Pseudojump inversion in special r. b. $\Pi_1^0$ classes

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Abstract

The Jump Inversion Theorem says that for every real $A \geq_T 0'$ there is a real $B$ such that $A \equiv_T B' \equiv_T B \oplus 0'$. A known refinement of this theorem says that we can choose $B$ to be a member of any special $\Pi_1^0$ subclass of $\{0, 1\}^\mathbb{N}$. We now consider the possibility of analogous refinements of two other well-known theorems: the Join Theorem – for all reals $A$ and $Z$ such that $A \geq_T Z \oplus 0'$ and $Z >_T 0$, there is a real $B$ such that $A \equiv_T B' \equiv_T B \oplus 0' \equiv_T B \oplus Z$ – and the Pseudojump Inversion Theorem – for all reals $A \geq_T 0'$ and every $e \in \mathbb{N}$, there is a real $B$ such that $A \equiv_T B \oplus W_e^B \equiv_T B \oplus 0'$. We show that in these theorems, $B$ can be found in some special $\Pi_1^0$ subclasses of $\{0, 1\}^\mathbb{N}$ but not in others.
1 Introduction

We begin with a well-known theorem.

**Theorem 1.1** (Jump Inversion Theorem, due to Friedberg, see [11, Theorem 13.3.IX]). Given a real \( A \geq T' \) we can find a real \( B \) such that \( A \equiv_T B' \equiv_T B \oplus 0' \).

Here \( 0' \) denotes the Halting Problem, \( B' \) denotes the Turing jump of \( B \), and \( \equiv_T \) denotes Turing equivalence. There are two important related theorems, which read as follows.

**Theorem 1.2** (Join Theorem, due to Posner and Robinson [10, Theorem 1], see also [3, Theorem 2.1]). Given reals \( A \) and \( Z \) such that \( A \geq_T Z \oplus 0' \) and \( Z >_T 0 \), we can find a real \( B \) such that \( A \equiv_T B' \equiv_T B \oplus 0' \equiv_T B \oplus Z \).

**Theorem 1.3** (Pseudojump Inversion Theorem, due to Jockusch and Shore [3, Theorem 2.1]). Given a real \( A \geq_T 0' \) and an integer \( e \in \mathbb{N} \), we can find a real \( B \) such that \( A \equiv_T J_e(B) \equiv_T B \oplus 0' \).

Here \( J_e \) denotes the \( e \)th pseudojump operator, defined by \( J_e(B) = B \oplus W^B_e \), where \( W^B_e \) is a fixed standard recursive enumeration of the \( B \)-recursively enumerable subsets of \( \mathbb{N} \).

Our results in this paper concern special \( \Pi^0_1 \) classes. Following [4] we define a *special \( \Pi^0_1 \) class* to be a nonempty \( \Pi^0_1 \) subset of \( \mathbb{N}^\mathbb{N} \) with no recursive elements. Here \( \mathbb{N}^\mathbb{N} \) is the Baire space, but as in [4, 5] we focus on special \( \Pi^0_1 \) subclasses of \( \{0,1\}^\mathbb{N} \), the Cantor space. On the other hand, because of [13, Theorem 4.10] our results concerning special \( \Pi^0_1 \) subclasses of \( \{0,1\}^\mathbb{N} \) will also apply to *recursively bounded* special \( \Pi^0_1 \) subclasses of \( \mathbb{N}^\mathbb{N} \). Following [5] we use “r. b.” as an abbreviation for “recursively bounded.”

We now recall another known theorem, which says that we can find the \( B \) for Theorem 1.1 in any special \( \Pi^0_1 \) subclass of the Cantor space.
Theorem 1.4 (due to Jockusch and Soare [5, just after the proof of Theorem 2.1]). Let $P$ be a special $\Pi^0_1$ subclass of $\{0, 1\}^\omega$. Given a real $A \geq_T 0'$, we can find a real $B \in P$ such that $A \equiv_T B' \equiv_T B \oplus 0'$.

This refinement of Theorem 1.1 suggests a question as to the existence or nonexistence of analogous refinements of Theorems 1.2 and 1.3. In order to state our results concerning this question, we make the following definition.

Definition 1.5. Let $P \subseteq \mathbb{N}$ be a subclass of the Baire space.

1. $P$ has the Join Property if for all reals $A$ and $Z$ such that $A \geq_T Z \oplus 0'$ and $Z >_T 0$, there exists $B \in P$ such that $A \equiv_T B' \equiv_T B \oplus 0' \equiv_T B \oplus Z$.

2. $P$ has the Pseudojump Inversion Property if for all reals $A \geq_T 0'$ and all $e \in \mathbb{N}$, there exists $B \in P$ such that $A \equiv_T J_e(B) \equiv_T B \oplus 0'$.

The question that we are asking is, which special $\Pi^0_1$ subclasses of the Cantor space $\{0, 1\}^\omega$ have one or both of the properties in Definition 1.5? The purpose of this paper is to present some partial results in this direction, as follows. Let CPA be the special $\Pi^0_1$ subclass of $\{0, 1\}^\omega$ consisting of all complete, consistent extensions of Peano Arithmetic. We show that CPA has both the Join Property and the Pseudojump Inversion Property. More generally, we show that any special $\Pi^0_1$ class which is Turing degree isomorphic to CPA has both of these properties. For example, this holds for the $\Pi^0_1$ class CZF consisting of all complete, consistent extensions of Zermelo-Fraenkel Set Theory (assuming that CZF is nonempty), and for the $\Pi^0_1$ class DNR consisting of all $\{0, 1\}$-valued diagonally nonrecursive functions. On the other hand, let $P$ be a special $\Pi^0_1$ subclass of $\{0, 1\}^\omega$ which is of positive measure. Citing a theorem of Nies and an observation of Patrick Lutz, we note that $P$ need not have the Join Property, but $P$ and indeed any special $\Pi^0_1$ class which is Turing degree isomorphic to $P$ has the Pseudojump Inversion Property. Finally, we construct a special $\Pi^0_1$ subclass $Q \subseteq \{0, 1\}^\omega$ which has neither the Join Property nor the Pseudojump Inversion Property. We do not know whether there exists a special $\Pi^0_1$ subclass of $\{0, 1\}^\omega$ which has the Join Property but not the Pseudojump Inversion Property.

Here is an outline of the paper. In §2 we present the classical proofs of Theorems 1.2 and 1.3. In §3 we use the Gödel-Rosser Incompleteness Theorem [8] to prove that CPA has the Join Property and the Pseudojump Inversion Property. In §4 we present some results about Turing degree isomorphism of $\Pi^0_1$ classes, and we apply these results to show that any $\Pi^0_1$ class which is Turing degree isomorphic to CPA has the Join Property and the Pseudojump Inversion Property. In §5 we show that special $\Pi^0_1$ classes consisting of Martin-Löf random reals have the Pseudojump Inversion Property but not the Join Property. In §6 we use a priority argument to construct a special $\Pi^0_1$ subclass $Q \subseteq \{0, 1\}^\omega$ which has neither the Join Property nor the Pseudojump Inversion Property.

Remark 1.6. The lattice $\mathcal{E}_w$ of all Muchnik degrees of nonempty $\Pi^0_1$ subclasses of $\{0, 1\}^\omega$ is of great interest; see for instance the references in [19]. Therefore, it is natural to ask about Muchnik degrees in $\mathcal{E}_w$ corresponding to the properties
in Definition 1.5. Our results in this paper shed some light on this question. First, it is known [13, Theorem 6.8] that any $\Pi^0_1$ subclass of $\{0,1\}^N$ which is Muchnik equivalent to CPA – i.e., of Muchnik degree 1, where 1 denotes the top degree in $E_w$ – is Turing degree isomorphic to CPA. Therefore, our results in §4 below imply that all $\Pi^0_1$ classes of Muchnik degree 1 have both of the properties in Definition 1.5. Second, it is easy to see that each degree in $E_w$ contains a $\Pi^0_1$ subclass of $\{0,1\}^N$ which includes CPA and hence again has both properties. On the other hand, by §§5 below we have a nonzero degree $\tau \in E_w$ containing a special $\Pi^0_1$ subclass of $\{0,1\}^N$ which has one property but not the other. Moreover, in §6 below we construct a nonzero degree in $E_w$ containing a special $\Pi^0_1$ subclass of $\{0,1\}^N$ which does not have either property.

In the remainder of this section, we fix some notation and terminology.

We write $f : \subseteq A \to B$ to mean that $f$ is a partial function from $A$ to $B$, i.e., a function with domain $\subseteq A$ and range $\subseteq B$. For $a \in A$ we write $f(a)_\downarrow$ and say that $f(a)$ converges or $f(a)$ is defined, if $a \in$ the domain of $f$. Otherwise we write $f(a)_\uparrow$ and say that $f(a)$ diverges or $f(a)$ is undefined. For $a \in A$ and $f, g : \subseteq A \to B$, we write $f(a) \simeq g(a)$ to mean that either $(f(a)_\downarrow)$ and $(g(a)_\downarrow)$ and $f(a) = g(a))$ or $(f(a)_\uparrow)$ and $(g(a)_\uparrow)$. We write $f(a)_\downarrow = b$ to mean that $f(a)_\downarrow$ and $f(a) = b$.

