Abstract

In this paper, we investigate the effects of electromagnetic field on the isotropic spherical gravastar models in metric $f(R, T)$ gravity. For this purpose, we have explored singularity-free exact models of relativistic spheres with a specific equation of state. After considering Reissner Nordström spacetime as an exterior region, the interior charged manifold is matched at the junction interface. Several viable realistic characteristics of the spherical gravastar model are studied in the presence of electromagnetic field through graphical representations. It is concluded that the electric charge has a substantial role in the modeling of proper length, energy contents, entropy and equation of state parameter of the stellar system. We have also explored the stable regions of the charged gravastar structures.

Keywords: Self-gravitation; Anisotropic fluid; Gravitational collapse.

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1 Introduction

Recent interesting outcomes from a few observational experiments like, cosmic microwave background (CMB) radiation, Supernovae type Ia, etc. [1] claimed that our cosmos is in
the phase of accelerating expansion. The observational ingredients from the BICEP2 experiment \[2–4\], the Planck satellite \[5–7\], and the Wilkinson Microwave anisotropy probe (WMAP) \[8, 9\], state that our cosmos is composed of only 5% baryonic matter, the rest is comprising of dark matter (DM) and dark energy (DE). Their percentages are observed to be 27% and 68%, respectively.

A popular approach to understand the structure formation and evolution of the Universe is modified gravity theories. Such theories are obtained by generalizing the usual Einstein-Hilbert action. The importance as well as the need of such theories have been discussed in detail by Nojiri and Odintsov \[10\]. The \(f(R)\) (\(R\) is the Ricci scalar) \[11\], \(f(T)\) (\(T\) is the torsion scalar) \[12\], \(f(R, \Box R, T)\) (\(\Box\) is the de Alember’s operator and \(T\) is the trace of energy momentum tensor) \[13\], \(f(G)\) (\(G\) is the Gauss-Bonnet term) \[14\] and \(f(G, T)\) \[15\] etc., are among the most attractive models of modified theories (for further reviews on such models, see, for instance, \[16\]). Harko et al. \[17\] introduced a notion of \(f(R, T)\) theory by introducing \(T\) in the well-known theory of \(f(R)\) gravity. This quantity \(T\) may be considered by exotic imperfect fluids or quantum effects. They presented dynamical equations and the associated equations of motion for a test particle. Houndjo \[18\] did reconstruction in order to solve cosmological issues in this gravity and found some models that could be useful to understand matter dominated eras of our universe. Jamil et al. \[19\] did the same process and found some well-consistent outcomes with low red-shifts surveys.

Adhav \[20\] studied the homogeneous and anisotropic cosmic model with the help of a constant deceleration parameter and presented a few analytical \(f(R, T)\) solutions under some conditions. Shabani and Farhoudi \[21\] examined the cosmological solutions of \(f(R, T)\) gravity for a perfect fluid using spatially Friedmann-Lemaître-Robertson-Walker universe. In order to simplify equations, they presented some dimensionless parameters and variables. Baffou et al. \[22\] analyzed the stability of power-law models with the help of de-Sitter cosmic models against linear perturbations. They concluded that these models could be taken as an efficient candidate for DE. Das and Ali \[23\] elaborated the anisotropic and homogeneous axially symmetric Bianchi type-I bulk viscous cosmological models using time-varying cosmological and gravitational constant. By using the Hubble parameter, they solved the field equations and discussed the kinematical and physical properties of the models. Kiran and Reddy \[24\] investigated the Bianchi type-III in the presence of viscous fluid and concluded that this model does not exist in \(f(R, T)\) gravity. Momeni et al. \[25\] analyzed the Noether symmetry problem for two types of modified theories. First is mimetic \(f(R)\) and second is a non-minimally coupled model which is known as \(f(R, T)\). Pankaj and Singh \[26\] discussed the viscous cosmology with matter creation under \(f(R, T)\) gravity. Sun and Huang \[27\] studied the issues of an isotropic and homogeneous universe under modified \(f(R, T)\) gravity with non-minimal coupling. Moraes et al. \[28\] studied the hydrostatic equilibrium conditions of compact objects by relating their pressure and density through an equation of state.
A gravastar (Gravitational Vacuum Star) is an astronomical substance hypothesized as a substitute to the black hole. The conception of gravastar was established from the theory of Mazur and Mottola, [29,30]. By increasing the idea of Bose-Einstein, this new form of solution was introduced as a consequence of gravitational collapse. Such kind of model is hypothesized to contain no event horizon. Though gravastar may look identical to a black hole, there are certain experiments, such as X-rays radiations by infalling matter, that may allow to discriminate them. As such, gravastars are of interest from a purely theoretical perspective. Such kind of stellar structures could be used to explain the role of DE in the accelerating expansion of the universe. These could be helpful to explain why some galaxies have a high or low DM concentrations. The gravastars could be described with the help of three different zones, in which I is the interior region \(0 \leq r, r < r_1\), II is the intermediate thin shell, with \(r_1 < r, r < r_2\), while III is an exterior region \(r_2 < r\). It is so happened that in the region I, the isotropic pressure produces a force of repulsion over the intermediate thin shell, which is equal to \(-\rho\) (where \(\rho\) is the energy density). This intermediate thin shell is supposed to be supported by a fluid pressure and ultrarelativistic plasma. However, the region III can be represented by the vacuum solution of the field equations. The pressure has zero value at this zone. It contains thermodynamically stable solution and maximum entropy under small fluctuations [29,30].

