Two-Loop $N_F = 1$ QED Bhabha Scattering Differential Cross Section

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Abstract

We calculate the two-loop virtual, UV renormalized corrections at order $\alpha^4(N_F = 1)$ in QED to the Bhabha scattering differential cross section, for arbitrary values of the squared c.m. energy $s$ and momentum transfer $t$, and on-shell electrons and positrons of finite mass $m$. The calculation is carried out within the dimensional regularization scheme; the remaining IR divergences appear as polar singularities in $(D - 4)$. The result is presented in terms of 1- and 2-dimensional harmonic polylogarithms, of maximum weight 3.

Key words: Feynman diagrams, Multi-loop calculations, Box diagrams, Bhabha scattering

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1 Introduction

The Bhabha scattering, $e^+e^- \rightarrow e^+e^-$, plays a special role in the phenomenology of particle physics, since it is employed to determine the luminosity of $e^+e^-$ colliders operating at both high ($\sim 100$ GeV) and intermediate ($\sim 1-10$ GeV) center of mass energies. The luminosity of such facilities is essentially determined by calculating the (inverse) ratio of the theoretical Bhabha scattering cross section to the number of Bhabha scattering events observed. The Bhabha scattering is a process particularly suited for luminosity monitoring purposes, since its cross section is large and QED dominated (this is true in specific kinematic regions: the small scattering angle region for what concerns high energy colliders, the large scattering angle configuration for machines operating in the $\sim 1-10$ GeV center of mass energy range). As a consequence, this process can be calculated in perturbation theory with a high degree of accuracy. Since the theoretical error on the Bhabha scattering cross section affects directly the precision of the luminosity of an $e^+e^-$ collider, a remarkable amount of work has been devoted to the study of the radiative corrections to this process, both virtual and real, in the last three decades. For an extensive list of references on the subject we refer the interested reader to the review in [1].

In this paper, we focus our attention on the technical problem of the diagrammatic calculation of the virtual corrections to the Bhabha scattering cross section. The one-loop radiative corrections to this process, together with the corresponding real corrections involving the emission of a soft photon, calculated within the framework of the full electroweak Standard Model, are well known [2]. At the two-loop level, the calculation of the virtual corrections becomes much more involved. Even in pure QED, the complete (non approximated) two-loop differential cross section, i.e. the two-loop amplitude interfered with the tree level matrix element and summed over all spins, has not yet been calculated. A large amount of work was devoted to the study of the contributions enhanced by factors of $\ln(s/m_e^2)$ [3]. In recent years, the complete two-loop QED virtual cross section has been obtained in the limit of zero electron mass [4]. The IR divergent structure of this result has been studied in detail in [5].

The main technical challenge encountered in calculating the complete, non approximated, set of two-loop virtual correction is represented by the evaluation of the box-diagrams. Recently, we proved that the calculation of the two-loop box diagrams involving a closed fermion loop can be reduced to the calculation of a set of 14 scalar Master Integrals (MIs) [6]. This result has been obtained by means of the (by now standard) Laporta algorithm [7], for the systematic exploitation of the Integration-By-Parts (IBPs) [8], Lorentz invariance (LIs) [9], and general symmetry identities. The MIs involved in the calculation have been evaluated in [6, 10], within the context of $D$-dimensional regularization scheme [11], and by employing the method of the differential equations in the external kinematical variables [12, 13, 14]. The result was conveniently expressed in terms of 1- and 2-dimensional harmonic polylogarithms (HPLs, 2dHPLs) [15, 16, 17, 18, 19], of maximal weight 3.

The results in [6, 10] completed the calculation of all the MIs that are necessary in order to evaluate the two-loop Bhabha scattering differential cross section in
pure QED, if we limit our analysis to diagrams involving a closed fermion loop (conventionally referred to as the $\alpha^4(N_F = 1)$ differential cross section). In fact, the relevant vacuum polarization diagrams have been calculated in [20]. The two-loop vertex corrections involving a closed electron loop have been obtained in [21, 22], where the authors present a complete calculation of the vertex form factors in QED, without neglecting the electron mass.

The purpose of the present paper is to provide the expression of the contribution of the virtual corrections to the Bhabha scattering differential cross section at order $\alpha^4(N_F = 1)$ in QED. The UV renormalization of all the two-loop diagrams is carried out within the $D$-continuous dimensional regularization in the on-shell renormalization scheme. The residual IR divergence in the final result appears as a pole in $(D - 4)$, to be canceled, as usual, by the corresponding divergence due to the real emission of soft photons in the complete expression of the physical cross section.

The paper is organized as follows. In Section 2 we describe the various steps of the calculation, introducing our notations and conventions. In Section 3 we recall the main points of the UV on-shell renormalization scheme. In Section 4 we re-derive, for completeness, the one-loop virtual corrections and the Bhabha scattering differential cross section at order $\alpha^3$. The main result of this paper, the virtual differential cross section at order $\alpha^4(N_F = 1)$, is presented in Section 5. In Appendix A we collect the explicit expressions of the auxiliary functions introduced in the paper in terms of HPLs, and we briefly review the analytical continuation of the HPLs from the Euclidean to the physical kinematic region. In Appendix B we give the expansions of the auxiliary functions in different kinematical regimes.

## 2 The Bhabha scattering

In this Section we summarize the strategy followed in carrying out the calculation. In order to fix the notation and conventions employed throughout the paper, we start by discussing the tree-level Bhabha scattering differential cross section.

We consider the following scattering process mediated by photon exchange:

$$e^-(p_1) + e^+(p_2) \rightarrow e^-(p_3) + e^+(p_4),$$

(1)

The incoming electron, incoming positron, outgoing electron and outgoing positron have momenta $p_1, p_2, p_3$ and $p_4$, respectively, and finite mass $m$. The process is well described in terms of the Mandelstam kinematic invariants $s$, $t$ and $u$, defined by the relations:

$$s = -P^2 \equiv -(p_1 + p_2)^2 = 4E^2,$$

(2)

$$t = -Q^2 \equiv -(p_1 - p_3)^2 = -4(E^2 - m^2) \sin^2 \frac{\theta}{2},$$

(3)

$$u = -V^2 \equiv -(p_1 - p_4)^2 = -4(E^2 - m^2) \cos^2 \frac{\theta}{2},$$

(4)

with

$$s + t + u = 4m^2,$$

(5)
and where $E$ is the particle energy in the center of mass frame of reference and $\theta$ is the scattering angle in the same frame.

The differential cross section for the Bhabha scattering process can be written, in perturbation theory, as an expansion in the fine-structure constant $\alpha$:

$$
\frac{d\sigma(s, t, m^2)}{d\Omega} = \frac{d\sigma_0(s, t, m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right) d\sigma_1(s, t, m^2) + \left(\frac{\alpha}{\pi}\right)^2 d\sigma_2(s, t, m^2) + O\left(\left(\frac{\alpha}{\pi}\right)^3\right),
$$

where $\sigma_0(s, t, m^2)$ is the tree-level (Born) cross section and $\sigma_i(s, t, m^2)$ ($i = 1, 2$) are the higher order contributions.

The $s$- and $t$-channel diagrams contributing to the scattering process at the tree level are shown in Fig. 1, where, as usual, the wavy lines represent photons and the arrows on the fermionic (solid) lines follow the flow of the negative charge.

The tree-level amplitude, $M_0$, is given by

$$
M_0 = A_t - A_s,
$$

where the matrix element of the $t$-channel ($s$-channel) diagram in Fig. 1 is indicated by $A_t$ ($A_s$). In Eq. (7) we indicated explicitly the reciprocal negative sign between the $s$- and $t$-channel diagrams; in the following, the expressions of the contribution of the $s$-channel virtual diagrams to the differential cross section are given including this extra minus sign.

After averaging the squared amplitude at order $\alpha^2$ over the spins of the initial states and summing over the spins of the final states, the differential Born cross section reads:

$$
\frac{d\sigma_0(s, t, m^2)}{d\Omega} = \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[ st + \frac{s^2}{2} + (t - 2m^2)^2 \right] + \frac{1}{t^2} \left[ st + \frac{t^2}{2} + (s - 2m^2)^2 \right] \right\}.
$$

The first, second, and third term between curly brackets represent the contribution of the squared $s$-channel diagram, the squared $t$-channel diagram, and the $s$-$t$-diagram interference, respectively.

If we consider only the virtual contributions at order $\alpha^3$, it is necessary to evaluate the diagrams shown in Fig. 2. The corresponding squared matrix element, summed
over the spin of the initial particles and averaged over the spins of the final ones, will be given by the interference of each diagram in Fig. 2 with the tree-level diagrams of Fig. 1. The differential cross section assumes the following form:

\[
\left( \frac{\alpha}{\pi} \right) d\sigma_1^V(s, t, m^2) \frac{d\Omega}{d\Omega} = \frac{s}{16} \sum_{\text{spin}} \left\{ (-1)^{\text{c.c.}} + \cdots \right\},
\]

where the superscript \( V \) stands for “virtual” (and the sum is performed on all spins). The contributions of the various terms on the r.h.s. of Eq. (9) to the differential cross section will be discussed in Section 4.

At order \( \alpha^4 \), if we limit our analysis to the diagrams containing a closed electron loop (\( N_F = 1 \)), it is necessary to consider the contributions coming from the interference between the diagrams shown in Fig. 3 and the tree-level amplitude, as well as the interference between the one-loop vacuum polarization diagrams with the other one-loop diagrams without a fermionic loop. Diagrammatically, we have:

\[
\left( \frac{\alpha}{\pi} \right)^2 d\sigma_2^V(s, t, m^2) \frac{d\Omega}{d\Omega} = \frac{s}{16} \sum_{\text{spin}} \left\{ \left( \begin{array}{c} -1 \\ \end{array} \right) \times \cdots + \text{c.c.} \right\},
\]

The expression of the various contributions to the differential cross section shown in Eq. (10) will be given in Section 5.
Figure 3: Two-loop $N_F = 1$ diagrams.
The evaluation of the virtual radiative corrections to the Bhabha scattering, Eqs. \((9,10)\), leads to the calculation of UV- and IR-divergent integrals, which need to be regularized. Both classes of divergences are regularized within the framework of Dimensional Regularization scheme \([11]\), in which the regularization parameter is the dimension of the space-time \(D\). In order to evaluate the virtual radiative corrections we follow a procedure consisting of three main steps:

- First, we consider the Dirac structure of the squared matrix elements; the sum over the spins generates traces over the Dirac indices, that can be easily evaluated in \(D\) dimensions.

- At this stage, the squared amplitude describing the interference of a one-loop or two-loop diagram with the tree-level matrix element, as well as the interference of a one-loop with another one-loop diagram, can be expressed as a sum of a large number of scalar integrals (several thousands). According to \([6, 10]\), it is possible to express all these integrals as a combination of 14 Master Integrals. This result was obtained by using the Laporta algorithm \([7]\) in order to exploit the information contained in various sets of identities (Integration-by-Parts, Lorentz invariance and other general symmetry identities) relating the scalar integrals among themselves. Once the reduction to the MIs has been carried out, we can expand the squared amplitude, which will depend on the dimensional regulator \(D\), around \(D = 4\). Using the expressions for the MIs also given in \([6, 10]\), we finally obtain an analytic result, expressed in terms of 1- and 2-dimensional harmonic polylogarithms, in which both UV and IR divergences are regularized by the same parameter \(D\) and appear as poles in \((D - 4)\).

- The last step concerns the UV renormalization. By subtracting the appropriate counterterm graphs, the poles in \((D - 4)\) corresponding to the unphysical UV divergences of the diagrams cancel out. We perform the subtraction of the UV poles in the on-shell renormalization scheme. The result obtained in this way still contains IR poles in \((D - 4)\), which are canceled in the physical cross section by the contribution of the real soft photon radiation.

### 3 Renormalization

We perform the UV renormalization in the on-shell scheme. The counterterm diagrams for the renormalization of the one-loop and two-loop cross section are shown in Fig. 4. The counterterms give rise to the following Feynman rules:

\[
\begin{align*}
\mathcal{Z}_{3}^{(1)} & = - \left( \frac{\alpha}{\pi} \right) Z_{3}^{(1)} (p^2 \delta_{\mu\nu} - p_{\mu}p_{\nu}) , \\
\mathcal{Z}_{3}^{(2)} & = - \left( \frac{\alpha}{\pi} \right)^2 Z_{3}^{(2)} (p^2 \delta_{\mu\nu} - p_{\mu}p_{\nu}) ,
\end{align*}
\]
Figure 4: Counterterm diagrams at one and two loops.

\( Z_2^{(1l)} = - \left( \frac{\alpha}{\pi} \right) Z_2^{(1l)} (i \not{p} + m) , \) \hspace{1cm} (13)

\( \delta m^{(1l)} = - \left( \frac{\alpha}{\pi} \right) \delta m^{(1l)} , \) \hspace{1cm} (14)

\( Z_1^{(1l)} = \left( \frac{\alpha}{\pi} \right) Z_1^{(1l)} , \) \hspace{1cm} (15)

\( Z_1^{(2l)} = \left( \frac{\alpha}{\pi} \right)^2 Z_1^{(2l)} . \) \hspace{1cm} (16)

The one- and two-loop renormalization constants are:

\( Z_1^{(1l)}(D) = \frac{3}{2(D-4)} - 1 + (D - 4) + \mathcal{O}((D - 4)^2) , \) \hspace{1cm} (17)

\( Z_2^{(1l)}(D) = Z_1^{(1l)}(D) \quad \text{(Ward identity)} , \) \hspace{1cm} (18)
\[ Z_3^{(1l)}(D) = \frac{2}{3(D-4)} , \]  
\[ \frac{\delta m^{(1l)}(D, m)}{m} = Z_1^{(1l)}(D) , \]  
\[ Z_1^{(2l, N_F=1)}(D) = -\frac{1}{8(D-4)} + \frac{947}{288} - 2\zeta(2) + O(D-4) . \]  
\[ Z_3^{(2l)}(D) = \frac{1}{4(D-4)} - \frac{15}{16} + O(D-4) . \]  

The renormalization constants are defined in such a way that the UV-renormalized quantity is obtained by adding to the UV-divergent diagrams the corresponding counterterm graphs.

At the one-loop level, it is necessary to renormalize the photon vacuum polarization graphs, shown in Fig. 2 (a) and (b), and the \( \gamma f \bar{f} \)-vertex graphs, shown in Fig. 2 (c)–(f). In the first case, we add the UV-divergent diagram of Fig. 2 (a) (or (b)) and the corresponding counterterm of Fig. 4 (a), ending up with the renormalized expression for the photon self-energy given in the next Section. For the one-loop vertex it is necessary to add the UV-divergent diagram of Fig. 2 (c) (or (d)–(f)) with the corresponding counterterm of Fig. 4 (b). The UV-renormalized form factors found in this way are given in Eqs. (85,86).

In the two-loop case, we have to renormalize the UV divergences occurring in the diagrams of Fig. 3.

The three two-loop vacuum polarization diagrams have to be added with the corresponding seven counterterm diagrams shown in Fig. 4 (c)–(h) and (l). Note that, because of the QED Ward identity, \( Z_1(D) = Z_2(D) \) and the contributions of diagrams (c), (d), (g) and (h) in Fig. 4 cancel among themselves. The renormalization of the two-loop vacuum polarization consists, therefore, only in the one-loop electron mass renormalization and in the subsequent cancellation of genuine two-loop UV divergence against the counterterm diagram (l). For the two-loop vacuum polarization vertices, Figs. 3 (g)–(j), we have to add the counterterms (i) and (k) of Fig. 4 the first one cancels the subdivergences due to the one-loop vacuum polarization insertion, while the second cancels the overall two-loop divergence. The box graphs, shown in Figs. 3 (k)–(r), do not present two-loop UV divergences and their subgraph divergence is renormalized by adding the corresponding counterterm diagrams of the kind shown in Fig. 4 (j). Finally, the diagrams in Fig. 3 (s)–(v) are renormalized by adding to them the corresponding counterterm graphs subtracting the one-loop divergences of the vacuum polarization insertion and of the vertex correction.

Note that, even after the UV renormalization procedure, the Bhabha scattering differential cross section still shows poles in \( (D-4) \). This is due to the fact that IR divergences are still present and they are regularized by the same parameter \( D \) that regularizes the UV ones. These divergences are not physical and will be removed adding to the process also the corresponding soft real emission.
The contributions to the differential cross section appearing in the remainder of this work are all UV-renormalized. In the following, whenever discussing the contribution of a UV divergent one- or two-loop diagram, we assume as understood that the appropriate counterterm graphs have already been added and therefore the quantities under study are UV finite.

4 One-Loop Differential Cross Section

For completeness, we report in this Section the contribution to the cross section given by the interference between the one-loop virtual graphs and the tree-level amplitude. The one-loop diagrams contributing to the Bhabha scattering process are of three kinds: vacuum polarization diagrams, vertex correction diagrams, and box diagrams. Their contribution are discussed separately in the following subsections.

4.1 One-Loop Vacuum Polarization Corrections

The two one-loop diagrams containing a vacuum polarization subgraph can be obtained by inserting a photon self energy correction in the photon propagator, as shown in Fig. 2 (a) and (b). The photon self energy can be written as

\[ \Pi^{(1)}_{\mu\nu}(-p^2) = \left( \frac{\alpha}{\pi} \right) (\delta_{\mu\nu}p^2 - p^\mu p^\nu)\Pi^{(1)}_0(-p^2) , \tag{23} \]

where \( p \) indicates the generic 4-momentum entering the self-energy graph. The UV renormalized \( \Pi^{(1)}(p^2) \) has the following Laurent expansion:

\[ \Pi^{(1)}_0(-p^2) = \Pi^{(1,0)}_0(-p^2) + (D - 4) \Pi^{(1,1)}_0(-p^2) + \mathcal{O}\left( (D - 4)^2 \right) ; \tag{24} \]

the explicit expression of the \( \Pi^{(1,i)}_0(-p^2) \), \( i = 0, 1 \) is given in Eqs. (82, 83).

The self-energy diagrams in Fig. 2 (a) and (b) factorize in the product of the tree-level amplitude times the function \( \Pi^{(1)}_0(-p^2) \), so that one finds:

\[ A^{(1,S)}_S = A^{(0)}_S \left( \frac{\alpha}{\pi} \right) \Pi^{(1)}_0(s) , \quad A^{(1,S)}_t = A^{(0)}_t \left( \frac{\alpha}{\pi} \right) \Pi^{(1)}_0(t) . \tag{25} \]

The superscript “S” in the equation above stands for “self-energy”.

This set of virtual QED corrections is IR finite; therefore, the contribution of these diagrams to the differential cross section at order \( \alpha^3 \) is:

\[
\frac{d\sigma_Y(s, t, m^2)}{d\Omega}|_{(1,S)} = \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[ st + \frac{s^2}{2} + (t - 2m^2)^2 \right] 2\text{Re}\Pi^{(1,0)}_0(s) \\
+ \frac{1}{t^2} \left[ st + \frac{t^2}{2} + (s - 2m^2)^2 \right] 2\Pi^{(1,0)}_0(t) \\
+ \frac{1}{st} \left[ (s + t)^2 - 4m^4 \right] \left( \text{Re}\Pi^{(1,0)}_0(s) + \Pi^{(1,0)}_0(t) \right) \right\} , \tag{26}
\]
4.2 One-Loop Vertex Corrections

At the one-loop level, there are four vertex correction diagrams (see Fig. 2 (c), (d), (e) and (f)). At first, we consider the diagram in Fig. 2 (c). The vertex correction to the electron current can be written as

\[ \Gamma^\mu(p_1, p_3) = \left( \frac{\alpha}{\pi} \right) \left[ F_1^{(1)}(t) \gamma^\mu + \frac{1}{2m} F_2^{(1)}(t) \sigma^{\mu\nu}(p_1 - p_3)_\nu \right], \]  

where \( \sigma^{\mu\nu} = -i/2 [\gamma^\mu, \gamma^\nu] \) and \( t = -(p_1 - p_3)^2 \) (space-like); the electron and positron spinors have been omitted from Eq. (27). The functions \( F_1^{(1)}(t) \) and \( F_2^{(1)}(t) \) are the UV renormalized form factors; their Laurent expansion reads

\[ F_1^{(1)}(t) = \frac{1}{(D - 4)} F_1^{(1,-1)}(t) + F_1^{(1,0)}(t) + \mathcal{O}((D - 4)), \]  
\[ F_2^{(1)}(t) = F_2^{(1,0)}(t) + \mathcal{O}((D - 4)). \]  

The renormalized form factor \( F_1^{(1)}(t) \) still shows an IR polar divergence in \( (D - 4) \); the explicit expressions in terms of harmonic polylogarithms are given in Eqs. (28, 29, 30) of Appendix A.