The set of all natural numbers is $\mathbb{N} = \{0,1,2,\ldots\}$. For any set $A$ let $A^\mathbb{N}$ denote the set of all sequences $X : \mathbb{N} \to A$. Thus $\mathbb{N}^\mathbb{N}$ and $\{0,1\}^\mathbb{N}$ are the Baire space and the Cantor space respectively. The points $X \in \mathbb{N}^\mathbb{N}$ or $X \in \{0,1\}^\mathbb{N}$ are sometimes called reals. We sometimes identify $\{0,1\}^\mathbb{N}$ with the powerset of $\mathbb{N}$ in the usual manner. For $X, Y \in A^\mathbb{N}$ the join $X \uplus Y \in A^\mathbb{N}$ is given by $(X \uplus Y)(2n) = X(n)$, $(X \uplus Y)(2n + 1) = Y(n)$, and for $P, Q \subseteq A^\mathbb{N}$ we write $P \times Q = \{X \uplus Y \mid X \in P \land Y \in Q\}$. Note in particular that $P \times Q \subseteq \mathbb{N}^\mathbb{N}$ whenever $P, Q \subseteq \mathbb{N}^\mathbb{N}$, and $P \times Q \subseteq \{0,1\}^\mathbb{N}$ whenever $P, Q \subseteq \{0,1\}^\mathbb{N}$.

For any set $A$ let $A^*$ be the set of strings, i.e., finite sequences, of elements of $A$. For $a_0, \ldots, a_{m-1} \in A$ we let $\langle a_0, \ldots, a_{m-1} \rangle$ denote the string $\sigma \in A^*$ of length $m$ given by $\sigma(i) = a_i$ for all $i < m$. In this case we write $|\sigma| = m$, and for each $k \leq m$ we write $\sigma[k] = \langle a_0, \ldots, a_k \rangle$. We also write $A_m = \{\sigma \in A^* \mid |\sigma| = m\}$ and $A_\geq m = \{\sigma \in A^* \mid |\sigma| \geq m\}$ and $A_\leq m = \{\sigma \in A^* \mid |\sigma| < m\}$. For strings $\sigma = \langle a_0, \ldots, a_{m-1} \rangle$ and $\tau = \langle b_0, \ldots, b_{n-1} \rangle$, their concatenation is $\sigma \tau = \langle a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1} \rangle$. Note that $|\sigma \tau| = |\sigma| + |\tau|$. We say that $\sigma$ is an initial segment of $\tau$, or equivalently $\tau$ is an extension of $\sigma$, written $\sigma \subseteq \tau$, if $\tau|_{|\sigma|} = \sigma$, i.e., if $\tau = \sigma \tau$ for some string $\rho$. We write $\sigma \subset \tau$ to mean that $\sigma \subseteq \tau$ and $\sigma \neq \tau$. We say that we say $\sigma$ is an initial segment of $X \in A^\mathbb{N}$, or equivalently $X$ is an extension of $\sigma$, written $\sigma \subset X$, if $X|_{|\sigma|} = \sigma$, i.e., if $\sigma = \langle X(0), \ldots, X(|\sigma| - 1)\rangle$.

Fix a standard recursive enumeration $\langle \psi^{(k)}_e \mid e \in \mathbb{N} \rangle$ of the $k$-place partial recursive functions $\psi : \subseteq \mathbb{N}^k \to \mathbb{N}$. Here $e$ is called an index of $\psi^{(k)}_e$. Likewise, for $X \in \mathbb{N}^\mathbb{N}$ we have a standard recursive enumeration $\langle \psi^{(k),X}_e \mid e \in \mathbb{N} \rangle$ of the $k$-place partial functions $\psi : \subseteq \mathbb{N}^k \to \mathbb{N}$ which are partial recursive relative to $X$. Here $e$ is again called an index of $\psi^{(k),X}_e$, while $X$ is called an oracle. We write
We write $W_e^X = \{ i \in \mathbb{N} \mid \varphi_e^{(1),X}(i) \downarrow \}$ and this gives a standard recursive enumeration of the subsets of $\mathbb{N}$ which are recursively enumerable relative to $X$. We also write $P_e = \{ X \in \mathbb{N}^N \mid \varphi_e^{(1),X}(0) \uparrow \}$ and this gives a standard recursive enumeration of the $\Pi^0_1$ subclasses of $\mathbb{N}^N$.

For $X \in \mathbb{N}^N$ let $X' \in \mathbb{N}^N$ denote the Turing jump of $X$. For definiteness we take $X'$ to be (the characteristic function of) the set

$$\{ e \in \mathbb{N} \mid 0 \in W_e^X \} = \{ e \in \mathbb{N} \mid X \not\in P_e \}. $$

For each $e \in \mathbb{N}$ we have a pseudojump operator $J_e : \mathbb{N}^N \to \mathbb{N}^N$ given by $J_e(X) = X \oplus W_e^X$. We write $\leq_T$ for Turing reducibility and $\equiv_T$ for Turing equivalence. We write $\leq_{tt}$ for truth-table reducibility\footnote{See for instance [11, §§8.3,9.6] and [13, §§4.5].} and $\equiv_{tt}$ for truth-table equivalence.

\section{The Join Theorem and the Pseudojump Inversion Theorem}

In this section we present the classical proofs of the Join Theorem and the Pseudojump Inversion Theorem. Later we shall build on these proofs in order to obtain our refinements.

We begin with the Join Theorem, Theorem 1.2.

\textbf{Proof of Theorem 1.2.} Let $A$ and $Z$ be reals such that $A \geq_T Z \oplus 0'$ and $Z >_T 0$. It will suffice to find a real $B$ such that $A \equiv_T B' \equiv_T B \oplus 0' \equiv_T B \oplus Z$. We shall construct a sequence of strings $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_j \subset \sigma_{j+1} \subset \cdots \subset \mathbb{N}$, and the desired real $B \in \mathbb{N}^N$ will be obtained as the limit $\bigcup_j \sigma_j$ of this sequence.

Let us regard $Z$ as a subset of $\mathbb{N}$. Since $Z$ is not recursive, at least one of $Z$ and its complement $\mathbb{N} \setminus Z$ is not $\Sigma^0_1$. And then, since $Z$ is Turing equivalent to $\mathbb{N} \setminus Z$, we may safely assume that $\mathbb{N} \setminus Z$ is not $\Sigma^0_1$.

Stage $j = 0$. Let $\sigma_0 = \langle \rangle$.

Stage $j = 2n + 1$. By induction we have $\sigma_{2n}$. Define

$$S_n = \{ i \in \mathbb{N} \mid \exists \sigma \ (\sigma \supseteq \sigma_{2n} \langle 0, \ldots, 0, 1 \rangle \text{ and } \varphi_{n+i}^{(1),\sigma}(0) \downarrow) \}$$

which is $\Sigma^0_1$. Since $\mathbb{N} \setminus Z$ is not $\Sigma^0_1$, there are infinitely many $i$ such that $i \in Z \iff i \in S_n$. Choose the first such $i$, and use this $i$ to define $\sigma_{2n+1}$ as follows. If $i \in Z$, then $i \in S_n$ so let $\sigma_{2n+1} = \text{the least } \sigma \supseteq \sigma_{2n} \langle 0, \ldots, 0, 1 \rangle$ such that $\varphi_{n+i}^{(1),\sigma}(0) \downarrow$. If $i \notin Z$, let $\sigma_{2n+1} = \sigma_{2n} \langle 0, \ldots, 0, 1 \rangle$.

Stage $j = 2n + 2$. Let $\sigma_{2n+2} = \sigma_{2n} \langle A(n) \rangle$.

This completes the definition of $\sigma_j$ for all $j \in \mathbb{N}$. We now show that $B = \bigcup_j \sigma_j$ has the desired properties. We begin with the following observations.
• If $\sigma_{2n}$ is known, then the $i$ in stage $2n+1$ can be found recursively in $Z \oplus 0'$ (because $S_n$ is computable from $0'$), or recursively in $B$ (because there is exactly one $i$ such that $\sigma_{2n}\overset{(0, \ldots, 0, 1)}{\equiv} B$). And then, using this $i$, $\sigma_{2n+1}$ can be found recursively in $Z$ (to test whether $i \in Z$), or recursively in $0'$ (to test whether $i \in S_n$).

• If $\sigma_{2n+1}$ is known, then $\sigma_{2n+2}$ can be found recursively in $A$ (by the definition of $\sigma_{2n+2}$), or recursively in $B$ (because $\sigma_{2n+2} = \sigma_{2n+1} \downarrow (B(\langle \sigma_{2n+1} \rangle ))$, hence also recursively in $B \oplus Z$ or in $B \oplus 0'$.

Combining these observations, we see that the entire sequence $\langle \sigma_j \mid j \in \mathbb{N} \rangle$ is $\leq_T B \oplus Z$, and $\leq_T B \oplus 0'$, and $\leq_T A$ (using the hypothesis $Z \oplus 0' \leq_T A$). We also have:

• $B \oplus Z \leq_T A$, because $B = \bigcup \sigma_j \leq_T A$ and by hypothesis $Z \leq_T A$.

• $B' \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle$, because $n \in B' \Leftrightarrow \forall_n \langle \sigma_{2n+1} \rangle (0) \downarrow$.

