Longest paths related to Steenrod length of real projective spaces
Khanh Nguyen Duc

To cite this version:
Khanh Nguyen Duc. Longest paths related to Steenrod length of real projective spaces. 2021. hal-03368805

HAL Id: hal-03368805
https://hal.science/hal-03368805
Preprint submitted on 7 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Longest paths related to Steenrod length of real projective spaces

Khanh Nguyen Duc

Abstract

We answer an open problem asked by Ravi Vakil in Homotopy theory: Find a combinatorial interpretation of a function $f(n)$ that appears in the Steenrod length of real projective $n$-space $\mathbb{RP}^n$.

2020 Mathematics Subject Classification. 05C05, 05-08 (primary), 55P42 (secondary).

Key words and phrases. Trees, longest paths, Steenrod length, Homotopy theory.

Contents

1 Introduction 1
  1.1 The problem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
  1.2 Motivation from Topology . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
  1.3 Statement of the main results . . . . . . . . . . . . . . . . . . . . . . . . . 2

2 Preliminary 3
  2.1 The maximum value of $f(n)$ . . . . . . . . . . . . . . . . . . . . . . . . . 3
  2.2 Binary representation classes . . . . . . . . . . . . . . . . . . . . . . . . . 4
  2.3 A program for $f(n)$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
  2.4 Key lemmas . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
  2.5 Canonical path to Vakil numbers . . . . . . . . . . . . . . . . . . . . . . . . 9

3 Proof of the main theorems 14
  3.1 Proof of Theorem 1.3 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
  3.2 Proof of Theorem 1.4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
  3.3 Proof of Theorem 1.5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18

4 Examples 19

Acknowledgements 20

1 Introduction

1.1 The problem

We consider the tree with vertices of the binary form of natural numbers: Let the binary form of a positive integer $n$ be

$$1 \cdots 110 \cdots 001 \cdots 11 \cdots 11 \cdots 0 \cdots 00(2).$$
From $n$, we make different branches, which are the binary forms of numbers $n - 2^s \geq 0$, where the coefficient of $2^s$ in the $n$’s summation of powers of 2 is zero. Redirection similarly works on leaves until all leaves have the form $1 \cdots 11$, we get a corresponding tree of $n$, denoted by $T_n$. Let $f(n)$ be the length of a longest path from $n$ to the leaves of $T_n$. What is the formula of $f(n)$?

**Example 1.1.** For $n = 10 = 2^3 + 2^1$. In case of $s = 0$, we get a branch $1010_2 \rightarrow 1001_2$ and for $s = 2$, we have $1010_2 \rightarrow 110_2$. By similar arguments on $1001_2, 110_2$, etc, we have the tree $T_{10}$ as follow:

```
10 = 1010_2
   /\        /\        /\        /\        /\        /\
5001_2 101_2 111_2 111_2 111_2 111_2 111_2
   V          V          V          V          V          V
11_2
```

We see that $f(10) = 3$ with two longest paths are

$$1010_2 \rightarrow 110_2 \rightarrow 101_2 \rightarrow 11_2$$

and

$$1010_2 \rightarrow 1001_2 \rightarrow 101_2 \rightarrow 11_2.$$ 

1.2 Motivation from Topology

Let $X$ be an object (such as a CW-complex) in the stable homotopy category. The Steenrod length mod 2 of $X$ is greater than the length of the longest chain of non-zero Steenrod operations in the mod 2 cohomology of $X$. The ghost length of $X$ is the number of wedges of sphere needed to build $X$. They are measures for the complexity of $X$. In [Chr97], [Chr98], Christensen said that the ghost length is bounded below by the Steenrod length. In particular, when $X$ are real projective $n$-spaces $\mathbb{RP}^n$, the Steenrod length is given by $g(n) + 1$, where $g(n) = \max_{q \leq n} f(q)$. For $2 \leq n \leq 19$, the Steenrod length and the ghost length are equal. In [Vak99], Ravi Vakil gave a formula of $g(n)$ (Theorem 2) without knowing about combinatorial interpretation for $f(n)$, except a recursive way to compute it. He asked an open question on finding the formula of $f(n)$. In this paper, we answer this problem.

1.3 Statement of the main results

For any positive integer $n$, we define the binary class of $n$ to be the set of all positive integers $m$ such that $m$ and $n$ have the same binary form after removing all 1’s at the end. The class would be represented by the minimum element, which we denote by $\pi n$. Suppose that

$$\pi = 1 \cdots 1 0 0 \cdots 0, 1 \cdots 1 0 0 \cdots 0, \cdots, 1 \cdots 1 0 0 \cdots 0_2,$$

then we present it in bracket form

$$\pi = [A, 0_1, \cdots, 0_a, B, 0_1, \cdots, b_b, \cdots, C, 0_1, \cdots, 0_c].$$
where \( A, B, \ldots, C \geq 1, 0 \) means the number zeroes which do not stand directly behind 1 and they are at position \( i \) (from left to right) in a block 0\( \cdots \)00. In general, \( \pi \) has form \([\alpha_k, \ldots, \alpha_1]\) for \( \alpha_i \geq 0 \). If \( \alpha_k > 0 \), then we say that \( \pi \) is a \( k \)-dimensional binary class. Set \( S(\pi) = \sum_{j=1}^{k} j \alpha_j \) and \( \Delta^n = f(n) - S(\pi) \). We say that \( n \) is a Vakil number with Vakil pair \((a, k)\) if \( \pi \) has form \( 2^{a+1}k \) with \( a \leq k \leq 2a + 1 \), \( k \in \mathbb{Z}_{\geq 0} \). The main results we obtained can be stated below. First, we obtain a key lemma that gives us a formula of \( \hat{n} \) for Vakil numbers. The next two theorems are simple formulas, which cover a huge number of binary classes. The last result is the general formula of \( f(n) \) for any positive numbers.

**Lemma 1.2.** Let \( n \) be a Vakil number with Vakil pair \((a, k)\). We have \( f(n) = k + \frac{a(a+1)}{2} \).

**Theorem 1.3.** Let \( n \) be a number of \( k \)-dimensional binary class. We have \( f(n) = S(\pi) + \Delta^k \), where \( \Delta^k = 0, 0, 0, 1, 2, 4, 7 \) for \( k = 1, 2, 3, 4, 5, 6, 7 \), respectively.

For a real number \( r \), we write \( z \approx r \) to mean that \( z \) is the largest integer such that \( z \leq r \).

**Theorem 1.4.** Let \( n \) be a number of \( k \)-dimensional binary class \([\alpha_k, \ldots, \alpha_1]\), \( k \geq 4 \). Set \( m = \lfloor \log_2 k \rfloor - 1 \), then for all \( \alpha_k \geq m \), we have

\[
f(n) = S(\pi) + (2^m - 1)2^h + \frac{(k-h)(k-h-1)}{2} - mk,
\]

where \( h \geq 1 \) such that \( (2^m - 1)2^h - 1 + h + 1 \approx k \), or equivalently, \( k \approx (2^m - 1)2^h + h + 1 \).

Denote \( \pi \sim \pi' \) to mean that there is a path from the first one to the second one in the tree \( T_\pi \). If \( \pi \sim \pi' \sim \pi'' \), we say that from \( \pi'' \) to \( \pi \) is closer than from \( \pi'' \) to \( \pi' \).

**Theorem 1.5.** Let \( n \) be a positive integer. We have \( f(n) = S(\pi) + \Delta^{\hat{n}} \) where \( \hat{n} \) is the closest Vakil number to \( \pi \) with Vakil pair \((a, k)\), \( 4|k \).

Though Theorem 1.5 is ultimate, it should be considered after trying Lemma 1.2, Theorems 1.3, 1.4 in practical. The way to find \( \hat{n} \) will be visualized in the last section.

The paper is organized as follows. in Section 2, we first introduce new concepts about binary classes, Vakil numbers, canonical paths, and recall some main results of Ravi Vakil. We write a program in Python to compute and check formulas for \( f(n) \). We show some key lemmas and facts about Vakil numbers which will be used in the proof of main theorems. In Section 3, we give the proof of the main results. In Section 4, we give some examples to compute \( f(n) \) in general.

## 2 Preliminary

### 2.1 The maximum value of \( f(n) \)

In this subsection, we recall some main results of the paper [Vak99].

**Lemma 2.1.** [Vak99, Lemma 1] Each non-negative integer \( n \) has a unique representation \( n = m2^p + k \), where \( m = m(n) \), \( p = p(n) \), \( k = k(n) \) are integers, and \( p > 0, p - 1 \leq m \leq 2p - 1, 0 \leq k < 2^p \).

