Abstract

Let $M$ be a complete Riemannian 3-manifold with sectional curvatures between 0 and 1. A minimal 2-sphere immersed in $M$ has area at least $4\pi$. If an embedded minimal sphere has area $4\pi$, then $M$ is isometric to the unit 3-sphere or to a quotient of the product of the unit 2-sphere with $\mathbb{R}$, with the product metric. We also obtain a rigidity theorem for the existence of hyperbolic cusps. Let $M$ be a complete Riemannian 3-manifold with sectional curvatures bounded above by $-1$. Suppose there is a 2-torus $T$ embedded in $M$ with mean curvature one. Then the mean convex component of $M$ bounded by $T$ is a hyperbolic cusp, i.e., it is isometric to $T \times \mathbb{R}$ with the constant curvature $-1$ metric: $e^{-2t}d\sigma_0^2 + dt^2$ with $d\sigma_0^2$ a flat metric on $T$.

Keywords: area of minimal sphere, rigidity of 3-manifolds, hyperbolic cusp.

1 Introduction

Consider a smooth ($C^\infty$) complete metric on the 2-sphere $S$ whose curvature is between 0 and 1. It is well known that a simple closed geodesic in $S$ has length at least $2\pi$ (see [1] or Klingenberg’s theorem in higher dimension [2]). It is less well known that when such an $S$ has a simple closed geodesic of length exactly $2\pi$, then $S$ is isometric to the unit 2-sphere $S^2_1$. This result is proved in [1], and the authors attribute the theorem to E. Calabi.

With this in mind, we consider what happens in a complete 3-manifold $M$ with sectional curvatures between 0 and 1 (henceforth we suppose this curvature condition on $M$, unless stated otherwise).

Let $\Sigma$ be an embedded minimal 2-sphere in $M$. Then the Gauss-Bonnet theorem and the Gauss equation tells us that the area of $S$ is at least $4\pi$: indeed we have

$$4\pi = \int_{\Sigma} \bar{K}_{\Sigma} = \int \text{det}(A) + K_{T\Sigma} \leq \int_{\Sigma} 1 = A(\Sigma)$$  \hspace{1cm} (1)

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with \(\det(A)\) the determinant of the shape operator which is non positive. We prove in Theorem 1 that when the area of \(\Sigma\) equals \(4\pi\), then \(M\) is isometric to the unit 3-sphere \(S^3_1\) or to a quotient of the product of the unit 2-sphere with \(\mathbb{R}\), \(S^2_1 \times \mathbb{R}\), with the product metric.

We remark that Theorem 1 does not hold for embedded minimal tori. Given \(\varepsilon\) greater than zero, there are Berger spheres with curvatures between 0 and 1, which contain embedded minimal tori of area less than \(\varepsilon\). But a minimal sphere always has area at least \(4\pi\).

It would be interesting to know what happens in higher dimensions. In the unit \(n\)-sphere \(S^n_1\), a compact minimal hyper-surface \(\Sigma\) always has volume at least the volume of the equatorial \(n-1\) sphere \(S^{n-1}_1\). Is there a rigidity theorem when one allows metrics on \(S^n (\equiv M)\), of sectional curvatures between 0 and 1? Two questions arise. First, does an embedded minimal hyper-sphere \(\Sigma\) in \(M\) have volume at least the volume of \(S^{n-1}_1\)? If this is so, and if \(\Sigma\) is an embedded minimal hyper-sphere with volume exactly the volume of \(S^{n-1}_1\), is \(M\) isometric to \(S^n_1\) or to \(S^{n-1}_1 \times \mathbb{R}\)?

In the same spirit as Theorem 1, we prove a rigidity theorem for hyperbolic cusps. We recall that a 3 dimensional hyperbolic cusp is a manifold of the form \(T \times \mathbb{R}\) with \(T\) a 2-torus and the hyperbolic metric \(e^{-2t}d\sigma_0^2 + dt^2\) with \(d\sigma_0^2\) a flat metric on \(T\). In Theorem 2, we prove that if \(M\) is a complete Riemannian manifold with sectional curvatures bounded above by \(-1\) and \(T\) is a constant mean curvature 1 torus embedded in \(M\) then the mean convex side of \(T\) in \(M\) is isometric to a hyperbolic cusp.

