ODD DECOMPOSITIONS OF EULERIAN GRAPHS

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Abstract

We prove that an eulerian graph $G$ admits a decomposition into $k$ closed trails of odd length if and only if and it contains at least $k$ pairwise edge-disjoint odd circuits and $k \equiv |E(G)| \pmod{2}$. We conjecture that a connected 2$d$-regular graph of odd order with $d \geq 1$ admits a decomposition into $d$ odd closed trails sharing a common vertex and verify the conjecture for $d \leq 3$. The case $d = 3$ is crucial for determining the flow number of a signed eulerian graph which is treated in a separate paper (arXiv:1408.1703v2). The proof of our conjecture for $d = 3$ is surprisingly difficult and calls for the use of signed graphs as a convenient technical tool.

Keywords: Eulerian graph, graph decomposition, signed graph, nowhere-zero flow.

1 Introduction

Eulerian graphs constitute a fundamental class of graphs extensively studied throughout the entire history of graph theory. Among the many problems concerning eulerian graphs, those related to decomposition into various types of subgraphs belong to most typical in this area [7][8]. In fact, one of the main pillars of the eulerian graph theory is the classical result of Veblen [13] that the existence of a circuit decomposition in a connected graph is equivalent for the graph to be eulerian. Decompositions of eulerian graphs into circuits or trails of restricted lengths, although natural to require, are generally difficult to find and the known results are scarce. In most cases, the graphs known to have such decompositions are complete multipartite graphs, thus having a very explicit structure [3][4][5][10][11].

In contrast, our paper focuses on the existence of closed trail decompositions in general eulerian graphs, with the only restriction that each member of the decomposition have an odd number of edges. It is easy to see that a connected graph $G$ admitting a decomposition into $k$ odd closed trails must be eulerian with $|E(G)| \equiv k \pmod{2}$ and has to contain at least $k$ edge-disjoint odd circuits, one for each closed trail. We show that this necessary condition is also sufficient (Theorem 3.1). In particular, every connected 2$d$-regular graph of odd order, with $d \geq 1$, admits a decomposition into $d$ odd closed trails (Corollary 3.2).
Our primary interest lies, however, in odd closed trail decompositions where all members are required to share a common vertex; we call such decompositions rooted. Equivalently, we ask whether an eulerian graph contains, for a given positive integer \( k \), an eulerian trail \( T \) and a vertex \( v \) which divides \( T \) into \( k \) closed trails of odd length based at \( v \). We show that for \( k = 2 \) the answer is positive if and only if the graph is non-bipartite and has an even number of edges (Theorem 4.1). As a consequence, the following is true.

**Theorem 1.** Every connected 4-regular graph of odd order has a rooted decomposition into two odd closed trails.

A substantial part of this paper is devoted to the problem of decomposing an eulerian graph into three odd closed trails with common origin. Our motivation comes from the area of nowhere-zero flows on signed graphs. In [16, Main Theorem (c)] we show that a signed eulerian graph admits a nowhere-zero integer 3-flow if and only if it has a rooted decomposition into three closed trails with an odd number of negative edges each (see also [14, Theorem 2.4]). After a series of natural reductions the proof amounts to proving the following theorem which is the main result of the present paper.

**Theorem 2.** Every connected 6-regular graph of odd order has a rooted decomposition into three odd closed trails.

The comparison of Theorem 1 and Theorem 2 suggests that the following might be true.

**Conjecture 1.** Every connected \( 2d \)-regular graph of odd order, with \( d \geq 1 \), has a rooted decomposition into \( d \) odd closed trails.

![Figure 1: None of these vertices can be a root of a decomposition into three odd trails](image)

If a graph admits a rooted decomposition into \( d \) odd closed trails, then there is an obvious question about the distribution of roots within the graph. Simple example show that in general the root cannot be chosen arbitrarily even in regular graphs. For example, none of the vertices depicted in Figure 1 can be a root of a decomposition into three odd closed trails.

However, it seems plausible, that in a 4-edge-connected 4-regular graph of odd order every vertex can be the root of a decomposition into two odd closed trails. Similarly, it is conceivable that for every vertex of a 6-edge-connected 6-regular there exists a rooted decomposition into three odd closed trails with root at that vertex. More generally, we propose the following conjecture.

**Conjecture 2.** In a 2\(d\)-edge-connected 2\(d\)-regular graph of odd order, with \( d \geq 1 \), every vertex is a root of some decomposition into \( d \) odd closed trails.

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The rest of our paper is organised as follows. In the next section we assemble the basic definitions needed in this paper. In Section 3 we begin the study of decompositions into odd closed trails in general and derive a necessary and sufficient condition for their existence. In Section 4 we introduce rooted decompositions and prove Theorem 1. The proof of Theorem 2 is divided into two parts which are contained in Sections 5 and 6 respectively. In the final section we briefly discuss a relationship between 3-odd decompositions and nowhere-zero integer 3-flows and derive a necessary and sufficient condition in for an eulerian graph to admit a rooted 3-odd decomposition.

2 Preliminaries

All graphs considered in this paper are finite and may have multiple edges and loops. The graph consisting of a single vertex and $d$ loops will be called the bouquet of $d$ circles and will be denoted by $B_d$.

We often write $e = uv$ for an edge with end-vertices $u$ and $v$, but this notation does not exclude the possibility that $u = v$ or that there is another edge $f$ with the same end-vertices as $e$.

An eulerian graph has all vertices of even degree and is always connected. A semi-eulerian graph is a connected graph with at most two vertices of odd degree. A $u$-$v$-eulerian graph is a connected graph where all vertices except possibly $u$ and $v$ have an even degree; if $u = v$, then the graph is eulerian.

Throughout this paper we make extensive use of various types of walks, trails, paths, and their segments. In accordance with the adopted terminology, a trail is a non-empty sequence $W = v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$ whose terms are alternately vertices and edges such that each edge $e_i$ joins the vertex $v_{i-1}$ to the vertex $v_i$, and all edge terms are distinct. More specifically, $W$ is a $v_0$-$v_k$-trail. If $v_0 = v_k$, we say that the trail is closed. A path is a trail in which all vertex terms are distinct. A circuit is a closed trail in which all inner vertex terms are distinct. If $W_1$ is a $u$-$v$-trail and $W_2$ is a $v$-$w$-trail, then $W_1W_2$ denotes the $u$-$w$-trail obtained by first traversing $W_1$ and then $W_2$. For subgraphs $H$ and $K$ of $G$ we define a $K$-$H$-path in $G$ as a $u$-$v$-path where the vertex $u$ is in $K$, the vertex $v$ is in $H$, and all other vertices lie outside $K \cup H$.

