On the dissociation number of Kneser graphs

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Abstract

A set $D$ of vertices of a graph $G$ is a dissociation set if each vertex of $D$ has at most one neighbor in $D$. The dissociation number of $G$, $\text{diss}(G)$, is the cardinality of a maximum dissociation set in a graph $G$. In this paper we study dissociation in the well-known class of Kneser graphs $K_{n,k}$. In particular, we establish that the dissociation number of Kneser graphs $K_{n,2}$ equals $\max\{n-1,6\}$. We show that for any $k \geq 2$, there exists $n_0 \in \mathbb{N}$ such that $\text{diss}(K_{n,k}) = \alpha(K_{n,k})$ for any $n \geq n_0$. We consider the case $k = 3$ in more details and prove that $n_0 = 8$ in this case. Then we improve a trivial upper bound $2\alpha(K_{n,k})$ for the dissociation number of Kneser graphs $K_{n,k}$ by using Katona’s cyclic arrangement of integers from $\{1, \ldots, n\}$. Finally we investigate the odd graphs, that is, the Kneser graphs with $n = 2k+1$. We prove that $\text{diss}(K_{2k+1,k}) = \binom{2k}{k}$.

Keywords: dissociation set, $k$-path vertex cover, Kneser graph, odd graphs, independence number

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1 Introduction

The Kneser graph, $K_{n,k}$, where $n, k$ are positive integers such that $n \geq 2k$, has the $k$-subsets of an $n$-set as its vertices, and two $k$-subsets are adjacent in $K_{n,k}$ if they are disjoint. The Erdős–Ko–Rado theorem \cite{8} determined the independence number $\alpha(K_{n,k})$ of the Kneser graph $K_{n,k}$ to be equal to $\binom{n-1}{k-1}$. Another famous result is Lovász’s proof of Kneser’s conjecture, which determines the chromatic number of Kneser graphs \cite{17}, see also Matoušek for a combinatorial proof of this result \cite{18}. Many other invariants were later considered in Kneser graphs by a number of authors. The diameter of a Kneser graph $K_{n,k}$ was computed in \cite{23} and the hamiltonicity was researched in \cite{7,22}. The domination number of Kneser graphs was also studied in several papers \cite{11,12,20}, but there is no such complete solution for domination number of Kneser graphs as it is the case with the chromatic and the independence number. Recently, the $P_3$-hull number of Kneser graphs was completely resolved for all Kneser graphs $K_{n,k}$ with the sole exception of odd graphs, that is, when $n = 2k+1$; see \cite{24}. The problem of independence in graphs can be rephrased as the search for a (largest) induced subgraph in which all components have only one vertex. In this paper, we extend the study to search for a largest induced subgraph of a Kneser graph in which all components have at most two vertices.

A set $D$ of vertices in a graph $G$ is called a dissociation set if the subgraph induced by vertices of $D$ has maximum degree at most 1. The cardinality of a maximum dissociation set $D$ in a graph $G$ is called the dissociation number of $G$, and is denoted by $\text{diss}(G)$. The dissociation number was introduced by

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Papadimitriou and Yannakakis [21] in relation with the complexity of the so-called restricted spanning tree problem. A dual concept to dissociation set can be generalized to \textit{m-path vertex cover}, which was introduced in [5] and studied in several papers [2][13]; it is defined as a set \( S \) of vertices in \( G \) such that \( G - S \) does not contain any path \( P_m \). The corresponding invariant, the \textit{m-path vertex cover number} of an arbitrary graph \( G \), is denoted by \( \psi_m(G) \). Note that dissociation sets are complements of 3-path vertex covers of \( G \), and so \( \text{diss}(G) = |V(G)| - \psi_3(G) \). The decision version of the \( m \)-path vertex cover number is NP-complete [5], moreover, in the case \( m = 3 \) it is NP-complete even in bipartite graphs which are \( C_4 \)-free and have maximum degree 3 [1]; see also [19] for further strengthening of this result and [14] for an approximation algorithm. Some variations of the problem were already studied as well (see e.g. [6][16]). We mention in passing that graphs in which all maximal dissociation sets are of the same size were studied in [3].

The independence number of a graph \( G \), \( \alpha(G) \), can be defined as the order of the largest induced subgraph of \( G \) with maximum degree 0. If 0 in this definition is replaced by 1, then we get a definition of the dissociation number of a graph. Since any independent set of a graph \( G \) is also a dissociation set of \( G \), the independence number of \( G \) is a lower bound for the dissociation number of \( G \). In addition, one can easily get the upper bound for the dissociation number of \( G \) as a function of \( \alpha(G) \). Let \( S \) be a dissociation set of \( G \) and \( A \subseteq S \) a maximum independent subset of \( S \). Then every vertex of \( S \setminus A \) has exactly one neighbor in \( A \) and any vertex of \( A \) has at most one neighbor in \( S \setminus A \). Therefore \( |S \setminus A| \leq |A| \). Hence we immediately get the following bounds for the dissociation number of \( G \):

\[
\alpha(G) \leq \text{diss}(G) \leq 2\alpha(G).
\]

