Sharp constant of Hardy operators corresponding to general positive measures

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Abstract
We investigate a new kind of Hardy operator \( H_\mu \) with respect to arbitrary positive measures \( \mu \) and prove that \( H_\mu \) is bounded on \( L^p(d\mu) \) with an upper constant \( p/(p-1) \). Moreover, we characterize a sufficient condition about the measure which makes \( p/(p-1) \) to be the \( L^p \)-norm of \( H_\mu \).

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1 Introduction
Let \( \mu \) be a positive measure on \([0, \infty)\) and \( f \) be a nonnegative \( \mu \)-measurable function. Define Hardy operator with respect to the measure \( \mu \) by

\[
H_\mu f(x) = \frac{1}{\mu([0,x])} \int_{[0,x]} f(t) d\mu(t),
\]

if \( 0 < \mu([0,x]) < \infty \), and set \( H_\mu f(x) = 0 \), if \( \mu([0,x]) = 0 \) or \( \infty \).

Observe that if \( \mu \) is Lebesgue measure, then \( H_\mu \) becomes the classical Hardy operator

\[
Hf(x) = \frac{1}{x} \int_{0}^{x} f(t) dt,
\]

and if \( \mu = \sum_{k=1}^{\infty} \delta_{k} \), then \( H_\mu \) becomes the discrete Hardy operator

\[
Hf(k) = \frac{f(1) + \cdots + f(k)}{k}.
\]

For \( 1 < p < \infty \), reference [1] showed that the two operators are bounded on \( L^p \) and \( F \) respectively. Moreover, for both, the best constants are \( p/(p-1) \) and the maximizing functions do not exist. We refer the reader to [2–6] for the background material and further references.

Hardy operator has a close relationship with Hardy–Littlewood maximal operator. From the point of rearrangement, \( Hf \) is equivalent to \( Mf \) (see reference [7]). In reference [8], Grafakos considered the \( L^p \)-boundedness for the maximal functions associated with general measures. In this paper, we shall discuss the sharp problems about \( H_\mu \). We will show that the operator \( H_\mu \) is bounded on \( L^p(d\mu) \) with an upper bound no more than

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Furthermore, we will characterize a sufficient condition about $\mu$ such that 
\[ \|H_\mu\|_{L^p\rightarrow L^p} = \frac{p}{p-1}. \]

From the definition about $H_\mu$, it is not necessary to consider the points $x$ such that $\mu([0,x]) = 0$ or $\infty$. Therefore, we let
\[ a = \inf\{x : \mu([0,x]) > 0\}, \]
and
\[ b = \begin{cases} \infty & \text{if } B = \emptyset, \\ \inf B & \text{if } B \neq \emptyset, \end{cases} \]
where $B$ denotes the set $\{x : \mu([0,x]) = \infty \text{ or } \mu([x,\infty)) = 0\}$. Then we call that the measure $\mu$ is supported in the interval $[a,b]$.

For the case of weak type inequality, the best constant from $L^p(d\mu)$ to $L^{p,\infty}(d\mu)$ is always $1$.

**Theorem 1.1** Let $\mu$ be a positive measure on $[0,\infty]$ and $1 \leq p < \infty$. Then we have
\[ \|H_\mu\|_{L^p(d\mu)\rightarrow L^{p,\infty}(d\mu)} = 1. \]

**Theorem 1.2** Suppose that $\mu$ is supported in $[a,b]$ and $f \in L^p(d\mu)$ with $1 < p < \infty$. For $f \neq 0$, define
\[ \mathcal{R}_\mu(f) = \frac{\|H_\mu f\|_{L^p(d\mu)}}{\|f\|_{L^p(d\mu)}}. \]
Then the following statements hold:
(i) $\|H_\mu f\|_{L^p(d\mu)} \leq \frac{p}{p-1}\|f\|_{L^p(d\mu)}$ holds for arbitrary positive measure $\mu$.
(ii) There exists no function $f$ such that $\mathcal{R}_\mu(f) = \frac{p}{p-1}$ holds.

**Theorem 1.3** If $\mu$ satisfies one of the following conditions:

Condition 1. $\mu([a,b]) = \infty$ and
\[ \lim_{x \to b} \frac{\mu([a,x])}{\mu([a,x])} = 1; \]

Condition 2. $[a]$ is not an atom of $\mu$, and
\[ \lim_{x \to a} \frac{\mu([a,x])}{\mu([a,x])} = 1, \]
then we have
\[ \sup_{f \in L^p(d\mu), f \neq 0} \mathcal{R}_\mu(f) = \frac{p}{p-1}. \]
We remark that there indeed exist some measures so that
\[ \sup_{f \in L^p(d\mu), f \neq 0} \mathcal{R}_\mu(f) < \frac{p}{p-1}. \] (3)
For example, it is easy to know that the Dirac measure $\delta_0$ satisfies inequality (3). In this paper, we will give some more complex counterexamples.

2 Preliminary and lemmas

In the study of sharp problems, the rearrangement of function is a very useful tool. Let

$$d_f(s) = \mu\left(\{|f| > s\}\right).$$

Then the rearrangement of $f$ is defined by

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}.$$

By the properties of the rearrangement, we can easily have

$$\|f\|_{L^p(d\mu)} = \|f^*\|_{L^p(d\mu')}.$$

We refer the reader to [9] for more properties of rearrangement. In reference [1], Hardy gave the following result.

**Lemma 2.1** (G.H. Hardy and J.E. Littlewood) Let $(X, \mu)$ be a measurable space. If $f, g \in \mathcal{M}(X, \mu)$, then

$$\int_X |fg| \, d\mu \leq \int_0^\infty f^*(t)g^*(t) \, dt$$

holds.

