A rational construction of Lie algebras of type $E_7$

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Abstract

We give an explicit construction of Lie algebras of type $E_7$ out of a Lie algebra of type $D_6$ with some restrictions. Up to odd degree extensions, every Lie algebra of type $E_7$ arises this way. Some applications to Tits algebras and Rost invariant are mentioned.

1 Introduction

In [13] Jacques Tits wrote the following: “It might be worthwhile trying to develop a similar theory for strongly inner groups of type $E_7$. For instance, can one give a general construction of such groups showing that there exist anisotropic strongly inner $K$-groups of type $E_7$ as soon as there exist central division associative 16-dimensional $K$-algebras of order 4 in $\text{Br} K$ whose reduced norm is not surjective?”

The goal of the present paper is to give such (and much more general) construction. We deal with Lie algebras; of course, the corresponding group is just the automorphism group of its Lie algebra. By rational constructions we mean those not appealing to the Galois descent, that is involving only terms defined over the base field.

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Let us recall several milestones in the theory. Freudenthal in [5] gave an elegant explicit construction of the split Lie algebra of type $E_7$. On the language of maximal Lie subalgebras it is a particular case of $A_7$-construction. Another approach was proposed by Brown in [3] (see also [6] for a recent exposition); this is an $E_6$-construction. It gives only isotropic Lie algebras. In full generality $A_7$-construction was described by Allison and Faulkner in [1] as a particular case of a Cayley-Dickson doubling; generically it produces anisotropic Lie algebras of type $E_7$. Another construction with this property was discovered by Tits in [13]; in our terms it is an $A_3 + A_3 + A_1$-construction. On the other hand, some Lie algebras of type $E_7$ can be obtained via the Freudenthal magic square, see [11] (or [7] for a particular case).

Our strategy is to define a Lie triple system structure on the (64-dimensional over $F$) simple module of the even Clifford algebra of a central simple algebra of degree 12 with an orthogonal involution under some restrictions. Then the embedding Lie algebra is of type $E_7$. Our construction is of type $D_6 + A_1$.

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2 Lie triple systems and quaternionic gifts

Let $F$ be a field of characteristic not 2. Recall that a Lie triple system is a vector space $W$ over $F$ together with a trilinear map

$$W \times W \times W \rightarrow W$$

$$(u, v, w) \mapsto [u, v, w] = D(u, v)w$$

satisfying the following axioms:

$$D(u, u) = 0$$
$$D(u, v)w + D(v, w)u + D(w, u)v = 0$$
$$D(u, v)[x, y, z] = [D(u, v)x, y, z] + [x, D(u, v)y, z] + [x, y, D(u, v)z].$$

A derivation is a linear map $D: W \rightarrow W$ such that

$$D[x, y, z] = [Dx, y, z] + [x, Dy, z] + [x, y, Dz].$$

The vector space of all derivations form a Lie algebra $\text{Der}(W)$ under the usual commutator map.
The vector space $\text{Der}(W) \oplus W$ under the map

$$[D + u, E + v] = [D, E] + D(u, v) + Dv - Eu$$

form a $\mathbb{Z}/2$-graded Lie algebra called the embedding Lie algebra of $W$. Conversely, degree 1 component of any $\mathbb{Z}/2$-graded Lie algebra is a Lie triple system under the triple commutator map.

Consider a $\mathbb{Z}$-graded Lie algebra

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$$

with one-dimensional components $L_{-2} = Ff$, $L_2 = Fe$, such that each $L_i$ is an eigenspace of the map $[[e, f], \cdot]$ with the eigenvalue $i$. Then $e$, $f$ and $[e, f]$ form an $\mathfrak{sl}_2$-triple, and the maps $[e, \cdot]$ and $[f, \cdot]$ are mutually inverse isomorphisms of $L_{-1}$ and $L_1$. Moreover, maps $[x, \cdot]$ with $x$ from $\langle e, f, [e, f] \rangle = \mathfrak{sl}_2$ defines a structure of left $M_2(F)$-module on $L_1 \oplus L_{-1}$, that by inspection coincides with the usual structure on $F^2 \otimes L_1$ (after identification of $L_{-1}$ and $L_1$ mentioned above).

