ABSTRACT. We study the problem of exact support recovery: given an (unknown) vector \( \theta \in \{-1,0,1\}^D \), we are given access to the noisy measurement
\[
y = X\theta + \omega,
\]
where \( X \in \mathbb{R}^{N \times D} \) is a (known) Gaussian matrix and the noise \( \omega \in \mathbb{R}^N \) is an (unknown) Gaussian vector. How small can \( N \) be and still reliably recover the support of \( \theta \)? We present RAWLS (Randomly Aggregated Unweighted Least Squares Support Recovery): the main idea is to take random subsets of the \( N \) equations, perform a least squares recovery over this reduced bit of information and then average over many random subsets. We show that the proposed procedure can provably recover an approximation of \( \theta \) and demonstrate its use in support recovery through numerical examples.

1. Introduction

1.1. Introduction. We study the problem of support recovery of high-dimensional sparse signals based on a small number of noisy observations. Specifically, let \( \theta \in \mathbb{R}^D \) be an unknown signal which we assume, for simplicity of exposition, to have each entry be either \( \pm 1 \) or 0, i.e. \( \theta \in \{-1,0,1\}^D \). Given a known measurement or design matrix \( X \in \mathbb{R}^{N \times D} \), we assume the \( N \) dimensional observation vector \( y \) is
\[
y = X\theta + \omega,
\]
where \( \omega \in \mathbb{R}^N \) is an unknown additive (Gaussian) noise vector. We want to recover the support of \( \theta \) with \( N \) as small as possible. We can assume that the number of nonzero entries \( k = \|\theta\|_0 \) is known and much smaller than \( D \).

Figure 1. We try to recover the support of \( \theta \) from the observations \( X \) and \( y \), where \( y = X\theta + \omega \). The (known) matrix \( X \) is a Gaussian random matrix, so is the (unknown) noise \( \omega \), we try to recover the support of \( \theta \) with few measurements.
The problem of support recovery, also termed feature selection, plays an important role in machine learning, signal processing, bioinformatics, and high-dimensional statistics. In some applications, identifying the support leads to direct benefits such as reduction of memory and computational costs [5] or identification of cancer risk genes [10]. In other tasks, such as image denoising [8], the coefficients \( \theta \) are of interest; based on the recovered support these could be estimated using least squares.

1.2. Existing results. In the regime \( N < D \), the problem is under-determined: we have less equations \( N \) than variables \( D \) and, moreover, the observations are contaminated by additive noise \( \omega \). However, in this setting sparsity would be a useful assumption and it would be natural to find \( \theta \) by minimizing

\[
\|y - X\theta\|_2^2 \quad \text{s.t.} \quad \|\theta\|_0 \leq k.
\]

Since optimizing over this equation is intractable, several authors have replaced the \( \ell_0 \) norm by the \( \ell_1 \), this leads to the well-known Least Absolute Shrinkage and Selection Operator (LASSO) [18]. The LASSO, typically formulated using a regularized version of the problem, enjoys efficient optimization schemes [14, 15] due to its convex nature. [20] showed that exact support recovery using the LASSO can occur with probability one if \( N > 2k \log(D - k) \). Iterative Support Detection (ISD) [21] improves upon this results by iterating over the following two steps: first estimating the support as in LASSO, then defining a refined \( \ell_1 \) penalty which is applied to the complement of the current support estimate. Other iterative methods include the iteratively reweighted least squares (IRLS) [6] and the iteratively reweighted \( \ell_1 \) minimization (IRL1) [4]. The problem has also been addressed using greedy methods such as Orthogonal Matching Pursuit (OMP) [19] and its extensions [7, 13], or non convex schemes such as Trimmed LASSO (TL) [2] or smoothly clipped absolute deviation (SCAD) [9]. The importance of the problem has made it quite impossible to give an accurate, complete summary of the literature: we refer to the surveys [1, 3, 11, 12].

