A dynamic model to predict the occurrence of skidding in wind-turbine bearings

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Abstract. Despite use of the best in current design practices, high-speed shaft (HSS) bearings, in a wind-turbine gearbox, continue to exhibit a high rate of premature failure. As HSS bearings operate under low loads and high speeds, these bearings are prone to skidding. However, most of the existing methods for analyzing skidding are quasi-static in nature and cannot be used to study dynamic operating conditions. This paper proposes a dynamic model, which includes gyroscopic and centrifugal effects, to study the skidding characteristics of angular-contact ball-bearings. Traction forces between rolling-elements and raceways are obtained using elastohydrodynamic (EHD) lubrication theory. Underlying gross-sliding mechanisms for pure axial loads, and combined radial and axial loads are also studied. The proposed model will enable engineers to improve bearing reliability at the design stage, by estimating the amount of skidding.

1. Introduction

Wind energy is the fastest growing renewable energy sector with an average annual growth rate of around 30% during last 10 years. In order to harvest energy most efficiently and reliably, various wind-turbine design concepts have been developed over the years. Most of the modern wind-turbine designs utilize a gearbox which connects the rotor-shaft to high-speed shaft, and increases rotational speed from 15-30 rpm (at blades) to 1000-1800 rpm – the speed required by most generators to produce electricity. As wind-turbines have grown larger, gearbox failure rates have gone up as well. Since gearbox is one of the most expensive components of a wind turbine, higher-than-expected failure rates increase the cost of energy production. For a typical turbine, 20% of the downtime is due to gearbox failures and an average gearbox failure takes about 250 hours to repair [1].

Most of the problems in wind turbine gearboxes appear to emanate from bearings[2]. Bearings supporting the high speed shaft exhibit a high rate of premature failure and are identified as one of the most critical components [1, 2]. These bearings operate under low loads and high speeds, and therefore, are prone to skidding, i.e., gross-sliding of rolling-elements on raceways. Sliding can lead to rolling surface distress and eventually to premature failure. Hence, skidding is an important design criterion for wind-turbine bearings. Both ball-bearings and roller-bearings are commonly used to support HSS. The focus of this work is on angular-contact ball-bearings.

Researchers have developed numerous analytical and numerical models of varying complexity to understand the skidding behaviour of bearings. Jones [3, 4] developed the first mathematical theory to analyze the motion of rolling-elements in ball bearings. He evaluated the frictional
forces resulting from interfacial slip at ball-race contacts using a dry friction model. One of the limitations of his theory is its dependence over raceway-control hypothesis to achieve a solution. According to this hypothesis, a ball is assumed to roll without spin on one race and roll with spin with respect to other race. Therefore, motion of the ball about its own axis and bearing axis is said to be controlled by the raceway at which no slip occurs. It is also further assumed that the gyroscopic moment acting on a ball is always resisted by frictional force acting at controlling raceway and no gyroscopic slippage takes place.

Harris [5, 6] questioned the validity of raceway-control hypothesis by formulating an analytical model for axially loaded angular contact ball bearings without using raceway control assumption. It was found that Harris’ model more closely approximated the measured data, reported by Poplawski and Mauriello [7], than the raceway-control method, which proves the inadequacy of raceway-control hypothesis. Boness and Gentle [8] also developed a quasi-static force equilibrium model of a ball bearing by using an analytical traction equation, for EHD contacts, derived by Gentle and Cameron [9].

Some skidding threshold criteria, to predict the minimum load required to avoid skidding, are also available in the literature. Hirano [10] carried out an experimental investigation on the ball motion inside an axially loaded angular contact ball bearing, and found that the gross-sliding of rolling-elements occurs when the value of the parameter \( z \frac{F_c}{F_a} \) exceeds 0.1, where \( z \), \( F_c \) and \( F_a \) denote the number of balls in the bearing, centrifugal force acting on the ball and applied thrust load respectively. However, it has been established by Poplawaski [7] and Boness [11] that this parameter alone is not sufficient to completely define the roll-slip behaviour of a ball bearing.

Recently, Liao and Lin [12] performed a geometric analysis of a ball bearing operating under combined axial and radial loads; and used a force balance approach to obtain axial and radial deformations. Since, the skidding maps produced by Liao and Lin [12] are based on the empirical criterion proposed by Hirano [10] for ball bearings under thrust load, their use can often be limited for combined loading conditions. Based on the experiments performed by Bujoreanu et al. [13], it was observed that the skidding damage in a bearing is related to the amount of heat generated inside a fluid film due to lubricant shearing. The onset of scuffing was estimated around \( 2 \times 10^{14} \) W/m³.

