Probabilistic state transfer, estimation and measures for optimal actuator/sensor placement for linear systems with packet dropouts

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Abstract—We consider linear systems subject to packet dropouts and obtain necessary and sufficient conditions for an arbitrary state transfer and state estimation over a finite time instance T. The data loss signal is modeled using the Bernoulli random variable. We leverage properties of the Hadamard product in our approach and use the derived necessary and sufficient conditions to compute the probability that an arbitrary state transfer is possible at a specified time instant. Similarly, the probability of finding an exact state estimate is found using the observability counterparts of the results. Using the necessary and sufficient conditions obtained for the invertibility of the Gramian, we give new probabilistic measures for optimal actuator and sensor placement problems and obtain optimal/sub-optimal solutions. We demonstrate by an example how the probabilities of packet dropouts influence the choice of an optimal actuator. We also discuss how to implement feedback laws and the LQR problem for these models involving packet dropouts.

I. INTRODUCTION

In many modern control systems, the plant and the controllers are geographically distributed and connected to each other via a communication network. One expects that there are disruptions in this communication network due to the presence of non-idealities such as packet losses in wireless communication. There could be time instances where no actuator input is available for control or no sensor output to observe when there are packet dropouts. This greatly influences system theoretic properties of control systems. We refer the reader to [1],[7],[13]-[16],[23] for details on optimal actuator and sensor placement problems.

In this section, we build some preliminaries to be used in the sequel.

Definition 1: Suppose that \( t \in \mathbb{N} \cup \{0\} \), the probability that no packet dropout occurs at \( t \) is \( p \). An admissible
switching signal $\sigma$ is defined by the Markovian model (Figure 1) where the two nodes are labeled as $s_0$ and $s_1$ and edges are labeled by pairs $(1, p)$ and $(0, 1-p)$. A sequence $\sigma(0)\sigma(1)\ldots$ is admissible if there exists a path in the graph above where the successive first component of the edge labels carries the sequence. The probability of occurrence of $\sigma$ is obtained by multiplying the second components of all the edge labels in the path above. The set of all admissible switching signals for length $t$ is denoted by $S^t$.

We now define the controllability matrix for (2). Rewriting (2) as

$$
\begin{align*}
x(t) &= Ax(t-1) + B\sigma(t-1)n(t-1) \\
&= A'x(0) + \sum_{i=0}^{t-1} A'^{-1}B\sigma(i)n(i) \\
&= A'x(0) + C_{\sigma(t-1)}(A,B)\tilde{u}
\end{align*}
$$

where $C_{\sigma(t-1)}(A,B) = \left[ A'^{-1}B\sigma(0) \ldots AB\sigma(t-2) B\sigma(t-1) \right]$ is the controllability matrix associated with a signal $\sigma$ at time $(t-1)$ and $\tilde{u} = [u^*(0) u^*(1) \ldots u^*(t-2) u^*(t-1)]^*$.

We now give an expression for the controllability Gramian for the system (2) for a fixed signal $\sigma$ and a fixed time $t$.

**Definition 2:**

$$W_{\sigma(t-1)} := C_{\sigma(t-1)}(A,B)C_{\sigma(t-1)}(A,B)^*$$

is the controllability Gramian associated with system (2) at time $(t-1)$ with respect to the switching signal $\sigma$.

Recall that for classical discrete LTI systems, the controllability Gramian is given by $W_x(A,B) = \sum_{i=0}^{t} A'^{i}BB^*(A^*)^i$. For a fixed time $t$ and a fixed signal $\sigma$, the controllability Gramian for (2) is also given by

$$W_{\sigma(t)} = \sum_{i=0}^{t} \sigma(t-i)A'^{i}BB^*(A^*)^i.$$  \hfill (9)

In the following proposition, we state how to compute the energy required for a state transfer from $x(0)$ to $x(t)$ for a fixed signal $\sigma$ using $W_{\sigma(t)}$.