The paper is organized as follows. In Section 2 we first present the exact result for the dissociation number of Kneser graphs \( K_{n,2} \). Then we prove that for any \( k \geq 2 \) there exists \( n_0 \in \mathbb{N} \) such that \( \text{diss}(K_{n,k}) = \alpha(K_{n,k}) \) for any \( n \geq n_0 \). Also, we find \( n_0 \) for \( k = 3 \): we prove that \( \text{diss}(K_{n,3}) = \alpha(K_{n,3}) \) if and only if \( n \geq 8 \). In Section 3 we use Katona’s cyclic arrangement of integers from his proof of Erdős-Ko-Rado theorem [15] to improve the upper bound \( 2\alpha(G) \) for the dissociation number for the case when \( G \) is a Kneser graph. In Section 4 we show that the dissociation number of odd graphs \( O_k \) (Kneser graphs \( K_{2k+1,k} \)) equals \((2k \choose k)\) for \( k \geq 2 \).

In the rest of this section we present the notation used throughout the paper and some basic results concerning the dissociation number of a graph.

Let \([n] = \{1, 2, \ldots, n\}\), where \( n \in \mathbb{N} \). For a graph \( G = (V, E) \) and \( S \subseteq V(G) \) we write \( G[S] \) for the subgraph of \( G \) induced by \( S \) and \( G - S \) for the subgraph of \( G \) induced by the set \( V(G) \setminus S \). The (open) neighborhood of \( v \in V(G) \), \( N_G(v) \), is the set of all neighbors of \( v \), while \( N_G[v] = N_G(v) \cup \{v\} \) denotes the closed neighborhood of \( v \). Similarly, for \( S \subseteq V(G) \), \( N_G[S] = \bigcup_{v \in S} N_G[v] \). The degree of a vertex \( v \) is \( |N_G(v)| \).

When the graph \( G \) is clear from the context we omit the subscripts. A matching \( M \) in a graph \( G \) is a set of edges in \( G \) having the property that no two edges in \( M \) have a common endvertex. For a matching \( M \) in \( G \), we denote by \( V(M) \) the set of endvertices of edges from \( M \). A set of pairwise non adjacent vertices in a graph \( G \) is called the independent set. The cardinality of the largest independent set of vertices in \( G \) is the independence number of \( G \) and is denoted by \( \alpha(G) \).

A center of a Kneser graph \( K_{n,k} \) is a set \( I(i) = \{x \in K_{n,k} : i \in x\} \), where \( i \in [n] \). Note that \( I(i) \) is an independent set of vertices of \( K_{n,k} \), and \( |I(i)| = \alpha(K_{n,k}) = \binom{n-1}{k-1} \). Note that for any \( n \geq 2k \), \( K_{2k,k} \) is an induced subgraph of \( K_{n,k} \) and hence \( \text{diss}(K_{n,k}) \geq \text{diss}(K_{2k,k}) = |V(K_{2k,k})| = \binom{2k}{k} \). We state this as follows.

**Proposition 1.1** For any \( n \geq 2k \), \( \text{diss}(K_{n,k}) \geq \binom{2k}{k} \).

## 2 Relations with the independence number

We start the study with the simplest non-trivial Kneser graphs, that is, \( K_{n,2} \), where \( n \geq 5 \). For the Petersen graph \( K_{5,2} \), one can easily see that \( D = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\} \) is a dissociation set. Indeed, \( D \) induces a subgraph with only three edges, namely \( \{1, 2\}\{3, 4\}, \{1, 3\}\{2, 4\} \) and \( \{1, 4\}\{2, 3\} \). It is
also a largest dissociation set, hence \( \text{diss}(K_{5,2}) = 6 \). The same construction is optimal also in \( K_{6,2} \) and \( K_{7,2} \), but not for \( K_{n,2} \), with larger \( n \), as the following result shows.

**Proposition 2.1** For \( n \geq 5 \), \( \text{diss}(K_{n,2}) = \max\{n - 1, 6\} \).

**Proof.** Note that a maximum independent set of a Kneser graphs \( K_{n,2} \), where \( n > 5 \), is a center, its size is \( n - 1 \), and it is also a dissociation set. By the above observation, we get \( \text{diss}(K_{n,2}) \geq \max\{n - 1, 6\} \). Suppose that \( D \) is a dissociation set, which is not independent. Without loss of generality, let \( \{\{1, 2\}, \{3, 4\}\} \subset D \). Note that \( V(K_{n,2}) \setminus N[D] = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\} \), which implies \( |D| \leq 6 \). Hence, if \( n > 7 \), a maximum dissociation set is independent, and the proposed equality follows. \( \square \)

We find a similar feature for Kneser graphs \( K_{n,k} \), where \( k > 2 \). Namely, as soon as \( n \) is large enough with respect to \( k \), we have \( \text{diss}(K_{n,k}) = \alpha(K_{n,k}) \).

