INTRINSIC DECAY RATE ESTIMATES FOR THE WAVE EQUATION WITH COMPETING VISCOELASTIC AND FRICTIONAL DISSIPATIVE EFFECTS

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ABSTRACT. Wave equation defined on a compact Riemannian manifold \((M, g)\) subject to a combination of locally distributed viscoelastic and frictional dissipations is discussed. The viscoelastic dissipation is active on the support of \(a(x)\) while the frictional damping affects the portion of the manifold quantified by the support of \(b(x)\) where both \(a(x)\) and \(b(x)\) are smooth functions. Assuming that \(a(x) + b(x) \geq \delta > 0\) for all \(x \in M\) and that the relaxation function satisfies certain nonlinear differential inequality, it is shown that the obtained decay estimates are intrinsic without any prior quantification of decay rates of both viscoelastic and frictional dissipative effects. This particular topic has been motivated by influential paper of Fabrizio-Polidoro [15] where it was shown that viscoelasticity with poorly behaving relaxation kernel destroys exponential decay rates generated by linear frictional dissipation. In this paper we extend these considerations to: (i) nonlinear dissipation with unquantified growth at the origin (frictional) and infinity (viscoelastic), (ii) more general geometric settings that accommodate competing nature of frictional and viscoelastic damping.

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1. Introduction.

1.1. Description of the problem. Let \((M, g)\) be a \(n\)-dimensional compact Riemannian manifold with boundary where \(g\) denotes a Riemannian metric of class \(C^\infty\). We denote by \(\nabla\) the Levi-Civita connection on \(M\) and by \(\Delta\) the Laplace-Beltrami operator on \(M\).

In this paper we consider a wave equation subjected to two damping mechanisms: frictional and viscoelastic. Our goal is to determine the effectiveness of each damping on the overall decay rates of the associated energy function. We shall also consider situations where the support of each damping mechanism have empty essential support. We begin by introducing the PDE model.

Let \(u(x, t)\) denote the displacement of the wave equation subjected to the following viscoelastic and frictional damping which is partially distributed:

\[
\begin{align*}
  u_{tt} &= \Delta u - \int_0^t g(t-s)\text{div}[a(x)\nabla u(s)]\,ds - b(x)f(u_t) & \text{on } M \times [0, \infty[, \\
  u &= 0 & \text{on } \partial M \times [0, \infty[, \\
  u(0) &= u^0, \quad u_t(0) = u^1 & \text{in } M,
\end{align*}
\]

where \(g\) is the relaxation function, \(f\) represents frictional damping and \(a(x), b(x)\) are responsible for the effectiveness of each damping mechanism through the assumption \(a(x) + b(x) \geq \delta > 0\) for all \(x \in M\). Thus, on the support of \(a(x)\), the viscoelastic dissipation prevails while on the support of \(b(x)\) the frictional damping prevails. Models of this sort are of interest in smart/intelligent material technology where diverse damping mechanisms are used in different parts of the vibrating/oscillating structure.

Our aim is to study the decay rates associated with the above model. Of particular interest is to quantify the effectiveness of each type of damping. This quantification is expressed via suitable decay rates. A natural question which arises in this context is: “what are the overall decay rates in the mixed configurations of both dampings where each of them may be characterized as producing certain decay rates in a solo configuration?” More specifically,

1. When both frictional and viscoelastic dampers are applied with a disjoint essential support, what is the overall decay rate?

2. What if both types of dampings act simultaneously? Is more the better?

While precise formulation of our results is given in the next section, roughly speaking, the solutions to the corresponding partial viscoelastic model, with viscoelasticity affecting the entire domain, decay uniformly to zero with the rates dictated by the viscoelastic dissipation. In the case when the viscoelastic dissipation does not cover the entire region, then the overall decay rates follow the worst possible scenario: the overall decay rates are the worst of the two. This provides the answer to question 1.

However, when both dampings are active and “competing” then the viscoelastic dissipation is dominant. This result was known in the configuration of linear frictional damping (yielding exponential decays for the pure wave equation) and polynomially decaying viscoelastic energy. The overall result of polynomial decay rates only was shown in the influential paper by Fabrizio-Polidoro [15]. We not only recover this result in a more general situation of partially localized dampers with general relaxation kernels and nonlinear frictional damping, but also we are able to show that, in the case when the frictional damping that is highly nonlinear (hence weak) and the viscoelastic damping is of full support, it is the viscoelastic
dissipation that dominates the game. In few words: in viscoelastic models, the frictional damping is unessential and plays a role only when the support of viscoelastic damping is restricted.

An additional feature of our work is the generality of the damping mechanisms where the frictional damping does not need to be quantified at the origin and the relaxation function is governed by a differential inequality with an arbitrary convex function. In order to obtain optimal results, in this framework of generality, we use the method introduced in [17] which reduces the study of decay rates for PDE to the analysis of decay rates of a solution to a constructed nonlinear ODE. This method has been recently extended in the context of viscoelasticity [18] and will be adapted in this paper in order to treat the simultaneous frictional and viscoelastic dissipation.

1.2. Past literature relevant to the problem studied. We shall provide a brief overview of the literature that is relevant to the problem studied in our paper. In what concerns the Euclidean setting and in the absence of viscoelastic effects, the linear or semilinear wave equation subject to locally distributed frictional damping has been extensively studied. Among the numerous works, we would like to mention the classical ones: [14], [19], [21], [25], [26], [29], [2], [31]. Regarding the propagation of the wave equation on compact manifolds we refer to the following papers: [4], [5], [10], [8], [9], [11], [16], [20], [22], [13], [27], [30]. On the other hand, there is a large number of works published concerning the viscoelastic wave equation in the Euclidean setting, where, in this context we can cite [24, 1, 3] and numerous references therein. However, very few are related to locally distributed viscoelastic effects, as, for example, [7], and [24], but, even so, they consider viscoelastic wave equation in the Euclidean setting and under restrictive growth and size assumptions imposed on the relaxation function. The aim of the present paper is to consider competition between the two different kinds of dissipation: viscoelastic versus frictional that affect locally the wave equation defined on a compact manifold. Both frictional and viscoelastic dissipations are of a priori unquantified growth: frictional at the origin and viscoelastic at the infinity. These are the critical regions responsible for the decay of solutions. The results presented and the techniques developed are independent on the “geometry of the compact manifold”. Indeed, the presence of viscoelasticity, even in small quantities (see assumption (6)), plays an essential role and it provides the dominant effect on the overall decay rates. On the contrary, without viscoelasticity, the well-known Geometric Control Condition (GCC) due to Bardos-Lebeau-Rauch and Taylor is necessary in securing the exponential decay rates for the energy function.

Our methods rely on suitably localized “frictional” and “viscoelastic” multipliers which provide “recovery” estimates for the total energy. These estimates depend on two functions $f$ and $g$ that describe the two kinds of dissipation. In order to translate the multipliers estimates into the ODE describing the overall decay rates for the energy function we adapt and further extend the method of [17], developed for frictional damping, and the method of [18] developed for the abstract wave equation with viscoelastic damping. This intrinsic approach allows to obtain new results by filling the gaps in the range of parameters which were intractable by the methods of previous literature; see Remark 5.
1.3. Assumptions and the main result. The following assumptions are made.

Assumption 1.
- The relaxation function \( g : [0, \infty] \rightarrow \mathbb{R}_+ \) is a \( C^1 \cap W^{1,1} \) decreasing function and it satisfies
  \[
g(0) > 0 \quad \text{and} \quad ||a||_{L^{\infty}} \int_0^{\infty} g(s) \, ds < 1.
  \]  
  In addition, we assume that \( g'(t) \leq -H_1(g(t)), \) for all \( t \geq 0, \)
  where \( H_1 \in C^1(\mathbb{R}^+), \) \( H_1(0) = 0 \) is a given strictly increasing and convex function.
- Function \( f(s) \) is a continuous, strictly increasing, \( f(0) = 0 \) and subject to the Sobolev’s growth at infinity:
  \[
k^{-1}s^2 \leq f(s)s \leq K|s|^{p+1}, |s| \geq 1,
  \]
  where \( H^1(M) \subset L^{p+1}(M) \) and \( k, K \) are such that \( 0 < k, K < \infty. \)

Remark 1. The conditions imposed in Assumption 1 on both dissipative mechanisms \( f \) and \( g \) are minimal. The frictional damping modeled by \( f \) is not required to satisfy any growth conditions at the origin (critical region for stability), the growth conditions imposed at infinity are known to be necessary for uniform decay rates with frictional damping [21], and relaxation function is very general due to the generality of \( H_1. \) Condition (3) was recently considered in [3, 23] along with other restrictions imposed on the relaxation function.

As seen above, the role of Assumption 1 is to quantify the critical behavior of frictional and viscoelastic damping via very general convex functions \( H_1 \) and \( H_2. \) In fact, we already know that when the equation is subjected to only one type of the damping then the decay rates of the energy corresponding to this damping are described “roughly” by the nonlinear ODE: \( s_i + H_i(s) = 0, \) \( i = 1, 2. \) The decay rates at infinity produced by these ODE’s allows us to talk about “weak” or “strong” effects of each of the damping.