**Proposition 1:** Assume that $C_{\sigma(t-1)}(A,B)$ is full rank. For a system of the form (2), the minimum input energy required to drive the state from $x(0)$ to $x(t)$ is

$$
(x(t) - A'x(0))^*W_{\sigma(t-1)}^{-1}(x(t) - A'x(0)). \hfill (10)
$$

**Proof:** Follows from the similar arguments used for the LTI case in [10], Chapter 3, Section 22, Theorem 1.

We now give an expression for the controllability Gramian using the Hadamard product of matrices ([8]).

**Assumption 1:** We assume that the discrete linear system $x(t+1) = Ax(t) + Bu(t)$ is controllable.

**Assumption 2:** We assume that $A$ is diagonalizable. We also assume that no two eigenvalues of $A$ have the same modulus. Furthermore, $0$ is not an eigenvalue of $A$.

The assumption of diagonalizability was also made in [4] where the decentralized control of discrete LTI systems is considered and also in [11] for discrete LTI systems.

Let $v_1, v_2, \ldots, v_n$ be the left eigenvectors of $A$ and $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. We define the following $n \times (t+1)$ matrix.

$$\Lambda_t := \begin{bmatrix}
1 & \lambda_1 & \cdots & \lambda_1^t \\
1 & \lambda_2 & \cdots & \lambda_2^t \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_n & \cdots & \lambda_n^t
\end{bmatrix}.$$  \hfill (11)

**Definition 3:** For a switching signal $\sigma$, we define

$$\Lambda_{\sigma(t)} := \begin{bmatrix}
\sigma(t) & \sigma(t-1)\lambda_1 & \cdots & \sigma(0)\lambda_1^t \\
\sigma(t) & \sigma(t-1)\lambda_2 & \cdots & \sigma(0)\lambda_2^t \\
\vdots & \vdots & \ddots & \vdots \\
\sigma(t) & \sigma(t-1)\lambda_n & \cdots & \sigma(0)\lambda_n^t
\end{bmatrix}. \hfill (12)$$

Let $\hat{\Lambda}_{\sigma(t)}$ be the matrix obtained from the above matrix by keeping only non-zero columns. Let $V^* = [v_1^* v_2^* \ldots v_n^*]$ be a matrix whose columns are right eigenvectors of $A^*$.

**Theorem 1:** Consider a discrete linear system of the form (2). Let $V$ be a non-singular matrix such that rows of $V$ form a set of left eigenvectors of $A$. Then, choosing rows of $V$ as a basis for $\mathbb{C}^n$, the controllability Gramian for (2) for a switching signal $\sigma$ is given by $(VBB^*V^*)\circ(\hat{\Lambda}_{\sigma(t)}\hat{\Lambda}_{\sigma(t)}^*)$ (where $\circ$ denotes the Hadamard product and $\hat{\Lambda}_{\sigma(t)}$ is obtained from (12) by dropping columns with all zeros).

**Proof:** Let $V$ be a matrix whose rows are left eigenvectors of $A$, hence $VA = DA^t V$ where $D_A$ is a diagonal matrix having eigenvalues of $A$. Consider a new basis for $\mathbb{C}^n$ given by the rows of $V$. Let $\hat{A} = VAV^{-1}, \hat{B} = VB$. Thus, the controllability Gramian $\hat{W}_{\sigma(t)} = \sum_{i=0}^{t} \sigma(t)A'^tBB^*(A^*)^i = V\sum_{i=0}^{t} \sigma(t)A'^tBB^*(A^*)^i V^* = \sum_{i=0}^{t} \sigma(t)\hat{D}_A'^iVBB^*V^*(\hat{A}_A^*)^i = (VBB^*V^*) \circ (\hat{\Lambda}_{\sigma(t)}\hat{\Lambda}_{\sigma(t)}^*)$.

III. PROBABILISTIC STATE TRANSFER AND STATE ESTIMATION

A. Probabilistic state transfer

We use properties of the Hadamard product to obtain the necessary and sufficient conditions for an arbitrary state transfer of (2). We need the following result from [8].

