Technical Uncertainty in Real Options with Learning

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Abstract

We introduce a new approach to incorporate uncertainty into the decision to invest in a commodity reserve. The investment is an irreversible one-off capital expenditure, after which the investor receives a stream of cashflow from extracting the commodity and selling it on the spot market. The investor is exposed to price uncertainty and uncertainty in the amount of available resources in the reserves (i.e. technical uncertainty). She does, however, learn about the reserve levels through time, which is a key determinant in the decision to invest. To model the reserve level uncertainty and how she learns about the estimates of the commodity in the reserve, we adopt a continuous-time Markov chain model to value the option to invest in the reserve and investigate the value that learning has prior to investment.

Keywords: Real Options; Investment under Uncertainty; Technical Uncertainty; Irreversibility

1. Introduction

What are the optimal market conditions to make an investment decision is an extensively studied problem in the academic literature and a key question at the heart of the valuation and execution of projects under uncertainty. Some investment projects are endowed with the option to delay decisions until market conditions are optimal. This option is valuable because decisions are made when the potential gains stemming from the decision are maximized. The classical work of McDonald and Siegel (1986) is the first to formalize the investment problem

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as a real option to invest in a project. In their work, the value $O_t$ of the option is calculated by comparing the difference in the value of investing now and the value of making the investment at a future time. Specifically, the value of the real option is

$$O_t = \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[ e^{-\rho(t-\tau)} (V_\tau - I_\tau)_{+} \right],$$

where $\mathcal{T}$ is the set of admissible $\mathcal{F}$-stopping (exercise) times, $\rho$ is the appropriate risk-adjusted discount rate, and $V_t$ and $I_t$ are the project value and (sunk) cost respectively. The project value and cost are traditionally modeled with a geometric Brownian motion (GBM). The solution to this problem shows that the optimal investment strategy is to invest when the ratio $V_t/I_t$ reaches a critical boundary $B$ (the problem of optimal scrapping/divesting is similar, with the roles of $V_t$ and $I_t$ reversed). More recently, several authors have studied this problem with mean-reverting project value and costs (see, e.g., Metcalf and Hasset (1995), Sarkar (2003), Jaimungal et al. (2013)).

In another classical paper, Brennan and Schwartz (1985) focus on the management of a mine (controlling output rate, opening/closing of mine, abandonment, and so on) rather than the optimal timing problem (see also Dixit (1989)). Management decisions are modulated by the output prices which are modeled as a GBM, while costs are known.

These classical works do not take into account the uncertainty associated with reserve levels. To account for such “technical uncertainty”, Pindyck (1980) develops a model where the demand and reserve levels fluctuate continuously with increasing variance. Furthermore, the optimal strategy is influenced by exploration and is introduced as a policy (i.e. control) variable in two distinct ways. The first allows exploratory effort to affect the level of “knowledge”, which reduces the variance of reserve fluctuations. The second assumes that reserves are discovered at a rate that depends on: how much has already been discovered in the past, the amount of current effort, and exogenous noise.

More recent approaches to the investment timing problem with technical uncertainty include those using Bayesian updating as in Armstrong et al. (2004), modeling project costs via Markov chains as in Elliot et al. (2009), and using proportionality to model learning as in Sadowsky (2005). Also, Cortázar et al. (2001) describes a comprehensive approach to valuing several-stage exploration, solving the timing problem, and provides investment management (closure, opening, etc.) decision rules – see also Brennan and Schwartz (1985). Other pieces of work that incorporate real option techniques in the valuation of flexibility and investment decisions in commodities and energy include that of Himpler and Madlener (2014) who look at the optimal timing of wind farm repowering; Taschini and Urech (2010) who look at the option to switch fuels under different scenarios and fuel incentives; the work of Fleten et al. (2011) that looks at the option to choose the capacity of an electricity interconnector between two locations, and Cartea and González-Pedraz (2012) who value an electricity interconnector as a stream of real options of the difference of prices in two locations.
This work adds to the literature by incorporating both market and reserve uncertainty, while allowing the agent to learn about the status of reserves. Reserve uncertainty is represented by a Markov chain model with transition rates that decay as time flows forward to mimic the notion of learning. The setup is developed in the context of oil exploration, however, it may be applied to other investment problems in commodities, such as mining for precious or base metals, and natural gas fields. We value the irreversible option to invest in the exploration by developing a version of Fourier space-time stepping, as in Jackson et al. (2008), Jaimungal and Surkov (2011), and Jaimungal and Surkov (2013) for equity, commodity and interest-rate derivatives, respectively.

If estimates on exploration costs and volume estimates are available, the calibration of the model is relatively simple. We demonstrate how the model can be used to assess whether exploration costs warrant the potential benefits from finding reserves and extracting them. Specifically, we show how to calculate the value of the option to delay investment and discuss the agent’s optimal investment threshold. This threshold, also referred to as exercise boundary, depends on an number of variables and factors including: the agent’s estimate of the volume in the reserve, the rate at which the agent learns about the volume of the reserve, the rate at which the agent extracts the commodity, and the expiry of the option. We show that the value of the option to wait-and-learn is high at the beginning and gradually decreases as expiry of the option approaches because the agent has little time left to learn.