2. The Idea and the Main Result

2.1. The Idea. Our basic idea is quite simple: to find \( \theta \) we will perform a least squares projection. It is easily seen that this is a bad idea since

\[
x^* = \arg \min_{x \in \mathbb{R}^d} \|Xx - y\|
\]

tends to require a fairly large number of queries \( N \) to stably reconstruct \( \theta \). The proposed scheme is based on the following observation: instead of running least squares on the full set of equations, we can use only a random subset of the equations. The underlying idea behind RAWLS (Randomly Aggregated Unweighted Least Squares Support Recovery) is that none of the equations are distinguished: taking merely a subset of them amounts to a loss of information but enables us to get a particularly unique point of view. However, since no particular subset of the equations is distinguished over any other subset, we average over a number of randomly selected subsets. Our analysis shows that this is indeed advantageous: while applying least squares using less equations leads to errors from the lack of

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1. ‘The natural distribution is neither just nor unjust; nor is it unjust that persons are born into society at some particular position. These are simply natural facts. What is just and unjust is the way that institutions deal with these facts.’ (John Rawls, ‘A Theory of Justice’ [16]).
information, these errors cancel (to some degree) when averaged. More precisely, let \( A \subset \{1, \ldots, N\} \), we define \( X_A \) to be the restriction of \( X \) onto the rows whose index is in the set \( A \) and likewise for \( y_A \). We then proceed to find

\[
x_A^* = \arg \min_{x \in \mathbb{R}^D} \| X_A x - y_A \|.
\]

We average this result over many subsets \( (A_i)_{i=1}^m \) which we assume, for some fixed \( n < \min(N, D) \), to be taken uniformly at random from all \( n \)-element subsets of \( \{1, 2, \ldots, N\} \) and use this as our estimate for a rescaling of \( \theta \). We hope that

\[
\frac{n}{D} \theta \sim \frac{1}{m} \sum_{i=1}^m x_{A_i}^*.
\]

Figure 2. The reconstructed vector is much larger on the support of \( \theta \) than off the support of \( \theta \) and correctly identifies its sign.

An example (see Fig. 2) is as follows: let us define \( \theta \in \mathbb{R}^{64} \) by setting the first \( k = 16 \) entries to be \( \pm 1 \) (randomly) and the rest to be 0. We take a random Gaussian matrix \( X \in \mathbb{R}^{64 \times 80} \), take subsets of size \( n = 58 \) equations and average the least-square recovery over \( m = 100 \) random choices of these 58 equations. We observe that the reconstructed vector is much larger on the actual support than it is off the support; moreover, it correctly identifies the sign of the entry of \( \theta \).

Figure 3. Reconstructing a noisy vector in \( \mathbb{R}^D, D = 64 \) (supported on the first 16 coordinates) using \( N = 30 \) equations (projected on \( n = \) 18–dimensional subsets) with RAWLS.
Once we go down to a smaller number of equations $N$, something remarkable happens. For simplicity of exposition, we consider the same problem as above (reconstruction of a vector in $D = 64$ dimensions) except now we only observe $N = 30$ equations and we average over random subsets of these equations of size $n = 18$. We emphasize that this quite the extreme setting; we are operating with very little information. This is reflected in the reconstructed vector (see Fig. 3): it is certainly not the case that the largest $k = 16$ entries (by absolute value) correspond to the support of $\theta$. However, what we observe in this setting is the largest entry is indeed located on the support of $\theta$: for this particular choice of parameters $(D, N, n) = (64, 30, 18)$, this happens in $\sim 90\%$ of all cases. This motivated our RAWLS-based peeling algorithm discussed in §3, where we iteratively remove the coordinate corresponding to the largest reconstructed vector. We note that correctly identifying the first coordinate is the most difficult task; after that we have reduced the problem by decreasing the size of the support, one less dimension $D \to D - 1$, and the same number of equations $N$. This is an easier problem.

2.2. The Result. We can show that this yields provably good results. Before formally stating the result, we will quickly outline its meaning. Instead of trying to reconstruct the vector $\theta$, we will try to reconstruct a rescaled version of it: we will try to reconstruct $(n/D)\theta$ via an average over random projections

$$\frac{1}{m} \sum_{i=1}^{m} \pi_{A_i}\theta,$$

where $\pi_{A_i}$ denotes the projection onto the subspace $A_i$, and the $A_i$ are, by an abuse of notation, subspaces of size $n$ chosen uniformly at random (subspaces spanned by the rows of $X$ indexed by $A_i$). However, we do not have access to $\theta \in \mathbb{R}^D$, we only have access to $y = X\theta + \omega$. Instead of taking a least squares projection of $y$, we will use the least squares projections of $y_{A_i}$ for random subsets of the equations in the hope that this approximately recovers $\theta$

