BLOW UP ANALYSIS FOR KELLER-SEGEL SYSTEM

HUA CHEN, JIAN-MENG LI AND KELEI WANG

Abstract. In this paper we develop a blow up theory for the parabolic-elliptic Keller-Segel system, which can be viewed as a parabolic counterpart to the Liouville equation. This theory is applied to the study of first time singularities, ancient solutions and entire solutions, leading to a description of the blow up limit in the first problem, and the large scale structure in the other two problems.

1. Introduction

1.1. Blow up analysis. In this paper we generalize the blow up analysis of Brezis-Merle [1] and Li-Shafrir [8] for the Liouville equation

\[ -\Delta u = e^u \]  

to a parabolic setting, that is, for the Keller-Segel system (see Keller-Segel [7])

\[
\begin{aligned}
  u_t &= \Delta u - \text{div}(u\nabla v), \\
  -\Delta v &= u.
\end{aligned}
\]

We will also apply this theory to the study of first time singularities in (1.2) and the large scale structure of ancient and entire solutions of (1.2).

Throughout this paper the spatial dimension is 2. Our first main result is about the convergence and blow up behavior for sequences of solutions to (1.2).

Theorem 1.1. Suppose \( u_i \) is a sequence of smooth, positive solutions of the Keller-Segel system (1.2) in the unit parabolic cylinder \( Q_1 := B_1 \times (-1, 1) \subset \mathbb{R}^2 \times \mathbb{R} \), satisfying

\[
\sup_{-1 < t < 1} \int_{B_1} u_i(x, t)\,dx \leq M \quad \text{for some constant } M.
\]

Then the followings hold.

(1) There exists a family of Radon measures \( \mu_t \) on \( B_1 \), \( t \in (-1, 1) \), such that after passing to a subsequence of \( i \), for any \( t \in (-1, 1) \),

\[ u_i(x, t)dx \rightharpoonup \mu_t \quad \text{weakly as Radon measures}; \]

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School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
Email: chenhua@whu.edu.cn (H. Chen), lijianmeng@whu.edu.cn (J. Li), wangkelei@whu.edu.cn (K. Wang).
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\( \mu_t \) is continuous in \( t \) with respect to the weak topology;
(3) for any \( t \in (-1, 1) \), there exist \( N(t) \) points \( q_j(t) \), where \( 0 \leq N(t) \leq M/(8\pi) \), and \( 0 \leq \rho(t) \in L^1(B_1) \) such that
\[
(1.3) \quad \mu_t = \sum_{j=1}^{N(t)} 8\pi \delta_{q_j(t)} + \rho(x, t)dx.
\]
(4) the blow up locus \( \Sigma := \cup_{t \in (-1, 1)} \cup_{j=1}^{N(t)} \{ (q_j(t), t) \} \) is relatively closed in \( Q_1 \);
(5) \( \rho \in C^\infty(Q_1 \setminus \Sigma) \) and satisfies (1.2) in this open set, where
\[
\nabla v(x, t) = -4 \sum_{j=1}^{N(t)} \frac{x - q_j(t)}{|x - q_j(t)|^2} - \frac{1}{2\pi} \int_{B_1} \frac{x - y}{|x - y|^2} \rho(y, t)dy;
\]
(6) if \( \rho \) is smooth in \( Q_1 \), then \( N(t) \equiv N \) for some \( N \in \mathbb{N} \), \( q_j(t) \in C^1(-1, 1) \) for each \( j = 1, \ldots, N \) and
\[
(1.4) \quad q_j(t) = 4 \sum_{k \neq j} \frac{q_k(t) - q_j(t)}{|q_k(t) - q_j(t)|^2} + \frac{1}{2\pi} \int_{B_1} \frac{y - q_j(t)}{|y - q_j(t)|^2} \rho(y, t)dy.
\]

Remark 1.2. (1) Eqn. (1.3) says that each Dirac measure in the singular limit has mass \( 8\pi \). Near this blow up point, \( u_i \) should look like a scaled bubble, that is,
\[
u_i(x, t) \sim \frac{8\lambda_{j,i}(t)^2}{(\lambda_{j,i}(t)^2 + |x - q_{j,i}(t)|^2)^2}, \quad \text{where} \quad \lambda_{j,i}(t) \to 0, \quad q_{j,i}(t) \to q_j(t),
\]
see Theorem 9.1 and Conjecture 1.11 below. This essentially means that there is only one bubble at each blow up point, or using the terminology of bubbling analysis, the blow up is isolated and simple.

It is also worth noticing that although these Dirac measures are separated for each \( t < 1 \), it is not claimed that they cannot converge to the same one as \( t \to 1 \). In fact, this is the picture for the first time singularity in Keller-Segel system as constructed in Seki-Sugiyama-Velázquez [13]. In this sense, this result corresponds to the quantization phenomena for blow ups of Liouville equations, where the mass of Dirac measures in the singular limit is \( 8\pi N \) for some \( N \in \mathbb{N} \) (Li-Shafrir [8]) and \( N \) could be larger than 1 (Chen [3]).

(2) The first term in the right hand side of (1.4) describes the interaction between different Dirac measures. If \( N = 1 \), (1.4) should be understood as without this term.

(3) If the diffusion part \( \rho(x, t)dx \) does not appear, then (1.4) is the gradient flow of the renormalized energy
\[
W(q_1, \cdots, q_N) := 4 \sum_{j \neq k} \log |q_j - q_k|.
\]
It arises as a renormalization of the free energy
\[
\mathcal{F}(u) := \int u(x) \log u(x)dx + \frac{1}{4\pi} \int \int \log |x - y|u(x)u(y)dxdy.
\]
Roughly speaking, if \( u \) is close to \( 8\pi \sum_{j=1}^{N} \delta_{q_j} \), then after subtracting the self-interaction terms from \( F \) (which could be very large), only those terms describing interaction between different Dirac measures are left, which is \( W \), see related discussions in [14, Chapter 2].

For the singular limit of a modified Keller-Segel system, a similar point dynamics was also established in Velázquez [17, 18].

The proof of Theorem 1.1 uses several tools such as symmetrization of test functions, \( \varepsilon \)-regularity theorems. The use of \( \varepsilon \)-regularity theorems is standard, just as in the study of blow up phenomena for many other PDE problems. The symmetrization of test function technique is mainly used to calculate the equation for the first and second momentum, which have also been used by many people in the study of Keller-Segel system. However, because our setting is local in nature, which is different from most literature on Keller-Segel system, a suitable localization of these calculations is necessary, see Remark 2.2 below for the definition of this localization procedure. In particular, the proof of (1.3) strongly relies on the localized calculation on the second momentum, which is also performed in a limiting form, see Section 5 for details. The derivation of (1.4) also uses a localized calculation on the first momentum, see Section 7.

In the literature, there are various notions of weak solutions about (1.2), see e.g. Suzuki [14, Chapter 13], which is related to the dynamical law (1.4), but we will not use them in this paper. Instead, to derive (1.4), we rely solely on explicit formulas derived from the convergence of sequences of smooth solutions.

1.2. First time singularity. Theorem 1.1 provides a convenient setting for the analysis of blow up behavior in Keller-Segel system. Our first application of Theorem 1.1 is on the analysis of first time singularities in the Keller-Segel system (1.2).

By standard parabolic theory, under suitable assumptions, there exists a local solution to the Cauchy problem of (1.2) on \( \mathbb{R}^2 \). The solution may not exist globally in time. For example, when the total mass is larger than the critical one, \( 8\pi \), the solution must blow up in finite time, see Dolbeault-Perthame [4]. Then it is natural to analyse the blow up behavior when the solution blows up at the first time. For the Keller-Segel system (1.2), blow up of solutions is caused by aggregation, that is, concentration of mass. The following is [14, Theorem 1.1] or [15, Theorem 1.1]. (Similar results also hold for initial-boundary value problems.)

**Theorem 1.3.** Suppose \( u \) blows up at finite time \( T \). Then as \( t \to T \),

\[
\lim_{t \to T} u(x,t)dx \to u_T(x)dx + \sum_{a_i} m_i \delta_{a_i}
\]

weakly as Radon measures, where \( \{a_i\} \), the set of blow up points, is a finite set of \( \mathbb{R}^2 \) and \( m_i \geq 8\pi \), \( u_T \in L^1(\mathbb{R}^2) \cap C(\mathbb{R}^2 \setminus \{a_i\}) \) is a nonnegative function.

For the construction of finite time blow up solutions, see Herrero-Velázquez [6], Velázquez [16] and Raphaël-Schwery [12].

By Theorem 1.3, to study first time singularites, we can work in the following local setting:
(H1): $u \in C^\infty(\overline{Q_1} \setminus \{(0,0)\})$ (here $Q_1^- := B_1 \times (-1,0)$ is the unit backward parabolic cylinder), $u > 0$ and

\begin{equation}
(1.5) \quad \sup_{t \in (-1,0)} \int_{B_1} u(x,t) dx \leq M;
\end{equation}

(H2): $u$ satisfies (1.2) in $Q_1^-$, where $\nabla v$ is given by

\begin{equation}
(1.6) \quad \nabla v(x,t) = -\frac{1}{2\pi} \int_{B_1} \frac{x-y}{|x-y|^2} u(y,t) dt, \quad \forall (x,t) \in Q_1^-;
\end{equation}

(H3): there exists a nonnegative function $u_0 \in L^1(B_1)$ and a positive constant $m$ such that as $t \to 0^-$,

$$u(x,t) dx \rightharpoonup u_0(x) dx + m \delta_0$$

weakly as Radon measures.

Under these hypothesis, we will examine the behavior of $u$ near the blow up point $(0,0)$. For this purpose, observe that the Keller-Segel system (1.2) is invariant under the scaling

\begin{equation}
(1.7) \quad u^\lambda(x,t) := \lambda^2 u(\lambda x, \lambda^2 t), \quad \nabla u^\lambda(x,t) = \lambda \nabla u(\lambda x, \lambda^2 t), \quad \forall \lambda > 0.
\end{equation}

Moreover, because the spatial dimension is 2, the $L^1$ norm of $u$ is invariant under this scaling. The small scale structure of $u$ near $(0,0)$ can be revealed by examining the convergence of $u^\lambda$ as $\lambda \to 0$. This is the blow up procedure.

By applying Theorem 1.1, we get the following result about the blowing up sequences $u^\lambda$.

**Theorem 1.4.** Under hypothesis (H1)-(H3), the followings hold.

(i) **Quantization:** There exists an $N \in \mathbb{N}$ such that $m = 8\pi N$.

(ii) **Blow-up limit and Self-similarity:** For any sequence $\lambda_i \to 0$, there exists a subsequence (not relabelling) and $N$ distinct points $p_j \in \mathbb{R}^2$ such that, for any $t < 0$,

$$u^{\lambda_i}(x,t) dx \rightharpoonup 8\pi \sum_{j=1}^N \delta_{\sqrt{-t}p_j} \quad \text{weakly as Radon measures.}$$

(iii) **Renormalized energy:** If $N = 1$, then $p_1 = 0$. If $N \geq 2$, then the $N$-tuple $(p_1, \cdots, p_N)$ is a critical point of the renormalized energy

\begin{equation}
(1.8) \quad \mathcal{W}(p_1, \cdots, p_N) := -\frac{1}{4} \sum_{j=1}^N |p_j|^2 + 4 \sum_{1 \leq j \neq k \leq N} \log |p_j - p_k|.
\end{equation}

**Remark 1.5.**

(1) In the converse direction, finite time blow up solutions with asymptotic behavior described as in this theorem have been constructed in Seki-Sugiyama-Velázquez [13] by using the matched asymptotics method.

(2) If $N = 2$, critical points of $\mathcal{W}$ must be a symmetric pair, that is, $(-p,p)$ for some $p \in \mathbb{R}^2$. 

In many parabolic equations, self-similarity of blow up limits is established with the help of a monotonicity formula. However, in the above theorem, this is proved by using the facts that, the ODE system (1.4) (if there is no diffusion part) is the gradient flow of $W$, and the critical energy levels of $W$ are discrete.

In the renormalized energy $W$, the first term comes from a self-similar transformation. As explained in Remark 1.2, the second term comes from a renormalization of the free energy $F$.

Because blow up limits are obtained by a compactness argument, we do not know if the $N$-tuple $(p_1, \cdots, p_N)$ is unique.

Most of these results are not new, cf. [14, Theorem 14.2] or [15, Chapter 1]. One main difference is the method of the proof. In [14] and [15], Suzuki uses the self-similar transformation, i.e. by considering similarity variables

$$y = \frac{x}{\sqrt{-t}}, \quad s = -\log(-t)$$

and then taking the transformation

$$z(y, s) := |t|u(x, t), \quad w(y, s) := v(x, t),$$

one gets the system

\begin{align*}
\begin{cases}
  z_s - \Delta z = -\text{div} \left[ z \left( \nabla w + \frac{y}{2} \right) \right], \\
  -\Delta w = z.
\end{cases}
\end{align*}

Then the proof is reduced to analyse the large time behavior of $(z, w)$. Our proof uses instead the blow up method, by considering the rescalings in (1.7) and then analyse its convergence as $\lambda \to 0$. These two convergence analysis are essentially equivalent, but the later one has an advantage: by examining the scalings $(u^\lambda, v^\lambda)$ at $t = 0$, we see that the trace of $u$ at $t = 0$, $u_0$, does not enter the singularity formation mechanism on the mass level. More precisely, Theorem 1.4 implies that

\begin{corollary}
In $Q_{1/2}^-$,

$$u(x, t) = o \left( \frac{1}{(|x| + \sqrt{-t})^2} \right).$$

In particular,

$$u_0(x) = o \left( \frac{1}{|x|^2} \right).$$

When the solution is radially symmetric, a rather precise asymptotic expansion of $u_0$ near the origin has been given in Herrero-Velázquez [5]. But at present it is still not known what further regularity on $u_0$ can be obtained in the above general setting. Such a knowledge will be helpful in defining the continuation after the blow up time.
1.3. Ancient solutions.

