On the Pareto Frontier of Regret Minimization and Best Arm Identification in Stochastic Bandits

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Abstract
We study the Pareto frontier of two archetypal objectives in stochastic bandits, namely, regret minimization (RM) and best arm identification (BAI) with a fixed horizon. It is folklore that the balance between exploitation and exploration is crucial for both RM and BAI, but exploration is more critical in achieving the optimal performance for the latter objective. To make this precise, we first design and analyze the BoBW-LIL’UCB(γ) algorithm, which achieves order-wise optimal performance for RM or BAI under different values of γ. Complementarily, we show that no algorithm can simultaneously perform optimally for both the RM and BAI objectives. More precisely, we establish non-trivial lower bounds on the regret achievable by any algorithm with a given BAI failure probability. This analysis shows that in some regimes BoBW-LIL’UCB(γ) achieves Pareto-optimality up to constant or small terms. Numerical experiments further demonstrate that when applied to difficult instances, BoBW-LIL’UCB outperforms a close competitor UCBα (Degene et al. (2019)), which is designed for RM and BAI with a fixed confidence.

1. Introduction
Consider a drug company Dandit (Drug Bandit) that wants to design an effective vaccine for a certain virus. It has a certain number of feasible options, say L = 10. Because Dandit has a limited budget, it can only test vaccines for a fixed number of times, say T = 1, 000. Using the limited number of tests, it wants to find the option that will lead to the “best” outcome, e.g., the maximum efficacy of the drug. At the same time, Dandit aims to protect individuals from potentially adverse side effects of the vaccines to be tested. How can Dandit find the optimal drug design and, at the same time, protect the health of participants? We design an algorithm BoBW-LIL’UCB that allows Dandit to balance between these two competing targets. In complement, we also show that it is impossible for Dandit to achieve optimal performances for both targets simultaneously, and Dandit has to settle for operating on the Pareto frontier of the two objectives.

To solve Dandit’s problem, we study the Regret Minimization (RM) and Best Arm Identification (BAI) problems for stochastic bandits with a fixed time horizon or budget. While most existing works only study one of these two targets (Auer et al., 2002a; Audibert and Bubeck, 2010), Degene et al. (2019) designed the UCBα algorithm for both RM and BAI with a fixed confidence. Therefore, these studies are not directly applicable to Dandit’s problem as Dandit is interested in obtaining the optimal item and minimizing the damage across a fixed number of tests. However, our setting dovetails neatly with company Dandit’s goals. Dandit can utilize our algorithm to sequentially and adaptively select different design options to test the vaccines and to eventually balance between choosing the optimal vaccine and, in the process, mitigating any physical damage on the participants. We also show that Dandit cannot achieve both targets optimally and simultaneously.

Beyond any specific applications, we believe that this problem is of fundamental theoretical importance in the broad context of multi-armed bandits (MAB). In order to design an efficient bandit algorithm, a well-known challenge is to balance between exploitation and exploration (Auer et al., 2002a; Lattimore and Szepesvári, 2020). Our work studies the Pareto frontier of RM and BAI, as well as the effects of exploitation and exploration on these two aims.

Main contributions. In stochastic bandits, there are L items with different unknown reward distributions. At each time step, a random reward is generated from each item’s distribution. Based on the previous observations, a learning agent selects an item and observes its reward. Given the number of time steps T ∈ N, the agent aims to maximize the cumulative rewards and to identify the optimal item with
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high probability.

Our first main contribution is the BoBW-LIL’UCB(γ) algorithm. BoBW-LIL’UCB(γ) is designed for both RM and BAI over a fixed time horizon, which achieves Pareto-optimality of RM and BAI in some regimes.

(i) On the other hand, we can shrink the confidence radius of each item by increasing γ, which encourages BoBW-LIL’UCB(γ) to pull items with high empirical mean rewards (exploitation) and leads to high rewards (i.e., small regret).

(ii) On the other hand, we can enlarge the confidence radius by decreasing γ to encourage the exploration of items that have not been sufficiently pulled in previous time steps (exploration); this will result in a high BAI success probability.

The parameter γ in BoBW-LIL’UCB(γ) can be tuned such that either its cumulative regret or its failure probability almost matches the corresponding state-of-the-art lower bound (Lai and Robbins, 1985; Carpentier and Locatelli, 2016). The performance of BoBW-LIL’UCB(γ) implies that exploitation is more critical in achieving the optimal performance for RM, while exploration is more crucial for BAI.

Moreover, we evaluate the Pareto frontier of RM and BAI theoretically. Lattimore and Szepesvári (2020, Note 33.3) mention that an asymptotically optimal algorithm for RM incurs a failure probability that decays only polynomially with T. This implies that if the cumulative regret of an algorithm is small, this may adversely affect its performance on a BAI task. We generalize this observation for all RM and BAI algorithms. In detail, given the BAI failure probability of an algorithm, we establish non-trivial lower bounds on its regret. We conclude that it is impossible for any algorithm to achieve asymptotically optimal performances for both RM and BAI simultaneously.

Furthermore, BoBW-LIL’UCB(γ) empirically outperforms a close competitor UCBα (Degene et al., 2019) in difficult scenarios in which the differences between the optimal and suboptimal items are small. While both algorithms identify the optimal item with high probability, UCBα, designed for the fixed-confidence case, requires a longer horizon to do so and also suffers from larger regret. This demonstrates the superiority of BoBW-LIL’UCB(γ) under the fixed-budget setting, which it is specifically designed for.

Novelty. (i) We are the first to design an algorithm for both RM and BAI with a fixed budget. We can adjust the proposed BoBW-LIL’UCB(γ) algorithm to perform (near-)optimally for both RM and BAI with proper choices of γ. (ii) The performance of BoBW-LIL’UCB(γ) implies that exploitation is more crucial to obtain a small regret, while exploration is more critical to shrink the BAI failure probability. (iii) We quantify the Pareto frontier of RM and BAI. We show that it is inevitable for any algorithm to compromise between RM and BAI in a fixed horizon setting.

Literature review. Both the RM and BAI problems have been studied extensively for stochastic multi-armed bandits. Firstly, an RM algorithm aims to maximize its cumulative rewards, i.e., to minimize its regret (the gap between the highest cumulative rewards and the obtained rewards). One line of seminal works on RM involve the class of Upper Confidence Bound (UCB) algorithms (Auer et al., 2002a; Garivier and Cappé, 2011), while another line of works study Thompson sampling (TS) algorithms (Agrawal and Goyal, 2012; Russo and Van Roy, 2014; Agrawal and Goyal, 2017). Lai and Robbins (1985) derived a lower bound on the regret of any online algorithm.

Secondly, there are two complementary settings for BAI: (i) given T ∈ N, the agent aims to maximize the probability of finding the optimal item in at most T steps (Audibert and Bubeck, 2010; Karlin et al., 2013; Zhong et al., 2021); (ii) given δ > 0, the agent aims to find the optimal item with the probability of at least 1 − δ in the smallest number of time steps (Bubeck et al., 2013; Kaufmann and Kalyanakrishnan, 2013). These two settings are known as the fixed-budget and fixed-confidence settings respectively. Moreover, Carpentier and Locatelli (2016) established a lower bound on the failure probability of any algorithm in a fixed time horizon.

While most existing works focus solely on RM or BAI, Degene et al. (2019) explored both goals with a fixed confidence and proposed the UCBα algorithm. However, to the best of our knowledge, there is no existing analysis of one algorithm for both RM and BAI in a fixed horizon. Our work fills in this gap by proposing the BoBW-LIL’UCB(γ) algorithm and proving that it achieves Pareto-optimality in some regimes. We also study the Pareto frontier of RM and BAI, which depends on the balance between exploitation and exploration. We show that a single algorithm cannot perform optimally for both RM and BAI simultaneously.

2. Problem Setup

For any n ∈ N, we denote the set {1, . . . , n} as [n]. Let there be L ∈ N ground items, contained in [L]. A random variable X (or its distribution) is σ-sub-Gaussian (σ-SG) if E[eσ(X−EX)] ≤ exp(σ²/2). Each item i ∈ [L] is associated with a σ-SG reward distribution νi, mean wi, and variance σ²i. The distributions {νi}i∈[L], means {wi}i∈[L], and variances {σ²i}i∈[L] are unknown to the agent. We let {g_{i,t}}_{t=1}^{T} be the i.i.d. sequence of rewards associated with item i during the T time steps; each g_{i,t} is an independent sample from νi.

We focus on instances with a unique item of the highest mean reward, and assume that w_{1} > w_{2} ≥ . . . ≥ w_{L}, so
the unique optimal item \(i^* = 1\). Note that the items can, in general, be arranged in any order; the ordering that \(w_i \geq w_j\) for \(i < j\) is employed to ease our discussion. We denote \(\Delta_{1,i} := w_1 - w_i\) as the optimality gap of item \(i\), and assume \(\Delta_{1,i} \leq 1\) for all \(i \in [L]\); this can be achieved by rescaling the instance if necessary. We define the minimal optimality gap

\[
\Delta := \min_{i \neq 1} \Delta_{1,i}.
\]

Clearly, \(\Delta > 0\). We characterize the hardness of an instance with the following canonical quantities:

\[
H_1 := \sum_{i \neq 1} \frac{1}{\Delta_{1,i}} \quad \text{and} \quad H_2 := \sum_{i \neq 1} \frac{1}{\Delta_{1,i}^2}.
\]

The agent uses an online algorithm \(\pi\) to decide the item \(i^*_T\) to pull at each time step \(t\), and the item \(i^*_t\) to output eventually. More formally, an online algorithm consists of a tuple \(\pi := ((\pi_t)_{t=1}^T, \pi_T^*)\), where

- the sampling rule \(\pi_t\), determined, based on the observation history, the item \(i^*_t\) to pull at time step \(t\). That is, the random variable \(i^*_t\) is \(F_{t-1}\)-measurable, where \(F_t := \sigma(i^*_1, g^*_1, i^*_2, g^*_2, \ldots, i^*_t, g^*_t)\);

- the recommendation rule \(\phi^*_T\) chooses an item \(\pi_T^*\), that is, by definition, \(\Phi_T\)-measurable.

Moreover, we define the pseudo-regret \(R_T\) of \(\pi\) as

\[
R_T(\pi) := \max_{1 \leq t \leq T} \left[ \sum_{i=1}^T g_{i,t} \right] - \mathbb{E} \left[ \sum_{i=1}^T g_{i^*_t,i} \right] = T \cdot w_1 - \mathbb{E} \left[ \sum_{i=1}^T w_{i^*_t,i} \right].
\]

The algorithm \(\pi\) aims to both minimize the pseudo-regret \(R_T(\pi)\) and at the same time, to identify the optimal item with high probability, i.e., to minimize the failure probability \(e_T(\pi) := \Pr(i^*_T \neq i^*_T) \neq 1\). We omit \(T\) and/or \(\pi\) in the superscript or subscript when there is no cause of confusion. We write \(R_T(\pi)\) as \(R_T(\pi,T)\), \(e_T(\pi)\) as \(e_T(\pi,T)\) when we wish to emphasize their dependence on both the algorithm \(\pi\) and the instance \(T\).

### 3. Discussion on Existing Algorithms

Although there is no existing work that analyzes a single algorithm for both RM and BAI in a fixed horizon, it is natural to question if an algorithm which is originally designed for RM can also perform well for BAI, and vice versa. In Table 3.1, we present the theoretical results from some existing works. We focus on algorithms that are with (potential) theoretical guarantees for both RM and BAI. We define

\[
H_p' := \max_{i \neq 1} \frac{i^p}{\Delta_{i,i}^2} \quad \text{and} \quad C_p := 2^{-p} + \sum_{r=2}^L r^{-p}
\]

for \(p > 0\) as in Shahrampour et al. (2017). We abbreviate SEQUENTIAL HALVING as SH, NONLINEAR SEQUENTIAL ELIMINATION with parameter \(p\) as NSE\((p)\), and UCB-E with parameter \(a\) as UCB-E\((a)\). Also see Appendix A for more discussions.

**Table 3.1.** Comparison among upper bounds for algorithms and lower bounds in stochastic bandits.

| Algorithm/Instance | Pseudo-regret \(R_T\) | Failure Probability \(e_T\) |
|---------------------|----------------------|--------------------------|
| SH                  | \(\Theta(T)\)        | \(\approx \exp \left( -\frac{T}{8H_2 \log L} \right)\)               |
| NSE\((p)\)          | \(\Theta(T)\)        | \(\approx \exp \left( -\frac{2(T-L)}{H_2 C_p} \right)\)               |
| UCB-E\((a \log T)\) |                     |                          |
| BoBW-LIU/UCB\((\gamma)\) (Theorem 4.1) |                     |                          |
| BoBW-LIU/UCB\((\gamma)\) |                     |                          |
| Stochastic Bandits | \(= 4H_1 \log T\)  | \(= \exp \left( -\frac{400T}{H_2 \log L} \right)\)               |

According to the discussions on RM and BAI in Lattimore and Szepesvári (2020), any algorithm with an asymptotically optimal regret would incur a failure probability lower bounded by \(\Omega(T(1+\alpha(1))(1+\epsilon)^2)\); this is much larger than the state-of-the-art lower bound \(\Omega(\exp(-400T/(H_2 \log L)))\) by Carpenter and Locatelli (2016). Therefore, we only include algorithms that were designed for BAI in Table 3.1.