If $\{V_1, V_2\}$ is a partition of the vertex set of $G$, then the set of all edges having an end-vertex in both partition sets is called a cut in $G$. A cut of size $n$ is an $n$-edge-cut. Recall that a graph $G$ is $k$-edge-connected if the removal of fewer than $k$ edges leaves $G$ connected. The edge-connectivity $\lambda(G)$ of $G$ is the largest integer $k$ for which $G$ is $k$-edge-connected. In order to have the parameter $\lambda(G)$ always finite we make an exception from the definition when $G$ has a single vertex. In this case $G$ coincides with $B_d$ for some $d \geq 0$, and we set $\lambda(B_d) = 2d$. Note that if $G$ is eulerian, then $\lambda(G)$ is an even number. We will be using this fact throughout the paper without mention.

Finally, if $G$ is a graph and $H$ and $K$ are subgraphs of $G$, we let $H - K$ denote the subgraph of $H$ obtained by the removal of the edges of $K$.

3 Odd eulerian decompositions

A decomposition of a graph $G$ is a set $D = \{G_1, G_2, \ldots, G_k\}$ of subgraphs of $G$ whose edge sets partition the edge set of $G$. The subgraphs $G_i$ constituting the decomposition will be called the constituents of $D$. A decomposition $D$ will be called eulerian if each constituent is an eulerian subgraph; equivalently, if each constituent is an closed trail.
Clearly, a connected graph that admits an eulerian decomposition must itself be eulerian.

The purpose of the present paper is to study eulerian decompositions where each constituent $G_i$ has an odd number of edges. Such a decomposition will be called odd. More specifically, a decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of an eulerian graph $G$ will be called $k$-odd if each $G_i$ is eulerian and has an odd number of edges.

Our first theorem characterizes eulerian graphs that admit a $k$-odd decomposition for a fixed integer $k \geq 1$.

**Theorem 3.1.** Let $G$ be an eulerian graph and let $k$ be a positive integer with $k \equiv |E(G)| \pmod{2}$. Then $G$ admits a $k$-odd decomposition if and only if $G$ contains at least $k$ pairwise edge-disjoint odd circuits.

**Proof.** To prove the necessity, let $\{G_1, G_2, \ldots, G_k\}$ be a $k$-odd decomposition of $G$. Each $G_i$ has an eulerian trail of odd length and since each closed walk of odd length contains an odd circuit, $G_i$ contains an odd circuit $C_i$. Hence $\{C_1, C_2, \ldots, C_k\}$ is a set of $k$ pairwise edge-disjoint circuits in $G$.

For the converse, let $\{C_1, C_2, \ldots, C_k\}$ be an arbitrary set of $k$ pairwise edge-disjoint circuits of $G$. Since each component of $G - (\bigcup_i C_i)$ is eulerian, $G$ admits a circuit decomposition $\mathcal{K}$ that includes all the circuits $C_1, C_2, \ldots, C_k$. Let us consider the intersection graph $J(\mathcal{K})$ of $\mathcal{K}$; its vertices are the elements of $\mathcal{K}$ and edges join pairs of elements that have a vertex of $G$ in common. Since $G$ is connected, so is $J(\mathcal{K})$.

It is obvious that every connected subgraph of $J(\mathcal{K})$, with vertex set a subset $\mathcal{L} \subseteq \mathcal{K}$, uniquely determines an eulerian subgraph of $G$. The latter subgraph will have an odd number of edges whenever $\mathcal{L}$ contains an odd number of odd circuits. Thus to finish the proof it is enough to show that $\mathcal{K}$ can be partitioned into $k$ subsets, each containing an odd number of odd circuits and each inducing a connected subgraph of $J(\mathcal{K})$. In fact, we may assume that $J(\mathcal{K})$ is a tree as the general case follows immediately with the partition of $\mathcal{K}$ obtained from a spanning tree of $J(\mathcal{K})$.

If $B$ is the set of all odd circuits from $\mathcal{K}$, then $|B| \equiv |E(G)| \equiv k \pmod{2}$ and $|B| \geq k$. Thus we may view $J(\mathcal{K})$ as a tree $T$ having a distinguished set $B$ of vertices such that $|B| \geq k$ and $|B| \equiv k \pmod{2}$. In this terminology, it remains to prove the following.

Claim 1. Let $T$ be a tree, $k \geq 1$ an integer, and $B \subseteq V(T)$ a subset with $|B| \geq k$ vertices. If $|B| \equiv k \pmod{2}$, then the vertex set of $T$ can be partitioned into $k$ pairwise disjoint subsets $V_1, V_2, \ldots, V_k$ such that each $V_i$ contains an odd number of vertices from $B$ and induces a subtree of $T$.

Proof of Claim 1. We proceed by induction on $k$, and for every fixed $k$ by induction on the number of vertices of $T$. If $k = 1$, then the conclusion is vacuously true for every tree $T$ and every subset $B \subseteq V(T)$ with an odd number of vertices. If $k = 2$, take the largest subtree $T_1$ of $T$ with an odd number of vertices from $B$, let $V_1$ be the vertex set of $T_1$, and let $V_2 = V(T) - V_1$. Clearly, the partition $\{V_1, V_2\}$ fulfils the conditions of the claim.

For the induction step assume that $T$ has $k \geq 3$ vertices. Let $T$ be any tree with a subset $B \subseteq V(T)$ such that $|B| \geq k$ and $|B| \equiv k \pmod{2}$. Clearly, $T$ has at least $k$ vertices. If $T$ has exactly $k$ vertices, then $B = V(T)$, and the partition into singletons is the sought partition of $V(T)$. Assume now that $T$ has more than $k$ vertices. At least two vertices of $T$ are leaves, say $u$ and $v$. There are two cases to consider.

