FINITELY PRESENTED GROUPS ASSOCIATED WITH EXPANDING MAPS

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Abstract. We associate with every locally expanding self-covering \( f : \mathcal{M} \to \mathcal{M} \) of a compact path connected metric space a finitely presented group \( \mathcal{V}_f \). We prove that this group is a complete invariant of the dynamical system: two groups \( \mathcal{V}_{f_1} \) and \( \mathcal{V}_{f_2} \) are isomorphic as abstract groups if and only if the corresponding dynamical systems are topologically conjugate. We also show that the commutator subgroup of \( \mathcal{V}_f \) is simple, and give a topological interpretation of \( \mathcal{V}_f / \mathcal{V}_f' \).

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1. Introduction

A dynamical system is \emph{finitely presented} if it can be represented as the factor of a shift of finite type by an equivalence relation that is also a shift of finite type, see [Fri87, CP93]. If we think of finite type as analogous to finite generation (of a group or of a normal subgroup), then the notion of a finitely presented dynamical system becomes analogous to the notion of a finitely presented group. But the relation is deeper than just a superficial analogy.

Condition of being finitely presented for a dynamical system is very closely related to dynamical hyperbolicity, see [Fri87]. For example, if \( f : J \rightarrow J \) is a locally expanding self-covering of a compact metric space, then \( f \) is finitely presented. Dynamical hyperbolicity is very closely related to Gromov hyperbolicity for groups, see [Gro87, CP93, Nek11a], and finite presentation is an important property of hyperbolic groups.

The aim of this paper is to show a new connection between expanding maps and finite presentations. We naturally associate with every finite degree self-covering \( f : M \rightarrow M \) of a path-connected space \( M \) a group \( V_f \) with the following property (see Theorem 5.9 and Theorem 7.1).

**Theorem 1.1.** If \( f : M \rightarrow M \) is a locally expanding self-covering of a compact path connected metric space then \( V_f \) is finitely presented.

If \( f_1 : M_1 \rightarrow M_1 \) and \( f_2 : M_2 \rightarrow M_2 \) are as above, then groups \( V_{f_1} \) and \( V_{f_2} \) are isomorphic if and only if \( f_1 \) and \( f_2 \) are topologically conjugate, i.e., there exists a homeomorphism \( \phi : M_1 \rightarrow M_2 \) such that \( f_1 = \phi^{-1} \circ f_2 \circ \phi \).

We also show that the commutator subgroup of \( V_f \) is simple, and give a dynamical interpretation of the abelianization \( V_f / V_f' \), see Theorem 4.7 and Proposition 5.7.

The groups \( V_f \) are defined in the following way. Let \( f : M \rightarrow M \) be a finite degree covering map, and suppose that \( M \) is path connected. Choose \( t \in M \), and consider the tree \( T_t \) of preimages of \( t \) under the iterations of \( f \). Its set of vertices is \( \bigcup_{n \geq 0} f^{-n}(t) \), and a vertex \( v \in f^{-n}(t) \) is connected to the vertex \( f(v) \in f^{-(n-1)}(t) \). Let \( \partial T_t \) be its boundary, which can be defined as the inverse limit of the discrete sets \( f^{-n}(t) \) with respect to the maps \( f : f^{-(n+1)}(t) \rightarrow f^{-n}(t) \).

Let \( \gamma \) be a path in \( M \) connecting a vertex \( v \in f^{-n}(t) \) to a vertex \( u \in f^{-m}(t) \) of the tree \( T_t \). Considering lifts of \( \gamma \) by the coverings \( f^k : M \rightarrow M \), we get an isomorphism \( S_{\gamma} : T_v \rightarrow T_u \) between subtrees \( T_v, T_u \) of \( T_t \). Namely, if \( \gamma_z \) is a lift of \( \gamma \) starting at \( z \in f^{-k}(u) \), then \( S_{\gamma}(z) \in f^{-k}(v) \) is the end of \( \gamma_z \). Denote by the same symbol \( S_{\gamma} \) the induced homeomorphism \( \partial T_v \rightarrow \partial T_u \) of the boundaries of the subtrees, seen as clopen subsets of \( \partial T_t \).

**Definition 1.1.** The group \( V_f \) is the group of all homeomorphisms \( \partial T_t \rightarrow \partial T_t \) locally equal to homeomorphisms of the form \( S_{\gamma} : \partial T_v \rightarrow \partial T_u \).

The group \( V_f \) is generated by the Higman-Thompson group \( G_{\deg f, 1} \) (see [Hig74]) acting on \( \partial T_t \) and the \emph{iterated monodromy group} \( \text{IMG} (f) \) of \( f : M \rightarrow M \). The
iterated monodromy group can be defined as the subgroup of $V_f$ consisting of homeomorphisms $S_\gamma : \partial T_v \to \partial T_v$, where $\gamma$ is a loop starting and ending at the basepoint $t$. It is an invariant of the topological conjugacy class of $f : M \to M$, and it becomes a complete invariant (in the expanding case), if we consider it as a self-similar group. Self-similarity is an additional structure on a group, and it can be defined using one of several equivalent approaches: virtual endomorphisms, wreath recursions, bisets, or structures of an automaton group. The fact that self-similar iterated monodromy group is a complete invariant of an expanding self-covering is one of the main topics of [Nek05].

From the point of view of group theory, $V_f$ seems to be a “cleaner” object, since no additional structure is needed to make it a complete invariant of a dynamical system. Besides, it is finitely presented, unlike the iterated monodromy groups, which are typically infinite presented. However, iterated monodromy groups have better functorial properties than the groups $V_f$, see [Nek08b].

A group $V_G$, analogous to $V_f$, can be defined for any self-similar group $G$, so that $V_f = V_{\text{IM}(f)}$. Groups of this type were for the first time studied by C. Röver [Röv99, Röv02]. In particular, he showed that if $G$ is the Grigorchuk group [Gri80], then $V_G$ is finitely presented, simple, and is isomorphic to the abstract commensurator of the Grigorchuk group. The case of a general self-similar group $G$ was studied later in [Nek04].

Several natural questions arise in connection with Theorem 1.1. For example, is the isomorphism problem solvable for the groups $V_f$? Equivalently, is the topological conjugacy problem for expanding maps algorithmically solvable? Note that expanding maps can be given in different ways by a finite amount of information: using finite presentations in the sense of [Fri87], using combinatorial models in the sense of [IS10, Nek08a], using iterated monodromy groups, e.t.c.

Another natural question is whether the groups $V_f$, similarly to the Higman-Thompson groups (see [Bro87]), satisfy the finiteness condition $F_\infty$, i.e., if they have classifying spaces with finite $n$-dimensional skeleta for all $n$. It would be also interesting to study homology of $V_f$ in relation with homological properties of the dynamical system.

The structure of the paper is as follows. In “Definition of the groups $V_f$” we give a review of terminology related to rooted trees, and define the groups $V_f$. In the next section “Symbolic coding” we encode the vertices of the tree of preimages $T_t$ by finite words over an alphabet $X$, and show that $V_f$ contains a natural copy of the Higman-Thompson group, and that $V_f$ is generated by the Higman-Thompson group and the iterated monodromy group. We also give a review of the basic notions of the theory of self-similar groups, and define the groups $V_G$ associated with self-similar groups, following [Nek04].

In Section 4 we prove that the commutator subgroup $V'_G$ of $V_G$ is simple for any self-similar group $G$. In particular, $V'_f$ is simple for any map $f$. Note that the fact that every proper quotient of $V_G$ is abelian, i.e., that every non-trivial normal subgroup of $V_G$ contains $V_G$ was already proved in [Nek04], and we use this fact in our proof. Later, in the next section we give an interpretation of $V_f/V'_f$ in topological terms. Namely, we prove the following (see Proposition 5.7).

**Proposition 1.2.** Suppose that $f : M_1 \to M$ is expanding, $M$ is path-connected and semi-locally simply connected, and $M_1 \subseteq M$.
If \( \deg f \) is even, then \( \mathcal{V}_f / \mathcal{V}'_f \) is isomorphic to the quotient of \( H_1(M) \) by the range of the endomorphism \( 1 - \iota_* \circ f^! \).

If \( \deg f \) is odd, then \( \mathcal{V}_f / \mathcal{V}'_f \) is isomorphic to the quotient of \( \mathbb{Z}/2\mathbb{Z} \oplus H_1(M) \) by the range of the endomorphism \( 1 - \sigma_1, \) where \( \sigma_1 = (t + \text{sign}(c), \iota_* \circ f^!(c)). \)

Here \( \iota_* : H_1(M_1) \rightarrow H_1(M) \) is the homomorphism induced by the identical embedding \( \iota : M_1 \rightarrow M, \) the homomorphism \( f^! : H_1(M) \rightarrow H_1(M_1) \) maps a cycle \( c \) to its full preimage \( f^{-1}(c), \) and \( \text{sign} : H_1(M) \rightarrow \mathbb{Z}/2\mathbb{Z} \) maps a cycle \( c \) defined by a loop \( \gamma \) to 1 if \( \gamma \) acts as an odd permutation on the fiber of \( f, \) and to 0 otherwise.

The main result of Section 5 is existence of finite presentation of \( \mathcal{V}_f \) when \( f \) is expanding. More generally, we show that \( \mathcal{V}_G \) is finitely presented, if \( G \) is a contracting self-similar group, see Theorem 5.9. We also give (in Subsection 5.2) a general definition of the groups \( \mathcal{V}_f \) for expanding maps \( f : M \rightarrow M \) (where \( M \) is not required to be path connected).

Section “Dynamical systems and groupoids” is an overview of the theory of limit dynamical systems of contracting self-similar groups and basic results of hyperbolic groupoids, following [Nek05] and [Nek11a]. These results are needed for the proof the fact that \( \mathcal{V}_f \) is a complete invariant of the dynamical system in the expanding case, which is proved (Theorem 7.1) in the last section of the paper.

The general scheme of the proof of Theorem 7.1 is as follows. First, we show, using a theorem of M. Rubin [Rub89], that two groups \( \mathcal{V}_{f_1} \) and \( \mathcal{V}_{f_2} \) are isomorphic if and only if their actions on the corresponding boundaries of trees are topologically conjugate. This implies, that the groupoid of germs \( \mathcal{G} \) of the action of the group \( \mathcal{V}_f \) on the boundary of the tree is uniquely determined by the group \( \mathcal{V}_f. \)

The groupoid \( \mathcal{G} \) is hyperbolic, and hence it uniquely determines the equivalence class of its dual (see [Nek11a]), which is the groupoid generated by the germs of \( f : M \rightarrow M. \) It remains to show that the dynamical system \( f : M \rightarrow M \) is uniquely determined (up to topological conjugacy) by the equivalence class of the groupoid of germs generated by it. This is proved using the techniques of hyperbolic groupoids. Connectedness of \( M \) is used in the proof in an essential way.

It is a natural question to ask if Theorem 7.1 is true in general (without the condition that \( M \) is path connected).

As a corollary of the proof of Theorem 7.1 we clarify the relation between the groupoid-theoretic equivalence of groupoids of germs associated with expanding dynamical systems, and their topological conjugacy, see Theorem 7.8.

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2. Definition of the groups \( \mathcal{V}_f \)

2.1. Rooted trees. Let \( T \) be a locally finite rooted tree, and let \( v, u \) be its vertices. We write \( v \leq u \) if the path connecting the root to \( u \) passes through \( v. \) This defines a partial order on the set of vertices of \( T, \) and \( T \) is its Hasse diagram (though we tend to draw rooted trees “upside down” with the root on top).
We denote by $T_v$ the sub-tree with root $v$ spanned by all vertices $u$ such that $v \preceq u$. We have $v \preceq u$ if and only if $T_v \supseteq T_u$. If $v$ and $u$ are incomparable, then $T_v$ and $T_u$ are disjoint.

Boundary $\partial T$ of the tree $T$ is the set of all infinite simple paths starting at the root of $T$. The boundary $\partial T_v$ is naturally identified with the set of paths $w \in \partial T$ passing through $v$. The sets $\partial T_v$ form a basis of open sets of a natural topology on $\partial T$. The subsets $\partial T_v$ are clopen (closed and open), and every clopen subset of $\partial T_v$ is disjoint union of a finite number of sets of the form $\partial T_v$.

The $n$th level of $T$ is the set of vertices that are on distance $n$ from the root of the tree. An antichain $A$ of a rooted tree $T$ is a set of pairwise incomparable vertices. A finite antichain $A$ is said to be complete if it is maximal, i.e., if every set $B$ of vertices of $T$ properly containing $A$ is not an antichain. For example, every level of $T$ is a complete antichain.

A set $A$ is an antichain if and only if the sets $\partial T_v$ for $v \in A$ are disjoint. It follows that $A$ is a complete antichain if and only if $\partial T$ is disjoint union of the sets $\partial T_v$ for $v \in A$.

Let $X$ be a finite set. We denote by $X^*$ the free monoid generated by $X$, i.e., the set of all finite words $x_1x_2\ldots x_n$ for $x_i \in X$, together with the empty word $\emptyset$. Length of a word $v \in X^*$ is the number of its letters.

We introduce on the set $X^*$ structure of a rooted tree coinciding with the left Cayley graph of the free monoid. Namely, two finite words are connected by an edge if and only if they are of the form $vx$ and $v$ for $v \in X^*$ and $x \in X$. The empty word is the root of the tree $X^*$. We have $v \preceq u$ for $v, u \in X^*$ if and only if $v$ is a beginning of $u$. The subtree $T_v$ of $T = X^*$ for $v \in X^*$ is the set $vX^*$ of all words starting with $v$. The $n$th level of $X^*$ is the set $X^n$ of words of length $n$.

The boundary of the tree $X^*$ is naturally identified with the space $X^\omega$ of right-infinite sequences $x_1x_2\ldots$ of elements of $X$. The topology on the boundary coincides with the direct product topology on $X^\omega$. 

![Figure 1. Rooted tree](image-url)
2.2. Definition. Let $\mathcal{M}$ be a topological space. A partial self-covering is a finite degree covering map $f : \mathcal{M}_1 \rightarrow \mathcal{M}$, where $\mathcal{M}_1 \subseteq \mathcal{M}$.

If $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ is a partial self-covering, then we can iterated it as a partial self-map of $\mathcal{M}$. Then the $n$th iteration $f^n : \mathcal{M}_n \rightarrow \mathcal{M}$ is also a partial self-covering. Here $\mathcal{M}_n \subset \mathcal{M}_{n-1} \subset \ldots \subset \mathcal{M}_1$ are domains of the iterations of $f$, defined by the condition $\mathcal{M}_{n+1} = f^{-1}(\mathcal{M}_n)$.

For a point $t \in \mathcal{M}$ denote by $T_t$ the tree of preimages of $t$ under the iterations of $f$, i.e., the tree with the set of vertices equal to the formal disjoint union $\bigsqcup_{n \geq 0} f^{-n}(t)$ of the sets of preimages of $t$ under the iterations $f^n : \mathcal{M}_n \rightarrow \mathcal{M}$. Here $f^{-0}(t) = \{t\}$ consists of the root of the tree, and a vertex $v \in f^{-n}(t)$ is connected by an edge to the vertex $f(v) \in f^{-(n-1)}(t)$, see Figure 2. If $v$ is a vertex of $T_t$, then the tree of preimages $T_v$ is in a natural way a sub-tree of the tree $T_t$, and our notation agrees with the notation of the previous subsection.

Assume now that $\mathcal{M}$ is path connected. Let $t_1, t_2 \in \mathcal{M}$, and let $\gamma$ be a path from $t_1$ to $t_2$ in $\mathcal{M}$. Then for every $n \geq 0$ and every $v \in f^{-n}(t_1)$ there is a unique lift by the covering $f^n : \mathcal{M}_n \rightarrow \mathcal{M}$ of $\gamma$ starting in $v$. Let $S_\gamma(v) \in f^{-n}(t_2)$ be its end. It is easy to see that the map $S_\gamma : T_{t_1} \rightarrow T_{t_2}$ is an isomorphism of the rooted trees, see Figure 3. It defines a homeomorphism of their boundaries $S_\gamma : \partial T_{t_1} \rightarrow \partial T_{t_2}$, which we will denote by the same letter.
Figure 4. Composition $S_{\gamma_1}S_{\gamma_2}$

**Definition 2.1.** Let $f : M_1 \to M$ be a partial self-covering, and let $t \in M$. Denote by $\mathcal{T}_f$ the semigroup of partial homeomorphisms of $\partial T_1$ generated by the homeomorphisms of the form $S_{\gamma} : \partial T_{v_1} \to \partial T_{v_2}$, where $\gamma$ is a path connecting points $v_1, v_2 \in \bigcup_{n \geq 0} f^{-n}(t)$.

The semigroup $\mathcal{T}_f$ contains the zero map between empty subsets of $T_t$. A product $S_{\gamma_1}S_{\gamma_2}$ is zero if the range of $S_{\gamma_2}$ is disjoint from the domain of $S_{\gamma_1}$.

Let $S_{\gamma_1} : \partial T_{v_1} \to \partial T_{v_2}$ and $S_{\gamma_2} : \partial T_{u_1} \to \partial T_{u_2}$ be two generators of $\mathcal{T}_f$. The product $S_{\gamma_1}S_{\gamma_2}$ is non-zero if and only if $T_{u_2} \supseteq T_{v_1}$ or $T_{u_2} \subseteq T_{v_1}$. If the first case, $v_1$ is a preimage of $u_2$ under some iteration $f^k$ of $f$. Let $\gamma'_2$ be the unique lift of $\gamma_2$ by $f^k$ that ends in $v_1$. Then it follows from the definition of the transformations $S_{\gamma}$ that

1. $S_{\gamma_1}S_{\gamma_2} = S_{\gamma_1\gamma'_2}$,

see Figure 4. Here and in the sequel, we multiply paths as we compose functions: in the product $\gamma_1\gamma'_2$ the path $\gamma'_2$ is passed before the path $\gamma_1$.

Similarly, if $T_{u_2} \subseteq T_{v_1}$, then $u_2$ is a $f^k$-preimage of $v_1$ for some $k \geq 0$, and

2. $S_{\gamma_1}S_{\gamma_2} = S_{\gamma'_1\gamma_2}$,

where $\gamma'_1$ is the lift of $\gamma_1$ by $f^k$ starting in $u_2$.

It follows that all non-zero elements of $\mathcal{T}_f$ are of the form $S_{\gamma}$ for some path $\gamma$ in $\mathcal{M}$ connecting two vertices of $T_t$. Note also that $\mathcal{T}_t$ is an inverse semigroup, where $S^*_{\gamma} = S_{\gamma^{-1}}$.

Let $A_1, A_2$ be two complete antichains of $T_t$ of equal cardinality. Choose a bijection $\alpha : A_1 \to A_2$ and a collection of paths $\gamma_a$ from $a \in A_1$ to the corresponding vertex $\alpha(a)$, see Figure 5. Let $g : \partial T_t \to \partial T_t$ be the map given by the rule

$$g(w) = S_{\gamma_a}(w), \quad \text{if } w \in \partial T_v.$$ 

It is easy to see that $g : \partial T_t \to \partial T_t$ is a homeomorphism. Denote by $\mathcal{V}_f$ the set of all such homeomorphisms.

We will represent homeomorphisms $g \in \mathcal{V}_f$ by tables of the form

$$\begin{pmatrix}
v_1 & v_2 & \ldots & v_n \\
\gamma_{\alpha(v_1)} & \gamma_{\alpha(v_2)} & \ldots & \gamma_{\alpha(v_n)} \\
\alpha(v_1) & \alpha(v_2) & \ldots & \alpha(v_n)
\end{pmatrix},$$

where the first row is the list of vertices of a complete antichain, and $g(w) = S_{\gamma_{\alpha(v)}}(w)$ for all $w \in \partial T_{\alpha(v)}$. 
An elementary splitting of such a table is the operation of replacing a column $\begin{pmatrix} u \\ \gamma_u \\ \alpha(u) \end{pmatrix}$ by the array $\begin{pmatrix} u_1 & u_2 & \ldots & u_d \\ \gamma_{u_1} & \gamma_{u_2} & \ldots & \gamma_{u_d} \\ w_1 & w_2 & \ldots & w_d \end{pmatrix}$, where $\{u_1, u_2, \ldots, u_d\} = f^{-1}(u)$, $\gamma_{u_i}$ is the lift of $\gamma_u$ by $f$ starting at $u_i$, and $w_i$ is the end of $\gamma_{u_i}$, see Figure 6.

A splitting of a table is the results of a finite sequence of elementary splittings.

It follows directly from the definition that splitting of a table does not change the homeomorphism $g \in \mathcal{V}_f$ that it defines. It is obvious that if $g_1$ and $g_2$ are defined by tables of the form

- $\begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ \gamma_{v_1} & \gamma_{v_2} & \ldots & \gamma_{v_n} \\ u_1 & u_2 & \ldots & u_n \end{pmatrix}$

and

- $\begin{pmatrix} w_1 & w_2 & \ldots & w_n \\ \gamma_{w_1} & \gamma_{w_2} & \ldots & \gamma_{w_n} \\ v_1 & v_2 & \ldots & v_n \end{pmatrix}$,

then the composition $g_1g_2$ is defined by the table

- $\begin{pmatrix} w_1 & w_2 & \ldots & w_n \\ \gamma_{v_1} \gamma_{w_1} & \gamma_{v_2} \gamma_{w_2} & \ldots & \gamma_{v_n} \gamma_{w_n} \\ u_1 & u_2 & \ldots & u_n \end{pmatrix}$.
Let

\[
\begin{pmatrix}
  a_1 & a_2 & \ldots & a_n \\
  \gamma_{a_1} & \gamma_{a_2} & \ldots & \gamma_{a_n} \\
  b_1 & b_2 & \ldots & b_n
\end{pmatrix},
\begin{pmatrix}
  c_1 & c_2 & \ldots & c_m \\
  \gamma_{c_1} & \gamma_{c_2} & \ldots & \gamma_{c_m} \\
  a'_1 & a'_2 & \ldots & a'_m
\end{pmatrix}
\]

be tables defining elements of \( V_f \). We can find a complete antichain \( A \) such that for every \( v \in A \) the subtree \( T_v \) is contained in a subtree \( T_{a_i} \) and a subtree \( T_{a'_j} \) for some \( i \) and \( j \). For example, we can take \( A \) to be equal to the \( k \)th level of the tree \( T_i \) for \( k \) big enough. Then there exists a splitting of the first table such that its first row is \( A \), and there exists a splitting of the second table such that its last row is \( A \).

