Conservation laws in generalized Riemann-Silberstein electrodynamics

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Starting from positive and negative helicity Maxwell equations expressed in Riemann-Silberstein vectors, we derive the ten usual and ten additional Poincaré invariants, the latter being related to the electromagnetic spin, *i.e.*, the intrinsic rotation, or state of polarization, of the electromagnetic fields. Some of these invariants have apparently not been discussed in the literature before.

The electromagnetic angular momentum and its relation to photon spin have been much debated [1, 2, 3, 4]. To help resolve this issue, we have derived conservation laws for Maxwell electrodynamics by using a generalized Riemann-Silberstein (RS) vector formalism [5, 6] that incorporates optical coherence theory [7, 8]. In terms of RS vectors \(G_\pm = E \pm iB\), where \(E, B\in \mathbb{C}^3\) are typically represented in a complex helical base, the Maxwell equations are

\[
\nabla \cdot G_\pm = \frac{\rho}{\varepsilon_0},
\]

\[
\nabla \times G_\pm = \pm i \left( \frac{1}{c} \frac{\partial G_\pm}{\partial t} + Z_0 j \right),
\]

where \(\rho, j \in \mathbb{C}^3\) are the charge and current densities, respectively, and \(Z_0 = \sqrt{\mu_0/\varepsilon_0}\) is the vacuum impedance. Defining the electromagnetic energy and momentum densities as \(H_\pm = \varepsilon_0 G_\pm \cdot G_\pm/2\) and \(K_\pm = \mp i0 \Im \{ G_\pm \times G_\pm^* \}/(2c)\), respectively, one can easily derive the conservation laws

\[
\frac{\partial H_\pm}{\partial t} + c^2 \nabla \cdot K_\pm + \Re \{ j \cdot G_\pm^* \} = 0,
\]

\[
\frac{\partial K_\pm}{\partial t} + \nabla \cdot T_\pm + f_{RS} = 0,
\]

where \(f_{RS} = \Re \{ c\rho G_\pm^* \pm ij \times G_\pm^* \}/c\) are electromechanical force densities, coined Riemann-Silberstein force densities, and the stress tensors are \(T_\pm = \partial^2 H_\pm - \varepsilon_0 \Re \{ G_\pm^* G_\pm^* \}/c\).

Introducing the field energy density \(u = \varepsilon_0 (E \cdot E + c^2 B \cdot B^*)/2\); the linear momentum density \(p = \varepsilon_0 \Re \{ E \times B^* \}\); the Maxwell stress tensor \(T = (T_+ + T_-)/2\); the Lorentz force density \(f_{\text{Lorentz}} = \Re \{ p E^* + j B^* \}\); the spin energy density \(w = \Im \{ (E \cdot B^*)^* \}/Z_0\); the spin momentum density \(V = -\varepsilon_0 \Im \{ (E \times E^* + c^2 B \times B^*)^* \}/(2c)\); the spin force density \(f_{\text{spin}} = \Im \{ cB^* \times j E^*/c\}\); and the spin stress tensor \(U = (T_+ - T_-)/2\), summing and subtracting the two conservation laws in \(\ref{eq:conservation}\), and summing and subtracting the two conservation laws in \(\ref{eq:conservation}\), we obtain the following conservation laws

\[
\frac{\partial u}{\partial t} + c^2 V \cdot p + \Re \{ j \cdot E^* \} = 0,
\]

\[
\frac{\partial p}{\partial t} + \nabla \cdot T + f_{\text{Lorentz}} = 0,
\]

\[
\frac{\partial w}{\partial t} + \nabla \cdot V + \Im \{ j \cdot B^* \} = 0,
\]

\[
\frac{\partial V}{\partial t} + \nabla \cdot U + f_{\text{spin}} = 0,
\]

respectively [9, 10, 11]. The conservation laws \(\ref{eq:conservation}\) and \(\ref{eq:conservation}\) involving the spin momentum density pseudovector \(V\), which is a three-dimensional generalization of the standard 2D Stokes parameter \(V\) [11, 12], seem to have been long unnoticed until now. Note that for linearly polarized fields, \(E, B \in \mathbb{R}^3\) and \(V = 0\). For circularly polarized fields \(E = \pm iB\) and \(p = \pm V\).

If the field is monochromatic, then the spin angular momentum (SAM) density is \(s = cV/\omega\) and the electromagnetic spin torque density is \(\tau_{\text{spin}} = cf_{\text{spin}}/\omega\) [13, 14, 15].