**Theorem 2.2** For any \( k \geq 2 \), there exists \( n_0 \in \mathbb{N} \) such that for all \( n, n \geq n_0 \), we have

\[
\text{diss}(K_{n,k}) = \alpha(K_{n,k}) = \binom{n-1}{k-1}.
\]

**Proof.** The result for \( k = 2 \) follows from Proposition 2.1. Fix \( k \geq 3 \), and suppose that a maximum dissociation set \( D \) is not an independent set. Assume without loss of generality that \( x = \{1, \ldots, k\} \) and \( y = \{k + 1, \ldots, 2k\} \) belong to \( D \). Let \( U = V(K_{n,k}) \setminus N[x,y] \). Note that every element in \( U \) is a \( k \)-set that contains at least one element from \( x \) and at least one element from \( y \). Setting \( z = \{2k + 1, \ldots, n\} \) note that any element in \( U \) consists of \( i \) elements from \( z \), where \( 0 \leq i \leq k - 2 \), \( j \) elements from \( x \), where \( 1 \leq j \leq k - i - 1 \), and consequently, \( k - i - j \) elements from \( y \).

Hence

\[
|U| = \sum_{i=0}^{k-2} \binom{n-2k}{i} \sum_{j=1}^{k-i-1} \binom{k}{j} \binom{k}{k-j-i}.
\]

Note that \( \sum_{j=1}^{k-i-1} \binom{k}{j} \binom{k}{k-j-i} \) is not dependent on \( n \), hence for a fixed \( k \) this is a constant, while \( \sum_{i=0}^{k-2} \binom{n-2k}{i} \) is a polynomial in \( n \) of degree \( k - 2 \). Hence \( |U| = \mathcal{O}(n^{k-2}) \), and note that \( |D| \leq 2 + |U| \). On the other hand, \( \alpha(K_{n,k}) = \binom{n-1}{k-1} \), hence the resulting independent (and dissociation) set is of size \( \Omega(n^{k-1}) \). Therefore, if \( n \) is big enough, \( D \) is not a maximum dissociation set, because its size is less than \( \binom{n-1}{k-1} \). \( \square \)

Note that the above proof relies on the fact that for any adjacent vertices \( x \) and \( y \) of a dissociation set \( D \) of \( G \), we have \( D \subseteq D_{x,y} = \{x, y\} \cup (V(G) \setminus (N[x] \cup N[y])) \). In fact, we show that for any \( k \geq 2 \) there exists \( n_0' \in \mathbb{N} \) such that for any adjacent vertices \( x, y \in V(K_{n,k}) \) we have \( \alpha(K_{n,k}) \geq |D_{x,y}| \) as soon as \( n \geq n_0' \). In particular, the smallest \( n_0' \) that appears in the statement of Theorem 2.2 may be much smaller than \( n_0' \) which is used in the proof. Note that \( |D_{x,y}| = 2 + \binom{n}{k} - 2\binom{n-k}{k} - (n-2k) \). For \( k = 2 \) the smallest \( n_0' \), for which \( \alpha(K_{n,2}) \geq |D_{x,y}| \) when \( n \geq n_0' \), is 7, which is by Proposition 2.1 also \( n_0 \) from Theorem 2.2 (that is, \( \text{diss}(K_{n,2}) = \alpha(K_{n,2}) \) as soon as \( n \geq n_0 = 7 \)). This is not the case for \( k > 2 \). For \( k = 3 \) one can easily compute that \( n_0' = 17 \) (by solving inequality \( \alpha(K_{n,k}) \geq |D_{x,y}| \) for \( k = 3 \)), but as we will see in Corollary 2.6 we have \( \text{diss}(K_{n,3}) = \alpha(K_{n,3}) \) already for \( n \geq 8 \). For \( k > 3 \), we do not know how large must \( n_0 \) be in Theorem 2.2 and we propose this as an open problem.

**Problem 1** Given an integer \( k \geq 4 \), what is the smallest integer \( n_0 \) such that for all \( n \geq n_0 \), \( \text{diss}(K_{n,k}) = \alpha(K_{n,k}) ? \)

From Proposition 2.1 we get the inequality that leads to the lower bound for \( n_0 \) from Theorem 2.2

**Lemma 2.3** The smallest integer \( n_0 \) for which \( \text{diss}(K_{n_0,k}) = \alpha(K_{n_0,k}) \), is at least \( 2k + 2 \).
Proof. Let $n \geq 2k$ and $\text{diss}(K_{n,k}) = \alpha(K_{n,k})$. Then \((n-1) \geq \binom{2k}{k}\) by Proposition\ref{prop:alpha}. Solving this inequality we infer $n \geq 2k+2$.

We follow with establishing the exact value of $\text{diss}(K_{8,3})$.

Lemma 2.4 $\text{diss}(K_{8,3}) = \alpha(K_{8,3})$.

Proof. For the purpose of contradiction assume that $\text{diss}(K_{8,3}) > 21$. Let $S$ be a maximum dissociation set. Since $|S| > \alpha(K_{8,3})$, $S$ is not independent. Without loss of generality we may assume that $x = \{1, 2, 3\}, y = \{4, 5, 6\}$ are two adjacent vertices contained in $S$. Since $S$ is a dissociation set, $S \cap (N[x] \cup N[y]) = \{x, y\}$. Let $H$ be the subgraph of $K_{8,3}$ induced by $V(K_{8,3}) \setminus (N[x] \cup N[y])$. We define the following sets

- $U = \{z \in V(H); 7 \in z\}, U' = \{z \in V(H); 8 \in z\}$;
- $\forall i \in [6], U_i = \{z \in U; i \in z\}, U'_i = \{z \in U'; i \in z\}$;
- $D = V(H) \setminus (U \cup U')$;
- $D' = \{z \in D; |z \cap x| = 2\}, D'' = \{z \in D; |z \cap y| = 2\}$;
- $\forall i, j \in [3], D'_{ij} = \{z \in D'; i, j \in z\}$;
- $\forall i, j \in \{4, 5, 6\}, D''_{ij} = \{z \in D''; i, j \in z\}$;
- $\forall i \in \{4, 5, 6\}, D'_i = \{z \in D'; i \in z\}$;
- $\forall i \in [3], D''_i = \{z \in D''; i \in z\}$.

