New spin generalisation for long range interaction models

N. Crampé

University of York, Department of Mathematics
Heslington, York YO10 5DD, United Kingdom
nc501@york.ac.uk

Dedicated to my PhD supervisor and friend D. Arnaudon

Abstract

We study new interactions between degrees of freedom for Calogero, Sutherland and confined Calogero spin models. These interactions are encoded by the generators of the Lie algebra $so(N)$ or $sp(N)$. We find the symmetry algebras of these new models: the half-loop algebra based on $so(N)$ or $sp(N)$ for the Calogero models and the Yangian of $so(N)$ or $sp(N)$ for the two types of other models. Surprisingly, these symmetry occur only for a specific value of the coupling constant.

MSC: 70H06, 81R12, 81R50 — PACS: 02.20.Uw, 02.30Ik, 03.65.Fd, 75.10.Pq

Key words: Calogero spin-models, Sutherland spin-models, Half-loop algebras of $so(N)$ or $sp(N)$, Yangians of $so(N)$ or $sp(N)$.

February 26, 2022
The Calogero and Sutherland models are one-dimensional many body problems with long range interactions \cite{1, 2}. The introduction of a $gl(N)$ internal degree of freedom in these models \cite{3–5} has proved to be fruitful in various physical and mathematical investigations. This is well illustrated in the study of their symmetries which turn out to be the half-loop algebra or the Yangian associated to $gl(N)$ \cite{6–9}. This letter is devoted to the introduction of new interactions between the internal degrees of freedom in these models and finding the symmetry algebra of these new models. The interactions are defined thanks to the fundamental representation of the generators of the Lie algebras $so(N)$ or $sp(N)$. We will call these new models $so(N)$ or $sp(N)$-spin models. At this point, to avoid ambiguity, let us remark that these models are different from the so-called $BC_N$ models \cite{10}. Indeed, in the latter case, it is the potential which is closely related to the root systems of the algebra $BC_N$ and such models possess reflection algebra symmetry \cite{11}.

The plan of the letter is as follows. In section 1, we introduce the definitions and different notations used in the letter. Then, in the three following sections which have the same structure, we introduce the Hamiltonian for Calogero $so(N)$ or $sp(N)$ spin model, Sutherland $so(N)$ or $sp(N)$ spin model and confined Calogero $so(N)$ or $sp(N)$ spin model. The main results of this letter consists in finding for each model the symmetry algebra. We finish by an appendix where technical details for computations are gathered.

## 1 General setting

Let $x_1, \ldots, x_L$ be the positions of $L$ particles on a one-dimensional space. We associate to each particle an internal degree of freedom or spin which will be considered as a vector belonging to $C^N$. The spin operators $E_{i}^{ab}$ ($1 \leq a, b \leq N$ and $1 \leq i \leq L$) are matrices with entry 1 in row $a$ and column $b$ and zero elsewhere which act on the spin space of the $i^{th}$ particle. They provide a representation of $\bigoplus_{i} gl_N$ and they satisfy the following commutation relations

$$[E_i^{ab}, E_j^{cd}] = \delta_{ij} \left( \delta_{bc} E_i^{ad} - \delta_{ad} E_i^{bc} \right). \quad (1.1)$$

Let $\theta_0 = \pm 1$. For each index $1 \leq a \leq N$, we introduce the following sign, for $N$ even,

$$\theta_a = \begin{cases} +1 & \text{for } 1 \leq a \leq \frac{N}{2}, \\ \theta_0 & \text{for } \frac{N}{2} + 1 \leq a \leq N, \end{cases} \quad (1.2)$$

and $\theta_a = +1$ if $N$ is odd. We introduce also the following conjugate index $\tilde{a}$

$$\tilde{a} = N + 1 - a \quad \text{for } 1 \leq a \leq N. \quad (1.3)$$

In particular $\theta_a \theta_{\tilde{a}} = \theta_0$.

