Dynamics of compressible edge and bosonization

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We work out the dynamics of the compressible edge of the quantum Hall system based on the electrostatic model of Chklovskii et al. We introduce a generalized version of Wen’s hydrodynamic quantization approach to the dynamics of sharp edge and rederive Aleiner and Glazman’s earlier result of multiple density modes. Bosonic operators of density excitations are used to construct fermions at the interface of the compressible and incompressible region. We also analyze the dynamics starting with the second-quantized Hamiltonian in the lowest Landau level and work out the time development of density operators. Contrary to the hydrodynamic results, the density modes are strongly coupled. We argue that the coupling suppresses the propagation of all acoustic modes, and that the excitations with large wavevectors are subject to decay due to coupling to the dissipative acoustic modes. A possible correction to the tunneling density of states is discussed.

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I. INTRODUCTION

The theory of edge excitations in the fractional quantum hall liquid had been first derived by Wen based on the gauge invariance of the Chern-Simons term for a bounded system. A more intuitive picture of the edge excitation was laid out in his subsequent paper where a classical energy and a classical equation of motion for the incompressible liquid is written down, and he stipulated that the conjugate variables in the corresponding Lagrangian satisfy the canonical commutation relations. In the absence of a more microscopic theory, such “phenomenological quantization” serves as a guideline in our understanding of the phenomena and not infrequently yields accurate results. Quite a lot of progress in the study of vortex motion in superfluid He and superconductors had been made in this way. Eventually, any phenomenological theory has to be based on the analysis of the underlying microscopic Hamiltonian. A good example is the plasma oscillation in a metal where one can show that a quantum-mechanical equation of motion of the density operator reduces to the (semi-)classical one in the regime where RPA is applicable. The strength of the microscopic analysis lies in the fact that it yields more information than can be garnered at the phenomenological level. Coupling of a plasma mode to single-particle excitations give rise to Landau damping and yields a correction to classical dispersion that can be systematically calculated using many-body theory.

With a sharp edge, there are so few degrees of freedom that one cannot distinguish between collective modes and single-particle excitations. This leads to bosonization of one-dimensional fluids for which the single-particle properties can be uniquely derived once we know the collective excitation spectrum.

The situation changes when one realizes that the physical edge is often not a sharp boundary separating an incompressible bulk from its surroundings. Chklovskii et al. have demonstrated that with the gate-confined 2DEG, the edge is a rather wide, smoothly varying region of density. While the idea of compressible edge had been actively adopted both experimentally and theoretically in the past few years, no attempt, to our knowledge, has been made to bridge it with the bosonization theory of the edge. A number of works have sprung up recently to address the effect of long-range Coulomb interaction on the properties of bosonized edge but the geometry is still that of a sharp edge as in Wen’s original paper. One interesting attempt to understand the dynamics of the compressible region was made by Aleiner and Glazman henceforth referred to as AG, who used classical hydrodynamics of charged fluid to derive the (necessarily collective) spectrum of the compressible edge analytically. They showed there are many density modes within the compressible edge, one of which is charged and higher in energy than the other, so-called acoustic modes, which are neutral. Their model, however, does not account for the boundary effect taking place at the interface of compressible and incompressible regions. We have tried in this paper to understand the gapless spectrum of a wide edge more rigorously from the quantum-mechanical standpoint.

In Sec. II we will show how the phenomenological quantization approach of Wen can be generalized to the dynamics of compressible edge including its boundary with the bulk, and recover AG’s result along the way. We also discuss the issue of bosonization in this kind of edge. In Sec. III, we start from the second-quantized Hamiltonian to address the same problem. It turns out that the density modes are strongly coupled because of the diffuse distribution of the electron occupation. In Sec. IV, we discuss various consequences of such coupling, including its possible effect on the tunneling density of states.
II. PHENOMENOLOGICAL APPROACH TO EDGE DYNAMICS

The geometry of the sample is that of a semi-infinite Hall bar where the boundary separating the compressible and incompressible liquid is defined as \( y = 0 \) and the compressible strip extends in the positive \( y \) direction (fig. 1(a)).

