On the sharp lower bounds of modular invariants and fractional Dehn twist coefficients

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Abstract  Modular invariants of families of curves are Arakelov invariants in arithmetic algebraic geometry. All the known uniform lower bounds of these invariants are not sharp. In this paper, we aim to give explicit lower bounds of modular invariants of families of curves, which is sharp for genus 2. According to the relation between fractional Dehn twists and modular invariants, we give the sharp lower bounds of fractional Dehn twist coefficients and classify pseudo-periodic maps with minimal coefficients for genus 2 and 3 firstly. We also obtain a rigidity property for families with minimal modular invariants, and other applications.

Keywords  lower bounds, fractional Dehn twists, modular invariants

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1 Introduction

1.1 Modular invariants

Without mention we always work on complex number field $\mathbb{C}$. A family of projective curves of genus $g$ is a surjective holomorphic morphism $f : S \to C$ whose general fiber is a smooth curve of genus $g$, where $S$ is a smooth projective surface, and $C$ is a smooth projective curve of genus $b$. Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus $g$, and $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$.

The intersection theory of divisors of $\overline{\mathcal{M}}_g$ is very beautiful. The intersection of rational divisor class $\gamma$ of $\overline{\mathcal{M}}_g$ with curves $D \subset \overline{\mathcal{M}}_g$ is also interested in the theory of birational geometry of $\overline{\mathcal{M}}_g$. Numerically, the intersection of $\gamma$ with $D$ can be regarded as the degree of $\gamma$ on $D$.

The modular invariant of $f$ corresponding to $\gamma$ is defined as the degree $\gamma(f) = \deg J_f^*(\gamma)$, where $J_f : C \to \overline{\mathcal{M}}_g$ is the induced moduli map (see [Ta10]). The modular invariant $\gamma(f)$ satisfies the base change property, i.e., if $\tilde{f} : \tilde{S} \to \tilde{C}$ is the pullback fibration of $f$ under a base change $\pi : \tilde{C} \to C$ of degree $d$, then $\gamma(\tilde{f}) = d \cdot \gamma(f)$. Modular invariants are also important in many other mathematical branches, such as arithmetic geometry ([Ja14]), low-dimensional topology ([Li21]), and ordinary differential equations ([Ta]). In particular, the modular invariants are generalized to be new invariants, Chern numbers, of ordinary differential equations ([Ta]) by the second author.

Let $\lambda$ be the Hodge divisor class of $\overline{\mathcal{M}}_g$, $\delta$ be the boundary divisor class, and their corresponding modular invariants be $\lambda(f)$ and $\delta(f)$. We also denote by $\kappa(f)$ the modular
invariant corresponding to $\kappa = 12\lambda - \delta$. By stable reduction theorem (see [Xi90]), we know that all these modular invariants are nonnegative rational numbers, and $\lambda(f) > 0$ for non-isotrivial family $f$. These kinds of modular invariants are called Arakelov invariants ([Ja14]) in number field case, and the modular invariant $\lambda(f)$ is Faltings height in particular.

The minimal uniform lower bounds for these invariants are interesting. In 1991, Mazur raised a question on the minimized Faltings heights of varieties in $\mathbb{P}^N$, which was studied by Zhang ([Zh96]) partly. Some uniform lower bounds for Faltings $\delta$-invariant of curves are also obtained by Faltings et al ([Fa84, Wi16]). But all these known bounds are not sharp.

In this paper, we consider the above uniform lower bounds problem in the case of curves over function fields. Our goal is to get sharp lower bounds depending only on $g$ and characterize families with these minimal lower bounds.

When $g = 1$, then the best lower bounds are $\kappa(f) = 0$, $\lambda(f) \geq \frac{1}{12}$ and $\delta(f) \geq 1$. We believe that these bounds of modular invariants are known to experts. Hence we only considered $g \geq 2$.

For $g = 2$, we have the following sharp lower bounds of modular invariants.

**Theorem 1.1.** Let $f : S \to C$ be a non-isotrivial fibration of genus 2, then

$$\lambda(f) \geq \frac{1}{60}, \quad \kappa(f) \geq \frac{1}{15}, \quad \delta(f) \geq \frac{1}{12}, \quad (1.1)$$

and each equality can be reached. Furthermore,

1) $\lambda(f) = \frac{1}{60}$ if and only if $\delta(f) = \frac{1}{12}$ if and only if all the singular fibers of $f$ have smooth reduction except one whose dual graph is either Figure (2-1a) or Figure (2-1b).

2) $\kappa(f) = \frac{1}{15}$ if and only if all the singular fibers of $f$ have smooth reduction except one whose dual graph is either Figure (2-0a) or Figure (2-0b).

We prove that the lower bounds in Theorem 1.1 are optimum by giving examples in Section 6.

**Theorem 1.2.** There exists a family of fibrations $(f_{\lambda,n} : S_{\lambda,n} \to \mathbb{P}^1)_{n \in \mathbb{N}}$ (resp. $(f_{\kappa,n} : S_{\kappa,n} \to \mathbb{P}^1)_{n \in \mathbb{N}}$) of genus 2 with $\lambda(f_{\lambda,n}) = \frac{1}{60}$, $\delta(f_{\lambda,n}) = \frac{1}{12}$ (resp. $\kappa(f_{\kappa,n}) = \frac{1}{15}$, $\lambda(f_{\kappa,n}) = \frac{1}{30}$, $\delta(f_{\kappa,n}) = \frac{1}{5}$), satisfying that

1) $f_{\lambda,n}$ (resp. $f_{\kappa,n}$) has $2n + 3$ singular fibers;

2) the image of $f_{\lambda,n}$ (resp. $f_{\kappa,n}$) in $\overline{M}_g$ by the moduli map $J : \mathbb{P}^1 \to \overline{M}_g$ is the same as that of $f_{\lambda,0}$ (resp. $f_{\kappa,0}$), for each $n \in \mathbb{N}$.

Moreover, if $n = 0$, then $S_{\lambda,0}$ and $S_{\kappa,0}$ are both rational surfaces.
Furthermore, if the fibred surface is rational, then we have the following rigidity property.

**Theorem 1.3.** There are only finitely many fibrations \( f : S \to C \) of genus 2 such that \( S \) is a rational surface, and \( \lambda(f) = \frac{1}{60} \).

For \( g = 3 \), we have the following results.

**Theorem 1.4.** Let \( f : S \to C \) be a fibration of genus 3 with \( \delta(f) \neq 0 \), then

\[
\lambda(f) \geq \frac{1}{105}, \quad \delta(f) \geq \frac{1}{30}, \quad \kappa(f) \geq \frac{8}{315}.
\]

Moreover, \( \delta(f) = \frac{1}{30} \) if and only if all singular fibers of \( f \) have periodic monodromy except one whose dual graph is one of figures (3-1a), (3-1b) and (3-1c) as follows.

\[
\begin{align*}
\text{(3-1a)} & \quad \begin{array}{c}
5 \\
2 \quad 10 \quad 3 \quad 1 \quad 0 \\
C_{v1} \quad C_{v2} \quad C_{v3} \quad C_{v4} \quad C_{v5}
\end{array} \\
\text{(3-1b)} & \quad \begin{array}{c}
1 \\
2 \\
5 \quad 6 \\
3 \quad 2 \quad 4 \quad 0 \\
C_{v1} \quad C_{v2} \quad C_{v3} \quad C_{v4}
\end{array} \\
\text{(3-1c)} & \quad \begin{array}{c}
1 \\
3 \\
6 \quad 5 \quad 3 \\
2 \quad 4 \quad 0 \\
C_{v1} \quad C_{v2} \quad C_{v3} \quad C_{v4}
\end{array}
\end{align*}
\]

Now, we will show that when \( g = 3 \), \( \lambda(f) = \frac{1}{105} \) can be “combinatorially reached”.

If all singular fibers of \( f \) are \( F_1, \ldots, F_s \), then we call \( (F_1, \ldots, F_s) \) the configuration of singular fibers of \( f \). Denote by \( F_a \) (resp. \( F_b \)) the singular fiber whose dual graph is Figure (3-1a) (resp. (3-1b)), and by (i2) the corresponding singular fiber in [AI02, p.202]. Then \( F_a \) (resp. \( F_b \)) is hyperelliptic, that is, \( F_a \) (resp. \( F_b \)) can be realized as a singular fiber of some hyperelliptic fibration ([Is04, pp.19-20]).

**Theorem 1.5.** Let \( S \) be a rational surface, and \( f : S \to C \) be a hyperelliptic fibration of genus 3. Assume that either \( F_a \) or \( F_b \) is a singular fiber of \( f \). Then \( \lambda(f) = \frac{1}{105} \) if and only if the configuration of singular fibers of \( f \) is one of the following:

\[
(F_a,(i2),(i4)), \quad (F_a,(i6),(i6)), \quad (F_a,(i26),(i44)), \quad (F_a,(i26),(i45)), \quad (F_a,(i26),(i46)), \quad (F_a,(i26),(i47),(i47)), \quad (F_b,(i26),(i47)).
\]

It is natural to consider which configuration in Theorem 1.5 can be realized. If \( S \) is not rational, we will obtain more possible combinatorial configurations (see the proof of Theorem 1.5), so we give the following conjecture.

**Conjecture 1.6.** There exists a hyperelliptic fibration \( f \) of genus 3 with \( \lambda(f) = \frac{1}{105} \).

For any \( g \geq 4 \), similarly as \( g = 3 \), it is possible to obtain the sharp lower bound of \( \lambda(f) \), see Remark 5.1. For uniform lower bounds for \( g \geq 4 \), we have the following lower bounds depending only on \( g \).

**Theorem 1.7.** Suppose \( f : S \to C \) is a fibration of genus \( g \geq 4 \), and \( \delta(f) \neq 0 \), then

\[
\lambda(f) \geq \frac{1}{16g(2g + 1)}, \quad \delta(f) \geq \frac{1}{4(g + 1)^2}, \quad \kappa(f) \geq \frac{g - 1}{4g^2(2g + 1)}.
\]

Let \( \lambda(g) \) be the sharp lower bound of \( \lambda(f) \) for non-isotrivial families of curves \( f \) of genus \( g \). From the above, we know that \( \lambda(1) = \frac{1}{12}, \lambda(2) = \frac{1}{60} \). Since modular invariants are heights in arithmetic algebraic geometry, we raise the following effective question which relates to finiteness of points on curves.

**Question 1.8.** Is there a positive real number \( r_0 > 0 \) with

\[
\liminf_{g \to \infty} \lambda(g) \geq r_0?
\]
1.2 Fractional Dehn twist coefficients

It is proved that the modular invariant \( \delta(f) \) is a summation of fractional Dehn twist coefficients ([Li21]). So to get results in Section 1.1 we need sharp lower bounds of fractional Dehn twist coefficients, which is an interesting problem in low-dimensional topology.

It is known that Dehn twists are the generators of the mapping class group, and fractional Dehn twist coefficients are also important in 3-manifolds. These coefficients were first studied by Gabai and Oertel in [GO99], and then applied in many aspects ([HKM07, HM18, ...]). The bounds of these coefficients are studied in many different contexts, see [HM18 Theorem 1], [IK17 Section 7], [KR13 Theorem 2.16], [Li21 Theorem 1.5].

For our purpose, we consider fractional Dehn twists coefficients in pseudo-periodic maps, and try to give their sharp uniform lower bounds which depends only on \( g \). Before we state our results, we will introduce some notations first.

Let \( \Sigma \) be a closed connected Riemann surface of genus \( g \geq 2 \). The mapping class group \( \text{Mod}(\Sigma_g) \) of \( \Sigma_g \) is the group of isotopy classes of orientation preserving homeomorphism of \( \Sigma_g \). The Nielsen-Thurston classification theorem says that any mapping class \( \phi \in \text{Mod}(\Sigma_g) \) is either periodic, pseudo-Anosov, or reducible. The homeomorphism \( \phi \) is reducible if there exist finite simple closed curves \( \mathcal{C} = \{\gamma_1, \ldots, \gamma_r\} \) on \( \Sigma_g \) such that the restriction of \( \phi \) on \( \Sigma_g - \mathcal{C} \) is either periodic or pseudo-Anosov. If \( \phi \in \text{Mod}(\Sigma_g) \) is periodic, or \( \phi \) is reducible and the restriction is periodic, then \( \phi \) is said to be pseudo-periodic. We may assume \( \mathcal{C} \) satisfies the following additional conditions: (i) \( \gamma_i \) does not bound a disk on \( \Sigma_g \), and (ii) \( \gamma_i \) and \( \gamma_j \) are disjoint, and \( \gamma_i \) is not parallel to \( \gamma_j \) if \( i \neq j \) ([MM11 Lemma 1.1]). Such \( \mathcal{C} \) is called an admissible system of cut curves.