We assume that the investment cost depends on the volume of the reserve. If the estimated volume is high (resp. low) the sunk cost to extract the commodity is high (resp. low). This has an effect on the optimal time to make the investment as well as the level of spot price commodity that justifies making the sunk cost. For example, we show that when the volume estimate is low, and the option to invest is far away from expiry, the agent sets a high investment threshold as a result of two effects which make the option to delay investment valuable. First, low volume requires a high commodity spot price to justify the investment. Second, far away from expiry the investor attaches high value to waiting and learning about the volume estimates of the reserve. On the other hand, as the option approaches expiry, these two effects become weaker. In particular, the value of learning is low because there is less time to learn about the reserve estimates, so the investment decision is merely based on whether costs will be recovered given the spot price of the commodity, the rate at which the commodity is extracted, and the uncertainty around it.

The remainder of this paper is organized as follows. In Section 2, we provide the details of our modeling framework, including how we model both technical uncertainty and the uncertainty in the underlying project. Moreover, we provide an approach for accounting for the agent’s learning of the reserve environment through exploration. Next, Section 3 provides an analysis of the Fourier space-time stepping approach for valuing the early exercise features in the irreversible investment with learning. Section 4 shows how the model can be calibrated to estimates of the cost of exploration and the expected benefits of those exploration. Section
5 provides some numerical experiments to demonstrate the efficacy of the approach and an analysis of the qualitative behavior of the model and its implications. Finally, we provide some concluding remarks and ideas for future lines of work.

2. Model Assumptions

In this section we provide models for the two sources of uncertainty that drive the value of the real option to explore and (irreversibly) invest in a project. The setting described here is tuned to some extent for oil exploration, however, it can be modified to deal with other activities including: mining, natural and shale gas, and other natural explorations. Another extension to our setup is to account for the option to mothball exploration and/or extraction (once extraction begins), as well as other managerial flexibilities that might arise in exploration and investment. See for example Dixit and Pindyck (1994), Trigeorgis (1999), Tsekrekos (2010), Jaimungal and Lawryshyn (2015), and Kobari et al. (2014). We first describe how we model technical uncertainty and then describe how we model project value uncertainty.

2.1. Technical Uncertainty

Let $V_t$ denote the estimated reserve volume (level) at time $t$ and $V_\infty$ be the true (unknown and random) reserve volume. We assume for simplicity that the possible reserve levels, and their estimates take on values from a finite set of reserve volumes. We model the reserve level as a continuous-time Markov chain because reserve estimates are updated as new information from exploration is obtained. Moreover, to capture the feature that the accuracy of estimates improves as more information becomes available through time, we assume that the transition between the volume estimate states decreases as time flows forward.

Reserve volume $V_t$ is modulated by a finite state, continuous-time, Markov chain $Z_t \in \{1, \ldots, m\}$ via

$$V_t = v(Z_t),$$

where the constants

$$\{v^{(1)}, \ldots, v^{(m)}\} \in \mathbb{R}^m_+$$

are the possible reserve volumes. The generator matrix of the Markov chain $Z_t$ is denoted by $G_t$ and assumed to be of the form

$$G_t = h_t A,$$

where $h_t$ is a deterministic, positive, and decreasing function of time, such that $h_t \xrightarrow{t \to \infty} 0$, and $A$ is a constant $m \times m$ matrix with $\sum_{j=1}^n A_{ij} = 0$ and $A_{ij} > 0$ for $i \neq j$. The states of the Markov chain correspond to various possible estimates for reserve level, thus capturing the uncertainty in those estimates.
The function $h_t$ captures how the agent learns about the volume or quantity of the commodity in the reserve. A decreasing $h$ implies that the transition rates are also decreasing, and hence the probability of changes in the estimated volumes decreases with time, and therefore the estimates become more accurate. Optimal policies for the irreversible investment to explore, and the subsequent value of the project based on the extraction of the commodity, naturally depend on the observed estimate of reserves – Section 4 discuss in detail the form of $h_t$ and how it is calibrated to data.

2.2. Market Uncertainty

The second source of uncertainty stems from the spot price of the commodity which we denote by $S_t$, and is modelled as the exponential of an Ornstein-Uhlenbeck (OU) process:

$$S_t = \exp\{\theta + X_t\}, \quad (5a)$$

where the OU process $X_t$ satisfies the stochastic differential equation (SDE)

$$dX_t = -\kappa X_t \, dt + \sigma \, dW_t, \quad (5b)$$

where $W_t$ is a standard Brownian motion, $\kappa > 0$ is the rate of mean-reversion, $\theta$ is the (log-)level of mean-reversion, and $\sigma$ is the (log-)volatility of the spot price. Such models of commodity spot prices have been widely used in the literature, see for example Cartea and Figueroa (2005), Weron (2007), Kiesel et al. (2009), Coulon et al. (2013).

