We propose a single crossover phase diagram applicable to 2D collinear Heisenberg antiferromagnets (AFMs) and ferromagnets (FMs), and show that the scaling regimes of AFMs and FMs are in one-to-one correspondence. The phase diagram is split into classical and quantum regions. Our two key results are: (i) in the classical region, the AFMs and FMs exhibit nearly identical behavior near their respective ordering wavevectors, which we observe for $S=1$ and higher using series expansions; and (ii) in the quantum region, quantum critical (QC) regime is present not only for the AFMs, but for FMs as well.

## 1. INTRODUCTION

We report a study of two-dimensional, collinear, spin-$S$ antiferromagnets (AFMs) and ferromagnets (FMs) aimed at better understanding the magnetic behavior of both, and in particular the role of quantum versus classical fluctuations in these systems. A number of quasi-2D experimental systems, including spin-$\frac{1}{2}$ AFMs La$_2$CuO$_4$ and Sr$_2$CuO$_2$Cl$_2$\cite{2}, spin-1 AFMs La$_2$NiO$_4$ and K$_2$NiF$_4$\cite{3}, spin-$\frac{3}{2}$ AFM Rb$_2$MnF$_4$\cite{4} as well as spin-$\frac{1}{2}$ FMs such as K$_2$CuF$_4$\cite{5} are well described over a range of temperatures by the 2D Heisenberg model

$$H = J \sum_{\langle ij \rangle} S_i S_j$$  \hspace{1cm} (1)

on a square lattice, which we study by high temperature series expansion methods. We expect much of our results to apply to other lattices having collinear long-range order at $T=0$.

Let us begin by considering the relevant energy scales in the problem. One important energy scale for both AFMs ($J>0$) and FMs ($J<0$) is the $T=0$ spin stiffness $\rho_s$, defined as rigidity with respect to a twist in the magnetic structure. Quantum $1/S$ corrections make this quantity different for AFMs and FMs with the same value of spin; for model (1), the order of magnitude is however the same, $\rho_s^{AFM} \sim JS^2$, $\rho_s^{FM} = JS^2$ (note that the FM value is exact).

As shown by Chakravarty, Halperin, and Nelson\cite{6} for AFMs, and by Kopietz and Chakravarty\cite{7} for FMs, the asymptotic $T \to 0$ magnetic behavior for these models obeys scaling and depends on only two dimensionful parameters: the energy scale $\rho_s$, and a quantity which sets the overall length scale and can be obtained from the $q \to 0$ limit of the spin wave dispersion $\epsilon(q)$. For AFMs, $\epsilon(q)$ is linear and one defines the $T=0$ spin wave velocity as $c = \lim_{q \to 0} \epsilon(q)/q \sim JSa$. For FMs, $\epsilon(q)$ is quadratic and one defines the $T=0$ spin wave stiffness $\rho = \lim_{q \to 0} \epsilon(q)/q^2 \sim JSa^2$. Note that the spin wave stiffness $\rho$ has the dimension of energy $\times$ length$^2$, while the spin stiffness $\rho_s$ has the dimension of energy.

For the Heisenberg model, the spin wave spectrum is well-defined for all $q$ and its upper bound, reached at the boundary of the Brillouin zone where $q \sim 1/a$, can be estimated as $\Lambda \sim c/a \sim JS$ for the AFMs, or $\Lambda \sim \rho/a^2 \sim JS$ for the FMs, i.e. not only $\rho_s$, but also $\Lambda \sim JS$ has the same order of magnitude in AFMs and FMs.

Whereas the $T \to 0$ behavior depends on $\rho_s$ and $c$ for AFMs, or $\rho_s$ and $\rho$ for FMs, the $T \to \infty$ behavior (the Curie-Weiss law) depends on $\rho_{CW} = JS(S+1)$ and the lattice constant $a$. In what follows, we show that this simple observation is part of a general picture where the low temperature behavior, which may contain several scaling regimes, at $T \sim \Lambda$ crosses over to the high-temperature behavior, which may also contain several scaling regimes. A similar crossover in 3D occurs below the long range ordering temperature (which is zero in 2D), and was studied by Vaks, Larkin, and Pikin\cite{8}.

