Gauge symmetries in Ashtekar’s formulation of general relativity

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It might seem that a choice of a time coordinate in Hamiltonian formulations of general relativity breaks the full four-dimensional diffeomorphism covariance of the theory. This is not the case. We construct explicitly the complete set of gauge generators for Ashtekar’s formulation of canonical gravity. The requirement of projectability of the Legendre map from configuration-velocity space to phase space renders the symmetry group a gauge transformation group on configuration-velocity variables. Yet there is a sense in which the full four-dimensional diffeomorphism group survives. Symmetry generators serve as Hamiltonians on members of equivalence classes of solutions of Einstein’s equations and are thus intimately related to the so-called “problem of time” in an eventual quantum theory of gravity.

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I. INTRODUCTION AND QUANTUM MOTIVATION

The symmetry of four-dimensional spacetime diffeomorphisms lies at the conceptual core of Einstein’s theory of classical general relativity. Any viable quantum theory of gravity must recognize and preserve this symmetry, at least in an appropriate semi-classical regime. In a recent series of papers we have shown that the full spacetime diffeomorphism group symmetry is present in phase space (cotangent bundle) versions of conventional general relativity [1], in Einstein-Yang-Mills theory [2], and in both real triad [3] and complex Ashtekar formulations of gravitation [4]. We constructed both infinitesimal and finite canonical gauge symmetry generators in a phase space which includes the gauge variables of the models. Rigid time translation is, however, not a gauge symmetry in phase space, and we shall begin to explore some of the profound implications of this fact below in the context of the Ashtekar loop approach to quantum gravity.

Foremost among current conceptual and technical problems with theories of quantum gravity is the “problem of time”. Time evolution and spacetime diffeomorphisms are inextricably linked; every spacetime diffeomorphism generator is, in a sense to be explained below, a generator of time evolution. Consequently the symmetries we display in this paper have a direct bearing on several aspects of the problem of time. Let us first focus on the implications of a choice of time foliation in the classical theory. Some authors have suggested that since the canonical gravitational Hamiltonian vanishes there is no time evolution in quantum gravity; time is said to be “frozen” [5]. Since rigid time translation is after all a symmetry in the Lagrangian formalism, these authors observe that we should not be dismayed with this fact. Since a foliation translates into a gauge choice in the quantum theory we need to inquire into the relation between gauge choices. Hájíček has shown in some simple cases that the canonical quantization procedure leads to unitarily inequivalent representations [6]. Perhaps an even more disquieting consequence of a time foliation in the canonical approach is the resulting either real or apparent quantum spatio-temporal asymmetry. To date only spatial discreteness (in area and volume) has emerged in the loop approach [7]. And we have no prescription for transforming to a new time slice.

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Finally, we seem to have no means of predicting the outcome of what must surely be one of the most basic thought experiments in quantum gravity: what is the spacetime separation between two timelike separated events? We could imagine that these events could be characterized, for example, by ambient matter. Surely we would in general expect a range of outcomes. No such quantum fluctuations arise in the current canonical approaches to quantum gravity.

II. PROJECTABILITY OF SYMMETRY VARIATIONS UNDER THE LEGENDRE MAP

All of these difficulties stem from efforts at excising gauge variables from the quantum theory. But these gauge variables play a fundamental role in the classical Hamiltonian symmetry structure. We have investigated conditions that must be fulfilled by gauge symmetry transformations in the original Lagrangian formalism which can be mapped under the Legendre map to phase space. (More precisely we require that phase space gauge variation pullbacks be configuration-velocity space variations.)

The Lagrangian density for vacuum gravity, where the configuration variables are the metric

\[ (g_{\mu\nu}) = \begin{pmatrix} -N^2 + N^c N^d g_{cd} & g_{ac} N^c \\ g_{bd} N^d & g_{ab} \end{pmatrix}, \] (2.1)

does not depend on the time derivatives of the lapse \( N \) and shift functions \( N^a \) (\( \mu, \nu \) are spacetime indices; latin indices from the beginning of the alphabet are spatial indices). Therefore projectable variations may not depend on these time derivatives \( \frac{\delta}{\delta N^a} \).

The projectable infinitesimal spacetime diffeomorphisms in conventional gravity are of the form \( x^\mu = x^\mu - \epsilon^\mu \) where the descriptor \( \epsilon^\mu \) contains a compulsory lapse and shift dependence and \( \xi^\mu \) is an arbitrary function:

\[ \epsilon^\mu = \delta^\mu_0 \xi^0 + n^\mu \xi^0 . \] (2.2)

The normal \( n^\mu \) to the fixed time hypersurface is expressed as follows in terms of the lapse and shift: \( n^\mu = (N^{-1}, -N^{-1} N^a) \).

