On a family of unitary representations of mapping class groups

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Abstract

For a compact surface $S = S_{g,n}$ with $3g+n \geq 4$, we introduce a family of unitary representations of the mapping class group $\text{Mod}(S)$ based on the space of measured foliations. For this family of representations, we show that none of them almost has invariant vectors. As one of applications, we obtain an inequality concerning the action of $\text{Mod}(S)$ on the Teichmüller space of $S$. Moreover, using the same method plus recent results about weakly equivalence, we also give a classification, up to weakly equivalent, for the unitary quasi-representations with respect to geometrical subgroups.

1 Introduction

Let $S = S_{g,n}$ be a compact, connected, orientable surface of genus $g$ with $n$ boundaries, the mapping class group $\text{Mod}(S)$ of $S$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of $S$ which preserving each boundary components (without the assumption that it should fix each boundary pointwise). Throughout this paper, $(g,n)$ is assumed to satisfy $3g+n \geq 4$ and a subsurface of $S$ is allowed to be disconnected.

Given a discrete group $G$, a unitary representation is a pair $(\pi, V)$ where $V$ is a Hilbert space and $\pi : G \to U(V)$ is a homomorphism from $G$ to the group of all unitary operators of $V$ [5]. Infinite dimensional unitary representations of mapping class groups $\text{Mod}(S)$ received a lot of attentions recently. In [20], the author considers the unitary representations given by actions of mapping class groups on the curve complexes associated to $S$. In [2], [1],[12], the authors construct unitary representations based on the action of mapping class groups on the representation varieties of the surface group. There are also many research on this topic from the perspective of TQFTs, for example, one remarkable announcement is [3].

The group $\text{Mod}(S)$ acts on the space of measured foliations, which is defined as the set of equivalence classes of measured foliations on $S$. As the action of $\text{Mod}(S)$ on $\mathcal{MF}(S)$ is ergodic with respect to generalized Thurston measures $\mu$ [16],[17],[14], [13] (see Section 3.1.1 for a brief description of the
measures), one obtains a family of unitary representations by considering
the action of $\text{Mod}(S)$ on the space $L^2(\mathcal{MF}(S),\mu)$. It is quite easy to see that
the unitary representation considered in [20] is a special one in this family.
However, unlike those studied in [20], Example 3.5 will show that some of
the representations considered here are reducible.

**Definition 1.1.** Let $(\pi,V)$ be a unitary representation of a discrete group
$G$. The representation $\pi$ is said to almost have invariant vectors if for every
finite set $K \subseteq G$ and every $\epsilon > 0$, there exists $v \in V$ such that
$$\sup_{g \in K} \|\pi(g)v - v\| < \epsilon \|v\|.$$  

**Remark 1.2.** In the language of [5], see also Section 5, this definition means
that the trivial representation is weakly contained in the representations $\pi$.

The main result of this paper is about the existence of almost invariant
vectors for the associated representations $\pi^\mu$ of the action of $\text{Mod}(S)$ on
$L^2(\mathcal{MF}(S),\mu)$. The existence of such vectors for other representations of
mapping class group has also been discussed in [3].

**Theorem 1.3** (Theorem 4.1). For a compact surface $S = S_{g,n}$ with $3g+n \geq 4$ and all of the generalized Thurston measures $\mu$, the associated representa-
tions $\pi^\mu$ of $\text{Mod}(S)$ does not have almost invariant vectors.

The first direct application of this theorem is the following:

**Corollary 1.1** (Corollary 4.1). Let $S = S_{g,n}$ be a compact surface with $3g+n \geq 4$ and $\mu$ be a generalized Thurston measure, then $\tilde{H}^1(\text{Mod}(S),\pi^\mu) = \overline{H}^1(\text{Mod}(S),\pi^\mu)$, where $\pi^\mu$ is the associated representations of $\text{Mod}(S)$.

For the second application, we will obtain a geometric inequality concerning
the action of $\text{Mod}(S)$ on the Teichmüller space $\text{Teich}(S)$ of independent
interests.

**Corollary 1.2** (Corollary 4.2). Let $\gamma$ be an isotopy class of essential simple
closed curve and $S = S_{g,n}$ be a compact surface with $3g+n \geq 4$. Then
there exists a finite subset $\{\phi_1, \ldots, \phi_n\}$ of $\text{Mod}(S)$ consisting pseudo-Anosov
mappings and a constant $\epsilon > 0$, such that, for every point $X$ in $\text{Teich}(S)$, we have:
$$\sup_i \left( \sum_{\alpha \in \text{Mod}(S), \gamma} e^{-2\ell_X(\alpha)} \left( e^{\Delta_{\alpha,\gamma}^\phi} - 1 \right)^2 \right) \geq \epsilon \sum_{\alpha \in \text{Mod}(S), \gamma} e^{-2\ell_X(\alpha)},$$  

where $\Delta_{\alpha,\gamma}^\phi = \ell_X(\alpha) - \ell_{\phi_i,\gamma}(\alpha)$ and $\ell_X(\alpha)$ is the geodesic length of $\alpha$.

For the unitary representations associated to the discrete measures on the
space of measured foliations, although some of them are irreducible and some
are reducible, we will make it clear that actually the irreducible components
appeared in reducible ones are essential those irreducible ones in \([20]\) (See Proposition 5.1). We then use the same method of the main theorem, combined with the results in \([8],[7],[4]\), to give a classification for a family of quasi-regular unitary representations, which is a stronger version of Corollary 5.5 in \([20]\). Recall that, given two unitary representations \((\pi, \mathcal{H}), (\phi, \mathcal{K})\) of a discrete group \(G\), \(\pi\) is weakly contained in \(\phi\) if for every \(\xi \in \mathcal{H}\), every finite subset \(Q\) of \(G\) and \(\epsilon > 0\), there exist \(\eta_1, \ldots, \eta_n\) in \(\mathcal{K}\) such that

\[
\max_{g \in Q} |\langle \pi(g)\xi, \xi \rangle - \sum_{i=1}^{n} \langle \phi(g)\eta_i, \eta_i \rangle | < \epsilon.
\]

If \(\pi\) is weakly contained in \(\phi\) and \(\phi\) is weakly contained in \(\pi\), then \(\phi\) and \(\phi\) are said to be weakly equivalent. We then have the following theorem.

