GLOBAL WELL-POSEDNESS IN ENERGY SPACE FOR THE
CHERN-SIMONS-HIGGS SYSTEM IN TEMPORAL GAUGE

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Abstract. The Cauchy problem for the Chern-Simons-Higgs system in the
(2+1)-dimensional Minkowski space in temporal gauge is globally well-posed
in energy space improving a result of Huh. The proof uses the bilinear space-
time estimates in wave-Sobolev spaces by d’Ancona, Foschi and Selberg, an
$L^6_t L^2_x$-estimate for solutions of the wave equation, and also takes advantage
of a null condition.

1. Introduction and main results
Consider the Chern-Simons-Higgs system in the Minkowski space $\mathbb{R}^{1+2} = \mathbb{R}_t \times \mathbb{R}^2$ with metric $g_{\mu\nu} = diag(1, -1, -1)$:

$$F_{\mu\nu} = 2\epsilon^{\mu\nu\rho} Im(\bar{\phi} D^\rho \phi)$$

$$D_\mu D^\mu \phi = -\phi V'(|\phi|^2),$$

with initial data

$$A_\mu(0) = a_\mu, \phi(0) = \phi_0, (\partial_t \phi)(0) = \phi_1,$$

where we use the convention that repeated upper and lower indices are summed, Greek indices run over 0,1,2 and Latin indices over 1,2. Here

$$D^\mu := \partial_\mu - i A_\mu$$

$$F_{\mu\nu} := D_\mu A_\nu - D_\nu A_\mu$$

$F_{\mu\nu} : \mathbb{R}^{1+2} \to \mathbb{R}$ denotes the curvature, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is a scalar field and $A_\nu : \mathbb{R}^{1+2} \to \mathbb{R}$ are the gauge potentials. We use the notation $\partial_\mu = \partial_{x^\mu}$, where we write $(x^0, x^1, ..., x^n) = (t, x^1, ..., x^n)$ and also $\partial_0 = \partial_t$ and $\nabla = (\partial_1, \partial_2)$. $\epsilon^{\mu\nu\rho}$ is the totally skew-symmetric tensor with $\epsilon^{012} = 1$, and the Higgs potential $V$ is assumed to fulfill $V \in C^\infty(\mathbb{R}^+, \mathbb{R})$, $V(0) = 0$ and all derivatives of $V$ have polynomial growth.

The energy $E(t)$ of the system is conserved, where

$$E(t) := \int_{\mathbb{R}^2} \left( \sum_{\mu=0}^2 |D_\mu \phi(t)|^2 + V(|\phi(t)|^2) \right) dx.$$ 

This model was proposed by Hong, Kim and Pac [HKP] and Jackiw and Weinberg [JW] in the study of vortex solutions in the abelian Chern-Simons theory.

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The equations are invariant under the gauge transformations

\[ A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi, \quad \phi \to \phi' = e^\chi \phi, \quad D_\mu \to D'_\mu = \partial_\mu - iA'_\mu. \]

The most common gauges are the Coulomb gauge \( \partial^i A_i = 0 \), the Lorenz gauge \( \partial^\mu A_\mu = 0 \) and the temporal gauge \( A_0 = 0 \). In this paper we exclusively study the temporal gauge for finite energy data.

Global well-posedness in the Coulomb gauge was proven by Chae and Choe \cite{CC} for data \( a_\mu \in H^a, \phi_0 \in H^K, \phi_1 \in H^{b-1} \) where \( (a,b) = (l,l+1) \) with \( l \geq 1 \), satisfying a compatibility condition and a class of Higgs potentials. Huh \cite{H} showed local well-posedness in the Coulomb gauge for \( (a,b) = (\frac{1}{2} + \epsilon, \frac{3}{2} + \epsilon) \) or \( (a,b) = (\frac{1}{2}, \frac{3}{2}) \), and in the Lorenz gauge for \( (a,b) = (\epsilon, 1 + \epsilon) \) and in the temporal gauge for \( (a,b) = (l,l) \) with \( l \geq \frac{5}{2} \).

He also showed global well-posedness in the temporal gauge for data \( \phi_0 \in H^2, \phi_1 \in H^1, a^{df}_i \in H^1, a^{\text{cf}}_i \in H^2 \), where \( a^{df} \) and \( a^{\text{cf}} \) denote the divergence-free and curl-free part of \( a \).

The local well-posedness result in the Lorenz gauge was improved to \( (a,b) = (l,l+1) \) and \( l > \frac{1}{2} \) by Bournaveas \cite{B} and by Yuan \cite{Y}. Also in Lorenz gauge the important global well-posedness result in energy space, where \( a_\mu \in \dot{H}^{\frac{3}{2}}, \phi_0 \in H^1, \phi_1 \in L^2 \), was proven by Selberg and Tesfahun \cite{ST} under a sign condition on the potential \( V \), and even unconditional well-posedness could be proven by Selberg and Oliveira da Silva \cite{SO}. In \cite{ST} the regularity assumptions on the data could also be lowered down in Lorenz gauge to \( (a,b) = (l,l+\frac{1}{2}) \) and \( l > \frac{1}{2} \). This latter result was improved to \( l > \frac{1}{2} \) by Huh and Oh \cite{HO}. Global well-posedness in energy space and local well-posedness for \( a_\mu \in \dot{H}^{\frac{3}{2}}, \phi_0 \in H^{\frac{3}{2} + \frac{1}{2}}, \phi_1 \in H^{\frac{1}{2} - \frac{1}{2}} \) for \( \frac{1}{2} \geq l > \frac{1}{2} \) in Coulomb gauge was recently obtained by Oh \cite{O}. For all these results up to the paper by Chae and Choe \cite{CC} and Oh \cite{O} it was crucial to make use of a null condition in the nonlinearity of the system.

A low regularity local well-posedness result in the temporal gauge for the Yang-Mills equations was given by Tao \cite{T1}.

In this paper we consider exclusively the temporal gauge. We show local well-posedness in energy space and above for potentials \( V \) of polynomial growth, more precisely for data \( \phi_0 \in H^1, \phi_1 \in L^2, |\nabla|^a a_j \in H^{\frac{3}{2} + \frac{1}{2}} (\epsilon > 0 \text{ small}) \), under the compatibility assumption \( \partial_1 a_2 - \partial_2 a_1 = 2Im(\phi_0 \phi_1) \). If \( V \) satisfies the sign condition \( V(r) \geq -\alpha^2 r^q \forall r \geq 0 \), where \( \alpha > 0 \), this solution exists globally in time.

Thus we directly show global well-posedness for finite energy data in temporal gauge, which was known before in the case of the Coulomb and the Lorenz gauge.

We use a contraction argument in \( X^{s,b} \)-type spaces adapted to the phase functions \( \tau \pm |\xi| \) on one hand and to the phase function \( \tau \) on the other hand. We also take advantage of a null condition which appears in the nonlinearity. Most of the crucial arguments follow from the bilinear estimates in wave-Sobolev spaces established by d’Ancona, Foschi and Selberg \cite{AFS}, which rely on the Strichartz estimates. Moreover we use an estimate for the \( L^p_0 L^2_\tau \)-norm for the solution of the wave equation which goes back to Tataru \cite{T2} and Tao \cite{T1}. When applying this estimate we partly follow Tao’s arguments in the case of the Yang-Mills equations. For the global existence part, which of course relies on energy conservation, we adapt the proof of Selberg-Tesfahun \cite{ST} for the Lorenz gauge to the temporal gauge.

We denote the Fourier transform with respect to space and time by \( \widehat{\cdot} \). The operator \(|\nabla|^a\) is defined by \(|\nabla|^a f)(\xi) = |\xi|^a (\mathcal{F} f)(\xi)\), where \( \mathcal{F} \) is the Fourier transform, and similarly \(|\nabla|\). The inhomogeneous and homogeneous Sobolev spaces are denoted by \( H^{s,p} \) and \( \dot{H}^{s,p} \), respectively. For \( p = 2 \) we simply denote them by \( H^s \) and \( \dot{H}^s \). We repeatedly use the Sobolev embeddings \( \dot{H}^{s,p} \subset L^q \) for \( 1 < p \leq q < \infty \).
and \( \frac{1}{q} = \frac{1}{p} - \frac{\epsilon}{2} \), and also \( \dot{H}^{1+} \cap \dot{H}^{1-} \subset L^\infty \) in two space dimensions.

\[ a_+ := a + \epsilon \] for a sufficiently small \( \epsilon > 0 \), so that \( a < a_+ < a_+ + \epsilon \), and similarly \( a_+ \) for \( a < a_- < a_+ \), and \( \epsilon \geq (1 + |\cdot|^2)^{\frac{1}{2}} \).

