C-parallel and C-proper Slant Curves of $S$-manifolds

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Abstract. In the present paper, we define and study $C$-parallel and $C$-proper slant curves of $S$-manifolds. We prove that a slant curve $\gamma$ in an $S$-manifold of order $r \geq 3$, under certain conditions, is $C$-parallel or $C$-parallel in the normal bundle if and only if it is a non-Legendre slant helix or Legendre helix, respectively. Moreover, under certain conditions, we show that $\gamma$ is $C$-proper or $C$-proper in the normal bundle if and only if it is a non-Legendre slant curve or Legendre curve, respectively. We also give two examples of such curves in $\mathbb{R}^{2n+1}$.

1. Introduction

Let $M^m$ be an integral submanifold of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$. Then $M$ is called integral $C$-parallel if $\nabla \perp B$ is parallel to the characteristic vector field $\xi$, where $B$ is the second fundamental form of $M$ and $\nabla \perp$ is given by

$$(\nabla \perp B)(X, Y, Z) = \nabla \perp B(Y, Z) - B(\nabla X Y, Z) - B(Y, \nabla X Z),$$

where $X, Y, Z$ are vector fields on $M$, $\nabla \perp$ and $\nabla$ are the normal connection and the Levi-Civita connection on $M$, respectively [8]. Now, let $\gamma$ be a curve in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$. Lee, Suh and Lee introduced the notions of $C$-parallel and $C$-proper curves along slant curves of Sasakian 3-manifolds in the tangent and normal bundles [12]. A curve $\gamma$ in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be $C$-parallel if $\nabla T H = \lambda \xi$, $C$-proper if $\Delta H = \lambda \xi$, $C$-parallel in the normal bundle if $\nabla \perp T H = \lambda \xi$, $C$-proper in the normal bundle if $\Delta \perp H = \lambda \xi$, where $T$ is the unit tangent vector field of $\gamma$, $H$ is the mean curvature vector field, $\Delta$ is the Laplacian, $\lambda$ is a non-zero differentiable function along the curve $\gamma$, $\nabla \perp$ and $\Delta \perp$ denote the normal connection and Laplacian in the normal bundle, respectively [12]. For a submanifold $M$ of an arbitrary Riemannian manifold $\tilde{M}$, if $\Delta H = \lambda H$, then $M$ is called submanifold with a proper mean curvature vector field $H$ [6]. If $\Delta \perp H = \lambda H$, then $M$ is said to be submanifold with a proper mean curvature vector field $H$ in the normal bundle [1].

Let $\gamma(s)$ be a Frenet curve parametrized by the arc-length parameter $s$ in an almost contact metric manifold $M$. The function $\theta(s)$ defined by $\cos[\theta(s)] = g(T(s), \xi)$ is called the contact angle function. A curve $\gamma$ is called a slant curve if its contact angle is a constant [7]. If a slant curve is with contact angle $\frac{\pi}{2}$, then it is called a Legendre curve [4].

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Lee, Suh and Lee studied C-parallel and C-proper slant curves of Sasakian 3-manifolds in [12]. As a generalization of this paper, in [9], the present authors studied C-parallel and C-proper slant curves in trans-Sasakian manifolds. In [14], the second author investigated C-parallel Legendre curves of non-Sasakian contact metric manifolds. In the present paper, our aim is to consider C-parallel and C-proper slant curves of S-manifolds.

The paper is organized as follows: In Section 2, we give a brief introduction about S-manifolds. Furthermore, we define the notions of C-parallel and C-proper curves in S-manifolds both in tangent and normal bundles. In Section 3, we consider C-parallel slant curves in S-manifolds in tangent and normal bundles, respectively. In Section 4, we study C-proper slant curves in S-manifolds in tangent and normal bundles, respectively. In the last section, we present two examples of these kinds of curves in $\mathbb{R}^{2m+s}$.