It follows that if \( g_1, g_2 \in V_f \), then their composition \( g_1g_2 \) also belongs to \( V_f \). As a corollary, we get the following proposition.

**Proposition 2.1.** The set \( V_f \) is a group.

We will use sometimes instead of tables the following notation. If \( F \) and \( G \) are two partial transformations with disjoint domains, then we denote by \( F + G \) their union, i.e., the map equal to \( F \) on the domain of \( F \) and equal to \( G \) on the domain of \( G \). Then the transformation defined by a table

\[
\begin{pmatrix}
  v_1 & v_2 & \ldots & v_n \\
  \gamma_1 & \gamma_2 & \ldots & \gamma_n \\
  u_1 & u_2 & \ldots & u_n
\end{pmatrix}
\]

is written \( S_{\gamma_1} + S_{\gamma_2} + \cdots + S_{\gamma_n} \).

An elementary splitting of a table is equivalent then to application of the identity

\[
S_\gamma = \sum_{\delta \in f^{-1}(\gamma)} S_\delta,
\]

where \( f^{-1}(\gamma) \) is the set of all lifts of \( \gamma \) by \( f \).

### 3. Symbolic coding

3.1. **Two trees.** Let \( \{t_1, t_2, \ldots, t_d\} = f^{-1}(t) \), and choose paths \( \ell_i \) in \( M \) from \( t \) to \( t_i \). Then \( S_{\ell_i} : T_i \to T_i \) is an isomorphism. The elements \( S_{\ell_i} \in T_f \) satisfy the relations:

\[
S_{\ell_i} S_{\ell_i} = 1, \quad \sum_{i=1}^d S_{\ell_i} S_{\ell_i}^* = 1.
\]

The \( C^* \)-algebra defined by such relations is called the Cuntz algebra \( \overline{\text{Cuntz}} \). If we denote by \( S \) the row \( (S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_d}) \), and by \( S^* \) the column \( (S_{\ell_1}^*, S_{\ell_2}^*, \ldots, S_{\ell_d}^*)^T \), then the relations can be written as matrix equalities

\[
S^* S = I_d, \quad S S^* = I_d,
\]

where \( I_n \) denotes the \( n \times n \) identity matrix. Rings given by these and similar defining relations were studied by W. Leavitt \[Lea56, Lea65\].

Let \( \Gamma_i \) be the graph with the set of vertices equal to the set of vertices of \( T_i \) in which two vertices \( v \in f^{-n}(t) \) and \( v \in f^{-(n+1)}(t) \) are connected by an edge if and only if they are connected by a path equal to a lift of a path \( \ell_i \) by the covering \( f^n \). In other words, the graph \( \Gamma_i \) is obtained by taking preimages of the paths \( \ell_i \) under all iterations of \( f \), see Figure 7.
It is easy to see that $\Gamma_t$ is a tree, and its $n$th level is equal to the $n$th level of the tree $T_t$. It follows that any two vertices of $T_t$ are connected by a unique simple path in $\Gamma_t$.

Let $i_{n-1} \ldots i_0 \in \{1, 2, \ldots, d\}^n$, and consider the product $S_{i_{n-1}} \ldots S_i S_0 \in T_t$. According to the multiplication rule \([2]\), it is equal to $S_{\gamma_{n-1} \cdots \gamma_0}$, where $\gamma = \ell_{i_0}$, and $\gamma_k$ is the lift of $\ell_k$ by $f^k$ starting at the end of $\gamma_{k-1}$. Denote by $\Lambda(i_{n-1} \ldots i_0)$ the end of the last path $\gamma_{n-1}$. Then $S_{i_{n-1}} \ldots S_i S_0 = S_{\gamma_{n-1} \cdots \gamma t}$, where $\gamma_{i_{n-1} \ldots i_2 t}$ is unique simple path in $\Gamma_t$ starting at the root and ending in $\Lambda(i_{n-1} \ldots i_0)$. The path $\gamma_{i_{n-1} \ldots i_2 t}$ and its end $\Lambda(i_{n-1} \ldots i_0)$ satisfy the recurrent rule:

$$\gamma_{i_{n-1} \ldots i_2 t} = \ell_{\Lambda(i_{n-2} \ldots i_0)}, i_{n-1} \cdot \gamma_{i_{n-2} \ldots i_2 t},$$

where $\ell_{\Lambda(i_{n-2} \ldots i_0)}, i_{n-1}$ is the lift of $\ell_{i_{n-1}}$ by $f^{n-1}$ starting at $\Lambda(i_{n-2} \ldots i_1 i_0)$ (and hence ending in $\Lambda(i_{n-1} \ldots i_1 i_0)$).

The map $\Lambda$ is a bijection between $\{1, 2, \ldots, d\}^n$ and the $n$th level $f^{-n}(t)$ of the trees $T_t$ and $\Gamma_t$. It follows directly from the description of the path $\gamma_{i_{n-1} \ldots i_0}$ that $\Lambda(i_{n-1} \ldots i_0)$ is adjacent to $\Lambda(i_{n-1} \ldots i_2 t)$ in $T_t$ and to $\Lambda(i_{n-2} \ldots i_1 i_0)$ in $\Gamma_t$. In other words, $T_t$ and $\Gamma_t$ are identified by $\Lambda$ with the right and the left Cayley graphs of the free monoid generated by $X = \{1, 2, \ldots, d\}$, respectively.

For any two sequences $i_1 i_2 \ldots i_n$ and $j_1 j_2 \ldots j_m \in X^*$, the product

$$S_{i_1} S_{i_2} \ldots S_{i_n} (S_{j_1} S_{j_2} \ldots S_{j_m})^*$$

is equal to $S_\gamma$, where $\gamma$ is the path inside $\Gamma_t$ from $\Lambda(j_1 j_2 \ldots j_m)$ to $\Lambda(i_1 i_2 \ldots i_n)$. This follows directly from the definitions of $\Gamma_t$, $\Lambda$, and rules \(4\) and \(2\).

We will use notation $S_x = \Lambda^{-1} S_\gamma$ and $S_{x_1 x_2 \cdots x_n} = S_{x_1} S_{x_2} \cdots S_{x_n}$, for $x, x_i \in X$. Then, by the definition of $\Lambda$, the transformations $S_{x_1 x_2 \cdots x_n}$ of $X^\omega$ are given by the rule

$$S_{x_1 x_2 \cdots x_n} (v) = x_1 x_2 \ldots x_n v.$$  

For $v, u \in X^*$, the transformation $S_u S_u^*$ is defined on $uX^\omega$, and acts by the rule

$$S_u S_u^*(vw) = u w.$$  

In particular, $\sum_{x \in X} S_x S_x^* = 1$, and we obviously have $S_x^* S_x = 1$. 

3.2. The Higman-Thompson group. Let $A_1$ and $A_2$ be complete antichains in $X^*$, and let $\alpha : A_1 \rightarrow A_2$ be a bijection. Then

$$g_\alpha = \sum_{v \in A_1} S_{\alpha(v)} S_v^*$$

is a homeomorphism of $X^\omega$ defined by the rule

$$g_\alpha(vw) = \alpha(v)w.$$
for all $v \in A_1$ and $w \in X^\omega$. The set of such homeomorphisms $g_\alpha$ is the Higman-Thompson group group $G_{1|X|,1}$, see [Hig74], which we will denote by $V_X$ of $V_d$, where $d = |X|.$

Its copy $\Lambda \cdot V_X \cdot \Lambda^{-1}$ in $V_f$ is the group defined by the paths in the tree $\Gamma_t$. Namely, for any bijection $\alpha : A_1 \rightarrow A_2$ between complete antichains of $T_t$ there exist unique simple paths $\gamma_\alpha$ connecting $v \in A_1$ to $\alpha(v) \in A_2$ inside the tree $\Gamma_t$.

Then the corresponding element of $V_f$ is equal to $\sum_{\alpha \in A_1} S_{\gamma_\alpha}$.

The following simple lemma will be useful later (for a proof, see, for example [Nek04] Lemma 9.12).

**Lemma 3.1.** Let $A_1, A_2 \subset X^*$ be finite incomplete (i.e., non-maximal) antichains, and let $\alpha : A_1 \rightarrow A_2$ be a bijection. Then there exists $g \in V_X$ such that $g(vw) = \alpha(v)w$ for all $v \in A_1$ and $w \in X^\omega$.

3.3. **The iterated monodromy group.** Every element $\gamma$ of the fundamental group $\pi_1(M_t)$ defines an element $S_\gamma : \partial T_t \rightarrow \partial T_t$ of $V_f$. We get in this way a natural homomorphism $\gamma \mapsto S_\gamma$ from $\pi_1(M_t)$ to $V_f$. Its image is called the **iterated monodromy group of $f$** and is denoted $\text{IMG}(f)$. It acts on $T_t$ by automorphisms, so that the action on the $n$th level coincides with the natural **monodromy action** associated with the covering $f^n : M_n \rightarrow M$, see [Nek05] Chapter 5, [BCG03] [Nek11b].

Let us choose paths $\ell_i$ connecting the root $t$ to the vertices of the first level $f^{-1}(t)$ of the tree $T_t$. Let $\Gamma_t$ be the tree obtained by taking lifts of the paths $\ell_i$ by iterations of $f$, as in Subsection 3.1.

For a vertex $v$ of $T_t$, denote by $\ell_v$ the unique simple path inside $\Gamma_t$ from $t$ to $v$.

Then for an arbitrary path $\gamma$ in $M$ starting in a vertex $v$ and ending in a vertex $u$ of $T_t$, the path $\ell_u^{-1} \gamma \ell_v$ is a loop based at $t$. Let $g = S_{\ell_u^{-1} \gamma \ell_v}$ be the corresponding element of $\text{IMG}(f)$. Then we have

$$S_\gamma = S_{\ell_u} S_{\ell_u^{-1} \gamma \ell_v} S_{\ell_v} = S_{\ell_u} g S_{\ell_v}^*.$$ 

Hence we get the following description of the elements of $V_f$.

**Lemma 3.2.** Let $g \in V_f$ be defined by a table $\left( \begin{array}{ccc} v_1 & v_2 & \cdots & v_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ u_1 & u_2 & \cdots & u_n \end{array} \right)$. Denote $g_i = S_{\ell_v}^* S_{\gamma_i} S_{\ell_u}$. Then $g_i \in \text{IMG}(f)$, and $g = \sum_{i=1}^n S_{\ell_u} g_i S_{\ell_v}^*$.

Let $g \in \text{IMG}(f)$ be defined by a loop $\gamma$, and let $x \in f^{-1}(t)$ be a vertex of the first level. Let $\gamma_x$ be the lift of $\gamma$ by $f$ starting at $x$, and let $y$ be its end. Then we have

$$g S_{\ell_x} = S_\gamma S_{\ell_x} = S_{\gamma_x} = S_{\ell_y} S_{\ell_y^{-1} \gamma_x} = S_{\ell_y} S_{\ell_y^{-1} \gamma_x \ell_x}.$$ 

Note that $\ell_y^{-1} \gamma_x \ell_x$ is a loop based at $t$, i.e., an element of $\pi_1(M_t)$, see Figure 8.

Let us conjugate the action of $V_f$ on $\partial T_t$ (and the action of $\text{IMG}(f)$ on $T_t$) to an action on $X^\omega$ (and $X^*$) using the isomorphism $\Lambda : X^* \rightarrow T_t$. We call the obtained actions of $V_f$ and $\text{IMG}(f)$ **standard**. Then formula (4) proves the following lemma, see also [Nek05] Proposition 5.2.2.

**Lemma 3.3.** For every $g \in \text{IMG}(f)$ and every $x \in f^{-1}$ there exist $h \in \text{IMG}(f)$ and $y \in f^{-1}$ such that

$$g S_x = S_y h.$$
Moreover, if \( g \) is defined by a loop \( \gamma \), then \( h \) is defined by the loop \( \ell^{-1}_y \gamma \ell_x \), where \( \gamma_x \) is the lift of \( \gamma \) by \( f \) starting at \( x \).

If \( gS_x = S_y h \) for \( g, h \in \text{IMG}(f) \) and \( x, y \in X \), then we denote \( h = g|_x \) and \( y = g(x) \). Note that the last equality agrees with the definition of \( g \) as an automorphism \( S_\gamma \) of the tree \( T_t \).

Since \( 1 = \sum_{x \in X} S_x S_x^* \), we have
\[
g = \sum_{x \in X} gS_x S_x^* = \sum_{x \in X} S_{g(x)} g|_x S_x^*,
\]
which gives us a splitting rule for the expressions for elements of \( \mathcal{V}_f \) given in Lemma 3.2.

Namely, we get the following description of \( \mathcal{V}_f \) in terms of \( \text{IMG}(f) \) and formula (5).

**Proposition 3.4.** The group \( \mathcal{V}_f \) is isomorphic to homeomorphisms of \( X^\omega \) of the form
\[
\sum_{v \in A_1} S_{\alpha(v)} g_v S_v^*,
\]
where \( g_v \in \text{IMG}(f) \), \( A_1 \) is a complete antichain in \( X^* \), and \( \alpha : A_1 \to A_2 \) is a bijection of \( A_1 \) with a complete antichain \( A_2 \). Two elements of \( \mathcal{V}_f \) given by expressions of the form (6) are equal if and only if they can be made equal after repeated applications of the splitting rules (5) to the elements \( g_v \).

Equivalently, we can use the table notation, and represent the element (6) by the table
\[
\begin{pmatrix}
  v_1 & v_2 & \cdots & v_m \\
  g_{v_1} & g_{v_2} & \cdots & g_{v_m} \\
  \alpha(v_1) & \alpha(v_2) & \cdots & \alpha(v_m)
\end{pmatrix},
\]
where the splitting rule is the operation of replacing a column by the array
\[
\begin{pmatrix}
  v \\
  g \\
  u
\end{pmatrix} \mapsto \begin{pmatrix}
  v x_1 & v x_2 & \cdots & v x_d \\
  g|x_1 & g|x_2 & \cdots & g|x_d \\
  u g(x_1) & u g(x_2) & \cdots & u g(x_d)
\end{pmatrix}.
\]

**Example 3.1.** Consider the self-covering \( f : x \mapsto 2x \) of the circle \( \mathbb{R}/\mathbb{Z} \). Take \( t = 0 \) as the basepoint. Its preimages are 0 and 1/2. Let \( \ell_0 \) be the trivial path at 0, and
let $\ell_1$ be the path from 0 to 1/2 equal to the image of the segment $[0, 1/2] \subset \mathbb{R}$. Let $\gamma$ be the generator of $\pi_1(\mathbb{R}/\mathbb{Z}, 0)$ equal to the image of the segment $[0, 1] \subset \mathbb{R}$ with the natural (increasing on $[0, 1]$) orientation. It has two lifts by the covering $f$:

$\gamma_0 = [0, 1/2]$, \hspace{1em} $\gamma_1 = [1/2, 1]$.

Note that $\gamma_0 = \ell_1$.

By (2),

(9) \hspace{1em} $S_\gamma S_{\ell_0} = S_{\gamma_0 \ell_0} = S_{\ell_1}$,

and

(10) \hspace{1em} $S_\gamma S_{\ell_1} = S_{\gamma_1 \ell_1} = S_{\ell_0 \gamma} = S_{\ell_0} S_\gamma$.

Let $X = \{0, 1\}$, and consider the corresponding standard actions on $X^*$ and $X^\omega$. Denote by $a$ the generator of $\text{IMG}(f)$ corresponding to $S_\gamma$. Then, by (9) and (10),

$a S_0 = S_1$, \hspace{1em} $a S_1 = S_0 a$.

In other words, the action of $a$ on $X^\omega$ is given by the recurrent formulas

$a(0v) = 1v$, \hspace{1em} $a(1v) = 0a(v)$.

We see that $a$ acts as the \textit{binary adding machine}, see [Nek05, Section 1.7.1].

The group $\mathcal{V}_f$ is generated by the Higman-Thompson group $\mathcal{V}_2$ (also coinciding with the Thompson group $V$, see [CFP96]) and an element $a$ satisfying the splitting rule $a = S_1 S_0^* + S_0 a S_1^*$, i.e.,

$$
\begin{pmatrix}
  v \\
  a \\
  u
\end{pmatrix} =
\begin{pmatrix}
  v0 & v1 \\
  1 & a \\
  u1 & u0
\end{pmatrix}.
$$

We see that $a$ acts as a self-covering of its Julia set, see Figure 10.

Example 3.2. Consider the complex polynomial $z^2 - 1$ as a partial self-covering $f : \mathcal{M}_1 \rightarrow \mathcal{M}$, where $\mathcal{M} = \mathbb{C} \setminus \{0, -1\}$, and $\mathcal{M}_1 = \mathbb{C} \setminus \{0, \pm 1\}$. Alternatively, we can consider it as a self-covering of its Julia set, see Figure 10.

Then $\mathcal{V}_{z^2 - 1}$ is generated by the Thompson group $\mathcal{V}_2$ and two elements $a, b$ satisfying

$a = S_1 S_0^* + S_0 b S_1^*$, \hspace{1em} $b = S_0 S_0^* + S_1 a S_1^*$.

For a detailed proof of the recurrent definitions of the elements $a$ and $b$, see [Nek05, Subsection 5.2.2].
3.4. Self-similar groups. The description of $V_f$ in terms of the iterated monodromy group given in Proposition 3.4 can be generalized in the following way.

**Definition 3.1.** Let $G$ be a group acting faithfully by automorphisms of the tree $X^*$. We say that $G$ is a self-similar group if for all $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for all $w \in X^*$.

We will usually denote self-similar groups as pairs $(G, X)$. Note that the equation in Definition 3.1 is equivalent to the equality

$$g \cdot S_x = S_y \cdot h,$$

of compositions of self-maps of $X^\omega$, where $S_x$ is, as before, the transformation $S_x(w) = xw$ of $X^\omega$.

**Definition 3.2.** Let $G$ be a self-similar group acting on $X^*$. The group $V_G$ is the set of all homeomorphisms $g$ of $X^\omega$ for which there exist complete antichains $A_1, A_2 \subset X^*$, a bijection $\alpha: A_1 \rightarrow A_2$, and elements $g_v \in G$, for $v \in A_1$ such that

$$g = \sum_{v \in A_1} S_{\alpha(v)} g_v S_v^*,$$

i.e.,

$$g(vw) = \alpha(v) g_v(w)$$

for all $v \in A_1$ and $w \in X^\omega$.

If $G$ is a self-similar group acting on $X^*$, then for every $v \in X^*$ there exists a unique element of $G$, denoted $g|_v$, such that

$$g(vw) = g(v) g|_v(w)$$

for all $w \in X^\omega$. We call $g|_v$ the section of $g$ in $v$.

The elements of $V_G$ are represented by tables of the form [4] with the splitting rule [5]. The following proposition follows now directly from the described constructions.

**Proposition 3.5.** Consider $\text{IMG}(f)$ as a self-similar group with respect to a standard action on $X^*$. Then $V_f$ (with the corresponding standard action on $X^\omega$) is equal to $V_{\text{IMG}(f)}$.

Let us describe more examples of groups $V_f$ and $V_G$. 

*Figure 10. Julia set of $z^2 - 1$*
Example 3.3. The Grigorchuk group $G$ is generated by the transformations
\[ a = S_1S_0^* + S_0S_1^*, \quad b = S_0S_0^* + S_1cS_1^*, \quad c = S_0aS_0^* + S_1dS_1^*, \quad d = S_0S_0^* + S_1bS_1^*, \]
see [Gri80].

The corresponding group $V_G$ was defined and studied by C. Röver in [Röv99, Röv02]. This was the first example of a group $V_G$. C. Röver proved that $V_G$ is isomorphic to the abstract commensurator of $G$, that it is finitely presented, and simple. We will study the last two properties of the groups $V_G$, generalizing the results of C. Röver for a wide class of self-similar groups.

Example 3.4. Let $f(z) = z^2 + c$ be a complex quadratic polynomial such that $f^n(0) = 0$ for some $n$ (we assume that $n$ is the smallest number with this property). Then $f$ is a self-covering of its Julia set, which is path connected. The iterated monodromy groups $IMG(f)$ associated with such polynomials were described in [BN08]. There exists a sequence $v = x_1x_2...x_{n-1} \in \{0,1\}^{n-1}$ such that $IMG(f)$ is isomorphic to the group $\mathfrak{K}_v$ generated by $n$ elements $a_0, a_1, \ldots, a_{n-1}$ given by the recurrent relations
\[ a_0 = S_1S_0^* + S_0a_{n-1}S_1^*, \]
and
\[ a_i = \begin{cases} S_0a_{i-1}S_0^* + S_1S_1^* & \text{if } x_i = 0 \\ S_0S_0^* + S_1a_{i-1}S_1^* & \text{if } x_i = 1, \end{cases} \]
for $i = 1, 2, \ldots, n-1$. For example, $IMG(z^2 - 1) = \mathfrak{K}_0$.

3.5. Wreath recursions. Let $(G,X)$ be a self-similar group. Every element $g \in G$ defines a permutation $g_f$ of $X = X^1 \subset X^*$, and an element of $G^X$ equal to the function $f_g : x \mapsto g|_x$. It is easy to check that the map $\psi : G \rightarrow Symm(X) \ltimes G^X$ mapping $g$ to $(g_f,g_g)$ is a homomorphism of groups, which we call the wreath recursion associated with the self-similar group.