• $A \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle$, because $A(n) = \sigma_{2n+2}(\langle \sigma_{2n+1} \rangle)$ for all $n$.

Thus $A \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle \leq_T B \oplus Z \leq_T A$, and $B' \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle \leq_T B \oplus 0' \leq_T B'$, hence $A \equiv_T B' \equiv_T B \oplus 0' \equiv_T B \oplus Z$, Q.E.D.

We now turn to Pseudojump Inversion, Theorem 1.3.

Proof of Theorem 1.3. Fix $e \in \mathbb{N}$, and let $A$ be a real such that $A \geq_T 0'$. It will suffice to find a real $B$ such that $A \equiv_T J_e(B) \equiv_T B \oplus 0'$. We shall construct a sequence of strings $\sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_j \subseteq \sigma_{j+1} \subseteq \cdots$ in $\mathbb{N}^*$, and the desired real $B \in \mathbb{N}^*$ will be obtained as the limit $\bigcup_j \sigma_j$ of this sequence.

Stage 0. Let $\sigma_0 = \langle \rangle$.

Stage $2n+1$. Let $\sigma_{2n+1}$ be the least string $\sigma \supseteq \sigma_{2n}$ for which $n \in W^e_{e,|\sigma|}$ if such a $\sigma$ exists. Otherwise let $\sigma_{2n+1} = \sigma_{2n}$.

Stage $2n+2$. Let $\sigma_{2n+2} = \sigma_{2n+1} \downarrow (A(n))$.

We now show that $B = \bigcup_j \sigma_j$ has the desired properties. We begin with the following observations.

• If $\sigma_{2n}$ is known, then in stage $2n+1$ the question of whether $\sigma$ exists can be answered recursively in $0'$ (because the question is $\Sigma^0_1$), or recursively in $J_e(B)$ (because $\sigma$ exists if and only if $n \in W^e_{e,|\sigma|}$), hence also recursively in $A$ (because by hypothesis $0' \leq_T A$), or recursively in $B \oplus 0'$. And once the existence or nonexistence of $\sigma$ is known, $\sigma_{2n+1}$ can be found recursively.

• If $\sigma_{2n+1}$ is known, then $\sigma_{2n+2}$ can be found recursively in $A$ (by the definition of $\sigma_{2n+2}$), or recursively in $B$ (because $\sigma_{2n+2} = \sigma_{2n+1} \downarrow (B(\langle \sigma_{2n+1} \rangle ))$, hence also recursively in $J_e(B)$ (because $B \leq_T J_e(B)$), or in $B \oplus 0'$.

Combining these observations, we see that the entire sequence $\langle \sigma_j \mid j \in \mathbb{N} \rangle$ is $\leq_T J_e(B)$, and $\leq_T B \oplus 0'$, and $\leq_T A$. We also have:
• \( B \oplus 0' \leq_T A \), because \( B = \bigcup_j \sigma_j \leq_T A \) and by hypothesis \( 0' \leq_T A \).
• \( J_e(B) \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle \), because \( n \in W^B_e \) if and only if \( n \in W^{\sigma_{2n+1}}_e \).
• \( A \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle \), because \( A(n) = \sigma_{2n+2}(|\sigma_{2n+1}|) \) for all \( n \).

Thus \( A \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle \leq_T B \oplus 0' \leq_T A \) and \( J_e(B) \leq_T \langle \sigma_j \mid j \in \mathbb{N} \rangle \leq_T J_e(B) \), hence \( A \equiv_T J_e(B) \equiv_T B \oplus 0' \), Q.E.D.

## 3 Completions of Peano Arithmetic

Let \( \text{PA} \) denote Peano Arithmetic, and let \( \text{Sent}_{\text{PA}} \) denote the set of sentences of the language of \( \text{PA} \). An extension of \( \text{PA} \) is a set \( T \subseteq \text{Sent}_{\text{PA}} \) which includes the axioms of \( \text{PA} \) and is closed under logical consequence. For any such \( T \), a completion of \( T \) is an extension \( X \) of \( \text{PA} \) which includes \( T \) and such that for each \( \varphi \in \text{Sent}_{\text{PA}} \) exactly one of the sentences \( \varphi \) and \( \neg \varphi \) belongs to \( X \). Given an extension \( T \) of \( \text{PA} \), let \( \text{Comp}_T \) denote the set of all completions of \( T \). By Lindenbaum’s Lemma, \( \text{Comp}_T \neq \emptyset \) if and only if \( T \) is consistent.

Fix a primitive recursive Gödel numbering \( \# : \varphi \mapsto \#(\varphi) : \text{Sent}_{\text{PA}} \rightarrow \mathbb{N} \). For convenience we shall assume that \( \# \) is a one-to-one correspondence between \( \text{Sent}_{\text{PA}} \) and \( \mathbb{N} \). This induces a one-to-one correspondence between subsets \( X \) of \( \text{Sent}_{\text{PA}} \) and subsets of \( \mathbb{N} \), namely \( X \mapsto \{ \#(\varphi) \mid \varphi \in X \} \). And of course subsets of \( \mathbb{N} \) are identified with their characteristic functions in \( \{0,1\}^\mathbb{N} \). Therefore, letting \( \text{CPA} \) be the subset of \( \{0,1\}^\mathbb{N} \) corresponding to \( \text{Comp}_{\text{PA}} \), we see that \( \text{CPA} \) is a special \( \Pi^0_1 \) subclass of \( \{0,1\}^\mathbb{N} \).

The purpose of this section is to prove that \( \text{CPA} \) has the Join Property and the Pseudojump Inversion Property. The Join Property will follow easily from Theorem 1.2 plus other known results, but for the Pseudojump Inversion Property we shall use a construction involving the Gödel-Rosser Incompleteness Theorem [8].

**Definition 3.1** (PA-degrees). By a \( \text{PA-degree} \) we mean the Turing degree \( \text{deg}_T(X) \) of some \( X \in \text{CPA} \). A set \( S \) of Turing degrees is said to be \emph{upwardly closed} if for all Turing degrees \( a \) and \( b \) such that \( a \leq b \), \( a \in S \) implies \( b \in S \).

**Lemma 3.2.** The set of all \( \text{PA-degrees} \) is upwardly closed.

**Proof.** This result is originally due to Robert M. Solovay. For a proof see [2, Theorem 2.21.3] or [13, Corollary 6.6].

**Lemma 3.3.** Let \( P \subseteq \{0,1\}^\mathbb{N} \) be a nonempty \( \Pi^0_1 \) class such that \( \{\text{deg}_T(X) \mid X \in P\} \) is upwardly closed. Then \( P \) has the Join Property.

**Proof.** Let \( A \) and \( Z \) be reals such that \( A \geq_T Z \oplus 0' \) and \( Z \geq_T 0 \). By Theorem 1.2 there exists a real \( C \) such that \( A \equiv_T C' \equiv_T C \oplus 0' \equiv_T C \oplus Z \). By the Low Basis Theorem [5, Theorem 2.1] relativized to \( C \), there exists \( B_0 \in P \) such that \( C' \equiv_T (B_0 \oplus C') \equiv_T B_0 \oplus C \oplus 0' \). Since \( B_0 \leq_T B_0 \oplus C \), we can find \( B \in P \) such that \( B \equiv_T B_0 \oplus C \), hence \( C' \equiv_T B' \equiv_T B \oplus 0' \). We now have \( C' \equiv_T \)
During the previous discussion, we defined a function $f: \mathbb{N} \times \{0, 1\} \to \mathbb{N}$ such that for all $i \in \mathbb{N}$, if $P_i$ is a nonempty $\Pi^0_1$ subclass of CPA then $P_{s(i,0)}$ and $P_{s(i,1)}$ are nonempty disjoint $\Pi^0_1$ subclasses of $P_i$.

**Lemma 3.7** (splitting property). There is a primitive recursive function $s : \mathbb{N} \times \{0, 1\} \to \mathbb{N}$ such that for all $i \in \mathbb{N}$, if $P_i$ is a nonempty $\Pi^0_1$ subclass of CPA then $P_{s(i,0)}$ and $P_{s(i,1)}$ are nonempty disjoint $\Pi^0_1$ subclasses of $P_i$.

**Proof.** Lemma 3.5 tell us that, given $i \in \mathbb{N}$, we can effectively find a recursively axiomatizable extension $T$ of $\text{PA}$ such that $\text{Comp}_T$ corresponds to $P_i \cap \text{CPA}$. We then apply the Gödel-Rosser Incompleteness Theorem [8] to effectively find a sentence $\psi \in \text{Sent}_{\text{PA}}$ such that if $T$ is consistent, then neither $\psi$ nor $\neg \psi$ belongs to $T$. Let $T_1$ (respectively $T_0$) be the closure of $T \cup \{\psi\}$ (respectively $T \cup \{\neg \psi\}$) under logical consequence. Clearly $\text{Comp}_{T_1}$ and $\text{Comp}_{T_0}$ are disjoint, and they are nonempty if $\text{Comp}_T$ is nonempty. Apply Lemma 3.5 to effectively find $s(i, 1), s(i, 0) \in \mathbb{N}$ such that $P_{s(i,1)}$ and $P_{s(i,0)}$ correspond to $\text{Comp}_{T_1}$ and $\text{Comp}_{T_0}$ respectively. This completes the proof. \qed
We shall now use this splitting property to redo Theorem 1.3 within CPA.

