Gauged supergravity from type IIB string theory on $Y^{p,q}$ manifolds

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Abstract

We first construct a consistent Kaluza-Klein reduction ansatz for type IIB theory compactified on Sasaki-Einstein manifolds $Y^{p,q}$ with Freund-Rubin 5-form flux giving rise to minimal $\mathcal{N} = 2$ gauged supergravity in five dimensions. We then investigate the $R$-charged black hole solution in this gauged supergravity, and in particular study its thermodynamics. Based on the gauge theory/string theory correspondence, this non-extremal geometry is dual to finite temperature strongly coupled four-dimensional conformal gauge theory plasma with a $U(1)_R$-symmetry charge chemical potential. We study transport properties of the gauge theory plasma and show that the ratio of shear viscosity to entropy density in this plasma is universal. We further conjecture that the universality of shear viscosity of strongly coupled gauge theory plasma extends to nonzero $R$-charge chemical potential.

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1 Introduction

The gauge theory/string theory correspondence of Maldacena [1–4] provides a valuable insight into the nonperturbative dynamics of strongly coupled gauge theory plasma. However, on the string theory side, this duality is tractable only for weakly curved backgrounds and with small string coupling, i.e., in the supergravity approximation. Following the Maldacena correspondence, this implies that the dual four dimensional gauge theory must be strongly coupled at all energy scales. Unfortunately, this latter condition excludes a dual supergravity description of real QCD. Nonetheless, one expects that certain universal properties of strongly coupled gauge theory plasma derived in the planar limit and for large ’t Hooft coupling would persist for real QCD as well.

A noteworthy example of such a universal property is the ratio of shear viscosity to the entropy density. In [5–7] a theorem was proven that, in the absence of chemical potentials, a gauge theory plasma in the planar limit and for infinitely large ’t Hooft coupling in theories that admit a dual supergravity description has a universal ratio of shear viscosity $\eta$ to the entropy density $s$:

$$\frac{\eta}{s} = \frac{1}{4\pi}. \quad (1.1)$$

Furthermore, other transport properties of the strongly coupled non-conformal gauge theory plasma, like the speed of sound $c_s$ and the bulk viscosity $\zeta$, while not universal, have a certain generic behavior in the near-conformal limit [10–12]

$$\frac{\zeta}{\eta} = -\kappa \left( c_s^2 - \frac{1}{3} \right) + \mathcal{O}\left( \left( c_s^2 - \frac{1}{3} \right)^2 \right), \quad \kappa \sim 2 \ldots 5. \quad (1.2)$$

Introduction of a nonzero chemical potential in the strongly coupled gauge theory plasma violates the sufficient condition under which the universal result (1.1) has been derived [5, 7]. On the other hand, it was recently shown [13–15] that the universality (1.1) persists in $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma even with nonzero $R$-charge chemical potentials$^2$. Thus it appears that the shear viscosity relation (1.1) is more robust than originally anticipated in [5–7]. However, further exploration of shear viscosity with chemical potential in strongly coupled four-dimensional gauge theory plasma violates the sufficient condition under which the universal result (1.1) has been derived [5, 7]. On the other hand, it was recently shown [13–15] that the universality (1.1) persists in $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma even with nonzero $R$-charge chemical potentials$^2$. Thus it appears that the shear viscosity relation (1.1) is more robust than originally anticipated in [5–7]. However, further exploration of shear viscosity with chemical potential in strongly coupled four-dimensional gauge theory plasma violated the sufficient condition under which the universal result (1.1) has been derived [5, 7]. On the other hand, it was recently shown [13–15] that the universality (1.1) persists in $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma even with nonzero $R$-charge chemical potentials$^2$. Thus it appears that the shear viscosity relation (1.1) is more robust than originally anticipated in [5–7]. However, further exploration of shear viscosity with chemical potential in strongly coupled four-dimensional gauge theory

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$^1$ Finite ’t Hooft coupling corrections to shear viscosity of the $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory plasma were discussed in [8, 9].

$^2$ The shear viscosity of the M2-brane plasma also appears to satisfy the universal relation (1.1) [16].
plasma has been hampered by the fact that, until recently, few models exist where the dual supergravity solutions are easily accessible.

The most well understood system is of course $\mathcal{N} = 4$ Yang-Mills plasma, which is holographically dual to IIB on $\text{AdS}_5 \times S^5$. From a five-dimensional point of view, this dual is given by $\mathcal{N} = 8$ gauged supergravity, which may be obtained by reduction of the ten-dimensional theory on $S^5$. The $\mathcal{N} = 4$ Yang-Mills theory has an $SO(6)$ $R$-symmetry which may correspondingly be identified with the isometries of $S^5$. It is then possible to turn on up to three commuting $R$-charges under the maximum abelian subgroup $U(1)^3 \subset SO(6)$. At the same time, this sub-sector of the full gauged supergravity admits a consistent truncation to $\mathcal{N} = 2$ supergravity coupled to two abelian vectors, commonly referred to as the STU model. As a result, explicit investigations of the nonzero $R$-charge sector of thermal $\mathcal{N} = 4$ Yang-Mills can be easily performed in the context of the STU model. In particular, non-extremal $R$-charged black holes were constructed in [17], and their thermodynamics were investigated in [18–21]. In addition, these five-dimensional solutions can be uplifted to the full IIB theory and investigated from the ten-dimensional point of view [22, 23].

The uplifting of five-dimensional gauged supergravity solutions to ten dimensions is closely related to the existence of a full non-linear Kaluza-Klein reduction ansatz of the original IIB theory. As it turns out, a complete proof of the consistency of the reduction of IIB theory on $S^5$ has yet to be given. Nevertheless, it is generally assumed to be consistent based on the linearized construction as well as explicit reductions to subsets of the full $\mathcal{N} = 8$ theory in five dimensions. In particular, the reduction to the STU model is given in [22], while reduction to $\mathcal{N} = 4$ in five dimensions was given in [24], and reduction of the metric and self-dual 5-form on $S^5$ was given in [25].

Although $\mathcal{N} = 4$ Yang-Mills has proven to be a fruitful system to investigate, it would be highly desirable to explore models with reduced supersymmetry. In this context, the explicitly constructed Sasaki-Einstein metrics $Y^{p,q}$ [26, 27] have led to rapid new developments in understanding their holographically dual $\mathcal{N} = 1$ Yang-Mills theories. At the same time, however, much of this progress has been restricted to either zero temperature or zero chemical potential (or both). What we aim to do below is to make progress towards the study of $\mathcal{N} = 1$ Yang-Mills plasma at finite $R$-charge chemical potential. To do so, we first construct the consistent non-linear Kaluza-Klein reduction of IIB on $Y^{p,q}$ giving rise to gauged $\mathcal{N} = 2$ supergravity in five dimensions, and then turn to the thermodynamics and hydrodynamics of the corresponding $R$-
charged black holes.

In general, the Kaluza-Klein reduction of a higher-dimensional theory on an internal manifold \( Y \) generally involves both ‘massless’ lower-dimensional fields (related to zero modes on \( Y \)) and the Kaluza-Klein tower of massive states. This distinction, however, is not entirely clear, except perhaps in the case where \( Y \) is a torus. Consistency of the reduction to the massless sector then depends on the decoupling of the Kaluza-Klein tower, and this is a highly non-trivial condition. For \( T^n \) reductions, truncation to the zero modes is guaranteed to be consistent, as this is a truncation to the charge singlet sector of \( U(1)^n \). However, for spheres or more complicated manifolds \( Y \), it is often the case that the states that are considered ‘massless’ (such as states in the lower-dimensional gauged supergravity multiplet) carry non-trivial internal charge. These charged states then have the potential of acting as sources to the Kaluza-Klein tower, thus rendering the truncation inconsistent unless some appropriate symmetry or conspiracy among fields is in place.

