Ponzano-Regge model revisited II: Equivalence with Chern-Simons

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Abstract

We provide a mathematical definition of the gauge fixed Ponzano-Regge model showing that it gives a measure on the space of flat connections whose volume is well defined. We then show that the Ponzano-Regge model can be equivalently expressed as Reshetikhin-Turaev evaluation of a colored chain mail link based on $\mathcal{D}(SU(2))$: a non compact quantum group being the Drinfeld double of $SU(2)$ and a deformation of the Poincare algebra. This proves the equivalence between spin foam quantization and Chern-Simons quantization of three dimensional gravity without cosmological constant. We extend this correspondence to the computation of expectation value of physical observables and insertion of particles.

Keywords:
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I. INTRODUCTION

The Ponzano-Regge model was the first model of quantum gravity ever proposed [1]. It gives a prescription for the computation of scalar product in 3-dimensional Euclidean quantum gravity. It is a state sum model over discretized geometries where the main building block is the Racah-Wigner 6j symbol of the classical SU(2) group and it has been mainly studied by physicists. The asymptotic of the 6j symbol was noticed to be given by the cosine of the Regge action [1]. Arguments in favor of this asymptotic were given [2] and a mathematical proof appeared in [3] (see also [4] for a geometrical simple proof). The relation between the Ponzano-Regge model and 3d gravity was further understood: It was proven that the Ponzano-Regge model realizes a projection on the space of flat SU(2) connections [5] and that it can be obtained from a discretization of a first order formulation of gravity ([6] and reference therein).

On the mathematical side, there is the Turaev-Viro model [7], which is the analog of Ponzano-Regge model but based on the quantum group SU(2)_q with q a root of unity q = exp(i2π/k). For this model, the sum over representations is finite and the invariance under refinement of the triangulation is exact and not formal. Moreover, it is well known that classically Euclidean 2+1 gravity with a positive cosmological constant is equivalent to a SO(4) Chern-Simons theory [8]. This equivalence has been proven to extend at the quantum level: Not only the Turaev-Viro partition function is the square of the SU(2) Chern-Simons partition function [9, 10], but there is also a one to one correspondence between the states of Turaev-Viro’s Hilbert space and Chern-Simons gauge invariant operators [11, 12]. This correspondence makes it clear that the Turaev-Viro model is related to Euclidean 3d gravity with a cosmological constant Λ = (2π/k)^2. This is consistent with the asymptotic of SU(2) 6j symbols [13, 14]. However, one should regret that there is not yet, a direct relationship between this model and a discretization of gravity as this is the case for the Ponzano-Regge model. Also, the relation between the Turaev-Viro and Ponzano-Regge model is not well understood: One would naively expect that the limit when k goes to ∞ of the Turaev-Viro model to reproduce the Ponzano-Regge amplitude. This is not true since the representations of spin comparable to the cut-off k still contributes in the limit [15, 16]. Eventually, 3d Euclidean or Lorentzian Gravity with or without cosmological constant is always equivalent classically to a Chern-Simons theory. However, it is only in the case of Euclidean gravity with positive cosmological constant, the case address by the Turaev-Viro model, that the Chern-Simons gauge group is compact SO(4) ∼ SU(2) × SU(2). In the other cases, Lorentzian gravity or Euclidean gravity with 0 or negative cosmological constant the gauge group is either a Poincare group or SO(3, 1) ∼ SL(2, C) or SO(2, 2) ∼ SL(2, R) × SL(2, R) hence a non-compact gauge group. It is then necessary, if we want to understand quantum Lorentzian geometry, to have a well defined mathematical model describing 3d quantum gravity based on a noncompact gauge group. The Ponzano-Regge model is the simplest model we know addressing this question1.

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1 see [17] for a state sum model describing Lorentzian 3d gravity
Therefore, we still need to have a direct mathematical understanding of the Ponzano-Regge model. This is the purpose of this paper. Two problems need to be addressed: first, the Ponzano-Regge was ill-defined as a mathematical model. There are divergences in the summation that need to be regularized, moreover, after regularization, the Ponzano-Regge model is not necessarily invariant under refinement of the triangulation [18]. It was only recently understood [19, 20] that these residual divergences were due to a residual ‘translational’ symmetry of the model. This symmetry being non-compact should be gauge fixed. In this paper we review the definition of the properly gauge fixed Ponzano-Regge model from a mathematical point of view. We show that, given a three manifold $M$, the Ponzano-Regge model defines a measure on the space of flat $SU(2)$ connections on $M$ and that the volume of this measure is truly independent of the triangulation and gauge fixing.

The second aspect of the theory addressed here is the equivalence with Chern-Simons theory at the quantum level. At the classical level, 2+1 gravity with no cosmological constant is equivalent to Chern-Simons theory based on the Poincare group. It was shown in [20] that there is an hidden quantum group symmetry in the Ponzano-Regge model: The braiding of particles being represented by the action of the braiding matrix of $\mathcal{D}(SU(2))$. This is a non-compact quantum group which is the quantum double of $SU(2)$ and a kappa deformation of the Poincare group [21, 22]. This quantum group is also known to be the building block of the combinatorial Chern-Simons quantization of 2+1 gravity without cosmological constant [23]. In this article, we show that the Ponzano-Regge partition function can be written as the Reshetikhin-Turaev evaluation, based on $\mathcal{D}(SU(2))$, of a chain mail link. This proves the equivalence at the quantum level between the spin foam and Chern-Simons quantization. We also prove that the evaluation of Chern-Simons Wilson lines reproduces the insertion of spinning particles which was proposed in [20].

We have included in the main text our definitions and main results. In order to simplify the reading we have put proofs of our statements in appendix for the more dedicated reader. We have also included a review of old and new results on the recoupling theory of $\mathcal{D}(SU(2))$.

The plan of the paper is as follows: In section II we give the definition of the regularized (gauge fixed) Ponzano-Regge invariant, and justification for this definition. We show that this invariant is the volume of the space of flat $SU(2)$ connection modulo gauge transformation with respect to measure depending on the triangulation and gauge fixing. We also exhibit a relationship between the singularity structure of this space and potential divergence of this volume. We present one of our main theorem stating the independence of this volume under change of triangulation and gauge fixing.

In section III we present the construction of the chain mail link associated to a 3d manifold and the chain mail link invariant associated with $\mathcal{D}(SU(2))$. We then prove the equivalence between this link invariant and the Ponzano-Regge model. The quantum group being non-compact, one has to carefully define such link invariant using special properties of the braiding matrices. We also present key properties such as sliding identities of the chain mail evaluation and give a chain mail proof of the independence under gauge fixing.

In section IV we show the equivalence between expectation value of Ponzano-Regge observables, like the Wilson lines or particle observables [20], and $\mathcal{D}(SU(2))$ evaluation of links. We use this correspondence to give an alternative chain mail proof of the independence of our invariant under change of gauge fixing.

In appendix A, B we recall well known facts about $SU(2)$ and $\mathcal{D}(SU(2))$. The appendix C contains all the key technical new results concerning the evaluation of $\mathcal{D}(SU(2))$ links and $\mathcal{D}(SU(2))$ fusion identities. In order to simplify the reading we have put in appendix D, E
the proofs of the main lemma and theorem of section II

II. THE GAUGE FIXED PONZANO-REGGE INVARIANT

In this section, we define an invariant that actually corresponds to a gauge fixed version of the Ponzano-Regge model [19, 20]. We will not present here the physical analysis that motivates this model but present instead a independent mathematical justification of our definition.

A. Definitions and notations

The invariant is defined using a triangulation $\Delta$ of a connected three dimensional compact manifold $M$ with boundary $\partial M$. $\Delta$ induces a 2d triangulation of the boundary denoted $\bar{\Delta} \equiv \partial \Delta$. We will also denote by $\Delta_i$ (resp. $\bar{\Delta}_i$) the i-skeleton of $\Delta$ (resp. $\bar{\Delta}$) consisting of open cells of dimension $i$. We will refer to the internal vertices, edges and faces (triangles) of this triangulation by $v \in \Delta_0, e \in \Delta_1, f \in \Delta_2$ and the boundary ones by $\bar{v}, \bar{e}, \bar{f} \in \bar{\Delta}$.

We also need to introduce the cellular complex $\tilde{\Delta}^*$ dual to the triangulation $\Delta$. This complex consists of two components: an interior component $\Delta^*$ and a boundary component $\bar{\Delta}^*$. The interior component is denoted $\Delta^*$ and its i-skeleton $\Delta^*_i$ is in one to one correspondence with $\Delta_{3-i}$. The union of the 0 and 1 skeleton of $\Delta^*$ is a graph with open ends. Its internal vertices are barycenters of tetrahedra, its internal edges are dual to faces sharing two tetrahedra and its open ends are dual to boundary triangles. The boundary part of $\tilde{\Delta}^*$ is denoted $\bar{\Delta}^*$ and is dual in $\partial M$ to the boundary triangulation. There is an isomorphism $\bar{\Delta}^*_i = \Delta_{2-i}$. The union of the 0 and 1 skeleton of $\Delta^*$ is a trivalent graph whose vertices are dual to boundary triangles. The full dual complex is the union of $\bar{\Delta}^*$ with $\Delta^*$, where the open ends of $\Delta^*$ are glued to the vertices of $\bar{\Delta}^*$. It is denoted by $\hat{\Delta}^*$ and there is an isomorphism $\hat{\Delta}^*_i = \Delta_{3-i} \cup \Delta_{2-i}$. We will refer to the internal vertices, edges and faces of the dual triangulation by $v^* \in \Delta^*_0, e^* \in \Delta^*_1, f^* \in \Delta^*_2$ and the boundary ones by $\bar{v}^*, \bar{e}^*, \bar{f}^* \in \hat{\Delta}$. This construction is illustrated in the figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dual_complex.png}
\caption{Dual complex around a tetrahedra and a boundary triangle.}
\end{figure}

\footnote{If the boundary is not empty we suppose for technical reason that none of its connected component is a 2-sphere, so $\sum_i g_{\Sigma_i} > 0$ where $g_{\Sigma_i}$ is the genus of $\Sigma_i$.}
In order to formulate the model, we will also need additional ingredients. First, we chose an orientation, denoted or, of the edges and faces of $\Delta$. This orientation induces an orientation of the dual faces and dual edges of $\Delta^*$. We denote by $\epsilon(f^*, e^*) = \pm 1$ the relative orientation of $f^*$ and $e^*$ so $\partial f^* = \sum_{e^* \subset f^*} \epsilon(f^*, e^*)e^*$. Also, for each dual face $f^* \in \tilde{\Delta}_2^*$, we pick up a dual vertex in it, that we call starting dual vertex of $f^*$ and we denote it $st(f^*)$.

We also need to a chose an internal maximal tree $T \subset \Delta$ and a dual maximal tree $T^* \subset \Delta^*$. A tree of a graph is a subgraph containing no loop. A tree is said to be maximal if it is touching all the vertices of the graph and is connected. We choose $T^*$ a maximal tree of $\Delta_1^*$. We choose $T$ to be a connected tree of $\Delta_1$ which touches every internal vertex of $\Delta_1$ and touches $\partial M$ in only one vertex of $\Delta_1$.

We denote this additional ingredients by $d \equiv (or, st, T, T^*)$ and denote $\Delta_d$ the triangulation decorated by these data. Given $\Delta_d$ we can first equipped the triangulated manifold with a discrete connection. A discrete connection is an assignment of group elements $g_{e^*} \in G = SU(2)$ to any oriented dual edge $e^* \in \Delta^*$. A change of dual edge orientation amounts to take the inverse group element: $g_{-e^*} = g_{e^*}^{-1}$. A discrete gauge group $G^{[\Delta_1]}$ is acting on this connection at the vertices of the dual triangulation

\[ g_{e^*} \rightarrow k_{e^*} g_{e^*} k_{s_{e^*}^{-1}}, \]  

\( s_{e^*}, t_{e^*} \) being the source and target of the edge $e^*$.

Given a maximal tree $T^*$ and using the gauge group action, we can fix the value of $g_{e^*}$ to be the identity for all edges $e^* \in T^*$. This gauge fixed connection $g^{T^*}$ is an element of $G^{\tilde{\Delta}^*}$ where $\tilde{\Delta}^* = \Delta_1^* \setminus T^*$ is the set of edges of $\Delta^*$ not belonging to $T^*$. The residual gauge group action on a gauge fixed connection is the diagonal adjoint action $g_{e^*} \rightarrow kg_{e^*}k^{-1}$.

Given a discrete connection we can assign to any face $f^* \in \tilde{\Delta}_2^*$, an holonomy

\[ G_{f^*} = \prod_{e^* \subset f^*} g_{e^*}^{\epsilon(e, e^*)}. \]  

The product is understood to be taken along the edges $e^*$ in the dual face $f^* \sim e$, starting from $st(f^*)$ and according to the orientation of $f$. $\epsilon(f^*, e^*) = \pm 1$ denotes the relative orientation of $e$ and the one induced by the orientation of $f \sim e^*$. If $f^*$ is in $\Delta_2^* \subset \tilde{\Delta}_2^*$ it is dual to an edge $e \in \Delta_1$ and in this case we will denote $G_e$ the holonomy around $f^* \sim e$.

The space of flat connections is denoted $\mathcal{A}_F$ and defined to be the sets of connection $g_{e^*}$ for which $G_{f^*} = 1$ for all dual faces $f^*$. A flat connection gives a representation of $\Pi_1(M)$ in $G$ so $\mathcal{A}_F = Rep(\Pi_1(M), G)$.

Given a decorated triangulation $\Delta_d$ with $d = (or, st, T, T^*)$ we have a map

\[ G_{\Delta_d} : G^{\tilde{\Delta}^*} \rightarrow G^{\Delta^*} \quad (g_{e^*})_{e^* \in \tilde{\Delta}^*} \rightarrow (G_e)_{e \in \Delta} \]  

where $\Delta^* = \Delta_1 \setminus T$ is the set of edges of $\Delta$ not in $T$. We then have the lemma

**Lemma 1**

\[ \mathcal{A}_F = G_{\Delta_d}^{-1}(1), \]  

and

\[ |\tilde{\Delta}^*| - |\Delta_t| = 1 + \sum_{\Sigma_i \in \partial M} (g_{\Sigma_i} - 1), \]  

\( \Sigma_i \) being the source and target of the edge $e^*$.
where the sum is over all connected components of $\partial M^3$ and $g_{\Sigma_i} > 0$ denotes the genus of each connected component $\Sigma_i \subset \partial M$. If there is no such component the RHS of (6) should be 0.

This lemma states that the conditions $G_{\Delta_d} = 1$ is a sufficient set of conditions implying $G_{f^*} = 1$ for all $f^*$, even the ones not in $\Delta_T$. The second part of the lemma suggest that the conditions $G_{\Delta_d} = 1$ are generically not redundant. The LHS of (6) computes the number of generators of $\Pi_1(M)$ minus the number of relations we impose on this generators. The RHS of (6) gives an bound on the minimal dimension of $\Pi_1(M)$: $\text{dim}(\Pi_1(M)) \geq 1 + \sum_i (g_{\Sigma_i} - 1)$ or $\text{dim}(\Pi_1(M)) \geq 0$ if $\partial M = \emptyset$. This bound is saturated for some manifolds. For instance when the manifold is closed and $\Pi_1(M)$ is a discrete set or when $M$ is a genus $g$ handlebody $H_g$ in which cases it is known that $\text{dim}(\Pi_1(H_g)) = g$. The proof of this lemma is presented in appendix D.

The previous lemma motivates the following definition.

Definition 1 (Gauge fixed Ponzano-Regge invariant) Given a triangulation $\Delta_d$ of a manifold $M$, equipped with an orientation $\epsilon$ of the edges and faces of $\Delta$, and a choice $st(f^*)$ of starting dual vertex for dual face and a choice of a maximal tree $T$ and a dual maximal tree $T^*$, we assign groups elements $g_e^* \in G$ to oriented dual edges of $\Delta^*$. We consider the quantity

$$PR[M_{\Delta_d}] = \int \prod_{e^* \in \Delta^*} dg_{e^*} \left( \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \left( \prod_{e \in T \cup \Delta} \delta(G_e) \right).$$

(7)

The measure $dg$ is the normalized Haar measure of $G = SU(2)$ and $\delta(g)$ denotes the corresponding Dirac distribution on the group.

In the following, we prove that this definition is actually an invariant of the 3-manifold $M$ not depending essentially on the triangulation additional data. When it is finite this invariant computes the volume of the moduli space of flat $SU(2)$ connection on $M$.

But first, lets focus on the definition which contains integrals over products of distributional delta function. The integrals can be potentially divergent and in this case $PR$ is infinite.

In order to understand this issue lets consider the following distribution

$$\delta_{\Delta_d}(x_e) \equiv \int \prod_{e^* \in \Delta^*} dg_{e^*} \left( \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \left( \prod_{e \in T \cup \Delta} \delta(G_ex_e^{-1}) \right).$$

(8)

The Ponzano-Regge invariant is defined as the value of this distribution at $x_e = 1$. In general the value of a distribution at a given point is ill-defined unless the distribution is sufficiently regular at the identity. Lets suppose that the data $d = (\Delta, T, T^*, \text{or})$ is such that the volume distribution $\delta_{\Delta_d}$ is regular (continuous or of class $C^k$). Under this condition it can be written as a limit$^4$

$$\delta_{\Delta_d}(x_e) = \lim_{\epsilon \to 0} \delta_{\Delta_d}^{(\epsilon)}(x_e) = \lim_{\epsilon \to 0} \int \prod_{e^* \in \Delta^* \setminus T^*} dg_{e^*} \left( \prod_{e \in \Delta \setminus T} \delta_e(G_ex_e^{-1}) \right),$$

(9)

$^3$ if there is no such component the RHS of (6) is 0

$^4$ if $f$ is a bounded continuous function one can write $f(x) = \lim_{\epsilon \to 0} \int \delta_e(xg^{-1})f(g)dg$
where $\delta_\epsilon(g)$ is the heat kernel on $SU(2)$ solution of $\partial_\epsilon \delta_\epsilon(g) = \Delta_G \delta_\epsilon(g)$, with $\Delta_G$ the invariant Laplacian on the group. Since this is a $C^\infty$ class function it can be expanded it terms of character as $\delta_\epsilon(g) = \sum_j d_j e^{-\epsilon c_j} \chi_j(g)$, where $d_j$ is the dimension, $\chi_j(g)$ the characters and $c_j$ the quadratic Casimir of the spin $j$ representation. The integral (9) can be compactly written as

$$\delta^{(e)}(x_e) = \int_{G^\Delta_{T^*}} \delta_\epsilon(G\Delta_d(g)x^{-1}) dg, \quad (10)$$

where $G\Delta_d$ is the map (3). When $\epsilon \to 0$ the integral localizes on $G\Delta_d^{-1}(x)$. More precisely, lets denote by $S \subset G^\Delta_{T^*}$ the set of critical points of $G\Delta_d$. If $S$ is of measure 0 in $G^\Delta_{T^*}$, then $\int_{G^\Delta_{T^*}} \delta_\epsilon(G\Delta_d(x)) dx = \int_{G\Delta_{T^*} \setminus S} \delta_\epsilon(G\Delta_d(x)) dx$ and taking the limit $\epsilon \to 0$ localizes the integral [24, 25]

$$\delta_{\Delta_d}(x_e) = \int_{G\Delta_d^{-1}(x_e) \setminus S} \frac{dx}{|\det(d_x G\Delta_d)^*|}, \quad (11)$$

where $d_x G\Delta_d$ denotes the differential of $G\Delta_d$ at $x$ and $(d_x G\Delta_d)^*$ the dual linear map. Since the set $S$ is by definition the locus of the zero of $|\det(d_x G\Delta_d)^*|$, this integral is indefinite and potentially infinite when $x_e$ is not a regular value. When $x_e = 1$ the integral localizes over the moduli space of flat $SU(2)$ connection $A_F = G\Delta_d^{-1}(1)$. This space is not a manifold when the identity is a singular value of $G\Delta_d$. However the space $A^0_F = A \setminus S$ is a (non-compact) smooth manifold. The Ponzano-Regge invariant computes, when $\delta_{\Delta_d}$ is regular at the identity, the volume of this manifold with the induced measure (11).

