Existence and boundary asymptotic behavior of large solutions of Hessian equations

Shanshan Ma\textsuperscript{a,*}, Dongsheng Li\textsuperscript{a}

\textsuperscript{a}School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China

Abstract

In this paper, we establish the existence of large solutions of Hessian equations and obtain a new boundary asymptotic behavior of solutions.

Keywords: Existence, Boundary asymptotic behavior, Hessian equations, large solutions

1. Introduction

Let $f \in C^1(0, \infty)$ be positive and nondecreasing, and $b \in C_1^{1,1}(\overline{\Omega})$ be positive in $\Omega$, where $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ is a bounded domain with boundary of class $C^2$. In this paper we investigate the following $k$-Hessian equation $(1 \leq k \leq n)$:

$$
\begin{align*}
\begin{cases}
S_k(D^2 u) = \sigma_k(\lambda) = b(x)f(u) & \text{in } \Omega, \\
 u = \infty & \text{on } \partial\Omega,
\end{cases}
\end{align*}
$$

(1.1)

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix $D^2 u$ and

$$
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}
$$

is the $k^{th}$ elementary symmetric function of $\lambda$. For completeness, we also set $\sigma_0(\lambda) = 1$ and $\sigma_k(\lambda) = 0$ for $k > n$. The boundary condition means $u(x) \to +\infty$ as $d(x) = \text{dist}(x, \partial\Omega) \to 0^+$. 

\textsuperscript{*}This work was supported by NSFC grant 11671316.
\textsuperscript{*}Corresponding author.

Email addresses: mss5221@stu.xjtu.edu.cn (Shanshan Ma ), lidsh@mail.xjtu.edu.cn (Dongsheng Li )
To work in the realm of elliptic operators, we have to restrict the class of functions and domains. Following [4], a function \( u \in C^2(\Omega) \) is called a \( k \)-admissible function if for any \( x \in \Omega \), \( \lambda(D^2u(x)) \) belongs to the cone given by

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \cdots, k \}.
\]

From [4] and [10], \( \Gamma_k \) is an open, convex, symmetric (under the interchange of any two \( \lambda_j \)) cone with vertex at the origin, and

\[
\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \{ \lambda \in \mathbb{R}^n : \lambda_1, \cdots, \lambda_n > 0 \}.
\]

In [4], it is also shown that

\[
\frac{\partial \sigma_k(\lambda)}{\partial \lambda_i} > 0 \text{ in } \Gamma_k \ \forall \ i
\]

and

\[
\sigma_k^{1/k}(\lambda) \text{ is a concave function in } \Gamma_k.
\]

Let

\[
S(\Gamma_k) = \{ A : A \in \mathbb{S}^{n \times n}, \lambda(A) \in \Gamma_k \},
\]

where \( \mathbb{S}^{n \times n} \) denotes the set of \( n \times n \) real symmetric matrices. Recall that \( S(\Gamma_k) \) is an open convex cone with vertex at the origin in matrix spaces. The properties of \( \sigma_k \) described above guarantee that

\[
\left( \frac{\partial S_k}{\partial A_{ij}} \right)_{n \times n} > 0 \ \forall \ A \in S(\Gamma_k)
\]

and

\[
\sigma_k^{1/k} \text{ is concave in } S(\Gamma_k).
\]

For an open bounded subset \( \Omega \subset \mathbb{R}^n \) with boundary of class \( C^2 \) and for every \( x \in \partial \Omega \), we denote by \( \rho(x) = (\rho_1(x), \cdots, \rho_{n-1}(x)) \) the principal curvatures of \( \partial \Omega \) (relative to the interior normal). Recall that \( \Omega \) is said to be \( l \)-convex \((1 \leq l \leq n - 1)\) if \( \partial \Omega \), regarded as a hypersurface in \( \mathbb{R}^n \), is \( l \)-convex, that is, for every \( x \in \partial \Omega \), \( \sigma_j(\rho(x)) \geq 0 \) with \( j = 1, 2, \cdots, l \). Respectively, \( \Omega \) is called strictly \( l \)-convex if \( \sigma_j(\rho(x)) > 0 \) with \( j = 1, 2, \cdots, l \).
Since we will consider viscosity solutions of (1.1), we first give the following definitions (See [27]). A function \( u \in C(\Omega) \) is said to be a viscosity subsolution (supersolution) of (1.1) if \( x_0 \in \Omega, A \) is an open neighborhood of \( x_0, \psi \in C^2(A) \) is \( k \)-admissible and \( u - \psi \) has a local maximum (minimum) at \( x_0 \), then

\[
S_k(D^2\psi(x_0)) \geq b(x_0)f(\psi(x_0)) \quad (\leq b(x_0)f(\psi(x_0))).
\]

A function \( u \in C(\Omega) \) is said to be a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

From now on we shall always assume that \( \Omega \) is bounded and strictly \((k - 1)\)-convex and consider viscosity solutions of (1.1).

In order to show the existence of the viscosity solution of (1.1), we need \( f \) to satisfy some conditions. Specifically, we assume that \( f \) satisfies:

(\( f_1 \)) \( f \in C^1(0, \infty), \ f(s) > 0 \), and is nondecreasing in \((0, \infty)\);

(\( f_2 \)) The function

\[
\Phi(s) = \int_s^\infty \frac{d\tau}{H(\tau)}
\]

is well defined for any \( s > 0 \), where

\[
H(\tau) = ((k + 1)F(\tau))^{1/(k+1)} \quad \forall \ \tau > 0
\]

and

\[
F(\tau) = \int_0^\tau f(s) ds \quad \forall \ \tau > 0.
\]

Since we will investigate the boundary behavior of \( u \) and \( u = \infty \) on the boundary, we only need to concern the behavior of \( F(\tau) \) and \( f(\tau) \) as \( \tau \) being sufficiently large. For convenience, we define by \( \varphi \) the inverse of \( \Phi \), i.e., \( \varphi \) satisfies

\[
\int_{\varphi(t)}^\infty \frac{d\tau}{H(\tau)} = t \quad \forall \ t > 0.
\]  

Our existence results are stated as follows:
Theorem 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial \Omega \in C^{3,1}$. Suppose that $f$ satisfies $(f_1)$ and $(f_2)$, and that $b \in C^{1,1}(\overline{\Omega})$ is positive in $\Omega$. Then problem (1.1) admits a viscosity solution $u \in C(\Omega)$.

The existence of viscosity solutions of Hessian equations with boundary blowup has been considered in Salani [24], where Salani obtained the existence of solutions by using radial function to constructed barrier functions. In this paper, we present a different proof from that in [24]. For more existence and nonexistence results, we refer to [8, 14, 17, 24] and the references therein.

To study the boundary behavior of the solution of (1.1), we need $f$ and $b$ to satisfy more conditions. Precisely, we assume that $f$ satisfies:

$(f_3)$ There exists $C_f > 0$ such that

$$\lim_{s \to \infty} H'(s) \int_s^\infty \frac{d\tau}{H(\tau)} = C_f,$$

where $H(\tau)$ is defined in $(f_2)$.