Given a pseudo-periodic map \( \phi \), a sufficiently high power \( \phi^m \) preserves each cut curve \( \gamma_1, \ldots, \gamma_r \). Denote by \( T_{\gamma_i} \) the (right-hand) Dehn twist of \( \Sigma_g \) along \( \gamma_i \), then there is a factorization of \( \phi \) into a commutative product \( \phi^m = T_{\gamma_i}^{k_1} \cdots T_{\gamma_r}^{k_r} \). The fractional Dehn twist coefficient of \( \phi \) along \( \gamma_i \) is defined to be \( c(\phi, \gamma_i) = k_i/m \) (see [Li17, Section 2.2.2]).

If \( \phi \in \text{Mod}(\Sigma_g) \) is a pseudo-periodic map of negative twist, that is, \( c(\phi, \gamma) < 0 \) for each \( \gamma \in \mathcal{C} \), then there exists a local family \( f_\phi : S \to \Delta \), whose monodromy homeomorphism around its central fiber is equal (up to isotopy and conjugation) to \( \phi \) ([MM11, Im09, Ta01]). Here, the local family \( f_\phi : S \to \Delta \) means a proper surjective holomorphic map from a complex surface \( S \) to the unit disk of the complex plane \( \Delta \), and only the central fiber \( F_\phi = f_\phi^{-1}(0) \) over the origin is singular. We also call \( F_\phi \) the singular fiber of \( \phi \).

It is known that the topological types of local families \( f \) of genus \( g \geq 2 \) are 1-1 correspondent to the conjugacy classes of pseudo-periodic maps \( \phi \) of negative twist ([MM11, Theorem 0.2]). Almost all the topological types of local families can be determined by dual graphs of their central fibers. (It is easy to check that the dual graphs in this paper determine topological types of the corresponding local families.) Hence we denote the conjugacy class of \( \phi \) by the dual graph \( G(F_\phi) \) of \( F_\phi \) for simplicity.

Now we give the sharp lower bounds of fractional Dehn twist coefficients \( |c(\phi, \gamma)| \) in fact), and classify the pseudo-periodic maps with these bounds.

**Theorem 1.9.** Let \( \phi \in \text{Mod}(\Sigma_g) \) be a pseudo-periodic map of negative twist, and \( \gamma \in \mathcal{C} \).

1. If \( g = 2 \), then
   \[
   |c(\phi, \gamma)| \geq \frac{1}{12},
   \]
   and the equality holds if and only if \( (G(F_\phi), \gamma) \) is either Figure (2-1a) or Figure (2-1b).
In each figure, we use thick edges to denote $\gamma$ using the correspondence in [Li21, Theorem 4.2].

(2) If $g = 3$, then

$$|c(\phi, \gamma)| \geq \frac{1}{30},$$

and the equality holds if and only if $(G(F_\phi), \gamma)$ is one of figures (3-1a), (3-1b) and (3-1c).

Moreover, we can obtain more precise results.

If $\gamma$ is a non-separated cut curve, then $\gamma$ is said to be of type 0. If $\gamma$ is separated, and the least genus of the two connected components is $i \geq 1$, then $\gamma$ is said to be of type $i$.

Theorem 1.10. Let $\phi \in \text{Mod}(\Sigma_g)$ be a pseudo-periodic map of negative twist, and $\gamma \in C$. Then the sharp lower bounds $c$ of $|c(\phi, \gamma)|$ are as follows.

| $g$ | type 0 | type 1 |
|-----|--------|--------|
| 2   | $\frac{1}{12}$ | $\frac{1}{3}$ |
| 3   | $\frac{1}{30}$ |

Furthermore,

(2-0) if $g = 2$, $\gamma$ is of type 0, then $|c(\phi, \gamma)| = \frac{1}{3}$ if and only if $(G(F_\phi), \gamma)$ is one of the four figures: Figure (2-0a) - Figure (2-0d).

(2-1) if $g = 2$, $\gamma$ is of type 1, then $|c(\phi, \gamma)| = \frac{1}{12}$ if and only if $(G(F_\phi), \gamma)$ is either Figure (2-1a) or Figure (2-1b).

(3-0) if $g = 3$, $\gamma$ is of type 0, then $|c(\phi, \gamma)| = \frac{1}{12}$ if and only if $(G(F_\phi), \gamma)$ is one of the following three figures

(3-0a)

(3-0b)

(3-0c)

(3-1) if $g = 3$, $\gamma$ is of type 1, then $|c(\phi, \gamma)| = \frac{1}{30}$ if and only if $(G(F_\phi), \gamma)$ is one of figures (3-1a), (3-1b) and (3-1c).

Remark that there are two edges between $C_{v_1}$ and $C_{v_2}$ in Figure (3-0a), and each edge corresponds to a cut curve $\gamma$ satisfying Theorem [1.10 (3-0)]. We only label one by a thick line for simplicity. For comparison, there are two edges between $C_v$ and the vertex $C'$ on the right of $C_v$ in Figure (2-0b). It is easy to see that $C'$ does not correspond to a connected component of $\Sigma_g - C$, and the two edges together correspond to a cut curve ([Li21]).

For general $g \geq 4$, there is a uniform lower bound of fractional Dehn twist coefficients in [Li21 Theorem 1.5] which is not sharp.
1.3 Effective Bogomolov conjecture

Though we have given a uniform lower bound of the effective Bogomolov conjecture for general $g$, see [LT17]. We now give a better bound for $g = 2, 3$.

Fix an algebraically closed field $k$ of characteristic zero and a smooth proper connected curve $Y/k$. Define $K$ to be the field of rational functions on $Y$. Let $C$ be a smooth proper geometrically connected curve of genus at least 2 over the function field $K$. Denote by $f : X \to Y$ the minimal regular model of the curve $C$ over $Y$, where $X$ is a smooth projective surface over $k$. Choose a divisor $D$ of degree 1 on $\overline{C} = C \times_K \overline{K}$ and consider the embedding of $C$ into its Jacobian $\text{Jac}(\overline{C}) = \text{Pic}^0(\overline{C})$ given on geometric points by $j_D(x) = [x] - D$. Define

$$a'(D) = \liminf_{x \in C(\overline{K})} \hat{h}(j_D(x)),$$

where $\hat{h}$ is the canonical Néron-Tate height on the Jacobian associated to the symmetric ample divisor $\Theta + [-1]^*\Theta$. As $C(\overline{K})$ may not be countable, the liminf is taken to mean the limit over the directed set of all cofinite subsets of $C(\overline{K})$ of the infimum of the heights of points in such a subset.

**Theorem 1.11.** Let $C/K$ be a smooth proper geometrically connected curve of genus $g \geq 2$, if the semistable reduction of $C$ is not smooth, then

$$\inf_{D \in \text{Div}^1(C)} a'(D) \geq \begin{cases} \frac{1}{2280}, & g = 2; \\ \frac{1}{3276}, & g = 3. \end{cases}$$

Remark that the bounds given in [LT17] are $\frac{1}{12160}$ for $g = 2$ and $\frac{1}{19656}$ for $g = 3$.

The organization of this paper is as follows.

In §2, we give notations of modular invariants $\delta_i(f)$ and valencies of periodic maps, which will be used in our proofs. In §3, we obtain the sharp lower bounds of rational Dehn twist coefficients (Theorem 1.9 and Theorem 1.10), using the theory of classification of singular fibers in [AI02]. We divide Theorem 1.10 into four parts in Section 3. Theorem 1.9 is in fact a corollary of Theorem 1.10. By the correspondence in [Li21], we obtain sharp lower bounds of $\delta_i(f)$. As an immediate application, we prove Theorem 1.11 at the end of §3. In §4 we prove results of lower bounds of modular invariants of fibrations of genus 2. We prove Theorem 1.1 first, and then prove the rigidity property (Theorem 1.3), using the theory of Chern numbers of fibers (see [Ta10]). Theorem 1.2, the optimum of the lower bounds in Theorem 1.1, is proved in §6. We prove results (Theorem 1.4, Theorem 1.5 and Theorem 1.7) for fibrations of genus $g \geq 3$ in §5.

2 Preliminaries

2.1 Modular invariants

Let $\Delta_0, \Delta_1, \ldots, \Delta_{[g/2]}$ be the boundary divisors of $\overline{\mathcal{M}}_g$, $\delta_i$ be the $\mathbb{Q}$-divisor classes corresponding to $\Delta_i$, and $\delta_i(f)$ be the modular invariants corresponding to $\delta_i$. Then

$$\delta(f) = \delta_0(f) + \delta_1(f) + \cdots + \delta_{[g/2]}(f).$$

If $g = 1$, then $\delta(f)$ is the number of poles of the $J$-function of the family (see [Li16] for generalization). When $g \geq 2$, it is shown ([Ta94], [Ta96]) that $\lambda(f) = 0$ if and only if
κ(φ) = 0 if and only if φ is an isotrivial family. In this paper, we always assume that φ is non-isotrivial, then λ(φ) and κ(φ) are positive rational number.

Let F be a singular fiber of f, and F be its d-th semistable model ([LT17, p.207]). Let p be a node of F. We say p is of type 0 if the normalization of F at p is connected. Otherwise, the normalization at p has two connected components, and we say p is of type i, where i is the minimum of the arithmetic genera of the two components. Denote by δi(F) the number of nodes of type i in F, then we define

\[ δ_i(F) := \frac{δ_i(F)}{d}, \quad (i = 0, 1, \ldots, [g/2]), \]

which is independent of the choice of the semistable model F of F. Let F1, . . . , Fs be all singular fibers of f. It is shown that, in [LT17],

\[ δ_i(f) = δ_i(F_1) + \cdots + δ_i(F_s), \quad i = 0, 1, \ldots, [g/2]. \] (2.3)

If we restrict f to a neighborhood of f(F) ∈ C, we can get a local family fF whose dual graph is G(F), and we denote by φF the pseudo-periodic map determined by fF. On the other hand, let φ ∈ Mod(Σγ) be a pseudo-periodic map of negative twist. Then, for each i ≥ 0, we have ([L321, Theorem 1.2])

\[ δ_i(F_φ) = δ_i(f_φ) = \sum_{γ ∈ C_i} |c(φ, γ)|, \] (2.4)

where C_i = {γ ∈ C : γ is of type i}. So, if δ(F_φ) = 0, then F_φ has smooth reduction, and φ has periodic monodromy.

2.2 Valencies of periodic maps

Let Σ be a connected real 2-dimensional manifold with or without boundary. When we emphasize its complex structure, we call Σ a Riemann surface.

Let φ : Σ → Σ be a periodic homeomorphism of order n ≥ 2, and p be a point on Σ. There is a positive integer m_p such that the points p, φ(p), . . . , φ^{m_p−1}(p) are mutually distinct and φ^{m_p}(p) = p. If m_p = n, we call the point p a simple point of φ, while if m_p < n, we call p a multiple point of φ.

Let γ be a cut curve in C and m = m_γ be the smallest positive integer such that φ^m(γ) = γ (i.e., φ^m(γ) = γ as a set, and φ^m preserving the orientation of γ). The restriction of φ^m to γ is a periodic map of order, say, λ ≥ 1. Let q be any point on γ, and suppose that the images of q under the iteration of φ^m are ordered (q, φ^m(q), . . . , φ^{(λ−1)m}(q)) viewed in the direction of γ, where σ is an integer with 0 ≤ σ ≤ λ − 1, gcd(σ, λ) = 1, and σ = 0 iff λ = 1. The triple (m, λ, σ) is called the valency of γ with respect to φ.

We define the valency of a boundary curve (i.e., a connected component of the boundary ∂Σ) as its valency with respect to φ, assuming it has the orientation induced by the surface Σ. The valency of a multiple point p is defined to be the valency of the boundary curve ∂D_p, oriented from the outside of a disk neighborhood D_p of p.

Let Σ be a surface of genus g with k boundary curves ∂_1, . . . , ∂_k. Let φ : Σ → Σ be an orientation-preserving homeomorphism which satisfies:

1. there is a disjoint union of simple closed curves C = ∪_{j=1}^k γ_j such that C and ∂Σ = ∪_{j=1}^k ∂_j do not intersect each other,
Then we can extend $\phi_{g, r, k}$ similarly to Lemma 2.1 (see [AI02, Section 2.2]).

Suppose $\Pi : \Sigma \to \Sigma'$ is the $n$-fold cyclic covering induced by $\phi$, where $\Sigma'$ is the quotient surface of $\Sigma$ with respect to $\phi$. Let $\{q_1, \ldots, q_l\} \subseteq \Sigma'$ be the set of branch points. If $\tilde{q}_i$ is a point of the pre-image $\Pi^{-1}(q_i)$ of $q_i$, and let the valency of $\tilde{q}_i$ be $(m_i, \lambda_i, \sigma_i)$. Then we know that $m_i$ is the number of points in $\Pi^{-1}(q_i)$ and $\lambda_i = n/m_i$. Since the valencies of points in $\Pi^{-1}(q_i)$ are the same, we can define the valency of $q_i$ to be the valency of $\tilde{q}_i$.

