On Concise Encodings of Preferred Extensions

Paul E. Dunne
Department of Computer Science
University of Liverpool
Liverpool L69 7ZF
United Kingdom
ped@csc.liv.ac.uk

Abstract

Argument Systems provide a rich abstraction within which diverse concepts of reasoning, acceptability and defeasibility of arguments, etc., may be studied using a unified framework. Much work has focused on the so-called preferred extensions of such systems, which define the maximal (with respect to \( \subseteq \)) collectively defeasible subsets of arguments within a given system of arguments and attack relationship. In this article we address problems related to the following issue. Identification and enumeration of preferred extensions of an argument system is (under the usual complexity theoretic assumptions) computationally demanding: there may be exponentially many and deciding if a given subset \( S \) of \( X \) defines a preferred set is \( \text{CO-NP-complete} \). For a domain which is questioned ‘frequently’ it may be acceptable to invest this computational effort once, but having done so it is desirable to encapsulate these data in a form which is compact and allows, for example, questions concerning the acceptability of specific arguments to be dealt with efficiently. In this article we consider two ‘plausible’ approaches to reducing the complexity of deciding if \( S \) is a preferred extension of a system \( \mathcal{H} \) both of which assume some initial potentially ‘expensive’ precomputation, invested to reduce time needed in subsequent queries to the system. The first approach examines ‘reasonable encoding’ approaches; the second is to determine the subset of all defeasible arguments providing these as additional data when attempting to decide if \( S \) is a preferred extension. It is shown that if certain properties are required of the encoding scheme, then the former approach is feasible only if \( \text{NP} = \text{CO-NP} \). In the latter case, we show that, even if provided with information regarding which arguments are credulously accepted, the question of whether a subset of arguments defines a preferred extension remains \( \text{CO-NP-complete} \).

Keywords: Argument Systems, Preferred Extension, Computational Complexity

1 Introduction

Since they were introduced by Dung [5], Argument Systems have provided a fruitful mechanism for studying reasoning in defeasible contexts. They have proved useful both to theorists who can use them as an abstract framework for the study and comparison of non-monotonic logics, e.g. [1], and for those who wish to explore more concrete contexts where defeasibility is central. In the study of reasoning in law, for example, they have been used to examine the resolution of conflicting norms, e.g. [11], especially where this is studied through the mechanism of a dispute between two parties, e.g. [9]. The basic definition below is derived from that given in [5].

Definition 1 An argument system is a pair \( \mathcal{H} = \langle X, A \rangle \), in which \( X \) is a set of arguments and \( A \subseteq X \times X \) is the attack relationship for \( \mathcal{H} \). Unless otherwise stated, \( X \) is assumed to be finite, and \( A \) comprises a set of ordered pairs of distinct arguments. A pair \( \langle x, y \rangle \in A \) is referred to as ‘\( x \) is attacked by \( y \)’ or ‘\( y \) attacks \( x \)’.

For \( R \), \( S \) subsets of arguments in the system \( \mathcal{H}(\langle X, A \rangle) \), we say that

a) \( s \in S \) is attacked by \( R \) if there is some \( r \in R \) such that \( \langle r, s \rangle \in A \).

b) \( x \in X \) is acceptable with respect to \( S \) if for every \( y \in X \) that attacks \( x \) there is some \( z \in S \) that attacks \( y \).

c) \( S \) is conflict-free if no argument in \( S \) is attacked by any other argument in \( S \).
d) A conflict-free set $S$ is admissible if every argument in $S$ is acceptable with respect to $S$.

e) $S$ is a preferred extension if it is a maximal (with respect to $\subseteq$) admissible set.

f) $S$ is a stable extension if $S$ is conflict free and every argument $y \not\in S$ is attacked by $S$.

g) $\mathcal{H}$ is coherent if every preferred extension in $\mathcal{H}$ is also a stable extension.

An argument $x$ is credulously accepted is there is some preferred extension containing it; $x$ is sceptically accepted if it is a member of every preferred extension.

The notation $\mathcal{P}E(\mathcal{H})$ is used to describe the set of all subsets of $\mathcal{X}$ which are preferred extensions of $\mathcal{H}$. Similarly, $\mathcal{S}E(\mathcal{H})$ denotes the set of all stable extensions of $\mathcal{H}$ and $\mathcal{BE}(\mathcal{H})$ refers to an arbitrary one of these sets. We use $n$ to denote $|\mathcal{X}|$.

The preferred extensions of an Argument System can be taken as being the consistent positions that can be adopted within the Argument System. Any argument that appears in all preferred extensions will be accepted in every consistent position, and any argument that appears in no preferred extension cannot be held in any consistent position. This means that the notion can be related to varieties of semantics for non-monotonic reasoning: credulously acceptable arguments will be those that appear in at least one preferred extension, and sceptically acceptable arguments will be those which appear in all preferred extensions. In the context of legal reasoning, the notion allows us to distinguish those arguments which must be accepted, those which can be defended, and those which are indefensible.

