Padé interpolation to $q$-Painlevé equations

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Abstract. We set up some interpolation problems related to $q$-Painlevé equations of types from $E_7^{(1)}$ to $(A_2 + A_1)^{(1)}$. By solving the interpolation problems, we derive the evolution equations, the scalar Lax pairs and the determinant formulae of special solutions for the corresponding $q$-Painlevé equations.

Key Words and Phrases: Padé method, Padé interpolation, $q$-Painlevé equation.

1 Introduction

Discrete Painlevé equations are discrete equations which are reduced to the Painlevé equations in a suitable limiting process. As integral systems, they are studied in various points of view (see [2] for example).

In Sakai’s theory [17], the discrete Painlevé equations have been classified on the basis of rational surfaces connected to extended affine Weyl groups. There exist three types of discrete Painlevé equations in the classification: elliptic difference ($e$-), multiplicative difference ($q$-) and additive difference ($d$-). In this paper, we will consider the discrete Painlevé equations of $q$-difference type, which is classified as follows:

$$
\text{mul.}(q-) \quad E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + A_1)^{(1)} \rightarrow A_1^{(1)}
$$

Concerning these equations, the results related to this paper are as follows:

\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
\text{} & $q$-$E_7^{(1)}$ & $q$-$E_6^{(1)}$ & $q$-$D_5^{(1)}$ & $q$-$A_4^{(1)}$ & $q$-($A_2 + A_1)^{(1)}$ \\
\hline
$q$-Painlevé & 3 & 15 & 7 & 10 & 10 & 10 & 10 & 17 & 17 & 17 \\
Lax pair & 21 & 19 & 7 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\
special solution & 11 & 5 & 16 & 4 & 13 & 13 & 13 & 13 & 13 & 13 \\
\hline
\end{tabular}
\end{center}

There exists a close connection between Painlevé equations and Padé approximations / interpolations. Padé method has been presented in [20]. By this method, starting from suitable problems of Padé approximation (e.g. [20]) or interpolations (e.g. [13], [5], [22]), one can obtain the Painlevé equations, the scalar Lax pairs and the determinant formulae of special solutions simultaneously.

Padé method for the discrete Painlevé equations are applied to the following cases:

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$e$-$E_8^{(1)}$ & $q$-$E_8^{(1)}$ & $q$-$E_6^{(1)}$ & $q$-$D_5^{(1)}$ \\
\hline
14 & 22 & 5 & 23 & 5 \\
grid & elliptic & $q$-quadric & $q$ & differential, $q$ \\
\hline
\end{tabular}
\end{center}
Here, the case $q$-$D_5^{(1)}$ is studied in both $q$-grid and differential grid (i.e. Padé approximation). The purpose of this paper is to apply Padé method to all the $q$-Painlevé equations of types from $q$-$E_7^{(1)}$ to $q$-$\phi(A_2 + A_1)^{(1)}$. As the main results, the following items are derived for each cases from $q$-$E_7^{(1)}$ to $q$-$\phi(A_2 + A_1)^{(1)}$.

(a) Setting of Padé problem,  
(b) Computation of contiguity relations,  
(c) Computation of $q$-Painlevé equation,  
(d) Computation of Lax pair,  
(e) Computation of special solutions.

This paper is organized as follows. In section 2 we will explain the methods of the derivation of the items (a)–(e). In section 3 we present the main results. In section 4 we will give a summary and discuss some future problems.

2 Padé interpolation method

In this section, we will explain the methods of the derivation of the items (a)–(e) in the main results given in section 3.

(a) Setting of Padé interpolation problem

In this paper, we will consider the following interpolation problem:

$$Y(q^s) = \frac{P_m(q^s)}{Q_n(q^s)} \quad (s = 0, 1, \cdots, m + n),$$

(2.1)

where $Y(x)$ is a given function and the interpolants $P_m(x), Q_n(x)$ are polynomials of degree $m, n$ determined by the interpolation condition (2.1). Then, we call this problem Padé interpolation problem associated with the generating function $Y(x)$.

The common normalization factor of the polynomials $P_m(x), Q_n(x)$ is not determined by the condition (2.1). However, this normalization factor is not essential to our arguments, i.e. the main results in section 3 (see Remark 2). The explicit expressions of $P_m(x), Q_n(x)$, which will be used in the computation of the item (e), were essentially given in [6] (see the item (e) below).

In this paper, we will set up the interpolation problem (2.1) by specifying the following generating functions $Y(x)$:

\[
\begin{array}{cccccc}
Y(x) & q$-E_7^{(1)}$ & q$-E_6^{(1)}$ & q$-D_5^{(1)}$ & q$-A_4^{(1)}$ & q$-(A_2 + A_1)^{(1)}$ \\
\hline
\prod_{i=1}^3 \frac{(a_i x, b_i)_\infty}{(a_i, b_i)_\infty} & \prod_{i=1}^2 \frac{(a_i x, b_i)_\infty}{(a_i, b_i)_\infty} & e^{\log_q x} \frac{(a_1 x, b_1)_\infty}{(a_1, b_1)_\infty} & e^{\log_q x} \frac{(b_1)_\infty}{(b_1 x)_\infty} & (d \sqrt{x/q})^{\log_q x} \\
Y_s = Y(q^s) & \prod_{i=1}^3 \frac{(b_i)_s}{(a_i)_s} & \prod_{i=1}^2 \frac{(b_i)_s}{(a_i)_s} & \frac{(b_1)_s}{(a_1)_s} e^s & c^s(b_1)_s & q^{(1)} d^s \\
\hline
\text{parameter} & 5 & 4 & 3 & 2 & 1 \\
q$-HGF$ & 4\varphi_3 & 3\varphi_2 & 2\varphi_1 & 2\varphi_1 & 1\varphi_1 \\
\end{array}
\]

