Highly clustered scale-free networks

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We propose a model for growing networks based on a finite memory of the nodes. The model shows stylized features of real-world networks: power law distribution of degree, linear preferential attachment of new links and a negative correlation between the age of a node and its link attachment rate. Notably, the degree distribution is conserved even though only the most recently grown part of the network is considered. This feature is relevant because real-world networks truncated in the same way exhibit a power-law distribution in the degree. As the network grows, the clustering reaches an asymptotic value larger than for regular lattices of the same average connectivity. These high-clustering scale-free networks indicate that memory effects could be crucial for a correct description of the dynamics of growing networks.

Many systems can be represented by networks, i.e. as a set of nodes joined together by links. Social networks, the Internet, food webs, distribution networks, metabolic and protein networks, the networks of airline routes, scientific collaboration networks and citation networks are just some examples of such systems [1–11]. Recently it has been observed that a variety of networks exhibit topological properties that deviate from those predicted by random graphs [12]. For instance, real networks display clustering higher than expected for random networks [13]. Also, it has been found that many large networks are scale-free. Their degree distribution decays as a power-law that cannot be accounted for by the Poisson distribution of random graphs [14–16]. The type of the degree distribution is of great importance for the functionality of the network [17–19]. Beside the degree distribution, other features of the growth dynamics of real-world networks are currently under investigation. For citation networks, the Internet, and collaboration networks of scientists and actors, it has been shown [17,18] that the probability for a node to obtain a new link is an increasing function of the number of links the node already has. This feature of the dynamics is called preferential attachment. Furthermore the aging of nodes is of particular interest [19]. In the network of scientific collaborations, every node stops receiving links a finite time after it has been added to the network, since scientists have a finite time span of being active. Similarly, in citation networks, papers cease to receive links (citations), because their contents are outdated or summarized in review articles, which are then cited instead. Whether a paper is still cited or not, depends on a collective memory containing the popularity of the paper.

In the current paper we address the study of growing complex networks from the perspective of the memory of the nodes. First, we present empirical evidence for the age dependence of the growth dynamics of the network of scientific citations. We find that old nodes are less likely to obtain links than nodes added to the network more recently. Second, motivated by this finding, we introduce a model of network self-organization that accounts for the three empirical features mentioned before: (1) power law distribution for the degree, (2) preferential attachment, and (3) negative correlation between age and attachment rate. The clustering of the generated networks is higher than in corresponding regular lattices, justifying the name highly clustered scale-free networks.

PREVIOUS MODELS

The earliest and most basic model generating scale-free networks has been introduced by Barabási and Albert [11], henceforth we use the acronym BA-model. This model explicitly incorporates the preferential attachment in the dynamical rules. At each time step a new node is added to the network and new links are attached from this new node to old nodes. The probability that a node obtains an additional link is proportional to its current degree. It can be interpreted as an application of Simon’s growth model in the context of networks [20,21], readily explaining the emergent scaling in the degree distribution. The BA-model has been successively modified reproducing the scale-free behavior of the connectivity distribution [22–24]. For the sake of clarity, in the remaining of the paper we will refer to the BA-model as a well-established model of growing scale-free networks.

Real-world networks have properties that cannot be accounted for by the BA-model. We find a discrepancy with respect to empirical data in the correlation between a node’s age and its rate of acquiring links. For the network of scientific citations this correlation is negative: the mean rate of citations a paper receives decreases with increasing age. This is supported by citation rate data of the years 1987-1998, shown in Figure 1. Except for the three first years prior to the publication year, the citation rate decreases with age [25]. In contradiction to this empirical result, in the BA-model the mean attachment rate is positively correlated with age. Here the attachment rate is proportional to the degree, being largest for the oldest nodes since these began accumulating links earliest. A further consequence of this feature is a strong positive correlation between the age of a node and its degree. This kind of correlation has not been found in the
The shortcomings indicated in the previous paragraph motivate our attempt to model self-organization of scale-free networks. The approach presented here is based on the degree-dependent deactivation dynamics of the nodes. Preferential attachment and the convergence to a power-law degree distribution are shown to be emergent properties of the dynamics.