**Theorem 3.8.** CPA has the Pseudojump Inversion Property.

**Proof.** Fix \( e \in \mathbb{N} \), and let \( A \in \{0,1\}^\mathbb{N} \) be a real such that \( A \geq_T 0' \). It will suffice to find a real \( B \in \text{CPA} \) such that \( A \equiv_T J_e(B) \equiv_T B \oplus 0' \). To find \( B \), we shall construct a sequence of \( \Pi^0_1 \) classes \( Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_j \supseteq \cdots \), by induction on \( j \) starting with \( Q_0 = \text{CPA} \subseteq \{0,1\}^\mathbb{N} \). The intersection \( \bigcap_j Q_j \) will be nonempty, and we shall have \( B \in \bigcap_j Q_j \). At the same time, we shall construct a function \( f : \mathbb{N} \to \mathbb{N} \) such that \( Q_j = P_{f(j)} \) for all \( j \).

Fix a primitive recursive splitting function \( s \) as in Lemma 3.7.

Stage 0. Fix \( f(0) \in \mathbb{N} \) such that \( P_{f(0)} = Q_0 = \text{CPA} \).

Stage 2n + 1. By induction we have \( Q_{2n} = P_{f(2n)} \). If \( P_{f(2n)} \cap \{X \mid n \notin J_e(X)\} = \emptyset \), let \( Q_{2n+1} = P_{f(2n)} \) and \( f(2n+1) = f(2n) \). Otherwise, let \( Q_{2n+1} = P_{f(2n)} \cap \{X \mid n \notin J_e(X)\} \) and choose \( f(2n+1) \) so that \( P_{f(2n+1)} = P_{f(2n)} \cap \{X \mid n \notin J_e(X)\} \). This \( f(2n+1) \) is found primitive recursively from \( f(2n) \) and \( n \) and \( e \).

Stage 2n + 2. By induction we have \( Q_{2n+1} = P_{f(2n+1)} \). Let \( f(2n+2) = s(f(2n+1), A(n)) \) and \( Q_{2n+2} = P_{s(f(2n+1), A(n))} \), where \( s \) is our splitting function. This \( f(2n+2) \) is found primitive recursively from \( f(2n+1) \) and \( A(n) \).

This completes the construction.

By construction each \( Q_j \) is nonempty, so by compactness of \( \{0,1\}^\mathbb{N} \) their intersection \( \bigcap_j Q_j \) is nonempty. It remains to show that \( B \in \bigcap_j Q_j \) has the desired properties. We begin with the following observations.

- If \( f(2n) \) is known then \( f(2n+1) \) can be found recursively in \( 0' \) (by checking whether the \( \Pi^0_1 \) class \( P_{f(2n)} \cap \{X \mid n \notin J_e(X)\} \) is empty or not), or recursively in \( J_e(B) \) (by checking whether \( n \in J_e(B) \) or not).
- If \( f(2n+1) \) is known then \( f(2n+2) \) can be found recursively in \( B \) (by finding the \( k \in \{0,1\} \) such that \( B \notin P_{s(f(2n+1), k)} \) ), or recursively in \( A \) (by evaluating \( s(f(2n+1), A(n)) \)).

Combining these observations, we see that \( f \) is \( \leq_T A \oplus 0' \equiv_T A \), and \( \leq_T J_e(B) \), and \( \leq_T B \oplus 0' \). Conversely, we also have:

- \( A \leq_T f \), because \( A(n) = i \) if and only if \( f(2n+2) = s(f(2n+1), i) \).
- \( J_e(B) \leq_T f \), because \( n \in J_e(B) \) if and only if \( f(2n+1) = f(2n) \).
- \( B \oplus 0' \leq_T f \), because \( B \leq_T J_e(B) \leq_T f \) and \( 0' \leq_T A \leq_T f \).

Thus \( f \equiv_T A \equiv_T J_e(B) \equiv_T B \oplus 0' \) and the proof is complete. \( \square \)

**Remark 3.9.** We have used the Splitting Lemma 3.7 to prove that CPA has the Pseudojump Inversion Property. Similarly, it would be possible to use the Splitting Lemma to prove directly that CPA has the Join Property. This direct proof would be in contrast to our shorter but indirect proof in Theorem 3.4 above. Note however that a version of the Splitting Lemma is used in our proof of Solovay’s Lemma 3.2; see [13, §6] and [14, §3].
4 Turing degree isomorphism

Definition 4.1 (Turing degree isomorphism). Following [4] we say that \( P, Q \subseteq \mathbb{N}^\mathbb{N} \) are **Turing degree isomorphic** if \( \{ \deg_T(X) \mid X \in P \} = \{ \deg_T(X) \mid X \in Q \} \).

The purpose of this section is to prove that the Join Property and the Pseudojump Inversion Property hold for all \( \Pi^0_1 \) subclasses of \( \{0, 1\}^\mathbb{N} \) which are Turing degree isomorphic to CPA. We also obtain some more general-looking results.

We begin with the Join Property.

**Theorem 4.2.** Let \( P \subseteq \{0, 1\}^\mathbb{N} \) be a \( \Pi^0_1 \) class. If \( P \) is Turing degree isomorphic to CPA, then \( P \) has the Join Property.

*Proof.* Let \( A \) and \( Z \) be reals such that \( A \geq_T Z \oplus 0' \) and \( Z >_T 0 \). By Theorem 3.4 let \( B \in \text{CPA} \) be such that \( A \equiv_T B' \equiv_T B \oplus 0' \equiv_T B \oplus Z \). Since \( P \) is Turing degree isomorphic to CPA, let \( C \in P \) be such that \( B \equiv_T C \). Then \( A' \equiv_T C' \equiv_T C \oplus 0' \equiv_T C \oplus Z \). Thus \( P \) has the Join Property, Q.E.D.

**Remark 4.3.** More generally, for \( P, Q \subseteq \mathbb{N}^\mathbb{N} \) we say that \( P \) is **Turing degree embeddable** into \( Q \) if \( \{ \deg_T(X) \mid X \in P \} \subseteq \{ \deg_T(X) \mid X \in Q \} \). The proof of Theorem 4.2 shows that if \( P \) has the Join Property and is Turing degree embeddable into \( Q \), then \( Q \) has the Join Property.

We now turn to Pseudojump Inversion. Unfortunately, we cannot simply imitate the proof of Theorem 4.2. This is because pseudojump operators are not invariant under Turing equivalence, i.e., \( X \equiv_T Y \) typically does not imply \( J_e(X) \equiv_T J_e(Y) \). Consequently, we do not know whether the Pseudojump Inversion Property is invariant under Turing degree isomorphism of special \( \Pi^0_1 \) subclasses of \( \{0, 1\}^\mathbb{N} \). However, we shall settle some interesting special cases of this question. As a first step, consider the following notion, which is a special case of Turing degree isomorphism.

**Definition 4.4** (recursive homeomorphism). For \( P, Q \subseteq \mathbb{N}^\mathbb{N} \), a **recursive homeomorphism** of \( P \) onto \( Q \) is a one-to-one onto mapping \( \Phi : P \to Q \) such that both \( \Phi \) and its inverse \( \Phi^{-1} : Q \to P \) are the restrictions of partial recursive functionals to \( P \) and \( Q \) respectively. We say that \( P \) and \( Q \) are **recursively homeomorphic** if there exists a recursive homeomorphism of \( P \) onto \( Q \).

**Lemma 4.5.** Let \( P, Q \subseteq \mathbb{N}^\mathbb{N} \) be recursively homeomorphic. If \( P \) has the Pseudojump Inversion Property, then so does \( Q \).

*Proof.* Assume that \( P \) has the Pseudojump Inversion Property. We shall prove that \( Q \) has the Pseudojump Inversion Property. Given \( e \in \mathbb{N} \) and \( A \geq_T 0' \), it will suffice to find \( C \in Q \) such that \( A \equiv_T J_e(C) \equiv_T C \oplus 0' \).

Let \( \Phi : P \to Q \) be a recursive homeomorphism. Let \( i \in \mathbb{N} \) be such that for all \( X \in P \) and all \( n \in \mathbb{N} \) we have \( \varphi_i^X(n) \simeq \varphi_i^{\Phi(X)}(n) \). Then for all \( X \in P \) we have \( W_i^X = \{ n \in \mathbb{N} \mid \varphi_i^X(n) \downarrow \} = \{ n \in \mathbb{N} \mid \varphi_i^{\Phi(X)}(n) \downarrow \} = W_i^{\Phi(X)} \). Since \( P \) has the Pseudojump Inversion Property, let \( B \in P \) such that \( A \equiv_T J_i(B) \equiv_T B \oplus 0' \). Let \( C = \Phi(B) \). Because \( \Phi \) is a recursive homeomorphism, we have \( B \equiv_T \Phi(B) = C \),
hence \( J_e(C) = C \oplus W_e^C = C \oplus W_e^{\Phi(B)} = C \oplus W_i^B \equiv_T B \oplus W_i^B = J_i(B) \), hence \( A \equiv_T J_e(C) \equiv_T C \oplus 0' \), Q.E.D.