**Definition 1.7.** If \( u, \nabla v \) are smooth in \( \mathbb{R}^2 \times (-\infty, 0] \) and satisfy (1.2), where the second equation is understood as

\[
\nabla v(x, t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} u(y, t) dt, \quad \forall (x, t) \in \mathbb{R}^2 \times (-\infty, 0],
\]

then it is called an ancient solution of (1.2).

Ancient solutions play an important role in the analysis of blow up phenomena for many nonlinear parabolic equations.

Similar to the blow up analysis used in the proof of Theorem 1.4, we can analyse the large scale structure of ancient solutions by the blowing down analysis, that is, to consider the convergence of the sequence defined in (1.7), but now with \( \lambda \to +\infty \).

**Theorem 1.8.** Assume that \((u, v)\) is an ancient solution, satisfying

\[
\sup_{t \leq 0} \int_{\mathbb{R}^2} u(x, t) dx < +\infty.
\]

Then the followings hold.

(i) **Quantization:** There exists an \( N \in \mathbb{N} \) such that

\[
\int_{\mathbb{R}^2} u(x, t) dx \equiv 8\pi N.
\]

(ii) **Blow-down limit:** For any sequence \( \lambda_i \to +\infty \), there exists a subsequence (not relabelling) and \( N \) distinct points \( p_j \in \mathbb{R}^2 \) such that, for any \( t < 0 \),

\[
u^{\lambda_i}(x, t) dx \rightharpoonup 8\pi \sum_{j=1}^{N} \delta_{\sqrt{-t}p_j} \text{ weakly as Radon measures.}
\]

(iii) **Renormalized energy:** If \( N = 1 \), then \( p_1 = 0 \). If \( N \geq 2 \), then the \( N \)-tuple \((p_1, \cdots, p_N)\) is a critical point of the renormalized energy \( W \).

As can be seen, the statement is almost the same with Theorem 1.4. In fact, the proof is also the same, which is still mainly an application of Theorem 1.1.

1.4. Entire solutions. Finally, we study the large scale structure of entire solutions.

**Definition 1.9.** If \( u, \nabla v \) are smooth in \( \mathbb{R}^2 \times \mathbb{R} \) and satisfy (1.2), where the second equation is understood as (1.10), then it is called an entire solution of (1.2).

Entire solutions appear as a suitable rescalings around blow up points, see Theorem 9.1. Therefore it is the micro-model of singularity formations at blow up points.

The large scale structure of entire solutions is described in the following theorem.
Theorem 1.10. Assume that \((u, v)\) is an entire solution, satisfying the finite mass condition
\[(1.11) \quad \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^2} u(x, t) dx < +\infty.\]

Then as \(\lambda \to +\infty\), for any \(t\),
\[u^\lambda(x, t) dx \to 8\pi \delta_0 \quad \text{weakly as Radon measures.}\]

For a related property on weak solutions, called weak Liouville property, see \([15, \text{Lemma 1.4}]\).

Because the blow-down limit for entire solutions is time-independent, it is natural to conjecture that the original entire solution is also time-independent. Then the parabolic equation \((1.2)\) is reduced to
\[
\begin{cases}
\Delta u - \text{div} (u \nabla v) = 0, \\
- \Delta v = u.
\end{cases}
\]

This is essentially the Liouville equation \((1.1)\). By the Liouville theorem of Chen-Li \([2]\), we thus have

Conjecture 1.11. Any nontrivial entire solution of \((1.2)\) satisfying the finite mass condition \((1.11)\) has the form
\[
\begin{cases}
u(x, t) \equiv -2 \log (\lambda^2 + |x - \xi|^2) + C,
\end{cases}
\]

for some \(\lambda > 0\), \(\xi \in \mathbb{R}^2\) and \(C \in \mathbb{R}\).

Notations: Throughout this paper, we use the following notations.

- A standard cut-off function \(\psi\) subject to two bounded smooth domains \(\Omega_1 \subset \Omega_2\) is a function satisfying \(\psi \in C_0^\infty(\Omega_2), 0 \leq \psi \leq 1, \psi \equiv 1\) in \(\Omega_1\) and for any \(k \geq 1\),
\[\nabla^k \psi \leq C_k \text{dist}(\Omega_1, \Omega_2)^{-k}.\]

- A parabolic cylinder is \(Q_r(x, t) := B_r(x) \times (t - r^2, t + r^2)\).

- We use standard notation of parabolic Hölder spaces, e.g. \(C^{\alpha, \alpha/2}\) denotes a function which is \(\alpha\)-Hölder in \(x\) and \(\alpha/2\)-Hölder in \(t\).

The organization of this paper will be as follows. Section 2-Section 7 is devoted to the blow up analysis, in which the proof of Theorem 1.1 will be given. We study first time singularities and prove Theorem 1.4 in Section 8. In Section 9 we prove Theorem 1.10 on entire solutions and explain how to construct entire solutions by rescalings. Finally, we give a brief remark on boundary blow up points in Section 10.
2. Setting

We will work in the following local setting, which is a little more general than (1.2): \( u, v \in C^\infty(Q_1), u > 0 \), and they satisfy

\[
\begin{aligned}
  \begin{cases}
    u_t = \Delta u - \text{div} [u \nabla (v + f)] + g, \\
    -\Delta v = u
  \end{cases}
\end{aligned}
\]

(2.1)

Here we assume

- there exists a constant \( M > 0 \) such that

\[
\sup_{t \in (-1, 1)} \int_{B_1} u(x, t) dx \leq M;
\]

- \( \nabla f \) and \( g \) are two given functions, where for some \( \alpha \in (0, 1) \), \( \nabla f \in C^{\alpha, \alpha/2}(Q_1) \), while \( g \in L^\infty(Q_1) \);

- the second equation is understood as

\[
\nabla v(x, t) = -\frac{1}{2\pi} \int_{B_1} \frac{x - y}{|x - y|^2} u(y, t) dy, \quad \forall t \in (-1, 1).
\]

The assumption on \( g \) can be weakened, for example, by denoting \( p := 2/(1 - \alpha) \), \( g \in L^p(Q_1) \) is sufficient for most arguments in this paper. But that will lead to some technical issues, and we will not pursue it here.

**Remark 2.1.** The equation (1.9) in self-similar coordinates is a special case of (2.1).

**Remark 2.2 (Localization).** The form of Eq. (2.1) is invariant under localization in the following sense. Take an \( \eta \in C^\infty_0(B_1), \eta \geq 0 \) and define \( \tilde{u} := u\eta \). Then \( \tilde{u} \) still satisfies (2.1), with \( \nabla v, f \) and \( g \) replaced respectively by

\[
\begin{aligned}
  \nabla \tilde{v}(x, t) &:= -\frac{1}{2\pi} \int_{B_1} \frac{x - y}{|x - y|^2} \tilde{u}(y, t) dy, \\
  \nabla \tilde{f}(x, t) &:= \nabla f(x, t) + \nabla v - \nabla \tilde{v}, \\
  \tilde{g}(x, t) &:= g(x, t)\eta(x) + u(x, t) [\nabla v(x, t) + \nabla f(x, t)] \cdot \nabla \eta(x) \\
  &\quad - 2\nabla u(x, t) \cdot \nabla \eta(x) - u(x, t) \Delta \eta(x).
\end{aligned}
\]

As will be seen below, it is very convenient that solutions can be localized, especially when studying concentration phenomena in (1.2).

3. Two basic tools

In this section we recall two tools, symmetrization of test functions and \( \varepsilon \)-regularity theorem. These two tools will be used a lot in this paper.

The first tool is symmetrization of test functions. For any \( \psi \in C^2(\mathbb{R}^2) \), define

\[
\Theta_\psi(x, y) := \frac{x - y}{|x - y|^2} \cdot [\nabla \psi(x) - \nabla \psi(y)].
\]

(3.1)

It belongs to \( L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \), with the estimate

\[
\|\Theta_\psi\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \leq C\|\psi\|_{C^2(\mathbb{R}^2)}.
\]

(3.2)
This symmetrized function $\Theta_\psi$ appears in the following estimate.

**Lemma 3.1.** For any $\psi \in C^2_0(B_1)$,

$$\frac{d}{dt} \int_{B_1} u \psi = \int_{B_1} u \Delta \psi + \int_{B_1} u \nabla f \cdot \nabla \psi + \int_{B_1} g \psi$$

$$- \frac{1}{4\pi} \int_{B_1} \int_{B_1} \Theta(x,y) u(x,t) u(y,t) dx dy.$$  

As a consequence, there exists a universal constant $C$ such that

$$\left| \frac{d}{dt} \int_{B_1} u(x,t) \psi(x) dx \right| \leq C \left( M^2 \| \nabla^2 \psi \|_{L^\infty(B_1)} + M \| \Delta \psi \|_{L^\infty(B_1)} \right)$$

$$+ C M \| \nabla f \|_{L^\infty(Q_1)} \| \nabla \psi \|_{L^\infty(B_1)}$$

$$+ C \| g \|_{L^\infty(Q_1)} \| \psi \|_{L^\infty(B_1)},$$

where $M$ is the mass bound in (2.2).

**Proof.** The proof is similar to the standard Keller-Segel system case, see e.g. [14, Lemma 5.1].

The second tool is $\varepsilon$-regularity theorem.

**Theorem 3.2.** There exist two small constants $0 < \varepsilon_* \ll \theta_* \ll 1$ (depending only on the constant $M$ in (2.2), $\| \nabla f \|_{L^\infty(Q_1)}$ and $\| g \|_{L^\infty(Q_1)}$) such that, if $(u, v)$ is a classical solution of (1.2) in $Q_1$, satisfying (2.2) and

$$\int_{B_1(0)} u(x,0) dx \leq \varepsilon_*,$$

then

$$\| u \|_{C^{1+\alpha,(1+\alpha)/2}(Q_{\theta_*})} \leq C.$$

The proof is still similar to the one for the standard Keller-Segel system. It will be given in Appendix A.

### 4. Convergence of solutions

In this section we study the convergence of sequences of smooth solutions to (2.1). We assume $u_i$ is a sequence of smooth solutions of (2.1) in $Q_1$, satisfying:

1. there exists a constant $M$ such that

$$\sup_{-1 < t < 1} \int_{B_1} u_i(x,t) dx \leq M;$$

2. $\nabla f_i$ are uniformly bounded in $C^{\alpha,\alpha/2}(Q_1)$, and $\nabla f_i \rightarrow \nabla f$ uniformly in $Q_1$;

3. $g_i$ are uniformly bounded in $L^\infty(Q_1)$, and $g_i$ converges to $g$ *-weakly in $L^\infty(Q_1)$.

Under these assumptions, we have
Theorem 4.1. (1) There exists a family of Radon measures \( \mu_t \) on \( B_1, t \in (-1, 1) \), such that after passing to a subsequence of \( i \),
\begin{equation}
(4.2) \quad u_i(x, t)dx \to \mu_t \quad \text{weakly as Radon measures, } \forall t \in (-1, 1);
\end{equation}
(2) \( \mu_t \) is continuous in \( t \) with respect to the weak topology;
(3) for any \( t \in (-1, 1) \), there exist \( N(t) \) points \( q_j(t) \), \( \varepsilon_\ast \leq m_i(t) \leq M \) and \( 0 \leq \rho(t) \in L^1(B_1) \) such that
\[
\mu_t = \sum_{j=1}^{N(t)} m_j(t)\delta_{q_j(t)} + \rho(x, t)dx.
\]
(4) the blow up locus \( \Sigma := \bigcup_{t \in (-1, 1)} \bigcup_{j=1}^{N(t)} \{ (q_j(t), t) \} \) is relatively closed in \( Q_1 \);
(5) \( \rho \in C^{1+\alpha, (1+\alpha)/2}(Q_1 \setminus \Sigma) \).

This covers most contents of Theorem 1.1, except that \( m_j(t) = 8\pi \) (the \( 8\pi \) phenomena) and the dynamical law (1.4). These two properties will be established below in Proposition 5.1 and Theorem 7.2 respectively.

Proof. (1) Choose a countable, dense subset \( \mathcal{D} \subset (-1, 1) \). By the mass bound (4.1) and a diagonal argument, we can choose a subsequence of \( i \) so that for each \( t \in \mathcal{D} \), \( u_i(t)dx \) converges weakly as Radon measures to a limit Radon measure \( \mu_t \). In the following, such a subsequence of \( i \) will be fixed, and by abusing notations, it will still be written as \( i \). We will prove that the convergence in (4.2) holds for this subsequence.