If $z \in V(K_{8,3})$ with $\{7, 8\} \subseteq z$, then $z \in N[x] \cup N[y]$. Thus $U \cap U' = \emptyset$, and so $[U, U', D]$ is a partition of $V(H)$. Also $[U_1, U_2, U_3]$ and $[U_4, U_5, U_6]$ are partitions of $U$, $[U'_1, U'_2, U'_3]$, $[U'_4, U'_5, U'_6]$ are partitions of $U'$, $[D'_{12}, D'_{13}, D'_{23}], [D'_4, D'_5, D'_6]$ are partitions of $D'$ and $[D''_{45}, D''_{46}, D''_{56}], [D''_1, D''_2, D''_3]$ are partitions of $D''$. Moreover, sets $U, U', D', D''$ are independent sets of cardinality 9. A spanning subgraph $H'$ of a graph $H$ is depicted in Figure\ref{fig:spanning_sub}. The edges of $H$ that are not in $H'$ are the edges between $U$ and $U'$, between $U$ and $D''$, between $D'$ and $D''$. Note that $H[U \cup D'] \cong H[U \cup D''] \cong H[U' \cup D'']$.

![Figure 1: Spanning subgraph of a graph H.](image-url)
Claim 1 \(|U_i \cap S| < 3\) for any \(i \in \{4, 5, 6\}\).

Proof. Suppose that there exists \(i \in \{4, 5, 6\}\) such that \(|U_i \cap S| = 3\) and let \(\{i, j, k\} = \{4, 5, 6\}\). Hence \((U'_j \cup U'_k \cup D''_{jk}) \cap S = \emptyset\), as each vertex of \(U'_j \cup U'_k \cup D''_{jk}\) has exactly 2 neighbors in \(U_i\). Let \(\ell\) be an arbitrary element of \([1, 2, 3]\) and let \(\{\ell, \ell', \ell''\} = \{1, 2, 3\}\). Since \(\{\ell', \ell'', j, k\}\) are both neighbors of \([i, i, 7] \in U_i \subseteq S\), \([\{\ell', \ell'', j\}, \{\ell', \ell'', k\}] \cap S \leq 1\). Note that subgraphs of \(H\) induced by \(U_j \cup \Delta'_j\) and \(U_k \cup D''_{ij}\) are isomorphic to \(C_6\) and thus \(|U_j \cup \Delta'_j \cap S|, |(U_k \cup D''_{ij}) \cap S| \leq 4\). If \(|(U_j \cup D''_{ij}) \cap S| = 4\), then \(S\) contains exactly 2 vertices from \(U_j\) and thus it contains at most 2 vertices from \(U_i\) (as \(H[U_j \cup U'_i] \cong C_6\)). Therefore \(|(U_j \cup D''_{ij} \cup U'_i) \cap S| \leq 6\). Hence, 
\[|S \cap V(H)| = |S \cap U_i| + |S \cap (U_j \cup U'_i \cup D''_{ij})| + |S \cap (U_j \cup D''_{ij} \cup U'_i)| + |S \cap (U_k \cup D''_{ij})| + |S \cap D'_{12}| + |S \cap D'_{13}| + |S \cap D'_{23}| \leq 3 + 0 + 6 + 4 + 2 + 2 + 2 = 19.\] We infer \(|S| \leq 21\), a contradiction. □

Analogous arguments imply

Claim 2 For any \(i \in \{4, 5, 6\}\) \(|U_i \cap S| < 3\).

Claim 3 For any \(i \in \{3\}\) let \(\{i, j, k\} = \{1, 2, 3\}\). Then \(D'_{jk} \cap S \neq \emptyset\).

Proof. Suppose that there exists \(i \in \{1, 2, 3\}\), such that \(|D'_{jk} \cap S| = 0\), where \(\{j, k\} = \{1, 2, 3\} \setminus \{i\}\). For \(\ell \in \{4, 5, 6\}\) and \(\{\ell', \ell''\} \in \{4, 5, 6\} \setminus \{\ell\}\) let \(A_{\ell'}\) be the subgraph of \(H\) induced by \(U_j \cup D''_{\ell', \ell''} \cup U_{\ell'}\). Note that the subgraphs of \(A_{\ell'}\) induced by \(U_j \cup D''_{\ell', \ell''} \cup U_{\ell'}\) are isomorphic to \(C_6\).