In the presence of a relatively weak local electric field fluctuation and a strong external magnetic field, one can write \( v_x = \delta E_y/B \) and \( v_y = -\delta E_x/B \) where the field is produced either by external potential or as a result of electron-electron interaction. In turn, the local field can be expressed as the gradient of the change in potential energy. To completely specify the motion, we need the following equations for the local density \( n(x,y) \).

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} v_x + \frac{\partial n}{\partial y} v_y &= 0, \\
n(x,y+h(x,y,t),t) &= n_0(y). 
\end{align*}
\]

(2.1) The first equation is the continuity equation for the divergence-free velocity field, \( \partial v_x/\partial x + \partial v_y/\partial y = 0 \). The second one parameterizes the dynamics by identifying points on the density contour \( n_0(y) \) with \( (x,y+h(x,y)) \). The situation is depicted in fig. 1(b). The boundary between the compressible and incompressible region is defined by \( n(x,y,0,t) = n_0(0) = \nu/2\pi l_0^2 \) where \( l_0 \) is the magnetic length. We will set both this and Planck’s constant, \( h \), equal to 1 in the rest of the section. If one imagined a collection of very long strings stacked along \( y \)-direction, each carrying a label \( n_0(y) \), the density fluctuation can be understood in terms of the strings wiggling about equilibrium, but never so violently enough that they cross other strings.

One can deduce a set of identities by differentiating Eq. (2.2) with respect to \( x, y, \) and \( t \), which can be put back into Eq.(2.2) to give

\[
\frac{\partial n_0}{\partial y} \left( \frac{\partial h}{\partial t} - v_y + v_x \frac{\partial h}{\partial x} \right) \left( 1 + \frac{\partial h}{\partial y} \right)^{-1} = 0. 
\]

(2.3)

We have \( \partial n_0/\partial y < 0 \) inside the compressible region. For small fluctuations, \( \partial h/\partial y \) should be small so that it is only the middle term which vanishes identically. By defining \( h(x,y,t) \equiv \partial n(x,y,t)/\partial x \), one arrives at the equation of motion of the compressible edge

\[
\frac{\partial^2 \eta}{\partial x^2} = v_y - v_x \frac{\partial^2 \eta}{\partial x^2} = \frac{1}{B} \frac{d}{dx} V(x,y + \partial n(x,y,t)/\partial x, t). 
\]

(2.4) where \( V(x,y+h) \) is the local potential(fluctuation) at a given time \( t \). One can also check that varying the following action reproduces the above equation of motion.

\[
S = \int \frac{1}{2} \frac{\partial n_0}{\partial y} \frac{\partial n_0}{\partial t} dxdydt - \int \frac{\partial n_0}{\partial y} \int_0^{\partial n/\partial x} eV(x,y + y')dy'dxdydt 
\]

(2.5)

with \( e \) being the electric charge. Quantization comes from the commutator,

\[
[\eta(x,y), \frac{\partial n_0}{\partial y} h(x',y')] = i\delta(x-x')\delta(y-y'). 
\]

(2.6)

Note that results derived in eqs. (2.4)-(2.6) reduce to known expressions for sharp edge by taking the limit \( \partial n_0(y)/\partial y = -(\nu/2\pi)\delta(y) \).

When one is solely interested in the dynamics inside the compressible edge, as AG were, one can regard \( V(x,y) \) as Coulomb potential due to the fluctuating density. In linear approximation, \( n(x,y) - n_0(y) = \delta n(x,y) \approx -(\partial n_0/\partial y)h(x,y) \), and the Hamiltonian is given, up to quadratic order in density, by

\[
H = \frac{e^2}{2\kappa} \int \frac{\partial n_0}{\partial y} \frac{h(x,y)h(x',y')}{\sqrt{(x-x')^2 + (y-y')^2}} dy'dxdy' \
\]

\[\kappa = 4\pi\epsilon_0\epsilon \]

(2.7)