To avoid repetition we will subsequently refer to the following decision problems:

**PREF-EXT** (Preferred Extension)

**Instance**: An argument system $\mathcal{H}(\mathcal{X}, \mathcal{A})$ and $S \subseteq \mathcal{X}$.

**Question**: Is $S \in \mathcal{P}E(\mathcal{H})$?

**STAB-EXT** (Stable Extension)

**Instance**: An argument system $\mathcal{H}(\mathcal{X}, \mathcal{A})$ and $S \subseteq \mathcal{X}$.

**Question**: Is $S$ a subset of some $T \in \mathcal{S}E(\mathcal{H})$, i.e. can $S$ be expanded to a stable extension?

**PREF-EXT-INF** (Preferred Extension given Information)

**Instance**: An argument system $\mathcal{H}(\mathcal{X}, \mathcal{A})$, $S \subseteq \mathcal{X}$, and $\alpha = (a_1, a_2, \ldots, a_n) \in \{\bot, \top\}^n$ an $n$-tuple of Boolean values such that $a_i = \top$ if and only if the argument $x_i$ is credulously accepted in $\mathcal{H}$.

**Question**: Is $S \in \mathcal{P}E(\mathcal{H})$?

**STAB-EXT-INF** (Stable Extension given Information)

**Instance**: An argument system $\mathcal{H}(\mathcal{X}, \mathcal{A})$, $S \subseteq \mathcal{X}$, and $\alpha = (a_1, a_2, \ldots, a_n) \in \{\bot, \top\}^n$ an $n$-tuple of Boolean values such that $a_i = \top$ if and only if the argument $x_i$ is credulously accepted in $\mathcal{H}$.

**Question**: Is $S$ a subset of some $T \in \mathcal{S}E(\mathcal{H})$, i.e. can $S$ be expanded to a stable extension?

Before proceeding it may be useful to consider in more detail the concepts of ‘preferred’ versus ‘stable’ extensions of an argument system. Both and offer a view of preferred extensions as providing a more general construct than stable extensions: thus any stable extension is also preferred but the converse is not always true. A significant difference between the two models is that whereas some argument systems may have no stable extension, it is always the case that a preferred extension extension exists since the empty set is always admissible. This difference raises a number of general questions that are discussed in some detail in . In particular,

1. Are there ‘natural’ or ‘meaningful’ argument systems with no stable extensions?

2. Are there ‘natural’ systems whose set of stable extensions form a strict (non-empty) subset of the set of preferred extensions?

Through consideration of a particular $n$-player game, , argues that studies of Lucas and Shubik suggest ‘stable semantics do not capture the intuitive semantics of every meaningful argumentation system’. As a further example, using a variant of the Stable Marriage Problem, , exhibits a concrete ‘natural’ system which has no stable extension. While both examples suggest a positive answer to the first of the two questions raised, neither treats the second question. Instead, , Defn. 31, p. 332] introduces the concept of coherence to describe systems for which every preferred extensions is also stable, discussing forms of argumentation system whose instantiation guarantees coherence. Recent work of Dunne and Bench-Capon, however, indicates that even when restricted to the context of finitely presented argument systems, deciding if a given system $\mathcal{H}(\mathcal{X}, \mathcal{A})$ is coherent is ‘likely’ to be extremely hard. One consequence of the proof employed in , is that it, naturally, gives rise to an infinite class of argument systems having a non-empty set of stable extensions but which are nonetheless incoherent: i.e. there is a supportable case that the answer to the second question raised is also positive.

A major difficulty that is encountered within these formalisms is the computational intractability of several decision problems that arise: results of Dimopoulos and Torres indicate that deciding if $p$ is credulously accepted in $\mathcal{H}$ is NP-complete and that PREF-EXT is CO-NP-complete (even when $S$ is the empty set). Similarly deciding if $\mathcal{H}$ has any stable extension (i.e. the decision problem $\mathcal{S}E(\mathcal{H}) \neq$
0?\) is \textit{NP}-complete (notice that this is the special case of \textit{STAB-EXT} when \(S = \emptyset\)). Dunne and Bench-Capon \cite{DBLP:journals/tcs/DunneB04} proved that deciding coherence is \(\Pi_{2}^{p}\)-complete, deducing the same complexity classification for sceptical acceptance as a consequence. Related work, described in \cite{conf/kr/StephanV99}, has shown that the sound and complete reasoning method for credulous argumentation introduced by Vreeswijk and Prakken \cite{DBLP:conf/kr/Prakken98} in which reasoning proceeds via a dialogue game requires an exponential number of moves to resolve some disputes. Examining differing concepts of ‘acceptance’ in various non-monotonic Logics, \cite{conf/kr/StephanV99,conf/kr/StephanV99}, indicate some disputes. Examining differing concepts of ‘acceptance’ as a consequence. Related work, described in \cite{conf/kr/StephanV99}, is superpolynomial in \(n\).