Now that we have specified the model for the stock of the commodity in the reserve and its market price, we need one final ingredient: the market value of the commodity in the reserve. We denote this value by $P_t$ and show how to calculate it in steps.

Suppose that investment is made at time $t$, which is followed by extraction of the commodity $\epsilon \geq 0$ later, and extraction continues until the random time $\tau = t + \epsilon + \Delta$. Here $\Delta > 0$ represents the future time by which the reserve has been depleted. The investor does not know how much of the commodity is in the reserve, so the time to depletion is random.

We assume that once extraction begins, the commodity is extracted at the rate

$$g(u) = \alpha e^{-\beta(u-(t+\epsilon))}, \quad u \in [t+\epsilon, t+\epsilon+\Delta], \quad (6)$$

where $\alpha \geq 0$, and $\beta \geq 0$. Figure 1 presents a stylized picture of the exponential extraction rate (6).

The random time to depletion can be re-cast in terms of the unknown total volume, denoted by $V_\infty$, of the reserve. However, engineering and physical limitations prevent the total amount of the commodity stored in the reserve from being extracted, and instead only
\( \gamma V_{\infty} \) is extractable, \( 0 < \gamma < 1 \). Thus, we set \( \int_{t+\epsilon}^{t+\epsilon+\Delta} g(u) \, du = \gamma V_{\infty} \) and under the specific extraction rate model in (6), the time to depletion can be written as

\[
\Delta = -\frac{1}{\beta} \log \left( 1 - \frac{\beta}{\alpha} \gamma V_{\infty} \right),
\]

which is a random variable because \( V_{\infty} \) is only known when time goes to infinity.

The value of the reserve when extraction begins is determined by a number of factors including: the prices of the commodity, the state of the Markov chain linked to reserve uncertainty, and the random time to exhaustion of the reserve. Specifically, the discounted value of the cash-flow generated from extracting the commodity at the rate \( g(u) \) and selling it at the spot price \( S_u \) is given by

\[
DCF_t = \int_{t+\epsilon}^{t+\epsilon+\Delta} e^{-\rho(u-t)} S_u g(u) \, du,
\]

where \( \rho \) is the agent’s discount factor for the level of risk she bears with the project.

To compute the expected discounted value of the reserve, which we denote by \( P_t \), we insert in (8): the spot price of the commodity (5), the time to extraction completion given in (7), and the extraction rate (6). Finally, we take expectations of \( DCF_t \) to obtain

\[
P_t = \mathbb{E} [DCF_t | \mathcal{F}_t] = \mathbb{E} \left[ \int_{t+\epsilon}^{t+\epsilon+\Delta} e^{-\rho(u-t)} F_t(u) g(u) \, du \bigg| \mathcal{F}_t \right],
\]

where \( \mathcal{F}_t \) is the natural filtration generated by both \( S \) and \( V \) (or equivalently the Markov chain \( Z \) introduced earlier), and

\[
F_t(u) = \mathbb{E} [S_u | \mathcal{F}_t] = \exp \left\{ \theta + e^{-\kappa(u-t)} x + \frac{1}{4\kappa} (1 - e^{-2\kappa(u-t)}) \right\}
\]

A good example is stored natural gas where there is always a residual that cannot be extracted from storage.
is the forward price of the underlying asset.

Thus, the expectation in (9) is over the random time to depletion $\Delta$. To compute this expectation we require the stationary distribution of the unknown reserve level $V_\infty$, that is we require $\mathbb{P}(V_\infty = v^{(i)} | \mathcal{F}_t)$. This is equivalent to determining the conditional distribution at time $t$ of the underlying Markov chain, $Z_\infty$. Specifically,

$$
\mathbb{P}(V_\infty = v^{(i)} | \mathcal{F}_t) = \mathbb{P}(Z_\infty = j | Z_t = i) = [e^{H_t A}]_{ij},
$$

where $H_t = \int_t^\infty h_u \, du$, and the notation $[\cdot]_{ij}$ denotes the $ij$ element of the matrix in the square brackets, and recall that the matrix $A$ is defined above in (4). Therefore, the final expression for the expected discounted value of the reserve is

$$
P_t := p^{(Z_t)}(t, X_t) = \sum_{j=1}^m [e^{H_t A}]_{Z_t,j} \int_{t+\epsilon}^{t+\epsilon + \beta \log \left( \frac{1}{1 - \frac{\beta}{\alpha} v^{(j)}} \right)} e^{-\rho(u-t)} F_t(u) g(u) \, du.
$$

This expression has two sources of uncertainty, the first stems from the spot price of the commodity, through the OU process $X_t$, and the second from the estimate of the reserve volume, through the state of the Markov chain $Z_t$. With the model of extraction rate being exponentially decaying in time, see (6), it is possible to write the integral appearing in the right-hand side of equation (10) in terms of special functions, however, such a rewrite does not add clarity so we opt to keep the integral as shown above.