These definitions allow us to deal simultaneously with the Lie algebras $so(N)$ and $sp(N)$, subalgebras of $gl(N)$. Let us define, for $1 \leq a, b \leq N$,

$$F_i^{ab} = E_i^{ab} - \theta_a \theta_b E_{i}^{\tilde{a}}. \quad (1.4)$$
The algebra $\mathfrak{g}^+(N)$ (resp. $\mathfrak{g}^-(N)$) spanned by these generators is isomorphic to $\mathfrak{so}(N)$ (resp. $\mathfrak{sp}(N)$) when $\theta_0 = +1$ (resp. $\theta_0 = -1$). Of course, the case $\theta_0 = -1$ occurs only when $N$ is even. These generators satisfy the symmetry relation

$$F^{ab} = -\theta_a \theta_b F^{ba}.$$  \hspace{1cm} (1.5)$$

To correctly define the structure constants and a non-degenerate metric tensor of $\mathfrak{g}^±(N)$, we need to restrict this set $\{F^{ab}\}$ of generators to a basis of $\mathfrak{g}^±(N)$. Let us define the subsets of indices $E^+ = \{(a, b) | \bar{a} > b\}$ and $E^- = \{(a, b) | \bar{a} \geq b\}$. The sets $\mathcal{B}^\pm = \{F^{ab} | (a, b) \in E^\pm\}$ form bases of Lie algebras $\mathfrak{g}^\pm(N)$. Then, the commutation relations of $\mathfrak{g}^\pm(N)$ can be written as follows, for $(a, b), (c, d) \in E^\pm$,

$$[F_i^{ab}, F_j^{cd}] = \delta_{ij} \sum_{(e, f) \in E^\pm} f_{ef}^{ab, cd} F_i^{ef},$$  \hspace{1cm} (1.6)$$

where the structure constants read

$$f_{ef}^{ab, cd} = \left[\delta^{bc}(\delta_e^a \delta_f^d - \theta_a \theta_d \delta_e^b \delta_f^d) - \delta^{ad}(\delta_f^b \delta_e^c - \theta_b \theta_e \delta_f^a \delta_e^b)\right] - \delta^{ae}(\theta_a \theta_b \delta_d^f \delta_e^c - \theta_c \theta_d \delta_e^a \delta_f^c) + \delta^{bf}(\theta_a \theta_b \delta_d^e \delta_c^a - \theta_c \theta_d \delta_e^a \delta_f^b)\right] H(\bar{e}, f).$$  \hspace{1cm} (1.7)$$

The function $H(i, j)$ is defined, for $(i, j) \in E^\pm$, as follows

$$H(i, j) = \begin{cases} 1 & \text{if } i > j, \\ \frac{1}{2} & \text{if } i = j. \end{cases}$$  \hspace{1cm} (1.8)$$

The factor $1/2$ in the function $H$ is relevant only in the case where we consider $\mathfrak{g}^-(N)$ and is due to the particular choice of the normalisation of the generators. We choose the non-degenerate metric tensor as follows, for $(a, b), (c, d) \in E^\pm$,

$$g^{ab, cd} = \frac{1}{2} \text{Tr}(F^{ab} F^{cd}).$$  \hspace{1cm} (1.9)$$

This metric will allow us to raise or lower the indices of the structure constants.

## 2 Calogero model

In this section, we will obtain the symmetry algebra of the Calogero $\mathfrak{g}^\pm(N)$-spin model which we defined through the following Hamiltonian

$$\mathcal{H}_C = -\sum_{j=1}^{L} \frac{\partial^2}{\partial x_j^2} + \sum_{j \neq k} \frac{\lambda^2 - \lambda P_{jk} + \lambda Q_{jk}}{(x_j - x_k)^2}.$$  \hspace{1cm} (2.1)$$