In order to derive uniform decay rates for the energy of the system (1) we impose assumptions of geometric nature which impose some lower bounds on the localization functions \( a(x), b(x) \).

Assumption 2. We assume that \( a \in C^1(M), b \in L^{\infty}(M) \) are nonnegative functions such that
  \[
  \text{meas} \{ x \in \partial M, a(x) > 0 \} > 0
  \]
  \[
a(x) + b(x) \geq \delta > 0 \text{ for all } x \in M.
  \]

Remark 2. The conditions imposed in Assumption 2 can be relaxed by requiring that the inequality (7) be satisfied only in a collar covering the boundary. Indeed, it is known that once the potential and kinetic energy are reconstructed in a full collar, then “flux multiplier” allows to propagate this reconstruction onto the full
domain. There is however one caveat to this strategy, namely the action of the flux multiplier on viscoelastic term introduces terms of the energy level which are non-local [6] (see also [24]). (Otherwise Carleman’s estimates could handle the local energy terms). In order to control these nonlocal terms one needs to assume either suitable “smallness” of the relaxation kernel, like in [6], or to impose active frictional damping in the area of spillower of the support of $a(x)$. This can be done at the price of additional technicalities which are tedious, however standard by now. For the sake of focusing on the “competing” nature of two kinds of dissipation, we choose to neglect this more general setup for this problem.

We define $\Sigma_T = M \times [0, T]$ and we set $H^1_0(M) := \{ v \in H^1(M); v|_{\partial M} = 0 \}$, which is a Hilbert space with the topology endowed by $H^1(M)$. The condition $v|_{\partial M} = 0$ is required to guarantee the Poincaré inequality,

$$\|h\|^2_{L^2(M)} \leq (\lambda_1)^{-1} \|\nabla h\|^2_{L^2(M)}, \quad \text{for all } h \in H^1_0(M), (8)$$

where $\lambda_1$ is the first eigenvalue of the Laplace-Beltrami operator for the Dirichlet problem.

The wellposedness of the dynamics described by (1) is classical by now and can be obtained by the methods used, for instance, in [1]. The corresponding results read:

**Theorem 1.1.** With $(w^0, w^1) \in [H^2(M) \cap H^1_0(M)] \times H^1_0(M)$ there exists a unique regular solution to problem (1) in the class

$$u \in L^\infty_{loc}(0, \infty; H^1_0(M) \cap H^2(M)), u_t \in L^\infty_{loc}(0, \infty; H^1_0(M)), \quad \text{(9)}$$

$$u_{tt} \in L^\infty_{loc}(0, \infty; L^2(M)).$$

With $(w^0, w^1) \in H^1_0(M) \times L^2(M)$ one also proves, by standard density argument, that problem (1) has a unique weak (variational) solution in the class

$$u \in C^0([0, \infty); H^1_0(M)) \cap C^1([0, \infty); L^2(M)). \quad (10)$$

We shall use standard notation for the following binary operators

$$(g * w)(t) := \int_0^t g(t-s)w(s) \, ds.$$  

$$(g \Box w)(t) := \int_0^t g(t-s)|w(t) - w(s)|^2 \, ds.$$  

$$(g \odot w)(t) := \int_0^t g(t-s)(w(t) - w(s)).$$

The following lemma establishes a helpful relationship between the operators introduced in section 2.

**Lemma 1.2.** For any $g, w \in C^1(\mathbb{R})$ we obtain the equality

$$2[g * w] w' = g' \Box w - g(t)|w|^2 - \frac{d}{dt} \left\{ g \Box w - \left( \int_0^t g \right) |w|^2 \right\}.$$  

**Proof.** The proof is obtained by differentiating the expression $g \Box w - \left( \int_0^t g(s) \, ds \right) |w|^2$. \hfill $\square$
Assuming that $u$ is the unique global weak solution to problem (1), we define the corresponding energy functional by

$$E(t) = \frac{1}{2} \int_M \left[ |u_t(x,t)|^2 + \kappa(x,t) |\nabla u(x,t)|^2 + a(x) g \Box \nabla u \right] dx,$$  \hspace{1cm} (11)

setting $\kappa(x,t) := 1 - a(x) \int_0^t g(s) \, ds$.

Note that, in view of (2) we have that

$$0 < l := 1 - ||a||_{L^\infty} \int_0^\infty g(s) \, ds \leq \kappa(x,t) \leq 1, \quad \forall (x,t) \in M \times \mathbb{R}_+.$$  \hspace{1cm} (12)

The energy function satisfies the following identity:

**Lemma 1.3.**

$$\frac{d}{dt} E(t) = \frac{1}{2} \int_M a(x) \left[ g' \Box \nabla u - g(t) |\nabla u|^2 \right] \, dx - \int_M b(x) f(u_t) u_t \, dx.$$  \hspace{1cm} (13)

**Proof.** Multiplying equation (1) by $u_t$, integrating by parts and using Lemma 1.2 yields the desired result. \hfill \Box

As a consequence of Lemma 1.3, every solution of (1) in the class (10) satisfies the following identity for all $t_2 > t_1 \geq 0$

$$E(t_2) - E(t_1) = \frac{1}{2} \int_{t_1}^{t_2} \int_M \left\{ a(x) \left[ g' \Box \nabla u - g(t) |\nabla u|^2 \right] - b(x) f(u_t) u_t \right\} \, dxdt,$$  \hspace{1cm} (14)

and therefore the energy is a non increasing function of the time variable $t$.

For simplicity, we shall denote the damping term:

$$D(t) := \frac{1}{2} \int_M \left\{ a(x) \left[ -g' \Box \nabla u + g(t) |\nabla u|^2 \right] + b(x) f(u_t) u_t \right\} \, dx.$$  \hspace{1cm} (15)

Our main result is the description of the decay rates for the energy function that is dissipated by both frictional and viscoelastic damping. The quantitative description of the decay rates will be described by the ODE equation that is driven by both $H_1$ and $H_2$ convex functions. In what follows below we shall built the appropriate functions.

To this end we denote by $\tilde{H}_1 : R^+ \rightarrow R^+$ a convex, continuous, increasing and zero at the origin function that satisfies

$$\tilde{H}_1(g \Box \nabla u)(t) \leq (-g' \Box \nabla u)(t), \quad t > T_0 > 0,$$  \hspace{1cm} (16)

for a given solution $u(t)$ to the equation and some $T_0 > 0$. We shall show later that such function can be constructed and has similar asymptotic properties (at the origin) as function $H_1(s)$. When $H_1$ is linear then clearly $\tilde{H}_1 = H_1$.

**Remark 3.** As shown in [18] a sufficient condition for (15) to hold true is the following condition introduced first in [23] : Let $D_0 \in C^1(R^+)$ be a positive, increasing function such that $H_1(D_0)$ is convex and

$$\frac{g}{H_1^{-1}(-g')} \in L_1(R^+).$$  \hspace{1cm} (17)

Then $\tilde{H}_1 = H_1(D_0)$. In typical applications function $D_0$ coincides with $D_0(s) = s^{\frac{1}{2}}, \alpha \in (0, 1)$. 
Theorem 1.4. Let us assume that Assumption 1, Assumption 2 and (15) are in place. Then, there exist positive constants \( \gamma \) and \( C_0 \) such that every weak solution of problem (1) satisfies \( E(t) \leq s(t) \) where \( s(t) \) satisfies the ODE

\[
s_t + p(s) = 0, s(0) = E(0), t \geq T_0
\]

and \( p(s) \sim [\hat{H}_1^{-1} + \hat{H}_2^{-1} + k]^{-1}(s) \) with \( \hat{H}_2(s) = c_2 H_2(d_2 s) + k_2 s \) and \( \hat{H}_1(s) = c_1 \hat{H}_1(d_1 s) + k_1 s \) for some intrinsic constants \( c_1, d_1, k_1, k_2, i = 1,2 \). In the case frictional damping is nonlinear, consistently with the nonlinear behavior, the above constants depend on \( E(0) \).

Remark 4. In particular, when \( H_1(s), H_2(s) \) are linear, then \( \hat{H}_1, \hat{H}_2 \) are linear as well, so is \( p(s) \). In that case we obtain the exponential bounds for the energy:

\[
E(t) \leq C_0 e^{-\gamma t} E(0), \quad \text{for all } t \geq T_0,
\]

with some constants \( C_0 > 0, \gamma > 0 \) and \( E(t) \) is given in (11).

If \( H_1(s) \) is polynomial: say \( H_1(s) \sim |s|^{q-1} s, q \in (1,2) \) then as shown in [18] \( \hat{H}_1(s) \sim |s|^{q-1} s \sim H_1(s) \). The above result is optimal on two accounts: (i) the decay rate of the energy reconstructs the decay rates of relaxation function, (ii) it applies to the full range of admissible parameters \( q \in [1,2) \). Previous results in the literature, including [23] cover only \( q \in [1,3/2) \).

