SYMPLECTIC GEOMETRY
OF $p$-ADIC TEICHMÜLLER UNIFORMIZATION
FOR ORDINARY NILPOTENT INDIGENOUS BUNDLES

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ABSTRACT. The aim of the present paper is to provide a new aspect of the $p$-adic Teichmüller theory established by S. Mochizuki. We study the symplectic geometry of the $p$-adic formal stacks $\mathcal{M}_g, \mathbb{Z}_p$ (= the moduli classifying $p$-adic formal curves of fixed genus $g > 1$) and $\mathcal{S}_g, \mathbb{Z}_p$ (= the moduli classifying $p$-adic formal curves of genus $g$ equipped with an indigenous bundle). A major achievement in the (classical) $p$-adic Teichmüller theory is the construction of the locus $\mathcal{N}^{\text{ord}}_g, \mathbb{Z}_p$ in $\mathcal{S}_g, \mathbb{Z}_p$ classifying $p$-adic canonical liftings of ordinary nilpotent indigenous bundles.

The formal stack $\mathcal{N}^{\text{ord}}_g, \mathbb{Z}_p$ embodies a $p$-adic analogue of uniformization of hyperbolic Riemann surfaces, as well as a hyperbolic analogue of Serre-Tate theory of ordinary abelian varieties.

In the present paper, the canonical symplectic structure on the cotangent bundle $T^*_\mathbb{Z}_p \mathcal{M}_g, \mathbb{Z}_p$ of $\mathcal{M}_g, \mathbb{Z}_p$ is compared to Goldman’s symplectic structure defined on $\mathcal{S}_g, \mathbb{Z}_p$ after base-change by the projection $\mathcal{N}^{\text{ord}}_g, \mathbb{Z}_p \to \mathcal{M}_g, \mathbb{Z}_p$. We can think of this comparison as a $p$-adic analogue of certain results in the theory of projective structures on Riemann surfaces proved by S. Kawai and other mathematicians. The key ingredient in our discussion is the $F$-crystal structure on the de Rham/crystalline cohomology associated to the adjoint bundle of each ordinary nilpotent indigenous bundle. We show that the slope decomposition of this $F$-crystal has a geometric interpretation, i.e., arises as the differential of the $p$-adic Teichmüller uniformization. This fact makes it clear how the two symplectic structures are related.

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The aim of the present paper is to provide a new aspect of the p-adic Teichmüller theory established by S. Mochizuki (cf. [11], [12]). We study the symplectic geometry of the p-adic formal stack defined as the moduli classifying p-adic formal curves equipped with an indigenous bundle. As a consequence of the present paper, we will propose a p-adic analogue of the comparison assertion concerning a canonical symplectic structure on the moduli space of projective structures.

Here, recall that a projective structure on a Riemann surface is a holomorphic atlas whose transition functions between coordinate charts may be expressed as Möbius transformations. Also, an indigenous bundle is defined as a sort of algebro-geometric counterpart of a projective structure (cf. §1.3 for the precise definition of an indigenous bundle). Indigenous bundles, or equivalently, projective structures, have provided a rich story in complex (i.e., the usual) Teichmüller theory for a long time. Canonical examples are constructed by means of various uniformizations such as Fuchsian, Bers, or Schottky. In other words, one may think of an indigenous bundle as an algebraic object encoding (analytic, i.e., non-algebraic) uniformization data for Riemann surfaces.

One subject in the theory of projective structures is to compare symplectic structures on the relevant spaces via uniformization. For example, we shall refer to works by S. Kawai, P. Arés-Gastesi, I. Biswas, B. Loustau, et al. (cf. [9]; [1]; [2]; [10]). To explain some of these works, let us consider the following spaces associated to a connected orientable closed surface Σ of genus g > 1:

\[ T^\text{an}_{g,\mathbb{C}} := \text{Conf}(\Sigma)/\text{Diff}^0(\Sigma) \quad (\text{resp.}, \quad S^\text{an}_{g,\mathbb{C}} := \text{Proj}(\Sigma)/\text{Diff}^0(\Sigma)) , \]

where Conf(Σ) (resp., Proj(Σ)) denotes the space of all holomorphic structures (resp., all projective structures) on Σ compatible with the orientation of Σ, and Diff^0(Σ) denotes the group of all diffeomorphisms of Σ homotopic to the identity map of Σ. (In particular, \( T^\text{an}_{g,\mathbb{C}} \) is nothing but the Teichmüller space associated to Σ.) It is well-known that \( T^\text{an}_{g,\mathbb{C}} \) admits the structure of a complex manifold of dimension 3g − 3 which is a universal covering of the moduli space \( \mathcal{M}^\text{an}_{g,\mathbb{C}} \) classifying connected compact Riemann surfaces of genus g. Also, \( S^\text{an}_{g,\mathbb{C}} \) admits the structure of a complex manifold of dimension 6g − 6 and moreover the structure of a relative affine space over \( T^\text{an}_{g,\mathbb{C}} \) modeled on (the total space of) the holomorphic cotangent bundle \( T^\vee_{\mathbb{C}} T^\text{an}_{g,\mathbb{C}} \).

Now, let us take a holomorphic section

\[ \sigma : T^\text{an}_{g,\mathbb{C}} \to S^\text{an}_{g,\mathbb{C}} \]

of the natural projection \( S^\text{an}_{g,\mathbb{C}} \to T^\text{an}_{g,\mathbb{C}} \). Because of the affine structure on \( S^\text{an}_{g,\mathbb{C}} \), this section may be extended to a unique biholomorphism \( \theta : T^\vee_{\mathbb{C}} T^\text{an}_{g,\mathbb{C}} \to S^\text{an}_{g,\mathbb{C}} \) compatible with the respective affine structures. It induces an isomorphism

\[ \Theta_\sigma : \theta^*_\sigma(\bigwedge^2 \Omega^*_{S^\text{an}_{g,\mathbb{C}}}) \cong \bigwedge^2 \Theta^*_\sigma(\Omega^*_{S^\text{an}_{g,\mathbb{C}}}) \cong \bigwedge^2 \Omega^*_{T^\vee_{\mathbb{C}} T^\text{an}_{g,\mathbb{C}}} . \]

Notice that \( T^\vee_{\mathbb{C}} T^\text{an}_{g,\mathbb{C}} \) admits a canonical holomorphic symplectic structure \( \omega^\text{Liou}_{g,\mathbb{C}} \) obtained as the differential of the tautological 1-form (i.e., the so-called Liouville form). Moreover, \( S^\text{an}_{g,\mathbb{C}} \) admits a holomorphic symplectic structure \( \omega^\text{PGL}_{g,\mathbb{C}} \) induced, via pull-back by the monodromy map, from
Goldman’s symplectic structure on the $\text{PGL}_2(\mathbb{C})$-character variety (cf. [1]). Thus, we obtain holomorphic symplectic manifolds

$$(T^\vee_{g,\mathbb{C}} \mathcal{J}^\text{an}_{g,\mathbb{C}}, \omega^{\text{Lio}}_{g,\mathbb{C}}) \quad \text{and} \quad (\mathcal{S}^\text{an}_{g,\mathbb{C}}, \omega^{\text{PGL}}_{g,\mathbb{C}}).$$

According to a pioneering result proved by S. Kawai, we can compare these symplectic structures. In fact, it follows from [9], Theorem, that if $\sigma$ is any Bers section, then $\theta_\sigma$ preserves the symplectic structure up to a constant factor; more precisely, the following equality holds:

$$(5) \quad \Theta_\sigma(\omega^{\text{PGL}}_{g,\mathbb{C}}) = \pi \cdot \omega^{\text{Lio}}_{g,\mathbb{C}}.$$ 

Also, B. Loustau proved (cf. [10], Theorem 6.10) this equality, which may be described as the equality $\Theta_\sigma(\omega^{\text{PGL}}_{g,\mathbb{C}}) = \sqrt{-1} \cdot \omega^{\text{Lio}}_{g,\mathbb{C}}$ with the conventions chosen by him. Moreover, by [1], Theorem 1.1 and Remark 3.2, the same equality holds for the case where $\sigma$ is taken as a section arising from either the Schottky uniformization or the Earle uniformization.

This article attempts to consider a $p$-adic analogue of these comparison results. Let $p$ be an odd prime and $\hat{\mathcal{M}}_{g,\mathbb{Z}_p}$ (cf. (29)) denote the $p$-adic formal stack defined as the moduli classifying (proper, smooth, and geometrically connected) $\mathbb{G}_m$-structures. In fact, it follows from [9], Theorem, that if $\sigma$ according to a pioneering result proved by S. Kawai, we can compare these symplectic structures. By the same manner as the complex case discussed above, $\hat{\mathcal{S}}_{g,\mathbb{Z}_p}$ and the cotangent bundle $T^\vee_{g,\mathbb{Z}_p} \hat{\mathcal{M}}_{g,\mathbb{Z}_p}$ of $\hat{\mathcal{M}}_{g,\mathbb{Z}_p}$ admit canonical structures

$$(6) \quad \hat{\omega}^{\text{Lio}}_{g,\mathbb{Z}_p} \in \Gamma(T^\vee_{g,\mathbb{Z}_p} \hat{\mathcal{M}}_{g,\mathbb{Z}_p}, \bigwedge^2 \Omega(T^\vee_{g,\mathbb{Z}_p} \hat{\mathcal{M}}_{g,\mathbb{Z}_p}), \quad \hat{\omega}^{\text{PGL}}_{g,\mathbb{Z}_p} \in \Gamma(\hat{\mathcal{S}}_{g,\mathbb{Z}_p}, \bigwedge^2 \Omega(\hat{\mathcal{S}}_{g,\mathbb{Z}_p})),$$

respectively (cf. (30) and (50)). $\hat{\omega}^{\text{PGL}}_{g,\mathbb{Z}_p}$ is, by definition, obtained by composing the Killing form on $\mathfrak{sl}_2$ and the cup product in the de Rham cohomology of the adjoint bundles on indigenous bundles. In this way, we obtain two symplectic $p$-adic formal stacks

$$(7) \quad (T^\vee_{g,\mathbb{Z}_p} \hat{\mathcal{M}}_{g,\mathbb{Z}_p}, \hat{\omega}^{\text{Lio}}_{g,\mathbb{Z}_p}) \quad \text{and} \quad (\hat{\mathcal{S}}_{g,\mathbb{Z}_p}, \hat{\omega}^{\text{PGL}}_{g,\mathbb{Z}_p}),$$

which have the natural projections onto $\hat{\mathcal{M}}_{g,\mathbb{Z}_p}$.

Here, we shall recall the main achievement of the (classical) $p$-adic Teichmüller theory studied in [11]. Denote by $\mathcal{N}^\text{ord}_{g,\mathbb{Z}_p}$ (cf. (51)) the locus in the stack $\mathcal{S}_{g,\mathbb{Z}_p} := \hat{\mathcal{S}}_{g,\mathbb{Z}_p} \otimes \mathbb{F}_p$ over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ classifying ordinary nilpotent indigenous bundles (cf. §1.6 for the precise definition of an ordinary nilpotent indigenous bundle). S. Mochizuki proved (cf. [11], Chap. II, §3, Corollary 3.8) that $\mathcal{N}^\text{ord}_{g,\mathbb{Z}_p}$ is a nonempty Deligne-Mumford stack which is étale and dominant over $\mathcal{M}_{g,\mathbb{F}_p} := \hat{\mathcal{M}}_{g,\mathbb{Z}_p} \otimes \mathbb{F}_p$. This implies that there exists the unique (up to isomorphism) $p$-adic formal stack

$$(8) \quad \hat{\mathcal{N}}^\text{ord}_{g,\mathbb{Z}_p}$$

(cf. (53)) over $\hat{\mathcal{M}}_{g,\mathbb{Z}_p}$ lifting $\mathcal{N}^\text{ord}_{g,\mathbb{Z}_p}$. Moreover, he also constructed (cf. Theorem 1.6.1 for the precise statement) a canonical $p$-adic lifting $\Phi_N : \hat{\mathcal{N}}^\text{ord}_{g,\mathbb{Z}_p} \rightarrow \hat{\mathcal{N}}^\text{ord}_{g,\mathbb{Z}_p}$ of the Frobenius endomorphism of $\mathcal{N}^\text{ord}_{g,\mathbb{Z}_p}$ together with an indigenous bundle $(\mathcal{E}_N, \nabla_{\mathcal{E}_N})$ on the universal family of curves $C_N$ over $\mathcal{N}^\text{ord}_{g,\mathbb{Z}_p}$ which is Frobenius invariant in the sense that $\mathbb{F}_p^\ast(\Phi_N^\ast(\mathcal{E}_N, \nabla_{\mathcal{E}_N})) \cong (\mathcal{E}_N, \nabla_{\mathcal{E}_N})$ (where $\mathbb{F}_p^\ast(\cdot)$ denotes renormalized Frobenius pull-back in the sense of §2.1). This result is used, via restriction to various points in $\mathcal{N}^\text{ord}_{g,\mathbb{Z}_p}$, to obtain $p$-adic canonical liftings of curves...
(endowed with an ordinary nilpotent indigenous bundle). We shall refer to $\hat{\mathcal{N}}^{\text{ord}}_{g,S_p}$ together with both $\Phi_N$ and $(\mathcal{E}_N, \nabla_{\mathcal{E}_N})$ as the classical ordinary $p$-adic Teichmüller uniformization.

These $p$-adic objects create a situation similar to the complex case. This means that the indigenous bundle $(\mathcal{E}_N, \nabla_{\mathcal{E}_N})$ determines its classifying morphism $\sigma: \hat{\mathcal{N}}^{\text{ord}}_{g,S_p} \rightarrow \hat{\mathcal{S}}_{g,S_p}$ (i.e., a section of the projection $\hat{\mathcal{S}}_{g,S_p} \rightarrow \hat{\mathcal{M}}_{g,S_p}$ defined on the étale dominant formal stack $\hat{\mathcal{N}}^{\text{ord}}_{g,S_p}$ over $\hat{\mathcal{M}}_{g,S_p}$). Moreover, since $\hat{\mathcal{S}}_{g,S_p}$ forms a relative affine space over $\hat{\mathcal{M}}_{g,S_p}$ modeled on $T_{g,S_p}^\vee \hat{\mathcal{M}}_{g,S_p}$, the morphism $\sigma$ gives a trivialization

$$\theta: T_{g,S_p}^\vee \hat{\mathcal{M}}_{g,S_p}|_\mathcal{N} \rightarrow (=: T_{g,S_p}^\vee \hat{\mathcal{M}}_{g,S_p} \times_{\hat{\mathcal{M}}_{g,S_p}} \hat{\mathcal{N}}^{\text{ord}}_{g,S_p}) \sim \hat{\mathcal{S}}_{g,S_p}|_\mathcal{N} : (\hat{\mathcal{S}}_{g,S_p} \times_{\hat{\mathcal{M}}_{g,S_p}} \hat{\mathcal{N}}^{\text{ord}}_{g,S_p})$$

(cf. (59)) of $\hat{\mathcal{S}}_{g,S_p}$ after base-change to $\hat{\mathcal{N}}^{\text{ord}}_{g,S_p}$. This trivialization induces an isomorphism

$$\Theta: (\bigwedge^2 \Omega_{g,S_p}|_\mathcal{N}) \cong (\bigwedge^2 \Omega_{\hat{\mathcal{S}}_{g,S_p}|_\mathcal{N}/S_p}) \sim (\bigwedge^2 \Omega_{T_{g,S_p}^\vee \hat{\mathcal{M}}_{g,S_p}|_\mathcal{N}/S_p})$$

Then, the main result of the present paper is described as the following theorem, asserting the comparison, via $\Theta$, between the pull-backs of symplectic structures

$$\hat{\omega}_{g,S_p}|_\mathcal{N} := \hat{\omega}_{g,S_p}|_{T_{g,S_p}^\vee \hat{\mathcal{M}}_{g,S_p}|_\mathcal{N}}, \quad \omega_{g,S_p}|_\mathcal{N} := \omega_{g,S_p}|_{\hat{\mathcal{S}}_{g,S_p}|_\mathcal{N}}$$

defined on $T_{g,S_p}^\vee \hat{\mathcal{M}}_{g,S_p}|_\mathcal{N}$, $\hat{\mathcal{S}}_{g,S_p}|_\mathcal{N}$ respectively. (We shall refer to [15] for the version of this theorem in the case of the moduli classifying dormant indigenous bundles $\mathcal{M}_g^{\text{zar}}$.)

**Theorem A** (= Theorem 1.7.1).

If $p > 3$, then the morphism $\theta$ preserves the symplectic structure, i.e., the following equality holds:

$$\Theta(\hat{\omega}_{g,S_p}|_\mathcal{N}) = \omega_{g,S_p}|_\mathcal{N}.$$ 

In particular, the image of $\sigma: \hat{\mathcal{N}}^{\text{ord}}_{g,S_p} \rightarrow \hat{\mathcal{S}}_{g,S_p}$ is Lagrangian with respect to the symplectic structure $\hat{\omega}_{g,S_p}|_\mathcal{N}$.

In this paragraph, we shall make a brief comment on the proof of the above theorem. The key ingredient in our discussion is the $F$-crystal structure of the cohomology associated to the adjoint bundle of each ordinary nilpotent indigenous bundle. Indeed, let us take a collection $(X_1, \mathcal{E}_1, \nabla_{\mathcal{E}_1})$ classified by $\mathcal{N}^{\text{ord}}_{g,S_p}$. Denote by $(X, \mathcal{E}, \nabla_{\mathcal{E}})$ the canonical $p$-adic lifting of $(X_1, \mathcal{E}_1, \nabla_{\mathcal{E}_1})$ arising from the c.o. $p$-Teich. uniformization and by $s_\infty$ its classifying point of $\hat{\mathcal{N}}^{\text{ord}}_{g,S_p}$. Then, the Frobenius invariance of $(\mathcal{E}, \nabla_{\mathcal{E}})$ gives rise to an $F$-crystal structure (cf. [17]) on the first de Rham cohomology $\mathbb{H}^1(K^* [\nabla_{\mathcal{E}}])$ (which is isomorphic to the first crystalline cohomology of the corresponding crystal) of the adjoint flat bundle associated to $(\mathcal{E}, \nabla_{\mathcal{E}})$. On the other hand, since $\mathbb{H}^1(K^* [\nabla_{\mathcal{E}}])$ is canonically isomorphic to the tangent space of $\hat{\mathcal{S}}_{g,S_p}$ at $s_\infty$ (cf. (33)), the differential of the embedding $\sigma: \hat{\mathcal{N}}^{\text{ord}}_{g,S_p} \rightarrow \hat{\mathcal{S}}_{g,S_p}$ determines a direct sum decomposition of $\mathbb{H}^1(K^* [\nabla_{\mathcal{E}}])$ (cf. (55)). One important observation is (cf. Corollary 2.5.1) that this decomposition of the $F$-crystal coincides with (i.e., gives the geometric interpretation of) the slope decomposition. It follows (cf. Corollary 2.5.2) that both $\hat{\omega}_{g,S_p}|_\mathcal{N}$ and $\hat{\omega}_{g,S_p}|_\mathcal{N}$ turn out to specify eigenvectors of the $F$-crystal structure defined on the second exterior power of
the dual $\mathbb{H}^1(K^\bullet[\nabla^{ad}])^\vee$. This fact makes it clear how the two symplectic structures are related via reduction modulo $p$. Thus, the proof of the main theorem will be reduced to an explicit computation of $\mathbb{H}^1(K^\bullet[\nabla^{ad}])$ in terms of the Čech double complex (cf. the discussion in §3.2).

In the Appendix of the present paper, we discuss crystals of torsors (equipped with a structure group) and prove the correspondence between flat torsors and them (cf. Theorem 4.4.2). This correspondence may be thought of as a generalization of the classical result (cf. [4], §6.6, Theorem) for crystals of vector bundles (i.e., of $\text{GL}_n$-torsors). Moreover, we observe (cf. Proposition 4.5.2) the relationship between the respective deformations of a prescribed flat torsor over distinct underlying spaces. Its application to the case of indigenous bundles (cf. Proposition 4.6.1) will be used in the proof of the main theorem. Throughout the present paper, we shall often refer to the Appendix for some definitions and facts involved.

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1. Symplectic structures on the moduli of indigenous bundles

In this section, we shall review various notions and facts concerning our discussion. In particular, the central characters of the present paper, i.e., the $p$-adic formal stacks and their symplectic structures displayed in (7) are defined precisely. Throughout the present paper, we fix an odd prime $p$ and an integer $g$ with $g > 1$.

1.1. Symplectic structures.

We begin by reviewing the notion of a symplectic structure. Let $R$ be a commutative ring with unit and $X$ a smooth Deligne-Mumford stack over $R$ of relative dimension $n > 0$. Denote by $\Omega_{X/R}$ the sheaf of 1-forms on $X$ relative to $R$ and by $T_{X/R}$ its dual. Hence, both $\Omega_{X/R}$ and $T_{X/R}$ are vector bundles (i.e., locally free coherent sheaves) on $X$ of rank $n$. A symplectic structure on $X$ is, by definition, a nondegenerate closed 2-form $\omega \in \Gamma(X, \bigwedge^2 \Omega_{X/R})$. Here, we shall say that a 2-form $\omega$ is nondegenerate if the $O_X$-linear morphism $\Omega_{X/R} \to T_{X/R}$ induced naturally by $\omega$ is an isomorphism. Let us fix a symplectic structure $\omega$ on $X$. Then, we shall say that a smooth substack $Y$ of $X$ is Lagrangian (with respect to $\omega$) if $\omega|_Y = 0$ and $\dim(Y) = \frac{n}{2}$.

Let us recall the canonical symplectic structure defined on the cotangent space. Let $X$ be as above. Denote the total space of $\Omega_{X/R}$ by

$$T^\vee_{X/R},$$

and call it the cotangent bundle of $X$ (over $R$). One obtains a symplectic structure

$$\omega_L \in \Gamma(T^\vee_{X/R}, \bigwedge^2 \Omega_{T^\vee_{X/R}})$$

(14)
on $T^\vee_R X$ defined as the differential of the tautological 1-form (called the Liouville form) on $T^\vee_R X$; it may be characterized uniquely by the condition that if $q_1, \ldots, q_n$ are any local coordinates in $X$ (relative to $R$) and $p_1, \ldots, p_n$ are the dual coordinates in $T^\vee_R X$, then $\omega^\text{Lio}_X$ has the local expression $\sum_{i=1}^n dp_i \wedge dq_i$.

Next, denote by

$$0_X : X \to T^\vee_R X$$

the zero section, whose image is immediately verified to be Lagrangian. The pull-back $0^*_X (\Omega_{T^\vee_R X/X})$ of $\Omega_{T^\vee_R X/X}$ is canonically isomorphic to $\Omega_{X/R}$. In what follows, let us describe the $\mathcal{O}_X$-bilinear map on $0^*_X (T^\vee_{T^\vee_R X/R})$ corresponding to the restriction $\omega^\text{Lio}_{X}|_{0_X}$ of $\omega^\text{Lio}_X$. Consider the short exact sequence

$$0 \longrightarrow 0^*_X (T^\vee_{T^\vee_R X/R}) \longrightarrow 0^*_X (T^\vee_{T^\vee_R X/R}) \longrightarrow T^\vee_{X/R} \longrightarrow 0$$

obtained by differentiating the projection $T^\vee_R X \to X$ and successively restricting it to $0_X$. The differential of $0_X : X \to T^\vee_R X$ specifies a split injection $T^\vee_{X/R} \hookrightarrow 0^*_X (T^\vee_{T^\vee_R X/R})$ of this short exact sequence. In other words, $0_X$ gives a direct sum decomposition

$$0^*_X (T^\vee_{T^\vee_R X/R}) \cong T^\vee_{X/R} \oplus \Omega_{X/R}.$$ 

Then, the $\mathcal{O}_X$-bilinear map on $0^*_X (T^\vee_{T^\vee_R X/R})$ corresponding to $\omega^\text{can}|_{0_X}$ is given by the natural pairing $\langle \cdot, \cdot \rangle : T^\vee_{X/R} \times \Omega_{X/R} \to \mathcal{O}_X$. More precisely, this bilinear map may be expressed, via (17), as the map given by assigning

$$((a, b), (a', b')) \mapsto \langle a, b' \rangle - \langle a', b \rangle$$

for local sections $a, a' \in T^\vee_{X/R}$ and $b, b' \in \Omega_{X/R}$.