We also assume that $b$ satisfies:

$(b_1)$ $b \in C^{1,1}(\overline{\Omega})$ is positive in $\Omega$;

$(b_2)$ There exist a positive and nondecreasing function $m(t) \in C^1(0, \delta_0)$ (for some $\delta_0 > 0$), and two positive constants $\overline{b}$ and $\overline{b}$ such that

$$\overline{b} = \liminf_{x \in \Omega} \frac{b(x)}{m^{k+1}(d(x))} \leq \limsup_{x \in \Omega} \frac{b(x)}{m^{k+1}(d(x))} = \underline{b},$$

where $d(x) = \text{dist}(x, \partial \Omega)$. Moreover, there exists $C_m \in [0, \infty)$ such that

$$\lim_{t \to 0^+} \left( \frac{M(t)}{m(t)} \right)' = C_m,$$

where $M(t) = \int_0^t m(s)ds < \infty$, $0 < t < \delta_0$.

The boundary behavior of the solution of (1.1) may involve the curvatures of $\partial \Omega$. We set

$$L_0 = \max_{\sigma \in \partial \Omega} \sigma_{k-1}(\rho(\sigma)) \quad \text{and} \quad l_0 = \min_{\sigma \in \partial \Omega} \sigma_{k-1}(\rho(\sigma)),$$

$(1.4)$
where $\rho(\mathbf{x}) = (\rho_1(\mathbf{x}), \rho_2(\mathbf{x}), \ldots, \rho_{n-1}(\mathbf{x}))$ are the principal curvatures of $\partial \Omega$ at $\mathbf{x}$. Observe that $0 < l_0 \leq L_0 < +\infty$ since $\Omega$ is bounded and strictly $(k-1)$-convex. The boundary estimates of the solution of (1.1) are related to $L_0$ and $l_0$.

Now, we state our boundary behavior results as follows.

**Theorem 1.2.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial \Omega \in C^{3,1}$. Suppose that $f$ satisfies $(f_1)$, $(f_2)$ and $(f_3)$, and that $b$ satisfies $(b_1)$ and $(b_2)$. If

$$C_f > 1 - C_m,$$

where $C_f$ and $C_m$ are the constants defined in $(f_3)$ and $(b_2)$ respectively, then the viscosity solution $u$ of (1.1) satisfies

$$1 \leq \liminf_{\mathbf{x} \in \Omega, d(\mathbf{x}) \to 0} \frac{u(x)}{\varphi(\xi M(d(x)))} \quad \text{and} \quad \limsup_{\mathbf{x} \in \Omega, d(\mathbf{x}) \to 0} \frac{u(x)}{\varphi(\xi M(d(x)))} \leq 1,$$

where $\varphi$ is defined by (1.2),

$$\xi = \left(\frac{b}{L_0(1 - C_f^{-1}(1 - C_m))}\right)^{1/(k+1)} \quad \text{and} \quad \xi = \left(\frac{b}{l_0(1 - C_f^{-1}(1 - C_m))}\right)^{1/(k+1)}.$$

Here $L_0$ and $l_0$ are the constants in (1.4).

**Remark 1.3.** Lemmas 2.1 and 2.2 of [28] showed that if $b$ satisfies $(b_1)$ and $(b_2)$, and $f$ satisfies $(f_1)$, $(f_2)$ and $(f_3)$, then $0 \leq C_m \leq 1$ and $C_f \geq 1$. Hence (1.5) holds if $C_f > 1$, or $C_f = 1$ with $C_m > 0$.

**Remark 1.4.** Theorem 1.2 contains the following interesting cases.

(i) $b \equiv 1$ and $f(s) = s^\gamma$, $\gamma > k$. In this case, choose $m(t) = 1$, and we get

$$M(t) = t \quad \text{and} \quad C_m = 1.$$

We also obtain $C_f = \frac{\gamma + 1}{\gamma - k}$,

$$\varphi(t) = \left(\frac{(k + 1)^k(\gamma + 1)}{\gamma - k)^{k+1}}\right)^{1/(\gamma - k)} t^{-(k+1)/(\gamma - k)},$$

5
ξ = \left( \frac{1}{L_0} \right)^{1/(k+1)} \quad \text{and} \quad \bar{\xi} = \left( \frac{1}{l_0} \right)^{1/(k+1)}.

Note that in this case (1.5) holds for any γ > k. Hence, by (1.6), the solution of (1.1) satisfies

1 \leq \liminf_{x \in \Omega, d(x) \to 0} \frac{u(x)}{\left( \frac{L_0(k+1)\gamma+1}{(\gamma-k)^{k+1}} \right)^{1/((\gamma-k))} d(x)^{-((k+1)/(\gamma-k))}}

and

\limsup_{x \in \Omega, d(x) \to 0} \frac{u(x)}{\left( \frac{L_0(k+1)\gamma+1}{(\gamma-k)^{k+1}} \right)^{1/((\gamma-k))} d(x)^{-((k+1)/(\gamma-k))}} \leq 1.

(ii) \( b = d(x)^{a(k+1)}, \alpha > 0, \) near \( \partial\Omega \) and \( f(s) = s^\gamma, \gamma > k. \) In this case, choose \( m(t) = t^\alpha, \) and we obtain

\[ M(t) = \frac{t^{a+1}}{\alpha + 1} \quad \text{and} \quad C_m = \frac{1}{\alpha + 1}. \]

We still have \( C_f = \frac{\gamma+1}{\gamma-k}, \)

\[ \varphi(t) = \left( \frac{(k+1)^{k+1}(\gamma+1)}{(\gamma-k)^{k+1}} \right)^{1/((\gamma-k))} t^{-(k+1)/(\gamma-k)}, \]

\[ \xi = \left( \frac{(\alpha+1)(\gamma+1)}{L_0(\gamma+1+\alpha k+\alpha)} \right)^{1/((k+1))} \quad \text{and} \quad \bar{\xi} = \left( \frac{(\alpha+1)(\gamma+1)}{l_0(\gamma+1+\alpha k+\alpha)} \right)^{1/((k+1))}. \]

Note that in this case (1.5) holds for any γ > k. Therefore, by (1.6), the solution of (1.1) satisfies

1 \leq \liminf_{x \in \Omega, d(x) \to 0} \frac{u(x)}{\left( \frac{L_0(\gamma+\alpha k+\alpha+1)(k+1)\gamma+1}{(\gamma-k)^{k+1}} \right)^{1/((\gamma-k))} d(x)^{-((k+1)(\alpha+1)/(\gamma-k))}}

and

\limsup_{x \in \Omega, d(x) \to 0} \frac{u(x)}{\left( \frac{L_0(\gamma+\alpha k+\alpha+1)(k+1)\gamma+1}{(\gamma-k)^{k+1}} \right)^{1/((\gamma-k))} d(x)^{-((k+1)(\alpha+1)/(\gamma-k))}} \leq 1.

Problem (1.1) is the Laplace equation for \( k = 1. \) The study of boundary blowup solutions of Laplace equation can be traced back to Bieberbach. The author considered \( \Delta u = e^u \) in a smooth bounded domain in \( \mathbb{R}^2. \) Since then many
papers have been dedicated to resolving existence, uniqueness and asymptotic behavior issues for solutions of blowup elliptic equations. See \([1, 2, 3, 6, 9, 18, 23]\) and their references.

For \(k = n\), problem (1.1) is the Monge-Ampère equation. There are many papers resolving existence, nonexistence, uniqueness and asymptotic behavior of boundary blowup solutions of Monge-Ampère equation. We refer to \([7, 13, 19, 21, 22, 28]\) and the references therein.

