Option Pricing with Lie Symmetry Analysis and Similarity Reduction Method

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Abstract

With some transformations, we convert the problem of option pricing under state-dependent volatility into an initial value problem of the Fokker-Planck equation with a certain potential. By using the Lie symmetry analysis and similarity reduction method, we are able to reduce the dimensions of the partial differential equation and find some of its particular solutions of the equation. A few case studies demonstrate that our new method can be used to produce analytical option pricing formulas for certain volatility functions.

Keywords: Option pricing; Lie symmetry analysis; Similarity reduction; Analytical solution

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1 Introduction

The landmark works of Black and Scholes (1973) and Merton (1973) have created a new field in quantitative finance. In the Black-Scholes/Merton framework, the price of an underlying asset is often modeled as a diffusion process. With a no-arbitrage argument, the price of a derivative contract written on the asset can be determined by solving an initial boundary value problem of a linear partial differential equation (PDE). In the classical Black-Scholes model, the volatility of the underlying asset, \( \sigma \) is assumed to be constant. In order to explain the empirical phenomenon of implied volatility smirk, see e.g., Zhang and Xiang (2008), researchers propose to use volatilities defined by deterministic functions of the underlying asset price and time. The corresponding PDE is often called the generalized Black-Scholes equation. Analytical formulas of the problem for the general case is not available. However, the problem of reducibility

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and solvability of the generalized Black-Scholes equation has been studied by Carr et al. (1999, 2002, 2006), Bouchouev (1998), Li and Zhang (2004), and Zhang and Li (2012). Haven (2005) suggested a solution technique for obtaining analytical solutions to the generalized Black-Scholes equation via an adiabatic approximation to the Schrödinger PDE.

In 1891, a famous mathematician, Sophus Lie, pointed out that, if an ordinary differential equation (ODE) is invariant under a one-parameter Lie group of transformations, the order of the ODE can be reduced constructively. The method of finding similarity reductions of a given PDE by using the Lie group method of infinitesimal transformation (sometimes called the method of group-invariant solutions) was originally developed by Lie (1891), see Olver (1993) for the recent developments. Bluman and Cole (1969) proposed a generalization of Lie’s method which is called the nonclassical method of group-invariant solutions. The method was further generalized by Olver and Rosenau (1986). A common feature of these methods is to determine Lie point transformations of a given PDE, i.e., transformations that depend only on the independent and dependent variables, see equation (13). After that, Lie group analysis was widely applied in solving differential equations in fluid mechanics and quantum mechanics. Nowadays Lie symmetry software packages are widely used in solving PDEs. Reviews and comparative studies of some of the earlier computer algebra packages have been carried out by Hereman (1997) and Butcher et al. (2003). More recently, Rocha Filho and Figueiredo (2011) presented the new MAPLE package SADe for the determination of symmetries and related properties of systems of differential equations. Vu et al. (2012) presented the new MAPLE symmetry package DESOLVII, an upgrade of DESOLV, which included the functionality to determine higher classical symmetries for both ordinary and partial differential equations. However, currently, in the situation that coefficient functions contain arbitrary functions, software packages for symmetry analysis cannot handle.

In fluid mechanics and quantum mechanics, the determination of the symmetry group of Fokker-Planck equations has a long history. Finkel (1999) completely classified the symmetries of the Fokker-Planck equation and constructed group-invariant solutions for a physically interesting family of Fokker-Planck equations in the case of two spatial dimensions, namely

$$u_t(x, y, t) - \frac{1}{2} \Delta u(x, y, t) + M(x, y, t)u = 0,$$

where $u$ is a dependent variable and $M(x, y, t)$ is a potential function. Building on Finkel’s result, Laurence and Wang (2005) found some closed-form fundamental solutions for a special family of Fokker-Planck equations. They showed how these results can be applied in finance to yield exact solutions for special affine and quadratic two-factor term structure models. In this paper, we not only show how to generate a series of new solutions with a given solution by using the last set of equations in Appendix A, but also perform similarity reductions of different cases.

In quantitative finance, Lo and Hui (2001) presented Lie-algebraic method for the valuation of financial derivatives with time-dependent parameters based upon the Wei-Norman theorem. Lo and Hui (2002) extended their Lie-algebraic approach for the valuation of multi-asset financial derivatives in a lognormal framework with time-dependent parameters (drift, standard-deviation, correlation), involving also
stochastic short-term interest rates. Lo and Hui (2006) proposed also a Lie-algebraic model for pricing more complex derivatives like moving barrier options with time-dependent parameters in a CEV framework. The difference between Lo and Hui’s Lie-algebraic approach and our Lie symmetry approach is as follows. In our Lie symmetry approach, we obtain similarity reduction by using a one-parameter invariant group of partial differential equations. Lie algebras are by-products after we obtain the vector fields in equation (20). However, Lo and Hui (2001) start from a Lie algebra, which is elevated to a group via an exponential mapping. Carr, Laurence and Wang (2006) performed the classification of driftless time and state dependent diffusions that are integrable in closed form via Lie’s equivalence transformations. However, the Lie symmetry analysis and similarity reduction of the generalized Black-Scholes equation (with general volatility function) are not available yet.

In this paper, we try to solve the problem of option pricing based on the theory of the Fokker-Planck equation. The 2-dimensional generalized Black-Scholes equation, arising from option pricing, can be transformed into the 2 + 1-dimensional Fokker-Planck equation. We demonstrate how to apply Lie symmetry analysis and similarity reduction to solve the option pricing problem for volatility as a function of underlying asset price.

Compared with Lo and Hui’s approach, our Lie symmetry approach is more systematic. The main purpose of this paper is to demonstrate the methodology by using a state-dependent volatility, $\sigma(S)$. If the volatility is state- and time-dependent, $\sigma(S,t)$, then the potential function, $M$, in equation (9) is also a function of time, i.e., $M(x,y) \rightarrow M(x,y,t)$. Our approach can be used to handle the case in principle as shown by equation (12). The application to the case of state-and-time-separable volatility, $\sigma(S,t) = \sigma_1(S)\sigma_2(t)$, will be reported in a subsequent research. For the case of only one CEV process, the parameter $\alpha$ in our Section 5.2 can take any non-negative value, while Lo and Hui (2001, 2006) focus on $0 < \beta < 2$, which is equivalent to our $0 \leq \alpha < 1$.

This paper is organized as follows. Section 2 discusses how to transform a typical option pricing problem into the Fokker-Planck equation like (1). Section 3 applies the Lie symmetry analysis to the equation. Section 4 presents the similarity reductions of different cases. Section 5 provides a few exact solutions of both 2-dimensional and 1-dimensional generalized Black-Scholes equation. Finally, section 6 concludes.

2 Typical option pricing problem

In the Black-Scholes (1973)/Merton’s (1973) framework, the prices of two stocks, $S_1$, $S_2$, are modeled by two pure diffusion processes

$$dS_i = \mu_i S_i dt + \tilde{\sigma}_i(S_i) S_i dB_i, \quad (i = 1, 2),$$

where $\mu_i$ is the drift, $\tilde{\sigma}_i(S_i)$ is the volatility of the stock $i$, and $B_i (i = 1, 2)$ are standard Brownian motions. The correlation coefficient between $B_1$ and $B_2$ is $\rho$. The correlation makes it harder to convert the equation to 2 + 1 dimensional Fokker-Planck equation like (1). In this paper, we only consider the

1 Lo and Hui’s (2001, 2006) time-dependent CEV, $\sigma(t)S^{\beta/2}$, is a special case of state-and-time-separable volatility.
case where the volatility, $\sigma$, is a deterministic function of the stock price, and leave the general case of time dependence for future research. Standard no-arbitrage theory shows that the price of a European style option, $c(S_1, S_2, t)$, satisfies the following generalized Black-Scholes equation

$$\frac{\partial c}{\partial t} + \frac{1}{2} \sigma_1^2(S_1) \frac{\partial^2 c}{\partial S_1^2} + \rho \sigma_1(S_1) \sigma_2(S_2) \frac{\partial^2 c}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2(S_2) \frac{\partial^2 c}{\partial S_2^2} + r S_1 \frac{\partial c}{\partial S_1} + r S_2 \frac{\partial c}{\partial S_2} - rc = 0, \quad (3)$$

$$c(S_1, S_2, T) = C(S_1, S_2), \quad (4)$$

where correlation coefficient $\rho$ and interest rate $r$ are assumed to be constant; $C(S_1, S_2)$ is the payoff function of the option on the maturity date ($t = T$). For brevity, we have used $\sigma_i(S_i) = \sigma_i(S_1)$.

