Fermionic Quasi-Particle Representations

for Characters of \( \frac{(G^{(1)\,1}) \times (G^{(1)\,1})}{(G^{(1)\,2})} \)

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Abstract

We present fermionic quasi-particle sum representations for some of the characters (or branching functions) of \( \frac{(G^{(1)\,1}) \times (G^{(1)\,1})}{(G^{(1)\,2})} \) for all simply-laced Lie algebras \( G \). For given \( G \) the characters are written as the partition function of a set of rank \( G \) types of massless quasi-particles in certain charge sectors, with nontrivial lower bounds on the one-particle momenta. We discuss the non-uniqueness of the representation for the identity character of the critical Ising model, which arises in both the \( A_1 \) and \( E_8 \) cases.
1. Introduction

Recently new sum representations for branching functions of the algebra $A^1_3$, which appear in the (non-diagonal) modular invariant partition function of the coset conformal field theory (CFT) $(A_{(1)}^1)_{1} \times (A_{(1)}^1)_{2}$, have been obtained from an analysis of Bethe’s equations for the antiferromagnetic 3-state Potts chain [1].

To explain this result consider the sum

$$S_G(q) \equiv \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{q^m C_G^{-1} m^t}{(q)_{m_1} \cdots (q)_{m_n}} \equiv \sum_{m=0}^{\infty} \frac{q^m C_G^{-1} m^t}{(q)_{m_1} \cdots (q)_{m_n}},$$

where $m = (m_1, \ldots, m_n)$, $(q)_m = \prod_{k=1}^{m} (1 - q^k)$ for $m > 0$, $(q)_0 = 1$, and $C_G$ is the Cartan matrix of the simply-laced Lie algebra $G$ of rank $n$. In ref. [1] it was found (and verified up to $O(q^{200})$) that

$$S_{A^1_3}(q) \equiv \sum_{m=0}^{\infty} \frac{q^{(3m_1^2 + 4m_2^2 + 3m_3^2 + 4m_1m_2 + 2m_1m_3 + 4m_2m_3)/4}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}}$$

$$= q^{1/24} [b^0_0(q; 3) + 2b^0_2(q; 3) + b^0_4(q; 3)],$$

where the $b^l_m(q; n)$ are the branching functions of $(A_{(1)}^1)_{1} \times (A_{(1)}^1)_{2}$, which are equal by level-rank duality [3] to the branching functions of $(A_{(1)}^1)_{n+1}/U(1)$ given in (2.4) below. Moreover, restricting the sum on the first line of (1.2) to

$$m_1 + 2m_2 + 3m_3 \equiv Q \pmod{4}, \quad \text{with } Q = 0, \pm 1, 2,$$

(1.3)
gives separately $b^0_0$, $b^0_2$, and $b^0_4$ (see eqs. (3.13) and (3.19) of [1], where the notation $m = (m_{2s}, m_{ns}, m_{-2s})$ was used). The sum representations of other characters in this theory are also presented in [1].

Here we present the $ADE$ generalization of (1.2) and (1.3), corresponding to the cosets $(G_{(1)}^1)_{1} \times (G_{(1)}^1)_{1}/(G_{(1)}^1)_{2}$, where $G$ is an arbitrary simply-laced Lie algebra. The possibly restricted sums of the form (1.1) again are branching functions, and we discuss their interpretation. Similar expressions for characters of the Virasoro minimal CFTs $M(2, 2n + 3)$ as the sum side of the Andrews-Gordon identities [9], which may be thought of as associated with the algebras $A^1_{2n}$, have appeared in ref. [10].
2. ADE Generalization

The (restricted) sums (1.1) will turn out to be characters of the identity and certain other extended primary fields in the corresponding coset CFTs, which are the first models in the \( W^G \) unitary series \[5\] \[11\]. In the \( A_n \) case these CFTs are also known to describe \( \mathbb{Z}_{n+1} \) parafermions \[12\] \[13\] \[14\] of central charge \( c_n = \frac{2n}{n+3} \). In the \( D_n \) case they are \[5\] \[15\] the points \( r = \sqrt{\frac{n}{2}} \) on the orbifold line (or gaussian line, depending on the modular invariant combination of characters in the partition function) of \( c = 1 \) CFTs. These CFTs are known to be minimal with respect to extended chiral algebras \[16\] whose characters are essentially theta functions, see below. The CFTs in the \( E_n \) case, with \( n = 6, 7, 8 \), coincide \[5\] \[11\] with the \( c = \frac{6}{7}, \frac{7}{10}, \frac{1}{2} \) Virasoro minimal models, respectively, and therefore the branching functions reduce to various Virasoro characters.

