Rotating Strings in Massive Type IIA Supergravity

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**ABSTRACT**

Massive type IIA supergravity admits a warped AdS$_6 \times S^4$ vacuum solution, which is expected to be dual to an $\mathcal{N} = 2$, $D = 5$ super-conformal Yang-Mills theory. We study solutions for strings rotating or spinning in this background. The warp factor plays no essential role when the string spins in the AdS$_6$, implying a commonality in the leading Regge trajectories between the $D = 4$ and $D = 5$ super-conformal field theories. The warp factor does, however, become important when the string rotates in the $S^4$, in particular for long strings, which have the the relation $E - \frac{3}{2}J = c_1 + c_2/J^5 + \cdots$, where the angular momentum $J$ is large. This relation is qualitatively different from that for long strings in the AdS$_5 \times S^5$ background. We also study Penrose limits of the AdS$_6 \times S^4$ solution, one of which gives rise to a free massive string theory with time-dependent masses.

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1 Introduction

Spinning extended object solutions, such as spherical membranes, were first constructed in supergravity some time ago [1–3]. The spin is necessary in order to prevent the membrane from collapsing. Inspired by the AdS/CFT correspondence, spinning string solutions were obtained in the AdS$_5 \times S^5$ background. These solutions correspond to string states on the leading Regge trajectory, with large angular momentum [4,5]. Subsequently, there has been a considerable interest in the subject [6-18]. (See a recent review [19].)

The majority of the focus has been on the AdS$_5 \times S^5$ and AdS$_5 \times T^4$ backgrounds of the type IIB theory. One nice feature of these solutions is that they are supported only by a Ramond-Ramond 5-form field strength, and so the bosonic sector of the sigma model action for the background is rather simple, with no Wess-Zumino term.

In this paper, we consider the warped AdS$_6 \times S^4$ background of the massive type IIA supergravity [20]. It is expected to be dual [21, 22] to an $\mathcal{N} = 2$, $D = 5$ superconformal Yang-Mills theory [23, 24]. This solution is also supported just by Ramond-Ramond field strengths, but now in addition with a non-constant dilaton. The AdS$_6 \times S^4$ metric is warped, rather than a direct product, with a warp factor that becomes singular on the equator of $S^4$, and so the geometry really corresponds to a hemisphere instead of the full $S^4$. The string coupling diverges at the equator; the metric is singular in the Einstein frame, but regular in the string frame.$^1$ The warp factor implies that the equator of the $S^4$ corresponds to an AdS$_5 \times S^3$ boundary [22].

We shall study spinning and rotating strings in this AdS$_6 \times S^4$ background. We consider two types of such solution. First, we study strings rotating in the $S^4$, for which the warp factor can be important. As one might expect, for a short string the warp factor plays little role, but its effect becomes pronounced for long strings with large R-charge $J$. The energy $E$ and angular momentum $J$ are now related by

$$E - \frac{3}{2} J \sim c_1 + \frac{c_2}{J^3} + \cdots$$

(1)

where $c_1$ and $c_2$ are certain constants. This behaviour is qualitatively different from the analogous relation for strings spinning in an AdS$_5 \times S^5$ background.

In the AdS$_5 \times S^5$ background, one can consider a string rotating in an $S^3$ inside the $S^5$ [9]. The $S^3$ lies within a great circle (or, more precisely, a “great 4-sphere”) in the $S^5$. Were it not for the warp factor, an analogous solution would also be possible in AdS$_6 \times S^4$, $^1$For brevity, we shall generally refer to the solution as AdS$_6 \times S^4$, with the understanding that the 4-sphere is cut at the equator.
with the $S^3$ inside the $S^4$. However, the only “great 3-sphere” in the northern hemisphere of $S^4$ is the equator itself, which is precisely where the warp factor diverges, and for this reason the analogous solution does not arise.

We also study situations where the string spins purely in the AdS$_6$ spacetime. In this case, the warp factor plays no essential role, since the gravitional repulsion from the boundary at the equator implies that the string can only be located at the north pole. We obtain results that are identical to those for a string spinning in AdS$_5$ in the type IIB theory, implying a commonality of the leading Regge trajectories for the $D = 5$ and $D = 4$ theories.

We then study two different Penrose limits of the AdS$_6 \times S^4$ solution. In the first of these, we arrive at a pp-wave which gives rise to a massive string theory with time-dependent masses. The singularity of the warp factor on the great 3-sphere in $S^4$ is reached in a finite string worldsheet time $x^+ = x_0^+$. The dilaton also depends on the worldsheet time, and in fact the string coupling becomes infinite at this limiting time $x_0^+$. This may signal a breakdown of the validity of the background. We also consider a second Penrose limit, which gives rise to an interacting string theory. Had there not been a warp factor in the AdS$_6 \times S^4$ solution, these two Penrose limits would have been equivalent.

