Controls that expedite scale-invariant transport in disordered systems

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Transport in disordered media, such as those involving charge carriers in amorphous semiconductors, or contaminants in hydrogeological systems, are often described by time-scale-free processes. We study the statistical properties of the first passage time of biased processes in different models, and employ the big jump principle that shows the dominance of the maximum trapping time on the first passage time. Inspired by the restart paradigm, we demonstrate that the removal of this maximum significantly expedites transport. As the disorder increases, the system enters a phase where the removal shows a dramatic effect. Our results show how we may speed up transport in strongly disordered systems exploiting scale invariance.

Introduction.—A “restart” of a stochastic search process may expedite the search time vastly, and hence this strategy was extensively studied [1–5], with applications in biological processes [6–8] and computer science [9], among other fields. The basic paradigm of restart is to consider a non-biased diffusive particle that is returned to its origin at a given rate. Under certain conditions, this repeated return to the origin will minimize the first passage time $t_f$ to a target [10]. When a bias is present, this strategy is not necessarily useful. Furthermore, the number of restarts can be large, and to pick up a particle at some position in space and return it to its origin is typically costly. Here, we present a new method to deal with such questions. Our developments exploit the scale-invariance of transport in disordered system [11–14] and the big jump principle [15–24] to find novel effects. Roughly speaking, along the path of a particle advancing in a disordered system, we identify a bottleneck where transport is slowed down. Namely, the particle is trapped and released many times along its path, and then the basic issue is: will the removal of one and only one of these trapping times dramatically reduce the first passage time? Thus, unlike classical restart theory, we do not send the particle back to its origin several times, and exploit the disorder to obtain a dramatic speed-up of the first passage time.

Tracer pathways, retention, and migration patterns in disordered environments are typically seen to be similar over a large range of length and time scales, being observed ubiquitously in nature, e.g. for the motion of a tracer in porous media [25–30], a colloidal particle in a glassy system [31–33], a charge carrier in a strongly disordered amorphous conductor [34–35], or an ion channel on the membrane of the cell [36–37]. In many cases, the probability density function of the transition times is heavy-tailed $\psi(t) \sim At^{-(1+\alpha)}$ with some amplitude $A$ and the scaling exponent $\alpha > 0$ [11–14]. A critical consequence is that extremely long trapping times occur in deep traps, narrow passages, or at major obstacles, hence, the motion slows down dramatically. We present a simple but effective concept that overcomes this slow down. One only needs to remove the single maximum trapping time $\tau_{\text{max}}$ along the tracer path to gain a surprisingly great effect on the transport behavior. This is vastly different from previous studies of restart strategy with scale free dynamics, [38–45]. Those papers use multiple and blind restarts, or use normal diffusion but with power law distributed times between restarts.

We demonstrate this removal technique for one of the most well-studied observable in stochastic dynamics, namely the time a tracer is advected through a system of length $L$, known as the first passage time $t_f$ [46–51]. A broad class of well-known transport models is considered, e.g. with rigorous theory for the unidirectional transport on a lattice serving as toy model, continuous time random walk [12, 25–27, 29, 52–64], and quenched trap model [31–33, 65–70], and with extensive numerical analysis for the simulation of tracer migration in porous media. The study for each model contains three parts: A) We establish the principle of the single long transition time for these different models, which is based on the principle of the single big jump [15–24]. The tracer path is described by trapping events, on a coarse grained scale. We may then define the longest trapping time $\tau_{\text{max}}$ (defined more precisely below). This time is clearly shorter than the total time $t_f$ it takes the particle to cross the system. Still, in scale free systems, as we show below, and for the slowest particles, $\tau_{\text{max}} \approx t_f$ (see precise definition below). This is the long transition time principle that we aim to establish, for widely applicable models. The question is now: how can we enhance the transport? B) We remove the maximum transition times from the associated particle trajectories, and find that the distribution of the modified first passage time

$$t_r = t_f - \tau_{\text{max}}.$$  

(1)

decays much faster compared to the original distribution of $t_f$. The index “r" stands for removal or restart. C) This transport speed-up is further quantified with the measure of gain

$$G = \frac{\langle t_r \rangle}{\langle t_f \rangle},$$  

(2)
where a small value indicates fast transport of the restarted process.

**Unidirectional transport.**—We start with a simple model, which still can capture some of the complexities in the following more challenging and realistic approaches. The transport is unidirectional on a one-dimensional lattice with length $L$. A particle starts at the lattice point $x = 1$ and jumps successively to the right neighbor positions until the absorbing boundary $x = L + 1$. At each lattice point, the particle waits randomly according to the waiting time distribution $\psi(t)$. The transition times $(\tau_1, \ldots, \tau_L)$ are independent and identically distributed. The first passage time $t_f = \sum_{x=1}^{L} \tau_x$ is related to the maximum transition time $\tau_{\text{max}} = \max(\tau_1, \ldots, \tau_L)$ via the well-known principle of the long transition time

$$\text{Prob}(t_f > t) \sim \text{Prob}(\tau_{\text{max}} > t) \sim L A \sqrt{t}^{-\alpha}$$

for large $t$ and any $L [13,24]$, i.e. the first passage time, when it is long, is solely determined by a single summand, the maximum transition time.

What is the effect of the elimination of the maximum transition time $\tau_{\text{max}}$ from the sequence $(\tau_1, \ldots, \tau_L)$? In the Supplemental Material (SM) Sec. II, we derive the asymptotic behavior of the distribution for the modified first passage time defined in Eq. (I)

$$\text{Prob}(t_r > t) \sim \text{Prob}(\tau_{\text{max}}^* > t) \sim \frac{1}{2} L (L-1) \left( \frac{A}{\alpha} \right)^2 t^{-2\alpha},$$

which is valid for large times $t$ and any $L$. Here, $\tau_{\text{max}}^*$ is the second longest transition time of the original sequence of transition times. The remarkable issue is the doubling effect of the exponent $-2\alpha$, whereas previously we had the $-\alpha$ decay in Eq. (3). Thus, the probability of large $t_r$ is drastically decreased compared to the probability of large $t_f$. In particular, if $\alpha < 1$, as is found in many disordered systems, the mean first passage time $\langle t_f \rangle$ diverges, but once we eliminate $\tau_{\text{max}}$ from each trajectory, the mean $\langle t_r \rangle$ will diverge only if $\alpha < 1/2$. We see a transition in the behavior of $G$ in Eq. (2) at the critical point $\alpha = 1$ similar to a dynamical phase transition. When the disorder becomes stronger $\alpha < 1$, the ratio $G$ is zero, which indicates a significant effect upon elimination. As an example, we present $G$ of Eq. (2) for Pareto-distributed transition times (see SM Sec. III)

$$G = \begin{cases} 0 & \text{for } 0 < \alpha < 1, \\ 1 - (-1)^L \frac{\alpha - 1}{\alpha} (L-1)! \frac{\Gamma(-L+\frac{1}{\alpha})}{\Gamma(\frac{1}{\alpha})} & \text{for } 1 < \alpha, \end{cases}$$

see Fig. I. In contrast, if we use exponential transition times (see SM Sec. II), we do not witness a critical transition. Finally, the large $L$ limit is discussed in SM Sec. III.

$\textbf{Continuous time random walk.}$—We now consider a basic model for anomalous transport, the continuous time random walk [12,53], which has found application in a vast number of systems. The first passage time problem in this model was studied extensively [47,49,51,71,73]. As for the unidirectional model from above, at each lattice point a particle takes a transition time according to $\psi(t) \sim At^{-(1+\alpha)}$. But now jumping to the left with probability $1 - p$ is permitted where $p$ is the probability of jumping to the right. A particle that starts at $x = 1$ has a random number $N$ of jumps at the absorption point $x = L + 1$. Therefore, we have to average over $N$ with a technique called subordination [47,50,74,75]. The idea is to consider a discrete time random walk with the same bias, initial conditions, and boundary conditions, as for the original model. Let $\phi_{\text{dis}}(n)$ be the probability that a particle performing this walk made $N = n$ jumps before its absorption, and the subscript “dis” stands for discrete time. For the biased case $p > 1/2$ the mean $\langle N \rangle = \sum_{n=1}^{\infty} n \phi_{\text{dis}}(n) < \infty$ is finite. We obtain

$$\text{Prob}(t_f > t) \sim \text{Prob}(\tau_{\text{max}} > t) \sim \langle N \rangle \frac{A}{\alpha} t^{-\alpha},$$

see SM Sec. IV. The difference compared to Eq. (3) is that we replace the fixed $N = L$ by $\langle N \rangle$ [20]. We calculate $\phi_{\text{dis}}(n)$ and then $\langle N \rangle$, which is easy to obtain $\langle N \rangle = \sum_{n=1}^{\infty} n \phi_{\text{dis}}(n)$.
\[ L/(p-q) \] (SM Sec. [IV]). Then Eq. (6) reads
\[ \text{Prob}(t_f > t) \sim \text{Prob}(\tau_{\text{max}} > t) \]
\[ \sim \frac{L}{p-q} \frac{A}{t^{-\alpha}} \]  
(7)
when \( p > 1/2 \). Similar results can be obtained for other models (SM Sec. [IV]), for example the case when we have a reflecting wall situated possibly far from the initial condition, or when we replace the lattice model with a continuous space version. In addition, the unbiased case \( p = 1/2 \) with infinite \( \langle N \rangle = \infty \) is presented in SM Sec. [IV].

![FIG. 2](image)

FIG. 2. The distributions of \( t_f \) (red circles), \( \tau_{\text{max}} \) (blue line), \( t_r \) (green circles) and \( \tau_{\text{max}}^\ast \) (purple line) compared with the theories Eq. (10) (upper black line) and Eq. (12) (lower black line) for the weakly biased quenched trap model with \( L = 10 \) and \( p = 0.7 \). We used \( 10^6 \) particles and the parameters \( T = T_g = 2 \) (so that \( \alpha = 1 \)) and \( t_0 = 1 \). We see the long transition time principle at work, the distributions of the maximum transition time and the first passage time follow the same laws for large arguments, further the effect of removal/restart is clearly strong.

The modified first passage time upon removal behaves as
\[ \text{Prob}(t_r > t) \sim \text{Prob}(\tau_{\text{max}}^\ast > t) \]
\[ \sim \frac{1}{2} \langle N(N-1) \rangle \left( \frac{A}{\alpha} \right)^2 t^{-2\alpha} \]  
(8)
with \( \langle N(N-1) \rangle = \sum_{n=1}^{\infty} n(n-1) \phi_{\text{dis}}(n) \). Compared to the result of the unidirectional model Eq. (4), we find again the same power law decay \(-2\alpha\) but the \( N\)-dependent prefactor must be averaged. For the example of the Pareto distributed transition times, \( G \) of Eq. (2) is
\[ G = \begin{cases} 
0 & , 0 < \alpha < 1, \\
1 - h_\alpha \sum_{n=1}^{\infty} \phi_{\text{dis}}(n)(-1)^\alpha n! \Gamma \left(-n + \frac{1}{\alpha}\right) & , 1 < \alpha
\end{cases} \]  
(9)
with \( h_\alpha = [(p-q)(\alpha-1)]/[\alpha L \Gamma(1/\alpha)] \), see Fig. 1. The phase like transition in \( G \), found when \( \alpha \) is varied, is insensitive to model details.

**Quenched trap model.**— In the unidirectional transport model and the continuous time random walk, the transition times are spatially homogeneous, i.e. independent of the lattice points. We remove this simplifying property now, considering the quenched trap model [31-33, 67, 69]. Similar to the simulation of the pore-scale system (see below), in the quenched trap model the disorder is fixed, more specifically the particle is performing a biased random walk with \( 1/2 < p < 1 \) in a random energy landscape. We deal with energetic traps \( E_x \) on the lattice points \( x \) with the distribution \( \text{Prob}(E_x > E) = \exp(-E/T_g) \) and \( T_g \) is a measure of disorder. As before, the particle starts at \( x = 1 \) and the absorption is at \( x = L + 1 \). According to the basic theory of activation, the distribution of the transition times is exponential \( \text{Prob}(\tau_x > t) = \exp(-t/\tau_x) \) with the well-known Arrhenius time \( \bar{\tau}_x = t_0 \exp(E_x/T) \) to escape from an energy trap, used in many activation processes. Here, \( T \) is the temperature of the system and \( t_0 \) is a well-studied timescale for dynamics in the bottom of the trap [76]. Averaging \( \text{Prob}(\bar{\tau}_x > t) \) over the disorder gives \( \psi(t) \) with \( \alpha = T/T_g \) [31].

We consider now the occupation time \( \tau_x \), which is the sum of all transition times \( \tau_x \) at a single trap \( x \). The full derivation of the asymptotic behaviors is non-trivial and can be found in SM Sec. [VIII]. We have to consider the joint probability \( \phi(n_x,k) \) with \( k \) visited traps and \( n_x \) visits at trap \( x \). For the average over the energy landscape, we get
\[ \langle \text{Prob}(t_f > t) \rangle_{en} \sim \langle \text{Prob}(\tau_{\text{max}} > t) \rangle_{en} \]
\[ \sim (t_0)^\alpha M_\alpha t^{-2\alpha} \]  
(10)
with the function
\[ M_\alpha = \sum_{x=-\infty}^{L} \left( \frac{\Gamma(N_x + \alpha)}{\Gamma(N_x)} \right) \]
\[ = \sum_{x=-\infty}^{L} \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \phi(n_x,k) \frac{\Gamma(n_x + \alpha)}{\Gamma(n_x)} \]  
(11)
see Fig. 2. The non-trivial function \( M_\alpha \) is obtained from the simulation of a discrete time and space random walk, i.e. without using any disorder (SM Sec. [VIII]). The strong bias \( p = 1 \) gives \( M_\alpha = \Gamma(1 + \alpha)/L \).

The idea is now to remove the deepest visited trap of each particle. The two probability distributions of the first passage time and of the maximum occupation time, both after the removal, behave as
\[ \langle \text{Prob}(t_r > t) \rangle_{en} \sim \langle \text{Prob}(\tau_{\text{max}}^\ast > t) \rangle_{en} \]
\[ \sim (t_0)^{2\alpha} M_\alpha t^{-2\alpha} \]  
(12)
with the function
\[ M^*_\alpha = \sum_{x=-\infty}^{L} \left( K - 1 \right) \frac{\Gamma(N_x + 2\alpha)}{\Gamma(N_x)} \]
\[ = \sum_{x=-\infty}^{L} \sum_{k=1}^{\infty} \frac{\phi(n_x,k)(k-1)}{\Gamma(n_x)} \frac{\Gamma(n_x + 2\alpha)}{\Gamma(n_x)}. \] (13)

see Fig. 2 and for the derivation SM Sec. IX. Both probabilities are related and experience a doubling effect in the power law exponent; compare with Eq. (10). We obtain \( M^*_\alpha \) similar to the method of finding \( M_\alpha \). For \( p = 1 \), the function is \( M^*_\alpha = \Gamma(1+2\alpha)L(L-1) \).

For the example in Fig. 2 with \( \alpha = 1 \), the gain quantifier \( G = \langle t_f \rangle_{en}/\langle t_f \rangle_{en} \) has the numerical value 0.151, showing once again that the elimination of the maximum contribution, here the deepest trap, yields a big gain. In Fig. 4 we show \( G \) for the case \( p = 1 \) (which is derived in SM Sec. IX). Finally, in SM Sec. XI we present a full discussion on the vastly different case of a single energy landscape when no averaging is performed. Nevertheless, in this highly non-trivial case, the removal method yields again striking performance.

![Graph](image.png)

**FIG. 3.** (a) The distributions of the first passage time \( t_f \) (red circles), the maximum of the transition times \( \tau_{\text{max}} \) (blue line), the modified first passage time \( t_\star \) (green circles) and the second longest transition time \( \tau_{\text{max}}^\star \) (purple line) for the simulation of transport in a pore-scale system, where we followed \( N = 30 \) transition times of \( 10^6 \) particles. The matching of the right tails of the respective distributions shows the single long transition time principle which is non-trivial if compared to the Chistyakov version 13, as we do not have here a power law tail. The removal yields a significant decrease of the tail.

**Simulation of pore-scale transport.**—Using extensive numerical simulations, we discuss the advection-dominated transport behavior in porous medium, thus, bringing our insights closer to truly complex systems. Over twenty years of field and laboratory experiments [25], numerical simulations [28] and theoretical studies [24] have shown how advective-dispersive tracer motion in hydrogeological systems is characterized by many time scales. In particular, the distribution of trapping times in these systems is very broad; for example for biased transport in a porous medium, the continuous time random walk, with power law sticking times, is a profound model [27].