Let suppose $M$ is an homology sphere (a closed manifold). In this case the trivial connection is isolated and let suppose that $A_F$ contains irreducible connections. By definition, the gauge group acts non trivially on irreducible connections. Therefore they are critical points of $G\Delta_d$. In this case the Ponzano-Regge invariant is infinite and in general we do not expect this invariant to be finite for topologically non trivial closed manifold. In the case of open manifold we know, on the contrary, some examples where the Ponzano-Regge invariant is well defined [20]. It is a very interesting problem to understand under which topological conditions the gauge fixed PR invariant is finite. It is also tempting to ask whether one can extend our definition to give a well defined value of the Ponzano-Regge invariant for all manifolds. For instance we can use the heat kernel regularization (9) to extract from $\delta^{(e)}(1)$ a finite part. The challenge is then to prove that such a procedure leads to a result which is independent of the triangulation as we show in the next section for our definition of. These questions are beyond the scope of our paper and we will not be addressed here.

**B. The invariance theorem**

To prove that this definition actually leads to an invariant of the manifold $M$, we have to prove that it is independent of the choice of all additional ingredients we introduced.

**Theorem 1** The definition of $PR[M_{\Delta_d}]$ is independent of: The choice of the orientation of the $e$, of the orientation of the $f$, of the choice of the $st(e)$. The choice of the maximal trees

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5 $x \in X$ is said to be a critical point of $G : X \to Y$, where $\dim(X) > \dim(Y)$, if $d_x G$ is not of maximal rank. $y \in Y$ is said to be a singular value if $G^{-1}(y)$ contains critical points, otherwise it is a regular value.

By Sard theorem the set of critical value is of measure 0
T and T*: It depends only on the homology class of T, T*. The choice of the triangulation ∆ of M. It is thus an invariant of M that we will denote PR(M).

Proof:
The detailed proof of this theorem is given in appendix E. The independence of the choice of orientation and starting vertices is based on the properties of the Haar measure and δ-function that insure invariance under reversing orientations or changing start vertices. This proof is given into proposition 13.

The proof of the independence of choice of trees relies on the fact that two homologous trees can always be related by a sequence of elementary moves (described appendix E lemma 2). We then prove that the properties of the integral definition (7) are such that two expressions obtained for trees related by an elementary move are actually the same. This is presented in propositions 15 and 14.

The proof of the independence of the choice of ∆ are done by proving the invariance under Pachner moves. A Pachner move lead from a triangulation ∆ to a triangulation ∆. To compute (7) defined with ∆, we need to use a maximal tree of ∆. The proof goes as follows: we choose maximal trees T, T* for ∆ and compute PR(M, ∆, T, T*). From these trees we explicitly construct maximal trees T̃, T̃* for ∆ and compute PR(M, T̃, T̃*) to prove that we obtain the same result. This is the result of proposition 16.

C. State-sum expression of the Ponzano-Regge invariant

In this section we introduce the notion of amplitude associated to a colored triangulation. A colored triangulation is obtained by assigning a half-integer to each edge of ∆ and element of $H/W = [0, 2\pi]$ to each face (dual edge), where $H/W$ denotes the Cartan subgroup of SU(2) quotiented by its Weyl group.

Definition 2 Let us consider a triangulation ∆ with trees T, T*, colored by $j_e$ for each edge $e \notin T$ and $\theta_{e^*}$ for dual edge $e^* \notin T^*$. We associated to this colored triangulation an amplitude

$$Z_{\Delta,T,T^*}((j_e), (\theta_{e^*})) = \left( \prod_{e^* \notin T^*} \int_{G/H} dx_{e^*} \right) \prod_{e \in T} \chi_{j_e} \left( \prod_{e^* \subset e} x_{e^*} h_{\theta_{e^*}}^{x_{e^*}} x_{e^*}^{-1} \right),$$

where $\chi_{j_e}$ is the character of the spin $j_e$ representation, and $h_{\theta_{e^*}} \in SU(2)$, is the representative element of $\theta_{e^*} \in U(1)$ given in appendix A.

We have seen in the previous section that The Ponzano-Regge invariant is given by the evaluation of the functional $\delta_{\Delta}(x_e)$ at the identity. Using repeatedly the Plancherel formula

$$\delta(G) \sim \sum_j d_j \chi_j(G),$$

This is called a combinatorial Euler structure. In [26] Turaev relates combinatorial Euler structures to smooth Euler structures given by a nowhere vanishing vector field on M normal to the boundary.
with \( d_j = 2j + 1 \) is the dimension of the spin \( j \) representation. We can compute the Fourier expansion of this functional and get

\[
\delta_{\Delta_d}(x_e) \sim \sum_{j_c} d_{j_c} \left( \prod_{e \in \delta T^*} \int dg_e^* \right) \prod_{e \in \delta T} \chi_{j_e} \left( \prod_{e^* \in e} g_e^* \right) x_e.
\]

This is an equality in the sense of distributions but this is not in general a pointwise equality between functions. If we restrict to a so called regular choice of \( M \) and \( \Delta \) for which \( \delta_{\Delta_d}(x_e) \) is regular (continuous) at the identity we have \( \delta_{\Delta_d}(x_e) = \lim_{\epsilon \to 0} \delta_{\Delta_d}(\epsilon x_e) \). Since \( \delta_{\Delta_d}(x_e) \) is a smooth function it is equal to its Fourier series. We can use the Weyl integration formula (cf appendix A) to decompose the group integrations

\[
\int_G dg f(g) = \int_0^{2\pi} \frac{d\theta}{\pi} \Delta^2(\theta) \int_{G/H} dx f(x h \theta x^{-1}),
\]

where \( \Delta(\theta) = \sin(\frac{\theta}{2}) \) and where \( dg \) and \( dx \) are normalized invariant measures on \( G \) and \( G/H \). We then have the following proposition:

**Proposition 1 (Ponzano-Regge invariant as a state sum)** The Ponzano-Regge invariant of a regular manifold \( M \) is given by

\[
PR(M_d) = \lim_{\epsilon \to 0} \left( \prod_{e \in \delta T} e^{-\epsilon c_j} d_{j_c} \right) \left( \prod_{e^* \in \delta T^*} \int_0^{2\pi} \frac{d\theta}{\pi} \Delta^2(\theta) \right) Z_{\Delta, T, T^*}(\{j_e\}, \{\theta_e^*\}).
\]

If the sum over \( j_c \) at \( \epsilon = 0 \) is absolutely convergent we can exchange the summation and limit process and forget about the \( \epsilon \to 0 \), limit. In the following we will restrict to the study of this case.

Note that if we perform the integrations over \( g_e^* \) in (14) or the \( \theta_e^* \) integration in (16) and uses recoupling identity one obtains the invariant in the Ponzano-Regge form

\[
PR(M) = \lim_{\epsilon \to 0} \sum_{j_c} e^{-\epsilon c_j} d_{j_c} \prod_{t \in \Delta_3} \prod_{t \in \Delta_3} \left\{ j_{e_{t1}} j_{e_{t2}} j_{e_{t3}} \right\} \delta_{j_c,0},
\]

where the summation is over all edges of \( \Delta \) and the product is over all tetrahedra, the weight associated to each tetrahedra is the normalized 6j symbol and \( e_{t} \) denotes the six edges belonging to the tetrahedra \( t \). \( d_j = 2j + 1 \) denotes the dimension of the spin \( j \) representation and \( \delta_{j_c,0} \) denotes the Kronecker delta symbol.

**III. CHAIN-MAIL INVARIANT USING \( D(SU(2)) \)**

The Turaev-Viro invariant has been proved by Roberts to be equal to its chain-mail invariant, built using Reshetikhin-Turaev evaluation for the quantum group \( U_q(SU(2)) \). In this part we propose to build the analogue of this equivalence for the Ponzano-Regge model, but using the evaluation with quantum double \( D(SU(2)) \).
A. Construction of the chain-mail link $L_\Delta$

1. Closed manifold

We explain here the construction by Roberts [10] of the chain-mail link $L_\Delta$ associated to a triangulation $\Delta$ of a closed manifold $M$. Consider a closed manifold $M$ and a triangulation $\Delta$ of it, equipped with orientation of edges and faces and a choice of a starting vertex for all dual faces $f^* \sim e$. To each face $f^* \sim e$ of the dual complex $\Delta^*$, we associate a circle component $C_e$ which is the boundary of $f^*$ slightly push inside $f^*$. To each edge of the dual complex $\Delta^*$ we associate a circle $C_e^*$ which is the boundary of a small disk transverse to the edge $e^*$ and encircling all the $C_e$ components associated with dual faces touching $e^*$ (see figure 2). We also equipped each component of the link with an orientation provided by the one of the corresponding face or edge. Finally, the starting vertex $st(f^*)$ is encoded in the marking of a point on each component $C_e^*$. The union of $C_e$ and $C_e^*$ is the chain mail link $L_\Delta$, see figure 2. Given trees $T, T^*$ we construct the Chain mail link $L_{\Delta, T, T^*}$ by deleting all the $C_e$ and $C_e^*$ components for $e \in T, e^* \in T^*$.

A more geometrical way to describe the construction of the link $L_\Delta$ uses a Heegard splitting of $M$. Such a splitting can be obtained from the triangulation $\Delta$ in the following way : we consider the 1-skeleton of $\Delta$ and thicken it to obtain a handlebody $H$ of boundary $\Sigma$. Now the dual 1-skeleton of $\Delta$ can be also thicken to get another handlebody $H^*$ whose boundary is also $\Sigma$. The boundaries of $H$ and $H^*$ are naturally identified and this gives a Heegard splitting $M = H \# \Sigma H^*$. \hspace{1cm} (18)

The link $L_\Delta$ is built from this splitting by considering the meridians of $H$ and $H^*$. These meridians are in one to one correspondence with edges of $\Delta_1$ and $\Delta_1^*$. We draw this meridians and equipped them with a framing given by thickening them inside the boundary $\Sigma$, and with an orientation induced by the corresponding orientation of the edges and faces of the triangulation. Pushing slightly the meridians of $H^*$ into $H$ gives a linking between all of these meridians. One thus get an oriented framed link $L_\Delta$ which has the following properties.

1. Each of its components is an unknot.

2. It is made of two different types of components, the $C$-components, being the meridians of $H$ and the $C^*$ components being the meridians of $H^*$.

3. By construction, components of the same type are not linked together.

Given the chain-mail link $L_\Delta$ and a choice of maximal trees $T$ and $T^*$ in the 1-skeleton and dual 1-skeleton of $\Delta$, the link $L_{\Delta, T, T^*}$ is obtained by removing from $L_\Delta$ the components coming from meridians around edges of $T$ and dual edges of $T^*$. If we denote by $V, E, F, T$ respectively the number of vertices, edges, faces and tetrahedra in $\Delta$. A maximal tree of the triangulation is by construction made of $V$ vertices and $V - 1$ edges, while a dual maximal tree is made of $T$ vertices and $T - 1$ edges. The number of $C$-components of $L_{\Delta, T, T^*}$ is thus $E - V + 1$ while the number of $C^*$-components is $F - T + 1$. For a closed orientable 3-manifold, these two numbers are equal since its Euler characteristic is zero

$$\chi(M) = V - E + F - T = 0. \hspace{1cm} (19)$$

In this case the link determines an element $F$ of the mapping class group of $\partial H$ which is such that under $F$ the sublink of $L_\Delta$ made of the $C$ components is mapped to the sublink
made of the $C^*$ component. $F$ is uniquely determined up to multiplication on the left and right by automorphism of $H$.

2. Open manifold

The chain mail link can be also defined in the case of a manifold with boundary as follows: To each face $f^* \sim e$ of the dual interior complex $\Delta^*$, we associate a circle component $C_e$ which is the boundary of $f^*$ slightly pushed inside $f^*$. To each edge of the full dual complex $\tilde{\Delta}^*$ we associate a circle $C^*_e$ which is the boundary of a small disk transverse to the edge $e^*$ and encircling all the $C_e$ components. The union of $C_e$ and $C^*_e$ is the Chain mail link $L_\Delta$, see figure 3.

Given trees $T, T^*$ we construct the Chain mail link $L_{\Delta,T,T^*}$ by annihilating all the $C_e$ and $C^*_e$ components for $e \in T, e^* \in T^*$. All the $C_e$ components are marked by the choice of a staring vertex $st(f^*)$.

B. $\mathcal{D}(SU(2))$ and its Reshetikhin-Turaev evaluation

The general idea of the Reshetikhin-Turaev [9] evaluation of a colored link using quasi-triangular Hopf algebra is the following. We consider a quantum group acting on representation spaces $V_\rho$. An oriented framed colored link $L(\{\rho_C\})$ is given by the assignment of representations $\rho_C$ of the quantum group to every component $C$ of an oriented framed
link \( L \). If one chose a point on each component of the link we can cut the link along this points and obtain a tangle which contains no closed loops. The original link is recovered as the closure of the tangle. The Reshetikhin-Turaev evaluation associates to every colored tangle an endomorphism of \( \bigotimes_b V_{\rho_C} \), where \( b \) denote the strands of the tangle and \( C_b \) the corresponding color. This endomorphism is obtained using the \( R \)-matrix structure of the quantum group and the ribbon element and gives an invariant of framed tangle. The evaluation of the colored link \( L(\{\rho_C\}) \) is then obtained from here by taking the (quantum) trace of this endomorphism and leads to a link invariant.

Our goal is to use the Reshetikhin-Turaev evaluation with the quantum group \( D(SU(2)) \). Basic facts about the quantum group \( D(SU(2)) \) are summarized in appendix B. This quantum group is constructed from \( SU(2) \) by using the Drinfeld double construction and is therefore a Ribbon Hopf algebra. We recall here few facts about its representation theory. The irreducible unitary representations of \( D(SU(2)) \) are labelled by a couple \((\theta,s)\) where \( \theta \in [0, 2\pi] \) and \( 2s \in \mathbb{N} \). We will denote the corresponding representations by \( (\theta,s)\Pi \) and the carrier spaces of representation by \( (\theta,s)V \). These carrier spaces are infinite dimensional if \( \theta \neq 0 \) and of dimension \( 2s + 1 \) if \( \theta = 0 \). The carrier space of the representation \((\theta,s), \theta \neq 0\) is given by

\[
(\theta,s)V = \bigoplus_{I - s \in \mathbb{N}} (\theta,s)V_I .
\]

A basis of \( V_I \) is given by the Wigner functions (matrix elements of representations)

\[
\{D^I_{m,s}(x) \forall I \geq n, -I \leq m \leq I\}.
\]

The carrier space of the representation \((0,s)\) is the usual \( SU(2) \) spin \( s \) representation and can be described as a functional space by

\[
(0,s)V = \{\phi \in \mathcal{L}^2(SU(2), V^s) \forall h \in SU(2), \phi(xh) = D^s(h^{-1})\phi(x)\} .
\]

The representation conjugate to \((\theta,s)\) is \((-\theta,-s)\). The \( R \)-matrix is acting on a tensor product of representation by

\[
(\theta_1,s_1) \otimes (\theta_2,s_2) \langle R \rangle \phi(x) \otimes \psi(y) = \phi(x) \otimes \psi(x h_{\theta_1}^{-1} x^{-1} y) .
\]

and the ribbon element is acting on \((\theta,s)V\) by multiplication with \( e^{is\theta} \). The braiding element is \( R = \sigma \otimes R \) with \( \sigma \) the permutation: \( \sigma(u \otimes v) = v \otimes u \).

Among all these representations, we will have a particular interest in the so-called simple representations. They are the representations of the form \((\theta,0)\) and \((0,j)\). The characteristic property of these representations comes from the fact that the ribbon element, which is central, is acting trivially on simple representations. It is important to note that if \( \theta_1 = 0 \)
the $R$-matrix is acting as the identity operator. In particular the representation of the braid group induced by this $R$ matrix is equivalent to the permutation group if we restrict to the representations $(0, j)$.

With this data we can easily construct the Reshetikhin-Turaev evaluation of $D$($SU(2)$) colored framed tangles. The key property that we need from the $R$ matrix is the fact that

The image of $\mathcal{V}_I \otimes \mathcal{V}_J$ under the action of $R$ is contained in $(\bigoplus_L \mathcal{V}_L) \otimes \mathcal{V}_J$ where the sum is over a finite number of $L$, therefore no infinities arise in open tangle evaluation.

An immediate and serious problem arise when we try to close the tangles to construct link invariant: $D$($SU(2)$) is a non compact quantum group and some of its unitary representations are infinite dimensional. In this case taking the trace is not allowed unless the tangle operator is proven to be trace class. In the case of a knot this can be easily circumvented [27]. It can be easily proven that the tangle of a colored knot $K_\rho$ is a central element and therefore acts on $V_\rho_{C_b}$ as a scalar and the value of this scalar is taken to be the knot invariant. The case of a general link is far more complicated and there is no known way to construct link invariants associated with infinite dimensional representations. We don’t know if it is possible to define a Reshetikhin-Turaev evaluation of any link using a non compact quantum group but we will show that we can define such an evaluation for colored Chain mail links. But first, we need the following notion of reduced trace.