The concern of this paper is to consider some further contrasts between computational properties of preferred and stable extension sets, arising from the following scenario. Suppose one is given a specific argument system \(\mathcal{H}(X, A)\) which describes a ‘frequently’ used application, for example one incorporating complex legal data in which the justification for various different positions may have to be assessed. Rather than deciding acceptability and support for an argument on each new query to \(\mathcal{H}\), we may be prepared to invest some computational effort once in the hope that the information elicited as a result may help in reducing the time taken for subsequent queries. For example, one could compute all sets in \(\mathcal{BE}(\mathcal{H})\) and then have queries on \(\mathcal{H}\) performed with respect to a \textit{representation} (encoding) of this set of subsets. There are, of course, several properties which such a representation should, ideally, satisfy. Suppose \(\eta(\mathcal{H})\) is some encoding of \(\mathcal{BE}(\mathcal{H})\). We define, informally, two such properties that are the main focus of this paper:

\begin{enumerate}
\item R1. \(\eta\) is \textit{terse}, i.e. \(|\eta(\mathcal{H})|\) – the number of \textit{bits} needed – is polynomially bounded in \(n\).
\item R2. \(\eta\) is \textit{extension tractable}, i.e. given any \(S \subseteq X\), the question \(S \in \mathcal{BE}\) can be decided from \(\eta(\mathcal{H})\) in time polynomial in \(|\eta(\mathcal{H})|\).
\end{enumerate}

One obvious representation scheme is simply to use a table \(T_{\mathcal{H}}\) of \(|\mathcal{BE}(\mathcal{H})|\) rows, each row being \(n\) bits in length, so that if \(\langle S_{1}, \ldots, S_{k}\rangle\) is an ordering of \(\mathcal{BE}(\mathcal{H})\), then \(T_{i,j} = 1\) if and only if \(x_{j} \in S_{i}\). While this representation meets the criterion specified by R2, it will fail to satisfy R1 in those cases where \(|\mathcal{BE}(\mathcal{H})|\) is superpolynomial in \(n\). We note that systems may be defined where this number is \(\Omega(3^{n^{2/3}})\). Alternatively, the system \(\mathcal{H}\) itself is a representation of \(\mathcal{BE}(\mathcal{H})\): while satisfying R1 it is, however, unlikely to satisfy R2 in the case \(\mathcal{BE} = \mathcal{PE}\) (assuming \(\text{NP} \neq \text{CO-NP}\)).

Given such examples, a natural question to raise is whether these extremes are inherent, or are there representation formalisms that are terse and \textit{extension tractable} – a property we subsequently refer to as \textit{concise}. In the next Section we formalise these concepts and, in Section 3, prove some basic results concerning them. In particular it is shown that in the case of \(\mathcal{PE}(\mathcal{H})\) ‘effective’ concise encodings are not possible, in general, unless \(\text{NP} = \text{CO-NP}\). In contrast, concise encoding schemes for \(\mathcal{SE}(\mathcal{H})\) are easy to construct. We note that this provides another example of a property which is considerably ‘easier’ under stable semantics than under preferred semantics for argument systems, cf. \cite{conf/kr/StephanV99,conf/kr/StephanV99}.

A further indication of the computational difficulties arising in considering preferred extensions is given in the concluding result of Section 3 where the problem \textit{PREF-EXT-INF} is shown to be \textit{CO-NP}-complete: thus, even if the definability status for every argument in \(X\) is supplied as part of an instance, the problem of deciding whether a given subset \(S\) is a preferred extension does not become any easier, i.e. remains \textit{CO-NP}-complete. Discussion and conclusions occupy Section 4.

\section{Definitions}

In the remainder of this paper the following notational conventions are used.

\(X_{n}\) is a set of \(n\) arguments \(\{x_{1}, x_{2}, \ldots, x_{n}\}\).

For an \(n\) element set \(X_{n}\), \(\varphi(X_{n})\) is the set of all subsets of \(X_{n}\).

The following definition formalises our abstract concept of \textit{encoding scheme} for the set \(\mathcal{BE}\) in an argument system \(\mathcal{H}\).

\textbf{Definition 2} A \(\mathcal{BE}\) encoding scheme is a pair \((\eta, P)\) where

\[\eta : \{\mathcal{H}(X_{n}, A) : \mathcal{H} \text{ is an argument system}\} \rightarrow \{0, 1\}^{*}\]

is a mapping from argument systems to finite binary words, and \(P\) is a deterministic Turing machine program, that takes as input a pair \((\eta(\mathcal{H}), S)\) in which \(S \subseteq X_{n}\), accepting if and only if \(S \in \mathcal{BE}(\mathcal{H})\).