We now formulate our main results and begin by defining the standard spaces \( X_{s,b}^\pm \) of Bourgain-Klainerman-Machedon type belonging to the half waves as the completion of the Schwarz space \( S(\mathbb{R}^3) \) with respect to the norm

\[
\| u \|_{X_{s,b}^\pm} = \| \langle \xi \rangle^s (\tau \pm |\xi|)^b \hat{u}(\tau, \xi) \|_{L^2_{\tau,\xi}}.
\]

Similarly, we define the wave-Sobolev spaces \( X_{s,b}^{\pm,\pm} \) with norm

\[
\| u \|_{X_{s,b}^{\pm,\pm}} = \| \langle \xi \rangle^s (|\tau| - |\xi|)^b \hat{u}(\tau, \xi) \|_{L^2_{\tau,\xi}}
\]

and also \( X_{s,b}^{\pm,0} \) with norm

\[
\| u \|_{X_{s,b}^{\pm,0}} = \| \langle \xi \rangle^s \hat{u}(\tau, \xi) \|_{L^2_{\tau,\xi}}.
\]

We also define \( X_{s,b}^{\pm,\pm}[0,T] \) as the space of the restrictions of functions in \( X_{s,b}^{\pm,\pm} \) to \([0,T] \times \mathbb{R}^2\) and similarly \( X_{s,b}^{\pm,0}[0,T] \) and \( X_{s,b}^{\pm,\pm}[0,T] \). We frequently use the estimates \( \| u \|_{X_{s,b}^{\pm,\pm}[0,T]} \leq \| u \|_{X_{s,b}^{\pm,\pm}[0,T]} \) for \( b \leq 0 \) and the reverse estimate for \( b \geq 0 \).

Our main theorems read as follows:

**Theorem 1.1.** Let \( \epsilon > 0 \) be sufficiently small, and \( V \in C^\infty(\mathbb{R}^+, \mathbb{R}) \), \( V(0) = 0 \), and all derivatives have polynomial growth. The Chern-Simons-Higgs system (1.1), (1.3) in temporal gauge \( A_0 = 0 \) with data \( \phi_0 \in H^1(\mathbb{R}^2) \), \( \phi_1 \in L^2(\mathbb{R}^2) \), \( \nabla|A_1 \in H^{\pm}(\mathbb{R}^2) \) satisfying the compatibility condition \( \partial_1 a_2 - \partial_2 a_1 = 2i m(\phi_0 \phi_1) \) has a unique local solution

\[ \phi \in C^0([0,T], H^1(\mathbb{R}^2)) \cap C^1([0,T], L^2(\mathbb{R}^2)) \), \( \nabla|A \in C^0([0,T], H^{\pm}(\mathbb{R}^2)) \).

More precisely, \( T \) only depends on the data norm

\[
\| \phi_0 \|_{H^1} + \| \phi_1 \|_{L^2} + \| \nabla|A^{\text{cf}}(0) \|_{H^{\pm}}
\]

and \( \phi = \phi_+ + \phi_- \) with \( \phi_\pm \in X_{s,b}^{1/2+\epsilon,\pm}[0,T] \). If \( A = A^{\text{cf}} + A^{\text{d}} = (-\Delta)^{-1} \nabla \text{div} A + (-\Delta)^{-1} \nabla \text{curl} \text{curl} A \) is the decomposition into its divergence-free and its curl-free part, one has \( \nabla A^{\text{cf}} \in X_{s,b}^{1/2+\epsilon,\pm}[0,T] \), \( \nabla|A^{\text{cf}} \in C^0([0,T], L^2(\mathbb{R}^2)) \), \( \nabla A^{\text{d}} \in X_{s,b}^{1/2-\epsilon,\pm}[0,T] \), \( \nabla|A^{\text{d}} \in C^0([0,T], L^2(\mathbb{R}^2)) \), and in these spaces uniqueness holds.

Moreover, one has \( \nabla A^{\text{d}} \in X_{s,b}^{1/2-\epsilon,\pm-\delta}[0,T] \) for \( 0 < \delta < \epsilon \), and higher regularity persists. In particular, the solution is smooth, if the data are smooth.

**Theorem 1.2.** Assume in addition that \( V(r) \geq \alpha^2 r \) for all \( r \geq 0 \) for some \( \alpha > 0 \). Then the solution of Theorem 1.1 exists globally in time.

**Remark:** 1. Under our assumptions on the data the energy \( E(0) \) is finite. We namely have \( \| D_j \phi(0) \|_{L^2} \lesssim \| \partial_j \phi(0) \|_{L^2} + \| a_j \phi_0 \|_{L^2} \lesssim \| \phi_0 \|_{H^1} + \| \nabla|A_1 \|_{H^{\pm}} \| \phi_0 \|_{H^1} < \infty \) and \( V(|\phi|^2) \in L^1 \), because \( H^1 \subset L^p \) for all \( 2 \leq p < \infty \).

2. Persistence of higher regularity is a standard fact for solutions constructed by a Picard iteration, so we omit its proof.
2. Reformulation of the problem

In the temporal gauge $A_0 = 0$ the Chern-Simons-Higgs system \[1\] \[2\] is equivalent to the following system
\[\partial_t A_j = 2\epsilon_{ij} \Im m(\bar{\phi} D^i \phi) \] \[\partial_t^2 \phi - D^i D_j \phi = -\phi V'(|\phi|^2) \] \[\Leftrightarrow \Box \phi = 2iA^i \partial_j \phi - i\partial_j A^i \phi + A^j A_j \phi - \phi V'(|\phi|^2) \] \[\partial_t A_2 - \partial_2 A_1 = 2\Im m(\bar{\psi} \theta_i \phi) , \] where $i, j = 1, 2$, $\epsilon_{12} = 1$, $\epsilon_{21} = -1$ and $\Box = \partial_x^2 - \partial_y^2 - \partial_z^2$.

We remark that (6) is fulfilled for any solution of \[1\] \[2\], if it holds initially, i.e., if the following compatibility condition holds, which we assume from now on:
\[\partial_t A_2(0) - \partial_2 A_1(0) = 2\Im m(\bar{\psi} \theta_i \phi)(0) . \] Indeed, we have by \[1\] and \[2\]:
\[\partial_t (\partial_t A_2 - \partial_2 A_1) = 2\Im m(\bar{\psi} (D_1^2 \phi + D_2^2 \phi)) = 2\Im m(\bar{\psi} \delta_1^2 \phi) = 2\partial_t \Im m(\bar{\psi} \phi) . \] Thus we only have to solve \[1\] and \[2\], and can assume that (6) is fulfilled. We make the standard decomposition of $A = (A_1, A_2)$ into its divergence-free part $A^{df}$ and its curl-free part $A^{cf}$, namely $A = A^{df} + A^{cf}$, where
\[A^{df} = (-\Delta)^{-1}(\partial_t \partial_2 A_2 - \partial_2 \partial_1 A_1) = (-\Delta)^{-1} \curl \curl A , \]
\[A^{cf} = (-\Delta)^{-1}(\partial_1 \partial_2 A_2 + \partial_2 \partial_1 A_1 + \partial_1^2 A_2) = -(-\Delta)^{-1} \nabla \div A . \] Let $B$ be defined by $A^{df}_1 = -\partial_2 B$, $A^{df}_2 = \partial_1 B$. Then by \[1\] and $\partial_t A^{cf}_2 - \partial_2 A^{cf}_1 = 0$ we obtain
\[\Delta B = \partial_1 A^{cf}_2 - \partial_2 A^{cf}_1 = 2\Im m(\bar{\psi} \theta_i \phi) , \] so that
\[A^{df}_1 = -2\Delta^{-1}\partial_2 \Im m(\bar{\psi} \theta_i \phi) , A^{df}_2 = 2\Delta^{-1} \partial_1 \Im m(\bar{\psi} \theta_i \phi) . \] Next we calculate $\partial_t A^{cf}$ for solutions $(A, \phi)$ of \[1\] \[2\]:
\[\partial_t A^{cf}_2 = -\Delta^{-1}\partial_1 (\partial_2 \partial_2 A_2 + \partial_1 \partial_1 A_1) \]
\[= 2\Delta^{-1}\partial_1 (\partial_2 \partial_1 \phi - \partial_1 \partial_2 \phi) \]
\[= 2\Delta^{-1}\partial_1 \Im m(\bar{\psi} \partial_1 \phi) - \partial_1 \Im m(\bar{\psi} \partial_2 \phi) \]
\[= 2\Delta^{-1}\partial_1 \Im m(\bar{\psi} \partial_1 \phi - \partial_1 \partial_2 \phi - \partial_2 A_2 - A_2 \theta_1 \phi) \]
\[= 2\Delta^{-1}\partial_1 \Im m(\bar{\psi} \partial_1 \phi - \partial_1 \partial_2 \phi) + 2\Delta^{-1}\partial_1 (A_2 \partial_1 |\phi|^2 - A_1 \partial_2 |\phi|^2) \]
\[+ 4\Delta^{-1}\partial_1 \Im m(\bar{\psi} \theta_i \phi) |\phi|^2 . \] Similarly
\[\partial_t A^{cf}_1 = 2\Delta^{-1}\partial_2 \Im m(\bar{\psi} \partial_2 \phi - \partial_1 \partial_2 \phi) + 2\Delta^{-1}\partial_2 (A_1 \partial_2 |\phi|^2 - A_2 \partial_1 |\phi|^2) \]
\[+ 4\Delta^{-1}\partial_2 \Im m(\bar{\psi} \theta_i \phi) |\phi|^2 . \] Moreover from \[2\] we obtain using $\partial^j A^{df}_j = 0$:
\[\Box \phi = 2iA^{cf} \nabla \phi + 2iA^{df} \nabla \phi - i\partial^j A^{cf}_j \phi + (A^{df}_1 \partial_1 + A^{cf}_1)(A^{df}_2 + A^{cf}_2) \phi - \phi V'(|\phi|^2) . \] We also obtain from \[3\] and \[5\]
\[\partial_t A^{df}_2 = 2\partial_1 \Delta^{-1}\partial_1 \Im m(\bar{\psi} \theta_i \phi) = 2\Delta^{-1}\partial_1 \Im m(\bar{\psi} D^j \partial_j \phi) . \]
Now
\[ Im(\bar{\phi}D^1D_1\phi) = Im(\bar{\phi}\partial^1\partial_j\phi - 2i\bar{\phi}A_j\partial^1\phi - i\bar{\phi}\partial^1A_j\phi) \]
\[ = Im(\bar{\phi}\partial^1\partial_j\phi - i\bar{\phi}A_j\partial^1\phi - iA_j\partial^1\bar{\phi} - i\bar{\phi}\partial^1A_j\phi) \]
\[ = \partial^1 Im(\bar{\phi}D_j\phi), \]
so that
\[ \partial_1 A^{(f)}_2 = 2\Delta^{-1}_1 \partial_1 \partial^1 \text{Im}(\bar{\phi}D_j\phi). \] (12)