$$\frac{1}{m} \sum_{i=1}^{m} x^*_{A_i} \sim \frac{n}{D}\theta.$$

**Theorem.** Let $\theta \in \mathbb{R}^D$ be an arbitrary vector. Then, by projecting onto subsets of $n < 0.9 \cdot D$ equations of the $N$ equations given by $y = X\theta + \omega$, we have

$$\mathbb{E}_{\mathbf{X},\omega} \left\| \frac{1}{m} \sum_{i=1}^{m} \pi_{A_i}\theta - \frac{1}{m} \sum_{i=1}^{m} x^*_{A_i} \right\|_{\ell^2} \lesssim \frac{n}{\sqrt{N}\sqrt{D-2}} + \frac{n}{D}.$$

Several remarks are in order.

(1) The statement is independent of $\theta$. In particular, there is no underlying assumption about the structure of $\theta$ that is being used (and $\theta$ need not be sparse). We also observe that the size of $\theta$ does not appear on the right-hand side. This respects the problem setup where instead of $X\theta$ we are given the (additive) noisy version $X\theta + \omega$, where $\omega$ is a standard Gaussian vector $\omega_i \sim \mathcal{N}(0,1)$.

(2) In the setting $n \leq N \ll D$ our analysis is accurate down to constants: in particular, both quantities on the right-hand side are asymptotically correct (in the sense of having the correct constant, 1, in front) if the scales separate more and more (see Fig. 4 and Fig. 5 and the Remark in §4.2).
(3) The randomness in the choice of the $A_i$ is not at all necessary. In fact, the proof suggests that one could simply pick completely deterministic subsets of the $N$ equations as long as none of the individual dimensions are featured too prominently and all are represented roughly an equal number of times in the projections. This is also substantiated by numerical evidence. This poses the question of whether there are ‘good’ deterministic choices of subsets or whether there is a natural weight one could assign to the outcome resulting from each subset of equations (some ‘measure of reliability’).

(4) The result suggests that picking $n$ smaller leads to a smaller error. However, it also leads to a smaller projection. We can compensate for that by inserting the appropriate scaling in our result from which we obtain

$$
\mathbb{E}_{X,\omega} \left\| \frac{1}{m} \frac{D}{n} \sum_{i=1}^{m} \pi_{A_i} \theta - \frac{1}{m} \frac{D}{n} \sum_{i=1}^{m} x^*_{A_i} \right\|_2 \lesssim \frac{\sqrt{D}}{\sqrt{N}} + 1.
$$

This shows that there is some flexibility in the choice of $n$. However, in practice we have found that $n = 0.6 \min(N, D)$ seems to be particularly suited (though not very different from, say, $n = 0.5 \min(N, D)$). The precise role of $n$ could be an interesting object for further study.

We conclude by showing Theorem 1 in a simple example. As mentioned above, the terms in the upper bound (without the implicit constant and constant 1 instead) correspond to the sharp asymptotic limiting case where $n \leq N \ll D$. We show the case where $D = 10000$, $N = 100$ and $1 \leq n \leq 100$. For each value of $n$, we sample over $m = 20$ random subsets of size $n$ of the $N$ equations. As for the vector $\theta$, it does not actually play a role, we chose it to be a Gaussian vector in $\mathbb{R}^D$. We observe that the prediction is quite accurate (and the proof explains why this would be the case – various quantities start concentrating tightly around their expectation).

**Figure 4.** The error bounds in Theorem 1 (orange; ignoring the implicit constant) compared to the actual error (blue) for $D = 10000$, $N = 100$ and $1 \leq n \leq 100$. 

We also quickly illustrate that the restriction $n < 0.9 \cdot D$ is not just an artifact of the proof but, in fact, necessary (this also explains why RAWLS is better at recovering $\theta$ than an application of least squares to the full set of equations). We consider $\theta$ to be a unit vector (obtained from normalizing an instance of a Gaussian vector) in $D = 200$ dimensions. We take $N = 200$ equations and see what happens for $1 \leq n \leq 195$ (see Figure 5). What we observe is that the theoretical error bound (with the implicit constant assumed to be 1) nicely dominates the error until $n$ starts getting very close to $D$ (we plot it for $1 \leq n \leq 195 < 200 = D$). We see that the error starts exceeding the size of the vector by many orders of magnitude. The proof will explain this as a degeneracy of the smallest singular value of a rectangular Gaussian matrix which becomes approximately square.