The model describes the growth dynamics of a network with directed links. By \( k_i \) we denote the in-degree of node \( i \), i.e., the number of links pointing to node \( i \). Each node of the network can be in two different states: active or inactive. A new node added to the network is always in the active state first. It receives links from subsequently generated nodes until it is deactivated. Then the node does not receive links any more. The transition of a node from the active to the inactive state can be interpreted as a collective “forgetting” of the node since new nodes do not connect to it any more. For the construction of the model we assume that the probability rate \( P \) of deactivation decreases with the in-degree of the node. Considering for instance the case of citation networks, this means that the more often a paper has been cited, the less likely it is forgotten. Specifically, we make the assumption that the deactivation probability can be written as \( P \propto (k + a)^{-1} \), where \( a > 0 \) is a constant bias.

At any step of the time-discrete dynamics \( m \) nodes in the network are active, all the other nodes are inactive. As the initial condition we use a network consisting of \( m \) active, completely connected nodes. Then the dynamics runs as follows:

1. Add a new node \( i \) to the network. The new node is disconnected at first, so \( k_i = 0 \) at this point.
2. Attach \( m \) outgoing links to the new node \( i \). Each node \( j \) of the \( m \) active nodes receives exactly one incoming link, thereby \( k_j \rightarrow k_j + 1 \).
3. Activate the new node \( i \).
4. Deactivate one of the active nodes. The probability that the node \( j \) is deactivated is given by
   \[
   P(k_j) = \frac{\gamma - 1}{a + k_j},
   \]  
   where \( a > 0 \) is a constant bias and the normalization factor is defined as \( \gamma = \frac{1}{\sum_{l \in A} \frac{1}{k_l}} \). The summation runs over the set \( A \) of the currently active nodes.
5. Resume at 1.

The average connectivity of the network is given by the number of outgoing links per node, \( m \). It is worth noting that a node receives incoming links during the lifetime \( T \) it is active, and once inactive it will not receive links any longer. Thus for each node \( i \) the time \( T_i \) spent in the active state and the in-degree \( k_i \) are equivalent. The deactivation mechanism strongly simplifies the dynamics of growing complex networks. Neither gradual aging nor possible reactivation are taken into account. For instance, in the context of citation networks, the model does not consider the rediscovery of “forgotten” papers.
Moreover, the functional form of the deactivation probability might well differ from Eq. (1). However, we will show that the model reproduces several features of real growing networks.

**DEGREE DISTRIBUTION**

The distribution \( N(k) \) of the in-degree \( k \) can be obtained analytically for the model defined above, considering the continuous limit of \( k \). Let us first derive the distribution \( p^{(t)}(k) \) of the in-degree of the active nodes at time \( t \). For \( k > 0 \), the time evolution is determined by the following master equation

\[
p^{(t+1)}(k+1) = (1 - P(k)) p^{(t)}(k)
\]

\[
= \left(1 - \frac{\gamma - 1}{a+k}\right) p^{(t)}(k)
\]

(2)

where \( a \) and \( \gamma \) are defined in step 4 of the model definition. The boundary value \( p(0) \) is a constant reflecting the constant rate of new nodes with initial \( k = 0 \).

Assuming that the fluctuations of the normalization \( \gamma - 1 \) are small enough, such that \( \gamma \) may be treated as a constant, the stationary case \( p^{(t+1)}(k) = p^{(t)}(k) \) of Eq. (2) yields

\[
p(k+1) - p(k) = \frac{\gamma - 1}{a+k} p(k)
\]

(3)

Treating \( k \) as continuous we write

\[
\frac{dp}{dk} = -\frac{\gamma - 1}{a+k} p(k)
\]

(4)

and obtain the solution

\[
p(k) = b(a+k)^{-\gamma+1}
\]

(5)

with appropriate normalization constant \( b \). In case the total number \( n \) of nodes in the network is large compared with the number \( m \) of active nodes, the overall degree distribution \( N(k) \) can be approximated by considering the inactive nodes only. Thus \( N(k) \) can be calculated as the rate of change of the degree distribution \( p(k) \) of the active nodes. We find

\[
N(k) = -\frac{dp}{dk} = c(a+k)^{-\gamma}
\]

(6)

with \( c = (\gamma - 1)a^{\gamma-1} \). The exponent \( \gamma \) is obtained from a self-consistency condition obtained from the average connectivity

\[
m = c \int_0^\infty \frac{k}{(a+k)^\gamma} \, dk
\]

(7)

which gives

\[
\gamma = 2 + \frac{a}{m}
\]

(8)

Thus the exponent \( \gamma \) depends only on the ratio \( a/m \).

