Essential Self-Adjointness of Klein-Gordon Type Operators on Asymptotically Static, Cauchy-Compact Spacetimes

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Abstract: Let \(X = \mathbb{R} \times M\) be the spacetime, where \(M\) is a closed manifold equipped with a Riemannian metric \(q_0\), and we consider a symmetric Klein-Gordon type operator \(\Box_g\) on \(X\), which asymptotically converges to \(\partial_t^2 - \Delta_{q_0}\) as \(|t| \to \infty\), where \(\Delta_{q_0}\) is the Laplace-Beltrami operator on \(M\). We prove the essential self-adjointness of \(\Box_g\) on \(C_0^\infty(X)\). The idea of the proof is partly related to a recent paper by the authors on the essential self-adjointness for Klein-Gordon operators on asymptotically flat spaces.

1. Introduction

Let \(M\) be an \(n\)-dimensional closed Riemannian manifold with a metric \(q_0\), and let \(X = \mathbb{R} \times M\) be our spacetime. We denote \(q_0(x) = \sum_{i,j=1}^n q_{ij}^0(x)dx_idx_j\) and its dual by \(\sum_{i,j=1}^n q_{ij}^0(x)dx_idx_j\) locally in \(x \in M\). We write \(x_0 = t\). We suppose

**Assumption A.** Let \(g\) be a Lorentzian metric on \(X\). Moreover, the matrix elements \(g(t,x) = \sum_{i,j=0}^n g_{ij}(t,x)dx_idx_j\) satisfy

\[
|\partial_{i,x}^a(g_{ij}(t,x) - \delta_{ij})| \leq C_a(1 + |t|)^{-1-\mu}, \quad i = 0, \ldots, n
\]

\[
|\partial_{i,x}^a(g_{ij}(t,x) - q_{0,ij}(x))| \leq C_a(1 + |t|)^{-1-\mu}, \quad i, j = 1, \ldots, n
\]

for each multi-index \(a\) and each local coordinate of the compact part \(M\).

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In particular, this assumption implies \( g = dt^2 - q_0 + O((1 + |t|)^{-1-\mu}) \). We note that if the coefficient of \( dx_0 dx_i \) vanishes for all \( i = 1, \ldots, n \), then \( g \) is globally hyperbolic. Let us denote the Laplace-Beltrami operator by \( \Box_g \), which is locally defined by
\[
\Box_g = \sqrt{G}^{-1} \sum_{j,k=0}^n \partial_x^j (\sqrt{G} g^{jk} \partial_x^k),
\]
where we write \( G(t,x) = |\det g(t,x)| \) locally. Moreover, we assume the following:

**Assumption B.** Let \( (t(s), x(s)) \) be a null geodesic of the Lorentzian metric \( g \), that is, \( g(t'(s), x'(s)) = 0 \). If \( (t(s), x(s)) \) is not constant in \( s \), then either \( t(s) \to \pm \infty \) as \( s \to \pm \infty \) or \( t(s) \to \mp \infty \) as \( s \to \pm \infty \).

**Remark 1.1.** For the relationship between null non-trapping condition and globally hyperbolicity, see [1, Proposition 3.13] and [8, Proposition 4.3].

Now we define a \( L^2 \)-space \( L^2(X, d\mu) \) on \( X \) induced from the metric \( g \) by \( d\mu = \sqrt{G(t,x)} dt dx \) locally. Then \( \Box_g \) is symmetric on \( C^\infty_0(X) \) with respect to the inner product of \( L^2(X, d\mu) \).

**Theorem 1.1.** Suppose Assumptions A and B. Then \( \Box_g \) is essentially self-adjoint on \( C^\infty_0(X) \) as a linear operator on \( L^2(X, d\mu) \).

In the proof, we will replace \( L^2(X, d\mu) \) by a \( L^2 \)-space with a time independent density, which is introduced in Sect. 2.2. This makes our proof much simpler. Moreover, we deal with more general second order differential operators on \( X \), see Sect. 2.4.

**Remark 1.2.** Sometimes a more general model is called asymptotically static, and our model may be called asymptotically ultra-static, see, e.g., [7]. The essential self-adjointness for the more general model is reduced to our result, and we discuss it in Appendix B.

One motivation to show the essential self-adjointness for Klein-Gordon operators on spacetimes is the construction of Feynman propagators on curved spacetime, which is crucial in the construction of quantum field theory. The Feynman propagator is formally constructed as a boundary value of the resolvent for the Klein-Gordon operator on the spacetime, but the self-adjointness of these Klein-Gordon operators had not been proved except for the stationary cases. See papers by Dereziński-Siemssen [3–5] and Gérard-Wrochna [8,9] about the background. The essential self-adjointness of Klein-Gordon operators is related to the uniqueness of the quantum propagation, and it was conjectured in these papers. Since then, it was proved for asymptotically flat spacetime by Vasy [15] and Nakamura-Taira [12]. Recently the authors obtained a simplified proof of this result [13], where the construction of the escaping functions (classically increasing observables) and the analytic argument to show regularities of solutions are significantly simplified. In this paper we prove the essential self-adjointness for asymptotically static spacetimes, which has been also conjectured and open so far. In the proof, we partly employ the idea of [13], but the geometry of the spacetime is significantly different, and hence the constructions of the escaping functions are rather different. In [13], the asymptotic flatness of the spacetime is the origin of the dispersive properties necessary to construct the escaping function, whereas in our model there is no such dispersive properties. Thus we use the fact that each null geodesics goes to \( \pm \infty \) (Assumption B) to construct the escaping functions. We note that in our model we need the short-range type decay assumptions on the perturbation, whereas in [13] we only need long-range type assumptions, because of the different geometries. The phase space decomposition to show the regularities of solutions is also slightly more complicated than in [13]
Lemma 3.2). For other motivations and related topics in scattering theory, we refer to these papers [12, 13, 15] and references therein.

We prove our main theorem in Sect. 3. After introducing the basic notations, we reduce the problem to regularity problem of complex eigenfunctions of $P$, and then explain that we only need to look at small areas in the phase space (Lemma 3.2). In Sect. 3.4, we construct the incoming escaping function. In Sect. 3.5, we show the microlocal regularities of eigenfunctions in incoming area, and the argument here is analogous to the proof in [13]. We use semiclassical pseudodifferential operator calculus to obtain non-semiclassical regularities, since it simplifies the proof which employs the classical mechanics. Then we show the local smoothness of the eigenfunctions using the propagation of singularity theorem. The proof of regularities in the outgoing area is analogous to the incoming area, and the necessary modifications are explained in Sect. 3.7. Combining them, we conclude the proof of Theorem 1.1.

A technical lemma is proved in Appendix A. Proof of the essential self-adjointness for a more general class of models is discussed in Appendix B.

2. Preliminary

2.1. Coordinate systems. In this paper, we fix an atlas of $M$ and $X$: Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be a finite open covering of $M$, $\{Z_\lambda\}_{\lambda \in \Lambda} \subset \mathbb{R}^n$ be open sets which are diffeomorphic to open balls and let $\varphi_\lambda : W_\lambda \to Z_\lambda$ be a diffeomorphism. In addition, we consider an atlas of $X$ as $U_\lambda := \mathbb{R} \times W_\lambda$, $V_\lambda := \mathbb{R} \times Z_\lambda$, and $\iota_\lambda := \text{id}_\mathbb{R} \times \varphi_\lambda$. Take a quadratic partition of unity $\{\psi_\lambda\}_{\lambda \in \Lambda} \subset C^\infty(M;[0, 1])$, that is, $\text{supp}\psi_\lambda \subset W_\lambda$ and $\sum_{\lambda \in \Lambda} \psi_\lambda^2 = 1$. We also regard $\psi_\lambda$ as a partition of unity on $X$: $\psi_\lambda(t, x) := \psi_\lambda(x)$. We set $G_{0, \lambda}(x) = | \det q_0(x) |$ and $G_\lambda(t, x) = | \det g(t, x) |$ for $(t, x) \in V_\lambda$, where we recall that $q_0$ is the Riemannian metric on $M$.