One can expand the operator \( h(x,y) \) in terms of its Fourier components in \( x, 0 < x < L \). Each component \( h_q(y) \) satisfies

\[
i\hbar h_q(y) = \frac{2e^2q}{\kappa} \int_0^\infty dy' \frac{\partial n_0}{\partial y'} K_0(|y-y'|)h_q(y'). 
\]

(2.8)

where \( K_0 \) is the modified Bessel function of zeroth order. For an eigenmode \( h_{qa}(y) \) whose time dependence is \( e^{-i\omega_\alpha t} \), it becomes a homogeneous integral equation. We can exploit the monotone nature of the function \( n_0(y) \) to use \( n_0 \) as
coordinate, and remove the partial derivative inside the integral. Hence we arrive at a Fredholm integral equation with a real, symmetric kernel, for which the eigenfunctions form a complete, orthogonal set and all eigenvalues are real. The kernel $K_0$ is expressed in terms of eigenvalues and eigenfunctions by

$$K_0(|qy(n_0) - qy(n'_0)|) = \frac{k}{2e^2 |q|} \sum_{\alpha=0}^{\infty} \omega_\alpha(q) h_{q\alpha}(n_0) h_{q\alpha}(n'_0). \quad (2.9)$$

The eqs. (2.8)-(2.9) summarize the results initially obtained by AG. When we have the boundary separating the incompressible and compressible (IC) region, density fluctuation in the compressible region necessarily implies some of the electrons on the incompressible side have to move as well. When they move, they are doing work against the field that confines them, and it costs additional energy. The amount of energy necessary is given by

$$H_{IC} = \frac{\nu v_F}{4\pi} \int h(x, 0)^2 dx. \quad (2.10)$$

where $y=0$ indicates the IC boundary. Indeed, this was the only term that appeared in the Hamiltonian for a sharp edge, without the electron-electron interaction.

The total Hamiltonian, $H_{tot} = H + H_{IC}$ is written in the basis of eigenfunctions as

$$h(x, y) = L^{-1/2} \sum_{q} c_{q\alpha} h_{q\alpha}(y) e^{-iqx},$$

$$H_{tot} = \sum_{q\alpha} \left( \frac{\omega_\alpha(q)}{2q} + \frac{\nu v_F}{4\pi} h_{q\alpha}^2(0) \right) c_{q\alpha}^* c_{q\alpha} + \sum_{q, \alpha \neq \beta} \frac{\nu v_F}{4\pi} h_{q\alpha}(0) h_{q\beta}(0) c_{q\alpha}^* c_{q\beta}. \quad (2.11)$$

The boundary term contributes a diagonal term which modifies the slope of the dispersion but, more importantly, there are off-diagonal terms which mix states that were thought to be orthogonal. The perturbation is quadratic, hence by rotating to a proper basis, one can again recover independent modes. One might as well think of Eq. (2.9) as containing the complete dynamical information, provided we use the eigenmodes and eigenvectors of the full Hamiltonian. Quantization condition, Eq. (2.6), becomes $[c_{q\alpha}^*, c_{q\alpha}] = q$ and the Hamiltonian becomes a sum of oscillator modes whose energy levels are set by $\omega_\alpha(q)$. Before mixing, $\omega_0(q) \propto q \log q$ is the most dominant mode while $\alpha \geq 1$ are nearly linear. The boundary will mix these and the linear modes become weakly logarithmic.

From the set of harmonic oscillator-type operators at our disposal, we can follow the standard recipe of bosonization in $1d$ and construct the following set of fermion operators at the IC boundary.

$$\psi_\alpha^\dagger(x) \propto \exp \left\{ \sum_{q > 0} \frac{2\pi}{\nu} \frac{1}{q} \left( c_{q\alpha} e^{-iqx} - c_{q\alpha}^* e^{iqx} \right) \right\} \quad (2.12)$$

Now $\psi_\alpha^\dagger(x)$ involves a coherent density change over the whole compressible region including the IC boundary and differs from Wen’s construction in which an electron is created at the perimeter of the incompressible fluid.

