Abstract. The interpretation of experimental and numerical data describing off-equilibrium aging dynamics crucially depends on the connection between spontaneous and induced fluctuations. The hypothesis that linear response fluctuations are statistically subordinated to irreversible outbursts of energy, so-called quakes, leads to predictions for the average values and the fluctuation spectra of physical observables in reasonable agreement with experimental results (see e.g. Sibani et al 2006 Phys. Rev. B 74 224407). Using simulational data from a simple but representative Ising model with plaquette interactions, direct statistical evidence supporting the subordination hypothesis is presented and discussed in this work. Both energy and magnetic fluctuations are analyzed, with and without an external magnetic field present. In all cases, fluctuation spectra have a Gaussian zero centered component. For large negative values, the energy spectrum additionally features an intermittent tail describing the quakes. In the magnetization spectrum, two intermittent tails are present. These are symmetric around zero for zero-field, but asymmetric in other cases. The field has thus a biasing effect on the spontaneous intermittent magnetic fluctuations. Furthermore, the field has a negligible effect on the energy fluctuation spectra. From the observed strict temporal correlation between quakes and intermittent magnetization fluctuations, it is possible to conclude that the linear response is controlled by the quakes and inherits their temporal statistics. On this basis, the information culled from intermittent linear response data can be analyzed in the same way as spontaneous thermal energy fluctuations. The latter have a central rôle in thermally activated dynamics, but are harder to measure than linear response data.
1. Introduction

How spontaneous fluctuations and linear response are related in off-equilibrium thermal
 dynamics is an open problem of considerable interest, as linear response measurements
 [1]–[4] are the main in-road into the rich phenomenology of aging systems. Highlighting
 the issue are recent observations that aging dynamics is intermittent [5]–[8]: rare but large
 fluctuations with an exponential size distribution punctuate much smaller equilibrium-like
 fluctuations with a Gaussian size distribution. Simulational [9]–[12] and experimental [8]
 evidence in different areas supports the idea that intermittent changes of magnetization and other
 observables are induced by, and hence statistically subordinated to, intermittent and irreversible
 outbursts of heat, so-called quakes. The same hypothesis leads to a widely discussed asymptotic
 logarithmic time re-parameterization of the aging dynamics (see e.g. [13]). To ascertain whether
 or not mechanisms other than subordination could be of importance, direct statistical evidence
 is called for.

In the following, statistical subordination and allied questions are numerically investigated,
 using as a test case a simple Ising model with plaquette interactions. In spite of its simplicity,
 the model has a metastable regime similar to the supercooled state of a glass former, and, at
 lower temperatures, a dynamical regime with the defining characteristics of aging [14, 15],
 e.g. slow dynamics and intermittency. The model can hence be considered as paradigmatic,
 and the conclusions drawn from the present investigations are expectedly of wider applicability
 to glassy systems.

Intermittent changes in thermal energy (the quakes) and intermittent magnetization
 fluctuations have strong temporal correlations. Since the magnetic field induced energy changes
 are negligible, these correlations can only be due to the intermittent magnetic fluctuations being
 triggered by the quakes, a property referred to as subordination. The fluctuation statistics is
 investigated in some detail, and found to be in good agreement with the results of previous
 investigations of intermittent heat flow [7, 11] and magnetic linear response intermittency
 [8, 12]: quakes are nearly uncorrelated events, and are Poisson distributed on a logarithmic time-
 scale. Additionally, the system-size dependence of the average number of quakes in a certain
 time interval is linear. This confirms that quakes are spatially localized events, a property also
 found directly via a real space analysis. The temperature dependence of the average number of
 quakes is very weak, except at the lowest temperatures. This hints at a hierarchical structure in
 the energy landscape of the thermalized domains spawning the quakes.
2. Model and method

In the model, }$N$ Ising spins, }$\sigma_i = \pm 1$ are placed on a cubic lattice with periodic boundary conditions. Unless otherwise specified, the system size used in the simulations is }$N = 16^3$. The spins interact through the plaquette Hamiltonian

$$H = - \sum_{P_{ijkl}} \sigma_i \sigma_j \sigma_k \sigma_l + H \eta(t_w - t) \sum_i \sigma_i.$$  

(1)