For general \(k\)-Hessian equation with boundary blowup, there are also several authors studying the asymptotic behavior. In \([24]\), Salani showed that: let \(\Omega\) be a bounded and strictly convex domain, which implies that there exist two positive numbers \(R_1 \leq R_2\) such that for any \(y \in \partial \Omega\), there exist two balls \(B_1^y\) and \(B_2^y\), with rays \(R_1\) and \(R_2\) respectively, with \(y \in \partial B_1^y \cap \partial B_2^y\) and \(B_1^y \subset \Omega \subset B_2^y\). Suppose that \(f\) satisfies \((f_1)\) and \((f_2)\), and \(c_1 \leq b(x) \leq c_2\) with \(c_1\) and \(c_2\) being two positive constants. Then the solution \(u\) of (1.1) satisfies

\[
c_1^{1/(k+1)} p(R_1) \leq \liminf_{x \in \Omega, d(x) \to 0} \frac{\Phi(u(x))}{d(x)} \leq \limsup_{x \in \Omega, d(x) \to 0} \frac{\Phi(u(x))}{d(x)} \leq c_2^{1/(k+1)} p(R_2),
\]

where \(\Phi\) is the function in \((f_2)\) and \(p(R) = (C_n^{k-1})^{-1/(k+1)} R^{(k-1)/(k+1)}\).

Later, Huang \([14]\) generalized the results of \([24]\). Let \(\Omega\) be a smooth, strictly \((k-1)\)-convex bounded domain. Suppose that \(f \in RV_q, q > k\), and \(b\) satisfies \((b_1)\) and \((b_2)\). Define

\[
\mathcal{P}(\tau) = \sup \left\{ \frac{f(y)}{y^k} : \theta \leq y \leq \tau \right\}, \quad \text{for } \tau \geq \theta, \ \theta \text{ sufficiently large,}
\]

and

\[
\mathcal{P}^+ (s) = \inf \left\{ \tau : \mathcal{P}(\tau) \geq s \right\}.
\]

Then the solution \(u\) of (1.1) satisfies

\[
\xi^- \leq \liminf_{d(x) \to 0} \frac{u}{\phi(d(x))} \quad \text{and} \quad \limsup_{d(x) \to 0} \frac{u}{\phi(d(x))} \leq \xi^+,
\]

where,

\[
\phi(t) = \mathcal{P}^+ \left( (M(t))^{-k-1} \right), \quad \text{for } t > 0 \text{ small,}
\]
and
\[
\frac{(\xi^+)^{k-q}}{b} \max_{\partial \Omega} \sigma_{k-1} = \frac{(\xi^-)^{k-q}}{b} \min_{\partial \Omega} \sigma_{k-1} = \frac{((q-k)/(k+1))^{k+1}}{1 + C_m(q-k)/(k+1)}.
\]

Here $b$, $b$ and $C_m$ are given by $b_2$.

In this paper, we investigate a new boundary behavior of solutions of (1.1). In fact, we generalize the asymptotic results for Monge-Ampère equation in [28] to $k$-Hessian equation. Our results are also more accurate than [14].

Theorem 1.1 and 1.2 will be proved in Section 2 and 3 respectively.

2. Proof of Theorem 1.1

The following comparison principle is a basic tool for proofs of both Theorem 1.1 and Theorem 1.2.

**Lemma 2.1** (The comparison principle). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose that $g(x, \eta)$ is positive and continuously differentiable, and is nondecreasing only with respect to $\eta$. If $u, v \in C(\Omega)$ are respectively viscosity subsolution and supersolution of
\[
S_k(D^2 u) = g(x, u)
\]
and $u \leq v$ on $\partial \Omega$, then we have
\[
u \leq v \quad \text{in} \, \Omega.
\]

**Proof.** We refer to Proposition 2.3 of [27] for the detailed proof.

From [15] and [16], we see, for any symmetric matrix $S$,
\[
S_k(S + \xi \times \xi) = S_k(S) + \frac{\partial S_k(S)}{\partial S_{ij}} \xi_i \xi_j, \quad 1 \leq k \leq n, \quad \xi \in \mathbb{R}^n.
\]

Similarly,
\[
S_k(S - \xi \times \xi) = S_k(S) - \frac{\partial S_k(S)}{\partial S_{ij}} \xi_i \xi_j, \quad 1 \leq k \leq n, \quad \xi \in \mathbb{R}^n.
\]

Then the following conclusion about composite functions can be obtained:
Lemma 2.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose $h \in C^2(\mathbb{R})$ and $g \in C^2(\Omega)$. Then

$$S_k(D^2h(g)) = (h'(g))^{k-1}h''(g)S^i_j(D^2g)g_ig_j + (h'(g))^kS_k(D^2g) \quad \text{in} \quad \Omega,$$

where $S^i_j(D^2g) = \frac{\partial S_k(D^2g)}{\partial g_{ij}}$.

Using the above lemma, we prove the following conclusion:

Lemma 2.3. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial \Omega \in C^{3,1}$. Suppose that $f$ satisfies $(f_1)$ and $(f_2)$, and that $b \in C^{1,1}(\Omega)$ is positive. Then there exists a $\overline{h} \in C^2(\Omega)$, $\overline{h}(x) \to \infty$ as $\text{dist}(x, \partial \Omega) \to 0$, such that for any $k$-admissible function $u \in C^2(\Omega) \cap C(\Omega)$ satisfying

$$S_k(D^2u) = b(x)f(u) \quad \text{in} \quad \Omega,$$

we have

$$u \leq \overline{h} \quad \text{in} \quad \Omega.$$

Proof. We assume that $w(w < 0)$ is the admissible solution of

$$\begin{cases} S_k(D^2w) = b(x) & \text{in} \quad \Omega; \\ w = 0 & \text{on} \quad \partial \Omega, \end{cases} \quad (2.1)$$

where $b \in C^{1,1}(\Omega)$ is positive. Indeed, from Theorem 1.1 of [25], (2.1) is uniquely solvable for admissible $w \in C^{3,\beta}(\Omega)$ for any $0 < \beta < 1$.

Define

$$\overline{h} = \varphi(-\varepsilon w) \quad \text{in} \quad \Omega,$$

where $\varphi$ is defined by (1.2) and $\varepsilon > 0$ is a constant to be chosen later, and we take the second derivative of $\overline{h}$,

$$\overline{h}_{ij} = (\varphi(-\varepsilon w))_{ij} = -\varepsilon \varphi'(-\varepsilon w)w_{ij} + \varepsilon^2 \varphi''(-\varepsilon w)w_iw_j.$$

By lemma 2.2, for $1 \leq l \leq k$,

$$S_l(D^2\overline{h}) = \varepsilon^l(-\varphi'(-\varepsilon w))^lS_l(D^2w) + \varepsilon^{l+1} \varphi''(-\varepsilon w)(-\varphi'(-\varepsilon w))^{l-1} S^i_j(D^2w)w_iw_j > 0 \quad \text{in} \quad \Omega.$$
since $-\varphi' > 0$, $\varphi'' \geq 0$ and $w$ is $k$-admissible. That is $\bar{h}$ is $k$-admissible. Specially,

$$S_k(D^2h) = \varepsilon \frac{-\varphi'(-\varepsilon w)\varphi''(-\varepsilon w)}{\varphi'(-\varepsilon w) + \varepsilon \frac{S^{ij}_k(D^2w)w_iw_j}{S_k(D^2w)}}.$$  

(2.2)

Let

$$M_\varepsilon(x) = \varepsilon \left[ \frac{-\varphi'(-\varepsilon w)\varphi''(-\varepsilon w)}{\varphi'(-\varepsilon w) + \varepsilon \frac{S^{ij}_k(D^2w)w_iw_j}{S_k(D^2w)}} \right].$$

We claim: for any $x \in \Omega$, $M_\varepsilon(x)$ is sufficiently small as $\varepsilon$ being sufficiently small.