**Case 1.** Both $u$ and $v$ belong to $B$. Consider the tree $T' = T - \{u, v\}$ and the set $B' = B - \{u, v\}$. Since $B'$ has at least $k - 2$ vertices, we can apply the induction hypothesis to $T'$ and $B'$ for $k' = k - 2$. From the induction hypothesis we get a partition $\{V_1, V_2, \ldots, V_{k-2}\}$ of $V(T')$ such that each $V_i$ contains an odd number of vertices from $B'$.
and induces a subtree of $T'$. Then $\{V_1, V_2, \ldots, V_{k-2}, \{u\}, \{v\}\}$ is the required partition for $T$.

**Case 2.** One of $u$ and $v$, say $u$, does not belong to $B$. In this case we set $T' = T - u$ and $B' = B$, and apply the induction hypothesis to $T'$ and $B'$ for $k' = k$. We conclude that $V(T')$ has a partition $\{V_1, V_2, \ldots, V_k\}$ such that each $V_i$ contains an odd number of vertices from $B'$ and induces a subtree of $T'$. One of the partition sets, say $V_1$, contains a neighbour of $u$. Then $\{V_1 \cup \{u\}, V_2, \ldots, V_k\}$ is the required partition for $T$.

This concludes the induction step and establishes the claim as well as the theorem. □

**Corollary 3.2.** Let $G$ be a connected $2d$-regular graph of odd order with $d \geq 1$. Then $G$ admits an $k$-odd decomposition for each $k \in \{1, 2, \ldots, d\}$ such that $k \equiv d \pmod{2}$.

**Proof.** By Petersen’s 2-factor theorem, $G$ can be decomposed into $d$ pairwise edge-disjoint 2-factors (see, for example, [6, Corollary 2.1.5]). Since $G$ has an odd order, each 2-factor contains an odd circuit, so $G$ contains at least $d$ pairwise edge-disjoint circuits. For each $k \leq d$ such that $k \equiv d \pmod{2}$ we have $k \equiv |E(G)| \pmod{2}$, and the conclusion now immediately follows from Theorem 3.1. □

## 4 Rooted odd decompositions

A $k$-odd decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of an eulerian graph $G$ is **rooted** if there is a vertex $v$ of $G$, called a **root** of $\mathcal{D}$, such that each constituent $G_i$ contains $v$. Clearly, a graph $G$ admits a rooted $k$-odd decomposition if and only if it contains an eulerian trail $T$ and a vertex $v$ that divides $T$ into $k$ closed subtrails $T_1, T_2, \ldots, T_k$ based at $v$, each of odd length.

Finding a rooted $k$-odd decomposition is a significantly more difficult task than just finding any $k$-odd decomposition, especially as $k$ increases. Nevertheless, every 2-odd decomposition is automatically rooted, so Theorem 3.1 also provides a characterisation of graphs that admit a 2-odd decomposition. The following theorem somewhat simplifies the condition and is provided with a short independent proof.

**Theorem 4.1.** An eulerian graph has a rooted 2-odd decomposition if and only if it is non-bipartite and has an even number of edges.

**Proof.** Each odd closed trail contains an odd circuit, so the necessary condition follows immediately. For the converse, let $G$ be a non-bipartite eulerian graph with an even number of edges. Then $G$ contains an odd circuit and hence an eulerian subgraph with an odd number of edges. Let $C$ be one that has the maximal number of edges. Since $G$ has an even number of edges, $C$ is a proper subgraph of $G$. Each component of $G - C$ is an eulerian subgraph of $G$ sharing a vertex with $C$. Since $C$ is maximal, there is only one non-trivial component, and it must have an odd number of edges. It follows that $\{C, G - C\}$ is a rooted 2-odd decomposition of $G$. □

We now derive Theorem [1] as a consequence of Theorem 4.1.

**Proof of Theorem [1].** Let $G$ be a connected 4-regular graph of odd order. Clearly, $G$ has an even number of edges. Furthermore, $G$ cannot be bipartite, because the partite sets of a connected regular bipartite graph have the same size, and hence such a graph has an even number of vertices. The result now follows from Theorem 4.1. □
Having characterised graphs that admit a rooted 2-odd decomposition we can proceed to rooted 3-odd decompositions. The situation here is much more complicated because there exist eulerian graphs that admit a 3-odd decomposition, but not a rooted one. One example, which can be easily extended to an infinite family, is displayed in Figure 2. It follows that a structural characterisation of graphs that admit a rooted 3-odd decomposition may be difficult. Our Theorem 2 shows that having an odd number of vertices (or edges) is a sufficient condition for a connected 6-regular graph to have a rooted 3-odd decomposition. Trivially, this condition is also necessary. The proof heavily depends on connectivity arguments and will be performed in two steps. First, in Section 5 we prove a special case of Theorem 2 for 6-edge-connected graphs. The general case will be treated in Section 6 after some additional preparation.

![Figure 2: An eulerian graph having no rooted 3-odd decomposition.](image)

5 Proof of Theorem 2: The 6-connected case

In this section we prove that every 6-edge-connected 6-regular graph of odd order has a rooted 3-odd decomposition. To make the proof easier we pass from ordinary graphs to signed graphs and prove a natural analogue of the required statement for signed graphs. The statement for unsigned graphs will follow as a trivial consequence of the signed graph version.

Recall that a *signed graph* is a graph $G$ together with a mapping, called the *signature* of $G$, which assigns $+1$ or $-1$ to each edge. An edge receiving value $+1$ is said to be *positive* while one with value $-1$ is said to be *negative*. The sign of each edge will usually be known from the immediate context, therefore no special notation for the signature will be required.

The signature of a signed graph is a means of introducing the concept of balance, which is more important than the signature itself. A circuit of a signed graph is said to be *balanced* if it contains an even number of negative edges, and is *unbalanced* otherwise. A signed graph in which all circuits are balanced is itself called *balanced*; an *unbalanced* signed graph is one that contains at least one unbalanced circuit. In general, the essence of any signed graph is constituted by the list of all balanced circuits. Two signed graphs with the same underlying graph are therefore considered to be *identical* if their lists of balanced circuits coincide. The corresponding signatures are called *equivalent*.