The representation is called the proper form of \( n \). Set \( g(n) = \max_{q \leq n} f(q) \).
Table 1: Table of $f(n)$ and $g(n)$ for small $n$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| $f(n)$ | 0 | 0 | 1 | 0 | 2 | 1 | 2 | 0 | 3 | 2 | 3 | 1 | 4 | 2 | 3 | 0 | 5 | 3 | 4 | 2 |
| $g(n)$ | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |

Table 2: Frequency table of $g(n)$ for small $n$.

| $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| $\#\{n \mid g(n) = s\}$ | 2 | 2 | 4 | 4 | 4 | 8 | 8 | 8 | 8 | 8 | 16 | 16 | 16 | 16 | 16 | 16 | 32 |

Theorem 2.2. [Vak99, Theorem 2] If $n = m2^p + k$ is the proper form of a non-negative integer $n$, then $g(n) = \frac{p(p-1)}{2} + m$.

Corollary 2.3. [Vak99, Corollary 3] The frequency table of $g(n)$ is a list of non-decreasing powers of 2, where $2^a$ appears $a + 1$ times ($a \geq 1$).

Given $n$, we can divide its binary representation into three parts $\alpha, \beta, \gamma$. Here, $\gamma$ is the block of 1’s at the end of $n$ and $l(\beta) - 1 \leq \alpha \leq 2l(\beta) - 1$ where $l(\beta)$ is the number of digits of $\beta$ and $\alpha$ is interpreted as an integer. The three parts are determined uniquely by $n$. For $n$ not of form $2^t - 1$, we can define the canonical edge from vertex $n$ to the edge corresponding to the leftmost zero in the rightmost block of zeros in the middle part $\beta$. A canonical edge has form $n \rightarrow n - 2^s$ for some unique number $s$ depend on $n$. We call this the canonical index $s(n)$ of $n$. We call the path with the starting point is $n$ and ending point of form 1...1 in the tree $T_n$ such that all edges are canonical the canonical path.

Remark 2.4. [Vak99, page 420] For $n$ not of form $2^t - 1$, $f(n)$ is equal the length of the canonical path of the tree $T_n$.

Example 2.5. For $n = 473 = 111011001_2$, we have $\alpha = 111_2 = 3$, $\beta = 01100$, $\gamma = 1$ hence $s = 2$. So, the first canonical edge is $473 \rightarrow 473 - 2^1 = 467 = 11101011_2$. Now for 469, we have $\alpha = 111_2 = 3$, $\beta = 01010$, $\gamma = 1$ hence $s = 1$. The second canonical edge is $469 \rightarrow 469 - 2^1 = 467 = 11101011_2$. By similar process, we get the canonical path of $T_{473}$ is $473 \rightarrow 469 \rightarrow 467 \rightarrow 659 \rightarrow 651 \rightarrow 619 \rightarrow 603 \rightarrow 595 \rightarrow 431 \rightarrow 399 \rightarrow 383 \rightarrow 319 \rightarrow 255 \rightarrow 127$. Hence $f(473) = 13$.

2.2 Binary representation classes

We introduce the concept of binary representation class of a positive integer and reduce the original problem on $f(n)$ to the problem on its binary class.

First of all, we see that each step of making a new branch of $n$ by subtraction $2^s$ can be redone in order:

1. Define the region from the number 0 at position $s$ to the position of the first number 1 on its left hand side.

2. Interchange all numbers 1 to 0 and 0 to 1.

Example 2.6. If $n = 110011000010011_2$ and we want to make a new branch at $s = 6$. We have

$$110011000010011_2 \rightarrow 110010111010011_2.$$
Now, let us show the reasons of above statement. Let the $n$’s summation of powers of 2 be
\[
n = a^a + \cdots + 2^b + 2^c + \cdots + 2^d.
\]
Without loss of generality, we can suppose that $a \geq \cdots \geq b \geq s \geq c \geq \cdots \geq d$. In the first case $b > s$, we have
\[
n - 2^s = 2^a + \cdots + 2^s(2^{b-s} - 1) + 2^c + \cdots + 2^d
\]
\[
= 2^a + \cdots + 2^s(2^{b-s} - 1 + 1) + 2^c + \cdots + 2^d
\]
\[
= 2^a + \cdots + 2^{b-1} + \cdots + 2^s + 2^c + \cdots + 2^d.
\]
And in the second case $s = b$, we just remove $2^b$ from $n$’s summation of powers of 2. Both cases show our rule of making new branches in binary representation is true.

Secondly, since the tail 111 in each binary form has no number 0, then deleting tail 111 of the binary form of $n$ does not change the value of $f(n)$. For example, $f(1100111_{(2)}) = f(11001_{(2)})$. It implies that we just need to consider problem of computing $f(n)$ on the binary equivalent class of $n$, that is
\[
\{m \mid m, n \text{ have the same binary form after removing the tails } 111\}.
\]
The class would be represented by the minimum element $\pi$.

Base on two above observations, if
\[
n = \underbrace{1 \cdots 1}_{A} \underbrace{0 \cdots 0}_{a} \underbrace{1 \cdots 1}_{B} \underbrace{0 \cdots 0}_{b} \cdots \underbrace{1 \cdots 1}_{C} \underbrace{0 \cdots 0}_{c} \underbrace{1}_{D} (2),
\]
then
\[
\overline{\pi} = \underbrace{1 \cdots 1}_{A} \underbrace{0 \cdots 0}_{a} \underbrace{1 \cdots 1}_{B} \underbrace{0 \cdots 0}_{b} \cdots \underbrace{1 \cdots 1}_{C} \underbrace{0 \cdots 0}_{c} (2).
\]
We denote $n$ and $\pi$ by
\[
n = [A, 0_1, \cdots, 0_a, B, 0_1, \cdots, b_b, \cdots, C, 0_1, \cdots, 0_c]D,
\]
\[
\pi = [A, 0_1, \cdots, 0_a, B, 0_1, \cdots, b_b, \cdots, C, 0_1, \cdots, 0_c].
\]
In which, $A, B, \cdots, C \geq 1$, $D \geq 0$, 0, means the number zeroes which do not stand directly behind 1 and they are at position $i$ (from left to right) in a block $0 \cdots 0$. We call the number $D$ the tail of $n$ and denote it by $t_n$. In the case $t_n = 0$, we do not need to write it in the block form of $n$. We also denote $T_{\pi}$ the corresponding tree of the binary equivalent class of $n$.

**Example 2.7.** $11001010100111110011_{(2)} = [2, 0_1, 0_2, 1, 1, 0_1, 0_2, 4, 0]$.

**Example 2.8.** The canonical path of $T_{\pi}$ in the Example 2.7 is $[3, 2, 0] \rightarrow [3, 1, 1] \rightarrow [3, 1, 0] \rightarrow [3, 0, 1] \rightarrow [3, 0, 0] \rightarrow [2, 1, 0] \rightarrow [2, 0, 1] \rightarrow [2, 0, 0] \rightarrow [1, 1, 0] \rightarrow [1, 0, 1] \rightarrow [1, 0, 0] \rightarrow [2] \rightarrow [1] \rightarrow [0]$.

We say a binary equivalent class is a $k$-dimension class (and hence all elements of the class have dimension $k$) if its minimum element has following form
\[
[\cdots, 0, 0, \alpha_k, \alpha_{k-1}, \cdots, \alpha_1] = [\alpha_k, \alpha_{k-1}, \cdots, \alpha_1],
\]
where $\alpha_i \in \mathbb{N}$ and $\alpha_k \geq 1$. 
2.3 A program for \( f(n) \)

Base on Remark 2.4, we write a program in Python to compute \( f(n) \). It will be very helpful to check the validity of our main results. The first program returns the minimum number in the binary class \( \overline{\pi} \).

```python
def r(n):
    B='0'+bin(n)[2:]
    while B[-1]=='1':
        B=B[:-1]
    return int(B,2)
```

The next programs to return the value of \( f(n) \). Here \( b(n) \) returns the part \( \beta \),

```python
def b(n):
    L=len(bin(n)[2:]
    i=0
    while i<L:
        if i-1 <= n//2^i <=2*i-1:
            break
        else:
            i=i+1
    return (bin(n)[2:][L-i:])
```

and \( s(n) \) returns canonical index \( s(n) \),

```python
def s(n):
    i=0
    L=len(b(n))
    B='0'+b(n)
    while B[-1]=='0' and i<L:
        i=i+1
        B=B[:-1]
    return i-1
```

and \( f(n) \) is for function \( f(n) \).

```python
def f(n):
    l=0
    n=r(n)
    while n>0:
        l=l+1
        n=r(n-2^s(n))
    return l
```

For example, we can check \( f(473) \)

```python
f(473)
```

The result is

```
13
```

2.4 Key lemmas

In this subsection, we present some important lemmas which are useful to compute the function \( f(n) \).

The first lemma tells us the length of the paths from the root of a tree \( T_\pi \) to the nodes of the same dimension.
Lemma 2.9. For \([\alpha'_k, \ldots, \alpha'_1] \in T_{[\alpha_k, \ldots, \alpha_1]}\), the length of the path

\[[\alpha_k, \ldots, \alpha_1] \rightarrow \cdots \rightarrow [\alpha'_k, \ldots, \alpha'_1]\]

is \(\sum_{j=1}^{k} j(\alpha_j - \alpha'_j)\), where \(\alpha_k, \alpha'_k \geq 1\).