2.2. Change of the density. We recall that $d\mu$ is the density induced from the Lorentzian metric $g$. Let $d\mu_0$ be a density induced from the the Lorentzian metric $dt^2 - q_0$. Then $d\mu_0 = \sqrt{G_{0, \lambda}(x)} dt dx$ locally. We define

$$U(t, x) := G_\lambda(t, x)^{1/2} / G_{0, \lambda}(x)^{1/2} \quad \text{for} \quad (t, p) \in V_\lambda,$$

which is a well-defined smooth function on $X$. In the following, we consider the $L^2$-space with the density $d\mu_0$:

$$L^2(X) := L^2(X, d\mu_0).$$

We note that $L^2(X) = L^2(X, d\mu)$ as sets and corresponding inner products are not the same but equivalent by Assumption A.

The multiplication operator $U$ is a unitary operator from $L^2(X, d\mu)$ to $L^2(X)$. Since $U$ preserves $C^\infty_0(X)$, essential self-adjointness of $\Box_g$ in $L^2(X, d\mu)$ is equivalent to that of

$$P_1 := -U \Box_g U^{-1} \quad \text{(2.1)}$$

in $L^2(X)$. By Assumption A, we have $| \partial^{\alpha}_t \delta_{t, \lambda}(U(t, x) - 1) | \leq C_\alpha |t|^{-1-\mu}$ for all multi-indices $\alpha$ and each coordinate $U_\lambda$. Hence $U[\Box_g, U^{-1}]$ is a first order differential order with coefficients decaying as $|t|^{-1-\mu}$. Moreover, it turns out that $-\Box_g - (D_t^2 + \triangle_{q_0})$ is
a second order differential order with coefficients decaying as \((t)^{-1-\mu}\). Therefore, we may write

\[
P_1 = D_t^2 + \Delta q_0 + Q, \tag{2.2}
\]

where \(Q\) is a second order differential order with coefficients decaying of \(O((t)^{-1-\mu})\).

2.3. Pseudodifferential operators. Following [11, Sect. 3.2], we define Weyl quantization \(O\_p(a)\) of \(a \in C^\infty(T^* X)\) by

\[
O\_p(a) = \sum_{\lambda \in \Lambda} \psi_{\lambda} \kappa_\lambda^w G_{0,\lambda}^{-\frac{1}{2}} a_\lambda^w (t, x, h D_t, h D_x) G_{0,\lambda}^{\frac{1}{2}} \kappa_\lambda^{-1} \ast \psi_{\lambda},
\]

where \(a_\lambda^w\) denote the usual Weyl quantization on \(\mathbb{R}^{n+1}\), see [16, (4.1.1)]. Although Weyl quantization on pseudo-Riemannian manifolds is discussed extensively and elegantly in [2], here we use a simpler definition as above. We also denote \(O(p) = O_1(p)\), i.e., the non-semiclassical Weyl quantization. We note that \(G_{0,\lambda} \) and \(\psi_{\lambda}\) in the definition of \(O\_p\) is independent of the time variable \(t\) and that the Weyl quantization of a real function is symmetric in \(L^2(X)\) due to the factor \(G_{0,\lambda}(x)^{\frac{1}{2}}\), which is very convenient in the following argument, see [11, Sect. 3.2].

For \(m, k \in \mathbb{R}\) and \(a \in C^\infty(T^* X)\), we write \(a \in S^{m,k}\) if

\[
|\partial_{t,x,\tau,\xi}^\alpha a(t, x, \tau, \xi)| \leq C_\alpha (1 + |t|)^k (1 + |\tau| + |\xi|)^m
\]

for each multi-index \(\alpha\) and on each coordinate \(U_\lambda \times \mathbb{R}^{n+1}\) defined above. We also denote symbol classes on \(T^*(\mathbb{R} \times \mathbb{R}^n)\) satisfying this property by the same notation \(S^{m,k}\). For a pair of symbols \(a(t, x, \tau, \xi)\) and \(b(t, x, \tau, \xi)\), we denote the Poisson bracket \([a, b] = H_{ab}\) by

\[
[a, b] = \frac{\partial a}{\partial \tau} \frac{\partial b}{\partial t} + \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial \tau} - \frac{\partial a}{\partial t} \frac{\partial b}{\partial \tau} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial \tau} - \frac{\partial a}{\partial \tau} \frac{\partial b}{\partial \tau} = \frac{d}{ds} \exp(s H_{ab}) b|_{s=0}.
\]

**Definition 1.** We say that a linear continuous operator \(A : C^\infty_0(X) \rightarrow C^\infty_0(X)\) belongs to a space of pseudodifferential operators \(\Psi_h^{m,k}\) if the following two properties hold:

(i) for each \(\lambda \in \Lambda\), there is \(a_\lambda \in S^{m,k}\) defined on \(T^*(\mathbb{R} \times \mathbb{R}^n)\) such that

\[
\kappa_\lambda^p \varphi A \psi (\kappa_\lambda^{-1} \ast \psi) = ((\kappa_\lambda^{-1}) \ast \varphi) a_\lambda^w (t, x, h D_t, h D_x)(\kappa_\lambda^{-1} \ast \psi)
\]

for \(\varphi, \psi \in C^\infty_0(X)\) supported in \(U_\gamma\), where \(a_\lambda^w\) denotes the usual Weyl quantization on \(\mathbb{R}^{n+1}\);

(ii) for all \(\varphi, \psi \in C^\infty_0(X)\) satisfying \(\text{dist}(\text{supp}[\varphi], \text{supp}[\psi]) > 0\), we have

\[
\| \langle t \rangle^{-k} \varphi A \psi \|_{H^{m+N}(X) \rightarrow H^N(X)} = O(h^\infty)
\]

for all \(N \in \mathbb{R}\).

For the definition of pseudodifferential operators on compact manifolds, see [16, Sect. 14.2.2]. Then, we have the following:

**Lemma 2.1.** (1) For \(a \in S^{m,k}\), we have \(O_\_p(a) \in \Psi_h^{m,k}\).
2.4. Second order differential operators on X. We recall that we need to construct an escaping function more carefully, see Remark 3.1. On the other hand, the order \( m \) is not better than the principal term unless symbols are functions of \( t, \tau \). This is different from the symbol class we have used in [12,13] previously. Therefore, we have\( \text{supp}_1 \subset (\tau, \xi) \), \( \text{dist}(\text{supp}_{a_{1}}, \text{supp}_{a_{2}}) > 0. \)

Moreover, we have \( \text{Op}_h(a_{1})\text{Op}_h(a_{2}) \in h^N\Psi_{h}^{m_{1}+m_{2},k_{1}+k_{2}} \) for each \( N \in \mathbb{R} \) when \( \text{supp}_{1} \subset (\tau, \xi) \).

(3) (The sharp Gårding inequality) If \( a \in S^{0,k} \) is real-valued and \( a \geq 0 \), then

\[
(u, \text{Op}_h(a)u)_{L^2} \geq -Ch\|t\|^k u_{L^2(X)}^2 \quad u \in C^\infty_0(X).
\]

(4) If \( a \in C^\infty(T^*X) \) depends only on \( (t, \tau) \), then

\[
\text{Op}_h(a) = a^w(t, hD_t),
\]

where \( a^w(t, hD_t) \) denotes the Weyl quantization on \( \mathbb{R} \).

These properties are proved similarly to the cases of compact manifolds since in our coordinate systems, the transition maps of the noncompact part \( \mathbb{R} \) are just the identity map. We refer, e.g., Dimassi-Sjöstrand [6], or Zworski [16] for the semiclassical pseudodifferential operator calculus. We remark that in our calculus, the decay order of lower order terms is not better than the principal term unless symbols are functions of \( t, \tau \). This is different from the symbol class we have used in [12,13] previously. Therefore, we need to construct an escaping function more carefully, see Remark 3.1. On the other hand, the order \( m \) is not essential in this paper since almost all symbols we use are compactly supported in \( (\tau, \xi) \).