For any \( \psi \in C_0^2(B_1) \), by Lemma 3.1, the linear functionals
\begin{equation}
(4.3) \quad \mathcal{L}_{i,t}(\psi) := \int_{B_1} u_i(x, t)\psi(x)dx
\end{equation}
are uniformly Lipschitz continuous on \((-1, 1)\). After passing to a subsequence, they converge to a limit function \( \mathcal{L}_t(\psi) \) uniformly on \((-1, 1)\), which is still Lipschitz continuous on \((-1, 1)\). Of course, if \( t \in \mathcal{D} \), because \( u_i(x, t)dx \to \mu_t \), we must have
\[
\mathcal{L}_t(\psi) = \int_{B_1} \psi(x)d\mu_t.
\]
Therefore, because \( \mathcal{D} \) is dense in \((-1, 1)\), the above convergence of \( \mathcal{L}_{i,t} \) holds for the entire sequence \( i \) and we do not need to pass to a further subsequence of \( i \).

For any \( t \in (-1, 1) \) (not necessarily in \( \mathcal{D} \)), for any further subsequence \( i' \) with \( u_{i'}(x, t)dx \) converging weakly to a Radon measure \( \mu_t \), because \( C_0^2(B_1) \) is dense in \( C_0(B_1) \), we must have
\[
\mu_t = \mathcal{L}_t,
\]
where we have used Riesz representation theorem to identify Radon measures with positive linear functionals on \( C_0(B_1) \). As a consequence, \( \mu_t \) is independent of the choice of subsequences, that is, for any \( t \) and the entire sequence \( i \),
\[
(4.4) \quad u_i(x, t)dx \to \mu_t.
\]
(2) By the Lipschitz continuity of \( \mathcal{L}_t(\psi) \) (when \( \psi \in C_0^2(B_1) \)), \( \mu_t \) is continuous in \( t \) with respect to the weak topology.
(3) For any \( t \in (-1, 1) \), define
\[
\Sigma_t := \{ x : \mu_t(\{x\}) \geq \varepsilon_s/2 \}.
\]
Hence there is an atom of \( \mu_t \) at each point in \( \Sigma_t \), whose mass is at least \( \varepsilon_s/2 \). Because \( \mu_t(B_1) \leq M \), \( \Sigma_t \) is a finite set.

If \( x \notin \Sigma_t \), then there exists an \( r < 1 - |x| \) such that \( \mu_t(B_r(x)) < \varepsilon_s/2 \). By the convergence of \( u_i(x,t)dx \), for all \( i \) large,
\[
\int_{B_r(x)} u_i(y,t)dy < \varepsilon_s.
\]

By Theorem 3.2, \( u_i \) are uniformly bounded in \( C^{1+\alpha,(1+\alpha)/2}(Q_{a,r}(x,t)) \). Hence they converge in \( C(Q_{a,r}(x,t)) \). As a consequence, \( \mu_t = \rho(x,t)dx \) in \( B_{a,r}(x) \) for some function \( \rho \in C^{1+\alpha,(1+\alpha)/2}(Q_{a,r}(x,t)) \). In conclusion, \( \mu_t \) is absolutely continuous with respect to Lebesgue measure outside \( \Sigma_t \).

(4) Since each \( \Sigma_t \) is a finite set, we need only to show that

**Claim.** If \( t_i \to t \), \( x_i \in \Sigma_{t_i} \) and \( x_i \to x \), then \( x \in \Sigma_t \).

Indeed, assume by the contrary that \( x \notin \Sigma_t \), then there exists an \( r > 0 \) such that
\[
\mu_t(B_r(x)) < \varepsilon_s/2.
\]
Take a cut-off function \( \psi \) subject to \( B_{r/2}(x) \subset B_r(x) \). By the Lipschitz continuity of \( L_t(\psi) \), there exists a small \( \delta > 0 \) such that for any \( s \in (t - \delta, t + \delta) \),
\[
\mu_s(B_{r/2}(x)) < \varepsilon_s/2.
\]
This implies that for all of these \( s \), \( B_{r/2}(x) \) is disjoint from \( \Sigma_s \). This is a contradiction with the fact that \( x_i \in \Sigma_{t_i} \) and \( t_i \to t \), and the claim follows.

(5) The regularity of \( \rho \) in \( Q_1 \setminus \Sigma \) has already been established in the proof of (3).

Because
\[
\nabla v_i(x,t) = -\frac{1}{2\pi} \int_{B_1} \frac{x-y}{|x-y|^2} u_i(y,t)dy,
\]
by the convergence of \( u_i(x,t)dx \) established in this theorem, we see \( \nabla v_i \) converges to
\[
\nabla v_\infty(x,t) := -\sum_{j=1}^{N(t)} \frac{m_j(t)}{2\pi} \frac{x-q_j(t)}{|x-q_j(t)|^2} - \frac{1}{2\pi} \int_{B_1} \frac{x-y}{|x-y|^2} \rho(y,t)dy.
\]

By standard estimates on Newtonian potential, this convergence is strong in \( L^q(B_1) \) for any \( q < 2 \), and uniform in any compact set of \( B_1 \setminus \Sigma_t \).

5. The 8\pi phenomena

This section is devoted to the proof of the following result, which says that the mass of each atom in \( \mu_t \) is exactly 8\pi. We keep the notations used in Theorem 4.1.

**Proposition 5.1.** For any \( t \in (-1, 1) \) and each \( q_j(t) \in \Sigma_t \), \( m_j(t) = 8\pi \).
\textbf{Proof.} Take an arbitrary \( t \in (-1, 1) \) and a point \( q_j(t) \in \Sigma_t \). Denote \( m := m_j(t) \). The remaining proof is divided into three steps.

\textbf{Step 1. Reduction to a simple setting.} For any \( r > 0 \) and \( s \in ((t+1)/r^2, (1-t)/r^2) \), define the Radon measure
\[
\hat{\mu}_s^r(A) := \mu_{t+r^2s} (q_j(t) + rA), \quad \forall A \subset \mathbb{R}^2 \text{ Borel.}
\]
Correspondingly, define
\[
\left\{\begin{array}{ll}
u_i^r(y, s) := r^2 u_i^r (q_j(t) + ry, t + r^2s), \\
\nabla v_i^r(y, s) := r \nabla v_i^r (q_j(t) + ry, t + r^2s), \\
\nabla f_i^r(y, s) := r \nabla f_i^r (q_j(t) + ry, t + r^2s), \\
g_i^r(y, s) := r^2 g_i^r (q_j(t) + ry, t + r^2s).
\end{array}\right.
\]
Then \( u_i^r(y, s) dy \) converges to \( \hat{\mu}_s^r \) in the sense of Theorem 4.1.

As in the proof of Theorem 4.1, we can take a subsequence \( r_i \to 0 \) so that for any \( s \in \mathbb{R} \), \( \hat{\mu}_s^r \) converges weakly to \( \hat{\mu}_s \) as finite Radon measures on \( \mathbb{R}^2 \). Then by a diagonal argument, we can take a further subsequence of \( i \) (not relabelling) so that \( u_i^r(y, s) dy \) converges to \( \hat{\mu}_s \) as finite Radon measures on \( \mathbb{R}^2 \). In the following we denote \( \hat{u}_i := u_i^r \).

Since \( \Sigma_t \) is a finite set, there exists an \( r > 0 \) such that \( \Sigma_t \cap B_r(q_j(t)) = \{q_j(t)\} \).

Then the structure of \( \mu_t \) in Theorem 4.1 (3), we get
\[
\hat{\mu}_0^r = m \delta_0 + r^2 \rho(rx, 0) dx.
\]
Because \( \rho \in L^1(B_1) \), sending \( r \to 0 \) we obtain
\[
\hat{\mu}_0 = m \delta_0.
\]

\textbf{Step 2. Clearing out.} Take a cut-off function \( \psi_1 \) subject to \( (B_{7/8} \setminus B_{1/4}) \subset (B_1 \setminus B_{1/8}) \). By Lemma 3.1 and our assumption on the uniform bound on \( \nabla f_i \) and \( g_i \),
\[
|\frac{d}{dt} \int_{B_1} \hat{u}_i(x, t) \psi_1(x) dx| \leq C(M + M^2) + o_i(1).
\]
Combining this inequality with (5.1), we deduce that for any \( t \) satisfying
\[
|t| \leq t^* := \frac{\varepsilon_*}{4C(M + M^2)},
\]
it holds that
\[
\int_{B_{7/8} \setminus B_{1/4}} \hat{u}_i(x, t) dx \leq \frac{\varepsilon_*}{3}.
\]
By Theorem 3.2 and Arzela-Ascoli theorem, \( \hat{u}_i \) converges uniformly to a function \( \hat{\rho} \) in \( (B_{3/4} \setminus B_{3/8}) \times (-t^*, t^*) \), that is, \( \hat{u}_i \) does not blow up in this annular cylinder.

\textbf{Step 3. Second momentum calculation.} Take a cut-off function \( \psi_2 \) subject to \( B_{1/2} \subset B_{5/8} \). Consider the second momentum
\[
M(s) := \int_{B_1} |x|^2 \psi_2(x) d\hat{\mu}_s.
\]
We claim that

**Claim.** $M(s)$ is differentiable in $(-t_*, t_*)$, and

$$ M'(0) = 4m - \frac{1}{2\pi} m^2. \tag{5.2} $$

By definition $M(s) \geq 0$, while by (5.1), $M(0) = 0$. The differentiability then implies that

$$ M'(0) = 0. $$

This identity combined with (5.2) implies that $m = 8\pi$. Hence the proof of this proposition is complete, provided this claim is true.

**Proof of the claim.** A direct calculation shows that

$$ \Theta_{|x|^2 \psi_2(x)}(x, y) = \frac{1}{2\pi} \psi_2(x) \psi_2(y) + \Phi(x, y), $$

where

$$ \Phi(x, y) := \frac{1}{2\pi} \frac{x - y}{|x - y|^2} \cdot [x \psi_2(x) (1 - \psi_2(y)) - y \psi_2(y) (1 - \psi_2(x))] $$

$$ + \frac{1}{4\pi} \frac{x - y}{|x - y|^2} \cdot \left[ |x|^2 \nabla \psi_2(x) - |y|^2 \nabla \psi_2(y) \right]. \tag{5.3} $$

Denote

$$ M_i(s) := \int_{B_1} |x|^2 \psi_2(x) \hat{u}_i(x, s) dx. $$

By (3.3), we obtain

$$ M'_i(s) = 4 \int_{B_1} \psi_2(x) \hat{u}_i(x, s) dx - \frac{1}{2\pi} \left[ \int_{B_1} \psi_2(x) \hat{u}_i(x, s) dx \right]^2 $$

$$ + \int_{B_1} \left[ |x|^2 \Delta \psi_2(x) + 4x \cdot \nabla \psi_2(x) \right] \hat{u}_i(x, s) dx $$

$$ + \int_{B_1} \nabla \hat{f}_i(x, s) \cdot \nabla (|x|^2 \psi_2(x)) \hat{u}_i(x, s) dx + \int_{B_1} \hat{g}_i(x, t) |x|^2 \psi_2(x) dx $$

$$ + \int_{\mathbb{R}^2} \Phi(x, y) \hat{u}_i(x, s) \hat{u}_i(y, s) dxdy $$

$$ =: I_i + II_i + III_i + IV_i + V_i + VI_i. $$

Let us analyse the convergence of these six terms one by one.

(1) By the convergence of $\hat{u}_i(x, s) dx$, $I_i$ converges uniformly in $(-t_*, t_*)$ to

$$ 4 \int_{B_1} \psi_2(x) d\hat{\mu}_s(x). $$

(2) In the same way, $II_i$ converges uniformly in $(-t_*, t_*)$ to

$$ -\frac{1}{2\pi} \left[ \int_{B_1} \psi_2(x) d\hat{\mu}_s(x) \right]^2. $$
(3) Similarly, III\(_i\) converges uniformly in \((-t_*, t_*)\) to
\[
\int_{B_{1/3}} \left[ |x|^2 \Delta \psi_2(x) + 4x \cdot \nabla \psi_2(x) \right] \, d\hat{\mu}_s.
\]

(4) Because \(\|\nabla \tilde{f}_i\|_{L^\infty} \leq Cr_i\), IV\(_i\) converges uniformly to 0.

(5) Because \(\|g_i\|_{L^\infty(Q_1)} \leq Cr_i^2\), V\(_i\) converges uniformly to 0.