**Definition 3 (Diagonal endomorphism and reduced trace)** The trace of an endomorphism $\mathcal{E}$ of $\mathcal{V}_{(\theta,0)}$ is the sum of the trace over each of the subspaces $\mathcal{V}_{(\theta,0)} I$. An endomorphism $\mathcal{E}$ is said to be **diagonal** when its trace over each $\mathcal{V}_{(\theta,0)} I$ is equal to $(2I + 1)T_\mathcal{E}$ where $T_\mathcal{E}$ is independent of $I$. We then define $T_\mathcal{E}$ as the **reduced trace** of $\mathcal{E}$ over $\mathcal{V}_{(\theta,0)}$.

**C. Definition of the chain-mail invariant for $D$($SU(2)$)**

In this section, we define the chain-mail invariant, using $D$($SU(2)$), associated to a manifold $M$. This construction depends also on the choice of a triangulation $\Delta$, and maximal trees $T$ and $T^*$. We will not directly prove that the result is independent of these ingredients but we are going to show that it is equivalent with the previously defined Ponzano-Regge invariant and we will refer to it as chain-mail invariant.

**Definition 4 (Amplitude for a colored link)** Let us label each $C$-component $e$ of $L_{\Delta,T,T^*}$ by a simple representation $(0, j_e)$ of $D$($SU(2)$) and each $C^*$-component $e^*$ by a simple representation $(\theta_{e^*}, 0)$. We obtain this way the colored oriented framed link $L_{\Delta,T,T^*}(\{j_e\}, \{\theta_{e^*}\})$.

We have the following key property: There exists a presentation of this link as a closure of a framed tangle such that the Reshetikhin-Turaev evaluation of this tangle gives an endomorphism of $\otimes_{e^*} \mathcal{V}$ which is diagonal for each factor $\mathcal{V}_{(\theta_{e^*},0)}$.

We define the evaluation $\text{eval}[L_{\Delta,T,T^*}(\{j_e\}, \{\theta_{e^*}\})]$ of this colored link to be the reduced trace of this tangle evaluation. Moreover since the representation are simple $\text{eval}[L_{\Delta,T,T^*}(\{j_e\}, \{\theta_{e^*}\})]$ does not depend on the framing of the link.
The fact that the endomorphisms of $V^{(\theta,0)}$ are all diagonal and that the notion of reduced trace apply will be proved later, when we will give an explicit computation of this quantity (see proposition 3). It should be noted that due to this notion of reduced trace, we are able for the first time to define an invariant using infinite dimensional representations. However we do not claim that such an invariant exist for every link. This makes sense only because we only have to trace diagonal endomorphism in the evaluation process. We conjecture that this is a property that holds for every link having the properties 1,2,3 listed above for the chain-mail link, namely the fact that it is made of unknot component that can be separated in two types, such that components of the same type are not linked together and such that one type of component is colored by finite dimensional representations.

**Definition 5 (Chain-mail invariant)** The chain-mail invariant is defined as the sum over representations labels (colors), of the evaluation of the colored link.

$$ CM(M, \Delta, T, T^*) = \left( \prod_{e \in T} \sum_{j_e} d_{j_e} \right) \left( \prod_{e^* \notin T^*} \int_{H/W} d\theta_e^* \Delta^2(\theta_e^*) \right) \text{eval}[L_{\Delta, T^*}(\theta_{e^*}, j_e)] $$  

(25)

**D. Main theorem**

So far we did not prove that $CM(M, \Delta, T, T^*)$ is independent of all the ingredients and is actually an invariant of $M$. This is proved by the fact that it is equal to the Ponzano-Regge invariant. This is the main result of this paper which gives rise to the following theorem

**Theorem 2** The Ponzano-Regge amplitude we defined for a colored triangulation is equal to the Reshetikhin-Turaev evaluation of the colored chain-mail link

$$ Z_{\Delta, T^*}(\{j_e\}, \{\theta_{e^*}\}) = \text{eval}(L_{\Delta, T^*}(\{j_e\}, \{\theta_{e^*}\})) $$  

(26)

It then follows from (16) and (25) that

$$ CM(M, \Delta, T, T^*) = PR(M_d). $$  

(27)

$CM(M, \Delta, T, T^*)$ is thus an invariant of $M$ that can be denoted $CM(M)$. It is equal to the gauge fixed Ponzano-Regge invariant.

To prove the theorem, we first rewrite the LHS of (26) in a convenient way, then we rewrite the RHS as the evaluation of a product of tangles that can be easily computed. In particular we prove that the endomorphism to be traced are diagonal.

**Proposition 2** Let us consider the following set of indices

$$ P = \{(e, e^*), \ s.t. \ e \subset e^*\} $$  

(28)

and we denote by $I = (e, e^*)$ an element of $P$. This is a finite set. For a choice of $I = (e, e^*)$, we denote $j_I = j_e$, $\theta_I = \theta_{e^*}$, $x_I = x_{e^*}$ and $\epsilon(I) = \epsilon(e, e^*)$. For $I = (e, e^*)$, we denote by $s(I)$ the index $(e, \tilde{e}^*)$ where $\tilde{e}^*$ is the successor of $e^*$ inside the dual face $f^* \sim e$, according to the cyclic order induced by orientation of $e$. The amplitude $Z_{\Delta, T^*}(\{j_e\}, \{\theta_{e^*}\})$ can be written as

$$ Z_{\Delta, T^*}(\{j_e\}, \{\theta_{e^*}\}) = \int_{G/H} \prod_{e^*} dx_{e^*} \prod_{I} D_{alg}^{ji}(x_I h_{\theta_I}^{e^*(I)} x_I^{-1}) \times \prod_{I} \delta_{br, \alpha_{s(I)}} $$  

(29)
Proof:
This comes from splitting the characters of definition 2 into matrix elements of representations of SU(2).

Proposition 3 Let us pick up an arbitrary order of the dual edges $e^*$. The link $L_{\Delta,T,T^*}$ can be written as the closure of the following product of colored tangles

\begin{equation}
T : \bigotimes_i V^{ji} \rightarrow \bigotimes_i V^{ji}, \quad T = \bigotimes_{e^*} T_{e^*}.
\end{equation}

where

- $T$ is a colored tangle

\begin{equation}
T_{e^*} : \bigotimes_{e \in C_{e^*}} V^{je} \rightarrow \bigotimes_{e \in C_{e^*}} V^{je}, \quad T_{e^*} = \theta
\end{equation}

The relative crossing between the lines depending on the relative orientation of $e^*$ and $e$. $T_{e^*}$ is the closure of a colored braid $B_{e^*}$. The Reshetikhin-Turaev evaluation of $B_{e^*}$ is a diagonal endomorphism of $V^{\otimes e \in C_{e^*}}V^{je}$, i.e $\text{tr}_{(\theta,0)}(B_{e^*})/(2I+1)$ is independent of $I$ and defines the evaluation of $T_{e^*}$ as an endomorphism of $\otimes_{e \in C_{e^*}} V^{je}$.

- $B$ is a colored braid

\begin{equation}
B : \bigotimes_i V^{ji} \rightarrow \bigotimes_i V^{j_{s(I)}},
\end{equation}

Proof:
The link $L_{\Delta}$ is such that the $C^*$-components carrying representations $\theta^*$ are unlinked together. We can thus isolate the $C^*$-components and put them according to the arbitrary order we choose for the $e^*$. We move the $C$-components in order that they all cross the $C^*$-components in the same direction. We then obtain the tangle $T$ as a product of $T_{e^*}$. Now $T$ is closed by a braid that permutes the strands of the $C$-components from $\bigotimes_i V^{ji}$ to $\bigotimes_i V^{j_{s(I)}}$. Since the action of the $R$-matrix is trivial on representations $(0,j)$ We don’t need to specify the detailed structure of this braid since it is only its permutation effect that will matters. Thus the Reshetikhin-Turaev evaluation of the colored braid $B$ is given by

\begin{equation}
\prod_i \delta_{b_i,a_{s(I)}}.
\end{equation}
$T_e$ is given in appendix C. It is proven there that the trace over the representation space $(\theta, 0)$ $\mathcal{V}$ involves a diagonal endomorphism. The evaluation can therefore be understood in terms of reduced trace and is equal to

$$
\begin{align*}
    \theta_{e^*} & \quad j_1 j_2 j_3 \quad j_{N-1} j_N \\
    \begin{array}{c}
        \cdots \cdots \cdots \cdots \cdots \cdots \\
    \end{array}
    = \int_{G/H} dx_{e^*} \bigotimes_{e \subset e^*} D^j_e (x_{e^*} e^{(e, e^*)}_e x_{e^*}^{-1})
\end{align*}
$$

(34)

The combination of these results then show that the evaluation of the colored link $L_{\Delta, T, T^*}(\{j_e\}, \{\theta_{e^*}\})$ gives the amplitude $Z_{\Delta, T, T^*}(\{j_e\}, \{\theta_{e^*}\})$ given by (29) and proves the theorem.

E. Properties of the chain-mail construction

In this section we investigate some properties of the chain-mail construction that could be used to give a direct proof of the invariance properties of the chain mail evaluation. To make contact with the chain mail notations of Roberts [10], one also introduce the representation $\Omega$ to color $E$-type components and the representation $\Omega^*$ to color $E^*$-type components. They correspond to formal linear combinations of the $(0, j)$ and $(\theta, 0)$ color

$$
\begin{align*}
    \Omega &= \sum_j d_j (0, j) \\
    \Omega^* &= \int_H d\theta \Delta(\theta)^2 (\theta, 0)
\end{align*}
$$

(35) (36)

With this definitions, we have the following result

**Proposition 4** The chain-mail invariant $CM(M)$ is equal to the evaluation of the link $L_{\Delta, T, T^*}$ where we colored all the $C$-components by the representation $\Omega$ and all the $C^*$-components by the representation $\Omega^*$.

F. Sliding, killing and fusion properties

We have already used some evaluations properties of $\mathcal{D}(SU(2))$ to prove the equality of amplitudes in theorem 2. Evaluation of elementary tangles are collected and proven in appendix C.

Particular features of Roberts chain-mail calculus are the so-called sliding and killing properties satisfied by colored link, and the fusion identities. They are exposed here and proved in appendix C.

**Proposition 5** The sliding properties are
The killing properties are

\[ j \begin{array}{c} \Omega' = \delta_{j,0} \\ \theta \end{array} = \begin{array}{c} \Omega \\ \theta \end{array} \]

The fusion properties are

\[ j \begin{array}{c} \Omega' = \delta_{j,0} \\ \theta \end{array} = \begin{array}{c} \Omega \\ \theta \end{array} \]

where

\[ j_1, j_2 \begin{array}{c} \theta_1 \theta_2 \\ \theta_3 \end{array} = \sum_{j_3} d_{j_3} \begin{array}{c} \theta_1 \theta_2 \\ \theta_3 \end{array} , \quad j \begin{array}{c} \theta_1 \theta_2 \\ \theta_3 \end{array} = \int d\nu(\theta_3) \sum_{s_3} (0, 0, \theta_3, s_3) (\theta_1, 0, \theta_2, 0) \quad (37) \]

respectively denotes the Clebsh-Gordan coefficient of the decomposition of the tensor product of representation \((0, j_1) \otimes (0, j_2)\) into \((0, j_3)\) (equal to the one of SU(2)), and the Clebsh-Gordan coefficient of the tensor product of representations \((\theta_1, 0) \otimes (\theta_2, 0)\) into \((\theta_3, s_3)\) \(^7\), \(d\nu(\theta_3) = \Delta(\theta_3) d\theta_3 = \sin(\theta_3/2)^2 d\theta_3\) being the Plancherel measure (explicit expression for the coefficient and the measure are given in appendix C).

IV. INSERTION OF WILSON LINES AND HILBERT SPACE

In this section we show that we can extend the construction of the Ponzano-Regge invariant in order to define insertion of Wilson loops, and insertion of what we will call particle graphs. For each case we present the additional data we need and the type of invariant we want to define. Then we present the chain-mail version of the corresponding invariant: A non-compact topological field theory\(^8\)

\(^7\) Note that contrary to \((0, j)\) case, the tensor product of two \((\theta, 0)\) representations does not decompose only on simple representations.

\(^8\) We call non-compact a topological field theory where we relax the axiom of having only finite dimensional Hilbert space.
We now consider the inclusion of Wilson loops (or rather Wilson graphs) into the model. In the continuum, Wilson loops are defined as the trace, in a certain representation, of the holonomy of the connection along a closed loop. This concept is generalized by the notion of spin-networks.

**Definition 6 (Spin-networks)** Let us consider an abstract oriented graph $\Gamma$ with edges $e$ colored by SU(2) representations $j_e$ and vertices colored by invariant tensors $\iota_v$ that intertwines the representations of corresponding edges according to their orientations. More precisely $\iota_v \in (\otimes_{e \in \Gamma} V_{j_e}^{e,v})$ where $\epsilon(e, v) = +1$ if $e$ is ingoing at $v$ and $\epsilon(e, v) = +1$ if $e$ is outgoing and $V^{+1} \equiv V, V^{+1} \equiv V^*$. The spin-network based on this colored graph is the function of $|e|$ group elements

$$\Phi_{\Gamma,j_e,\iota_v} : \{g_e\} \rightarrow \left( \bigotimes_e D^{j_e}(g_e) \right) \left( \bigotimes_v \iota_v \right).$$

(39)

Where the pairing is the natural pairing between $\bigotimes_e (V_{j_e} \otimes V_{j_e}^*)$ and $\bigotimes_v (\otimes_{e \in \Gamma} V_{j_e}^{e,v})$. It should be noted that in the case of a trivalent graph $\Gamma$, intertwiners do not need to be specified since it exists only zero or one normalized intertwiner for three representations of SU(2).

The spin-networks based on a graph $\Gamma$ are functions of $|e|$ group elements of SU(2), and form an orthogonal basis of the Hilbert space $\mathcal{H}_\Gamma \equiv \mathcal{L}^2(\text{SU}(2)^{|e|}/\text{SU}(2)^{|v|}, d^{[e]} g)$, where $d^{[e]} g$ is the product of normalized Haar measure. And the action of SU(2)$^{|v|}$ on SU(2)$^{|e|}$ is given by $g_e \rightarrow k_{t_e} g_k k_{s_e}$, with $s_e, t_e$ the starting and terminal vertex of $e$.

**A. Ponzano-Regge invariant with Wilson graph**

We define a colored Wilson graph $\Gamma_C = (\Gamma, j_{e*}, \iota_{v*})$ as subgraph of the dual 1-complex of $\Delta^*$, with edges $e^*$ colored by representations $j_{e*}$ of SU(2), and vertices $v^*$ colored by intertwiners $\iota_{v*}$. To define the Ponzano-Regge invariant with insertion of a Wilson graph, we take the same prescription as before for the maximal trees $T$ and $T^*$.

**Definition 7** Consider a decorated triangulation $\Delta_d$ of $M$, with $d = (or, st, T, T^*)$ We define $\Gamma_C = (\Gamma, j_{e*}, \iota_{v*})$ to be a colored subgraph of $\Delta^*$ if $\Gamma \in \Delta^*$ is a subgraph of the dual 1-complex of $\Delta^*$, with edges $e^*$ colored by representations $j_{e*}$ of SU(2), and vertices $v^*$ colored by intertwiners $\iota_{v*}$. We also define

$$\text{PR}[M, \Delta_d, \Gamma_C] = \left( \prod_{e^* \in \Delta^*} \int_G dg_{e^*} \right) \prod_{e^* \in T^*} \delta(g_{e^*}) \prod_{e \in T} \delta(G_e) \Phi_{\Gamma_C}(\{g_{e^*}\})$$

(40)

In section II A we have seen that the data $\Delta_d$ defines a measure $\mu$ (9,11) on the space $\mathcal{A}_F$ of SU(2) flat connection. Given a colored graph we can construct a spin network functional $\Phi_{\Gamma_C}$ on $\mathcal{A}_F$ and $\text{PR}[M, \Delta_d, \Gamma_C]$ then compute the integral $\mu(\Phi_{\Gamma_C})$. It is easy to prove using techniques of appendix D that $\text{PR}[M, \Delta_d, \Gamma_C]$ depends only on the homology class of $\Gamma$. A more challenging question is wether one can generalize theorem 1 and show that not only the total volume of $\mathcal{A}_F$ is invariant under the change of decorated triangulation but the measure $\mu$ itself. We will not answer this question here but focus on the relation of this definition with a chain mail evaluation.
Definition 8. Let consider a decorated triangulation $\Delta_d$ of $M$, colored by $j_e$ for each edge $e \notin T^*$ and $\theta_e^*$ for dual edge $e^* \notin T^*$, together with $\Gamma_C$, a colored subgraph of $\Delta^*$. We associate to these data the amplitude

$$Z_{\Delta_d,\Gamma_C}(\{j_e\}, \{\theta_e^*\}) = \int_{G/H} \prod_{e} dx_e^* \prod_e \chi_{j_e} \left( \prod_{e^* \subset C_e} x_{e^*} h_{\theta_{e^*}} x_{e^*}^{-1} \right) \Psi_{\Gamma_C}(\{x_e, h_{\theta_e}, x_e^{-1}\}),$$

(41)

where $\chi_{j_e}$ is the character of the spin $j_e$ representation, and $h_{\theta_e^*} \in SU(2)$, is the representative element of $\theta_e^* \in U(1)$ given in appendix A.

Given the data $\Delta, \Gamma_C$ we can construct a graph $L_{\Delta, \Gamma} = L_{\Delta} \cup \Gamma$ which consist of the chain mail link $L_{\Delta}$ knotted with $\Gamma$ as follows: As described previously, the chain mail link $L_{\Delta}$ is obtained, for a closed manifold, from $\Delta$ by considering the handlebodies $H, H^*$ which are the thickening of the one skeleton of $\Delta$ and $\Delta^*$. In the boundary surface of $H^*$ we draw the collection $C_{e^*}$ of $H^*$'s meridians and the collection $C_e^*$ of $H$'s meridians, moreover in the core of $H^*$ we put the graph $\Gamma$. The component $C_e$ and $C_e^*$ intersects each other on $\partial H^*$. In order to obtain a link we push the $C_e$ components inside $H^*$ in such a way that these components are not link with the graph $\Gamma$ in the core of $H^*$. The resulting object is $L_{\Delta, \Gamma}$.