2 General equations

The massive type IIA supergravity theory supports a warped AdS$_6 \times S^4$ background, which arises as the near-horizon geometry [22] of a semi-localised D4-D8 system [25]. The solution is given by

$$
\begin{align*}
\text{ds}^2 &= \frac{1}{2} W(\xi)^2 \left[ 9(- \cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_4^2) + 4(d\xi^2 + \sin^2 \xi \, d\Omega_3^2) \right], \\
F_{(4)} &= \frac{20 \sqrt{2}}{3} (\cos \xi)^{\frac{1}{3}} \sin^3 \xi \, d\xi \wedge \Omega_{(3)}, \quad e^\phi = (\cos \xi)^{-\frac{5}{6}},
\end{align*}
$$

(2)

where $d\Omega_4^2$ and $d\Omega_3^2$ are the metrics for the unit $S^4$ and $S^3$, which we choose to parameterise as follows:

$$
\begin{align*}
d\Omega_4^2 &= \, d\theta_1^2 + \cos \theta_1^2 \left( d\theta_2^2 + \cos^2 \theta_2 \, d\phi_1^2 + \sin^2 \theta_2 \, d\phi_2^2 \right), \\
d\Omega_3^2 &= \, d\theta^2 + \cos^2 \theta \, d\psi_1^2 + \sin^2 \theta \, d\psi_2^2,
\end{align*}
$$

(3)

and $\Omega_{(3)}$ is the volume form of $d\Omega_3^2$. The solution preserves half of the maximal supersymmetry, and remarkably, it corresponds to the vacuum of a consistent Kaluza-Klein reduction on the warped $S^4$ that gives rise [26] to the $D = 6, N = 2$ gauged supergravity constructed by Romans [27].
The warp factor \( W(\xi) \) in the metric (2) is given by \( W = (\cos \xi)^{-1/6} \) in the string frame, and \( W = (\cos \xi)^{-1/3} \) in the Einstein frame. The coordinate \( \xi \) runs from 0 to \( \frac{1}{2} \pi \), at which the metric in the Einstein frame becomes singular, while in the string frame the metric remains regular there. To be precise, the metric describes a warped product of the AdS\(_6\) with the upper hemisphere of the \( S^4 \). Since we will use this solution as background geometry in the string sigma-model action, we should work with the string-frame metric. The equator \( \xi = \frac{1}{2} \pi \) can be viewed as an AdS\(_6\) \( \times S^3 \) boundary.

It is straightforward to write down the bosonic sector of the string \( \sigma \)-model action, since the “cosmological term” and the 4-form \( F_{(4)} \) are R-R fields, which enter the action only through fermion bilinears, as does the dilaton. Choosing the conformal gauge, we can make the following consistent ansatz that describes strings spinning in the AdS\(_6\) \( \times S^4 \) background:

\[
t = \kappa \tau, \quad \psi_1 = \omega_1 \tau, \quad \psi_2 = \omega_2 \tau, \quad \phi_1 = \omega_3 \tau, \quad \phi_2 = \omega_4 \tau
\]
\[
\rho = \rho(\sigma), \quad \xi = \xi(\sigma) \quad \theta = \theta(\sigma), \quad \theta_1 = \theta_1(\sigma), \quad \theta_2 = \theta_2(\sigma),
\]
where a prime denotes a derivative with respect to the world-sheet coordinate \( \sigma \). The action becomes

\[
I = \frac{1}{4\pi \alpha'} \int d\sigma \mathcal{L}
\]

where

\[
\mathcal{L} = 9W^2 \left( -\kappa^2 \cosh^2 \rho - \rho'^2 + \sinh^2 \rho \left( -\theta_1'^2 - \cos^2 \theta_1 \theta_2'^2 + \omega_3^2 \cos^2 \theta_1 \cos^2 \theta_2 + \omega_4^2 \sin^2 \theta_2 \right) \right)
\]
\[
+ 4W^2 \left( -\xi'^2 + \sin^2 \xi \left( -\theta_2'^2 + \omega_1^2 \cos^2 \theta + \omega_2^2 \sin^2 \theta \right) \right),
\]

(6)

Together with the conformal gauge constraint

\[
9W^2 \left( -\kappa^2 \cosh^2 \rho + \rho'^2 + \sinh^2 \rho \left( \theta_1'^2 + \cos^2 \theta_1 \theta_2'^2 + \omega_3^2 \cos^2 \theta_1 \cos^2 \theta_2 + \omega_4^2 \sin^2 \theta_2 \right) \right)
\]
\[
+ 4W^2 \left( \xi'^2 + \sin^2 \xi \left( \theta_2'^2 + \omega_1^2 \cos^2 \theta + \omega_2^2 \sin^2 \theta \right) \right) = 0,
\]