### III. MICROSCOPIC APPROACH TO EDGE DYNAMICS

Our starting point is the Coulomb Hamiltonian,

$$H = \frac{e^2}{2\kappa} \int \frac{1}{|r - r'|} \psi^\dagger(r) \psi^\dagger(r') \psi(r') \psi(r) d^2r d^2r'. \quad (3.1)$$

We are interested in the dynamics within the lowest Landau level, therefore we can expand $\psi(r)$ as follows:

$$\psi(r) = (\sqrt{\pi L})^{-1/2} \sum_k e^{ikx} e^{-\frac{\kappa}{2}(|y-k|^2 e^{-ikx} c_k. \quad (3.2)$$

When it is substituted into Eq. (3.1), we obtain
\[
H \approx \frac{e^2}{kL} \sum_{k_1k_2q} e^{-\frac{i}{2}q^2} K_0(\{qk_1-qk_2\}) e_{k_1+\frac{q_1}{2} c_{k_1-q_1} c_{k_2+\frac{q_2}{2}}}
\]
\[
= \frac{1}{2} \sum_{q \alpha} \omega_\alpha(q)e^{-\frac{i}{2}q^2} P^\dagger_\alpha(q)P_\alpha(q),
\]
\[
P^\dagger_\alpha(q) = \frac{1}{\sqrt{|q|L}} \sum_k h_{q\alpha}(k)c^\dagger_{k+q/2}c_{k-\frac{q}{2}} = P_\alpha(-q).
\]

The difference between the normal-ordered form, Eq. (3.1), and the above Hamiltonian is an infinitely large constant which can be cancelled by assuming a compensating positive charge background. The Hamiltonian is approximate in the sense that we have replaced the convolution \( \int f(y)e^{-(y-k)^2/2}dy \) with \( \sqrt{2\pi} f(k) \) or, in other words, identified \( y \) with \( l_0^2 k \). We have used the expansion of \( K_0 \), Eq. (2.9), and it should be understood that the effect of the IC boundary can be included by resorting to appropriate generalizations of \( \omega_\alpha(q) \) and \( h_{q\alpha} \).

More can be understood about the form of the above Hamiltonian by recalling that for a homogeneous system, the Coulomb Hamiltonian takes the form \( 1/2 \sum_q V(q)\rho(q)\rho(-q) \), provided one discards the exchange term. Because the \( y \)-direction is not translationally invariant, we have \( \alpha \) taking over the role of a momentum, labeling the underlying classical transverse modes.

The hydrodynamic result of the previous section is tantamount to having each density operator, \( P^\dagger_\beta(p) \), oscillating at a definite frequency, \( \omega_\beta(q) \). We will examine the Heisenberg equation of motion of the density operators as follows:

\[
-iP^\dagger_\beta(p) = \frac{1}{2} \sum_{q \alpha} \omega_\alpha(q)\{ P^\dagger_\beta(p)P_\alpha(q), P^\dagger_\beta(p)P_\alpha(q) \} + \{ P_\alpha(q), P^\dagger_\beta(p)P^\dagger_\alpha(q) \}.
\]

We have absorbed the overlap integral \( e^{-q^2/2} \) into \( \omega_\alpha(q) \). The r.h.s. of the above equation contains the hydrodynamic term, \( \omega_\beta(p)P^\dagger_\beta(p) \), but there are also off-diagonal components present. Evaluation of those terms must rely on some kind of approximation scheme. We will solve Eq. (3.4) by linearizing the r.h.s. so that it reduces to a set of coupled linear equations for \( P^\dagger_\beta(p) \).