In the general case, analytical formulae of the problem (3) and (4) cannot be obtained. Practitioners rely on numerical methods such as finite difference, binomial trees, or Monte Carlo simulation. However, in the way of reducibility and solvability of the 1+1-dimensional generalized Black-Scholes equation, Li and Zhang (2004) determined the boundary condition and the nature of the eigenvalues and eigenfunctions with Weyl-Titchmarsh theory. The solution can be written analytically in a Stieltjes integral. Zhang and Li (2012) provide a systematic way of finding the volatility function, $\sigma(S)$, for a given solvable potential function.

Analytical solutions for the generalized Black-Scholes equation are of paramount importance to practitioners as they allow a better qualitative understanding of the solution behavior. More significantly, volatility functions are typically fitted to market data in empirical research. Parametric volatility models that produce analytical solutions are in very high demand.

For certain volatility functions, e.g., a volatility being a quadratic function of asset price studied by Zühlsdorff (2001), the generalized Black-Scholes equation can be transformed into the standard heat equation, which, in turn, can be solved analytically. Even for the case where the problem cannot be reduced to the standard heat equation, it is still possible to solve the problem analytically for some particular volatility functions. This paper pushes further along this direction.

With the following transformation

$$\begin{cases}
x = \sqrt{\frac{2}{1 + \rho}} \left( \int_0^{S_2} \frac{1}{\sigma_2(S)} dS + \int_{S_1}^{S_2} \frac{1}{\sigma_1(S)} dS \right), \\
y = \sqrt{\frac{2}{1 - \rho}} \left( \int_0^{S_2} \frac{1}{\sigma_2(S)} dS - \int_0^{S_1} \frac{1}{\sigma_1(S)} dS \right),
\end{cases} \quad (5)$$

equations (3) and (4) become

$$\frac{\partial c}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) + Q_1 \frac{\partial c}{\partial x} + Q_2 \frac{\partial c}{\partial y} - rc = 0, \quad (6)$$

$$c(x, y, T) = C(x, y), \quad (7)$$

where

$$\begin{cases}
Q_1 = \sqrt{\frac{2}{1 + \rho}} \left( \frac{r S_1}{\sigma_1} + \frac{r S_2}{\sigma_2} \right) - \frac{2}{1 + \rho} \left( \frac{\sigma_{1x}}{\sigma_1} + \frac{\sigma_{2x}}{\sigma_2} \right) + \frac{2}{\sqrt{1 - \rho^2}} \left( \frac{\sigma_{1y}}{\sigma_1} - \frac{\sigma_{2y}}{\sigma_2} \right), \\
Q_2 = \sqrt{\frac{2}{1 - \rho}} \left( \frac{r S_1}{\sigma_1} + \frac{r S_2}{\sigma_2} \right) - \frac{2}{\sqrt{1 - \rho^2}} \left( \frac{\sigma_{1x}}{\sigma_1} - \frac{\sigma_{2x}}{\sigma_2} \right) - \frac{2}{1 - \rho} \left( \frac{\sigma_{1y}}{\sigma_1} + \frac{\sigma_{2y}}{\sigma_2} \right),
\end{cases}$$

$$c(S_1, S_2, t),$$
$\sigma_{ix}$ and $\sigma_{iy}$ ($i = 1, 2$) stand for partial derivative of $\sigma_i$ with respect to $x$ and $y$ respectively, $S_i$ is a function of $x, y$, which can be solved by equation 5. Consequently, $\sigma_i(S_i)$ converts to $\sigma_i(x,y)$. For brevity, we have replaced $\sigma_i(x,y)$ by $\sigma_i$.

We introduce the following transformation

$$c(x, y, t) = e^{\omega(x,y) - rt} u(x, y, \tau), \quad \tau = T - t,$$

where $\nabla \omega(x,y) = -(Q_1, Q_2)$. Here we need following compatibility condition

$$\frac{\partial Q_2}{\partial x} = \frac{\partial Q_1}{\partial y}.$$

By a simple calculation, equations (6) and (7) become 2+1-dimensional Fokker-Planck equation like (1):

$$\frac{\partial u}{\partial \tau} - \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + M(x,y)u = 0,$$

$$u(x,y,\tau) |_{\tau=0} = u_0(x,y),$$

where the coefficient $M(x,y)$, regarded as a potential function, reads

$$M(x,y) = \frac{1}{2} \left( \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + Q_1^2 + Q_2^2 \right).$$

Similarly, we can convert the 1-dimensional generalized Black-Scholes equation to 1+1-dimensional Fokker-Planck equation. Carr, Laurence and Wang (2006) exploit a remarkable intertwining with the inhomogeneous Burger’s equation in the time dependent and state dependent one dimensional case via point transformations. By using the separating variable method, Li and Zhang (2004), and Zhang and Li (2012) transformed the option pricing problem into a Schrödinger equation which is similar to the 1+1-dimensional Fokker-Planck equation studied here.

### 3 Lie point symmetries

We now perform Lie symmetry analysis for the 2+1-dimensional Fokker-Planck equation. Let us consider a 2+1-dimensional equation

$$F(u) = u_t(x,y,t) - \frac{1}{2} \Delta u(x,y,t) + M(x,y,t)u.$$

### Footnotes

2If the condition is not satisfied, the generalized Black-Scholes equation will be converted into a general case of Fokker-Planck equation, instead of the irrotational case studied in this paper. It is possible to study the solution of the general case of Fokker-Planck equation by using Lie symmetry approach. The result will be reported in a subsequent research.

3Li and Zhang (2004), and Zhang and Li (2012) study the pricing of European options written on a single asset, while we are studying case of two assets. Their transformation is similar to a single-asset case of ours here without the drift of risk-free rate.
and a one-parameter Lie group of infinitesimal transformation

\[ t \rightarrow t + \epsilon T(x, y, t, u), \]
\[ x \rightarrow x + \epsilon X(x, y, t, u), \]
\[ y \rightarrow y + \epsilon Y(x, y, t, u), \]
\[ u \rightarrow u + \epsilon U(x, y, t, u). \]  

(13)

With a small parameter \( \epsilon \ll 1 \), the vector field associated with the group of transformations (13) can be written as

\[ u = T \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + U \frac{\partial}{\partial u}. \]  

(14)

or equivalently in the symmetry form

\[ \sigma = U - T u_t - X u_x - Y u_y. \]  

(15)

We wish to determine all possible coefficient functions \( X, Y, T \) and \( U \), so that the corresponding one-parameter group is a symmetry group of the Fokker-Planck equation. The symmetry equation, i.e. the corresponding infinitesimal criterion becomes

\[ \frac{\partial}{\partial \epsilon} F(u + \epsilon \sigma) \bigg|_{\epsilon=0}^{F(u)=0} = 0. \]  

(16)

Based on (1), i.e. \( F(u) = 0 \), substituting \( u_t \) by \( \frac{1}{2} \Delta u - Mu \) whenever it occurs gives an equation, of which left hand side is a polynomial with \( u, u_x, u_y, u_{xx}, u_{yy}, u_{xy} \) and right hand side is 0. Taking the coefficients of the various monomials in the first and second order partial derivatives of \( u \) in the polynomial be 0, we find the determining equations for the symmetry group of the Fokker-Planck equation.

By solving them, an invariance of equation (1) under transformation (13) leads to the expressions for the functions \( T, X, Y, U \) of the form (throughout this paper we use symbolic package MAPLE to perform all calculations)

\[
\begin{align*}
T &= f_1, \\
X &= \frac{1}{2} \left( \frac{\partial f_1}{\partial t} \right) x + k y + f_2, \\
Y &= \frac{1}{2} \left( \frac{\partial f_1}{\partial t} \right) y - k x + f_3, \\
U &= - \left[ \frac{1}{4} \left( \frac{\partial^2 f_1}{\partial t^2} \right) (x^2 + y^2) + \left( \frac{\partial f_3}{\partial t} \right) x + \left( \frac{\partial f_3}{\partial t} \right) y + f_4 \right] u + g,
\end{align*}
\]  

(17)

and the compatibility condition

\[ T_t M + X M_x + Y M_y + U_u M = 0. \]  

(18)

where \( k \) is arbitrary constant, and \( f_i, \ i = 1, \ldots, 4 \) are arbitrary functions of \( t \), which satisfy the condition (18), and \( g \) is the solution of the original equation (1). Similar mathematical results were given by Finkel (1999) and Laurence and Wang (2005) by using a prolongation method. Nowadays computer algebra

\[ \text{More explanation of the treatment and the meaning of the variables can be found in Chapters 2 and 3 of Olver’s (1993) book.} \]
packages are widely used in determination of symmetries of differential equations. However, in this case, 
$T, X, Y, U$ contain the arbitrary functions of $t$ and the arbitrary function of $x, y, t$, which current software
packages for symmetry analysis, such as DESOLV, DESOLVII and SADE, cannot handle. Therefore, we
will manually use Lie symmetry analysis to deal with the Fokker-Planck equation.

