GENERALIZED HARNACK’S INEQUALITY FOR NONHOMOGENEOUS ELLIPTIC EQUATIONS

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Abstract. This paper is concerned with nonlinear elliptic equations in nondivergence form
\[ F(D^2u, Du, x) = 0 \]
where \( F \) has a drift term which is not Lipschitz continuous. Under this condition the equations are nonhomogeneous and nonnegative solutions do not satisfy the classical Harnack’s inequality. This paper presents a new type of generalization of the classical Harnack’s inequality for such equations. As a corollary we obtain the optimal Harnack type of estimate for \( p(x) \)-harmonic functions which quantifies the strong minimum principle.

1. Introduction

The famous Krylov-Safonov theorem [12], [13] states that a nonnegative solution \( u \in C(B_{2R}(x_0)) \) of a linear, uniformly elliptic equation
\[ \text{Tr}(A(x)D^2u) = 0 \]
with measurable and bounded coefficients satisfies the Harnack’s inequality
\[ \sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u \]
where \( C \) is a universal constant. This result is important since it quantifies the strong minimum principle and gives Hölder estimate. There are numerous generalizations of the Krylov-Safonov theorem e.g. by Trudinger [15],[8] for quasilinear operators and by Caffarelli [5],[6] for fully nonlinear operators.

In this paper we study nondivergence form elliptic equations
\[ F(D^2u, Du, x) = 0. \]
The operator \( F \) is assumed to be elliptic in the sense that there are \( 0 < \lambda \leq \Lambda \) such that
\[ \lambda \text{Tr}(Y) \leq F(X, p, x) - F(X + Y, p, x) \leq \Lambda \text{Tr}(Y) \]
for every symmetric matrices \( X, Y \) where \( Y \) is positive semidefinite, and for every \( x \in B_{2R}(x_0) \) and \( p \in \mathbb{R}^n \). We assume that \( F \) has a drift term which has modulus of continuity and asymptotic behaviour given by function \( \phi : [0, \infty) \to [0, \infty) \), i.e.,
\[ |F(0, p, x)| \leq \phi(|p|) \]
for every \( x \in B_{2R}(x_0) \) and \( p \in \mathbb{R}^n \). This condition implies that constants are solutions, i.e., \( F(0, 0, x) = 0 \) for every \( x \). Since the second order term is 1-homogeneous and the first order drift term is not, the equations are nonhomogeneous and it is known that nonnegative solutions do not satisfy (1.1) with a uniform constant \( C \). In this paper we introduce a new type of Harnack’s inequality for such equations. This inequality is a natural generalization of (1.1) since it quantifies the strong minimum principle (whenever it is true) in a precise way, and the constant in the inequality is independent of the solution itself. We make some assumptions on the asymptotic behaviour of \( \phi \). Under these conditions the inequality also gives Hölder continuity estimate for the solutions.

The quantification of the strong minimum principle turns out to be a rather delicate issue. A naive example shows that when \( \phi \) is merely Hölder continuous, i.e., \( \phi(t) = t^\alpha \) for \( t \in [0,1] \) and
\( \alpha < 1 \), the strong minimum principle is not true. In fact, the strong minimum principle can only be true if \( \phi \) satisfies the so called Osgood condition
\[
(1.2) \quad \int_0^\varepsilon \frac{1}{\phi(t)} \, dt = \infty \quad \text{for every } \varepsilon > 0.
\]
Indeed, this can be easily seen by defining \( v : (-2R, 2R) \to [0, \infty) \) such that \( v(x) = 0 \) for \( x \leq 0 \) and
\[
x = \int_0^{v(x)} \frac{dt}{\phi(t)} \quad \text{for } x \geq 0.
\]
The function \( u(x) = \int_0^x v(t) \, dt \) satisfies \( u'' = \phi(u') \) and violates the strong minimum principle. Hence, if we require the generalized Harnack’s inequality to quantify the strong minimum principle, we have to take into account the Osgood condition (1.2).

Let us now state precisely our main result. Following the idea of Caffarelli [5] we replace the equation \( F(D^2u, Du, x) = 0 \) by two inequalities which follow from the ellipticity condition and the modulus of continuity of the drift term. In other words we assume that \( u \in C(B_{2R}(x_0)) \) is a viscosity supersolution of
\[
(1.3) \quad \mathcal{P}_{\lambda, \Lambda}^+(D^2u) \geq -\phi(|Du|)
\]
and a viscosity subsolution of
\[
(1.4) \quad \mathcal{P}_{\lambda, \Lambda}^-(D^2u) \leq \phi(|Du|)
\]
in \( B_{2R}(x_0) \). Here \( \mathcal{P}_{\lambda, \Lambda}^+, \mathcal{P}_{\lambda, \Lambda}^- \) are the usual Pucci operators defined in the next section. The function \( \phi : [0, \infty) \to [0, \infty) \) is assumed to be of the form \( \phi(t) = \eta(t)t \) and to satisfy the following conditions:

(P1) \( \phi : [0, \infty) \to [0, \infty) \) is increasing, locally Lipschitz continuous in \( (0, \infty) \) and \( \phi(t) \geq t \) for every \( t \geq 0 \). Moreover, \( \eta : (0, \infty) \to \mathbb{R} \) is nonincreasing on \( (0, 1) \) and nondecreasing on \( [1, \infty) \).

(P2) \( \eta \) satisfies
\[
\lim_{t \to \infty} \frac{t\eta'(t)}{\eta(t) \log(\eta(t))} = 0.
\]

(P3) There is a constant \( \Lambda_0 \) such that
\[
\eta(st) \leq \Lambda_0 \eta(s)\eta(t).
\]
for every \( s, t \in (0, \infty) \).

Roughly speaking we assume that \( \eta \) is slowly increasing function. We say that a constant is universal if it depends only on \( \lambda, \Lambda, \phi \) and the dimension of the space. Note that \( \phi(t) = t \) leads to the homogeneous case and it is well known that a nonnegative \( u \) which is a supersolution of (1.3) and a subsolution of (1.4) satisfies (1.1), [14].

Our main result is the generalization of the Harnack’s inequality.

**Theorem 1.1.** Let a nonnegative function \( u \in C(B_{2R}(x_0)) \), for \( R \leq 1 \), be a viscosity supersolution of (1.3) and a viscosity subsolution of (1.4) in \( B_{2R}(x_0) \). Denote \( m := \inf_{B_R(x_0)} u \) and \( M := \sup_{B_R(x_0)} u \). There is a universal constant \( C \) such that
\[
(1.5) \quad \int_m^M \frac{dt}{R^2 \phi(t/R) + t} \leq C.
\]

A couple of remarks are in order. If \( \phi \) satisfies (1.2), the inequality (1.5) quantifies the strong minimum principle. A naive example indicates that this estimate is sharp (see Section 3). On the other hand if \( \phi(t) = t \), (1.5) reduces to (1.1). The fact that the inequality (1.5) depends on the radius is clear from the following scaling argument. If \( u \in C(B_{2R}(x_0)) \) satisfies (1.3) and (1.4) in \( B_{2R}(x_0) \), then the rescaled function \( u_R(x) := u(Rx) \) satisfies the same inequalities in \( B_{2R}(x_0) \) with \( \phi_R(t) = R^2 \phi(t/R) \) instead of \( \phi \) on the right hand side. In the next section we show that the condition (P2) implies that \( R^2 \phi(t/R) \to 0 \) locally uniformly in \( [0, \infty) \) as \( R \to 0 \). Therefore (1.5) asymptotically converges to (1.1) as the radius approaches to zero. This observation leads to the following Hölder estimate.
Corollary 1.2. Let $u \in C(\Omega)$ be a viscosity supersolution of (1.3) and a viscosity subsolution of (1.4) in $\Omega$. Then $u$ is locally $\alpha$-H"older continuous with a uniform $\alpha \in (0,1)$, and for every ball $B_{R_0}(x) \subset \subset \Omega$ and $R \leq R_0$ we have

$$\text{osc}_{B_R(x)} u \leq C \left( \frac{R}{R_0} \right)^\alpha$$

where the constant $C$ depends on $\sup_{B_{R_0}(x)} |u|$.

I need to assume that $\phi$ is of the form $\phi(t) = \eta(t)t$ and $\eta$ satisfies (P2), which implies that $\eta$ is slowly increasing function. However, I see no reason why Theorem 1.1 could not be true for any increasing and continuous function $\phi$. On the other hand, (P2) is only a condition of the asymptotic behaviour at the infinity. It is not a condition for small values of $\phi$ and thus plays no role in the strong minimum principle.

As a further application of Theorem 1.1 we obtain the sharp Harnack’s type of inequality for so called $p(x)$-harmonic functions. These are local minimizers of the energy

$$\int_{\Omega} \frac{1}{p(x)} |Du|^{p(x)} \, dx,$$

where $1 < p(x) < \infty$, and therefore they are weak solutions of the Euler-Lagrange equation

$$(1.6) \quad - \text{div} \left( |Du(x)|^{p(x)-2} Du(x) \right) = 0.$$  

This problem was first considered by Zhikov [17] and has recently received a lot of attention. The regularity is rather well understood [1] and the strong minimum principle was obtained in [7], see also [9]. However, finding the sharp Harnack’s inequality which quantifies the strong minimum principle has been an open problem [2, 10, 16].

We assume that $p \in C^1(\mathbb{R}^n)$ and that there are numbers $1 < p^- \leq p^+ < \infty$ such that $p^- \leq p(x) \leq p^+$ for every $x \in \mathbb{R}^n$. In [11] it was shown that the weak solutions of (1.6) coincide with the viscosity solutions of the same equation which can be written in nondivergence form as

$$(1.7) \quad - \Delta u -(p(x)-2)\Delta_{\infty} u - \log |Du| |Dp(x), Du| = 0$$

at least when $Du \neq 0$. Here $\Delta_{\infty} u = \langle D^2u, D_u |Du| \rangle$ denotes the infinity Laplace operator. It is not difficult to show that if $u$ is a viscosity solution of (1.7), then it is a viscosity supersolution of (1.3) and a viscosity subsolution of (1.4) for $\lambda = \min\{1, p^- - 1\}$, $\Lambda = \max\{1, p^+ - 1\}$ and for

$$\phi(t) = C(|\log t| + 1)t,$$

where $C$ is the $C^1$-norm of $p(\cdot)$ (see Lemma 5.2). Note that the above function satisfies the Osgood condition (1.2). Hence we have the following corollary.