Among the various BAI algorithms, SH and NSE\((p)\) perform almost the best. However, their bounds on the failure probabilities are incomparable in general. The comparison among more BAI algorithms is provided in Table A.1. Due to the designs of SH and NSE\((p)\), we surmise their regrets grow linearly with \(T\), which is vacuous for the RM task.

Although UCB-E\((a \log T)\) has upper bounds on both pseudo-regret and failure probability, its bound on the latter, which decays only polynomially fast with \(T\) when \(a\) is an absolute constant, is clearly suboptimal vis-à-vis the state-of-the-art lower bound by Carpenter and Locatelli (2016). In order to achieve an exponentially decaying upper bound on \(e_T\) (i.e., \(\exp(-\Theta(T))\)), we need to set \(\alpha = O(T/\log T)\), and hence the regret bound (see Corollary A.1 in the supplementary) will be \(O(T^2/\log T)\), which is vacuous.

The discussion above raises a natural question. Is it possible to provide a non-trivial bound on the regret for an algorithm that performs optimally for BAI over a fixed horizon?
This motivates us to design the BoBW-LIL’UCB algorithm, which can be tuned to perform near-optimally for both RM and BAI.

4. The BoBW-LIL’UCB Algorithm

We design and analyze BoBW-LIL’UCB(γ) (Best of Both Worlds-Law of Iterated Logs-UCB), an algorithm for both RM and BAI in a fixed horizon. By choosing parameter γ judiciously, the guarantees of BoBW-LIL’UCB(γ) match those of the state-of-the-art algorithms for both RM (up to log factors) and BAI (concerning the exponential term).

Algorithm 1 BoBW-LIL’UCB(γ)

1: Input: time budget T, size of ground set of items L, scale σ > 0, ε ∈ (0, 1), β ≥ ε, and γ ∈ (0, 1).
2: Sample i_t = i for t = 1, ..., L and set t = L.
3: For all i ∈ [L], compute N_{i,t}, g_{i,t}, C_{i,t,γ}, U_{i,t,γ}:
   \[N_{i,t} = \sum_{u=1}^{t} 1\{i_u = i\}, \quad g_{i,t} = \sum_{u=1}^{t} 1\{i_u = i\} / N_{i,t},\]
   \[C_{i,t,γ} = 5σ(1 + √ε) \sqrt{2(1+ε)/N_{i,t}} \log \left( \frac{\log(β+(1+ε)N_{i,t})}{γ} \right),\]
   \[U_{i,t,γ} = g_{i,t} + C_{i,t,γ} \]
4: For t = L + 1, ..., T do
5: Pull item i_t = arg max_{i∈[L]} U_{i,t-1,γ}.
6: For all i ∈ [L], update N_{i,t}, g_{i,t}, C_{i,t,γ}, U_{i,t,γ}.
7: end for
8: Output i_{out} = arg max_{i∈[L]} g_{i,T}.

Design of algorithm. We design BoBW-LIL’UCB in the spirit of the law of the iterated logarithm (LIL) (Darling and Robbins, 1967; Jamieson et al., 2014). We remark that it is a variation of the LIL’UCB algorithm proposed by Jamieson et al. (2014). The three differences are:

(i) to construct the confidence radius C_{i,t,γ}, we replace \((1+β)\) and \(δ\) in LIL’UCB by 5 and \(γ\) in BoBW-LIL’UCB(γ) respectively;
(ii) in the design of C_{i,t,γ}, we also replace log(\((1+ε)N_{i,t}\)) by log(\((β+(1+ε)N_{i,t})\));
(iii) BoBW-LIL’UCB(γ), which is designed for both RM and BAI in a fixed horizon, involves no stopping rule since it proceeds for exactly T time steps; while LIL’UCB is designed for BAI with a fixed confidence. Although our algorithm depends on the choices of ε, β, and γ, we term it as BoBW-LIL’UCB(γ) instead of the more verbose BoBW-LIL’UCB(ε, β, γ) because we scale the confidence radius by only varying γ which adjusts the performance of the algorithm. More precisely, inspired by the LIL (see Theorem B.1), we design item i’s confidence radius C_{i,t,γ} with \(N_{i,t}\) (the number of time steps when item i is pulled up to and including the \(t^{th}\) time step) and \(g_{i,t}\) (the empirical mean of item i at time step t), and its upper confidence bound \(U_{i,t,γ}\) accordingly.

The design of BoBW-LIL’UCB(γ) allows us to shrink \(C_{i,t,γ}\), the confidence radius of each item i, by increasing γ; and vice versa. Moreover, with a fixed γ, if item i is rarely pulled in previous time steps, it has a small \(N_{i,t}\) and hence a large \(C_{i,t,γ}\); and vice versa.

(i) Therefore, when γ increases, the dominant term in \(U_{i,t,γ} = g_{i,t} + C_{i,t,γ}\) becomes the empirical mean \(\hat{g}_{i,t}\). Since BoBW-LIL’UCB pulls the item with the largest \(U_{i,t-1,γ}\) at time step t, the algorithm tends to pull the item with the largest empirical mean in this case. In other words, a large γ encourages exploitation.

(ii) When γ decreases, the confidence radius \(C_{i,t,γ}\) dominates \(U_{i,t,γ}\). Consequently, BoBW-LIL’UCB is likely to pull items with large \(C_{i,t,γ}\), i.e., the rarely pulled items with small \(N_{i,t}\). This indicates that a small γ encourages exploration.

Altogether, we can scale \(U_{i,t,γ}\) by adjusting γ, which allows us to balance exploitation and exploration and trade-off between the twin objectives — RM and BAI.

Analysis for RM. We first derive problem-dependent and problem-independent bounds on the pseudo-regret of BoBW-LIL’UCB(γ).

Theorem 4.1 (Bounds on the pseudo-regret of BoBW-LIL’UCB). Let \(ε ∈ (0, 1), β ≥ ε, and γ ∈ (0, log(β + 1 + ε)/ε). The pseudo-regret of BoBW-LIL’UCB(γ) satisfies

\[R_T ≤ O\left(\sigma^2 \cdot \sum_{i≠j} \frac{\log(1/γ)}{Δ_{1,i}}\right),\]
\[R_T ≤ O\left(\sigma^2 \sqrt{TL} \log \left(\frac{\log(T/(Lγ))}{γ}\right)\right).

Furthermore, we can set γ = 1/√T to obtain

\[R_T ≤ O\left(\sigma^2 \cdot \sum_{i≠j} \frac{\log T}{Δ_{1,i}}\right), \quad R_T ≤ O(σ^2\sqrt{TL} \log T).

We observe that the order of the problem-dependent upper bound on the pseudo-regret of BoBW-LIL’UCB(1/√T) almost matches that of the lower bound (Lai and Robbins, 1985). Moreover, the worst-case (problem-independent) upper bound of BoBW-LIL’UCB(1/√T) is \(O(√TL)\) (Bubeck et al., 2012) up to log factors. This implies that we can tune the parameter γ in the BoBW-LIL’UCB(γ) algorithm to obtain close-to-optimal performance for RM.

We remark that when the optimal item is not unique, we can also derive analogous upper bounds on the pseudo-regret of BoBW-LIL’UCB(γ) using a similar line of analysis (see Proposition C.1).
Analysis for BAI. Next, we upper bound the failure probability of BoBW-LIL’UCB(γ).

Theorem 4.2 (Bounds on the failure probability of BoBW-LIL’UCB). Let ε ∈ (0, 1), β ≥ ε, and γ ∈ (0, log(β + 1 + ε)/ε). Let Δ_i = max{Δ, Δ_{1,i}} for all i ∈ [L]. The failure probability of BoBW-LIL’UCB(γ) satisfies

\[
e_T \leq \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma}{\log(1 + \varepsilon)} \right)^{1+\varepsilon},
\]

(4.1)

if

\[
\frac{T - L}{1 + \varepsilon} \geq \sum_{i=1}^{L} \frac{72\sigma^2}{\Delta_i^2} \cdot \log \left( \frac{2.8}{\gamma^2} \log \left( \frac{11\sigma(1 + \varepsilon)^2}{\Delta_i} + \beta \right) \right).
\]

In particular, the bound on \( e_T \) in (4.1) holds when γ ≥ γ_1(Δ, H_2), where

\[
\gamma_1(\Delta, H_2) = \sqrt{\frac{2.8 \log \left( \frac{6\sqrt{2} \sigma(1 + \varepsilon)^2}{\Delta} + \beta \right)}{\Delta}} \cdot \exp \left( -\frac{T - L}{144\sigma^2(1 + \varepsilon)^3(H_2 + \Delta^{-2})} \right).
\]

When γ assumes its lower bound \( \gamma_1(\Delta, H_2) \), we have

\[
e_T \leq \hat{O} \left( L \exp \left( -\frac{T - L}{144\sigma^2(1 + \varepsilon)^2(H_2 + \Delta^{-2})} \right) \right). \quad (4.2)
\]

When \( T \gg L \), the gap between our upper bound in (4.2) and \( \Omega(\exp(-400T/(H_2 \log L))) \), the state-of-the-art lower bound (Carpentier and Locatelli, 2016), is manifested by the (pre-exponential) term \( L \) as well as the constant in the exponent. This indicates that BoBW-LIL’UCB(γ) can be adjusted to perform near-optimally for BAI over a fixed horizon.

Further observation. As discussed earlier, BoBW-LIL’UCB(γ) encourages more exploitation than exploration when γ is large (e.g. γ = 1/\sqrt{T}) and it stimulates more exploration when γ is small (e.g. γ = γ_1(Δ, H_2)). Besides, Theorems 4.1 and 4.2 imply that the pseudo-regret of BoBW-LIL’UCB(γ) decreases with γ while its failure probability increases with γ. Therefore, to minimize the regret, we should increase γ to stimulate exploitation; and we should decrease γ to encourage exploration for obtaining a small failure probability. This indicates that an optimal RM algorithm encourages more exploitation compared to an optimal BAI one, and vice versa.

5. Pareto Frontier of RM and BAI

Theorems 4.1 and 4.2 together suggest that BoBW-LIL’UCB(γ) cannot perform optimally for both RM and BAI simultaneously with a universal (or single) choice of γ. In this section, we prove that no algorithm can perform optimally for these two objectives simultaneously. Given a certain failure probability of an algorithm, our goal is to establish a non-trivial lower bound on its pseudo-regret.

We first consider bandit instances in which items have bounded rewards. Let \( B_1(\Delta, \overline{R}) \) denote the set of stochastic instances where (i) the minimal optimality gap \( \Delta \geq \Delta \); and (ii) there exists \( R_0 \in \mathbb{R} \) such that the rewards are bounded in \([R_0, R_0 + \overline{R}]\). Let \( B_2(\Delta, \overline{R}, \overline{H}_2) \) denote the set of instances that (i) belong to \( B_1(\Delta, \overline{R}) \), and (ii) have hardness quantities \( H_2 \leq \overline{H}_2 \).

Theorem 5.2. Let \( \phi_T, \Delta, \overline{V}, \overline{H}_2 > 0 \). Let π be any algorithm with \( e_T(\pi, I) \leq \exp(-\phi_T)/4 \) for all I ∈ \( B_1(\Delta, \overline{V}) \). Then

\[
\sup_{I \in B_1(\Delta, \overline{V})} R_T(\pi, I) \geq \phi_T \cdot \frac{(L - 1)\overline{V}}{2\Delta},
\]

\[
\sup_{I \in B_2(\Delta, \overline{V}, \overline{H}_2)} R_T(\pi, I) \geq \phi_T \cdot \frac{\Delta\overline{H}_2^2\overline{V}}{4}.
\]

By characterizing stochastic rewards with different statistics, Theorems 5.1 and 5.2 provide different lower bounds on the pseudo-regret. We observe that when the rewards of items are bounded in \([R_0, R_0 + \overline{R}]\) for some \( R_0 \in \mathbb{R} \), the variances of the rewards are bounded by \( \overline{V}^2/4 \). Therefore,

\[
B_1(\Delta, \overline{R}) \subset B_1' \left( \Delta, \frac{\overline{R}^2}{4} \right),
\]

\[
B_2(\Delta, \overline{R}, \overline{H}_2) \subset B_2' \left( \Delta, \frac{\overline{R}^2}{4} \overline{H}_2 \right).
\]
Besides, it is clear that
\[ B_1(\Delta, \overline{R}), B_2(\Delta, \overline{R}, h), B'_2(\Delta, \frac{\overline{R}^2}{4}, h) \subset B'_1(\Delta, \frac{\overline{R}^2}{4}). \]
Due to the relationship among these four sets of instances, we let \( \tau \) be an algorithm with \( e^{-\tau} \leq \exp(-\phi_T)/4 \) in any instance of \( B'_1(\Delta, \frac{\overline{R}^2}{4}) \), and compare the derived lower bounds on its pseudo-regret \( R_T \) in Table 5.1.

| Instance Set | Bound on \( R_T \) |
|--------------|-------------------|
| \( B_1(\Delta, \overline{R}) \) | \( \phi_T \cdot (L - 1)\overline{R}/(8\Delta) \) |
| \( B_2(\Delta, \overline{R}, \overline{H}_2) \) | \( \phi_T \cdot \Delta^2 \overline{H}_2^3/8 \) |
| \( B'_2(\Delta, \frac{\overline{R}^2}{4}, \overline{H}_2) \) | \( \phi_T \cdot \Delta \overline{H}_2^3/8 \) |

Table 5.1 indicates that

- when the bound for \( B_1(\Delta, \overline{R}) \) (first row of Table 5.1) holds for \( B'_2(\Delta, \frac{\overline{R}^2}{4}) \), the quantities \( L \) and \( \Delta \) are of the same order in the bounds derived for \( B_1(\Delta, \overline{R}) \) and \( B'_1(\Delta, \frac{\overline{R}^2}{4}) \); respectively;
- similarly, when the bound for \( B_2(\Delta, \overline{R}, \overline{H}_2) \) (second row of Table 5.1) holds for \( B'_2(\Delta, \frac{\overline{R}^2}{4}, \overline{H}_2) \), the quantities \( L \) and \( \Delta \) are of the same order in the bounds for \( B_2(\Delta, \overline{R}, \overline{H}_2) \) and \( B'_2(\Delta, \frac{\overline{R}^2}{4}, \overline{H}_2) \).

