A GEOMETRIC REALIZATION OF THE $m$-CLUSTER CATEGORY OF TYPE $\tilde{A}$

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Abstract. We give a geometric realization of a subcategory of the $m$-cluster category $\mathcal{C}^m_{\tilde{A}}$ by using $(m + 2)$-angulations of an annulus with $p + q$ marked points. We also give a bijection between an equivalence class of $(m + 2)$-angulations and the mutation class of coloured quivers of type $\tilde{A}_{p,q}$.

Introduction

The cluster category was defined in [BMRRT] in general and in [CCS] for Dynkin Type $A$. Their motivation was to categorify the combinatorics of cluster algebras defined by Fomin and Zelevinsky [FZ1]. In [CCS] the authors defined a category with indecomposable objects the diagonals in a regular polygon. The geometric model was later extended to Dynkin type $D$ by Schiffler [S] and to Dynkin type $\tilde{A}$ by Brüstle and Zhang [BZ].

The $m$-cluster category (see for example [K, T, W, ZZ]) generalizes the cluster category. Baur and Marsh extended the geometric models of the cluster category of type $A$ [BM2] and $D$ [BM1] to $m$-cluster categories. For example, in the $A$-case, they consider $m$-diagonals of regular polygons. In this paper we will consider $m$-cluster categories of type $\tilde{A}$.

The results in this paper are a part of the author’s PhD thesis from December 2010 [To4].

1. $m$-CLUSTER CATEGORIES

In [BMRRT] the cluster category was defined as an orbit category of the derived category. Let $H = kQ$ be a finite dimensional hereditary algebra over an algebraically closed field $k$, where $Q$ is a quiver. The cluster category $\mathcal{C}_H$ is the orbit category $D^b(H)/\tau^{-1}[1]$, where $\tau$ is the Auslander-Reiten translation, $[1]$ is the shift functor and $D^b(H)$ the bounded derived category of $\text{mod } H$. We can also consider the orbit category $D^b(H)/\tau^{-1}[m]$, and this is the $m$-cluster category. The $m$-cluster category is a Krull-Schmidt category for all $m$, and it has an AR-translate $\tau$. From [K] we also know that it is a triangulated category for all $m$.

The $m$-cluster categories come equipped with a class of objects called $m$-cluster tilting objects. An $m$-cluster tilting object $T$ is an object with the property that $X$ is in $\text{add } T$ if and only if $\text{Ext}^i_{\mathcal{C}_H}(T, X) = 0$ for all $i \in \{1, 2, ..., m\}$. An object $X$ is called maximal $m$-rigid if it has the property that $X$ is in $\text{add } T$ if and only if $\text{Ext}^i_{\mathcal{C}_H}(T \oplus X, T \oplus X) = 0$ for all $i \in \{1, 2, ..., m\}$. A maximal $m$-rigid object is an $m$-cluster tilting object [W, ZZ], and an $m$-cluster tilting object has always $n$ non-isomorphic indecomposable summands [Z]. The algebra $\text{End}_{\mathcal{C}_H}(T)$ is called an $m$-cluster tilted algebra when $T$ is $m$-cluster tilting.

If $\bar{T}$ is an object in $\mathcal{C}_H^m$ with $n - 1$ non-isomorphic indecomposable direct summands, such that $\text{Ext}^i_{\mathcal{C}_H}(\bar{T}, T) = 0$ for $i \in \{1, 2, ..., m\}$, there exist exactly $m + 1$ non-isomorphic objects $T'$ (called complements) such that $\bar{T} \oplus T'$ is an $m$-cluster tilting object [W, ZZ]. The object $\bar{T}$ is called an almost complete $m$-cluster tilting
object. Let $T_k^{(c)}$, where $c \in \{0, 1, 2, ..., m\}$, be the complements of $\bar{T} = T/T_k$. Then we know from [BY] that the complements are connected by $m + 1$ exchange triangles

$$T_k^{(c)} \rightarrow B_k^{(c)} \rightarrow T_k^{(c+1)} \rightarrow,$$

where $B_k^{(c)}$ is in add $\bar{T}$.

2. Quiver mutation

Quiver mutation was defined by Fomin and Zelevinsky in their work with cluster algebras. Buan and Thomas extended quiver mutation to a class of coloured quivers to model mutation in $m$-cluster categories. Let $T$ be an $m$-cluster tilting object. In [BT] they associate to $T$ a coloured quiver $Q_T$ in the following way. There is a vertex in $Q_T$ for every indecomposable summand of $T$. The arrows have colours chosen from the set $\{0, 1, 2, ..., m\}$. If $T_i$ and $T_j$ are two indecomposable summands of $T$ corresponding to vertex $i$ and $j$ in $Q_T$, there are $r$ arrows from $i$ to $j$ of colour $c$, where $r$ is the multiplicity of $T_j$ in $B_i^{(c)}$.

They show that quivers obtained in this way have no loops. Also, if there is an arrow from $i$ to $j$ with colour $c$, then there is no arrow from $i$ to $j$ with colour $c' \neq c$. If there are $r$ arrows from $i$ to $j$ of colour $c$, then there are $r$ arrows from $j$ to $i$ of colour $m - c$.

Coloured quiver mutation keeps track of the exchange of indecomposable summands of $m$-cluster tilting objects. The mutation of $Q_T$ at vertex $j$ is defined as the quiver $\mu_j(Q_T)$ obtained as follows.

1. For each pair of arrows $\xymatrix{i \ar@/_1pc/[rr]_{(c)} & j \ar[l]_{(0)} & k}$, where $i \neq k$ and $c \in \{0, 1, ..., m\}$, add an arrow from $i$ to $k$ of colour $c$ and an arrow from $k$ to $i$ of colour $m - c$.
2. If there exist arrows of different colours from a vertex $i$ to a vertex $k$, cancel the same number of arrows of each colour until there are only arrows of the same colour from $i$ to $k$.
3. Add one to the colour of all arrows that goes into $j$, and subtract one from the colour of all arrows going out of $j$.

In [BT] they prove that if $T = \oplus_{i=1}^n T_i$ is an $m$-cluster tilting object in $\mathcal{C}_H^m$ and $T' = T/T_i \oplus T_j^{(1)}$ is an $m$-cluster tilting object where there is an exchange triangle $T_j \rightarrow B_j^{(0)} \rightarrow T_j^{(1)} \rightarrow$, then $Q_{T'} = \mu_j(Q_T)$. We note that this was already known for $m = 1$ [BM2]. In [ZZ] it was shown that any $m$-cluster tilting object can be reached from any other $m$-cluster tilting object via iterated mutation. And in [BT] the authors show that for an $m$-cluster category $\mathcal{C}_H^m$, where $H = kQ$, all quivers of $m$-cluster tilted algebras are given by repeated mutation of $Q$.

When $m = 1$, it was shown in [BR] that the mutation class of an acyclic quiver $Q$ is finite if and only if the underlying graph of $Q$ is either Dynkin, extended Dynkin or $Q$ has at most two vertices. This was generalized in [Lo2]. A coloured quiver $Q$ corresponding to an $m$-cluster tilting object, has finite mutation class if and only if $Q$ is mutation equivalent to a quiver $Q'$, where the quiver obtained from $Q'$ by removing all arrows with colour $\neq 0$ has underlying graph Dynkin or extended Dynkin, or it has at most two vertices, and there are only arrows of colour $0$ and $m$ in $Q'$.

3. Geometric descriptions of $m$-cluster categories

We will not go into detail about the various geometric descriptions of $m$-cluster categories, since we will do it in detail for $m$-cluster categories of type $\tilde{A}$. We refer to the papers [CCS] [S] [BM1] [BM2] [BZ].
In [CCS] the authors defined the cluster category of type $A_n$ by using regular polygons and diagonals between vertices on the border of the polygons. In [BM2] they generalized this to $m$-cluster categories. Baur and Marsh considered an $(nm+2)$-gon $P_{nm+2}$ and $m$-diagonals between vertices on the border of $P_{nm+2}$. An $m$-diagonal is a diagonal that divides $P_{nm+2}$ into two parts with number of vertices congruent to 2 modulo $m$. They defined an additive category where the indecomposable objects are the $m$-diagonals. The morphisms are spanned by certain elementary moves. The authors showed that this category is equivalent to the $m$-cluster category of type $A_n$. A set of $m$-diagonals that divides $P_{nm+2}$ into $(m+2)$-gons is called an $(m+2)$-angulation, and such a set always has $n$ elements. We have that an $(m+2)$-angulation $\Delta$ is an $m$-cluster tilting object in this category, and we can define mutation on $m$-diagonals in $\Delta$. Also, given an $(m+2)$-angulation $\Delta$, we can define a coloured quiver $Q_\Delta$ (see [BM2] [BT]), and mutation on $\Delta$ commutes with mutation on $Q_\Delta$. We mention that the Dynkin case $A$ has been considered in [BZ] for $m = 1$.