The presence of these arbitrary functions leads to an infinite-dimensional Lie algebra of symmetries. A
general element of this algebra is written as

$$
\mathfrak{g} = \mathfrak{u}_1 k + \mathfrak{u}_2 (f_1) + \mathfrak{u}_3 (f_2) + \mathfrak{u}_4 (f_3) + \mathfrak{u}_5 (f_4) + \mathfrak{u}_6 (g).
$$

Let $\varphi_i$ be arbitrary functions of $t, \psi$ and $\phi$ be arbitrary functions of $x, y, t$, then

$$
\begin{align*}
\mathfrak{u}_1 &= \psi \frac{\partial}{\partial t} - x \frac{\partial}{\partial y}, \\
\mathfrak{u}_2 (\varphi_i) &= \varphi_i \frac{\partial}{\partial t} + \frac{1}{2} \phi_i x \frac{\partial}{\partial x} + \frac{1}{2} \phi_i y \frac{\partial}{\partial y} + \frac{1}{4} \phi_i (x^2 + y^2) u \frac{\partial}{\partial u}, \\
\mathfrak{u}_3 (\varphi_i) &= \phi_i \frac{\partial}{\partial x} + \phi_i y u \frac{\partial}{\partial u}, \\
\mathfrak{u}_4 (\varphi_i) &= \phi_i \frac{\partial}{\partial y} + \phi_i x u \frac{\partial}{\partial u}, \\
\mathfrak{u}_5 (\varphi_i) &= \phi_i u \frac{\partial}{\partial u}, \\
\mathfrak{u}_6 (\psi) &= \psi \frac{\partial}{\partial u}.
\end{align*}
$$

The commutation relations between all these vector fields are given by Table 1.

| $\mathfrak{u}_1$ | $\mathfrak{u}_2 (\varphi_j)$ | $\mathfrak{u}_3 (\varphi_j)$ | $\mathfrak{u}_4 (\varphi_j)$ | $\mathfrak{u}_5 (\varphi_j)$ | $\mathfrak{u}_6 (\phi)$ |
|-----------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 0               | $\mathfrak{u}_2 (\varphi_j)$ | $\mathfrak{u}_3 (\varphi_j)$ | $\mathfrak{u}_4 (\varphi_j)$ | $\mathfrak{u}_5 (\varphi_j)$ | $\mathfrak{u}_6 (\phi)$ |
| $\mathfrak{u}_2 (\varphi_i)$ | $\mathfrak{u}_2 (\varphi_j - \phi_i \varphi_j)$ | $\mathfrak{u}_3 (\varphi_j)$ | $\mathfrak{u}_4 (\varphi_j)$ | $\mathfrak{u}_5 (\varphi_j)$ | $\mathfrak{u}_6 (\phi)$ |
| $\mathfrak{u}_3 (\varphi_i)$ | $\mathfrak{u}_5 (\varphi_i \varphi_j - \phi_i \varphi_j)$ | 0 | $\mathfrak{u}_5 (\varphi_j)$ | $\mathfrak{u}_6 (\phi)$ |
| $\mathfrak{u}_4 (\varphi_i)$ | $\mathfrak{u}_5 (\varphi_i \varphi_j - \phi_i \varphi_j)$ | 0 | 0 | $\mathfrak{u}_6 (\phi)$ |
| $\mathfrak{u}_5 (\varphi_i)$ | $\mathfrak{u}_5 (\varphi_i \varphi_j - \phi_i \varphi_j)$ | 0 | 0 | 0 |
| $\mathfrak{u}_6 (\psi)$ | $\mathfrak{u}_6 (\varphi_i \varphi_j)$ | 0 | $\mathfrak{u}_6 (\varphi_i \varphi_j)$ | 0 | 0 |
| $\mathfrak{u}_6 (\phi)$ | $\mathfrak{u}_6 (\varphi_i \varphi_j)$ | 0 | $\mathfrak{u}_6 (\phi)$ | 0 | 0 |

The entry in row $i$ and column $j$ representing $[\mathfrak{u}_i, \mathfrak{u}_j]$.

From Table 1, we see that $\mathfrak{u}_2 (\varphi), \mathfrak{u}_3 (\varphi), \mathfrak{u}_4 (\varphi), \mathfrak{u}_5 (\varphi)$ constitute a subalgebra. And there exist some types of interesting subalgebras, For instance, Virasoro algebra and $\omega_\infty$-type algebra.

Furthermore, we find that the transform $[\mathfrak{u}_1, \mathfrak{u}_6 (\psi)] = \mathfrak{u}_6 (y \psi_x - x \psi_y)$ is invariant, if $M(x, y)$ satisfies the type $C \cdot (x^2 + y^2)$, where $C$ is an arbitrary constant. In other words, if $g$ is a solution of the Fokker-Planck equation like this, then $yg_x - xg_y$ is another solution of the same equation.

Moreover, we get a series of transformations of the solution. New solutions can be generated through them with a known solution. The one-parameter groups generated by $\mathfrak{u}_i$ and the transformations are included in the Appendix A for the readers with an interest in the details of applying the theory.
4 Similarity reductions

After determining the infinite-dimensional algebra of symmetries, the similarity variables can be found by solving the characteristic equations

\[
\frac{dt}{T} = \frac{dx}{X} = \frac{dy}{Y} = \frac{du}{U}.
\]

(21)

By solving the ordinary differential equations (21), we can obtain integration constants \( \xi, \eta, P \). Substituting \( \xi, \eta, P \) for \( x, y, t, u \) in original equation (1), we can reduce the equation from 2 + 1-dimensional to 2-dimensional finally. This process is called similarity reductions.

Since there are many arbitrary functions in \( T, X, Y, U \), it is hard to solve the equations (21) in the general case. Likewise, it is also hard to solve them by substituting generators (20). Finkel (1999) completely classified the symmetries of the Fokker-Planck equation based on the compatibility condition (18). For simplicity, he has dropped out the two trivial infinitesimal symmetries \( \partial_t \) and \( u \partial_u \) in his classification result. It means that constant terms are omitted in the forms for \( f_1 \) and \( f_4 \).

In this subsection, we will list some cases in details for reductions as the classification done by Finkel, which are helpful for the following subsections. Other cases of the Fokker-Planck equation for reductions are included in Appendix B.

- Case 1.1a

\[
\begin{align*}
M &= \frac{C_0}{x^2} + by + c_0, \quad C_0 \neq 0, \\
\xi &= \frac{x}{\sqrt{\delta_3^2 t^2 + \delta_1 t}}, \\
\eta &= \frac{2y\delta_1^2 - b_0^2 \delta_1^2 t^2 + 4(2\beta_0 \delta_2 - \beta_1 \delta_1) t + 4\beta_0 \delta_1}{2\delta_1^2 \sqrt{\delta_3^2 t^2 + \delta_1 t}}, \\
P &= u \cdot \exp \left\{ \frac{1}{\delta_1} \left( \eta \delta_1^2 \sqrt{\delta_2 t^2 + \delta_1 t} (\delta_1^2 b t + 2\delta_1 \beta_1 - 4\delta_2 \beta_0) + \delta_1 \ln(\delta_2 t + \delta_1)(\delta_1^2 + 2\delta_1 \beta_0 \beta_1 - 2\delta_2 \beta_0^2) \right. \\
&\quad \left. + \beta_0 \delta_1 \ln(t)(2\beta_0 \delta_2 - 2\beta_1 \delta_1) + \frac{1}{3} b^2 \delta_1^4 t^3 + 2b_0 \delta_2^2 (\delta_1 \beta_1 - 2\delta_2 \beta_0) t^2 \\
&\quad + [8\beta_0^2 \delta_2^2 - 8\beta_0 \beta_1 \delta_1 \delta_2 - 2b_0 \delta_0 \delta_1^3 + \delta_1^4 c_0 + 2\beta_0^2 \delta_1^3 + \frac{1}{2} \delta_1^4 \delta_2 (\xi^2 + \eta^2)] t \right\}.
\end{align*}
\]

(22)

and the reduced PDE becomes

\[
\delta_1^3 \xi^2 (P_{\xi \xi} + P_{\eta \eta}) + \delta_1^3 \xi^3 P_{\xi} + \delta_1^3 \xi^2 P_{\eta} + (4\delta_2 \beta_0^2 \xi^2 - 4\delta_1 \beta_0 \beta_1 \xi^2 - 2C_0 \delta_1^2) P = 0.
\]

(24)

We get the solution by the method of separation of variables

\[
P = F_1(\xi) F_2(\eta),
\]

(25)
where $F_1(\xi)$ and $F_2(\eta)$ is
\[
\begin{align*}
F_1(\xi) &= \frac{e^{-2\frac{\xi^2}{\delta_1}}}{\sqrt{\delta_1}} \left[ C_1 \text{WhittakerM} \left( \frac{c_1}{2\delta_1} - 1, \frac{1}{4}, \frac{\sqrt{8c_1+1}}{4}, \frac{\delta_1 \xi^2}{2} \right) + C_2 \text{WhittakerW} \left( \frac{c_1}{2\delta_1} - 1, \frac{1}{4}, \frac{\sqrt{8c_1+1}}{4}, \frac{\delta_1 \xi^2}{2} \right) \right], \\
F_2(\eta) &= \frac{e^{-2\frac{\eta^2}{\delta_2}}}{\sqrt{\delta_2}} \left[ C_3 \text{WhittakerM} \left( \frac{c_1}{2\delta_2} - 2\frac{\beta_1}{\delta_1^2} + \frac{2\beta_2}{\delta_2^2} - 1, \frac{1}{4}, \frac{\delta_2 \eta^2}{2} \right) + C_4 \text{WhittakerW} \left( \frac{c_1}{2\delta_2} - 2\frac{\beta_1}{\delta_1^2} + \frac{2\beta_2}{\delta_2^2} - 1, \frac{1}{4}, \frac{\delta_2 \eta^2}{2} \right) \right].
\end{align*}
\]
where $c_1, C_1, C_2, C_3, C_4$ are arbitrary constants.