Before proceeding there are several points that should be noted. First we observe that it is not insisted that schemes be \textit{uniform}, i.e. it \textit{is} not required that there is an algorithm which given \(\mathcal{H}\) computes \(\eta(\mathcal{H})\). Of course, in ‘practical’ schemes, one would wish to have some mechanism for automating this translation. In order to capture some sense of ‘practical’ scheme, we introduce the notion of \textit{verifiability}.

\textbf{Definition 3} A verifiable \(\mathcal{BE}\) encoding scheme is a triple \((\eta, P, Q)\) in which \((\eta, P)\) is a \(\mathcal{BE}\) encoding scheme and \(Q\) a (non-deterministic) Turing Machine program, that is given \(w \in \{0, 1\}^{*}\) and \(\mathcal{H}(X, A)\) as input, accepting if \(w = \eta(\mathcal{H})\).
Secondly, the definition provides a foundation for introducing more ‘sophisticated’ schemes other than binary words.

With this abstract idea of encoding scheme we can formalise the notions of terse and extension tractable outlined earlier.

**Definition 4** Let \( \langle \eta, P, Q \rangle \) be a verifiable \( BE \) encoding scheme. We say \( \langle \eta, P \rangle \) is terse if there is a constant \( k \), such that for all \( H(X, A) \), \( |\eta(H)| \leq n^k \); it is extension tractable if there is a constant \( k \) such that: given \( \eta(H) \) and \( S \in \psi(X_a) \) as input, \( P \) decides if \( S \in BE(H) \) taking at most \((n + |\eta(H)|)^k \) steps. Finally, \( \langle \eta, P, Q \rangle \) is effective if there is a constant \( k \) for which \( Q \) has an accepting computation of \( w = \eta(H) \) in non-deterministic time bounded by \((n + |w|)^k \). A \( BE \) encoding scheme \( \langle \eta, P \rangle \) is concise if it is both terse and extension tractable. A verifiable \( BE \) encoding scheme \( \langle \eta, P, Q \rangle \) is usefully concise if it is terse, extension tractable, and effective.

In ‘practical’ terms usefully concise verifiable \( BE \) encoding schemes, define the ‘ideal’ representation form: if \( \langle \eta, P, Q \rangle \) is usefully concise then one can describe \( BE(H) \) in its entirety using only a ‘small’ amount of space – since \( \eta \) is terse; one can determine efficiently (in terms of \( |\eta(H)| \) and \( |S| \)) if \( S \in BE(H) \) – since the scheme is extension tractable; and, finally, one may test if an arbitrary \( w \) does, indeed, describe the encoding \( \eta(H) \) for a given \( H \).

We now present some examples of encoding approaches.

### 2.0.1 Tabular Representation

Given \( H \), \( tab(H) \) is the \( n|BE(H)| \)-bit word in which bit \( i_{(i-1)n+j} = 1 \) if and only if \( x_i \in S \), where \( 1 \leq j \leq n \) and \( \{S_1, S_2, \ldots, S_r\} \) is an ordering of \( BE(H) \). If the algorithm \( P \) in \( \langle \eta, P \rangle \) is chosen to be an appropriate table look up method, then \( \langle \eta, P \rangle \) is extension tractable. It is not, however, terse.

### 2.0.2 Representations via Propositional Logic Functions

Given any \( H(X_a, A) \) there is a unique propositional logic function, \( f_H(X_a) \) definable from \( BE(H) \) as follows. For any \( S \in \psi(X_a) \) let the instantiation, \( \alpha_S \) of the propositional variables \( X_a = x_i = \top \) if \( x_i \in S \) and \( x_i = \bot \) if \( x_i \notin S \). The function \( f_H(X_a) \) takes the value \( \top \) on exactly those instantiations \( \alpha_S \) for which \( S \in BE(H) \).

Given this approach, any representation formalism for arbitrary \( n \)-argument propositional logic functions serves as a basis for a \( BE \) encoding scheme, e.g. truth-tables, propositional formulae over a finite complete basis, etc.

### 3 Properties of Usefully Concise Encoding Schemes

We first observe that construction of usefully concise encoding schemes for stable extensions is trivial: the problem of deciding, given \( H(X, A) \) and \( S \subseteq X \) whether \( S \in SE(H) \) is polynomial-time solvable; therefore since \( H(X, A) \) already defines a terse encoding of \( SE(H) \) with an appropriate decision algorithm we have a concise \( SE \) encoding scheme. This can be extended to give a usefully concise scheme, by encoding \( H \) as its \( n^2 \) element adjacency matrix, so that \( w = \eta(H) \) is decided in \( |w| \) steps.