![Figure 5. The error bounds in Theorem 1 (orange; ignoring the implicit constant) compared to the actual error (blue) for $D = 200 = N$ and $1 \leq n \leq 195$.](image)

The error observed around $n = 195 \sim N = D$ demonstrates why least squares using the full set of equations does not work; we obtain similar results also for $N \ll D$, the averaging has a natural stabilizing effect. We refer to the Remark in §4.2. for a prediction for what one would expect the error to look like when, say, $n = 0.99D$.

2.3. Open Problems. Theorem 1 raises a lot of open questions. Is there a particularly natural choice of subspaces on which to project? We are investigating the case of random projections but the proof does not seem to require this; are there natural ‘adapted’ subspaces that one can derive from a given matrix $X$?

We conclude with a particularly interesting question. We recall that the random projections reduce the size of the resulting vector. We can compensate for that by inserting the appropriate scaling in our result from which we obtain

$$
\mathbb{E}_{X, \omega} \left\| \frac{1}{m} \frac{D}{n} \sum_{i=1}^{m} \pi_{A_i} \theta - \frac{1}{m} \frac{D}{n} \sum_{i=1}^{m} x_{A_i}^* \right\|_{\ell^2} \approx \theta \left( \frac{\sqrt{D}}{\sqrt{N}} + 1 \right).
$$

In the case where we assume $\theta \in \{-1, 0, 1\}^D$, we want to make sure that we are properly able to distinguish two different vectors of that type and this can then be
seen to require $N \sim D$ (not entirely surprising, we are not making any assumptions on the sparsity of $\theta$). However, a more refined approach is conceivable: ultimately, we are using the entries of our approximating vector to derive statements about the support. As such, the $\ell^2$ is perhaps not the only interesting quantity and estimates on $\ell^\infty$ would be quite desirable. In particular, what we observe in practice (and what motivated the peeling algorithm) is that very large entries (either very large or very small) in the recovered approximation is a good indicator for $\theta$ having support in that coordinate. This simple observations forms the basis of the algorithm discussed in §3. It would be interesting to have results in that direction.

We also emphasize that the idea underlying RAWLS might have many other applications: it is ultimately an $\ell^2$-based concept and as such many natural variations seem conceivable. One such applications, a nonlinear variant that is shown to work particularly well in the support recovery problem, is discussed in the next section. A second question, outside the scope of this paper, is whether other methods used for support recovery could conceivable be merged with our philosophy: running it on random subsets of the equations and hoping that the averaging effects compensates for the loss of information.

3. Support Recovery with RAWLS

3.1. The Idea. If it is indeed the case that

$$\frac{1}{m} \sum_{i=1}^{m} \pi_{A_i} \theta \approx \frac{n}{D} \theta + \text{some error}$$

and if the error is nicely random (as one usually expects in these cases), then the largest (or smallest) entries of the vector should be contained in the support of $\theta$.

In a more elementary formulation, if we are given $v \in \{-1, 0, 1\}^D$ (such that $\|v\|_0$ is not too small compared to $D$) and add a random Gaussian vector $g$ to it, then the largest (absolute) entry of $v + g$ will be attained (with high probability) on the support of $v$. This is a simple consequence of the rapid decay of the Gaussian, and motivates the Algorithm proposed in the following section.

3.2. Peeling with RAWLS.

(1) Compute the approximation

$$x^* = \frac{1}{m} \sum_{i=1}^{m} x_{A_i}^*.$$ 

(2) Find the largest absolute value of $x^*$. If it is positive, then we assume that $\theta = 1$ in that coordinate; if negative, we assume $\theta = -1$ in that coordinate.

(3) Having gained some knowledge, remove the corresponding column from the matrix $X$ and update the right-hand side $y$.

(4) Return to (1) until $k$ entries of $\theta$ are estimated.

This algorithm is thus a fairly simple greedy algorithm that identifies likely candidates for the support of $\theta$ by looking for particularly large entries in the RAWLS-reconstruction of $\theta$. We emphasize again that for this type of iterative algorithm, each step is more difficult than the next one: having found a correct entry, the problem is reduced to a simpler problem $D \rightarrow D - 1$ while maintaining the same
Figure 6. Numerical evaluation for the probability of exact support recovery vs. number of measurements $N$. We compare Peeling with RAWLS to several baselines for $\sigma = 0.5$.