(6) Concerning VI\(_i\), first because we have established the uniform convergence of \(\hat{u}_i\) in \((B_{3/4} \setminus B_{3/8}) \times (-t_*, t_*)\) in Step 2, an application of the dominated convergence theorem gives us the uniform in \(s \in (-t_*, t_*)\) convergence
\[
\int_{B_{3/4} \setminus B_{3/8}} \int_{B_{3/4} \setminus B_{3/8}} \Phi(x, y) \hat{u}_i(x, s) \hat{u}_i(y, s) \, dx \, dy
\]
\[
\to \int_{B_{3/4} \setminus B_{3/8}} \int_{B_{3/4} \setminus B_{3/8}} \Phi(x, y) \hat{\mu}_s(x) \hat{\mu}_s(y).
\]

Next, by the form of \(\Phi\) in (5.3) and using the facts that \(\psi_2 \equiv 0\) in \(B_{1/2}\), \(\psi_2 \equiv 1\) outside \(B_{5/8}\), we see \(\Phi\) is continuous outside \((B_{3/4} \setminus B_{3/8}) \times (B_{3/4} \setminus B_{3/8})\).\(^1\)

Then by the weak convergence of \(\hat{u}_i\), the remaining part in VI\(_i\) converges uniformly in \((-t_*, t_*)\) to
\[
\int \int \left( \mathbb{R}^2 \times \mathbb{R}^2 \right) \left[ (B_{3/4} \setminus B_{3/8}) \times (B_{3/4} \setminus B_{3/8}) \right] \Phi(x, y) \, d\hat{\mu}_s(x) \, d\hat{\mu}_s(y).
\]

In conclusion, VI\(_i\) converges uniformly to
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Phi(x, y) \, d\hat{\mu}_s(x) \, d\hat{\mu}_s(y).
\]

Putting these six facts together, we deduce that \(M_i(s)\) converges to \(M(s)\) in \(C^1(-t_*, t_*)\). Moreover,
\[
M'(s) = 4 \int_{B_1} \psi_2(x) \, d\hat{\mu}_s(x) - \frac{1}{2\pi} \left[ \int_{B_1} \psi_2(x) \, d\hat{\mu}_s(x) \right]^2
\]
\[
+ \int_{B_1} \left[ |x|^2 \Delta \psi_2(x) + 4x \cdot \nabla \psi_2(x) \right] \, d\hat{\mu}_s
\]
\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Phi(x, y) \, d\hat{\mu}_s(x) \, d\hat{\mu}_s(y).
\] (5.4)

Evaluating this identity at \(s = 0\), and then using (5.1) and the definition of \(\psi_2\), we get (5.2). The proof of the claim is thus complete. \(\square\)

**Remark 5.2.** By the clearing out property established in the above proof, in the following we will assume without loss generality that \(\Sigma \subset B_{1/2} \times (-1, 1)\).

\(^1\)In fact, the only trouble appears on the diagonal \(\{x = y\}\), but when \((x, y)\) belongs to this domain, it is directly checked that \(\Phi(x, y) = 0\) if \(|x - y| \leq 1/16\).
6. Local structure of blow up locus

In this section we study the local structure of the blow up locus, $\Sigma$. Since we are only interested in interior case, by Remark 5.2, we will assume $\Sigma \subset B_{1/2} \times (-1, 1)$. The main result of this section is

**Proposition 6.1.** For any $r < 1$, there exists an $N \in \mathbb{N}$ and $N$ functions $\xi_i : [-r, r] \to B_{1/2}$ which are $1/2$-Hölder, such that $\Sigma \cap \{|t| \leq r\} \subset \bigcup_{i=1}^{N} \{(\xi_i(t), t)\}$.

First we need two technical lemmas, which are global version of Proposition 6.1. Below they will be used to study blow up limits of $\mu_t$. The first lemma is about a backward problem.

**Lemma 6.2.** Suppose a family of Radon measures on $\mathbb{R}^2$, $\mu_t$, $t \in (-\infty, 0]$ is a weak limit given in Theorem 4.1, from a sequence of solutions $u_i$, where in the equation of $u_i$, the two terms $\nabla f_i$ and $g_i$ converge to 0 uniformly. Assume there exists a constant $M$ such that

$$\mu_t(\mathbb{R}^2) \leq M, \quad \forall t \in (-\infty, 0].$$

If $\mu_0 = 8\pi \delta_0$, then for any $t \leq 0$, $\mu_t = 8\pi \delta_t$.

**Proof.**

**Step 1.** Let $\rho$ be the regular part in the Lebesgue decomposition of $\mu_t$ as in Theorem 4.1. We claim that $\rho \equiv 0$.

In fact, in the open set $\Omega := (\mathbb{R}^2 \times (-\infty, 0)) \setminus \Sigma$, $\rho$ is a classical solution of a linear parabolic equation (i.e. the first equation in (1.2)). By Theorem 4.1, $\Omega$ is connected. By our assumption, $\rho(x, 0) = 0$ in $\mathbb{R}^2 \setminus \{0\}$. Hence we can apply the strong maximum principle, which implies that $\rho \equiv 0$ in $\Omega$. Because for each $t$, $\rho(\cdot, t) \in L^1(\mathbb{R}^2)$ and $\Sigma_t$ is a finite set, we get the claim.

**Step 2.** For each $R > 0$, take a cut-off function $\psi_R$ subject to $B_R \subset B_{2R}$. By Lemma 3.1 and Theorem 4.1,

$$\left| \frac{d}{dt} \int_{\mathbb{R}^2} \psi_R(x) d\mu_t(x) \right| \leq \frac{C(M + M^2)}{R^2}.$$

Integrating this inequality in $t$, we obtain

$$\left| \int_{\mathbb{R}^2} \psi_R(x) d\mu_t(x) - 8\pi \right| \leq \frac{C(M + M^2)}{R^2} |t|.$$

Letting $R \to +\infty$, we deduce that

$$\mu_t(\mathbb{R}^2) = 8\pi, \quad \forall t \in (-\infty, 0].$$

Combining this fact with Proposition 5.1 and the result in Step 1, we deduce that for each $t$, there exists a $q(t) \in \mathbb{R}^2$ such that $\mu_t = 8\pi \delta_{q(t)}$.

**Step 3.** Take the cut-off function $\psi_R$ as before and define

$$M_R(t) := \int_{\mathbb{R}^2} |x|^2 \psi_R(x) d\mu_t(x) = 8\pi |q(t)|^2 \psi_R(q(t)).$$

By the same derivation of (5.4) as in the proof of Proposition 5.1, we get

$$M'_R(t) = 4 \int_{\mathbb{R}^2} \psi_R(x) d\mu_t(x) - \frac{1}{2\pi} \left[ \int_{\mathbb{R}^2} \psi_R(x) d\mu_t(x) \right]^2.$$
\[ + \int_{\mathbb{R}^2} \left[ |x|^2 \Delta \psi_R(x) + 4x \cdot \nabla \psi_R(x) \right] d\mu_t(x) \]

\[ + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Phi(x, y) d\mu_t(x) d\mu_t(y), \]

where \( \Phi(x, y) \) is defined in (5.3).

For any \( T > 0 \), if \( R \) has been chosen to be large enough so that for any \( t \in [-T, 0] \), \( q(t) \in B_R \), by using the fact that \( \mu_t = 8\pi \delta_{q(t)} \) and the form of \( \Phi \) etc., we deduce that

\[ M'_R(t) \equiv 0, \quad \forall t \in [-T, 0]. \]

By the assumption that \( q(0) = 0 \), we also have \( M_R(0) = 0 \), so \( M_R(t) \equiv 0 \). This then implies that \( q(t) \equiv 0 \). \( \square \)

The second lemma is about a forward problem.

**Lemma 6.3.** Suppose a family of Radon measures on \( \mathbb{R}^2 \), \( \mu_t, t \in [0, +\infty) \) is a weak limit given in Theorem 4.1, from a sequence of solutions \( u_i \), where in the equation of \( u_i \), the two terms \( \nabla f_i \) and \( g_i \) converge to 0 uniformly. Assume there exists a constant \( M \) such that \( \mu_t(\mathbb{R}^2) \leq M, \forall t \in [0, +\infty) \).

If \( \mu_0 = 8\pi \delta_0 \), then for any \( t \geq 0 \), \( \mu_t = 8\pi \delta_0 \).

**Proof.** First as in Step 2 in the proof of the previous lemma, we still have (6.1) (for any \( t \in [0, +\infty) \)). Combining this fact with Theorem 4.1, we see for any \( t > 0 \),

- **Alternative I:** either there exists a \( q(t) \in \mathbb{R}^2 \) such that \( \mu_t = 8\pi \delta_{q(t)} \);
- **Alternative II:** or there exists a positive smooth function \( \rho(t) \) on \( \mathbb{R}^2 \) such that \( \mu_t = \rho(x, t) dx \).

Define

\[ T_* := \sup \{ t : \text{Alternative I holds at } t \}. \]

Similar to Step 1 in the proof of the previous lemma, we deduce that if \( T_* > 0 \), then for any \( t \in [0, T_*] \), Alternative I holds.

To finish the proof, we need to show that

\[ (6.2) \quad T_* = +\infty. \]

**Proof of (6.2)** Assume by the contrary that \( T_* < +\infty \). Then in \( \mathbb{R}^2 \times (T_*, +\infty) \), \( \mu_t = \rho(x, t) dx \), where \( \rho \) is a smooth solution of (1.2). For any \( t > T_* \) and \( R > 1 \), take a cut-off function \( \psi_R \) subject to \( B_R \subset B_{2R} \) and define

\[ M_R(t) := \int_{\mathbb{R}^2} |x|^2 \psi_R(x) \rho(x, t) dx, \quad E_R(t) := \int_{B_R} \rho(x, t) dx. \]

By (3.1), we have

\[ M'_R(t) = 4 \int_{\mathbb{R}^2} \psi_R(x) \rho(x, t) dx - \frac{1}{2\pi} \left[ \int_{\mathbb{R}^2} \psi_R(x) \rho(x, t) dx \right]^2 \]

\[ + \int_{\mathbb{R}^2} \left[ 4x \cdot \nabla \psi_R(x) + |x|^2 \Delta \psi_R(x) \right] \rho(x, t) dx \]
By (6.1),
\[ \int_{\mathbb{R}^2} \rho(x,t) \, dx = 8\pi. \]
Plugging this identity into (6.3), we obtain
\[ |M'_R(t)| \leq C \mathcal{E}_R(t). \]
Because \( M_R(T_*) = 0 \), this inequality implies that for any fixed \( t > T_* \),
\[ M_R(t) \leq C(t - T_*) \max_{T_* \leq s \leq t} \mathcal{E}_R(s). \]
By the definition of \( \psi_R \), \( M_R(t) \) is non-decreasing in \( R \). On the other hand, we have
\[ \lim_{R \to +\infty} \max_{T_* \leq s \leq t} \mathcal{E}_R(s) = 0. \]
Thus
\[ M_R(t) \leq \lim_{R \to +\infty} M_R(t) \leq C(t - T_*) \lim_{R \to +\infty} \max_{T_* \leq s \leq t} \mathcal{E}_R(s) = 0. \]
This is possible only if \( \mu_t \) is a Dirac measure at 0, which is a contradiction with our assumption. In other words, (6.2) must hold.

Finally, once we have shown that \( \mu_t = 8\pi \delta_0(t) \) for any \( t > 0 \), following the argument in Step 3 in the proof of the previous lemma, we deduce that for any \( t > 0 \), \( M_R(t) = 0 \) for all \( R \) large enough. This implies that \( q(t) \equiv 0 \).

\[ \square \]

**Remark 6.4.** The above proof can be used to show that, there does not exist forward self-similar solutions of the Keller-Segel system (1.2) with mass not smaller than \( 8\pi \). More precisely, there does not exist solutions satisfying the following three conditions:

1. \( u \in C^{2,1}(\mathbb{R}^n \times (0, +\infty)) \) is a solution of (1.2);
2. it is forward self-similar in the sense that \( u(\lambda x, \lambda^2 t) = \lambda^2 u(x, t), \ \forall \lambda > 0; \)
3. the total mass satisfies \( \int_{\mathbb{R}^2} u(x,t) \, dx \equiv m \geq 8\pi. \)

Indeed, under these assumptions, the initial condition is \( u(0) = m \delta_0 \). The above calculation in the proof of (6.2) then leads to a contradiction. On the other hand, if \( m < 8\pi \), such forward self-similar solutions do exist, see Naito-Suzuki [11].
For any \((q_j(t), t) \in \Sigma\), define \(\hat{\mu}_s^r\) as in the first step in the proof of Proposition 5.1, that is,
\[
\hat{\mu}_s^r(A) := \mu_{t+r^2 s}(q_j(t) + r A), \quad \forall A \subset \mathbb{R}^2 \text{ Borel.}
\]
We have the following characterization of its limit as \(r \to 0\).

**Lemma 6.5.** For any \(s \in \mathbb{R}\), \(\hat{\mu}_s^r \to 8\pi \delta_0\) as \(r \to 0\).

**Proof.** First, as in the proof of Theorem 4.1, we can assume (after passing to a subsequence of \(r \to 0\)) that, for any \(s \in \mathbb{R}\), \(\hat{\mu}_s^r\) converges weakly to \(\mu_s^0\) as Radon measures. Moreover, \(\mu_s^0\), as a family of Radon measures, is a weak limit of a sequence of solutions \(\hat{u}_i\), where in the equation of \(\hat{u}_i\), the two terms \(\nabla \hat{f}_i\) and \(\hat{g}_i\) converge to 0 uniformly.

By Theorem 4.1 and Proposition 5.1, for any \(R > 0\),
\[
\hat{\mu}_0^r(B_R) = \mu_0(B_{Rr}(q_j(t))) = 8\pi + \int_{B_{Rr}(q_j(t))} \rho(x, t) dx \to 8\pi \quad \text{as } r \to 0.
\]
Thus \(\mu_0^0 = 8\pi \delta_0\).