Proof. Since \(\alpha_k, \alpha'_k \geq 1\), the dimensions of two classes are the same. Then each step in the path

\[[\alpha_k, \ldots, \alpha_1] \rightarrow \cdots \rightarrow [\alpha'_k, \ldots, \alpha'_1]\]

is of form

\[[\beta_k, \ldots, \beta_{j-1}, \beta_j] \rightarrow [\beta_k, \ldots, \beta_j - 1, \beta_{j-1} + 1, \ldots, \beta_1]\].

If we regard \(\beta_j\) as a power of \(x_j\) in the polynomial \(x_k^{\beta_k} \cdots x_1^{\beta_1}\), then each step in the branching progress which does not change the dimension is equivalent to one of the multiplications \(x_j / x_{j+1}\) for \(j \in [1, k - 1]\) or \(1 / x_{1}\). So, if

\([\alpha_k, \ldots, \alpha_1] \xrightarrow{t \text{ times}} [\alpha'_k, \ldots, \alpha'_1]\),

then \(t = t_1 + \cdots + t_k\), where \(t_j \geq 0\) and

\[\prod_{j=1}^{k} x_j^{\alpha_j} \cdot \left(\frac{x_{k-1}}{x_k}\right)^{t_k} \cdots \left(\frac{x_1}{x_2}\right)^{t_2} \cdot \left(\frac{1}{x_1}\right)^{t_1} = \prod_{j=1}^{k} x_j^{\alpha'_j} \cdot \left(\frac{x_{k-1}}{x_k}\right)^{t_k} \cdots \left(\frac{x_1}{x_2}\right)^{t_2} \cdot \left(\frac{1}{x_1}\right)^{t_1}\]

Or equivalently

\[
\begin{align*}
\alpha_k - t_k &= \alpha'_k, \\
\alpha_{k-1} + t_k - t_{k-1} &= \alpha'_{k-1}, \\
\alpha_{k-2} + t_{k-1} - t_{k-2} &= \alpha'_{k-2}, \\
&\quad \ldots \\
\alpha_2 + t_3 - t_2 &= \alpha'_2, \\
\alpha_1 + t_2 - t_1 &= \alpha'_1.
\end{align*}
\]

So, we have

\[t = \sum_{j=1}^{k} t_j = kt_k + \sum_{j=1}^{k-1} j (t_j - t_{j+1}) = \sum_{j=1}^{k} j (\alpha_j - \alpha'_j).\]

Obviously, \(t\) is the length of the path \([\alpha_k, \ldots, \alpha_1] \rightarrow \cdots \rightarrow [\alpha'_k, \ldots, \alpha'_1]\). Our conclusion is clear. \(\square\)

Suppose that \(\mathbf{N} = [\alpha_k, \ldots, \alpha_1]\), set \(S(N) = \sum_{j=1}^{k} j \alpha_j\) and \(\Delta^N = f(N) - S(N)\). Denote \([\alpha_k, \ldots, \alpha_1] \sim [\beta_1, \ldots, \beta_1], \ k \geq l\) to mean that there is a path from the first one to the second one in the tree \(T_{[\alpha_k, \ldots, \alpha_1]}\).

Lemma 2.10. \([\alpha_k, \ldots, \alpha_1] \sim [\beta_k, \ldots, \beta_1]\) if and only if \(\sum_{j=i}^{k} \alpha_j \geq \sum_{j=i}^{k} \beta_j\) for all \(1 \leq i \leq k\).
Corollary 2.13. The statement is a corollary of Lemma 1.2.

Proof. As we see in the proof of Lemma 2.9, it is equivalent to say that there exists \( t_i \geq 0 \) such that

\[
\alpha_k - t_k = \beta_k, \\
\alpha_{k-1} + t_k - t_{k-1} = \beta_{k-1}, \\
\alpha_{k-2} + t_{k-1} - t_{k-2} = \beta_{k-2}, \\
\vdots \\
\alpha_2 + t_3 - t_2 = \beta_2, \\
\alpha_1 + t_2 - t_1 = \beta_1.
\]

This system has solution \( t_i \geq 0 \) if and only if \( t_i = \sum_{j=i}^{k} \alpha_j - \sum_{j=i}^{k} \beta_j \geq 0. \)

Lemma 2.11. If \( [\alpha_k, \ldots, \alpha_l] \sim [\beta_k, \ldots, \beta_1] \), then \( \Delta^{[\alpha_k, \ldots, \alpha_l]} \geq \Delta^{[\beta_k, \ldots, \beta_1]} \).

Proof. We have \( f([\alpha_k, \ldots, \alpha_l]) \geq \sum_{j=1}^{k} j(\alpha_j - \beta_j) + f([\beta_k, \ldots, \beta_1]) \) by Lemma 2.9. The inequality implies our conclusion.

The next lemma tells us the formula of \( f(n) \) for some special numbers \( n \).

Lemma 2.12. For \( n = 2^a + 1 k \), where \( k \in [a, 2a + 1] \), we have \( f(n) = k + \frac{a(a+1)}{2} \).

Proof.

- We first show that the minimum number \( n \) such that \( f(n) = l \) is \( n = 2^a + 1 (l - \frac{a(a+1)}{2}) \), where \( \frac{a(a+3)}{2} \leq l < \frac{(a+1)(a+4)}{2} \). Indeed, the minimum number \( n \) such that \( f(n) = l \) is \( \sum_{i=0}^{l} F(i) \), where \( F(s) = \# \{ n \mid g(n) = s \} \). By Corollary 2.3, in the frequency table of \( g(n) \), \( 2^a \) appears \( a+1 \) and \( F(0) = 2^1 \). Hence \( F(s) = 2^a + 1 \) if and only if \( 2^a + 1 \cdot (a+1) \leq s < 2^a + 1 \cdot (a+2) \), or equivalently \( \frac{a(a+3)}{2} \leq s < \frac{(a+1)(a+4)}{2} \).

Now, for \( s \in \left[ \frac{a(a+3)}{2}, \frac{(a+1)(a+4)}{2} \right) \) we have

\[
\sum_{i=0}^{s} F(i) = \sum_{j=1}^{a} 2^j (j + 1) + 2^a + 1 (s - \frac{a(a+3)}{2}) = 2^a + 1 (s - \frac{a(a+1)}{2}).
\]

Hence \( n = 2^a + 1 (l - \frac{a(a+1)}{2}) \), where \( \frac{a(a+3)}{2} \leq l < \frac{(a+1)(a+4)}{2} \).

- We have \( n = 2^a + 1 k = 2^a + 1 (l - \frac{a(a+1)}{2}) \), where \( l = k + \frac{a(a+1)}{2} \in \left[ \frac{a(a+3)}{2}, \frac{(a+1)(a+4)}{2} \right) \) since \( k \in [a, 2a + 1] \). So we have \( f(n) = l = k + \frac{a(a+1)}{2} \).

For a real number \( r \), we denote \( z \approx r \) to mean that \( z \) is the largest integer such that \( z \leq r \) and \( z \approx r \) to mean that \( z \) is the largest integer such that \( z < r \). The following statement is a corollary of Lemma 1.2.

Corollary 2.13. For \( i \geq 2 \) and \( n < \log_2 i \), we have

\[
f([n, 0_1, \ldots, 0_{i-1}]) = (2^n - 1)2^k + \frac{(i-k)(i-k-1)}{2},
\]

where \( k \geq 1 \) such that \( (2^n - 1)2^{k-1} + k + 1 \approx i \), or equivalently, \( i \approx (2^n - 1)2^k + k + 1 \).
Suppose that $N_k$ prove that the edge $2.5$ Canonical path to Vakil numbers

For $i$, we will prove that after finite steps of going on canonical $Vakil$ path, a number $N$ is a $Vakil$ number if and only if $\beta$ is a $Vakil$ number if and only if $\beta = 0^b \cdots 10 \cdots 0$. We already know the formula of $f(N)$ such that $N$ is a $Vakil$ number by Lemma 1.2. We will prove that after finite steps of going on canonical path, a number $N$ which is not a $Vakil$ number will meet the first $Vakil$ number $N'$ of the same dimension. Hence, by Remark 2.4 and Lemma 2.9, we have $\Delta^N = \Delta^{N'}$ - which we can compute exactly by Lemma 1.2.

**Theorem 2.14.** If $N$ is not a $Vakil$ number, then some first steps in canonical path of $N$ are

\[
N = N_0 = 1 \cdots * \underbrace{1000 \ldots 0}_{l(\beta)} (2) \\
N_1 = 1 \cdots * \underbrace{0100 \ldots 0}_{l(\beta)} (2) \\
N_2 = 1 \cdots * \underbrace{0010 \ldots 0}_{l(\beta)} (2) \\
N_3 = 1 \cdots * \underbrace{0001 \ldots 0}_{l(\beta)} (2) \\
\cdots \\
N_{b-1} = 1 \cdots * \underbrace{0000 \ldots 10}_{l(\beta)} (2) \\
N_b = 1 \cdots * \underbrace{0000 \ldots 01}_{l(\beta)} (2)
\]

with $l(\beta(N_k)) = l(\beta)$ for $0 \leq k \leq b - 1$ and $l(\beta) - 1 \leq l(\beta(N_b)) \leq l(\beta)$.