Moreover, when \( \overline{R} > 1 \), we can apply the analysis of \( B'_2(\Delta, \frac{\overline{R}^2}{4}, \overline{H}_2) \) to obtain a better bound (higher lower bound) for \( B'_2(\Delta, \frac{\overline{R}^2}{4}, \overline{H}_2) \).

In any set of instances studied in Theorems 5.1 or 5.2,

- when \( \phi_T \) linearly grows with \( T \), which is typical in the bounds on \( e^{-\tau} \) (Karnin et al., 2013; Carpentier and Locatelli, 2016), the corresponding bound on \( R_T \) grows linearly with \( T \) (vacuous);
- when the bound on \( R_T \) grows with \( \log T \) as in Garivier and Cappé (2011) and Lai and Robbins (1985), \( \phi_T \) grows logarithmically with \( T \) (i.e., if \( \phi_T \) decays polynomially).

Thus, we cannot achieve optimal performances for both RM and BAI using any algorithm with fixed parameters. Alternatively, we can apply BoBW-LIL’UCB(\( \gamma \)) to achieve the best of both objectives with proper choices of the single parameter \( \gamma \).

6. On the Tightness of the Upper and Lower Bounds

To evaluate the tightness of the lower bounds derived in Section 5, we compare the upper and lower bounds on the pseudo-regret of BoBW-LIL’UCB(\( \gamma \)) when the horizon \( T \) is allowed to be arbitrarily large. We show that the parameter \( \gamma \) is crucial for BoBW-LIL’UCB(\( \gamma \)) to achieve the Pareto frontier (up to constant or small terms) in some regimes.

**Corollary 6.1.** Let \( \pi_1 \) be the online algorithm BoBW-LIL’UCB(\( 1/\sqrt{T} \)) and \( \pi_2 \) be the online algorithm BoBW-LIL’UCB(\( \gamma_1(\Delta, \overline{H}_2) \)). Then

\[
\sup_{\mathcal{I} \in B_2(\Delta, \overline{H}_2)} R_T(\pi_1, \mathcal{I}) \in \Omega \left( \Delta \log \left( \frac{T}{L} \right) \right) \cap O \left( \frac{L \log T}{\Delta} \right).
\]

\[
\sup_{\mathcal{I} \in B_2(\Delta, \overline{H}_2)} R_T(\pi_2, \mathcal{I}) \in \Omega \left( \frac{(T - L) \cdot \Delta \overline{H}_2}{(\overline{H}_2^2 + \Delta^2)} \right) \cap O \left( \frac{L(T - L)}{\Delta(\overline{H}_2^2 + \Delta^2)} \right).
\]

We observe from Corollary 6.1, which combines Theorems 4.1, 4.2, and 5.1, that

- For both \( \pi_1 \) and \( \pi_2 \), the upper and lower bounds on the regret match in their dependence on \( T \);
- For both \( \pi_1 \) and \( \pi_2 \), the gaps between the upper and lower bounds depend on the term \( \Delta \overline{H}_2 \) in the lower bound and \( L/\Delta \) in the upper bound. As \( H_2 \leq (L - 1)\Delta^{-2} \) for any instance, when \( \Delta = \Delta_1 \) for all \( i \neq 1 \) (all suboptimal items have the same suboptimality gap), equality holds, and hence the bounds for \( \pi_2 \) match exactly, while the bounds for \( \pi_1 \) match up to a small \( \log(1/L) \) term.

Corollary 6.1 also implies that the parameter \( \gamma \) in BOBW-LIL’UCB(\( \gamma \)) is essential in tuning the algorithm such that it can perform optimally for either RM or BAI. This analysis shows that in some regimes, BOBW-LIL’UCB(\( \gamma \)) achieves Pareto-optimality up to constant or small (e.g., \( \log(1/L) \)) terms.

Furthermore, Corollary 6.1 also suggests that the lower bound in Theorem 5.1 is almost tight, as it can be achieved by BoBW-LIL’UCB(\( \gamma \)). Hence, up to terms logarithmic in the parameters such as \( L \), we have quantified the Pareto frontier for the trade-off between RM and BAI in stochastic bandits.

7. Sketches of the Proofs

We sketch the analysis of Theorems 4.1 to 5.2. The detailed proofs are postponed to Appendices C and D.

7.1. Proof sketch of Theorem 4.1

First, as usual, we write the pseudo-regret as

\[
R_T = \sum_{i=2}^{L} \Delta_{i, i} \cdot \mathbb{E}[N_{i, T}].
\]

It is clear that to upper bound \( R_T \), it suffices to upper bound \( \mathbb{E}[N_{i, T}] \) for each item \( i \neq 1 \).
Concentration. By the law of the iterated logarithm (see Theorem B.1), $U_{i,t,\gamma}$ upper bounds $w_i$ for each item $i$ with high probability during the whole horizon.

Sufficient observations. Since (i) BoBW-LIL’UCB pulls the item with the largest $U_{i,t,\gamma}$ at time step $t > L$ and (ii) the confidence radius $C_{i,t-1,\gamma} = U_{i,t-1,\gamma} - w_i$ shrinks as item $i$ has more observations, when $N_{i,T}$ is sufficiently large for item $i \neq 1$, we have $C_{i,t-1,\gamma} < \Delta_{1,i}/2$ and $U_{i,t-1,\gamma} < U_{i,t-1,\gamma}$. Therefore, item $i \neq 1$ will no longer be pulled in the subsequent time steps. This observation allows us to provide an almost sure upper bound on $N_{i,T}$, which completes the analysis of the problem-dependent bound on the pseudo-regret.

Worst-case bound. To obtain this, we divide the ground set $[L]$ into two classes depending on whether the optimality gap $\Delta_{1,i} \geq \sqrt{L/T}$, which leads to a worst-case bound on the pseudo-regret.

7.2. Proof sketch of Theorem 4.2

We upper bound the failure probability of BoBW-LIL’UCB($\gamma$) with a similar analysis to that for UCB-E(a) (Audibert and Bubeck, 2010). The key difference results from the different designs of confidence radii in the two algorithms.

In short, we show that when the empirical mean $\hat{g}_{i,t}$ approaches the mean $w_i$ for each item $i \in [L]$ with high probability, all items are observed for sufficiently many times, and the algorithm can identify the optimal item. To analyze the number of times we observe each item, we diligently apply the method of induction twice.

7.3. Proof sketch of Theorems 5.1 and 5.2

Design of instances. To begin with, we design $L$ instances such that the difference of instance 1 and 2 is $\ell$ in instance 1.

By the pigeonhole principle, there exists an instance $2 \leq \ell \leq L$ such that we can upper bound the expected number of observations of item $\ell$ with the pseudo-regret. Therefore, for algorithms with bounded BAI failure probabilities, we can derive a minimax lower bound over the $L$ instances on their pseudo-regrets.

Conclusion. Lastly, we classify an instance depending on its (i) minimal optimality gap; (ii) support of rewards (Theorem 5.1) or variance of rewards (Theorem 5.2); and (iii) hardness. Over a suitably constructed set of instances, we establish a lower bound on the regrets of algorithms with bounded failure probabilities.

8. Numerical Experiments

We numerically compare BoBW-LIL’UCB($\gamma$) and UCB$_\alpha$ as they are the only algorithms that can be tuned to perform (near-)optimally for both RM and BAI. Since BoBW-LIL’UCB($\gamma$) is designed for the fixed-budget setting and UCB$_\alpha$ is for the fixed-confidence setting, there cannot be a completely fair comparison between them. However, we attempt to perform fair comparisons as much as possible.

For BoBW-LIL’UCB($\gamma$), we fix $\varepsilon = 0.01$, $\beta = e$, and vary $\gamma$. For UCB$_\alpha$, we fix $\delta = 0.01$ and vary $\alpha$. We set $w_i = 0.5$, and $w_i = 0.5 - \delta$ for all $i \neq 1$. We let $\text{Bern}(\alpha)$ denote the Bernoulli distribution with parameter $\alpha$. We consider either Bernoulli or Gaussian bandits, i.e., $\nu_i = \text{Bern}(w_i)$ or $\nu_i = \mathcal{N}(w_i, 1)$.

We run BoBW-LIL’UCB($\gamma$) for $T = 10^5$ time steps, when the horizon (stopping time) of UCB$_\alpha$ depends on its stopping rule and the instance. Due to the difference between the fixed-horizon and fixed-confidence settings, the regrets of each algorithm may be accumulated over different time horizons. For each choice of algorithm and instance, we run 100 independent trials.

In our experiments, each algorithm can identify the optimal item with high probability ($\geq 95\%$) in each instance. Therefore, we focus on the comparison on (i) the time horizon each algorithm runs; and (ii) the regret incurred over its corresponding horizon.

We present the averages and standard deviations of the time horizons and regrets of each algorithm in Figures 8.1 and 8.2. We display the numerical results for Bernoulli and Gaussian instances in Figures 8.1 and 8.2 respectively. More numerical results that reinforce the conclusions herein are presented in Appendix E.

Under each instance presented in Figures 8.1 and 8.2, the regret of BoBW-LIL’UCB($\gamma$) is reduced when $\gamma$ grows, which corroborates Theorem 4.1. Both the regret and the
stopping time of $\text{UCB}_\alpha$ grow with $\alpha$, which corroborates Degene et al. (2019, Theorem 3). Moreover, we observe that the standard deviations of the regrets are larger for $\text{UCB}_\alpha$ compared to $\text{BoBW-LIL'}\text{UCB}(\gamma)$, which suggests that $\text{BoBW-LIL'}\text{UCB}(\gamma)$ is more statistically robust and consistent in terms of the regret.

Note that a larger $\Delta$ means that the difference between the optimal and suboptimal items is more pronounced, resulting in an easier instance and commensurately better performances across all algorithms. Indeed, when $\Delta$ grows, the average pseudo-regrets of both $\text{BoBW-LIL'}\text{UCB}(\gamma)$ and $\text{UCB}_\alpha$ decrease. Concerning the time horizon, for the Bernoulli instances presented in Figure 8.1, we observe that the average time horizon of $\text{UCB}_\alpha$ is larger than $10^5$ when $\Delta = 0.05$ and is about $10^5$ when $\Delta = 0.1$. Thus, in a difficult instance with a small $\Delta$ (e.g. $\Delta = 0.05$), $\text{BoBW-LIL'}\text{UCB}(\gamma)$ can identify the best item with high probability within the fixed horizon $T$, when $\text{UCB}_\alpha$ takes longer to do so, and also suffers from a larger regret. This observation, which can also be gleaned from Figure 8.2, suggests that given a fixed horizon $T$, $\text{BoBW-LIL'}\text{UCB}(\gamma)$ is superior in handling difficult instances (with small $\Delta$) compared to $\text{UCB}_\alpha$.

9. Conclusion and Future Work

In summary, we explore the Pareto frontier of RM and BAI over a fixed horizon. The performance of our $\text{BoBW-LIL'}\text{UCB}$ algorithm sheds light on the different emphases of RM and BAI. More precisely, exploitation is more critical than exploration for achieving the optimality of RM, while exploration is more crucial for BAI. Moreover, we prove that no algorithm can simultaneously perform optimally for both objectives. To mitigate this limitation, we propose $\text{BoBW-LIL'}\text{UCB}$, an algorithm that nearly achieves Pareto-optimality in some parameter regimes.

In real-life applications, it may be unrealistic to obtain i.i.d. stochastic rewards. Hence, the stochastic bandit model cannot be directly applied. This brings the study of adversarial bandits (Auer et al., 2002b; Abbasi-Yadkori et al., 2018) to the fore. Here, the rewards of each item are not necessarily drawn independently from the same distribution. In adversarial bandits, while there exists a lower bound on the regret of any algorithm (Gerchinovitz and Lattimore, 2016), there is no lower bound on the failure probability for BAI. Furthermore, there is no existing analysis of a single algorithm that is applicable to both RM and BAI in adversarial bandits. This serves as an interesting direction for future work. More ambitiously, as Abbasi-Yadkori et al. (2018) studied BAI in both stochastic and adversarial bandits, we wonder whether it is possible to design one algorithm for both RM and BAI for both stochastic and adversarial bandits. Lastly, we surmise that the Pareto frontier of RM and BAI and the trade-off between exploitation and exploration in adversarial bandits are similar to those in their stochastic counterparts.
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A. Detailed discussion on existing algorithms

While most existing works only aim to perform either RM or BAI, Degenne et al. (2019) designed and analyzed an algorithm called UCB$_{\alpha}$ for both RM and BAI under the fixed-confidence setting. Given any $\delta$, UCB$_{\alpha}$ aims to minimize the number of time steps $T$ so that $e_T \leq \delta$, and, at the same time, the incurred regret $R_T$ can also be upper bounded. Therefore, the focus of Degenne et al. (2019) differs from that of our work. We aim to study the pseudo-regret of an algorithm which can identify the best item with high probability in a fixed horizon $T$ in this work.