The quasi-static analysis techniques and skidding threshold formulations described above provide a good insight into the frictional behaviour of ball bearings and also show the existence of gross-sliding. However, these methods cannot be used to analyze combined radial and thrust loads or time-varying operating conditions, both of which are crucial for wind turbine applications. The work presented herein details a dynamic model formulation, which takes into account the centrifugal and gyroscopic effects. The frictional forces at contact interfaces are calculated for a Newtonian fluid using EHD lubrication theory. The model is used to investigate the skidding mechanisms for axial as well as combined axial and radial loading conditions, and to quantify the effect of operating parameters on gross-sliding behaviour.

2. Model Description
The analysis approach consists of two stages. In the first stage, a quasi-static method is used to determine bearing internal load distribution; and during the second stage these loads are used in a dynamic model to analyze rolling-element motion. The two methods are described in the following paragraphs.

2.1. Determination of Internal Load Distribution
In bearings, load is transmitted from one raceway to another through rolling-elements. The magnitude of load carried by an individual rolling-element depends upon the internal geometry of a bearing, number of rolling-elements in contact and location of a rolling-element inside the load-zone at a given time. In this study, Hertz elastic theory [14] is used to determine the contact
force between rolling-elements and raceways. According to Hertz theory, the contact load \( F \) between two elastic solids can be expressed in terms of maximum deformation \( \delta \) at the centre of contact ellipse as
\[
F = K \delta^{3/2}
\]  
Here, \( K \) is the stiffness parameter given by:
\[
K = \frac{\pi \kappa E'}{3\xi} \sqrt{\frac{3\varepsilon R}{\xi}}
\]  
where, \( E' \) is the effective modulus, \( R \) is the effective radius of curvature, \( \kappa \) is ellipticity parameter, and \( \xi \) and \( \varepsilon \) are the elliptical integrals of first and second kind respectively. Simplified expressions for \( \kappa \), \( \xi \) and \( \varepsilon \), derived using linear regression, can be found in Brewe and Hamrock [15].

If \( K_{\text{inn}} \) and \( K_{\text{out}} \) are the stiffness parameters for inner and outer raceway contacts respectively, defined by equation 2, then effective stiffness parameter for a rolling-element can be calculated as [16]:
\[
K_{\text{eff}} = \frac{1}{\left\{ \left( \frac{1}{K_{\text{inn}}} \right)^{2/3} + \left( \frac{1}{K_{\text{out}}} \right)^{2/3} \right\}^{3/2}}
\]  
Assuming that the raceways are rigid, for a rolling-element located at an angle \( \theta_i \) (see Figure 1a), the deformation along the contact-line can be calculated as:
\[
\delta_i = \left[ C_{\alpha} C_{\beta}, S_{\alpha} C_{\beta}, S_{\beta} \right] \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}
\]  
where, \( C_{\alpha} = \cos(\alpha) \), \( S_{\alpha} = \sin(\alpha) \), \( \beta \) is the contact angle, and \( (\delta_x i + \delta_y j + \delta_z k) \) is the inner-race displacement vector.

Contact forces acting between rolling-elements and raceways can be obtained by the following expressions.
\[
F_{\text{inn}} = F_{\text{out}} = K_{\text{eff}} \delta_i^{3/2}
\]  
Element loads obtained from equations 5, are based on an assumption that contact angle between rolling-elements and raceways remain unaffected by bearing rotational velocity. However, at high rotational speeds, the centrifugal force acting on a rolling-element, forces contact angle to change from its nominal value, and this creates a differential between inner and

Figure 1: (a) Coordinate system for quasi-static analysis (b) Forces acting on a rolling element
outer contact angles. Since, the bearings supporting the HSS of a wind-turbine, operate at fairly moderate speeds (1200-1800 rpm), therefore, change in the contact angle due to centrifugal force would be very small [10].