Visser [31] developed a simple mathematical model for the description of Mazur-Mottola scenario and described the stability of gravastars after exploring some realistic values of EoS parameter. Cattoen et al. [32] extended their results for the case of anisotropic gravastar structures. They calculated the anisotropic factor \(\Delta\) from the equations of motion for the spherically symmetric spacetime and analyzed the inclusion of pressure anisotropy could be useful to support relatively high compact gravastars. Based upon the ranges of the parameters involved, Carter [33] studied the stability of the gravastar and checked the existence of thin shell. He used Israel junction condition in order to join de-Sitter spacetime (interior) with a Reissner-Nordström (RN) exterior metric. He also analyzed the role of EoS parameter in the modeling of gravastar structures.

Horvat et al. [34] presented two different theoretical models of the gravastars in the presence of an electromagnetic field. After joining the interior metric with an appropriate exterior vacuum geometry, they obtained viability constraints for the stability of gravastar though dominant energy conditions. They also studied the effects of electromagnetic field on the formulations as well as graphical representations of EoS, the speed of sound and the surface red-shift. Rahaman et al. [35] discussed the existence of charged gravastar in an environment of \((2+1)\)-dimensional spacetime. They studied various physical properties, like, length and energy within the thin shell and entropy of the charged gravastars and claimed that their solutions are nonsingular that could present a viable alternative to the black hole.

De Felice et al. [36] found exact solutions of the spherically symmetric spacetimes in
the presence of electric charge and also compared their results with the existing black holes models. It can be concluded that one can mollify the phenomenon of gravitational collapse to a great extent in the presence of a electric charge. Yousaf and Bhatti [37] investigated the modeling of relativistic structures in the presence of electromagnetic field. They concluded that the electric charge has greatly weakens the influence of modified gravity leading to produce a repulsive field. Turimov et al. [38] studied the slowly rotating magnetized gravastars in the presence of electromagnetic field.

Rahaman et al. [39] studied the 3-dimensional neutral spherically symmetry model of gravitational vacuum star whose exterior region is elaborated by Bañados-Teitelboim-Zanelli metric. He presented a non-singular and stable model and discussed various physical features, for example, length, energy conditions, entropy, and junction conditions of the spherical distribution by using static spherically symmetric matter distribution as an interior spacetime. Lobo and Garattini [40] performed linearized stability analysis with noncommutative geometry of gravastars and investigated few exact solutions of gravastar after exploring their physical features and characteristics.

Usmani et al. [41] studied charged gravastar experiencing conformal motion and elaborated the dynamics of the formation of thin shell and the entropy of the system. Herrera and de León [42] discussed the role of anisotropic/isotropic pressure on charged spheres and found some exact solutions of the non-linear field equations by assuming spherical symmetry spacetime. The same authors [43] also analyzed the dynamics of anisotropic spheres by introducing one parameter group of conformal motions. They inferred that on the boundary of matter, all of their calculated solutions can exist on the Schwarzschild exterior metric. Esculpi and Aloma [44] studied the conformal motion of the charged fluid sphere with linear equation of state (EoS). They also discussed dynamical stability analysis of the relativistic structures. The process of dynamical instability [45] and the regularity of certain physical quantities [46] on the surface of evolving matter distribution are also discussed in literature. Ray and Das [47] discussed the electromagnetic mass model with the help of conformal killing vector. By considering the existence of a one parameter group, the corresponding conformal motion have been described for the charged strange quark star model.

This paper is devoted to understand the existence of gravastar under spherically symmetric spacetime in the realm of Maxwell-\( f(R, T) \) gravity. The paper is arranged as follows. In section 2, we shall describe the basic frame work of \( f(R, T) \) theory and their conservation equation. Section 3 is based on the formulation of field/conservation equations and gravitational mass of the static spherically symmetric manifold. In section 4, we shall calculate the mass of thin shell by using certain matching conditions. The section 5 is aimed to discuss the effects of charge on the various physical features of gravastar. Finally, we summarize our findings.
\section{\textit{f}(R, T)\ Gravity}

The action of the \textit{f}(R, T)\ theory is given as follows

\[ S = \frac{1}{16\pi} \int f(R, T)\sqrt{-g}\, d^4x + \int L_m\sqrt{-g}\, d^4x, \quad (1) \]

where \( f(R, T)\) is the arbitrary function of \( R \) and \( T \) with \( R \) as the Ricci scalar and \( T \) as the trace of the energy-momentum tensor, \( L_m\) is the Lagrangian matter density, and \( g \) is the determinant of metric tensor \( g_{\zeta\eta}\). In this paper, we have consider \( G = c = 1 \). By varying the action of \( f(R, T)\) with respect to the metric \( g_{\zeta\eta}\), we can deduce the field equations of \( f(R, T)\) of gravity as

\[ f_R(R, T)R_{\zeta\eta} - \frac{1}{2} f(R, T)g_{\zeta\eta} - (\nabla_\zeta \nabla_\eta - g_{\zeta\eta}\Box)f(R, T) + f_T(R, T)(T_{\zeta\eta} + \Theta_{\zeta\eta}) = 8\pi Tg_{\zeta\eta}, \quad (2) \]

where \( f_R(R, T)\) is the derivative of generic function \( f \) with respect to the Ricci scalar \( R \), \( f_T(R, T)\) is the derivative of generic function with respect to trace of the energy-momentum tensor \( T \), \( \Box \) is the product of contravariant and covariant derivative,

\[ \Theta_{\zeta\eta} = g_{\zeta\eta}\frac{\partial T_{\zeta\eta}}{\partial g_{\zeta\eta}} \quad (3) \]

where the stress-energy tensor is defined below

\[ T_{\zeta\eta} = g_{\zeta\eta}L_m - 2\frac{\partial L_m}{\partial g_{\zeta\eta}}. \quad (4) \]