There is a convenient way of turning one signature into an equivalent one. Let $G$ be a signed graph and let $U$ be a set of vertices of $G$. If we change the sign of each edge
with exactly one end in \( U \), then the product of signs on every circuit does not change and hence the new signature is equivalent to the previous one. This operation is called \textit{switching} at \( U \). Note that switching the signature at \( U \) has the same effect as switching at all the vertices of \( U \) in a succession. It is easy to see that by successive vertex switching we can turn any spanning tree of \( G \) into an all-positive subgraph. This fact readily implies that two signatures are equivalent if and only if they are \textit{switching-equivalent}, that is, if they can be transformed into each other by a sequence of vertex switchings [19, Proposition 3.2]. In particular, a signed graph is balanced if and only if its signature is equivalent to the all-positive signature. For a more detailed introduction to signed graphs we refer the reader to Zaslavsky [19].

Observe that switching the signature of a signed eulerian graph does not change the parity of the number of negative edges. Therefore all signed eulerian graphs fall into two natural subclasses depending on whether the number of negative edges is even or odd. Accordingly, a signed eulerian graph \( G \) will be called \textit{even} if it has an even of negative edges, otherwise \( G \) will be called \textit{odd}. It is easy to see that even eulerian graphs can be balanced as well as unbalanced. In contrast, odd eulerian graphs are necessarily unbalanced.

A decomposition \( D = \{G_1, G_2, \ldots, G_k\} \) of a signed eulerian graph \( G \) will be called \textit{odd}, or more specifically \( k \)-\textit{odd}, if each \( G_i \) is an odd eulerian signed subgraph of \( G \). It may be useful to realise that a decomposition of an unsigned graph \( G \) is odd if and only if it is odd for the signed graph obtained from \( G \) by assigning \(-1\) to each edge.

We proceed to the main result of this section which gives a sufficient condition for a 6-regular signed graph to have a rooted 3-odd decomposition. Clearly, every signed eulerian graph that admits a 3-odd decomposition must have an odd number of negative edges, and must contain at least three pairwise edge-disjoint unbalanced circuits, one for each constituent. We show that for 6-edge-connected 6-regular graphs these necessary conditions are also sufficient. In fact, we can show that it is enough to require two edge-disjoint unbalanced circuits – the parity of the number of negative edges will ensure the existence of three.

**Lemma 5.1.** Let \( G \) be a signed eulerian graph with an odd number of negative edges which contains two edge-disjoint unbalanced circuits. Then \( G \) contains at least three pairwise edge-disjoint unbalanced circuits.

**Proof.** Take two edge-disjoint unbalanced circuits \( C_1 \) and \( C_2 \) of \( G \) and form the signed graph \( G' = G - (C_1 \cup C_2) \). The total number of negative edges in components of \( G' \) is odd, so \( G' \) has a component \( K \) with an odd number of negative edges. Since \( K \) is eulerian, it contains an unbalanced circuit \( C_3 \). The circuits \( C_1, C_2, \) and \( C_3 \) are obviously pairwise edge-disjoint, as required.

Now we are ready for the main result of this section.

**Theorem 5.2.** Let \( G \) be a 6-edge-connected 6-regular signed graph with an odd number of negative edges which contains two edge disjoint unbalanced circuits. Then \( G \) has a rooted 3-odd decomposition.

**Proof.** We prove the result by induction on the number of vertices. If \( G \) has a single vertex, then \( G \) must be a bouquet of three negative loops, and the conclusion for \( G \) is clearly holds.

For the induction step, and throughout the rest of the proof, let \( G \) be a 6-edge-connected 6-regular signed graph with an odd number of negative edges which contains two edge disjoint unbalanced circuits on \( n \geq 2 \) vertices.
Consider an arbitrary vertex $v$ of $G$ and let $e_1, e_2, \ldots, e_6$ be the edges incident with $v$ listed in a certain fixed order, and let $v_i$ denote the other end of $e_i$. Let us form a graph $G'$ of order $n - 1$ by removing $v$ from $G$ and by adding three new edges $e'_i = v_iv_{i+3}$ for $i \in \{1, 2, 3\}$ to $G - v$; note that by doing this we may introduce parallel edges and loops. We define the signature for $G'$ in a natural way: the sign of each edge $v_iv_{i+3}$ will be obtained by multiplying the sign of $v_iv$ with the sign of $vv_{i+3}$; the signs of all other edges being directly inherited from $G$. We will say that the signed graph $G'$ is obtained by splitting off the vertex $v$ from $G$.

In 1992, Frank [9], Theorem 1A, generalising an earlier result of Lovász [13], proved that every vertex of even degree $d \geq 4$ in a 2-edge-connected graph $K$ can be split off in a similar manner as defined above to produce a graph $K'$ that has the same edge-connectivity as $K$; that is, $\lambda(K') = \lambda(K)$. Using this fact we can prove the following.

**Claim 1.** Every vertex of $G$ can be split in such a way that the resulting signed graph $G'$ is 6-edge-connected, 6-regular, and has an odd number of negative edges.

**Proof of Claim 1.** Given a vertex $v$ of $G$, let us perform splitting in the way guaranteed by the result of Frank [9]. It is obvious that $G'$ is 6-regular and 6-edge-connected. Observe that the signature for $G'$ has been defined in such a way that the number of negative edges in $G'$ has the same parity as that of $G'$. Therefore $G'$ is odd and thus has all the properties stated. This proves Claim 1.

**Claim 2.** Let $G'$ be a connected 6-regular signed graph with an odd number negative edges obtained from $G$ by splitting off a vertex $v$ of $G$. If $G'$ admits a rooted 3-odd decomposition, then so does $G$.

**Proof of Claim 2.** The graph $G$ can be reconstructed from $G'$ by first subdividing each edge $e'_i = v_iv_{i+3}$, with a new vertex, then by identifying the three new vertices into one – the vertex $v$ – and by reinstating the original signature of $G$ on the newly formed edges of $G$. Let $\{G'_1, G'_2, G'_3\}$ be a rooted 3-odd decomposition of $G'$. The process in which $G$ arises from $G'$ produces from each $G'_i$ an eulerian subgraph $G_i$ of $G$ with the parity of the number of negative edges preserved. Therefore $\{G_1, G_2, G_3\}$ is a rooted 3-odd decomposition of $G$, and Claim 2 is proved.

