Three-spheres theorem for $p$-harmonic mappings

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Abstract Let $u = (u^1, \ldots, u^n)$ be a $p$-harmonic mapping in a domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$. We investigate level sets for compositions of coordinate functions $u^i$ with convex functions satisfying growth conditions and derive the de Giorgi-type estimates. Our main result is the arithmetic three-spheres theorem for coordinate functions of mapping $u$. The discussion is illustrated by radial $p$-harmonics.

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1 Introduction

Consider a subharmonic function $u$ in a planar domain and let $M(r)$ denote the maximum of $u$ over a circle $x^2 + y^2 = r^2$ concentric with two other circles with radii satisfying $r_1 < r < r_2$. Then, the classical Hadamard three-circles theorem asserts that $M(r)$ is a convex function of $r$, see e.g. Chapter 12 in Protter–Weinberger [19] (Hadamard formulated this result for analytic functions without providing a proof, see Hadamard [13]). Namely it holds that

$$M(r) \leq \frac{\log(r_2/r)}{\log(r_2/r_1)} M(r_1) + \frac{\log(r/r_1)}{\log(r_2/r_1)} M(r_2).$$

The result can be further generalized in the following directions: by considering higher dimensional analogs, by studying equations more general than the Laplace equation and by investigating different types of inequalities involving $M(r)$ or the norms of functions in subject. Indeed, one studies the setting of concentric spheres in $\mathbb{R}^n$, $n \geq 3$ see e.g. Theorem 30
in [19] or spheres which need not be concentric, see Arakelian–Matevosyan [5]. The three-circles (or the three-spheres) theorem can be extended to the setting of more general elliptic equations, see discussion in [19, Chapter 12], Brummelhuis [7] for a discussion in the setting of second-order linear elliptic equations, Miklyukov–Rasila–Vuorinen [18] for $p$-harmonic equations, Výborný [22] for quasilinear equations with Lipschitz coefficients; see also Granlund–Marola [12] for studies in the setting of $(A, B)$-equations of Riccati type and for further references. Finally, instead of the above inequality, one studies estimates involving $L^2$ or $L^\infty$ norms of solutions to elliptic equations, see e.g. Lin–Nagayasu–Wang [17] and Alessandrini–Rondi–Rosset–Vessella [4]. In the latter publication, among other topics the authors discuss the role of the three-spheres theorems in the studies of the unique continuation problems and ill-posed problems. For related topics and estimates we refer to Colding–De Lellis–Minicozzi [8] and Garofalo–Lin [9].

The main goal of this paper is to prove a variant of three-spheres theorem in the context of coupled elliptic systems of equations represented by $p$-harmonic systems of equations. According to our best knowledge three-spheres theorems have not yet been studied for systems of equations.

A mapping $u = (u^1, \ldots, u^n) \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is called a $p$-harmonic mapping if it is a solution to the following system of equations:

$$\text{div}(|Du|^{p-2}Du) = 0, \quad u = (u^1, \ldots, u^n) : \Omega \subset \mathbb{R}^N \to \mathbb{R}^n, \quad 1 < p < \infty,$$

(1)

where $Du$ denotes the Jacobi matrix of $u$, i.e. $Du = (\nabla u^1, \ldots, \nabla u^n)^T$. Equivalently, this system can be written as follows:

$$\begin{cases}
\text{div}(|Du|^{p-2}\nabla u^1) = 0, \\
\vdots \\
\text{div}(|Du|^{p-2}\nabla u^n) = 0.
\end{cases}$$

(2)

The $p$-harmonic system of equations is the Euler-Lagrange system of the associated $p$-Dirichlet energy functional

$$\int_\Omega |Du|^p.$$

In the weak form (2) reads

$$\int_\Omega |Du|^{p-2}\langle \nabla u^i, \nabla \phi^i \rangle = 0 \quad \text{for} \quad i = 1, \ldots, n,$$

(3)

where $\phi^i \in C_0^\infty(\Omega)$ are test functions. In what follows we will consider the case of $N = n$. For $p = 2$ the system reduces to the well-known harmonic system of equations (such a system is uncoupled). Therefore, one may view a $p$-harmonic system as a natural generalization of the harmonic system to the nonlinear setting. If we let the dimension of the target space be $n = 1$, then we retrieve the scalar $p$-harmonic equation. The above system is strongly coupled by the appearance of the full differential $Du$. As a consequence many methods of PDEs known in the linear (harmonic) setting fail for $p \neq 2$. This in turn stimulates the development of new methods and new approaches to handle the nonlinear problems.

The $p$-harmonic systems and their generalizations appear naturally in differential geometry, see e.g. Hardt–Lin [14] or in relation to differential forms and quasiregular maps, see e.g. Bonk–Heinonen [6]. As for the applied sciences, the second order coupled elliptic systems
are studied in nonlinear elasticity theory, e.g. Iwaniec–Onninen [15], nonlinear fluid dynamics, as well as in astrophysics or climate sciences and several other areas (see Adamowicz [3] for the list of further references).

We will now introduce notation, describe our approach to the three-spheres estimates and formulate auxiliary results and the main result of the paper. The proofs are presented in the remaining sections.

Let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^2 \) convex function such that there exists a subinterval \( I \subset \mathbb{R} \) with properties:

\[
(\phi'(x))^2 \leq \phi''(x) \quad \text{for all } x \in I
\]

(4)

either (i) \( \phi'(x) > 0 \) in \( I \) or (ii) \( \phi'(x) < 0 \) in \( I \)

(5)

In what follows we will deal several times with the expression \(|Du|/|\nabla u_1|\), its reciprocal and integrals involving it. In particular, we will need some conditions to ensure that

\[
\int_{B_r} \left( \frac{|Du|}{|\nabla u_1|} \right)^{p-2} \, dx < \infty,
\]

(6)

where \( B_r \) is a ball of radius \( r \). The case of \( 1 < p < 2 \) is easy, since then \((|\nabla u_1|/|Du|)^{2-p} < 1\) and by employing also continuity of \( p \)-harmonic mappings (see e.g. Tolksdorf [20]) we get that the above integral exists and is finite. The case of \( p = 2 \) is trivial. For \( p > 2 \) we need, however, to be more careful at critical points of \( u_1 \).

First, recall that results due to Tolksdorf [20, Theorem, Formula (1.14)] and Uhlenbeck [21, Theorem 3.2] applied to \( p \)-harmonic systems give us, respectively, that

\[
\sup_{B_{2r}} |Du| \leq C(p, n) \left( |B_{2r}|^{1/p} + \|Du\|_{L^p(B_{2r})} \right) \quad \text{for } 1 < p < 2
\]

(7)

\[
\sup_{B_{2r}} |Du| \leq C(p, n) \|Du\|_{L^p(B_{2r})} \quad \text{for } p \geq 2.
\]

(8)

The discrepancy between formulas (7) and (8) is a consequence of different nature of \( p \)-harmonic mappings for \( 1 < p < 2 \) (singular case) and \( p > 2 \) (degenerate elliptic case). In fact (7) holds for \( 1 < p < \infty \), cf. statement of Theorem and Section 3 in [20]. Nevertheless, the fact that inequality (8) is scale invariant with respect to \( u \) and is independent of the size of \( B_{2r} \) makes (8) still of interest to us and, therefore, in what follows we will appeal to both estimates. The above discussion allows us to continue estimation at (6) for \( p > 2 \) in a following way:

\[
\int_{B_r} \left( \frac{|Du|}{|\nabla u_1|} \right)^{p-2} \, dx \leq \frac{C(p, n)}{r^n(1-\frac{2}{p})} \int_{B_r} \frac{1}{|\nabla u_1|^{p-2}}. \]

(9)

Suppose that a ball \( B_r \) is centered at \( x_0 \) and that \( B_{2r} \subset \Omega \). Furthermore, assume that for \( p > 2 \)

\[
|\nabla u_1(x)| \geq |x - x_0|^{\alpha} \quad \text{for } x \in B_r \text{ and some } \alpha < \frac{n}{p - 2}.
\]

(10)

Then, by (9) we have:

\[
\int_{B_r} \left( \frac{|Du|}{|\nabla u_1|} \right)^{p-2} \, dx \leq \frac{C(p, n)}{r^n(1-\frac{2}{p})} \int_0^r t^{-(p-2)+n-1} \, dt < \infty.
\]

(11)
For the sake of simplicity and clarity of discussion all the results in the paper are stated for $u^1$, the first coordinate function of a $p$-harmonic mapping $u$. However, the reader should keep in mind that all the presented results hold as well for all coordinate functions $u^i$, for $i = 1, \ldots, n$ upon the necessary reformulations of results. Denote by

$$M(r) = \sup_{|x-x_0|=r} u^1(x)$$

and

$$m(r) = \inf_{|x-x_0|=r} u^1(x).$$

In Sect. 2 we show the Caccioppoli-type estimates for a composition of a convex function with a coordinate function of a $p$-harmonic mapping (Lemma 1). According to our best knowledge, such estimates and such an approach to $p$-harmonic mappings has not been studied in the literature so far.