(7)

where \( W \equiv (\cos \xi)^{-1/6} \). The equations of motion for this system are given by

\[
(W^2 \rho')' + \frac{1}{2} W^2 \sinh 2\rho \left( -\kappa^2 - \theta_1'^2 + \cos^2 \theta_1 (-\theta_2'^2 + \omega_3^2 \cos^2 \theta_2 + \omega_4^2 \sin^2 \theta_2) \right) = 0,
\]

\[
(W^2 \sinh^2 \rho \theta_1')' - \frac{1}{2} W^2 \sinh^2 \rho \sin(2\theta_1) (-\theta_2'^2 + \omega_3^2 \cos^2 \theta_2 + \omega_4^2 \sin^2 \theta_2) = 0,
\]

\[
(W^2 \sinh^2 \rho \cos^2 \theta_1 \theta_2')' + \frac{1}{2} W^2 \sinh^2 \rho \cos^2 \theta_1 \sin 2\theta_2 (\omega_1^2 - \omega_3^2) = 0,
\]

\[
(W^2 \xi')' + \frac{1}{2} \tan \xi \mathcal{L} + \frac{1}{2} W^2 \sin 2\xi (-\theta_2'^2 + \omega_1^2 \cos \theta^2 + \omega_2^2 \sin^2 \theta) = 0,
\]

\[
(W^2 \sin^2 \xi \theta')' + \frac{1}{2} W^2 \sin^2 \xi \sin 2\theta (\omega_2^2 - \omega_1^2) = 0.
\]

(8)
It is unlikely that one can obtain the most general solutions to these equations explicitly, especially since the warp factor \( W \) introduces fractional powers of \( \cos \xi \). We shall consider restricted cases in which analytical solutions can be obtained.

### 3 Rotation in the warped \( S^4 \)

We first consider rotations in the warped \( S^4 \). The angular momentum in this case can be related to the \( R \)-charges of operators in the corresponding \( D = 5 \) conformal field theory. There exist a maximum of two commuting \( U(1) \) isometries in \( S^4 \). Here, we shall consider just one angular momentum, by setting the parameters associated with these two charges equal. To turn off the contribution from the AdS\(_6\) directions, we can either set \( \rho = 0 \), or set \( \rho \) to be a constant and take \( \kappa = \omega_3 = \omega_4 \); both yield the same result. In the \( S^4 \) direction, we set \( \omega_2 = \omega_1 \equiv \omega \), which implies

\[
\theta' = \frac{c}{W^2 \sin^2 \xi}.
\]

Substituting this into the constraint equation, we have

\[
\xi'^2 = \frac{9}{4} \kappa^2 - \omega^2 \sin^2 \xi - \frac{c^2}{W^4 \sin^2 \xi}.
\]  

When \( c \neq 0 \), \( \xi \) has two turning points, \( 0 < \xi_1 < \xi_2 \), but when \( c = 0 \) there is only one turning point \( \xi_2 \), and the coordinate \( \xi \) oscillates between \( -\xi_2 \) and \( \xi_2 \). \(^2\) There does not seem to exist an analytical solution with \( c \neq 0 \), and so we shall specialise to the case \( c = 0 \). The parameters \( \kappa \) and \( \omega \) then satisfy the constraint

\[
4 \int_0^{\xi_0} \frac{d\xi}{\sqrt{\frac{9}{4} \kappa^2 - \omega^2 \sin^2 \xi}} = \int d\sigma = 2\pi.
\]  

Defining \( \eta = 2\omega/(3\kappa) \), then for \( \eta \geq 1 \) there exists a turning point \( \xi_0 \) given by \( \sin \xi_0 = 1/\eta \). Note that \( \xi \) is a compact coordinate, running from 0 to \( \frac{1}{2}\pi \). Clearly, if \( \eta >> 0 \) then \( \xi_0 \) is close to 0, and we have a short string. On the other hand, if \( \eta \sim 1^+ \) we have \( \xi_0 \) close to \( \pi/2 \), and the string is “long”. In terms of \( \eta \), the constraint (11) reads

\[
\eta^{-\frac{1}{2}} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \eta^{-1}\right) = \frac{3}{2\kappa}.
\]  

\(^2\)The hemisphere of \( S^4 \) is normally specified by \( 0 \leq \xi \leq \frac{1}{2}\pi \). When the oscillating solution passes into the region with \( \xi < 0 \), this should really be re-interpreted via a coordinate transformation under which \( \xi \rightarrow -\xi \) together with an antipodal mapping on the \( S^3 \) foliating surfaces. This is analogous to re-interpreting motion into the region \( \theta < 0 \) along a polar great circle on \( S^2 \) with standard spherical polar coordinates \((\theta, \phi)\) via the coordinate transformation \((\theta, \phi) \rightarrow (-\theta, \phi + \pi)\).
The energy and the angular momentum are given by

\[ E = \int \frac{d\sigma}{2\pi} P_t = \frac{12}{\pi} \int_0^{\xi_0} (\cos \xi)^{-1/3} (1 - \eta \sin^2 \xi)^{-1/2} \]

\[ J = \int \frac{d\sigma}{2\pi} P_\phi = \frac{2\sqrt{\eta}}{\pi} \int_0^{\xi_0} d\xi (\cos \xi)^{-1/3} \sin^2 \xi (1 - \eta \sin^2 \xi)^{-1/2} \]

\[ = \frac{2}{\eta} F_1[\frac{2}{3}, \frac{3}{2}; \eta] \]

(13)

We can easily determine the form of the relation between energy and spin in each of the short and the “long” string limits.