3. Real Option Valuation

Now that we have a model for the value of the reserve, we focus on the cost required to exploit the reserve of the commodity and the value of the flexibility to decide when to make the investment. The cost of investing in the reserve is irreversible and denoted by $I^{(k)}$, where $k$ is the regime of the reserve volume estimate. Here we assume that the cost $I^{(k)}$ is linked to the volume estimate because extracting a large reserve will likely require a larger up-front investment than that required to extract the commodity from a small reserve. We assume that the investment cost is

$$
I^{(k)} = c_0 + c_1 v^{(k)},
$$

where $c_0 \geq 0$ is a fixed cost, $c_1 \geq 0$, and recall that $v^{(k)}$ are the possible reserve volumes, see (3).

We denote the value of the option by $L_t = \ell^{(Z_t)}(t, X_t)$, where the collection of functions $\ell^{(1)}(t, x), \ldots, \ell^{(m)}(t, x)$ represent the value of the real option conditional on the state $Z_t = 1, \ldots, m$ (indexed by the superscript) and $X_t = x$. The agent must make a decision by time $T$, otherwise loses the option to make the investment and exploit the reserve. Standard theory
implies that the value of the option to irreversibly invest in the reserve, and begin extraction, is given by the optimal stopping problem

\[ L_t = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\rho \tau} \max \left( P_\tau - I^{(Z_\tau)}, 0 \right) \mid \mathcal{F}_t \right] \]  

(12a)  

\[ = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\rho \tau} \max \left( P_\tau - I^{(Z_\tau)}, 0 \right) \mid Z_t, X_t \right] . \]  

(12b)

Here, \( T \) denotes the set of admissible stopping times, taken to be the finite collection of \( F \)-stopping times restricted to \( t_i = i\Delta t, i = 0, \ldots, N \) with \( t_N \leq T \). In other words, the agent is restricted to making the investment decision on days \( t_i \). In the interim time, the agent can acquire more information to improve her volume reserve estimates.

For notational convenience we define the deflated value process

\[ \overline{\ell}^{(j)}(t, x) := e^{-rt} \ell^{(j)}(t, x), \]  

(13)

and observe that in between the investment dates, the deflated value processes \( \overline{\ell}^{(j)}(t, x) \) for \( j = 1, \ldots, m \), are martingales. In addition, since in between the investment dates there is no opportunity to exercise the option, \( \overline{\ell}^{(j)}(t, x) \) is the same as a European claim with payoff equal to the value at the next exercise date. Thus,

\[ \overline{\ell}^{(j)}(t_i, x) = \max \left( \lim_{t \downarrow t_i} \overline{\ell}^{(j)}(t, x); e^{-r t_i} \left( p^{(j)}(t, x) - I^{(j)} \right) \right), \]  

(14)

where \( p^{(j)}(t, x) \) is as in (10), and recall that \( j = 1, \ldots, m \) represents the state of the regime.

Finally, in the interval \( t \in (t_i, t_{i+1}] \) the processes \( \overline{\ell}^{(j)}(t, x) \) satisfy the coupled system of PDEs

\[ (\partial_t + \mathcal{L}) \overline{\ell}^{(j)}(t, x) + h_t \sum_{j=1}^{m} A_{jk} \overline{\ell}^{(k)}(t, x) = 0, \quad t \in (t_i, t_{i+1}], \]  

(15)

where \( \mathcal{L} = -\kappa x \partial_x + \frac{1}{2} \sigma^2 \partial_{xx} \) is the infinitesimal generator of the process \( X_t \).

The maximization in (14) represents the agent’s option to hold on to the investment option at time \( t_i \) or to invest immediately. If the second argument attains the maximum, then the agent exercises her option to invest in the reserve, at a cost of \( I^{(j)} \), and receives the expected discounted value of the cash-flow \( p^{(j)}(t, x) \), which results from extracting and selling the commodity on the spot market. This investment decision is tied to the reserve volume estimate through the regime \( j \). Different regimes \( j \) will result in different exercise policies and we explore this relationship in the next section.

Motivated by the work of Jaimungal and Surkov (2011), who study options on multiple commodities driven by Lévy processes, we solve the system of PDEs (15) recursively by employing the Fourier transform of \( \overline{\ell}^{(j)}(t, x) \) with respect to \( x \), which we denote by \( \tilde{\ell}^{(j)}(t, \omega) \).
Specifically, we write
\[
\bar{\ell}^{(j)}(t, \omega) = \int_{-\infty}^{\infty} e^{-i \omega x} \bar{\ell}^{(j)}(t, x) \, dx, \quad \text{and} \quad \bar{\ell}^{(j)}(t, x) = \int_{-\infty}^{\infty} e^{i \omega x} \tilde{\ell}^{(j)}(t, \omega) \frac{d\omega}{2\pi},
\]
where \( i = \sqrt{-1} \). Applying the Fourier transform to (15), we obtain a new PDE, without the parabolic term, which depends on the state variable \( \omega \) rather than the state variable \( x \), i.e.:

\[
[\partial_t + (\kappa - \frac{1}{2} \sigma^2 \omega^2) + \kappa \omega \partial_\omega] \bar{\ell}^{(j)}(t, \omega) + h_t \sum_{j=1}^m A_{jk} \bar{\ell}^{(k)}(t, \omega) = 0. \tag{17}
\]