Figure 7. Numerical evaluation for the probability of exact support recovery vs. number of measurements $N$. We compare RAWLS to several baselines for $\sigma = 1$. 

amount of information $N \to N$. We point out that the method, just as other methods, should also be highly suitable for partial recovery: finding a set of $k$ entries that has large overlap with the ground truth, we do not pursue this here. We do not have any theoretical guarantees for the success rate of the peeling algorithm at this point and consider this to be an interesting problem. Perhaps the most interesting question at this stage is whether there are other implementations of these underlying ideas that can yield even better results.
3.3. **Numerical Performance.** In this section we support the effectiveness of RAWLS using numerical simulations. We focus on the task of exact support recovery using a random Gaussian design matrix $X$ and random additive Gaussian noise $\omega$. We test this on a $k = 10$-sparse vector in $D = 64$ dimensions. As baselines, we compare the method to LASSO [18], IRL1 [4], TL [2], OMP [19] and STG [22]. To evaluate the probability of exact support recovery we run each method 100 times and count the portion successful estimations. A successful estimation of the support is counted if $S(\theta) = S(x^*)$, where $S(\theta) := \{i \in 1, ..., D|\theta_i \neq 0\}$. For LASSO the regularization parameter $\lambda$ is set as in [20]. To improve the stability of LASSO, after each run we select the top $k$ coefficients of $\theta$ as the estimated support.

**Figure 8.** Numerical evaluation for the probability of exact support recovery vs. number of measurements $N$. We compare Peeling with RAWLS to several baselines for $\sigma = 1.5$.

4. **Proof of the Theorem**

4.1. **Setup.** We first recall the setting. Let $\theta \in \{-1, 0, 1\}^D$ be a sparse vector with support $\|\theta\|_0 = k$ and let $X \in \mathbb{R}^{N \times D}$ be a matrix all of whose entries are distributed following i.i.d. $\mathcal{N}(0, 1)$ random variables. We will also use the notation $g = (g_i)_{i=1}^N$ to denote the Gaussian vectors in $\mathbb{R}^N$ dimensions that are forming the rows. We are given

$$y = X\theta + \omega,$$

where each entry of $\omega$ is i.i.d. normally distributed $\omega_i \sim \mathcal{N}(0, 1)$. We try to understand how our algorithm performs on this data. Let now $A \subset \{1, \ldots, N\}$ be a random subset of size $|A| = n$. We are trying to understand the least squares solution

$$x^*_A = \arg \min_{x \in \mathbb{R}^D} \|X_Ax - y_A\|^2,$$

where $X_A$ denotes the restrictions onto the rows of $X$ indexed by $A$ and likewise for $y_A$. If $n \leq D$, then the system has more variables than equations and always has a solution: we are interested in the solution with the smallest $\ell^2$-norm and will denote it by $x^*_A$. 
4.2. A Single Projection. The purpose of this statement is to provide the analysis of a single projection onto a random subspace spanned by a random subset of the rows. The main insight is that this projection can be approximately deconstructed into the projection of the ground truth, a highly structured Gaussian error on top of that and a relatively small error term.

Lemma. Let \( \theta \in \mathbb{R}^D \) be fixed, let \( X \in \mathbb{R}^{N \times D} \) be a random Gaussian matrix and let \( A \subseteq \{1, 2, \ldots, N\} \) be a randomly chosen subset of size \( |A| = n < 0.9 \cdot D \). Then the orthogonal (noisy) projection of \( \theta \) onto the subspace spanned by the rows indexed by \( A \) (given by \( y = X\theta + \omega \)) satisfies

\[
x_A = \pi_A \theta + \left( \sum_{a \in A} \frac{g_a}{\|g_a\|^2} \omega_a \right) + e,
\]

where the vector \( e \) satisfies, with high likelihood,

\[
E_{X,\omega} \|e\| \lesssim \frac{n}{D}.
\]

The purpose of this Lemma is to show that the (noisy) projection of \( \theta \) onto a random subspace (this is one interpretation of \( y = X\theta + \omega \)) leads to substantial distortions; however, these distortions are not arbitrary and follow a fairly regular pattern up to a small error. The second term is not necessarily all that small, however, its explicit form will allow us to show that averaging it over multiple subspaces will further decrease the size. We emphasize that in the case \( n \ll D \) our estimate is actually rather sharp and we expect \( \|e\| \sim n/D \) with tight concentration and a small error (this could be made precise in the regime where \( D/n \) becomes large).