We assume perfect fluid as the energy-momentum tensor

\[ T_{\zeta\eta} = (\rho + p)U_\zeta U_\eta - pg_{\zeta\eta}, \quad (5) \]

where \( \rho \) is the density, \( p \) is the pressure and \( U_\zeta \) is the 4-velocity vector. It is interesting to mention that here the equations of motion is dependant on the role of fluid distribution. Therefore, one can take specific equations of motion by choosing \( L_m\). In literature, researchers chose \( L_m = p \) and \( L_m = \rho \) \cite{17,18,22,48,49}. Here, we are interested to study the charged isotropic fluid, therefore we shall take \( L_m = -(\rho + \mathcal{F}) \), where \( \mathcal{F} \) shows the contribution of electromagnetic field and is defined through Maxwell tensor \( (F_{\alpha\beta}) \) as

\[ \mathcal{F} = \frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\sigma}g^{\alpha\gamma}g^{\beta\sigma}. \]

The Maxwell field tensor is defined through four potential \( \phi_\alpha \) as \( F_{\alpha\beta} = \phi_{\beta,\alpha} - \phi_{\alpha,\beta} \). The Maxwell equations of motion are

\[ F^{\xi\beta}_{\gamma\beta} = K_0J_\xi, \quad F_{[\xi\gamma\beta]} = 0, \quad (6) \]
in which $J^\xi$ is the four current and $K_0$ indicates the permeability of the magnetic field. In view of this scenario, Eq. (3) reduces to

$$\Theta_{\zeta\eta} = -2T_{\zeta\eta} - pg_{\zeta\eta} - \mathcal{F}g_{\zeta\eta}.$$  

In this work, we use functional form of $f(R,T) = 2\chi T + R$. Substituting this relation in Eq. (2), we get

$$G_{\zeta\eta} = 8\pi(T_{\zeta\eta} + E_{\zeta\eta}) + \chi T g_{\zeta\eta} + 2\chi(T_{\zeta\eta} + E_{\zeta\eta} + pg_{\zeta\eta} + \mathcal{F}g_{\zeta\eta}),$$

(7)

where $G_{\zeta\eta}$ is the Einstein tensor and and $E_{\zeta\eta}$ is the energy momentum tensor for an electromagnetic field and is given by [50]

$$E_{\xi\beta} = \frac{1}{4\pi} \left( F_{\xi\gamma} F_{\beta\gamma} - \frac{1}{4} F_{\gamma\delta} F_{\gamma\delta} g_{\alpha\xi} \right).$$

(8)

In this paper, we consider a scenario in which the system is evolving by keeping the charged particles at the state of rest. This will give zero contribution to the magnetic field. Therefore,

$$\phi_\xi = \Phi(t,r)\delta_\xi^0, \quad J^\xi = K_1(t,r)V^\xi,$$

where $\Phi$ represents the corresponding scalar potential and the and $K_1$ indicates charge density.

To understand $f(R,T)$ theory as an appropriate gravitational theory, one must consider a viable and effective distribution of $f(R,T)$ function. Besides its physical consistency with the observations of current cosmic acceleration, it should pass the stability tests and should meet the viability requirements from solar and terrestrial static/non-static systems. Usually, the $f(R,T)$ models are presented in the following different ways:

1. $f(R,T) = R + 2g(T)$. Such kind of selection in the geometric part of the Lagrangian describes cosmological constant $\Lambda$ as a time-dependent entity and hence represents the $\Lambda$CDM model.

2. $f(R,T) = f_1(R) + f_2(T)$. This type of choice corresponds to the minimal knowledge to understand modified relativistic dynamics. This could be regarded as the corrections to the notable $f(R)$ theory. By considering any linear combination of $f_2$, various distinct results can be obtained from the choices of $f(R)$ function.

3. $f(R,T) = f_1(R) + f_2(T)f_3(R)$. This explicitly describes non-minimal coupled matter-geometry theory of gravity. The comparison of results found from this selection may be different from the minimal interacting choices.
The criteria for understanding the viability of \( f_1(R) \) model are as follows

- For positive value of \( f_{1R}(R) \) with \( R > R \); where \( R \) represents today’s choice of the Ricci scalar. This condition is necessary to prevent the appearances of a ghost state. Ghosts often appear which notify that DE is responsible for cosmic acceleration under modified gravity theories. This state could be induced by a mysterious force which creates a repulsion between the supermassive or massive stellar object. For retaining the attractive feature of gravity, the constraint should maintain a positive sign with a consistent gravitational constant, \( G_{\text{eff}} = G/f_{1R}. \)

- The positive value of \( f_{1RR}(R) \) with \( R > R \). This requirement is introduced for making the evolving system not to conceive situations in which tachyons appears. A hypothetical object that could move faster than the speed of light is known as tachyon.

