INTEGRABLE MODELS IN TWO-DIMENSIONAL DILATON GRAVITY

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We briefly present two-dimensional dilaton gravity from the point of view of integrable systems.

Recently, two-dimensional dilaton gravity (DG) models have been extensively investigated both from classical and quantum points of view because of their connection to string theory, dimensional reduced models, black holes, and gravitational collapse. (For short reviews and references see e.g. \cite{1,2}) In this talk I would like to illustrate another important aspect of DG: its relation to integrable systems. DG is described by the action

\[ S[\phi, g_{\mu\nu}] = \int d^2 x \sqrt{-g} U(\phi) R(g) + V(\phi) + W(\phi)(\nabla \phi)^2 + S_M[\phi, g_{\mu\nu}, f_i], \] (1)

where \( U, V, \) and \( W \) are arbitrary functions of the dilaton, \( R \) is the two-dimensional Ricci scalar, and \( S_M \) represents the contribution of matter fields \( f_i \) which include any field but the dilaton \( \phi \) and the graviton \( g_{\mu\nu} \). Classically we may always choose \( U(\phi) = \phi \) and locally set \( W(\phi) = 0 \) by a Weyl-rescaling of the metric so, for a given \( S_M \), Eq. (1) describes a family of models whose elements are identified by the choice of the dilatonic potential. For instance, when \( S_M = 0 \) (pure DG) \( V(\phi) = \text{const.} \) identifies the (pure) Callan-Giddings-Harvey-Strominger model (CGHS), \( V(\phi) = \phi \) identifies the Jackiw-Teitelboim model, and \( V(\phi) = 2/\sqrt{\phi} \) describes the two-dimensional sector of the four-dimensional spherically-symmetric Einstein gravity after having integrated on the two-sphere.

According to their integrability properties, DG models can be roughly divided in three classes: i) Completely Integrable Models, i.e. models that can be expressed in terms of free fields by a canonical transformation. Remarkable examples are the pure DG and the CGHS models; ii) Completely Solvable Models, i.e. models that cannot be analytically solved in terms of free fields but whose general solution is known. Two-dimensional effective generalized theory of 2+1 cylindrical gravity minimally coupled to a massless scalar field and DG with constant or linear dilatonic potential minimally coupled to massless Dirac fermions belong to this class; iii) Partially Integrable Models, i.e. models that
are integrable in a 0+1 dimensional sector only, namely after reduction to a
finite number of degrees of freedom. In this category we find, for example,
DG minimally coupled to massless Dirac fermions with arbitrary potential
and two-dimensional effective models describing uncharged black p-branes in
N dimensions.

Completely integrable models are of particular interest from the quantum
point of view. In this case one is able to quantize exactly the theory (in the
free-field representation) and, hopefully, to discuss quantization subtleties and
non-perturbative quantum effects. (See e.g. Refs. 2, 3, 8 for a review of the state
of the art of the CGHS model.) Often these models can be used to describe
black holes and/or gravitational collapse, so the quantization program is worth
exploring. In the remaining part of this contribution I will briefly illustrate
how far one can go in the quantization program for the simple case of pure
dG.

An immediate consequence of the complete integrability of pure DG is that
both the metric and the dilaton can be expressed in terms of a D'Alembert
field and of a local integral of motion independent of the coordinates. So,
using the gauge in which the free field is one of the coordinates, one finds
that all solutions depend on a single coordinate. This property is nothing
else than the generalization of the classical Birkhoff theorem. (For spherically-
symmetric Einstein gravity the “local integral of motion independent of the
coordinates” is just the Schwarzschild mass.) The reduction of the theory
to a finite-dimensional dynamical system signals that pure DG is actually a
topological theory. Hence, the model can be quantized using two alternative, a
priori non-equivalent, approaches. In the first approach the theory is quantized
by first reducing it to a dynamical system with a finite number of degrees of
freedom, namely using first the Birkhoff theorem and then the quantization
algorithm. Conversely, in the second approach the theory is quantized in the
full 1+1 sector and the topological nature of the system must be recovered a
posteriori (Quantum Birkhoff Theorem): 8