To the best of our knowledge, there is no existing work that analyzes a single algorithm for both RM and BAI under the fixed-budget setting. However, it is natural to question if an algorithm which is originally designed for RM can also perform well for BAI, and vice versa. We study some algorithms that are originally designed to achieve optimal performance for either RM or BAI.

RM. According to the discussions on RM and BAI in Lattimore and Szepesvári (2020) (see the second point in Note 33.3), for any algorithm with a regret that (nearly) matches the state-of-the-art lower bound (Carpentier and Locatelli, 2016):

$$\liminf_{T \to \infty} \frac{R_T(\pi)}{\log T} \geq \sum_{i \neq 1} \frac{\Delta_{1,i}}{\text{KL}(\nu_i \| \nu_1)},$$

we can construct two instances $I$ and $I'$ with

$$w_T^I > w_T^2 \geq \ldots \geq w_T^L, \quad w_T^{I'} = w_T^I + \varepsilon(w_T^I - w_T^L), \text{ for some } \varepsilon > 0$$

such that

$$e_T(\pi, I) + e_T(\pi, I') \geq \Omega(T^{(1+o(1))(1+\varepsilon)^2}). \tag{A.1}$$

This serves as a basic observation on the limitation for BAI of an algorithm that performs (near-)optimally for RM.

BAI. Audibert and Bubeck (2010) were the first to explore the BAI problem under the fixed-budget setting. Carpentier and Locatelli (2016) provided a lower bound on the failure probability of any algorithm.

In the spirit of UCB1 (Auer et al., 2002a), Audibert and Bubeck (2010) designed UCB-E for BAI. We let UCB-E$(\alpha)$ denote the UCB-E algorithm when it is run with parameter $\alpha$. When $T$ is sufficiently large, we can upper bound the pseudo-regret of UCB-E$(\alpha \log T)$ ($\alpha \geq 2$) with a similar analysis as that for UCB1 (see Proof of Theorem 1 in Auer et al. (2002a)). Besides, we can upper bound its failure probability with Theorem 1 in Audibert and Bubeck (2010).

**Corollary A.1.** Let $\alpha > 12.5$. Assume that $g_{i,t} \in [0, 1]$ for all $i \in [L]$, and $\alpha \log T \leq 25(T - L)/(36H_2)$. UCB-E$(\alpha \log T)$ satisfies

$$R_T \leq 2\alpha^2 \sum_{i \neq 1} \left( \frac{\log T}{\Delta_{1,i}} \right) + \left( 1 + \frac{\pi^2}{3} \right) \cdot \left( \sum_{i \neq 1} \Delta_{1,i} \right),$$

$$e_T \leq 2LT^{(1-2\alpha/25)}.$$

When the horizon $T$ grows, Corollary A.1 indicates that the BAI failure probability of UCB-E$(\alpha \log T)$ decays only polynomially fast. In order to achieve the upper bound on $e_T$ as $\exp(-\Theta(T))$, we need to set $\alpha = O(T/\log T)$, and hence the regret bound as shown in Corollary A.1 will be $O(T^2/\log T)$, which is vacuous.

A.1. Existing results under the fixed-budget setting of BAI

We abbreviate sequential rejects as SR, sequential halving as SH, nonlinear sequential elimination with parameter $p$ as NSE$(p)$. Besides, we simplify the bounds for algorithms which were initially analyzed for more general problems than identification of the optimal item $i^*$. We define

$$H_p' := \max_{i \neq 1} \frac{i^p}{\Delta_i}, \quad C_p := 2^{-p} + \sum_{i=2}^L i^{-p}$$

for $p > 0$ as in Shahrampour et al. (2017). We let UGAPEb$(\alpha)$ denote the UGAPEb algorithm when it is run with parameter $\alpha$. In Table A.1, we present existing bounds from some seminal works. The algorithms are listed in chronological order.
Table A.1. Comparison under the fixed-budget setting of BAI: upper bounds for algorithms and lower bounds in stochastic bandits.

| Algorithm/Instance | Reference | Failure probability $e_T$ |
|--------------------|-----------|--------------------------|
| UCB-E \(\frac{25(T - L)}{36H_2}\) | Audibert and Bubeck (2010) | \(2TL \exp -\frac{T - L}{18H_2}\) |
| SR | Audibert and Bubeck (2010) | \(L(L - 1) \exp -\frac{T - L}{(1/2 + \sum_{i=2}^{L} 1/i)H_2}\) |
| UGAPEB \(\frac{T - L}{16H_2}\) | Gabillon et al. (2012) | \(2TL \exp -\frac{T - L}{8H_2}\) |
| SAR | Bubeck et al. (2013) | \(2L^2 \exp -\frac{T - L}{8(1/2 + \sum_{i=2}^{L} 1/i)H_2}\) |
| SH | Karnin et al. (2013) | \(3 \log_2 L \cdot \exp -\frac{T}{8H_2 \log_2 L}\) |
| NSE($p$) | Shahrampour et al. (2017) | \((L - 1) \exp -\frac{2(T - L)}{H_p C_p}\) |
| Stochastic Bandits | Carpentier and Locatelli (2016) | \(\frac{1}{6} \exp -\frac{400T}{H_2 \log L}\) (Lower Bound) |

As discussed in Shahrampour et al. (2017), $H_p C_p \leq H_2 \log L$ in some special cases. Therefore, SH is better than NSE($p$) if we disregard the sub-exponential term, while NSE($p$) is better in some cases in its dependence on the exponential term. However, they are incomparable in general.

B. Useful facts

B.1. Concentration

**Theorem B.1** (Non-asymptotic law of the iterated logarithm; Jamieson et al. (2014), Lemma 3). Let $X_1, X_2, \ldots$ be i.i.d. zero-mean sub-Gaussian random variables with scale $\sigma > 0$; i.e. $\mathbb{E}[e^{\lambda X_i}] \leq \exp(\lambda^2 \sigma^2/2)$. For all $\varepsilon \in (0, 1)$ and $\gamma \in (0, \log(1 + \varepsilon)/\varepsilon)$, we have

$$\Pr(\forall \tau \geq \frac{T}{\sigma} \sum_{s=1}^{\tau} X_s \leq \sigma(1 + \sqrt{\varepsilon}) \sqrt{\frac{2(1+\varepsilon)}{\tau}} \cdot \log \left( \frac{\log((1+\varepsilon)\tau)}{\gamma} \right)) \geq 1 - \frac{2 + \varepsilon}{\varepsilon} \gamma \left( \frac{\log(1 + \varepsilon)}{\varepsilon} \right)^{1+\varepsilon}.$$  

**Theorem B.2** (Multiplicative variant of the Chernoff-Hoeffding bound; Dubhashi and Panconesi (2009), Theorem 1.1). Suppose that $X_1, \ldots, X_T$ are independent $[0, 1]$-valued random variables, and let $X = \sum_{t=1}^{T} X_t$. Then for all $\varepsilon \in (0, 1)$,

$$\Pr(X - \mathbb{E}X \geq \varepsilon \mathbb{E}X) \leq \exp \left( -\frac{\varepsilon^2}{3} \mathbb{E}X \right), \quad \Pr(X - \mathbb{E}X \leq -\varepsilon \mathbb{E}X) \leq \exp \left( -\frac{\varepsilon^2}{3} \mathbb{E}X \right).$$

B.2. Change of measure

**Lemma B.3** (Tsybakov (2008), Lemma 2.6). Let $P$ and $Q$ be two probability distributions on the same measurable space. Then, for every measurable subset $A$ (whose complement we denote by $\bar{A}$),

$$P(A) + Q(\bar{A}) \geq \frac{1}{2} \exp(-\text{KL}(P \| Q)).$$

**Lemma B.4** (Gerchinovitz and Lattimore (2016), Lemma 1). Consider two instances 1 and 2. We let $N_{i,t}$ denote the number of pulls of item $i$ up to and including time step $t$. Under instance $j (j = 1, 2)$,
• we let \((g^i_t)_{t=1}^T\) be the sequence of rewards of item \(i\) and \(i^*_t\) be the pulled item at time step \(t\), and let \(P_{j,i}\) denote the distribution of the gain of item \(i\);
• we assume \(\{g^i_t = (g^i_{1,t}, g^i_{2,t}, \ldots, g^i_{L,t})\}_{t=1}^T\) is an i.i.d. sequence, i.e., \(g^i_t\) and \(g^j_t\) are i.i.d. for \(t_1 \neq t_2\) but \(\{g^i_t\}_{t=1}^L\) can be independent.
• we let \(i^*_t\) be the pulled item at time step \(t\), and let \(\mathbb{P}_j\) denote the probability law of the process \(\{i^*_t, g^j_{i^*_t}\}_{t=1}^T\).

Then, we have

\[
\text{KL}(\mathbb{P}_1 \mid \mathbb{P}_2) = \sum_{i=1}^L \mathbb{E}_{\mathbb{P}_i}[\mathcal{N}(i,t)] \cdot \text{KL}(P_{1,i} \mid P_{2,i}).
\]

**B.3. KL divergence**

**Theorem B.5** (Pinsker’s and reverse Pinsker’s inequality; Götze et al. (2019), Lemma 4.1). Let \(P\) and \(Q\) be two distributions that are defined in the same finite space \(A\) and have the same support. We have

\[
\delta(P, Q)^2 \leq \frac{1}{2} \text{KL}(P, Q) \leq \frac{1}{\alpha_Q} \delta(P, Q)^2
\]

where \(\delta(P, Q) = \sup\{|P(A) - Q(A)| \mid A \subseteq A\} = \frac{1}{2} \sum_{x \in A} |P(x) - Q(x)|\) is the total variational distance, and \(\alpha_Q = \min_{x \in X: Q(x) > 0} Q(x)\).

**Lemma B.6** (KL divergence between two Gaussian distributions). Let \(P_1 = \mathcal{N}(\mu_1, \sigma_1^2), P_2 = \mathcal{N}(\mu_2, \sigma_2^2)\). Then

\[
\text{KL}(P_1 \mid P_2) = \log \left( \frac{\sigma_2}{\sigma_1} \right) + \frac{\sigma_2^2 + (\mu_1 - \mu_2)^2}{2\sigma_2^2} - \frac{1}{2}.
\]

**C. Analysis of BoBW-LIL’UCB(\(\gamma\))**

**Proposition C.1** (Bounds on the pseudo-regret of BoBW-LIL’UCB(\(\gamma\))). Assume the distribution \(\nu_i\) is sub-Gaussian with scale \(\sigma > 0\) for all \(i \in [L]\), and \(w_1 \geq w_2 \geq \ldots \geq w_L\). Let \(\varepsilon \in (0, 1), \beta \geq \varepsilon, \gamma \in (0, \log(\beta + 1 + \varepsilon)/\varepsilon)\). The pseudo-regret of BoBW-LIL’UCB(\(\gamma\)) satisfies

\[
R_T \leq O\left(\sigma^2 (1 + \varepsilon)^3 \cdot \sum_{i: \Delta_1, i > 0} \frac{\log(1/\gamma)}{\Delta_1, i}\right), \quad R_T \leq O\left(\sigma^2 (1 + \varepsilon)^3 \sqrt{TL} \log \left( \frac{\log(T/L\gamma)}{\gamma} \right) \right).
\]

Furthermore, we can set \(\gamma = 1/\sqrt{T}\) to obtain

\[
R_T \leq O\left(\sigma^2 (1 + \varepsilon)^3 \cdot \sum_{i: \Delta_1, i > 0} \frac{\log T}{\Delta_1, i}\right), \quad R_T \leq O(\sigma^2 (1 + \varepsilon)^3 \sqrt{TL} \log T).
\]

**C.1. Proof of Theorem 4.1**

**Theorem 4.1** (Bounds on the pseudo-regret of BoBW-LIL’UCB). Let \(\varepsilon \in (0, 1), \beta \geq \varepsilon, \gamma \in (0, \log(\beta + 1 + \varepsilon)/\varepsilon)\). The pseudo-regret of BoBW-LIL’UCB(\(\gamma\)) satisfies

\[
R_T \leq O\left(\sigma^2 \cdot \sum_{i \neq 1} \frac{\log(1/\gamma)}{\Delta_1, i}\right), \quad R_T \leq O\left(\sigma^2 \sqrt{TL} \log \left( \frac{\log(T/(L\gamma))}{\gamma} \right) \right).
\]

Furthermore, we can set \(\gamma = 1/\sqrt{T}\) to obtain

\[
R_T \leq O\left(\sigma^2 \cdot \sum_{i \neq 1} \frac{\log T}{\Delta_1, i}\right), \quad R_T \leq O(\sigma^2 \sqrt{TL} \log T).
\]
Lemma C.2 (Concentration of \( \hat{g}_{i,t} \)). Fix any \( \varepsilon \in (0, 1) \) and \( \gamma \in (0, \log(\beta + 1 + \varepsilon)/\varepsilon) \). We have

\[
\Pr \left( \bigcap_{i=1}^L \mathcal{E}_{i,\gamma} \right) \geq 1 - \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma}{\log(1 + \varepsilon)} \right)^{1+\varepsilon}.
\]

