Closure Under Minors of Undirected Entanglement

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Abstract

Entanglement is a digraph complexity measure that origins in fixed-point theory. Its purpose is to count the nested depth of cycles in digraphs. In this paper we prove that the class of undirected graphs of entanglement at most \( k \), for arbitrary fixed \( k \in \mathbb{N} \), is closed under taking minors. Our proof relies on the game theoretic characterization of entanglement in terms of Robber and Cops games.

Key words. Entanglement, minors, Robber and Cops games.

1 Introduction

Entanglement is a complexity measure of finite directed graphs introduced in [5] as a tool to analyze the descriptive complexity of the propositional modal \( \mu \)-calculus [6]. This measure has shown its use in solving the variable hierarchy problem\(^1\) for the modal \( \mu \)-calculus [6] and for the lattice \( \mu \)-calculus [4]. Roughly speaking, the entanglement of a \( \mu \)-formula (viewed as a graph) gives the minimum number of bound variables (i.e. fixed-point variables) required in any equivalent \( \mu \)-formula. From these considerations, the entanglement is considered as the combinatorial counterpart of the variable hierarchy.

Leaving the logical motivations in the background, recent works have been devoted to a graph theoretic study of entanglement [3, 10, 1], and in particular to characterizing the structure of graphs of entanglement at most \( k \). However, only partial results are known: the structure of directed graphs for \( k = 1 \) [5], \( k = 2 \) [10]; and of undirected graphs for \( k = 2 \) [3], and partially for \( k = 3 \) [1]. Furthermore, the exact complexity of deciding the entanglement of a graph is not yet known. By using general algorithms [9] it was argued in [2] that deciding whether a graph has entanglement at most \( k \), for fixed \( k \), is a problem in \( \text{PTIME} \). In particular, using the structural characterizations mentioned above,

\(^1\)This problem asks whether the expressive power of a given fixed-point logic increases with the number of bound variables.
this problem is in NLOGSPACE for directed graphs and \( k = 1 \) \cite{5}. This problem can be solved in linear time for the undirected graphs and \( k = 2 \) \cite{3} and in cubic time for directed graphs and \( k = 2 \) \cite{10}.

In this paper we prove a fundamental result of the undirected entanglement: the class of undirected graphs of bounded entanglement is closed under minors. Our working definition of the entanglement of a graph \( G \) is the minimum number of \( k \)-cops required to catch Robber in some games \( E(G, k) \) on \( G \) \cite{5}. Since the other definition \cite{5} in terms of a certain unfolding into trees with back edges can not be used in an easy way. Our proof technique to show that the entanglement of a (undirected) graph \( H \) is greater than the entanglement of its minor \( G \) is largely inspired by \cite{8, 7}: a move of Robber in the game \( E(G, k) \) is simulated by a move or a sequence of Robber’s moves in the game \( E(H, k) \) and in turn, Cops’ response in \( E(H, k) \) is mapped to \( E(G, k) \) in the desired way and soon. This sort of back-and-forth simulation reminds the back-and-forth games of \cite{7}.

Wagner’s conjecture (proved in a series of papers by Robertson and Seymour \cite{12}), states that for every infinite set of graphs, one of its members is a minor of an other. Thus, every class of graphs that is closed under taking minors can be characterized by a finite set of excluded minors. Since the class of undirected graphs of bounded entanglement is minor closed, Theorem 3.2, then it follows that this class can be characterized by a finite set of excluded minors. Therefore, testing weather an undirected graph has entanglement at most \( k \) can be checked in cubic time.

Finally we point out that only the set of excluded minors characterizing the graphs of entanglement \( \leq 2 \) is known \cite{3}, or see \cite{2, §7} for more details. In the case of entanglement, the number of excluded minors is relatively large because an excluded minor may contain articulation points. The main challenge consists in finding a compact representation of the excluded minors.

Preliminaries and notations

Throughout this paper, an undirected graph is called simply a graph, and a directed graph is called a digraph. A graph \( G \) is a minor of a graph \( H \) if \( G \) can be obtained from \( H \) by successive application of the following operations on it: (i) delete an edge, (ii) contract an edge, (iii) delete an isolated vertex.