The matrix $P_{jk}$ permutes the spins of the $j^{th}$ and $k^{th}$ particles and can be written in terms of the spin operators as

$$P_{jk} = \sum_{a, b=1}^{N} E_{j}^{ab} E_{k}^{ba}.$$  \hspace{1cm} (2.2)$$
The operator $Q_{jk}$ is defined by

$$Q_{jk} = \sum_{a,b=1}^{N} \theta_a \theta_b E^a_j E^b_k. \quad (2.3)$$

They satisfy, in particular, the useful properties $P_{jk} = P_{kj}$ and $Q_{jk} = Q_{kj}$. These two operators are the crucial elements to construct the R-matrix associated to the Yangian of $so(N)$ or $sp(N)$ [12–15].

The introduction in the Hamiltonian of the operator $Q_{jk}$ modifies the interaction between the degrees of freedom of the particles as compared with the $gl(N)$-spin model. Note that we can write the new interactions in terms of the generators of $g^{\pm}(N)$ as follows

$$P_{jk} - Q_{jk} = \frac{1}{2} \sum_{a,b=1}^{N} F^a_j F^b_k. \quad (2.4)$$

We have used in the previous formula the conventional notation $(F_j F_k)^{ab} = \sum_{c=1}^{N} F^{ac}_j F^{cb}_k$.

It is well-known that the symmetry algebra of $A_N$ Calogero $gl(N)$-spin model is the half-loop algebra of $gl(N)$ [6–9]. We shall show how this symmetry algebra is modified for Calogero $g^{\pm}(N)$-spin model. We introduce the following operators, for $(a, b) \in E^\pm$,

$$J^{ab}_0 = \sum_{j=1}^{L} F^a_j , \quad (2.5)$$

$$J^{ab}_1 = \sum_{j=1}^{L} F^a_j \frac{\partial}{\partial x_j} - \lambda \sum_{j \neq k} (F_j F_k)^{ab} \frac{1}{x_j - x_k}. \quad (2.6)$$

After a straightforward computation, we can show that these operators satisfy the following relations

$$[J^{ab}_0, J^{cd}_0] = f^{ab,cd}_{ef} J^{ef}_0 , \quad (2.7)$$

$$[J^{ab}_0, J^{cd}_1] = f^{ab,cd}_{ef} J^{ef}_1 , \quad (2.8)$$

$$[J^{ab}_1, [J^{cd}_0, J^{ef}_1]] + [J^{ef}_1, [J^{ab}_0, J^{cd}_1]] + [J^{cd}_1, [J^{ef}_0, J^{ab}_1]] = 0 , \quad (2.9)$$

for the following particular value of the coupling constant

$$\lambda = \frac{2}{N - 4\theta_0}. \quad (2.10)$$

In relations (2.7) and (2.8), we have used the Einstein’s notation for the repeated pair of indices $(e, f)$ but the sums are only for $(e, f) \in E^\pm$ (for example, we write explicitly this sum in (1.6)). From now on, we use this convention for the repeated indices.

For the particular choice $N = 4\theta_0$, the denominator in (2.10) vanishes and therefore $J^{ab}_0, J^{ab}_1$ are not well-defined. However, this corresponds to the case where we consider the algebra $so(4)$ which is a non-simple Lie algebra.

The higher level generators, $J^{ab}_2, J^{ab}_3, \ldots$, are defined recursively from $J^{ab}_0$ and $J^{ab}_1$. Relation (2.9), called Serre relation, guarantees that these generators are well-defined. We have the commutation relations

$$[J^{ab}_n, J^{cd}_m] = f^{ab,cd}_{ef} J^{ef}_{n+m}. \quad (2.11)$$
Relations (2.7)-(2.9) define the half-loop algebra (also called Gaudin algebra) associated to the Lie algebra $\mathfrak{g}^\pm(N)$.