**Lemma 1** Let $u^1$ be the first coordinate function of a $p$-harmonic mapping $u$ in a domain $\Omega$ for $p > 1$. Assume, furthermore, that a convex function $\phi : I \rightarrow \mathbb{R}$ satisfies conditions (4) and (5). Then the following estimates hold for every ball $B_r \subset \Omega$ and $0 < \delta < 1$:

If $1 < p < 2$, then

$$\int_{B_r(1-\delta)} \left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p} |\nabla \phi(u^1)|^p \leq \frac{c(p, n)}{\delta^p} p^{n-p}. \quad (12)$$

If $p \geq 2$, then

$$\int_{B_r(1-\delta)} |\nabla \phi(u^1)|^p \leq \frac{c(p)}{\delta^p} \int_{B_r} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} (\phi(u^1) - k)^p. \quad (13)$$

Furthermore, for $p > 2$ the growth condition (10) ensures finitness of the last integral.

In the next result we study the behavior of $\phi(u^1)$ over the level sets and investigate the de Giorgi type estimates. Such estimates are well-known for solutions of elliptic equations, see e.g. Giusti [10]. In the setting of vector functions and systems of equations such estimates require extra attention and effort. The results of Lemma 2 can be used in further analysis of level sets for coordinate functions of $p$-harmonic mappings (see Sect. 3 for the proof of Lemma 2).

Let $k \geq 0$. Upon the above notation we define

$$A_{k,r} := \{x \in B_r : \phi(u^1(x)) > k\}.$$

**Lemma 2** Let $u^1$ be the first coordinate function of a $p$-harmonic mapping $u$ in a domain $\Omega \subset \mathbb{R}^n$ for $1 < p \leq n$. Assume, furthermore, that conditions (4) and (5) hold for a convex function $\phi : I \rightarrow \mathbb{R}$. Then the following estimates hold for every ball $B_r \subset \Omega$ and $0 < \delta < 1$:

$$\int_{A_{k,(1-\delta)r}} \left(\frac{|\nabla u^1|}{|Du|}\right)^{2-p} |\nabla \phi(u^1)|^p \leq \frac{c(p)}{\delta^p} \int_{A_{k,r}} (\phi(u^1) - k)^p \quad \text{for} \quad 1 < p \leq 2, \quad (14)$$

$$\int_{A_{k,(1-\delta)r}} |\nabla \phi(u^1)|^p \leq \frac{c(p)}{\delta^p} \int_{A_{k,r}} \left(\frac{|Du|}{|\nabla u^1|}\right)^{p-2} (\phi(u^1) - k)^p \quad \text{for} \quad p \geq 2 \text{ and } n > 2. \quad (15)$$

Furthermore, the following inequality holds for $1 < p < 2$:

$$\left(\sup_{B_r(1-\delta)} \phi(u^1)\right)^p \leq \frac{C^n}{\delta^n} \int_{B_r} (\phi(u^1))^p + r^{p-n}, \quad (16)$$
where the constant
\[ C = C(p, n, c_S) \left[ 1 + \|\phi\|_{L^\infty(B_r)}^p \left( r^n + \|Du\|_{L^p(B_{2r})}^p \right) \right]. \] (17)

If, additionally, there exists a positive constant \( c \), such that
\[ |\nabla u^1| > c \quad \text{in} \quad B_r, \] (18)
then the following inequality holds for \( p > 2 \) and \( n > 2 \):
\[ \left( \sup_{B(1-3r)} \phi(u^1) \right)^p \leq \frac{C_n^p}{(\delta r)^n} \int_{B_r} (\phi(u^1))^p, \] (19)
with constant
\[ C = c(p, n, c_S) \left( \frac{\|Du\|_{L^p(B_{2r})}}{cr^n} \right)^{\frac{n}{p}(p-2)}. \] (20)

Here, \( c_S \) stands for the constant in the Sobolev embedding theorem. In the harmonic case \( p = 2 \), assumption (18) can be neglected and (19) holds with constant \( C = c(p, n, c_S) \) in (20).

The main result of this paper is the following version of the arithmetic three-spheres theorem for coordinate functions of \( p \)-harmonic mappings. We prove Theorem in Sect. 4 as well as comment on the existence of \( p \)-harmonic mappings satisfying assumptions of Theorem.

**Theorem** (The arithmetic three-spheres theorem) Let \( 1 < p \leq n \) and \( u = (u^1, \ldots, u^n) : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a \( p \)-harmonic mapping. Consider three concentric balls centered at \( x_0 \in \Omega \) with radii \( 0 < r_1 < r < r_2 \) such that \( \overline{B_{2r_2}} \subseteq \Omega \) and
\[ 0 < c \leq \frac{r_1}{r}, \quad \frac{r_1}{r_2} < c < 1, \] (21)for some fixed \( c \).

If \( 1 < p < 2 \), then let us assume that for a given \( \alpha > 0 \), the coordinate function \( u^1 \) satisfies the following growth condition:
\[ |u^1(x) - u^1(x_0)| \geq C|x - x_0|^\alpha \quad \text{for} \quad x \in \Omega \setminus B_{r_2}. \] (22)
If \( 2 < p < n \), then let us assume instead that there is a positive constant \( c_1 \) such that
\[ |\nabla u^1| > c_1 \quad \text{in} \quad B_{r_2}. \] (23)

Then there exist a constant \( C \) and a radius \( r_3 \) such that if \( \overline{B_{r_3}} \subseteq \Omega \), then the following inequalities hold:
\[ M(r) \leq CM(r_1) + (1 - C)M(r_3), \]
\[ m(r) \geq Cm(r_1) + (1 - C)m(r_3). \]

For \( 1 < p < 2 \) constant \( C \) depends on \( n, p, c_S \) the constant in the Sobolev embedding theorem, \( r_1, r_2, c, \alpha \) and \( \|Du\|_{L^p(B_{2r_2})} \), while \( r_3 > r_2 \) depends on \( p, n, r_2, \alpha \), and \( \|Du\|_{L^p(B_{2r_2})} \). For \( 2 < p \leq n \) constant \( C \) depends on \( n, p, c_S, r_1, r_2, c, c_1 \) and \( \|Du\|_{L^p(B_{2r_2})} \), while \( r_3 = r_2 \). For \( p = 2 < n \) condition (23) is obsolete, \( r_3 = r_2 \) and constant \( C \) depends only on \( n, p, c_S \) and \( c, r_1, r_2 \).
In the statement of Theorem one may consider local solutions in \(\mathbb{R}^n\) of the \(p\)-harmonic system (2) instead in the domain \(\Omega\). In such a case one may neglect the assumption that \(\overline{B}_r \subset \Omega\).