We label each $C$-component of $L_{\Delta, \Gamma}$ by a simple representation $(\theta_{e^*}, 0)$ of $D(SU(2))$ and each $C^*$-component $e^*$ by a simple representation $(0, j_e)$ and we color the graph $\Gamma$ by $C \equiv ((0, j_e^*), l_{\theta_e^*})$. We obtain in this way a graph colored by $D(SU(2))$ representations and intertwiners denoted $L_{\Delta, \Gamma}(\{j_e\}, \{\theta_{e^*}\}, C)$.

**Proposition 6.** The Wilson graph amplitude $Z_{\Delta_d,\Gamma_C}(\{j_e\}, \{\theta_{e^*}\})$ amplitude is equal to the Reshetikhin-Turaev evaluation of $L_{\Delta, \Gamma}(\{j_e\}, \{\theta_{e^*}\}, C)$.

This proof is similar to the proof of theorem 2. It was shown that the evaluation of the chain mail link gives an integral over $\{x_{e^*}\}_{e^* \in \Delta^*}, x_{e^*} \in G/H$ of $\prod_e \chi_{j_e} \left( \prod_{e^* \subset C_e} x_{e^*} h_{\theta_{e^*}} x_{e^*}^{-1} \right)$. Now each edge $e^*$ of the graph $\Gamma$ colored by $j_{e^*}^\Gamma$ is linked with a component $C_{e^*}$ colored by $\theta_{e^*}$. As shown in (34, C13) the Reshetikhin-Turaev evaluation of such a linking produces in the integrand an additional term $D^j(x_{e^*} h_{\theta_{e^*}} x_{e^*}^{-1})$, the contraction of this terms with the vertex intertwiners reproduces $Z_{\Delta_d,\Gamma_C}(\{j_e\}, \{\theta_{e^*}\})$.

### B. Ponzano-Regge invariant with Particle graph

In reference [20] a new class of Ponzano-Regge observables allowing the insertion of interacting spinning particles coupled to gravity have been considered. In this section we show that the expectation value of such observables can be reproduced by a Reshetikhin-Turaev evaluation.

A particle graph, denoted $\mathcal{T}$ is a subgraph of $\Delta^*$ submitted to a marking of bivalent vertices. Generically a subgraph of $\Delta^*$ contains bivalent vertices and (true) multivalent vertices. Each bivalent vertex $v^*$ belong to several\(^9\) dual faces, only one of which do not

---

\(^9\) A dual vertex is dual to a tetrahedra, dual faces which meet in $v^*$ are dual to the edges of this tetrahedra, there are therefore 6 such dual faces.
Physically, the subgraph $\tilde{\gamma}$ with $H$ on the boundary of which we draw the meridians of $H$ intersects $\gamma$. We can associate a unique edge of $\Delta$ to a bivalent vertex of $\tilde{\gamma}$.

A marking of $\tilde{\gamma}$ is a choice of bivalent vertices of $\tilde{\gamma}$ such that the corresponding set of edges form a subgraph of $\Delta$, denoted $\tilde{\gamma}$, which is isotopic to $\gamma$. The graph $\tilde{\gamma}$ provides $\tilde{\gamma}$ with a framing. We restrict the graph and the marking to the case where two different marked vertex corresponds to different edges of $\tilde{\gamma}$.

A coloring $C$ of $\tilde{\gamma}$ is a choice of $SU(2)$ representation $I_{e^*}$ for each edge $e^*$ of $\tilde{\gamma}$, a choice of $SU(2)$ intertwiner $i_e$ for each unmarked vertex and a choice of a $D(SU(2))$ representation $(m_e,s_e)$ to each marked bivalent vertex $e \in \tilde{\gamma}$.

**Definition 9 (Particle graph functional)** Given a decorated particle graph $\gamma_C$, we construct a functional depending on $|e^*| + |e|$ group elements where $|e^*|$ is the number of edges of $\tilde{\gamma}$ and $|e|$ the number of marked vertices (or edge of $\tilde{\gamma}$):

$$
\Pi_{\gamma_C} : \{(g_{e^*}),\{u_e\}) \rightarrow \langle \bigotimes_{e^*} D^{I_{e^*}}(g_{e^*}) | \bigotimes_{s_e,t_e} (u_e) \bigotimes v_e^* \rangle. \quad (42)
$$

Where $s_e,t_e$ are the two edges of $\tilde{\gamma}$ meeting at the marked vertex $e$ and

$$
\Pi_{I,J}(u) = \sqrt{d_I d_J} D^I(u^{-1}) |s,I\rangle \langle J,s| D^J(u) \in V_I \otimes V_J^* \quad (43)
$$

with $|s,I\rangle$ the normalized vector of spin $s$ in $V_I$. The pairing in $(43)$ is the natural pairing between $\bigotimes_{e^*}(V_{I_{j_e^*}} \otimes V_{j_e^{*^*}})$ and $\bigotimes_{e^*}(V_{I_{s_e}} \otimes V_{t_e}^*) \bigotimes v_{e^*}^* \bigotimes (\otimes_{e^{*^*}} V_{I_{e^{*^*}}})$.

Given these data we can construct the following amplitude

$$
Z_{\Delta,\gamma_C}(\{j_e\}, \{\theta_{e^*}\}) = \int_{G/H} \prod_{e^*} dx_{e^*} \int_G du_e \prod_{e \notin \gamma} \chi_{j_e} \left( \prod_{e^* \in \gamma} x_{e^*} h_{e^*}^{(e^*,e)} x_{e^*}^{-1} \right) \times \prod_{e \in \gamma} \chi_{j_e} \left( \prod_{e^{*^*} \in \gamma} x_{e^{*^*}} h_{e^{*^*}}^{(e,e^*)} x_{e^{*^*}}^{-1} u_e h_{m_e} u_e^{-1} \right) \Pi_{\gamma_C}(\{x_{e^{*^*}} h_{e^*}, x_{e^*}^{-1}\}, \{u_e\}). \quad (45)
$$

Given a particle graph we can construct a graph $L_{\Delta,\gamma}$ which is the union of the chain mail $L_{\Delta}$ with $\gamma$. As before in order to obtain this graph we consider the Handlebody $H^*$, on the boundary of which we draw the meridians of $H$ and $H^*$ and we put $\tilde{\gamma}$ in the core of $H^*$. We push the $C_e$ components slightly inside $H^*$ or the $C_e^*$ components slightly outside $H^*$. Moreover, if $e$ correspond to an edge of $\tilde{\gamma}$ we link the part of $C_e$ lying between $C_{s_e}$ and $C_{t_e}$ with $\tilde{\gamma}$ by letting $C_e$ go once around the core of $H^*$ (see figure 4).

We label each $C$-component $e$ of $L_{\Delta,\gamma}$ by a simple representation $(0,j_e)$ of $D(SU(2))$ and each $C^*$-component $e^*$ by a simple representation $(\theta_{e^*},0)$ and we color the edges of $\gamma$ by $(0,j_e)$ the vertices by $i_e^*$ and the marked bivalent vertices by $D(SU(2))$ representations $(m_e,j_e)$. We obtain in this way a graph colored by $D(SU(2))$ representations and intertwiners denoted $L_{\Delta,\gamma}(\{j_e\}, \{\theta_{e^*}\}, C)$.

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10 Physically, the subgraph $\tilde{\gamma}$ of $\Delta$ associated with the particle graph $\gamma$ consist of particle worldlines, and $(m,s)$ denote the mass and spin of the particles (see [20]).
Proposition 7 The Wilson graph amplitude $Z_{\Delta, \gamma_c}(\{j_e\}, \{\theta_{e^*}\})$ amplitude is equal to the Reshetikhin-Turaev evaluation of $L_{\Delta, \gamma}(\{j_e\}, \{\theta_{e^*}\}, C)$.

This proof is similar to the proof of the previous proposition and the theorem 2. For edges $e$ which do not belong to $\tilde{\gamma}$ the evaluation of the chain mail link gives an integral over $\{x_{e^*}\}_{e^* \in \Delta^*_c}, x_{e^*} \in G/H$ of $\prod_e \chi_{j_e}(\prod_{e^* \subset e} x_{e^*} h_{\theta_{e^*}}^{(e^*, e)} x_{e^*}^{-1})$. For edges $e$ which belong to $\tilde{\gamma}$ there is an additional linking of $C_e$ with an edge of $\tilde{\gamma}$ colored by $(m_e, s_e)$. By construction $C_e$ is linked with only one edge of $\tilde{\gamma}$. As shown in proposition 11 such a linking insert in the chain mail an integral over $u_e$, an element $u_e h_{m_e} u_e^{-1}$ in the trace over the spin $j$ representation, giving a factor $\chi_{j_e}(\prod_{e^* \subset e} x_{e^*} h_{\theta_{e^*}}^{(e^*, e)} x_{e^*}^{-1} u_e h_{m_e} u_e^{-1})$. This linking also insert in the evaluation of $L_{\Delta, \gamma}(\{j_e\}, \{\theta_{e^*}\}, C)$ a factor $\Pi_{\delta j_e \delta \ell_e} (u_e)$. Eventually, each edge $e^*$ of the graph $\tilde{\gamma}$ colored by $j^{\ell_e}$ is linked with a component $C_{e^*}$ colored by $\theta_{e^*}$. As shown in (34, C13) the Reshetikhin-Turaev evaluation of such a linking produces in the integrand an additional term $D^j(x_{e^*} h_{\theta_{e^*}} x_{e^*}^{-1})$ which is contracted with $\Pi$ and the graph intertwiners. Putting all these terms together we recover $Z_{\Delta, \gamma_c}(\{j_e\}, \{\theta_{e^*}\})$.

C. Gauge fixing operators and their evaluation

If we start from the definition of the gauge fixed Ponzano-Regge invariant (7), we can perform only the Plancherel decomposition given in prop. 1 to get the following expression

$$PR(M) = \left( \prod_e \sum_{j_e} d_{j_e} \int_{SU(2)} dg_{e^*} \right) \left( \prod_{e \in T} \delta_{j_e \ell_e} \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \left( \prod_e \chi_{j_e}(\prod_{e^* \subset e} g_{e^*}^{(e, e^*)}) \right).$$  (46)
This makes apparent the fact that the regularization we proposed, using maximal trees, can be performed by inserting the expression

\[
\left( \prod_{e \in T} \delta_{j_e,0} \prod_{e^* \in T^*} \delta(g_{e^*}) \right), \tag{47}
\]

into the non-gauge fixed expression. This insertion of \( \delta \)-functions allows to avoid redundant sum and integral that cause divergences. This expression is a particular case of the notion of \textit{evaluation of an operator} that we define now.

\textbf{Definition 10 (Operator and its evaluation)} We define as an operator every function \( \mathcal{O}(j_e, g_{e^*}) \) of the labels \( j_e \) and group elements \( g_{e^*} \) for edges \( e \) and dual edges \( e^* \) of \( \Delta \). The evaluation of an operator is defined to be

\[
\langle \mathcal{O} \rangle = \left( \prod_e \sum_{j_e} d_{j_e} \prod_{e^*} \int_{SU(2)} dg_{e^*} \right) \mathcal{O}(j_e, g_{e^*}) \left( \prod_e \chi_{j_e} \left( \prod_{e^* \subset e} g_{e^*}^{(e,e^*)} \right) \right). \tag{48}
\]

According to this definition, the gauge fixed expression (46) corresponds then to the evaluation of the operator (47). This type of operator can be generalized to the so-called \textit{gauge fixing operators}.

\textbf{Definition 11 (Gauge fixing operators)} For a choice of a graph \( \Gamma \) of \( \Delta \), a dual graph \( \Gamma^* \), and an assignment of labels \( j_\Gamma^e \) and \( \theta_{\Gamma^*}^{e^*} \), we define the gauge fixing operator

\[
\mathcal{O}_{\Delta,\Gamma,\Gamma^*,j_\Gamma^e,\theta_{\Gamma^*}^{e^*}}(j_e, g_{e^*}) = \prod_{e \in \Gamma} \frac{\delta_{j_e,j_\Gamma^e}}{d_{j_e}^2} \prod_{e^* \in \Gamma^*} \delta_{\theta_{\Gamma^*}^{e^*}}(g_{e^*}), \tag{49}
\]

where \( \delta_{\theta_{\Gamma^*}^{e^*}} \) is defined by

\[
\int_G f(g) \delta_{\theta}(g) = \int_{G/H} du f(uh, u^{-1}) \tag{50}
\]

projects \( g_{e^*} \) onto the conjugacy class \( \theta_{\Gamma^*}^{e^*} \) of \( SU(2) \).

\textbf{D. Examples of gauge fixing operators}

Among the possible gauge fixing operators, we notice the following

As we have seen, the gauge fixed Ponzano-Regge invariant can be rewritten as the evaluation of the operator obtained by taking \( \Gamma = T \) with \( j_e^T = 0 \) and \( \Gamma^* = T^* \) with \( \theta_{e^*}^{T^*} = 0 \).

\[
PR(\Delta, T, T^*) = \langle \mathcal{O}_{\Delta, T, T^*, 0, 0} \rangle \tag{51}
\]

The meaning of the invariance of the choices of trees proved in theorem 1 is that this evaluation does not depend on the choice of \( T \) and \( T^* \)

\[
\langle \mathcal{O}_{\Delta, T_1, T_1^*, 0, 0} \rangle = \langle \mathcal{O}_{\Delta, T_2, T_2^*, 0, 0} \rangle \tag{52}
\]

for every choices of \( (T_1, T_1^*) \) and \( (T_2, T_2^*) \).
We consider the operator where $\Gamma$ is the 1-skeleton $\Delta_1$ of $\Delta$, $\Gamma^*$ its dual 1-skeleton $\Delta_1^*$. If we choose labels $j_e^T = 0$ on $T$ and $\theta_e^{T^*} = 0$ on $T^*$ and generic labels $j_e^T \theta_e^{T^*}$ elsewhere, we get the amplitude

$$\langle O_{\Delta, \Delta_1, j_e, \theta_e^*} \rangle = Z_{\Delta, T, T^*}(\{j_e^T\}, \{\theta_e^{T^*}\}).$$ (53)

The notion of evaluation of operator leads us to the possibility of modifying the definition (46) by putting labels fixing on the trees other than the zero. We consider the operator $\mathcal{O}_{\Delta, T, T^*, j_e^T, \theta_e^{T^*}}$ instead of $\mathcal{O}_{\Delta, T, T^*, 0, 0}$. Its evaluation is given by the next theorem.

**Theorem 3** We consider the gauge fixing operator $\mathcal{O}_{\Delta, T, T^*, j_e^T, \theta_e^{T^*}}$ whose graphs are maximal trees, and labels are generic labels $j_e^T$ and $\theta_e^{T^*}$.

- The evaluation of operator $\langle \mathcal{O}_{\Delta, T, T^*, j_e^T, \theta_e^{T^*}} \rangle$ is equal to the Reshetikhin-Turaev evaluation of the chain-mail link $L_\Delta$ colored by representations $\Omega$ and $\Omega^*$, except the components on $T$ and $T^*$ that are colored by $j_e^T$ and $\theta_e^{T^*}$.

- We then have the following equality

$$\langle \mathcal{O}_{\Delta, T, T^*, j_e^T, \theta_e^{T^*}} \rangle = \langle \mathcal{O}_{\Delta, T, T^*, 0, 0} \rangle = PR(M_d)$$ (54)

**Proof:**
As described in appendix A, we can relate $\delta_\theta(g)$ to a delta function on $H/W$, we get

$$\delta_\theta(h_\phi) = \frac{\pi}{\Delta^2(\theta)} \delta_\theta(\phi).$$ (55)

Let us rewrite $\langle \mathcal{O}_{\Delta, T, T^*, j_e^T, \theta_e^{T^*}} \rangle$ in terms of explicit formula, using the Weyl formula, we get

$$\langle \mathcal{O}_{\Delta, T, T^*, j_e^T, \theta_e^{T^*}} \rangle = \left( \prod_{e \notin T} D_{j_e} \prod_{e^* \in T^*} \frac{d\theta_{e^*}}{\pi} \int_{H/W} \Delta^2(\theta_{e^*}) \right) \prod_{e^* \in G/H} \int_{e^*} \chi_{j_e}(\prod_{e^* \in e} g_{e^*}(e, e^*))$$ (56)

where $g_{e^*}$ is understood to be $x_{e^*} h_{\theta_{e^*}} x_{e^*}^{-1}$ if $e^* \not\in T^*$ and $x_{e^*} h_{\theta_{e^*}} x_{e^*}^{-1}$ if $e^* \in T^*$. This shows that this is the evaluation of $L_\Delta$ where we colored the $e^* \not\in T^*$ by $\Omega^*$, the $e^* \in T^*$ by $\theta_e^{T^*}$, the $e \not\in T$ by $\Omega$ and the $e \in T$ by $j_e^T$. This is the first result.

Now the second result can be proved using sliding identities. We just proved that the LHS is the evaluation of a certain colored link. We can apply sliding identities to this colored link. Our claim is that the components corresponding to edges of $T$ and dual edges of $T^*$ can be independently slid out of the link. The consequence is that $\langle \mathcal{O}_{\Delta, T, T^*, j_e^T, \theta_e^{T^*}} \rangle$ is equal to the evaluation of a colored link made of the union of unknot components carrying representations $j_e^T$ and $\theta_e^{T^*}$ and of the link $L_{\Delta, T, T^*}$ colored by $\Omega$ and $\Omega^*$. The evaluation of this one gives $PR(M)$ while the evaluation of the unknots components give the volume factor 1 and prove the formula.

To prove the sliding property, we will use the specificity of the tree structure. We pick up an arbitrary vertex that is called the root. Each vertex can be assigned a number called its level, and representing the number of edges between it and the root (root has level 0). By extension, we assign to an edge the level of its vertex of highest level.
We start with the tree $T^*$, and present the chain-mail link as it arise from the triangulation, i.e to each tetrahedron is associated a piece of link (see figure 2). Each component of the link corresponding to faces (circles) carry a color $\theta$ if they are in $T$ or $\Omega^*$ if not. We will prove that we can slide out the component of $T^*$ by proving the two following statements

- A circle enclosing only edges of maximal level can be slid out of the link;
- A circle enclosing edges of level $k$ can be slid to enclose edges of level $k + 1$;

Let us prove the first statement. If a circle enclose only edges of maximal level, it means that around the target vertices of each this edges, there is only $\Omega^*$ color - the contrary would mean that there is another edge in the tree touching this vertices, and contradicts the fact that we are at the maximum level. At each such vertex we can slide the link this way

The link can be thus liberated from all edges of maximal level.