Let us consider the case of formal stacks. Let $X$ a smooth $p$-adic formal stack over $\mathbb{Z}_p$; it may be given as $X = \lim_{\leftarrow n} X_n$, where each $X_n$ $(n \geq 1)$ is a smooth stack over $\mathbb{Z}/p^n\mathbb{Z}$ such that $X_n = X_m \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ (if $n < m$). We shall write $T^\vee_{X/Z_p} := \lim_{\leftarrow n} T^\vee_{X_n/(\mathbb{Z}/p^n\mathbb{Z})}$ and $\Omega_{X/Z_p} := \lim_{\leftarrow n} \Omega_{X_n/(\mathbb{Z}/p^n\mathbb{Z})}$, that are rank $n$ vector bundles on $X$. By a symplectic structure on $X$, we mean a collection $\omega := (\omega_n)_{n \geq 1}$, where each $\omega_n$ denotes a symplectic structure on $X_n$ such that $\omega_n|_{X_n} = \omega_n$ (if $n < m$). Since the natural morphism $\Gamma(X, \bigwedge^2 \Omega_{X/Z_p}) \to \lim_{\leftarrow n} \Gamma(X_n, \bigwedge^2 \Omega_{X_n/(\mathbb{Z}/p^n\mathbb{Z})})$ is an isomorphism (cf. [2], Chap.8, §8.2, Corollary 8.2.4), each symplectic structure $\omega$ on $X$ may be considered as an element of $\Gamma(X, \bigwedge^2 \Omega_{X/Z_p})$.

Denote by

$$T^\vee_{Z_p} X$$

the (smooth) $p$-adic formal stack defined as $T^\vee_{Z_p} X := \lim_{\leftarrow n} T^\vee_{Z_p/(\mathbb{Z}/p^n\mathbb{Z})} X_n$. Then, the collection

$$\omega^\text{Lio}_X := (\omega^\text{Lio}_{X_n})_{n \geq 1} \in \Gamma(T^\vee_{Z_p} X, \bigwedge^2 \Omega_{T^\vee_{Z_p} X/Z_p})$$

forms a symplectic structure on $T^\vee_{Z_p} X$. The fiber of the projection $T^\vee_{Z_p} X \to X$ over each point in $X(\mathbb{Z}_p)$ is Lagrangian.
1.2. Moduli of algebraic curves.

We shall introduce some notation concerning algebraic curves and their moduli. By a curve (of genus \( g \)) over a fixed scheme \( S \), we mean a geometrically connected, proper, and smooth scheme \( f : X \to S \) over \( S \) of relative dimension 1 such that \( f_\ast(\Omega_{X/S}) \) is locally free of constant rank \( g \). We shall denote by
\[ M_{g,R} \]
the moduli stack classifying curves of genus \( g \) over \( R \), which is a geometrically connected smooth Deligne-Mumford stack over \( R \) of relative dimension \( 3g - 3 \). Also, denote by
\[ f_{g,R} : C_{g,R} \to M_{g,R} \]
the universal family of curves over \( M_{g,R} \). In what follows, we fix a specific choice of an \( \mathcal{O}_{M_{g,R}} \)-linear isomorphism
\[ \int_{C_{g,R}} : \mathbb{R}^1 f_{g,R\ast}(\Omega_{C_{g,R}/M_{g,R}}) \cong \mathcal{O}_{M_{g,R}} \]
(i.e., the trace map) obtained by Serre duality. For any family of curves \( f : X \to S \) of genus \( g \), we shall write
\[ \int_X : \mathbb{R}^1 f_\ast(\Omega_{X/S}) \cong \mathcal{O}_S \]
for the pull-back of the isomorphism \( \int_{C_{g,R}} \) via the classifying morphism \( S \to M_{g,R} \) of this curve. Here, recall (cf. [7], Corollary 5.6) that if \( d \) denotes the universal derivation \( \mathcal{O}_X \to \Omega_{X/S} \) (namely, the trivial connection on \( \mathcal{O}_X \) over \( S \)), then \( \mathbb{R}^1 f_\ast(\Omega_{X/S}) \) is canonically isomorphic to \( \mathbb{R}^2 f_\ast(K^\bullet[d]) \) (cf. §4.5 for the definition of \( K^\bullet[-] \)) via the Hodge to de Rham spectral sequence of \( K^\bullet[d] \). Accordingly, we sometimes consider \( \int_X \) as an \( \mathcal{O}_S \)-linear isomorphism \( \mathbb{R}^2 f_\ast(K^\bullet[d]) \cong \mathcal{O}_S \).

Also, write
\[ \oint_X : f_\ast(\Omega_{X/S}^\otimes 2) \cong \mathbb{R}^1 f_\ast(\mathcal{T}_{X/S})^\vee \]
for the isomorphism arising from the bilinear map
\[ \oint_X : f_\ast(\Omega_{X/S}^\otimes 2) \otimes \mathbb{R}^1 f_\ast(\mathcal{T}_{X/S}) \cong \mathbb{R}^1 f_\ast(\Omega_{X/S}) \xrightarrow{f_X} \mathcal{O}_S. \]

By well-known generalities on the deformation theory of curves, there exists a canonical isomorphism of \( \mathcal{O}_S \)-modules
\[ \mathcal{T}_{M_{g,R}/R}|_S \cong \mathbb{R}^1 f_\ast(\mathcal{T}_{X/S}) \]
(i.e., the Kodaira-Spencer map), where we use the notation “\( |_S \)” to denote pull-back by the classifying morphism \( S \to M_{g,R} \). This isomorphism gives the following composite isomorphism:
\[ f_\ast(\Omega_{X/S}^\otimes 2) \xrightarrow{f_X^\ast} \mathbb{R}^1 f_\ast(\mathcal{T}_{X/S})^\vee \xrightarrow{\text{[27]}} ((\mathcal{T}_{M_{g,R}/R}|_S)^\vee) \cong \Omega_{M_{g,R}/R}|_S. \]
Next, by a \textit{\textbf{p-adic formal curve of genus}} \textit{g} over a \textit{\textbf{p-adic formal scheme}} \textit{S}, we mean a flat \textit{\textbf{p-adic formal scheme}} \textit{X} over \textit{S} whose reduction modulo \textit{p^n} (for each \textit{n} \geq 1) is a curve (of genus \textit{g}) over \textit{S} \otimes (\mathbb{Z}/p^n\mathbb{Z}). Denote by
\begin{equation}
\widehat{M}_{g,p}
\end{equation}
the smooth \textit{\textbf{p-adic formal stack}} defined as \textit{\textbf{\widehat{M}_{g,p} := \lim_{n \to \infty} M_{g,p^n}}} \textit{p}. Then, \textit{\textbf{\widehat{M}_{g,p}}} may be identified with the moduli classifying \textit{\textbf{p-adic formal curves}} of genus \textit{g}. By the discussion in the previous subsection, we obtain a symplectic structure
\begin{equation}
\widehat{\omega}_{g,p} \quad \text{(:= \omega}_{g,p})
\end{equation}
on \textit{T}_{ \widehat{M}_{g,p}}.

\subsection*{1.3. Indigenous bundles.}

We shall recall the notion of an indigenous bundle. Some definitions and notation concerning connections on torsors are introduced in the Appendix of the present paper. Suppose that \textit{2} is invertible in \textit{R}. Denote by \textit{B} the Borel subgroup of \textit{PGL}_2 (:= the projective linear group of rank 2 over \textit{R}) defined to be the image (via the quotient \textit{GL}_2 \to \textit{PGL}_2) of upper triangular matrices. Let \textit{S} be a scheme over \textit{R} and \textit{f : X} \to \textit{S} a curve of genus \textit{g} over \textit{S}. Recall from, e.g., \cite{11}, Chap. I, \S 2, Definition 2.2, or \cite{15}, Definition 2.1.1, that an \textit{indigenous bundle} on \textit{X} of rank 2 over \textit{S} is a flat \textit{PGL}_2-torsor \textit{E} over \textit{X} (cf. \cite{11}, \S 1), i.e., a pair consisting of a \textit{PGL}_2-torsor \textit{\pi : E} \to \textit{X} over \textit{X} and an \textit{S-connection \mathcal{T}_{X/S}} \to \textit{E}, which induces an inclusion \textit{\mathcal{T}_{E/S}} \hookrightarrow \textit{E}, such that the composite

\begin{equation}
\text{KS}_{(E, \nabla_E)} : \mathcal{T}_{X/S} \nabla_E \to \mathcal{T}_{E/S} \to \mathcal{T}_{E/S} / \mathcal{T}_{E/B/S}
\end{equation}
is an isomorphism. If (\textit{E}, \nabla_E) is an indigenous bundle, then a \textit{\textbf{B-reduction}} \textit{E}_B of \textit{E} satisfying the above condition is uniquely determined (up to isomorphism); we shall refer to it as the \textbf{Hodge reduction} of (\textit{E}, \nabla_E). An \textit{isomorphism} (\textit{E}, \nabla_E) \sim (\textit{E}', \nabla_{E'}) between indigenous bundles on \textit{X} of rank 2 over \textit{X} is defined as an isomorphism \textit{E} \sim \textit{E}' of \textit{PGL}_2-torsors compatible with the respective connections \nabla_E and \nabla_{E'}.

Let (\textit{E}, \nabla_E) be an indigenous bundle on \textit{X} of rank 2. \nabla_E induces an \textit{S-connection}
\begin{equation}
\nabla^\text{ad} : \text{Ad}(E) \to \Omega_{X/S} \otimes \text{Ad}(E)
\end{equation}(cf. \cite{138}) on the adjoint bundle \text{Ad}(E) (:= \textit{E} \times^{\text{PGL}_2} \mathfrak{sl}_2). According to \cite{15}, the discussion in \S 2.2 (or \cite{11}, Chap. I, \S 1, the discussion following Definition 1.8), there exist canonical injection and surjection
\begin{equation}
\zeta^\sharp : \Omega_{X/S} \hookrightarrow \text{Ad}(E), \quad \zeta^\flat : \text{Ad}(E) \twoheadrightarrow \mathcal{T}_{X/S}
\end{equation}(i.e., \nabla^\sharp and \nabla^\flat defined in \cite{15}, \S 2.2) satisfying the equalities \text{Im}(\zeta^\flat) = \mathcal{E}_B \times^B \mathfrak{n} and \text{Ker}(\zeta^\flat) = \mathcal{E}_B \times^B \mathfrak{b}, where \mathfrak{b}, \mathfrak{n} (\subseteq \mathfrak{sl}_2) denote the Lie algebras of \textit{B}, \text{[B, B]} (\subseteq \text{PGL}_2) respectively. In particular, if we set
\begin{align*}
\text{Ad}(E)^0 & := \text{Ad}(E), \\
\text{Ad}(E)^1 & := \text{Ker}(\zeta^\flat), \\
\text{Ad}(E)^2 & := \text{Im}(\zeta^\flat), \\
\text{Ad}(E)^3 & := 0,
\end{align*}
then $\{\text{Ad}(\mathcal{E})^j\}_{j=0}^3$ forms a 3-step decreasing filtration on $\text{Ad}(\mathcal{E})$ by subbundles whose subquotients are line bundles with $\text{Ad}(\mathcal{E})^j/\text{Ad}(\mathcal{E})^{j+1} \cong \Omega^{\otimes (j-1)}_{X/S}$ ($j = 0, 1, 2$).

1.4. Moduli of indigenous bundles.

Denote by

$$
\mathcal{S}_{g,R}
$$

the moduli stack classifying collections of data $(X, \mathcal{E}, \nabla_{\mathcal{E}})$ consisting of a curve $X$ of genus $g$ over $R$ and an indigenous bundle $(\mathcal{E}, \nabla_{\mathcal{E}})$ on it. By forgetting the data of an indigenous bundle, we obtain a projection $\mathcal{S}_{g,R} \to \mathcal{M}_{g,R}$. According to [11], Chap. I, §2, Corollary 2.9 (or [15], Proposition 2.7), $\mathcal{S}_{g,R}$ admits canonically the structure of a relative affine space over $\mathcal{M}_{g,R}$ modeled on $f_{g,R}^{*}(\Omega_{C_{g,R}/\mathcal{M}_{g,R}}^{\otimes 2})$ (i.e., modeled on $T^{\vee}_{R/\mathcal{M}_{g,R}}$ under isomorphism (23)) that is compatible with base-change over $R$. In particular, $\mathcal{S}_{g,R}$ is a geometrically connected smooth Deligne-Mumford stack over $R$ of relative dimension $6g - 6$.

Next, we shall write

$$
\hat{\mathcal{S}}_{g,p}
$$

for the $p$-adic formal stack over $\mathbb{Z}_p$ defined as $\hat{\mathcal{S}}_{g,p} := \lim_{\longrightarrow n} \mathcal{S}_{g,p^n}$. By an indigenous bundle on a $p$-adic formal curve $X := \lim_{\longrightarrow n} X_n$ (where $X_n := X \otimes (\mathbb{Z}/p^n\mathbb{Z})$), we shall mean a collection $((\mathcal{E}_n, \nabla_{\mathcal{E},n})_{n \geq 1}$, where each $(\mathcal{E}_n, \nabla_{\mathcal{E},n})$ denotes an indigenous bundle on $X_n$ such that $(\mathcal{E}_m, \nabla_{\mathcal{E},m})|_{X_n} \cong (\mathcal{E}_n, \nabla_{\mathcal{E},n})$ (if $n < m$). Then, $\hat{\mathcal{S}}_{g,p}$ may be identified with the moduli classifying $p$-adic formal curves over $\mathbb{Z}_p$ together with an indigenous bundle on it. Moreover, the affine space structures on $\mathcal{S}_{g,p^n}$’s carry the structure of a relative affine space over $\hat{\mathcal{M}}_{g,p}$ modeled on $T^{\vee}_{Z_{\mathbb{Z}_p}}\hat{\mathcal{M}}_{g,p}$.

Let $f : X \to S$ be as in §1.3 and $(\mathcal{E}, \nabla_{\mathcal{E}})$ an indigenous bundle on $X/S$. The collection $(X, \mathcal{E}, \nabla_{\mathcal{E}})$ determines its classifying morphism $S \to \mathcal{S}_{g,R}$. Let us consider the complex of sheaves $K^*|_{\nabla_{\mathcal{E}}}$ on $X$. It follows from [11], Chap. I, §2, Theorem 2.8 (or [15], §2.2), that

$$
\mathbb{R}^0 f^*_s(K^*|_{\nabla_{\mathcal{E}}}) = \mathbb{R}^2 f^*_s(K^*|_{\nabla_{\mathcal{E}}}) = 0
$$

and the sequence

$$
0 \longrightarrow f^*_s(\Omega^{\otimes 2}_{X/S}) \overset{\xi^f}{\longrightarrow} \mathbb{R}^1 f^*_s(K^*|_{\nabla_{\mathcal{E}}}) \overset{\xi^p}{\longrightarrow} \mathbb{R}^1 f^*_s(T_{X/S}) \longrightarrow 0
$$

is exact, where $\xi^f$ and $\xi^p$ denote the morphisms arising from $\text{id}_{\Omega^{\otimes 2}_{X/S}} \otimes \xi^f : \Omega^{\otimes 2}_{X/S} \to \Omega^{\otimes 2}_{X/S} \otimes \text{Ad}(\mathcal{E})$ and $\xi^p : \text{Ad}(\mathcal{E}) \to T_{X/S}$ respectively. In particular, $\mathbb{R}^1 f^*_s(K^*|_{\nabla_{\mathcal{E}}})$ is a vector bundle on $S$ of rank $6g - 6$. Moreover, according to [15], Proposition 2.8.1, there exists a canonical isomorphism

$$
\mathcal{T}_{S_{g,R}/S} \cong \mathbb{R}^1 f^*_s(K^*|_{\nabla_{\mathcal{E}}})
$$

of $\mathcal{O}_S$-modules fitting into the following isomorphism of short exact sequences:

$$
0 \longrightarrow \mathcal{T}_{S_{g,R}/\mathcal{M}_{g,R}/S} \longrightarrow \mathcal{T}_{S_{g,R}/S} \longrightarrow \mathcal{T}_{\mathcal{M}_{g,R}/S} \longrightarrow 0
$$

$$
0 \longrightarrow f^*_s(\Omega^{\otimes 2}_{X/S}) \overset{\xi^f}{\longrightarrow} \mathbb{R}^1 f^*_s(K^*|_{\nabla_{\mathcal{E}}}) \overset{\xi^p}{\longrightarrow} \mathbb{R}^1 f^*_s(T_{X/S}) \longrightarrow 0,
$$

of $\mathcal{O}_S$-modules fitting into the following isomorphism of short exact sequences:
where the left-hand vertical arrow arises from the affine structure on \( S_{g,R} \) mentioned above and the upper horizontal sequence is obtained by differentiating the projection \( S_{g,R} \to M_{g,R} \).

1.5. Symplectic structure on the moduli of indigenous bundles.

Next, we shall construct a canonical symplectic structure on \( S_{g,R} \). Let \( f : X \to S, (\mathcal{E}, \nabla_\mathcal{E}) \) be as above. Recall that the Killing form on the Lie algebra \( \mathfrak{sl}_2 \) (defined over \( R \)) is a nondegenerate symmetric bilinear map \( \kappa : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to R \) defined by \( \kappa(a, b) = \frac{1}{4} \cdot \text{tr}(\text{ad}(a) \cdot \text{ad}(b)) \) (where \( \text{tr} \) denotes the trace) for any \( a, b \in \mathfrak{sl}_2 \). By the change of structure group via \( \kappa \), the PGL_2-torsor \( \mathcal{E} \) induces a symmetric \( \mathcal{O}_X \)-bilinear morphism \( \kappa(\mathcal{E}, \nabla_\mathcal{E}) : \text{Ad}(\mathcal{E}) \otimes \text{Ad}(\mathcal{E}) \to \mathcal{O}_X \), which is nondegenerate, i.e., the associated morphism

\[
\kappa^2(\mathcal{E}, \nabla_\mathcal{E}) : \text{Ad}(\mathcal{E}) \to \text{Ad}(\mathcal{E})^\vee
\]

is an isomorphism. Let us write \( (42) \) as above. Recall that the Killing form on the Lie algebra \( \mathfrak{sl}_2 \) is nondegenerate, i.e., the associated morphism \( (41) \) induced naturally by \( \nabla_\mathcal{E} \) for any \( \mathcal{E} \). The morphism \( \kappa(\mathcal{E}, \nabla_\mathcal{E}) \) is compatible with the respective connections \( \nabla^{\mathcal{E} \otimes \mathcal{E}} \) and \( d \). By composing \( \kappa(\mathcal{E}, \nabla_\mathcal{E}) \) and the cup product in the de Rham cohomology, we obtain a skew-symmetric \( \mathcal{O}_S \)-bilinear morphism on \( \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \):

\[
\oint_{X,(\mathcal{E},\nabla_\mathcal{E})} : \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \otimes \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \xrightarrow{\kappa(\mathcal{E}, \nabla_\mathcal{E})} \mathbb{R}^2 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad\otimes 2}]) \xrightarrow{f_X} \mathbb{R}^2 f_*(\mathcal{K}^\bullet[d]) \xrightarrow{f_X} \mathcal{O}_S.
\]

Denote by

\[
\oint^2_{X,(\mathcal{E},\nabla_\mathcal{E})} : \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \to \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}])^\vee
\]

the morphism induced by \( \oint_{X,(\mathcal{E},\nabla_\mathcal{E})} \); i.e., the morphism determined by the condition that \( \oint^2_{X,(\mathcal{E},\nabla_\mathcal{E})}(a \otimes b) = (\oint^2_{X,(\mathcal{E},\nabla_\mathcal{E})} a)(b) \) for any local sections \( a, b \in \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \).

**Proposition 1.5.1.**

The morphism \( \oint^2_{X,(\mathcal{E},\nabla_\mathcal{E})} \) fits into the following morphism of short exact sequences:

\[
0 \longrightarrow f_*(\Omega^{\mathcal{E} \otimes \mathcal{E}}) \xrightarrow{\xi} \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \xrightarrow{\xi^\vee} \mathbb{R}^1 f_*(\mathcal{T}_{X/S}) \longrightarrow 0
\]

\[
0 \longrightarrow \mathbb{R}^1 f_*(\mathcal{T}_{X/S})^\vee \xrightarrow{(\xi^\vee)^\vee} \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}])^\vee \xrightarrow{(\xi^\vee)^\vee} f_*(\Omega^{\mathcal{E} \otimes \mathcal{E}})^\vee \longrightarrow 0.
\]

In particular, \( \text{Im}(\xi^\vee) \) (\( \subseteq \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \)) is isotropic with respect to \( \oint^2_{X,(\mathcal{E},\nabla_\mathcal{E})} \).

**Proof.** Note that \( \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \) may be, locally on \( S \), described as the total cohomology of the \( \check{C} \)ech double complex \( \check{C}^\bullet(\mathcal{U}, \mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}]) \) (for an affine open covering \( \mathcal{U} := \{U_a\}_a \) of \( X \)) associated to \( \mathcal{K}^\bullet[\nabla_\mathcal{E}^{ad}] \). Since \( \kappa(\mathcal{E}, \nabla_\mathcal{E})|_{\text{Ad}(\mathcal{E})^2 \otimes \text{Ad}(\mathcal{E})^1} = 0 \) (because of the definition of \( \{\text{Ad}(\mathcal{E})^j\}_{j=0}^3 \)), this
explicit description of $\mathbb{R}^1 f_\ast(\mathcal{K}^\bullet[\nabla^\text{ad}])$ implies that $\text{Im}(\xi^2) \ (= \text{Ker}(\xi^3))$ is isotropic with respect to $\xi$. In particular, we obtain two morphisms
\begin{align}
\text{Im}(\xi^2) \to (\mathbb{R}^1 f_\ast(\mathcal{K}^\bullet[\nabla^\text{ad}]))^\vee,
\mathbb{R}^1 f_\ast(\mathcal{K}^\bullet[\nabla^\text{ad}]))/\text{Im}(\xi^2) \to \text{Im}(\xi^3)^\vee
\end{align}
arising as a restriction and a quotient of $\xi$ respectively. Moreover, the following diagram is verified to be commutative:
\begin{align}
\text{Ad}(\mathcal{E})^2 \times (\text{Ad}(\mathcal{E})/\text{Ad}(\mathcal{E})^1) \xrightarrow{(\zeta^2) \times \zeta^b} \Omega_{X/S} \times T_{X/S}
\end{align}
where
- $(-, -)$ denotes the natural paring $\Omega_{X/S} \times T_{X/S} \to \mathcal{O}_X$;
- $\zeta^2 : \Omega_{X/S} \xrightarrow{\sim} \text{Ad}(\mathcal{E})^2$ and $\zeta^b : \text{Ad}(\mathcal{E})/\text{Ad}(\mathcal{E})^1 \xrightarrow{\sim} T_{X/S}$ denote the isomorphisms induced naturally by $\zeta^2$ and $\zeta^b$ respectively;
- $f_{(\xi, \nabla^\xi)}$ denotes the morphism $\text{Ad}(\mathcal{E})^2 \times (\text{Ad}(\mathcal{E})/\text{Ad}(\mathcal{E})^1) \to \mathcal{O}_X$ induced by $\kappa_{(\xi, \nabla^\xi)}$.

