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Almost Paracontact Almost Paracomplex Riemannian Manifolds with a Pair of Associated Schouten–van Kampen Connections

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Abstract: Two correlated Schouten–van Kampen affine connections on an almost paracontact almost paracomplex Riemannian manifold are introduced and investigated. The considered manifolds are characterized by virtue of the presented non-symmetric connections. Curvature properties of the studied connections are obtained. A family of examples on a Lie group are given in confirmation of the obtained results.

Keywords: distribution; Schouten–van Kampen affine connection; almost paracontact almost paracomplex Riemannian manifold; paracontact distribution

MSC: 53C25; 53C07; 53C50; 53D15

1. Introduction

In [1], the concept of an almost paracontact structure on a smooth $(2n + 1)$-dimensional manifold is presented. In [2], the restricted almost product structure on a paracontact distribution is studied. On a manifold equipped with an almost paracontact structure, two kinds of compatible metrics can be considered—the induced transformations are isometry or anti-isometry on the paracontact distribution of the tangent space, respectively. In the first case, [3–5], the manifold has an almost paracontact Riemannian structure, while in the second case, [6,7], an almost paracontact metric manifold equipped with a semi-Riemannian metric of type $(n + 1, n)$ is considered.

The objects of our consideration are the almost paracontact almost paracomplex Riemannian manifolds. The restriction on the paracontact distribution of the introduced almost paracontact structure is an almost paracomplex structure. From [8], these manifolds are known as almost paracontact Riemannian manifolds of type $(n + 1, n)$. Moreover, they are classified in the cited paper. The investigation of the considered manifolds is continued in [9,10].

The Schouten–van Kampen connection preserves, by parallelism, a pair of complementary distributions on a smooth manifold equipped with an affine connection [11–13]. The author of [14] uses the considered connection and studies hyperdistributions in Riemannian manifolds. In [15,16], the Schouten–van Kampen connection is investigated and adapted to an almost (para)contact metric structure and an almost contact B-metric structure, respectively. The studied connection is generally not a natural connection on these manifolds.

In the present paper, we introduce and investigate two correlated Schouten–van Kampen connections which are associated to the pair of Levi-Civita connections of an almost
paracontact almost paracomplex Riemannian manifold, and adapted to its paracontact distribution. We characterize the classes of considered manifolds by means of the constructed non-symmetric connections, and we obtain some curvature properties.

The present paper is organized as follows: Following the present introductory Section 1, in Section 2, we present some definitions and facts about the almost paracontact almost paracomplex Riemannian manifold. Section 3 is devoted to the study of remarkable metric connections regarding the paracontact distribution on the considered manifolds—the Schouten–van Kampen connections associated to the Levi-Civita connections. In Sections 4 and 5, we obtain some torsion and curvature properties, respectively, of the considered connections. In the final Section 6, we consider a family of proper examples which confirm the statements proven in previous sections.

2. Almost Paracontact Almost Paracomplex Riemannian Manifolds

Let \((\mathcal{M}, \phi, \xi, \eta, g)\) be an almost paracontact almost paracomplex Riemannian manifold (abbr. apapR manifold). Therefore, \(\mathcal{M}\) is a smooth manifold of dimension \((2n + 1)\) \((n \in \mathbb{N})\), \(g\) is a compatible Riemannian metric, and \((\phi, \xi, \eta)\) is an almost paracontact structure. Then, the endomorphism \(\phi\) of the tangent bundle \(T\mathcal{M}\), the characteristic vector field \(\xi\), and its dual 1-form \(\eta\) satisfy the following identities:

\[
\begin{align*}
\phi^2 &= 0, \quad \phi^2 = I - \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \quad \text{tr}\phi = 0, \\
g(\phi x, \phi y) &= g(x, y) - \eta(x)\eta(y), \quad g(x, \xi) = \eta(x),
\end{align*}
\]

denoting the identity transformation on \(T\mathcal{M}\) by \(I\) [1,9]. Both here and further on, we shall denote by \(x, y, z,\) and \(w\) arbitrary vector fields from \(\mathfrak{X}(\mathcal{M})\) or vectors in \(T\mathcal{M}\) at a fixed point of \(\mathcal{M}\).

Almost paracontact almost paracomplex Riemannian manifolds are also known as almost paracontact Riemannian manifolds of type \((n, n)\), using the term introduced by Sasaki. These manifolds are classified in [8], where 11 basic classes \(F_1, F_2, \ldots, F_{11}\) are determined, taking into account the properties with respect to the structure of a \((0,3)\)-tensor \(F\). This tensor is defined by

\[
F(x, y, z) = g((\nabla x\phi)y, z)
\]

for the Levi-Civita connection \(\nabla\) of \(g\), and \(F\) has the following basic properties:

\[
\begin{align*}
F(x, y, z) &= F(x, z, y) = -F(x, \phi y, \phi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi), \\
(\nabla x\eta)y &= (\nabla x\xi)y = -F(x, \phi y, \xi).
\end{align*}
\]

The Lee forms associated with \(F\) are defined by:

\[
\theta = g^{ij}F(e_i, e_j, \cdot), \quad \theta^* = g^{ij}F(e_i, \phi e_j, \cdot), \quad \omega = F(\xi, \xi, \cdot),
\]

where \((g^{ij})\) is the inverse matrix of the matrix \((g_{ij})\) of \(g\) with respect to a basis \(\{\xi; e_i\}\) \((i = 1, 2, \ldots, 2n)\) of \(T\mathcal{M}\).