Then by Lemma 6.2, for any \(t \leq 0\), \(\mu_t^0 = 8\pi \delta_0\), and by Lemma 6.3, for any \(t \geq 0\), \(\mu_t^0 = 8\pi \delta_0\). \(\square\)

**Corollary 6.6.** For any \((q_j(t), t) \in \Sigma\), as \(r \to 0\), \(\Sigma \cap Q_r(q_j(t), t)\) belongs to an \(o(r)\) neighborhood of \(\{q_j(t)\} \times (t - r^2, t + r^2)\).

Finally, let us prove Proposition 6.1. Recall that we always assume that \(\Sigma \subset B_{1/2} \times (-1, 1)\).

**Proof of Proposition 6.1.** We need a fact about the extension of Hölder functions: If \(f\) is an \(\alpha\)-Hölder continuous function (for any \(\alpha \in (0, 1)\)) on a closed subset \(D \subset \mathbb{R}^n\), then it can be extended to the whole \(\mathbb{R}^n\). Indeed, if \(f\) is scalar, the extension can be defined as
\[
\tilde{f}(x) := \inf_{y \in D} [f(y) + L|x - y|^\alpha],
\]
where \(L\) is the Hölder constant of \(f\), while if \(f\) is vector valued, we can take this extension for each component.

Take an arbitrary \(t \in [-r, r]\) and \(q_j(t) \in \Sigma_t\). By Theorem 4.1 and Proposition 5.1, there exists an \(r > 0\) such that
\[
\begin{align*}
\{\Sigma_t \cap B_r(q_j(t)) &= \{q_j(t)\}, \\
8\pi \leq \mu_t(B_r(q_j(t))) < 8\pi + \varepsilon_s/4.
\end{align*}
\]
By the weak continuity of \(\mu_t\), there exists a \(\delta > 0\) such that for any \(s \in (t - \delta, t + \delta)\),
\[
8\pi - \varepsilon_s/2 < \mu_s(B_{r/2}(q_j(t))) < 8\pi + \varepsilon_s/2.
\]
By Proposition 5.1, for these \(s\), there exists at most one atom of \(\mu_s\) in \(B_{r/2}(q_j(t))\).

Let \(I \subset (t - \delta, t + \delta)\) be the set containing those \(s\) such that \(\mu_s\) consists of an atom inside \(B_{r/2}(q_j(t))\). Because \(\Sigma\) is a closed set, \(I\) is relatively closed in \((t - \delta, t + \delta)\). Then
\[
\Sigma \cap (B_r(q_j(t)) \times (t - \delta, t + \delta)) = \{(q_j(s), s) : s \in I\}.
\]
By Corollary 6.6, $q_j$ is 1/2-Hölder on $I$. We extend it to $(t - \delta, t + \delta)$ by the above remark about extension of Hölder continuous functions. Because $\Sigma_t$ consists of $N(t)$ points, the number of these curves is $N(t)$. Moreover, these curves do not intersect (perhaps after shrinking $\delta$).

Because $\Sigma$ is a closed set, the above discussion implies that for any $t \in [-r, r]$, there exists an interval $(t - \delta(t), t + \delta(t))$ such that $\Sigma \cap (t - \delta(t), t + \delta(t))$ is contained in these 1/2-Hölder curves emanating from $\Sigma_t$. These open intervals form a covering of $[-r, r]$. Take a finite sub-covering from it, denoted by $\{(t_j^-, t_j^+)\}$. Let $N_j := N((t_{j-1} + t_j)/2)$, so $\Sigma \cap (t_{j-1}, t_j)$ is contained in $N_j$ curves.

In the intersection of two neighboring intervals, that is, $(t_j^+, t_{j+1}^-)$, define a continuous, nonnegative function $\eta_j$ such that $\eta_j(t_j^+) = 1$, $\eta_j(t_{j+1}^-) = 0$. In $(t_j^-, t_j^+)$, $\Sigma$ is contained in $N_j$ curves $\gamma_k^1 := \{(q_k^1(t), t)\}$, $k = 1, \ldots, N_j$, while in $(t_{j+1}^-, t_{j+1}^+)$, $\Sigma$ is contained in $N_{j+1}$ curves $\gamma_k^2 := \{(q_k^2(t), t)\}$, $\ell = 1, \ldots, N_{j+1}$. In each group, the distance between any curve and other curves has a positive lower bound. Because these curves are graphs over the $t$-axis, each $\Gamma_k^1$ has nonempty intersection with at most one $\Gamma_k^2$, and vice versa. If two such curves, say $\gamma_k^1$ and $\gamma_{\ell}^2$, do intersect, define their joint as

$$q_{kk}^2(t) = \eta_j(t)q_k^1(t) + [1 - \eta_j(t)]q_{k}^2(t).$$

If there is no curves with such nonempty intersection, we stop here (either in the left or the right end). In this way, we glue all of those local 1/2-Hölder curves to be defined on a maximal interval. Some of these maximal curves are not defined on the entire interval $[-r, r]$, but we can extend it suitably to $[-r, r]$ so that after extension, they still do not intersect with each other. These curves are the desired one. Moreover, from the construction, we see that the number of these curves is at most $\sum_j N_j$. \hfill \Box

**Remark 6.7.** It is possible that $N(t)$ (the number of points in $\Sigma_t$) is not constant in time. If finite time blow up solutions can be continued past the singular time, then it implies that $N(t)$ will jump up at the singular time. On the other hand, it seems highly impossible that a singularity can disappear once it has been created.\(^2\) Hence it is natural to conjecture that $N(t)$ is non-decreasing and right continuous in $t$, but we do not know how to prove this.

7. Limit Equations

By the smooth convergence of $u_i$ in $Q_1 \setminus \Sigma$, we see $\rho$ satisfies

$$\rho_t - \Delta \rho = -\text{div} [\rho (\nabla v_\infty + \nabla f)] + g \quad \text{in } Q_1 \setminus \Sigma.$$  \hfill (7.1)

Here by Proposition 5.1, the definition of $\nabla v_\infty$ (cf. (4.4)) is

$$\nabla v_\infty(x, t) = -4 \sum_{j=1}^{N(t)} \frac{x - q_j(t)}{|x - q_j(t)|^2} - \frac{1}{2\pi} \int_{B_1} \frac{x - y}{|x - y|^2} \rho(y, t) dy.$$  \hfill (7.2)

\(^2\)In a global setting, this is exactly what Lemma 6.3 says, which claims that if the initial value is $8\pi\delta_0$, then the solution will always be $8\pi\delta_0$, that is, it cannot be smoothed.
In this section, we will establish the dynamical law for $\mu_t$ only in the case that $\rho$ is smooth enough and it satisfies (7.1) in the entire $Q_1$. The reason to add this assumption is that, only under this assumption, we can show that the blow up locus $\Sigma$ is regular in the following sense.

**Lemma 7.1.** If $\rho \in C(Q_1)$, then there exists an $N \in \mathbb{N}$ such that $N(t) \equiv N$, that is, for any $t \in (-1, 1)$, $\Sigma_t$ consists of exactly $N$ distinct points, $q_1(t), \ldots, q_N(t)$. Moreover, each $q_j(t)$ is continuous.

**Proof.** Because $\rho$ is continuous, $\rho(x,t)dx$, viewed as Radon measures, is continuous in the weak topology. By Theorem 4.1 (2), $\mu_t$ is also continuous in the weak topology. Hence by Theorem 4.1 (3) and Proposition 5.1,

$$\mu_t - \rho(x,t)dx = \sum_{j=1}^{N(t)} 8\pi \delta_{q_j(t)}$$

is continuous in the weak topology. By testing this weak continuity with suitable compactly supported, continuous functions, we deduce that its total mass, which is $8\pi N(t)$, is constant in time. In other words, there exists an $N \in \mathbb{N}$ such that for any $t \in (-1, 1)$, $\Sigma_t$ consists of exactly $N$ distinct points, $q_1(t), \ldots, q_N(t)$. The continuity of $q_j(t)$ also follows from this weak continuity. \hfill \Box

Next we establish the dynamical law for these $q_j(t)$ under this assumption.

**Theorem 7.2.** For each $j = 1, \ldots, N$, $q_j$ is differentiable in $(-1, 1)$. Moreover, it satisfies

$$(7.3) \quad q_j'(t) = 4 \sum_{k \neq j} \frac{q_k(t) - q_j(t)}{|q_k(t) - q_j(t)|^2} + \frac{1}{2\pi} \int_{B_1} \frac{y - q_j(t)}{|y - q_j(t)|^2} \rho(y,t)dy + \nabla f(q_j(t),t).$$

If $f = 0$, this is (1.4) in Theorem 1.1. The proof of Theorem 1.1 is thus complete.

Before going into the details of the proof, we explain briefly the idea. The main task is to single out the interaction between different atoms and the effect of the diffusion part $\rho$ to these atoms. In the following proof we achieve this goal by taking a localization (as in Remark 2.2) around a selected atom. We can also use Lemma 3.1 directly, by calculating a local first momentum as in the proof of Proposition 5.1. However, this will need a detailed knowledge of $\Theta_{x_\eta}$ (with $\eta$ a suitable cut-off function). Nevertheless, by the localization argument, the self-interaction term, the interaction terms between different atoms as well as the interaction between the selected atom and the diffusion part can be revealed in a clear way, so we will use this approach in the following proof.

**Proof.** Take an arbitrary $t_0 \in (-1, 1)$ and a point $q_j(t_0) \in \Sigma_0$. By Lemma 7.1, there exists an $r > 0$ such that

$$(7.4) \quad \Sigma \cap Q_{2r}(q_j(t_0), t_0) = \{(q_j(t), t)\},$$

By Lemma 6.5, after shrinking $r$ further, we may assume for any $t \in (t_0 - r^2, t_0 + r^2)$, $q_j(t) \in B_{r/4}(q_j(t_0))$. 

Take a cut-off function \( \eta \) subject to \( B_{r/2}(q_j(t_0)) \subset B_r(q_j(t_0)) \). Following the localization procedure in Remark 2.2, set \( \tilde{u}_i := u_i \eta \), and then define \( \nabla \tilde{v}_i, \nabla \tilde{f}_i, \tilde{g}_i \) accordingly. By the convergence of \( u_i \) etc., we have the following convergence results.

1. For any \( t \in (t_0 - r^2, t_0 + r^2) \), \( \tilde{u}_i(x,t)dx \rightarrow 8\pi \delta_{q_j(t)} + \tilde{\rho}(x,t)dx \), where \( \rho(x,t) := \rho(x,t)\eta(x) \).

2. By the convergence of \( \tilde{v}_i \), \( \nabla \tilde{v}_i \) converges to

\[
\nabla \tilde{v}_\infty(x,t) = -\frac{1}{2\pi} \int_{B_1} \frac{x-y}{|x-y|^2} \tilde{\rho}(y,t)dy.
\]

in \( L^q(B_1) \) for any \( q < 2 \), and uniformly in any compact set of \( Q_r(q_j(t_0), t_0) \setminus \{(q_j(t_0), t_0)\} \). Here

\[
\nabla \tilde{v}_\infty(x,t) := \frac{\partial \tilde{v}}{\partial x} - 4 \frac{x - q_j(t)}{|x - q_j(t)|^2}
\]

Because

\[
\nabla \tilde{v}_i(x,t) - \nabla \tilde{v}_i(x,t) = -\frac{1}{2\pi} \int_{B_1} \frac{x-y}{|x-y|^2} u_i(y,t)[1 - \eta(y)]dy.
\]

and in particular, \( u_i \) converges uniformly to \( \rho \) in \( (B_{2r}(q_j(t_0)) \setminus B_{r/4}(q_j(t_0))) \times (t_0 - r^2, t_0 + r^2) \), by standard estimates on Newtonian potential, \( \nabla \tilde{v}_i(x,t) - \nabla \tilde{v}_i(x,t) \) converges uniformly in \( Q_r(q_j(t_0), t_0) \) to

\[
-4 \sum_{k \neq j} \frac{x - q_k(t)}{|x - q_k(t)|^2} - \frac{1}{2\pi} \int_{B_1} \frac{x-y}{|x-y|^2} \rho(y,t)[1 - \eta(y)]dy.
\]

Then by the convergence of \( \nabla f_i \), we deduce that \( \nabla \tilde{f}_i \) converges uniformly in \( Q_r(q_j(t_0), t_0) \) to

\[
\nabla \tilde{f}(x,t) := \nabla f(x,t) - 4 \sum_{k \neq j} \frac{x - q_k(t)}{|x - q_k(t)|^2} - \frac{1}{2\pi} \int_{B_1} \frac{x-y}{|x-y|^2} \rho(y,t)[1 - \eta(y)]dy.
\]

4. In the support of \( \nabla \eta, u_i, \nabla u_i \text{ and } \nabla v_i \) converge uniformly to \( \rho, \nabla \rho \text{ and } \nabla v_\infty \) respectively, so \( \tilde{g}_i(x,t) \) converges uniformly to

\[
\tilde{g}(x,t) := g(x,t)\eta(x) + \rho(x,t)[\nabla v_\infty(x,t) + \nabla f(x,t)] \cdot \nabla \eta(x)
\]

\[
-2\nabla \rho(x,t) \cdot \nabla \eta(x) - \rho(x,t) \Delta \eta(x).
\]

By definition, for any \( t \in (t_0 - r^2, t_0 + r^2) \), \( \tilde{u}_i(x,t) \) is compactly supported in \( B_r(q_j(t_0)) \). Plugging \( \psi(x) = x \) as test function into (3.3), by noting that \( \Delta \psi \equiv 0 \) and \( \Theta_\psi \equiv 0 \), we obtain

\[
\frac{d}{dt} \int_{B_1} \psi(x)\tilde{u}_i(x,t)dx = \int_{B_1} \left[ \nabla \tilde{f}_i(x,t) \cdot \nabla \psi(x)\tilde{u}_i(x,t) + \tilde{g}_i(x,t)\psi(x) \right] dx.
\]
Sending $i \to +\infty$, by the weak convergence of $\tilde{u}_i$ and the uniform convergence of $\nabla \tilde{f}_i$ and $\tilde{g}_i$, we get

$$
\frac{d}{dt} \left[ \int_{B_1} \psi(x)\tilde{\rho}(x,t)dx + 8\pi q_j(t) \right]
= \int_{B_1} \nabla \tilde{f}(x,t) \cdot \nabla \psi(x,t)\tilde{\rho}(x,t)dx + \int_{B_1} \tilde{g}(x,t)\psi(x)dx + 8\pi \nabla \tilde{f}(q_j(t),t).
$$

Note that this differential identity should be understood in the distributional sense.