The contribution of the diagram in Fig. 2 (c) to the Bhabha differential cross section at order \( \alpha^3 \) can be written as

\[ \frac{d\sigma_\gamma^\gamma(s, t)}{d\Omega} \bigg|_{(V(c))} = \frac{\alpha^2}{s} \left[ \frac{1}{2t} V_1^{(1)}(s, t) + \frac{1}{t^2} V_2^{(1)}(s, t) \right], \]

where

\[ V_1^{(1)}(s, t) = \left[ 2 \left( st + \frac{1}{2} s^2 + \frac{1}{2} t^2 - 2m^2 \right) + \frac{1}{2}(D - 4)(st - 2sm^2 + s^2) - 2tm^2 + t^2 \right] \text{Re} F_1^{(1)}(t) + 2 \left( st - \frac{4}{3} tm^2 + \frac{3}{4} t^2 \right) \text{Re} F_2^{(1)}(t), \]
\[ V_2^{(1)}(s, t) = \left[ 2 \left( st - 4sm^2 + s^2 + \frac{1}{2} t^2 + 4m^4 \right) + \frac{1}{2}(D - 4)t^2 \right] \text{Re} F_1^{(1)}(t) + 2 \left( tm^2 + \frac{1}{2} t^2 \right) \text{Re} F_2^{(1)}(t). \]

Due to the IR pole present in \( F_1^{(1)}(t) \), the two functions \( V_i^{(1)}(s, t), i = 1, 2 \) have the Laurent expansions

\[ V_i^{(1)}(s, t) = \frac{1}{(D - 4)} V_i^{(1,-1)}(s, t) + V_i^{(1,0)}(s, t) + \mathcal{O}((D - 4)). \]  

The explicit values of the Laurent coefficients can then be expressed in terms of the Laurent coefficients of the form factors \( F_i^{(1)}(t) \) in Eqs. (28, 29, 30):

\[ V_i^{(1,-1)}(s, t) = c_{i,1}(s, t) \text{Re} F_1^{(1,-1)}(t) \]
\[ V_i^{(1,0)}(s, t) = c_{i,1}(s, t) \text{Re} F_1^{(1,0)}(t) + c_{i,2}(s, t) \text{Re} F_1^{(1,-1)}(t) + c_{i,3}(s, t) \text{Re} F_2^{(1,0)}(t), \]

where \( c_{i,j}(s, t) \) are constants.
with

\[ c_{1,1}(s,t) = (s + t)^2 - 4m^4, \]  \hfill (36)
\[ c_{1,2}(s,t) = \frac{1}{2}(st - 2sm^2 + s^2 - 2tm^2 + t^2), \]  \hfill (37)
\[ c_{1,3}(s,t) = 2\left(st - \frac{4}{3}m^2 + \frac{3}{4}t^2\right), \]  \hfill (38)
\[ c_{2,1}(s,t) = 2\left(st - 4sm^2 + s^2 + \frac{1}{2}t^2 + 4m^4\right), \]  \hfill (39)
\[ c_{2,2}(s,t) = \frac{1}{2}t^2, \]  \hfill (40)
\[ c_{2,3}(s,t) = t(t + 2m^2). \]  \hfill (41)

For convenience of later use in the above equations we write \( \text{Re}F^{(1\ell)}_i(t) \) \((i = 1, 2)\) even if the \( F^{(1\ell)}_i(t) \) are real functions for space-like \( t \). The first term between square brackets in Eq. \((30)\) comes from the interference between the tree level \( s \)-channel diagram and the diagram in Fig. 2 (c); the second term originates in the interference between the same vertex correction and the tree level \( t \)-channel. The terms proportional to \((D - 4)\) in \( V_1(s,t) \) and \( V_2(s,t) \) give rise, when multiplied by the IR pole in \( F^{(1\ell)}_1(t) \), to the finite contributions proportional to \( c_{i,2}(s,t) \) in Eq. \((35)\). We observe that in \( V_1(s,t) \), the term \( c_{1,1}(s,t) \) of Eqs. \((34, 35)\) is identical to the coefficient of the \( s-t \)-interference term in the tree-level, Eq. \((3)\), while the term \( c_{2,1}(s,t) \) is twice the coefficient of the \( 1/t^2 \) term in Eq. \((3)\). Those relations are expected since \( F^{(1\ell)}_1(t) \) appears as a multiplicative factor of the tree-level QED vertex in the amplitude to be squared.

The diagram in Fig. 2 (d) can be obtained from the diagram in Fig. 2 (c) with the transformations \( p_2 \leftrightarrow -p_3 \) and \( p_4 \leftrightarrow -p_1 \). Since these transformations leave the Mandelstam invariants \( s \) and \( t \) unchanged, the contribution of the diagram (d) to the cross section is equal to the contribution of diagram (c).

The contributions of diagrams (e) and (f) in Fig. 2 are also identical, since the latter can be obtained from the former by exchanging \( p_1 \leftrightarrow -p_4 \) and \( p_2 \leftrightarrow -p_3 \), which leave \( s \) and \( t \) unchanged. Moreover, diagram (e) can be obtained from diagram (c) by the replacement \( p_2 \leftrightarrow -p_3 \) which exchanges \( s \leftrightarrow t \), so that the contribution of diagram (e) to the differential cross section at order \( \alpha^3 \) will be

\[
\left. \frac{d\sigma^V(s,t)}{d\Omega} \right|_{(V_i)} = \frac{\alpha^2}{s} \left[ \frac{1}{s^2}V_2^{(1\ell)}(t,s) + \frac{1}{st}V_1^{(1\ell)}(t,s) \right]. \quad \text{(42)}
\]

According to the definitions in Eqs. \((31, 32)\), the expression above involves the functions \( F^{(1\ell)}_i(s) \) \((i = 1, 2)\); above threshold \((s > 4m^2)\) they develop an imaginary part, which obviously cancels out in the cross section at this order. In view of that, in writing Eqs. \((31, 32)\) we explicitly indicated that only the real part of the functions \( F^{(1\ell)}_i \) contributes to the cross section.

The total contribution of the four vertex correction diagrams of Fig. 2 (c)–(f) to
the differential cross section is

\[
\frac{d\sigma^V_{(s,t)}}{d\Omega}_{(1l,V)} = \frac{\alpha^2}{s} \left[ \frac{1}{s^2} V_2^{(1l)}(t,s) + \frac{1}{t^2} V_2^{(1l)}(s,t) + \frac{1}{st} \left( V_1^{(1l)}(s,t) + V_1^{(1l)}(t,s) \right) \right].
\]  

(43)

4.3 One-Loop Box Corrections

The one-loop virtual corrections to Bhabha scattering in QED include four box diagrams: the direct box diagrams in the \( t \) - and \( s \) -channel (shown in Fig. 2 (g) and (h), respectively), and the corresponding crossed graphs (shown in Fig. 2 (i) and (j)). The interference of the diagram in Fig. 2 (g) with the tree-level matrix element is given by

\[
\frac{d\sigma^V_{(s,t)}}{d\Omega}_{(B,(g))} = \frac{\alpha^2}{4s} \left[ \frac{m^2}{s} \text{Re} B_1^{(1l)}(s,t) + \frac{m^2}{t} \text{Re} B_2^{(1l)}(s,t) \right],
\]  

(44)

where the quantities \( B_1^{(1l)}(s,t) \) and \( B_2^{(1l)}(s,t) \) are functions of the \( s \) and \( t \) Mandelstam invariants (and of the electron mass); they are UV finite, but develop an IR pole in \((D - 4)\) (to be canceled out in the physical cross section once the real soft radiation is accounted for) so that their Laurent expansion reads

\[
B_i^{(1l)}(s,t) = \frac{1}{(D - 4)} B_i^{(1l,-1)}(s,t) + B_i^{(1l,0)}(s,t) + O\left((D - 4)\right). 
\]  

(45)

The explicit expression of the various functions is given by Eqs. (87,88,89,90) in Appendix A.

The diagram (h) in Fig. 2 is the same as diagram (g), in the same figure, once we exchange \( p_2 \leftrightarrow -p_3 \), which is equivalent to the exchange \( s \leftrightarrow t \). As a consequence, the contribution of diagram (h) to the differential cross section at order \( \alpha^3 \) is:

\[
\frac{d\sigma^V_{(s,t)}}{d\Omega}_{(B,(h))} = \frac{\alpha^2}{4s} \left[ \frac{m^2}{s} \text{Re} B_2^{(1l)}(t,s) + \frac{m^2}{t} \text{Re} B_1^{(1l)}(t,s) \right].
\]  

(46)

Interfering the diagram (i) in Fig. 2 with the tree-level graphs one finds:

\[
\frac{d\sigma^V_{(s,t)}}{d\Omega}_{(B,(i))} = \frac{\alpha^2}{4s} \left[ \frac{m^2}{s} B_3^{(1l)}(u,t) - \frac{m^2}{t} B_2^{(1l)}(u,t) \right],
\]  

(47)

where \( B_3^{(1l)}(u,t) \) is another auxiliary function, with the same Laurent expansion as Eq. (45), whose explicit expression is given in Eqs. (91,92) of Appendix A. In Eq. (47), the functions \( B_3^{(1l)}(u,t) \) and \( B_2^{(1l)}(u,t) \) do not develop an imaginary part in the physical region, so that \( \text{Re} B_3^{(1l)}(u,t) = B_3^{(1l)}(u,t) \), and \( \text{Re} B_2^{(1l)}(u,t) = B_2^{(1l)}(u,t) \).

The diagram (j) of Fig. 2 can be obtained from diagram (i) of the same figure, by exchanging \( p_2 \leftrightarrow -p_3 \), so that \( s \leftrightarrow t \); the contribution of this diagram to the cross
section can then be obtained from Eq. (47) by replacing \( t \) with \( s \) in the arguments of the functions \( B_2^{(11)} \) and \( B_3^{(11)} \):

\[
\frac{d\sigma^V_1(s, t)}{d\Omega}_{(B,(j))} = -\frac{\alpha^2}{4s} \left[ \frac{m^2}{s} \text{Re}B_2^{(11)}(u, s) - \frac{m^2}{t} \text{Re}B_3^{(11)}(u, s) \right].
\] (48)

Summarizing, the contribution of the interference of the 4 diagrams in Fig. 2 (g)–(j) with the tree-level matrix element to the Bhabha scattering differential cross section is given by:

\[
\frac{d\sigma^V_1(s, t)}{d\Omega}_{(1l,B)} = \frac{\alpha^2}{4s} \left[ \frac{m^2}{s} \left( \text{Re}B_1^{(11)}(s, t) + \text{Re}B_2^{(11)}(t, s) + B_3^{(11)}(u, t) \right) \\
- \text{Re}B_2^{(11)}(u, s) \right] + \frac{m^2}{t} \left( \text{Re}B_2^{(11)}(s, t) + \text{Re}B_1^{(11)}(t, s) \\
- B_2^{(11)}(u, t) + \text{Re}B_3^{(11)}(u, s) \right).\] (49)

We checked the expression of the IR pole for the direct \( t \)-channel diagram of Fig. 2 (g) against the corresponding result given in [23], where the author calculates the one-loop virtual corrections to electron-muon scattering, without neglecting the two fermion masses and using a small photon mass \( \lambda \) for regularizing the IR divergences. Setting the mass of the muon equal to the mass of the electron in [23], we find full agreement with our result provided that the dimensional IR pole is replaced by

\[
\frac{1}{(D - 4)} \rightarrow \log \left( \frac{m}{\lambda} \right).
\] (50)

We compared also the contribution of the \( s \)-channel diagrams to the differential cross section (pole and finite part), in the limit \( m \rightarrow 0 \), with the results in [24], where the \( \lambda \)-regularization is used and the Dirac algebra is in 4 dimensions. If we calculate the contribution of the diagrams in Fig. 2 (h) and (j) to the cross section using \( D \)-continuous dimensions for the MIs but 4 dimensions for the Dirac algebra, in the limit \( m \rightarrow 0 \) we find again agreement with [24], provided the correspondence of Eq. (50) is used. When calculating all the Dirac traces also in \( D \) dimensions, we obtain, compared to the results in [24], an extra finite term proportional to the coefficient of the IR pole in \( (D - 4) \).

Let us recall here that the expression of the one-loop scalar box integral in the \( D \)-regularization, given in [6] and used in this paper, is the same as the result of Appendix E (b) of [25] (where a small photon mass is still used), once the replacement (50) is carried out.

### 5 Two-Loop \( N_F = 1 \) Differential Cross Section

In this Section we discuss the virtual diagrams contributing to the Bhabha scattering differential cross section at order \( \alpha^4 \) (summed over the spin of the final states and averaged over the spin of the initial ones). We limit our analysis to the diagrams
that include a closed electron loop. Beside the genuine two-loop diagrams, we have to consider the interference of the one-loop vacuum polarization graphs with the one-loop vertex and box corrections.

5.1 Two-Loop Vacuum Polarization Corrections

The two-loop photon self-energy can be written in a way completely analogous to the one-loop case:

$$\Pi_{\mu\nu}^{(2l)}(-p^2) = \left(\frac{\alpha}{\pi}\right)^2 \left(p^\mu p^\nu - \delta_{\mu\nu} p^2\right) \Pi_0^{(2l)}(-p^2).$$  \hspace{1cm} (51)

The explicit expression of the UV-renormalized function $\Pi_0^{(2l)}(-p^2)$ in terms of HPLs is given in Eq. (93) of Appendix A.

The contribution of the two-loop vacuum polarization diagrams in Fig. 3 (a)–(f) can be obtained by replacing $\Pi_0^{(1l)}$ by $\Pi_0^{(2l)}$ in Eq. (26):

$$\frac{d\sigma^V_2}{d\Omega}_{(2l,S)} = \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[ st + \frac{s^2}{2} + (t - 2m^2)^2 \right] 2\text{Re}\Pi_0^{(2l)}(s) \\
+ \frac{1}{t^2} \left[ st + \frac{t^2}{2} + (s - 2m^2)^2 \right] 2\Pi_0^{(2l)}(t) \\
+ \frac{1}{st} \left[ (s + t)^2 - 4m^4 \right] \left( \text{Re}\Pi_0^{(2l)}(s) + \Pi_0^{(2l)}(t) \right) \right\}. \hspace{1cm} (52)$$

5.2 Two-Loop Vertex Corrections

The two-loop $N_F = 1$ virtual corrections to Bhabha scattering include the four diagrams shown in Fig. 3 (g)–(j) (and the corresponding counter terms), obtained by including a vacuum polarization correction in the photon propagator of the one-loop vertex diagram. The Lorentz structure of these corrections is identical to the one-loop case, Eq. (27), with the form factors $F_1^{(1l)}(-p^2)$ ($i = 1, 2; -p^2 = s, t$) replaced by $F_1^{(2l)}(-p^2)$ ($i = 1, 2; -p^2 = s, t$). The explicit expressions of the UV renormalized form factors $F_i^{(2l)}(-p^2)$ are given in Eqs. (94,95) of Appendix A at variance with the one loop case, the two-loop contributions of the graphs in Fig. 3 (g)–(j) are infrared finite, so that no expansion in $(D - 4)$ is needed.

The full set of two-loop $N_F = 1$ vertex corrections to the squared amplitude can be written, similarly to Eq. (43), as

$$\frac{d\sigma^V_2}{d\Omega}_{(2l,V)} = 2\frac{\alpha^2}{s} \left\{ \frac{1}{s^2} V_1^{(2l)}(t, s) + \frac{1}{t^2} V_2^{(2l)}(s, t) \\
+ \frac{1}{st} \left[ V_1^{(2l)}(s, t) + V_2^{(2l)}(t, s) \right] \right\}, \hspace{1cm} (53)$$

where the functions $V_1^{(2l)}$ and $V_2^{(2l)}$ are given by

$$V_1^{(2l)}(s, t) = c_{1,1}(s, t)\text{Re}F_1^{(2l)}(t) + c_{1,3}(s, t)\text{Re}F_2^{(2l)}(t), \hspace{1cm} (54)$$
$$V_2^{(2l)}(s, t) = c_{2,1}(s, t)\text{Re}F_1^{(2l)}(t) + c_{2,3}(s, t)\text{Re}F_2^{(2l)}(t), \hspace{1cm} (55)$$
and the coefficients $c_{1,1}(s, t)$ etc. are the same as in Eqs. (36–41).

5.3 Two-Loop Box Corrections

There are 8 two-loop box diagrams including a closed fermion loop, shown in Fig. 3 (k)–(r).

Let us consider the diagram (l) in Fig. 3 once we transform the external momenta according to $p_4 \leftrightarrow -p_1$ and $p_2 \leftrightarrow -p_3$, this diagram becomes identical to diagram (k). Since this set of transformations leaves unchanged the Mandelstam variables $s$ and $t$, the contributions of diagrams (k) and (l) to the differential cross section at order $\alpha^4(N_F = 1)$ is identical. Similar considerations hold for the pairs of diagrams (m)-(n), (o)-(p), and (q)-(r). The diagrams (k), (m), (o), and (q) in Fig. 3 are linked by the same symmetry relations that connect the four one-loop box diagrams in Fig. 2 (g), (i), (h), and (j), respectively (see the discussion in Section 4.3). Consequently, in strict analogy to the one-loop case, the contributions of the two-loop box diagrams in Fig. 3 to the Bhabha scattering differential cross section can be written, similarly to Eq. (49), as

$$
\frac{d\sigma^V_2(s, t, m^2)}{d\Omega}_{(2l,B)} = -2\frac{\alpha^2}{4s} \left[ \frac{m^2}{t} \left( \text{Re} B_1^{(2l)}(s, t) + \text{Re} B_2^{(2l)}(t, s) + B_3^{(2l)}(u, t) \right) - \text{Re} B_2^{(2l)}(u, s) \right] + \frac{m^2}{t} \left( \text{Re} B_2^{(2l)}(s, t) + \text{Re} B_1^{(2l)}(t, s) \right) - B_3^{(2l)}(u, t) + \text{Re} B_2^{(2l)}(u, s) \right].
$$

(56)

We choose to factor out in Eq. (56) an overall multiplicative factor $-2$ accounting for the minus sign coming from the closed fermion loops and the identity of the contributions from the pairs of diagrams (k)-(l), (m)-(n), (o)-(p), and (q)-(r) of Fig. 3.

The Laurent expansion of the UV-renormalized functions $B_i^{(2l)}$ reads:

$$
B_i^{(2l)}(s, t) = \frac{1}{(D-4)} B_i^{(2l,-1)}(s, t) + B_i^{(2l,0)}(s, t) + O((D-4)) \right).
$$

(57)

The expressions of the coefficients of the Laurent expansion, $B_i^{(2l,j)} (j = -1, 0)$, are given in Eqs. (96–101) in Appendix A.

5.4 Two-Loop Reducible Corrections

The complete set of two-loop diagrams to the Bhabha scattering process containing a closed electron loop includes the four 1-loop reducible diagrams, shown in Fig. 3 (s)–(v). They can be obtained from the one-loop vertex diagrams by inserting a one-loop photon self-energy correction in the photon propagator connecting the two fermionic currents. Since the vacuum polarization corrections amount to a multiplicative factor $\Pi_0^{(1l)}(-p^2)$, the contribution of such diagrams to the the differential cross section at order $\alpha^4$ is easily obtained by appending the appropriate $\Pi_0^{(1l)}(-p^2) (p^2 =
−s, −t) to the various terms in Eq. (43) and properly accounting for the \((D − 4)\) expansion. In close analogy to Eq. (43) we write therefore

\[
\frac{d\sigma^V_{2}(s, t, m^2)}{d\Omega} \bigg|_{(2l, R)} = 2\frac{\alpha^2}{s} \left[ \frac{1}{s^2} V^R_2(t, s) + \frac{1}{t^2} V^R_2(s, t) + \frac{1}{st} (V^R_1(s, t) + V^R_1(t, s)) \right]
\]

where the \(V^R_i\) are to be Laurent expanded as in Eq. (33). The coefficients of the Laurent expansions are

\[
V^{(R,−1)}_i(s, t) = c_{i,1}(s, t) \Re \left( F^{(I,−1)}_1(t) \Pi^{(I,0)}_0(t) \right),
\]

\[
V^{(R,0)}_i(s, t) = c_{i,1}(s, t) \Re \left( F^{(I,0)}_1(t) \Pi^{(I,0)}_0(t) + F^{(I,−1)}_1(t) \Pi^{(I,1)}_0(t) \right) + c_{i,2}(s, t) \Re \left( F^{(I,−1)}_1(t) \Pi^{(I,0)}_0(t) \right) + c_{i,3}(s, t) \Re \left( F^{(I,0)}_2(t) \Pi^{(I,0)}_0(t) \right)
\]

where the coefficients \(c_{i,j}(s, t)\) are given by Eqs. (36–41), the Laurent coefficients of \(\Pi^{(I)}_0(t)\) and of the \(F^{(I)}_1(t)\) by the Eqs. (82–86).

### 5.5 Products of Two One-Loop Corrections

Finally, to conclude our catalog of contributions to the Bhabha scattering cross section at order \(\alpha^4(N_F = 1)\), we need to discuss the interference between the one-loop vacuum polarization diagrams of Fig. 2 (a), (b) and the one-loop vertex and box graphs of Fig. 2 (c)–(f) and (g)–(j). Since the one-loop vacuum polarization insertion factorizes in the tree-level diagram and the multiplicative factor \(\Pi^{(I)}_0(−p^2) (−p^2 = s, t)\), the contribution of these interferences among one-loop diagrams is easily obtained by replacing the tree-level photon propagators \(1/s\) and \(1/t\) in Eqs. (43) and (49) by \(\Re \Pi^{(I)}_0(s)/s\) and \(\Re \Pi^{(I)}_0(t)/t\), respectively.

In this way one obtains for the interference of the self mass graphs in Fig. 2 (a), (b) and the vertex graphs in Fig. 2 (c)–(f)

\[
\frac{d\sigma^V_{2}(s, t, m^2)}{d\Omega} \bigg|_{(S,V)} = 2\frac{\alpha^2}{s} \left[ \frac{1}{s^2} V^{(I)}_2(t, s) \left( \Pi^{(I)}_0(s) \right)^* + \frac{1}{t^2} V^{(I)}_2(s, t) \left( \Pi^{(I)}_0(t) \right)^* + \frac{1}{st} V^{(I)}_1(s, t) \left( \Pi^{(I)}_0(s) \right)^* + \frac{1}{st} V^{(I)}_1(t, s) \left( \Pi^{(I)}_0(t) \right)^* \right]
\]

where the functions \(V^{(I)}_i(s, t)\) are the same as in Eqs. (33, 31, 32). As they have the Laurent expansion of Eq. (33), the above cross section can be Laurent expanded in
the same way. One has for instance, in the notation of Eqs. (33) and (24):

\[ V^{(1l)}_{2}(t, s)(\Pi^{(1l)}_{0}(s))^{*} = \frac{1}{(D - 4)} V^{(1l, -1)}_{2}(t, s)(\Pi^{(1l, 0)}_{0}(s))^{*} \]

\[ + V^{(1l, 0)}_{2}(t, s)(\Pi^{(1l, 0)}_{0}(s))^{*} + V^{(1l, -1)}_{2}(t, s)(\Pi^{(1l, 1)}_{0}(s))^{*} + O((D - 4)) , \] (64)

and similar expressions for all the other terms of Eq. (63).