Remark 5. One can provide explicit (modulo some constants) calculations of \( \hat{H}_1 \) in terms of \( H_1 \). For instance \( H_1(s) = H_1(s^{1/\alpha}) \) where \( \alpha \in (0,1] \) is such that \( g^{1-\alpha} \in L_1(0,\infty) \). However, the above construction is not optimal and can be improved as in [18]. In fact, as shown in [18] in the case of polynomial behavior of \( H_1(s) \sim s^p, p \in [1,2) \) one obtains the same asymptotic behavior for \( \hat{H}_1 \) and \( \hat{H}_1 \) and for the full range of the admissible parameters \( p \). We note that previous results in the literature, even in the case of just single viscoelastic damping, apply only to the interval \( p \in (1,3/2) \). (see [3], [23])

In order to achieve optimality of the viscoelastic decay rates it is important that the condition imposed in (15) is dynamic (depending on the solution) and not static as in (16) which is only a sufficient condition. To illustrate: let’s assume that the relaxation kernel \( g \) satisfies (16), which is probably the most general available “static” condition imposed in the prior literature, and which was introduced in [23]:

As shown in [18], under condition (16), our standing hypothesis (15) is satisfied with \( \hat{H}_1 = H_0 \). However, this choice can never be optimal (i.e. we can not have \( H_1 \approx H_0 \) which means \( D_0 \approx I \)). To see this, it suffices to notice that in such case

\[
\frac{g}{H_0(\cdot - g')} \sim \frac{g}{H_1(\cdot - g')} \sim 1
\]

where the latter can never be in \( L_1(R^+) \). It is essential for this criterion (16) to apply, that the decay rates be compromised by an additional nontrivial (different from the identity) operator \( D_0 \). In the case of polynomial decay rates and under the assumption that \( g^{1-\alpha} \in L_1 \), the operator \( D_0 \) coincides with \( s^{1/\alpha} \). Thus, the optimal value \( \alpha = 1 \) can never be achieved by this method.

On the other hand, our dynamic hypotheses allows for reiteration of the argument, until we reach \( \alpha = 1 \), which can be achieved in finitely many steps. This procedure is presented in [18]. In fact, this is the key feature of our method that allows to obtain optimal decay rates for the polynomial viscoelastic damping with \( H_1(s) \sim s^p \) with \( p \) covering the entire admissible set of parameters \( p \in [1,2) \).

Remark 6. We note that the decay rates of the ODE depend on the behavior of \( p(s) \) at the origin. Thus, the ultimate rate will be dictated by the highest growth at
the origin of the corresponding functions $\hat{H}_i$, $i = 1, 2$. When both functions $H_i$ are of polynomial growth and the essential supports of $a(x)$ and $b(x)$ are disjoint, then the ultimate decay rates are polynomial with its lower order (the worst scenario case). If viscoelastic function $H_1$ is of a polynomial growth, then regardless the support of $b(x)$ and $a(x)$ and regardless the growth of $H_2$, the overall decay rates can be at most polynomial and dictated by $H_1$. This confirms the result of Fabrizio-Polidoro [15] which was derived for linear frictional damping and polynomial viscoelastic damping acting simultaneously. Our results generalize this finding to any nonlinear frictional damping. The effect of overdamping caused by the presence of the second stronger damper does not improve the compromised decay rates caused by the viscoelastic damping. The reason for this is that we consider the total energy consisting of mechanical (elastic) and viscoelastic energy. It is viscoelastic part that is entirely controlled by the viscoelastic damping, i.e. decay rates of relaxation function. Thus, no matter how strong is the frictional damping, this has no much effect on the decay of the viscoelastic energy. This property is displayed by our condition (15).

On the other hand, when the viscoelastic damping is stronger than the frictional damping, our proof shows that under the assumption that $a(x) \geq c > 0$ in $M$, the decay rates are “essentially” driven by the viscoelastic damping. For instance, one has exponential relaxation kernel and logarithmic decays due to $f(s)$. The overall decays are exponential, provided $a(x) \geq c > 0$, $x \in M$. This latter condition simply means that viscoelastic damping is present regardless the effects of frictional damping. The above observation confirms the effectiveness of viscoelastic damping that dominates dissipative characteristics of the entire model.

**Corollary 1.** Assume:

- $a(x) \geq \delta > 0$, $x \in M$ along with all the hypotheses imposed on the relaxation function $g(s)$ and quantified by function $H_1$.
- $f \in C^1(R)$, is monotone, zero at the origin and satisfies the growth condition at the infinity $|f(s)| \leq K|s|^{p+1} + 1$, where $H^1(M) \subset L_{p+1}(M)$.

Then the conclusion of Theorem 1.4 is satisfied with $p(s) = [\hat{H}_1^{-1} + kI]^{-1}(s)$.

The reminder of this paper is devoted to the proofs. The main idea behind the proofs is to “build” differential equations describing the decay rates for each case. This method has been introduced in [17] in the case of frictional damping and in [18] in the case of viscoelastic damping.

In closing this introduction we wish to point out that the strategy developed in this manuscript could also potentially be pursued within the framework of weighted energy methods [3]. This approach may lead to more accurate description of decay through better control of the constants entering the weighted estimates. It would be interesting to compare the corresponding results.

2. **Proof of Theorem 1.4.** Our aim is to prove that the following inequality holds:

**Lemma 2.1.** Let $\hat{H}_1, \hat{H}_2$ be the functions from Theorem 1.4. Then, there exist positive constants $T_0 > 0$, $k_T$ such that the following inequality holds.

$$E((n+1)T) \leq (\hat{H}_1^{-1} + \hat{H}_2^{-1} + kI) \left\{ \int_{nT}^{(n+1)T} D(t) \, dt \right\},$$

for all $T > T_0$ and for all $n \in \mathbb{N}$, where the constant $k_T$ depends on $T$ but does not depend on $n$.  

Before proving (17) in Lemma 2.1 we need few technical results introduced in the Appendix of this manuscript.

In what follows and in order to achieve inequality (17) our task is reduced to the reconstruction of the full energy in terms of the dissipation. This is done, as usual, by employing suitable multipliers. Reconstruction of each part of the energy: kinetic, potential and viscoelastic will be linked to a suitable multiplier. For brevity, we shall denote: \((u,v)_{L^2(M)} = (u,v)\) and \(\|u\|_{L^2(M)} = \sqrt{(u,u)} = \|u\|\).

2.1. Recovering the kinetic energy. We shall first recover kinetic energy on the support of \(a(x)\). For this purpose we shall multiply equation (1) by viscoelastic multiplier \((g \circ (\varphi u))(t) = \int_0^t g(t-s)\varphi(x)(u(t) - u(s))\,ds\) to obtain:

\[
\int_{nT}^{(n+1)T} (u_{tt}(t) - \Delta u(t) + g \ast \text{div}[a(x)\nabla u(s)]) \, dt
+ b(x)f(u_t), g \circ (\varphi u)(t) \, dt = 0. \tag{18}
\]

We shall analyze the four above terms separately. We have for the first term:

\[
\int_{nT}^{(n+1)T} (u_{tt}(t), g \circ (\varphi u)(t)) \, dt \tag{19}
= (u_t(t), \int_0^t g(t-s)\varphi(\cdot)(u(t) - u(s))\,ds)|_{nT}^{(n+1)T}
- \int_{nT}^{(n+1)T} (u_t(t), \int_0^t g'(t-s)\varphi(\cdot)(u(t) - u(s))\,ds)\,dt
- \int_{nT}^{(n+1)T} \left(\int_0^t g(\xi)\,d\xi\right) \int_M \varphi(x)|u_t|^2\,dxdt.
\]

For the second term we deduce:

\[
\int_{nT}^{(n+1)T} (-\Delta u(t), g \circ (\varphi u)(t)) \, dt \tag{20}
= \int_{nT}^{(n+1)T} (\nabla u(t), \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s))\,ds) \, dt
+ \int_{nT}^{(n+1)T} (\nabla u(t), \int_0^t g(t-s)\varphi(\cdot)\nabla (u(t) - u(s))\,ds) \, dt.
\]

For the third term we infer:

\[
\int_{nT}^{(n+1)T} \left(\int_0^t g(t-s)\text{div}[a(\cdot)\nabla u(s)]\,ds, g \circ (\varphi u)(t)\right) \, dt \tag{21}
= \int_{nT}^{(n+1)T} \left(\int_0^t g(t-s)\text{div}[a(\cdot)\nabla u(s)]ds, \int_0^t g(t-s)\varphi(\cdot)(u(t) - u(s))ds\right)\,dt
dt
= - \int_{nT}^{(n+1)T} \left(\int_0^t g(t-s)a(\cdot)\nabla u(s)ds, \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s))ds\right)\,dt
- \int_{nT}^{(n+1)T} \left(\int_0^t g(t-s)a(\cdot)\nabla u(s)ds, \int_0^t g(t-s)\varphi(\cdot)\nabla (u(t) - u(s))ds\right)\,dt.
\]
Finally, for the fourth one, we obtain:

\[
\int_{nT}^{(n+1)T} (b(\cdot)f(u_t), g \circ (\varphi u)(t)) \, dt \\
= \int_{nT}^{(n+1)T} (b(\cdot)f(u_t), \int_0^t g(t-s)\varphi(\cdot)(u(t) - u(s)) \, ds) \, dt
\]