Corollary 1.3. Let $p \in C^1(\mathbb{R}^n)$ be a function such that $1 < p^- \leq p(x) \leq p^+ < \infty$. Let $u \in C(B_{2r}(x_0))$ be a nonnegative $p(x)$-harmonic function in $B_{2R}(x_0)$ for $R \leq 1$. Denote $m := \inf_{B_{2r}(x_0)} u$ and $M := \sup_{B_{2r}(x_0)} u$. Then the following Harnack’s inequality holds

$$\int_m^M \frac{dt}{(\log t + 1)t} \leq C,$$

where the constant $C$ depends on the dimension, on $L^\infty$-norm of $\nabla p$ and on the numbers $p^-, p^+$.  

The previous estimate can be written more explicitly as

$$\min\{M, M^{1+CR}\} \leq C \max\{m, m^{1+CR}\}$$

by possibly enlarging the constant $C$. At the end of Section 5 we show that this result is optimal and thus solves the problem of finding the optimal generalization of the Harnack’s inequality for $p(x)$-harmonic functions. Note that we may relax the assumption of $p \in C^1(\mathbb{R}^n)$ to $p$ being merely Lipschitz continuous by a standard approximation argument.

The paper is organized as follows. In the next section we recall some standard definitions and results. In the third section we prove Theorem 1.1 in dimension one. This easy proof clarifies where the estimate in the theorem comes from. Section 4 is devoted to the proof of the main result.
The main difficulties are, of course, due to the fact that the equation is not scaling invariant. The idea is to replace the Calderón-Zygmund decomposition with a more refined scaling and covering argument where we take into account the scaling of the equation. This leads to estimate the decay of the level sets, which turns out to be far more involved than in the homogeneous case. In the last section we show how Corollary 1.3 follows from Theorem 1.1.

2. Preliminaries

Let $X \in \mathbb{S}^{n \times n}$ be a symmetric $n$-by-$n$ matrix with eigenvalues $e_1, e_2, \ldots, e_n$. The Pucci extremal operators $\mathcal{P}_{\lambda, \Lambda}^+$ and $\mathcal{P}_{\lambda, \Lambda}^-$ with ellipticity constants $0 < \lambda \leq \Lambda$ are defined by

$$\mathcal{P}_{\lambda, \Lambda}^+(X) := -\lambda \sum_{e_i \geq 0} e_i - \Lambda \sum_{e_i < 0} e_i$$

and

$$\mathcal{P}_{\lambda, \Lambda}^-(X) := -\Lambda \sum_{e_i \geq 0} e_i - \lambda \sum_{e_i < 0} e_i.$$ 

For elementary properties of the Pucci operators see [6].

We recall the definitions of a viscosity supersolution of (1.3) and a viscosity subsolution of (1.4) .

**Definition 2.1.** A function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of (1.3) in $\Omega$ if it is lower semicontinuous and the following holds: if $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ is such that $\varphi \leq u$ and $\varphi(x_0) = u(x_0)$ then

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2 \varphi(x_0)) \geq -\phi(|D\varphi(x_0)|).$$

A function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (1.4) in $\Omega$ if it is upper semicontinuous and the following holds: if $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ is such that $\varphi \geq u$ and $\varphi(x_0) = u(x_0)$ then

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2 \varphi(x_0)) \leq \phi(|D\varphi(x_0)|).$$

Finally we recall the definition of slowly increasing function. We will not need this property but it is important to connect the condition (P2) to the theory of regularly varying functions.

**Definition 2.2.** A locally Lipschitz continuous $f : [a, \infty) \to \mathbb{R}$ is slowly increasing function if

$$\lim_{t \to \infty} \frac{t f'(t)}{f(t)} = 0.$$ 

The condition (P2) implies that $\eta$ is slowly increasing. In the next proposition we use this condition to study the asymptotic behaviour of $\eta$. For more about the subject of regular variation see [4].

**Proposition 2.3.** Let $\eta$ satisfy the conditions (P1)-(P3). Then it holds:

(i) For every $c > 0$ we have

$$\lim_{t \to \infty} \frac{\eta(ct)}{\eta(t)} = 1.$$ 

(ii) For every $\gamma > 0$ we have

$$\lim_{t \to \infty} \frac{\eta(t)}{t^\gamma} = 0.$$ 

(iii) There is a constant $\Lambda_1$ such that for every $t > 0$ it holds

$$\eta(\eta(t)t) \leq \Lambda_1 \eta(t).$$ 

(iv) There is a constant $\Lambda_2$ such that for every $t > 0$ and $0 < r < s$ it holds

$$r \eta(t/r) \leq \Lambda_2 s \eta(t/s).$$ 

**Proof.** The property (i) is called slowly varying property and it follows from the Representation Theorem ([4, Theorem 1.3.1]), since $\eta$ is slowly increasing. The proof of (ii) is elementary (see [4, Proposition 1.3.6]).

It follows from the assumption (P2) that

$$\lim_{t \to \infty} \frac{\eta(\eta(t)t)}{\eta(t)} = 1,$$ 

see ([4, Proposition 2.3.2]) and ([4, Theorem 2.3.3]). This implies (iii) for every $t \geq 1$. Note that $\phi(t) \geq t$ yields $\eta(t) \geq 1$. Therefore we obtain (iii) for every $t \in (0,1)$ by the monotonicity condition (P1).

We are left with (iv). By changing $t/s \mapsto t$ and $r/s \mapsto r$ the claim is equivalent to
\[ r\eta(t/r) \leq \Lambda_2 \eta(t) \]
for every $t > 0$ and $r < 1$. The part (ii) yields $\frac{\eta(1/r)}{r} \leq C$ for some constant $C$. The condition (P3) then gives
\[ r\eta(t/r) \leq \Lambda_0 \frac{\eta(1/r)}{1/r} \eta(t) \leq \Lambda_2 \eta(t). \]

In fact, Proposition 2.3 remains true even without the condition (P3).

3. WARM UP, THE ONE-DIMENSIONAL CASE

In this section we formally prove Theorem 1.1 in dimension one. This is of course far more trivial than the general case, but the proof will give the reader a clear picture of why the result is true and where the estimate comes from. We also construct a naive example which indicates that Theorem 1.1 is sharp.

Proof of Theorem 1.1 in dimension one. Let $R = 1$, $x_0 = 0$ and let $u \in C^{1,1}\text{loc}((-2,2))$ be a non-negative supersolution of (1.3) and subsolution of (1.4) in $(-2,2)$. In one-dimension this simply means that
\[
|u''(x)| \leq \lambda^{-1} \phi(|u'(x)|) \quad \text{a.e. } x \in (-2,2).
\]

Let us recall that the goal is to show
\[
\int_m^M \frac{dt}{\phi(t) + t} \leq C
\]
where
\[
m := \min_{-1 \leq x \leq 1} u(x) \quad \text{and} \quad M := \max_{-1 \leq x \leq 1} u(x).
\]

In order to simplify the proof let us restrict to the case when $u$ is monotone, say nondecreasing. Then we have $m = u(-1)$, $M = u(1)$ and $u(-2) \leq m$. By the mean value theorem there exist $\xi_1, \xi_2 \in (-2,1)$ such that
\[
m - u(-2) = u' (\xi_1) \quad \text{and} \quad M - u(-2) = 3u' (\xi_2).
\]

Note that if $M/6 \leq m$ then the classical Harnack’s inequality holds and the claim follows. Thus we treat the case $M/6 \geq m$. We denote $a := u(-2)$ and use the above estimate to deduce
\[
[m - a, \frac{M - a}{3}] \subset u'((-2,1)).
\]

Note that $M/6 \geq m$ implies that the length of the interval $[m - a, M/3 - a/3]$ is bigger that $M/6$. We use the monotonicity of $\phi$, (3.1) and (3.2) to obtain
\[
\int_m^{M/3} \frac{dt}{\phi(t)} \leq \int_{m-a}^{M-a} \frac{dt}{\phi(t)} \leq \int_{u'((-2,1))}^{1} \frac{dt}{\phi(t)} \leq \int_{-2}^{1} \frac{|u''(x)|}{\phi(u'(x))} dx \leq 3\lambda^{-1}.
\]

Since $M/3 - m \geq M/6$, the monotonicity of $\phi$ and the above estimate yield
\[
\int_{km/6}^{(k+1)m/6} \frac{dt}{\phi(t)} \leq 3\lambda^{-1}
\]
for $k = 2, 3, 4, 5$. This implies the result in the case $R = 1$. The general case $R \leq 1$ follows by a simple scaling argument. □
Let us construct an example which indicates that the result is sharp. To that aim we assume that $\phi$ is of the form $\phi(t) = \eta(t)t$, it is $C^1$-regular, it satisfies the Osgood condition (1.2) and the monotonicity condition (P1), and that the converse of Proposition 2.3 (iii) holds

\begin{equation}
\eta(t) \leq C\eta(\eta(t)t) \quad \text{for } t > 0.
\end{equation}

By the inverse function theorem, for every $k \in \mathbb{N}$, there is a positive increasing function $u_k : (-2, 2) \rightarrow (0, \infty)$ such that $u_k(0) = \frac{1}{k}$ and

\begin{equation}
x = \int_{\frac{1}{k}}^{u_k(x)} \frac{dt}{\phi(t)} \quad x \in (-2, 2).
\end{equation}

I claim that $u_k$ satisfies $|u_k''(x)| \leq C\phi(u_k'(x))$ for $x \in (-2, 2)$.

Indeed, by differentiating (3.4) we obtain

\begin{equation}
\frac{u_k'}{\phi(u_k)} = 1 \quad \text{and} \quad u_k'' = \phi'(u_k)u_k'.
\end{equation}

Because $\eta$ is slowly increasing and nonincreasing in $(0, 1)$ we have $\phi'(t) = \eta'(t)t + \eta(t) \leq C\eta(t)$ for every $t > 0$. The assumption (3.3) and (3.5) imply

$\phi'(u_k) \leq C\eta(u_k) \leq C\eta(\eta(u_k)u_k) = C\eta(\phi(u_k)) = C\eta(u_k)$.

Therefore (3.5) gives $u_k'' \leq C\eta(u_k')u_k' = C\phi(u_k')$. On the other hand, because $u_k$ and $\phi$ are nondecreasing, (3.5) yields $u_k'' \geq 0$. This shows that $u_k$ satisfies

$|u_k''(x)| \leq C\phi(u_k'(x)) \quad x \in (-2, 2)$.