2 Minimal spheres of area \(4\pi\) and rigidity of 3-manifolds

In this section, we prove a rigidity result for a Riemannian 3-manifold \(M\) whose sectional curvatures are between 0 and 1. As explained in the introduction, any minimal sphere in such a manifold has area at least \(4\pi\).

We denote by \(S^n_1\) the sphere of dimension \(n\) with constant sectional curvature 1. We then have the following result.

**Theorem 1.** Let \(M\) be a complete Riemannian 3-manifold whose sectional curvatures satisfy \(0 \leq K \leq 1\). Assume that there exists an embedded minimal sphere \(\Sigma\) in \(M\) with area \(4\pi\). Then the manifold \(M\) is isometric either to the sphere \(S^3_1\) or to a quotient of \(S^2_1 \times \mathbb{R}\).

**Proof.** Let \(\Phi\) be the map \(\Sigma \times \mathbb{R} \to M, (p,t) \mapsto \exp_p(tN(q))\) where \(N\) is a unit normal vector field along \(\Sigma\). In the following, we focus on \(\Sigma \times \mathbb{R}_+\); by
symmetry of the configuration, the study is similar for $\Sigma \times \mathbb{R}_-$. 

$\Sigma$ is compact, so there is an $\varepsilon$ such that $\Phi$ is an immersion and even an embedding on $\Sigma \times [0, \varepsilon)$. Let us define 

$$
\varepsilon_0 = \sup \{ \varepsilon > 0 | \Phi \text{ is an immersion on } \Sigma \times [0, \varepsilon) \};
$$

$\varepsilon_0$ can be equal to $+\infty$. Using $\Phi$, we pull back the Riemannian metric of $M$ to $\Sigma \times [0, \varepsilon_0)$. This metric can be written $ds^2 = d\sigma^2_t + dt^2$ where $d\sigma^2_t$ is a smooth family of metrics on $\Sigma$. With this metric, $\Phi$ becomes a local isometry from $\Sigma \times [0, \varepsilon_0)$ to $M$ and $(\Sigma \times [0, \varepsilon_0), ds^2)$ has sectional curvatures between 0 and 1. Moreover, $\Sigma_0$ is minimal and has area $4\pi$. Actually, we will prove the following facts.

**Claim.** The metric $d\sigma^2_0$ has constant sectional curvature 1 so $(\Sigma, d\sigma^2_0)$ is isometric to $S^2_1$. Moreover, we have two cases

1. $\varepsilon_0 = \pi/2$ and $d\sigma^2_t = \sin^2 t d\sigma^2_0$ or
2. $\varepsilon_0 = +\infty$ and $d\sigma^2_t = d\sigma^2_0$

Let us denote by $\Sigma_t = \Sigma \times \{t\}$ the equidistant surfaces. We denote by $H(p, t)$ the mean curvature of $\Sigma_t$ at the point $(p, t)$ with respect to the unit normal vector $\partial_t$. We also define $\lambda(p, t) \geq 0$ such that $H + \lambda$ and $H - \lambda$ are the principal curvature of $\Sigma_t$ at $(p, t)$. We notice that $\lambda = 0$ if $\Sigma_t$ is umbilical at $(p, t)$.

The surfaces $\Sigma_t$ are spheres so, using the Gauss equation, the Gauss-Bonnet formula implies:

$$
4\pi = \int_{\Sigma_t} \bar{K}_{\Sigma_t} = \int_{\Sigma_t} (H + \lambda)(H - \lambda) + K_t = \int_{\Sigma_t} H^2 - \lambda^2 + K_t
$$

where $\bar{K}_{\Sigma_t}$ is the intrinsic curvature of $\Sigma_t$ and $K_t$ is the sectional curvature of the ambient manifold of the tangent space to $\Sigma_t$. Since $K_t \leq 1$, we obtain the following inequality

$$
\int_{\Sigma_t} \lambda^2 = \int_{\Sigma_t} H^2 + K_t - 4\pi \leq \int_{\Sigma_t} H^2 + A(\Sigma_t) - 4\pi \quad (2)
$$

where $A(\Sigma_t)$ is the area of $\Sigma_t$. In the following, we denote by $F(t)$ the right hand side of this inequality.