When \( \eta >> 1 \), implying a short string, both \( E \) and \( J \) approach zero. They obey the relation

\[ E = 3\sqrt{2}J (1 + \frac{1}{2\pi} J + \frac{1}{1152} J^2 - \frac{7}{82944} J^3 + \cdots) \]

(14)

If, on the other hand, \( \eta^{-1} = 1 - \epsilon \) with \( \epsilon \) approaching zero from above, the string is long. In this case, both \( E \) and \( J \) approach infinity with power-law dependences on \( \epsilon^{-1} \), and \( E \) and \( J \) are related by

\[ E - \frac{3}{2} J = \frac{6\Gamma(\frac{5}{6})}{\sqrt{\pi} \Gamma(\frac{1}{3})} - \frac{3072\Gamma(\frac{1}{6})^6}{5\pi^3 \Gamma(\frac{2}{3})^6} J^5 + \frac{2}{\pi^5} \frac{82944 \Gamma(\frac{1}{6})^6}{\Gamma(\frac{2}{3})^3} J^6 + \cdots \]

(15)

An alternative expression for the above relation is

\[ E - \frac{3}{2\sin \xi_0} J = E - \frac{3}{2} \sqrt{\eta} J = \frac{3}{\sqrt{\eta}} F_1[\frac{2}{3}, \frac{1}{2}, 2; \eta^{-1}] \]

\[ = \frac{6\Gamma(\frac{5}{6})}{\sqrt{\pi} \Gamma(\frac{1}{3})} (1 + \frac{3}{2} \epsilon + \frac{57}{56} \epsilon^2 + \cdots) \]

\[ + \frac{3\Gamma(\frac{5}{6})}{\sqrt{\pi} \Gamma(\frac{1}{3})} \epsilon^6 (1 + \frac{13}{22} \epsilon + \frac{677}{1196} \epsilon^2 + \cdots) \]

(16)

Comparison with AdS\(_5 \times S^5\):

Having obtained the results for strings rotating in the \( S^4 \), it is instructive to compare them with the previously-obtained results for strings rotating in the \( S^5 \) in an AdS\(_5 \times S^5\) background, where there is no warp factor \([4,5]\). The metric of the \( S^5 \) can be parameterised as

\[ d\Omega_5^2 = d\xi^2 + \sin^2 \xi \left( d\theta_1^2 + \cos^2 \theta_1 (d\theta_2^2 + \cos^2 \theta_2 d\phi_1^2 + \sin^2 \theta_2 d\phi_2^2) \right) \]

(17)
To make a direct comparison with our warped $S^4$ result, we consider the string spinning in an $S^4$ section ($\theta_1 = 0$) of the $S^5$. The solution is given by $t = \kappa \tau$, $\phi_1 = \omega \tau = \phi_2$, $\theta = \text{const.}$, where $\xi = \xi(\sigma)$ satisfies the constraint equation

$$\xi'^2 = \kappa^2 - \omega^2 \sin^2 \xi. \quad (18)$$

This implies that $\kappa$ and $\omega$ are related by

$$\kappa = \frac{1}{\sqrt{\eta}} \, 2F_1\left[\frac{1}{2}, \frac{1}{2}, 1; \eta^{-1}\right] = \frac{2}{\pi \sqrt{\eta}} \, K(\eta^{-1}), \quad (19)$$

where $K$ is the complete elliptic integral of the first kind. The energy and the angular momentum can be easily obtained, given by

$$E = \kappa = \frac{1}{\sqrt{\eta}} \, 2F_1\left[\frac{1}{2}, \frac{1}{2}, 1; \eta^{-1}\right] = \frac{2}{\pi \sqrt{\eta}} \, K(\eta^{-1}), \quad J = \frac{1}{2\eta} \, 2F_1\left[\frac{1}{2}, \frac{3}{2}, 2; \eta^{-1}\right], \quad (20)$$

For short strings, where $\eta \gg 1$, the energy and spin both approach to zero, and they obey the relation

$$E = \sqrt{2J} \left(1 + \frac{1}{8} J + \frac{3}{128} J^2 + \frac{1}{1024} J^3 + \cdots\right). \quad (21)$$

