C\textsuperscript{2}-Cofiniteness of Cyclic-Orbifold Models

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Received: 12 February 2014 / Accepted: 24 August 2014
Published online: 26 November 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: We prove an orbifold conjecture for conformal field theory with a solvable automorphism group. Namely, we show that if \( V \) is a \( C\textsuperscript{2} \)-cofinite simple vertex operator algebra and \( G \) is a finite solvable automorphism group of \( V \), then the fixed point vertex operator subalgebra \( V^G \) is also \( C\textsuperscript{2} \)-cofinite, where \( C\textsuperscript{2} \)-cofiniteness is equivalent to the condition that \( V \) has only finitely many isomorphism classes of simple \( V \)-modules (including weak modules) and all finitely generated \( V \)-modules have composition series. This result offers a mathematically rigorous background to orbifold theories with solvable automorphism groups.

1. Introduction

In order to explain the moonshine phenomenon on the monster simple group and the modular functions, Borcherds [1] has introduced a concept of vertex (operator) algebra as an algebraic version of conformal field theory. It is a quadruple \( (V, \ Y, \mathbf{1}, \omega) \) satisfying the several axioms, where \( V \) is a graded vector space \( V = \bigoplus_{i=-\infty}^{\infty} V_i \), \( Y(v, z) = \sum_{m \in \mathbb{Z}} v_{m} z^{-m-1} \in \text{End}(V)[[z, z^{-1}]] \) denotes a vertex operator of \( v \in V \) on \( V \) which satisfies Borcherds identity (2.1), \( \mathbf{1} \in V_0 \) and \( \omega \in V_2 \) are specified elements called the vacuum and the Virasoro element of \( V \), respectively. We set \( Y(\omega, z) = \sum_{n \in \mathbb{Z}} (n) z^{-n-2} \).

One of the main targets in the research of vertex operator algebras (shortly VOA) is a construction of VOAs of finite type, that is, all modules (including weak modules) have a composition series consisting of only finitely many isomorphism classes of simple modules. If \( V \) is a VOA and \( \sigma \) is a finite automorphism of order \( p \), then a fixed point subVOA \( V^\sigma \) is called an orbifold model, (see [3,4]). So-called “orbifold conjecture” says that if \( V \) is of finite type, then so is \( V^\sigma \). It is revealed that the above finiteness condition is equivalent to the \( C\textsuperscript{2} \)-cofiniteness by [2,8]. Here a \( V \)-module \( W \) is called to

Supported by the Grants-in-Aids for Scientific Research, No. 22654002, The Ministry of Education, Science and Culture, Japan.
be $C_2$-cofinite when $C_2(W) = \text{Span}_C \{ v_{-2}u \mid v \in V, u \in W, \text{wt}(v) > 0 \}$ has a finite co-dimension in $W$. This condition was originally introduced by [13] as a technical condition to prove the modular invariance property. It is very important and the most general theorems require this condition. For example, the author in [9] mentioned that if the orbifold model $V^\sigma$ is $C_2$-cofinite, then we are able to get all the information of (weak) $V^\sigma$-modules from (twisted and ordinary) $V$-modules and every simple $V^\sigma$-module is a submodule of a (twisted or ordinary) $V$-module. Therefore, the $C_2$-cofiniteness on $V^\sigma$ offers a mathematically rigorous background to all orbifold theories.

For the orbifold conjecture, there are partial answers. For example, T. Abe has proved it for a permutation automorphism of order $p = 2$. For lattice VOAs, Yamakulna [12] has shown the case $p = 2$ and the author [10] has shown the case $p = 3$, which was used to construct a new holomorphic VOA of central charge 24. In this paper, we will prove all cases for any finite order $p$ with the powerful help of the Borcherds identity (2.1) and the skew-symmetry (4.1).

**Main Theorem.** Let $V$ be a $C_2$-cofinite simple VOA of CFT-type and $\sigma \in \text{Aut}(V)$ of finite order $p$. Then a fixed point vertex operator subalgebra $V^\sigma$ is also $C_2$-cofinite.