Given a graph \( G \) and an edge \( e \), edge deletion results in a graph \( G \setminus e \) with the same vertex set as \( G \) and the edge set \( E_G \setminus \{e\} \); edge contraction results in a graph \( \partial e \) with the vertex set obtained by replacing the end-vertices of \( e \) in \( G \) by a new vertex \( z \), the latter inherits all the neighbors of the two replaced vertices. We shall write \( N(v) \) for the neighbors of vertex \( v \). We denote by \( G \setminus v \) the vertex deletion.

A class \( C \) of graphs is closed under minors if \( G \in C \) then for every minor \( H \) of \( G \) we have that \( H \in C \).
2 Entanglement

The entanglement of a finite digraph $G$, denoted $E(G)$, was defined in [5] by means of some games $E(G, k)$, $k = 0, \ldots, |V_G|$. The game $E(G, k)$ is played on the graph $G$ by Robber against Cops, a team of $k$ cops. The rules are as follows. Initially all the cops are placed outside the graph, Robber selects and occupies an initial vertex of $G$. After Robber’s move, Cops may do nothing, may place a cop from outside the graph onto the vertex currently occupied by Robber, may move a cop already on the graph to the current vertex. In turn Robber must choose an edge outgoing from the current vertex whose target is not already occupied by some cop and move there. If no such edge exists, then Robber is caught and Cops win. Robber wins if he is never caught. It will be useful to formalize these notions.

Definition 2.1. The entanglement game $E(G, k)$ of a digraph $G$ is defined by:

- Its positions are of the form $(v, C, P)$, where $v \in V_G$, $C \subseteq V_G$ and $|C| \leq k$, $P \in \{\text{Cops, Robber}\}$.
- Initially Robber chooses $v_0 \in V_G$ and moves to $(v_0, \emptyset, \text{Cops})$.
- Cops can move from $(v, C, \text{Cops})$ to $(v, C', \text{Robber})$ where $C'$ can be
  1. $C$: Cops skip,
  2. $C \cup \{v\}$: Cops add a new Cop on the current position,
  3. $(C \setminus \{x\}) \cup \{v\}$: Cops move a placed Cop to the current position.
- Robber can move from $(v, C, \text{Robber})$ to $(v', C, \text{Cops})$ if $(v, v') \in E_G$ and $v' \notin C$.

Every finite play is a win for Cops, and every infinite play is a win for Robber.

The entanglement of $G$, denoted by $E(G)$, is the minimum $k \in \{0, \ldots, |V_G|\}$ such that Cops have a winning strategy in $E(G, k)$.

The following Proposition provides a useful variant of entanglement games, see also [4].

Proposition 2.2. Let $\tilde{E}(G, k)$ be the game played as the game $E(G, k)$ apart that Cops are allowed to retire a number of cops placed on the graph. That is, Cops moves are of the form

- $(g, C, \text{Cops}) \rightarrow (g, C', \text{Robber})$ (generalized skip move),
- $(g, C, \text{Cops}) \rightarrow (g, C' \cup \{g\}, \text{Robber})$ (generalized replace move),

where in both cases $C' \subseteq C$. Then Cops have a winning strategy in $E(G, k)$ if and only if they have a winning strategy in $\tilde{E}(G, k)$.  

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3 Closure under minor of undirected entanglement

Lemma 3.1. If $G$ is a subgraph of $H$ then $\mathcal{E}(G) \leq \mathcal{E}(H)$.

Proof. Let $k = \mathcal{E}(G)$, then clearly, if Robber has a winning strategy in $\mathcal{E}(G, k)$ then he can use it to win in $\mathcal{E}(H, k)$ by restricting his moves on $G$. \qed

Theorem 3.2. The class of graphs of entanglement at most $k$, for arbitrary fixed $k \in \mathbb{N}$, is minor closed, that is if $G$ is a minor of $H$ then $\mathcal{E}(G) \leq \mathcal{E}(H)$.