We simulated particle transport in a two-dimensional heterogeneous system, based on the flow solution of the Navier-Stokes Equations and use of a particle tracking technique along streamlines [28] (SM Sec. XI). In Fig. 3 we present the distributions for \( t_f \) and \( \tau_{\text{max}} \). As expected from previous works, these distributions are very wide (note the log-log scale). The striking behavior is that we observe excellent matching in the right part of the tails of the distributions, i.e., when \( t_f \) and the maximum are large. Thus, we find the principle of the single long transition time in a non-trivial system, as in Eq. (3). This indicates that the longest first passage times are dominated by the longest transition times, and not by a sequence of many relatively long sticking times. An interesting effect is found in Fig. 3 the tails of both distributions, of \( t_f \) and \( \tau_{\text{max}} \), are not smooth, but exhibit step-like structures. The transport is dominated by the disorder configuration of the system. Namely, while the specific pattern of the distributions depends on the details of the porous medium model at hand, the single long transition principle holds also in this fixed background system.

Now we eliminate \( \tau_{\text{max}} \) from each trajectory and study \( t_\star \). In Fig. 3 we present the distribution of \( t_\star \) (green). Clearly, we see a dramatic improvement in the transport of \( t_f \) (red). In the same figure, we also show the distribution for \( \tau_{\text{max}}^\star \) after elimination (purple). The figure illustrates the second level of the long transition time principle, namely the tails of the distribution of \( t_\star \) and of \( \tau_{\text{max}}^\star \) match, where the global maximum distribution is compared with the distribution of \( t_f \); compare also to Eq. (4). The matching of the two distributions for \( t_\star \) and \( \tau_{\text{max}}^\star \) is valid even for a relatively complicated structure of the right tail.

Practically the most important observation is that the elimination leads to a significant truncation of the first passage times. Viewing the tracers as contamination, clearly, the removal or treatment of the long transition time has a dramatic effect on the leaching of the system (see the original data in red and the data after removal in green). Comparing the mean first passage times before and after elimination, as in Eq. (2), we obtain the value \( G \approx 0.6085 \); thus the elimination of \( \tau_{\text{max}} \) leads to expedited transport by about 39%. In SM Sec. XI we discuss a cost-efficient method which treats only a fraction of critical trajectories while maintaining a significant gain.

**Summary.**—Modifying the restart strategy, we exploited the strong disorder to expedite transport. For all transport models, we found a fundamental change of the first passage time statistics upon removal of...
the maximum transition time, which demonstrates a drastic speed-up of the transport process. The mean first passage time, see Fig. 1 and the dispersion are drastically reduced, see Eq. (4), (9) and (12), and Fig. 2 and [3]. The results open a field with many applications, e.g., to address chemical contamination in porous media and single molecule tracking. Our method equips experimentalists and practitioners with identification of controls to actively expedite transport in disordered systems while maintaining manageable costs.

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Supplemental Material

We derive the results from the manuscript in this Supplemental Material (SM); these include rigorous calculations for the unidirectional transport, the continuous time random walk and the quenched trap model. For the pore-scale transport, we explain the simulation method and the details of the data analysis.

I. INTRODUCTION

Here, we summarize the content of the SM. For the unidirectional transport, we calculate the relationship between the modified first passage time $t_r$ and $\tau_{\text{max}}^*$ in SM Sec. II and the measure of gain $G$ in SM Sec. III. Furthermore, SM Sec. VII considers the relationship between the $t_r$ and the transition times upon removal, which is not discussed in the main text, but necessary for a more complete picture of the removal method.

For the continuous time random walk, we provide the principle of the single long transition time is derived in SM Sec. IV, where we also present boundary conditions not discussed in the main text. Also, not in the main text, but an important class, is the case of continuous jumps in SM Sec. V for which we provide the principle of the single long transition time. The measure $G$ can be found in SM Sec. VI, and the relationship between the modified first passage time and the transition times upon removal in SM Sec. VII (not in the main text).

For the quenched trap model, the principle of the single long transition time is presented in SM Sec. VIII. The elimination of the deepest trap and the relevant statistics are calculated in SM Sec. IX. The special case of one dimension can be found in SM Sec. X.

For the simulation of the pore-scale transport, we explain the simulation technique, the transition time identification and cost-efficient removal method (not in the main text) in SM Sec. XI.

II. ELIMINATION OF THE LONGEST TRANSITION TIMES IN THE UNIDIRECTIONAL TRANSPORT MODEL

For the unidirectional transport, we derive the asymptotic behaviors of the distributions of the modified first passage time after removal of the maximum transition time, and its relationship to the second longest transition time, which is presented in the main text in Eq. (4). Note that we generalize this result here by removing the $s$ longest transition times, where $s = 1$ is the case discussed in the main text.

We remind that a particle starts at $x = 1$ and moves to the right until $x = L + 1$, see Fig. I. At each lattice point $x \in \{1, \ldots, L\}$, a particle takes a random transition time $\tau_n$ ($n$ is the step number $n = x$) before jumping to the right neighbour. The $N = L$ transition times are independent and identically distributed following the probability density function

$$\psi(t) \sim At^{-(1+\alpha)}$$  \hspace{1cm} (II.1)

with the scaling exponent $\alpha > 0$ and the amplitude $A$. This function is defined as $\psi(t) = -d/dt \text{Prob}(\tau_n > t)$. In the following, we will usually use the notation $p_{\tau_n}(t) = \psi(t)$ for the probability density function, also for other random variables. In the index, we write the random variable and $t$ is the value. Further, we will also use the cumulative distribution function $P_{\tau_n}(t) = 1 - \text{Prob}(\tau_n > t)$.

So, after $N = L$ steps the particle reaches $L + 1$. We have the set of $N$ transition times $(\tau_1, \ldots, \tau_N)$. The idea is to employ order statistics [1] and order the set

$$\tau_{(1)} < \cdots < \tau_{(N)}.$$  \hspace{1cm} (II.2)
Fig. II.1. (a) Conceptual figure of the unidirectional transport model on a lattice. The first passage time $t_f$ (red) to reach some boundary is the sum of the transition times between lattice points. The maximum transition time $\tau_{\text{max}}$ (blue) is also marked. (b) Conceptual figure of the same transport model as in (a) but $\tau_{\text{max}}$ has been removed from the trajectory. This shows the modified first passage time $t_r = t_f - \tau_{\text{max}}$ (green) and the second maximum $\tau^*_{\text{max}}$ (purple). (c) The distributions of $t_f$ (red circles), $\tau_{\text{max}}$ (blue line), $t_r$ (green circles) and $\tau^*_{\text{max}}$ (purple line) obtained from Monte Carlo simulations for the unidirectional model. We used the Pareto distribution with $\alpha = 0.7$ for the transition times with $L = 5$ and $10^6$ trajectories. As shown, the right tails before and after the elimination of $\tau_{\text{max}}$ from each trajectory match for the maximum and the first passage time distributions, as predicted by the principle of the single long transition time Eq. (3) and the relationship after elimination Eq. (4). Importantly, the elimination decreases the power law of the $t_f$ distribution by the factor 2. This shows clearly that the elimination of the maximum transition time dramatically modifies the statistics in the tail distribution by orders of magnitude.

Obviously, it is $\tau(N) = \tau_{\text{max}}$ and $\tau(1) = \min(\tau_1, \ldots, \tau_N)$. Now, we eliminate the $s = 1, \ldots, N - 1$ longest transition times. So we are left with $(\tau(1), \ldots, \tau(N-s))$, see Fig. II.1 for $s = 1$. The first passage time after elimination of the $s$ largest values is

$$t_r(s) = \sum_{q=1}^{N-s} \tau(q) \quad (\text{II.3})$$

and the longest transition time after removal is

$$\tau^*_{\text{max}}(s) = \tau(N-s). \quad (\text{II.4})$$

When $s = 1$, the two quantities

$$t_r(1) = t_r \quad (\text{II.5})$$

and

$$\tau^*_{\text{max}}(1) = \tau_{\text{max}} \quad (\text{II.6})$$

are mainly considered in the main text.

In order to get the asymptotic distributions of both quantities for any $s$, we need the probability density of the $q$-th order statistics $\tau(q)$ \cite{1}

$$p_{\tau(q)}(t) = \frac{N!}{(q-1)!(N-q)!} p_{\tau_n}(t) [P_{\tau_n}(t)]^{q-1} [1 - P_{\tau_n}(t)]^{N-q}. \quad (\text{II.7})$$
For the maximum after removal, we get the asymptotics

\[ P_{\tau_{\text{max}}^s}(t) \sim \frac{N!}{s!(N-s-1)!} p_{\tau_n}(t)[1 - P_{\tau_n}(t)]^s, \quad (\text{II.8}) \]

which is explicitly

\[ p_{\tau_{\text{max}}^s}(t) \sim \frac{N!}{s!(N-s-1)!} \frac{A^{s+1}}{\alpha^s} t^{-(s+1)\alpha} \quad (\text{II.9}) \]

and in terms of the probability distribution

\[ \text{Prob}(\tau_{\text{max}}^s > t) \sim \frac{N!}{(s+1)!(N-s-1)!} \left( \frac{A}{\alpha} \right)^{s+1} t^{-(s+1)\alpha}. \quad (\text{II.10}) \]

For the modified first passage time, we first need the joint probability density function of the \( N-s \) smallest transition times \((\tau_1, \ldots, \tau_{N-s})\). We integrate the joint probability density of all order statistics \( \frac{N!}{s!} p_{\tau_1}(t_1) \ldots p_{\tau_N}(t_N) \) over the eliminated transition times \((\tau_{N-s+1}, \ldots, \tau_{N})\) and get

\[ p_{\tau_1, \ldots, \tau_{N-s}}(t_1, \ldots, t_{N-s}) = \frac{N!}{s!} [1 - P_{\tau_{N-s}}(t)]^s \prod_{r=1}^{N-s} p_{\tau_r}(t_r) \Theta(t_r - t_{r-1}) \quad (\text{II.11}) \]

with the Heaviside function \( \Theta \). The probability density function of \( t_r(s) \) is

\[ p_{t_r(s)}(t) = (\delta(t - t_r(s))), \quad (\text{II.12}) \]

which is the generalised convolution given by the \((N-s-1)\)-multiple integral

\[ p_{t_r(s)}(t) = \int_0^\infty \! \! \int_0^\infty \! \! \cdots \! \! \int_0^\infty \! \! dt_1 \ldots dt_{N-s-1} \times p_{\tau_{r_1}, \ldots, \tau_{r_{N-s-1}}}(t_1, \ldots, t_{N-s-1}, t - \|t\|_{N-s-1}). \quad (\text{II.13}) \]

We use the notation \( \|t\|_{N-s-1} = \sum_{i=1}^{N-s-1} t_i \). Following, we wish to derive the large \( t \) behavior of \( p_{t_r(s)}(t) \). The \( r = N-s \) term is asymptotically

\[ p_{\tau_n}(t - \|t\|_{N-s-1})[1 - P_{\tau_n}(t - \|t\|_{N-s-1})] \sim p_{\tau_n}(t)[1 - P_{\tau_n}(t)]^s \quad (\text{II.14}) \]

which we can put in front of the integrals. This can be verified by using the Binomial Theorem and considering only the asymptotic behavior. Furthermore, the lower limits of the integrals are set by \( \Theta(t_r - t_{r-1}) \) in Eq. \( \text{(II.11)} \) for all \( r \). Thus, we have asymptotically

\[ p_{t_r(s)}(t) \sim \frac{N!}{s!} p_{\tau_n}(t)[1 - P_{\tau_n}(t)]^s \prod_{t_0}^{t_{N-s-2}} dt_1 \ldots dt_{N-s-1} p_{\tau_n}(t_1) \ldots p_{\tau_n}(t_{N-s-1}). \quad (\text{II.15}) \]

Without loss of generality, we work with the Pareto probability density function \( p_{\tau_n}(t) = \alpha t_0^\alpha t^{-1-\alpha} \) with \( t \geq t_0 \). Generally, one uses the Binomial theorem. Let us start with the first inner integral

\[ \int_{t_{N-s-2}}^\infty dt_{N-s-1} p_{\tau_n}(t_{N-s-1}) = (t_0)^\alpha (t_{N-s-2})^{-\alpha}. \quad (\text{II.16}) \]

The second inner integral gives

\[ (t_0)^\alpha \int_{t_{N-s-3}}^{t_{N-s-2}} dt_{N-s-2} p_{\tau_n}(t_{N-s-2})(t_{N-s-2})^{-\alpha} = \frac{1}{2}(t_0)^{2\alpha} (t_{N-s-3})^{-2\alpha}. \quad (\text{II.17}) \]

We can continue similar with the other integrals. Generally, the \( k \)-th inner integral (with \( k = 1, \ldots, N-s-1 \)) is

\[ \frac{(t_0)^{k-1}}{(k-1)!} \int_{t_{N-s-k-1}}^{t_{N-s-k}} dt_{N-s-k} p_{\tau_n}(t_{N-s-k})(t_{N-s-k})^{-(k-1)\alpha} = \frac{1}{k!}(t_0)^{k\alpha} (t_{N-s-k-1})^{-k\alpha}. \quad (\text{II.18}) \]
FIG. II.2. Distributions of \( t_r(s) \) (red circles) and \( \tau_{\text{max}}(s) \) (black lines) with \( s = 0, 1, 2, 3 \) (from top to bottom) for the unidirectional transport model with \( N = 5 \). We used Pareto transition times with \( t_0 = 1 \) and \( \alpha = 0.5 \) and Monte-Carlo simulations with \( 10^5 \) particles. We find the matching predicted by Eq. (II.21).

In particular, the final inner integral (the outer integral) with \( k = N - s - 1 \) is

\[
\frac{(t_0)^{N-s-2}}{(N-s-2)!} \int_{t_0}^\infty dt_1 p_{\tau_n}(t_1)(t_1)^{-(N-3)\alpha} = \frac{1}{(N-s-1)!} (t_0)^{(N-s-1)\alpha} (t_0)^{-(N-s-1)\alpha} = \frac{1}{(N-s-1)!}. \tag{II.19}
\]

So, we have finally

\[
 p_{\tau_r}(s)(t) \sim \frac{N!}{s!(N-s-1)!} P_{\tau_n}(t) [1 - P_{\tau_n}(t)]^s. \tag{II.20}
\]

which is the same as Eq. (II.8) with \( A = \alpha(t_0)^\alpha \). This asymptotic equivalence shows the principle of the single long transition time

\[
\text{Prob}(t_r(s) > t) \sim \text{Prob}(\tau_{\text{max}}(s) > t), \tag{II.21}
\]

which is presented for \( s = 1 \) in the main text Eq. (4) and Fig. II.1. In Fig. II.2 we present Eq. (II.21) for the four cases \( s = 0, 1, 2 \) and 3. When \( s = 0 \), we have the original first passage time and maximum transition time without any removal, see main text Eq. (3).

III. THE MEASURE OF GAIN \( G \) FOR THE UNIDIRECTIONAL TRANSPORT MODEL

Let us shortly consider the removal of the maximum with \( s = 1 \) (then consider again any \( s \)). When the transition time distribution follows a power law with the exponent \( \alpha < 1 \) Eq. (II.1), then \( \langle t_f \rangle \) and \( \langle \tau_{\text{max}} \rangle \) are both infinite. What does this imply for the mean first passage time after elimination

\[
\langle t_r \rangle = \langle t_f \rangle - \langle \tau_{\text{max}} \rangle? \tag{III.1}
\]

We discuss this problem here, i.e. we calculate \( \langle t_r \rangle \). In general, we consider any \( \alpha > 0 \) and any \( s \geq 1 \). Exponential transition times are considered in addition. With the results of this section, we provide the derivations for the measure of gain \( \langle G \rangle = \langle t_r \rangle / \langle t_f \rangle \) in the main text Eq. (5). The thermodynamic limit \( N \to \infty \) is discussed in addition,
The basis of all following calculations is again that we use the order statistics of the transition times. The mean first passage time after elimination

$$\langle t_r(s) \rangle = \sum_{q=1}^{N-s} \langle \tau(q) \rangle$$

(III.2)

and the quantifier of the elimination (Eq. (2) in the main text) is

$$G(s) = \frac{\langle t_r(s) \rangle}{\langle t_f \rangle}$$

(III.3)

for \(s = 1, \ldots, N - 1\). When \(s = 0\), it is \(t_r(0) = t_f\). When \(s = N\), we set \(t_r(N) = 0\).