A further ambiguity in the quantization procedure is related to gauge invar-
ance. Indeed, due to the coordinate reparametrization invariance of the the-
ory, the standard operator quantization of the system can be implemented according to two different methods that are a priori non-equivalent – the *Dirac method* (quantization of the constraints followed by gauge fixing) and the *reduced canonical method* (classical gauge fixing followed by quantization in the reduced space).

\[
\begin{array}{ccc}
\text{Classical (Gauge) Theory} & \xrightarrow{\text{Gauge Fixing}} & \text{Classical Reduced System} \\
\text{Quantization Algorithm} & & \text{Quantization Algorithm} \\
\text{Quantum (Gauge) Theory} & \xrightarrow{\text{Gauge Fixing}} & \text{Quantum Physical Theory}
\end{array}
\]

(3)

It is really surprising that both diagrams can be closed and the equivalence of the different approaches proved. Let me briefly illustrate this point.

In the 0+1 approach the closure of the second diagram is obvious. Since the model is integrable we can solve the finite gauge transformations generated by the (single) constraint \(H = 0\) and find a maximal set of gauge-invariant canonical variables \(\{M, P_M, H, T\}\). Then \(T\) can be used to fix the gauge since its transformation properties for the gauge transformation imply that time defined by this variable covers once and only once the symplectic manifold. The quantization becomes trivial and both Dirac and reduced approaches lead to the same Hilbert space. This program has been implemented in detail in Ref. [9] for the case of spherically-symmetric Einstein gravity but can be easily generalized to an arbitrary \(V(\phi)\). The resulting Hilbert space is spanned by the eigenvectors of the (gauge invariant) “mass operator” \(M\).

Let us consider now the reduced quantization of the 1+1 theory. We may find a canonical chart \(\{M, \pi_M, \phi, \pi_\phi\}\) such that the ADM super-Hamiltonian and super-momentum constraints read

\[
\mathcal{H} = [N(\phi) - M] \pi_\phi \pi_M + [N(\phi) - M]^{-1} \phi' M', \quad \mathcal{P} = -\phi' \pi_\phi - M' \pi_M,
\]

where ‘ means differentiation w.r.t. the spatial coordinate \(x_1\). Eventually, the canonical action must be complemented by a boundary term at the spatial infinities of the form \(S_\partial = - \int dx_0 (M_+ \alpha_+ + M_- \alpha_-)\) where \(M_\pm \equiv M(x_0, x_1 = \pm \infty)\) and \(\alpha_\pm(x_0)\) parametrize the action at infinities. [1]

Solving the constraints \((\pi_\phi = 0, M' = 0)\) the effective Hamiltonian coincides with the boundary term and the Hilbert space is spanned by the eigenfunctions of \(M \equiv M(x_0)\) with eigenvalue \(m\). The Hilbert space coincides with the Hilbert space obtained in the 0+1 approach. This proves the equivalence of the 0+1 and 1+1 reduced methods of quantization.

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The equivalence between the 0+1 and the 1+1 Dirac methods can be proved using a canonical transformation to free fields. This has been done in detail in Ref. 8 for the (pure) CGHS model. In this case an explicit canonical transformation to free fields is known and the quantization is carried out by use of the standard Gupta-Bleuler method. This is possible because the constraints can be linearized, due to positivity conditions that are present in the model, leading to an anomaly-free quantum theory. Again, the only gauge invariant operators are the mass and its conjugate momentum and the vacuum must be labeled by the eigenvalue of the mass operator. So there are infinite vacua, differing by the eigenvalue of the mass, and one recovers the quantum mechanics of the previous approaches.

Acknowledgements

This work has been supported by a Human Capital and Mobility grant of the European Union, contract no. ERBFMRX-CT96-0012.

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