Moreover, the theory of rearrangement plays an important role in proving the existence of maximizing function. This is because of the following lemma introduced by Lieb [10].

**Lemma 2.2** Suppose that $(M, \Sigma, \mu)$ and $(M', \Sigma', \mu')$ are two measure spaces. Let $X$ and $Y$ be $L^p(M, \Sigma, \mu)$ and $L^q(M', \Sigma', \mu')$ with $1 \leq p < q < \infty$. Let $A$ be a bounded linear operator from $X$ to $Y$. For $f \in X$ with $f \neq 0$, set

$$\mathcal{R}(f) = \frac{\|Af\|_Y}{\|f\|_X}$$

and

$$N = \sup\{\mathcal{R}(f) : f \neq 0\}.$$

Let $\{f_j\}$ be a uniform norm-bounded maximizing sequence for $N$, and assume that $f_j \to f \neq 0$ and that $A(f_j) \to A(f)$ pointwise almost everywhere. Then $f$ maximizes, i.e., $\mathcal{R}(f) = N$.

3 The boundedness of weak-$L^p$

In this section, we first prove Theorem 1.1. For the sake of clarity, we define a function as

$$F_\mu(x) := \mu([0, x]).$$
Obviously $F_\mu$ increases as $x \to \infty$. It follows from Lemma 2.1 and the definition of $H_\mu$ that

$$H_\mu f(x) = \frac{1}{\mu([0,x])} \int_{[0,x]} f(t) \, d\mu(t)$$

$$\leq \frac{1}{F_\mu(x)} \int_{[0,F_\mu(x)]} f^*(t) \, dt$$

$$= Hf^*(F_\mu(x)).$$  \hspace{1cm} (4)

Let

$$E_\mu^\ast (\lambda) := \{ x : Hf^*(F_\mu(x)) > \lambda \}.$$

Note that $f^*$ decreases, so we easily have that $Hf^*$ decreases as well. If we take

$$x_0 = \sup \{ x : Hf^*(F_\mu(x)) > \lambda \},$$

then it implies that

$$E_\mu^\ast (\lambda) = [0,x_0).$$

Thus, we can obtain that

$$\{ x : Hf^*(x) > \lambda \} \supset [0,F_\mu(x_0)).$$

We conclude that

$$\mu(\{ x : Hf^*(F_\mu(x)) > \lambda \}) = \mu(\{ x : Hf^*(x) > \lambda \}),$$

$$\text{where } | \cdot | \text{ denotes the Lebesgue measure.}$$

It follows from inequalities (4) and (5) that

$$\frac{\sup_{\lambda > 0} \lambda \mu(\{ x : Hf(x) > \lambda \})^{\frac{1}{p}}}{\| f^* \|_{L^p(dm)}} \leq \frac{\sup_{\lambda > 0} \lambda \mu(\{ x : Hf^*(F_\mu(x)) > \lambda \})^{\frac{1}{p}}}{\| f^* \|_{L^p(dm)}}$$

$$\leq \frac{\sup_{\lambda > 0} \lambda \mu(\{ x : Hf^*(x) > \lambda \})^{\frac{1}{p}}}{\| f^* \|_{L^p(dm)}}.$$  \hspace{1cm} (6)

Since $f^* \in L^p(dm)$, by Hölder’s inequality, we have that

$$Hf^*(x) = \frac{1}{x} \int_0^x f^*(t) \, dt \leq \left( \frac{1}{x} \int_0^x |f^*(t)|^p \, dt \right)^{\frac{1}{p}} \leq x^{-\frac{1}{p}} \| f^* \|_{L^p(dm)}.$$  \hspace{1cm} (7)

Thus it is obvious to obtain that

$$\left| \{ x : Hf^*(x) > \lambda \} \right| \leq \left| \{ x : x^{-\frac{1}{p}} \| f^* \|_{L^p(dm)} > \lambda \} \right| = \frac{\| f^* \|_{L^p(dm)}^p}{\lambda^p}. \hspace{1cm} (8)$$

From inequality (6) and inequality (8), we have

$$\frac{\sup_{\lambda > 0} \lambda \mu(\{ x : H_\mu f(x) > \lambda \})^{\frac{1}{p}}}{\| f \|_{L^p(dm)}} \leq 1.$$
That is,
\[
\frac{\|H_\mu f\|_{L^p(\{0,\infty\}, d\mu)}}{\|f\|_{L^p(d\mu)}} \leq 1
\]  
(9)
holds. This is equivalent to
\[
\|H_\mu\|_{L^p(d\mu) \to L^p(\{0,\infty\}, d\mu)} \leq 1.
\]  
(10)

Next it suffices to show that the constant 1 is sharp for inequality (10).
Take \(0 < x_1 < x_2 < \infty\) such that \(0 < \mu([x_1, x_2]) < \infty\). Let \(g = \chi_{(x_1, x_2)}\). It is easy to obtain
\[
\|H_\mu g\|_{L^p(\{0,\infty\}, d\mu)} = \|g\|_{L^p(d\mu)}.
\]

The proof is completed.

4 \(L^p\)-boundedness of the operator \(H_\mu\) with upper bound \(p/(p - 1)\)

Now we will show the results (i) and (ii) of Theorem 1.2.