Our results are as follows:

\( A_n \) (\( n \geq 1 \)): Here the quadratic form in (1.1) reads

\[
\mathbf{m} C_{A_n}^{-1} \mathbf{m}^t = \frac{1}{n+1} \left( \sum_{a=1}^{n} a(n+1-a)m_a^2 + 2 \sum_{1 \leq a < b \leq n} a(n+1-b)m_a m_b \right) .
\] (2.1)

For this algebra the generalization of (1.2) is

\[
S^Q_{A_n}(q) = q^{c_n/24} b^0_{2Q}(q; n) ,
\] (2.2)

where the superscript \( Q \) on \( S \) indicates restriction of the summation in (1.1) to

\[
\sum_{a=1}^{n} m_a a \equiv Q \pmod{n+1}
\] (2.3)

with \( Q = 0, 1, \ldots, n \). The branching function on the rhs of (2.2) corresponds to the \( \mathbb{Z}_{n+1} \)-parafermionic primary field of weight \( Q(1 - \frac{Q}{n+1}) \). In general \[8\],

\[
b^l_m(q; n) = q^{l(l+2)/4(n+3)-m^2/4(n+1)+1/4(n+3)} \eta^{-2} \times \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{r(r+1)/2+s(s+1)/2+rs(n+2)} \times \left\{ q^{r(l+m)/2+s(l-m)/2} - q^{n+2-l+r|n+2-2-(l+m)/2|+s[n+2-(l-m)/2]} \right\}
\] (2.4)
when \( l - m \in 2\mathbb{Z} \), \( b_{m}^{l} = 0 \) otherwise, and

\[
\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)
\]

is the Dedekind eta function. In the \( n = 1, 2 \) cases the branching functions coincide with certain characters \( \chi_{r,s}^{(m)}(q) \) of the highest weight \( \Delta_{r,s}^{(m)} = \frac{(r(m+1)-sm)^2-1}{6m(m+1)} \) irreducible representations of the Virasoro algebra at central charge \( c = 1 - \frac{6}{m(m+1)} \) \[17\]

\[
\chi_{r,s}^{(m)}(q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} \left( q^{\Delta_{r+2km,s}^{(m)}} - q^{\Delta_{r+2km,-s}^{(m)}} \right), \quad (2.6)
\]

namely,

\[
\begin{align*}
S_{A_1}^{0}(q) &= \sum_{m=0}^{\infty} \frac{q^{m^2/2}}{(q)_m} = q^{1/48} \chi_{1,1}^{(3)}(q), \quad S_{A_1}^{1}(q) = \sum_{m=1}^{\infty} \frac{q^{m^2/2}}{(q)_m} = q^{1/48} \chi_{1,3}^{(3)}(q), \\
S_{A_2}^{0}(q) &= q^{1/30} [\chi_{1,1}^{(5)}(q) + \chi_{1,5}^{(5)}(q)], \quad S_{A_2}^{\pm 1}(q) = q^{1/30} \chi_{1,3}^{(5)}(q). \quad (2.7)
\end{align*}
\]

When considered as characters of \((A_1^{(1)})_{n+1}/U(1)\), the formula (2.2) is that of Lepowski and Primc \[13\].