This implies that, under the identifications
\begin{align}
f_\ast(\mathcal{O}^{(2)}_{X/S}) \xrightarrow{\sim} \text{Im}(\xi^2), \quad \mathbb{R}^1 f_\ast(\mathcal{K}^\bullet[\nabla^\text{ad}]))/\text{Im}(\xi^2) \xrightarrow{\sim} \mathbb{R}^1 f_\ast(T_{X/S})
\end{align}
determined by $\xi^2$, $\zeta^b$ respectively, the isomorphisms in (46) coincide with $\xi$ and $\xi^\vee$ respectively. This completes the proof of the assertion.

By considering the $\mathcal{O}_S$-bilinear maps $\xi$ (together with the isomorphism (39) for various schemes $S$ over $\mathcal{S}_{g,R}$, we obtain a 2-form
\begin{align}
\omega^{\text{PGL}}_{g,R} \in \Gamma(\mathcal{S}_{g,R}, \bigwedge^2 \Omega_{\mathcal{S}_{g,R}/R}).
\end{align}
This 2-form is, by construction, compatible with base-change over $R$. Moreover, it follows from [15], Proposition 4.2.2, that $\omega^{\text{PGL}}_{g,R}$ specifies a symplectic structure on $\mathcal{S}_{g,R}$. Thus, the collection
\begin{align}
\hat{\omega}^{\text{PGL}}_{g,\mathbb{Z}_p} := (\omega^{\text{PGL}}_{g,\mathbb{Z}/p^n\mathbb{Z}})_{n \geq 1} \in \Gamma(\hat{\mathcal{S}}_{g,\mathbb{Z}_p} \bigwedge^2 \Omega_{\hat{\mathcal{S}}_{g,\mathbb{Z}_p}/\mathbb{Z}_p})
\end{align}
specifies a symplectic structure on $\hat{\mathcal{S}}_{g,\mathbb{Z}_p}$. One verifies immediately that the fiber of the projection $\hat{\mathcal{S}}_{g,\mathbb{Z}_p} \to \hat{\mathcal{M}}_{g,\mathbb{Z}_p}$ over each point in $\hat{\mathcal{M}}_{g,\mathbb{Z}_p}(\mathbb{Z}_p)$ is Lagrangian.

1.6. Ordinary nilpotent indigenous bundles.

Now, we shall consider certain indigenous bundles in characteristic $p$ playing central roles in the (classical) $p$-adic Teichmüller theory. Let $S$ be an $\mathbb{F}_p$-scheme and $f : X \to S$ a curve of genus $g$ over $S$. Write $\Phi_f : S \to S$ for the absolute Frobenius morphism of $S$, $f^{(1)} : X^{(1)} \to S$ for the Frobenius twist of $X$ relative to $S$, and $\Phi_{X/S} : X \to X^{(1)}$ for the relative Frobenius morphism (cf. §4.2). Also, let us fix an indigenous bundle $(\mathcal{E}, \nabla^\xi)$ on $X/S$. The connection $\nabla^\xi$ determines its $p$-curvature $\psi_{(\xi, \nabla^\xi)} : \Phi_{X/S}^*(\nabla_{X^{(1)}/S}) \to \text{Ad}(\mathcal{E})$ (cf. §4.2).
Recall that \((\mathcal{E}, \nabla_{\mathcal{E}})\) is called nilpotent (cf. [11], Chap. II, §2, Definition 2.4) if the composite
\[
\Phi_{X/S}(T_{X(1)/S}) \xrightarrow{\psi(\mathcal{E}, \nabla_{\mathcal{E}})} \text{Ad}(\mathcal{E}) \xrightarrow{\kappa^2(\mathcal{E}, \nabla_{\mathcal{E}})} \text{Ad}(\mathcal{E}) \xrightarrow{\psi(\mathcal{E}, \nabla_{\mathcal{E}})} \Phi_{X/S}(T_{X(1)/S})^\vee
\]
coincides with the zero map. In particular, if \((\mathcal{E}, \nabla_{\mathcal{E}})\) is nilpotent in the sense discussed in §4.2 and hence, corresponds to a crystal of \(\text{PGL}_2\)-torsors over the crystalline site \(\text{Crys}(X/S)\) (cf. Remark 4.3.2 (ii) and Theorem 4.4.2).

Next, let us consider the composite
\[
\Phi_{X/S}(T_{X(1)/S}) \xrightarrow{\psi(\mathcal{E}, \nabla_{\mathcal{E}})} \text{Ad}(\mathcal{E}) \xrightarrow{\kappa^2(\mathcal{E}, \nabla_{\mathcal{E}})} \text{Ad}(\mathcal{E}) \xrightarrow{\psi(\mathcal{E}, \nabla_{\mathcal{E}})} \Phi_{X/S}(T_{X(1)/S})^\vee
\]
(cf. [11] for the definition of \(\psi(\mathcal{E}, \nabla_{\mathcal{E}})\)). By applying the functor \(\mathbb{R}^1 f_*(-)\) to this composite, we obtain an \(\mathcal{O}_S\)-linear morphism
\[
\Phi_{X/S}^\vee(\mathbb{R}^1 f_*(T_{X/S})) \xrightarrow{\kappa^2(\mathcal{E}, \nabla_{\mathcal{E}})} \mathbb{R}^1 f_!(T_{X(1)/S}) \xrightarrow{\psi(\mathcal{E}, \nabla_{\mathcal{E}})} \Phi_{X/S}^\vee(\mathbb{R}^1 f_*(T_{X/S})).
\]

Then, recall that \((\mathcal{E}, \nabla_{\mathcal{E}})\) is called ordinary (cf. [11], Chap. II, §3, Definition 3.1) if the morphism (53) is an isomorphism.

Denote by
\[
\mathcal{N}^\text{ord}_{g, \mathbb{F}_p}
\]
the substack of \(\mathcal{S}_{g, \mathbb{F}_p}\) classifying ordinary nilpotent indigenous bundles. It follows from [11], Chap. II, §3, Corollary 3.8, that \(\mathcal{N}^\text{ord}_{g, \mathbb{F}_p}\) is a nonempty smooth Deligne-Mumford stack over \(\mathbb{F}_p\) and the projection \(\mathcal{N}^\text{ord}_{g, \mathbb{F}_p} \to \mathcal{M}_{g, \mathbb{F}_p}\) is étale and quasi-finite (and hence, since \(\mathcal{M}_{g, \mathbb{F}_p}\) is irreducible, it is dominant when restricted to each component of \(\mathcal{N}^\text{ord}_{g, \mathbb{F}_p}\)). Therefore, there exists a unique (up to isomorphism) smooth \(p\)-adic formal stack
\[
\widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p}
\]
whose reduction modulo \(p\) is \(\mathcal{N}^\text{ord}_{g, \mathbb{F}_p}\). Let us recall here the following assertion, which is one of the main results in [11].

**Theorem 1.6.1** (cf. [11], Chap. III, §2, Theorem 2.8).

Denote by \(f_{\mathcal{N}} : C_{\mathcal{N}} \to \widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p}\) the universal family of curves over \(\widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p}\). Then, there exists a canonical endomorphism
\[
\Phi_{\mathcal{N}} : \widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p} \to \widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p}
\]
of \(\widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p}\) together with a canonical indigenous bundle \((\mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{E}_{\mathcal{N}}})\) on \(C_{\mathcal{N}}/\widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p}\) satisfying the following properties:

- \(\Phi_{\mathcal{N}}\) is a Frobenius lifting over \(\mathbb{Z}_p\) (i.e., the reduction modulo \(p\) of \(\Phi_{\mathcal{N}}\) coincides with the absolute Frobenius morphism of \(\widehat{\mathcal{N}}^\text{ord}_{g, \mathbb{Z}_p}\));
- The reduction modulo \(p\) of \((\mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{E}_{\mathcal{N}}})\) is isomorphic to the indigenous bundle on \((C_{\mathcal{N}} \otimes \mathbb{F}_p)/\mathcal{N}^\text{ord}_{g, \mathbb{F}_p}\) classified by the natural immersion \(\mathcal{N}^\text{ord}_{g, \mathbb{F}_p} \hookrightarrow \mathcal{S}_{g, \mathbb{F}_p}\);
- There exists an isomorphism \(\mathbb{F}^* \Phi_{\mathcal{N}}(\mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{E}_{\mathcal{N}}}) \xrightarrow{\sim} (\mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{E}_{\mathcal{N}}})\) of indigenous bundles, where \(\mathbb{F}^*(-)\) denotes renormalized Frobenius pull-back (cf. [11], Chap. III, §2, Definition 2.4, or §2.4 in the present paper) and \(\Phi_{\mathcal{N}}^*(-)\) denotes base-change by \(\Phi_{\mathcal{N}}\).
Moreover, the collection of data \((\Phi_N, \mathcal{E}_N, \nabla_{\mathcal{E}_N})\) is uniquely characterized (up to isomorphism) by the above properties.

1.7. Statement of the main theorem.

In this subsection, we shall describe the main theorem in the present paper. In what follows, we shall write \(\mathcal{N} := \tilde{\mathcal{N}}_{\text{ord}}^g, \mathbb{Z}_p\) for simplicity. The indigenous bundle \((\mathcal{E}_N, \nabla_{\mathcal{E}_N})\) obtained in Theorem 1.6.1 determines its classifying morphism

\[
\sigma : \mathcal{N} \rightarrow \hat{S}_{g, \mathbb{Z}_p}
\]

over \(\mathcal{M}_{g, \mathbb{Z}_p}\), which turns out to be an immersion; it gives, after base-change by \(\mathcal{N} \rightarrow \mathcal{M}_{g, \mathbb{Z}_p}\), a trivialization of the affine space structure on \(\hat{S}_{g, \mathbb{Z}_p}\) (modeled on \(T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}\)). More precisely, there exists a unique isomorphism

\[
\theta : T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}|_\mathcal{N} \rightarrow \hat{S}_{g, \mathbb{Z}_p}|_\mathcal{N} =: \hat{S}_{g, \mathbb{Z}_p} \times_{\mathcal{M}_{g, \mathbb{Z}_p}} \mathcal{N}
\]

which extends \(\sigma\) and is compatible with the affine space structures pulled-back from \(T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}\) and \(\hat{S}_{g, \mathbb{Z}_p}\) respectively. It induces an isomorphism

\[
\theta^*(\Omega_{\hat{S}_{g, \mathbb{Z}_p}|_\mathcal{N}/\mathbb{Z}_p}) \sim \Omega_{T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}|_\mathcal{N}/\mathbb{Z}_p},
\]

and hence, an isomorphism

\[
\Theta : \theta^*(\bigwedge^2 \Omega_{\hat{S}_{g, \mathbb{Z}_p}|_\mathcal{N}/\mathbb{Z}_p}) \sim \bigwedge^2 \theta^*(\Omega_{\hat{S}_{g, \mathbb{Z}_p}|_\mathcal{N}/\mathbb{Z}_p}) \rightarrow \bigwedge^2 \Omega_{T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}|_\mathcal{N}/\mathbb{Z}_p}.
\]

Since \(\mathcal{N}\) is étale over \(\mathcal{M}_{g, \mathbb{Z}_p}\), the projections \(T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}|_\mathcal{N} \rightarrow T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}\) and \(\hat{S}_{g, \mathbb{Z}_p}|_\mathcal{N} \rightarrow \hat{S}_{g, \mathbb{Z}_p}\) are étale. Therefore, the 2-form

\[
\hat{\omega}^{\text{Liou}}_{g, \mathbb{Z}_p}|_\mathcal{N} \quad \text{(resp., } \hat{\omega}^{\text{PGL}}_{g, \mathbb{Z}_p}|_\mathcal{N}\text{)}
\]

on \(T^\vee_{\mathbb{Z}_p} \mathcal{M}_{g, \mathbb{Z}_p}|_\mathcal{N}\) (resp., \(\hat{S}_{g, \mathbb{Z}_p}|_\mathcal{N}\)) defined as the pull-back of \(\hat{\omega}^{\text{Liou}}_{g, \mathbb{Z}_p}\) (resp., \(\hat{\omega}^{\text{PGL}}_{g, \mathbb{Z}_p}\)) specifies a symplectic structure. (Notice that \(\hat{\omega}^{\text{Liou}}_{g, \mathbb{Z}_p}|_\mathcal{N} = \omega^{\text{Liou}}_{\mathcal{N}}\).) The main result of the present paper is the following Theorem 1.7.1 which describes the relationship between \(\hat{\omega}^{\text{Liou}}_{g, \mathbb{Z}_p}|_\mathcal{N}\) and \(\hat{\omega}^{\text{PGL}}_{g, \mathbb{Z}_p}|_\mathcal{N}\). (The proof will be given in \(\S 3.2\))

**Theorem 1.7.1** (= Theorem A).

*If \(p > 3\), then the morphism \(\theta\) preserves the symplectic structures, i.e., the following equality holds:*

\[
\Theta(\hat{\omega}^{\text{PGL}}_{g, \mathbb{Z}_p}|_\mathcal{N}) = \hat{\omega}^{\text{Liou}}_{g, \mathbb{Z}_p}|_\mathcal{N}.
\]

*In particular, the image of \(\sigma : \mathcal{N} \rightarrow \hat{S}_{g, \mathbb{Z}_p}\) is Lagrangian with respect to the symplectic structure \(\hat{\omega}^{\text{PGL}}_{g, \mathbb{Z}_p}\).*
2. F-crystals associated to ordinary nilpotent indigenous bundles

Before proving Theorem 1.7.1 we shall study, in this section, a certain F-crystal structure (cf. (70)) on the cohomology associated to the adjoint bundle of an ordinary nilpotent indigenous bundle. One important observation is (cf. Corollary 2.5.1) that the direct sum decomposition of this cohomology determined by (the differential of) \( \sigma : \tilde{\mathcal{H}}_{g,\mathbb{Z}_p} \to \tilde{S}_{g,\mathbb{Z}_p} \) coincides with (i.e., gives the geometric interpretation of) the slope decomposition. It follows (cf. Corollary 2.3.2 and (129)) that both \( \tilde{\mathcal{H}}_{g,\mathbb{Z}_p}\lvert_{\mathcal{N}} \) and \( \tilde{\mathcal{S}}_{g,\mathbb{Z}_p}\lvert_{\mathcal{N}} \) turn out to specify eigenvectors of the F-crystal structure defined on the second exterior power of the dual \( \mathbb{H}^1(K^\bullet[\nabla_{\mathcal{E}}])^\vee \). This fact, being an essential point of our proof of Theorem 1.7.1 makes it clear how the two symplectic structures are related via reduction modulo \( p \).

2.1. Renormalized Frobenius pull-back.

First, let us recall (cf. [11], Chap. III, § 2, the discussion preceding Definition 2.4) the definition of renormalized Frobenius pull-back, that appeared in the statement of Theorem 1.6.1. In what follows, we shall denote, for each positive integer \( m \), the reductions of objects over \( \mathbb{Z}_p \) to \( \mathbb{Z}/p^m\mathbb{Z} \) by means of a subscripted \( m \). Let \( S \) be a \( p \)-adic formal scheme. Also, let \( X \) and \( Y \) be curves of genus \( g \) over \( S \) such that \( Y \) is a \( p \)-adic lifting of \( X^{(1)}_1 \) \( := (X_1)^{(1)} = (X^{(1)})_1 \).

We shall fix a flat \( \text{PGL}_2 \)-torsor \( (\mathcal{E}, \nabla_{\mathcal{E}}) \) over \( Y/S \) whose reduction modulo \( p \) (i.e., \( (\mathcal{E}_1, \nabla_{\mathcal{E},1}) \)) forms a nilpotent indigenous bundle on \( X^{(1)}_1 \) \( := Y_1/S_1 \). Let \( n \) be an integer with \( n > 1 \), and assume tentatively that there exists a rank 2 flat vector bundle \( (\mathcal{V}_n, \nabla_{\mathcal{V},n}) \) on \( Y_n/S_n \) (i.e., a pair of a rank 2 vector bundle \( \mathcal{V}_n \) on \( Y_n \) and an \( S_n \)-connection \( \nabla_{\mathcal{V},n} \) on \( \mathcal{V}_n \)) whose projectivization is isomorphic to \( (\mathcal{E}_n, \nabla_{\mathcal{E},n}) \). Denote by \( \mathcal{L}_1 \) the line subbundle of \( \mathcal{V}_1 \) \( := (\mathcal{V}_n)_1 \) corresponding to the Hodge reduction of the indigenous bundle \( (\mathcal{E}_1, \nabla_{\mathcal{E},1}) \). Since \( \nabla_{\mathcal{V},1} \) has nilpotent \( p \)-curvature, \( (\mathcal{V}_n, \nabla_{\mathcal{V},n}) \) corresponds to a crystal \( \mathcal{V}_n^\phi \) of vector bundles on the crystalline site \( \text{Crys}(X^{(1)}_1/S_n) \). Moreover, it induces a crystal \( \Phi_{X_1/S_1}^*(\mathcal{V}_n^\phi) \) on \( \text{Crys}(X_1/S_n) \) defined as the pull-back of \( \mathcal{V}_n^\phi \) via the relative Frobenius \( \Phi_{X_1/S_1} \). One may obtain a crystal \( \mathcal{F}^*(\mathcal{V}_n^\phi) \) defined as the subsheaf of \( \Phi_{X_1/S_1}^*(\mathcal{V}_n^\phi) \) consisting of sections whose reduction modulo \( p \) are contained in the subsheaf \( \Phi_{X_1/S_1}^*(\mathcal{L}_1) \) \( \subset \Phi_{X_1/S_1}^*(\mathcal{V}_1) \). If \( \nabla' \) denotes the \( S_n \)-connection on the \( \mathcal{O}_{X_n} \)-module \( \mathcal{F}^*(\mathcal{V}_n^\phi)_{X_n} \) (i.e., the evaluation of \( \mathcal{F}^*(\mathcal{V}_n^\phi) \) at \( X_n \)) corresponding to this crystal, then its reduction \( \mathcal{F}^*(\mathcal{V}_n^\phi)_{X_{n-1}, \nabla'_{n-1}} \) modulo \( p^{n-1} \) turns out to form a flat vector bundle on \( X_{n-1}/S_{n-1} \). We shall write

\[
(63) \quad \mathcal{F}^*(\mathcal{V}_n^\phi)
\]

for the crystal of vector bundles on \( \text{Crys}(X_1/S_{n-1}) \) corresponding to this flat vector bundle. The (isomorphism class of the) flat \( \text{PGL}_2 \)-torsor over \( X_{n-1}/S_{n-1} \) defined as the projectivization of (the flat vector bundle corresponding to) \( \mathcal{F}^*(\mathcal{V}_n^\phi) \) is independent of the choice of \( (\mathcal{V}_n, \nabla_{\mathcal{V},n}) \) (i.e., depends only on \( (\mathcal{E}_n, \nabla_{\mathcal{E},n}) \)); thus it makes sense to use the notation \( \mathcal{F}^*(\mathcal{E}_n, \nabla_{\mathcal{E},n}) \) to denote this flat \( \text{PGL}_2 \)-torsor. Moreover, the independence of the choice \( (\mathcal{V}_n, \nabla_{\mathcal{V},n}) \) implies that we can construct \( \mathcal{F}^*(\mathcal{E}_n, \nabla_{\mathcal{E},n}) \) without the existence assumption of \( (\mathcal{V}_n, \nabla_{\mathcal{V},n}) \) imposed above. By applying the above argument to all \( n \), we obtain a flat \( \text{PGL}_2 \)-torsor

\[
(64) \quad \mathcal{F}^*(\mathcal{E}, \nabla_{\mathcal{E}})
\]
over $X/S$, which we call the renormalized Frobenius pull-back of $(\mathcal{E}, \nabla_{\mathcal{E}})$. Denote by

$$F^*(\mathcal{E}, \nabla_{\mathcal{E}})^{\Diamond}$$

the crystal over Crys$(X_1/S)$ (i.e., the compatible system consisting of crystals over Crys$(X_1/S_n)$ for various $n$) corresponding to $F^*(\mathcal{E}, \nabla_{\mathcal{E}})$. Note that (the isomorphism class of) $F^*(\mathcal{E}, \nabla_{\mathcal{E}})^{\Diamond}$ does not depend on the choice of the $p$-adic lifting $X$ of $X_1$.

2.2. Frobenius structure on the cohomology of the adjoint bundle.

In this subsection, we shall construct a Frobenius structure on the cohomology associated with the adjoint bundle of each ordinary nilpotent indigenous bundle. Let $S$, $\Phi_S$, and $X$ be as in the previous subsection, and let us keep some notational convention as needed (e.g., $(-)_n$ and $(-)^{(1)}$, etc.). Assume that $S$ is endowed with a morphism $S \to \mathcal{N}$ via which $\Phi_S$ is compatible with $\Phi_X$. Denote by $(\mathcal{E}, \nabla_{\mathcal{E}})$ the ordinary nilpotent indigenous bundle on $X/S$ classified by this morphism. Since the reduction modulo $p$ of $\nabla_{\mathcal{E}}$ (as well as $\nabla_{\mathcal{E},n}^{ad}$) has nilpotent $p$-curvature, the flat vector bundle $(\text{Ad}(\mathcal{E}_n), \nabla_{\mathcal{E},n}^{ad})$ determines a crystal $\text{Ad}(\mathcal{E}_n)^{\Diamond}$ on Crys$(X_1/S_n)$. In particular, we obtain the relative crystalline cohomology sheaf $\mathbb{R}^1 f_{\text{crys}}(\text{Ad}(\mathcal{E}_n)^{\Diamond})$ on $S_n$ associated to $\text{Ad}(\mathcal{E}_n)^{\Diamond}$, and hence, obtain the $\mathcal{O}_T$-module

$$\mathbb{R}^1 f_{\text{crys}}(\text{Ad}(\mathcal{E}_n)^{\Diamond}) := \lim_{\rightarrow n} \mathbb{R}^1 f_{\text{crys}}(\text{Ad}(\mathcal{E}_n)^{\Diamond}).$$

Suppose tentatively that there exists a crystal $\mathcal{V}_{n}^{\Diamond}$ of rank 2 vector bundles on Crys$(X_1/S_n)$ whose projectivization corresponds to $(\mathcal{E}_n, \nabla_{\mathcal{E},n})$. Write $\Phi_S^{\ast}(\mathcal{E}_n, \nabla_{\mathcal{E},n})$ (resp., $\Phi_S^{\ast}(\mathcal{V}_{n}^{\Diamond})$) for the base-change of $(\mathcal{E}_n, \nabla_{\mathcal{E},n})$ (resp., $\mathcal{V}_{n}^{\Diamond}$) by $\Phi_{S,n}: S_n \to S_n$, which forms an indigenous bundle on $X_n^{(1)}/S_n$ (resp., a crystal on Crys$(X_1^{(1)}/S_n)$). Here, notice that $\text{Ad}(\mathcal{E}_n)^{\Diamond}$ may be identified with the crystal which assigns to each $(U \hookrightarrow T, \delta)$ in Crys$(X_1/S_n)$, the sheaf $\mathcal{E}\text{nd}^{0}(\mathcal{V}_{n,T}^{\Diamond})$ of $\mathcal{O}_T$-linear endomorphisms of $\mathcal{V}_{n,T}$, with vanishing trace.