**Lemma 1:** If $X,Y$ are positive semi-definite, then so is $X \circ Y$. If, in addition, $Y$ is positive definite and $X$ has no diagonal entry equal to 0, then $X \circ Y$ is positive definite. In particular, if both $X$ and $Y$ are positive definite, then so is $X \circ Y$.

**Proof:** We refer the reader to Theorem 5.2.1 of [8].

**Theorem 2:** Let $p$ be the probability that $\sigma(t) = 1$ where $t \in \mathbb{N}$. Consider a single input system of the form (2). Suppose $A$ is non-singular and $(A, B)$ controllable. Then,

1) $W_{\sigma(t)}$ is positive definite if and only if $\sigma$ is non-zero for at least $n$ time instances.

2) The probability that an arbitrary state transfer from $x_0 \in \mathbb{C}^n$ to $x_f \in \mathbb{C}^n$ is possible for (2) is given by $P(T) = \sum_{t=n}^{\infty} \binom{t}{n} p^t (1-p)^{t-n}$. 

Proof: Note that after a change of basis, \( W_{\sigma(t)} = (VBB^*V^*) \circ (\hat{\Lambda}_{\sigma(t)} \hat{\Lambda}_{\sigma(t)}^*) \). By Assumption 1, \( VBB^*V^* \) has non-zero diagonal entries. With reference to the Lemma 1, \( W_{\sigma(t)} \) is positive definite if the matrix \( \hat{\Lambda}_{\sigma(t)} \hat{\Lambda}_{\sigma(t)}^* \) is positive definite. For a single input system, \( VBB^*V^* \) has rank one. It is shown in [8] that rank \((X \circ Y) \leq \text{rank}(X) \cdot \text{rank}(Y)\). Thus, for \( VBB^*V^* \circ (\hat{\Lambda}_{\sigma(t)} \hat{\Lambda}_{\sigma(t)}^*) \) to be of full rank, \( \hat{\Lambda}_{\sigma(t)} \hat{\Lambda}_{\sigma(t)}^* \) must be full rank. By Assumption 1, \( \hat{\Lambda}_{\sigma(t)} \) is controllable and by Assumption 2, \( \hat{\Lambda}_{\sigma(t)} \) is diagonalizable.

Remark 1: For multi-input systems or the case where \( A \) is diagonalizable with repeated real eigenvalues, it could happen that \( W_{\sigma(t)} \) is full rank but both \( VBB^*V^* \) and \( \hat{\Lambda}_{\sigma(t)} \) are not full rank. Furthermore, when \( A \) has repeated eigenvalues, \( \hat{\Lambda}_{\sigma(t)} \) is never full rank. Thus, we can not apply Theorem 2 to characterize signals for which \( W_{\sigma(t)} \) is positive definite.

We give the following result from [9] which is required in our next result for the case of repeated eigenvalues of \( A \).

Proposition 2: Suppose \((A, B)\) is controllable. Let \( m \) be the number of inputs and \( k \) be the cyclic index of \( A \) (i.e. the number of invariant factors of \( A \)). Let \( B \) be the column span of \( B = [b_1, b_2, \ldots, b_m] \). Let \( B_i = [b_1, \ldots, b_i] (1 \leq i \leq m) \) denotes the span of the corresponding vectors. Let \( \alpha_1, \ldots, \alpha_k \) be the invariant factors of \( A \). Then, there exists \( A \)-invariant subspaces \( X_i \subset \mathbb{C}^n \) and subspaces \( B_i \subset B \) such that:

\[
\begin{align*}
\mathbb{C}^n &= X_1 \oplus \ldots \oplus X_k, \\
A \text{ restricted to } X_i &= \text{ cyclic with minimal polynomial } \alpha_i, \\
(A_i, B_i) &= X_1 \oplus \ldots \oplus X_i.
\end{align*}
\]

Proof: We refer the reader to Theorem 1.2 of [9].

From Proposition 2, it can be shown that there exists a basis such that \( A, B \) can be transformed as

\[
A = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A_k
\end{bmatrix}, \quad B = \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1k} \\
b_{21} & b_{22} & \cdots & b_{2k} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & b_{kk}
\end{bmatrix}
\]

where \((A_i, b_{ii})\) is controllable ([9], page 44) (note that entries * in the matrix \( B \) denote all the remaining columns of \( B \)).