The first sum runs over the elementary plaquettes of the lattice, including for each the product of the four spins located at its corners. The second term describes the coupling of the total magnetization }$\sum \sigma_i$ to an external magnetic field. As conveyed by the Heaviside step function }$\eta(t_w - t)$, the field changes instantaneously at }$t = t_w$ from zero to a value }$H$. Previous investigations of the model’s properties in zero field show a low temperature aging regime [14, 15], during which energy leaves the system intermittently, and at a rate falling off as the inverse of time [11]. A more widely studied model of glassy behavior, the Edwards–Anderson spin-glass model, has a Hamiltonian with two-spin nearest neighbor interactions and quenched random couplings. The randomness creates (static) frustration and, concomitantly, a large number of local energy minima in configuration space. The present model lacks quenched randomness, and is in this respect similar to structural glasses. Self-induced frustration is nevertheless generated by highly constrained spins, which, for a limited time, act as an effective random coupling for the other spins in the same plaquettes. Unlike the Edwards–Anderson model, the ground states of the plaquette model are easily found: the fully polarized state is one ground state. Starting from this state, new ground states are generated by reversing the direction of all the spins within a plane normal to one of the principal axes, }$x$, }$y$ or }$z$. This procedure leaves the energy unchanged, as an even number of spins (zero, two or four) changes direction in each plaquette. With }$L$ denoting the linear size of the cubic lattice, the global orientation of the spins within }$3L$ different planes can be chosen independently, resulting in }$2^{3L}$ degenerate ground states. At non-zero temperatures, the degeneracy leads to a multitude of domains. In each of these, the spins are arranged in a way similar to a particular ground state configuration.

The present simulations are all performed within the aging regime, i.e. in the temperature range }$0.5 < T < 2.5$, using the rejectionless waiting time algorithm (WTM) [16]. The ‘intrinsic’ time unit of the WTM approximately corresponds to one Monte Carlo sweep. By choosing a high-energy random configuration as the initial state for low-temperature isothermal simulations, an effectively instantaneous thermal quench is achieved. For each set of physical parameters, probability distribution functions (PDFs) are collected over 2000 independent runs, and other statistical data are collected over 1000 independent runs. The symbol }$t$ stands for the time elapsed from the initial quench (and from the beginning of the simulations). The symbol }$t_w$ is the time at which the field is switched on, while }$t_{obs} = t - t_w$ stands for the ‘observation’ time, during which data are collected. The external field is set to }$H = 0.3$ for }$t > t_w$. This particular value is large enough to produce a clear magnetic response, but small enough to ensure that its effects on the energy are hardly noticeable. The thermal energy is denoted by }$E$, and the magnetization by }$M$. The average energy per spin is denoted by }$\mu_e$. The PDF of fluctuations in energy and magnetization are constructed using finite time differences of }$E$ and }$M$, taken over short time intervals of length }$\delta t \ll t_{obs}$. 

New Journal of Physics 10 (2008) 033013 (http://www.njp.org/)
Figure 1. Left panel: the average energy per spin, $\mu_e$, is plotted versus the system age $t$. Six data sets are shown, where the magnetic field is switched on at different times, i.e. $t_w = 200, 400, 600, 800, 1000$ and $t_w = 2000$, respectively. The magnetic field is seen to have a quite small effect on $\mu_e$. Right panel: the PDF of the energy changes $\delta E$ over a time $\delta t = 100$, with (stars) and without (circles) a magnetic field applied at $t_w = 1000$. Both PDFs feature a central Gaussian part of zero average and intermittent tails. The PDFs are nearly identical, except for the largest and rarest events. Again, the magnetic field has a negligible effect. The simulation temperature is $T = 1.5$ for all the plots shown.

3. Results

An applied perturbation can act as a probe of the unperturbed dynamics if it does not significantly alter the behavior of the system. For example, in the present case, the average energy versus time and the fluctuation spectra of both energy and magnetization should not undergo qualitative changes when the external field is applied. Figure 1, left panel, shows the average energy as a function of time, for six different choices of the time $t_w$ at which the perturbation is switched on. The values of $t_w$ are $t_w = 200, 400, \ldots, 1000$ and $t_w = 2000$. Consider now two data sets with e.g. $t_w = 400$ and $t_w = 600$. Up to $t = 400$, the data are taken under identical conditions and will therefore coincide, except for statistical errors. In the interval $400 \leq t \leq 600$, a magnetic field acts in one case but not in the other, and a split between the traces builds up. For $t \geq 600$ a magnetic field is present in both cases, and the further energy decay occurs then in a nearly parallel fashion. Similar considerations apply to any other pair of data traces. How exactly the energy depends on the field is however not our present concern. The main point is simply that the field has in all cases a small effect on the average energy, as can be appreciated by visual inspection. The right panel of figure 1 shows, on a logarithmic scale, the PDF of the energy fluctuations in zero field (circles) and in a field switched on at $t_w = 1000$ (stars). Gaussian energy fluctuations of zero average are in both cases flanked on the left by an intermittent tail, which carries the net energy flow out of the system. Again, only a minor difference is visible between the unperturbed and the perturbed fluctuation spectra, and this only for the (numerically) largest and rarest intermittent fluctuations. In conclusion, quakes dissipate the excess energy entrapped in the initial configuration, see figure 1, with or without an applied field.