Let $X = \{1,2,\ldots,d\}$. We will write elements of $Symm(d) \ltimes G^d = Symm(X) \ltimes G^X$ as products $\sigma(g_1,g_2,\ldots,g_d)$, where $\sigma \in Symm(d)$ and $(g_1,g_2,\ldots,g_d) \in G^d$. Multiplication rule for elements of the wreath product $G \ltimes Symm(d) = Symm(d) \ltimes G^d$ is given by the formula
\[ \sigma(g_1,g_2,\ldots,g_d)\pi(h_1,h_2,\ldots,h_d) = \sigma\pi(g_{\pi(1)}h_1,g_{\pi(2)}h_2,\ldots,g_{\pi(d)}h_d). \]

The wreath recursion completely describes the self-similar group $G$ by giving recurrent formulas for the action of its elements on $X^*$.

Example 3.5. The adding machine, see Example 3.1, is given by the recursion $a = \sigma(1,a)$, where $\sigma$ is the transposition $(0,1)$. The generators of $IMG(z^2 - 1)$ are given by
\[ a = \sigma(1,b), \quad b = (1,a), \]
see Example 3.2.

Any homomorphism $\psi : G \rightarrow Symm(d) \ltimes G^d$ defines an action of $G$ on $X^*$ (for $X = \{1,2,\ldots,d\}$) by the recurrent rule:
\[ g(xw) = \sigma(x)g_x(w), \]
where $\sigma$ and $g_x$ are defined by the condition $\psi(g) = \sigma(g_1,g_2,\ldots,g_d)$. This action is not faithful in general. The quotient of $G$ by the kernel of its action on $X^*$ is called the faithful quotient of $G$, and it is a self-similar group in the sense of Definition 3.1.
Let $\psi : G \rightarrow \text{Symm}(d) \ltimes G^d$ be an arbitrary homomorphism. Then we can define the group $\mathcal{V}_\psi$ associated with it in the same way as the groups $\mathcal{V}_G$ were defined for self-similar groups. Namely, elements of $\mathcal{V}_\psi$ are defined by tables of the form

$$
\begin{pmatrix}
  v_1 & v_2 & \ldots & v_n \\
  g_1 & g_2 & \ldots & g_n \\
  u_1 & u_2 & \ldots & u_n
\end{pmatrix},
$$

where $g_i \in G$, and $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_n\}$ are complete antichains of $X^*$. Two tables define the same element if they can be made equal (up to permutations of the columns) by iterated replacement of a column $\begin{pmatrix} v \\ g \\ u \end{pmatrix}$ by the columns

$$
\begin{pmatrix}
  v_1 & v_2 & \ldots & vd \\
  g_1 & g_2 & \ldots & gd \\
  u\sigma(1) & u\sigma(2) & \ldots & u\sigma(d)
\end{pmatrix},
$$

where $\psi(g) = \sigma(g_1, g_2, \ldots, g_d)$. Multiplication of the tables is defined by the rule

$$
\begin{pmatrix}
  w_1 & \ldots & w_n \\
  g_1 & \ldots & g_n \\
  u_1 & \ldots & u_n
\end{pmatrix} \cdot
\begin{pmatrix}
  v_1 & \ldots & v_n \\
  h_1 & \ldots & h_n \\
  w_1 & \ldots & w_n
\end{pmatrix} =
\begin{pmatrix}
  v_1 & \ldots & v_n \\
  g_1h_1 & \ldots & g_nh_n \\
  u_1 & \ldots & u_n
\end{pmatrix}.
$$

### 3.6. Bisets

A formalism equivalent to wreath recursions is provided by the notion of a covering biset. If $(G, X)$ is a self-similar group, then the set of transformations $S_xg : w \mapsto xg(w)$ of $X^\omega$ is invariant under the left and right multiplications by elements of $G$:

$$
S_xg \cdot h = S_x(gh), \quad h \cdot S_xg = S_{h(x)}(h|_xg).
$$

We get therefore commuting left and right actions of $G$ on the set $\Phi = \{S_xg : x \in X, g \in G\}$. We will write elements $S_x \cdot g$ of $\Phi$ just as $x \cdot g$.

We adopt the following definition.

**Definition 3.3.** Let $G$ be a group. A $G$-biset is a set $\Phi$ together with commuting left and right $G$-actions. It is called a covering biset if the right action is free (i.e., if $x \cdot g = x$ for $x \in \Phi$ and $g \in G$ implies $g = 1$) and has a finite number of orbits.

Let $\Phi_1, \Phi_2$ be $G$-bisets. Then their tensor product $\Phi_1 \otimes \Phi_2$ is defined as the quotient of the set $\Phi_1 \times \Phi_2$ by the identifications

$$(x \cdot g) \otimes y = x \otimes (g \cdot y), \quad g \in G.$$

Let $\Phi$ be the biset $\{x \cdot g : x \in X, g \in G\}$ associated with a self-similar group. Then every element of $\Phi^{\otimes n}$ can be uniquely written in the form $x_1 \otimes x_2 \otimes \cdots \otimes x_n \cdot g$, where $x_i = x_i^{-1}$ are elements of $X$. It follows that $n$th tensor power $\Phi^{\otimes n}$ is naturally identified with the set of pairs $v \cdot g$, for $v \in X^n$ and $g \in G$, with the actions

$$
h \cdot (v \cdot g) = h(v) \cdot (h|_xg), \quad (v \cdot g) \cdot h = v \cdot (gh).
$$

Let $\Phi$ be an arbitrary covering $G$-biset. Choose a transversal $X \subset \Phi$ of the orbits of the right action. Then every element of $\Phi$ is uniquely written in the form $x \cdot g$ for $x \in X$ and $g \in G$. For every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$
g \cdot x = y \cdot h,$$
and the elements $y, h$ are uniquely determined by $g$ and $x$. We get hence a homomorphism $\psi : G \rightarrow \text{Symm}(X) \ltimes G^X$, called the \textit{wreath recursion} associated with $\Phi$ and $X$. Namely, $\psi(g) = \sigma \cdot f$, where $\sigma \in \text{Symm}(X)$ and $f \in G^X$ satisfy

$$g \cdot x = \sigma(x) \cdot f(x)$$

for all $x \in X$ (where $G^X$ is seen as the set of functions $X \rightarrow G$). If we change the orbit transversal $X$ to an orbit transversal $Y$, then the homomorphism $\psi : G \rightarrow \text{Symm}([X]) \ltimes G^{|X|}$ is composed with an inner automorphism of the wreath product (after we identify $X$ with $Y$ by a bijection). Note that the biset $\Phi$ is uniquely determined, up to an isomorphism of biset, by the homomorphism $\psi$.

\textbf{Definition 3.4.} Two self-similar actions $(G, X_1)$ and $(G, X_2)$ of a group $G$ are called \textit{equivalent} if their associated bisets $\Phi_1 = X_1 \cdot G$ are isomorphic, i.e., if there exists a bijection $F : \Phi_1 \rightarrow \Phi_2$ such that $F(g_1 \cdot a \cdot g_2) = g_1 \cdot F(a) \cdot g_2$ for all $g_1, g_2 \in G$ and $a \in \Phi_1$. Two self-similar actions of groups $G_1, G_2$ are equivalent if they become equivalent after identification of the groups $G_1, G_2$ by an isomorphism $G_1 \rightarrow G_2$.

The wreath recursion can be defined invariantly, without a choice of the orbit transversal. Namely, let $\text{Aut}(\Phi_G)$ be the automorphism group of the right $G$-set $\Phi$, i.e., the set of all bijections $\alpha : \Phi \rightarrow \Phi$ such that $\alpha(x \cdot g) = \alpha(x) \cdot g$. Then $\text{Aut}(\Phi_G)$ is isomorphic to the wreath product $\text{Symm}([d]) \ltimes G^d$, where $d$ is the number of the orbits of the right action on $\Phi$, since the right $G$-set $\Phi$ is free and has $d$ orbits, i.e., is isomorphic to the disjoint union of $d$ copies of $G$. For every element $g \in G$ the map $\psi(g) : x \mapsto g \cdot x$ is an automorphism of the right $G$-set $\Phi$. Then $\psi : G \rightarrow \text{Aut}(\Phi_G)$ is the wreath recursion. For more on wreath recursions and bisets, see [Nek05, Nek08b, Nek08a].

\textbf{Example 3.6.} Let $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ be a self-covering map, and let $\iota : \mathcal{M}_1 \rightarrow \mathcal{M}$ be a continuous map (for example, $f$ is a partial self-covering, and $\iota$ is the identical embedding).

Suppose that $\mathcal{M}$ is path-connected. Choose a basepoint $t \in \mathcal{M}$, and consider the set $\Phi$ of pairs $(z, \ell)$, where $z \in f^{-1}(t)$, and $\ell$ is a homotopy class of a path in $\mathcal{M}$ from $t$ to $\iota(z)$. Then $\pi_1(\mathcal{M}, t)$ acts on $\Phi$ be appending loops to the beginning of the path $\ell$:

$$(z, \ell) \cdot \gamma = (z, \ell \gamma).$$

It also acts by appending images of lifts of $\gamma$ to the end of the path $\ell$:

$$\gamma(z, \ell) = (z', \iota(\gamma_z)\ell),$$

where $\gamma_z$ is the lift of $\gamma$ by $f$ starting at $z$, and $z'$ is the end of $\gamma_z$. Here, as before, we multiply paths as functions (second path in a product is passed first).

Then $\Phi$ is a covering $\pi_1(\mathcal{M}, t)$-biset. The associated wreath recursion coincides with the wreath recursion associated with the standard action of $\text{IMG}(f)$.

Let us show a more canonical definition of the groups $\mathcal{V}_w$ in terms of bisets. Let $\Phi$ be a covering $G$-biset. Consider the biset $\Phi^*$ equal to the disjoint union of the bisets $\Phi^n$ for all integers $n \geq 0$. Here $\Phi^0$ is the group $G$ with the natural $G$-biset structure. The set $\Phi^*$ is a semigroup with respect to the tensor product operation.

Let us order the semigroup $\Phi^*$ with respect to the left divisibility, i.e., $v \preceq u$ if and only if there exists $w$ such that $u = v \otimes w$. It is easy to check that $\Phi^*$ is left-cancellable, i.e., that $v \otimes w_1 = v \otimes w_2$ implies that $w_1 = w_2$. 

The quotient of $\Phi^*$ by the right $G$-action is a rooted $d$-regular tree, and the image of $\preceq$ under the quotient map is the natural order on the rooted tree $\Phi^*/G$. If $X$ is a right orbit transversal of $\Phi$, then $X^{\otimes n} = X^n$ is a right orbit transversal of $\Phi^{\otimes n}$, and the identical embedding of $X^* = \bigcup_{n \geq 0} X^{\otimes n}$ into $\Phi^*$ induces an isomorphism of the rooted tree $X^*$ with $\Phi^*/G$.

The left action of $G$ on $\Phi^*$ permutes the orbits of the right action and preserves the relation $\preceq$, hence $G$ acts on the tree $\Phi^*/G$ by automorphisms. The corresponding action on $X^*$ is the self-similar action defined by the wreath recursion associated with $X$.

Let $A_1, A_2 \subseteq \Phi^*$ be finite maximal antichains with respect to the divisibility order $\preceq$. Note that a subset $A \subseteq \Phi^*$ is a finite maximal antichain if and only if its image in $\Phi^*/G$ is a maximal antichain. Choose a bijection $\alpha : A_1 \to A_2$.

If $w \in \Phi^*$ is such that $v \geq w$ for some $v \in A_1$, then there exists a unique $u \in \Phi^*$ such that $w = v \otimes u$. Consider then the transformation $h_{A_1, \alpha, A_2} : w \mapsto \alpha(v) \otimes u$. The map $h_{A_1, \alpha, A_2}$ is defined for all elements of $\Phi^*$, hence for all elements of $\Phi^{\otimes n}$, where $n$ is big enough. We will identify to transformation $h_{A_1, \alpha, A_2}$ and $h_{A_1', \alpha', A_2'}$ if their actions on the sets $\Phi^{\otimes n}$ agree for all $n$ big enough. It is not hard to prove that the set of equivalence classes of such maps is a group, which we will denote $\mathcal{V}_\Phi$. It is also straightforward to show that $\mathcal{V}_\Phi$ coincides with $\mathcal{V}_\psi$, where $\psi$ is the wreath recursion associated with $\Phi$, and that if $\Phi$ is the usual biset associated with a self-similar group $G$, then $\mathcal{V}_\Phi$ coincides with $\mathcal{V}_G$.

3.7. **Epimorphism onto the faithful quotient.** Consider a covering $G$-biset $\Phi$ and the corresponding group $\mathcal{V}_\Phi$. The faithful quotient $\overline{\mathcal{V}_\Phi}$ is a self-similar group acting on $X^*$. We have, therefore two groups: $\mathcal{V}_\Phi$ and $\overline{\mathcal{V}_\Phi}$, which are non-isomorphic in general (the group $\overline{\mathcal{V}_\Phi}$ is a homomorphic image of $\mathcal{V}_\Phi$).

**Proposition 3.6.** Let $\Phi$ be a covering biset. Denote by $K_n$ the subgroup of elements of $G$ acting trivially from the left on $\Phi^{\otimes n}$, i.e., the kernel of the wreath recursion associated with the biset $\Phi^{\otimes n}$. Then $K_n \supseteq K_{n-1}$. If $\bigcup_{n \geq 0} K_n$ is equal to the kernel of the epimorphism $G \to \overline{\mathcal{V}_\Phi}$, then the natural epimorphism $\mathcal{V}_\Phi \to \overline{\mathcal{V}_\Phi}$ is an isomorphism.

**Proof.** Suppose that an element $g$ of the kernel of $\mathcal{V}_\Phi \to \overline{\mathcal{V}_\Phi}$ is defined by a table $\begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ g_1 & g_2 & \ldots & g_n \end{pmatrix}$. Then the table $\begin{pmatrix} \overline{v_1} & \overline{v_2} & \ldots & \overline{v_n} \\ \overline{g_1} & \overline{g_2} & \ldots & \overline{g_n} \end{pmatrix}$ represents the trivial element of $\overline{\mathcal{V}_\Phi}$, where $g \mapsto \overline{g}$ is the epimorphism $G \to \overline{\mathcal{V}_\Phi}$. But this means that $v_i = u_i$ and $\overline{g_i} = 1$ for all $i$. Consequently, there exists $k$ such that $g_i \in K_k$ for all $i$. It follows that after applying elementary splittings $k$ times to all column of the table defining $g$, we will get a table defining the trivial element of $\mathcal{V}_\Phi$, which means that $g = 1$. \qed

4. **Simplicity of the commutator subgroup**

4.1. **Some general facts.** Let $G$ be a group acting faithfully by homeomorphisms on an infinite Hausdorff space $X$. For an open subset $U \subseteq X$, denote by $G(U)$ the subgroup of elements of $G$ acting trivially on $X \setminus U$. Denote by $R_U$ the normal closure in $G$ of the derived subgroup $G(U)' = [G(U), G(U)]$. 

The following simple lemma has appeared in many papers in different forms, see, for example [BGS03 Lemma 5.3], [Mat06 Theorem 4.9].

**Lemma 4.1.** Let \( N \) be a non-empty open subgroup of \( G \). Then there exists a non-empty open subset \( U \subset X \) such that \( R_U \leq N \).

**Proof.** It is sufficient to prove that there exists an open subset \( U \) such that \( G(U) \) is equal to its normal subgroup \( U \) and non-trivial, hence contains \( N \). Let \( g \in N \setminus \{1\} \), and let \( x \in X \) be such that \( g(x) \neq x \). Then there exists an open subset \( U \) such that \( x \in U \) and \( U \cap gU = \emptyset \). For example, find disjoint neighborhoods \( U_x \) and \( U_{g(x)} \) of \( x \) and \( g(x) \), and set \( U = U_x \cap g^{-1}(U_{g(x)}) \).

Let \( h_1, h_2 \in G(U) \). Then \( gh_1^{-1}g^{-1} \) acts trivially outside \( gU \). Consequently, \([g^{-1}, h_1] = gh_1^{-1}g^{-1}h_1 \) acts as \( h_1 \) on \( U \), since \( gh_1^{-1}g^{-1} \) on \( gU \), and trivially outside \( U \). It follows that \([g^{-1}, h_1, h_2] \) acts as \([h_1, h_2] \) on \( U \) and trivially outside \( U \). On the other hand, \([g^{-1}, h_1, h_2] \in N \), since \( N \) is normal and \( g \in N \). It follows that \([h_1, h_2] \in N \) for all \( h_1, h_2 \in G(U) \), i.e., that \( G(U) \) is normal and contains \( N \).

**Lemma 4.2.** Let \( U \) be an open subset of \( X \) such that its \( G \)-orbit is a basis of topology of \( X \). Then every non-trivial normal subgroup of \( G \) contains \( R_U \).

**Proof.** For any \( g \in G \) and \( U \subset X \), we have \( gG(U)g^{-1} = G(gU) \). Consequently, \( gG(U)g^{-1} = G(gU) \)' and \( R_U \) is equal to the group generated by \( \bigcup_{g \in G} G(gU) \)' in particular, \( R_{gU} = R_U \) for all \( g \in G \).

Since \( \{gU : g \in G\} \) is a basis of topology, for every open subset \( W \subset X \) there exists \( g \in G \) such that \( gU \subset W \), hence \( R_U = R_{gU} \leq R_W \). This implies, by Lemma 4.1 that \( R_U \) is contained in every non-trivial normal subgroup of \( G \).

**Proposition 4.3.** Suppose that \( U \) is an open subset of \( X \) such that its \( R_U \)-orbit is equal to its \( G \)-orbit and is a basis of topology of \( X \). Then \( R_U \) is simple and is contained in every non-trivial normal subgroup of \( G \).

**Proof.** The group \( G(U) \) is non-trivial, since otherwise its normal closure \( R_U \) is trivial, which contradicts the fact that the \( R_U \)-orbit of \( U \) is a basis of topology. It follows that there exists \( g \in G(U) \) moving a point \( x \in X \). We have then \( x \in U \) and \( g(x) \in U \). There exists a neighborhood \( W \) of \( x \) such that \( W \cap gW = \emptyset \). There exists an element \( h \in R_U \) such that \( hU \subset W \), and there exists a non-trivial element \( g' \in G(U) \) of \( G(U) \) contained in \( N \). We have \( [g, g'] \neq 1 \), since \( g' \) and \( g'g^{-1} \) have disjoint supports. This shows that \( G(U) \) is non-abelian, i.e., that \( G(U)'' \) is non-trivial.

The subgroup \( R_U \) is contained in every non-trivial normal subgroup of \( G \), by Lemma 4.2. We also have that the normal closure in \( R_U \) of the group \( (R_U)(U) \) is contained in every normal subgroup of \( R_U \).

Note that \( G(U) \)' is contained in \( G(U) \cap R_U = (R_U)(U) \), hence \( G(U)'' \leq (R_U)(U) \). Conjugating by an element \( g \in G \) (and using that \( R_U \) is normal in \( G \)), we get that \( G(gU)'' \leq (R_U)(gU) \).

It follows that for any non-trivial subgroup \( N \leq R_U \) we have

\[
N \supseteq \bigcup_{g \in R_U} (R_U)(gU) \supseteq \bigcup_{g \in R_U} G(gU)'' = \bigcup_{g \in G} G(gU)''.
\]

Consequently, \( N \) contains the group generated by the set \( \bigcup_{g \in G} G(gU)'' \), which is normal in \( G \) and non-trivial, hence contains \( R_U \). We have proved that every non-trivial normal subgroup of \( R_U \) contains \( R_U \), i.e., that \( R_U \) is simple.
4.2. Simplicity of \( \mathcal{V}_G \). Let \((G, X)\) be a self-similar group, and let \( \mathcal{V}_G \) be the corresponding group of homeomorphisms of the Cantor set \( X^\omega \).

Fix a linear ordering "\( \leq \)" of the elements of \( X \). Extend it to the lexicographic ordering on \( X^* \). Namely, if \( x_1 x_2 \ldots x_n \) and \( y_1 y_2 \ldots y_m \) are incomparable with respect to the order \( \leq \), then \( x_1 x_2 \ldots x_n < y_1 y_2 \ldots y_m \) if and only if \( x_i < y_i \), where \( i \) is the smallest index such that \( x_i \neq y_i \). If a word \( v \) is a beginning of a word \( w \), then \( v \leq w \).

Suppose that \( \left( \begin{array}{cccc} v_1 & v_2 & \ldots & v_n \\ u_1 & u_2 & \ldots & u_n \end{array} \right) \) is a table defining an element \( g \in \mathcal{V}_X \), i.e., that \( g = \sum_{i=1}^{n} S_{u_i} S_{v_i} \). Assume that its first row is ordered in the increasing lexicographic order. If the permutation putting the second row into the increasing lexicographic order is even, then we say that the table is even.

Note that if \( d = |X| \) is even, then every table has an even splitting. On the other hand, if \( d \) is odd, then the set of elements defined by even tables is a subgroup of index 2 in \( \mathcal{V}_d \). The following is proved in [Hig74].

**Theorem 4.4.** If \( d \) is even, then \( \mathcal{V}_d \) is simple. If \( d \) is odd, then the commutator subgroup \( \mathcal{V}_d' \) is the group of elements defined by even tables, and is a simple subgroup of index 2 in \( \mathcal{V}_d \).

The following theorem is a proved in [Nek04, Theorem 9.11].

**Theorem 4.5.** All proper quotients of \( \mathcal{V}_G \) are abelian.

Let us describe the \( \mathcal{V}_d \)-orbits of clopen subsets of \( X^\omega \).

**Proposition 4.6.** Let \( d = |X| \), and let \( U \) be a clopen subset of \( X^\omega \). Let us decompose \( U \) into a disjoint union \( \bigsqcup_{i=1}^{n} v_i X^\omega \) for \( v_i \in X^* \). Then the residue of \( n \) modulo \( d - 1 \) does not depend on the decomposition. Let us denote it \( m(U) \in \mathbb{Z}/(d-1)\mathbb{Z} \).

Let \( U_1, U_2 \subset X^\omega \) be non-empty clopen proper subsets. Then the following conditions are equivalent.

1. \( U_1 \) and \( U_2 \) belong to one \( \mathcal{V}_X \)-orbit.
2. \( U_1 \) and \( U_2 \) belong to one \( \mathcal{V}_X' \)-orbit.
3. \( m(U_1) = m(U_2) \).