Theorem 3: Let \( \alpha_i (1 \leq i \leq k) \) be the invariant factors of \( A \) such that \( \alpha_{i+1} \mid \alpha_i \) (where \( k \) is the cyclic index of \( A \)). Let \( n_1 \) be the degree of the minimal polynomial \( \alpha_1 \). Let \( m \) be the number of inputs such that \( k \leq m \leq n \). If the switching signal \( \sigma \) has at least \( n_1 \) non-zero entries, then \( W_{\sigma(t)} \) is positive definite.

Proof: For simplicity, we assume that there are just two invariant factors \( \alpha_1 \) and \( \alpha_2 \). The general case follows in exactly similar manner. Let

\[
A = \begin{bmatrix}
A_1 & 0 & 0 \\
0 & A_2 & 0 \\
0 & 0 & A_k
\end{bmatrix}, \quad B = \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1k} \\
b_{21} & b_{22} & \cdots & b_{2k} \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
\hat{\Lambda}_{\sigma(t)} = \begin{bmatrix}
\hat{\Lambda}_{\sigma(t)}(A_1) & 0 \\
0 & \hat{\Lambda}_{\sigma(t)}(A_2)
\end{bmatrix}, \quad V = \begin{bmatrix}
V_1 & 0 \\
0 & V_2
\end{bmatrix}
\]

where

\[
V_1 \equiv V_{11} \hat{\Lambda}_{\sigma(t)}(A_1), \quad V_2 \equiv V_{22} \hat{\Lambda}_{\sigma(t)}(A_2)
\]

Let \( P = \hat{\Lambda}_{\sigma(t)} \hat{\Lambda}_{\sigma(t)}^* \).

From Theorem 1,

\[
W_{\sigma(t)} = (VBB^*V^*) \circ (\hat{\Lambda}_{\sigma(t)} \hat{\Lambda}_{\sigma(t)}^*).
\]

Let \( b_1 = \begin{bmatrix} b_{11} \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \) and \( B_3 = \begin{bmatrix} * \\ * \end{bmatrix} \). Observe that

\[
BB^* = b_1 b_1^* + b_2 b_2^* + B_3 B_3^*.
\]

(17)

(18)

(19)

(20)

(21)

Let \( P = \hat{\Lambda}_{\sigma(t)} \hat{\Lambda}_{\sigma(t)}^* \). Note that

\[
P = \begin{bmatrix}
\hat{\Lambda}_{\sigma(t)}(A_1) \hat{\Lambda}_{\sigma(t)}^*(A_1) \\
\hat{\Lambda}_{\sigma(t)}(A_2) \hat{\Lambda}_{\sigma(t)}^*(A_2)
\end{bmatrix}
\]

(16)

(17)

(18)

(19)

(20)

(21)

(19)
sum of \(k + 1\) positive semidefinite matrices and apply the same trick used above. If \(m = k\), then we define \(M_{k+1} = 0\) and the same arguments work.

**Example 1:** Let

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}, B = \begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 1 \\
0 & 1
\end{bmatrix}.
\]

Observe that \(A\) has two invariant factors of degree 3 and 2 respectively. Thus, \(n_1 = 3, n_2 = 2\). Let \(\sigma = \{1, 0, 0, 1, 0, 0, 1\}\). We observe that the condition of Theorem 3 is satisfied and \(W_\sigma\) is positive definite.

Note that using the notation used in the above theorem, we can write \(BB^T = \sum_{i=1}^{k+1} b_i b_i^T + B_{k+1} B_{k+1}^T\). The controllability Gramian is given by

\[
W_{\sigma(t)} = \sum_{i=1}^{k+1} M_i
\]

where \(M_i\) is as defined in the proof of Theorem 3. The following corollary gives necessary and sufficient conditions for controllability of multi-input systems.