As a bridge from Turing degree isomorphism to recursive homeomorphism, we have the following lemma.

**Lemma 4.6.** Let \( P, Q \subseteq \{0, 1\}^\mathbb{N} \) be nonempty \( \Pi^0_1 \) classes. If \( P \) is Turing degree embeddable into \( Q \), then there exist nonempty \( \Pi^0_1 \) subclasses \( \tilde{P} \subseteq P \) and \( \tilde{Q} \subseteq Q \) such that \( \tilde{P} \) and \( \tilde{Q} \) are recursively homeomorphic.

**Proof.** We shall draw on some facts about hyperimmune-freeness and truth-table reducibility. For this background, see [11, §§8.3,9.6] and [13, §4].

By the Hyperimmune-Free Basis Theorem [5, Theorem 2.4] (see also [13, Theorem 4.19]), let \( X_0 \in P \) be hyperimmune-free. Since \( P \) is Turing degree embeddable into \( Q \), let \( Y_0 \in Q \) be such that \( X_0 \equiv_T Y_0 \). Then \( Y_0 \) is also hyperimmune-free, and by hyperimmune-freeness there exist truth-table functionals \( \Phi, \Psi : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N} \) such that \( \Phi(X_0) = Y_0 \) and \( \Psi(Y_0) = X_0 \). Let \( \tilde{P} = \{ X \in P \mid \Phi(X) \in Q \text{ and } \Psi(\Phi(X)) = X \} \). Then \( \tilde{P} \) is a \( \Pi^0_1 \) subclass of \( P \), and it is nonempty because it contains \( X_0 \). Moreover \( \tilde{Q} = \{ \Phi(X) \mid X \in \tilde{P} \} \) is also a \( \Pi^0_1 \) subclass of \( Q \), and we have a recursive homeomorphism \( \Phi|\tilde{P} \) of \( \tilde{P} \) onto \( \tilde{Q} \).

**Lemma 4.7.** Any nonempty \( \Pi^0_1 \) subclass of CPA is recursively homeomorphic to CPA.

**Proof.** See [14, §3].

We can now prove the following theorem, which is parallel to Theorem 4.2.

**Theorem 4.8.** Let \( P \subseteq \{0, 1\}^\mathbb{N} \) be a \( \Pi^0_1 \) class. If \( P \) is Turing degree isomorphic to CPA, then \( P \) has the Pseudojump Inversion Property.

**Proof.** Let \( P \) be Turing degree isomorphic to CPA. By Lemma 4.6 there are nonempty recursively homeomorphic \( \Pi^0_1 \) classes \( \tilde{P} \subseteq P \) and CPA \( \subseteq \tilde{P} \). By Lemma 4.7 CPA is recursively homeomorphic to CPA. By Theorem 3.8 CPA has the Pseudojump Inversion Property, so by Lemma 4.5 CPA and hence \( \tilde{P} \) have the Pseudojump Inversion Property. But then, since \( \tilde{P} \subseteq P \), it follows that \( P \) has the Pseudojump Inversion Property, Q.E.D.

Next we prove a more general-looking result. Recall from [13, 16, 17, 18, 19] that a nonempty \( \Pi^0_1 \) class \( P \subseteq \{0, 1\}^\mathbb{N} \) is said to be Muchnik complete if every nonempty \( \Pi^0_1 \) class \( Q \subseteq \{0, 1\}^\mathbb{N} \) is Muchnik reducible to \( P \), i.e., for all \( X \in P \) there exists \( Y \in Q \) such that \( Y \leq_T X \).

**Lemma 4.9.** A \( \Pi^0_1 \) class \( P \subseteq \{0, 1\}^\mathbb{N} \) is Muchnik complete if and only if it is Turing degree isomorphic to CPA. Moreover, for such a \( P \) the set of Turing degrees \( \{\deg_T(X) \mid X \in P\} \) is upwardly closed.

**Proof.** See [13, §§3,6].
Theorem 4.10. Let $P \subseteq \{0, 1\}^N$ be a nonempty $\Pi_1^0$ class. If $P$ is Muchnik complete, then $P$ has the Join Property and the Pseudojump Inversion Property.

Proof. This is immediate from Lemma 4.9 and Theorems 4.2 and 4.8. □

An even more general-looking result reads as follows.

Theorem 4.11. Let $P \subseteq \{0, 1\}^N$ be a nonempty $\Pi_1^0$ class such that $\{\deg_T(Y) \mid Y \in P\}$ is upwardly closed. Then any $P_1 \subseteq N^N$ which includes $P$ has the Join Property and the Pseudojump Inversion Property.

Proof. By Lemma 4.9 CPA is Turing degree embeddable into $P$. Our result then follows by Remark 4.3 and Theorems 4.2 and 4.8. □

5 $\Pi_1^0$ classes of positive measure

For $\sigma \in \{0, 1\}^*$ we write $[\sigma] = \{X \in \{0, 1\}^N \mid \sigma \subset X\}$. Let $\mu$ be the fair coin measure on $\{0, 1\}^N$, defined by letting $\mu([\sigma]) = 2^{-|\sigma|}$ for all $\sigma \in \{0, 1\}^*$. This $\mu$ is a Borel probability measure on $\{0, 1\}^N$, a $\Pi_1^0$ class $P \subseteq \{0, 1\}^N$ is said to be of positive measure if $\mu(P) > 0$. In this section we note that such a $P$ must have the Pseudojump Inversion Property but need not have the Join Property. We also obtain the same results for $\Pi_1^0$ subclasses of $\{0, 1\}^N$ which are Turing degree isomorphic to such a $P$.

To prove these results we need some basic facts about Martin-Löf randomness and LR-reducibility. We cite our semi-expository papers [13, 15, 17, 18] but one can also consult the treatises of Downey and Hirschfeldt [2] and Nies [9]. Let $\text{MLR} = \{X \in \{0, 1\}^N \mid X$ is Martin-Löf random\}.

Lemma 5.1. A $\Pi_1^0$ subclass of $\{0, 1\}^N$ is of positive measure if and only if it includes a nonempty $\Pi_1^0$ subclass of $\text{MLR}$. 

Proof. This is because $\text{MLR} \subseteq \{0, 1\}^N$ is a $\Sigma_2^0$ class of full measure. See for instance [13, §8]. □

Lemma 5.2 (due to Nies [9]). Let $P \subseteq \{0, 1\}^N$ be a $\Pi_1^0$ class of positive measure. Then $P$ has the Pseudojump Inversion Property.

Proof. See [9, Theorem 6.3.9] or a simpler proof in [15, Theorem 5.1]. □

Theorem 5.3. Let $P \subseteq \{0, 1\}^N$ be a $\Pi_1^1$ class of positive measure, and let $Q \subseteq \{0, 1\}^N$ be a $\Pi_1^0$ class which is Turing degree isomorphic to $P$. Then $Q$ has the Pseudojump Inversion Property.

Proof. By Lemma 5.1 let $P_0$ be a nonempty $\Pi_1^0$ subclass of $P \cap \text{MLR}$. Then $P_0$ is Turing degree embeddable in $Q$, so by Lemma 4.6 we can find nonempty $\Pi_1^0$ classes $\bar{P} \subseteq P_0$ and $\bar{Q} \subseteq Q$ which are recursively homeomorphic. By Lemma 5.1 $\bar{P}$ is of positive measure, so by Lemma 5.2 $\bar{P}$ has the Pseudojump Inversion Property. It then follows by Lemma 4.5 that $\bar{Q}$ and hence $Q$ have the Pseudojump Inversion Property. □
Relativizing the notion of Martin-Löf randomness, for any real \( Y \) we write
\[
\text{MLR}^Y = \{ X \in \{0,1\}^\mathbb{N} | X \text{ is Martin-Löf random relative to } Y \}.
\]
If \( \text{MLR}^Y \subseteq \text{MLR}^Z \) we say that \( Z \) is LR-reducible to \( Y \), abbreviated as \( Z \leq_{LR} Y \). Intuitively this means that \( Y \) has at least as much “derandomizing power” as \( Z \). Clearly \( Z \leq_T Y \) implies \( Z \leq_{LR} Y \), but the converse does not hold:

**Lemma 5.4.** There exists a nonrecursive real \( Z \) such that \( Z \leq_{LR} 0 \).

*Proof.* See [15, Theorem 6.1] or [2, 9].

On the other hand, we have:

**Lemma 5.5.** If \( Z \leq_{LR} Y \) then \( Z' \leq_T Z \oplus Y' \).

*Proof.* See [15, Theorem 8.8] or [2, 9].

**Lemma 5.6** (Lutz [6]). If \( Z \leq_{LR} 0 \) then \( X \oplus Z \leq_{LR} X \) for all \( X \in \text{MLR} \).

*Proof.* Assume \( Z \leq_{LR} 0 \) and \( X \in \text{MLR} \). We must prove that \( \text{MLR}^X \subseteq \text{MLR}^{X \oplus Z} \). Given \( X_1 \in \text{MLR}^X \), it follows by Van Lambalgen’s Theorem ([15, Theorem 3.6], see also [2, Corollary 6.9.3]) that \( X_1 \oplus X \in \text{MLR} \). But then \( X_1 \oplus X \in \text{MLR}^Z \), so by Van Lambalgen’s Theorem relative to \( Z \) we have \( X_1 \in \text{MLR}^{X \oplus Z} \), Q.E.D.