Proof. If $G$ is obtained from $H$ by edge-deletion then the statement obviously holds by Lemma 3.1. Otherwise, if $G$ is obtained by edge-contraction i.e. $G = \partial_{ab} H$ for some $ab \in E_H$, then this allows to define a total function $f : V_H \rightarrow V_G$ as follows:

$$f(v) = \begin{cases} z & \text{if } v \in \{a, b\}, \\ v & \text{otherwise.} \end{cases}$$

Let $k = \mathcal{E}(H)$, using the function $f$ we shall construct a Cops’ winning strategy in the game $\tilde{\mathcal{E}}(G, k)$ out of a Cops’ winning strategy in $\mathcal{E}(H, k)$. To this goal, every position $(g, C_G, P)$ of $\tilde{\mathcal{E}}(G, k)$ is matched with the position $(h, C_H, P)$ of $\mathcal{E}(H, k)$, where $P \in \{\text{Robber, Cops}\}$, such that the following invariants hold:

1. $g = f(h)$ and $C_G = f(C_H)$, \quad \text{(COPS)}
2. if $g = z$ (hence $h \in \{a, b\}$) and $P = \text{Robber}$, then $z \in C_G$ and $h \in C_H$; moreover $|C_H \cap \{a, b\}| = 1$. \quad \text{(Robber-Z)}

The invariant (Robber-Z) may be understood as follows: whenever Robber will move from $z$ then $z$ must be occupied by a cop. At this moment, in $\mathcal{E}(H, k)$, either $a$ or $b$ must be occupied by a cop but not both.

We simulate every Robber’s move of the form

$$M_G = (v, C_G, \text{Robber}) \rightarrow (w, C_G, \text{Cops})$$

of $\tilde{\mathcal{E}}(G, k)$ either by a move or a sequence of moves in $\mathcal{E}(H, k)$ according to the locality of Robber’s move $M_G$:

1. If $M_G$ is outside $z$, i.e. $v, w \neq z$ then in this case $M_G$ is simulated by the same move in $\mathcal{E}(H, k)$.

2. If $M_G$ is entering to $z$, i.e. $w = z$ and $vw \in E_G$. Assume $v \in \mathcal{N}(a)^2$. In this case, the move $M_G$ is simulated by a finite alternation of Robber between $a$ and $b$ until Cops put a cop on $a$ or $b$, and then the simulation is halted. That is, the move $M_G$ is simulated by the finite alternating sequence $M_H^*$ of moves that is the following sequence apart the last move:

\footnote{The case $v \in \mathcal{N}(b) \setminus \mathcal{N}(a)$ is similar; recall that $\mathcal{N}(v)$ are just the neighbors of $v$.}
\[ M'_H = (v, C_H, Robber) \rightarrow (a, C_H, Cops) \rightarrow (a, C_H, Robber) \rightarrow (b, C_H, Cops) \rightarrow \ldots \rightarrow (x, C_H, Robber) \rightarrow (y, C_H, Cops) \rightarrow (y, C'_H, Robber) \]

Such that \( \{ x, y \} = \{ a, b \} \) and \( C'_H \neq C_H \). Clearly \( y \in C'_H \). Observe that this sequence is possible i.e. \( b \notin C_H \), because if \( b \in C_H \) then it follows by the invariant \((COPS)\) that \( f(b) = z \in f(C_H) = C_G \), that is \( z \in C_G \), which cannot happen because we have assumed that the move \( M_G \) is possible. The particular case of Robber’s first move to \( z \) is simulated by a similar finite alternating sequence of moves between \( a \) and \( b \), apart that \( \boxed{C_H = C_G = \emptyset} \).

3. If \( M_G \) is leaving \( z \), i.e. \( v = z \) and \( vw \in E_G \). Assume that the position \((z, C_G, Robber)\) is matched with \((a, C_H, Robber)\). Recall that \( z \in C_G \) and \( a \in C_H \), by the invariant \((Robber-Z)\).

(a) If \( w \in N(a) \) then the move \( M_G \) is simulated by the same move of \( E(H, k) \).