2.4. Second order differential operators on X. We recall that \( (q^{ij}_0(x)) \) is the inverse of the Riemannian metric \( q_0 \) on \( M \). We define

\[
q_0(x, \xi) := \sum_{j,k=1}^{n} q^{jk}_0(x) \xi_j \xi_k
\]

on each local coordinate \( (x, \xi) \in T^*Z_{\lambda} = Z_{\lambda} \times \mathbb{R}^n \). Then \( q_0 \) is a well-defined function on \( T^*M \). Set \( Q_0 = \text{Op}(q_0) \). Then, we write

\[
Q_0 = -\Delta_{q_0} + V, \tag{2.4}
\]

where \( V \in C^\infty(M) \) is a real-valued function. Since \( M \) is compact, it is well-known that \( Q_0 \) is essentially self-adjoint on \( C^\infty_0(X) \) in \( L^2(X) \), where \( L^2(X) \) is defined in Sect. 2.2. We denote the unique self-adjoint extension by the same symbol \( Q_0 \).

Now we set \( P := P_1 - V \). From (2.2) and (2.4), we may write

\[
P = D_t^2 - Q_0 + Q, \tag{2.5}
\]

where \( Q \) is defined in (2.2). Since \( q_0 \) is a quadratic form, we have \( h^2Q_0 = \text{Op}_h(q_0) \).

By Lemma 2.1, we have \( h^2D_t^2 = \text{Op}_h(\tau^2) \). Using the remark after (2.2), we have \( h^2Q = \text{Op}_h(q) \), where \( q = q_h \) is a polynomial of degree 2 with respect to \( (\tau, \xi) \). Moreover, \( q = q_h \) satisfies

\[
q_h \in S^{2-1-\mu} \tag{2.6}
\]
and is uniformly bounded in $S^{2,-1-\mu}$ with respect to $h \in (0, 1]$. As a consequence, we may write
\[
h^2 P = Op_h(\tau^2 - q_0 + q) = (hD_t)^2 - h^2 Q_0 + Op_h(q).
\] (2.7)

Finally, Let $p$ be a principal symbol of $P$ in the microlocal sense. Since principal symbols of $-\Box_g$, $P_1$ and $P$ are the same, we have
\[
p(t, x, \tau, \xi) = \sum_{j,k=0}^{n} g^{jk}(t, x)\xi_j\xi_k,
\]
where we set $\xi_0 = \tau$. Then our null non-trapping assumption (Assumption B) implies that each integral curve $(t(s), x(s), \tau(s), \xi(s))$ of $H_p$ with $p(t(s), x(s), \tau(s), \xi(s)) = 0$ satisfies either $t(s) \to \pm \infty$ as $s \to \pm \infty$ or $t(s) \to \mp \infty$ as $s \to \pm \infty$.

Remark 2.1. Although essential self-adjointness of $P_1$ is equivalent to that of $P$, the commutator arguments in the next section do not work for $P_1$. In fact, in Lemma 3.3, $i[B^*B, V]$ might not decay with respect to $t$, and hence this term cannot be regarded as a lower order term.

3. Proof of Theorem 1.1

3.1. Essential self-adjointness of $P$. To prove Theorem 1.1, it suffices to see that $P$ defined in (2.5) is essentially self-adjoint on $L^2(X)$. In fact, as is mentioned in Sect. 2.2, we only need to show essential self-adjointness of $P_1$ on $L^2(X)$. This is also equivalent to the essential self-adjointness of $P$ since $P = P_1 - V$ with a bounded real-valued smooth function $V$.

Properties which we use in the following proof are (2.5), (2.6), (2.7) and null non-trapping condition of the principal symbol $p$ only.

We denote $\chi_1(s) \in C^\infty(\mathbb{R}; \mathbb{R})$ such that $\chi_1(s) = 1$ for $s \leq -1$, supp[$\chi_1$] $\subset (-\infty, 0]$, and $\chi_1'(s) \leq 0$ for all $s \in \mathbb{R}$. We also denote $\chi_2 \in C_0^\infty(\mathbb{R}; \mathbb{R})$ such that $\chi_2(s) = 1$ for $s \in [-1, 1]$, supp[$\chi_2$] $\subset [-2, 2]$, and $s\chi_2'(s) \leq 0$ for all $s \in \mathbb{R}$.

3.2. The first reduction. By the basic criterion for the essential self-adjointness, it suffices to show $\text{Ker}(P^* - z_{\pm}) = \{0\}$ for $z_{\pm} \in \mathbb{C}$, $\pm \text{Im}z_{\pm} > 0$ [14, Theorem X.1]. We note $(P^* - z)\psi = 0$ is equivalent to $(P - z)\psi = 0$ in the distribution sense, since the domain of $P$ is $\mathcal{D} = C_0^\infty(X)$. Thus we prove that if $(P - z_{\pm})\psi = 0$ for $\psi \in L^2(X)$ in the distribution sense then $\psi = 0$. Here is a simple but useful condition to show $\psi = 0$.

We denote the Sobolev space of order $s$ on $X$ by $H^s(X)$.

Lemma 3.1. Let $\psi \in L^2(X)$ such that $(P - z)\psi = 0$ with $z \in \mathbb{C}$. If, moreover, $\langle t \rangle^{-1} \psi \in H^1(X)$, then $\psi = 0$.

The proof is a simple commutator computation, and we omit it. See, e.g., Appendix A of [13]. We also note that actually the condition $\langle t \rangle^{-1/2} \psi \in H^{1/2}(X)$ is sufficient, and this is used in [12,15], though our condition is sufficient for our purpose and the proof is slightly more elementary.

In the following, we will show that if $(P - z)\psi = 0$ with $z \in \mathbb{C}$, $\text{Im}z > 0$ then $\langle t \rangle^\gamma \psi \in H^N(X)$ with any $\gamma, N > 0$. The case $\text{Im}z < 0$ is similar, and we mostly concentrate on the case $\text{Im}z > 0$. Moreover, we may assume $0 < \mu \leq 1$ without loss of generalities.
3.3. Remarks on microlocal regularities. We use the well-known semiclassical characterization of the microlocal regularities. Let \( \psi \in \mathcal{D}'(\mathbb{R}^n) \) and \((x_0, \xi_0) \in \mathbb{R}^{2n} \) with \( \xi_0 \neq 0 \). Then \((x_0, \xi_0) \notin \text{WF}(\psi)\) if there is \( a \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}) \) such that \( a(x_0, \xi_0) \neq 0 \) and
\[
\|\text{Op}_h(a)\psi\|_{L^2} = \|a(x, hD_x)\psi\|_{L^2} = O(h^{\infty}), \quad \text{as } h \to 0,
\]
where \(\text{WF}(\psi)\) is the wave front set of \(\psi\). It is also easy to show, under the same notation, \(\psi\) is microlocally \(H^s\) near \((x_0, \xi_0)\) if \(\|\text{Op}_h(a)\psi\|_{L^2} = O(h^{s'})\) as \(h \to 0\) with some \(s' > s\). This characterization works globally also. For example, suppose \(a \in \mathcal{S}^0_{1,0}\) and \(a(x, \xi) \geq c_0 > 0\) if \(|\xi| - 1| < \delta\) with \(\delta > 0\), and if \(\|\text{Op}_h(a)\psi\|_{L^2} = O(h^{s'})\) as \(h \to 0\), then \(\psi \in H^s\) for \(s < s'\). In the following, we use the semiclassical analysis to study regularities of complex eigenfunctions of the non-semiclassical operator \(P\). Thus the semiclassical parameter \(h\) sometimes appears in slightly nonstandard form, especially in the commutator formulas, e.g., in Lemma 3.3.

