Energy, Hamiltonian, Noether Charge, and Black Holes

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Abstract

It is shown that in general the energy $E$ and the Hamiltonian $H$ of matter fields on the black hole exterior play different roles. $H$ is a generator of the time evolution along the Killing time while $E$ enters the first law of black hole thermodynamics. For non-minimally coupled fields the difference $H - E$ is not zero and is a Noether charge $Q$ analogous to that introduced by Wald to define the black hole entropy. If fields vanish at the spatial boundary, $Q$ is reduced to an integral over the horizon. The analysis is carried out and an explicit expression for $Q$ is found for general diffeomorphism invariant theories. As an extension of the results by Wald et al, the first law of black hole thermodynamics is derived for arbitrary weak matter fields.

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1 Introduction

There may be two definitions of the energy of matter fields on an external time independent background. The first one defines the energy in terms of the stress-energy tensor $T_{\mu\nu}$ as

$$E(m) = -\int_{\Sigma_t} \sqrt{-g} d^3 x T_{00}^0,$$

(1.1)

where $\Sigma_t$ is a hypersurface of a constant time $t = \text{const}$, $g$ is the determinant of the metric tensor $g_{\mu\nu}$ of the background space-time. The definition of the stress-energy tensor is standard

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \delta I(m) / \delta g^{\mu\nu},$$

(1.2)

in terms of the action $I(m)$ of matter fields. Another possibility is to identify the energy with the Hamiltonian

$$H(m) = \int_{\Sigma_t} \sqrt{-g} d^3 x \left[ \partial L(m) / \partial \dot{\phi} - L(m) \right],$$

(1.3)

where $L(m)$ is the Lagrangian of fields $\phi$, and $\dot{\phi}$ are the time derivatives of $\phi$.

The energies $E(m)$ and $H(m)$ do not depend on $t$ (or on the choice of the hypersurface $\Sigma_t$) provided $\phi$ obey the equations of motion and vanish at the spatial boundary $C_t$ of $\Sigma_t$ or at asymptotic infinity. It can be shown that $E(m)$ and $H(m)$ either coincide or differ by a surface term on $C_t$. In the most physical situations, however, the boundary conditions can be chosen in such a way to eliminate the boundary terms and make $E(m)$ and $H(m)$ equal.

The black holes are an exclusion from this rule. For a system in the black hole exterior, the surfaces $\Sigma_t$ meet at the bifurcation surface $\Sigma$ of the black hole horizon. It is important that $\Sigma$ is an internal boundary of $\Sigma_t$ where fields obey no conditions but regularity. Thus, if $E(m)$ and $H(m)$ differ by a surface term the contribution from $\Sigma$ cannot be eliminated.

On a static black hole background one can write

$$H(m) - E(m) = \frac{\kappa}{2\pi} Q$$

(1.4)

where $\kappa$ is the surface gravity of the Killing horizon and $Q$ is an integral over $\Sigma$. Later we show that for theories where $L(m)$ does not include the derivatives of the metric higher than second order

$$Q = 8\pi \int_{\Sigma} \sqrt{\sigma} d^2 y \left( \partial L(m) / \partial R_{\mu\nu\lambda\rho} \right) p_\mu l_\nu p_\lambda l_\rho.$$

(1.5)

Here $R_{\mu\nu\lambda\rho}$ is the Riemann tensor of the background space-time and $p_\mu$, $l_\mu$ are two unit mutually orthogonal vectors normal to $\Sigma$. It follows from (1.5) that $Q$ is not zero when fields have couplings with the curvature in the Lagrangian.

Because $E(m)$ and $H(m)$ are different they are related to the different physical properties of the system. According to (1.3), $H(m)$ is associated to the generator of the time evolution.

\footnote{We follow conventions of book [1].}
The energy $\mathcal{E}(m)$ is obtained from the observable stress-energy tensor and should be a physical energy of fields $\phi$ on $\Sigma_t$. In the case of a black hole, the role of $\mathcal{E}(m)$ as the physical energy becomes evident after examining the first law of black hole thermodynamics. In the Einstein gravity the variation of the mass $M$ of a black hole under small excitation of matter fields $\phi$ with the energy $\mathcal{E}(m)$ can be represented as:

$$
\delta M = \frac{\kappa}{8\pi G} \delta A + \mathcal{E}(m),
$$

where $G$ is the Newton constant and $A$ is the surface area of the horizon.

Thus, in case of black holes, $\mathcal{H}(m)$ is related to the time evolution while $\mathcal{E}(m)$ is related to the thermodynamical properties of the system. For the sake of simplicity we call $\mathcal{H}(m)$ and $\mathcal{E}(m)$ the Hamiltonian and the energy, respectively.

It should be noted that the difference between $\mathcal{E}(m)$ and $\mathcal{H}(m)$ may be crucial. For instance, this difference draw an attention under studying statistical-mechanical interpretation of the Bekenstein-Hawking entropy in the models of induced gravity \[3\]-\[5\]. Non-minimal couplings of the constituent fields with the curvature are an important feature of these models. As was shown in \[4\], \[5\], the Noether charge $Q$, Eq. (1.5), is not trivial in such models and appears in the formula for the Bekenstein-Hawking entropy $S^{BH}$

$$
S^{BH} = S^{SM} - Q,
$$

where $S^{SM}$ is the statistical-mechanical (or entanglement) entropy of the constituents. According to Eq. (1.6), the spectrum of the black hole mass $M$ can be related to the spectrum of the energy $\mathcal{E}(m)$ of the constituents. On the other hand, to calculate $S^{SM}$ one uses the spectrum of the Hamiltonian $\mathcal{H}(m)$. These two facts were used in \[3\] to explain the subtraction of $Q$ in Eq. (1.7).

The present paper has two aims. The first one is to carry out a general analysis of the energy and the Hamiltonian in diffeomorphism invariant theories on an external black hole background and derive an expression for $Q$ (which is reduced to Eq. (1.3) in the particular case). This also implies studying stationary geometries with axial symmetry and corresponding different definitions of the angular momentum of the system. Our second aim is to demonstrate that $\mathcal{H}(m)$ and $\mathcal{E}(m)$ play different roles and prove an analog of variational formula (1.6) for black holes in arbitrary diffeomorphism invariant theories of gravity.

It should be noted that $Q$ is closely related to the Noether charge which was introduced by Wald \[3\] for description of the black hole entropy and studied in Refs. \[4\]-\[5\]. In many respects our consideration will be parallel to that of work by Iyer and Wald \[7\].

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\footnote{This variational formula can be found, for instance, in book \[2\]. We omit in (1.6) the term proportional to $T^{\mu\nu}\delta g_{\mu\nu}$ by assuming that it is of the second order in perturbations.}
The paper is organized as follows. Section 2 outlines the Noether charge construction [3],[7]. The definition of \(E_m\) and \(H_m\) and a general form of \(Q\) are given in Section 3. In Section 4 we prove that our definition of \(H_m\) gives the generator of time translations. Then, in Section 5 we obtain a generalization of variational formula (1.6) for the energy \(E_m\). Our comments regarding axisymmetric space-times and an extension of the results to rotating black holes can be found in Section 6. We finish with a summary and a brief discussion in Section 7.