**A. The measure of gain \(G\) for exponentially distributed transition times**

For the exponentially distributed transition times \(p_{\tau_n}(t) = \exp(-t/\langle \tau_n \rangle)/\langle \tau_n \rangle\), the mean order transition time is

$$\langle \tau(q) \rangle = \langle \tau_n \rangle \sum_{j=1}^{q} \frac{1}{N - j + 1},$$

(III.4)

see [2]. The mean first passage time after elimination can be calculated as

$$\langle t_r(s) \rangle = \langle \tau_n \rangle \sum_{q=1}^{N-s} \sum_{j=1}^{q} \frac{1}{N - j + 1} = \langle \tau_n \rangle \sum_{j=1}^{N-s} \left(1 - \frac{s}{N - j + 1}\right).$$

(III.5)

Using the harmonic number \(H(K) = \sum_{j=1}^{K} 1/j\), we write

$$\langle t_r(s) \rangle = \langle \tau_n \rangle [N - s - s(H(N) - H(s))].$$

(III.6)

Therefore, the improvement quantifier is exactly given by

$$G(s) = 1 - \frac{s}{N} [1 - H(s) + H(N)].$$

(III.7)

Now, we consider the limit \(N \to \infty\). For fixed \(s\), the large \(N\) limit is

$$G(s) \sim 1.$$

(III.8)

For a fixed fraction \(f\) with \(s = fN\), \(0 < f < 1\), the large \(N\) limit is

$$G(s) \sim 1 - f[1 - \ln(f)],$$

(III.9)

which is independent of \(\langle \tau_n \rangle\).

**B. The measure of gain \(G\) for Pareto distributed transition times**

For the Pareto distributed transition times \(p_{\tau_n}(t) = \alpha(t_0)^{N-1}t^{-1-\alpha}, t \geq t_0\), the mean order transition time can be calculated as

$$\langle \tau(q) \rangle = \frac{(-1)^q t_0 N! \Gamma(-N + 1/\alpha) \Gamma(q)}{(N-q)! (q-1)! \Gamma(-N + q + 1/\alpha)}$$

(III.10)

for \(\alpha > 1/(N - q + 1)\). This result has been obtained with Mathematica using the formula \(\langle \tau(q) \rangle = \int_0^\infty t p_{\tau(q)}(t)dt\). From that, we obtain the mean first passage time after elimination

$$\langle t_r(s) \rangle = N t_0 \frac{\alpha}{\alpha - 1} - \frac{(-1)^{N-s+1}(1 + \alpha s) t_0 N! \Gamma(-N + 1/\alpha)}{(-1 + \alpha)(-1 + s)! \Gamma(1 - s + 1/\alpha)}$$

(III.11)
for $\alpha > 1/(s+1)$. Thus, we have $G$ taking the ratio of this formula between the general $s$ and $s=0$ case. The latter is $\langle t_r(0) \rangle = \langle t_f \rangle = N t_0 \alpha / (\alpha - 1)$ for $\alpha > 1$ and infinite otherwise. The quantifier $G$ with $s=1$ is presented in the main text Eq. (5).

Next, we calculate the large $N$ limit of $\langle t_r(s) \rangle$ for $s = f N$ with the fixed fraction $0 < f < 1$. Each $N$-dependent term of Eq. (III.11) behaves as

\begin{align*}
N! &\sim \sqrt{2\pi} e^{-N} N^{N+1/2}, \\
(s-1)! &\sim \sqrt{2\pi} e^{-s} s^{s-1/2}, \\
\Gamma(-N+1/\alpha) &\sim \sqrt{\pi/2} e^{-N} N^{-N-1/2+1/\alpha} \csc[\pi(-N+1/\alpha)] = (-1)^N \sqrt{\pi/2} e^{-N} N^{-N-1/2+1/\alpha} \csc[\pi/\alpha], \\
\Gamma(-s+1+1/\alpha) &\sim \sqrt{\pi/2} e^{-s} s^{-s+1/2+1/\alpha} \csc[\pi(-s+1+1/\alpha)] = (-1)^s \sqrt{\pi/2} e^{-s} s^{-s+1/2+1/\alpha} \csc[\pi(1+1/\alpha)].
\end{align*}

Putting these terms together yields

\begin{equation}
\langle t_r(s) \rangle \sim N t_0 \frac{\alpha}{\alpha - 1} - f^{-1/\alpha} t_0 \frac{1 - \alpha f N}{1 - \alpha}.
\end{equation}

Since $s \to \infty$ the existence of this mean is ensured for $\alpha > 0$. Finally, the improvement quantifier for $\alpha > 1$ scales as

\begin{equation}
G(s) \sim 1 - f^{1-1/\alpha}
\end{equation}

and for $0 < \alpha < 1$, we have $G(s) \sim 0$.

**IV. PRINCIPLE OF THE SINGLE LONG TRANSITION TIME FOR THE CONTINUOUS TIME RANDOM WALK**

Particles in the continuous time random walk, see Fig. [IV.1], start at $x = 1$ and are absorbed at $x = L + 1$. The probability to jump left is $q$ and right $p = 1 - q$ with right bias $p > q$. The number of jumps $N$ is random. Each transition time follows the probability density function Eq. (II.1). We provide the principle of the single long transition time Eq. (6) and (7) presented in the main text. We also calculate different boundary conditions here.
The probability density functions of $t_f = \sum_{n=1}^{N} \tau_n$ and $\tau_{\text{max}} = \max(\tau_1, \ldots, \tau_N)$ obey

$$p_{t_f}(t) = \sum_{n=1}^{N} \phi_{\text{dis}}(n)p_{t_f|N}(t|n),$$

$$p_{\tau_{\text{max}}}(t) = \sum_{n=1}^{N} \phi_{\text{dis}}(n)p_{\tau_{\text{max}}|N}(t|n)$$

(IV.1)

where $\phi_{\text{dis}}(n)$ is the probability that a particle made $N = n$ jumps before its absorption. The conditional probability density functions on the right hand side consider fixed $N = n$.

Under the assumption of a finite mean number $\langle N \rangle < \infty$, we can utilize the principle of the single long transition time of the unidirectional model (see Eq. (4) in the main text). We get in generality

$$\text{Prob}(t_f > t) \sim \text{Prob}(\tau_{\text{max}} > t) \sim \langle N \rangle \frac{A}{\alpha} t^{-\alpha}$$

(IV.2)

with the mean $\langle N \rangle = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n)n$. This result is presented in Eq. (6) in the main text. The same technique is used to get the principle after removal, which is presented in the main text Eq. (8), namely averaging main text Eq. (4) over $N$.

A. Principle of the single long transition time with finite $\langle N \rangle$ - Second approach

We derive the principle of the single long transition time Eq. (IV.2) where $\langle N \rangle$ is finite with an alternative approach, which is needed for the unbiased situation (when $\langle N \rangle$ is infinite). Here, we use a generating function approach. The generating function is defined as

$$\tilde{\phi}_{\text{dis}}(z) = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n)z^n.$$  

(IV.3)

From Eq. (IV.1), we see that we can relate the Laplace transform of $p_{t_f}(t)$ (defined as $\hat{p}_{t_f}(s) = \mathcal{L}_{t \to s} [p_{t_f}(t)] = \int_{0}^{\infty} p_{t_f}(t) \exp(-st)dt$) and the cumulative distribution function of $\tau_{\text{max}}$ to the generating function

$$\hat{p}_{t_f}(s) = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n)[\hat{p}_{\tau_{\text{max}}}(s)]^n,$$

$$\text{Prob}(\tau_{\text{max}} \leq t) = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n)[\text{Prob}(\tau_{\text{n}} \leq t)]^n.$$  

(IV.4)

The $\hat{p}_{\tau_{\text{n}}}(s) \approx 1$ expansion of the first formula of Eq. (IV.4) is

$$\hat{p}_{t_f}(s) \sim 1 + \langle N \rangle (\hat{p}_{\tau_{\text{n}}}(s) - 1) + \ldots.$$  

(IV.5)

Inserting the $s \approx 0$ expansion $\hat{p}_{\tau_{\text{n}}}(s) \sim 1 + A\Gamma(-\alpha)s^{\alpha}$ yields

$$\hat{p}_{t_f}(s) \sim 1 + \langle N \rangle A\Gamma(-\alpha)s^{\alpha}.$$  

(IV.6)

The small $s$ behavior of the Laplace transform gives the large $t$ behavior of the probability density function, which is known as the Tauberian theorem [3,4]. Inverse Laplace transform gives

$$p_{t_f}(t) \sim \langle N \rangle At^{-1-\alpha}.$$  

(IV.7)

The $\text{Prob}(\tau_{\text{n}} \leq t) \approx 1$ expansion of the second formula of Eq. (IV.4) is

$$\text{Prob}(\tau_{\text{max}} \leq t) \sim 1 + \langle N \rangle (\text{Prob}(\tau_{\text{n}} \leq 1) - 1) + \ldots.$$  

(IV.8)

Inserting the large $t$ behavior $\text{Prob}(\tau_{\text{n}} \leq t) \approx 1 - At^{-\alpha}/\alpha$ and derivation gives

$$p_{\tau_{\text{max}}}(t) \sim \langle N \rangle At^{-1-\alpha}$$  

(IV.9)

which is the same asymptotic behavior as Eq. (IV.2). We now calculate $\langle N \rangle$ rigorously. We will also consider $\langle N \rangle = \infty$ later (that is for example the case of no bias).
B. Weak bias $p > q$ with no left boundary condition

We consider weak bias $p > q$ and calculate $\langle N \rangle$, thus, the principle of the single long transition time Eq. (IV.2). Here, there is no left boundary condition and the lattice points are $x = \{\ldots, 0, 1, \ldots, L\}$ with start at $x = 1$ and absorption at $L + 1$.

We use a generating function approach. The generating function of the mean number of jumps $\tilde{\phi}_{\text{dis}}(z)$ defined in Eq. (IV.3). For the special case $L = 1$ (i.e. absorption at $x = 2$) the generating function is

$$
\tilde{\phi}_{\text{dis}}(z)|_{L=1} = \frac{1 - \sqrt{1 - 4pqz^2}}{2qz},
$$

(IV.10)

which can be derived from the renewal equation [4,5]. For general $L$, we can use the following argument. Let us say we have now $L = 2$ (i.e. absorption at $x = 3$). Once the particle reaches $x = 2$, it is the problem of Eq. (IV.10) where both the initial starting point and the absorbing boundary are shifted one lattice point to the right. The same picture holds for any $L$. Thus, the general generating function is

$$
\tilde{\phi}_{\text{dis}}(z) = [\tilde{\phi}_{\text{dis}}(z)|_{L=1}]^L = \left[\frac{1 - \sqrt{1 - 4pqz^2}}{2qz}\right]^L.
$$

(IV.11)

The $z = 0$ expansion for the case $L = 1$ of Eq. (IV.10) is

$$
\tilde{\phi}_{\text{dis}}(z)|_{L=1} = \sum_{n=1,3,\ldots} \frac{(2n+1-3)!!}{(n+1)!} \frac{2^n}{p^{n+1}} \frac{1}{q^{n+1}} \frac{q^{n+1}}{2} \frac{z^{n+1}}{2} \frac{1}{1 - 4pqz^2},
$$

(IV.12)

$$
= \sum_{i=1}^{\infty} \frac{(2i-3)!!}{i!} p^{i-1} q^{i-1} 2i z^{2i-1} = \frac{pz}{q^2} + \frac{p^2 q^3 z^3}{2} + \frac{2p^3 q^5 z^5}{6} + \frac{5p^4 q^7 z^7}{24} + \ldots
$$

where the step number is $n = 2i - 1$. Clearly, for the strongly biased transport $p = 1$, we have only one term in the expansion corresponding to the case where the particle is only jumping to the right and absorbed after one step, namely $\phi_{\text{dis}}(1) = 1$ and otherwise zero. From the last formula, we can read $\phi_{\text{dis}}(n)$ for $L = 1$ as

$$
\phi_{\text{dis}}(n)|_{L=1} = \begin{cases} 
\frac{(2n+1-3)!!}{(n+1)!} \frac{2^n}{p^{n+1}} \frac{1}{q^{n+1}} \frac{q^{n+1}}{2} \frac{z^{n+1}}{2} \frac{1}{1 - 4pqz^2}, & \text{for } n = 1,3,5,\ldots, \\
0, & \text{for } n = 2,4,6,\ldots.
\end{cases}
$$

(IV.13)

The mean $\langle N \rangle = \sum_{n=1}^{\infty} n \phi_{\text{dis}}(n)$ for $L = 1$ is

$$
\langle N \rangle|_{L=1} = \sum_{i=1}^{\infty} (2i - 1) \frac{(2i-3)!!}{i!} p^{i-1} q^{i-1} = \frac{1}{p-q}.
$$

(IV.14)

Here, we see that when $p \to 1/2$ this mean diverges, as it is well-known. Therefore, we can conclude the mean jump number for general $L$ as

$$
\langle N \rangle = L \langle N \rangle|_{L=1} = \frac{L}{p-q}.
$$

(IV.15)

Another way, we can also read the mean from

$$
\langle N \rangle = \frac{d\phi_{\text{dis}}(z)}{dz}|_{z=1}.
$$

(IV.16)

Thus, using the $z \approx 1$ expansions

$$
\phi_{\text{dis}}(z)|_{L=1} \sim 1 + \frac{1}{p-q} (z-1) + O((z-1)^2)
$$

(IV.17)

and

$$
\phi_{\text{dis}}(z) \sim 1 + \frac{L}{p-q} (z-1) + O((z-1)^2),
$$

(IV.18)

we find again the mean numbers of jumps in the second terms. With this exact formula for $\langle N \rangle$, we present the principle of the single long transition time in Eq. (7) in the main text.
C. No bias $p = 1/2$ with no left boundary condition

We consider no bias $p = q = 1/2$. Here, there is no left boundary and the lattice points are $x = \{\ldots, 0, 1, \ldots, N\}$ with start at $x = 1$ and absorption at $L + 1$. Since the bias is zero, the mean number of jumps $(N)$ diverges. So the principle of Eq. (IV.2) does not hold. However, we can use again the approach of SM Sec. IV A to find the modification of the principle.

We start with the generating function for $L = 1$, which is

$$\tilde{\phi}_{\text{dis}}(z)|_{L=1} = \frac{1 - \sqrt{1 - z^2}}{z} \quad \text{(IV.19)}$$

where we used $p = q = 1/2$ in Eq. (IV.10). The $z \approx 1$ expansion is

$$\tilde{\phi}_{\text{dis}}(z)|_{L=1} \sim 1 - \frac{2}{z} + \ldots \quad \text{(IV.20)}$$

For general $L$, the $z \approx 1$ expansion is

$$\tilde{\phi}_{\text{dis}}(z) \sim 1 - L \sqrt{2(1 - z)} + \ldots \quad \text{(IV.21)}$$

where we used $\tilde{\phi}_N(z) = [\tilde{\phi}_N(z)|_{L=1}]^L$, see Eq. (IV.11).

In the following, we consider $0 < \alpha < 1$. First, we discuss $\hat{p}_{tf}(s)$ where $\hat{p}_{tf}(s)$ given in Eq. (IV.4). We put the $s \approx 0$ expansion of $\hat{p}_{\tau_n}(s)$ into the $s \approx 0$ expansion of $\hat{p}_{tf}(s)$ and get

$$\hat{p}_{tf}(s) \sim 1 - L \sqrt{2A} |\Gamma(-\alpha)| s^{\alpha/2} \quad \text{(IV.22)}$$

Inverse Laplace transform obtains

$$p_{tf}(t) \sim \sqrt{\frac{2A |\Gamma(-\alpha)|}{|\Gamma(-\alpha/2)|}}Lt^{-1-\alpha/2} \quad \text{(IV.23)}$$

valid for long times. On the other hand, the large $t$ behavior of $p_{\tau_{\max}}(t)$ is obtained as follows. We put the large $t$ behavior of $\text{Prob}(\tau_n \leq t)$ into the $\text{Prob}(\tau_n \leq t) \approx 1$ expansion of $\text{Prob}(\tau_{\max} \leq t)$, so

$$\text{Prob}(\tau_{\max} \leq t) \sim 1 - L \sqrt{2A/\alpha}t^{-\alpha/2} \quad \text{(IV.24)}$$

with the derivation

$$p_{\tau_{\max}}(t) \sim \sqrt{\frac{\alpha A}{2Lt^{1-\alpha/2}}} \quad \text{(IV.25)}$$

This as the same power law decay as Eq. (IV.23) but with a different prefactor. To formulate the single long transition time principle we define a rescaled maximum transition time $\tilde{\tau}_{\max} = \Delta_\alpha \tau_{\max}$ with the $\alpha$-dependent factor

$$\Delta_\alpha = \left(\frac{2\sqrt{|\Gamma(-\alpha)|}}{|\Gamma(-\alpha/2)|}\right)^{2/\alpha} \quad \text{(IV.26)}$$

In Fig. [IV.2a], this rescaling factor is presented. Clearly $\Delta_\alpha > 1$, hence $\tilde{\tau}_{\max} > \tau_{\max}$. For $\alpha \to 0$, we get back the original maximum $\tilde{\tau}_{\max} \to \tau_{\max}$ while for $\alpha \to 1$, the scaling factor $\Delta_\alpha$ diverges. Now this rescaled long transition time $\tilde{\tau}_{\max}$ is related to the first passage time by the asymptotics

$$\text{Prob}(t_f > t) \sim \text{Prob}(\tilde{\tau}_{\max} > t) \sim (\Delta_\alpha)^{\alpha/2} \sqrt{\frac{2A}{\alpha}}t^{-\alpha/2} \quad \text{(IV.27)}$$

see Fig. [IV.2b]. The fact that here we rescale $\tau_{\max}$ with the factor $\Delta_\alpha$ means that the previously discussed long transition time principle Eq. (6) still holds but with a renormalised definition of the maximum transition time. But
in contrast to Eq. (6), in Eq. (IV.27) the power law decay is $t^{-\alpha/2}$. The first passage time is always larger than the original $\tau_{\text{max}}$, that is why we rescaled the latter with $\Delta_\alpha > 1$. Thus, we call Eq. (IV.27) the principle of the extended long transition time. In the limit $\alpha \to 0$, the first passage time scales as the longest transition time without rescaling. On the other hand, in the limit $\alpha \to 1$, we have $\Delta_\alpha \to \infty$ so that the principle breaks down as we discuss now.