In fact, by the choice of $\varphi$, we have

$$\frac{-\varphi'(-\varepsilon w)\varphi''(-\varepsilon w)}{\varphi'(-\varepsilon w)} = \frac{(k+1)F(\varphi(-\varepsilon w))^{k+k+1}}{f(\varphi(-\varepsilon w))}.$$

From [12] and [22], we know that if $f$ satisfies $(f_2)$, then

$$\lim_{t \to \infty} \frac{F(t)^{k+k+1}}{f(t)} = 0.$$  

(2.3)

Note that $\varphi(0) = \lim_{t \to 0^+} \varphi(t) = +\infty$, we see

$$\lim_{\varepsilon \to 0^+} \frac{-\varphi'(-\varepsilon w)\varphi''(-\varepsilon w)}{\varphi'(-\varepsilon w)} = 0.$$  

(2.4)

Furthermore, since $w \in C^{3,\beta}(\overline{\Omega})$ is $k$-admissible and the matrix $\{S^{ij}_k(D^2w)\} > 0$, we have,

$$\frac{S^{ij}_k(D^2w)w_iw_j}{S_k(D^2w)} > 0$$

is bounded in $\Omega$.

This combining with (2.3) imply that our claim holds.

From [24], we have, for sufficiently small $\varepsilon$,

$$S_k(D^2h) \leq b(x)f(\bar{h}) \text{ in } \Omega.$$  

Note that, by the definition of $\bar{h}$, $\bar{h}(x) \to \infty$ as $\text{dist}(x, \partial \Omega) \to 0$. Therefore, for any $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satsifying

$$S_k(D^2u) = b(x)f(u) \text{ in } \Omega,$$
we have \( u \leq \overline{h} \) on \( \partial \Omega \) and then by Lemma 2.1,
\[
u \leq \overline{h} \quad \text{in } \Omega.
\]

**Proof of Theorem 1.1.** Since (2.3) holds, we see, by (f2), for sufficiently large \( s > 0 \),
\[
\Psi(s) = \int_{s}^{\infty} \frac{1}{f(\tau)^{1/k}} d\tau
\]
is well defined. Let \( \psi \) be the inverse of \( \Psi \), i.e., \( \psi \) satisfies
\[
\psi(s) = \int_{s}^{\infty} \frac{1}{f(\tau)^{1/k}} d\tau.
\]

Then we have
\[
\psi(0) = \lim_{t \to 0^+} = \infty, \quad \psi'(s) = -(f(\psi(s)))^{1/k}, \quad \psi''(s) = \frac{1}{k} (f(\psi(s)))^{(2-k)/k} f'(\psi(s)).
\]

Assume that \( w \) is the admissible solution of (2.1) with \( b \in C^{1,1}(\overline{\Omega}) \) being positive in \( \Omega \) (See [25]). We define
\[
h(x) = \psi(-w(x)), \quad x \in \Omega,
\]
and for \( j = 1, 2, \cdots, \)
\[
\Omega_j = \{ x \in \Omega : h(x) < j \}.
\]

Since \( w \) is \( k \)-admissible, we see that \( \Omega_j \) is strictly \( (k-1) \)-convex (see [26]).

Consider
\[
\begin{aligned}
S_k(D^2u) &= b(x)f(u), & x &\in \Omega_j; \\
u &= j, & x &\in \partial \Omega_j.
\end{aligned}
\tag{2.5}
\]

We show that (2.4) has a \( k \)-admissible solution \( u_j \). By Theorem 4.1 of [20], we only need to prove that (2.4) has a \( k \)-admissible subsolution, and we will show that \( h \) is actually a \( k \)-admissible subsolution of (2.5).
By direct computation,

\[ h_{ij} = (\psi(-w))_{ij} \]
\[ = -\psi'(-w)w_{ij} + \psi''(-w)w_iw_j. \]

Since \( f \) is positive and nondecreasing, \( \psi' < 0 \) and \( \psi'' \geq 0 \). Then according to that the matrix \( \{w_iw_j\} \) is nonnegative, we have

\[ D^2 h \geq -\psi'(-w)D^2 w. \]

This implies that for any \( 1 \leq j \leq k \),

\[ S_j(D^2 h) \geq S_j(-\psi'(-w)D^2 w) > 0, \quad x \in \Omega, \]
i.e., \( h \) is \( k \)-admissible, and

\[ S_k(D^2 h) \geq (-\psi'(-w))^k S_k(D^2 w) = (-\psi'(-w))^k b(x) = b(x) f(h), \quad x \in \Omega \]
since \( w \) is \( k \)-admissible and satisfies (2.1). By the construction of \( \Omega_j \),

\[ h = j, \quad x \in \partial \Omega_j. \]

That is, we have shown that \( h \) is a \( k \)-admissible subsolution of (2.5). Therefore, (2.5) has a \( k \)-admissible solution \( u_j \).

Since, for any \( j = 1, 2, \cdots, \)

\[ h = u_j, \quad x \in \partial \Omega_j, \]
we have, by Lemma 2.1,

\[ h \leq u_j, \quad x \in \Omega_j. \]  \quad (2.6)

Furthermore, since

\[ u_j = h \leq u_{j+1}, \quad x \in \partial \Omega_j, \]
we see,

\[ u_j \leq u_{j+1}, \quad x \in \Omega_j. \]  \quad (2.7)
For any $\Omega' \subset \subset \Omega$ with $\Omega'$ being strictly $(k-1)$-convex and $\partial \Omega' \in C^{3,1}$, by Lemma 2.3, there exists a $\rho \in C^2(\Omega'), \rho(x) \to \infty$ as $\text{dist}(x, \partial \Omega') \to 0$, such that for sufficiently large $j, \Omega' \subset \Omega_j$, and
\[ u_j \leq \rho, \quad x \in \Omega'. \]
This combining with (2.7) imply that for any $x \in \Omega'$, the limit function
\[ u(x) = \lim_{j \to \infty} u_j(x) \]
exists. Then by the diagonal rule, for any $x \in \Omega$, $u(x)$ exists. Moreover, $u(x) \to \infty$ as $d(x, \partial \Omega) \to 0$ and is a viscosity solution of (1.1).

\[ \square \]

### 3. Proof of Theorem 1.2

To study the boundary behavior of solutions of (1.1), we need the asymptotic estimate of functions in (b2) and (b2) as $t \to 0$. The following two lemmas describe those asymptotic behaviors.