Resultant load acting on the inner-race can be determined by a vector summation of $F_{\text{inn}}$, for all the rolling-elements which are in contact with inner-race. Considering centrifugal force, inner-race forces can be calculated as

$$\begin{array}{c}
\left\{ \begin{array}{c}
F_x \\
F_y \\
F_z
\end{array} \right\} = - \sum_{i=1}^{z} \left( \frac{K_{\text{eff}} \delta_i^{3/2}}{2} - \frac{1}{2} m r_p \omega^2 \cos(\beta) \right) [C_\theta C_\beta, S_\theta C_\beta, S_\beta]^T
\end{array}$$

where, $m$ is the mass of a rolling-element, $r_p$ is the pitch radius, $\omega_c$ is the rotational speed of cage, and $z$ is the number of rolling-elements. Equations 4 to 6 are solved iteratively, using Newton-Raphson method, until calculated loads are equal to the applied ones. The method, outlined above, does not account for changes in the load distribution which can occur due to variation in numbers and positions of rolling-elements inside load-zone. However, from a detailed experimental study [17], it was concluded that the fluctuations in deflection and stiffness due to these factors can be less than 0.5% of the total value, for a given load.

2.2. Dynamic Model to Analyze Rolling-Element Motion
The model consists of two reference frames. The first reference frame $X'Y'Z'$ is fixed at bearing centre with $X'$ and $Y'$ axis lying in the bearing plane. The second reference frame, $xyz$, is a moving frame with its origin attached to the centre of a rolling-element (Figure 2). Each rolling-element has four degrees of freedom: three rotational degrees of freedom about its centre ($\omega_x, \omega_y$ and $\omega_z$), and one rotational degree of freedom about bearing centre ($\omega_c$).

The equations governing the rolling-element motion are derived using Euler’s equations and are given by:

$$\begin{array}{c}
\left\{ \begin{array}{c}
\Sigma M_x \\
\Sigma M_y \\
\Sigma M_z
\end{array} \right\} = \text{diag} \left[ \frac{2}{5} m r^2 \right] \left\{ \begin{array}{c}
\omega_x \\
\omega_y \\
\omega_z
\end{array} \right\} + \left[ \begin{array}{ccc}
0 & -\omega'_z & \omega'_y \\
\omega'_z & 0 & -\omega'_x \\
-\omega'_y & \omega'_x & 0
\end{array} \right] \text{diag} \left[ \frac{2}{5} m r^2 \right] \left\{ \begin{array}{c}
\omega_x \\
\omega_y \\
\omega_z
\end{array} \right\}
\end{array}$$

where, $r$ is the ball radius, $\Sigma M_{x\ell} + \Sigma M_{y\ell} + \Sigma M_{z\ell}$ is the moment vector acting on a ball, $\omega_{x\ell} + \omega_{y\ell} + \omega_{z\ell} (= \omega_b)$ is the ball angular velocity vector in $xyz$-frame, and $\omega'_x + \omega'_y + \omega'_z$ is the angular velocity vector of the frame $xyz$ with respect to $X'Y'Z'$.

For the system shown in the figure 2, reference frame $xyz$ is constrained to rotate about $Z'$ axis with angular velocity $\omega_c$. Therefore, $\omega'_x = \omega'_y = 0$ and $\omega'_z = \omega_c$. Putting these values into equation 7 gives

$$\begin{array}{c}
\Sigma M_x = I(\omega_x - \omega_c \omega_y) \\
\Sigma M_y = I(\omega_y + \omega_c \omega_z) \\
\Sigma M_z = I \omega_z
\end{array}$$

where, $I = 2 m r^2 / 5$. Calculation of moment terms in equation 8 is described in section 2.3.

To determine the equation governing rolling-element motion about bearing axis, interaction between rolling-element and cage must be considered. A very basic approach has been adopted here to define this interaction. Springs of very high stiffness ($k_{\text{cage}} = 10^8$) are inserted in between rolling-elements so that their motion about bearing axis can be coupled and cage
forces are determined by calculating the compression or elongation in these springs (figure 3). Deformations in the left and right springs ($\delta_{(+)}$ and $\delta_{(-)}$) can be written as:

$$\delta_{(+)} = r_p \sqrt{\left\{ \cos(\theta_{i+1}^c) - \cos(\theta_i^c) \right\}^2 + \left\{ \sin(\theta_{i+1}^c) - \sin(\theta_i^c) \right\}^2 - 2r_p \sin \left( \frac{\pi}{z} \right)}$$ (9a)

$$\delta_{(-)} = r_p \sqrt{\left\{ \cos(\theta_{i-1}^c) - \cos(\theta_i^c) \right\}^2 + \left\{ \sin(\theta_{i-1}^c) - \sin(\theta_i^c) \right\}^2 - 2r_p \sin \left( \frac{\pi}{z} \right)}$$ (9b)