Now, let us take a divided power thickening $(U \hookrightarrow T, \delta)$ in Crys$(X_1/S_n)$ and an $\mathcal{O}_T$-linear endomorphism $h$ of $\mathcal{V}_{n,T}$, with vanishing trace (i.e., a global section of $\mathcal{E}\text{nd}^{0}(\mathcal{V}_{n,T}^{\Diamond})$). After possibly replacing $U$ with its open covering, we suppose that $T$ admits an endomorphism $\Phi_T: T \to T$ compatible with $\Phi_{S,n}$ whose reduction modulo $p$ coincides with the absolute Frobenius $\Phi_U$. The endomorphism $\Phi_T^{\ast}(h)$ of $\Phi_T^{\ast}(\mathcal{V}_{n,T}^{\Diamond})$ restricts to an endomorphism of $\widehat{\mathbb{F}}^{\ast}(\Phi_T^{\ast}(\mathcal{V}_{n,T}^{\Diamond}))_{T} \subseteq \Phi_T^{\ast}(\mathcal{V}_{n,T}^{\Diamond})_{T} = F_{X_1/S_1}(\Phi_T^{\ast}(\mathcal{V}_{n,T}^{\Diamond}))_{T}$; we shall denote its reduction modulo $p^{n-1}$ by $\mathbb{F}^{\ast}(h)$, which lies in $\mathcal{E}\text{nd}^{0}(\mathbb{F}^{\ast}(\Phi_T^{\ast}(\mathcal{V}_{n,T}^{\Diamond})))_{T}$ (where $T := T_{n-1}$). Since the assignment $\mathbb{F}^{\ast}(-)$ is compatible with base-change over the parameter spaces of underlying families of curves, it follows from Theorem I.6.1 (and the assumption that $\Phi_S$ is compatible with $\Phi_X$) that $\mathbb{F}^{\ast}(\Phi_T^{\ast}(\mathcal{E}, \nabla_{\mathcal{E}}))$ is isomorphic to $(\mathcal{E}, \nabla_{\mathcal{E}})$. Hence, $\mathbb{F}^{\ast}(\Phi_T^{\ast}(\mathcal{V}_{n,T}^{\Diamond}))_{T}$ is isomorphic to $\mathcal{V}_{n,T}^{\Diamond}$ up to tensoring with a line bundle, and $\mathbb{F}^{\ast}(h)$ determines a well-defined section of $\mathcal{E}\text{nd}^{0}(\mathcal{V}_{n,T}^{\Diamond})$ (resp., $\text{Ad}(\mathcal{E}_{n-1}^{\Diamond})$). Given a basis $(e_1, e_2)$ of $\mathcal{V}_{n,T}^{\Diamond}$ such that $e_1$ mod $p$ generates $\mathcal{L}_1$, we can describe locally $\mathbb{F}^{\ast}(h)$ by means of this basis. Indeed, if the matrix representation of $h$ with respect to the basis $(e_1, e_2)$ is of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then $\mathbb{F}^{\ast}(h)$ may be expressed as $\begin{pmatrix} p \cdot \Phi_T^{\ast}(a) & p^2 \cdot \Phi_T^{\ast}(b) \\ \Phi_T^{\ast}(c) & -p \cdot \Phi_T^{\ast}(a) \end{pmatrix} \pmod{p^{n-1}}$ with respect to the basis $(\Phi_T^{\ast}(e_1), p \cdot \Phi_T^{\ast}(e_2)) \pmod{p^{n-1}}.$
Denote by
\begin{equation}
(\Phi_{X_1})_{\text{crys}} : (X_1/S_{n-1})_{\text{crys}} \to (X_1/S_n)_{\text{crys}}
\end{equation}
the morphism of topoi induced by the absolute Frobenius endomorphism \(\Phi_{X_1}\) of \(X_1\) covering the PD morphism \(\Phi_{S_{n-1}|S_{n-1}} : S_{n-1} \to S_n\). Then, the assignment \(h \mapsto \mathbb{P}^n(h)\) (for each \((U \to T, \delta)\) and \(h\) as above) determines a morphism of crystals
\begin{equation}
\text{Ad}(\mathbb{P}^n) : \text{Ad}(\mathcal{E}_n) \to (\Phi_{X_1})_{\text{crys}}(\text{Ad}(\mathcal{E}_{n-1})).
\end{equation}
It induces a \(\Phi_{S_{n-1}}\)-linear morphism \(\mathbb{P}^n f_{n,\text{crys}}(\text{Ad}(\mathcal{E}_n)) \to \mathbb{P}^n f_{n-1,\text{crys}}(\text{Ad}(\mathcal{E}_{n-1}))\), or equivalently, an \(\mathcal{O}_{S_n}\)-linear morphism
\begin{equation}
F_{(\mathcal{E}_n, \nabla_{\mathcal{E}_n})}^\circ : \Phi_{S_n}^*(\mathbb{P}^n f_{n,\text{crys}}(\text{Ad}(\mathcal{E}_n))) \to \mathbb{P}^n f_{n-1,\text{crys}}(\text{Ad}(\mathcal{E}_{n-1})).
\end{equation}
One verifies immediately that this morphism is independent of the choice of \(V_n^\circ\) (i.e., depends only on \((\mathcal{E}_n, \nabla_{\mathcal{E}_n})\)), which implies that we can remove the existence assumption of \(V_n^\circ\) imposed above. By applying the above argument to all \(n\), we obtain from \(F_{(\mathcal{E}_n, \nabla_{\mathcal{E}_n})}^\circ\)’s an \(\mathcal{O}_{S}\)-linear morphism
\begin{equation}
F_{(\mathcal{E}, \nabla_{\mathcal{E}})}^\circ : \Phi_{S}^*(\mathbb{P}^n f_{n,\text{crys}}(\text{Ad}(\mathcal{E}))) \to \mathbb{P}^n f_{n-1,\text{crys}}(\text{Ad}(\mathcal{E})).
\end{equation}

Here, recall from [3], Theorem 7.1, that there exists a canonical isomorphism
\begin{equation}
\mathbb{P}^n f_{n,\text{crys}}(\text{Ad}(\mathcal{E})) \cong \mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}}])
\end{equation}
of \(\mathcal{O}_{S_n}\)-modules. By taking the inverse limit \(\lim_{n} (-)\), we obtain an isomorphism
\begin{equation}
\mathbb{P}^n f_{n,\text{crys}}(\text{Ad}(\mathcal{E})) \cong \mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}}]) \left( := \lim_{n} \mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}}]) \right).
\end{equation}
The isomorphism \(F_{(\mathcal{E}, \nabla_{\mathcal{E}})}^\circ\) becomes, via (72), an \(\mathcal{O}_{S}\)-linear morphism
\begin{equation}
F_{(\mathcal{E}, \nabla_{\mathcal{E}})} : \Phi_{S}^*(\mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}}])) \to \mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}}]).
\end{equation}
In particular, we apply this construction of \(F_{(\mathcal{E}, \nabla_{\mathcal{E}})}\) to the universal case (i.e., the case where the collection \((S, \Phi_{S}, \mathcal{E}, \nabla_{\mathcal{E}})\) is taken as \((\mathcal{N}, \Phi_{\mathcal{N}}, \mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{E}_{\mathcal{N}}})\)), and obtain an \(\mathcal{O}_{\mathcal{N}}\)-linear morphism
\begin{equation}
F : \Phi_{\mathcal{N}}^*(\mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}_{\mathcal{N}}}]))) \to \mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}_{\mathcal{N}}}]).
\end{equation}

2.3. Relationship between \(f_{X, (\mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{N}})}\) and \(F\).

By the following assertion, we can see that the morphism \(F\) defined above is compatible, up to multiplication by “\(p^3\)”, with the bilinear morphism \(f_{X, (\mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{N}})}\).

**Proposition 2.3.1.**

*The following square diagram is commutative:*

\begin{equation}
\begin{array}{ccc}
\Phi_{\mathcal{N}}^*(\mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}_{\mathcal{N}}}]))) \otimes \Phi_{\mathcal{N}}^*(\mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}}])) & \xrightarrow{F \otimes F} & \Phi_{\mathcal{N}}^*(\mathcal{O}_{\mathcal{N}}) \cong \mathcal{O}_{\mathcal{N}} \\
\downarrow{F \otimes F} & & \downarrow{[p^3]} \\
\mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}_{\mathcal{N}}}])) \otimes \mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}}]) & \xrightarrow{f_{X, (\mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{N}})}} & \mathbb{P}^n f_{n*}(\mathcal{K}^* [\nabla_{\mathcal{E}_{\mathcal{N}}}]))) \end{array}
\end{equation}
where the right-hand vertical arrow $[p^3]$ denotes multiplication by $p^3$.

Proof. Let us keep the notation in §2.2. The Killing form $\kappa$ on $\mathfrak{sl}_2$ induces, for each $n$, a morphism of crystals $\kappa_{(\mathcal{E}_n, \nabla_{\mathcal{E}_n})} : \mathrm{Ad}^{\diamond}(\mathcal{E}_n) \times \mathrm{Ad}(\mathcal{E}_n) \to \mathcal{O}_{X_1/S_n}$. Moreover, $\kappa_{(\mathcal{E}_n, \nabla_{\mathcal{E}_n})}$ induces an $\mathcal{O}_{S_n}$-bilinear morphism

$$
\int \mathcal{O}_{X_n, (\mathcal{E}_n, \nabla_{\mathcal{E}_n})} : \mathbb{R}^1 f_{n, \text{crys}}(\mathrm{Ad}(\mathcal{E}_n)^{\diamond}) \otimes \mathbb{R}^1 f_{n, \text{crys}}(\mathrm{Ad}(\mathcal{E}_n)^{\diamond}) \to \mathcal{O}_{S_n},
$$

which is compatible with $\int \mathcal{O}_{X_n, (\mathcal{E}_n, \nabla_{\mathcal{E}_n})}$ via the isomorphism (71). Also, by taking account of the local description of $\mathbb{R}^*(\cdot)$ discussed in §2.2, we see that the following square diagram is commutative:

(77)

$$
\begin{array}{ccc}
\mathrm{Ad}(\mathcal{E}_n)^{\diamond} \times \mathrm{Ad}(\mathcal{E}_n)^{\diamond} & \xrightarrow{\kappa_{(\mathcal{E}_n, \nabla_{\mathcal{E}_n})}} & \mathcal{O}_{X_1/S_n} \\
\mathrm{Ad}(\mathbb{P}^*) \times \mathrm{Ad}(\mathbb{P}^*) & \downarrow & \\
(\Phi_{X_1})_{\text{crys}}(\mathrm{Ad}(\mathcal{E}_n)^{\diamond}) \times (\Phi_{X_1})_{\text{crys}}(\mathrm{Ad}(\mathcal{E}_n-1)^{\diamond}) & \xrightarrow{(\Phi_{X_1})_{\text{crys}}(\kappa_{(\mathcal{E}_n-1, \nabla_{\mathcal{E}_n-1})})} & (\Phi_{X_1})_{\text{crys}}(\mathcal{O}_{X_1/S_n})
\end{array}
$$

(cf. (67) for the definition of $(\Phi_{X_1})_{\text{crys}}$), where the right-hand vertical arrow $p^2 \cdot \Phi_{X_1/S_1}$ denotes $p^2$ times the morphism $\Phi_{X_1/S_1} : \mathcal{O}_{X_1/S_n} \to (\Phi_{X_1})_{\text{crys}}(\mathcal{O}_{X_1/S_n-1})$ induced by $\Phi_{X_1/S_1}$. Here, recall from [3], Theorem 6.12, that we have

(78)

$$
\mathbb{R}^2 f_{1, \text{crys}}(\mathcal{O}_{X_1/S_m}) \sim \left(\mathbb{R}^2 f_{m*}(\mathcal{K}^*[d]) \sim \mathcal{O}_{S_m}\right)
$$

($m = 1, 2, \cdots$). The morphism

(79)

$$
\Phi_{S,n}^{*} (\mathbb{R}^2 f_{1, \text{crys}}(\mathcal{O}_{X_1/S_n})) \to \mathbb{R}^2 f_{1, \text{crys}}(\mathcal{O}_{X_1/S_{n-1}})
$$

induced by $F_{X_1/S_1} : \mathcal{O}_{X_1/S_n} \to (\Phi_{X_1})_{\text{crys}}(\mathcal{O}_{X_1/S_{n-1}})$ coincide, via (78), with the composite of the natural quotient $\Phi_{S,n}^{*} (\mathcal{O}_{S_n}) (\cong \mathcal{O}_{S_n}) \to \mathcal{O}_{S_{n-1}}$ and multiplication by $p$ (cf. [3], Chap.VII, §3, Proposition 3.2.4). Hence, the diagram (77) gives rise to a commutative diagram of the form

(80)

$$
\begin{array}{ccc}
\Phi_{S,n}^{*} (\mathbb{R}^1 f_{1, \text{crys}}(\mathrm{Ad}(\mathcal{E}_n)^{\diamond})) \otimes \Phi_{S,n}^{*} (\mathbb{R}^1 f_{1, \text{crys}}(\mathrm{Ad}(\mathcal{E}_n)^{\diamond})) & \xrightarrow{\Phi_{S,n}^{*} (f_{X_n, (\mathcal{E}_n, \nabla_{\mathcal{E}_n})})} & \Phi_{S,n}^{*} (\mathcal{O}_{S_n}) (\cong \mathcal{O}_{S_n}) \\
\mathbb{R}^1 f_{1, \text{crys}}(\mathrm{Ad}(\mathcal{E}_n-1)^{\diamond}) \otimes \mathbb{R}^1 f_{1, \text{crys}}(\mathrm{Ad}(\mathcal{E}_n-1)^{\diamond}) & \xrightarrow{f_{X_{n-1}, (\mathcal{E}_n-1, \nabla_{\mathcal{E}_n-1})}} & \mathcal{O}_{S_{n-1}}.
\end{array}
$$

The diagram (76) may be obtained, via (12), as the inverse limit (over $n$) of the diagrams (80) in the universal case (i.e., the case where the collection $(S, X, \mathcal{E}, \nabla_{\mathcal{E}})$ is taken to be $(\mathcal{N}, C_{\mathcal{N}}, \mathcal{E}_{\mathcal{N}}, \nabla_{\mathcal{E}_{\mathcal{N}}})$). This implies the required commutativity, and completes the proof of the assertion. □
In what follows, we shall give a restatement of the above proposition. By passing the isomorphism $\mathcal{T}_{S,g,z_p}|_N \sim \mathbb{R}^1 f_{N*}(\mathcal{K}^*[\nabla^{ad}_{E_N}])$ (cf. (39) in the universal case over $S = \mathcal{N}$), we obtain, from $F$, an $\mathcal{O}_N$-linear morphism

$$F^{\mathbb{P}G L} : \bigwedge^2 \Omega_{S,g,z_p}/z_p|_N \rightarrow \left( \bigwedge^2 \Phi_N^*(\Omega_{S,g,z_p}/z_p|_N) \right) \approx \Phi_N^*(\bigwedge^2 \Omega_{S,g,z_p}/z_p|_N).$$

Let us consider $\Gamma(N, \bigwedge^2 \Omega_{S,g,z_p}/z_p|_N)$ as a submodule of $\Gamma(N, \Phi_N^*(\bigwedge^2 \Omega_{S,g,z_p}/z_p|_N))$ via pull-back by $\Phi_N$. Then, Proposition 2.3.1 implies the following assertion, which is essential to complete our proof of Theorem 1.7.1 (Theorem 2.4.1 described in the next subsection may be thought of as another essential ingredient of the proof.)

**Corollary 2.3.2.**
The following equality holds:

$$F^{\mathbb{P}G L}([\omega]_{S,g,z_p}|_N) = p^3 \cdot [\omega]_{S,g,z_p}|_N.$$

2.4. **Slope decomposition of** $(\mathbb{R}^1 f_{N*}(\mathcal{K}^*[\nabla^{ad}_{E_N}]), F)$.

Consider the composite isomorphism

$$\mathcal{T}_{N/z_p} \tilde{\rightarrow} \mathcal{T}_{\bar{M}_{g,z_p}/z_p}|_N \rightarrow \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N}),$$

where the first arrow arises from the étaleness of $N/\bar{M}_{g,z_p}$. It induces the composite isomorphism

$$\Omega_{N/z_p} \left( = \mathcal{T}_{N/z_p}^\vee \right) \rightarrow \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N})^\vee \rightarrow f_{N*}(\Omega^2_{C_N/N}).$$

Denote by

$$\Upsilon : \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N}) \oplus f_{N*}(\Omega^2_{C_N/N}) \tilde{\rightarrow} \mathbb{R}^1 f_{N*}(\mathcal{K}^*[\nabla^{ad}_{E_N}])$$

the unique isomorphism making the following diagram commute:

$$\begin{align*}
0_N^*(\mathcal{T}_{S,g,z_p}/z_p|_N) & \xrightarrow{\mathbb{L}} \mathcal{T}_{N/z_p} \oplus \Omega_{N/z_p} \xrightarrow{\mathbb{L}} \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N}) \oplus f_{N*}(\Omega^2_{C_N/N}) \\
& \xrightarrow{\text{d}\theta|_{0_N}} \mathcal{T}_{S,g,z_p}/z_p|_N \xrightarrow{\sim} \mathbb{R}^1 f_{N*}(\mathcal{K}^*[\nabla^{ad}_{E_N}]),
\end{align*}$$

where the left-hand vertical arrow $\text{d}\theta|_{0_N}$ denotes the differential of $\theta$ at $0_N$ (i.e., the dual of (59) restricted to $0_N$). That is to say, the direct sum decomposition $\Upsilon$ arises from the classical ordinary $p$-adic Teichmüller uniformization (cf. (37) or Introduction).

We shall denote by

$$F^z := p^2 \cdot \Phi_N : \Phi_N^*(f_{N*}(\Omega^2_{C_N/N})) \rightarrow f_{N*}(\Omega^2_{C_N/N})$$

the morphism defined as $p^2$ times the morphism $\Phi_N^* : \Phi_N^*(\Omega_{N/z_p}) \rightarrow \Omega_{N/z_p}$ induced naturally by $\Phi_N$ under the identification $\Omega_{N/z_p} \sim f_{N*}(\Omega^2_{C_N/N})$ (cf. (34)). Here, notice that since the
reduction modulo \( p \) of \( \Phi_N \) coincides with the Frobenius endomorphism, \( \Phi_N^* \) is divisible by \( p \). According to [11], Chap. III, § 2, Proposition 2.3, the morphism \( \frac{1}{p} \cdot \Phi_N^* \) (i.e., \( \Phi_N^* \) divided by \( p \)) is an isomorphism. Thus, we obtain a morphism
\[
F^\circ : \Phi_N^*(\mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N})) \rightarrow \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N})
\]
defined to be the inverse to the dual of \( \frac{1}{p} \cdot \Phi_N^* \) under the identification \( \mathcal{T}_{N/\mathbb{Z}_p} \rightarrow \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N}) \) (cf. (83)). The Frobenius structure \( F \) on \( \mathbb{R}^1 f_{N*}(\mathcal{K}{\bullet}^{[\nabla_{\Xi_N}]}) \) will turn out to be compatible with the Frobenius structures \( F^g, F^\circ \), as described below. (The proof will be given in §3.1)

**Theorem 2.4.1.**

(i) Let us consider the short exact sequence
\[
0 \rightarrow f_{N*}(\Omega_{C_N/N}^{\otimes 2}) \xrightarrow{\xi_N^*} \mathbb{R}^1 f_{N*}(\mathcal{K}{\bullet}^{[\nabla_{\Xi_N}]}) \xrightarrow{\xi_N^*} \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N}) \rightarrow 0
\]
defined as the inverse limit (over \( n \geq 1 \)) of the sequence (38) of the case where \((\mathcal{E}, \nabla_\mathcal{E})\) is taken to be \((\mathcal{E}_{N,n}, \nabla_{\Xi_N,n})\). Then, the morphisms \( F^\circ, F^g \), and \( F \) are compatible with the morphisms in this short exact sequence. More precisely, the following diagram is commutative:
\[
\begin{array}{ccc}
\Phi_N^*(f_{N*}(\Omega_{C_N/N}^{\otimes 2})) & \xrightarrow{\Phi_N^*(\xi_N^*)} & \Phi_N^*(\mathbb{R}^1 f_{N*}(\mathcal{K}{\bullet}^{[\nabla_{\Xi_N}]}) ) \\
0 \rightarrow f_{N*}(\Omega_{C_N/N}^{\otimes 2}) & \xrightarrow{\xi_N^*} & \mathbb{R}^1 f_{N*}(\mathcal{K}{\bullet}^{[\nabla_{\Xi_N}]}) \xrightarrow{\xi_N^*} \mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N})) \\
\end{array}
\]

(ii) The direct sum decomposition \( \Upsilon \) (cf. (39)) is compatible with \( F^\circ \oplus F^g \) and \( F \). More precisely, the following square diagram is commutative:
\[
\begin{array}{ccc}
\Phi_N^*(\mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N})) & \oplus & \Phi_N^*(f_{N*}(\Omega_{C_N/N}^{\otimes 2})) \\
\mathbb{R}^1 f_{N*}(\mathcal{T}_{C_N/N}) & \oplus & \mathbb{R}^1 f_{N*}(\Omega_{C_N/N}^{\otimes 2}) \\
\end{array}
\]

2.5. \( F \)-crystals associated to canonical liftings of indigenous bundles.

In this subsection, we shall describe a consequence of the above theorem obtained by restricting \( \mathbb{R}^1 f_{N*}(\mathcal{K}{\bullet}^{[\nabla_{\Xi_N}]}) \) to the fiber over each Frobenius invariant point in \( N \). Let \( k \) be an algebraically closed field of characteristic \( p \). Write \( W \) for the ring of Witt vectors over \( k \) and write \( \Phi_W \) for the absolute Frobenius automorphism of \( \text{Spf}(W) \). Let us take an arbitrary \( k \)-rational point \( s_1 \in N_{g_{\Xi, p}}(k) \). Then, there exists a canonical lifting \( s_\infty : \text{Spf}(W) \rightarrow N \) of \( s_1 \) characterized uniquely by the equality \( \Phi_N^* \circ s_\infty = s_\infty \circ \Phi_W \) (cf. [11], Chap. III, the discussion preceding Definition 1.9). The point \( s_\infty \) classifies a curve \( X \) over \( W \) and an indigenous bundle \((\mathcal{E}, \nabla_{\Xi})\) on it. Denote by
\[
H^1(\mathcal{K}{\bullet}^{[\nabla_{\Xi}]}) = \lim_{\rightarrow} H^1(\mathcal{K}{\bullet}^{[\nabla_{\Xi_n}]})
\]
the first hypercohomology associated to the complex $K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack$ (which may be obtained by restricting $\mathbb{R}^1 f_{N^*}(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack)$ to $s_{\infty}$). The isomorphism $\Upsilon$ restricts to a direct sum decomposition

\begin{equation}
\Upsilon_{(\mathcal{E}, \nabla_{\mathcal{E}})} \colon := s_{\infty}^*(\Upsilon) : H^1(X, T_{X/W}) \oplus \Gamma(X, \Omega^2_{X/W}) \xrightarrow{\sim} \mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack) .
\end{equation}

Next, consider the following sequence of isomorphisms

\begin{equation}
s_{\infty}^*(\Phi_N^*(\mathbb{R}^1 f_{N*}(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack))) \xrightarrow{\sim} (\Phi_N \circ s_{\infty})^*(\mathbb{R}^1 f_{N*}(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack))
\xrightarrow{\sim} (s_{\infty} \circ \Phi_W)^*(\mathbb{R}^1 f_{N*}(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack))
\xrightarrow{\sim} \Phi_W^*(s_{\infty}^*(\mathbb{R}^1 f_{N*}(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack))).
\end{equation}

By means of this composite isomorphism, the morphism $F$ restricts to a $W$-linear morphism

\begin{equation}
F_{(\mathcal{E}, \nabla_{\mathcal{E}})} \colon := s_{\infty}^*(F) : \Phi_W^*(\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack)) \to \mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack).
\end{equation}

Also, the morphism $F^0$ (resp., $F^3$) restricts to a $W$-morphism

\begin{equation}
F^0_{(\mathcal{E}, \nabla_{\mathcal{E}})} : \Phi_W^*(H^1(X, T_{X/W})) \to H^1(X, T_{X/W})
\end{equation}

\begin{equation}
\text{resp., } F^3_{(\mathcal{E}, \nabla_{\mathcal{E}})} : \Phi_W^*(\Gamma(X, \Omega^2_{X/W})) \to \Gamma(X, \Omega^2_{X/W}),
\end{equation}

for which the pair

\begin{equation}
(H^1(X, T_{X/W}), F^0_{(\mathcal{E}, \nabla_{\mathcal{E}})}) \text{ (resp., } (\Gamma(X, \Omega^2_{X/W}), F^3_{(\mathcal{E}, \nabla_{\mathcal{E}})}))
\end{equation}

forms an isoclinic $F$-crystal over $k$ of rank $3g - 3$ and its unique Newton slope is $0$ (resp., $3$). Then, Theorem 2.4.1 asserted above implies the following assertion, which gives a geometric interpretation of the slope decomposition on $\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack)$ with respect to $F_{(\mathcal{E}, \nabla_{\mathcal{E}})}$.