For brevity’s sake, if we have the data of valencies $(n/\lambda_i, \lambda_i, \sigma_i)$ ($1 \leq i \leq l$), we symbolically write $\sigma_1/\lambda_1 + \cdots + \sigma_l/\lambda_l$ which is called the total valency. We also write the order $n$ of the map and the genus $g'$ of $\Sigma'$. However if $g' = 0$, the genus is omitted. A periodic map can be represented by its total valency. For the reader’s convenience, we list the classification of periodic maps in [AI02, Lemma 1.4] here.

**Lemma 2.1.** Non-identical conjugacy classes of periodic maps of closed surfaces of genus $1 \leq g \leq 2$ are classified as follows:

(i) $g = 1$

1. $n = 6$: $1/6 + 1/3 + 1/2, 5/6 + 2/3 + 1/2$.
2. $n = 4$: $1/4 + 1/4 + 1/2, 3/4 + 3/4 + 1/2$.
3. $n = 3$: $1/3 + 1/3 + 1/3, 2/3 + 2/3 + 2/3$.
4. $n = 2$: $1/2 + 1/2 + 1/2 + 1/2$.
5. $g' = 1, n$ is arbitrary and $\Pi : \Sigma \to \Sigma'$ is an unramified covering.

(ii) $g = 2$

1. $n = 10$: $1/10 + 2/5 + 1/2, 3/10 + 1/5 + 1/2, 7/10 + 4/5 + 1/2, 9/10 + 3/5 + 1/2$.
2. $n = 8$: $1/8 + 3/8 + 1/2, 5/8 + 7/8 + 1/2$.
3. $n = 6$: $1/6 + 1/6 + 2/3, 5/6 + 5/6 + 1/3, 1/3 + 2/3 + 1/2 + 1/2$.
4. $n = 5$: $1/5 + 1/5 + 3/5, 1/5 + 2/5 + 2/5 + 2/5 + 4/5 + 4/5 + 3/5 + 3/5 + 3/5 + 3/5 + 3/5 + 3/5$.
5. $n = 4$: $1/4 + 3/4 + 1/2 + 1/2$.
6. $n = 3$: $1/3 + 1/3 + 2/3 + 2/3$.
7. $g' = 1, n = 2, 1/2 + 1/2$.

**2.3 Representation of a pseudo-periodic map**

Let $\phi : \Sigma_g \to \Sigma_g$ be a pseudo-periodic map, and $\mathcal{C} = \{\gamma_1, \ldots, \gamma_r\}$ be the admissible system of cut curves. Then the restriction of $\phi$ on $\mathcal{B} = \Sigma_g - \mathcal{C}$ is isotopic to a periodic map.

Now we use a weighted graph $G_{\mathcal{C}}$ to denote the decomposition $\Sigma = \mathcal{B} \cup \mathcal{C}$. A vertex $v$ in $G_{\mathcal{C}}$ corresponds to a connected component $B_v$ of $\mathcal{B}$, and an edge $e$ corresponds to a separated cut curve $\gamma_e$ in $\mathcal{C}$, where $\gamma_e$ is adjacent to two connected components of $\mathcal{B}$. We define the weight of a vertex $v$ to be $g(B_v) + \rho(v)$, where $g(B_v)$ is the genus of $B_v$, and $\rho(v)$ is the number of cut curves only adjacent to $B_v$. We use a small circle to denote a vertex, and the number inside the small circle means $g(B_v) + \rho(v)$. We omit the number when it is zero.

Note that a weighted graph may represent different decompositions. For example, the graph (II) in Lemma 3.1 represents four types of decompositions, that is, the component
corresponding to \( v_1 \) has genus \( i_1 \) and is adjacent to \( 1 - i_1 \) non-separating curves in \( \mathcal{C} \), and the component corresponding to \( v_2 \) has genus \( i_2 \) and is adjacent to \( 1 - i_2 \) non-separating curves in \( \mathcal{C} \) (\( 0 \leq i_1 \leq 1, \ 0 \leq i_2 \leq 1 \)).

The map \( \phi : \Sigma_g = \mathcal{B} \cup \mathcal{C} \rightarrow \Sigma_g = \mathcal{B} \cup \mathcal{C} \) induces an automorphism \( \sigma_\phi \) on the weighted graph \( G_\mathcal{C} \). Here an automorphism of \( G_\mathcal{C} \) means an automorphism of the graph such that the weight \((g(B_v), \rho(v))\) coincides with \((g(B_{\sigma(v)}), \rho(\sigma(v)))\) for each vertex \( v \) of \( G_\mathcal{C} \), see [Ai02, Section 3.3] for an example.

For each cut curve \( \gamma \in \mathcal{C} \), there exists a minimal positive integer \( \alpha \) such that \( \phi^\alpha(\gamma) = \gamma \). The curve \( \gamma \) is said to be amphidrome if \( \alpha \) is even and \( \phi^{\alpha/2}(\gamma) = -\gamma \), (where \( \gamma \) and \( -\gamma \) denote the same \( \gamma \) with the opposite directions assigned) and non-amphidrome otherwise. There exists a minimal positive integer \( L \) such that the restriction of \( \phi^L \) to an annulus of \( \gamma \) is isotopic to a Dehn twist of \( e \) times \((e \in \mathbb{Z})\). The rational number \( e\alpha/L \) is called the screw number of \( \phi \) about \( \gamma \), and is denoted by \( s(\gamma) \) (see [Ni44]). We may always assume that \( s(\gamma) \neq 0 \) for each \( \gamma \in \mathcal{C} \) (see [MM11, p.5]).

For each \( \gamma \in \mathcal{C} \), denote by \( m_\gamma \) the length of the cyclic orbit of \( \gamma \) under the permutation caused by \( \phi \), that is,
\[
m_\gamma = \#\{\phi^k(\gamma) : k \in \mathbb{N}\}.
\]

Our classification is based on the following theorem in [MM11].

**Theorem 2.2.** The conjugacy class of a pseudo-periodic map \( \phi : \Sigma_g \rightarrow \Sigma_g \) of negative twist is determined by the following data: an admissible system \( \mathcal{C} \) of cut curves, the induced automorphism \( \sigma_\phi \) of \( G_\mathcal{C} \), the screw numbers \( s(\gamma) \) for each \( \gamma \in \mathcal{C} \), and the valency data of the periodic maps which stabilize the connected components of \( \Sigma_g - \mathcal{C} \).

Now we give the formula of fractional Dehn twist coefficients in [Li21, Theorem 4.5] and the formula of screw number (see [Ai02, Section 2.1]). Let \( \mathcal{A}_\gamma \) be the annular neighborhood of \( \gamma \). Let \( (m_1, \lambda_1, \sigma_1) \) and \( (m_2, \lambda_2, \sigma_2) \) be the valencies of the two boundary curves of \( \mathcal{A}_\gamma \). If \( \gamma \) is non-amphidrome, then \( m_1 = m_2 = m_\gamma \), and
\[
|c(\phi, \gamma)| = \frac{|s(\gamma)|}{m_\gamma} = \frac{1}{m_\gamma} \left( \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + K \right), 
\]
where \( K \geq -1 \) is an integer, and \( \mu_i \) are integers with
\[
\sigma_i \mu_i \equiv 1 \mod \lambda_i, \quad 0 \leq \mu_i \leq \lambda_i - 1.
\]
If \( \gamma \) is amphidrome, then the two boundary curves have the same valency \((2m_\gamma, \lambda, \sigma)\) where \( 2m_\gamma = \alpha \), and
\[
|c(\phi, \gamma)| = \frac{|s(\gamma)|}{2m_\gamma} = \frac{1}{m_\gamma} (\mu + K),
\]
where \( K \geq 0 \) is an integer, and \( \mu \) is an integer with
\[
\sigma \mu \equiv 1 \mod \lambda, \quad 0 \leq \mu \leq \lambda - 1.
\]

Let \( \gamma \) be a cut curve adjacent to connected components \( B_{v_1}, B_{v_2} \) with \( B_{v_1} \neq B_{v_2} \). Suppose that \( \gamma_k := \phi^k(\gamma) \) \((k = 1, \ldots, m_\gamma)\) are all adjacent to \( B_{v_i} \) \((i = 1, 2)\). Denote the valencies of the two boundary curves of \( \mathcal{A}_{\gamma_k} \) by \( (m_1, \lambda_{v_1, \gamma}, \sigma_{v_1, \gamma}) \) and \( (m_2, \lambda_{v_2, \gamma}, \sigma_{v_2, \gamma}) \) respectively. If \( a \) is the minimal positive integer such that \( \phi^a|B_{v_i} = B_{v_i} \), then the restriction \( \phi^a|_{B_{v_i}} \) of \( \phi^a \) to \( B_{v_i} \) is periodic. Let \( n(B_{v_i}) \) be the order of \( \phi^a|_{B_{v_i}} \).
If $\phi$ interchanges $B_{v_i}$ ($i = 1, 2$), then $\phi(\gamma_k) = -\gamma_{k+1}$ ($k = 1, \ldots, m_\gamma$), where $\gamma_{m_\gamma+1} := \gamma_1 = \gamma$. So $\phi^n(\gamma) = (-1)^{m_\gamma} \gamma$. Moreover, if $m_\gamma$ is even, then $\gamma$ is non-amphidrome and $m_\gamma = m_\gamma$; if $m_\gamma$ is odd, then $\gamma$ is amphidrome and $m_\gamma = 2m_\gamma$. Since $\phi$ interchanges $B_{v_i}$, $n(B_{v_i}) = m_\gamma \lambda_i \gamma / 2$. Moreover, if $\phi$ does not interchange $B_{v_i}$, then $\gamma$ is non-amphidrome. Thus we have that, for $B_{v_1} \neq B_{v_2}$,

$$\lambda_{v_i, \gamma} = \begin{cases} \frac{n(B_{v_i})}{m_\gamma}, & \text{if } \phi \text{ does not interchange } B_{v_i}; \\ \frac{n(B_{v_i})}{m_\gamma}, & \text{if } \phi \text{ interchanges } B_{v_i} \text{ and } \gamma \text{ is amphidrome}; \\ \frac{2n(B_{v_i})}{m_\gamma}, & \text{if } \phi \text{ interchanges } B_{v_i} \text{ and } \gamma \text{ is non-amphidrome}. \end{cases} \quad (2.7)$$

In particular, if $m_\gamma$ is odd, then $\lambda_{v_i, \gamma} = \frac{n(B_{v_i})}{m_\gamma}$.

3 Bounds of fractional Dehn twist coefficients

3.1 Genus 2 case

First we give the classification of decompositions of Riemann surfaces of genus two.

**Lemma 3.1.** The decompositions of a Riemann surface of genus two by an admissible system of cut curves can be classified in terms of weighted graphs (I)-(III) as follows.

(I) \hspace{1cm} (II) \hspace{1cm} (III)

\[
\begin{array}{c}
\text{(2)} \\
v_1
\end{array} \quad \frac{1}{e} \quad \frac{1}{v_2} \quad \frac{1}{e_2} \quad \frac{1}{v_2} \\
\begin{array}{c}
\text{(2)} \\
v_1
\end{array}
\]

**Proof.** This problem is equivalent to classifying stable curves of genus two, which is trivial.

**Theorem 3.2.** Let $\phi \in \text{Mod}(\Sigma_2)$ be a pseudo-periodic map of negative twist, and $\gamma$ be a cut curve of type 1, then

$$|c(\phi, \gamma)| \geq \frac{1}{12},$$

and the equality holds if and only if $(G(F_\phi), \gamma)$ is either Figure (2-1a) or Figure (2-1b).

**Proof.** The possible cut curve $\gamma$ is $e$ in (II), and $m_\gamma = 1$.

(II1). Assume that $\phi$ does not interchange $B_{v_i}$. Then $\gamma$ is non-amphidrome.

Claim: if $\rho(v_i) = 1, g(B_{v_i}) = 0$ for some $i$, then $|c(\phi, \gamma)| > \frac{1}{12}$.