If a model of \( f_1(R) \) does not fulfill those conditions, it would not be considered viable. Haghani et al. \[51\] as well as Odintsov and D. Sáez-Gómez \[52\] proposed that Dolgov-Kawasaki instability in \( f(R,T) \) gravity requires a similar sort of limitations as in \( f(R) \) gravity and one needs to satisfy \( 1 + f_T > 0 \) with \( G_{\text{eff}} > 0 \). Thus, in the realm of \( f(R,T) \) models, the following conditions should be fulfilled

\[
f_R > 0, \quad 1 + f_T > 0, \quad f_{RR} > 0, \quad R > R
\]

Thus, throughout in our paper, we assume that \( 1 + 2\chi > 0 \). One thing which must be taken into account is that the divergence of the energy-momentum tensor is not zero in \( f(R,T) \) gravity and is defined as

\[
\nabla^\zeta T_{\zeta \eta} = \frac{f_T}{8\pi} - \frac{f_T}{f_T}[(T_{\zeta \eta} + \Theta_{\zeta \eta})\nabla^\zeta \ln f_T + \nabla^\zeta \Theta_{\zeta \eta} - \frac{1}{2}g_{\zeta \eta} \nabla^\zeta T]. \quad (9)
\]

The non-zero value of divergence of energy momentum tensor causes the breaking of all equivalence principle in \( f(R,T) \) gravity. According to the weak equivalence principle, “All test particles in a given gravitational field will undergo the same acceleration, independent of their properties, including their rest mass.” In this modified theory, the equation of motion is based on those feature of the particle that are thermodynamic in nature, e.g., pressure, energy density, etc. Further, the strong equivalence principle states that “The gravitational motion of a small test body depends only on its initial position and velocity, and not on its configuration.” \[53\]. This principle also does not hold in \( f(R,T) \) theory thus causing the particles to experience non-geodesic motion along the world lines. In the background of quantum theory, one can relate the non-zero divergence of the effective energy-momentum tensor with the violation of energy conservation in the scattering phenomenon.
In this theory, the energy non-conservation can cause an energy flow between the four-dimensional spacetime and a compact extra-dimensional metric \([54]\). It is worth notable that the constraint \(f(T) = 0\) in Eq.(9) would reduce our dynamics to that of \(f(R)\) gravity. One can write Eq.(9) as follows

\[
\nabla^\zeta T_{\zeta\eta} = \frac{-2\chi}{8\pi + 2\chi} \left[ \nabla^\zeta (pg_{\zeta\eta}) + \nabla^\zeta (F g_{\zeta\eta}) + \frac{1}{2} g_{\zeta\eta} \nabla^\zeta T \right].
\]

(10)

It is worthy to stress that Eq.(9) corresponds to a general \(f(T)\) part, while Eq.(10) describes the covariant divergence of stress-energy tensor with a linear contribution on \(f(T)\) model.

3 Spherically Symmetric Spacetime Models

This section is devoted to explore modified equations of motion, including field and conservation laws. By simultaneous solving these equation, we evaluate evolution equation. After using a specific combination of EoS, we shall evaluate the value of the corresponding scale factors that would eventually leads to gravitational mass of the relativistic structure. We consider irrotational static form of the spherically spacetime as follows

\[
ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

(11)

The nonzero components of the Einstein tensor for above equation are

\[
G^{00} = \frac{\lambda' r + e^\lambda - 1}{r^2 e^\nu e^\lambda},
\]

(12)

\[
G^{11} = \frac{\nu' r - e^\lambda + 1}{r^2 e^\lambda},
\]

(13)

\[
G^{22} = \frac{-2\lambda' - 2\nu' + (2\nu'' + \nu'^2 - \nu' \lambda') r}{4r^3 e^\lambda}.
\]

(14)

The nonzero components of Eq.(8) are given as

\[
E_{00} = 2\pi E^2 e^\nu, \quad E_{11} = 2\pi E^2 e^\lambda, \quad E_{22} = 2\pi E^2 r^2,
\]

(15)

where \(E\) is the electric intensity, which is defined via electric charge \((q)\) as

\[
E = \frac{q}{4\pi r^2}.
\]

After using Eqs.(12) - (15) in Eq.(6), we get

\[
\lambda' r + e^\lambda - 1 = r^2 e^\lambda \left[ 8\pi \rho + \chi (3\rho - p) + \frac{q^2}{r^4} + \frac{\chi}{r^4} \left( \frac{1}{4\pi} + 1 \right) q^2 \right],
\]

(16)
\[
-n'r + e^\lambda - 1 = r^2e^\lambda \left[ -8\pi p + \chi(\rho - 3p) + \frac{q^2}{r^4} + \frac{\chi}{r^4} \left( \frac{1}{4\pi} + 1 \right) q^2 \right],
\]

\[
-\frac{r}{2}(\nu' - \nu') + \frac{r^2}{4}(\nu' + 2\nu'' - \nu'^2) = r^2e^\lambda \left[ -8\pi p + \chi(\rho - 3p) - \frac{q^2}{r^4} + \frac{\chi}{r^4} \left( -\frac{1}{4\pi} + 1 \right) q^2 \right].
\]