**Theorem 1.4 (Theorem 5.3).** Let \(S = S_{g,n}\) be a compact surface with \(3g + n \geq 4\). Let \(\gamma, \delta\) be two geometric multi-curves (i.e., unions of pairwise distinct isotopy classes of essential simple closed curves which have zero geometric intersection numbers) with the number of geometric components is equal to \(k, l\), respectively. Then

1. If at least one of \(k, l\) is not \(3g - 3 + n\), then the associated unitary representation \(\pi_\gamma, \pi_\delta\) is weakly equivalent if and only if \(\gamma, \delta\) are in the same type.

2. Suppose \(S\) is not \(S_{0,1}, S_{1,1}, S_{1,2}, S_{2,0}\). If the number of geometric components of \(\gamma\) is \(3g - 3 + n\), then \(\pi_\gamma\) is weakly equivalent to the regular representation \(\lambda_S\). Therefore, if the number of geometric components of \(\gamma\) is not \(3g - 3 + n\), \(\pi_\gamma\) is not weakly contained in \(\lambda_S\).

This paper is organized as follows. Section 2 is devoted to preliminary for group cohomology with coefficient in unitary representations. The proof of the main theorem is given in Section 4. The proof in the case of discrete measures is elementary, however, the non-discrete cases need a technical statement, see Proposition 3.2, and then we can complete our proof via the proof for discrete measures. Section 3 is mainly devoted to this proposition and Section 5 is for irreducible decomposition and classification up to weakly containment.

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2 Cohomology with coefficients in representations

Cohomology and reduced cohomology For a discrete group \( G \) and a unitary representation \((V, \pi)\), one can talk about both cohomology and reduced cohomology group of \( G \) with coefficients in \( \pi \). Definitions of cohomology and reduced cohomology of discrete group with coefficients in a representation \( \pi \) are standard, so we only refer to [15],[2],[5]. Briefly speaking, one defines the following vector spaces for a unitary representation \((V, \pi)\):

\[
Z_1^1(G, \pi) \doteq \{ b : G \to V | b(gh) = b(g) + \pi(g)b(h), \text{ for all } g, h \in G \};
\]

\[
B_1^1(G, \pi) \doteq \{ b \in Z_1^1(G, \pi) | \text{there exists } v \in V, \text{ such that for all } g \in G, b(g) = \pi(g)v - v \};
\]

\[
H_1^1(G, \pi) \doteq Z_1^1(G, \pi)/B_1^1(G, \pi);
\]

\[
\overline{H}_1^1(G, \pi) \doteq Z_1^1(G, \pi)/B_1^1(G, \pi),
\]

where the closure in the last one is understood in the sense of uniform convergence. The vector space \( H_1^1(G, \pi) \) (resp. \( \overline{H}_1^1(G, \pi) \)) is the first (resp. reduced) cohomology group with coefficient in \( \pi \).

Almost invariant vectors Given a unitary representation \((V, \pi)\) of \( G \), it is not easy to determine whether \( H_1^1(G, \pi) \) equal to \( \overline{H}_1^1(G, \pi) \), however, the following theorem, which back to Guichardet, provide a way to determine it.

**Theorem 2.1** ([15]). Let \( G \) be a finite generated discrete group, \((V, \pi)\) be a unitary representation without nonzero invariant vectors. Then the following are equivalent:

1. The associated first reduced cohomology is the same as the first cohomology, that is, \( H_1^1(G, \pi) = \overline{H}_1^1(G, \pi) \);

2. The representation \( \pi \) does not have almost invariant vectors.

One observation is that not having almost invariant vectors is closed under taking limit, more precisely, we have the following lemma.

**Lemma 2.2.** Let \((V, \pi)\) be a unitary representation of \( G \), \( W \) be a \( G \)-invariant vector subspace of \( V \) such that the closure \( \overline{W} = V \). Then \( \pi \) does not have almost invariant vectors if and only if \( \pi|_W \) does not have almost invariant vectors.

**Proof.** Suppose that the pair \((K, \epsilon)\), where \( K \) is a finite subset of \( G \) and \( \epsilon > 0 \), is given by the condition that \( \pi|_W \) does not have almost invariant vector. Given any element \( \xi \in V - W \), there is a sequence of elements \( \{\xi_n\} \subseteq W \) such that \( \xi_n \to \xi \) as \( n \to \infty \). Then, for \( n \) enough large, we have:

\[
\sup_{g \in K} \| \pi(g)\xi - \xi \| = \sup_{g \in K} \| \pi(g)\xi - \pi(g)\xi_n + \pi(g)\xi_n - \xi_n + \xi_n - \xi \|
\]
\[ \geq \sup_{g \in K} \| \pi(g)\xi_n - \xi_n \| - 2\sup_{g \in K} \| \xi_n - \xi \| \geq \epsilon \| \xi \| - \delta. \]

Now \( \delta \) can be enough small, so

\[ \sup_{g \in K} \| \pi(g)\xi - \xi \| \geq \epsilon \| \xi \|, \]

Which complete the proof of one direction. The opposite direction is obvious. \( \square \)

Another easy observation is that, in order to show a representation of group does not have almost invariant vectors, one only need to pass to a subgroup. That is,

**Lemma 2.3.** A unitary representation \((\pi, V)\) of a group \(G\) does not have almost invariant vectors iff there exists a subgroup \(H\) of \(G\) such that the unitary representation \((\pi|_H, V)\) of \(H\) does not have almost invariant vectors.

**Amenable groups** A basic strategy in this article is to use the regular representation of free group \(F_2\) of rank 2, so the following theorem is of fundamental importance.

**Theorem 2.4 ([9]).** For the left regular representation \(\pi\) of a finitely generated discrete group \(G\) on \(\ell^2(G)\), \(\pi\) almost has invariant vectors if and only if \(G\) is amenable.

**Remark 2.5.** Since \(F_2\) is not amenable, the left regular representation of \(F_2\) on \(\ell^2(F_2)\) does not have almost invariant vectors. We will regard \(\ell^2(F_2)\) as \(\ell^2\)-functions on vertices of the Cayley graph of \(F_2\) with respect to some chosen generators, and thus further identify \(\ell^2(F_2)\) with the vector space \(V\), where

\[ V = \{ \sum_i \alpha_i g_i \mid \sum_i |\alpha_i|^2 < \infty, \alpha_i \in \mathbb{C}, g_i \in F_2 \}. \]

### 3 Generalized Thurston measures and dynamics on measured foliation spaces

In this section we will describe the integral theory on the space of measured foliations and the action of groups of mapping class on the space of measured foliations. A subgroup of \(Mod(S)\) in which all elements except identity are pseudo-Anosov mappings will be called a pseudo-Anosov subgroup.
3.1 Measures and $L^2$–theory on $\mathcal{MF}(S)$

3.1.1 Generalized Thurston measures on $\mathcal{MF}(S)$

The space of measured foliations $\mathcal{MF}(S)$ of $S$ is the set of equivalence classes of transversal measured (singular) foliations on $S$. Using train tracks, one can show that $\mathcal{MF}(S)$ has a piecewise linear integral structure such that $\text{Mod}(S)$ acts on it as automorphisms (that is, preserve this piecewise linear integral structure)\[21\]. Therefore, in the local PL coordinates, $\text{Mod}(S)$ acts as linear transformations.