(2.2)

where $\frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1$ is a constraint for the parameters in case $q$-$E_7^{(1)}$, and the $q$-shifted factorials
are defined by
\[(a_1, a_2, \cdots, a_r)_j = \prod_{k=0}^{j-1} (1 - a_1 q^k)(1 - a_2 q^k) \cdots (1 - a_r q^k), \quad (2.3)\]

and the \(q\)-HGF (the \(q\)-hypergeometric functions) \(\binom{a}{b}_q\) are defined by
\[
\binom{a}{b}_q \left( \begin{array}{c} a_1, \ldots, a_r \cr b_1, \ldots, b_s \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s, q)_n} \left[ (-1)^n q^\frac{n(n-1)}{2} \right] ^{1+s-r} x^n, \quad (2.4)
\]

with \(\binom{n}{2} = n(n-1)/2\).

**Remark 1** The \(q\)-HGFs \(\binom{a}{b}_q\) written in the table \((2.2)\) appear as a special solution for the \(q\)-Painlevé equations. Their expressions are closely related to those of the sequences \(Y_s\), in fact, we choose \(Y_s\) by using the \(q\)-HGFs as a hint. Then, we also choose the generating functions \(Y(x)\), which is equal to the \(Y_s\) at \(x = q^a\).

In this paper, we will consider yet another Padé problem where some parameters \(a_i, b_i, m, n, \text{etc.}\) in \(Y(x)\) is shifted. The parameter shift operators \(T\) are given as follows:

| Parameter | \(q-E_7^{(1)}\) | \(q-E_6^{(1)}\) | \(q-D_5^{(1)}\) | \(q-A_3^{(1)}\) | \(q-(A_2 + A_1)^{(1)}\) |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \((a_1, a_2, a_3, b_1, b_2, b_3, m, n)\) | \((a_1, a_2, a_3, b_1, b_2, b_3, m, n)\) | \((a_1, a_2, b_1, b_2, b_3, m, n)\) | \((a_1, b_1, c, m, n)\) | \((b_1, c, m, n)\) | \((d, m, n)\) |

Here, we call the operators \(T\) the time evolutions, because they specify the directions of the time evolutions for \(q\)-Painlevé equations.

**(b) Computation of contiguity relations**

We will consider a pair of linear \(q\)-difference equations \(L_2(x), L_3(x)\) for unknown function \(y(x)\), which are the main object in our study.

\[
L_2(x) : \left| \begin{array}{ccc} y(x) & y(qx) & \overline{y}(x) \\ P_m(x) & P_m(qx) & \overline{P}_m(x) \\ Y(x)Q_n(x) & Y(qx)Q_n(qx) & \overline{Y}(x)\overline{Q}_n(x) \end{array} \right| = 0,
\]

\[
L_3(x) : \left| \begin{array}{ccc} y(x) & \overline{y}(x) & \overline{y}(x/q) \\ P_m(x) & \overline{P}_m(x) & \overline{P}_m(x/q) \\ Y(x)Q_n(x) & \overline{Y}(x)\overline{Q}_n(x) & \overline{Y}(x/q)\overline{Q}_n(x/q) \end{array} \right| = 0. \quad (2.6)
\]

The pair of the \(q\)-difference equations \((2.6)\) are called contiguity relations. Here, we denote by \(F = T(F)\) and \(\overline{F} = T^{-1}(F)\) and \(T\) is the shift operator acting on parameters given in the table \((2.5)\).

Put \(y(x) = \left[ \frac{P_m(x)}{Y(x)Q_n(x)} \right]\) and define Casorati determinants \(D_i(x)\) by
\[
D_1(x) = \det[y(x), y(qx)] , \quad D_2(x) = \det[y(x), \overline{y}(x)] , \quad D_3(x) = \det[y(qx), \overline{y}(x)]. \quad (2.7)
\]
Then contiguity relations (2.6) are rewritten as follows:
\[
\begin{align*}
L_2(x) & : D_1(x)\overline{y}(x) - D_2(x)y(qx) + D_3(x)y(x) = 0, \\
L_3(x) & : D_1(x/q)y(x) + D_3(x/q)\overline{y}(x) - D_2(x)\overline{y}(x/q) = 0.
\end{align*}
\]
(2.8)

Define basic quantities \(G(x), K(x), H(x)\) (e.g. (3.4), (3.17)) by
\[
G(x) = Y(qx)/Y(x), \quad K(x) = \overline{Y}(x)/Y(x), \quad H(x) = \text{L.C.M}(G_{\text{den}}(x), K_{\text{den}}(x)),
\]
where \(G_{\text{den}}(x), G_{\text{num}}(x)\) are polynomials of the denominator and the numerator of \(G(x)\) respectively, and \(K_{\text{den}}(x), K_{\text{num}}(x)\) are also similar.

Substituting the quantities into the equations (2.7), we obtain the determinants
\[
\begin{align*}
D_1(x) &= \frac{Y(x)}{G_{\text{den}}(x)}\{G_{\text{num}}(x)P_m(x)Q_n(qx) - G_{\text{den}}(x)P_m(qx)Q_n(x)\}, \\
D_2(x) &= \frac{K_{\text{den}}(x)}{Y(x)}\{K_{\text{num}}(x)P_m(x)\overline{Q}_n(x) - K_{\text{den}}(x)\overline{P}_m(x)Q_n(x)\}, \\
D_3(x) &= \frac{Y(x)}{H(x)}\frac{K_{\text{den}}(x)}{K_{\text{num}}(x)}\{K_{\text{num}}(x)P_m(qx)\overline{Q}_n(x) - K_{\text{den}}(x)\overline{Q}_m(x)Q_n(qx)\}.
\end{align*}
\]
(2.10)

Using the interpolation condition (2.1) and the form of the basic quantities, we can investigate positions of zeros and degrees of the polynomials in the braces \{ \} of the equations (2.10). Then, we can simply compute the determinants \(D_i(x)\) (e.g. (3.3), (3.18)) except for some factors such as \(1 - f x \) in \(D_1(x)\) and \(1 - x/g \) in \(D_3(x)\), where \(f, g\), etc. are some constants with respect to \(x\). In this way, we obtain the contiguity relations \(L_2(x), L_3(x)\) (e.g. (3.6), (3.19)).