**Claim 1.** $F$ is vanishing on $[0, \varepsilon_0)$. 


Since $\Sigma_0$ is minimal and has area $4\pi$, we have $F(0) = 0$. We notice that this implies that $\lambda(p, 0) = 0$ so $\Sigma_0$ is umbilical and $K_{T\Sigma_0} = 1$. Thus $(\Sigma_0, d\sigma_0)$ is isometric to $S^2_1$.

We have the usual formula:
\[
\frac{\partial}{\partial t} A(\Sigma_t) = -\int_{\Sigma_t} 2H \quad \text{and} \quad \frac{\partial H}{\partial t} = \frac{1}{2}(Ric(\partial_t) + |A_t|^2)
\]
where $A_t$ is the shape operator of $\Sigma_t$ and $Ric$ is the Ricci tensor of $\Sigma \times [0, \varepsilon_0)$. Since the sectional curvatures of $M \times [0, \varepsilon_0)$ are non-negative, $Ric$ is non-negative. So the second formula above implies that $H$ is increasing and thus $H \geq 0$ everywhere. Let us now compute and estimate the derivative of $F$:
\[
F'(t) = \int_{\Sigma_t} (2H \frac{\partial H}{\partial t} - 2H^2) - \int_{\Sigma_t} 2H
\]
\[
= \int_{\Sigma_t} H(Ric(\partial_t) + |A_t|^2 - 2H^2 - 2)
\]
\[
= \int_{\Sigma_t} H((Ric(\partial_t) - 2) + ((H + \lambda)^2 + (H - \lambda)^2 - 2H^2))
\]
\[
= \int_{\Sigma_t} H((Ric(\partial_t) - 2) + 2\lambda^2)
\]
\[
\leq 2 \int_{\Sigma_t} H\lambda^2
\]
where the last inequality comes from $Ric(\partial_t) - 2 \leq 0$ because of the hypothesis on the sectional curvatures. If we choose $\varepsilon < \varepsilon_0$, there is a constant $C \geq 0$ such that $H \leq C$ on $\Sigma \times [0, \varepsilon]$. So for $t \in [0, \varepsilon]$, using the inequality (2), we get $F'(t) \leq 2CF(t)$. Then $F(t) \leq F(0)e^{2Ct} = 0$ on $[0, \varepsilon]$. So $F \leq 0$ on $[0, \varepsilon_0)$ and, because of (2), $F = 0$ on $[0, \varepsilon_0)$; this finishes the proof of Claim 1.

The first consequence of Claim 1 is that all the equidistant surfaces $\Sigma_t$ are umbilical (see inequality (2)); so $\lambda \equiv 0$. In the computation of the derivative of $F$, this implies that
\[
\int_{\Sigma_t} H(Ric(\partial_t) - 2) = 0
\]
Since $H(Ric(\partial_t) - 2) \leq 0$ everywhere, we obtain
\[
H(Ric(\partial_t) - 2) = 0 \text{ everywhere.}
\]
Moreover the umbilicity and (3) implies that $\frac{\partial H}{\partial t} = \frac{1}{2}Ric(\partial_t) + H^2$. We now prove the following claim
Claim 2. Let \((p, t) \in \Sigma \times [0, \varepsilon_0) \ (t > 0)\) be such that \(H(p, t) > 0\) then \(H(q, t) > 0\) for any \(q \in \Sigma\).

In other words, when the mean curvature is positive at a point of an equidistant, it is positive at any point of this equidistant. We recall that \(H\) is increasing in the \(t\) variable so when it becomes positive it stays positive.