Let us prove the second statement. If a link encloses an edge of level $k$, around the target vertex there are edges which are not in the tree, enclosed by circle with $\Omega^*$ representations, and edges in the tree, which are by definition edges of level $k + 1$. One can thus slide our circle on the available $\Omega^*$ representations in order to make it enclose only edges of level $k + 1$ – as in this example with only one $\Omega^*$:

The combination of the two statements show that every component of the link corresponding to edges in $T^*$ can be slid out of the link.

The same way, we can now present the link as a gluing of elementary pieces associated to each vertex of $\Delta$ (see figure 5) and to same thing with the tree $T$. The differences with the previous case being the fact that the vertex can have any valence, and each edge is bounded by any number of edges. However it does not affect the previous procedure. The key point is that this procedure does not depend of whether some of the component of the other type are still there or not, in other words the procedure for $T^*$ and for $T$ do not interfere and therefore the two operations commute. ■
FIG. 5: Piece of link coming from a vertex of $\Delta$

[1] G. Ponzano and T. Regge, “Semiclassical limit of Racah coefficients” in Spectroscopic and group theoretical methods in physics (Bloch ed.), North-Holland, 1968.

[2] Neville (Temple U.), J. Math. Phys. 12, 2438-2453 (1971) - “A Technique for solving recurrence relations”. Klaus Schulten (then at Harvard) and Roy G. Gordon (Harvard), J. Math. Phys. 16, 1971-1988 (1975) - “Semiclassical approximations to the 3j and 6j coefficients for quantum-mechanical coupling of angular momenta”.

[3] J. Roberts, “Classical 6j-symbols and the tetrahedron”, Geom. Topol. 3 (1999), 21-66, [arXiv:math-ph/9812013].

[4] L. Freidel and D. Louapre, “Asymptotics of 6j and 10j symbols,” Class. Quant. Grav. 20, 1267 (2003) [arXiv:hep-th/0209134].

[5] H. Ooguri and N. Sasakura, “Discrete and continuum approaches to three-dimensional quantum gravity,” Mod. Phys. Lett. A 6, 3591 (1991) [arXiv:hep-th/9108006].

[6] L. Freidel and K. Krasnov, “Spin foam models and the classical action principle,” Adv. Theor. Math. Phys. 2, 1183 (1999) [arXiv:hep-th/9807092].

[7] V.G. Turaev and O.Y. Viro, State sum invariants of 3-manifolds and quantum 6j symbols. Topology 31 (4), 885-902, 1992.

[8] A. Achucarro and P. K. Townsend, “A Chern-Simons Action For Three-Dimensional Anti-De Sitter Supergravity Theories,” Phys. Lett. B 180, 89 (1986). E. Witten, “(2+1)-Dimensional Gravity As An Exactly Soluble System,” Nucl. Phys. B 311, 46 (1988).

[9] V.G. Turaev, “Quantum invariants of knots and 3-manifolds”, de Gruyter studies in math., vol 18, Walter de Gruyter, Berlin, 1994.

[10] J. D. Roberts, “Skein theory and Turaev-Viro invariants,” Topology 34 (1995), 771–787.

[11] M. Karowski and R. Schrader, “A Combinatorial approach to topological quantum field theories and invariants of graphs,” Commun. Math. Phys. 151, 355 (1993).

[12] L. Freidel and K. Krasnov, “2D conformal field theories and holography,” arXiv:hep-th/0205091.

[13] S. Mizoguchi and T. Tada, “Three-dimensional gravity from the Turaev-Viro invariant,” Phys. Rev. Lett. 68, 1795 (1992), arXiv:hep-th/9110057.

[14] Y. U. Taylor and C. T. Woodward, “Non Euclidean tetrahedra and 6j symbols for $U_q(su(2))$, [arXiv:math.QA/0305113].

[15] Y. U. Taylor and C. T. Woodward, “Spherical Tetrahedra and Invariants of 3-manifolds”, [arXiv:math.GT/0406228]

[16] L. Freidel, D. Louapre, K. Noui and P. Roche “Duality formulas and 6j symbols” To appear.
[17] L. Freidel, “A Ponzano-Regge model of Lorentzian 3-dimensional gravity,” Nucl. Phys. Proc. Suppl. 88, 237 (2000) [arXiv:gr-qc/0102098].
[18] J. Barrett, Personal communication.
[19] L. Freidel and D. Louapre, “Diffeomorphisms and spin foam models,” Nucl. Phys. B 662, 279 (2003), [arXiv:gr-qc/0212001].
[20] L. Freidel and D. Louapre, “Ponzano-Regge model revisited. I: Gauge fixing, observables and interacting spinning particles,” arXiv:hep-th/0401076.
[21] F. A. Bais and N. M. Muller, “Topological field theory and the quantum double of SU(2),” Nucl. Phys. B 530, 349 (1998) [arXiv:hep-th/9804130].
[22] L. Freidel, J. Kowalski-Glikman and L. Smolin, “2+1 gravity and doubly special relativity,” arXiv:hep-th/0307085.
[23] F. A. Bais, N. M. Muller and B. J. Schroers, “Quantum group symmetry and particle scattering in (2+1)-dimensional quantum gravity,” Nucl. Phys. B 640, 3 (2002) [arXiv:hep-th/0205021].
C. Meusburger and B. J. Schroers, “Poisson structure and symmetry in the Chern-Simons formulation of (2+1)-dimensional gravity,” Class. Quant. Grav. 20, 2193 (2003) [arXiv:gr-qc/0301108].
C. Meusburger and B. J. Schroers, “The quantisation of Poisson structures arising in Chern-Simons theory with gauge group G x g*,” Adv. Theor. Math. Phys. 7, 1003 (2004) [arXiv:hep-th/0310218].
[24] K. Liu “Heat kernels, symplectic geometry, moduli spaces and finite groups”, [arxiv:math.DG/991010]
[25] A. Sengupta, “The Volume Measure for Flat Connections as Limit of the Yang–Mills Measure”, Journal of Geometry and Physics, 47 (2003) 398-426.
[26] V. Turaev “Euler structures, non singular vector fields, and Torsion of reidemeister type”, Izvestia 34 (1990) 627-662.
[27] J. F. Martins, “Knot Theory With The Lorentz Group”, math.QA/0309162.
[28] L. Freidel and E. R. Livine, “Spin networks for non-compact groups,” J. Math. Phys. 44, 1322 (2003) [arXiv:hep-th/0205268].
[29] T. H. Koornwinder, F. A. Bais and N. M. Muller, “Tensor product representations of the quantum double of a compact group,” Commun. Math. Phys. 198, 157 (1998) [arXiv:q-alg/9712042].
[30] W. B. R. Lickorish, “3 manifolds and Temperley-Lieb algebra”, Math. Ann. 290, 657-67, (1991). L. Kauffman and S. Lins, Temperley Lieb recoupling theory and invariants of 3-manifolds, P.U.P. (1994).

APPENDIX A: NOTATIONS ON SU(2)

We consider the group SU(2). We define the following $2 \times 2$ matrices

$$
h_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \quad (A1)
$$

$$
a_\theta = \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix} \quad (A2)
$$

With these definitions, we consider the Euler-angle parametrization of SU(2)

$$
g(\phi, \theta, \psi) = h_\phi a_\theta h_\psi, \quad (A3)
$$
with the domain
\begin{align}
0 & \leq \phi \leq 2\pi \quad (A4) \\
0 & \leq \theta \leq \pi \quad (A5) \\
-2\pi & \leq \psi \leq 2\pi. \quad (A6)
\end{align}

The Cartan subgroup $H$ of $SU(2)$ is $U(1)$ can be realized with matrices $h_\phi$ for $-2\pi \leq \psi \leq 2\pi$ and the Weyl group $W$ consists of the identity and the reflection $h_\phi \to h_{-\phi}$. The quotient space $G/H$ can be realized into $G$ using a choice of section. The Euler angle parametrization gives the parametrization $x(\phi, \theta) = h_\phi a_\theta$ for this section. The set $Conj$ of conjugacy class of $SU(2)$ is $[0, 2\pi]$, and an explicit realization of it is also given by the $h$ matrices. The representations of $SU(2)$ are labelled by a non-negative half-integer $j$ and realized on the carrier space $V_j \sim \mathbb{C}^{2j+1}$. The representation is given by the Wigner matrices $D_j(g)$. We consider the Haar measure $dg$ on $SU(2)$. The functions \( \{\sqrt{d_j}D_j^{m'n'}(g), \ 2j \in \mathbb{N}, \ -j \leq n, n' \leq j\} \) give an orthonormal basis of the space $L^2(SU(2), dg)$. This orthonormality relation can be written in terms of convolution product for characters
\begin{equation}
\int_G \chi_{j'}(g)\chi_j(gx)dg = \delta_{j,j'}\frac{\chi_j(x)}{d_j}, \quad (A7)
\end{equation}
from which it is clear that the group delta function can be written as
\begin{equation}
\delta(G) = \sum_j d_j \chi_j(G). \quad (A8)
\end{equation}

Let us choose a normalized Haar measure on the group, the integration over the group can be written as an integration over the conjugacy classes using the Weyl integration formula
\begin{equation}
\int_G df(g) = \int_{H/W} \Delta(\theta)^2 \left( \int_{G/H} f(xh_\theta x^{-1})dx \right) \frac{d\theta}{\pi}, \quad (A9)
\end{equation}
where $H = U(1)$ denote the Cartan subgroup, $\Delta(\theta) = \sin(\theta/2)$ and $W$ is the Weyl group, and $dx$ is the normalized invariant measure on $G/H$.

We define the distribution $\delta_\phi(g)$ to be the distribution forcing $g$ to be in the conjugacy class of $h_\phi$. It is invariant under conjugation $\delta_\phi(g) = \delta_\phi(xgx^{-1})$ and normalized by
\begin{equation}
\int_G \delta_\phi(g)f(g)dg = \int_{G/H} f(xh_\phi x^{-1})dx. \quad (A10)
\end{equation}
We can write this distribution in terms of characters
\begin{equation}
\delta_\phi(g) = \sum_j \chi_j(h_\phi)\chi_j(g). \quad (A11)
\end{equation}
The Weyl integration formula imply that
\begin{equation}
\int_{H/W} d\phi \pi \Delta^2(\phi)\delta_\phi(h_\phi) = 1. \quad (A12)
\end{equation}
This means that we can relate this distribution to $\delta_\phi(\theta)$ the delta function on $H/W$,
\begin{equation}
\delta_\phi(h_\theta) = \frac{\pi}{\Delta^2(\phi)} \delta_\phi(\theta). \quad (A13)
\end{equation}
APPENDIX B: THE DOUBLE QUANTUM GROUP OF SU(2)

1. Drinfeld double of a group

The Drinfeld double of a finite group $G$ is the quasi-triangular algebra defined by the following settings

- **Vector space structure** : $\mathcal{D}(G) = \mathcal{C}(G) \otimes \mathbb{C}[G]$ with $\mathcal{C}(G)$ the space of group functions and $\mathbb{C}[G]$ the group algebra. A general element can be represented as a linear combination of elements $(f \otimes g)$ where $f$ is a function over $G$ and $g \in G$.

- **Product** : $(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = f_1(\cdot)f_2(g_1^{-1} \cdot g_1) \otimes g_1g_2$
  In particular we have $(1 \otimes g) \cdot (\delta_k \otimes e) = \delta_k(g^{-1} \cdot g) \otimes g$ and $(\delta_k \otimes e) \cdot (1 \otimes g) = \delta_k \otimes g$.

- **Coproduct** : $\Delta(f \otimes g)(x_1, x_2) = f(x_1x_2)g \otimes g$

- **Unit** : $(1 \otimes e)$;

- **Co-unit** : $\epsilon(f \otimes g) = f(e)$

- **Antipode** : $S(f \otimes g)(x) = f(gx^{-1}g^{-1}) \otimes g^{-1}$

- **Star structure** : $(f \otimes g)^* = \overline{f}(xgx^{-1}) \otimes g^{-1}$

- **R-matrix** : $R = \sum_g (\delta_g \otimes e) \otimes (1 \otimes g)$, and in particular $R^{-1} = \sum_g (\delta_g \otimes e) \otimes (1 \otimes g^{-1})$

To give a ribbon structure to this Hopf algebra, we can consider the Drinfeld element

$$ u = \int dg S(1 \otimes g) \cdot (\delta_g \otimes e) = \int dg (\delta_g \otimes g^{-1}). \quad (B1) $$

which is central. This element satisfies $S(u) = u$ and $u$ itself is thus a square root of $S(u)u$ that we chose as the ribbon element.

In principle, for reasons of convergence, these definitions hold only for the case of a finite group $G$. This construction has been generalized to the case of locally compact group by Bais, Koornwinder and Muller [29]. For a group like SU(2), all these settings can be translated using the isomorphism $\mathcal{D}(G) = \mathcal{C}(G) \otimes \mathbb{C}[G] \sim \mathcal{C}(G \times G)$

$$ \mathcal{C}(G) \otimes \mathbb{C}[G] \leftrightarrow \mathcal{C}(G \times G) \quad (B2) $$

$$ (f \otimes g) \rightarrow f(x)\delta_g(y) \quad (B3) $$

$$ \sum_g F(\cdot, g) \otimes g \leftrightarrow F(x, y) \quad (B4) $$

However, to keep the notations clearer, we will use the notations with $\mathcal{D}(G) = \mathcal{C}(G) \otimes \mathbb{C}[G]$, keeping in mind that all the computations can be put under a rigorous form using this isomorphism.
2. Unitary Representations of $\mathcal{D}(\text{SU}(2))$

The classification of the representations of the double of a compact group has been done in [21, 29]. The representations of $\mathcal{D}(\text{SU}(2))$ are labelled by a pair made of a conjugacy class and a representation of the corresponding centralizer. In the case of SU(2), the conjugacy classes are labelled by a $\theta \in [0; 2\pi]$, see appendix A. For $\theta \neq 0, 2\pi$, the centralizer is $U(1)$, and the representations $s$ of $U(1)$, $\rho_s(h_\theta) = e^{is\theta}$ are such that $2s \in \mathbb{Z}$. If $\theta = 0$, the centralizer is SU(2) itself. The representations of $\mathcal{D}(\text{SU}(2))$ are thus of the form $(\theta, s)$ or $(0, j)$. Among them we will distinguish the so-called simple representations. They are representations of the form $(\theta, 0)$ or $(0, j)$.

Representations $(\theta, s)$:

The corresponding carrier space is given by

$$\mathcal{V}^{(\theta, s)} = \{ \phi \in L^2(G, \mathbb{C}) | \forall \xi \in [-2\pi, 2\pi], \phi(xh_\xi) = e^{is\xi}\phi(x) \}, \quad (B5)$$

where the representation of an element $(f \otimes g) \in \mathcal{D}(\text{SU}(2))$ is given by

$$\left[\Pi^{(\theta, s)}(f \otimes g)\phi\right](x) = f(xh_\theta)\phi(g^{-1}x). \quad (B6)$$

The ribbon element is acting diagonally

$$\Pi^{(\theta, s)}(u) = e^{is\theta}. \quad (B7)$$

An orthonormal basis of $\mathcal{V}^{(\theta, s)}$ is given by the Wigner functions (matrix elements of representations)

$$\mathcal{V}^{(\theta, s)} = \bigoplus_{2(1-i-s) \in \mathbb{N}} \mathcal{V}^{(\theta, s)}_I \quad (B8)$$

$$\mathcal{V}^{(\theta, s)}_I = \{ D^I_{ms}(x), -I \leq m \leq I \} \quad (B9)$$

where the matrix index $s$ is kept fixed. The finite dimensional subspaces $\mathcal{V}^{(\theta, s)}_I$ are invariant under the SU(2) subgroup of $\mathcal{D}(\text{SU}(2))$ composed of elements of the form $1 \otimes g$.

Representations $(0, j)$:

The carrier space of the representation $(0, j)$ is given by

$$\mathcal{V}^{(0, j)} = \{ \phi \in \mathcal{C}(G, V^j) | \forall g \in \text{SU}(2), \phi(xg) = D^j(g^{-1})\phi(x) \} \quad (B10)$$

One can see that $\mathcal{V}^{(0, j)}$ is actually isomorphic to $V^j$ since such a function $\phi$ is completely determined by its value $\phi(e) = v \in V^j$. The ribbon element is acting trivially on in this representation. An element $(f \otimes g)$ is represented on $\mathcal{V}^{(0, j)} \sim V^j$ by

$$\Pi^{(0, j)}(f \otimes g) = f(e)D^j(g^{-1}). \quad (B11)$$
APPENDIX C: RESHETIKHIN-TUARAEV EVALUATION OVER $\mathcal{D}(SU(2))$

In this appendix, we present the proofs of the main properties of the chain-mail evaluation based on simple representations of $\mathcal{D}(SU(2))$. In particular we present the computation of simple colored tangles and of more involved tangles needed in our constructions. We then expose the proofs for the basic properties of chain-mail evaluation: the so-called sliding properties, killing properties and fusion identities. Most of these results are new.

The representation spaces $(\theta, s) V$ are spaces of functions over the group $SU(2)$. We will denote generically their elements $\phi(x), \psi(y)$... where $x, y, ... \in SU(2)$. We will thus write the endomorphisms of these representation spaces by describing their action on these generic elements. The representation spaces $(0, j) V$ are isomorphic to the representation spaces $V^j$ of $SU(2)$. We will thus write the corresponding endomorphisms using matrix elements of representations of $SU(2)$.

1. Simples braidings

We begin with the computation of simple braidings directly obtained from the $R$-matrix of $\mathcal{D}(SU(2))$.

**Proposition 8 (Simple braidings)**

\[
\begin{align*}
\theta \alpha & = \phi(x) \otimes \psi(y) \rightarrow \psi(xh^{-1}_\theta x^{-1}y) \otimes \phi(x) \quad \text{(C1)} \\
\alpha \theta & = \psi(y) \otimes \phi(x) \rightarrow \phi(x) \otimes \psi(xh_\theta x^{-1}y) \quad \text{(C2)} \\
\begin{array}{c}
\alpha \\
\beta
\end{array} & = P_{12} \quad \text{(C3)}
\end{align*}
\]

where $P_{12}$ denotes the permutation $V^{j_1} \otimes V^{j_2} \rightarrow V^{j_2} \otimes V^{j_1}$ and the action is taken to go from top to bottom.