The first method we try is a version of RPA, in which one only keeps those bilinear terms with the momentum transfer equal to \( p \), i.e. terms proportional to \( c^\dagger_{k+p}c_k \). To leading order in momenta, we get

\[
P^\dagger_\alpha(q)[P_\alpha(q), P^\dagger_\beta(p)] \approx \delta_{\alpha\beta}\delta_{pq}P^\dagger_\beta(p) - \frac{2}{p^{1/2}qL^{3/2}} \sum_k h_{qa}(p\hbar h^\prime_{\alpha\beta} + qh_{qa}h^\prime_{\beta\beta})n_k c^\dagger_{k+p/2}c_{k-p/2}.
\]

The primes indicate derivatives with respect to \( k \). In deriving the above result, we have used

\[
\{ P_\alpha(q), P^\dagger_\beta(p) \} \approx \langle 1/\sqrt{|q|L} \rangle \sum_k \{ ph^\prime_{\alpha\beta}h_{p\beta} + qh_{qa}h^\prime_{p\beta} \} c^\dagger_{k+(p-q)/2}c_{k-(p-q)/2}.
\]

The diagonal term in Eq. (3.5), when multiplied by \( \omega_\alpha(q)/2 \) and summed over \( q \) and \( \alpha \), gives half of \( \omega_\beta(p)P^\dagger_\beta(p) \). The other half comes from \( \{ P_\alpha(q), P^\dagger_\beta(p) \} P^\dagger_\beta(q) \). In evaluating the second term in Eq. (3.5), we assume that various eigenfunctions do not depend on \( q \) as much as they do on \( \alpha \) so that we can simply label them \( h_{q\alpha} \). The integration over \( q \) extends from 0 to \( \infty \), but due to rapidly decaying \( e^{-q^2/2} \), there is a cutoff at large momenta. Since our interest lies in estimation, we conveniently choose the region of integration for \( q \) to be \([0,p]\). The Eq. (3.5) becomes

\[
\omega_\beta(p)P^\dagger_\beta(p) - \sum_{\alpha} \omega_\alpha(p)\frac{1}{(pL)^{1/2}} \sum_k \frac{p}{2\pi} (h^\prime_{\alpha\beta}h^\prime_{\beta\alpha}) n_k c^\dagger_{k+p/2}c_{k-p/2} = \omega_\beta(p)P^\dagger_\beta(p) - \sum_{\alpha,\gamma} \omega_\alpha(p)A_{\alpha\beta\gamma}P^\dagger_\gamma(p)
\]

where \( A_{\alpha\beta\gamma} \) is a constant of order \( l_0^2/aL \), \( a \) being the width of the compressible edge. The size of coupling of \( P^\dagger_\beta(p) \) to \( P^\dagger_\gamma(p) \) is the sum \( \sum_\alpha \omega_\alpha(p)A_{\alpha\beta\gamma} \sim (l_0^2/aL) \sum_\alpha \omega_\alpha(p) \sim (l_0/L)\langle \omega_\alpha(p) \rangle_{avg} \). In arriving at the last relation, we assumed the net number of modes involved are roughly \( a/l_0 \). So the coupling between modes is generally weak.

The underlying assumption of the above analysis, or indeed in the original RPA theory, is that there is a stable collective excitation. For a stable excitation, \( p \) is a good quantum number, and it may be safe to keep just those terms that preserve \( p \). In the theory of Nozières and Pines, it is a posteriori justified by calculating various decay rates of a plasmon which turns out to be small compared to the energy scale of a plasmon. In our problem, there is no obvious energy scale to set collective modes apart from single particle excitations and there may be a large coupling between
the two. This may in turn show up as a coupling between modes of different momenta and if it is significant, will destroy the assumption of the RPA.

As another way to linearize Eq. \(3.4\), we propose to replace the commutators with their expectation values in the ground state. The resulting equation will range over differing momenta as well as \(\alpha\). Taking the expectation value of Eq. \(3.6\) requires the knowledge of \(\langle c^\dagger_{k+p} c_k \rangle\). For \(p = 0\), it is just the density, \(n_k\). For \(p \neq 0\), we can argue the following. If the state \(k\) was unoccupied in the ground state, \(c_k\) acting on the ground state would destroy it. Likewise, if the state \(k+p\) had been occupied, there would be no place for an added electron to go, hence the expectation value will be again zero. So a plausible answer is \(\langle c^\dagger_{k+p} c_k \rangle = n_k(1-n_{k+p}) \approx n_k(1-n_k)\) for small \(p\). We get precisely this result for the form of the ground state \(\mid g.s. \rangle = \prod_k (\sqrt{1-n_k} + \sqrt{n_k} c^\dagger_k)\mid vac \rangle\). There are as yet no definitive answers in the literature on how the ground state of the edge should behave at \(T = 0\). Our guess is that if one is at a sufficiently high temperature that any many-body correlations which may exist at absolute zero are destroyed, but still low enough that quantum hall effect survives, the ground state should be approximately described by the single particle state wavefunction given above.