2 Review of the Noether charge and black hole entropy

We begin with a diffeomorphism invariant theory of gravity which includes matter fields \(\phi\). It is assumed that \(\phi\) may have tensor and spinor indices. We suppose that the system is defined on a space-time \(M\) with a time-like boundary \(\partial M\). The matter fields are assumed to vanish at the past and future infinities. The dynamical equations of the theory are determined by the action

\[
I[g, \phi] = \int_M \sqrt{-g} d^4x L[g, \phi] - \int_{\partial M} \sqrt{-h} d^3x B[g, \phi] .
\]

(2.1)

Here \(L[g, \phi]\) is the Lagrangian of the theory which can be represented as a function of metric \(g_{\mu\nu}\), Riemann tensor \(R_{\mu\nu\lambda\rho}\), matter fields \(\phi\) and symmetrized covariant derivatives of \(R_{\mu\nu\lambda\rho}\) and \(\phi\). (For the proof see Ref. [7] and [5].) \(h\) is the determinant of the metric induced on \(\partial M\). The variation of the action has the form

\[
\delta I[g, \phi] = \int_M \sqrt{-g} d^4x [E^{\mu\nu} \delta g_{\mu\nu} + E_\phi \delta \phi + \nabla_\mu \theta^\mu (\delta g, \delta \phi)] - \delta \left[ \int_{\partial M} \sqrt{-h} d^3x B[g, \phi] \right] .
\]

(2.2)

Quantities \(E^{\mu\nu}\) and \(E_\phi\) depend on the background fields \(g_{\mu\nu}\) and \(\phi\) only. The components \(\theta^\mu\) of a one-form depend on the background fields, variations \(\delta \phi\), \(\delta g\) and their covariant derivatives. To have a well-defined variational procedure it is assumed that the following equality

\[
\int_{\partial M} \sqrt{-h} d^3x u_\mu \theta^\mu (\delta g, \delta \phi) = -\delta \left[ \int_{\partial M} \sqrt{-h} d^3x B[g, \phi] \right] \tag{2.3}
\]

takes place for given boundary conditions\(^3\). Here \(u_\mu\) is a unit inward-pointing vector normal to the boundary \(\partial M\). The action has an extremum when \(g_{\mu\nu}\) and \(\phi\) obey the equations

\[
E^{\mu\nu} = 0 ,
\]

(2.4)

\[
E_\phi = 0 .
\]

(2.5)

Equation (2.5) is the equation of motion of the matter fields. Consider now an infinitesimal transformation of coordinates and fields

\[
x'^\mu = x^\mu - \xi^\mu(x) ,
\]

(2.6)

\(^3\)Note that Eq. (2.3) is necessary but not sufficient condition to fix the functional \(B\).
\[
\phi'(x) = \phi(x) + \mathcal{L}_\xi \phi(x) ,
\]
\[
g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{L}_\xi g_{\mu\nu}(x) ,
\]
where \(\mathcal{L}_\xi\) is the Lie derivative along the vector field \(\xi^\mu\). When \(\xi^\mu(x)\) has a compact support, it follows from (2.2), (2.6)–(2.8) that
\[
\delta_\xi I = \int_M \sqrt{-g} d^4x \left[ E_{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} + E_\phi \mathcal{L}_\xi \phi + \nabla_\mu (\theta^\mu(\xi) - \xi^\mu L) \right] + O(\xi^2) ,
\]
where
\[
\theta^\mu(\xi) \equiv \theta^\mu(\mathcal{L}_\xi g, \mathcal{L}_\xi \phi) .
\]
In a diffeomorphism invariant theory \(\delta_\xi I = 0\) and each coordinate transformation generates a Noether current
\[
J^\mu(\xi) = \theta^\mu(\xi) - \xi^\mu L
\]
which conserves
\[
\nabla^\mu J_\mu(\xi) = 0
\]
provided if equations of motion (2.4), (2.5) are satisfied. By taking into account that (2.12) holds for any \(\xi^\mu(x)\) one can prove [10] that there is such a tensor \(Q^{\mu\nu}(\xi) = -Q^{\nu\mu}(\xi)\) that
\[
J^\mu(\xi) = \nabla_\nu Q^{\mu\nu}(\xi)
\]
The quantity \(Q^{\mu\nu}(\xi)\) is called the Noether potential. The integral over a two-dimensional surface \(\Sigma\)
\[
Q(\Sigma, \xi) = c \int_\Sigma Q^{\mu\nu}(\xi) d\sigma_{\mu\nu}
\]
is called the Noether charge. Here \(c\) is a normalization constant which will be fixed later.

Suppose now that \(M\) is a stationary asymptotically flat black hole background. Denote by \(\xi^\mu\) a Killing vector
\[
\xi^\mu = t^\mu + \Omega_H \varphi^\mu
\]
where \(t^\mu\) is a time-like Killing vector corresponding to time translations of the system and \(\varphi^\mu\) is a Killing vector corresponding to rotations. The coefficient \(\Omega_H\) coincides with the angular velocity near the horizon where \(\xi^2 = 0\). The bifurcation surface \(\Sigma\) is determined by the condition \(\xi^\mu = 0\).

The total energy \(M\) and angular momentum \(\mathcal{J}\) of the system conserve. On equations of motion (2.4) and (2.5) these quantities are reduced to surface integrals on a two-dimensional spatial boundary \(C_t\) of \(\Sigma_t\) (\(C_t\) is the intersection of \(\Sigma_t\) and \(\partial M\)). According to [7],
\[
M = \int_{C_t} \sqrt{\gamma} d^2y (Q^{\mu\nu}(t)n_\mu u_\nu + BN) ,
\]
\[
\mathcal{J} = -\int_{C_t} \sqrt{\gamma} d^2y Q^{\mu\nu}(\varphi)n_\mu u_\nu .
\]
Here \(N = \sqrt{-g_{00}}\), \(\gamma\) is the metric induced on \(C_t\), and \(n^\mu\) is the future-directed unit vector normal to \(\Sigma_t\). When \(\Sigma_t\) is an infinite surface we will write \(C_\infty\) instead of \(C_t\) for
the boundary at asymptotic infinity. By following [7] we assume that $\varphi^\mu$ is everywhere tangent at $C_\infty$. One can show [7] that in the Einstein gravity the boundary function $B$ can be chosen so that $M$, calculated at $C_\infty$, coincides with the ADM mass of the black hole.

Hypersurfaces of constant time $t$ intersect at the bifurcation surface $\Sigma$ of the Killing horizons. $\Sigma$ is an inner boundary of $\Sigma_t$. Let us define the following quantity

$$S = \frac{2\pi}{\kappa} \int_{\Sigma} \sqrt{\sigma} d^2 z Q^{\mu\nu}(\xi) l_\mu p_\nu$$

(2.18)

where $\kappa$ is the surface gravity of the horizon and $l_\mu$ and $p_\mu$ are unit mutually orthogonal vectors normal to $\Sigma$. We assume that $l_\mu$ is inward-pointing and $p_\mu$ is future-directed vectors normalized as $l^2 = -p^2 = 1$. The important result by Wald [6] relates the variation of the integral (2.18) on $\Sigma$ to the variation of the total energy and momentum computed on $C_t$. Namely,

$$\delta M = T_H \delta S + \Omega_H \delta J$$

(2.19)

where $T_H = \kappa/(2\pi)$. This equation can be interpreted as a first law of black hole thermodynamics (mechanics) in a general diffeomorphism invariant theory of gravity. The quantity $T_H$ is associated to the temperature of a black hole, and $S$ plays the role of the black hole entropy. Formula (2.19) holds when the background fields are the solutions of Eqs. (2.4), (2.5) while variations $\delta g$ and $\delta \phi$ obey the linearized equations of motion. Note that it is not required that $\delta g$ and $\delta \phi$ preserve the symmetries of the background solutions (i.e., that their Lie derivatives along $\xi^\mu$ vanish).

