GLOBAL STRONG SOLUTIONS TO THE 3D FULL COMPRESSIBLE NAVIER-STOKES SYSTEM WITH VACUUM IN A BOUNDED DOMAIN

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ABSTRACT. In this short paper we establish the global well-posedness of strong solutions to the 3D full compressible Navier-Stokes system with vacuum in a bounded domain \( \Omega \subset \mathbb{R}^3 \) by the bootstrap argument provided that the viscosity coefficients \( \lambda \) and \( \mu \) satisfy that \( 7 \lambda > 9 \mu \) and the initial data \( \rho_0 \) and \( u_0 \) satisfy that \( \| \rho_0 \|_{L^\infty(\Omega)} \) and \( \| \rho_0 | u_0 |^5 \|_{L^1(\Omega)} \) are sufficient small.

1. Introduction

In this short paper, we consider the following initial and boundary problem to the 3D full compressible Navier-Stokes system in a bounded domain \( \Omega \subset \mathbb{R}^3 \):

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho u) = 0 \quad \text{on} \quad \Omega \times (0, \infty),
\]

\[
\frac{\partial (\rho u)}{\partial t} + \text{div} (\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla p = 0 \quad \text{on} \quad \Omega \times (0, \infty),
\]

\[
C_V \{ \frac{\partial (\rho \theta)}{\partial t} + \text{div} (\rho u \theta) \} - \lambda \nabla \theta + p \text{div} u
= \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\text{div} u)^2 \quad \text{on} \quad \Omega \times (0, \infty),
\]

\[
u = 0, \quad \frac{\partial \theta}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty),
\]

\[
(\rho, \rho u, \rho \theta) (\cdot, 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0) \quad \text{in} \quad \Omega.
\]

Here the unknowns \( \rho, u, \theta \) denote the density, velocity and temperature of the fluid, respectively. The pressure \( p := R \rho \theta \) and the internal energy \( e := C_V \theta \) with positive constants \( R \) and \( C_V \). \( \lambda \) and \( \mu \) are two viscosity constants satisfying \( \mu > 0 \) and \( \lambda + \frac{2}{3} \mu \geq 0 \). \( n \) is the unit outward normal vector to the smooth boundary \( \partial \Omega \) of \( \Omega \).

If the initial density \( \rho_0 \) has a positive lower bound, the global existence of small smooth solutions to the problem (1.1)-(1.5) was obtained in [6, 10] three decades ago.

If the initial data may contain vacuum, Cho and Kim [1] proved the local well-posedness of strong solutions to the problem (1.1)-(1.5) under some compatibility
conditions:

\[-\mu \Delta u_0 - (\lambda + \mu) \nabla \text{div} u_0 + R \nabla (\rho_0 \theta_0) = \sqrt{\rho_0} g_1, \quad (1.6)\]
\[\Delta \theta_0 + \frac{\mu}{2} |\nabla u_0 + \nabla u_0^t|^2 + \lambda (\text{div} u_0)^2 = \sqrt{\rho_0} g_2, \quad (1.7)\]

with \(g_1, g_2 \in L^2(\Omega)\).

Recently, Huang and Li [5] prove that the global well-posedness of strong solutions to the full compressible Navier-Stokes equations in the whole space \(\mathbb{R}^3\) with smooth initial data which are of small energy but possibly large oscillations where the initial density is allowed to vanish, see also [9]. However, the methods developed in [5, 9] can not applied directly to bounded domain case.

The aim of this paper is to prove that, although the initial density may contain vacuum, the problem (1.1)-(1.5) still has a unique global strong solution for small initial data. Our results reads as

**Theorem 1.1.** Let \(0 \leq \rho_0 \in W^{1,6}(\Omega), u_0 \in H^1_0(\Omega) \cap H^2(\Omega), 0 \leq \theta_0 \in H^2(\Omega)\) with \(\frac{\partial u_0}{\partial n} = 0\) on \(\partial \Omega\) and (1.6), (1.7) hold true. If

\[7\lambda > 9\mu \quad \text{and} \quad \|\rho_0\|_{L^\infty} + \|\rho_0 u_0\|_{L^1}^5\]

is sufficient small, then the problem (1.1)-(1.5) has a unique global-in-time strong solution.