Similarly we obtain
\[ \partial_1 A^{(f)}_1 = -2\Delta^{-1}_1 \partial_2 \partial^1 \text{Im}(\bar{\phi}D_j\phi). \] (13)

Reversely defining \( A := A^{(f)} + A^{(c)} \) we show that our new system \([8],[9],[10],[11]\) implies \([4],[5]\) and also \([9]\), provided the compatibility condition \([7]\) is fulfilled. \([3]\) is obvious. \([8]\) is fulfilled because by use of \([9]\) and \([10]\) one easily checks
\[ \partial_1 A^{(f)}_2 - \partial_2 A^{(f)}_1 = 0, \]
so that by \([8]\)
\[ \partial_1 (\partial_1 A_2 - \partial_2 A_1) = \partial_1 (\partial_1 A^{(f)}_2 - \partial_2 A^{(f)}_1) = \text{Im}(\bar{\phi}\partial_1 \phi). \]
Thus \([6]\) is fulfilled, if \([7]\) holds. Finally we obtain
\[ \partial_1 A_1 = \partial_1 A^{(f)}_1 + \partial_1 A^{(c)}_1 \]
\[ = 2\Delta^{-1}_1 \partial_1 (\partial_1 \text{Im}(\bar{\phi}D_1\phi)) - 2\Delta^{-1}_1 \partial_2 (\partial_1 \text{Im}(\bar{\phi}D_2\phi)) \]
\[ = -2\text{Im}(\bar{\phi}D_2\phi), \]
where we used \([9]\) and also \([13]\), which was shown to be a consequence of \([5]\) and \([11]\). Similarly we also get
\[ \partial_1 A_2 = 2\text{Im}(\bar{\phi}D_1\phi), \]
so that \([11]\) is shown to be satisfied.

Summarizing we have shown that \([4],[5],[6]\) are equivalent to \([8],[9],[10],[11]\) (which also implies \([12],[13]\)).

Concerning the initial conditions assume we are given initial data for our system \([4],[5],[6]\):
\[ A_j(0) = a_j, \phi(0) = \phi_0, (\partial_t \phi)(0) = \phi_1 \]
satisfying \(|\nabla|a_j \in H^{1/2} , \phi_0 \in H^1 , \phi_1 \in L^2 \) and \([7]\). Then by \([8]\) and \([7]\) we obtain
\[ A^{(f)}_1(0) = -2\Delta^{-1}_1 \partial_2 \text{Im}(\bar{\phi}_0 \phi_1) = -\Delta^{-1}_1 \partial_2 (\partial_1 a_2 - \partial_2 a_1) \]
\[ A^{(f)}_2(0) = 2\Delta^{-1}_1 \partial_1 \text{Im}(\bar{\phi}_0 \phi_1) = \Delta^{-1}_1 (\partial_1 a_2 - \partial_2 a_1) \]
and
\[ A^{(c)}_j(0) = a_j - A^{(f)}_j(0), \]
thus
\[ |\nabla|A^{(c)}_j(0) \in H^{1/2} , \text{Im}(\bar{\phi}_0 \phi_1) \in H^{1/2}. \]

In the sequel we construct a solution of the Cauchy problem for \([8],[9],[10],[11]\) with data \( \phi_0 \in H^1, \phi_1 \in L^2 \), \(|\nabla| A^{(c)}_j(0) \in H^{1/2} \). We have shown that whenever we have a local solution of this system with data \( \phi_0, \phi_1 \) and \( A^{(f)}_j(0) = a_j - A^{(f)}_j(0) \), we also have that \( (\phi, A) \) with \( A := A^{(f)} + A^{(c)} \) is a local solution of \([4],[5]\) with data \( (\phi_0, \phi_1, a_1, a_2) \). If \([7]\) holds then \([6]\) is also satisfied.

Defining
\[ \phi_\pm = \frac{1}{2} (\phi \pm i^{-1} (\nabla)^{-1} \partial_t \phi) \iff \phi = \phi_+ + \phi_- , \partial_t \phi = i (\nabla) (\phi_+ - \phi_-) \]
the equation (11) transforms to

\[
(i\partial_t \pm \langle \nabla \rangle)\phi = \pm 2^{-1} \langle \nabla \rangle^{-1} (2iA'^j \nabla \phi + 2iA''^j \nabla \phi - i\partial^j A''^j_\phi) \\
+ (A'^j_\phi + A''^j_\phi)(A'^j_\phi + A''^j_\phi) - \phi V'(|\phi|^2) + \phi
\]  

(14)

Fundamental for the proof of our theorem are the following bilinear estimates in wave-Sobolev spaces which were proven by d’Ancona, Foschi and Selberg in the two-dimensional case \( n = 2 \) in [AFS] in a more general form which include many limit cases which we do not need.

**Theorem 2.1.** Let \( n = 2 \). The estimate

\[
\|uv\|_{X^{s,0,-b}_\infty} \lesssim \|u\|_{X^{s_1,b_1}_\infty} \|v\|_{X^{s_2,b_2}_\infty}
\]

holds, provided the following conditions hold:

\[
b_0 + b_1 + b_2 > \frac{1}{2} \\
b_0 + b_1 \geq 0 \\
b_0 + b_2 \geq 0 \\
b_1 + b_2 \geq 0 \\
s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2) \\
s_0 + s_1 + s_2 > 1 - \min(b_0, b_1, b_0 + b_2, b_1 + b_2) \\
s_0 + s_1 + s_2 > \frac{1}{2} - \min(b_0, b_1, b_2) \\
s_0 + s_1 + s_2 > \frac{3}{4} \\
(s_0 + b_0) + 2s_1 + 2s_2 > 1 \\
2s_0 + (s_1 + b_1) + 2s_2 > 1 \\
2s_0 + 2s_1 + (s_2 + b_2) > 1 \\
s_1 + s_2 \geq \max(0, -b_0) \\
s_0 + s_2 \geq \max(0, -b_1) \\
s_0 + s_1 \geq \max(0, -b_2).
\]

We also need the following

**Proposition 2.1.** The following estimates hold

\[
\|u\|_{L^6_t L^6_x} \lesssim \|u\|_{X^{s_1,b_1}_\infty}^{\frac{1}{2}} \|u\|_{X^{s_2,b_2}_\infty}^{\frac{1}{2}} \\
\|u\|_{L^6_t L^2_x} \lesssim \|u\|_{X^{s_1,b_1}_\infty}^{\frac{1}{2}} \|u\|_{X^{s_2,b_2}_\infty}^{\frac{1}{2}}
\]\n
(15) (16)

**Proof.** [15] is the original Strichartz estimate [Str] combined with the transfer principle. [16] goes back to [KMBT], Thm. 3.2:

\[
\|\mathcal{F}_t u\|_{L^6_t L^6_x} \lesssim \|u_0\|_{H^{\frac{1}{2}}}
\]

if \( u = e^{it|\nabla|}u_0 \) and \( \mathcal{F}_t \) denotes the Fourier transform with respect to time. By Plancherel and Minkowski’s inequality we obtain

\[
\|u\|_{L^6_t L^6_x} = \|\mathcal{F}_t u\|_{L^6_t L^6_x} \lesssim \|\mathcal{F}_t u\|_{L^6_t L^6_x} \lesssim \|u_0\|_{H^{\frac{1}{2}}}
\]