Given an $(m+2)$-angulation $\Delta$, there exist, as we mentioned above, a coloured quiver $Q_\Delta$. In [LX1] it was shown that there exist a bijection between the set of triangulations, where two triangulations are equivalent if they are rotations of each other, and the mutation class of quivers of Dynkin type $A$. This was generalized in [LX2] to $m$-coloured quivers and $(m+2)$-angulations of $P_{nm+2}$. A similar result was obtained for Dynkin type $D$ in [BT1]. In the second part of this paper, we will obtain a similar result for Dynkin type $A$.

4. $(m+2)$-ANGULATIONS OF $P_{p,q,m}$

Let $m \geq 1$, $p \geq 2$ and $q \geq 2$ be integers, and set $n = p + q$. Let $P_{p,q,m}$ be a regular $mp$-gon, with a regular $mq$-gon at its center, cutting a hole in the interior of the outer polygon. When $m = 1$ we just write $P_{p,q}$. Denote by $P_{p,q,m}^0$ the interior between the outer and inner polygon. Label the vertices on the outer polygon $O_0, O_1, \ldots, O_{mp-1}$ in the counterclockwise direction, and label the vertices of the inner polygon $I_0, I_1, \ldots, I_{mq-1}$, in the clockwise direction. See Figure 1. If one of the polygons has 2 vertices, we draw the polygon as a circle with two marked points. For simplicity, we always draw the polygons such that the vertices $O_0$ and $I_0$ are as close as possible.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{$P_{8,6}$ and $P_{5,2}$ for $m = 1$.}
\end{figure}

Let $\delta_{i,k}$ be the path in the counterclockwise direction from $O_i$ to $O_{i+k-1}$ along the border of the outer polygon, where $k$ is the number of vertices that $\delta_{i,k}$ runs through (including the start and end vertex). If $k = pm + 1$, the path runs around
the polygon exactly once. If \( k > pm + 1 \), the path crosses itself. Similarly, we denote by \( \gamma_{i,k} \) the path in the clockwise direction from \( I_i \) to \( I_{i+k-1} \), along the border of the inner polygon, where \( k \) is the number of vertices the path runs through. Of course, here we always compute modulo \( pm \) and \( qm \).

Now we consider paths of the following types.

- **Type 1:** A path in \( P_{p,q,m}^0 \) between a vertex on the outer polygon and a vertex on the inner polygon, i.e. a path between \( O_i \) and \( I_j \) for some \( i \) and \( j \).
- **Type 2:** A path \( \alpha \) from \( O_i \) to \( O_{i+k-1} \) in \( P_{p,q,m}^0 \), such that \( \alpha \) is homotopic to \( \delta_{i,k} \) for some \( k \geq 3 \).
- **Type 3:** A path \( \alpha \) from \( I_i \) to \( I_{i+k-1} \) in \( P_{p,q,m}^0 \), such that \( \alpha \) is homotopic to \( \gamma_{i,k} \) for some \( k \geq 3 \).

![Figure 2. Some examples of diagonals in \( P_{8,6} \).](image)

See Figure 2 for some examples. Note that the winding number of a path of any type can be greater than 1. Also, a path of Type 1 is homotopic to a path that never crosses itself.

Now, we define two paths to be equivalent if they start in the same vertex, end in the same vertex and they are homotopic. We call these equivalence classes *diagonals* in \( P_{p,q,m} \). Let \( O_{i,k} \) denote the diagonals homotopic to \( \delta_{i,k} \), and let \( I_{i,k} \) be the diagonals homotopic to \( \gamma_{i,k} \).

We define the *crossing number* of any two diagonals, \( \alpha \) and \( \beta \), to be

\[
\epsilon(\alpha, \beta) = \min\{|\alpha \cap \beta \cap P_{p,q,m}^0|\}.
\]

The crossing number of a diagonal \( \alpha \) is defined as the minimal number of times the diagonal crosses itself in the interior. The crossing number of any diagonal of Type 1 is always 0.

We say that two diagonals cross if the crossing number is not 0, and we say that a diagonal crosses itself if the crossing number of the diagonal is not 0. This enables us to define a triangulation, which is a maximal set of diagonals which do not cross. See Figure 3 for some examples of triangulations.

We have the following easy lemma

**Lemma 4.1.** In any triangulation of \( P_{p,q,m} \), there exist at least one diagonal of Type 1.

Using the lemma, we can prove that the number of diagonals in any triangulation is given by \( p + q \). More generally, see [FST] for an annulus of \( n_1 + n_2 \) marked points.

**Proposition 4.2.** Any triangulation of \( P_{p,q,m} \) consists of exactly \( p + q \) diagonals.

**Proof.** Let \( \Delta \) be a triangulation, and let \( \alpha \) be a diagonal of Type 1, which exist by Lemma 4.1, say from \( O_i \) to \( I_j \). See Figure 4 where we cut the polygon along \( \alpha \) and fold it out. We obtain a polygon with \( p + q + 2 \) vertices, and such a polygon can be triangulated by \( p + q - 1 \) diagonals by [FST] [CCS]. Then \( \Delta \) has \( p + q \) diagonals, counting the diagonal \( \alpha \). \( \square \)
Figure 3. Examples of triangulations of $P_{2,2}$ and $P_{3,3}$.

Figure 4. See proof of Lemma 4.2.

An $m$-diagonal in $P_{p,q,m}$ is a diagonal of the types above, but with the following restrictions:

- The $m$-diagonals of Type 2 (Type 3) are of the form $O_{i,km+2} (I_{i,km+2})$, where $k \geq 1$, for all $i$.

- If $\alpha$ is an $m$-diagonal of Type 1 between $O_i$ and $I_j$, then $i$ is congruent to $j$ modulo $m$.

We say that a set of $m$-diagonals cross if they intersect in the interior $P_{p,q,m}^0$ (i.e. their crossing number as diagonals is not 0). A set of non-crossing $m$-diagonals that divides $P_{p,q,m}$ into $(m + 2)$-gons is called an $(m + 2)$-angulation. When $m = 1$ this is a triangulation as described above.

We also have the following.

**Proposition 4.3.** Any $(m + 2)$-angulation of $P_{p,q,m}$ consists of exactly $p + q$ $m$-diagonals, and there exist at least one $m$-diagonal of Type 1.

See examples of $(m + 2)$-angulations in Figure 5.

Let $\alpha$ be a diagonal, and let $x$ be the point at the center of the inner (and hence outer) polygon. The winding number of $\alpha$ is an integer denoting how many times $\alpha$ travels around $x$. If $\alpha$ travels around $s$ times but not $s + 1$ times, we say that the winding number is $s$. We do not care about orientation, so the winding number is always $\geq 0$. 
5. The quiver corresponding to an \((m+2)\)-angulation

For an \((m+2)\)-angulation \(\Delta\) of \(P_{p,q,m}\), we define a corresponding coloured quiver \(Q_\Delta\) with \(p+q\) vertices in the following way. The vertices are the \(m\)-diagonals. There is an arrow between \(i\) and \(j\) if the \(m\)-diagonals bound a common \((m+2)\)-gon. The colour of the arrow is the number of edges forming the segment of the boundary of the \((m+2)\)-gon which lies between \(i\) and \(j\), counterclockwise from \(i\). This is the same definition as in [BT] in the Dynkin \(A\) case, and it is easy to see that such a quiver satisfy the conditions described in [BT] for coloured quivers. See Figure 6 for an example. If \(\alpha\) is an \(m\)-diagonal in an \((m+2)\)-angulation, we always denote by \(v_\alpha\) the corresponding vertex in \(Q_\Delta\).

\[\text{Figure 5. 4-angulation of } P_{3,3,2} \text{ and 5-angulation of } P_{4,2,3}.\]

\[\text{Figure 6. Example of a triangulation } \Delta \text{ of } P_{2,2} \text{ and its corresponding quiver } Q_\Delta.\]

It is known from [FST] that a quiver obtained in this way, for \(m=1\), is a quiver of a cluster-tilted algebra of type \(\tilde{A}_{p,q}\). We also know that all quivers of cluster-tilted algebras of type \(\tilde{A}_{p,q}\) can be obtained this way.

If \(\alpha\) is a diagonal in a triangulation \(\Delta\), the mutation of \(\Delta\) at \(\alpha\) is the triangulation \(\Delta'\) obtained by replacing \(\alpha\) with the unique other diagonal \(\alpha'\) such that \(\Delta' = (\Delta - \alpha) \cup \alpha'\) is a triangulation. See Figure 7. It is known that this operation commutes with quiver mutation, i.e. mutating at \(\alpha\) corresponds to mutating at \(v_\alpha\).