Because $\rho$ is a solution of (7.1) in $Q_1$, $\tilde{\rho} = \rho \eta$ satisfies

$$
\tilde{\rho}_t - \Delta \tilde{\rho} = -\text{div} \left[ \tilde{\rho} \left( \nabla \tilde{v}_\infty - 4 \frac{x - q_j(t)}{|x - q_j(t)|^2} + \nabla \tilde{f} \right) \right] + \tilde{g}.
$$

Then by Lemma 3.1, we have

$$
\frac{d}{dt} \int_{B_1} \psi(x)\tilde{\rho}(x,t)dx
= \int_{B_1} \left[ -4 \frac{x - q_j(t)}{|x - q_j(t)|^2} + \nabla \tilde{f}(x,t) \right] \cdot \nabla \psi(x,t)\tilde{\rho}(x,t)dx + \int_{B_1} \tilde{g}(x,t)\psi(x)dx.
$$

In view of the form of $\nabla \tilde{f}$ in (7.5), subtracting this identity from (7.6) gives (7.3).

Finally, although by now (7.3) is only understood in the distributional sense, by Lemma 7.1 and our assumption on $\rho$, the right hand side of (7.3) is continuous in $t$. Then by standard regularity theory for ODEs, we deduce that $q_j$ is $C^1$ in $t$ and it satisfies (7.3) in the classical sense. □

8. Blow-up limits: Proof of Theorem 1.4

Now we turn to the study of first time singularities and the proof of Theorem 1.4.

Recall that $u$ satisfies (H1-H3) and the blowing up sequence is defined as

$$
u^\lambda(x,t) := \lambda^2 u(\lambda x, \lambda^2 t), \quad \nabla v^\lambda(x,t) = \lambda \nabla v(\lambda x, \lambda^2 t).
$$

These functions are defined on $Q_{\lambda^{-1}}$, which tends to $\mathbb{R}^2 \times (-\infty, 0]$ as $\lambda \to 0$. By Theorem 4.1, we may assume that for a subsequence (not relabelling) $\lambda \to 0$,

$$
u^\lambda(x,t)dx \to \mu_t, \quad \forall t \leq 0.
$$

First we notice the following two facts. Recall that Hypothesis (H3) says that $u(x,t)$ forms a Dirac measure of mass $m$ at the origin as $t \to 0^-$. 

**Lemma 8.1.** $\mu_0 = m\delta_0$.

**Proof.** For any $\lambda > 0$, by (H3), as $t \to 0$,

$$
u^\lambda(x,t)dx \to m\delta_0 + u_0^\lambda(x)dx, \quad \text{where} \quad u_0^\lambda(x) := \lambda^2 u_0(\lambda x).
$$

Because $u_0 \in L^1(B_1)$, as $\lambda \to 0$,

$$u_0^\lambda(x)dx \to 0.$$
Then by the uniform Lipschitz continuity of \(L_{\lambda}^t(\psi)\) (for \(\psi \in C^2_0(\mathbb{R}^2)\), see (4.3)), we deduce that

\[
\lim_{t \to 0} \lim_{\lambda \to 0} u^\lambda(x, t) dx = m \delta_0.
\]

\[\square\]

Lemma 8.2. \(\rho \equiv 0\) in \((\mathbb{R}^2 \times (-\infty, 0)) \setminus \Sigma\).

Proof. In the open set \(\Omega := (\mathbb{R}^2 \times (-\infty, 0)) \setminus \Sigma\), \(\rho\) is a classical solution of a linear parabolic equation (i.e. the first equation in (1.2)). By Theorem 4.1, \(\Omega\) is connected. By Lemma 8.1, \(\rho(x, 0) = 0\) in \(\mathbb{R}^2 \setminus \{0\}\). The desired claim follows by applying the strong maximum principle. \[\square\]

By this lemma and Lemma 7.1, there exists an \(N \in \mathbb{N}\) such that for each \(t < 0\), \(\Sigma_t\) consists of exactly \(N\) distinct points, \(q_1(t), \ldots, q_N(t)\), and

\[
\left\{\begin{array}{l}
\mu_t = 8\pi \sum_{j=1}^{N} \delta_{q_j(t)}, \\
\nabla u_{\infty}(x, t) = -4 \sum_{j=1}^{N} \frac{x - q_j(t)}{|x - q_j(t)|^2}.
\end{array}\right.
\]

Furthermore, each \(q_j(t)\) is 1/2-Hölder in \(t\) and \(q_j(0) = 0\).

As a consequence we obtain the quantization of mass, \(m = 8N\pi\). This also implies that, even though the blow-up limit may be not unique, the number of Dirac measures in every blow-up limit is \(N\).

Because \(\rho \equiv 0\), an application of Theorem 7.2 gives

Proposition 8.3. For each \(j\), \(q_j(t) \in C((-\infty, 0], \mathbb{R}^2) \cap C^1((-\infty, 0), \mathbb{R}^2)\). Moreover,

1. if \(N = 1\), then \(q_1(t) \equiv 0\);
2. if \(N \geq 2\), then the vector valued function \((q_j(t))\) satisfies

\[
q'_j(t) = 4 \sum_{k \neq j} \frac{q_k(t) - q_j(t)}{|q_k(t) - q_j(t)|^2}, \quad q_j(0) = 0.
\]

To finish the proof of Theorem 1.4, it remains to show that

Proposition 8.4. There exist \(N\) distinct points \(p_1, \ldots, p_N \in \mathbb{R}^2\) such that

\(q_j(t) \equiv \sqrt{-t}p_j, \quad \forall t \leq 0\).

Moreover, \((p_j)\) is a critical point of the renormalized energy \(W\).

If \(N = 1\), it has been established in Proposition 8.3 that \(q_1(t) \equiv 0\), so we will assume \(N \geq 2\) in the remaining part of this section.

We first prove several technical estimates on \(q_j(t)\).

Lemma 8.5. For any \(t < 0\),

\[
\sum_{j=1}^{N} q_j(t) = 0,
\]

\[
\sum_{j=1}^{N} |q_j(t)|^2 = 2N(N - 1)|t|.
\]
Proof. By (8.1), we have
\[
\begin{cases}
\frac{d}{dt} \sum_{j=1}^{N} q_j(t) = 4 \sum_{j=1}^{N} \sum_{k \neq j} \frac{q_k(t) - q_j(t)}{|q_k(t) - q_j(t)|^2} = 0, \\
\sum_{j=1}^{N} q_j(0) = 0.
\end{cases}
\]
Then (8.2) follows by continuity.
In the same way, we obtain
\[
\begin{cases}
\frac{d}{dt} \sum_{j=1}^{N} |q_j(t)|^2 = 8 \sum_{j=1}^{N} q_j(t) \cdot \sum_{k \neq j} \frac{q_k(t) - q_j(t)}{|q_k(t) - q_j(t)|^2} = 2N(N - 1), \\
\sum_{j=1}^{N} |q_j(0)|^2 = 0.
\end{cases}
\]
Then (8.3) follows by an integration in \(t\).

Corollary 8.6. For any \(t < 0\),
\[
\max_{1 \leq j \leq N} |q_j(t)| \leq \sqrt{2N(N - 1)}\sqrt{-t}.
\]

Lemma 8.7. There exists a constant \(c_\ast > 0\) such that for any \(t < 0\),
\[
\min_{1 \leq j \neq k \leq N} |q_j(t) - q_k(t)| \geq c_\ast \sqrt{-t}.
\]

Proof. Assume by the contrary, there exists a sequence of \(t_i < 0\), and two indicies \(j \neq k\) such that
\[
\lim_{i \to +\infty} \frac{|q_j(t_i) - q_k(t_i)|}{\sqrt{-t_i}} = 0.
\]
Set
\[
\mu_i(A) := \mu_{|t_i|} \left( \frac{A}{\sqrt{-t_i}} \right), \quad \forall A \subset \mathbb{R}^2 \text{ Borel.}
\]
As in the first step in the proof of Theorem 4.1, after passing to a subsequence, we may assume \(\mu_i \rightharpoonup \mu_\infty\) for any \(t \leq 0\). By a diagonal argument as in the proof of Proposition 5.1 (see Step 1 therein), we deduce that \(\mu_\infty\) is also a blow up limit of \(u\) at \((0, 0)\).

By the definition of \(\mu_i\),
\[
\mu_{i-1} \geq 8\pi \delta_{q_j(t_i)/\sqrt{-t_i}} + 8\pi \delta_{q_k(t_i)/\sqrt{-t_i}}.
\]
By Corollary 8.6, we may take a subsequence of \(i\) so that both \(q_j(t_i)/\sqrt{-t_i}\) and \(q_k(t_i)/\sqrt{-t_i}\) converge. By (8.4), their limit points coincide, which is denoted by \(q_\infty\).
Passing to the limit in (8.5), we obtain
\[
\mu_{\infty-1} \geq 16\pi \delta_{q_\infty}.
\]
This is a contradiction with Proposition 5.1.

In view of these estimates, it is better to normalize the blow-up limits \((q_j(t))\).

Definition 8.8 (Renormalized blow-up limits). By setting \(s := -\log(-t)\), \(s \in \mathbb{R}\), we define the renormalized blow-up limit to be
\[
(p_j(s)) := \left( \frac{1}{\sqrt{-t}} q_j(t) \right).
\]
The following lemma gives a characterization of the energy level of remormalized blow-up limits at infinite time, by using the renormalized energy \( W \) (see (1.8)).

**Lemma 8.9.**

1. For any renormalized blow up limit \( (p_j(s)) \) constructed from a blowing up sequence at scales \( \lambda_i \to 0 \), it is a solution to the ODE system

\[
p_j'(s) = \frac{1}{2}p_j(s) + 4 \sum_{k \neq j} \frac{p_k(s) - p_j(s)}{|p_k(s) - p_j(s)|^2}, \quad s \in \mathbb{R}.
\]

2. The two limits

\[
W_\pm(\{\lambda_i\}) := \lim_{s \to \pm\infty} W(p_1(s), \cdots, p_N(s))
\]

exist, and they are critical energy levels of \( W \).

3. There exists a constant \( C(N) \) depending on \( N \) only such that

\[
-C(N) \leq W_-(\{\lambda_i\}) \leq W_+(\{\lambda_i\}) \leq C(N).
\]

**Proof.** The ODE (8.6) follows by taking a change of variables in (8.1). This system of ODEs is the gradient flow of the function \( W \). Hence \( W(p_1(s), \cdots, p_N(s)) \) is non-increasing in \( s \).

By Corollary 8.6,

\[
\max_{1 \leq j \leq N} \sup_{s \in \mathbb{R}} |p_j(s)| \leq \sqrt{2N(N-1)}.
\]

By Lemma 8.7,

\[
\min_{1 \leq j \neq k \leq N} \inf_{s \in \mathbb{R}} |p_j(s) - p_k(s)| \geq c_*.
\]

Hence

\[
W_\pm(\{\lambda_i\}) := \lim_{s \to \pm\infty} W(p_1(s), \cdots, p_N(s))
\]

are two well-defined, finite constants. This also implies that the \( \omega \)-limit points and \( \alpha \)-limit points of \( (p_j(s)) \) are critical points of \( W \). They may not be unique, but the corresponding energy levels are \( W_\pm(\{\lambda_i\}) \) respectively.

