On the Statistics of Urban Street Networks

Jérôme Benoit and Saif Eddin Jabari
New York University Abu Dhabi, Abu Dhabi, United Arab Emirates
jerome.benoit@nyu.edu

1 Introduction

We seek to understand the statistics of urban street networks. Such an understanding will improve urban policies in general and urban transportation in particular. In our work here we investigate urban street networks as a whole within the frameworks of information physics [9] and statistical physics [7].

Although the number of times that a natural road crosses another one has been widely observed to follow a discrete Pareto probability distribution [3] among self-organized cities [1,4,8], very few efforts have focused on deriving the statistics of urban street networks from fundamental principles. Here a natural road (or road) denotes an accepted substitute for a “named” street [8].

Our approach explicitly emphasizes the road-junction hierarchy of the initial urban street network rather than implicitly splitting it accordingly in two dual but distinct networks. Most of the investigations indeed seek to cast the initial urban street network into a road-road network [8] and to describe its valence probability distribution.

This holistic viewpoint adopted by the urban community [1,2] also fits with the mindset of information physics [9], which is built upon partial order relations [5,9]. Here the partial order relation derives from the road-junction incidence relation. Applying then information physics enables us to envisage urban street networks as evolving social systems subject to an entropic equilibrium similar to the Paretian one effectively observed among cities of a same country [6].

2 Method

The relation that ties natural roads and junctions is bijectively reduced into an algebraic structure known as Galois lattice [5]. Then, by imposing natural consistency constraints, information physics [9] enables us not only to evaluate urban street networks but also to assess a probability distribution and, ultimately, an information measure. It appears that urban street networks effortlessly reduce to Galois lattices with two nontrivial layers: the natural roads form the lower layer and the junctions form the upper one, while the partial ordering relation is “passing through.” This causes urban street networks to become a toy model for the emerging paradigm.

The passage from Galoisean hierarchy to Paretian coherence is then achieved by invoking Jaynes’s Maximum Entropy principle [7] with the first logarithmic moment as the sole characterizing constraint and our complete ignorance as initial knowledge. The corresponding most plausible probability distribution expresses for each natural road
or junction, indiscriminately, the likeliness of possessing a given number of equally likely states; it is a discrete Pareto probability distribution whose entropy is the imposed first logarithmic moment, as desired. This probability distribution must ultimately be decomposed with respect to the structure of the Galois lattice and the algebraic rules imposed by information physics theory. In other words, a physical meaning remains to be given to the evaluation.

Hypothesizing a crude asymptotic binomial paired-agent model with the spirit of the city model [6] allows us finally to predict the statistics of urban street networks. Here each natural road or junction is envisioned as an intranetwork whose very survival relies on the ability of each of its agent to preserve a crucial number of intraconnections. Thereby each urban street network becomes characterized by two generalized binomial combination numbers which asymptotically rise up as two characterizing exponents beside the Pareto exponent $\lambda$: the numbers of vital connections $\nu_r$ and $\nu_j$. 

![Relative Frequency Distributions (RFD) for the urban street network of London](image)

**Fig. 1.** Relative Frequency Distributions (RFD) for the urban street network of London: circles represent relative frequencies for the valences of the road-road topological network; crosses represent relative frequencies for the valences of the junction-junction topological network. The red fitted curve for the natural road statistics describes the Maximum Likelihood Estimate (MLE) for the discrete Pareto probability distribution (1a) estimated according to the state of the art [3] ($n_r = 4, 2\lambda \nu_r = 2.610(0.065), n = 5000$ samples, $p$-value $= 0.929$). The green fitted curve for the junction statistics shows the best Nonlinear Least-Squares Fitting (NLSF) for the nonstandard discrete probability distribution (1b) with $n_r$ and $2\lambda \nu_r$ fixed to their respective MLE value ($2\lambda \nu_r = -1.3$); since fast evaluation of the normalizing function $W$ has yet to be found, no MLE approach can be used for now. Having a number of vital connections $\nu_j$ negative means that the associated generalized binomial combination number is smaller than one, i.e., that the number of agent intraconnections for junctions is relatively much smaller than the one for natural roads.
3 Results and Discussion

Our approach recovers the discrete Pareto probability distribution widely observed for natural roads evolving in self-organized cities, and foresees a nonstandard bell-shaped distribution with a Paretian tail for their joining junctions. The probability for a natural road to cross \( n_r \) natural roads is

\[
\Pr(n_r) = \frac{n_r^{-2\lambda \upsilon_r}}{\zeta(2\lambda \upsilon_r ; n_r)} \tag{1a}
\]

where \( \zeta(\alpha; n) = \sum_{n=1}^{\infty} n^{-\alpha} \) is the generalized zeta function, and the probability for a junction to see \( n_j \) junctions through its joining natural roads reads

\[
\Pr(n_j) = \sum_{m,n} m^{-\alpha} n^{-\alpha} \omega(m,n)^2 n_j^{-2\lambda \upsilon_j} \tag{1b}
\]

where \( \omega(\alpha, \beta, \gamma; n) = \sum_{m,n} m^{-\alpha} n^{-\beta} (m+n)^{-\gamma} \) is the two-dimensional generalized Mordell-Tornheim-Witten zeta function; the number of junctions per natural road \( n_r \) is assumed to span from some minimal value \( n^* \) for practical reasons [3].

Figure 1 exhibits the urban street network of London as a case study. The probability distribution for natural roads \( \Pr(n_r) \) (1a) is highly plausible, as expected for any recognized self-organized city [1,8]. The validation of the probability distribution for junctions \( \Pr(n_j) \) (1b) appears more delicate for the time being. Meanwhile a crude data analysis is not conclusive enough. Interestingly, this case study reveals that the number of intraconnections for junctions might be relatively much smaller than the one for natural roads in self-organized cities.

Thus the statistical model for urban street networks (1) appears fine enough to study urban macro behaviours with the exponents \( \lambda, \upsilon_r, \) and \( \upsilon_j \) as complexity parameters. Future work includes (i) finding patterns via the ratio \( \upsilon_r/\upsilon_j \) among self-organized cities, (ii) extending the model to designed cities, (iii) applying the paradigm to more intricate systems, and (iv) full investigation of the resulting Paretian statistical physics.

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