**Corollary 2.5.1.**

The pair

\begin{equation}
(\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack), F_{(\mathcal{E}, \nabla_{\mathcal{E}})})
\end{equation}

forms an $F$-crystal over $k$ of rank $6g - 6$ and all its Newton slopes are $0$ and $3$. Moreover, the following equalities hold:

\begin{equation}
\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack)^{F=p^0} = \Upsilon_{(\mathcal{E}, \nabla_{\mathcal{E}})}(H^1(X, T_{X/W})),
\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack)^{F=p^3} = \Upsilon_{(\mathcal{E}, \nabla_{\mathcal{E}})}(\Gamma(X, \Omega^2_{X/W})),
\end{equation}

where $\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack)^{F=p^m}$ (for an integer $m$) denotes the isoclinic component of $\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack)$ of slope $m$ (with respect to $F_{(\mathcal{E}, \nabla_{\mathcal{E}})}$). In particular, $\Upsilon_{(\mathcal{E}, \nabla_{\mathcal{E}})}$ may be regarded as the slope decomposition of $(\mathbb{H}^1(K^\bullet \lbrack \nabla_{\mathcal{E}}^{\text{ad}} \rbrack), F_{(\mathcal{E}, \nabla_{\mathcal{E}})})$.

**Remark 2.5.2.**

In this remark, we shall describe $F^0_{(\mathcal{E}, \nabla_{\mathcal{E}})}$ in terms of deformations of $(\mathcal{E}, \nabla_{\mathcal{E}})$. Let us keep the above notation. Denote by $T_{N^*/\mathbb{Z}_p, s_{\infty}}$ the tangent space of $N$ (over $\mathbb{Z}_p$) at $s_{\infty}$, which may be identified with the deformation space of $(\mathcal{E}, \nabla_{\mathcal{E}})$ over $W_{\epsilon} := W[\epsilon]/(\epsilon^2)$. Let us consider the differential

\begin{equation}
d\Phi_N|_{s_{\infty}} : T_{N^*/\mathbb{Z}_p, s_{\infty}} \to (\Phi_N^*(T_{N^*/\mathbb{Z}_p, s_{\infty}}) =) \Phi_W^*(T_{N^*/\mathbb{Z}_p, s_{\infty}})
\end{equation}
of ΦN at s∞. Under the identification \( T_N/\mathbb{Z}_p, s_\infty \cong H^1(X, T_X/\mathbb{Z}) \) obtained by restricting (83), the equality \( (F^p_{(E, V)} = \frac{1}{p} \cdot d\Phi_N|_{s_\infty} \) holds. In particular, we have

\[
\text{Im}(d\Phi_N|_{s_\infty}) = p \cdot \Phi_W(T_N/\mathbb{Z}_p, s_\infty) = \Phi_W(pT_N/\mathbb{Z}_p, s_\infty) \quad (\subseteq \Phi_W(T_N/\mathbb{Z}_p, s_\infty)),
\]

and the morphism \( T_N/\mathbb{Z}_p, s_\infty \to \Phi_W(pT_N/\mathbb{Z}_p, s_\infty) \) induced by \( d\Phi_N|_{s_\infty} \) is an isomorphism.

Now, let us take an element \( v \in T_N/\mathbb{Z}_p, s_\infty \) (resp., \( v' \in pT_N/\mathbb{Z}_p, s_\infty \)). By the above discussion, one may find a unique \( \tilde{v} \in T_N/\mathbb{Z}_p, s_\infty \) with \( d\Phi_N|_{s_\infty}(\tilde{v}) = v' \), or equivalently, \( \Phi_N \circ \tilde{v} = v' \circ \Phi_{\mathbb{C}_p} \), where \( \Phi_{\mathbb{C}_p} \) denotes the base-change of \( \Phi \) over \( \mathbb{C}_p \). Denote by \( (X^v_\mathbb{C}, E^v_\mathbb{C}, V^v_\mathbb{C}) \), \( (X^v_\mathbb{C}, E^v_\mathbb{C}, V^v_\mathbb{C}) \), and \( (X^v_\mathbb{C}, E^v_\mathbb{C}, V^v_\mathbb{C}) \) the collections classified by \( v \), \( v' \), and \( \tilde{v} \) respectively. In particular, \( (X^v_\mathbb{C})_1 \cong (X^v_\mathbb{C})_1^{(1)} \). Then, we have the following sequence of isomorphisms over \( X^v_\mathbb{C} \):

\[
\text{(E)}^v_{\mathbb{C}} \cong \tilde{v}^* (E_N, V^v_\mathbb{C}) \equiv \tilde{v}^* (\mathbb{F}^* (\Phi_N^*(E_N, V^v_\mathbb{C}))) \equiv \mathbb{F}^* ((v' \circ \Phi_{\mathbb{C}_p})^*(E_N, V^v_\mathbb{C})) \equiv \mathbb{F}^* (\Phi_{\mathbb{C}_p}^*(E_N, V^v_\mathbb{C})).
\]

It follows consequently that \( (p \cdot (F^p_{(E, V)} = \frac{1}{p} \cdot d\Phi_N|_{s_\infty}(v) = v' \text{ (or equivalently, } v = \tilde{v}) \text{ if and only if } (E^v_\mathbb{C}, V^v_\mathbb{C}) \cong \mathbb{F}^* (\Phi_{\mathbb{C}_p}^*(E^v_\mathbb{C}, V^v_\mathbb{C})).
\]

2.6. Relationship between \( p \)-curvature and the differential of \( \sigma \).

In this subsection, we shall prove a proposition (cf. Proposition 2.6.1 asserted below), which describes the reduction modulo \( p \) of the differential of \( \sigma \) by means of \( p \)-curvature. That proposition will be used in the proof of Theorem 1.7.1 (cf. the proof of Lemma 3.2.1).

Let \( k \) be as before and \( s_1 \) a \( k \)-rational point of \( N^\text{ord}_{g, \mathbb{F}_p} \) and \( (X, E, V) \) the data classified by \( s_1 \). The morphism \( \psi_{(E, V)}^{(\text{ord})} \) (cf. (11)) induces a morphism of complexes \( \Phi_{X/k}^{-1} (T_{X^1/k}[0] \to K^\bullet [V^\text{ad}] \) (cf. §4.5 for the definition of \((-)[0]) \). By applying the functor \( \mathbb{H}^1(-) \) to this morphism, we obtain a morphism

\[
\mathbb{H}^1(\psi_{(E, V)}^{(\text{ord})}) : H^1(X^{(1)}, T_{X^1/k}) \to \mathbb{H}^1(K^\bullet[V^\text{ad}]).
\]

On the other hand, let us consider the composite

\[
\text{d} \sigma|_{s_1} : H^1(X, T_{X/k}) \hookrightarrow H^1(X, T_{X/k}) \oplus \Gamma(X, \Omega^\otimes_{X/k}) \twoheadrightarrow \mathbb{H}^1(K^\bullet[V^\text{ad}])),
\]

where the first arrow denotes the inclusion into the first factor and the second arrow denotes the restriction to \( s_1 \) of the direct sum decomposition (83). It may also be obtained as the differential of the immersion \( N^\text{ord}_{g, \mathbb{F}_p} \to S_{g, \mathbb{F}_p} \) at \( s_1 \) under the identifications (82) and (the reduction modulo \( p \)) of (83). Then, the following lemma holds.
Proposition 2.6.1.

The following square diagram is commutative:

\[ \Phi^*_k(H^1(X, T_{X/k})) \xrightarrow{\sim} H^1(X^{(1)}, T_{X^{(1)}/k}) \]

\[ -F^\flat_{(E, V_E)} \downarrow \iota \quad \xrightarrow{\iota} \quad H^1(\psi^\vee_{(E, V_E)}) \]

\[ H^1(X, T_{X/k}) \xrightarrow{d\sigma|_{s_1}} H^1(\mathcal{K}^\bullet[\nabla_{E}^{\text{ad}}]), \]

where \( \varsigma \) denotes the natural isomorphism induced by \( \text{id}_X \times \Phi_k : X^{(1)} \to X \) and \( F^\flat_{(E, V_E)} \) denotes the restriction of \( F^\flat \) to \( s_1 \). In particular, \( H^1(\psi^\vee_{(E, V_E),1}) \) is injective.

Proof. By definition, the composite

\[ \xi^\flat \circ H^1(\psi^\vee_{(E, V_E)}) \circ \varsigma : \Phi^*_k(H^1(X, T_{X/k})) \to H^1(X, T_{X/k}) \]

coincides with the morphism \( (53) \) in the present case. Hence, it follows from \[11\], Chap. III, §2, Proposition 2.3, that

\[ \xi^\flat \circ (H^1(\psi^\vee_{(E, V_E),1}) \circ \varsigma) = -F^\flat_{(E, V_E),1} \left( = -\xi^\flat \circ (d\sigma|_{s_1} \circ F^\flat_{(E, V_E),1}) \right). \]

This implies that, to complete the proof of the assertion, it suffices to prove the equality

\[ \text{Im}(H^1(\psi^\vee_{(E, V_E)})) = \text{Im}(d\sigma|_{s_1}). \]

First, let \( v \) be an element of \( \text{Im}(H^1(\psi^\vee_{(E, V_E)})) \), and denote by \( (X^v, E^v, \nabla^v_{E,v}) \) the deformation over \( k_e := k[\varepsilon]/(\varepsilon^2) \) of \( (X, E, \nabla_E) \) corresponding to \( (\eta^\text{ad}_B)^{-1}(v) \in H^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E,B}]) \) (cf. \[17\]) for the definition of \( \eta^\text{ad}_B \). By the definition of \( \psi^\vee_{(E, V_E)} \), \( v \) may be represented as the data \( (156) \) with \( b_\alpha = 0 \) (for any \( \alpha \)). It follows from the construction of the bijection \( (152) \) that \( (E^v, \nabla^v_{E,v}) \) is, locally on \( X \), isomorphic to the trivial deformation. Hence, it has nilpotent \( p \)-curvature, and hence, forms a nilpotent indigenous bundle. According to \[11\], Chap. II, §2, Theorem 2.13, \( \mathcal{N}^{\text{ord}}_{g,F_p} \) coincides with the étale locus (relative to \( \mathcal{M}_{g,F_p} \)) in the substack of \( S_{g,F_p} \) classifying nilpotent indigenous bundles. This implies that the element \( v \), when considered as an element of \( S_{g,F_p}(k_e) \), lies in \( \mathcal{N}^{\text{ord}}_{g,F_p}(k_e) \). That is to say, \( v \) is contained in \( \text{Im}(d\sigma|_{s_1}) \).

Conversely, let \( u \) be an element of \( \text{Im}(d\sigma|_{s_1}) \) and denote by \( (X^u, E^u, \nabla^u_{E,v}) \) the deformation of \( (X, E, \nabla_E) \) over \( k_e \) determined by \( (\eta^\text{ad}_B)^{-1}(u) \in H^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E,B}]) \). Recall here the notion of the FL-(vector) bundle introduced in \[11\], Chap. II, §1, Definition 1.3. Denote by \( (G^u, \nabla_{G,u}) \) (resp., \( (G^v, \nabla_{G,v}) \)) the FL-bundle on \( X \) (resp., \( X^v \)) corresponding to \( (E, \nabla_E) \) (resp., \( (E^v, \nabla^v_{E,v}) \)) (cf. \[11\], Chap. II, §2, Proposition 2.5). The flat bundle \( (G^u, \nabla_{G,u}) \) forms an extension of \( (\mathcal{O}_{X^v}, d) \) (i.e., the trivial flat bundle) by \( (\Phi^\flat_{X^u/k_e}(T_{X^u(k^v)})/k_e), \nabla^\text{can}_{(X^u(k^v))} \) (cf. \[13\]) for the definition of \( \nabla^\text{can}_{(\cdot)} \).

Let us consider the short exact sequence

\[ 0 \to H^1((X^u)^{(1)}, T_{(X^u)^{(1)}/k_e}) \xrightarrow{\nu} H^1(\mathcal{K}^\bullet[\nabla^\text{can}_{(X^u)^{(1)}}]) \xrightarrow{\mu} \Gamma((X^u, \mathcal{O}_{X^u})/k_e = k_e) \to 0 \]

discussed in \[11\], Chap. II, §1, Proposition 1.1, in the case where the curve \( ^*X^{\text{log}}/S^{\text{log}} \) in \textit{loc. cit.} is taken to be \( X^u/k_e \); it is obtained by applying the functor \( H^1(\cdot) \) to the following short exact sequence of complexes:

\[ 0 \to \Phi^{-1}_{X^u/k_e}(T_{X^u(k^v)})[0] \to \mathcal{K}^\bullet[\nabla^\text{can}_{(X^u)^{(1)}}] \to \Phi^{-1}_{X^u/k_e}(\mathcal{O}_{(X^u)^{(1)}})[1] \to 0 \]
where the lower horizontal arrow is (109) (i.e., the upper horizontal sequence) with \( \Phi \) (cf. [13], Proposition 1.2.4 together with the discussion following that proposition). If \( \text{ex}(G^u, \nabla^u_G) \) denotes the extension class determined by \( (G^u, \nabla^u_G, \alpha) \), then it follows from the definition of an FL-bundle that \( \mu(\text{ex}(G^u, \nabla^u_G)) \in k^x_\epsilon \). Also, let us denote by \( \text{ex}(G^u, \nabla^u_G) \) the extension class determined by the trivial deformation \( (G^u, \nabla^u_G) \).

In what follows, we shall denote the base-changes to \( k_\epsilon \) of objects over \( k \) by means of a subscript \( \epsilon \). Let us take an affine open covering \( U = \{ U_\alpha \} \) of \( X \). In the Čech double complex \( \text{Tot}^*(\tilde{C}(U, K^*[[\nabla]])) \), the element \( u \) may be represented by \( \{ \{ a_{\alpha \beta} \}_{\alpha \beta}, \{ b_{\alpha} \}_\alpha \} \) as in (156). Since \( X^u_\alpha \) and \( E^u_\alpha \) may be obtained by gluing together \( U_{\alpha \epsilon} \)'s and \( E_{\alpha \epsilon} \)'s respectively, there exist natural isomorphisms \( \iota_{X, \alpha} : U_{\alpha \epsilon} \cong X^u_{\epsilon} \mid U_{\alpha \epsilon} \) and \( \iota_{E, \alpha} : E_{\alpha \epsilon} \cong E^u_{\epsilon} \mid U_{\alpha \epsilon} \) (for each \( \alpha \in I \)) respectively, whose reductions modulo \( \epsilon \) are the identity morphisms. Under the isomorphism \( \iota_{E, \alpha} \), the restriction \( (E^u_{\epsilon} \mid U_{\alpha \epsilon}, \nabla^u_{\epsilon} \mid U_{\alpha \epsilon}) \) may be identified with \( (E_{\alpha \epsilon} \mid U_{\alpha \epsilon}, \nabla_{\epsilon \alpha} + \epsilon \cdot b_{\alpha}) \).

Let us fix \( \alpha \in I \), and consider the following morphism of short exact sequences induced by restriction via \( U_{\alpha \epsilon} \xrightarrow{\iota_{E, \alpha}} X^u_{\epsilon} \mid U_{\alpha \epsilon} \xrightarrow{} X^u_{\epsilon} \):

\[
\begin{array}{c}
0 \rightarrow H^1((X^u_\alpha)^{(1)}, T(X^u_\alpha) \mid k_\epsilon) \xrightarrow{\nu} \bigoplus \left( \text{H}^1(K^*[[\nabla]](1)) \right) \xrightarrow{\mu} k_\epsilon \rightarrow 0 \\
0 \rightarrow H^1(U_{\alpha \epsilon}, T(U_{\alpha \epsilon}) \mid k_\epsilon) \xrightarrow{\nu_a} \bigoplus \left( \text{H}^1(K^*[[\nabla]](1)) \right) \xrightarrow{\mu_a} \Gamma(U_{\alpha \epsilon}^{(1)}, O_{U_{\alpha \epsilon}}^{(1)}) \rightarrow 0,
\end{array}
\]

where the lower horizontal arrow is (109) (i.e., the upper horizontal sequence) with \( X^u_\alpha \) and \( \nabla^u_\alpha \mid (X^u_\alpha) \) replaced by \( U_{\alpha \epsilon} \) and \( \nabla^u_{\alpha \epsilon} \mid U_{\alpha \epsilon} \) respectively. Since \( \nu_a = 0 \) (which implies that \( \mu_a \) is an isomorphism), \( k_\epsilon \left( \subseteq \Gamma(U_{\alpha \epsilon}^{(1)}, O_{U_{\alpha \epsilon}}^{(1)}) \right) \) may be thought, via \( \mu_a \), of as a submodule of \( \bigoplus \left( \text{H}^1(K^*[[\nabla]](1)) \right) \). The restriction \( \text{ex}(G^u, \nabla^u_G, \alpha) \mid U_{\alpha \epsilon} \) lies in \( k^x_\epsilon \). On the other hand, the restriction \( \text{ex}(G^u, \nabla^u_G, \alpha) \mid U_{\alpha \epsilon} \) of \( \text{ex}(G^u, \nabla^u_G) \) to \( U_{\alpha \epsilon} \) lies in \( k^x_\epsilon \). Hence, \( \text{ex}(G^u, \nabla^u_G, \alpha) \mid U_{\alpha \epsilon} \) and \( \text{ex}(G^u, \nabla^u_G) \mid U_{\alpha \epsilon} \) differ at most by a constant factor in \( k^x_\epsilon \), which implies that the restrictions \( (G^u_\alpha \mid U_{\epsilon \alpha}, \nabla^u_{\alpha \epsilon} \mid U_{\epsilon \alpha}) \) and \( (G^u_\alpha \mid U_{\epsilon \alpha}, \nabla^u_{\epsilon \alpha} \mid U_{\epsilon \alpha}) \) are isomorphic. The resulting isomorphism \( (\iota_{E, \alpha} \mid U_{\alpha \epsilon}, \nabla_{\alpha \epsilon} + \epsilon \cdot b_{\alpha}) \) may be expressed as \( \text{id}_{\epsilon \alpha} \mid U_{\alpha \epsilon} + \epsilon \cdot c_{\alpha} \) for some \( c_{\alpha} \in \Gamma(U_{\alpha \epsilon}, \text{Ad}(\mathcal{E})) \). By the definition of \( \nabla^u_G \), the equality \( \nabla^u_G(c_{\alpha}) = b_{\alpha} \) holds. Hence, if \( \nabla^u_{\text{Im}} \) denotes the morphism \( \text{Ad}(\mathcal{E}) \rightarrow \text{Im}(\nabla^u_G) \) obtained from \( \nabla^u_G \) by restricting its codomain, then \( \epsilon \) lies in the image of the morphism \( \bigoplus \left( \text{H}^1(K^*[[\nabla]](1)) \rightarrow \bigoplus \left( \text{H}^1(K^*[[\nabla]](1)) \right) \) induced by the natural injection \( K^*[[\nabla]](1) \rightarrow K^*[[\nabla]](1) \). Observe that the morphism \( H^1(X, \text{Ker}(\nabla^u_G)) \rightarrow \bigoplus \left( \text{H}^1(K^*[[\nabla]](1)) \right) \) induced by the natural morphism \( \text{Ker}(\nabla^u_G) \rightarrow K^*[[\nabla]](1) \) is an isomorphism. Moreover, it follows from [11], Chap. II, § 2, Proposition 2.7, that \( \psi^{\nabla}_{\epsilon, \alpha} \) restricts to an isomorphism \( \Phi^{-1}_{X/k}(T(X^{(1)}) \mid k) \cong \text{Ker}(\nabla^u_G) \), which induces an isomorphism \( H^1(X^{(1)}, T(X^{(1)}) \mid k) \cong \bigoplus \left( \text{H}^1(K^*[[\nabla]](1)) \right) \).
Since this composite is nothing but $H^1(\psi_{(E,\nabla^\varepsilon)}$, we have $u \in \text{Im}(H^1(\psi_{(E,\nabla^\varepsilon)}))$. This completes the proof of the assertion. □

3. Proofs of theorems

This section is devoted to prove Theorem 1.7.1 and Theorem 2.4.1 described earlier.

3.1. Proof of Theorem 2.4.1

First, we prove Theorem 2.4.1. Let $k$, $W$, $s_1$, $s_\infty$, $X$, and $(E, \nabla^\varepsilon)$ be as in §2.5. By considering various $s_1$, we see that, in order to prove assertions (i) and (ii), it suffices to verify the required commutativities of the diagrams (100), (101) restricted to $s_\infty \in N(W)$. Write $\pi_B : E_B \to X$ for the Hodge reduction of $(E, \nabla^\varepsilon)$, and write $\nabla^\text{ad}_{E_B} := \lim_n \nabla^\text{ad}_{E_B,n}$, $\nabla^\text{ad} := \lim_n \nabla^\text{ad}_n$. Denote by $\Phi_{W_*}$ the base-change of $\Phi_W$ over $W_\varepsilon := W[\varepsilon]/(\varepsilon^2)$.