A consequence of this PL structure is that $\mathcal{MF}(S)$ can be equipped with a $\text{Mod}(S)$–invariant measure $\mu_{\text{Th}}$, called the Thurston measure. Moreover, this measure can be generalized to obtain a family of locally finite, ergodic $\text{Mod}(S)$–invariant measures $\mu_{\text{Th}}^{[[\mathcal{R},\gamma]]}$ on $\mathcal{MF}(S)$ which will be called the generalized Thurston measures and those measures are classified by the following theorem:

**Theorem 3.1** (Hamenstädt\[13\], Lindenstrauss-Mirzakhani\[14\]). Any locally finite $\text{Mod}(S)$–invariant ergodic measure on $\mathcal{MF}(S)$, up to a constant multiple, is one of $\mu_{\text{Th}}^{[[\mathcal{R},\gamma]]}$.

We present a brief summary of the construction of generalized Thurston measures according to \[14\]. For any (not necessary connected) subsurface $\mathcal{R}$ of $S$ with boundary that smooth embedded in $S$ and no component of $S - \mathcal{R}$ homeomorphic to a disk, define

$$\mathcal{MF}(\mathcal{R}) = \prod_i \mathcal{MF}^*(S_i)$$

where $S_i$ is the finite connected component of $\mathcal{R}$ and $\mathcal{MF}^*(S_i) = \mathcal{MF}(S_i) \cup 0_{S_i}$ in which $0_{S_i}$ is the zero foliation on $S_i$. The space $\mathcal{MF}(\mathcal{R})$ can be $\text{Mod}(\mathcal{R})$–embedded on $\mathcal{MF}(S)$ via enlarging. Denote by $\mathcal{M}(\mathcal{R})$ the image of this embedding. This set is endowed with a measure from the product measure. A pair $(\mathcal{R}, \gamma)$ is said to be a complete pair if $\gamma$ is a multicurve and $\mathcal{R}$ is a union of connected components of the surface obtained by cutting along the support of $\gamma$. Now, given any complete pair $(\mathcal{R}, \gamma)$, define

$$M(\mathcal{R}, \gamma) = \{ F + \gamma : F \in \mathcal{M}(\mathcal{R}) \} \subseteq \mathcal{MF}(S).$$

This set gives rise to the measure $\mu_{\text{Th}}^{[[\mathcal{R},\gamma]]}$ on $\mathcal{MF}(S)$ supported on the set of $\text{Mod}(S)$–orbits of $M(\mathcal{R}, \gamma)$. Special cases are the cases when $\mathcal{R} = \emptyset$ and $\gamma$ is an isotopy class of non-separating curve, or when $\mathcal{R} = S$ and $\gamma = \emptyset$. The corresponding measure in the case of $\mathcal{R} = \emptyset$ is the discrete measure, denoted by $\mu_1$, supported the orbit the subset of $\mathcal{MF}(S)$ corresponding to $\text{Mod}(S).\gamma$, while in the case of $\gamma = \emptyset$ is exactly the Thurston measure $\mu_{\text{Th}}$. 


3.1.2 Associated $L^2$–theory over $\mathcal{MF}(S)$

Discrete measures case Recall that when $\mathcal{R} = \emptyset$, $\mu^{(R,\gamma)}_{Th}$ is the discrete measure supported on the set $\text{Mod}(S)\gamma$, where $\text{Mod}(S)\gamma$ is regarded as a subset of $\mathcal{MF}(S)$. We will first deal with the case that $\gamma$ is an isotopy class of an essential simple closed curve and denote the measure by $\mu_\gamma$.

Let $X_\gamma = \mathcal{C}_\gamma^0(S)$ be the subset of vertices of the curve complex consisting of $\text{Mod}(S)\gamma$. By considering the discrete measure, one can define the Hilbert space $\ell^2(X_\gamma)$. It is clear that $\ell^2(X_\gamma)$ is $\text{Mod}(S)$–equivariantly isomorphic to $L^2(\mathcal{MF}(S), \mu_\gamma)$. On the other hand, Let $G_\gamma = \text{Mod}(S, \gamma)$ be the set of all elements that fix $\gamma$, then $\ell^2(X_\gamma)$ can be further $\text{Mod}(S)$–equivariantly identified with $\ell^2(\text{Mod}(S)/G_\gamma)$. Each of them give the same unitary representation of $\text{Mod}(S)$, actually we have

**Theorem 3.2** (Paris[20]). The infinite dimensional unitary representation of $\text{Mod}(S)$ given by $\ell^2(\text{Mod}(S)/G_\gamma)$ is irreducible.

**Remark 3.3.** Actually, this theorem was proved in more general setting, that is, $\gamma$ is a collection of pairwise distinct isotopy classes of essential simple closed curves which have zero geometric intersection numbers.

Thus, in particular, this representation does not have non-zero invariant vectors. Meanwhile, the irreducibility also allows us to describe $\ell^2(\text{Mod}(S)/G_\gamma)$ more geometrically.

The first description of $\ell^2(\text{Mod}(S)/G_\gamma)$ is classical. For $f \in \ell^2(X_\gamma)$, let $\text{Supp}(f) = \{v \in X_\gamma : f(v) \neq 0\}$. The function $f$ is compactly supported if the cardinal of $\text{Supp}(f)$ is finite. Define the subspace $W$ of $\ell^2(X_\gamma)$ as the set of all of elements in $\ell^2(X_\gamma)$ which is compactly supported. As $X_\gamma$ is discrete, following notation will be used to represent $f \in W$: $f = \sum_{i=1}^n k_i \alpha_i$. Note that $W$ is $\text{Mod}(S)$–invariant and the closure $\overline{W}$ of $W$ in $\ell^2(X_\gamma)$ is then $\ell^2(X_\gamma)$ itself. This description will be used in the proof of the main theorem in the case of discrete measures.

The second description of $\ell^2(\text{Mod}(S)/G)$ needs more explanations. Let $\text{Teich}(S)$ be the Teichmüller space of $S$, and for each point $x$ of $\text{Teich}(S)$, define a function on $X_\gamma$ by

$$f_x(\alpha) = e^{-\ell_x(\alpha)}, \alpha \in X_\gamma$$

where $\ell_x(\alpha)$ is the length of the unique geodesic in the homotopic class $\alpha$.