For a “long” string where $\eta^{-1} \sim 1 - \epsilon$, $E$ and $J$ both diverge logarithmically as $\log \epsilon$, while the difference $E - J$ approaches a constant:

$$E - J = \frac{2}{\pi} - \frac{8}{\pi} e^{-\pi J - 2} + \cdots \quad (22)$$

Thus we see that for short strings, located near the north pole where the effect of any warp factor is negligible, the energy-angular momentum relations are qualitatively the same for the warped AdS$_6 \times S^4$ and the AdS$_5 \times S^5$ backgrounds. In fact, (21) is the same as (14), in the first three leading terms, after sending $J \to J/3$ and $E \to E/(3\sqrt{3})$. For long strings, on the other hand, although at the leading order $E - \tilde{\Delta} J$ approaches a constant for both the AdS$_6 \times S^4$ and the AdS$_5 \times S^5$ backgrounds, the next-to-leading order is quite different in the two cases. Note that $\tilde{\Delta}$ is the conformal dimension for the R-charge operators.

In the AdS$_5 \times S^5$ background, solutions describing a string rotating in an $S^3$ submanifold of $S^5$ were also obtained [9]. In our AdS$_6 \times S^4$ case, since the $S^4$ can be viewed as a foliation by $S^3$ surfaces, one might expect such a solution also to exist. However, owing to a repulsion from the equatorial boundary implied by the warp factor, any solution with constant latitude coordinate $\xi$ necessarily lies at the north pole of the $S^4$, which renders a rotation only in the $S^3$ impossible. Thus we see that the warp factor implies that the energy/R-charge relation obtained in the $D = 4$ Yang-mills theory corresponding to the AdS$_5 \times S^5$ background is significantly different from the one for the $D = 5$ Yang-Mills theory from AdS$_6 \times S^4$. 6
4 Penrose limit of $\text{AdS}_6 \times S^4$

In the case of the $\text{AdS}_5 \times S^5$ background, long strings rotating in $S^5$ correspond to states with large RR-charge, which are closely related to pp-waves via the Penrose limit. The situation is more complicated if we look for a direct analogue in the warped $\text{AdS}_6 \times S^4$ background, owing to the presence of the warp factor. We shall first consider the Penrose limit using the method outlined in [28].

We begin by expressing the metric (2) (after a constant scaling) as

$$ds^2 = \frac{1}{2} \lambda^2 (\cos \xi)^{-\frac{3}{2}} \left[ 9(-d\tau^2 + \sin^2 \tau \, d\Omega_{H^5}^2) + 4 (d\xi^2 + \sin^2 \xi \, d\Omega_3^2) \right], \quad (23)$$

where $d\Omega_{H^5}^2$ is a unit hyperbolic 5-plane. Making the coordinate transformation $u = \xi + \frac{3}{2} \tau$ and $v = \lambda^2 (\xi - \frac{3}{2} \tau)$, and then sending $\lambda \to \infty$, we obtain a pp-wave in the form

$$ds^2 = 2(du \, dv + \frac{9}{4} \sin^2 (\frac{1}{3} u) \, dy^i dy^i + \sin^2 (\frac{1}{3} u) \, dz^m dz^m), \quad (24)$$

where the metrics $\lambda^2 d\Omega_{H^5}^2$ and $\lambda^2 d\Omega_3^2$, which become flat in the limit $\lambda \to \infty$, are written as $dy^i dy^i$ and $dz^m dz^m$ respectively. After the further coordinate transformations (see [28])

$$\bar{x}^+ = \frac{1}{3} u, \quad \bar{x}^- = v - \frac{1}{3} \left( \frac{3}{2} \bar{y}_i \sin \frac{2}{3} u + \bar{z}_m \sin u \right),$$

$$\bar{y}^i = \frac{3}{\sqrt{2}} \bar{y}^i \sin \frac{1}{3} u, \quad \bar{z}^m = \sqrt{2} \bar{z}^m \sin \frac{1}{2} u, \quad (25)$$

the pp-wave metric takes the form

$$ds^2 = (\cos \bar{x}^+)^{-\frac{1}{2}} \left[ 4d\bar{x}^+ d\bar{x}^- - \left( \frac{4}{9} \bar{y}^2 + \bar{z}_m^2 \right) (d\bar{x}^+)^2 + d\bar{y}^2 + d\bar{z}_m^2 \right]. \quad (26)$$

where there are five coordinates $\bar{y}^i$ and three coordinates $\bar{z}^m$. We can now perform the coordinate transformation

$$W^2 d\bar{x}^+ = dx^+, \quad d\bar{x}^- = x^- + \frac{W'}{4W} (y^i y^i + z^m z^m),$$

$$\bar{y}_i = \frac{y_i}{W}, \quad \bar{z}^m = \frac{z^m}{W}, \quad (27)$$

where $W = (\cos \bar{x}^+)^{-\frac{1}{6}}$, and a prime denotes a derivative with respect to $x^+$. The metric becomes

$$ds^2 = 4dx^+ dx^- - \left[ \left( \frac{4}{9W^4} - \frac{W''}{W} \right) y^i y^i + \left( \frac{1}{W^4} - \frac{W''}{W} \right) z^m z^m \right] + dy^i dy^i + dz^m dz^m. \quad (28)$$