**Proof.** Denote $\alpha_k, \beta_k$ as $\alpha, \beta$ parts of $N_k$ and $s_k = s(N_k)$, we have $N_k = \alpha_k \beta_k$. First, we prove that the edge

\[
N = 1 \cdots * \underbrace{1000 \ldots 00}_{b \geq 2} (2) \rightarrow N' = 1 \cdots * \underbrace{0100 \ldots 00}_{\alpha' = \alpha} (2)
\]
is canonical with $s = b - 1$ and $s' = s - 1$. Indeed, $s = b - 1 \geq 1$ by definition. We have $l(\beta') = l(\beta)$. Indeed, $\frac{\alpha' \beta'}{2^\epsilon} = \frac{\alpha \beta - 2^\epsilon}{2^\epsilon} = \alpha + \beta - 2^\epsilon \epsilon$ for some $\epsilon \in [0, 1)$ because $2^\epsilon \leq \beta < 2^{l(\beta)}$. Hence

$$l(\beta) - 1 \leq \frac{\alpha' \beta'}{2^{l(\beta)}} < 2l(\beta)$$

because of $l(\beta) - 1 \leq \alpha < 2l(\beta)$. It implies that $l(\beta') = l(\beta)$. By $\beta' = \ast \ast \ast 010 \ldots 0$, we have $s' = b - 2 = s - 1$.

Now apply on $N_0 = \alpha_0 \beta_0$, we get the canonical edge $N_0 \to N_1$. Do the same thing for $N_1$, we get $N_1 \to N_2$. So on, we get $N_0 \to N_1 \to \cdots \to N_{b-1}$. Because $s_{b-1} = 0$ we get the edge $N_{b-1} \to N_b = N_0/2 - 2^{b-1}$.

To prove that $l(\beta) - 1 \leq l(\beta(N_b)) \leq l(\beta)$, we just need to prove that $2^{l(\beta)} - 2^{l(\beta)} - 2 \leq N_0/2 - 2^{b-1} < 2^{l(\beta)}+1l(\beta)$. But that is obviously true since $2^{l(\beta)}(l(\beta) - 1) \leq N_b < 2^{l(\beta)}+1l(\beta)$.

We see that $l(\beta_k) = l(\beta)$ for all $0 \leq k < b$, hence $N_b$ can not be a Vakil number because $N_b/2^{l(\beta)} \not\in \mathbb{Z}$. But $N_b$ can be a Vakil number in some cases. Moreover, if $N_0$ and $N_c$ (for some $c \geq b$) have the same dimension and $N_c$ reaches $N_b$ through a canonical path, then $\Delta N_c = \Delta N_b$. In the language of equivalent classes, if $N_0 = [\alpha_k, \ldots, \alpha_j, 0_1, \ldots, 0_{j-1}]$, where $\alpha_j, j \geq 1$, then $N_b = [\alpha_k, \ldots, \alpha_j - 1, 0_1, \ldots, 0_{j-1}]$. So, we can compute $\Delta^N$ by reduction process below:

1. Write $N = [\alpha_k, \ldots, \alpha_1]$.
2. Reduce $\alpha_i \geq 1$ by following way from top to bottom, left to right. If $\alpha_i = 0$ we do not make a reduction.

$$\alpha_1 \to \alpha_1 - 1 \to \cdots \to 0$$
$$\alpha_2 \to \alpha_2 - 1 \to \cdots \to 0$$
$$\ldots$$
$$\alpha_k \to \alpha_k - 1 \to \cdots \to 0$$

3. The process will finish when we first reach a Vakil number $N'$ of the same dimension. The existence of $N'$ is guaranteed by Lemma 2.15 below.

4. $\Delta^N = \Delta^{N'}$.

The number $N'$ described in this method is exactly the $k$-Vakil number which is closest to $N$.

**Lemma 2.15.** The number $2^d$ is a $d$-Vakil number with $V(2^d) = (d - x, 2^x - 1)$ or $(d - x - 1, 2^x)$, where $x = \lfloor \log_2 d \rfloor$.

**Proof.** Since $x = \lfloor \log_2 d \rfloor$, we have $d - x - 1 \leq 2^x$ and $2^x - 1 \leq 2(d - x) + 1$. If $2^x - 1 + x < d$, then $2^x \leq 2(d - x - 1) + 1$, hence $V(2^d) = (d - x - 1, 2^x)$. If $2^x - 1 + x \geq d$, then $2^x - 1 \geq d - x$, hence $V(2^d) = (d - x, 2^x)$.

**Remark 2.16.** The process of reduction in step 2. counts canonical edges from $N$ to the first Vakil number $N'$ but not the whole canonical path of $T_N$. The reason this that Theorem 2.14 does not apply for Vakil numbers. So if $N''$ is the second Vakil number we meet in the reduction process, we have $\Delta^N \geq \Delta^{N''}$ but they do not guarantee equality. For example, we will see in Tableau 3 later, there are some Vakil number that can come to other Vakil number but the value of $\Delta$ strictly decrease. On the other hand, we will see in Theorem 2.20 below, that equality still often appears.
Before computing all values of $\Delta$ of $k$-Vakil numbers, we need the following lemma.

**Lemma 2.17.** Set $S'(\alpha,k,\alpha_1,\ldots,\alpha_t) = \sum_{j=1}^{k} \alpha_j$. If $N = [\alpha_k,\ldots,\alpha_2,\alpha_1]a_0$, then

\begin{align*}
& a) \ 2^{a+1}N = [\alpha_k,\ldots,\alpha_0,0,1,\ldots,0_a] \text{ and } S(2^{a+1}N) = S([\alpha_k,\ldots,\alpha_0]) + S'([\alpha_k,\ldots,\alpha_0])a, \\
& b) \ N + 1 = \begin{cases} 
[\alpha_k,\ldots,\alpha_1+1,0,1,\ldots,0_{a_0-1}] & \text{if } \alpha_0 \geq 1, \\
[\alpha_k,\ldots,\alpha_2](\alpha_1 + 1) & \text{if } \alpha_0 = 0.
\end{cases}
\end{align*}

*Proof.* Easy computation. \hfill $\square$

**Proposition 2.18.** $2^{a+1}N$ and $2^{a+1}(N+1)$ have the same positive dimension if and only if $a' = a - t_N + 1_N$, where $1_N$ is equal 1 if $N + 1$ is not a power of 2 and it is equal 0 otherwise.

*Proof.* Suppose that $N = [\alpha_k,\ldots,\alpha_1]a_0$. We consider two cases:

- If $N$ is even ($\alpha_0 = 0$), by Lemma 2.17 we have
  \[2^{a+1}(N + 1) = [\alpha_k,\ldots,\alpha_2,\alpha_1 + 1,0,1,\ldots,0_a].\]
  There exists $\alpha_j > 0$ for some $j \geq 1$ because $2^{a+1}N$ has positive dimension. Hence $2^{a+1}N$ and $2^{a+1}(N' + 1)$ have the same dimension if and only if $a' = a + 1$.

- If $N$ is odd ($\alpha_0 > 0$), we have
  \[2^{a+1}(N + 1) = [\alpha_k,\ldots,\alpha_1 + 1,0,1,\ldots,0_{a_0-1},0,1,\ldots,0_a].\]
  If $N = [0]a_0$, then $2^{a+1}(N + 1)$ and $2^{a+1}N$ have the same dimension if and only if $a' = a - a_0$. Otherwise, $a' = a - a_0 + 1$.

So $a' = a - t_N + 1_N$. \hfill $\square$

**Proposition 2.19.** Suppose that $N$ and $N'$ are $d$-Vakil numbers with $V(N) = (a,k)$, $V(N') = (a',k+1)$. If $k + 1$ is not power of 2, then $\Delta^N - \Delta^{N'} = \frac{k(k-1)}{2}$. 