Combining (18), (19), (20), (21) and (22), we arrive at

\[
(u_t(t), \int_0^t g(t-s)\varphi(\cdot)(u(t) - u(s)) \, ds)|_{nT}^{(n+1)T}
\]

\[
- \int_{nT}^{(n+1)T} (u_t(t), \int_0^t g(t-s)\varphi(\cdot)(u(t) - u(s)) \, ds) \, dt \\
- \int_{nT}^{(n+1)T} \left( \int_0^t g(\xi) \, d\xi \right) \int_M \varphi(\cdot)|u_t|^2 \, dx \, dt \\
+ \int_{nT}^{(n+1)T} (\nabla u(t), \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s)) \, ds) \, dt \\
+ \int_{nT}^{(n+1)T} (\nabla u(t), \int_0^t g(t-s)\varphi(\cdot)\nabla(u(t) - u(s)) \, ds) \, dt \\
- \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(s) \, ds, \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt \\
- \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(s) \, ds, \int_0^t (t-s)\varphi(\cdot)\nabla(u(t) - u(s)) \, ds \right) \, dt \\
+ \int_{nT}^{(n+1)T} (b(\cdot)f(u_t), \int_0^t g(t-s)\varphi(\cdot)(u(t) - u(s)) \, ds) \, dt = 0.
\]

On the other hand it is convenient to observe that

\[
\int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(t) \, ds, \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt \\
= \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(t) - u(s) \, ds, \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt \\
+ \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(s) \, ds, \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt.
\]

Analogously,

\[
\int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(t) \, ds, \int_0^t g(t-s)\varphi(\cdot)\nabla(u(t) - u(s)) \, ds \right) \, dt \\
= \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(t) - u(s) \, ds, \int_0^t g(t-s)\varphi(\cdot)\nabla(u(t) - u(s)) \, ds \right) \, dt \\
+ \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s)a(\cdot)\nabla u(s) \, ds, \int_0^t g(t-s)\varphi(\cdot)\nabla(u(t) - u(s)) \, ds \right) \, dt.
\]
Substituting (24) and (25) in (23) we conclude that

\[
\int_{nT}^{(n+1)T} \left( \int_0^t g(\xi) d\xi \right) \int_M \varphi(x) |u_t|^2 \, dx \, dt
\]

\[=(u_t(t), \int_0^t g(t-s)\varphi(-)(u(t) - u(s)) \, ds) \right|_{nT}^{(n+1)T}
\]

\[- \int_{nT}^{(n+1)T} (u_t(t), \int_0^t g'(t-s)\varphi(-)(u(t) - u(s)) \, ds) \, dt
\]

\[+ \int_{nT}^{(n+1)T} (\nabla u(t), \int_0^t g(t-s)\varphi(-)(u(t) - u(s)) \, ds) \, dt
\]

\[+ \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s) a(\cdot) \nabla (u(t) - u(s)) \, ds, \int_0^t g(t-s) \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt
\]

\[- \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s) a(\cdot) \nabla u(t) \, ds, \int_0^t g(t-s) \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt
\]

\[+ \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s) a(\cdot) \nabla (u(t) - u(s)) \, ds, \int_0^t g(t-s) \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt
\]

\[- \int_{nT}^{(n+1)T} \left( \int_0^t g(t-s) a(\cdot) \nabla u(t) \, ds, \int_0^t g(t-s) \varphi(\cdot)(u(t) - u(s)) \, ds \right) \, dt
\]

\[+ \int_{nT}^{(n+1)T} (b(\cdot)f(u_t), \int_0^t g(t-s)\varphi(-)(u(t) - u(s)) \, ds) \, dt
\]

\[= J_1 + J_2 + \cdots + J_8 + J_9.
\]

Estimate for $J_1$. We have,

\[J_1 = (u_t((n+1)T), \int_0^{(n+1)T} g((n+1)T - s)\varphi(-)(u((n+1)T) - u(s)) \, ds)
\]

\[- (u_t(nT), \int_0^{nT} g(nT - s)\varphi(-)(u(nT) - u(s)) \, ds).
\]

Now, let $m \in \mathbb{N}$ an arbitrary natural number. Thus, from inequality (60) of Lemma 2.3 and having in mind the definition of the energy in (11), we deduce

\[(u_t(mT), \int_0^{mT} g(mT - s)\varphi(-)(u(mT) - u(s)) \, ds)
\]

\[\leq \int_0^{mT} g(mT - s) ||u_t(mT)|| ||\varphi(-)(u(mT) - u(s))|| \, ds
\]

\[\leq \int_0^{mT} g(mT - s) \left[ \frac{1}{2} ||u_t(mT)||^2 + \frac{1}{2} ||\varphi(-)(u(mT) - u(s))||^2 \right] \, ds
\]

\[\leq \frac{1}{2} ||g||_{L^1(0,\infty)} ||u_t(mT)||^2 + \frac{C}{2} \int_0^{mT} g(mT - s) ||\nabla a(\cdot)(\nabla u(mT) - \nabla u(s))||^2 \, ds
\]

\[\leq \frac{1}{2} ||g||_{L^1(0,\infty)} ||u_t(mT)||^2 + \frac{C}{2} \int a(x)(g(\nabla u)(mT)) \, dx
\]

\[\leq ||g||_{L^1(0,\infty)} E(mT) + CE(mT).
\]
Returning to (27) taking (28) into account, we deduce,

\[ |J_1| \leq C[E((n + 1)T) + E(nT)], \tag{29} \]

where the constant \( C \) depends on \( g, \varphi \) but does not depend on \( n \), which is crucial for the proof.

*Estimate for \( J_2 \).* Employing Lemma 2.4 and Lemma 2.3, one has,

\[
\begin{align*}
|J_2| & \leq \int_{nT}^{(n+1)T} ||u_t(t)|| \int_0^t \left| g'(t-s)\varphi(\cdot)(u(t) - u(s)) \right| ds \, dt \tag{30} \\
& \leq \varepsilon \int_{nT}^{(n+1)T} ||u_t(t)||^2 \, dt \\
& \quad + \frac{1}{4\varepsilon} \int_{nT}^{(n+1)T} \left( \int_0^t \left| g'(t-s)\varphi(\cdot)(u(t) - u(s)) \right|^2 \, ds \right) \, dt \\
& \leq \varepsilon \int_{nT}^{(n+1)T} ||u_t(t)||^2 \, dt \\
& \quad + \frac{1}{4\varepsilon} ||g'||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t \left| g'(t-s)\varphi(\cdot)(u(t) - u(s)) \right|^2 \, ds \, dt \\
& \quad \leq \varepsilon \int_{nT}^{(n+1)T} ||u_t(t)||^2 \, dt \\
& \quad - \frac{C}{4\varepsilon} ||g'||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s) \left| \sqrt{a(\cdot)}(\nabla u(t) - \nabla u(s)) \right|^2 \, ds \, dt,
\end{align*}
\]

where \( \varepsilon \) is an arbitrary positive constant and the constant \( C \) depends on \( \varphi \) and does not depend on \( n \).

*Estimate for \( J_3 \).* Repeating exactly the same arguments used when estimating \( J_2 \), we deduce,

\[
\begin{align*}
|J_3| & \leq \varepsilon \int_{nT}^{(n+1)T} ||\nabla u(t)||^2 \, dt \tag{31} \\
& \quad + \frac{C}{4\varepsilon} ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s) \left| \sqrt{a(\cdot)}(\nabla u(t) - \nabla u(s)) \right|^2 \, ds \, dt.
\end{align*}
\]

*Estimate for \( J_4 \).* In the same manner we infer,

\[
\begin{align*}
|J_4| & \leq \varepsilon \int_{nT}^{(n+1)T} ||\nabla u(t)||^2 \, dt \tag{32} \\
& \quad + \frac{C}{4\varepsilon} ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s) \left| \sqrt{a(\cdot)}(\nabla u(t) - \nabla u(s)) \right|^2 \, ds \, dt.
\end{align*}
\]
Estimate for \( J_5 \). We have,
\[
|J_5| \leq \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)a(\cdot)\nabla(u(t) - u(s))ds \right| dt
\]
\[
\leq \frac{1}{2} \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s))ds \right| dt
\]
\[
+ \frac{1}{2} \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s))ds \right|^2 dt
\]
\[
\leq \frac{1}{2} \| g \|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s))ds \right|^2 dt
\]
\[
+ \frac{1}{2} \| g \|_{L^1(0,\infty)} \| a \| \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\sqrt{a(\cdot)}\nabla(u(t) - u(s))ds \right|^2 dt
\]
\[
= \frac{1}{2} \| g \|_{L^1(0,\infty)} (\| a \| + C) \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\sqrt{a(\cdot)}\nabla(u(t) - u(s))ds \right|^2 dt.
\]