Since $u$ is nondecreasing we have $m = \inf_{-1 < x < 1} u_k(x) = u(-1)$ and $M = \sup_{-1 < x < 1} u_k(x) = u_k(1)$, and therefore (3.4) gives

$$2 = \int_{\frac{1}{k}}^{u_k(1)} \frac{dt}{\phi(t)} - \int_{\frac{1}{k}}^{u_k(-1)} \frac{dt}{\phi(t)} = \int_{m}^{M} \frac{dt}{\phi(t)}.$$ 

This estimate is therefore optimal. In particular, it is not true that the ratio

$$\frac{u_k(1)}{u_k(-1)}$$

is uniformly bounded for $k \in \mathbb{N}$.

For a more concrete example we can choose $\phi(t) = (|\log(t)| + 1)t$. Then $u_k(x) = e^{-e^{x+k}}$ satisfies

$|u_k''(x)| \leq 2\phi(|u_k'(x)|) \quad x \in (-2, 2)$

and the ratio $\frac{u_k(1)}{u_k(-1)}$ is clearly not uniformly bounded. This will be discussed more in Section 5.

### 4. Proof of the Harnack’s Inequality

In this section we prove Theorem 1.1. We begin by following the proof of the Krylov-Safonov theorem found by Caffarelli ([5], [6]) and obtain a decay estimate on small scales (Lemma 4.2). Then we replace the Calderón-Zygmund decomposition with a more refined argument where we take into account the scaling of the equation (Lemma 4.6). By iterating this lemma we get an estimate for the decay of the level sets.

We begin with two lemmata which are more or less standard. In the first lemma we construct a barrier function. The proof can be found in the Appendix A.

**Lemma 4.1.** There is a smooth function $\varphi$ in $\mathbb{R}^n$ and universal constants $L_1$ and $r_0 \in (0, 1]$ such that

\begin{enumerate}
  \item \( \varphi = 0 \) on $\partial B_{2r_0}$,
  \item $\varphi \leq -2$ in $B_{r_0}$,
  \item \( \mathcal{P}^{\alpha}_{\lambda, \lambda}(D^2\varphi) \geq \phi(|D\varphi|) - C\xi \) in $B_{2r_0}$,
\end{enumerate}

where $0 \leq \xi \leq 1$ is a continuous function such that supp$\xi \subset B_{2r_0}$. Moreover, $\varphi \geq -L_1$ in $B_{2r_0}$ and $|D\varphi| \geq L_1^{-1}$ in $B_{2r_0} \setminus B_{2r_0}$. 

In the second lemma we use the Alexandroff-Bakelman-Pucci estimate and the previous barrier function to obtain a decay estimate on small scales. The proof is again in the Appendix A.

**Lemma 4.2.** Let \( u \in C(B_2) \) be a nonnegative supersolution of (1.3) in \( B_2 \). If \( \inf_{B_1} u \leq 1 \) then it holds
\[
|\{ x \in B_2 \mid u(x) \leq L_1 \}| > \mu,
\]
for universal constants \( L_1 > 1 \) and \( \mu > 0 \).

**Remark 4.3.** Suppose that \( v \in C(B_2R(x_0)) \) satisfies the assumptions of Theorem 1.1. Then the function \( v_R \in C(B_2) \)
\[
v_R(x) := \frac{1}{R}v(Rx + x_0)
\]
is a supersolution of
\[
(4.1) \quad \mathcal{P}^+_{\lambda, \Lambda}(D^2u) \geq -R\phi(\|Du\|)
\]
and a subsolution of
\[
(4.2) \quad \mathcal{P}^-_{\lambda, \Lambda}(D^2u) \leq R\phi(\|Du\|).
\]

Therefore in order to prove Theorem 1.1 we need to show that a nonnegative function \( u \in C(B_2) \) which is a viscosity supersolution of (4.1) and a viscosity subsolution of (4.2) in \( B_2 \) satisfies
\[
\int_m^M \frac{dt}{R\phi(t) + t} \leq C,
\]
where \( m = \inf_{B_1} u \) and \( M = \sup_{B_1} u \).

We now come to the point where the proof of Theorem 1.1 truly differs from the proof of Krylov-Safonov theorem. Since the equation (1.3) is not scaling invariant we can not simply iterate Lemma 4.2. We overcome this problem by the following scaling argument. If \( u \) is a supersolution of (1.3) and \( A > 0 \) is given we can find \( r > 0 \) such that the rescaled function
\[
\tilde{u}(x) := \frac{u(rx)}{A}
\]
is again a supersolution of (1.3). This will be done in the next lemma, which will be later used frequently. In this lemma we need the assumption (P2) on \( \phi \).

**Lemma 4.4.** Let \( u \in C(B_2) \) be a supersolution of (4.1) in \( B_2 \) for \( R \leq 1 \). There exists a universal constant \( L_2 \) such that for every
\[
ar \leq \frac{A}{L_2 R\phi(A) + A} = \frac{1}{L_2(R\eta(A) + 1)}
\]
the rescaled function
\[
\tilde{u}(x) := \frac{u(rx)}{A}
\]
is a supersolution of (1.3) in its domain, i.e., \( \tilde{u} \) is a supersolution of (4.1) for \( R = 1 \).

**Proof.** We have
\[
D\tilde{u}(x) = \frac{r}{A}Du(rx) \quad \text{and} \quad D^2\tilde{u}(x) = \frac{r^2}{A}Du(rx).
\]
Hence, \( \tilde{u} \) is a supersolution of
\[
\mathcal{P}^+_{\lambda, \Lambda}(D^2\tilde{u}(x)) \geq -\frac{r^2R}{A}\phi \left( \frac{A}{r}|D\tilde{u}(x)| \right)
\]
in its domain. Therefore the claim follows if we can show
\[
\frac{r^2R}{A}\phi \left( \frac{At}{r} \right) \leq \phi(t) \quad \text{for every} \ t > 0.
\]
Since \( \phi(t) = \eta(t)t \) this is equivalent to
\[
nR\eta(At/r) \leq \eta(t).
\]
Noting that it holds \( \eta(t) \geq 1 \).
Proposition 2.3 (iii) and the monotonicity condition (P1) imply
\[
\frac{\eta((R\eta(A)+1)A)}{\eta(A)} \leq C_0
\]
for a universal constant \(C_0\). We use Proposition 2.3 (iv) and the conditions (P1) and (P3) to conclude that for some constant \(C_0\) it holds
\[
r R \eta\left(\frac{At}{r}\right) \leq \frac{RA}{L_2(R\phi(A) + A)}\eta(L_2(R\phi(A) + A)t) \leq \frac{1}{L_2\eta(A)}\eta(L_2(R\eta(A)+1)A) \leq C_0 \frac{\eta(L_2)}{L_2} \frac{\eta((R\eta(A)+1)A)}{\eta(A)} \eta(t) \leq C_0 \frac{\eta(L_2)}{L_2} \eta(t).
\]
By Proposition 2.3 (ii) we may choose \(L_2\) such that
\[
\frac{\eta(L_2)}{L_2} \leq \frac{1}{C_0}
\]
and (4.3) follows. \(\square\)

We denote the open \(\delta\)-neighbourhood of a set \(S \subset \mathbb{R}^n\) by
\[(4.4) \quad \mathcal{I}_\delta(S) = \{x \in \mathbb{R}^n \mid \text{dist}(x, S) < \delta\}.
\]
We need the following corollary of the relative isoperimetric inequality.

**Lemma 4.5.** There is a dimensional constant \(c_n > 0\) such that for every set \(E \subset B_R\) it holds
\[
|\mathcal{I}_\delta(\partial E \cap B_R) \cap E| \geq c_n \min\left\{\delta |B_R \setminus E|^\frac{n-1}{n}, \delta |E|^\frac{n-1}{n}, |E|\right\}
\]
for every \(\delta \leq 1\).

**Proof.** We may assume that \(|E| \leq |B_R \setminus E|\), for in the case \(|E| \geq |B_R \setminus E|\) the argument is similar. Let us denote
\[E_s := \{x \in E \mid d_{\partial E}(x) > s\}\]
where \(d_{\partial E}(x) = \text{dist}(x, \partial E \cap B_R)\). Notice that \(E_t \subset E_s\) for every \(s < t\) and \(|E_s| \leq \frac{1}{2}|B_R|\). If \(|E_s| \leq \frac{1}{2}|E|\), we have
\[|\mathcal{I}_\delta(\partial E \cap B_R) \cap E| = |E \setminus E_\delta| = |E| - |E_s| \geq \frac{1}{2}|E|
\]
and the claim follows. Let us then treat the case \(|E_s| \geq \frac{1}{2}|E|\). Note that then \(|E_s| \geq \frac{1}{2}|E|\) for every \(s \in (0, \delta)\).

Lipschitz continuity of \(d_{\partial E}\) implies that for almost every \(s \in (0, \delta)\) the set \(E_s\) has finite perimeter [3, Theorem 3.40]. By the relative isoperimetric inequality [3, Remark 3.50] we have
\[(4.5) \quad P(E_s, B_R) \geq c_n \min\{|B_R \setminus E_s|^\frac{n-1}{n}, |E_s|^\frac{n-1}{n}\} \geq c_n |E|^\frac{n-1}{n}
\]
for some dimensional constants \(c_n\) and \(c_n\), where the last inequality follows from \(|E_s| \leq |B_R \setminus E_s|\).
Here \(P(E_s, B_R)\) denotes the perimeter of \(E_s\) in the ball \(B_R\). Since \(|\nabla d_E(x)| = 1\) for almost every \(x \in E_s\), the coarea formula in \(BV\) [3, Theorem 3.40] and (4.5) yield
\[
|\mathcal{I}_\delta(\partial E \cap B_R) \cap E| = |E \setminus E_\delta| = \int_{E \setminus E_\delta} |\nabla d_E(x)| \, dx \\
\geq \int_0^\delta P(E_s, B_R) \, ds \geq c_n \delta |E|^\frac{n-1}{n}.
\]
\(\square\)
In the next lemma we study the decay of the level-sets
\[ \{ x \in B_{\frac{3}{2}} \mid u(x) > t \}. \]
From now on we assume that \( u \) is a positive supersolution of (4.1) for \( R \leq 1 \) in \( B_2 \) and denote \( m = \inf_{B_1} u > 0 \). We study the sets
\[ A_k := \{ x \in B_{\frac{3}{2}} \mid u(x) > L^k m \} \]
and choose a scaling factor for \( k \in \mathbb{N} \) as
\[ a_k := \frac{L^{k-1}m}{R \phi(L^k m) + L^k m} = \frac{1}{L} \frac{1}{R \eta(L^k m) + 1}, \]
where \( L \) is a uniform constant which will be chosen later.