Suppose that \(|A_{\ell'} \cap S| > 4\). Then \(|D''_{\ell', \ell''} \cap S| \neq \emptyset\), as \(|U_j \cap S| \leq 2\) and \(|U_i \cap S| \leq 2\) by Claims 1 and 2. Without loss of generality we may assume that \(\{1, \ell', \ell''\} \subseteq S \cap D''_{\ell', \ell''}\). Since \(\{2, 6, 8\}, \{3, 6, 8\}, \{2, 7, 6\}, \{3, 7, 6\}\) are all neighbors of \(\{1, \ell', \ell''\} \subseteq S\), we get \(|\{2, 6, 8\}, \{3, 6, 8\}, \{2, 7, 6\}, \{3, 7, 6\}\} \cap S \leq 1\). Since \(\{1, 6, 8\}, \{1, 7\}\) are both neighbors of \(\{2, \ell', \ell''\}\) and \(\{3, \ell', \ell''\}\), the following statements hold.

- If \(\{\{2, \ell', \ell''\}, \{3, \ell', \ell''\}\} \subseteq S\), then \(S \cap \{\{1, 6, 8\}, \{1, 7\}\} = \emptyset\);
- If \(|S \cap \{\{1, 6, 8\}, \{1, 7\}\}| = 2\), then \(S \cap \{\{2, \ell', \ell''\}, \{3, \ell', \ell''\}\} = \emptyset\).

Thus \(S\) contains at most two vertices from \(\{\{2, \ell', \ell''\}, \{3, \ell', \ell''\}, \{1, 6, 8\}, \{1, 7\}\}\) and consequently \(|A_{\ell'} \cap S| \leq 4\), a contradiction. Hence \(|A_{\ell'} \cap S| \leq 4\) for any \(\ell \in \{4, 5, 6\}\). Since \(D'_{jk} \cap S = \emptyset\), \(|S \cap D'_{ij}| \leq 6\) and thus 
\[|V(H) \cap S| = \sum_{\ell=4}^{6} |A_{\ell} \cap S| + |D' \cap S| \leq 3 \cdot 4 + 6 = 18,\] a contradiction. □

For any \(i \in \{3\}\) define \(X''_i\) as a subgraph of \(H\) induced by veritces of \(D''_{ij} \cup U_i \cup U'_i \cup D'_{jk}\), where \(\{j, k\} = \{3\} \setminus \{\ell\}\).

Let \(i \in \{3\}\) and \(\{i, j, k\} = \{1, 2, 3\}\). Since by Claim 4 \(D'_{jk} \cap S \neq \emptyset\), we have three possibilities.

If \(|D'_{jk} \cap S| = 1\), let \(\ell \in \{4, 5, 6\}\) such that \(\{j, k, \ell\} \subseteq S\) and let \(\{4, 5, 6\} \setminus \{\ell\} = \{\ell', \ell''\}\). Since \(\{j, k, \ell\}\) is adjacent to all vertices from \(A = \{\{i, \ell', \ell''\}, \{i, \ell', 7\}, \{i, \ell'', 7\}, \{i, 6, 8\}, \{i, 6', 8\}\}\), \(|S \cap A| \leq 1\). Since \(|X''_i \cap (A \cup D'_{jk})| = 4\), \(|S \cap A| \leq 1\) and \(|S \cap D'_{jk}| = 1\) we get \(|S \cap V(X''_i)| \leq 6\).

If \(|D'_{jk} \cap S| = 2\), let \(\ell, \ell' \in \{4, 5, 6\}\) such that \(\{j, k, \ell\} \subseteq S\) and let \(\{4, 5, 6\} \setminus \{\ell, \ell'\} = \{\ell''\}\). Since \(i, 6', 7\} \{i, \ell''\} \subseteq S\) and \(i, 6, 8\) \(\notin S\). Since \(A = \{\{i, \ell', \ell''\}, \{i, 6', 7\}, \{i, 6, 8\}\}\) is the set of neighbors in \(X''_i\) of \(\{j, k, \ell\} \subseteq D'_{jk} \cap S\), at most one vertex from \(A\) can be contained in \(S\). Similarly at most one vertex from \(B = \{\{i, \ell, \ell''\}, \{i, 6', 7\}, \{i, 6, 8\}\}\) can be contained in \(S\), as all vertices of \(B\) are neighbors of \(\{j, k, \ell\}\). Since \(|X''_i \cap (A \cup B \cup D'_{jk})| = 1\), \(|S \cap V(X''_i)| \leq |S \cap D'_{jk}| + |S \cap A| + |S \cap B| + 1 \leq 2 + 1 + 1 + 5 = 15\).

Finally, let \(S \cap D'_{jk} = D'_{jk}\). Then \(U_i \cap S = U'_i \cap S = \emptyset\) and thus \(|V(X''_i) \cap S| \leq 6\).

We have proved that for any \(i \in \{3\}\) it holds \(|V(X''_i) \cap S| \leq 6\). Therefore 
\[|S \cap V(H)| = |S \cap V(X''_i)| + |S \cap V(X''_j)| + |S \cap V(X''_k)| \leq 18,\] which is a final contradiction. □
Proposition 2.5 Let \( n \geq 9 \). If \( \text{diss}(K_{n-1,3}) = \alpha(K_{n-1,3}) \) then \( \text{diss}(K_{n,3}) = \alpha(K_{n,3}) \).