Finally, (8.7) is a consequence of (8.8) and (8.9). \( \square \)

In the following we denote, for any \( s \in \mathbb{R} \),

\[
\overline{u}^s(y) := e^{-s}u(e^{-s/2}y - e^{-s}),
\]

which is just the slice of the blowing up sequence \( u e^{-s/2} \) at time \(-1\). By the blow up analysis developed so far, we have

**Lemma 8.10.** There exists a \( T_2 > 0 \) such that for any \( s > T_2 \), there exist \( N \) points \( \overline{p}_1(s), \ldots, \overline{p}_N(s) \) satisfying

\[
\min_{j \neq k} |\overline{p}_j(s) - \overline{p}_k(s)| \geq c_*/2 \quad \text{and} \quad \sum_{j=1}^N |\overline{p}_j(s)|^2 \leq 2N^2,
\]

such that \( \overline{u}^s(y) dy \) is close to \( 8\pi \sum_{j=1}^N \delta_{\overline{p}_j(s)} \) weakly as Radon measures.
Next we give a canonical construction of the Dirac measures by using an optimal approximation procedure. For any \((p_1, \cdots, p_N) \in \mathbb{R}^{2N}\) satisfying \((c_*)\) as in Lemma \ref{tightness})
\[
\min_{j \neq k} |p_j - p_k| \geq c_*/4 \quad \text{and} \quad \sum_{j=1}^{N} |p_j|^2 \leq 2N^2 + c_*^2,
\]
define a positive smooth function \(\omega(x; p_1, \cdots, p_N)\) such that it equals \(|x - p_j|^2\) in \(B_{c_*/8}(p_j)\), it is bounded below by \(c_*^2/64\) outside \(\bigcup_{j=1}^{N} B_{c_*/4}(p_j)\) and it is equal to a positive constant outside \(B_{2N+2c_*}\).

**Lemma 8.11.** There exists a \(T_* > 0\) such that for any \(s > T_*\), the following minimization problem
\[
\min_{(p_1, \cdots, p_N) \in \mathbb{R}^{2N}} \int_{B_{2N+3c_*}} \omega(x; p_1, \cdots, p_N) \tilde{u}^s(x) dx
\]
has a unique minimizer \((p_{j,s}(s))\). Moreover,

(1) for any \(s > T_*\),
\[
(8.11) \quad \min_{j \neq k} |p_{j,s}(s) - p_{k,s}(s)| \geq c_*/8 \quad \text{and} \quad \sum_{j=1}^{N} |p_{j,s}(s)|^2 \leq 2N^2 + 2c_*^2;
\]

(2) for each \(j\), \(p_{j,s} \in C^1(T_*, +\infty)\).

**Proof.** Denote
\[
\mathcal{J}^s(p_1, \cdots, p_N) := \int_{B_{2N+3c_*}} \omega(x; p_1, \cdots, p_N) \tilde{u}^s(x) dx.
\]
It is smooth in \(p_1, \cdots, p_N\) and positive everywhere.

By Lemma \ref{asymptotics}, we have

(1) \(\lim_{s \to +\infty} \mathcal{J}^s(\tilde{p}_1(s), \cdots, \tilde{p}_N(s)) = 0\);

(2) for any \(\varepsilon > 0\), there exist \(T(\varepsilon) > 0\) and \(\delta(\varepsilon) > 0\) such that, for any \(s > T(\varepsilon)\), if \((p_j)\) satisfies
\[
\sum_{j=1}^{N} |p_j - \tilde{p}_j(s)| > \delta(\varepsilon),
\]
then
\[
\mathcal{J}^s(p_1, \cdots, p_N) > \varepsilon.
\]

Thus there exists a \(T_* > 0\) such that for any \(s \geq T_*\), the minima of \(\mathcal{J}^s\) is attained at some point. Denote it by \((p_{j,s}(s))\). By Property 1, we have
\[
\lim_{s \to +\infty} \sum_{j=1}^{N} |p_{j,s}(s) - \tilde{p}_j(s)| = 0.
\]
Combining this relation with (8.10), we obtain (8.11), perhaps after taking a larger \(T_*\).

By Lemma \ref{asymptotics}, there exists a small constant \(\delta > 0\) such that for any \(s > T_*\) and \((p_j)\) satisfying \(\sum_{j=1}^{N} |p_j - \tilde{p}_j(t)| < \delta\), it holds that
\[
(8.12) \quad \sum_{j,k=1}^{N} \left| \int_{B_{2N+3c_*}} \frac{\partial^2 \omega}{\partial p_j \partial p_k}(x; p_1, \cdots, p_N) \tilde{u}^s(x) dx - \delta_{jk} \right| \leq \delta,
\]
where $\delta_{jk}$ is the Kronecker delta symbol. This implies that $J^s$ is strictly convex in a fixed neighborhood of $(\tilde{p}_j(s))$. Therefore its minima point is unique. By the implicit function theorem, $p_{js}(s)$ is continuously differentiable in $s$.

**Lemma 8.12.** For any sequence $T_i \to +\infty$, there exists a subsequence (not relabelling) so that $(p_{js}(s + T_i))$ converges to a renormalized blow-up limit in $C^1_{loc}(\mathbb{R})$.

**Proof.** Let $\lambda_i := e^{-T_i/2}$. Take a subsequence of $\lambda_i$ so that the blowing up sequence $u^{\lambda_i}$ converges to a blow-up limit $8\pi \sum_{j=1}^N \delta_{q_i(t)}$ (in the sense of Theorem 4.1). Let $(p_j(s))$ be the corresponding renormalized blow up limit. By the definition of $J^s$, we have

$$
\lim_{T_i \to +\infty} J^{s + T_i}(p_1(s), \ldots, p_N(s)) = 0.
$$

Then by the minimality of $(p_{js}(s))$ and the strict convexity of $J^{s + T_i}$ near $(p_j(s))$, we deduce that $(p_{js}(s + T_i))$ converges to $(p_j(s))$ uniformly on any compact set of $\mathbb{R}$.

Next we prove the uniform convergence of $(p_{js}'(s + T_i))$. The proof of the previous lemma implies that for all $t < 0$, the minimization problem

$$
\int_{B_{2N+3c_a}} \omega(x; p_1, \ldots, p_N)u^{\lambda_i}(x, t)dx
$$

has a unique minimizer $(p_{js}^{\lambda_i}(t))$. In fact, the uniqueness implies that

$$
(p_{js}^{\lambda_i}(t)) = (p_{js}(-\log(-t) + T_i)).
$$

Hence it suffices to establish the uniform convergence for the derivative of $p_j^{\lambda_i}$.

By the minimization condition for $(p_j^{\lambda_i}(t))$, we have

$$
\int_{B_{2N+3c_a}} \frac{\partial \omega}{\partial p_j}(x; p_1^{\lambda_i}(t), \ldots, p_N^{\lambda_i}(t))u^{\lambda_i}(x, t)dx = 0, \ \forall j = 1, \ldots, N.
$$

Differentiating this identity in $t$, we obtain

$$
(8.13) \quad \sum_{k=1}^N \left( \int_{B_{2N+3c_a}} \frac{\partial^2 \omega}{\partial p_j \partial p_k}(x; p_1^{\lambda_i}(t), \ldots, p_N^{\lambda_i}(t))u^{\lambda_i}(x, t)dx \right) \frac{dp_k^{\lambda_i}}{dt}(t) = -\int_{B_{2N+3c_a}} \frac{\partial \omega}{\partial p_j}(x; p_1^{\lambda_i}(t), \ldots, p_N^{\lambda_i}(t))\partial_t u^{\lambda_i}(x, t)dx
$$

$$
= -\int_{B_{2N+3c_a}} \Delta x \frac{\partial \omega}{\partial p_j}(x; p_1^{\lambda_i}(t), \ldots, p_N^{\lambda_i}(t))u^{\lambda_i}(x, t)dx + \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Theta \frac{\partial \omega}{\partial p_j}(x, y)u^{\lambda_i}(x, t)u^{\lambda_i}(y, t)dxdy.
$$

In the above, we have used Lemma 3.1 and the fact that $\frac{\partial \omega}{\partial p_j}(x; p_1^{\lambda_i}(t), \ldots, p_N^{\lambda_i}(t))$ is compactly supported in $x$. In view of (8.12), we can solve $dp_k^{\lambda_i}/dt$ from this equation. By the convergence of $u^{\lambda_i}(x, t)dx$, the last two integrals converge uniformly on any compact set of $(-\infty, 0)$. This then gives the uniform convergence of $dp_k^{\lambda_i}/dt$ on any
compact set of \((-\infty, 0)\), and consequently, the uniform convergence of \((p'_{j,*}(s + T))\) on any compact set of \(\mathbb{R}\).

Let

\[ E(s) := W(p_{1,*}(s), \ldots, p_{N,*}(s)). \]

**Lemma 8.13.** The limit \(\lim_{s \to +\infty} E(s)\) exists, and it is a critical energy level of \(W\) in \([-C(N), C(N)]\).

**Proof.** First, by (8.11) and the definition of \(W\), \(E(s)\) is a continuous function on \([T_*, +\infty)\). Moreover, for any \(s \geq T_*\),

\[
- C(N) \leq E(s) \leq C(N).
\]

Next, let us recall a fact about the critical energy levels of \(W\). Because critical points of \(W\) are identical to those of \(e^{-W}\), and \(e^{-W}\) is a real analytic function, by the Lojasiewicz inequality for real analytic functions (see Lojasiewicz [10]), its critical energy levels are discrete. Hence there are only finitely many critical energy levels of \(W\) in the interval \([-C(N), C(N)]\). Denote them by \(E_1, \ldots, E_K\) for some \(K \in \mathbb{N}\). Let

\[
\delta := \min_{1 \leq i \neq j \leq K} |E_i - E_j| > 0
\]

be the gap between these energy levels. Then for any non-static gradient flow \((p_j(s))\) of \(W\), if \(W(p_1(s), \ldots, p_N(s)) \in [-C(N), C(N)]\), we must have

\[
\lim_{s \to -\infty} W(p_1(s), \ldots, p_N(s)) - \lim_{s \to +\infty} W(p_1(s), \ldots, p_N(s)) \geq \delta.
\]

For any \(\varepsilon \in (0, \delta/10)\), set

\[
I_{\varepsilon} := \{ s : s > T_*, \ \min_{j=1,\ldots,K} |E(s) - E_j| > \varepsilon \}.
\]

It is an open subset of \((T_*, +\infty)\). Assume

\[
I_{\varepsilon} = \cup_0 I_{\varepsilon, \alpha},
\]

where \(I_{\varepsilon, \alpha} = (t_{\varepsilon, \alpha}^-, t_{\varepsilon, \alpha}^+)\) is an open interval. We claim that

**Claim.** The number of these intervals is finite.

By Lemma 8.12, for any \(T > 0\) and any \(s \geq T_* + T\), \((p_{j,*})\) is close to a gradient flow of \(W\) in \([s - T, s + T]\). Because in each \(I_{\varepsilon, \alpha}\), \(E(s)\) is not close to any critical energy level \(E_j\), in every \(I_{\varepsilon, \alpha}\) (with at most finitely many exceptions), \((p_{j,*})\) cannot be close to any static solution of (8.6) (i.e. a critical point of \(W\)). In view of (8.16), this is possible only if (at least when \(t_{\varepsilon, \alpha}^-\) is sufficiently large)

\[
E(t_{\varepsilon, \alpha}^-) > E(t_{\varepsilon, \alpha}^+) + \delta - 4\varepsilon.
\]

Then because

\[
\sup_{s_1, s_2 \in [t_{\varepsilon, \alpha}^+, t_{\varepsilon, \alpha}^-]} |E(s_1) - E(s_2)| \leq 2\varepsilon,
\]

we get a discrete monotonicity relation

\[
E(t_{\varepsilon, \alpha}^+) - E(t_{\varepsilon, \alpha}^-) \geq \delta - 6\varepsilon \geq \frac{\delta}{8}.
\]
Combining this inequality with (8.14), we deduce that there are only finitely many open intervals in \( I_\varepsilon \). This finishes the proof of this claim.

This claim can be reformulated as the statement that, for any \( \varepsilon > 0 \), there exists \( T(\varepsilon) > 0 \) such that for any \( s \geq T(\varepsilon) \),

\[
\min_{j=1,\ldots,K} |E(s) - E_j| \leq \varepsilon.
\]

By the gap between different critical energy levels in (8.15) and the continuity of \( E(s) \), we then deduce that there exists a fixed \( j \) such that for any \( \varepsilon \in (0, \delta/10) \) and \( s \geq T(\varepsilon) \),

\[
|E(s) - E_j| \leq \varepsilon.
\]

Completion of the proof of Theorem 1.4. By the previous lemma, there exists an energy critical level \( E \) of \( W \) such that, for any renormalized blow-up limit \((p_j(s))\),

\[
W(p_1(s), \cdots, p_N(s)) \equiv E.
\]

Because \((p_j(s))\) is the gradient flow of \( W \), this is possible only if it does not depend on \( s \). Hence it must be a critical point of \( W \).

Remark 8.14. Because blow up limits are obtained by a compactness argument and critical points of \( W \) are not discrete, it is not clear if the renormalized blow-up limit is unique.

9. Entire solutions

In this section we prove Theorem 1.10. We will also explain, in the setting of Section 4, how to obtain entire solutions from suitable rescalings around a blow up point.