**Proposition 3.1.** The function defined above is actually in $\ell^2(X_\gamma)$

**Proof.** (F.Labourie) It amounts to say

$$\sum_{\alpha \in X_\gamma} e^{-2\ell_x(\alpha)} < \infty.$$ 

Thus this proposition is a corollary of the result of [6] or [19] about the polynomial growth of simple closed geodesics. \qed
Let $W'$ be the subspace of $\ell^2(X_\gamma)$ which consisting all of finite linear combinations of such kind of elements. It is also easy to see that this subspace is $\text{Mod}(S)$–invariant. Also by using the irreducibility, the closure $\overline{W'}$ of $W'$ is $\ell^2(X_\gamma)$.

**Remark 3.4.** The second description give rise to a parametrization for $\ell^2(X_\gamma)$ via the Teichmüller space, thus it can be viewed as some kind of answer, in the present setting, for Problem 2.5 in [12].

For the case of $R = \emptyset$ and $\gamma$ is a general integral multicurve, the result as in Theorem 3.2 may not be true, see an example below.

**Example 3.5.** Consider the genus 2 closed surface $S$, regarded as a quotient along boundaries of holed sphere with four disjoint open disks deleted. Let $\gamma = 2\gamma_1 + 3\gamma_2, \delta = \gamma_1 + \gamma_2$, where $\gamma_i$ is the isotopic classes of two distinct images of boundaries. Obviously, there is a mapping class $s$ that permutes the $\gamma_i$’s. Denote $H = \text{Stab}_{\text{Mod}(S)}(\gamma)$ and $H' = \text{Stab}_{\text{Mod}(S)}(\delta)$, then we have the exact sequence:

$$1 \to H \to H' \to \mathbb{Z}_2 \to 1.$$  

That is, $H$ is a normal subgroup of $H'$ of index 2. This exact sequence allows us to define a self-map of the left cosets $\{fH\}$ as follows. Write $H'$ as $H \sqcup sH$. There are two $\text{Mod}(S)$–invariant bijections:

$$\text{Mod}(S).\gamma \leftrightarrow \{ [g] = gH \},$$

$$\text{Mod}(S).\delta \leftrightarrow \{ [f] = fH' \}.$$  

As $fH' = fH \sqcup fsH$, the set $\{gH\}$ can be re-wrote as $\{fH, fsH\}$, this formulation induce a well-defined inversion $i : fH = [f] \mapsto [fs] = fsH$.

A function $\phi$ on $G/H = \{gH\}$ is called even if for every $[g] \in G/H$, $\phi([g]) = \phi(i([g]))$ and a function $\varphi$ on $G/H$ is called odd if for every $[g] \in G/H$,  $\varphi([g]) = -\phi(i([g]))$.

Define $V_1$ to be the subset of $\ell^2(G/H)$ consisting of even functions and $V_2$ to be the subset of $\ell^2(G/H)$ consisting of odd functions. It is easy to see that these two vector spaces are non-empty, closed and $\text{Mod}(S)$–invariant subspace of $\ell^2(G/H)$.

**Remark 3.6.** For any discrete measures mentioned above (including the case that all of the connected components of $R$ are $S_{0,3}$), all of the associated unitary representations have no nonzero invariant vectors.

**Non-discrete measures case** For non-discrete measures case, we mention one remark.

**Remark 3.7.** If $R$ is nontrivial, ergodicity of the action shows that the corresponding unitary representations all have no nonzero invariants.
3.2 Actions of subgroups of $\text{Mod}(S)$ on $\mathcal{M} \mathcal{F}(S)$

Almost properly discontinuous action

We introduce a concept for a group acts on a Borel space (that is, a topological space endowed with a Radon measure) which is weaker than usual properly discontinuous action.

**Definition 3.8.** Let $G$ be a group and $(X, \mu)$ be a Borel space. Suppose that $G$ acts on $X$ by measure-preserving homeomorphisms. We say that $G$ acts on $X$ **almost properly discontinuously** if there exists a $G$-invariant subset $K$ with $\mu(K) = 0$ such that $G$ acts on $X - K$ properly discontinuously.

**Example 3.9.** Let $H \leq \text{SL}(2, \mathbb{Z})$ be a Schottky group, then its limit set $\Lambda(H) \subseteq S^1$ has zero Lebesgue measure, and thus it acts on $S^1$ almost properly discontinuously.

Although the action of $\text{Mod}(S)$ on $\mathcal{M} \mathcal{F}(S)$ is ergodic with respect to the generalized Thurston measures, the action of subgroups of $\text{Mod}(S)$ on $\mathcal{M} \mathcal{F}(S)$ is not always ergodic. The following proposition allows us to use properties of the “properly discontinuous” action.

**Proposition 3.2.** For each complete pair $(\mathcal{R}, \gamma)$, there exists a free pseudo-Anosov subgroup $H$ of $\text{Mod}(S)$ acts on $\mathcal{M} \mathcal{F}(S)$ almost properly discontinuous with respect to the generalized Thurston measures.

Any such free group will be called a *p-rank 2 free subgroup*.

The first case is when $\mathcal{R} = \emptyset$ or each components of $\mathcal{R}$ is $S_{0,3}$, then this proposition is obvious by taking $H$ to be any free pseudo-Anosov group generated by two pseudo-Anosov maps (this works same for non-integral multicurves as for integral multicurves). For other cases, we prove this proposition through several lemmas.

**Lemma 3.10.** There exists a subgroup $H$ of $\text{Mod}(S)$ that acts on $\mathcal{M} \mathcal{F}(S)$ almost properly discontinuous with respect to the Thurston measure $\mu_{Th}$.

**Proof.** If $S = S_{0,4}$ or $S_{1,1}$, then, in both cases, $\mathcal{M} \mathcal{F}(S)$ can be identified with $R^2 - (0,0)$ and $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ can be identified with $S^1$. Moreover, there is a finite index subgroup of $\text{Mod}(S)$ such that the action of this subgroup on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ is equivalent to the action of $\text{PSL}(2, \mathbb{Z})$ on $S^1$, see [10], Chapter 15] for the case of $S_{0,4}$. By taking $H$ to be any subgroup given in Example 3.9 and considering the set $Y = P^{-1}_r(\Lambda(H))$, where $P_r : \mathcal{M} \mathcal{F}(S) \rightarrow \mathcal{P} \mathcal{M} \mathcal{F}(S)$ is the projection, the action of $H$ on $\mathcal{M} \mathcal{F}(S)$ is thus almost properly discontinuous and $\mu_{Th}(Y) = 0$.