**Theorem 5.7** (Lutz [6]). \( \text{MLR} \) does not have the Join Property.

*Proof.* By Lemma 5.4 let \( Z >_T 0 \) be such that \( Z \leq_{LR} 0 \). For any \( X \in \text{MLR} \) we have \( X \oplus Z \leq_{LR} X \) by Lemma 5.6, hence \( (X \oplus Z)' \leq_T X' \oplus Z \) by Lemma 5.5. If \( \text{MLR} \) had the Join Property, there would be an \( X \in \text{MLR} \) such that \( X' \equiv_T X \oplus Z \), hence \( (X \oplus Z)' \leq_T X \oplus Z \), a contradiction.

**Theorem 5.8** (Lutz [6]). There is a \( \Pi^0_1 \) class \( P \subseteq \{0,1\}^\mathbb{N} \) of positive measure which does not have the Join Property.

*Proof.* This is immediate from Theorem 5.7 and Lemma 5.1.

6 \( \Pi^0_1 \) classes constructed by priority arguments

In this section we construct a special \( \Pi^0_1 \) class \( Q \subseteq \{0,1\}^\mathbb{N} \) which has neither the Join Property nor the Pseudojump Inversion Property. In addition, this \( \Pi^0_1 \) class \( Q \) has some other interesting features, which we also discuss.

We begin by stating a technical theorem which embodies our construction of \( Q \). Recall from §1 that \( (P_e)_{e \in \mathbb{N}} \) is a fixed enumeration of all \( \Pi^0_1 \) classes.

**Theorem 6.1.** There is a nonempty perfect\(^2\) \( \Pi^0_1 \) class \( Q \subseteq \{0,1\}^\mathbb{N} \) with the following properties. For all \( e, n \in \mathbb{N} \) there exists \( m \in \mathbb{N} \) such that for all pairwise distinct \( \tau_1, \ldots, \tau_n \in \{0,1\}^* \) with \( |\tau_1| = \cdots = |\tau_n| \geq m \), the \( \Pi^0_1 \) class

\(^2\)A topological space is said to be perfect if it has no isolated points, i.e., there is no open set consisting of exactly one point. In particular, a set \( P \subseteq \{0,1\}^\mathbb{N} \) is perfect if and only if there is no \( \tau \in \{0,1\}^* \) such that \( P \cap [\tau] = \{X\} \) for some \( X \in \{0,1\}^\mathbb{N} \).
\[(Q \cap [t_1]) \times \cdots \times (Q \cap [t_n])\]
is either disjoint from \(P_e\) or included in \(P_e\). Moreover, this \(m\) can be computed from \(e\) using an oracle for \(0'\).

Before proving Theorem 6.1, we spell out some of its consequences which are of more general interest. The next theorem summarizes these features of \(Q\).

**Theorem 6.2.** Let \(Q \subseteq \{0, 1\}^\mathbb{N}\) be a nonempty \(\Pi^0_1\) class as in Theorem 6.1.

1. \(Q\) is thin, i.e., for every \(\Pi^0_1\) subclass \(P\) of \(Q\), the complement \(Q \setminus P\) is again a \(\Pi^0_1\) subclass of \(Q\). More generally, for any finite sequence \(Q_1, \ldots, Q_n\) of pairwise disjoint\(^3\) \(\Pi^0_1\) subclasses of \(Q\), the \(\Pi^0_1\) class \(Q_1 \times \cdots \times Q_n\) is thin.

2. \(Q\) is special, i.e., no \(X \in Q\) is recursive. More generally, the Turing degrees of members of \(Q\) are independent, i.e., no \(X \in Q\) is \(\leq_T\) the join of any finitely many other members of \(Q\).

3. Every \(X \in Q\) is of minimal truth-table degree. Consequently, every hyperimmune-free \(X \in Q\) is of minimal Turing degree.

4. Every finite sequence \(X_1, \ldots, X_n \in Q\) is generalized low, i.e., \((X_1 \oplus \cdots \oplus X_n)' \equiv_T X_1 \oplus \cdots \oplus X_n \oplus 0'\).

**Proof.** We prove parts 1 through 4 in that order.

1. To see that \(Q\) is thin, let \(P\) be a \(\Pi^0_1\) subclass of \(Q\). Fix \(e \in \mathbb{N}\) such that \(P = P_e\), and let \(m\) be as in Theorem 6.1. Then for each \(\tau \in \{0, 1\}^m\) the \(\Pi^0_1\) class \(Q \cap [\tau]\) is either disjoint from \(P\) or included in \(P\). Hence \(Q \setminus P = \bigcup_{\tau}(Q \cap [\tau])\) where the union is taken only over all \(\tau \in \{0, 1\}^m\) such that \(Q \cap [\tau]\) is disjoint from \(P\). Thus \(Q \setminus P\) is a union of finitely many \(\Pi^0_1\) classes, so it too is a \(\Pi^0_1\) class.

For the generalization, let \(P\) be a \(\Pi^0_1\) subclass of \(Q_1 \times \cdots \times Q_n\) where \(Q_1, \ldots, Q_n\) are pairwise disjoint \(\Pi^0_1\) subclasses of \(Q\). Fix \(e \in \mathbb{N}\) such that \(P = P_e\), and let \(m\) be as in Theorem 6.1. We then have

\[(Q_1 \times \cdots \times Q_n) \setminus P = \bigcup_{\tau_1, \ldots, \tau_n}(Q_1 \cap [\tau_1]) \times \cdots \times (Q_n \cap [\tau_n])\]

where the union is taken over all pairwise distinct sequences \(\tau_1, \ldots, \tau_n \in \{0, 1\}^m\) such that \((Q_1 \cap [\tau_1]) \times \cdots \times (Q_n \cap [\tau_n])\) is disjoint from \(P\). Thus \((Q_1 \times \cdots \times Q_n) \setminus P\) is a union of finitely many \(\Pi^0_1\) classes, so it too is a \(\Pi^0_1\) class. Q.E.D.

2. Assume for a contradiction that \(X \in Q\) is recursive. Fix \(e \in \mathbb{N}\) such that \(P_e = \{X\}\), let \(m \in \mathbb{N}\) be as in Theorem 6.1, and let \(\tau = X[m]\). Clearly \(X \in Q \cap [\tau]\), hence \(Q \cap [\tau]\) is not disjoint from \(P_e\), hence \(Q \cap [\tau]\) is included in \(P_e\), hence \(X\) is the unique member of \(Q \cap [\tau]\). Thus \(X\) is an isolated point of \(Q\), contradicting the fact that \(Q\) is perfect.

For the generalization, it will suffice to show that \(X \not\leq_T X_1 \oplus \cdots \oplus X_n\) for all pairwise distinct \(X, X_1, \ldots, X_n \in Q\). Let \(\Psi : \{0, 1\}^\mathbb{N} \rightarrow \{0, 1\}^\mathbb{N}\) be a partial recursive functional, and assume for a contradiction that \(X = \Psi(X_1 \oplus \cdots \oplus X_n)\). Let \(e \in \mathbb{N}\) be such that

3The hypothesis of pairwise disjointness is essential. For instance, if \(P = \{X \oplus X \mid X \in Q\}\) then \((Q \times Q) \setminus P\) is not \(\Pi^0_1\), so \(Q \times Q\) is not thin.
\[ P_e = \{ Y \oplus Y_1 \oplus \cdots \oplus Y_n \mid \forall i \forall j \text{ (if } \Psi(Y_1 \oplus \cdots \oplus Y_n)(i) \downarrow = j \text{ then } Y(i) = j)\}. \]

Let \( m \) be as in Theorem 6.1 and sufficiently large so that \( \tau = X \mid m, \tau_1 = X_1 \mid m, \ldots, \tau_n = X_n \mid m \) are pairwise distinct. The \( \Pi^0_1 \) class

\[(Q \cap [\tau]) \times (Q \cap [\tau_1]) \times \cdots \times (Q \cap [\tau_n])\]

contains \( X \oplus X_1 \oplus \cdots \oplus X_n \) and is therefore not disjoint from \( P_e \), so it is included in \( P_e \). In particular, for all \( Y \in Q \cap [\tau] \) and all \( i \in \mathbb{N} \) we have \( \Psi(X_1 \oplus \cdots \oplus X_n)(i) \downarrow = X(i) = Y(i) \), hence \( X = Y \). Thus \( Q \cap [\tau] = \{ X \} \), so again \( X \) is an isolated point of \( Q \), contradicting the fact that \( Q \) is perfect.

3. Let \( X \in Q \) be given. We have already seen that \( X \) is not recursive. To prove that \( X \) is of minimal truth-table degree, it remains to show that \( X \leq_{tt} \Psi(X) \) for any truth-table functional \( \Psi \) such that \( \Psi(X) \) is not recursive. Given such a functional \( \Psi \), let \( e \) be such that

\[ P_e = \{ X_0 \oplus X_1 \in X \times X \mid \Psi(X_0) = \Psi(X_1) \}. \]

By Theorem 6.1 let \( m \) be such that for all \( \tau \in \{0,1\}^{\geq m} \) the \( \Pi^0_1 \) class \((Q \cap [\tau \upharpoonright 0]) \times (Q \cap [\tau \upharpoonright 1])\) is either disjoint from \( P_e \) or included in \( P_e \). If it is disjoint from \( P_e \), let us say that \( \tau \) is splitting.