2. Preliminaries

Let $(M,g)$ be a $(2m+s)$-dimensional Riemann manifold. $M$ is called a framed metric manifold [17] with a framed metric structure $(\varphi, \varepsilon, \eta^\alpha, g), \alpha \in \{1, ..., s\}$, if this structure satisfies the following equations:

$$q^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \varepsilon_\alpha, \quad \eta^\alpha(\varepsilon_\beta) = \delta^\alpha_\beta, \quad \varphi(\varepsilon_\alpha) = 0, \quad \eta^\alpha \circ \varphi = 0$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y),$$

$$d\eta^\alpha(X,Y) - g(\varphi X, \varphi Y) = -d\eta^\alpha(Y,X), \quad \eta^\alpha(X) = g(X, \varepsilon),$$

where, $\varphi$ is a $(1,1)$ tensor field of rank $2m$; $\varepsilon_1, ..., \varepsilon_s$ are vector fields; $\eta^1, ..., \eta^s$ are 1-forms and $g$ is a Riemannian metric on $M$; $X, Y \in TM$ and $\alpha, \beta \in \{1, ..., s\}$. $(M^{2m+s}, \varphi, \varepsilon, \eta^\alpha, g)$ is also called a framed $\varphi$-manifold [13] or an almost r-contact metric manifold [16]. $(\varphi, \varepsilon, \eta^1, g)$ is said to be an S-structure, if the Nijenhuis tensor of $\varphi$ is equal to $-2d\eta^\alpha \otimes \varepsilon_\alpha$, where $\alpha \in \{1, ..., s\}$ [3, 5].

When $s = 1$, a framed metric structure turns into an almost contact metric structure and an S-structure turns into a Sasakian structure. For an S-structure, the following equations are satisfied [3, 5]:

$$(\nabla_X \varphi)Y = \sum_{\alpha=1}^{s} \left\{ g(\varphi X, \varphi Y)\varepsilon_\alpha + \eta^\alpha(Y)q^2X \right\},$$

$$\nabla_X \varepsilon_\alpha = -\varphi X, \quad \alpha \in \{1, ..., s\}. \quad (5)$$

If $M$ is Sasakian ($s = 1$), (5) can be directly calculated from (4).

Firstly, we give the following definition:

**Definition 2.1.** Let $\gamma : I \rightarrow (M^{2m+s}, \varphi, \varepsilon, \eta^1, g)$ be a unit speed curve in an S-manifold. Then $\gamma$ is called

i) C-parallel (in the tangent bundle) if

$$\nabla^T H = \lambda \sum_{\alpha=1}^{s} \varepsilon_\alpha,$$

ii) C-parallel in the normal bundle if

$$\nabla^N H = \lambda \sum_{\alpha=1}^{s} \varepsilon_\alpha,$$

iii) C-proper (in the tangent bundle) if

$$\Delta H = \lambda \sum_{\alpha=1}^{s} \varepsilon_\alpha.$$
iv) C-proper in the normal bundle if

\[ \Delta^+ H = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}, \]

where \( H \) is the mean curvature field of \( \gamma \), \( \lambda \) is a real-valued non-zero differentiable function, \( \nabla \) is the Levi-Civita connection, \( \nabla^+ \) is the Levi-Civita connection in the normal bundle, \( \Delta \) is the Laplacian and \( \Delta^+ \) is the Laplacian in the normal bundle.

Let \( \gamma : I \rightarrow M \) be a curve parametrized by arc length in an \( n \)-dimensional Riemannian manifold \((M,g)\). Denote by the Frenet frame and curvatures of \( \gamma \) by \( \{E_1, E_2, ..., E_r\} \) and \( \kappa_1, ..., \kappa_{r-1} \), respectively. We know that (see [1])

\[
\nabla_T H = -\kappa_1 E_1 + \kappa'_1 E_2 + \kappa_1 \kappa_2 E_3,
\]
\[
\nabla^+_T H = \kappa'_2 E_2 + \kappa_1 \kappa_2 E_3,
\]
\[
\Delta H = -\nabla_T \nabla_T \nabla_T T = 3\kappa_1 \kappa'_1 E_1 + (\kappa_1^2 + \kappa_2^2 - \kappa''_1) E_2 - (2\kappa_1 \kappa_2 + \kappa_1 \kappa'_2) E_3 - \kappa_1 \kappa_2 \kappa_3 E_4,
\]

and

\[
\Delta^+ H = -\nabla^+_T \nabla^+_T \nabla^+_T T = (\kappa_1 \kappa_2^2 - \kappa''_1) E_2 - (2\kappa_1 \kappa_2 + \kappa_1 \kappa'_2) E_3 - \kappa_1 \kappa_2 \kappa_3 E_4.
\]