In contrast to the easy construction above, usefully concise encoding schemes for preferred extensions are ‘unlikely’ to exist.

**Theorem 1** If \( NP \neq CO-NP \) then usefully concise \( PE \) encoding schemes do not exist.

**Proof.** Suppose \( NP \neq CO-NP \) and that for the sake of contradiction, \( \langle \eta, P, Q \rangle \) is a usefully concise \( PE \) encoding scheme. We show that \( \langle \eta, P, Q \rangle \) can be used as the basis of an \( NP \) decision method for \( PRE-EXT \). Since this problem is \( CO-NP \)-complete it follows that the existence of such a decision method would imply \( NP = CO-NP \). Since \( \langle \eta, P, Q \rangle \) is terse there is some constant \( k \) such that \( |\eta(H)| \leq n^k \) for any \( n \)-argument system \( H \). Our \( NP \) algorithm is as follows: given an instance \( \langle H(X, A), S \rangle \) of \( PRE-EXT \) non-deterministically choose a sequence \( \beta(H) \) of \((at most) |X|^k \) bits. Then simulate \( Q \) on input \( \langle H, \beta(H) \rangle \). If \( \beta(H) = \eta(H) \) \( Q \) will have an accepting computation of polynomial length (since \( Q \) is effective). Finally, the program, \( P \), is run with input \( \langle \beta(H), S \rangle \). Since \( |\beta(H)| \leq |X|^k \) and \( P \) is a deterministic polynomial time computation, the (non-deterministic) algorithm runs in time polynomial in \( X \). To see that the algorithm accepts instances for which \( S \in PE(H) \) it suffices to observe that if \( S \in PE(H) \) then there is some choice of \( \beta(H) \) that will correspond to \( \eta(H) \), be accepted by \( Q \) and on which the extension tractable algorithm \( P \) will accept \( \langle \beta(H), S \rangle \).

It should be noted that the argument used in the proof requires the assumption that \( \langle \eta, P, Q \rangle \) is effective. The reason being that if \( P \) were invoked directly on the word \( \beta(H) \) then every instance would be accepted: given a subset \( S \) there is certainly some argument system, \( G \) for which \( PE(G) = \{S\} \) – the system with \( |S| \) isolated arguments – and if \( \beta(H) = \eta(G) \) then the instance is accepted regardless of whether \( S \in PE(H) \).

**Theorem 1** indicates that even if one is prepared to invest considerable computational effort in constructing an encoding \( \eta(H) \), such effort will not aid in testing \( S \in PE(H) \) if the encoding form is terse and effective.
The problem, PREF-EXT-INF in allowing knowledge regarding the set of credulously accepted arguments to be given for free, can be seen as defining an alternative ternar encoding scheme. We note that this scheme is not effective (in our usage) (assuming NP ≠ CO-NP) since in the encoding $\langle H, \alpha \rangle$, should any bit of $\alpha$ be $\perp$ indicating the associated argument is not credulously accepted, we cannot test $w = \eta(H)$ using an NP computation.

Thus, since determining the set of credulously accepted arguments in a system, may at worst involve similar computational expenditure to that of enumerating preferred extensions, the complexity of the problem PREF-EXT-INF is of some interest: the implied encoding scheme is not one which is within the scope of Theorem [3]. Our next result shows that PREF-EXT-INF is no easier than PREF-EXT.

**Theorem 2**  
PRE-F-EXT-INF is CO-NP–complete.

**Proof.** Membership in CO-NP is immediate from the fact that PROF-EXT ∈ CO-NP. To show that PROF-EXT-INF is CO-NP–hard, we give a reduction from the problem of deciding is a propositional formula in 3–CNF is unsatisfiable: 3-UNSAT. Let

$$\Phi(X_n) = \bigwedge_{i=1}^{m} C_i = \bigwedge_{i=1}^{m} (y_{i,1} \lor y_{i,2} \lor y_{i,3})$$

be an instance of 3-UNSAT, so that each $y_{i,j}$ is a literal from $\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$. Let $x_{n+1}, x_{n+2}$ be two new propositional variables (i.e. not belonging to $X_n$) and consider the CNF formula $\Psi(X_n, x_{n+1}, x_{n+2})$ defined as,

$$\bigwedge_{i=1}^{m} (C_i \lor \neg x_{n+1} \lor x_{n+2}) \land (C_i \lor x_{n+1} \lor \neg x_{n+2})$$

The following properties of $\Psi(X_n, x_{n+1}, x_{n+2})$ are easily verified.

a) For each variable $y \in \{x_n, x_{n+1}, x_{n+2}\}$ there is some satisfying instantiation for $\Psi$ under which $y = \top$.

b) For each variable $y \in \{x_n, x_{n+1}, x_{n+2}\}$ there is some satisfying instantiation for $\Psi$ under which $y = \bot$.

c) There is a satisfying instantiation for $\Psi$ under which $x_{n+1} = \top$ and $x_{n+2} = \bot$ if and only if $\Phi(X_n)$ is satisfiable.