Let \(\Delta\) be any \((m+2)\)-angulation of \(P_{p,q,m}\), and let \(\alpha \in \Delta\). By removing the \(m\)-diagonal \(\alpha\) from \(\Delta\), we obtain an inner \((2m+2)\)-gon in the "almost complete"
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Figure 7. Example of mutation of a triangulation at a diagonal and the corresponding quiver.

$(m + 2)$-angulation $\Delta - \alpha$. There are exactly $m + 1$ possible $m$-diagonals, say $\alpha = \alpha_0, \alpha_1, \alpha_2, ..., \alpha_m$, such that $(\Delta - \alpha) \cup \alpha_i$ is an $(m + 2)$-angulation. These possible $m$-diagonals are called diameters of the inner $(2m + 2)$-gon, because they geometrically connect two opposite vertices in the $(2m + 2)$-gon. Consequently, they can all be obtained from $\alpha$ by rotating the inner $(2m + 2)$-gon. For an $m + 2$-angulation $\Delta$, we define the mutation at $\alpha$ to be the $(m + 2)$-angulation $\mu_\alpha(\Delta)$ obtained by rotating the $(2m + 2)$-gon corresponding to $\alpha$ clockwise. This definition is similar to the one given in [BT] for the Dynkin case $A$ and $(m + 2)$-angulations of regular polygons.

We note that this operation is well-defined, for if $\alpha$ is an $m$-diagonal between $O_i$ and $I_j$, then $i$ and $j$ are congruent modulo $m$, and of course $i - 1$ is congruent to $j - 1$ modulo $m$. It is straightforward to check that the operation is well-defined on $m$-diagonals of Type 2 and 3. Also, it is well-defined for mutation that takes an $m$-diagonal of a certain type to an $m$-diagonal of another type.

Let us fix a particular $(m + 2)$-angulation $\Delta^0_{p,q,m}$ of $P_{p,q,m}$ and the corresponding quiver $Q_{\Delta^0_{p,q,m}}$. See Figure 3. The $(m + 2)$-angulation $\Delta^0_{p,q,m}$ gives rise to the coloured quiver $Q_{\Delta^0_{p,q,m}}$ of type $\tilde{A}_{p,q}$. The proof of the following proposition is straightforward and similar to the Dynkin type $A$ case [BT], and we leave the proof to the reader.

**Proposition 5.1.** Mutation at any $m$-diagonal $\alpha$ in an $(m + 2)$-angulation $\Delta$, corresponds to coloured quiver mutation at $v_\alpha$ in $Q_{\Delta}$.

Any $(m + 2)$-angulation can be obtained from $\Delta^0_{p,q,m}$ by a finite sequence of mutations. This can be seen by noting that any $(m + 2)$-angulation consisting of only $m$-diagonals of Type 1 can be reached from $\Delta^0_{p,q,m}$ by a finite sequence of mutations, and any $(m + 2)$-angulation can be mutated into an $(m + 2)$-angulation with $m$-diagonals of only Type 1. It follows that any coloured quiver of type $\tilde{A}_{p,q}$ can be obtained from an $(m + 2)$-angulation and that a coloured quiver corresponding to an $(m + 2)$-angulation of $P_{p,q,m}$, is a coloured quiver of an $m$-cluster-tilted algebra of type $\tilde{A}_{p,q}$.
Figure 8. The triangulation $\Delta^0_{p,q,m}$ of $P_{p,q,m}$ and the corresponding quiver $Q_{\Delta^0_{p,q,m}}$. Note that we have only drawn the arrows of colour 0, and there are only arrows of colour 0 and $m$. Also note that all $m$-diagonals are of Type 1.

Let $T_{p,q,m}$ be the set of all $(m+2)$-angulations of $P_{p,q,m}$, and let $M_{p,q,m}$ be the mutation class of $m$-coloured quivers of type $\tilde{A}_{p,q}$. By the above we have a surjective function

$$\sigma_{p,q,m}: T_{p,q,m} \to M_{p,q,m},$$

where $\sigma_{p,q,m}(\Delta) = Q_\Delta$. The function $\sigma_{p,q,m}$ commutes with mutation.

6. The category of $m$-diagonals

Let $\alpha$ be an $m$-diagonal in $P_{p,q,m}$ and $s$ a positive integer. We define $\alpha[s]$ to be the $m$-diagonal obtained by rotating the outer polygon $s$ steps clockwise and the inner polygon $s$ step counterclockwise. More precisely,

- $\alpha[s]$, where $\alpha$ is a path from $O_i$ to $I_j$ of Type 1, is obtained by continuously moving the endpoint of the path at $O_i$ to $O_{i-s}$ and the endpoint of $I_j$ to $I_{j-s}$;
- $O_{i,k}[s] = O_{i-s,k}$;
- $I_{i,k}[s] = I_{i-s,k}$.
We always compute modulo \( m_p \) and \( m_q \) when we refer to vertices on the outer and inner polygon respectively. Obviously we can define the opposite operation, and we denote it by \([-s]\). Certainly this operation is well-defined, for if \( \alpha \) is an \( m \)-diagonal, then \( \alpha[s] \) is also an \( m \)-diagonal for all integers \( s \). Set \( \tau = [m] \). We have the following lemma, which follows directly from the definition.

**Lemma 6.1.** We have that \( O_{i,k}[m_p] = \tau^s O_{i,k} = O_{i,k} \) and \( I_{i,k}[m_q] = \tau^s I_{i,k} = I_{i,k} \). Furthermore, \( \alpha[s] \neq \alpha \) for all \( s \), when \( \alpha \) is of Type 1.

If \( \Delta \) is an \((m + 2)\)-angulation, it is clear that if \( \Delta[s] \) is the \((m + 2)\)-angulation obtained from \( \Delta \) by applying \([s]\) on each diagonal in \( \Delta \), we obtain a new \((m + 2)\)-angulation. It is also clear that \( Q_\Delta = Q_{\Delta[s]} \) for all \( s \), since \( m \)-diagonals bounding a common \((n + 2)\)-gon in \( \Delta \) also bound a common \((m + 2)\)-gon in \( \Delta[s] \). The function \( \sigma_{p,q,m} : \mathcal{T}_{p,q,m} \to \mathcal{M}_{p,q,m} \) from the previous section is therefore not an injection.

We want to define a category of \( m \)-diagonals, and the construction is motivated by [CCS], where they defined the cluster category of type \( A_n \) using diagonals of regular polygons. This construction was generalized in \[BM2\] to \( m \)-cluster categories.

First we define elementary moves of \( m \)-diagonals, which are certain operations that send one \( m \)-diagonal to another \( m \)-diagonal, and it is easy to check that the operation is well-defined. The operation should be considered as continuously moving the endpoints of the \( m \)-diagonals. We consider several cases.

- **\( m \)-diagonals of Type 1**: If \( \alpha \) is an \( m \)-diagonal between \( O_i \) and \( I_j \), there are exactly two elementary moves:

  \[
  \begin{array}{c}
  \alpha \\
  \downarrow \\
  \beta \\
  \end{array}
  \begin{array}{c}
  \downarrow \\
  \epsilon \\
  \end{array}
  \]

  The \( m \)-diagonal \( \beta \) is the \( m \)-diagonal obtained from \( \alpha \) by continuously moving the endpoint of \( \alpha \) at \( O_i \) counterclockwise \( m \) steps to \( O_i + m \). The \( m \)-diagonal \( \epsilon \) is the \( m \)-diagonal obtained from \( \alpha \) by continuously moving the endpoint of \( \alpha \) at \( I_j \) clockwise \( m \) steps to \( I_j + m \).

- **\( m \)-diagonals of Type 2**:
  - If \( k = m + 2 \), there is exactly one elementary move,

    \[
    O_{i,k} \to O_{i,k+m}.
    \]
   
  - If \( k > m + 2 \), there are exactly two elementary moves:

    \[
    \begin{array}{c}
    O_{i,k} \\
    \downarrow \\
    O_{i,k+m} \\
    \end{array}
    \begin{array}{c}
    O_{i,m,k-m} \\
    \downarrow \\
    O_{i+m,k-m} \\
    \end{array}
    \]

- **\( m \)-diagonals of Type 3**:
  - If \( k = m + 2 \), there is exactly one elementary move,

    \[
    I_{i,k} \to I_{i,k+m}.
    \]
   
  - If \( k > m + 2 \), there are exactly two elementary moves:

    \[
    \begin{array}{c}
    I_{i,k} \\
    \downarrow \\
    I_{i,k+m} \\
    \end{array}
    \begin{array}{c}
    I_{i+m,k-m} \\
    \downarrow \\
    I_{i+m,k-m} \\
    \end{array}
    \]
Similarly when \( \alpha \) and \( \beta \) are of Type 1 as rotating the outer and inner polygon \( m \) steps counterclockwise and clockwise respectively. Note that rotating only the outer polygon gives rise to another \((m + 2)\)-angulation that preserves the corresponding coloured quiver. Similarly for the inner polygon.