For the interference of the self mass graphs in Fig. 2 (a), (b) and the box graphs in Fig. 2 (g)-(j) one finds, in analogy with Eq. (49),

\[ \frac{d\sigma^{V}_{2}(s, t, m^{2})}{d\Omega} \bigg|_{(s, B)} = \frac{\alpha^{2}}{4s} \text{Re} \left[ \frac{1}{s} \left( B^{(1l)}_{1}(s, t) + B^{(1l)}_{2}(t, s) \right. \right. \]

\[ \left. \left. + B^{(1l)}_{3}(u, t) - B^{(1l)}_{2}(u, s) \right) \left( \Pi^{(1l)}_{0}(s) \right)^{*} + \frac{1}{t} \left( B^{(1l)}_{2}(s, t) + B^{(1l)}_{1}(t, s) \right. \right. \]

\[ \left. \left. - B^{(1l)}_{2}(u, t) + B^{(1l)}_{3}(u, s) \right) \left( \Pi^{(1l)}_{0}(t) \right)^{*} \right] \right] , \] (65)

where the functions \( B^{(1l)}_{i}(s, t) \) are the same as in Eqs. (44, 47), with the Laurent expansion of Eq. (45). A similar expansion holds for Eq. (65); in analogy with Eq. (64), one has for instance:

\[ B^{(1l)}_{1}(s, t)(\Pi^{(1l)}_{0}(s))^{*} = \frac{1}{(D - 4)} B^{(1l, -1)}_{1}(s, t)(\Pi^{(1l, 0)}_{0}(s))^{*} \]

\[ + B^{(1l, 0)}_{1}(s, t)(\Pi^{(1l, 0)}_{0}(s))^{*} + B^{(1l, -1)}_{1}(s, t)(\Pi^{(1l, 1)}_{0}(s))^{*} + O((D - 4)) , \] (66)

and similar expressions for all the other terms of Eq. (65).

### 6 Conclusions

In this paper, we considered the virtual radiative corrections to the Bhabha scattering differential cross section (summed over the spins of the final states and averaged over the spins of the initial ones) in pure QED; we calculated the contribution of all the one-loop diagrams (order \( \alpha^{3} \)) and of the subset of two-loop diagrams involving an electron loop (order \( \alpha^{4}(N_{F} = 1) \)).

The provided result does not rely on any kind of approximation; the full dependence of the cross section on the electron mass \( m \) and on the Mandelstam invariants \( s \) and \( t \) was retained.

Both UV and IR divergences were regularized within the framework of dimensional regularization. The UV renormalization was performed in the on-shell renormalization scheme.
The reduction of the integrals to the Master Integrals was carried out employing the Laporta algorithm. The MIs involved in the calculation of the cross section were evaluated in two previous works \[6, 10\], by means of the differential equation method.

All the results are written in terms of 1- and 2-dimensional HPLs of maximum weight 3.

The differential cross sections at the one- and two-loop level can be obtained adding up the various contributions discussed in the paper. In particular, the one-loop virtual cross section is the sum of the contributions given in Eqs. (26, 43, 49) respectively:

\[
\frac{d\sigma_1^V(s, t, m^2)}{d\Omega} = \frac{d\sigma_1^V(s, t, m^2)}{d\Omega}_{(1l,S)} + \frac{d\sigma_1^V(s, t, m^2)}{d\Omega}_{(1l,V)} + \frac{d\sigma_1^V(s, t, m^2)}{d\Omega}_{(1l,B)}; \quad (67)
\]

the two-loop differential cross section is the sum of the expressions given in Eqs. (52, 53, 56, 58, 63, 65):

\[
\frac{d\sigma_2^V(s, t, m^2)}{d\Omega} = \frac{d\sigma_2^V(s, t, m^2)}{d\Omega}_{(2l,S)} + \frac{d\sigma_2^V(s, t, m^2)}{d\Omega}_{(2l,V)} + \frac{d\sigma_2^V(s, t, m^2)}{d\Omega}_{(2l,B)} + \frac{d\sigma_2^V(s, t, m^2)}{d\Omega}_{(2l,R)} + \frac{d\sigma_2^V(s, t, m^2)}{d\Omega}_{(S,V)} + \frac{d\sigma_2^V(s, t, m^2)}{d\Omega}_{(S,B)}; \quad (68)
\]

The differential cross section still includes single poles in \((D - 4)\), of IR origin, to be canceled by the inclusion of the contribution of the diagrams involving the emission of a real soft photon, not discussed in this paper.

7 Acknowledgments

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A  Auxiliary Functions

In this Appendix, we give the explicit expressions of the auxiliary functions used throughout the paper. They involve 1- and 2-dimensional Harmonic PolyLogarithms (HPLs), introduced and discussed in [15, 16, 17, 18, 19, 6], of arguments $x, y$ related to the kinematical Mandelstam variables $s, t$ by the relations

$$-s = (p_1 + p_2)^2 = P^2 = m^2 \frac{(1 - x)^2}{x}, \quad x = \frac{\sqrt{P^2 + 4m^2} - \sqrt{P^2}}{\sqrt{P^2 + 4m^2} + \sqrt{P^2}}; \tag{69}$$

$$-t = (p_1 - p_3)^2 = Q^2 = m^2 \frac{(1 - y)^2}{y}, \quad y = \frac{\sqrt{Q^2 + 4m^2} - \sqrt{Q^2}}{\sqrt{Q^2 + 4m^2} + \sqrt{Q^2}}. \tag{70}$$

For completeness, we introduce also

$$-u = (p_1 - p_4)^2 = V^2 = m^2 \frac{(1 - z)^2}{z}, \quad z = \frac{\sqrt{V^2 + 4m^2} - \sqrt{V^2}}{\sqrt{V^2 + 4m^2} + \sqrt{V^2}}; \tag{71}$$

the variable $z$ is indeed related to $x$ and $y$ through the relation

$$-u = s + t - 4m^2 = m^2 \frac{1}{x} (x + y) \left( x + \frac{1}{y} \right) = m^2 \frac{(1 - z)^2}{z}. \tag{72}$$

If $s = -P^2$ is space-like ($s$ negative, $P^2$ positive), $x$ as given by Eq. (69) is real and positive, varying from $x = 1$ at $s = 0$ to $x = 0$ at $s = -\infty$, and similarly for the variable $y$.

In the explicit formulas which will follow, the one-dimensional HPLs are written, according to [15], as $H(0, 1; x), H(-1, 0; y)$ etc.; examples of two-dimensional HPLs are $G(-1/y; x), G(-y, 0, 0; x)$ etc., for their definition see [6].

The arguments of the auxiliary functions will be taken to be space-like, ($-P^2$ in the case of a single argument, $(-P^2, -Q^2)$ if the functions depend on two variables) so that the corresponding $x$ and $y$ vary both in the $(0, 1)$ range and all HPLs are real. The time-like regions can be recovered by analytic continuation.

Let us consider, for definiteness, the variable $s = -P^2$. The continuation to the time-like region is performed, according to the usual $i\epsilon$-prescription, by setting

$$P^2 = -s - i\epsilon, \tag{73}$$

when $s > 0$. For $0 < s < 4m^2$, $\sqrt{-s - i\epsilon} = -i\sqrt{s}$ and Eq. (69) gives $x = r$, where $r$ is the phase factor

$$r = \frac{\sqrt{4m^2 - s} + i\sqrt{s}}{\sqrt{4m^2 - s} - i\sqrt{s}} = e^{i2\phi}, \tag{74}$$

with

$$\phi = \arctan \sqrt{\frac{s}{4m^2 - s}}. \tag{75}$$

Above threshold, $s > 4m^2$, we define

$$x' = \frac{\sqrt{s} - \sqrt{s - 4m^2}}{\sqrt{s} + \sqrt{s - 4m^2}}. \tag{76}$$
with $x' = 1$ at $s = 4m^2$ and $x' = 0$ at $s = \infty$ and the continuation in $x$ is performed by the replacement
\begin{equation}
x = -x' + i\epsilon .
\end{equation}

Remembering that
\begin{equation}
H(0; x) = \ln x ,
\end{equation}
the continuation from space-like $s$ to $s$ above threshold is then performed by
\begin{equation}
H(0; x) = H(0; -x' + i\epsilon) = H(0; x') + i\pi .
\end{equation}

Another less trivial example is
\begin{align*}
H(-1, 0; x) &= H(-1; x)H(0; x) - H(0, -1; x) \\
&= H(-1; -x' + i\epsilon)H(0; -x' + i\epsilon) - H(0, -1; -x' + i\epsilon) \\
&= -H(1; x')\left(H(0; x') + i\pi\right) + H(0, 1; x') ,
\end{align*}
where an algebraic property of the 1-dimensional HPLs (shuffle algebra) has been used in the first line and the relation reversing the sign of the argument in the last line. For more details on the analytic continuation of the HPLs we refer to the already quoted papers.

The contribution of the two-loop box diagrams to the differential cross section includes a subset of the 2-dimensional HPLs introduced in [6]; this subset can be expressed in terms of Nielsen polylogarithms (whose argument depends however on both $x$ and $y$, i.e. on both $s$ and $t$), explicitly given in Eqs. (C.21,C.22,C.27-C.30) of [6]. As an example of analytic continuation which applies to those cases, consider for instance $\text{Li}_2(-x/y)$; if $x, y$ correspond to $s, t$ when they are both spacelike, the continuation to physical (timelike) $s$ and spacelike $t$ implies $x \to -x' + i\epsilon$, with $x' < y$, so that one has simply $\text{Li}_2(-x/y) \to \text{Li}_2(x'/y)$. The case of timelike $t$, spacelike $s$, on the other hand, implies $y \to -y' + i\epsilon$, with $y' < x$, so that the continuation is given by
\begin{align*}
\text{Li}_2\left(-\frac{x}{y}\right) &\to \text{Li}_2\left(\frac{x}{y'} + i\epsilon\right) = -\text{Li}_2\left(\frac{y'}{x}\right) - \frac{1}{2} \ln^2\left(\frac{y'}{x}\right) + 2\zeta(2) - i\pi \ln \left(\frac{y'}{x}\right) .
\end{align*}

Similarly for $\text{Li}_3$ we have:
\begin{align*}
\text{Li}_3\left(-\frac{x}{y}\right) &\to \text{Li}_3\left(\frac{x}{y'} + i\epsilon\right) = \text{Li}_3\left(\frac{y'}{x}\right) + \frac{1}{6} \ln^3\left(\frac{y'}{x}\right) - 2\zeta(2) \ln \left(\frac{y'}{x}\right) \\
&\quad + i\pi \frac{1}{2} \ln^2\left(\frac{y'}{x}\right) .
\end{align*}

In the following subsections we provide the explicit expressions of the auxiliary functions.
A.1 Auxiliary Functions for the One-Loop Cross Section

\[
\Pi_0^{(1\ell,0)}(-P^2) = -\left\{ \frac{5}{9} - \frac{4}{3(1-x)^2} + \frac{4}{3(1-x)^3} \right\} - \left[ \frac{1}{3} + \frac{4}{3(1-x)^3} \right] H(0; x),
\]

\[
\Pi_0^{(1\ell,1)}(-P^2) = -\left\{ \frac{14}{27} - \frac{16}{9(1-x)^2} + \frac{16}{9(1-x)} \right\} + \left[ \frac{2}{3(1-x)^3} - \frac{1}{(1-x)^2} + \frac{1}{6} \right] \zeta(2)
- \left[ \frac{5}{18} - \frac{16}{9(1-x)^3} - \frac{8}{3(1-x)^2} + \frac{1}{3(1-x)} \right] \zeta(2) H(0; x)
+ \left[ \frac{1}{6} + \frac{2}{3(1-x)^3} - \frac{1}{(1-x)^2} \right] \zeta(2) H(0, 0; x)
+ \left[ \frac{1}{3} + \frac{4}{3(1-x)^3} - \frac{2}{(1-x)^2} \right] \zeta(2) H(-1, 0; x),
\] (82)

\[
F_1^{(1\ell,-1)}(-P^2) = 1 - \left[ 1 - \frac{1}{1-x} - \frac{1}{1+x} \right] H(0; x),
\]

\[
F_1^{(1\ell,0)}(-P^2) = -1 - \frac{1}{2} \left[ 1 - \left( \frac{1}{1-x} - \frac{1}{1+x} \right) \right] H(0; x)
- \frac{1}{2} \left[ 1 - \frac{1}{1-x} - \frac{1}{1+x} \right] \zeta(2) - 2H(0; x) - H(0, 0; x)
+ 2H(-1, 0; x),
\]

\[
F_2^{(1\ell,0)}(-P^2) = -\frac{1}{2} \left[ \frac{1}{1-x} - \frac{1}{1+x} \right] H(0; x).
\]

\[
B_1^{(1\ell,-1)}(-P^2, -Q^2) = \left( -48 - \frac{8}{x^2(1-y)^2} + \frac{8}{x^2(1-y)} + \frac{32}{x(1-y)^2} - \frac{32}{x(1-y)} \right)
- \frac{16}{x} - \frac{32}{32x} \right) + \frac{32}{32x} + 16x + \frac{8x^2}{(1-y)^2} - \frac{8x^2}{(1-y)}
- \frac{8}{y(1+x)} - \frac{8}{y(1-x)} + \frac{8y}{y(1-x)} - \frac{8y}{(1-x)} + 8y
- \frac{96}{(1+x)(1-y)^2} + \frac{96}{(1+x)(1-y)} + \frac{80}{16}
+ \frac{32}{(1-x)(1-y)} - \frac{32}{(1-x)(1-y)} + \frac{16}{(1-x)}
+ \frac{32}{(1-y)^2} - \frac{32}{(1-y)} \right) H(0; x),
\]

\[
B_1^{(1\ell,0)}(-P^2, -Q^2) = \zeta(2) \left( \frac{8}{x^2(1+y)} - \frac{8}{x^2(1-y)} - \frac{32}{x(1+y)^2} + \frac{48}{x(1+y)^2} \right)
\]
\[-\frac{88}{x(1+y)} + \frac{40}{x(1-y)} + \frac{12}{x} - \frac{32x}{(1+y)^3} + \frac{48x}{(1+y)^2}\]
\[-\frac{88x}{(1+y)} + \frac{40x}{(1-y)} + 20x + \frac{8x^2}{(1+y)} - \frac{8x^2}{(1-y)}\]
\[-\frac{4}{y(1-x)} - \frac{6}{y} - \frac{4y}{(1-x)} + 10y + \frac{32}{(1+x)}\]
\[+ \frac{64}{(1+y)^3} - \frac{96}{(1+y)^2} + \frac{208}{(1+y)} - \frac{48}{(1-y)} - 80\]
\[+ \left( -\frac{4}{x^2(1-y)^2} + \frac{4}{x^2(1-y)} + \frac{8}{x(1-y)^2} - \frac{8}{x(1-y)} - \frac{2}{x} \right)\]
\[+ \frac{8x}{(1-y)^2} + \frac{8x}{(1-y)} + 2x + \frac{4x^2}{(1-y)^2} - \frac{4x^2}{(1-y)}\]
\[+ \frac{8}{y(1+x)} - \frac{2}{y} + \frac{4y}{(1+x)} - \frac{8y}{(1-x)}\]
\[+ \frac{16}{(1+y)^2} H(0; x) + \left( 4 - \frac{8}{x(1+y)^2} + \frac{8}{x(1+y)} \right)\]
\[+ \frac{2x - y}{(1+y)^2} + \frac{8x}{(1+y)} - 2x - \frac{2}{y} - 2y + \frac{16}{(1-y)^2}\]
\[+ \frac{16}{(1+y)} H(0; y) + \left( 8 - \frac{16}{x(1+y)^2} + \frac{16}{x(1+y)} \right)\]
\[+ \frac{4}{x - (1+y)^2} + \frac{16x}{(1+y)} - \frac{4x}{y} - \frac{4y}{(1+y)^2}\]
\[+ \frac{32}{(1+y)^2} H(1; y) + \left( -32 - \frac{8}{x} + 8x - \frac{8x}{y(1-x)} + \frac{4}{y} \right)\]
\[+ \frac{8y}{(1-x)} + 4y + \frac{64}{(1+x)} H(-1; 0; x) + \left( 8 + \frac{4}{x^2(1-y)^2} \right)\]
\[- \frac{x^2(1-y)}{1} - \frac{16}{x(1-y)^2} + \frac{16}{x(1-y)} + \frac{4}{x} + \frac{16x}{(1-y)^2}\]
\[- \frac{4x}{(1-y)^2} - \frac{4x^2}{(1-y)^2} + \frac{4x^2}{(1-y)} + \frac{4}{y(1+x)} - \frac{2}{y}\]
\[+ \frac{4y}{(1+x)} - 2y + \frac{48}{(1+y)^2} - \frac{48}{(1+y)(1-y)}\]
\[+ \frac{8}{(1+x)} - \frac{16}{(1-x)(1-y)^2} + \frac{16}{(1-x)(1-y)} - \frac{8}{(1-x)}\]
\[+ \frac{16}{(1-y)^2} + \frac{16}{(1-y)} H(0; x) H(0; y) + \left( \frac{16 + \frac{8}{x^2(1-y)^2} }{ } \right)\]
\[+ \frac{8}{x^2(1-y)} - \frac{32}{x(1-y)^2} + \frac{32}{x(1-y)} + \frac{8}{x} - \frac{32x}{(1-y)^2}\]
\[
\begin{align*}
&-\frac{32x}{(1-y)} - 8x - \frac{8x^2}{(1-y)^2} + \frac{8x^2}{(1-y)} + \frac{8}{y(1+x)} - \frac{4}{y} \\
&+ \frac{8y}{(1+x)} - 4y + \frac{96}{(1+x)(1-y)^2} - \frac{96}{(1+x)(1-y)} \\
&- \frac{16}{(1+x)} - \frac{32}{(1-x)(1-y)^2} + \frac{32}{(1-x)(1-y)} - \frac{16}{(1-x)} \\
&- \frac{32}{(1-y)^2} + \frac{32}{(1-y)} H(0;x) H(1;y) + \left(16 + \frac{4}{x} - 4x\right) \\
&+ \frac{4}{y(1-x)} - \frac{2}{y} + \frac{4y}{(1-x)} - 2y - \frac{32}{(1+x)} H(0,0;x) \\
&+ \left(-16 + \frac{2}{x^2(1+y)} - \frac{2}{x^2(1-y)} - \frac{8x}{(1+y)^3} + \frac{12}{x(1+y)^2} \\
&- \frac{22}{x(1+y)} + \frac{10x}{x(1-y)} + \frac{4}{x} - \frac{8x}{(1+y)^3} + \frac{12x}{(1+y)^2} \\
&- \frac{22x}{(1+y)} + \frac{10x}{(1-y)} + \frac{4x}{(1+y)} - \frac{2x^2}{(1-y)} - \frac{2}{y} + 2y \\
&+ \frac{16}{(1+y)^3} - \frac{24}{(1+y)^2} + \frac{52}{(1+y)} - \frac{12}{(1-y)} H(0,0;y) \\
&+ \left(-32 + \frac{4}{x^2(1+y)} - \frac{4}{x^2(1-y)} - \frac{16x}{(1+y)^3} + \frac{24}{x(1+y)^2} \\
&- \frac{44}{x(1+y)} + \frac{20}{x(1-y)} + \frac{8}{x} - \frac{16x}{(1+y)^3} + \frac{24x}{(1+y)^2} \\
&- \frac{44x}{(1+y)} + \frac{20x}{(1-y)} + \frac{8x}{(1+y)} - \frac{4x^2}{(1-y)} - \frac{4}{y} + 4y \\
&+ \frac{32}{(1+y)^3} - \frac{48}{(1+y)^2} + \frac{104}{(1+y)} - \frac{24}{(1-y)} \right) H(0,1;y), (88)
\end{align*}
\]

\[
B_2^{(11,-1)}(-P^2,-Q^2) = \left(-48 + \frac{16}{x^2(1-y)^2} + \frac{16}{x^2(1-y)} - \frac{16}{x} + 16x + \frac{16x^2}{(1-y)^2} \\
- \frac{16x^2}{(1-y)} + \frac{8}{y(1+x)} - \frac{8}{y} + \frac{8}{y} \right) H(0;x),
\]

\[
\zeta(2) \left(-16 + \frac{16}{x^2(1+y)} - \frac{16}{x^2(1-y)} - \frac{64}{x(1+y)^3} + \frac{96}{x(1+y)^2} \right)
\]

\[
B_2^{(11,0)}(-P^2,-Q^2) = \zeta(2) \left(-16 + \frac{16}{x^2(1+y)} - \frac{16}{x^2(1-y)} - \frac{64}{x(1+y)^3} + \frac{96}{x(1+y)^2} \right)
\]
\[-\frac{80}{x(1 + y)} + \frac{16}{x(1 - y)} + \frac{12}{x} - \frac{64x}{(1 + y)^3} + \frac{96x}{(1 + y)^2}
\]
\[-\frac{80x}{(1 + y)} + \frac{16x}{(1 - y)} + 20x + \frac{16x^2}{(1 + y)} - \frac{16x^2}{(1 - y)} - \frac{2}{y(1 + x)}
\]
\[-\frac{2y}{y(1 - x)} - \frac{6}{y} - \frac{2y}{(1 + x)} + 10y + \frac{200}{(1 + x)}
\]
\[+ \frac{4}{(1 - x)} + \frac{128}{(1 + y)^3} - \frac{192}{(1 + y)^2} + \frac{256}{(1 + y)} - \frac{32}{(1 - y)} - 92\]
\[+ \left( -8 + \frac{2}{x} - 2x - \frac{4}{y(1 - x)} + \frac{2}{y} - \frac{4y}{(1 - x)} + 2y \right)
\]
\[+ \frac{16}{(1 - x)} ) H(0; x) + \left( 12 - \frac{16}{x(1 + y)^2} + \frac{16}{x(1 + y)} - \frac{2}{x} \right)
\]
\[-\frac{16x}{(1 + y)^2} + \frac{16x}{(1 + y)} - 2x - \frac{2}{y} - 2y + \frac{32}{(1 + y)^2}
\]
\[-\frac{32}{(1 + y)} ) H(0; y) + \left( 24 - \frac{32}{x(1 + y)^2} + \frac{32}{x(1 + y)} - \frac{4}{x} \right)
\]
\[-\frac{32x}{(1 + y)^2} + \frac{32x}{(1 + y)} - 4x - \frac{4}{y} - 4y + \frac{64}{(1 + y)^2}
\]
\[-\frac{64}{(1 + y)} ) H(1; y) + \left( -24 - \frac{8}{x} + 8x - \frac{4}{y(1 + x)} - \frac{4}{y(1 - x)} \right)
\]
\[+ \frac{4}{y} - \frac{4y}{(1 + x)} - \frac{4y}{(1 - x)} + 4y + \frac{40}{(1 + x)} + \frac{8}{(1 - x)} \right) H(-1, 0; x)
\]
\[+ \left( 12 + \frac{8}{x^2(1 - y)^2} - \frac{8}{x^2(1 - y)} + \frac{4}{x} - 4x - \frac{8x^2}{(1 - y)^2} \right)
\]
\[+ \frac{8x^2}{(1 - y)} + \frac{2}{y(1 + x)} + \frac{2}{y(1 - x)} - \frac{2y}{(1 + x)} + \frac{2y}{(1 - x)}
\]
\[\frac{-2y}{(1 + x)(1 - y)^2} - \frac{32}{(1 - x)(1 - y)} - \frac{4}{(1 - x)} - \frac{32}{(1 - y)^2}
\]
\[+ \frac{32}{(1 - y)} ) H(0; x) H(0; y) + \left( 24 + \frac{16}{x^2(1 - y)^2} - \frac{16}{x^2(1 - y)} \right)
\]
\[+ \frac{8}{x} - 8x - \frac{16x^2}{(1 - y)^2} + \frac{16x^2}{(1 - y)} + \frac{4}{y(1 + x)} + \frac{4}{y(1 - x)}
\]
\[+ \frac{4y}{(1 + x)} + \frac{4y}{(1 - x)} - 4y + \frac{64}{(1 + x)(1 - y)^2}
\]
\[-\frac{64}{(1 + x)(1 - y)} - \frac{40}{(1 + x)} + \frac{64}{(1 - x)(1 - y)^2}
\]
\[-\frac{64}{(1 + x)(1 - y) - \frac{8}{(1 - x)} + \frac{64}{(1 - y)^2} + \frac{64}{(1 - y)} ) H(0; x) H(1; y)
\]