We use $\Psi(X_n, x_{n+1}, x_{n+2})$ to build an argument system $\mathcal{H}_\Psi(V, A)$. The argument set $V$ contains $2(m + n + 3)$ arguments labelled

$$V = \{\Psi, \chi\} \cup \{x_i, \bar{x}_i : 1 \leq i \leq n + 2\} \cup \{C^{(1)}_i, C^{(2)}_j : 1 \leq j \leq m\}$$

The attack relationship, $A$, comprises

1. $\langle x_i, \bar{x}_i \rangle \land \langle x_i, x_i \rangle : 1 \leq i \leq n + 2$
2. $\langle x_i, \bar{x}_i \rangle \land \langle x_i, x_i \rangle : 1 \leq i \leq n$
3. $\langle y_{i,j}, C^{(k)}_i \rangle : 1 \leq k \leq 2, 1 \leq i \leq m, y_{i,j} \in C_i$
4. $\langle C^{(1)}_i, \Psi \rangle \land \langle C^{(2)}_j, \Psi \rangle : 1 \leq i \leq m$
5. $\langle \bar{x}_{n+1}, C^{(1)}_i \rangle \land \langle x_{n+2}, C^{(1)}_i \rangle : 1 \leq i \leq m$
6. $\langle x_{n+1}, C^{(2)}_i \rangle \land \langle \bar{x}_{n+2}, C^{(2)}_i \rangle : 1 \leq i \leq m$
7. $\langle \Psi, \chi \rangle$

In the system $\mathcal{H}_\Psi(V, A)$ all except the arguments $\{\chi\} \cup \{C^{(1)}_i, C^{(2)}_j : 1 \leq j \leq m\}$ are credulously accepted. To see this first observe that a satisfying instantiation $\alpha$ of $\Psi(X_n, x_{n+1}, x_{n+2})$ induces a preferred extension of $\mathcal{H}_\Psi$ containing $\Psi$ together with the $n+2$ arguments corresponding to literals evaluating to $\top$ under $\alpha$. From properties (a) and (b) above we deduce that $\Psi$ is credulously accepted and each of the arguments $x_i, \bar{x}_i$ for $1 \leq i \leq n + 2$. In order for an argument $C^{(k)}_i$ to be credulously accepted, an admissible set containing it, would have to contain $y_{i,j}$ for each $y_{i,j} \in C_i$. These arguments, however, are attacked by $\chi$: $\Psi$ (the only attacker of $\chi$) cannot be included in an admissible set containing $C^{(k)}_i$. Similarly $\chi$ cannot be credulously accepted, since the only attackers of its attacker $- \Psi$ are $C^{(k)}_i$.

The instance of PREF-EXT-INF constructed from $\Phi(X_n)$ is $\langle H_\Psi, \{x_{n+1}, \bar{x}_{n+2}\}, \alpha_C \rangle$ where $\alpha_C$ is the $2(m + n + 3)$ tuple indicating the credulously accepted arguments in $\mathcal{H}_\Psi$ as described in the preceding paragraph.

We claim that $\{x_{n+1}, \bar{x}_{n+2}\} \in \mathcal{PE}(\mathcal{H}_\Psi)$ if and only if $\Phi(X_n)$ is unsatisfiable. First suppose that $\{x_{n+1}, \bar{x}_{n+2}\} \in \mathcal{PE}(\mathcal{H}_\Psi)$. Assume, for the sake of contradiction, that $\Phi(X_n)$ is satisfiable and let $\alpha$ be a satisfying instantiation of $X_n$ for $\Phi$. Consider the subset, $S_\alpha$ of $V$ given by,

$\{x_{n+1}, \bar{x}_{n+2}\} \cup \{\Psi\} \cup \{x_i : \alpha_i = \top\} \cup \{\bar{x}_i : \alpha_i = \bot\}$

We claim that $S_\alpha$ is an admissible set (in fact $S_\alpha \in \mathcal{PE}(\mathcal{H}_\Psi)$). To see this first observe that $S_\alpha$ is conflict-free and consider any argument $p \in V$ that attacks some argument of $S_\alpha$. If $p = \chi$, then $\Psi \in S_\alpha$ attacks $p$; if $p = y \in \{x_i : 1 \leq i \leq n + 2\}$, then $\bar{y} \in S_\alpha$ attacks $p$; similarly if $p = \bar{y} \in \{x_i : 1 \leq i \leq n + 2\}$ then $y \in S_\alpha$ attacks $p$. If $p \in \{C^{(1)}_i, C^{(2)}_j : 1 \leq i \leq m\}$ then $x_{n+1}$ attacks $p$. We are left with the case $p \in \{C^{(1)}_i\}$. Since $\alpha$ satisfies $\Phi$ some literal $y_{i,j}$ of $C_i$ must take the value $\top$ under $\alpha$. Now we find an attack on $C^{(1)}_i$ with the corresponding $x_i$ or $\bar{x}_i$ argument in $S_\alpha$. We deduce that $\Phi(X_n)$ satisfiable would contradict the assumption $\{x_{n+1}, \bar{x}_{n+2}\} \in \mathcal{PE}(\mathcal{H}_\Psi)$. 