Thus we see that string theory on this pp-wave background becomes a massive free string, but with time-dependent masses $m_1$ and $m_2$ in the $y^i$ and $z^m$ directions respectively, given by

$$m_1^2 = \frac{8 \cos^2 x^+ - 7}{36 \cos^2 x^+} + \frac{2}{3} \left( \cos x^+ \right)^{\frac{2}{3}}, \quad m_2^2 = \frac{8 \cos^2 x^+ - 7}{36 \cos^2 x^+} + \frac{2}{3} \left( \cos x^+ \right)^{\frac{2}{3}}. \quad (29)$$
The coordinates $\bar{x}^+$ and $x^+$ are related by
\[ x^+ = -\frac{3}{2} (\cos \bar{x}^+) \frac{2}{3} \mathrm{ Hypergeometric}_2 \left[ \frac{1}{2}, \frac{4}{3}; \cos^2 \bar{x}^+ \right]. \tag{30} \]

The coordinate $\bar{x}^+$ runs from 0 to $\frac{1}{2} \pi$, at which point the string coupling constant $g_s = e^\phi = (\cos \bar{x}^+)^{-5/6}$ becomes infinite. The coordinate $x^+$ runs from $-\frac{3}{2} \sqrt{\pi} \frac{1}{3} \Gamma(\frac{4}{3})/\Gamma(\frac{5}{6})$ to 0.

Some aspects of massive string theory with time dependent masses were discussed in [29]. It is also of interest to examine the Penrose limit described in [30], since this makes a direct analogy with the rotating strings we obtained earlier. In this procedure, one magnifies the null geodesics along a great circle of the internal space. This is rather problematic in our case, since the great 3-sphere is the equator of $S^4$, on which the solution becomes singular.

In order to obtain a regular solution, it is necessary to scale the fields appropriately using global symmetries of the theory. We begin by making the coordinate transformations
\[ t \to x^+ - \frac{x^-}{9 \lambda^2}, \quad \psi_1 \to \frac{2}{3} x^+ - \frac{x^-}{6 \lambda^2}, \quad \rho \to \frac{\rho}{3 \lambda}, \quad \theta \to \frac{\theta}{2 \lambda}, \quad \xi \to \frac{1}{2} \pi - \frac{\xi}{2 \lambda}. \tag{31} \]

The solution (2), after a scaling of fields in which the metric is multiplied by 2 for convenience, becomes
\[ ds^2 = 2^{-\frac{2}{3} \lambda^{-\frac{5}{3}}} ds^2, \quad \bar{F}_{(4)} = 2^{-\frac{1}{3} \lambda^{-\frac{10}{3}}} F_{(4)}, \quad e^{\bar{\phi}} = (2 \lambda)^\frac{5}{6} e^\phi, \tag{32} \]

where
\[ ds^2 = \xi^{-\frac{1}{3}} \left[ -4 dx^+ dx^- - \left[ \rho^2 + \frac{9}{4} (\xi^2 + \theta^2) \right] (dx^+)^2 + 2 \rho^2 d\Omega_4^2 + d\xi^2 + d\theta^2 + \theta^2 d\psi_2^2 \right], \]
\[ F_{(4)} = 5 \xi^{\frac{1}{3}} d\xi \wedge d\theta \wedge d\psi_2 \wedge dx^+, \quad e^\phi = \xi^{-\frac{5}{6}}. \tag{33} \]

Normally, when one takes a Penrose limit of AdS$_5 \times S^5$, AdS$_4 \times S^7$ or AdS$_7 \times S^4$, one just uses the homogeneous global scaling symmetry of the supergravity equations of motion in order to absorb the singular $\lambda$ scaling factors in the limiting forms analogous to (32). In the present case, in which the dilaton is excited and also has a $\lambda$ scaling, it is necessary also to make use of the dilaton shift symmetry of the type IIA theory, in conjunction with the homogeneous scaling symmetry. These two symmetries take the form
\[ \bar{g}_{\mu\nu} = \kappa^2 \Lambda^2 g_{\mu\nu}, \quad e^{\bar{\phi}} = \Lambda^3 e^\phi, \quad \bar{F}_{(3)} = \kappa^2 \Lambda^2 F_{(3)}, \]
\[ \bar{F}_{(4)} = \kappa^3 F_{(4)}, \quad \bar{F}_{(2)} = \kappa \Lambda^{-2} F_{(2)}, \quad \bar{m} = \kappa^{-1} \Lambda^{-4} m, \tag{34} \]

where $\kappa$ and $\Lambda$ are the associated global parameters. To absorb the $\lambda$ dependences in (32) we therefore choose these parameters to be
\[ \kappa = 2^{-\frac{1}{3}} \lambda^{-\frac{10}{3}}, \quad \Lambda = (2 \lambda)^\frac{5}{18}. \tag{35} \]
The rescaled pp-wave solution is then given simply by (33). The canonical momenta $p^\pm$ in this case can be related to the rotating string solutions via

$$p^- = \frac{i}{2} \left( \partial_t + \frac{3}{2} \partial_{\psi_1} \right) = \frac{i}{2} (E - \frac{3}{2} J), \quad p^+ = \frac{i}{18 \lambda^2} \left( \partial_t - \frac{3}{2} \partial_{\psi_1} \right) = \frac{1}{18 \lambda^2} (E + \frac{3}{2} J)$$