We now consider our spacetime model. We note that away from the characteristic set:
\[
\text{Char}(P) = \{(t, x, \tau, \xi) \mid p(t, x, \tau, \xi) = 0\},
\]
the operator \(P\) is elliptic, and hence \((P - z)\psi = 0\) and \((t_0, x_0, \tau_0, \xi_0) \notin \text{Char}(P)\) imply \((t_0, x_0, \tau_0, \xi_0) \notin \text{WF}(\psi)\). We note moreover that, on the conic set:
\[
\left\{(t, x, \tau, \xi) \mid |p(t, x, \tau, \xi)| > \delta (\tau^2 + q_0(x, \xi))\right\}, \quad \delta > 0,
\]
the operator \(P\) is uniformly elliptic, and hence \(\psi \in H^{\infty}\) there with a suitable microlocal cut-off. Thus, in order to show the smoothness of \(\psi\), it suffices to study microlocal smoothness in a neighborhood of \(\text{Char}(P)\).

We first consider the area \(|t| \gg 0\). As \(|t| \to \infty\), by Assumption A, \(p(x, \xi) \sim \tau^2 - q_0(x, \xi)\), and hence \(\tau^2 \sim q_0(x, \xi)\) on \(\text{Char}(P)\). On the other hand, when we study the regularities of \(\psi\) in a neighborhood of \((t, x, \tau, \xi)\) using the semiclassical method, we consider the area \(\tau^2 + q_0(x, \xi) \sim c_0 h^{-2}\) with some \(c_0 > 0\), e.g., \(c_0 = 2\). Now, if \(\tau^2 \sim q_0(x, \xi)\) and \(\tau^2 + q_0(x, \xi) \sim 2 h^{-2}\), then we conclude \(\tau^2 \sim h^{-2}\) and \(q_0(x, \xi) \sim h^{-2}\). This informal argument suggests that it suffices to study the behavior of \(\|\text{Op}_h(a(t, x, \tau, \xi))\psi\|_{L^2}\) as \(h \to 0\) in order to obtain regularities for \(|t| \gg 0\), where \(a\) is supported in a neighborhood of \(\{(t, x, \tau, \xi) \mid \tau^2 = 1, q_0(x, \xi) = 1\}\). We actually use the following lemma:

**Lemma 3.2.** Let \(\delta > 0\), and suppose \(\psi \in L^2\) with \((P - z)\psi = 0\). Then there is \(T_0 > 0\) such that the following holds: Let
\[
a(\delta, T; t, x, \tau, \xi) = \chi_2(|\tau^2 - 1|/\delta) \chi_2(|q_0(x, \xi) - 1|/\delta) \chi_1(1 - |t|/T).
\]
If \(\|\text{Op}_h(a(\delta, T))\psi\|_{L^2} = O(h^{s'})\) as \(h \to 0\) with \(T > T_0\), then \(\psi \in H^{s}((\mathbb{R} \setminus [-T - 1, T + 1]) \times M)\) for \(s < s'\).

Intuitively, the claim is straightforward from the above observation. The proof is elementary but slightly involved, and it is given in Appendix A.
3.4. Incoming observable. In this subsection, we construct an operator $B$ which is microlocally supported in the incoming region, and monotone decreasing (non-increasing) along the null geodesics in the support.

For $T \geq T_0$ in Lemma 3.2, we set

$$\zeta_1^\pm(t) = \chi_1((\mp t/T) + 1)$$

so that $\zeta_1^\pm(t) = 1$ if $\mp t \geq 2T$ and $\zeta_1^\pm(t) = 0$ if $\mp t \leq T$.

For $0 < \delta \ll 1$, we introduce a smooth positive function $\lambda(t)$ such that

$$\lambda(t) = 2\delta - \delta|t|^{-\nu},$$

for $|t| \geq 1$, where $0 < \nu < \mu$. Note $\lambda(t) \geq \delta$ and $\lambda(t) \to 2\delta$ as $|t| \to \infty$, monotonically in $(-\infty, -1]$, and in $[1, \infty)$, respectively. We then set

$$\zeta_2^\pm(t, \tau) = \chi_2((\pm \tau - 1)/\lambda(t)).$$

We now set $b_1^\pm(t, \tau) := |t|^\gamma \zeta_1^\pm(t) \zeta_2^\pm(t, \tau), b_2 = \chi_2(\lambda(t)^{-1}(q_0(x, \xi) - 1))$ and

$$B_1^\pm = \text{Op}_h(b_1^\pm), B_2 = \chi_2(\lambda(t)^{-1}(h^2 Q_0 - 1)), B = B(\delta, T) = (B_1^+ + B_1^-) B_2.$$

We note that $B_1^\pm$ and $B_2^\pm$ are microlocally supported in

$$\{(t, \tau) \mid \mp t \geq T, |\pm \tau - 1| \leq 2\lambda(t)\} \text{ and } \{(t, x, \tau, \xi) \mid |q_0(x, \xi) - 1| \leq 2\lambda(t)\},$$

respectively. Since $\delta \leq \lambda(t) \leq 2\delta$ with our small constant $\delta > 0$, $B_1^\pm$ and $B_2^\pm$ are microlocal cut-off’s to small neighborhoods of $\{\mp t \gg 0, |\pm \tau - 1| \leq 1\}$ and $\{q_0(x, \xi) = 1\}$, respectively. From the proof of [16, Theorem 14.8], we have $B_2 - \text{Op}_h(b_2) \in h\Psi^0_1$. Moreover, there is $b_3 \in C^\infty(T^* M)$ which is supported in $\text{supp} b_2$ such that $B_2 - \text{Op}_h(b_3)$ has a smooth kernel decaying of $O(h^\infty)$.

Remark 3.1. Crucial property of $B$ with this definition is that $B$ commutes with $Q_0$. This property does not hold if we define $B$ by straightforward quantization of the principal symbol

$$b_0(t, x, \tau, \xi) = \sum_{\pm} |t|^\gamma \zeta_1^\pm \zeta_2^\pm \zeta_3 = \sum_{\pm} b_1^\pm b_2.$$

Lemma 3.3. Let $B = B^\pm(\delta, T)$ as above. Then $[B^* B, i h^2 P] \in h\Psi^0_{1,2\gamma -1}$ and

$$[B^* B, i h^2 P] - h\text{Op}_h((b_0^2, \tau^2 + q)) \in h^2\Psi^0_{1,2\gamma -1}. \tag{3.1}$$

Moreover,

$$-\{\tau^2 + q, b_0^2\} \geq c_0|t|^{-1} b_0^2, \tag{3.2}$$

with some constant $c_0 > 0$ if $T > 0$ is sufficiently large.

Remark 3.2. The mapping property $[B^* B, i h^2 P] \in h\Psi^0_{1,2\gamma -1}$ is not a direct consequence of the standard pseudodifferential calculus. In fact, Lemma 2.1 (2) only implies $[B^* B, i h^2 P] \in h\Psi^0_{2\gamma}$. So we need to calculate this operator more carefully. On the other hand, in the following calculation, the order $m$ in $\Psi^{m,k}$ can be taken freely since $b_0$ has a compact support in the fiber of $T^* X$. 
We note \((B_1^\pm)^* = B_1^\pm, B_2^\mp = B_2\). By Lemma 2.1 (4) and the support property of \(b_1^\pm\), we have \(B_2 B_1^\pm B_1^\mp B_2 \in h^N \Psi_{h}^{-N,-N}\) for each \(N \in \mathbb{R}\). Using this property and \([B^* B, Q_0] = 0\), we have
\[
 i[B^* B, h^2 P] = i[B_2 B_1^\pm B_2, (h D_t)^2 + \text{Op}_h(q)] \notag \\
 = \sum_\pm B_2[(B_1^\pm)^2, i((h D_t)^2 + \text{Op}_h(q))]B_2 \notag \\
 + 2\text{Re}[B_2, i((h D_t)^2 + \text{Op}_h(q))] (B_1^\pm)^2 B_2 \notag
\]
up to errors of \(h^N \Psi_h^{-N,-N}\). In the following, we deal with the + case only. For simplicity, we write \(b_1 = b_1^+\) and \(B_1 = B_1^+\).