The transfer principle gives (16). \( \square \)
The following easy consequences are obtained by interpolation between (15), (16) and the trivial identity

\[ \|u\|_{L^2_{|\tau|}}^2 = \|u\|_{X^{0,0}_{|\tau|=|\ell|}}^2 \quad (17) \]

\[ \|u\|_{L^2_{|\tau|} L^2_{|\tau|}} \lesssim \|u\|_{X^{0,0}_{|\tau|=|\ell|}} \quad \text{(interpolate (15) and (16))}, \quad (18) \]

\[ \|u\|_{L^2_{|\tau|} L^2_{|\tau|}} \lesssim \|u\|_{X^{0,0}_{|\tau|=|\ell|}} \quad \text{(interpolate (15) and (17))}, \quad (19) \]

\[ \|u\|_{L^2_{|\tau|} L^2_{|\tau|}} \lesssim \|u\|_{X^{0,0}_{|\tau|=|\ell|}} \quad \text{(interpolate (15) and (17))}, \quad (20) \]

\[ \|u\|_{L^2_{|\tau|} L^2_{|\tau|}} \lesssim \|u\|_{X^{0,0}_{|\tau|=|\ell|}} \quad \text{(interpolate (15) and (17))}, \quad (21) \]

\[ \|u\|_{L^2_{|\tau|} L^2_{|\tau|}} \lesssim \|u\|_{X^{0,0}_{|\tau|=|\ell|}} \quad \text{(interpolate (15) and (17))}, \quad (22) \]

\[ \|u\|_{L^2_{|\tau|} L^2_{|\tau|}} \lesssim \|u\|_{X^{0,0}_{|\tau|=|\ell|}} \quad \text{(interpolate (15) and (17))}, \quad (23) \]

3. Proof of Theorem 1.1

Taking the considerations of the previous section into account Theorem 1.1 reduces to the following proposition and its corollary.

**Proposition 3.1.** Let \( \epsilon > 0 \) be sufficiently small. The system

\[ (iD_\tau \pm (\nabla)\phi_\pm = \pm 2^{-1}(\nabla)^{-1}(2iA^{\tau\rho} \nabla \phi + 2iA^\rho \nabla \phi - i D^{\tau} A^{\tau\rho} \phi) + (A^{\tau\rho} + A^{\rho\tau})(A^{\tau\rho} + A^{\rho\tau})\phi - \phi V'(\phi^2) + \phi) \]

\[ A^{\tau\rho}_2 = -2\Delta^{\tau} \partial_2 A_1(\partial \phi_\rho), \quad A^{\rho\tau}_2 = 2\Delta^{\tau} \partial_1 A_2(\partial \phi_\rho) \]

\[ \partial_\tau A^{\tau\rho}_1 = 2\Delta^{\tau} \partial_1 A_1(\partial_1 \phi_\rho), \quad \partial_\tau A^{\rho\tau}_2 = 2\Delta^{\tau} \partial_2 A_2(\partial_2 \phi_\rho) \]

with data \( \phi_\pm(0) \in H^1 \text{ and } |\nabla| A^{\tau\rho}(0) \in H^{\frac{3}{2}} \) has a unique local solution

\[ \phi_\pm \in X^{1,\frac{3}{2}+\epsilon}_{|\tau|=|\ell|}[0, T], \text{ with } \nabla A^{\tau\rho} \in X^{0, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|}[0, T], \text{ and } \nabla A^{\rho\tau} \in C^0([0, T], L^2). \]

Here \( \phi = \phi_+ + \phi_- \), \( \partial_\tau \phi = i(\nabla)(\phi_+ - \phi_-) \). Moreover \( A^{\tau\rho} \) satisfies \( \nabla A^{\tau\rho} \in X^{1, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|}[0, T], \text{ with } \nabla A^{\tau\rho} \in C^0([0, T], L^2) \) and also \( \nabla A^{\rho\tau} \in X^{1, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|}[0, T] \) for \( 0 < \delta < \epsilon \).

We obtain immediately

**Corollary 3.1.** The solution has the property \( \phi \in C^0([0, T], H^1) \cap C^1([0, T], L^2), \]

\( |\nabla| A^{\tau\rho} \in C^0([0, T], H^{\frac{3}{2}}) \text{ and } |\nabla| A^{\rho\tau} \in C^0([0, T], H^{1-\epsilon}) \).

**Proof.** We want to apply the contraction mapping principle for

\[ \phi_\pm \in X^{1, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|}[0, T], \text{ with } \nabla A^{\tau\rho} \in X^{0, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|}[0, T], \text{ and } \nabla A^{\rho\tau} \in C^0([0, T], L^2). \]

By well-known arguments this is reduced to the estimates of the right hand sides of (24), (25) and (27) stated as claims 1-9 below. We start to control \( \nabla A^{\tau\rho} \) in \( X^{0, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|} \).

**Claim 1:**

\[ \|\partial_\tau \partial_\rho \phi - \partial_\rho \partial_\tau \phi\|_{X^{0, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|}} \lesssim \|\nabla \phi\|^2 \]

\[ \|\partial_\tau \partial_\rho \phi - \partial_\rho \partial_\tau \phi\|_{X^{0, \frac{3}{2}+\epsilon}_{|\tau|=|\ell|}} \lesssim \|\nabla \phi\|^2 \]
Let \( \pm_1 \) and \( \pm_2 \) denote independent signs. Using
\[
\partial_x \bar{\phi} \partial_y \psi - \partial_y \bar{\phi} \partial_x \psi = \sum_{\pm_1, \pm_2} (\partial_x \bar{\phi}_{\pm_1} \partial_y \phi_{\pm_2} - \partial_y \bar{\phi}_{\pm_1} \partial_x \phi_{\pm_2})
\]
it suffices to show
\[
\| \partial_x \bar{\phi} \partial_y \psi - \partial_y \bar{\phi} \partial_x \psi \|_{X^{r, s}_{x, y}} \lesssim \| \nabla \phi \|_{Y^{r, s}_{x, y}} \| \nabla \psi \|_{X^{r, s}_{x, y}}
\]
We now use the null structure of this term in the form that for vectors \( \xi = (\xi^1, \xi^2) \), \( \eta = (\eta^1, \eta^2) \in \mathbb{R}^2 \) the following estimate holds
\[
|\xi \eta^j - \xi^j \eta^i| \leq |\xi| |\eta| \angle (\xi, \eta),
\]
where \( \angle (\xi, \eta) \) denotes the angle between \( \xi \) and \( \eta \). The following lemma gives the decisive bound for the angle:

**Lemma 3.1.** (ST, Lemma 2.1 or ST, Lemma 3.2)
\[
\angle (\pm_1 \xi_1, \pm_2 \xi_2) \lesssim \left( \frac{\langle \tau_1 \pm_1 |\xi_1| \rangle + \langle \tau_2 \pm_2 |\xi_2| \rangle}{\min(\langle \xi_1, \langle \xi_2 \rangle)} \right)^{\frac{1}{2} - \frac{1}{2} + \varepsilon} + \left( \frac{\langle |\tau_3| - |\xi_3| \rangle}{\min(\langle \xi_1, \langle \xi_2 \rangle)} \right)^{\frac{1}{2} - \frac{1}{4} + \varepsilon}
\]
(28)
\forall \xi_1, \xi_2, \xi_3 \in \mathbb{R}^2, \tau_1, \tau_2, \tau_3 \in \mathbb{R} with \( \xi_1 + \xi_2 + \xi_3 = 0 \) and \( \tau_1 + \tau_2 + \tau_3 = 0 \).

Thus the claimed estimate reduces to
\[
\left| \int \langle \hat{u}_1(\tau_1, \xi_1) \rangle \frac{\hat{u}_2(\tau_2, \xi_2)}{\langle \tau_1 \pm_1 |\xi_1| \rangle} \frac{\hat{u}_3(\tau_3, \xi_3)}{\langle \tau_2 \pm_1 |\xi_2| \rangle} \angle (\pm_1 \xi_1, \pm_2 \xi_2) \right| \lesssim \prod_{i=1}^{3} \| u_i \|_{L^2_x},
\]
(29)
where \( * \) denotes integration over \( \xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3 \) with \( \xi_1 + \xi_2 + \xi_3 = 0 \) and \( \tau_1 + \tau_2 + \tau_3 = 0 \). We assume without loss of generality that \( |\xi_1| \leq |\xi_2| \) and the Fourier transforms are nonnegative. We distinguish three cases according to which of the terms on the right hand side of (29) is dominant.

**Case 1:** The last term in (29) dominant. In this case (29) reduces to
\[
\left| \int \langle \hat{u}_1(\tau_1, \xi_1) \rangle \frac{\hat{u}_2(\tau_2, \xi_2)}{\langle \tau_1 \pm_1 |\xi_1| \rangle} \frac{\hat{u}_3(\tau_3, \xi_3)}{\langle \tau_2 \pm_1 |\xi_2| \rangle} \angle (\pm_1 \xi_1, \pm_2 \xi_2) \right| \lesssim \prod_{i=1}^{3} \| u_i \|_{L^2_x},
\]

1.1: \( |\tau_3| \geq \frac{|\xi_1|}{2} \). In this case (29) reduces to
\[
\left| \int \langle \hat{u}_1(\tau_1, \xi_1) \rangle \frac{\hat{u}_2(\tau_2, \xi_2)}{\langle \tau_1 \pm_1 |\xi_1| \rangle} \frac{\hat{u}_3(\tau_3, \xi_3)}{\langle \tau_2 \pm_1 |\xi_2| \rangle} \angle (\tau_3, |\xi_3|) \right| \lesssim \prod_{i=1}^{3} \| u_i \|_{L^2_x},
\]
which follows from Theorem 2.1

1.2: \( |\tau_3| \leq \frac{|\xi_1|}{4} \Rightarrow |\langle \tau_3 \rangle - |\xi_3| | \sim |\xi_3| \).