**Lemma 3.1.** Let $m$ and $M$ be the functions given by (b2). Then
\[ M(0) = \lim_{t \to 0^+} M(t) = 0, \]
\[ \lim_{t \to 0^+} \frac{M(t)}{m(t)} = 0, \]
and
\[ \lim_{t \to 0^+} \frac{M(t)m'(t)}{m^2(t)} = 1 - C_m. \]

**Lemma 3.2.** Assume that $f$ satisfies (f1), (f2) and (f3), and $\varphi$ satisfies (f2). Then we have
\[ (i) \varphi(t) > 0, \quad \varphi(0) = \lim_{t \to 0^+} \varphi(t) = +\infty, \quad \varphi'(t) = -((k+1)F(\varphi(t)))^{1/(k+1)}, \]
and $\varphi''(t) = ((k+1)F(\varphi(t)))^{(1-k)/(k+1)} f(\varphi(t));$
\[ (ii) \lim_{t \to 0^+} \frac{-\varphi'(t)}{\varphi''(t)} = \lim_{t \to 0^+} \frac{((k+1)F(\varphi(t)))^{1/(k+1)}}{f(\varphi(t))} = \frac{1}{C_f}. \]
For detailed proofs of Lemmas 3.1 and 3.2, we refer to Lemmas 2.1 and 2.3 of [28], respectively. More characterization of functions in (f_2 ) and (b_2 ) are also provided there.

We also need to recall some results of the distance function. Let \( d(x) = \text{dist}(x, \partial \Omega) = \inf_{y \in \partial \Omega} |x - y| \). For any \( \delta > 0 \), we define

\[
\Omega_\delta = \{ x \in \Omega : 0 < d(x) < \delta \}.
\]  
(3.1)

If \( \Omega \) is bounded and \( \partial \Omega \in C^2 \), by Lemma 14.16 of [11], there exists \( \delta_1 > 0 \) such that

\[
d \in C^2(\Omega_{\delta_1}).
\]

Let \( \overline{x} \in \partial \Omega \), satisfying \( \text{dist}(x, \partial \Omega) = |x - \overline{x}| \), be the projection of the point \( x \in \Omega_{\delta_1} \) to \( \partial \Omega \), and \( \rho_i(\overline{x})(i = 1, \ldots, n-1) \) be the principal curvatures of \( \partial \Omega \) at \( \overline{x} \). Then, in terms of a principal coordinate system at \( \overline{x} \), we have, by Lemma 14.17 of [11],

\[
\begin{align*}
Dd(x) &= (0, 0, \ldots, 1), \\
D^2d(x) &= \text{diag}\left(\frac{-\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \ldots, \frac{-\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}, 0\right).
\end{align*}
\]  
(3.2)

**Proof of Theorem 1.1.** Under the assumptions in Theorem 1.2, we have the following conclusions: For any \( \varepsilon > 0 \), we choose \( \delta_2 > 0 \) small enough such that

- (a_1) \( m(t) \) satisfies (b_2) for \( 0 < t < \delta_2 \);
- (a_2) \( d(x) \in C^2(\Omega_{2\delta_2}) \), where \( \Omega_{2\delta_2} \) is defined by (3.1);
- (a_3) \( (\bar{b} - \varepsilon)m^{k+1}(d(x)) \leq b(x) \leq (\bar{b} + \varepsilon)m^{k+1}(d(x)) \) in \( \Omega_{2\delta_2} \);  
- (a_4) For any \( 1 \leq j \leq k - 1, \sigma_j\left(\frac{\rho_j(\overline{x})}{1-d(x)\rho_j(\overline{x})}, \ldots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) > 0 \) in \( \Omega_{2\delta_2} \).

Recall that \( \rho_i(\overline{x}) \) \( (i = 1, 2, \ldots, n-1) \) denote the principal curvatures of \( \partial \Omega \) at \( \overline{x} \), where \( \overline{x} \in \partial \Omega \) satisfies \( d(x) = |x - \overline{x}| \);

- (a_5) \( (1 - \varepsilon)l_0 \leq \sigma_k\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \ldots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) \leq (1 + \varepsilon)L_0 \) in \( \Omega_{2\delta_2} \);  
- (a_6) \( \sigma_k\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \ldots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) \) is bounded in \( \Omega_{2\delta_2} \).
Fix $0 < \varepsilon < \frac{b}{2}$ and we choose

$$
\xi = \left( \frac{b - 2\varepsilon}{(1 + \varepsilon)L_0 (1 - C_f^{-1}(1 - C_m))} \right)^{1/(k+1)},
$$

(3.3)

and

$$
\xi = \left( \frac{b + 2\varepsilon}{(1 - \varepsilon)L_0 (1 - C_f^{-1}(1 - C_m))} \right)^{1/(k+1)},
$$

(3.4)

where $b, b, C_m, L_0, l_0$ and $C_f$ are given by (b2), (1.4) and (f3) respectively. Let $\delta_\varepsilon$ be a small enough constant such that the above (a1 - a6) hold and choose $0 < \sigma < \delta_\varepsilon$. We define

$$
d_1(x) = d(x) - \sigma, \quad d_2(x) = d(x) + \sigma
$$

(3.5)

and

$$
\begin{cases}
\overline{u}_\varepsilon(x) = \varphi(\xi M(d_1(x))) \text{ in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma, \\
v_\varepsilon(x) = \varphi(\xi M(d_2(x))) \text{ in } \Omega_{2\delta_\varepsilon} - \sigma.
\end{cases}
$$

(3.6)

We divide the proof of Theorem 1.2 into three steps.

**Step 1.** We prove that $\overline{u}_\varepsilon$ is $k$-admissible and

$$
S_k(D^2\overline{u}_\varepsilon (x)) \leq b(x)f(\overline{u}_\varepsilon (x)) \text{ in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma
$$

(3.7)

as $\delta_\varepsilon$ being sufficiently small.

First, we show that $\overline{u}_\varepsilon$ is a $k$-admissible function in $\Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma$. That is, for $1 \leq j \leq k$,

$$
S_j(D^2u_\varepsilon) > 0 \text{ in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma.
$$

(3.8)

In view of (3.6), we see, obviously,

$$
\overline{u}_\varepsilon(x) > 0 \text{ in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma \text{ and } \overline{u}_\varepsilon(x) = \infty \text{ on } \partial\Omega_\sigma.
$$

By straightforward computations,

$$
(\overline{u}_\varepsilon (x))_{\alpha\beta} = (\varphi(\xi M(d_1(x))))_{\alpha\beta}
$$

$$
= \xi \left[ \xi \varphi''(\xi M(d_1(x))) m^2(d_1(x)) + \varphi'(\xi M(d_1(x))) m'(d_1(x)) \right] d_\alpha d_\beta
$$

$$
+ \xi \varphi'(\xi M(d_1(x))) m(d_1(x)) d_\alpha \beta.
$$
Using (3.2) and Lemma 3.2, we derive that for $1 \leq j \leq k$,

\[
S_j(D^2 u_\sigma) = \sum_{j=1}^{S_j} (d_1(x)) \sum_{j=1}^{S_j} (d_1(x)) f(\varphi(\xi M(d_1(x)))) (k + 1) F \left( \varphi \left( \xi M(d_1(x)) \right) \right) (j-k)/(k+1)
\]

\[
\times \left[ \left( 1 - \frac{M(d_1(x)) m'(d_1(x)) (k+1) F(\varphi(\xi M(d_1(x))))^{k/(k+1)}}{m^2(d_1(x)) \xi M(d_1(x)) f(\varphi(\xi M(d_1(x))))} \right) \right]
\]