Case 1: For all sufficiently large \( \tau \subset X \), \( \tau \) is splitting. Then for all sufficiently large \( \tau \subset X \) and all \( X_0 \in Q \cap [\tau \upharpoonright 0] \) and \( X_1 \in Q \cap [\tau \upharpoonright 1] \), we have \( \Psi(X_0) \neq \Psi(X_1) \). In particular we have \( \Psi(X) \neq \Psi(Y) \) for all \( Y \in Q \cap [\tau] \) such that \( X \neq Y \). From this it follows that \( X \) is truth-table reducible to \( \Psi(X) \).

Case 2: There are arbitrarily large \( \tau \subset X \) such that \( \tau \) is not splitting. Let \( \tau \subset X \) be non-splitting with \( |\tau| \geq m \). Then \( Q \cap [\tau \upharpoonright 0] \) and \( Q \cap [\tau \upharpoonright 1] \) are nonempty, and for all \( X_0 \in Q \cap [\tau \upharpoonright 0] \) and \( X_1 \in Q \cap [\tau \upharpoonright 1] \), we have \( \Psi(X_0) = \Psi(X_1) \). In particular, letting \( i, j \in \{0,1\} \) be such that \( X \in Q \cap [\tau \upharpoonright i] \) and \( i + j = 1 \), we have \( \Psi(X) = \Psi(Y) \) for all \( Y \in Q \cap [\tau \upharpoonright j] \). From this it follows that \( \Psi(X) \) is recursive, Q.E.D.

A general property of hyperimmune-free reals \( X \) is that the Turing degrees \( \leq \deg_T(X) \) are the same as the truth-table degrees \( \leq \deg_{tt}(X) \). Since every \( X \in Q \) is of minimal truth-table degree, it follows that every hyperimmune-free \( X \in Q \) is of minimal Turing degree.

4. Up until now we have not used the “moreover” clause of Theorem 6.1, but now we shall use it. We may safely assume that \( X_1, \ldots, X_n \in Q \) are pairwise distinct. Recall that we have defined the Turing jump \( X' \in \{0,1\}^\mathbb{N} \) of \( X \in \{0,1\}^\mathbb{N} \) to be (the characteristic function of) the set \( \{ e \in \mathbb{N} \mid X \not\in P_e \} \). To compute \( (X_1 \oplus \cdots \oplus X_n)' \) we proceed as follows. First we use our oracle for \( X_1 \oplus \cdots \oplus X_n \) to find \( l \in \mathbb{N} \) such that \( X_1[l], \ldots, X_n[l] \) are pairwise distinct. Then, given \( e \in \mathbb{N} \), we use our oracle for \( 0' \) to compute \( m \geq l \) as in Theorem 6.1. Letting \( \tau_i = X_i \mid m \) for \( i = 1, \ldots, n \), we know that the \( \Pi^0_1 \) class

\[(Q \cap [\tau_1]) \times \cdots \times (Q \cap [\tau_n])\]

is either disjoint from \( P_e \) or included in \( P_e \). Using our oracle for \( 0' \) again, we can decide whether this \( \Pi^0_1 \) class is disjoint from \( P_e \) or not. The answer to this question tells us whether \( e \in (X_1 \oplus \cdots \oplus X_n)' \) or not. Thus \( (X_1 \oplus \cdots \oplus X_n)' \) is computable from \( X_1 \oplus \cdots \oplus X_n \oplus 0' \), Q.E.D.
Remark 6.3. A plausible generalization of part 3 of Theorem 6.2 would say that for every pairwise distinct finite sequence \(X_1, \ldots, X_n \in Q\), the \(tt\)-degrees \(\leq \deg_{tt}(X_1 \oplus \cdots \oplus X_n)\) should form a lattice isomorphic to the powerset of \(\{1, \ldots, n\}\). From this it would follow that the same holds for Turing degrees, provided \(X_1 \oplus \cdots \oplus X_n\) is hyperimmune-free.

Theorem 6.4. The special \(\Pi^0_1\) class \(Q \subseteq \{0, 1\}^N\) of Theorem 6.1 has neither the Join Property nor the Pseudojump Inversion Property.

Proof. We rely mainly on part 4 of Theorem 6.2. To see that the Join Property fails for \(Q\), fix \(Z \in Q\). Then \(Z \geq_T 0\) but for all \(B \in Q\) we have \((B \oplus Z)' \equiv_T B \oplus Z\), hence \(0' \nleq_T B \oplus Z\). To see that Pseudojump Inversion fails for \(Q\), consider a p.s. jump operator \(J_e\) with the property\(^4\) that \(X <_T J_e(X)\) and \((J_e(X))' \equiv_T X'\) for all \(X \in \mathbb{N}^N\). Then for all \(B \in Q\) we have \((J_e(B))' \equiv_T B' \equiv_T B \oplus 0'\), hence \(0' \nleq_T J_e(B)\).

The rest of this section is devoted to the proof of Theorem 6.1. We use a priority construction in the vein of Martin/Pour-El [7] and Jockusch/Soare [5, Theorem 4.7].

Our construction will be presented in terms of treemaps. A treemap is a mapping \(T : \{0, 1\}^* \to \{0, 1\}^*\) such that \(T(\sigma^\sim(i)) \supseteq T(\sigma)^\sim(i)\) for all \(\sigma \in \{0, 1\}^*\) and all \(i \in \{0, 1\}\). Note that for any treemap \(T\) and \(\sigma, \rho \in \{0, 1\}^*\) we have \(\sigma \subset \rho\) if and only if \(T(\sigma) \subset T(\rho)\). Note also that there is a one-to-one correspondence between treemaps \(T\) and nonempty perfect closed subsets of \(\{0, 1\}^N\), given by \(T \mapsto [T] = \{T(X) | X \in \{0, 1\}^N\}\) where \(T(X) = \bigcup_{n \in \mathbb{N}} T(X|n)\).

Recall that any \(\Pi^0_1\) class \(P \subseteq \{0, 1\}^N\) is closed. Therefore, if \(P\) is also nonempty and perfect, there is a unique treemap \(T_P\) such that \(P = [T_P]\).

Lemma 6.5. Let \(P \subseteq \{0, 1\}^N\) be a nonempty perfect \(\Pi^0_1\) class. Then, the treemap \(T = T_P\) corresponding to \(P\) is computable using \(0'\) as an oracle.

Proof. Given \(\tau \in \{0, 1\}^*\) we can use our oracle for \(0'\) to decide whether \(P \cap [\tau]\) is empty or not. Thus, if already know \(T(\sigma)\) for some \(\sigma \in \{0, 1\}^*\), we can then compute \(T(\sigma^\sim(i))\) for \(i \in \{0, 1\}\), because \(T(\sigma^\sim(i))\) is the smallest \(\tau \supseteq T(\sigma)^\sim(i)\) such that \(P \cap [\tau^\sim(j)]\) is nonempty for all \(j \in \{0, 1\}\). Thus \(T \leq_T 0'\), Q.E.D.

In our proof of Theorem 6.1, the treemap \(T = T_Q\) corresponding to \(Q\) will be uniform, in the sense that \(|T(\sigma)|\) will depend only on \(|\sigma|\). This feature of \(T\) will be for convenience only, but \(T\) will also have another key property, which reads as follows. For all \(e, l, n \in \mathbb{N}\) with \(e \leq l\) and \(n \leq 2^l\) and all pairwise distinct \(\sigma_1, \ldots, \sigma_n \in \{0, 1\}^l\), either \([T(\sigma_1)] \times \cdots \times [T(\sigma_n)] \cap P_e = 0\) or \((Q \cap [T(\sigma_1)]) \times \cdots \times (Q \cap [T(\sigma_n)]) \subseteq P_e\). Thus by Lemma 6.5 the “moreover” clause of Theorem 6.1 will hold with \(m = m_e = |T(\sigma)|\) for all \(\sigma \in \{0, 1\}^e\).

\(^4\)The existence of such pseudojump operators is well known [20, §VII.1]. One way to obtain such an operator is to combine part 4 of Theorem 6.2 with the R. E. Basis Theorem [4, Theorem 3] to get an \(e \in \mathbb{N}\) such that \(0 <_T W_e\) and \((W_e)' \equiv_T 0'\). The desired operator \(J_e\) is then obtained by uniform relativization to an arbitrary Turing oracle \(X\).
Proof of Theorem 6.1. We shall construct \( T = T_Q \) as the limit \( T = \lim_s T_s \) of a recursive sequence of recursive uniform treemaps \( T_s, s \in \mathbb{N} \). This will be a pointwise limit as \( s \to \infty \), in the sense that for each \( \sigma \in \{0,1\}^* \) we shall have \( T(\sigma) = T_s(\sigma) \) for all sufficiently large \( s \). Also, the treemaps \( T_s \) will be nested, in the sense that for all \( s \in \mathbb{N} \) and all \( \sigma \in \{0,1\}^* \) there will be a \( \rho \in \{0,1\}^* \) such that \( T_{s+1}(\sigma) = T_s(\rho) \). From this it follows that \( [T_{s+1}] \subseteq [T_s] \) for all \( s \), and that \( [T] = \bigcap_s [T_s] \). Thus \( Q = [T] \) will be a nonempty perfect \( \Pi^0_1 \) subclass of \( \{0,1\}^\mathbb{N} \).