While a magnetic field induces a non-zero average magnetization, it does not change the overall structure of the magnetic fluctuation spectra: the left panel of figure 2 shows the PDF of
Figure 2. Left panel: the PDF of the spontaneous magnetic fluctuations has a Gaussian central part and two symmetric intermittent wings. Data are sampled in the interval \( t_w, t_w + t_{\text{obs}} \). Right panel: the same quantity (outer graph, circles) when a field \( H = 0.3 \) is turned on at \( t_w = 1000 \). The left intermittent wing is clearly reduced relative to the no-field case, and the right intermittent wing is correspondingly amplified. The inner, almost Gaussian shaped, PDF (stars) is the conditional PDF obtained by excluding the magnetic fluctuations which happen in unison with the quakes. The simulation temperature is \( T = 1.5 \) for all the plots shown.

the spontaneous magnetic fluctuations occurring in the interval \([t_w, t_w + t_{\text{obs}}]\). Intermittent wings symmetrically extend the central Gaussian part of the PDF. The outer curve (circles) in the right panel of the same figure depicts the PDF obtained in a field which is turned on at \( t_w = 1000 \). The positive intermittent tail is enhanced, the negative tail is reduced and the Gaussian part is not affected. Thus, the net average magnetization, i.e. the linear response, arises through a field induced biasing effect on the distribution of the spontaneous intermittent magnetic fluctuations.

We now show that intermittent magnetic fluctuations and quakes occur almost simultaneously. To this end, we construct a conditional PDF, which only includes the magnetic fluctuations which are time-wise well separated from the quakes: all magnetic fluctuations which occur within the same \( \delta t \) as a quake, or within a \( \delta t \) immediately following a quake are discarded. The filtering threshold utilized to decide whether a certain energy fluctuation qualifies as a quake is \( \delta E \leq -5 \), i.e. as seen in figure 1, near the onset of the intermittent behavior of the heat flow PDF. The filtering utilizes an energy criterion, but nonetheless removes all the intermittent magnetic fluctuations, as seen from the nearly Gaussian form of the resulting conditional PDF, shown in the right panel of figure 2 as the inner curve (stars). On this basis, we conclude that intermittent magnetic fluctuations and quakes are nearly synchronous. Considering the simplicity of the model, quakes and magnetic fluctuations are then very likely causally related. However, as the magnetic field has nearly no effect on the quake statistics (see figure 1) the quakes induce the intermittent magnetic fluctuations, and not vice versa. In other words, the intermittent magnetic fluctuations, and hence the linear response, are subordinated to the quakes, which embody the unperturbed dynamics of the system.

We now turn to a discussion of the temporal statistics and other properties of the quakes. Visual inspection shows that energy traces of the plaquette model have a fluctuating
part superimposed onto a monotonic step-wise decay, the latter given by the function \( r_E(t) = \min_{y < t} E(y) \). This function is called record signal, or best-so far (BSF) energy. The BSF energy is plotted in the main left panel of figure 3. The insert shows the full trace for a shorter interval of time. The right panel of figure 3 is a pictorial rendering of the real space positions of spins assessed to participate in one or more quakes during the time interval \([10^5, 1.1 \times 10^6]\). In the plot, the ‘active’ spins are shown in color, and all others are omitted. By definition, active spins change their absolute average value over five consecutive time units by exactly 2 (blue) or by 1.6 (green). Most spins are excluded as they either fluctuate with no average change or retain a fixed value during all \( \delta \) updates.

The quake statistics should be amenable to description as a Poisson process [7], [17]–[19], a process fully characterized by its average, i.e. in our case the average number \( n_I \) of quakes falling in the interval \((t', t)\). From previous investigations, we expect

\[
n_I(t', t) = \alpha \ln(t/t'),
\]

where the pre-factor \( \alpha \) depends linearly on system size, and weakly on the temperature. The theoretical justification of the above equation relies on record sized thermal fluctuations inducing the quakes.