Similar expressions have been obtained for a version of the BA-model with directed links [23,24]. Although the growth and deactivation model has been formulated for directed networks, it can be easily applied also to generate undirected networks.

![FIG. 2. Comparison of the degree distribution obtained for the undirected networks following the BA (dashed line) and the growth and deactivation model (solid line). In (a) the complete networks are considered after 5 \times 10^4 \text{ time steps. In contrast, in (b) only the network formed by the newest nodes and their links is taken into account. In (c) we plot the maximum degree, } k_{\text{max}}, \text{ observed in the truncated network against the truncation ratio } \Delta. \text{ In the BA model, } k_{\text{max}} \text{ scales as a power law with } \Delta. \text{ However, the degree distribution in the new model shows a power law distribution of degree, whose cutoff is only slightly affected by the finite size of the truncated network. All curves are averages over 100 independent simulation runs.}

**Numerical results**

Figure 2(a) shows the cumulative distribution of the total degree \( k' = (m+k) \) obtained by simulating the model for \( 5 \times 10^4 \text{ time steps. We obtain a power law scaling for several decades, in agreement with the analytical result in Eq. (1). The exponent found numerically is 1.9, slightly below the analytical result } \gamma - 1 = 2 + a/m - 1 = 2 \text{ for the case } a = m. \text{ The deviation can be explained by the con-}
tinuous limit used in the theoretical derivation of $\gamma$ and the assumption that $\gamma$ is a constant. Conducting further simulations for various values of $m$ and $a$, we find that the fluctuations of $\gamma$ become smaller when increasing $m$ and/or $a$. Then the discrepancy between analytical and numerical results decreases. Figure (a) also shows corresponding simulation results for the BA model, using $m = 10$ and $5 \times 10^4$ time steps as well. In the range $k' < 1000$ we obtain almost the same distribution as for the growth and deactivation model. However, the main difference between both models is the presence of a cutoff at a lower value for the BA-model.

Up to this point we have considered degree distributions including all nodes of the network. However, in many cases empirical data contain only those nodes and links of the network that have been created most recently. For instance, studies on scientific citation networks [9] are restricted to papers that are not older than 20 years, thereby ignoring the major part of the initial network. A pronounced power law regime is observed in the degree distribution of these truncated networks. Therefore it is important to investigate the robustness of the scale-free networks obtained from models under truncation in time. Figure (b) shows the cumulative degree distributions analogous to Fig. (a), but now regarding the truncated network where the fraction $\Delta = 50\%$ of oldest nodes and all their links are disregarded. Concerning the BA-model the effect of truncation is drastic. The truncated network does not exhibit a scale-free range in the degree distribution. This is different for the growth and deactivation model. The influence of the truncation on the degree distribution is a slight shift of the cutoff for high $k'$. In order to view systematically the effect of truncation, we consider the largest degree $k'_{\text{max}}$, occurring in the truncated network, as a function of the fraction $\Delta$ of disregarded nodes. According to Fig. (c), $k'_{\text{max}}$ decays as a power law (with an approximate exponent of 0.5, $k'_{\text{max}} \sim \Delta^{-0.5}$) for the BA-model. On the other hand, the new model introduced here exhibits only a weak dependence of the maximum degree on the truncation.

**LINEAR PREFERENTIAL ATTACHMENT**

Another relevant dynamical property is the degree-dependent attachment rate $\Pi(k)$. It is measured as follows: Consider the set $K$ of nodes with degree $k$ at a certain time $t$. Measure the average degree $k + \Delta k$ of the nodes in $K$ at a later time $t + \Delta t$. Then let $\Pi(k) = \Delta k/\Delta t$. In recent studies of various growing networks, it has been found empirically that $\Pi(k)$ is an increasing function [7]15]18]27]. This phenomenon is called preferential attachment. For the Internet and citation networks the preferential attachment is linear, $\Pi(k) \propto k$.

We can calculate $\Pi(k)$ for the model introduced in the present Paper. At a time $t$, the network contains $t$ nodes. $tN(k)$ of these have degree $k$. The number of active nodes with degree $k$ is $mp(k)$. A time step later, $\Delta t = 1$, each of the active nodes has increased its degree by 1, whereas the degree of the inactive nodes remains unchanged. Thus, according to Eqs. (3) and (3), the average increase of the degree is

$$\Pi(k) = \frac{mp(k)}{tN(k)} \propto (a + k). \quad (9)$$

The model shows linear preferential attachment as an emergent property of the degree-dependent attachment dynamics.