To finish the proof of the theorem it suffices to prove the following claim.

**Claim 3.** At least one of the following statements holds for $G$:

1. $G$ admits a rooted 3-odd decomposition.

2. $G$ contains a vertex that can be split in such a way that the resulting signed graph $G'$ is 6-edge-connected 6-regular signed graph with an odd number of negative edges which contains two edge disjoint unbalanced circuits.

**Proof of Claim 3.** To prove the claim it is enough to show that either $G$ admits a rooted 3-odd decomposition or $G$ has a vertex such that its splitting off from $G$ in accordance with Claim 1 produces a signed graph with two edge-disjoint unbalanced circuit. The remaining conditions are automatically fulfilled.

In order to do it we first recall that, by Lemma 5.1, $G$ contains three edge-disjoint unbalanced circuits $C_1, C_2, C_3$. We will analyse the possible positions of $\{C_1, C_2, C_3\}$ within $G$ and in each case we show that either (1) or (2) holds.

If $G$ contains a vertex $v$ that belongs to at most one of $C_1, C_2, C_3$, we split $v$ according to Claim 1. The resulting graph $G'$ is 6-regular, 6-edge-connected, retains at
least two edge-disjoint unbalanced circuits, and has an odd number of negative edges. In other words, (2) holds. Thus we may assume that each vertex of \( G \) is contained in at least two circuits from \( \{ C_1, C_2, C_3 \} \). Consider the subgraph \( H \) of \( G \) obtained from \( G \) by removing all edges of \( C_1 \cup C_2 \cup C_3 \) and by deleting isolated vertices that may result. Since \( G \) is 6-regular, each component of \( H \) is a circuit whose vertices lie in \( C_1 \cup C_2 \cup C_3 \). Moreover, \( G = C_1 \cup C_2 \cup C_3 \cup H \).

Next, assume that \( G \) contains a vertex \( v \) belonging to all three circuits \( C_1, C_2, \) and \( C_3 \). If \( H \) is balanced, then for each \( i \in \{ 1, 2, 3 \} \) we can form a subgraph \( G_i \) by taking the union of \( C_i \) with some of the circuits of \( H \) that share a vertex with \( C_i \). Clearly, each \( G_i \) is an odd eulerian signed graph. Furthermore, if each circuit of \( H \) is absorbed into precisely one \( G_i \), then \( \{ G_1, G_2, G_3 \} \) becomes a 3-odd decomposition of \( G \) with root at \( v \). This verifies (1). If \( H \) is unbalanced, \( \bar{H} \) contains a pair of disjoint unbalanced circuits \( D_1 \) and \( D_2 \) because the total number of negative edges in \( H \) is even. Clearly, both of \( D_1 \) and \( D_2 \) are disjoint from \( v \). We now split \( v \) according to Claim 1, so \( D_1 \) and \( D_2 \) will be inherited into the resulting graph \( G' \). Taking into account Claim 1 we see that \( G' \) has all the required properties. Hence (2) is satisfied.

For the rest of the proof we may assume that every vertex of \( G \) lies on precisely two of the circuits \( C_1, C_2, \) and \( C_3 \). In particular, \( H \) is a 2-factor. By parity, \( H \) contains an even number of unbalanced circuits. If \( H \) contains at least two unbalanced circuits, say \( D_1 \) and \( D_2 \). Take any vertex \( v \) of \( D_1 \) and split it in accordance with Claim 1 to produce a graph \( G' \). There exists exactly one circuit \( C_j \in \{ C_1, C_2, C_3 \} \) that does not contain \( v \). Furthermore, \( D_2 \) also does not contain \( v \) because \( D_1 \cap D_2 = \emptyset \). Therefore both \( C_j \) and \( D_2 \) are inherited to \( G' \), which establishes (2).

Thus we may assume that \( H \) is balanced. Recall that each vertex of \( G \) is of one of three types, depending on which pair of circuits from \( \{ C_1, C_2, C_3 \} \) it belongs to. If \( H \) contains a circuit \( B \) with vertices of different types, we construct a 3-odd decomposition of \( G \) as follows. Without loss of generality we may assume that \( B \) has a vertex \( v \) that belongs to \( C_1 \cap C_2 \) and a vertex \( w \) that belongs to \( C_1 \cap C_3 \). In this situation \( C_1, B \cup C_2, \) and \( C_3 \) are three closed trails that share the vertex \( v \). We now extend \( \{ C_1, C_2, B \cup C_3 \} \) to a decomposition \( \{ G_1, G_2, G_3 \} \) of \( G \) by adding each component \( K \) of \( H \) to a member of \( \{ C_1, C_2, B \cup C_3 \} \) which is intersected by \( K \). It is easy to see that \( \{ G_1, G_2, G_3 \} \) is indeed a 3-odd decomposition of \( G \) rooted at \( v \), which verifies (1).

We are left with the case where \( H \) is balanced and each component of \( H \) contains vertices of the same type. Pick an arbitrary vertex \( v \) of \( G \). It belongs to exactly two circuits from \( \{ C_1, C_2, C_3 \} \), say to \( C_1 \) and \( C_2 \). There is an edge \( e \) of \( H \) incident with \( v \). Since \( G \) is 6-edge-connected, \( e \) cannot be a loop, so \( e = vw \) for some vertex \( w \neq v \). The vertex \( w \) must have the same type as \( v \), so \( w \) also belongs to \( C_1 \cap C_2 \); in particular, \( e \) is a chord of both \( C_1 \) and \( C_2 \). Assume that \( w \) is not a neighbour of \( v \) in at least one of \( C_1 \) and \( C_2 \), say in \( C_1 \). Then \( e \) divides \( C_1 \) into two edge-disjoint \( v-w \)-paths \( P \) and \( Q \) of length at least 2. Since \( C_1 \) is unbalanced, exactly one of the circuits \( Pe \) and \( eQ \) is unbalanced, say \( Pe \). Now \( Pe, C_2, \) and \( C_3 \) are three pairwise edge-disjoint unbalanced circuits. Furthermore, any inner vertex \( u \) of \( Q \) belongs to exactly two members of \( \{ C_1, C_2, C_3 \} \) one of which is \( C_1 \). It follows that \( u \) belongs to only one of the circuits \( \{ Pe, C_2, C_3 \} \), and by the case already treated this vertex can be split off to fulfil (2). Hence, if \( G \) fails to satisfy (1) or (2), then \( w \) is a neighbour of \( v \) in both \( C_1 \) and \( C_2 \). In such a case, however, we can repeat the same reasoning for \( w \) in place of \( v \), and so on all around the same component of \( H \). As a result of this consideration we deduce that \( G = C_1 \cup C_2 \cup H \), which is absurd. This contradiction shows that either (1) or (2) holds for \( G \). This completes the proof of Claim 3 as well as that of the theorem. \( \square \)
Corollary 5.3. Every 6-edge-connected 6-regular graph of odd order has a rooted 3-odd decomposition.