1.2.1: \( |\tau_2| \ll |\xi_2| \). We have to show
\[
\left| \int \langle \hat{u}_1(\tau_1, \xi_1) \rangle \frac{\hat{u}_2(\tau_2, \xi_2)}{\langle \tau_1 \pm_1 |\xi_1| \rangle} \frac{\hat{u}_3(\tau_3, \xi_3)}{\langle \tau_2 \pm_1 |\xi_2| \rangle} \angle (\tau_3, \xi_3) \right| \lesssim \prod_{i=1}^{3} \| u_i \|_{L^2_x},
\]

1.2.2: \( |\tau_2| \gg |\xi_2| \). We have to show
\[
\left| \int \langle \hat{u}_1(\tau_1, \xi_1) \rangle \frac{\hat{u}_2(\tau_2, \xi_2)}{\langle \tau_1 \pm_1 |\xi_1| \rangle} \frac{\hat{u}_3(\tau_3, \xi_3)}{\langle \tau_2 \pm_1 |\xi_2| \rangle} \angle (\tau_3, |\xi_3|) \right| \lesssim \prod_{i=1}^{3} \| u_i \|_{L^2_x},
\]
But by (20) we obtain
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim ||v_1||_{L^2_x L^2_t} ||v_2||_{L^2_x L^2_t} ||v_3||_{L^2_x L^2_t}^2
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{H^\frac{5}{2} L^2_t} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]
as desired.

1.2.2: $|\tau_2| \gtrsim |\xi_2|$. In this case we use $\tau_1 + \tau_2 + \tau_3 = 0$ to estimate
\[
1 \lesssim \frac{\langle \tau_2 \rangle^{\frac{1}{2}}}{\langle \xi_2 \rangle^{\frac{1}{2}}} \lesssim \frac{\langle \tau_1 \rangle^{\frac{1}{2}}}{\langle \xi_2 \rangle^{\frac{1}{2}}} + \frac{\langle \tau_3 \rangle^{\frac{1}{2}}}{\langle \xi_2 \rangle^{\frac{1}{2}}}
\]
The second term on the right hand side is taken care of by
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim ||v_1||_{L^2_x L^2_t} ||v_2||_{L^2_x L^2_t} ||v_3||_{L^2_x L^2_t}^2
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]
which follows from (20) which gives
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim ||v_1||_{L^2_x L^2_t} ||v_2||_{L^2_x L^2_t} ||v_3||_{L^2_x L^2_t}^2
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]

We have to show
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim ||v_1||_{L^2_x L^2_t} ||v_2||_{L^2_x L^2_t} ||v_3||_{L^2_x L^2_t}^2
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]
Thus we need
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim ||v_1||_{L^2_x L^2_t} ||v_2||_{L^2_x L^2_t} ||v_3||_{L^2_x L^2_t}^2
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]

For the first term we consider two subcases.

1.2.2.1: $|\tau_1| \lesssim |\xi_1| \Rightarrow \frac{\langle \tau_1 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^{\frac{1}{2}}} \lesssim \frac{\langle \tau_1 \rangle^{\frac{1}{2}}}{\langle \xi_2 \rangle^{\frac{1}{2}}}
\]

This follows from (20) which gives
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim ||v_1||_{L^2_x L^2_t} ||v_2||_{L^2_x L^2_t} ||v_3||_{L^2_x L^2_t}^2
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]

1.2.2.2: $|\tau_1| \gg |\xi_1| \Rightarrow \frac{\langle \tau_1 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^{\frac{1}{2}}} \sim \frac{\langle \tau_1 \rangle^{\frac{1}{2}}}{\langle \xi_2 \rangle^{\frac{1}{2}}}
\]

Thus we need
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim ||v_1||_{L^2_x L^2_t} ||v_2||_{L^2_x L^2_t} ||v_3||_{L^2_x L^2_t}^2
\]
\[
\lesssim ||v_1||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_2||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}} ||v_3||_{X^{\frac{4}{3} + \frac{3}{2} + \frac{3}{2}}_{|t|=|x|}}
\]
which gives the desired estimate.

Case 2: The first term in (25) is dominant (and $|\xi_1| \lesssim |\xi_2|$).

We have to show

$$\left| \int \frac{\tilde{u}_1(\tau_1, \xi_1)}{|\tau_1| - |\xi_1|^{\sigma-}(\xi_1)^\frac{\sigma}{2}} \frac{\tilde{u}_2(\tau_2, \xi_2)}{|\tau_2| - |\xi_2|^{\sigma+(\xi_2)^{\frac{\sigma}{2}}} - (\xi_3)^{\frac{\sigma}{2} - (\tau_3)^{\frac{\sigma}{2}}} - (\tau_3)^{\frac{\sigma}{2}}} \right| \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_x},$$

2.1: $|\tau_2| \ll |\xi_2| \Rightarrow (|\tau_2| - |\xi_2|) \sim (\xi_2)$.

The last estimate is reduced to

$$\left| \int \frac{\tilde{u}_1(\tau_1, \xi_1)}{|\tau_1| - |\xi_1|^{\sigma-}(\xi_1)^\frac{\sigma}{2}} \frac{\tilde{u}_2(\tau_2, \xi_2)}{|\tau_2| - |\xi_2|^{\sigma+(\xi_2)^{\frac{\sigma}{2}}} - (\xi_3)^{\frac{\sigma}{2} - (\tau_3)^{\frac{\sigma}{2}}} - (\tau_3)^{\frac{\sigma}{2}}} \right| \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_x},$$

which follows by Sobolev’s embedding from the estimate

$$\left| \int v_1v_2v_3dxdt \right| \lesssim \|v_1\|_{L^2_x L^4_t} \|v_2\|_{L^6_t L^6_x} \|v_3\|_{L^\infty_x L^\infty_t} \lesssim \|v_1\|_{X_{|\tau| = |\xi|}^{\frac{3}{4}, 0}} \|v_2\|_{X_{|\tau| = |\xi|}^{\frac{3}{4}, 0}} \|v_3\|_{X_{|\tau| = |\xi|}^{\frac{3}{4}, 0}}.$$

2.2: $|\tau_2| \geq |\xi_2|$.

2.2.1: $|\tau_1| \lesssim |\xi_1|$. We use

$$1 \lesssim \frac{\langle \tau_2 \rangle^\frac{\sigma}{2}}{\langle \xi_2 \rangle^\frac{\sigma}{2}} \lesssim \frac{\langle \tau_1 \rangle^\frac{\sigma}{2} - \langle \tau_3 \rangle^\frac{\sigma}{2}}{\langle \xi_2 \rangle^\frac{\sigma}{2}}.$$

If the first term is dominant we reduce to

$$\left| \int \frac{\tilde{u}_1(\tau_1, \xi_1)}{|\tau_1| - |\xi_1|^{\sigma-}(\xi_1)^\frac{\sigma}{2}} \frac{\tilde{u}_2(\tau_2, \xi_2)}{|\tau_2| - |\xi_2|^{\sigma+(\xi_2)^{\frac{\sigma}{2}}} - (\xi_3)^{\frac{\sigma}{2} - (\tau_3)^{\frac{\sigma}{2}}} - (\tau_3)^{\frac{\sigma}{2}}} \right| \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_x}.$$

It is a consequence of the following estimate which follows from (18) and Sobolev:

$$\left| \int v_1v_2v_3dxdt \right| \lesssim \|v_1\|_{L^2_x L^4_t} \|v_2\|_{L^6_t L^6_x} \|v_3\|_{L^\infty_x L^\infty_t} \lesssim \|v_1\|_{X_{|\tau| = |\xi|}^{\frac{3}{4}, 0}} \|v_2\|_{X_{|\tau| = |\xi|}^{\frac{3}{4}, 0}} \|v_3\|_{X_{|\tau| = |\xi|}^{\frac{3}{4}, 0}}.$$

If the second term is dominant we need

$$\left| \int \frac{\tilde{u}_1(\tau_1, \xi_1)}{|\tau_1| - |\xi_1|^{\sigma-}(\xi_1)^\frac{\sigma}{2}} \frac{\tilde{u}_2(\tau_2, \xi_2)}{|\tau_2| - |\xi_2|^{\sigma+(\xi_2)^{\frac{\sigma}{2}}} - (\xi_3)^{\frac{\sigma}{2} - (\tau_3)^{\frac{\sigma}{2}}} - (\tau_3)^{\frac{\sigma}{2}}} \right| \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_x},$$

which follows from Theorem 2.1

2.2.2: $|\tau_1| \gg |\xi_1| \Rightarrow (|\tau_1| - |\xi_1|) \sim (\tau_1)$.