The following conditions define the basic classes, and for convenience, we replace some of the equalities given in [8] by their equivalent ones:

\[
\begin{align*}
F_1 : \quad & F(x, y, z) = \frac{1}{2n} \{g(\phi x, \phi y)\theta(\phi^2 z) + g(\phi x, \phi z)\theta(\phi^2 y) \\
& - g(x, \phi y)\theta(\phi z) - g(x, \phi z)\theta(\phi y)\}; \\
F_2 : \quad & F(\xi, y, z) = F(x, \xi, z) = 0, \quad \theta = 0; \\
F_3 : \quad & F(\xi, y, z) = F(x, \xi, z) = 0, \quad (\nabla_{\phi y}z)^{\xi} F(x, y, z) = 0; \\
F_4 : \quad & F(x, y, z) = \frac{1}{2n} \theta(\xi) \{g(\phi x, \phi y)\eta(z) + g(\phi x, \phi z)\eta(y)\}; \\
F_5 : \quad & F(x, y, z) = \frac{1}{2n} \theta^*(\xi) \{g(x, \phi y)\eta(z) + g(x, \phi z)\eta(y)\};
\end{align*}
\]
\[ \mathcal{F}_6 : \ F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \]
\[ F(x, y, \xi) = F(y, x, \xi), \ \theta = \theta^* = 0; \]
\[ \mathcal{F}_7 : \ F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \]
\[ F(x, y, \xi) = -F(y, x, \xi), \ F(\phi x, \phi y, \xi); \]
\[ \mathcal{F}_8 : \ F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \]
\[ F(x, y, \xi) = F(y, x, \xi) = F(\phi x, \phi y, \xi); \]
\[ \mathcal{F}_9 : \ F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \]
\[ F(x, y, \xi) = -F(y, x, \xi) = F(\phi x, \phi y, \xi); \]
\[ \mathcal{F}_{10} : \ F(x, y, z) = \eta(x)F(\xi, \phi y, \phi z); \]
\[ \mathcal{F}_{11} : \ F(x, y, z) = \eta(x)(\eta(y)\omega(z) + \eta(z)\omega(y)), \]

where \( \mathcal{S} \) is the cyclic sum by three arguments, \( x, y, z \).

It is said that a manifold \( (M, \phi, \xi, \eta, g) \) of the considered type belongs to the class \( \mathcal{F}_i \) \( (i \in \{1, 2, ..., 11\}) \) if the tensor \( F \) on \( M \) satisfies the corresponding conditions of \( \mathcal{F}_i \) given in (3).

The intersection of the basic classes is the special class \( \mathcal{F}_0 \), which is determined by the condition \( F = 0 \).

The metric \( \tilde{g} \), defined by \( \tilde{g}(x, y) = g(x, \phi y) + \eta(x)\eta(y), \) is the associated metric of \( g \) on \( (M, \phi, \xi, \eta, g) \). Then, \( \tilde{g} \) is compatible with \( (M, \phi, \xi, \eta) \) as \( g \) in (1), but \( \tilde{g} \) is indefinite of signature \( (n + 1, n) \).

The relations between the Lee forms and the divergences \( \text{div} \) and \( \text{div} \) regarding \( g \) and \( \tilde{g} \), respectively, follow directly from (2) and they have the form

\[ \theta(\xi) = -\text{div}(\eta), \quad \theta^*(\xi) = -\text{div}(\eta). \]

Taking into account (3), the covariant derivative of \( \xi \) with respect to \( \nabla \) is determined in each class as follows:

\[ \mathcal{F}_1 : \ \nabla \xi = 0; \quad \mathcal{F}_2 : \ \nabla \xi = 0; \quad \mathcal{F}_3 : \ \nabla \xi = 0; \]
\[ \mathcal{F}_4 : \ \nabla \xi = \frac{1}{2\lambda} \text{div}(\eta)\phi; \quad \mathcal{F}_5 : \ \nabla \xi = \frac{1}{2\lambda} \text{div}(\eta)\phi^2; \]
\[ \mathcal{F}_6 : \ g(\nabla x \xi, y) = g(\nabla \phi x \xi, \phi x) = g(\nabla \phi y \xi, \phi y), \quad \text{div}(\eta) = \text{div}(\eta) = 0; \]
\[ \mathcal{F}_7 : \ g(\nabla x \xi, y) = g(\nabla \phi y \xi, \phi x) = g(\nabla \phi y \xi, \phi y); \]
\[ \mathcal{F}_8 : \ g(\nabla x \xi, y) = g(\nabla x \xi, \phi x) = g(\nabla x \xi, \phi y); \]
\[ \mathcal{F}_9 : \ g(\nabla x \xi, y) = g(\nabla x \xi, \phi x) = g(\nabla x \xi, \phi y); \]
\[ \mathcal{F}_{10} : \ \nabla \xi = 0; \quad \mathcal{F}_{11} : \ \nabla \xi = \eta \otimes \phi \omega^\sharp, \]

where \( \sharp \) is the musical isomorphism of \( T^*M \) in \( TM \) with respect to \( \tilde{g} \).