Proof. Let us endow $G$ with the all-negative signature. Since $G$ has an odd number of edges, as a signed graph $G$ is odd. Petersen’s 2-factor theorem further implies that $G$ can be decomposed into three pairwise edge-disjoint 2-factors, and since the number of vertices of $G$ is odd, each of these 2-factors contains an odd circuit. Thus $G$ contains at least three unbalanced circuits. Theorem 5.2 now yields that under the all-negative signature $G$ has a rooted 3-odd decomposition. This decomposition is clearly 3-odd also in the unsigned sense, and the result is proved.

Remark 5.4. The assumption of Theorem 5.2 requiring a 6-regular graph $G$ to be 6-edge-connected is essential and cannot be relaxed. Figure 3 displays odd signed 6-regular graphs $G_1$ and $G_2$ with $\lambda(G_1) = 2$ and $\lambda(G_2) = 4$ neither of which admits a rooted 3-odd decomposition (edges not labelled are positive). This shows that Theorem 2 does not directly generalise to signed 6-regular graphs without additional assumptions.

Remark 5.5. The formulation of Theorem 5.2 can be improved by using concepts and results from [16]. Namely, the assumption requiring a signed eulerian graph $G$ to have at least two edge-disjoint unbalanced circuits can be replaced by the condition that $G$ cannot be turned into a balanced graph by deleting a single edge. To be more precise, let us call a signed graph $G$ tightly unbalanced if it is unbalanced and contains an edge $e$ such that $G - e$ is balanced. If $G$ is unbalanced but not tightly unbalanced we say that it is amply unbalanced. Amply and tightly unbalanced signed graphs are important classes of signed graphs which naturally appear in the study of various problems, see for example [15, 16, 17]. In [16, Corollary 3.4] it is shown that an eulerian signed graph admits a nowhere-zero integer flow if and only if it is amply unbalanced. Further, the equivalence (a)$\iff$(c) in Theorem 4.2 from [16] states that a signed eulerian graph contains two edge-disjoint unbalanced circuits if and only if it is amply unbalanced. The proof of this equivalence is non-trivial.

6 Proof of Theorem 2: The general case

In this section we prove Theorem 2. We begin with two simple lemmas.
Lemma 6.1. If a connected graph has exactly $2k$ vertices of odd degree, then it contains a set of $k$ pairwise edge-disjoint paths whose ends cover all odd-degree vertices.

Proof. This is an immediate consequence of the following classical result due to Listing and Lucas (see König [12, Satz 4, p. 22]): Every connected graph $G$ with exactly $2k$ vertices of odd degree can be decomposed into $k$ open trails, and each such decomposition contains at least $k$ open trails.

Consider a vertex $v$ of a graph $G$ and a sequence $(v_1, v_2, \ldots, v_k)$ of vertices of $G$ not containing $v$; note that we permit $v_i = v_j$ for $i \neq j$. A $(v_1, v_2, \ldots, v_k)$-fan in $G$ is a collection $F = \{F_1, F_2, \ldots, F_k\}$ of $k$ edge-disjoint paths such that the path $F_i$ joins $v$ to $v_i$ for each $i \in \{1, 2, \ldots, k\}$.

Lemma 6.2. Let $G$ be a $k$-edge-connected graph. Then for every vertex $v$ and an arbitrary sequence $v_1, v_2, \ldots, v_k$ of vertices of $G$ there exists a $v$-$(v_1, v_2, \ldots, v_k)$-fan in $G$.

Proof. Consider the graph $G^+$ arising from $G$ by adding a new vertex $w$ together with $k$ new edges $wv_1, wv_2, \ldots, wv_k$. We first show that $G^+$ is $k$-edge-connected. Suppose not. Then $G^+$ contains an $m$-edge-cut $S$ with $m < k$ separating $w$ from some vertex $z$ of $G$. At least one of the edges incident with $w$ does not belong to $S$, say $wv_i$. Since $G$ is $k$-edge-connected, it contains $k$ pairwise edge-disjoint $v_i$-$z$-paths. As $m < k$, one of the paths, say $Q$, includes no edge from $S$. It follows that $wv_iQ$ is a $w$-$z$-path in $G^+ - S$, contradicting the choice of $S$. To finish the proof observe that the edge-connectivity version of Menger’s Theorem implies that for any vertex $w \neq v$ there exist $k$ pairwise edge-disjoint $v$-$w$ paths in $G^+$. The removal of $w$ from $G^+$ leaves in $G$ the required $k$ paths which form a $v$-$(v_1, v_2, \ldots, v_k)$-fan in $G$.

Now we are in position to prove Theorem 2.

Proof of Theorem 2. Let $G$ be a smallest counterexample to Theorem 2. Below, in Propositions 6.3 and 6.4 we show that $G$ contains neither a 2-edge-cut nor a 4-edge-cut and therefore must be 6-edge-connected. However, by Corollary 5.3 this is impossible. What remains is to prove Propositions 6.3 and 6.4.

Proposition 6.3. A smallest counterexample to Theorem 2 must be 4-edge-connected.