The hydrostatic equilibrium condition can be evaluated with the help of conservation law as

\[
\frac{\nu'}{2}(\rho + p) + \frac{dp}{dr} + \frac{3q^2}{8\pi r^5} + \frac{\chi}{4\pi + \chi} \left[ \frac{2q^2}{r^5} + \frac{1}{2}(\rho' - p') \right] = 0.
\]

With the help of Misner-Sharp formula and \( G_{00} \) component of the Einstein tensor, the corresponding component of line element \( g_{00} \) becomes

\[
e^{-\lambda} = 1 - \frac{2m}{r} - \chi \left( \rho - \frac{p}{3} \right) r^2 + \frac{2q^2}{r^4} + \frac{\chi}{r^4} \left( \frac{1}{4\pi} + 1 \right) q^2.
\]

Using Eq. (17) in Eq. (19), we get

\[
\frac{dp}{dr} = -\frac{\frac{2q^2}{r^5} - \frac{3q^2}{8\pi r^5} - \frac{\nu'/(\rho + p)}{1 + \frac{\chi}{2(4\pi + \chi)} \left( 1 - \frac{dp}{dp} \right)}}{1 + \frac{\chi}{2(4\pi + \chi)} \left( 1 - \frac{dp}{dp} \right)},
\]

where

\[
\nu' = \frac{r \left[ 8\pi p - \chi(\rho - 3p) - \frac{q^2}{r^4} - \frac{\chi}{r^4} \left( \frac{1}{4\pi} + 1 \right) q^2 \right] + \frac{1}{r} \left[ \frac{2m}{r} + \chi \left( \rho - \frac{p}{3} \right) r^2 + \frac{2q^2}{r^4} + \frac{\chi}{r^4} \left( \frac{1}{4\pi} + 1 \right) q^2 \right]}{1 + \frac{2q^2}{r^4} + \frac{2m}{r} - \chi \left( \rho - \frac{p}{3} \right) r^2 + \frac{2q^2}{r^4} + \frac{\chi}{r^4} \left( \frac{1}{4\pi} + 1 \right) q^2}.
\]

Gravastars consists of three regions characterized by an EoS \( p = \omega \rho \), where \( \omega \) is constant. Here, we assume that the interior region is filled with an enigmatic gravitational source. The corresponding EoS for dark energy model is given as

\[
p = -\rho, \quad \text{with } \omega = -1.
\]

Using, \( \rho = \rho_0 \) (constant) in Eq. (22), we get

\[
p = -\rho_0.
\]

After using Eq. (22) in Eq. (16), it follows that

\[
e^{-\lambda} = 1 - \frac{4r^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{r^4} + \frac{\chi}{r^4} \left( \frac{1}{4\pi} + 1 \right) q^2 + H \frac{r}{r},
\]

9
where $H$ is an integration constant, whose value, after applying the regularity condition is found to be zero. Therefore, Eq. (25) becomes

$$e^{-\lambda} = 1 - \frac{4r^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{r^2} + \frac{\chi}{r^2} \left( \frac{1}{4\pi} + 1 \right) q^2. \quad (25)$$

Substituting an EoS in Eqs. (16) and (17), we have

$$e^{-\lambda} = I e^\nu, \quad (26)$$

where $I$ is an integration constant. The gravitational mass $M(D)$ can be found as follows

$$M(D) = \int_0^{r=D} 4\pi \left( \rho_0 + \frac{q^2}{2r^3} \right) r^2 dr = 2\pi D \left( \frac{2}{3} D^2 + q^2 \right), \quad (27)$$

where $\rho_0$ is the constant density. Equation (27) describes that the interior gravitational mass and radius of the stellar system are directly proportional to each other. This is the characteristic feature of the stellar compact object. Furthermore, the above equation also states the substantial dependence of $M$ on the specific value $r = D$ in the presence of electric charge. This integral becomes improper on substituting $r = \infty$. However, this choice is not realistic as one can not consider the infinite radius of the stellar body.

### 4 Intermediate Shell of the Charged Gravastar

In this section, we tend to discuss the effect of electromagnetic charge on the formulation of intermediate shell of the corresponding gravastars. We shall also explore the smooth matching conditions for the joining of interior and exterior manifolds of the gravastar structures by using Darmois-Israel formalism. For this purpose, we assume that the intermediate shell is formed by an ultrarelativistic fluid with non-vacuum background with an equation of state $p = -\rho$. It is hard to calculate the solution of the cumbersome set of field equations in the non-vacuum region. To avoid this query, we will use some approximation and find analytical solution, i.e., $0 < e^{-\lambda} \ll 1$. By solving field Eqs. (16)-(18) under EoS, we end up with the following two equations as

$$\frac{de^{-\lambda}}{dr} = -\frac{2q^2}{r^3} - \frac{2\chi}{r^3} \left( \frac{1}{4\pi} + 1 \right) q^2 + 2 \frac{2}{r}, \quad (28)$$

$$\left( \frac{3}{2r} + \frac{\nu'}{4} \right) \frac{de^{-\lambda}}{dr} = -\frac{2\chi q^2}{r^2} + \frac{1}{r^2}. \quad (29)$$
Integrating Eq. (28), we get
\[ e^{-\lambda} = \frac{q^2}{r^2} + \frac{2}{r^2} \left( \frac{1}{4\pi} + 1 \right) q^2 + 2\ln r + B, \] (30)
where \( B \) is an integration constant and \( r \) is the radius belonging to \( D \ll r \ll D + \epsilon \), under \( \epsilon \ll 1 \). To get analytical values of pressure and radius of thin shell, we use Eqs. (28) and (29) in Eq. (19), we get the behavior of pressure with respect to radius, which are shown in Fig. (1).

