2m-2}\}
\]

and henceforth \( \left\{ \left( (\sqrt{\varepsilon}t)^{k} \rho_0(x) \right) \xi_{i,\varepsilon} \bigg| 1 \leq k \leq m \right\} \perp \{e_{2m}\} \). From (3.12) it follows that

\[
\left| \left( e_{2m}, \sum_{n=0}^{M} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon}t)^{2n+1} \rho_0(x) \xi_{i,\varepsilon} \right) \right| = \left| \sum_{n=0}^{M} \left( e_{2m}, \frac{1}{2n+1} x^{2m} \xi_{i,\varepsilon} \right) \right| = |(e_{2m},x^{2m} \xi_{i,\varepsilon})| \sum_{n=0}^{M} \frac{1}{2n+1} \to \infty \quad (M \to \infty).
\]

Hence \( \sum_{n=0}^{M} \frac{(-1)^n}{2n+1} (\sqrt{\varepsilon}t)^{2n+1} \rho_0(x) \xi_{i,\varepsilon} \) does not converge as \( M \to \infty \) and \( \rho_0(x) \xi_{i,\varepsilon} \notin D(S_\varepsilon) \).

\[\square\]

Corollary 3.11  Let \( e_n \) be the \( n \)th eigenvector of \( h_\varepsilon \). Then \( e_n \notin D(S_\varepsilon^\#) \).

Proof: Since \( e_n \in \mathfrak{r}_\varepsilon \), from Theorem 3.10, it follows that \( e_n \notin D(S_\varepsilon^\#) \).

\[\square\]

4  Matrix representations

4.1  Ultra-weak time operators

We firstly confirm that \( S_\varepsilon^\# \) is a conjugate operator of \( h_\varepsilon \). Let \( \mathfrak{m}_{\varepsilon,0} \) and \( \mathfrak{m}_{\varepsilon,1} \) be (2.5) and (2.6), respectively.

Lemma 4.1  It follows that \( [h_\varepsilon, S_\varepsilon^\#] = -i \mathbb{1} \) on \( \mathfrak{m}_{\varepsilon,\#} \).

Proof: We shall show \( [h_\varepsilon, S_\varepsilon] = -i \mathbb{1} \) on \( \mathfrak{m}_{\varepsilon,0} \). It is sufficient to show that, for \( n \in \mathbb{N} \), \( \alpha \in (0,1) \) and \( \xi_{\alpha i,\varepsilon} = e^{-\alpha x^2/(2\sqrt{\varepsilon})} \),

\[
[h_\varepsilon, S_\varepsilon] x^{2n} \xi_{\alpha i,\varepsilon} = -i x^{2n} \xi_{\alpha i,\varepsilon}.
\]
From Lemma 3.5, we see that
\[
\sqrt{\varepsilon} h_\varepsilon S_\varepsilon x^{2n} \xi_{\alpha,\varepsilon} = \frac{i}{4} \left( \left( (1 - \alpha^2) x^2 + \alpha \right) t_{\alpha,\varepsilon}^n - 2n(1 - 2\alpha x^2) t_{\alpha,\varepsilon}^{n-1} - n(n - 1) x^2 t_{\alpha,\varepsilon}^{n-2} \right) \log \left( \frac{1 + \alpha}{1 - \alpha} \right) \xi_{\alpha,\varepsilon},
\]
\[
\sqrt{\varepsilon} S_\varepsilon h_\varepsilon x^{2n} \xi_{\alpha,\varepsilon} = \frac{i}{4} \left( \left( 1 - \alpha^2 \right) t_{\alpha,\varepsilon}^{n+1} + (4n + 1) \alpha t_{\alpha,\varepsilon}^n - 2n(2n - 1) t_{\alpha,\varepsilon}^{n-1} \right) \log \left( \frac{1 + \alpha}{1 - \alpha} \right) \xi_{\alpha,\varepsilon}.
\]

Therefore, by Lemma 4.7, (4.5) and (4.6) below we have \([h_\varepsilon, S_\varepsilon] x^{2n} \xi_{\alpha,\varepsilon} = -i x^{2n} \xi_{\alpha,\varepsilon} \). In a similar way to this it follows that \([h_\varepsilon, S_\varepsilon^*] = -i S_\varepsilon \xi_{\alpha,\varepsilon} \). □

Intuitively we expect that
\[
T_\varepsilon = \frac{1}{2} (S_\varepsilon + S_\varepsilon^*)
\]
is a time operator of \(h_\varepsilon\). We notice that \(\mathcal{M}_{\varepsilon,0} \subset L_0^2\) and \(\mathcal{M}_{\varepsilon,1} \subset L_1^2\). Furthermore
\[
t^n : \mathcal{M}_{\varepsilon,0} \to L_0^2, \quad (t^*)^n : \mathcal{M}_{\varepsilon,1} \to L_1^2,
\]
but \(D(t^n) \cap \mathcal{M}_{\varepsilon,1} = \{0\}\) and \(D((t^*)^n) \cap \mathcal{M}_{\varepsilon,0} = \{0\}\). From this it is not obvious to find a dense domain of \(T_\varepsilon\) and to construct a dense domain \(D\) such that \([h_\varepsilon, T_\varepsilon] = -i \varepsilon \). So, instead of considering time operators, we define an ultra-weak time operator of \(h_\varepsilon\). By the decomposition \(h_\varepsilon = h_{\text{even}} \oplus h_{\text{odd}}\) it is enough to construct ultra-weak time operators \(t_0\) and \(t_1\) of \(h_{\text{even}}\) and \(h_{\text{odd}}\), respectively. Hence \(t_0 \oplus t_1\) becomes an ultra-weak time operator of \(h_\varepsilon\).