We need the following proposition.

**Proposition 6.2.** Let \( \alpha \) and \( \beta \) be \( m \)-diagonals. Then there is an elementary move \( \alpha \rightarrow \beta \) if and only if there is an elementary move \( \tau \beta \rightarrow \alpha \).

**Proof.** Suppose \( f : \alpha \rightarrow \beta \) is an elementary move.

First, assume that \( \alpha \) and \( \beta \) are of Type 1, say \( \alpha \) is an \( m \)-diagonal between \( O_i \) and \( I_j \) and \( \beta \) is an \( m \)-diagonal between \( O_i \) and \( I_j' \). Then either \( i' = i + m \) and \( j' = j \) or \( i' = i \) and \( j' = j + m \). If \( i' = i + m \) and \( j' = j \), then \( \tau \beta \) is an \( m \)-diagonal between \( O_i \) and \( I_{j-m} \). Then, by definition, there is an elementary move \( \tau \beta \rightarrow \alpha \).

Similarly when \( i' = i \) and \( j' = j + m \).

Next, suppose \( \alpha \) and \( \beta \) are of Type 2, say \( \alpha = O_{i,k} \) and \( \beta = O_{i',k'} \). If \( k = m + 2 \), we have that \( i' = i \) and \( k' = k + m = 2m + 2 \). Then \( \tau \beta = \tau O_{i',k'} = \tau O_{i,2m+2} = O_{i-m,2m+2} \), and by definition, there is an elementary move \( \tau \beta = O_{i-m,2m+2} \rightarrow \alpha = O_{i,2m+2-m} = O_{i,m+2} = O_{i,k} \).

If \( k \geq m + 2 \), then either \( i' = i \) and \( k' = k + m \) or \( i' = i + m \) and \( k' = k - m \). If \( i' = i \) and \( k' = k + m \), we have \( \tau \beta = \tau O_{i',k'} = \tau O_{i,k+m} = O_{i-m,k+m} \). Then, by definition, there is an elementary move \( \tau \beta = O_{i-m,k+m} \rightarrow \alpha = O_{i,k} \). Similarly if \( i' = i + 1 \) and \( k' = k - 1 \).

In the same way we can show that this holds for \( m \)-diagonals of Type 3. The converse is similar.

Let \( K \) be an algebraically closed field, and let \( C_{p,q}^m \) be the \( K \)-linear additive category defined as follows. The indecomposable objects are the \( m \)-diagonals, so the
objects in $C_{p,q}^m$ are direct sums of the $m$-diagonals. Morphisms between indecomposable objects $X$ and $Y$ are the vector space over $K$ spanned by the elementary moves modulo certain mesh relations which we define below.

Let $\alpha$ be an indecomposable object (an $m$-diagonal) in $C_{p,q}^m$. If $f : \beta \to \alpha$ is an elementary move, there exist an elementary move $f' : \tau \alpha \to \beta$ by the proposition. Let $L$ be the set of all elementary moves ending in $\alpha$. Then the mesh relation is defined as

$$\sum_{f \in L} ff'.$$

Consider the following situation, where $f_1, f_2, \ldots, f_n$ are all elementary moves ending in $\alpha$.

This means that the sum of compositions $f_i f'_i$ is 0. In our case there are at most two elementary moves, so $t = 1$ or $t = 2$. If $t = 1$, the diagonals are of Type 2 ($O_{i,k}$) or Type 3 ($I_{i,k}$) and $k = m + 2$. This means that the compositions $O_{i,m+2} \to O_{i,2m+2} \to O_{i+m,m+2}$ and $I_{i,m+2} \to I_{i,2m+2} \to I_{i+m,m+2}$ are 0. See Figure 10. If $t = 2$, we have equalities of compositions of elementary moves. See Figure 11.

![Figure 10](image-url)

**Figure 10.** The composition $O_{i,m+2} \to O_{i,2m+2} \to O_{i+m,m+2}$ is 0.

The translation $\tau$ is clearly an equivalence on this category. In fact $[s]$ is an equivalence for all $s$.

7. The $m$-cluster category and the category of $m$-diagonals

Given $P_{p,q,m}$ and the category of $m$-diagonals $C_{p,q}^m$, we define a quiver in the following way. The vertices are the indecomposable objects (i.e. the $m$-diagonals), and there is an arrow from the indecomposable object $\alpha$ to the indecomposable object $\beta$ if an only if there is an elementary move $\alpha \to \beta$. We call this the AR-quiver of $C_{p,q}^m$, and we will see that it is isomorphic to a subquiver of the AR-quiver of the $m$-cluster category obtained from $\tilde{A}_{p,q}$.

Let $\alpha$ be an $m$-diagonal. If $\alpha$ is of Type 1, we say that $\alpha$ is in level $d$, where $0 \leq d < m$, if we can obtain the $m$-diagonal between $O_d$ and $I_d$ with winding number 0 from $\alpha$ by applying a finite sequence of $\tau$ and elementary moves. If $\alpha$ is of Type 2 (or 3), we say that $\alpha$ is in level $d$ if we can obtain the $m$-diagonal $O_{d,m}$ (or $I_{d,m}$) from $\alpha$ by a finite sequence of $\tau$ and elementary moves.
It is straightforward to show that every \( m \)-diagonal is in some level. Given an \( m \)-diagonal in level \( d \), we can not reach another \( m \)-diagonal in a level \( \neq d \) by a finite sequence of \( \tau \) and elementary moves. It follows that the quiver of \( m \)-diagonals consists of at least \( m \) components. It is also clear that there is no sequence of \( \tau \) and elementary moves between \( m \)-diagonals of different types. Also, there exist a sequence of elementary moves and \( \tau \) between any two \( m \)-diagonals in the same level and of the same type. In fact, if \( \alpha \) is of Type 1 in level \( d \), then \( \alpha \) is of the form \( \tau^s \alpha' [d] \), where \( \alpha' \in \Delta^0_{p,q,m} \). It follows that the quiver consists of \( 3m \) components. It is easy to see that the components consisting of \( m \)-diagonals of the same type are isomorphic.

Let \( T^d_p \) be the component containing objects of Type 2 in level \( d \) and let \( T^d_q \) be the component containing objects of type Type 3 in level \( d \). Denote by \( S^d \) the component consisting of \( m \)-diagonals of Type 1 in level \( d \). See Figure 12 for an example. We draw the translation \( \tau \) as dotted arrows.

Now we want to define an additive functor \( F : C^m_{p,q} \rightarrow C^m \), where \( C^m \) is the \( m \)-cluster category of type \( \tilde{A}_{p,q} \). It is enough to define the functor on indecomposable objects and elementary moves. We also want that \( F \) induces a quiver isomorphism between the AR-quiver of the category of \( m \)-diagonals and a subquiver of the AR-quiver of the \( m \)-cluster category.

First we consider the objects in the components \( S^d \). Denote by \( S^d \) the component in the AR-quiver of the \( m \)-cluster category consisting of objects of the form \( \tau_C^s P[d] \), where \( P \) is a projective and \( \tau_C \) is the Auslander-Reiten translation. We want to show that \( S^d \) is isomorphic to \( S^d \) via the functor \( F \), which we will define below. When there is no confusion, we write the AR-translation \( \tau_C \) as \( \tau \). Similarly with the shift functor \([i]\).
First we make a choice, and it is natural to let the \((m+2)\)-angulation \(\Delta^{0}_{p,q,m}\) and the quiver \(Q_{\Delta^{0}_{p,q,m}}\) in Figure 8 correspond to the \(m\)-cluster tilting object \(T = \oplus P_{i}\), where \(P_{i}\) is the projective corresponding to vertex \(i\) in the quiver. The \(m\)-diagonal \(\alpha_{i}\) in \(\Delta^{0}_{p,q,m}\) corresponding to vertex \(i\) is mapped to \(P_{i}\). Also, we define \(F(\tau^{s}\alpha_{i}) = \tau^{s}P_{i}\) for all integers \(s\). By Lemma 6.1 and by the fact that all elements in \(S^{0}\) are of the form \(\tau^{s}\alpha_{i}\), this gives a bijection between the \(m\)-diagonals in \(S^{0}\) and the objects in the component \(S^{0}\) of the AR-quiver of the \(m\)-cluster category.