Remark 2 When the common normalization factor of \(P_m(x), Q_n(x)\) is changed, an \(x\)-independent gauge transformation of \(y(x)\): \(y(x) \rightarrow G y(x)\), the coefficients of \(\overline{y}(x), y(x/q)\), \(y(x)\) and \(y(x), \overline{y}(x), \overline{y}(x/q)\) in (2.8) change as follows:
\[
\begin{align*}
(D_1(x) : D_2(x) : D_3(x)) & \rightarrow (GD_1(x)/G : D_2(x) : D_3(x)) \\
(D_1(x/q) : D_3(x/q) : D_2(x)) & \rightarrow (GD_1(x/q)/G : D_3(x/q) : D_2(x))
\end{align*}
\]
(2.11)

The coefficients \(C_0, C_1\) in \(L_2(x), L_3(x)\) (e.g. (3.6), (3.19)) are defined as the normalization factors of the coefficients of \(\overline{y}(x), y(x)\) respectively. Then, \(C_0, C_1\) change under the gauge transformation, but the product \(C_0C_1\) is the gauge invariant quantity. Moreover, \(C_0, C_1\) do not appear in the final form of the \(q\)-Painlevé equations. □

(c) Computation of \(q\)-Painlevé equation

We can derive the \(q\)-Painlevé equation from the compatibility condition of the contiguity relations \(L_2(x), L_3(x)\). Computing the compatibility condition, we obtain expressions of \(g, \overline{y}, C_0C_1\) in terms of \(f, g\). The first and the second equations are the \(q\)-Painlevé equation (e.g. (3.8), (3.20)). The third equation is a constraint for the product \(C_0C_1\) (e.g. (3.10), (3.22)).

Remark 3 We use variables \(f, g\) in two different meanings. The first meaning is the \(f, g\) which are explicitly determined in terms of parameters \(a_i, b_i, m, n\) by Padé problem, and the second meaning is the \(f, g\) which are unknown functions in the \(q\)-Painlevé equation. In items (c) and (d), we consider \(f, g\) in the second meaning.

Similarly, we use parameters \(m, n\) in two meanings. In the first meaning, \(m, n\) are integers, and in the second meaning, \(m, n\) are generic complex parameters. In items (c) and (d), we consider \(m, n\) in the second meaning. Then, the result of the compatibility of \(L_2(x), L_3(x)\) also holds in the second meaning. □
(d) Computation of Lax pair

We will consider the following pair of linear $q$-difference equations for unknown function $y(x)$:

\[
\begin{align*}
L_1(x) & : A_1(x)y(x/q) + A_2(x)y(x) + A_3(x)y(qx) = 0, \\
L_2(x) & : A_4(x)\overline{f}(x) + A_5(x)y(x) + A_6(x)y(qx) = 0,
\end{align*}
\]

such that their compatibility condition gives a $q$-Painlevé equation. Then, the pair of the $q$-difference equations is called a scalar Lax pair.

The Lax pair $L_1(x), L_2(x)$, which satisfies this definition, is derived by using the results of the items (a)–(c) as follows. The $L_2(x)$ equation in the item (d) (e.g. \[3.11\], \[3.23\]) is the same as the $L_2(x)$ in the item (b). We can obtain the Lax equation $L_1(x)$ as follows. First, combining the contiguity relations $L_2(x), L_3(x)$ (e.g. \[3.11\], \[3.23\], etc.), one obtains the equation between three terms $y(qx), y(x), y(x/q)$, whose coefficient functions depend on variables $f, g, \overline{f}, C_0, C_1$. However, the variables $C_0, C_1$ appear through the product $C_0C_1$. Then, expressing $\overline{f}, C_0C_1$ in terms of $f, g$ only, one obtains the Lax equation $L_1(x)$ (e.g. \[3.11\], \[3.23\]).

\[
\begin{array}{c}
\overline{f}(x/q) \\
L_3(x) \\
L_2(x) \\
L_2(qx) \\
y(x/q) \\
L_1(x) \\
y(x) \\
y(qx)
\end{array}
\]

(e) Computation of special solutions

By construction, the expressions of $f, g$ in the first meaning in Remark 3 gives a special solution for the $q$-Painlevé equation. We present how to compute determinant formulae of the special solutions.

The following formulae are essentially written in \[6\]. For a given sequence $Y_s$, the polynomials $P_m(x), Q_n(x)$ of degree $m, n$ for an interpolation problem:

\[
Y_s = P_m(x_s)/Q_n(x_s) \quad (s = 0, 1, \ldots, m + n),
\]

are given by the following determinant expressions:

\[
P_m(x) = F(x) \det \left[ \sum_{s=0}^{m+n} u_s x_{s}^{i+j} \right]_{i,j=0}^{n}, \quad Q_n(x) = \det \left[ \sum_{s=0}^{m+n} u_s x_{s}^{i+j}(x - x_s) \right]_{i,j=0}^{n-1},
\]

where $u_s = Y_s/F'(x_s), F(x) = \prod_{i=0}^{m+n} (x - x_i)$.