We define
\[
t_{\varepsilon,0}[\psi, \phi] = \frac{1}{2} (\psi, S_\varepsilon \phi) + (S_\varepsilon \psi, \phi), \quad \psi, \phi \in D(S_\varepsilon), \quad (4.1)
\]
\[
t_{\varepsilon,1}[\psi, \phi] = \frac{1}{2} (\psi, S_\varepsilon^* \phi) + (S_\varepsilon^* \psi, \phi), \quad \psi, \phi \in D(S_\varepsilon^*). \quad (4.2)
\]

**Lemma 4.2** The sesqui-linear form \(t_{\varepsilon,0}\) (resp. \(t_{\varepsilon,1}\)) is an ultra-weak time operator of \(h_{\text{even}}\) (resp. \(h_{\text{odd}}\)) with the symmetric domain \(D(S_\varepsilon)\) (resp. \(D(S_\varepsilon^*)\)) and the ultra-weak CCR-domain \(\mathcal{M}_{\varepsilon,0}\) (resp. \(\mathcal{M}_{\varepsilon,1}\)).

Proof: From Lemma 4.1 it follows that \(t_{\varepsilon,0}[h_{\text{even}} \psi, \phi] = t_{\varepsilon,0}[\psi, h_{\text{even}} \phi] = -i (\psi, \phi)\) for \(\psi, \phi \in \mathcal{M}_{\varepsilon,0}\) and \(t_{\varepsilon,1}[h_{\text{odd}} \psi, \phi] = t_{\varepsilon,1}[\psi, h_{\text{odd}} \phi] = -i (\psi, \phi)\) for \(\psi, \phi \in \mathcal{M}_{\varepsilon,1}\). The lemma is proven. □

Let \(\mathcal{M}_\varepsilon = \mathcal{M}_{\varepsilon,0} \oplus \mathcal{M}_{\varepsilon,1}\) and \(\mathcal{D}_\varepsilon = D(S_\varepsilon) \oplus D(S_\varepsilon^*)\). The sesqui-linear form \(t_\varepsilon\) is defined by
\[
t_\varepsilon = t_{\varepsilon,0} \oplus t_{\varepsilon,1}
\]
with the domain \(\mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon\). We see that
\[
t_\varepsilon[\psi_0 \oplus \psi_1, \phi_0 \oplus \phi_1] = t_{\varepsilon,0}[\psi_0, \phi_0] + t_{\varepsilon,1}[\psi_1, \phi_1]
\]
for \(\psi_0 \oplus \psi_1, \phi_0 \oplus \phi_1 \in \mathcal{D}_\varepsilon\).
Theorem 4.3 The sesqui-linear form $t_ε$ is an ultra-weak time operator of $h_ε$ under the decomposition $h_ε = h_{even} ⊕ h_{odd}$ with the symmetric domain $D_ε$ and the ultra-weak CCR-domain $M_ε$.

Proof: Let $ϕ = ϕ_0 ⊕ ϕ_1, ψ = ψ_0 ⊕ ψ_1 ∈ M_ε$. By Lemma 4.2, we have

$$t_ε[h_εϕ, ψ] - t_ε[ϕ, h_εψ] = h_ε[f_{even}ϕ_0, ψ_0] - h_ε[ϕ_0, h_{even}ψ_0] + t_ε,1[h_{odd}ϕ_1, ψ_1] - t_ε,1[ϕ_1, h_{odd}ψ_1]$$

$$= -i(ϕ_0, ψ_0) - i(ϕ_1, ψ_1)$$

Then the theorem is proven. \qed

4.2 Matrix representations of $t_ε$ for $0 < α < 1$

We are interested in computing $t_ε[f, g]$ for $f, g ∈ M_ε$. More precisely we want to see the explicit form of the function $K_{ab}$ such that

$$t_ε[x^aξ_{αi, ε}, x^bξ_{βi, ε}] = (ξ_{αi, ε}, K_{ab}ξ_{βi, ε}), \quad a, b ∈ N, \quad α, β ∈ (0, 1).$$

(4.3)

Theorem 4.4 Fix $α, β ∈ (0, 1)$. Let $f_a = x^aξ_{αi, ε}$ and $f_b = x^bξ_{βi, ε}$. Then

$$t_ε[f_a, f_b] = \begin{cases} 
\frac{i}{4\sqrt{ε}}\left(ξ_{αi, ε}, \begin{cases} x^{2m_i}t^n_{αi, ε}log\frac{1 + α}{1 - α} - x^{2n_i}t^n_{βi, ε}log\frac{1 + β}{1 - β} \end{cases}ξ_{βi, ε}\right) & \text{if } a = 2n, \quad b = 2m, \\
\frac{i}{4\sqrt{ε}}\left(ξ_{αi, ε}, \begin{cases} x^{2m_i+2}t^n_{αi, ε}log\frac{1 + α}{1 - α} - x^{2n_i+2}t^n_{βi, ε}log\frac{1 + β}{1 - β} \end{cases}ξ_{βi, ε}\right) & \text{if } a = 2n + 1, \quad b = 2m + 1, \\
0 & \text{otherwise.}
\end{cases}$$

Proof: If $a + b$ is odd, then $t_ε[f_a, f_b] = 0$ by the definition of $t_ε$. If $a + b$ is even, then $t_ε[f_a, f_b] = \frac{1}{2}\{(S_f, f_b) + (f_s, S_f_b)\}$ for $a = 2n, b = 2m$ and $t_ε[f_a, f_b] = \frac{1}{2}\{(S^*_f, f_b) + (f_s, S^*_f)\}$ for $a = 2n + 1, b = 2m + 1$. Then the theorem follows from Lemmas 3.6 and 3.7. \qed

We can also see the corollary below.