Step 2: Bound on \( N_{i,T} \) for \( i \neq 1 \). Next, for all \( t > L \), when

\[
\{ \hat{g}_{i,t-1} > w_1 - C_{1,t-1,\gamma}, \quad \hat{g}_{i,t-1} < w_i + C_{i,t-1,\gamma}, \quad \Delta_{1,i} > 2C_{i,t-1,\gamma}, \quad \forall i \neq 1 \}
\]

holds, we have

\[
\{ U_{1,t-1,\gamma} = \hat{g}_{1,t-1} + C_{1,t-1,\gamma} > w_1 = w_i + \Delta_{1,i} > w_i + 2C_{i,t-1,\gamma} > \hat{g}_{i,t-1} + C_{i,t-1,\gamma} = U_{i,t-1,\gamma}, \quad \forall i \neq 1 \},
\]

which indicates \( i_t = 1 \). In other words, when \( i_t = i \neq 1 \) for \( t > L \), one of the following holds:

\[
\hat{g}_{1,t-1} \leq w_1 - C_{1,t-1,\gamma}, \quad \hat{g}_{i,t-1} \geq w_i + C_{i,t-1,\gamma}, \quad \Delta_{1,i} \leq 2C_{i,t-1,\gamma},
\]

We see that

\[
\Delta_{1,i} \leq 2C_{i,t-1,\gamma} \leq 10\sigma(1 + \sqrt{\varepsilon}) \sqrt{\frac{2(1 + \varepsilon)}{N_{i,t-1}}} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right) - \frac{\Delta_{1,i}^2}{\Lambda_{1,i}^2} \cdot 200\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon) \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right).
\]

In order to bound \( N_{i,t-1} \), we derive the following lemma:

Lemma C.3. For all \( \tau > 0, b \geq e \), we have

\[
\tau \leq c \log \left( \frac{\log(a \tau + b)}{\rho} \right) \Rightarrow \tau \leq c \log \left( \frac{1}{\rho} \log \left( \frac{1.4 \rho}{\log(1.4 \rho + b)} \right) \right).
\]

We apply Lemma C.3 with

\[
c = \frac{200\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_{1,i}^2}, \quad a = 1 + \varepsilon, \quad \rho = \gamma, \quad \text{and} \quad a_1 = 1.4
\]

to obtain

\[
N_{i,t-1} \leq \frac{200\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_{1,i}^2} \cdot \log \left( \frac{a_1}{\gamma} \log \left( \frac{200a_1\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)^2}{\Delta_{1,i}^2} + \beta \right) \right) \equiv \tilde{N}_{i,\gamma}.
\]

Therefore, when \( t > L \), \( \bigcap_{i=1}^L \mathcal{E}_{i,\gamma} \) holds and \( N_{i,t-1} > \tilde{N}_{i,\gamma} \) for all \( i \neq 1 \), we always have \( i_t = 1 \).

Step 3: Conclusion. Consequently,

\[
R_T = \mathbb{E} \left[ \sum_{t=1}^T g_{1,t} - g_{i_t, t} \right] = \mathbb{E} \left[ \left( \sum_{t=1}^T g_{1,t} - g_{i_t, t} \right) \cdot 1 \left( \bigcap_{i=1}^L \mathcal{E}_{i,\gamma} \right) \right] + \mathbb{E} \left[ \left( \sum_{t=1}^T g_{1,t} - g_{i_t, t} \right) \cdot 1 \left( \bigcap_{i=1}^L \mathcal{E}_{i,\gamma} \right) \right]
\]

\[
\leq \mathbb{E} \left[ \left( \sum_{t=1}^T g_{1,t} - g_{i_t, t} \right) \cdot 1 \left( \bigcap_{i=1}^L \mathcal{E}_{i,\gamma} \right) \right] + \Pr \left( \bigcap_{i=1}^L \mathcal{E}_{i,\gamma} \right)
\]
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\[
\sum_{j \neq 1} \mathbb{E} \left[ \left( \sum_{i=1}^{T} g_{1,i} - g_{i,j} \right) \cdot 1 \left( i = j, \bigcap_{i=1}^{L} \mathcal{E}_{i,\gamma} \right) \right] + \Pr \left( \bigcap_{i=1}^{L} \mathcal{E}_{i,\gamma} \right)
\]

\[
\leq \sum_{j \neq 1} \Delta_{1,j} \cdot \mathbb{E} \left[ \sum_{i=1}^{T} 1(i = j) \right] \bigcap_{i=1}^{L} \mathcal{E}_{i,\gamma} + \Pr \left( \bigcap_{i=1}^{L} \mathcal{E}_{i,\gamma} \right)
\]

\[
= \sum_{j \neq 1} \Delta_{1,j} \cdot \mathbb{E} \left[ N_{j,T} \left| \bigcap_{i=1}^{L} \mathcal{E}_{i,\gamma} \right. \right] + \Pr \left( \bigcap_{i=1}^{L} \mathcal{E}_{i,\gamma} \right)
\]

\[
\leq \sum_{j \neq 1} \Delta_{1,j} \cdot (2 + \bar{N}_{j,\gamma}) + \Pr \left( \bigcap_{i=1}^{L} \mathcal{E}_{i,\gamma} \right)
\]

We see that \( a_1 = 1.4 \leq 2 \). If we divide the ground set into two classes depending on whether \( \Delta_{1,i} \geq \sqrt{L/T} \), we have

\[
R_T \leq T \cdot \sqrt{\frac{L}{T}} + 2L + \frac{200\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{L/T} \log \left( \frac{4\gamma}{\log \left( \frac{20\sigma(1 + \sqrt{\varepsilon})(1 + \varepsilon)}{\sqrt{L/T}} \right) + \beta} \right)
\]

In short, we have

\[
R_T \leq O \left( \sigma^2(1 + \varepsilon)^3 \sum_{i \neq 1} \frac{\log(1/\gamma)}{\Delta_{1,i}} \right), \quad R_T \leq O \left( \sigma^2(1 + \varepsilon)^3 \sqrt{TL} \log \left( \frac{\log(T/L\gamma)}{\gamma} \right) \right)
\]

Let \( \gamma = 1/\sqrt{T} \), we have

\[
R_T \leq O \left( \sigma^2(1 + \varepsilon)^3 \sum_{i \neq 1} \frac{\log T}{\Delta_{1,i}} \right), \quad R_T \leq O(\sigma^2(1 + \varepsilon)^3 \sqrt{TL} \log T).
\]

C.2. Proof of Theorem 4.2

**Theorem 4.2** (Bounds on the failure probability of BoBW-LIL'UCB). Let \( \varepsilon \in (0, 1) \), \( \beta \geq \varepsilon \), and \( \gamma \in (0, \log(\beta + 1 + \varepsilon))/\varepsilon \). Let \( \Delta_i = \max\{\Delta, \Delta_{1,i}\} \) for all \( i \in [L] \). The failure probability of BoBW-LIL'UCB(\( \gamma \)) satisfies

\[
e_T \leq \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma}{\log(1 + \varepsilon)} \right)^{1+\varepsilon}, \tag{4.1}
\]
\[
\frac{T - L}{(1 + \varepsilon)^3} \geq \sum_{i=1}^{L} \frac{72\sigma^2}{\Delta_i^2} \cdot \log \left( \frac{2.8}{\gamma^2} \log \left( \frac{11\sigma(1 + \varepsilon)^2}{\Delta_i + \beta} \right) \right).
\]

In particular, the bound on \(e_T\) in (4.1) holds when \(\gamma \geq \gamma_1(\Delta, H_2)\), where
\[
\gamma_1(\Delta, H_2) = \sqrt{2.8 \log \left( \frac{6\sqrt{2.8}\sigma(1 + \varepsilon)^2}{\Delta} + \beta \right)} \cdot \exp \left( - \frac{T - L}{144\sigma^2(1 + \varepsilon)^3(H_2 + \Delta^{-2})} \right).
\]

When \(\gamma\) assumes its lower bound \(\gamma_1(\Delta, H_2)\), we have
\[
e_T \leq \tilde{O}\left( L \exp \left( - \frac{T - L}{144\sigma^2(1 + \varepsilon)^3(H_2 + \Delta^{-2})} \right) \right).
\]

**Proof.** Recall that we assume \(w_1 > w_2 \geq \ldots \geq w_L\). We let \(\Delta_1 = w_1 - w_2\) and \(\Delta_i = w_1 - w_i\) for all \(i \neq 1\). Then \(\Delta = \Delta_1\) and \(\Delta_{1,i} = \Delta_i\) for \(i \neq 1\).

**Step 1: Concentration.** Let \(E_{i,\gamma}^t := \{\forall t \geq L_i | \hat{g}_{i,t} - w_i \leq C_{i,t,\gamma}/5\}\) for all \(i \in [L]\). Similarly to Lemma C.2, we can apply Theorem B.1 to show that
\[
\Pr \left( \bigcap_{i=1}^{L} E_{i,\gamma}^t \right) \geq 1 - \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma}{\log(1 + \varepsilon)} \right)^{1+\varepsilon}.
\]

In the following, we prove that conditioning on the event \(\bigcap_{i=1}^{L} E_{i,\gamma}^t\), we have \(i_{out} = 1\), which concludes the proof.

We assume \(\bigcap_{i=1}^{L} E_{i,\gamma}^t\) holds from now on. Since \(i_{out}\) is the item with the largest empirical mean, we have
\[
\hat{g}_{i_{out},t} \geq \hat{g}_{i,t} \quad \forall i \neq i_{out}, \quad \hat{g}_{i_{out},t} \geq w_{i_{out},t} - C_{i_{out},T,\gamma}/5, \quad w_i + C_{i,T,\gamma}/5 \geq \hat{g}_{i,t} \quad \forall i \neq i_{out}.
\]

Consequently, to show \(i_{out} = 1\), it is sufficient to show that
\[
\frac{C_{i,T,\gamma}}{5} \leq \frac{\Delta_i}{2} \Leftrightarrow \Delta_i \geq \frac{2C_{i,T,\gamma}}{5} = 2\sigma(1 + \sqrt{\varepsilon}) \sqrt{\frac{2(1 + \varepsilon)}{N_{i,T}} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,T})}{\gamma} \right)}
\]
\[
\Leftrightarrow N_{i,T} \geq \frac{8\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_i^2} \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,T})}{\gamma} \right) \quad \forall i \in [L]. \quad (C.1)
\]

**Step 2: Upper bound \(N_{i,t}\) \((i \neq 1)\).** To begin with, we prove by induction that
\[
N_{i,t} \leq \frac{72\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_i^2} \cdot \log \left( \frac{a_1}{\gamma} \log \left( \frac{72a_1\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)^2}{\Delta_i^2} + \beta \right) \right) + 1 \quad \forall i \neq 1. \quad (C.2)
\]

Clearly, this inequality holds for all \(i \neq 1\) when \(1 \leq t \leq L\). Now we assume that the inequality holds for all \(i \neq 1\) at time \(t - 1(t > L)\). If \(i_{t} \neq i\), we have \(N_{i,t} = N_{i,t-1}\) and the inequality still holds for \(i\). Otherwise, we have \(i_{t} = i\) and in particular \(U_{i,t-1,\gamma} \geq U_{i,t-1,\gamma}\). Since
\[
U_{i,t-1,\gamma} = \hat{g}_{i,t-1} + C_{i,t-1,\gamma} \leq w_i + \frac{6C_{i,t-1,\gamma}}{5}, \quad U_{i,t-1,\gamma} = \hat{g}_{i,t-1} + C_{i,t-1,\gamma} \geq w_1 + \frac{4C_{i,t-1,\gamma}}{5} \geq w_1 = w_i + \Delta_i,
\]
we have
\[
\frac{6C_{i,t-1,\gamma}}{5} \geq \Delta_i \Leftrightarrow N_{i,t-1} \leq \frac{72\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_i^2} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right) \quad \Rightarrow \quad N_{i,t-1} \leq \frac{72\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_i^2} \cdot \log \left( \frac{a_1}{\gamma} \log \left( \frac{72a_1\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)^2}{\Delta_i^2} + \beta \right) \right).
\]

We obtain (a) using Lemma C.3 with \(a_1 = 1.4\):
Lemma C.3. For all \( \tau > 0, b \geq e \), we have
\[
\tau \leq c \log \left( \frac{\log(a \tau + b)}{\rho} \right) \Rightarrow \tau \leq c \log \left( \frac{1.4}{\rho} \log \left( \frac{1.4ac}{\rho + b} \right) \right).
\]

Subsequently, by using \( N_{i,t} = N_{i,t-1} + 1 \), we obtain (C.2).