- **Case 1.2b**

\[
\begin{align*}
M &= \frac{C(\theta)}{r^2} + cr^2 + c_0, \quad c \neq 0, \\
f_1 &= \delta_1 e^{2\sqrt{\delta_1} t} + \delta_2 e^{-2\sqrt{\delta_1} t}, \quad k = 0, \quad f_2 = f_3 = 0, \\
f_4 &= (\sqrt{2c} + c_0)\delta_1 e^{2\sqrt{\delta_1} t} - (\sqrt{2c} - c_0)\delta_2 e^{-2\sqrt{\delta_1} t}.
\end{align*}
\]
where $C(\theta) \neq (c_1 \cos \theta + c_2 \sin \theta)^{-2}, C'(\theta) \neq 0,$ and $r = \sqrt{x^2 + y^2}.$

We have the similarity variables $\xi, \eta, P$:
\[
\begin{align*}
\xi &= \frac{xe^{\sqrt{\delta_1} t}}{\sqrt{\delta_1 e^{2\sqrt{\delta_1} t} + \delta_2}}, \\
\eta &= \frac{ye^{\sqrt{\delta_1} t}}{\sqrt{\delta_1 e^{2\sqrt{\delta_1} t} + \delta_2}}, \\
P &= e^{-c_0 t}\sqrt{\delta_1 e^{2\sqrt{\delta_1} t} + \delta_2} u \exp \left\{ \frac{1}{2} \left( (\delta_1 e^{2\sqrt{\delta_1} t} - \delta_2 e^{-2\sqrt{\delta_1} t}) (\xi^2 + \eta^2) - 2t \right) \right\},
\end{align*}
\]
and the reduced PDE becomes
\[
(\xi^2 + \eta^2)(P_{\xi\xi} + P_{\eta\eta}) + 2[4c_1\delta_1\xi^2 + \eta^2)^2 - C(\theta)]P = 0. \tag{28}
\]

With the transformation $\xi = \varrho \cos \theta, \eta = \varrho \sin \theta,$ \ref{28} becomes
\[
\varrho^2 P_{\varrho\varrho} + \varrho P_{\varrho} + P_{\theta\theta} + 2[4c_1\delta_1\varrho^4 - C(\theta)]P = 0. \tag{29}
\]

We can get the solution by the method of separation of variables
\[
P = F_1(\varrho)F_2(\theta), \tag{30}
\]
where $F_1(\varrho), F_2(\theta)$ is the solution of
\[
\begin{align*}
\frac{d^2 F_1(\varrho)}{d\varrho^2} + \frac{1}{\varrho} \frac{dF_1(\varrho)}{d\varrho} + \left( 8c_1\delta_2 \varrho^2 - \frac{c_1}{\varrho^2} \right) F_1(\varrho) &= 0, \\
\frac{d^2 F_2(\theta)}{d\theta^2} + (c_1 - 2C(\theta))F_2(\theta) &= 0,
\end{align*}
\]
where $c_1$ is an arbitrary constant. Given $C(\theta)$, the ODE systems \ref{31} can be solved directly.

- **Case 1.4b**

\[
\begin{align*}
M &= \frac{C_0}{r^2} + cr^2 + ax + by + c_0, \\
f_1 &= \delta_1 e^{2\sqrt{\delta_1} t} + \delta_2 e^{-2\sqrt{\delta_1} t}, \quad k = 0, \quad f_2 = f_3 = 0, \\
f_4 &= (\sqrt{2c} + c_0)\delta_1 e^{2\sqrt{\delta_1} t} - (\sqrt{2c} - c_0)\delta_2 e^{-2\sqrt{\delta_1} t},
\end{align*}
\]

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where \( r = \sqrt{x^2 + y^2} \). If \( \delta_1 \neq 0, \delta_2 \neq 0 \), then \( a = b = 0 \) should be held to make \( M \) satisfy the compatibility condition \([18]\). Obviously, this is the simplification of Case 1.2b. We have the similarity variables \( \xi, \eta, P \) same as \([27]\), then the reduced PDE becomes

\[
(\xi^2 + \eta^2)(P_{\xi\xi} + P_{\eta\eta}) + 2[4c\delta_2(\xi^2 + \eta^2)^2 - C_0]P = 0. \tag{33}
\]

With the transformation \( \xi = q \cos \theta, \eta = q \sin \theta \), we can get the solution by the method of separation of variables

\[
P = F_1(q)F_2(\theta), \tag{34}
\]

where \( F_1(q) \) and \( F_2(\theta) \) is

\[
\begin{align*}
F_1(q) &= C_1J\left(\frac{\sqrt{c_1}}{2}, \sqrt{2\delta_1}q c \rho^2\right) + C_2Y\left(\frac{\sqrt{c_1}}{2}, \sqrt{2\delta_1}q c \rho^2\right), \\
F_2(\theta) &= C_3 \sin(\theta \sqrt{c_1 - 2C_0}) + C_4 \cos(\theta \sqrt{c_1 - 2C_0}),
\end{align*}
\]

where \( c_1, C_1, C_2, C_3, C_4 \) are arbitrary constants, and \( J(\nu, z) \) and \( Y(\nu, z) \) are the Bessel functions of the first and second kinds, respectively.

## 5 Case studies

We now study a few cases, most of which are not well known in the financial literature. Our purpose here is to demonstrate the procedure of producing analytical option pricing formulas with the method of similarity reduction.

### 5.1 2–dimensional: Double CEV Model

In the traditional CEV Model (Cox 1975, Cox and Ross 1976, Schroder 1989), \( \sigma(S) = \sigma S^\alpha \). Base on their work, we try to build a Double CEV Model, which has two assets. Assuming

\[
\sigma_i(S_i) = \sigma_i S_i^{\alpha_i}, \tag{35}
\]

where \( \sigma_i > 0, \alpha_i > 0 \). From \([19]\), we have

\[
\begin{align*}
x &= \sqrt{\frac{2}{1 + \rho}} \left( \frac{S_1^{1-\alpha_1}}{\sigma_1(1 - \alpha_1)} + \frac{S_2^{1-\alpha_2}}{\sigma_2(1 - \alpha_2)} \right), \\
y &= \sqrt{\frac{1 - \rho}{1 + \rho}} \left( \frac{S_2^{1-\alpha_1}}{\sigma_1(1 - \alpha_1)} - \frac{S_1^{1-\alpha_2}}{\sigma_2(1 - \alpha_2)} \right),
\end{align*}
\]

and the following transformation \([8]\), where

\[
\begin{align*}
Q_1 &= \frac{r(1 - \alpha_1)}{2} \left( \frac{x - \sqrt{1 - \rho}}{1 + \rho} y \right) + \frac{r(1 - \alpha_2)}{2} \left( \frac{x + \sqrt{1 - \rho}}{1 + \rho} y \right) \\
&\quad - \frac{1 - \alpha_1}{\alpha_1} \left( \frac{1 + \rho}{2} x - \sqrt{1 - \rho^2} \right) - \frac{1 - \alpha_2}{\alpha_2} \left( \frac{1 + \rho}{2} x + \sqrt{1 - \rho^2} \right), \\
Q_2 &= \frac{r(1 - \alpha_2)}{2} \left( \frac{1 + \rho}{2} x + y \right) - \frac{r(1 - \alpha_1)}{2} \left( \frac{1 + \rho}{2} x - y \right) \\
&\quad - \frac{1 - \alpha_1}{\alpha_1} \left( \frac{\sqrt{1 - \rho^2}}{2} x - \frac{1 - \rho}{2} y \right) - \frac{1 - \alpha_2}{\alpha_2} \left( \frac{\sqrt{1 - \rho^2}}{2} x + \frac{1 - \rho}{2} y \right),
\end{align*}
\]

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and
\[ \omega(x, y) = 2\left(\frac{\alpha_1 \ln(x - y)}{1 - \alpha_1} + \frac{\alpha_2 \ln(x + y)}{1 - \alpha_2}\right) + \frac{r}{4}\left[\alpha_1(x - y)^2 + \alpha_2(x + y)^2 - 2(x^2 + y^2)\right] + C_0, \]
where \(C_0\) is an arbitrary constant.