Corollary 4.5 Fix $α, β ∈ (0, 1)$. Let $H_a$ be the ath Hermite polynomial given by (3.2). Let $f_a = H_aξ_{αi, ε}$ and $f_b = H_bξ_{βi, ε}$. Then $t_ε[f_a, f_b] = (ξ_{αi, ε}, L_{ab}ξ_{βi, ε})$, where

$$L_{ab} = \begin{cases} 
\frac{i}{4\sqrt{ε}}\left(H_{2m}(x)h_m(t^n_{αi, ε})log\frac{1 + α}{1 - α} - H_{2n}(x)h_m(t^n_{βi, ε})log\frac{1 + β}{1 - β}\right) & \text{if } a = 2n, \quad b = 2m, \\
\frac{i}{4\sqrt{ε}}\left(xH_{2m+1}(x)k_m(t^n_{αi, ε})log\frac{1 + α}{1 - α} - xH_{2n+1}(x)k_m(t^n_{βi, ε})log\frac{1 + β}{1 - β}\right) & \text{if } a = 2n + 1, \quad b = 2m + 1, \\
0 & \text{otherwise.}
\end{cases}$$

Here $h_n$ and $k_n$ are defined by $H_{2n}(x) = h_n(x^2)$ and $H_{2n+1}(x) = k_n(x^2)x$.

Remark 4.6 We give a comment on a matrix representation for $α = 1$. The matrix representation of $t_ε$ with respect to $\{e_n\}$ is formally defined by $t_ε[e_n, e_m]$. However $t_ε[e_n, e_m]$ diverges to infinity by Corollary 3.11. Nevertheless in [11] a matrix representation on $\{e_n\}$ is discussed through an analytic continuation.
4.3 Analytic continuations of matrix representations

In this section, we set $\varepsilon = 1$. We denote $\xi_{\alpha i, \varepsilon = 1} = \xi_{\alpha i}$, $t_{\varepsilon = 1} = t$, $S^#_{\varepsilon = 1} = S^#$ and $h_{\varepsilon = 1} = h$ for notational simplicity. In Section 3.2 we show that $S\rho(x^2)\xi_{\alpha i} = \rho(t_{\alpha i})\arctan(\alpha i)\xi_{\alpha i}$ for $\alpha \in (0, 1)$. We can extend $\alpha i$ to $z \in \mathbb{C}$. Let

$$\xi_z = e^{+izx^2/2} \z \in \mathbb{C}. $$

$\xi_z \in L^2(\mathbb{R})$ if and only if $\text{Im} z > 0$. We also have $t\xi_z = z\xi_z$ for $z \in \mathbb{C}$.

Let $C^\varepsilon : \mathbb{C} \setminus \{-i\} \to \mathbb{C}$ be the Cayley transform defined by

$$C^\varepsilon(z) = \frac{z - i}{z + i}. $$

We sometimes omit the variable $z$ and write $C^\varepsilon(z)$ simply as $C^\varepsilon$. In terms of the Cayley transform we have

$$\log 1 + \alpha = \log(-C(\alpha i)). $$

In particular, $2\arctan z = -i\log(-C(z))$ follows. Here and in what follows we define $\log z = \log|z| + i\arg z$, $-\pi \leq \arg z < \pi$.

Hence $\log z$ is analytic on $\mathbb{C} \setminus \{-\infty, 0\}$. Let

$$\mathbb{H} = \{z \in \mathbb{C} | \text{Im} z > 0\}$$

be the open upper half plane and

$$\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$$

be the open unit disc in $\mathbb{C}$. The Cayley transform maps as $C(\mathbb{H}) = \mathbb{D}$, $C(\mathbb{R}) = \partial\mathbb{D}$, $C(i[1, \infty)) = [0, 1)$, $C(i[0, 1)) = [-1, 0)$, $C(i(-1, 0]) = (-\infty, -1]$ and $C(i(-\infty, -1)) = (1, \infty)$, where $\partial\mathbb{D}$ denotes the unit circle in $\mathbb{C}$. Thus it is known that $z \mapsto \log(-C(z))$ is analytic on $\mathbb{C} \setminus \{\alpha i | \alpha \in (-\infty, -1] \cup [1, \infty)\}$. Let

$$\mathfrak{N}_0 = \text{LH}\{\rho(x^2)\xi_z | \rho \text{ is a polynomial}, z \in \mathbb{H} \setminus \{i\}\},$$

$$\mathfrak{N}_1 = \text{LH}\{\rho(x^2)x\xi_z | \rho \text{ is a polynomial}, z \in \mathbb{H} \setminus \{i\}\},$$

$$\mathfrak{N} = \mathfrak{N}_0 \oplus \mathfrak{N}_1.$$

We also define $t_z$ by

$$t_z = 2\left(\frac{x^2}{2} - i\frac{d}{dz}\right). $$

Now we define $\hat{S}^#$ by

$$\hat{S}\rho(x^2)\xi_z = \frac{i}{2} (\rho(t_z) \log(-C)) \xi_z,$$

$$\hat{S}^*\rho(x^2)x\xi_z = \frac{i}{2} (\rho(t_z) \log(-C)) x\xi_z. $$

We see that, for $z = \alpha i$ and $\alpha \in (0, 1)$, the actions of $\hat{S}^#$ and $S^#$ on $\rho(x^2)\xi_{\alpha i}$ are identical:

$$\hat{S}^#\rho(x^2)\xi_{\alpha i} = S^#\rho(x^2)\xi_{\alpha i}. $$
Lemma 4.7 It follows that \([h, \hat{S}^\#] = -i \mathbb{1} \text{ on } \mathfrak{N}_d\).