What happens for $\alpha > 1$ for the unbiased case $p = 1/2$? The probability distribution $\text{Prob}(t_f > t)$ called survival probability decays like $t^{-1/2}$ which is a well-known result in the theory of diffusion [5]. Because the mean transition time is finite, $\alpha$ plays no role in the decay of this survival probability. This is vastly different from the distribution of the maximum $\text{Prob}(\tau_{\text{max}} > t)$ which decays in the continuous time random walk model like $t^{-\alpha/2}$. Hence, for $\alpha > 1$ and $p = 1/2$, there is no principle of the single long transition time nor of the extended version.

![Graph](a) The factor $\Delta_\alpha$ given in Eq. (IV.26) to obtain the rescaled transition time $\tilde{\tau}_{\text{max}} = \Delta_\alpha \tau_{\text{max}}$. (b) The distributions of $t_f$ (red circles) and $\tilde{\tau}_{\text{max}}$ (blue line) compared with the theory Eq. (IV.27) (black line) for the continuous time random walk with no bias $p = 0.5$ and $L = 5$. We used Pareto transition times with $t_0 = 1$ and $\alpha = 0.5$ for the Monte-Carlo simulations with $10^5$ particles. Due to possible very long left excursions, we cut off the simulations once $t_f > 10^{14}$. We find perfect agreement with the theory of the extended long transition time principle Eq. (IV.27).

D. Left bias $p < q$ with no left boundary condition

We consider a left bias $p < q$. Here, there is no left boundary and the lattice points are $x = \{\ldots, 0, 1, \ldots, N\}$ with start at $x = 1$ and absorption at $L + 1$. Since the bias pushes the particles away from the absorbing wall, the mean number of jumps $\langle N \rangle$ diverges. So the principle of Eq. (IV.2) does not hold. However, we can use again the approach of SM Sec. [IV.A] to find the modification of the principle.

We start with the generating function for $L = 1$. The $z \approx 1$ expansion is

$$\hat{\phi}_{\text{dis}}(z)|_{L=1} \sim \frac{p}{q} + \frac{1 - \sqrt{1 - 4pq}}{2q \sqrt{1 - 4pq}} (z - 1) + O((z - 1)^2)$$ (IV.28)

For general $L$, the $z \approx 1$ expansion is

$$\hat{\phi}_{\text{dis}}(z) \sim \left(\frac{p}{q}\right)^L + L \frac{1 - \sqrt{1 - 4pq}}{2q \sqrt{1 - 4pq}} (z - 1) + O((z - 1)^2)$$ (IV.29)

where we used $\hat{\phi}_N(z) = [\hat{\phi}_N(z)|_{L=1}]^L$, see Eq. (IV.11). We use the definition of the generating function, namely $\hat{\phi}_N(z)|_{z=1} = \sum_{n=1}^{\infty} \phi_N(n)$ hence the leading term $(p/q)^L < 1$ is the probability to reach the absorbing boundary $L + 1$. In other words, that absorption is not guaranteed due to the left bias $p < q$. 

![Diagram](FIG. IV.2. (a) The factor $\Delta_\alpha$ given in Eq. (IV.26) to obtain the rescaled transition time $\tilde{\tau}_{\text{max}} = \Delta_\alpha \tau_{\text{max}}$. (b) The distributions of $t_f$ (red circles) and $\tilde{\tau}_{\text{max}}$ (blue line) compared with the theory Eq. (IV.27) (black line) for the continuous time random walk with no bias $p = 0.5$ and $L = 5$. We used Pareto transition times with $t_0 = 1$ and $\alpha = 0.5$ for the Monte-Carlo simulations with $10^5$ particles. Due to possible very long left excursions, we cut off the simulations once $t_f > 10^{14}$. We find perfect agreement with the theory of the extended long transition time principle Eq. (IV.27).)

[Diagram Description]:

- **Part (a)**: The graph illustrates the function $\Delta_\alpha$ as a function of $\alpha$. The x-axis represents $\alpha$ ranging from 0 to 1, and the y-axis represents $\Delta_\alpha$ ranging from 0 to 10. The curve shows a smooth transition from 0 to 10 as $\alpha$ increases from 0 to 1.

- **Part (b)**: Two distributions are compared: $t_f$ (red circles) and $\tilde{\tau}_{\text{max}}$ (blue line) along with the theory curve (black line) for the continuous time random walk with no bias $p = 0.5$ and $L = 5$. The distributions are shown over a range of time values $t$ from $10^0$ to $10^{10}$. The survival probability decays like $t^{-1/2}$, and the principle of the extended long transition time is evident in the comparison with theory.
In what follows, we consider an ensemble of trajectories that are conditioned on absorption, namely those trajectories that are never absorbed are discarded. We indicate this with “(abs)”. Following the procedure of SM Sec. IV A, we have

\[ p_{t}^{(\text{abs})}(t) \sim p_{\text{max}}^{(\text{abs})}(t) \sim \langle N \rangle^{(\text{abs})} A t^{-1-\alpha} \]  

(IV.30)

with

\[ \langle N \rangle^{(\text{abs})} = \frac{1 - \sqrt{1 - 4pq}}{2q \sqrt{1 - 4pq}L}. \]  

(IV.31)

The mean \( \langle N \rangle^{(\text{abs})} \) here is the mean with respect to the ensemble of trajectories that were eventually absorbed.

**E. No bias \( p = q = 1/2 \) with left reflecting boundary**

We discuss the problem unbiased situation \( p = q = 1/2 \) in the presence of a reflecting boundary at \( R < 1 \). In this case, the mean \( \langle N \rangle \) is finite, in contrast to the unbiased case with no right wall \( (R \rightarrow -\infty) \). The problem for \( L = 1 \) was discussed in [6], and the generating function for \( L = 1 \) is

\[ \tilde{\phi}_{\text{dis}}(z)|_{R,L=1} = \frac{z/2}{1 - \frac{z^2}{4}} H_+ \Lambda_+ + H_- \Lambda_- - R - 1, \]  

(IV.32)

with the \( z \)-dependent functions

\[ \Lambda_{\pm} = \frac{1 \pm \sqrt{1 - z^2}}{2}, \quad B_{-} = \frac{1 - z^2/2 - \Lambda_{-}}{\Lambda_{-} - \Lambda_{+}}, \quad B_{+} = 1 - B_{-}. \]  

(IV.33)

For any \( L > 1 \), we obtain the generating function as follows. Let’s consider first \( L = 2 \). When the particle reaches \( x = 2 \) for the first time, it is now the \( L = 1 \) problem but the reflecting boundary \( R \) is moved one lattice point to the left. Similar, for any \( L \). So the generating function is

\[ \tilde{\phi}_{\text{dis}}(z)|_{R,L} = \prod_{R'=0}^{L-1} \tilde{\phi}_{\text{dis}}(z)|_{R-R',L=1} = \prod_{R'=0}^{L-1} \frac{z/2}{1 - \frac{z^2}{4}} H_+ \Lambda_+ + H_- \Lambda_- - R - 1. \]  

(IV.34)

The generating function \( \tilde{\phi}_{\text{dis}}(z)|_{R-R',L=1} \) with absorbing boundary \( x = L + 1 = 2 \) was derived in [6]. For \( L = 1 \) the \( z \approx 1 \) expansion of \( \tilde{\phi}_{\text{dis}}(z)|_{R-R',L=1} \) is

\[ \tilde{\phi}_{\text{dis}}(z)|_{R-R',L=1} \sim 1 + (3 - 2R - 2R') (z - 1) + \ldots \]  

(IV.35)

from which we see the mean number of jumps for \( L = 1 \)

\[ \langle N \rangle|_{R-R',L=1} = 3 - 2R - 2R'. \]  

(IV.36)

Therefore, the mean for any \( L \) is

\[ \langle N \rangle|_{R,L} = \sum_{R'=0}^{L-1} \langle N \rangle|_{R-R',L=1} = (2 + L + 2|R|)L. \]  

(IV.37)

Thus, we have the explicit statement the principle of the single long transition time Eq. (IV.2).
F. Bias $p \neq q$ with left reflecting boundary

We discuss the problem $p \neq q$ (which includes left and right bias) in the presence of a reflecting boundary at $R < 1$. In this case, the mean $\langle N \rangle$ is finite. Again, the particle starts at $x = 1$ and absorption is at $x = L + 1$. The problem for $L = 1$ was discussed in [6], and the generating function for $L = 1$ is

$$\tilde{\phi}_{\text{dis}}(z)|_{R,L=1} = \frac{pz}{1 - pqz^2}$$

with the $z$-dependent functions

$$\lambda_\pm = \frac{1 \pm \sqrt{1 - 4pqz^2}}{2},$$
$$B_+ = \frac{1 - (\lambda_-)^2 - pz^2}{\lambda_+ - \lambda_-},$$
$$B_- = 1 - B_+.$$  \tag{IV.39}

For any $L > 1$, the generating function is

$$\tilde{\phi}_{\text{dis}}(z)|_{R,L} = \prod_{R'=0}^{L-1} \tilde{\phi}_{\text{dis}}(z)|_{R-R',L=1} = \prod_{R'=0}^{L-1} \frac{pz}{1 - pqz^2} \frac{B_+ \lambda_+^{R-R'-1} + B_- \lambda_-^{R-R'-1}}{B_+ \lambda_+^{R-R'} + B_- \lambda_-^{R-R'}}.$$  \tag{IV.40}

For $L = 1$ the $z \approx 1$ expansion of $\tilde{\phi}_N(z)|_{R-R',L=1}$ is

$$\tilde{\phi}_{\text{dis}}(z)|_{R-R',L=1} \sim 1 + \frac{d\tilde{\phi}_{\text{dis}}(z)|_{R-R',L=1}}{dz}|_{z=1} (z - 1) + \ldots$$

from which we see the mean number of jumps for $L = 1$

$$\langle N \rangle|_{R-R',L=1} = \frac{d\tilde{\phi}_{N}(z)|_{R-R',L=1}}{dz}|_{z=1}.$$  \tag{IV.41}

Therefore, the mean for any $L$ is

$$\langle N \rangle|_{R,L} = \sum_{R'=0}^{L-1} \langle N \rangle|_{R-R',L=1} = \sum_{R'=0}^{L-1} \frac{d\tilde{\phi}_{N}(z)|_{R-R',L=1}}{dz}|_{z=1} = \frac{d\tilde{\phi}_{N}(z)|_{R,L}}{dz}|_{z=1}.$$  \tag{IV.42}

We suggest to use Mathematica, in order to calculate numerical values of $\langle N \rangle|_{R,L}$ for given parameters $p$, $q$, $R$ and $L$. With that, we have the principle of the single long transition times Eq. [IV.2].

V. PRINCIPLE OF THE SINGLE LONG TRANSITION TIME FOR THE CONTINUOUS TIME RANDOM WALK WITH CONTINUOUS JUMPS

For the continuous time random walk, the principle of the single long transition time Eq. [6] (main text) and [IV.2] depends on the mean number of jumps $\langle N \rangle$ (when $\langle N \rangle$ exists). Here, we discuss the case of continuous jumps $\Delta x_n$ which is not presented in the main text but still serves as an important example. When the transport is biased, the mean jump size is positive $\langle \Delta x_n \rangle > 0$ so that $\langle N \rangle$ exists, and then the principle Eq. [6] and [IV.2] still holds.

We discuss exemplary the unidirectional transport. The particles start at $x = 0$ and are absorbed at $x = L$. The first passage at $L$ happens at the $N$-th jump event under the condition $\sum_{n=1}^{N-1} \Delta x_n < L \leq \sum_{n=1}^{N} \Delta x_n$. The position at the step $N - 1$ before passing $L$ is $\sum_{n=1}^{N-1} x_n$. The last step $\Delta x_N$ can be seen as the survival jump size. Thus, the probability of $N = n$ jumps in Laplace space is $\tilde{\phi}_{\text{dis}}(n) = L_{L \to s} [\phi_{\text{dis}}(n)] = [\tilde{p}_{\Delta x_n}(s)]^{n-1} [1 - \tilde{p}_{\Delta x_n}(s)]/s$. For example, for an exponential jump size distribution, it can be solved exactly and we find the mean number $\langle N \rangle = 1 + L/\langle \Delta x_n \rangle$, which gives us an explicit expression for the principle Eq. [6] and [IV.2].
VI. MEASURE OF GAIN $G$ FOR THE WEAKLY BIASED $(p > 1/2)$ CONTINUOUS TIME RANDOM WALK

The measure of gain $G = \langle t_r \rangle / \langle t_f \rangle$ for the biased continuous time random walk $(p > 1/2)$ with Pareto distributed transition times is presented in Eq. (9) in the main text. We obtained this formula because with the help of the two mean first passage times

$$
\langle t_f \rangle = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n) \langle t_f | n \rangle,
$$

$$
\langle t_r \rangle = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n) \langle t_r | n \rangle.
$$

(VI.1)

The conditional means on the right hand side assume fixed $N = n$ number of jumps. We use now Pareto transition times. That means for the first formula $\langle t_f | n \rangle = n t_0 \alpha / (\alpha - 1)$ when $\alpha > 1$ and therefore $\langle t_f \rangle = \langle N \rangle t_0 \alpha / (\alpha - 1)$. For the second formula, we use $\langle t_r | n \rangle$ from Eq. (III.11) (when $s = 1$), which is valid for $\alpha > 1/2$. Putting both results together gives $G$, which is Eq. (9) in the main text, namely

$$
G = \begin{cases} 
0, & 0 < \alpha < 1, \\
1 - h_\alpha \sum_{n=1}^{\infty} \phi_{\text{dis}}(n)(-1)^n n! \Gamma \left(-n + \frac{1}{\alpha}\right), & 1 < \alpha
\end{cases}
$$

(VI.2)

with $h_\alpha = [(p - q)(\alpha - 1)] / [\alpha L \Gamma(1/\alpha)]$ where we used $\langle N \rangle = L / (p - q)$ Eq. (IV.15). In order to obtain numerical values of $G$, we have to calculate the sum (for $1 < \alpha$). We use the analytical expression of the generating function $\tilde{\phi}_{\text{dis}}(z)$ Eq. (IV.11) and obtain numerical values of $\phi_{\text{dis}}(z)$ Eq. (IV.3) using Mathematica. This allows us to get $G$ numerically. In Fig. VI.1 we compare $G$ between the unidirectional model and the continuous time random walk.

FIG. VI.1. The quantifiers $G$ of the unidirectional model Eq. (5) (black circles) and the continuous time random walk Eq. (9) (red crosses). The transition times follow the Pareto distribution with $\alpha = 1.1$ and $t_0 = 1$. To make the comparison we take $N$ in the unidirectional model equal to $L / (p - q)$, which is the mean number of steps before absorption in the continuous time random walk, namely $\langle N \rangle$. Furthermore, the continuous time random walk has the bias parameter $p = 3/4$. 

VI.1
VII. ELIMINATION PRINCIPLE DEPENDING ON THE TRANSITION TIME DISTRIBUTION AFTER ELIMINATION

The principle of the single long transition time of the unidirectional model (Eq. (3) in the main text) can also be written as

$$\text{Pr}(t_f > t) \sim N \text{Pr}(\tau_n > t),$$  \hspace{1cm} (VII.1)

i.e. the first passage time distribution is asymptotically distributed as $N$ times the transition time distribution. Here, we want to derive this relationship after we removed the maximum transition time for the unidirectional transport and the continuous time random walk. This is not discussed in the main text but necessary for a more complete picture of the mathematical structure of the removal technique.