The explicit form of $S$ is determined the general structure of the Noether potential found out by Iyer and Wald [7]

$$Q^{\mu\nu}(\xi) = 2E^{\mu\nu\lambda\rho} \nabla_\lambda \xi_\rho + W^{\mu\nu\lambda} \xi_\lambda$$

(2.20)

where the quantities $E^{\mu\nu\lambda\rho}$ and $W^{\mu\nu\lambda}$ do not depend on $\xi^\mu$. The second term in (2.20) does not contribute to $S$ because $\xi^\mu = 0$ on $\Sigma$, and one can show that

$$S = -8\pi \int_{\Sigma} \sqrt{\sigma} d^2 z E^{\mu\nu\lambda\rho} p_\mu l_\nu p_\lambda l_\rho$$

(2.21)

The quantity $E^{\mu\nu\lambda\rho}$ is [7]

$$E^{\mu\nu\lambda\rho} = \hat{X}^{\mu\nu\lambda\rho} L$$

(2.22)

$$\hat{X}^{\mu\nu\lambda\rho} \equiv \frac{\partial}{\partial R^{\mu\nu\lambda\rho}} - \nabla_{\gamma_1} \frac{\partial}{\partial \nabla_{\gamma_1} R^{\mu\nu\lambda\rho}} + \ldots + (-1)^m \nabla_{(\gamma_1 \ldots \gamma_m)} \frac{\partial}{\partial \nabla_{(\gamma_1 \ldots \gamma_m)} R^{\mu\nu\lambda\rho}}.$$  

(2.23)

Here $m$ is the highest derivative of the Riemann tensor in the Lagrangian $L$ of the theory, and symbol $\nabla_{(\gamma_1 \ldots \gamma_m)}$ denotes symmetrization of the covariant derivatives. The partial derivatives in (2.23) are uniquely fixed by requiring them to have the same tensor symmetries as varied quantities.
It should be noted that there is a freedom in the definition of the Noether potential \( Q^{\lambda \rho} \). First, one can add to the Lagrangian \( L \) a total derivative \( \nabla_\mu \). As a result, \( Q^{\lambda \rho} \) changes as
\[
\Delta Q^{\lambda \rho}(\xi) = \mu^\lambda \xi^\rho - \mu^\rho \xi^\lambda .
\]

Second, Eq. (2.9) does not determine the Noether current (2.11) uniquely. One can add to \( J^\mu(\xi) \) the term \( \nabla_\lambda Y^{\mu \lambda} \), where \( Y^{\mu \lambda} = -Y^{\mu \lambda} \) is linear in the varied fields. This results in a change of the potential
\[
\Delta Q^{\lambda \rho}(\xi) = Q^{\lambda \rho}(g, \phi, \mathcal{L}_\xi g, \mathcal{L}_\xi \phi) .
\]

Finally, Eq. (2.13) allows the freedom to add the term
\[
\Delta Q^{\lambda \rho}(\xi) = \nabla_\nu Z^{\lambda \rho \nu} ,
\]
where \( Z^{\lambda \rho \nu} \) is a totally antisymmetric tensor. It is important that the black hole entropy \( S \) is defined on the bifurcation surface \( \Sigma \) where \( \xi^\mu = 0 \) and so adding terms (2.24) and (2.25) to the potential does not change \( S \). Regarding term (2.26), it vanishes in \( S \) because \( \Sigma \) is a closed surface.

### 3 Energy and Hamiltonian of matter fields

Let us define now the energy and the Hamiltonian of matter fields on an external gravitational background. We start with the total action (2.1) and split it onto two parts
\[
I[g, \phi] = I_{(g)}[g] + I_{(m)}[g, \phi] ,
\]
\[
I_{(g)}[g] = \int_M \sqrt{-g} d^4x L_{(g)}[g] - \int_{\partial M} \sqrt{-h} d^3x B_{(g)}[g] ,
\]
\[
I_{(m)}[g, \phi] = \int_M \sqrt{-g} d^4x L_{(m)}[g, \phi] - \int_{\partial M} \sqrt{-h} d^3x B_{(m)}[g, \phi] .
\]

Functional \( I_{(g)} \) represents a pure gravitational action without the matter ("in vacuum"), while \( I_{(m)} \) is the action of matter fields \( \phi \) in an external gravitational field \( g_{\mu \nu} \). Without loss of generality we assume that
\[
I_{(m)}[g, \phi = 0] = 0 .
\]

According to Eq. (3.1), the form \( \theta^\mu(\xi) \), introduced in (2.2), can be written as
\[
\theta^\mu(\xi) = \theta^\mu_{(g)}(\xi) + \theta^\mu_{(m)}(\xi) .
\]

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\(^4\)This addition does not change equations of motion (2.4), (2.5). However, it is equivalent to a modification of the boundary functional \( B \) by the term \( u^{\lambda \mu \lambda}_{(g)} \) on \( \partial M \). This modification must not violate the variational procedure of the action \( I \), otherwise it results in a change of the boundary conditions imposed on the fields.
The quantities \( \theta^\mu_{(g)}(\xi) \) and \( \theta^\mu_{(m)}(\xi) \) are determined by the higher order derivatives of \( g_{\mu\nu} \), \( \phi \) in the Lagrangians \( L_{(g)} \) and \( L_{(m)} \), respectively. \( \theta^\mu_{(g)}(\xi) \) and \( \theta^\mu_{(m)}(\xi) \) are linear in variations \( L_\xi g \) and \( L_\xi \phi \).

Note that equations of motion (2.3) of the fields \( \phi \) are determined by the variation of the matter action only,

\[
\frac{1}{\sqrt{-g}} \frac{\delta I_{(m)}}{\delta \phi} = E_\phi = 0 \ .
\] (3.6)

Consider now diffeomorphism transformations (2.6)–(2.8) in the matter action. By assuming that the background fields \( \phi \) obey (3.6) and their variations have a compact support one finds

\[
\delta_\xi I_{(m)} = \int_M \sqrt{-g} d^4x \left[ \frac{1}{2} T^{\mu\nu} L_\xi g_{\mu\nu} + \nabla_\mu \left( \theta^\mu_{(m)}(\xi) - \xi^\mu L_{(m)} \right) \right] \ .
\] (3.7)

Here we introduced the stress-energy tensor of the matter

\[
T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I_{(m)}}{\delta g_{\mu\nu}} \ .
\] (3.8)

In a diffeomorphism invariant theory \( \delta_\xi I_{(m)} = 0 \) and we find from (3.7)

\[
\nabla_\mu \left( \theta^\mu_{(m)}(\xi) - \xi^\mu L_{(m)} + \xi_\nu T^{\mu\nu} - \xi^\nu T^{\nu\mu} ; \mu \right) = 0 \ .
\] (3.9)

On equations of motion (3.6) the divergence of \( T^{\mu\nu} \) vanishes \[^5\]

\[
T^{\mu\nu} ; \nu = 0 \ .
\] (3.10)

Thus, there is a conservation law

\[
\nabla_\mu J^\mu_{(m)}(\xi) = 0 \ ,
\] (3.11)

for the following vector

\[
J^\mu_{(m)}(\xi) = -\theta^\mu_{(m)}(\xi) + \xi^\mu L_{(m)} - T^{\mu\nu} \xi_\nu \ .
\] (3.12)

Let us emphasize that for Eq. (3.11) to hold only equations of matter fields (3.6) are required while the background metric \( g_{\mu\nu} \) can be arbitrary. Thus, \( J^\mu_{(m)}(\xi) \) is the Noether current of the matter in an external gravitational field.