As corollaries, we have:

**Theorem 1.** Let $V$ be a $C_2$-cofinite simple VOA of CFT-type and $G \leq \text{Aut}(V)$ finite solvable. Then a fixed point vertex operator subalgebra $V^G$ is also $C_2$-cofinite.

**Corollary 2.** Let $V$ be a $C_2$-cofinite VOA and a subVOA $U$ is isomorphic to a lattice VOA. Then the commutant $E$ of $U$ is $C_2$-cofinite.

**Corollary 3.** If $V$ is $C_2$-cofinite and a subVOA $U$ is isomorphic to a 2-dim. Ising model $L\left(\frac{1}{2}, 0\right)$ of central charge $\frac{1}{2}$, then the commutant $E$ of $U$ is $C_2$-cofinite.

Here the commutant $E$ of $U$ is defined by $\{ v \in V \mid u_m v = 0 \text{ for all } u \in U, m \geq 0 \}$. We note that it is a subVOA.

**Remark 1.** In this paper, we assume that $V$ is of CFT-type. This is because of simplifying the proof. From our proof, it is not difficult to see that we have the same conclusion without the assumption of CFT-type.

We close this introduction by acknowledging with thanks a number of communications with Yu-ichi Tanaka and Shigeki Akiyama. The author thanks Toshiyuki Abe, Hiroshi Yamauchi and Atsushi Matsuo for reviewing the manuscript and their suggestions about the shorter proofs. He also thanks the organizers of the conference held at Taitung University in March 2013 for their hospitality.

**2. Truncation Property**

From the axiom of VOAs, for $v \in V_r$ and $u \in V_n$, we have $v_m u \in V_{r-m-1+n}$. Hence there is an integer $N$ such that $v_n u = 0$ for any $n > N$. This is called a truncation property. To simplify the notation, we will say that $v$ is truncated on $u$.

Set $V^* = \text{Hom}(V, \mathbb{C})$ and define a pairing $\langle \cdot, \cdot \rangle$ by $\langle v, \xi \rangle = \xi(v)$ for $\xi \in V^*$ and $v \in V$. For $v \in V$ and $m \in \mathbb{Z}$, actions $v_m$ on $V^*$ are defined by

$$\langle w, Y^*(v)(z)\xi \rangle = \langle Y(e^{L(1)}z(-z^{-2})^{L(0)}v, z^{-1})w, \xi \rangle$$
for $w \in V, \xi \in \text{Hom}(V, \mathbb{C})$, where $Y^*(v, z) = \sum_{m \in \mathbb{Z}} v_m z^{-m-1}$ is called an adjoint operator of $v$. An important fact is that $(\bigoplus_{m=0}^{\infty} \text{Hom}(V_m, \mathbb{C}), Y^*)$ becomes a $V$-module, see [6] for the proof. This module is called a restricted dual of $V$ which is denoted by $V'$. In particular, $Y^*(\cdot, z)$ satisfy the Borcherds identity:

$$\sum_{i=0}^{\infty} \binom{m}{i} (u_{r+i} v)_{m+n-i} \xi = \sum_{j=0}^{\infty} (-1)^j \binom{r}{j} (u_{r+m-j} v_{n+j} \xi - (-1)^j v_{r+n-j} u_{m+j} \xi)$$

(2.1)

for any $m, n, r \in \mathbb{Z}, v, u \in V, \xi \in V'$. Since $V^* = \prod_n \text{Hom}(V_n, \mathbb{C})$, we can express $\xi \in V^*$ by $\prod_n \xi^{(n)}$ with $\xi^{(n)} \in \text{Hom}(V_n, \mathbb{C})$. We call $\xi \in V^*$ $L(0)$-free if $\dim \mathbb{C}[L(0)]\xi = \infty$, that is, $\xi^{(n)} \neq 0$ for infinitely many $n$. We note that if $V$ is $C_2$-cofinite, then any (weak) module does not contain $L(0)$-free elements.