**Proof:**

Let us prove the first braiding. The $R$-matrix of $\mathcal{D}(SU(2))$ is given by

\[
R = \int_G dg \ (\delta_g \otimes e) \otimes (1 \otimes g).
\]

its representation on the space $^{(\theta,0)} V \otimes ^{(0,0)} V$ is, according to the representations described in appendix B

\[
\phi(x) \otimes \psi(y) \rightarrow \int_G dg\phi(x)\delta_g(xh_\theta x^{-1})\psi(g^{-1}y) = \phi(x) \otimes \psi(xh^{-1}_\theta x^{-1}y) \quad \text{(C5)}
\]

The braiding $R = PR$ is obtained by composing with the permutation. The second braiding is $R^{-1}$ and the other braidings are computed in the same way, by representing the element $R$ on other representation spaces. ■
2. Elementary tangles

We now present the computation of an elementary tangle involving trace over one representation.

**Proposition 9 (Elementary tangles)** We have the following evaluations

\[
\theta_j \mapsto \phi(x) \rightarrow \chi_j(h^{-1}_\theta \phi(x)) = \phi(x) \rightarrow \chi_j(h_\theta \phi(x)) \tag{C6}
\]

\[
\theta_j \mapsto \int dx D^j(xh^{-1}_\theta x^{-1}) = \int dx D^j(xh_\theta x^{-1}) \tag{C7}
\]

**Proof:**

Let us compute the evaluation of the following tangle

We presented this tangle as the closure of a braid. The universal element of \( \mathcal{D}(SU(2)) \otimes \mathcal{D}(SU(2)) \) associated to this braid is

\[
Q = R^2 = R_{21}R_{12} = \int dg_1dg_2 [(1 \otimes g_1)(\delta_{g_2} \otimes e)] \otimes [(\delta_{g_1} \otimes e)(1 \otimes g_2)]
\]

\[
= \int dg_1dg_2 [\delta_{g_2}(g_1^{-1} \cdot g_1) \otimes g_1] \otimes [\delta_{g_1} \otimes g_2] \tag{C8}
\]

where we use the rule of product of elements of \( \mathcal{D}(SU(2)) \), see appendix B. If we first represent \( Q \) on the tensor product of representation spaces \((\theta, 0) \otimes (0, j)\). We get

\[
(\Pi \otimes \Pi) \cdot Q : v \otimes \phi(x) \rightarrow D^j(xh^{-1}_\theta x^{-1})v \otimes \phi(x), \tag{C9}
\]

where \( v \) is any vector in \( V^j \). The trace over \( V^j \) then gives the first result of \( (C6. \)

We now represent \( Q \) on the tensor product of representation spaces \((\theta, 0) \otimes \mathcal{V} \) .

\[
(\Pi \otimes (0,j)) \cdot Q^{(1)} : \phi(x) \otimes v \rightarrow \phi(x) \otimes D^j(xh^{-1}_\theta x^{-1}) \tag{C10}
\]

To take the trace over the representation space \((\theta, 0)\), we use its orthonormal basis given by the Wigner functions \(\{\sqrt{a_k} D^k_{m_0}\}\). We want to prove that this endomorphism is diagonal...
and that the reduced trace can be taken. Let us compute the trace over \( \mathcal{V}_k \) in the sense of def.3.

\[
\text{tr}_{\mathcal{V}_k}^{(\theta,0)} \left[ \prod_{m=-k}^{k} (\Pi \otimes \Pi) \cdot Q \right] = \sum_{m=-k}^{k} \int dx D^j(x h_\theta^{-1} x^{-1}) d_k D_{m0}^k(x) D_{m0}^k(x) (C11)
\]

\[
= d_k \left( \int dx D^j(x h_\theta^{-1} x^{-1}) \right) (C12)
\]

This shows that the endormorphism is diagonal and that its reduced trace is \( \int dx D^j(x h_\theta^{-1} x^{-1}) \). The other tangles are evaluated in the same way. ■

3. Complex colored tangles

In this part, we present the evaluation of complex colored tangles that are extensively used in our constructions.

**Proposition 10**

\[
\begin{align*}
\theta \theta \theta & \quad \theta \theta \theta \\
\theta & \quad \theta \theta \theta \\
\theta & = \bigotimes_{i=1}^{N} \phi_i(x_i) \rightarrow \chi_j \left( \prod_{i=1}^{N} x_i h_\theta^{-\epsilon_i}, x_i^{-1} \right) \bigotimes_{i=1}^{N} \phi_i(x_i) (C14)
\end{align*}
\]

and

\[
\begin{align*}
\theta \theta \theta & \quad \theta \theta \theta \\
\theta & \quad \theta \theta \theta \\
\theta & = \bigotimes_{i=1}^{N} \phi_i(x_i) \rightarrow \chi_j \left( \prod_{i=1}^{N} x_i h_\theta^{-\epsilon_i}, x_i^{-1} \right) \bigotimes_{i=1}^{N} \phi_i(x_i) (C14)
\end{align*}
\]

where \( \epsilon_i = \pm 1 \) denotes the relative orientation of the \( i \)-th strand and the circle component carrying representation \( j \) (On this picture the first strand is \( \epsilon = +1 \) and the second \( \epsilon = -1 \)).

**Proof:**

We denote \( N \) the number of strands. We represent the tangle as a closure of a braid, with the closure taking place on the first strand of the braid. One can show that the action of the universal element of \( D(SU(2)) \otimes (N+1) \), representing the braid associated to this tangle, on \( \phi(x) \otimes_{i=1}^{N} \phi_i(x_i) \) is given by

\[
\int_{SU(2)^{2N}} dg_1 dg_2 \prod_{i=1}^{N} dy_i \delta_g_1 (y_1 \cdots y_n) \times (C15)
\]

\[
(\delta_{g_1 g_2^{-1}} \otimes g_1) \phi(x) \otimes_{i=1}^{N} \left\{ \begin{array}{ll}
(\delta_{y_i^{-1} g_2 g_2^{-1}}) & \text{if } \epsilon_i = +1 \\
(\delta_{y_i^{-1} g_2 g_2^{-1}} g_1) & \text{if } \epsilon_i = -1 \end{array} \right\} \phi_i(y_i) (C16)
\]
The representation of this element on the representation space \( \mathcal{V} \otimes_i \mathcal{V} \) leads to

\[
\phi(x) \rightarrow \int_{SU(2)^{2N}} dg_1 dg_2 \prod_{i=1}^N dy_i \delta_{g_1}(y_1 \cdots y_n) \times \]

\[
\phi(g_1^{-1}x) \delta_{g_1g_2g_1^{-1}}(xh_\theta^{-1}x^{-1}) \otimes \prod_{i=1}^N \delta y_i(g_2^{-1}) \quad \text{if} \quad \epsilon_i = +1
\]

\[
\delta g_2^{-1}y_{g_2}^{-1}(e) \delta g_2^{-1}y_{g_2}^{-1}(e) \quad \text{if} \quad \epsilon_i = -1 \quad .
\] (C17)

The integrals can now be easily computed, giving the value of the endomorphism of \( \mathcal{V} \otimes_i \mathcal{V} \) associated with the total braid.

\[
\phi(x) \rightarrow \phi(x) \otimes \prod_{i=1}^N D^{j_i}(xh_\theta^{-\epsilon_i}x^{-1})
\] (C19)

This endomorphism is diagonal on \( (\theta,0) \mathcal{V} \) and its reduced trace is

\[
\int_{G/H} dx \otimes \prod_{i=1}^N D^{j_i}(xh_\theta^{-\epsilon_i}x^{-1}),
\] (C20)

which proves C13.

If one now evaluate (C15) in the representation \( (0,j) \mathcal{V} \otimes_i (\theta,s) \mathcal{V} \) one gets

\[
\otimes_{i=1}^N \phi_i(x_i) \rightarrow \int_{SU(2)^{2N}} dg_1 dg_2 \prod_{i=1}^N dy_i \delta_{g_1}(y_1 \cdots y_n) \times
\]

\[
D^{j_i}(g_1^{-1}) \delta_{g_1g_2g_1^{-1}}(e) \otimes \prod_{i=1}^N \delta y_i(g_2^{-1}) |x_ih_\theta^{-\epsilon_i}x_i^{-1}\rangle \phi_i(x_i),
\] (C21)

which gives after integration

\[
\otimes_{i=1}^N \phi(x_i) \rightarrow \prod_{i=1}^n x_ih_\theta^{-\epsilon_i}x_i^{-1} \bigg) \otimes \otimes_{i=1}^N \phi(x_i) ,
\] (C23)

By taking the trace on \( (0,j) \mathcal{V} \) we obtain the RHS of (C14). ■

So far we considered only representations spaces of the form \( (0,j) \) or \( (\theta,0) \). Now we also consider the representation space \( (m,S) \), but for this one we will consider the discrete basis (B8), whose elements are labelled by \( (I,n) \) with \( I - S \in \mathbb{N} \) and \( -I \leq n \leq I \). With these notations, we have the following evaluation

\[
(m,S)
\]

\[
\mathcal{J} = \int_G d\chi^j(uh_m^{-1}u^{-1}) \sqrt{d_1} \sqrt{d_T} D_{nS}(u) D_{S'}(u^{-1}).
\] (C24)
This is proved by representing the element (C8) on $\mathcal{V} \otimes (m, S)$ of $\mathcal{V}$. Let us compute the action of the corresponding endomorphism on an element $\sqrt{dT}D^I_{nS}(u)$. We get

$$\int dg_1 dg_2 \delta_{g_2}(e) D^j(g_1^{-1}) \delta_{g_1}(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(g_2 u) = D^j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u).$$

We then take the trace over $(0, j)$ and get an endomorphism of $(m, S)$. To express this endomorphism in the discrete basis of $(m, S)$, we need to decompose the given result on this basis. We get

$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

We can now state the following proposition, which is proved in the same way:

**Proposition 11** As before, we consider the representations spaces $(\theta, 0)$ and $(0, j)$ whose elements are functions $\phi(x)$ over $SU(2)$ with the covariance property $\phi(xh) = \phi(x)$, $\forall h \in H$, and the representation space $(m, S)$ with the discrete basis whose elements are labelled by $(I, n)$. With these notations, we have the following evaluation

$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

We then take the trace over $(0, j)$ and get an endomorphism of $(m, S)$. To express this endomorphism in the discrete basis of $(m, S)$, we need to decompose the given result on this basis. We get

$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

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$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

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$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

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$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

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$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

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$$\int_G du \chi_j(uh_m u^{-1}) \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u) = \int_G du \chi_j(uh_m u^{-1}) dT \sqrt{dT}D^I_{nS}(u) \sqrt{dT}D^I_{nS}(u).$$

We can now state the following proposition, which is proved in the same way:
Using the evaluation (C7) the LHS of the first identity can be written

\[ \int_{H/W} \frac{d\theta}{\pi} \Delta^2(\theta) \int_{G/H} dx D^j(xh_0 x^{-1}) = \int dg D^j(g) = \delta_{j,0} \quad (C30) \]

we where use that the integration is a full group integration by the Weyl integration formula which projects over the trivial representation \( j = 0 \), we then get the first identity. Using the evaluation (C6) and the Plancherel decomposition

\[ \delta(g) = \sum_j d_j \chi_j^j(g), \quad (C31) \]

the LHS of the second graph is

\[ \delta(h_\theta) \Id_{(\theta,0)}, \quad (C32) \]

and we get the second identity.

5. **Proof of the sliding properties**

The sliding properties are

\[ \begin{align*}
\underline{\Omega} & \quad \underline{\Omega} \\
\underline{\Omega} & \quad \underline{\Omega} \\
\underline{\Omega} & \quad \underline{\Omega}
\end{align*} \]

where

\[ (\Omega) = \sum_j d_j (0,j), \quad (\Omega^*) = \int_H d\theta \Delta(\theta)^2 (\theta,0). \quad (C33) \]

To prove them, we start from the braid

\[ \begin{align*}
\begin{array}{c}
\quad \\
\quad \\
\quad
\end{array}
\end{align*} \]

The associated abstract element is

\[ P = \int \prod_{i=1}^{4} [(1 \otimes g_2)(\delta_{g_3} \otimes e)] \otimes [(1 \otimes g_1)(\delta_{g_4} \otimes e)] \otimes [(\delta_{g_1} \otimes e)(\delta_{g_2} \otimes e)(1 \otimes g_3)(1 \otimes g_4)] \]

\[ = \int dg_2 dg_3 dg_4 \left[ \delta_{g_2 g_3 g_2^{-1}} \otimes g_2 \right] \otimes \left[ \delta_{g_2 g_3 g_2^{-1}} \otimes g_2 \right] \otimes [\delta_{g_2} \otimes g_3 g_4] \quad (C34) \]
where use the product rules in $D(SU(2))$. We first represent this element on $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$

$$\left[ (0,j) \otimes (0,k) \otimes (\theta,s) \right] \cdot P \phi(x) = D^j(x h_\theta^{-1} x^{-1}) \otimes D^k(x h_\theta^{-1} x^{-1}) \phi(x)$$

(C35)

We take the trace over $k$ representations and sum over them with the factor $d_k$. We get

$$\delta(h_\theta) D^j(1) \otimes Id(\theta,s)_{\mathcal{V}}$$

(C36)

which is the LHS of the first sliding identity. To prove the second identity, we represent $P$ over $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$, we obtain

$$\phi_1(x_1) \otimes \phi_2(x_2) \rightarrow \phi_1(x_1) \otimes \phi_2(x_2) \otimes D^j(x_1 h_{\theta_1}^{-1} x_1^{-1}) D^j(x_2 h_{\theta_2}^{-1} x_2^{-1})$$

(C37)

Tracing over $(\theta_2,0)$ (according to the reduce trace prescription) gives the endomorphism

$$\phi(x) \rightarrow \chi_j(h_{\theta_2}) \phi(x) \otimes D^j(x h_{\theta_1} x^{-1})$$

(C38)

Integrating over the $\theta_2$ allows to built $g_2 = x_2 h_{\theta_2} x_2^{-1}$

$$\int dg_2 D^j(x_1 h_{\theta_1}^{-1} x_1^{-1}) \delta_{y_1}(x_1) D^j(g_2) = \delta_{y_1}(x_1)$$

(C39)

and prove the second sliding identity. It should be noted that for this identity, the Lickorish proof [30] does not work since the fusion of two $(\theta,0)$ representations contains also non-simple representations. ■

6. Fusion properties and Clebsh-Gordan coefficient

In this section we recall the construction [21, 29] of Clebsh-Gordan coefficient associated with simple representations of $D(SU(2))$ and then establish the orthonormality property satisfied by these coefficients.

a. Tensor product $(0,j_1) \otimes (0,j_2)$

The tensor product decomposition of simple representations of type $(0,j)$ is given

$$(0,j_1) \otimes (0,j_2) = \bigoplus_{j_3 = |j_1 - j_2|}^{j_1 + j_2} (0,j_3).$$

(C40)

Each representation on the RHS appears with multiplicity 1, and the corresponding Clebsh-Gordan coefficient is given by the $SU(2)$ normalized Clebsh-Gordan coefficient

$$\mathcal{C}^{(0,j_1)(0,j_2)(0,j_3)}_{m_1 m_2 m_3} = \mathcal{C}^{j_1 j_2 j_3}_{m_1 m_2 m_3}.$$  

(C41)

Let us remark that simple representations of type $(0,j)$ decompose only on simple representations of the same type.
b. **Tensor product** \((\theta_1, 0) \otimes (\theta_2, 0)\)

The tensor product of two simple representations of type \((\theta, 0)\) is more involved. The tensor product \((\theta_1, 0) \otimes (\theta_2, 0)\) decomposes on representations \((\theta_3, n_3)\) with multiplicity 1.

We want to express the corresponding Clebsh-Gordan coefficient

\[
\begin{array}{c}
\sum_{\theta_3, n_3} C_{x_1 x_2 x_3}^{(\theta_1, 0)(\theta_2, 0)(\theta_3, 0), n_3} \\
\end{array}
\]

This coefficient defines an invariant map \(\mathcal{V} \otimes \mathcal{V} \to \mathcal{V}\)

\[
\phi_1(x_1) \otimes \phi_2(x_2) \to \int dx_1 dx_2 C_{x_1 x_2 x_3}^{(\theta_1, 0)(\theta_2, 0)(\theta_3, n_3)} \phi_1(x_1) \phi_2(x_2).
\]

described as follows. Consider three conjugacy class \(\theta_1, \theta_2, \theta_3\), satisfying

\[
|\theta_1 - \theta_2| \leq \theta_3 \leq \theta_1 + \theta_2, \quad \theta_1 + \theta_2 + \theta_3 \leq 4\pi.
\]

It exists a unique \(\theta \in U(1) / SU(2) / U(1) \sim [0, \pi]\) such that \(\theta_3\) is the conjugacy class of \(h_\theta a_\theta^{} h_\theta a_\theta^{-1}\) (see appendix A for the notation \(a_\theta\)). This \(\theta\) is a function of \(\theta_1, \theta_2, \theta_3\) and parametrizes the relation between the 3 conjugacy classes. Taking the trace of this defining relation we get

\[
\cos \frac{\theta_3}{2} = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} - \cos \theta \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}.
\]

Now let us denote by \(w_\theta\) the unique (up to a choice of section) element of \(G/H\) such that

\[
h_\theta a_\theta h_\theta a_\theta^{-1} = w_\theta h_\theta a_\theta^{-1}.
\]

From [29] we know that the Clebsh-Gordan coefficient is given by

\[
C_{x_1 x_2 x_3}^{(\theta_1, s_1)(\theta_2, s_2)(\theta_3, s_3)} = (4 \sin(\theta_1/2) \sin(\theta_2/2) \sin(\theta_3/2))^{-1/2} \times
\]

\[
\int_{U(1)} d\xi_1 d\xi_2 d\xi_3 e^{-i\xi_3 \delta_{x_2 h_\xi_3} (x_1 h_\xi_1 w_\theta)} \delta_{x_2 h_\xi_2} (x_1 h_\xi_1 a_\theta)
\]

where \(d\xi\) is normalized on \(U(1)\) and the normalization coefficient is chosen in order to simplify the orthogonality relations. Let us recall that in this expression, \(\theta, a_\theta\) and \(w_\theta\) are uniquely determined in terms of \(\theta_1, \theta_2\) and \(\theta_3\). If \(\theta_1, \theta_2, \theta_3\) do not satisfy the relation (C44) we define the Clebsh-Gordan coefficient to be zero.