Equation (3) becomes

\[ u_t(x, y, t) - \frac{1}{2}\Delta u(x, y, t) + M(x, y, t)u = 0, \tag{37} \]

where
\[ M(x, y, t) = \frac{48(x^2 + y^2)}{(x^2 - y^2)^2} + r^2(x^2 + y^2) - 18r. \tag{38} \]

For brevity we have taken \(\rho = 0, \alpha_1 = \alpha_2 = 2\).

(If \(\rho \neq 0\), the compatibility condition (18) is also satisfied with the following \(f_i, C\) and \(g\). Moreover, the similarity variables and the solution of the reduced PDE can be obtained. Here, taking \(\rho = 0\) is just for brevity.)

Obviously, the function \(M\) belongs to the **Case 1.2b**. Therefore, we take

\[
\begin{align*}
f_1 &= \delta_1 e^{2\sqrt{2}rt} + \delta_2 e^{-2\sqrt{2}rt}, & k = 0, \quad f_2 = f_3 = 0, \\
f_4 &= (\sqrt{2}r - 18r)\delta_1 e^{2\sqrt{2}rt} - (\sqrt{2}r + 18r)\delta_2 e^{-2\sqrt{2}rt}.
\end{align*}
\]

We have the similarity variables \(\xi, \eta, P\),

\[
\begin{align*}
\xi &= \frac{xe^{\sqrt{2}rt}}{\sqrt{\delta_1 e^{4\sqrt{2}rt} + \delta_2}}, \\
\eta &= \frac{ye^{\sqrt{2}rt}}{\sqrt{\delta_1 e^{4\sqrt{2}rt} + \delta_2}}, \\
P &= e^{-18rt}\sqrt{\delta_1 e^{4\sqrt{2}rt} + \delta_2} \cdot \exp\left\{\frac{1}{\sqrt{2}}r[(\delta_1 e^{2\sqrt{2}rt} - \delta_2 e^{-2\sqrt{2}rt})(\xi^2 + \eta^2) - 2t]\right\},
\end{align*}
\]

and the reduced PDE becomes

\[
(P_{\xi\xi} + P_{\eta\eta}) + 8r^2\delta_1\delta_2(\xi^2 + \eta^2)P - \frac{96(\xi^2 + \eta^2)}{(\xi^2 - \eta^2)^2}P = 0. \tag{40}
\]

With the transformation \(\xi = \varrho \cos \theta, \eta = \varrho \sin \theta\), (40) becomes

\[
\varrho^2 P_{\varrho \varrho} + \varrho P_{\varrho} + P_{\theta \theta} + 8r^2\delta_1\delta_2 \varrho^4 P - \frac{96P}{\cos^2 2\theta} = 0. \tag{41}
\]

The solution can be written as

\[
P = F_1(\varrho)F_2(\theta), \tag{42}
\]

where \(F_1(\varrho)\) and \(F_2(\theta)\) are

\[
\begin{align*}
F_1(\varrho) &= C_1 J\left(\frac{\sqrt{C_1}}{2}, \sqrt{2\delta_1\delta_2 r^2}\varrho^2\right) + C_2 Y\left(\frac{\sqrt{C_1}}{2}, \sqrt{2\delta_1\delta_2 r^2}\varrho^2\right), \\
F_2(\theta) &= \frac{(2 \cos(\theta) - 2)^\frac{1}{2}}{\sqrt{\sin(\theta)}} \left\{ C_3 \left(\cos(\theta) + 1\right)^\frac{1}{2} + \frac{\varrho^2}{\sqrt{\varrho}} \text{Hypergeom}\left[\left[\frac{3 + \sqrt{17}}{4}, \frac{3 + \sqrt{17}}{4}\right], \left[\frac{3 - \sqrt{17}}{4}, \frac{3 - \sqrt{17}}{4}\right]\right], \\
&\quad \left[1 + \frac{\varrho^2}{\sqrt{\varrho}}, \cos(\theta) + 1\right] + C_4 \left(\cos(\theta) + 1\right)^\frac{1}{2} - \frac{\varrho^2}{\sqrt{\varrho}} \text{Hypergeom}\left[\left[\frac{3 - \sqrt{17}}{4}, \frac{3 - \sqrt{17}}{4}\right], \left[\frac{3 + \sqrt{17}}{4}, \frac{3 + \sqrt{17}}{4}\right]\right], \\
&\quad \left[1 - \frac{\varrho^2}{\sqrt{\varrho}}, \cos(\theta) + 1\right] \right\},
\end{align*}
\]
where $c_1$, $C_1$, $C_2$, $C_3$, $C_4$ are arbitrary constants, and Hypergeom is generalized hypergeometric function. We can get the original solution $c$ of generalized Black-Scholes equation (3) through substituting $P$ with the transformation (39), (8) and (5).

For 1-dimensional generalized Black-Scholes equation, similarity reduction method can be used to reduce the PDE to an ODE which is easier to solve. Except the time dependent cases, we can also use this method to reduce all equations Carr, Laurence and Wang (2006) transformed, which are associated with the 1-dimensional simplification of Case 1.4b.

5.2 1-dimensional: CEV Model

Assuming

$$\sigma(S) = \sigma S^\alpha,$$  \hspace{1cm} (43)

from the transformation, we know that the corresponding $M(x)$

$$M = \alpha \sigma^2 \left( \frac{\alpha \sigma^2}{(\alpha - 1)^2} - \frac{1}{2(\alpha - 1)} \right) \frac{1}{x^2} + \frac{r^2(\alpha - 1)^2}{\sigma^4} x^2 - 2r \alpha - \frac{r(\alpha - 1)}{2\sigma^2},$$  \hspace{1cm} (44)

is the one dimensional case of Case 1.4b. Therefore the solution can be written as

$$P(\xi) = c_1 \xi^{1/4} - \frac{m}{\sqrt{\xi}}.$$  \hspace{1cm} (45)

We can get the original solution $c$ by substituting $P$ with the transformation (27), (5) and (8) (1-dimensional form).

5.3 1-dimensional: Exponentially Decreasing Volatility

Assuming

$$\sigma(S) = e^{-S},$$  \hspace{1cm} (46)

from the transformation, we know that the corresponding $M(x)$

$$M = \frac{1}{2x^2},$$  \hspace{1cm} (47)

where for brevity we let $r = 0$ \hspace{1cm} \footnote{Similar transformation can be found in Li and Zhang (2004), and Zhang and Li (2012).} is the one dimensional case of Case 1.1a. Therefore the solution can be written as

$$P(\xi) = e^{\frac{\delta_1 \xi^2}{\sqrt{\xi}}} \left[ c_1 \text{WhittakerM}\left( \frac{\delta_2 t}{2\delta_1}, -\frac{1}{4}, \frac{1}{2}, \frac{\sqrt{5}}{4}, -\frac{\delta_1 \xi^2}{2} \right) + c_2 \text{WhittakerW}\left( \frac{\delta_2 t}{2\delta_1} - \frac{1}{4}, \frac{\sqrt{5}}{4}, \frac{\delta_1 \xi^2}{2} \right) \right].$$  \hspace{1cm} (48)

where WhittakerM and WhittakerW are the Whittaker function $M$ and $W$, respectively. We can get the original solution $c$ through substituting $P$ with the transformation (23), (8) and (5) (1-dimensional form).
With a proper re-scaling transformation, the exponential decreasing function volatility function can be converted to

$$\sigma(S_t) = \sigma_0 S_0 e^{\alpha \left(1 - \frac{S_t}{S_0}\right)}$$

where $S_0$ stands for the initial stock price, then

$$\tilde{\sigma}(S_t) = \sigma_0 S_0 e^{\alpha \left(1 - \frac{S_t}{S_0}\right)}$$

$$= \sigma_0 \left\{ 1 - (1 + \alpha) \left(\frac{S_t}{S_0} - 1\right) + \left(1 + \alpha + \frac{1}{2} \alpha^2\right) \left(\frac{S_t}{S_0} - 1\right)^2 + O \left[\left(\frac{S_t}{S_0} - 1\right)^3\right]\right\},$$

where $O(\epsilon)$ is the order of $\epsilon$. The function is negatively skewed for positive $\alpha$, can be used to produce the phenomenon of the implied volatility smirk observed by Zhang and Xiang (2008), see also, Zhang and Li (2012).

6 Conclusion

With some transformation, we convert the problem of option pricing under state-dependent volatility into an initial value problem of the Fokker-Planck equation with a certain potential. By using the Lie symmetry analysis and similarity reduction method, we are able to write the solution analytically.