If \(p = 2\), then the \(p\)-harmonic system (2) reduces to the uncoupled system of harmonic equations satisfied by coordinate functions \(u^i\) for \(i = 1, \ldots, n\). In such a case the arithmetic three-spheres theorem holds for each \(u^i\), see Theorem 30 in Protter–Weinberger [19, Chapter 12].

We point out that the case of scalar \(p\)-harmonic functions for \(1 < p < \infty\) can be handled by the approach very similar to the one for harmonic functions, as one has on the disposal comparison principles and the Harnack-type estimates. Such tools are not known in the setting of coordinate functions of \(p\)-harmonic mappings.

For the case \(p = n\), Granlund–Marola [12] proved a variant of the three-spheres theorem in the setting of \((A, B)\)-quasilinear equations, in particular for a \(p\)-harmonic equation, cf. Theorem 5.4 [12]. However, their approach is based on the existence of the strong maximum principle and the Harnack inequality for solutions of the considered \((A, B)\)-equation. Similar results are not known in the setting of coupled \(p\)-harmonic mappings \((p \neq 2)\).

2 The proof of Lemma 1

In this section we prove Lemma 1 and then illustrate the discussion by the class of radial \(p\)-harmonic mappings.

Proof We begin the proof as in Granlund–Marola [12]. For the sake of simplicity, let us assume that (5) (i) holds, i.e. \(\phi'(u^1(x)) > 0\) for all \(x \in B_r\). Take a nonnegative function \(\xi \in C_0^\infty(B_r)\) and define a test function \(\eta(x) = \phi'(u^1(x))^{p-1}\xi^p(x)\) for \(x \in B_r\). Then

\[
\nabla \eta = (p - 1)(\phi'(u^1))^p \phi''(u^1)\xi^p \nabla u^1 + p\xi^{p-1}(\phi'(u^1))^{p-1} \nabla \xi.
\]

We use \(\nabla \eta\) in the first equation of \(p\)-harmonic system (3) and by using (4) together with (5) we obtain

\[
(p - 1) \int_{B_r} |Du|^p - 2|\nabla u^1|^2 \phi'(u^1)^p \xi^p \leq p \int_{B_r} |Du|^p - 2|\nabla u^1| \phi'(u^1)^{p-1} \xi^{p-1} |\nabla \xi|.
\]  

(24)

For some \(0 < \epsilon < 1\), whose value will be determined later, we rewrite the right-hand side of (24) and apply the Young inequality:

\[
p \int_{B_r} |Du|^p - 2|\nabla u^1| \phi'(u^1)^{p-1} \xi^{p-1} |\nabla \xi| \\
= p \int_{B_r} \left( \epsilon |Du|^{(p-2)^+ - 1} |\nabla u^1|^{2 \frac{p-1}{p}} \phi'(u^1)^{p-1} \xi^{p-1} \right) \left( \frac{1}{\epsilon} |Du|^{(p-2)^+} |\nabla u^1|^2 |\nabla \xi| \right) \\
\leq (p - 1)\epsilon \int_{B_r} |Du|^p - 2|\nabla u^1|^2 \phi'(u^1)^p \xi^p + p\epsilon^{-p} \int_{B_r} \left( \frac{|Du|}{|\nabla u^1|} \right)^{p-2} |\nabla \xi|^p.
\]  

(25)

In the last inequality we also estimated \(\epsilon^{p-1} \leq \epsilon\). Now, we use (25) in (24) and by taking e.g. \(\epsilon = \frac{1}{2}\) we may include the first integral on the right-hand side of (25) into the left-hand side of (24). In a consequence we arrive at the following estimate:

\[
\int_{B_r} |Du|^p - 2|\nabla u^1|^2 \phi'(u^1)^p \xi^p \leq \frac{p^{2+1}}{p-1} \int_{B_r} \left( \frac{|Du|}{|\nabla u^1|} \right)^{p-2} |\nabla \xi|^p.
\]  

(26)
Denote \(c(p) := \frac{p^{2p+1}}{p-1}\).

Case 1: \(1 < p < 2\). The left-hand side of the above inequality can be written as follows.

\[
\int_{B_r} |Du|^2 |\nabla u|^2 |\phi'(u^1)|^p \xi^p = \int_{B_r} \left( \frac{|\nabla u|^2}{|Du|} \right)^{2-p} |\nabla \phi(u^1)|^p \xi^p. \tag{27}
\]

Using (27) and the fact that \(|\nabla u| \leq |Du|\) in \(\Omega\) we observe that for \(1 < p < 2\) inequality (26) becomes:

\[
\int_{B_r} \left( \frac{|\nabla u|^2}{|Du|} \right)^{2-p} |\nabla \phi(u^1)|^p \xi^p \leq c(p) \int_{B_r} |\nabla \xi|^p. \tag{28}
\]

Case 2: \(p \geq 2\). We have that \(|\nabla u|^2 \leq |Du|^2\) in \(\Omega\) and hence (26) takes the following form.

\[
\int_{B_r} |\nabla \phi(u^1)|^p \xi^p \leq c(p) \int_{B_r} \left( \frac{|Du|^2}{|\nabla u|^2} \right)^{p-2} |\nabla \xi|^p. \tag{29}
\]

If we additionally assume (10), then (11) holds and the last integral is finite. In the definition of \(\eta\) we take \(\xi\) such that \(0 \leq \xi \leq 1\), \(\text{supp} \xi \subset B_r\), \(\xi \equiv 1\) in \(B_{(1-\delta)r}\) and \(|\nabla \xi| \leq \frac{c}{\delta r}\) in \(B_r\).

Using such \(\xi\) in (28) and (29) we arrive at claims (12) and, respectively, (13) of Lemma 1.

If (5) (ii) holds, i.e. \(\phi'(u^1(x)) < 0\) for all \(x \in B_r\), then as a test function we take \(\eta(x) = (\phi'(u^1(x)))^{p-1} \xi^p(x)\) for \(x \in B_r\) and claims of the lemma follow the same way as previously.

\(\square\)

We remark that assertions (28) and (29) of Lemma 1 can be further refined. Namely, the following remark holds by Lemma 1 and the discussion at formulas (7) and (8).

**Remark 1** (A) Let \(1 < p < 2\). Then (7) implies that for \(x \in B_r\)

\[
\frac{1}{|Du(x)|} \geq \frac{1}{\sup_{B_r} |Du|} \geq C(p, n, \text{diam}(\Omega), \|Du\|_{L^p(B_{2r})}) \frac{n}{r^\gamma}.
\]

In such a case (28) reads:

\[
\int_{B_r} \left( \frac{|\nabla u|^2}{r^\frac{p}{2}} \right)^{2-p} |\nabla \phi(u^1)|^p \xi^p \leq c(p) \int_{B_r} |\nabla \xi|^p.
\]

Similarly, for \(p > 2\) the Uhlenbeck inequality (8) results in the following estimate.

\[
\int_{B_r} |\nabla \phi(u^1)|^p \xi^p \leq C(p, n, \|Du\|_{L^p(B_{2r})}) \int_{B_r} \frac{1}{r^{n(1-\frac{2}{p})}} |\nabla u|^{p-2} |\nabla \xi|^p.
\]

(B) In fact estimate (26) gives rise to the following inequality for \(p > 1\):

\[
\int_{B_r} \left( \frac{|Du|^2}{|\nabla u|^2} \right)^{p-2} |\nabla \phi(u^1)|^p \xi^p \leq \frac{p^{p+1}}{p-1} \int_{B_r} \left( \frac{|Du|^2}{|\nabla u|^2} \right)^{p-2} |\nabla \xi|^p.
\]

We now turn to considering a special class of \(p\)-harmonic mappings, namely radial transformations in the form

\[
u(x) = H(|x|)(x_1, \ldots, x_n), \quad \text{for } x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n,
\]
where $H \in C^2(\Omega)$ and $|x|$ stands for the magnitude of $x$. For such mappings $p$-harmonic system (2) becomes a nonlinear second order ODE:

$$(p - 1)H''(H')^2r^3 + (2p + n - 3)(H')^3r^2 + 2(p - 1)HH''r^2$$

$$+ (np + 3p - 4)H(H')^2r + (p + n - 2)H^2H''r + (n + 1)(p + n - 2)H^2H' = 0.$$  

(30)

The problem of finding suitable examples is the general feature of the $p$-harmonic world, as we know only few classes of $p$-harmonic maps and few explicit solutions of the $p$-harmonic system of equations, namely affine, radial and quasiradial, see e.g. Adamowicz [2,3], Iwaniec–Onninen [15, Part 1] for various applications of radial $p$-harmonics, and Adamowicz [1, Chapter 2] for the definition of quasiradial $p$-harmonic mappings.