In model (1), the ratio $\Lambda/\rho_s \sim 1/S$ can be made arbitrarily small by increasing $S$, but it cannot be made arbitrarily large as $S \geq \frac{1}{2}$. Nevertheless, the case $\Lambda/\rho_s \gg 1$ is of great interest. First, there exist models where $\rho_s$ goes to zero while $\Lambda$ does not, in which case $\Lambda/\rho_s \gg 1$ applies rigorously; an example of such a system is the two-layer Heisenberg model\cite{9,10,11} near the critical point where the Néel AFM long-range order vanishes. Second, for the $S = 1/2$ AFM model (1), the ratio $\Lambda/\rho_s \sim 10 \gg 1$ is quite large; furthermore, there exists evidence\cite{9,10,11,12,13,14,15}, first pointed out by Chubukov and Sachdev, that at intermediate temperatures this model is indeed in the quantum critical limit $\Lambda/\rho_s \gg 1$. 

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[9] See reference [8].  
[10] See reference [9].  
[11] See reference [10].  
[12] See reference [11].  
[13] See reference [12].  
[14] See reference [13].  
[15] See reference [14].
Quantum Critical Scaling

Quantum Crossover

Classical Crossover

Renormalized Classical Scaling

Curie−Weiss

FIG. 1. A phase diagram of spin-S quantum antiferromagnets (AFMs) and ferromagnets (FMs); see Table I for the corresponding scaling and crossover expressions for the correlation length. All regime boundaries are gradual crossovers rather than phase transitions, and their positions can only be defined within numerical factors of order unity. The behavior in the $T/\Lambda \gg 1$ region is in the universality class of 2D classical magnets, where magnetic properties of AFMs and FMs for the same value of $S$ are the same near their respective ordering wavevectors, and depend on $T/\rho_{\text{CL}}$, where $\rho_{\text{CL}} = JS(S+1)$. The classical behavior includes Curie-Weiss regime ($T \gg \max(\rho_{\text{CL}}, \Lambda)$) and classical scaling regime ($\Lambda \ll T \ll \rho_{\text{CL}}$), which are separated by a fairly wide classical crossover regime for $\Lambda \ll T \sim \rho_{\text{CL}}$, where scaling does not hold, but nevertheless $\xi_{\text{AFM}} \approx \xi_{\text{FM}} \approx a\psi_{\text{CL}}(T/\rho_{\text{CL}})$. In the region $T/\Lambda \ll 1$ quantum effects are important and therefore AFMs and FMs behave differently. For the AFMs, this region is in the universality class of the QNL$\sigma$ model. Here these scaling regimes are found for both AFMs and FMs: renormalized classical (RC) regime for $T \ll \min(\rho_s, \Lambda)$; quantum critical (QC) regime for $\rho_s \ll T \ll \Lambda$; and a wide quantum crossover regime for $T \sim \rho_s \ll \Lambda$, where the correlation length remains a universal function (different for AFMs and FMs) of the thermal de Broglie wavelength for spin waves ($\lambda_T = c/T$ or $\lambda_T = (\rho/T)^{1/2}$) and the ratio $T/\rho_s$.

| Regime                              | Correlation Length                                      |
|-------------------------------------|---------------------------------------------------------|
| Curie-Weiss                         | $\xi_{\text{AFM}} = \xi_{\text{FM}} \simeq a$          |
| $T \gg \max(\rho_{\text{CL}}, \Lambda)$ | Properties depend on $T/\rho_{\text{CL}}$               |
| Classical Crossover                 | $\xi_{\text{AFM}} = \rho_{\text{CL}} \gg \Lambda$     |
| $T \sim \rho_{\text{CL}} \gg \Lambda$ | $\xi_{\text{AFM}} = a\psi_{\text{CL}}(T/\rho_{\text{CL}})$ |
| $\rho_{\text{CL}} \gg T \gg \Lambda$ | $\xi_{\text{AFM}} \sim a(T/\rho_{\text{CL}})$ $\times \exp(2\pi\rho_{\text{CL}}/T)$ |
| Quantum Critical                    | $\xi_{\text{AFM}} \sim c/T,$ $\rho_s \ll T \ll \Lambda$ |
| $\rho_s \ll T \ll \Lambda$        | $\xi_{\text{FM}} \sim (\rho/T \log(T/\rho_s))^{1/2}$  |
| Quantum Crossover                  | $\xi_{\text{AFM}} = (c/T)\phi_{\text{AFM}}(T/\rho_s)$ |
| $T \sim \rho_s \ll \Lambda$       | $\xi_{\text{FM}} = (\rho/T)^{1/2}\phi_{\text{FM}}(T/\rho_s)$ |
| Renormalized Classical             | $\xi_{\text{AFM}} \sim (c/\rho_s) \exp(2\pi\rho_s/T)$ |
| $T \ll \min(\rho_s, \Lambda)$    | $\xi_{\text{FM}} \sim (\rho/T)^{1/2}\exp(2\pi\rho_s/T)$ |