**D_\text{n} (n \geq 3):** In this case

\[
\begin{align*}
\mathbf{m}C_{D_n}^{-1} \mathbf{m}^t &= \sum_{a=1}^{n-2} a m_a^2 + \frac{n}{4} (m_{n-1}^2 + m_n^2) + 2 \sum_{1 \leq a < b \leq n-2} a m_a m_b \\
& \quad + \sum_{a=1}^{n-2} a m_a (m_{n-1} + m_n) + \frac{n-2}{2} m_{n-1} m_n , \quad (2.8)
\end{align*}
\]

and the result is

\[
S_{D_n}^{Q}(q) = q^{1/24} f_{n,nQ}(q) \quad (2.9)
\]

with \( Q = 0, 1 \), where summation is restricted to

\[
m_{n-1} + m_n \equiv Q \pmod{2}, \quad (2.10)
\]

and the characters

\[
f_{n,j}(q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} q^{(k+\frac{j}{2n})^2} \quad (2.11)
\]
with \( j = 0, 1, \ldots, n \) correspond to the extended primary fields of weight \( \frac{j^2}{2n} \). Note that due to the coincidence \( D_3 = A_3 \) the expressions (2.2) and (2.9) are related when \( n = 3 \) by (cf. [1] [18]) \( S^0_{D_3} = S^0_{A_3} + S^2_{A_3} \) and \( S^1_{D_3} = 2S^1_{A_3} \).

**E_6**: With a suitable labeling of roots we have

\[
C^{-1}_{E_6} = \begin{pmatrix}
4/3 & 2/3 & 1 & 4/3 & 5/3 & 2 \\
2/3 & 4/3 & 1 & 5/3 & 4/3 & 2 \\
1 & 1 & 2 & 2 & 2 & 3 \\
4/3 & 5/3 & 2 & 10/3 & 8/3 & 4 \\
5/3 & 4/3 & 2 & 8/3 & 10/3 & 4 \\
2 & 2 & 3 & 4 & 4 & 6
\end{pmatrix}.
\]  

(2.12)

Here we find

\[
S^0_{E_6}(q) = q^{c/24} \left[ \chi_{1,1}^{(6)}(q) + \chi_{5,1}^{(6)}(q) \right], \quad S^\pm_{E_6}(q) = q^{c/24} \chi_{3,1}^{(6)}(q),
\]

(2.13)

with the restrictions

\[
m_1 - m_2 + m_4 - m_5 \equiv Q \pmod{3}
\]

(2.14)

and \( c = 6/7 \). The Virasoro characters are as in (2.6), the relevant weights being \( \Delta_{1,1}^{(6)} = 0 \), \( \Delta_{5,1}^{(6)} = 5 \), and \( \Delta_{3,1}^{(6)} = 4/3 \).

**E_7**: Here we can write

\[
C^{-1}_{E_7} = \begin{pmatrix}
3/2 & 1 & 3/2 & 2 & 2 & 5/2 & 3 \\
1 & 2 & 2 & 2 & 3 & 3 & 4 \\
3/2 & 2 & 7/2 & 3 & 4 & 9/2 & 6 \\
2 & 2 & 3 & 4 & 4 & 5 & 6 \\
2 & 3 & 4 & 4 & 6 & 6 & 8 \\
5/2 & 3 & 9/2 & 5 & 6 & 15/2 & 9 \\
3 & 4 & 6 & 6 & 8 & 9 & 12
\end{pmatrix}
\]

(2.15)

and

\[
S^0_{E_7}(q) = q^{c/24} \chi_{1,1}^{(4)}(q), \quad S^1_{E_7}(q) = q^{c/24} \chi_{3,1}^{(4)}(q).
\]

(2.16)

The restrictions in this case are

\[
m_1 + m_3 + m_6 \equiv Q \pmod{2},
\]

(2.17)

\( c = 7/10 \), and \( \Delta_{1,1}^{(4)} = 0 \), \( \Delta_{3,1}^{(4)} = 3/2 \).
$E_8$: Finally in this case

$$C^{-1}_{E_8} = \begin{pmatrix} 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 4 & 4 & 5 & 6 & 7 & 8 & 10 \\ 3 & 4 & 6 & 6 & 8 & 8 & 10 & 12 \\ 3 & 5 & 6 & 8 & 9 & 10 & 12 & 15 \\ 4 & 6 & 8 & 9 & 12 & 12 & 15 & 18 \\ 4 & 7 & 8 & 10 & 12 & 14 & 16 & 20 \\ 5 & 8 & 10 & 12 & 15 & 16 & 20 & 24 \\ 6 & 10 & 12 & 15 & 18 & 20 & 24 & 30 \end{pmatrix} \quad (2.18)$$

and

$$S_{E_8}(q) = q^{c/24} \chi^{(3)}_{1,1}(q) , \quad (2.19)$$

is the character of the identity ($\Delta = 0$) representation of the Virasoro algebra with $c = 1/2$.