Suppose now that the background space \( M \) is invariant with respect to time translations defined by the Killing vector field \( t^\mu \). Let \( \Sigma_t \) be the surface of the constant time \( t \) and define on this surface the quantities

\[
\mathcal{H}_{(m)} = -\int_{\Sigma_t} \left( \theta^\mu_{(m)}(t) - t^\mu L_{(m)} \right) d\Sigma_\mu \ .
\] (3.13)
\[ \mathcal{E}(m) = \int_{\Sigma_t} T^{\mu\nu} t_\mu d\Sigma_\nu , \]

where \( d\Sigma^\mu \) is the future-directed vector of the volume element on \( \Sigma_t \). We call the functionals \( \mathcal{H}(m) \) and \( \mathcal{E}(m) \) the Hamiltonian and the energy, respectively.

Let us note that the forms \( \theta^{\mu}(m)(t) \) depend on the variations of the fields \( \phi \) only. Variations of the background metric vanish \( \mathcal{L}_t g_{\mu\nu} = 0 \). It is easy to see that definition of \( \mathcal{E}(m) \) coincides with standard formula (3.1). To justify the definition of the Hamiltonian let us consider a theory with non-degenerate Lagrangian \( L(m) \) which does not contain the derivatives of \( \phi \) higher than the first order. In such theories

\[ n_\mu \theta^{\mu}(m)(t) = n_\mu \frac{\partial L(m)}{\partial \phi} \mathcal{L}_t \phi = -N \frac{\partial L(m)}{\partial \dot{\phi}} \mathcal{L}_t \phi , \]

where \( \dot{\phi} = d\phi/dt \). By taking into account (3.15) and the identities \( N\tilde{h}^{1/2} = \sqrt{-g} \), \( t^\mu n_\mu = -N \) one obtains \( \mathcal{H}(m) \) in the standard form (1.3). In the next Section we will show explicitly that \( \mathcal{H}(m) \) corresponds to the generator of canonical evolution of the matter fields along the Killing time \( t \).

It is worth considering the following ”currents”

\[ J^{\mu}(m)(\xi) = T^{\mu\nu} \xi_\nu \ , \quad q^{\mu}(m)(\xi) = -\theta^{\mu}(m)(\xi) + \xi^\mu L . \]

When \( \xi^\mu = t^\mu \) one can verify the conservation laws

\[ \nabla_\mu J^{\mu}(m)(t) = 0 \ , \quad \nabla_\mu q^{\mu}(m)(t) = 0 \ , \]

which follow from equations (3.10)–(3.12). Equations (3.17) imply that the energy \( \mathcal{E}(m) \) and the Hamiltonian \( \mathcal{H}(m) \) do not change when the surface \( \Sigma_t \) undergoes variations of a compact support. Moreover, if the boundary terms \( \int J^{\mu}(m)(t) dB_\mu \) and \( \int q^{\mu}(m)(t) dB_\mu \) on \( \partial M \) are zero \( \mathcal{E}(m) \) and \( \mathcal{H}(m) \) do not depend on the time parameter \( t \) which labels the foliation \( \Sigma_t \).

Let us find now the relation between the Hamiltonian and the energy. According to Eq. (3.12),

\[ \mathcal{H}(m) - \mathcal{E}(m) = \int_{\Sigma_t} J^{\mu}(m)(t)d\Sigma_\mu . \]

One can introduce a Noether potential \( Q^{\mu\nu}(m) \) corresponding to the Noether current \( J^{\mu}(m) \). For any \( \xi^\mu \),

\[ J^{\mu}(m)(\xi) = \nabla_\nu Q^{\mu\nu}(m)(\xi) . \]

Then Eq. (3.18) can be rewritten as

\[ \mathcal{H}(m) - \mathcal{E}(m) = \int_{\Sigma_t} \sqrt{-g} d^3 x Q^{\mu\nu}(m)(t) u_\mu n_\nu . \]

---

6. \( d\Sigma^\mu = n^\mu \tilde{h}^{1/2} d^3 x \) where \( \tilde{h} \) is the Jacobian of the metric on \( \Sigma_t \) and \( n^\mu \) is the unit vector orthogonal to \( \Sigma_t \), \( n_\mu = (-N, 0, 0, 0) \).

7. Here \( dB^\mu \) is the vector of the volume element of the boundary surface \( \partial M \). The boundary terms vanish, for instance, when \( \partial M \) is at asymptotic infinity and the fields decay fast enough in this region.
Here $C_t$ is the boundary of $\Sigma_t$, i.e., the intersection of $\Sigma_t$ and $\partial M$. As in Eqs. (2.16), (2.17), $u_\mu$ is a unit inward normal of $\partial M$ and $n_\mu$ is future-directed normal of $\Sigma_t$. (Both vectors are unit mutually orthogonal normals of surface $C_t$). The quantity in r.h.s. of (3.20) is a Noether charge defined with respect to the surface $C_t$ and the Killing vector $t^\mu$. Therefore, we proved that the difference of the energy and the Hamiltonian is a surface term. In many physical situations $M$ has the topology $R \times \Sigma$ and the surface $C_t$ is chosen at the spatial infinity. If the fields rapidly decay at infinity the boundary term in r.h.s of (3.20) vanishes and $H(m)$ and $E(m)$ coincide.

Consider the case of black hole space-times. In this instance we assume that the space-time is static and the Killing vector $t^\mu = 0$ on the bifurcation surface $\Sigma$. All surfaces $\Sigma_t$ of the foliation meet at $\Sigma$. Although $\Sigma$ plays a role of the inner boundary of $\Sigma_t$ the fields subject no conditions on this surface except regularity. The region of $M$ between $\Sigma$ and $\partial M$ is an exterior region of a black hole which has the topology $R^2 \times \Sigma$. For the black hole exterior relation (3.20) can be written as

$$H(m) - E(m) = \int_{\Sigma} \sqrt{\sigma} d^2 z Q_{\mu\nu}(m) (t) l_\mu p_\nu + \int_{C_t} \sqrt{\gamma} d^2 y Q_{\mu\nu}(m) (t) u_\mu n_\nu ,$$

(3.21)

where $l^\mu$ and $p^\mu$ are normals of $\Sigma$ defined as in Eq. (2.18). The last term in r.h.s. disappears when fields vanish on $C_t$ and one gets

$$H(m) - E(m) = \int_{\Sigma} \sqrt{\sigma} d^2 z Q_{\mu\nu}(m) (t) l_\mu p_\nu .$$

(3.22)

Thus, we can conclude that the energy and the Hamiltonian are always different for the theories where the Noether potential $Q_{\mu\nu}(m) (t)$ at the bifurcation surface is not zero.

In fact, an explicit expression for the Noether charge in r.h.s. of (3.22) follows from a consideration similar to that of Section 2. First, let us note that the only difference between the forms of the Noether current for matter fields $J_\mu^\mu (m) (\xi)$, Eq. (3.12), and the current of the complete theory $J_\mu (\xi)$, Eq. (2.11), is the term $-T^{\mu\nu} \xi_\nu$ in $J_\mu^\mu (m) (\xi)$. The matter current does not include the derivatives of the vector $\xi^\mu$ higher than the third order. Thus, by taking into account Eq. (3.19) one can write

$$Q_{\mu\nu}(m) (\xi) = -2 E^{\mu\lambda\rho}_{(m)} \nabla_\lambda \xi_\rho - W^{\mu\nu}_{(m)} \xi_\lambda ,$$