Within the interval \((t_k, t_{k+1}]\), we introduce a moving coordinate system and write \( \hat{\ell}^{(j)}(t, \omega) = \tilde{\ell}^{(j)}(t, e^{-\kappa(t_{k+1}-t)} \omega) \), which removes the derivative in \( \omega \) and we find that the functions \( \hat{\ell}^{(j)} \) satisfy the coupled linear system of ODEs

\[
\partial_t \hat{\ell}^{(j)}(t, \omega) + (\kappa - \frac{1}{2} \sigma^2 \omega^2 e^{-2\kappa(t_{k+1}-t)}) \hat{\ell}^{(j)}(t, \omega) + h_t \sum_{j=1}^m A_{jk} \hat{\ell}^{(k)}(t, \omega) = 0. \tag{18}
\]

By writing \( A = UDU^{-1} \) where \( U \) is the matrix of eigenvectors of \( A \), and \( D \) the diagonal matrix of eigenvalues of \( A \), the above coupled system of ODEs can be recast as independent ODEs, which in vector form reads

\[
\partial_t \left(U^{-1} \hat{\ell}(t, \omega)\right) + \left(\psi(\omega e^{-\kappa(t_{k+1}-t)}) \mathbb{I} + h_t D\right) U^{-1} \hat{\ell}(t, \omega) = 0, \tag{19}
\]

where \( \hat{\ell}(t, \omega) = (\hat{\ell}^{(1)}(t, \omega), \ldots, \hat{\ell}^{(n)}(t, \omega))^\prime, \psi(\omega) = \kappa - \frac{1}{2} \sigma^2 \omega^2 \) and \( \mathbb{I} \) is the \( n \times n \) identity matrix. These uncoupled ODEs have solution

\[
U^{-1} \hat{\ell}(t^+_{k+1}, \omega) = \exp\left\{ \int_{t_k}^{t_{k+1}} \psi(\omega e^{-\kappa(s-t)} \omega) \, ds \, \mathbb{I} + \int_{t_k}^{t_{k+1}} h_s \, ds \, D \right\} U^{-1} \hat{\ell}(t_{k+1}, \omega), \tag{20}
\]

where \( \hat{\ell}(t_{k+1}, \omega) = \lim_{t\downarrow t_k} \hat{\ell}(t, \omega) \).

Next, we left-multiply by \( U \) to obtain

\[
\hat{\ell}(t^+_{k}, \omega) = \exp\left\{ \int_{t_k}^{t_{k+1}} \psi(\omega e^{-\kappa(s-t)} \omega) \, ds \right\} \exp\left\{ \int_{t_k}^{t_{k+1}} h_s \, ds \, A \right\} \hat{\ell}(t_{k+1}, \omega), \tag{21}
\]

and writing this expression in terms of the original coordinate system. Thus, the Fourier transform of the deflated value of the option to irreversibly invest is

\[
\bar{\ell}(t^+_{k}, \omega) = \exp\left\{ \int_{0}^{t_{k+1}-t_k} \psi(\omega e^{\kappa s}) \, ds \right\} \exp\left\{ \int_{t_k}^{t_{k+1}} h_s \, ds \, A \right\} \tilde{\ell}(t_{k+1}, \omega e^{\kappa(t_{k+1}-t_k)}). \tag{22}
\]

This result has a few interesting features. The first is that the role of mean-reversion decouples from the Markov chain driving the volume estimates. The second is that the value
at time $t_k^+$ at frequency $\omega$ depends on the value at time $t_{k+1}$ at frequency $\omega \ e^{\kappa(t_{k+1}-t_k)}$. This requires an extrapolation in the frequency space as the algorithm to calculate the option value steps backward in time. When we discretize the state space, such extrapolations could lead to inaccurate results since the edges of the state space are the most important contributions to the extrapolated values. Instead, we make use of the inverse relationship between frequencies and real space in Fourier transforms

$$\int_{-\infty}^{\infty} e^{i\omega x} g(x/a) \, dx = \int_{-\infty}^{\infty} e^{(i\omega x) x} g(x) \, dx/a = \frac{1}{a} \tilde{g}(a \omega),$$

to write

$$\tilde{\ell}(t_k^+, \omega) = \exp \left\{ \int_0^{t_{k+1} - t_k} \psi(\omega e^{\kappa s}) \, ds \right\} \exp \left\{ \int_{t_k}^{t_{k+1}} h_s \, ds \, A \right\} \tilde{\ell}(t_{k+1}, \omega),$$

where $\tilde{\ell}(t_{k+1}, x) = \ell(t_{k+1}, xe^{-\kappa(t_{k+1}-t_k)})$ and $\tilde{\ell}$ denotes the Fourier transform of $\ell$. Thus the value of the right-limit of the real option to invest at $t_k^+$ is determined in terms of interpolation in $x$ (rather than extrapolation in $\omega$). In Figure 2 we summarize the approach for valuing the real option to irreversibly invest in the reserve.