We first compute \(i B_2[B_1^\pm, (h D_t)^2 + \text{Op}_h(q)]B_2\). By a direct calculation, we have \((b_1^\pm, \tau^2) \in S^{0,2\gamma-1}\). Lemma 2.1 (4) implies \((h D_t)^2 = \text{Op}_h(\tau^2)\) and \([B_1^\pm, i \text{Op}_h(q)] = -h \text{Op}_h([b_1^\pm, \tau^2]) \in h^2 \Psi_h^{0,2\gamma-2}\). By Lemma 2.1 (2), we have
\[
 B_2[B_1^\pm, i(h D_t)^2]B_2 \in h^2 \Psi_h^{0,2\gamma-1}, \tag{3.3}
\]
\[
 B_2[B_1^\pm, i(h D_t)^2]B_2 - h B_2 \text{Op}_h([b_1^\pm, \tau^2])B_2 \in h^2 \Psi_h^{0,2\gamma-1}. \tag{3.4}
\]
where the orders of regularity can be taken as 0 since \(\text{supp}[b_1] \cap \text{supp}[b_2]\) is compact in \((\tau, \xi)\)-variable. We use this technique in the following frequently. Moreover, it follows from \(q \in S^{2,-1-\mu}\) and Lemma 2.1 (2) that
\[
 B_2[B_1^\pm, i \text{Op}_h(q)]B_2 \in h^2 \Psi_h^{0,2\gamma-1-\mu}, \tag{3.5}
\]
\[
 B_2[B_1^\pm, i \text{Op}_h(q)]B_2 - \text{Op}_h(b_2(b_1^\pm, \tau^2))B_2 \in h^2 \Psi_h^{0,2\gamma-1-\mu}. \tag{3.6}
\]

Now we calculate the symbol \(b_2[b_1^\pm, \tau^2 + q]\) = \(-2b_1 b_2\{\tau^2 + q, b_1\}\). It is easy to see
\[
-\{\tau^2 + q, |t|^\gamma\} = -(2\tau + \partial_\tau q) \tau \text{sgn}(t)|t|^{\gamma-1} = 2\gamma |t|^{\gamma-1}(|t| + O(|t|^{-1-\mu}))
\]
on the support of \(b_0\), and this part gives us the positivity of the commutator. It remains to show the contribution from other terms are nonnegative.

As well as the above computation, we have
\[
-\{\tau^2 + q, \zeta_1^\mp(t)\} = \mp(2\tau + \partial_\tau q) T^{-1} \lambda^{-1}(\pm \text{sgn}(t)|t|^{\gamma-1} + 1) \notag \\
= 2T^{-1} |\lambda^{-1}(\pm \text{sgn}(t)|t|^{\gamma-1} + 1)|(|t| + O(|t|^{-1-\mu}))
\]
on the support of \(\zeta_1^\mp(t)\zeta_2^\pm(t, \tau)\), which is nonnegative. Then, using \(\lambda'(t) = \text{sgn}(t) \delta v \notag |t|^{-1-v}\), we compute
\[
-\{\tau^2 + q, \zeta_2^\pm(t, \tau)\} = -(2\tau + \partial_\tau q) \partial_\tau \zeta_2^\pm + \partial_\tau q \partial_\tau \zeta_2^\pm \notag \\
= (2\tau + \partial_\tau q) \frac{\pm \tau - 1}{\lambda(t)} \lambda'(t) \pm \partial_\tau q \lambda^{-1}(\pm \tau) \lambda'(t) \notag \\
= \lambda^{-1}(t) \lambda'(t) \frac{\pm \tau - 1}{\lambda(t)} ((2\tau + \partial_\tau q) \delta v |t|^{-1-v} \frac{\pm \tau - 1}{\lambda(t)} \pm \partial_\tau q(t, x, \xi)) \notag \\
= \lambda^{-1}(t) \lambda'(t) \frac{\pm \tau - 1}{\lambda(t)} |2 \delta v \notag |t|^{-1-v} |\lambda^{-1}(t)| + O((|t|^{-1-\mu}))
\]
on the support of \(b_0\), since \(s\mathcal{X}'_2(s) \leq 0\). Since \(v < \mu\) and \(|\tau| \sim 1\) on the support, we learn the right hand side is nonnegative on the support provided \(T\) is chosen sufficiently large. Combining these, we have

\[
-\{\tau^2 + q, b_1\} = -\{\tau^2 + q, |t|^\gamma \xi_1^\top \xi_2^{\pm}\} \geq 2\gamma |t|^{\gamma - 1} \xi_1^\top (t) \xi_2^{\pm} = 2\gamma |t|^{-1} b_1
\]

(3.7)
on the support of \(b_0\), provided \(T\) is chosen sufficiently large.

Next, we deal with the term \(iB_2B_1^2[B_2, (hD_t)^2 + \text{Op}_h(q)]\). We then compute

\[
i[B_2, (hD_t)^2] = -h(hD_t)[\partial_t, \mathcal{X}_2(h^2 Q_0 - 1)] - h[\partial_t, \mathcal{X}_2(h^2 Q_0 - 1)](hD_t)
\]

using the functional calculus. Since \(\lambda'(t) = O(|t|^{-1-v})\),

\[
[\partial_t, \mathcal{X}_2(h^2 Q_0 - 1)] = -\frac{\lambda'(t)}{\lambda(t)^2}(h^2 Q_0 - 1)\mathcal{X}_2(h^2 Q_0 - 1) \in \Psi_{h}^{0,-1-v}.
\]

Thus we have \(iB_2B_1^2[B_2, (hD_t)^2] \in \Psi_{h}^{0,2\gamma - 1-v}\) and

\[
iB_2B_1^2[B_2, (hD_t)^2] - h\text{Op}_h(b_1^2 \mathcal{B}_2 \mathcal{B}_4) \in h^2 \Psi_{h}^{0,2\gamma - 1-v}
\]

by Lemma 2.1, where we set

\[
\zeta_4(t, x, \tau, \xi) := \{b_2, \tau^2\} = 2\tau \lambda(t)^{-1} \lambda'(t) \lambda(t)^{-1}(q_0 - 1) \mathcal{X}_2'(\lambda(t)^{-1}(q_0 - 1)).
\]

Noting that on the support of \(b_1\), \(\lambda'(t) = \delta v |t|^{-1-v} \text{sgn}(t), \tau \text{sgn}(t) < 0\), and that \(s\mathcal{X}'_2(s) \leq 0\), we learn \(b_1^2 \mathcal{B}_2 \mathcal{B}_4 \geq 0\).