So we can directly state the following Proposition:

**Proposition 2.2.** Let \( \gamma : I \rightarrow (M^{\text{dim}+}, q, \xi_{\alpha}, \eta_{\alpha}, g) \) be a unit speed curve in an S-manifold. Then

i) \( \gamma \) is C-parallel (in the tangent bundle) if and only if

\[ -\kappa_1^2 E_1 + \kappa'_1 E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}, \quad (6) \]

ii) \( \gamma \) is C-parallel in the normal bundle if and only if

\[ \kappa'_2 E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}, \quad (7) \]

iii) \( \gamma \) is C-proper (in the tangent bundle) if and only if

\[ 3\kappa_1 \kappa'_1 E_1 + (\kappa'_1 + \kappa_1 \kappa_2^2 - \kappa''_1) E_2 - (2\kappa_1 \kappa_2 + \kappa_1 \kappa'_2) E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}, \quad (8) \]

iv) \( \gamma \) is C-proper in the normal bundle if and only if

\[ (\kappa_1 \kappa_2^2 - \kappa''_1) E_2 - (2\kappa_1 \kappa_2 + \kappa_1 \kappa'_2) E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}. \quad (9) \]
Now, our aim is to apply Proposition 2.2 to slant curves in $S$-manifolds. Let $\gamma: I \rightarrow (M^{2m+s}, \varphi, \xi, \eta, g)$ be a slant curve. Then, if we differentiate
\[ \eta^a(T) = \cos \theta, \]
we get
\[ \eta^a(E_2) = 0, \]
where $\theta$ denotes the constant contact angle satisfying
\[ -\frac{1}{\sqrt{s}} \leq \cos \theta \leq \frac{1}{\sqrt{s}}. \]
The equality case is only valid for geodesics corresponding to the integral curves of
\[ T = \pm \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_{\alpha}, \]
(see [10]).

3. C-parallel Slant Curves of $S$-manifolds

Our first Theorem below is a result of Proposition 2.2 i).

Theorem 3.1. Let $\gamma: I \rightarrow M^{2m+s}$ be a unit-speed slant curve. Then $\gamma$ is C-parallel (in the tangent bundle) if and only if it is a non-Legendre slant helix of order $r \geq 3$ satisfying
\[ \sum_{\alpha=1}^{s} \xi_{\alpha} \in \text{sp}\{T, E_3\}, \]
\[ \varphi T \in \text{sp}\{E_2, E_4\}, \]
\[ \kappa_2 = -\kappa_1 \sqrt{1 - s \cos^2 \theta} \sqrt{\frac{1}{s} \cos \theta}, \quad \kappa_2 \neq 0, \]
\[ \lambda = -\frac{\kappa_1^2}{s \cos \theta} = \text{constant}, \]
and moreover if $\kappa_3 = 0$, then
\[ \kappa_1 = -s \cos \theta \sqrt{1 - s \cos^2 \theta}, \quad \kappa_2 = \sqrt{s} \left(1 - s \cos^2 \theta\right). \]
(10) \hspace{1cm} (11)