On the other hand, suppose that \( \Phi(X_n) \) is unsatisfiable. We
show that \( \{x_{n+1}, \bar{x}_{n+2}\} \in \mathcal{PE}(H_{\Phi}) \).

Certainly \( \{x_{n+1}, \bar{x}_{n+2}\} \) is admissible. Consider any \( S \subset \mathcal{V} \)
for which \( \{x_{n+1}, \bar{x}_{n+2}\} \subset S \) and with \( S \) admissible. Let \( p \in S/\{x_{n+1}, \bar{x}_{n+2}\} \). If \( p = y \in \{x_i, \bar{x}_i : 1 \leq i \leq n\} \), then in
order to counter-attack the attack by \( \chi \) on \( y \), the argument \( \Psi \)
must be in \( S \). If \( \Psi \in S \), then for each argument \( C_i^{(\ell)} \), \( S \) must
contain some argument corresponding to a literal \( y_i \) of \( C_i \). Since
\( S \) is assumed admissible, it follows that the set of literals identified do not conflict. Choosing an instantiation of \( X_n \) which makes each of these literals take the value \( \top \)
will satisfy \( \Phi \). This, however, contradicts the assumption
that \( \Phi \) was unsatisfiable. We deduce that \( S/\{x_{n+1}, \bar{x}_{n+2}\} = \emptyset \)
and therefore \( \{x_{n+1}, \bar{x}_{n+2}\} \in \mathcal{PE}(H_{\Phi}) \) as claimed.

The following Corollary is easily obtained,

**Corollary 1** STAB-EXT-INF is \( \mathcal{NP} \)-complete.

**Proof.** Using the construction of the Theorem, \( \{x_{n+1}, \bar{x}_{n+2}\} \) can be developed to a stable extension, if and
only if \( \Phi(X_n) \) is satisfiable.

It is stressed that Theorem 3 and its corollary are addressing
different decision problems from their counterparts PREF-EXT and STAB-EXT: for the latter problems an instance comprises an argument system \( H(X, A) \) and
a subset \( S \) of \( X \); in the problems PREFERENCES-INF and STAB-
EXT-INF an instance additionally provides \(|X|\) bits of information, \( \alpha_X \), delineating which arguments of \( X \) are credulously accepted. One indication of the different nature of these problems can be seen by considering the case when \( S = \emptyset \): PREFERENCES-EXT is \( \mathcal{CO-NP} \)-complete for this case, how-
ever PREFERENCES-INF has an easy polynomial time algorithm by simply checking if any bit of \( \alpha_X \) is \( \top \). It is of interest to note, however, that the complexity of STAB-EXT-INF when \( S = \emptyset \) is less clear: while we can deduce the absence
of any stable extension in \( H \) from \( \alpha_X = (\bot)^{|X|} \), we cannot deduce that one does exist if \( \alpha_X \) is not of this form. We
conjecture that, in fact, STAB-EXT-INF is \( \mathcal{NP} \)-complete even for instances \( (H, \emptyset, \alpha_X) \). We note that if correct, this provides a rare, albeit arguably ‘unnatural’, example of a
problem where a decision concerning preferred extensions is ‘easier’ than the corresponding decision regarding stable
extensions.

4 Conclusions

The principal focus of this article has been in deriving neg-
ative results concerning various mechanisms for reducing
the complexity of deciding \( S \in \mathcal{PE}(H) \) though ‘expensive’ precomputation. Thus for a rather general notion of ‘useful encoding scheme’ it has been shown that such approaches are unlikely to succeed. There remains, of course, the possibility that specific sub-classes of argument system are amenable to concise encoding approaches.

Another direction for further work arises from the fact
that the requirement for \( \langle q, P, Q \rangle \) to be effective is rather strong and, as we have seen in Theorem 3 does not apply to what might be regarded as otherwise ‘reasonable’ approaches. One possible encoding approach concerning which Theo-
rem 3 will not in general apply is the following.