For the class of radial $p$-harmonic mappings, Lemma 1 can be refined. Indeed, for radial mappings we can formulate simple conditions for integrability of ratio $|\nabla u^1|/|Du|$ and in turn estimates (12) and (13) reduce to the following result.

**Proposition 1** (Radial Lemma 1) Let $u^1$ be the first coordinate function of a radial $p$-harmonic mapping $u$ in $\Omega \subset \mathbb{R}^n$, $n > 1$ for $p > 1$. Assume that a convex function $\phi : I \rightarrow \mathbb{R}$ satisfies conditions (4) and (5). Furthermore, assume the following:

1. if $x$ is such that $H(x) = 0$, then $H'(x) \neq 0$,
2. there exist constant $c_\Omega$, $c'_\Omega > 0$ such that $x_1^2 < c_\Omega(|x|^2 - x_1^2)$ and $x_1 > c'_\Omega$
   
   for all $x \in \Omega$,  

(31)

3. there exists $C > 0$ such that $\eta(x) := \frac{H'(|x|)}{H(|x|)}|x| < C$ for all $x \in \Omega$.  

(32)

Then the following estimate holds for every ball $\overline{B}_r \subset \Omega$ and $0 < \delta < 1$:

$$\int_{B_{(1-\delta)}r} |\nabla \phi(u^1)||^p \leq \frac{c(\text{diam}(\Omega), p, n, c_\Omega, c'_\Omega)}{\delta^p}x_{n-p}.$$  

(33)

Conditions (31) and (32) geometrically mean that we require domain $\Omega$ to lie inside a cone symmetric about $x_1$-axis and to consist of points with a positive distance to the hyperplane $\{x_1 = 0\}$.

**Proof** Let $u(x) = H(|x|)(x_1, \ldots, x_n)$ be a radial $p$-harmonic mapping in $\Omega$. Denote $r := |x|$ the magnitude of vector $x$. Then $u^i(x) = H(r)x_i$ for $i = 1, \ldots, n$ and

$$u^i_{x_j}(x) = \begin{cases}
H'(r)\frac{x_i x_j}{r}, & \text{for } j \neq i, \\
H'(r)\frac{x_i^2}{r} + H(r), & \text{for } j = i.
\end{cases}$$

Hence $|\nabla u^i|^2 = \sum_{j \neq i}(H')^2\frac{x_i x_j}{r}^2 + (H')^2x_i^2 + 2HH'\frac{x_i^2}{r} + H^2$. Upon computing $|\nabla u^i|^2/|\nabla u^1|^2$ and summing over $i = 1, \ldots, n$ we get that

$$g := \frac{|Du|^2}{|\nabla u^1|^2} = \frac{|\nabla u^1|^2 + \ldots + |\nabla u^n|^2}{|\nabla u^1|^2} = \frac{(H')^2r^2 + 2HH'r + nH^2}{(H')^2x_1^2 + 2HH'\frac{x_1^2}{r} + H^2}.$$  

$\square$ Springer
Let $x_0 \in B_r$ and consider the following cases.

Case 1: $H(x_0) = 0$. Since, by assumptions we have that $H'(\{|x_0|\}) \neq 0$, then $g(x_0) = \frac{r^2}{\{|x_0|\}^2}$.

Under the second part of assumption (31) we obtain that $|g(x_0)| \leq c(\text{diam}(\Omega), c_\Omega)$.

Case 2: $H(x_0) \neq 0$ and $H'(x_0) = 0$. Then $g(x_0) = n$.

Case 3: $H(x_0) \neq 0$ and $H'(x_0) \neq 0$. Then for $\eta := \eta(x_0) = \frac{H'(\{|x_0|\})}{H'(\{|x_0|\})} \{|x_0|\}$ we have

$$g(x_0) = \frac{\eta (\eta + 2) + n}{|\{x_0\}|^2} \eta (\eta + 2) + 1.$$  

Depending on the sign of $\eta(\eta + 2)$ we distinguish two cases:

(a) If $\eta \geq 0$ or $\eta \leq -2$, then $g(x_0) \leq \eta(\eta + 2) + n = (\eta + 1)^2 + n - 1$ and assumption (32) results in $g(x_0) \leq (C + 1)^2 + n - 1$.

(b) If $-2 \leq \eta \leq 0$, then $|g(x_0)| \leq n \frac{|\{x_0\}|^2}{|\{x_0\}|^2 - |\{x_0\}|^2} \leq nc_\Omega'$. In the last inequality we also used the first part of assumption (31).

Therefore we conclude that $g$ is bounded by a constant $c(\text{diam}(\Omega), n, c_\Omega, c_\Omega')$ for all $x \in \overline{B_r}$.

Then assertion (33) follows immediately from (12) and (13).  

\[ \square \]

3 The proof of Lemma 2

The first part of the proof is similar to the one of Lemma 1 (see also Granlund [11, Section 2] and Granlund–Marola [12, Lemma 2.6]). Let $k \geq 0$ and define $\psi(x) = \max[\phi(u^1(x)) - k, 0]$ for $x \in B_r$ with $B_r \subset \Omega$. Take $\xi \in C_0^\infty(\Omega)$ and define a test function $\eta(x) = \psi(x)(\phi'(u^1(x)))^{p-1} \xi^p$.

Then

$$\nabla \eta = \phi'(u^1)(\phi'(u^1))^{p-1} \xi^p \nabla u^1 + (p - 1) \psi \phi''(u^1)(\phi'(u^1))^{p-2} \xi^p \nabla u^1 + p \psi \phi'(u^1))^{p-1} \xi^{p-1} \nabla \xi.$$  

Using $\nabla \eta$ in the first equation of $p$-harmonic system (3) we get the following inequality.

\[
\int_{B_r} |Du|^p - 2|\nabla u^1|^2 (\phi'(u^1))^{p-1} \xi^p + (p - 1) \int_{B_r} |Du|^p - 2|\nabla u^1|^2 \psi \phi''(u^1)(\phi'(u^1))^{p-2} \xi^p \\
\leq p \int_{B_r} |Du|^p - 2|\nabla u^1| \psi \phi'(u^1))^{p-1} \xi^{p-1}. \tag{34}
\]

We invoke property (4) of function $\phi$ and use it in the second integral on the left-hand side of the inequality. Since $\psi \geq 0$ in $B_r$ we can drop the aforementioned integral. The Young inequality applied to the integral on the right-hand side gives us the estimate (cf. inequality (25)):

\[
p \int_{B_r} |Du|^p - 2|\nabla u^1| \psi \phi'(u^1))^{p-1} \xi^{p-1} \\
= p \int_{B_r} \left( |Du|^{p-2} \frac{p-1}{p} |\nabla u^1|^2 \phi'(u^1)^{p-1} \xi^{p-1} \right) \left( |Du|^{p-2} |\nabla u^1|^{p-2} \frac{p-1}{p} \psi |\nabla \xi| \right) \\
\leq (p - 1) c \int_{B_r} \left( |Du|^{p-2} |\nabla u^1|^2 \phi'(u^1)^{p-1} \xi^{p-1} \right) + \epsilon - p \int_{B_r} \left( |Du| \psi |\nabla \xi| \right)^{p-2} \psi \psi |\nabla \xi|^p.
\]