Table I. Correlation length in the scaling and crossover regimes shown in the phase diagram of Fig. 1, after Refs. [14,6,7,11], and this work. The sign $\sim$ indicates the presence of a numerical prefactor. The functions $\phi_{\text{AFM}}$ and $\phi_{\text{FM}}$ are universal; the function $\psi_{\text{CL}}$ is not, for instance, it explicitly depends on the ratio of the next-nearest and nearest-neighbor exchange couplings.
II. CURIE-WEISS REGIME

At high temperatures $T \gg \max(\rho_s, \Lambda)$, the spins are weakly correlated and one can keep only a few leading terms in the high temperature series expansion. For model $[\mathbb{H}]$, this temperature range corresponds to $T \gg \max(JS, JS^2)$. The corresponding mean-field theory yields the familiar Curie-Weiss law.

III. RENORMALIZED CLASSICAL REGIME

In the opposite limit of $T \ll \min(\rho_s, \Lambda)$, the behavior is of the renormalized classical (RC) type. This regime was studied in detail in Refs. [14,15], where magnetic properties were calculated by mapping the system onto the classical nonlinear $\sigma$-model and setting the momentum-integration cutoff to be proportional to the thermal de Broglie wave vector $q_T$, defined such that $\epsilon(q_T) \sim T$. The predicted $\xi(T)$ has the same form for AFMs ($J > 0$) and FMs ($J < 0$), when expressed in terms of their respective thermal de Broglie wavelengths $\lambda_T = 1/q_T$:

$$\xi_{RC} = \frac{c}{8} \lambda_T \frac{T}{2\pi \rho_s} \exp\left(\frac{2\pi \rho_s}{T}\right), \quad (2)$$

(here the exact value of the prefactor is obtained from the correlation length of the classical $O(3)$ nonlinear-$\sigma$ model in the minimal subtraction scheme $[14,15]$). However, not only the values of $\rho_s$ differ for FMs and AFMs, but also $\lambda_T$ has qualitatively different temperature dependences:

$$\lambda_T = \begin{cases} \frac{c}{T} & \text{for AFMs} \\ \frac{\sqrt{\rho_s}}{T} & \text{for FMs} \end{cases} \quad (3)$$

An important question is upto what temperatures should this RC expression hold? We recall that when $T$ is larger than the upper bound of the $T = 0$ spin wave spectrum, $\Lambda$, there can be no spin waves with wavelengths that are comparable to the thermal de Broglie wavelength. Since such spin waves are important for RC theory, one would expect it to be applicable only for $T \ll \min(\Lambda, \rho_s)$, which for model $[\mathbb{H}]$ translates into $T \sim \min(JS, JS^2)$ (here, we ignore numerical factors of order unity). At higher temperatures new physics must arise.

The Curie-Weiss regime occurs for $T \gg \max(\rho_s, \Lambda)$ and the RC regime for $T \ll \min(\rho_s, \Lambda)$. These two regimes exist for all $S$. We now turn to the discussion of the regimes where the temperature is larger than one of the scales $\rho_s$ or $\Lambda$, but smaller than the other.

IV. CLASSICAL AND CLASSICAL CROSSOVER REGIMES

In our earlier work in collaboration with Greven and Birgeneau [17], we proposed a scaling crossover scenario to describe substantial deviations from RC behavior observed in the neutron scattering experiments for AFMs with spin-one and larger. This scenario calls for a crossover to the behavior of the $S \to \infty$ classical system when spin wave energies for all wavevectors in the Brillouin zone become smaller than the temperature, i.e. for $T \gtrsim \Lambda$. The correlation length at this RC to classical boundary, obtained from Eqs.(2), is exponentially large for large spin,

$$\xi \sim \frac{\exp(S)}{S} \quad (4)$$

In [17], the arguments in favor of RC to classical crossover were based on data collapse for large spin when plotted versus $T/(JS(S+1))$. The studied values of spin were small enough ($S = 1/2$ to $S = 5/2$) to make the results sensitive to the choice of the renormalized temperature as $T/(JS(S+1))$ or $T/JS^2$.

Here, we resolve this arbitrariness by studying antiferro- and ferromagnets simultaneously. AFMs and
FMs have the same classical limit, but the quantum effects differ and therefore the difference between AFMs and FMs can be used as a probe for the strength of quantum effects. It turns out that for any $\rho$ where $F\rho_s$ can be used as a probe for the strength of quantum effects differ and therefore the difference between AFMs and FMs have the same classical limit, but the quantum effects are nearly the same for AFMs and FMs with the same value of $s$. For the other hand, for $S = 1/2$ the difference between $\xi_{AFM}$ and $\xi_{FM}$ is always large and temperature-dependent, which indicates the importance of quantum physics for this value of $S$. Spin-one appears to be a borderline case.