The study on a few cases demonstrates that our new method can be used to produce analytical option pricing formulas for certain volatility functions. A few exact solutions of the corresponding cases provided in this paper can be regarded as contributions to the option pricing literature.

The comparison with Finkel (1999), and Laurence and Wang (2005) is as follows. In terms of the method, Finkel (1999) studied 2 + 1-dimensional Fokker-Planck equation in general by using the prolongation of vector-field, but he did not discuss the applications in finance. Laurence and Wang (2005) used the same method as Finkel’s and applied Finkel’s results in finance. Our method presented in Section 3 is more succinct. In terms of the results, Finkel (1999) provided the vector fields of group invariants in the symmetry reduction and group invariant solutions in the particular case of 1.1a. On the top of Finkel (1999), Laurence and Wang (2005) provided the group invariant solutions via subgroups generated by particular subalgebras in cases of 1.1ab, 1.2ab, 1.4ab, 1.5ab, 1.7ab. We perform similarity reduction, and provide the group invariant solutions in the cases of 1.3, 1.6 and 1.8ab. In terms of finance application, Laurence and Wang (2005) only studied the case of the generalized Black-Scholes equation on a single asset. We point out that the problem of independent double-CEV can be reduced to case 1.2b. Even for the case of a single asset, the examples in our Section 5.2, 5.3, were not studied in Laurence and Wang (2005).

In finance, it is an open problem to find a closed form solution for the option on two correlated CEV assets. The case of independent double-CEV has been studied in Section 5.1 in this paper. In order to study the case of correlated double CEV, we need to use the general case of Fokker-Planck equation. In principle, we can find generators of invariant groups by using Lie symmetry approach. With Finkel’s (1999) classification, we can then obtain reduced equation by using similarity reduction method.
quantitative finance, we are interested in a solution of the generalized Black-Scholes equation with a particular final condition, i.e., payoff function. Constructing a solution of relevance in quantitative finance by using some particular solutions seems not straightforward. This problem is left for further research.

It is also an interesting topic to explore the application of current approach to the pricing of path-dependent derivatives.

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A Transforms of the solution

Given a vector field \( \mathbf{u} \), the corresponding one-parameter group of infinitesimal transformation \( G : (x, y, t, u) \rightarrow (\bar{x}, \bar{y}, \bar{t}, \bar{u}) \) can be obtained by solving the ODE

\[
\begin{align*}
\frac{d}{de}(\bar{x}, \bar{y}, \bar{t}, \bar{u}) &= (X, Y, T, U)(x, y, t, u), \\
(\bar{x}, \bar{y}, \bar{t}, \bar{u})|_{e=0} &= (x, y, t, u).
\end{align*}
\]

They are

\[
\begin{align*}
G_1 : (x, y, t, u) &\rightarrow (x \cos(\epsilon) + y \sin(\epsilon), -x \sin(\epsilon) + y \cos(\epsilon), t, u), \\
G_2 : (x, y, t, u) &\rightarrow (x, y, t, u), \\
G_3 : (x, y, t, u) &\rightarrow (x + f_2 \epsilon, y, t, u e^{f_2(\frac{1}{4}f_2^2 + x \epsilon)}), \\
G_4 : (x, y, t, u) &\rightarrow (x, y + f_3 \epsilon, t, u e^{f_3(\frac{1}{4}f_3^2 + y \epsilon)}), \\
G_5 : (x, y, t, u) &\rightarrow (x, y, t, u e^{f_4 \epsilon}), \\
G_6 : (x, y, t, u) &\rightarrow (x, y, t, u + ge),
\end{align*}
\]

where \( \epsilon \) is an arbitrary constant, and \( f_i, i = 1, \ldots, 4 \) are arbitrary functions of \( t \), which satisfy the compatibility condition \([13]\), and \( g \) is the solution of the original equation \([1]\). Solving \( G_2 \) is feasible only when given the definite form of \( f_1 \). We have tried to solve it with two forms (polynomial function and exponential function)\(^6\):

\[
f_1 = \delta_2 t^2 + \delta_1 t, \quad f_1 = \delta_1 e^{2\sqrt{2} \epsilon t} + \delta_2 e^{-2\sqrt{2} \epsilon t}.
\]

Due to the space limitation, we only consider a special case here: \( f_1 = \delta t \).

\[
G_2 : (x, y, t, u) \rightarrow (xe^{\frac{1}{4}f_2^2 \epsilon t - 1}, ye^{\frac{1}{4}f_2^2 \epsilon t - 1}, te^{\delta \epsilon}, u e^{\frac{1}{4}f_2^2 \epsilon t - 1} - \delta(2x^2 + y^2)),
\]

We observe that \( G_1 \) is a rotation, \( G_3 \) and \( G_4 \) are compositions of space translation and Galileo boost, \( G_5 \) is a Galileo boost, \( G_6 \) shows that the solution of original equation \([1]\) is linear, which is consistent with the fact that the equation itself is linear, \( G_2 \) is a Galileo boost when \( f_1 = \delta t \). The entire symmetry group is obtained by combining the six subgroups \( G_i, i = 1, \ldots, 6 \).

Furthermore, if \( u = \phi(x, y, t) \) is the solution of Fokker-Planck equation, then so are the functions \( u^{(1)}, u^{(2)}, \ldots, u^{(6)} \),

\[
\begin{align*}
u^{(1)} &= \phi(x \cos(\epsilon) - y \sin(\epsilon), x \sin(\epsilon) + y \cos(\epsilon), t), \\
u^{(2)} &= e^{\frac{1}{4}f_2^2(t^2 e^{\delta \epsilon t - 1})} \phi(x f_2^2 e^{\delta \epsilon t - 1}, ye^{\frac{1}{4}f_2^2 \epsilon t - 1}, te^{\delta \epsilon}), \\
u^{(3)} &= e^{f_2 (\frac{1}{4}f_2^2 e^{\delta \epsilon t - 1})} \phi(x - f_2 \epsilon, y, t), \\
u^{(4)} &= e^{f_3 (\frac{1}{4}f_3^2 e^{\delta \epsilon t - 1})} \phi(x, y - f_3 \epsilon, t), \\
u^{(5)} &= e^{-f_4 \epsilon \phi(x, y, t)}, \\
u^{(6)} &= \phi(x, y, t) - ge.
\end{align*}
\]

\(^6\)They are associated with \( f_1 \) in Section 4 and Appendix B.
By using some one-parameter groups of transformation $G_i$, a new solution can be generated. Moreover, we can use groups $G_1, G_2, \ldots, G_6$ compositely by taking different constant $\epsilon_1, \epsilon_2, \ldots, \epsilon_6$, to obtain a series of new solutions.
B Similarity reductions of Fokker-Planck equation

- Case 1.1b

\[
\begin{align*}
M = \frac{C_0}{x^2} + cr^2 + by + c_0, \quad C_0 \neq 0, c \neq 0, \\
f_1 = \delta_1 e^{2\sqrt{2}c t} + \delta_2 e^{-2\sqrt{2}c t}, \quad k = 0, \quad f_2 = 0, \\
f_3 = \frac{b_1}{\sqrt{2c}} e^{2\sqrt{2}c t} - \frac{b_1}{\sqrt{2c}} e^{-2\sqrt{2}c t} + \beta_1 e^{\sqrt{2}c t} + \beta_2 e^{-\sqrt{2}c t}, \\
f_4 = (\sqrt{2c} + c_0 + \frac{b_1}{4c})\delta_1 e^{2\sqrt{2}c t} - (\sqrt{2c} - c_0 - \frac{b_1}{4c})\delta_2 e^{-2\sqrt{2}c t} + \frac{b_1}{\sqrt{2c}} e^{\sqrt{2}c t} - \frac{b_1}{\sqrt{2c}} e^{-\sqrt{2}c t},
\end{align*}
\]

where \( r = \sqrt{x^2 + y^2} \).