Upon choosing small enough value of $0 < \epsilon < 1$, e.g. $\epsilon = \frac{1}{2(p-1)}$ we include the first integral on the right-hand side of the inequality into the integral on the left-hand side of (34) and arrive at the following estimate (cf. (26) in Lemma 1):

\[ \square \]
\[
\int_{B_r} |Du|^{p-2} |\nabla u^1|^2 \phi'(u^1)^p \xi^p \leq c(p) \int_{B_r} \left( \frac{|Du|}{|\nabla u^1|} \right)^{p-2} \psi^p |\nabla \xi|^p. \tag{35}
\]

We choose function \( \xi \) so that it satisfies: \( \text{supp } \xi \subset B_r, 0 \leq \xi \leq 1, \xi \equiv 1 \) in \( B_{(1-\delta)r} \) and \( |\nabla \xi| \leq \frac{\delta}{\delta r} \) in \( B_r \). Note also, that by definition \( \psi \equiv 0 \) in \( B_r \setminus A_{k,r} \). This and the choice of \( \xi \) in (35) lead us to the following inequality (cf. estimate (26) in Lemma 1):

\[
\int_{A_{k,(1-\delta)r}} |Du|^{p-2} |\nabla u^1|^2 \phi'(u^1)^p \leq \frac{c(p)}{(\delta r)^p} \int_{A_{k,r}} \left( \frac{|Du|}{|\nabla u^1|} \right)^{p-2} \psi^p.
\]

The discussion similar to the one for Cases 1 and 2 in Lemma 1 (cf. inequalities (28) and (29)) gives us assertions (14) and (15).

The proofs of supremum estimates (16) (in both cases) require extra attention due to the appearance of expression \( g := \left( \frac{|Du|}{|\nabla u^1|} \right)^{p-2} \) under integrals (14) and (15) exploited in derivation of the supremum estimate. Note, that when the \( p \)-harmonic system \((1)\) reduces to a single \( p \)-harmonic equation, then \(|Du| = |\nabla u^1|\), and so \( g \equiv 1 \). In such a case we retrieve estimates from Lemma 2.6 in Granlund–Marola [12]. Here, we instead follow the method from a book by Giusti [10, Theorem 7.2] and adapt it to the vectorial case.

Using the notation analogous to [10], let \((1-\delta)r \leq \sigma r \leq \tau r \leq r\). At this point the discussion splits into four cases: (1) \( 1 < p < 2 \), (2) \( p > 2 \) according to estimates (14) and (15), respectively, (3) \( p = 2 \) and (4) \( p = n \).

Case 1: \( 1 < p < 2 \).

Let \( \eta \in C_0^\infty (B_{\frac{\tau}{\tau-r},r}) \) such that \( \eta \equiv 1 \) on \( B_{\sigma r} \) and \( |\nabla \eta| \leq \frac{c}{(\tau - \sigma)r} \). Define \( \xi(x) = \eta(x)\psi(x) \) for function \( \psi \) as in the first part of the proof. By the Hölder and the Sobolev inequalities we get

\[
\int_{A_{k,\sigma r}} \psi^p \leq \int_{A_{k,\sigma r}} \xi^p \leq \left( \int_{A_{k,\sigma r}} \xi^{\frac{np}{\pi-p}} \right)^{\frac{\pi}{n}} |A_{k,\sigma r}|^{\frac{p}{n}} \leq c_S |A_{k,\tau r}|^{\frac{p}{n}} \int_{A_{k,\sigma r}} |\nabla \xi|^p. \tag{36}
\]

Using the definition of \( \xi \) we compute that \( \nabla \xi = \psi \nabla \eta + \eta \nabla \psi \) and hence

\[
\int_{A_{k,\sigma r}} \psi^p \leq c_S |A_{k,\tau r}|^{\frac{p}{n}} \left( \int_{A_{k,\frac{\tau}{\tau-r},r}} |\nabla \phi(u^1)|^p + \frac{1}{(\tau - \sigma)^p \rho p} \int_{A_{k,\frac{\tau}{\tau-r},r}} \psi^p \right). \tag{37}
\]

Let \( \alpha = \frac{p}{4} (2 - p) \) and \( \beta = p (1 - \frac{p}{4}) \). Then, the Young inequality applied with exponents \( \frac{2-p}{\alpha} = \frac{4}{p} \) and its conjugate \( \left( \frac{2-p}{\alpha} \right)' = \frac{2-p}{2-p-\alpha} = \frac{4-p}{4} \) gives us the following:

\[
\int_{A_{k,\frac{\tau}{\tau-r},r}} |\nabla \phi(u^1)|^p = \int_{A_{k,\frac{\tau}{\tau-r},r}} \left( |\nabla \phi(u^1)|^{p-\beta} \left( \frac{|\nabla u^1|}{|Du|} \right)^{\alpha} \right) \left( |\phi'(u^1)|^\beta |Du|^p \left( \frac{|\nabla u^1|}{|Du|} \right)^{\beta-\alpha} \right) \leq \frac{p}{4} \int_{A_{k,\frac{\tau}{\tau-r},r}} \left( \frac{|\nabla u^1|}{|Du|} \right)^{2-p} |\nabla \phi(u^1)|^p + (1 - \frac{p}{4}) \int_{A_{k,\frac{\tau}{\tau-r},r}} \left( \frac{|\nabla u^1|}{|Du|} \right)^{2-p} |\phi'(u^1)|^p |Du|^p. \tag{38}
\]
In order to estimate further (38) for $p$ in the given range we appeal to Tolksdorf’s estimate [20] and note that $\left(\frac{|\nabla u^1|}{Du}|^p \right)^{\frac{2p}{4-p}} < 1$. Then

$$\int_{A_k,\frac{\sigma+t}{r}} \left(\frac{|\nabla u^1|}{Du}\right)^{\frac{2p}{4-p}} |\phi'(u^1)|^p |Du|^p$$

$$\leq C(p, n) \|\phi'\|_{L^\infty(B_{\frac{\sigma+t}{r}}^2)} \left(\frac{2}{(\sigma + \tau)r}\right)^n \left(\|B(\sigma+\tau)r\| + \|Du\|_{L^p(B_{(\sigma+\tau)r})}\right) |A_k,\frac{\sigma+t}{r}|.$$

We use this inequality in the second integral on the right-hand side of (38). Moreover, we observe that under assumptions on $\sigma$ and $\tau$ it holds that $\frac{\sigma+t}{2} \leq \tau$ and thus $A_k,\frac{\sigma+t}{r} \subset A_{k,\tau r}$. We apply estimate (14) with $r := \tau r$ and $\delta$ such that $(1-\delta)r \tau := \frac{\sigma+t}{r}$ in the first integral on the right-hand side of (38). In a consequence, estimate (37) takes the following form:

$$\int_{A_k,\sigma r} \psi^p \leq C(p, n, c_S) |A_{k,\tau r}| \frac{p}{n} \left(\frac{2^p}{(\tau-\sigma)p r^p}\right) \int_{A_k,\tau r} \psi^p + \|\phi'\|_{L^\infty(B_{\frac{\sigma+t}{r}}^2)}^{\frac{p}{n}} \left(\int_{A_k,\frac{\sigma+t}{r}} r^n + \|Du\|_{L^p(B_{\frac{\sigma+t}{r}})} \right) + \frac{1}{(\tau-\sigma)^p r^p} \int_{A_k,\frac{\sigma+t}{r}} \psi^p \right). \hspace{1cm} (39)$$

If $h < k$, then

$$(k-h)^p |A_{k,\tau r}| \leq \int_{A_{h,\tau r}} (\phi(u^1) - h)^p \quad \text{and} \quad \int_{A_{k,\tau r}} (\phi(u^1) - k)^p \leq \int_{A_{h,\tau r}} (\phi(u^1) - h)^p.$$  

Denote

$$C = C(p, n, c_S) \left[ 1 + \|\phi'\|_{L^\infty(B_r)} (r^n + \|Du\|^p_{L^p(B_{2r})}) \right].$$

