Weyl symmetry for curve counting invariants via spherical twists

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Gromov-Witten invariants

Given a smooth projective variety $X$, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X, \beta)]^{vir} \in A_{virdim}(M_g(X, \beta)).$$

A special case is when $X$ is a Calabi-Yau 3-fold (CY3): the virtual dimension of is 0 for all $g \geq 0$, $\beta \in H_2(X; \mathbb{Z})$ so we get numbers

$$GW^X_{g, \beta} = \int_{[M_g(X, \beta)]^{vir}} 1 \in \mathbb{Q}.$$ 

Goal:

Compute all numbers $GW^X_{g, \beta}$. Equivalently, understand the partition function

$$Z_X = \exp \left( \sum_{g, \beta} GW^X_{g, \beta} u^{2g-2} z^{\beta} \right).$$
Stable pairs

Stable pairs provide an alternative approach to curve counting on CY3.

**Definition (Pandharipande-Thomas ’09)**

A stable pair on $X$ is an object $\{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where $F$ is a coherent sheaf and $s$ a section satisfying the following two stability conditions:

1. $F$ is pure of dimension 1: every non-trivial coherent sub-sheaf of $F$ has dimension 1.
2. The cokernel of $s$ has dimension 0.

We associate two discrete invariants:

$$\beta = [\text{supp}(F)] \in H_2(X; \mathbb{Z}) \text{ and } n = \chi(X, F).$$

The space $P_n(X, \beta)$ parametrizing stable pairs with fixed discrete invariants is a projective fine moduli space.
Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X, \beta)$ also has a virtual fundamental class, and when $X$ is a CY3 its virtual dimension is 0, producing again numbers

\[ \text{PT}_n^X = \int_{[P_n(X, \beta)]^\text{vir}} 1 \in \mathbb{Z}. \]

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande ’06)

The Gromov-Witten and Pandharipande-Thomas invariants determine each other:

\[ \exp \left( \sum_{g, \beta} \text{GW}_{g, \beta}^X u^{2g-2} z^\beta \right) = \sum_{n, \beta} \text{PT}_n^X (-q)^n z^\beta \]

after the change of variables $q = e^{iu}$. 
Rationality and symmetry

To even make sense of the change of variables $q = e^{iu}$ an important structural result is required:

**Theorem (Bridgeland ’16)**

For each $\beta$ the generating function

$$\sum_{n \in \mathbb{Z}} PT_{n, \beta}^X (-q)^n$$

is the expansion of a rational function $f_\beta$ satisfying the symmetry

$$f_\beta(1/q) = f_\beta(q).$$

Typical example (contribution of isolated rational curve):

$$f(q) = \frac{q}{(1 - q)^2}.$$
Proof of rationality

The proof of rationality illustrates a very general principle:

Symmetry of the derived category \( \phi \in \text{Aut}(D^b(X)) \)

\[ \Downarrow \]

Constraints on curve counting on \( X \).

The proof of rationality uses the derived dual

\[ \phi = \mathbb{D} = \text{RHom}(\cdot, \mathcal{O}_X)[2]. \]

Basic idea: use wall-crossing in the derived category to relate

\[ P_n(X, \beta) \leftrightarrow \phi(P_n(X, \beta)) \subseteq D^b(X). \]
Let $Y$ be a Calabi-Yau 3-fold containing a smooth divisor $E \subseteq Y$ isomorphic to a Hirzebruch surface (so $E$ is a $\mathbb{P}^1$ bundle $E \to C = \mathbb{P}^1$). Let $B = [\mathbb{P}^1] \in H_2(Y; \mathbb{Z})$ be the curve class of the fibers of $E \to C$.

(Key examples: $Y = K_E$, $Y$ elliptic fibration over $E$, $Y = STU$)
A key source of examples are elliptic fibrations (with section) over Hirzebruch surface $E$. Let $\pi : Y \to E$ be the fibration and $F$ the fiber class. Each fiber $\pi^{-1}(B)$ is a $K3$ surface. The monodromy of $K3$ implies the symmetry

$$GW^Y_{g,hF+iB} = GW^Y_{g,hF+(h-i)B}.$$ 

For more general $\beta$, our work is about some symmetry relating

$$GW^Y_{g,\beta} \sim GW^Y_{g,\beta'},$$

where $\beta' = \beta + (E \cdot \beta)B$ (note that $\beta \mapsto \beta'$ is an involution since $E \cdot B = -2$).
Weyl symmetry for PT invariants