**Proposition 12** We have the orthogonality relation

\[
\int_{0}^{2\pi} \sin(\theta_3/2)^2 d\theta_3 \sum_{n_3 \in \mathbb{Z}} \int_{G/H} dx_3 C_{x_1 x_2 x_3}^{(\theta_1, 0)(\theta_2, 0)(\theta_3, n_3)} \overline{C_{x_1 x_2 x_3}^{(\theta_1, 0)(\theta_2, 0)(\theta_3, n_3)}} = \bar{\delta}_{x_1}(x_1) \bar{\delta}_{x_2}(x_2),
\]

with

\[
\bar{\delta}_x(g) \equiv \int_{H} dh \delta_{x h}(g).
\]
Proof:
The LHS is
\[
\int_0^{2\pi} d\nu(\theta_3) \int_{G/H} dx_3 \sum_{n_3} \prod_{i=1}^3 d\xi_i \ e^{i n_3 \xi_3 \delta_{x_1 h_{\xi_3}} (x_3 h_{\xi_3} w_{\theta}^{-1}) \delta_{x_2 h_{\xi_2}} (x_1 h_{\xi_1} a_{\theta})}
\]
\[
\int_0^{2\pi} d\nu(\theta_3) \int_{G/H} dx_3 \sum_{n_3} \prod_{i=1}^3 d\xi_i \ e^{-i n_3 \xi_3 \delta_{z_1 h_{\xi_3}} (x_3 h_{\xi_3} w_{\theta}^{-1}) \delta_{x_2 h_{\xi_2}} (z_1 h_{\xi_1} a_{\theta})}, \quad (C51)
\]
with
\[
d\nu(\theta_3) = \frac{\sin(\theta_3/2)}{4 \sin(\theta_1/2) \sin(\theta_2/2)} d\theta_3 \quad (C52)
\]
We perform the sum over \(n_3\), this imposes \(\delta(\xi_3 - \xi_3')\), we then rearrange the arguments of the delta functions and the result can be written as
\[
\int_0^{2\pi} d\nu(\theta_3) \int_{G/H} dx_3 \int_H d\xi_1 d\xi_2 d\xi_3 \delta_{x_1 h_{\xi_1}} (x_3 h_{\xi_3} w_{\theta}^{-1}) \delta_{x_2 h_{\xi_2}} (x_1 h_{\xi_1} a_{\theta}) \tilde{\delta}_{z_1} (x_1) \tilde{\delta}_{x_2} (x_2) \quad (C53)
\]
We can put together the variables \(x_3 \in G/H\) and \(\xi_3 \in H\) to build an element of \(G\) and integrate the first delta function. It remains
\[
\int_0^{2\pi} d\nu(\theta_3) \int_H d\xi_1 d\xi_2 \delta_{x_1^{-1} x_2} (h_{\xi_1} a_{\theta} h_{\xi_2}^{-1}) \tilde{\delta}_{z_1} (x_1) \tilde{\delta}_{x_2} (x_2) \quad (C54)
\]
The delta function \(\delta_{x_1^{-1} x_2} (h_{\xi_1} a_{\theta} h_{\xi_2}^{-1})\) imposes \(\theta_3\) to be the conjugacy class of \(x_1 h_{\theta_3} x_1^{-1} x_2 h_{\theta_3} x_2^{-1}\). In order to compute the integral we need to compute the jacobian of the transformation from \(\theta\) to \(\theta_3\). This jacobian from \(\theta\) to \(\theta_3\) is easily computed from the relations (C45, C52) and reads
\[
d\nu(\theta_3) = \frac{\sin \theta}{2} d\theta. \quad (C55)
\]
The integral in equation (C54) reads
\[
\int_0^{\pi} \frac{\sin \theta}{2} d\theta \int_H d\xi_1 d\xi_2 \delta_{x_1^{-1} x_2} (h_{\xi_1} a_{\theta} h_{\xi_2}^{-1}) = \int_G dg \delta_{x_1^{-1} x_2} (g) = 1 \quad (C56)
\]
where we have recognized the normalized Haar measure written in terms of the Euler angles (A3).

7. Fusion properties

We prove here the two fusion identities for simple representations of the same kind.
a. Fusion properties of the \((0, j)\) representations

The fusion identity for \((0, j_1) \otimes (0, j_2)\) reads

\[
\theta_{j_1 j_2} = \sum_{j_3} d_{j_3} \theta_{j_1 j_2}
\]  

(C57)

To prove it, we consider the following evaluation of the LHS

\[
\int dx \ [D_{j_1} \otimes D_{j_2}](xh\theta^{-1} x^{-1})
\]

(C58)

We can decompose the SU(2) representations into Clebsh-Gordan series

\[
\sum_{j_3} d_{j_3} \int dx \ [C_{j_1 j_2 j_3} D_{j_3}(xh\theta^{-1} x^{-1})C_{j_1 j_2 j_3}]
\]

(C59)

where \(C_{j_1 j_2 j_3}\) are SU(2) normalized Clebsh-Gordan coefficients. Due to the fact that the fusion of two representations \((0, j_1)\) and \((0, j_2)\) decomposes only on the \((0, j)\) representations, with Clebsh-Gordan coefficients \(C_{j_1 j_2 j_3} = C_{j_1 j_2 j_3}\) which are those of SU(2), (C59) is the RHS of (C57). This result can be extended to the fusion of an arbitrary number of \((0, j)\).

b. Fusion of two \(\theta\)

The fusion identity for \((\theta_1, 0) \otimes (\theta_2, 0)\) reads

\[
\theta_{\theta_1 \theta_2} = \int d\nu(\theta_3) \sum_{s_3} (0, 2) \otimes (\theta_1, 0) \otimes (\theta_2, 0)
\]

(C60)

where

\[
C_{x_1, x_2, x_3}^{(\theta_1, 0)(\theta_2, 0)(\theta_3, s_3)}
\]

(C61)

denotes the Clebsh-Gordan coefficient of the tensor product of representations \((\theta_1, 0) \otimes (\theta_2, 0)\) into \((\theta_3, s_3)\)

\footnote{Note that contrary to \((0, j)\) case, the tensor product of two \((\theta, 0)\) representations does not decompose only on simple representations.}
are given in appendix B). We have already computed the LHS in propositionprop:Testar1, its action on \( \phi_1(x_1) \otimes \phi_2(x_2) \) gives
\[
\chi_j(x_1 h_{\theta_1}^{-1} x_1^{-1} x_2 h_{\theta_2}^{-1} x_2^{-1}) \phi_1(x_1) \otimes \phi_2(x_2)
\]
(C62)
On the basis of delta functions, the evaluation of the RHS is then
\[
\chi_j(x_1 h_{\theta_1}^{-1} x_1^{-1} x_2 h_{\theta_2}^{-1} x_2^{-1}) \delta_g_1(x_1) \delta_g_2(x_2)
\]
(C63)
Now looking at the proof of orthogonality of the Clebsh-Gordan coefficients (see Prop. 12), we can easily modify this proof, inserting a \( \chi_j(\theta_3) \) functions, to get the following evaluation of the LHS
\[
\int d\theta_3 \Delta(\theta_3) \sum_{n_3} \int_{G/H} d x_3 \ C^{(\theta_1,0)(\theta_2,0)(\theta_3,n_3)}_{x_1 x_2 x_3} \chi_j(\theta_3) \sum_{i} C^{(\theta_1,0)(\theta_2,0)(\theta_3,n_3)}_{x_1 x_2 x_3} =
\]
(C64)
\[
\chi_j(x_1 h_{\theta_1} x_1^{-1} x_2 h_{\theta_2} x_2^{-1}) \delta_g_1(x_1) \delta_g_2(x_2)
\]
(C65)
which proves the identity. ■

APPENDIX D: PROOF OF LEMMA 1

Note that \( T^* \) possess \( |\Delta_3| + |\Delta_2| \) vertices and \( |\Delta_3| + |\Delta_2| - 1 \) edges. Also \( |\tilde{\Delta}_2^*| = |\Delta_3| + |\Delta_2| - 1 \). Therefore the cardinal of \( \tilde{\Delta}_T^* \) is given by
\[
|\tilde{\Delta}_T^*| = |\Delta_2| + |\tilde{\Delta}_1| - |\Delta_3| - |\tilde{\Delta}_2| - 1.
\]
(D1)
\( T \) possesses \( |\Delta_0| - |\Delta_0| + 1 \) vertices and \( |\Delta_0| - |\Delta_0| - 1 \) edges if \( \partial M = \bigcup_{i=1}^{n} \Sigma_i \) is non empty. The cardinal of \( \Delta_T \) is then given by\( ^{12} \)
\[
|\Delta_T| = |\Delta_1| - |\Delta_0| + |\Delta_0|
\]
(D2)
the difference LHS of (6) gives
\[
|\tilde{\Delta}_T^*| - |\Delta_T| = \chi(M) - \chi(\partial M) + 1,
\]
(D3)
where \( \chi(\Sigma) \) denotes the Euler characteristic
\[
\chi(M) = \sum_{i=0}^{3} (-1)^i |\Delta_i|, \chi(\partial M) = \sum_{i=0}^{2} (-1)^i |\tilde{\Delta}_i|.
\]
(D4)
It is well known that for an open 3 dimensional manifold we have
\[
2\chi(M) - \chi(\partial M) = \chi(M^*_{\partial M}),
\]
(D5)
Moreover,
\[
\chi(\partial M) = \chi(\bigcup_{i} \Sigma_i) = \sum_{i} \chi(\Sigma_i) = \sum_{i} (2 - 2g_{\Sigma_i}).
\]
(D6)
\(^{12} \) If \( \partial M = \), \( T \) possesses one edge less and \( |\Delta_T| = |\Delta_1| - |\Delta_0| + |\Delta_0| + 1 \)
All this result imply that (D3) is equal to \(1 + \sum_i (g_{e_i} - 1)\) which proves the second part of the lemma.

In order to prove the first part of the Lemma we need to use the Bianchi Identity, whose proof is recalled in lemma 3 of appendix E: For every set of dual faces \(f^*\) forming a closed surface \(S^*\) having the topology of a 2-sphere, it exists some group elements \(k_{f^*}\) (functions of the \(g_{e^*}\)), some \(\sigma_{f^*} = \pm 1\) and an order of the faces \(f^*\) such that

\[
\prod_{f^* \in S^*} k_{f^*} g_{f^*}^{\sigma_{f^*}} k_{f^*}^{-1} = 1 \quad (D7)
\]

We now chose the connection to satisfy \(G_{\Delta_d} = 1\). Suppose that \(f^* \in \tilde{\Delta}_2^\ast\) is a boundary face. This face belongs to a unique elementary three cell \(c^* \in \tilde{\Delta}_3\). If this three cell is not intersecting the tree \(T\) then all group elements, except \(G_{f^*}\), associated to faces belonging to \(c^*\) are equal to one if \(G_{\Delta_d,T^*} = 1\). By the Bianchi identity this means that \(G_{f^*} = 1\).

If \(c^*\) intersects the tree, \(f^*\) is said to be connex to the tree\(^{13}\). In this case, one can consider the surface \(\sigma_T \in \tilde{\Delta}_2\) which is the boundary of the tubular neighborhood of \(T\). Explicitly this surface can be built as the union of boundary faces connex to the tree and the union of the \(f^*\) for \(f^* \sim e\), with \(e\) touching the vertices of \(T\) without belonging to \(T\). Since \(T\) is a tree of \(\Delta\), its tubular neighborhood has the topology of a 3-ball\(^{14}\) and its boundary \(\sigma_T\) has the topology of a 2-sphere. We can then use the Bianchi identity on \(\Sigma_T\) to conclude that \(G_{f^*} = 1\).

Now that we have proven \(G_{f^*} = 1\) for all boundary faces we consider \(f^* \in \Delta_2^\ast\) an internal dual face intersecting \(T\) (\(f^* \sim e\), \(e \in T\)). \(f^*\) cut the tree tubular neighborhood surface \(\sigma_T\) in two halves \(\Sigma_{T,f^*}, \Sigma_{T,f^*}\). Moreover by the previous argument and the hypothesis \(G_{\Delta_d} = 1\) all group elements except \(G_{f^*}\) are equal to one. Using the Bianchi identity on \(\Sigma_{T,f^*}\) one conclude \(G_{f^*} = 1\). This conclude the proof of the Lemma.

APPENDIX E: PROOF OF THE INVARIANCE OF THE PONZANO-REGGE INARIANT

In this appendix, we prove that the Ponzano-Regge invariant defined by equation (7) is independent of all the ingredients and thus is actually an invariant \(PR(M)\) of the manifold \(M\).

1. Independence from the orientations

**Proposition 13** The definition of \(PR[M, \Delta, T, T^*]\) does not depend on the choice of the orientations of edges, faces, and of the choice of starting dual vertex \(st(e)\).

**Proof:**
Let us reverse the orientation of an edge \(e\) of \(\Delta\). This will change the signs of all the \(\epsilon(e, e^*)\)

\(^{13}\) By construction there is only one face with such property

\(^{14}\) This is true only if \(\Delta\) is a triangulation of a real manifold (not a pseudo-manifold). In this case the neighborhood of every vertex \(v\) is a ball
for the \( e^* \subset e \) and the order of the product. The effect on the integrand is to reverse \( G_e \) into \( G_e^{-1} \). This does not affect the value of the integral since \( \delta(G^{-1}) = \delta(G) \) for every group element \( G \).

Let us reverse the orientation of a dual edge \( e^* \). This will change the signs of all the \( \epsilon(e^*, e) \) for \( e \supset e^* \). The effect on the integrand is to reverse every \( g_{e^*} \) into \( g_{e^*}^{-1} \). This does not affect the integral since the Haar measure is invariant by inversion \( dg^{-1} = dg \).

Let us choose another starting vertex \( st(f^*) \) for a given dual face \( f^* \sim e \). The effect is to conjugate \( G_e \) by adjoint action of a group element. This does not affect the integral since \( \delta(k^{-1}G_ek) = \delta(G_e) \).

2. Independence from choice of trees

To prove the invariance under changes of trees, we will need the following lemma regarding the relation between different choices of maximal trees [28].

**Lemma 2 (Elementary moves of maximal tree)** Two homologous maximal trees \( T_1 \) and \( T_2 \) in a graph \( \Gamma \) be related by a sequence of elementary moves. \( T_2 \) is said to be related to \( T_1 \) by an elementary move if \( T_2 \) is obtained from \( T_1 \) in the following way: Pick up a vertex \( v \) in \( T_1 \), an edge \( e \) in \( T_1 \) and an edge \( \tilde{e} \) not in \( T_1 \), both edges touching \( v \). Create \( T_2 \) by removing \( e \) from \( T_1 \) and add \( \tilde{e} \) to it.

**Proposition 14** The definition of \( PR[M, \Delta, T, T^*] \) does not depend on the choice of \( T^* \).

**Proof**:
The proof of the independence in the choice of \( T^* \) is based on two ingredients. The first one is the invariance of the Haar measure by left (right) translation and inverse. The second one is the particular structure of the partition function i.e the way the \( G_e \) are built from the \( g_{e^*} \).

We consider first the invariance under an elementary move of the tree \( T^* \). We pick up a vertex \( v^* \), an edge \( e^* \in T \) and an edge \( \tilde{e}^* \notin T^* \), see figure 6. We want to perform the move that remove \( e^* \) from \( T^* \) and add \( \tilde{e}^* \) to it. In terms of the explicit expression, we want to
prove that
\[
\int_G \prod_{e^*} dg_{e^*} \left( \prod_{e^* \in T^*, e^* \neq e^*} \delta(g_{e^*}) \right) \delta(g_{e^*}) \prod_{e \in T} \delta(G_e)
\]
\[
= \int_G \prod_{e^*} dg_{e^*} \left( \prod_{e^* \in T^*, e^* \neq e^*} \delta(g_{e^*}) \right) \delta(g_{e^*}) \prod_{e \in T} \delta(G_e)
\]
(E1)

The strategy is to start from the first expression, to perform a change of variable and to show that it leads to the second expression.

Change of variable: For simplicity, we will consider that all the edges touching \( v^* \) are oriented toward \( v^* \). We denote \( R^* \) the other vertex of \( \tilde{e}^* \). We consider it as the root of the tree i.e. it induces a partial order on the vertices, in particular we will talk about the vertices \( w^* \preceq v^* \) which are under \( v^* \) in the tree (including \( v^* \) itself). We define \( \sigma(e^*) = 1 \) if the source of \( e^* \) is a vertex \( w^* \preceq v^* \), and \( \sigma(e^*) = 0 \) if not. We define \( \tau(e^*) = 1 \) if the target of \( e^* \) is a vertex \( w^* \preceq v^* \), and \( \tau(e^*) = 0 \) if not. We perform the following change of variable

\[
g_{e^*} \rightarrow g_{e^*} \quad g_{\tilde{e}^*} \rightarrow g_{\tilde{e}^*} g_{e^*}^{-1} g_{e^*}
\]
\[
\forall e^* \text{ s.t. } e^* \neq e^*, e^* \neq \tilde{e}^*, \quad g_{e^*} \rightarrow k^{-\sigma(e^*)} g_{e^*} k^{\tau(e^*)}, \quad \text{with } k \equiv g_{\tilde{e}^*}^{-1} g_{e^*}.
\]  
(E2)

Effect of the change of variable: Let us examine the effect of the change of variable on the integral expression. Our claim is the following

\[
\int_G \prod_{e^*} dg_{e^*} \left( \prod_{e^* \in T^*, e^* \neq e^*} \delta(g_{e^*}) \right) \delta(g_{e^*}) \prod_{e \in T} \delta(G_e)
\]
\[
= \int_G \prod_{e^*} dg_{e^*} \left( \prod_{e^* \in T^*, e^* \neq e^*} \delta(g_{e^*}) \right) \delta(g_{e^*}) \prod_{e \in T} \delta(\tilde{G}_e).
\]  
(E3)

where the elements \( \tilde{G}_e \) are the elements \( G_e \) where the role of \( g_{e^*} \) and \( g_{\tilde{e}^*} \) are exchanged. Let us prove this claim by a detailed inspection of the effect of the change of variable (E2) on the RHS of (E3). We examine separately each term and make a detailed discussion for each type of elements \( G_e \).

- The measure is unaffected by this change of variable since the Haar measure is invariant by left/right translation and inverse.

- The term
\[
\prod_{e^* \in T^*, e^* \neq e^*} \delta(g_{e^*}),
\]  
(E4)

is unaffected. The reason is that these \( e^* \) have either 1) not any vertex under \( v^* \) and \( g_{e^*} \) is unaffected, or 2) both vertices under \( v^* \) and \( g_{e^*} \) is conjugated into \( k^{-1} g_{e^*} k \) which does not change \( \delta(g_{e^*}) \). The only edge of \( T^* \) that has exactly one vertex under \( v^* \) is \( e^* \) and it is excluded from this product.