**Step 3: Lower bound** \( N_{i,T} \) \((i \neq 1)\). Next, we again prove by induction that
\[
N_{i,t} \geq 200\sigma^2(1 + \varepsilon)^2 \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t})}{\gamma} \right) \cdot \min \left\{ \frac{1}{25\Delta_i^2}, \frac{1}{36(C_{1,t-1,\gamma})^2} \right\}, \forall i \neq 1. \tag{C.3}
\]
Clearly, this inequality holds for all \( i \neq 1 \) when \( 1 \leq t \leq L \). Now we assume that these inequalities hold for all \( i \neq 1 \) at time \( t-1 \) \((t > L)\). If \( i_t \neq 1 \), we have
\[
N_{i,t} \geq N_{i,t-1} \quad \forall i \neq 1, \quad N_{1,t} = N_{1,t-1},
\]
which implies that the inequalities still hold for all \( i \neq 1 \). Otherwise, \( i_t = 1 \) indicates that \( U_{1,t-1,\gamma} \geq U_{i,t-1,\gamma} \) for all \( i \neq 1 \). Since
\[
U_{1,t-1,\gamma} = \hat{g}_{1,t-1} + C_{1,t-1,\gamma} \leq w_1 + \frac{6C_{1,t-1,\gamma}}{5}, \quad U_{i,t-1,\gamma} = \hat{g}_{i,t-1} + C_{i,t-1,\gamma} \geq w_i + \frac{4C_{i,t-1,\gamma}}{5},
\]
we have
\[
\frac{4C_{i,t-1,\gamma}}{5} \leq \Delta_i + \frac{6C_{i,t-1,\gamma}}{5} \Rightarrow C_{i,t-1,\gamma} = 5\sigma(1 + \sqrt{\varepsilon}) \cdot \sqrt{\frac{2(1 + \varepsilon)}{N_{i,t-1}}} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right) \leq \frac{5\Delta_i + 6C_{i,t-1,\gamma}}{4}.
\]
\[
\Leftrightarrow \frac{200\sigma^2(1 + \varepsilon)^2}{5\Delta_i + 6C_{1,t-1,\gamma}} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right) \leq \frac{N_{i,t-1}}{2(1 + \varepsilon)}.
\]
\[
\Leftrightarrow 400\sigma^2(1 + \varepsilon)^2 \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right) \leq N_{i,t-1} \cdot \frac{2(1 + \varepsilon)}{(5\Delta_i + 6C_{1,t-1,\gamma})^2} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right).
\]
We apply \( u + v \leq 2 \max \{ u, v \} \) and \( N_{i,t} = N_{i,t-1} \) for all \( i \neq 1 \) to obtain (C.3).

**Step 4: Lower bound on** \( N_{i,T} \). Recall that we want to show (C.1). (i) To show (C.1) holds for all \( i \neq 1 \), (C.3) indicates that it is sufficiently to show that
\[
\frac{200\sigma^2(1 + \varepsilon)(1 + \sqrt{\varepsilon})}{36(C_{1,T-1,\gamma})^2} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,T})}{\gamma} \right) \geq \frac{8\sigma^2(1 + \sqrt{\varepsilon})^2}{\Delta_i^2} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,T})}{\gamma} \right).
\]
Moreover, since \( \Delta_1 = \min_{i \in [L]} \Delta_i \), it is sufficient to show that
\[
\frac{25}{36(C_{1,T-1,\gamma})^2} \geq \frac{1}{\Delta_i^2} \Leftrightarrow C_{1,T-1,\gamma} \leq \frac{5\Delta_1}{6}.
\]
(ii) In order to show (C.1) holds for all \( i \in [L] \), it is sufficient to show that
\[
C_{1,T-1,\gamma} \leq \frac{5\Delta_1}{6}.
\]
This is implied by
\[
N_{1,T-1} \geq \frac{72\sigma^2(1 + \sqrt{\varepsilon})^2}{\Delta_i^2} \cdot \log \left( \frac{a_1}{\gamma} \log \left( \frac{72a_1\beta\sigma^2(1 + \sqrt{\varepsilon})^2}{\Delta_i^2} \right) \right).
\]
We obtain the inequality in the above display by applying Lemma C.3. Meanwhile, (C.2) and $t = \sum_{i=1}^{L} N_{i,t}$ implies that

$$N_{1,T} = T - \sum_{i \neq 1} N_{i,T} \geq T - (L - 1) - \sum_{i \neq 1} \frac{72\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_i^2} \cdot \log \left( \frac{a_i}{\gamma} \log \left( \frac{72a_i \sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)^2}{\Delta_i^2 \gamma} + \beta \right) \right).$$

Altogether, we complete the proof with

$$\sum_{i=1}^{L} \frac{72\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_i^2} \cdot \log \left( \frac{a_i}{\gamma} \log \left( \frac{72a_i \sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)^2}{\Delta_i^2 \gamma} + \beta \right) \right) \leq T - L + 1. \quad (C.4)$$

**Step 5: Conclusion.** Since

$$\frac{72\sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)}{\Delta_i^2} \cdot \log \left( \frac{a_i}{\gamma} \log \left( \frac{72a_i \sigma^2(1 + \sqrt{\varepsilon})^2(1 + \varepsilon)^2}{\Delta_i^2 \gamma} + \beta \right) \right) \leq \frac{72\sigma^2(1 + \varepsilon)^3}{\Delta_i^2} \cdot \log \left( \frac{2a_i}{\gamma^2} \log \left( \frac{6\sqrt{2a_i} \cdot \sigma(1 + \varepsilon)^2}{\Delta_i^2 \gamma} + \beta \right) \right),$$

To show (C.4), it is sufficient to have

$$\sum_{i=1}^{L} \frac{72\sigma^2(1 + \varepsilon)^3}{\Delta_i^2} \cdot \log \left( \frac{2a_i}{\gamma^2} \log \left( \frac{6\sqrt{2a_i} \cdot \sigma(1 + \varepsilon)^2}{\Delta_i^2 \gamma} + \beta \right) \right) \leq T - L + 1$$

$$\Leftrightarrow \sum_{i=1}^{L} \frac{72\sigma^2(1 + \varepsilon)^3}{\Delta_i^2} \cdot \log \left( \frac{2a_i}{\gamma^2} \right) \leq T - L + 1 - \sum_{i=1}^{L} \frac{72\sigma^2(1 + \varepsilon)^3}{\Delta_i^2} \cdot \log \left( \log \left( \frac{6\sqrt{2a_i} \cdot \sigma(1 + \varepsilon)^2}{\Delta_i^2 \gamma} + \beta \right) \right)$$

$$\Leftrightarrow \gamma \geq \sqrt{2a_1} \cdot \exp \left( - \frac{T - L + 1 - \sum_{i=1}^{L} 72\sigma^2\Delta_i^{-2} \cdot (1 + \varepsilon)^3 \log \left( \log \left( \frac{6\sqrt{2a_i} \cdot \sigma(1 + \varepsilon)^2}{\Delta_i^2 \gamma} + \beta \right) \right) }{\sum_{i=1}^{L} 144\sigma^2(1 + \varepsilon)^3\Delta_i^{-2}} \right).$$

Recall the definition of $H_2$ in (2.1):

$$H_2 = \sum_{i \neq 1} \frac{1}{\Delta_i^2}.$$

Furthermore, it suffices to have

$$\gamma \geq \sqrt{2a_1} \cdot \exp \left( - \frac{T - L}{144\sigma^2(1 + \varepsilon)^3(H_2 + 1/\Delta_{1,2}^2)} + \frac{1}{2} \log \left( \log \left( \frac{6\sqrt{2a_1} \cdot \sigma(1 + \varepsilon)^2}{\Delta_{1,2} \gamma} + \beta \right) \right) \right)$$

$$= \sqrt{2.8} \cdot \log \left( \frac{6\sqrt{2.8} \cdot \sigma(1 + \varepsilon)^2}{\Delta_{1,2} \gamma} + \beta \right) \exp \left( - \frac{T - L}{144\sigma^2(1 + \varepsilon)^3(H_2 + 1/\Delta_{1,2}^2)} \right) := \gamma_1(\Delta_{1,2}, H_2).$$

Note that $\Delta = \Delta_{1,2}$. When $\gamma = \gamma_1(\Delta, H_2)$,

$$\epsilon_T \leq \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma_1(\Delta, H_2)}{\log(1 + \varepsilon)} \right)^{1+\varepsilon}$$

$$= \frac{2L(2 + \varepsilon)}{\varepsilon \left( \log(1 + \varepsilon) \right)^{1+\varepsilon}} \cdot 2.8 \log \left( \frac{6\sqrt{2.8} \cdot \sigma(1 + \varepsilon)^2}{\Delta} + \beta \right)^{(1+\varepsilon)/2} \cdot \exp \left( - \frac{T - L}{144\sigma^2(1 + \varepsilon)^2(H_2 + 1/\Delta^2)} \right).$$
C.3. Proof of Lemma C.2

**Lemma C.2 (Concentration of \( \hat{g}_{i,t} \)).** Fix any \( \varepsilon \in (0, 1) \) and \( \gamma \in (0, \log(\beta + 1 + \varepsilon)/\varepsilon) \). We have

\[
\Pr\left( \bigcap_{i=1}^{L} E_{i,\gamma} \right) \geq 1 - \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma}{\log(1 + \varepsilon)} \right)^{1+\varepsilon}.
\]

**Proof.** Let

\[ E_{i,\gamma} := \{ \forall t \geq L, |\hat{g}_{i,t} - w_i| \leq C_{i,t,\gamma} \}. \]

Then

\[
\Pr(E_{i,\gamma}) = \Pr\left( \forall t \geq L, |\hat{g}_{i,t} - w_i| \leq 5\sigma(1 + \sqrt{\varepsilon}) \frac{2(1 + \varepsilon)}{N_{i,t}} \cdot \log \left( \frac{\log(\beta + (1 + \varepsilon)N_{i,t-1})}{\gamma} \right) \right).
\]

When \( \varepsilon \in (0, 1) \) and \( \gamma \in (0, \log(\beta + 1 + \varepsilon)/\varepsilon) \), Theorem B.1 indicates that

\[
\Pr(E_{i,\gamma}) \geq 1 - \frac{2(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma}{\log(1 + \varepsilon)} \right)^{1+\varepsilon}.
\]

Furthermore,

\[
\Pr\left( \bigcap_{i=1}^{L} E_{i,\gamma} \right) = 1 - \Pr\left( \bigcap_{i=1}^{L} E_{i,\gamma} \right) = 1 - \Pr\left( \cup_{i=1}^{L} E_{i,\gamma} \right)
\geq 1 - \sum_{i=1}^{L} \Pr\left( E_{i,\gamma} \right) \geq 1 - \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{\gamma}{\log(1 + \varepsilon)} \right)^{1+\varepsilon}.
\]

\[ \square \]

C.4. Proof of Lemma C.3

**Lemma C.3.** For all \( \tau > 0, b \geq c \), we have

\[
\tau \leq c \log \left( \frac{\log(\alpha \tau + b)}{\rho} \right) \Rightarrow \tau \leq c \log \left( \frac{1.4 \log(1.4ac \rho + b)}{\rho} \right).
\]

**Proof.** Let

\[ f(\tau) = c \log \left( \frac{\log(\alpha \tau + b)}{\rho} \right), \quad \tau_{a_1,a_2} = c \log \left( \frac{a_1}{\rho} \log \left( \frac{a_2c}{\rho} + b \right) \right). \]

Then

\[
\tau_{a_1,a_2} \geq f(\tau_{a_1,a_2}) \Leftrightarrow c \log \left( \frac{a_1}{\rho} \log \left( \frac{a_2c}{\rho} + b \right) \right) \geq c \log \left( \frac{1}{\rho} \log \left( \frac{a_1}{\rho} \log \left( \frac{a_2c}{\rho} \right) + b \right) \right) \Leftrightarrow a_1 \log \left( \frac{a_2c}{\rho} + b \right) \geq \log \left[ ac \log \left( \frac{a_1}{\rho} \log \left( \frac{a_2c}{\rho} \right) + b \right) \right].
\]
Let $a_1 \geq 1.4$, then $x^{a_1} \geq x \log x$ for all $x \geq 1$. To obtain $\tau_{a_1,a_2} \geq f(\tau_{a_1,a_2})$, it suffices to have
\[
\left( \frac{a_2 c}{\rho} + b \right) \cdot \log \left( \frac{a_2 c}{\rho} + b \right) \geq ac \log \left( \frac{a_1}{\rho} \log \left( \frac{a_2 c}{\rho} \right) \right) + b,
\]
which is implied by
\[
\frac{a_2 c}{\rho} \cdot \log \left( \frac{a_2 c}{\rho} + b \right) \geq \frac{ac a_1}{\rho} \log \left( \frac{a_2 c}{\rho} \right).
\]
The last inequality holds when $a_2 \geq a \cdot a_1$. Since $\tau - f(\tau)$ is monotonically increasing in $\tau$, and $\tau_{a_1,a_2} \geq f(\tau_{a_1,a_2})$, i.e., $\tau_{a_1,a_2} - f(\tau_{a_1,a_2}) \geq 0$, we have
\[
\tau \geq \tau_{1.4,1.4a} \Rightarrow \tau - f(\tau) \geq 0.
\]
In other words,
\[
\tau \leq f(\tau) \Rightarrow \tau \leq \tau_{1.4,2.8} = c \log \left( \frac{1.4}{\rho} \log \left( \frac{1.4ac}{\rho} \right) \right).
\]

\section*{D. Analysis of the Pareto frontier of RM and BAI}

\subsection*{D.1. Proof of Theorem 5.1}

\textbf{Theorem 5.1.} Let $\phi_T, \triangle, \overline{R}, \overline{H}_2 > 0$. Let $\pi$ be any algorithm with $e_T(\pi, \mathcal{I}) \leq \exp(-\phi_T)/4$ for all $\mathcal{I} \in \mathcal{B}_1(\triangle, \overline{R})$. Then
\[
\sup_{\mathcal{I} \in \mathcal{B}_1(\triangle, \overline{R})} R_T(\pi, \mathcal{I}) \geq \phi_T \cdot \frac{(L - 1)\overline{R}}{8\triangle},
\]
\[
\sup_{\mathcal{I} \in \mathcal{B}_1(\triangle, \overline{R}, \overline{H}_2)} R_T(\pi, \mathcal{I}) \geq \phi_T \cdot \frac{\Delta \overline{H}_2 \overline{R}^3}{8}.
\]

\textbf{Proof.} \textbf{Step 1: Construct instances.} To begin with, we fix any $\sigma > 0, d_\ell \in (0, 1/4]$ for all $2 \leq \ell \leq L$. We let $\text{Bern}(a)$ denote the Bernoulli distribution with parameter $a$. We define the following distributions:
\[
\nu_1 := \text{Bern}(1/2), \quad \nu_\ell := \text{Bern}(1/2 - d_\ell) \quad \forall 1 < \ell \leq L;
\]
\[
\nu'_1 := \text{Bern}(1/2), \quad \nu'_\ell := \text{Bern}(1/2 + d_\ell) \quad \forall 1 < \ell \leq L.
\]

We construct $L$ instances such that under instance $\ell$ ($1 \leq \ell \leq L$), the stochastic reward of item $i$ is drawn from distribution
\[
\nu'_\ell := b \cdot (\nu_1 \{i \neq \ell\} + \nu'_1 \{i = \ell\}),
\]
where $b > 0$. Under instance $\ell$ ($1 \leq \ell \leq L$), we see item $\ell$ is optimal, and we define several other notations as follows:

(i) We let $g_{i,t}^\ell$ be the random reward of item $i$ at time step $t$. Then $g_{i,t}^\ell \in \{0, b\}$.