Proof of Theorem 1.10. If \( u \) is an entire solution, by Theorem 1.1, the blowing down sequence converges to a limit \( \mu_t \) in \( \mathbb{R}^2 \times \mathbb{R} \). First by Theorem 1.8, there exists an \( N \in \mathbb{N} \) such that \( \mu_0 = 8\pi N \). On the other hand, because the blowing down sequence converges in \( Q_1 \), an application of Proposition 5.1 at \( t = 0 \) shows that \( N = 1 \). Then by Lemma 6.2 and Lemma 6.3, we conclude the proof of Theorem 1.10.

Next, we use the convergence theory in Section 4 to show how entire solutions arise as micro-models of singularity formations.

We work under the same assumptions of Theorem 4.1. After localization (see Remark 2.2), we may assume that

\[
u_i(x, 0)dx \to 8\pi \delta_0 + \rho(x, 0)dx \quad \text{in} \quad B_1.
\]

This implies that

\[
\max_{B_1} u_i(x, 0) \to +\infty,
\]

and it is attained at an interior point, say \( x_i \). Because \( u_i \) converges to \( \rho \) in \( C_{loc}(B_1 \setminus \{0\}) \),

\[
|x_i| \to 0.
\]

Denote \( R_i := u_i(x_i, 0)^{1/2} \). Set \( \tilde{u}_i(x, t) := R_i^{-2}u_i(x_i + R_i^{-1}x, R_i^{-2}t) \) and define \( \nabla \tilde{v}_i \) as in (1.6) by using \( \tilde{u}_i \). Then we have
**Theorem 9.1.** As $i \to +\infty$, $\tilde{u}_i$ converges to a limit $\tilde{u}_\infty$ in $C^{2,1}_{loc}(\mathbb{R}^2 \times \mathbb{R})$, where $\tilde{u}_\infty$ is an entire solution of (1.2).

**Proof.** By Theorem 4.1, for each $t \in \mathbb{R}$, there exist finitely many points $q_j(t)$, $j = 1, \ldots, N(t)$, such that as Radon measures on $\mathbb{R}^2$,

$$
\int \tilde{u}_i(x, t) dx \to 8\pi \sum_{j=1}^{N(t)} \delta_{q_j(t)} + \tilde{u}_\infty(x, t) dx.
$$

By (9.1) and Lemma 3.1, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $t \in (-\delta, \delta)$,

$$
\limsup_{i \to +\infty} \int_{B_{\delta}} u_i(x, t) dx < 8\pi + \varepsilon.
$$

After a scaling, this is transformed into

$$
\limsup_{i \to +\infty} \int_{B_{\delta} R_i} \tilde{u}_i(x, t) dx \leq 8\pi + \varepsilon, \quad \forall t \in (-\delta R_i^2, \delta R_i^2).
$$

Combining this upper bound with (9.2), we see $N(t) \leq 1$. Moreover, by Theorem 4.1 and Proposition 5.1,

- **Alternative I:** either $N(t) = 1$ and $\tilde{u}_\infty(t) \equiv 0$;
- **Alternative II:** or $N(t) = 0$ and $\tilde{u}_\infty(t) \in C^2(\mathbb{R}^2)$.

Let $\mathcal{I}$ be the set of those $t$ satisfying Alternative I. By the same strong maximum principle argument used in the proof of Lemma 8.2, we deduce that there exists a $T \leq +\infty$ such that

$$
\mathcal{I} = (-\infty, T].
$$

On the other hand, by definition, we have

$$
\max_{B_{R_i}} \tilde{u}_i(x, 0) = \tilde{u}_i(0, 0) = 1.
$$

Then by Theorem 3.2 (applied to the cylinder $Q_r(x, 0)$, for any $x \in \mathbb{R}^2$ and a fixed, sufficiently small $r$), there exists a $\delta > 0$ such that $\tilde{u}_i \in C^{2,1}(\mathbb{R}^2 \times (-\delta, \delta))$. As a consequence, $0 \notin \mathcal{I}$, or equivalently, $T < 0$.

If $T > -\infty$, $\tilde{u}_\infty$ is smooth in $\mathbb{R}^2 \times (T, +\infty)$ and at time $T$ it is a Dirac measure with mass $8\pi$. Then the same proof of Lemma 6.3 (more precisely, the proof of (6.2)) leads to a contradiction. This contradiction implies that $T = -\infty$, that is, Alternative II holds for all $t$. Then by Theorem 4.1, $\tilde{u}_i$ converges uniformly to $\tilde{u}_\infty$ on any compact set of $\mathbb{R}^2 \times \mathbb{R}$. By standard parabolic regularity theory, we deduce that $\tilde{u}_\infty$, with the corresponding $\nabla \tilde{v}_\infty$, is a classical solution of (1.2) on $\mathbb{R}^2 \times \mathbb{R}$, i.e. it is an entire solution. Moreover, by passing to the limit in (9.3), we see $\tilde{u}_\infty$ satisfies the finite mass condition (1.11). \qed
10. Boundary blow up points

Finally, we give a remark on boundary blow up points. Let \( \Omega \subset \mathbb{R}^2 \) be a smooth domain. Consider the initial-boundary value problem

\[
\begin{aligned}
  u_t &= \Delta u - \text{div}(u \nabla v), & \text{in } \Omega \times (0, T), \\
  - \Delta v + v &= u, & \text{in } \Omega \times (0, T), \\
  \partial_\nu u &= \partial_\nu v = 0, & \text{on } \partial\Omega \times (0, T),
\end{aligned}
\]

where \( \nu \) is the outward unit normal vector of \( \partial\Omega \).

Solutions to this problem could also blow up in finite time, which is still caused by the concentration of \( u \), see [14, Chapter 11]. The blow up points could lie on the boundary. The blow up analysis can also be used to study boundary blow up points. More precisely, similar to Theorem 1.4, we have

**Theorem 10.1.** Assume \( T \) is the first blow up time, and \( a \in \partial\Omega \) is a boundary blow up point. Let

\[
u_{\lambda}(x,t) := \lambda^2 u(a + \lambda x, T + \lambda^2 t), \quad \lambda \to 0.
\]

For any sequence \( \lambda_i \to 0 \), there exists a subsequence (not relabelling), an half space \( H \) and \( N \) distinct points \( p_j \in H \) such that, for any \( t < 0 \),

\[
u_{\lambda_i}(x,t)dx \rightharpoonup 8\pi \sum_{j=1}^{N} \delta_{\sqrt{-t}p_j} \text{ weakly as Radon measures.}
\]

We only briefly explain the proof of Theorem 10.1. We can still work in the local setting as in Theorem 1.4: first take a small ball \( B_r(a) \) around \( a \), then take a diffeomorphism to flatten \( \partial\Omega \cap B_r(a) \). After scaling the radius of this ball to be 1, extend \( u \) and \( v \) evenly to the whole \( B_1 \). Denote by \( g \) the Riemannian metric obtained by pushing forward the original Euclidean metric through these transformations. Then we are in the following local setting:

- \( u \in C^\infty(Q_1^- \setminus \{(0,0)\}), \ u > 0 \) and

  \[
  \sup_{t \in (-1,0)} \int_{B_1} u(x,t)dx \leq M;
  \]

- \( u \) satisfies

  \[
  u_t - \Delta_g u = -\text{div}_g( u \nabla_g v + \nabla_g f) \quad \text{in } Q_1^-,
  \]

  where \( \Delta_g \) is the Beltrami-Laplace operator with respect to the Riemannian metric \( g \), \( \text{div}_g \) is the corresponding divergence operator, \( f \) is a smooth function in \( Q_1 \), \( \nabla v \) is given by

  \[
  \nabla v(x,t) = \int_{B_1} \left[ -\frac{1}{2\pi} \frac{x-y}{|x-y|^2} + \nabla R(x,y) \right] u(y,t)dt, \quad \forall (x,t) \in Q_1^-,
  \]

  with \( R \) a smooth function of \((x,y)\) (the regular part of the Green function);

- there exists a nonnegative function \( u_0 \in L^1(B_1) \) and a positive constant \( m \) such that as \( t \to 0^- \),

  \[
  u(x,t)dx \rightharpoonup u_0(x)dx + m\delta_0 \quad \text{weakly as Radon measures.}
  \]
After blowing up, the inhomogeneity caused by the Riemannian metric $g$ will disappear. The remaining proof is exactly the same as the one of Theorem 1.4.

**Appendix A. Proof of Theorem 3.2**

In this appendix we prove Theorem 3.2. It is a consequence of the following theorem.

**Theorem A.1.** Suppose $(u, v)$ is a classical solution of (2.1) in $Q_1$, satisfying

\begin{equation}
\sup_{|t|<1} \int_{B_1} u(x, t) dx < 2\varepsilon^* ,
\end{equation}

then

\begin{equation}
\|u\|_{C^{1+\alpha, (1+\alpha)/2}(Q_{1/2})} \leq C.
\end{equation}

The only difference with Theorem 3.2 is that now we make a stronger assumption (A.1). However, (A.1) follows by combining (3.5) with Lemma 3.1, perhaps after restricting to a smaller cylinder and then scaling this cylinder to the unit one.

Before proving this theorem, we recall several inequalities. The first two inequalities are taken from [14, Lemma 4.2 and Lemma 11.1].

**Lemma A.2.** For any $\psi \in C_0^\infty(B_1)$, the following inequalities hold for any $s > 1$, where $C > 0$ is a constant determined by $\psi$ only:

\begin{equation}
\int_{B_1} u^2 \psi dx \leq C\|u\|_1 \left( \int_{B_1} u^{-1} |\nabla u|^2 \psi dx \right) + C\|u\|_1^2 ,
\end{equation}

\begin{equation}
\int_{B_1} u^3 \psi dx \leq \frac{C}{\log s} \left( \int_{B_1} (u \log u + e^{-1}) dx \right) \left( \int_{B_1} |\nabla u|^2 \psi dx \right) + C\|u\|_1^3 + 10s^3. \tag{A.3}
\end{equation}

The next two inequalities are similar to the corresponding ones for (1.2) (see [14, Eqns. (11.15) and (11.16)]). Although there are additional terms $\nabla f$ and $g$ in (2.1), due to their lower order nature, they only produce terms which can be incorporated into the constant term.

**Lemma A.3.** For any $\psi \in C_0^2(B_1)$,

\begin{equation}
\frac{d}{dt} \int_{B_1} (u \log u) \psi dx \leq -\frac{1}{2} \int_{B_1} u^{-1} |\nabla u|^2 \psi dx + 2 \int_{B_1} u^2 \psi dx + C. \tag{A.5}
\end{equation}

\begin{equation}
\frac{d}{dt} \int_{B_1} u^2 \psi dx \leq -\int_{B_1} |\nabla u|^2 \psi dx + 3 \int_{B_1} u^3 \psi dx + C. \tag{A.6}
\end{equation}

**Proof of Theorem A.1.** Take two functions $\psi_1 \in C_0^\infty(B_1)$, $\psi_2 \in C_0^\infty(B_{3/4})$ such that $0 \leq \psi_1, \psi_2 \leq 1$, $\psi_1 \equiv 1$ in $B_{3/4}$ and $\psi_2 \equiv 1$ in $B_{2/3}$. Denote

\begin{equation}
I_1(t) := \int_{B_1} (u \log u + e^{-1}) \psi_1 dx, \quad J_1(t) := \int_{B_1} u^2 \psi_1 dx
\end{equation}
and

\[ I_2(t) := \int_{B_1} u^2 \psi_2 \, dx, \quad J_2(t) := \int_{B_1} u^3 \psi_2 \, dx. \]

**Step 1.** In view of (A.1), combining (A.5) and (A.3) gives

\[ I'_1(t) \leq -cJ_1(t) + C. \]

By the Hölder inequality, we have

\[ I_1(t) \leq CJ_1(t)^{\frac{1}{3}} + C. \]

Hence we get

\[ I'_1(t) \leq -cI_1(t) + C. \]

Because \( I_1(t) \geq 0 \) for any \( t \in (-1, 1) \), this differential inequality implies that

\[ (A.7) \quad I_1(t) \leq C, \quad \forall t \in [-7/8, 7/8]. \]

**Step 2.** In view of (A.7), combining (A.4) and (A.6) (where we choose a sufficiently large \( s \), determined by the upper bound of \( J_1 \) in (A.7)) gives

\[ I'_2(t) \leq -cJ_2(t) + C. \]

By the Hölder inequality, we have

\[ I_2(t) \leq CJ_2(t)^{\frac{1}{3}} + C. \]

Hence we get

\[ I'_2(t) \leq -cI_2(t) + C. \]

Because \( I_2(t) \geq 0 \) for any \( t \in (-7/8, 7/8) \), this differential inequality implies that

\[ (A.8) \quad I_2(t) \leq C, \quad \forall t \in [-3/4, 3/4]. \]

**Step 3.** We have shown that \( u \in L^\infty(-3/4, 3/4; L^2(B_{3/4})) \). Then by standard \( W^{2,p} \) estimate and Sobolev embedding theorem, \( \nabla v \in L^\infty(-3/4, 3/4; L^3(B_{2/3})) \). This implies that \( u \nabla v \in L^\infty(-3/4, 3/4; L^{6/5}(B_{3/4})) \). Then we can lift the regularity of \( u \) and \( v \) by bootstrapping standard \( W^{2,p} \) estimate and Schauder estimate (see [9, Theorem 4.8]).

\[ \square \]

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