A. Unidirectional transport

For the unidirectional transport of Sec. II, the transition times after elimination of the maximum at step number $m$, i.e. $\tau_{max} = \tau_m$ are $(\tau_1, \ldots, \tau_{m-1}, \tau_{m+1}, \ldots, \tau_N)$. We call an element of this set $\tau^\star_n$. The probability density function is the average

$$p_{\tau^\star_n}(t) = \frac{1}{N-1} \sum_{q=1}^{N-1} p_{\tau(q)}(t)$$  \hspace{1cm} (VII.2)

with $p_{\tau(q)}(t)$ from Eq. [II.7]. The large $t$ behavior is

$$p_{\tau^\star_n}(t) \sim N p_{\tau_n}(t)[1 - P_{\tau_n}(t)].$$  \hspace{1cm} (VII.3)

Comparison with the large $t$ behavior of the $t_r$ distribution Eq. (II.20) yields

$$p_{t_r}(t) \sim (N - 1)p_{\tau^\star_n}(t),$$  \hspace{1cm} (VII.4)

i.e. the transition time distribution after elimination is multiplied with the number of jumps $N - 1$.

B. Continuous time random walk with finite $\langle N \rangle$

We consider the continuous time random walk of Sec. IV with finite moments of $N$, e.g. the weakly biased $p > q$ case. The transition times after elimination of $\tau_{max} = \tau_m$ are $(\tau_1, \ldots, \tau_{m-1}, \tau_{m+1}, \ldots, \tau_N)$. Importantly, $N$ is random. An immediate question is the following; does Eq. (VII.4) still hold when we average the prefactor $p_{t_r}(t) \sim \langle N - 1 \rangle p_{\tau^\star_n}(t)$?

Let us start with a straightforward definition of $p_{\tau^\star_n}(t)$, namely the average over $N$ of Eq. (VII.2), namely

$$p_{\tau^\star_n}(t) = \langle p_{\tau^\star_n|N}(t|N) \rangle \sim \langle N \rangle p_{\tau_n}(t)[1 - P_{\tau_n}(t)].$$  \hspace{1cm} (VII.5)

The mean is $\langle \phi \rangle = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n) \phi$ and $\langle p_{\tau^\star_n|N}(t|N) \rangle$ is given in Eq. (VII.2) and (VII.3). How to obtain an estimate of $p_{\tau^\star_n}(t)$ from simulation? Given $R$ trajectories. For each trajectory $r \in \{1, \ldots, R\}$, we have the set of transition times $\tau^\star(r) = (\tau_1, \ldots, \tau_{m-1}, \tau_{m+1}, \ldots, \tau_N)$. We calculate the histogram of $\tau^\star(r)$ denoted as $h_{\tau^\star_n}(t; r)$. For large enough $R$, the sample mean $\frac{1}{R} \sum_{r=1}^{R} h_{\tau^\star_n}(t; r) \approx p_{\tau^\star_n}(t)$ estimates our first definition of $p_{\tau^\star_n}(t)$.

The asymptotic relationship to the $t_r$ distribution is

$$p_{t_r}(t) \sim \frac{\langle N(N - 1) \rangle}{\langle N \rangle} p_{\tau^\star_n}(t),$$  \hspace{1cm} (VII.6)

which is obtained via averaging Eq. (II.20) (with $s = 1$) over $N$ and comparing with Eq. (VII.5). Thus, the prefactor is not the averaged prefactor of Eq. (VII.4).
Now we consider a second definition
\[
p_{\tau^*}(t) = \frac{\langle (N - 1)p_{\tau^*|N}(t|N) \rangle}{\langle N(N - 1) \rangle} \sim \frac{\langle N(N - 1) \rangle}{(N - 1)} - p_{\tau_n}(t)[1 - P_{\tau_n}(t)].
\] (VII.7)

Again, the mean is \(\langle \circ \rangle = \sum_{n=1}^{\infty} \phi_{\text{dis}}(n) \circ\) and \(p_{\tau^*|N}(t|N)\) is given in Eq. (VII.2) and (VII.3). This second definition weights \(p_{\tau^*|N}(t|N)\) with the number of jumps \(N - 1\) in the numerator and divides by the mean number of jumps \(\langle N - 1 \rangle\).

How to obtain an estimate of \(p_{\tau^*}(t)\) from simulation? Given \(R\) trajectories. For each trajectory \(r \in \{1, \ldots, R\}\), we have the set of transition times \(\tau^*(r) = (\tau_1, \ldots, \tau_{m-1}, \tau_{m+1}, \ldots, \tau_N)\). We do not calculate the histogram \(h_{\tau^*}(t; r)\) of this set as for the first definition. Instead, we first collect all transition times (from every trajectory) so that we have \(\{\tau^*_1, \ldots, \tau^*_R\}\). And now we calculate the histogram of this set denoted as \(h_{\tau^*}(t; 1, \ldots, R) \approx p_{\tau^*}(t)\). For large enough \(R\), this histogram approximates the second definition of the density \(h_{\tau^*}(t; 1, \ldots, R) \approx p_{\tau^*}(t)\).

The asymptotic relationship to the \(t_r\) distribution is
\[
p_{t_r}(t) \sim \langle N - 1 \rangle p_{\tau^*}(t),
\] (VII.8)
which is obtained via averaging Eq. (II.20) (with \(s = 1\)) over \(N\) while using Eq. (VII.7). Thus, the prefactor is the averaged prefactor of Eq. (VII.4). So the second definition Eq. (VII.7) can be seen as way to obtain a straightforward generalisation of the unidirectional transport relationship Eq. (VII.4). However, also the first definition Eq. (VII.5) can still be useful for data analysis where one needs to estimate asymptotic behaviors.

VIII. PRINCIPLE OF THE SINGLE LONG TRANSITION TIME FOR THE QUENCHED TRAP MODEL

Here, we discuss the quenched trap model, see Fig. IV.1 with the case \(p > 1/2\) when averaged over the energy landscape, which is also presented in the main text.

A. Basic setting

We start the analysis of the transport in a one-dimensional energy landscape. The lattice points are \(x \in \{\ldots, 0, 1, \ldots, L\}\) with \(L \geq 1\), there is no left boundary condition. The start is at \(x = 1\) and \(x = L + 1\) is the absorbing wall. Each lattice point \(x\) has a fixed energetic trap \(E_x\), which was coined by the probability density function
\[
p_E(E_x) = \frac{1}{T_g} e^{-E_x/T_g}.
\] (VIII.1)

At each trap \(x\), the transition time depends on trap depth \(E_x\). We consider the probability density function
\[
p_{\tau_x}(t) = \frac{1}{\tau_x} e^{-t/\tau_x}
\] (VIII.2)
with \(\tau_x = t_0 \exp(E_x/T)\) and the temperature \(T\). A particle starting at \(x = 1\) has the span \(\{L - K + 1, \ldots, L\}\) of visited lattice points. The number of visited traps \(K \geq L\) is random, and its moments exists (since we assume right bias \(p > 1/2\)). The furthest point visited is \(L - K + 1 \leq 1\).

B. Occupation time

Before we discuss the two important quantities, namely the first passage time and the maximum occupation time (see definition below), we have to think about the transition times. A trap can be visited multiple times by the same
particle. Each visit takes a new transition time distributed from the same distribution Eq. (VIII.2). The occupation time is the total time of all visits, that is the sum of all transition times of one particle at the same trap

\[ \tau_x = \sum_{n_x=0}^{N_x} \tau_x^{(n_x)}. \] (VIII.3)

Here, \( N_x \) is the number of visits at \( x \), \( n_x = 0, \ldots, N_x \) is the specific number of the visit at \( x \), and \( \tau_x^{(n_x)} \) denotes the transition time of the \( n_x \)-th visit.

The number of visited traps \( K \) and the number of visits \( N_x \) at each trap \( x \) are both random. Thus, we have to consider the marginal probability density function obtained from summing up the joint probability density function

\[ p_{\tau_x}(t) = \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} p_{\tau_x,N_x,K}(t,n_x,k) = \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k)p_{\tau_x|N_x,K}(t|n_x,k). \] (VIII.4)

Here, \( \phi_{N_x,K}(n_x,k) \) is the probability of having \( k \) traps and \( n_x \) visits at \( x \). The conditional probability density function on the right hand side describes the occupation time for \( n_x \) visits, thus we have the \( n_x \)-fold convolution

\[ p_{\tau_x|N_x,K}(t|n_x,k) = [p_{\tau_x} \ast \ldots \ast p_{\tau_x}]^{(n_x)}(t). \] (VIII.5)

Note that the \( k \)-dependency is implicit because the distribution of \( N_x \) depends on \( K \). The 2-fold convolution is \([p_{\tau_x} \ast p_{\tau_x}]^{(2)}(t) = \int_0^t p_{\tau_x}(t')p_{\tau_x}(t-t')dt'\) and higher orders are obtained subsequently. In Laplace space the \( n_x \)-fold convolution is the \( n_x \)-multiple product \([\hat{p}_{\tau_x}(s)]^{(n_x)}\), and we can calculate it

\[ \hat{p}_{\tau_x|N_x,K}(s|n_x,k) = \left( \frac{1}{1 + \tau_x s} \right)^{n_x}. \] (VIII.6)

Inverse Laplace transform yields the gamma probability density function

\[ p_{\tau_x|N_x,K}(t|n_x,k) = \frac{t^{n_x-1} \exp(-t/\tau_x)}{\Gamma(n_x)}, \] (VIII.7)

which will be used frequently in the following calculations.

C. Occupation time distribution averaged over disorder

Here, we calculate the average over disorder for the conditional probability density function Eq. (VIII.7). Generally, this average is performed over the full energy landscape. Since Eq. (VIII.7) describes the statistics at a specific lattice point \( x \), the average is performed over this point

\[ \langle \phi \rangle_{en} = \int_0^\infty dE_x p_E(E_x)\phi, \] (VIII.8)

while the average over the remaining traps gives one. The index “en” refers to “energy landscape”. From Eq. (VIII.4), we have to calculate

\[ \langle p_{\tau_x}(t) \rangle_{en} = \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \langle \phi_{N_x,K}(n_x,k)p_{\tau_x|N_x,K}(t|n_x,k) \rangle_{en}. \] (VIII.9)

The average over disorder of the conditional probability density function is

\[ \langle p_{\tau_x|N_x,K}(t|n_x,k) \rangle_{en} = \int_0^\infty dE_x p_E(E_x)p_{\tau_x|N_x,K}(t|n_x,k) \sim \alpha(t_0)^{\alpha} t^{-1-\alpha} \Gamma(n_x + \alpha) / \Gamma(n_x) \] (VIII.10)
with the exponent
\[ \alpha = \frac{\tau}{\Gamma}. \] (VIII.11)

The same asymptotic behavior is derived from Laplace space
\[ \langle \hat{p}_{\tau|N_x,K}(s|n_x,k) \rangle \sim \frac{1}{1 + \tau s} s^{n_x} \sim 1 + \alpha s^{\alpha} (t_0)^{\alpha} \Gamma(n_x + \alpha) \Gamma(n_x) \] (VIII.12)

Now with Eq. (VIII.10), the asymptotics of Eq. (VIII.9) are
\[ \langle \hat{p}_{\tau|N_x}(t) \rangle \sim \alpha (t_0)^{-1 - \alpha} \frac{\Gamma(n_x + \alpha)}{\Gamma(n_x)} \] (VIII.13)

with the mean
\[ \frac{\Gamma(n_x + \alpha)}{\Gamma(n_x)} = \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k) \frac{\Gamma(n_x + \alpha)}{\Gamma(n_x)} = \sum_{n_x=0}^{\infty} \phi_{N_x}(n_x) \frac{\Gamma(n_x + \alpha)}{\Gamma(n_x)} \] (VIII.14)

where we performed the sum over \( k \) in the last step. This mean plays an important role in the principle of the single long transition time (see Eq. (10) in the main text).

**D. First passage time in the quenched trap model**

The first passage time is
\[ t_f = \sum_{x=L+1-k}^{L} \hat{\tau}_x \] (VIII.15)

when the number of visited traps is \( K = k \). The probability density function is
\[ p_{t_f}(t) = \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} p_{t_f,N_x,K}(t,n_x,k). \] (VIII.16)

We write \( p_{t_f}(t) \) now in a more convenient way. It is useful to perform the average over \( n_x \) in Eq. (VIII.16)
\[ p_{t_f}(t) = \sum_{k=1}^{\infty} p_{t_f,K}(t,k) = \sum_{k=1}^{\infty} \phi_K(k) p_{t_f|K}(t|k) \] (VIII.17)

because, for given \( K = k \), the marginal probability density function on the right hand is the \( k \)-fold convolution
\[ p_{t_f|K}(t|k) = [p_{\hat{\tau}_{L-K+1}K} \ast \ldots \ast p_{\hat{\tau}_{L}K}]^{(k)}(t), \] (VIII.18)

see Eq. (VIII.15). Here, \( \phi_K(k) \) is the marginal probability that a particle visited \( K = k \) traps. Thus, in Laplace space Eq. (VIII.17) is
\[ \hat{p}_{t_f}(s) = \sum_{k=1}^{\infty} \phi_K(k) \hat{p}_{t_f|K}(s|k) = \sum_{k=1}^{\infty} \phi_K(k) \prod_{x=L-k+1}^{L} \hat{p}_{\hat{\tau}_x|K}(s|k) \] (VIII.19)

The convolution in Eq. (VIII.18) depends on the conditional probability density function
\[ p_{\hat{\tau}_x|K}(t|k) = \sum_{n_x=0}^{\infty} p_{\hat{\tau}_x,N_x|K}(t,n_x,k) = \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k) p_{\hat{\tau}_x,N_x,K}(t|n_x,k) \] (VIII.20)
where we find \( p_{\xi}|_{N_0,K}(t|n_x,k) \) from Eq. (VIII.7). In Laplace space, this is

\[
\hat{p}_{\xi}|_{K}(s|k) = \sum_{n_x=0}^{\infty} \phi_{N_0|K}(n_x|k) \hat{p}_{\xi}|_{N_0,K}(s|n_x,k). \tag{VIII.21}
\]

So the Laplace transform of \( p_{\xi}(t) \) can be written as

\[
\hat{p}_{\xi}(s) = \sum_{k=1}^{\infty} \phi_K(k) \prod_{x=L-k+1}^{L} \left( \sum_{n_x=0}^{\infty} \phi_{N_0|K}(n_x|k) \hat{p}_{\xi}|_{N_0,K}(s|n_x,k) \right) \tag{VIII.22}
\]

where we put Eq. (VIII.21) into (VIII.19). We will use these result next when averaging over disorder.

### E. First passage time distribution averaged over disorder

We want to derive the asymptotics of \( \langle p_{\xi}(t) \rangle_{en} \). The average over disorder \( \langle \cdot \rangle \) is performed over all traps. We work in Laplace space and take Eq. (VIII.22)

\[
\langle \hat{p}_{\xi}(s) \rangle_{en} = \sum_{k=1}^{\infty} \phi_K(k) \prod_{x=L-k+1}^{L} \left( \sum_{n_x=0}^{\infty} \phi_{N_0|K}(n_x|k) \langle \hat{p}_{\xi}|_{N_0,K}(s|n_x,k) \rangle_{en} \right) \tag{VIII.23}
\]

where we can see that the average over disorder on the right hand side is performed over \( K = k \) traps

\[
\langle \cdot \rangle_{en} = \int_0^\infty dE_{L-k+1}p_E(E_{L-k+1}) \cdots \int_0^\infty dE_Lp_E(E_L) \circ . \tag{VIII.24}
\]

We are interested in the asymptotics of Eq. (VIII.23), therefore considering the small \( s \) behavior of \( \langle \hat{p}_{\xi}|_{N_0,K}(s|n_x,k) \rangle_{en} \sim 1+C_{n_x} s^\alpha \) with \( C_{n_x} = \alpha(t_0)^\alpha \Gamma(-\alpha)\Gamma(n_x+\alpha)/\Gamma(n_x) \) as found in Eq. (VIII.12). So Eq. (VIII.23) behaves as

\[
\langle \hat{p}_{\xi}(s) \rangle_{en} \sim \sum_{k=1}^{\infty} \phi_K(k) \prod_{x=L-k+1}^{L} \left( \sum_{n_x=0}^{\infty} \phi_{N_0|K}(n_x|k) [1 + C_{n_x} s^\alpha] \right)
\]

\[
= \sum_{k=1}^{\infty} \phi_K(k) \left( 1 + s^\alpha \sum_{x=L-k+1}^{L} \sum_{n_x=0}^{\infty} \phi_{N_0|K}(n_x|k) C_{n_x} \right)
\]

\[
= 1 + s^\alpha \sum_{k=1}^{\infty} \phi_K(k) \sum_{x=L-k+1}^{L} \sum_{n_x=0}^{\infty} \phi_{N_0|K}(n_x|k) C_{n_x}
\]

\[
= 1 + s^\alpha \sum_{k=1}^{\infty} \sum_{x=L-k+1}^{L} \sum_{n_x=0}^{\infty} \phi_{N_0,K}(n_x,k) C_{n_x}. \tag{VIII.25}
\]

The triple sum can be written as

\[
\sum_{k=1}^{\infty} \sum_{x=L-k+1}^{L} \sum_{n_x=0}^{\infty} \phi_{N_0,K}(n_x,k) C_{n_x} = \sum_{x=-\infty}^{L} \sum_{n_x=0}^{\infty} \phi_{N_0,K}(n_x,k) C_{n_x} \tag{VIII.26}
\]

because \( \phi_{N_0,K} = 0 \) for \( x \leq L-k \). So we finally get the asymptotics via inverse Laplace transform of the last step in Eq. (VIII.25)

\[
\langle p_{\xi}(t) \rangle_{en} \sim \alpha(t_0)^\alpha M_\alpha t^{-1-\alpha} \tag{VIII.27}
\]

where we defined the function

\[
M_\alpha = \sum_{x=-\infty}^{L} \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \phi_{N_0,K}(n_x,k) \frac{\Gamma(n_x+\alpha)}{\Gamma(n_x)} = \sum_{x=-\infty}^{L} \left( \frac{\Gamma(n_x+\alpha)}{\Gamma(n_x)} \right), \tag{VIII.28}
\]
compare with Eq. (VIII.14). Before we consider the maximum transition time, we provide an alternative derivation of the \( \langle p_t(t) \rangle_{\text{en}} \) asymptotics, which will be used later for the modified first passage time \( t_r \) calculations when removing the deepest trap (see SM Sec. IX).