We use

$$1 \lesssim \frac{\langle \tau_2 \rangle^\frac{\sigma}{2}}{\langle \xi_2 \rangle^\frac{\sigma}{2}} \lesssim \frac{\langle \tau_1 \rangle^\frac{\sigma}{2} - \langle \tau_3 \rangle^\frac{\sigma}{2}}{\langle \xi_2 \rangle^\frac{\sigma}{2}}.$$

The first term reduces to

$$\left| \int \frac{\tilde{u}_1(\tau_1, \xi_1)}{|\tau_1| - |\xi_1|^{\sigma-}(\xi_1)^\frac{\sigma}{2}} \frac{\tilde{u}_2(\tau_2, \xi_2)}{|\tau_2| - |\xi_2|^{\sigma+(\xi_2)^{\frac{\sigma}{2}}} - (\xi_3)^{\frac{\sigma}{2} - (\tau_3)^{\frac{\sigma}{2}}} - (\tau_3)^{\frac{\sigma}{2}}} \right| \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_x}.$$
We estimate using (23) and Sobolev:
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim \|v_1\|_{L^1_t L^{2+}_x} \|v_2\|_{L^2_x L^2_t} \|v_3\|_{L^\infty_t L^2_x} \\
\lesssim \|v_1\|_{H^2_x H^0_t} \|v_2\|_{X^{1+\epsilon}_x \times \{t=0\}} \|v_3\|_{H^2_x H^0_t} \\
\lesssim \|v_1\|_{X^{1+\epsilon}_x \times \{t=0\}} \|v_2\|_{X^{1+\epsilon}_x \times \{t=0\}} \|v_3\|_{X^{1+\epsilon}_x \times \{t=0\}} ,
\]
which gives the desired bound.

If the second term is dominant we have to show
\[
\left| \int \frac{\hat{u}_1(\tau_1, \xi_1) \hat{u}_2(\tau_2, \xi_2) \hat{u}_3(\tau_3, \xi_3)}{\langle |\tau_1|^\epsilon - |\xi_1| \rangle^\epsilon \langle |\tau_2|^\epsilon - |\xi_2| \rangle^\epsilon \langle |\tau_3|^\epsilon - |\xi_3| \rangle^\epsilon} \right| \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{\tau_i}},
\]
which follows from
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim \|v_1\|_{L^2_{\tau_1} L^1_x} \|v_2\|_{L^2_{\tau_2} \frac{L^{2+}_x}{L^2_t}} \|v_3\|_{L^2_{\tau_3} L^\infty_x} \\
\lesssim \|v_1\|_{H^2_x H^0_t} \|v_2\|_{H^2_x L^1_t} \|v_3\|_{H^2_x H^0_t} \\
\lesssim \|v_1\|_{X^{1+\epsilon}_x \times \{t=0\}} \|v_2\|_{X^{1+\epsilon}_x \times \{t=0\}} \|v_3\|_{X^{1+\epsilon}_x \times \{t=0\}},
\]
where we used Sobolev’s embedding.

Case 3: The second term in (23) is dominant (and $|\xi_1| \leq |\xi_2|$).

In this case we reduce to
\[
\left| \int \frac{\hat{u}_1(\tau_1, \xi_1) \hat{u}_2(\tau_2, \xi_2) \hat{u}_3(\tau_3, \xi_3)}{\langle |\tau_1|^\epsilon - |\xi_1| \rangle^\epsilon \langle |\tau_2|^\epsilon - |\xi_2| \rangle^\epsilon \langle |\tau_3|^\epsilon - |\xi_3| \rangle^\epsilon} \right| \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{\tau_i}},
\]
which follows by Strichartz’ (15) and Sobolev’s estimates by
\[
\left| \int v_1 v_2 v_3 dx dt \right| \lesssim \|v_1\|_{L^\infty_x L^1_t} \|v_2\|_{L^2_x L^2_t} \|v_3\|_{L^2_x L^2_t} \\
\lesssim \|v_1\|_{X^{1+\epsilon}_x \times \{t=0\}} \|v_2\|_{X^{1+\epsilon}_x \times \{t=0\}} \|v_3\|_{X^{1+\epsilon}_x \times \{t=0\}}.
\]

Claim 1 is now proven.

Before proceeding we estimate $A_i^{\text{diff}}$. By Sobolev’s embedding $H^{1-\epsilon, \frac{2\epsilon}{3}} \subset L^2$ we obtain:
\[
\|\nabla^a A_i^{\text{diff}}\|_{L^\infty_t L^2_x} \lesssim \|\phi \partial_t \phi\|_{L^\infty_t H^{1+\epsilon} \times \{t=0\}} \lesssim \|\phi \partial_t \phi\|_{L^\infty_t L^\frac{2}{1-\epsilon}} \\
\lesssim \|\phi\|_{L^\infty_t L^\frac{2}{1-\epsilon}} \|\partial_t \phi\|_{L^\infty_t L^2_x} \lesssim \|\phi\|_{X^{1+\epsilon}_x \times \{t=0\}} \|\partial_t \phi\|_{X^{0+\epsilon}_x \times \{t=0\}}
\]
and for $\epsilon > 0$:
\[
\|\nabla A_i^{\text{diff}}\|_{X^{0+\epsilon}_x \times \{t=0\}} \lesssim \|\phi \partial_t \phi\|_{X^{0+\epsilon}_x \times \{t=0\}} \lesssim \|\phi\|_{X^{1+\epsilon}_x \times \{t=0\}} \|\partial_t \phi\|_{X^{0+\epsilon}_x \times \{t=0\}}
\]
by Theorem 2.1 which for $0 < \delta < \epsilon$ also implies
\[
\|\nabla A_i^{\text{diff}}\|_{X^{1-\delta, \frac{2\delta}{3}}_x \times \{t=0\}} \lesssim \|\phi \partial_t \phi\|_{X^{1-\delta, \frac{2\delta}{3}}_x \times \{t=0\}} \lesssim \|\phi\|_{X^{1+\epsilon}_x \times \{t=0\}} \|\partial_t \phi\|_{X^{0+\epsilon}_x \times \{t=0\}}.
\]

The cubic terms are easier to handle, because they contain one derivative less.
Claim 2:
\[ \|A_i \partial_j (|\phi|^2)\|_{L^2_t H^{\frac{1}{2} + \epsilon}_x} \lesssim \left( \|\nabla A_i^{\alpha \beta} \|_{X^{\frac{3}{4}}_{|\tau| = |\xi|}} + \|\nabla A_i^{\alpha \beta} \|_{L^\infty_t (L^2_x)} \right) \|\phi\|_{X^{\frac{3}{4}}_{|\tau| = |\xi|}}^2 + \|\phi\|_{X^{\frac{3}{4}}_{|\tau| = |\xi|}}^3 \|\partial_\xi \phi\|_{X^{\frac{3}{4}}_{|\tau| = |\xi|}}. \]

We split \( A = A^{\alpha \beta} + A^{\alpha \beta} \), and moreover \( A_i^{\alpha \beta} = A_i^{\alpha \beta, h} + A_i^{\alpha \beta, l} \) as well as \( A^{\alpha \beta} = A_i^{\alpha \beta, h} + A_i^{\alpha \beta, l} \) into their low and high frequency parts, i.e. \( \operatorname{supp} A_i^{\alpha \beta, h} \subset \{ |\xi| \geq 1 \} \), \( \operatorname{supp} A_i^{\alpha \beta, l} \subset \{ |\xi| \leq 1 \} \) and similarly \( A^{\alpha \beta} \).

For the high frequency parts we obtain by (15) and Sobolev
\[
\|A_i^{\alpha \beta, h} \|_{L^2_t H^{\frac{1}{2} + \epsilon}_x} \lesssim \|A_i^{\alpha \beta, h} \|_{L^2_t L^{\frac{1}{2} + \epsilon}_x} \|\phi\|_{X^{\frac{3}{4}}_{|\tau| = |\xi|}}^2 \|\partial_\xi \phi\|_{X^{\frac{3}{4}}_{|\tau| = |\xi|}}.
\]

This follows from \( \partial_\xi \phi \) by Sobolev's embedding \( H^{\frac{1}{2} + \epsilon}_x \cap H^{-\frac{1}{2} - \epsilon}_x \subset L^\infty_x \), which implies
\[
\int P(\nabla \phi \nabla \phi) \, dx = \int \nabla \phi \nabla \phi P \, dx \lesssim \|\nabla \phi\|_{L^2_x}^2 \|P \|_{L^\infty} \lesssim \|\nabla \phi\|_{L^2_x}^2 \|\nabla^{1-\epsilon} w\|_{L^2} \lesssim \|\nabla \phi\|_{L^2_x}^2 \|\nabla^{1-\epsilon} w\|_{L^2}.
\]
A frequency part is easily estimated as follows: This estimate would follow if we prove
\[ \| \phi \|_{L^\infty L^2} \lesssim T \| \nabla^\alpha A^f \|_{L^\infty L^2} \lesssim T \| \nabla^\alpha A^f \|_{L^\infty L^2} \]
by (30).