1. Set the real and frequency space grids
   $x = [-x : \Delta x : x], \ \tilde{x} = e^{-\kappa \Delta t} x$ and $\omega = [0 : \Delta \omega : \omega]$.
2. Place terminal conditions: $\ell(t_n, x) = e^{-\rho t_n}(P(t_n, x) - I)_+.$
3. Set $k = n$.
4. Step backwards from $t_{k+1}$ to $t_k$:
   (a) $\tilde{\ell}_{t_{k+1}} = \text{interp}(x, \ell_{t_{k+1}}, \tilde{x})$
   (b) $\tilde{\ell}_{t_{k+1}}(x) = F[\ell_{t_{k+1}}(x)]$
   (c) $\tilde{\ell}_{t_k}(x) = F^{-1} \left[ e^{t_{k+1} - t_k} \psi(\omega e^{\kappa s}) \, ds \right] e^{t_{t_k+1}} h_s \, ds \, A \tilde{\ell}_{t_{k+1}}(\omega)$
   (d) $\tilde{\ell}_{t_k} = \max (\tilde{\ell}_{t_k}(x) ; e^{-\rho t_k}(P(t_k, x) - I)_+)$
5. Set $k \rightarrow k - 1$, if $k \geq 0$ goto step 4.

Figure 2: Algorithm for computing the value of the option to irreversibly invest in the reserve.

4. Calibration and learning

Armed with the model in Section 2, and the valuation procedure developed in Section 3, we discuss in detail the investor’s learning function $h_t$, see (4). First, we need an approach for calibrating the model to estimates of the reserve volume, and that is the purpose of this section.
At time $t = 0$ the agent has an estimate of the expected reserve volume, denoted by $\mu$, and the volatility of the estimate, denoted by $\sigma$. Thus

$$\mu = \mathbb{E}[V_\infty | F_0] \quad \text{and} \quad \sigma^2_0 = \mathbb{E}[(V_\infty)^2 | F_0] - (\mathbb{E}[V_\infty | F_0])^2. \quad (23)$$

As time passes, the agent gathers more and better quality information about the volumes of the commodity in the reserve. Thus, the variance of the estimated volume in the reserve decreases from $\sigma^2_0$ to $\sigma^2_T < \sigma^2_0$ by time $T$. Specifically

$$\sigma^2_T = \mathbb{E}[(V_\infty)^2 | F_T] - (\mathbb{E}[V_\infty | F_T])^2. \quad (24)$$

The first step in the calibration procedure is to select the states of reserve volume conditional on the Markov chain state, i.e. to select $v^{(1)}, \ldots, v^{(m)}$. Since the $t = 0$ estimate of the reserve volume represents an unbiased estimator of the reserves, the Markov chain should be symmetric around the current estimate of the reserve volume. To ensure this symmetry, we assume that (i) the cardinality of the states of the Markov chain $Z_t$ is odd, i.e., $m = 2L + 1$ for some positive integer $L$; (ii) $v^{(L+1)} = \mu$ where $\mu$ is the current estimate of the reserve volume, (iii) $v^{(k)}$ is increasing in $k$ and (iv)

$$v^{(L+1+i)} - \mu = \mu - v^{(L+1-i)}, \quad \forall \; i = 1, \ldots, L. \quad (25)$$

We further assume that the agent’s estimate of the volume is Normally distributed (this assumption can be modified to any distribution the agent considers to represent their prior knowledge)

$$V_\infty | F_0 \sim \mathcal{N}(\mu, \sigma^2_0), \quad (26)$$

Therefore, we choose $v^{(k)}$ to be equally spaced over $n\sigma$ of the normal random variable support, i.e., we select

$$v^{(k)} = \mu - n \sigma_0 + (k-1) \frac{2n \sigma_0}{m}, \quad \forall \; k = 1, \ldots, 2m + 1. \quad (27)$$

Placing symmetry on the states of the reserve volume estimator is not sufficient to ensure symmetry in its distribution. We further require the symmetry in the base generator rate matrix $A$, and assume that

\begin{align}
A^\lambda_{1,1} &= -\lambda_1, \quad &A^\lambda_{1,2} &= \lambda_1, \quad (28a) \\
A^\lambda_{i-1,i} &= 1 - \lambda_i, \quad &A^\lambda_{i,i} &= -1, \quad &A^\lambda_{i,i+1} &= \lambda_i, \quad \forall \; i = 1, \ldots, L, \quad (28b) \\
A^\lambda_{i-1,i} &= \lambda_{2L-i}, \quad &A^\lambda_{i,i} &= -1, \quad &A^\lambda_{i,i+1} &= 1 - \lambda_{2L-i}, \quad \forall \; i = L + 1, \ldots, 2L, \quad (28c) \\
A^\lambda_{2L+1,2L} &= \lambda_1, \quad &A^\lambda_{2L+1,2L+1} &= -\lambda_1, \quad (28d)
\end{align}