**Corollary 3.1:** For a multi-input system \((A, B)\), \(W_{\sigma(t)}\) is positive definite for a signal \(\sigma\) at time \(t \Leftrightarrow \bigcap_{i=1}^{k+1} \ker(M_i) = 0\).

**Proof:** Note that since \(M_i \geq 0 (i = 1, \ldots, k + 1)\), from Equation (22), \(\ker(W_{\sigma(t)}) = 0\) implies that \(\bigcap_{i=1}^{k+1} \ker(M_i) = 0\) and conversely.

Thus, given a signal \(\sigma\) and a fixed time \(t\), we obtain necessary and sufficient conditions for \(W_{\sigma(t)}\) to be positive definite for single input as well as multi-input systems. Note that the sufficient condition of Theorem 3 is not necessary. For example, in Example [1] if \(B = I_2\), then for any non-trivial \(\sigma\), \(W_\sigma\) is positive definite for \(\sigma\) even when the number of its non-zero entries are strictly less than the degree of the minimal polynomial of \(A\).

**Lemma 2:** Let \(m\) be the number of inputs of \((A, B)\) and \(n\) be the dimension of the state space. Let \(l = \left\lfloor \frac{n}{m} \right\rfloor\). Then, \(W_\sigma\) is singular if the number of non-zero entries of \(\sigma\) is strictly less than \(l\).

**Proof:** If the number of non-zero entries of \(\sigma\) are strictly less than \(l\), then the number of columns of \(C_\sigma(A, B)\) are less than \(n\). Hence, \(W_\sigma\) is singular.

The above lemma says that for \(W_\sigma\) to be non-singular, \(\sigma\) must have at least \(l\) non-zero entries. Thus, we have a necessary condition of \(\sigma\) which can be checked by looking at the entries of \(\sigma\). Again this necessary condition is not sufficient as the following example shows.

**Example 2:** Let \(A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}, B = \begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 1 \\
0 & 1
\end{bmatrix}.
\]

Since \(n = 4\) and \(m = 2\), \(\left\lfloor \frac{n}{m} \right\rfloor = 2\). Observe that if \(\sigma\) has two non-zero entries (say \(\sigma = \{1, 0, 0, 1\}\), \(W_\sigma\) still remains singular.

**Remark 2:** Let \(n_\sigma\) be the number of non-zero entries of \(\sigma\) and \(n_1\) be the degree of the minimal polynomial of the system matrix \(A\). Then we observe that

- if \(n_\sigma \geq \left\lceil \frac{n}{m} \right\rceil\), then \(W_\sigma\) is positive definite.
- if \(\left\lceil \frac{n}{m} \right\rceil \leq n_\sigma < n_1\), then we need Corollary 3.1.

**Theorem 4:** Let \(p\) be the probability that \(\sigma(t)\) is singular for all \(t \in \mathbb{N}\). Consider a multi-input system of the form (4). Assume that \(A\) is non-singular and \((A, B)\) controllable. Let \(m\) be the number of inputs and \(n_1\) be the degree of the minimal polynomial of \(A\). Let \(n\) be the dimension of state space and \(l = \left\lceil \frac{n}{m} \right\rceil\). Let \(P(T)\) be the probability that an arbitrary state transfer is possible at time \(T\). Then,

\[
\sum_{i=n_1}^{T} \binom{T}{i} p^i(1-p)^{T-i} \leq P(T) \leq 1 - \sum_{i=1}^{T-1} \binom{T}{i} p^i(1-p)^{T-i}.
\]

**Proof:** It follows from Theorem 3 that if the switching signal \(\sigma\) has at least \(n_1\) non zero entries, then \(\sigma(t)\) is controllable. Thus, \(\sum_{i=n_1}^{T} \binom{T}{i} p^i(1-p)^{T-i} \leq P(T)\). From Lemma 2 it is clear that for a switching signal \(\sigma\), an arbitrary state transfer is not possible for \(\sigma\) if the number of non zero entries of the switching signal \(\sigma\) is strictly less than \(l\). Therefore, \(P(T) \leq 1 - \sum_{i=1}^{T-1} \binom{T}{i} p^i(1-p)^{T-i}\).