For other $S$, we also deduce this lemma by first passing to $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ and use the result of [18] on limit sets. As in [11], Exposé 13], a rank 2 free pseudo-Anosov subgroup $H$ can be constructed such that the limit set $\Lambda(H)$ that is defined to be the closure of the set of fixed points (which are all linear foliations) of elements of $H$ is a subset of a circle $\mathcal{C}$ in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ (In [11], a
construction is described in the case of closed surface and it also mentioned a construction in the case of general compact surface). Moreover, \( \Lambda(H) \) consists of pseudo-Anosov foliations and enlarging of multicurves. On the other hand, one can define the zero set \( Z(\Lambda(H)) \subseteq \mathcal{PMF}(S) \) of \( \Lambda(H) \) [18]. By combining the fact that \( Z(\Lambda(H)) - \Lambda(H) \) consists of no uniquely ergodic foliations and uniquely ergodic foliation has full \( \mu_{Th} \) measure with the fact that \( \Lambda(H) \) has Lebesgue dimension at most 1, dimension counting implies \( P_r^{-1}(Z(\Lambda(H))) \) has \( \mu_{Th} \) - measure zero, which complete the proof.

A complete pair \((\mathcal{R}, \gamma)\) is called a **middle type** if \( \mathcal{R} \neq \emptyset \), no connected components is \( S_{0,3} \) and \( \mathcal{R} \neq S \).

**Lemma 3.11.** For a complete pair \((R, \gamma)\) of middle type, there exists a subgroup \( H \) of \( \text{Mod}(S) \) acts on \( \mathcal{MF}(S) \) almost properly discontinuous with respect to the measure \( \mu_{Th}([R, \gamma]) \).

**Proof.** We will follow the idea of [14], Lemma 3.1 to prove this lemma. Fix any hyperbolic structure \( X \) on \( S \) and consider the continuous function \( \ell_X : \mathcal{MF}(S) \to R_+ \) extending the geodesic length function. Thus

\[
\mathcal{MF}(S) = \lim_{L_1 \to 0, L_2 \to \infty} B_{L_2}^{L_1}(X),
\]

\[
B_{L_2}^{L_1}(X) = \{ \nu \in \mathcal{MF}(S) : \ell_X(\nu) \in [L_1, L_2] \}.
\]

\( B_{L_2}^{L_1}(X) \) is a compact set and as pointed out in the proof of [14], Lemma 3.1, \( B_{L_2}^{L_1}(X) \cap (\bigcup_{g \in \text{Mod}(S)} g \cdot M(\mathcal{R}, \gamma)) \) is equal to \( B_{L_2}^{L_1}(X) \cap (\bigcup_{i=1}^{n} g_i \cdot M(\mathcal{R}, \gamma)) \), for some finite set \( \{g_1, ..., g_n\} \subset \text{Mod}(S) \). Fix a free pseudo-Anosov subgroup \( H \) of \( \text{Mod}(S) \) and take any compact subset \( K \subset \bigcup_{g \in \text{Mod}(S)} g \cdot M(\mathcal{R}, \gamma) \). The set \( K \) is then a compact subset of \( \mathcal{MF}(S) \). Taking \( L_1 \) small enough and \( L_2 \) large enough, one can assume \( K \subset B_{L_2}^{L_1}(X) = B \). We now claim that

\[
\{ h \in H : h \cdot B \cap B \neq \emptyset \} < \infty.
\]

Since every element in \( B \) can be written as \( \gamma + \nu \) such that \( \ell_X(\gamma) \) is bounded. If \( h \cdot B \cap B \neq \emptyset \), then \( h(\gamma) \) also has bounded \( \ell_X \) - length and all of those bounds can be chosen to be uniform. Form a weight curve complex (that is, a curve complex with a positive number at each vertex) \( Z = \text{Mod}(S), \gamma \) and thus \( \ell_X : Z \to R_+ \) is a proper map (that is, the inverse of compact set is also compact). Then if one fixing compact \( K' \subset Z \), then \( \{ h \in H : h \cdot B \cap B \neq \emptyset \} \subset \{ h \in H : h \cdot K' \cap K' \} \). By the discussion of the case \( \mathcal{R} = \emptyset \), the last set is finite. Taking the zero measure set to be \( Y = \mathcal{MF}(S) - \bigcup_{g \in \text{Mod}(S)} g \cdot M(\mathcal{R}, \gamma) \) completes the proof.

**H-related covering** Given a generalized Thurston measure \( \mu \) such that at least one connected component of the corresponding surface \( \mathcal{R} \) is neither empty nor \( S_{0,3} \), Proposition 3.2 gives a free pseudo-Anosov subgroup \( H \) and
a zero $\mu$-measure set $Y$. For any compact subset $K$ of $\mathcal{MF}(S) - Y$, we will describe a “nice” covering of $K$. Since $\mathcal{MF}(S) - Y$ is the domain of discontinuity of $H$, there is an open neighbourhood $\mathcal{U}_p$ of $p$ in $K$ with finite nonzero $\mu$-measure such that for all $h \in H, h\mathcal{U}_p \cap \mathcal{U}_p = \emptyset$ for every $p$ in $K$. Thus there is now an open cover of $K$. By compactness of $K$, choose a finite sub-cover of this cover. Label this sub-cover by $\mathcal{U}_1,...,\mathcal{U}_n$ and for each $i \in 1,...,n$, consider $A_i = \{ h\mathcal{U}_i | h \in H \}$. Starting from $i = 1$, form a family $B_1 = \{ X_k \in A_1 | X_k \cap K \neq \emptyset \}$ as well as $C_1 = \{ Y_k | Y_k = X_k \cap K, X_k \in B_1 \}$. Delete $\bigcup_{Y_k \in C_1} X_k$ from $K$, denote the resulting compact set by $K_1$. Then for $K_1$, there is a family $B_2 = \{ X_k \in A_2 | X_k \cap K_1 \neq \emptyset \}$ as well as $C_2 = \{ Y_k | Y_k = X_k \cap K_1, X_k \in B_2 \}$. Delete $\bigcup_{Y_k \in C_2} X_k$ from $K_2$, denote the resulting compact set by $K_3$. Continuing this process, there is a cover of $K$ which can be written in the following formula:

$$K \subseteq \bigcup_{k=1}^{n} \bigcup_{Y_k \in C_k} Y_k.$$

So $K$ can be covered by finite many nonzero $\mu$-measurable set such that each pair of them are disjoint. This will be called a $H$-related covering of $K$, since, for each $k$, $C_k$ is a family of disjoint set that inside $H$-orbit of some set.