Proof of Claim: We may assume that $\rho(v_1) = 1, g(B_{v_1}) = 0$. Let $\gamma'$ be the cut curve only adjacent to $B_{v_1}$, and $\mathcal{A}_{\gamma'}$ be the annular neighbourhood of $\gamma'$. Then $\phi$ maps boundary curves of $\mathcal{A}_{\gamma'}$ (resp. $\mathcal{A}_{\gamma'}$) to boundary curves of $\mathcal{A}_{\gamma'}$ (resp. $\mathcal{A}_{\gamma'}$). Thus $n(B_{v_1}) \leq 2$, since the automorphism of Riemann sphere with three fixed points is identity. Hence $\lambda_{v_1, \gamma} = n(B_{v_1}) \leq 2$. Now consider $B_{v_2}$, we have that $n(B_{v_2}) \leq 6$: if $\rho(v_2) = 0$, then $g(B_{v_2}) = 1$ and $n(B_{v_2}) \leq 6$ by Lemma 2.4(i); if $\rho(v_2) = 1$, then $g(B_{v_2}) = 0$ and $n(B_{v_2}) \leq 2$ as above. Thus, by (2.5),

$$|c(\phi, \gamma)| = \frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_2, \gamma}}{\lambda_{v_2, \gamma}} + K \geq \frac{1}{\text{lcm}(\lambda_{v_1, \gamma}, \lambda_{v_2, \gamma})} > \frac{1}{12},$$
and we finish the proof of Claim.

Now we only need to prove the case that \( \rho(v_i) = 0, g(B_{v_i}) = 1, i = 1, 2 \). Since there is one edge \( e \) adjacent to \( v_i \) \((i = 1, 2)\), we know that the restriction of \( \phi \) on \( B_{v_i} \) can not induce an unramified covering of degree \( n \geq 2 \). Thus \( n(B_{v_i}) \leq 6 \) and the valency data are classified in Lemma \( 2.1 \) (i). So, by \( (2.5) \),

\[
|c(\phi, \gamma)| = \frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_2, \gamma}}{\lambda_{v_2, \gamma}} + K \geq \frac{1}{12}.
\]

If the equality holds, then \( K = -1 \). Furthermore, by Lemma \( 2.1 \) and Theorem \( 2.2 \), the cut curves and the pseudo-periodic maps with the lowest bound are classified as follows:

\[
\begin{align*}
(2-1a) & \quad B_{v_1} : \frac{1}{3} + \frac{2}{3} + \frac{1}{3}, \quad B_{v_2} : \frac{3}{4} + \frac{2}{4} + \frac{1}{2}, \quad K = -1; \\
(2-1b) & \quad B_{v_1} : \frac{2}{5} + \frac{2}{5} + \frac{1}{2}, \quad B_{v_2} : \frac{1}{4} + \frac{1}{4} + \frac{1}{2}, \quad K = -1.
\end{align*}
\]

Here we write valency data of \( \gamma \) by bold face characters. For the reader’s convenience, we take the case (2-1b) as an example, the valencies of the two boundary curves of \( \mathcal{F}_\gamma \) are \((m_{v_1, \gamma}, \lambda_{v_1, \gamma}, \sigma_{v_1, \gamma}) = (1, 6, 5)\) and \((m_{v_2, \gamma}, \lambda_{v_2, \gamma}, \sigma_{v_2, \gamma}) = (1, 4, 1)\). Thus \( \mu_{v_1, \gamma} = 5, \mu_{v_2, \gamma} = 1, \) and

\[
|c(\phi, \gamma)| = \frac{5}{6} + \frac{1}{4} + (-1) = \frac{1}{12}.
\]

Using the correspondence in \([L21] \) Section 4, it is easy to check that the dual graphs of the above two pseudo-periodic maps are Figure (2-1a) and Figure (2-1b) respectively. In the following, we always use same labels (for example, (2-1a)) for pseudo-periodic maps and their corresponding dual graphs without mention.

**Corollary 3.3.** Let \( f \) be a family of curves of genus 2 with \( \delta_1(f) \neq 0 \), then

\[
\delta_1(f) \geq \frac{1}{12},
\]

and the equality holds if and only if all the singular fibers of \( f \) have smooth reduction except one whose dual graph is either Figure (2-1a) or Figure (2-1b).

**Proof.** Let \( F \) be a singular fiber of \( f \), then

Claim (*) \( \delta_1(F) \geq \frac{1}{12} \), and \( \delta_1(F) = \frac{1}{12} \) if and only if the dual graph of \( F \) is either Figure (2-1a) or Figure (2-1b).

Proof of Claim (*): By \( (2.4) \) and Theorem \( 3.2 \) we have

\[
\delta_1(F) = \sum_{\gamma \in \mathcal{F}_{\phi_F}, 1} |c(\phi_F, \gamma)| \geq \frac{1}{12}.
\]

If \( \delta_1(F) = \frac{1}{12} \), then \( \phi_F \) has only one cut curve \( \gamma \) of type 1, and the possible dual graphs of \( (\phi_F, \gamma) \) are Figure (2-1a) and Figure (2-1b). Hence we obtain the claim.
Since $\delta_1(f) \neq 0$, there is a singular fiber of $f$, say $F_1$, with $\delta_1(F_1) \neq 0$ by (2.3). So

$$\delta_1(f) \geq \delta_1(F_1) \geq \frac{1}{12},$$

and $\delta_1(f) = \delta_1(F_1) = \frac{1}{12}$ if and only if $\delta_1(F_1) = \frac{1}{12}$ and $\delta_1(F_i) = 0$, $i \geq 2$. Then we complete the proof by Claim (*). \hfill \Box

**Theorem 3.4.** Let $\phi \in \text{Mod}(\Sigma_2)$ be a pseudo-periodic map of negative twist, and $\gamma$ be a cut curve of type 0, then

$$|c(\phi, \gamma)| \geq \frac{1}{3},$$

and the equality holds if and only if $(G(F_\phi), \gamma)$ is one of figures: (2-0a) – (2-0d).

**Proof.** The possible weighted graphs $G_\phi$ are (I), (II) and (III) classified in Lemma 3.1. 

Case (I) In this case, $\gamma$ is adjacent to $B_{v_1}$ with $g(B_{v_1}) + \rho(v_1) = 2$ and $\rho(v_1) \geq 1$. Let $(m_1, \lambda_1, \sigma_1)$ and $(m_2, \lambda_2, \sigma_2)$ be the valencies of the boundary curves of $\mathcal{A}_\gamma$.

(II). First consider $\rho(v_1) = 1$. Then $g(B_{v_1}) = 1$ and $m_\gamma = 1$. If $\gamma$ is amphidrome, then $\lambda = n(B_{v_1})/2 \leq 3$. Thus, by (2.6) and Lemma 2.1

$$|c(\phi, \gamma)| = \frac{\mu}{\lambda} + K \geq \frac{1}{3}.$$  

Furthermore, the equality holds if and only if $K = 0$ and the valency data of the boundary curves of $\mathcal{A}_\gamma$ is

(2-0a) $B_{v_1} : \frac{1}{2} + \frac{1}{3} + \frac{1}{2}.$

If $\gamma$ is non-amphidrome, then $\lambda_1 = \lambda_2 = n(B_{v_1})$. By (2.5), we have that

$$|c(\phi, \gamma)| = \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + K \geq \frac{1}{3}.$$  

Furthermore, the equality holds if and only if $K = -1$ and the valency data of the boundary curves of $\mathcal{A}_\gamma$ is

(2-0b) $B_{v_1} : \frac{2}{3} + \frac{2}{3} + \frac{2}{3}.$

(II). Now we consider $\rho(v_1) = 2$, then $g(B_{v_1}) = 0$. Let the two cut curves be $\gamma = \gamma_1$ and $\gamma_2$. Then $m_\gamma \leq 2$. Note that $\phi|_{B_{v_1}}$ maps $\mathcal{A}_{\gamma_i}$ to $\mathcal{A}_{\gamma_j}$, and maps boundary curves of $\mathcal{A}_{\gamma_i}$ to those of $\mathcal{A}_{\gamma_j}$, where $1 \leq i, j \leq 2$. If $m_\gamma = 1$, then $\lambda \leq n(B_{v_1}) \leq 2$, and thus $|c(\phi, \gamma)| \geq \frac{1}{2}$. If $m_\gamma = 2$, then $\lambda = 1$ which is independent of the action of $\phi$ on boundary curves (note that if $\gamma$ is amphidrome, then $\lambda = n(B_{v_1})/2m_\gamma$). Thus $|c(\phi, \gamma)| \geq \frac{1}{2}$, by (2.5) and (2.6).

Case (III) In this case, we may assume that $\gamma$ is adjacent to $B_{v_1}$ only and $m_\gamma \leq 2$, then we have $\rho(v_1) = 1, g(B_{v_1}) = 0$. We have that $\lambda = 1$ which is independent of the action of $\phi$ on boundary curves, and thus $|c(\phi, \gamma)| \geq \frac{1}{2}$.

Case (III) We may assume that $\gamma = e_1$.

(III). Assume that $m_\gamma = 3$. If $\phi$ does not interchange $B_{v_1}$, then $\gamma$ is non-amphidrome, $n(B_{v_1}) = 3$ and $\lambda_{v_1, \gamma} = n(B_{v_1})/m_\gamma = 1$ for $i = 1, 2$. So $|c(\phi, \gamma)| \geq \frac{1}{3}$ by (2.5). Furthermore, the equality holds if and only if $K = 1$ and the valencies of the boundary curves of $\mathcal{A}_\gamma$ are

(2-0c) $B_{v_1} : \frac{1}{3} + \frac{2}{3} + 1, \quad B_{v_2} : \frac{1}{3} + \frac{2}{3} + 1.$

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If \( \phi \) interchanges \( B_{v_1} \), then \( \gamma \) is amphidrome, \( n(B_{v_1}) = 3 \) and \( \lambda_{v_1, \gamma} = n(B_{v_1})/m_\gamma = 1 \), see (2.7). So \( |c(\phi, \gamma)| \geq \frac{1}{3} \) by (2.6). Furthermore, the equality holds if and only if \( K = 1 \) and the valency data of the boundary curves of \( A_\gamma \) is

\[(2.0d) \ B_{v_1} : \frac{1}{3} + \frac{2}{3} + 1.\]

(III2). Assume that \( m_\gamma = 2 \). If \( \phi \) does not interchange \( B_{v_1} \), then \( \gamma \) is non-amphidrome, \( n(B_{v_1}) = 2 \) and \( \lambda_{v_1, \gamma} = 1 \); if \( \phi \) interchanges \( B_{v_1} \), then \( \gamma \) is also non-amphidrome, \( n(B_{v_1}) = 1 \) and \( \lambda_{v_1, \gamma} = 1 \) by (2.7). Thus, we always have that \( \lambda_{v_1, \gamma} = 1 \) and \( |c(\phi, \gamma)| \geq \frac{1}{3} \).

(III3). Assume that \( m_\gamma = 1 \). If \( \phi \) does not interchange \( B_{v_1} \), then \( \gamma \) is non-amphidrome, \( \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 2 \); if \( \phi \) interchanges \( B_{v_1} \), then \( \gamma \) is amphidrome and \( n(B_{v_1}) \leq 2 \). So \( \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 2 \) by (2.7). Thus \( |c(\phi, \gamma)| \geq \frac{1}{2} \).

\[\square\]

**Corollary 3.5.** Let \( f \) be a fibration of genus 2 with \( \delta_0(f) \neq 0 \), then

\[ \delta_0(f) \geq \frac{1}{3}, \]

and the equality holds if and only if all the singular fibers of \( f \) have smooth reduction except one whose dual graph is either Figure (2-0a) or Figure (2-0b).

**Proof.** Similar to the proof of Corollary 3.3

\[\square\]

### 3.2 Genus 3 case

In this subsection, we use the same method as Section 3.1 to discuss lower bounds of fractional Dehn twist coefficients in genus 3 case.

**Lemma 3.6.** The decompositions of a Riemann surface of genus three by an admissible system of cut curves can be classified in terms of weighted graphs (A)-(O) as follows.

(D) 

(A)  

(B)  

(C)  

(D)  

(E)  

(F)  

(G)  

(H)  

(I)  

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Proof. See [AI02, Lemma 3.2]. □

**Theorem 3.7.** Let \( \phi \in \text{Mod}(\Sigma_3) \) be a pseudo-periodic map of negative twist, and \( \gamma \) be a cut curve of type 1, then

\[
|c(\phi, \gamma)| \geq \frac{1}{30},
\]

and the equality holds if and only if \((G(F_\phi), \gamma)\) is one of figures (3-1a), (3-1b) and (3-1c).