Next we define \(F(\alpha_{i}[d]) = P_{i}[d]\) and \(F(\tau^{s}\alpha_{i}[d]) = \tau^{s}P_{i}[d]\) for all integers \(t\). This takes care of all \(m\)-diagonals of Type 1, and by Lemma 6.1 and by the fact that all elements in \(S^{d}\) are of the form \(\tau^{s}\alpha_{i}[d]\), this is a bijection between the set of \(m\)-diagonals of Type 1 and the set of indecomposable objects in the transjective components in the AR-quiver of the \(m\)-cluster category.

Next we want to show that elementary moves (or arrows in the quiver of \(m\)-diagonals) correspond to irreducible morphisms (arrows in the AR-quiver of the \(m\)-cluster category). Suppose there is an elementary move between two \(m\)-diagonals \(\alpha \to \beta\). Then there exist an integer \(s\) such that \(\tau^{s}\alpha\) is of the form \(\alpha_{i}[d]\) for some integer \(d\) and \(\alpha_{i} \in \Delta^{0}_{p,q,m}\). Also, there is an elementary move \(\tau^{s}\beta \to \tau^{s}\beta\). We have \(F(\tau^{s}\alpha) = F(\alpha_{i}[d]) = P_{i}[d]\), where \(P_{i}\) is the projective corresponding to the vertex \(i\) in \(Q_{\Delta^{0}_{p,q,m}}\). By considering the possibilities for \(\tau^{s}\beta\), it is now straightforward to check that there is an arrow \(F(\tau^{s}\alpha) = P_{i}[d] \to F(\tau^{s}\beta)\), and hence there is an arrow \(F(\alpha) \to F(\beta)\). We leave the details to the reader. The converse is similar.

**Proposition 7.1.** The component \(S^{d}\) in the AR-quiver of \(C^{m}_{p,q}\) is isomorphic to the component \(S^{d}\) in the AR-quiver of the \(m\)-cluster category \(C^{m}_{p,q}\) of type \(\tilde{A}_{p,q}\), for all integers \(d\), with \(0 \leq d < m\).

Now we consider the objects in the components \(T^{d}_{p}\) and \(T^{d}_{q}\) in the AR-quiver of \(C^{m}_{p,q}\). Let \(T^{d}_{p}\) and \(T^{d}_{q}\) be the tubes of rank \(p\) and \(q\) respectively in the \(m\)-cluster category of type \(\tilde{A}_{p,q}\), where \(d\) denotes the degree. The tubes of rank \(p\) are isomorphic, and they are isomorphic to \(ZA_{\infty}/(\tau^{d})\). The tubes of rank \(q\) are isomorphic to \(ZA_{\infty}/(\tau^{d})\). We know that the tube \(T^{d}_{p}\) has \(p\) quasi-simple objects, say \(Q^{0}_{1}, Q^{1}_{1}, ..., Q^{p-1}_{1}\). Then the quasi-simple objects in \(T^{d}_{q}\) are of the form \(Q^{0}_{s}, Q^{1}_{s}, ..., Q^{p-1}_{s}\). There is only one arrow to and one arrow from each of these objects in the AR-quiver. Let the indecomposable objects in \(T^{d}_{p}\) be denoted \(Q^{i}_{s}[d]\), where \(s\) is the quasi-length of \(Q^{i}_{s}\). For a given \(i\), a ray is the sequence of irreducible maps

\[
Q^{i}_{s}[d] \to Q^{i}_{s+1}[d] \to Q^{i}_{s+2}[d] \to Q^{i}_{s}[d] \to \ldots,
\]

and a coray is the sequence

\[
\ldots \to Q^{i}_{s-2}[d] \to Q^{i}_{s-1}[d] \to Q^{i}_{s}[d] \to Q^{i}_{s+1}[d].
\]

We are in the situation shown in Figure 8.

For a fixed \(d\), with \(0 \leq d < m\), we have that \(\tau^{s}Q^{i}_{s}[d] = Q^{i}_{s-1}[d]\), and that \(\tau^{n}Q^{i}_{s}[d] = Q^{i}_{s}[d]\) for all \(s\) and \(i\). It is clear, by Lemma 6.1, that the quasi-simple objects have to correspond to diagonals of the form \(O_{i,m+2}\), since there are exactly one elementary move from and to \(O_{i,m+2}\) and their \(\tau\)-orbit is exactly \(p\).

For a fixed \(d\), we define \(F(O_{0,m+2}[d]) = Q^{0}_{1}[d]\), \(F(O_{0,2m+2}[d]) = Q^{0}_{0}[d]\) and so on, i.e. \(F(O_{0,m+2}[d]) = Q^{0}_{1}[d]\). This corresponds to the ray starting in \(Q^{0}_{1}[d]\). We have \(\tau^{-1}O_{0,m+2}[d] = O_{0,m+2}[d]\), so we set \(F(\tau^{-1}O_{0,m+2}[d]) = F(O_{m+2}[d]) = \tau^{-1}Q^{0}_{1}[d] = Q^{1}_{1}[d]\). Then we continue as above, and the pattern is clear, so in general we define \(F(O_{0,m+2}[d]) = Q^{1}_{1}[d]\). This takes care of all \(m\)-diagonals of Type 2. We do similarly with the \(m\)-diagonals of Type 3, and they correspond to objects in the tubes of rank \(q\). We leave the details to the reader.
Figure 12. The quiver of $C_{2,2}^1$; component $S$, $T_{p=2}$ and $T_{q=2}$ respectively.
if there is an elementary move $O$ to $\tau Q^j i$. Proposition 7.2.

The components $\tau Q$ of the $m$- and $G$ are not full, since we do not have any maps in $C$.

Theorem 7.3. The functor $\tau Q$ induces a quiver isomorphism between the AR-quiver of $C_{p,q}$ and the AR-quiver of $C'$ consisting of all objects not in a homogeneous tube, we get the induced functor $G : C_{p,q} \to C'$. Then $G$ is dense and faithful. The functor is not full, since we do not have any maps in $C_{p,q}$ between the objects in different components in the AR-quiver, i.e. no maps corresponding to maps in the infinite radical.

The following theorem summarizes.

Proposition 7.2. The components $T_{p}^d$ and $T_{q}^e$ of the quiver of $C_{p,q}$ are isomorphic to $T_{p}^d$ and $T_{q}^e$ respectively in the $m$-cluster category.

Next we show that we have an irreducible morphism $Q^i[d] \to Q^j[d]$ if and only if there is an elementary move $O_{m, sm+2}[d] \to O_{jm, tm+2}$.

If $Q^i[d]$ is quasi-simple, $s = 1$, and so hence $t = 2$ and $j = i$. Then we have an elementary move

$$O_{im, sm+2}[d] = O_{im, m+2}[d] \to O_{jm, tm+2}[d] = O_{im, 2m+2}[d].$$

If $Q^i$ is not quasi-simple, we have $s > 1$, $t = s + 1$ and $j = i$ or $t = s - 1$ and $j = i + 1$. Then we have elementary moves

$$O_{im, sm+2}[d] \to O_{jm, tm+2}[d] = O_{im, (s+1)m+2}[d] = O_{im, (sm+2+m)}[d]$$

and

$$O_{im, sm+2}[d] \to O_{jm, tm+2}[d] = O_{i(m+1)(s-1)m+2}[d] = O_{im+m, sm+2-m}[d].$$

The converse is similar.

We see that given any object $Q^i$, which corresponds to $O_{im, sm+2}$, we have that $\tau Q^i = Q^i_{i-1}$ corresponds to $\tau O_{im, sm+2} = O_{im-m, sm+2}$.

Proposition 7.2. The components $T_{p}^d$ and $T_{q}^e$ of the quiver of $C_{p,q}$ are isomorphic to $T_{p}^d$ and $T_{q}^e$ respectively in the $m$-cluster category.

It is now easy to see that $F$ is in fact a functor. We know that the AR-quiver of the $m$-cluster category has tubes of rank 1, i.e. non-rigid homogeneous objects. The functor $F$ is therefore not dense. However, if $C'$ is the full subcategory of the $m$-cluster category $C^m$ consisting of all objects not in a homogeneous tube, we get the induced functor $G : C_{p,q} \to C'$. Then $G$ is dense and faithful. The functor is not full, since we do not have any maps in $C_{p,q}$ between the objects in different components in the AR-quiver, i.e. no maps corresponding to maps in the infinite radical.

The following theorem summarizes.

Theorem 7.3. The functor $G$ induces a quiver isomorphism between the AR-quiver of $C_{p,q}$ and the AR-quiver of $C' \subset C^m$. Furthermore $G$ is dense and faithful, and we have that $F(\alpha[s]) = F(\alpha)[s]$, for any $m$-diagonal $\alpha$ and all integers $s$. 