(36)

5 String spinning in AdS$_6$

Here we consider a string spinning purely in the AdS$_6$ spacetime, in which case $\xi$ is a constant. In fact the equations of motion require that $\xi = 0$, since there is effectively a gravitational repulsion because of the warp factor, which forces the string to be at the north pole. Since there are only two commuting $U(1)$ isometries in the AdS$_6$, there can at most be two commuting angular momenta. For simplicity, we shall consider only solutions with one angular momentum parameter, by setting the two angular momenta equal. Furthermore, we shall focus on the solution where $\theta_1 = 0$, and where we set $\omega_3 = \omega_4 \equiv \omega$. This leads to

$$\theta_2' = \frac{c}{\sinh^2 \rho},$$

(37)

where $c$ is a constant. Substituting this into the conformal constraint, we obtain

$$\rho^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - \frac{c^2}{\sinh^2 \rho}.$$  

(38)

Periodicity in $\sigma$ implies that we must have

$$n \int_{\rho_1}^{\rho_2} \frac{d\rho}{\sqrt{\kappa^4 \cosh^2 \rho - \omega^2 \sinh^2 \rho - c^2/\sinh^2 \rho}} = \int d\sigma = 2\pi, $$

(39)

where $\rho_1$ and $\rho_2$ are the turning points of the oscillatory solution. If $c \neq 0$, these two turning points both occur at positive values of $\rho$, and we take $n = 2$, since a complete period of the oscillation runs from $\rho_1$ to $\rho_2$ and then back to $\rho_1$. If instead $c = 0$, the motion runs from the turning point at $\rho = \rho_2$, passes through zero, and turns again at $\rho = -\rho_2$. Thus in this case we can take $\rho_1 = 0$ (where it is now the mid-point, rather than a turning point, of the oscillatory motion) and set $n = 4$, since a complete period consists of four segments between $\rho = 0$ and $\rho = \rho_2$. (See footnote 2.) By making appropriate coordinate transformations, the above integral can be expressed as a hypergeometric function. To see this, let us define

$$t = -\cosh^2 \rho/(y_+ - y_-),$$

where

$$y_{\pm} = \frac{2\omega^2 - \kappa^2 \pm \sqrt{\kappa^4 + 4c^2 (\kappa^2 - \omega^2)}}{2(\omega^2 - \kappa^2)}.$$  

(40)

For $\omega^2 > \kappa^2$, we have $y_+ > y_- > 1$, ensuring that the corresponding $\rho$ is real. When $c = 0$ we have $y_- = 1$, corresponding $\rho = 0$, which is no longer a turning point, and then as
discussed above we then take $n = 4$ instead of $n = 2$. In terms of the new coordinate $t$ we have

$$
2\pi = \frac{n}{2\sqrt{y_+ (\omega^2 - \kappa^2)}} \int_0^1 \frac{dt}{t^{\frac{1}{2}} (1 - t)^{\frac{1}{2}} (1 - \lambda t)^{\frac{1}{2}}}
$$

$$
= \frac{n \pi}{2\sqrt{y_+ (\omega^2 - \kappa^2)}} {}_2F_1[\frac{1}{2}, \frac{1}{2}, 1; \lambda],
$$

(41)

where

$$
\lambda = \frac{y_+ - y_-}{y_+}.
$$

(42)

It is straightforward to obtain the energy and angular momentum for this system, given by

$$
E = \frac{9n \kappa}{2\pi} \int \frac{\cosh^2 \rho d\rho}{\sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - c^2 / \sinh^2 \rho}},
$$

$$
S = \frac{9n \omega}{2\pi} \int \frac{\sinh^2 \rho d\rho}{\sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - c^2 / \sinh^2 \rho}}
$$

(43)

Thus $E$ and $J$ can also be expressed as hypergeometric functions, given by

$$
E = \frac{9n \kappa \pi \sqrt{y_+}}{2\sqrt{\omega^2 - \kappa^2}} {}_2F_1[\frac{1}{2}, \frac{1}{2}, 1; \lambda],
$$

$$
\frac{E}{\kappa} - \frac{S}{\omega} = {}_2F_1[\frac{1}{2}, \frac{1}{2}, 1; \lambda].
$$

(44)

Not surprisingly, the result is the same as that for a string rotating in the $\text{AdS}_5 \times S^5$ background. This is because we have considered a solution in which $\theta_1 = 0$, which is an $\text{AdS}_5$ slice of $\text{AdS}_6$. From the AdS/CFT point of view, the $\mathcal{N} = 1$ $D = 4$ Yang-Mills can be obtained from a Kaluza-Klein truncation of the $\mathcal{N} = 2$ conformal field theory in $D = 5$.