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\[
B_{3}^{(11, -1)}(-P^2, -Q^2) = \left( \frac{8}{x^2(1-y)^2} - \frac{8}{x^2(1-y)} + \frac{32}{x(1-y)^2} - \frac{32}{x(1-y)} \right)
- \frac{32}{(1-y)^2} + \frac{32}{(1-y)} - \frac{8x^2}{(1-y)^2} + \frac{8x^2}{(1-y)}
- \frac{32}{(1+x)(1-y)^2} + \frac{32}{(1+x)(1-y)} + \frac{96}{(1-x)(1-y)^2}
- \frac{96}{(1-x)(1-y)} - \frac{32}{(1-y)^2} + \frac{32}{(1-y)} \right) H(0; x),
\]

\[
B_{3}^{(11, 0)}(-P^2, -Q^2) = \zeta(2) \left( -\frac{8}{x^2(1+y)} + \frac{8}{x^2(1-y)} + \frac{32}{x(1+y)^3} - \frac{48}{x(1+y)^2} \right)
- \frac{24}{x(1+y)} + \frac{24}{x(1-y)} + \frac{32x}{x(1+y)^3} - \frac{48x}{x(1+y)^2} - \frac{8x}{x(1+y)}
+ \frac{24x}{8x^2} + \frac{8x^2}{8x^2} + \frac{8x^2}{8x^2} - \frac{2}{2}
+ \frac{2y}{(1+y)} - \frac{2y}{(1-y)} + \frac{12}{y(1+x)} - \frac{y(1-x)}{4} + \frac{64}{y(1-x)}
+ \frac{96}{(1+y)^2} - \frac{48}{(1+y)} - \frac{16}{(1-y) + 12} + \left( 8 + \frac{4}{x^2(1-y)^2} \right)
\]
\[- \frac{4}{x^2(1-y)} + \frac{8}{x(1-y)^2} - \frac{8}{x(1-y)} + \frac{4}{x} - \frac{8x}{(1-y)^2} \]
\[+ \frac{8x}{(1-y)} - \frac{4x}{(1-y)^2} + \frac{4x}{x(1-y)} + \frac{8}{y(1+x)} - \frac{4}{y} \]
\[+ \frac{16}{(1+y)^2} \bigg( \frac{16}{(1-y)^2} + \frac{16}{(1-y)} \bigg) H(0; x) + \left( -8 + \frac{8}{x(1+y)^2} - \frac{8}{x(1+y)} \right) \]
\[+ \frac{8x}{(1+y)^2} - \frac{8x}{(1+y)} - \frac{16}{(1+y)^2} + \frac{16}{(1+y)} \bigg) H(0; y) \]
\[+ \left( -16 + \frac{16}{x(1+y)^2} - \frac{16}{x(1+y)} + \frac{16x}{(1+y)^2} - \frac{16x}{(1+y)} \right) \]
\[+ \left( \frac{32}{(1+y)^2} + \frac{32}{(1+y)} \bigg) H(1; y) + \left( -8 + \frac{4}{y(1+x)} \right) \]
\[+ \left( -4 - \frac{4}{x^2(1-y)^2} + \frac{4}{x^2(1-y)} - \frac{16}{x(1-y)^2} + \frac{16}{x(1-y)} \right) \]
\[+ \frac{16x}{(1-y)^2} - \frac{16x}{(1-y)} + \frac{16x}{(1-y)^2} - \frac{16}{y(1+x)} \]
\[+ \frac{2}{y(1-x)} + \frac{2y}{(1+x)} - \frac{2y}{(1-x)} + \frac{16}{(1+x)(1-y)^2} \]
\[+ \frac{48}{(1+x)(1-y) + (1+x)} - \frac{16}{(1-y)^2} - \frac{16}{(1-y)} \bigg) H(0; x) H(0; y) + \left( -8 \right) \]
\[+ \frac{8}{x^2(1-y)^2} + \frac{8}{x(1-y)^2} - \frac{32}{x(1-y)} + \frac{32}{x(1-y)} + \frac{32x}{(1-y)^2} \]
\[+ \frac{32}{(1-y)^2} + \frac{32}{(1-y)} \bigg) H(0; x) H(1; y) + \left( 4 - \frac{2}{y(1+x)} \right) \]
\[+ \frac{2}{y(1-x)} - \frac{2y}{(1+x)} + \frac{2y}{(1-x)} - \frac{12}{(1+x)} + \frac{4}{(1-x)} \bigg) H(0, 0; x) \]
\[+ \left( 4 - \frac{2}{x^2(1+y)} + \frac{2}{x^2(1-y)} + \frac{8}{x(1+y)^3} - \frac{12}{x(1+y)^2} \right) \]

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\[ \Pi_2^{(2)}(-P^2) = -\frac{5}{24} + \frac{13}{6(1-x)^2} - \frac{13}{6(1-x)} + \left(1 - \frac{4}{(1-x)^4} + \frac{8}{(1-x)^3}\right) + \left(-\frac{4}{(1-x)^2}\right)\zeta(3) + \left(\frac{1}{4} + \frac{3}{2(1-x)^2} + \frac{1}{(1-x)}\right)H(0; x) + \left(-\frac{4}{3} - \frac{16}{3(1-x)^3} + \frac{8}{3(1-x)^2}\right)H(-1, 0; x) + \left(2 - \frac{7}{3(1-x)^4}\right) + \left(\frac{26}{3(1-x)^3} - \frac{23}{3(1-x)^2} - \frac{2}{3(1-x)}\right) + \left(\frac{2}{3} + \frac{8}{3(1-x)^3}\right) - \frac{4}{(1-x)^2}H(1, 0; x) + \left(\frac{4}{3} - \frac{16}{3(1-x)^4} + \frac{32}{3(1-x)^3}\right) - \frac{16}{3(1-x)^2}\left[H(0, -1, 0; x) - H(-1, 0, 0; x)\right] + \left(2 - \frac{8}{3(1-x)^4}\right) + \frac{16}{3(1-x)^3} - \frac{8}{3(1-x)^2}\left[H(1, 0, 0; x) - H(0, 1, 0; x)\right], \tag{93} \]

\[ F_1^{(2)}(-P^2) = \frac{383}{108} + \frac{49}{9(1+x)^2} - \frac{49}{9(1+x)} + \left(-\frac{1}{4} + \frac{98}{3(1+x)^4} - \frac{196}{3(1+x)^3}\right) + \frac{229}{6(1+x)^2} - \frac{11}{2(1+x)}\zeta(2) + \left(-\frac{265}{216} + \frac{89}{9(1+x)^3} - \frac{89}{6(1+x)^2}\right)H(0; x) + \left(\frac{19}{36} + \frac{62}{9(1+x)^4} - \frac{124}{9(1+x)^3}\right) \]

\[ + \frac{563}{108(1+x)} + \frac{59}{27(1-x)}\right)H(0; x) + \left(-\frac{1}{6} - \frac{6}{(1+x)^5} + \frac{15}{(1+x)^4}\right) \]

\[ A.2 \quad \text{Auxiliary Functions for the Two-Loop Cross Section} \]
\[ F_2^{(2)}(-P^2) = -\frac{11}{(1+x)^3} + \frac{3}{2(1+x)^2} + \frac{5}{12(1+x)} + \frac{5}{12(1-x)} \left[ \zeta(2) H(0; x) + H(0,0,0;x) \right], \quad (94) \]

\[ B_1^{(2l-1)}(-P^2, -Q^2) = -\left[ \left( \frac{56}{3} - \frac{16}{64} + \frac{32}{128} - \frac{28}{16} \right) \right] \]
\[ - \left( \frac{9x^2(1-y)}{16} + \frac{3x(1-y)^4}{64} - \frac{3x(1-y)^3}{128} + \frac{9x(1-y)^2}{16} \right) \]
\[ + \left( \frac{176}{9} - \frac{40}{64} - \frac{128y}{16x} - \frac{9y(1-y)^4}{32x^2} + \frac{9y(1-y)^3}{28x^2} \right) \]
\[ + \left( \frac{20x^2}{9} + \frac{3x(1-y)^4}{32x} - \frac{9y(1-y)^3}{28x^2} - \frac{20y}{9} \right) \]
\[ + \left( \frac{9x + 1}{9} - \frac{20y}{64} \right) \]
\[ + \left( \frac{112}{80} \right) \]
\[ - \left( \frac{9(1-x)(1-y)^2}{64} + \frac{9(1-x)(1-y)}{64} - \frac{9(1-x)}{88} \right) \]
\[ - \left( \frac{128}{9(x+1)} + \frac{3x(1-y)^4}{16} - \frac{3(1-y)^3(x+1)}{80} \right) \]
\[ - \left( \frac{208}{3(1-y)(x+1)} + \frac{272}{9(x+1)} - \frac{248}{9(1-y)^2} \right) \]
\[ H(0; x) + \left( -8 \right) \]
\[ - \left( \frac{3x^2(1-y)^5}{16} + \frac{3x^2(1-y)^4}{40} - \frac{x^2(1-y)^3}{8} - \frac{3x^2(1-y)^2}{160} \right) \]
\[ + \left( \frac{3x^2(1-y)}{64} + \frac{3x(1-y)^5}{16} - \frac{3x(1-y)^4}{8} - \frac{3x(1-y)^3}{64} \right) \]
\[ - \left( \frac{3x(1-y)^2}{3x(1-y)} - \frac{3x(1-y)}{16} \right) 
\[ \right] 
\[ \right] 
\[ = \left( -8 \right) \]
\[ + \left( \frac{3x(1-y)^2}{3} - \frac{3x(1-y)}{8} \right) \]
\[ + \left( \frac{3x(1-y)^4}{3} - \frac{3x(1-y)^3}{16} \right) 
\[ 28 \]
\[
B_{1}^{(2\ell_{0})}(-P^{2},-Q^{2}) = -\left\{ \begin{array}{l}
-\frac{64}{3(1-y)^{3}} - \frac{64}{3(1-y)^{2}} + \frac{16}{3(1-y)} + \frac{8x}{3} + \frac{16x^{2}}{3(1-y)^{5}} \\
-\frac{40x^{2}}{3(1-y)^{4}} + \frac{8x^{2}}{(1-y)^{3}} + \frac{4x^{2}}{(1-y)^{2}} - \frac{4x^{2}}{(1-y)} + \frac{4}{3y(x+1)} - \frac{4}{3y} - \frac{4x}{3(1-x)} - \frac{4x}{3(x+1)} + \frac{4}{3}
\end{array} \right. \\
+ \frac{3y(x+1)}{64} - \frac{160}{3y} - \frac{160}{3(1-x)} - \frac{160}{3(x+1)} + \frac{32}{3} \\
+ \frac{3(1-x)(1-y)^{5}}{16} - \frac{3(1-x)(1-y)}{32} - \frac{3(1-x)}{3(1-y)} \\
- \frac{3(1-x)(1-y)^{2}}{64} + \frac{(1-y)^{5}(x+1)}{160} + \frac{(1-y)^{4}(x+1)}{80} - \frac{(1-y)^{2}(x+1)}{112} \\
+ \frac{32}{3(1-y)(x+1)} + \frac{40}{3(x+1)} \right) H(0;x) H(0;y), 
\]