Proof: Here we summarise formulae used in computations.

(1) \(t_z \log(-\mathcal{E}) = x^2 \log(-\mathcal{E}) + \frac{4}{1 + z^2}\),

(2) \([h, t_z] = -1 - 2ixp, \quad [[h, t_z], t_z] = -4x^2\),

(3) \([h, t^n_z] = nt^n_z(-1 - 2ixp) + \frac{1}{2} n(n - 1)t^{n-2}_z(-4x^2),\)

(4) \(hx z^n = \frac{1}{2} ((1 + z^2)x z^n + i(4n + 1)x z^n - 2n(2n - 1)x z^{n-2})\)

We define the complex differential operator \(Q_n\) by

\[
Q_n = \begin{cases} 
1 + z^2 & n = 0, \\
(1 + z^2)t_z - i4z & n = 1, \\
(1 + z^2)t^n_z - i4nzt^{n-1}_z - 4n(n - 1)t^{n-2}_z & n \geq 2.
\end{cases}
\] (4.4)

Using formulae (1)-(4) and \(\hat{S} z^n = \frac{-i}{2}(1 + z^2)\log(-\mathcal{E})\), we can directly see that

\[
h \hat{S} z^n = \frac{i}{4} \left( \{ (1 + z^2)x^2 t^n_z - izt^n_z - 2n(1 + 2ixz^2)t^{n-1}_z - 4n(n - 1)x^2 t^{n-2}_z \} \log(-\mathcal{E}) \right) z^n.
\] (4.5)

\[
\hat{S} z^n = \frac{i}{4} \left( \{ (1 + z^2)t^{n+1}_z - i(4n + 1)zt^n_z - 2n(2n - 1)t^{n-1}_z \} \log(-\mathcal{E}) \right) z^n.
\] (4.6)

Note that \(t^n_z \log(-\mathcal{E}) = x^2 t^{n-1}_z \log(-\mathcal{E}) + t^{n-1}_z \frac{4}{1 + z^2}, m = n + 1, n, n - 1\). Inserting these identities into the right hand side of \(\hat{S} z^n\), we can see that

\[
[h, \hat{S}] z^n = -i \left( Q_n \frac{1}{1 + z^2} \right) z^n \quad n \geq 0.
\] (4.7)

The similar identity is obtained for \(\hat{S}^*\):

\[
[h, \hat{S}^*] z^n = -i \left( Q_n \frac{1}{1 + z^2} \right) x z^n \quad n \geq 0.
\] (4.8)

Next we shall show that

\[
Q_n = t^n_z (1 + z^2)
\] (4.9)

as an operator, i.e., \(Q_n f = t^n_z (1 + z^2) f\). For \(n = 0, 1\) one can show (4.9) directly. Note that \([t_z, z] = -2i\) and \([t^2_z, z^2] = -4iz\). Thus we have

\[
(1 + z^2)t^n_z = t^n_z (1 + z^2) + 4iznt^{n-1}_z + 4n(n - 1)t^{n-2}_z.
\]

Then (4.9) follows. Together with them we have

\[
[h, \hat{S}] z^n = -i \left( t^n_z (1 + z^2) \frac{1}{1 + z^2} \right) z^n = -i (t^n_z) z^n = -ix z^n.
\]

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and similarly \([h, \hat{S}^*]x^{2n+1}\xi_z = -ix^{2n+1}\xi_z\). Then the lemma follows. \(\square\)

Let \(G_0(\alpha i) = (f, \hat{S}\rho(x^2)\xi_{\alpha i})\) for \(f \in L^2(\mathbb{R})\). The function \(G_0\) can be extended to \(z \in \mathbb{D} \cap \mathbb{H}\). Actually we can see that \(\rho(x^2)\xi_z \in D(\hat{S})\) and

\[
G_0(z) = \left( f, \hat{S}\rho(x^2)\xi_z \right) = \left( f, \frac{i}{2} (\rho(t_z) \log(-\mathcal{C})) \xi_z \right), \quad z \in \mathbb{D} \cap \mathbb{H}. \tag{4.10}
\]

This can be proven in a similar way to Lemma 3.8.

**Lemma 4.8** Let \(f \in L^2(\mathbb{R})\). Define \(F_0(z) = (f, \hat{S}\rho(x^2)\xi_z)\) for \(z \in \mathbb{H} \setminus \{i\}\). Then the function \(F_0\) is analytic on \(\mathbb{H} \setminus \{\alpha i \mid \alpha \in [1, \infty)\}\). In particular \(F_0\) is the analytic continuation of \(G_0\) given by (4.10) to \(\mathbb{H} \setminus \{\alpha i \mid \alpha \in [1, \infty)\}\).