In $q$-grid case of the problem \[2.13\] (i.e. the case of the problem \[2.11\]), the formulae \[2.14\] take the following form:

\[
P_m(x) = \frac{F(x)}{(q)^{m+n+1}} \det \left[ \sum_{s=0}^{m+n} Y_s (q^{-n(s+n)}Y_s q^{s(i+j+1)}) \right]_{i,j=0}^{n},
\]

\[
Q_n(x) = \frac{1}{(q)^{m+n}} \det \left[ \sum_{s=0}^{m+n} Y_s (q^{-n(s+n)}Y_s q^{s(i+j+1)}(x - q^s)) \right]_{i,j=0}^{n-1}.
\]

5
In the derivation of (2.15), we have used the following relations:

\[ F'(x_s) = (x_s - x_0) \cdots (x_s - x_{s-1})(x_s - x_{s+1}) \cdots (x_s - x_{m+n}) \]
\[ = (-1)^s \frac{q^{(s-1)s/2}(q)_{s(m+n-s)}}{s(m+n-s)}(q)_{m+n-s} \]
\[ = (q)_s (q)_{m+n}/q^s(q^{-m-n})_s. \]  

Moreover, substituting the values of \( Y_s, F'(x_s) \) into the formulae (2.14), then one obtains the determinant formulae (2.15).

The expressions of the special solutions \( f, g \) can be derived by comparing \( D_i(x) \) in (2.10) and \( D_i(x) \) in item (b) as the identity with respect to variable \( x \). For example, the computation for the case \( q-E_7^{(1)} \) is as follows. Substituting \( x = 1/a_i \) into \( D_1(x) \) in (2.10) and \( D_1(x) \) in (3.5), we obtain the expression of the \( f \) in the first equation of (3.12) by comparing the two expressions for \( D_1(x) \). Similarly, substituting \( x = 1/b_i \) into the third equation of (2.10) and \( D_3(x) \) in (3.5), we obtain the expression of the \( g \) in the second equation of (3.12) by comparing the two expressions for \( D_3(x) \).

### 3 Main results

In this section, we will present the results obtained through the method, which is written for each cases from \( q-E_7^{(1)} \) to \( q-(A_2 + A_1)^{(1)} \) in section 2.

We use the following notations:

\[ a_1 a_2 \cdots a_n/b_1 b_2 \cdots b_n = \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n}, \]
\[ \mathcal{N}(x) = \prod_{i=0}^{m+n-1} (1 - x/q_i^n), \]
\[ T_{a_i}(F) = F|_{a_i \rightarrow qa_i}, \quad T_{a_i}^{-1}(F) = F|_{a_i \rightarrow a_i/q}. \]  

for any quantity (or function) \( F \) depending on variables \( a_i, b_i \).

#### 3.1 Case \( q-E_7^{(1)} \)

(a) Setting of Padé interpolation problem

The generating function, the interpolated sequence and the constraint (2.2):

\[ Y(x) = \prod_{i=1}^{3} \frac{(a_i x, b_i)_\infty}{(a_i, b_i x)_\infty}, \quad Y_s = \prod_{i=1}^{3} \frac{(b_i)_s}{(a_i)_s}, \quad \frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1. \]  

The time evolution (2.5):

\[ T : (a_1, a_2, a_3, b_1, b_2, b_3, m, n) \rightarrow (qa_1, a_2, qa_3, b_1, b_2, qb_3, m - 1, n). \]  

(b) Contiguity relations

The basic quantities:

\[ G(x) = \prod_{i=1}^{3} \frac{1 - b_i x}{(1 - a_i x)}, \quad K(x) = \frac{1 - b_3 x}{1 - b_3} \prod_{i=1, 3} \frac{(1 - a_i x)}{(1 - a_i x)}, \quad H(x) = (1 - b_3) \prod_{i=1}^{3} (1 - a_i x). \]  

\[ \text{(3.3)} \]
The Casorati determinants:

\[
D_1(x) = \frac{c_0(x(1-x)N(x)Y(x))}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1(1-b_3x/a_2q^m)N(x)Y(x))}{K_{\text{den}}(x)}, \\
D_3(x) = \frac{c_1(1-b_3x)(1-x/g)N(x)Y(x))}{H(x)},
\]

where \( f, g, c_0, c_1 \) are some constants with respect to \( x \).

The contiguity relations:

\[
L_2(x) : C_0x(1-xf)\overline{y}(x) - (1-a_2x)(1-b_3x/a_2q^m)yg(x) + (1-b_3x)(1-x/g)y(x) = 0, \\
L_3(x) : C_1x(1-x\overline{f}/q)y(x) + A_2(x)\left(1-x/q \overline{y}(x) - \frac{A_1(x/q)}{1-a_2x/q}(1-b_3x/a_2q^m)\overline{y}(x/q) = 0,
\]

where

\[
A_1(x) = (1-a_2x)(1-qx) \prod_{i=1,2} (1-b_ix), \\
A_2(x) = (1-b_3x)(1-x/q^n) \prod_{i=1,3} (1-a_ix),
\]

and \( C_0 = c_0(1-b_3)/c_1, C_1 = \overline{c}_0(1-a_1)(1-a_3)/qc_1 \).

(c) \( q \)-Painlevé equation

The compatibility gives the following equations:

\[
\frac{(fg - 1)(fg - 1)}{(fg-b_3/a_2q^m)(fg-b_3/a_2q^{m+1})} = \frac{A_1(1/f)}{A_2(1/f)} , \\
\frac{(1-fg)(1-f\overline{g})}{(1-a_2q^mfg/b_3)(1-a_2q^{m+1}f\overline{g}/b_3)} = \frac{A_1(g)}{A_2(a_2q^m g/b_3)}.
\]

These equations are equivalent to the \( q \)-Painlevé equation of type \( E_7^{(1)} \) given in [3][15][8].