Recall that \( f_H(X_n) \) is the propositional logic function for
which \( f_H(\alpha_S) = \top \) if and only the subset \( S \) of \( X \) indicated
by \( \alpha_S \) is in \( \mathcal{PE}(H) \). Given \( f_H(X_n) \) one might represent this
using a suitable propositional formula. Of course, there are
(infinitely) many equivalent formulae in this regard. Sup-
pose for a propositional formula \( \Phi(X_n) \) over the (binary op-
eration) basis of \( \{\land, \lor, \neg\} \) we define the length of \( \Phi(X_n) \)
as its total number of occurrences of literals., denoting this
\( |\Phi(X_n)| \). Now consider the following measures.

\[
L(H) \xlongleftarrow{def} \min \{ |\Phi(X_n)| : \Phi(X_n) \text{ represents } f_H(X_n) \}
\]

\[
L(n) = \max_{H(X_n, A)} \{ L(H) \}
\]

Informally, \( L(n) \) is given by: for each different \( n \)-argument
\( H \) identify the shortest formula representing \( f_H \); \( L(n) \) is then the maximum of these values. The function \( L(n) \) is
well-defined (and computable, albeit by highly infeasible
mechanisms). It is certainly the case that encoding \( \mathcal{PE}(H) \)
by a propositional formula \( \Phi(X_n) \) is an extension tractable
approach: to test \( S \in \mathcal{PE}(H) \) simply evaluate \( \Phi(\alpha_S) \). This
is unlikely to be effective: given \( w \in \{0, 1\}^* \), even though
(assuming some standard encoding of propositional formu-
lae, e.g. \( \Phi \) p.273) one could determine whether \( w \) encodes
some \( \Phi \), it is unlikely that one can test within \( \mathcal{NP} \) if this
represents \( f_H \). In summary we have open the possibility that
propositional formulae offer concise encodings of \( \mathcal{PE}(H) \)
since such are not ruled out by Theorem 3.

**Problem 1** Do propositional formulae admit concise \( \mathcal{PE} \)
encodings, i.e. is there any \( k \in \mathbb{N} \) such that \( L(n) = O(n^k) \)?

There are two points worth considering concerning Problem 3. First, the classic information-theoretic argument of Riordan and Shannon [12], (cf [6, pp.273–274]) does not help in proving superpolynomial lower bounds: even if it is assumed that each distinct \( n \) argument system encodes a different preferred extension set, the lower bound on \( L(n) \) implied by this is only \( n^2 / \log n \). A second point concerns the class of propositional functions being addressed: work of Lupanov[10] indicates how formulae for proposi-
tional functions satisfying certain ‘inheritance’ properties can be constructed. This approach – the so-called ‘Prin-
ciple of Local Coding’ – allows ‘small’ formulae to be
built for suitable classes of functions provided that specific
small formulae used in the approach can also be built. 
An overview of the mechanism is given in [4, Chapter 3, pp. 136–8]. If one considers the class $\mathcal{G} = \bigcup_{n=1}^{\infty} \{\mathcal{F}_n\}$ in which $\mathcal{F}_n$ is the set of $n$-variable propositional functions $f_{n}$ for systems $\mathcal{H}$ of $n$ arguments, then it may be possible to show that $\mathcal{G}$ has the required ‘inheritance’ property and that this, given suitable subsidiary formula constructions might lead to concise encoding schemes. It should be noted, however, even if this route is possible, it is likely to be the case that generating and verifying the correctness of resulting formulae may well be computationally demanding (although such a process need only be performed once with respect to any given $\mathcal{H}$).

As a final open question we mention the following decision problem. As our starting point for building an encoding of $\mathcal{PE}(\mathcal{H})$ we have assumed that the argument system $\mathcal{H}$ is provided as the instance. One might ‘relax’ this and assume that an arbitrary subset of $\wp(\mathcal{X}_n)$ is given and we wish to encode only those subsets that correspond to $\mathcal{PE}(\mathcal{H})$ for some $\mathcal{H}(\mathcal{X}, \mathcal{A})$. Thus, we have the following decision problem:

**REALISABLE**

**Instance:** $\mathcal{S} = \{s_1, s_2, \ldots, s_k\} \subseteq \wp(\mathcal{X}_n)$.

**Question:** Does there exist an argument system $\mathcal{H}(\mathcal{X}_n, \mathcal{A})$ for which $\mathcal{PE}(\mathcal{H}) = \mathcal{S}$?

**Problem 2** Determine the complexity classification of REALISABLE.

We conjecture that REALISABLE $\in$ P, which would follow by proving that the following condition (which is easily shown to be necessary) is also sufficient for $\mathcal{S}$ to be realisable.

$$\forall T \subseteq \mathcal{X}_n \forall \{x, y\} \subseteq T \Rightarrow (\exists s_i \in \mathcal{S}, s_i \subseteq T) \Rightarrow (\forall s_i \in \mathcal{S} \{x, y\} \not\subseteq s_i)$$

This condition can be tested in polynomial-time simply by restricting $T$ to range over those supersets of $s_i \in \mathcal{S}$ formed by adding a single new argument.

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