Let

$$PT_\beta(q, Q) = \sum_{n,j \in \mathbb{Z}} P_{n,\beta+jB} (-q)^n Q^j.$$  

The generating series $PT_0$ of multiples of $B$ is computed (for example via the topological vertex) as

$$PT_0(q, Q) = \prod_{j \geq 1} (1 - q^j Q)^{-2j}.$$
Weyl symmetry for PT invariants

Theorem (Buelles-M. ’21)

Let $Y$ be a Calabi-Yau 3-fold containing a smooth divisor $E$ isomorphic to a Hirzebruch surface and satisfying a few assumptions (to explain later). Then

$$\frac{\text{PT}_\beta(q, Q)}{\text{PT}_0(q, Q)} \in \mathbb{Q}(q, Q)$$

is the expansion of a rational function $f_\beta(q, Q)$ which satisfies the functional equations

$$f_\beta(q^{-1}, Q) = f_\beta(q, Q) \text{ and } f_\beta(q, Q^{-1}) = Q^{-E \cdot \beta} f_\beta(q, Q).$$
Corollary

For all \((g, \beta) \neq (0, mB), (1, mB)\) the series

\[
\sum_{j \in \mathbb{Z}} \text{GW}_{g, \beta + jB} Q^j
\]

is the expansion of a rational function \(f_\beta(Q)\) with functional equation

\[
f_\beta(Q^{-1}) = Q^{-E \cdot \beta} f_\beta(Q).
\]

Predicted by physics, at least in the local case \(K_E\) (Katz-Klemm-Vafa '97).

If \(f_\beta\) were a Laurent polynomial (as in the case of \(K3\) classes), the functional equation means symmetry holds on the nose

\[
\text{GW}^Y_{g, \beta} = \text{GW}^Y_{g, \beta'}.
\]
Assumptions on $Y$

Our proofs at the moment assume the following:

- The curve $B$ generates an extremal ray in the cone of curves of $Y$. I.e. there is a nef divisor $A$ such that
  \[
  \ker \left( A_1(Y) \xrightarrow{A \cdot} \mathbb{Q} \right) = \mathbb{Q} \cdot B.
  \]
  Holds for any elliptic fibration.

- $-K_E$ is nef, i.e. $E \cong \mathbb{F}_r$ with $r = 0, 1, 2$ (probably not really necessary).

- For the Gromov-Witten corollary we assume the GW/PT correspondence holds.
Example

Let $Y = \mathbb{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$ and let $C$ be the other $\mathbb{P}^1$ in the product. A computation with the topological vertex shows:

$$\frac{\text{PT}_C(q, Q)}{\text{PT}_0(q, Q)} = \frac{2q}{(1-q)^2(1-Q)^2}$$

$$\frac{\text{PT}_{2C}(q, Q)}{\text{PT}_0(q, Q)} = \frac{2q^4}{(1-q)^2(1-q^2)^2(1-qQ)^2(1-Q)^2} + \frac{2q^4}{(1-q)^2(1-q^2)^2(q-Q)^2(1-Q)^2} + \frac{2q^4}{(1-q)^4(1-qQ)^2(q-Q)^2}.$$
Spherical twists

The main ingredient of our symmetry is the existence of a certain anti-equivalence \( \rho \in \text{Aut}(D^b(Y)) \) promoting the involution

\[
\beta \mapsto \beta' = \beta + (E \cdot \beta)B
\]
on \( H_2(Y; \mathbb{Z}) \) to the derived category. Its construction uses spherical twists.

**Definition**

An object \( G \in D^b(Y) \) is a spherical object if

\[
\text{Ext}^i(G, G) = \begin{cases} 
\mathbb{C} & \text{if } i = 0, 3 \\
0 & \text{otherwise}
\end{cases}
\]

Given a spherical object \( G \), Seidel-Thomas define a spherical twist \( \text{ST}_G \in \text{Aut}(D^b(Y)) \) by the exact triangle

\[
\bigoplus_i \text{Ext}^i(F, G) \otimes G[-i] \to F \to \text{ST}_G(F).
\]
Denote by $C \subseteq E \subseteq Y$ the class of one of the sections of the projection $E \to C$. For every $k \in \mathbb{Z}$,

$$\mathcal{O}_E(-C + kB) \in D^b(Y)$$

is a spherical object.