Estimate for \( J_6 \). One has,
\[
|J_6| \leq \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)a(\cdot)\nabla(u(t))ds \right| dt
\]
\[
\leq \epsilon \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)a(\cdot)\nabla u(t)ds \right|^2 dt
\]
\[
+ \frac{1}{4\epsilon} \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\nabla \varphi(\cdot)(u(t) - u(s))ds \right|^2 dt
\]
\[
\leq \epsilon \| g \|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)|a(\cdot)\nabla u(t)|ds \right|^2 dt
\]
\[
+ \frac{C}{4\epsilon} \| g \|_{L^1(0,\infty)} \| a \| \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\sqrt{a(\cdot)}\nabla(u(t) - u(s))ds \right|^2 dt
\]
\[
\leq \epsilon \| g \|_{L^1(0,\infty)} (\| a \| + C) \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\sqrt{a(\cdot)}\nabla(u(t) - u(s))ds \right|^2 dt.
\]

Estimate for \( J_7 \). Analogously to \( J_5 \), we deduce,
\[
|J_7| \leq \frac{1}{2} \| g \|_{L^1(0,\infty)} (\| a \| + C) \int_{nT}^{(n+1)T} \left| \int_0^t g(t-s)\sqrt{a(\cdot)}\nabla(u(t) - u(s))ds \right|^2 dt.
\]
Estimate for \( J_8 \). Analogously to \( J_6 \), we infer,

\[
|J_8| \leq \epsilon ||g||_{L^1(0,\infty)} ||a|| \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a}-\nabla u(t)||^2 \text{dsdt} \tag{36}
\]

\[+ \frac{C}{4\epsilon} ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a}-\nabla(u(t)-u(s))||^2 \text{dsdt}. \tag{37}\]

Estimate for \( J_9 \). One has, (as in the case of potential energy recovery by splitting \( u_t \) small and \( u_t \) big and considering the viscoelastic integral term as potential energy building \( E(t) \). Here are some details.

\[
\int_{nT}^{(n+1)T} (b(\cdot)f(u_t),g \circ (\varphi u))dt = \int_{\Sigma_A} + \int_{\Sigma_B},
\]

where \( \Sigma_A \equiv \{ t \in (nT,(n+1)T), x \in M, |u_t(t,x)| \leq 1 \} \) and \( \Sigma_B \) is the complement of \( \Sigma_A \) in \((nT,(n+1)T) \times M\).

On the set \( \Sigma_B \) we use that,

\[
||\varphi(\cdot)\int_0^t g(t-s)(u(t)-u(s))ds||_{L^{p+1}(M)} \leq C||\varphi(\cdot)\int_0^t g(t-s)(u(t)-u(s))ds||_{H^1(M)} \leq C||\varphi(\cdot)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds|| \leq C\int a(x)(g\nabla u(t))dx \frac{1}{2} \leq CE^{1/2}(0),
\]

where we have used Sobolev’s embedding \( H^1(M) \hookrightarrow L^{p+1}(M) \), Poincaré inequality, \( (60)-(61) \) and energy inequality. Thus using Hölder inequality, the above estimative and \( (4) \), we can conclude for \( T \) large enough, that

\[
\int_{\Sigma_B} \leq C \int_{nT}^{(n+1)T} \left[ \int_{x \in M,|u_t|>1} |b(x)f(u_t)| b^{p+1} dx \right]^{\frac{p}{p+1}} dt \\
\leq \tilde{C} \int_{nT}^{(n+1)T} \int_M kb(x)f(u_t)u_t dx dt, \tag{38}\]

where \( \tilde{C} \) depends on \( T,E(0),g,b,k \) and \( K \) but it does not depend on \( n \).

On the set \( \Sigma_A \) we use instead Cauchy-Schwarz inequality along with the definition of function \( H_2 \),

\[
f^2(u_t) \leq H_2^{-1}(u_t f(u_t)) \tag{39}\]

Combining \( \Sigma_A \) and \( \Sigma_B \) integrals leads to the estimate

\[
|J_9| \leq \left( \frac{1}{4\epsilon} ||b||_{\infty} + \tilde{C} \right) \int_{nT}^{(n+1)T} \int_M b(x)[H_2^{-1} + kI](f(u_t)u_t)dx dt \tag{40}\]

\[+ \epsilon ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a}-\nabla(u(t)-u(s))||^2 \text{dsdt}. \]
In order to recover the full kinetic energy, we need to add the part that corresponds to the support of $b(x)$. This corresponds to

$$ \int_{nT}^{(n+1)T} \int_M b(x)|u_t|^2 \, dx \, dt = \int_{\Sigma_A} + \int_{\Sigma_B}. $$

Applying $|u_t|^2 \leq kf(u_t)u_t, (t,x) \in \Sigma_B$ and $|u_t|^2 \leq H_2^{-1}(f(u_t)), (t,x) \in \Sigma_A$, recalling that $||a||_\infty ||g||_{L^1(0,\infty)} < 1$ and combining (26), (29)-(36) and (40), we write

$$ \int_{nT}^{(n+1)T} \left( \left( \int_0^t g(\xi) \, d\xi \right) \int_M (\varphi(x) + b(x))|u_t|^2 \, dx \, dt \leq C[E((n+1)T) + E(nT)] 
+ \frac{1}{4\varepsilon} ||b||_\infty + \hat{C} + ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_M b(x)[H_2^{-1} + kI][f(u_t)u_t] \, dx \, dt 
+ \varepsilon \int_{nT}^{(n+1)T} ||u_t(t)||^2 \, dt 
- \frac{C}{4\varepsilon} ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}(\nabla u(t) - \nabla u(s))||^2 \, ds \, dt 
+ 2\varepsilon \int_{nT}^{(n+1)T} ||\nabla u(t)||^2 \, dt 
+ \frac{C}{2\varepsilon} ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}(\nabla u(t) - \nabla u(s))||^2 \, ds \, dt 
+ ||g||_{L^1(0,\infty)}(||a||_\infty + C) \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}(\nabla u(t) - \nabla u(s))||^2 \, ds \, dt 
+ 2\varepsilon \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}\nabla u(t)||^2 \, ds \, dt 
+ \frac{C}{2\varepsilon} ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}(u(t) - u(s))||^2 \, ds \, dt 
+ \varepsilon C ||g||_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}(u(t) - u(s))||^2 \, ds \, dt. $$

Since $g(0) > 0$ we can select a point $t_1 < T$ ($t_1$ close to zero) such that for all $t \geq t_1, \int_0^t g(s) \, ds \geq t_1 g(t_1) = C_0$. With this in mind, employing inequality (59), namely, $\varphi(x) + b(x) \geq \delta/2$ for all $x \in M$, and for $\varepsilon < C_0\delta^2$, from (41) we obtain the recovery of the entire kinetic energy

$$ \int_{nT}^{(n+1)T} \int_M |u_t|^2 \, dx \, dt \leq C[E((n+1)T) + E(nT)] 
+ C \int_{nT}^{(n+1)T} \int_M b(x)[H_2^{-1} + kI + KI](f(u_t)u_t) \, dx \, dt $$
Remark 7. In comparing the effectiveness of frictional and viscoelastic damping, it is of interest to ask a question what happens when the viscoelastic damping is fully supported on $M$. This means that $a(x) \geq \delta$ on $M$. In such scenario it is easy to see that the frictional damper has no impact on the decay rates, provided, it is differentiable near the origin. Indeed, the estimate responsible for deterioration of decay rates due to frictional damping are the ones corresponding to the estimates in term $J_0$ on the support of $\Sigma_A$. But in that case, under differentiability condition, we can simply write $f^2(u_t) \leq L f(u_t) u_t$ for $u_t \in \Sigma_A$ with $L$ Lipschitz constant at the origin. Since support of $\varphi$ is full, viscoelastic multiplier recovers the full kinetic energy which results in the final estimate

$$
- C \| g' \|_{L^1(0, \infty)} \int_{nT}^{(n+1)T} \int_0^t \left| g'(t-s) \right| \sqrt{a(c)} \| \nabla (u(t) - u(s)) \|^2 \, ds \, dt \\
+ 2 \varepsilon \int_{nT}^{(n+1)T} \| \nabla u(t) \|^2 \, dt \\
+ C \int_{nT}^{(n+1)T} \int_0^t \left| g(t-s) \right| \sqrt{a(c)} \| \nabla (u(t) - u(s)) \|^2 \, ds \, dt \\
+ 2 \varepsilon \int_{nT}^{(n+1)T} \int_0^t \left| g(t-s) \right| \sqrt{a(c)} \| \nabla u(t) \|^2 \, ds \, dt \\
+ \varepsilon C \| g \|_{L^1(0, \infty)} \int_{nT}^{(n+1)T} \int_0^t \left| g(t-s) \right| \sqrt{a(c)} \| \nabla (u(t) - u(s)) \|^2 \, ds \, dt,
$$

for all $t \geq t_1$ and for some positive constant $C$ that does not depend on $n$.