We have the similarity variables \( \xi, \eta, P \):

\[
\begin{align*}
\xi &= \frac{x e^{\sqrt{2}c t}}{4 \sqrt{\delta_1 e^{4\sqrt{2}c t} + \delta_2} e^{\sqrt{2}c t}}, \\
\eta &= \frac{x e^{\sqrt{2}c t}(4\delta_1 \delta_2 + 2b_1 \delta_2 - \delta_1 \beta_2 \sqrt{2c} e^{\sqrt{2}c t} + \delta_2 \beta_1 \sqrt{2c} e^{-\sqrt{2}c t})}{c_0 \delta_2 \sqrt{\delta_1 e^{4\sqrt{2}c t} + \delta_2}}, \\
P &= u \cdot \exp \left\{ \frac{1}{16 \sqrt{c_1 \delta_1 \delta_2}} \left[ 8 \eta e^{\frac{3}{2}(\delta_1 \delta_2)} \sqrt{\delta_1 e^{4\sqrt{2}c t} + \delta_2 (\beta_1 \delta_2 + \beta_2 \delta_1) \arctan \left( \frac{\sqrt{2}e^{2\sqrt{2}c t}}{c_0 - 4 \sqrt{2c} - b^2} \right) + 4 \sqrt{c} (\delta_1 \delta_2) \frac{1}{2} \left( 4c_0 - 4 \sqrt{2c} - b^2 \right) e^{2\sqrt{2}c t} + \sqrt{2c} \delta_1 \frac{1}{2} \sqrt{\delta_1^2 + 8c_0 (\delta_1 \delta_2 + \beta_2 \delta_1)} (\xi^2 + \eta^2) \right] e^{4\sqrt{2}c t} - \sqrt{2c} \delta_1 \frac{1}{2} \sqrt{\delta_1^2 + 8c_0 (\delta_1 \delta_2 + \beta_2 \delta_1)} (\xi^2 + \eta^2) \right\}.
\end{align*}
\]

and the reduced PDE becomes

\[
\delta_1 \delta_2 \xi^2 (P_{\xi \xi} + P_{\eta \eta}) + [8c_0 (\delta_1 \delta_2 \xi^2 (\xi^2 + \eta^2) - \xi^2 (\beta_2 \delta_2 + \beta_2 \delta_1) - 2c_0 \delta_1 \delta_2) P] = 0.
\]

We get the solution by the method of separation of variables

\[
P = F_1(\xi) F_2(\eta),
\]

where \( F_1(\xi) \) and \( F_2(\eta) \) is

\[
\begin{align*}
F_1(\xi) &= \frac{1}{\sqrt{\pi}} \left[ C_1 \text{WhittakerM} \left( \frac{\sqrt{2c}}{16 \sqrt{\delta_1 \delta_2}}, \frac{\sqrt{8c_0 + 1}}{4}, i 2 \sqrt{2\delta_1 \delta_2 c} \xi^2 \right) + C_2 \text{WhittakerW} \left( \frac{\sqrt{2c}}{16 \sqrt{\delta_1 \delta_2}}, \frac{\sqrt{8c_0 + 1}}{4}, i 2 \sqrt{2\delta_1 \delta_2 c} \xi^2 \right) \right], \\
F_2(\eta) &= \frac{1}{\sqrt{\pi}} \left[ C_3 \text{WhittakerM} \left( \frac{\sqrt{2c}}{16 \sqrt{\delta_1 \delta_2}}, \frac{\sqrt{8c_0 + 1}}{4}, i 2 \sqrt{2\delta_1 \delta_2 c} \eta^2 \right) + C_4 \text{WhittakerW} \left( \frac{\sqrt{2c}}{16 \sqrt{\delta_1 \delta_2}}, \frac{\sqrt{8c_0 + 1}}{4}, i 2 \sqrt{2\delta_1 \delta_2 c} \eta^2 \right) \right],
\end{align*}
\]

where \( c_1, C_1, C_2, C_3, C_4 \) are arbitrary constants, and WhittakerM and WhittakerW are the Whittaker function \( M \) and \( W \), respectively, and \( i = \sqrt{-1} \).

- Case 1.2a

\[
\begin{align*}
M &= \frac{C}{r^2} + c_0, \\
f_1 &= \delta_2 t^2 + \delta_1 t, \quad k = 0, \quad f_2 = f_3 = 0, \\
f_4 &= c_0 \delta_2 t^2 + (\delta_2 + c_0 \delta_1) t.
\end{align*}
\]
where \( C(\theta) \neq (c_1 \cos \theta + c_2 \sin \theta)^{-2}, \) \( C(\theta) \neq 0, \) and \( r = \sqrt{x^2 + y^2}. \)

We have the similarity variables \( \xi, \eta, P, \)

\[
\begin{align*}
\xi &= \frac{x}{\sqrt{\delta_2 t^2 + \delta_1 t}}, \\
\eta &= \frac{y}{\sqrt{\delta_2 t^2 + \delta_1 t}}, \\
P &= (\delta_2 t + \delta_1)u \cdot \exp\left\{ -\frac{\delta_2}{2}(\xi^2 + \eta^2)t - c_0t \right\},
\end{align*}
\]

and the reduced PDE becomes

\[
(\xi^2 + \eta^2)\left( P_{\xi\xi} + P_{\eta\eta} \right) + \delta_1(\xi^2 + \eta^2)(\xi P_\xi + \eta P_\eta) - 2C(\theta)P = 0.
\]

With the transformation \( \xi = \rho \cos \theta, \eta = \rho \sin \theta, \) it becomes

\[
\rho^2 P_{\xi\xi} + (\delta_1 \rho^3 + \rho)P_\rho + P_{\eta\theta} - 2C(\theta)P = 0.
\]

We can get the solution by the method of separation of variables

\[
P = F_1(\rho)F_2(\theta),
\]

where \( F_1(\rho), F_2(\theta) \) is the solution of

\[
\begin{align*}
\frac{d^2 F_1(\rho)}{d\rho^2} + \frac{\delta_1 \rho^2 + 1}{\rho} \frac{dF_1(\rho)}{d\rho} - \frac{c_1 F_1(\rho)}{\rho^2} = 0, \\
\frac{d^2 F_2(\theta)}{d\theta^2} + (c_1 - 2C(\theta))F_2(\theta) = 0.
\end{align*}
\]

where \( c_1 \) is an arbitrary constant. Given \( C(\theta), \) the above ODE systems can be solved directly.

**Case 1.3**

\[
\begin{align*}
M &= \frac{C(\lambda \ln r + \theta)}{r^2} + c_0, \quad C_0 \neq 0, \\
f_1 &= \frac{2k}{\lambda}, \quad f_2 = f_3 = 0, \quad f_4 = \frac{2kc_0}{\lambda}t.
\end{align*}
\]

where \( C'(\theta) \neq 0 \neq \lambda, \) and \( r = \sqrt{x^2 + y^2}. \)

We have the similarity variables \( \xi, \eta, P, \)

\[
\begin{align*}
\xi &= -\sqrt{\frac{1}{t}}\left( x \cos(\frac{1}{2}\lambda \ln t) - y \sin(\frac{1}{2}\lambda \ln t) \right), \\
\eta &= \sqrt{\frac{1}{t}}\left( x \sin(\frac{1}{2}\lambda \ln t) + y \cos(\frac{1}{2}\lambda \ln t) \right), \\
P &= ue^{\xi t},
\end{align*}
\]

and the reduced PDE becomes

\[
(\xi^2 + \eta^2)[P_{\xi\xi} + P_{\eta\eta} + (\xi - \lambda \eta)p_\xi + (\eta + \lambda \xi)p_\eta] - 2C(\lambda \ln r + \theta)P = 0.
\]

**Case 1.4a**
where \( r = \sqrt{x^2 + y^2} \). If \( \delta_1 \neq 0, \delta_2 \neq 0 \), then \( a = b = 0 \) should be held to make \( M \) satisfy the compatibility condition [18].

Obviously, this is the simplification of Case 1.2a. We have the same similarity variables \( \xi, \eta, P \), then the reduced PDE becomes

\[
(\xi^2 + \eta^2)(P_{\xi\xi} + P_{\eta\eta}) + \delta_1(\xi^2 + \eta^2)(\xi P_\xi + \eta P_\eta) - 2C_0P = 0.
\]

With the transformation \( \xi = \varphi \cos \theta, \eta = \varphi \sin \theta \), we can get the solution by the method of separation of variables

\[
P = F_1(\varphi)F_2(\theta),
\]

where \( F_1(\varphi) \) and \( F_2(\theta) \) is

\[
F_1(\varphi) = \frac{\delta_1 \varphi^2}{4} \left[ C_1 I \left( \frac{\sqrt{c_1}}{2}, \frac{\delta_1 \varphi^2}{4} \right) + C_1 I \left( \frac{\sqrt{c_1} + 1}{2}, \frac{\delta_1 \varphi^2}{4} \right) \right],
\]

\[
F_2(\theta) = C_3 \sin(\theta \sqrt{c_1 - 2C_0}) + C_4 \cos(\theta \sqrt{c_1 - 2C_0}),
\]

where \( c_1, C_1, C_2, C_3, C_4 \) are arbitrary constants, and \( I(\nu, z) \) and \( K(\nu, z) \) are the modified Bessel functions of the first and second kinds respectively.