Using the conservation laws Eqs. \(\ref{eq:conservation}\) and \(\ref{eq:conservation}\), the energy and momentum can be written as \(H_\pm = u \pm V, K_\pm = p \pm V\) and \(f_{RS} = f_{\text{Lorentz}} \pm f_{\text{spin}}\). This elucidates the difference in handedness between \(G_\pm\) and \(G_{\mp}\), which are linearly independent under Lie transformations [16]. We interpret \(G_\mp\) as wave fields of positive and negative helicity \(\chi = V \cdot p/|p|^2\), manifesting the well-known fact that left and right handed polarized fields represent two different helicities [17, 18].

The interpretation of the RS vector as a photon wave function was suggested by many authors [17, 18, 19, 20]. However, for \(E, B \in \mathbb{R}^3\), the wave functions \(G_\pm\) collapse since in that case \(G_+ = G_-\). This special case has been studied [17, 21, 22, 23, 24, 25] in relation to RS vortices, which are solutions to \((E + icB)^2 = 0\), referred to as vortex lines.

Let us consider a right-hand circularly polarized wave impinging upon two different optical elements, as depicted in Fig. \(\ref{fig:diagram}\) (a) a \(\lambda/2\) plate, and (b) a reflector. According to Eq. \(\ref{eq:conservation}\), the Lorentz force interaction corresponds to a change in EM linear momentum density, \(\Delta p\), and Eq. \(\ref{eq:conservation}\) yields a spin force interaction corresponding to the change in SAM density, \(\Delta s\). The interaction with the \(\lambda/2\) plate is an example of a non-Lie transformation; the corresponding Jones matrix [20].
of this, we propose that Beth’s experiments \cite{14} be repeated. This example corresponds to the Beth experiment \cite{14}. From their sum, one obtains the well-known EM field angular momentum conservation law

$$\frac{\partial}{\partial t}(r \times K_{\pm}) + \nabla \cdot (r \times T_{\pm}) + r \times f_{\text{RS}} = 0. \tag{9}$$

From their sum, one obtains the well-known EM field angular momentum conservation law

$$\frac{\partial}{\partial t} J + \nabla \cdot M + r \times f_{\text{Lorentz}} = 0, \tag{10}$$

where $J = r \times p$ is the angular momentum density, and $M = r \times T$ is the angular momentum flux tensor \cite{27}. By taking the difference of the two equations in Eq. (9), and introducing $N = r \times V$, which we interpret as the spin-orbit angular momentum (SOAM) density, and $O = r \times U$, which we interpret as the SOAM flux tensor, we obtain the conservation law

$$\frac{\partial N}{\partial t} + \nabla \cdot O + r \times f_{\text{spin}} = 0. \tag{11}$$

which, to the best of our knowledge, has not been derived before. As in solid body mechanics, the spin angular momentum can be viewed as the intrinsic rotation of the fields and the orbital angular momentum (OAM) as their precession. The SOAM would then correspond to their nutation.

FIG. 1: Right-hand circularly polarized light, depicted by spirals, impinging from the left onto (a) a $\lambda/2$ plate, (b) a reflector. The $\mathbf{p}$ arrows indicate the changes in linear momentum and the $\mathbf{V} = k \mathbf{s}$ arrows indicate the spin angular momentum changes due to the interactions.

In analogy with Eq. (3.30) in Ref. \cite{27}, one can derive the virial theorem

$$\frac{\partial (r \cdot K_{\pm})}{\partial t} + \nabla \cdot (r \cdot T_{\pm}) - H_{\pm} + r \cdot f_{\text{RS}} = 0. \tag{12}$$

Summation and subtraction of the two helicity components yield

$$\frac{\partial (r \cdot p)}{\partial t} + \nabla \cdot (r \cdot T) - u + r \cdot f_{\text{Lorentz}} = 0, \tag{13}$$

$$\frac{\partial (r \cdot V)}{\partial t} + \nabla \cdot (r \cdot U) - w + r \cdot f_{\text{spin}} = 0. \tag{14}$$