## 4 Nonexistence of almost invariants

Let $\mathcal{H}(\mu) = L^2(\mathcal{MF}(S), \mu)$, where $\mu$ is a generalized Thurston measure explained in Section 3.1.1, and $\pi^\mu$ be corresponding unitary representation of $\text{Mod}(S)$. The main result of this section is the following:

**Theorem 4.1.** For a compact surface $S = S_{g,n}$ with $3g + n \geq 4$ and all of the generalized Thurston measures $\mu$, the associated representations $\pi^\mu$ of $\text{Mod}(S)$ does not have almost invariant vectors.

By using the Theorem 2.1, Remark 3.6 and Remark 3.7, we have:

**Corollary 4.1.** Let $S = S_{g,n}$ be a compact surface with $3g + n \geq 4$ and $\mu$ be a generalized Thurston measure, then $\mathcal{H}^1(\text{Mod}(S), \pi^\mu) = \overline{\mathcal{H}^1(\text{Mod}(S), \pi^\mu)}$, where $\pi^\mu$ is the associated representations of $\text{Mod}(S)$.

Let $\gamma$ be an isotopy class of essential simple closed curve, $X = \text{Mod}(S), \gamma$ and $\mathcal{X} \in \text{Teich}(S)$. Denoting $\Delta_{\phi,\mathcal{X}}(\alpha) = \ell_{\mathcal{X}}(\alpha) - \ell_{\phi,\mathcal{X}}(\alpha)$, where $\alpha \in X$, and using second description of $\ell^2(X)$ in Section 3.1.2, the following inequality is easy to show:

**Corollary 4.2.** Let $\gamma$ be an isotopy class of essential simple closed curve and $S = S_{g,n}$ be a compact surface with $3g + n \geq 4$. Then there exists a
finite subset \( \{ \phi_1, \ldots, \phi_n \} \) of \( \text{Mod}(S) \) consisting pseudo-Anosov mappings and a constant \( \epsilon > 0 \), such that, for every point \( X \) in \( \text{Teich}(S) \), we have:

\[
\sup_i \left( \sum_{\alpha \in \text{Mod}(S), \gamma} e^{-2\ell_X(\alpha)}(e^\Delta x^\alpha(\alpha) - 1)^2 \right) \geq \epsilon \sum_{\alpha \in \text{Mod}(S), \gamma} e^{-2\ell_X(\alpha)}.
\]

We now prove Theorem 4.1. First we prove a lemma whose direct application is the proof for the discrete cases of the theorem.

**Lemma 4.2.** Let \( G \) be a discrete countable group, \( X \) be a discrete set equipped with a \( G \)-action. Suppose that there is a rank 2 free subgroup \( H \) of \( G \) such that, for every \( x \) in \( X \), the stabilizer \( \text{Stab}_H(x) \) of \( H \) at \( x \) is trivial. Then the unitary representation \( \pi = \ell^2(X) \) of \( G \) associated to the action of \( G \) on \( X \) does not have almost invariant vectors.

**Remark 4.3.** This lemma have a very easy proof, the reason we give such elementary proof is to indicate the aspect of discretization of the main theorem.

**Proof.** By Lemma 2.3, we can pass to subgroups. For any point \( p \in X \), consider the image of \( H \) under the map given by

\[
h \mapsto h.p.
\]

Since the stabilizer \( \text{Stab}_H(x) \) of \( H \) at \( x \) is trivial, this map is injective. This image will be called the 2-tree based at \( p \).

Let \( W \) be the subspace of \( \ell^2(X) \) which consisting of functions of finite supports. As \( W \) is \( G \)-invariant, by Lemma 2.2, it is enough to show that \( (\pi, W) \) does not have almost invariant vectors. That is, we have to find \( (K, \epsilon) \) with the property that

\[
\sup_{g \in K} \| \pi(g)f - f \|^2 \geq \epsilon \| f \|^2, \quad \text{for all } f \in W.
\]

Since \( H \cong \mathbb{F}_2 \), as mentioned in Remark 2.5, the left regular representation \( \ell^2(H) \) does not have almost invariant, thus such a pair \( (K, \epsilon) \) exists. Fix such pair once and for all. Here are two facts.

**Facts:**
1. For every 2-tree \( T \) based on some points, \( \ell^2(T) \) is \( H \)-equivariantly isomorphic to \( \ell^2(H) \).
2. Different 2-trees are disjoint and thus, if \( f_1, f_2 \in \ell^2(X) \) such that the support \( A_1, A_2 \) are located in different 2-trees, then they are orthogonal.

These two facts imply that we only need to deal with \( \ell^2 \)-functions on \( X \) with finite support so that the support is contained in one 2-tree. In fact, for every \( f \in W \), if we decompose its support \( K_f \) as

\[
K_f = \bigcup_{i=1}^n K_{f_i},
\]

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where $K_f$, lie in different 2-trees and $f_i$ is defined to be the restriction of $f$ in those different 2-trees, then

$$f = \sum_{i=1}^{n} f_i,$$

$$\|\pi(g)f - f\|^2 = \sum_{i=1}^{n} \|\pi(g)f_i - f_i\|^2, \text{ for all } g \in K.$$  

Note that $K \subseteq H$ is fixed. If the support of $f_i$ is contained in one 2-tree $T_i$, by Remark 2.5, there exists $g_i \in K$ such that

$$\|\pi(g_i)f_i - f_i\| \geq \epsilon \|f_i\|.$$  

Now for every $f_i$, one can assign $g_i$ satisfying the above inequality. If two 2-trees $f_i, f_j$ corresponding to the same $g_i = g_j$, then $f_i + f_j$ also satisfies that inequality. As $K$ is finite, assume $\#K = m$, $f$ can be decomposed in another way, that is, $f = f'_1 + f'_2 + \cdots + f'_s, (s \leq m)$, such that $f'_k = \sum_{j} f_{jk}$, where $f_{jk} \in \{f_1, \ldots, f_n\}$ and $\{f_{jk}\}_j$ corresponding to the same $g_k \in K$. We claim that

there exists $g_l \in K$ such that $\|\pi(g_l)f - f\|^2 \geq \epsilon \|f\|^2.$

Otherwise, since

$$\|\pi(g_i)f - f\|^2 \geq \|\pi(g_i)f_i - f_i\|^2 \geq \epsilon \|f_i\|^2,$$  

(4.1) then

$$\epsilon \|f\|^2 = \sum_{i=1}^{m} \frac{\epsilon}{m} \|f_i\|^2 \geq \sum_{i=1}^{m} \|\pi(g_i)f - f\|^2 \geq \sum_{i=1}^{s} \|\pi(g_i)f - f\|^2 \geq \sum_{i=1}^{s} \epsilon \|f_i\|^2 = \epsilon \|f\|^2.$$  