The weight of the terms in (2.1) for $\xi \in \text{Hom}(V_t, \mathbb{C})$ and that for $\xi \in \text{Hom}(V_s, \mathbb{C})$ are different when $t \neq s$. We also have that the both sides of (2.1) are well-defined for each $\xi \in \text{Hom}(V_t, \mathbb{C})$. Therefore the Borcherds’ identity is also well-defined on $V^*$, as Haisheng Li has pointed out in [7]. However, $V^*$ is not a $V$-module. The problem is a failure of truncation properties.

**Lemma 4.** If $u$ and $v$ are truncated on $\xi$, then $v_m u$ is also truncated on $\xi$ for any $m$. In particular, if $V$ is generated by $\Omega \subseteq V$ as a vertex algebra and all elements in $\Omega$ are truncated on $\xi$, then all elements in $V$ are truncated on $\xi$.

**Proof.** We may assume $u_n \xi = v_n \xi = u_n v = 0$ for $n \geq N$. We assert that for $s \in \mathbb{N}$ and $n \geq 2N + s$, we have $(u_{N-s} v)_n \xi = 0$. Suppose false and let $s$ be a minimal counterexample. Substituting $r = N - s, n = N + s + p, m = N + q$ in (2.1) with $p, q \geq 0$, the left side equals

$$\text{LH} = \sum_{i=0}^{\infty} \binom{N + q}{i} (u_{N-s+i} v)_{2N+q+s+p-i} \xi = \sum_{i=0}^{s} \binom{N + q}{i} (u_{N-(s-i)} v)_{2N+s-i+p+q} \xi$$

$$= (u_{N-s} v)_{2N+s+p+q} \xi$$

by the minimality of $s$. On the other hand, the right side is

$$\text{RH} = \sum_{i=0}^{\infty} (-1)^i \binom{N - s}{i} (u_{2N-s+q-i} v_{N+s+p+i} \xi - (-1)^{N-s} v_{2N-s+p-i} u_{N+q+i} \xi) = 0,$$

which contradicts the choice of $s$. □

Since $v_n u_m \xi = u_m v_n \xi + \sum_{i=0}^{\infty} \binom{n}{i} (v_i u)_{n+m-i} \xi$, Lemma 2 (see also Li [7]) implies:

**Lemma 5.** If $v, u \in V$ are truncated on $\xi \in V^*$, then $v$ is truncated on $u_m \xi$ for any $m$. In particular, if all elements of $V$ are truncated on $\xi$, then $\text{Span}_{\mathbb{C}} \{u_1^{m_1} \cdots u_k^{m_k} \xi \mid u^i \in V, m_i \in \mathbb{Z}\}$ is a $V$-module.
3. General Setting

Let \((V, Y, 1, \omega)\) be a \(C_2\)-cofinite VOA and \(\sigma\) an automorphism of \(V\) of order \(p\). Viewing \(V\) as a \(<\sigma>\)-module, we decompose

\[
V = V^{(0)} \oplus V^{(1)} \oplus \cdots \oplus V^{(p-1)}
\]

where \(V^{(m)} = \{v \in V \mid v^{\sigma} = e^{2\pi \sqrt{-1}m/p}v\}\). If \(V^{(1)}\) and \(V^{(p-1)}\) are \(C_2\)-cofinite, then so is \(V^{(0)}\) by the main theorem in [11]. Therefore we assume that \(V^{(1)}\) is not \(C_2\)-cofinite. For \(A, B \subseteq V\) and \(m \in \mathbb{Z}\), \(A(m)B\) denotes a subspace \(\text{Span}_\mathbb{C}\{a_mb \mid a \in A, b \in B\}\).