The proof makes use of the following basic fact in linear algebra that we recall for the convenience of the reader: let \( (g_a)_{a=1}^n \) be \( n \) vectors in \( \mathbb{R}^D \) with \( D > n \) and let \( v \in \text{span}\{g_1, \ldots, g_n\} \). Then

\[
\sigma_{\text{min}}(G)^2 \|v\|^2 \leq \sum_{a=1}^n |\langle g_a, v \rangle|^2 \leq \sigma_{\text{max}}(G)^2 \|v\|^2,
\]

where \( \sigma \) denotes the singular values of the matrix \( G \) obtained by collecting the \( (g_a) \) as column vectors (or, alternatively, the largest and smallest eigenvectors of \( G^T G \)). This follows easily from observing that

\[
\sum_{a=1}^n |\langle g_a, v \rangle|^2 = \|G^T v\|^2.
\]

This is well-known in frame theory: the frame constants for finite-dimensional problems are given by the singular values of the associated matrix.

Proof of the Lemma. We will use \( x_A \) to denote the \( \ell^2 \)-smallest vector satisfying

\[
x_A = \arg \min_x \|X_A x - y_A\|.
\]

This solutions is contained in the vector space \( V = \text{span}\{g_a : a \in A\} \) (if \( x \) had a component that was orthogonal to these rows, then it would not have any effect in the matrix multiplication \( X_A x \) and removing that component would result in a smaller \( \ell^2 \)-norm). Since the number of variables, \( D \), is larger than the number of equations, \( n \), and \( X \) is Gaussian we know that the minimum is 0 with likelihood 1.
Thus $X_A x_A = y_A = X_A \theta + \omega_A$. We will analyze this equation for a single row. For any $a \in A$,
\[
\langle g_a, x_A \rangle = \langle g_a, \theta \rangle + \omega_a = \left\langle g_a, \theta + \frac{\omega_a}{\|g_a\|^2} g_a \right\rangle.
\]
We will use this equation for all $a \in A$. By definition of the orthogonal projection, we have, for all $a \in A$,
\[
\langle g_a, \theta \rangle = \langle g_a, \pi_A \theta \rangle,
\]
and thus the identity
\[
\langle g_a, x_A \rangle = \left\langle g_a, \pi_A \theta + \frac{\omega_a}{\|g_a\|^2} g_a \right\rangle.
\]
This is an interesting way of interpreting the introduction of additive noise: the error that we are given makes it seem as if the inner product was not with $\pi_A \theta$ but instead with $\pi_A \theta$ and a small additional multiple of $g_a$. In practice, if $n \ll D$, then the Gaussian vectors are almost orthogonal and almost form an orthogonal basis of the space that they span. This motivates the ansatz
\[
x_A = \pi_A \theta + \sum_{a \in A} \frac{g_a}{\|g_a\|^2} \omega_a + e,
\]
where $\pi_A$ is the orthogonal projection onto the vector space $V = \text{span} \{g_a : a \in A\}$ and $e \in \mathbb{R}^D$ is an error term whose size we try to investigate. We plug in our ansatz and obtain, for all $a \in A$,
\[
\left\langle g_a, e + \sum_{a \neq i \in A} \frac{g_i}{\|g_i\|^2} \omega_i \right\rangle = 0.
\]
Alternatively, we obtain
\[
\langle g_a, e \rangle = -\left\langle g_a, \sum_{a \neq i \in A} \frac{g_i}{\|g_i\|^2} \omega_i \right\rangle.
\]
We emphasize that, since the $g_a$ span $V$ with probability 1, these $n$ equations uniquely identify $e \in V$ with probability 1. We first try to understand the quantity on the right-hand side. We have
\[
\left\langle g_a, \sum_{a \neq i \in A} \frac{g_i}{\|g_i\|^2} \omega_i \right\rangle = \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle}{\|g_i\|^2} \omega_i.
\]
The inner product of two random Gaussians is a random variable at scale $\langle g_i, g_a \rangle \sim \sqrt{D}$, the size of an individual Gaussian vector is at scale $\mathbb{E} \|g_i\|^2 \sim D + O(\sqrt{D})$ with high likelihood. The $\omega_i \sim \mathcal{N}(0,1)$ have an additional randomization effect. The sum runs over $n - 1$ elements. Altogether, we expect the quantity to be a random variable at scale
\[
\left| \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle}{\|g_i\|^2} \omega_i \right| \sim \frac{\sqrt{n}}{\sqrt{D}}.
\]
An explicit computation shows that
\[ E \left| \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle^2}{\|g_i\|^4} \omega_i \right|^2 = E \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle^2}{\|g_i\|^4} \omega_i^2 + E \sum_{a \neq i \neq i' \in A} \frac{\langle g_i, g_a \rangle \langle g_i, g_{i'} \rangle}{\|g_i\|^2 \|g_{i'}\|^2} \omega_i \omega_{i'}. \]