**Proof.** The possible cut curves are in (B), (C), (D), (F), (G), (I), and (K).

**Case (B)** In this case, \( \gamma = e_1 \) and \( m_\gamma = 1 \). Similar to Claim 1 in the proof of Theorem 3.2, we may assume that \( g(Bv_1) \leq 2 \), \( g(Bv_2) \leq 1 \), and thus \( n(Bv_1) \leq 10 \) and \( n(Bv_2) \leq 6 \) classified in Lemma 2.1. By (2.5),

\[
|c(\phi, \gamma)| = \frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_2, \gamma}}{\lambda_{v_2, \gamma}} + K \geq \frac{1}{30}.
\]

If the equality holds, then \( K = -1 \). Furthermore, by Lemma 2.1 the cut curves and the pseudo-periodic maps with the lowest bound are classified as follows:

- (3-1a) \( Bv_1 : \frac{3}{10} + \frac{1}{5} + \frac{1}{2} \), \( Bv_2 : \frac{1}{3} + \frac{1}{5} + \frac{1}{3} \), \( K = -1 \);
- (3-1b) \( Bv_1 : \frac{1}{5} + \frac{2}{5} + \frac{2}{5} \), \( Bv_2 : \frac{5}{6} + \frac{2}{3} + \frac{1}{2} \), \( K = -1 \);
- (3-1c) \( Bv_1 : \frac{1}{5} + \frac{1}{3} + \frac{3}{5} \), \( Bv_2 : \frac{5}{6} + \frac{2}{3} + \frac{1}{2} \), \( K = -1 \).

The dual graphs of the above three pseudo-periodic maps are Figure (3-1a), Figure (3-1b), and Figure (3-1c) respectively.

**Case (C)** We may assume that \( \gamma = e_1 \). Then \( m_\gamma \leq 2 \). We may assume that \( g(Bv_i) = 1 \) and \( n(Bv_i) \leq 6 \), where \( i = 1, 3 \). Then, by Lemma 2.1 (i) and (2.5), we have that

\[
|c(\phi, \gamma)| \geq \frac{1}{2} \left( \frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_3, \gamma}}{\lambda_{v_3, \gamma}} + K \right) \geq \frac{1}{24}.
\]

**Case (D)** In this case, we may assume that \( \gamma = e_1 \). If \( m_\gamma \geq 2 \), then \( n(Bv_4) = m_\gamma \leq 3 \), and \( \lambda_{v_4, \gamma} = n(Bv_4)/m_\gamma = 1 \); if \( m_\gamma = 1 \), then \( \lambda_{v_4, \gamma} = n(Bv_4) \leq 2 \). Since \( \lambda_{v_1, \gamma} \leq 6 \), we know
that

\[ |c(\phi, \gamma)| \geq \frac{1}{m_\gamma} \left( \frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_2, \gamma}}{\lambda_{v_2, \gamma}} + K \right) \geq \frac{1}{18}. \]

The rest cases are similar, and we omit their proofs.

\[ \square \]

**Corollary 3.8.** Let \( f \) be a fibration of genus 3 with \( \delta_1(f) \neq 0 \), then

\[ \delta_1(f) \geq \frac{1}{30}, \]

and the equality holds if and only if all the singular fibers of \( f \) have smooth reduction except one whose dual graph is one of figures (3-1a), (3-1b) and (3-1c).

**Proof.** Similar to the proof of Corollary 3.3. \[ \square \]

**Lemma 3.9.** Let \( \phi \in \text{Mod}(\Sigma_3) \) be a pseudo-periodic map of negative twist, and \( \gamma \) be a cut curve of type 0 which is adjacent to one component only. Then

\[ |c(\phi, \gamma)| \geq \frac{1}{5}, \]

and the equality holds if and only if \((G(F_\phi), \gamma)\) is one of the following figures (3-0-0a), (3-0-0b) and (3-0-0c).

\[ \begin{array}{ccc}
1 & 3 & 5 \\
\gamma & \omega & 4 \\
\end{array} \quad \begin{array}{ccc}
1 & 5 & 2 \\
\gamma & \omega & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 10 & 5 \\
\gamma & \omega & 3 & 2 \\
\end{array} \\
(3-0-0a) & (3-0-0b) & (3-0-0c)
\]

**Proof.** The proof is similar to that of Case (I) in Theorem 3.4.

Assume \( \gamma \) is adjacent to \( B_v \) with \( g(B_v) + \rho(v) \leq 3 \) and \( \rho(v) \geq 1 \). Let \((m_1, \lambda_1, \sigma_1)\) and \((m_2, \lambda_2, \sigma_2)\) be the valencies of the boundary curves of \( \mathcal{A}_\gamma \).

If \( \gamma \) is non-amphidrome, then \( \lambda_1 = \lambda_2 = n(B_v)/m_\gamma \). Since \( \rho(v) \geq 1 \) and \( g = 3 \), by Lemma 2.1 we have that

\[ |c(\phi, \gamma)| = \frac{1}{m_\gamma} \left( \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + K \right) \geq \frac{1}{5}. \]

Furthermore, the equality holds if and only if \( K = -1, g(B_v) = 2, m_\gamma = \rho(v) = 1 \), and the valencies of the boundary curves of \( \mathcal{A}_\gamma \) are one of the following cases:

- (3-0-0a) \( B_v : \frac{4}{5} + \frac{3}{5} + \frac{3}{5} \).
- (3-0-0b) \( B_v : \frac{2}{5} + \frac{2}{5} + \frac{1}{5} \).

If \( \gamma \) is amphidrome, then the valencies of the boundary curves of \( \mathcal{A}_\gamma \) are the same \((2m_\gamma, \lambda, \sigma)\). Here \( \lambda = n(B_v)/2m_\gamma \). Since \( \rho(v) \geq 1 \), \( g(B_v) \leq 2 \), we have \( n(B_v) \leq 10 \). Thus

\[ |c(\phi, \gamma)| = \frac{1}{m_\gamma} \left( \frac{\mu}{\lambda} + K \right) \geq \frac{2}{n(B_v)} \geq \frac{1}{5}. \]
Furthermore, the equality holds if and only if $K = 0$, $g(B_v) = 2$, $m_\gamma = \rho(v) = 1$, and the valencies of the boundary curves of $\mathcal{A}_\gamma$ are

\[(3\text{-}0\text{-}0\text{c}) B_v : \frac{3}{10} + \frac{1}{5} + \frac{1}{2}. \]

\[\square\]

**Theorem 3.10.** Let $\phi \in \text{Mod}(\Sigma_3)$ be a pseudo-periodic map of negative twist, and $\gamma$ be a cut curve of type 0. Then

\[|c(\phi, \gamma)| \geq \frac{1}{12},\]

and the equality holds if and only if $(G(F_\phi), \gamma)$ is one of figures (3-0a), (3-0b), (3-0c).

**Proof.** By Lemma 3.9 we may assume that $\gamma$ is adjacent to two connected components. Thus $\gamma$ is in (E)-(O).

Case (E) Without loss of generality, we may assume $\gamma = e_1$. Then $m_\gamma \leq 2$.

**Case (E1).** Assume that $m_\gamma = 1$. If $\phi$ interchanges $B_{v_i}$, then $\gamma$ is amphidrome. So $\lambda_{\gamma,v_i} = n(B_{v_i})$ for $i = 1, 2$ by (2.7), and $n(B_{v_i}) \leq 4$ by Lemma 2.1(i). Thus $|c(\phi, \gamma)| \geq \frac{1}{3}$ by (2.6).

If $\phi$ does not interchanges $B_{v_i}$, then $\gamma$ is non-amphidrome. So $\lambda_{\gamma,v_i} = n(B_{v_i})$ for $i = 1, 2$, and $n(B_{v_i}) \leq 4$. Thus, by (2.5),

\[|c(\phi, \gamma)| = \frac{\mu_{\psi_{\gamma,v_i}}}{\lambda_{\psi_{\gamma,v_i},\gamma}} + \frac{\mu_{\psi_{\gamma,v_2}}}{\lambda_{\psi_{\gamma,v_2},\gamma}} + K \geq \frac{1}{12}.\]

Furthermore, the equality holds if and only if $K = -1$ and the valencies of the boundary curves of $\mathcal{A}_\gamma$ are

\[(3\text{-}0\text{a}) B_{v_1} : \frac{3}{8} + \frac{3}{4} + \frac{1}{2}, \quad B_{v_2} : \frac{1}{8} + \frac{1}{2}.\]

**Case (E2).** Assume that $m_\gamma = 2$. If $\phi$ interchanges $B_{v_i}$, then $\gamma$ is non-amphidrome, and $\lambda_{\gamma,v_i} = 2n(B_{v_i})/m_\gamma = n(B_{v_i})$ $(i = 1, 2)$ by (2.7). Similarly, $\lambda_{\gamma,v_j} = n(B_{v_j})$ for $i = 1, 2, j = 1, 2$. By Lemma 2.1(i), we have that $n(B_{v_i}) \leq 4$. So $|c(\phi, \gamma)| \geq \frac{1}{5}$ by (2.5). If $\phi$ does not interchange $B_{v_i}$, then $\gamma$ is also non-amphidrome, and $\lambda_{\gamma,v_i} = n(B_{v_i})/2 \leq 3$. Thus

\[|c(\phi, \gamma)| = \frac{1}{2} \left( \frac{\mu_{\psi_{\gamma,v_i}}}{\lambda_{\psi_{\gamma,v_i},\gamma}} + \frac{\mu_{\psi_{\gamma,v_2}}}{\lambda_{\psi_{\gamma,v_2},\gamma}} + K \right) \geq \frac{1}{12}.\]

Furthermore, the equality holds if and only if $K = -1$ and the valencies of the boundary curves of $\mathcal{A}_\gamma$ are one of the following two cases:

\[(3\text{-}0\text{b}) B_{v_1} : \frac{3}{8} + \frac{3}{2} + \frac{1}{2}, \quad B_{v_2} : \frac{1}{8} + \frac{1}{4} + \frac{1}{2};\]

\[(3\text{-}0\text{c}) B_{v_1} : \frac{3}{8} + \frac{3}{4} + \frac{1}{2}, \quad B_{v_2} : \frac{3}{8} + \frac{1}{4} + \frac{1}{2}.\]

Though the rest cases are similar as above, we give the detail proof here for the reader’s convenience.

Case (F) We may assume that $\gamma = e_1$. In this case, the rational component $B_{v_i}$ $(i = 3, 4)$ is adjacent to three edges, $m_\gamma \leq 2$, and $n(B_{v_i}) \leq 2$. If $m_\gamma = 1$, then $\lambda_{\gamma,v_i} = n(B_{v_i}) = 1$ by (2.7). If $m_\gamma = 2$, then $\lambda_{\gamma,v_i} \leq 2n(B_{v_i})/m_\gamma \leq 2$ by (2.7). So, we always have that $\lambda_{\gamma,v_i} \leq 2$. By (2.5) and (2.6),

\[|c(\phi, \gamma)| \geq \frac{1}{4}.

Case (G) We may assume that $\gamma = e_1$. Then $m_\gamma \leq 2$. 

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(G1). If \( m_\gamma = 1 \), then \( n(B_{v_2}) = 1 \) and \( \lambda_{v_2, \gamma} = 1 \). Since \( \lambda_{v_1, \gamma} = \lambda_{v_1, e_2} = n(B_{v_1}) \), we know that \( \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 4 \). Thus
\[
|c(\phi, \gamma)| \geq \frac{1}{4}.
\]

(G2). If \( m_\gamma = 2 \), then \( n(B_{v_2}) = 2 \) and \( \lambda_{v_2, \gamma} = 1 \). Since \( \lambda_{v_1, \gamma} = \lambda_{v_1, e_2} = \frac{1}{2} n(B_{v_1}) \leq 3 \), we know that
\[
|c(\phi, \gamma)| \geq \frac{1}{6}.
\]

Case (H) We may assume that \( \gamma = e_1 \). Then \( m_\gamma \leq 3 \).

(H1). If \( m_\gamma \geq 2 \), then \( n(B_{v_2}) = m_\gamma \), \( \lambda_{v_2, \gamma} = n(B_{v_2})/m_\gamma = 1 \), and \( \lambda_{v_1, \gamma} = n(B_{v_1})/m_\gamma \). Since \( n(B_{v_1}) \leq 6 \), we have that
\[
|c(\phi, \gamma)| = \frac{1}{m_\gamma} \left( \frac{\mu_1}{\lambda_{v_1, \gamma}} + \frac{\mu_2}{\lambda_{v_2, \gamma}} + K \right) \geq \frac{1}{m_\gamma \lambda_{v_1, \gamma}} \geq \frac{1}{6}.
\]

(H2). If \( m_\gamma = 1 \), then \( \lambda_{v_2, \gamma} = n(B_{v_2}) = 2 \) and \( \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 6 \). So
\[
|c(\phi, \gamma)| \geq \frac{1}{6}.
\]

Case (I) We may assume that \( \gamma = e_1 \). Then \( m_\gamma \leq 3 \). In this case, we always have that
\( \lambda_{v_1, \gamma} = \lambda_{v_2, \gamma} \leq 2 \), similarly as Case (H). So
\[
|c(\phi, \gamma)| \geq \frac{1}{6}.
\]

Case (J) If \( \gamma \) is either \( e_1 \) or \( e_2 \), then \( |c(\phi, \gamma)| \geq \frac{1}{4} \), similarly as Case (F). Now we may assume that \( \gamma = e_3 \). Then \( m_\gamma \leq 2 \).