**Lemma 6.** \((V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)}\) has a finite co-dimension in \(V^{(1)}\).

*Proof.* Suppose false, i.e. \(V^{(1)}_{m}/((V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)})_m \neq 0\) for infinitely many \(m\). Then there is a \(L(0)\)-free element \(\xi \in (V^{(1)})^*\) such that \((V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)}, \xi) = 0\). In other words,

\[
0 = \langle u_{-2N}v, \xi \rangle = \langle u_{-2N}u, \xi \rangle
\]

for any \(v \in V^{(1)}\) and \(u \in V^{(0)}\). Since \(V^{(1)}\) is a direct summand of \(V\), we may view \((V^{(1)})^* \subseteq V^*\). By taking adjoint operators, we have:

\[
\langle u, v_{2\omega(v)+2N}\rangle = \langle (-1)^{\omega(v)} \sum_{s=0}^{\infty} \frac{1}{s!} (L(1)^s v)_{-2-s-2N}u, \xi \rangle = 0
\]

\[
\langle v, u_{2\omega(u)+2N}\rangle = \langle (-1)^{\omega(u)} \sum_{s=0}^{\infty} \frac{1}{s!} (L(1)^s u)_{-2-s-2N}v, \xi \rangle = 0,
\]

which imply that \(v \in V^{(1)}\) and \(u \in V^{(0)}\) truncate on \(\xi\). However, since \(V\) is simple, \(V^{(1)} + V^{(0)}\) generates a \(C_2\)-cofinite VOA \(V\) by normal products, which contradicts that \(\xi\) is \(L(0)\)-free. \(\Box\)

So, there is a finite dimensional subspace \(P\) of \(V^{(1)}\) such that \(V^{(1)} = (V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)} + P\). We may assume that \(P\) is a direct sum of homogeneous spaces.

**Proposition 7.** \(V^{(1)} = (V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(1)]P\).

*Proof.* Suppose false and we choose \(0 \neq w \in V^{(1)} - ((V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(1)]P)\) with minimal weight. Since \(V^{(1)} = (V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)} + P\), we may assume \(w \in (V^{(1)})_{(-2)}V^{(0)}\). We may also assume \(w = a-2u\) with \(a \in V^{(1)}\) and \(u \in V^{(0)}\). Then by the skew-symmetry (4.1), we have

\[
w = -u_{-2}a - \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} L(-1)^j u_{-2+j}a.
\]

Since \(\omega(u_{-2+j}a) < \omega(u_{-2}a) = \omega(w)\) for \(j \geq 1\), we have

\[
w \in (V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)](V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)]P
\]

\[
\subseteq (V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)]P
\]

by the minimality of \(\omega(w)\), which contradicts the choice of \(w\). \(\Box\)
4. The Coefficient Functions

Since $L(-1)C_2(V^{(1)}) \subseteq C_2(V^{(1)})$, $V^{(1)}/C_2(V^{(1)})$ is a finitely generated $\mathbb{C}[L(-1)]$-module by Proposition 7 and so the $L(-1)$-torsion part has a finite dimension. Let $\tilde{T}$ be the inverse image in $V^{(1)}$ of the $L(-1)$-torsion submodule of $V^{(1)}/C_2(V^{(1)})$. Since $\dim V^{(1)}/C_2(V^{(1)}) = \infty$, there is a set of free generators $\{\alpha^i : i = 1, \ldots, t\}$ such that

$$V^{(1)} = \left( \bigoplus_{i=1}^{t} \mathbb{C}[L(-1)]\alpha^i \right) \oplus \tilde{T}.$$

Set $\tilde{T} = \left( \bigoplus_{i=2}^{t} \mathbb{C}[L(-1)]\alpha^i \right) \oplus \tilde{T}$ and $\alpha = \alpha^1$, then $\tilde{T}$ is $\mathbb{C}[L(-1)]$-invariant and

$$V^{(1)} = \mathbb{C}[L(-1)]\alpha \oplus \tilde{T}.$$

We may assume that $\alpha$ is a homogeneous element.