As an example of this inconsistency, consider for example the compactification of IIB on \( T^{1,1} \). The resulting five-dimensional theory is \( \mathcal{N} = 2 \) gauged supergravity with a full Kaluza-Klein spectrum which was obtained in [28]. Because the isometry of \( T^{1,1} \) is \( SU(2) \times SU(2) \times U(1) \), the massless gauge bosons transform under the identical group. Since the \( U(1) \) gauge boson is clearly the \( \mathcal{N} = 2 \) graviphoton coupling to \( U(1)_R \), the massless sector may be described as \( \mathcal{N} = 2 \) gauged supergravity coupled to \( SU(2) \times SU(2) \) vector multiplets. It turns out, however, that it is inconsistent to retain these \( SU(2) \times SU(2) \) vector multiplets in any truncation to the massless sector [29]; the only consistent truncation is to pure \( \mathcal{N} = 2 \) gauged supergravity. This, however, is sufficient for our needs, as the pure supergravity sector is all that is needed for uplifting the \( R \)-charged configurations of interest.

The more general case of IIB reduced on \( Y^{p,q} \) is similar. For generic \( Y^{p,q} \), its isometry is \( SU(2) \times U(1) \times U(1) \), and the resulting Kaluza-Klein reduction yields \( \mathcal{N} = 2 \) gauged supergravity coupled to \( SU(2) \times U(1) \) vector multiples in the massless sector. Only the truncation to minimal \( \mathcal{N} = 2 \) gauged supergravity is consistent, and the resulting geometries are holographically dual to certain strongly coupled superconformal quiver gauge theories [30, 31]. Working at finite temperature and \( R \)-charge in these gauge theories corresponds to taking an \( R \)-charged black hole in the supergravity description. Such black holes carrying \( U(1)_R \) graviphoton charge have been discussed in [17], and simply correspond to STU black holes with three equal charges. Given
these black holes and the Kaluza-Klein reduction ansatz on $Y^{p,q}$, we may then study the thermodynamics and hydrodynamics of such objects. In particular, we examine the shear viscosity and the speed of sound in the strongly coupled superconformal plasma which is holographically dual to the black hole background. We find that the shear viscosity and entropy density in the quiver gauge theories [30, 31] continues to satisfy the universality relation (1.1), even in the presence of a nonzero chemical potential. Although we do not provide a proof, we believe that this universality (1.1) is likely to be true in any gauge theory plasma with $R$-charge chemical potentials turned on.

This paper is organized as follows. In the following section, we present the Kaluza-Klein reduction ansatz for IIB on $Y^{p,q}$ giving rise to minimal $\mathcal{N} = 2$ gauged supergravity in five dimensions. We then take up the thermodynamics of $R$-charged black hole solutions of this $\mathcal{N} = 2$ gauged supergravity in section 3 and the hydrodynamics in section 4. Finally, we present our conclusions in section 5.

2 $\mathcal{N} = 2$ gauged sugra from type IIB string theory on $Y^{p,q}$ manifolds with 5-form fluxes

A five-dimensional Sasaki-Einstein manifold $Y$ is an Einstein manifold preserving some fraction of supersymmetry such that the cone over $Y$ is (non-compact) Calabi-Yau. It is well known that a Sasaki-Einstein manifold $Y$ admits a constant norm Killing vector field, known as the Reeb vector. The existence of the Reeb vector allows the metric on $Y$ to be written as a $U(1)$ bundle over a four-dimensional Kähler-Einstein base $B$

$$ds^2(Y) = ds^2(B) + \frac{1}{9}(d\psi + A)^2,$$  \hspace{1cm} (2.1)

where $dA = 6J$ and $J$ is the Kähler form on $B$.

The Freund-Rubin compactification of IIB on $S^5$ admits a straightforward generalization where $S^5$ is replaced by the Sasaki-Einstein manifold $Y$. Taking only the metric and self-dual 5-form, the only ten-dimensional IIB equations of motion that we are concerned with are the Einstein and 5-form equations

$$R_{MN} = \frac{1}{2} \cdot \frac{1}{4!} F_{MPQRS} F_N^{PQRS}, \quad dF^{(5)} = 0, \quad F^{(5)} = *F^{(5)}.$$  \hspace{1cm} (2.2)

In addition, the IIB gravitino variation is given by

$$\delta\psi_M = \left[ \nabla_M + \frac{i}{16 \cdot 5!} F_{NPQRS} \Gamma^{NPQRS} \Gamma_M \right] \epsilon.$$  \hspace{1cm} (2.3)
The resulting Freund-Rubin ansatz is then of the form

\[ ds_{10}^2 = ds^2(AdS_5) + \frac{1}{g^2} ds^2(Y), \quad F(5) = (1 + \star) G(5), \quad G(5) = 4g\epsilon(5), \quad (2.4) \]

where \( \epsilon(5) \) is the five-dimensional volume form of \( AdS_5 \), and \( g \) is the coupling constant of the five-dimensional gauged supergravity (corresponding to the inverse \( AdS_5 \) radius). The five-dimensional theory is described by \( \mathcal{N} = 2 \) gauged supergravity, and is holographically dual to an \( \mathcal{N} = 1 \) superconformal gauge theory.

Since \( Y \) has the form of (2.1), there is a natural symmetry corresponding to the \( U(1) \) fiber. On the gravity side, the gauge boson under this \( U(1) \) is the \( \mathcal{N} = 2 \) graviphoton, while on the SCFT side this symmetry is just the \( U(1)_R \) symmetry. In order to retain the graviphoton, the Freund-Rubin metric ansatz (2.4) may be extended in the obvious manner

\[ ds_{10}^2 = g_{\mu\nu} dx^\mu dx^\nu + \frac{1}{g^2} \left( ds^2(B) + \frac{1}{2} (d\psi + A + A_\mu dx^\mu)^2 \right), \quad (2.5) \]

where \( g_{\mu\nu} \) is the five-dimensional metric and \( A_\mu \) is the graviphoton. The reduction of \( F(5) \) has the form

\[ F(5) = (1 + \star) G(5), \quad G(5) = 4g\epsilon(5) - \frac{1}{3g^2} J(2) \wedge \star F(2), \quad (2.6) \]

where \( J(2) \) is the Kähler form on \( B \) and \( F(2) = dA(1) \) with \( A(1) = A_\mu dx^\mu \). At a linearized order in gauge potential \( A_\mu \), and without the backreaction of its field strength on the metric, the ansatz (2.5), (2.6) appeared previously in [32]. We claim here that (2.5) and (2.6) are in fact consistent at a nonlinear level.

The above Kaluza-Klein reduction ansatz, (2.5) and (2.6), gives rise to equations of motion that may be obtained from the five-dimensional Lagrangian

\[ e^{-1} \mathcal{L}_5 = R + 12g^2 - \frac{1}{12} F_{\mu\nu}^2 + \frac{1}{108} \epsilon^{\mu\nu\rho\lambda\sigma} F_{\mu\nu} F_{\rho\lambda} A_\sigma. \quad (2.7) \]

This is the bosonic sector of minimal \( \mathcal{N} = 2 \) gauged supergravity in five dimensions. As written, the graviphoton is non-canonically normalized. However canonical normalization may be achieved by the rescaling \( A_\mu \rightarrow \sqrt{3} A_\mu \). Note, also, that the five-dimensional Newton’s constant \( G_5 \) of this \( \mathcal{N} = 2 \) gauged supergravity is related to the ten dimensional Newton’s constant \( G_{10} \) according to

\[ G_5 = \frac{G_{10}}{\text{volume } (Y)}. \quad (2.8) \]
Although the above results hold in general, much of the interest in this system is due to the availability of explicit Sasaki-Einstein metrics $Y^{p,q}$ that were originally constructed in [26, 27]. When IIB is compactified on $Y^{p,q}$, the resulting AdS$_5$ backgrounds are holographically dual to explicit $\mathcal{N} = 1$ superconformal quiver gauge theories SCFT$_{p,q}$ [30, 31]. The special case of $Y^{1,0}$ corresponds to $T^{1,1}$, and the dual gauge theory was investigated in [33]. To make the above more explicit, we will first work out the $T^{1,1}$ reduction in detail, and then follow with the extension to general $Y^{p,q}$ manifolds.