In presenting our construction of \( T = \lim_s T_s \), we shall use the notation

\[
P_{e,s} = \{ X \in \{0,1\}^\mathbb{N} | \varphi_e(s) \downarrow \}.
\]

Note that \( P_{e,s} = \bigcup_\tau [\tau] \) where the union is taken over a finite subset of \( \{0,1\}^s \).

Note also that \( P_{e,s+1} \subseteq P_{e,s} \) for all \( s \in \mathbb{N} \), and that

\[
P_e = \{ X \in \{0,1\}^\mathbb{N} | \varphi_e(\downarrow) \} = \bigcap_s P_{e,s}.
\]

We now offer a preliminary account of the construction and proof. To each \( e, l, n \in \mathbb{N} \) with \( e \leq l \) and \( n \leq 2^l \) and each pairwise distinct sequence \( \sigma_1, \ldots, \sigma_n \in \{0,1\}^l \), we associate a requirement \( R(e, \sigma_1, \ldots, \sigma_n) \) at level \( l \). Intuitively, the purpose of this requirement is to insure that \( ([T(\sigma_1)] \times \cdots \times [T(\sigma_n)]) \cap P_e = \emptyset \) “if possible.” The strategy here will be to attempt to arrange that \( ([T(\sigma_1)] \times \cdots \times [T(\sigma_n)]) \cap P_e = \emptyset \) for all sufficiently large \( s \), “if possible.” We shall argue that if this attempt fails, then \( (Q \cap [T(\sigma_1)]) \times \cdots \times (Q \cap [T(\sigma_n)]) \subseteq P_{e,s} \) for all sufficiently large \( s \), hence \( (Q \cap [T(\sigma_1)]) \times \cdots \times (Q \cap [T(\sigma_n)]) \subseteq P_e \).

We now give the detailed construction of \( T_s \) for all \( s \in \mathbb{N} \). Begin by fixing a recursive linear ordering of all of the requirements, called the priority ordering. Arrange this ordering so that for each \( l \in \mathbb{N} \), all requirements at level \( l \) are of lower priority than all requirements at level \( l < l \). The idea here is that requirements at level \( l \) can be “injured” by requirements at level \( l < l \) but will never be “injured” by requirements at level \( l > l \). Note that for each \( l \) there are only finitely many requirements at level \( l \). We proceed by induction on \( s \).

Stage 0. Let \( T_0(\nu) = \nu \) for all \( \nu \in \{0,1\}^* \). Clearly \( T_0 \) is a uniform treemap.

Stage \( s + 1 \). Assume inductively that we have defined a uniform treemap \( T_s \). Our task at this stage is to define \( T_{s+1} \). A requirement \( R(e, \sigma_1, \ldots, \sigma_n) \) at level \( l \) is said to be requesting attention at stage \( s \) if \( ([T_e(\sigma_1)] \times \cdots \times [T_e(\sigma_n)]) \cap P_{e,s} \neq \emptyset \) but there exist \( k \leq s \) and \( \rho_1, \ldots, \rho_n \in \{0,1\}^k \) such that \( \rho_i \supseteq \sigma_i \) for all \( i = 1, \ldots, n \) and \( ([T_e(\rho_1)] \times \cdots \times [T_e(\rho_n)]) \cap P_{e,s} = \emptyset \). If no requirements are requesting attention, do nothing, i.e., let \( T_{s+1}(\nu) = T_s(\nu) \) for all \( \nu \in \{0,1\}^* \).

Otherwise, let \( R_e \) be the requirement of highest priority which is requesting attention. For this requirement only, choose \( k > l \geq e \) and \( \rho_1, \ldots, \rho_n \) as above and define \( T_{s+1} \) as follows. For each \( \nu \in \{0,1\}^l \) let \( T_{s+1}(\nu) = T_s(\nu) \). For each \( i = 1, \ldots, n \) let \( T_{s+1}(\sigma_i) = T_s(\rho_i) \). For each \( \sigma \in \{0,1\}^l \) other than \( \sigma_1, \ldots, \sigma_n \), let \( T_{s+1}(\sigma) = T_s(\sigma \downarrow (0,\ldots,0)) \). For each \( \sigma \in \{0,1\}^l \) and each \( \nu \in \{0,1\}^* \), let

\[
T_{s+1}(\sigma \downarrow \nu) = T_s(\rho \downarrow \nu) \quad \text{where} \quad \rho \text{ is such that } T_{s+1}(\sigma) = T_s(\rho) \text{. Clearly } T_{s+1} \text{ is a treemap, and it is uniform because } T_s \text{ is uniform and for each } \sigma \in \{0,1\}^l \text{ we have } T_{s+1}(\sigma) = T_s(\rho) \text{ for some } \rho \in \{0,1\}^k.
\]
We now have a recursive nested sequence of uniform treemaps $T_s$. As $s$ goes to infinity, consider the history of a single requirement $R = R(e, \sigma_1, \ldots, \sigma_n)$ at level $l$. Let us say that $R$ is satisfied at stage $s$ if \((\big{[} T_s(\sigma_1) \big{]} \times \cdots \times \big{[} T_s(\sigma_n) \big{]} ) \cap P_{e,s} = \emptyset\), otherwise unsatisfied at stage $s$. By construction, if $R = R_s$ then $R$ is unsatisfied at stage $s$ but becomes satisfied at stage $s + 1$. And if $R$ is satisfied at stage $s$, it remains satisfied at stage $s + 1$ unless $R_s$ is at level $< l$. Let us say that $R$ is injured at stage $s$ if the latter case holds, i.e., if the level of $R_s$ is less than the level of $R$. By induction along the priority ordering, we now see that there are only finitely many stages $s$ at which $R$ is injured, and only finitely many stages $s$ at which $R$ requests attention. This holds for all of the finitely many requirements at level $\leq l$, so for all $\sigma \in \{0, 1\}^l$ and all sufficiently large $s$ we have $T_{s+1}(\sigma) = T_s(\sigma)$. We now have a uniform treemap $T$ defined by $T(\sigma) = \lim_{s \to \infty} T_s(\sigma)$, so letting $Q = [T] = \bigcap_s [T_s]$ we have a nonempty perfect $\Pi^0_1$ class $Q \subseteq \{0, 1\}^N$.

We claim that for each requirement $R$ at level $l$ as above, either \((\big{[} T(\sigma_1) \big{]} \times \cdots \times \big{[} T(\sigma_n) \big{]} ) \cap P_e = \emptyset\) or \((Q \cap \big{[} T(\sigma_1) \big{]} ) \times \cdots \times (Q \cap \big{[} T(\sigma_n) \big{]} ) \subseteq P_e\). Suppose not. Then \((\big{[} T(\sigma_1) \big{]} ) \times \cdots \times \big{[} T(\sigma_n) \big{]} ) \cap P_e \neq \emptyset\) and \((Q \cap \big{[} T(\sigma_1) \big{]} ) \times \cdots \times (Q \cap \big{[} T(\sigma_n) \big{]} ) ) \setminus P_e \neq \emptyset\).

For $i = 1, \ldots, n$ fix $X_i \in Q \cap [T(\sigma_i)]$ so that $X_1 \oplus \cdots \oplus X_n \notin P_e$. Let $m$ be so large that \((\big{[} X_1[m] \big{]} \times \cdots \times \big{[} X_n[m] \big{]} ) \cap P_e = \emptyset\). Let $k > l$ and $\rho_1, \ldots, \rho_n \in \{0, 1\}^k$ be such that $T(\rho_i) \subset X_i$ and $|T(\rho_i)| \geq m$ for all $i = 1, \ldots, n$. Thus we have $\rho_i \supseteq \sigma_i$ for all $i = 1, \ldots, n$, and \((\big{[} T(\rho_1) \big{]} ) \times \cdots \times \big{[} T(\rho_n) \big{]} ) \cap P_e = \emptyset\). It follows by compactness that for all sufficiently large $s$ we have \((\big{[} T(\rho_1) \big{]} ) \times \cdots \times \big{[} T(\rho_n) \big{]} ) \cap P_{e,s} = \emptyset\). And we can also let $s$ be so large that $e \leq l < k \leq s$ and $T_s(\sigma_i) = T(\sigma_i)$ and $T_s(\rho_i) = T(\rho_i)$ for all $i = 1, \ldots, n$. So now we see that \((\big{[} T_s(\sigma_1) \big{]} ) \times \cdots \times \big{[} T_s(\sigma_n) \big{]} ) \cap P_{e,s} \neq \emptyset\) and \((\big{[} T_s(\rho_1) \big{]} ) \times \cdots \times \big{[} T_s(\rho_n) \big{]} ) \cap P_{e,s} \neq \emptyset\) for all sufficiently large $s$. Thus our requirement $R$ is requesting attention at all sufficiently large stages $s$. This contradiction proves our claim.

The above claim gives us the principal conclusion of Theorem 6.1. And then, as we have seen, Lemma 6.5 gives us the “moreover” clause with $m = m_e$. The proof of Theorem 6.1 is now complete.

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