We also introduce an equivalent relation $\equiv$ on $V^{(1)}$ by modulo $\tilde{T}$. Under this setting, for any $n \in \mathbb{N}$ and any homogeneous elements $a \in V^{(k)}$ and $b \in V^{(p-k+1)}$, there are complex numbers $f^{a,b}(n)$ such that

$$a \equiv n - \text{wt}(a) + \text{wt}(b) - \text{wt}(a) \equiv f^{a,b}(n)\alpha - n\mathbb{1} \pmod{\tilde{T}}.$$

We note $\frac{L(-1)^n}{n!} \alpha = \alpha_n \mathbb{1}$ for $n \in \mathbb{N}$. From now on, for $a, b \in V$, we always use $M$ to denote $\text{wt}(a) + \text{wt}(b) - \text{wt}(a)$ for simplifying the notation. We view $f^{a,b}$ as a map from $\mathbb{N}$ to $\mathbb{C}$. We note that since $\text{wt}(a) - \text{wt}(b) < \text{wt}(a)$ for $n \in \mathbb{Z}_{<0}$, we have $a_{-n+M-1}b \in T$ by the choice of $\alpha$ and so we may also consider $f^{a,b}(n) = 0$ for $n \in \mathbb{Z}_{<0}$.

For $k = 0, \ldots, p-1$, we set

$$\mathcal{F}_k = \text{Span}_{\mathbb{C}} \left\{ f^{a,b} \mid a \in V^{(k)}, b \in V^{(p-k+1)} \right\}.$$

For a map $f : \mathbb{N} \to \mathbb{C}$, we define a map $xf$ by $(xf)(n) = nf(n)$.

**Lemma 8.** $\mathcal{F}_k$ are all $\mathbb{C}[x]$-invariant.

**Proof.** Clearly, $\mathcal{F}_k$ is a vector space. Since

$$(L(-1)a)_{-n+M}b = (n - M)a_{-n+M-1}b \equiv (n - M)f^{a,b}(n)\alpha - n\mathbb{1} \pmod{\tilde{T}},$$

we have $xf^{a,b} = f^{L(-1)a,b} + Mf^{a,b} \in \mathcal{F}_k$. \qed

**Lemma 9.** For $f \in \mathcal{F}_0$, $Q_f = \{n \in \mathbb{Z} \mid f(n) \neq 0\}$ is a finite set. We will call such a function finite type.

**Proof.** For $a \in V^{(0)}$, $b \in V^{(1)}$, we have $a_{-n+M-1}b \equiv f^{a,b}(n)\alpha - n\mathbb{1} \pmod{\tilde{T}}$. Therefore, $a_{-n+M-1}b \in C_2(V^{(1)})$ and $f^{a,b}(n) = 0$ for $n \geq M + 1$. Since all elements in $\mathcal{F}_0$ are linear combinations of such elements, we have the desired result. \qed

For a map $f : \mathbb{N} \to \mathbb{C}$, we introduce two operators $S$ and $T$ as follows:

$$Sf(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) \quad \text{for } n \in \mathbb{N}$$

$$Tf(n) = (-1)^n f(n) \quad \text{for } n \in \mathbb{N}.$$

Clearly, $S^2 = T^2 = \text{id}$. We also have the following by induction.
Lemma 10.

\[(ST)^k f(n) = \sum_{j=0}^{n} \binom{n}{j} k^j f(n-j) \quad \text{for } k = 1, \ldots \]

The following is a key result.

Lemma 11. For \( k = 1, 2, \ldots, p - 1 \), we have:

\[ \mathcal{F}_{1+p-k} = S(\mathcal{F}_k) \quad \text{and} \quad \mathcal{F}_{p-k} = T(\mathcal{F}_k). \]

Proof. The operator \( S \) comes from the skew-symmetry. In fact, for \( a \in V^{(k)}, b \in V^{(p-k+1)} \) and \( a_{-n+M-1}b \equiv f^{a,b}(n)\alpha_{-n-1}1 \) for \( x \in \mathbb{N} \), then

\[ b_{-n+M-1}a \equiv (-1)^{n+M} \sum_{k=0}^{\infty} \frac{L(-1)^k}{k!} (-1)^k a_{-(n-k)M-1}b \]

\[ \equiv (-1)^{n+M} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f^{a,b}(n-k)\alpha_{-n-k-1}1 \]

since \( L(-1)^k a_{-n+M-1}b \in \mathcal{T} \) for \( k > n \),

\[ \equiv (-1)^{n+M} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f^{a,b}(n-k)\alpha_{-n-1}1 \]

\[ \equiv (-1)^{M} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f^{a,b}(n)\alpha_{-n-1}1 \equiv (-1)^M Sf^{a,b}(n)\alpha_{-n-1}1. \] (4.1)