**Definition**

Let

$$\rho = 
\mathbb{D} \circ \text{ST}_{O_E(-C+kB)} \circ \text{ST}_{O_E(-C+(k+1)B)} \in \text{Aut}(D^b(Y)).$$

(the definition doesn't depend on $k$)
Properties of $\rho$

1. $\rho$ is an involution, i.e. $\rho \circ \rho = \text{id}$.
2. $\rho(\mathcal{O}_Y) = \mathcal{O}_Y[2]$.
3. If $F$ is supported away from $E$ then $\rho(F) = \mathbb{D}(F)$.
4. $\rho(\mathcal{O}_B(-2)) = \mathcal{O}_B(-2)[1]$ and $\rho(\mathcal{O}_B(-1)) = \mathcal{O}_B(-1)[-1]$.
5. If $F$ is a sheaf of dimension 1 and $\text{ch}_2(F) = \beta, \chi(F) = n$ then

$$\text{ch}_2(\rho(F)) = \beta + (E \cdot \beta)B$$

$$\chi(\rho(F)) = -n.$$
Orbifold inspiration

When $Y$ arises as a crepant resolution $Y \to X$ of an orbifold with $\mathbb{Z}/2$-singularities along a $\mathbb{P}^1$ so that $E$ is the exceptional divisor (and the fibers $B$ are contracted to points), the main result is a consequence of the DT crepant resolution conjecture proven by Beentjes-Calabrese-Rennemo ('18).

![Diagram showing orbifold inspiration](image-url)
Their proof uses $\mathbb{D}^\mathcal{X}$ to prove the symmetry of PT invariants in $\mathcal{X}$.

**Proposition**

*Under the McKay correspondence*

$$\Phi : D^b(Y) \xrightarrow{\sim} D^b(\mathcal{X})$$

the derived dual $\mathbb{D}^\mathcal{X}$ corresponds to $\rho$, i.e.

$$\rho = \Phi^{-1} \circ \mathbb{D}^\mathcal{X} \circ \Phi.$$

Important examples (e.g. the STU) don’t arise as such crepant resolution.
Stable pairs are equivalently described as follows:

**Proposition**

Let $I \in \langle \mathcal{O}_Y[1], \text{Coh}_{\leq 1} \rangle_{\text{ex}}$. Then $I$ is a stable pair if and only if $\text{rk}(I) = -1$ and

\[
\text{Hom}(\text{Coh}_0(Y), I) = 0 = \text{Hom}(I, \text{Coh}_1(Y)).
\]

Bridgeland's proof of rationality with the derived dual uses

\[
\mathbb{D}(\text{Coh}_1(Y)) = \text{Coh}_1(Y) \quad \text{and} \quad \mathbb{D}(\text{Coh}_0(Y)) = \text{Coh}_0(Y)[-1].
\]

Gives description of the image of $\mathbb{D}(P_n(X, \beta))$ and helps finding wall-crossing back to $P_n(X, \beta)$.
The derived equivalence $\rho$ doesn’t respect $\text{Coh}(Y)$ and the dimension filtration so well.

**Example**

If $x \in E$ is a point in the divisor lying in a fiber $B$ then

$$\rho(\mathcal{O}_x) = \{ \mathcal{O}_B(-1)[-1] \to \mathcal{O}_B(-2) \}.$$  

We use instead a tilting of $\text{Coh}(Y)$.

$$\mathcal{T} = \{ T \in \text{Coh}(Y) : R^1 p_* T|_E = 0 \}$$

$$\mathcal{F} = \{ F \in \text{Coh}(Y) : \text{Hom}(\mathcal{T}, F) = 0 \}$$

$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\text{ex}}.$$  

$\mathcal{A}$ is a heart of $D^b(Y)$. 
Dimension filtration

Together with $\mathcal{A}$ comes a modified dimension defined by:

$$\dim(F) = \max\{\dim(\text{supp}(F|_{Y\setminus E})), \dim(\rho(\text{supp}(F|_{E})))\}$$

The modified dimension is used to define $\mathcal{A}_0, \mathcal{A}_1$ which are analogous to $\text{Coh}_0(Y), \text{Coh}_1(Y)$:

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$

**Example**

1. $\text{Coh}_0 \subseteq \mathcal{A}_0$;
2. $\mathcal{O}_B(-1), \mathcal{O}_B(-2)[1] \in \mathcal{A}_0$;
3. If $F \in \text{Coh}_1(Y)$ and $F|_{E}$ is 0-dimensional then $F \in \mathcal{A}_1$;
4. $\mathcal{O}_E(-C), \mathcal{O}_E(-2C)[1] \in \mathcal{A}_1$. 
Perverse stable pairs

**Definition**

A perverse stable pair is an object \( I \in \langle \mathcal{O}_Y[1], \mathcal{A}_{\leq 1} \rangle_{\text{ex}} \) such that \( \text{rk}(I) = -1 \) and

\[
\text{Hom}(\mathcal{A}_0, I) = 0 = \text{Hom}(I, \mathcal{A}_1).
\]

We define the virtual counts of perverse stable pairs: for

\[
\gamma = (\beta, \ell[E]) \in H_2(Y) \oplus \mathbb{Z} \cdot [E]
\]

we have

\[
p\PT_{n, \gamma} \in \mathbb{Z},
\]

\[
p\PT_{\gamma}(q, Q) = \sum_{n, j \in \mathbb{Z}} p\PT_{n, \gamma + jB} (-q)^n Q^j.
\]
Rationality for $p_{PT}$

**Theorem**

The series $p_{PT,\gamma}(q, Q)$ is the expansion of a rational function $f_\gamma \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\gamma(q^{-1}, Q^{-1}) = Q^{-E \cdot \beta + 2\ell} f_\gamma(q, Q).$$

- Rationality of $PT_\beta(q)$
- Anti-equivalence $\mathbb{D}$
- Torsion pair $\langle \text{Coh}_0, \text{Coh}_1 \rangle$
- Usual slope stability
- Vanishing of Poisson brackets
  $$\{ \text{Coh}_{\leq 1}, \text{Coh}_{\leq 1} \} = 0$$
- Rationality of $p_{PT,\gamma}(q, Q)$
- Anti-equivalence $\rho$
- Torsion pair $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$
- Nironi slope stability
- No vanishing, extra combinatorial difficulty (dealt with in [BCR]).
We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.

**Proposition**

*For any $\beta \in H_2(Y; \mathbb{Z})$ we have the following identity of rational functions:*

$$p^{PT}_\beta(q, Q) = \frac{PT_\beta(q, Q)}{PT_0(q, Q)}.$$  

The wall-crossing establishing the equality has two steps and uses the counting of a third type of objects: Bryan-Steinberg invariants.
When $Y$ arises as a crepant resolution $Y \to \mathcal{X}$, Bryan-Steinberg introduced ('12) invariants $BS_{n,\beta}$. Roughly speaking, they count sheaves + sections $\{O_X^s \to F\}$ but allowing the cokernel to have support on fibers of $B$.

They provide a natural interpretation for the quotient $PT_\beta/PT_0$ via a DT/PT type wall-crossing.

**Proposition**

$$BS_\beta(q, Q) \equiv \sum_{n,j \in \mathbb{Z}} BS_{n,\beta+jB}(-q)^n Q^j = \frac{PT_\beta(q, Q)}{PT_0(q, Q)}.$$ 

Unlike $pPT$, BS are defined using the heart $\text{Coh}(Y)$, no need to tilt.
Final step is comparing $p_{PT}$ and $BS$.

**Proposition**

We have the following identity of rational functions:

$$BS_{\beta}(q, Q) = p_{PT}(q, Q).$$

The identity above is strictly of rational functions, the coefficients are not the same on the nose. When we cross a wall in the path of stability conditions we change the direction in which we expand the same rational function.
Example

The rational function $\frac{1}{q - Q}$ can be expanded in two different ways:

$$
\frac{1}{q - Q} = \frac{q^{-1}}{1 - Qq^{-1}} = \sum_{i \geq 0} Q^i q^{1-i}
$$

$$
\frac{1}{q - Q} = -\frac{Q^{-1}}{1 - Q^{-1}q} = -\sum_{i \geq 0} Q^{1-i} q^i.
$$
Thank you!

\[ \text{PT} \xleftarrow{\text{quotient}} \text{BS} \xrightarrow{\text{re-expansion}} p\text{PT} \]

\[ \rho(p\text{PT}) \]