Having established that in the temperature range $T \gg T_o$ a finite-spin models behave similarly to the $S \to \infty$ classical magnet, we now comment on the behavior of the latter. The asymptotic low-temperature behavior of the classical 2D Heisenberg antiferromagnet is given by [16]:

$$\frac{\xi_{CL}}{a} = \frac{e - \pi/2}{8\sqrt{2}} \frac{T}{2\pi \rho_{CL}} \exp \left(\frac{2\pi \rho_{CL}}{T}\right),$$

where $\rho_{CL} = JS(S + 1)$. It turns out that because of the very small numerical value of the prefactor ($\sim 0.01$), this formula becomes accurate only for very low temperatures, $T/\rho_{CL} \lesssim 0.6$ ($\xi \gtrsim 100$). We label the intermediate temperature range where neither Curie-Weiss nor classical scaling behavior apply the classical crossover regime. In this regime, neither large nor small-temperature asymptotic expressions describe the correlations accurately, nevertheless, all properties of the model are nearly the same for AFMs and FMs, and depend on $T$ through $T/\rho_{CL}$ only.

V. QUANTUM CRITICAL AND QUANTUM CROSSOVER REGIMES

We call quantum critical that behavior which can formally be obtained for $\rho_s \ll T \ll \Lambda$. For the antiferromagnets, this regime is the asymptotic quantum critical (QC) regime [11]. The correlation length $\xi$ is proportional to $\lambda_T = c/T$ with corrections that depend universally on $T/\rho_s$, and to leading order the dominant frequency scale for spin fluctuations is $\omega \sim T$.

For the FMs the spin wave spectrum is quadratic, which causes infrared ($q \to 0$) log divergences for $\rho_s \ll T \ll \Lambda$. Such divergences (not present for AFMs) are cutoff by $\rho_s$, and lead to the following multiplicative correction to the correlation length:

$$\xi_{FM} \sim (\rho/T)^{1/2} \log^{-1/2}(T/\rho_s),$$

in either the sigma model or the Schwinger boson formalism.

Note that such a behavior cannot be the true quantum critical behavior because the limit $\rho_s \to 0$ leads to singular $\xi$ at finite $T$. Therefore, Eq. (5) must fail as $\rho_s \to 0$, and the $T = 0$ quantum ferromagnet-paramagnet phase transition is likely to belong to a different universality class [15]. Nevertheless, lacking a better name and for compatibility with the AFM nomenclature, we call this behavior a ferromagnetic quantum critical regime.

The applicability of the QC description to model $S = 1/2$ AFM is widely discussed in the literature [2,11,12,13]. A similar analysis for the ferromagnets requires detailed calculations of their properties in the QC regime, and is beyond the framework of this paper.

VI. CONCLUSION

In this paper, we present a complete phase diagram of spin-$S$ antiferromagnets (AFMs) and ferromagnets (FMs). We show that comparing AFMs and FMs allows one to elucidate crossover effects from the quantum regimes (including the renormalized classical regime where, despite its name, quantum fluctuations are essential) to the purely classical regimes, where AFMs and FMs exhibit identical behavior near their respective ordering wavevectors. For a detailed description of the phase diagram and the proposed new regimes, we refer the reader to Fig. 2 and Table 1.

For the antiferromagnets, both our series expansion calculations and the neutron data [2,17] suggest that the regime boundaries in Fig. 2 are positioned such that the universal $T \ll \Lambda$ scaling theory, free of any lattice corrections, is not obeyed for spin-one and larger at any numerically or experimentally accessible values of the correlation length. The data also highlights the disagreement in the magnitude of spin correlations even for $S = 1/2$ at all accessible temperatures with the existing RC predictions, whereas $\xi(T)$ was found [2] to agree remarkably well with the RC theory.

Our results suggest that an accurate analytical theory for $S \geq 1$ in the experimentally relevant temperature range can be constructed by taking into account leading quantum corrections about the $S = \infty$ limit on a lattice. The important step here would be to obtain an analytic approximation which is valid in the classical crossover regime. On the other hand, any theory based on a purely continuum description, such as the QNL$\sigma$ model without lattice corrections, is clearly inadequate for any spin larger than $S = 1/2$.

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