Therefore, \( f^{b,a}(n) = (-1)^M Sf^{a,b}(n) \) and \( \mathcal{F}_{1+p-k} = S(\mathcal{F}_k) \).

For any \( m \in \mathbb{Z} \) and \( h \in \mathbb{N} \), by substituting \( n = -m - 2 - h \) and \( r = -x + N + h \) in the Borechers’ identity (2.1), we have

\[ 0 \equiv \sum_{i=0}^{\infty} \binom{m}{i} (u_{-n+N+h+i}v)_{-2-h-j} \xi \]

\[ = \sum_{j=0}^{\infty} \binom{-n+N+h}{j} (-1)^j u_{-n+N+h+m-j}v_{-m-2-h+j} \xi \]

\[ - (-1)^{-n} \sum_{j=0}^{\infty} \binom{-n+N+h}{j} (-1)^{j+N+h} v_{-n+N-m-2-j} u_{m+j} \xi \]

for \( u \in V^{(k)}, v \in V^{(p-k)}, \xi \in V^{(1)} \), where \( N = \text{wt}(\xi) + \text{wt}(u) + \text{wt}(v) \). We note \( u_{m+j} \xi = 0 \) for \( j \geq Q = \text{wt}(u) + \text{wt}(\xi) - m \). Since we will treat only \( v_{-n+N-m-2} u_{m} \xi \) later, we may assume \( u_{m} \xi \neq 0 \) and so \( Q \geq 1 \). Let us consider a \( Q \times Q \)-matrix

\[ A := ((-1)^{h-j+N} \binom{-x+h+N}{j})_{h,j=0,\ldots,Q-1} \]

consisting of coefficients of \( (-1)^x v_{-x-2+N-m} u_{m+j} \xi \), where we view \( n \) as a variable \( x \). It is easy to see \( \det A = \pm 1 \) since \( \binom{s+1}{j} - \binom{s}{j} = \binom{s}{j-1} \). Therefore, there are
polynomials $\lambda^m_n(x) \in \mathbb{C}[x]$ for $0 \leq h < Q$ such that

$$(-1)^x v_{-x-2+N-mu_m \xi} = \sum_{h=0}^{Q-1} \lambda^m_n(x) \left( \sum_{j=0}^{N+h+m+2} \binom{-x+h+N}{j} (-1)^j u_{-x+h+m+N-j}(v_{-2-m-h+j} \xi) \right).$$

Since the coefficients of the right side at $\alpha_{-x-1}$ are all in $\mathcal{F}_k$ by Lemma 8, the above equation implies that a function defined by $v_{-x-2+N-mu_m \xi}$ is in $T(\mathcal{F}_k)$ for any $v \in V^{(p-k)}$, $u \in V^{(k)}$, $\xi \in V^{(1)}$ and $m \in \mathbb{Z}$. Since $V^{(1+k)}$ is a simple $V^{(0)}$-module, $V^{(1+k)}$ is spanned by elements with the form $u_m \xi$ with $u \in V^{(k)}$, $\xi \in V^{(1)}$ and $m \in \mathbb{Z}$ and so we have $T(\mathcal{F}_{p-k}) \subseteq \mathcal{F}_k$ for any $k$. Since $T^2 = 1$, we have the equality $T(\mathcal{F}_k) = \mathcal{F}_{p-k}$.