Proof. Let $G$ be a smallest counterexample to Theorem 2. Suppose that $G$ contains a 2-edge-cut $S$. Then $S$ separates $G$ into two components $H$ and $K$. One of them, say $H$, has an odd number of vertices, and therefore an even number of edges. Consequently, $K$ has an odd number of edges. Let $S = \{a_1b_1, a_2b_2\}$ where $\{a_1, a_2\} \subseteq V(H)$ and $\{b_1, b_2\} \subseteq V(K)$. Note that we do not exclude the possibility that $a_1 = a_2$ or $b_1 = b_2$, or both. The graph $H' = H + a_1a_2$ is a 6-regular graph of odd order smaller than $G$. By the induction hypothesis, $H'$ has a rooted 3-odd decomposition $D = \{H_1, H_2, H_3\}$ with root at some vertex $v$ of $H$. One of the constituents of $D$, say $H_1$, contains the edge $a_1a_2$. By setting $H_1' = (H_1 - a_1a_2) \cup S \cup K$ we get a rooted 3-odd decomposition $\{H_1', H_2, H_3\}$ of the entire $G$ with root at the same vertex $v$. This contradiction proves that $G$ is 4-edge-connected.

Proposition 6.4. A smallest counterexample to Theorem 2 must be 6-edge-connected.

Proof. Again, let $G$ be a smallest counterexample to Theorem 2. By Proposition 6.3 $G$ is 4-edge-connected. Suppose that $G$ contains a 4-edge-cut $S$. Then $G - S$ has two components $H$ and $K$, one of which, say $H$, has an even number of edges. It follows that $H$
has even order and that $K$ has odd. Let $S = \{a_1b_1, a_2b_2, a_3b_3, a_4b_4\}$ with $\{a_1, a_2, a_3, a_4\} \subseteq V(H)$ and $\{b_1, b_2, b_3, b_4\} \subseteq V(K)$. Note that some vertices within both $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ may coincide. Set $A = \{a_1, a_2, a_3, a_4\}$.

To establish the result it suffices to show that $H$ can be decomposed into two odd semi-eulerian subgraphs, an $a_k$-$a_l$-eulerian subgraph $H_{k,l}$ and an $a_m$-$a_n$-eulerian subgraph $H_{m,n}$, where $\{k, l, m, n\} = \{1, 2, 3, 4\}$. Having such a decomposition of $H$, we can construct a rooted 3-odd decomposition of $G$ as follows. We add the edges $b_kb_l$ and $b_mb_n$ to $K$ to form a 6-regular graph $G'$ of odd order. Since $H$ contains at least two vertices, the order of $G'$ is smaller than that of $G$. By the induction hypothesis, $G'$ has a rooted 3-odd decomposition $D = \{K_1, K_2, K_3\}$ with root at some vertex $v$ of $K$. We replace the edge $b_kb_l$ with $H_{k,l}$ and the edge $b_mb_n$ with $H_{m,n}$ in the corresponding constituents of $D$ thereby producing a rooted 3-odd decomposition of the entire $G$ with root at the same vertex – a contradiction. What remains is to find the subgraphs $H_{k,l}$ and $H_{m,n}$. We distinguish two cases depending on whether $H$ is or is not bipartite.

**Case 1.** The subgraph $H$ is bipartite. Let $\{V_1, V_2\}$ be the bipartition of $H$. Let $x$ denote the number of indices $i$ for which $a_i$ belongs to $V_1$; obviously, the number of indices $i$ for which $a_i$ belongs to $V_2$ is $4 - x$. Since $G$ is bipartite, we have $6|V_1| - x = 6|V_2| - (4 - x)$, which implies that $x \equiv 2 \pmod{6}$ and therefore $x = 2$.

Without loss of generality we may assume that $a_1$ and $a_2$ lie in $V_1$ and $a_3$ and $a_4$ lie in $V_2$. (Note that the vertices $a_1$ and $a_2$ may coincide as well as $a_3$ and $a_4$ may coincide.) Since $H$ is connected, there exists an $a_1$-$a_3$-path $P_{1,3}$ in $H$. Clearly $P_{1,3}$ has odd length. The graph $H - P_{1,3}$ has precisely two vertices of odd degree, namely $a_2$ and $a_4$. It follows that $a_2$ and $a_4$ lie in the same component of $H - P_{1,3}$, so there exists an $a_2$-$a_4$-path $P_{2,4}$ in $H - P_{1,3}$, which again must have an odd number of edges. The graph $H - (P_{1,3} \cup P_{2,4})$ is bipartite and has all its vertices of even degree. It follows that $H - (P_{1,3} \cup P_{2,4})$ can be decomposed into a collection of even circuits; in particular, each component of $H - (P_{1,3} \cup P_{2,4})$ has an even number of edges. Moreover, each of these components is incident with at least one of the paths $P_{1,3}$ and $P_{2,4}$. Thus we can add each component of $H - (P_{1,3} \cup P_{2,4})$ to either $P_{1,3}$ or $P_{2,4}$ to produce a decomposition of $H$ into an $a_1$-$a_3$-eulerian subgraph $H_{1,3}$ and an $a_2$-$a_4$-eulerian subgraph $H_{2,4}$. As the number of edges in each of $H_{1,3}$ and $H_{2,4}$ is odd, $\{H_{1,3}, H_{2,4}\}$ is the required decomposition of $H$.

**Case 2.** The subgraph $H$ is not bipartite. As before, we wish to construct suitable semi-eulerian subgraphs $H_{k,l}$ and $H_{m,n}$ that decompose $H$. To this end, the following technical tool will be useful.

**Claim 2.** Let $Y$ be an eulerian subgraph of $H$ and let $B_Y$ be the union of all nontrivial components of $H - Y$ that contain a vertex of $A$. Then $B_Y$ can be decomposed into two semi-eulerian subgraphs $B_{k,l}$ and $B_{m,n}$ such that $B_{k,l}$ is $a_k$-$a_l$-eulerian, $B_{m,n}$ is $a_m$-$a_n$-eulerian, both intersect $Y$, and $\{k, l, m, n\} = \{1, 2, 3, 4\}$.