Proof Following the proof of (5), we obtain
\[
\int_{\{0,\infty\}} \left(\frac{1}{\mu([0, x])}\right)^p d\mu(x) \leq \int_{\{0,\infty\}} f^p(x) \, dx.
\]  
(11)
By inequality (11), we conclude that
\[
\|H_\mu f\|_{L^p(\mathbb{R}, d\mu)} = \left(\int_{\mathbb{R}} \left(\frac{1}{\mu([0, x])}\right) \left|\int_{[0, x]} f(t) \, d\mu(t)\right|^p \, d\mu(x)\right)^{\frac{1}{p}}
\]  
\[
\leq \left(\int_{\mathbb{R}} \left|f^*(t)\right|^p \, d\mu(x)\right)^{\frac{1}{p}}
\]  
\[
= \left(\int_{\mathbb{R}} \left|\int_{[0, x]} f^*(t) \, dt\right|^p \, d\mu(x)\right)^{\frac{1}{p}}
\]  
\[
\leq \left(\int_{\mathbb{R}} \left|f^*(x)\right|^p \, dx\right)^{\frac{1}{p}}.
\]  
(12)
It follows from the inequality of classical Hardy operator that
\[
\left(\int_{\mathbb{R}} \left|f^*(x)\right|^p \, dx\right)^{\frac{1}{p}} \leq \frac{p}{p - 1} \|f^*\|_{L^p(d\mu)} = \frac{p}{p - 1} \|f\|_{L^p(d\mu)}.
\]  
(13)
Combining inequality (12) with inequality (13), we have
\[
\|H_\mu f\|_{L^p(\mathbb{R}, d\mu)} \leq \frac{p}{p - 1} \|f\|_{L^p(d\mu)}.
\]
Since the sharp function for the classical Hardy operator does not exist, it is easy to know from inequality (12) that there exists no function \(f\) such that \(\mathcal{R}_\mu(f) = \frac{p}{p - 1}\). The proof of the result (ii) of Theorem 1.2 is completed. 
\(\square\)
5 A characterization of the measure $\mu$ which ensures $\sup_{f \neq 0} R_\mu(f) = p/(p - 1)$

In this section, we try to characterize the measure $\mu$ which ensures $\sup_{f \neq 0} R_\mu(f) = p/(p - 1)$. We regard $\mu$ as a complete atom measure by giving an appropriate partition on $[0, \infty]$. We first present a partition on $[0, \infty]$ by giving an appropriate partition on $[0, \infty]$.

**Lemma 5.1** Let $\mu$ be a positive measure that is supported on $[0, \infty]$. If $\mu([0, \infty]) = \infty$ and

$$\lim_{x \to \infty} \frac{\mu([x])}{\mu([0,x])} = 0,$$

then there exists a partition on $[0, \infty]$ as

$$I_0 = [0, x_1], \quad I_1 = (x_1, x_2], \ldots, \quad I_k = (x_k, x_{k+1}], \ldots,$$

such that

$$\mu(I_{k+1}) \geq \mu(I_k),$$

and

$$\lim_{k \to \infty} \frac{\mu(I_k)}{\mu([0,x_{k+1}])} = 0.$$

**Proof** Let $x_1$ be any positive number. Denote $I_0 = [0, x_1]$. Since $\mu$ is supported on $[0, \infty]$, we have

$$\mu(I_0) > 0.$$ 

For $k = 2$, we let

$$x_2 = \inf \{x : \mu((x_1, x]) \geq \mu([0, x_1])\}.$$ 

For $k > 2$, we let

$$x_k = \inf \{x : \mu((x_{k-1}, x]) \geq \mu((x_{k-2}, x_{k-1}])\}.$$ 

Denote $I_k = [x_{k-1}, x_k]$ with $k = 2, 3, \ldots$. Since $\mu([0, \infty]) = \infty$, we easily have

$$\lim_{k \to \infty} x_k = \infty.$$ 

Thus, $\{I_k\}$ obviously constitutes a partition of $[0, \infty]$.

We first show that

$$\mu(I_k) \geq \mu(I_{k-1})$$

and

$$\mu((x_k, x_{k+1}]) \leq \mu(I_{k-1}). \quad (14)$$
By our construction, for any \( x > x_{k+1} \), it follows that
\[
\mu((x_k, x]) \geq \mu((I_{k-1})).
\]
Thus the property of measure implies that
\[
\mu(I_k) = \lim_{x \to x_{k+1}} \mu((x_k, x]) \geq \mu(I_{k-1}).
\]
Moreover, if \( x_k < x < x_{k+1} \), then \( \mu([x_k, x]) < \mu(I_{k-1}) \). Thus, it follows that
\[
\mu((x_k, x_{k+1})) = \lim_{x \to x_{k+1}} \mu((x_k, x]) \leq \mu(I_{k-1}).
\]
To complete the proof, it remains to show that
\[
\lim_{k \to \infty} \frac{\mu(I_{k-1})}{\mu([0, x_k])} = 0.
\]
This is equivalent to prove that, for any \( \epsilon > 0 \), there is an integer \( N > 0 \) such that
\[
\frac{\mu(I_{k-1})}{\mu([0, x_k])} \leq 2\epsilon
\]
holds for \( k \geq N \).
In order to prove this result, we divide the set \( \mathbb{Z}^+ \setminus \{1\} \) into two parts:
\[
F_\epsilon := \left\{ k \in \mathbb{Z} : k \geq 2, \frac{\mu([x_k])}{\mu([x_k, x_{k-1}])} < \epsilon \right\}
\]
and
\[
G_\epsilon := \left\{ k \in \mathbb{Z} : k \geq 2, \frac{\mu([x_k])}{\mu([x_k, x_{k-1}])} \geq \epsilon \right\}. \tag{16}
\]
By definition (16), if \( k \in G_\epsilon \), then we have
\[
\mu(I_{k-1}) \leq \left( 1 + \frac{1}{\epsilon} \right) \mu([x_k]). \tag{17}
\]
We discuss the problem in two cases:
Case I. \( G_\epsilon \) is not a finite set.
Case II. \( G_\epsilon \) is a finite set.
If \( G_\epsilon \) is not a finite set, then by equality \( \lim_{x \to \infty} \frac{\mu([x])}{\mu([0,x])} = 0 \), there exists an integer \( N \in G_\epsilon \) such that, for any \( k \geq N \),
\[
\frac{\mu([x_k])}{\mu([0, x_k])} < \frac{\epsilon^2}{1 + \epsilon}. \tag{18}
\]
Thus if \( k > N \) and \( k \in G_\epsilon \), then by inequalities (17) and (18), we have

\[
\frac{\mu(I_{k-1})}{\mu([0,x_k])} \leq \epsilon. \tag{19}
\]