Proof. Let us assume that $\gamma$ is C-parallel (in the tangent bundle). Then, if we take the inner product of equation (6) with $E_2$, we find $\kappa_1' = 0$, that is, $\kappa_1 = \text{constant}$. Now, taking the inner product of equation (6) with $T$, we have
\[ \lambda s \cos \theta = -\kappa_1^2. \]
Here, $\theta \neq \frac{\pi}{2}$ since $\kappa_1 \neq 0$. Hence, $\gamma$ is non-Legendre slant. So, we get
\[ \lambda = -\frac{\kappa_1^2}{s \cos \theta} = \text{constant}. \]
Equation (6) can be rewritten as
\[ \sum_{a=1}^{s} \xi_{a} = -\frac{\kappa_{1}}{\lambda} T + \frac{\kappa_{1}k_{2}}{\lambda} E_{3}, \]
which is equivalent to
\[ \sum_{a=1}^{s} \xi_{a} = s \cos \theta T - \frac{\kappa_{2}s \cos \theta}{\kappa_{1}} E_{3}. \]
(12)
If we calculate the norm of both sides, we obtain
\[ \kappa_{2} = -\frac{\kappa_{1}}{s \cos \theta} \sqrt{1 - s \cos^{2} \theta}. \]
(13)
If we assume \( \kappa_{2} = 0 \), then \( \sum_{a=1}^{s} \xi_{a} \) is parallel to \( T \). Hence \( \kappa_{1} \neq 0 \) and \( r \geq 3 \). If we write equation (13) in (12), we get
\[ \sum_{a=1}^{s} \xi_{a} = s \cos \theta T + \sqrt{s} \sqrt{1 - s \cos^{2} \theta} E_{3}. \]
If we differentiate this last equation along the curve \( \gamma \), we find
\[ \varphi_{T} = -\frac{\kappa_{1}}{s \cos \theta} E_{2} - \frac{\kappa_{3}}{\sqrt{s}} \sqrt{1 - s \cos^{2} \theta} E_{4}. \]
(14)
If we calculate \( g(\varphi T, \varphi T) \), we have
\[ s \cos \theta \left( 1 - s \cos^{2} \theta \right) \left( s \cos \theta - \kappa_{3}^{2} \right) = \kappa_{1}^{2}, \]
which gives \( \kappa_{3} \) =constant. In particular, if \( \kappa_{3} = 0 \), then we find equations (10) and (11). If \( \kappa_{3} \neq 0 \), we differentiate equation (14) along the curve \( \gamma \) and find that \( \kappa_{4} \) =constant. If we continue differentiating and calculating the norm of both sides, we easily obtain \( \kappa_{i} \) =constant for all \( i = 1, T, \) that is, \( \gamma \) is a slant helix of order \( r \). Thus, we have just proved the necessity.

To prove sufficiency, if \( \gamma \) satisfies the equations given in the Theorem, then it is easy to show that equation (6) is satisfied. So, \( \gamma \) is C-parallel (in the tangent bundle).

For C-parallel slant curves in the normal bundle, we have the following Theorem:

**Theorem 3.2.** Let \( \gamma : I \to M^{2m+1} \) be a unit-speed slant curve. Then \( \gamma \) is C-parallel in the normal bundle if and only if it is a Legendre helix of order \( r \geq 3 \) satisfying
\[ \sum_{a=1}^{s} \xi_{a} = \sqrt{s} E_{3}, \]
\[ \varphi_{T} = \frac{\kappa_{2}}{\sqrt{s}} E_{2} - \frac{\kappa_{3}}{\sqrt{s}} E_{4}, \]
\[ \kappa_{2} \neq 0, \quad \lambda = \frac{\kappa_{1}k_{2}}{\sqrt{s}}, \]
and moreover if \( \kappa_{3} = 0 \), then
\[ \kappa_{2} = \sqrt{s}, \quad \varphi_{T} = E_{2}. \]
Proof. Let us assume that $\gamma$ is $C$-parallel in the normal bundle. Then, if we take the inner product of equation (7) with $T$, we have $\eta'(T) = 0$, so $\gamma$ is Legendre. Next, we take the inner product with $E_2$ and find $\kappa_1 =$constant. Thus, equation (7) becomes

$$\kappa_1 \kappa_2 E_3 = \lambda \sum_{a=1}^{s} \xi_a,$$

which gives us

$$E_3 = \frac{1}{\sqrt{s}} \sum_{a=1}^{s} \xi_a,$$  \hspace{1cm} (15)

If we differentiate equation (15), we get

$$\varphi T = \frac{\kappa_2}{\sqrt{s}} E_2 - \frac{\kappa_3}{\sqrt{s}} E_4.$$  \hspace{1cm} (16)

If we differentiate this last equation, we obtain

$$V_T \varphi T = \sum_{a=1}^{s} \xi_a + \kappa_1 \varphi E_2$$

$$= \frac{\kappa_2'}{\sqrt{s}} E_2 + \frac{\kappa_2}{\sqrt{s}} (-\kappa_1 T + \kappa_2 E_3) - \frac{\kappa_3'}{\sqrt{s}} E_4 - \frac{\kappa_3}{\sqrt{s}} (-\kappa_3 E_3 + \kappa_4 E_5).$$

If we take the inner product of both sides with $E_2$, we find $\kappa_2 =$constant. Then, the norm of equation (16) gives us $\kappa_3 =$constant. In particular, if $\kappa_3 = 0$, from equation (16), we have

$$\kappa_2 = \sqrt{s}, \quad \varphi T = E_2.$$  \hspace{1cm} (18)

Otherwise, from the norm of both sides in (17), we also have $\kappa_4 =$constant. If we continue differentiating equation (17), we find that $\gamma$ is a helix of order $r$.