The 8 singular points are on the two curves \( fg = 1, f = b_3/a_2q^m \).

\[
(f, g) = (a_2, 1/a_3), (b_1, 1/b_1), (b_2, 1/b_2), (g, 1/q), \\
(a_1, b_3/a_1a_2q^m), (b_3, 1/a_2q^m), (1/q^{m+n}, b_3q^n/a_2), (a_3, b_3/a_2a_3q^m).
\]

The product \( C_0C_1 \):

\[
C_0C_1 = \frac{A_1(g)(1-b_3/a_2q^m)(1-b_3/a_2q^{m-1})}{q(1-fg)(1-f\overline{g})g^2}.
\]

(d) Lax pair

\[
L_1(x) : \frac{(b_3 - a_2q^m)x^2}{a_2b_3q^m g} \begin{bmatrix} A_1(g) & A_2(a_2q^m g/b_3) \\ A_2(a_2q^m g/b_3) & A_1(g) \end{bmatrix} y(x) \\
+ \frac{(q - b_3x)(q - f x)}{(1 - a_2x)A_2(x)} \begin{bmatrix} y(x/q) - \frac{a_2q^m(g/q - x)(q - b_3x)}{b_3(a_2q^{m+1} g/b_3 - x)} y(x) \\ \frac{a_2q^m(g/q - x)(q - b_3x)}{b_3(a_2q^{m+1} g/b_3 - x)} y(x) \end{bmatrix} = 0,
\]

\[
L_2(x) : C_0x(1-xf)\overline{y}(x) - (1-a_2x)(1-b_3x/a_2q^m)yg(x) + (1-b_3x)(1-x/g)y(x) = 0.
\]

The scalar Lax pair (3.11) is equivalent to that in [21] by using the suitable gauge transformation of \( y(x) \), if the typo \( \overline{f}g - t^2 \) is corrected as \( \overline{f}gq - t^2 \) in the second equation of (36) in [21].
(e) Special solutions

\[
\frac{1 - f/a_1}{1 - f/a_2} = \frac{\gamma_1 T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})}{\gamma_2 T_{a_2}(\tau_{m,n})T_{a_2}^{-1}(\tau_{m+1,n-1})}, \quad \frac{1 - 1/b_1 g}{1 - 1/b_2 g} = \frac{\omega_1 T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})}{\omega_2 T_{b_2}^{-1}(\tau_{m,n})T_{b_2}(\tau_{m+1,n-1})},
\]

(3.12)

where

\[
\tau_{m,n} = \text{det} \left[ \phi_3 \left( b_1, b_2, b_3, q^{-(m+n)} \right) \begin{pmatrix} 1 & 1 \\ \alpha_1 & \alpha_2, \alpha_3 \end{pmatrix} : q^{i+j+1} \right]_{i,j=0}^n,
\]

(3.13)

\[
\gamma_i = \frac{a_i (1 - a_i q^{m+n}) (1 - a_i q/m)}{(1 - a_i)^{n+1}} \prod_{k=1}^3 (1 - b_k/a_i), \quad \omega_i = \frac{(1 - a_i/b_i) (1 - b_i)^n}{(1 - b_i/q)^n}, \quad \text{for } i = 1, 2.
\]

(3.14)

These determinant formulae of hypergeometric solutions are expected to be equivalent to the terminating case of that in [11].

3.2 Case \( q_{-E_6}^{(1)} \)

(a) Setting of Padé interpolation problem

The generating function and the interpolated sequence (2.2):

\[
Y(x) = \frac{(a_1 x, a_2 x, b_1, b_2)_{\infty}}{(a_1, a_2, b_1 x, b_2 x)_{\infty}}, \quad Y_s = \frac{(b_1, b_2)_{\infty}}{(a_1, a_2)_{\infty}}.
\]

(3.15)

The time evolution (2.5):

\[
T: (a_1, a_2, b_1, b_2, m, n) \mapsto (qa_1, a_2, b_1, b_2, m - 1, n).
\]

(3.16)

(b) Contiguity relations

The basic quantities:

\[
G(x) = \prod_{i=1}^2 \frac{(1 - b_i x)}{(1 - a_i x)}, \quad K(x) = \frac{1 - a_i}{1 - a_1 x}, \quad H(x) = \prod_{i=1}^2 (1 - a_i x).
\]

(3.17)

The Casorati determinants:

\[
D_1(x) = \frac{c_0 x(1 - x f) \overline{N}(x) Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1 N(x) Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_1 (1 - x/g) \overline{N}(x) Y(x)}{H(x)},
\]

(3.18)

where \( f, g, c_0, c_1 \) are some constants with respect to \( x \).

The contiguity relations:

\[
L_2(x) : C_0 x(1 - x f) \overline{\overline{Y}}(x) - (1 - a_2 x) y(q x) + (1 - x/g) y(x) = 0,
\]

\[
L_3(x) : C_1 x(1 - x f/q) y(x) + (1 - a_1 x)(1 - x/q^{m+n})(1 - x/q g) \overline{\overline{Y}}(x)
\]

\[
- (1 - x)(1 - b_1 x/q)(1 - b_2 x/q) \overline{\overline{Y}}(x/q) = 0,
\]

(3.19)

where \( C_0 = c_0/c_1, C_1 = \overline{\overline{c}_0}(1 - a_1)/qc_1 \).