By the Koszul equality for \( \tilde{g} \) and its Levi-Civita connection \( \tilde{\nabla} \), using (1) and (2), we obtain the relation between \( \tilde{F} \) and \( \tilde{F}(x, y, z) = \tilde{g}(\tilde{\nabla} \phi y \phi z, y, z) \), as well as the expression of the potential \( \Phi(x, y, z) = \nabla x y - \nabla x y \) in terms of \( \tilde{F} \) as follows:

\[ 2\tilde{F}(x, y, z) = F(\phi y, z, x) - F(y, \phi z, x) + F(\phi z, \phi y, x) - F(z, \phi y, x) \]
\[ + \eta(x)\{F(y, z, \xi) - F(\phi z, \phi y, \xi) + z, \phi y, \xi) - F(\phi y, \phi z, \xi)\} \]
\[ + \eta(y)\{F(z, x, \xi) - F(\phi z, \phi x, \xi) + F(x, \phi z, \xi)\} \]
\[ + \eta(z)\{F(x, y, \xi) - F(\phi y, \phi x, \xi) + F(x, \phi y, \xi)\}, \]

\[ 2\Phi(x, y, z) = F(x, y, \phi z) + F(y, x, \phi z) - F(\phi z, x, y) \]
\[ - \eta(x)\{F(y, z, \xi) - F(\phi y, \phi z, \xi)\} \]
\[ - \eta(y)\{F(z, x, \xi) - F(\phi z, \phi x, \xi)\} \]
\[ - \eta(z)\{F(x, y, \xi) - F(\phi y, \phi y, \xi) + F(x, \phi y, \xi) - \omega(\phi y)\eta(y) \]
\[ - F(x, y, \xi) + F(y, \phi x, \xi) - \omega(\phi y)\eta(x)\}, \]

where \( \Phi(x, y, z) = g(\Phi(x, y, z), z) \).
Obviously, $\mathcal{F}_0$ is determined by any of the conditions: $F = 0$, $\tilde{F} = 0$, $\Phi = 0$, and $\nabla = \tilde{\nabla}$.

3. Pair of Associated Schouten–van Kampen Connections

Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be an apapR manifold. Using the structure $(\xi, \eta)$ on $\mathcal{M}$, the following two distributions in $T\mathcal{M}$ are determined:

$$H = \ker(\eta), \quad V = \text{span}(\xi),$$

known as horizontal and vertical ones. They are mutually complementary in $T\mathcal{M}$ and orthogonal with respect to $g$ and $\tilde{g}$, that is, $H \oplus V = T\mathcal{M}$ and $H \perp V$; moreover, $H$ is also known as the paracontact distribution.

Bearing in mind $\phi^2 = I - \eta \otimes \xi$ from (1), we consider the corresponding projectors $h : T\mathcal{M} \to H$ and $\nu : T\mathcal{M} \to V$, defined by

$$x^h = \phi^2 x, \quad x^\nu = \eta(x)\xi. \quad (8)$$

3.1. The Schouten–van Kampen Connections Associated to the Levi-Civita Connections

The Schouten–van Kampen connections (abbr. SvK) $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$ for $\nabla$ and $\tilde{\nabla}$, respectively, and adapted to $(H, V)$ on $(\mathcal{M}, \phi, \xi, \eta, g)$ are defined by [12,13]

$$\nabla^\parallel y = (\nabla_x y^h)^h + (\nabla_x y^\nu)^\nu,$$

$$\tilde{\nabla}^\parallel y = (\tilde{\nabla}_x y^h)^h + (\tilde{\nabla}_x y^\nu)^\nu. \quad (9)$$

Hence, (9) implies the parallelism of $H$ and $V$ with respect to $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$. Taking into account (8), we express the formulae of $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$ in terms of $\nabla$ and $\tilde{\nabla}$, respectively, as follows [14]:

$$\nabla^\parallel x = \nabla_x - \eta(y)\nabla_x \xi + (\nabla_x \eta)(y) \xi, \quad (10)$$

$$\tilde{\nabla}^\parallel x = \tilde{\nabla}_x - \eta(y)\tilde{\nabla}_x \xi + (\tilde{\nabla}_x \eta)(y) \xi. \quad (11)$$

Obviously, $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$ exist on $(\mathcal{M}, \phi, \xi, \eta, g)$ in each class regarding $F$.

Let us consider the potentials $Q^\parallel$ of $\nabla^\parallel$ with respect to $\nabla$, $\tilde{Q}^\parallel$ of $\tilde{\nabla}^\parallel$ with respect to $\tilde{\nabla}$ and the torsions $T^\parallel$ of $\nabla^\parallel$, $\tilde{T}^\parallel$ of $\tilde{\nabla}^\parallel$ defined by

$$Q^\parallel(x,y) = \nabla^\parallel x - \nabla^\parallel y, \quad \tilde{Q}^\parallel(x,y) = \tilde{\nabla}^\parallel x - \tilde{\nabla}^\parallel y,$$

$$T^\parallel(x,y) = \nabla^\parallel x - \nabla^\parallel y - [x,y], \quad \tilde{T}^\parallel(x,y) = \tilde{\nabla}^\parallel x - \tilde{\nabla}^\parallel y - [x,y].$$

Then, we obtain the following expressions:

$$Q^\parallel(x,y) = -\eta(y)\nabla_x \xi + (\nabla_x \eta)(y) \xi, \quad (12)$$

$$\tilde{Q}^\parallel(x,y) = -\eta(y)\tilde{\nabla}_x \xi + (\tilde{\nabla}_x \eta)(y) \xi, \quad (13)$$

$$T^\parallel(x,y) = \eta(x)\nabla_y \xi - \eta(y)\nabla_x \xi + d\eta(x,y) \xi, \quad (14)$$

$$\tilde{T}^\parallel(x,y) = \eta(x)\tilde{\nabla}_y \xi - \eta(y)\tilde{\nabla}_x \xi + d\eta(x,y) \xi. \quad (15)$$

**Theorem 1.** The unique affine connections having torsions of forms (14) and (15) and preserving the structure $(\xi, \eta, g)$ are the SvK connections $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$, respectively.