The proof is based on an observation that for a fixed \( k \), due to a scaling argument and Lemma 4.4, we may use a rescaled version of Lemma 4.2 in a \( \delta \)-neighborhood of \( \partial A_k \) for \( \delta = a_k \). A standard covering argument then implies that a part of the \( a_k \)-neighborhood of \( \partial A_k \) in \( A_k \) does not belong to the next level set \( A_{k+1} \). We then use Lemma 4.5 to estimate the size of the set \( A_k \setminus A_{k+1} \).

**Lemma 4.6.** Let \( u \) be a positive supersolution of (4.1) for \( R \leq 1 \) in \( B_2 \) and let \( L = \max \{ L_1, L_2, 6 \} \), where \( L_1, L_2 \) are the constants from Lemma 4.2 and Lemma 4.4. Let us denote \( m = \inf_{B_1} u \) and suppose that the set \( A_k \) is defined in (4.6) and the number \( a_k \) in (4.7). Then it holds
\[ |A_k \setminus A_{k+1}| \geq c_0 \min \{ a_k |B_{\frac{3}{2}} \setminus A_k|^\frac{n-1}{n}, a_k |A_k|^\frac{n-1}{n}, |A_k| \}, \]
for a uniform constant \( c_0 > 0 \).

**Proof.** Let us fix \( k \geq 1 \). Note that
\[ A_k \setminus A_{k+1} = \{ x \in B_{\frac{3}{2}} : L^k m < u \leq L^{k+1} m \}. \]
We denote the open \( a_k \)-neighborhood of \( \partial A_k \) in \( A_k \) by
\[ D_k := I_{a_k}(\partial A_k \cap B_{\frac{3}{2}}) \cap A_k = \{ x \in A_k : \text{dist}(x, \partial A_k \cap B_{\frac{3}{2}}) < a_k \}. \]
Note that if \( A_k \cap B_{\frac{3}{2}} = \emptyset \) the claim is trivially true. Let \( B \) be collection of balls \( B_{2r}(x) \subset B_2 \) such that \( x \in A_k \) and
\[ r = \text{dist}(x, \partial A_k \cap B_{\frac{3}{2}}) \leq a_k \leq \frac{1}{6}. \]
Then \( B \) is a cover of \( D_k \). By Vitali’s covering theorem we may choose a countable subcollection from \( B \), say \( B_{2r_i}(x_i) \), which are disjoint and the balls \( B_{10r_i}(x_i) \) still cover \( D_k \). Lemma 4.5 implies
\[ 5^n \sum \limits_{i} |B_{2r_i}(x_i)| = \sum \limits_{i} |B_{10r_i}(x_i)| \geq |D_k| \]
\[ \geq c_n \min \{ a_k |B_{\frac{3}{2}} \setminus A_k|^\frac{n-1}{n}, a_k |A_k|^\frac{n-1}{n}, |A_k| \}. \]
Let us fix a ball \( B_{2r_i}(x_i) \) which belongs to the Vitali cover and rescale \( u \) by
\[ \tilde{u}(x) := \frac{u(r_i x + x_i)}{L^k m}. \]
Then \( \tilde{u} \) is nonnegative in \( B_2 \) and \( \inf_{B_1} \tilde{u} \leq 1 \), which follows from \( \partial B_{r_i}(x_i) \cap A_k \neq \emptyset \). Moreover, since
\[ r_i \leq a_k = \frac{L^{k-1}m}{R \phi(L^k m) + L^k m} \leq \frac{L^k m}{L_2 R \phi(L^k m) + L^k m} \]
we deduce from Lemma 4.4 that \( \tilde{u} \) is a supersolution of (1.3) in \( B_2 \). We may thus apply Lemma 4.2 to conclude that
\[ \frac{|B_{2r_i}(x_i) \setminus A_{k+1}|}{|B_{2r_i}|} = \frac{|\{ x \in B_{2r_i}(x_i) : u(x) \leq L^{k+1} m \}|}{|B_{2r_i}|} \]
\[ = |\{ y \in B_2 : \tilde{u}(y) \leq L \}| \]
\[ \geq |\{ y \in B_2 : \tilde{u}(y) \leq L_1 \}| \geq \mu. \]
Since $B_{2r_i}(x_i) \subset D_k \subset A_k$ are disjoint, we obtain from (4.8) and (4.9) that

$$|A_k \setminus A_{k+1}| \geq |D_k \setminus A_{k+1}| \geq \sum_i |B_{2r_i}(x_i) \setminus A_{k+1}|$$

$$\geq \mu \sum_i |B_{2r_i}(x_i)|$$

$$\geq 5^{-n} \epsilon_n \mu \min \left\{ a_k |B_\frac{3}{2} \setminus A_k|^{\frac{n-1}{n}}, a_k |A_k|^{\frac{n-1}{n}}, |A_k| \right\}.$$ \hfill \Box

We continue by iterating the estimate from Lemma 4.6. Due to the relative isoperimetric inequality the iteration behaves differently depending on the size of $A_k$.

**Lemma 4.7.** Let the function $u$ be as in Lemma (4.6) and suppose the sets $A_k \subset B_\frac{3}{2}$ and the numbers $a_k \in (0, 1)$ are given by (4.6) and (4.7).

(a) If there is $\delta \in (0, 1)$ such that $|A_j| \geq \delta$ for every $j = 0, 1, \ldots, k$ where $k \geq 1$, then it holds

$$|A_k| \leq |B_\frac{3}{2}| - c \left( \sum_{j=0}^{k-1} a_j \right)^n$$

for a constant $c > 0$ which depends on $\delta$.

(b) If there is $k_0 \in \mathbb{N}$ such that $a_j^n \leq |A_j| \leq \frac{1}{2^n}$ for every $j = k_0, k_0 + 1, k_0 + 2, \ldots, k$ for $k > k_0$, then it holds

$$|A_k| \leq \frac{1}{n^n} \left( 1 - c \sum_{j=k_0}^{k-1} a_j \right)^n$$

for a universal constant $c > 0$.

**Proof.** Let us first prove (a). Since $|A_k| \geq \delta$, we may use Lemma 4.6 to deduce

$$|A_k \setminus A_{k+1}| \geq \epsilon \alpha_k \left( |B_\frac{3}{2}| - |A_k| \right)^{\frac{n-1}{n}}$$

for $\epsilon = \frac{\alpha_k}{|A_k|}$, where $\alpha_0$ is the constant from Lemma 4.6. Moreover, we may assume that $\epsilon \leq \mu$, where $\mu$ is from Lemma 4.2, by possibly decreasing $\epsilon$.

We make a few observations on sequence $(a_k)$ defined in (4.7). First of all, since $\phi$ is increasing we have

$$\frac{a_k}{a_{k-1}} = \frac{L \phi(L^{k-1}m) + L^{k-1}m}{L \phi(L^km) + L^km} \leq L$$

for every $k = 1, 2, 3, \ldots$. In particular, it holds

$$a_k \leq L \sum_{j=0}^{k-1} a_j$$

for every $k = 1, 2, 3, \ldots$. Therefore we can find $N > 0$ such that

$$\left( \sum_{j=0}^{k-1} a_j + a_k \right)^n = \sum_{i=0}^n \binom{n}{i} \left( \sum_{j=0}^{k-1} a_j \right)^{n-i} a_k^i$$

$$\leq \left( \sum_{j=0}^{k-1} a_j \right)^n + Na_k \left( \sum_{j=0}^{k-1} a_j \right)^{n-1}.$$ \hfill (4.13)

Let us prove the claim by induction for $c_1 = \left( \frac{L}{\mu} \right)^n$, where $\epsilon$ is the constant from (4.12). We begin by observing that the only information from the set $A_0 = \{ x \in B_\frac{3}{2} : u(x) > m \}$ is that there
is a point \( \hat{x} \in \bar{B}_1 \) such that \( u(\hat{x}) = m \). Let us choose \( x_0 \in B_1 \) such that \( \hat{x} \in \bar{B}_{a_0}(x_0) \subset \bar{B}_1 \). Note that since \( a_0 \leq \frac{1}{6} \) we have \( B_{2a_0}(x_0) \subset \bar{B}_2 \). We argue as in the previous lemma and rescale \( u \) by

\[
\tilde{u}(x) = \frac{u(a_0x + x_0)}{m}.
\]

Then \( \tilde{u} \) is nonnegative in \( B_2 \), \( \inf_{B_2} \tilde{u} \leq 1 \) and by Lemma 4.4 it is a supersolution of (1.3) in \( B_2 \). We may thus apply Lemma 4.2 to conclude that

\[
\frac{|B_{-\frac{3}{2}} \setminus A_1|}{|B_{2a_0}|} \geq \frac{|\{ x \in B_{2a_0}(x_0) : u(x) \leq Lm \}|}{|B_{2a_0}|}
= |\{ y \in B_2 : \tilde{u}(y) \leq L \}|
\geq \mu.
\]

Therefore we have

\[
|A_1| \leq |B_{-\frac{3}{2}}| - \mu |B_{2a_0}|
\leq |B_{-\frac{3}{2}}| - \left( \frac{\tilde{c}}{N} \right)^n a_0^n
\]

where the last inequality follows from \( \tilde{c} \leq \mu \). Hence, the claim holds for \( k = 1 \).

We assume that the claim holds for \( k > 1 \), i.e.,

\[
(A.14) \quad |A_k| \leq |B_{-\frac{3}{2}}| - \left( \frac{\tilde{c}}{N} \right)^n \left( \sum_{j=0}^{k-1} a_j \right)^n.
\]

We have by (4.12), (4.13) and (4.14) that

\[
|A_{k+1}| \leq |A_k| - \tilde{c}a_k \left( |B_{-\frac{3}{2}}| - |A_k| \right)^{\frac{n-1}{n}}
\leq |B_{-\frac{3}{2}}| - \left( \frac{\tilde{c}}{N} \right)^n \left( \sum_{j=0}^{k-1} a_j \right)^n + Na_k \left( \sum_{j=0}^{k-1} a_j \right)^{n-1}
\leq |B_{-\frac{3}{2}}| - \left( \frac{\tilde{c}}{N} \right)^n \left( \sum_{j=0}^{k} a_j \right)^n,
\]

which proves (4.10).