Proof. Suppose that there exists \( n \geq 9 \) with \( \text{diss}(K_{n-1,3}) = \alpha(K_{n-1,3}) \) and \( \text{diss}(K_{n,3}) > \alpha(K_{n,3}) = \binom{n-1}{2} \). Let \( S \) be a maximum dissociation set in \( K_{n,3} \). Since \( |S| > \alpha(K_{n,3}) \), \( S \) is not an independent set. Note that since \( n \geq 9 \), \( |S| \geq \binom{n-1}{2} + 1 \geq 29 \). Without loss of generality we may assume that \( x = \{1, 2, 3\} \), \( y = \{4, 5, 6\} \in S \). Since \( S \) is a dissociation set, \( (S \cap (N[x] \cup N[y]) \setminus \{x, y\} = \emptyset \). Let \( H \) be the subgraph of \( K_{n,3} \) induced by \( V(K_{n,3}) \setminus (N[x] \cup N[y]) \). Let \( U = \{z \in V(H) \mid z \not\in S\} \) and \( D = \{z \in V(H) \mid n \not\in z\} \). Since each vertex \( z \in U \) contains exactly one element from \( x \) and exactly one element from \( y \), \( |U| = 9 \).

If \( D \cap S = \emptyset \), then \( |S| \leq 2 + 9 = 11 \), a contradiction. Hence we may assume that \( D \cap S \neq \emptyset \). Let \( z \in D \cap S \). Then at least one element from \( x \), say \( i \), and at least one element from \( y \), say \( \ell \), is contained in \( z \). Denote \( z = \{i, \ell, w\} \).

Suppose first that \( w \notin x \cup y \). Let \( \{1, 2, 3\} = \{i, j, k\} \), \( \{4, 5, 6\} = \{\ell, \ell', \ell''\} \). Then all vertices of \( A = \{\{j, \ell, n\}, \{j, \ell', n\}, \{k, \ell', n\}, \{k, \ell'' , n\}\} \subseteq U \) are neighbors of \( z \in S \). Hence \( |S \cap A| \leq 1 \) and therefore \( |S \cap U| \leq 6 \). Since vertices of \( N[x] \cup N[y] \) that do not contain \( n \) together with \( D \) induce \( K_{n-1,3} \), \( |S| \leq \alpha(K_{n-1,3}) + |U \cap S| \leq \binom{n-2}{2} + 6 \). Hence for any \( n \geq 8 \) we get \( |S| \leq \alpha(K_{n,3}) \), a contradiction.

Hence \( w \notin x \cup y \) or, in other words, \( S \) does not contain vertices \( \{i, j, z\} \), where \( i \in \{1, 2, 3\} \), \( j \in \{4, 5, 6\} \), \( z \in \{\ell, \ell', \ell''\} \). Thus if \( d \in S \cap D \), then \( d = \{i_1, i_2, \ell\} \) or \( d = \{i_1, \ell_1, \ell_2\} \), where \( i_1, i_2 \in \{1, 2, 3\} \), \( \ell, \ell_1, \ell_2 \in \{4, 5, 6\} \). Hence \( |S \cap D| \leq 18 \). If \( S \) contains \( \{i_1, i_2, \ell\} \) for \( \{i_1, i_2\} = \{1, 2, 3\} \), \( \{\ell_1, \ell_2\} = \{4, 5, 6\} \), then \( S \) cannot contain both \( \{i_1, \ell_1, n\}, \{i_2, \ell_2, n\} \subseteq U \) and thus \( |S \cap (U \cup D)| \leq 26 \). Therefore \( |S| \leq 28 \), a contradiction.

We suspect that Proposition 2.5 also holds for \( k \) bigger than 3, and pose it as a problem.

Problem 2 Is it true that \( \text{diss}(K_{n,k}) = \alpha(K_{n,k}) \) implies \( \text{diss}(K_{n+1,k}) = \alpha(K_{n+1,k}) \) for all \( k \geq 2 \) and \( n \geq 2k + 2 \)?

As a direct corollary of Lemma 2.3, Lemma 2.4 and Proposition 2.5 we get that \( \text{diss}(K_{n,3}) = \alpha(K_{n,3}) \) for any \( n \geq 8 \).

Corollary 2.6 For \( k = 3 \), \( \text{diss}(K_{n,3}) = \alpha(K_{n,3}) \) if and only if \( n \geq 8 \).

3 An upper bound for \( \text{diss}(K_{n,k}) \)

In this section, we consider upper bounds for the dissociation number of Kneser graphs \( K_{n,k} \). We already know that \( \alpha(K_{n,k}) \leq \text{diss}(K_{n,k}) \leq 2\alpha(K_{n,k}) \) and that if \( n \) is large enough, then the dissociation number coincide with the independence number. Thus the bound is interesting only when \( n \leq n_0 \), where \( n_0 \) is the integer that appears in Theorem 2.2 and Problem 1. Our aim of this section is to improve the upper bound \( 2\alpha(K_{n,r}) \) for \( n < n_0 \).

We will improve the upper bound by using Katona’s cyclic arrangement of integers from his proof of Erdős-Ko-Rado theorem. Let \( D \) be a maximum dissociation set of \( K_{n,k} \). We count in two different ways the number of ordered pairs \( (D, C) \), where \( D \in D \) and \( C \) is a cyclic arrangement of integers from \( [n] \) in which \( D \) appears as a substring.