**Proof.** Using (10), we directly obtain $\nabla^\parallel \xi = \nabla^\parallel \eta = \nabla^\parallel g = 0$, that is, $\xi$, $\eta$, and $g$ are parallel with respect to $\nabla^\parallel$. Since $\nabla^\parallel$ is a metric connection, it is completely determined.
by its torsion $T^\parallel$. The isomorphism between the spaces $\{T\}$ and $\{Q\}$ is known, and this bijection is given by [17]

\begin{align}
T(x, y, z) &= Q(x, y, z) - Q(y, x, z), \\
2Q(x, y, z) &= T(x, y, z) - T(y, z, x) + T(z, x, y).
\end{align}

We verify directly that the potential $Q^\parallel$ and the torsion $T^\parallel$ of $\nabla^\parallel$, determined by (12) and (14), respectively, satisfy the latter equalities. This completes the proof for $\nabla^\parallel$. Similarly, we prove for $\tilde{\nabla}^\parallel$. □

We state the following:

**Theorem 2.** Let $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$ be the SvK connections on $(\mathcal{M}, \phi, \xi, \eta, g)$. Then, the following statements are equivalent:

1. $\nabla^\parallel$ coincides with $\nabla$;
2. $\nabla^\parallel$ coincides with $\tilde{\nabla}$;
3. $(\mathcal{M}, \phi, \xi, \eta, g)$ belongs to $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$.

**Proof.** According to (10), $\nabla^\parallel$ coincides with $\nabla$ if, and only if $\nabla \xi^x = 0$ for any $x$. Having in mind (5), this vanishing holds only in the class $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_{10}$. Similarly, the connection $\tilde{\nabla}^\parallel$ coincides with $\tilde{\nabla}$ if, and only if $\tilde{\nabla} \xi^x$ vanishes, which holds if, and only if $\tilde{\nabla}$ satisfies the conditions of $\tilde{\nabla}$ in (3) for $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_9$, taking into account (6). Bearing in mind the two direct sums from above, we complete the proof. □

Therefore, we have the following:

**Corollary 1.** Let $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$ be the SvK connections on $(\mathcal{M}, \phi, \xi, \eta, g)$. A sufficient condition for coincidence of the four connections $\nabla^\parallel$, $\tilde{\nabla}^\parallel$, $\nabla$ and $\tilde{\nabla}$ is $\nabla^\parallel \equiv \nabla$ or $\tilde{\nabla}^\parallel \equiv \tilde{\nabla}$.

Obviously, the coincidence of the four connections $\nabla^\parallel$, $\tilde{\nabla}^\parallel$, $\nabla$, and $\tilde{\nabla}$ is equivalent to the condition $(\mathcal{M}, \phi, \xi, \eta, g)$ to be from $\mathcal{F}_0$.

We establish the truthfulness of the following:

**Theorem 3.** The necessary and sufficient condition of the SvK connections $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$ to coincide is $(\mathcal{M}, \phi, \xi, \eta, g)$, to be from the class $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}$.

**Proof.** We obtain the following relation between $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$, using (11) and $\Phi$,

\[ \nabla^\parallel_2 y = \nabla^\parallel_2 y + \Phi(x, y) - \eta(\Phi(x, y))\xi \eta(\Phi(x, y)). \]

The two connections $\tilde{\nabla}^\parallel$ and $\nabla^\parallel$ coincide if, and only if the following condition is valid: $\Phi(x, y) = \eta(\Phi(x, y))\xi + \eta(y)\Phi(x, \xi)$. Bearing in mind that $\Phi$ is symmetric, the latter equality is equivalent to $\Phi(x, y) = \eta(\Phi(x, y))\xi + \eta(x)\eta(y)\Phi(\xi, \eta)$. By virtue of (7) and the latter expression of $\Phi$, we obtain

\[ F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \]

which determines the class $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}$. □

### 3.2. The Conditions for Natural Connections $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$

It is known that a connection is called natural for a structure $(\phi, \xi, \eta, g)$ when all of the structure tensors are parallel with respect to this connection. Obviously, if a connection is parallel for $(\phi, \xi, \eta, g)$, then $\tilde{\phi}$ is also parallel with respect to it. According to Theorem 1, $\nabla^\parallel$ preserves $(\xi, \eta, g)$. However, $\nabla^\parallel$ is not a natural connection for the studied structures, because $\nabla^\parallel \phi$ (therefore $\nabla^\parallel \tilde{\phi}$, too) is generally not zero.
Theorem 4. The necessary and sufficient condition for the SoK connection $\nabla^h$ to be natural is $(\mathcal{M}, \phi, \xi, \eta, g)$, to be from the class $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}$.

Proof. Bearing in mind (10), we obtain the covariant derivative of $\phi$ with respect to $\nabla^h$ as follows:

$$\left(\nabla_x^h\phi\right)y = (\nabla_x \phi) y + \eta(y) \phi \nabla_x \xi - \eta(\nabla_x \phi y) \xi. \quad (20)$$

Then, $\nabla^h \phi$ vanishes if, and only if $(\nabla_x \phi) y = -\eta(y) \phi \nabla_x \xi + \eta(\nabla_x \phi y) \xi$ holds, which is equivalent to (19). Bearing in mind the proof of Theorem 3, we find that the class with natural connection $\nabla^h$ is the class in the statement. \(\square\)

Taking into account Theorems 2 and 4, we obtain the following:

Corollary 2. The class of all apapR manifolds can be decomposed orthogonally to the subclass of the manifolds with coinciding connections $\nabla^h$ and $\nabla$ and the subclass of manifolds with natural $\nabla^h$.