Now $L$ defines two kind of structures: one is a Lie triple structure on $L_1 \oplus L_{-1}$, and the other is a ternary system considered by Faulkner in [4] on $L_1$ (roughly speaking, it is an asymmetric version of a Freudenthal triple system). Namely, define maps $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot, \cdot \rangle$ by formulas

$$[u, v] = \langle u, v \rangle e$$
$$\langle u, v, w \rangle = [[[f, u], v], w].$$

Note that $\langle \cdot, \cdot \rangle$ allows to identify the dual $L_1^*$ with $L_1$, and so the map

$$L_1 \otimes L_1 \to \text{End}(L_1)$$

corresponding to $\langle \cdot, \cdot, \cdot \rangle$ produces a linear map

$$\pi: \text{End}(L_1) \to \text{End}(L_1),$$

namely

$$\pi(\langle \cdot, u \rangle v) = \langle u, v, w \rangle.$$

By the Morita equivalence, we can consider $\pi$ as a map

$$\text{End}_{M_2(F)}(F^2 \otimes L_1) \to \text{End}_{M_2(F)}(F^2 \otimes L_1).$$
Also, the same equivalence gives rise to a Hermitian (with respect to the canonical symplectic involution on \( M_2(F) \)) form

\[
\phi \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \begin{pmatrix} \langle u_1, v_2 \rangle - \langle u_1, v_1 \rangle \\ \langle u_2, v_2 \rangle - \langle u_2, v_1 \rangle \end{pmatrix}.
\]

Now we want to relate the two structures on \( V_1 \oplus V_{-1} \simeq F^2 \otimes V_1 \). Direct calculation shows that

\[
D(u, v) = \frac{1}{2} \left( \pi(\phi(\cdot, u)v - \phi(\cdot, v)u) + \phi(v, u) - \phi(u, v) \right).
\]

This description admits a Galois descent. Namely, let \( Q \) be a quaternion algebra over \( F \), \( W \) be a left \( Q \)-module equipped with a Hermitian (with respect to the canonical involution on \( Q \)) form \( \phi \) and a linear map \( \pi: \text{End}_Q(W) \to \text{End}_Q(W) \).

Assume that \( \phi \) and \( \pi \) become maps as above over a splitting field of \( Q \). In terms of [7] this means that \( \text{End}_Q(W) \) together with \( \pi \) and the symplectic involution adjoint to \( \phi \) form a gift (an abbreviation for a generalized Freudenthal (or Faulkner) triple); one can state the conditions on \( \pi \) and \( \phi \) as a list of axioms not appealing to the descent (Garibaldi assumes that \( W \) is of dimension 28 over \( Q \), but this really doesn’t matter, at least under some additional restrictions on the characteristic of \( F \)). Then equation (*) defines on \( W \) a structure of a Lie triple system, hence the embedded Lie algebra \( \text{Der}(W) \oplus W \).

3 \( D_6 + A_1 \)-construction

We say that a map of functors \( A \to B \) from fields to sets is surjective at 2 if for any field \( F \) and \( b \in B(F) \) there exists an odd degree separable extension \( E/F \) and \( a \in A(E) \) such that the images of \( a \) and \( b \) in \( B(E) \) coincide.

We enumerate simple roots as in [2]. Erasing vertex 1 from the extended Dynkin diagram of \( E_7 \) we see that the simply connected split group \( E_7^{sc} \) contains a subgroup of type \( D_6 + A_1 \), namely \( (\text{Spin}_{12} \times \text{SL}_2)/\mu_2 \). Its image in the adjoint group \( E_7^{ad} \) is \( (\text{HSpin}_{12} \times \text{SL}_2)/\mu_2 \), which we denote by \( H \) for brevity.

**Theorem 1.** The map \( H^1(F, H) \to H^1(F, E_7^{ad}) \) is surjective at 2.
Proof. Note that \( W(D_6 + A_1) \) and \( W(E_7) \) has the same Sylow 2-subgroup. Then the result follows by repeating the argument from the proof of Proposition 14.7, Step 1 in [8] (this argument is a kind of folklore).

The long exact sequence

\[
H^1(F, \mu_2) \to H^1(F, H) \to H^1(\text{PGO}_{12}^+ \times \text{PGL}_2) \to H^2(F, \mu_2)
\]

shows that the orbits of \( H^1(F, H) \) under the action of \( H^1(F, \mu_2) \) are the isometry classes of central simple algebras of degree 12 with orthogonal involutions \((A, \sigma)\) and fixed isomorphism \( \text{Cent}(C_0(A, \sigma)) \simeq F \times F, \) with \([C_0^+(A, \sigma)] = [Q]\) in \( \text{Br}(F) \) for some quaternion algebra \( Q \), where \( C_0 \) stands for the Clifford algebra and \( C_0^+ \) for its first component (see [9, §8] for definitions). Now \( C_0^+(A, \sigma) \simeq \text{End}_Q(W) \)

for a 16-dimensional space \( W \) over \( Q \), and the canonical involution on \( C_0^+(A, \sigma) \) induces a Hermitian form \( \phi \) on \( W \) up to a scalar factor. It is not hard to see that \( H^1(F, H) \) parametrizes all the mentioned data together with \( \phi \) (and not only its similarity class), and \( H^1(F, \mu_2) \) multiplies \( \phi \) by the respective constant. Over a splitting field of \( Q \) the 32-dimensional half-spin representation carries a structure of Faulkner ternary system, so we are in the situation of Section 2. The resulting embedding \( 66 + 3 + 64 = 133 \)-dimensional Lie algebra \( \text{Der}(W) \oplus W \) is the twist of the split Lie algebra of type \( E_7 \) obtained by a cocycle representing the image in \( H^1(F, E_{7a}^d) \). Theorem 1 shows that any Lie algebra of type \( E_7 \) over \( F \) arises this way up to an odd degree extension.