The second expectation is clearly 0 since \( \omega_i \sim \mathcal{N}(0, 1) \) and these are independent of each other. It remains to evaluate the first expectation. Since \( E\|g_i\|^2 \sim D + \mathcal{O}(\sqrt{D}) \) with exponentially decaying tail, we get
\[ E X, \omega \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle^2}{\|g_i\|^4} \omega_i^2 = E X \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle^2}{\|g_i\|^4}. \]

This sum can be decoupled into two parts
\[ E X \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle^2}{\|g_i\|^4} = E X \sum_{a \neq i \in A} \left( \frac{g_i}{\|g_i\|} \right)^2 \frac{1}{\|g_i\|^2}. \]

We observe that \( g_i/\|g_i\| \) is a random vector on the unit sphere (this follows from the rotational symmetry of Gaussian vectors); as such, it is completely independent of its length \( \|g_i\| \) allowing us to treat both quantities as independent random variables. However, the first term is simply an inner product of a Gaussian vector against a vector of length 1 and thus
\[ \left\langle \frac{g_i}{\|g_i\|}, g_a \right\rangle \text{ is a Gaussian variable and } E \left\langle \frac{g_i}{\|g_i\|}, g_a \right\rangle^2 = 1. \]

The remaining quantity is the mean of an inverse \( \chi \)-distribution which is \( 1/(D-2) \) for \( D \geq 3 \) and thus
\[ E X \sum_{a \neq i \in A} \frac{\langle g_i, g_a \rangle^2}{\|g_i\|^4} = E X \sum_{a \neq i \in A} \frac{1}{\|g_i\|^2} = \frac{n-1}{D-2} \lesssim \frac{n}{D}. \]

Summing up, we obtain
\[ E \sum_{a \in A} |\langle g_a, e \rangle|^2 \lesssim \frac{n^2}{D}. \]

However, since \( e \in \text{span} \{g_a : a \in A\} \), we have the basic inequality
\[ \sigma^2_{\min} \|e\|^2 \leq \sum_{a \in A} |\langle g_a, e \rangle|^2 \leq \sigma^2_{\max} \|e\|^2. \]

The smallest singular value of a random rectangular Gaussian matrix was determined by Silverstein [17] who showed that we can expect, in the limit, that
\[ \sigma_{\min} \sim \sqrt{D} - \sqrt{n}. \]

Combining all these results shows that we expect, in the regime where \( d \) has a bounded gap from \( D \), say \( n \leq 0.9 \cdot D \), that
\[ \|e\| \lesssim \frac{n}{D}. \]

□
Remark. We observe that the first part of the argument is fairly tight, in particular, we expect
\[ E \sum_{a \in A} |(g_a, e)|^2 \sim \frac{n^2}{D} \]
with tight concentration. The second part of the argument is not precise down to constants but it becomes tight if we have \( n \ll D \). We observe that if \( n \ll D \), then we actually have \( \sigma_{\min} \sim \sigma_{\max} \) since the singular are expected to be in the interval \( [\sqrt{D} - \sqrt{n}, \sqrt{D} + \sqrt{n}] \). Since all the estimates we carried out are actually quite tightly concentrated, we thus expects, with a fair degree of accuracy,
\[ \|e\| \sim \frac{n}{\sqrt{D}}. \]
More precise, estimates are conceivable: if \( e \) is uniformly distributed across all singular vectors, then we could hope that
\[ \frac{1}{\|e\|^2} \sum_{a \in A} |(g_a, e)|^2 \sim Z^2, \]
where \( Z \) is the Marchenko-Pastur distribution modeling the singular values of the random matrix \( X \). When \( n \ll D \), then \( Z \sim \sqrt{D} \pm \sqrt{n} \sim \sqrt{D} \) and we recover the usual estimate. As soon as \( n \) starts approaching \( D \), the distribution of \( Z \) gets closer and closer to 0 and the inverse distribution \( \frac{1}{Z^2} \) spreads over many scales. However, in principle, if \( e \) is uniformly distributed over the singular vectors, then one could use this heuristic to predict the sharp constant to be expected when, for example \( n = 0.99 \cdot D \). Basic numerics seems to indicate that this is a reasonable assumption.