The above generalizations of $[1]$ have been verified using Mathematica$^\text{TM}$ for all simply-laced Lie algebras of rank $n \leq 8$ up to order $q^{100}$.

It would of course be most desirable to have direct proofs of the $D_n$ and $E_n$ sum representations for the characters, as for the $A_n$ case $[13]$.

3. Quasi-Particle Representation

All the above representations for branching functions may by put in a fermionic quasi-particle form by following the reverse of the procedure of $[1]$. Let $P_d(m, N)$ denote the number of ways that a positive integer $N$ can be additively partitioned into $m$ distinct positive integers. (We set $P_d(0, 0) = 1$ and $P_d(m, 0) = P_d(0, N) = 0$ for $m, N > 0$.) Then using

$$\sum_{N=0}^{\infty} P_d(m, N) q^N = \frac{q^{m(m+1)/2}}{(q)_m} \quad (3.1)$$

we rewrite (1.1) as

$$S_G(q) = \sum_{N=0}^{\infty} \sum_{m=0}^{\infty} \left( \prod_{a=1}^{N} P_d(m_a, N_a) \right) q^{\sum_{a=1}^{N} [N_a + m_a - \frac{m_a^2}{2}]} \quad (3.2)$$

Fermionic quasi-particles are characterized by a set of single particle energy levels $e_a(P)$ and an additive composition law for multi-particle energy gaps above the ground state

$$\hat{E}({\{P}\}) \equiv E_{\text{ex}}({\{P}\}) - E_{GS} = \sum_{a, j_a, \text{rules}} e_a(P_a, j_a) \quad , \quad (3.3)$$
and their momenta

\[ P(\{P\}) - P_{GS} = \sum_{a,j_a}{P_{a,j_a}}. \]  

(3.4)

One of the rules of composition is the fermionic exclusion rule

\[ P_{a,i_a} \neq P_{a,j_a} \quad \text{for} \quad i_a \neq j_a. \]  

(3.5)

To specify the \( P_{a,j_a} \) we consider the system to be in a box of size \( L \) which serves as an infrared cutoff and quantizes the \( P_i \) in units of \( 2\pi/L \). Taking a continuum limit, the massless single particle energies \( e_a \) all reduce to

\[ e_a(P) = v|P|, \]  

(3.6)

where \( v \) is called the speed of sound (or light). Considering the partition function for left moving excitations, with the ground state energy scaled out,

\[ Z(q) = \sum e^{-\hat{E}/kT}, \]  

(3.7)

and setting

\[ q = e^{-2\pi v/LkT}, \]  

(3.8)

we see that (3.2) results if the \( P_{a,j_a} \) are restricted by

\[ P_{a,j_a} = P_{a}^{\text{min}}(m) + \frac{2\pi}{L} k_{j_a} \]  

(3.9)

where \( k_{j_a} \) are distinct nonnegative integers and

\[ P_{a}^{\text{min}}(m) = \frac{2\pi}{L} \left[ \frac{1}{2} + \frac{1}{2} \sum_{b=1}^{n} N_{ab} m_b \right], \]  

(3.10)

in terms of the matrix \( N = 2C_G^{-1} - 1 \). This generalizes the infrared anomaly rule (3.15) of [1].

A particularly interesting example of the quasi-particle representation is provided by comparing the \( E_8 \) sum with the \( Q = 0 \) sector of the \( A_1 \) sum, both of which give the identity character of the critical Ising model. This demonstrates the important fact that the fermionic quasi-particle representation is not unique, in general. In the \( A_1 \) case \( C_{A_1} = 2 \), so the momentum restrictions (3.10) are simply those of a free fermion, \( P^{\text{min}} = \).
π/L independent of m. In contrast, the E₈ representation involves 8 quasi-particles with nontrivial momentum restrictions. We compare these representations in tables 1 and 2, where we show the particle content up to order q³⁰. The order q³⁰ is chosen because this is the lowest order where all E₈ quasi-particle types make a nonvanishing contribution. One pattern that is seen to emerge from the two tables, namely that the lowest level where a 2N-particle state appears in the A₁ spectrum is also the lowest level a state of N quasi-particles of type 1 appears in the E₈ spectrum, is easily shown to persist for all N. In table 3 we give more details about the structure of the E₈ representation up to order q¹⁴.