where, $\theta_i^c$ is the position angle of the rolling-element, at time $t$, for which cage force is being calculated (= $\int_0^t \omega_c dt$), $\theta_{i+1}^c$ and $\theta_{i-1}^c$ are position angles of neighboring elements at time $t$. Now, the remaining differential equation governing the cage velocity can be formulated as:

$$I_c \ddot{\omega}_c = -(f_A r_i + f_B r_o) - k_{cage} (\delta_{(-)} - \delta_{(+)}) r_p - F_D$$ (10)

where, $F_D$ is rolling-element drag, $r_i$ and $r_o$ are inner and outer radii, and $I_c$ is the moment of inertia of the ball about $Z'$ axis and is given by: $I_c = \frac{2}{5}mr_i^2 + mr_o^2$.

Equations 8 and 10 are the first-order non-linear differential equations and are solved numerically using Matlab’s ODE solver.
2.3. Traction Equations for Elliptical Contact

Traction forces in a lubricant film are generated due to shearing effect which also produces frictional heat (given by the product of shear stress and strain rate) inside the film. Crook [18, 19] investigated the effect of temperature rise due to film shearing on lubricant viscosity and traction properties. The investigation was based on a Newtonian fluid model according to which shear stress in a lubricant film is proportional to shear-strain rate. The dependency of lubricant viscosity on pressure and temperature is described by well known Barus equation [20]:

$$\eta = \eta_0 e^{c_{\eta p}p - c_{\eta T}(T - T_R)}$$  \hspace{1cm} (11)

where $\eta_0$ is the reference viscosity at reference temperature $T_R$, $p$ is the hertzian pressure, $T$ is the lubricant temperature, and $c_{\eta p}$ and $c_{\eta T}$ are pressure and temperature coefficients respectively. Values of $c_{\eta p}$ and $c_{\eta T}$ are generally determined from the viscosity-pressure and viscosity-temperature curves (measured experimentally) [21].

Consider a fluid film trapped between two contacting solids (figure 4). Thickness of the film is $h$ and linear velocities of contacting surfaces are $u_1$ and $u_2$. Shear stress ($\tau$) in the film can be expressed by the following expression (see Crook [18] for derivation).

$$\tau(x, y) = \eta_m(x, y) \frac{\Delta u(x, y)}{h}$$  \hspace{1cm} (12)

where, $\Delta u$ is the slip velocity and $\eta_m$ is the fluid viscosity at $(x, y)$ and is described as

$$\eta_m(x, y) = \eta(x, y) \frac{\ln \left( \sqrt{\psi + 1} + \sqrt{\psi} \right)}{\sqrt{\psi} (\psi + 1)}$$  \hspace{1cm} (13)

$\eta(x, y) = \eta_0 e^{c_{\eta p}p(x, y)}$, $\psi = \frac{\eta(x, y) c_{\eta T} \Delta u^T}{8K_c}$ and $K_c$ is the thermal conductivity of lubricant.

At any point $(x, y)$ on the contact patch, $x$-component of the spin velocity is $\omega_s y$ and $y$-component is $\omega_s x$. If linear slip velocities are $\Delta u_x^x$ and $\Delta u_y^y$ in $x$ and $y$ direction respectively then the slip velocity vector at $(x, y)$ can be written as

$$\Delta u(x, y) = (\Delta u_x^x - \omega_s y) \hat{i} + (\Delta u_y^y + \omega_s x) \hat{j}$$  \hspace{1cm} (14)

For a contact patch shown in figure 4, the average traction force vector ($\vec{f} = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$) and traction moment vector ($\vec{M} = M_z \hat{k}$) can be calculated by integrating the equation 12.
Therefore,

\[
\bar{f} = \frac{1}{h} \int_{-a}^{a} \int_{-b}^{b} \eta_m(x,y) \Delta u(x,y) dxdy
\]

\[
\bar{M} = \frac{1}{h} \int_{-a}^{a} \int_{-b}^{b} \eta_m(x,y)(x i + y j) \times \Delta u(x,y) dxdy
\]

\[
= \frac{1}{h} \int_{-a}^{a} \int_{-b}^{b} \eta_m(x,y) \left\{ \Delta u_x^p - \Delta u_y^p y + \omega_s \left( x^2 + y^2 \right) \right\} dxdy
\]