**Proof.** Let \( U = \bigsqcup_{i=1}^{n} v_i X^\omega \) be a decomposition of \( U \). We can split it by replacing one set \( v_i X^\omega \) by the collection of \( d \) sets \( v_i x X^\omega \), for \( x \in X \). This way we increase the number of sets in the decomposition by \( d - 1 \). It is easy to see that for any two decompositions of \( U \) into cylindrical sets there exist sequences of successive splittings of each of the decompositions leading to the same decomposition. This implies that \( m(U) \) is well defined.

It is obvious that \( m(U) \) is preserved under the action of \( \mathcal{V}_X \). Suppose that \( m(U_1) = m(U_2) \) for non-empty proper clopen subsets of \( X^\omega \). Then there exist decompositions \( U_1 = \bigsqcup_{i=1}^{n} v_i X^\omega \) and \( U_2 = \bigsqcup_{i=1}^{n} u_i X^\omega \) of the sets \( U_i \) into equal number of cylindrical subsets. The sets \( \{v_i\}_{i=1}^{n} \) and \( \{u_i\}_{i=1}^{n} \) are incomplete antichains in \( X^* \), hence (see Lemma 1.3.1) there exists \( g \in \mathcal{V}_X \) such that \( g(v_i X^\omega) = u_i X^\omega \) for all \( i \). If \( g \notin \mathcal{V}_X' \), then we can compose it with an element \( h \in \mathcal{V}_X \setminus \mathcal{V}_X' \) acting trivially on \( \bigsqcup_{i=1}^{n} v_i X^\omega \), and get an element \( gh \in \mathcal{V}_X' \) such that \( gh(v_i X^\omega) = u_i X^\omega \) for all \( i \). It follows that \( U_1 \) and \( U_2 \) belong to one \( \mathcal{V}_X' \)-orbit and to one \( \mathcal{V}_X \)-orbit.

It easily follows from the definitions that \( \mathcal{V}_X \)-orbits of clopen subsets of \( X^\omega \) coincide with their \( \mathcal{V}_G \)-orbits for any self-similar group \( (G, X) \).
Theorem 4.7. The commutator subgroup of $V_G$ is simple.

Proof. Let $U = xX^a$ for some $x \in X$. Then the $V_G$-orbit of $U$ coincides with its $V_X$-orbit and its $V_X'$-orbit, and is a basis of the topology on $X^a$, see Proposition 4.3. The group $V_X'$ is contained in $R_U$, since the normal closure of $(V_X)_{U}'$ in $V_X$ is equal to $V_X'$, by Theorem 4.4. It follows that the $R_U$-orbit of $U$ is equal to its $G$-orbit, and is a basis of the topology.

Consequently, by Proposition 4.3, $R_U$ is simple and is contained in every nontrivial normal subgroup of $V_G$. On the other hand, by Theorem 4.5, every nontrivial normal subgroup of $V_G$ contains the commutator subgroup $V_G'$. It follows that $V_G'' = R_U$, and $V_G''$ is simple. \[\square\]

Abelianization of $V_G$ is described in [Nek04, Theorem 9.14].

Theorem 4.8. Let $(G, X)$ be a self-similar group. Let $\pi : G \rightarrow G/G'$ be the abelianization epimorphism.

Suppose that $d$ is even. Then $V_G/V_G'$ is isomorphic to the quotient of $G/G'$ by the relations $\pi(g) = \sum_{x \in X} \pi(g|x)$ for all $g \in G$.

Suppose that $d$ is odd. Then $V_G/V_G'$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus G/G'$ modulo the relations $\pi(g) = \text{sign}(g) \oplus \sum_{x \in X} \pi(g|x)$ for all $g \in G$; where $\text{sign}(g) = 0$ if $g$ defines an even permutation on the first level $X$ of $X^*$, and $\text{sign}(g) = 1$ otherwise.

Proof of the following lemma follows directly from the multiplication rule (I).

Lemma 4.9. Let $(G, X)$ be a self-similar group, and let $\pi : G \rightarrow G/G'$ be the canonical homomorphism. Then $\sigma : \pi(g) \mapsto \sum_{x \in X} \pi(g|x)$ is a well defined endomorphism of $G/G'$.

For odd $d$ denote by $\sigma_1$ the endomorphism $(t, g) \mapsto (t + \text{sign}(g), \sigma(g))$ of $\mathbb{Z}/2\mathbb{Z} \oplus G/G'$, where $\text{sign} : G/G' \rightarrow \mathbb{Z}/2\mathbb{Z}$ gives the parity of the action of any preimage of $g$ in $G$ on the first level $X$ of the tree $X^*$.

Then Theorem 4.8 can be reformulated as follows. (We denote here the identical automorphism of a group by 1.)

Theorem 4.10. If $d$ is even, then $V_G/V_G'$ is isomorphic to the quotient of $G/G'$ by the range of the endomorphism $1 - \sigma$.

If $d$ is odd, then $V_G/V_G'$ is isomorphic to the quotient of $\mathbb{Z}/2\mathbb{Z} \oplus G/G'$ by the range of the endomorphism $1 - \sigma_1$.

Example 4.1. Consider the double self-covering $f(x) = 2x$ of the circle $\mathbb{R}/\mathbb{Z}$. Then $\text{IMG}(f)$ is generated by the adding machine $a = S_1S_0^* + S_0aS_1^*$. It follows that $V_f/V_f'$ is the quotient of $\mathbb{Z} = \text{IMG}(f) / \text{IMG}(f)'$ by the range of $1 - \sigma$, where $\sigma(a) = 0 + a = a$ is the identity isomorphism. Hence, $1 - \sigma = 0$, and $V_f/V_f' \cong \mathbb{Z}$.

Example 4.2. Let $G = R_{x_1x_2...x_{n-1}}$ be the iterated monodromy group of a quadratic polynomial, as in Example 3.4. It is proved in [BN08, Proposition 3.3] that $G/G'$ is the free abelian group $\mathbb{Z}^n$ freely generated by the images of the generators $a_i$ of $G$. The recursive relations defining $R_{x_1x_2...x_{n-1}}$ show that $V_G/V_G'$ is $\mathbb{Z}^n = (\pi(a_0), \ldots, \pi(a_{n-1}))$ modulo the relations $\pi(a_0) = \pi(a_1)$, and $\pi(a_i) = \pi(a_{i-1})$ for all $i = 1, 2, \ldots, n - 1$. Consequently, $V_G/V_G'$ is isomorphic to $\mathbb{Z}$.

Example 4.3. Let $G$ be the Grigorchuk group, see Example 3.3. It is generated by

$$a = S_1S_0^* + S_0S_1^*, b = S_0aS_0^* + S_1cS_1^*, c = S_0aS_0^* + S_1dS_1^*, d = S_0bS_0^* + S_1bS_1^*.$$
5. Expanding maps and finite presentation

5.1. Contracting self-similar groups and expanding maps.

**Definition 5.1.** Let \((G, X)\) be a self-similar group. We say that it is *contracting* if there exists a finite set \(N \subset G\) such that for every \(g \in G\) there exists \(n \) such that \(g^n \in N\) for all \(v \in X^*\), |\(v| \geq n\).

More generally, let \(\Phi\) be a covering \(G\)-biset, and let \(X\) be a right orbit transversal. Define then \(g^n\) for \(g \in G\) and \(v \in X^n \subset \Phi^\otimes n\) as the unique element of \(G\) such that \(g \cdot v = u \cdot g^n\), for some (also unique) \(u \in X^n\). We say that \(\Phi\) is contracting (or hyperbolic) if there exists a finite set \(N\) such that for every \(g \in G\) there exists \(n\) such that \(g^n \in N\) for all \(v \in X^*\) such that \(|v| \geq n\). One can show (see [Nek05, Corollary 2.11.7]) that this property depends only on \(\Phi\) (but the set \(N\) will depend on \(X\)).

The smallest set \(N\) satisfying the conditions of Definition 5.1 is called the *nucleus* of the self-similar action.

**Definition 5.2.** Let \(f : \mathcal{M}_1 \longrightarrow \mathcal{M}\) be a partial self-covering such that \(\mathcal{M}\) is compact. The covering is called expanding if there exist \(L > 1\), \(\epsilon > 0\), a positive integer \(n\), and a metric \(|x - y|\) on \(\mathcal{M}\) such that

\[ |f^n(x) - f^n(y)| \geq L|x - y| \]

for all \(x, y \in \mathcal{M}_n\) such that \(|x - y| < \epsilon\).

For example, if \(\mathcal{M}\) is a connected Riemann manifold, and \(\|D f^n(\xi)\| \geq C L^n \|\xi\|\) for all \(n \geq 1\) and every tangent vector \(\xi\), then \(f\) is expanding.

The following proposition is proved in [Nek05, Theorem 5.5.3] for the case when \(\mathcal{M}\) is a complete length metric space with finitely generated fundamental group. We will repeat here the proof in a more general situation (but avoiding orbispaces, which was one of the technical issues in [Nek05]).

**Proposition 5.1.** Let \(f : \mathcal{M}_1 \longrightarrow \mathcal{M}\) be an expanding partial self-covering of a compact path connected space. Then \(IMG(f)\) is a contracting self-similar group (with respect to any standard action).

**Proof.** Let \(\{\ell_i\}_{i=1,...,d}\) be paths connecting the basepoint \(t \in \mathcal{M}\) to the preimages \(z_i \in f^{-1}(t)\). We consider the standard action of \(IMG(f)\) on \(X^*\) defined by these connecting paths, where \(X = \{1,\ldots,d\}\).

It follows from Definition 5.2 that there exist \(\epsilon > 0, L > 1, C > 1\) such that for any subset \(A \subset \mathcal{M}\) of diameter less than \(\epsilon\) and every \(n \geq 1\), the set \(f^{-n}(A)\) is a disjoint union of \(d^n\) sets \(A_i\) such that \(f^n : A_i \longrightarrow A\) are homeomorphisms, and diameters of \(A_i\) are less than \(CL^{-n}\).
Let \( g \in \text{IMG}(f) \) be defined by a loop \( \gamma \). Since we can represent \( \gamma \) as a union of sub-paths of diameter less than \( \epsilon \), there exists \( C_\gamma > 1 \) such that diameter of any lift of \( \gamma \) by \( f^n \) is less than \( C_\gamma L^{-n} \).

By Lemma 3.3, section \( g|_{i_1 i_2 \ldots i_n} \) is defined by the loop \( \ell^{-1}_{j_1 j_2 \ldots j_m} \gamma_{i_1 i_2 \ldots i_m} \ell_{i_1 i_2 \ldots i_n} \), where \( \gamma_{i_1 i_2 \ldots i_n} \) is a lift of \( \gamma \) by \( f^n \), and \( \ell_{i_1 i_2 \ldots i_n} \) are paths inside the tree \( \Gamma_i \) formed by lifts of the connecting paths \( \ell_i \). Since diameters of lifts of paths by \( f^n \) exponentially decrease with \( n \), there exists \( n_0 \) (depending only on the connecting paths \( \ell_i \)) such that if \( n_0 \leq n < m \), then for all sequences \( i_1 i_2 \ldots i_m \in \mathbb{X}^* \), the path \( \ell_{i_1 i_2 \ldots i_m} \) is a lift of the connecting paths \( \ell_i \).

There exists \( n_1 \geq n_0 \) (depending on \( \gamma \)) such that if \( n \geq n_1 \), then all lifts of \( \gamma \) by \( f^n \) have diameters less than \( \epsilon/6 \). Then for every \( n \geq n_1 \), the section \( g|_{i_1 i_2 \ldots i_n} \) is defined by a loop of the form \( \beta = \ell^{-1}_{j_1 j_2 \ldots j_m} \alpha \ell_{i_1 i_2 \ldots i_n} \), where \( \alpha \) is a path of diameter less than \( \epsilon/2 \). If \( \beta' = \ell^{-1}_{j_1 j_2 \ldots j_n} \alpha' \ell_{i_1 i_2 \ldots i_{n_0}} \) is another path such that \( \alpha' \) has diameter less than \( \epsilon/2 \), then

\[
\beta' \beta^{-1} = \ell^{-1}_{j_1 j_2 \ldots j_n} \alpha' \alpha^{-1} \ell_{j_1 j_2 \ldots j_{n_0}},
\]

where \( \alpha' \alpha^{-1} \) is a loop of diameter less than \( \epsilon \). By the choice of \( \epsilon \), all lifts of \( \alpha' \alpha^{-1} \) by iterations of \( f \) are loops, which implies that \( \beta \) and \( \beta' \) define equal elements of \( \text{IMG}(f) \). Consequently, \( g|_{i_1 i_2 \ldots i_n} \) is uniquely determined by the pair \((i_1 i_2 \ldots i_{n_0}, j_1 j_2 \ldots j_{n_0})\). Since \( n_0 \) does not depend on \( g \), it follows that \( \text{IMG}(f) \) is contracting.

### 5.2. General definition of \( \mathcal{V}_f \) for expanding maps

The group \( \mathcal{V}_f \) can be defined for an expanding self-covering \( f : \mathcal{M} \rightarrow \mathcal{M} \), even if \( \mathcal{M} \) is not path connected.

Let \( f : \mathcal{M} \rightarrow \mathcal{M} \) be an expanding self-covering of a compact metric space \( \mathcal{M} \), such that for every \( t \in \mathcal{M} \) the set \( \bigcup_{n \geq 0} f^{-n}(t) \) is dense in \( \mathcal{M} \). This condition is always satisfied for a path-connected space \( \mathcal{M} \).

Let \( \delta > 0 \) be such that for any two points \( t_1, t_2 \in \mathcal{M} \) such that \( t_1 \neq t_2 \) and \( f(t_1) = f(t_2) \) we have \( |t_1 - t_2| > \delta \). It is easy to prove that such \( \delta \) exists for any self-covering \( f : \mathcal{M} \rightarrow \mathcal{M} \) of a compact metric space. It follows from the definition of an expanding self-covering, that there exists \( \epsilon > 0 \) such that for any two points \( z_1, z_2 \in \mathcal{M} \) and for any \( n \geq 1 \) there exists an isomorphism \( S_{z_1, z_2} : T_{z_1} \rightarrow T_{z_2} \) of the trees of preimages such that \( |S_{z_1, z_2}(v) - v| < \delta/2 \) for all \( v \in T_{z_1} \). Moreover, it is easy to prove (by induction on the level number) that the isomorphism \( S_{z_1, z_2} \) is unique.

Fix a basepoint \( t \in \mathcal{M} \), and define \( \mathcal{V}_f \) as the group of homeomorphisms of \( \partial T_t \) piecewise equal to the isomorphisms \( S_{z_1, z_2} : \partial T_{z_1} \rightarrow \partial T_{z_2} \) for \( z_1, z_2 \in T_t \).

Note that if \( \gamma : [0, 1] \rightarrow \mathcal{M} \) is a path in \( \mathcal{M} \), then there exists \( n \) such that all lifts of \( \gamma \) by \( f^n \) for \( m \geq n \) have diameter less than \( \delta/2 \). This implies that if \( \mathcal{M} \) is path connected, then our original definition of \( \mathcal{V}_f \) agrees with the given definition for expanding maps.

### Example 5.1

Consider the one-sided shift \( s : X^\omega \rightarrow X^\omega \)

\[
s(x_1 x_2 \ldots) = x_2 x_3 \ldots
\]

Consider the metric \( |w_1 - w_2| = 2^{-n} \), where \( n \) is the smallest index for which \( x_n \neq y_n \), where \( w_1 = x_1 x_2 \ldots \) and \( w_2 = y_1 y_2 \ldots \).

Then \( |s(w_1) - s(w_2)| = 2 |w_1 - w_2| \) whenever \( |w_1 - w_2| \leq 1/2 \). Consequently, \( s \) is expanding.
For any \( w \in X^\omega \) the set \( s^{-n}(w) \) is equal to the set of sequences of the form \( vw \), where \( v \in X^n \). Hence, we can identify the \( n \)th level of the tree \( T_w \) with the set \( X^n \) by the map \( vw \mapsto v \). Note that the tree \( T_w \) after this identification becomes the left Cayley graph of the monoid \( X^* \): two vertices are connected by an edge if and only if they are of the form \( v, xv \) for \( v \in X^*, x \in X \). In particular, the boundary \( \partial T_w \) is naturally identified with the space \( X^{-\omega} \) of left-infinite sequences \( x_2x_1 \).

It follows directly from the definitions that the maps \( S_{wv_1,wv_2} : T_{wv_1} \to T_{wv_2} \) act by the rule
\[
S_{wv_1,wv_2}(uv_1) = uv_2.
\]
It follows that the homeomorphism \( \ldots x_2x_1 \mapsto x_1x_2 \ldots \) of \( X^{-\omega} = \partial T_w \) with \( X^\omega \) conjugates the action of \( \mathcal{V}_x \) with the Higman-Thompson group \( \mathcal{V}_X \).

**Example 5.2.** If \( \mathcal{M} \) is not connected, then a covering \( f : \mathcal{M} \to \mathcal{M} \) needs not to be of constant degree. For example, \( f : \mathcal{M} \to \mathcal{M} \) can be a one-sided shift of finite type. The corresponding group \( \mathcal{V}_f \) is the topological full groups of a shift of finite type (the dual of \( f \)). These groups were studied in [Mat12].

### 5.3. Abelianization of \( \mathcal{V}_f \) in expanding case

Self-similar contracting groups acting faithfully on \( X^* \) are typically infinitely presented. On the other hand, for every contracting group \( G \) there exists a finitely presented group \( \tilde{G} \) and a hyperbolic covering \( \tilde{G} \)-biset \( \Phi \) such that the faithful quotient of \( \tilde{G} \) is \( G \). More precisely, we have the following description of \( \tilde{G} \), given in [Nek05 Section 2.13.2].

**Proposition 5.2.** Let \((G, X)\) be a contracting group. Suppose that the nucleus \( N \) generates \( G \).

Let \( \tilde{G} \) be the group given by the presentation \( \langle N \mid R \rangle \), where \( R \) is the set of all relations \( g_1g_2g_3 = 1 \) of length at most 3 that hold for elements of \( N \) in \( G \). Let \( \Phi \) be the \( \tilde{G} \)-biset of pairs \( x \cdot g \), for \( x \in X \) and \( g \in \tilde{G} \) with the actions given by the usual rules:
\[
(x \cdot g) \cdot h = x \cdot (gh), \quad h \cdot (x \cdot g) = h(x) \cdot (h|_x g),
\]
where \( g \in \tilde{G}, h \in N, x \in X; \) and \( h(x) \in X, h|_x \in N \) are defined as in \( G \). Then \( \Phi \) is contracting.

Note that for any contracting group \( G \), the group generated by the nucleus \( N \) is self-similar contracting, and \( \mathcal{V}_{\langle N \rangle} = \mathcal{V}_G \).

The following is proved in [Nek05 Proposition 2.13.2].

**Proposition 5.3.** Let \( \Phi \) be a contracting \( G \)-biset. Let \( \rho : G \to \overline{G} \) be the canonical epimorphism onto the faithful quotient of \( G \). If \( \rho(g) \neq 1 \) for every non-trivial element of the nucleus of \( G \) (defined using some right orbit transversal \( X \)), then the kernel of \( \rho \) is equal to the union of the kernels \( K_n \) of the left actions of \( G \) on \( \Phi^{\otimes n} \).

Let \( \Phi \) be a \( G \)-biset, and let \( d \) be the number of orbits of the right action of \( G \) on \( \Phi \). Choose a right orbit transversal \( X \subset \Phi \), and define, for \( g \in G \) and \( x \in X \), the section \( g|_x \) as the unique element of \( G \) such that \( g \cdot x = y \cdot g|_x \) for \( y \in X \). Let \( \pi : G \to G/G' \) be the abelianization epimorphism.

By the same arguments as in Lemma 4.9, the map \( \sigma : \pi(g) \mapsto \sum_{x \in X} \pi(g|_x) \) is a well defined endomorphism of \( G/G' \). It is also checked directly that it does not depend on the choice of the right orbit transversal \( X \). If \( d \) is odd, then define homomorphism sign : \( G/G' \to \mathbb{Z}/2\mathbb{Z} \) as in Theorem 4.8.

The following is a direct corollary of Propositions 5.3, 3.6 and Theorem 4.10.
Corollary 5.4. Let $Φ$ be a $G$-biset satisfying the conditions of Proposition 5.3.

If $d$ is even, then $V_Φ / V_Φ'$ is isomorphic to the quotient of $G / G'$ by the range of the homomorphism $1 - σ$. If $d$ is odd, then $V_Φ / V_Φ'$ is isomorphic to the quotient of $\mathbb{Z} / 2\mathbb{Z} \oplus G / G'$ by the range of the endomorphism $1 - σ_1$, where $σ_1(t, g) = (t + \text{sign}(g), σ(g))$.

Proposition 5.5. Suppose that $f : M_1 → M$ is expanding, $M$ is path-connected and semi-locally simply connected. Then the $π_1(M)$-biset associated with $f$ is contracting.

If $g ∈ π_1(M)$ has trivial image in $\text{IMG}(f)$, then there exists $n$ such that $g$ acts trivially from the left on $Φ_f^n$.

The first paragraph of the proposition is proved in the same way as Proposition 5.4. The second paragraph follows directly from exponential decreasing of diameters of lifts of paths by iterations of $f$ and the condition that $M$ is semi-locally simply connected.

Corollary 5.6. Suppose that $f : M_1 → M$ satisfies the conditions of Proposition 5.4 and let $Φ$ be the $π_1(M)$-biset associated with $f$. Then $V_Φ$ is isomorphic to $V_f = V_{\text{IMG}(f)}$.

Let $f : M_1 → M$ be a partial self-covering satisfying the conditions of Proposition 5.4. Let $ι : M_1 → M$ be the identical embedding.

The group $π_1(M) / π_1(M)'$ is naturally isomorphic to the first homology group $H_1(M)$. The map $σ : H_1(M) → H_1(M)$ from Corollary 5.4 is equal to $ι_σ \circ f'$, where $f' : H_1(M) → H_1(M_1)$ is the map (called the transfer map) given by the condition that image of a chain $c$ is equal to its full preimage $f^{-1}(c)$.