Let $(E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon})$ be a deformation of $(E, \nabla^\varepsilon)$ over $X_\varepsilon := X \times_W W_\varepsilon$ classified by $pH^1(\mathcal{K}^\bullet[\nabla^\text{ad}])$ ($\subseteq H^1(\mathcal{K}^\bullet[\nabla^\text{ad}]))$. Then, $(\text{since } F^\ast(\Phi_W^* (E, \nabla^\varepsilon)) \cong (E, \nabla^\varepsilon))$ the flat PGL$_2$-torsor $F^\ast(\Phi_W^* (E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon}))$ over $X_\varepsilon/W_\varepsilon$ forms a deformation of $(E, \nabla^\varepsilon)$, i.e., specifies an element of $H^1(\mathcal{K}^\bullet[\nabla^\text{ad}])$. The assignment $(E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon}) \mapsto F^\ast(\Phi_W^* (E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon}))$ defines a $W$-linear map

$$\phi^\varepsilon : \Phi_W^* (\mathcal{K}^\bullet[\nabla^\text{ad}]) \to H^1(\mathcal{K}^\bullet[\nabla^\text{ad}]).$$

On the other hand, let $(X_\varepsilon^1, E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon})$ be the deformation of $(X, E, \nabla^\varepsilon)$ over $W_\varepsilon$ classified by $pH^1(\mathcal{K}^\bullet[\nabla^\text{ad}])$ ($\subseteq H^1(\mathcal{K}^\bullet[\nabla^\text{ad}]))$. According to Corollary 1.6.2 there exists a unique deformation $(X_\varepsilon^1, E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon})$ classified by $H^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E_B}])$ such that $(E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon}) \cong F^\ast(\Phi_W^* (E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon}))_{X_\varepsilon^1}$. The assignment $(X_\varepsilon^1, E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon}) \mapsto (X_\varepsilon^1, E_\varepsilon^1, \nabla^\text{ad}_{E_\varepsilon})$ defines a $W$-linear map

$$\phi_{E_B} : \Phi_W^* (\mathcal{K}^\bullet[\nabla^\text{ad}_{E_B}]) \to H^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E_B}]).$$

By construction, the following square diagram is commutative:

$$\begin{array}{ccc}
\Phi_W^* (pH^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E_B}])) & \xrightarrow{\phi_{E_B}} & H^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E_B}]) \\
\eta^\text{ad}_{E_B} \downarrow & & \downarrow \eta^\text{ad}_{E_B} \\
\Phi_W^* (pH^1(\mathcal{K}^\bullet[\nabla^\text{ad}])) & \xrightarrow{\phi^\varepsilon} & H^1(\mathcal{K}^\bullet[\nabla^\text{ad}]).
\end{array}$$

Observe that the composite

$$H^1(X, \mathcal{T}_{X/W}) \hookrightarrow H^1(X, \mathcal{T}_{X/W}) \oplus \Gamma(X, \Omega^2_{X/W}) \xrightarrow{\tau_{(E,\nabla^\varepsilon)}} H^1(\mathcal{K}^\bullet[\nabla^\text{ad}]) \xrightarrow{(\eta^\text{ad})^{-1}} H^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E_B}])$$

is the identity. Therefore, the composite

$$H^1(X, \mathcal{T}_{X/W}) \hookrightarrow H^1(X, \mathcal{T}_{X/W}) \oplus \Gamma(X, \Omega^2_{X/W}) \xrightarrow{\tau_{(E,\nabla^\varepsilon)}} H^1(\mathcal{K}^\bullet[\nabla^\text{ad}]) \xrightarrow{(\eta^\text{ad})^{-1}} H^1(\mathcal{K}^\bullet[\nabla^\text{ad}_{E_B}])$$

is also the identity.
coincides (via (83) and (169)) with the differential of \( \sigma : \mathcal{N} \to \mathcal{S}_{g, z_p} \) at the point \( s_\infty \). If we consider \( H^1(X, \mathcal{T}_{X/W}) \) as a submodule of \( \mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2) \) via this composite, then, by the definition of \( \tilde{\phi}_{\mathcal{E}_B} \), we have \( \tilde{\phi}_{\mathcal{E}_B}(\Phi^1_W(pH^1(X, \mathcal{T}_{X/W}))) \subseteq H^1(X, \mathcal{T}_{X/W}) \). Moreover, it follows from the discussion in Remark 2.5.2 that the restriction \( \Phi^*_W(pH^1(X, \mathcal{T}_{X/W})) \to H^1(X, \mathcal{T}_{X/W}) \) of \( \tilde{\phi}_{\mathcal{E}_B} \) coincides with the inverse of the differential \( d\Phi^*_N|_{s_\infty} \) (cf. (100)) of \( \Phi_N \) at \( s_\infty \). That is to say, the equality

\[
F^0_{(\varepsilon, \nabla_\varepsilon)} = \tilde{\phi}_{\mathcal{E}_B} \circ \Phi^*_W([p])
\]

holds, where, for each \( W \)-module \( H \), we shall write \([p]\) for the morphism \( H \to pH (\subseteq H) \) given by multiplication by \( p \). Also, if we consider \( \Gamma(X, \Omega^\otimes_{X/W}) \) as a submodule of \( \mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2) \) via the composite injection \((\eta_B^ad)^{-1} \circ \xi^2 : \Gamma(X, \Omega^\otimes_{X/W}) \to \mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2) \) (cf. (83)), then the definition of \( \tilde{\phi}_{\mathcal{E}_B} \) implies that \( \tilde{\phi}_{\mathcal{E}_B}(\Phi^1_W(p\Gamma(X, \Omega^\otimes_{X/W}))) \subseteq \Gamma(X, \Omega^\otimes_{X/W}) \). We shall write

\[
\phi_0 : \Phi^*_W(\Gamma(X, \Omega^\otimes_{X/W})) \to \Gamma(X, \Omega^\otimes_{X/W})
\]

for the composite of \( \Phi^*_W([p]) : \Phi^*_W(\Gamma(X, \Omega^\otimes_{X/W})) \to \Phi^*_W(p\Gamma(X, \Omega^\otimes_{X/W})) \) and the restriction of \( \tilde{\phi}_{\mathcal{E}_B} \) to \( \Phi^*_W(p\Gamma(X, \Omega^\otimes_{X/W})) \). By the above discussion, the following square diagram turns out to be commutative:

\[
\begin{array}{ccc}
\Phi^*_W(H^1(X, \mathcal{T}_{X/W})) \oplus \Phi^*_W(\Gamma(X, \Omega^\otimes_{X/W})) & \xrightarrow{\Phi^*_W(\Gamma(\mathcal{T}_{X/W}))} & \Phi^*_W(\mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2)) \\
F^0_{(\varepsilon, \nabla_\varepsilon)} \oplus \phi_0 & \downarrow & \phi_0 \circ \Phi^*_W([p]) \\
H^1(X, \mathcal{T}_{X/W}) \oplus \Gamma(X, \Omega^\otimes_{X/W}) & \xrightarrow{\Gamma(\mathcal{T}_{X/W})} & \mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2). \\
\end{array}
\]

But, it follows from the definitions of \( \phi_\varepsilon \) and \( F_{(\varepsilon, \nabla_\varepsilon)} \) that \( \phi_\varepsilon \circ (\Phi^*_W([p])) \) (i.e., the right-hand vertical arrow in the above diagram) coincides with \( F_{(\varepsilon, \nabla_\varepsilon)} \). Thus, to obtain the required commutativities of the diagrams, it suffices to verify the equality \( \phi_0 = F^0_{(\varepsilon, \nabla_\varepsilon)} \).

The commutative diagram (153) induces a commutative square diagram of the form

\[
\begin{array}{ccc}
\Phi^*_W(\mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2)) & \xrightarrow{\Phi^*_W(\xi^2)} & \Phi^*_W(\mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2)) \\
F_{(\varepsilon, \nabla_\varepsilon)} \downarrow & & \downarrow F_{(\varepsilon, \nabla_\varepsilon)} \\
\mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2) & \xrightarrow{\xi^2} & \mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2), \\
\end{array}
\]

where \([p^3]\) denotes multiplication by \( p^3 \). The submodule \( \Gamma(X, \Omega^\otimes_{X/W}) \subseteq \mathbb{H}^1(\mathcal{K}^* \mathcal{N}_{ad}^2) \) is isotropic with respect to \( \xi^2_{(\varepsilon, \nabla_\varepsilon)} \) (cf. Proposition 1.5.1), and the above diagram restricts to a commutative square diagram of the form

\[
\begin{array}{ccc}
\Phi^*_W(\Gamma(X, \Omega^\otimes_{X/W})) & \xrightarrow{\Phi^*_W(\xi^2)} & \Phi^*_W(\mathbb{H}^1(X, \mathcal{T}_{X/W})) \\
\phi_0 \downarrow & & \downarrow F_{(\varepsilon, \nabla_\varepsilon)} \\
\Gamma(X, \Omega^\otimes_{X/W}) & \xrightarrow{\xi^2} & \mathbb{H}^1(X, \mathcal{T}_{X/W}). \\
\end{array}
\]
Since \((F^\text{proj}_{(e,\overline{e})})^{-1} \circ [p^3] = F^z_{(e,\overline{e})}\) via \(f^z_X\), the above diagram implies the equality \(\phi_0 = F^z_{(e,\overline{e})}\), as desired. This completes the proof of the assertion.

### 3.2. Proof of Theorem [1.7.1]

Next, we shall prove Theorem [1.7.1]. Denote by

\[(123) \quad 0^*_N(\hat{\varpi}_{g,z_p}|\mathcal{N}) \in \Gamma(N, \bigwedge^2 0^*_N(\Omega^{Z_p}_N|/Z_p)) \quad \text{(resp., } \sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N})) \in \Gamma(N, \bigwedge^2 \sigma^*(\Omega^{S_{g,z_p}}|/Z_p))\]

the restriction of \(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}\) (resp., \(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}\)) via the zero section \(0_N : \mathcal{N} \rightarrow T^{Z_p}_N\) (resp., \(\sigma : \mathcal{N} \rightarrow \hat{S}_{g,z_p}\)). Also, denote by

\[(124) \quad \Lambda : \Gamma(N, \bigwedge^2 \sigma^*(\hat{S}_{g,z_p}^{PGL}|\mathcal{N})) \rightleftharpoons \Gamma(N, \bigwedge^2 0^*_N(\Omega^{Z_p}_N|/Z_p))\]

the morphism obtained by restricting \((60)\) to \(0_N\). Then, let us consider the following lemma.

**Lemma 3.2.1.**

The following equality holds:

\[(125) \quad \Lambda(\sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N})) = 0^*_N(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N})\]

**Proof.** Let us consider \(F^\text{proj} \oplus F^z\) as a morphism \(\Phi_N^*(0^*_N(\mathcal{T}^{Z_p}_N|/Z_p)) \rightarrow 0^*_N(\mathcal{T}^{Z_p}_N|/Z_p)\) via \((17)\). This morphism induces a morphism

\[(126) \quad F^{\text{PGL}} : \bigwedge^2 0^*_N(\Omega^{Z_p}_N|/Z_p) \rightarrow \bigwedge^2 \Phi_N^*(0^*_N(\Omega^{Z_p}_N|/Z_p)) \cong \Phi_N^*(0^*_N(\Omega^{Z_p}_N|/Z_p)).\]

Let us consider \(\Gamma(N, \bigwedge^2 0^*_N(\Omega^{Z_p}_N|/Z_p))\) as a submodule of \(\Gamma(N, \bigwedge^2 0^*_N(\Omega^{Z_p}_N|/Z_p))\) via pull-back by \(\Phi_N\). By the definitions of \(F^\text{proj} \), \(F^z\), and the explicit description of \(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N})\), the following equality holds:

\[(127) \quad F^{\text{PGL}}(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) = p^3 \cdot 0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}).\]

Here, let us write \(H := \Gamma(N, \bigwedge^2 0^*_N(\Omega^{Z_p}_N|/Z_p))\), and suppose that \(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) = \Lambda(\sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N})) \in p^m H\) for some \(m > 0\), i.e., \(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) = \Lambda(\sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N})) \in p^m H\) for some \(h \in H\). Then,

\[(128) \quad p^3 \cdot 0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) - \Lambda(\sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N})) = p^3 \cdot 0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) - \Lambda(p^3 \cdot \sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}))
\]

\[= F^{\text{PGL}}(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) - \Lambda(p^3 \cdot \sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}))
\]

\[= F^{\text{PGL}}(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) - \Lambda(\sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}))
\]

\[= F^{\text{PGL}}(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) - \Lambda(\sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}))
\]

\[= F^{\text{PGL}}(0^*_N((\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}) - \Lambda(\sigma^*(\hat{\varpi}_{g,z_p}^{PGL}|\mathcal{N}))
\]

\[= F^{\text{PGL}}(p^m \cdot h)
\]

\[= p^m \cdot F^{\text{PGL}}(h).\]
This implies that $0^*_N(\widehat{\omega}_{g,z_p}^\text{Liou}|_N) - \Lambda(\sigma^*(\widehat{\omega}_{g,z_p}^\text{PGL}|_N)) \in p^{m-3}H \subseteq p^{m+1}H$ (where we recall the assumption $p > 3$). By induction on $m$, we see that

\begin{equation}
0^*_N(\widehat{\omega}_{g,z_p}^\text{Liou}|_N) - \Lambda(\sigma^*(\widehat{\omega}_{g,z_p}^\text{PGL}|_N)) \in \bigcap_{m>0} p^m H = \{0\},
\end{equation}

that is to say, the equality $0^*_N(\widehat{\omega}_{g,z_p}^\text{Liou}|_N) = \Lambda(\sigma^*(\widehat{\omega}_{g,z_p}^\text{PGL}|_N))$ holds. Thus, in order to complete the proof of the assertion, it suffices to verify the equality (125) modulo $p$.

Denote by $\langle -,- \rangle_{\text{PGL}}$ the bilinear map on $\mathbb{R}^1 f_{N_1*}(\mathcal{T}_{C_{N_1/1}/N_1}) \oplus f_{N_1*}(\Omega^\otimes_{C_{N_1/1}/N_1})$ (where $N_1 := N_{g,z_p}^\text{ord}$) corresponding to $\sigma^*(\widehat{\omega}_{g,z_p}^\text{PGL}|_N)$ mod $p$ via $\Upsilon$ (cf. (129) and (132)). Let us fix local sections $a,b \in \mathbb{R}^1 f_{N_1*}(\mathcal{T}_{C_{N_1/1}/N_1})$ and $b',b'' \in f_{N_1*}(\Omega^\otimes_{C_{N_1/1}/N_1})$. The result of Proposition 1.5.1 implies that

\begin{equation}
\langle a,b',b'' \rangle_{\text{PGL}} = \langle a,a' \rangle_{\text{PGL}} + \langle a,b' \rangle_{\text{PGL}} + \langle b',a' \rangle_{\text{PGL}} + \langle b',b'' \rangle_{\text{PGL}}
= \langle a,a' \rangle_{\text{PGL}} + \langle a,b' \rangle - \langle a',b \rangle,
\end{equation}

where $\langle -,- \rangle$ denotes the natural pairing $\mathbb{R}^1 f_{N_1*}(\mathcal{T}_{C_{N_1/1}/N_1}) \times f_{N_1*}(\Omega^\otimes_{C_{N_1/1}/N_1}) \to \mathcal{O}_{N_1}$. Also, according to Proposition 2.6.1 any local section of $\mathbb{R}^1 f_{N_1*}(\mathcal{T}_{C_{N_1/1}/N_1})$ (considered as a local section of $\mathbb{R}^1 f_{N_1*}(\mathcal{K}^*\left[\nabla^\text{ad}_{\mathcal{V}_f} \right])$ via $\Upsilon$) may be represented, locally on $N_1$, by a collection $\{a_{\alpha}\}_{\alpha}$ as in (156) with $b_{\alpha} = 0$ (for any $\alpha$). In particular, $\mathbb{R}^1 f_{N_1*}(\mathcal{T}_{C_{N_1/1}/N_1})$ is isotropic with respect to $\langle -,- \rangle_{\text{PGL}}$, and hence, $\langle a,a' \rangle_{\text{PGL}} = 0$. Thus, by (130), the equality

\begin{equation}
\langle a + b,a' + b' \rangle_{\text{PGL}} = \langle a,b' \rangle - \langle a',b \rangle
\end{equation}

holds. It follows from the definition of $0^*_N(\widehat{\omega}_{g,z_p}^\text{Liou}|_N)$ modulo $p$ (cf. (18)) and the above equality that the equality (125) modulo $p$ holds, as desired. This completes the proof of the assertion.

Now, let $n$ be a positive integer and write $R = \mathbb{Z}/p^n\mathbb{Z}$. Let $S$ be an $R$-scheme admitting an étale morphism $v : S \to N_n := \mathcal{N} \otimes R$ which dominates any component of $N_n$. Denote by $\widehat{\omega}_{g,z_p}^\text{Liou}|_S$, $\widehat{\omega}_{g,z_p}^\text{PGL}|_S$, and $\Theta_S$ the base-changes by the composite $S \xleftarrow{v} N_n \to \mathcal{N}$ of $\widehat{\omega}_{g,z_p}^\text{Liou}|_N$, $\widehat{\omega}_{g,z_p}^\text{PGL}|_N$, and $\Theta$ respectively. Since the natural map

$$
\Gamma(T_R^*\mathcal{N}_n, \bigwedge^2 \Omega^* T^*_{\mathcal{N}_n/R}) \to \Gamma(T_R^*S, \bigwedge^2 \Omega^* T^*_{\mathcal{N}_n/R})
$$

induced by $v$ is injective, the proof of Theorem 1.7.1 may be reduced to proving the equality (62) restricted to $S$, i.e., the equality

\begin{equation}
\Theta_S(\widehat{\omega}_{g,z_p}^\text{PGL}|_S) = \widehat{\omega}_{g,z_p}^\text{Liou}|_S
\end{equation}

(for any $n$ and $S$). Moreover, for the same reason, we are always free to replace $S$ by any étale covering of $S$.

Next, let us take $A \in \Gamma(S, \Omega_{S/R})$. We denote by $\sigma_A : S \to T^*_{\mathcal{N}_n/S}$ the section corresponding to $A$. It follows from an argument similar to the argument in 15), §5.3 (and §5.1), that the following equality holds:

\begin{equation}
\sigma_A^*(\widehat{\omega}_{g,z_p}^\text{PGL}|_S) - 0^*_S(\widehat{\omega}_{g,z_p}^\text{Liou}|_S) = \sigma_A^*(\Theta_S(\widehat{\omega}_{g,z_p}^\text{PGL}|_S) - \widehat{\omega}_{g,z_p}^\text{Liou}|_S),
\end{equation}

where $\sigma_S$ denotes the morphism $S \to \mathcal{N} \times_{\mathcal{N}_n} S$ induced by $\sigma$. After possibly replacing $S$ by its étale covering, we may assume that $S$ is affine and the vector bundle $\Omega_{S/R}$ is free. Under
this assumption, \( \Theta^*_S(\omega^\text{PGL}_{g,Z_p}|_S) \) \( - \omega^\text{Liou}_{g,Z_p}|_S = 0 \) if and only if \( \sigma^*_S(\omega^\text{PGL}_{g,Z_p}|_S) - 0^*_S(\omega^\text{Liou}_{g,Z_p}|_S) = 0 \) for all \( A \in \Gamma(S,\Omega_{S/R}) \). Thus, in order to verify the equality (132), it suffices (by (133)) to prove the equality \( \sigma^*_S(\omega^\text{PGL}_{g,Z_p}|_S) = 0^*_S(\omega^\text{Liou}_{g,Z_p}|_S) \). In particular, it suffices to prove the equality

(134) \[
\sigma^*(\omega^\text{PGL}_{g,Z_p}|_\mathcal{N}) = 0^*_X(\omega^\text{Liou}_{g,Z_p}|_\mathcal{N}).
\]

But, this equality holds by Lemma 3.2.1. This completes the proof of Theorem 1.7.1.

4. Appendix (Crystals of torsors and connections)

In this Appendix, we study crystals of torsors (equipped with a structure group) and prove the bijective correspondence between crystals of torsors and quasi-nilpotent flat torsors (cf. Theorem 4.4.2). This correspondence enable us to understand the relationship (cf. Proposition 4.5.2) between the respective deformations of a prescribed flat torsor over distinct underlying spaces. Notice that its application to the case of indigenous bundles (cf. Proposition 4.6.1) was used in the proof of the main theorem.

4.1. Connections on torsors.

Let \( R \) be a commutative ring with unit, \( G \) a geometrically connected smooth algebraic group over \( R \) with Lie algebra \( \mathfrak{g} \), \( S \) a scheme over \( R \), and \( f : X \to S \) a smooth scheme over \( S \) of relative dimension \( n > 0 \). Suppose that we are given a \( G \)-torsor \( \pi : \mathcal{E} \to X \) over \( X \). Denote by

(135) \[
\text{Ad}(\mathcal{E}) \quad (:= \mathcal{E} \times^G \mathfrak{g})
\]

the adjoint vector bundle on \( X \) associated to \( \mathcal{E} \) (i.e., the vector bundle obtained from \( \mathcal{E} \) by the change of structure group via the adjoint representation \( G \to \text{GL}(\mathfrak{g}) \)). Also, denote by

(136) \[
\tilde{T}_{\mathcal{E}/S} \quad (:= (\pi_*({\mathcal{T}_{\mathcal{E}/S}}))^{G})
\]

the subsheaf of \( \pi_*({\mathcal{T}_{\mathcal{E}/S}}) \) consisting of \( G \)-invariant sections. Then, the differential of \( \pi \) induces a surjection \( \tilde{T}_{\mathcal{E}/S} \to \mathcal{T}_{X/S} \), which we denote by \( d\pi \). The kernel of \( d\pi \) may be naturally identified with \( \text{Ad}(\mathcal{E}) \quad (\cong (\pi_*({\mathcal{T}_{\mathcal{E}/X}}))^{G}) \). Thus, we have a short exact sequence

(137) \[
0 \to \text{Ad}(\mathcal{E}) \to \tilde{T}_{\mathcal{E}/S} \xrightarrow{d\pi} \mathcal{T}_{X/S} \to 0.
\]

An \textbf{S-connection} on \( \mathcal{E} \) is, by definition, a split injection of (137), i.e., an \( \mathcal{O}_X \)-linear morphism \( \nabla_\mathcal{E} : \mathcal{T}_{X/S} \to \tilde{T}_{\mathcal{E}/S} \) satisfying the equality \( d\pi \circ \nabla_\mathcal{E} = \text{id}_{\mathcal{T}_{X/S}} \). If \( G = \text{GL}_n \) for some \( n > 0 \), then the above definition of an \textbf{S-connection} is equivalent to the classical definition of an \textbf{S-connection} on the corresponding vector bundle \( \mathcal{V} := \mathcal{E} \times^\text{GL}_n R^\otimes_n \) (cf. [16], §4.2), i.e., an \( f^{-1}(\mathcal{O}_S) \)-linear morphism \( \mathcal{V} \to \Omega_{X/S} \otimes \mathcal{V} \) satisfying the Leibniz rule. In this case, we shall not distinguish between these definitions of an \textbf{S-connection}. Denote by

(138) \[
\nabla_{\mathcal{E}}^\text{ad} : \text{Ad}(\mathcal{E}) \to \Omega_{X/S} \otimes \text{Ad}(\mathcal{E})
\]

the \textbf{S-connection} on the vector bundle \( \text{Ad}(\mathcal{E}) \) induced by \( \nabla_\mathcal{E} \) by the change of structure group via the adjoint representation \( G \to \text{GL}(\mathfrak{g}) \).
Let us fix an $S$-connection $\nabla_\mathcal{E}$ on $\mathcal{E}$. The curvature of $\nabla_\mathcal{E}$ is, by definition, the $\mathcal{O}_X$-linear morphism $\wedge^2 T_{X/S} \to \text{Ad}(\mathcal{E}) \subseteq \tilde{T}_{\mathcal{E}/S}$ determined by assigning $\partial_1 \wedge \partial_2 \mapsto [\nabla_\mathcal{E}(\partial_1), \nabla_\mathcal{E}(\partial_2)] = \nabla_\mathcal{E}([\partial_1, \partial_2])$ (for any local sections $\partial_1, \partial_2 \in T_{X/S}$). We shall say that $\nabla_\mathcal{E}$ is flat if its curvature vanishes identically on $X$. (If $X/S$ is of relative dimension 1, which implies that $T_{X/S}$ is a line bundle, then any $S$-connection on $\mathcal{E}$ is automatically flat.) It is verified that $\nabla_\mathcal{E}$ is flat if and only if $\nabla_\mathcal{E}^\text{ad}$ is flat. By a flat $G$-torsor over $X/S$, we mean a pair $(\mathcal{E}, \nabla_\mathcal{E})$ consisting of a $G$-torsor over $X$ and a flat $S$-connection $\nabla_\mathcal{E}$ on $\mathcal{E}$. An isomorphism of flat $G$-torsors from $(\mathcal{E}, \nabla_\mathcal{E})$ to $(\mathcal{E}', \nabla_\mathcal{E}')$ is an isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ of $G$-torsors compatible with the respective connections $\nabla_\mathcal{E}$ and $\nabla_\mathcal{E}'$. Thus, flat $G$-torsors over $X/S$ and isomorphisms between them form a groupoid.