**Assumption 3.** Let us assume that $a(x) \geq \delta > 0$ on $M$

- $f \in C(R)$ is monotone increasing, differentiable at the origin with $f(0) = 0$.
- $|f(s)| \leq K |s|^{p+1}$, for $|s| \geq 1$.

There is no need for a lower bound $m$ at the infinity nor the use of function $H_2$.

### 2.2. Recovering the potential energy.

Having obtained the reconstruction of the kinetic energy, we proceed to the recovery of the potential energy. This is done by the usual “partition” of energy procedure which is accomplished with the use...
of multiplier $u$. We shall thus multiply equation (1) by $u$ and we integrate over $M \times (nT,(n+1)T)$, in order to infer

$$\int_{nT}^{(n+1)T} (u_t(t) - \Delta u(t) + \int_0^t g(t-s) \text{div}[a(\cdot)\nabla u(s)] \, ds + b(\cdot)f(u_t), u(t)) \, dt = 0. \quad (41)$$

After performing some integration by parts we obtain

$$- \int_{nT}^{(n+1)T} ||u_t(t)||^2 \, dt + (u_t(t), u(t))_{nT}^{(n+1)T} + \int_{nT}^{(n+1)T} ||\nabla u(t)||^2 \, dt \quad (42)$$

$$- \int_{nT}^{(n+1)T} \int_0^t g(t-s) (a(\cdot)\nabla u(s), \nabla u(t)) \, dsdt = - \int_{nT}^{(n+1)T} (b(\cdot)f(u_t), u(t)) \, dt. \quad (43)$$

Estimate for $I_1 := - \int_{nT}^{(n+1)T} \int_0^t g(t-s) (a(\cdot)\nabla u(s), \nabla u(t)) \, dsdt$.

Employing Cauchy-Schwarz inequality combined with the inequality $ab \leq \frac{1}{4\varepsilon}a^2 + \varepsilon b^2$, gives

$$|I_1| \leq \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}\nabla u(s)||||\sqrt{a(\cdot)}\nabla u(t)|| \, dsdt \leq \int_{nT}^{(n+1)T} \int_0^t g(t-s) \left[||\sqrt{a(\cdot)}\nabla (u(s) - u(t))|| \right.$$

$$+ ||\sqrt{a(\cdot)}\nabla u(t)||] ||\sqrt{a(\cdot)}\nabla u(t)|| \, dsdt = \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}\nabla (u(s) - u(t))|| ||\sqrt{a(\cdot)}\nabla u(t)|| \, dsdt + \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}\nabla u(t)||^2 \, dsdt$$

$$\leq \frac{1}{4\varepsilon} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}\nabla u(t)||^2 \, dsdt + (\varepsilon + 1) \int_{nT}^{(n+1)T} \int_0^t g(t-s)||\sqrt{a(\cdot)}\nabla u(t)||^2 \, dsdt,$$

where $\varepsilon$ is an arbitrary positive constant.

**Estimate for $I_2 := - \int_{nT}^{(n+1)T} (b(\cdot)f(u_t), u(t)) \, dt$**. Considering Cauchy-Schwarz and Poincaré inequalities combined with the inequality $ab \leq \frac{1}{4\varepsilon}a^2 + \varepsilon b^2$, we deduce that

$$\int_{nT}^{(n+1)T} (b(\cdot)f(u_t), u(t))dt = \int_{|u_t| \leq 1} + \int_{|u_t| \geq 1} = I_A + I_B.$$

As before, let $\Sigma_A = \{(t,x) \in [nT, (n+1)T] \times M; |u_t(t,x)| \leq 1\}$ and $\Sigma_B = \{(t,x) \in [nT, (n+1)T] \times M; |u_t(t,x)| > 1\}$. Then,

$$|I_A| \leq \frac{\lambda^{-1}}{4\varepsilon} \int_{nT}^{(n+1)T} \int_M b(x)H^{-1}_2(f(u_t)u_t)dx + \varepsilon \int_{nT}^{(n+1)T} ||\nabla u(t)||^2 \, dt, \quad (44)$$

where we have used (39).
As for large frequencies we employ Hölder’s inequality, Sobolev’s embedding $H^1(M) \hookrightarrow L^{p+1}(M)$, Poincaré inequality, (4) and energy inequality, 

\[
|I_B| \leq \int_{nT}^{(n+1)T} \|b(\cdot)f(u_t)\|_{L^{p+1}(\{x \in M; |u_t| > 1\})} \|u(t)\|_{L^{p+1}(\{x \in M; |u_t| > 1\})} \quad (45)
\]

\[
\leq \tilde{C} \int_{nT}^{(n+1)T} \int_M b(x)kf(u_t)u_t dx dt,
\]

where $\tilde{C}$ depends on $l, b, \lambda_1, K, k$ and $E(0)$, but it does not depend on $n$. Hence, (44) and (45) yield 

\[
\int_{nT}^{(n+1)T} (b(\cdot)f(u_t), u_t) dt \leq \varepsilon \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2 dt + C \int_{nT}^{(n+1)T} \int_M b(x)[H^{-1}_2 + kI](f(u_t)u_t) dx dt.
\]

Estimate for $I_3 := (u_t(t), u(t))|_{nT}^{(n+1)T}$. One has, 

\[
|I_3| \leq \frac{\lambda_1^{-1/2}}{2} \left[ \|u_t((n+1)T)|^2 + \|\nabla u((n+1)T)|^2 \right] + \frac{\lambda_1^{-1/2}}{2} \left[ \|u_t(nT)|^2 + \|\nabla u(nT)|^2 \right].
\]

From the last inequality and from the fact that $\frac{1}{2}||\nabla u(t)||^2 \leq l^{-1}E(t)$ for all $t \geq 0$, where $l = 1 - ||a||_{\infty} \int_0^\infty g(s) ds$, we infer,

\[
|I_3| \leq C[E((n+1)T) + E(nT)], \quad (47)
\]

where the constant $C$ does not depend on $n$.

Combining (41), (42), (43), (46) and (47) we can write 

\[
\int_{nT}^{(n+1)T} \|\nabla u(t)\|^2 dt - \int_{nT}^{(n+1)T} ||u_t(t)||^2 dt \leq C[E((n+1)T) + E(nT)]
\]

\[
+ \frac{1}{4\varepsilon} \int_{nT}^{(n+1)T} \int_0^t g(t-s)||a(\cdot)\nabla(u(s) - u(t))||^2 ds dt \quad (48)
\]

\[
+ (\varepsilon + 1) \int_{nT}^{(n+1)T} \int_0^t g(t-s)||a(\cdot)\nabla u(t)||^2 ds dt
\]

\[
+ \frac{\lambda_1^{-1}||b||_{\infty}}{4\varepsilon} + C \int_{nT}^{(n+1)T} \int_M b(x)[H^{-1}_2 + kI](f(u_t)u_t) dx dt
\]

\[
+ \varepsilon \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2 dt.
\]

Remark 8. As before, we ask what happens if the support of the viscoelastic damping is full, this means $a(x) \geq \delta$ on $M$. Also, as before, we conclude that the frictional damping has no effect as long as it is differentiable at the origin and bounded from above at infinity by $K|s|^p$. Indeed, the relevant estimate is in term
I_2 which requires only the bounds specified in (3). The resulting estimate becomes

\[ \int_{nT}^{(n+1)T} ||u(t)||^2 \, dt - \int_{nT}^{(n+1)T} ||u_t(t)||^2 \, dt \]
\[ \leq C[E((n + 1)T) + E(nT)] \]
\[ + \frac{1}{4\varepsilon} \int_{nT}^{(n+1)T} \int_0^t g(t-s) ||\sqrt{a(\cdot)}\nabla(u(s) - u(t))||^2 \, ds \, dt \]
\[ + (\varepsilon + 1) \int_{nT}^{(n+1)T} \int_0^t g(t-s) ||\sqrt{a(\cdot)}\nabla u(t)||^2 \, ds \, dt \]
\[ + \left( \frac{\lambda_1^{-1} ||b||_\infty}{4\varepsilon} + C \right) \int_{nT}^{(n+1)T} \int_M b(x)[Kl + LI](f(u_t)u_t) \, dx \, dt \]
\[ + \varepsilon \int_{nT}^{(n+1)T} ||\nabla u(t)||^2 \, dt. \]