Let \( n = 2k + r \), thus we consider \( K_{2k+r,k} \), where \( r \geq 1 \). If one takes any set from \( D \), then it appears as a substring in \( k!(n-k)! \) different cyclic arrangements. Thus, altogether there are \( |D|k!(n-k)! \) such ordered pairs \( (D, C) \). Second, note that there are \( (n-1)! \) different cyclic arrangements of integers from \( [n] \). Next, let us bound from above the number of sets from \( D \) that appear as substrings in any given cyclic arrangement.

We distinguish two cases. First, let \( r > k - 2 \). We claim that in any given cyclic arrangement there are at most \( k + 1 \) elements from \( D \) that appear as its substrings. Suppose that all elements of \( D \) that appear as substring in \( C \) pairwise intersect. Then, it is easy to see that at most \( k \) elements from \( D \) appear as substrings in \( C \). Without loss of generality, let \( D_1 : 1, 2, \ldots, k \), and \( D_2 : k + t, k + t + 1, 2k + t - 1, t \in [n-2k+1] \), be the substrings in \( C \) that correspond to elements of \( D \). Since the sets that correspond to \( D_1 \) and \( D_2 \) form an edge in \( K_{n,r} \), we infer that all other sets of \( D \) that appear as substrings in \( C \) must intersect both \( D_1 \) and
$D_2$. Since $r > k - 2$, we infer that there are at most $k - 1$ such substrings of length $k$ that intersect both $D_1$ and $D_2$. This implies that there are at most $k + 1$ elements from $D$ that appear as substrings, as claimed. Thus, when $n > 3k - 2$, we get
\[
\text{diss}(K_{n,k}) \leq \frac{k + 1}{k} \binom{n-1}{k-1}.
\] (1)

The second case is $r \leq k - 2$. Again, let $D_1 : 1, 2, \ldots, k$ be a substring in a cyclic arrangement $C$ that corresponds to an element of $D$ (by abuse of language, we denote this element by $D_1$ as well). Clearly, there is at most one set in $D$ that does not intersect $D_1$. Note that a set $D$ in $D \setminus \{D_1\}$ can intersect $D_1$ in two different ways, either $1$ is at most one set in $D$. We denote by $A_1$ the substring in $C$ for which $A_1 \cap D_1 = [i]$, and by $B_i$ the substring in $C$ for which $B_i \cap D_1 = [k] \setminus [i]$. If for some $i \in [n-1], A_i \in D$ and $B_i \in D$, then $i$ is a double point. On the other hand, if just one of $A_i \in D$ or $B_i \in D$ holds, then $i$ is a single point. Let $d$ be the number of double points and $s$ the number of single points. Note that the number of substrings of $C$ that correspond to elements of $D$ is bounded from above by $2 + s + 2d$, where $2$ corresponds to $D_1$ and possibly one more element from $D$ that does not intersect $D_1$.

Suppose that $i$ is a double point. Since $D$ is a dissociation set, any set in $D \setminus \{A_i, B_i\}$ must intersect both $A_i$ and $B_i$. This in turn implies that $A_1, \ldots, A_{\max(i-r,1)}$ do not belong to $D$ and also $B_{i+1}, \ldots, B_{\max(i+r,k)}$ do not belong to $D$. In other words, a double point can appear at most in every $2r + 1$ turn, that is, at most $\left\lceil \frac{k}{2r+1} \right\rceil$ times. Hence, the number of substrings of $C$ that correspond to elements of $D$ is bounded from above by $2 + s + 2d = 2 + k - 1 + \left\lceil \frac{k}{2r+1} \right\rceil \leq 2 + k + \frac{k}{2r+1}$. This yields
\[
|D|(n-k)|k|! \leq (2 + k + \frac{k}{2r+1})(n-1)!
\]
which implies
\[
\text{diss}(K_{2k+r,k}) \leq 2^{rk} + 2r + k + 1 \binom{n-1}{k-1}.
\] (2)

4 Dissociation number of odd graphs

In this section, we use the famous Hall’s marriage theorem, which we next formulate. Let $G$ be a bipartite graph, where a bipartition of $V(G)$ is $[X,Y]$. A matching $M$ in $G$ is an $X$-matching of $G$ if every vertex in $X$ is incident with an edge of $M$.

**Theorem 4.1** [10] A bipartite graph $G$ with $V(G) = [X,Y]$ has an $X$-matching if and only if for every subset $W \subset X$ we have $|N(W)| \geq |W|$.

Perhaps the most interesting class of Kneser graphs is that of odd graphs, $O_k = K_{2k+1,k}$. The dissociation number of odd graph $O_k$ is by Proposition [11] bounded below by $\binom{2k}{k}$. Proposition [2.1] implies that this bound is also an upper bound for $k = 2$. In the next result we prove that the bound is the exact value also for $k > 2$.

**Theorem 4.2** For any $k \geq 2$, $\text{diss}(O_k) = \binom{2k}{k}$.

**Proof.** By Proposition [11] $\text{diss}(O_k) \geq \binom{2k}{k}$.

Let $S$ be a maximum dissociation set of $O_k$. Let $D = \{x \in V(O_k); 2k+1 \notin x\}$ and $U = V(O_k) \setminus D$, that is, $U = \{x \in V(O_k); 2k+1 \in x\}$. Note that $O_k[D]$ is isomorphic to $K_{2k,k}$ which is in turn isomorphic to $\frac{1}{2}(\binom{2k}{k})K_2$. On the other hand, $U$ is a center $I(2k+1)$ of $K_{2k+1,k}$, hence an independent set.