We now prove the claim (b). In this case Lemma 4.6 implies

\[
|A_k \setminus A_{k+1}| \geq c_0 a_k |A_k|^{\frac{n-1}{n}}
\]

for all \( k \geq k_1 \). In other words

\[
(A.15) \quad |A_{k+1}| \leq |A_k| - c_0 a_k |A_k|^{\frac{n-1}{n}}.
\]

By possibly decreasing \( c_0 \) we may assume that \( c_0 \leq 1 - 2^{-1/n} \). Therefore it follows from \( a_{k_1} \leq \frac{1}{6} \) that

\[
|A_{k_1+1}| \leq \frac{1}{2n} \leq \frac{1}{n}(1 - c_0 a_{k_1})^n.
\]

Hence the claim holds for \( k = k_1 + 1 \).

Let us assume that the claim holds for \( k > k_1 + 1 \), i.e.,

\[
(A.16) \quad |A_k| \leq \frac{1}{n} \left( 1 - c_0 \sum_{j=k_1}^{k-1} a_j \right)^n.
\]

Notice first that the assumption \( |A_k| \geq a_k^n \) implies

\[
c_0 a_k \leq |A_k|^\frac{1}{n} \leq \frac{1}{n} \left( 1 - c_0 \sum_{j=k_1}^{k-1} a_j \right).\]
Next we remark that if there are positive numbers \( a \) and \( b \) such that \( b \leq \frac{1}{n}a \), then it holds
\[
(a - b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i}(-b)^i \geq a^n - na^{n-1}b.
\]
The previous two inequalities then yield
\[
(4.17) \quad \left(1 - c_0 \sum_{j=k_1}^{k-1} a_j - c_0 a_k\right)^n \geq \left(1 - c_0 \sum_{j=k_1}^{k-1} a_j\right) - n c_0 a_k \left(1 - c_0 \sum_{j=k_1}^{k-1} a_j\right)^{n-1}.
\]
Notice that the function \( t \mapsto t - c_0 a_k t^{\frac{n-1}{n}} \) is increasing in \([a_k^n, 1]\). Since \( |A_k| \geq a_k^n \) we have by (4.15) and (4.16) that
\[
|A_{k+1}| \leq |A_k| - c_0 a_k |A_k|^{\frac{n-1}{n}} \leq \frac{1}{n^n} \left(1 - c_0 \sum_{j=k_1}^{k-1} a_j\right) - n c_0 a_k \left(1 - c_0 \sum_{j=k_1}^{k-1} a_j\right)^{n-1} \leq \frac{1}{n^n} \left(1 - c_0 \sum_{j=k_1}^{k-1} a_j\right)^n,
\]
where the last inequality follows from (4.17).

The next lemma asserts that when the level sets \( A_k \) are very small they start to decay as in the homogeneous case. Roughly speaking this means that the asymptotic behaviour of an unbounded supersolution of (1.3) is completely determined by the second order operator, not the lower order drift term.

**Lemma 4.8.** Let \( u \) be a positive supersolution of (4.1) for \( R \leq 1 \) in \( B_2 \), and suppose the sets \( A_k \subset B_{\frac{1}{2}} \) and the numbers \( a_k \in (0,1) \) are given by (4.6) and (4.7). Let the constant \( c_0 \) be as in Lemma 4.6 and denote \( m = \inf_{B_2} u \). There is a universal constant \( C_1 \) such that either \( \sum_{j=0}^{\infty} a_j \leq C_1 \) or there is an index \( k_1 \in \mathbb{N} \) such that
\[
(4.18) \quad \sum_{j=0}^{k_1} a_j \leq C_1 \quad \text{and} \quad |A_k| \leq (1 - c_0)^{k-k_1} a_k^{n}, \quad k \geq k_1.
\]
In the latter case we have
\[
(4.19) \quad \{|x \in B_{\frac{1}{2}} : u(x) > tL^{k_1} m_1\} \leq c_\varepsilon t^{-\varepsilon} a_k^n \quad \text{for} \quad t \geq 1
\]
where \( c_\varepsilon \) and \( \varepsilon > 0 \) are universal constants.

Before the proof I would like to point out that there is no bound for the index \( k_1 \). The point is that the sum (4.18) is uniformly bounded.

**Proof.** Let us begin with a few preparations. Let \( c_0 \) and \( L \) be the constants from Lemma 4.6. By Proposition 2.3 (i) there is a number \( t_L \geq 1 \) such that
\[
(4.20) \quad \sup_{t \geq t_L} \frac{\eta(Lt) + R^{-1}}{\eta(t) + R^{-1}} \leq \sup_{t \geq t_L} \frac{\eta(Lt)}{\eta(t)} \leq \left(\frac{1}{(1 - c_0)}\right)^{1/n}.
\]
Moreover, Proposition 2.3 (ii) implies that
\[
(4.21) \quad \delta_0 := \inf_{j \in \mathbb{N}} \left(1 - c_0\right)^{-j} \left(\min_{t \in [L^{-1}, L t_L]} \frac{1}{(R \eta(t) + 1)^n}\right) > 0.
\]
We begin by using Lemma 4.7 (a) for $\delta = \min\{\delta_0, \frac{1}{2n^2}\}$, where $\delta_0$ is defined in (4.21). We conclude that there is $C_0$ such that either $\sum_{j=0}^{\infty} a_j \leq C_0$ or there is $k_0 \in \mathbb{N}$ such that
\[
\sum_{j=0}^{k_0-1} a_j \leq C_0 \quad \text{and} \quad |A_k| \leq \delta
\]
for every $k \geq k_0$. In the latter case Lemma 4.7 (b) implies that there is $C_1$ such that either $\sum_{j=k_0}^{\infty} a_j \leq C_1$ or there is $k_1 > k_0$ such that
\[
\sum_{j=k_0}^{k_1-1} a_j \leq C_1 \quad \text{and} \quad |A_{k_1}| \leq a_k^0.
\]
For every $k \geq k_1$ Lemma 4.6 gives
\[
|A_k \setminus A_{k+1}| \leq c_0 \min\{a_k|A_k|^{1/n}, |A_k|\}.
\]
In particular, we have
\[
|A_{k+1}| \leq (1 - c_0)|A_{k_1}|.
\]
We divide the proof into two cases.

**Case 1:** $L^k m \geq t_L$.

In this case it follows from (4.20) that
\[
\frac{a_k}{a_{k+1}} = \frac{\eta(L^{k+1}m) + R^{-1}}{\eta(L^k m) + R^{-1}} \leq \left( \frac{1}{(1 - c_0)} \right)^{1/n}
\]
for every $k \geq k_1$. I claim that it holds
\[
|A_{k+1}| \leq (1 - c_0)|A_k| \quad \text{and} \quad |A_k| \leq a_k^0
\]
for every $k \geq k_1$, which implies the claim in the first case.

Indeed, (4.23) implies that (4.25) holds for $k = k_1$. Assume that (4.25) holds for $k > k_1$. It follows from the induction assumption and (4.24) that
\[
|A_{k+1}| \leq (1 - c_0)|A_k| \leq (1 - c_0)a_k^0 \leq a_{k+1}^0.
\]
Hence, (4.22) yields
\[
|A_{k+2}| \leq (1 - c_0)|A_{k+1}|
\]
and (4.25) follows.

**Case 2:** $L^k m \leq t_L$. Let us prove that also in this case we have
\[
|A_k| \leq (1 - c_0)^{k - k_1}|A_{k_1}|
\]
for $k \geq k_1$. Again (4.23) implies that the claim holds for $k = k_1 + 1$. Moreover, let us recall that we have
\[
|A_{k_1}| \leq \frac{\delta_0}{L^0}
\]
where $\delta_0$ is given by (4.21).

Assume (4.26) is true for $k > k_1 + 1$. Let us first treat the case when $L^k m < 1/L$. Since $\eta$ is nonincreasing in $(0, 1)$ we have
\[
a_k = \frac{1}{L^k \eta(L^k m) + 1} \geq \frac{1}{L^k \eta(L^{k+1} m) + 1} = a_{k_1} \geq |A_{k_1}|^{1/n} \geq |A_k|^{1/n}.
\]
Therefore (4.22) gives
\[
|A_{k+1}| \leq (1 - c_0)|A_k| \leq (1 - c_0)^{k+1 - k_1}|A_{k_1}|.
\]
On the other hand, if \( L^k m \geq 1/L \) we use (4.21) and the assumption \( L^k m \leq t_L \) to deduce that

\[
a_k^n = \frac{1}{L^n} \frac{1}{R_\eta(L^k m + 1)^n} \geq (1 - c_0)^{k - k_1} \frac{\delta_0}{L^n} \geq (1 - c_0)^{k - k_1} |A_{k_1}| \geq |A_k|,
\]

where the last inequality follows from the induction assumption. Hence, (4.22) yields

\[
|A_{k+1}| \leq (1 - c_0)|A_k| \leq (1 - c_0)^{k - k_1} |A_{k_1}|
\]

which proves (4.26).

\[\square\]

Theorem 1.1 follows from Lemma 4.8 and the following result which is similar to the one in [6].

**Lemma 4.9.** Let \( u \in C(B_2) \) be a supersolution of (4.1) and a subsolution of (4.2) for \( R \leq 1 \) in \( B_2 \). Suppose that \( A_k \subset B_{3/2} \), \( a_k \in \mathbb{R} \) and \( L \) are as in Lemma 4.6, and that (4.19) from Lemma 4.8 holds for an index \( k_1 \) and \( \epsilon \). Denote \( m = \inf_{B_1} u \). There are universal numbers \( L_0 \) and \( \sigma \) such that for \( \nu = \frac{L_0}{L_0 - 1/2} \) the following holds: if \( x_0 \in B_{4/3} \) and \( l \in \mathbb{N} \) are such that

\[
u l \leq \frac{L_0 L^{k_1} m}{a_{k_1}}.
\]

then it holds

\[
\sup_{B_{2r_l}(x_0)} u \geq \nu^{l+1} L_0 L^{k_1} m
\]

where \( r_l = \sigma \nu^{-(l+1)} e/n(L_0/2) - 1/n a_{k_1} \).