Taking into account (18), we get the following relation between the covariant derivatives of $\phi$ with respect to $\nabla^h$ and $\nabla^\parallel$:

$$\left(\nabla_x^\parallel \phi\right)y = \left(\nabla_x^h \phi\right)y + \Phi(x, \phi y) - \Phi(x, \xi) + \eta(y) \phi \Phi(x, \xi) - \eta(\Phi(x, \phi y)) \xi. \quad (21)$$

Therefore, we have that $\nabla^\parallel \phi$ and $\nabla^h \phi$ coincide if, and only if

$$\Phi(x, \phi^2 y, \phi^2 z) = -\Phi(x, \phi y, \phi z)$$

holds, which is fulfilled only when $(\mathcal{M}, \phi, \xi, \eta, g)$ is in $\mathcal{F}_3 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_11$, and therefore we prove the following:

Theorem 5. The necessary and sufficient condition of the covariant derivatives of $\phi$ with respect to the SoK connections $\nabla^h$ and $\nabla^\parallel$ to coincide is $(\mathcal{M}, \phi, \xi, \eta, g)$, to be from the class $\mathcal{F}_3 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_{11}$.

Bearing in mind (7), (20), and (21), we obtain that $\nabla^\parallel \phi = 0$ is equivalent to

$$F(\phi y, \phi z, x) + F(\phi^2 y, \phi^2 z, x) - F(\phi z, \phi y, x) - F(\phi^2 z, \phi^2 y, x) = 0.$$  

Then, the latter equality and (3) imply the truthfulness of the following:

Theorem 6. The necessary and sufficient condition of the SoK connection $\nabla^h$ to be natural is $(\mathcal{M}, \phi, \xi, \eta, g)$, to be from the class $\mathcal{F}_3 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_{11}$.

Combining Theorems 4–6, we get the validity of the following:

Theorem 7. The necessary and sufficient condition of the SoK connections $\nabla^h$ and $\nabla^\parallel$ to be natural is $(\mathcal{M}, \phi, \xi, \eta, g)$, to be from the class $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_{11}$.

4. Torsion Properties of the Pair of Connections $\nabla^h$ and $\nabla^\parallel$

The shape operator $S : \mathcal{H} \rightarrow \mathcal{H}$ for $g$ is defined as usual by $S(x) = -\nabla_x \xi$. Then, using (12), (14), (16), (17), we have that the properties of $T^h$, $Q^h$ and $S$ are related.

Horizontal and vertical components of $Q^h$ and $T^h$ given in (12) and (14), respectively, are the following:

$$Q^h = S \otimes \eta, \quad T^h = -\eta \wedge S, \quad Q^\parallel = -S^h \otimes \xi,$$

$$T^\parallel = -2\text{Alt}(S^h) \otimes \xi, \quad (22)$$

where $S^h(x, y) = g(S(x), y)$, that is, $S^h = -\nabla \eta$, whereas $\wedge$ and $\text{Alt}$ denote the exterior product and the alternation, respectively.
Using the vertical components of \( Q^\parallel \) and \( T^\parallel \) from (22), we immediately obtain the following.

**Theorem 8.** The following properties are equivalent:
1. \( \nabla \eta \) is symmetric
2. \( \eta \) is closed, that is, \( d\eta = 0 \)
3. \( S \) is self-adjoint regarding \( g \)
4. \( S^\flat \) is symmetric
5. \( Q^\parallel^v \) is symmetric
6. \( T^\parallel^v \) vanishes
7. \( (M, \phi, \zeta, \eta, g) \in F_1 \oplus F_2 \oplus F_3 \oplus F_4 \oplus F_5 \oplus F_6 \oplus F_9 \oplus F_{10} \).

**Theorem 9.** The following properties are equivalent:
1. \( \xi \) is Killing with respect to \( g \), that is, \( \mathcal{L}_\xi g = 0 \)
2. \( \nabla \eta \) is skew-symmetric
3. \( S \) is anti-self-adjoint regarding \( g \)
4. \( S^\flat \) is skew-symmetric
5. \( Q^\parallel^v \) is skew-symmetric
6. \( (M, \phi, \zeta, \eta, g) \in F_1 \oplus F_2 \oplus F_3 \oplus F_7 \oplus F_8 \oplus F_{10} \).

**Theorem 10.** The following properties are equivalent:
1. \( \nabla \xi = 0 \)
2. \( \nabla \eta = 0 \)
3. \( d\eta = \mathcal{L}_\xi g = 0 \)
4. \( \nabla^\parallel = \nabla \)
5. \( S = 0 \)
6. \( S^\flat = 0 \)
7. \( (M, \phi, \zeta, \eta, g) \in F_1 \oplus F_2 \oplus F_3 \oplus F_10 \).

In the same manner, we obtain similar linear relations between the torsion \( \tilde{T}^\parallel \) and the potential \( \tilde{Q}^\parallel \) of \( \tilde{\nabla}^\parallel \), as well as the shape operator \( \tilde{S} \) for \( \tilde{\nabla} \).