On the other hand, if \( k > N \) and \( k \in F_\epsilon \), since \( G_\epsilon \) is not a finite integer and \( N \in G_\epsilon \), we can find a series of integers \( k_0, k_0 + 1, \ldots, k \) such that \( k_0 \in G_\epsilon \), and

\[
k_0 + 1, \ldots, k \in F_\epsilon.
\]

By the definition of \( F_\epsilon \) and inequality (14), we can conclude that if \( i \in F_\epsilon \), then

\[
\mu\left((x_{i-1}, x_i]\right) = \mu\left((x_{i-1}, x_i]\right) + \mu\left([x_i, x_i]\right)
\leq (1 + \epsilon)\mu\left((x_{i-1}, x_i]\right)
\leq (1 + \epsilon)\mu\left((x_{i-2}, x_{i-1}]\right). \tag{20}
\]

It immediately implies from inequality (20) that

\[
\mu\left((x_{k_0}, x_k]\right) = \sum_{i=k_0+1}^{k} \mu\left((x_{i-1}, x_i]\right)
\geq \sum_{i=k_0+1}^{k} (1 + \epsilon)^{i-k} \mu\left((x_{k-1}, x_k]\right)
= \mu\left((x_{k-1}, x_k]\right) \frac{1 - (1 + \epsilon)^{k-k_0}}{1 - (1 + \epsilon)}. \tag{21}
\]

Thus, by inequality (21), we have

\[
\frac{\mu(I_{k-1})}{\mu([0,x_k])} \leq \frac{\mu((x_{k-1}, x_k])}{\mu((x_{k_0-1}, x_k])} \leq \frac{1 - \frac{1}{1+\epsilon}}{1 - (\frac{1}{1+\epsilon})^{k-k_0}}
\leq \frac{\epsilon}{1 - (\frac{1}{1+\epsilon})^{k-k_0}}. \tag{22}
\]

Since \( k_0 \in G_\epsilon \), inequalities (14) and (20) imply

\[
\frac{\mu(I_{k-1})}{\mu([0,x_k])} \leq (1 + \epsilon)^{k-k_0} \frac{\mu(I_{k_0-1})}{\mu([0,x_k])} \leq (1 + \epsilon)^{k-k_0} \epsilon. \tag{23}
\]

If \( (1 + \epsilon)^{k-k_0} > 2 \), by inequality (22), we have

\[
\frac{\mu(I_{k-1})}{\mu([0,x_k])} \leq 2\epsilon.
\]

If \( (1 + \epsilon)^{k-k_0} \leq 2 \), by inequality (23), we have

\[
\frac{\mu(I_{k-1})}{\mu([0,x_k])} \leq 2\epsilon.
\]
At last, we conclude that if $k > N$ and $k \in F_{\varepsilon}$, then
\[
\frac{\mu(I_{k-1})}{\mu([0,x_k])} \leq 2\varepsilon. \quad (24)
\]
The proof of Case I is complete.

If $G_{\varepsilon}$ is a finite set, then we can find an integer $k_0$ such that $k \in F_{\varepsilon}$ for $k > k_0$. Then, by inequality (22), we can find a big enough integer $N$ such that
\[
\frac{\mu(I_{k-1})}{\mu([0,x_k])} \leq 2\varepsilon
\]
if $k \geq N$. The proof is completed. □

**Lemma 5.2** Suppose that $\mu$ is supported in $[0,\infty]$. If $\mu([0]) = 0$ and $\lim_{x \to 0} \frac{\mu([0,x])}{\mu([0,1])} = 1$, then there exists a partition on $[0,1]$,
\[
(x_1, 1], (x_2, x_1], \ldots, (x_k, x_{k-1}], \ldots,
\]
such that $\lim_{k \to \infty} x_k = 0$ and
\[
\lim_{k \to \infty} \frac{\mu((x_k, x_{k-1}])}{\mu([0, x_{k-1}])} = 0.
\]

**Proof** Without loss of generality, suppose
\[
\mu([0,1]) = \sum_{k=1}^{\infty} \frac{1}{k^2}.
\]
If $\mu([1]) < 1$, then we set $k_0 = 0$. If $\mu([1]) \geq 1$, then we set
\[
k_0 = \max \left\{ m : \sum_{k=1}^{m} \frac{1}{k^2} \leq \mu([1]) \right\}.
\]
It is easy to see that
\[
\sum_{k=1}^{k_0+1} \frac{1}{k^2} > \mu([1]).
\]
Then we can find a positive real number $x_1 < 1$ such that
\[
x_1 = \sup \left\{ x : \mu((x, 1]) \geq \sum_{k=1}^{k_0+1} \frac{1}{k^2} \right\}.
\]
Proceeding in this way, we set
\[
k_i = \max \left\{ m : \sum_{k=1}^{m} \frac{1}{k^2} \leq \mu([x_i, 1]) \right\} \quad (25)
\]
and

$$x_{i+1} = \sup \left\{ x : \mu((x,1]) \geq \sum_{k=1}^{k_i+1} \frac{1}{k^2} \right\}$$  \hspace{1cm} (26)$$

for \(i \geq 1\). By (27), (25), and (26), we can conclude

$$\sum_{k=1}^{k_i} \frac{1}{k^2} \leq \mu([x_i,1]) \leq \mu((x_{i+1},1]) \leq \sum_{k=1}^{k_i+1} \frac{1}{k^2} \leq \mu([x_{i+1},1]).$$  \hspace{1cm} (27)$$