So assume that \(H(p, t) > 0\) and consider \(\Omega = \{q \in \Sigma | H(q, t) > 0\}\) which is a nonempty open subset of \(\Sigma\). Let \(q \in \Omega\). Since \(H(q, t) > 0\), \(\text{Ric}(\partial_t)(q, t) = 2\) by \([3]\). Thus \(\text{Ric}(\partial_t)(r, t) = 2\) for any \(r \in \Omega\). So if \(r \in \Omega\), \(\text{Ric}(\partial_t)(r, s) > 0\) for \(s < t\), close to \(t\) and, by \([3]\), this implies that \(H(r, t) > 0\) and \(r \in \Omega\). So \(\Omega\) is closed and \(\Omega = \Sigma\). This finishes the proof of Claim 2.

Let us assume that there is an \(\varepsilon_1 > 0\) such that \(H(p, t) = 0\) for \((p, t) \in \Sigma \times [0, \varepsilon_1]\) and \(H(p, t) > 0\) for any \((p, t) \in \Sigma \times (\varepsilon_1, \varepsilon_0)\). Because of the evolution equation of \(H\), this implies that \(\text{Ric}(\partial_t) = 0\) on \(\Sigma \times [0, \varepsilon_1]\). On \(\Sigma \times (\varepsilon_1, \varepsilon_0)\), we have \(\text{Ric}(\partial_t) = 2\) because of \([3]\). So by continuity of \(\text{Ric}(\partial_t)\), we get a contradiction and then we have two possibilities

1. \(H = 0\) on \(\Sigma \times [0, \varepsilon_0)\) and \(\text{Ric}(\partial_t) = 0\) on \(\Sigma \times [0, \varepsilon_0)\).
2. \(H > 0\) on \(\Sigma \times (0, \varepsilon_0)\) and \(\text{Ric}(\partial_t) = 2\) on \(\Sigma \times (0, \varepsilon_0)\).

In the first case, this implies that the sectional curvature of any 2-plane orthogonal to \(\Sigma_t\) is zero. Thus \(d\sigma_t^2 = d\sigma_0^2\). Since the map \(\Phi\) ceases to be an immersion only if \(d\sigma_t^2\) becomes singular this implies that \(\varepsilon_0 = +\infty\). Thus \(\Sigma \times \mathbb{R}_+\) with the induced metric is isometric to \(S_1^2 \times \mathbb{R}_+\) and \(\Phi\) is a local isometry from \(S_1^2 \times \mathbb{R}_+\) to \(M\).

In the second case, the sectional curvature of any 2-plane orthogonal to \(\Sigma_t\) is equal to 1. Thus \(d\sigma_t^2 = \sin^2 t d\sigma_0\) and \(\varepsilon_0 = \pi/2\). This also implies that \(\Phi(p, \pi/2)\) is a point. So \(\Sigma \times [0, \pi/2]\) with the metric \(ds^2\) is isometric to a hemisphere of \(S_1^3\) and the map \(\Phi\) is a local isometry from that hemisphere to \(M\).

Doing the same study for \(\Sigma \times \mathbb{R}_-\), we get in the first case a local isometry \(\Phi : S_1^2 \times \mathbb{R} \to M\) and in the second case a local isometry \(\Phi : S_1^3 \to M\). Since \(S_1^2 \times \mathbb{R}\) and \(S_1^3\) are simply connected, \(\Phi\) is then the universal cover of \(M\) and \(M\) is then isometric to a quotient of \(S_1^2 \times \mathbb{R}\) or \(S_1^3\). Since \(\Phi\) is injective on \(\Sigma\) this implies that in the second case, \(\Phi\) is actually injective and then a global isometry.