Suppose that $c ∈ H_1(M)$ is defined by a loop $γ$. Then parity of the monodromy action of $γ$ on fibers of $f$ is well defined and generates a homomorphism from $H_1(M)$ to $\mathbb{Z} / 2\mathbb{Z}$. Let us denote it by $\text{sign} : H_1(M) → \mathbb{Z} / 2\mathbb{Z}$.

Then the following description of $V_f / V_f'$ follows directly from Corollary 5.4.

Proposition 5.7. Suppose that $f : M_1 → M$ is expanding, $M$ is path-connected and semi-locally simply connected.

If $\deg f$ is even, then $V_f / V_f'$ is isomorphic to the quotient of $H_1(M)$ by the range of the endomorphism $1 - ι_σ \circ f'$.

If $\deg f$ is odd, then $V_f / V_f'$ is isomorphic to the quotient of $\mathbb{Z} / 2\mathbb{Z} \oplus H_1(M)$ by the range of the endomorphism $1 - σ_1$, where $σ_1(t, c) = (t + \text{sign}(c), σ(1) \circ f'(c))$.

5.4. Example: Hyperbolic rational functions. Let $f$ be a complex rational function, and let $C_f$ be the set of critical points of $f : \hat{C} → \hat{C}$. The post-critical set of $f$ is the union $P_f = \bigcup_{n \geq 1} f^n(C_f)$ of forward orbits of critical values. Suppose that $P_f$ is finite (we say then that $f$ is post-critically finite).

Let us additionally suppose that every cycle of $f : P_f → P_f$ contains a critical point. Then $f$ is hyperbolic, i.e., is expanding on a neighborhood of its Julia set, see [Mil06, Section 19].

One can find disjoint open topological discs around points of $P_f$ such that if $M$ is the complement of the union of these discs in the Riemann sphere, then $M$ contains the Julia set of $f$, $M_1 = f^{-1}(M) ⊂ M$, and there exists a metric on $M$ such that $f : M_1 → M$ is strictly expanding. For instance, one can take discs bounded by the equipotential lines of the basins of attraction, see [Mil06, Section 9].
Proposition 5.8. Let $f$ of even length, which is equal to parity of $|f|$ of local degrees is equal to deg the monodromy action are equal local degrees of $f$ at a small simple loop around $z$ is an odd permutation. Namely, lengths of cycles of the monodromy action are equal local degrees of $f$ in the preimages of $z$. The sum of local degrees is equal to deg $f$, i.e., is odd, hence the number of odd local degrees is odd. Parity of the monodromy action is equal to parity of the number of cycles of even length, which is equal to parity of $|f^{-1}(z)|$ minus the number of odd local degrees, which is equal to parity of $|f^{-1}(z)| + 1$.

**Proposition 5.8.** Let $f$ be a hyperbolic post-critically finite rational function. Let $k$ be the number of attracting cycles of $f$. Let $l$ be the greatest common divisor of their lengths.

If deg $f$ is even, then $V_f/V'_f$ is isomorphic to $\mathbb{Z}^{k-1} \oplus \mathbb{Z}/l\mathbb{Z}$.

If deg $f$ is odd, and there exists an attracting cycle $C$ such that the number of critical values mod 2 whose forward $f$-orbits are attracted to $C$ is odd, then $V_f/V'_f$ is also isomorphic to $\mathbb{Z}^{k-1} \oplus \mathbb{Z}/l\mathbb{Z}$. Otherwise, $V_f/V'_f \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{k-1} \oplus \mathbb{Z}/l\mathbb{Z}$.

Note that $\mathbb{Z}^{k-1} \oplus \mathbb{Z}/l\mathbb{Z}$ coincides with the $K_1$-group of the $C^*$-algebraic analog of $V_f$, see [Nek09]. Its $K_0$-group $\mathbb{Z}/(d - 1)\mathbb{Z}$ has also appeared in our paper, see Proposition 4.6.

**Proof.** Every attracting cycle of $f$ is superattracting (i.e., contains critical points of $f$), hence it belongs to the post-critical set $P_f$.

Suppose at first that deg $f$ is even. We say that $z$ is a critical value mod 2 if $|f^{-1}(z)|$ is even. Note that if $z$ is a critical value mod 2, then it is a critical value, since then $|f^{-1}(z)| \neq \text{deg } f$. In particular, all critical values mod 2 belong to $P_f$. It is also easy to see that $z$ is a critical value mod 2 if and only if the monodromy action of a small simple loop around $z$ is an odd permutation. Namely, lengths of cycles of the monodromy action are equal local degrees of $f$ in the preimages of $z$. The sum of local degrees is equal to deg $f$, i.e., is odd, hence the number of odd local degrees is odd. Parity of the monodromy action is equal to parity of the number of cycles of even length, which is equal to parity of $|f^{-1}(z)|$ minus the number of odd local degrees, which is equal to parity of $|f^{-1}(z)| + 1$.

If $C$ is a cycle of $f : P_f \rightarrow P_f$, then the images of $a_z$ in $V_f/V'_f$, for $z \in C$, satisfy the relations $\pi(a_z) = \pi(a_{f(z)})$, since we have $\sigma(a_{f(z)}) = a_z$. It follows that $V_f/V'_f$ is the quotient of $H_1(M)$ by the relations making elements corresponding to the points of each cycle of $f : P_f \rightarrow P_f$ equal, and making equal to zero all elements corresponding to elements of $P_f$ not belonging to cycles. It follows that $V_f/V'_f$ is the quotient of the free abelian group $\mathbb{Z}^k = \langle e_1, e_2, \ldots, e_k \rangle$ by the relation $l_1e_1 + l_2e_2 + \cdots + l_ke_k = 0$, where $l_i$ are the lengths of the corresponding cycles of $f : P_f \rightarrow P_f$. Consequently, $V_f/V'_f \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/l\mathbb{Z}$, where $l$ is the g.c.d. of $l_1, l_2, \ldots, l_k$.

Suppose now that deg $f$ is odd. Then $V_f/V'_f$ is isomorphic to the quotient of $\mathbb{Z}/2\mathbb{Z} \oplus H_1(M)$ by the relations $\sigma_1(a) = a$, where $\sigma_1(t, g) = (t + \text{sign}(g), \sigma(g))$, where sign($g$) is the parity of the monodromy action of $g$ for the covering map $f : M_1 \rightarrow M$. 

Proof. Every attracting cycle of $f$ is superattracting (i.e., contains critical points of $f$), hence it belongs to the post-critical set $P_f$.
Figure 11. A post-critical cycle

It follows that $\sigma_1$ acts on the elements of the form $(0, a_z)$, for $z \in P_f$, by the rule

$$
\sigma_1(0, a_z) = \begin{cases} 
(1, \sum_{y \in f^{-1}(z) \cap P_f} a_y, a_z) & \text{if } z \text{ is a critical value mod } 2, \\
(0, \sum_{y \in f^{-1}(z) \cap P_f} a_y) & \text{otherwise}.
\end{cases}
$$

Suppose that $z \in P_f$ is such that no point of $\bigcup_{n \geq 0} f^{-n}(z)$ is a critical value mod 2, and $z$ does not belong to a cycle. Then there exists $n$ such that $\sigma^n_1(0, a_z) = 0$, hence image of $(0, a_z)$ in $\mathcal{V}_f/\mathcal{V}_f'$ is isomorphic to $\mathbb{Z}$.

If $z \in P_f$ is a critical value mod 2, but no point of $\bigcup_{n \geq 1} f^{-n}(z)$ is a critical value mod 2, then $\sigma_1(0, a_z) = (1, \sigma(a_z))$, and hence the image of $(0, a_z)$ in $\mathcal{V}_f/\mathcal{V}_f'$ is equal to the image of $(1, 0) \in \mathbb{Z}/2\mathbb{Z} \oplus H_1(\mathcal{M})$.

It follows by induction that if $z \in P_f$ does not belong to a cycle, then the image of $(0, a_z)$ in $\mathcal{V}_f/\mathcal{V}_f'$ is equal to the image of $(m, 0) \in \mathbb{Z}/2\mathbb{Z} \oplus H_1(\mathcal{M})$, where $m$ is the parity of the number of critical values mod 2 in the set $\bigcup_{n \geq 0} f^{-n}(z)$. In particular, $\mathcal{V}_f/\mathcal{V}_f'$ is a quotient of $\mathbb{Z}/2\mathbb{Z} \oplus H$, where $H \leq H_1(\mathcal{M})$ is the subgroup generated by $a_z$ for $z$ belonging to cycles of $P_f$.

Suppose now that $C$ is a cycle of length $r$ of the map $f : P_f \to P_f$. For $z \in C$, denote by $z'$ the unique element of $C$ such that $f(z') = z$, and by $t_z$ the parity of the number of critical values mod 2 in the set $B_z = \{z\} \cup \bigcup_{y \in f^{-1}(z) \cap C} \bigcup_{n \geq 0} f^{-1}(y)$, see Figure 11. Then $\mathcal{V}_f/\mathcal{V}_f'$ is isomorphic to the quotient of $\mathbb{Z}/2\mathbb{Z} \oplus H$ by the relations $(0, a_z) = (t_z, a_z)$.

Note that $t_C = \sum_{z \in C} t_z$ is the number of points $y$ that are critical values mod 2 and $f^n(y) \in C$ for all $n$ big enough. If $t_C$ is odd, then we have a relation $(0, a_z) = (1, a_z)$, which implies that $(1, 0)$ belongs to the kernel of the epimorphism $\mathbb{Z}/2\mathbb{Z} \oplus H_1(\mathcal{M}) \to \mathcal{V}_f/\mathcal{V}_f'$, and the arguments for even deg $f$ show that $\mathcal{V}_f/\mathcal{V}_f' \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$.

Suppose that $t_C$ is even for every cycle $C$ of $P_f$. Order elements of every cycle $C \subset P_f$ into a sequence $z_0, z_1, \ldots, z_{r-1}$ so that $f(z_i) = z_{i-1}$ for all $i = 1, 2, \ldots, r-1$, and $f(z_0) = z_{r-1}$. Denote then $b_{z_0} = (0, a_{z_0}), b_{z_1} = (t_{z_0}, a_{z_1}), b_{z_2} = (t_{z_0} + t_{z_1}, a_{z_2}), \ldots, b_{z_{r-1}} = (t_{z_0} + t_{z_1} + \cdots + t_{z_{r-2}}, a_{z_0})$. Then $\mathcal{V}_f/\mathcal{V}_f'$ is the quotient of $\mathbb{Z}/2\mathbb{Z} \oplus H$ by the relations $b_{z_i} = b_{z_{i+1}}$ and $b_{z_{r-1}} = b_{z_0}$. It follows that $\mathcal{V}_f/\mathcal{V}_f'$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$.

\[ \Box \]
Example 5.3. If $f(z) = z^2 + c$ is a hyperbolic post-critically finite quadratic polynomial, then it has two attracting cycles: $\{\infty\}$ and the orbit of the critical point 0. It follows that $\mathcal{V}_f / \mathcal{V}'_f$ is isomorphic to $\mathbb{Z}$.

Example 5.4. Suppose now that $f$ is a hyperbolic post-critically finite cubic polynomial. If it has only one finite critical point $c$, then $|f^{-1}(f(c))| = 1$, hence there are no critical values mod 2.

If $f$ has two critical points $c_1, c_2$, then $f(c_1)$ and $f(c_2)$ are critical values mod 2, and we have one of the following possibilities:

(a) forward orbits of $c_1$ and $c_2$ are disjoint cycles;
(b) both points $c_1$ and $c_2$ belong to a common cycle;
(c) one of the critical points belongs to a cycle $C$, and the forward orbit of the other critical point eventually belongs to $C$.

It follows now from Proposition 5.8 that $\mathcal{V}_f / \mathcal{V}'_f \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ if the number of finite attracting cycles of $f$ is 1, and $\mathcal{V}_f / \mathcal{V}'_f \cong \mathbb{Z}^2$ if it is 2.

5.5. Finite presentation.

Theorem 5.9. If $G$ is a contracting self-similar group, then the group $\mathcal{V}_G$ is finitely presented.

Corollary 5.10. Let $f : \mathcal{M} \to \mathcal{M}$ be an expanding self-covering of a compact path connected metric space. Then the group $\mathcal{V}_f$ is finitely presented.

Proof. Let $\mathcal{N}$ be the nucleus of $G$. We may assume that $\mathcal{N}$ is a generating set of $G$, since otherwise we can replace $G$ by $\langle \mathcal{N} \rangle$ without changing $\mathcal{V}_G$.

For $v \in X^*$, and $g \in \mathcal{V}_G$, denote by $L_v(g)$ the element of $\mathcal{V}_G$ defined by the rule

$$L_v(g)(w) = \begin{cases} vg(u) & \text{if } w = vu \text{ for some } u \in X^w \\ w & \text{if } w \text{ does not start with } v. \end{cases}$$

The following is straightforward.

Proposition 5.11. For every $v \in X^*$ the map $L_v : \mathcal{V}_G \to \mathcal{V}_G$ is a group monomorphism. If $v, u \in X^*$ are not comparable, then the subgroups $L_v(\mathcal{V}_G)$ and $L_u(\mathcal{V}_G)$ of $\mathcal{V}_G$ commute. If $v, u \in X^*$ are non-empty, and $h \in \mathcal{V}_G$ is such that $h(vw) = uw$ for all $w \in X^w$, then $h \cdot L_v(g) \cdot h^{-1} = L_u(g)$ for all $g \in \mathcal{V}_G$.

Fix a letter $x_1 \in X$. We will denote $L(g) = L_{x_1}(g)$. For every pair $x, y \in X$ choose elements $A_{x,y}$ and $B_x$ of $\mathcal{V}_X$ such that

$$A_{x,y}(yw) = xyw, \quad B_x(x_1w) = wx,$$

for all $w \in X^w$. We assume that $B_{x_1} = 1$.

Let $\langle S \mid R \rangle$ be a finite presentation of the Higman-Thompson group $\mathcal{V}_X$, see [Hig74]. Let $S_1$ be the set of elements of $\mathcal{V}_G$ of the form $L(g)$ for $g \in \mathcal{N}$.

Lemma 5.12. The set $S \cup S_1$ generates $\mathcal{V}_G$.

Proof. For every non-empty $v \in X^*$ we can find an element $h_v \in \mathcal{V}_X$ such that $h_v(vw) = x_1w$ for all $w \in X^w$, see Lemma 5.1. Then $h_v^{-1}L(g)h_v = L_v(g)$ for all $g \in \mathcal{V}_G$. It follows that $L_v(g) \in \langle S \cup S_1 \rangle$ for all $g \in \mathcal{N}$ and $v \in X^* \setminus \{\varnothing\}$. Every element $g \in \mathcal{V}_G$ can be represented by a table

$$\begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ g_1 & g_2 & \ldots & g_n \\ u_1 & u_2 & \ldots & u_n \end{pmatrix},$$

where $v_i, u_i$
are non-empty, and \( g_i \in \mathcal{N} \). But then
\[
g = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_n \end{pmatrix} L_{u_1}(g_1) L_{u_2}(g_2) \cdots L_{u_n}(g_n) \in \langle S \cup S_1 \rangle.
\]
\[
\]
Represent each \( A_{x,y}, B_x \) as group words \( \overline{A}_{x,y}, \overline{B}_x \) in \( S \), and denote, for \( y_1y_2 \cdots y_n \in X^n \) and \( y \in X \),
\[
\overline{A}_{y_1y_2 \cdots y_n,y} = \overline{A}_{y_1,y_2} \cdots \overline{A}_{y_{n-1},y_n} \overline{A}_{y_n,y}.
\]
Let \( A_{v,y} \) be the image of \( \overline{A}_{v,y} \) in \( V_X \). Then \( A_{v,y} \) satisfies
\[
A_{v,y}(yw) = vyw
\]
for all \( w \in X^{\omega} \).

For every word \( v = y_1y_2 \cdots y_n \) of length at least 2 and every \( g \in V_G \) we have
\[
L_v(g) = A_{y_1y_2 \cdots y_{n-1},y_n} B_{y_n} \overline{L}(g) B_{y_n}^{-1} A_{y_1y_2 \cdots y_{n-1},y_n}^{-1}.
\]
For every \( v = y_1y_2 \cdots y_n \in X^* \) and \( g \in \mathcal{N} \) denote by \( \overline{T}_v(g) \) the word
\[
\overline{T}_{y_1y_2 \cdots y_{n-1},y_n} L(g) \overline{T}_{y_1y_2 \cdots y_{n-1}} \overline{T}_{y_1y_2 \cdots y_{n-1},y_n}^{-1}
\]
in generators \( S \cup S_1 \). Also denote by \( \overline{T}_x(g) \) the word \( \overline{B}_x L(g) \overline{B}_x^{-1} \).

Denote by \( \text{Symm}(d^n) \) the subgroup of \( V_X \) consisting of all elements of the form \( \sum_{i=1}^{d^n} T_{v_1,v_2,\ldots,v_{d^n}} \), where \( \{v_1,v_2,\ldots,v_{d^n}\} = \{u_1,u_2,\ldots,u_{d^n}\} = X^n \). It is isomorphic to the symmetric group of degree \( d^n \). Here \( d = |X| \).

For every \( x \in X \) choose a finite generating set \( W_x \) (as a set of group words in \( S \)) of the group \( (V_X)/(X^* \backslash X^+) \) of elements of \( V_X \) acting trivially on \( xX^\omega \). This group is isomorphic to the Higman-Thompson group \( G_{d,d-1} \), hence is finitely generated (see [Hig74]).

Let \( R_1 \) be the union of the following sets of relations.

(C) **Commutation.** Relations of the form
\[
\overline{T}(g_1), \overline{T}(g_2) = [\overline{T}(v_1), \overline{T}(v_2)] = [\overline{L}(g), h] = 1
\]
for all \( g, g_1, g_2 \in \mathcal{N} \), \( x, y \in X \), \( v_1, v_2 \in X^2 \), \( h \in W_{x_1} \), where \( x \neq y \) and \( v_1 \neq v_2 \).

(N) **Nucleus.** Relations of the form
\[
\overline{L}(g_1) \overline{L}(g_2) = 1
\]
for all \( x \in X \) and \( g_1, g_2, g_3 \in \mathcal{N} \) such that \( g_1g_2g_3 = 1 \) in \( G \).

(S) **Splitting.** Relations of the form
\[
\overline{L}(g) = \overline{h} \cdot \overline{T}_{x_1y_1}(g|y_1) \overline{T}_{x_1y_2}(g|y_2) \cdots \overline{T}_{x_1y_d}(g|y_d),
\]
for all \( g \in \mathcal{N} \), where \( \overline{h} \) is a word in the generators \( S \) representing an element \( h \in \text{Symm}(d^2) \) such that \( \overline{L}(g) = h \overline{L}_{x_1y_1}(g|y_1) \overline{L}_{x_1y_2}(g|y_2) \cdots \overline{L}_{x_1y_d}(g|y_d) \).

Let us show that the set \( R \cup R_1 \) is a set of defining relations for the group \( V_G \). Denote by \( V_G \) the group defined by the presentation \( \langle S \cup S_1 \mid R \cup R_1 \rangle \). All relations \( R \cup R_1 \) hold in \( V_G \), hence \( V_G \) is a quotient of \( \hat{V}_G \), and it is enough to show that all relations in \( V_G \) also hold in \( \hat{V}_G \).
Note that since $R$ is a set of defining relations of $\mathcal{V}_X$, a group word in $S$ is trivial in $\mathcal{V}_X$ if and only if it is trivial in $\mathcal{V}_X$. We will identify, therefore, the elements of the subgroup $\langle S \rangle \leq \mathcal{V}_G$ with the corresponding elements of $\mathcal{V}_X$.

**Lemma 5.13.** Suppose that $h \in \mathcal{V}_X$ and $u,v \in X^*$ are such that $h(uv) = vw$ for all $w \in X^\omega$. Then $h \mathcal{T}_u(g) h^{-1} = \mathcal{T}_v(g)$ holds in $\mathcal{V}_G$.

**Proof.** Let $u = a_1 a_2 \ldots a_n$ and $v = b_1 b_2 \ldots b_m$ for $a_i, b_i \in X$. Then

$$\mathcal{T}_u(g) = A_{a_1 a_2 \ldots a_n} B_{a_n} L(g) B_{a_n}^{-1} A_{a_1 a_2 \ldots a_n}^{-1}$$

and

$$\mathcal{T}_v(g) = A_{b_1 b_2 \ldots b_m} B_{b_m} L(g) B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1}$$

by definition. We have dropped the lines above the letters $A$ and $B$, because the corresponding elements belong to $\mathcal{V}_X$.

We have then

$$h \mathcal{T}_u(g) h^{-1} = h A_{a_1 a_2 \ldots a_n} B_{a_n} L(g) B_{a_n}^{-1} A_{a_1 a_2 \ldots a_n}^{-1} h$$

$$= A_{b_1 b_2 \ldots b_m} B_{b_m}^{-1} B_{b_m} \cdot B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1} h A_{b_1 b_2 \ldots b_m} B_{b_m} \cdot B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1} L(g) \cdot$$

$$B_{a_n}^{-1} A_{a_1 a_2 \ldots a_n}^{-1} h A_{b_1 b_2 \ldots b_m} B_{b_m} \cdot B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1} B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1} h A_{b_1 b_2 \ldots b_m} B_{b_m} \cdot B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1}$$

The element

$$f = B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1} h A_{b_1 b_2 \ldots b_m} B_{b_m} \cdot B_{b_m}^{-1} A_{b_1 b_2 \ldots b_m}^{-1}$$

satisfies

$$f(x_1 w) = A_{v \cdot y_m}^{-1} h A_{u \cdot x_n} a \cdot y_m \cdot x_n (y_m w) = x_1 w,$$

for all $w \in X^\omega$. Hence, by relations (C), $L(g)$ commutes with $f$, i.e., $f \cdot L(g) \cdot f^{-1} = L(g)$ in $\mathcal{V}_G$, which finishes the proof.