It is easy to see that \(x_i > x_{i+1}\) and

$$\lim_{i \to \infty} \mu([x_i,1]) \geq \lim_{i \to \infty} \sum_{k=1}^{k_i} \frac{1}{k^2} = \mu((0,1)).$$

Thus we have \(\lim_{i \to \infty} x_i = 0\). It is easy to see that

\((x_1,1],(x_2,x_1],\ldots,(x_k,x_{k-1}],\ldots,\)

divide \((0,1]\). It can be implied from inequality (27) that

$$\mu([x_i,1]) + \frac{1}{(k_i+1)^2} \geq \sum_{k=1}^{k_i+1} \frac{1}{k^2} \geq \mu((x_{i+1},1]).$$  \hspace{1cm} (28)$$

To prove this partition satisfying the requirement of the lemma, we define two integer sets:

$$F_\epsilon = \left\{ k \geq 1 : \frac{\mu([x_k])}{\mu((x_{k+1},x_k])} < \epsilon \right\}$$

and

$$G_\epsilon = \left\{ k \geq 1 : \frac{\mu([x_k])}{\mu((x_{k+1},x_k])} \geq \epsilon \right\},$$

where \(\epsilon\) is an arbitrary positive real number. Since \(\lim_{x \to 0} \frac{\mu([0,x])}{\mu([0,x])} = 1\), we have \(\lim_{x \to 0} \frac{\mu([0,x])}{\mu([0,x])} = 0\). It is easy to find an integer \(N\) such that

$$\frac{\mu([x_{i-1}])}{\mu([0,x_{i-1}])} < 2\epsilon^2$$

for any integer \(i > N\). Thus, by the construction of \(G_\epsilon\), if \(i > N\) and \(i \in G_\epsilon\), we have

$$\frac{\mu((x_i,x_{i-1}])}{\mu([0,x_{i-1}])} < 2\epsilon.$$  \hspace{1cm} (29)$$

If \(i \in F_\epsilon\), then we have

$$\mu((x_{i+1},x_i]) \leq \frac{1}{1-\epsilon} \mu((x_{i+1},x_i)).$$
By inequalities (28) and (29), we have
\[
\frac{\mu((x_{i+1}, x_i))}{\mu([0, x_i])} \leq \frac{1}{1 - \epsilon} \frac{\mu((x_{i+1}, x_i))}{\mu([0, x_i])} = \frac{1}{1 - \epsilon} \frac{\mu((x_{i+1}, 1)) - \mu([x_i, 1])}{\mu((0, 1)) - \mu([x_i, 1])} \leq \frac{1}{1 - \epsilon} \frac{1/(k_i + 1)^2}{\sum_{k=k_i+2}^{\infty} 1/k^2}.
\]
Thus we can find a sufficiently large integer which is still denoted by \( N \) such that, for any integer \( i > N \) and \( i \in F_{\epsilon} \), there is
\[
\frac{\mu((x_i, x_{i-1}))}{\mu([0, x_{i-1}])} < 2\epsilon.
\]
Since \( \epsilon \) is an arbitrary real number, we have
\[
\lim_{k \to \infty} \frac{\mu((x_i, x_{i-1}))}{\mu([0, x_{i-1}])} = 0.
\]
The proof is completed.

After finishing our preparations, we can give the proof of the result (iii) of the main theorem.

**Proof** Let
\[
T_{a,b}(x) = \begin{cases} 
\tan(\frac{\pi}{2}(\frac{x-a}{b-a})), & 0 < b < \infty; \\
x - a, & b = \infty.
\end{cases} \tag{30}
\]
By equality (30), we can obtain a new measure denoted by \( \mu_T \) which is supported in \([0, \infty]\) so that, for any open interval \((x, y)\), we have
\[
\mu_T((x, y)) = \mu\left((T_{a,b}^{-1}(x), T_{a,b}^{-1}(y))\right).
\]
Then it is easy to get
\[
\sup\{\mathcal{R}_\mu f \mid f \in L^p(d\mu)\} = \sup\{\mathcal{R}_{\mu_T} f \mid f \in L^p(d\mu_T)\}.
\]
Thus it is enough to assume that the measure \( \mu \) is supported in \([0, \infty]\).

We first consider Condition 1.

By Lemma 5.1, we can divide \( \mathbb{R}^+ \) into a series of intervals
\[
[0, x_1], (x_1, x_2], \ldots, (x_k, x_{k+1}], \ldots,
\]
such that
\[
\lim_{k \to \infty} \frac{\mu((x_k, x_{k+1}])}{\mu([0, x_k])} = 0.
\]
For any $\epsilon > 0$, if we can find a function $f_{\epsilon}$ such that $R(f_{\epsilon}) \geq \frac{p}{p-1} - O(\epsilon)$, then the proof is completed.