(15)

The film thickness, \( h \), between the contacting surfaces is assumed to be constant throughout the contact patch, and is calculated using the central film-thickness formula provided by Hamrock and Dowson [16]

\[
\frac{h}{R_x} = 2.69U^{0.67}G^{0.53}W^{-0.067} \left( 1 - 0.61e^{-0.73a} \right)
\]

(16)

where, \( U = \frac{U_{avg}}{E'} \), \( G = \frac{E'}{\nu_p} \), and \( W = \frac{F}{E'R_x^2} \) are the dimensional parameters for speed, material and load respectively, \( R_x \) is the effective radius along \( X'' \)-axis, and \( u_{avg} \) is the mean velocity of sliding surfaces.

Now, the moment terms of equation 8 can be described in terms of traction forces as

\[
\Sigma M_x = r \left( f_y^B - f_y^A \right)
\]

\[
\Sigma M_y = rsin\beta \left( f_x^B - f_x^A \right) + (M_x^A + M_x^B) \cos\beta
\]

\[
\Sigma M_z = rcos\beta \left( f_y^B - f_y^A \right) - (M_y^A + M_y^B) \sin\beta
\]

(18a)

(18b)

(18c)

3. Results and Discussion

The skidding phenomenon in angular contact ball bearings is demonstrated using an example bearing, parameters of which are listed in the table 1a and 1b. The gross-sliding is represented by cage-slip, which is the deviation of actual cage speed from its corresponding theoretical value

\[
\left( \frac{\omega_c - \omega_{th}^c}{\omega_c + \omega_{th}^c} \right) \times 100\%
\]

where \( \omega_c \) is the actual cage speed and \( \omega_{th}^c \) is the one calculated from inner-race speed (\( \omega_i \)), using pure-rolling condition (equation 19)

\[
\omega_{th}^c = \left( 1 - \frac{\cos\beta}{r_p/r} \right) \left( \frac{\omega_i}{2} \right)
\]

(19)

3.1. Skidding Under Constant Axial Load

Figure 5a shows that at lower axial loads, high cage slip is present. As the applied load is increased, the value of cage-slip decreases. When the applied load is increased above a critical value, which is 1.3kN for this example bearing, cage-slip becomes less than 1% and almost no gross-sliding takes place. This critical load (1.3kN) can be considered as the minimum load required to prevent skidding; and corresponding rolling-element load is 125N. Note that the value of cage-slip in figure 5a never goes to zero, this is for the reason that in order to generate traction forces, some amount of relative sliding is required between two contacting solids.

Ball orientation angles, \( \beta_{YZ} \) and \( \beta_{XZ} \) (figure 5b), are the angles which spin-axis of a ball (or vector \( \omega_e \)) makes with \( yz \) and \( xz \) planes respectively. In an angular contact ball bearing, ball
Table 1: Parameters defining the bearing geometry and lubricant properties

(a) Bearing

| Parameter                      | Value |
|--------------------------------|-------|
| Number of rolling-elements (z) | 16    |
| Contact angle (β)              | 40°   |
| Ball radius (r)                | 12.5 mm |
| Pitch radius (r_p)             | 77.5 mm |
| Ball mass (m)                  | 64 grams |
| Material                       | Steel |

(b) Lubricant

| Parameter                              | Value                          |
|----------------------------------------|--------------------------------|
| Dynamic viscosity (η_0)                | 0.04 Pa.s                      |
| Reference temperature (T_R)            | 30°C                           |
| Viscosity-Pressure coefficient (c_{np})| 1.2 x 10^{-8} Pa^{-1}         |
| Viscosity-Temperature coefficient (c_{OT})| 0.05°C^{-1}            |
| Thermal conductivity (K_c)             | 0.125 J/(kgK)                  |
| Density (ρ)                            | 860 kg/m^3                     |

Figure 5: Simulation results for axial loading (a) Cage-slip variation with axial load (b) Ball orientation angles; Inner-race speed ω_i = 1800rpm

spins about an axis passing through its centre, at an angle β from the bearing axis. This spinning ball is also forced to rotate about the bearing axis. As the ball rotates around bearing axis, the direction of angular momentum vector changes continuously, and this change in angular momentum generates a gyroscopic couple which is balanced by the traction forces acting at contact interfaces. At low loads, traction forces are not enough to provide required gyroscopic couple and ball spin-axis changes its orientation and becomes almost parallel to the bearing axis, thereby reducing the required gyro-couple (figure 5b). As the applied load is increased, traction forces increase as well, and ball spin-axis approaches its theoretical orientation.