(ii) We let $\Delta_{i,t}^\ell := \mathbb{E}[(\sum_{t=1}^T g_{i,t}^\ell - g_{i,t}^\ell)]/T$ denote the gap between item $i$ and $j$. Then
\[
\Delta_{1,j}^\ell = b \cdot d_j \quad \forall 2 \leq j \leq L, \quad \Delta_{\ell,1}^\ell = b \cdot d_\ell, \quad \Delta_{\ell,j}^\ell = b \cdot d_\ell + b \cdot d_j \quad \forall 2 \leq j, \ell \leq L, j \neq \ell.
\]

(iii) We denote the difficulty of the instance with
\[
H_2(\ell) := \sum_{j \neq \ell} (\Delta_{\ell,j}^\ell)^{-2}.
\]

Then $H_2(1) = \max_{1 \leq \ell \leq L} H_2(\ell) \leq (L - 1)b^{-2} \cdot \max_{2 \leq \ell \leq L} d_\ell^{-2}$. 


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(iv) We let $i_t^\ell$ be the pulled item at time step $t$, and $O_\ell^T = \{i_t^\ell, g_{i_t^\ell, t}\}_{t=1}^T$ be the sequence of pulled items and observed rewards up to and including time step $t$.

(v) We let $\mathbb{P}_\ell^T$ be the measure on $O_\ell^T$, and let $P_{\ell, i}$ be the measure on the rewards of item $i$.

For simplicity, we abbreviate $\mathbb{P}_\ell^T$, $O_\ell^T$ as $\mathbb{P}_\ell$, $O_\ell$ respectively. Moreover, we let $N_i(t)$ denote the number of pulls of item $i$ up to and including time step $t$.

**Step 2: Change of measure.** First of all, we apply Lemmas B.3 and B.4 obtain that for all $1 \leq \ell \leq L$,

$$\Pr_{O_1}(i_{\text{out}} \neq 1) + \Pr_{O_\ell}(i_{\text{out}} = 1) \geq \frac{1}{2} \exp(-\mathbb{KL}(\mathbb{P}_\ell \parallel \mathbb{P}_1)) = \frac{1}{2} \exp(-\mathbb{E}_{P_1}[N_\ell(T)] \cdot \mathbb{KL}(P_1, \ell \parallel P_{\ell, \ell})) - \Pr_{O_\ell}(i_{\text{out}} = 1).

Suppose the expected pseudo-regret is upper bounded by $\overline{\text{Reg}}$, we have

$$\overline{\text{Reg}} \geq \mathbb{E}_{P_1} \left[ \sum_{t=1}^T 1\{i_t^1 \neq 1\} \cdot (g_{1, t}^1 - g_{i_t^1, t}) \right] = \sum_{t=1}^T \mathbb{E}_{P_1}[1\{i_t^1 \neq 1\} \cdot (g_{1, t}^1 - g_{i_t^1, t})]

= \sum_{\ell=2}^L \sum_{t=1}^T \mathbb{E}_{P_\ell}[1\{i_t^1 = \ell\} \cdot (g_{1, t}^1 - g_{i_t^1, t})] = \sum_{\ell=2}^L \sum_{t=1}^T \mathbb{E}_{P_\ell}[((g_{1, t}^1 - g_{i_t^1, t})i_t^1 = \ell) \cdot \mathbb{E}_{P_\ell}[1\{i_t^1 = \ell\}]

= \sum_{\ell=2}^L \sum_{t=1}^T \Delta_{1, \ell}^t \cdot \mathbb{E}_{P_\ell}[1\{i_t^1 = \ell\}] = \sum_{\ell=2}^L b \cdot d_{\ell} \cdot \mathbb{E}_{P_\ell}[N_\ell(T)].

Since $H_2(\ell) = \sum_{\ell \neq \ell}(\Delta_{\ell, j})^{-2}$, we have

$$\frac{\overline{\text{Reg}}}{H_2(1)} = \sum_{\ell=2}^L b \cdot d_{\ell} \cdot \mathbb{E}_{P_\ell}[N_\ell(T)] = \frac{\sum_{\ell=2}^L b \cdot d_{\ell} \cdot \mathbb{E}_{P_\ell}[N_\ell(T)]}{b^{-3} \cdot \sum_{\ell=2}^L d_{\ell}^{-2}}.

Thus, by the pigeonhole principle, there exists $2 \leq \ell_1 \leq L$ such that

$$b^3 d_{\ell_1}^3 \mathbb{E}_{P_1}[N_{\ell_1}(T)] = \frac{b \cdot d_{\ell_1} \cdot \mathbb{E}_{P_1}[N_{\ell_1}(T)]}{b^{-3} \cdot d_{\ell_1}^{-2}} \leq \frac{\overline{\text{Reg}}}{H_2(1)} \Leftrightarrow \mathbb{E}_{P_1}[N_{\ell_1}(T)] \leq \frac{\overline{\text{Reg}}}{b^3 d_{\ell_1}^3 H_2(1)}.

Since $d_{\ell} \in (0, 1/4)$ for all $1 \leq \ell \leq L$, we apply Theorem B.5 to obtain

$$\Pr_{O_1}(i_{\text{out}} \neq 1) + \Pr_{O_{\ell_1}}(i_{\text{out}} = 1) \geq \frac{1}{2} \exp(-\mathbb{E}_{P_1}[N_{\ell_1}(T)] \cdot \mathbb{KL}(P_{1, \ell_1} \parallel P_{\ell_1, \ell_1})) \geq \frac{1}{2} \exp \left( - \frac{\overline{\text{Reg}}}{b^3 d_{\ell_1}^3 H_2(1)} \cdot \frac{(2d_{\ell_1})^2}{1/4} \right).

Since $\Pr_{O_{\ell_1}}(i_{\text{out}} \neq \ell_1) \geq \Pr_{O_{\ell_1}}(i_{\text{out}} = 1)$, we have

$$\max_{1 \leq \ell \leq L} \Pr_{O_\ell}(i_{\text{out}} \neq \ell) \geq \frac{1}{4} \exp \left( - \frac{8\overline{\text{Reg}}}{H_2(1)b^3 \cdot \min_{2 \leq \ell \leq L} d_{\ell}} \right).

**Step 3: Conclusion.** We define

$$\ell_2 := \arg \max_{1 \leq \ell \leq L} \Pr_{O_\ell}(i_{\text{out}} \neq \ell).

Suppose algorithm $\pi$ satisfies that

$$\Pr_{O_{\ell_2}}(i_{\text{out}} \neq \ell_2) \leq \frac{1}{4} \exp(-\phi_T),

then we can lower bound its pseudo-regret as follows:

$$R_T \geq \phi_T \cdot \frac{H_2(1)b^3 \cdot \min_{2 \leq \ell \leq L} d_{\ell}}{8}.$$
When \( d = d_\ell > 0 \) for all \( 2 \leq \ell \leq L \), we have \( H_2(1) = (L - 1)/(b^2d^2) \).

**Step 4: Classification of instances.** Suppose algorithm \( \pi \) satisfies that \( e_T(\pi) \leq \exp(-\phi_T)/4 \). Let \( B_1(\Delta, R) \) denote the set of stochastic instances where (i) the minimal optimality gap \( \Delta \geq \Delta \); and (ii) there exists \( R_0 \in \mathbb{R} \) such the rewards are bounded in \([R_0, R_0 + R]\). Then

\[
\sup_{\pi \in B_1(\Delta, R)} R_T(\pi, I) \geq \phi_T \cdot \frac{(L - 1)R}{8\Delta} \quad \forall \Delta, R > 0.
\]

Let \( B_2(\Delta, R, H_2^*) \) denote the set of stochastic instances that (i) belong to \( B_1(\Delta, R) \), and (ii) are with hardness parameter \( H_2 \leq H_2^* \). Then, we have

\[
\sup_{\pi \in B_2(\Delta, R, H_2^*)} R_T(\pi, I) \geq \phi_T \cdot \frac{\Delta H_2^* R}{8} \quad \forall \Delta, R, H_2^* > 0.
\]

\( \square \)

**D.2. Proof of Theorem 5.2**

**Theorem 5.2.** Let \( \phi_T, \Delta, V, V_2 > 0 \). Let \( \pi \) be any algorithm with \( e_T(\pi, I) \leq \exp(-\phi_T)/4 \) for all \( I \in B_1^*(\Delta, V) \). Then

\[
\sup_{\pi \in B_1^*(\Delta, V)} R_T(\pi, I) \geq \phi_T \cdot \frac{(L - 1)V}{2\Delta},
\]

\[
\sup_{\pi \in B_1^*(\Delta, V_2)} R_T(\pi, I) \geq \phi_T \cdot \frac{\Delta V_2^2}{2}.
\]

**Proof.**

**Step 1: Construct instances.** To begin with, we fix any \( \sigma > 0, d_\ell > 0 \) for all \( 2 \leq \ell \leq L \). We define the following distributions:

\[
\nu_1 := N(1/2, \sigma^2), \quad \nu_\ell := N(1/2 - d_\ell, \sigma^2) \quad \forall 1 < \ell \leq L;
\]

\[
\nu'_1 := N(1/2, \sigma^2), \quad \nu'_\ell := N(1/2 + d_\ell, \sigma^2) \quad \forall 1 < \ell \leq L.
\]

We construct \( L \) instances such that under instance \( \ell \) (\( 1 \leq \ell \leq L \)), the stochastic reward of item \( i \) is drawn from distribution

\[
\nu''_\ell := \nu_\ell 1\{i \neq \ell\} + \nu'_\ell 1\{i = \ell\}.
\]

Under instance \( \ell \) (\( 1 \leq \ell \leq L \)), we see item \( \ell \) is optimal, and we define several other notations as follows:

(i) We let \( g_{i, t}^\ell \) be the random reward of item \( i \) at time step \( t \).

(ii) We let \( \Delta_{i, j}^\ell := \mathbb{E}[(\sum_{t=1}^T g_{i, t}^\ell - g_{k, t, t}^\ell)/T] \) denote the gap between item \( i \) and \( j \). Then

\[
\Delta_{1, j}^\ell = d_j \quad \forall 2 \leq j \leq L, \quad \Delta_{\ell, 1}^\ell = d_\ell, \quad \Delta_{\ell, j}^\ell = d_\ell + d_j \quad \forall 2 \leq j, \ell \leq L, j \neq \ell.
\]

(iii) We denote the difficulty of the instance with

\[
H_2(\ell) := \sum_{j \neq \ell}(\Delta_{i, j}^\ell)^{-2}.
\]

Then \( H_2(1) = \max_{1 \leq \ell \leq L} H_2(\ell) \leq (L - 1) \cdot \frac{d_\ell}{2} \).

(iv) We let \( i_t^\ell \) be the pulled item at time step \( t \), and \( O_{\ell t} = \{i_{u, t}^\ell, g_{i_{u, t}^\ell, t}^\ell\}_{u=1} \) be the sequence of pulled items and observed rewards up to and including time step \( t \).

(v) We let \( P_\ell^\ell \) be the measure on \( O_{\ell t} \), and let \( R_{\ell, i} \) be the measure on the rewards of item \( i \).
For simplicity, we abbreviate \( \mathbb{P}_t^T, \mathbb{O}_t^T \) as \( \mathbb{P}_t, \mathbb{O}_t \) respectively. Moreover, we let \( N_i(t) \) denote the number of pulls of item \( i \) up to and including time step \( t \).