By the property of the partition, there exists an integer $N$ satisfying

$$\frac{\mu([(x_k, x_{k+1}])}{\mu([0, x_k])} < \epsilon$$

for $k \geq N$. This inequality is equivalent to

$$\frac{\mu([0, x_{k+1}])}{\mu([0, x_k])} < 1 + \epsilon. \quad (31)$$

Let

$$f_{\epsilon} = \sum_{k=N}^{\infty} \mu([0, x_{k+1}])^{-\frac{1}{p} - \epsilon} x_{(x_k, x_{k+1})}.$$  

First we estimate the norm of $f_{\epsilon}$

$$\|f_{\epsilon}\|_{L^p(d\mu)} = \left( \sum_{k=N}^{\infty} \mu([0, x_{k+1}])^{-1-p\epsilon} \mu((x_k, x_{k+1})) \right)^{\frac{1}{p}}$$

$$\geq \left( \frac{1}{1 + \epsilon} \right)^{\frac{1}{p} + \epsilon} \left( \sum_{k=N}^{\infty} \mu([0, x_{k+1}])^{-1-p\epsilon} \mu((x_k, x_{k+1})) \mu([0, x_{k+1}])^{-1-p\epsilon} dt \right)^\frac{1}{p}$$

$$\geq \left( \frac{1}{1 + \epsilon} \right)^{\frac{1}{p} + \epsilon} \left( \int_{\mu([0, x_N])}^{\infty} t^{-1-p\epsilon} dt \right)^\frac{1}{p}$$

$$\geq \left( \frac{1}{1 + \epsilon} \right)^{\frac{1}{p} + \epsilon} \left( \frac{1}{p\epsilon} \right)^\frac{1}{p} \mu([0, x_N])^{-\epsilon}. \quad (32)$$

Next, we estimate the value of $H_{\mu}f_{\epsilon}(x)$. When $k \geq N$ and $x_k < x \leq x_{k+1}$, we have

$$H_{\mu}f_{\epsilon}(x) = \frac{1}{\mu([0, x])} \int_{[0, x]} f_{\epsilon}(t) d\mu(t)$$

$$\geq \frac{1}{\mu([0, x_{k+1}])} \int_{[0, x_k]} f_{\epsilon}(t) d\mu(t)$$

$$= \frac{1}{\mu([0, x_{k+1}])} \sum_{i=N}^{k-1} \mu([0, x_{i+1}])^{-\frac{1}{p} - \epsilon} \mu((x_i, x_{i+1}))$$

$$\geq \left( \frac{1}{1 + \epsilon} \right)^{\frac{1}{p} + \epsilon} \frac{1}{\mu([0, x_{k+1}])} \sum_{i=N}^{k-1} \mu([0, x_i])^{-\frac{1}{p} - \epsilon} \mu((x_i, x_{i+1}))$$

$$= \left( \frac{1}{1 + \epsilon} \right)^{\frac{1}{p} + \epsilon} \frac{1}{\mu([0, x_{k+1}])} \sum_{i=N}^{k-1} \int_{\mu([0, x_i])}^{\mu([0, x_{i+1}])} \mu([0, x_i])^{-\frac{1}{p} - \epsilon} dt$$
\[ \geq \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{\mu([0, x_k])} \int_{\mu([0, x_N])} t^{-\frac{1}{p} \epsilon} \, dt \]
\[ \geq \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \left( \mu([0, x_k]) \right)^{1 - \frac{1}{p} \epsilon} \frac{1}{\mu([0, x_k])} \left( \frac{\mu([0, x_N])^{1 - \frac{1}{p} \epsilon}}{\mu([0, x_k])} \right). \] (33)

Set

\[ f^{(1)}_\epsilon = \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \sum_{k=N}^{\infty} \mu([0, x_k]) \left( \frac{1}{\mu([0, x_k])} x_{(x_k, x_{k+1})} \right) \]
\[ = \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} f \]

and

\[ f^{(2)}_\epsilon = \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \sum_{k=N}^{\infty} \mu([0, x_k]) \left( \frac{1}{\mu([0, x_k])} x_{(x_k, x_{k+1})} \right). \]

Then we have

\[ \| f^{(2)}_\epsilon \|_{L^p(d\mu)} = \left( \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \sum_{k=N}^{\infty} \mu([0, x_k]) \left( \frac{1}{\mu([0, x_k])} x_{(x_k, x_{k+1})} \right) \right) \]
\[ \leq \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \left( \sum_{k=N}^{\infty} \mu([0, x_k]) \left( \frac{1}{\mu([0, x_k])} x_{(x_k, x_{k+1})} \right) \right)^{\frac{1}{p}} \]
\[ \leq \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \left( \frac{1}{1 - \frac{1}{p} - \epsilon} \right)^{\frac{1}{p}} \mu([0, x_N])^{-\epsilon}. \] (34)

By inequality (33), we have

\[ \| H_\epsilon f \|_{L^p(d\mu)} \geq \| f^{(1)}_\epsilon \|_{L^p(d\mu)} - \| f^{(2)}_\epsilon \|_{L^p(d\mu)}. \]

From this result and inequalities (32) and (34), we can get

\[ \mathcal{R}(f_\epsilon) \geq \frac{\| f^{(1)}_\epsilon \|_{L^p(d\mu)} - \| f^{(2)}_\epsilon \|_{L^p(d\mu)}}{\| f_\epsilon \|_{L^p(d\mu)}} \]
\[ = \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} - \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \left( \frac{1}{1 - \frac{1}{p} - \epsilon} \right)^{\frac{1}{p}} \]
\[ \geq \left( \frac{1}{1 + \epsilon} \right)^{1 + \frac{1}{p} \epsilon} \frac{1}{1 - \frac{1}{p} - \epsilon} \left( \frac{1}{1 + \epsilon} \right)^{\frac{1}{p} \epsilon} \left( \frac{1}{1 - \frac{1}{p} - \epsilon} \right)^{\frac{1}{p}}. \] (35)

Since \( \epsilon \) is arbitrary, it is easy to imply \( \sup_{\epsilon \neq 0} \mathcal{R}(f) = \frac{p}{p-1} \).