To finish the proof of the symmetry, we show that $J^{ab}_0$ and $J^{ab}_1$ are conserved operators i.e.

$$[H_C, J^{ab}_0] = 0 \quad \text{and} \quad [H_C, J^{ab}_1] = 0.$$  \hfill (2.12)

The particular value (2.10) of $\lambda$ is necessary and sufficient to prove the second relation in (2.12) whereas the first one holds for any value of $\lambda$. We have used, in particular, the two following properties

$$P_{jk} F^{ab}_j = F^{ab}_k P_{jk} \quad \text{and} \quad Q_{jk} F^{ab}_j = -Q_{jk} F^{ab}_k.$$  \hfill (2.13)

Therefore, we have therefore shown that the symmetry algebra of the model described by the Hamiltonian (2.1) is the half-loop algebra associated to $\mathfrak{g}^\pm(N)$. To obtain the symmetry, it was necessary to constrain the coupling constant. This feature is new in comparison with the Calogero $gl(N)$-spin model where the coupling constant remains arbitrary free.

## 3 Sutherland model

In this section, we introduce a new Sutherland spin model, called Sutherland $\mathfrak{g}^\pm(N)$-spin model, with Hamiltonian given by

$$H_S = -\sum_{j=1}^{L} \left( x_j \frac{\partial}{\partial x_j} \right)^2 + \sum_{j \neq k} (\lambda^2 - \lambda P_{jk} + \lambda Q_{jk}) \frac{x_j x_k}{(x_j - x_k)^2}$$  \hfill (3.1)

and exhibit its symmetry algebra.

It is well-known that the symmetry algebra of Sutherland $gl(N)$-spin model is the Yangian of $gl(N)$ and we show that for this model it is the Yangian of $\mathfrak{g}^\pm(N)$. The end of this section consists in proving this statement. For convenience, let us define the symmetriser of any three elements $K^A, K^B, K^C$ by

$$\{K^A, K^B, K^C\} = \frac{1}{24} \sum_{\sigma \in S_3} K^{\sigma(A)} K^{\sigma(B)} K^{\sigma(C)},$$  \hfill (3.2)

where $S_3$ is the group of the permutations of order 6. The Yangian of $\mathfrak{g}^\pm(N)$ is the associative algebra generated by $\{K_0^{ab}, K_1^{ab}: (a, b) \in \mathcal{E}^\pm\}$ constrained by the following commutation relations [16], for $(a, b), (c, d) \in \mathcal{E}^\pm$,

$$[K_0^{ab}, K_0^{cd}] = f^{ab,cd} \, K_0^{ef}, \quad [K_0^{ab}, K_1^{cd}] = f^{ab,cd} \, K_1^{ef},$$  \hfill (3.3)  

$$[K_1^{ab}, [K_0^{cd}, K_1^{ef}]] + [K_1^{ef}, [K_0^{ab}, K_1^{cd}]] + [K_1^{cd}, [K_0^{ef}, K_1^{ab}]]$$

$$= \lambda^2 f^{ab,\alpha\beta,ij} f^{cd,\gamma\delta,kl} f^{ef,\epsilon\phi,mn} f^{ij,kl, mn} \{K_0^{\alpha\beta}, K_0^{\gamma\delta}, K_0^{\epsilon\phi}\}.$$  \hfill (3.5)
We recall that we use the Einstein’s notation for repeated indices but that the skipped sums are running over $E^\pm$. The Lie algebra indices are lowered or raised by the invariant non-degenerate metric tensor defined by relation (1.9).

The following operators, for $(a,b) \in E^\pm$,

\[ K_0^{ab} = \sum_{j=1}^{L} F_j^{ab}, \]

\[ K_1^{ab} = \sum_{j=1}^{L} F_j^{ab} x_j \frac{\partial}{\partial x_j} - \lambda \sum_{j \neq k} (F_j F_k)^{ab} x_j + x_k, \]

give a representation of the Yangian of $g^\pm(N)$ provided that the coupling constant $\lambda$ (which is also the deformation parameter of the Yangian) takes the particular value (2.10). The first two relations (3.3) and (3.4) are easily proven by direct computation. We give some details for the computation of the Serre relation (3.5) in the appendix.