**Lemma 5.14.** If $v, u \in X^*$ are incomparable, then $\mathcal{T}_v(g_1)$ and $\mathcal{T}_u(g_2)$ commute in $\mathcal{V}_G$ for all $g_1, g_2 \in \mathcal{N}$.

**Proof.** Let $x, y \in X$ be a pair of different letters. Then $[\mathcal{T}_{xx}(g_1), \mathcal{T}_{xy}(g_2)] = 1$ in $\mathcal{V}_G$, by (C). Since $v, u$ are incomparable, either they both have length 1, or they form an incomplete antichain. In the first case commutation of $\mathcal{T}_v(g_1)$ and $\mathcal{T}_u(g_2)$ is a part of relations (C). In the second case, there exists $a \in \mathcal{V}_X$ such that $a(uw) = xuw$ and $a(vw) = xwy$ for all $w \in X^\omega$ (see Lemma 3.1). Then, by Lemma 5.13

$$[\mathcal{T}_v(g_1), \mathcal{T}_u(g_2)] = a[\mathcal{T}_{xx}(g_1), \mathcal{T}_{xy}(g_2)] a^{-1} = 1$$

in $\mathcal{V}_G$.

Let us prove now that any group word in $S \cup S_1$ that is trivial in $\mathcal{V}_G$ is trivial in $\mathcal{V}_d$. Note that relations (S) and Lemma 5.13 imply relations (S')

$$\mathcal{T}_v(g) = h \cdot \mathcal{T}_{vy_1}(g|y_1) \mathcal{T}_{vy_2}(g|y_2) \cdots \mathcal{T}_{vy_d}(g|y_d)$$

for all $g \in \mathcal{N}$ and non-empty $v \in X^*$, where $h$ is an element of $\mathcal{V}_d$ such that $L_v(g) = h L_{vy_1}(g|y_1) L_{vy_2}(g|y_2) \cdots L_{vy_d}(g|y_d)$. 
Every element of $\hat{\mathcal{V}}_G$ can be written in the form $h L(g_1)^{h_1} \cdots L(g_n)^{h_n}$ for $h, h_1 \in \mathcal{V}_X$, and $g_i \in \mathcal{N}$.

Let $n_1$ be such that the element $h_n$ can be written as $\sum_{i=1}^{d^{n_1}} S_{u_i} S_{v_i}$, where $\{u_1, u_2, \ldots, u^{d^{n_1}}\} = X^{n_1}$. Using relations $(S')$ and Lemma 5.13 we can rewrite the element $L(g_i)$ in the form $\alpha \prod_{i \in X^{n_1-1}} T_{x_{i,v}}(g_{n \mid v})$ for $\alpha \in \Symm(d^{n_1})$. (Note that the factors $T_{x_{i,v}}(g_{n \mid v})$ commute with each other, by Lemma 5.14) Then for every $v \in X^{n_1-1}$ there exists $i$ such that $x_i \mid v = u_i$, and then by Lemma 5.13 we have

$$T_{x_{i,v}}(g_{n \mid v})^{h_n} = T_{v_l}(g_{n \mid v}),$$

so that $L(g_n)^{h_n}$ can be rewritten as a product of $\alpha^{h_n}$ followed by a product of elements of the form $T_{v_l}(g_{v,n})$ for some $v \in X^{n_1}$ and $g_{v,n} \in \mathcal{N}$.

It follows by induction that every element of $\hat{\mathcal{V}}_G$ can be written in the form

$$(12) \quad g = h T_{v_1}(g_1) \cdots T_{v_m}(g_m)$$

for some $v_1 \in X^*$, $g_i \in \mathcal{N}$, and $h \in \mathcal{V}_X$.

Suppose that not all words $v_i$ are of the same length. Let $v_i$ be the shortest, and let $k > |v_i|$ be the shortest length of words $v_j$ strictly longer than $v_i$. Using relations $(S')$, we can rewrite $T_{v_i}(g_i)$ as $\alpha_i \prod_{k \in X^{k-|v_i|}} L_{v_i}(g_i(u))$, and then, using Lemma 5.13 move $\alpha_i \in \Symm(d^k)$ to the beginning of the product $(12)$. This procedure will increase the length of the shortest word $v_i$ in $(12)$ without changing the length of the longest one. Repeating this procedure, we will change $(12)$ to a product of the same form, but in which all words $v_i$ are of the same length.

Therefore, we may assume that in $(12)$ all words $v_i$ are of the same length $k$. Note that then $T_{v_1}(g_1) \cdots T_{v_m}(g_m)$ does not change the beginning of length $k$ in any word $w \in X^\omega$. Since $g$ is trivial in $\mathcal{V}_G$, this implies that $h$ does not change the beginning of length $k$ in any word. It follows that we can write $h$ as a product $\prod_{v \in X^k} L_v(h_v)$ for some $h_v \in \mathcal{V}_X$. Using Lemmas 5.13 and 5.14 we can now rearrange the factors of $(12)$ in such a way that $g = \prod_{v \in X^k} f_v$, where $f_v = L_v(h_v) T_{v_1}(g_{v,1}) \cdots T_{v_m}(g_{v,m})$ for $g_{v,1} \in \mathcal{N}$ and $h_v \in \mathcal{V}_X$. Note that $f_v$ are trivial in $\mathcal{V}_G$. The latter implies that $h_v \in \Symm(d^k)$ for some $l$, and that the action of $h_{v_1} g_{v_2} \cdots g_{v,m}$ on $X^l$ is trivial. Consequently, using relations $(S')$, we can rewrite $f_v$ as a product of elements of the form $T_{v,u}(g)$ for $u \in X^l$. Therefore, we may assume that $h_v$ are trivial. Then $g_{v,1} g_{v,2} \cdots g_{v,m}$ is trivial in $G$. Relations (N), (S), and Propositions 5.2, 5.3 finish the proof. \hfill \Box

6. DYNAMICAL SYSTEMS AND GROUPOIDS

This section is an overview of relations between expanding dynamical systems and self-similar groups, basic definitions of the theory of étale groupoids, and properties of hyperbolic groupoids. For more details and proofs, see [Nek05, Nek08a, Nek08b, Ren00a, Pat99, Hae01, Nek11a].

6.1. Limit dynamical system of a contracting group. Let $(G, X)$ be a contracting self-similar group. Denote by $X^{-\omega}$ the space of all left-infinite sequences $\ldots x_2 x_1$ of elements of $X$ with the direct product topology.

**Definition 6.1.** Sequences $\ldots x_2 x_1, y_2 y_1 \in X^{-\omega}$ are asymptotically equivalent if there exists a finite set $N \subset G$ and a sequence $g_n \in N$ such that

$$g_n(x_n \ldots x_2 x_1) = y_n \ldots y_2 y_1,$$
for all $n \in \mathbb{N}$.

Denote by $J_G$ the quotient of the space $X^\omega$ by the asymptotic equivalence relation.

The asymptotic equivalence relation on $X^\omega$ is invariant with respect to the shift $x_2x_1 \mapsto \ldots x_3x_2$. Therefore, the shift induces a continuous map $f : J_G \to J_G$. The dynamical system $(f, J_G)$ is called the limit dynamical system of $G$.

The map $f : J_G \to J_G$ is expanding in the sense of Definition 5.2 (even though it is not a covering in general). Namely, we can represent $J_G$ as the boundary of a naturally defined Gromov hyperbolic graph (see [Nek03] and [Nek05, Section 3.8]), and some iteration of $f$ will be locally uniformly expanding with respect to the visual metric on the boundary.

**Definition 6.2.** A self-similar group $(G, X)$ is said to be regular if for every $g \in G$ there exists a positive integer $n$ such that for every $v \in X^n$ either $g(v) \neq v$, or $g|_v = 1$.

Note that it is enough to check the conditions of Definition 6.2 for elements $g$ of the nucleus of $G$.

The following proposition is proved in [Nek09, Proposition 6.1].

**Proposition 6.1.** The shift map $f : J_G \to J_G$ is a covering if and only if $G$ is regular.

**Definition 6.3.** A self-similar action of $G$ on $X^\ast$ is said to be self-replicating (recurrent in [Nek05]) if the left action of $G$ on the associated biset is transitive, i.e., if for every $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y \cdot 1$.

An automorphism group $G$ of the rooted tree $X^\ast$ is said to be level-transitive if it is transitive on $X^n$ for every $n$.

Note that every self-replicating group is level transitive.

**Theorem 6.2.** Let $G$ be a contracting group. The space $J_G$ is connected if and only if $G$ is level-transitive. It is path connected if and only if $G$ is self-replicating. If $G$ is self-replicating, then $J_G$ is also locally path connected.

**Proof.** Proof of connectedness, local connectedness, and path connectedness of $J_G$ under the appropriate conditions is given in [Nek05, Theorem 3.6.3.].

If $x_2x_1 \mapsto \ldots y_2y_1$ are asymptotically equivalent elements of $X^\omega$, then for any $n$ the words $x_n \ldots x_2x_1$ and $y_n \ldots y_2y_1$ belong to the same $G$-orbit. If there exists $n$ such that the action of $G$ on $X^n$ is not transitive, then partition of $X^n$ into $G$-orbits defines a partition of $X^\omega$ into clopen sets such that the asymptotic equivalence relation identifies only points inside the sets of the partition. This implies that $J_G$ is disconnected.

The same arguments shows that if $G$ is not self-replicating, then $X_G$ is disconnected. Moreover, if $v_1, v_2 \in X^n \cdot G$ belong to different orbits of the left action, then for any $k, m \geq 0$, and any $u_1, u_2 \in X^k \cdot G$ and $w_1, w_2 \in X^m \cdot G$ the elements $u_1 \otimes v_1 \otimes w_1$ and $u_2 \otimes v_2 \otimes w_2$ belong to different orbits of the left action. It follows that the set of connected components of $X_G$ is then uncountable. Consequently, the set of path connected components of $X_G$ is uncountable, and since $G$ is countable, the set of path connected components of $J_G = X_G/G$ is also uncountable. □
The following theorem is proved in [Nek05, Sections 5.3, 5.5] (in the context of length metric spaces, but all the arguments remain to be valid in the general case, if we use diameters of paths instead of their lengths, as in the proof of Proposition 5.1.1).

**Theorem 6.3.** Suppose that \( f : \mathcal{J} \rightarrow \mathcal{J} \) is an expanding self-covering of a path connected space. Then \( \text{IMG} (f) \) is contracting, regular, self-replicating, and the limit dynamical system of \( \text{IMG} (f) \) is topologically conjugate to \((f, \mathcal{J})\).

Let \( G \) be a contracting regular self-replicating group. Then it is equivalent, as a self-similar group (see Definition 3.4), to the iterated monodromy group of its limit dynamical system.

**Corollary 6.4.** Let \( f_i : \mathcal{J}_i \rightarrow \mathcal{J}_i \), for \( i = 1, 2 \), be expanding self-coverings of path connected compact spaces. Then \((f_1, \mathcal{J}_1)\) and \((f_2, \mathcal{J}_2)\) are topologically conjugate if and only if \( \text{IMG} (f_1) \) and \( \text{IMG} (f_2) \) are equivalent as self-similar groups.

### 6.2. Limit solenoid and the limit \( G \)-space.

Let \( X^\mathbb{Z} \) be the space of all bi-infinite sequences \( \ldots x_{-2}x_{-1}x_0x_1 \ldots \), where the dot denotes the place between the coordinates number 0 and -1. Sequences \( (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}} \in X^\mathbb{Z} \) are asymptotically equivalent if there exists a finite set \( N \subset G \) and a sequence \( g_n \in N \) such that

\[
g_n(x_nx_{n+1} \ldots) = y_ny_{n+1} \ldots
\]

for all \( n \in \mathbb{Z} \).

The quotient \( S_G \) of \( X^\mathbb{Z} \) by the asymptotic equivalence relation is called the limit solenoid of the group \( G \). The shift \( \ldots x_{-2}x_{-1}x_0x_1 \ldots \mapsto \ldots x_{-3}x_{-2}x_{-1}x_0 \ldots \) induces a homeomorphism of \( S_G \), which we will denote by \( \hat{f} \).

It is shown in [Nek05, Proposition 5.7.8.] that the space \( S_G \) is naturally homeomorphic to the inverse limit of the backward iterations of the limit dynamical system \( f : \mathcal{J}_G \rightarrow \mathcal{J}_G \):

\[
\mathcal{J}_G \leftarrow \mathcal{J}_G \leftarrow \mathcal{J}_G \leftarrow \cdots,
\]

and the map \( \hat{f} \) is conjugate to the map induced by \( f \) on the inverse limit. In other words, \((\hat{f}, S_G)\) is the natural extension of the limit dynamical system \((f, \mathcal{J}_G)\). The point of \( S_G \) represented by a sequence \( \ldots x_{-2}x_{-1}x_0x_1 \ldots \in X^\mathbb{Z} \) corresponds to the point of the inverse limit represented by the sequence

\[
\ldots \mapsto \ldots x_{-2}x_{-1}x_0x_1 \mapsto \ldots x_{-2}x_{-1}x_0 \mapsto \ldots x_{-2}x_{-1} \mapsto \ldots x_{-2}x_1.
\]

Another natural dynamical system associated with a contracting group \( G \) is the limit \( G \)-space \( X_G \). Consider the topological space \( X^{-\omega} \times G \), where \( G \) is discrete. Two pairs \((\ldots x_2x_1, g), (\ldots y_2y_1, h) \in X^{-\omega} \times G \) are asymptotically equivalent if there exists a sequence \( g_n \in G \) taking a finite set of values such that for all \( n \geq 1 \)

\[
g_n \cdot x_n \ldots x_2x_1 \cdot g = y_n \ldots y_2y_1 \cdot h
\]

in the \( n \)th tensor power \( \Phi^\otimes n \) of the associated \( G \)-biset, i.e., if

\[
g_n(x_n \ldots x_2x_1) = y_n \ldots y_2y_1, \quad g_n|_{x_n \ldots x_2x_1} g = h.
\]

The quotient of \( X^{-\omega} \times G \) by the asymptotic equivalence relation is called the limit \( G \)-space \( X_G \). We represent the points of the space \( X_G \) by the sequences \( \ldots x_2x_1 \cdot g \), where \( x_i \in X \) and \( g \in G \).

The asymptotic equivalence relation on \( X^{-\omega} \times G \) is invariant with respect to the right action of \( G \) on the second factor of the direct product. It follows that this
action induces a right action of $G$ on $X_G$ by homeomorphisms. The action of $G$ on $X_G$ is proper, and the space of orbits $X_G/G$ is naturally homeomorphic to $\mathcal{J}_G$.

The spaces $\mathcal{J}_G, S_G, X_G$ and the corresponding dynamical systems depend only on the biset $\Phi$ associated with the self-similar group. For example, $X_G$ can be constructed in the following way.

Let $\Omega$ be the direct limit of the spaces $A^\omega$, where $A$ runs through all finite subsets of $\Phi$. We write a sequence $(\ldots, a_2, a_1) \in A^\omega$ as $\ldots \otimes a_2 \otimes a_1$. Two sequences $\ldots \otimes a_2 \otimes a_1, \ldots \otimes b_2 \otimes b_1 \in \Omega$ are said to be equivalent if there exist a sequence $g_n \in G$ taking values in a finite set, such that

$$g_n \cdot a_n \otimes \cdots \otimes a_2 \otimes a_1 = b_n \otimes \cdots \otimes b_2 \otimes b_1$$

in $\Phi^\otimes n$ for all $n$.

The quotient of $\Omega$ by this equivalence relation is naturally homeomorphic to $X_G$. Moreover, the homeomorphism conjugates the natural action on $X_G$ with the action induced by the natural right action of $G$ on $\Omega$.

For every $v \cdot g \in X^n \cdot G = \Phi^{\otimes |v|}$ we have the map $w \mapsto w \otimes (v \cdot g)$ on $\Omega$, given in terms of $X^{-\omega} \times G$ by the rule

$$\ldots x_2 x_1 \cdot h \mapsto \ldots x_2 x_1 h(v) : h(v).$$

It induces a continuous map $F_{v,g} : X_G \rightarrow X_G$. If $G$ is regular, then $F_{v,g}$ is a covering map.

Since the limit dynamical system $f : \mathcal{J}_G \rightarrow \mathcal{J}_G$ is induced by the shift on $X^{-\omega}$, the maps $F_{v,g} : X_G \rightarrow X_G$ are lifts of $f^{||v||}$ by the quotient map $P : X_G \rightarrow X_G/G = \mathcal{J}_G$, i.e., we have equality $f^{||v||} \circ P \circ F_{v,g} = P$.

6.3. Groupoids of germs.

**Definition 6.4.** Let $\mathcal{X}$ be a topological space. A pseudogroup acting on $\mathcal{X}$ is a set $\bar{\mathcal{G}}$ of homeomorphisms between open subset of $\mathcal{X}$ that is closed under

1. **compositions:** if $F_1 : U_1 \rightarrow V_1$ and $F_2 : U_2 \rightarrow V_2$ are elements of $\bar{\mathcal{G}}$, then $F_1 \circ F_2 : F_2^{-1}(V_2 \cap U_1) \rightarrow F_1(V_2 \cap U_1)$ is an element of $\bar{\mathcal{G}}$;
2. **taking inverse:** if $F : U \rightarrow V$ is an element of $\bar{\mathcal{G}}$, then $F^{-1} : V \rightarrow U$ is an element of $\bar{\mathcal{G}}$;
3. **restriction** to an open subset: if $F : V \rightarrow U$ is an element of $\bar{\mathcal{G}}$ and $V'$ is an open subset of $V$, then $F|_{V'} \in \bar{\mathcal{G}}$;
4. **unions:** if for a homeomorphism $F : U \rightarrow V$ between open subsets of $\mathcal{X}$ there exists a covering $U_i$ of $U$ by open subsets, such that $F|_{U_i} \in \bar{\mathcal{G}}$ for all $i$, then $F \in \bar{\mathcal{G}}$.

We always assume that the identical homeomorphism $\mathcal{X} \rightarrow \mathcal{X}$ belongs to the pseudogroup.

Let $\bar{\mathcal{G}}$ be a pseudogroup acting on $\mathcal{X}$. A germ of $\bar{\mathcal{G}}$ is equivalence class of a pair $(F, x)$, where $F \in \bar{\mathcal{G}}$, and $x$ is a point of the domain of $F$. Two pairs $(F_1, x_1)$ and $(F_2, x_2)$ represent the same germ (are equivalent) if and only if $x_1 = x_2$ and there exists a neighborhood $U$ of $x_1$ such that $F_1|_{U} = F_2|_{U}$.

The set of all germs of $\bar{\mathcal{G}}$ has a natural topology whose basis consists of sets of the form \{$(F, x) : x \in \text{dom}(F)$\}, where $F \in \bar{\mathcal{G}}$. 


If \((F_1, x_1)\) and \((F_2, x_2)\) are such germs that \(F_2(x_2) = x_1\), then we can compose them:

\[(F_1, x_1)(F_2, x_2) = (F_1 \circ F_2, x_2).\]

**Inverse** of a germ \((F, x)\) is the germ \((F^{-1}, F(x))\). The set \(\mathfrak{G}\) of all germs of \(\tilde{\mathfrak{G}}\) is a groupoid with respect to these operations (i.e., a small category of isomorphisms).

The groupoid \(\mathfrak{G}\) is **topological**, i.e., the operations of composition and taking inverse are continuous.

**Example 6.1.** If \(f : X \to X\) is a covering map, then restrictions of \(f\) to open subsets \(U \subset X\) such that \(f : U \to f(U)\) is a homeomorphism generate a pseudogroup. Its groupoid of germs \(\mathfrak{F}\) will be called **pseudogroup generated by** \(f\). Every element of \(\mathfrak{F}\) can be represented as a product \((f^n, x)^{-1}(f^m, y)\) for some \(x, y \in X\) such that \(f^n(y) = f^m(x)\).

**Example 6.2.** If \(G\) is a group acting on a topological space \(X\), then every germ of the pseudogroup generated by \(G\) is a germ of an element of \(G\). Therefore, the **groupoid of germs of** \(G\) is the set of equivalence classes of pairs \((g, x) \in G \times X\), where \((g_1, x_1)\) and \((g_2, x_2)\) are equivalent if and only if \(x_1 = x_2\), and \(g_1^{-1}g_2\) fixes all points of a neighborhood of \(x_1\). This groupoid is in general different from the groupoid of the action, which is equal as a set to \(G \times X\).

If \(\mathfrak{G}\) is a groupoid of germs of a pseudogroup acting on a space \(X\), then we identify the germ \((1, x)\) of the identical homeomorphism \(1 : X \to X\) with the point \(x\) of \(X\), and call elements of the form \((1, x)\) the **units** of the groupoid. We will sometimes denote \(X\) by \(\mathfrak{G}^{(0)}\), as the space of units of \(\mathfrak{G}\).

For \((F, x) \in \mathfrak{G}\), we denote by \(o(F, x) = x\) and \(t(F, x) = F(x)\) the **origin** and **target** of the germ. Two germs \(g_1, g_2 \in \mathfrak{G}\) are **composable** (i.e., \(g_1g_2\) is defined) if and only if \(t(g_2) = o(g_1)\).

We say that points \(x, y \in \mathfrak{G}^{(0)}\) **belong to one orbit** if there exists \(g \in \mathfrak{G}\) such that \(x = o(g)\) and \(y = t(g)\). This is an equivalence relation on \(\mathfrak{G}^{(0)}\), \(X\), and this notion of orbits coincides with the natural notion of orbits of pseudogroups. A set \(A \subset \mathfrak{G}^{(0)}\) is a \(\mathfrak{G}\)-**transversal** if it intersects every \(\mathfrak{G}\)-orbit.

If \(A\) is a subset of \(\mathfrak{G}^{(0)}\), then **restriction** \(\mathfrak{G}|_A\) of \(\mathfrak{G}\) to \(A\) is the groupoid of all elements \(g \in \mathfrak{G}\) such that \(o(g), t(g) \in A\). The **isotropy** group \(\mathfrak{G}_x\), for \(x \in \mathfrak{G}^{(0)}\), is the group of elements \(g \in \mathfrak{G}\) such that \(o(g) = t(g) = x\).