*Proof.* Suppose that $k = [\alpha_l,\ldots,\alpha_1]a_0$. By Lemma 2.17 we have

\[N = [\alpha_l,\ldots,\alpha_1,a_0,0,1,\ldots,0_a], \quad N' = [\alpha_l,\ldots,\alpha_1+1,0,1,\ldots,0_{a+1}].\]

By Lemma 1.2, we have

\[\Delta^N = k + \frac{a(a+1)}{2} - S([\alpha_l,\ldots,\alpha_0]) - S'([\alpha_l,\ldots,\alpha_0])a, \quad \Delta^{N'} = (k + 1) + \frac{a'(a'+1)}{2} - S([\alpha_l,\ldots,\alpha_1+1,0]) - S'([\alpha_l,\ldots,\alpha_1+1,0])a.\]

Since $k + 1$ is not a power of 2, Proposition 2.18 give us $a' = a - t_k + 1$. Hence $\Delta^N - \Delta^{N'} = \frac{k(k-1)}{2}$. \hfill $\square$

**Theorem 2.20.** For $0 \leq i \leq 4$, let $N_i = 2^{a+1}k_i$, where $(a_i,k_i)$ be the pairs such that $k_{i+1} = 1 + k_i$, $a_{i+1} = a_i - t_{k_i} + 1_{k_i}$ and $4|k_0$, $k_4$ is not a power of 2. We have

\begin{align*}
& a) \ a_1 = a_2 = a_0 + 1, a_3 = a_0 + 2, a_4 = a_0 - t_{k_0/4} + 1.
\end{align*}
b) If $N_0$ is a Vakil number with a Vakil pair $(a_0, k_0)$ then $N_i$ are also a Vakil number with a Vakil pair $(a_i, k_i)$ for $i = 1, 2, 3$ and $\Delta^{N_0} = \Delta^{N_1} = \Delta^{N_2} = \Delta^{N_3}$. In addition, if $N_4$ is a Vakil number then

$$\Delta^{N_4} = \Delta^{N_0} + \left[ \frac{(t_{k_0/4}+1)(t_{k_0/4}+2)}{2} \left( (a_4 + 1) - \frac{k_4}{2} \right) \right] \quad \text{if } V(N_4) = (a_4, k_4),$$

$$\Delta^{N_4} = \Delta^{N_0} \quad \text{if } V(N_4) = (a_4 + 1, k_4/2).$$

(3)

c) If $N_0$ is a Vakil number with a Vakil pair $(a_0 + 1, k_0/2)$ then $N_2$ are also Vakil numbers with Vakil pairs $(a_2 + 1, k_2/2)$ and $\Delta^{N_0} = \Delta^{N_2}$. In addition, if $N_1, N_3$ are Vakil numbers with Vakil pairs $(a_i, k_i)$, then $\Delta^{N_1} = \Delta^{N_3} = \Delta^{N_0}$.

**Proof.**

a) Easy computation with remark that $t_{k_3} = t_{k_0/4} + 2$.

b) By Proposition 2.18, The numbers $N_i$ have the same degree. Since $N_0$ is a Vakil number, the pair $(a_0, k_0)$ satisfies the inequality $a \leq k \leq 2a + 1$. It makes the same thing happens on pairs $(a_i, k_i)_{i=1,2,3}$. Hence $N_i$ are Vakil numbers with Vakil pairs $(a_i, k_i)$ for $i = 1, 2, 3$ and $\Delta^{N_0} = \Delta^{N_1} = \Delta^{N_2} = \Delta^{N_3}$. If $N_4$ is a Vakil number with pair $(a_4, k_4)$, we have $\Delta^{N_4} - \Delta^{N_3} = \frac{(t_{k_0/4}+1)(t_{k_0/4}+2)}{2}$. As we see in the proof of Proposition 2.19, the value of $\Delta^{N_4}$ when we replace the pair $(a_4, k_4)$ by $(a_4 + 1, k_4/2)$ just changes $\frac{k_4}{2} + \frac{(a_4+1)(a_4+2)}{2} - k_4 = (a_4 + 1) - \frac{k_4}{2}$.

c) The first conclusion is easy computation. For the second conclusion, if $N_i$ is a Vakil number with pair $(a_i, k_i)$ for $i = 1, 3$, then $k_0 = 2(a_0 + 1)$. The value of $\Delta^{N_0}$ when we replace the pair $(a_0, k_0)$ by $(a_0 + 1, k_0/2)$ just changes $(a_0 + 1) - \frac{k_0}{2} = 0$. So, by part b) we have $\Delta^{N_0} = \Delta^{N_1}$. By the same argument, we get $\Delta^{N_2} = \Delta^{N_3}$.

\[ \blacksquare \]

**Proposition 2.21.** If $N$ is a $d$-Vakil number with $V(N) = (a, k)$ and $V(2^d) = (a_0, k_0)$, then $a \geq a_0$.

**Proof.** We first show that if $k + \frac{a(a+1)}{2} \leq h + \frac{b(b+1)}{2}$ with $k \in [a, 2a+1]$ and $h \in [b, 2b+1]$, then $a \leq b$. Indeed, if $b \leq a - 1$, then

$$2b + 1 + \frac{b(b+1)}{2} \leq a + \frac{a(a+1)}{2} - 1 < k + \frac{a(a+1)}{2} \leq h + \frac{b(b+1)}{2} \quad \text{(contradiction)}.$$ 

Since $N \sim 2^d$, we have $f(N) \geq f(2^d)$. Hence $a \geq a_0$. \[ \blacksquare \]

**Theorem 2.22.** For $d > 4$, let $x = \lceil \log_2 d \rceil$.

1. If $2^{x-1} + x < d$, then

$$\{d\text{-Vakil numbers}\} \subset \{2^{a+1}k \text{ of dimension } d \text{ and } k \in [2^x, 2^{x+1})\}.$$ 

For those pairs $(a, k)$, if $(a, k)$ is not a Vakil pair, then $V(2^{a+1}k) = (a + 1, k/2)$ when $k$ is even. If $(a, k)$ is not a Vakil pair and $k$ is odd, then $2^{a+1}k$ is not a Vakil number.

2. If $2^{x-1} + x \geq d$, then

$$\{d\text{-Vakil numbers}\} = \{2^{a+1}k \text{ of dimension } d \text{ and } k \in [2^{x-1}, 2^x)\}.$$
Proof.

1. We have $x = \lfloor \log_2 d \rfloor \Leftrightarrow 2^{x-1} < d \leq 2^x$. If $2^{x-1} + x < d$, then $V(2^d) = (d - x - 1, 2^x)$. Suppose that $N$ is a $d$-Vakil number with $V(N) = (a', k')$, then $a' \geq d - x - 1$. So $k' \in [2^{x-1}, 2^{x+1})$ because

\[
k \geq a' \geq d - x - 1 \geq 2^{x-1},
\]

\[
k \leq 2a' + 1 \leq 2(d - 1) + 1 \leq 2(2^x - 1) + 1 < 2^{x+1}.
\]

If $k' \in [2^{x-1}, 2^x)$, then $2k' \in [2^x, 2^{x+1})$.

- Now, when $k$ is even, if $(a, k)$ is not a Vakil pair, then $k \geq 2a + 2$ because the speed of increasing from $2^x$ to $k$ is larger than the speed of increasing from $d - x - 1$ to $a$ (by Proposition 2.18) and $2^x \geq d - x - 1$. Hence $k/2 \geq a + 1$. So, if $(a + 1, k/2)$ is not a Vakil pair, we must have $k/2 > 2(a + 1) + 1$ or $k > 4a + 6$. It makes

\[
2^{x+1} > k > 4a + 6 \geq 4(d - x - 1) + 6 \geq 4(d - x) + 2 \Rightarrow 2^x > 2(d - x) + 1.
\]

So, we have $2^x \geq 2(d - x + 1) \Leftrightarrow 2^{x-1} \geq d - x + 1 \Rightarrow 2^{x-1} + x \geq d + 1 > d$ (contradiction). It implies that $V(2^{a+1}k) = (a + 1, k/2)$.

- Because $k$ is odd, if $V(2^{a+1}k) = (a', k')$, then $k' > k$, hence $a' < a$. So $k' > k > 2a + 1 > 2a' + 1$ (contradiction).

2. First, we prove that all pair $(a, k)$ such that $2^{a+1}k$ has degree $d$ and $k \in [2^{x-1}, 2^x)$ are Vakil pairs. Indeed, since $2^{x-1} + x \geq d$ we have $V(2^d) = (d - x, 2^{x-1})$. Like the first part, the speed of increasing from $2^{x-1}$ to $k$ is larger than the speed of increasing from $d - x$ to $a$, hence $k \geq a$. We must show that $k \leq 2a + 1$.