Now, taking into account (13), (15), and \( \tilde{S}(x) = -\nabla_x \xi \), we express the horizontal and vertical components of \( \tilde{Q}^\parallel \) and \( \tilde{T} \) of \( \tilde{\nabla}^\parallel \) as follows:

\[
\begin{align*}
\tilde{Q}^\parallel^h &= \tilde{S} \otimes \eta, & \quad \tilde{Q}^\parallel^v &= -\tilde{S}^\flat \otimes \xi, \\
\tilde{T}^\parallel^h &= -\eta \wedge \tilde{S}, & \quad \tilde{T}^\parallel^v &= -2\text{Alt}(\tilde{S}^\flat) \otimes \xi,
\end{align*}
\]

(23)

where we denote \( \tilde{S}^\flat(x, y) = \tilde{g}(\tilde{S}(x), y) \).

Bearing in mind that \( (\nabla_x \eta)(y) = (\nabla_y \eta)(y) - \eta(\Phi(x, y)) \) and \( \nabla_x \xi = \nabla_x \Phi + \Phi(x, \xi) \), we get

\[
\begin{align*}
\tilde{S}(x) &= S(x) - \Phi(x, \xi), & \quad \tilde{S}^\flat(x, y) &= S^\flat(x, \phi y) - \Phi(\xi, x, \phi y).
\end{align*}
\]

(24)

Moreover, (12), (14), (22), (13), (15), and (23) imply the following relations

\[
\begin{align*}
\tilde{Q}^\parallel^h &= Q^\parallel^h - (\xi \otimes \Phi) \otimes \eta, & \quad \tilde{Q}^\parallel^v &= Q^\parallel^v - (\eta \circ \Phi) \otimes \xi, \\
\tilde{T}^\parallel^h &= T^\parallel^h + \eta \wedge (\xi \otimes \Phi), & \quad \tilde{T}^\parallel^v &= T^\parallel^v,
\end{align*}
\]

(25)

where \( \circ \) denotes the interior product.
Subsequently, (24) and (25) imply the following formulae:
\[
\begin{align*}
\tilde{Q}^\parallel &= Q^\parallel + (\tilde{\mathcal{S}} - \mathcal{S}) \odot \eta - (\tilde{\mathcal{S}} - \mathcal{S}) \odot \xi, \\
\tilde{T}^\parallel &= T^\parallel + (\tilde{\mathcal{S}} - \mathcal{S}) \wedge \eta;
\end{align*}
\]
\[
\begin{align*}
\tilde{Q}^\parallel h &= Q^\parallel h + (\tilde{\mathcal{S}} - \mathcal{S}) \odot \eta, \\
\tilde{T}^\parallel h &= T^\parallel h + (\tilde{\mathcal{S}} - \mathcal{S}) \wedge \eta, \\
\tilde{Q}^\parallel v &= Q^\parallel v - (\tilde{\mathcal{S}} - \mathcal{S}) \odot \xi, \\
\tilde{T}^\parallel v &= T^\parallel v.
\end{align*}
\]

By virtue of the obtained results, we get the truthfulness of the following:

**Theorem 11.** The following properties are equivalent:
1. \(\tilde{\nabla} \eta\) is symmetric
2. \(\eta\) is closed
3. \(\tilde{S}\) is self-adjoint regarding \(\tilde{g}\)
4. \(\tilde{\mathcal{S}}\) is symmetric
5. \(\tilde{Q}^\parallel v\) is symmetric
6. \(\tilde{T}^\parallel v\) vanishes
7. \((\mathcal{M}, \phi, \xi, \eta, g)\) \(\in \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_9 \oplus \mathcal{F}_9\).

**Theorem 12.** The following properties are equivalent:
1. \(\xi\) is Killing with respect to \(\tilde{g}\), that is, \(\Sigma_2 \tilde{g} = 0\)
2. \(\tilde{\nabla} \eta\) is skew-symmetric
3. \(\tilde{S}\) is anti-self-adjoint regarding \(\tilde{g}\)
4. \(\tilde{\mathcal{S}}\) is skew-symmetric
5. \(\tilde{Q}^\parallel v\) is skew-symmetric
6. \((\mathcal{M}, \phi, \xi, \eta, g)\) \(\in \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_7 \oplus \mathcal{F}_9\).

**Theorem 13.** The following properties are equivalent:
1. \(\tilde{\nabla} \xi = 0\)
2. \(\tilde{\nabla} \eta = 0\)
3. \(d\eta = \Sigma_2 \tilde{g} = 0\)
4. \(\tilde{\nabla}^\parallel = \tilde{\nabla}\)
5. \(\tilde{S} = 0\)
6. \(\tilde{\mathcal{S}} = 0\)
7. \((\mathcal{M}, \phi, \xi, \eta, g)\) \(\in \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_9\).

5. **Curvature Properties of the Pair of Connections** \(\tilde{\nabla}^\parallel\) and \(\tilde{\nabla}^\parallel\)

Let the curvature tensor of \(\tilde{\nabla}\) be denoted by \(R\), that is, \(R = [\tilde{\nabla} , \tilde{\nabla} ] - \tilde{\nabla} [ , ]\) and \(R(x, y, z, w) = g(R(x, y)z, w)\). The traces \(\rho(y, z) = g^{ij}R(e_i, y, z, e_j)\) and \(\tau = g^{ij}\rho(e_i, e_j)\) define the Ricci tensor \(\rho\) and the scalar curvature \(\tau\), respectively.