By direct computation, we can also prove that

\[ [H_S, K_0^{ab}] = 0 \quad \text{and} \quad [H_S, K_1^{ab}] = 0. \]

The first relation in (3.8) is true for any $\lambda$ whereas the second one holds only and only if $\lambda$ is equal to the particular value (2.10). Therefore, we have proved that the symmetry algebra of the model described by the Hamiltonian (2.1) is the Yangian of $g^\pm(N)$ (for the deformation parameter equal to $\frac{2}{N - 4\theta_0}$).

4 Confined Calogero model

This section is devoted to studying the symmetry algebra of the confined Calogero $g^\pm(N)$-spin model which is described by the following Hamiltonian

\[ H_{CC} = H_C + \omega^2 \sum_{j=1}^{L} x_j^2. \]

The operator $H_C$ is the Hamiltonian of the Calogero model introduced in (2.1). Let us remark that the introduction of this harmonic potential breaks translation invariance.

We shall prove that the linear combinations introduced in [7] to obtain the symmetry algebra of the confined Calogero $gl(N)$-spin chain model are also relevant in our case to obtain the symmetry algebra. Let us define a new set of operators, for $(a,b) \in E^\pm$

\[ \mathcal{J}_0^{ab} = J_0^{ab}, \]

\[ \mathcal{J}_1^{ab} = J_2^{ab} - \omega^2 \mathcal{O}_2^{ab}, \]
where we have introduced the new operators $O^a_n = \sum_{j=1}^L F^a_j x^j$. We can easily show that the set of operators $\{O^a_n\}$ satisfy the relations of the half-loop algebra (2.11). By computing $[J^a_{1}, J^c_{1}]$, we find the following explicit form for $J^a_{2}$,

$$J^a_{2} = \sum_{j=1}^L F^a_j \frac{\partial^2}{\partial x_j^2} - \lambda \sum_{j \neq k} \frac{(F^a_j F^b_k)^{ab}\partial}{(x_j - x_k)^2} + \lambda \sum_{j \neq k} \frac{(E_j E_k)^{ab} - \theta_a \theta_b (E_j E_k)^{ba} - \lambda F^a_j}{(x_j - x_k)^2} \partial^2 \partial x_j^2 \partial x_k^2 - \lambda^2 \sum_{j \neq k \neq \ell} \frac{(F^a_j F^b_k F^c_\ell)^{ab}}{(x_j - x_k)(x_j - x_\ell)},$$

(4.4)

where we have used this following contraction $(F^a_j F^b_k F^c_\ell)^{ab} = \sum_{\alpha, \beta=1}^N F^a_{\alpha} F^b_{\beta} F^c_{\alpha}$. We can prove that the operators $J^a_0$ and $J^a_1$ provide a representation of the Yangian of $g^\pm(N)$

$$[J^a_0, J^c_0] = f^{a,c}_{0,ef} J^e_0,$$

(5.5)

$$[J^a_0, J^c_1] = f^{a,c}_{0,ef} J^e_1,$$

(5.6)

$$[J^c_1, [J^a_0, J^c_0]] + [J^c_0, [J^a_1, J^c_0]] + [J^c_0, [J^c_0, J^a_1]] = 4\lambda^2 \omega^2 f^{a,b}_{0,ijkl} f^{c,d}_{ef,km} f^{ij}_{ijkl} f^{ef}_{0,km} \{J^0_0, J^0_0, J^0_0\}.$$

(4.7)

Once again, these relations are satisfied if and only if the parameter $\lambda$ takes the particular value (2.10) whereas the parameter $\omega$ remains free. Obviously, in the limit $\omega$ tends to 0, we recover the half-loop algebra for the corresponding Calogero model. The commutation relations (4.5) and (4.6) are computed directly. Some details for the computation of the Serre relations are presented in the appendix.