2.1 $T^{1,1}$ reduction of IIB

The well known case of IIB on AdS$_5 \times T^{1,1}$ was originally investigated in [33], and this has led to numerous important variations. Furthermore, this was one of only a few explicitly known examples of reduced supersymmetry systems, at least until the recent construction of an infinite family of $Y^{p,q}$ manifolds. As indicated in the introduction, although $T^{1,1}$ admits an $SU(2) \times SU(2) \times U(1)$ isometry, it is only consistent to truncate to minimal $\mathcal{N} = 2$ supergravity with the $U(1)$ gauged by the $\mathcal{N} = 2$ graviphoton.

Following (2.5) and (2.6), and expressing $T^{1,1}$ as $U(1)$ bundled over $S^2 \times S^2$, the reduction ansatz is as follows

$$
\begin{align*}
\text{ds}_{10}^2 &= g_{\mu\nu}dx^\mu dx^\nu + \frac{1}{g^2}\left[\frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{6}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 + gA_{(1)})^2\right], \\
F_5 &= (1 + *) \left[4g\epsilon_{(5)} + \frac{1}{18g^2}(\sin \theta_1 d\theta_1 \wedge d\phi_1 + \sin \theta_2 d\theta_2 \wedge d\phi_2) \wedge *_5F_2\right].
\end{align*}
$$

(2.9)

Before proceeding with the reduction, it is natural to introduce a vielbein basis for the $T^{1,1}$ coordinates

$$
\begin{align*}
e^1 &= \frac{1}{\sqrt{6}}d\theta_1, \quad e^2 = \frac{1}{\sqrt{6}}\sin \theta_1 d\phi_1, \quad e^3 = \frac{1}{\sqrt{6}}d\theta_2, \quad e^4 = \frac{1}{\sqrt{6}}\sin \theta_2 d\phi_2, \\
e^5 &= \frac{1}{3}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 + gA_{(1)}).
\end{align*}
$$

(2.10)

Note that the self-dual Kähler form on the $S^2 \times S^2$ base is given by $J_{(2)} = -e^1 \wedge e^2 - e^3 \wedge e^4$, where the sign is chosen to match (2.1) and (2.6). Using the relations

$$
\begin{align*}
de^1 &= 0, \quad de^2 = \sqrt{6} \cot \theta_1 e^1 \wedge e^2, \quad de^3 = 0, \quad de^4 = \sqrt{6} \cot \theta_2 e^3 \wedge e^4, \\
de^5 &= 2J_{(2)} + \frac{1}{3}gF_{(2)},
\end{align*}
$$

(2.11)
we may obtain the spin-connections
\begin{align*}
\omega_{12} &= -\sqrt{6} \cot \theta e^2 + e^5, & \omega_{34} &= -\sqrt{6} \cot \theta_2 e^4 + e^5, \\
\omega_{15} &= e^2, & \omega_{25} &= -e^1, & \omega_{35} &= e^4, & \omega_{45} &= -e^3,
\end{align*}
(2.12)
on $T^{1,1}$, as well as
\begin{align*}
\omega_{\alpha\beta} &= \omega^{(5)\alpha\beta} - \frac{1}{6g} F^{\alpha\beta} e^5, & \omega^{\alpha 5} &= -\frac{1}{6} F^{\alpha}_\beta e^\beta,
\end{align*}
(2.13)
for the mixed components.

The above vielbeins are naturally given for the unit radius $T^{1,1}$. Hence the rewriting of (2.9) in terms of the above vielbeins introduces several factors of $g$ as follows:
\begin{align*}
ds_{10}^2 &= g_{\mu\nu} dx^\mu dx^\nu + \frac{1}{g^2} \sum_i (e^i)^2, \\
F_{(5)} &= (1 + *) \left[ 4g \epsilon_{(5)} - \frac{1}{3g^2} J_{(2)} \wedge \star F_{(2)} \right] \\
&= 4 \left( g \epsilon_{(5)} + \frac{1}{g} e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \right) - \frac{1}{3g^2} J_{(2)} \wedge \left( \star F_{(2)} + \frac{1}{g} e^5 \wedge F_{(2)} \right).
\end{align*}
(2.14)
Since the Kähler form is closed, $dJ_{(2)} = 0$, it is now easy to see that the ten-dimensional Bianchi identity $dF_{(5)} = 0$ yields the five-dimensional Bianchi identity and equation of motion for $F_{(2)}$
\begin{align*}
dF_{(2)} &= 0, & d\star F_{(2)} &= -\frac{1}{3} F_{(2)} \wedge F_{(2)}. 
\end{align*}
(2.15)
Note that, to verify the Bianchi identity, we also need the fact that $J_{(2)} \wedge J_{(2)}$ gives twice the volume form on the Einstein-Kähler base, namely $J_{(2)} \wedge J_{(2)} = 2e^1 \wedge e^2 \wedge e^3 \wedge e^4$.

Turning next to the ten-dimensional Einstein equation, we note that it splits into several components, $\mu\nu$, $ij$, and $\mu i$. We find that the $\mu\nu$ components gives the five-dimensional Einstein equation
\begin{align*}
R^{(5)}_{\mu\nu} &= -4g^2 g_{\mu\nu} + \frac{1}{6} (F_{\mu\lambda} F^{\nu}_\lambda - \frac{1}{6} g_{\mu\nu} F^2),
\end{align*}
(2.16)
while the internal $ij$ components are identically satisfied. In addition, the mixed $\mu 5$ component yields the 2-form equation of motion
\begin{align*}
\nabla^\lambda F_{\lambda\mu} &= -\frac{1}{12} \epsilon_{\mu\nu\rho\lambda\sigma} F^{\nu\rho} F^{\lambda\sigma},
\end{align*}
(2.17)
which is identical to the one given in (2.15) in form notation. As indicated at the beginning of this section, the combined equations of motion (2.15) and (2.16) may be obtained from the Lagrangian (2.7) for minimal gauged supergravity in five dimensions.

Although we have only focused on the bosonic fields of the reduction, it ought to be apparent that the full Kaluza-Klein reduction onto $\mathcal{N} = 2$ gauged supergravity including the gravitino is a consistent one. To see this, we may highlight the reduction of the IIB gravitino variation (2.3). In order to proceed, we first introduce a decomposition of the ten-dimensional Dirac matrices

$$
\Gamma^A = \{\gamma^\alpha \otimes 1 \otimes \sigma^1, 1 \otimes \tilde{\gamma}^a \otimes \sigma^2\}.
$$

By convention, we take the product of spacetime Dirac matrices to be $\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4 = -i$ and internal Dirac matrices to be $\tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 \tilde{\gamma}^4 \tilde{\gamma}^5 = 1$. In addition, the IIB gravitino $\psi_\mu$ and supersymmetry parameter $\epsilon$ are ten-dimensional Weyl spinors satisfying the chirality projection $\Gamma^{11} \epsilon = \epsilon$ where $\Gamma^{11} = \Gamma^0 \ldots \Gamma^9 = 1 \otimes 1 \otimes \sigma^3$. As a result, such chiral IIB spinors may be written as

$$
\epsilon = \epsilon \otimes \eta \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
$$

Given this decomposition of the Dirac matrices, we now proceed with the reduction. The gravitino variation (2.3) is perhaps most straightforwardly investigated in tangent space. Using (2.14) for the 5-form, and taking chirality into account, we note that

$$
F_{BCDEFG} \Gamma_{BCDEFG} \alpha = -40i[24g\gamma_\alpha - (\tilde{\gamma}^{12} + \tilde{\gamma}^{34})F_{\beta\gamma}\gamma^\beta\gamma^\gamma\gamma_\alpha],
$$

$$
F_{BCDEFG} \Gamma_{BCDEFG} \alpha = 40[24g\tilde{\gamma}_\alpha - (\tilde{\gamma}^{12} + \tilde{\gamma}^{34})F_{\beta\gamma}\tilde{\gamma}^\beta\tilde{\gamma}^\gamma\tilde{\gamma}_\alpha].
$$