To prove condition (ii), by Lemma 5.2, we can part the intervals \([0, 1]\) to

\([x_1, 1], [x_2, x_1], \ldots, [x_{k+1}, x_k], \ldots\)
such that
\[
\lim_{k \to \infty} \frac{\mu((x_{k+1}, x_k])}{\mu((0,x_k]))} = 0.
\]

Then, for any \( \epsilon > 0 \), there is a sufficiently large integer \( N \) such that
\[
\frac{\mu((x_{k+1}, x_k])}{\mu((0,x_k]))} < \epsilon
\]
for \( k \geq N \).

Thus, we have
\[
\mu((0,x_k])) - \frac{1}{p+\epsilon} \geq 1 - \frac{1}{p+\epsilon}
\]
for \( k \geq N \).

Let \( f_\epsilon = \sum_{k=N}^{\infty} \mu((0,x_k]))^{-\frac{1}{p+\epsilon}} \chi_{(x_{k+1}, x_k]} \). Then, for \( x_{k+1} < x \leq x_k \) and \( k \geq N \), we have
\[
H_{\mu}f_\epsilon(x) = \frac{1}{\mu((0,x_k]))} \int_{(0,x_k] \setminus (0,x_{k+1}]} f_\epsilon(t) d\mu(t)
\]
\[
\geq \frac{1}{\mu((0,x_k]))} \int_{(0,x_{k+1}]} f_\epsilon(t) d\mu(t)
\]
\[
= \frac{1}{\mu((0,x_k]))} \sum_{i=k+1}^{\infty} \mu((0,x_i]))^{-\frac{1}{p+\epsilon}} \mu((x_{i+1}, x_i])
\]
\[
\geq \frac{1}{\mu((0,x_k]))} \left( \mu((0,x_k])^{1-\frac{1}{p+\epsilon}} \right)
\]
\[
= \frac{(1-\epsilon)^{1-\frac{1}{p+\epsilon}}}{1-\frac{1}{p+\epsilon}} f_\epsilon(x).
\]
\[
(36)
\]
\[
(37)
\]

It follows from inequality (36) that
\[
\mathcal{R}(f_\epsilon) \geq \frac{(1-\epsilon)^{1-\frac{1}{p+\epsilon}}}{1-\frac{1}{p+\epsilon}} f_\epsilon(x).
\]

Because \( \epsilon \) is arbitrary, it is easy to know \( \sup \mathcal{R}(f) \geq \frac{p}{p-1} \). The proof of the suffice part of Theorem 1.2 is then completed.

6 Counterexample

In this section we give some counterexamples that make \( \sup_{x \in \mathbb{R}} \mathcal{R}(f) < p/(p-1) \). The following two lemmas tell us that we can limit our discussion to a special function set.

Lemma 6.1 Suppose \( \mu \) is a positive measure on \( \mathbb{R} \), and it has an atom \( x_0 \). If \( \{f_n\}, n = 1, 2, \ldots \) is a series of functions satisfying \( f_n(x_0) = 1 \) and
\[
\lim_{n \to \infty} \mathcal{R}_p(f_n) = \frac{p}{p-1}.
\]
then we have
\[
\lim_{n \to \infty} \| f_n \|_{L^p(d\mu)} = \infty.
\]

Proof Without loss of generality, we assume that \( \mu(\{x_0\}) = 1 \). If the assertion does not hold, then we can assume that there exists a constant \( C \) satisfying \( \| f_n \|_{L^p(d\mu)} \leq C \). Let \( f_n^* \) be the decreasing rearrangement of \( f_n \), then it is easy to get \( \| f_n^* \|_{L^p(d\mu)} \leq C \) and \( f_n^*(1) \geq 1 \). Thus we have \( f^*(x) \geq 1 \) for \( 0 < x \leq 1 \). By Helly’s theorem, we can assume \( \lim_{n \to \infty} f_n^* = f^* \) almost everywhere. Since \( f_n^* \) is decreasing, we have
\[
C^p \geq \| f_n^* \|_{L^p(d\mu)}^p \geq \int_{[0,R]} |f_n^*(t)|^p \, dt \geq R |f_n^*(R)|^p,
\]
which is equivalent to \( f_n^*(R) \leq CR^{-\frac{1}{p}} \). Thus, by the control convergence theorem,
\[
\lim_{n \to \infty} Hf_n^*(x) = \lim_{n \to \infty} \frac{1}{x} \int_{[0,x]} f_n^*(t) \, dt = Hf^*(x). \quad (38)
\]
However, by inequality (12), we have
\[
\mathcal{R}_m(f_n^*) \geq \mathcal{R}_\mu(f_n),
\]
it obviously shows that \( \{ f_n^* \} \) is a maximizing sequence for \( H \), i.e.,
\[
\lim_{n \to \infty} \mathcal{R}(f_n^*) = \frac{p}{p-1}.
\]
By \( \lim_{n \to \infty} f_n^* = f^* \) and equality (38), using Lemma 2.2, we get \( \mathcal{R}_m(f^*) = \frac{p}{p-1} \), which contradicts the result about Hardy operator we have known. The proof is completed. \( \square \)