**Definition 4:** Let \(S_T^z\) be the set of switching signals for which an arbitrary state transfer is possible in time \(T\). Let \(S_T^{z_n}\) denote the set of switching signals of length \(T\) with the number of non zero entries greater than or equal to \(n\) and \(S_T^{z_n}\) denote the set of switching signals of length \(T\) with the number of non zero entries less than or equal to \(n\).

**Remark 3:** It is clear that for single input systems, \(S_T^z = S_T^{z_n}\). For multi-input systems, we do not have an exact enumeration of \(S_T^z\). However, \(S_T^{z_m} \subset S_T \setminus S_T^{z_{\lceil \frac{n}{m} \rceil}}\) where \(m\) is the number of inputs.

**Definition 5:** Let \(P(\sigma)\) be the probability of occurrence of \(\sigma\). The average control input energy to go from \(x_0\) to \(x_f\) in \(T\) time steps over the set \(S_T^z\) is

\[
E_{av}(x_0, x_f, T) := \sum_{\sigma \in S_T^z} P(\sigma)(x_f - x_0)^TW_{\sigma(T)}^{-1}(x_f - x_0).
\]

**Theorem 5:** Let \(n\) be the degree of the characteristic polynomial of \(A\) and let \(n_1\) be the degree of minimal polynomial of \(A\). Let \(P(\sigma)\) be the probability of occurrence of \(\sigma\). Then,

- For single input systems (2), \(E_{av}(x_0, x_f, T) = \sum_{\sigma \in S_T^{z_n}} P(\sigma)(x_f - x_0)^TW_{\sigma(T)}^{-1}(x_f - x_0)\).

- For multi-input systems (2), \(E_{av}(x_0, x_f, T) \geq \sum_{\sigma \in S_T^{z_{\lceil \frac{n}{m} \rceil}}} P(\sigma)(x_f - x_0)^TW_{\sigma(T)}^{-1}(x_f - x_0)\).

**Proof:** Follows from Definition 5, Theorem 2 and Theorem 4.

Next, we consider the dual state estimation problem to obtain the probability that state estimation is possible from measured outputs with time going from 0 to \(T\).

**B. Probabilistic state estimation**

The next natural step is to consider non-idealities in the transmission of measurements obtained by sensors over a communication network for discrete LTI systems [1], [15],
Consider the following discrete linear system subject to packet dropouts
\[
\begin{align*}
    x(t+1) &= Ax(t) \\
y(t) &= \begin{cases} Cx(t), & \text{if } \sigma(t) = 1, \\
0, & \text{if } \sigma(t) = 0.
\end{cases}
\end{align*}
\] (27)

where the switching signal \(\sigma\) is random signal with Bernoulli distribution, taking values 0 and 1. The observability Gramian associated with discrete LTI systems is defined as \(W_o^\tau := \sum_{i=0}^\tau (A^i)^*C^*CA^i\) (17). If \(W_o^\tau\) is singular, then the states in the null space of \(W_o^\tau\) are unobservable.

Define the associated observability Gramian for \(\tau\) as
\[
W_o^\tau := \sum_{i=0}^\tau \sigma(i)(A^i)^*C^*CA^i.
\] (28)

Note that, the observability matrix \(O_{\sigma(t)}(C, A) = \begin{bmatrix} \sigma(0)C^* & \sigma(1)(CA)^* & \cdots & \sigma(\tau)(CA^\tau)^* \end{bmatrix}^*\).

Let \(y = \left[ y(0)^* \ y(1)^* \cdots \ y(\tau)^* \right]^*\) be a vector of observed outputs. Then using \(y(t) = C\sigma(t)x(t)\), we get
\[
x(0) = (W_o(\sigma(t))^*O_{\sigma(t)}(C, A)y).
\]

The exact counterpart of the arbitrary state transfer Theorems (Theorem 1, Theorem 2, Theorem 3) Theorem 4, hold for the observability Gramian and the state estimation for a given switching signal.