The proof of the previous lemma can be found in the Appendix. We now give the proof of the main result.

**Proof of Theorem 1.1.** Let us assume that \( u \in C(B_2) \) is a supersolution of (4.1) and subsolution of (4.2) in \( B_2 \) for \( R \leq 1 \). By Remark 4.3 we need to show that

\[
\int_m^M \frac{dt}{R \phi(t) + t} \leq C,
\]

where \( m = \inf_{B_1} u \) and \( M = \sup_{B_1} u \). Since \( a_k \) are given by (4.7) and \( \phi \) is increasing we have

\[
\int_m^{L^{k_1} m} \frac{dt}{R \phi(t) + t} \leq L \sum_{j=0}^{k_1 - 1} \frac{L^j m}{R \phi(L^j m) + L^j m} = L \sum_{j=0}^{k_1 - 1} a_j
\]

for every \( k = 1, 2, 3, \ldots \). Therefore in the case

\[
\sum_{j=0}^{\infty} a_j \leq C_1
\]

the claim is trivially true. Let us treat the case when we have (4.18), i.e., there is an index \( k_1 \) such that

\[
\sum_{j=0}^{k_1 - 1} a_j \leq C_1
\]

where \( C_1 \) is a uniform constant and

\[
|A_{k_1+1}| \leq (1 - c_0)^l a_{k_1}^{n_l}, \quad l = 0, 1, 2, \ldots
\]

Suppose that \( r_l = \alpha \nu^{-l/2} L_0 \epsilon/n a_{k_1} \) are as in Lemma 4.9. Then there is a uniform index \( l_0 \) such that

\[
\sum_{j=l_0}^{\infty} r_j \leq 1/3.
\]
I claim that it holds

(4.28) \( \sup_{B_1} u \leq \nu^{\beta_0} L_0 L^{k_1} m. \)

Indeed, if this were not true there would be a point \( x_{l_0} \in B_1 \) such that

\[ u(x_{l_0}) \geq \nu^{\beta_0} L_0 L^{k_1} m. \]

By Lemma 4.9 there is \( x_{l_0+1} \in B_{r_{l_0+1}}(x_{l_0}) \) such that

\[ u(x_{l_0+1}) \geq \nu^{\beta_0+1} L_0 L^{k_1} m. \]

We may repeat this process since at every step we have \( |x_{j+1} - x_j| \leq r_j \) and therefore for every \( l > l_0 \) it holds

\[ |x_l| \leq |x_{l_0}| + \sum_{j=l_0}^{l-1} |x_{j+1} - x_j| < 1 + \sum_{j=l_0}^{\infty} r_j \leq \frac{4}{3}. \]

Hence, \( u(x_l) \geq \nu^{\beta_0} L_0 L^{k_1} m \) and \( x_l \in B_{\frac{4}{3}} \) for every \( l > l_0 \). Hence, \( u \) is unbounded in \( \overline{B}_{\frac{4}{3}} \) which contradicts the continuity of \( u \) in \( B_2 \).

From (4.27) and (4.28) we deduce

\[ \int_m^M \frac{dt}{R^{\phi(t)} + t} \leq \int_{L^{k_1} m}^{\nu^{\beta_0} L_0 L^{k_1} m} t^{-1} dt + \int_m^{L^{k_1} m} \frac{dt}{R^{\phi(t)} + t} \]

\[ \leq \log (\nu^{\beta_0} L_0) + L \sum_{j=0}^{k_1-1} a_j \]

\[ \leq \log (\nu^{\beta_0} L_0) + LC_1 \]

and the result follows. \( \square \)

We conclude the section with a proof of Corollary 1.2.

**Proof of Corollary 1.2.** Let us show that if \( v \in C(B_{R_0}(x_0)) \) is a nonnegative viscosity supersolution of (1.3) and a subsolution of (1.4) in \( B_{R_0}(x_0) \) such that \( \sup_{B_{R_0}(x_0)} v \leq M_0 \), then there is \( \hat{R} \leq \hat{R} \) depending on \( M_0 \) such that for every \( R \leq \hat{R} \) it holds

(4.29) \( \sup_{B_R} v \leq C(\inf_{B_R} v + \sqrt{R}) \)

for a uniform constant \( C \). The Hölder continuity of \( u \) then follows from (4.29) by a standard iteration argument ([8, Chapter 8.9]).

To show (4.29) we denote \( m = \inf_{B_R} v \) and \( M = \sup_{B_R} v \). By Proposition 2.3 there is \( \hat{R} \) such that for every \( R \leq \hat{R} \) it holds

\[ \sqrt{\hat{R}} \eta \left( \frac{M_0}{R} \right) \leq \frac{1}{M_0}. \]

Therefore for every \( R \leq \hat{R} \) and \( t \leq M_0 \) we have

\[ R^2 \phi(t/R) \leq R^2 \phi(M_0/R) = M_0 R \eta \left( \frac{M_0}{R} \right) \leq \sqrt{R}. \]

Hence, the bound \( M \leq M_0 \) yields

\[ \int_m^M \frac{dt}{R^2 \phi(t/R)} + t \geq \int_m^M \frac{dt}{\sqrt{R} + t} = \log \left( \frac{M + \sqrt{R}}{m + \sqrt{R}} \right). \]

Theorem 1.1 implies

\[ M \leq C(m + \sqrt{R}). \]

Hence we have (4.29). \( \square \)
5. On \( p(x)\)-harmonic functions

In this section we discuss how Theorem 1.1 implies Corollary 1.3. Moreover, we will see that this inequality is optimal. Let us recall that a function \( u \in W_{\text{loc}}^{1,1}(\Omega) \) is \( p(x)\)-harmonic in \( \Omega \) if it has locally finite energy
\[
\int_{\Omega} \frac{1}{p(x)} |Du|^{p(x)} \, dx < \infty \quad \text{for every } \Omega' \subset \subset \Omega
\]
and it is weak solution of the corresponding Euler-Lagrange equation (1.6), i.e.,
\[
(5.1) \quad \int_{\Omega} |Du(x)|^{p(x)-2} \langle Du(x), D\varphi(x) \rangle \, dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega).
\]
We assume that \( p \in C^1(\mathbb{R}^n) \) and that there are numbers \( 1 < p^- \leq p^+ < \infty \) such that \( p^- \leq p(x) \leq p^+ \) for every \( x \in \mathbb{R}^n \). For more about \( p(x)\)-harmonic functions see [1] and the references therein.

It follows from [1] that under these conditions on \( p(\cdot) \) the weak solutions of (5.1) are locally \( C^{1,\alpha}\)-regular. In [11] it was shown that the weak solutions of the equation (1.6) coincide with the viscosity solutions. We need only the “easy” part of this result, i.e., that the weak solutions of (5.1) are viscosity solutions of the same equation. In order to formulate this result more precisely we define the following operator
\[
\Delta_{p(x)} \varphi(x) := \Delta \varphi(x) + (p(x) - 2) \Delta_{\infty} \varphi(x) + \log |D\varphi(x)| \langle Dp(x), D\varphi(x) \rangle,
\]
where \( \Delta_{\infty} \varphi = \langle D^2 \varphi \frac{D_p x}{|D_p x|}, \frac{D_p x}{|D_p x|} \rangle \) denotes the infinity Laplace operator. This operator is well defined whenever \( D\varphi(x) \neq 0 \). The following result is from [11].

**Proposition 5.1.** Let \( u \in C(\Omega) \) be a weak solution of (1.6). If \( \varphi \in C^2(\Omega) \) is such that \( \varphi(x_0) = u(x_0) \) at \( x_0 \in \Omega \), \( D\varphi(x_0) \neq 0 \) and \( \varphi \leq u \) then it holds
\[
-\Delta_{p(x)} \varphi(x_0) \geq 0,
\]
and if \( \varphi \geq u \) then
\[
-\Delta_{p(x)} \varphi(x_0) \leq 0.
\]

Corollary 1.3 follows immediately from Theorem 1.1 once we show that \( p(x)\)-harmonic functions are viscosity supersolutions of (1.3) and subsolutions of (1.4) for \( \phi(t) = C(|\log t| + 1)t \) for some \( C \). This is the assertion of the next lemma.

**Lemma 5.2.** Let \( u \in C(\Omega) \) be \( p(x)\)-harmonic in \( \Omega \) and let \( \phi(t) = C(|\log t| + 1)t \) where \( C = \|p\|_{C^1(\Omega)} < \infty \). If \( \varphi \in C^2(\Omega) \) is such that \( \varphi \leq u \) and \( \varphi(x_0) = u(x_0) \) at \( x_0 \in \Omega \) then it holds
\[
P^+_{\lambda,\Lambda}(D^2 \varphi(x_0)) \geq -\phi(|D\varphi(x_0)|),
\]
i.e., it is a viscosity supersolution of (1.3), and if \( \varphi \geq u \) then it holds
\[
P^-_{\lambda,\Lambda}(D^2 \varphi(x_0)) \leq \phi(|D\varphi(x_0)|),
\]
i.e., it is a viscosity subsolution of (1.4). Here \( \lambda = \min\{1, p^- - 1\} \) and \( \Lambda = \max\{1, p^+ - 1\} \).

**Proof.** We only prove that \( u \) is a viscosity supersolution of (1.3), for the subsolution property is similar. Let \( \varphi \in C^2(\Omega) \) be such that \( \varphi(x_0) = u(x_0) \) at \( x_0 \in \Omega \) and \( \varphi \leq u \) in a neighborhood of \( x_0 \). Without loss of generality we may assume that \( x_0 = 0 \), \( u(0) = 0 \), \( \varphi(x) = \langle Ax, x \rangle + \langle b, x \rangle \) for a symmetric matrix \( A \) and a vector \( b \), and that \( \varphi(x) < u(x) \) for \( x \neq 0 \) in \( B_{\rho} \) for some small \( \rho > 0 \). The goal is to show that
\[
P^+_{\lambda,\Lambda}(D^2 \varphi(0)) + \phi(|D\varphi(0)|) \geq 0.
\]
Note that if \( D\varphi(0) \neq 0 \) then the claim follows from Proposition 5.1 after some calculations. Therefore we need to treat the case \( D\varphi(0) = 0 \) to conclude the proof. Note that in this case \( b = 0 \).