Note that the pseudogroup \(\mathfrak{G}\) can be reconstructed from the groupoid of its germs \(\mathfrak{G}\). Namely, a **bisection** is a subset \(F \subset \mathfrak{G}\) of the groupoid, such that \(o : F \to o(F)\) and \(t : F \to t(F)\) are homeomorphisms. Every open bisection \(F\) defines a homeomorphism \(o(F) \to t(F)\) by the rule \(o(g) \mapsto t(g)\) for \(g \in F\). The set \(\mathfrak{G}\) of all open bisections is a pseudogroup, and if \(\mathfrak{G}\) is the groupoid of germs of a pseudogroup, then the pseudogroup of bisections coincides with \(\mathfrak{G}\). We say that \(\mathfrak{G}\) is the **associated pseudogroup** of the groupoid.

**Definition 6.5.** Let \(\mathfrak{G}_1, \mathfrak{G}_2\) be groupoids of germs. We say that they are **equivalent** if there exists a groupoid \(\mathfrak{G}\) such that \(\mathfrak{G}^{(0)}\) is the disjoint union \(\mathfrak{G}_1^{(0)} \sqcup \mathfrak{G}_2^{(0)}\), restrictions of \(\mathfrak{G}\) to \(\mathfrak{G}_i^{(0)}\) is equal to \(\mathfrak{G}_i\) for every \(i = 1, 2\), and the sets \(\mathfrak{G}_i^{(0)}\) are \(\mathfrak{G}\)-transversals.

The following procedure is a standard way of constructing a groupoid equivalent to a given one. Namely, let \(p : Y \to \mathfrak{G}^{(0)}\) be a local homeomorphism, i.e., for
every $y \in Y$ there exists a neighborhood $U$ of $y$ such that $p : U \rightarrow p(U)$ is a homeomorphism. Suppose that $p(Y)$ is a $\mathcal{G}$-transversal. Then lift of $\mathcal{G}$ by $p$ is the groupoid of germs of the pseudogroup generated by all local homeomorphisms of the form $p' \circ F \circ p : U \rightarrow W$, where

- $U$ is such that $p : U \rightarrow p(U)$ is a homeomorphism,
- $p(U)$ is contained in the domain of $F$,
- $W$ is such that $p : W \rightarrow F(p(U))$ is a homeomorphism,
- $p'$ is the inverse of $p : W \rightarrow F(p(U))$.

Then the map $p$ induces a morphism from the lift of $\mathcal{G}$ to $\mathcal{G}$, mapping the germ of $p' \circ F \circ p$ at $x$ to the germ of $F$ at $p(x)$.

**Example 6.3.** Consider the trivial groupoid on a manifold $M$, i.e., the groupoid consisting of units only. Let $\pi : \tilde{M} \rightarrow M$ be the universal covering. Then lift of the trivial groupoid by $\pi$ is the groupoid of germs of the action of the fundamental group on $\tilde{M}$. (In this case it coincides with the groupoid of the action.)

**Definition 6.6.** Let $\mathcal{G}$ be a groupoid of germs. It is said to be proper if the map $(o, t) : \mathcal{G} \rightarrow \mathcal{G}(0) \times \mathcal{G}(0)$ is proper, i.e., if preimages of compact subsets of $\mathcal{G}(0) \times \mathcal{G}(0)$ under this map are compact.

The groupoid $\mathcal{G}$ is proper if and only if for every compact subset $C$ of $\mathcal{G}(0)$ the set of elements $g \in \mathcal{G}$ such that $o(g), t(g) \in C$ is compact.

If $\mathcal{G}$ is proper, then for every $x \in \mathcal{G}(0)$ the isotropy group $\mathcal{G}_x$ is finite.

Every groupoid equivalent to a proper groupoid is proper. If $\mathcal{G}$ is proper, then the space of orbits of $\mathcal{G}$ is Hausdorff.

Let $\mathcal{G}$ be a groupoid of germs. Its topological full group $[[\mathcal{G}]]$ is the set of all bisections $F$ such that $o(F) = t(F) = \mathcal{G}(0)$, i.e., the set of homeomorphisms $F : \mathcal{G}(0) \rightarrow \mathcal{G}(0)$ such that all germs of $F$ belong to $\mathcal{G}$. See [GPS99], where the notion of a topological full group (for a groupoid of germs generated by one homeomorphism) was introduced.

**Example 6.4.** Let $f : M_1 \rightarrow M$ be a partial self-covering. Then $\mathcal{V}_f$ is the full topological group of the groupoid of germs of the local homeomorphisms $S_t$, of the boundary of the tree $T_t$ for $t \in M$.

**Example 6.5.** Let $G$ be a self-similar group acting on $X^\omega$. Let $\mathcal{G}$ be the groupoid of germs of the pseudogroup generated by the action of $G$ and the germs of the homeomorphisms $S_t(x_1, x_2, \ldots) = xx_1x_2\ldots$ for $x \in X$. It is easy to see that the topological full group of $\mathcal{G}$ is the group $\mathcal{V}_G$.

### 6.4. Hyperbolic groupoids.

Here we present a very short overview of the basic definitions and results of the paper [Nek11a].

Let $\mathcal{G}$ be a groupoid of germs. A compact generating pair of $\mathcal{G}$ is a pair $(S, X_1)$, where $S \subset \mathcal{G}$ and $X_1 \subset \mathcal{G}(0)$ are compact, $X_1$ contains an open $\mathcal{G}$-transversal, $S \subset \mathcal{G}|_{X_1}$, and for every $g \in \mathcal{G}|_{X_1}$ there exists $n$ such that $(S \cup S^{-1})^n$ is a neighborhood of $g$ in $\mathcal{G}|_{X_1}$.

A groupoid is compactly generated if it has a compact generating pair. See a variant of this definition in [Hae02]. A groupoid equivalent to a compactly generated groupoid is also compactly generated.

Let $(S, X_1)$ be a compact generating pair of $\mathcal{G}$. Let $x \in X_1$. Then the Cayley graph $\Gamma(x, S)$ is the oriented graph with the set of vertices

$$\{g \in \mathcal{G} : o(g) = x, t(g) \in X_1\},$$
Then, for all \( n \) \( \leq 0 \) the spaces agree where they overlap, and locally coincide with the maps given such that the map \( \partial U \) of \( \partial F \) by \( \partial \) pair \((g,s)\) such that \( g \in F \) is equal to a germ \((x,S)\) and a sequence \( (x_1, x_2, \ldots, x_n) \) is said to be a \( C \)-quasi-geodesic where \( C > 1 \) is a constant if \( \sum_{i=1}^{n} |x_i - x_{i+1}| \geq C^{-1} |i - j| + C \) for all \( i, j \).

**Definition 6.7.** A Hausdorff groupoid of germs \( \mathfrak{G} \) is hyperbolic if there is a compact generating pair \((S, \mathcal{X}_1)\) of \( \mathfrak{G} \), a metric \( |x - y| \) defined on a neighborhood of \( \mathcal{X}_1 \), and constants \( L, C > 1, \Delta > 0 \) such that

1. Every element \( g \in S \) is a germ of a homeomorphism \( F \in \mathfrak{G} \) such that \( |F(x) - F(y)| \leq L^{-1} |x - y| \) for all \( x, y \in \text{Dom} F \).
2. For every \( x \in \mathcal{X}_1 \) the Cayley graph \( \Gamma(x, S) \) is Gromov \( \Delta \)-hyperbolic.
3. For every \( x \in \mathcal{X}_1 \) there exists a point \( \omega_x \) of the boundary of \( \Gamma(x, S) \) such that every oriented path in the Cayley graph \( \Gamma(x, S^{-1}) \) is a \( C \)-quasi-geodesic converging to \( \omega_x \).
4. \( \partial(S) = t(S) = \mathcal{X}_1 \).
5. All elements of the pseudogroup \( \mathfrak{G} \).

**Example 6.6.** Let \( f : J \rightarrow J \) be an expanding self-covering of a compact metric space. Then the groupoid of germs generated by \( f \) is hyperbolic. The corresponding generating pair is \((S, J)\), where \( S \) is the set of germs of \( f^{-1} \). The corresponding Cayley graphs \( \Gamma(x, S) \) are trees. The special point \( \omega_x \) of the boundary is the limit of the forward germs \( (f^n, x) \) for \( n \rightarrow +\infty \). See more in [Nek11a, Section 5.2]

Let \( \mathfrak{G} \) be a hyperbolic groupoid. For every \( x \in \mathfrak{G}^{(0)} \) there exists a generating pair \((S, \mathcal{X}_1)\) satisfying the conditions of Definition 6.7 and such that \( x \in \mathcal{X}_1 \). Denote by \( \partial \mathfrak{G}_x \) the boundary of the Cayley graph \( \Gamma(x, S) \) minus the point \( \omega_x \). The space \( \partial \mathfrak{G}_x \) does not depend on the generating pair.

Let \((S, \mathcal{X}_1)\) be a generating pair satisfying the conditions of Definition 6.7. Find a finite set of contractions \( \mathcal{F} \subseteq \mathfrak{G} \) such that \( S \subseteq \bigcup_{F \in \mathcal{F}} F \), i.e., every element \( s \in S \) is a germ of a contraction \( F \in \mathcal{F} \). Every point \( \xi \in \partial \mathfrak{G}_x \) can be represented as the limit of a sequence vertices of the Cayley graph of the form \( g, s_1 g, s_2 g, \ldots, s_n g \), where \( g \in \mathfrak{G} \) and \( s_i \in S \). There exists \( \epsilon > 0 \) (not depending on \( \xi \)) and a sequence \( F_i \in \mathcal{F} \) such that \( s_i \) is equal to a germ \((F_i, x_i)\), and the \( \epsilon \)-neighborhood of \( x_i = \text{Dom} F_i \) belongs to the domain of \( F_i \). Then there exists \( \delta \) (also depending only on \( S \) and \( \mathcal{F} \)) such that the \( \delta \)-neighborhood of \( t(g) \) belongs to the domain of \( F_n \circ \cdots \circ F_2 \circ F_1 \) for all \( n \).

Let \( F \in \mathfrak{G} \) be such that \( g \in F \) belongs to the \( \delta \)-neighborhood of \( t(g) \). Then \( \partial(F_n \circ \cdots \circ F_2 \circ F_1 \circ F) = \partial(F) \) for every \( n \).

It is shown in [Nek11a] that there exists a topology on the disjoint union \( \partial \mathfrak{G} \) of the spaces \( \partial \mathfrak{G}_x, x \in \mathfrak{G}^{(0)} \), which agrees with the topology on its subsets \( \partial \mathfrak{G}_x \) and such that the map

\[
(y, \xi) \mapsto \lim_{n \rightarrow \infty} (F_n \circ F_{n-1} \circ \cdots \circ F_1 \circ F, y) \in \partial \mathfrak{G}_y
\]

is a well-defined homeomorphism (if \( U = \text{Dom} F \) is small enough) from the direct product of \( U = \text{Dom} F \) with a subset of \( \partial \mathfrak{G}_x \) to a subset of \( \partial \mathfrak{G} \), see Figure 12.

Moreover, these homeomorphisms agree with a natural local product structure of \( \partial \mathfrak{G} \). Namely, a basis of the topology on \( \partial \mathfrak{G} \) consists of rectangles, i.e., sets with a decomposition into a direct product of topological spaces, such that the decompositions agree where they overlap, and locally coincide with the maps given by \( (13) \) and...
The groupoid $\mathfrak{G}$ acts on the space $\partial \mathfrak{G}$ from the right. Namely, every $g \in \mathfrak{G}$ defines a natural homeomorphism $\partial \mathfrak{G}_{t(g)} \to \partial \mathfrak{G}_{o(g)}$ mapping the limit of a sequence $g_n \in \Gamma(t(g), S)$ to the limit of the sequence $g_n g \in \Gamma(o(g), S)$. This action is an action of the topological groupoid $\mathfrak{G}$ on the topological space $\partial \mathfrak{G}$ over the projection map $P : \partial \mathfrak{G} \to \mathfrak{G}^{(0)}$ mapping all points of $\partial \mathfrak{G}$ to $x$.

The action of $\mathfrak{G}$ on $\partial \mathfrak{G}$ (i.e., the associated action of $\tilde{\mathfrak{G}}$ on $\partial \mathfrak{G}$ by local homeomorphisms) preserves with the local product structure of $\partial \mathfrak{G}$. Naturally defined projection of the action of $\mathfrak{G}$ onto the first coordinate of the local product decomposition is equivalent to $\mathfrak{G}$, while the projection onto the second coordinate is the dual groupoid of $\mathfrak{G}$.

Let us give an equivalent, and maybe more intuitive, definition of the dual groupoid.

**Definition 6.8.** Let $\Gamma(x, S)$ and $\Gamma(y, S)$ be the Cayley graphs of $\mathfrak{G}$ with adjoined boundaries $\partial \mathfrak{G}_x$ and $\partial \mathfrak{G}_y$. A homeomorphism $F : U \to V$ between open neighborhoods $U \subset \Gamma(x, S)$ and $V \subset \Gamma(y, S)$ of points of $\partial \mathfrak{G}_x$ and $\partial \mathfrak{G}_y$ is an asymptotic morphism if for every sequence of pairwise different edges $(g_1, h_1), (g_2, h_2), \ldots$ in $U$ the distance between $g_i h_i^{-1}$ and $F(g_i)F(h_i)^{-1}$ goes to zero.

Note that $g_i h_i^{-1}$ and $F(g_i)F(h_i)^{-1}$ belong to a compact subset of $\mathfrak{G}$, hence the notion of convergence of their distance to zero does not depend on the choice of a metric on $\mathfrak{G}$.

**Definition 6.9.** The groupoid $\mathfrak{dG}$ of germs of restrictions of the asymptotic morphisms to the spaces $\partial \mathfrak{G}_x$, $x \in \mathfrak{G}^{(0)}$, is the dual groupoid of $\mathfrak{G}$.

The space of units of $\mathfrak{dG}$ is the topologically disjoint union of the spaces $\partial \mathfrak{G}_x$. (In particular, it is not separable.) If $\mathfrak{G}$ is minimal (i.e., if all orbits are dense), then $\partial \mathfrak{G}_x$ is an open transversal of the dual groupoid for any $x \in \mathfrak{G}^{(0)}$, hence the dual groupoid can be defined as the groupoid of germs at $\partial \mathfrak{G}_x$ of the asymptotic morphisms. We will denote it $\mathfrak{dG}_x$.

We will denote by $\mathfrak{G}^\top$ any groupoid equivalent to $\mathfrak{dG}$. The following theorem is proved in [Nek11a].

**Theorem 6.5.** Let $\mathfrak{G}$ be a minimal Hausdorff hyperbolic groupoid. Then the dual groupoid $\mathfrak{G}^\top$ is minimal, Hausdorff, and hyperbolic, and $(\mathfrak{G}^\top)^\top$ is equivalent to $\mathfrak{G}$.

6.5. **Groupoid of germs generated by an expanding self-covering.** Let $f : \mathcal{J} \to \mathcal{J}$ be an expanding self-covering of a path connected compact metric space. Denote by $\mathfrak{F}$ the groupoid of germs generated by $f$. Every element of $\mathfrak{F}$ can be written as a product $(f^n, x)^{-1}(f^m, y)$, for $n, m \in \mathbb{N}$, and $x, y \in \mathcal{J}$ such that $f^n(x) = f^m(y)$.
A natural extension of $f$ is the inverse limit $\hat{J}$ of the maps $f$ together with the homeomorphism $\hat{f}$ of $\hat{J}$ induced by $f$, see [3, 6.2]. Let $P_\mathcal{S} : \hat{J} \to J$ be the natural projection.

For every point $x \in J$ there exists a neighborhood $U$ that is evenly covered by each map $f^n : J \to J$, since $f$ is expanding. It follows that the set $P^{-1}(U)$ is naturally decomposed into the direct product of $U$ with the boundary $\partial T_x$ of the tree of preimages of any point $x \in U$.

The groupoid $\mathfrak{G}$ is hyperbolic, and we can consider the space $\partial \mathfrak{G}$ together with the projection $P : \partial \mathfrak{G} \to \mathfrak{G}^{(0)} = J$. Let us use the generating set $S$ of $\mathfrak{G}$ equal to the set of germs of the inverse map $f^{-1}$. Then the Cayley graphs $\Gamma(x, S)$ are regular trees such that every vertex has one incoming and $d = \deg f$ outgoing arrows. The fiber $\partial \mathfrak{G}_x$ is equal to the boundary of this tree minus the limit of the path $(f^n, x)$, $n \geq 0$. In other words, it is the natural inductive limit of the boundaries of the preimage trees $T_{f^n}(x)$ for $n \geq 0$.

Let $\nu : \mathfrak{G} \to \mathbb{Z}$ be the homomorphism (cocycle) given by the rule $\nu(f, x) = -1$, so that $\nu((f^n, x)^{-1}(f^m, y)) = n - m$. See Figure 13 where the Cayley graph $\Gamma(x, S)$ together with the levels of the cocycle $\nu$ are shown.

Every point of $\partial \mathfrak{G}_x$ can be uniquely represented as the limit of a sequence $s_n \cdots s_2 s_1 \cdot g$, where $s_i \in S$, and $g \in \mathfrak{G}$ is such that $\sigma(g) = x$ and $\nu(g) = 0$. Note that the set of limits of the sequences $s_n \cdots s_2 s_1 \cdot g$ for a fixed $g$ and all possible choices of $s_i \in S$ is naturally identified with the fiber $P_{\mathfrak{S}}^{-1}(t(g))$ of the solenoid $\mathfrak{J}$ (i.e., with $\partial T_{\mathfrak{J}}(t(g))$). It follows then directly from the definitions that the space $\partial \mathfrak{G}$ is homeomorphic to the subset $\{(\zeta, g) : P_\mathcal{S}(\zeta) = t(g)\}$ of the direct product $\hat{J} \times \mathfrak{S}_0$, where $\mathfrak{S}_0$ is the subgroupoid $\nu^{-1}(0) \subset \mathfrak{G}$. The action of $\mathfrak{G}$ on $\partial \mathfrak{G}$ is given in these terms by the rules

$$
(\zeta, g) \cdot h = \begin{cases} 
(\hat{f}^n(\zeta), f^n \circ g h), & \text{if } \nu(h) = n > 0, \\
(\hat{f}^{-n}(\zeta), s_n \cdots s_2 s_1 g h), & \text{if } \nu(h) = -n < 0, \\
(\zeta, gh), & \text{if } \nu(h) = 0,
\end{cases}
$$

where $s_i \in S$ are such that $\zeta = \lim_{m \to \infty} s_m \cdots s_2 s_1$.

We get hence the following description of the natural extension $\hat{f} : \hat{J} \to \hat{J}$ in terms of the $\mathfrak{G}$-space $\partial \mathfrak{G}$.

---

**Figure 13.** Cayley graph of $\mathfrak{G}$
Proposition 6.6. The quotient of the space $\hat{G}$ by the action of $G_0 = \nu^{-1}(0)$ is homeomorphic to $\hat{J}$. If $F \in \hat{G}$ is such that $\nu(F) = \{n\}$, then the germs of the map induced by $F$ on the quotient space $\hat{J} = \hat{G}/G_0$ are germs of the map $\hat{f}^{-n}$.

We say that points $\xi, \zeta \in \hat{J}$ are unstably equivalent if distance between $\hat{f}^{-n}(\xi)$ and $\hat{f}^{-n}(\zeta)$ goes to zero as $n \to +\infty$. They are said to be stably equivalent if distance between $\hat{f}^n(\xi)$ and $\hat{f}^n(\zeta)$ goes to zero as $n \to +\infty$.

A leaf of $\hat{J}$ is its path connected component. (Recall that we assume that $J$ is path connected.) Every leaf is an equivalence class of the unstable equivalence relation on $\hat{J}$.

Each leaf is dense in $\hat{J}$, and it is more natural to consider it with the inductive limit topology. Namely, a subset $U$ of a leaf $L$ is open if and only if its intersection with every compact subset $C \subset L$ is open in $C$. Note that for every compact set $C \subset L$ the map $P : C \to \hat{J}$ is finite-to-one.

Restriction of the map $P : \hat{J} \to J$ to any leaf $L$ of $\hat{J}$ is a covering map. Let $\hat{G}_L$ be the lift of the groupoid $G$ to the leaf $L$ by this covering. Then the groupoid $\hat{G}_L$ is equivalent to $\hat{G}$. The cocycle $\nu$ lifts to the cocycle $\nu \circ P$ on $\hat{G}_L$, which we will denote by $\nu_L$ or just $\nu$.

It follows then from the definition of $\partial G$ for a hyperbolic groupoid $G$, that the space $\partial \hat{G}_L$ is the fiber product of the maps $P_S : L \to J$ and $P : \partial \hat{G} \to J$, i.e., the subset $\{(x, y) : P_S(x) = P(y)\}$ of $L \times \partial \hat{G}$. We get the following corollary of Proposition 6.6.

Corollary 6.7. Let $L$ be a leaf of $\hat{J}$, and let $\hat{G}_L$ be the lift of $\hat{G}$ by the covering $P_S : L \to J$. Then the quotient of the space $\partial \hat{G}_L$ by the action of $\partial \hat{G}_{L,0} = \nu_L^{-1}(0)$ is homeomorphic to $\hat{J}$. If $F \in \hat{G}_L$ is such that $\nu_L(F) = \{n\}$, then the germs of the map induced by $F$ on the quotient space $\hat{J} = \hat{G}_L/\hat{G}_{L,0}$ are germs of the map $\hat{f}^{-n}$.

Let $G$ be a contracting regular self-replicating group. Let $G$ be the groupoid of germs of the action of the group $\mathcal{V}_G$ on $X^\omega$. It is generated by the groupoid of germs of $G$ and the germs of the maps $S_x : x_1x_2 \ldots \to xx_1x_2 \ldots$. It is shown in [Nek11a, Subsection 5.3] that $G$ is hyperbolic, and its dual is the groupoid generated by the limit dynamical system $f : \mathcal{J}_G \to \mathcal{J}_G$.