**AGE DISTRIBUTION**

Let us now consider the distribution of the age $\tau$ of nodes receiving a new link. We define the time-dependent age distribution $h(\tau, t)$ as the probability that a new link created at time $t$ attaches to a node of age $\tau$, i.e. to a node created at time $t - \tau$. For the model defined here, the age distribution $h$ is easy to obtain. Only active nodes receive links, and for these nodes their age $\tau$ and their indegree $k$ have the same value. Therefore the probability that the node of age $\tau$ obtains a new link is the same as the probability for a node with $\tau$ links to be active, given by equation (3). It is independent of $t$:

$$h(\tau) \propto (a + \tau)^{-\gamma + 1}. \quad (10)$$

For comparison we calculate the age distribution for the BA-model. Apart from small deviations, the total degree of the node $i$ created at time $t_i$ is [7]11]:

$$k_i' = m \left( \frac{t}{t_i} \right)^{0.5} = m \left( \frac{t}{t - \tau} \right)^{0.5}, \quad (11)$$

where the second equality is due to the substitution $t_i = \tau - t$. The probability of obtaining a new link is proportional to the total degree, thus we find

$$h(\tau, t) = \frac{1}{2mt} m \left( \frac{t}{t - \tau} \right)^{0.5} = \frac{1}{2} \left[ t(t - \tau) \right]^{-0.5}. \quad (12)$$

In the BA-model the probability of receiving a new link increases with the age of the node. In sharp contrast, the growth and deactivation model displays a forgetting of old nodes where the rate of forgetting is a power-law, Eq. (10). Figure 3 shows plots of the age distributions for both models, to be compared with the empirical data in Fig. 1. The age distribution of the growth and deactivation model decays with $\tau$. This agrees with the empirical data on citation networks except for the first 3 years after publication.
The clustering coefficient $C$ is one of the parameters used to characterize the topology of complex networks. It is a local property measuring the probability with which two neighbors of a node are also neighbors to each other (nodes $i$ and $j$ are neighbors if there is a link between $i$ and $j$). It has been found that many real world networks present a clustering coefficient much larger than the corresponding random graph, which scales with the system size $N$ as $C_{\text{rand}} \sim \langle k \rangle/N$.

Fig. 3(a) shows that for the growth and deactivation model the clustering coefficient tends towards an asymptotic value ($\approx 0.83$) as the network grows. The analytical derivation of $C$ is facilitated by the observation, that the clustering $C_i$ of a node merely depends on the node’s in-degree $k_i$. A detailed calculation gives an asymptotic value $C = 5/6$ for the case of $a = m$ considered here. Thus the model generates networks with a higher clustering than the corresponding one-dimensional regular lattices, $C_{1D} < 3/4$. The large value of the clustering coefficient and the fact that it does not decrease with network size is in qualitative agreement with recent data on the Internet [28]. For the sake of comparison, in Fig. 3(b) the clustering coefficient of the BA-model is plotted for several network sizes $N$. Here the clustering clearly decays with increasing $N$. The quantitative behavior of the decay can be described by $C \sim (\ln N)^2/N$. The detailed derivation of the clustering coefficients for both models is included in (Klemm and Eguiluz, unpublished work).

CONCLUSIONS

The analysis of citation networks suggests a negative correlation between the age of a node and its probability to obtain further links. Older nodes are less likely to increase their connectivity than those added to the network more recently. Motivated by this finding, we have proposed and analyzed a new approach based on nodes with
one degree of freedom, a memory, indicating the ability of the node to attract further links. We have found that with the simple setting of the model the degree distribution converges to a power law, where the exponent can be obtained analytically. As emergent properties of the model, (1) preferential attachment is obtained, a feature observed recently in various real growing networks, and (2) the correlation between age and linking probability is negative, in agreement also with the empirical results mentioned above. Unlike previous models, degree and age of nodes are uncorrelated in the model introduced here. Therefore the networks retain the power-law distribution of the degree even though only the most recent nodes are considered. This agrees with the fact that also truncated real-world networks are observed to be scale-free. Finally it is worth noting the resemblance of the grown networks to regular lattices. The highly clustered scale-free networks make a connection between scale-free networks and regular lattices. They define a new class of scale-free networks. Interesting extensions of the model include the introduction of random links, similarly to models of small-world networks. We expect to find a connection between scale-free growing networks and the small-world transition from regular lattices. Research along this line is in progress.

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