Assuming that quakes can be treated as instantaneous events, which occur at times \( t_1, \ldots, t_q, \ldots \), an easily testable property equivalent to equation (2), is whether the logarithmic differences (rather than linear differences) \( \tau_q \) are independent and identically distributed stochastic variables, sharing the exponential distribution

\[
\text{Prob}(\tau_q > x) = \exp(-\alpha x).
\]
Figure 4. Left panel: the empirical cumulative distribution (stars) of the ‘logarithmic waiting times’ $\tau_k = \ln(t_k) - \ln(t_{k-1})$ is plotted on a log scale. The full line is a least square fit to $y \propto \exp(-\alpha x)$, using the data points between the vertical lines. Right panel: the normalized correlation between $\tau_k$ and $\tau_{k+n}$ (stars) is plotted versus $n$ on a logarithmic scale. The line is a guide to the eye. We see a strong decorrelation occurring between the first two data points, and that the correlation decays exponentially for $n > 1$.

A simple choice to evaluate the $\tau_q$ is to identify the quake times $t_q$ with the steps of the BSF energy. Admittedly, this introduces a spurious dependence on the sampling frequency: when the latter is too high a single transition from one metastable configuration to the next is counted multiple times. This leads to an over-counting of the quakes, and as discussed below, to systematic deviations from equation 3, and to spurious correlations appearing between successive quakes.

From the empirical series of $\{\tau_q\}_{q=1,2,...}$ collected for each trajectory, the correlation function is estimated as $C(k) = \langle \tau_q \tau_{q+k} \rangle_q - \langle \tau_q \rangle_q \langle \tau_{q+k} \rangle_q$. The result is then averaged over all trajectories. The theoretically predicted statistical independence of the $\tau_k$ values implies a testable property, namely whether the corresponding correlation function is a Kronecker $\delta$, i.e. ideally, $C(k)/C(0) = \delta_{k,0}$. Deviations from this form indicate, as anticipated, the presence of both systematic and statistical errors in the quake identification and collection procedures. In the left panel of figure 4, the distribution of the $\tau_q$ (dots) is seen to be well approximated by an exponential decay over nearly three decades (line). For large values of the abscissa, deviations seen are likely due to a statistical under-sampling of rare events. For $x \approx 0$ (the first three points are excluded from the fit), deviations indicate that successive intervals of equal or nearly equal duration appear more frequently than theoretically expected. This can arise when the interval between successive events is spuriously affected by the sampling time. The correlation function $C(k)$ of the logarithmic time differences, normalized to $C(k = 0) = 1$, is shown in the right panel of figure 4. After a drop in a single step to approximately 1/10 of its initial value, the correlation tapers off exponentially with increasing $k$. The residual correlation arises as discussed due to successive measurements being occasionally misclassified as quakes.

A spatially extended glassy system with short range interactions is expected to contain a number, say $\alpha$, of independently relaxing thermalized domains, with a small and slowly growing characteristic size [20, 21]. The same is true for the present model, whose domains locally resemble different ground states. Assuming that quakes originate independently within
these domains, the observed energy (and magnetic) fluctuations will integrate the effect of $\alpha$ independent Poisson processes. The value of $\alpha$ can be found as the slope of the energy versus the logarithm of time. The equivalent procedure presently followed is to estimate $\alpha$ from the exponential distribution of the logarithmic waiting times $\tau_q$, see figure 4.

The number of domains, and hence $\alpha$, increases linearly with the system size, as demonstrated in the left panel of figure 5. This simply shows that domains are spatially localized, as also seen directly from a real space analysis. The temperature dependence (or lack thereof) of $\alpha$ is related to the geometrical properties of the energy landscape of each domain. For record sized thermal fluctuations to elicit attractor changes, the energy landscape must be scale invariant [12]. As a consequence, changing the temperature should not change $\alpha$ at all. Note, however, that scale invariance will fail below a cut-off value where the granularity of the energy values makes itself felt. In the present model, the numerically smallest energy change following a single spin flip is $\delta E = \pm 4$, i.e. unlike models with Gaussian quenched disorder, the granularity is important. The right panel of figure 5 shows that $\alpha$ has a modest temperature dependence for $1 \leq T \leq 2$, i.e. for a major part of the range where aging behavior is observed. For lower temperatures, a clear $T$ dependence is visible.