The second inequality comes from the contradiction assumption and the last inequality is from inequality 4.1. Thus there exists a pair $(K, \eta = \frac{\epsilon}{m})$ such that

$$\sup_{g \in K} \|\pi(g)f - f\|^2 \geq \eta \|f\|^2, \text{ for all } f \in W.$$  

So the proof of the lemma is finished. \hfill \square

Hence, for the case that $\mathcal{H}(\mu_G) = \ell^2(X)$, where $X = \text{Mod}(S) \cdot \gamma$ for a multicurve $\gamma$, $\mu_G$ does not have almost invariant vectors.

**Proof Theorem 4.1.** We only need to deal with the case that $\mathcal{R} = S$ or $\mathcal{R}$ is of middle type. Also by Lemma 2.3, we pass to subgroups. Fix the choice of a p-rank 2 free subgroup $H$ constructed in Proposition 3.2. For any point $p \in \mathcal{MF}(S)$, consider the image of $H$ under the map given by

$$h \mapsto h.p.$$
Since $H$ is a pure pseudo-Anosov group, this map is injective. This image also will be called the 2-tree based at $p$. Define $W$ to be subspace of $\mathcal{H}(\mu)$ consisting all of the function $f \in \mathcal{H}(\mu)$ that have compact support in $\mathcal{MF}(S)$. Here compact subsets is considered as a subset of $\mathcal{MF}(S) - Y$. Thus $\overline{W} = \mathcal{H}(\mu)$. So as before, we only needs to prove the theorem in the case of $(W, \pi^\mu)$.

For each $f \in W$ with the property that its compact support set $K_f$ contained in a disjoint union of sets in one $H-$orbit, that is,

$$K_f \subseteq \bigcup_{h \in H} h.U,$$

fix a point $p$ in $U$, and associate a element $A_f \in \ell^2(\mathbb{T})$, where $\mathbb{T}$ is the 2-tree based on $p$, via

$$A_f(h.p) = (\int_{h.U} |f|^2 d\mu)^{\frac{1}{2}}.$$

Define

$$K' = \{ g \in H | g \text{ or } g^{-1} \in K \},$$

where $K$ is the same finite subset of $H$ as in Case 1. For such $f$, one has:

$$\int_{K_f} |\pi(g)f - f|^2 d\mu = \sum_{h} \int_{h.U} |\pi(g)f - f|^2 d\mu$$

$$\geq \sum_{h} (\int_{h.U} |\pi(g)f|^2 d\mu)^{\frac{1}{2}} - (\int_{h.U} |f|^2 d\mu)^{\frac{1}{2}}^2$$

$$= \sum_{h} |A_{\pi(g)f}(h.p) - A_f(h.p)|^2$$

$$= \sum_{h} |(\pi(g^{-1})A_f)(h.p) - A_f(h.p)|^2,$$

where the second inequality is the result of triangle inequality. By the result of Lemma 4.2, then

$$\sup_{g \in K'} \|\pi(g)f - f\|^2 \geq \sup_{g \in K'} \sum_{h} \|\pi(g)A_f(h.p) - A_f(h.p)\|^2$$

$$= \sup_{g \in K'} \|\pi(g)A_f - A_f\|^2 \geq \eta \|A_f\|^2$$

$$= \epsilon'' \|f\|^2,$$

where $\epsilon$ is a multiple of the constant $\eta$ in Lemma 4.2, since in this case $\#K' = 2\#K$. For the case that the compact set $K_f$ is not contained in one $H-$orbit, take a $H$-related covering of $K_f$, by the orthogonality similar to the Fact 2 in Lemma 4.2 and same proof in the last few lines as Lemma 4.2, the proof of the whole theorem can be completed once the pair $(K', \epsilon'')$ has been chosen, where $\epsilon''$ possibly be a constant multiple of the old $\epsilon'$. \[\square\]
Remark 4.4. The same trick can be used to show that the representation of mapping class groups in the space of $L^2$–functions on the Teichmüller space with respect to the measure given by the Weil-Petersson volume also doesn’t have almost invariant vectors. As one can show that this representation does not have non-trivial invariant vectors, we also have the similar conclusion about the corresponding cohomology group.

5 Classification of quasi-regular representations up to weakly containment

Irreducible decomposition  As pointed out in the Section 3.1.2, For the unitary representations of mapping class group associated to the discrete measures on the space of measured foliations, both reducible and irreducible ones exist. By examining Example 3.5 carefully, reducible representation have a irreducible decomposition. Given any multi-curves $\gamma = \sum_{i=1}^{k} c_i \gamma_i$, where $c_i > 0$ for $1 \leq i \leq k$ and $\{\gamma_i\}$ is a collection of pairwise distinct isotopy classes of essential simple closed curves which have zero geometric intersection numbers. Form $\delta = \sum_{i=1}^{k} \gamma_i$. As before, denote $G_\gamma$ and $G_\delta$ the corresponding subgroups of $\text{Mod}(S)$. Thus $G_\gamma$ is a subgroup of $G_\delta$ of finite index.

Proposition 5.1. Let $S = S_{g,n}$ be a compact surface with $3g + n \geq 4$ and $\gamma, \delta$ as above. Then

1. If the index of $G_\gamma$ in $G_\delta$ is one, then the associated representation $\ell^2(\text{Mod}(S)/G_\gamma)$ of $\text{Mod}(S)$ is irreducible.

2. If the index of $G_\gamma$ in $G_\delta$ is $n > 1$, then associated representation $\ell^2(\text{Mod}(S)/G_\gamma)$ of $\text{Mod}(S)$ is reducible and it can be decomposed as the sum of $n$ irreducible unitary representation each of which is equivalent to the unitary representation $\ell^2(\text{Mod}(S)/G_\delta)$.

Proof. It is obvious for the case of index is equal to one, since the representation $\ell^2(\text{Mod}(S)/G_\gamma) = \ell^2(\text{Mod}(S)/G_\delta)$ is irreducible by Remark 3.3. Now assume $[G_\delta : G_\gamma] = n > 1$. Let $X_\gamma = \text{Mod}(S).\gamma$ and $Y_\delta = \text{Mod}(S).\delta$, then $X_\gamma$ is a $\text{Mod}(S)$–equivariant covering $Y_\delta$ of $n$ sheets. So every $\ell^2$–function on $Y_\delta$ define a $\ell^2$–function on $X_\gamma$, and this corresponding produce a proper closed $\text{Mod}(S)$–invariant subspace of $\ell^2(X_\gamma)$ which shows the reducibility.