Let \( \rho > 0 \) be small. For \( y \in B_{\rho} \) we define
\[
\varphi_y(x) = \varphi(x - y).
\]
For every $y$ there is a number $c_y$ such that the function $\varphi_y + c_y$ touches $u$ from below, say at a point $x_y$. It is clear that $x_y \to 0$ as $|y| \to 0$. If there exists a sequence $(y_k)$ such that $|y_k| \to 0$ and at the associated contact points $(x_k)$ it holds $D\varphi(x_k) \neq 0$, Proposition 5.1 implies
\[
P^+_{\lambda, \Lambda}(D^2\varphi(x_k)) + \phi(|D\varphi(x_k)|) \geq -\Delta_p(x)\varphi(x_k) \geq 0.
\]
The claim then follows by continuity by letting $k \to \infty$. Hence, we need to treat the case when there exists $r > 0$ such that for every $y \in B_r$ at every associated contact point of $x_y$ it holds $D\varphi(x_y) = 0$.

Since $\varphi_y(x) = \varphi(x-y) = \left< A(x-y), (x-y) \right> + c_y$ and $D\varphi_y(x_y) = 0$ we have that $x_y = y$ for every $y \in B_r$. This means that at every point $y \in B_r$ we may touch the graph of $u$ from below with a paraboloid
\[
P(x) = -|A||x-y|^2 + u(y).
\]
This implies that $u$ is semi-convex in $B_r$. In particular, $u$ is locally Lipschitz continuous in $B_r$ and therefore it is differentiable at almost every point in $B_r$. By the previous estimate the gradient of $u$ is zero almost everywhere. Hence, $u$ is constant in $B_r$ and the claim is trivially true.

We conclude this section by constructing a naive example which verifies that Corollary 1.3 is indeed sharp. To that aim let us denote the interval $I_r(k) = (k - r, k + r)$. We consider the function $u : (0, \infty) \to (0, 1)$,
\[
u(x) = e^{-ex^r}.
\]
Below we show that for every $k \in \mathbb{N}$ there exists $p_k \in C^1(I_2(k))$ which satisfies the assumptions of Corollary 1.3 such that $u$ is a solution of the $p_k(x)$-Laplace equation in $I_2(k)$. Note that the function $u$ does not satisfy the classical Harnack’s inequality, since
\[
\frac{\sup_{I_2(k)} u}{\inf_{I_2(k)} u} = e^{-e^{k-1}} \frac{e^{k(e-e^{-1})}}{e^{e^{k-1}}} \to \infty \quad \text{as } k \to \infty.
\]
On the other hand Corollary 1.3 implies
\[
\sup_{I_2(k)} u^C \leq C \inf_{I_2(k)} u
\]
for a constant $C > 1$ which is the optimal estimate.

Let us fix $k \in \mathbb{N}$ and find the function $p_k \in C^1(I_2(k))$. We construct $p_k$ such that it satisfies $1 + C^{-1} \leq p_k \leq C$ and $||p_k||_{C^1(I_2(k))} \leq C$ for a constant $C$ which is independent of $k$. It turns out that it is more convenient to work with the function $q_k(x) = p_k(x) - 1$. Because $u' < 0$ we may write the equation (1.6) in nondivergence form as
\[
\Delta_{p_k(x)} u(x) = q_k(x) u''(x) + q_k'(x) \log |u'(x)| u'(x)
\]
\[
= (q_k(x)(e^x - 1) + q_k'(x)(e^x - x)) e^x e^{-e^x}.
\]
If $q_k$ is a solution of
\[
q_k'(x) + \left( \frac{e^x - 1}{e^x - x} \right) q_k(x) = 0 \quad x \in I_2(k)
\]
then $u$ is a solution of $p_k(x)$-Laplace equation in $I_2(k)$. If $q_k$ is a solution of (5.2) in $I_2(k)$ with a condition
\[
q_k(k) = 1
\]
then it is easy to see that we have $C^{-1} \leq q_k \leq C$ in $I_2(k)$ for a constant $C$ which is independent of $k$. The bound for $|q_k'|$ follows from the previous estimate and from the equation (5.2).
APPENDIX A. PROOF OF THE LEMMATA OF SECTION 4

Proof of Lemma 4.1. For every \( x \in B_{2r_0} \setminus B_{\frac{r_0}{2}} \) the function is defined as
\[
\varphi(x) = M_1 - M_2|\alpha|^{-\alpha}
\]
where \( \alpha = \max\{\frac{2(\alpha-1)\Lambda}{\Lambda}, 1\} \) and \( M_1, M_2 \) are such that \( \varphi(x) = 0 \) when \( |x| = 2r_0 \) and \( \varphi(x) = -2 \) when \( |x| = r_0 \). In other words
\[
M_2 = \frac{2r_0^{\alpha}}{1 - 2^{-\alpha}}.
\]
Note that while \( \alpha \) is already fixed, the radius \( r_0 \) is still to be chosen. If we can show that there is \( r_0 \) such that
\[
(A.1) \quad \mathcal{P}_{\lambda,\Lambda}^- (D^2 \varphi(x)) \geq \phi(|D \varphi(x)|) \quad x \in B_{2r_0} \setminus B_{\frac{r_0}{2}}
\]
we ma extend \( \phi \) smoothly to the whole ball \( B_{2r_0} \) in such a way that it will satisfy all the required conditions.

Let us find \( r_0 \) which satisfies (A.1). For \( \frac{r_0}{2} \leq |x| \leq 2r_0 \), we have
\[
D \varphi(x) = \alpha M_2|\alpha|^{-\alpha-2}x \quad \text{and} \quad D^2 \varphi(x) = \alpha M_2|\alpha|^{-\alpha-2} \left( I - (\alpha + 1) \frac{x}{|x|} \otimes \frac{x}{|x|} \right).
\]
Therefore it holds
\[
\mathcal{P}_{\lambda,\Lambda}^- (D^2 \varphi) \geq \alpha M_2|\alpha|^{-\alpha-2} (\lambda(\alpha + 1) - \Lambda(n - 1))
\]
\[
\geq n\alpha M_2|\alpha|^{-\alpha-2}
\]
\[
\geq \frac{n\alpha}{2(2^{\alpha-1})} r_0^{-2}
\]
by the choices of \( \alpha \) and \( M_2 \). For \( \frac{r_0}{2} \leq |x| \leq 2r_0 \) the monotonicity of \( \phi \) yields
\[
\phi(|D \varphi(x)|) = \phi \left( \frac{2\alpha r_0}{1 - 2^{-\alpha}} |\alpha|^{-\alpha-1} \right) \leq \phi \left( 2^{\alpha+3} \alpha r_0^{-1} \right).
\]
Therefore in order to show (A.1) we only need to find \( r_0 \) which satisfies
\[
(A.2) \quad \frac{n\alpha}{2(2^{\alpha-1})} r_0^{-2} \leq \phi \left( 2^{\alpha+3} \alpha r_0^{-1} \right).
\]
Writing \( \phi(t) = \eta(t)t \) (A.2) reads as
\[
\frac{n\alpha}{2^{\alpha+4}(2^{\alpha-1})} r_0^{-1} \geq \eta \left( 2^{\alpha+3} \alpha r_0^{-1} \right).
\]
By Proposition 2.3 (ii) we have
\[
\lim_{t \to \infty} \frac{\eta(t)}{t} = 0
\]
and therefore (A.2) follows by choosing \( r_0 \) small enough. \( \square \)

Proof of Lemma 4.2. By approximating \( u \) with infimal convolution
\[
u_\varepsilon(x) = \inf_{y \in B_1} \left( u(y) + \frac{1}{\varepsilon} |x - y|^2 \right)
\]
we may assume that \( u \) is semiconcave.

Let \( \hat{x} \in B_1 \) be a point where \( u(\hat{x}) \leq 1 \). Let \( r_0 \leq 1 \) be as in Lemma 4.1 and choose \( x_0 \in \partial B_1 \) such that \( \hat{x} \in B_{r_0}(x_0) \subset B_1 \). Let \( \varphi \) be the barrier function from Lemma 4.1 and define \( v : B_{2r_0} \to \mathbb{R} \),
\[
v(x) = u(x) + \varphi(x - x_0).
\]
By Lemma 4.1 (ii) we have \( \inf_{B_{r_0}(x_0)} v \leq v(\hat{x}) \leq -1 \).

Since \( u \) is nonnegative and \( \varphi(x - x_0) \geq 0 \) for \( x \in \partial B_{2r_0}(x_0) \) (Lemma 4.1 (i)), we have \( v \geq 0 \) on \( \partial B_{2r_0}(x_0) \). Moreover, by the monotonicity of \( \phi \), by elementary properties of the Pucci-operators and by Lemma 4.1 (iii) we obtain that \( v \) is a viscosity supersolution of
\[
P^+(D^2 v(x)) \geq -\phi(|D v(x)| + |D \varphi(x - x_0)|) + \phi(|D \varphi(x - x_0)|) - C \xi(x - x_0),
\]

in \( B_{2r_0}(x_0) \). Here \( \xi \) is a continuous function such that \( 0 \leq \xi \leq 1 \) and \( \text{supp} \xi \subset B_{\frac{1}{2}} \).

Let us extend \( v \) by 0 outside \( B_{2r_0}(x_0) \) and denote the convex envelope of \(-v^- = \min\{v, 0\}\) in \( B_{3r_0}(x_0) \) by \( \Gamma_v \), i.e.,

\[
\Gamma_v(x) := \sup_{p \in \mathbb{R}^n} \inf_{y \in B_{3r_0}(x_0)} (p \cdot (x - y) - v^-(y)).
\]

We denote the contact set by \( \{ v = \Gamma_v \} := \{ x \in B_{3r_0}(x_0) : -v^-(x) = \Gamma_v(x) \} \). Since \( v \) is semiconcave we have \( \Gamma_v \in C^{1,1}(\{ v = \Gamma_v \}) \), see [6, Theorem 5.1]. In particular, \( \Gamma_v \) is twice differentiable almost everywhere on \( \{ v = \Gamma_v \} \). Since \( v \) is a viscosity supersolution of (A.3) and \( \Gamma_v \leq v \) in \( B_{2r_0}(x_0) \) we have

\[
(\text{A.4}) \quad \mathcal{P}^+(D^2\Gamma_v(x)) \geq -\phi(|D\Gamma_v(x)| + |D\varphi(x - x_0)|) + \phi(|D\varphi(x - x_0)|) - C\xi(x - x_0)
\]

for a.e. \( x \in \{ v = \Gamma_v \} \).