The function \( M_\alpha \) can be obtained from the simulation of a discrete time and space random walk with the bias \( p \). We generate on a computer a trajectory of the discrete time and space random walk which starts at \( x = 1 \) and is absorbed at \( x = L + 1 \). With this trajectory we count for every simulated trajectory the number of visits \( n_x \) at each lattice point \( x < L + 1 \). Then averaging \( \Gamma(n_x + \alpha)/\Gamma(n_x) \) gives us \( M_\alpha \). In Fig. (VIII.1b), we plot \( M_\alpha \) versus \( \alpha \) and in Fig. (VIII.1c), the same function versus \( p \). Clearly, for \( p \to 0.5 \), the value of this parameter blows up, indicating the breakdown of the long transition time principle. The physical reason for this is that when \( p = 1/2 \), the particle can explore in principle a very large number of traps as the motion becomes non-biased (somewhat similar, but far less trivial as the case found with \( p = 1/2 \) for the continuous time random walk).

![Figure VIII.1](image)

**FIG. VIII.1.** (a) The distributions of \( t_f \) (red circles), \( \tilde{\tau}_{\text{max}} \) (blue line), \( t_r \) (green circles) and \( \tilde{\tau}^*_{\text{max}} \) compared with the theories Eq. (VIII.57) (upper black line) and Eq. (IX.21) (lower black line) for the weakly biased quenched trap model with \( L = 10 \) and \( p = 0.7 \). The functions \( M_\alpha \) and \( M^*_{\alpha} \) were obtained from a discrete time and space random walk, as explained in the main text. We used the parameters \( T = T_g = 2 \) so that \( \alpha = 1 \), \( t_0 = 1 \) and \( 10^6 \) particles. (b) \( M_\alpha \) of Eq. (VIII.28) versus \( \alpha \) is plotted for \( p = 0.55, 0.75 \) and 1 (three curves with black circles from top to bottom). When \( p = 1 \) we get \( M_\alpha = L\Gamma(1 + \alpha) \) as in Eq. (VIII.68) which is also shown (red line). We used \( L = 10 \) and \( 10^4 \) particles were simulated to obtain \( M_\alpha \). (c) \( M_\alpha \) of Eq. (11) versus \( p \) is plotted for \( \alpha = 1 \) (black circles). We used the same parameters for the simulation as in (b).

**F. First passage time distribution - Order statistics approach**

A second approach to get the large \( t \) behavior of \( \langle p_t(t) \rangle_{\text{en}} \) uses order statistics. We order the set of the \( K = k \) visited traps \( \{E_{L-k+1}, \ldots, E_L\} \) according

\[
E_{(1)} < \ldots < E_{(k)}.
\]
The $q$-th order has the probability density function

$$p_{E(q)}(E) = \frac{k!}{(q-1)![(k-q)!]} p_E(E)[P_E(E)]^{q-1}[1-P_E(E)]^{k-q}$$

(VIII.30)

with $q = 1, \ldots, k$. The first passage time is

$$t_f = \sum_{q=1}^{k} \tilde{\tau}_{x[E(q)]}$$

(VIII.31)

where $x[E(q)]$ is the lattice point of trap $E(q)$. For the sake of simplicity we write

$$x[E(q)] = y[q]$$

(VIII.32)

in the following. We repeat Eq. (VIII.17)

$$p_{t_j}(t) = \sum_{k=1}^{\infty} \phi_K(k)p_{t_j|K}(t|k)$$

(VIII.33)

where $p_{t_j|K}(t|k)$ given in Eq. (VIII.18) is now replaced by

$$p_{t_j|K}(t|k) = [p_{x[q]|K} * \ldots * p_{x[q]|K}]^{(k)}(t)$$

(VIII.34)

Each single conditional probability density function is given as

$$p_{x[q]|K}(t|k) = \sum_{n_y[q]=0}^{\infty} \phi_{N_{y[q]}|K}(n_y[q]|k)p_{x[q]|N_{y[q]}},K}(t|n_y[q],k).$$

(VIII.35)

Therefore, the Laplace of $p_{t_j}(t)$ is

$$\hat{p}_{t_j}(s) = \sum_{k=1}^{\infty} \phi_K(k)\hat{p}_{t_j|K}(s|k) = \sum_{k=1}^{\infty} \phi_K(k) \prod_{q=1}^{k} \left( \sum_{n_y[q]=0}^{\infty} \phi_{N_{y[q]}|K}(n_y[q]|k)\hat{p}_{x[q]|N_{y[q]}}(s|n_y[q],k) \right)$$

(VIII.36)

compare with Eq. (VIII.22). We will use this formula for the average over disorder. But first, we need to discuss the conditional probability density function on the right hand side of Eq. (VIII.35), which is the gamma function

$$p_{x[q]|N_{y[q]}},K}(t|n_y[q],k) = (\tilde{\tau}_{y[q]} - n_y[q]t^{n_y[q]} - 1)\exp(-t/\tilde{\tau}_{y[q]}) / \Gamma(n_y[q])$$

(VIII.37)

with $\tilde{\tau}_{y[q]} = t_0\exp(E(q)/T)$, see Eq. (VIII.7). We perform the average over disorder

$$\langle p_{x[q]|N_{y[q]}},K}(t|n_y[q],k) \rangle_{en} = \int_0^{\infty} dE(q)p_{E(q)}(E(q)) \frac{(\tilde{\tau}_{y[q]} - n_y[q]t^{n_y[q]} - 1)\exp(-t/\tilde{\tau}_{y[q]})}{\Gamma(n_y[q])}.$$
so that
\[
\langle p_{\gamma_q}|N_{y_q}|K(t|n_{y_q}, k)\rangle_{en} = \frac{k!}{(q-1)!(k-q)!} \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j \int_0^\infty dE_q \frac{1}{T_g} \exp[-(1+j+k-q)E_q/T_g] \frac{\left(\tau_{y_q}\right)^{-n_{y_q}} t^{n_{y_q}-1} \exp(-t/\tau_{y_q})}{\Gamma(n_{y_q})}.
\]

(VIII.41)

Using Mathematica finds the integral \( I \) as
\[
I = \frac{t^{-1+n_{y_q}(t_0)-n_{y_q}|\Gamma(\alpha'+n_{y_q})|F_1(\alpha'+n_{y_q}, \alpha'+n_{y_q}+1, -t/t_0)\right)}{\Gamma(n_{y_q})\Gamma(\alpha'+n_{y_q}+1)}
\]

(VIII.42)

with \( \alpha' = \alpha(1+j+k-q) \) and the Kummer confluent hypergeometric function \( \Phi \). The asymptotic behavior is
\[
I \sim \alpha(t_0)^{\alpha'} t^{-1-\alpha'} \frac{\Gamma(\alpha' + n_{y_q})}{\Gamma(n_{y_q})}
\]

(VIII.43)

where we see that we only need \( j = 0 \) for the large \( t \) behavior. Therefore, the large \( t \) behavior is found
\[
\langle p_{\gamma_q}|N_{y_q}|K(t|n_{y_q}, k)\rangle_{en} \sim J_q t^{-1-\alpha(k+1-q)}.
\]

(VIII.44)

The prefactor is
\[
J_q = \alpha(t_0)^{\alpha(k+1-q)} \frac{k!}{(q-1)!} \frac{\Gamma(\alpha(k+1-q) + n_{y_q})}{\Gamma(n_{y_q})}.
\]

(VIII.45)

The small \( s \) behavior in Laplace space of Eq. (VIII.44) is
\[
\langle \hat{p}_{\gamma_q}|N_{y_q}|K(s|n_{y_q}, k)\rangle_{en} \sim 1 + J_q (\alpha(k+1-q)) s^{\alpha(k+1-q)}.
\]

(VIII.46)

which we use to calculate \( \langle \hat{p}_{\gamma_q}(s)\rangle_{en} \) with \( \hat{p}_{\gamma_q}(s) \) taken from Eq. (VIII.36)
\[
\langle \hat{p}_{\gamma_q}(s)\rangle_{en} \sim \sum_{k=1}^{\infty} \phi_K(k) \prod_{q=1}^{\infty} \phi_{N_{y_q}|K(n_{y_q}|k)} \left(1 + J_q (\alpha(k+1-q)) s^{\alpha(k+1-q)}\right).
\]

(VIII.47)

We are interested in the large \( t \) behavior of \( \langle \hat{p}_{\gamma_q}(t)\rangle_{en} \) so that only \( q = k \) is required. That is basically the principle of the single long transition time because \( q = k \) refers to the deepest trap \( E_{(k)} \), see discussion in SM Sec. (VIII.3). Hence, it follows
\[
\langle \hat{p}_{\gamma_q}(s)\rangle_{en} \sim \sum_{k=1}^{\infty} \phi_K(k) \sum_{n_{y_q}|k=0}^{\infty} \phi_{N_{y_q}|K(n_{y_q}|k)} (1 + J_k (\alpha) s^\alpha)
\]

(VIII.48)

\[
= 1 + s^\alpha \Gamma(-\alpha) \sum_{k=1}^{\infty} \sum_{n_{y_q}|k=0}^{\infty} \phi_{N_{y_q}|K(n_{y_q}|k)} J_k
\]

with
\[
J_k = \frac{\alpha(t_0)^{\alpha} \Gamma(\alpha + n_{y_q}|k)}{\Gamma(n_{y_q}|k)}.
\]

(VIII.49)

We finally get the asymptotic behavior
\[
\langle p_{\gamma_q}(t)\rangle_{en} \sim \alpha(t_0)^{\alpha} t^{-1-\alpha} \sum_{k=1}^{\infty} \sum_{n_{y_q}|k=0}^{\infty} \phi_{N_{y_q}|K(n_{y_q}|k)} \frac{\Gamma(\alpha + n_{y_q}|k)}{\Gamma(n_{y_q}|k)} k.
\]

(VIII.50)
The average on the right hand side picks a specific location of the deepest trap $y[k] = x[E(k)]$ (see Eq. (VIII.32)). However, this location is random, moreover, uniformly distributed over the span $\{L-k+1, \ldots, L\}$ with probability $1/k$. When we take out the average over this location $\sum_{x=L-k+1}^{L} 1/k$, we obtain

$$
\langle p_{t_j}(t) \rangle_{en} \sim \alpha(t_0)^{\alpha-1} \sum_{k=1}^{\infty} \sum_{x=L-k+1}^{L} \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k) \frac{\Gamma(\alpha+n_x)}{\Gamma(n_x)}
$$

$$
= \alpha(t_0)^{\alpha-1} \sum_{x=-\infty}^{L} \sum_{k=1}^{L} \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k) \frac{\Gamma(\alpha+n_x)}{\Gamma(n_x)}
$$

(VIII.51)

where we used Eq. (VIII.26) to rewrite the triple sum. The result is exactly Eq. (VIII.27). The purpose of this second approach here is that we will need it below for the large $t$ behavior of $\langle p_{t_k}(t) \rangle_{en}$ and now for the occupation time in the deepest trap.

G. Occupation time in the deepest trap

The occupation time in the deepest trap $E_{\max} = E(k) = \max(E_{L-k+1}, \ldots, E_L)$ is

$$
\tau_{\max} = \tau_{y[k]}
$$

(VIII.52)

when the number of visited traps is $K = k$. Here, $y[q]$ is the location of the deepest trap Eq. (VIII.32), and $E(k)$ can be found in Eq. (VIII.29). Since $K$ is random, we have to consider the marginal probability density function

$$
p_{\tau_{\max}}(t) = \sum_{k=1}^{\infty} \phi_K(k)p_{\tau_{y[k]}K}(t|k) = \sum_{k=1}^{\infty} \phi_K(k) \sum_{n_y[q]=0}^{\infty} \phi_{N_y[q]|K}(n_y[q],k)p_{\tau_{y[q]}|N_y[q],K}(t|n_y[q],k).
$$

(VIII.53)

Note that we implicitly suppose the location of the deepest trap $x = y[q]$. The average over the disorder of Eq. (VIII.53) is

$$
\langle p_{\tau_{\max}}(t) \rangle_{en} = \sum_{k=1}^{\infty} \phi_K(k) \sum_{n_y[q]=0}^{\infty} \phi_{N_y[q]|K}(n_y[q],k)\langle p_{\tau_{y[q]}|N_y[q],K}(t|n_y[q],k) \rangle_{en}.
$$

(VIII.54)

In Eq. (VIII.44), we already calculated the large $t$ behavior of $\langle p_{\tau_{y[q]}|N_y[q],K}(t|n_y[q],k) \rangle_{en}$ for any $q \leq k$. Using $q = k$, we obtain

$$
\langle p_{\tau_{\max}}(t) \rangle_{en} \sim \alpha(t_0)^{\alpha-1} \sum_{k=1}^{\infty} \sum_{n_y[k]=0}^{\infty} \phi_{N_y[k]|K}(n_y[k],k) \frac{\Gamma(\alpha+n_y[k])}{\Gamma(n_y[k])},
$$

(VIII.55)

which is exactly the same asymptotic behavior as $\langle p_{t_j}(t) \rangle_{en}$ presented in Eq. (VIII.50). Following the discussion around Eq. (VIII.51) of rewriting the double sum, we can also use

$$
\langle p_{\tau_{\max}}(t) \rangle_{en} \sim \alpha(t_0)^{\alpha-1} \sum_{x=-\infty}^{L} \sum_{k=1}^{L} \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k) \frac{\Gamma(\alpha+n_x)}{\Gamma(n_x)}
$$

(VIII.56)

which is the same expression found for $\langle p_{t_j}(t) \rangle_{en}$ in Eq. (VIII.27). We summarize this asymptotic equivalence in the next subsection.

H. Principle of the single long transition time for weak bias $1/2 < p < 1$

The asymptotic equivalence between $\langle p_{t_j}(t) \rangle_{en}$ Eq. (VIII.27) and $\langle p_{\tau_{\max}}(t) \rangle_{en}$ Eq. (VIII.56) is the principle of the single long transition time

$$
\langle p_{t_j}(t) \rangle_{en} \sim \langle p_{\tau_{\max}}(t) \rangle_{en}
$$

(VIII.57)
for the quenched trap model with weak bias $1/2 < p < 1$, which we presented in the main text in Eq. (10). With Eq. (VIII.13), we can write the principle also as

$$\langle p_f(t) \rangle_{en} \sim \sum_{x=-\infty}^{\infty} \langle \hat{\tau}_x(t) \rangle_{en},$$

which is the analog to the reference case of the unidirectional transport $p_f(t) \sim N \tau_n(t)$ in Eq. (VII.1).