Combining this with the covariant derivatives formed from the spin connections (2.12) and (2.13), we obtain the spacetime variation

$$
\delta \psi_\alpha = [\nabla^{(5)}_\alpha - gA_\alpha \partial_\psi + \frac{i}{12} F_{\alpha\beta\gamma}\gamma^\gamma\tilde{\gamma}^5 - \frac{1}{48} F_{\beta\gamma}\gamma^\beta\gamma^\gamma\gamma_\alpha (\tilde{\gamma}^{12} + \tilde{\gamma}^{34}) + \frac{1}{2} g\gamma_\alpha] \epsilon.
$$
as well as the $T^{1,1}$ variations

\[
\begin{align*}
\delta \psi_1 &= \left[\sqrt{6} \partial_\theta_1 - \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma_1 (-\tilde{\gamma}^{12} + \tilde{\gamma}^{34}) + \frac{i}{2} \gamma^1 (1 - i \tilde{\gamma}^{34}) \right] \epsilon, \\
\delta \psi_2 &= \left[\sqrt{6} \csc \theta_1 \partial_{\phi_1} - \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma_2 (-\tilde{\gamma}^{12} + \tilde{\gamma}^{34}) + \frac{i}{2} \gamma^2 (1 - i \tilde{\gamma}^{34}) \right] \epsilon \\
& \quad - \sqrt{6} \cot \theta_1 (\partial_\psi + \frac{1}{2} \tilde{\gamma}^{12}) \epsilon, \\
\delta \psi_3 &= \left[\sqrt{6} \partial_\theta_2 - \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma_3 (\tilde{\gamma}^{12} - \tilde{\gamma}^{34}) + \frac{i}{2} \gamma^3 (1 - i \tilde{\gamma}^{12}) \right] \epsilon, \\
\delta \psi_4 &= \left[\sqrt{6} \csc \theta_2 \partial_{\phi_2} - \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma_4 (\tilde{\gamma}^{12} - \tilde{\gamma}^{34}) + \frac{i}{2} \gamma^4 (1 - i \tilde{\gamma}^{12}) \right] \epsilon \\
& \quad - \sqrt{6} \cot \theta_2 (\partial_\psi + \frac{1}{2} \tilde{\gamma}^{34}) \epsilon, \\
\delta \psi_5 &= \left[3 \partial_\phi - \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} (2 - i \tilde{\gamma}^{12} - i \tilde{\gamma}^{34}) + \frac{1}{2} (\tilde{\gamma}^{12} + \tilde{\gamma}^{34} + i \tilde{\gamma}^5) \right] \epsilon. 
\end{align*}
\]

The internal Dirac matrices provide a spinor representation of the $SO(5)$ tangent space of $T^{1,1}$. This 4 of $SO(5)$ corresponds to taking the independent weights $i \tilde{\gamma}^{12} = \pm 1$ and $i \tilde{\gamma}^{34} = \pm 1$. Examining the variations (2.22), we see that the Killing spinor must take the weights $i \tilde{\gamma}^{12} = i \tilde{\gamma}^{34} = 1$. Furthermore, $\tilde{\gamma}^5$ is not independent, and must have corresponding eigenvalue $\tilde{\gamma}^5 = -1$. Thus this $T^{1,1}$ background preserves 1/4 of the supersymmetries, and the Killing spinor $\eta$ of $T^{1,1}$ satisfies the projections

\[
i \tilde{\gamma}^{12} \eta = i \tilde{\gamma}^{34} \eta = -\tilde{\gamma}^5 \eta = \eta. \tag{2.23}
\]

The $T^{1,1}$ Killing spinor equation is now trivial to solve, and the resulting solution is

\[
\eta = e^{\frac{1}{2} \psi} \eta_0, \tag{2.24}
\]

with $\eta_0$ satisfying the above projections. This demonstrates that the Killing spinor is charged along the $U(1)$ fiber. This charge is of course expected, and gives rise to a charged gravitino in five dimensions as appropriate for the gauged supergravity theory. Inserting this Killing spinor into (2.21), and rewriting this variation using curved indices, we finally obtain

\[
\delta \psi_\mu = \left[\nabla^{(5)}_\mu - \frac{i}{2} g A_\mu + \frac{i}{24} (\gamma_\mu ^{\nu \lambda} - 4 \delta_\mu ^{\nu \lambda} F_{\nu \lambda} + \frac{1}{2} g \gamma_\mu \right] \epsilon, \tag{2.25}
\]

which is the appropriate gravitino variation for minimal $\mathcal{N} = 2$ gauged supergravity in five dimensions.
2.2 $Y^{p,q}$ reduction of IIB

We now turn to the case of $Y^{p,q}$. As expected, the reduction of IIB on $Y^{p,q}$ is almost identical to that for $T^{1,1}$. We start with the metric on $Y^{p,q}$ written as

$$
\begin{align*}
\text{ds}_{10}^2 &= g_{\mu\nu}dx^\mu dx^\nu + \frac{1}{g^2} \left[ \frac{1}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{wv} dy^2 + \frac{wv}{36} (d\beta + \cos \theta d\phi)^2 \\
&\quad + \frac{1}{9} (d\psi - \cos \theta d\phi + y(d\beta + \cos \theta d\phi) + gA_1)^2 \right],
\end{align*}
$$

where

$$
\begin{align*}
w &= \frac{2(a - y^2)}{1 - y}, & v &= \frac{a - 3y^2 + 2y^3}{a - y^2},
\end{align*}
$$

and introduce the natural vielbein basis

$$
\begin{align*}
e^1 &= \sqrt{\frac{1 - y}{6}} d\theta, & e^2 &= \sqrt{\frac{1 - y}{6}} \sin \theta d\phi, \\
e^3 &= \frac{1}{\sqrt{6H}} dy, & e^4 &= \frac{H}{\sqrt{6}} (d\beta + \cos \theta d\phi), \\
e^5 &= \frac{1}{3} (d\psi - \cos \theta d\phi + y(d\beta + \cos \theta d\phi) + gA_1).
\end{align*}
$$

For later convenience, we have defined the functions

$$
H = \sqrt{\frac{wv}{6}} = \sqrt{\frac{a - 3y^2 + 2y^3}{3(1 - y)}}, \quad K = \frac{H}{2(1 - y)} = \sqrt{\frac{a - 3y^2 + 2y^3}{12(1 - y)^3}},
$$

which satisfy the relation

$$
\frac{dH}{dy} = K - \frac{y}{H}.
$$

In terms of the vielbeins, the 5-form ansatz takes the same form as that given above for the $T^{1,1}$ case. In particular

$$
F_5 = (1 + *) \left[ 4g\epsilon(5) - \frac{1}{3g^2} J_{(2)} \wedge *_5 F_{(2)} \right],
$$

where the Kähler form is given by $J_{(2)} = e^1 \wedge e^2 + e^3 \wedge e^4$. Using the relations

$$
\begin{align*}
de^1 &= \sqrt{6K} e^1 \wedge e^3, & de^2 &= \sqrt{6K} e^2 \wedge e^3 + \sqrt{\frac{6}{1 - y}} \cot \theta e^1 \wedge e^2, \\
de^3 &= 0, & de^4 &= \sqrt{6} \left( K - \frac{y}{H} \right) e^3 \wedge e^4 - 2\sqrt{6K} e^1 \wedge e^2, \\
de^5 &= 2(e^1 \wedge e^2 + e^3 \wedge e^4) + \frac{1}{3} F_{(2)},
\end{align*}
$$

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we see that
\[ d(\varepsilon^1 \wedge \varepsilon^2) = -2\sqrt{6}K \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3, \quad d(\varepsilon^3 \wedge \varepsilon^4) = 2\sqrt{6}K \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3. \quad (2.33) \]
As a result, \(d(\varepsilon^1 \wedge \varepsilon^2 + \varepsilon^3 \wedge \varepsilon^4) = 0\), giving an explicit demonstration that the Kähler form is closed, namely \(dJ_{(2)} = 0\). As a result, the ten-dimensional Bianchi identity \(dF_5 = 0\) again reduces to the five-dimensional Bianchi identity and equation of motion (2.15). Similarly, the reduction of the ten-dimensional Einstein equation follows just as in the \(T^{1,1}\) case. The resulting bosonic fields \(g_{\mu \nu}\) and \(A_\mu\) are then described by the \(\mathcal{N} = 2\) Lagrangian (2.7).