If $S \cap D = \emptyset$, then $|S| \leq |U| = \binom{2k}{k-1} \leq \binom{2k}{k}$ and the proof is done. If $S \cap U = \emptyset$, then $|S| \leq |D| = \binom{2k}{k}$ which also completes the proof. Thus it remains to consider the case when $D \cap S \neq \emptyset$ and $L = U \cap S \neq \emptyset$. Set $\ell = |L|$. Since $|D| = \binom{2k}{k}$ and $|S \cap U| = \ell$, it suffices to prove that at least $\ell$ vertices from $D$ are not contained in $S$. 


Let $E$ be the set of edges having one endvertex in $L$ and the other endvertex in $N_{O_k}(L)$, where $N_{O_k}(L)$ is a subset of $D$. Now, $|E| = (k+1)|L|$, since any $u \in L$ has exactly $k+1$ neighbors in $D$. On the other hand, any $x \in N_{O_k}(L)$ has exactly $k$ neighbors in $U$ and hence $|E| \leq k|N_{O_k}(L)|$. Therefore $|N_{O_k}(L)| \geq \frac{k+1}{k}|L| > |L|$. The same argument applies for any subset of $L$: that is, if $L' \subset L$, then $|N_{O_k}(L')| \geq \frac{k+1}{k}|L'| > |L'|$. Thus, by Theorem [4.1] there is an $L$-matching $M = \{u_1x_1, u_2x_2, \ldots, u_\ell x_\ell\}$ in a bipartite graph $G = (L \cup N_{O_k}(L), E)$, where $u_i \in L$ and $x_i \in N_{O_k}(L)$ for any $i \in \ell$. Let $[M', M'']$ be a partition of $M$, where $M'$ is the set of edges in $M$ with both endvertices contained in $S$. Since $L \subseteq S$, exactly one endvertex of each edge in $M''$ (that is, the endvertex from $L$) is contained in $S$. Denote by $Z$ (resp. $A'$) the set of endvertices in $L$ (resp. $N_{O_k}(L)$) of edges in $M'$ and let $W$ (resp. $B'$) be the set of endvertices of edges in $M''$ that are contained in $L$ (resp. $N_{O_k}(L)$). Furthermore, let $A'' = (N_{O_k}(L) \cap S) \setminus A'$ and $B'' = (N_{O_k}(L) \setminus S) \setminus B'$. This definitions directly imply that $[A', A'', B', B'']$ is a partition of $N_{O_k}(L)$. Note that $A' \cup A''$ consists exactly of the vertices of $N_{O_k}(L)$ that are contained in $S$, and $B' \cup B''$ contains the vertices from $N_{O_k}(L)$ not contained in $S$.

To complete the proof, we count the number of edges between $L$ and $B' \cup B''$. Denote the set of those edges by $E'$. Since $Z \cup A' \cup A''$ is a subset of a dissociation set $S$ and $u_i x_i \in M$ is an edge in $O_k[S]$, $x_i$ is the only neighbor of $u_i \in Z$ that is contained in $S$. Hence all other $k$ neighbors of $u_i$ in $D$ are from $B' \cup B''$. Since $W \cup A' \cup A'' \subseteq S$, any vertex $u_i \in W$ has at most one neighbor in $A''$ and all other $k$ neighbors of $u_i$ in $D$ are from $B' \cup B''$. Thus $|E'| \geq k(|Z| + |W|) = k|L|$. Since each vertex $x \in B' \cup B''$ has exactly $k$ neighbors in $U$ and as $L$ is a subset of $U$, we get $|E'| \leq k(|B'| + |B''|)$. Consequently $|B'| + |B''| \geq |L|$. Hence $B' \cup B'' \subseteq D \setminus S$ is a set of at least $\ell$ vertices in $D$ that are not contained in $S$.

\section{5 Concluding remarks}

In this paper, we found the dissociation number of several families of Kneser graphs $K_{n,k}$. This includes the cases $k \in \{2, 3\}$ and $n = 2k + 1$. As proved in Theorem [2.2] when $n$ is large enough with respect to $k$, then the dissociation number equals the independence number of the corresponding Kneser graph. The point when this happens for a given $k$, the integer $n_0$, is open, and we give a lower bound for it. Two problems posed in Section [2] are related to $n_0$. Establishing exact values of $\text{diss}(K_{2k+r,k})$, where $r$ is a small integer greater than 1, is another challenge.

The dissociation number is dual invariant to the 3-path vertex cover number (as is the independence number dual to the (2-path) vertex cover number). A natural problem is to consider the $m$-path vertex cover number of Kneser graphs, for any given $m > 3$. An alternative extension of the problem studied in this paper is the following. Since a dissociation set induces a subgraph with maximum degree at most 1, it would be interesting to find the largest size of a subset of vertices in $K_{n,k}$ that induces a subgraph with maximum degree $\Delta$, for any given $\Delta > 2$.

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