(3.23)

where quantities $E^{\mu\lambda\rho}_{(m)}$ and $W^{\mu\nu}_{(m)}$ depend on the background fields only. Here $Q_{\mu\nu}(m)$ is presented in the same form as $Q_{\mu\nu}$, see (2.21). Because the second term in r.h.s. of (3.23) vanishes on $\Sigma$ we find from (3.22)

$$H(m) - E(m) = \frac{K}{2\pi} Q(m) ,$$

(3.24)

$$Q(m) = 8\pi \int_{\Sigma} \sqrt{\sigma} d^2 z E^{\mu\nu\lambda}_{(m)} p_\mu l_\nu p_\lambda l_\rho ,$$

(3.25)

Note that potentials $Q_{\mu\nu}(m)$ and $Q_{\mu\nu}$ have different signs. Such a difference is a matter of our convention of the definition of $J_\mu^\mu (m)$. We follow Refs. [4],[5] where this convention was used first.
To obtain Eqs. (3.24) and (3.25) we took into account that $t_{\mu\nu} = \kappa (p_{\mu} l_{\nu} - p_{\nu} l_{\mu})$ on $\Sigma$, where $\kappa$ is the surface gravity of the Killing horizon. Obviously, the tensor $E_{(m)}^{\mu\nu\lambda\rho}$ does not depend on the presence of the term $-T^{\mu\nu} \xi_{\mu}$ in $J_{(m)}^{\mu}$. This implies that $E_{(m)}^{\mu\nu\lambda\rho}$ has precisely the same form as the tensor $E^{\mu\nu\lambda\rho}$ in $Q^{\mu\nu}(\xi)$. Thus, as follows from (2.22),

$$E_{(m)}^{\mu\nu\lambda\rho} = \hat{X}^{\mu\nu\lambda\rho} L_{(m)} .$$

$L_{(m)}$ is the Lagrangian of matter fields, see (3.3), and the operator $\hat{X}^{\mu\nu\lambda\rho}$ is defined by (2.23).

According to equations (3.24)–(3.26), a necessary condition for the energy to differ from the Hamiltonian is the presence of non-minimal couplings of matter fields with the curvature of the space-time. To put it in another way, $L_{(m)}$ has to depend on $R_{\mu\nu\lambda\rho}$ and its derivatives. In the case when $L_{(m)}$ does not include the derivatives of $R_{\mu\nu\lambda\rho}$ the Noether charge $Q_{(m)}$ takes simple form (1.5).

By comparing Eqs. (2.21), (2.22) with (3.25), (3.26) one makes another observation: the charge $Q_{(m)}$ is the contribution of the matter fields to the entropy of a black hole. Indeed, it follows from the decomposition of the total action onto gravitational and matter parts, Eqs. (3.1)–(3.3), that the black hole entropy is represented as

$$S = S_{(g)} + S_{(m)} ,$$

$$S_{(m)} = -Q_{(m)} .$$

The term $S_{(g)}$ is the pure gravitational part of the entropy determined by the Lagrangian $L_{(g)}$

$$S_{(g)} = -8\pi \int_{\Sigma} \sqrt{\sigma} d^{2}z E_{(g)}^{\mu\nu\lambda\rho} p_{\mu} l_{\nu} p_{\lambda} l_{\rho} ,$$

$$E_{(g)}^{\mu\nu\lambda\rho} = \hat{X}^{\mu\nu\lambda\rho} L_{(g)} .$$

Also, it should be noted that the Noether potential $Q_{(m)}^{\mu\nu}$ is not determined uniquely. It changes, for instance, when one adds a total divergence to the matter Lagrangian $L_{(m)}$. Possible changes of $Q_{(m)}^{\mu\nu}$ are analogous to changes of $Q^{\mu\nu}$ and are described by Eqs. (2.24)–(2.26). It is important, however, that the charge $Q_{(m)}$ is defined on the bifurcation surface $\Sigma$ and does not depend on this freedom.

Finally, let us emphasize that in the above analysis we assumed that the space-time is static. A generalization of these results to stationary axisymmetric spacetimes including rotating black holes will be given in Section 6.

## 4 $\mathcal{H}_{(m)}$ as a generator of canonical transformations

In what follows we study properties of the Hamiltonian $\mathcal{H}_{(m)}$ and the energy $\mathcal{E}_{(m)}$. In this section we prove that $\mathcal{H}_{(m)}$ is the generator of the canonical transformation of the
system along the Killing time $t$. To define generators of canonical transformations of a field system on a curved background we choose the formalism developed by DeWitt in [11].

Let us first assume that $\mathcal{M}$ has the topology $\mathbb{R} \times \Sigma_t$. We also suppose that fields vanish fast enough at $t \to \pm \infty$ and consider "observables" which are functionals of $\phi$ on $\mathcal{M}$. The Poisson bracket of two "observables" represented by functionals $A[\phi]$ and $B[\phi]$ can be defined as [12], [11]

$$(A, B) = \int d^4x d^4y \frac{\delta A}{\delta \phi(x)} \tilde{G}(x, y) \frac{\delta B}{\delta \phi(y)}.$$  \hspace{1cm} (4.1)

Here $\tilde{G}$ is the Pauli-Jordan function of the theory which is expressed in terms of the advanced, $G^+$, and retarded, $G^-$, Green functions

$$\tilde{G}(x, y) = G^+(x, y) - G^-(x, y).$$  \hspace{1cm} (4.2)

Functions $G^\pm$ obey the equation

$$\int d^4y \frac{\delta^2 I_{(m)}[g, \phi]}{\delta \phi(x) \delta \phi(y)} G^\pm(y, z) = -\delta^{(4)}(x - z),$$  \hspace{1cm} (4.3)

and additional conditions which in local theories are

$$G^+(x, y) = 0 \hspace{1cm} t_x > t_y,$$

$$G^-(x, y) = 0 \hspace{1cm} t_x < t_y.$$  \hspace{1cm} (4.4, 4.5)

where $t_x, t_y$ are the time coordinates of the points $x$ and $y$, respectively. We assume that $\delta^2 I_{(m)}/\delta \phi^2$ is a non-degenerate operator and $G^\pm$ can be uniquely defined by Eq. (4.3) under chosen boundary conditions. A simple example which illustrates (4.3) is a theory of a free massive non-minimally coupled scalar field described by the action

$$I_{(m)}[g, \phi] = -\frac{1}{2} \int \sqrt{-g} d^4x \left((\nabla \phi)^2 + m^2 \phi^2 + \xi R \phi^2\right).$$  \hspace{1cm} (4.6)

For this theory equations (4.3) are reduced to

$$\nabla^\mu \nabla_\mu - m^2 - \xi R)G^\pm(x, y) = -(-g(x))^{-1/2} \delta^{(4)}(x - y).$$  \hspace{1cm} (4.7)

As follows from (4.1), the Poisson bracket of two fields $\phi(x)$ and $\phi(y)$ is

$$(\phi(x), \phi(y)) = \tilde{G}(x, y).$$  \hspace{1cm} (4.8)

In quantum theory this relation is replaced by the commutator (or anticommutator) of the corresponding operators

$$[\hat{\phi}(x), \hat{\phi}(y)] = i\tilde{G}(x, y).$$  \hspace{1cm} (4.9)

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Our analysis can be also carried out for degenerate operators which appear in gauge theories. The problem of singular operators is resolved by imposing gauge conditions [11]. Because there are no other changes we do not pay here a special attention to this case.
We demonstrate now that $H(m)$, as defined by Eq. (3.13), is the generator of canonical transformations of the form $\delta \phi(x) = L_t \phi(x)$. Let us fix the hypersurface $\Sigma_t$ in (3.13) at some constant time $t$ and split the action onto two parts

$$I_{(m)}[g, \phi] = I_{(m)}^+[g, \phi] + I_{(m)}^-[g, \phi] .$$

(4.10)