Next in order to estimate \( \| \phi \|_{X_{\{r\}=|\xi|}} \) and \( \| \partial_t \phi \|_{X_{\{r\}=|\xi|}} \), we have to control the right hand side of (13).

**Claim 5:**
\[ \| A^f \nabla \phi \|_{X_{\{r\}=|\xi|}} \lesssim \left( \| \nabla^\alpha A^f \|_{L^\infty L^2} + \| \nabla A^f \|_{X_{\{r\}=|\xi|}} \right) \| \nabla \phi \|_{X_{\{r\}=|\xi|}} \lesssim \| \phi \|_{X_{\{r\}=|\xi|}} \]

The last estimate follows from (29) and (31). For the first estimate we consider the low and high frequency parts of \( A^f \) as follows:
\[ \| A^{f,a} \nabla \phi \|_{L^2} \lesssim T^{\frac{1}{2}} \| A^{f,a} \|_{L^\infty L^2} \| \nabla \phi \|_{L^\infty L^2} \]
\[ \lesssim T^{\frac{1}{2}} \| \nabla A^{f,a} \|_{L^\infty L^2} \| \nabla \phi \|_{L^\infty L^2} \lesssim T^{\frac{1}{2}} \| \nabla A^{f,a} \|_{L^\infty L^2} \| \nabla \phi \|_{X_{\{r\}=|\xi|}} \]
\[ \| A^{f,h} \nabla \phi \|_{X_{\{r\}=|\xi|}} \lesssim \| A^{f,h} \|_{X_{\{r\}=|\xi|}} \| \nabla \phi \|_{X_{\{r\}=|\xi|}} \]

where the last estimate follows by Theorem 2.1.

**Claim 6:**
\[ \| A^f \nabla \phi \|_{L^2} \lesssim \left( \| \nabla A^f \|_{X_{\{r\}=|\xi|}} + \| \nabla A^f \|_{X_{\{r\}=|\xi|}} \right) \| \nabla \phi \|_{X_{\{r\}=|\xi|}} \]

where we split \( A^f \) into its low and high frequency parts \( A^{f,a} \) and \( A^{f,h} \). The low frequency part is easily estimated as follows:
\[ \| A^{f,a} \nabla \phi \|_{L^2} \lesssim \| A^{f,a} \|_{L^\infty L^2} \| \nabla \phi \|_{L^\infty L^2} \lesssim \| \nabla A^{f,a} \|_{L^\infty L^2} \| \nabla \phi \|_{X_{\{r\}=|\xi|}} \]

For the high frequency part we want to show:
\[ \| A^{f,h} \nabla \phi \|_{L^2} \lesssim \| A^{f,h} \|_{X_{\{r\}=|\xi|}} \| \nabla \phi \|_{X_{\{r\}=|\xi|}} \]

This estimate would hold if we prove
\[ \int m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \tilde{u}_1(\xi_1, \tau_1) \tilde{u}_2(\xi_2, \tau_2) \tilde{u}_3(\xi_3, \tau_3) d\xi dt \lesssim \prod_{i=1}^3 \| u_i \|_{L^2_{\xi \tau}}, \]
where
\[
m = \frac{1}{(|\tau_2| - |\xi_2|)^{\frac{1}{2}} + (\xi_3)^{\frac{1}{2}} + (\tau_3)^{\frac{1}{2}}}.
\]

The following argument is closely related to the proof of a similar estimate in [T1].
By two applications of the averaging principle ([T], Prop. 5.1) we may replace \( m \) by
\[
n' = \frac{\chi(|\tau_2| - |\xi_2| = 1, \xi_3 \sim 1)}{(\xi_3)^{\frac{1}{2}} + (\tau_3)^{\frac{1}{2}}}.
\]

Let now \( \tau_2 \) be restricted to the region \( \tau_2 = T + O(1) \) for some integer \( T \). Then \( \tau_1 \)
is restricted to \( \tau_1 = -T + O(1) \), because \( \tau_1 + \tau_2 + \tau_3 = 0 \), and \( \xi_2 \) is restricted to \( |\xi_2| = |T| + O(1) \). The \( \tau_1 \)-regions are essentially disjoint for \( T \in \mathbb{Z} \) and similarly the \( \tau_2 \)-regions. Thus by Schur’s test ([T], Lemma 3.11) we only have to show

\[
\sup_{T \in \mathbb{Z}} \int \chi_{\tau_1 = -T + O(1)} \chi_{\tau_2 = T + O(1)} \chi_{|\tau_3| \sim 1} \frac{\alpha_1(\xi_1, \tau_1) \alpha_2(\xi_2, \tau_2) \alpha_3(\xi_3, \tau_3)}{(\xi_3)^{\frac{1}{2}} + (\tau_3)^{\frac{1}{2}}} d\xi d\tau \lesssim \prod_{i=1}^3 \| u_i \|_{L^2_{x,t}}.
\]

The \( \tau \)-behaviour of the integral is now trivial, thus we reduce to
\[
\sup_{T \in \mathbb{N}} \int \chi_{|\xi_2| = |T| + O(1)} \frac{\alpha_1(\xi_1, \tau_1) \alpha_2(\xi_2, \tau_2) \alpha_3(\xi_3, \tau_3)}{(\xi_3)^{\frac{1}{2}} + (\tau_3)^{\frac{1}{2}}} d\xi \lesssim \prod_{i=1}^3 \| f_i \|_{L^2}.
\]

It only remains to consider the following two cases:

**Case 1:** \( |\xi_1| \sim |\xi_3| \gtrsim T \). We obtain in this case

\[
L.H.S. of (33) \lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^{\frac{1}{2}}} \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_2 \|_{L^2} \| \chi_{|\xi| = T + O(1)} \hat{f}_2(\xi) \|_{L^\infty(\mathbb{R}^2)} \lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^{\frac{1}{2}}} \| f_1 \|_{L^2} \| f_3 \|_{L^2} \| \chi_{|\xi| = T + O(1)} \hat{f}_2(\xi) \|_{L^1(\mathbb{R}^2)} \lesssim \sup_{T \in \mathbb{N}} \frac{T^{\frac{1}{2}}}{T^{\frac{1}{2}}} \prod_{i=1}^3 \| f_i \|_{L^2} \lesssim \prod_{i=1}^3 \| f_i \|_{L^2}.
\]

**Case 2:** \( |\xi_1| \sim |\xi_3| \gtrsim T \). An elementary calculation shows that

\[
L.H.S. of (33) \lesssim \sup_{T \in \mathbb{N}} \| \chi_{|\xi| = T + O(1)} \frac{\alpha_1(\xi_1, \tau_1) \alpha_2(\xi_2, \tau_2) \alpha_3(\xi_3, \tau_3)}{(\xi_3)^{\frac{1}{2}} + (\tau_3)^{\frac{1}{2}}} \prod_{i=1}^3 \| f_i \|_{L^2} \lesssim \prod_{i=1}^3 \| f_i \|_{L^2},
\]

so that the desired estimate follows.

**Claim 7:**
\[
\| \nabla A^{cf} \phi \|_{X_{\tau = 0, \xi = 0, \cdots}} \lesssim \| \nabla A^{cf} \|_{X_{\tau = 0, \xi = 0, \cdots}}\| \phi \|_{X_{\tau = 0, \xi = 0, \cdots}}.
\]

By duality this is equivalent to
\[
\| w \phi \|_{X_{\tau = 0, \xi = 0, \cdots}} \lesssim \| w \|_{X_{\tau = 0, \xi = 0, \cdots}} \| \phi \|_{X_{\tau = 0, \xi = 0, \cdots}}.
\]

We use the estimate \((\xi)\|_{X_{\tau = 0, \xi = 0, \cdots}} \lesssim (|\xi| - |\tau|)\|_{X_{\tau = 0, \xi = 0, \cdots}}\| \phi \|_{X_{\tau = 0, \xi = 0, \cdots}}\| \phi \|_{X_{\tau = 0, \xi = 0, \cdots},}
\]

where the last estimate follows from Theorem 25.1 with \( s_0 = 0 \), \( b_0 = -\frac{1}{2} + \epsilon \), \( s_1 = 0 \), \( s_2 = 1 \), \( b_1 = \frac{1}{2} - \epsilon + \), \( b_2 = \frac{1}{2} + \epsilon - \).