Since \(q \in S^{2,-1-\mu}\), we have \(iB_2B_1^2[B_2, \text{Op}_h(q)] \in h\Psi_{h}^{0,2\gamma - 1-\mu}\) and

\[
iB_2B_1^2[B_2, \text{Op}_h(q)] - h\text{Op}_h(b_1^2 \mathcal{B}_2 \mathcal{B}_4(b_2, q)) \in h^2 \Psi_{h}^{0,2\gamma - 1-\mu}
\]

by Lemma 2.1.

\[
\mathcal{B}_2 \mathcal{B}_1(b_2, q) = \frac{\partial q_0 \cdot \partial x q - \partial q_0 \cdot \partial \xi q}{\lambda(t)} \frac{\mathcal{X}_2'(\frac{q_0 - 1}{\lambda(t)})}{\lambda(t)^2} = \frac{\lambda'(t)(q_0 - 1)}{\lambda(t)^2} \mathcal{X}_2'(\frac{q_0 - 1}{\lambda(t)}) \partial \tau q
\]

\[
= \mathcal{X}_2'(\frac{q_0 - 1}{\lambda(t)}) \cdot O((t, \xi))^{3}|t|^{-1-\mu}.
\]

Consequently, we obtain \(iB_2B_1^2[B_2, (hD_t)^2 + \text{Op}_h(q)] \in h\Psi_{h}^{0,2\gamma - 1-v}\) and

\[
iB_2B_1^2[B_2, (hD_t)^2 + \text{Op}_h(q)] - h\text{Op}_h(b_1^2 \mathcal{B}_2 \mathcal{B}_4(b_2, \tau^2 + q)) \in h^2 \Psi_{h}^{0,2\gamma - 1-v}.
\]

Moreover, the symbol \(b_2 b_1^2 (b_2, \tau^2 + q)\) is equal to

\[
b_2 b_1^2 \left(2|\tau|^{\delta v} \frac{q_0 - 1}{\lambda(t)} |t|^{-1-v} + O((\xi, \tau)^3)|t|^{-1-\mu}\right) \mathcal{X}_2'(\frac{q_0 - 1}{\lambda(t)})
\]

(3.8)
and it is nonnegative provided \(T\) is chosen sufficiently large. Combining them with (3.3), (3.4), (3.5) and (3.6), we conclude \([B^* B, i\hbar^2 P] \in h\Psi_{h}^{0,2\gamma - 1}\) and (3.1). The estimate of the symbol (3.2) follows from (3.7) and (3.8). \(\square\)
3.5. Incoming regularity. The discussions in this subsection is analogous to [13] Subsection 3.2, and we show the $H^N$-regularity of $ψ ∈ \text{Ker}(P^* − z)$ in the incoming region. The proof is mostly self-contained, but we refer to [13] Subsection 3.2 for technical details.

Let $δ < \tilde{δ}, \tilde{T} < T$, and set $B = B(δ, T)$ and $\tilde{B} = B(\delta, \tilde{T})$. Then by the sharp Gårding inequality and Lemma 3.3, we learn there are $c, c’ > 0$ such that

$$i[B^*B, P] ≥ \frac{c}{h} B^*(t)^{-1}B − c’ \tilde{B}^* (t)^{-1} \tilde{B} − E^*E \quad (3.9)$$

where $\|E\| = O(h^∞)$ as $h → 0$. We note that the essential support of $\tilde{B}$ is slightly larger than that of $B$, and hence we can estimate the error term of the Gårding inequality by constant times $\tilde{B}^* (t)^{-1} \tilde{B}$ from below, up to errors of $O(h^∞)$.

We set

$$δ_0 < δ_1 < ··· < δ_∞ ≪ 1, \quad T_0 > T_1 > T_2 > ··· > T_∞ \gg 0$$

and let $B_j = B(δ_j, T_j)$. We apply the above observation to show that

$$i[B_j^*B_j, P] ≥ \frac{c_j}{h} B_j^*(t)^{-1}B_j − c’_j B_{j+1}^* (t)^{-1} B_{j+1} − E_j E_j$$

for each $j$ with some $c_j, c’_j > 0$ and $\|E_j\| = O(h^∞)$. Then, by commutator computations, we obtain

$$\frac{c_j}{2h} \| (t)^{-1/2} B_j ϕ \|^2 + 2(\text{Im } \psi) \| B_j ϕ \|^2 ≤ 2h \frac{c’_j}{c_j} \| (t)^{1/2} B_j (P − z) ϕ \|^2 + c’_j \| (t)^{-1/2} B_{j+1} ϕ \|^2 + \| E_j ϕ \|^2. \quad (3.10)$$

This (more or less standard) commutator computation is essentially algebraic, and at least formally straightforward, but we need to justify that this inequality holds for any $ϕ$ such that the right hand side is well-defined. This technical point is discussed in Appendix B of [13]. We note the constants in the above inequality are independent of $z$, and this fact plays a crucial role when it is applied to the scattering theory and Feynman propagators.

Lemma 3.4. Suppose Assumptions A and B, $(P − z)ψ = 0$ where $ψ ∈ L^2(X)$ and $\text{Im } z > 0$, and let $B_0$ as above. Then $\| B_0 ψ \|_{L^2} = O(h^N)$ with any $N$ as $h → 0$.

Proof. We use the standard bootstrap argument to show $\| B_0 ψ \| = O(h^N)$ with any $N$ as $h → 0$. At first we note, by (3.10),

$$\frac{c_j}{2h} \| (t)^{-1/2} B_j ψ \|^2 + 2(\text{Im } \psi) \| B_j ψ \|^2 ≤ c’_j \| (t)^{-1/2} B_{j+1} ψ \|^2 + \| E_j ψ \|^2 \quad (3.11)$$

for each $j$. Since $ψ ∈ L^2(X)$ and $\| E_j \| = O(h^∞)$, we have, in particular,

$$\| (t)^{-1/2} B_j ψ \|^2 ≤ h(2c’_j/c_j) \| (t)^{-1/2} B_{j+1} ψ \|^2 + C_N h^{2N} \| ψ \|^2.$$

We recall $(t)^{-1/2} B_j$ are bounded in $L^2(X)$ since $0 < γ < 1/2$. Thus, by setting $j = 2N$, we learn $\| (t)^{-1/2} B_{2N} ψ \| = O(\sqrt{h})$. Then, iterating this procedure, we obtain $\| (t)^{-1/2} B_{2j+1} ψ \| = O(h^{j/2})$ for each $j = 1, 2, \ldots, 2N$, and in particular, $\| (t)^{-1/2} B_1 ψ \| = O(h^N)$. We now apply (3.11) again with $j = 0$ to conclude $\| B_0 ψ \| = O(h^N)$, since $\text{Im } z > 0$. □
Now this implies that \( \psi \) is smooth in the incoming area. We set \( \xi_5^\pm(\tau) = \chi_1(1 \mp \tau) \) so that \[
\xi_5^\pm(\tau) = \begin{cases} 1 & \text{if } \pm \tau \geq 1, \\ 0 & \text{if } \pm \tau \leq 0. \end{cases}
\]

We then set \( \Pi^\pm = \text{Op}(\chi_1(1)(\xi_5^-)^2(\tau) + (\xi_5^+)^2(\tau)) \), with \( T = T_0 \) so that \( \Pi^- \) is a projection to microlocally incoming area (with \( |t| > T \)), and \( \Pi^+ \) is the projection to outgoing area. Here we use the usual (non semiclassical) quantization by \( \text{Op}(\cdot) \), i.e., the quantization with \( h = 1 \). Recalling the symbol of \( b_0 \), we learn that Lemma 3.4 implies the regularity of \( \Pi^- \psi \) by Lemma 3.2. More precisely, we have the following incoming regularity:

**Lemma 3.5.** Under the same setting as above, \( \langle t \rangle^N \Pi^- \psi \in H^N(X) \) with any \( N \). In particular, \( \Pi^- \psi \in C^\infty(X) \).

### 3.6. Overall smoothness.

Combining the result of the previous subsection with the propagation of singularities theorem and Assumption B, we learn \( \psi \) is smooth on \( X \). At first we need a simple lemma:

**Lemma 3.6.** Suppose Assumption A and let \( 0 < c_1 < 1 \). Then there is \( T_1 > 0 \) such that if \( p(t, x, \tau, \xi) = 0 \) and \( |t| \geq T_1 \), then \( |2\tau - \partial_t p(t, x, \tau, \xi)| \leq c_1|\tau| \). In particular, \( \tau \) has the same sign as \( \partial_t p(t, x, \tau, \xi) \) if \( |t| \geq T_1 \).

**Proof.** It suffices to show \( |2\tau - \partial_t p(t, x, \tau, \xi)| \leq c_1|\tau| \) if \( p(t, x, \tau, \xi) = 0 \). In fact, we then have

\[
\tau(2 - c_1|\tau|) \leq \partial_t p(t, x, \tau, \xi) \leq \tau(2 + c_1|\tau|/|t|).
\]

Since \( 2 \pm c_1|\tau|/|t| \geq 2 - c_1 > 0 \), this implies \( \tau \) has the same sign as \( \partial_t p(t, x, \tau, \xi) \).