3.2. Skidding Under Combined Axial and Radial Loads

Rolling-element motion in the presence of combined loads is remarkably different from the axially loaded bearings, because of the formation of loaded and unloaded zones. Figure 6 shows the angular velocity components and slip velocity of a rolling-element in the example bearing (table 1a) subjected to an axial load of 2.2kN and a radial load of 2kN. The maximum contact force between a rolling-element and raceway is 650N. The motion is also illustrated using a graphical representation (figure 7). During the unloaded zone (C → D), the angular momentum of the rolling-element remains nearly constant (in the absence of contact force) and the components of angular velocity vary sinusoidally in the local coordinate system. As the rolling-element enters the load-zone, slip velocity starts to decrease due to the application of
Figure 6: Simulation results for combined loading; $F_z = -2.2\text{kN}$, $F_y = 2\text{kN}$, $\omega_i = 1800\text{rpm}$

3.3. Influence of Radial Load on Skidding Behaviour
Firstly, in the presence of radial load maximum skidding damage takes place at the entry to load-zone; whereas, damage is uniformly distributed under axial load. Secondly, a bearing with radial load requires a larger rolling-element force to minimize skidding than a bearing with pure axial load. Finally, if an unloaded-zone is created inside a bearing (by applying radial load), then it is not possible to completely eliminate skidding, but the length of skid zone (inside loaded region) can be reduced by increasing the applied load.

3.4. Measures to Prevent Skidding
The most effective way to avoid skidding in bearings is to provide a static preload. The amount of preload must be chosen such that the operating load on a rolling-element must be greater than the minimum load required to prevent skidding. Skidding can also be minimized by using a high-traction lubricant or by reducing the number of rolling-elements.
4. Conclusions and Future Work

A dynamic model, considering EHD lubrication theory and gyroscopic effects, is formulated and used to analyze the skidding characteristics of angular-contact ball bearings. The findings indicate that the gross-sliding mechanism for combined loading conditions is substantially different from the one observed for pure-axial loads. Future work will include the implementation of a detailed cage interaction model, and consideration of cage clearance and frictional effects.

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References

[1] Ribrant J and Bertling L 2007 IEEE Power Engineering Society General Meeting, 2007 pp 1–8
[2] Musial W, Butterfield S and McNiff B 2007 Proceedings of the European Wind Energy Conference (Citeseer)
[3] Jones A B 1959 ASME Trans 81 1–12
[4] Jones A B 1960 Journal of Basic Engineering 82 309–320
[5] Harris T A 1971 ASME Journal of Lubrication Technology 93 17–24
[6] Harris T A 1971 Journal of Lubrication Technology, Transactions of the ASME 32–38
[7] Poplawski J V and Mauriello A ASME Paper No 69-Lubs-20
[8] Boness R J and Gentle C R 1975 Wear 35 131–148
[9] Gentle C R and Cameron A 1974 Wear 27 71–81
[10] Hirano F 1965 Tribology Transactions 8 425–434
[11] Boness R J 1981 Journal of lubrication technology 103 35
[12] Liao N T and Lin J F 2002 Mechanism and Machine Theory 37 113
[13] Bujoreanu C, Crețu S and Nelias D 2003 FASCICLE VIII, Tribology ISSN 1221-4590
[14] Hertz H 1881 J. Reine Angew. Mathematik 92 156–171
[15] Brew E D and Hamrock B J 1977 ASME, Transactions, Series F-Journal of Lubrication Technology 99 485–487
[16] Hamrock B J and Dowson D 1981 Ball bearing lubrication (Wiley New York)
[17] While M F 1979 Journal of Applied Mechanics 46 677–684
[18] Crook A W 1961 Phil. Trans. Roy. Soc., London. Series A, Mathematical and Physical Sciences 254 237–258
[19] Crook A W 1963 Phil. Trans. Roy. Soc., London. Series A, Mathematical and Physical Sciences 255 281–312
[20] Barus C 1893 Am. J. Sci 45 87–96
[21] Evans C R and Johnson K L 1986 Proceedings of the Institution of Mechanical Engineers. Part C. Mechanical engineering science 200 303–312