**Claim 8:**
\[
\| A^{cf} A^{cf} \|_{L^2_{x,t}} \lesssim (\| \nabla A^{cf} \|_{L^2_{x,t}}^2 + \| \nabla A^{cf} \|_{L^2_{x,t}}^2) \| \phi \|_{X_{\tau = 0, \xi = 0, \cdots}}.
\]
Splitting $A^{cf} = A^{cf,h} + A^{cf,l}$ we first consider
\[ \|A^{cf,h} A^{cf,h} \phi\|_{L^2_{x,t}} \lesssim \|A^{cf,h}\|_{L^{\infty}_{x,t}}^2 \lesssim \|A^{cf,h}\|_{X_{\tau=0}^{1,1}} \phi_{X_{|\tau|=\epsilon}}^{1,1} \mathcal{L}. \]

Next we consider
\[ \|A^{cf,l} A^{cf,l} \phi\|_{L^2_{x,t}} \lesssim \|A^{cf,l}\|_{L^{\infty}_{x,t}}^2 \|\phi\|_{L^2_{x,t}} \lesssim \|\nabla A^{cf,l}\|_{L^{2}_{x,t}}^2 \|\phi\|_{X_{|\tau|=\epsilon}}^{1,1} \mathcal{L}, \]
and also
\[ \|A^{cf,l} A^{cf,h} \phi\|_{L^2_{x,t}} \lesssim \|A^{cf,l}\|_{L^{\infty}_{x,t}}^2 \|A^{cf,l}\|_{L^2_{x,t}} \|A^{cf,h}\|_{L^2_{x,t}} \|\phi\|_{L^2_{x,t}} \lesssim \|\nabla A^{cf,l}\|_{L^{2}_{x,t}} \|A^{cf,h}\|_{L^2_{x,t}} \|\phi\|_{X_{|\tau|=\epsilon}}^{1,1} \mathcal{L}, \]
which completes the proof of claim 8.

If one combines similar estimates with (30) and (31) we also obtain the required bounds for $A^{df} A^{df} \phi\|_{L^2_{x,t}}$ and $A^{df} A^{cf} \phi\|_{L^2_{x,t}}$.

Claim 9: For a suitable $N \in \mathbb{N}$ the following estimate holds:
\[ \|\phi V'(\phi^2)\|_{L^2_{x,t}} \lesssim \|\phi\|_{X_{|\tau|=\epsilon}}^{1,1} (1 + \|\phi\|_{X_{|\tau|=\epsilon}}^{N,1,1}). \]

Using the polynomial bound of $V'$ we obtain:
\[ \|\phi V'(\phi^2)\|_{L^2_{x,t}} \lesssim \|\phi\|_{L^2_{x,t}} + \|\phi\|_{L^2_{x,t}}^{N+1,1} \lesssim \|\phi\|_{X_{|\tau|=\epsilon}}^{1,1} (1 + \|\phi\|_{X_{|\tau|=\epsilon}}^{N,1,1}) \mathcal{L}. \]

Now the contraction mapping principle applies. The claimed properties of $A^{df}$ follow immediately from (30), (31) and (32). The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

Proof. We follow the arguments of Selberg-Tesfahun [ST] in the case of the Lorenz gauge. Define
\[ I(t) = \|\phi(t)\|_{L^2} + \sum_{j=1}^{2} \|D_j \phi(t)\|_{L^2} + \|\partial_t \phi(t)\|_{L^2}. \]

The first step is to show that the local existence time in Theorem 1.1 in fact only depends on $I(0)$.

Even in the temporal gauge $A_0 = 0$ there is some freedom left for the choice of the gauge. We apply a gauge transformation with
\[ \chi(x) = (-\Delta)^{-1} div A(0, x) = (-\Delta)^{-1} div a(x). \]

In the new gauge we obtain $\chi(t) = A_0 + \partial_t \chi = A_0 = 0$ and
\[ (A')^{cf}(0) = A^{cf}(0) + (\nabla \chi)^{cf}(0) = (-\Delta)^{-1} \nabla div A(0) + (-\Delta)^{-1} \nabla div A(0) = 0. \]

By this transformation the regularity of the data and of a solution is preserved, as we now show. The same holds for its inverse obtained by replacing $\chi$ by $-\chi$. We namely have $\|\phi\|_{L^2} = \|\phi\|_{L^2}$ and $\|\partial_t \phi\|_{L^2} = \|\partial_t \phi\|_{L^2}$ as well as
\[ \|\partial_j \phi\|_{L^2} \lesssim \|\partial_j (e^{i\chi} \phi)\|_{L^2} \lesssim \|\partial_j \chi\| e^{i\chi} \phi_{L^2} \lesssim \|\partial_j \chi\| e^{i\chi} \phi_{L^2} \lesssim \|\chi\|_{L^2} \|\phi\|_{H^1} \lesssim \bigl( \|\nabla' a\|_{L^2} + 1 \bigr) \|\phi\|_{H^1} < \infty, \]
\[ \|\nabla' A\|_{H^{1,2}} \lesssim \|\nabla' A\|_{H^{1,2}} + \|\nabla' \chi\|_{H^{1,2}} \lesssim \|\nabla' A\|_{H^{1,2}} + \|\nabla' a\|_{H^{1,2}} < \infty. \]
Moreover the compatibility condition is obviously preserved. An elementary computation also shows that $I(t)$ as well as $E(t)$ is preserved, because

$$
\|D_\mu \phi\|_{L^2} = \|(\partial_\mu - i(A_\mu + \partial_\mu \chi))(e^{iX}\phi)\|_{L^2} \\
= \|ie^{iX}\partial_\mu \chi \phi + e^{iX}\partial_\mu \phi - iA_\mu e^{iX} \phi - i\partial_\mu \chi e^{iX} \phi\|_{L^2} = \|D_\mu \phi\|_{L^2}.
$$

We apply Theorem 1.2 to the transformed problem and obtain a solution on $[0, T]$, where $T$ depends only on $\|\phi'(0)\|_{H^1} + \|\partial_t \phi'(0)\|_{L^2}$, where we used that $(A')^{-f}(0) = 0$. We now show that this quantity is controlled by $I(0)$. Trivially we have $\|\phi'(0)\|_{L^2} = \|\phi(0)\|_{L^2}$ and $\|(\partial_t \phi')(0)\|_{L^2} = \|(\partial_t \phi)(0)\|_{L^2}$. Furthermore using $(A')^{-f}(0) = 0$ we obtain

$$
\|\partial_t \phi'(0)\|_{L^2} \leq \|D_\mu \phi'(0)\|_{L^2} + \|(A')^{-f}(0)\|_{L^2} = \|D_\mu \phi'(0)\|_{L^2} + \|(A')^{-f}(0)\|_{L^2}.
$$

But now we obtain by (3)

$$
(A'_1)^{df}(0) = -2\Delta^{-1}\partial_2 Im(\overline{\phi'(0)}(\partial_2 \phi'(0)) \\
= -2\Delta^{-1}\partial_2 Im(e^{-iX}\overline{\phi'(0)} e^{iX}(\partial_2 \phi(0))) = A'_1(0)
$$

and similarly $(A'_2)^{df}(0) = A'_2(0)$. By the covariant Sobolev inequality (cf. (6.4))

$$
\|\phi'(0)\|_{L^2} \lesssim \|\phi'(0)\|^\frac{1}{2} \left(\sum_{j=1}^{2} \|D^j \phi'(0)\|_{L^2}\right)^\frac{1}{2} = \|\phi(0)\|^\frac{1}{2} \left(\sum_{j=1}^{2} \|D^j \phi(0)\|_{L^2}\right)^\frac{1}{2} \lesssim I(0)
$$

and similarly $\|\phi(0)\|_{L^2} \lesssim I(0)$, and thus by (33) and (35)

$$
\|\partial_t \phi'(0)\|_{L^2} \lesssim \|D_\mu \phi(0)\|_{L^2} + \|A^{df}(0)\|_{L^2} \|\phi'(0)\|_{L^2} \\
\lesssim I(0) + \|\nabla^{-1} \phi(0)(\partial_t \phi(0))\|_{L^2} I(0) \\
\lesssim I(0)(1 + \|\phi(0)\|_{L^2} \|\partial_t \phi(0)\|_{L^2}) \\
\lesssim I(0)(1 + \|I(0)\|^\frac{1}{2}_{L^2}).
$$

We conclude that $T$ only depends on $I(0)$. Finally we reverse the gauge transform to obtain the solution $(\phi(t), A(t))$ on $[0, T]$.

What we need to obtain a global solution is an a priori bound of $I(t)$ on every finite time interval. Of course we use energy conservation $E(t) = E(0)$. Under our sign assumption $V(r) \geq -\alpha^2 r$ $\forall r \geq 0$ we obtain

$$
\sum_{\mu=0}^{2} \|D_\mu \phi(t)\|_{L^2}^2 = E(t) - \int V(|\phi|^2) dx \leq |E(0)| + \alpha^2 \|\phi(t)\|_{L^2}^2.
$$

This implies

$$
\frac{d}{dt} (\|\phi(t)\|_{L^2}^2) = \int 2Re(\overline{\phi(t)}(D_\mu \phi(t))) dx \\
\leq 2\|\phi(t)\|_{L^2} \|D_\mu \phi(t)\|_{L^2} \\
\leq 2\|\phi(t)\|_{L^2} (|E(0)| + \alpha^2 \|\phi(t)\|_{L^2}^2) \leq \alpha^{-1} |E(0)| + 2\alpha \|\phi(t)\|_{L^2}^2,
$$

hence by Gronwall’s lemma

$$
\|\phi(t)\|_{L^2}^2 \leq e^{2\alpha t}(|\phi(0)|_{L^2}^2 + |t|\alpha^{-1}|E(0)|).
$$

By (36) and (37) we obtain the desired a priori control of $I(t)$, so that Theorem 1.2 is proved. □
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