Functionals $I_{(m)}^\pm$ are defined in the regions $M^\pm$ of $M$. Points of $M^+$ and $M^-$ are in the future or in the past of $\Sigma_t$, respectively. We have

$$I_{(m)}^\pm[g, \phi] = \int_{M^\pm} \sqrt{-g}d^4x L_{(m)}[g, \phi] - \int_{\partial M^\pm} \sqrt{-h}d^3x B_{(m)}[g, \phi] .$$

(4.11)

Note that $M^\pm$ have two boundaries: the time-like boundary $\partial M^\pm$ and $\Sigma_t$. By using diffeomorphism invariance of the action one obtains the equality

$$\delta_t I_{(m)}^\pm = \int_{M^\pm} \sqrt{-g}d^4x \left[ E_\phi L_t \phi + \nabla_\mu \left( \theta^\mu_{(m)}(t) - t^\mu L_{(m)} \right) \right]
- \delta_t \left[ \int_{\partial M^\pm} \sqrt{-h}d^3x B_{(m)} \right] = 0 .$$

(4.12)

As a result of the definition of $B_{(m)}$, see Eq. (2.3),

$$\int_{\partial M^\pm} \sqrt{-h}d^3x u_\mu \theta^\mu_{(m)}(t) = -\delta_t \left[ \int_{\partial M^\pm} \sqrt{-h}d^3x B_{(m)} \right]$$

(4.13)

and the boundary terms on $\partial M^\pm$ are canceled. From (4.12) we find that

$$\int_{M^\pm} \sqrt{-g}d^4x E_\phi L_t \phi = \pm H_{(m)} .$$

(4.14)

Let us make a variation of the both parts of (4.14) over the field $\phi$ and replace afterwards the background field $\phi$ by a solution of classical equations (3.6). One gets

$$\int_{M^\pm} d^4y \delta^2 I_{(m)} \delta \phi(x) \delta \phi(y) = \pm \frac{\delta H_{(m)}}{\delta \phi(x)} .$$

(4.15)

We can consider (4.13) as an equation which defines $L_t \phi$. By using definition (4.3) and conditions (4.4), (4.5) we find

$$L_t \phi(x) = (\phi(x), H_{(m)}) ,$$

(4.16)

where the signs ”$\mp$” correspond to the field $L_t \phi$ in the regions $M^\pm$. Finally, by taking into account (4.2) we have

$$L_t \phi(x) = (\phi(x), H_{(m)}) ,$$

(4.17)

where the Poisson bracket is defined by (4.1). As a consequence, variation with time of any ”observable” $A$ is generated by $H_{(m)}$

$$\delta_t A = \int_M d^4x \frac{\delta A}{\delta \phi(x)} L_t \phi(x) = (A, H_{(m)}) .$$

(4.18)
Therefore, we demonstrated that $H^{(m)}$ coincides with the generator of the canonical transformations along the Killing time $t$. As was shown in [11], the formalism based on using brackets (4.1) is equivalent to the standard canonical formalism. According to (4.9), in a quantized theory Eq. (4.17) becomes

$$i\mathcal{L}_t \hat{\phi}(x) = [\hat{\phi}(x), \hat{H}^{(m)}]$$

(4.19)

which is the standard relation.

Generalization of this formalism to black hole space-times requires some care because the surface $\Sigma_t$ has the internal boundary $\Sigma$ where the Killing vector $t^\mu$ vanishes. To avoid this complication let us consider the surface $\Sigma'_t$ which is obtained from $\Sigma_t$ by cutting the region near $\Sigma$. We thus assume that $\Sigma'_t$ has an inner boundary which lies near $\Sigma$ at a proper distance $\epsilon$, where $\epsilon$ is a small parameter. The foliation of the surfaces $\Sigma'_t$ with different $t$ represents a region $\mathcal{M}_\epsilon$ with an inner boundary $\mathcal{B}_\epsilon$ near the Killing horizon. Consider now Eq. (4.12) in this region. It results in identity analogous to (4.14)

$$\int_{(\mathcal{M}_\epsilon)^\pm} \sqrt{-g} d^4 x E \mathcal{L}_t \phi = \mp H^{(m)} + \int_{(\mathcal{B}_\epsilon)^\pm} \sqrt{-h} d^3 x u^\mu (m) \theta^\mu (m) (t)$$

(4.20)

The last term in r.h.s. of (4.20) is the contribution from the inner boundary, $u^\mu$ is inward pointing normal of $\mathcal{B}_\epsilon$. One can now take the limit $\epsilon \to 0$ in (4.20). In this limit $H^{(m)}$ coincides with the Hamiltonian defined on $\Sigma_t$. When $\epsilon$ tends to zero, $\sqrt{-h} \simeq \epsilon$ on $\mathcal{B}_\epsilon$ and (4.20) is reduced to (4.14). Thus, in the presence of a Killing horizon Eq. (4.17) preserves its form and $\mathcal{H}^{(m)}$ does generate canonical transformations along $t$.

By using relation (3.24) it is not difficult to predict what will be the field equations if one tries to determine the time evolution by the energy rather than by the Hamiltonian. The Poisson bracket of $\mathcal{E}^{(m)}$ with $\phi$ contains an extra term which comes out from the bracket of $\phi$ and $Q^{(m)}$. The Pauli-Jordan function $\tilde{G}$ vanishes outside the light cone. For this reason, for the field in the black hole exterior the extra term is not zero only on the horizon. Thus, the time evolution generated by the energy is different from that generated by the Hamiltonian. The former corresponds to equations of motion modified by the term which can be interpreted as a specific interaction of fields with the horizon due to the non-minimal coupling.

5 $\mathcal{E}^{(m)}$ and black hole thermodynamics

We now show that $\mathcal{E}^{(m)}$ is the energy of matter fields which is related to thermodynamical properties of a black hole. To this aim we compare a black hole with zero matter fields (an analog of a vacuum configuration) to the corresponding black hole with ”excited” fields $\phi$. We will assume that the contribution of the fields is so small that the back reaction effect can be described by linearized equations. In what follows we denote with a bar all
quantities for a black hole with $\phi \neq 0$. Let $M$ and $\bar{M}$ be the mass of a black hole with $\phi = 0$ and $\phi \neq 0$, respectively. The black hole metric obeys the equation
\[ E_{(g)}^{\mu\nu} = -\frac{1}{2} T^{\mu\nu} , \] (5.1)
\[ E_{(g)}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta I_{(g)}}{\delta g_{\mu\nu}} , \] (5.2)
where $I_{(g)}$ is pure gravitational part (3.2) of the action and $T^{\mu\nu}$ is the stress-energy tensor (3.8) of matter fields. Note, we consider only equations for the metric, but do not require matter fields to obey the equations of motion.

In the Einstein gravity
\[ I_{(g)} = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} d^4x R , \] (5.3)
where $G$ is the Newton constant and Eq. (5.1) takes a familiar form
\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu} . \] (5.4)

For a black hole in vacuum, $T^{\mu\nu} = 0$ and $S = S_{(g)}$, where $S_{(g)}$ is determined by Eq. (3.29). When matter fields are present $T^{\mu\nu} \neq 0$ and the vacuum black hole metric $g_{\mu\nu}$ changes to $\bar{g}_{\mu\nu}$. The variation of a matter field from zero to some value $\phi$ has two effects: the black hole mass $M$ changes to $\bar{M}$ and the black hole entropy $S_{(g)}$ changes to $\bar{S}_{(g)}$. In the Einstein gravity $S_{(g)} = \mathcal{A}/(4G)$ and matter fields result in a change of the area $\mathcal{A}$ of the black hole horizon. The total entropy when $\phi \neq 0$ is $\bar{S} = \bar{S}_{(g)} + \bar{S}_{(m)}$, where $\bar{S}_{(m)}$ is the contribution of the matter fields due to the non-minimal coupling.