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**Figure 13.** A tube $T_p$ of rank $p$ in the cluster category.
8. Bijection between the mutation class and an equivalence class of triangulations

Let $T_{p,q,m}$ be the set of all $(m+2)$-angulations of $P_{p,q,m}$, and let $M_{p,q,m}$ be the mutation class of $m$-coloured quivers of type $\mathbb{A}_{p,q}$. Recall that we have surjective function

$$\sigma_{p,q,m} : T_{p,q,m} \to M_{p,q,m},$$

where $\sigma_{p,q,m}(\Delta) = Q_\Delta$. As we have already pointed out, this function commutes with mutation, i.e. $\sigma_{p,q,m}(\mu_i(\Delta)) = \mu_i(\sigma_{p,q,m}(\Delta)) = \mu_i(Q_\Delta)$. We also know that $\sigma_{p,q,m}$ is not injective, since for example $\sigma_{p,q,m}(\Delta) = \sigma_{p,q,m}(\tau(\Delta))$.

We want to find an equivalence relation $\sim$ on $T_{p,q,m}$ such that we obtain a bijection induced from $\sigma_{p,q,m}$,

$$\tilde{\sigma}_{p,q,m} : (T_{p,q,m}/\sim) \to M_{p,q,m}.$$

We define two functions, $r_O$ and $r_I$, on the set of diagonals. Let $\alpha$ be a diagonal. Define $r_O\alpha$ to be the diagonal obtained from $\alpha$ by rotating the outer polygon 1 step clockwise and $r_I\alpha$ to be the diagonal obtained by rotating the inner polygon 1 step counterclockwise. This function is not well-defined on $m$-diagonals when $m > 1$, for if $\alpha$ is a $m$-diagonal, then $r_O\alpha$ and $r_I\alpha$ are not necessarily $m$-diagonals when $\alpha$ is of Type 1. However, we note that the functions $r_Or_I = r_Ir_O$, $r_O^2$ and $r_I^2$ are well-defined on $m$-diagonals, and also that $r_Or_I = [1]$ and consequently $r_O^m r_I^m = \tau$. We let $r_O^{-1}$ and $r_I^{-1}$ be rotating in the opposite direction.

For simplicity, we will in this section consider sets of diagonals that divides $P_{p,q,m}$ into $(m+2)$-gons. In other words, we will allow diagrams that do not necessarily satisfy the restrictions for $m$-diagonals. This is just for simplicity, and if we can find the desired equivalence relation on this set, it induces an equivalence relation on the set of $(m+2)$-angulations consisting of only $m$-diagonals. We will still call elements in this bigger set $(m+2)$-angulations. We can associate to an element in this set a coloured quiver as before.

If $\Delta$ is an $(m+2)$-angulation, we define $r_O\Delta$ ($r_I\Delta$) to be the set of diagonals obtained from $\Delta$ by applying $r_O$ ($r_I$) on each diagonal in $\Delta$. Similarly we define $r_I\Delta$. Note that by rotating the outer or inner polygon, $(m+2)$-gons and hence the corresponding coloured quivers are preserved.

We define another function on the set of $(m+2)$-angulations that sends an $(m+2)$-angulation of $P_{p,q,m}$ to an $(m+2)$-angulation of $P_{q,p,m}$. Given an $(m+2)$-angulation, we can "flip" it by making the outer polygon the inner polygon and the inner polygon the outer polygon. We shall see that this operation corresponds to reversing every arrow in the corresponding quiver. We can visualize this operation as continuously stretching the $(m+2)$-angulation in three dimensions, such that the interior together with the diagonals become the side surface of a cylinder, the inner polygon becomes the top of the cylinder and the outer polygon becomes the bottom. Then we push the cylinder back into the plane, making the top of the cylinder the outer polygon and the bottom the inner polygon, thus "flipping" the $(m+2)$-angulation. We denote the flipped $(m+2)$-angulation $\Delta$ by $\epsilon(\Delta)$. Clearly $\epsilon(\epsilon(\Delta)) = \Delta$.

**Lemma 8.1.** If $\Delta$ is an $(m+2)$-angulation of $P_{p,q,m}$, then $\epsilon(\Delta)$ is an $(m+2)$-angulation of $P_{q,p,m}$. The quiver $Q_{\epsilon(\Delta)}$ is obtained from $Q_{\Delta}$ by reversing all arrows.

**Proof.** It is clear that $\epsilon(\Delta)$ is an $(m+2)$-angulation of $P_{q,p,m}$. Also, it is easy to see that any inner $(m+2)$-gon is preserved by the flip, but that it changes orientation (it is turned upside down). Hence the arrows in $Q_{\epsilon(\Delta)}$ are reversed. □
We know that the mutation classes of $\tilde{A}_{p,q}$ and $\tilde{A}_{q,p}$ are equal up to isomorphism of quivers. Suppose $p \neq q$. Let $Q_\Delta$ be the quiver consisting of a cycle with $p$ arrows of colour 0 clockwise and $q$ arrows of colour 0 counterclockwise, i.e., we have fixed the quiver $Q_\Delta$ in the plane. Then we can not reach the quiver $Q_{\ell(\Delta)}$ from the quiver $Q_\Delta$ by a finite sequence of mutations.

If $p = q$, we may have that flipping a triangulation $\Delta$ preserves the corresponding quiver up to isomorphism, i.e. $Q_\Delta$ may be isomorphic to $Q_{\ell(\Delta)}$. This is not always the case, but we do have that $Q_{\ell(\Delta)}$ can be reached from $Q_\Delta$ by a finite sequence of mutations, since any orientation of $A_{p,p}$ is in the mutation class. Let $Q$ be a coloured quiver. We say that $Q$ is reflection-symmetric if the quiver obtained from $Q$ by reversing every arrow is isomorphic to $Q$. Clearly, if $Q_\Delta$ is reflection-symmetric then $Q_{\ell(\Delta)}$ is isomorphic to $Q_{\ell(\Delta)}$. If $Q_\Delta$ is isomorphic to $Q_{\ell(\Delta)}$, we say that $\Delta$ is reflection-symmetric.

For $p \neq q$, we define an equivalence relation on $T_{p,q,m}$ by letting two $(m + 2)$-angulations $\Delta$ and $\Delta'$ be equivalent if and only if $\Delta' = r_{ij}^k \Delta$ for some integers $i$ and $j$. If $p = q$, we define the two $(m + 2)$-angulations to be equivalent if and only if $\Delta' = r_{ij}^k \Delta$ for some integers $i, j$ and $k \in \{0, 1\}$, where $k = 0$ if $\Delta$ is not reflection-symmetric. We write $(T_{p,q,m}/\sim)$ for the class of equivalent $(m + 2)$-angulations thus obtained. Now, clearly this gives a map

$$\tilde{\sigma}_{p,q,m} : (T_{p,q,m}/\sim) \to M_{p,q,m},$$

induced from $\sigma_{p,q,m}$, and we claim that this map is bijective. By the discussion above we already have surjectivity.

Recall that for an $m$-diagonal $\alpha$ in an $(m + 2)$-angulation $\Delta$, we always denote by $v_\alpha$ the corresponding vertex in $Q_\Delta$. If $\Delta$ is an $(m + 2)$-angulation and $\alpha$ an $m$-diagonal, we want to investigate the procedure of factoring out the corresponding vertex $v_\alpha$ in $Q_\Delta$. We say that $\alpha$ is close to the border of the outer polygon if $\alpha$ is an $m$-diagonal of Type 2 and homotopic to $\delta_{i,m+2}$ for some $i$, i.e. $\alpha$ is in an $(m + 2)$-gon together with edges only on the border. Similarly $\alpha$ is close to the border of the inner polygon if $\alpha$ is of Type 3 and homotopic to $\gamma_{i,m+2}$ for some $i$. Recall that $m$-diagonals close to the border corresponds to quasi-simple objects in the $m$-cluster category, and all $m$-diagonals of Type 2 and 3 corresponds to objects in the tubes. Objects in the transjective components corresponds to $m$-diagonals of Type 1.

**Lemma 8.2.** If $\Delta$ is an $(m + 2)$-angulation containing an $m$-diagonal $\alpha$ of Type 2 (Type 3), then there exist an $m$-diagonal close to the border of the outer (inner) polygon.

**Proof.** Suppose $\alpha$ is of Type 2 and not close to the border of the outer polygon. Then $\alpha$ divides the polygon into two parts $A$ and $B$, where one part, say $A$, contains the inner polygon. Then $B$ is just an $(m + 2)$-angulation of a regular polygon, so there exist an $m$-diagonal that divides $B$ into two smaller parts. By induction, there exist an $m$-diagonal close to the border. The proof for $m$-diagonals of Type 3 is similar. \qed

If $\alpha$ is close to the border of the outer (inner) polygon, we define $\Delta/\alpha$ to be the $(m + 2)$-angulation obtained from $\Delta$ by letting $\alpha$ be a border edge of the outer (inner) polygon and leaving all the other $m$-diagonals unchanged. We say that we factor out $\alpha$. See Figure 14.