The commutator part can now be worked out straightforwardly. For \(p = q\), it is equal to \(\delta_{\alpha\beta}\). When \(p \neq q\),

\[
\langle [P_{\alpha}(q), P_{\beta}(p)] \rangle = \frac{p+q}{4\pi \sqrt{pq}} S_{\alpha\beta} + \frac{q-p}{4\pi \sqrt{pq}} A_{\alpha\beta},
\]

where \(S_{\alpha\beta} = \int_0^\nu h_\alpha h_\beta(1-2n_k)dn_k\), and \(A_{\alpha\beta} = \int_0^\nu (h_\beta \partial h_\alpha / \partial n_k - h_\alpha \partial h_\beta / \partial n_k)(n_k-n_k^2)dn_k\). The two integrals contribute numbers of order unity as shown in the Appendix. We find it more convenient to carry out the analysis in terms of \(P_{\alpha}(p)/\sqrt{p}\) and will refer to it as \(P_{\alpha}(p)\). We write \(\omega_\alpha(p) = v_\alpha \nu\) where it is understood that the velocities have weak logarithmic dependence and \(v_0/v_1 \approx -\log(p\alpha) \approx \log 10^{1/4}\) With \(\nu = 1\) and assuming the existence of just two modes, \(\alpha = 0, 1\), we have

\[
-i \dot{P}_{0}^\dagger(p) = \omega_0(p) P_0^\dagger(p) + \frac{4\sqrt{2}}{\pi^2} \omega_1(p) \sum_q P_1^\dagger(q)
\]

\[
-i \dot{P}_{1}^\dagger(p) = \omega_1(p) P_1^\dagger(p) + \frac{4\sqrt{2}}{\pi^2} \sum_q \omega_0(q) P_0^\dagger(q).
\]

What we have now is a coupled equation without the presence of small dimensionless parameters, \(l_0/\alpha\), or \(a/L\). This is the crucial feature that arises as a result of the microscopic calculation. The modes which were thought to be independent in the hydrodynamic model are in fact strongly coupled. One can do a perturbative analysis of the above equation using the fact that \(\omega_1(p)/\omega_0(p) = v_1/v_0\) is small,

\[
P_{1}^\dagger(p) \approx i \frac{4\sqrt{2}}{\pi^2} \sum_q \omega_0(q) P_1^\dagger(q) \approx \sum_q P_0^\dagger(q)
\]

\[
-i \dot{P}_0^\dagger(p) \approx \omega_0(p) P_0^\dagger(p) + \left(\frac{4\sqrt{2}}{\pi^2}\right)^2 \omega_1(p) \sum_q \sum_{q'} P_0^\dagger(q').
\]

Up to this order of \(v_1/v_0\), the dynamical behavior of \(P_1^\dagger(p)\) is completely determined by that of \(P_0^\dagger(p)\) and results in renormalization of the equation of motion for \(P_0^\dagger(p)\). With \(\nu = 1/3\), there is also coupling within the same mode,

\[
-i \dot{P}_0^\dagger(p) = \omega_0(p) P_0^\dagger(p) + \sum_q \frac{p+q}{3} v_0 P_0^\dagger(q) + \sum_q \frac{\sqrt{2}}{3} v_1(p-q+4p/\pi^2) P_1^\dagger(q)
\]

\[
-i \dot{P}_1^\dagger(p) = \omega_1(p) P_1^\dagger(p) + \sum_q \frac{p+q}{3} v_1 P_1^\dagger(q) + \sum_q \frac{\sqrt{2}}{3} v_0(q-p+4q/\pi^2) P_0^\dagger(q).
\]