We denote by \( E \) the subset of \( B_{2r_0}(x_0) \cap \{ v = \Gamma_v \} \) where the gradient of \( \Gamma_v \) is less than one

\[
(\text{A.5}) \quad E := \{ v = \Gamma_v \} \cap \{ x \in B_{2r_0}(x_0) : |D\Gamma_v(x)| \leq 1 \}.
\]

I claim that it holds

\[
(\text{A.6}) \quad \mathcal{P}^+(D^2\Gamma_v(x)) \geq -b|D\Gamma_v(x)| - C_1\xi_1(x)
\]
a.e. on \( E \), for some universal constants \( C_1, b \) and a continuous function \( \xi_1 \) with \( 0 \leq \xi_1 \leq 1 \) and \( \text{supp} \xi_1 \subset B_{r_0}(x_0) \).

Indeed, denote \( \tilde{L} := \sup_{x \in B_{2r_0}} |D\varphi(x)| \). By Lemma 4.1 it holds \( |D\varphi(x)| \geq L_1^{-1} \) for \( x \in B_{2r_0} \setminus B_{\frac{1}{2}} \). Since \( |D\Gamma_v(x)| \leq 1 \) for \( x \in E \), we have by the local Lipschitz continuity of \( \phi \) that

\[
(\text{A.7}) \quad \phi(|D\Gamma_v(x)| + |D\varphi(x - x_0)|) - \phi(|D\varphi(x - x_0)|) \leq b|D\Gamma_v(x)|
\]
a.e. on \( E \setminus B_{\frac{1}{2}}(x_0) \), where

\[
b = \max\{|D\phi(p)| : L_1^{-1} \leq |p| \leq \tilde{L} + 1\}.
\]

Since \( \text{supp} \xi \subset B_{\frac{1}{2}} \) we conclude from (A.4) and (A.7) that

\[
\mathcal{P}^+(D^2\Gamma_v(x)) \geq -b|D\Gamma_v(x)|
\]
a.e. on \( E \setminus B_{\frac{1}{2}}(x_0) \). On the other hand, we may trivially estimate from (A.4) that

\[
\mathcal{P}^+(D^2\Gamma_v(x)) \geq -\phi(|D\Gamma_v(x)| + |D\varphi(x - x_0)|) + \phi(|D\varphi(x - x_0)|) - C\xi_1(x - x_0)
\]

\[
\geq -\phi(1 + \tilde{L}) - C
\]
a.e. on \( E \cap B_{\frac{1}{2}}(x_0) \). Hence we have (A.6).

Next we use (A.6) to deduce

\[
0 \leq \det(D^2\Gamma_v(x)) \leq C_2(|D\Gamma_v(x)|^n + \xi_1(x))
\]
a.e. \( x \in E \) for some universal constant \( C_2 \). The previous inequality, the fact that \( \text{supp} \xi_1 \subset B_{r_0}(x_0) \) and the coarea formula yield

\[
\int_{D\Gamma_v(E)} \frac{dp}{|p|^n + \delta} \leq \int_E \frac{\det(D^2\Gamma_v)}{|D\Gamma_v|^n + \delta} dx
\]

\[
\leq C_2 \int_E \frac{|D\Gamma_v|^n + \xi_1}{|D\Gamma_v|^n + \delta} dx
\]

\[
\leq C_2 |B_{2r_0}| + \frac{C_2}{\delta} |B_{r_0}(x_0) \cap \{ v = \Gamma_v \}|,
\]

where \( \delta > 0 \) is a small number which will be chosen later.

Let us recall that \( v = 0 \) in \( B_{3r_0}(x_0) \setminus B_{2r_0}(x_0) \) and \( \inf_{B_{r_0}(x_0)} v \leq -1 \). I claim that it holds

\[
(\text{A.9}) \quad B_{1/4} \subset D\Gamma_v(E),
\]

where the set \( E \) is defined in (A.5). Indeed, let us choose \( p \in B_{1/4} \). The function \( w(x) = -v^-(x) - p \cdot (x - x_0) + 3r_0|p| \) is nonnegative on \( \partial B_{3r_0}(x_0) \) and \( w(x_0) < 0 \). Recall that \( -v^- = \min\{v, 0\} \).

Therefore \( w \) attains its minimum in \( B_{3r_0}(x_0) \), say at \( \tilde{x} \). In particular, \( \tilde{x} \) belongs to the contact set
\{\Gamma_v = v\} and \(p = D\Gamma_v(\tilde{x})\). Notice that it holds \(v(\tilde{x}) < 0\) and therefore \(\tilde{x} \in B_{2r_0}(x_0)\), since \(v = 0\) in \(B_{3r_0}(x_0) \setminus B_{2r_0}(x_0)\). Moreover we have \(|D\Gamma_v(\tilde{x})| = |p| \leq 1/4\). Hence, \(\tilde{x} \in E\) which proves (A.9).

The estimate (A.9) yields
\[
\int_{D\Gamma_v(E)} |p|^n + \delta \geq \int_{B_{1/4}} |p|^n + \delta = \omega_n \int_0^{1/4} \rho^{n-1} \rho^n + \delta.
\]
Since the above integral diverges as \(\delta \to 0\), we may choose \(\delta\) such that
\[
\omega_n \int_0^{1/4} \rho^{n-1} \rho^n + \delta \geq C_2 |B_{2r_0}| + 1.
\]
The estimate (A.8) then implies
\[
|B_{r_0}(x_0) \cap \{v = \Gamma_v\}| > \mu.
\]
for some \(\mu > 0\).

To conclude the proof we notice that if \(x\) belongs to the contact set \(\{v = \Gamma_v\}\), then it holds \(v(x) \leq 0\). Therefore \(u(x) = -\varphi(x - x_0) \leq L_1\) for every \(x \in B_{r_0} \cap \{v = \Gamma_v\}\) and the previous inequality yields
\[
|\{u(x) \leq L_1 : x \in B_{r_0}(x_0)\}| > \mu.
\]
The claim follows since \(B_{r_0}(x_0) \subset B_2\).

**Proof of Lemma 4.9.** Let \(\mu\) be the constant from Lemma 4.2. Let us choose \(\sigma^n > \frac{c_n}{2\omega_n\mu}\), where \(c_n\) is the constant from (4.19) and \(\omega_n\) is the volume of the unit ball.

We argue by contradiction and assume that \(\sup_{B_{2r_1}(x_0)} u < \nu^l + 1 L_0 L^{k_1} m\). By choosing \(L_0\) large enough we have that \(r_1 \leq \frac{1}{5}\). The estimate (4.19) from Lemma 4.8 gives
\[
|\{x \in B_{2r_1}(x_0) : u(x) \geq \nu^l + 1 \frac{L_0}{2} L^{k_1} m\}| \leq |\{x \in B_{\frac{2r_1}{5}} : u(x) \geq \nu^l + 1 \frac{L_0}{2} L^{k_1} m\}|
\]
(A.10)
\[
\leq c_n \nu^{-(l+1)\varepsilon} \left( \frac{L_0}{2} \right)^{-\varepsilon} a_{k_1}^{-1}.
\]

We define a positive function \(v : B_2 \to \mathbb{R}\) by
\[
v(x) := \frac{\nu}{(\nu - 1)} - \frac{u(r_1 x + x_0)}{(\nu - 1) l L_0 L^{k_1} m}.
\]
Denote \(A = (\nu - 1) l L_0 L^{k_1} m\). I claim that it holds
\[
r_1 \leq \frac{A}{R(\phi(A) + A)} = \frac{1}{R\eta((\nu - 1) l L_0 L^{k_1} m) + 1}.
\]
Indeed, this is equivalent to
(A.11)
\[
R\eta((\nu - 1) l L_0 L^{k_1} m) + 1 \leq \sigma^{-1} \nu^{l+1}\varepsilon/n (L_0/2)^{\varepsilon/n} a_{k_1}^{-1}.
\]
Since \(a_{k_1}^{-1} \geq R\eta(L^{k_1} m) + 1\) we have by the condition (P3), by \(\eta \geq 1\) and by the choice of \(\nu\) that
\[
R\eta((\nu - 1) l L_0 L^{k_1} m) + 1 \leq L_0 \eta((\nu - 1) l L_0) (R\eta(L^{k_1} m) + 1) \leq C\eta(l) a_{k_1}^{-1}.
\]
By Proposition 2.3 (ii) it holds \(\lim_{l \to \infty} \frac{\eta(l)}{l} = 0\) and therefore we have
\[
\sigma^{-1} \geq \frac{C}{(L_0/2)^{\varepsilon/n} \left( \sup_{l \geq 1} \frac{\eta(l)}{l^{l+1}\varepsilon/n} \right)}
\]
when \(L_0\) is chosen large enough. This proves (A.11).

By Lemma 4.4 we deduce that \(v\) is a positive supersolution of (1.3) in \(B_2\). Hence, Lemma 4.2 yields
(A.12)
\[
|\{x \in B_{2r_1}(x_0) : u(x) < \nu^l + 1 \frac{L_0}{2} L^{k_1} m\}| = |B_{2r_1}| |\{x \in B_2(x_0) : v(x) > L_0\}|
\[
\leq (1 - \mu)|B_{2r_1}|
when $L_0 \geq L$. Combining (A.10) and (A.12) yields
\[
|B_{2r_1}| = |\{x \in B_{2r_1}(x_0) : u(x) > \frac{L_0}{2} L^{k_1+1} m\}| + |\{x \in B_{2r_1}(x_0) : u(x) \leq \frac{L_0}{2} L^{k_1} m\}| \leq c_\sigma \nu^{-(l+1)\varepsilon} \left( \frac{L_0}{2} \right)^{-\varepsilon} a_{k_1}^n + (1 - \mu)|B_{2r_1}|.
\]
In other words
\[
2^n \omega_n r_1^n \leq \frac{c_\sigma}{\mu} \nu^{-(l+1)\varepsilon} \left( \frac{L_0}{2} \right)^{-\varepsilon} a_{k_1}^n.
\]
Since $r_1 = \sigma \nu^{-(l+1)\varepsilon/n}(L_0/2)^{-\varepsilon/n} a_{k_1}$ this implies
\[
\sigma^n \leq \frac{c_\sigma}{2^n \omega_n} \frac{1}{\mu}
\]
which contradicts the choice of $\sigma$. \qed

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