This suggests us that to show (3.8), we only need to prove that for $1 \leq j \leq k$,

\[
\left( 1 - \frac{M(d_1(x)) m'(d_1(x)) (k+1) F(\varphi(\xi M(d_1(x))))^{k/(k+1)}}{m^2(d_1(x)) \xi M(d_1(x)) f(\varphi(\xi M(d_1(x))))} \right)
\]

\[
\times \sigma_j \left( \frac{\rho_1(x)}{1-d(x)\rho_1(x)}, \ldots, \frac{\rho_{n-1}(x)}{1-d(x)\rho_{n-1}(x)} \right)
\]

\[
+ \frac{M(d_1(x)) m'(d_1(x)) (k+1) F(\varphi(\xi M(d_1(x))))^{k/(k+1)}}{m^2(d_1(x)) \xi M(d_1(x)) f(\varphi(\xi M(d_1(x))))} \sigma_j \left( \frac{\rho_1(x)}{1-d(x)\rho_1(x)}, \ldots, \frac{\rho_{n-1}(x)}{1-d(x)\rho_{n-1}(x)} \right)
\]

\[
> 0 \text{ in } \Omega_{2\delta_\sigma} 
\]

as $\delta_\sigma > 0$ being sufficiently small.

Actually, since, by Lemma 3.1, Lemma 3.2(i_2) and (1.5),

\[
\lim_{x \in \Omega_{\delta_\sigma} \to 0} \frac{M(d(x)) (k+1) F(\varphi(\xi M(d(x))))^{k/(k+1)}}{m^2(d(x)) \xi M(d(x)) f(\varphi(\xi M(d(x))))} = 0
\]

and

\[
1 - \lim_{x \in \Omega_{\delta_\sigma} \to 0} \frac{M(d(x)) m'(d(x)) (k+1) F(\varphi(\xi M(d(x))))^{k/(k+1)}}{m^2(d(x)) \xi M(d(x)) f(\varphi(\xi M(d(x))))} = 1 - \frac{C}{1-C} > 0,
\]

we have, for sufficiently small $\delta_\sigma > 0$,

\[
\frac{M(d(x)) (k+1) F(\varphi(\xi M(d(x))))^{k/(k+1)}}{m^2(d(x)) \xi M(d(x)) f(\varphi(\xi M(d(x))))} \text{ is small enough in } \Omega_{2\delta_\sigma}
\]

and

\[
1 - \frac{M(d(x)) m'(d(x)) (k+1) F(\varphi(\xi M(d(x))))^{k/(k+1)}}{m^2(d(x)) \xi M(d(x)) f(\varphi(\xi M(d(x))))} > 0 \text{ in } \Omega_{2\delta_\sigma}.
\]
Then to show (3.16), we only need to prove

\[
0 < d_1(x) < 2\delta_\varepsilon - \sigma \quad \text{in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma,
\]

we see that (3.13) and (3.14) still holds with \( d(x) \) being replaced by \( d_1(x) \).
Therefore, we obtain (3.10) from (a4) and (a6). By (3.9), we have (3.8).

Next, we show (3.7).
Since \( m \) is nondecreasing, we have, by (a3) and (3.5),

\[
(b - \varepsilon)m^{k+1}(d_1(x)) \leq (b - \varepsilon)m^{k+1}(d(x)) \leq b(x) \quad \text{in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma.
\]

Hence, to show (3.7), we only need to prove

\[
S_k(D^2\bar{u}_\varepsilon(x)) \leq (b - \varepsilon)m^{k+1}(d_1(x)) f(\bar{u}_\varepsilon(x)) \quad \text{in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma.
\]

By (3.9),

\[
S_k(D^2\bar{u}_\varepsilon(x)) - (b - \varepsilon)m^{k+1}(d_1(x)) f(\bar{u}_\varepsilon(x))
= \xi^{k+1}m^{k+1}(d_1(x)) f(\varphi(\xi M(d_1(x))))
\times \left( 1 - \frac{M(d_1(x)) m'(d_1(x)) ((k+1) F(\varphi(\xi M(d_1(x)))) )^{k/(k+1)} }{m^2(d_1(x)) \xi M(d_1(x)) f(\varphi(\xi M(d_1(x))))} \right)
\times \sigma_{k-1} \left( \frac{\rho_1(\overline{\rho})}{1-d(x)\rho_1(\overline{\rho})}, \ldots, \frac{\rho_{n-1}(\overline{\rho})}{1-d(x)\rho_{n-1}(\overline{\rho})} \right)
+ \frac{M(d_1(x)) ((k+1) F(\varphi(\xi M(d_1(x)))) )^{k/(k+1)} }{m(d_1(x)) \xi M(d_1(x)) f(\varphi(\xi M(d_1(x))))} \sigma_k \left( \frac{\rho_1(\overline{\rho})}{1-d(x)\rho_1(\overline{\rho})}, \ldots, \frac{\rho_{n-1}(\overline{\rho})}{1-d(x)\rho_{n-1}(\overline{\rho})} \right) - (b - \varepsilon)m^{k+1}(d_1(x)) f(\varphi(\xi M(d_1(x))))
\]

Then to show (3.10), we only need to prove

\[
\xi^{k+1} \left[ \left( 1 - \frac{M(d_1(x)) m'(d_1(x)) ((k+1) F(\varphi(\xi M(d_1(x)))) )^{k/(k+1)} }{m^2(d_1(x)) \xi M(d_1(x)) f(\varphi(\xi M(d_1(x))))} \right)
\times \sigma_{k-1} \left( \frac{\rho_1(\overline{\rho})}{1-d(x)\rho_1(\overline{\rho})}, \ldots, \frac{\rho_{n-1}(\overline{\rho})}{1-d(x)\rho_{n-1}(\overline{\rho})} \right)
+ \frac{M(d_1(x)) ((k+1) F(\varphi(\xi M(d_1(x)))) )^{k/(k+1)} }{m(d_1(x)) \xi M(d_1(x)) f(\varphi(\xi M(d_1(x))))} \sigma_k \left( \frac{\rho_1(\overline{\rho})}{1-d(x)\rho_1(\overline{\rho})}, \ldots, \frac{\rho_{n-1}(\overline{\rho})}{1-d(x)\rho_{n-1}(\overline{\rho})} \right) \right)
- (b - \varepsilon) \leq 0 \quad \text{in } \Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma
\]

for sufficiently small \( \delta_\varepsilon > 0 \).

Since, by (3.3),

\[
\xi^{k+1} \left[ (1 + \varepsilon) L_0(1 - C_f^{-1}(1 - C_m)) \right] - (b - \varepsilon) = -\varepsilon,
\]

(3.19)
we see from (3.11), (3.12), (3.15) and \((a_6)\) that

\[
\xi^{k+1} \leq (1 + \varepsilon) L_0 \left( 1 - \frac{M(d_1(x)) \nu'(d_1(x))}{m^*(d_1(x))} \right) \frac{(k+1) F(\xi M(d_1(x)))}{\xi M(d_1(x))} \left( \frac{(k+1) F(\xi M(d_1(x)))}{\xi M(d_1(x))} \right)^{b/(k+1)} \\
+ \frac{M(d_1(x))}{m(d_1(x))} \left( \frac{(k+1) F(\xi M(d_1(x)))}{\xi M(d_1(x))} \right)^{b/(k+1)} \sigma(\frac{\rho_1(\xi)}{\rho_1(\xi)}, \ldots, \frac{\rho_{n-1}(\xi)}{1-d(x)\rho_{n-1}(\xi)}) \\
\leq \frac{\rho_1(\xi)}{\rho_1(\xi)}, \ldots, \frac{\rho_{n-1}(\xi)}{1-d(x)\rho_{n-1}(\xi)}
\]

for sufficiently small \(\delta > 0\). This and \((a_5)\) imply (3.18). Therefore, by (3.17), we obtain (3.19).