Why is there so much mixing? For a sharp edge, the occupation number makes a step function, and we get \(\langle c^\dagger_{k+p} c_k \rangle = \delta_0 n_k\). Compressible edge has a diffuse distribution of electrons, giving a fairly large contribution to \(\langle c^\dagger_{k+p} c_k \rangle\) for \(p\) different than zero, and hence to the coupling of modes. In the light of the above analysis, our initial RPA assumption to keep only the momentum-conserving terms seems unjustifiable.
IV. DISCUSSION

The density excitation at the edge is often called the edge magneto-plasmon (EMP) and to distinguish the different modes, the $\alpha \geq 1$ are called the “acoustic” modes. The Eqs. (3.9) and (3.11) then represent the coupling of the EMP modes with the acoustic modes. Since experiments are often done at the fundamental frequency corresponding to $p = 2\pi/L$, one can keep just $P_0(2\pi/L)$ and $P_1(2\pi/L)$ in Eqs. (3.9) and (3.11) and disregard higher modes. Note that the Gaussian factor implicit in the frequency $\omega_0$ tends to suppress the coupling to higher momentum modes. We also introduce damping of the acoustic mode, $\omega_1(p) = (v_1 + i\gamma)p$ where $\gamma$ is comparable or larger than $v_1$ but still quite small compared to $v_0$. This is the problem of two coupled harmonic oscillators one of which is heavily damped. The resulting frequencies are easily found to be $v_0 \rightarrow v_0 + \lambda^2(v_1 + i\gamma)$ and $v_1 \rightarrow (1 - \lambda^2)(v_1 + i\gamma)$, with $\lambda = 4\sqrt{2\gamma}/\pi^2$. As a result of coupling, the initially dissipationless mode becomes weakly dissipative, and the acoustic mode propagates more slowly.

The acoustic modes, $\alpha \geq 1$, were predicted to be essentially unobservable due to dissipation in currently available samples with mobilities $\mu \approx 100$ m$^2$/Vsec. A number of EMP experiments both in the integer and fractional fillings have been unable to detect the diversity of modes either. Our analysis shows the mode coupling works in favor of slowing down the speed of propagation of the acoustic mode, thus making its detection more difficult.

For higher momentum states, the effect of coupling should be more significant, as one can see from the second half of Eq. (3.10). Since $\omega_1$ is complex, the more terms added in the summation will increase the dissipation. We therefore expect the states with large $p$ to decay easily compared to the modes close to the fundamental frequency.

In the experimental study of Luttinger liquid behavior of the edge, it is necessary that the edge be as sharp as possible in order to both reduce the number of active modes in the edge, and to eliminate coupling between them. Recent experiment of Chang et al. was designed to create a very sharp edge profile, and the experiment seems to vindicate many predictions of the chiral Luttinger liquid theory. They see exponents in the I-V measurement which are not totally integral, as expected from the theory, but rather show deviations of the order of 10%. In the light of this result, it is instructive to examine the possible correction of the tunneling density of states away from the ideal sharp-edge, short-range interaction limit. The long-range character of the dispersion makes the evaluation of the spectral function difficult, but it has been successfully solved in the recent work of Züliche and MacDonald. They conclude the long-range interaction results in the logarithmic correction to the tunneling density of states. We focus instead on the consequence of the coupling of modes.