Conversely, let $\gamma$ be a Legendre helix of order $r \geq 3$ satisfying the stated equations. Then, it is easy to show that equation (7) is verified. Thus, $\gamma$ is $C$-parallel in the normal bundle. $\square$

4. $C$-proper Slant Curves of $S$-manifolds

For $C$-proper slant curves in the tangent bundle, we can state the following Theorem:

**Theorem 4.1.** Let $\gamma: I \to M^{2m+3}$ be a unit-speed slant curve. Then $\gamma$ is $C$-proper (in the tangent bundle) if and only if it is a non-Legendre slant curve satisfying

$$\sum_{a=1}^{s} \xi_a \in \text{sp} \{T, E_3, E_4\},$$

$$\varphi T \in \text{sp} \{E_2, E_3, E_4, E_5\},$$

$$\kappa_1 \neq \text{constant, } \kappa_2 \neq 0,$$

$$\lambda = \frac{3\kappa_1 \kappa_1'}{s \cos \theta}.$$  \hspace{1cm} (18)
\[ \kappa_1^2 + \kappa_2^2 = \frac{\kappa''}{\kappa_1} \]  
(19)

\[ \lambda s \eta'(E_3) = -2(\kappa_1' \kappa_2 + \kappa_1 \kappa_2'), \]  
(20)

\[ \lambda s \eta'(E_4) = -\kappa_1 \kappa_2 \kappa_3, \]  
(21)

\[ \eta'(E_3)^2 + \eta'(E_4)^2 = \frac{1 - s \cos^2 \theta}{s} \]  
(22)

and moreover if \( \kappa_3 = 0 \), then

\[ \phi T = \sqrt{1 - s \cos^2 \theta} E_2, \]  
(23)

\[ E_3 = \frac{1}{\sqrt{s} \sqrt{1 - s \cos^2 \theta}} \left(-s \cos \theta T + \sum_{\alpha=1}^s \xi_\alpha \right), \]  
(24)

\[ \kappa_2 = \sqrt{s} \left(1 + \frac{\kappa_1 \cos \theta}{\sqrt{1 - s \cos^2 \theta}} \right). \]  
(25)

**Proof.** Let \( \gamma \) be \( C \)-proper (in the tangent bundle). If we take the inner product of equation (8) with \( T \), we find

\[ \lambda s \cos \theta = 3 \kappa_1 \kappa_1'. \]

Let us assume that \( \gamma \) is Legendre. Then we have \( \kappa_1' = 0 \), that is, \( \kappa_1 \) = constant. If we take the inner product of equation (8) with \( E_2 \), we get

\[ 0 = \kappa_1' + \kappa_1 \kappa_2' = \kappa_1 \left(\kappa_2^2 + \kappa_2^2\right), \]

which gives us \( \kappa_1 = 0 \). Then equation (8) becomes

\[ \lambda \sum_{\alpha=1}^s \xi_\alpha = 0, \]

which is a contradiction. Thus, \( \gamma \) is non-Legendre slant and \( \kappa_1 \neq \text{constant} \). We find equations (18), (19), (20) and (21) taking the inner product with \( T, E_2, E_3 \) and \( E_4 \), respectively. Then, we write these equations in (8) and calculate the norm of both sides to obtain equation (22). Now, let us assume \( \kappa_2 = 0 \). Then, from equation (8), we have

\[ \lambda \sum_{\alpha=1}^s \xi_\alpha = 3 \kappa_1 \kappa_1' T, \]

which is only possible when

\[ T = \frac{1}{\sqrt{s}} \sum_{\alpha=1}^s \xi_\alpha. \]

If we calculate \( V_T T \), we find \( \kappa_1 = 0 \), which is a contradiction. Hence, \( \kappa_2 \neq 0 \). Differentiating equation (8), we can easily see that

\[ \phi T \in \text{sp} \{E_2, E_3, E_4, E_5\}. \]

In particular, if \( \kappa_3 = 0 \), we obtain equations (23), (24) and (25). See our paper [10], Case III, equation (4.9), which is also valid when \( \kappa_1 \) and \( \kappa_2 \) are not constants.