The proof of the symmetry is provided by the two following relations

$$[H_{CC}, J^{a, b}_0] = 0 \quad \text{and} \quad [H_{CC}, J^{a, b}_1] = 0.$$

(4.8)

Let us remark that the first relation in (4.8) holds for any $\lambda$ whereas the second one requires that $\lambda$ takes the particular value (2.10). Therefore we have proved that the symmetry of the model described by the Hamiltonian (4.1) is the Yangian of $g^\pm(N)$ with the deformation parameter equals to $\frac{2\omega N}{N - 4\theta_0}$.

**Conclusion**

In this letter, we studied new interaction between degree of freedom for different models with long range interaction such as Calogero model, Sutherland model or confined Calogero model. For each one, we obtain the symmetry algebra. Several questions remain open. The Lax pair as well as the Dunkl operators are not computed for these new models. These different approaches may allow one to deeply understand the constraint on the coupling constant which appears here. Another problem consists in computing the eigenfunctions and the eigenvalues of these models. The knowledge of the symmetry may help for their resolution. Indeed, for previous cases like the Sutherland $gl(N)$-spin
model whose the symmetry is the Yangian of \( gl(N) \), the algebra symmetry is crucial to find the spectrum (see e.g. [6, 8, 17]). Finally, we want to point out the problem of the "freezing" method. Indeed when the coupling constant is not constrained, it is possible to obtain non-dynamical spin chains known as Frahm-Polychronakos or Haldane-Shastry spin chains [18–21]. This method seems not to work for the models studied in this letter.

Acknowledgements: This work is supported by the TMR Network ‘EUCLID’ Integrable models and applications: from strings to condensed matter’, contract number HPRN-CT-2002-00325.

A Appendix

In this appendix, we give some details about the proof of the Serre relations (3.5) and (4.7).

Sutherland model

Using relation (2.8), the left-hand side of relation (3.5) can be written as

\[
LHS = f_{\alpha\beta}^{cd,ef} [K_1^{ab}, K_1^{\alpha\beta}] + f_{\alpha\beta}^{ab,cd} [K_1^{ef}, K_1^{\alpha\beta}] + f_{\alpha\beta}^{ef,ab} [K_1^{cd}, K_1^{\alpha\beta}].
\]

(A.1)

By direct computation, we obtain

\[
[K_1^{ab}, K_1^{\alpha\beta}] = f_{\alpha\beta}^{ab,\alpha\beta} m_{ab} m_{\alpha\beta} + X_{ab,\alpha\beta}.
\]

(A.2)

The explicit form of \( K_{mn}^{ab} \) is not relevant because the Jacobi identity for the structure constants implies that these terms vanish in (A.1). The operator \( X_{ab,\alpha\beta} \) takes the following form,

\[
X_{ab,\alpha\beta} = -\frac{\lambda}{2} \sum_{j \neq k} f(x_j, x_k) \left( \theta_a \theta_b F_j E_{\alpha\beta} E_k - \theta_a \theta_b F_j E_{\beta\alpha} E_k + \theta_b \theta_a E_j E_{\alpha\beta} - \theta_a \theta_b E_j E_{\beta\alpha} \right) + \frac{\lambda^2}{4} \sum_{j \neq k \neq \ell} \left( F_{j}^{ab} (F_k F_\ell)^{\alpha\beta} - F_{j}^{\alpha\beta} (F_k F_\ell)^{ab} + \theta_a \theta_b F_j^{a\beta} (F_k F_\ell)^{\alpha\beta} - \theta_a \theta_b F_j^{\beta\alpha} (F_k F_\ell)^{a\beta} \right),
\]

(A.3)

where

\[
f(x_j, x_k) = \left( \frac{\lambda(N - 4\theta_0)(x_j + x_k)^2 - 8x_j x_k}{2(x_j - x_k)^2} \right).
\]

(A.4)

Since the right-hand side of (3.5) does not depend on the position, the function \( f(x_j, x_k) \) must be constant. This constraint implies that \( \lambda = 2(N - 4\theta_0) \) and \( f(x_j, x_k) = 1 \).