\footnote{In principle, we could develop a model that can calibrate to a sequence of times and variances, however, the one step reduction is enough to illustrate the essential ideas in this reduction.}
for some set of $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_L\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_L > 0$. This ensures that the Markov chain generates an invariant distribution equal to a discrete approximation of the original estimate of the reserve volume distribution. The form of $A^{\lambda}$ ensures transitions only occur between neighboring states and ensures symmetry in the transition rates of states that are across the mean estimate. The parameters $\lambda$ are calibrated so the invariant distribution of $V_\infty$ without learning coincides with a discrete approximation of a normal random variable with mean $\mu$ and variance $\sigma_0^2$. Formally, let $P^{\lambda} = e^{A^{\lambda}}$ denote the transition probability after one unit of time, and let $\pi^{\lambda}$ denote the invariant distribution of $P$, i.e., $\pi^{\lambda}$ solves the eigenproblem $P^{\lambda} \pi^{\lambda} = \pi^{\lambda}$. Then, we choose $\lambda$ such that

$$\pi^{\lambda}_1 = \Phi_{\mu,\sigma_0} \left( \frac{1}{2} (v^{(1)} + v^{(2)}) \right),$$

$$\pi^{\lambda}_2 = \Phi_{\mu,\sigma_0} \left( \frac{1}{2} (v^{(i+1)} + v^{(i)}) \right) - \Phi \left( \frac{1}{2} (v^{(i)} + v^{(i-1)}) \right), \quad i = 2, \ldots, 2L,$$

$$\pi^{\lambda}_m = 1 - \Phi_{\mu,\sigma_0} \left( \frac{1}{2} (v^{(m)} + v^{(m-1)}) \right),$$

where $\Phi_{\mu,\sigma_0}(\cdot)$ denotes the normal cumulative density of a normal with mean $\mu$ and variance $\sigma_0^2$.

Next, for parsimony, we assume that the agent’s learning function is of the form

$$h_t = ae^{-bt} \quad \text{for some } a, b > 0,$$

where the parameter $a$ represents the initial transition rates, i.e. $h_0 = a$, between the states of the Markov chain, and hence reflects the uncertainty in the initial estimates of reserves. The learning parameter $b$ represents the rate at which the agent learns – the larger (resp. smaller) is $b$ the quicker is the learning process because large (resp. small) values of $b$ make the transition rates decay faster (slower) through time and therefore reserve estimates become stable quickly (resp. slowly). Recall that the learning rate function plays a key role in the generator matrix of the Markov chain, see (4), in that it captures how the agent learns how much volume of the commodity is in the reserve.

The parameters in the learning rate function $h$ are calibrated to obey the constraints (23) and (24). Due to the symmetry in the base transition rates $A$, and the symmetry in the reserve volume states $v^{(k)}$, we automatically satisfy the mean constraints

$$\mathbb{E}[V_\infty \mid V_0 = \mu] = \mu, \quad \text{and} \quad \mathbb{E}[V_\infty \mid V_T = \mu] = \mu.$$  \hspace{1cm} (30)

To satisfy the variance constraints we require the transition probabilities of the Markov chain from an arbitrary state at time $t$ to its infinite horizon state, which we denote $p_{t,ij} := P(Z_\infty = j \mid Z_t = i)$, to be

$$p_{t,ij} = \left[ \exp \left\{ \int_t^\infty h_u du \right\} A \right]_{ij} = \left[ \exp \left\{ \frac{a}{b} e^{-bt} A \right\} \right]_{ij}.$$  \hspace{1cm} (31)

Note that the right-hand side of the equation above is a matrix exponential and, as before, the notation $[\cdot]_{ij}$ denotes the $ij$ element of the matrix in the square brackets. Now we must
solve the two coupled system of non-linear equations for the parameters $a$ and $b$:

$$
V_0 [V_\infty | V_0 = \mu] = \sum_{k=1}^{2L+1} v^{(k)} (p_{0,Lk} - \mu)^2 = \sigma_0^2, \tag{32a}
$$

$$
V_T [V_\infty | V_T = \mu] = \sum_{k=1}^{2L+1} v^{(k)} (p_{T,Lk} - \mu)^2 = \sigma_T^2. \tag{32b}
$$

After which, all technical uncertainty model parameters are calibrated to the distributional properties of the initial reserve volume estimates and the reduction in variance as a result of learning.

The value of the irreversible option to invest in the reserve with learning can now be valued using the approach in Section 3, a summary of which is presented in the algorithm shown in Figure 2. The value of exploration, which improves the variance of the estimators of the reserve volume, can be assessed by considering the option value when there is no learning and comparing it to the option value without learning.

5. Numerical Results

In this section we investigate the optimal exercise policies of the agent and assess the value of learning. Throughout we use the following parameters and modelling choices:

- **Investment costs.** In the state where the volume of the commodity in the reserve is highest, the cost is 75% of the value of extracting and selling the commodity in that state. Similarly, in the state with the lowest reserve value, the investment cost is 50% of the value of extracting the commodity and selling it in the spot market in that state. The cost of the inner states are obtained as a straight line between the low and high states.

- **Expiry of option.** $T = 5$ years which consists of 255 weeks.