Conversely, if \( \gamma \) is a non-Legendre slant curve satisfying the stated equations, then Proposition 2.2 iii) is valid. So, \( \gamma \) is \( C \)-proper (in the tangent bundle).  \( \blacksquare \)
Finally, we give the following Theorem for $C$-proper slant curves in the normal bundle:

**Theorem 4.2.** Let $\gamma: I \rightarrow M^{2m+s}$ be a unit-speed slant curve. Then $\gamma$ is $C$-proper in the normal bundle if and only if it is a Legendre curve satisfying

\[
\sum_{\alpha=1}^{s} \xi_{\alpha} \in sp\{E_{3}, E_{4}\},
\]

\[\varphi T \in sp\{E_{2}, E_{3}, E_{4}, E_{5}\},\]

\[\kappa_{1} \neq \text{constant}, \quad \kappa_{2} \neq 0,\]

\[\kappa_{1}\kappa_{2}^{2} - \kappa_{1}'' = 0,\]

\[\lambda s \eta^\alpha (E_{3}) = -(2\kappa_{1}'\kappa_{2} + \kappa_{1}\kappa_{2}'),\]

\[\lambda s \eta^\alpha (E_{4}) = -\kappa_{1}\kappa_{2}\kappa_{3},\]

\[\eta^\alpha (E_{3})^{2} + \eta^\alpha (E_{4})^{2} = \frac{1}{s}\]

and moreover if $\kappa_{3} = 0$, then

\[
\sum_{\alpha=1}^{s} \xi_{\alpha} = \sqrt{s}E_{3},
\]

\[\kappa_{2} = \sqrt{s}, \quad \varphi T = E_{2}.
\]

**Proof.** The proof is similar to the proof of Theorem 4.1. For the case $\kappa_{3} = 0$, we refer to [15].

5. Examples

In this section, we give the following two examples in the well-known $S$-manifold $R^{2m+s}(-3s)$. For more information on $R^{2m+s}(-3s)$, see [11].

**Example 5.1.** Let us consider $R^{2m+s}(-3s)$ with $m = 2$ and $s = 2$. The curve $\gamma: I \rightarrow R^{6}(-6)$ given by

\[\gamma(t) = (\sin t, 2 + \sin t, -\cos t, 3 - \cos t, -2t - \sin t \cos t, 1 - 2t - \sin t \cos t)\]

is a unit-speed non-Legendre slant helix with

\[\kappa_{1} = \kappa_{2} = \frac{1}{\sqrt{2}}, \quad \theta = \frac{2\pi}{3}.
\]

It has the Frenet frame field

\[\left\{ T, \sqrt{2}\varphi T, \left( T + \sum_{\alpha=1}^{2} \xi_{\alpha}\right) \right\}
\]

and it is $C$-parallel (in the tangent bundle) with $\lambda = \frac{1}{2}$. 

Example 5.2. Let us consider $\mathbb{R}^{2m+s}(-3s)$ with $m = 1$ and $s = 4$. We define real valued functions on an open interval $I$ as

$$
\gamma_1(t) = 2 \int_0^t \cos(e^{2u})du, \quad \gamma_2(t) = -2 \int_0^t \sin(e^{2u})du,
$$

$$
\gamma_3(t) = ... = \gamma_6(t) = -4 \int_0^t \cos(e^{2u}) \left( \int_0^u \sin(e^{2v})dv \right)du.
$$

The curve $\gamma : I \to \mathbb{R}^6(-12)$, $\gamma(t) = (\gamma_1(t), ..., \gamma_6(t))$ is a unit-speed Legendre curve with

$$
\kappa_1 = 2e^{2t}, \quad \kappa_2 = 2, \quad r = 3,
$$

$$
\phi T = E_2, \quad E_3 = \frac{1}{2} \sum_{\alpha=1}^4 \xi_\alpha
$$

and it is C-proper in the normal bundle with $\lambda = -8e^{2t}$.

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