Using the condition \( p(t, x, \tau, \xi) = 0 \), we have \( \tau^2 - q_0(x, \xi) = q(t, x, \tau, \xi) \), and hence, recalling the ellipticity of \( q_0(x, \xi) \), we learn

\[
q_0(x, \xi) = \tau^2 - |q(t, x, \tau, \xi)| \leq \tau^2 + C(t)^{-1-\mu}(\tau^2 + q_0(x, \xi)).
\]

This implies \( q_0(x, \xi) \leq 2\tau^2 \) if \( |t| \gg 0 \). Now we have

\[
|\partial_t q(t, x, \tau, \xi)| \leq C(t)^{-1-\mu}(|\tau| + |\xi|) \leq C'(t)^{-1-\mu}|\tau|
\]

and this implies \( |2\tau - \partial_t p(t, x, \tau, \xi)| \leq c_1|\tau| \), provided \( |t| \gg 0 \). \( \square \)

Let \( (t(s), x(s), \tau(s), \xi(s)) \) be a non-constant null geodesics, that is an integral curve of \( H_p \). We recall the velocity is given by \( (\dot{t}(s), \dot{x}(s)) = (\partial_t p, \partial_\xi p) \). By Assumption B, there are \( s_b \ll 0 \) such that either \( t(s_b) < -\max(T_0, T_1) \) and \( \dot{t}(s_b) > 0 \), or \( t(s_b) > \max(T_0, T_1) \) and \( \dot{t}(s_b) < 0 \). These imply, by Lemma 3.6, \( |t(s_b)| > T_0 \) and \( t(s_b)\tau(s_b) < 0 \), i.e., incoming in the sense of the previous subsection. Hence, by Lemma 3.5, \( (t(s_b), x(s_b), \tau(s_b), \xi(s_b)) \notin \text{WF}(\psi) \). Now we use the propagation of singularities theorem ([10] Theorem 23.2.9), and we learn that

\[
(t(s), x(s), \tau(s), \xi(s)) \notin \text{WF}(\psi)
\]

for all \( s \). Thus we have proved:

**Lemma 3.7.** Let \( \psi \in L^2(X) \) such that \( (P - z)\psi = 0 \) with \( \text{Im } z > 0 \), then \( \psi \in C^\infty(X) \).
3.7. Outgoing observable and the regularity. While we now have the overall smoothness of $\psi$, we actually need Sobolev estimates. We already have the Sobolev estimate for the incoming region, and it remains to show the Sobolev estimate for the outgoing region. We employ observables which are similar to those used in the incoming estimate, but somewhat different.

We redefine the positive smooth function $\lambda(t)$ as follows:

$$\lambda(t) = \delta + \delta |t|^{-\nu}$$

for $|t| > 1$, where $0 < \delta \ll 1$ and $0 < \nu < \mu$ as well as in the incoming case. We note $\delta \leq \lambda(t) \leq 2\delta$, and $\lambda(t) \to \delta$ as $|t| \to \infty$ monotonically in $t \in (1, \infty)$ and $(-\infty, -1)$, respectively. Then $\xi_1^\pm$ and $\xi_2^\pm$ are defined by the same expression but with the above $\lambda(t)$, i.e.,

$$\xi_1^\pm(t) = \chi_1((\mp t/T) + 1), \quad \xi_2^\pm(t, \tau) = \chi_2((\pm \tau - 1)/\lambda(t)).$$

We now set

$$B = B(\delta, T) = \sum_{\pm} \text{Op}_h (|t|^{-\nu} \xi_1^\pm \xi_2^\pm) \chi_2((\pm \tau - 1)/\lambda(t)),$$

where $h > 0$ is the small semiclassical parameter as before, and $T \gg 0$. We write the principal symbol by $b_0(t, x, \tau, \xi)$, i.e.,

$$b_0(t, x, \tau, \xi) = \sum_{\pm} |t|^{-\nu} \xi_1^\pm(t) \xi_2^\pm(t, \tau) b_2(t, x, \xi),$$

where $b_2(t, x, \xi) = \chi_2((q_0(x, \xi) - 1)/\lambda(t))$.

We have a symbol estimate for this operator $B$, analogously to Lemma 3.3, but slightly different.

**Lemma 3.8.** Let $B = B(\delta, T)$ as above. Then $[B^* B, ih^2 P] \in h^\Psi_{h}^{0, -2\gamma - 1}$. Moreover, there is $\tilde{f} \in S^{0, -1 - 2\gamma}$ supported in $\{(t, x, \tau, \xi) \mid T \leq t \leq 2T\} \cap \text{supp}[b_0]$ such that

$$[B^* B, ih^2 P] - h\text{Op}_h ([b_0^2, \tau^2 + q]) - h\text{Op}_h (\tilde{f}) \in h^\Psi_{h}^{0, -2\gamma - 1}$$

and

$$-(\tau^2 + q, b_0^2) \geq c_0 |t|^{-1} b_0^2 - \tilde{f},$$

with some constant $c_0 > 0$ if $T > 0$ is sufficiently large.

**Proof.** The proof is almost the same as that of Lemma 3.8, but there are several differences, and we sketch them here.

At first, the Poisson bracket $-(\tau^2 + q, \xi_1^\pm)$ is not necessarily nonnegative on supp $[\xi_1^\pm \xi_2^\pm]$. In fact, it is nonpositive, and hence we set

$$\tilde{f}(t, x, \xi) = \{\tau^2 + q, (\xi_1^\pm)^2\}|t|^{-2\gamma} (\xi_2^\pm)^2 b_2$$

to compensate it.

The other computations are almost the same as Lemma 3.3, except for the fact that $\lambda'(t) = -\text{sgn}(t) \delta \nu |t|^{-\nu - 1}$, i.e., the sign is opposite from the incoming case. However it is consistent since $t \tau > 0$ on supp $[\xi_1^\pm \xi_2^\pm]$, and almost the same computations are carried out with changes of several signs. We omit the detail.  

$\Box$
We then proceed to show regularity in the outgoing region. As well as in the incoming case, in particular as (3.9), we let $\delta < \delta, \tilde{T} < T$, and set $B = B(\delta, T)$ and $\tilde{B} = B(\tilde{\delta}, \tilde{T})$. Then by Lemma 3.8 and the sharp Gårding inequality, we learn there are $c_0, c' > 0$ such that

$$i[B^* B, P] \geq \frac{c_0}{h} B^*(t)^{-1} B - c'B^*(t)^{-1} \tilde{B} - E^* E - F$$

where $F = \text{Op}_h(\tilde{f})$ and $\|E\| = O(h^\infty)$ as $h \to 0$. As well as (3.10), this implies,

$$\frac{c_0}{2h} \|\langle t \rangle^{1/2} B\varphi\|^2 + 2(\text{Im}z) \|B\varphi\|^2 \leq \frac{2h}{c_0} \|\langle t \rangle^1 B(P - z)\varphi\|^2 + c' \|\langle t \rangle^{-1/2} \tilde{B}\varphi\|^2 + \|E\varphi\|^2 + \langle \varphi, F\varphi \rangle$$

for $\varphi \in C_0^\infty(X)$. This estimate is easily extended to $\psi \in L^2$ with $(P - z)\psi = 0$, and we learn

$$\frac{c_0}{2h} \|\langle t \rangle^{-1/2} B\psi\|^2 + 2(\text{Im}z) \|B\psi\|^2 \leq c' \|\langle t \rangle^{-1/2} \tilde{B}\psi\|^2 + \|E\psi\|^2 + \langle \psi, F\psi \rangle.$$ 

We observe $\|E\psi\| = O(h^N)$ with any $N$ by the construction of $E$. We now recall the support of the symbol of $F$ is contained in $\{T \leq t \leq 2T\} \cap \text{supp}[b_0]$, and in particular, compactly supported in $X$. By this and the global regularity, i.e., Lemma 3.7, we learn $\langle \psi, F\psi \rangle = O(h^N)$ with any $N$. Thus we can use the same iteration procedure as in the proof of Lemma 3.4 to conclude $\|B_0\psi\|_{L^2} = O(h^N)$ with any $N$ as $h \to 0$. Now we have the following outgoing regularity result as well as the incoming case, i.e., Lemma 3.5:

**Lemma 3.9.** Suppose $\psi \in L^2(X)$ and $(P - z)\psi = 0$ where $\text{Im}z > 0$. Then $\langle t \rangle^{-\gamma} \Pi^* \psi \in H^N(X)$ with any $N$, provided $T$ is sufficiently large.