Now we are able to complete the proof of Main Theorem. As we have shown, every elements in $\mathcal{F}_0$ is of finite type. In particular, since $1_{-x-1} \alpha = \delta_{0,x} \alpha$, we have $\delta_{0,x} \in \mathcal{F}_0$. On the other hand, by Lemma 11, we have

$$\mathcal{F}_0 \downarrow T \not\mathcal{F}_1 \downarrow T \not\mathcal{F}_2 \downarrow T \not\cdots \not\mathcal{F}_{p-1} \not\mathcal{F}_0$$

where $p$ is the order of $\sigma$. In particular, we have $(ST)^p(\mathcal{F}_0) = \mathcal{F}_0$. However, since $(ST)^p(\delta_{0,x})(n) = p^n$ is not of finite type. We have a contradiction.

This completes the proof of Main Theorem. □

As last, we will prove corollaries of Main Theorem.

Proof of Theorem 1. Let $V$ be a $C_2$-cofinite simple VOA and $G$ a finite solvable subgroup of $\text{Aut}(V)$. We will prove Theorem 1 by the induction on $|G|$. Since $G$ is solvable, $G$ has a normal abelian subgroup $A \neq 1$. We first assume that $G = A$ and let $1 \neq \sigma \in G$ be an element of prime order. Then $V^\sigma$ is $C_2$-cofinite by Main Theorem. Furthermore, $V^\sigma$ is simple by [5]. Therefore, $V^G = (V^\sigma)^{G/\langle \sigma \rangle}$ is also $C_2$-cofinite by the induction, which proves the assertion of Theorem 1. So, we have $A < G$. By the minimality of $|G|$, $V^A$ is $C_2$-cofinite and it is also simple by [5]. Therefore, by the minimality of $|G|$, $V^G = (V^A)^{G/A}$ is also $C_2$-cofinite.

Proof of Corollary 1. Assume $U \cong V_L$ for some lattice $L$ and set $L^* = \{a \in \mathbb{Q}L \mid \langle a, L \rangle \subseteq \mathbb{Z}\}$. We view $V$ as a $V_L$-module. Since $V_L$ is rational and the category of $V_L$-modules have a $L^*/L$-module structure, the actions of $G = \text{Hom}(L^*/L, \mathbb{C}^\times)$ on $V$ are induced from this structure. Then $V^G \cong U \otimes E$, where $E$ is a commutant of $U$ in $V$. By Theorem 1, $U \otimes E$ is $C_2$-cofinite. If $E$ is not $C_2$-cofinite, then $E$ has a weak module $B$ containing L(0)-free element by [8] and so $U \otimes E$ has a weak module $U \otimes B$ containing L(0)-free elements, which contradicts the $C_2$-cofiniteness on $U \otimes E$.

Proof of Corollary 2. Let $U \cong L(\frac{1}{2}, 0)$ and we view $V$ as a $U$-module. Since $L(\frac{1}{2}, 0)$ is rational, $V$ is a direct sum of simple $U$-modules. Then $\tau$ defined by $\text{Id}$ on $L(\frac{1}{2}, 0)$ and $L(\frac{1}{2}, \frac{1}{2})$ and $-\text{Id}$ on $L(\frac{1}{2}, -\frac{1}{16})$ as $U$-modules, respectively, becomes an automorphism of $V$ by the fusion rules of $L(\frac{1}{2}, 0)$-modules. Then $V^\tau$ is $C_2$-cofinite by Main Theorem and simple by [5]. We then view it as a $U$-module, whose compositions are isomorphic to $L(\frac{1}{2}, 0)$ or $L(\frac{1}{2}, \frac{1}{2})$ as $U$-modules. Then $\sigma$ defined by $\text{Id}$ on $L(\frac{1}{2}, 0)$ and $-\text{Id}$ on $L(\frac{1}{2}, \frac{1}{2})$
as $U$-modules, respectively, is again an automorphism of $V^\tau$. Then $(V^\tau)'^\alpha = U \otimes E$, where $E$ is a commutant of $U$ in $V$. By Main Theorem, $U \otimes E$ is $C_2$-cofinite and so is $E$ by the same argument as above.

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Communicated by Y. Kawahigashi