Put $d = 2^{x-1} + \epsilon_1$ for some $\epsilon_1 \in (0, x]$ since $2^{x-1} < d \leq 2^{x-1} + x$. And $a = d - x + \epsilon_2$ for some $\epsilon_2 \geq 0$, $k = 2^{x-1} + \epsilon_3$ for some $\epsilon_3 \in (0, 2^{x-1})$. We have

\[
k \leq 2a + 1 \Leftrightarrow \frac{\epsilon_3 - 1}{2} - \epsilon_2 \leq 2^{x-2} - x + \epsilon_1.
\]

(4)

Because $\epsilon_1 - x \geq 1 - x$, we should prove that

\[
\frac{\epsilon_3 - 1}{2} - \epsilon_2 \leq 2^{x-2} - x + 1.
\]

(5)

By part b) Theorem 2.20, we just need to prove that $(a, k)$ are Vakil pairs for $4|k$. Since $d > 4$ we have $4|2^{x-1}$, hence we just prove that (5) is true for $4|\epsilon_3$. We realize that $\frac{\epsilon_3 - 1}{2} - \epsilon_2$ is increasing when $\epsilon_3$ increasing such that $4|\epsilon_3$ because

\[
\frac{\epsilon_3 - 1}{2} - \epsilon_2 \leq \frac{(\epsilon_3 + 4) - 1}{2} - \epsilon_2' \Leftrightarrow \epsilon_2' < \epsilon_2 + 2.
\]

The last inequality is true by part a) Theorem 2.20 ($\epsilon_2, \epsilon_2'$ are corresponding to $a_0, a_4$). The maximum value of $\epsilon_3$ such that $4|\epsilon_3$ is $2^{x-1} - 4$ and the corresponding value of $\epsilon_2$ is $x - 3$ (To compute $\epsilon_2$ we consider the pair $(d - x + \epsilon_2, 2^x - 4)$ as the pair $(a_0, k_0)$ and $(d - x - 1, 2^x)$ as the pair $(a_4, k_4)$ in Theorem 2.20.
with remark that $k_4$ is a power of 2. Hence, $a_4 = a_0 - tk_{k_0}/4$. It gives us $\epsilon_2 = -1 + t x_{x-4} = x - 3$. So, for $4|\epsilon_3$ and $\epsilon_3 \in \{0, 2^{x-1}\}$ we have

$$\frac{\epsilon_3 - 1}{2} - \epsilon_2 \leq \frac{2^{x-1} - 5}{2} - (x - 3) = 2^{x-2} - x + \frac{1}{2} < 2^{x-1} - x + 1.$$ 

So $(a, k)$ are Vakil pairs for all $4|k \in [2^{x-1}, 2^x]$, hence for all $k \in [2^{x-1}, 2^x]$.

- Second, we prove that if $(a, k)$ is Vakil pair, then $k \in [2^{x-1}, 2^x)$. We have $k \in [2^{x-2}, 2^{x+1})$ because

$$2^{x+1} > k \geq a \geq d - x \geq 2^{x-1} - x \geq 2^{x-2}.$$ 

So, $k \in [2^{x-2}, 2^{x-1}) \cup [2^{x-1}, 2^x) \cup [2^x, 2^{x+1})$. If $k \in [2^{x-2}, 2^{x-1})$, then $2k \in [2^{x-1}, 2^x)$. As we know, $(a - 1, 2k)$ is a Vakil pair hence $k \notin [2^{x-2}, 2^{x-1})$. In the domain $[2^x, 2^{x+1})$, we will prove that the inequality (4) is not true with $\epsilon_3 \geq 2^{x-1}$ (in this case $\epsilon_2 \geq -1$). That is

$$\frac{\epsilon_3 - 1}{2} - \epsilon_2 > 2^{x-2} - x + \epsilon_1.$$ 

Because $\epsilon_1 - x \leq 0$, we should prove that $\frac{\epsilon_3 - 1}{2} - \epsilon_2 > 2^{x-2}$. We just need to prove that the inequality for $4|\epsilon_3$ because if $\frac{\epsilon_3 - 1}{2} - \epsilon_2 > 2^{x-2}$, then

$$\frac{(\epsilon_3 + 1) - 1}{2} - (\epsilon_2 + 1) = \frac{\epsilon_3 - 1}{2} - \epsilon_2 - \frac{1}{2} > 2^{x-2},$$

$$\frac{(\epsilon_3 + 2) - 1}{2} - (\epsilon_2 + 1) = \frac{\epsilon_3 - 1}{2} - \epsilon_2 > 2^{x-2},$$

$$\frac{(\epsilon_3 + 3) - 1}{2} - (\epsilon_2 + 2) = \frac{\epsilon_3 - 1}{2} - \epsilon_2 - \frac{1}{2} > 2^{x-2}.$$ 

They imply that if $(a, k)$ is not a Vakil pair for $4|k$, then $(a + 1, k + 1), (a + 1, k + 2), (a + 2, k + 3)$ are not Vakil pair. Because $\frac{\epsilon_3 - 1}{2} - \epsilon_2$ increasing when $\epsilon_3$ increasing and $4|\epsilon_3$ and $\epsilon_2 = -1$ when $\epsilon_3 = 2^{x-1}$, we have

$$\frac{\epsilon_3 - 1}{2} - \epsilon_2 \geq \frac{2^{x-1} - 1}{2} + 1 = 2^{x-2} + \frac{1}{2} > 2^{x-2}.$$ 

So $k \notin [2^x, 2^{x+1})$. In conclusion, we have

$$\{d-Vakil\ number\} = \{2^{a+1}k\ of\ degree\ d\ and\ k \in [2^{x-1}, 2^x)]\}.$$

\[\square\]

3 Proof of the main theorems

3.1 Proof of Theorem 1.3

In this subsection, we present the proof of Theorem 1.3 by using Lemma 2.9.

Proof. Let $\overline{a} = [\alpha_k, \cdots, \alpha_1]$ with $\alpha_k \geq 1$.

1. For $k = 1, \alpha_1 \geq 1$, we have $[\alpha_1] \rightarrow [\alpha_1 - 1]$ and $f([1]) = 1$. So, we implies that $f([\alpha_1]) = \alpha_1$.  

14
2. For $k = 2, \alpha_2, \alpha_1 \geq 1$, we have

$$[\alpha_2, \alpha_1] \rightarrow [\alpha_2 - 1, \alpha_2 + 1] \text{ or } [\alpha_2, \alpha_1 - 1]$$
$$[\alpha_2, 0_1] \rightarrow [\alpha_2 - 1, 1] \text{ or } [\alpha_2 - 1]$$

So, we just consider two cases ($\alpha_1 \geq 0$)

$$[\alpha_2, \alpha_1] \rightarrow \cdots \rightarrow [0_1, \alpha_1'] = [\alpha_1']$$
$$[\alpha_2, \alpha_1] \rightarrow \cdots \rightarrow [\alpha_2', 0_1] \rightarrow [\alpha_2' - 1]$$

So

$$f([\alpha_2, \alpha_1]) = \max \begin{cases} 2(\alpha_2 - \alpha_2') + \alpha_1 + 1 + f([\alpha_2' - 1]) \\ 2\alpha_2 + \alpha_1 - \alpha_1' + f([\alpha_1']) \end{cases}$$

$$= \max \begin{cases} 2\alpha_2 + \alpha_1 - \alpha_2' \\ 2\alpha_1 + \alpha_2 \end{cases}$$

$$= 2\alpha_2 + \alpha_1.$$

3. For $k = 3$, similar to $k = 2$, we just consider four cases

$$[\alpha_3, \alpha_2, \alpha_1] \rightarrow \cdots \rightarrow [0_1, \alpha_2', \alpha_1'] = [\alpha_2', \alpha_1']$$
$$[\alpha_3, \alpha_2, \alpha_1] \rightarrow \cdots \rightarrow [\alpha_3', 0_1, \alpha_1'] \rightarrow [\alpha_3' - 1, \alpha_1' + 2]$$
$$[\alpha_3, \alpha_2, \alpha_1] \rightarrow \cdots \rightarrow [\alpha_3', \alpha_2', 0_1] \rightarrow [\alpha_3', \alpha_2' - 1]$$
$$[\alpha_3, \alpha_2, \alpha_1] \rightarrow \cdots \rightarrow [\alpha_3', 0_1, 0_2] \rightarrow [\alpha_3' - 1]$$

So, we have

$$f([\alpha_3, \alpha_2, \alpha_1])$$

$$= \max \begin{cases} 3\alpha_3 + 2(\alpha_2 - \alpha_2') + (\alpha_1 - \alpha_1') + f([\alpha_2', \alpha_1']) \\ 3(\alpha_3 - \alpha_3') + 2\alpha_2 + (\alpha_1 - \alpha_1') + 1 \\ + f([\alpha_3' - 1, \alpha_1' + 2]) \end{cases}$$

$$= \max \begin{cases} 3\alpha_3 - \alpha_3' + 2(\alpha_2 - \alpha_2') + \alpha_1 + 1 \\ + f([\alpha_3', \alpha_2' - 1]) \end{cases}$$

$$= \max \begin{cases} 3\alpha_3 - \alpha_3' + 2\alpha_2 + \alpha_1 + 1 + f([\alpha_3' - 1]) \\ 3\alpha_3 + 2\alpha_2 + \alpha_1 \end{cases}$$

$$= \max \begin{cases} 3\alpha_3 + 2\alpha_2 + \alpha_1 \\ 3\alpha_3 + 2\alpha_2 + \alpha_1 + 1 - \alpha_3' \\ 3\alpha_3 + 2\alpha_2 - \alpha_2' - \alpha_3' \\ 3\alpha_3 - 2\alpha_3' \end{cases}$$

$$= 3\alpha_3 + 2\alpha_2 + \alpha_1.$$
4. For \( k = 4 \), we just consider following cases

\[
\begin{align*}
[a_4, \ldots, a_1] & \rightarrow \cdots \rightarrow [0_1, a'_3, a'_2, a'_1] \\
[a_4, \ldots, a_1] & \rightarrow \cdots \rightarrow [a'_4, 0_1, a'_2, a'_1] \\
& \rightarrow [a'_4 - 1, a'_2 + 2, a'_1] \\
[a_4, \ldots, a_1] & \rightarrow \cdots \rightarrow [a'_4, a'_3. 0_1, a'_1] \\
& \rightarrow [a'_4, a'_3 - 1, a'_1 + 2] \\
[a_4, \ldots, a_1] & \rightarrow \cdots \rightarrow [a'_4, a'_3, a'_2, 0_1] \\
& \rightarrow [a'_4, a'_3, a'_2 - 1] \\
[a_4, \ldots, a_1] & \rightarrow \cdots \rightarrow [a'_4, 0_1, 0_2, a'_1] \\
& \rightarrow [a'_4 - 1, a'_1 + 3] \\
[a_4, \ldots, a_1] & \rightarrow \cdots \rightarrow [a'_4, a'_3, 0_1, 0_2] \\
& \rightarrow [a'_4, a'_3 - 1] \\
[a_4, \ldots, a_1] & \rightarrow \cdots \rightarrow [a'_4, 0_1, 0_2, 0_3] \\
& \rightarrow [a'_4 - 1]
\end{align*}
\]