\[
\frac{9}{352} - \frac{16}{32} + \frac{27}{1216} + \frac{3x^{2}(1-y)^{4}}{3x^{2}} + \frac{3x^{2}(1-y)^{3}}{3x^{2}} - \frac{9x^{2}(1-y)^{2}}{3x^{2}} - \frac{9x^{2}(1-y)}{3x^{2}} + \frac{9x}{3x^{2}} + \frac{176}{388} + \frac{16}{16} + \frac{14}{16} + \frac{112}{112}
\]
\[
\begin{align*}
- \frac{224x}{3(1-y)^2} + & \frac{124x}{9(1-y)^2} + \frac{212x}{9(1-y)} + \frac{16x}{3(y+1)^2} - \frac{16x}{3(y+1)} \\
- \frac{26x}{9} - & \frac{16x^2}{3(1-y)^4} + \frac{32x^2}{3(1-y)^3} - \frac{28x^2}{9(1-y)^2} - \frac{20x^2}{9(1-y)} \\
+ & \frac{9y(1-x)}{22} - \frac{3y(x+1)^4}{2} + \frac{3y(x+1)^3}{10y} - \frac{3y(x+1)^2}{224y} \\
+ & \frac{9y(x+1)}{116y} + \frac{2y}{9} + \frac{9(1-x)}{3(x+1)^4} + \frac{3(x+1)^3}{32} \\
- & \frac{3(x+1)^2}{64} + \frac{9(x+1)}{9} + \frac{3(1-x)(1-y)^4}{40} \\
- & \frac{3(1-x)(1-y)^3}{44} + \frac{9(1-x)(1-y)^2}{192} + \frac{9(1-x)(1-y)}{160} - \frac{3(y+1)}{304} \\
+ & \frac{9(1-x)}{384} - \frac{(1-y)^4(x+1)^2}{30} + \frac{(1-y)^4(x+1)}{6} + \frac{3(y+1)^4}{6} \\
- & \frac{(1-y)^2(x+1)^4}{184} + \frac{(1-y)^2(x+1)^3}{244} - \frac{(1-y)^2(x+1)^2}{64} \\
+ & \frac{3(y+1)^2}{128} - \frac{9(y+1)^2}{208} + \frac{(1-y)(x+1)^4}{296} \\
- & \frac{(1-y)(x+1)^3}{668} + \frac{32}{64} - \frac{(1-y)(x+1)^2}{344} + \frac{3(y+1)^2}{868} \\
- & \frac{3(y+1)^2}{32} + \frac{3(y+1)}{80} \zeta(2) + \left( - \frac{124}{3} + \frac{40}{3x^2(1-y)^4} \right) \\
- & \frac{3x^2(1-y)^3}{352} + \frac{27x^2(1-y)^2}{452} + \frac{27x^2(1-y)}{1132} - \frac{3x(1-y)^4}{242} \\
+ & \frac{3x(1-y)^3}{176x} - \frac{27x(1-y)^2}{352x} - \frac{27x(1-y)}{452x} + \frac{27x^2}{1132} - \frac{27x}{242} \\
+ & \frac{3(1-y)^4}{80x^2} - \frac{3(1-y)^3}{278x^2} + \frac{3(1-y)^2}{82x^2} - \frac{27}{358} \\
- & \frac{27y}{70} + \frac{9(y+1)^3}{112} + \frac{9(y+1)^2}{56} - \frac{27y(x+1)}{358} \\
+ & \frac{27y}{130} - \frac{27(1-x)}{70y} - \frac{9(y+1)^3}{112y} + \frac{3(y+1)^2}{56y} - \frac{27y(x+1)}{358} \\
+ & \frac{27}{130} - \frac{(1-x)(1-y)^4}{64} + \frac{(1-x)(1-y)^3}{128} \\
- & \frac{27(1-x)(1-y)^2}{1664} + \frac{27(1-x)(1-y)}{64} + \frac{464}{27(1-x)}
\end{align*}
\]
\[
\begin{align*}
&\frac{512}{3(1 - y)^4(x + 1)} - \frac{160}{3(1 - y)^4} - \frac{1024}{3(1 - y)^3(x + 1)} \\
&\frac{3(1 - y)^3}{32} - \frac{3(1 - y)^2(x + 1)^3}{616} + \frac{(1 - y)^2(x + 1)^2}{64} \\
&\frac{3(1 - y)^2(x + 1)}{32} + \frac{27(1 - y)^2}{496} + \frac{3(1 - y)(x + 1)^3}{2056} \\
&\frac{32}{(x + 1)^2} - \frac{16}{3(1 - y)(x + 1)} - \frac{1912}{27(x + 1)} \bigg) H(0; x) + \left( \frac{46}{45} \\
&\frac{16}{45x^2(1 - y)^5} + \frac{8}{9x^2(1 - y)^4} - \frac{8}{9x^2(1 - y)^2} + \frac{2}{9x^2(1 - y)} \\
&\frac{2}{15x^2} + \frac{4y}{15x} + \frac{45x(1 - y)^5}{64x} - \frac{9x(1 - y)^4}{32x} - \frac{45x(1 - y)^3}{208x} \\
&\frac{16x}{15(1 - y)} + \frac{8x}{3(y + 1)} - \frac{148x}{45} - \frac{16x^2}{8x^2} + \frac{8x^2}{3(1 - y)} - \frac{45(1 - y)^3}{128} + \frac{2y}{15} + \frac{2y^2}{15} - \frac{32}{45(1 - y)^5} \\
&\frac{16}{9(1 - y)^2} + \frac{15(1 - y)^3}{656} - \frac{5(1 - y)}{52} \\
&- \frac{16}{3(y + 1)} \bigg) H(0; y) + \left( -\frac{28}{3} - \frac{8}{3} - \frac{20}{3} \right) \\
&\frac{x}{2} - \frac{3x^2(1 - y)^3}{32} + \frac{3x^2(1 - y)}{32} - \frac{3x(1 - y)^5}{3} - \frac{8}{8} \\
&- \frac{3x^3(1 - y)^4}{32} + \frac{3x(1 - y)^3}{3} - \frac{3x(1 - y)^2}{32x} + \frac{8x}{8} \\
&- \frac{3x^3(1 - y)^5}{32x} + \frac{3(1 - y)^4}{32x} - \frac{3(1 - y)^3}{32x} - \frac{3(1 - y)^2}{3(1 - y)} + \frac{2x^2}{3} \\
&+ \frac{8x^2}{3} - \frac{20x}{3} + \frac{4x^2}{3} + \frac{2x^2}{3} \\
&- \frac{2x^2}{3} - \frac{2x^2}{3} + \frac{2y}{3(1 - x)} - \frac{2y}{3(1 - x)} - \frac{2y}{3(1 - x)} + \frac{2y}{3(1 - x)} \\
&+ \frac{4y}{32} + \frac{8}{3(1 - x)(1 - y)^5} - \frac{3(1 - x)(1 - y)^4}{16} + \frac{(1 - x)(1 - y)^3}{16} + \frac{8}{3(1 - x)(1 - y)^2} - \frac{16}{3(1 - x)(1 - y)} + \frac{4}{3(1 - x)} \\
&- \frac{(1 - y)^5(x + 1)}{32} + \frac{3(1 - y)^5}{80} + \frac{(1 - y)^3(x + 1)}{80} - \frac{80}{3(1 - y)^4} \\
\end{align*}
\]
\[
\begin{align*}
&- \frac{80}{3(1-y)^3(x+1)} + \frac{16}{3(1-y)^3} - \frac{40}{(1-y)^2(x+1)} + \frac{56}{3(1-y)^2} \\
&+ \frac{3(1-y)(x+1)}{32} + \frac{52}{3(x+1)} \zeta(2) H(0; x) + \left( -\frac{16}{3} \right) \\
&- \frac{3\sqrt{1-y}^5}{4} + \frac{3\sqrt{1-y}^4}{128} - \frac{8}{3x(1-y)^3} - \frac{32}{x(1-y)^5} - \frac{320}{3x(1-y)^4} \\
&+ \frac{x^2(1-y)}{128} + \frac{3x(y+1)}{128} - \frac{8}{x} - \frac{8}{x} + \frac{8}{x} \zeta(2) H(0; x) + \left( -\frac{16}{3} \right) \\
&- \frac{3(1-y)^3}{8x} + \frac{3(1-y)^2}{8x} + \frac{(1-y)^3}{3x(y+1)} + \frac{3(y+1)^3}{3x(y+1)} \\
&- \frac{8}{(y+1)^2} + \frac{44}{8x^2} + \frac{8x}{3(1-y)^3} + \frac{32x^2}{4x^2} + \frac{80x^2}{4x^2} \\
&+ \frac{(1-y)^3}{8x} - \frac{3(1-y)^2}{3x(y+1)} - \frac{3(1-y)}{3x(y+1)} + \frac{3(y+1)}{3y(1-x)} \\
&+ \frac{3(1-x)(1-y)^5}{128} + \frac{3(1-x)(1-y)^4}{64} + \frac{16}{3} \\
&+ \frac{3(1-x)(1-y)^2}{128} + \frac{3(1-x)(1-y)^4}{64} + \frac{16}{3} \\
&- \frac{(1-y)^5(x+1)}{320} + \frac{3(1-y)^5}{320} + \frac{(1-y)(x+1)}{64} + \frac{160}{3} \\
&- \frac{3(1-y)^4}{224} + \frac{3(1-y)^3(x+1)}{64} + \frac{3(1-y)^3}{8} + \frac{80}{3x(x+1)} \\
&- \frac{32}{3(y+1)^3} + \frac{16}{(y+1)^2} - \frac{104}{3(y+1)} \zeta(2) H(0; y) + \left( \frac{32}{15} + \frac{2}{15x^2} \right) \\
&+ \frac{4}{15xy} - \frac{4y}{15x} + \frac{16}{15x} + \frac{4x}{15y} + \frac{4xy}{15} - \frac{16x}{15} + \frac{2x^2}{15} + \frac{2}{15y^2} \\
&- \frac{16}{15x} - \frac{16y}{15} + \frac{2y^2}{15} H(1; y) + \left( \frac{112}{9} + \frac{56}{3x^2(1-y)^5} \right) \\
&- \frac{140}{3x^2(1-y)^4} + \frac{32}{x^2(1-y)^3} - \frac{8x(x+1)}{3x^2(1-y)^2} - \frac{3x^2(1-y)^2}{3x^2(1-y)} \\
&- \frac{x(1-y)^5}{80} + \frac{x(1-y)^4}{200} - \frac{9x(1-y)^3}{896} + \frac{3x(1-y)^2}{152} \\
&+ \frac{68}{3x(1-y)} + \frac{80x}{9x} + \frac{200x}{(1-y)^5} + \frac{896x}{9(1-y)^3}
\end{align*}
\]
\[
\begin{align*}
\frac{152x}{3(1-y)^2} & = \frac{68x}{3(1-y)} - \frac{34x}{9} - \frac{56x^2}{3(1-y)^3} + \frac{140x^2}{3(1-y)^4} \\
- \frac{32x^2}{(1-y)^3} & + \frac{4x^2}{3(1-y)^2} + \frac{8x^2}{3(1-y)} - \frac{4}{9y(1-x)} - \frac{4}{y(x+1)} \\
+ \frac{20}{9y} + \frac{4y}{(x+1)} & + \frac{4y}{9(1-x)} + \frac{20y}{9} - \frac{20y}{3(1-x)(1-y)^5} \\
+ \frac{3(1-x)(1-y)}{304} & - \frac{9(1-x)(1-y)^3}{1408} + \frac{3(1-x)(1-y)^2}{64} \\
+ \frac{9(1-x)(1-y)}{1760} & - \frac{3(1-x)^3}{560} + \frac{3(1-y)^3(x+1)}{2368} - \frac{224}{3(1-y)^3} \\
+ \frac{104}{(1-y)^2(x+1)} & - \frac{3(1-y)^2}{320} - \frac{(1-y)(x+1)}{80} + \frac{208}{(1-y)} \\
- \frac{104}{9(x+1)} & H(0; x)H(0; y) + \left( - \frac{40}{9} + \frac{16}{3x(1-y)^4} \\
- \frac{3x(1-y)^3}{160x} & + \frac{x(1-y)^2}{52x} + \frac{x(1-y)}{188x} - \frac{9x}{9} + \frac{3x(1-y)^4}{16x} \\
- \frac{160x^2}{32x^2} + \frac{9(1-y)^2}{28x^2} + \frac{9(1-y)}{20x^2} - \frac{9}{10} + \frac{112}{3(1-y)^3} \\
+ \frac{3(1-y)^3}{224} & - \frac{9y(x+1)^2}{116y} - \frac{9y(x+1)}{116y} + \frac{10}{y} - \frac{9y(1-x)}{10y} \\
- \frac{9(x+1)^3}{32} + \frac{9(x+1)^3}{64} - \frac{9(x+1)^2}{3(x+1)} + \frac{56}{9} \\
- \frac{3(1-x)(1-y)^4}{40} & + \frac{3(1-x)(1-y)^3}{44} - \frac{9(1-x)(1-y)^2}{64} \\
- \frac{9(1-x)(1-y)}{96} + \frac{9(1-x)}{48} - \frac{(1-y)^4(x+1)}{128} \\
+ \frac{(1-y)^4(x+1)}{192} & - \frac{(1-y)^4}{96} + \frac{(1-y)^3(x+1)^2}{64} \\
- \frac{(1-y)^3(x+1)}{128} + \frac{(1-y)^3}{368} - \frac{3(1-y)^2(x+1)^4}{152} \\
+ \frac{3(1-y)^2(x+1)^3}{28} & - \frac{9(1-y)^2(x+1)^2}{64} + \frac{9(1-y)^2(x+1)}{128} \\
- \frac{9(1-y)^2}{208} + \frac{3(1-y)(x+1)^4}{712} - \frac{3(1-y)(x+1)^3}{404} \\
- \frac{3(1-y)(x+1)^2}{32} & + \frac{9(1-y)(x+1)}{712} - \frac{9(1-y)}{404} + \frac{3(x+1)^4}{33}
\end{align*}
\]
\[\begin{align*}
- \frac{16}{9x^2(1-y)^4} &+ \frac{32}{9x^2(1-y)^3} + \frac{7}{9x^2(1-y)^2} - \frac{23}{9x(1-y)} \\
+ \frac{9x(1-y)^4}{80} &- \frac{9x(1-y)^3}{160} - \frac{9x(1-y)^2}{56} + \frac{9x(1-y)}{136} \\
+ \frac{3x(y+1)^2}{56x} &- \frac{3x(y+1)}{136x} + \frac{9x}{16x} + \frac{9(1-y)^4}{16x} - \frac{9(1-y)^3}{16x} \\
- \frac{9(1-y)^2}{9(1-y)} &+ \frac{32x}{32} + \frac{9(1-y)}{32x} - \frac{3(y+1)^2}{3(y+1)} \quad \text{for } y \neq 0, x \neq 0 \\
+ \frac{16x^2}{9} - \frac{9(1-y)^4}{9(1-y)^3} + \frac{32x^2}{64} + \frac{9(1-y)^2}{9(1-y)^2} - \frac{23x^2}{74} \\
+ \frac{25y}{9y} + \frac{25y}{9} &- \frac{32}{3(1-y)^4} + \frac{32}{3(1-y)^3} + \frac{3(1-y)^2}{3(1-y)} \quad \text{for } y \neq 0, x \neq 0 \\
- \frac{32}{3(y+1)^2} + \frac{32}{3(y+1)} &H(0,0;y) + \left( \frac{56}{3} - \frac{16}{3} \right) x^2(1-y)^4 \\
+ \frac{3x^2(1-y)^3}{128} &- \frac{9x^2(1-y)^2}{16} - \frac{9x^2(1-y)}{176} + \frac{3x(1-y)^4}{40} + \frac{9x}{64x} \\
- \frac{3x(1-y)^3}{32x^2} &+ \frac{9x(1-y)^2}{28x^2} + \frac{9x(1-y)}{176x} - \frac{9}{20} + \frac{3(1-y)^4}{16x^2} \\
+ \frac{20y}{9y} + \frac{20y}{9(1-x)} &+ \frac{20y}{9(x+1)} - \frac{20y}{9} + \frac{3(1-x)(1-y)^4}{64} \\
- \frac{3(1-x)(1-y)^3}{88} + \frac{9(1-x)(1-y)^2}{64} &+ \frac{9(1-x)(1-y)}{128} \\
\frac{9(1-x)}{128} &+ \frac{(1-y)^4(x+1)}{16} + \frac{3(1-y)^3}{80} + \frac{(1-y)^3(x+1)}{208} \\
\frac{3(1-y)^3}{272} &- \frac{248}{9(1-y)(x+1)} H(-1,0;x) + \left( -\frac{16}{3} - \frac{4}{3} + \frac{4x}{3} \right) \\
+ \left( \frac{4}{3y(1-x)} + \frac{2}{3y} - \frac{4y}{3(1-x)} + \frac{2y}{3} + \frac{32}{3(1-x)} \right) H(0,0,0;x) \\
\left( \frac{x^2(1-y)^4}{3x^2(1-y)^3} + \frac{2}{3x^2(1-y)^4} - \frac{4}{x^2(1-y)} \right) \\
- \frac{3x^2(1-y)^2}{80} &+ \frac{2}{3x^2(1-y)} - \frac{3x^2(y+1)}{32} + \frac{3x(1-y)^5}{32} \\
- \frac{3x(1-y)^4}{3x(1-y)^3} &+ \frac{3x(1-y)^2}{3x(1-y)^2} - \frac{28}{3x(1-y)} \\
\end{align*}\]
\[
\begin{align*}
+ \frac{16}{3x(y+1)^3} - \frac{8}{x(y+1)^2} + \frac{44}{3x(y+1)} - \frac{4}{x} - \frac{32x}{3(1-y)^5} \\
+ \frac{8x}{32} - \frac{x(1-y)^4}{32x} - \frac{(1-y)^2}{4x} - \frac{4x}{3(1-y)^5} - \frac{2}{3(1-y)^4} \\
- \frac{8x}{(y+1)^2} + \frac{44x}{3(y+1)} - \frac{4x}{3} + \frac{8x^2}{(1-y)} + \frac{2x^2}{3(1-y)^5} - \frac{20x^2}{3(1-y)^4} \\
+ \frac{1-y)^3}{2} + \frac{3(1-y)^2}{2} + \frac{3(1-y)}{3} - \frac{3(y+1)}{3(1-x)} \\
+ \frac{3y(1-x)}{32} + \frac{3y(x+1)}{80} + \frac{2y}{3y} - \frac{2y}{3} - \frac{2y}{3} \\
+ \frac{3(1-x)(1-y)^5}{8} - \frac{3(1-x)(1-y)^4}{16} + \frac{(1-x)(1-y)^5}{4} \\
+ \frac{(1-y)^4(x+1)}{32} - \frac{(1-y)^4(x+1)}{32} - \frac{(1-y)^4(x+1)}{80} \\
- \frac{3(1-y)^3(x+1)}{32} - \frac{3(1-y)^3(x+1)}{80} - \frac{(1-y)^2(x+1)}{16} \\
+ \frac{3(1-y)^2}{16} + \frac{3(1-y)(x+1)}{16} + \frac{(1-y)}{8} + \frac{20}{3(1-x)} \\
- \frac{32}{3(y+1)^3} + \frac{16}{(y+1)^2} - \frac{104}{3(y+1)} H(0,0,0; y) + \left(\frac{64}{3}\right) \\
+ \frac{3x^2(1-y)^5}{8} - \frac{3x^2(1-y)^4}{128} + \frac{x^2(1-y)^3}{320} - \frac{3x^2(1-y)^2}{128} \\
- \frac{3x^2(1-y)}{128} - \frac{3x^2(1-y)^5}{32x} + \frac{3x^2(1-y)^4}{320x} - \frac{3x^2(1-y)^2}{32x} \\
+ \frac{3(1-y)^3}{32} - \frac{3(1-y)^2}{32x} - \frac{3(1-y)^3}{32x} - \frac{3(1-y)^2}{32x} \\
- \frac{3(1-y)^4}{80x^2} + \frac{(1-y)^3}{16x^2} - \frac{3(1-y)^2}{8x^2} + \frac{3(1-y)^3}{8x^2} - \frac{3(1-y)^4}{8x^2} \\
- \frac{8}{3y(x+1)} + \frac{2}{y} + \frac{4y}{(1-x)} + \frac{8y}{3y} - \frac{10y}{3} \\
- \frac{3(1-x)(1-y)^5}{32} - \frac{3(1-x)(1-y)^4}{64} - \frac{(1-x)(1-y)^5}{16} \\
- \frac{3(1-x)(1-y)^2}{128} + \frac{3(1-x)(1-y)^5}{320} - \frac{3(1-x)}{320} \\
+ \frac{(1-y)^5(x+1)}{320} - \frac{(1-y)^4(x+1)}{64} + \frac{(1-y)^4(x+1)}{160} - \frac{3(1-y)^4}{224} \\
+ \frac{3(1-y)^3(x+1)}{64} - \frac{(1-y)^2(x+1)}{32(1-y)^2} 
\end{align*}
\]
\[
- \frac{64}{3(1-y)(x+1)} - \frac{112}{3(x+1)} ) H(0,0;y)H(0;x) + \left( 4 + \frac{8}{20(1-y)^5} - \frac{3x^2(1-y)^4}{32} + \frac{4}{x^2(1-y)^3} + \frac{2}{3x(1-y)^2} \\
+ \frac{3x^2(1-y)^4}{32} - \frac{3x(1-y)^5}{3} + \frac{3x(1-y)^4}{32x} - \frac{3x(1-y)^3}{80x} \\
- \frac{3x(1-y)^2}{8} - \frac{3x(1-y)^5}{8x} + \frac{3x(1-y)^4}{32x} - \frac{8y}{8x^2} \\
+ \frac{3(1-y)^3}{2} + \frac{3(1-y)^2}{8} - \frac{3(1-y)}{32x} - \frac{3(1-y)}{8x^2} \\
+ \frac{3(1-y)^3}{2} - \frac{(1-y)^3}{3} - \frac{3(1-y)^2}{2} + \frac{3(1-y)}{2y} - \frac{2y}{3} \\
- \frac{3(1-y)^5}{3y(1-x)} + \frac{3(1-y)^4}{3y(x+1)} - \frac{(1-y)^4}{4} \\
- \frac{8(1-y)^7(x+1)}{3(1-y)^2} + \frac{3(1-y)^5}{32} - \frac{3(1-y)^4}{80} \\
+ \frac{3(1-y)^4}{80} - \frac{(1-y)^5(x+1)}{56} - \frac{3(1-y)^3}{16} \\
+ \frac{3(1-y)^4}{40} - \frac{(1-y)^2(x+1)}{3(1-y)^2} - \frac{9x^2(1-y)^2}{9x^2(1-y)} \\
- \frac{40}{3(1-y)^4} \right) \left[ (G(-y,0,0;x) - G(-1/y,0,0;x)) \\
+ (G(-y,0;x) + G(-1/y,0,x))H(0;y) + \left( G(-1/y;x) \\
- G(-y;x) \right) \left( H(0,0;y) + 3\zeta(2) \right) + 2H(0;x)H(1,0;y) \\
- 2H(-1,0;x)H(0;y) - 6H(-1,0;y)H(0;x) \right] . \tag{97}
\]