(c) \( q \)-Painlevé equation

The compatibility gives the following equations:

\[
\frac{(fg - 1)(fg - 1)}{gg} = \frac{(f - a_2)(f - b_1)(f - b_2)(f - q)}{(f - a_1)(f - 1/q^{m+n})}, \quad \frac{(fg - 1)(fg - 1)}{ff} = \frac{(g - 1/a_2)(g - 1/b_1)(g - 1/b_2)(g - 1/q)}{(g - 1/a_2 q^{m})(g - a_1/b_1 b_2 q^n)}.
\]

(3.20)
These equations are equivalent to the \( q \)-Painlevé equation of type \( E_6^{(1)} \) given in \([15,8,5]\). The 8 singular points are on the two lines \( f = 0, g = 0 \) and one curve \( fg = 1 \).

\[
(f, g) = (a_2, 1/a_2), (b_1, 1/b_1), (b_2, 1/b_2), (q, 1/q), (a_1, 0), (1/q^{m+n}, 0), (0, 1/a_2 q^m), (0, a_1/b_1 q^n).
\]

The product \( C_0 C_1 \):

\[
C_0 C_1 = \frac{(1 - a_2 g)(1 - b_1 g)(1 - b_2 g)(1 - qg)}{qq^2(1 - fg)(1 - fg)}.
\]

(d) Lax pair

\[
L_1(x) : \frac{g^{m+n}g(1 - x)(q - b_1 x)(q - b_2 x)}{g - f} \left[ y(x/q) - \frac{g(q - a_2 x)}{qg - x} \frac{y(x)}{g(1 - a_2 x)} \right] x \left[ \frac{(a_2 g^m - 1)(b_1 b_2 q^n - a_1)}{g^{m+n}(a_2 g - 1)(b_2 g - 1)(gq - 1)} \right] y(x) = 0,
\]

\[
L_2(x) : C_0 x(1 - x)\phi(x) - (1 - a_2 x)y(qx) + (1 - x/q)y(x) = 0.
\]

The scalar Lax pair \([3.23]\) is equivalent to the \( 2 \times 2 \) matrix ones in \([19,24]\), and the scalar ones in \([21,5]\) by using the suitable gauge transformations of \( y(x) \). We note that there exist some typos in the equations (30), (31) in \([5]\), while the equation (29) is correct.

(e) Special solutions

\[
\frac{1 - f/a_1}{1 - f/a_2} = \frac{\gamma_1 T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})}{\gamma_2 T_{a_2}(\tau_{m,n})T_{a_2}^{-1}(\tau_{m+1,n-1})}, \quad \frac{1 - 1/b_1 g}{1 - 1/b_2 g} = \frac{\omega_1 T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})}{\omega_2 T_{b_2}^{-1}(\tau_{m,n})T_{b_2}(\tau_{m+1,n-1})},
\]

where

\[
\gamma_i = \frac{\gamma_i f/a_1(1 - a_i q^{m+n})(1 - a_i/q)^n \prod_{k=1}^{2}(1 - b_k/a_k)}{(1 - a_i)^{n+1}}, \quad \omega_i = \frac{(1 - a_i/b_1)(1 - b_i)^n}{(1 - b_i/q)^n}, \quad \text{for } i = 1, 2.
\]

These determinat formulæ of hypergeometric solutions are equivalent to that in \([5]\), if the typo \( T_{a_2} T_{a_3}(\tau_{m,n-1}) \) is corrected as \( T_{a_2} T_{a_3}(\tau_{m,n-1}) \) in the equation (38) in \([5]\).

### 3.3 Case \( q-D_5^{(1)} \)

(a) Setting of Padé interpolation problem

The generating function and the interpolated sequence \([2.2]\):

\[
Y(x) = e^{\log x^a(a_1 x, b_1)_\infty}, \quad Y_s = e^s b_1^s (a_1)_s.
\]

The time evolution \([2.3]\):

\[
T : (a_1, b_1, c, m, n) \mapsto (qa_1, b_1, c, m - 1, n).
\]

(b) Contiguity relations
The basic quantities:

\[
G(x) = \frac{(1 - b_i x)c}{1 - a_1 x}, \quad K(x) = \frac{1 - a_1}{1 - a_1 x}, \quad H(x) = 1 - a_1 x.
\] (3.29)

The Casorati determinants:

\[
D_1(x) = \frac{c_0(1 - x f)N(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1 N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2 N(x)Y(x)}{H(x)},
\] (3.30)

where \(f, c_0, c_1, c_2\) are some constants with respect to \(x\).

The contiguity relations:

\[
L_2(x) : C_0(1 - x f)\overline{y}(x) - y(q x) + y(x)/g = 0, \\
L_3(x) : C_1(1 - x f/q)y(x) - (1 - a_1 x)(1 - x/q^{m+n})\overline{y}(x)/g + c(1 - x)(1 - b_1 x/q)\overline{y}(x/q) = 0,
\] (3.31)

where \(C_0 = c_0/c_1, C_1 = -c_0(1 - a_1)/c_1, g = c_1/c_2\).

(c) \(q\)-Painlevé equation

The compatibility gives the following equations:

\[
gy = \frac{(f - a_1)(f - 1/q^{m+n})}{c(f - b_1)(f - g)}, \quad f \overline{y} = \frac{q b_1 (g - 1/q^m)(g - a_1/b_1 q^n c)}{(g - 1)(g - 1/c)}.
\] (3.32)

These equations are equivalent to the \(q\)-Painlevé equation of type \(D_5^{(1)}\) given in [7] [8].