**Step 2: Change of measure.** First of all, we apply Lemmas B.3 and B.4 obtain that for all \( 1 \leq \ell \leq L \),

\[
\Pr_{\mathbb{O}_t}(i_{\text{out}} \neq 1) + \Pr_{\mathbb{O}_t}(i_{\text{out}} = 1) \geq \frac{1}{2} \exp(-\text{KL}(\mathbb{P}_1 \parallel \mathbb{P}_t)) = \frac{1}{2} \exp(-\mathbb{E}_{\mathbb{P}_1}[N_1(T)] \cdot \text{KL}(\mathbb{P}_{1,t} \parallel \mathbb{P}_{t,t}).
\]

Suppose the expected pseudo-regret is upper bounded by \( \overline{\text{Reg}} \), we have

\[
\overline{\text{Reg}} \geq \mathbb{E}_{\mathbb{P}_1} \left[ \sum_{t=1}^{T} \mathbb{1}_{\{i_t^1 \neq 1\}} \cdot (g_{i_t^1,t} - g_{i_t^1,t}) \right] = \sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_1} \left[ \mathbb{1}_{\{i_t^1 \neq 1\}} \cdot (g_{i_t^1,t} - g_{i_t^1,t}) \right]
= \sum_{t=1}^{T} \sum_{\ell=2}^{L} d_{\ell} \cdot \mathbb{E}_{\mathbb{P}_1} \left[ \mathbb{1}_{\{i_t^\ell \neq 1\}} \cdot (g_{i_t^\ell,t} - g_{i_t^\ell,t}) \right] = \sum_{t=1}^{T} d_{t} \cdot \mathbb{E}_{\mathbb{P}_1} \left[ N_1(T) \right].
\]

Since \( H_2(\ell) = \sum_{j \neq \ell} (\Delta_{1,j})^{-2} \), we have

\[
\frac{\overline{\text{Reg}}}{H_2(1)} = \frac{\sum_{t=1}^{T} d_t \cdot \mathbb{E}_{\mathbb{P}_1} \left[ N_1(T) \right]}{\sum_{t=1}^{T} (\Delta_{1,t})^{-2}} = \frac{\sum_{t=1}^{T} d_t \cdot \mathbb{E}_{\mathbb{P}_1} \left[ N_1(T) \right]}{\sum_{t=1}^{T} d_t^{-2}}.
\]

Thus, by the pigeonhole principle, there exists \( 2 \leq \ell_1 \leq L \) such that

\[
d_{\ell_1} \cdot \mathbb{E}_{\mathbb{P}_1} \left[ N_{\ell_1}(T) \right] \leq \frac{\overline{\text{Reg}}}{H_2(1)} \iff \mathbb{E}_{\mathbb{P}_1} \left[ N_{\ell_1}(T) \right] \leq \frac{\overline{\text{Reg}}}{d_{\ell_1} H_2(1)}.
\]

Further, we apply Lemma B.6 to obtain

\[
\Pr_{\mathbb{O}_t}(i_{\text{out}} \neq 1) + \Pr_{\mathbb{O}_t}(i_{\text{out}} = 1) \geq \frac{1}{2} \exp(-\mathbb{E}_{\mathbb{P}_1}[N_{\ell_1}(T)] \cdot \text{KL}(\mathbb{P}_{1,\ell_1} \parallel \mathbb{P}_{t,\ell_1})) \geq \frac{1}{2} \exp \left( -\frac{\overline{\text{Reg}}}{d_{\ell_1} H_2(1)} \cdot \frac{(2d_{\ell_1})^2}{2\sigma^2} \right).
\]

Since \( \Pr_{\mathbb{O}_t}(i_{\text{out}} \neq \ell_1) \geq \Pr_{\mathbb{O}_t}(i_{\text{out}} = 1) \), we have

\[
\max_{1 \leq \ell \leq L} \Pr_{\mathbb{O}_t}(i_{\text{out}} \neq \ell) \geq \frac{1}{4} \exp \left( -\frac{2\overline{\text{Reg}}}{H_2(1)\sigma^2 \cdot \min_{2 \leq \ell \leq L} d_{\ell}} \right).
\]

**Step 3: Conclusion.** We define

\[
\ell_2 := \arg \max_{1 \leq \ell \leq L} \Pr_{\mathbb{O}_t}(i_{\text{out}} \neq \ell).
\]

Suppose algorithm \( \pi \) satisfies that

\[
\Pr_{\mathbb{O}_{\ell_2}}(i_{\text{out}} \neq \ell_2) \leq \frac{1}{4} \exp(-\phi_T),
\]

then we can lower bound its pseudo-regret as follows:

\[
R_T \geq \phi_T \cdot \frac{H_2(1)\sigma^2 \cdot \min_{2 \leq \ell \leq L} d_{\ell}}{2}.
\]

When \( d = d_{\ell} > 0 \) for all \( 2 \leq \ell \leq L \), we have \( H_2(1) = (L-1)/d^2 \).
Step 4: Classification of instances. Suppose algorithm $\pi$ satisfies that $e_T(\pi) \leq \exp(-\phi_T)/4$. Let $B'_i(\Delta, \mathcal{V})$ denote the set of stochastic instances where (i) the minimal optimality gap $\Delta \geq \Delta_i$; (ii) for each item $i$, the variance $\sigma^2_i \leq \mathcal{V}$. Then

$$\sup_{\pi \in B'_i(\Delta, \mathcal{V})} R_T(\pi, T) \geq \phi_T \cdot \frac{(L - 1)\mathcal{V}}{2\Delta} \quad \forall \Delta, \mathcal{V} > 0.$$ 

Let $B'_2(\Delta, \mathcal{V}, \mathcal{H}_2)$ denote the set of stochastic instances (i) that belong to $B'_i(\Delta, \mathcal{V})$, and (ii) are with the hardness $H_2 \leq \mathcal{H}_2$. We have

$$\sup_{\pi \in B'_2(\Delta, \mathcal{V}, \mathcal{H}_2)} R_T(\pi, T) \geq \phi_T \cdot \frac{\Delta \mathcal{H}_2 \mathcal{V}}{2} \quad \forall \Delta, \mathcal{V}, \mathcal{H}_2 > 0.$$ 

D.3. Proof of Corollary 6.1

Corollary 6.1. Let $\pi_1$ be the online algorithm BoBW-LIL’UCB($1/\sqrt{T}$) and $\pi_2$ be the online algorithm BoBW-LIL’UCB($\gamma_1(\Delta, \mathcal{H}_2)$). Then

$$\sup_{\pi \in B_2(\Delta, \mathcal{V}, \mathcal{H}_2)} R_T(\pi, T) \in \Omega\left(\frac{\Delta \mathcal{H}_2 \log \left(\frac{T}{L}\right)}{\mathcal{V}}\right) \cap O\left(\frac{\log T}{\Delta}\right),$$

$$\sup_{\pi \in B_2(\Delta, \mathcal{V}, \mathcal{H}_2)} R_T(\pi, T) \in \Omega\left(\frac{(T - L) \cdot \Delta \mathcal{H}_2}{(\mathcal{H}_2 + \Delta^{-2})}\right) \cap O\left(\frac{L(T - L)}{\Delta(\mathcal{H}_2 + \Delta^{-2})}\right).$$

Proof. We consider the stochastic instances in $B_2(\Delta, \mathcal{V}, \mathcal{H}_2)$. By the classification of instances in Theorem 5.1, these instances satisfy the conditions

$$g_{i,t} \in [0, 1] \quad \forall i, t \quad \text{and} \quad H_2 \leq \mathcal{H}_2.$$ 

Therefore, the distribution $\nu_i$ is sub-Gaussian with scale $\sigma = 1/2$ for all $i \in [L]$. We assume $T$ is sufficiently large such that

$$\frac{1}{\sqrt{T}} \geq \gamma_1 = \sqrt{2.8\log \left(\frac{6\sqrt{2.8}(1 + \varepsilon)^2}{\Delta} + \beta\right)} \cdot \exp\left(-\frac{T - L}{144\sigma^2(1 + \varepsilon)^3(\mathcal{H}_2 + 1/\Delta^2)}\right)$$

$$= \sqrt{2.8\log \left(\frac{3\sqrt{2.8}(1 + \varepsilon)^2}{\Delta} + \beta\right)} \cdot \exp\left(-\frac{T - L}{36(1 + \varepsilon)^3(\mathcal{H}_2 + \Delta^{-2})}\right).$$

As a result, for all instance in $B_2(\Delta, \mathcal{H}_2)$, since $\Delta \geq \Delta$ and $H_2 \leq \mathcal{H}_2$, we have

$$\frac{1}{\sqrt{T}} \geq \sqrt{2.8\log \left(\frac{3\sqrt{2.8}(1 + \varepsilon)^2}{\Delta} + \beta\right)} \cdot \exp\left(-\frac{T - L}{36(1 + \varepsilon)^3(\mathcal{H}_2 + \Delta^{-2})}\right).$$

We let $\pi_1$ denote BoBW-LIL’UCB($1/\sqrt{T}$) and $\pi_2$ denote BoBW-LIL’UCB($\gamma_1$). One one hand, fix any instance in $B_2(\Delta, \mathcal{H}_2)$, Theorem 4.1 implies that

$$R_T(\pi_1) \in O\left((1 + \varepsilon)^3 H_1 \log T\right),$$

and

$$R_T(\pi_2) \in O\left((1 + \varepsilon)^3 \sum_{i \neq 1} \log(1/\gamma_1) \cdot \frac{\Delta_1,i}{\Delta_{i,1}}\right) = O\left(\sum_{i \neq 1} \frac{1}{\Delta_1,i} \cdot \frac{T - L}{\mathcal{H}_2 + \Delta^{-2}}\right).$$
On the other hand, Theorem 4.2 implies that

\[
e_T(\pi_1) \leq \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{1}{\log(1 + \varepsilon)} \right)^{1+\varepsilon} \cdot T^{- (1+\varepsilon)/2},
\]

\[
e_T(\pi_2) \leq \frac{2L(2 + \varepsilon)}{\varepsilon} \left( \frac{1}{\log(1 + \varepsilon)} \right)^{1+\varepsilon} \cdot \left[ 2.8 \log \left( \frac{3\sqrt{4(1 + \varepsilon)^2}{\Delta}}{\Delta} + \beta \right) \right]^{(1+\varepsilon)/2} \cdot \exp \left( - \frac{T - L}{36(1 + \varepsilon)^2(H_2 + \Delta^{-2})} \right).
\]

Moreover, we can apply Theorem 5.1 to obtain that

\[
\sup_{\mathcal{I} \in \mathcal{B}_2(\Delta, 1, \pi_2)} R_T(\pi_1, \mathcal{I}) \in \Omega \left( (1 + \varepsilon) \cdot \frac{\Delta H_2}{L} \log \left( \frac{T}{L} \right) \right)
\]

\[
\sup_{\mathcal{I} \in \mathcal{B}_2(\Delta, 1, \pi_2)} R_T(\pi_2, \mathcal{I}) \in \Omega \left( \frac{(T - L) \cdot \Delta H_2}{(1 + \varepsilon)^2(H_2 + \Delta^{-2})} \right).
\]

Altogether, we have

\[
\sup_{\mathcal{I} \in \mathcal{B}_2(\Delta, 1, \pi_2)} R_T(\pi_1, \mathcal{I}) \in \Omega \left( (1 + \varepsilon) \cdot \frac{\Delta H_2}{L} \log \left( \frac{T}{L} \right) \right) \cap O \left( \frac{(1 + \varepsilon)^3(L - 1) \log T}{\Delta} \right),
\]

\[
\sup_{\mathcal{I} \in \mathcal{B}_2(\Delta, 1, \pi_2)} R_T(\pi_2, \mathcal{I}) \in \Omega \left( \frac{(T - L) \cdot \Delta H_2}{(1 + \varepsilon)^2(H_2 + \Delta^{-2})} \right) \cap O \left( \frac{(L - 1)(T - L)}{\Delta(H_2 + \Delta^{-2})} \right).
\]

Neglecting the constant terms in $1 + \varepsilon$ completes the proof of Corollary 6.1.

\[\square\]

E. Additional Numerical Results

In this section, we present more numerical results for a larger instance with $L = 128$ items. These figures yield the same conclusions as in Section 8. In particular, we see that when the gap $\Delta$ is small (Figs. E.1 and E.4), BoBW-LIL’UCB($\gamma$) performs much better, in terms of the regret and the average stopping time, compared to its closest competitor UCB$_\alpha$. When the gap is large, however, UCB$_\alpha$ can outperform BoBW-LIL’UCB($\gamma$). Hence, these two algorithms work well in different regimes but since stochastic bandits with small gaps correspond to “harder”, and thus more interesting, instances, BoBW-LIL’UCB($\gamma$) is arguably a fundamental contribution to the study of stochastic bandits. Additionally, as we have shown, BoBW-LIL’UCB($\gamma$) almost achieves the Pareto frontier of RM and BAI.

![Figure E.1. L = 128, \Delta = 0.05, \nu_i = Bern(w_i).](image-url)
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Figure E.2. $L = 128, \Delta = 0.1, \nu_i = \text{Bern}(\omega_i)$.

Figure E.3. $L = 128, \Delta = 0.2, \nu_i = \text{Bern}(\omega_i)$.

Figure E.4. $L = 128, \Delta = 0.1, \nu_i = \mathcal{N}(\omega_i, 1)$.

Figure E.5. $L = 128, \Delta = 0.2, \nu_i = \mathcal{N}(\omega_i, 1)$. 
Figure E.6. $L = 128$, $\Delta = 0.4$, $\nu_i = N(w_i, 1)$. 