(b) If \( w \in N(b) \setminus N(a) \), then the move \( M_G \) is simulated by the following sequence of moves:

\[ (a, C_H, Robber) \rightarrow (b, C_H, Cops) \rightarrow (b, C'_H, Robber) \rightarrow (w, C'_H, Cops) \]

This sequence is possible, i.e. \( b \notin C_H \) because already \( a \in C_H \), therefore \( b \notin C_H \), by the invariant \((Robber-Z)\). At this point, the ending position of \( M_G \) – which is the position \((w, C'_H, Cops)\) – is matched with the position \((w, C'_H, Cops)\) of \( E(H, k) \), we emphasize that Cops’ next move \((w, C'_H, Cops) \rightarrow (w, C''_H, Robber)\) in \( E(H, k) \) should be mapped to the move

\[ (w, C'_G, Cops) \rightarrow (w, f(C''_H), Robber) \]

in \( \tilde{E}(G, k) \), and the main technical part is to prove that the latter move respects the rules of the game.

A Cops’ move in \( E(H, k) \) is mapped to a Cops’ move in \( \tilde{E}(G, k) \) as follows. Assume that the position \((g, C_G, Cops)\) of \( \tilde{E}(G, k) \) is matched with the position \((h, C_H, Cops)\) of \( E(H, k) \) and moreover Cops have moved to

\[ (h, C_H, Cops) \rightarrow (h, C'_H, Robber) \] (1)
Therefore Cops in \( \tilde{E}(G, k) \) should move to
\[
(g, C_G, \text{Cops}) \rightarrow (g, f(C'_H), \text{Robber})
\] (2)
the aim is to prove that the latter move is legal w.r.t the rules of the game \( \tilde{E}(G, k) \).

We distinguish three cases according to the manner by which \( g \) has been reached by Robber in \( \tilde{E}(G, k) \) in the previous round of simulation.

1. If \( g \) has been reached by an outside move, hence \( g \neq z, g = h \) (\( g \) is the vertex considered in the move (2), and \( h \) is considered in the move (1)), then in this case, \( C'_H \) may be written: \( C'_H = (C_H \setminus A) \cup B \), where \( \emptyset \subseteq B \subseteq \{ g \} \) and \( |A| \leq 1 \). (To be more precise we have \( |A| \leq |B| \).) Therefore
\[
f(C'_H) = [f(C_H \setminus A)] \cup f(B)
\]
\[
= \begin{cases} 
f(C_H) \cup f(B) & \text{if } a, b \in C_H \text{ and } A \subseteq \{ a, b \}, \\
(f(C_H) \setminus f(A)) \cup f(B) & \text{otherwise}
\end{cases}
\]
It is easy to see that this is a legal move.

2. If \( g \) has been reached by an entering move, hence \( g = z \) and \( h \in \{ a, b \} \) (again \( g \) is the vertex considered in the move (2), and \( h \) is considered in the move (1)), then in this case \( z \notin C_G \) and therefore \( a, b \notin C_H \). We shall argue that the move \( (z, C_G, \text{Cops}) \rightarrow (z, f(C'_H), \text{Robber}) \) respects the rules of the game. Assume that \( h = a \). In this case \( C'_H \) is of the form
\[
C'_H = (C_H \setminus A) \cup B
\]
where \( 0 \leq |A| \leq 1 \) with \( a, b \notin A \) and \( \emptyset \subseteq B \subseteq \{ a \} \), therefore
\[
f(C'_H) = f[(C_H \setminus A) \cup B]
\]
\[
= [f(C_H) \setminus f(A)] \cup f(B)
\]
Observe that \( z \notin f(A) \) and \( \emptyset \subseteq f(B) \subseteq \{ z \} \). Hence the move in question respects the rules of the game.

3. If \( g \) has been reached by a leaving move, hence \( h = g \) and \( hz \in E_G \), then in this case \( z \in C_G \) and either \( a \in C_H \) or \( b \in C_H \) but not both, by the invariant (Robber-Z). We distinguish two cases:

Case (i). If \( h \) has been reached by a single Robber’s move in \( \mathcal{E}(H, k) \) in the previous round of simulation, then one can check easily that every Cops’ move from position \( (h, C_H, \text{Cops}) \) in \( \mathcal{E}(H, k) \) is mapped to the same move from \( (h, C_G, \text{Cops}) \) in \( \tilde{E}(G, k) \).