**Lemma 8.3.** Let $\Delta$ be an $(m + 2)$-angulation of $P_{p,q,m}$ and let $\alpha$ be close to the border of the outer (inner) polygon. Then the quiver $Q_\Delta/v_\alpha$, obtained from $Q_\Delta$ by factoring out the vertex $v_\alpha$ is connected and of type $\tilde{A}_{p-1,q}$ ($\tilde{A}_{p,q-1}$). Furthermore, we have that $Q_{\Delta/\alpha} = Q_\Delta/v_\alpha$. 


Proof. We refer to [To3] and the proof there for Dynkin type $A$. The proof in this case is a straightforward adaption. □

Next we consider factoring out vertices that correspond to $m$-diagonals not close to the border.

**Lemma 8.4.** Let $\Delta$ be an $(m+2)$-angulation. If we factor out a vertex in $Q\Delta$ corresponding to an $m$-diagonal not close to the border and not of Type 1 (an $m$-diagonal between the outer and inner polygon), then the resulting quiver is disconnected.

**Proof.** In this case $\alpha$ divides the polygon into two parts $A$ and $B$, where one part, say $A$, contains the inner polygon. Let $\beta$ be an $m$-diagonal in $A$ and $\beta'$ an $m$-diagonal in $B$. If $\beta$ and $\beta'$ would determine a common $(m+2)$-gon, the third $m$-diagonal would have to cross $\alpha$, hence there is no arrow between the subquiver determined by $A$ and the subquiver determined by $B$, except those passing through $v_\alpha$. Thus factoring out $v_\alpha$ disconnects the quiver. □

Let $\Delta$ be an $(m+2)$-angulation in $T_{p,q,m}$. We know by Lemma 4.1 that $\Delta$ contains at least one $m$-diagonal $\alpha$ of Type 1. We want to define a function on $\Delta$, which correspond to factoring out the corresponding vertex $v_\alpha$ in the quiver $Q\Delta$, when $\alpha$ is of Type 1. By rotating the outer and inner polygon we can assume that we are in the situation shown in the first picture in Figure 14, and hence that the winding number of any diagonal in $\Delta$ is $\leq 1$. We cut the $(m+2)$-angulation along $\alpha$ as shown in Figure 14. We obtain two new border edges $\alpha'$ and $\alpha''$ in a regular polygon. All the other diagonals are left unchanged.

We have the following results.

**Lemma 8.5.** Let $\Delta$ be an $(m+2)$-angulation of $P_{p,q,m}$ and let $\alpha$ be an $m$-diagonal of Type 1. Then factoring out $\alpha$ gives an $(m+2)$-angulation of the regular polygon with $mp + mq + 2$ vertices.

**Proof.** Clearly the resulting polygon has $mp + mq + 2 = m(p+q) + 2$ vertices. It is known, for example by [BM2], that any $(m+2)$-angulation of the regular polygon with $mn + 2$ vertices has exactly $n - 1$ $m$-diagonals. There are exactly $p + q - 1$
diagonals left after factoring out $\alpha$, hence we have obtained an $(m+2)$-angulation of the regular polygon with $m(p+q)+2$ vertices.

Lemma 8.6. Let $\Delta$ be an $(m+2)$-angulation of $P_{p,q,m}$. If $\alpha$ is a diagonal of Type 1, then the quiver $Q_{\Delta}/v_\alpha$ is connected and of Dynkin type $A_{p+q-1}$. Furthermore, factoring out $\alpha$ corresponds to factoring out the corresponding vertex in $Q_\Delta$, i.e. $Q_{\Delta/\alpha} = Q_\Delta/v_\alpha$.

Proof. Factoring out $\alpha$ does not affect the inner $(m+2)$-gons in $\Delta$, and so hence the arrows between vertices not equal to $v_\alpha$ stay the same. The arrows from and to $v_\alpha$ are removed. □

We note that this procedure is reversible. Let $\Delta'$ be an $(m+2)$-angulation of a regular polygon with $mp+mq+2$ vertices. Suppose we want an $(m+2)$-angulation of $P_{p,q}$. Pick any border edge $e$. Then there are two possible border edges we can identify with $e$ to obtain an $(m+2)$-angulation of $P_{p,q}$.

Summarizing we obtain the following proposition.

Proposition 8.7. Let $\Delta$ be an $(m+2)$-angulation of $P_{p,q,m}$ and $\alpha$ an $m$-diagonal. Then the $m$-coloured quiver $Q_\Delta/v_\alpha$ is connected and of Dynkin type $A_{p+q-1}$ if and only if $\alpha$ is of Type 1. Also $Q_{\Delta}/v_\alpha$ is connected and of Dynkin type $\tilde{A}_{p-1,q}$ ($\tilde{A}_{p,q-1}$) if and only if $\alpha$ is close to the border of the outer (inner) polygon. Else $Q_{\Delta}/v_\alpha$ is disconnected.

Let $\Delta$ be an $(m+2)$-angulation of $P_{p,q,m}$, and let $e$ be a border edge between $i$ and $j$ on the outer or inner polygon. We define another $(m+2)$-angulation $\Delta'(e)$ of $P_{p+1,q,m}$ (if $e$ is on the outer polygon) or $P_{p,q+1,m}$ (if $e$ is on the inner polygon), called the extension of $\Delta$ at $e$. The new $(m+2)$-angulation $\Delta'(e)$ is obtained from $\Delta$ by adding $m$ new border vertices between $i$ and $j$ and letting $e$ be an $m$-diagonal close to the border in $\Delta(e)$. See Figure 16.

Now we are ready to prove the bijection between $T_{p,q,m}/\sim$ and $M_{p,q,m}$. We look at two different cases, namely the case when $(m+2)$-angulations only contain $m$-diagonals of Type 1 and the case when $(m+2)$-angulations contain at least one $m$-diagonal of Type 2 or 3.

Suppose $\Delta$ is an $(m+2)$-angulation that contains only $m$-diagonals of Type 1, i.e. $m$-diagonals between the outer and inner polygon. Then the corresponding coloured quiver $Q_\Delta$ is a cycle of length $p+q$, with arrows of possibly different colours going both ways. Given such a quiver $Q$, we want to show that there is a unique $(m+2)$-angulation in $T_{p,q,m}/\sim$ that maps to $Q$.

Lemma 8.8. Let $Q$ be a coloured quiver consisting of a cycle. If $\Delta$ and $\Delta'$ are two $(m+2)$-angulations such that $\sigma_{p,q,m}(\Delta) = \sigma_{p,q,m}(\Delta') = Q$, then $\Delta = \Delta'$ in $(T_{p,q,m}/\sim)$. 

Figure 15. Factoring out a diagonal $\alpha$ of Type 1.
Proof. We sketch a proof. Start with an \( m \)-diagonal \( \alpha \) between \( I_i \) and \( O_j \). Choose a vertex \( v_\alpha \) in \( Q \) corresponding to \( \alpha \). Suppose there is an arrow \( r \) from \( v_\alpha \) to \( v_\beta \) of colour \( c \). Then \( \beta \) is either a diagonal between \( I_{i+m-c} \) and \( O_{i+c} \) if \( r \) goes counterclockwise or a diagonal between \( I_{i+c} \) and \( O_{j+m-c} \) if \( r \) goes clockwise. By induction we can continue. If \( p = q \) and if the quiver obtained from \( Q \) by reversing all arrows is isomorphic to \( Q \), then the corresponding \((m+2)\)-angulations are flips and rotations of each other. \( \square \)

For the case when the \((m+2)\)-angulation contains \( m \)-diagonals of Type 2 or 3, we need two more lemmas.

**Lemma 8.9.** Let \( \Delta \) be an \((m+2)\)-angulation. Suppose that \( Q_\Delta \simeq Q_{\Delta'} \) for some \((m+2)\)-angulation \( \Delta' \) implies \( \Delta = \Delta' \) in \( (T_{p,q,m}/\sim) \).

Let \( \alpha \) be an \( m \)-diagonal in \( \Delta \). Suppose there is an isomorphism \( Q_\Delta \xrightarrow{\theta} Q_{\Delta'} \) that sends \( v_\alpha \) to \( v_\alpha' \). Then \( \alpha' = r^{i}_{O}r^{j}_{I}c^{k}\alpha \) and \( \Delta' = r^{i}_{O}r^{j}_{I}c^{k}\Delta \) for some integers \( i, j \) and \( k \in \{0, 1\} \), where \( k = 0 \) if \( Q_\Delta \) is not reflection-symmetric.

**Proof.** If \( \alpha \) is of Type 2 or 3, the proof is a straightforward adaption of the proof in \([103]\) for the Dynkin \( A \) case, and we omit it.