(J1). If \( \phi \) does not interchange \( B_{v_i} \) (\( i = 1, 2 \)), then \( m_\gamma = 1 \), \( \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 2 \) and \( \lambda_{v_3, \gamma} \leq 6 \). So
\[
|c(\phi, \gamma)| \geq \frac{1}{6}.
\]

(J2). Assume that \( \phi \) interchanges \( B_{v_i} \). Then \( m_\gamma = 2 \) and \( \lambda_{v_3, \gamma} = n(B_{v_3})/2 \leq 3 \). In this case, we always have that \( \phi^2(\vec{e}_i) = \vec{e}_i \) (\( i = 1, 2 \)). Thus \( n(B_{v_3}) = 1 \) (\( i = 1, 2 \)) and \( \lambda_{v_3, \gamma} = 1 \). So
\[
|c(\phi, \gamma)| \geq \frac{1}{6}.
\]

Similarly, we can obtain results for Cases (K), (M), (N) and (O).

Case (L) We may assume that \( \gamma = e_1 \). Then \( m_\gamma \leq 4 \).

(L1). Assume that \( m_\gamma \geq 2 \). If \( m_\gamma \) is even and \( \phi \) interchanges \( B_{v_i} \), then \( \gamma \) is non-amphidrome, \( n(B_{v_i}) = m_\gamma/2 \), and \( \lambda_{v_i, \gamma} = 2n(B_{v_i})/m_\gamma = 1 \) by (2.7). If \( m_\gamma \) is even and \( \phi \) does not interchanges \( B_{v_i} \), then \( \gamma \) is also non-amphidrome, \( n(B_{v_i}) = m_\gamma \), and \( \lambda_{v_i, \gamma} = n(B_{v_i})/m_\gamma = 1 \). If \( m_\gamma \) is odd, that is, \( m_\gamma = 3 \), then \( n(B_{v_i}) = 3 \) and \( \lambda_{v_i, \gamma} = n(B_{v_i})/m_\gamma = 1 \) by (2.7). So, we always have that \( \lambda_{v_1, \gamma} = 1 \) and \( |c(\phi, \gamma)| \geq \frac{1}{4} \).

(L2). If \( m_\gamma = 1 \), then \( \lambda_{v_1, \gamma} \leq n(B_{v_1}) \leq 3 \). So \( |c(\phi, \gamma)| \geq \frac{1}{6} \).

\[ \square \]

Corollary 3.11. Let \( f \) be a fibration of genus 3 with \( \delta_0(f) \neq 0 \), then
\[
\delta_0(f) \geq \frac{1}{6},
\]
and the equality holds if and only if all the singular fibers of \( f \) have smooth reduction except one whose dual graph is one of figures (3-0a), (3-0b) and (3-0c).
Proof. Let $F$ be a singular fiber of $f$, then we claim that:

Claim: $\delta_0(F) \geq \frac{1}{6}$, and $\delta_0(F) = \frac{1}{6}$ if and only if the dual graph of $F$ is one of figures (3-0a),(3-0b) and (3-0c).

Proof of Claim: By Lemma 3.9 and (2.4), we may assume that the dual graph of the stable model $\tilde{F}$ of $F$ is one of (E)–(O). For each of these graphs, there are at least two cut curves of type 0 adjacent to two connected components. So, by (2.4) and Theorem 3.10, we have

$$\delta_0(F) = \sum_{\gamma \in \mathcal{E}_{\phi,0}} |c(\phi, \gamma)| \geq 2 \cdot \frac{1}{12} = \frac{1}{6}.$$ 

If $\delta_0(F) = \frac{1}{6}$, then $F$ has exactly two cut curves $\gamma$ of type 0 with $|c(\phi, \gamma)| = \frac{1}{12}$. Thus we obtain Claim.

The rest of the proof is similar to that of Corollary 3.3.

Remark 3.12. From the proof of Theorem 3.10, we know that if $\gamma$ is a cut curve of type 0 in (F)–(O), and $\gamma$ is adjacent to two connected components, then $|c(\phi, \gamma)| \geq \frac{1}{6}$. Similarly to the proof of Corollary 3.11, we have that:

If the dual graph of the stable model $\tilde{F}$ of $F$ is one of (F)–(O), then $\delta_0(F) \geq \frac{1}{3}$.

Now we can give an immediate application of the above results to the effective Bogomolov conjecture [LT17].

Proof of Theorem 1.11. Denote by $f$ the family of curves corresponding to $C/K$. By the assumption, we know that the semistable model of $f$ is not smooth. So either $\delta_0(f) \neq 0$ or $\delta_1(f) \neq 0$. By [Ci11 Theorem 2.4],

$$\inf_{D \in \text{Div}^1(C)} a'(D) \geq \frac{1}{2(2g+1)} \left(\frac{(g-1)^2}{2g(7g+5)} \delta_0(f) + \sum_{i \in \{0, g/2\}} \frac{2i(g-i)}{g} \delta_i(f) \right).$$

If $\delta_1(f) \neq 0$, then

$$\inf_{D \in \text{Div}^1(C)} a'(D) \geq \begin{cases} \frac{1}{120}, & \text{if } g = 2, \\ \frac{1}{315}, & \text{if } g = 3. \end{cases}$$

If $\delta_0(f) \neq 0$, then

$$\inf_{D \in \text{Div}^1(C)} a'(D) \geq \begin{cases} \frac{1}{2520}, & \text{if } g = 2, \\ \frac{1}{2070}, & \text{if } g = 3. \end{cases}$$

Comparing these two cases, we get our result.

4 Lower bounds of modular invariants for genus 2

4.1 Lower bounds

Now we can use results in Section 3 to prove Theorem 1.1, Theorem 1.4, and Theorem 1.7.
Proof of Theorem 1.1. Since \( f \) is a family of curves of genus 2, we know that ([Mu83, p.317])

\[
\lambda(f) = \frac{1}{10} \delta_0(f) + \frac{1}{5} \delta_1(f), \quad \kappa(f) = \frac{1}{5} \delta_0(f) + \frac{7}{5} \delta_1(f).
\]

(4.1)

Because \( f \) is non-isotrivial, either \( \delta_0(f) \neq 0 \) or \( \delta_1(f) \neq 0 \). So there are the following two cases.

Case 1: \( \delta_0(f) \neq 0 \). By (4.1) and Corollary 3.3, we know that

\[
\lambda(f) \geq \frac{1}{10} \delta_0(f) \geq \frac{1}{10} \times \frac{1}{3} = \frac{1}{30}, \quad \delta(f) \geq \delta_0(f) \geq \frac{1}{3}, \quad \kappa(f) \geq \frac{1}{5} \delta_0(f) \geq \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}. \quad (4.2)
\]

Case 2: \( \delta_1(f) \neq 0 \). Similarly, we have that

\[
\lambda(f) \geq \frac{1}{5} \times \frac{1}{12} = \frac{1}{60}, \quad \delta(f) \geq \frac{1}{12}, \quad \kappa(f) \geq \frac{7}{5} \times \frac{1}{12} = \frac{7}{60}.
\]

From the above two cases, it is easy to see that

\[
\lambda(f) \geq \min\{\frac{1}{30}, \frac{1}{60}\} = \frac{1}{60}, \quad \delta(f) \geq \min\{\frac{1}{3}, \frac{1}{12}\} = \frac{1}{12}, \quad \kappa(f) \geq \min\{\frac{7}{15}, \frac{7}{60}\} = \frac{1}{15}.
\]

Moreover, \( \lambda(f) = \frac{1}{60} \) if and only if \( \delta(f) = \frac{1}{12} \) if and only if \( \delta_1(f) = \frac{1}{12} \) and \( \delta_0(f) = 0 \). So we obtain Theorem 1.1 (1) by Corollary 3.3. Similarly, we can get Theorem 1.1 (2).

Now we have completed the proof except that each equality of (1.1) can be reached, which will be proved by examples in Section 5.

\[\square\]

Remark 4.1. From Theorem 1.1, we know that if \( f \) has \( \lambda(f) = \frac{1}{60} \), then \( \delta(f) = \frac{1}{12} \), and \( \kappa(f) = \frac{7}{60} \neq \frac{1}{15} \) by Noether equality. Hence there does not exist a non-isotrivial family \( f \) such that both \( \lambda(f) \) and \( \kappa(f) \) are minimal.

4.2 Proof of rigidity, Theorem 1.3

Now we want to study rigidity properties of non-isotrivial families of curves with minimal modular invariants.

Let \( f : S \to C \) be a relative minimal fibration of genus \( g \geq 2 \), and \( C \) be a smooth curve of genus \( b \). We have three fundamental relative invariants which are non-negative,

\[
K_f^2 = K_{S/C}^2 = K_S^2 - 8(g - 1)(b - 1),
\]

\[
e_f = \chi_{\text{top}}(S) - 4(g - 1)(b - 1),
\]

\[
\chi_f = \deg f_* \omega_{S/C} = \chi(\mathcal{O}_S) - (g - 1)(b - 1).
\]

(4.3)

If \( f \) is semistable, then

\[
\lambda(f) = \chi_f, \quad \delta(f) = e_f, \quad \kappa(f) = K_f^2.
\]

(4.4)

Moreover, if \( f \) is semistable and \( e_f \neq 0 \), then \( \chi_f \) and \( K_f^2 \) are positive.

Chern numbers \( c_1^2(F), c_2(F), \chi_F \) of a singular fiber \( F \) are defined as follows (see [Ta10]),

\[
\begin{cases}
    c_1^2(F) = 4N_F + F_{\text{red}}^2 + \alpha_F - \beta_F^-,
    \\
    c_2(F) = 2N_F + \mu_F - \beta_F^+,
    \\
    12\chi_F = 6N_F + F_{\text{red}}^2 + \alpha_F + \mu_F - \beta_F.
\end{cases}
\]

(4.5)
For the notations here, we refer to [Ta10 §1] and [LT13 §2]. By (1.5), it is easy to get Chern numbers of the extremal fibers in Theorem [1.1]. For similar detailed computation, we refer to [LT13]. In the following, denote by \( F_{2-1a} \) the singular fiber whose dual graph is Figure (2-1a) in Theorem [1.1] and others are similar.

| \( F \) | \( N_F \) | \( F^2_{\text{red}} \) | \( \mu_F \) | \( \alpha_F \) | \( \beta^\tau_F \) | \( \beta^\circ_F \) | \( \chi_F \) | \( c^2_1(F) \) | \( c_2(F) \) |
|---|---|---|---|---|---|---|---|---|---|
| \( F_{2-1a} \) | 2 | -4 | 7 | 0 | 23/12 | 1/12 | 2 | 13/12 | 25/12 | 131/12 |
| \( F_{2-1b} \) | 2 | -4 | 7 | 0 | 23/12 | 1/12 | 2 | 13/12 | 25/12 | 131/12 |
| \( F_{2-0a} \) | 1 | -2 | 5 | 1 | 7/3 | 1/3 | 2 | 2/3 | 4/3 | 20/3 |
| \( F_{2-0b} \) | 1 | -1 | 4 | 0 | 2/3 | 1/3 | 1 | 2/3 | 7/3 | 17/3 |

Figure 1: Chern numbers of extremal singular fibers

The relative invariants can be obtained from modular invariants and Chern numbers of singular fibers (see [Ta94, Ta96]), i.e.,

\[
\begin{align*}
K^2_f &= \kappa(f) + \sum_{i=1}^s c^2_1(F_i), \\
e_f &= \delta(f) + \sum_{i=1}^s c_2(F_i), \\
\chi_f &= \lambda(f) + \sum_{i=1}^s \chi F_i.
\end{align*}
\]

(4.6)

Let \( F \) be a singular fiber of genus 2 with smooth reduction, then \( F \) is of elliptic type [1] in NU73. The Chern numbers of these fibers have been computed in [GLT16]. We rewrite the obtained table in [GLT16 §5.1] as Figure 3 in the following for convenience, where \( \chi_F = \frac{1}{12}(c^2_1(F) + c_2(F)) \).

| \( F \) | [I^0_{0-0-0}] | [II] | [III] | [IV] | [V] | [V'] |
|---|---|---|---|---|---|---|
| \( (c^2_1, c_2, \chi) \) | (2, 10, 1) | (2, 4, \( \frac{1}{2} \)) | (2, 10, 1) | (3, 9, 1) | (1, 5, \( \frac{1}{2} \)) | (3, 15, \( \frac{5}{2} \)) |
| \( F \) | [VI] | [VII] | [VII*] | [VIII-1] | [VIII-2] | [VIII-3] |
| \( (c^2_1, c_2, \chi) \) | (2, 10, 1) | (1, 5, \( \frac{1}{2} \)) | (3, 15, \( \frac{5}{2} \)) | (12, \( \frac{5}{2} \)) | (12, \( \frac{5}{2} \)) | (12, \( \frac{5}{2} \)) |
| \( F \) | [VIII-4] | [IX-1] | [IX-2] | [IX-3] | [IX-4] |
| \( (c^2_1, c_2, \chi) \) | (\( \frac{15}{2} \), 16, \( \frac{5}{2} \)) | (\( \frac{9}{2} \), 8, \( \frac{3}{2} \)) | (\( \frac{9}{2} \), 6, \( \frac{1}{2} \)) | (\( \frac{14}{2} \), 14, \( \frac{7}{2} \)) | (\( \frac{12}{2} \), 12, \( \frac{9}{2} \)) |

Figure 2: Chern numbers of singular fibers of genus 2 with smooth reduction

From Figure 3 we know that \( \chi_F \geq \frac{2}{5} \). Furthermore if \( \chi_F \neq \frac{2}{5} \), then \( \chi_F \geq \frac{1}{2} \).