4. Summary and conclusions

Direct numerical evidence has been provided that intermittent magnetic fluctuations are statistically subordinated to a certain type of events, the quakes, which dissipate the excess energy trapped in the initial configuration. The external field does not alter the temporal statistics of either the quakes or the spontaneous magnetic fluctuations. It only slightly biases the size distribution of the latter. Therefore, the field can rightly be considered as a probe of the unperturbed off-equilibrium aging dynamics. In agreement with previous numerical [12] and experimental [8] investigations, the temporal statistics of intermittent energy and magnetization changes is shown to be well described by a Poisson process.

Considering that aging dynamics is widely insensitive to details of the microscopic interactions, it is reasonable to assume that the findings described above are valid beyond the plaquette model investigated. On this basis, a wide range of intermittent linear response data
from complex dynamical systems can be statistically analyzed precisely as done for intermittent heat flow data in [7, 11].

Acknowledgments

Financial support from the Danish Natural Sciences Research Council is gratefully acknowledged. We are indebted to the Danish Center for Super Computing (DCSC) for computer time on the Horseshoe Cluster, where most of the simulations were carried out.

References

[1] Svedlindh P, Granberg P, Nordblad P, Lundgren L and Chen H S 1987 Relaxation in spin glasses at weak magnetic fields Phys. Rev. B 35 268–73
[2] Vincent E, Hammann J, Ocio M, Bouchaud J-P and Cugliandolo L F 1996 Slow dynamics and aging in spin-glasses SPEC-SACLAY-96/048
[3] Jonason K, Vincent E, Hammann J, Bouchaud J P and Nordblad P 1998 Memory and chaos effects in spin glasses Phys. Rev. Lett. 81 3243–6
[4] Yoshino H, Komori T and Takayama H 2000 Numerical study on aging dynamics in Ising spin-glass models. Temperature change protocols J. Phys. Soc. Japan 69 228–37
[5] Bissig H, Romer S, Cipelletti L, Trappe V and Schurtenberger P 2003 Intermittent dynamics and hyper-aging in dense colloidal gels Phys. Chem. Commun. 6 21–3
[6] Buisson L, Bellon L and Ciliberto S 2003 Intermittency in aging J. Phys. Condens. Matter. 15 S1163
[7] Sibani P and Jensen H J 2005 Intermittency, aging and extremal fluctuations Europhys. Lett. 69 563–9
[8] Sibani P, Rodriguez G F and Kenning G G 2006 Intermittent quakes and record dynamics in the thermoremanent magnetization of a spin-glass Phys. Rev. B 74 224407
[9] Oliveira L P, Jensen H J, Nicodemi M and Sibani P 2005 Record dynamics and the observed temperature plateau in the magnetic creep rate of type II superconductors Phys. Rev. B 71 104526
[10] Sibani P 2006 Mesoscopic fluctuations and intermittency in aging dynamics Europhys. Lett. 73 69–75
[11] Sibani P 2006 Aging and intermittency in a p-spin model Phys. Rev. E 74 031115
[12] Sibani P 2007 Linear response in aging glassy systems, intermittency and the Poisson statistics of record fluctuations Eur. Phys. J. B 58 483–91
[13] Castillo H E, Chamon C, Cugliandolo L F, Iguain J L and Kenneth M P 2003 Spatially heterogeneous ages in glassy systems Phys. Rev. B 68 134442
[14] Lipowski A and Johnston D 2000 Cooling-rate effects in a model of glasses Phys. Rev. E 61 6375–82
[15] Swift M R, Bokil H, Travasso R D M and Bray A J 2000 Glassy behavior in a ferromagnetic p-spin model Phys. Rev. B 62 11494–8
[16] Dall J and Sibani P 2001 Faster Monte Carlo simulations at low temperatures. The waiting time method Comput. Phys. Commun. 141 260–7
[17] Sibani P and Littlewood P B 1993 Slow dynamics from noise adaptation Phys. Rev. Lett. 71 1482–5
[18] Sibani P and Dall J 2003 Log-Poisson statistics and pure aging in glassy systems Europhys. Lett. 64 8–14
[19] Anderson P, Jensen H J, Oliveira L P and Sibani P 2004 Evolution in complex systems Complexity 10 49–56
[20] Rieger H 1993 Non-equilibrium dynamics and aging in the three dimensional Ising spin-glass model J. Phys. A: Math. Gen. 26 L615–21
[21] Yoshino H, Komori T and Takayama H 1999 Numerical study on aging dynamics in the 3D Ising spin-glass model. I. Energy relaxation and domain coarsening J. Phys. Soc. Japan 68 3387–93

New Journal of Physics 10 (2008) 033013 (http://www.njp.org/)