In order to discuss the structure of gravastar, we take static Schwarzschild spacetime as an exterior geometry given as follows,
\[ ds^2 = \left( 1 - \frac{2M}{r} + \frac{q^2}{D^2} \right) dt^2 - \left( 1 - \frac{2M}{r} + \frac{q^2}{D^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (31)

Darmois [56] and Israel [57] introduced conditions for the matching of interior and exterior geometries over the surface. The metric coefficients are continuous at the junction surface \((\Sigma)\), i.e., their derivatives might not be continuous at interior surfaces. The surface tension and surface stress energy of the joining surface \( S \) may be resolved from the discontinuity of the extrinsic curvature of \( S \) at \( r = D \). The field equation of intrinsic surface is defined by Lanczos equation as
\[ S_\alpha^\beta = -\frac{1}{8\pi}(\nabla_\alpha^\beta - \delta_\alpha^\beta \Gamma_\kappa^\kappa), \] (32)
where \( S_{ji} \) is the stress-energy tensor for surface, \( \Upsilon_{\alpha\beta} = \eta^+_{\alpha\beta} - \eta^-_{\alpha\beta} \) tells the extrinsic curvatures or second fundamental forms and \((+)\) sign indicates the interior surface while \((-)\) sign indicates the exterior surface. The second fundamental forms connect interior and exterior surfaces of the thin shell and are defined as follows

\[
\eta^\pm_{\mu\nu} = -n^\pm_i \left[ \frac{\partial^2 x_i}{\partial \xi^\mu \partial \xi^\nu} + \Gamma^i_{\gamma\delta} \frac{\partial x^\gamma}{\partial \xi^\mu} \frac{\partial x^\delta}{\partial \xi^\nu} \right] \Sigma,
\]

where \( \xi^\mu \) represents the coordinate of intrinsic metric and \( n^\pm_i \) describes the unit normals on the surface of gravastar.

\[
n^\pm_i = \pm \left| g^{\alpha\beta} \frac{\partial f(r)}{\partial x^\alpha} \frac{\partial f(r)}{\partial x^\beta} \right|^{-\frac{1}{2}} \frac{\partial f(r)}{\partial x^i}, \quad n_j n^j = 1,
\]

where \( f(r) \) illustrates the coordinate of exterior metric. Using the Lanczos equations, we can get surface energy density \( \varphi \) and surface pressure \( \psi \) as

\[
\varphi = -\frac{1}{4\pi D} \left[ \sqrt{f(r)} \right]^\pm,
\]

\[
\psi = -\frac{\varphi}{2} + \frac{1}{16\pi} \left[ \frac{f(r)'}{\sqrt{f(r)}} \right]^+.
\]

Making use of Eqs. (35) and (36), we get

\[
\varphi = -\frac{1}{4\pi D} \left[ \sqrt{1 - \frac{2M}{D} + \frac{q^2}{D^2}} - \sqrt{1 - \frac{4D^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{D^2} + \frac{\chi}{D^2} \left( \frac{1}{4\pi} + 1 \right) q^2} \right],
\]

\[
\psi = \frac{1}{8\pi D} \left[ \frac{(1 - \frac{M}{D})}{\sqrt{1 - \frac{2M}{D} + \frac{q^2}{D^2}}} - \left[ 1 - \frac{8D^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{2D^2} + \frac{\chi}{2D^2} \left( \frac{1}{4\pi} + 1 \right) q^2 \right] \right].
\]

One can find mass of the intermediate thin shell by using areal density as

\[
m_s = 4\pi D^2 \varphi = -D \left[ \sqrt{1 - \frac{2M}{D} + \frac{q^2}{D^2}} - \sqrt{1 - \frac{4D^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{D^2} + \frac{\chi}{D^2} \left( \frac{1}{4\pi} + 1 \right) q^2} \right],
\]

where

\[
M = \frac{m_s}{D} \sqrt{1 - \frac{4D^2\rho_0}{3}(2\pi + \chi) + \frac{q^2}{D^2} + \frac{\chi}{D^2} \left( \frac{1}{4\pi} + 1 \right) q^2} - \frac{m_s^2}{2D} + \frac{2\rho_0 D^3}{3}(2\pi + \chi).
\]
Figure 2: Stability regions of the charged gravastar in terms of $\eta = \frac{\psi'}{\varphi'}$. We have chosen $\chi = 0.2$, $M = 1.2$, $\rho = 0.002$ at $q = 1$ and $q = 1.5$.

$$- \frac{\chi}{2D} \left( \frac{1}{4\pi} + 1 \right) q^2,$$

represents the total mass of the gravitational vacuum star with $m_s = m$. It can be noticed that one can calculate the value of $M$, once the mass of intermediate thin shell ($m_s$), the radial distance ($D$) as well as the value of $f(R,T)$ correction term ($\chi$) or electric charge ($q$) are known. It also indicates that the physical quantities $m_s$, $D$ and $q$ have dominated their influence over the $f(R,T)$ dark source terms. This is because one can diminish the role of $\chi$ by substituting zero to $m_s$, $D$ and $q$. This situation could be different, if one consider the Palatini $f(R,T)$ gravity \[58\] instead of metric $f(R,T)$ gravity \[17\].