Now, let us focus on the right-hand side. It contains a sum of \( 8^4 = 4096 \) terms. By hand, it would be about impossible to deal with this number of terms. Fortunately, formal computations with Maple allow us to reduce this number. Finally, the right-hand side can be written as

\[
RHS = \lambda^2 \left( f_{\alpha\beta}^{cd,ef} Y^{ab,\alpha\beta} + f_{\alpha\beta}^{ab,cd} Y^{ef,\alpha\beta} + f_{\alpha\beta}^{ef,ab} Y^{cd,\alpha\beta} \right).
\]

(A.5)
where
\[ Y^{ab,\alpha\beta} = \sum_{\ell=1}^{N} \left( \{ K_0^{ab}, K_0^{\alpha\ell}, K_0^{\ell\beta} \} - \{ K_0^{a\beta}, K_0^{\alpha\ell}, K_0^{\ell b} \} + \alpha_{a\beta} \theta_{\alpha\beta} \{ K_0^{ab}, K_0^{\alpha\ell}, K_0^{\ell b} \} - \theta_{\alpha\beta} \{ K_0^{a\beta}, K_0^{\alpha\ell}, K_0^{\ell b} \} \right). \]

(A.6)

We recall that \( \{., ., .\} \) is defined by relation (3.2). To simplify the notation, we introduce also new generators, for \( (a,b) \in \mathcal{E}^\pm \),
\[ K_0^{ab} = -\theta_{a\beta} K_0^{ab}. \]

(A.7)

Finally, we compute \( Y^{ab,\alpha\beta} \) using explicit expression (3.6) of \( K_0^{ab} \) to show that RHS given by (A.5) is equal to LHS given by (A.1) which finishes the proof of the Serre relation (3.5).

Confined Calogero model

The computation of relation (4.7) is simplified by remarking that its right-hand side is identical to the one of relation (3.5) (up to a factor \( 4 \omega^2 \)) which has been computed previously. Its left-hand side (divided by \( -\omega^2 \)) can be reduced to
\[ f^{cd,ef}_{\alpha\beta} ([J_2^{ab}, O_2^{\alpha\beta}] + [O_2^{ab}, J_2^{\alpha\beta}]) + f^{ab,cd}_{\alpha\beta} ([J_2^{ef}, O_2^{\alpha\beta}] + [O_2^{ef}, J_2^{\alpha\beta}]) + f^{ef,ab}_{\alpha\beta} ([J_2^{cd}, O_2^{\alpha\beta}] + [O_2^{cd}, J_2^{\alpha\beta}]) \]

(A.8)

by remarking, in particular, that the generators \( O_2^{ab} \) and \( J_2^{ab} \) satisfy the Serre relations (2.9). Now, by direct computation of the commutator \( [J_2^{ab}, O_2^{\alpha\beta}] \), using the explicit form (4.4) for \( J_2^{ab} \), we can prove that the Serre relation (4.7) is satisfied.

References

[1] F. Calogero, Solution of a Three-Body Problem in One Dimension, J. Math. Phys. 10 (1969) 2191. Ground State of a One-Dimensional N-Body System, J. Math. Phys. 10 (1969) 2197. Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials, J. Math. Phys. 12 (1971) 419.

[2] B. Sutherland, Exact Results for a Quantum Many-Body Problem in One Dimension, Phys. Rev. A4 (1971) 2019. Exact Results for a Quantum Many-Body Problem in One Dimension. II Phys. Rev. A5 (1972) 1372. Exact Ground-State Wave Function for a One-Dimensional Plasma Phys. Rev. Lett. 34 (1975) 1083.

[3] Z.N.C. Ha and F.D.M. Haldane, On Models with Inverse-Square Exchange, Phys. Rev. B46 (1992) 9359 and cond-mat/9204017.