In order to probe the non-trivial properties of the $D = 5$ conformal field theory, it would be necessary to look for solutions with $\theta_1 \neq 0$. However, what we have illustrated implies that there are common features in the leading Regge trajectories for the two theories, whilst in their large R-charge expansions the two theories diverge more significantly.

6 Conclusions

In this paper, we have contructed spinning and rotating string solutions in the massive type IIA warped $\text{AdS}_6 \times S^4$ background. String theory on such a background is expected to be dual to an $\mathcal{N} = 2$, $D = 5$ super-conformal Yang-Mills theory.

In the case of a string spinning purely in the $\text{AdS}_6$, the gravitional repulsion from the equatorial boundary implies that the string can only lie at the north pole of the $S^4$. Besides
this, the warp factor plays no essential role. The resulting $E$ and $S$ relation is the same as that obtained in the AdS$_5 \times S^5$ background, implying the same behaviour of the leading Regge trajectories of the two theories.

In the case of a string rotating in the $S^4$, the warp factor plays a more significant role. First, it rules out a string rotating only in the $S^3$ that foliates the $S^4$, which implies that the analogous energy vs. R-charge relation in the $D = 4$ Yang-Mills theory associated with AdS$_5 \times S^5$ is absent in the $D = 5$ Yang-Mills theory. We obtained analytic solutions for the rotating string extending in the latitude coordinate of the $S^4$. Owing to the presence of the warp factor, the energy and angular momentum relation is qualitatively different for the long string at the sub-leading order, in comparison to the relation in AdS$_5 \times S^5$.

We also studied Penrose limits of the AdS$_6 \times S^4$ background. In one limit, we obtained a massive string theory with masses that depend upon the worldsheet time coordinate $x^+$ in the light-cone gauge. The range of the time coordinate is restricted, owing to the presence of the warp factor, which diverges at a finite value of $x^+$. At the same time, the string coupling constant becomes infinite, which may signal a breakdown of the validity of the solution. A second, inequivalent Penrose limit was also constructed, which yields a time-independent interacting string theory. The two Penrose limits arise from schemes that would have given identical results in a situation such as AdS$_5 \times S^5$ where there is no warp factor.

In comparison to the AdS$_5 \times S^5$ solution of the type IIB theory, the AdS$_6 \times S^4$ solution of the massive type IIA theory exhibits a number of undesirable features, all of which stem from the warp factor which becomes singular on the equator of the $S^4$. In particular, the string probe senses this singularity, in any motion that approaches the equator. Interestingly, however, a 4-brane probe is insensitive to the warp factor. In other words, if one makes a Weyl rescaling of the metric to the “4-brane frame,” then it becomes purely a direct product of AdS$_6$ and $S^4$. By definition, the rescaling of the metric to the 4-brane frame is such that the dual field strength $F_{(6)} \equiv e^{\frac{1}{2} \phi} * F_{(4)}$, to which a fundamental 4-brane couples, arises in the massive type IIA supergravity Lagrangian with the same dilaton coupling as for the $\sqrt{-g} R$ term, namely

$$L = e^{-\frac{2}{5} \phi} \sqrt{-g} (R - \frac{1}{60} F_{(6)}^2 + \cdots).$$  \hspace{1cm} (45)

Thus the 4-brane metric is related to the Einstein metric by $ds^2_{4\text{-brane}} = e^{\frac{1}{10} \phi} ds^2_{\text{Ein}}$. We then see from (2) that in the 4-brane frame, the metric becomes a direct product of AdS$_6 \times S^4$. It would seem that 4-branes might therefore be more natural candidates for probing the geometry of the background. Unfortunately, however, the necessity of wrapping a spinning or rotating 4-brane in all the available $U(1)$ circle isometries of AdS$_6$ and $S^4$ appears to
exclude the possibility of obtaining simple solutions.

The warped $\text{AdS}_6 \times S^4$ background of the massive type IIA supergravity, as a near-horizon geometry of the D4-D8 system, may provide the simplest arena for studying superconformal field theories beyond $D = 4$. The preliminary analysis of our paper suggests that there are many common features with the $D = 4$ Yang-Mills theory. The new features, other than dimensionality, are principally due to the warped nature of the $\text{AdS}_6 \times S^4$ product. In particular, this leads to energy vs. R-charge relationships that differ from those in the $D = 4$ Yang-Mills theory. It would be of interest to study this further from the standpoint of the five-dimensional superconformal field theory.

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