Using these in (39) we obtain the following estimate:

$$\int_{A_k,\sigma r} (\phi(u^1) - k)^p \leq \frac{C |A_{k,\tau r}|}{(\sigma - \tau)p r^p} \left( \int_{A_{h,\tau r}} (\phi(u^1) - h)^p + \frac{(\sigma - \tau)p r^p}{(\sigma + \tau)p r^p} |A_{k,\tau r}| \right)$$

$$\leq \frac{C}{(k-h)^\frac{p}{n}} \frac{(\sigma - \tau)p r^p}{(\sigma + \tau)p r^p} \left( \int_{A_{h,\tau r}} (\phi(u^1) - h)^p \right)^{1+\frac{p}{n}} \left[ 1 + \frac{1}{(k-h)^p} \frac{(\sigma - \tau)p r^p}{(\sigma + \tau)p r^p} \right]. \hspace{1cm} (40)$$

We are in a position to use the iteration scheme as in Lemma 7.1 in Giusti book [10]. Indeed, for some $d > 0$, to be determined later, let us consider the following quantities:

$$k_i := 2d(1 - 2^{-i}), \quad \text{for } i = 0, 1, \ldots, \quad k = k_{i+1}, \quad h = k_i, \quad k - h = \frac{d}{2^i}$$

$$\sigma_i := \delta + (1-\delta)2^{-i}, \quad \text{for } i = 0, 1, \ldots, \quad \sigma = \sigma_{i+1}, \quad \tau = \sigma_i.$$  

Hence

$$\tau - \sigma = (1-\delta)2^{-i-1}, \quad \tau + \sigma = 2\delta + (1-\delta)3 \cdot 2^{-i-1}.$$  

Finally, let

$$\phi_i := \int_{A_{k_i,\sigma_i}} (\phi(u^1) - k_i)^p.$$
With this notation inequality (40) reads:

$$\phi_i + 1 \leq C \left( 1 + (1 - \delta) d - p \right) \left( 1 + \frac{p}{n} \right) \left( 1 + \frac{p}{n} \right) \phi_i + \frac{p}{n}, \quad \text{for } i = 0, 1, \ldots.$$  

The claim of the second part of Lemma 2 for $1 < p < 2$ follows from [10, Lemma 7.1], cf. the proof of Theorem 7.2 in [10]. Indeed, we set that for some $a > 0$,

$$1 + \frac{p}{n} \leq a.$$  

By taking $B = 2^{p(1 + p/n)}$, $\alpha = \frac{p}{n}$ and $c = \frac{C}{(1 - \delta) d - p}$ we verify that the assumption of [10, Lemma 7.1] that $\phi_0 \leq c^{-1/\alpha} B^{-1/(\alpha^2)}$ leads to the following conditions:

$$d \geq \frac{(C \alpha)^{\frac{n}{p}}}{(1 - \delta)^{\frac{n}{p}}} \int_{B_r} (\phi(u^1)) \quad \text{and} \quad (a - 1)d \geq r^{-p/n},$$

as $A_0 = B_r$. Thus, by taking e.g. $a = 2$ we get that the above inequalities for $d \geq C \alpha^{\frac{n}{p}}$ and $(a - 1)d \geq r^{-p/n}$.

and the claim follows, since by [10, Lemma 7.1] $\lim_{i \to \infty} \phi_i = 0$ and so $\sup_{B_r} \phi(u^1) \leq 2d$.

Case 2: $2 < p < n$.

We proceed similarly to the previous case. Upon using estimate (15) in (37) together with the fact that $|Du|$ is bounded by (8) and our assumptions, we obtain that

$$\int_{A_k, o_r} (\phi(u^1) - k)^p \leq \frac{C_{\sup}}{(\sigma - \tau)^r} \left( 1 \left( (k - h)^p \int_{A_{k, \tau}} (\phi(u^1) - h)^p \right) 1 + \frac{p}{n} \right).$$  

(41)

Constant $C_{\sup} = c(p, n, c_S) \left( \frac{\|Du\|_{L^p(B_{2r})}}{c} \right)^{\frac{n}{p}(p - 2)}$. The reasoning similar to the previous case gives us the claim for $2 < p < n$. However, in this case we get the homogeneous estimate (19).

Case 3: $p = 2$.

If $p = 2$ and $n > 2$, then we follow the proof for $2 < p < n$ and since $\left( \frac{\|Du\|_{L^p(B_{2r})}}{c} \right)^{p - 2} \equiv 1$, assumption (18) and discussion at (38) are not needed and we obtain (19) with constant $C = c(p, n, c_S)$ in (20).

If $p = 2$ and $n = 2$, then in (36) we use a variant of the Sobolev inequality (see e.g. Corollary 1.57 in Malý–Ziemer [16]) and get

$$\int_{A_k, o_r} \psi^2 \leq \int_{A_k, o_r} \xi^2 \leq |A_{k, \tau}| \int_{A_{k, o_r}} |
\nabla \xi|^2.$$  

This leads to the estimate similar to (41) and (19), while the resulting constant $C$ depends on $p, n$ and $c_S$.

Case 4: $p = n$.

As in the previous case, we use [16, Corollary 1.57] and obtain the following

$$\int_{A_k, o_r} \psi^n \leq \int_{A_k, o_r} \xi^n \leq |A_{k, \tau}| \int_{A_{k, o_r}} |
\nabla \xi|^n.$$  

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We follow the proof for the case of $2 < p < n$ and get $C_{\sup} = c(n) \left( \frac{\|Du\|_{L^p(B_{cR})}}{cr} \right)^{n-2}$.

Hence, the proof of Lemma 2 is completed. \hfill \Box

4 The proof of Theorem

In the proof of Theorem we use the doubling property of the Lebesgue measure. This means, that for any ball $B_R \subset \mathbb{R}^n$ it holds that $C^n(B_{2R}) \leq C \mathcal{L}^n(B_R)$, where $C = 2^n$. Below, we also appeal to the $(1, p)$-Poincaré inequality: if $v \in W^{1,p}_{loc}(\Omega)$, then

$$\int_{B_r} |v - v_{B_r}|^p \leq CR^p \int_{B_r} |\nabla v|^p,$$

where $v_{B_r}$ denotes the mean value of $v$ over the ball $B_r$ and $C$ depends on $n$ and $p$.

Finally, the following auxiliary result is used in the proof of Theorem as well, see Theorem 4.20 in Adamowicz [1] and Appendix A.2 in Adamowicz [3].

Lemma 3 [3, Observation 2] Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a $p$-harmonic mapping in the domain $\Omega \subset \mathbb{R}^n$. If for some $u^i$, $i = 1, \ldots, n$ there exists $k \in \mathbb{R}$ such that $u^i \leq k$ on $\partial \Omega$, then $u^i \leq k$ in $\Omega$.

Proof of Theorem  Our approach is based on Theorem 1.3 in Granlund–Marola [12]. However, the fact that now we are in the setting of mappings instead of scalar functions requires new type of the de Giorgi estimates (cf. Lemmas 1 and 2). Moreover, the dependence of those estimates on $\|Du\|_{L^p}$ and $\|\phi\|_{L^\infty}$ requires additional efforts and caution.