**Remark 1.1.** It is interesting to note that the initial temperature \(\theta_0\) need not be small in our results.

**Remark 1.2.** It is possible to establish a similar result for the full compressible magnetohydrodynamical system.

**Remark 1.3.** When \(\Omega := \mathbb{R}^3\) and consider the isentropic Navier-Stokes system, a similar result can be proved when \(\|\rho_0\|_{L^p}\) is small for some large \(p\) by the method developed here and a blow-up criterion

\[\rho \in L^p(\mathbb{R}^3 \times (0, T))\]

proved in [8].

**Remark 1.4.** A similar result holds true when the boundary condition \(\frac{\partial \theta}{\partial n} = 0\) on \(\partial \Omega\) is replaced by \(\theta = 0\) on \(\partial \Omega\).

To prove Theorem 1.1, we will use the following abstract bootstrap argument or continuity argument ([7], Page 20).

**Lemma 1.2 ([7]).** Let \(T > 0\). Assume that two statements \(C(t)\) and \(H(t)\) with \(t \in [0, T]\) satisfy the following conditions:

(a) If \(H(t)\) holds for some \(t \in [0, T]\), then \(C(t)\) holds for the same \(t\);
(b) If \(C(t)\) holds for some \(t_0 \in [0, T]\), then \(H(t)\) holds for \(t\) in a neighborhood of \(t_0\).
(c) If \( C(t) \) holds for \( t_m \in [0, T] \) and \( t_m \to t \), then \( C(t) \) holds;

(d) \( C(t) \) holds for at least one \( t_1 \in [0, T] \).

Then \( C(t) \) holds for all \( t \in [0, T] \).

We will also use the following regularity criterion ([4]):

**Lemma 1.3.** Let \( 0 \leq \rho_0 \in W^{1,6}(\Omega), u_0 \in H^1_0(\Omega) \cap H^2(\Omega), 0 \leq \theta_0 \in H^2(\Omega) \) with \( \frac{\partial \theta_0}{\partial n} = 0 \) on \( \partial \Omega \) and (1.6), (1.7) hold true. If \( \rho \) and \( u \) satisfy

\[
\rho \in L^\infty(\Omega \times (0,T)) \quad \text{and} \quad u \in L^5(\Omega \times (0,T)),
\]

then

\[
\|\rho\|_{L^\infty(0,T;W^{1,6}(\Omega))} + \|u\|_{L^5(0,T;H^2(\Omega))} + \|\theta\|_{L^\infty(0,T;H^2(\Omega))} + \|u\|_{L^2(0,T;W^{2,6}(\Omega))} \leq C_1. \tag{1.11}
\]

The remainder of this paper is to the proof of Theorem 1.1. Our proof is very short due to that it heavily depends on using Lemma 1.3.

### 2. Proof of Theorem 1.1

We will use the bootstrap argument and regularity criterion (1.10) to prove Theorem 1.1.

Let \( \delta > 0 \) be a fixed number, say

\[
2\|\rho_0\|_{L^\infty} + 2\|\rho_0|u_0|^5\|_{L^1} \leq \delta. \tag{2.1}
\]

Denote by \( H(t) \) the statement that, for \( t \in [0, T] \),

\[
\|\rho\|_{L^\infty(\Omega \times [0,t])} + \|u\|_{L^5(\Omega \times [0,t])} \leq \delta \tag{2.2}
\]

and \( C(t) \) the statement that

\[
\|\rho\|_{L^\infty(\Omega \times [0,t])} + \|u\|_{L^5(\Omega \times [0,t])} \leq \frac{\delta}{2}. \tag{2.3}
\]

The conditions (b)-(d) in Lemma 1.2 are clearly true and it remains to verify (a) under the condition (1.8). Once this is verified, then the bootstrap argument would imply that \( C(t) \), or (2.3) actually holds for any \( t \in [0, T] \) and thus (1.11) holds true.