So, we have

\[
f([a_4, \ldots, a_1]) = \max \begin{cases}
4a_4 + \sum_{j=1}^{3} j(a_j - a'_j) + f([a'_3, a'_2, a'_1]) \\
4(a_4 - a'_4) + 3a_3 + 2(a_2 - a'_2) + (a_1 - a'_1) + 1 + f([a'_4 - 1, a'_2 + 2, a'_1]) \\
4(a_4 - a'_4) + 3(a_3 - a'_3) + 2a_2 + (a_1 - a'_1) + 1 + f([a'_4, a'_3 - 1, a'_1 + 2]) \\
4(a_4 - a'_4) + 3(a_3 - a'_3) + 2(a_2 - a'_2) + a_1 + 1 + f([a'_4, a'_3, a'_2 - 1]) \\
4(a_4 - a'_4) + 3a_3 + 2a_2 + (a_1 - a'_1) + 1 + f([a'_4 - 1, a'_1 + 3]) \\
4(a_4 - a'_4) + 3(a_3 - a'_3) + 2a_2 + a_1 + 1 + f([a'_4, a'_3 - 1]) \\
4(a_4 - a'_4) + 3a_3 + 2a_2 + a_1 + 1 + f([a_4 - 1])
\end{cases}
\]
\[
\begin{align*}
\sum_{j=1}^{4} j\alpha_j \\
\sum_{j=1}^{4} j\alpha_j - \alpha'_4 + 2 \\
\sum_{j=1}^{4} j\alpha_j - (\alpha'_4 + \alpha'_3 + \alpha'_2) \\
\sum_{j=1}^{4} j\alpha_j - 2\alpha'_4 + 2 \\
\sum_{j=1}^{4} j\alpha_j - 2(\alpha'_4 + \alpha'_3) \\
\sum_{j=1}^{4} j\alpha_j - 3\alpha'_4 \\
= \sum_{j=1}^{4} j\alpha_j + 1
\end{align*}
\]

5. For \(k = 5, 6, 7\), the arguments are similar.

\[\square\]

### 3.2 Proof of Theorem 1.4

In this subsection, we will show a proof of Theorem 1.4. We first need to present some lemmas and go to the proof in the end.

**Lemma 3.1.** For \(n \geq \log_2 i, i \geq 2\), we have

\[\Delta^{[n,0_1,\ldots,0_{i-1}]} = \Delta^{[\log_2 i-1,0_1,\ldots,0_{i-1}]} .\]  \hfill (6)

**Proof.** Denote \([n,0_1,\ldots,0_{i-1}]\) by \((n,i)\)'. Since \(2^i(2^n - 1)\) is not a Vakil number, Theorem 2.14 says that in the canonical path of \(T_{(n,i)'}\) we have

\[(n,i)' \xrightarrow{i \text{ steps}} (n-1,i)' \xrightarrow{i \text{ steps}} \cdots \xrightarrow{i \text{ steps}} (n',i)',\]

where \(1 \leq n' \approx \log_2 i\). The number \(2^i(2^{n'} - 1)\) is the first Vakil number we meet in reduction process from \((n,i)'\) with

\[n' = \begin{cases} 
\lfloor \log_2 i \rfloor & \text{if } \log_2 i \notin \mathbb{N}, \\
\lfloor \log_2 i \rfloor - 1 & \text{if } \log_2 i \in \mathbb{N}.
\end{cases}\]

By Remark 2.4, we have \(\Delta^{[n,0_1,\ldots,0_{i-1}]} = \Delta^{[n',0_1,\ldots,0_{i-1}]}\). In the case \(\log_2 i \notin \mathbb{N}\), by Corollary 2.13, we have

\[\Delta^{[n'-1,0_1,\ldots,0_{i-1}]} = 2^{n'+1} + \frac{i(i-5)}{2} - 1 - n'i = \Delta^{[n',0_1,\ldots,0_{i-1}]} ,\]

since the number \(k \geq 1\) such that \(i \approx (2^{n'-1} - 1)2^k + k + 1\) is \(k = 2\), and the number \(k \geq 1\) such that \(i \approx (2^{n'} - 1)2^k + k + 1\) is \(k = 1\). \(\square\)
We are now going to the proof of Theorem 1.4.

**Proof.** By Lemma 2.10, we have
\[ \left[ \sum_{j=1}^{k} \alpha_j, 0, \ldots, 0_{k-1} \right] \sim [\alpha_k, \ldots, \alpha_1] \sim [m, 0, \ldots, 0_{k-1}] . \]

By Lemmas 2.11, 3.1, we have
\[ \Delta_{[m, 0_1, \ldots, 0_{k-1}]} = \Delta_{[\sum_{j=1}^{k} \alpha_j, 0_1, \ldots, 0_{k-1}]} \geq \Delta_{[\alpha_k, \ldots, \alpha_1]} \geq \Delta_{[m, 0_1, \ldots, 0_{k-1}]} . \]

Hence by Corollary 2.13, we have
\[ \Delta_{[\alpha_k, \ldots, \alpha_1]} = \Delta_{[m, 0_1, \ldots, 0_{k-1}]} = (2^m - 1)^{2^h} + \frac{(k - h)(k - h - 1)}{2} - m k , \]
where \( h \geq 1 \) such that \( (2^m - 1)^{2^h - 1} + h + 1 \approx k \), or equivalently, \( k \approx (2^m - 1)^{2^h} + h + 1 \). \( \square \)

### 3.3 Proof of Theorem 1.5

**Proof.** By Remark 2.4 and Lemma 2.9, we have \( \Delta^n = \Delta^\pi = \Delta^{n'} \) where \( n' \) is the closest Vakil number to \( \pi \). By Theorem 2.20, \( \Delta^{n'} = \Delta^n \) where \( n \) is the closest Vakil number with Vakil pair \( (a, k), 4|k \). Indeed, with the notations in Theorem 2.20, the binary class representation of \( N_i \) are

\[
N_0 = [a, \ldots, b, 0, \ldots, 0], \\
N_1 = [a, \ldots, b, 1, \ldots, 0], \\
N_2 = [a, \ldots, b, c + 1, \ldots, 0], \\
N_3 = [a, \ldots, b, c + 2, \ldots, 0], \\
N_4 = [a, \ldots, b + 1, 0, 0, \ldots, 0],
\]

where \( k_0 = [a, \ldots, b, c, 0] \). It is clear that if \( n' \) is one of \( N_0, N_1, N_2, N_3, N_4 \) then \( n = N_0 \), and if \( n' = N_4 \) then \( n = N_4 \). It is also clear that \( n \) is exactly the closest Vakil number to \( \pi \) with Vakil pair \( (a, k), 4|k \). So, we have proven the theorem. \( \square \)

We can describe a way to compute \( f(n) \) using Theorem 1.5 as follows:

- Suppose that \( \pi \) has dimension \( d \). Let \( x = \lfloor \log_2 d \rfloor \).
- First, we create a short tableau of Vakil pairs of dimension \( d \):
  - If \( 2^{x-1} + x < d \), then we start with the first Vakil pair \( V(2^d) = (d - x - 1, 2^x) \). By Lemma 1.2, we can compute \( f(2^d) \) and \( \Delta^{2^d} \). From now, to find remaining Vakil pairs of dimension \( d \) and the corresponding value of \( \Delta \), we add up 4 to \( 2^x \) while the sum does not get over \( 2^{x+1} \) (Theorem 2.22). Use Theorem 2.20 to get the new pair: \( (a_0, k_0) \rightarrow (a_4, k_4) \). We compute the Vakil pair \( (a', k') \) for each number \( 2^{x+1}k \) and skip all Vakil pairs \( (a', k') \) such that \( 4 \not\mid k' \). The corresponding value of \( \Delta \) is given by the second and third parts of Theorem 2.20. We note that they are inferred directly from Lemma 1.2, but Theorem 2.20 is a faster way for this mission.
  - If \( 2^{x-1} + x \geq d \), do the same thing with the Vakil pair \( V(2^d) = (d - x, 2^{x-1}) \) while the sum of adding up 4 does not get over \( 2^x \). But in this case, we know that all pairs we made are Vakil pairs by the second part of Theorem 2.22.
Now, we use above tableau to compute \( f(n) \): By Lemma 2.10, we see the list of classes that \( \pi \) can reach in the tableau. The closest class to \( \pi \) is \( \hat{n} \). Hence \( \Delta \pi = \Delta \hat{n} \) and \( f(n) = S(\pi) + \hat{n} \).