**Proof of Theorem 1.3** Since \( S \) is a rational surface, \( q(S) = p_g(S) = 0, \chi(O_S) = 1 \), and \( C \cong \mathbb{P}^1 \). Thus

\[ \chi_f = \chi(O_S) + 1 = 2. \]

Since \( \lambda(f) = \frac{1}{69} \), there is only one singular fiber, say \( F_1 \), whose dual graph is either Figure (2-1a) or Figure (2-1b) by Theorem 1.1. From Figure 1 we know that \( \chi_{F_1} = \frac{13}{12} \).
By (1.6),
\[ 2 = \chi_f = \lambda(f) + \sum_{i=1}^{s} \chi_{F_i} = \frac{1}{60} + \frac{13}{12} + \sum_{i=2}^{s} \chi_{F_i} \geq \frac{11}{10} + (s - 1) \frac{2}{5}. \]
Hence \( s \leq 3 \). If \( f \) has only two singular fibers, then \( f \) is isotrivial (see [Be81]), and \( \lambda(f) = 0 \). Therefore \( f \) has three singular fibers exactly. We may assume that the singular fibers are over three fixed points of \( \mathbb{P}^1 \) by projective transformation. Then the desired finiteness is from the solved Shafarevich conjecture (see [Ar71, Pa68]).

5 Lower bounds of modular invariants for \( g \geq 3 \)

In order to see our method for \( g \geq 3 \) clearly, we prove Theorem 1.7 first.

Proof of Theorem 1.7 Since \( \delta(f) \neq 0 \), there exists \( 0 \leq i \leq \lfloor g/2 \rfloor \) such that \( \delta_i(f) \neq 0 \).

From [LT17, Theorem 1.4], we know that if \( \delta_i(f) \neq 0 \), then
\[ \delta_i(f) \geq \begin{cases} \frac{1}{4g}, & \text{if } i = 0, \\ \frac{1}{(4i+2)(4(g-i)+2)}, & \text{if } i \geq 1. \end{cases} \]

In this proof, we use the following Moriwaki’s inequality (see [Mo98, Theorem D])
\[ (8g + 4)\lambda(f) \geq g\delta_0(f) + \sum_{i=1}^{\lfloor g/2 \rfloor} 4i(g - i)\delta_i(f). \]  \hspace{1cm} (5.1)

If \( \delta_0(f) \neq 0 \), then, by (5.1),
\[ \lambda(f) \geq \frac{1}{8g + 4} \cdot g \cdot \frac{1}{4g^2}. \]

If \( \delta_i(f) \neq 0 \) for some \( i \geq 1 \), then
\[ \lambda(f) \geq \frac{1}{8g + 4} \cdot 4i(g - i) \cdot \frac{1}{(4i+2)(4(g-i)+2)}. \]

Hence, combining all these cases, we have that
\[ \lambda(f) \geq \frac{1}{8g + 4} \cdot \min \left\{ g \cdot \frac{1}{4g^2}, \frac{4(g - 1)}{(4 + 2)(4(g - 1) + 2)}, \ldots, \frac{4\left[ \frac{g}{2} \right](g - \left[ \frac{g}{2} \right])}{(4\left[ \frac{g}{2} \right] + 2)(4(g - \left[ \frac{g}{2} \right]) + 2)} \right\} \]
\[ \geq \frac{1}{16g(2g + 1)}. \]

By Cornalba-Harris-Xiao’s slope inequality ([Xi87, Theorem 2]), we have that
\[ \kappa(f) \geq \frac{4g - 4}{g} \cdot \lambda(f) \geq \frac{g - 1}{4g^2(2g + 1)}. \]
Applied the proof of Theorem 1.7 to the case $g = 3$, we get Theorem 1.4 by Corollary 3.8 and Corollary 3.11 directly. So we omit the proof of Theorem 1.4.

**Proof of Theorem 1.5.** If $f : S \to C$ is a hyperelliptic fibration of genus $g \geq 2$, then

$$(8g + 4)\lambda(f) = g\xi_0(f) + \sum_{j=1}^{[g-1/2]} 2(j+1)(g-j)\xi_j(f) + \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i(f).$$  \hfill (5.2)

See [CH88, Li16] for the notation $\xi_j$.

The hyperelliptic singular fibers $F(i_k)$ of genus three with periodic monodromy are classified in [Is04, Lemma 1.1], and we list them in Figure 3 using the same notations. It is easy to know (Li16) that $\xi_j(F_a) = \xi_j(F_b) = \xi_j(F(i_k)) = 0 \ (j \geq 0)$, $\delta_1(F_a) = \frac{1}{30}$, $\delta_1(F(i_k)) = 0$.

Let $F_1, F_2, \ldots, F_s$ be all singular fibers of $f$. By our assumption, we may assume that $F_1$ is either $F_a$ or $F_b$. If $\lambda(f) = \frac{1}{105}$, then $\xi_0(F_i) = \xi_1(F_i) = 0 \ (l \geq 2)$ by (5.2), i.e., $F_l \ (l \geq 2)$ have periodic monodromy. Moreover, if $F_l \ (l \geq 2)$ have periodic monodromy, then $\lambda(f) = \frac{1}{105}$ by (5.2). In this case, we have that

$$\xi_0(f) = 0, \ \delta_1(f) = \frac{1}{30}, \ \xi_1(f) = 0,$$

and

$$\chi_f = \lambda(f) + \sum_{l=1}^{s} \chi_{F_l} = \frac{8}{28} + \frac{1}{30} + \sum_{l=1}^{s} \chi_{F_l} = \frac{1}{105} + \sum_{l=1}^{s} \chi_{F_l}.$$

Since $S$ is rational, $\chi(O(S)) = 1$ and $C \cong \mathbb{P}^1$. Then $\chi_f = \chi(O(S)) - (g-1)(b-1) = 3$.

It is easy to calculate that $\chi_{F_a} = \frac{17}{15}$, $\chi_{F_b} = \frac{49}{30}$, and we give the Chern number $\chi_{F(i_k)}$ for each fiber $F(i_k)$ in Figure 3. Thus we can classify all possible configurations directly.

| $F$ | (i1) | (i2) | (i3) | (i4) | (i5) | (i6) | (i7) | (i8) | (i21) |
|-----|------|------|------|------|------|------|------|------|------|
| $\chi_F$ | $\frac{25}{14}$ | $\frac{17}{12}$ | $\frac{33}{28}$ | $\frac{9}{14}$ | $\frac{29}{12}$ | $\frac{13}{12}$ | $\frac{9}{7}$ | $\frac{3}{7}$ | $\frac{7}{7}$ |
| $F$ | (i22) | (i23) | (i24) | (i25) | (i26) | (i29) | (i30) | (i31) | (i32) |
| $\chi_F$ | $\frac{5}{4}$ | $\frac{9}{4}$ | $\frac{3}{4}$ | $\frac{5}{7}$ | $\frac{15}{7}$ | $\frac{6}{7}$ | $\frac{12}{7}$ | $\frac{9}{7}$ | $\frac{11}{7}$ |
| $F$ | (i33) | (i38) | (i39) | (i40) | (i43) | (i44) | (i45) | (i46) | (i47) |
| $\chi_F$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{7}{2}$ | $\frac{5}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{7}{2}$ |

Figure 3: $\chi_F$ of hyperelliptic singular fibers of genus 3 with smooth reduction

**Remark 5.1.** Note that the equality in (5.1) holds for hyperelliptic fibrations $f$ with $\xi_j(f) = 0 \ (j > 0)$ (see (5.2)). For $g \geq 4$, if there exists a hyperelliptic singular fiber with minimal fractional Dehn twist coefficient of corresponding type, then we shall obtain the sharp lower bound for $\lambda(f)$ which can be reached “combinatorially”, similarly as $g = 3$.  

\hfill \square
6 Proof of optimum, Theorem 1.2

Before the proof of Theorem 1.2, we introduce the notation of ramification index.
A reduced divisor $D$ of $S$ is called vertical, if $f(D)$ is a point. If $D$ contains no vertical component, then $f$ induces a morphism $\phi : D \to C$. Let

$$\rho = \rho_1 \circ \cdots \circ \rho_r : (\tilde{S}, \tilde{D}) = (S_r, D_r) \to (S_{r-1}, D_{r-1}) \to \cdots \to (S_1, D_1) \to (S_0, D_0) = (S, D)$$

be the resolution of $D$, where $D_i$ is the strict transform of $D_{i-1}$, $\tilde{D}$ is smooth, and $\rho_i$ is a blow-up at a singularity of $D_{i-1}$ with multiplicity $m_i$. Then the relative ramification index of $\phi$ is defined to be

$$r(D) := \deg \tilde{R} + \sum_{i=1}^{r} m_i(m_i - 1), \quad (6.1)$$

where $\deg \tilde{R}$ is the ramification index of the induced morphism $\tilde{\phi} : \tilde{D} \to C$. Then

$$r(D) = K_{S/C}D + D^2, \quad (6.2)$$

which is a generalized Riemann-Hurwitz formula (see [Xi92, Lemma 2.4.8]).

Now we give our examples.

**Proposition 6.1.** There is a family of fibrations $(f_{\lambda,n} : S_n \to \mathbb{P}^1)_{n \in \mathbb{N}}$ of genus 2 with

1. $\lambda(f_{\lambda,n}) = \frac{1}{60}$, $\delta(f_{\lambda,n}) = \frac{11}{12}$ satisfying that
2. the image of $f_{\lambda,n}$ in $\overline{M}_g$ by the moduli map $J : \mathbb{P}^1 \to \overline{M}_g$ is the same as that of $f_{\lambda,0}$, for each $n \in \mathbb{N}$.

**Proof.** Let $\Gamma_t$ be the fiber over $t \in \mathbb{P}^1$ of the second projection

$$p_2 : P = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad p_2((x,t)) = t.$$ 

Let $R_h$ be the divisor on $P$, whose affine equation is

$$h(x,t) = x^6 + (15x^4 + 40x^3)t - (45x^2 + 24x)t^2 + 5t^3. \quad (6.3)$$

Let $R_{\lambda,n} = R_h + \Gamma_\infty + \sum_{i=1}^{2n} \Gamma_i$, where $n \geq 0$ and $\Gamma_i$'s are generic fibers of $p_2$. Here, when $n = 0$, the sum means that there is no generic fiber. Then there is an invertible sheaf $\delta_{\lambda,n}$ with $O_P(R_{\lambda,n}) \cong \delta_{\lambda,n}^{\otimes 2}$, and there is a double cover $\pi_n : S_n \to P$ whose branch locus is $R_{\lambda,n}$, see [BPV84, §117]. Taking birational transforms, we can obtain a relative minimal fibration $f_{\lambda,n} : S_n \to \mathbb{P}^1$ induced by the second projection $p_2$.

**Case 1:** $n = 0$. Denote $f_{\lambda,0}$ (resp. $R_{\lambda,0}$) by $f_{\lambda}$ (resp. $R_{\lambda}$) for brief.

**Claim A:** There are exactly three singular fibers $F_0 = f^{-1}_\lambda(0)$, $F_{-1} = f^{-1}_\lambda(-1)$ and $F_\infty = f^{-1}_\lambda(\infty)$ in $f_{\lambda}$. Moreover, $F_0$ is the fiber in Theorem 1.1(1), $F_{-1}$ is of type [VIII-1] and $F_\infty$ is of type [II], see [NU73] for the notations.