I. The strongly biased case $p = 1$

When the bias is strong $p = 1$, the particles move directly from $x = 1$ to the right until being absorbed at $x = L + 1$. The traps are $\{(E_1, ..., E_L)\}$. A particle visits each trap at $x$ only once $N_x = 1$, and the total number of traps is $K = L$. Obviously, the occupation time at $x$ Eq. (VIII.3) is the only transition time

$$\hat{\tau}_x = \tau_x,$$

the first passage time Eq. (VIII.15) is

$$t_f = \sum_{x=1}^{L} \tau_x,$$

and the occupation time in the deepest trap Eq. (VIII.52) is simply the only transition time in the deepest trap

$$\hat{\tau}_{\text{max}} = \tau_{\text{max}} = \tau_{y[k]}$$

with $y[k]$ as the location of the deepest trap $E_{\text{max}} = E(k)$. The probability functions of the latter two are

$$p_{t_f}(t) = [p_{\tau_1} * \ldots * p_{\tau_L}]^{(L)}(t),$$

$$p_{\tau_{\text{max}}}(t) = p_{\tau_{y[q]}}(t).$$

Compare the first equation with Eq. (VIII.16) to (VIII.18) while

$$\phi_{N_x,K}(n_x, k) = \delta_{n_x,1} \delta_{k,L}$$

because $K = L$ and $N_x = 1$ are fixed. The first equation in Laplace space and the second equation are

$$\hat{p}_{t_f}(s) = \prod_{x=1}^{L} (1 + \hat{\tau}_x s)^{-1},$$

$$p_{\tau_{\text{max}}}(t) = \frac{1}{\bar{\tau}_{y[q]}} \exp(-t/\bar{\tau}_{y[q]}),$$

which is Eq. (X.1) in the main text. The solution (and consequently the large $t$ behavior) of the first equation is derived in the main text. We get the principle of the single long transition time

$$\text{Prob}(t_f > t) \sim Q_m \text{Prob}(\tau_{\text{max}} > t) = Q_m \exp\left(-\frac{t}{\bar{\tau}_m}\right)$$

with the prefactor

$$Q_{y[k]} = (\bar{\tau}_{y[k]})^{L-1} \prod_{x=1, x \neq y[k]}^{L} [\bar{\tau}_{y[k]} - \bar{\tau}_x]^{-1},$$

compare with Eq. (X.2) in the main text.
When averaging over the disorder, we already found the principle of the single long transition time Eq. (VIII.57) with the exact expression Eq. (VIII.27). So we have here

\[ \langle p_t(t) \rangle_{en} \sim \langle p_{\tau_{\text{max}}}(t) \rangle_{en} \sim \alpha(t_0)^{\alpha}(1 + \alpha)Lt^{-1-\alpha} \]  

(VIII.67)

where \( \phi_{N_x,K}(n_x,k) = \delta_{n_x,1}\delta_{k,L} \) Eq. (VIII.63) was used to calculate the function \( M_\alpha \) Eq. (VIII.28)

\[ M_\alpha = \sum_{x=-\infty}^{L} \left( \frac{\Gamma(N_x + \alpha)}{\Gamma(N_x)} \right) = \sum_{x=1}^{L} \frac{\Gamma(1 + \alpha)}{\Gamma(1)} = L \Gamma(1 + \alpha). \]  

(VIII.68)

In Fig. VIII.2 we compare the theory with the numerics.

**FIG. VIII.2.** The effect of elimination of the deepest trap on transport is studied for the quenched trap model. The plot shows the distributions of \( t_f \) (upper red circles), \( \tau_{\text{max}} \) (upper blue line), \( t_r \) (lower red circles) and \( \tau^\star_{\text{max}} \) (lower blue line) for the strongly biased (\( \rho = 1 \)) quenched trap model with \( L = 10 \). An average over the disorder was performed. We find perfect agreement with the theory, i.e. the principle of the single long transition time Eq. (VIII.67) (upper black line) and the relationship after elimination Eq. (IX.17) (lower black line). We used the parameters \( T = T_g = 2 \) such that \( \alpha = 1 \), \( t_0 = 1 \). We used \( 10^6 \) particles in the Monte-Carlo simulations.

**J. Principle of the single long transition time for deterministic transition times**

The content of this subsection is not presented in the main text. However, one often uses deterministic transition times

\[ p_{\tau_x}(t) = \delta(t - \bar{\tau}_x) \]  

(VIII.69)

with \( \bar{\tau}_x = t_0\exp(E_x/T) \) instead of the exponential distribution Eq. (VIII.2). For that case, we get the conditional probability density function Eq. (VIII.5) as

\[ p_{\tau_x|N_x,K}(t|n_x,k) = \delta(t - n_x\mu_x). \]  

(VIII.70)

Now we proceed as in Eq. (VIII.10), i.e. we calculate the average over the disorder

\[ \langle p_{\tau_x|N_x,K}(t|n_x,k) \rangle_{E_x} = \int_0^\infty dE_x p_E(E_x)p_{\tau_x|N_x,K}(t|n_x,k) = \int_0^\infty dE_x p_E(E_x)\delta(t - n_x\mu_x) = \frac{p_E(E)}{|g'(E)|} \]  

(VIII.71)
Here, \( \hat{E} \) is the only zero of the function \( g(E) = t - n_x \hat{t}_x \). We repeat the previous calculations, and find the principle of the single long transition time

\[
\langle p_{t_f} (t) \rangle_{en} \sim \langle p_{\hat{t}_{\text{max}}} (t) \rangle_{en} \sim \alpha(t_0)^{\alpha - 1} \sum_{x = -\infty}^{L} \langle (N_x)^{\alpha} \rangle.
\] (VIII.72)

In contrast to the principle Eq. (VIII.57) where the transition times are exponentially distributed, we perform here the sum on \( \langle (N_x)^{\alpha} \rangle \) and not on \( (\Gamma(N_x + \alpha)/\Gamma(N_x)) \).

### IX. ELIMINATION OF THE DEEPEST TRAP IN THE QUENCHED TRAP MODEL

For the quenched trap model, we presented in the main text the result Eq. (12), which considers the statistics after removing the deepest trap. We derive these results here. Generally, the following calculations assume any bias \( p > 1/2 \). The special case of strong bias \( p = 1 \) can be found below in SM Sec. [IX.D](#).

#### A. Modified first passage time in the quenched trap model

A particle visits \( K = k \) traps. We remove the maximum of these visited traps \( E_{\text{max}} = E_{(k)} = \max(E_{L-k+1}, \ldots, E_L) \) and deal with \( E_{(1)} < \ldots < E_{(k-1)} \). The order statistics of the traps were defined in Eq. (VIII.29) and (VIII.30). The modified first passage time is

\[
t_r = t_f - \hat{t}_{\text{max}} = \sum_{q=1}^{k-1} \hat{t}_{x[E(q)]},
\] (IX.1)

compare with Eq. (VIII.31). In the following, we basically repeat the steps of SM Sec. VIII.F We are looking for the marginal probability density function

\[
p_{t_r}(t) = \sum_{k=1}^{\infty} \phi_K(k)p_{t_r|K}(t|k)
\] (IX.2)

and its average over disorder. The conditional probability density function is the \((n-1)\)-fold convolution

\[
p_{t_r|K}(t|k) = [p_{\hat{t}_{E(1)}|K} \ast \ldots \ast p_{\hat{t}_{E(k-1)}|K}]^{(k-1)}(t).
\] (IX.3)

Each single conditional probability density function is given as

\[
p_{\hat{t}_{E(q)}|K}(t|k) = \sum_{n_{y[q]} = 0}^{\infty} \phi_{N_{y[q]}|K}(n_{y[q]}, k)p_{\hat{t}_{E(q)}|N_{y[q]}, K}(t|n_{y[q]}, k),
\] (IX.4)

which was already shown in Eq. (VIII.35). Following the steps after Eq. (VIII.35), we get again the small \( s \) behavior of the Laplace transform Eq. (VIII.47)

\[
\langle \hat{p}_{t_r}(s) \rangle_{en} \sim \sum_{k=1}^{\infty} \phi_K(k) \prod_{q=1}^{k-1} \left( \sum_{n_{y[q]} = 0}^{\infty} \phi_{N_{y[q]}|K}(n_{y[q]}, k) \left( 1 + J_q \Gamma(-\alpha(k + 1 - q) + s \alpha(k+1-q)) \right) \right).
\] (IX.5)

but the product ends at \( k-1 \) instead of \( k \). We are interested in the large \( t \) behavior of \( \langle p_{t_r}(t) \rangle_{en} \) so that only \( q = k-1 \) is required. Hence, it follows

\[
\langle \hat{p}_{t_r}(s) \rangle_{en} \sim 1 + s^{2\alpha} \Gamma(-2\alpha) \sum_{k=1}^{\infty} \sum_{n_{y[k-1]} = 0}^{\infty} \phi_{N_{y[k-1]}|K}(n_{y[k-1]}, k)J_{k-1}
\] (IX.6)

with

\[
J_{k-1} = \frac{\alpha(t_0)^{2\alpha} \Gamma(2\alpha + n_{y[k-1]})}{\Gamma(n_{y[k-1]})} k(k-1).
\] (IX.7)
We finally get the asymptotic behavior
\[
\langle p_t(t) \rangle_{en} \sim \alpha(t_0)^{2\alpha} t^{-1-2\alpha} \sum_{k=1}^{\infty} \sum_{n_y[k-1]=0}^{\infty} \phi_{N_y[k-1],k}(n_y[k-1],k) \frac{\Gamma(2\alpha + n_y[k-1])}{\Gamma(n_y[k-1])} k(k-1). \tag{IX.8}
\]

The average on the right hand side picks a specific location of the second deepest trap \(y[k-1] = x[E_{k-1}]\) (see Eq. (VIII.32)). However, this location is random, moreover, uniformly distributed over the span \(\{L - k + 1, \ldots, L\}\) with probability \(1/k\). When we take out the average over this location \(\sum_{x=L-k+1}^{L} 1/k\), we obtain
\[
\langle p_t(t) \rangle_{en} \sim \alpha(t_0)^{2\alpha} t^{-1-2\alpha} \sum_{k=1}^{L} \sum_{x=L-k+1}^{\infty} \sum_{n_x=0}^{\infty} \phi_{N_x,k}(n_x,k) \frac{\Gamma(2\alpha + n_x)}{\Gamma(n_x)} (k-1) \tag{IX.9}
\]
where we used Eq. (VIII.26) to rewrite the triple sum. Next, we derive the occupation time in the second deepest trap.

**B. Occupation time in the second deepest trap**

After removing \(E_{\text{max}} = E_{(k)}\) from a given number of visited traps \(K = k\), the occupation time in the newly deepest trap \(E_{(k-1)}\) is
\[
\hat{\tau}_{\text{max}}^* = \hat{\tau}_{y[k-1]}^* \tag{IX.10}
\]
We repeat the steps of SM Sec. VIII G to obtain the asymptotics of \(\langle p_{\tau_{\text{max}}^*}(t) \rangle_{en}\). From Eq. (VIII.54) we learn that
\[
\langle p_{\tau_{\text{max}}^*}(t) \rangle_{en} = \sum_{k=1}^{\infty} \phi_K(k) \sum_{n_y[k-1]=0}^{\infty} \phi_{N_y[k-1],K}(n_y[k-1],k) \langle p_{\tau_{y[k-1]}^*}(t) | n_y[k-1],K \rangle_{en}. \tag{IX.11}
\]
where we replaced the location of the deepest trap \(y[k]\) with \(y[k-1]\). In Eq. (VIII.44), we already calculated the large \(t\) behavior of \(\langle p_{\tau_{y[q]}^*}(t) | n_y[q],K \rangle_{en}\) for any \(q \leq k\). Using \(q = k - 1\), we obtain
\[
\langle p_{\tau_{\text{max}}^*}(t) \rangle_{en} \sim \alpha(t_0)^{2\alpha} t^{-1-2\alpha} \sum_{k=1}^{\infty} \sum_{n_y[k-1]=0}^{\infty} \phi_{N_y[k-1],K}(n_y[k-1],k) \frac{\Gamma(2\alpha + n_y[k-1])}{\Gamma(n_y[k-1])} k(k-1), \tag{IX.12}
\]
which is exactly the same asymptotic behavior as \(\langle p_t(t) \rangle_{en}\) presented in Eq. (VIII.50). As explained above, we can also write
\[
\langle p_{\tau_{\text{max}}^*}(t) \rangle_{en} \sim \alpha(t_0)^{2\alpha} t^{-1-2\alpha} \sum_{x=-\infty}^{L} \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k) \frac{\Gamma(2\alpha + n_x)}{\Gamma(n_x)} (k-1) \tag{IX.13}
\]
which is the same expression used for \(\langle p_t(t) \rangle_{en}\) in Eq. (IX.9). We summarize this asymptotic equivalence in the next subsection.

**C. Relationship between \(t_r\) and \(\hat{\tau}_{\text{max}}^*\) for the bias \(p > 1/2\)**

From the equivalence of the asymptotics in Eq. (IX.9) and (IX.13), we see the relationship between the modified first passage time and the occupation time in the second deepest trap
\[
\langle p_t(t) \rangle_{en} \sim \langle p_{\tau_{\text{max}}^*}(t) \rangle \sim \alpha(t_0)^{2\alpha} t^{-1-2\alpha} M_\alpha^* \tag{IX.14}
\]
with the function
\[
M_\alpha^* = \sum_{x=-\infty}^{L} \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \phi_{N_x,K}(n_x,k) \frac{\Gamma(2\alpha + n_x)}{\Gamma(n_x)} (k-1). \tag{IX.15}
\]
This result is shown in Eq. (12) in the main text.
D. Asymptotics in the strongly biased case \( p = 1 \)

The bias \( p = 1 \) implies that a particle only jumps from left to right through the \( K = L \) traps \( \{E_1, \ldots, E_L\} \) until absorption at \( L + 1 \). Each trap is visited only once. Therefore, the function Eq. (IX.15) is

\[
M^\star_{\alpha} = \sum_{x=1}^{L} \sum_{k=1}^{\infty} \sum_{n_x=0}^{\infty} \delta_{k,L} \delta_{n_x,1} \Gamma(2\alpha + 1)(k - 1) = L(L - 1)\Gamma(2\alpha + 1).
\]

(IX.16)

So the relationship in Eq. (IX.21) becomes

\[
\langle p_t \rangle_{en} \sim \langle \bar{p}_{\tau,\max} \rangle \sim \alpha(t_0)^{2\alpha} L(L - 1)\Gamma(2\alpha + 1)t^{-1-2\alpha},
\]

(IX.17)

see also Fig. [VIII.2]

E. Measure of gain \( G \) in the strongly biased case \( p = 1 \)

Here, we derive \( G = \langle \bar{t}_r \rangle_{en} / \langle \bar{t}_f \rangle_{en} \) for \( p = 1 \). The bias is strong \( p = 1 \) so that the particles move from left to right through the traps \( \{E_1, \ldots, E_L\} \). First, we calculate the mean first passage time in this one-dimensional environment

\[
\bar{t}_f = \sum_{x=1}^{L} \bar{\tau}_x = \sum_{x=1}^{L} t_0 \exp(E_x/T).
\]

(IX.18)

The average over disorder yields

\[
\langle \bar{t}_f \rangle_{en} = \sum_{x=1}^{L} \langle \bar{\tau}_x \rangle_{en} = \sum_{x=1}^{L} \int_{0}^{\infty} dE_x p_{E_x}(E_x) t_0 \exp(E_x/T) = L t_0 \frac{\alpha}{\alpha - 1} = \alpha/\alpha - 1
\]

under the assumption \( \alpha = T/T_g > 1 \).

Secondly, the mean of the modified first passage time in the one-dimensional environment is

\[
\bar{t}_r = \sum_{q=1}^{L-1} \bar{\tau}_{y[q]} = \sum_{q=1}^{L-1} t_0 \exp(E_{(q)}/T).
\]

(IX.20)

The average over disorder is

\[
\langle \bar{t}_r \rangle_{en} = \sum_{q=1}^{L-1} \langle \bar{\tau}_{y[q]} \rangle_{en}
\]

\[
= \sum_{q=1}^{L-1} \int_{0}^{\infty} dE_{(q)} p_{E_{(q)}}(E_{(q)}) t_0 \exp(E_{(q)}/T)
\]

\[
= \sum_{q=1}^{L-1} \frac{L!}{(q-1)!} \int_{0}^{\infty} dE_{(q)} p_{E_{(q)}}(E_{(q)}) [P_{E_{(q)}}]^q [1 - P_{E_{(q)}}]^{L-q} t_0 \exp(E_{(q)}/T).
\]

(IX.21)

We use the Binomial theorem for the term

\[
[P_{E_{(q)}}]^q = \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j \exp(-j E_{(q)}/T_g).
\]

(IX.22)
FIG. X.1. The distributions of $t_f$ (red circles) and $\tau_{\text{max}}$ (blue lines) for the strongly biased quenched trap model with $L = 25$. Here, we consider a specific realization of the disorder, corresponding to an experimental situation where no averaging over disorder is made. We find excellent agreement with theory Eq. (X.2). The two upper lines represent a unique realization of the disorder with $E_{\text{max}} \approx 10.61$ and the two lower lines represent another realization with $E_{\text{max}} \approx 8.86$. We used the parameters $T = T_g = 2$ so that $\alpha = 1$ and $t_0 = 1$. To generate the figure, we used $10^5$ trajectories for each disorder. These simulations take into consideration the thermal fluctuations, namely the activation process happens at random times. The fact that different sets of disorder do not produce the same behavior is an indication for non-self averaging for the observables of interest, however, the single long transition time principle clearly holds.