The 8 singular points are on the four lines \(f = 0, f = \infty, g = 0, g = \infty\).

\[(f, g) = (a_1, 0), (1/q^{m+n}, 0), (0, 1/q^m), (0, a_1/b_1 c q^n), (\infty, 1), (\infty, 1/c), (b_1, \infty), (g, \infty)\]. (3.33)

The product \(C_0 C_1\):

\[
C_0 C_1 = -(1 - g)(1 - c g)/g^2.
\] (3.34)

(d) Lax pair

\[
L_1(x) : \frac{g(x - q^{m+n})(a_1 x - 1)}{fx - 1} \left[ y(q x) - \frac{y(x)}{g} \right] + \frac{c q^{m+n} g(x - 1)(b_1 x - q)}{fx - q} \left[ \frac{x}{q} y(x) - g y(x) \right] \\
+ \left[ (q^{m+1} - 1)(b_1 c q^n g - a_1 x) f \right] y(x) = 0,
\]

\[
L_2(x) : C_0(1 - x f)\overline{y}(x) - y(q x) + y(x)/g = 0.
\] (3.35)

The scalar Lax pair [3.35] is equivalent to the \(2 \times 2\) matrix ones in [7] [12], the scalar one in [21] by using the suitable gauge transformations of \(y(x)\).

(e) Special solutions

\[
f = \left(1 - a_1(b_1 - a_1)/(1 - a_1 q^{m+n})/(1 - a_1/q)\right)^n \frac{\tau_{m,n,1,1}}{\tau_{m,n,1,1}}, \quad \frac{\tau_{m,n,1,n-1,1}}{\tau_{m,n,1,n-1,1}},
\]

\[
g = \left(1 - a_1 q^{m+n}(1 - b_1/q^n)\right)^n \frac{\tau_{m,n,1,1}}{\tau_{m,n,1,1}}, \quad \frac{\tau_{m,n,1,n-1,1}}{\tau_{m,n,1,n-1,1}},
\]

where

\[
\tau_{m,n,k} = \text{det} \left[ 2 \varphi_1(b_1 q^{-(m+n)}; a_1 q^{i+j+k})_i_j = 0 \right].
\] (3.37)

These determinant formulae of hypergeometric solutions are expected to be equivalent to the terminating case of that in [16].
3.4 Case $q$-$A_4^{(1)}$

(a) Setting of Padé interpolation problem
The generating function and the interpolated sequence (2.2):

$$Y_s = c^s(b_1)_s, \quad Y(x) = c^{\log_q x} \frac{(b_1)_\infty}{(b_1 x)_\infty}$$

(3.38)

The time evolution (2.5):

$$T : (b_1, c, m, n) \mapsto (b_1, c, m - 1, n).$$

(3.39)

(b) Contiguity relations
The basic quantities:

$$G(x) = (1 - b_1 x)c, \quad K(x) = 1, \quad H(x) = 1.$$ 

(3.40)

The Casorati determinants:

$$D_1(x) = \frac{c_0 (1 - xf) N(x) Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1 N(x) Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2 N(x) Y(x)}{H(x)}.$$ 

(3.41)

where $f, c_0, c_1, c_2$ are some constants with respect to $x$.

The contiguity relations:

$$L_2(x) : C_0 (1 - xf) \bar{g}(x) - y(qx) + y(x)/g = 0,$$

$$L_3(x) : C_1 (1 - x f) \bar{g}(x) - (1 - x / q^{m+n}) \bar{g}(x)/g + c(1 - x)(1 - b_1 x/q) \bar{g}(x/q) = 0,$$

(3.42)

where $C_0 = c_0 / c_1, C_1 = -b_0 / c_1, g = c_1 / c_2$.

(c) $q$-Painlevé equation
The compatibility gives the following equations:

$$gg = \frac{f(f - 1/q^{m+n})}{c(f - b_1)(f - q)}, \quad f\bar{g} = \frac{q b_1 g(1 - 1/q^n)}{(g - 1)(g - 1/c)}.$$ 

(3.43)

These equations are equivalent to the $q$-Painlevé equation of type $A_4^{(1)}$ given in [10] [8].

The 8 singular points are on the four lines $f = 0, f = \infty, g = 0, g = \infty$ and $(0, 0)$ is a double point.

$$(f, g) = (1/q^{m+n}, 0), (0, 1/q^m), (\infty, 1), (\infty, 1/c), (b_1, \infty), (q, \infty), (0, 0) \leftrightarrow g/f = -1/b_1 q^n.$$ 

(3.44)

The product $C_0 C_1$:

$$C_0 C_1 = -(1 - g)(1 - cg)/g^2.$$ 

(3.45)

(d) Lax pair

$$L_1(x) : \frac{cgq^{m+n} (1 - x)(q - b_1 x)}{q - fx} [y(x/g) - gg(x)] + \frac{g(q^{m+n} - x)}{1 - fx} [y(qx) - y(x)/g]$$

$$+ q^n [q^{m}(1 - q)(1 - cg) + b_1 cg(1 - q^n g) x/f] y(x) = 0,$$

$$L_2(x) : C_0 (1 - xf) \bar{g}(x) - y(qx) + y(x)/g = 0.$$ 

(3.46)

The scalar Lax pair (3.46) is equivalent to the $2 \times 2$ matrix one for the $q$-Painlevé equation of type $q$-$P(A_4)$ in [12] by using the suitable gauge transformation of $y(x)$.

(e) Computation of special solutions

$$f = \frac{b_1 c}{(c - 1) q^n} \tau_{m,n,1}^{\tau_{m,n,1}^{m+1,n-1,1}}, \quad g = \frac{(1 - b_1 / q)^n}{q^n (1 - b_1)^n} \tau_{m,n,1}^{\tau_{m,n,1}^{m+1,n-1,1}} T_{b_1}^{-1} (\tau_{m,n,1}) T_{b_1} (\tau_{m,n+1,1}),$$

(3.47)
where
\[ \tau_{m,n,k} = \det \left[ 2_{2}^{\Phi_{1}} \left( b_{i}, q^{-m+n}; c_{q}^{i+j+k} \right) \right]^{i,j=0}. \] (3.48)

These determinant formulae of hypergeometric solutions are expected to be equivalent to the terminating case of that in [4].

