Synchronization in Complex Networks: a Reply on a recent Comment

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We clarify a number of points raised in [Matias, arXiv:cond-mat/0507471v2 (2005)].

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In a previous comment on cond-mat\textsuperscript{1}, two of our recent publications\textsuperscript{2,3} were compared with researches simultaneously carried out by the Potsdam group and published in\textsuperscript{4,5}. It was argued that the conclusions of both sets of papers are closely related. To obviate confusion, we will in the following point out a number of facts which will help to better understand the differences in the results and conclusions of the involved works.

1. The root for the improvement observed in\textsuperscript{4,5} is the normalization of the diagonal terms of the connectivity matrix. Such a normalization is limited in\textsuperscript{4,5} to the optimal case $\beta = 1$, whereas in our papers it is realized for all values of $\alpha$ and $\theta$. Furthermore, the optimal condition $\beta = 1$ in\textsuperscript{4,5} corresponds to $\alpha = 0$ in\textsuperscript{2} and to $\theta = 0$ in\textsuperscript{3}. Therefore, any further synchronizability enhancement observed, e.g., in Figs. 2.3 of\textsuperscript{2} and in Figs. 1.2 of\textsuperscript{3} cannot be related to any of the mechanisms reported in\textsuperscript{4,5}. It is therefore not appropriate to compare the enhancement of\textsuperscript{4,5} with respect to a non-weighted configuration to the further improvements that our original approaches provide. For instance, the very nice analytical study of\textsuperscript{4,5} (duly quoted by us in\textsuperscript{2}) is restricted to a series of explicit hypotheses that require a sufficiently ”random” configuration to be satisfied. Our studies manifestly show (see e.g. Fig. 1 and 2c of\textsuperscript{2}) that while for Erdös-Rényi graphs the maximal synchronizability is attained in the condition of\textsuperscript{4,5} (that is for $\theta = 0$), the interplay between hierarchy and weighting is able to further improve this value for other network’s configurations with the same property of constant total strength of input connections. This extra enhancement cannot be extracted by the arguments raised in\textsuperscript{4,5}.

2. A weighted wiring does not necessarily imply asymmetric connections between elements. For instance, the weighting procedure of\textsuperscript{4,5} (based on powers of the node degrees $k_i^\beta$) yields always (i.e. for all $\beta$) a symmetric connection between connectivity peers (as e.g. it can frequently occur in assortative networks). At variance, in\textsuperscript{2} we are able to modulate and direct the interaction between any pair of nodes by the parameter $\theta \in (-1,1)$ (from a unidirectional to a bidirectional symmetrical one), accordingly to the hierarchical (age) ordering of the nodes. The conclusion that a weighted configuration based on a hierarchical structure can enhance synchronizability is, therefore, original.

3. Our studies\textsuperscript{2,3} report a detailed analysis of the propensity for synchronization in parameter spaces. Namely, in\textsuperscript{2} (\textsuperscript{3}), we detail in Fig. 2 (Figs. 1,2) the behavior of $\lambda_N/\lambda_2$ ($\lambda_N^2/\lambda_2^2$) in the parameter space $[0,\beta]$ ($\theta$), showing that the enhancement due to a weighting procedure based on the link loads (the age ordering) is a generic feature in parameter space. In contrast to what was argued in\textsuperscript{1}, we in fact carefully considered the dependence of our results on the network size $N$ in both our papers. In particular, we show via Gershgorin’s circle theorem that the normalization procedure yields an upper bound for $\lambda_N$ for any network size and topology. Notice that, in\textsuperscript{4,5}, when the weight is selected to be $k_i^ \beta$, $\lambda_N$ is in general size dependent for $\beta < 0$, and one has $\lim_{N \to \infty} \lambda_N = \infty$.

4. The Author of\textsuperscript{1} is incorrect when stating that the main conclusion of\textsuperscript{2} is that the enhancement in synchronization is evident when the coupling direction is from the younger to the older nodes. Our Letter explicitly claims that one has to combine this property (relaxed also in\textsuperscript{4,5} for $\beta > 0$) with a structure of interconnected hubs induced by the hierarchical (age) ordering. The second part of our Letter (as well as Fig. 2c of\textsuperscript{2}), is entirely dedicated to study a series of ad-hoc modified networks with the aim of pointing out how such an interplay works in scale-free networks. Furthermore, it is straightforward to show that for any network size and topology one can construct a coupling scheme based solely on age ordering and able to give $\lambda_N/\lambda_2 = 1$ (the minimal as possible value). This can be realized by constructing a unidirectional tree structure, by means of which one
randomly selected node forces a cascade of nodes spanning the whole networks. Using properly normalized weights for each uni-directional connection (i.e. normalizing the connection strength to the in-degree of the slave node), and imposing a zero-row sum configuration induces a triangular matrix whose diagonal elements are \((0,1,1,1,1,1,1,\ldots,1)\), giving the following set of eigenvalues \(\lambda_1 = 0\), \(\lambda_i = 1, i = 2, \ldots N\).

5. It is well known that a crucial difference in topological information content exists between the node degree and the load of the link or edge betweennesses, the former retaining only information on the local structure of the network, the latter providing a measure of centrality of the node in the global topological structure. It is also well known (e.g. in the study of network’s community structures) that nodes with high degrees do not necessarily have links with high loads, and links with high loads do not necessarily connect two nodes with high degrees. An important and original result of our studies is therefore that (under the assumption of input normalization) the knowledge of global topological properties of the network can be used to improve the propensity of synchronization over the limit obtained in Refs. 4, 5, where only local properties were used. This improvement is not closely related to the ideas or conclusions of Refs. 4, 5.

In summary we have shown that the results and conclusions in Refs. 2, 3, 4 are independent and complementary. However, though from a completely different point of view than that suggested by 1, a relationship between these works indeed exists. Taken together, the approaches of 2, 3, 4, 5 have undoubtedly and independently established weighted networks as a promising framework for the further study of synchronized behavior. While the arguments provided in the present Manuscript demonstrate that it is erroneous trying to deduce the conclusions of one approach from the other, we believe that much effort will be stimulated in the forthcoming years to investigate other ad-hoc weighted configurations, for assessing more deeply the underlying mechanisms for the formation of collective dynamics in all those circumstances where the weighting features can be directly extracted from available data.

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