It will be very useful to understand the stability of gravastars by defining a parameter ($\eta$) as the ratio of the derivatives of $\psi$ and $\varphi$ as follows

$$\eta(a) = \frac{\psi'(a)}{\varphi'(a)} \bigg|_{r=a_0}.$$

The stability regions can be explored by analyzing the behavior of $\eta$ as a function of $r = a_0$. Övgün \textit{et al.} \[59\] considered static form of the spherically symmetric spacetime and analyzed the the stability of a charged thin-shell gravastar with the help of a similar parameter as defined above. We have investigated the stable regimes of gravastars with specific choices of parameters involved. The letter $S$ in Figs\[2\] and \[3\] describe the stable epochs of spherically symmetric gravastar structures.
Figure 3: Stability regions of the charged gravastar in terms of $\eta = \frac{\psi}{\phi}$. We have chosen $\chi = 0.2$, $M = 1.2$, $\rho = 0.002$ at $q = 2$ and $q = 2.5$.

5 Some Features of Gravastars

This section is devoted to examine the impact of electromagnetic field on different physical features of gravastar. In this context, we shall calculate proper length of the thin shell as well as energy of relativistic structure. After examining the entropy of gravastars, the role of EoS parameter will be analyzed on the dynamical formulation of gravastars. We shall also describe our results by drawing various diagrams and graphs.

5.1 Proper Length of the Thin Shell

In this subsection we shall consider $r = D$ for describing the radius of an interior region, while $r = D + \epsilon$ (with $\epsilon \ll 1$) and $\epsilon$ represent the radius of exterior region and the thickness of the intermediate thin shell, respectively. The proper thickness between two surfaces can be described mathematically as

$$\ell = \int_{D}^{D+\epsilon} \sqrt{e^\lambda} dr = \int_{D}^{D+\epsilon} \frac{1}{\sqrt{\frac{q^2}{r^2} + \frac{2\chi}{r} (\frac{1}{4\pi} + 1) q^2 + 2lnr + C}} dr,$$

The analytic solution of the above expression with Maxwell-$f(R, T)$ gravity corrections is not possible. We shall solve it by numerical method and examine the behavior of charge. The behavior of length of the shell vs its thickness has been shown in Fig.(4).
5.2 Energy of the Charged Gravastar

The energy content within the shell is given as
\[ \varepsilon = \int_{D}^{D+\epsilon} 4\pi \rho r^2 dr \]
which is found after using the corresponding values from the equation of motion as follows
\[ \varepsilon = \frac{4\pi H}{7} \left[ (D + \epsilon)^7 - D^7 \right] + 2q^2 \epsilon \]  
(41)

The graphical representation about the role of energy and thickness of the intermediate shell is being explored from the above equation and is shown in Fig. 5. The graph (5) shows the linear relationship between energy and thickness of the shell, while energy of the system tends to increase by increasing the corresponding charge values.

5.3 Entropy of the Charged Gravastars

Entropy is the disorderness within the body of a gravastar. It is found in the literature that the entropy density of the interior region of the charged gravastar is zero. The entropy relation for the shell can be calculated through the following formula
\[ S = \int_{D}^{D+\epsilon} 4\pi r^2 S(r) \sqrt{e^\lambda} dr, \]  
(42)
Figure 5: Plot of energy $\varepsilon$ with thickness of the shell $\epsilon$ (km). Fixing $H = 0.00001$, $D = 1$.

where

$$S(r) = \frac{\alpha^2 K_B^2 T(r)}{4\pi h^2} = \alpha \left( \frac{K_B}{h} \right) \sqrt{\frac{P}{2\pi}}, \quad (43)$$

describes entropy density corresponding to a specific temperature $T(r)$. In the above expression, $\alpha$ is a constant term that has no any dimension. It is noteworthy that we are using geometrical ($G = C = 1$) as well as Planck units ($K_B = h = 1$) in our computation, therefore $S(r)$ becomes

$$S(r) = \alpha \sqrt{\frac{P}{2\pi}}. \quad (44)$$

Then, Eq.(42) turns out to be

$$S = (8\pi H)^\frac{1}{2} \alpha \int_D^{D+\epsilon} r^4 \sqrt{\frac{q^2}{r^2} + 2\frac{1}{r^2} + 2\ln r + C} dr. \quad (45)$$

The above equation contains the contribution of charge as well as corrections from $f(R, T)$ gravity. The analytical solutions of the above is not possible. After using numerical method, we have drawn graphs to examine the behavior of an electric charge which are given in Fig.(6).
5.4 EoS Parameter

At a particular radius \( r = D \), the EoS can be written as

\[
\omega(D) = \frac{\nu}{\sigma}. \tag{46}
\]