### 4.2. $p$-curvature of flat torsors.

In this subsection, suppose further that $p \cdot \mathcal{O}_S = 0$. Write $\Phi_S : S \to S$ for the absolute Frobenius morphism of $S$, $f^{(1)} : X^{(1)} := X \times_{\Phi_S, S} S \to S$ for the Frobenius twist of $X$ relative to $S$, and $\Phi_{X/S} : X \to X^{(1)}$ for the relative Frobenius morphism. Recall that for each $\mathcal{O}_{X^{(1)}}$-module $\mathcal{F}$, the pull-back $\Phi_{X/S}^* (\mathcal{F})$ admits a canonical $S$-connection
\begin{equation}
\nabla^\text{can}_{\mathcal{F}} : \Phi_{X/S}^* (\mathcal{F}) \to \Omega_{X/S} \otimes \Phi_{X/S}^* (\mathcal{F})
\end{equation}
determined uniquely by the condition that the sections of $\Phi_{X^{(1)}/S}^{-1}(\mathcal{F})$ are horizontal.

Now, let us fix a flat $G$-torsor $(\mathcal{E}, \nabla_\mathcal{E})$ over $X/S$. The $p$-curvature of $(\mathcal{E}, \nabla_\mathcal{E})$ is defined to be the $\mathcal{O}_X$-linear morphism
\begin{equation}
\psi_{(\mathcal{E}, \nabla_\mathcal{E})} : \Phi_{X/S}^* (T_{X^{(1)}/S}) \to \text{Ad}(\mathcal{E}) \subseteq \tilde{T}_{\mathcal{E}/S}
\end{equation}
determined uniquely by $F_{X/S}^{-1}(\partial) \mapsto \nabla_\mathcal{E}(\partial)^{[p]} - \nabla_\mathcal{E}(\partial^{[p]})$ for any local section $\partial \in T_{X/S}$ (cf. [8], §5.0). Here, $(-)^{[p]}$ denotes the operator on vector fields given by taking the $p$-th iterates of the corresponding derivations, by which both $T_{X^{(1)}/S}$ and $\tilde{T}_{\mathcal{E}/S}$ form sheaves of $p$-Lie algebras. As is well-known, $\psi_{(\mathcal{E}, \nabla_\mathcal{E})}$ is compatible with the respective connections $\nabla^\text{can}_{T_{X^{(1)}/S}}$ and $\nabla^\text{ad}_{\mathcal{E}}$. In particular, if
\begin{equation}
\psi_{(\mathcal{E}, \nabla_\mathcal{E})}^\text{ad} : \Phi_{X/S}^{-1} (T_{X^{(1)}/S}) \to \text{Ad}(\mathcal{E})
\end{equation}
denotes the restriction of $\psi_{(\mathcal{E}, \nabla_\mathcal{E})}$, then its image lies in $\ker(\nabla^\text{ad}_{\mathcal{E}})$.

Finally, we shall say that $(\mathcal{E}, \nabla_\mathcal{E})$ is $p$-nilpotent if $\psi_{(\mathcal{E}, \nabla_\mathcal{E})}$ has nilpotent image. It is verified that $(\mathcal{E}, \nabla_\mathcal{E})$ is $p$-nilpotent if and only if $(\text{Ad}(\mathcal{E}), \nabla^\text{ad}_{\mathcal{E}})$ is $p$-nilpotent.

### 4.3. Quasi-nilpotence and crystals.

In this subsection, we shall suppose that $p^N \cdot \mathcal{O}_S = 0$ for some $N > 0$. Fix a flat $G$-torsor $(\mathcal{E}, \nabla_\mathcal{E})$ over $X/S$. Let $U_+ := (U, \{x_i\}_{i=1}^n)$ be a collection, where $U$ denotes an open subscheme of $X$ and $\{x_i\}_{i=1}^n \subseteq \Gamma(U, \mathcal{O}_X)$ denotes a local coordinate system defined on $U$ relative to $S$; we shall refer to such a collection as a coordinate chart for $X/S$. Given a coordinate chart $U_+ := (U, \{x_i\}_{i=1}^n)$ for $X/S$, we shall consider the following condition:
\((*)_{U_+}:\) For each \(s \in \Gamma(U, \text{Ad}(\mathcal{E}))\), there exist an open covering \(\{U_\alpha\}_\alpha\) of \(U\) and a set of positive integers \(\{e_{i,\alpha}\}_\alpha\) such that \(\text{ad}(\nabla_\mathcal{E}(\frac{\partial}{\partial x_i}))^{e_{i,\alpha}}(s|_{U_\alpha}) = 0\) for all \(i\) and \(\alpha\), where \(\text{ad}(v)\) (for each \(v \in \mathcal{T}_{E/S}\)) denotes the adjoint operator \([v, -]: \mathcal{T}_{E/S} \to \mathcal{T}_{E/S}\).

**Definition 4.3.1.**

We shall say that \((\mathcal{E}, \nabla_\mathcal{E})\) is **quasi-nilpotent** if there exists a collection \(\mathcal{U}_+ := \{U_{\gamma,+}\}_\gamma\), where each \(U_{\gamma,+}\) denotes a coordinate chart \((U_{\gamma}, \{x_{\gamma,i}\}_{i=1}^n)\) for \(X/S\) satisfying the condition \((*)_{U_{\gamma,+}}\) such that \(\{U_\gamma\}_\gamma\) forms an open covering of \(X\).

**Remark 4.3.2.**

(i) Suppose that \(\nabla_\mathcal{E}\) is flat. Then, it is verified that any coordinate chart \(U_+ := (U, \{x_i\}_{i=1}^n)\) for \(X/S\) satisfies the condition \((*)_{U_+}\) (cf. [11], Remark 4.11).

(ii) It is verifies that \(\nabla_\mathcal{E}\) is quasi-nilpotent if and only if its reduction modulo \(p\) is quasi-nilpotent. Also, the quasi-nilpotence of a connection may be related to the \(p\)-nilpotence of its reduction modulo \(p\). In fact, let \(X_1/S_1\) and \((\mathcal{E}_1, \nabla_{\mathcal{E},1})\) be the reductions modulo \(p\) of \(X/S\) and \((\mathcal{E}, \nabla_\mathcal{E})\) respectively. Let us fix a coordinate chart \(U_+ := (U, \{x_i\}_{i=1}^n)\) for \(X_1/S_1\). Since \((\frac{\partial}{\partial x_i})^p = 0\), the following sequence of equalities holds:

\[
\text{ad}(\psi_{\mathcal{E}_1, \nabla_{\mathcal{E},1}})(F_{X_1/S_1}^{-1}((\frac{\partial}{\partial x_i}))) = \text{ad}(\nabla_{\mathcal{E},1}((\frac{\partial}{\partial x_i}))^p) = \text{ad}(\nabla_{\mathcal{E},1}(\frac{\partial}{\partial x_i}))^p.
\]

Hence, \((\mathcal{E}, \nabla_\mathcal{E})\) is quasi-nilpotent (or equivalently, \((\mathcal{E}_1, \nabla_{\mathcal{E},1})\) is quasi-nilpotent) if and only if (\(\text{ad}(\nabla_{\mathcal{E},1}(\frac{\partial}{\partial x_i}))\) is nilpotent for any \(i\), or equivalently) the reduction of \((\mathcal{E}, \nabla_\mathcal{E})\) modulo \(p\) is \(p\)-nilpotent.

Let \((S, I, \gamma)\) be a PD scheme over \(R\) (with \(I\) a quasi-coherent ideal) and \(X\) an \(S\)-scheme to which \(\gamma\) extends. Denote by

\[
\text{Crys}(X/S)
\]

the crystalline site, which is the site whose objects are divided power thickenings, i.e., pairs \((U \hookrightarrow T, \delta)\), where \(U\) is a Zariski open subscheme of \(X\), \(U \hookrightarrow T\) is a closed \(S\)-immersion defined by an ideal \(J\), and \(\delta\) is a PD structure on \(J\) which is compatible with \(\gamma\) in the evident sense. We shall often abuse notation by writing \((U, T, \delta)\) for \((U \hookrightarrow T, \delta)\), or even by just writing \(T\) for the whole thing. (We shall call \((U \hookrightarrow T, \delta)\) an \(S\)-PD thickening of \(U\).) The morphisms and the covering families in \(\text{Crys}(X/S)\) are defined in the usual manner. Recall that a **crystal of \(G\)-torsors** over \(\text{Crys}(X/S)\) (resp., a **crystal** of vector bundles on \(\text{Crys}(X/S)\)) is a cartesian section of the fibered category of \(G\)-torsors over \(\text{Crys}(X/S)\) (resp., the fibered category of vector bundles on \(\text{Crys}(X/S)\)). For each crystal \(\mathcal{F}^0\) (of either \(G\)-torsors or vector bundles) on \(\text{Crys}(X/S)\) and each divided power thickening \(T\) in \(\text{Crys}(X/S)\), we shall write \(\mathcal{F}_T^0\) for the evaluation of \(\mathcal{F}^0\) on this thickening.
4.4. **Correspondence between crystals and quasi-nilpotent flat torsors.**

Let us suppose that the following condition \((**)_G\) on \(G\) is satisfied:
\begin{align*}
\text{(**)}_G : \text{\(G\) is a simple algebraic group over \(R\) of adjoint type satisfying the inequality \(p > h\), where \(h\) denotes the Coxeter number of \(G\).}
\end{align*}
(For instance, \(G = \text{PGL}_n\) with \(n < p\)). In particular, the morphism of algebraic groups \(G \rightarrow \text{Aut}^0(\mathfrak{g})\) obtained as the adjoint representation of \(G\) is an isomorphism, where \(\text{Aut}^0(\mathfrak{g}) \subseteq \text{GL}(\mathfrak{g})\) denotes the identity component of the group of Lie algebra automorphisms of \(\mathfrak{g}\). (Indeed, it follows from a result in [14] that the reduction modulo \(p\) of this morphism is an isomorphism.)

Let \(X\) be a smooth scheme over \(S\) and denote by \(D_{X/S}(1)\) the divided power envelope of \(X\) in \(X \times_S X\) via the diagonal embedding \(\Delta : X \rightarrow X \times_S X\). For \(i = 1, 2\), we shall write \(\text{pr}_i : D_{X/S}(1) \rightarrow X\) for the composite of the natural morphism \(D_{X/S}(1) \rightarrow X \times_X X\) and the projection \(X \times_X X \rightarrow X\) onto the \(i\)-th factor.

**Definition 4.4.1.**
Let \(E\) be a \(G\)-torsor over \(X\). An **HPD stratification** on \(E\) is an isomorphism \(e : \text{pr}_2^*(E) \xrightarrow{\sim} \text{pr}_1^*(E)\) of \(G\)-torsors over \(D_{X/S}(1)\) whose restriction to \(X\) via \(\Delta\) coincides with the identity morphism of \(E\) and which satisfies the cocycle condition in the usual sense (cf. [14], §2, the comment following Definition 2.10).

**Theorem 4.4.2.**
Let \(J\) be a sub-PD ideal of \(I\). Write \(S\) the closed subscheme of \(S\) defined by \(J\) and write \(\overline{X} := X \times_S S\). Then, the following categories are naturally equivalent:
\begin{enumerate}
\item The category of crystals of \(G\)-torsors on \(
\text{Crys}(\overline{X}/S)\);
\item The category of \(G\)-torsors \(E\) over \(X\) together with an HPD stratification \(\text{pr}_2^*(E) \xrightarrow{\sim} \text{pr}_1^*(E)\) on it;
\item The category of \(G\)-torsors \(E\) over \(X\) together with an HPD stratification \(\text{pr}_2^*(\text{Ad}(E)) \xrightarrow{\sim} \text{pr}_1^*(\text{Ad}(E))\) on its adjoint bundle \(\text{Ad}(E)\) (in the sense of [14], §4, Definition 4.3H) that is compatible with the respective Lie bracket structures pulled-back from \(\text{Ad}(E)\);
\item The category of \(G\)-torsors \(E\) over \(X\) together with a quasi-nilpotent flat \(S\)-connection on \(\text{Ad}(E)\) compatible with the Lie bracket structure.
\item The category of quasi-nilpotent flat \(G\)-torsors over \(X/S\).
\end{enumerate}

**Proof.** The equivalence of (ii) and (iii), as well as (iv) and (v), follows from the isomorphism \(G \xrightarrow{\sim} \text{Aut}^0(\mathfrak{g})\). The equivalent of (iii) and (iv) follows from [14], §4, Theorem 4.12 (and its proof). Thus, it suffices to check the equivalence of (i) and (ii). First, let \(E^\circ\) be a crystal of \(G\)-torsors on \(\text{Crys}(\overline{X}/S)\). The morphism \(\text{pr}_i : D_{X/S}(1) \rightarrow X\) (for each \(i = 1, 2\)) induces an isomorphism \(e_i : \text{pr}_i^*(E^\circ_X) \xrightarrow{\sim} E^\circ_{D_{X/S}(1)}\). Then, the composite isomorphism \(e_\circ := e_1^{-1} \circ e_2 : \text{pr}_2^*(E^\circ_X) \xrightarrow{\sim} \text{pr}_2^*(E^\circ_X)\) specifies an HPD stratification on the \(G\)-torsor \(E^\circ_X\).

Conversely, given a \(G\)-torsor \(E\) over \(X\) together with an HPD stratification \(e : \text{pr}_2^*(E) \xrightarrow{\sim} \text{pr}_1^*(E)\), we construct a crystal \(E^\circ\) of \(G\)-torsors on \(\text{Crys}(\overline{X}/S)\). To this end, it suffices to specify \(E^\circ_T\) for sufficiently small \((U, T, \delta) \in \text{Ob}(\text{Crys}(\overline{X}/S))\), e.g., so that there exists an \(S\)-morphism \(h : T \rightarrow X\) over \((S, I, \gamma)\) extending the open immersion \(U \hookrightarrow \overline{X}\). Then, let us define \(E^\circ_T\) to be the pull-back \(h^*(E)\). For a morphism \(h' : T \rightarrow X\) as \(h\), we obtain \((h, h') : T \rightarrow D_{X/S}(1)\). The pull-back of \(e\) via this morphism determines an isomorphism \((h, h')^*(e) : h^*(E) \xrightarrow{\sim} h'^*(E)\). The
fact that $\mathcal{E}_e^\diamondsuit$ does not depend on $h$ (up to canonical isomorphism) comes from the isomorphisms $(h, h')^*(e)$ (for $(h, h')$’s). Thus, $\mathcal{E}_e^\diamondsuit$’s for various $(T, T, \delta)$’s forms a crystal $\mathcal{E}_e^\diamondsuit$ of $G$-torsors. One verifies immediately that the assignments $\mathcal{E}_e^\diamondsuit \mapsto e_{\mathcal{E}_e^\diamondsuit}$, $e \mapsto \mathcal{E}_e^\diamondsuit$ define the equivalence of (i) and (ii). This completes the proof of the assertion. □

For each quasi-nilpotent flat $G$-torsors $(\mathcal{E}, \nabla_\mathcal{E})$ over $X/S$, we shall write

$$ (\mathcal{E}, \nabla_\mathcal{E})^\diamondsuit $$(144)

for the crystal of $G$-torsors corresponding to $(\mathcal{E}, \nabla_\mathcal{E})$ via the equivalence of categories between (i) and (v) in Theorem 4.4.2.

**Remark 4.4.3.**
Let us consider the case where $G = \text{PGL}_n$ with $n < p$. Then, the equivalences of categories obtained in Theorem 4.4.2 are compatible, via projectivization, with those obtained in the corresponding classical result for crystals of rank $n$ vector bundles (i.e., crystals of $\text{GL}_n$-torsors) described, e.g., [2], § 6, Theorem 6.6. To be precise, let $(\mathcal{V}, \nabla_\mathcal{V})$ be a quasi-nilpotent flat vector bundle of rank $n$ and denote by $\mathcal{V}^\diamondsuit$ the corresponding crystal obtained by the result in *loc. cit.* Then, the flat $\text{PGL}_n$-torsor obtained from $(\mathcal{V}, \nabla_\mathcal{V})$ via projectivization (i.e., via the change of structure group by the quotient $\text{GL}_n \twoheadrightarrow \text{PGL}_n$) corresponds, via the equivalence of categories in Theorem 4.4.2, to the crystal given by assigning, to each $(U \twoheadrightarrow T, \gamma) \in \text{Ob}(\text{Crys}(X/S))$, the projectivization of $\mathcal{V}_T^\diamondsuit$.

Let us describe the following two corollaries of Theorem 4.4.2. We shall keep the notation in that theorem.

**Corollary 4.4.4.**
There exists an equivalence of categories between the category of crystals of $G$-torsors on $\text{Crys}(\overline{X}/S)$ and the category of crystals of $G$-torsors on $\text{Crys}(X/S)$.

**Corollary 4.4.5.**
Suppose that we are given another smooth scheme $X'$ over $S$ whose reduction modulo $J$ is isomorphic to $\overline{X}$. Then, for each quasi-nilpotent flat $G$-torsor $(\mathcal{E}, \nabla_\mathcal{E})$ over $X/S$, there exists a unique (up to isomorphism) quasi-nilpotent flat $G$-torsor

$$ (\lambda_{X'}(\mathcal{E}), \lambda_{X'}(\nabla_\mathcal{E})) $$(145)

over $X'/S$ such that the crystals of $G$-torsors over $\text{Crys}(\overline{X}/S)$ corresponding, via the equivalence of categories in Theorem 4.4.2, to $(\mathcal{E}, \nabla_\mathcal{E})$ and $(\lambda_{X'}(\mathcal{E}), \lambda_{X'}(\nabla_\mathcal{E}))$ are isomorphic. Moreover, the assignment $(\mathcal{E}, \nabla_\mathcal{E}) \mapsto (\lambda_{X'}(\mathcal{E}), \lambda_{X'}(\nabla_\mathcal{E}))$ determines an equivalence of categories between the category of quasi-nilpotent flat $G$-torsors on $X/S$ and the category of quasi-nilpotent flat $G$-torsors on $X'/S$. 
4.5. Deformation space of flat torsors.

In this subsection, we describe the change of flat torsors \((\mathcal{E}, \nabla_{\mathcal{E}}) \mapsto (\lambda_{X^{\prime}}(\mathcal{E}), \lambda_{X^{\prime}}(\nabla_{\mathcal{E}}))\) obtained in Corollary 4.4.5 in terms of de Rham cohomology of complexes. For each morphism of sheaves \(\nabla : \mathcal{K}^0 \to \mathcal{K}^1\) on \(X\), we shall write \(\mathcal{K}^\bullet[\nabla]\) for \(\nabla\) regarded as a complex concentrated at degree 0 and 1. Also, for each sheaf \(\mathcal{F}\) on \(X\) and each \(n \in \mathbb{Z}\), we shall write \(\mathcal{F}[n]\) for \(\mathcal{F}\) considered as a complex concentrated at degree \(n\).

Let us keep the notation in the previous subsection and suppose that \(S = \text{Spec}(R)\) and \(X\) is a curve over \(S\) (cf. §112). Denote by

\[
\tilde{\nabla}^\text{ad}_{\mathcal{E}} : \tilde{T}_{X/R} \to \Omega_{X/R} \otimes \text{Ad}(\mathcal{E})
\]

the unique \(R\)-linear morphism determined by the condition that

\[
\langle \partial_1, \tilde{\nabla}^\text{ad}_{\mathcal{E}}(\partial_2) \rangle = [\nabla_{\mathcal{E}}(\partial_1), \partial_2] - \nabla_{\mathcal{E}}([\partial_1, d\pi(\partial_2)])
\]

(cf. (137) for the definition of \(d\pi\)) for any local sections \(\partial_1 \in T_{X/R}\) and \(\partial_2 \in \tilde{T}_{X/R}\), where \(\langle -, - \rangle\) denotes the pairing \(T_{X/R} \times (\Omega_{X/R} \otimes \text{Ad}(\mathcal{E})) \to \text{Ad}(\mathcal{E})\) arising from the natural pairing \(T_{X/R} \times \Omega_{X/R} \to \mathcal{O}_X\). The short exact sequence (137) induces naturally the following short exact sequence of complexes:

\[
0 \longrightarrow \mathcal{K}^\bullet[\nabla_{\mathcal{E}}] \longrightarrow \mathcal{K}^\bullet[\tilde{\nabla}^\text{ad}_{\mathcal{E}}] \longrightarrow T_{X/R}[0] \longrightarrow 0.
\]

This sequence gives rise to a short exact sequence

\[
0 \longrightarrow H^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}}]) \xrightarrow{\iota^\mathbb{H}} H^1(\mathcal{K}^\bullet[\tilde{\nabla}^\text{ad}_{\mathcal{E}}]) \xrightarrow{\pi^\mathbb{H}} H^1(X, T_{X/R}) \longrightarrow 0.
\]

By passing to the injection \(\iota^\mathbb{H}\), we shall consider \(H^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}}])\) as a submodule of \(H^1(\mathcal{K}^\bullet[\tilde{\nabla}^\text{ad}_{\mathcal{E}}])\).