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Assume Claim A firstly, then we have that

$$\lambda(f_{\lambda}) = \frac{1}{60}, \quad \delta(f_{\lambda}) = \frac{1}{12},$$

by Theorem 1.1 for both $F_{-1}$ and $F_{\infty}$ have smooth reduction. Furthermore, $\kappa(f_{\lambda}) = \frac{7}{60}$ by Noether equality $12\lambda(f_{\lambda}) = \kappa(f_{\lambda}) + \delta(f_{\lambda})$. From Equation (4.6) and the Chern numbers of fibers in Figure 1 and Figure 3, we know that

$$K_{f_{\lambda}}^2 = \kappa(f_{\lambda}) + c_1^2(F_0) + c_1^2(F_{-1}) + c_1^2(F_{\infty}) = \frac{7}{60} + \frac{25}{12} + \frac{4}{5} + 2 = 5,$$

$$\chi_{f_{\lambda}} = \lambda(f_{\lambda}) + \chi_{F_0} + \chi_{F_{-1}} + \chi_{F_{\infty}} = \frac{1}{60} + \frac{13}{11} + \frac{2}{5} + \frac{1}{2} = 2.$$

Since $K_{f_{\lambda}}^2 = 5 < 6$, $S_0$ is a ruled surface by Theorem 0.2 in [TTZ05]. Furthermore, $S_0$ is a rational surface for $q(S_0) = q_{f_{\lambda}} = 0$ by Theorem 1.1.

Case 2: $n \geq 1$.

Comparing $R_{f_{\lambda},n}$ with $R_{f_{\lambda}}$, it is easy to see that $f_{\lambda,n}$ has $2n + 3$ singular fibers, three of them are the same as singular fibers of $f_{\lambda}$ and the rest are all of type $[I^*_{0-0-\infty}]$. Hence

$$\lambda(f_{\lambda,n}) = \frac{1}{60}, \quad \kappa(f_{\lambda,n}) = \frac{7}{60}, \quad \delta(f_{\lambda,n}) = \frac{1}{12},$$

by Theorem 1.1.

For each integer $n > 0$, the family $f_{\lambda,n}$ is the same as $f_{\lambda}$ except for a finite number of fibers. So the image of $f_{\lambda,n}$ in $\overline{M}_g$ induced by the moduli map $J_{f_{\lambda,n}} : \mathbb{P}^1 \rightarrow \overline{M}_g$ is the same as that of $f_{\lambda}$. Hence we will complete our proof after proving Claim A.

Proof of Claim A: See Figure 4 for the branch locus $R_{\lambda}$ in $P$.

Denote by $r_i(R_h)$ the contribution of the point $(x, t) = (i, i)$ $(i = -1, 0, \infty)$ to the relative ramification index $r(R_h)$.

$F_0$: Let $p$ be the point $(x, t) = (0, 0)$. The local equation of $R_h$ near $p$ is $h(x, t)$ in (6.3), thus

![Figure 4: Branch locus of $f_{\lambda}$](image)
The root $x = 0$ of $h(x, 0) = 0$ is with multiplicity 6.

(2) The point $p$ is a singularity of $R_h$ with multiplicity $m_1 = 3$, and the vertical direction is a tangent line of $R_h$ with multiplicity 2.

(3) From the following figure of resolution of $p$, we have that $r_0(R_h) = m_1 (m_1 - 1) + m_2 (m_2 - 1) + 3 = 11$, where 3 comes from the contribution of smooth ramification points (see (6.1)).

F₀: The local equation of $R_h$ near $(x, t) = (-1, -1)$ is
\[ h_{-1}(u, s) := h(u - 1, s - 1) = u^6 - 6u^5 + (15u^4 - 20u^3 + 60u^2 - 72u + 32)s + (-45u^2 + 66u - 36)s^2 + 5s^3. \]

So $R_h$ is smooth near $(u, s) = (0, 0)$, and $u = 0$ is a root of $f(u, 0) = u^6 - 6u^5$ with multiplicity 5. Thus $r_{-1}(R_h) = 4$.

The local equation of $R_h$ near $(x, t) = (-1, -1)$ is the same as $y^2 = x^3 + t$, and $F_{-1}$ is of type $\text{[VIII-1]}$ whose dual graph is Figure 5(a). (See Figure 5 where \bullet denotes a smooth elliptic curve.)

F∞: Let $q$ be the point $(x, t) = (\infty, \infty)$. The local equation of $R_h$ near $q$ is
\[ h_{\infty}(w, r) := w^6 r^3 h\left( \frac{1}{w}, \frac{1}{r} \right) = 5w^6 - (24w^5 + 45w^4)r + (40w^3 + 15w^2)r^2 + r^3. \quad (6.4) \]

Then we know that

(1) The root $w = 0$ of $h_{\infty}(w, 0) = 0$ is of multiplicity 6.

(2) The point $q$ is a singularity of $R_h$ with multiplicity $m_1 = 3$, and the vertical direction is a tangent line of $R_h$ with multiplicity 3.

(3) From the following figure of resolution of $q$, we have that $r_{\infty}(R_h) = m_1 (m_1 - 1) + m_2 (m_2 - 1) + 3 = 15$. 

Figure 5: Singular fibers in fibrations with minimal modular invariants.
Hence the local equation of $R_\lambda$ near $q$ is the same as $y^2 = t\Pi_{i=1}^3(x^2 + \alpha_it)$, $F_\infty$ is of type $[\Pi]$ and the dual graph of $F_\infty$ is Figure 5(b).

Now we know that the relative ramification of $R_h$ is

$$r(R_h) = K_{P/P_1}R_h + R_h^2 = 30 \geq r - 1(R_h) + r_0(R_h) + r_\infty(R_h) = 30.$$ 

So $f_\lambda$ has no other singular fibers.

**Proposition 6.2.** There is a family of fibrations $(f_{\kappa,n} : X_n \to \mathbb{P}^1)_{n \in \mathbb{N}}$ of genus 2 with $\kappa(f_{\kappa,n}) = \frac{1}{15}$, $\lambda(f_{\kappa,n}) = \frac{1}{3}$, $\delta(f_{\kappa,n}) = \frac{1}{3}$, satisfying that

1) $f_{\kappa,n}$ has $2n + 3$ singular fibers;

2) the image of $f_{\kappa,n}$ in $\overline{\mathcal{M}}_g$ by the moduli map $J : \mathbb{P}^1 \to \overline{\mathcal{M}}_g$ is the same as that of $f_{\kappa,0}$, for each $n \in \mathbb{N}$.

**Proof.** This proof is similar to that of Proposition 6.1.

Let $\Gamma_t$ be the fiber over $t \in \mathbb{P}^1$ of the second projection $p_2 : \mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, $p_2((x,t)) = t$.

Let $R_g$ be the divisor on $\mathbb{P}$, whose affine equation is

$$g(x,t) = 5x^6 - 18x^5 + (15x^4 + 20x^3)t + (-45x^2 + 30x - 16)t^2 + 9t^3. \quad (6.5)$$

Let $R_{\kappa,n} = R_g + \Gamma_\infty + \sum_{i=1}^{2n} \Gamma_i$, where $n \geq 0$ and $\Gamma_i$‘s are generic fibers of $p_2$. Combining with the second projection $p_2$, let $f_{\kappa,n} : X_n \to \mathbb{P}^1$ be the relative minimal fibration determined by the double cover over $\mathbb{P}$ whose branch locus is $R_{\kappa,n}$.

Case 1: $n = 0$. Denote $f_{\kappa,0}$ (resp. $R_{\kappa,0}$) by $f_\kappa$ (resp. $R_\kappa$) for brief.

**Claim B:** There are exactly three singular fibers $F_1 = f_\kappa^{-1}(1)$, $F_0 = f_\kappa^{-1}(0)$ and $F_\infty = f_\kappa^{-1}(\infty)$ in $f_\kappa$. Moreover, $F_1$ is the fiber in Theorem 1.1 (2), $F_0$ is of type $[IX-1]$ and $F_\infty$ is of type $[\Pi]$, see [NU73] for the notations.

Assume Claim B firstly, then we have that

$$\kappa(f_\kappa) = \frac{1}{15}, \quad \delta(f_\kappa) = \frac{1}{3}.$$
by Theorem 1.1, for both \( F_0 \) and \( F_\infty \) have smooth reduction. Furthermore, \( \lambda(f_\kappa) = \frac{1}{30} \) by Noether equality. From Equation (4.6) and the Chern numbers of fibers in Figure 4 and Figure 3, we know that

\[
K_2^2 = \kappa(f_\kappa) + c_1^2(F_1) + c_1^2(F_0) + c_1^2(F_\infty) = \frac{1}{15} + \frac{4}{3} + \frac{8}{5} + 2 = 5,
\]

\[
\chi(f_\kappa) = \lambda(f_\kappa) + \chi(F_1) + \chi(F_0) + \chi(F_\infty) = \frac{1}{30} + \frac{4}{3} + \frac{1}{5} + \frac{1}{2} = 2.
\]

Since \( K_2^2 = 5 < 6 \), \( X_0 \) is a ruled surface by Theorem 0.2 in [TTZ05] Furthermore, \( X_0 \) is a rational surface for \( q(X_0) = q_{f_\kappa} = 0 \) by Theorem 1.1.

Case 2: \( n \geq 1 \).
Comparing \( R_{\kappa,n} \) with \( R_\kappa \), it is easy to see that \( f_{\kappa,n} \) has \( 2n + 3 \) singular fibers, three of them are the same as singular fibers of \( f_\kappa \) and the others are all of type \([I_{0}^* - 0 - 0]\). Hence

\[
\lambda(f_{\kappa,n}) = \frac{1}{30}, \quad \kappa(f_{\kappa,n}) = \frac{1}{15}, \quad \delta(f_{\kappa,n}) = \frac{1}{3},
\]

by Theorem 1.1.

For each integer \( n > 0 \), the family \( f_{\kappa,n} \) is the same as \( f_\kappa \) except for a finite number of fibers. So the image of \( f_{\kappa,n} \) in \( \mathcal{M}_g \) induced by the moduli map is the same as that of \( f_\kappa \). Hence we will complete the proof after proving Claim B.

Proof of Claim B: See Figure 6 for the branch locus \( R_\kappa \) in \( P \).

![Figure 6: Branch locus of \( f_\kappa \)](image)

Denote by \( r_i(R_g) \) the contribution of the point \( (x, t) = (i, i) \) \( (i = 0, 1, \infty) \) to the relative ramification index \( r(R_g) \).

\( F_0 \): Let \( p \) be the point \( (x, t) = (0, 0) \). The local equation of \( R_g \) near \( p \) is \( g(x, t) \) in (6.5), then we know that

1. The root \( x = 0 \) of \( g(x, 0) = 0 \) is with multiplicity 5.
2. The point \( p \) is a singularity of \( R_g \) with multiplicity \( m_1 = 2 \), and the vertical direction is a tangent line of \( R_g \) with multiplicity 2.
3. From the following figure of resolution of \( p \), we have that \( r_0(R_g) = m_1(m_1 - 1) + m_2(m_2 - 1) + 4 = 8 \), where 4 comes from the contribution of smooth ramification points.
Hence the local equation of $R_g$ near $p$ is the same as $y^2 = x^5 + t^2$, $F_0$ is of type $[IX-1]$ and the dual graph of $F_0$ is Figure 3(c).

$F_1$: Let $q$ be the point $(x, t) = (1, 1)$. The local equation of $R_g$ near $q$ is

$$g_1(u, s) := g(u + 1, s + 1) = 5u^6 + 12u^5 + (15u^4 + 80u^3 + 60u^2)r - (45u^2 + 60u + 4)r^2 + 9r^3.$$  

Then we know that

1. The root $u = 0$ of $g_1(u, 0) = 0$ is of multiplicity 5.
2. The point $q$ is a singularity of $R_g$ with multiplicity $m_1 = 2$, and the vertical direction is a tangent line of $R_g$ with multiplicity 2.
3. From the following figure of resolution of $q$, we have that

$$r_1(R_g) = m_1(m_1 - 1) + m_2(m_2 - 1) + 3 = 7,$$  

where 3 comes from the contribution of smooth ramification points.

Hence the local equation of $R_g$ near $q$ is the same as $y^2 = (x^2 + t)(x^3 + t)$, and the dual graph of $F_1$ is Figure (2-0a).

$F_\infty$: The local equation of $R_g$ near $(x, t) = (\infty, \infty)$ is

$$
g_\infty(w, r) = w^6r^3g(1/w, 1/r)$$

$$= 9w^6 + (-16w^5 + 30w^5 - 45w^4)r + (20w^3 + 15w^2)r^2 + (-18w + 5)r^3. \quad (6.6)$$

It is easy to see that this is the same as $F_\infty$ in Proposition 6.1. In particular, $r_\infty(R_g) = 15$.

Now we know that the relative ramification of $R_g$ is

$$r(R_g) = K_{P/P_1}R_g + R_g^2 = 30 \geq r_0(R_g) + r_1(R_g) + r_\infty(R_g) = 30.$$  

So $f_\kappa$ has no other singular fibers.

\[\square\]
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