Now we assume that (2.2) holds true for some \( t \in [0, T] \). By Lemma 1.3, we have

\[
\|\rho\|_{L^\infty(0,t;W^{1,6})} + \|u\|_{L^\infty(0,t;H^2)} + \|\theta\|_{L^\infty(0,t;H^2)} + \|u\|_{L^2(0,t;W^{2,6})} \leq C_1. \tag{2.4}
\]

Testing (1.1) by \( \rho^{q-1} \quad  (q > 2) \) and using (2.4), we see that

\[
\frac{d}{dt}\|\rho\|_{L^q} \leq \left(1 + \frac{1}{q}\right)\|\text{div } u\|_{L^\infty}\|\rho\|_{L^q} \leq C_0\|u\|_{W^{2,6}}\|\rho\|_{L^q},
\]

which yields

\[
\|\rho\|_{L^q} \leq \|\rho_0\|_{L^q} \exp\left(C_0 \int_0^T \|u\|_{W^{2,6}} dt\right) \leq \|\rho_0\|_{L^q} \exp(C_0 \sqrt{T} C_1).
\]
Taking $q \to +\infty$ and letting
\[
\|\rho_0\|_{L^\infty} \text{ be small,}
\]
we arrive at
\[
\|\rho\|_{L^\infty(\Omega \times [0, t])} \leq \|\rho_0\|_{L^\infty} \exp(C_0 \sqrt{T} C_1) \leq \frac{\delta}{4}. \tag{2.5}
\]

When $7\mu > 9\lambda$, we can adopt a technique of Hoff [3] to bound the velocity in $L^5(\Omega \times [0, t])$ as follows. Setting $q = 5$ and testing (1.2) by $q|u|^{q-2}u$ and using (1.11), we derive
\[
\frac{d}{dt} \int \rho|u|^q \, dx + \int \{ q|u|^{q-2} [\mu|\nabla u|^2 + (\lambda + \mu)(\text{div } u)^2 + \mu(q - 2)|\nabla |u||^2 ] \\
+ q(\lambda + \mu)(|\nabla |u|^{q-2}) \cdot \text{div } u \} \, dx \\
= q \int \rho \text{div}(|u|^{q-2}u) \, dx = q \int \rho \theta |\nabla |u|^{q-2} \, dx \leq C_2 \int \rho |u|^{q-2}|\nabla u| \, dx \\
\leq \epsilon \int |u|^{q-2}|\nabla u|^2 \, dx + \frac{C_0^2}{\epsilon} \|\rho\|_{L^\infty}^2. \tag{2.6}
\]

Now, noticing that $|\nabla |u|| \leq |\nabla u|$ and the condition $7\mu > 9\lambda$, we have, after a straight calculation that (2.5):
\[
\text{the second term on the left hand side of (2.6)} \geq q \int |u|^{q-2} \left[ \mu(q - 1) - \frac{\lambda + \mu}{4}(q - 2)^2 \right] |\nabla u|^2 \, dx \\
\geq C_0 \int |u|^{q-2}|\nabla u|^2 \, dx.
\]

Inserting the above inequality into (2.6) and taking $\epsilon = \frac{C_0^2}{2}$, we have
\[
\int \rho|u|^q \, dx + \frac{C_0}{2} \int_0^t \int |u|^{q-2}|\nabla u|^2 \, dx \, ds \leq \int \rho_0|u_0|^q \, dx + \frac{C_0^2}{\epsilon} \|\rho\|_{L^\infty}^2 T
\]
which gives
\[
\int_0^t \int |u|^5 \, dx \, ds \leq C_0 \int_0^t \int |u|^{q-2}|\nabla u|^2 \, dx \, ds \\
\leq C_0 \int \rho_0|u_0|^5 \, dx + \frac{C_0^2}{\epsilon} T \|\rho\|_{L^\infty}^2 \\
\leq \frac{\delta}{4}. \tag{2.7}
\]

Summing up (2.5) and (2.7) gives (2.3). Thus we have (1.10). Hence we can apply Lemma 1.3 to complete the proof of Theorem 1.1. \hfill \Box

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