Here, we shall write \(R_\epsilon := R[\epsilon]/(\epsilon^2)\), in which the ideal \(IR_\epsilon\) is endowed with a divided power structure extended from \(I \subseteq R\). In what follows, we shall denote the base-changes to \(R_\epsilon\) of objects over \(R\) by means of a subscripted \(\epsilon\). It is well-known that there exists a canonical bijection

\[
H^1(X, T_{X/R}) \xrightarrow{\sim} \text{Def}_{R_\epsilon}(X)
\]

between the set \(H^1(X, T_{X/R})\) and the set \(\text{Def}_{R_\epsilon}(X)\) consisting of isomorphism classes of deformations of \(X\) over \(R_\epsilon\). Also, denote by

\[
\text{Def}_{R_\epsilon}(X, \mathcal{E}, \nabla_{\mathcal{E}})
\]

the set of isomorphism classes of deformations over \(R_\epsilon\) of \((X, \mathcal{E}, \nabla_{\mathcal{E}})\) (as a data consisting of a curve and a flat \(G\)-torsor over it). By well-known generalities on the deformation theory of connections, there exists a canonical bijections

\[
H^1(X, T_{X/R}) \xrightarrow{\sim} \text{Def}_{R_\epsilon}(X, \mathcal{E}, \nabla_{\mathcal{E}})
\]

making the following square diagram commute:

\[
\begin{array}{ccc}
\mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}}]) & \xrightarrow{\sim} & \text{Def}_{R_\epsilon}(X, \mathcal{E}, \nabla_{\mathcal{E}}) \\
\pi^\mathbb{H} & \downarrow & \\
H^1(X, T_{X/R}) & \xrightarrow{\sim} & \text{Def}_{\epsilon}(X),
\end{array}
\]
where the right-hand vertical arrow denotes the projection induced by forgetting the data of a deformation of \((\mathcal{E}, \nabla_\mathcal{E})\). Moreover, denote by
\[
\text{Def}_{R_\epsilon}(\mathcal{E}, \nabla_\mathcal{E})
\]
the set of isomorphism classes of deformations over \(X_\epsilon\) of \((\mathcal{E}, \nabla_\mathcal{E})\) (as a flat \(G\)-torsor), which may be thought of as a subset of \(\text{Def}_{R_\epsilon}(X, \mathcal{E}, \nabla_\mathcal{E})\), i.e., the subset consisting of deformations whose underlying curves are the trivial deformation of \(X\). One verifies immediately that the bijection \((152)\) restricts to a bijection
\[
\mathbb{H}^1(\mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}})| \sim \text{Def}_{R_\epsilon}(\mathcal{E}, \nabla_\mathcal{E}).
\]

**Remark 4.5.1.**

In this remark, we describe the bijection \((152)\) in terms of Čech cohomology. Let us take an affine open covering \(\mathcal{U} := \{U_\alpha\}_{\alpha \in I}\) (where \(I\) is an index set) of \(X\). We shall write \(I_2\) for the set of pairs \((\alpha, \beta)\) \(\in I \times I\) with \(U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset\). One may calculate \(\mathbb{H}^1(\mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}})|\) as the total cohomology of the Čech double complex \(\text{Tot}^*(\tilde{C}^*(\mathcal{U}, \mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}})))\) associated to \(\mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}}\). Each element \(v\) of \(\mathbb{H}^1(\mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}})|\) may be given by a collection of data
\[
v = ((a_{\alpha\beta})_{\alpha, \beta}, \{b_\alpha\}_\alpha)
\]
consisting of a Čech 1-cocycle \((a_{\alpha\beta})_{\alpha, \beta} \in \tilde{C}^1(\mathcal{U}, \tilde{T}_{\mathcal{E}/R})\) (where \(a_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \tilde{T}_{\mathcal{E}/R})\)) and a Čech 0-cochain \(\{b_\alpha\}_\alpha \in \tilde{C}^0(\mathcal{U}, \Omega_{X/R} \otimes \text{Ad}(\mathcal{E}))\) (where \(b_\alpha \in \Gamma(U_\alpha, \Omega_{X/R} \otimes \text{Ad}(\mathcal{E})) = \text{Hom}\_{\mathcal{O}_{U_\alpha}}(\mathcal{T}_{U_\alpha/R}, \text{Ad}(\mathcal{E})|_{U_\alpha})\)) which agree under \(\nabla_\mathcal{E}^{\text{ad}}\) and the Čech coboundary map. (The elements in \(\mathbb{H}^1(\mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}})|\) may be represented by \(v\) as above such that \(d\pi(a_{\alpha\beta}) = 0\) for any pair \((\alpha, \beta)\).) The \(R_\epsilon\)-schemes \(U_{\alpha, \epsilon}\) (for various \(\alpha \in I\)) may be glued together by means of the isomorphisms
\[
\tau_{X, \alpha\beta}^\epsilon := \text{id}|_{U_{\alpha, \epsilon}} + \epsilon \cdot d\pi(a_{\alpha\beta}) : U_{\beta, \epsilon}|_{U_{\alpha\beta}} \sim U_{\alpha, \epsilon}|_{U_{\alpha\beta}}
\]
(for \((\alpha, \beta) \in I_2\)). The resulting \(R_\epsilon\)-scheme, which we denote by \(X_\epsilon^v\), specifies the deformation corresponding to \(v\) via \((150)\). Moreover, the flat \(G\)-torsors \((\mathcal{E}_\epsilon|_{U_\alpha}, \nabla_\mathcal{E}_\epsilon|_{U_\alpha} + \epsilon \cdot b_\alpha)\) may be glued together by means of the isomorphisms
\[
\tau_{X, \alpha\beta}^\epsilon := \text{id}|_{U_{\alpha, \epsilon}} + \epsilon \cdot a_{\alpha\beta} : (\mathcal{E}_\epsilon|_{U_{\beta}}, \nabla_\mathcal{E}_\epsilon|_{U_{\beta}} + \epsilon \cdot b_\beta)|_{U_{\alpha\beta}} \sim (\mathcal{E}_\epsilon|_{U_\alpha}, \nabla_\mathcal{E}_\epsilon|_{U_\alpha} + \epsilon \cdot b_\alpha)|_{U_{\alpha\beta}}
\]
(over \(\tau_{X, \alpha\beta}^\epsilon\) for \((\alpha, \beta) \in I_2\)). The data consisting of \(X_\epsilon^v\) and the resulting flat \(G\)-torsor, which we denote by \((\mathcal{E}_\epsilon^v, \nabla_\mathcal{E}_\epsilon^v)\), specifies the deformation of \((X, \mathcal{E}, \nabla_\mathcal{E})\) corresponding to \(v\) via \((152)\).

The assignment \(v \mapsto (X_\epsilon^v, \mathcal{E}_\epsilon^v, \nabla_\mathcal{E}_\epsilon^v)\) obtained in this way gives the bijection \((152)\).

Since \(\tilde{\nabla}_\mathcal{E}^{\text{ad}} \circ \nabla_\mathcal{E} = 0\), the pair of \(\nabla_\mathcal{E}\) and the zero map \(0 \to \Omega_{X/R} \otimes \text{Ad}(\mathcal{E})\) induces a morphism of complexes \(\mathcal{T}_{X/R}[0] \to \mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}}|\). By applying this morphism to the functor \(\mathbb{H}^1(-)\), we obtain a split injection of \((149)\):
\[
\nabla_\mathcal{E}^\mathbb{H} : H^1(X, \mathcal{T}_{X/R}) \hookrightarrow \mathbb{H}^1(\mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}})|.
\]

**Proposition 4.5.2.**

Let us take an element \(v \in \mathbb{H}^1(\mathcal{K}^*|\nabla_\mathcal{E}^{\text{ad}})|\) and denote by \((X_\epsilon^v, \mathcal{E}_\epsilon^v, \nabla_\mathcal{E}_\epsilon^v)\) the deformation over \(R_\epsilon\) of \((X, \mathcal{E}, \nabla_\mathcal{E})\) determined by \(v\) via \((152)\). (In particular, \(X_\epsilon^v\) is the deformation of \(X\) determined by \(\pi_\mathbb{H}(v) \in H^1(X, \mathcal{T}_{X/R})\) via \((150)\).) Also, let us take an element \(s \in JH^1(X, \mathcal{T}_{X/R})\) (where we
recall that deformation (cf. Corollary 4.4.5 for the definition of the curve over \( \mathbb{C} \vDash \text{C} \) 1-cocycle in Remark 4.5.1 and apply the discussion there to the present \( v \) be obtained by gluing together \( U_{\alpha,\beta} \)'s by means of the isomorphisms \( (\text{id} + \epsilon \cdot s_{\alpha,\beta}) \circ \tau_{X,\alpha,\beta} \) and \( \epsilon \cdot s_{\alpha,\beta} \)'s (cf. (157)). Since \( \text{id}_{U_{\alpha,\beta,\epsilon}} + \epsilon \cdot s_{\alpha,\beta} + \text{id}_{a_{\alpha,\beta}} \) may be obtained by gluing together \( U_{\alpha,\beta,\epsilon} \)'s by means of the isomorphisms \( (\text{id} + \epsilon \cdot s_{\alpha,\beta}) \circ \tau_{X,\alpha,\beta} \) (for various \( \alpha, \beta \in I_2 \)). Now, let \( e : \text{pr}^2_2(\mathcal{E}_\epsilon) \xrightarrow{\sim} \text{pr}^1_1(\mathcal{E}_\epsilon) \) be the HPD stratification on \( \mathcal{E}_\epsilon \) corresponding to \( \nabla_{X,\epsilon} \) via the equivalence of categories between (ii) and (v) in Theorem 4.4.2.

Proof. Let us fix an affine open covering \( \mathfrak{U} := \{ U_i \}_i \) of \( X \) and take a representative \( \{ (a_{\alpha,\beta})_\alpha, \beta, \} \) of the class \( v \in \mathbb{H}^1(K^\bullet[\nabla_{X,\epsilon}]) \) as displayed in (156). Also, \( s \) may be represented by a Čech 1-cocycle \( \{ s_{\alpha,\beta} \}_\alpha, \beta \in \mathcal{C}(\mathfrak{U}, T_X) \). In the following, we shall use the notation in Remark 4.5.1 and apply the discussion there to the present \( v \). In particular, \( X_\epsilon^v \) denotes the curve over \( \epsilon \) obtained by gluing together \( U_\alpha,\epsilon \)'s by means of the isomorphisms \( \tau_{X,\alpha,\epsilon} \) (cf. (157)). Since \( \text{id}_{U_{\alpha,\beta,\epsilon}} + \epsilon \cdot s_{\alpha,\beta} + \text{id}_{a_{\alpha,\beta}} \) may be obtained by gluing together \( U_{\alpha,\beta,\epsilon} \)'s by means of the isomorphisms \( (\text{id} + \epsilon \cdot s_{\alpha,\beta}) \circ \tau_{X,\alpha,\beta} \) (for various \( \alpha, \beta \in I_2 \)). Now, let \( e : \text{pr}^2_2(\mathcal{E}_\epsilon) \xrightarrow{\sim} \text{pr}^1_1(\mathcal{E}_\epsilon) \) be the HPD stratification on \( \mathcal{E}_\epsilon \) corresponding to \( \nabla_{X,\epsilon} \) via the equivalence of categories between (ii) and (v) in Theorem 4.4.2.

If \( i_{\alpha,\beta} : D_{U_{\alpha,\beta,1}/R}(1) \to D_{X/1}(1) \) (for each \( \alpha, \beta \in I_2 \)) denotes the morphism arising from the natural open immersion \( U_{\alpha,\beta,1} \to X_\epsilon \), then the following equality holds:

\[
(\epsilon \circ (\text{id}_{U_{\alpha,\beta,\epsilon}} + \epsilon \cdot s_{\alpha,\beta})) \circ (\epsilon) = \text{id}_{\epsilon \circ U_{\alpha,\beta}} + (\nabla_{X,\epsilon} \circ U_{\alpha,\beta} + \epsilon \cdot b_{\alpha}) \circ \epsilon = \text{id}_{\epsilon \circ U_{\alpha,\beta}} + \epsilon \cdot \nabla_{X,\epsilon} \circ s_{\alpha,\beta},
\]

where \( (\text{id}_{U_{\alpha,\beta,\epsilon}}, \text{id}_{U_{\alpha,\beta,\epsilon}} + \epsilon \cdot s_{\alpha,\beta}) \) denotes the unique morphism \( U_{\alpha,\beta,\epsilon} \to D_{U_{\alpha,\beta,1}/R}(1) \) whose composites with the first and the second projections \( D_{U_{\alpha,\beta,1}/R}(1) \to U_{\alpha,\beta,1} \) coincide with \( \text{id}_{U_{\alpha,\beta,\epsilon}} \) and \( \text{id}_{U_{\alpha,\beta,\epsilon}} + \epsilon \cdot s_{\alpha,\beta} \) respectively. The following square diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{E}_\epsilon \circ U_{\alpha,\beta} & \xrightarrow{(\text{id} + \epsilon \cdot \nabla_{X,\epsilon} \circ s_{\alpha,\beta}) \circ \tau_{X,\alpha,\beta}} & \mathcal{E}_\epsilon \circ U_{\alpha,\beta} \\
\downarrow & & \downarrow \\
U_{\alpha,\beta} \circ U_{\alpha,\beta} & \xrightarrow{(\text{id} + \epsilon \cdot s_{\alpha,\beta}) \circ \tau_{X,\alpha,\beta}} & U_{\alpha,\beta} \circ U_{\alpha,\beta},
\end{array}
\]

where the both sides of the vertical arrows denote the natural projections. It follows from the definition of \( \lambda_{(-)}(-) \) that \( (\lambda_{X,\epsilon} \circ \mathcal{E}_\epsilon, \lambda_{X,\epsilon} \circ \nabla_{X,\epsilon}) \) may be obtained by gluing together \( (\mathcal{E}_\epsilon \circ U_{\alpha,\beta}, \nabla_{X,\epsilon} \circ U_{\alpha,\beta} + \epsilon \cdot b_{\alpha}) \) by means of the isomorphisms

\[
(\text{id}_{\mathcal{E}_\epsilon \circ U_{\alpha,\beta}} + \epsilon \cdot \nabla_{X,\epsilon} \circ s_{\alpha,\beta}) \circ \tau_{X,\alpha,\beta} = \text{id}_{\mathcal{E}_\epsilon \circ U_{\alpha,\beta}} + \epsilon \cdot (a_{\alpha,\beta} + \nabla_{X,\epsilon} \circ s_{\alpha,\beta}),
\]

for various \( \alpha, \beta \in I_2 \). Hence, the element of \( \mathbb{H}^1(K^\bullet[\nabla_{X,\epsilon}]) \) corresponding to the deformation \( (X_\epsilon^v, \lambda_{X,\epsilon} \circ \mathcal{E}_\epsilon, \lambda_{X,\epsilon} \circ \nabla_{X,\epsilon}) \) may be represented by the collection of data

\[
\{ a_{\alpha,\beta} + \nabla_{X,\epsilon} \circ s_{\alpha,\beta} \}_{\alpha, \beta}, \{ b_\alpha \}_\alpha,
\]

which specifies \( v + \nabla_{X,\epsilon}^H(s) \). This completes the proof of the assertion. \( \square \)
4.6. Deformation space of indigenous bundles.

Suppose further that \( G = \text{PGL}_2 \) and \((\mathcal{E}, \nabla_\mathcal{E})\) forms an indigenous bundle on the curve \( X/R \). Denote by \( \pi_B : \mathcal{E}_B \rightarrow X \) the Hodge reduction of \((\mathcal{E}, \nabla_\mathcal{E})\) and by

\[
\nabla_{\mathcal{E}_B}^\text{ad} : \tilde{\mathcal{T}}_{\mathcal{E}_B/R} \rightarrow \Omega_{X/R} \otimes \text{Ad}(\mathcal{E})
\]

the morphism obtained from \( \nabla_{\mathcal{E}}^\text{ad} \) by restricting its domain. Then, we obtain the following morphism of short exact sequences:

\[
0 \longrightarrow \tilde{\mathcal{T}}_{\mathcal{E}_B/R} \overset{\text{incl.}}{\longrightarrow} \tilde{\mathcal{T}}_{\mathcal{E}/R} \overset{d\pi}{\longrightarrow} \mathcal{T}_{X/R} \longrightarrow 0
\]

where \( d\pi \) denotes the composite of the quotient \( \tilde{\mathcal{T}}_{\mathcal{E}/R} \rightarrow \tilde{\mathcal{T}}_{\mathcal{E}_B/R} \) and the inverse of \( \text{KS}(\mathcal{E}, \nabla_\mathcal{E}) : \mathcal{T}_{X/R} \overset{\sim}{\rightarrow} \tilde{\mathcal{T}}_{\mathcal{E}_B/R} \) (cf. (13)). Since \( H^0(X, \mathcal{T}_{X/R}) = 0 \), this morphism of sequences induces the following short exact sequence:

\[
0 \longrightarrow \mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}_B/R}^\text{ad}]) \overset{i^\mathbb{H}}{\longrightarrow} \mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}/R}^\text{ad}]) \overset{\mathbb{H}}{\longrightarrow} H^1(X, \mathcal{T}_{X/R}) \longrightarrow 0.
\]

If we consider \( \mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}_B/R}^\text{ad}]) \) as a submodule of \( \mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}/R}^\text{ad}]) \) via \( i^\mathbb{H} \), then the elements of \( \mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}_B/R}^\text{ad}]) \) classify the deformations equipped with a deformation of the \( B \)-reduction \( \mathcal{E}_B \). This implies that if

\[
\text{Def}^\text{ind}_{R_e}(X, \mathcal{E}, \nabla_\mathcal{E})
\]

the set of isomorphism classes of deformations over \( R_e \) of \((X, \mathcal{E}, \nabla_\mathcal{E})\) (as a data consisting of a curve and an indigenous bundle on it), then the bijection (152) restricts to a bijection

\[
\mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}_B}^\text{ad}]) \overset{\sim}{\rightarrow} \text{Def}^\text{ind}_{R_e}(X, \mathcal{E}, \nabla_\mathcal{E}).
\]

Denote by

\[
\eta : \tilde{\mathcal{T}}_{\mathcal{E}/R} \rightarrow \tilde{\mathcal{T}}_{\mathcal{E}/R}
\]

the \( \mathcal{O}_X \)-linear endomorphism of \( \tilde{\mathcal{T}}_{\mathcal{E}/R} \) given by \( s \mapsto s - (\nabla_\mathcal{E} \circ d\pi)(s) \) for any local section \( s \in \tilde{\mathcal{T}}_{\mathcal{E}/R} \). One verifies that \( \eta \) is an isomorphism and its inverse may be given by \( s - (\nabla_\mathcal{E} \circ d\pi)(s) \). Since the equality \( \nabla^\text{ad}_\mathcal{E} \circ \eta = \nabla^\text{ad}_\mathcal{E} \) holds, the pair of morphisms \( (\eta, \text{id}_{\Omega_{X/R} \otimes \text{Ad}(\mathcal{E})}) \) specifies an automorphism of the complex \( \mathcal{K}^\bullet[\nabla^\text{ad}_\mathcal{E}] \). Moreover, one verify that this automorphism restricts to an isomorphism \( \mathcal{K}^\bullet[\nabla^\text{ad}_\mathcal{E}_B] \overset{\sim}{\rightarrow} \mathcal{K}^\bullet[\nabla^\text{ad}_\mathcal{E}] \). By applying the functor \( \mathbb{H}^1(\_\)\) to these isomorphisms of complexes, we obtain the following square diagram:

\[
\begin{array}{ccc}
\mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}_B}^\text{ad}]) & \overset{i^\mathbb{H}_{\text{ad}}}{\longrightarrow} & \mathbb{H}^1(\mathcal{K}^\bullet[\nabla^\text{ad}_\mathcal{E}]) \\
\downarrow \iota^\mathbb{H} & & \downarrow \iota^\mathbb{H} \\
\mathbb{H}^1(\mathcal{K}^\bullet[\nabla_{\mathcal{E}}^\text{ad}]) & \overset{\sim}{\rightarrow} & \mathbb{H}^1(\mathcal{K}^\bullet[\nabla^\text{ad}_\mathcal{E}]).
\end{array}
\]
In particular, by restricting $\eta_{B,J}^{\text{ad}}$, we obtain an isomorphism
\begin{equation}
\eta_{B,J}^{\text{ad}} : J^{\text{H}}(K^*_{\text{\ad}}([\nabla_{E_B}])) \cong J^{\text{H}}(K^*_{\text{\ad}}([\nabla_{E}])).
\end{equation}

Now, let us introduce some notation to describe the statement of Proposition 4.6.1 below. Fix an element $u$ of $\mathbb{H}^{1}(K^*_{\text{\ad}}([\nabla_{E_B}])) \otimes \overline{R}$, where $\overline{R} := R/J$. Denote by
\begin{equation}
\text{Def}_{R_{\text{c}}}^{\text{ind}}(X, E, \nabla_{E})_{u}
\end{equation}
the subset of $\text{Def}_{R_{\text{c}}}^{\text{ind}}(X, E, \nabla_{E})$ consisting of deformations whose reductions modulo $J$ correspond to $u$ via the bijection (169). Notice that the $\mathbb{H}^{1}(K^*_{\text{\ad}}([\nabla_{E_B}]))$-action on $\text{Def}_{R_{\text{c}}}^{\text{ind}}(X, E, \nabla_{E})$ arising from (169) gives the structure of $J\mathbb{H}^{1}(K^*_{\text{\ad}}([\nabla_{E_B}]))$-torsor on $\text{Def}_{R_{\text{c}}}^{\text{ind}}(X, E, \nabla_{E})_{u}$. Also, let us fix an element $s$ of $H^{1}(X, \mathcal{T}_{X/R})$ whose image in $H^{1}(X, \mathcal{T}_{X/R}) \otimes \overline{R}$ coincides with $(\pi^{\mathbb{H}} \otimes \text{id}_{\overline{R}})(u)$. Denote by
\begin{equation}
\text{Def}_{R_{\text{c}}}(X, E, \nabla_{E})_{u,s}
\end{equation}
the subset of $\text{Def}_{R_{\text{c}}}(X, E, \nabla_{E})$ consisting of deformations which are contained in $(\pi^{\mathbb{H}})^{-1}(s)$ via (152) and whose reductions modulo $J$ correspond to $u$. By means of (152), it has canonically the structure of $J\mathbb{H}^{1}(K^*_{\text{\ad}}([\nabla_{E_B}]))$-torsor. Then, the following assertion holds.

**Proposition 4.6.1.**

Let us keep the notation in Proposition 4.5.2. Suppose that $(E, \nabla_{E})$ is quasi-nilpotent. Then, the assignment
\begin{equation}
(X_{e}^{v}, E_{e}^{v}, \nabla_{E_{e}}^{v}) \mapsto (X_{e}^{v} \circ s \circ v, \lambda_{X_{e}^{v} \circ s \circ v}(E_{e}^{v}), \lambda_{X_{e}^{v} \circ s \circ v}(\nabla_{E_{e}}^{v})),
\end{equation}
where $s \circ v := s - \pi^{\mathbb{H}}(v)$, defines a bijection
\begin{equation}
\text{Def}_{R_{\text{c}}}^{\text{ind}}(X, E, \nabla_{E})_{u} \xrightarrow{\cong} \text{Def}_{R_{\text{c}}}(X, E, \nabla_{E})_{u,s},
\end{equation}
that is compatible with the respective torsor structures via $\eta_{B,J}^{\text{ad}}$ (cf. (172)).

**Proof.** By definition, the assignment (175) is compatible with the respective torsor structures on $\text{Def}_{R_{\text{c}}}^{\text{ind}}(X, E, \nabla_{E})_{u}$ and $\text{Def}_{R_{\text{c}}}(X, E, \nabla_{E})_{u,s}$. Thus, the assertion follows from the fact that $\eta_{B,J}^{\text{ad}}$ is bijective.

By Proposition 4.6.1 we obtain the following corollary, which was used in the proof of Theorem 2.4.1

**Corollary 4.6.2.**

Denote by $X_{e}^{s}$ deformation over $R_{e}$ corresponding to $s$ and by $(\overline{X}_{e}^{u}, E_{e}^{u}, \nabla_{E_{e}}^{u})$ the deformation over $\overline{R}_{e} := R_{e} \times_{R} \overline{R}$ of $(X, E, \nabla_{E})$ corresponding to $u$. (Hence, the reduction modulo $J$ of $X_{e}^{s}$ is isomorphic to $\overline{X}_{e}^{u}$, and $(\overline{E}_{e}^{u}, \nabla_{E_{e}}^{u})$ forms an indigenous bundle on $\overline{X}_{e}^{u}$.) Now, suppose that $(E, \nabla_{E})$ is quasi-nilpotent. Also, let $E^{\diamond}$ be a crystal of $\text{PGL}_{2}$-torsors on $\text{Crys}(\overline{X}_{e}^{u}/R_{e})$ such that the associated crystal on $\text{Crys}(\overline{X}_{e}/R_{e})$ corresponds to $(\overline{E}_{e}^{u}, \nabla_{E_{e}}^{u})$ via the equivalence of categories between (i) and (v) in Theorem 4.4.2. Then, there exists a unique (up to isomorphism) deformation over $X_{e}^{s}$ of $(\overline{E}_{e}^{u}, \nabla_{E_{e}}^{u})$ which forms an indigenous bundle and corresponds to $E^{\diamond}$ (via the equivalence between (i) and (v) as above).
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