\[
B_2^{(2i,-1)}(-P^2, -Q^2) = - \left[ \left( \frac{56}{3} - \frac{32}{3x^2(1-y)^4} + \frac{64}{3x(1-y)^3} - \frac{64}{9x^2(1-y)^2} - \frac{32}{9x^2(1-y)} \\
+ \frac{32}{3x(1-y)} - \frac{32}{3x^2(1-y)^4} + \frac{40}{3x(1-y)^3} + \frac{32x}{3(1-y)^2} - \frac{32x}{3(1-y)} - \frac{40x}{9} \\
+ \frac{32x^2}{9y(x+1)} - \frac{64x^2}{9y} + \frac{56x^2}{9y(x+1)} - \frac{40x^2}{9} \\
+ \frac{20}{9y} + \frac{20y}{9(1-x)} + \frac{20y}{9(x+1)} - \frac{20y}{9} \right) \left[ (G(-y,0,0;x) - G(-1/y,0,0;x)) \\
+ (G(-y,0;x) + G(-1/y,0,x))H(0;y) + \left( G(-1/y;x) \\
- G(-y;x) \right) \left( H(0,0;y) + 3\zeta(2) \right) + 2H(0;x)H(1,0;y) \\
- 2H(-1,0;x)H(0;y) - 6H(-1,0;y)H(0;x) \right] . \tag{97}
\]
\[
B_{2}^{(21,0)}(-P^{2},-Q^{2}) = - \left\{ \frac{64}{9} - \frac{32}{45x^{2}(1-y)^{4}} + \frac{64}{45x^{2}(1-y)^{3}} + \frac{208}{135x^{2}(1-y)^{2}} \right. \\
- \frac{64}{304} - \frac{15x(1-y)^{2}}{15x(1-y)} + \frac{15x(1-y)}{135x} - \frac{15(1-y)^{2}}{16} + \frac{32x^{2}}{32x} + \frac{64x^{2}}{64x} + \frac{304x}{304} - \frac{32x^{2}}{32x} \\
- \frac{45(1-y)^{3}}{45(1-y)^{3}} + \frac{135(1-y)^{2}}{135(1-y)^{2}} - \frac{135(1-y)}{3y(x+1)^{2}} - \frac{3y(x+1)^{2}}{3y(x+1)^{2}} - \frac{8}{8} + \frac{8y}{8y} + \frac{152y}{152y} - \frac{8y}{8y} \\
- \frac{64}{64} + \frac{152}{135y} - \frac{3(x+1)^{2}}{3(x+1)^{2}} + \frac{3(x+1)^{2}}{3(x+1)^{2}} + \frac{128}{128} - \frac{64}{64} \\
- \frac{45(1-y)^{4}}{45(1-y)^{4}} - \frac{9(1-y)^{2}(x+1)^{2}}{9(1-y)^{2}(x+1)^{2}} \right\} 
\]
\[
\begin{align*}
&\frac{64}{9(1-y)^2(x+1)} + \frac{128}{135(1-y)^2} + \frac{64}{9(1-y)(x+1)^2} \\
&- \frac{9(1-y)(x+1)}{64} - \frac{27(1-y)}{128} + \frac{3(x+1)^2}{16} - \frac{3(x+1)}{16} \\
&+ \left( \frac{4}{3} - \frac{3x^2(1-y)^4}{64} + \frac{3x^2(1-y)^3}{128} - \frac{9x^2(1-y)^2}{8} \\
&- \frac{80}{9x^2(1-y)} + \frac{32}{x(1-y)^4} - \frac{64}{x(1-y)^3} + \frac{32}{x(1-y)^2} + \frac{16}{x(1-y)} - \frac{16}{8} \\
&+ \frac{3x(1-y)}{64x} + \frac{3x(y+1)^2}{32x^2} - \frac{3x(y+1)}{32x} + \frac{14}{9x} + \frac{32x}{(1-y)^4} \\
&- \frac{64x}{(1-y)^3} + \frac{32}{3(1-y)^2} + \frac{56x}{3(1-y)} - \frac{64}{3(y+1)^2} - \frac{56x^2}{3(y+1)} \\
&- \frac{3(y+1)}{640} - \frac{9y(x+1)}{40x} - \frac{2}{y(x+1)^4} + \frac{16}{y} - \frac{16y}{9} + \frac{16y}{9(1-x)} \\
&+ \frac{32y}{9(1-x)} - \frac{20y}{(x+1)^2} + \frac{46y}{9(x+1)} - \frac{2}{y} + \frac{64}{9} - \frac{3(1-x)(1-y)}{128} - \frac{9(1-x)(1-y)^2}{112} - \frac{9(1-x)(1-y)}{80} \\
&- \frac{44}{9(1-x)} - \frac{224}{(1-y)^4(x+1)} + \frac{3(1-y)^4(x+1)}{640} \\
&- \frac{3(1-y)^4}{448} + \frac{(1-y)^3(x+1)^2}{128} - \frac{3(1-y)^3(x+1)}{256} \\
&+ \frac{3(1-y)^3}{160} - \frac{3(1-y)^2(x+1)^4}{128} + \frac{3(1-y)^2(x+1)^3}{256} \\
&- \frac{9(1-y)^2(x+1)^2}{176} + \frac{9(1-y)^2(x+1)}{9} - \frac{9(1-y)^2}{104} \\
&+ \frac{3(1-y)(x+1)}{784} - \frac{3(1-y)(x+1)^3}{568} - \frac{3(1-y)(x+1)^2}{32} \\
&+ \frac{40(x+1)^2}{9(1-y)(x+1)} - \frac{9(1-y)}{9(1-y)^4} + \frac{64}{3(y+1)^2} + \frac{64}{3(y+1)} \right) \zeta(2) \\
&+ \left( \frac{308}{x^2(1-y)^4} - \frac{196}{x^2(1-y)^3} + \frac{272x^2(1-y)^2}{x(1-y)} + \frac{272}{27x} - \frac{32x}{32x} \\
&+ \frac{32x}{1-2} - \frac{272x}{(1-y)^4} + \frac{64x^2}{(1-y)^3} - \frac{640x^2}{27(1-y)^2} \right)
\end{align*}
\]
\[\begin{align*}
- \frac{224x^2}{27(1-y)} & - \frac{118}{27y(1-x)} - \frac{16}{3y(x+1)^3} + \frac{8}{y(x+1)^2} \\
- \frac{214}{27y(x+1)} + \frac{130}{27y} - \frac{118y}{27(1-x)} - \frac{16y}{3(x+1)^3} + \frac{8y}{(x+1)^2} \\
= \frac{214y}{27(x+1)} + \frac{130y}{27} + \frac{(1-x)(1-y)^3}{128} - \frac{(1-x)(1-y)^3}{800} + \frac{256}{356} \\
\frac{27(1-x)(1-y)^2}{27(1-x)(1-y)^2} + \frac{27(1-x)(1-y)^2}{27(1-x)(1-y)^2} + \frac{27(1-x)}{27(1-x)} \\
\frac{(1-y)^4(x+1)}{(1-y)^4(x+1)} - \frac{(1-y)^4}{(1-y)^4} - \frac{(1-y)^3(x+1)}{(1-y)^3(x+1)} \\
\frac{256}{256} - \frac{9(1-y)^2(x+1)^3}{1184} + \frac{3(1-y)^2(x+1)^3}{3(1-y)^2(x+1)^3} \\
- \frac{27(1-y)^2(1-x)}{27(1-y)^2(1-x)} - \frac{9(1-y)(1-x)^4}{27(1-y)^2 + 9(1-y)(1-x)^4} \\
- \frac{3(1-y)(x+1)^2}{3(1-y)(x+1)^2} + \frac{27(1-y)(x+1)}{27(1-y)(x+1)} - \frac{27(1-y)}{27(1-y)} \\
\frac{32}{3(x+1)^3} - \frac{16}{(x+1)^2} - \frac{1636}{27(x+1)} + \left(\frac{42}{5}\right) \\
- \frac{45x^2(1-y)^5}{4x} + \frac{9x^2(1-y)^4}{64} - \frac{9x^2(1-y)^2}{32} + \frac{9x^2(1-y)}{112} \\
+ \frac{15x^2}{15} - \frac{15x(1-y)^3}{64x} + \frac{5x(1-y)^2}{32x} - \frac{45x(1-y)}{112x} \\
+ \frac{16}{4y} - \frac{124}{15} - \frac{15(1-y)^3}{64x} + \frac{32x^2}{16x^2} \\
- \frac{45(1-y)}{45(1-y)} + \frac{3(y+1)}{45} - \frac{124}{46y} + \frac{2}{2} - \frac{16x^2}{64} \\
\frac{16x^2}{9(1-y)^2} + \frac{4x^2}{9(1-y)} + \frac{4x^2}{15} + \frac{2}{15} - \frac{16x^2}{64} \\
\frac{32}{32} - \frac{45(1-y)^3}{45(1-y)^3} - \frac{45(1-y)^2}{45(1-y)^2} - \frac{5(1-y)}{8} \\
- \frac{32}{3(y+1)} - \frac{16}{3(y+1)} + \left(-8 - \frac{16}{3(y+1)} + \frac{40}{3(y+1)}\right) \\
\frac{x^2(1-y)^3}{8} - \frac{3x^2(1-y)^2}{8} + \frac{3x^2(1-y)}{3} - \frac{3x(1-y)^3}{4} \\
\frac{x(1-y)^2}{3} + \frac{16x}{3} + \frac{8x}{3} + \frac{8x}{3} + \frac{16x^2}{3} \\
- \frac{8x^2}{3} + \frac{3(1-y)^3}{3} - \frac{(1-y)^2}{3} + \frac{(1-y)^2}{3} + \frac{(1-y)^2}{3} + \frac{(1-y)^2}{3} \\
- \frac{4y}{3} - \frac{3(1-x)(1-y)^5}{3} + \frac{3(1-x)(1-y)^4}{160}
\end{align*}\]
\[
\begin{align*}
&- \frac{32}{(1-x)(1-y)^3} - \frac{16}{3(1-x)(1-y)^2} + \frac{8}{3(1-x)(1-y)} \\
&+ \frac{3(1-x)}{160} - \frac{3(1-y)^5(x+1)}{32} + \frac{3(1-y)^5}{64} + \frac{3(1-y)^4(x+1)}{112} \\
&+ \frac{3(1-y)^4}{64} - \frac{3(1-y)^3(x+1)}{32} + \frac{3(1-y)^3}{64} - \frac{3(1-y)^2(x+1)}{40} \\
&+ \frac{3(1-y)^2}{8} + \frac{3(1-y)(x+1)}{3(1-y)} - \frac{8}{3(1-y)} + \frac{3(x+1)}{32} \zeta(2) H(0; x) \\
&+ \left( - \frac{8}{3} - \frac{64}{3x^2(1-y)^5} + \frac{160}{3x^2(1-y)^4} - \frac{32}{x^2(1-y)^3} - \frac{16}{3x^2(1-y)^2} + \frac{8}{x^2(1-y)} - \frac{8}{3x^2(y+1)} + \frac{16}{3x(y+1)} + \frac{3x(y+1)^3}{40} \\
&- \frac{8}{x} + \frac{3x(1-y)}{64x} - \frac{3x(y+1)^3}{32x} - \frac{x(y+1)^2}{8x} - \frac{x(y+1)}{32x} - \frac{3(x+1)}{16x} - \frac{(y+1)^2}{16x^2} \\
&+ \frac{32x^2}{3(1-y)^3} - \frac{16x^2}{3(1-y)^2} - \frac{8x^2}{3(1-y)} - \frac{8x^2}{3(y+1)} + \frac{8}{8y} \\
&+ \frac{8}{3y(x+1)} - \frac{3y}{3(1-x)} - \frac{3(y+1)}{3(x+1)} + \frac{3}{3y(1-x)} + \frac{3y(1-x)}{256} \\
&+ \frac{3(1-x)(1-y)^4}{32} - \frac{3(1-x)(1-y)^3}{16} - \frac{3(1-x)(1-y)^2}{256} \\
&+ \frac{3(1-x)(1-y)}{3(1-x)} + \frac{3(1-x)^5(x+1)}{640} - \frac{3(1-x)^4}{640} - \frac{3(1-x)^3(x+1)}{128} - \frac{3(1-x)^2(x+1)}{448} \\
&+ \frac{3(1-x)^3}{16} - \frac{3(1-x)^2(x+1)}{64} + \frac{3(1-x)^2}{32} - \frac{3(1-y)(x+1)}{40} - \frac{3(y+1)}{3(y+1)^2} \\
&- \frac{128}{3(y+1)} \zeta(2) H(0; y) + \left( - \frac{8}{15} + \frac{4x}{15y} + \frac{4xy}{15} - \frac{8x}{15} + \frac{4x^2}{15} + \frac{2}{15y^2} - \frac{16}{15y} - \frac{16y}{15} \\
&+ \frac{2y^2}{15} \right) H(1; y) + \left( - \frac{44}{3} + \frac{128}{3x^2(1-y)^5} - \frac{3x^2(1-y)^4}{3} - \frac{3x^2(1-y)^2}{20} - \frac{3x^2(1-y)}{352} \\
&+ \frac{x^2(1-y)^3}{3x^2(1-y)^2} - \frac{x^2(1-y)}{3x^2(1-y)} + \frac{9x(1-y)^3}{3x^2(1-y)^2}
\right).
\end{align*}
\]
\[
\begin{align*}
- \frac{176}{3x(1-y)^2} &+ \frac{32}{3x(1-y)} + \frac{40}{9x} - \frac{352x}{9(1-y)^3} + \frac{176x}{3(1-y)^2} \\
- \frac{32x}{3(1-y)} &- \frac{128x^2}{9} - \frac{4x^2}{3(1-y)^5} + \frac{320x^2}{3(1-y)^4} - \frac{72x^2}{(1-y)^3} \\
+ \frac{3(1-y)^2}{20x^2} &+ \frac{3(1-y)}{9y(1-x)} - \frac{20}{9y(x+1)} - \frac{20}{3(1-x)(1-y)^5} \\
+ \frac{20}{9y} &+ \frac{20y}{9(1-x)} + \frac{20y}{9(x+1)} - \frac{20y}{9} + \frac{512}{3(1-x)(1-y)^5} \\
- \frac{3(1-x)(1-y)^4}{1280} &+ \frac{(1-x)(1-y)^3}{52} + \frac{3(1-y)^5(x+1)}{512} - \frac{3(1-x)(1-y)^2}{1280} \\
- \frac{3(1-y)^4(x+1)}{1888} &+ \frac{3(1-y)^4}{688} + \frac{9(1-y)^3(x+1)}{112} - \frac{9(1-y)^4(x+1)}{520} \\
+ \frac{328}{9(1-y)} &- \frac{212}{9(x+1)} \cdot H(0;y) \cdot H(0;x) + \left( \frac{-28}{3} + \frac{32}{3x(1-y)^4} \right) \\
- \frac{3x(1-y)^3}{64x} &+ \frac{8}{x(1-y)^2} + \frac{8}{3x(1-y)} - \frac{26}{9x} + \frac{32x}{3(1-y)^4} \\
- \frac{3(1-y)^3}{64x^2} &- \frac{3(1-y)^2}{56x^2} + \frac{3(1-y)}{40x^2} + \frac{9}{3(1-y)} - \frac{32x^2}{3(1-y)^4} \\
+ \frac{3(1-y)^3}{16} &- \frac{9(1-y)^2}{32} - \frac{9(1-y)}{20} - \frac{9y(1-x)}{3(1-x)(1-y)} \\
- \frac{3y(x+1)^4}{10} &+ \frac{3y(x+1)^3}{10y} + \frac{3y(x+1)^2}{16y} - \frac{32y}{3(1-x)(1-y)^3} + \frac{9y(x+1)}{3(1-x)^2} \\
+ \frac{2y}{9(1-x)} &+ \frac{10y}{3(x+1)^4} + \frac{64}{3(x+1)^3} + \frac{32y}{3(x+1)^3} - \frac{20y}{3(x+1)^2} \\
+ \frac{9(1-x)(1-y)^2}{128} &+ \frac{9(1-x)(1-y)}{64} + \frac{44}{9(1-x)} + \frac{9(1-y)(x+1)}{160} \\
- \frac{3(1-y)^4(x+1)^2}{256} &+ \frac{(1-y)^4(x+1)}{128} - \frac{3(1-y)^4}{320} \\
+ \frac{3(1-y)^3(x+1)^2}{128} &- \frac{(1-y)^3(x+1)}{256} + \frac{3(1-y)^5}{160} \\
- \frac{9(1-y)^2(x+1)^4}{16} &+ \frac{9(1-y)^2(x+1)^3}{56} - \frac{9(1-y)(x+1)^4}{128} - \frac{9(1-y)(x+1)^2}{224} + \frac{9(1-y)(x+1)^4}{208} \\
- \frac{9(1-y)(x+1)^3}{256} &- \frac{9(1-y)(x+1)^2}{3(1-y)(x+1)} + \frac{208}{3(1-y)(x+1)}
\end{align*}
\]
\[-\frac{424}{9(1-y)} + \frac{32}{3(x+1)^4} - \frac{64}{3(x+1)^3} + \frac{40}{3(x+1)^2}
\]
\[+\frac{100}{9(x+1)} H(0, 0; x) + \left(\frac{-142}{9} - \frac{32}{9x^2(1-y)^4} + \frac{64}{9x^2(1-y)^3}\right)
\]
\[+\frac{9x^2(1-y)^2}{40} - \frac{9x^2(1-y)}{8} + \frac{9x(1-y)^4}{32} - \frac{9x(1-y)^3}{32} - \frac{9x(1-y)^2}{8} \]
\[+\frac{9x + 9(1-y)^4}{64x} - \frac{32}{3(y+1)^2} + \frac{9(1-y)^3}{3(y+1)^2} + \frac{9(1-y)^2}{3(y+1)^2} + \frac{9(1-y)}{8x} \]
\[+\frac{32x}{3(y+1)^2} - \frac{32x}{3(y+1)} + \frac{32x}{9} - \frac{32x^2}{9(1-y)^4} + \frac{64x^2}{9(1-y)^3} \]
\[+\frac{46x^2}{9(1-y)^3} - \frac{25}{9y} + \frac{25y}{9} - \frac{9(1-y)^4}{9(1-y)^3} \]
\[+\frac{20}{64} - \frac{124}{3(y+1)^2} + \frac{64}{3(y+1)^2} + \frac{64}{9(1-y)^3} \]
\[+\frac{56}{3} - \frac{32}{3x^2(1-y)^4} + \frac{32x^2(1-y)^3}{x} - \frac{32x}{3x^2(1-y)^3} + \frac{9x^2(1-y)^2}{9x^2(1-y)^3} \]
\[+\frac{9x^2(1-y)}{40} - \frac{32}{3(y+1)^2} + \frac{3x(1-y)^2}{3(y+1)^2} + \frac{9x}{3(y+1)^2} \]
\[+\frac{32x}{3(y+1)} + \frac{32x^2}{9} - \frac{32x^2}{3(y+1)^3} + \frac{64x^2}{9(y+1)^3} + \frac{56x^2}{9(y+1)^3} \]
\[+\frac{20}{9y(y+1)} + \frac{20}{9y(y+1)} - \frac{20}{9y} + \frac{20y}{9(1-y)} \]
\[+\frac{20y}{20y} - \frac{9}{20y} - \frac{3(1-y)(1-y)^4}{160} + \frac{3(1-y)^3}{160} \]
\[+\frac{224}{9(1-x)(1-y)^2} - \frac{9(1-x)(1-y)}{128} - \frac{9(1-x)}{256} \]
\[+\frac{3(1-y)^4(x+1)}{256} + \frac{3(1-y)^4}{160} + \frac{3(1-y)^3}{32} + \frac{9(1-y)^2}{3(y+1)} \]
\[+\frac{544}{9(1-y)^2(x+1)} + \frac{352}{9(1-y)^2} - \frac{248}{9(x+1)} \]
\[+\left(\frac{4}{3x} + \frac{4x}{3y(1-x)} - \frac{2}{3y(x+1)} + \frac{2}{3y} - \frac{2y}{3(1-x)} \right) \]
\[+\left(\frac{2y}{3(x+1)} + \frac{2y}{3} + \frac{4}{3(1-x)} + \frac{20}{3(x+1)} \right) \]
\[+\frac{28}{3} - \frac{16}{3x^2(1-y)^5} + \frac{40}{3x^2(1-y)^4} - \frac{8}{x^2(1-y)^3} \]
\[-\frac{4}{3x^2(1-y)^2} + \frac{4}{x^2(1-y)} - \frac{8}{3x(y+1)} - \frac{16}{3x(1-y)^3} \]
\[+ \frac{8}{x(1-y)^2} - \frac{3x(1-y)}{16x} + \frac{3x(y+1)^3}{8x} - \frac{8}{3(1-y)} \]
\[+ \frac{32x}{3y(y+1)} - \frac{16x}{y(1+y)^2} + \frac{40}{3(y+1)^2} - \frac{4x}{3} + \frac{2y}{16x^2} \]
\[- \frac{3(1-y)^4}{3(y+1)^3} + \frac{(1-y)^3}{3(1-y)^2} + \frac{3(1-y)^2}{3(1-y)} \]
\[- \frac{8}{3(y+1)} - \frac{2}{3y(1-y)} + \frac{2}{3y(x+1)} - \frac{2y}{3(1-x)} \]
\[- \frac{3(x+1)}{32} - \frac{3}{3(1-x)(1-y)^5} + \frac{3(1-x)(1-y)^4}{8} \]
\[- \frac{3(1-y)^4}{3(1-y)^3} - \frac{3(1-x)(1-y)^2}{3(1-x)(1-y)} \]
\[- \frac{3(1-x)}{160} - \frac{3(1-y)x}{32} \]
\[- \frac{3(1-y)^2}{64} + \frac{3(1-y)(x+1)}{8} + \frac{3(1-y)}{3(x+1)} \]
\[- \frac{32}{3(y+1)^3} + \frac{128}{3(y+1)} - H(0, 0; y) \]
\[+ \left(20 + \frac{3x^2(1-y)^5}{3x^2(1-y)^4} - \frac{3x^2(1-y)^4}{x^2(1-y)^3} \right) \]
\[- \frac{16}{3(1-y)^3} + \frac{20}{3(1-y)^2} - \frac{2x}{16x^2} \]
\[- \frac{32x}{16x^2} + \frac{16x^2}{3(1-y)^4} - \frac{2}{3} \]
\[- \frac{2}{3} + \frac{10y}{3(1-x)} + \frac{10y}{3(x+1)} - \frac{10y}{3} + \frac{3(1-x)(1-y)^5}{256} \]
\[- \frac{3(1-x)(1-y)^4}{32} - \frac{3(1-x)}{20} + \frac{3(1-y)(x+1)}{256} \]
\[- \frac{3(1-y)^4}{3(1-y)(x+1)} + \frac{3(1-y)^3}{3(1-y)^3(x+1)} \]
\[- \frac{256}{3(1 - y)^3} + \frac{448}{3(1 - y)^2(x + 1)} - \frac{256}{3(1 - y)^2} \]
\[- \frac{32}{3(1 - y)(x + 1)} + \frac{32}{3(1 - y)} - \frac{100}{3(x + 1)} \right) H(0, 0; y) H(0; x) \]
\[+ \left( \frac{4}{3} + \frac{16}{3x^2(1 - y)^5} - \frac{8}{3x^2(1 - y)^4} + \frac{16}{3x^2(1 - y)^3} - \frac{2}{3x^2(1 - y)^2} \right) \]
\[\frac{16}{x(1 - y)^2} + \frac{4}{3x} - \frac{16}{3x} \]
\[\frac{8}{3(1 - y)^3} + \frac{4x^2}{3(1 - y)} - \frac{4x^2}{3(1 - y)^3} + \frac{8}{3(1 - y)^2} \]
\[\frac{2}{3(1 - y)^2} + \frac{2}{3(1 - y)} - \frac{2y}{3(1 - x)(1 - y)^5} \]
\[\frac{160}{3(1 - x)(1 - y)^4} + \frac{4}{3(1 - x)(1 - y)^3} + \frac{16}{3(1 - x)(1 - y)^2} \]
\[\frac{8}{3(1 - x)} + \frac{32}{3(1 - x)(1 - y)^5} \]
\[- \frac{160}{3(1 - x)(1 - y)^4} + \frac{160}{3(1 - x)(1 - y)^3} + \frac{32}{3(1 - x)(1 - y)^2} \]
\[\frac{64}{3(1 - y)^3} + \frac{160}{3(1 - y)^2(x + 1)} - \frac{160}{3(1 - y)(x + 1)} \]
\[\frac{32}{3(1 - y)^3} - \frac{32}{3(1 - y)(x + 1)} \]
\[\frac{8}{3(1 - y)} - \frac{20}{3(x + 1)} \right) \left[ G(-y, 0; 0; x) - G(-1/y, 0, 0; x) \right] \]
\[+ \left( G(-y, 0; x) + G(-1/y, 0, x) \right) H(0; y) + \left( G(-1/y; x) \right. \]
\[- G(-y; x) \left( H(0, 0; y) + 3\zeta(2) \right) + 2H(0; x) H(1, 0; y) \]
\[-2H(-1, 0; x) H(0, 0; y - 6H(-1, 0; y) H(0; x) \right] \}