**Step 2.** We prove that \(u_x\) is \(k\)-admissible and

\[
S_k(D^2 u_x(x)) \geq b(x)f(u_x(x)) \text{ in } \Omega_{2\delta_x-\sigma}
\]

as \(\delta_x\) being sufficiently small.

The proof is similar to **Step 1**.

First, we show that \(u_x\) is a \(k\)-admissible function in \(\Omega_{2\delta_x-\sigma}\). That is, for \(1 \leq j \leq k\),

\[
S_j(D^2 u_x) > 0 \text{ in } \Omega_{2\delta_x-\sigma}.
\]

As in the proof of **Step 1**, by straightforward computations, we derive that for \(1 \leq j \leq k\),

\[
S_j(D^2 u_x) = \xi^{j+1} m^{j+1}(d_2(x)) + \left( \frac{(k+1) F(\xi M(d_2(x)))}{\xi M(d_2(x))} \right)^{b/(k+1)} \sigma(\frac{\rho_1(\xi)}{\rho_1(\xi)}, \ldots, \frac{\rho_{n-1}(\xi)}{1-d(x)\rho_{n-1}(\xi)})
\]

Since

\[
\sigma < d_2(x) < 2\delta_x \text{ in } \Omega_{2\delta_x-\sigma},
\]

18
by the same argument as in Step 1, we also have, for \(1 \leq j \leq k\),
\[
\left(1 - \frac{M(d_2(x))m'(d_2(x))}{m_2(d_2(x))} \left(\frac{(k+1)}{k} \frac{F(\varphi(\xi, M(d_2(x))))}{\xi, M(d_2(x))f(\varphi(\xi, M(d_2(x))))}\right)^{k/(k+1)}\right) \\
\times \sigma_{j-1} \left(\frac{\rho_1(\overline{\tau})}{1-d(x)\rho_1(\overline{\tau})}, \ldots, \frac{\rho_{n-1}(\overline{\tau})}{1-d(x)\rho_{n-1}(\overline{\tau})}\right) \\
+ \frac{M(d_2(x))}{m_2(d_2(x))} \left(\frac{(k+1)}{k} \frac{F(\varphi(\xi, M(d_2(x))))}{\xi, M(d_2(x))f(\varphi(\xi, M(d_2(x))))}\right)^{k/(k+1)} \\
> 0 \text{ in } \Omega_{2\delta_x - \sigma}
\]

(3.25)
as \(\delta_x > 0\) being sufficiently small. This combining with \((3.23)\) imply \((3.22)\).

Next, we show (3.21).

Since \(m\) is nondecreasing, we have, by (a3) and (3.5),
\[
(\overline{b} + \varepsilon)m^{k+1}(d_2(x)) \geq (\overline{b} + \varepsilon)m^{k+1}(d(x)) \geq b(x) \text{ in } \Omega_{2\delta_x - \sigma}.
\]

Hence, (3.21) is an easy consequence of
\[
S_k(D^2 u_\varepsilon(x)) \geq (\overline{b} + \varepsilon)m^{k+1}(d_2(x))f(u_\varepsilon(x)) \text{ in } \Omega_{2\delta_x - \sigma}.
\]

(3.26)

It follows from (3.23) that to show (3.20), we only need to prove
\[
\tilde{\xi}_\varepsilon^{k+1} \left(\frac{(1 - \overline{b} + \varepsilon)}{k} \frac{F(\varphi(\xi, M(d_2(x))))}{\xi, M(d_2(x))f(\varphi(\xi, M(d_2(x))))}\right)^{k/(k+1)} \\
\times \sigma_{k-1} \left(\frac{\rho_1(\overline{\tau})}{1-d(x)\rho_1(\overline{\tau})}, \ldots, \frac{\rho_{n-1}(\overline{\tau})}{1-d(x)\rho_{n-1}(\overline{\tau})}\right) \\
+ \frac{M(d_2(x))}{m_2(d_2(x))} \left(\frac{(k+1)}{k} \frac{F(\varphi(\xi, M(d_2(x))))}{\xi, M(d_2(x))f(\varphi(\xi, M(d_2(x))))}\right)^{k/(k+1)} \\
- (\overline{b} + \varepsilon) \geq 0 \text{ in } \Omega_{2\delta_x - \sigma}.
\]

(3.27)

Since, by (3.3),
\[
\tilde{\xi}_\varepsilon^{k+1} \left[(1 - \varepsilon)(1 - C_f^{-1})(1 - C_m)\right] - (\overline{b} + \varepsilon) = \varepsilon,
\]
we see from (3.11), (3.12) (note here that \(\xi_\varepsilon\) is replaced by \(\tilde{\xi}_\varepsilon\), (3.24) and (a6)
that
\[
\xi^{k+1}_\varepsilon \left[ (1 - \varepsilon)l_0 \left( 1 - \frac{M(d_2(x)m'(d_2(x))}{m^2(d_2(x))} \left( \frac{(k+1)F\left( \varphi(x,M(d_2(x))) \right)}{m_d(x)f\left( \varphi(x,M(d_2(x))) \right)} \right)^{1/(k+1)} \right) \\
+ \frac{M(d_2(x))}{m(d_2(x))} \left( \frac{(k+1)F\left( \varphi(x,M(d_2(x))) \right)}{m_d(x)f\left( \varphi(x,M(d_2(x))) \right)} \right)^{1/(k+1)} \right] \sigma_k \left( \frac{\rho_1(x)}{1-d(x)\rho_1(x)}, \cdots, \frac{\rho_{p_n-1}(x)}{1-d(x)\rho_{p_n-1}(x)} \right)
\]
\[-(\overline{b} + \varepsilon) \geq 0 \text{ in } \Omega_{2\delta_\varepsilon}\]  \hspace{1cm} (3.29)

for sufficiently small \(\delta_\varepsilon > 0\). This combining with (a5) imply (3.27). Consequently, we have (3.26) and then (3.21).