Then just as functions can be decomposed as even and odd functions, we can decomposed $\ell^2(X_\gamma)$ as the sum of $n$ irreducible unitary representation each of which is equivalent to the unitary representation $\ell^2(\text{Mod}(S)/G_\delta)$.  

Classification up to weakly containment  We first fix some notations. We denote $\gamma, \delta$ union of pairwise distinct isotopy classes of essential simple closed curves which have zero geometric intersection numbers, we
will call such unions geometric multi-curves and a isotopy classes in the union a geometric component. Denote $G_\gamma, G_\delta$ the corresponding subgroups of $\text{Mod}(S), \pi_\gamma, \pi_\delta$ are the associated unitary representations $\ell^2(\text{Mod}(S)/G_\gamma), \ell^2(\text{Mod}(S)/G_\delta)$ and $\lambda_S$ the regular representation of the mapping class group $\text{Mod}(S)$ of $S$. We first recall some definitions which can be found in [20], [5], [4].

Let $G$ be a countable discrete group, $H$ a subgroup of $G$, the commensurator of $H$ is defined to be

$$\text{Com}_G(H) = \{g \in G : gHg^{-1} \cap H \text{ has finite index in both } H \text{ and } gHg^{-1}\}.$$ 

A discrete group is said to be C*-simple if every unitary representation which is weakly contained in the regular representation of $G$ is weakly equivalent to the regular representation. Let $\gamma, \delta$ are geometric multi-curves, then $\gamma, \delta$ is in the same type if there is an element $f$ in $\text{Mod}(S)$ such that $f(\gamma) = \delta$. We say a subgroup $H$ of $G$ has the spectral gap property if the unitary representation associated to the action $H \acts \times = G/H - H$ does not have almost invariant vectors.

**Lemma 5.1.** Given a geometric multi-curve $\gamma$, let $m$ be the number of its geometric components.

1. If $m = 3g - 3 + n$, then $G_\gamma$ is amenable.
2. If $1 \leq m < 3g - 3 + n$, then $G_\gamma$ has the spectral gap property.

**Proof.** If $m = 3g - 3 + n$, then $G_\gamma$ is virtually abelian, thus it is amenable. For other cases, as $m < 3g - 3 + n$, one can cut $S$ along geometric components with the resulting possible disconnecting surfaces has at least one component which admit two pseudo-Anosov mappings that generated a rank 2 pseudo-Anosov subgroup. Assume the components which admit pseudo-Anosov mappings are labelled as $T_1, ..., T_k$, and the two pseudo-Anosov mappings on each $T_i$ and the group it generated are also labelled as $\varphi_i, \psi_i, H_i$. Note that this mappings all fix the boundaries. Then define two mappings $\varphi, \psi$ on $S$ (thus its isotopy class) by extending $\varphi = \prod_i \varphi_i, \psi = \prod_i \psi_i$. Then the subgroup $H$ generated by $\varphi, \psi$ is a rank 2 free group. Moreover the action of $H$ on the set $X_\gamma - \gamma$ has trivial stabilizer. For if a element $\phi$ in $H$ fix $\delta \in X_\gamma - \gamma$, then by the construction of $H$, the geometric intersection number of $\delta$ and $\gamma$ is nonzero and it should intersec one of $T_i$. We can cut $S$ along $\gamma$ so that $\delta$ becomes a family of isotopy classes of arcs. Since $\phi$ fixes $\delta$, up to some power of $\phi$, it fixes each isotopy class of arcs, but then it can be show that, for some $i$, there is an element in $H_i$ fix an isotopy class of essential simple closed curve which contradict the assumption that $H_i$ is a pseudo-Anosov subgroup. The use the Lemma 4.2, we can show that $G_\gamma$ has the spectral gap property.  \[ \square \]
Lemma 5.2 (Theorem A in [4]). Let $G$ be a countable discrete group and $H$ be a subgroup of $G$ that has the spectral gap property. Suppose $L$ is subgroup of $G$ satisfying $\text{Com}_G(L) = L$, then the two unitary representations $\ell^2(G/H), \ell^2(G/L)$ of $G$ are weakly equivalent if and only if $L$ is conjugate to $H$.

Theorem 5.3. Let $S = S_{g,n}$ be a compact surface with $3g + n \geq 4$. Let $\gamma, \delta$ be two geometric multi-curves with the number of geometric components is equal to $k, l$, respectively. Then

1. If at least one of $k, l$ is not $3g - 3 + n$, then the associated unitary representation $\pi_\gamma, \pi_\delta$ is weakly equivalent if and only if $\gamma, \delta$ are in the same type.

2. Suppose $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If the number of geometric components of $\gamma$ is $3g - 3 + n$, then $\pi_\gamma$ is weakly equivalent to the regular representation $\lambda_S$. Therefore, if the number of geometric components of $\gamma$ is not $3g - 3 + n$, $\pi_\gamma$ is not weakly contained in $\lambda_S$.

Proof. For any geometric multi-curves $\gamma$, $\text{Com}_{\text{Mod}(S)}(G_\gamma) = G_\gamma$ (see [20]). Given two geometric multi-curves $\gamma, \delta$ as the assumption, if at least of $k, l$ is not $3g - 3 + n$, by Lemma 5.1, Lemma 5.2 and the fact that $G_\gamma$ is conjugate to $G_\delta$ if and only if $\gamma, \delta$ are in the same type, we complete the proof of the first part. For the second part, By [8], If $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$, the group $\text{Mod}(S)$ is C*-simple. By the result of [7] which states that a discrete group is C*-simple if and only if, for any amenable subgroup $M$ of $G$, quasi-regular representation $\ell^2(G/M)$ is weakly equivalent to the regular one. So combine with Lemma 5.1, we complete the proof of the second part.

Remark 5.4. The only if part of the first result is a stronger version of Corollary 5.5 in [20].

Remark 5.5. If $S$ is one of $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$, it is easy to show that, if the number of components of $\gamma$ is $3g - 3 + n$, then $\pi_\gamma$ is weakly contained in the regular representation $\lambda_S$. However we don't know for other type of $\gamma$, whether it is weakly contained in $\lambda_S$. And we don't know what can we say about the unitary representation corresponding to non-discrete measures on the space of measured foliations.

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