Define the following function (keeping in mind that the exact value of $r_3$ will be determined later):

$$\phi(t) := - \log \left( \frac{M(r_3) - t + \epsilon}{M(r_3) - M(r_1) + \epsilon} \right) \quad \text{for } t \in (-\infty, M(r_3)), \quad (42)$$

for any but fixed $\epsilon > 0$. It is easy to check that $\phi$ is $C^2$ convex and satisfies conditions (4) and (5). Indeed, by computations we see that

$$\phi'(t) = \frac{1}{M(r_3) - t + \epsilon} > 0 \quad \text{and} \quad \phi''(t) = \frac{1}{(M(r_3) - t + \epsilon)^2} \quad \text{in } (-\infty, M(r_3)). \quad (43)$$

Furthermore, since $\phi(u^i) < 0$ on $B_{r_1}$, it holds that function $\psi := \max\{\phi(u^i), 0\}$ satisfies $\psi \equiv 0$ on $B_{r_1}$, thus also the mean value of $\psi$ vanishes, $\psi_{B_{r_1}} = 0$. This together with the $(1, p)$-Poincaré inequality and the doubling property of the Lebesgue measure (with doubling constant $2^n$) implies

$$\int_{B_{r+r_2}} \psi^p \leq \int_{B_{r+r_2}} \left| \psi - \psi_{B_{(r+r_2)/2}} + \left( \psi_{B_{(r+r_2)/2}} - \psi_{B_{r_1}} \right) \right|^p \leq 2^p \int_{B_{r+r_2}} \left| \psi - \psi_{B_{(r+r_2)/2}} \right|^p + 2^{p+n} \left( \int_{B_{r+r_2}} |\nabla \psi| \right)^p \leq C(p, n) \left( \frac{r + r_2}{2} \right)^p \int_{B_{r+r_2}} |\nabla \psi|^p. \quad (44)$$
We consider two cases.

Case 1: $1 < p < 2$. Recall the supremum estimate (16) from Lemma 2:

$$\left( \sup_{B_{r+\varepsilon}} \phi(u^1) \right)^p \leq C_{\sup} \frac{(\delta R)^n}{(\delta R)^n} \int_{B_R} \phi(u^1)^p + R^{p-n},$$

where $C_{\sup}$ is constant in (17), cf. statement of Lemma 2. We apply this estimate with $R = (r + r_2)/2$ and $0 < \delta = (r_2 - r)/(r_2 + r) < 1$, then use the definition of function $\psi$ and Poincaré-type estimate (44).

$$\left( \sup_{B_r} \phi(u^1) \right)^p \leq C_{\sup} \frac{1}{(r_2 - r)^n} \int_{B_{r+\varepsilon}} \phi(u^1)^p + \left( \frac{r + r_2}{2} \right)^{p-n}$$

$$\leq C_{\sup} \frac{1}{(r_2 - r)^n} \left( \frac{r + r_2}{2} \right)^{p-n} \int_{B_{r+\varepsilon}} |\nabla \phi(u^1)|^p + \left( \frac{r + r_2}{2} \right)^{p-n}$$

$$\leq C_{\sup} \frac{1}{c + 1} \frac{1}{r_2^{n-p}} \int_{B_{r+\varepsilon}} |\nabla \phi(u^1)|^p + \left( \frac{2}{c + 1} \right) \frac{1}{r_2^{n-p}}. \quad (45)$$

In the last step we included expression with $r_1$, $r_2$, $c$ into the constant $C_{\sup}$. In order to use estimate (12) in (45) we need to use first the Young inequality with exponents $\frac{1}{\alpha} = 1 + \frac{1}{\varepsilon}$ and $\beta = 1 + \varepsilon$, for some $\varepsilon \in (0, 1)$:

$$\int_{B_{r+\varepsilon}} |\nabla \phi(u^1)|^p = \int_{B_{r+\varepsilon}} \frac{|\nabla u^1|}{|Du|}^{(2-p)\alpha} |\nabla \phi(u^1)|^p \alpha \left( \frac{|\nabla u^1|}{|Du|} \right)^{(p-2)\alpha} |\nabla \phi(u^1)|^{(1-\alpha)}$$

$$\leq \int_{B_{r+\varepsilon}} \left( \frac{|\nabla u^1|}{|Du|} \right)^{2-p} \left( \frac{|\nabla \phi(u^1)|}{|Du|} \right)^{2-p} \left( \frac{|Du|}{|\nabla u^1|} \right)^{2-p} |\nabla \phi(u^1)|^p.$$

The second integral on the right-hand side of (46) can be easily estimated as follows:

$$\int_{B_{r+\varepsilon}} \left( \frac{|Du|}{|\nabla u^1|} \right)^{(2-p)\varepsilon} |\nabla \phi(u^1)|^p \leq \|\phi\|_{L^\infty(B_{r_2})}^p \|Du\|_{L^p(B_{r_2})}^p.$$

We use (12) from Lemma 1 with with $r_2$ and $\delta = \frac{r_2 - r}{2r_2}$ together with properties of radii (21). Then estimate (46) for the $p$-energy of $\phi(u^1)$ takes the following form:

$$\int_{B_{r+\varepsilon}} |\nabla \phi(u^1)|^p \leq \left( \frac{2c}{cr_2 - r_1} \right)^p r_2^{n-p} + \|\phi\|_{L^\infty(B_{r_2})}^p \|Du\|_{L^p(B_{r_2})}^p.$$

The expression on the right-hand side is similar to the one in $C_{\sup}$. We use this observation and the last inequality in estimate (45) and obtain the following bound, which we in turn estimate using properties of function $\phi$, see (43):
We use this inequality on the right-hand side of (47) and notice that by taking sufficiently
large \( r \), we get that the right-hand side of (47) can now be estimated by
\[
\left( \sup_{B_r} \phi(u^1) \right)^p \leq C_{\text{sup}} \frac{2^{p-n}}{(c + 1)^{p-n}} \frac{1}{r_2^{n-p}} \left( C(p, r_1, r_2, c) r_2^{n-p} + \| \phi \|^p_{L^\infty(B_{r_2})} \left\| D u \right\|^p_{L^p(B_{r_2})} \right) + \frac{2^{p-n}}{(c + 1)^{p-n}} \frac{1}{r_2^{n-p}} \\
\leq \frac{C}{r_2^{n-p}} \left\{ \left( 1 + \| \phi \|^p_{L^\infty(B_{r_2})} \right) \left( r_2^{n-p} + \| \phi \|^p_{L^\infty(B_{r_2})} \left\| D u \right\|_{L^p(B_{r_2})} + 1 \right) \right\} \\
\leq C \left( 1 + \| \phi \|^p_{L^\infty(B_{r_2})} \right) \left( r_2^{n-p} + \| \phi \|^p_{L^\infty(B_{r_2})} \left\| D u \right\|_{L^p(B_{r_2})} \right) \max\{1, r_2^{p-n}\} \\
\leq C \left( 1 + \frac{1}{|M(r_3) - M(r_2)|^{2p}} \right) \left( r_2^{n-p} + \| \phi \|^p_{L^\infty(B_{r_2})} \left\| D u \right\|_{L^p(B_{r_2})} \right) \max\{1, r_2^{p-n}\}. 
\]  

(47)

Here \( C = C(p, n, c, c_S, r_1, r_2) \). By the weak maximum principle in Lemma 3, it holds that
\( M(r_2) < M(r_3) \) for a non-constant \( u^1 \) and some \( r_3 > r_2 \). The continuity of \( u \) implies that
maxima \( M(r_2) \) and \( M(r_3) \) are attained at some points \( x_3 \in S_{r_3} \) and \( x_2 \in S_{r_2} \). By the mean value theorem and the Tolksdorf estimate we get that
\[
|u^1(x_2) - u^1(x_0)| \leq \sup_{B_{r_2}} |D u| \cdot |x_2 - x_0| \leq C(p, n) (r_2^{1-p} + \| D u \|_{L^p(B_{r_2})}) r_2^{1-n/p}.
\]

We now appeal to growth condition (22), to obtain the following estimate:
\[
|M(r_3) - M(r_2)| = |u^1(x_3) - u^1(x_2)| \geq |u^1(x_3) - u^1(x_0)| - |u^1(x_2) - u^1(x_0)| \\
\geq C r_3^\alpha - C(p, n) (r_2^{1-p} + \| D u \|_{L^p(B_{r_2})}) r_2^{1-n/p}.
\]

We use this inequality on the right-hand side of (47) and notice that by taking sufficiently
large \( r_3 \), for instance such that
\[
C r_3^\alpha \geq (1 + r_2^{2n} + \| D u \|^p_{L^p(B_{r_2})})^{1/(2p)} + C(p, n) (r_2^{1-p} + \| D u \|_{L^p(B_{r_2})}) r_2^{1-n/p}
\]
we get that the right-hand side of (47) can now be estimated by \( A := C(p, n, c, c_S, r_1, r_2) \max\{1, r_2^{p-n}\} \). Observe that \( r_3 \) depends on \( n, p, r_2 \) and \( \| D u \|_{L^p(B_{r_2})} \), but not on \( r \).