Lemma 6.2 Suppose \( \mu \) is a positive measure on \( \mathbb{R} \), and it has an atom \( x_0 \). If
\[
\sup \{ \mathcal{R}_\mu(f) : f \in L^p(d\mu) \} = \frac{p}{p-1},
\]
then there exists a series of functions \( \{ f_k \} \), \( k = 1, 2, \ldots \), and \( f_k(x_0) = 0 \) such that
\[
\lim_{k \to \infty} \mathcal{R}_\mu(f_k) = \frac{p}{p-1}.
\]

Proof It is obvious that we can assume there exists a series of functions \( g_k, g_k(x_0) = 1 \), such that
\[
\lim_{k \to \infty} \mathcal{R}_\mu(g_k) = \frac{p}{p-1}.
\]
Let
\[
f_k(x) = \begin{cases} 
    g_k(x), & x \neq x_0, \\
    0, & x = x_0.
\end{cases}
\]
Then we have
\[
H_{\mu}f_k(x) = \begin{cases} 
H_\mu g_k(x), & x < x_0; \\
H_\mu g_k(x) - \mu(\{x_0\})/\mu([0,x]), & x \geq x_0.
\end{cases}
\] (39)

By equality (39), we can get
\[
\|H_{\mu}f_k\|_{L^p} \geq \|H_\mu g_k\|_{L^p} - \left\| \frac{\mu(\{x_0\})}{\mu([0,x])} \chi_{[x_0,\infty]} \right\|_{L^p}. 
\] (40)

On the other hand, it is easy to obtain
\[
\|f_k\|_{L^p} \leq \|g_k\| + \mu(\{x_0\})^{\frac{1}{p}}. 
\] (41)

By Lemma 6.1, we know that \(\lim_{k \to \infty} \|g_k\|_{L^p(\mu)} = \infty\) and \(\lim_{k \to \infty} \mathcal{R}_{\mu}(g_k) = p/(p-1)\). By this result, together with inequalities (40) and (41), we can have
\[
\lim_{k \to \infty} \mathcal{R}_{\mu}(f_k) = \frac{p}{p-1}. 
\]

Now we can give some counterexamples.

**Example 6.3** Suppose that \(\mu\) is supported in \([a,b]\), \(\mu(\{a\}) > 0\), and \(\mu(\mathbb{R}_+) < \infty\). Then \(\sup_{f \neq 0} \mathcal{R}_{\mu}(f) < p/(p-1)\).

**Proof** Suppose that the result is not valid. By Lemma 6.2, we can find a series of functions \(\{f_k\}, f_k(a) = 0\), such that
\[
\lim_{k \to \infty} \mathcal{R}_{\mu}(f_k) = \frac{p}{p-1}. 
\]

Let \(A = \mu(\{a\})\), \(B = \mu(\mathbb{R}_+)\), and \(\mu_1 = \mu - A\delta_a\). Then we have
\[
H_\mu f_k(x) = \frac{\mu_1([0,x])}{\mu([0,x])} \frac{1}{\mu_1([0,x])} \int_{[0,x]} f_k \, d\mu_1 \leq \frac{B-A}{B} H_{\mu_1} f_k(x)
\] (42)
and
\[
\|f_k\|_{L^p(\mu)} = \|f_k\|_{L^p(\mu_1)}. 
\] (43)

By inequalities (42) and (43), we obtain
\[
\mathcal{R}_{\mu}(f_k) \leq \frac{B-A}{B} \mathcal{R}_{\mu_1}(f_k) \leq \frac{B-A}{B} \frac{p}{p-1}.
\]

It contradicts with \(\lim_{k \to \infty} \mathcal{R}_{\mu}(f_k) = p/(p-1)\). Then the counterexample is valid.

**Example 6.4** If \(\mu = \sum_{k=-\infty}^{\infty} \lambda^k \delta_{k}\) with \(\lambda > 1\), then \(\sup_{f \in L^p(\mu)} \mathcal{R} f < \frac{p}{p-1}\).
Proof. By the definition of $\mu$, we have

$$H_\mu f(\lambda_k) = \frac{1}{\mu((0, \lambda_k])} \int_{(0, \lambda_k]} f(t) d\mu(t)$$

$$= (\lambda - 1) \sum_{i=-\infty}^{k} \lambda^i f(\lambda^i)$$

$$= \frac{\lambda - 1}{\lambda} \sum_{i=-\infty}^{0} \lambda^i f(\lambda^i)$$

(44)

and

$$\|f(\lambda^i)\|_{L^p(d\mu)} = \left( \sum_{k=-\infty}^{\infty} |f(\lambda^{i+k})|^p \lambda^k \right)^{\frac{1}{p}}$$

$$= \left( \sum_{k=-\infty}^{\infty} |f(\lambda^k)|^p \lambda^{-k} \right)^{\frac{1}{p}}$$

$$= \lambda^{-\frac{1}{p}} \|f\|_{L^p(d\mu)}.$$  

(45)

By inequalities (44), (45), and Minkowski’s inequality, it follows

$$\|H_\mu f\|_{L^p(d\mu)} = \left\| \frac{\lambda - 1}{\lambda} \sum_{i=-\infty}^{0} \lambda^i f(\lambda^i) \right\|_{L^p(d\mu)}$$

$$\leq \frac{\lambda - 1}{\lambda} \sum_{i=-\infty}^{0} \lambda^i \|f(\lambda^i)\|_{L^p(d\mu)}$$

$$= \frac{\lambda - 1}{\lambda} \sum_{i=-\infty}^{0} \lambda^i \|f\|_{L^p(d\mu)}$$

$$= \frac{\lambda - 1}{\lambda} \|f\|_{L^p(d\mu)}.$$  

(46)

It is easy to get $\frac{\lambda - 1}{\lambda} < \frac{p}{p+1}$. The proof is completed. $\Box$

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