[4] J.A. Minahan and A.P. Polychronakos, Integrable Systems for Particles with Internal Degrees of Freedom, Phys. Lett. B302 (1993) 265 and hep-th/9206046.

[5] A.P. Polychronakos, Exchange operator formalism for integrable systems of particles, Phys. Rev. Lett 69 (1992) 703 and hep-th/9202057.
[6] D. Bernard, M. Gaudin, F.D.M. Haldane, V. Pasquier, Yang-Baxter equation in spin chains with long range interactions, J. Phys. A26 (1993) 5219 and hep-th/9301084.

[7] D. Bernard, K. Hikami and M. Wadati, The Yangian Deformation of the W-Algebras and the Calogero-Sutherland model, Proceeding of 6-th Nankai Workshop and hep-th/9412194.

[8] K. Hikami, Symmetry of the Calogero model confined in the harmonic potential-Yangian and W algebra, J. Phys. A28 (1995) 131, Yangian symmetry and Virasoro character in a lattice spin with long range interaction, Nucl. Phys. B441 (1995) 530.

[9] F. Finkel, D. Gómez-Ullate, A. González-López, M.A. Rodríguez and R. Zhdanov, $A_N$-type Dunkl operators and new spin Calogero-Sutherland models, Comm. Math. Phys. 221 (2001) 477 and hep-th/0102039.

[10] M.A. Olshanetsky and A.Perelomov, Quantum integrable systems related to the Lie algebras, Phys. Rep. 94 (1983) 313.

[11] V.Caudrelier and N.Crampé, Integrable N-particle Hamiltonians with Yangian or Reflection Algebra Symmetry, J. Phys. A37 (2004) 6285 and math-ph/0310028.

[12] Al. B. Zamolodchikov, Al. B. Zamolodchikov, Relativistic factorized S-matrix in two dimensions having $O(N)$ isotropic symmetry, Nucl. Phys. B133 (1978) 525 and Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models, Ann. Phys. 120 (1979) 253.

[13] P.P. Kulish, E.K. Sklyanin, Solutions of the Yang–Baxter equation, Zap. Nauchn. Sem. LOMI, 95 (1980) 129 and J. Sov. Math. 19 (1982) 1596.

[14] A.P. Isaev, Quantum groups and Yang–Baxter equations, Phys. Part. Nucl. 26 (1995) 501.

[15] D. Arnaudon, J. Avan, N. Crampé, L. Frappat and É. Ragoucy, R-matrix presentation for (super)-Yangians $Y(g)$, J. Math. Phys. 44 (2003) 302 and math.QA/0111325.

[16] V.G. Drinfel’d, Hopf algebras and the quantum Yang–Baxter equation, Soviet. Math. Dokl. 32 (1985) 254, Quantum Groups, Proceedings Int. Cong. Math. Berkeley, California, USA (1986) 798, A new realization of Yangians and quantized affine algebras, Soviet. Math. Dokl. 36 (1988) 212.

[17] K. Takemura and D. Uglov, The orthogonal eigenbasis and norms of eigenvectors in the Spin Calogero-Sutherland Model, J. Phys. A30 (1997) 3685 and solv-int/9611006.

[18] F.D.M. Haldane, Exact Jastrow-Gutzwiller resonating-valence-bond ground state of the spin-1/2 antiferromagnetic Heisenberg chain with 1/r2 exchange, Phys. Rev. Lett. 60 (1988) 635.

[19] B.S. Shastry, Exact solution of an $S=1/2$ Heisenberg antiferromagnetic chain with long-ranged interactions, Phys. Rev. Lett. 60 (1988) 639.
[20] A.P. Polychronakos, *Lattice integrable systems of Haldane-Shastry type*, Phys. Rev. Lett. **70** (1993) 2329 and hep-th/9210109.

[21] H. Frahm, *Spectrum of a spin chain with inverse square exchange*, J. Phys **A26** (1993) L473 and cond-mat/9303050.