Case 2: \( 2 \leq p < n \). We start from the estimate similar to (45). Namely, the supremum estimate (19) leads to the following inequality:
\[
\left( \sup_{B_r} \phi(u^1) \right)^p \leq C_{\text{sup}} \left( \frac{2}{c + 1} \right)^{p-n} \frac{1}{r_2^{n-p}} \int_{B_{r_2}} |\nabla \phi(u^1)|^p.
\]

We use Lemma 1 with \( r_2 \) and \( \delta = \frac{r_2 - r}{2r_2} \) together with constant (20) from Lemma 2 and
properties of radii (21) to obtain the following inequalities:
\[
\left( \sup_{B_r} \phi(u^1) \right)^p \leq C_{\text{sup}} \left( \frac{2}{c + 1} \right)^{p-n} \frac{1}{r_2^{n-p}} \int_{B_{r_2}} |\nabla \phi(u^1)|^p \\
\leq C_{\text{sup}} \left( \frac{2}{c + 1} \right)^{p-n} \frac{1}{r_2^{n-p}} \left( \frac{2}{r_2 - r} \right)^p \int_{B_{r_2}} \left( \frac{|D u|}{|\nabla u^1|} \right)^{p-2}
\]
The reasoning analogous to Case 2 gives us that $p$ only on inequality by A. Observe that for $p = 2$ the constant on the right-hand side of (48) depends only on $p$, $n$, $c_S$ and $c$, $r_1$, $r_2$ due to (21).

Case 3: $p = n$. We discuss this case separately due to the importance of $n$-harmonic mappings in nonlinear analysis. As in the previous case we obtain the estimate similar to (45):

$$\left( \sup_{B_r} \phi(u^1) \right)^n \leq C \left( \frac{2}{r_2 - r} \right)^n \int_{B_{r+2r}} |\nabla \phi(u^1)|^n. $$

The reasoning analogous to Case 2 gives us that

$$\left( \sup_{B_r} \phi(u^1) \right)^n \leq c(n) \left( \frac{\|Du\|_{L^n(B_{2r})}}{c_1} \right)^{\frac{2n-2}{n}} \left( \frac{r_1}{c} + r_2 \right)^2 \frac{1}{r_2^{n-2}}. $$

As in the previous cases, we denote the constant on the right-hand side of the above inequality by A.

We are now in a position to complete the proof of Theorem. By our assumptions $\phi$ is strictly increasing and so we have that

$$\log \left( \frac{M(r_3) - M(r) + \epsilon}{M(r_3) - M(r_1) + \epsilon} \right) = - \sup_{B_r} \phi(u^1) \leq -A. $$

Note that in both cases: $1 < p < 2$ and $p \geq 2$, constant $A$ is independent of $\epsilon$. Upon simplifying this inequality we arrive at the following:

$$M(r) \leq e^{-A} M(r_1) + (1 - e^{-A}) M(r_3) + (1 - e^{-A}) \epsilon. $$

Letting $\epsilon \to 0^+$ we reach the first assertion of theorem.

In order to prove the second assertion, we define a function

$$\phi(t) = -\log \left( \frac{t - m(r_3) + \epsilon}{m(r_1) - m(r_3) + \epsilon} \right) \text{ for } t \in [m(r_3), \infty). $$

Similarly to the proof of the first assertion, we verify that $\phi$ is $C^2$ convex and satisfies conditions (4) and (5). Indeed, by computations we see that

$$\phi'(t) = -\frac{1}{t - m(r_3) + \epsilon} < 0 \quad \text{and} \quad \phi''(t) = \frac{1}{(t - m(r_3) + \epsilon)^2} \quad \text{in } [m(r_3), \infty).$$

As in the case of maxima, we introduce a function $\psi := \max \{ \phi(u^1), 0 \}$ and show that $\psi \equiv 0$ on $B_{r_1}$. Then, following the steps of the proof for $M(r)$ we reach conclusion that

$$\log \left( \frac{u^1(x) - m(r_3) + \epsilon}{m(r_1) - m(r_3) + \epsilon} \right) = - \sup_{x \in B_r} \phi(u^1(x)) \geq -A \quad x \in B_r. $$

Thus,

$$u^1(x) \geq e^{-A} m(r_1) + (1 - e^{-A}) m(r_3) - (1 - e^{-A}) \epsilon. $$
The second assertion of the theorem now follows from taking $\epsilon \to 0^+$ and the proof of Theorem is completed. \hfill \square

Example 1 Let us comment on the existence of $p$-harmonic mappings satisfying assumptions (22) and (23) of Theorem. In order to do so, we employ radial $p$-harmonic mappings, cf. Lemma 1 and discussion in Sect. 2. Under the notation of Theorem, let us suppose that $1 < p < 2$ and $u = H(|x|)x$ is a radial $p$-harmonic mapping in $\Omega \subset \mathbb{R}^n$ such that $H(x_0) = 0$ for $x_0 \in \Omega$. Then (22) reads:

$$|H(|x|)|x_1| \geq C|x|^\alpha \quad \text{for } x \in \mathbb{R}^n \setminus B_2(x_0).$$

For instance, let $\Omega$ be such that $\text{dist}(\Omega, \{ x \in \mathbb{R}^n : x_1 = 0 \}) > c$ and $H(|x|) = |x|^{\frac{2 - p - n}{p - 1}} + 1$, then computations at (30) reveal that $u = H(|x|)x$ is $p$-harmonic in $\Omega$ and the above condition holds for $\alpha = \frac{2 - p - n}{p - 1}$, see also Adamowicz [1, Chapter 4.1] for further discussion on radial $p$-harmonics.

As for $p > 2$ and assumption (23), recall that by the proof of Lemma 1 we have that

$$|\nabla u|^2 = (H')^2x_1^2 + 2HH'\frac{x_1}{r} + H^2 = \frac{x_1}{r}^2 \left( \frac{H'}{H} + 1 \right)^2 + 1 - \frac{x_1}{r^2} \geq 1 - \frac{x_1}{r^2}.$$

From this we infer, that $|\nabla u| > c$ follows from $(1 - \frac{x_1^2}{r^2})^{1/2} > c$, which in turn is satisfied e.g. if $\Omega$ is contained in cone-type domain $\{ x \in \mathbb{R}^n : \frac{x_1^2}{r^2} < 1 - c \}$ provided that $0 < c < 1$.

Remark 2 For $1 < p < 2$, Theorem can be proven in a modified version with radius $r_3 = r_2$ and without imposing the growth condition (22). Namely, for the proof of the first assertion we define function [cf. (42)]:

$$\phi(t) := -\log \left( \frac{M(r_3) - t + 1}{M(r_3) - M(r_1) + 1} \right) \quad \text{for } t \in (-\infty, M(r_3)),$$

and the analogous function for the proof of the second assertion, cf. (49). Then $\|\phi'\|_{L^\infty} < 1$ and estimate (47) simplifies as follows:

$$\left( \sup_{Br} \phi(u^1) \right)^p \leq C(p, n, c, C_S) \left( 1 + r_2^p + \|Du\|_{L^p(B_{2r_2})}^2 \right) \max\{1, r_2^{p-n} \}.$$

In such a case no additional growth restriction on $u^1$ is needed. However, the first assertion of Theorem takes the form:

$$M(r) \leq CM(r_1) + (1 - C)M(r_2) + 1 - C,$$

where $C = C(n, p, c, C, r_1, r_2, c, \|Du\|_{L^p(B_{2r_2})}^2)$.

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