\[B_3^{(2l-1)}(-p^2, -Q^2) = - \left[ \left( \frac{16}{3x^2(1 - y)^4} - \frac{32}{3x^2(1 - y)^3} + \frac{28}{9x^2(1 - y)^2} + \frac{20}{9x^2(1 - y)} \right) \right. \]
\[+ \left( \frac{64}{3x(1 - y)^4} - \frac{128}{3x(1 - y)^3} + \frac{112}{9x(1 - y)^2} + \frac{80}{9x(1 - y)} \right) \]
\[- \frac{3(1 - y)^4}{16x^2} + \frac{32}{3x^2} - \frac{9x^2}{80x} \]
\[\frac{28x^2}{16x^2} + \frac{28x^2}{20x^2} \]
\[- \frac{3(1 - y)^4}{64} + \frac{3(1 - y)^3}{128} - \frac{9x^2}{112} + \frac{9x^2}{112} \]
\[+ \frac{(1 - x)(1 - y)^4}{(1 - x)(1 - y)^3} + \frac{3(1 - x)(1 - y)^2}{3(1 - x)(1 - y)^2} \]
\[
B_3^{(2,0)}(-P^2, -Q^2) = \left\{ \frac{16}{3} + \frac{16}{45x^2(1-y)^4} - \frac{32}{45x^2(1-y)^3} - \frac{104}{135x^2(1-y)^2} \right. \\
+ \frac{152}{135x^2(1-y)} + \frac{64}{45x(1-y)^4} - \frac{128}{45x(1-y)^3} \\
+ \frac{64}{135x(1-y)^2} + \frac{128}{135x(1-y)} + \frac{64x}{45(1-y)^4} - \frac{128x}{45(1-y)^3} \\
+ \frac{104x^2}{135(1-y)^2} + \frac{152x^2}{135(1-y)} + \frac{32}{45(1-y)^4} - \frac{32}{45(1-y)^3} \\
- \frac{135(1-y)^2}{32y} + \frac{135(1-y)}{32y} - \frac{9y(x+1)^2}{32} + \frac{9y(x+1)}{64} \\
- \frac{9(x+1)^2}{32} + \frac{9(x+1)}{32} - \frac{45(1-y)^4}{45(1-y)^3} - \frac{992}{992} \\
- \frac{9(1-y)^2(x+1)^2}{9(1-y)^2(x+1)} + \frac{135(1-y)^2}{135(1-y)^2}
\right\}, \quad (100)
\]
\[
\frac{32}{9(1-x)(x+1)^2} - \frac{9(1-y)(x+1)}{64} - \frac{896}{135(1-y)} \\
+ \left( \frac{20 + \frac{32}{3x^2(1-y)^4} - \frac{3x^2(1-y)^3}{16} + \frac{9x^2(1-y)}{96}}{124} \right) - \frac{9x^2(1-y)}{124} - \frac{3(1-y)^3}{32x} \\
+ \left( \frac{9x(y+1)}{32x} - \frac{9(1-y)}{8} + \frac{9(1-y)}{3(y+1)^2} + \frac{9(1-y)}{4x} + \frac{9(1-y)}{3(1-y)^4} + \frac{3(1-y)^3}{28x^2} \right) - \frac{9x(1-y)}{32x} - \frac{9(1-y)}{8} \\
+ \left( \frac{3(1-y)^4}{64} - \frac{3(1-y)^4}{256} + \frac{3(1-y)^3}{32} - \frac{3(1-y)^3}{32x^2} + \frac{3(y+1)^2}{16x} + \frac{3(y+1)^4}{64} + \frac{3(y+1)^3}{128} - \frac{3y(x+1)^2}{64} - \frac{3y(x+1)^3}{128y} \right) - \frac{3y(x+1)}{8y} - \frac{3(x+1)^4}{32} + \frac{3(x+1)^3}{3y(x+1)} \\
+ \left( \frac{3(1-x)(1-y)^3}{64} - \frac{3(1-x)(1-y)^4}{160} + \frac{3(1-x)(1-y)^2}{16} - \frac{3(1-y)^4(x+1)^2}{104} + \frac{3(1-y)^3(x+1)^2}{128} \right) - \frac{3(1-x)(1-y)^2}{140} + \frac{3(1-y)^2(x+1)^3}{64} - \frac{3(1-y)^2(x+1)^2}{128} - \frac{3(1-y)(x+1)^4}{224} \\
+ \left( \frac{224}{3(x+1)} + \frac{3(1-x)(1-y)^2}{104} - \frac{3(1-y)(x+1)^4}{100} + \frac{3(1-y)(x+1)^2}{224} \right) \zeta(2) + \left( -\frac{88}{9} \right) - \frac{3x^2(1-y)^3}{176} \\
+ \left( \frac{3x(1-y)^3}{352} - \frac{27x^2(1-y)^2}{1172} - \frac{3x(1-y)^3}{352} \right) - \frac{e^{27x^2(1-y)^2}}{412} - \frac{27x(1-y)^2}{176x} - \frac{9x(1-y)^4}{176} \\
+ \left( \frac{9y(x+1)^3}{80x^2} + \frac{27(1-y)^2}{278x^2} + \frac{9y(x+1)^3}{32} - \frac{27(1-y)^2}{58} - \frac{27(1-y)^2}{10x} + \frac{9y(x+1)^3}{3y(1-x)} + \frac{27(1-y)^2}{2} + \frac{3(1-y)^4}{16} \right) + \frac{9y(x+1)^3}{9y(x+1)^2} + \frac{9y(x+1)^3}{9} + \frac{9y(x+1)^3}{3y(1-x)} \\
\]
\[
\begin{align*}
\frac{64y}{9(x+1)^3} &+ \frac{32y}{3(x+1)^2} - \frac{58y}{9(x+1)} + \frac{10y}{9} - \frac{512}{3(1-x)(1-y)^4} \\
+ \frac{3(1-x)(1-y)^3}{20} - \frac{9(1-x)(1-y)^2}{64} - \frac{9(1-x)(1-y)}{160} \\
+ \frac{9(1-x)}{128} + \frac{(1-y)^4(x+1)}{64} + \frac{3(1-y)^4}{64} \\
- \frac{(1-y)^3(x+1)}{32} - \frac{3(1-y)^3}{1520} - \frac{9(1-y)^2(x+1)^3}{16} \\
+ \frac{3(1-y)^2(x+1)^2}{64} + \frac{27(1-y)^2(x+1)}{32} + \frac{27(1-y)^2}{208} \\
+ \frac{472}{27(1-y)} + \frac{52}{3(x+1)} \cdot H(0; x) + \left( -\frac{332}{45} + \frac{16}{45x^2(1-y)^5} \right) \\
- \frac{9x^2(1-y)^4}{32} + \frac{9x^2(1-y)^2}{32} - \frac{9x^2(1-y)}{32} - \frac{16}{15x^2} \\
- \frac{45x(1-y)^5}{32} - \frac{9x(1-y)^4}{32} + \frac{9x(1-y)^3}{32} - \frac{9x(1-y)^2}{32} \\
+ \frac{9x(1-y)}{32x} - \frac{3x(y+1)}{16x} - \frac{8x}{32x} - \frac{8x^2}{16x} + \frac{8x^2}{32x} - \frac{9x(1-y)^4}{8x} \\
+ \frac{9(1-y)^3}{8x^2} - \frac{8x^2}{32x} - \frac{9(1-y)^2}{8x^2} + \frac{8x^2}{32x} - \frac{2x^2}{15} \\
+ \frac{16(1-y)^5}{8x^2} - \frac{9(1-y)^4}{8x^2} + \frac{9(1-y)^3}{8x^2} - \frac{9(1-y)^2}{8x^2} + \frac{(1-y)^4}{15} \\
+ \frac{16}{3(y+1)} \cdot H(0; y) + \left( -\frac{4}{3} + \frac{8}{3x^2(1-y)^5} - \frac{20}{3x^2(1-y)^4} \right) \\
+ \frac{2}{3x(1-y)^3} + \frac{2}{3x^2(1-y)^2} - \frac{2}{3x^2(1-y)} + \frac{2}{3x(1-y)^5} \\
- \frac{3x(1-y)^4}{32x} + \frac{x(1-y)^3}{80x} - \frac{3x(1-y)^2}{16x} - \frac{8x}{3}(1-y) \\
- \frac{3(1-y)^5}{8x^2} + \frac{3(1-y)^4}{20x^2} - \frac{(1-y)^3}{4x^2} - \frac{3(1-y)^2}{2x^2} + \frac{3(1-y)}{2x^2} \\
- \frac{2}{3y(1-x)} + \frac{2}{3y(y+1)} - \frac{2}{3y(1-x)} + \frac{2}{3y(x+1)} \\
+ \frac{3(1-x)(1-y)^5}{32} - \frac{(1-x)(1-y)^4}{80} + \frac{(1-x)(1-y)^3}{48} \\
+ \frac{(1-x)(1-y)^2}{3(1-x)(1-y)^3} - \frac{(1-x)(1-y)^2}{3(1-x)} - \frac{3(1-y)^5(x+1)}{32} \\
\end{align*}
\]
\[
\begin{align*}
\frac{32}{3(1-y)^5} + & \frac{80}{3(1-y)^4(x+1)} + \frac{80}{3(1-y)^4} - \frac{16}{(1-y)^3(x+1)} \\
- \frac{8}{(1-y)^3} - & \frac{3(1-y)^2(x+1)}{8} - \frac{3(1-y)^2}{8} + \frac{3(1-y)^2(x+1)}{32} \\
+ & \frac{3(1-y)}{8} + \frac{4}{(x+1)} \left( \zeta(2)H(0; x) + \left( \frac{8}{3} + \frac{32}{3x^2(1-y)^5} \right) \right) \\
+ & \frac{80}{3x^2(1-y)^4} + \frac{16}{x^2(1-y)^3} + \frac{8}{3x^2(1-y)^2} - \frac{64}{x^2(1-y)} \\
+ & \frac{3x^2(y+1)}{32} - \frac{3x(1-y)^2}{128} - \frac{3x(1-y)^2}{320} + \frac{x(y+1)^2}{8} + x(1-y)^2 \\
+ & \frac{3x(y+1)}{32} - \frac{3(1-y)^5}{3(1-y)} + \frac{3(1-y)^4}{32x} - \frac{3(1-y)^4}{80x^2} + \frac{(y+1)^2}{8x} \\
+ & \frac{3(y+1)}{8x^2} - \frac{3(1-y)^5}{3(1-y)} + \frac{3(1-y)^4}{4x^2} - \frac{3(1-y)^4}{168} + \frac{(1-x)(1-y)^5}{32} \\
+ & \frac{3(1-y)^2}{320} + \frac{3(1-y)}{128} + \frac{3(1-y)}{192} + \frac{(1-x)(1-y)^3}{32} - \frac{3(1-y)^5}{32} \\
+ & \frac{3(1-y)^4(x+1)}{320} + \frac{3(1-y)^4}{320} - \frac{(1-y)^3(x+1)}{32} + \frac{3(1-y)^4}{32} \\
- & \frac{16}{3(1-y)^3} - \frac{3(1-y)^2(x+1)}{8} - \frac{3(1-y)^2}{32} + \frac{3(1-y)^2(x+1)}{32} \\
+ & \frac{40}{3(1-y)} + \frac{32}{3(y+1)^3} - \frac{16}{(y+1)^2} + \frac{8}{(y+1)^2} \left( \zeta(2)H(0; y) \\ \\
- & \left( \frac{4}{15} + \frac{2}{15x^2} + \frac{8}{15x} + \frac{8x}{15} + \frac{2x^2}{15} \right) \right)H(1; y) + \left( \frac{20}{9} \right) \\
- & \frac{56}{3x^2(1-y)^5} + \frac{140}{3x^2(1-y)^4} - \frac{32}{x^2(1-y)^3} + \frac{32x^2(1-y)^2}{400} \\
+ & \frac{3x^2(1-y)}{8} - \frac{x(1-y)^5}{80} + \frac{x(1-y)^4}{200x} - \frac{3x(1-y)^3}{400x} \\
+ & \frac{12x}{x(1-y)} + \frac{2}{3x} + \frac{(1-y)^5}{3(1-y)^4} - \frac{(1-y)^4}{3(1-y)^4} + \frac{3(1-y)^3}{3(1-y)^4} \\
- & \frac{12x}{(1-y)} - \frac{2x}{3} + \frac{56x^2}{3(1-y)^5} - \frac{140x^2}{3(1-y)^4} + \frac{32x^2}{(1-y)^3} \\
\end{align*}
\]
\[-\frac{4x^2}{3(1-y)^2} - \frac{8x^2}{3(1-y)} + \frac{10}{9y(1-x)} - \frac{22}{9y(x+1)} + \frac{2}{3y}\]

\[-\frac{10y}{9(1-x)} + \frac{22y}{9(x+1)} - \frac{2y}{3} - \frac{3(1-x)(1-y)^5}{3568} - \frac{1760}{416} + \frac{3(1-x)(1-y)^4 - 9(1-x)(1-y)^3 + (1-x)(1-y)^2}{8} + \frac{3(1-y)^4}{44} + \frac{224}{256} + \frac{3(1-y)^3(1-x)}{24} + \frac{1072}{1424} + \frac{9(1-y)^3(1-x)}{8} + \frac{9(1-y)^3(1-x)}{224}\]

\[-\frac{32}{3(1-y)} + \frac{28}{3(x+1)} H(0; x)H(0; y) + \frac{16}{32x(1-y)^4}\]

\[-\frac{3x(1-y)^3}{76x} - \frac{x(1-y)^2}{68x} - \frac{3x(1-y)}{16x^2} - \frac{(1-y)^4}{32x^2} - \frac{9(1-y)^2}{28x^2}\]

\[-\frac{20x^2}{8} - \frac{9(1-y)}{64y} + \frac{9y(x+1)^4}{128y} - \frac{9y(x+1)^2}{56y} - \frac{9y(x+1)}{8y}\]

\[-\frac{9y(x+1)}{32} - \frac{(1-x)(1-y)^4}{64} - \frac{(1-x)(1-y)^3}{32} - \frac{3(1-x)(1-y)^2}{32}\]

\[-\frac{3(1-x)(1-y)}{32} - \frac{3(1-y)^4(x+1)}{128} + \frac{(1-y)^4(x+1)}{64}\]

\[-\frac{3(1-y)^4}{32} - \frac{9(1-y)^2(x+1)^4}{64} + \frac{9(1-y)^2(x+1)^3}{128}\]

\[-\frac{9(1-y)^2(x+1)^2}{208} + \frac{9(1-y)^2(x+1)}{200} + \frac{9(1-y)^2}{28}\]

\[-\frac{9(1-y)(x+1)^4}{128} - \frac{9(1-y)(x+1)^3}{224} + \frac{9(1-y)(x+1)^2}{224}\]

\[-\frac{9(1-y)(x+1)^2}{8} + \frac{224}{224} + \frac{9(1-y)(x+1)}{9(x+1)} H(0,0; x)\]

\[-\frac{68}{9} + \frac{16}{9x^2(1-y)^4} - \frac{32}{9x^2(1-y)^3} - \frac{7}{9x^2(1-y)^2}\]

\[-\frac{23}{64} + \frac{16}{9x^2(1-y)^4} - \frac{32}{3x(y+1)^2} - \frac{16x}{3x(y+1)^4} - \frac{32x}{3(1-y)^3}\]
\[
\frac{16x}{9(1 - y)^2} + \frac{64y}{3(1 + y)^2} + \frac{16x}{3(y + 1)} + \frac{16x^2}{9(1 - y)^4} - \frac{32x^2}{9(1 - y)^3} - \frac{7x^2}{9(1 - y)^2} + \frac{23x^2}{9(1 - y)} - \frac{32}{9(1 - y)^4} + \frac{64}{9(1 - y)^3} + H(0, 0; y)
\]
\[
+ \left( \frac{22}{3(1 - y)^2} - \frac{98}{9(1 - y)^3} + \frac{32}{3(y + 1)^2} - \frac{32}{3(y + 1)} \right) H(0, 0; y)
\]
\[
+ \left( \frac{16}{3x^2(1 - y)^4} - \frac{3x^2(1 - y)^3}{3x(1 - y)^3} + \frac{9x^2(1 - y)^2}{9x(1 - y)^2} + \frac{20}{9x(1 - y)^3} \right) H(0, 0; y)
\]
\[
+ \frac{64x}{3(1 - y)^2} + \frac{128x}{3(1 - y)^3} - \frac{112x}{9(1 - y)^2} - \frac{80x}{9(1 - y)}
\]
\[
- \frac{16x^2}{3(1 - y)^4} + \frac{32x^2}{3(1 - y)^3} - \frac{28x^2}{9(1 - y)^2} - \frac{20x^2}{9(1 - y)}
\]
\[
+ \left( \frac{128}{1 - x}(1 - y)^4 - \frac{64}{1 - x}(1 - y)^3 + \frac{32}{3(1 - x)(1 - y)^2} \right) H(-1, 0; x)
\]
\[
- \frac{112}{3(1 - y)^4} + \frac{3(1 - y)^4(x + 1)}{80} - \frac{80}{3(1 - y)^4}
\]
\[
+ \frac{(4 + \frac{2}{3y(1 - x)} - \frac{2}{3y(x + 1)} + \frac{2y}{3(1 - x)} - \frac{2y}{3(x + 1)}}{3(1 - x)} H(0, 0, 0; x) + \left( \frac{-8}{3} + \frac{8}{2} \right) \frac{3x^2(1 - y)^5}{4}
\]
\[
- \frac{3x^2(1 - y)^4 + \frac{3x^2(1 - y)^3}{3x(1 - y)^2} + \frac{3x^2(1 - y)^2}{x^2(1 - y)}}{80} - \frac{16}{16}
\]
\[
+ \frac{3x^2(1 - y)^4 + \frac{3x(1 - y)^5}{32} - \frac{3x(1 - y)^4}{80 - \frac{16}{8}} + \frac{x(x + 1)^2}{16}}{8}
\]
\[
+ \frac{3x(1 - y)^2 - \frac{3x(1 - y)}{32}}{8x - \frac{32}{16x}} + \frac{3x(1 - y)^3}{8x} - \frac{3x(y + 1)}{16x}
\]
\[
+ \frac{3x(y + 1)}{8x} - \frac{3x(y + 1)^2}{8x} - \frac{3x(y + 1)^3}{3(y + 1)} - \frac{3x(y + 1)^4}{3(y + 1)}
\]
\[
+ \frac{3x(y + 1)}{3(y + 1)} + \frac{3x(1 - y)^5}{4x} - \frac{(1 - y)^3}{2x} - \frac{3x(1 - y)^2}{3(1 - y)} - \frac{3x(1 - y)^4}{3(1 - y)}
\]
\[
+ \frac{4x^2}{3(y + 1)} + \frac{(1 - x)(1 - y)^5}{(1 - x)(1 - y)^4} - \frac{1}{80}
\]

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$$\frac{48}{(1 - x)(1 - y)^3} + \frac{8}{(1 - x)(1 - y)^2} - \frac{8}{(1 - x)(1 - y)}$$

$$- \frac{3(1 - y)^5(x + 1)}{80} - \frac{3(1 - y)^5}{16} + \frac{3(1 - y)^4(x + 1)}{16}$$

$$+ \frac{3(1 - y)^4}{8} - \frac{(1 - y)^3(x + 1)}{16} - \frac{(1 - y)^3}{16} + \frac{3(1 - y)^2(x + 1)}{32}$$

$$- \frac{3(1 - y)^2}{8} + \frac{3(1 - y)(x + 1)}{16} + \frac{3(1 - y)}{8} + \frac{3(y + 1)^3}{32}$$

$$- \frac{16}{(y + 1)^2} + \frac{8}{(y + 1)} H(0, 0; 0) + \left(\frac{4}{3} - \frac{32}{3x^2(1 - y)^5}\right)$$

$$+ \frac{3x^2(1 - y)^4}{128} - \frac{x^2(1 - y)^3}{320} - \frac{3x^2(1 - y)^2}{64} + \frac{3x^2(1 - y)}{32}$$

$$- \frac{3x(1 - y)^5}{32} + \frac{3x(1 - y)^4}{128x} - \frac{x(1 - y)^3}{320x} + \frac{64x}{16x^2} + \frac{3x(1 - y)^2}{32x}$$

$$- \frac{3(1 - y)}{32x} + \frac{3(1 - y)^5}{32x} - \frac{3(1 - y)^4}{80x^2} + \frac{(1 - y)^3}{16x^2} + \frac{3(1 - y)^2}{8x^2}$$

$$- \frac{8x^2}{128} + \frac{2}{3y(x - 1)} - \frac{2}{3y(x + 1)} + \frac{2y}{3(x - 1)} - \frac{2y}{3(x + 1)}$$

$$- \frac{(1 - x)(1 - y)^5}{32} + \frac{(1 - x)(1 - y)^4}{32} - \frac{(1 - x)(1 - y)^3}{320}$$

$$+ \frac{3(1 - y)^2}{128} + \frac{(1 - x)(1 - y)}{128} + \frac{3(1 - x)}{320}$$

$$+ \frac{3(1 - y)^6(x + 1)}{320} + \frac{3(1 - y)^6}{64} - \frac{3(1 - y)^4(x + 1)}{32}$$

$$- \frac{3(1 - y)^4}{32} + \frac{(1 - y)^3(x + 1)}{32} - \frac{(1 - y)^3}{32} + \frac{3(1 - y)^2(x + 1)}{32}$$

$$+ \frac{3(1 - y)^2}{32} - \frac{3(1 - y)(x + 1)}{32} - \frac{3(1 - y)}{32} - \frac{4}{(x + 1)} \times$$

$$\times H(0, 0; y) H(0; x) + \left(\frac{8}{3x^2(1 - y)^5} + \frac{20}{3x^2(1 - y)^4}\right)$$

$$- \frac{x^2(1 - y)^3}{80} - \frac{3x^2(1 - y)^2}{16} + \frac{3x^2(1 - y)}{8} - \frac{3x(1 - y)^5}{8}$$

$$+ \frac{3x(1 - y)^4}{16} - \frac{x(1 - y)^3}{80x} + \frac{8x}{16x} + \frac{3x(1 - y)}{8x}$$

$$+ \frac{3(1 - y)^5}{32x} - \frac{3(1 - y)^4}{80x} + \frac{(1 - y)^3}{16x} + \frac{3(1 - y)^2}{8x}$$

$$- \frac{8x}{3(1 - y)} + \frac{8x^2}{3(1 - y)^5} - \frac{20x^2}{3(1 - y)^4} + \frac{4x^2}{3(1 - y)^3} + \frac{2x^2}{3(1 - y)^2}$$

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B Asymptotic Expansions

In this Appendix, we provide the asymptotic expansions of the auxiliary functions introduced in the paper. For the functions related to the self-energy and vertex correction, that depend on a single kinematic invariant, we give the expansion in the case in which the kinematic invariant is significantly larger than the electron mass $m$. For the functions $B^{(1i)}_i$ and $B^{(2i)}_i$ ($i = 1, 2, 3$), related to the box graphs, that depend on two kinematic invariants, we discuss the expansion in two different cases:

i) both arguments of the functions $B^{(1i)}_i$ and $B^{(2i)}_i$ are larger than $m^2$; no specific hierarchy between the two arguments is assumed;

ii) both arguments are much larger than the electron mass, but the first (second) argument is also much larger than the second (first).

Employing these results, in combination with the expression for the cross section in terms of the auxiliary functions, it is possible to obtain the Bhabha scattering cross section in the kinematic regimes in which $s \sim t \sim u \gg m^2$, $s \sim t \gg u \gg m^2$, and $s \sim u \gg t \gg m^2$.

The expansions of the auxiliary functions are truncated in such a way that the full cross section can be recovered up to terms of order $m^2/R^2$ (where $R^2 = -s, -t, -u$) with $R^2 \gg m^2$. In particular, for the functions $\Pi^{(1i)}_0, \Pi^{(2i)}_0, F^{(1i)}_1, F^{(2i)}_1, F^{(1i)}_2$ and $F^{(2i)}_2$ one needs only the term of order $(m^2/R^2)^0$. On the contrary, for the functions $B^{(1i)}_1$ and $B^{(2i)}_1$ ($i = 1, 2, 3$) one needs only the terms proportional to $(m^2/R^2)^{−1}$, since these functions appear in the cross section always multiplied by $m^2/s$ or $m^2/t$.

\[
\frac{2x^2}{3(1-y)} - \frac{32}{8} \frac{1}{(1-x)(1-y)^3} - \frac{80}{80} \frac{1}{(1-x)(1-y)^2} + \frac{1}{(1-x)(1-y)} + \frac{3(1-y)^5(x+1)}{80} + \frac{16}{16} \frac{3(1-y)^5}{(1-y)^3} + \frac{8}{8} \frac{3(1-y)^4}{(1-y)^3} + \frac{8}{8} \frac{3(1-y)^4(x+1)}{3(1-y)} \left[ (G(-y, 0, 0; x) - G(-1/y, 0, 0; x) + (G(-y, 0; x) + G(-1/y, 0, x))H(0; y) + (G(-1/y; x) - G(-y; x)) \left( H(0, 0; y) + 3\zeta(2) \right) + 2H(0; x)H(1, 0; y) - 2H(-1, 0; x)H(0; y) - 6H(-1, 0; y)H(0; x) \right].
\]
In the following equations we use the definitions

\[ L_{P^2} = \ln \left( \frac{m^2}{P^2} \right), \quad L_{Q^2} = \ln \left( \frac{m^2}{Q^2} \right), \quad (102) \]

In the expansions we will keep \( P^2 \) and \( Q^2 \) both in the Euclidean region; the continuation to timelike values is trivial; if, for instance, \( P^2 \to -(s + i\epsilon) \), then

\[ L_{P^2} \to \ln \left( \frac{m^2}{s} \right) + i\pi. \quad (103) \]

