Galerkin FEM for Fractional Order Parabolic Equations with Initial Data in $H^{-s}$, $0 \leq s \leq 1$

Bangti Jin, Raytcho Lazarov, Joseph Pasciak, and Zhi Zhou

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

Abstract. We investigate semi-discrete numerical schemes based on the standard Galerkin and lumped mass Galerkin finite element methods for an initial-boundary value problem for homogeneous fractional diffusion problems with non-smooth initial data. We assume that $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ is a convex polygonal (polyhedral) domain. We theoretically justify optimal order error estimates in $L^2$-and $H^1$-norms for initial data in $H^{-s}(\Omega)$, $0 \leq s \leq 1$. We confirm our theoretical findings with a number of numerical tests that include initial data $v$ being a Dirac $\delta$-function supported on a $(d-1)$-dimensional manifold.

1 Introduction

We consider the initial–boundary value problem for the fractional order parabolic differential equation for $u(x,t)$:

\[
\partial_t^\alpha u(x,t) + \mathcal{L} u(x,t) = f(x,t) \quad \text{in } \Omega, \quad T \geq t > 0,
\]

\[
u(x,t) = 0 \quad \text{in } \partial \Omega, \quad T \geq t > 0,
\]

\[
u(x,0) = v(x) \quad \text{in } \Omega,
\]

where $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) is a bounded convex polygonal domain with a boundary $\partial \Omega$, and $\mathcal{L}$ is a symmetric, uniformly elliptic second-order differential operator. Integrating the second order derivatives by parts (once) gives rise to a bilinear form $a(\cdot, \cdot)$ satisfying

\[
a(v, w) = (\mathcal{L} v, w) \forall v \in H^2(\Omega), w \in H^1_0(\Omega),
\]

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$. The form $a(\cdot, \cdot)$ extends continuously to $H^1_0(\Omega) \times H^1_0(\Omega)$ where it is symmetric and coercive and we take $\|u\|_{H^1} = a(u, u)^{1/2}$, for all $u \in H^1_0(\Omega)$. Similarly, $\mathcal{L}$ extends continuously to an operator from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ (the set of bounded linear functionals on $H^1_0(\Omega)$) by

\[
\langle \mathcal{L} u, v \rangle = a(u, v) \forall u, v \in H^1_0(\Omega).
\]

Here $\langle \cdot, \cdot \rangle$ denotes duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. We assume that the coefficients of $\mathcal{L}$ are smooth enough so that solutions $v \in H^1_0(\Omega)$ satisfying

\[
a(v, \phi) = (f, \phi) \forall \phi \in H^1_0(\Omega)
\]

with $f \in L^2(\Omega)$ are in $H^2(\Omega)$. 

I. Dimov, I. Faragó, and L. Vulkov (Eds.): NAA 2012, LNCS 8236, pp. 24–37, 2013. © Springer-Verlag Berlin Heidelberg 2013
Here $\partial_t^\alpha u$ ($0 < \alpha < 1$) denotes the left-sided Caputo fractional derivative of order $\alpha$ with respect to $t$ and it is defined by (cf. [1] p. 91 or [2] p. 78)

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - \tau)^{-\alpha} \frac{d}{d\tau} v(\tau) \, d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function. Note that as the fractional order $\alpha$ tends to unity, the fractional derivative $\partial_t^\alpha u$ converges to the canonical first order derivative $\frac{d}{dt} u$, and thus (1) reproduces the standard parabolic equation. The model (1) captures well the dynamics of subdiffusion processes in which the mean square variance grows slower than that in a Gaussian process [3] and has found a number of practical applications. A comprehensive survey on fractional order differential equations arising in viscoelasticity, dynamical systems in control theory, electrical circuits with fractance, generalized voltage divider, fractional-order multipoles in electromagnetism, electrochemistry, and model of neurons is provided in [4]; see also [2].

The goal of this study is to develop, justify, and test a numerical technique for solving (1) with non-smooth initial data $v \in H^{-s}(\Omega)$, $0 \leq s < 1$, an important case in various applications and typical in related inverse problems; see e.g., [5], [6, Problem (4.12)] and [7,8]. This includes the case of $v$ being a delta-function supported on a $(d-1)$–dimensional manifold in $\mathbb{R}^d$, which is particularly interesting from both theoretical and practical points of view.

The weak form for problem (1) reads: find $u(t) \in H_0^1(\Omega)$ such that

$$(\partial_t^\alpha u, \chi) + a(u, \chi) = (f, \chi) \quad \forall \chi \in H_0^1(\Omega), \ T \geq t > 0, \ u(0) = v. \quad (3)$$

The following two results are known, cf. [6]: (1) for $v \in L_2(\Omega)$ the problem (1) has a unique solution in $C([0, T]; L_2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ [6, Theorem 2.1]; (2) for $f \in L_\infty(0, T; L_2(\Omega))$, problem (1) has a unique solution in $L_2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ [6, Theorem 2.2].

To introduce the semidiscrete FEM for problem (1) we follow standard notation in [9]. Let $\{T_h\}_{0 < h < 1}$ be a family of regular partitions of the domain $\Omega$ into $d$-simplexes, called finite elements, with $h$ denoting the maximum diameter. Throughout, we assume that the triangulation $T_h$ is quasi-uniform, i.e., the diameter of the inscribed disk in the finite element $\tau \in T_h$ is bounded from below by $h$, uniformly on $T_h$. The approximation $u_h$ will be sought in the finite element space $X_h : X_h(\Omega)$ of continuous piecewise linear functions over $T_h$:

$$X_h = \{ \chi \in H_0^1(\Omega) : \chi \text{ is a linear function over } \tau \ \forall \tau \in T_h \}.$$

The semidiscrete Galerkin FEM for problem (1) is: find $u_h(t) \in X_h$ such that

$$(\partial_t^\alpha u_h, \chi) + a(u_h, \chi) = (f, \chi) \quad \forall \chi \in X_h, \ T \geq t > 0, \ u_h(0) = v_h, \quad (4)$$

where $v_h \in X_h$ is an approximation of $v$. The choice of $v_h$ will depend on the smoothness of $v$. For smooth data, $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we can choose $v_h$ to