Suppose \( \alpha \) is of Type 1. Then \( \alpha' \) is also of Type 1 by Proposition \([87]\) and there exist some \( i, j \) and \( k \) such that \( \alpha' = r^{i}_{O}r^{j}_{I}c^{k}\alpha \). Take any \( m \)-diagonal \( \beta \) in \( \Delta \) such that \( v_\alpha \) has an arrow of colour \( c \) to \( v_\alpha' \), i.e. \( \alpha \) and \( \beta \) are in a common \((m+2)\)-gon in \( \Delta \). Suppose \( \theta(v_{\beta}) = v_{\beta'} \). Then \( \beta' \) and \( \alpha' \) are in a common \((m+2)\)-gon in \( \Delta' \), and \( v_{\alpha'} \) has an arrow of colour \( c \) to \( v_{\beta'} \).

Factoring out \( \alpha \) and \( \alpha' \), as described right before Lemma \([87]\), gives \((m+2)\)-angulations \( \Delta/\alpha \) and \( \Delta'/\alpha' \) of the regular polygon with \( mp + mq + 2 \) vertices, and corresponding quivers of type \( A_{p+q-1} \). Then, clearly, \( Q_\Delta/v_\alpha = Q_{\Delta/\alpha} \simeq Q_{\Delta'/\alpha'} = Q_{\Delta'/v_\alpha'} \), and from \([103]\) it follows that there exist some integer \( i' \) such that \( \beta' = r^{i'}\beta \) and \( \Delta'/\alpha' = r^{i'}\Delta/\alpha \), where \( r^{i'} \) is rotating the \((m+2)\)-angulation of the regular polygon \( i' \) steps in the counterclockwise direction. Since \( v_{\beta} \) has an arrow of colour \( c \) to \( v_{\alpha} \) and \( v'_{\beta} \) has an arrow of colour \( c \) to \( v'_{\alpha} \), the claim follows. \( \square \)

**Lemma 8.10.** Let \( \Delta \) be an \((m+2)\)-angulation, and suppose that \( Q_\Delta \simeq Q_{\Delta'} \) for some \((m+2)\)-angulation \( \Delta' \) implies \( \Delta = \Delta' \) in \( (T_{p,q,m}/\sim) \).

![Figure 16. Extension of \( \Delta \) at the border edge \( e \) on the outer and inner polygon respectively.](image-url)
Let $\Delta(e)$ and $\Delta(e')$, with $e \neq e'$, be two extensions of $\Delta$. Then $\Delta(e) = \Delta(e')$ in $(\mathcal{T}_{p,q,m}/\sim)$ if and only if $Q_{\Delta(e)} \simeq Q_{\Delta(e')}$. 

Proof. Suppose $Q_{\Delta(e)} \simeq Q_{\Delta(e')}$. The coloured quiver $Q_{\Delta(e)}$ is connected, so suppose $v_\alpha$ has an arrow of colour $e$ to $v_{\gamma}$, where $\alpha$ is an $m$-diagonal in $\Delta(e)$. This means that $e$ is an edge in some $(m+2)$-gon together with $\alpha$. Then there is some $v_\beta$ in $Q_{\Delta(e')}$ such that exist an isomorphism $Q_{\Delta(e)} \to Q_{\Delta(e')}$ sending $v_\alpha$ to $v_\gamma$ and $v_\sigma$ to $v_\beta$. By Proposition 8.11 we have that $\beta$ is close to the border in $\Delta(e')$, since $e$ is close to the border in $\Delta(e)$. Then we have that 

$$Q_\Delta \simeq Q_{\Delta(e)/e} \simeq Q_{\Delta(e)}/v_\alpha \simeq Q_{\Delta(e')}/v_\beta \simeq Q_{\Delta(e')}/\beta,$$

so hence, by assumption and Lemma 8.9 $\Delta(e')/\beta = r_\alpha^i r_\gamma^j e^k \Delta$ and $\gamma = r_\alpha^i r_\gamma^j e^k \alpha$, where $k = 0$ if $\Delta$ is not reflection-symmetric.

Suppose the $m$-diagonal $\alpha$ is of Type 2. Then $\alpha$ divides the polygon into two parts $A$ and $B$, where, say, $B$ contains the inner polygon. Then also $\gamma$ divides the polygon into two parts $A'$ and $B'$, where, say, $B'$ contains the inner polygon. If $e$ lies in $A$, then $\beta$ lies in $A'$, and if $e$ lies in $B$, then $\beta$ lies in $B'$. There is only one way to extend $\Delta(e')/\beta$ in $A'$ (or $B'$) such that the new vertex has an arrow of colour $e$ to $v_\gamma$. Hence $\beta = r_\alpha^i r_\gamma^j e^k$, and $r_\alpha^i r_\gamma^j e^k \Delta(e) = \Delta(e')$.

If $\alpha$ is of Type 3, we do similarly.

Suppose the $m$-diagonal $\alpha$ is of Type 1. Then $\gamma$ is also of Type 1. There is only one way to extend $\Delta(e')/\beta$ (on the outer or inner polygon) such that the new vertex has an arrow of colour $e$ to $v_\gamma$, and as above we are done. \hfill $\Box$

Now we can prove the main theorem in this section.

Theorem 8.11. The function $\tilde{\sigma}_{p,q,m} : (\mathcal{T}_{p,q,m}/\sim) \to \mathcal{M}_{p,q,m}$ is a bijection.

Proof. We only need to show injection. Suppose $\tilde{\sigma}_{p,q,m}(\Delta) = \tilde{\sigma}_{p,q,m}(\Delta')$. We want to show that $\Delta = \Delta'$ in $(\mathcal{T}_{p,q,m}/\sim)$. If $\Delta$ does not contain a diagonal close to the border, we are finished by Lemma 5.8 and 5.9 so we can assume that there exist a diagonal $\alpha$ close to the border of the inner or outer polygon. If $\alpha$ is close to the border of the inner polygon, we can consider $e(\Delta)$ instead, hence we can also assume that $\alpha$ is close to the border of the outer polygon. It is straightforward to verify that $\tilde{\sigma}_{2,2,m} : (\mathcal{T}_{2,2,m}/\sim) \to \mathcal{M}_{2,2,m}$ is bijective.

Fix $q \geq 2$ and let $p > 2$. Suppose that $\tilde{\sigma}_{p-1,q,m} : (\mathcal{T}_{p-1,q,m}/\sim) \to \mathcal{M}_{p-1,q,m}$ is injective for all $m$.

Let $\alpha$ be close to the border in $\Delta$. Then the $m$-diagonal $\alpha'$ in $\Delta'$ corresponding to $v_\alpha$ in $Q$ is also close to the border. By hypothesis $\Delta/\alpha = \Delta'/\alpha'$ in $\mathcal{T}_{p-1,q,m}/\sim$. We can obtain $\Delta$ and $\Delta'$ from $\Delta/\alpha = \Delta'/\alpha'$ by extension. By Lemma 5.10 all possible extensions of $\Delta/\alpha$ and $\Delta'/\alpha'$ give non-isomorphic quivers, unless $\Delta = \Delta'$. We do the same induction step on $q$, and as above we are done. \hfill $\Box$

We obtain the following corollary.

Corollary 8.12. The number $\tilde{a}(p,q,m)$ of elements in the mutation class of any $m$-coloured quiver of type $\tilde{A}_{p,q}$ is equal to the number of $(m+2)$-angulations of $P_{p,q,m}$, up to rotation of the outer and inner polygon, and up to “flip” if $p = q$ and the $(m+2)$-angulation is reflection-symmetric.

We mention that these numbers have already been determined in [BPRLS] for $m = 1$, and from these results we get the following corollary.

Corollary 8.13. The number of triangulations of $P_{p,q}$, up to rotation and flip of reflection symmetric triangulations, is given by
\[ \tilde{a}_{p,q} = \begin{cases} \frac{1}{2} \sum_{k \mid p, k \neq q} \phi(k) \left( \frac{2p}{k} \right) \left( \frac{2q}{k} \right) & \text{if } p \neq q \\ \frac{1}{2} \left( \frac{1}{2} \left( \frac{2p}{p} \right) + \sum_{k \mid p} \phi(k) \left( \frac{2p}{k} \right)^2 \right) & \text{if } p = q, \end{cases} \]

where \( \phi(k) \) is the Euler function.

Furthermore, we can also consider the number of triangulations of the annulus with \( p + q \) marked points, i.e., two triangulations are equivalent if and only if they are rotations of eachother.

**Corollary 8.14.** The number of triangulations of \( P_{p,q} \), up to rotation, is given by

\[ \frac{1}{2} \sum_{k \mid p, k \neq q} \phi(k) \left( \frac{2p}{k} \right) \left( \frac{2q}{k} \right), \]

where \( \phi(k) \) is the Euler function.

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