Our aim now is to find the relation between variation of $M$ and $S_{(g)}$ and the energy $\bar{E}_{(m)}$ of the fields. We assume that $\Sigma_t$ is an infinite hypersurface and fields vanish at spatial infinity. First, let us note that according to (2.13) and (2.16) the black hole mass $\bar{M}$ can be written as
\[ \bar{M} = -\int_{\Sigma_t} \bar{J}^\mu(t)d\Sigma_\mu - \int_{\Sigma_t} \sqrt{\sigma} d^2z Q^{\mu\nu}(t) p_\mu J_\nu + \int_{C_\infty} \sqrt{\gamma} d^2y B N \]
\[ = -\int_{\Sigma_t} \bar{J}^\mu(t)d\Sigma_\mu + \frac{\bar{K}}{2\pi} \bar{S} + \int_{C_\infty} \sqrt{\gamma} d^2y B N . \] (5.5)
The total entropy $\bar{S}$ is defined by (2.18). By using Eqs. (3.13) and (3.20) we get
\[ \bar{M} - \bar{E}_{(m)} = -\int_{\Sigma_t} \left( \bar{J}^\mu(t) - \bar{\theta}^\mu_{(m)}(t) + t^\mu \bar{L}_{(m)} \right) d\Sigma_\mu \]
\[ + \frac{\bar{K}}{2\pi} \left( \bar{S} + \bar{Q}_{(m)} \right) + \int_{C_\infty} \sqrt{\gamma} d^2y B N . \] (5.6)
We can now use the fact that, according to Eqs. (3.1) and (3.3)
\[ \bar{J}^\mu(t) = \bar{\theta}^\mu_{(g)}(t) + \bar{\theta}^\mu_{(m)}(t) - t^\mu (\bar{L}_{(g)} + \bar{L}_{(m)}) . \] (5.7)
From (3.28) and (5.7) it follows that

$$M - \bar{E}_{(m)} = -\int_{\Sigma_t} \left( \theta_{(g)}^\mu (t) - t^\mu L_{(g)} \right) d\Sigma_\mu + \frac{\overline{F}}{2\pi} \bar{S}_{(g)} + \int_{C_\infty} \sqrt{\gamma} d^2y B N .$$  \hspace{1cm} (5.8)

Let us compare this expression to that when \( \phi = 0 \). One easily finds that

$$\delta M - \bar{E}_{(m)} = -\delta \left[ \int_{\Sigma_t} \left( \theta_{(g)}^\mu (t) - t^\mu L_{(g)} \right) d\Sigma_\mu \right] + \delta \left( \frac{\kappa}{2\pi} S_{(g)} \right) + \delta \left[ \int_{C_\infty} \sqrt{\gamma} d^2y B N \right] .$$  \hspace{1cm} (5.9)

Where \( \delta \) denotes the difference between the quantities corresponding to the two black hole solutions. For instance, \( \delta M = M - \bar{M}, \delta S_{(g)} = \bar{S}_{(g)} - S_{(g)} \).

To proceed with (5.9) we neglect by all terms which are of the second order and higher in the variation of the black hole metric \( \delta g_{\mu\nu} = \bar{g}_{\mu\nu} - g_{\mu\nu} \). By taking into account Eqs. (5.1) and (5.2) we find

$$\delta \left[ \int_{\Sigma_t} t^\mu L_{(g)} d\Sigma_\mu \right] = -\delta \left[ \int_{\Sigma_t} \sqrt{-g} d^3x L_{(g)} \right] = \int_{\Sigma_t} \sqrt{-g} d^3x \left[ \frac{1}{2} T^\mu{}_{\nu} \delta g_{\mu\nu} - \nabla_\mu \theta_{(g)}^\nu (\delta g) \right] ,$$  \hspace{1cm} (5.10)

where we used the fact that \( t^\mu d\Sigma_\mu = -\sqrt{-g} d^3x \). Obviously, the first term in the r.h.s. of (5.10) can be neglected because it is of the second order in perturbations.

Other terms in Eq. (5.9) can be also transformed. Note that the vacuum metric is static, \( \mathcal{L}_t g_{\mu\nu} = 0 \), and so

$$\delta \theta_{(g)}^\mu (t) \equiv \delta \theta_{(g)}^\mu (L_t g) = \mathcal{L}_t \left( \theta_{(g)}^\mu (\delta g) \right) .$$  \hspace{1cm} (5.11)

Thus, by using Eqs. (5.10) and (5.11) and leaving only the terms linear in the perturbations one has

$$-\delta \left[ \int_{\Sigma_t} \left( \theta_{(g)}^\mu (t) - t^\mu L_{(g)} \right) d\Sigma_\mu \right] = -\int_{\Sigma_t} \left[ \mathcal{L}_t \left( \theta_{(g)}^\mu (\delta g) \right) - t^\mu \nabla_\nu \theta_{(g)}^\nu (\delta g) \right] d\Sigma_\mu \hspace{1cm}$$

$$= -\int_{\Sigma_t} \left[ t^\nu \nabla_\nu \theta_{(g)}^\mu (\delta g) - (\nabla_\nu t^\mu) \theta_{(g)}^\nu (\delta g) - t^\mu \nabla_\nu \theta_{(g)}^\nu (\delta g) \right] d\Sigma_\mu \hspace{1cm}$$

$$= -\int_{\Sigma_t} \nabla_\nu \left[ t^\nu \theta_{(g)}^\mu (\delta g) - t^\mu \theta_{(g)}^\nu (\delta g) \right] d\Sigma_\mu = \int_{C_\infty} \sqrt{\gamma} d^2y N u_\nu \theta_{(g)}^\nu (\delta g) .$$  \hspace{1cm} (5.12)

In the last line we took into account that contribution from the bifurcation surface \( \Sigma \), where \( t^\mu = 0 \), is zero. We also assume that \( u^\nu t_\nu = 0 \) on \( C_\infty \). Matter fields vanish at \( C_\infty \). This guarantees that \( B = B_{(g)} \) in (5.9). Beside that,

$$\int_{\partial M} \sqrt{-h} d^3x u_\mu \theta_{(g)}^\mu (\delta g) = -\delta \left[ \int_{\partial M} \sqrt{-h} d^3x B_{(g)} \right] .$$  \hspace{1cm} (5.13)

As a result of (5.12) and (5.13), Eq. (5.9) is reduced to

$$\delta M - \bar{E}_{(m)} = \delta \left( \frac{\kappa}{2\pi} S_{(g)} \right) .$$  \hspace{1cm} (5.14)

In the given approximation, \( \bar{E}_{(m)} = E_{(m)} \). From the condition \( \delta t^\mu = 0 \) on \( \Sigma \) it also follows that \( \delta \kappa = 0 \). Thus, our final expression can be represented as

$$\delta M = \frac{\kappa}{2\pi} \delta S_{(g)} + E_{(m)} .$$  \hspace{1cm} (5.15)
This is the direct generalization of Eq. (1.6) of black hole thermodynamics in the Einstein gravity to arbitrary diffeomorphism invariant theories. Let us emphasize that variations of the metric in (5.15) should obey the linearized equations but they are not required to be static.

The quantity $S(g)$ depends only on geometrical characteristics of the black hole solution and it can be interpreted as the proper black hole entropy. As distinct from $S(g)$, the entropy $S$ depends on the non-minimal couplings on the horizon and it is not a pure black hole characteristic. Equation (5.15) is an analogue of the first law of black hole thermodynamics (mechanics) [13]. It relates the change of the total mass $M$ to the change of the entropy $S(g)$ and the energy $E_{(m)}$ of the matter in the black hole exterior.