- **Model parameters**
  - The reserve volume parameters are: $m = 31$ states, $\mu = 10^7$, $\sigma_0^2 = 5 \times 10^6$, $\sigma_T^2 = 2 \times 10^6$.
  - Underlying resource model parameters are: $\kappa = 0.5$, $\theta = \log(100)$, $\sigma_X = 0.5$.
  - Discount rate $\rho = 0.05$.
  - Extraction rate parameters are: $\alpha = 1$, $\beta = 0.05$, $\gamma = 0.9$, $\epsilon = 2$. 


Finally, we compare the agent’s optimal policies when her learning rate is high and low. Specifically, fast learning is driven by the parameter choice $a_1 = 33.06, b_1 = 0.83$, and the parameters for slow learning are $a_2 = 0.04, b_2 = 10^{-3}$.

Recall that $m = 1$ is the state with the lowest volume estimate, and the estimates increase monotonically in $m$. Moreover, the middle state $m = 16$ of the Markov chain maps to the initial reserve estimate of $\mu$.

Figure 3 shows the optimal exercise boundary for an agent who learns at a low rate and for different volume estimates (i.e. different states of the Markov chain). The $y$-axis of the figure shows the spot price of the commodity, and the $x$-axis is the time elapsed measured in weeks. The figure shows that as the agent’s volume estimate increases, the exercise boundary shifts down because a larger reserve requires a lower commodity spot price to justify the investment.

Figure 4 shows the exercise boundaries, for different volume states, when the agent’s learning rate is high. Observe that the exercise boundaries do not exhibit the same simple nearly parallel shifts as the non-learning cases as a function of the volume states. The exercise boundaries shift downwards as we move to higher volume states of the reserve, and the shift is more pronounced for low volume states when the option is far away from expiry.

For instance, state $m = 5$ is one in which the volume estimate is low and the figure shows that far away from expiry the agent sets a high investment threshold for two reasons. First, low volume requires a high commodity spot price to justify the investment. Second, far away from expiry the investor attaches high value to waiting and learning about the volume estimates
of the reserve. Clearly, this option to wait-and-learn is very valuable at the beginning and it gradually decreases as expiry approaches because the agent has little time left to learn, so investment becomes optimal so long as costs are covered.

This last point is again depicted in Figures 3 and 4 where we clearly see that the agent with the high learning rate is more sensitive to the state of the volume estimate. When the expiry of the option is far away from expiry, and the volume estimate is low, the value attached to waiting and gathering more information is higher for the agent who can learn about the volume of the reserves than when the learning rate is low. Note that if the agent’s learning rate is low, waiting will not improve her view of the potential gains from exploiting the reserve, thus the exercise boundary is low and does not vary much, as expiry is approached, as that of the fast learner.

In Figures 5, 6, 7 we compare the exercise boundaries that trigger investment when the agent’s learning rate is high and low, for three different volume estimates in states $m = \{5, 16, 25\}$. The figures show that for low volume estimate the exercise boundary for the quick learner is always above that of the slow learner.

When the state is $m = 5$ the boundary for the quick learner is always above that of the slow learner. The distance between the fast and slow learner’s exercise boundary decreases as the option approaches expiry, which results from the learner’s exercise boundary quickly decaying as the value attached to learning diminishes as expiry approaches.

When the state is $m = 16$ the boundary for the quick learner starts above that of the slow
Figure 5: State $m = 5$, i.e. low volume estimate. Comparison of exercise boundaries for low and high learning rates.

Figure 6: State $m = 16$, i.e. low volume estimate. Comparison of exercise boundaries for low and high learning rates.
learner, then crosses it and past week 70 (approx) stays below. At the other extreme, when the volume state is high, i.e. \( m = 25 \), the exercise boundary of the high rate learner starts high, then dips, and then increases before it decreases as time to expiry approaches, and it always lies below the slow rate learner’s exercise boundary.

6. Conclusions

In this paper we show how to incorporate technical uncertainty into the decision to invest in a commodity reserve. This uncertainty stems from not knowing the volume of the commodity stored in the reserve, and compounds with the uncertainty of the value of the reserve because future spot prices are unknown.

The agent has the option to wait-and-see before making the irreversible investment to exploit the commodity reserve. In our model, as time goes by, the agent learns about the volume of the commodity stored in the reserve, so the option to delay investment is valuable because it allows the agent to learn and to wait for the optimal market conditions (i.e. spot price of the commodity) before sinking the investment.

We adopt a continuous-time Markov chain to model the reserve volume and the technical uncertainty. In our model the agent learns about the volume in the reserve as time goes by and the accuracy of the estimates is also improved with time. We show how to calculate the value of the option to delay investment and discuss the agent’s optimal investment threshold.
We show how the exercise boundary depends on the agent’s estimate of the volume (which depends on the state of the Markov chain) and how this boundary depends on: the rate at which she refines her estimates of the reserve, and the uncertainty of the price of the commodity. For example, we show that when the option to invest is far away from expiry, and the volume estimate is low, the value attached to waiting and gathering more information is higher for the agent who can quickly learn about the volume of the reserves than for an agent who learns at a very low rate.
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