Turning to the IIB gravitino variation, we first compute the spin connections from (2.32)
\[
\begin{align*}
\omega^{12} &= -\sqrt{\frac{6}{1 - y}} \cot \theta \varepsilon^2 + \sqrt{6}K \varepsilon^4 - \varepsilon^5, \\
\omega^{13} &= -\sqrt{6}K \varepsilon^1, \\
\omega^{14} &= \sqrt{6}K \varepsilon^2, \\
\omega^{23} &= -\sqrt{6}K \varepsilon^2, \\
\omega^{24} &= -\sqrt{6}K \varepsilon^1, \\
\omega^{34} &= -\sqrt{6} \left( K - \frac{y}{H} \right) \varepsilon^4 - \varepsilon^5, \\
\omega^{15} &= -\varepsilon^2, \\
\omega^{25} &= \varepsilon^1, \\
\omega^{35} &= -\varepsilon^4, \\
\omega^{45} &= \varepsilon^3,
\end{align*}
\]
and
\[
\omega^{\alpha \beta} = \omega^{(5) \alpha \beta} - \frac{1}{6g} F^{\alpha \beta} \varepsilon^5, \\
\omega^{\alpha 5} = -\frac{1}{6} F^{\alpha \beta} \varepsilon^5.
\]
Since the 5-form reduction (2.31) has the same form as for the \(T^{1,1}\) case, the reduction of \(F \cdot \Gamma_A\) results in an expression identical to (2.20), however with the replacement \(F_{\beta \gamma} \rightarrow -F_{\beta \gamma}\) due to the opposite sign convention for the Kähler form \(J_{(2)}\).

The IIB gravitino variation (2.3) then breaks up into the spacetime component
\[
\delta \psi_\alpha = [\nabla^{(5)}_\alpha - gA_\alpha \partial_\psi + \frac{i}{12} F_{\alpha \beta \gamma} \gamma^5 + \frac{1}{48} F_{\beta \gamma \delta} \gamma^\delta \gamma_\alpha (\lambda^{12} + \lambda^{34}) + \frac{1}{2} g \gamma_\alpha |\epsilon|, \quad (2.36)
\]
and \(Y^{p,q}\) components
\[
\begin{align*}
\delta \psi_1 &= \left[ \sqrt{6}(1 - y)^{-1/2} \partial_\theta + \frac{1}{2} \sqrt{6}K \gamma^{23}(\lambda^{12} - \lambda^{34}) + \frac{1}{2} \hat{\gamma}^1 (1 + i \gamma^{34}) \\
&\quad - \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma^1 (\lambda^{12} - \lambda^{34}) \right] |\epsilon|, \\
\delta \psi_2 &= \left[ \sqrt{6}(1 - y)^{-1/2} (\csc \theta \partial_\phi - \cot \theta \partial_\beta + \cot \theta (\partial_\psi - \frac{1}{2} \lambda^{12})) - \frac{1}{2} \sqrt{6}K \gamma^{24} (\lambda^{12} - \lambda^{34}) \\
&\quad + \frac{1}{2} \gamma^2 (1 + i \gamma^{34}) + \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma^2 (\lambda^{12} - \lambda^{34}) \right] |\epsilon|, \\
\delta \psi_3 &= \left[ \sqrt{6} H \partial_\gamma + \frac{1}{2} \gamma^3 (1 + i \gamma^{12}) + \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma^3 (\lambda^{12} - \lambda^{34}) \right] |\epsilon|, \\
\delta \psi_4 &= \left[ \sqrt{6} H^{-1} \partial_\beta - \sqrt{6} y H^{-1} (\partial_\phi - \frac{1}{2} \gamma^{34}) + \frac{1}{2} \sqrt{6} K (\lambda^{12} - \lambda^{34}) + \frac{1}{2} \gamma^4 (1 + i \gamma^{12}) \\
&\quad + \frac{1}{48} F_{\alpha \beta} \gamma^{\alpha \beta} \gamma^4 (\lambda^{12} - \lambda^{34}) \right] |\epsilon|, \\
\delta \psi_5 &= \left[ 3 \partial_\psi - \frac{1}{2} (\lambda^{12} + \lambda^{34} - i \gamma^{5}) - \frac{i}{48} F_{\alpha \beta} \gamma^{\alpha \beta} (\lambda^{12} + \lambda^{34} - 2i) \right] |\epsilon|.
\end{align*}
\]
It is clear from the above that the Killing spinor $\eta$ on $Y^{p,q}$ is given by

$$\eta = e^{\frac{1}{2} \psi} \eta_0, \quad i \tilde{\gamma}^{12} \eta_0 = i \tilde{\gamma}^{34} \eta_0 = \tilde{\gamma}^5 \eta_0 = -\eta_0,$$

(2.38)

and hence the spacetime component (2.36) reduces to the expected $\mathcal{N} = 2$ gravitino variation (2.25).

Although the derivation of this reduction ansatz is straightforward (since in essence we are only performing a Kaluza-Klein reduction along a $U(1)$ fiber), the exact Kaluza-Klein ansatz is extremely useful in that it enables us to greatly expand the investigations of gauge theories at finite $R$-charge chemical potential. This is what we now turn to in the following sections.

# 3 $R$-charged black holes and their thermodynamics

Working in the Poincaré patch of AdS$_5$, we may consider the following black hole ansatz in the minimal gauged supergravity described by (2.7):

$$ds^2_5 = -c_1^2 (dt)^2 + c_2^2 (d\vec{x})^2 + c_3^2 (dr)^2.$$  

(3.1)

Here the functions only depend on the radial coordinate $r$, namely $c_i = c_i(r)$. In order to work in a given $R$-charge sector, we must also consider turning on a gauge field potential

$$A_t = a(r).$$  

(3.2)

Solving the equations of motion (2.15), (2.16) within the ansatz (3.1), (3.2) yields

$$c_1 = r f^{1/2}, \quad c_2 = r, \quad c_3 = c_1^{-1}, \quad a' = -\frac{2Qc_1c_3}{c_2^3},$$

(3.3)

where the prime denotes a derivative with respect to $r$, $\mu$ is the non-extremality parameter and the parameter $Q$ is related to the $R$ charge of the black hole.

Let us compute the renormalized (in the sense of [34]) Euclidean gravitational action $I_E$ of (2.7). First, we regularize (2.7) by introducing a boundary $\partial M_5$ at fixed (large)
with the unit orthonormal space-like vector $n^\mu \propto \delta^\mu_r$

$$S_5^r = \frac{1}{16\pi G_5} \int_{r_+}^r dr \int_{\partial \mathcal{M}_5} d^4\xi \sqrt{g_E} \mathcal{L}_E = -\frac{1}{16\pi G_5} \int_{r_+}^r dr \int_{\partial \mathcal{M}_5} d^4\xi \sqrt{-g} \mathcal{L}$$

$$= \frac{1}{16\pi G_5} \int_{r_+}^r dr \left[ \frac{2c_2^2c_1c_3^2}{c_3} \right]^{\prime} \int_{\partial \mathcal{M}_5} d^4\xi$$

$$= \frac{\beta V_3}{16\pi G_5} \left[ \frac{2c_2^2c_1c_3^2}{c_3} \right]^{\prime} \bigg|_{r_+}^{r} ,$$

where the subscript $E$ indicates that all the quantities are to be computed in Euclidean signature, and $r_+$ is the outer black hole horizon, i.e., the largest positive root of

$$f(r_+) = 0 .$$

The black hole temperature and entropy are given by

$$T_H = \frac{\mu}{\pi r_+^3} - \frac{Q^2}{6\pi r_+^5} , \quad S_{BH} = \frac{r_+^3 V_3}{4G_5} .$$