Remark 1. In the proof, since \(\Phi\) is injective on \(\Sigma\), the possible quotients of \(S_1^2 \times \mathbb{R}\) are either \(S_1^2 \times \mathbb{R}\) or its quotient by the subgroup generated by an isometry of the form \(S_1^2 \times \mathbb{R} \to S_1^2 \times \mathbb{R}; (p, t) \mapsto (\alpha(p), t + t_0)\) with \(\alpha\) an isometry of \(S_1^2\) and \(t_0 \neq 0\).
Remark 2. Something can be said about constant mean curvature $H_0$ spheres in a Riemannian 3-manifold with sectional curvatures between 0 and 1. Indeed, the computation (1) implies that the area of $\Sigma$ is larger than $\frac{4\pi}{1+H_0^2}$, which is the area of a geodesic sphere in $S^3_1$ of mean curvature $H_0$. Moreover, if $\Sigma$ has area $\frac{4\pi}{1+H_0^2}$, the above proof can be adapted to prove that the mean convex side of $\Sigma$ is isometric to a spherical cap of $S^3_1$ with constant mean curvature $H_0$ (see Theorem 2 below, for a similar result in the hyperbolic case).

Remark 3. Let $M$ be a Riemannian $n$-manifold whose sectional curvatures are between 0 and 1 and let $\Sigma$ be a minimal 2-sphere in $M$. A computation similar to (1) proves also that the area of $\Sigma$ is larger than $4\pi$. It also implies that, if $\Sigma$ has area $4\pi$, $\Sigma$ is totally geodesic and isometric to $S^2_1$.

3 Existence of hyperbolic cusps

Let $(\mathbb{T}^2, g)$ be a flat 2 torus, the manifold $\mathbb{T}^2 \times \mathbb{R}_+$ with the complete Riemannian metric $e^{-2t}g + dt^2$ is a hyperbolic 3-dimensional cusp. $\mathbb{T}^2 \times \mathbb{R}$ is actually isometric to the quotient of a horoball of $\mathbb{H}^3$ by a $\mathbb{Z}^2$ subgroup of isometries of $\mathbb{H}^2$ leaving the horoball invariant. Any $\mathbb{T}^2 \times \{t\}$ has constant mean curvature 1. The following theorem says that, in certain 3-manifolds, a constant mean curvature 1 torus is necessarily the boundary of a hyperbolic cusp.

**Theorem 2.** Let $M$ be a complete Riemannian 3-manifold with its sectional curvatures satisfying $K \leq -1$. Assume that there exists a constant mean curvature 1 torus $T$ embedded in $M$. Then $T$ separates $M$ and its mean convex side is isometric to a hyperbolic cusp.

As a consequence, the existence of this torus implies that $M$ can not be compact. The proof uses the same ideas as in Theorem 1.

**Proof.** Let us consider the map $\Phi : T \times \mathbb{R}_+ \to M, (p, t) \mapsto \exp_p(tN(p))$ where $N$ is the unit normal vector field normal to $T$ such that $N$ is the mean curvature vector of $T$. Let us define

$$\varepsilon_0 = \sup \{ \varepsilon > 0 | \Phi \text{ is an immersion on } T \times [0, \varepsilon) \}.$$  

Using $\Phi$, we pull back the Riemannian metric of $M$ to $T \times [0, \varepsilon_0)$; it can be written $ds^2 = dt^2 + d\sigma^2$. We define $T_t = T \times \{t\}$ the equidistant surfaces to $T_0$. We also denote by $H(p, t)$ the mean curvature of the equidistant surfaces.
at \((p, t)\) with respect to \(\partial_t\). We finally define \(\lambda(p, t)\) such that \(H + \lambda \) and \(H - \lambda\) are the principal curvatures of \(T_t\) at \((p, t)\).