Proof: We see that \(F_0\) is differentiable on \(z \in \mathbb{H} \setminus \{\alpha i \mid \alpha \in [1, \infty)\}\) with

\[
\frac{d}{dz} F_0(z) = \left( f, \frac{i}{2} \rho(t_z) \frac{d}{dz} \log(-\mathcal{C}) \cdot \xi_z \right) + \left( f, \frac{i}{2} \rho(t_z) \log(-\mathcal{C}) \cdot \frac{x^2}{2} \xi_z \right).
\]

Then \(F\) is analytic on \(\mathbb{H} \setminus \{\alpha i \mid \alpha \in [1, \infty)\}\) and the analytic continuation of the analytic function \(G\) on \(D \cap \mathbb{H}\). \(\square\)

In a similar manner we can see that \(\rho(x^2)\xi_z \in D(\hat{S}^*)\) for \(z \in \mathbb{D} \cap \mathbb{H}\), and

\[
G_1(z) = \left( f, \hat{S}^*\rho(x^2)\xi_z \right) = \left( f, \frac{i}{2} (\rho(t_z) \log(-\mathcal{C})) x^2 \xi_z \right), \quad z \in \mathbb{D} \cap \mathbb{H}. \tag{4.11}
\]

**Lemma 4.9** Let \(f \in L^2(\mathbb{R})\). Define \(F_1(z) = (f, \hat{S}^*\rho(x^2)\xi_z)\) for \(z \in \mathbb{H} \setminus \{i\}\). Then \(F_1\) is analytic on \(\mathbb{H} \setminus \{\alpha i \mid \alpha \in [1, \infty)\}\). In particular \(F_1\) is the analytic continuation of \(G_1\) given by (4.11) to \(\mathbb{H} \setminus \{\alpha i \mid \alpha \in [1, \infty)\}\).

We define sesqui-linear forms \(i_0\) and \(i_1\) by

\[
i_0[\psi, \phi] = \frac{1}{2} \left\{ \left( \hat{S}\psi, \phi \right) + \left( \psi, \hat{S}\phi \right) \right\}, \quad \psi, \phi \in \mathfrak{N}_0,
\]

\[
i_1[\psi, \phi] = \frac{1}{2} \left\{ \left( \hat{S}^*\psi, \phi \right) + \left( \psi, \hat{S}^*\phi \right) \right\}, \quad \psi, \phi \in \mathfrak{N}_1.
\]

The sesqui-linear form \(\hat{i}\) is defined by

\[
\hat{i} = i_0 \oplus i_1
\]

with the domain \(\mathfrak{N} \times \mathfrak{N}\).

**Theorem 4.10** The sesqui-linear form \(\hat{i}\) is an ultra-weak time operator of \(h\) with the symmetric domain \(\mathfrak{N}\) and the ultra-weak CCR-domain \(\mathfrak{N}\).

Proof: The theorem follows from Lemma 4.7. \(\square\)

Now we consider the matrix representation of \(\hat{i}\). We prove that \(t[f_a, f_b] = t[x^a \xi_z, x^b \xi_z]\) diverges at \(z = i\) in Theorem 3.10. The next theorem is however to guarantee the existence of the analytic continuation of the map \(z \mapsto t[f_a, f_b]\).
Theorem 4.11 Let \( z \in \mathbb{H} \setminus \{i\}. \) Let \( f_a = x^a \xi_z \) and \( f_b = x^b \xi_z. \)

1. It follows that
   \[
   \hat{t}[f_a, f_b] = \begin{cases} 
   -\frac{i}{4} \left( \xi_z, \left\{ \left(t^n z x^{2m} - x^{2n} t^m z \right) \log(-\mathcal{C}) \right\} \xi_z \right) & \text{if } a = 2n \text{ and } b = 2m, \\
   -\frac{i}{4} \left( \xi_z, \left\{ \left(t^n z x^{2m+2} - x^{2n+2} t^m z \right) \log(-\mathcal{C}) \right\} \xi_z \right) & \text{if } a = 2n + 1 \text{ and } b = 2m + 1, \\
   0 & \text{otherwise.}
   \end{cases}
   \]

2. Let \( z = \alpha i \) with \( \alpha \in (0, 1). \) Then \( \hat{t}[f_a, f_b] = t[f_a, f_b]. \)

3. The map \( z \mapsto \hat{t}[f_a, f_b] \) is analytic on \( \mathbb{H} \setminus \{\alpha i | \alpha \in [1, \infty)\}. \) In particular, \( \hat{t}[f_a, f_b] \) is the analytic continuation of \( t[f_a, f_b]. \)

Proof: (1) can be shown from the definition of \( \hat{t}. \) (2) can be derived from \( \hat{S}^\# \rho(x^2) \xi_z = S^\# \rho(x^2) \xi_z \) for \( z \in i(0, 1). \) Finally (3) follows from Lemmas 4.8 and 4.9.

5 Continuum limits

In this section we consider the continuum limit of \( t_\varepsilon \) as \( \varepsilon \to 0. \) \( T_\varepsilon \) is formally written as
   \[
   T_\varepsilon = -\frac{1}{2} \frac{1}{\sqrt{\varepsilon}} (\arctan(\sqrt{\varepsilon} x) + \arctan(\sqrt{\varepsilon} y)).
   \]

Since
   \[
   \lim_{\varepsilon \to 0} \frac{1}{2 \sqrt{\varepsilon}} (\arctan(\sqrt{\varepsilon} x) + \arctan(\sqrt{\varepsilon} y)) = \frac{1}{2} (x + y) \quad (x, y \in \mathbb{R}),
   \]

it can be expected that \( \lim_{\varepsilon \to 0} T_\varepsilon = -\frac{1}{2} (t + t^*) \) in some sense. Since subspaces \( \mathcal{M}_{\varepsilon, \#} \) depend on \( \varepsilon, \) to consider the continuum limit of \( t_\varepsilon \) we introduce \( \mathcal{M}_{\#} \) by \( \mathcal{M}_{\varepsilon, \#} \) with \( \varepsilon \) replaced by 1:

\[
\mathcal{M}_0 = \text{LH} \left\{ x^{2n} e^{-\alpha x^2 / 2} \left| n \in \mathbb{N}, \alpha \in (0, 1) \right. \right\},
\]

\[
\mathcal{M}_1 = \text{LH} \left\{ x^{2n+1} e^{-\alpha x^2 / 2} \left| n \in \mathbb{N}, \alpha \in (0, 1) \right. \right\}.
\]