4.3. Multiple Projections. We now discuss the effect of averaging quantities like
\[ \sum_{a \in A} \frac{g_a}{\|g_a\|^2} \omega_a \]
over multiple randomly chosen sets \( A \).

Lemma. Let \( X \in \mathbb{R}^{N \times D} \) be a matrix with i.i.d. standard \( \mathcal{N}(0, 1) \) entries and let \( \omega \in \mathbb{R}^N \) be a random vector all of whose entries are i.i.d. \( \mathcal{N}(0, 1) \). Let \( A \subset \{1, \ldots, N\} \) denote a random set of size \( n \) (chosen uniformly at random among all \( n \)-element subsets of \( A \)). Then
\[ E_{X, \omega} \lim_{\ell \to \infty} \left\| \frac{1}{\ell} \sum_{i=1}^{\ell} \sum_{a \in A_i} \frac{g_a}{\|g_a\|^2} \omega_a \right\| \leq \frac{n}{N \sqrt{D - 2}}. \]

Proof. We observe that the vectors \( g_a \) are, albeit Gaussian random vectors, fixed once given and so are the \( \omega_a \). Thus, the law of large numbers implies that averaging over many randomly chosen subsets \( A \subset \{1, 2, \ldots, N\} \) of size \( n \) results, ultimately, in each coordinate being picked the same number of times and thus
\[ \lim_{\ell \to \infty} \frac{1}{\ell} \sum_{i=1}^{\ell} \sum_{a \in A_i} \frac{g_a}{\|g_a\|^2} \omega_a = \frac{n}{N} \sum_{a=1}^{N} \frac{g_a}{\|g_a\|^2} \omega_a. \]

We have
\[ \frac{n}{N} \sum_{a=1}^{N} \frac{g_a}{\|g_a\|^2} \omega_a = \frac{n}{N} \sum_{a=1}^{N} \frac{g_a}{\|g_a\|} \omega_a. \]
We interpret this as follows: the vector $g_a/\|g_a\|$ is uniformly distributed over the unit sphere in $\mathbb{R}^D$ (a consequence of the radial symmetry of the Gaussian distribution), the vector $\omega^* = (\omega_a/\|g_a\|)_{a=1}^N$ is interpreted as a random vector. Again, as a consequence of the radial symmetry, the vector $g_a/\|g_a\|$ and the size $\|g_a\|$ can be interpreted as independent random variables. We compute

$$E_{X,\omega} \left\| \sum_{a=1}^N \frac{g_a}{\|g_a\|} \frac{\omega_a}{\|g_a\|} \right\|^2 = \sum_{a_1,a_2=1}^N E_{X,\omega} \left( \frac{g_{a_1}}{\|g_{a_1}\|} \frac{\omega_{a_1}}{\|g_{a_1}\|}, \frac{g_{a_2}}{\|g_{a_2}\|} \frac{\omega_{a_2}}{\|g_{a_2}\|} \right)$$

$$= \sum_{a=1}^N E_{X,\omega} \frac{\omega_a^2}{\|g_a\|^2}$$

$$= \sum_{a=1}^N E_X \frac{1}{\|g_a\|^2} = E \frac{N}{\|g\|^2}.$$

This quantity is the mean of an inverse $\chi^2$-distribution which is $1/(D-2)$ for $D \geq 3$. Thus, using the Cauchy-Schwarz inequality, we get

$$E_{X,\omega} \frac{n}{N} \left\| \sum_{a=1}^N \frac{g_a}{\|g_a\|} \frac{\omega_a}{\|g_a\|} \right\| \leq \frac{n}{N} \frac{\sqrt{N}}{\sqrt{D-2}} = \frac{n}{\sqrt{N}\sqrt{D-2}}.$$

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