3.8. **Proof of Theorem 1.1.** Let $\psi \in L^2(X)$ and $(P - z)\psi = 0$ with $\text{Im}z > 0$. Then, combining Lemmas 3.5, 3.7 and 3.9, we learn that $\langle t \rangle^{-\gamma} \psi \in H^N(X)$ with any $N$. Then, by Lemma 3.1, we conclude $\psi = 0$. Thus we have proved $\text{Ker}(P^* - z) = \{0\}$. By similar arguments, we can also show $\text{Ker}(P^* - z) = \{0\}$ when $\text{Im}z < 0$. These implies the essential self-adjointness of $P$ on $C_0^\infty(X)$ (see, e.g., [14, Theorem X.1]). \hfill \Box

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Appendix A: Proof of Lemma 3.2

Let \( \eta(t, x, \tau, \xi) = \chi_2((\tau^2 + q_0(x, \xi) - 2)/\gamma) \) with \( \gamma = \delta/4 \). It suffices to show 
\[
\|\text{Op}_h(\eta)\psi\|_{L^2(\tilde{X})} = O(h^{s'}) \quad \text{as} \quad h \to 0,
\]
where \( \tilde{X} = (\mathbb{R} \setminus [-T - 1, T + 1]) \times M \) with \( T > T_0 \). We choose \( T_0 \) so that 
\[
|q(t, x, \tau, \xi)| \leq \alpha(\tau^2 + q_0(x, \xi)) \quad \text{for} \quad |t| \geq T_0
\]
with \( \alpha = \delta/(4+\delta) \) (see Assumption A). By the assumption, we know 
\[
\|\text{Op}_h(\eta)\text{Op}_h(a(\delta, T))\psi\|_{L^2(\tilde{X})} = O(h^{s'}),
\]
and hence it remains to show 
\[
\|\text{Op}_h(\eta)\text{Op}_h(1-a(\delta, T))\psi\|_{L^2(\tilde{X})} = O(h^{s'}), \tag{A.1}
\]
where \( T \geq T_0 \). We note the symbol of \( \text{Op}_h(\eta)\text{Op}_h(1-a(\delta, T)) \) is essentially supported in the support of \( \eta(1-a(\delta, T)) \), and if \( P \) is elliptic on this support, the property (A.1) follows from the equation \((P-z)\psi = 0\) by the standard elliptic estimate (with any \( s' > 0 \)). Thus, in order to prove (A.1), it is sufficient to show 
\[
|p(t, x, \tau, \xi)| \geq \delta > 0 \quad \text{on} \quad \text{supp}[\eta(1-a(\delta, T))].
\]

By the construction, we have 
\[
|\tau^2 + q_0 - 2| \leq 2\gamma \quad \text{on} \quad \text{supp}[\eta]. \tag{A.2}
\]

We also have 
\[
|q| \leq \alpha(\tau^2 + q_0) \leq \alpha(2 + 2\gamma) \quad \text{on} \quad \text{supp}[\eta] \cap \{|t| \geq T_0\}. \tag{A.3}
\]

using the choice of \( T_0 \) and (A.2).

On the other hand, we have 
\[
|\tau^2 - 1| \geq \delta \quad \text{or} \quad |q_0 - 1| \geq \delta \quad \text{on} \quad \text{supp}[1-a] \cap \{|t| \geq T\}. \tag{A.4}
\]

If \( |\tau^2 - 1| \geq \delta \), then we have 
\[
|p| = |\tau^2 - q_0 + q| \geq |\tau^2 - q_0| - |q| \\
\geq |\tau^2 - (2 - \tau^2)| - 2\gamma - \alpha(2 + 2\gamma) \\
= 2|\tau^2 - 1| - (2\gamma + \alpha(2 + 2\gamma)) = 2|\tau^2 - 1| - \delta \geq \delta,
\]

where we have used (A.3) and (A.4) in the second inequality. We note \( 2\gamma + \alpha(2 + 2\gamma) = \delta \) by our choice of constants \( \alpha, \gamma \). Similarly, if \( |q_0 - 1| \geq \delta \), then we have 
\[
|p| = |\tau^2 - q_0 + q| \geq |\tau^2 - q_0| - |q| \\
\geq |(2 - q_0) - q_0| - 2\gamma - \alpha(2 + 2\gamma) \\
= 2|q_0 - 1| - (2\gamma + \alpha(2 + 2\gamma)) = 2|q_0 - 1| - \delta \geq \delta,
\]

using (A.3) and (A.4) again. These inequalities imply \( P \) is elliptic on \( \text{supp}[\eta(1-a(\delta, T))] \cap \{|t| \geq T\} \), and this completes the proof of Lemma 3.2. \( \square \)
Appendix B: More general models

We refer the metric \( g \) satisfying Assumption A as **asymptotically static**, but sometimes more general operators are called asymptotically static. For example, in [7], a Lorentzian metric \( g \) is called asymptotically static if

\[
g = c(t, x)^2 \tilde{g},
\]

where \( \tilde{g} \) satisfy Assumption A and \( c(t, x) \in C^\infty(X; \mathbb{R}_+) \) such that it satisfies, for any indices \( k \) and \( \alpha \),

\[
\left| \partial_t^k \partial_x^\alpha (c(t, x) - c_{\pm}(x)) \right| \leq C_{k,\alpha} (t)^{-1-\mu} \quad \text{for} \quad \pm t \geq 0.
\]

with some \( c_{\pm} \in C^\infty(M; \mathbb{R}_+) \). One can prove such operators are essentially self-adjoint under the same additional condition, Assumption B. The proof is reduced to the essential self-adjointness of our model using a conformal transform. We sketch the proof below.

We set the conformal transform of \( g \) by:

\[
\tilde{g} = c(t, x)^{-2} g.
\]

It is known that the modified Laplacian \( P_M = -\Box_g + \frac{n-1}{4n} R \) is conformal covariant, where \( R \) is the scaler curvature. Namely, if we write \( P_M \) and \( \tilde{P}_M \) be the modified Laplacians corresponding to \( g \) and \( \tilde{g} \), respectively, then

\[
\tilde{P}_M = c^{\frac{n+3}{2}} P_M c^{-\frac{n-1}{2}}.
\]

In particular, we learn \( \Box_{\tilde{g}} \) is a conformal transform of \( \Box_g \) up to a bounded smooth function, i.e.,

\[
\Box_{\tilde{g}} = c^{\frac{n+3}{2}} \Box_g c^{-\frac{n-1}{2}} + V \tag{B.1}
\]

where \( V \) is a smooth bounded function.

It is straightforward to show \( \tilde{g} \) satisfies Assumption A, and \( \tilde{g} \) also satisfies Assumption B since the null geodesics are invariant under conformal transforms. Hence, by Theorem 1.1, \( \tilde{P} \) is essentially self-adjoint on \( C^\infty_0(X) \).

Now in order to prove the essential self-adjointness of \( \Box_g \), it suffices to see that \( C^\infty_0(X) \) is dense in a Banach space \( D(P^*) := \{ u \in L^2(X) \mid \Box_g u \in L^2(X) \} \) equipped with the graph norm. This follows easily from (B.1) and the boundedness of \( c \) and \( c^{-1} \).

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