4 Examples

We first give an example of using Theorem 1.5.

Example 4.1. Let compute \( f(n) \) for \( \pi = [1, 1, 2, 1, 3^2, 0, 0, 0, 2^9, 4, \ldots, 0] \) of dimension 53. We will proceed to do it step by step as follows. First, we create a short list of Vakil pairs of dimension 53 (see Tableau 3):

- We have \( \lceil \log_2 53 \rceil = 6, 2^5 + 6 = 38 < 53 \). Hence \( V(2^{53}) = (46, 64) \). By Lemma 1.2, we have \( f(2^{53}) = 64 + 46 \cdot 47 \) and \( \Delta 2^{53} = f(2^{53}) - S(2^{53}) = 1092 \).

- From now, to find remaining Vakil pairs of dimension 53 and the corresponding value of \( \Delta \), we add up 4 to \( k \) until we get \( 2^7 - 4 \) (Theorem 2.22). The corresponding value of \( a \) for \( k + 4 \) is then computed by the first part of Theorem 2.20. We compute the Vakil pair \((a', k')\) for each number \( 2^a + 1 \) and skip all Vakil pairs \((a', k')\) such that \( 4 \nmid k' \). The corresponding value of \( \Delta \) is given by Theorem 2.20.

| \( a \) | \( k \) | \( t_{k/4} \) | Vakil pair | \( \Delta \) | representation of \( 2^{a+1}k \) |
|---|---|---|---|---|---|
| 46 | 64 | 0 | (46,64) | \( 64 + \frac{46 \cdot 47}{2} - 53 = 1092 \) | \([1,0,\ldots,0]\) |
| 47 | 68 | 1 | (47,68) | \( 1092 + \frac{1 \cdot 2}{2} = 1093 \) | \([1,0,0,1,0,\ldots,0]\) |
| 47 | 72 | 0 | (47,72) | \( 1093 + \frac{2 \cdot 2}{2} = 1096 \) | \([1,0,1,0,\ldots,0]\) |
| 48 | 76 | 2 | (48,76) | \( 1096 + \frac{4 \cdot 2}{2} = 1097 \) | \([1,0,2,0,\ldots,0]\) |
| 47 | 80 | 0 | (47,80) | \( 1097 + \frac{4 \cdot 4}{2} = 1103 \) | \([1,1,0,\ldots,0]\) |
| 48 | 84 | 1 | (48,84) | \( 1103 + \frac{8 \cdot 2}{2} = 1104 \) | \([1,1,1,0,\ldots,0]\) |
| 48 | 88 | 0 | (48,88) | \( 1104 + \frac{2 \cdot 8}{2} = 1107 \) | \([1,2,0,\ldots,0]\) |
| 49 | 92 | 3 | (49,92) | \( 1107 + \frac{1 \cdot 2}{2} = 1108 \) | \([1,3,0,\ldots,0]\) |
| 47 | 96 | 0 | (48,96) | \( 1108 + \frac{4 \cdot 16}{2} + 48 - 48 = 1118 \) | \([2,0,\ldots,0]\) |
| 48 | 100 | 1 | (49,100), skip | | |
| 48 | 104 | 0 | (49,104) | \( 1118 + \frac{1 \cdot 2}{2} = 1119 \) | \([2,1,\ldots,0]\) |
| 49 | 108 | 2 | (50,108), skip | | |
| 48 | 112 | 0 | (49,112) | \( 1119 + \frac{2 \cdot 2}{2} + 49 - 56 = 1122 \) | \([3,0,\ldots,0]\) |
| 49 | 116 | 1 | (50,116), skip | | |
| 49 | 120 | 0 | (50,120) | \( 1122 + \frac{4 \cdot 2}{2} = 1123 \) | \([4,0,\ldots,0]\) |
| 50 | 124 | | (51,124), skip | | |

Table 3: Short list tableau of \( \Delta \) for 53-dimensional binary classes.

Now, we use above tableau to compute \( f(n) \): By Lemma 2.10, we know that \( \pi \) only can reach the first six classes in the tableau. The closest class to \( \pi \) is \([1, 1, 1, 0, \ldots, 0]\). Hence \( \Delta \pi = 1104 \). So, we have \( f(n) = S(\pi) + 1104 \).

The example below shows us how to use Lemma 1.2, Theorem 1.3, Theorem 1.4. Theorem 1.5 is powerful but it should be considered if we can not apply previous results.

Example 4.2.

1. If \( n = 69632 \), then \( n = \pi = 2^{12}, 17 \) is a Vakil number with Vakil pair \((11, 17)\). Hence, by Lemma 1.2, \( f(69632) = 17 + \frac{11 \cdot 2}{2} = 83 \).
2. If \( n = 473 \), then \( n = 111011001_{(2)} \) and \( \pi = 11101100_{(2)} = 236 = 2^2.59 \). We see that 473 is not a Vakil number because 59 > 3. Since 236 = [3, 2, 0] has dimension 3, by Theorem 1.3, we have \( f(473) = S([3, 2, 0]) = 13. \)

3. If \( n = 8923773549686799 \) then \( n = 11111101101000001111111000001111100000000001111_{(2)} \) and \( n = 1111110110100000111111111000001111100001100000000111_{(2)} = 236 = 2^8.2178655651779. \)

Hence \( n \) is not a Vakil number. We see that \( n = [6, 2, 1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \) has dimension 24. Since \( \alpha_{24} = 6 \geq \lceil \log_2 24 \rceil - 1 = 3 \), by Theorem 1.4, we have \( f(n) = S(\pi) + (2^3 - 1)2^2 + \frac{(24 - 2)(24 - 3)}{2} - 3.24 = 628. \)

4. If \( n = 12737511856113 \), then \( n = 101110010101101011101101111010010111111001 \), \( \pi = 10111001010110101110110111101001011111111001_{(2)} = 6368755928056 = 2^3.796094491007. \)

Hence \( n \) is not a Vakil number. We see that \( \pi = [1, 3, 0, 1, 1, 2, 1, 3, 2, 4, 1, 0, 1, 7, 0, 0] \)

has dimension 16, and \( \alpha_{16} = 1 < \lceil \log_2 16 \rceil - 1. \) So we can not apply directly Lemma 1.2, Theorems 1.3, 1.4, but Theorem 1.5. Since \( \lceil \log_2 16 \rceil = 4, 2^3 + 4 = 12 < 16, \) we have \( V(2^{16}) = (11,16). \)

| \( a \) | \( k \) | \( t_{k/4} \) | Vakil pair | \( \Delta \) | representation of \( 2^{a+1}k \) |
|---|---|---|---|---|---|
| 11 | 16 | 0 | (11, 16) | \( 16 + \frac{11.12}{2} - 16 = 66 \) | \([1,0,\ldots,0]\) |
| 12 | 20 | 1 | (12, 20) | \( 66 + \frac{1.2}{2} = 67 \) | \([1,1,0,\ldots,0]\) |
| 12 | 24 | 0 | (12, 24) | \( 67 + \frac{2.3}{2} = 70 \) | \([2,0,\ldots,0]\) |
| 13 | 28 | skip | (14, 14) (skip) | | \([3,0,\ldots,0]\) |

Table 4: Short list tableau of \( \Delta \) for 16-dimensional binary classes.

By Lemma 2.10, \( \pi \) only can reach the first two classes in the Tableau 4. The closest class to \( \pi \) is \([1,1,0,\ldots,0]\). Hence \( \Delta^\pi = 67. \) So, we have \( f(n) = S(\pi) + 67 = 287. \)

Acknowledgments

The author would like to express his sincere gratitude to his former teacher Prof. Ha Le Minh to introduce the problem. He is grateful for the Visiting Fellowship supported by MathCoRe and Prof. Petra Schwer at Otto-von-Guericke University Magdeburg. He would also like to thank Prof. Cristian Lenart for his strong encouragement and support.
References

[Chr97] J Daniel Christensen. *Ideals in triangulated categories: phantoms, ghosts and skeleta*. PhD thesis, Massachusetts Institute of Technology, June 1997.

[Chr98] J Daniel Christensen. Ideals in triangulated categories: phantoms, ghosts and skeleta. *Advances in Mathematics*, 136(2):284–339, 1998.

[Vak99] Ravi Vakil. On the steenrod length of real projective spaces: finding longest chains in certain directed graphs. *Discrete mathematics*, 204(1-3):415–425, 1999.

Otto-von-Guericke University Magdeburg, IAG, Postschließfach 4120, 39016 Magdeburg, Germany
E-mail: khanh.mathematic@gmail.com