A commonly accepted interpretation of Beth’s experimental results \cite{14} was given in Ref. \cite{28} where it is argued that the finite extent of the waveplate leads to sharp intensity gradients, and thus strong parallel field components, attributed to a non-vanishing angular momentum \cite{29}. However, a boundary effect of this kind would be geometry dependent, which is physically unsatisfactory. E.g., in Feynman’s example of circularly polarized light interacting with a free atom \cite{30}, it is difficult to even define a boundary. Yet, an absorption is followed by an emission of light with unchanged polarization, just as in our reflector example in Fig. 1b. Another example is radio, where wave polarization can be measured in a single point with an infinitesimally small antenna. Hence, SAM can be detected even though the sensor is much smaller than the wavelength. In the model presented here, the result can be explained as a transfer of SAM. So far, one has not been able to separate spin angular momentum and orbital angular momentum other than for beam geometries \cite{31}. But Eqs. (8) and (10) show that, classically, angular momentum and SAM are conserved independently of each other, so that the separation is indeed always possible. An alternative method of separation is to directly make use of $G_{\pm}$, where the spin is embedded. The only separation is then with respect to helicity $\chi = \pm 1$. Thereafter, electromagnetic energy, momentum, and angular momentum can be unambiguously defined through their respective conservation laws, Eqs. (5), (4), and (7).

The remaining three Poincaré invariants are contained in the center of energy (CE) vector \cite{32}. Two CE conservation laws for positive and negative helicity fields can be derived by multiplying Eqs. (3) with $r$ and (4) with $c^2 t$, which yields

$$\frac{\partial}{\partial t} \left( rH_{\pm} - c^2 rK_{\pm} \right) + c^2 \nabla \cdot (rK_{\pm} - rT_{\pm})$$

$$= \Re \left\{ \frac{i \mathbf{p}}{\varepsilon_0} \mathbf{G}^*_{\pm} + i \mathbf{z}_0 j \times \mathbf{G}^*_{\pm} - \mu_0 r (\mathbf{j} \cdot \mathbf{G}^*_{\pm}) \right\}. \tag{15}$$

In vacuum, expressions for the energy and momentum propagation velocities can be derived \cite{27}. Assuming them equal, it follows that both right- and left-handed photons propagate with the speed of light, $c$. By forming linear combinations of
the CE conservation laws in Eq. (15), the field and spin CE conservation laws in vacuum are found to be

$$\frac{\partial}{\partial t} (ru - c^2 ip) + c^2 \nabla \cdot (rp - e T) = 0, \quad (16)$$

$$\frac{\partial}{\partial t} (rv - c^2 qv) + c^2 \nabla \cdot (rv - e U) = 0. \quad (17)$$

As demonstrated above, using the framework of generalized RS electrodynamics we have been able to derive all Poincaré invariants. However, there exist other quadratic forms of the RS fields that should be mentioned. Reactive but Lorentz invariant observables, obeying non-conservation laws [27], can be derived by examining forms which are quadratic in $G_\pm$ and $G_\mp$. The Lorentz scalars $e_0 (E \cdot E' - c^2 B \cdot B')/2$ and Re $\{E \cdot B'\}/Z_0$, and the imaginary part of the complex linear momentum vector $e_0 \text{Im} \{E \times B'\}$ are important examples. Similarly, “instantaneous” quantities, conserved and non-conserved, can be derived by considering forms quadratic in $G_\pm$, $G_\mp$, and $G_\pm$, $G_\mp$, respectively. The theory can be generalized to incorporate a magnetic charge density, $\rho_m$, and current density, $j_m$, by introducing $\rho_\pm = \rho_e \pm i \rho_m/c$ and $j_\pm = j_e \pm i j_m/c$ [33], sometimes referred to as the Beltrami charge and current densities [34].

In conclusion, based on the assumption of an analytic continuation of the fields so that $E, B \in \mathbb{C}^3$, we have introduced generalized RS vectors $G_\pm$ that are interpreted as wave functions describing fields of positive and negative helicity. This has allowed us to derive the two sets of Poincaré invariants $\{H_\pm, K_\pm, r \times K_\pm, (r i H_\pm - c^2 r K_\pm)\}$ and their associated conservation laws, Eqs. (3), (5), (9), and (15), respectively. All well-known EM observables are contained in these sets as linear combinations of the two versions, but some are less well known or seem to have gone unnoticed. Eq. (7), the spin-energy equivalent to the Poynting theorem Eq. (5), are among these. The SOAM conservation law, Eq. (11), and the spin CE conservation law Eq. (17), as well as the spin virial theorem Eq. (14), are, to the best of our knowledge, given here for the first time.

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