As usual, to have a well-defined variational problem in the presence of a boundary requires the inclusion of the Gibbons-Hawking $S_{GH}$ term

$$S_{GH} = -\frac{1}{8\pi G_5} \int_{\partial \mathcal{M}_5} d^4\xi \sqrt{h_E} \nabla_{\mu} n^\mu$$

$$= -\frac{\beta V_3}{8\pi G_5} \left[ \frac{(c_1c_3^2)^{\prime}}{c_3} \right] ,$$

where $\nabla_{\mu}$ is the induced metric on $\partial \mathcal{M}_5$

$$\nabla_{\mu} n^\nu \equiv g_{\mu\nu} - n_\mu n_\nu .$$

Finally, as in [34], we supplement the combined regularized action ($S_5^r + S_{GH}$) by the appropriate boundary counterterms constructed out of metric invariants on the boundary $\partial \mathcal{M}_5$

$$S_{\text{counter}} = \frac{1}{16\pi G_5} \int_{\partial \mathcal{M}_5} d^4\xi \frac{6\sqrt{h_E}}{6} .$$

In the present case the boundary curvature vanishes, so there are no extra counterterms. The renormalized Euclidean action $I_E$ defined as

$$I_E \equiv \lim_{r \to \infty} \left( S_5^r + S_{GH} + S_{\text{counter}} \right) ,$$

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is finite.

We now proceed to the computation of the ADM mass for the background (3.1). Following [34] we define

\[ M = \int_{\Sigma} \sqrt{\sigma} N_\Sigma \epsilon, \]

(3.11)

where \( \Sigma \equiv S^3 \) is a spacelike hypersurface in \( \partial M_5 \) with a timelike unit normal \( u^\mu \), \( N_\Sigma \) is the norm of the timelike Killing vector in (3.1), \( \sigma \) is the determinant of the induced metric on \( \Sigma \), and \( \epsilon \) is the proper energy density

\[ \epsilon = u^\mu u^\nu T_{\mu\nu}. \]

(3.12)

The quasilocal stress tensor \( T_{\mu\nu} \) for our background is obtained from the variation of the full action

\[ S_{\text{tot}} = S_5^r + S_{\text{GH}} + S_{\text{counter}}, \]

(3.13)

with respect to the boundary metric \( \delta h_{\mu\nu} \)

\[ T_{\mu\nu} = 2 \frac{\delta S_{\text{tot}}}{\sqrt{-h} \delta h_{\mu\nu}}. \]

(3.14)

An explicit computation yields

\[ T_{\mu\nu} = \frac{1}{8\pi G_5} \left[ -\Theta_{\mu\nu} + \Theta h^\mu - 3h_{\mu\nu} \right], \]

(3.15)

where

\[ \Theta_{\mu\nu} = \frac{1}{2} (\nabla^\mu n^\nu + \nabla^\nu n^\mu), \quad \Theta = \text{Tr} \Theta_{\mu\nu}. \]

(3.16)

Again, the renormalized stress energy tensor is finite.

Inserting the black hole solution into the above action (3.13) and renormalized stress tensor (3.15), we obtain the action and mass

\[ I_E = -\frac{\beta \mu V_3}{16\pi G_5}, \]

\[ M = \frac{3\mu V_3}{16\pi G_5}. \]

(3.17)

Notice that, by using (3.6) and (3.17), we find the expected thermodynamical relation

\[ I_E = \beta (M - \mu \tilde{q}) - S_{\text{BH}}, \]

(3.18)

where \( \mu \tilde{q} \) is the chemical potential conjugate to the physical black hole charge density

\[ \tilde{q} = \frac{Q}{\sqrt{3}}. \]

(3.19)
related to the gauge potential \( A_t (3.2) \) at the horizon, \( r = r_+ \)

\[
\mu \tilde{q} = \frac{V_3}{8\pi G_5} \left| \frac{A_t}{\sqrt{3}} \right|_{r = r_+} = \frac{V_3}{8\pi G_5} \frac{Q}{\sqrt{3} r_+^2}.
\] (3.20)

The factors of \( \sqrt{3} \) in (3.19) and (3.20) come from the noncanonical normalization of the gauge field in (2.7).

We may now identify the thermodynamical quantities of the charged black hole \( \{ I_E, M, S_{BH}; T_H, \mu_\tilde{q} \} \) with the appropriate gauge theory quantities \( \{ \Omega, E, S, T, \mu_J \} \)

\[
\{ I_E, M, S_{BH}; T_H, \mu_\tilde{q} \} \longleftrightarrow \{ \Omega/T, E, S; T, \mu_J \},
\] (3.21)

where the thermodynamic potential \( \Omega \) is related to the Helmholtz free energy \( F \) in the standard way

\[
\Omega = F - \mu_J J = E - T S - \mu_J J.
\] (3.22)

On can explicitly verify that with the identification (3.21) the first law of thermodynamics for the grand canonical ensemble with \( \{ T, \mu_J \} \) as independent variables

\[
d\Omega = -S \, dT - J \, d\mu_J.
\] (3.23)

is satisfied automatically. To check (3.23) it is useful to use

\[
d(f(r_+)) = 0,
\] (3.24)

which is simply the statements that given \( \{ \mu, Q \} \), the radius of the outer horizon of the BH is given by (3.5).

## 4 Hydrodynamics

In this section, we examine the hydrodynamics of the \( \mathcal{N} = 2 \) Yang-Mills plasma at finite \( R \)-charge which is dual to the black hole solution of (3.1), (3.2) and (3.3). In particular, using prescription [35], we compute the retarded Green’s function of the boundary stress-energy tensor \( T_{\mu \nu}(t, x) \) (\( \mu = \{ t, x^\alpha \} \)) at zero spatial momentum, and in the low-energy limit \( \omega \to 0 \):

\[
G^{R}_{12,12}(\omega, 0) = -i \int dt \, d^3 x \, e^{i\omega t} \theta(t) \langle [T_{12}(t, x), T_{12}(0, 0)] \rangle.
\] (4.1)
Computation of this Green’s function allows a determination of the shear viscosity \( \eta \) through the Kubo relation
\[
\eta = \lim_{\omega \to 0} \frac{1}{2\omega^2} \left[ G_A^{12,12}(\omega, 0) - G_R^{12,12}(\omega, 0) \right],
\]
(4.2)
where the advanced Green’s function is given by \( G_A(\omega, 0) = (G_R(\omega, 0))^* \).

Although the extraction of the retarded Green’s function from the \( R \)-charged black hole background is somewhat involved, the result turns out to be independent of charge and chemical potential. As demonstrated below, we find
\[
G_R^{12,12}(\omega, 0) = -\frac{i\omega s}{4\pi} \left( 1 + \mathcal{O}\left( \frac{\omega}{T} \right) \right),
\]
(4.3)
where
\[
s = \frac{r^3}{4G_5}
\]
(4.4)
is the Bekenstein-Hawking entropy density of the black hole. Inserting this expression into (4.2) then yields the universal ratio
\[
\frac{\eta}{s} = \frac{1}{4\pi}.
\]
(4.5)

### 4.1 Computation of the retarded Green’s function

We begin the computation of (4.1) by recalling that the coupling between the boundary value of the graviton and the stress-energy tensor of a gauge theory is given by \( \delta g_{12} T_1^2 / 2 \). According to the gauge/gravity prescription, in order to compute the retarded thermal two-point function (4.1), we should add a small bulk perturbation \( \delta g_{12}(t, y) \) to the metric (3.1), and compute the on-shell action as a functional of its boundary value \( \delta g_{12}^b(t) \). Symmetry arguments [36] guarantee that for a perturbation of this type and metric and gauge potential of the form (3.1) and (3.2), all other components of a generic perturbation \( \delta g_{\mu\nu} \) along with the gauge potential perturbations \( \delta A_\mu \) can be consistently set to zero.