4 Tits algebras and Rost invariant

Recall that the class of Tits algebra of a cocycle class from \( H^1(F, E_{7a}^d) \) is its image under the connecting map of the long exact sequence

\[
H^1(F, E_{7a}^{sc}) \to H^1(F, E_{7a}^{ad}) \to H^2(F, \mu_2).
\]

The sequence fits in the following diagram:

\[
\begin{array}{cccc}
H^1(F, (\text{Spin}_{12} \times \text{SL}_2)/\mu_2) & \to & H^1(F, H) & \to & H^2(F, \mu_2) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(F, E_{7a}^{sc}) & \longrightarrow & H^1(F, E_{7a}^{ad}) & \longrightarrow & H^2(F, \mu_2).
\end{array}
\]

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Since the middle vertical arrow is surjective at 2, we obtain the following result:

**Theorem 2.** The class of the Tits algebra in \( \text{Br}(F) \) of the class in \( H^1(F, E^\text{ad}_7) \) corresponding to \( \text{Der}(W) \oplus W \) is \([A] + [Q]\). For any cocycle class from \( H^1(F, E^\text{ad}_7) \) there is an odd degree extension \( E/F \) such that the class of the Tits algebra in \( \text{Br}(E) \) is a sum of three symbols.

**Proof.** Indeed, by the fundamental relation for groups of type \( D_6 \) (see [9, 9.14]) the image of a cocycle class from \( H^1(F, H) \) in \( H^2(F, \mu_2) \) is \([A] + [Q]\). Here \( Q \) is a quaternion algebra, and \( A \) over an odd degree extension is Brauer-equivalent to an algebra of degree 4 and exponent 2, that is to a biquaternion algebra. The second claim follows from Theorem 1. \( \square \)

Now we reproduce a construction from [13] in our terms. Let \( D \) be an algebra of degree 4 and \( \mu \) be a constant from \( F^\times \). By the exceptional isomorphism \( A_3 = D_3 \), the group \( \text{PGL}_1(D) \) defines a 3-dimensional anti-Hermitian form \( h \) over \( Q \) up to a constant, where \([Q] = 2[D]\) in \( \text{Br}(F) \). Consider the algebra \( M_6(Q) \) with the orthogonal involution \( \sigma \) adjoint to the 6-dimensional form \( h \perp -\mu h \). One of the component of \( C_0(M_6(Q), \sigma) \) is trivial in \( \text{Br}(F) \) and the other is Brauer-equivalent to \( Q \). Choose \( \phi \) on \( W = Q^{16} \); by Theorem 2 the class of the Tits algebra of the corresponding cocycle class in \( H^1(F, E^\text{ad}_7) \) is trivial, so the cocycle class comes from some \( \xi \in H^1(F, E^\text{sc}_7) \).

Let us compute the Rost invariant of \( \xi \) (see [8] or [9, § 31] for definitions).

**Theorem 3.** For \( D \) and \( \mu \) as above, there is a cocycle from \( H^1(F, E^\text{sc}_7) \) whose Rost invariant is \((\mu) \cup [D]\). In particular, if this element cannot be written as a sum of two symbols from \( H^3(F, \mathbb{Z}/2) \) with a common slot, then there is a strongly inner anisotropic group of type \( E_7 \) over \( F \).

**Proof.** Consider \( \xi \) as above. Over the function field \( F(\text{SB}(Q)) \) the image of the Rost invariant of \( \xi \) equals to the Arason invariant of the 12-dimensional quadratic form Morita-equivalent to \( h - \mu h \). Explicitly, over \( F(\text{SB}(Q)) \) the algebra \( D \) becomes a biquaternion algebra \((a, b) \otimes (c, d)\), and the quadratic form is Witt equivalent to \((\langle \mu \rangle)(\langle \langle a, b \rangle \rangle - \langle \langle c, d \rangle \rangle)\), so the Arason invariant equals the image of \((\mu) \cup [D]\) over \( F(\text{SB}(Q)) \). It follows that the Rost invariant is \((\mu) \cup [D] + (\lambda) \cup [Q]\) for some \( Q \). Changing \( \phi \) to \( \lambda \phi \) adds \((\lambda) \cup [Q]\) (cf. [9, p. 441]), so there is a cocycle class from \( H^1(F, E^\text{sc}_7) \) whose Rost invariant is \((\mu) \cup [D]\). The last claim follows from the easy computation of the Rost invariant of cocycles corresponding to isotropic groups of type \( E_7 \) (cf. [8, Appendix A, Proposition]). \( \square \)
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