More explicitly, let $w \in X^\omega$. Then the boundary $\partial G_w$ is the leaf of the limit solenoid $S_G$ consisting of points representable by sequences $\ldots x_{-2}x_{-1} \cdot x_0x_1 \ldots$, where $x_0x_1 \ldots$ belongs to the $G$-orbit of $w$. The groupoid $G_w$ is equal to the lift of the groupoid generated by the limit dynamical system $f : \mathcal{J}_G \to \mathcal{J}_G$ to the leaf $\partial G_w$ (by the covering induced by the projection $S_G \to \mathcal{J}_G$ of the natural extension onto $\mathcal{J}_G$).

7. Reconstruction of the Dynamical System from $\mathcal{V}_f$

The main result of this section is the following classification of the groups $\mathcal{V}_f$.

Theorem 7.1. Let $f_i : \mathcal{J}_i \to \mathcal{J}_i$, for $i = 1, 2$, be expanding self-coverings of path connected compact metric spaces. Then $\mathcal{V}_{f_1}$ and $\mathcal{V}_{f_2}$ are isomorphic as abstract groups if and only if the dynamical systems $(f_1, \mathcal{J}_1)$ and $(f_2, \mathcal{J}_2)$ are topologically conjugate.

Example 7.1. One can show that the limit dynamical systems of two groups $\mathcal{V}_{v_i}$, $i = 1, 2$, are topologically conjugate if and only if either $v_1 = v_2$, or $v_1$ can be
obtained from \( v_2 \) by replacing each 0 by 1 and each 1 by 0. Namely, the sequence of letters of \( v_1 \) can be interpreted as a *kneading sequence* of the dynamical system, which in turn can be defined in purely topological terms. This gives a complete classification of the groups \( V_{\mathcal{A}_v} \) up to isomorphism.

7.1. **M. Rubin’s theorem.** Recall that if \( G \) is a group acting on a topological space \( \mathcal{X} \), and \( U \subseteq \mathcal{X} \) is an open subset, then we denote by \( G(\mathcal{U}) \) the group of elements \( g \in G \) acting trivially outside of \( U \), see Subsection 4.1.

The following theorem is proved in [Rub89, Theorem 0.2].

**Theorem 7.2.** Let \( G_i \), for \( i = 1, 2 \), be groups acting faithfully by homeomorphisms on Hausdorff topological spaces \( \mathcal{X}_i \). Suppose that the following conditions hold for both pairs \( (G, \mathcal{X}) = (G_i, \mathcal{X}_i) \), \( i = 1, 2 \).

1. For every non-empty open subset \( U \subseteq \mathcal{X} \) the group \( G(\mathcal{U}) \) is non-trivial.
2. For every non-empty open subset \( U \subseteq \mathcal{X} \) there exists a non-empty open subset \( U_1 \subseteq U \) such that if \( V, W \subseteq U_1 \) are open sets such that there exists \( g \in G \) such that \( g(V) \cap W \neq \emptyset \), then there exists \( g \in G(\mathcal{U}) \) such that \( g(V) \cap W \neq \emptyset \).

Then for every isomorphism \( \phi : G_1 \to G_2 \) there exists a homeomorphism \( F : \mathcal{X}_1 \to \mathcal{X}_2 \) inducing it, i.e., such that \( \phi(g) = F \circ g \circ F^{-1} \) for all \( g \in G \).

We say that a group \( G \) acting on a topological space \( \mathcal{X} \) is *locally transitive* if there exists a basis of open sets \( \mathcal{U} \) such that for every \( U \in \mathcal{U} \) the group \( G(\mathcal{U}) \) has a dense orbit in \( U \).

The following is a direct corollary of Theorem 7.2.

**Corollary 7.3.** If \( G_i \) are locally transitive groups of homeomorphisms of topological spaces \( \mathcal{X}_i \), then every isomorphism \( \phi : G_1 \to G_2 \) is induced by a homeomorphism \( F : \mathcal{X}_1 \to \mathcal{X}_2 \).

Similar results (with simpler proofs), which can be applied to many groups \( \mathcal{V}_G \), are proved in [GPS99, Med11, Mat12].

It is easy to see that the Higman-Thompson group \( \mathcal{V}_{\mathcal{X}_1} \) acting on the space \( \mathcal{X}_1^\omega \) is locally transitive. It follows that every group of homeomorphisms of \( \mathcal{X}_1^\omega \) containing the Higman-Thompson group is locally transitive, which implies the following fact.

**Theorem 7.4.** Let \( G_i \) be groups acting on the Cantor sets \( \mathcal{X}_i^\omega \) and containing the Higman-Thompson groups \( \mathcal{V}_{\mathcal{X}_i} \). Then every isomorphism \( \phi : G_1 \to G_2 \) is induced by a homeomorphism \( F : \mathcal{X}_1^\omega \to \mathcal{X}_2^\omega \).

7.2. **Proof of Theorem 7.1** If \( f_1 : \mathcal{J}_1 \to \mathcal{J}_1 \) and \( f_2 : \mathcal{J}_2 \to \mathcal{J}_2 \) are topologically conjugate self-coverings of path-connected spaces, then the groups \( \mathcal{V}_{f_1} \) and \( \mathcal{V}_{f_2} \) are obviously isomorphic, since they were defined in purely topological terms.

Let us prove the converse implication for expanding maps. By Theorem 7.3 if groups \( \mathcal{V}_{f_1} \) and \( \mathcal{V}_{f_2} \) are isomorphic, then their action on the corresponding spaces \( \mathcal{X}_1^\omega \) are topologically conjugate, hence the groupoid of germs of the action of \( \mathcal{V}_{f_1} \) on \( \mathcal{X}_1^\omega \) are isomorphic.

Therefore, it is enough to show that if \( f : \mathcal{J} \to \mathcal{J} \) is an expanding self-covering of a compact path-connected metric space, then the dynamical system \( (f, \mathcal{J}) \) can be reconstructed from the topological groupoid \( \mathcal{G} \) of germs of the action of \( \mathcal{V}_f \) on \( \mathcal{X}_1^\omega \).
Denote by $\mathcal{F}$ the groupoid of germs generated by $f : J \to J$. We identify $J$ with the limit space $\mathcal{J}_G$ of the self-similar group $G = \text{IMG}(f)$, and hence encode points of $J$ by sequences $\ldots x_2x_1 \in X^{-\omega}$. Recall that $f$ acts then by the shift $\ldots x_2x_1 \mapsto \ldots x_3x_2$. Let $\nu : \mathcal{F} \to \mathbb{Z}$ be the cocycle (groupoid homomorphism) defined by the condition that $\nu(f, x) = -1$ for all $x \in J$.

The groupoids $\mathcal{F}$ and $\mathcal{G}$ are hyperbolic and mutually dual. Let $w \in X^\omega$ be an arbitrary point, and denote $\mathcal{N} = \partial \mathcal{G}_w$ and $\mathcal{H} = \partial \mathcal{G}_w = \mathcal{N}^{(0)}$. It is enough to show that $(f, J)$ is uniquely determined (up to a topological conjugacy) by the groupoid $\mathcal{N}$.

Denote by $\Omega_w$ the set of bi-infinite sequences $\ldots x_{-2}x_{-1}x_0x_1 \ldots$ such that $x_0x_1 \ldots$ belongs to the $G$-orbit of $w$. Note that $J$ is path connected, $G$ is self-replicating, hence $G$-orbits coincide with the $V_f = V_G$-orbits. We consider $\Omega_w$ with the topology of the disjoint union of the set of the form $X^{-\omega}x_0x_1 \ldots$.

Then the space $\mathcal{H} = \partial \mathcal{G}_w$ is naturally identified with the quotient of the space $\Omega_w$ by the asymptotic equivalence relation (defined in the same way as on $X\hat{\mathcal{G}}$, see Subsection 5.2). Let $P_S : \mathcal{H} \to J$ be the natural projection induced by $\ldots x_{-2}x_{-1}x_0x_1 \ldots \mapsto \ldots x_{-2}x_{-1}$. It is a covering map, and $\mathcal{N}$ is the lift of $\mathcal{F}$ by $P_S$. We will also denote by $P_S$ the corresponding functor (homomorphism of groupoids) $P_S : \mathcal{N} \to \mathcal{F}$.

Let us show at first that the cocycle $\nu : \mathcal{N} \to \mathbb{Z}$ (equal to the lift of the cocycle $\nu : \mathcal{F} \to \mathbb{Z}$) is uniquely determined by the structure of the topological groupoid $\mathcal{N}$.

**Proposition 7.5.** Let $\mathcal{C}$ be a connected component of $\mathcal{N}$. Then $\mathcal{C} : \mathcal{C} \to \mathcal{H}$, $\mathcal{T} : \mathcal{T} \to \mathcal{H}$ are coverings.

If $\nu(\mathcal{C}) \neq 0$, then $\mathcal{C}$ contains a non-trivial element of infinite order in the isotropy group $\mathcal{N}_v$ of a point.

If $\nu(\mathcal{C}) = 0$, then the groupoid generated by $\mathcal{C}$ is proper.

**Proof.** Let $\mathcal{X}_G$ be the limit $G$-space. The action of $G$ on $\mathcal{X}_G$ is free, and the maps $F_v : \xi \mapsto \xi \otimes v$ are coverings for all $v \in X^* \cdot G$.

For $w \in X^\omega$, the leaf $\partial \mathcal{G}_w = \mathcal{H}$ is the image of $\mathcal{X}_G$ under the map $P_w : \xi \mapsto \xi \cdot w$. This map coincides with the quotient of $\mathcal{X}_G$ by the action of the stabilizer $G_w$.

Let $\mathcal{X}$ be the groupoid of germs with the space of units $\mathcal{X}_G$ generated by the germs of the action of $G$ and the germs of the maps $F_v(\xi) = \xi \otimes v$ for $v \in X^* \cdot G$. Then $\mathcal{X}$ is the lift of $\mathcal{N}$ by the quotient map $P_w : \mathcal{X}_G \to \mathcal{H}$.

Every element of $\mathcal{N}$ is a germ of the transformation

$$F_{v,g,u,h} : \xi \otimes v \cdot g(w) \mapsto \xi \otimes u \cdot h(w),$$

for some $g, h \in G$ and $u, v \in X^*$.

The germ $(F_{v,g,u,h}, \xi \otimes v \cdot g(w))$ can be lifted to the germ $(\tilde{F}_{v,g,u,h}, \xi \otimes v \cdot g)$ of the local homeomorphism

$$\tilde{F}_{v,g,u,h} : \xi \otimes g \mapsto \xi \otimes u \cdot h$$

of $\mathcal{X}$. It follows that every element of $\mathcal{N}$ is a germ of $P_w F_{u,h} F_v^{-1} P_w^{-1}$. The space $\mathcal{X}_G$ is connected, the maps $F_{u,h}, F_{v,g}, P_w$ are coverings, hence if $\mathcal{C}$ is the connected component of the germ $(F_{v,g,u,h}, \xi \otimes v \cdot g(w))$, then $\mathcal{C} : \mathcal{C} \to \mathcal{H}$ and $\mathcal{T} : \mathcal{T} \to \mathcal{H}$ are covering maps.

Suppose that $\nu(\mathcal{C}) \neq 0$. It means that every germ $(F_{v,g,u,h}, \xi \otimes v \cdot g(w)) \in \mathcal{C}$ is such that $|v| \neq |u|$. Without loss of generality, we may assume that $|u| > |v|$. Let $u = u_1 v_1$, where $|v_1| = |v|$. Since $G$ is self-replicating, there exists $g_1 \in G$ such that
$g_1 \cdot v \cdot g = v_1 \cdot h$ in the biset. Then a lift of the germ $(F_{v,g,u,h}, \zeta \otimes v.g(w))$ to $X$ is a germ of the transformation

$$\xi \otimes v \cdot g \mapsto \xi \otimes u_1 \otimes v_1 \cdot h = \xi \otimes u_1 \cdot g_1 \otimes v \cdot g.$$ 

The point $\zeta = \ldots u_1 \cdot g_1 \otimes u_1 \cdot g_1 \otimes u_1 \cdot g_1 \in \mathcal{X}_G$ is well defined (as it is the image of a point of $\Omega$, see Subsection 6.2), and it satisfies $\zeta = \zeta \otimes u_1 \cdot g_1$. Then the germ of the transformation

$$\xi \otimes v \cdot g \mapsto \xi \otimes u_1 \cdot g \otimes v \cdot g$$

at $\zeta \otimes v \cdot g$ is a non-trivial contracting element of the isotropy group of $\zeta \otimes v \cdot g$. It is contained in the connected component of the germs of the transformation $\xi \otimes v \cdot g \mapsto \xi \otimes u \cdot h$. Mapping everything to $\mathcal{H}$ by $P_u$, we find a non-trivial contracting (hence infinite order) element of an isotropy group.

If $\nu(C) = 0$, then elements of $C$ are germs of transformations of the form $\xi \otimes v.g(w) \mapsto \xi \otimes u.h(w)$, where $g, h \in G$ and $v, u \in X^*$ are such that $|v| = |u|$. There exists $g_1 \in G$ such that $v \cdot h = g_1 \cdot v \cdot g$ in $X^\ast \setminus G$. It follows that elements of $C$ are lifted by $P_{\nu} F_{v,g} : X \to \mathcal{H}$ to the action of $g_1$ on $\mathcal{X}_G$. It follows that the groupoid generated by $C$ lifts by $P_{\nu} F_{v,g}$ to a subgroupoid of the action of $G$ on $\mathcal{X}_G$, and hence is proper.

Proposition 7.6. The cocycle $\nu : \mathcal{H} \to \mathbb{Z}$ is uniquely determined by the topological groupoid $\mathcal{H}$.

Proof. It follows from Proposition 7.5 that the value of $\nu$ on a connected component $C$ of $\mathcal{H}$ is zero if and only if $C$ generates a proper groupoid. (Since isotropy groups of a proper groupoid are finite.)

Let $g_1, g_2$ be arbitrary elements of $\mathcal{H}$. By the first claim of Proposition 7.5, there exist $g_1', g_2'$ in the components of $g_1$ and $g_2$, respectively, such that the product $g_1' g_2'$ is defined. Note that the connected component of $g_1' g_2'$ depends only on the connected components of $g_1$ and $g_2$. It follows that the set of connected components of $\mathcal{H}$ is a group. The quotient of this group by the subgroup of components on which $\nu$ is zero is isomorphic to $\mathbb{Z}$. Since the set of components on which $\nu$ is zero is uniquely determined by the topological groupoid, we conclude that the set $\{\nu, -\nu\}$ is uniquely determined by the structure of the topological groupoid $\mathcal{H}$. But we can distinguish between $\nu$ and $-\nu$ using Proposition 3.4.1.

The next statement follows now directly from Proposition 7.6 and Corollary 6.7.

Proposition 7.7. The natural extension $\hat{f} : \hat{J} \to \hat{J}$ is uniquely determined, up to topological conjugacy, by the groupoid $\mathcal{G}$.

Suppose that $f_1 : J_1 \to J_1$ and $f_2 : J_2 \to J_2$ are two expanding homeomorphisms with the same natural extension $\hat{f} : S \to S$. It remains to prove that $(f_1, J_1)$ and $(f_2, J_2)$ are topologically conjugate.

Denote by $P_1 : S \to J_1$ the corresponding projections. Let $\hat{J}$ be the image of $S$ in $J_1 \times J_2$ under the map $(P_1, P_2)$. It is compact and connected, since so is $S$. We will denote by $\hat{P}_1 : \hat{J} \to J_1$ the restrictions of the projections $J_1 \times J_2 \to J_1$.

Since $\hat{P}_1$ locally are projections on the unstable coordinate of the local product decomposition of $S$ (which depends only on the conjugacy class of $(\hat{f}, S)$), for every $\xi \in S$ there exists a rectangular neighborhood $U \ni \xi$ such that $\hat{P}_1 : U \to \pi_i(U)$ is decomposed into the composition of projection of $U$ onto its unstable direction
The groupoid of germs $f$ mines $(V)$. Equivalence of (7) and (1) is proved in [Nek05].

Let $(\xi_1, \xi_2) \in \tilde{\mathcal{J}}$, i.e., there exists $\xi \in \mathcal{S}$ such that $\xi_i = P_i(\xi)$. There exists a rectangular neighborhood $U$ of $\xi$ such that $P_i$ are projections onto the unstable direction composed with a homeomorphism, and the unstable direction of $U$ is connected. If $U$ is small enough, then $(\tilde{f})^{-1}(U)$ is decomposed into a union of a finite set $R$ of rectangles on which each of $P_i$ is a homeomorphism with projection onto the unstable direction. Consider the sets $(P_1, P_2)(R)$ for $R \in \mathcal{R}$. We get a finite number of components of $(\tilde{f})^{-1}((P_1, P_2)(U))$ such that $\tilde{f}$ is a homeomorphism on each of them. It follows that $\tilde{f}$ is a finite degree covering map.

For every $\xi \in \mathcal{J}_1$ the set $P_1^{-1}(\xi)$ is a compact subset of $\mathcal{S}$ contained in one stable equivalence class. Consequently, there exists $n_0$ such that $P_2(f^n(P_1^{-1}(\xi)))$ is a single point for all $n \geq n_0$. It follows that there exists a small neighborhood $U$ of $\xi$ such that the map $P_2 \circ f^{n_0} \circ P_1^{-1} = f_2^{n_0} \circ P_2 \circ P_1^{-1}$ is a homeomorphism on $U$. By compactness, there exists $n_1$ such that $P_2 \circ f^{n_1} \circ P_1^{-1} = f_2^{n_1} \circ P_2 \circ P_1^{-1}$ is a well-defined covering map from $\mathcal{J}_1$ to $\mathcal{J}_2$.

It follows that the projections $\tilde{P}_i : \tilde{\mathcal{J}} \to \mathcal{J}_i$ are finite degree covering maps. For every point $\tilde{T}_i(t) \in \mathcal{P}_i^{-1}(t_i)$ we have the corresponding tree $T_{i(t)}$ of preimages under iterations of $\tilde{f}$. They are disjoint (more precisely, for every $n$ the sets $\tilde{f}^{-n}(T_{i(t)})$ are disjoint for different $T_{i(t)}$).

By the arguments above, there exists $n_1$ such that $\tilde{P}_1(z_1) = \tilde{P}_1(z_2)$ implies $\tilde{f}^{n_1}(z_1) = \tilde{f}^{n_1}(z_2)$. But this contradicts the fact that the trees $T_i$ are disjoint. It follows that $\tilde{P}_i$ have degree 1, i.e., are homeomorphisms conjugating $f$ with $f_i$.

7.3. Equivalence of groupoids.

**Theorem 7.8.** Let $f_i : \mathcal{J}_i \to \mathcal{J}_i$, for $i = 1, 2$, be expanding self-coverings of connected and locally connected compact metric spaces. Then the following conditions are equivalent.

1. The dynamical systems $(f_1, \mathcal{J}_1)$ and $(f_2, \mathcal{J}_2)$ are topologically conjugate.
2. The groupoids generated by germs of $f_1$ and $f_2$ are equivalent.
3. The natural extensions of $f_1$ and $f_2$ are topologically conjugate.
4. The natural extensions of $f_1$ and $f_2$ generate equivalent groupoids of germs.
5. The actions of $\mathcal{V}_{f_1}$ and $\mathcal{V}_{f_2}$ on the corresponding Cantor sets are topologically conjugate.
6. The groupoids of germs generated by the actions of $\mathcal{V}_{f_1}$ and $\mathcal{V}_{f_2}$ on the corresponding Cantor sets are equivalent.
7. The self-similar groups $\text{IMG}(f_1)$ and $\text{IMG}(f_2)$ are equivalent.
8. The groups $\mathcal{V}_{f_1}$ and $\mathcal{V}_{f_2}$ are isomorphic as abstract groups.

**Proof.** The groupoid of germs $\mathfrak{G}_i$ generated by $f_i$, the groupoid of germs $\mathfrak{G}_i$ generated by $\mathcal{V}_{f_i}$, and the groupoid of germs generated by the natural extension uniquely determine each other, up to equivalence of groupoids, since the first two are mutually dual hyperbolic groupoids, and the third one is their geodesic flow, see [Nek11a].

Equivalence of (7) and (1) is proved in [Nek05].

It remains, therefore, to prove that the equivalence class of $\mathfrak{G}_i$ uniquely determines $(f_i, \mathcal{J}_i)$. 

and a homeomorphism of this direction with $P_i(U)$. Moreover, since $\mathcal{S}$ is compact, we can cover $\mathcal{S}$ by a finite number of such rectangles $U$.

The map $\tilde{f} : \mathcal{S} \to \mathcal{S}$ induces a map $\tilde{f} : \tilde{\mathcal{J}} \to \tilde{\mathcal{J}}$ by the rule $\tilde{f}(\xi_1, \xi_2) = (f_1(\xi_1), f_2(\xi_2))$. The projections $\hat{P}_i$ are semi-conjugacies of $\tilde{f}$ with $f_i$.

The dynamical systems $(f_1, \mathcal{J}_1)$ and $(f_2, \mathcal{J}_2)$ are topologically conjugate.
Suppose that $w_1$ and $w_2$ belong to one orbit of the groupoid $\mathfrak{G}$ from Definition 6.5. Let $g \in \mathfrak{G}$ be such that $o(g) = w_2$ and $t(g) = w_1$. Then the map $h \mapsto hg$ is a quasi-isometry between the Cayley graphs of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ based at $w_1$ and $w_2$ respectively, inducing an isomorphism $\partial \mathfrak{G}_{w_1} \rightarrow \partial \mathfrak{G}_{w_2}$. We have shown during the proof of Theorem 7.1 that the dynamical systems $(f_i, J_i)$ can be uniquely reconstructed from the topological groupoids $\partial \mathfrak{G}_{w_i}$, which implies that $(f_1, J_1)$ and $(f_2, J_2)$ are topologically conjugate. \(\square\)

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