**Proof of Claim 2.** Consider the graph $G/K$ obtained from $G$ by contracting $K$ into a single vertex $b$. The contraction transforms each edge $a_ib_i$ from $S$ into the edge $a_ib$ of $G/K$. Since $G/K$ is 4-edge-connected, by Lemma 6.2 it contains four pairwise edge-disjoint $b$-$Y$-paths, one through each edge $a_ib$. Take the path containing the edge $a_ib$ and denote by $P_i$ its segment starting at $a_i$; let $a'_i$ be the end-vertex of $P_i$ in $B_Y \cap Y$. Thus $P_1$, $P_2$, $P_3$, and $P_4$ are four pairwise edge-disjoint $A$-$Y$-paths entirely contained in $B_Y$. Let $A' = \{a'_1, a'_2, a'_3, a'_4\}$; again, some of these vertices may coincide.

Consider the subgraph $B' = B_Y - \bigcup P_i$. Observe that each odd-degree vertex of $B'$ lies in $A'$. By Lemma 6.1, $B'$ contains an $a'_k$-$a'_l$-path $P_{k,l}$ and an $a'_m$-$a'_n$-path $P_{m,n}$ such that $P_{k,l}$ and $P_{m,n}$ are edge-disjoint and $\{k, l, m, n\} = \{1, 2, 3, 4\}$. If some vertex $a'_k \in A'$
has even degree in $B'$, then there exists $l \neq k$ such that $a_i' = a_k'$; in this case we can choose the path $P_{k,l}$ to be trivial. Set $Q_{k,l} = P_k \cup P_{k,l} \cup P_l$ and $Q_{m,n} = P_m \cup P_{m,n} \cup P_n$. Then $Q_{k,l}$ is an $a_k-a_l$-eulerian subgraph and $Q_{m,n}$ is an $a_m-a_n$-eulerian subgraph of $B_Y$, and the subgraph $B'' = H - (Q_{k,l} \cup Q_{m,n})$ has all vertices of even degree. Each component of $B''$ is eulerian and has at least one vertex in either $Q_{k,l}$ or $Q_{m,n}$. We extend $Q_{k,l}$ and $Q_{m,n}$ to subgraphs $B_{k,l}$ and $B_{m,n}$, respectively, by attaching components of $B''$ to either $Q_{k,l}$ and $Q_{m,n}$ in such a way that the resulting subgraphs $B_{k,l}$ and $B_{m,n}$ are connected. It is clear from the construction that $B_{k,l}$ is an $a_k-a_l$-eulerian subgraph of $B_Y$ and $B_{m,n}$ is an $a_m-a_n$-eulerian subgraph of $B_Y$. They intersect $Y$ and form a decomposition of $B_Y$. This establishes Claim 2.

We continue with the proof of Case 2. Recall that $H$ is now non-bipartite. It means that $H$ contains an odd circuit, and hence an eulerian subgraph with an odd number of edges. Let $C$ be one with maximum number of edges. Let $B = B_C$ be the union of all nontrivial components of $H - C$ that contain a vertex from $A = \{a_1, a_2, a_3, a_4\}$ and let $D$ be the union of all other nontrivial components. Note that each of $B$ and $D$ may be empty, but since $H$ has an even number of edges, $B \cup D$ contains at least one component. Each component of $B \cup D$ has at least one vertex in $C$ because $H$ is connected.

The maximality of $C$ implies that $B \cup D$ contains at most one eulerian component, which necessarily has an odd number of edges. Furthermore, $B$ has at most two components, which may be either eulerian or semi-eulerian. We consider two subcases according to whether $B$ has an even or an odd number of edges.

**Subcase 2.1.** The subgraph $B$ has an even number of edges. Since $C$ is odd, $D$ must be nonempty. We now apply Claim 2 with $Y = D$. This implies that $B_Y = B \cup C$ and Claim 2 guarantees a decomposition of $B \cup C$ into semi-eulerian subgraphs $B_{k,l}$ and $B_{m,n}$. Since $B \cup C$ is odd, one of $B_{k,l}$ and $B_{m,n}$ is odd, say $B_{k,l}$. Furthermore, $B_{m,n}$ intersects $D$, so $B_{m,n} \cup D$ is connected and is also odd. Thus we can set $H_{k,l} = B_{k,l}$ and $H_{m,n} = B_{m,n} \cup D$, which is the required decomposition of $H$.

**Subcase 2.2.** The subgraph $B$ has an odd number of edges. Then $D$ must be empty. Let us take $Y = C$ thereby obtaining $B_Y = B$. By Claim 2, we can decompose $B$ into two semi-eulerian subgraphs $B_{k,l}$ and $B_{m,n}$, one of which is odd, say $B_{k,l}$. As $B_{m,n}$ intersects $C$, we can set $H_{k,l} = B_{k,l}$ and $H_{m,n} = B_{m,n} \cup C$, which is the required decomposition of $H$. This establishes Subcase 2.1 and completes the proof of Proposition 6.4. 

[7] Concluding remarks

As mentioned in Introduction, the existence of rooted 3-odd decompositions of eulerian graphs is closely related to the existence of nowhere-zero integer 3-flows in signed eulerian graphs. Part (c) of Main Theorem in [16] states that a signed eulerian graph admits a nowhere-zero integer 3-flow but not a nowhere-zero 2-flow if and only if it admits a rooted decomposition into three eulerian subgraphs with an odd number of negative edges each. This result can be used to derive a necessary and sufficient condition for an unsigned eulerian graph to admit a rooted 3-odd decomposition. For this purpose, let us define an undirected nowhere-zero integer $k$-flow on a graph $G$ as a mapping $\phi : E(G) \rightarrow \mathbb{Z}$ such that for each edge $e$ one has $|\phi(e)| < k$ and $\phi(e) \neq 0$. The concept of an undirected integer flow is easily seen to be equivalent to an integer flow on the signed graph obtained by equipping $G$ with the all-negative signature. Let us note that under the term zero-sum flow undirected integer flows were studied by Akbari et al. in [1] [2].
By using items (b) and (c) of Main Theorem of [16] restricted to the all-negative signatures we obtain the following result.

**Theorem 7.1.** An eulerian graph admits a rooted 3-odd decomposition if and only if it has an odd number of edges and admits an undirected nowhere-zero integer 3-flow.

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