**B.1 One- and Two-Loop Auxiliary Functions in the Small Mass Limit \( P^2, Q^2 \gg m^2 \)**

The first region of interest is the one in which \( P^2, Q^2 \gg m^2 \), \( x, y \ll 1 \), relevant for the large angle Bhabha scattering. We find:

\[ \Pi_0^{(1,0)}(-P^2) = -\left( \frac{5}{9} + \frac{1}{3}L_{P^2} \right), \quad (104) \]

\[ \Pi_0^{(1,1)}(-P^2) = -\left( \frac{14}{27} - \frac{1}{6}\zeta(2) + \frac{5}{18}L_{P^2} + \frac{1}{12}L_{P^2} \right), \quad (105) \]

\[ F_1^{(1,1)}(-P^2) = 1 + L_{P^2}, \quad (106) \]

\[ F_1^{(1,0)}(-P^2) = -1 + \frac{1}{2}\zeta(2) - \frac{3}{4}L_{P^2} - \frac{1}{4}L_{P^2}, \quad (107) \]

\[ F_2^{(1,0)}(-P^2) = 0, \quad (108) \]

\[ B_1^{(1,0)}(-P^2, -Q^2) = -16 \frac{P^2}{m^2}L_{P^2} - 8 \frac{Q^2}{m^2}L_{P^2} - 8 \frac{P^4}{m^2Q^2}L_{P^2}, \quad (109) \]

\[ B_1^{(1,0)}(-P^2, -Q^2) = -\frac{P^2}{m^2} \left[ 20\zeta(2) + 2L_{P^2} - 2L_{P^2}^2 - 4L_{P^2}L_{Q^2} + 2L_{Q^2} + 2L_{Q^2} \right] - \frac{Q^2}{m^2} \left[ 10\zeta(2) + 2L_{P^2} - L_{P^2}^2 - 2L_{P^2}L_{Q^2} + 2L_{Q^2} + L_{Q^2} \right] - \frac{P^4}{m^2Q^2} \left[ 16\zeta(2) + 4L_{P^2} - 4L_{P^2}L_{Q^2} + 2L_{Q^2} \right], \quad (110) \]

\[ B_2^{(1,0)}(-P^2, -Q^2) = -16 \frac{P^2}{m^2}L_{P^2} - 8 \frac{Q^2}{m^2}L_{P^2} - 16 \frac{P^4}{m^2Q^2}L_{P^2}, \quad (111) \]

\[ B_2^{(1,0)}(-P^2, -Q^2) = -\frac{P^2}{m^2} \left[ 20\zeta(2) - 2L_{P^2} - 2L_{P^2}^2 - 4L_{P^2}L_{Q^2} + 2L_{Q^2} + 2L_{Q^2} \right] - \frac{Q^2}{m^2} \left[ 10\zeta(2) + 2L_{P^2} - L_{P^2}^2 - 2L_{P^2}L_{Q^2} + 2L_{Q^2} + L_{Q^2} \right] - \frac{P^4}{m^2Q^2} \left[ 32\zeta(2) - 8L_{P^2}L_{Q^2} + 4L_{Q^2} \right], \quad (112) \]
\[ B_3^{(11,-1)}(-P^2, -Q^2) = 8 \frac{P^4}{m^2 Q^2} L_{p^2}, \]

\[ B_3^{(11,0)}(-P^2, -Q^2) = 4 \frac{P^2}{m^2} L_{p^2} + 4 \frac{Q^2}{m^2} L_{p^2} + \frac{P^4}{m^2 Q^2} \left[ 16 \zeta(2) + 4 L_{p^2} - 4 L_{p^2} L_{Q^2} + 2 L_{Q^2} \right], \]

and for the two-loop functions:

\[ \Pi_0^{(2l,0)}(-P^2) = -\frac{5}{24} + \zeta(3) - \frac{1}{4} L_{p^2}, \]

\[ F_1^{(2l,0)}(-P^2) = \frac{383}{108} - \frac{1}{4} \zeta(2) + 265 \frac{2}{216} L_{p^2} + \frac{1}{6} \zeta(2) L_{p^2} + \frac{19}{72} L_{p^2}^2 + \frac{1}{36} L_{p^2}^3, \]

\[ F_2^{(2l,0)}(-P^2) = 0, \]

\[ B_1^{(2l,-1)}(-P^2, -Q^2) = -\left\{ \frac{P^2}{m^2} \left[ \frac{40}{9} L_{p^2} + \frac{8}{3} L_{p^2} L_{Q^2} \right] + \frac{Q^2}{m^2} \left[ \frac{20}{9} L_{p^2} + \frac{4}{3} L_{p^2} L_{Q^2} \right] \right\} + \frac{P^4}{m^2 Q^2} \left[ \frac{20}{9} L_{p^2} + \frac{4}{3} L_{p^2} L_{Q^2} \right], \]

\[ B_1^{(2l,0)}(-P^2, -Q^2) = -\left\{ \frac{P^2}{m^2} \left[ \frac{68}{27} + \frac{14}{9} \zeta(2) + 4 \zeta(2) \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{242}{27} L_{p^2} - \frac{13}{9} L_{p^2}^2 \right. \right. \]

\[ + \frac{2}{3} L_{p^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{2}{9} L_{p^2}^2 - \frac{4}{3} L_{p^2} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) \]

\[ - \frac{34}{9} L_{p^2} L_{Q^2} - 2 L_{p^2} L_{Q^2}^2 + \frac{4}{3} L_{p^2} L_{Q^2} \left( -\frac{Q^2}{P^2} \right) + \frac{20}{9} L_{Q^2} \]

\[ + 8 \zeta(2) L_{Q^2} + \frac{16}{9} L_{Q^2}^2 + \frac{2}{3} L_{Q^2}^3 \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{2}{3} L_{Q^2}^3 \]

\[ \left. \left. + \frac{Q^2}{m^2} \left[ \frac{34}{27} + \frac{22}{9} \zeta(2) + 2 \zeta(2) \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{130}{27} L_{p^2} - \frac{5}{9} L_{p^2} \right. \right. \right. \]

\[ + \frac{1}{3} L_{p^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{1}{9} L_{p^2}^3 - \frac{2}{3} L_{p^2} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) \]

\[ - \frac{20}{9} L_{p^2} L_{Q^2} - L_{p^2} L_{Q^2}^2 + \frac{2}{3} L_{p^2} L_{Q^2} \left( -\frac{Q^2}{P^2} \right) + 4 \zeta(2) L_{Q^2} \]

\[ + 2 L_{Q^2} + \frac{25}{18} L_{Q^2}^2 + \frac{1}{3} L_{Q^2}^3 \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{1}{3} L_{Q^2}^3 \]

\[ \left. \left. \left. - \frac{2}{3} L_{Q^2} L_{Q^2} \left( -\frac{Q^2}{P^2} \right) + \frac{2}{3} L_{Q^2} \left( -\frac{Q^2}{P^2} \right) \right]\right\} + \frac{P^4}{m^2 Q^2} \left[ \frac{34}{27} + \frac{40}{9} \zeta(2) + 2 \zeta(2) L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{82}{27} L_{p^2} \right]. \]
\[ B^{(2l,-1)}_2(-P^2, -Q^2) = - \left\{ \frac{P^2}{m^2} \left[ \frac{40}{9} L_{P^2} + \frac{8}{3} L_{P^2} L_{Q^2} \right] + \frac{Q^2}{m^2} \left[ \frac{20}{9} L_{P^2} + \frac{4}{3} L_{P^2} L_{Q^2} \right] - \frac{4}{3} L_{P^2} L_{Q^2}^2 - 2 L_{P^2} L_{Q^2}^2 + \frac{4}{3} L_{P^2} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{4}{9} L_{P^2} - \frac{2}{3} L_{P^2} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{20}{9} L_{Q^2} \right\} + \frac{Q^2}{m^2} \left[ \frac{34}{27} + \frac{22}{9} \zeta(2) + 2 \zeta(2) \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{130}{27} L_{P^2} - \frac{5}{9} L_{P^2}^2 \right\] + \frac{Q^2}{m^2} \left[ \frac{80}{9} \zeta(2) + 4 \zeta(2) \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{4}{3} \zeta(2) L_{P^2} \right\] - \frac{20}{9} L_{P^2} L_{Q^2} - L_{P^2} L_{Q^2}^2 + \frac{2}{3} L_{P^2} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{2}{3} L_{Q^2} L_{P^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{2}{3} L_{Q^2} L_{P^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{25}{18} L_{Q^2}^2 + \frac{1}{3} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{1}{3} L_{Q^2}^3 \right\} + \frac{P^4}{m^2 Q^2} \left[ \frac{68}{27} + \frac{80}{9} \zeta(2) + 4 \zeta(2) \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{4}{3} \zeta(2) L_{P^2} \right\] - \frac{224}{27} L_{P^2} + \frac{2}{3} L_{P^2} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{20}{3} L_{P^2} L_{Q^2}^2 + \frac{8}{3} L_{P^2} L_{Q^2} - \frac{8}{3} L_{P^2} L_{Q^2}^2 + \frac{4}{9} L_{Q^2} \]
\[ B_{3}^{(2,-1)}(-P^2, -Q^2) = \frac{P^4}{m^2 Q^2} \left[ \frac{20}{3} L_{p2} + \frac{4}{3} L_{p2} L_{Q^2} \right], \tag{122} \]

\[ B_{3}^{(2,0)}(-P^2, -Q^2) = \frac{P^2}{m^2} \left[ \frac{10}{9} L_{p2} + \frac{2}{3} L_{p2} L_{Q^2} \right] + \frac{Q^2}{m^2} \left[ \frac{10}{9} L_{p2} + \frac{2}{3} L_{p2} L_{Q^2} \right] - \frac{P^4}{m^2 Q^2} \left[ -\frac{34}{27} - \frac{40}{9} \zeta(2) - 2\zeta(2) \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{82}{27} L_{p2} - \frac{2}{3} \zeta(2) L_{p2} - \frac{1}{3} L_{p2}^2 \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{8}{9} L_{p2} L_{Q^2} \right] \]

\[ -\frac{2}{3} L_{p2} L_{Q^2} \ln \left( 1 + \frac{Q^2}{P^2} \right) + \frac{4}{3} L_{p2} L_{Q^2}^2 - \frac{2}{9} L_{Q^2} \]

\[ -\frac{1}{3} L_{Q^2}^2 \ln \left( 1 + \frac{Q^2}{P^2} \right) - \frac{5}{9} L_{Q^2}^3 + \frac{2}{3} L_{Q^2} L_{p2} \left( -\frac{Q^2}{P^2} \right) \]

\[ -\frac{2}{3} L_{Q^2}^3 \left( -\frac{Q^2}{P^2} \right) \]  \tag{123}

B.2 One- and Two-Loop Auxiliary Functions in the Limit

\( P^2 \gg Q^2 \gg m^2 \) and \( Q^2 \gg P^2 \gg m^2 \)

In this limit, the Eqs. \[104\text{10, 120, 122}\] remain valid, while Eqs. \[104, 120, 122\] must be replaced, in the case in which \( P^2 \gg Q^2 \gg m^2 \), by:

\[ B_{1}^{(2,0)}(-P^2, -Q^2) = -\left\{ \frac{P^4}{m^2 Q^2} \left[ \frac{34}{27} + \frac{40}{9} \zeta(2) - \frac{82}{27} L_{p2} + \frac{2}{3} \zeta(2) L_{p2} - \frac{8}{3} L_{p2} L_{Q^2} \right] \right. \]

\[ -\frac{4}{3} L_{p2} L_{Q^2}^2 + \frac{2}{9} L_{Q^2}^3 - \frac{16}{3} L_{Q^2}^2 \zeta(2) - \frac{23}{18} L_{Q^2}^3 + \frac{5}{9} L_{Q^2}^3 \left. \right] \]

\[ + \frac{P^2}{m^2} \left[ \frac{50}{27} + \frac{32}{9} \zeta(2) - \frac{260}{27} L_{p2} - \frac{10}{9} L_{p2}^2 - \frac{2}{9} L_{p2}^3 - \frac{40}{9} L_{p2} L_{Q^2} \right] \]

\[ -2 L_{p2} L_{Q^2}^2 + \frac{26}{9} L_{Q^2}^3 + 8 \zeta(2) L_{Q^2} + \frac{19}{9} L_{Q^2}^3 + \frac{2}{3} L_{Q^2}^3 \]

\[ -\frac{Q^4}{m^2 P^2} \left[ \frac{85}{162} - \frac{2}{3} \zeta(2) + \frac{11}{27} L_{p2} - \frac{1}{9} L_{p2}^2 + \frac{2}{9} L_{p2} L_{Q^2} \right] \]

\[ -\frac{11}{27} L_{Q^2}^2 - \frac{1}{9} L_{Q^2}^3 \]

\[ \right\} \]
\[ + \frac{Q^6}{m^2 P^4} \left[ \frac{115}{2592} - \frac{1}{6} \zeta(2) + \frac{13}{216} L_{P^2} - \frac{1}{36} L_{P^2}^2 + \frac{1}{18} L_{P^2} L_{Q^2} \right. \\
\left. - \frac{13}{216} L_{Q^2} - \frac{1}{36} L_{Q^2}^2 \right] \\
+ \frac{Q^2}{m^2} \left[ \frac{1}{108} + \frac{49}{9} \zeta(2) - \frac{323}{54} L_{P^2} - \frac{1}{18} L_{P^2}^2 - \frac{1}{9} L_{P^2}^3 \\
- \frac{29}{9} L_{P^2} L_{Q^2} - L_{P^2} L_{Q^2}^2 + \frac{19}{6} L_{Q^2} + 4 \zeta(2) L_{Q^2} + \frac{17}{9} L_{Q^2}^2 \\
+ \frac{1}{3} L_{Q^2}^3 \right] \right), \quad (124) \]

\[ B_2^{(2l,0)}(-P^2, -Q^2) = \left\{ \frac{P^4}{m^2 Q^2} \left[ \frac{68}{27} + \frac{80}{9} \zeta(2) - \frac{224}{27} L_{P^2} + \frac{4}{3} \zeta(2) L_{P^2} - \frac{20}{3} L_{P^2} L_{Q^2} \right. \\
\left. - \frac{8}{3} L_{P^2} L_{Q^2}^2 + \frac{4}{9} L_{Q^2} + \frac{32}{3} \zeta(2) L_{Q^2} + \frac{23}{9} L_{Q^2}^2 + \frac{10}{9} L_{Q^2}^3 \right] \\
+ \frac{P^2}{m^2} \left[ \frac{32}{27} + \frac{50}{9} \zeta(2) - \frac{308}{27} L_{P^2} - \frac{7}{9} L_{P^2}^2 - \frac{2}{9} L_{P^2}^3 - \frac{52}{9} L_{P^2} L_{Q^2} \right. \\
- 2 L_{P^2} L_{Q^2}^2 + \frac{32}{9} L_{Q^2} + 8 \zeta(2) L_{Q^2} + \frac{22}{9} L_{Q^2}^2 + \frac{2}{3} L_{Q^2}^3 \right) \\
- \frac{Q^4}{m^2 P^2} \left[ \frac{89}{162} - \frac{4}{3} \zeta(2) + \frac{13}{27} L_{P^2} - \frac{2}{9} L_{P^2}^2 + \frac{4}{9} L_{P^2} L_{Q^2} \right. \\
\left. - \frac{13}{27} L_{Q^2} - \frac{2}{9} L_{Q^2}^2 \right] \\
+ \frac{Q^6}{m^2 P^4} \left[ \frac{71}{1296} - \frac{2}{3} \zeta(2) + \frac{11}{108} L_{P^2} - \frac{1}{9} L_{P^2}^2 + \frac{2}{9} L_{P^2} L_{Q^2} \right. \\
\left. - \frac{11}{108} L_{Q^2} - \frac{1}{9} L_{Q^2}^2 \right] \\
+ \frac{Q^2}{m^2} \left[ \frac{5}{54} + \frac{40}{9} \zeta(2) - \frac{157}{27} L_{P^2} - \frac{2}{9} L_{P^2}^2 - \frac{1}{9} L_{P^2}^3 - \frac{26}{9} L_{P^2} L_{Q^2} \right. \\
- L_{P^2} L_{Q^2}^2 + 3 L_{Q^2}^2 + 4 \zeta(2) L_{Q^2} + \frac{31}{18} L_{Q^2}^2 + \frac{1}{3} L_{Q^2}^3 \right] \right\}, \quad (125) \]

\[ B_3^{(2l,0)}(-P^2, -Q^2) = \left\{ \right. \\
- \frac{P^4}{m^2 Q^2} \left[ \frac{34}{27} + \frac{40}{9} \zeta(2) - \frac{82}{27} L_{P^2} + \frac{2}{3} \zeta(2) L_{P^2} - \frac{8}{3} L_{P^2} L_{Q^2} \right. \\
\left. - \frac{4}{3} L_{P^2} L_{Q^2}^2 + \frac{2}{9} L_{Q^2} + \frac{16}{3} \zeta(2) L_{Q^2} + \frac{23}{18} L_{Q^2}^2 + \frac{5}{9} L_{Q^2}^3 \right] \\
+ \frac{P^2}{m^2} \left[ \frac{2}{3} L_{P^2}^2 - \frac{4}{9} L_{P^2} - \frac{1}{3} L_{P^2}^2 - \frac{2}{3} L_{P^2}^2 - \frac{1}{3} L_{P^2}^2 \right. \\
\left. + \frac{Q^4}{m^2 P^2} \left[ \frac{2}{81} - \frac{2}{3} \zeta(2) + \frac{2}{27} L_{P^2} - \frac{1}{9} L_{P^2}^2 + \frac{2}{9} L_{P^2} L_{Q^2} \right] \right\}, \quad (125) \]

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In the case in which $Q^2 \gg P^2 \gg m^2$, Eqs. (119, 121, 123) must be replaced by:

\begin{align*}
B_{1}^{(2l,0)}(-P^2, -Q^2) &= -\left\{ \frac{P^2}{m^2} \left[ \frac{50}{27} + \frac{32}{9} \zeta(2) - \frac{224}{27} L_{P^2} + \frac{4}{3} \zeta(2) L_{P^2} - \frac{10}{9} L_{P^2}^2 - \frac{4}{9} L_{P^2}^3 ight.ight. \\
&\quad + \frac{2}{3} L_{P^2}^2 L_{Q^2} - \frac{40}{9} L_{P^2} L_{Q^2} - \frac{8}{3} L_{P^2} L_{Q^2}^2 + \frac{8}{9} L_{Q^2}^2 \\
&\quad + \frac{20}{3} \zeta(2) L_{Q^2} + \frac{19}{9} L_{Q^2}^2 + \frac{8}{9} L_{Q^2}^3 \\
&\quad + \frac{Q^2}{m^2} \left[ \frac{34}{27} + \frac{22}{9} \zeta(2) - \frac{130}{27} L_{P^2}^2 + \frac{2}{3} \zeta(2) L_{P^2}^2 - \frac{5}{9} L_{P^2}^2 - \frac{2}{9} L_{P^2}^3 ight.ight. \\
&\quad + \frac{1}{3} L_{P^2}^2 L_{Q^2} - \frac{20}{9} L_{P^2} L_{Q^2} - \frac{4}{3} L_{P^2} L_{Q^2}^2 + 2 L_{Q^2} \\
&\quad + \frac{10}{3} \zeta(2) L_{Q^2} + \frac{25}{18} L_{Q^2}^2 + \frac{4}{9} L_{Q^2}^3 \\
&\quad - \frac{P^6}{m^2 Q^2} \left[ \frac{85}{162} - \frac{2}{3} \zeta(2) - \frac{11}{27} L_{P^2}^2 - \frac{1}{9} L_{P^2}^2 + \frac{2}{9} L_{P^2} L_{Q^2} ight. \\
&\quad + \frac{11}{27} L_{Q^2}^2 - \frac{1}{9} L_{Q^2}^2 \\
&\quad + \frac{P^4}{m^2 Q^2} \left[ \frac{1}{108} + \frac{67}{9} \zeta(2) - \frac{101}{54} L_{P^2}^2 + \frac{4}{3} \zeta(2) L_{P^2}^2 + \frac{1}{2} L_{P^2}^2 ight. \\
&\quad - \frac{1}{9} L_{P^2}^3 + \frac{1}{3} L_{P^2}^2 L_{Q^2} - \frac{11}{3} L_{P^2} L_{Q^2} - \frac{5}{3} L_{P^2} L_{Q^2}^2 \\
&\quad - \frac{17}{18} L_{Q^2}^2 + \frac{14}{3} \zeta(2) L_{Q^2}^2 + \frac{16}{9} L_{Q^2}^2 + \frac{2}{3} L_{Q^2}^3 \right] \right\} \quad (126)
\end{align*}

In the case in which $Q^2 \gg P^2 \gg m^2$, Eqs. (119, 121, 123) must be replaced by:

\begin{align*}
B_{2}^{(2l,0)}(-P^2, -Q^2) &= -\left\{ \frac{P^2}{m^2} \left[ \frac{50}{27} + \frac{32}{9} \zeta(2) - \frac{254}{27} L_{P^2}^2 + \frac{4}{3} \zeta(2) L_{P^2}^2 - \frac{10}{9} L_{P^2}^2 - \frac{4}{9} L_{P^2}^3 ight.ight. \\
&\quad + \frac{2}{3} L_{P^2}^2 L_{Q^2} - \frac{46}{9} L_{P^2} L_{Q^2} - \frac{8}{3} L_{P^2} L_{Q^2}^2 + \frac{14}{9} L_{Q^2}^2 \\
&\quad + \frac{20}{3} \zeta(2) L_{Q^2} + \frac{19}{9} L_{Q^2}^2 + \frac{8}{9} L_{Q^2}^3 \right] \right\} \quad (127)
\end{align*}
\[
+ \frac{Q^2}{m^2} \left[ \frac{34}{27} + \frac{22}{9} \zeta(2) - \frac{130}{27} L_{p2} + \frac{2}{3} \zeta(2)L_{p2} - \frac{5}{9} L_{p2}^2 \right. \\
\left. - \frac{2}{9} L_{p2}^3 + \frac{1}{3} L_{p2}^2 L_{Q2} - \frac{20}{9} L_{p2} L_{Q2}^2 - \frac{4}{3} L_{p2} L_{Q2}^2 + 2 L_{Q2} \right] \\
+ \frac{10}{3} \zeta(2) L_{Q2}^2 + \frac{25}{18} L_{Q2}^2 + \frac{4}{9} L_{Q2}^3 \right] \\
- \frac{P^6}{m^2 Q^2} \left[ \frac{193}{162} - \frac{8}{3} \zeta(2) - \frac{29}{27} L_{p2} + \frac{4}{9} L_{p2}^2 + \frac{8}{9} L_{p2} L_{Q2} \\
+ \frac{29}{27} L_{Q2}^2 - \frac{4}{9} L_{Q2}^3 \right] \\
+ \frac{P^4}{m^2 Q^2} \left[ \frac{137}{108} + \frac{107}{9} \zeta(2) - \frac{385}{54} L_{p2} + \frac{8}{3} \zeta(2) L_{p2} + \frac{1}{2} L_{p2}^2 \\
- \frac{2}{9} L_{p2}^3 + \frac{2}{3} L_{p2}^2 L_{Q2} - \frac{23}{3} L_{p2} L_{Q2}^2 - \frac{10}{3} L_{p2} L_{Q2}^2 \\
- \frac{13}{18} L_{Q2}^2 + \frac{28}{3} \zeta(2) L_{Q2}^2 + \frac{55}{18} L_{Q2}^2 + \frac{4}{3} L_{Q2}^3 \right] \right], \quad (128)
\]

\[
B_3^{(2l,0)}(-P^2, -Q^2) = - \left\{ -\frac{P^2}{m^2} \left[ \frac{10}{9} L_{p2} + \frac{2}{3} L_{p2} L_{Q2} \right] - \frac{Q^2}{m^2} \left[ \frac{10}{9} L_{p2} + \frac{2}{3} L_{p2} L_{Q2} \right] \\
+ \frac{P^6}{m^2 Q^2} \left[ \frac{2}{3} - 2 \zeta(2) - \frac{2}{3} L_{p2} - \frac{1}{3} L_{p2}^2 + \frac{2}{3} L_{p2} L_{Q2}^2 + \frac{2}{3} L_{Q2}^2 \\
- \frac{1}{3} L_{Q2}^2 \right] \\
- \frac{P^4}{m^2 Q^2} \left[ \frac{34}{27} + \frac{40}{9} \zeta(2) - \frac{82}{27} L_{p2} + \frac{4}{3} \zeta(2) L_{p2} - \frac{1}{9} L_{p2}^3 \\
+ \frac{1}{3} L_{p2} L_{Q2}^2 - \frac{8}{3} L_{p2} L_{Q2}^2 - \frac{5}{3} L_{p2} L_{Q2}^2 + \frac{2}{9} L_{Q2}^2 \\
+ \frac{14}{3} \zeta(2) L_{Q2}^2 + \frac{23}{18} L_{Q2}^2 + \frac{2}{3} L_{Q2}^3 \right] \right\}, \quad (129)
\]

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