Let \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) and \( \mathcal{D} = \text{D}(S) \oplus \text{D}(S^*). \) Then the sesqui-linear form \( t_\varepsilon \) is also an ultra-weak time operator with the symmetric domain \( \mathcal{D} \) and the ultra-weak CCR domain \( \mathcal{M}. \) We also extend the Aharonov-Bohm operator \( T_{AB} = \frac{1}{2} (t + t^*) \) to an ultra-weak time operator. Let

\[
t_{AB,0}[\psi, \phi] = -\frac{1}{2} \left\{ (\psi, t \phi) + (t \psi, \phi) \right\}, \quad \psi, \phi \in \mathcal{M}_0,
\]

\[
t_{AB,1}[\psi, \phi] = -\frac{1}{2} \left\{ (t^* \psi, \phi) + (\psi, t^* \phi) \right\}, \quad \psi, \phi \in \mathcal{M}_1.
\]

Define \( t_{AB} \) by

\[
t_{AB} = t_{AB,0} \oplus t_{AB,1}.
\]

The sesqui-linear form \( t_{AB} \) is an ultra-weak time operator of \( \frac{1}{2} q^2 \) with the symmetric domain \( \mathcal{M} \) and the ultra-weak CCR-domain \( \mathcal{M}. \)
Theorem 5.1 Let \( \phi, \psi \in \mathcal{M} \). Then it follows that

\[
\lim_{\varepsilon \to 0} t_{\varepsilon} \big[ \psi, \phi \big] = t_{AB} \big[ \psi, \phi \big].
\]

Proof: Let \( c_1, c_2, d_1, d_2 \in \mathbb{C}, \ f_1 = 2k \varepsilon^{-\alpha x^2/2}, \ f_2 = 2l \varepsilon^{-\beta x^2/2}, \ g_1 = 2m \varepsilon^{-\gamma x^2/2} \) and \( g_2 = 2n \varepsilon^{-\delta x^2/2} \). Set \( \phi = c_1 f_1 + d_1 g_1 \) and \( \psi = c_2 f_2 + d_2 g_2 \). It is sufficient to show that \( \lim_{\varepsilon \to 0} t_{\varepsilon} \big[ \phi, \psi \big] = t_{AB} \big[ \phi, \psi \big] \). From Theorem 4.4 it can be seen that

\[
t_{\varepsilon,0} \big[ f_1, f_2 \big] = \frac{i}{4 \sqrt{\varepsilon}} \left( \xi_{\alpha,i,1} \left\{ x^{2l} t_{\alpha,i,1} \log \frac{1 + \sqrt{\varepsilon} \alpha}{1 - \sqrt{\varepsilon} \alpha} - x^{-2k} t_{\beta,i,1} \log \frac{1 + \sqrt{\varepsilon} \beta}{1 - \sqrt{\varepsilon} \beta} \right\} \xi_{\beta,i,1} \right).
\]

Expanding \( t_{\alpha,i,1} \log \frac{1 + \sqrt{\varepsilon} \alpha}{1 - \sqrt{\varepsilon} \alpha} \) on \( x \), we have

\[
x^{2l} t_{\alpha,i,1} \log \frac{1 + \sqrt{\varepsilon} \alpha}{1 - \sqrt{\varepsilon} \alpha} = x^{2(l+k)} \log \frac{1 + \sqrt{\varepsilon} \alpha}{1 - \sqrt{\varepsilon} \alpha} - k \sqrt{\varepsilon} x^{2(l+k-1)} \frac{4}{1 - \varepsilon \alpha^2} + R_{l,i,\alpha,\varepsilon}(x),
\]

where \( R_{l,i,\alpha,\varepsilon}(x) \) is a real valued function on \( \mathbb{R} \) such that \( \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} R_{l,i,\alpha,\varepsilon}(x) = 0 \) for all \( x \in \mathbb{R} \). Since \( \frac{1}{\sqrt{\varepsilon}} \log \frac{1 + \sqrt{\varepsilon} x}{1 - \sqrt{\varepsilon} x} \to 2X \) and \( (1 - \varepsilon X)^{-1} \to 1 \) as \( \varepsilon \to 0 \), we see that

\[
\lim_{\varepsilon \to 0} t_{\varepsilon,0} \big[ f_1, f_2 \big] = \frac{i}{2} \left( \xi_{\alpha,i,1} x^{2(l+k-1)} \left\{ \alpha x^2 - 2k - (\beta x^2 - 2l) \right\} \xi_{\beta,i,1} \right)
\]

\[
= \frac{i}{2} \left( \left( (\alpha x^2 - 2k) x^{2(l+k-1)} \xi_{\alpha,i,1} x^{2l} \xi_{\beta,i,1} \right) - \left( x^{2k} \xi_{\alpha,i,1} x^{2l} \xi_{\beta,i,1} \right) \right)
\]

\[
= -\frac{1}{2} \left( (tf_1, f_2) + (f_1, tf_2) \right)
\]

\[
= t_{AB,0} \big[ f_1, f_2 \big].
\]

In the same way as above, we obtain \( \lim_{\varepsilon \to 0} t_{\varepsilon} \big[ g_1, g_2 \big] = t_{AB} \big[ g_1, g_2 \big] \). Hence, for any \( \psi, \phi \in \mathcal{M} \), we conclude that \( \lim_{\varepsilon \to 0} t_{\varepsilon} \big[ \phi, \psi \big] = t_{AB} \big[ \phi, \psi \big] \). \( \square \)