The sectional curvature \(k(a; p)\) of an arbitrary non-degenerate 2-plane \(a\) in \(T_p\mathcal{M}\) with respect to the metric \(g\) is defined by \(k(a; p) = \frac{R(x, y, z, w)}{\pi_1(x, y, z, w)}\), where \(\{x, y, z\}\) is an arbitrary basis of \(a\) and \(\pi_1(x, y, z, w) = g(y, z)g(x, w) - g(x, z)g(y, w)\). Let us remark that \(\pi_1(x, y, y, x)\) does not vanish for any non-degenerate 2-plane \(a\) with respect to a Riemannian metric \(g\). In the case of the pseudo-Riemannian metric \(\tilde{g}\), the respective sectional curvature \(\tilde{k}\) is defined in a similar way, that is, \(\tilde{k}(a; p) = \frac{\tilde{R}(x, y, z, w)}{\tilde{\pi}_1(x, y, z, w)}\), and the tensor \(\tilde{\pi}_1(x, y, y, x) = \tilde{g}(x, y)^2\) also does not vanish because of the non-degeneracy of the 2-plane \(a\) with respect to \(\tilde{g}\).

In the set of 2-planes \(a\), we distinguish three special types with respect to the structure: a \(\xi\)-section \((\xi \in a)\), a \(\phi\)-holomorphic section \((a = \phi a)\), and a \(\phi\)-totally real section \((a \perp \phi a\) regarding \(g\)). The third kind exists for \(\dim \mathcal{M} \geq 5\).
Let $R^\parallel$, $\rho^\parallel$, $\tau^\parallel$, $k^\parallel$, $\tilde{R}$, $\tilde{\rho}$, $\tilde{\tau}$, $\tilde{k}$ and $\tilde{R}^\parallel$, $\tilde{\rho}^\parallel$, $\tilde{\tau}^\parallel$, $\tilde{k}^\parallel$ denote the corresponding quantities for the connections $\nabla^\parallel$, $\tilde{\nabla}$ and $\tilde{\nabla}^\parallel$, respectively.

**Theorem 14.** The curvature tensors $R^\parallel$ and $R$ ($\tilde{R}^\parallel$ and $\tilde{R}$, respectively) satisfy the following relations

$$
\begin{align*}
R^\parallel(x, y, z, w) &= R(x, y, \phi^2 z, \phi^2 w) + \tau_1(S(x), S(y), z, w), \\
\tilde{R}^\parallel(x, y, z, w) &= \tilde{R}(x, y, \phi^2 z, \phi^2 w) + \tilde{\tau}_1(\tilde{S}(x), \tilde{S}(y), z, w),
\end{align*}
$$

(26)

**Proof.** Using (10) and $g(\nabla x \xi, \xi) = 0$ for arbitrary $x$ and $\nabla^\parallel \xi = 0$, we get

$$
R^\parallel(x, y)z = R(x, y)z - \eta(z)R(x, y)\xi - \eta(R(x, y)z)\xi
$$

$$
= g(\nabla_x \xi, z)\nabla^\parallel \xi + g(\nabla^\parallel \xi, z)\nabla_x \xi,
$$

which implies the first equality in (26).

In a similar way, we obtain the second equality in (26). \(\square\)

Let $\tilde{\text{tr}}$ denote the trace with respect to $\tilde{g}$. We get the following:

**Corollary 3.** The Ricci tensors $\rho^\parallel$ and $\rho$ ($\tilde{\rho}^\parallel$ and $\tilde{\rho}$, respectively) satisfy the following relations:

$$
\begin{align*}
\rho^\parallel(y, z) &= \rho(y, z) - \eta(z)\rho(y, \xi) - R(\xi, y, z, \xi) \\
&\quad - g(S^2(y), z) + \text{tr}(S)g(S(y), z), \\
\tilde{\rho}^\parallel(y, z) &= \tilde{\rho}(y, z) - \eta(z)\tilde{\rho}(y, \xi) - \tilde{R}(\xi, y, z, \xi) \\
&\quad - \tilde{g}(S^2(y), z) + \tilde{\text{tr}}(\tilde{S})\tilde{g}(\tilde{S}(y), z).
\end{align*}
$$

(27)

Using (7) and (2), we get that $g^{ij}\Phi(\xi, e_i, e_j) = 0$. Therefore, bearing in mind (4) and the definitions of $S$ and $\tilde{S}$, we have that $\text{tr}(S) = \tilde{\text{tr}}(\tilde{S}) = -\text{div}(\eta)$. The definition of the shape operator implies $R(x, y)\xi = -(\nabla_x S)y + (\nabla_y S)x$, which, together with $S(\xi) = -\nabla^\parallel \xi = -\phi^2\xi$, leads to the following expression:

$$
R(\xi, y, z, \xi) = g\left((\nabla_x S)y - (\nabla_y S)\xi, z\right) = g\left((\nabla_x S)y - \nabla_y (S(\xi)) - S(S(y)), z\right).
$$

(28)

After that, the relations

$$
\text{div}(\omega \circ \phi) = g^{ij}(\nabla \omega \circ \phi)e_j = g^{ij}\tilde{g}\left(\nabla \omega \circ \phi^2, e_j\right) = -\text{div}(S(\xi)),
$$

and the trace of (28) imply

$$
\rho(\xi, \xi) = \text{tr}(\nabla^\parallel S) - \text{div}(S(\xi)) - \text{tr}(S^2).
$$

(29)

Similarly, we get

$$
\tilde{\rho}(\xi, \xi) = \tilde{\text{tr}}(\tilde{\nabla}^\parallel \tilde{S}) - \tilde{\text{div}}(\tilde{S}(\xi)) - \tilde{\text{tr}}(S^2).
$$

(30)

Taking into account (27), (29), and (30), we obtain the following:

**Corollary 4.** The scalar curvatures $\tau^\parallel$ and $\tau$ ($\tilde{\tau}^\parallel$ and $\tilde{\tau}$, respectively) satisfy the following relations

$$
\begin{align*}
\tau^\parallel &= \tau - 2\rho(\xi, \xi) - \text{tr}(S^2), \\
\tilde{\tau}^\parallel &= \tilde{\tau} - 2\tilde{\rho}(\xi, \xi) - \tilde{\text{tr}}(S^2).
\end{align*}
$$

Moreover, using Theorem 14, we get the following:
Corollary 5. The sectional curvatures $k^\parallel(a; p)$ and $k(a; p)$ ($\tilde{k}^\parallel(a; p)$ and $\tilde{k}(a; p)$, respectively) satisfy the following relations

$$
k^\parallel(a; p) = k(a; p) + \frac{\pi_1(S(x), S(y), y, x) - \eta(x)R(x, y, y, z) - \eta(y)R(x, y, z, x)}{\pi_1(x, y, y, x)},
$$

$$
\tilde{k}^\parallel(a; p) = \tilde{k}(a; p) + \frac{\tilde{\pi}_1(S(x), S(y), y, x) - \tilde{\eta}(x)\tilde{R}(x, y, y, z) - \tilde{\eta}(y)\tilde{R}(x, y, z, x)}{\tilde{\pi}_1(x, y, y, x)}.
$$

Let $\alpha_x$ be a $z$-section at $p \in M$ with a basis $\{x, z\}$. Then, using (31), $g(S(x), z) = 0$ and $\tilde{g}(S(x), z) = 0$ for arbitrary $x$, we get that

$$
k^\parallel(\alpha_x; p) = 0, \quad \tilde{k}^\parallel(\alpha_x; p) = 0.
$$

Let $\phi_x$ be a $\phi$-section at $p \in M$ with a basis $\{x, y\}$. Then, using (31) and $\eta(x) = \eta(y) = 0$, we obtain that

$$
k^\parallel(\phi_x; p) = k(\phi_x; p) + \frac{\pi_1(S(x), S(y), y, x)}{\pi_1(x, y, y, x)},
$$

$$
\tilde{k}^\parallel(\phi_x; p) = \tilde{k}(\phi_x; p) + \frac{\tilde{\pi}_1(S(x), S(y), y, x)}{\tilde{\pi}_1(x, y, y, x)}.
$$

Let $\alpha_\perp$ denote a $\phi$-totally real section which is orthogonal to $z$ and $\{x, y\}$ is its basis. Then, using (31) and $\eta(x) = \eta(y) = 0$, we get that

$$
k^\parallel(\alpha_\perp; p) = k(\alpha_\perp; p) + \frac{\pi_1(S(x), S(y), y, x)}{\pi_1(x, y, y, x)},
$$

$$
\tilde{k}^\parallel(\alpha_\perp; p) = \tilde{k}(\alpha_\perp; p) + \frac{\tilde{\pi}_1(S(x), S(y), y, x)}{\tilde{\pi}_1(x, y, y, x)}.
$$

Let us remark that, when $a$ is a $\phi$-totally real section which is non-orthogonal to $z$ with respect to $g$ or $\tilde{g}$, then (31) is valid.

6. A Family of Lie Groups as Manifolds of the Studied Type

Let us consider as an example the $(2n + 1)$-dimensional apapR manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ given in [9]. Then, the Lie algebra of a real connected Lie group $\mathcal{L}$ is defined by

$$
[E_0, E_i] = -a_i E_i - a_{n+i} E_{n+i}, \quad [E_0, E_{n+i}] = -a_{n+i} E_i + a_i E_{n+i},
$$

where $a_1, \ldots, a_{2n}$ are real constants and $[E_j, E_k]$ vanishes in the other cases for a global basis $\{E_0, E_1, \ldots, E_{2n}\}$ of left invariant vector fields on $\mathcal{L}$. The apapR structure $(\phi, \xi, \eta, g)$ is determined as follows:

$$
\phi E_0 = 0, \quad \phi E_i = E_{n+i}, \quad \phi E_{n+i} = E_i,
$$

$$
\xi = E_0, \quad \eta(E_0) = 1, \quad \eta(E_i) = \eta(E_{n+i}) = 0,
$$

$$
g(E_0, E_0) = g(E_i, E_i) = g(E_{n+i}, E_{n+i}) = 1, \quad g(E_0, E_i) = g(E_{n+i}, E_i) = 0,
$$

where $i \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots, 2n\}, j \neq k$.

The constructed manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ is a $(2n + 1)$-dimensional apapR manifold. In [9], the case of dimension 3, that is, for $n = 1$, is considered. It is proved that $(\mathcal{L}, \phi, \xi, \eta, g)$ belongs to $F_2 \oplus F_9$ ($a_1 \neq 0, a_2 \neq 0$).

Using (10) and (11), we obtain that all components $\nabla^\parallel E_j$ and $\tilde{\nabla}^\parallel E_j$ vanish. That means the pair of associated SvK connections $\nabla^\parallel$ and $\tilde{\nabla}^\parallel$ coincide with the so-called
Weitzenböck connection, the connection of the parallelization. Therefore, $\nabla_\parallel$ and $\tilde{\nabla}_\parallel$ have vanishing curvature, but generally non-vanishing torsion, because it equals to $-\left[\cdot, \cdot\right]$. A quick review of the statements proved in previous sections shows that the example in this section confirms them.

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