Let \(S_o^\tau\) be the set of switching signals for which the observability Gramian becomes full column rank. Using counterparts of Theorem 2 and Theorem 4, we can find the probability that state estimation is possible from the measured outputs \(y(0), \ldots, y(\tau)\) with packet dropouts.

Note that \(y^*y = x^*(0)W_o^\tau x(0)\). The average output energy for a fixed time \(T \) over the set \(S_o^T\) is defined as
\[
E_{\text{av}}(x(0), T) = \sum_{\sigma \in S_o^T} P(\sigma)x^*(0)W_o^\tau x(0).
\] (29)

It follows that the counterpart of Theorem 5 holds.

IV. PROBABILISTIC MEASURES FOR OPTIMAL ACTUATOR/SENSOR PLACEMENT

We use the results obtained in the previous section to address optimal actuator/sensor placement problems for linear systems with packet dropouts. The following definition gives a new probabilistic measure for our models.

Definition 6: Let \(T\) be fixed and
\[
\mu(T) := \sum_{\sigma \in S_o^T} P(\sigma)\det(W_o(\sigma(T))).
\] (30)

Remark 4: We can use (30) as a controllability metric for the optimal actuator placement problem. Equation (30) gives average volume reached over all signals in \(S_o^T\) using unit energy inputs. Note that by Remark 3 for single input systems, \(S_o^T = S_n^T\). Hence, for single input systems, \(\mu(T) = \sum_{\sigma \in S_n^T} P(\sigma)\det(W_o(\sigma(T)))\) thus, one can obtain the optimal solution as \(S_n^T\) can be enumerated for a fixed time instance \(T\).

For multi-input systems, \(S_{n1}^T \subset S_o^T\). Hence,
\[
\hat{\mu}(T) := \sum_{\sigma \in S_{n1}^T} P(\sigma)\det(W_o(\sigma(T))) \leq \mu(T).
\] (31)

Thus, we can use \(\hat{\mu}(T)\) as a lower bound for optimal actuator placement problem for multi-input systems. In other words, we consider the sub-optimal solution for multi-input systems obtained by considering \(\hat{\mu}(T)\) as a selection criterion.

The following example demonstrates that the choice of an optimal actuator could be different for classical systems and systems with packet dropouts. It can be observed that the probabilities of packet dropouts associated with the communication network for each actuator plays a role in deciding the optimal actuator.

Example 3: Let
\[
A = .04 \begin{bmatrix} 2 & 16 & 4 \\
5 & 10 & 15 \\
1 & 20 & 12 \end{bmatrix}, 
B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 2 \\
4 & 1 & 1 \\
2 & 3 & 4 \end{bmatrix}.
\]

Note that \((A, b_i)\) is controllable for \(i = 1, 2, 3\). Suppose we want to choose an optimal actuator from \(b_1, b_2, b_3\). Let \(T = 3\), and let \(\det(W_o(A, b))\) be the controllability metric. The values of the determinant of the Gramian for three actuators are 9.1718, 315.2886 and 2.6837 respectively, implying that \(b_2\) gives the optimal actuator for the classical case.

Now suppose each actuator is connected to the system by a different communication network. Let \(p_1 = 0.8, p_2 = 0.4\) and \(p_3 = 0.5\) be the probabilities of packet dropouts corresponding to the three actuators \(b_1, b_2\) and \(b_3\) respectively. Using Theorem 2 with (30) as a controllability metric, the values of \(\hat{\mu}(3)\) for the three actuators are 4.6959, 2.5223 and 0.3355 respectively. Thus, \(b_1\) is the optimal actuator.

This demonstrates the role of probabilities of packet dropouts in the communication network deciding the choice of the optimal actuator.

Remark 5: Note that for single output systems, \(S_o^T = S_n^T\) and for multi-output systems, \(S_o^T \subset S_n^T\). We can define similar probabilistic measure as mentioned in Definition 6 and Remark 4 for the optimal sensor placement problem. It is clear from Example 3 that with the probabilistic measure, the choice of the optimal sensor is different from the classical observability measure.