Case (ii). If \( h \) has been reached by a sequence of moves in \( \mathcal{E}(H, k) \), then let us go back to the previous round of the simulation. The previous move in \( \tilde{E}(G, k) \) was indeed of the form
\[
(z, C_G, \text{Robber}) \rightarrow (h, C_G, \text{Cops})
\]
and its related simulation moves in $E(H, k)$ are of the form

$$(a, C_H^{-1}, \text{Robber}) \rightarrow (b, C_H^{-1}, \text{Cops}) \rightarrow (b, C_H, \text{Robber}) \rightarrow (h, C_H, \text{Cops})$$

In $E(H, k)$, if Cops move to $(h, C_H, \text{Cops}) \rightarrow (h, C'_H, \text{Robber})$ then this move is obviously mapped to Cops’ move $(h, C_G, \text{Cops}) \rightarrow (h, f(C'_H), \text{Robber})$ in $E(G, k)$. Note that $C'_H = (C_H^{-1} \setminus A) \cup B$ where $\emptyset \subseteq B \subseteq \{b, h\}$ and $A \subseteq V_H$ with $0 \leq |A| \leq 2$, let us compute $C'_G = f(C'_H)$ in terms of $C_G$:

$$f(C'_H) = [f(C_H^{-1} \setminus A)] \cup f(B) = [(f(C_H^{-1}) \setminus f(A)) \cup Z] \cup f(B)$$

where $\emptyset \subseteq Z \subseteq \{z\}$ and $\emptyset \subseteq f(B) = B' \subseteq \{z, h\}$, therefore

$$f(C'_H) = [f(C_H^{-1}) \setminus f(A)] \cup (Z \cup B') = (C_G \setminus f(A)) \cup B''$$

where still $\emptyset \subseteq B'' = Z \cup B' \subseteq \{z, h\}$. Recall that $z \in C_G$ by the invariant (Robber-Z) and hence the move in question respects the rules of the game.

Finally, the invariants (COPS) and (Robber-Z) are preserved by construction. This ends the proof of Theorem 3.2. □

A similar Proposition to the following one concerning the tree-width instead of the entanglement, has been proved in [11].

**Proposition 3.3.** If $G$ is a direct minor of $H$ then $E(H) - 1 \leq E(G)$

**Proof.** We need the following Claim.

**Claim 3.4.** To prove that $E(H) - 1 \leq E(G)$ it suffices to prove that $E(H \setminus v) \leq E(G)$, for some $v \in V_H$.

**Proof.** Assume that $E(H \setminus v) \leq E(G)$, and let $k = E(G)$. This implies that if Cops have a winning strategy in $E(G, k)$ then they have a winning strategy $S_1$ in $E(H \setminus v, k)$. Out of the winning strategy $S_1$ they can construct a winning strategy in $E(H, k+1)$ as follows: if Robber restricts his moves on $V_H \setminus v$ then play with $S_1$, and if Robber goes to $v$ then put the $(k+1)^{th}$ cop on $v$ and never move it. This ends the proof of the Claim. □

If $G$ is obtained from $H$ by deleting some edge $ab$, then observe that $H \setminus a$ is a subgraph of $G$, therefore from Lemma 3.1 we get $E(H \setminus a) \leq E(G)$. We conclude – according to the Claim – that $E(H) - 1 \leq E(G)$. If $G$ is obtained from $H$ by contracting some edge $ab$, then $H \setminus a$ is again a subgraph of $G$, and the argument is similar to the above one. □
The following Corollary provides a useful indication for searching the minimal set of excluded minors characterizing graphs of bounded entanglement.

**Corollary 3.5.** Let $F_k$ be the minimal excluded minors for the class of graphs of entanglement at most $k$. Then, every graph in $F_k$ has exactly entanglement $k + 1$.

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