- \( \delta(g_{e^*}) \) is unaffected since \( g_{e^*} \) is not changed.
• if $e$ is such that $e^* \not\subset e$ and $\bar{e}^* \not\subset e$, then $G_e$ becomes

$$G_e \to \tilde{G}_e = \prod_{e^* \in e} (k^{-\sigma(e^*)}g_{e^*}k^{\tau(e^*)})^{\epsilon(e,e^*)}.$$ (E5)

One can now check that the factors $\sigma$ and $\tau$ inside the ordered product exactly compensate, since we are taking the product along a dual face $f^*$. The only possible effect is to conjugate $G_e$, which does not affect $\delta(G_e)$.

• if $e^* \subset e$ and $\bar{e}^* \not\subset e$ then $G_e$ becomes

$$G_e \to \tilde{G}_e = g_{e^*} \prod_{e^* \in e, e^* \neq \bar{e}^*} (k^{-\sigma(e^*)}g_{e^*}k^{\tau(e^*)})^{\epsilon(e,e^*)}$$ (E6)

We choose the starting vertex at $v^*$ and the orientation of the $e$ that is compatible with $e^*$. Once again one can use the fact that the $\sigma$ and $\tau$ are compensate each other inside the product. We just to worry about what happens at the beginning and end of the product. We now have the fact that one of the vertex of $e^*$ is $v^*$ and the other is not $\preceq v^*$ by construction. We thus have

$$\tilde{G}_e = g_{e^*}k^{-1} \left( \prod_{e^* \in e, e^* \neq \bar{e}^*} g_{e^*}^{\epsilon(e,e^*)} \right) = g_{e^*} \prod_{e^* \in e, e^* \neq \bar{e}^*} g_{e^*}^{\epsilon(e,e^*)}$$ (E7)

Then $\tilde{G}_e$ is truly $G_e$ with the role of $g_{e^*}$ and $g_{\bar{e}^*}$ exchanged.

• if $\bar{e}^* \subset e$ and $e^* \not\subset e$, $G_e$ becomes

$$G_e \to \tilde{G}_e = g_{e^*}g_{\bar{e}^*}^{-1}g_{e^*}k^{-1} \prod_{e^* \in e, e^* \neq \bar{e}^*} (k^{-\sigma(e^*)}g_{e^*}k^{\tau(e^*)})^{\epsilon(e,e^*)}$$ (E8)

We use a similar argument than the previous case. Due to the fact that one of the vertex of $\bar{e}^*$ is $v^*$ and the other $R^*$ (hence not under $v^*$) we get

$$\tilde{G}_e = g_{e^*}g_{\bar{e}^*}^{-1}g_{e^*}k^{-1} \prod_{e^* \in e, e^* \neq \bar{e}^*} g_{e^*}^{\epsilon(e,e^*)}$$ (E9)

which is $G_e$ with the role of $g_{e^*}$ and $g_{\bar{e}^*}$ exchanged.

• if $\bar{e}^* \subset e$ and $e^* \subset e$, we orient $f^* \sim e$ starting from $v^*$ in a compatible way with $e^*$, $G_e$ becomes

$$G_e \to \tilde{G}_e = g_{e^*}g_{\bar{e}^*}^{-1}g_{e^*}g_{\bar{e}^*}^{-1} \left( \prod_{e^* \in e, e^* \neq \bar{e}^*, e^*} k^{-\sigma(e^*)}g_{e^*}k^{\tau(e^*)} \right)^{\epsilon(e,e^*)} = g_{\bar{e}^*}g_{e^*}^{-1} \left( \prod_{e^* \in e, e^* \neq \bar{e}^*, e^*} g_{e^*}^{\epsilon(e,e^*)} \right)$$ (E10)

Renaming $g_e$ and $g_{\bar{e}^*}$: The final result of the investigation is that the $\tilde{G}_e$ are the $G_e$ with the roles of $g_{\bar{e}^*}$ and $g_{e^*}$ exchanged which proves the claim made in (E3). Now in the RHS of this expression, we can rename $g_{\bar{e}^*}$ and $g_{e^*}$ and get (E1) which is the move we wanted to perform.
FIG. 7: Elementary move of $T$ at a vertex $v$. The tree is in thick lines.

**Proposition 15** The definition of $PR[M, \Delta, T, T^*]$ does not depend on the choice of $T$.

**Proof**: We will prove that $PR$ is invariant under an elementary move of the tree $T$. For that we choose a vertex $v$ in $T$, an edge $\epsilon \in T$, an edge $\tilde{\epsilon} \notin T$ and we try to prove the following equality

$$
\left( \prod_{e^*} \int_G dg_{e^*} \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \prod_{e \notin T, e \neq \tilde{\epsilon}} \delta(G_e) \delta(G_{\tilde{\epsilon}}) = \left( \prod_{e^*} \int_G dg_{e^*} \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \prod_{e \notin T, e \neq \tilde{\epsilon}} \delta(G_e) \delta(G_{\epsilon})
$$

(E11)

The root of the tree is chosen to be the other vertex of $e$, and denoted $R$, see figure 7. Here again, we will need an ingredient which is the fact that if the integrand contains a $\delta(g)$, then the group element $g$ can be inserted anywhere, since it is imposed to be identity. The property of the integrand we will need is the following one

**Lemma 3 (Bianchi identity)** For every set of dual faces $f^*$ forming a closed surface $S^*$ having the topology of a 2-sphere, it exists some group elements $k_{f^*}$ (functions of the $g_{e^*}$), some $\sigma_{f^*} = \pm 1$ and an order of the faces $f^*$ such that

$$
\prod_{f^* \in S^*} k_{f^*} G_{f^*}^{\sigma_{f^*}} k_{f^*}^{-1} = 1
$$

(E12)

Two particular cases are the following

- If $\Delta$ is a triangulation of a real manifold (not a pseudo-manifold), the neighborhood of every vertex $v$ is a ball, and the set of $f^*$ dual to the edges $e \supset v$ form a 2-sphere. We then have

$$
\prod_{e \supset v} k_{e} G_{e}^{\sigma_{e}} k_{e}^{-1} = 1
$$

(E13)

- If $T$ is a tree of $\Delta$ (for a real manifold), its tubular neighborhood has the topology of a 3-ball and its boundary has the topology of a 2-sphere. This surface can be built has the union of the $f^*$ for $f^* \sim e$, with $e$ touching the vertices of $T$ without belonging to $T$. 

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Proof:
Let us consider the dual faces \( f^* \) forming a closed surface \( S^* \) with the topology of a 2-sphere. It is easy to see that, if we adapt the starting vertices and orientations, it exists a way to take the product of the \( U_e \) such that the constitutive \( g_e^* \) cancels each other and the net result is just the identity. The elements \( k \) and the signs \( \sigma \) express the difference between starting vertices and orientation we need to perform this operation, and those chosen for definition of the model. Let us remark that to write the ordered product, we have a freedom in the choice the starting dual face \( f^* \).

Let us now apply this lemma to the tree \( T_v \) which is the subtree of \( T \) made by all edges and vertices under \( v \) (\( v \) is thus the root of this \( T_v \)). Choosing the dual face to \( \tilde{e} \) as start, the Bianchi identity in the neighborhood of \( T_v \) can be written as

\[
k_{\tilde{e}} G_{\tilde{e}} k_{\tilde{e}}^{-1} = \left( \prod_{e \in S_1} k_e G_e^{\sigma(e)} k_e^{-1} \right) k_{\tilde{e}} G_{\tilde{e}} k_{\tilde{e}}^{-1} \left( \prod_{e \in S_2} k_e G_e^{\sigma(e)} k_e^{-1} \right)
\]

(E14)

where the products on the RHS are on edges touching vertices under \( v \), without belonging to \( T_1 \). These edges are split into two sets \( S_1 \) and \( S_2 \) that we do not need to describe. Due to this equality, one can write

\[
\delta(G_{\tilde{e}}) = \left[ \left( \prod_{e \in S_1} k_e G_e^{\sigma(e)} k_e^{-1} \right) k_{\tilde{e}} G_{\tilde{e}}^{\sigma(e)} k_{\tilde{e}}^{-1} \left( \prod_{e \in S_2} k_e G_e^{\sigma(e)} k_e^{-1} \right) \right]
\]

(E15)

Now all the \( G_e \) involved in the RHS are out of \( T \) and are not \( \tilde{e} \), thus if we multiply both LHS and RHS by \( \prod_{e \notin T, e \neq \tilde{e}} \delta(G_e) \) we get

\[
\prod_{e \notin T, e \neq \tilde{e}} \delta(G_e) \delta(G_{\tilde{e}}) = \prod_{e \notin T, e \neq \tilde{e}} \delta(G_e) \delta(G_{\tilde{e}})
\]

(E16)

which achieves the elementary move. ■

3. Pachner moves

**Proposition 16**  The definition of \( Z_{PR}[M, \Delta] \) does not depend on \( \Delta \).

**Proof:**
We are now in position to prove that our definition does not depend on the choice of triangulation. This is done by proving its invariance under the Pachner moves. We want to consider a triangulation \( \tilde{\Delta} \) obtained from \( \Delta \) by Pachner move, and prove

\[
PR(\Delta) = PR(\tilde{\Delta})
\]

But we need a choice of trees to explicitly write such invariant, and the previous proposition says the choices are all equivalent. So what we need to prove is

\[
PR(\Delta, T, T^*) = PR(\tilde{\Delta}, \tilde{T}, \tilde{T}^*)
\]

(E17)

for any choice of trees. The strategy is the following, we start from a triangulation \( \Delta \) and choose arbitrary trees \( T \) and \( T^* \). Performing a move on \( \Delta \) leads to a new triangulation \( \tilde{\Delta} \),
but we have to specify a choice of trees for this triangulation. We explicitly described such trees $\tilde{T}$ and $\tilde{T}^*$ as extension of $T$ and $T^*$. It should be noted that we truly prove the equality (E17) without any additional multiplicative (infinite) factor that appears under the move and has to be divided out.

We write all the associated $\delta(U_e)$, without taking care of the fact that some of them will not actually be in the integrand (for $e \in T$) and without taking care of the fact that certain group elements living on the dual edges will be imposed to identity by a delta function (those for dual edges in $T^*$). Performing a move on $\Delta$ leads to a new triangulation $\tilde{\Delta}$, but we have to specify a choice of trees for this triangulation. We explicitly described such trees $\tilde{T}$ and $\tilde{T}^*$ as extension of $T$ and $T^*$. We then prove that $PR(\Delta, T, T^*) = PR(\tilde{\Delta}, \tilde{T}, \tilde{T}^*)$. It should be noted that we truly prove this equality without any additional multiplicative (infinite) factor that appears under the move and has to be divided out.

A Pachner move add or remove vertices, edges, faces and tetrahedra. To prove that $PR(\Delta, T, T^*) = PR(\tilde{\Delta}, \tilde{T}, \tilde{T}^*)$, we need to show that

$$ \left( \int_G \prod_{e^* \in \Delta^*} d g_{e^*} \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \prod_{e \in \Delta} \delta(G_e) $$

$$ = \left( \int_G \prod_{e^* \in \Delta^*} d g_{e^*} \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \prod_{e \in \tilde{\Delta}} \delta(G_e) $$

(E18)

But in this equality, most of the integrals and $\delta$-functions are identical in the RHS and LHS, since the Pachner move affects the triangulation locally. We just have to consider the objects that are actually affected by the move, and we just need to prove the equality of the distributions

$$ \left( \int_G \prod_{e^* \in \Delta^*} d g_{e^*} \prod_{e^* \in T^*} \delta(g_{e^*}) \right) \prod_{e \in \Delta, e \notin \tilde{T}} \delta(G_e) $$

$$ = \left( \int_G \prod_{e^* \in \Delta^*} d g_{e^*} \prod_{e^* \in \tilde{T}^*} \delta(g_{e^*}) \right) \prod_{e \in \tilde{\Delta}, e \notin \tilde{T}} \delta(G_e) $$

(E19)

where $(e \ \text{modified})$ means that the edge is either removed from $\Delta$, or added in $\tilde{\Delta}$, or such that $G_e$ is modified by the move due to faces added or removed. These notations will become more clear on the specific examples of moves.

**Move (1,4) :** Let us first look at the (1,4) move (see figure 8). This move

- Remove one tetrahedra $[(1234)]$
- Add 1 vertex $[(5)]$
- Add 4 edges $[(15),(25),(35),(45)]$
- Add 6 faces $[(125),(135),(145),(235),(245),(345)]$
- Add 4 tetrahedra $[(1235),(1245),(1345),(2345)]$. 

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There is one more vertex, this means that we have to add one edge to $T$ to get a $\tilde{T}$, and three more tetrahedra, so we have to add three edges to $T^*$ to get a $\tilde{T}^*$. Let us add the edge (15) to extend $T$ into $\tilde{T}$ and the faces (dual edges) (235), (245), (345) to extend $T^*$ into $\tilde{T}^*$. Let us add the edge (15) to extend $T$ into $\tilde{T}$ and the faces (dual edges) (235), (245), (345) to extend $T^*$ into $\tilde{T}^*$.

Let us explicitly write the LHS and RHS of the equality (E19) we want to prove. All the edges represented on the figure are affected. We add some superscript to the $\delta$-functions to make explicit reference to the edge they come from. We get

$$LHS = \delta^{(12)}(g_{124}g_{123}\tilde{G}_{12})\delta^{(13)}(g_{123}^{-1}g_{134}\tilde{G}_{13})\delta^{(14)}(g_{134}^{-1}g_{124}\tilde{G}_{14})$$
$$\delta^{(23)}(g_{234}g_{123}\tilde{G}_{23})\delta^{(24)}(g_{124}g_{234}\tilde{G}_{24})\delta^{(34)}(g_{234}g_{134}\tilde{G}_{34})$$

(E20)

where the $\tilde{G}_{ij}$ denote the product of other group elements for faces containing the edge $(ij)$. They are six more faces after the move, so 6 more group integrals. They are four new edges but one of the corresponding $\delta$-function is removed due to the extension of the tree $T$ into $\tilde{T}$, so we have 3 more $\delta$ functions. We also add three dual edges to get $\tilde{T}^*$ so we have 3 other $\delta$-functions. We get for the RHS

$$RHS = \int dg_{125}dg_{135}dg_{145}dg_{235}dg_{245}dg_{345} \delta(g_{235})\delta(g_{245})\delta(g_{345})$$
$$\delta^{(12)}(g_{124}g_{125}g_{123}\tilde{G}_{12})\delta^{(13)}(g_{123}^{-1}g_{135}g_{134}\tilde{G}_{13})\delta^{(14)}(g_{134}^{-1}g_{145}g_{124}\tilde{G}_{14})$$
$$\delta^{(23)}(g_{234}g_{235}g_{123}\tilde{G}_{23})\delta^{(24)}(g_{124}g_{245}g_{234}\tilde{G}_{24})\delta^{(34)}(g_{234}g_{134}g_{134}\tilde{G}_{34})$$
$$\delta^{(25)}(g_{125}g_{235}^{-1}g_{245}^{-1})\delta^{(35)}(g_{135}g_{345}^{-1}g_{235})\delta^{(45)}(g_{145}g_{245}g_{345})$$

(E21)

Now one can explicitly solve the first line of $\delta$-functions, then the last line and check we get LHS.

**Move (2,3):** Let us now look at the (2,3) move (see figure 9). This move

- Removes one face (123)
- Remove two tetrahedra [(1243) and (1235)]
- Add three tetrahedra [(1245), (1345) and (2345)]
- Add three faces [(145), (245) and (345)].
FIG. 9: Pachner moves (2 → 3) and its dual

We have one more tetrahedron so we need to add one edge to $T^*$ to get $\tilde{T}^*$. We choose to add $(145)$. The LHS of (E19) reads

$$\text{LHS} = \int dg_{123} \delta^{(12)}(g_{125}g_{123}g_{124}\tilde{G}_{12})\delta^{(23)}(g_{235}g_{123}g_{234}\tilde{G}_{23})\delta^{(13)}(g_{134}g_{123}g_{135}\tilde{G}_{13})$$

$$\delta^{(14)}(g_{124}g_{134}\tilde{G}_{14})\delta^{(24)}(g_{234}g_{124}\tilde{G}_{24})\delta^{(34)}(g_{134}g_{234}\tilde{G}_{34})$$

$$\delta^{(15)}(g_{135}g_{125}\tilde{G}_{15})\delta^{(25)}(g_{125}g_{235}\tilde{G}_{25})\delta^{(35)}(g_{235}g_{135}\tilde{G}_{35})$$

(E22)

The RHS is

$$\text{RHS} = \int dg_{145}dg_{245}dg_{345}\delta(g_{145})$$

$$\delta^{(12)}(g_{125}g_{124}\tilde{G}_{12})\delta^{(23)}(g_{235}g_{234}\tilde{G}_{23})\delta^{(13)}(g_{134}g_{135}\tilde{G}_{13})$$

$$\delta^{(14)}(g_{124}g_{145}\tilde{G}_{14})\delta^{(24)}(g_{234}g_{245}g_{124}\tilde{G}_{15})\delta^{(34)}(g_{134}g_{345}g_{234}\tilde{G}_{34})$$

$$\delta^{(15)}(g_{135}g_{125}\tilde{G}_{15})\delta^{(25)}(g_{125}g_{245}g_{235}\tilde{G}_{25})\delta^{(35)}(g_{235}g_{345}g_{135}\tilde{G}_{35})\delta^{(45)}(g_{145}g_{345}g_{245}\tilde{G}_{45})$$

(E23)

We first solve $\delta(g_{145})$. We can then solve $\delta^{(45)}$ and get $g_{245} = g_{345}^{-1}$, we will call $g_{123}$ this common group element for purpose of comparison. We get

$$\text{RHS} = \int dg_{123}\delta^{(12)}(g_{125}g_{124}\tilde{G}_{12})\delta^{(23)}(g_{235}g_{234}\tilde{G}_{23})\delta^{(13)}(g_{134}g_{135}\tilde{G}_{13})$$

$$\delta^{(14)}(g_{124}g_{134}\tilde{G}_{14})\delta^{(24)}(g_{234}g_{123}g_{124}\tilde{G}_{15})\delta^{(34)}(g_{134}g_{123}g_{234}\tilde{G}_{34})$$

$$\delta^{(15)}(g_{135}g_{125}\tilde{G}_{15})\delta^{(25)}(g_{125}g_{123}g_{125}\tilde{G}_{25})\delta^{(35)}(g_{235}g_{123}g_{135}\tilde{G}_{35})$$

(E25)

One can now use the following translation

$$g_{125} \to g_{125}g_{123}$$

(E26)

$$g_{234} \to g_{123}g_{234}$$

(E27)

$$g_{135} \to g_{123}g_{135}^{-1}$$

(E28)

and get the LHS.