Thus, we get

$$\langle \tilde{t}_r \rangle_{en} = \sum_{q=1}^{L-1} \frac{L!}{(q-1)!(L-q)!} t_0 \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j \frac{\alpha}{\alpha(1 + L - q + j) - 1}$$

under the assumption $\alpha > 1/2$. This assumption comes from Eq. (IX.21) where the integral exist for $\alpha > 1/(1+L-q)$. Finally, we obtain

$$\langle \tilde{t}_r \rangle_{en} = L t_0 \frac{\alpha}{\alpha - 1} + \frac{(-1)^{L+1} t_0 L! \Gamma(-L + 1/\alpha)}{\Gamma(1/\alpha)}.$$

The ratio between Eq. (IX.24) and (IX.19) gives the measure $G$.

X. ONE-DIMENSIONAL QUENCHED TRAP MODEL

A. Principle of the single long transition time

We consider the strong bias case $p = 1$ where the particles move only to the right, namely a constant strong driving force acts on the system. And we consider the first passage time without averaging over the disorder, namely we treat a system with a specified realization of the energies $\{E_1, ..., E_L\}$ and the absorption at $x = L + 1$. This corresponds to a situation when the experiment has one realization of the disordered system. The first passage time $t_f$ is a sum of the microscopic transition times at the traps with $N = L$. The maximum transition time, in the quenched trap
model, is defined as the transition time in the deepest trap $E_{\text{max}} = \max(E_1, \ldots, E_L)$. Let’s say the deepest trap is at the lattice point $x = m$. Then $\tau_{\text{max}} = \tau_m$. We consider the probability density function of the first passage time $p_\tau(t) = \langle \delta(t - [\tau_1 + \ldots + \tau_L]) \rangle$. Its Laplace transform $\hat{p}_\tau(s) = \int_0^\infty p_\tau(t)e^{-st}dt$ and the maximum probability are

$$\hat{p}_\tau(s) = \prod_{x=1}^{L}(1 + \bar{\tau}_x s)^{-1}, \quad \text{(X.1)}$$

$$\text{Prob}(\tau_{\text{max}} > t) = \exp \left(-\frac{t}{\tau_{\text{max}}} \right).$$

Note the equal sign in the second equation due to the assumption that $\tau_{\text{max}}$ is the transition time from the deepest trap. Asymptotically, when the maximum transition time is very long, as found in strongly disordered systems, then it always happens in the deepest trap. The inverse Laplace transform of the first formula can be calculated exactly

$$p_\tau(t) = \sum_{x=1}^{L} Q_x \tau_x(t) \quad \text{with} \quad Q_x = \bar{\tau}_x^{-1} \prod_{y=1, y\neq x}^{L} [\bar{\tau}_x - \bar{\tau}_y]^{-1}. \quad \text{From here we find}$$

$$\text{Prob}(t_f > t) \sim Q_m \text{Prob}(\tau_{\text{max}} > t)$$

$$= Q_m \exp \left(-\frac{t}{\tau_{\text{max}}} \right) \quad \text{(X.2)}$$

with the prefactor $Q_m = \bar{\tau}_m^{-1} \prod_{x=1, x\neq m}^{L} [\bar{\tau}_m - \bar{\tau}_x]^{-1}$. The decay of both distributions for $t_f$ and $\tau_{\text{max}}$ is exponential, unlike the power laws found previously. The reason is that in each trap we have exponentially distributed trapping times and hence naturally the distribution cannot be fat tailed, as the system is finite. What is remarkable, is that the principle of the single long transition time still holds, in the sense that the exponential decays of the two probabilities are the same though note the prefactor $Q_m$ in Eq. (X.2). In Fig. X.1 we demonstrate Eq. (X.2) and compare its prediction with Monte-Carlo simulations. Eq. (X.2) is an indication that the removal of the deepest trap, is going to qualitatively change the statistical properties of the time to cross the system $t_f$.

### B. Elimination of the deepest trap

We now study the effect of elimination of the maximum transition time on the statistics of the first passage time for the strongly biased model. The idea is to remove the deepest trap max($E_1, \ldots, E_L$) from the set of traps $\{E_1, \ldots, E_L\}$ and study the effect on the transport.

We apply the methods of order statistics, i.e., we order the traps according to $E_{(1)} < \ldots < E_{(L)}$ and remove $E_{(L)} = \max(E_1, \ldots, E_L)$ from this set. Let $x[E_{(q)}]$ be the lattice point of the q-th deepest trap $E_{(q)}$ with $q = 1, \ldots, L$. The first passage time and the long transition time both after elimination of $\tau_{\text{max}} = \tau_m$ (remember that $m$ is the location of the deepest trap $E_m = E_{(L)}$) are $t_r = \sum_{q=1}^{L-1} \tau_x[E_{(q)}] = t_f - \tau_x[E_{(L)}]$ and $\tau_{\text{max}}^* = \tau_x[E_{(L-1)}]$. So clearly $\tau_{\text{max}}^*$ is the time spent in the trap whose depth is ranked second in the sequence. The Laplace transform $\hat{p}_\tau(s)$ of the probability density function $p_\tau(t) = \langle \delta(t - [\tau_x[E_{(1)}] + \ldots + \tau_x[E_{(L-1)}]]) \rangle$ and the probability of the maximum in the second deepest trap are

$$\hat{p}_\tau(s) = \prod_{q=1}^{L-1}(1 + \bar{\tau}_x[E_{(q)}] s)^{-1}, \quad \text{(X.3)}$$

$$\text{Prob}(\tau_{\text{max}}^* > t) = \exp \left(-\frac{t}{\bar{\tau}_x[E_{(L-1)}]} \right)$$

We can analyze Eq. (X.3) for the one-dimensional random environment (i.e. one channel of energy traps) just as Eq. (X.1), thus, finding after the removal

$$\text{Prob}(t_r > t) \sim Q_x[E_{(L-1)}] \text{Prob}(\tau_{\text{max}}^* > t)$$

$$= Q_x[E_{(L-1)}] \exp \left(-\frac{t}{\bar{\tau}_x[E_{(L-1)}]} \right) \quad \text{(X.4)}$$
with the prefactor $Q_x(E_{(L-1)}) = (\bar{r}_{x[E_{(L-1)}]})^{L-1} \prod_{x=1}^{L-1} [\bar{r}_{x[E_{(L-1)}]} - \bar{r}_x]^{-1}$. The exponential decay of both distributions is the same, namely the second deepest trap with the rate $-1/\bar{r}_{x[E_{(L-1)}]}$ takes control, which is faster than the decay $-1/\bar{r}_{x[E_{(L)}]}$ found previously without the elimination in Eq. (X.2) because $E_{(L-1)} < E_{(L)}$. Thus, removing the deepest trap yields a gain depending on the particular values of the energies $E_{(L-1)}$ and $E_{(L)}$. Recall that the Arrhenius times are related to the energies $\bar{r}_{x[E_{(L-1)}]} = t_0 \exp(E_{(L-1)}/\bar{r}_x)$ and $\bar{r}_{x[E_{(L)}]} = t_0 \exp(E_{(L)}/\bar{r}_x)$, so the times in Eq. (X.2) and (X.4) are mapped to the energies as usual.

We now consider the two examples in Fig. XI.1(b) where the energy landscape was generated with $\alpha = 1$. We find before elimination $\text{Prob}(t_f > t) \propto \exp(-t/201)$ and $\exp(-t/84)$ while after elimination $\text{Prob}(t_f > t) \propto \exp(-t/18)$ in the first example and $\propto \exp(-t/11)$ in the second. The gain is clearly enormous, and if we would consider $\alpha < 1$, we expect an even larger typical gain. However, obviously, since we did not average over disorder, this result is specific for a realization of disorder. To quantify the effect we consider below the ensemble averages. The measure of gain is $G = \bar{t}_f/\bar{t}_f$ with the averages $\bar{t}_f = \sum_{x=1}^{L} \bar{r}_x$ and $\bar{t}_f = \sum_{x=1}^{L} \bar{r}_{x[E_{(L)}]}$. For the two examples of Fig. XI.1 we get $G = 0.25$ and 0.46. Note that while $G$ is a measure of gain based on the mean, the above discussion on the exponential decay is focusing on large times. Both $G$ and the exponential tails show remarkable sensitivity after the removal. However, $G$ is roughly speaking a statistical measure of typical events, while the tails are naturally sensitive to the longest transition times.

XI. SIMULATION OF PORE-SCALE TRANSPORT

A. Simulation method

In this section, we discuss the advection-dominated transport behavior in porous medium. We have generated a two-dimensional heterogeneous system, by randomly distributing in space solid circular grains from a log-normal distribution, see Fig. XI.1(a), with a mean diameter of $\lambda = 1\text{ mm}$ and a standard deviation of $\lambda/2$. The system overall dimensions are $L_x = 50\lambda$ and $L_y = 40\lambda$, with an average porosity of $\phi = 0.35$. A lognormal distribution can characterize the grain size distribution of different natural soils [7], and therefore was used here as a representative distribution. Fluid flow though this system was determined by solving the Stokes equation (using COMSOL Multiphysics®): $\mu \nabla^2 U = \nabla p$, where $U$ is the pore-scale (local) velocity vector, $p$ the fluid pressure, and $\mu = 10^{-3}\text{Pa-s}$ the fluid dynamic viscosity of water, coupled with mass conversation, $\nabla U = 0$. No-slip boundary conditions were applied on the perimeters of the solid objects (fluid-solid interface) and on the external-vertical
Particle transport was modelled by following an ensemble of particles that move according to the flow field \( \mathbf{U} \) (Fig. XI.1(b). The spatiotemporal displacement of each particle was determined by a streamline-based method [9], which computes the time and distance needed for a particle to exit its current element (within the numerical grid) and arrive to the adjacent element. Particles (~10^6) were distributed randomly within a small vertical region at the bottom boundary of the flow domain as an initial condition, and then moved according to the equation of motion, \( d\mathbf{x} = \mathbf{U}(\mathbf{x}(t))dt \); where \( \mathbf{x} \) is the particle location vector and \( dt \) is the time step [10]. Here, the transport mechanism takes into account only the advective component, while neglecting the occurrence of molecular diffusion. In practice, this scenario is suitable for advection-dominated flow regimes [8,9].

To relate the transport of particles within the system to the principle of the single long transition time, we first need to characterize the transition times. To do so, we use a standard method [9] from single particle tracking to define the transition times, using a length scale roughly the size of the grain diameter (\( \lambda \)); see below for more information. We use \( N = 30 \) as a representative number of transitions in the numerical simulations in order to capture the fixed \( N \) situation of the previous model. Thus, the typical travel distance of the particles is \( L \approx \lambda N \). Smaller values of \( N \) were examined (not shown) and showed the same behavior.

In Fig. XI.1(b) inset, we show an example of a single particle trajectory within the domain, where the locations of transitions are marked in black dots, and the maximum transition time \( \tau_{\text{max}} \) is marked in red. From this example, it can be seen that the maximum transition time occurs when a particle is transported within a narrow pore, perpendicular to the pressure gradient (main flow direction). As a result, the particle velocity magnitude (\( \parallel \mathbf{U} \parallel \)) is small (see the color bar in Fig. XI.1(b), and therefore the transition time \( \sim \lambda/\parallel \mathbf{U} \parallel \) becomes large.

### B. Transition times for the pore-scale transport simulation

We obtained the transition times as follows. To link the advective transport of particles within the random walk framework, one must consider the spatial and temporal transitions of each particle trajectory. To characterize the particle transition times from the transport simulations, we use a length scale of \( \lambda \) (the mean grain diameter) as a constant space transition [9]. Specifically, for each transition, we define a circle of diameter \( \lambda \) around the initial position of the particle. The particle is then transported according to the flow field (\( \mathbf{U} \)). Once it leaves its current circle, the center of the circle is moved to the “new” particle coordinates. In doing so, we assume that the velocity field correlations (in space) breaks (on average) after the particle leaves the vicinity of the previous grain [11].

### C. Cost-efficient removal method

In the main text, we showed that we gain nearly 40% increase of efficiency of transport by the elimination method. However, the method we used is costly, as it demands the elimination of the maximum transition time from each trajectory. To move closer to real applications, we address the following protocol. We chose to remove the longest transition times from a finite percentage \( 0 < C < 1 \) of the trajectories (\( C \) is for cost). In general, the idea is that in transport we may discover a few pivotal regions or hot-spots where a critical number of very large \( \tau_{\text{max}} \) occur, and then we need to treat/eliminate only these spots, to expedite the transport. First, we order the maxima according to \( \tau_{\text{max},(1)} \leq \ldots \leq \tau_{\text{max},(R)} \) where \( R \) is the number of trajectories. Then we eliminate the \( C \times R \) largest maxima \( \{\tau_{\text{max},(R(1-C))} \leq \ldots \leq \tau_{\text{max},(R)}\} \) and obtain the modified first passage time \( t_f^\star \). The remaining \( R(1-C) - 1 \) trajectories (with low maxima) remain with \( t_f \). The total elimination \( C = 1 \) is clearly costly (but as shown very efficient) while \( C = 0 \) is the limit of zero cost but also clearly not useful for our purpose. In Fig. XI.2 we compare the distribution of \( t_f^\star \) with \( C = 0.05 \) with the two distributions of the original \( t_f \) with \( C = 0 \) and \( t_f^\star \) with \( C = 1 \). Remarkably, for large first passage times, the distributions of \( t_f^\star \) with \( C = 0.05 \) and 1 are similar. Thus, the far right tail of the distribution of \( t_f^\star \), for the case of partial removal of merely 5% (\( C = 0.05 \)) is nearly as efficient as the costly case with \( C = 1 \). Thus, because of the scale-free nature of the process, it is sufficient to use a relatively inexpensive method, and small \( C \) performs well.
FIG. XI.2. The distribution of the first passage time after eliminating only $C = 5\%$ of the largest maxima (black crosses) as explained in the text, compared with the distributions of the zero cost $C = 0\%$ first passage time $t_f$ (red circles) and the costly eliminated $t^\star_f$ (green circles) with $C = 100\%$. Clearly, the low level removal $C = 5\%$ performs as well as the costly removal $C = 100\%$ (the removal of each maximum transition time from its associated trajectory) in the far tails of the distribution. Namely, for the slow movers, which in the context of contaminant spreading are those who leave long-lasting effects, we can use the cheaper strategy. Note that in our example, the longest transition times we sample are of the order $2 \times 10^4$ while after removal we find this to be reduced by a factor of 10, indicating a large benefit time-wise.

To further quantify these observations we plot in Fig. XI.3(a) the gain quantifier $G$ versus the percentage treated trajectories, as mentioned when $C = 1$ we get $G \approx 0.61$ while clearly if $C = 0$ then $G = 1$. We find that already for the relatively small value $C \approx 0.1$, $G$ converged almost to the fixed value 0.61. It means that any additional elimination above $C \approx 0.1$ does not yield further significant gain.

Finally, we can quantify the gain also with the variance. In Fig. XI.3(b), we show the ratio between the variance of the first passage time after elimination of the longest sticking time, and the variance of the original data set. This ratio is denoted as $G_2$, and we perform the elimination as before with some percentage $C$. We find that for a small value $C \approx 0.01\%$ the quantifier $G_2$ dropped quickly. After that, the convergence is very slow until $G_2 \approx 0.25\%$ for $C = 100\%$. We see from the sharp drop in $G_2$, that the variance of the first passage time is very sensitive to the removal. For the advection-diffusion model in the porous medium under study, this is because the variance is by far more sensitive to the shape of the distribution at its fat tail if compared to the mean. And this is also related to the fact that here the disorder is not too strong, namely the mean of the transition times is finite, for the simulation in Fig. XI.1 we found $\langle \tau \rangle \approx 0.96$. More specifically, the fact that the ratio of the variances is so small is important, because reducing the variance makes the system more homogeneous, and hence predictable (we will discuss this in a future publication). Thus, the transport behavior tends toward Fickian behavior as we remove more maxima.
FIG. XI.3. (a) The quantifier $G$ Eq. (2) versus the cost factor $C$ for the porous medium simulation presented in Fig. XI.1. As explained in the text, $C$ is the percentage of trajectories where the $\tau_{\text{max}}$ was removed. (b) The quantifier $G_2$ for the variance versus $C$ for the same simulation.

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