Instead of working directly with \( \delta g_{12} \), we find it convenient to introduce the field \( \phi = \phi(t, r) \) according to
\[
\phi = \frac{1}{2} g^{11} \delta g_{12} = \frac{1}{2} c_2^{-2} \delta g_{12}.
\]
(4.6)
The retarded correlation function \( G_R^{12,12}(\omega, 0) \) can be extracted from the (quadratic) boundary effective action \( S_{\text{boundary}} \) for the metric fluctuations \( \phi^b \) given by
\[
S_{\text{boundary}}[\phi^b] = \int \frac{d^4k}{(2\pi)^4} \phi^b(-\omega) \mathcal{F}(\omega, r) \phi^b(\omega) \bigg|_{\text{horizon}},
\]
(4.7)
where
\[
\phi^b(\omega) = \left. \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t} \phi(t, r) \right|_{\partial M_5}.
\] (4.8)

In particular, the Green’s function is given simply by
\[
G^R_{12,12}(\omega, 0) = \lim_{\partial M_5^r \to \partial M_5} 2 \mathcal{F}(\omega, r),
\] (4.9)

where \( \mathcal{F} \) is the kernel of (4.7). The boundary metric functional is defined as
\[
S_{\text{boundary}}[\phi^b] = \lim_{\partial M_5^r \to \partial M_5} \left( S^r_{\text{bulk}}[\phi] + S_{\text{GH}}[\phi] + S_{\text{counter}}[\phi] \right),
\] (4.10)

where \( S^r_{\text{bulk}} \) is the bulk Minkowski-space effective supergravity action (2.7) on a cutoff space \( M_5^r \) (where \( M_5 \) in (3.1) is regularized by the compact manifold \( M_5^r \) with a boundary \( \partial M_5^r \)). Also, \( S_{\text{GH}} \) is the standard Gibbons-Hawking term over the regularized boundary \( \partial M_5^r \). The regularized bulk action \( S^r_{\text{bulk}} \) is evaluated on-shell for the bulk metric fluctuations \( \phi(t, r) \) subject to the following boundary conditions:
\[
(a) : \lim_{\partial M_5^r \to \partial M_5} \phi(t, r) = \phi^b(t),
\]
\[
(b) : \phi(t, r) \text{ is an incoming wave at the horizon}.
\] (4.11)

The purpose of the boundary counterterm \( S_{\text{counter}} \) is to remove divergent (as \( \partial M_5^r \to \partial M_5 \)) and \( \omega \)-independent contributions from the kernel \( \mathcal{F} \) of (4.7).

We find that the effective bulk action for \( \phi(t, r) \) in the supergravity background (3.1), (3.2) takes the form
\[
S_{\text{bulk}}[\phi] \equiv \frac{1}{16\pi G_5} \int d^5x \mathcal{L}_5 = \frac{1}{16\pi G_5} \int d^5x \left[ c_1 c_2^3 c_3 \left\{ \frac{1}{2c_1^2} (\partial_t \phi)^2 - \frac{1}{2c_3^2} (\partial_r \phi)^2 \right\} \right. \]
\[
\left. + \left\{ -\partial_t \left( \frac{2c_3^3 c_2}{c_1} \phi \partial_t \phi \right) + \partial_r \left( \frac{2c_2^3 c_1}{c_3} \phi \partial_r \phi + \frac{c_1 c_2^2 c'_2}{c_3} \phi^2 \right) \right\} \right].
\] (4.12)

The second line in (4.12) is the effective action for a minimally coupled scalar in the geometry (3.1), while the third line is a total derivative. Thus the bulk equation of motion for \( \phi \) is that of a minimally coupled scalar in (3.1). The latter equation is simplified by introducing a new radial coordinate\(^3\)
\[
x \equiv \frac{c_1}{c_2},
\] (4.13)

\(^3\)With such a definition, \( x \) has a range \( x \in [0, 1] \) with \( x \to 0_+ \) being the horizon and \( x \to 1_- \) the boundary.
which using (3.3) can be inverted to give the near horizon expansion
\[ c_2(x) = r_+ - \frac{9r_+^7}{2(Q^2 - 18r_+^6)} x^2 + \frac{81r_+^{13}(11Q^2 - 90r_+^6)}{8(Q^2 - 18r_+^6)^3} x^4 + O(x^6). \] (4.14)

Decomposing \( \phi \) as
\[ \phi(t, x) = e^{-i\omega t} \phi_\omega(x), \] (4.15)
we find that the equation of motion reduces to
\[ 0 = \phi''_\omega - \left(\frac{18c_2^8 - c_2^2 Q^2 + 6xc'_2c_2Q^2 + 12x^2 (c'_2)^2 Q^2}{xc_2^2(Q^2 - 18c_2^6)}\right) \phi'_\omega - \frac{9\omega^2 c_2^2 c'_2 (c_2 + 2c'_2 x)}{x^3(Q^2 - 18c_2^6)} \phi_\omega, \] (4.16)
where primes denote derivatives with respect to \( x \).

A low-frequency solution of (4.16) which is an incoming wave at the horizon, and which near the boundary satisfies
\[ \lim_{x \to 1^-} \phi_\omega(x) = 1, \] (4.17)
can be written as
\[ \phi_\omega(x) = x^{-i\omega} \left( F_0(x) + i\omega F_\omega(x) + O(\omega^2) \right), \] (4.18)
where \( \omega \)
\[ \omega = \frac{\omega}{2\pi T}. \] (4.19)
The functions \( \{F_0, F_\omega\} \), which are smooth at the horizon, satisfy the following differential equations:
\[ 0 = F_0'' - \left(\frac{18c_2^8 - c_2^2 Q^2 + 6xc'_2c_2Q^2 + 12x^2 (c'_2)^2 Q^2}{xc_2^2(Q^2 - 18c_2^6)}\right) F_0', \]
\[ 0 = F_\omega'' - \left(\frac{18c_2^8 - c_2^2 Q^2 + 6xc'_2c_2Q^2 + 12x^2 (c'_2)^2 Q^2}{xc_2^2(Q^2 - 18c_2^6)}\right) F_\omega' - \frac{2}{x} F_0' + \frac{6c'_2 Q^2(c_2 + 2xc'_2)}{xc_2^2(Q^2 - 18c_2^6)} F_0. \] (4.20)

Notice that for \( Q = 0 \) the dependence on the specific background geometry (\( c_2(x) \) dependence) drops out and (4.20) coincides with universal equations derived in [7]. In the present case, however, since the \( Q \) dependence appears highly non-trivial, universality is far from assured.
The general solution of the first equation in (4.20) takes the form
\[ F_0 = C_1 + C_2 \int dx \exp \left\{ \int_x^y \frac{18c_2^6 - c_2^2 Q^2 + 6yc_2 c_2 Q^2 + 12y^2 (c_2')^2 Q^2}{yc_2^2 (Q^2 - 18c_2^6)} \right\}, \] (4.21)
where \( C_i \) are integration constants. Using the near horizon expansion (4.14), we find as \( x \to 0_+ \)
\[ F_0 = C_1 + C_2 \left( \ln x - \frac{27 Q^2 r_+^6}{2(Q^2 - 18r_+^6)^2} x^2 + O(x^4) \right). \] (4.22)
Thus nonsingularity of \( F_0 \) at the horizon requires \( C_2 = 0 \). The boundary condition (4.17) further specifies
\[ F_0(x) = 1. \] (4.23)
The solution of the second equation in (4.20) is a bit more complicated and will be discussed in the Appendix. We note here that unlike the universal case [7], given (4.23) and the boundary conditions, this solution is non-trivial. Altogether we have
\[ \phi(t, x) = e^{-i\omega t} x^{-i\omega} (1 + i\omega F_\omega(x) + O(\omega^2)). \] (4.24)

Once the bulk fluctuations are on-shell (i.e., satisfy equations of motion) the bulk gravitational Lagrangian becomes a total derivative. From (4.12) we find (without dropping any terms)
\[ \mathcal{L}_5 = \partial_t J^t + \partial_r J^r, \] (4.25)
where
\[ J^t = -\frac{3c_2^3 c_3}{2c_1} \phi \partial_t \phi, \]
\[ J^r = \frac{3c_2^3 c_1}{2c_3} \phi \partial_r \phi + \frac{c_2^3 c_1 c_3}{c_3} \phi'^2. \] (4.26)
Additionally, the Gibbons-Hawking term provides an extra contribution so that
\[ J^r \rightarrow J^r - \frac{2c_2^3 c_1}{c_3} \phi \partial_r \phi - \frac{(c_1 c_3')}{c_3} \phi'^2. \] (4.27)