The surfaces \(T_t\) are tori so, by the Gauss equation and the Gauss-Bonnet formula, we have

\[
0 = \int_{T_t} K_{T_t} = \int_{T_t} H^2 - \lambda^2 + K_t
\]

where \(K_t\) is the sectional curvature of the ambient manifold of the tangent space to \(T_t\). Since \(K_t \leq -1\), we obtain the inequality

\[
\int_{T_t} \lambda^2 = \int_{T_t} H^2 + K_t \leq \int_{T_t} H^2 - A(T_t)
\]

Let \(F(t)\) denote the right hand term of the above inequality. By hypothesis, \(H(p, 0) = 1\) so \(F(0) = 0\) and \(F(t) \geq 0\) for any \(t \geq 0\). Let us compute the derivative of \(F\)

\[
F'(t) = \int_{T_t} (2H \frac{\partial H}{\partial t} - 2H^3) + \int_{T_t} 2H
\]

\[
= \int_{T_t} H(Ric(\partial_t)) + |A_t|^2 - 2H^2 + 2
\]

\[
= \int_{T_t} H((Ric(\partial_t) + 2) + 2\lambda^2)
\]

Since \(H(p, 0) = 1\), we can consider \(\varepsilon \in (0, \varepsilon_0)\) such that \(0 < H \leq C\) on \(T \times [0, \varepsilon]\). Since \(Ric(\partial_t) + 2 \leq 0\) we get:

\[
F'(t) \leq \int_{T_t} 2H \lambda^2 \leq 2CF(t)
\]

Thus \(F(t) \leq F(0)e^{2Ct}\) for \(t \in [0, \varepsilon]\); this implies \(F(t) = 0\) on that segment. We then obtain \(\lambda = 0\) on \(T \times [0, \varepsilon]\) (the equidistant surfaces are umbilical) and \(Ric(\partial_t) = -2\) since \(H > 0\). Thus \(H\) satisfies the differential equation

\[
\frac{\partial H}{\partial t} = -2 + 2H^2.
\]

This gives that \(H = 1\) on \(T \times [0, \varepsilon]\) since \(H = 1\) on \(T_0\). Thus we can let \(\varepsilon\) tend to \(\varepsilon_0\) to obtain that \(F(t) = 0\) on \([0, \varepsilon_0]\) and \(Ric(\partial_t) = -2\) and \(H = 1\) on \(T \times [0, \varepsilon_0]\). Since \(0 = \int_{T_t} H^2 + K_t\) and \(K_t \leq -1\), it follows that \(K_t = -1\) for all \(t\) in the interval. We then have proved that the sectional curvature of \(T \times [0, \varepsilon_0]\) with the metric \(ds^2\) is equal to \(-1\) for any 2-plane. Moreover, we get that \(d\sigma_0^2\) is flat and that \(d\sigma_t^2 = e^{-2t}d\sigma_0^2\). This implies that \(\Phi\) is actually an immersion on \(T \times \mathbb{R}_+\) \((\varepsilon_0 = +\infty)\) and \(T \times \mathbb{R}_+\) is isometric to a hyperbolic cusp. \(\Phi\) is then a local isometry from this hyperbolic cusp to \(M\).
To finish the proof, let us prove that $\Phi$ is in fact injective. If this is not the case, let $\varepsilon_1 > 0$ be the smallest $\varepsilon$ such that $\Phi$ is not injective on $T \times [0, \varepsilon]$.
This implies that there exist $p$ and $q$ in $T$ such that

- either $\Phi(p, 0) = \Phi(q, \varepsilon_1)$
- or $\Phi(p, \varepsilon_1) = \Phi(q, \varepsilon_1)$ (with $p \neq q$ in this case).

Let $U$ and $V$ be respective neighborhoods of $(p, 0)$ (or $(p, \varepsilon_1)$) in $T_0$ (or $T_{\varepsilon_1}$) and $(q, \varepsilon_1)$ in $T_{\varepsilon_1}$ such that $\Phi$ is injective on them. Since $\varepsilon_1$ is the smallest one, $\Phi(U)$ and $\Phi(V)$ are two constant mean curvature 1 surfaces in $M$ that are tangent at $\Phi(q, \varepsilon_1)$. Moreover, in the first case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ so by the maximum principle $\Phi(U) = \Phi(V)$. Thus $\Phi(T_0)$ would be equal to $\Phi(T_{\varepsilon_1})$ which is impossible since these two surfaces do not have the same area. In the second case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ and then $\Phi$ is not injective on $T_s$ for $s$ near $t$ $s < t$, which is a contradiction.

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