If gauge symmetries exist beyond those induced by diffeomorphisms one obtains additional projectability conditions. We have shown that in a real triad approach to gravity in which the configuration variables are taken to be a densitized inverse triad to \( t^a_i \), from which one forms the 3-metric \( g_{ab} = t^a_i t^b_j \delta_{ij} \). Over- and underrides label the integer weight of the density under spatial diffeomorphisms. Latin indices from the middle of the alphabet range from 1 to 3 and label the triad vectors. These indices are raised and lowered with the Kronecker delta. \( T^a_i \) is the inverse triad to \( t^a_i \). It turns out then that spacetime diffeomorphism-induced gauge variations are not by themselves projectable; an \( SO(3, R) \) triad rotation fixed by the arbitrary function \( \xi^0 \) must be added to them. The infinitesimal descriptor of the required rotation is \( \xi^i = (\Omega_i^\mu N^\mu \xi^0) \) where \( \Omega_i^\mu = (\epsilon^i_k \Omega_i^k) \) are the 3-dimensional Ricci rotation coefficients. (The infinitesimal \( SO(3, R) \) variation of a triad vector corresponding to a descriptor \( \xi^i \) is \( \delta_R(\xi^i) t^a_i = -\epsilon^i_k \xi^k t^a_i \). \( \Omega_i^k \) transforms as a spacetime connection: \( \delta_R(\xi^i) \Omega_i^\mu = -\epsilon^i_\mu + \epsilon^i_j \xi^j \Omega_i^\mu \).

These triad variables are in fact among the set of configuration variables in Ashtekar’s complex connection approach to general relativity. The connection is formed with the 4-dimensional Ricci rotation coefficients \( \Omega_i^\mu \):

\[ A_i^\mu = \epsilon^i_k \Omega_i^k + i \Omega_i^\mu . \] (2.3)

(Indices \( I, J \) range from 0 to 3 and are tetrad labels.) Since the action is independent of the time derivatives of the connection components \( A_i^\mu \), projectable symmetry variations must be independent of this time derivative. Thus it turns out once again that in order to be projectable, variations induced by infinitesimal spacetime diffeomorphisms, which already require the same lapse and shift dependence as above, must be accompanied in general by \( SO(3, C) \) triad rotations \( \Omega_i^\mu \). The functional form of the required infinitesimal descriptor, \( \xi^i = A_i^\mu N^\mu \xi^0 - i N^{-1} T^a_i N^a \xi^0 \), differs from the real triad case, but the required rotations of course agree in the real triad sector of the Ashtekar theory.

III. SYMMETRY GENERATORS

Phase space in the Ashtekar theory is coordinatized by the canonical pairs \{\( \tilde{T}^a_i, i A_i^\mu \}\}, plus the gauge functions \{\( N, N^a, -A_i^\mu \)\} =: \( N^A \), with their canonical momenta, which are primary constraints: \{\( P, P_a, -P_i \)\} =: \( P_A \). The
physical phase space is further constrained by secondary constraints \( \{ \tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_a, \tilde{\mathcal{H}}_i \} =: \mathcal{H}_A \). These constraints generate symmetry variations of the non-gauge variables. The complete generators (complete in the sense that they also generate variations of the gauge variables), with infinitesimal descriptors \( \{ \xi^0, \xi^a, \xi^i \} =: \xi^A \), are of the form
\[
G[\xi] = P_A \dot{\xi}^A + (\mathcal{H}_A + P_C N^B c_{C}^{' Prim}_{AB}) \xi^A, \tag{3.1}
\]
where the structure functions are obtained from the closed Poisson bracket algebra
\[
\{ \mathcal{H}_A, \mathcal{H}_{B'} \} =: c_{AB}^{'' Prim}_{C} \mathcal{H}_C', \tag{3.2}
\]
and where spatial integrations over corresponding repeated capital indices are assumed.

### IV. RIGID TIME TRANSLATION AND A QUANTUM PROPOSAL

We observe that \( G[\xi] \delta t \) effects rigid time translations on those solution trajectories satisfying
\[
N = t \xi^0, \quad N^a = \xi^a, \quad -A^0_0 + A^a_i N^a = \xi^i, \tag{4.1}
\]
where the descriptors \( \xi^A \) are here taken to be finite. Thus every generator \( G[\xi] \delta t \) with non-vanishing and positive \( \xi^0 \) is in this sense a generator of time evolution. Of course, on solutions whose gauge functions are not related to the descriptors as in (4.1) the engendered variation is more general.

The general finite generator is
\[
\mathcal{T} \exp \left( \int_{t_0}^{t_0 + \tau} dt \left\{ -, G[\xi] \right\} \right), \tag{4.2}
\]
where \( \mathcal{T} \) is the time ordering operator. This suggests a tentative implementation of this much larger symmetry in quantum gravity. We propose to retain the gauge variables in an expanded loop structure. In so doing we will be able to construct true spacetime holonomies (with the full spacetime connection \( A^\mu_i \)), and since the lapse and shift will constitute quantum operators, quantum fluctuations in the full spacetime metric will emerge. Physical states can be constructed in principle in this formalism by integrating out the full spacetime diffeomorphism gauge freedom, generalizing an expression proposed by Rovelli [8]; we propose a functional integral projector onto physical states of the form
\[
\mathcal{T} \left( [D\xi] e^{-i \int dt G[\xi]} \right). \tag{4.3}
\]