It also follows from (5.13) that the energy $E_{(m)}$ can be found by studying the back-reaction effects caused by excitations of matter fields. To this aim, one has to find the variation of the black hole mass at spatial infinity and determine the variation of the geometry near the black hole horizon. To put it in another way: the energy $E_{(m)}$ connects the change of the black hole mass $M$ with the change of the proper black hole entropy, i.e. the quantity at spatial infinity with the quantity at the horizon.

It should be emphasized that in derivation of (5.15) only equations of motion for the metric were used. The matter fields were assumed to be weak but arbitrary. Thus, Eq. (5.13) is a generalization of the first law (2.19) to the case where matter fields are off shell.

6 Rotating black hole space-times

We now comment on a generalization of our results to space-times $M$ with axial symmetry and to the case of rotating black holes in particular. Let $\varphi^{\mu}$ be a Killing vector which generates rotations. A conserved charge associated to the rotational symmetry is the angular momentum of the system. In the analogy with the Hamiltonian and the energy it is possible to give two different definitions of the angular momentum of matter fields on an external background. We consider the canonical angular momentum $J^{C}_{(m)}$

$$J^{C}_{(m)} = \int_{\Sigma_t} (\theta^{\mu}_{(m)}(\varphi) - \varphi^{\mu} L_{(m)}) d\Sigma_{\mu} \ ,$$  

and the angular momentum $J^{E}_{(m)}$ defined in terms of the stress-energy tensor

$$J^{E}_{(m)} = - \int_{\Sigma_t} T^{\mu\nu} \varphi_{\mu} d\Sigma_{\nu} \ .$$  

Note that the form $\theta_{(m)}(\varphi)$ depends only on the variations $L_{\phi}\phi$ of the matter fields. Variations of the background metric vanish $L_{\phi}g_{\mu\nu} = 0$. It follows from (3.12) that the difference between two angular momenta is determined by the Noether current $J^{(m)}_{(\varphi)}$ associated to the Killing vector $\varphi^{\mu}$

$$J^{C}_{(m)} - J^{E}_{(m)} = - \int_{\Sigma_t} J^{(m)}_{(\varphi)}(\varphi) d\Sigma_{\mu} \ .$$  

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For a space-time without horizons the canonical angular momentum $J^C_m$ is the generator of rotations along the vector $\varphi^\mu$. The proof of this fact is analogous to the proof that $\mathcal{H}_m$ generates time translations, see Section 4.

When $\mathcal{M}$ is a black hole space-time additional comments are in order. The black hole horizon is determined as a region where the Killing vector $\xi^\mu = t^\mu + \Omega_H \varphi^\mu$ is null. (Here $\Omega_H$ is the angular velocity of the horizon). The bifurcation surface $\Sigma$ is a region where $\xi^\mu = 0$. The black hole horizon is inside of the static limit surface $S_{st}$ on which the Killing vector $t^\mu$ is null. $S_{st}$ is the boundary of the ergosphere where $t^\mu$ is space-like. Inside the ergosphere $\mathcal{H}_m$ and $\mathcal{E}_m$ cannot be interpreted as an energy. For this reason, instead of these quantities it is more appropriate to consider the conserved charges corresponding to the vector $\xi^\mu$

$$G^C_m = -\int_{\Sigma} \left( \theta^\mu_m(\xi) - \xi^\mu L_m \right) d\Sigma = \mathcal{H}_m - \Omega_H J^C_m ,$$  \hspace{1cm} (6.4)

$$G^E_m = \int_{\Sigma} T^{\mu\nu} \xi_\nu d\Sigma = \mathcal{E}_m - \Omega_H J^E_m .$$ \hspace{1cm} (6.5)

The quantity $G^C_m$ is the generator of canonical transformations along the Killing field $\xi^\mu$ and by repeating arguments of Section 4 one proves that for a matter field $\phi$ on the black hole exterior

$$\mathcal{L}_\xi \phi(x) = (G^C_m, \phi(x)) \hspace{1cm} (6.6)$$

Generalization of relation (3.24) to the case of rotating black holes is

$$G^C_m - G^E_m = (\mathcal{H}_m - \mathcal{E}_m) - \Omega_H (J^C_m - J^E_m) = \frac{\kappa}{2\pi} Q_m .$$ \hspace{1cm} (6.7)

When the fields vanish on the spatial boundary, the Noether charge $Q_m$ is given by Eq. (3.25).

Finally, one can find a generalization of the first law (5.15). By assuming that $C_\infty$ is everywhere tangent to $\varphi^\mu$ it is not difficult to show that

$$\delta M = \frac{\kappa}{2\pi} \delta S(g) + \Omega_H \delta J + \mathcal{E}_m - \Omega_H J^E_m .$$ \hspace{1cm} (6.8)

To prove this one has to repeat the analysis of Section 5 by replacing $\mathcal{E}_m$ by $G_m$ and $t^\mu$ by $\xi^\mu$. From $(6.8)$, we see that the angular momentum $J^E_m$ together with the energy is related to thermodynamical properties of a black hole.

7 Summary and discussion

We have shown that the two definitions of the energy of matter fields in the presence of a black hole correspond to different objects. The Hamiltonian $\mathcal{H}_m$ is the generator of the time evolution, while the energy $\mathcal{E}_m$ is the quantity which appears in the first law

\footnote{For a discussion of rotating black holes see Refs. [1] and [2].}
of black thermodynamics. The difference between $H_{(m)}$ and $E_{(m)}$ is the Noether charge $Q_{(m)}$ which is not zero when fields are non-minimally coupled. We derived a formula for $Q_{(m)}$ valid for an arbitrary diffeomorphism invariant theory and demonstrated its relation to the Noether charge introduced by Wald [6]. As a by product, we found out the first law of black hole thermodynamics in case of weak off-shell matter fields. This may be considered as a further development of the results obtained in Refs. [6], [7]. Equations (5.15) and (6.8) may be especially useful for studying black hole thermodynamics in the presence of quantum fields, i.e. with the renormalized quantum stress-energy tensor in the r.h.s. of (5.1).

Now a comment about another possible application of these results is in order. As we pointed out, non-minimally coupled fields are crucial for constructing ultraviolet finite models of induced gravity. Calculations of [3], [5] show that the entropy $S^{BH}$ of a static black hole in induced gravity is related to statistical-mechanical entropy $S^{SM}$ and the Noether charge of non-minimally coupled constituents by Eq. (1.7). This can be explained as follows [4]. The black hole entropy is connected with the spectrum of the black hole mass $M$. According to Eq. (5.15), if the geometry near the horizon is fixed the spectrum of $M$ is equivalent to the spectrum of the energy of matter fields. On the other hand, $S^{SM}$ is computed by using the canonical Hamiltonian. Thus, the entropies $S^{BH}$ and $S^{SM}$ are related to the different energies and for this reason they do not coincide.

Our results give a strong support to the above interpretation of formula (1.7). The results concern arbitrary diffeomorphism invariant theories and suggest that Eq. (1.7) is universal and does not depend on the choice of the concrete induced gravity model. If the model is ultraviolet finite $S^{BH}$ and $S^{SM}$ always differ by the Noether charge of non-minimally coupled constituents. Moreover, by taking into account the results of Section 6, we may speculate that (1.7) holds for rotating black holes as well. It would be interesting to check this hypothesis by computations.

Acknowledgements: I am very grateful to Valeri Frolov for helpful discussions and for the hospitality during my stay in the University of Alberta, Canada.
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