Remark 6: Another possible controllability metric is
\[
\gamma(T) := \sum_{\sigma \in S_o^T} P(\sigma)\text{trace}(W_o(\sigma(T))).
\]
We can similarly compare actuator placement problems for both classical and probabilistic models. In future, we wish to generalize some of the classical controllability metrics for models considered here.

A. Feedback laws and LQR

Consider the following model
\[
\begin{align*}
x(t+1) &= Ax(t) + B\sigma(t)u(t) \\
y(t) &= C\sigma(t)x(t).
\end{align*}
\] (32)

Observe that the same switching signal is used for the measured sensor outputs and the actuator inputs. Therefore, it is clear that if the state estimation is possible for a switching signal, then state feedback laws can be implemented for that particular switching signal. Suppose the observability matrix \(O_{\sigma(T)}(C, A)\) is full column rank for a particular signal \(\sigma\). Hence, \(x(0)\) is uniquely determined. Thus, from the input-state equation \(x(t+1) = Ax(t) + B\sigma(t)u(t)\), one can find
the current state \( x(t) \) which allows us to implement state feedback laws.

One can consider the finite horizon LQR problem for each switching signal \( \sigma \) as follows:

\[
\min J_{\sigma} := x^*(T)Q_jx(T) + \sum_{t=0}^{T-1} x^*(t)Qx(t) + u^*(t)Ru(t) \tag{33}
\]

where \( Q, Q_j \geq 0, R > 0 \). Suppose the initial condition \( x(0) \) is fixed. For a fixed \( \sigma \), we can consider \( J_{\sigma} \) as a linear time varying system and consider the LQR problem for linear time varying systems by choosing \( A(t) = A \) and \( B(t) = B\sigma(t) \). By solving the difference Riccati equation with time varying coefficient \( B(t) \), we can obtain a state feedback \( u(t) = -K(t)x(t) \) (for a fixed signal \( \sigma \)) as a solution of the LQR problem. From the solution of the difference Riccati equation, we can compute the optimal cost say \( J_{\sigma} \) for each \( \sigma \). Thus, we can compute the average LQR cost \( J_{avg} = \sum_{\sigma \in \mathcal{S}} P(\sigma)J_{\sigma} \) by considering all switching signals \( \mathcal{S}^f \) (or \( \mathcal{S}_O^f \)) for which the controllability and the observability matrix becomes full rank. The average LQR cost can also be used as a selection metric for optimal actuator placement problem for a fixed initial state.

In future, we wish to consider different models of switching signals for actuators and sensors instead of the model considered here.

V. CONCLUSION AND FUTURE WORK

We found necessary and sufficient conditions on the admissible signals \( \sigma \) (which models systems with a packet loss) such that the controllability Gramian \( W_{\sigma(t)} \) is positive definite for a fixed \( t \). This allows us to obtain necessary and sufficient conditions for an arbitrary state transfer for our models. We considered the analogous state estimation problem as well. We introduced a notion of average input/output energy and defined a new probabilistic measure which allowed us to have a new selection criterion for optimal actuator/sensor placement problem for single input/multi-input systems with packet dropouts. We stated how feedback laws and LQR problem can be considered for these models.

In future, we wish to extend the results obtained for more general systems by relaxing a few assumptions made here. We wish to develop efficient algorithms/heuristics to solve the optimal actuator/sensor placement problem and extend the other classical controllability metrics to systems with packet dropouts using similar ideas. Moreover, we wish to analyze the performance of the probabilistic measure defined in this article by using the notion of tight frames used in [6]; where we expect that the tight frames would lead to an optimal solution [6]. Furthermore, we wish to study the optimal actuator placement problem subject to energy bounds considered in [5]. There are networks where the probability that a packet dropout could be a function of system states such as transmission rate. In future, we wish to consider such state-dependent switching signals to obtain trade-offs between the allowable packet loss probability and the transmission rate such that the system remains controllable. Moreover, we also wish to incorporate time delays in our switching signals.

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