Substituting the values of \( \nu \) and \( \sigma \) in Eq. (46), we get

\[
\omega(D) = \frac{1}{\sqrt{1 - \frac{2M}{D} + \frac{q^2}{D^2}}} \left[ \frac{1 - \frac{8D^2}{3} \rho_o (2\pi + \chi) + \frac{q^2}{2D^2} + \frac{\chi}{2D^2} (\frac{1}{\pi} + 1) q^2}{\sqrt{1 - \frac{4D^2}{3} \rho_o (2\pi + \chi) + \frac{q^2}{D^2} + \frac{\chi}{D^2} (\frac{1}{4\pi} + 1) q^2}} \right] - 2 \left[ \sqrt{1 - \frac{2M}{D} + \frac{q^2}{D^2}} - \frac{1}{\sqrt{1 - \frac{4D^2}{3} \rho_o (2\pi + \chi) + \frac{q^2}{D^2} + \frac{\chi}{D^2} (\frac{1}{4\pi} + 1) q^2}} \right]. \tag{47}
\]

To get real solutions of the above equation, we have a tendency to use some approximations, i.e., \( \frac{2M}{D} < 1 \) and \( \frac{4(2\pi + \chi) \rho_o D^2}{3} < 1 \). We use binomial expansion for avoiding square root terms which is responsible for producing increments in the sensitivity of equation and some approximations, i.e., \( M \frac{D}{D} < 1 \), \( \frac{4(2\pi + \chi) \rho_o D^2}{3} < 1 \) and \( \frac{q^2}{2D^2} \ll 1 \), then we get

\[
\omega(D) \approx -3Dq^2 + 3Mq^2 + 12(2\pi + \chi) \rho_o D^5 - 2(2\pi + \chi) q^2 \rho_o D^2 [5 + 6(\frac{1}{4\pi} + 1) \chi], \tag{48}
\]

which can be rewritten as

\[
\omega(D) \approx \frac{\phi_1 - \phi_2}{8D^5(2\pi + \chi) \rho_o [\phi_3 - 1]}, \tag{49}
\]
where

\[
\begin{align*}
\phi_1 &= 3Mq^2 + 12(2\pi + \chi)\rho_oD^5, \\
\phi_2 &= 2(2\pi + \chi)q^2\rho_oD^2[5 + 6\left(\frac{1}{4\pi} + 1\right)\chi], \\
\phi_3 &= \frac{3\chi(\frac{1}{4\pi} + 1)q^2 + 6M}{4(2\pi + \chi)\rho_oD^4}.
\end{align*}
\]

The sign of EoS parameter is being controlled by the signs of numerator and denominator. The EoS parameter becomes positive, if \(\phi_1 > \phi_2\) along with \(\phi_3 > 1\) or \(\phi_1 < \phi_2\) with \(\phi_3 < 1\). However, if during evolution, stellar system satisfy the the constraints \(\phi_1 > \phi_2, \phi_3 < 1\) or \(\phi_1 < \phi_2, \phi_3 > 1\), then the \(\omega\) will enter into a negative phase. For instance, the choice \(\omega = -1\) incorporates the DE effects of the cosmological constant \(\Lambda\). This scenario could be helpful to understand the theoretical modeling of gravastars.

6 Conclusion

In this work, we have investigated the role of electromagnetic field on an isotropic stellar model with extra degrees of freedom coming from \(f(R, T)\) gravity. Gravastar is the short form of Gravitationally vacuum stars, that take up a new idea in the gravitational system. Such kind of stellar model can be considered as an alternative to black holes. Gravastar can be described through three different regions, first is the interior region with radius \(r\), second is the intermediate thin shell with thickness \(\epsilon\) and third is the exterior region with radius \(r^+\). The evolution of fluid is dealt with by a specific EoS. We have worked out a set of singularity-free solution of gravastar that represents different features of the isotropic relativistic system. Some of the discussed properties of our systems are described below.

(1) Pressure – density profile: The relationship between pressure and density of the ultrarelativistic fluid within the intermediate thin shell is shown in Fig.1 against the radial coordinate \(r\). We can see the effect of electromagnetic charge on pressure and density.

(2) Proper length of thin shell: Figure 4 is plotted between proper length of the shell and the thickness of the shell. We can conclude from the graph that if charge within the gravastar is increasing then length of thin shell is decreasing and if charge within the gravastar is decreasing then length of thin shell is increasing. The electromagnetic field and thickness of the shell are having an inverse relation.

(3) Energy content: Energy within the shell and thickness of the shell are directly proportional to each other. It can be seen from Fig.5 that the increase of charge energy would directly increase the thickness of the shell. Furthermore, the thickness can be enhanced by including huge amount of charge in gravastars.

(4) Entropy: In order to see the role of entropy, thickness and electric charge, we have drawn
a graph mentioned in Fig. 6. This graph shows the linear relationship between entropy and thickness of the shell. By studying the effect of electromagnetic charge, we can conclude that if charge in the gravastar is increasing then shell’s entropy is decreasing and vice versa.

(5) Equation of state: We use some approximation on binomial expressions to required a real solution of $\omega(D)$. The constraints depends upon the electric charge, $f(R, T)$ corrections, mass and radius of the metric.

We have an overall observation regarding the contribution of $f(R, T)$ gravity is that unlike GR the involvement of extra degrees of freedom coming from $\chi$ has made our analysis quite different in both mathematical and graphical point of view. On assigning zero value to this coupling constant would eventually provide the limiting case of GR.

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