6 Comparison of \( T_{\varepsilon} \) to time operators by POVM

Let \( \mathcal{H} \) be a Hilbert space and \( (\Omega, \mathcal{B}_\Omega) \) a measurable space. An operator-valued set function \( P \) on \( \mathcal{B}_\Omega \) is a POVM if and only if for every \( A \in \mathcal{B}_\Omega \), \( P(A) \) is a bounded non-negative self-adjoint operator, and

\[
\mathcal{B}_\Omega \ni A \mapsto \frac{f, P(A)f}{\|f\|^2} \in [0,1]
\]

is a probability measure for any non-zero \( f \in \mathcal{H} \). Note that \( P \) is not necessarily projection-valued.

In this section we compare \( T = T_{\varepsilon=1} \) to the time operator \( T_{\varepsilon} \) derived through the POVM associated with \( h = \frac{1}{4}(p^2 + q^2) \). The normalized eigenvector of \( h \) associated with the eigenvalue \( n + \frac{1}{2} \) is denoted by \( e_n \). For \( t \in \mathbb{R} \) and \( N \in \mathbb{N} \) we define

\[
\varphi_t^{(N)} = \sum_{n=0}^{N} e^{-i\hbar t} e_n.
\]
and we set
\[ P_t^{(N)} = \frac{1}{2\pi} (\varphi_t^{(N)}, \cdot) \varphi_t^{(N)}. \]

Let \( \mathcal{B} \) be the Borel \( \sigma \)-field on \([0, 2\pi]\). The operator-valued set function \( P_{(N)} \) on \( \mathcal{B} \) is defined by
\[ \mathcal{B} \ni A \mapsto (f, P_{(N)}(A)g) = \int_{[0,2\pi]} \mathbb{1}_A(t)(f, P_t^{(N)} g) dt. \]

Let \( A, A' \in \mathcal{B} \) and \( A \subset A' \). Then \( 0 \leq (f, P_{(N)}(A)f) \leq (f, P_{(N)}(A')f) \).

**Lemma 6.1** For any \( A \in \mathcal{B} \) and any \( f, g \in \mathcal{H} \), \( \lim_{N \to \infty} (f, P_{(N)}(A)g) \) exists.

Proof: It is immediate to see that
\[ (f, P_{(N)}(A)g) = \frac{1}{2\pi} \int_A \sum_{n=0}^N (f, e_n)(e_n, g) dt + \frac{1}{2\pi} \int_A \sum_{n \neq m} e^{-i(n-m)}(f, e_n)(e_m, g) dt. \] (6.1)

Suppose that \( f = g \) and \( A = [0, 2\pi] \). Then \( (f, P_{(N)}([0, 2\pi])f) = (2\pi)^{-1} \int_0^{2\pi} |(f, e_n)|^2 dt \leq \|f\|^2 \). Using the polarization identity we can also see that for any \( A \in \mathcal{B} \), \( \{(f, P_{(N)}(A)g)\}_N \) is a Cauchy sequence in \( \mathbb{C} \). Then the lemma follows. \( \square \)

For each \( A \in \mathcal{B} \), by the Riesz representation theorem there exists a unique bounded non-negative self-adjoint operator \( P(A) \) such that
\[ (f, P(A)g) = \lim_{N \to \infty} (f, P_{(N)}(A)g). \] (6.2)

Note that \( (f, P([0, 2\pi])f) = \|f\|^2 \). Then \( P \) is a POVM on \(([0, 2\pi], \mathcal{B})\).

**Definition 6.2** We call \( P(\cdot) \) the POVM associated with \( h \).

In a similar manner we can construct POVMs associated with self-adjoint operators with purely discrete spectrum.

Now the sesqui-linear form \( t_G[f, g] \) is defined by
\[ t_G[f, g] = \lim_{N \to \infty} \int_{[0,2\pi]} t(f, P_t^{(N)} g) dt \] (6.3)
with the domain \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \).

**Lemma 6.3** It follows that \(|t_G[f, g]| \leq 2\pi \|f\| \|g\| \) for \( f, g \in L^2(\mathbb{R}) \).

Proof: By a similar computation to (6.1), we obtain that
\[ t_G[f, g] = \pi(f, g) + i \sum_{n=0}^\infty \left( \sum_{m \neq n} (e_m, g) (f, e_n) \right) \]
and \(|t_G[f, g]| \leq (2\pi) \|f\| \|g\| \). Here we used the inequality \([24, \text{Theorem 294}]\):
\[ \left| \sum_{n=0}^\infty \sum_{m \neq n} x_n x_m / (n - m) \right| \leq \pi \left( \sum_{n=0}^\infty x_n^2 \right)^{1/2} \left( \sum_{n=0}^\infty y_n^2 \right)^{1/2}. \]
Then the lemma is proven. \[\square\]

The Riesz representation theorem yields again that there exists a unique bounded self-adjoint operator \(T_G\) such that \(\|T_G\| \leq 2\pi\) and

\[tg_G[f, g] = (f, T_Gg).\] (6.4)

Formally writing \(P(A) = \int_A dP_t\), we can express \(T_G\) as

\[T_G = \int_{[0,2\pi]} t dP_t.\] (6.5)

We shall show that \(T_G\) is a time operator of \(h\).