- **Case 1.5a**

\[
\begin{align*}
M &= ax + by + c_0, \\
f_1 &= \delta_2 t^2 + \delta_1, \\
f_2 &= \frac{a \delta_2}{2} t^3 + \frac{1}{4} (3a \delta_1 - 2bk) t^2 + \alpha_1 t + \alpha_0, \\
f_3 &= \frac{b \delta_2}{2} t^3 + \frac{1}{4} (3b \delta_1 + 2ak) t^2 + \beta_1 t + \beta_0, \\
f_4 &= \frac{1}{8} (a^2 + b^2) \delta_2 t^4 + \frac{1}{4} (a^2 + b^2) \delta_1 t^3 + \frac{1}{2} (a \alpha_1 + b \beta_1) + c_0 \delta_2] t^2 + (\delta_2 + c_0 \delta_1 + a \alpha_0 + b \beta_0) t.
\end{align*}
\]

Taking \( k = 0 \) for brevity, we have the similarity variables \( \xi, \eta, P \),

\[
\begin{align*}
\xi &= \frac{(2x - a^2 t) \delta_1^2 + 4(a_0 - \alpha_1 t) \delta_1 + 8a_0 \delta_2 t}{2 \sqrt{\delta_2 t^2 + \delta_1}}, \\
\eta &= \frac{2(y - b^2 t) \delta_1^2 + 4(\beta_0 - \beta_1 t) \delta_1 + 8\beta_0 \delta_2 t}{2 \sqrt{\delta_2 t^2 + \delta_1}}, \\
P &= u \cdot \exp \left\{ \frac{1}{\delta_1^2} \left\{ \delta_1^2 \{2 \xi^2 (a \xi + b \eta) + 2 \xi (a_0 \xi + \beta_1) \} \ln(\delta_2 t + \delta_1) - 2 \delta_1 \{2 a_0 \xi + \beta_0 \beta_1 \} \right\} \right\} \\
&+ \delta_1 \{\delta_1^2 + 2 \delta_1 (a_0 \alpha_1 + \beta_0 \beta_1) - 2 \delta_2 (a_0^2 + \beta_0^2) \} \ln(\delta_2 t + \delta_1) - 2 \delta_1 \{2 \delta_2 (a_0 \alpha_1 + \beta_0 \beta_1) \} \\
&- \delta_2 (a_0^2 + \beta_0^2) \} \ln(t^2 (a^2 + b^2) + 3 \delta_2 (\xi^2 + \eta^2) + 6 \delta_1^2 c_0) \\
&+ 12 \delta_1^2 \delta_1 (a_0 \alpha_1 + \beta_0 \beta_1) - 6 \delta_1 (a \alpha_0 + b \beta_0) + (a_1^2 + \beta_0^2)] - 24 \delta_1^2 \delta_2 (a_0 \alpha_1 + \beta_0 \beta_1) \\
&+ 48 \delta_2 (a_0^2 + \beta_0^2) + \delta_1 (a_0 \alpha_1 + \beta_0 \beta_1) \right\}.
\end{align*}
\]
and the reduced PDE becomes
\[\delta^2_1(P_{\xi\xi} + P_{\eta\eta}) + \delta^3_1(P_\xi + P_\eta) - 4[\delta_1(\alpha_0\alpha_1 + \beta_0\beta_1) + \delta_2(\alpha_0^2 + \beta_0^2)]P = 0.\]

We obtain the solution by the method of separation of variables
\[P = F_1(\xi)F_2(\eta),\] (50)
where \(F_1(\xi)\) and \(F_2(\eta)\) is
\[
\begin{align*}
F_1(\xi) &= C_1e^{\frac{\xi}{2}(\sqrt{\delta_1^2 + 4c_1} - \delta_1)} + C_2e^{\frac{\xi}{2}(\sqrt{\delta_1^2 + 4c_1} + \delta_1)}, \\
F_2(\eta) &= C_3e^{\frac{\eta}{2\delta_1}(\sqrt{\delta_1^2 - 4c_1\delta_1^2 + 16[\delta_1(\alpha_0\alpha_1 + \beta_0\beta_1) - \delta_2(\alpha_0^2 + \beta_0^2)]} - \delta_1^2)} + C_4e^{\frac{-\eta}{2\delta_1}(\sqrt{\delta_1^2 - 4c_1\delta_1^2 + 16[\delta_1(\alpha_0\alpha_1 + \beta_0\beta_1) - \delta_2(\alpha_0^2 + \beta_0^2)]} + \delta_1^2)},
\end{align*}
\]
where \(c_1, C_1, C_2, C_3, C_4\) are arbitrary constants.

- **Case 1.6**
\[
\begin{align*}
M &= C(r) + d\theta, \\
f_1 &= f_2 = f_3 = 0, \quad f_4 = -dkt,
\end{align*}
\]
where \(r = \sqrt{x^2 + y^2}\). If \(d = 0\), then \(C(r) \neq C_0r^{-2} + C_1r^2 + c_0\) should be held to make \(M\) satisfy the compatibility condition.

We have the similarity variables \(\xi, \eta, P\),
\[
\begin{align*}
\xi &= x^2 + y^2, \\
\eta &= t, \\
P &= ue^{d\theta t},
\end{align*}
\]
and the reduced PDE becomes
\[4\xi P_{\xi\xi} + 2\xi(2P_\xi - P_\eta) + (d^2\eta^2 - 2\xi C(r))P = 0.\]

- **Case 1.8a**
\[
\begin{align*}
M &= C(x) + by, \\
f_1 &= f_2 = 0, \quad k = 0, \\
f_3 &= \beta_1t + \beta_0, \quad f_4 = \frac{b\beta_1}{2}t^2 + b\beta_0t,
\end{align*}
\]
where \(C(x) \neq C_0x^2 + ax + c_0\) and \(C(x) \neq \frac{c_0}{x^2} + c_0\).

We have the similarity variables \(\xi, \eta, P\),
\[
\begin{align*}
\xi &= x, \\
\eta &= t, \\
P &= u \cdot \exp \left\{ \frac{(\beta_1 y + b\beta_1 t^2 + 2b\beta_0 t)y}{2(\beta_1 t + \beta_0)} \right\},
\end{align*}
\]
and the reduced PDE becomes

\[4(\beta_1 \eta + \beta_0)^2 P_{\xi \xi} - 8(\beta_1 \eta + \beta_0)^2 P_{\eta} + b^2 \eta^2 (\beta_1 \eta + 2\beta_0)^2 P - 4\beta_1 (\beta_1 \eta + \beta_0) P - 8C(\xi)(\beta_1 \eta + \beta_0)^2 p = 0.\]

We can get the solution by the method of separation of variables

\[P = F_1(\xi)F_2(\eta),\]

where \(F_1(\varrho), F_2(\theta)\) is the solution of

\[
\begin{align*}
\frac{d^2 F_1(\xi)}{d\xi^2} + [c_1 - 2C(\xi)]F_1(\xi) &= 0, \\
\frac{dF_2(\eta)}{d\eta} + \frac{1}{2} - \frac{b^2 \eta^2 (\beta_1 \eta + 2\beta_0)^2 - 4\beta_1 (\beta_1 \eta + \beta_0)}{8(\beta_1 \eta + \beta_0)^2} F_2(\eta) &= 0.
\end{align*}
\]

where \(c_1\) is an arbitrary constant. Given \(C(\xi)\), the above ODE systems can be solved directly.

- **Case 1.8b**

\[
\begin{align*}
M &= C(x) + cy^2 + by, \\
f_1 &= f_2 = 0, \quad k = 0, \\
f_3 &= \beta_1 e^{\sqrt{2ct}} + \beta_2 e^{-\sqrt{2ct}}, \quad f_4 = \frac{b\beta_1}{\sqrt{2c}} e^{\sqrt{2ct}} - \frac{b\beta_2}{\sqrt{2c}} e^{-\sqrt{2ct}},
\end{align*}
\]

where \(C(x) \neq C_0 x^2 + ax + c_0\) and \(C(x) \neq \frac{C_0}{\sqrt{x}} + c_0\).

We have the similarity variables \(\xi, \eta, P,\)

\[
\begin{align*}
\xi &= x, \\
\eta &= t, \\
P &= u \cdot \exp \left\{ \frac{\beta_1 e^{\sqrt{2ct}} - \beta_2 e^{-\sqrt{2ct}} y(cy + b)}{\beta_1 e^{\sqrt{2ct}} + \beta_2 e^{-\sqrt{2ct}} \sqrt{2c}} \right\},
\end{align*}
\]

and the reduced PDE becomes

\[P_{\xi \xi} - 2P_{\eta} - 2C(\xi)P = 0.\]

We can get the solution by the method of separation of variables

\[P = F_1(\xi)F_2(\eta),\]

where \(F_1(\varrho), F_2(\theta)\) is the solution of

\[
\begin{align*}
\frac{d^2 F_1(\xi)}{d\xi^2} + [c_1 - 2C(\xi)]F_1(\xi) &= 0, \\
\frac{dF_2(\eta)}{d\eta} + \frac{1}{2} F_2(\eta) &= 0.
\end{align*}
\]

where \(c_1\) is an arbitrary constant. Given \(C(\xi)\), the above ODE systems can be solved directly.
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