We assume a much simplified model with a single-mode edge excitation, such as given by $P_1(q)$ whose excitation is undamped for $p$ ranging from 0 to $p_c$. The modes with $p > p_c$ are damped with a constant coefficient, $\omega = p + i\gamma p(v_0 = 1)$. The source of damping will be the coupling to the invisible $P_1$ mode as discussed above. The density of states is extracted from the imaginary part of the retarded electron Green’s function, $G_R(xt) = -i\langle\psi(xt)\psi^\dagger(0)\rangle$, and the total tunneling density of states at energy $\omega$ is proportional to the imaginary part of $G_R(x = 0, \omega)$. A straightforward calculation shows that

$$G_R(0, \omega) \propto \omega^{m-1} \int_0^\infty \exp\left[ it + m \int_0^{p/\omega} dq \frac{dq}{q} \frac{iqt - q\gamma t + (1 - e^{-q\gamma t})}{(t + t\gamma t - i0^+)^m} \right] dt$$

where $m = 1/\nu$. For $p_c = 0$ (constant rate of damping for all momenta) the tunneling density of states is proportional to $\omega^{m-1}$, as is the case for a non-dissipative system given by $\gamma = 0$. With $p_c \neq 0$, the limit of integration explicitly depends on $\omega$ and the naive scaling result will be modified. For $p_c/\omega \ll 1$, one can check that the first correction term goes like $(p_c + \omega)^{m-2}$. One should note, however, that other than this kind of dissipative correction and the effect of long-range interaction on the spectral density, the compressible edge will not pose an essential modification of the original sharp-edge theory.

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APPENDIX: EVALUATION OF $S_{\alpha\beta}$ AND $A_{\alpha\beta}$

One needs a specific expression of $h_{q}\alpha$ in order to evaluate $S_{\alpha\beta}$ and $A_{\alpha\beta}$. We follow AG and use $h_{q}\alpha(n_k) = \sqrt{2\pi/\nu}$, $h_{q}\alpha(n_k) = 2\sqrt{\pi/\nu}\cos(\pi\alpha n_k/\nu)$ for $\alpha \geq 1$. One can check this is an exact set of solutions once the charged
mode, $\alpha = 0$, is so large in energy that it decouples from all of the lower modes. There is also mixing due to the IC boundary as mentioned in Sec. II, and the true eigenstates are a complicated mixture of the above set. We have not considered the mixing of states, but do not expect that it will significantly change the result that follows.

Using the above eigenfunctions, we obtain

$$S_{\alpha\beta} = S_{\beta\alpha} = \begin{cases} \alpha = \beta, & 2\pi (1 - \nu) \\ \alpha = 0 \neq \beta, & (4\sqrt{2}/\pi^2) (1 - (-1)^\beta) \\ \alpha \neq \beta \neq 0, & (8\pi/\nu) \{(\alpha^2 + \beta^2)/(\alpha^2 - \beta^2\}^2 (1 - (-1)^{\alpha + \beta}) \end{cases}$$  \quad (A1)

and

$$A_{\alpha\beta} = -A_{\alpha\beta} = \begin{cases} \alpha = 0 \neq \beta, & 2\sqrt{2}\pi \{(2\nu/\beta^2) (1 - (-1)^\beta) - (1 - \nu)(-1)^\beta \} \\ \alpha \neq \beta \neq 0, & 4\pi(1 - \nu) \{(\alpha^2 + \beta^2)/(\alpha^2 - \beta^2)^2\} (-1)^{\alpha + \beta} + (4\nu/\pi) \{(\beta - \alpha)/(\beta + \alpha)^3 + (\beta + \alpha)/(\beta - \alpha)^3\} (1 - (-1)^{\alpha + \beta}) \end{cases}$$ \quad (A2)

Some specific results for $\nu = 1$ and $\nu = 1/3$ are

- $S_{00} = 0 = S_{11}$, $S_{01} = 8\sqrt{2}/\pi$, $A_{01} = 8\sqrt{2}/\pi$, and $S_{00} = 4\pi/3 = S_{11}$, $S_{01} = 8\sqrt{2}/3\pi$, $A_{01} = 4\sqrt{2}/3\pi + 8\sqrt{2}/3\pi$ \quad ($\nu = 1$)

FIG. 1. Density profile given by $n$ of the edge in equilibrium, (a), and out of equilibrium, (b). The shaded region indicates incompressible bulk of filling factor $\nu$ and $z$ is the width of the edge. The constant density $n_0(y)$ is given by a straight, horizontal line in equilibrium, but is displaced by $h(x, y)$ at each point $(x, y)$ in (b).