Lemma 6.4 Let \(f, g \in D(h)\). Then we have

\[\left( f, [h, P_t^{(N)}] g \right) = -\frac{1}{2\pi} \frac{d}{dt} \left( f, P_t^{(N)} g \right).\]

Proof: Since \(he^{-ith}e_n = -\frac{1}{i} \frac{d}{dt} e^{-ith}e_n\) in the strong sense, we have

\[\left( f, [h, P_t^{(N)}] g \right) = -\frac{1}{2\pi} \sum_{1 \leq n, m \leq N} \left\{ \frac{1}{i} \frac{d}{dt} \left( f, e^{-ith}e_m \cdot (e^{-ith}e_n, g) \right) + \left( f, e^{-ith}e_m \cdot \frac{1}{i} \frac{d}{dt} \left( e^{-ith}e_n, g \right) \right) \right\} = -\frac{1}{i} \frac{d}{dt} \left( f, P_t^{(N)} g \right).\]

Then the lemma follows. \[\square\]

We define the unbounded operator \(P_0\) by

\[P_0 = \lim_{N \to \infty} P_0^{(N)} = \lim_{N \to \infty} \frac{1}{2\pi} \left( \sum_{m=0}^N e_m, \right) \sum_{n=0}^N e_n.\]

Lemma 6.5 It follows that \([h, T_G] = -i(1 - 2\pi P_0)\) on \(D(h)\).

Proof: Let \(f, g \in D(h)\). By Lemma 6.4 we have

\[\left( f, [h, T_G] g \right) = \lim_{N \to \infty} \int_{[0,2\pi]} t \left( f, [h, P_t^{(N)}] g \right) dt = -\lim_{N \to \infty} \int_{[0,2\pi]} \frac{1}{i} \frac{d}{dt} \left( f, P_t^{(N)} g \right) dt\]

\[= -\lim_{N \to \infty} \left\{ \frac{1}{i} \left[ t f, P_t^{(N)} g \right]_{0}^{2\pi} - \int_{[0,2\pi]} \frac{1}{i} \left( f, P_t^{(N)} g \right) dt \right\}\]

\[= i \lim_{N \to \infty} \left( f, 2\pi P_0^{(N)} g \right) - i(f, g).\]

Then the lemma follows. \[\square\]

Lemma 6.6 It follows that \([h, T_G] = -i1\) on \(LH\{e_n - e_m \mid n \neq m\}\). I.e., \(T_G\) is a bounded self-adjoint time operator of \(h\).

Proof: Since \(P_0(e_n - e_m) = 0\), the lemma follows from Lemma 6.5. \[\square\]
Remark 6.7 By the definition of $T_G$ we can see that

$$T_G = i \sum_{n=0}^{\infty} \left( \sum_{m \neq n} \frac{(e_m, \cdot)}{n-m} e_n \right) + \pi \mathbb{I}.$$ 

This can be extended to a general self-adjoint operator $H$ possessing purely discrete spectrum: $\text{Spec}(H) = \{E_n\}_{n=0}^{\infty}$ and $H f_n = E_n f_n$. The time operators of $H$ of the form

$$T_H = i \sum_{n=0}^{\infty} \left( \sum_{m \neq n} \frac{(f_m, \cdot)}{E_n - E_m} f_n \right)$$

are studied in e.g., [6, 19, 18].

We can also define the sesqui-linear form associated with $T_G$ by

$$t_G[\phi, \psi] = \frac{1}{2} \{(\phi, T_G \psi) + (T_G \phi, \psi)\}.$$ 

The sesqui-linear form $t_G$ is also an ultra-weak time operator of $h$ with the symmetric domain $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and the ultra-weak CCR domain $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Note that $t_G$ is a bounded sesqui-linear form.

Theorem 6.8 It follows that $t \neq t_G$.

Proof: The sesqui-linear form $t_G$ is bounded, but $t$ is unbounded. Then the theorem follows. □

7 Conclusion

In this paper we discuss two time operators. One is the angle operator of the form

$$T_\varepsilon = -\frac{1}{2} \frac{1}{\sqrt{\varepsilon}} \left( \arctan(\sqrt{\varepsilon}q^{-1}p) + \arctan(\sqrt{\varepsilon}pq^{-1}) \right)$$

and the other is the operator of the form

$$T_G = i \sum_{n=0}^{\infty} \left( \sum_{m \neq n} \frac{(e_m, \cdot)}{n-m} e_n \right) + \pi \mathbb{I}.$$ 

We give the firm definition of $T_\varepsilon$ as a sesqui-linear form and its matrix representation. Furthermore the analytic continuation of the matrix representation is obtained. We also show that $T_\varepsilon$ converges to the Aharonov-Bohm operator $T_{AB}$ as $\varepsilon \to 0$ in the sense of sesqui-linear forms. Finally it is shown that $T_{\varepsilon=1} \neq T_G$ in the sense of sesqui-linear forms. As far as we know these are new.

We introduce the phase operator $\hat{\phi}$ in (1.7) as a time operator of the number operator $N$. The number operator $N$ and $\hat{\hbar} = \frac{1}{2}(p^2 + q^2)$ are simply related as $N + \frac{1}{2} \mathbb{I} = \hat{\hbar}$. The rigorous definitions of both $\log a$ and $\log a^*$ are however not established. In particular it is
not obvious to define $\log a^*$ as a densely defined operator in $L^2(\mathbb{R})$. Let $\hat{T}$ be (1.4). One can expect however in e.g., [40] that $\dot{\phi} = \hat{T} + G(\hbar)$ with some function $G$. Needless to say it is far away from mathematics. We shall discuss this in the second paper [26]. Furthermore we shall also show in [26] the existence of the interpolation connecting the angle operator $\hat{T}_{\varepsilon=1}$ with $T_G$.

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