2.3. Recovering the viscoelastic energy \( E(t) \). Our last step is to recover the viscoelastic energy. Combining (48), (41) and adding and subtracting the terms

\[- \int_{nT}^{(n+1)T} \int_M \left( \int_0^t g(s) \, ds \right) a(x)||\nabla u(t)||^2 \, dx \, dt \quad \text{and} \quad \int_{nT}^{(n+1)T} \int_M a(x)(g\Box u)(t) \, dx \, dt,\]

in order to recover the energy \( E(t) \), we obtain

\[ (1 - 5\varepsilon) \int_{nT}^{(n+1)T} \int_M \left( 1 - a(x) \int_0^t g(s) \, ds \right) ||\nabla u(t)||^2 \, dx \, dt \]
\[ + \int_{nT}^{(n+1)T} ||u_t(t)||^2 \, dt \]
\[ + \int_{nT}^{(n+1)T} \int_M a(x)(g\Box u)(t) \, dx \, dt \]
\[ \leq C[E((n + 1)T) + E(nT)] + C \int_{nT}^{(n+1)T} \int_M a(x)(g\Box u)(t) \, dx \, dt \]
\[ + C \int_{nT}^{(n+1)T} \int_M b(x)[H_2^{-1} + kI + LI](f(u_t)u_t) \, dx \, dt \]
\[ + C \int_{nT}^{(n+1)T} \int_M a(x)k_1(-g\Box u)(t) \, dx \, dt. \]

From (50), choosing \( \varepsilon \) sufficiently small, \( k_1 > 0 \) and \( T \) large enough we obtain the following observability inequality:

\[ \int_{nT}^{(n+1)T} E(t) \, dt \leq C[E((n + 1)T) + E(nT)] \]
\[ + C \int_{nT}^{(n+1)T} \int_M a(x)(g\Box u)(t) \, dx \, dt \]
\[ + C \int_{nT}^{(n+1)T} \int_M b(x)[H_2^{-1} + kI + LI](f(u_t)u_t) \, dx \, dt \]
\[ + C \int_{nT}^{(n+1)T} \int_M a(x)k_1(-g\Box u)(t) \, dx \, dt. \]
In the last step we need to relate the viscoelastic energy with the viscoelastic damping. In the case when the relaxation function obeys the linear equation, this relation is straightforward and expressed by a suitable multiplication. However, in the case of general decays additional arguments are used. Here we follow [18]. From the assumption made on viscoelastic kernel $g$ given in (15), we obtain

$$(g \Box \nabla u)(t) \leq \hat{H}_1^{-1}(-g' \Box \nabla u)(t), \ t \in [nT, (n+1)T].$$  

From (52) and taking (51) into account we deduce

$$\int_{nT}^{(n+1)T} E(t) \, dt \leq C[E((n+1)T) + E(nT)]$$  

$$+ C \int_{nT}^{(n+1)T} \int_M a(x)[\hat{H}_1^{-1} + k_1](-g' \Box \nabla u)(t) \, dx dt$$

$$+ C \int_{nT}^{(n+1)T} \int_M b(x)[H_2^{-1} + kI + K](f(u_t)u_t) \, dx dt.$$  

We shall employ next the following version of Jensen’s inequality applied to measures and convex functions $F$.

- Let $F$ be a convex increasing function on $[a, b]$, $f : \Omega \to [a, b]$ and $h$ be an integrable function such that $h(x) \geq 0$ and $\int_\Omega h(x) \, dx = h_0 > 0$. Then

$$\int_\Omega F^{-1}(f(x))h(x) \, dx \leq h_0 F^{-1}[h_0^{-1} \int_\Omega f(x)h(x) \, dx].$$  

We shall use (54) in order to bring the functions $H_i$ in front of the integrals. Let us denote

$$\int_M a(x) = a_0, \int_M b(x) = b_0,$$

where we can assume $a_0, b_0 > 0$. We note that the functions $\hat{H}_1^{-1} + k_1, \ H_2^{-1} + k + K$ are concave. Thus

$$\int_{nT}^{(n+1)T} \int_M a(x)[\hat{H}_1^{-1} + k_1](-g' \Box \nabla u)(t) \, dx dt$$

$$\leq a_0 T[\hat{H}_1^{-1} + k_1][a_0^{-1}T^{-1} \int_{nT}^{(n+1)T} \int_M a(x)(-g' \Box \nabla u)(t) \, dx dt],$$

$$\int_{nT}^{(n+1)T} \int_M b(x)[H_2^{-1} + kI + K](f(u_t)u_t) \, dx dt$$

$$\leq b_0 T[H_2^{-1} + kI + K][b_0^{-1}T^{-1} \int_{nT}^{(n+1)T} \int_M b(x)f(u_t)u_t \, dx dt].$$

On the other hand, from the identity of the energy (13) we can write

$$E((n+1)T) - E(nT)$$

$$= \frac{1}{2} \int_{nT}^{(n+1)T} \int_M \{a(x) [g' \Box \nabla u - g(t) |\nabla u|^2] - b(x)f(u_t)u_t \} \, dx dt.$$
2.4. Total recovery of the energy. Substituting $E(nT)$ given in (55) in the inequality (53) having in mind the notation (14) we obtain for $T$ large enough

\[ \int_{nT}^{(n+1)T} E(t) \, dt \leq CE((n+1)T) + CH^{-1}\left[\int_{nT}^{(n+1)T} D(t) \, dt\right], \]

where $C$ is a positive constant which does not depend on $n$, while $H = \left[H_2^{-1} + kI + \hat{H}_1^{-1} + k_1I\right]^{-1}$.

**Remark 9.** When viscous damping is fully supported, then under the assumptions (3) we obtain

\[ \int_{nT}^{(n+1)T} E(t) \, dt \leq CE((n+1)T) + CH^{-1}\left[\int_{nT}^{(n+1)T} D(t) \, dt\right], \]

but with $H = [LI + KI + \hat{H}_1^{-1} + kI]^{-1}$, thus it does not depend on the compromised dissipation due to frictional damping.

Since $E(t)$ is non-increasing from the last inequality we deduce

\[ (T - C)E((n+1)T) \leq CH^{-1}\left[\int_{nT}^{(n+1)T} D(t) \, dt\right], \]

which implies, for $T$ sufficiently large that

\[ E((n+1)T) \leq CH^{-1}\left[\int_{nT}^{(n+1)T} D(t) \, dt\right]. \]

The above inequality proves Lemma 2.1. The rest of the proof follows now known procedure [17]

\[ H(C^{-1}E((n+1)T)) \leq \int_{nT}^{(n+1)T} D(t) \, dt = E(nT) - E((n+1)T). \]

The above gives

\[ E((n+1)T) + H(C^{-1}E(n+1)T) \leq E(nT), \quad n = 1, 2, \ldots \]

From [17] we infer that the asymptotic behavior of $E(t)$ can be compared to solutions of an appropriate ODE driven by

\[ p(s) \sim H(C^{-1}s), \]

where the function $H$ given above depends explicitly on $\hat{H}_1, \hat{H}_2$ describing the viscoelastic and the frictional damping.

**Remark 10.** When $a(x)$ is fully supported in $M$ and the hypotheses in Assumption 3 are satisfied, then the function $H(s)$ does not depend on $H_2(s)$. This is to say that the decay rates are driven by viscoelastic damping. This fact is stated in the Corollary.

2.5. Exponential or polynomial stability.

By (17), (14) and (55) we have for $T > T_0$,

\[ E((n+1)T) \leq C \int_{nT}^{(n+1)T} D(t) \, dt = -CE((n+1)T) + CE(nT), \]

where the constant $C$ depends on $T$ but does not depend on $n$. The last inequality yields

\[ E((n+1)T) \leq \frac{C}{C + 1} E(nT), \quad \text{for all } n \in \mathbb{N}, \quad (56) \]
which implies the exponential stability. Indeed, from (56) we infer

\[
E(T) \leq \frac{C}{1 + Ce} E(0) = \frac{1}{1 + \frac{C}{e}} E(0), \quad \text{for all } T > T_0. \tag{57}
\]

Repeating the above process from \( T \) to \( 2T \) we obtain

\[
E(2T) \leq \frac{1}{1 + \frac{C}{e}} E(T) \leq \frac{1}{1 + \frac{C}{e}}^2 E(0).
\]

In general we have that

\[
E(nT) \leq \frac{1}{(1 + \frac{C}{e})^n} E(0).
\]

Since any number \( t \) can be written as \( t = nT + r \) where \( 0 \leq r < T \) and \( E(t) \) is a decreasing function, one has

\[
E(t) \leq E(t - r) \leq \frac{1}{(1 + \frac{C}{e})^{nT}} E(0) = C_0 e^{-\gamma t} E(0),
\]

where \( C_0 = e^{\frac{C}{e} \ln(1 + \frac{1}{e})} \) and \( \gamma = \frac{\ln(1 + \frac{1}{e})}{T} \), and the exponential decay follows.

If we assume now that the function \( H_1(s) \sim |s|^{q-1} s \) with \( q \in [1, 2) \) and \( H_2(s) \sim |s|^{r-1} s \) for some \( 1 \leq r \leq 2 \) and \( r \geq q \geq 1 \). This corresponds to polynomial damping of \( q(t) \) and \( f(s) \). Then evaluating asymptotics at the origin, we conclude that \( H(s) \sim |s|^{r-1} q \), when \( q \in [1, 2) \).