**Step 3.** We show (1.6).

Let \(u \in C(\Omega)\) be a viscosity solution of (1.1) and let \(T > 0\) (depending on \(\delta_\varepsilon\)) sufficiently large such that for any \(0 < \sigma < \delta_\varepsilon\),

\[
u \leq \nu_\varepsilon + T \text{ on } \Lambda_1 = \{x \in \Omega : d(x) = 2\delta_\varepsilon\}\] \hspace{1cm} (3.30)

and

\[
u_\varepsilon \leq u + T \text{ on } \Lambda_2 = \{x \in \Omega : d(x) = 2\delta_\varepsilon - \sigma\}.\] \hspace{1cm} (3.31)

We observe that

\[
u \leq \nu_\varepsilon + T = \infty \text{ on } \Lambda_2 = \{x \in \Omega : d(x) = \sigma\}.\] \hspace{1cm} (3.32)

and

\[
u_\varepsilon \leq u + T = \infty \text{ on } \partial\Omega.\] \hspace{1cm} (3.33)

Since, \(f\) is nondecreasing and \(\nu_\varepsilon\) satisfies (3.7), we have

\[S_k(D^2(\nu_\varepsilon + T)) = S_k(D^2\nu_\varepsilon) \leq b(x)f(\nu_\varepsilon) \leq b(x)f(\nu_\varepsilon + T) \text{ in } \Omega_{2\delta_\varepsilon}/\Omega_\sigma.\]

Note that in the viscosity sense,

\[S_k(D^2u)) = b(x)f(u) \text{ in } \Omega.\]

Therefore, by Lemma 2.1, we deduce from (3.30) and (3.32) that

\[u \leq \nu_\varepsilon + T \text{ in } \Omega_{2\delta_\varepsilon}/\Omega_\sigma.\] \hspace{1cm} (3.34)
Similarly, since in the viscosity sense

\[ S_k(D^2(u + T)) = S_k(D^2u) = b(x)f(u) \leq b(x)f(u + T) \text{ in } \Omega_{2\delta, -\sigma}, \]

and \( u_\varepsilon \) satisfies (3.21), we deduce from (3.31) and (3.33) that

\[ u_\varepsilon \leq u + T \text{ in } \Omega_{2\delta, -\sigma}. \quad (3.35) \]

Substituting (3.34) into (3.31) and (3.35) respectively, we have

\[ \frac{u}{\varphi(\xi_\varepsilon M(d_1(x)))} \leq 1 + \frac{T}{\varphi(\xi_\varepsilon M(d_1(x)))} \text{ in } \Omega_{2\delta, \sigma}, \]

and

\[ 1 - \frac{T}{\varphi(\xi_\varepsilon M(d_2(x)))} \leq \frac{u}{\varphi(\xi_\varepsilon M(d_2(x)))} \text{ in } \Omega_{2\delta, -\sigma}. \]

Let \( \sigma \to 0 \),

\[ \frac{u}{\varphi(\xi_\varepsilon M(d(x)))} \leq 1 + \frac{T}{\varphi(\xi_\varepsilon M(d(x)))} \text{ in } \Omega_{2\delta, \sigma}, \]

and

\[ 1 - \frac{T}{\varphi(\xi_\varepsilon M(d(x)))} \leq \frac{u}{\varphi(\xi_\varepsilon M(d(x)))} \text{ in } \Omega_{2\delta, -\sigma}. \]

Note that

\[ \varphi(\xi_\varepsilon M(d(x))) = \infty \text{ on } \partial \Omega \]

and

\[ \varphi(\xi_\varepsilon M(d(x))) = \infty \text{ on } \partial \Omega. \]

We obtain

\[ \limsup_{\delta(x) \to 0} \frac{u(x)}{\varphi(\xi_\varepsilon M(d(x)))} \leq 1 \]

and

\[ 1 \leq \liminf_{\delta(x) \to 0} \frac{u(x)}{\varphi(\xi_\varepsilon M(d(x)))}. \]

Let \( \varepsilon \to 0 \) and then we conclude (1.6).

The proof of Theorem 1.1 is complete.
References

[1] C. Bandle, M. Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, J. Anal. Math. 58 (1992) 9-24.

[2] C. Bandle, M. Marcus, Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (1995) 155-171.

[3] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, Math. Ann. 77 (1916) 173-212.

[4] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations III. Functions of the eigenvalues of the Hessian, Acta Math. 155 (1985) 261-301.

[5] F. Cîrstea, V. Rădulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorption, C. R. Math. Acad. Sci. Paris 335 (2002) 447-452.

[6] F. Cîrstea, V. Rădulescu, Asymptotics for the blow-up boundary solution of the logistic equation with absorption, C. R. Math. Acad. Sci. Paris 336 (2003) 231-236.

[7] F.C. Cîrstea, C. Trombetti, On the Monge-Ampère equation with boundary blow-up: existence, uniqueness and asymptotics, Calc. Var. Partial Differential Equations 31 (2008) 167-186.

[8] A. Colesanti, P. Salani, E. Francini, Convexity and asymptotic estimates for large solutions of Hessian equations, Differential Integral Equations 13 (2000) 1459-1472.

[9] J. García-Melián, Nondegeneracy and uniqueness for boundary blow-up elliptic problems, J. Differential Equations 223 (2006) 208-227.
[10] L. Gårding, An inequality for hyperbolic polynomials, J. Math. Mech. 8 (1959) 957-965.

[11] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order. third ed., Springer, Berlin, 2001.

[12] Gladiali F, Porru G. Estimates for explosive solutions to p-Laplace equations, Progress in Partial Differential Equations (Pont-à-Mousson 1997), Vol.1, Pitman Res. Notes Math. Series, Longman, 383 (1998) 117-127.

[13] B. Guan, H.Y. Jian, The Monge-Ampère equation with infinite boundary value, Pacific J. Math. 216 (2004) 77-94.

[14] Y. Huang, Boundary asymptotical behavior of large solutions to Hessian equations, Pacific J. Math. 244 (2010) 85-98.

[15] N. Ivochkina, Second order equations with d-elliptic operators. Tr. Mat. Inst. Steklova 147 (1980) 40-56 (in Russian); English transl.: Proc. SteklovInst. Math. 147 (1981) 37-54.

[16] N. Ivochkina, N. Filimonenkova, On the backgrounds of the theory of m-Hessian equations, Comm. Pure. Appl. Anal. 12 (2013) 1687-1703.

[17] H. Jian, Hessian equations with infinite Dirichlet boundary value, Indiana Univ. Math. J. 55 (2006) 1045-1062.

[18] J.B. Keller, On solutions of \( \Delta u = f(u) \), Comm. Pure Appl. Math. 10 (1957) 503-510.

[19] A.C. Lazer, P.J. McKenna, On singular boundary value problems for the Monge-Ampère operator, J. Math. Anal. Appl. 197 (1996) 341-362.

[20] Y.Y. Li, Some existence results for fully nonlinear elliptic equations of Monge-Ampère type, Comm. Pure Appl. Math. 43 (1990) 233-271.

[21] C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in: Contributions to Analysis (A
Collection of Papers Dedicated to Lipman Bers), Academic Press, New York, 1974: 245-272.

[22] A. Mohammed, On the existence of solutions to the Monge-Ampère equation with infinite boundary values, Proc. Amer. Math. Soc. 135 (2007) 141-149.

[23] R. Osserman, On the inequality $\Delta u \geq f(u)$, Pacific J. Math. 7 (1957) 1641-1647.

[24] P. Salani, Boundary blow-up problems for Hessian equations, Manus. Math. 96 (1998) 281-294.

[25] N.S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995) 151-164.

[26] N.S. Trudinger, On new isoperimetric inequalities and symmetrization, J. Reine Angew. Math. 488 (1997) 203-220.

[27] J.I.E. Urbas, On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations, Indiana Univ. Math. J. 39 (1990) 355-382.

[28] Z.J. Zhang, Boundary behavior of large solutions to the Monge-Ampère equations with weights, J. Differential Equations 259 (2015) 2080-2100.