We are now ready to extract the kernel \( F \) of (4.7). The regularized boundary effective action for \( \phi \) is
\[ S_{\text{boundary}}[\phi]^r = \frac{1}{16 \pi G_5} \int_{\partial M_5} dt d^3x \left( -\frac{c_2^3 c_1}{2c_3} \phi \partial_r \phi \right) + c.t., \] (4.28)
where as prescribed in [35], we need only keep the boundary contribution. In (4.28) c.t. stands for (finite) contact terms that will not be important for computations. Substituting (4.24) into (4.28) we can obtain $\mathcal{F}^r(\omega, r)$:

$$
\mathcal{F}^r(\omega, r) = -\frac{i\omega}{32\pi G_5} (1 + \mathcal{O}(\omega)) \times \frac{c_3^2(r)c_1(r)}{c_3(r)} \left( \frac{c_1(r)}{c_2(r)} \right) ^{'} \times \left\{ 1 - \frac{dF_\omega(x)}{dx} \right\},
$$

(4.29)

where we have recalled the definition of $x$ in (4.13). Thus, to order $\mathcal{O}(\omega^2)$, we have

$$
\lim_{r \to \infty} \mathcal{F}^r(\omega, r) = -\frac{i\omega \mu}{32\pi G_5} \times \lim_{x \to 1^-} \left\{ 1 - \frac{dF_\omega(x)}{dx} \right\},
$$

(4.30)

where we used (3.3). Now, given (A.1) and (A.8), we finally obtain

$$
\lim_{r \to \infty} \mathcal{F}^r(\omega, r) = -\frac{i\omega \mu}{16\pi G_5} \left( -\frac{\pi T r_+^3}{\mu} \right) = -\frac{i\omega}{8\pi} \frac{r_+^3}{4G_5} = -\frac{i\omega s}{8\pi},
$$

(4.31)

where we have used the expression for the entropy density (4.4) as well as the definition (4.19). Using (4.9) to extract the Green’s function from $\mathcal{F}^r$ then gives

$$
G_{12,12}^R(\omega, 0) \approx -\frac{i\omega s}{4\pi},
$$

(4.32)

at least in the low frequency limit $\omega \to 0$. This is the result claimed in (4.3), giving rise to the universal ratio of shear viscosity to entropy density (4.5).

4.2 The sound speed

We conclude this section by computing the speed of sound in the SCFT$^{p,q}$ plasma. Using the fact that the thermodynamic potential is

$$
\Omega = -PV_3
$$

(4.33)

where $P$ is the pressure, we find from (3.21) and (3.17) that

$$
P = \frac{1}{3} \frac{E}{V_3},
$$

(4.34)

which further implies that the speed of sound is

$$
c_s^2 = \frac{1}{V_3} \frac{\partial P}{\partial E} = \frac{1}{3},
$$

(4.35)

independent of the value for the chemical potential.
5 Conclusion

In the first half of this paper, we have derived explicit Kaluza-Klein reduction ansätze of IIB on $T^{1,1} \equiv Y^{1,0}$ and $Y^{p,q}$, yielding in both cases minimal $\mathcal{N} = 2$ gauged supergravity in five dimensions. Although the Kaluza-Klein spectra of these reductions include additional vector multiplets in the massless sector ($SU(2) \times SU(2)$ for $T^{1,1}$ or $SU(2) \times U(1)$ for generic $Y^{p,q}$), these vectors cannot be retained in a consistent truncation [29]. In fact, this inconsistency even precludes the retention of the vectors in the $U(1)^2$ subgroups of the above groups. As a result, we see that it is in fact not possible to realize the $\mathcal{N} = 2$ STU model from $T^{1,1}$ or $Y^{p,q}$ reduction, even though this attractive possibility is otherwise suggested from the linearized Kaluza-Klein analysis.

Our main result is a demonstration that the shear viscosity of SCFT$^{p,q}$ plasma with nonzero $U(1)_R$ symmetry charge chemical potential is universally related to the entropy density (1.1). For $p = q$, a cone over $Y^{p,p}$ is a $\mathbb{Z}_{2p}$ orbifold of $\mathbb{C}^3$, and hence the dual superconformal quiver plasma is just a $\mathbb{Z}_{2p}$ orbifold of the $\mathcal{N} = 4$ Yang-Mills theory. It is probably not surprising that the orbifold quivers of $\mathcal{N} = 4$ Yang-Mills plasma have a universal shear viscosity with nonzero chemical potential, much like the parent gauge theory [13–16]. What is rather unexpected, however, is that the universality (1.1) is also true for the $Y^{1,0} \equiv T^{1,1}$ superconformal gauge quiver, which arises as a nontrivial superconformal infrared fixed point of the renormalization group flow from the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ supersymmetric Yang-Mills in the ultraviolet [33]. Thus it is natural to conjecture that the ratio $\eta/s$ is a constant along the renormalization group flow, and as such, must be true for any strongly coupled gauge theory plasma with nonzero chemical potential which allows for a (gauged) supergravity dual. Needless to say, it would be very interesting to prove this conjecture, and if it is true, understand its applications for the charged black holes in string theory. In particular, one should try to understand the shear viscosity of the gauge theory plasma with a finite chemical potential but with $c_s \neq 1/\sqrt{3}$. An example of exactly such a model would be the cascading gauge theory plasma [37–40].

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A Solution for $F_\omega$

As is clear from (4.30), what we really need for the computation of the shear viscosity is not $F_\omega$, but rather it’s derivative. We thus introduce

$$F(x) = -\frac{1}{x} + \frac{dF_\omega(x)}{dx}. \quad (A.1)$$

Given the analytical solution for $F(x)$, $F_\omega$ can be found by integrating (A.1)

$$F_\omega = -\int_x^1 dy F(y) + \ln x. \quad (A.2)$$

We now note that nonsingularity of $F_\omega$ at the horizon $x \to 0_+$ implies that

$$F = \frac{1}{x} + O(1). \quad (A.3)$$

Using the definition (A.1), one finds from the second equation in (4.20)

$$0 = \frac{dF}{dx} - \frac{18c_2^6 - c_2^5 Q^2 + 6xc_2c_2' Q^2 + 12x^2 (c_2')^2 Q^2}{xc_2'(Q^2 - 18c_2')^2} F. \quad (A.4)$$

In terms of the $r$ coordinate [see (4.13)] (A.4) takes form

$$0 = \frac{dF}{dr} - \frac{(5Q^4 + 18r^6 Q^2 - 54Q^2 \mu r^2 + 108\mu^2 r^4)}{r(Q^2 - 6\mu r^2)(9r^6 + Q^2 - 9\mu r^2)} F, \quad (A.5)$$

where we used the explicit background solution (3.3). A general solution of (A.5) takes the form

$$F = \frac{Cr^5}{(9r^6 + Q^2 - 9\mu r^2)^{1/2}(Q^2 - 6\mu r^2)}, \quad (A.6)$$

where $C$ is an arbitrary integration constant. Using $r = r(x) = c_2(x)$ given by (4.14), the boundary condition (A.2) determines

$$C = \frac{18r_+^6 - Q^2}{r_+^2} = 18\pi T r_+^3, \quad (A.7)$$

during the final stages of this work. JTL would like to thank the National Center for Theoretical Sciences (Taiwan) and the National Taiwan University Department of Physics for hospitality. AB research at Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MEDT. AB gratefully acknowledges further support by an NSERC Discovery grant. JTL is supported in party by the US Department of Energy under grant DE-FG02-95ER40899.
where we used the expression for the temperature (3.6) and (3.5). Using (A.6) and (A.7), we can finally evaluate

\[
\lim_{x \to 1^-} F(x) = \lim_{r \to \infty} F(r) = -\frac{C}{18\mu} = -\frac{\pi T r_+^3}{\mu}.
\]

(A.8)

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