3.5 Case $q$-($A_{2} + A_{1}$)$^{(1)}$

(a) Setting of Padé interpolation problem

The generating function and the interpolated sequence (2.2):
\[ Y_{s} = q^{(s)} d^{s}, \quad Y(x) = (d \sqrt{x/q})^{\log_{q} x}. \] (3.49)

The time evolution (2.3):
\[ T : (d, m, n) \mapsto (d, m - 1, n). \] (3.50)

(b) Contiguity relations

The basic quantities:
\[ G(x) = dx, \quad K(x) = 1, \quad H(x) = 1. \] (3.51)

The Casorati determinants:
\[ D_{1}(x) = \frac{c_{0}(1 - xf)N(x)Y(x)}{G_{\text{den}}(x)}, \quad D_{2}(x) = \frac{c_{1}N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_{3}(x) = \frac{c_{2}N(x)Y(x)}{H(x)}, \] (3.52)

where $f, c_{0}, c_{1}, c_{2}$ are some constants with respect to $x$.

The contiguity relations:
\[ L_{2}(x) : C_{0}(1 - xf)\tilde{y}(x) - y(qx) + y(x)/q = 0, \]
\[ L_{3}(x) : C_{1}(1 - x\tilde{f}/q)y(x) + (1 - x/q^{m+n})\tilde{y}(x)/q - dx(1 - x)\tilde{y}(x)/q = 0, \] (3.53)

where $C_{0} = c_{0}/c_{1}, C_{1} = \tau_{0}/c_{1}, g = c_{1}/c_{2}$.

(c) $q$-Painlevé equation

The compatibility gives the following equation:
\[ gg = \frac{f(f - 1/q^{m+n})}{d(f - q)}, \quad \tilde{f} \tilde{g} = \frac{qdg(g - 1/q^{m})}{(g - 1)}. \] (3.54)

These equations are equivalent to the $q$-Painlevé equation of type $P_{III}$ given in [10] [8].

The 8 singular points are on the four lines $f = 0, f = \infty, g = 0, g = \infty$ and $(0, 0), (\infty, \infty)$ are double points.
\[ (f, g) = (1/q^{m+n}, 0), (0, 1/q^{m}), (\infty, 1), (q, \infty), \]
\[ (0, 0) \leftrightarrow g/f = 1/dq^{n}, (\infty, \infty) \leftrightarrow g/f = 1/d. \] (3.55)

The product $C_{0}C_{1}$:
\[ C_{0}C_{1} = (1 - g)/g^{2}. \] (3.56)

(d) Lax pair
\[ L_{1}(x) : \frac{d(q^{m+n}(1 - x))}{fx - q} \left[ y(x/q) - gy(x) \right] + \left( \frac{q^{m+n} - x}{fx - 1} \right) y(qx - y(x)/g) \]
\[ + q^{n} (q^{m}(g - 1) - dg(q^{m} - 1)x/f) y(x) = 0, \]
\[ L_{2}(x) : C_{0}(1 - xf)\tilde{y}(x) - y(qx) + y(x)/g = 0. \] (3.57)
The scalar Lax pair (3.57) is equivalent to the 2 × 2 matrix one for the q-Painlevé equation of type $q-P(A_5)$ in [12] by using the suitable gauge transformation of $y(x)$.

(e) Special solutions

$$f = \frac{d}{q^m} \frac{\tau_{m,n,1}\tau_{m+1,n-1}}{\tau_{m,n,0}\tau_{m+1,n-2}}, \quad g = \frac{1}{q^{m+n}} \frac{\tau_{m,n,1}\tau_{m,n-1}}{\tau_{m,n,0}\tau_{m,n-2}}, \quad (3.58)$$

where

$$\tau_{m,n,k} = \det \left[ \psi_1 \left( \frac{q^{-(m+n)}}{0}; -d q^{i+j+k} \right) \right]_{i,j=0}^n. \quad (3.59)$$

These determinant formulae of hypergeometric solutions are expected to be equivalent to the terminating case of that in [13].

4 Conclusion

4.1 Summary

In this paper, we set up the interpolation problems with the generating functions $Y(x)$ and the time evolutions $T$ for all the $q$-Painlevé equations of types from $q-E_7^{(1)}$ to $q-(A_2 + A_1)^{(1)}$. Then, for the $T$, we derive the evolution equations, the scalar Lax pairs and the determinant formulae of the special solutions for the corresponding $q$-Painlevé equations. The main results are given in section 3.

4.2 Problems

Some open problems are related to the results of this paper as follows:

1. In this paper, by choosing one time evolution $T$, we apply Padé method to each cases from $q-E_7^{(1)}$ to $q-(A_2 + A_1)^{(1)}$. By choosing the other time evolutions, we can perform similar computations. It may be interesting to study the relation between Padé method for the various time evolutions and Bäcklund transformations of the affine Weyl group, for example case $q-E_6^{(1)}$ in [5].

2. By the results of this paper and previous ones [13] [22] [5], it turns out that all the discrete Painlevé equations of types from $e-E_8^{(1)}$ to $q-(A_2 + A_1)^{(1)}$ can be obtained by using Padé method. It may be interesting to study the degenerations of these results.

3. In this paper, we derive the $q$-Painlevé equation of type $q-D_5^{(1)}$ by using Padé method with $q$-grid. On the other hand, the $q-P_{\text{IV}}$ equation was also obtained by using the method with differential grid (i.e. Padé approximation) in [23] [5]. It may be interesting to study whether the $q$-Painlevé equations of other types can be also obtained by using the method with deferential grid.

4. It may be interesting to study whether Padé method can be further applied to the other generalized Painlevé systems, for example $q$-Garnier system in [18].

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