In what follows we shall provide several specific examples.

- Assuming that \( r > q \) and the support of \( a(x) \) is strictly contained in \( M \), then we have \( E(t) \leq c_1 \frac{1}{t^{r-1}} \).
- Assuming that \( r < q \) and the support of \( a(x) \) is strictly contained in \( M \), then we have \( E(t) \leq c_2 \frac{1}{t^{q-1}} \).
- Assuming that \( r > q \) and \( a(x) \geq \delta > 0 \) in \( M \), then we have \( E(t) \leq c_1 \frac{1}{t^{r-1}} \).
- Assuming that \( r < q \) and \( a(x) \geq \delta > 0 \) in \( M \), then we have \( E(t) \leq c_2 \frac{1}{t^{q-1}} \).

**Non-differentiable friction**: Assuming that

\[
f(s) = \begin{cases} s^\alpha, & s \in [0, 1], \\ c|s|^{m-1}s, & s \geq 1, 1 < m \leq p, \end{cases}
\]

for some \( 0 < \alpha < 1 \) and \( H_1(s) = s \) with \( a(x) \geq \delta > 0 \) on \( M \). In this case we have \( H_2(s) \sim s^{\frac{\alpha+1}{\alpha}} \) and the decay rates become (as long as \( q < \frac{\alpha+1}{2} \)) \( E(t) \leq \frac{c_1}{t^{r-1}} \). Thus, in this case the frictional damping dominates.

**Differentiable friction**: Assuming that

\[
f(s) = \begin{cases} |s|^{k-1}s, & s \in [0, 1], k > 1 \\ |s|^{m-1}s, & s \geq 1, 1 \leq m \leq p \end{cases}
\]

and \( H_1(s) = |s|^{q-1} s \) with \( q \in [1, 2) \) and \( a(x) \geq \delta > 0 \) on \( M \). In this case we have \( H_2(s) \sim s^{\frac{k+1}{k}} \) and the decay rates become \( E(t) \leq \frac{c_1}{t^{k-1}} \). If, instead, \( a(x) = 0 \) on some essential support of \( M \) and \( q < \frac{k+1}{2} \), then \( E(t) \leq \frac{c_2}{t^{q-1}} \).
**Lemma 2.3.** The following inequalities hold:

\[
\varphi(x) \geq \delta/2 \quad \text{if} \quad x \in a^{-1}([\delta/2, \infty[),
\]
\[
0 \leq \varphi(x) \leq \delta/2 \quad \text{if} \quad x \in a^{-1}([\delta/4, \delta/2]),
\]
\[
\varphi(x) = 0 \quad \text{if} \quad x \in a^{-1}([0, \delta/4]).
\]

By construction, \(\text{supp}(\varphi) \subset \text{supp}(a)\). In fact, we have more, if \(x \in \text{supp}(\varphi)\) then \(a(x) \geq \delta/4\), or, in other words, the function \(a(x)\) is bounded from below by \(\delta/4\) for all \(x \in \text{supp}(\varphi)\).

Observe that if \(a(x) \leq \delta/2\), for all \(x \in M\), this implies that \(b(x) > \delta/2\), for all \(x \in M\), since, if this is not the case, if \(b(x) \leq \delta/2\), for some \(x \in M\), then
\[
a(x) + b(x) \leq \delta/2 + \delta/2 = \delta,
\]
which contradicts the assumption (7), namely,
\[
a(x) + b(x) > \delta \quad \text{for all} \quad x \in M.
\]

Consequently, \(a(x) \leq \delta/2\), for all \(x \in M\), which implies that \(b(x) > \delta/2\) for all \(x \in M\). Therefore, we have the frictional damping acting on the whole \(M\). Analogously, \(b(x) \leq \delta/2\), for all \(x \in M\), implies that \(a(x) > \delta/2\), for all \(x \in M\), which shows us that the viscoelastic damping acts on the whole \(M\). Now, when one has \(a(x) > \delta/2\) for some \(x \in M\), and, having in mind that \(a\) is a continuous function, then, \(a(x) > \delta/2\) holds for a neighborhood \(W\) of \(M\) (which can be considered the maximal one satisfying the property \(a(x) > \delta/2\), for all \(x \in W\)). This means, at least, that \(b(x) > \delta/2\) in \(M \setminus W\). The most interesting case occurs when one has simultaneous but complementary damping effects.

We have the first technical lemma.

**Lemma 2.2.** We have
\[
\varphi(x) + b(x) \geq \frac{\delta}{2}, \quad \text{for all} \quad x \in M.
\]

**Proof.**

(i) \(x \in a^{-1}([\delta/2, +\infty[)\). In this case, since \(\varphi(x) \geq \delta/2\) and \(b(x) \geq 0\), we obtain
\[
\varphi(x) + b(x) \geq \delta/2.
\]

(ii) \(x \notin a^{-1}([\delta/2, +\infty[)\). We have \(0 \leq a(x) < \delta/2\) which implies that \(-a(x) > -\delta/2\). From this last inequality and taking assumption (7) into account, we deduce
\[
\varphi(x) + b(x) \geq b(x) \geq \delta - a(x) > \delta - \delta/2 = \delta/2,
\]
which proves the inequality. 

Next, we announce the second useful result.

**Lemma 2.3.** The following inequalities hold:

\[
\int_M \left((\varphi(x))^2 + |\nabla \varphi(x)|^2\right)w^2 \, dx \leq C \int_M a(x)|\nabla w|^2 \, dx, \tag{60}
\]
\[
\int_M \left((\varphi(x))^2 + |\nabla \varphi(x)|^2\right)|\nabla w|^2 \, dx \leq C \int_M a(x)|\nabla w|^2 \, dx, \tag{61}
\]

for all \(w \in H_0^1(M)\) and for some positive constant \(C\) which depends on \(\varphi, a_0\).

**Proof.** Indeed, before proving inequality (60) let us remember an useful result which is, in fact, a variant of Poincaré inequality, namely: Let \(\Omega_1, \Omega_2\) be subsets of \(M\) with
positive measure and such that $\Omega_1 \subset \Omega_2$. Then, assuming that $\text{meas}(\partial \Omega_2 \cap \partial M) \neq 0$, we have:
\[
\int_{\Omega_1} |\omega|^2 \, dx \leq C \int_{\Omega_2} |\nabla \omega|^2 \, dx; \quad \forall \omega \in H^1_0(M),
\]
where $C$ is a positive constant. The proof of the last inequality is immediate. Indeed, it is sufficient to observe that $\omega|_{\partial \Omega_2 \cap \partial M} = 0$ and $\text{meas}(\partial \Omega_2 \cap \partial M) > 0$.

On the other hand, from assumption (6) and since $a$ is continuous there exist $\varepsilon_0 > 0$ and $V \subset M$, neighborhood of $\partial M$ such that $\text{meas}(\partial V \cap \partial M) > 0$ and $a(x) \geq \varepsilon_0$ for all $x \in V$. Setting, $\Omega_1 := \text{supp}(\varphi)$, $\Omega_2 := \{ x \in M; a(x) > \max\{\delta/4, \varepsilon_0\} := a_0 \}$ and considering $\omega \in H^1_0(M)$, from the above statements, we deduce that
\[
\int_M ((\varphi(x))^2 + |\nabla \varphi(x)|^2)w^2 \, dx = \int_{\Omega_1} (\varphi(x))^2 + |\nabla \varphi(x)|^2)w^2 \, dx
\leq C a_0^{-1} \int_{\Omega_2} a(x)|\nabla w|^2 \, dx,
\]
which concludes the proof of (60). The proof of inequality (61) is immediate.

Before announcing the next result let us introduce for short, the notation
\[
(g \circ v)(t) := \int_0^t g(t - s)||v(t) - v(s)||^2_{L^2(M)} \, ds.
\]

We also need in the proof of the main theorem the third technical lemma.

**Lemma 2.4.** Let $u$ be a solution of (1), $\psi \in L^1(0, \infty)$ and $\varphi = \varphi(x)$ a smooth function. Then,
\[
\|(\psi \circ (\varphi u))\|_{L^2(M)}^2 \leq ||\psi||_{L^1(0, \infty)}(\psi \circ (\varphi u))(t).
\]

**Proof.** We have, making using of Cauchy-Schwarz inequality and Fubini’s theorem,
\[
\|(\psi \circ (\varphi u))\|_{L^2(M)}^2 = \int_M \left( \int_0^t \frac{\psi(t) - s}{\sqrt{\psi(t) - s}} \varphi(x)(u(t) - u(s)) \, ds \right)^2 \, dx
\leq \int_M \left( \int_0^t \psi(x) \, dx \right) \int_0^t \psi(t) \varphi(x)^2(u(t) - u(s))^2 \, ds \, dx
\leq ||\psi||_{L^1(0, \infty)} \int_0^t \psi(t) \left( \varphi(\cdot)(u(t) - u(s)) \right)^2_{L^2(M)} \, dx \, ds,
\]
which proves the lemma.

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