The meaning of the restrictions (2.3), (2.10), (2.14), (2.17) is understood in the above picture if we assign the quasi-particles charges with respect to a symmetry of the Dynkin diagram of the affine extension G⁽¹⁾ of G. In the Aₙ case this symmetry is Z₂ × Zₙ₊₁ (except for the n=1 case where it is just Z₂) and the quasi-particle of type a carries a Zₙ₊₁ charge a, quasi-particles a and n + 1 − a being C-conjugates of each other. Thus Q = ∑ₙ₌₁ⁿ mₐₐ is the total charge of the system, and S₊₉(q) is the partition function in the corresponding charge sector. In the Dₙ case we need to utilize just a Z₂ subgroup of the full symmetry of the D⁽¹⁾ₙ Dynkin diagram, under which the quasi-particles n − 1 and n are charged while all others are neutral. In the E₆ case the symmetry is Z₂ × Z₃, with the Z₃-charge assignment 1, −1, 0, 1, −1, 0 for the quasi-particles a = 1, . . . , 6, respectively. Finally, the E₇ system exhibits a Z₂ symmetry, under which quasi-particles 1, 3, and 6 are odd, the rest being even. It can easily be seen that mC⁻¹₉m₉(mod 1) is constant in every charge sector — as it should, since the branching functions have a power series expansion in q (up to an overall possibly fractional power) — while generically they are different in sectors of different charge.

4. Extracting the Central Charge

The leading behavior of any character χ(q) of a CFT for q → 1⁻ is determined by the effective central charge \( \tilde{c} = c - 12d_{\text{min}} \) (where \( d_{\text{min}} \) is the smallest scaling dimension in the theory):

\[
\chi(q) \approx \tilde{q}^{-\tilde{c}/24} \quad \text{as} \quad q \to 1^-,
\]

(4.1)
where \( \tilde{q} = e^{-2\pi LkT/v} \) for \( q \) given by (3.8). In the cases at hand, where the characters are of the form (1.1), \( \tilde{c} \) can be evaluated along the lines of [21]. The result can be written as follows (cf. [20]): For an \( n \times n \) symmetric matrix \( B \),

\[
\sum_{m=0}^{\infty} \frac{q^{mBm^t}}{(q)_{m_1} \cdots (q)_{m_n}} \simeq \tilde{q}^{-c/24} \quad \text{as } q \to 1^- ,
\]

where

\[
\tilde{c} = \frac{6}{\pi^2} \sum_{a=1}^{n} \mathcal{L} \left( \frac{x_a}{1 + x_a} \right) .
\]

Here

\[
\mathcal{L}(z) = -\frac{1}{2} \int_{0}^{z} dt \left[ \ln \frac{1}{1-t} + \frac{\ln(1-t)}{t} \right]
\]

is the Rogers dilogarithm function [22], and the \( x_a \) satisfy the equations

\[
x_a = \prod_{b=1}^{n} (1 + x_b)^{-N_{ab}} ,
\]

with \( N = 2B - 1 \).

In the cases of interest here, where \( B \) is the inverse Cartan matrix of a simply-laced algebra \( G \), eqs. (4.3) and (4.5) have been encountered and solved previously in the context of 1) purely elastic scattering theories and their ultraviolet limits [13] and 2) the finite-temperature thermodynamics of RSOS spin chains [23]. Eqs. (4.5) have a unique real solution. Inserting this solution into (4.3) and using appropriate sum rules [24][23][15] for the Rogers dilogarithm the known central charges of the \( \frac{(G^{(1)})_1 \times (G^{(1)})_2}{(G^{(1)})_2} \) coset CFTs are obtained.

In [20] eqs. (4.1)–(4.5) were applied to the non-unitary Virasoro minimal models \( M(2,2n+3) \) of central charge \( c = -\frac{2n(6n+5)}{2n+3} \). The sum side of the Andrews-Gordon identities [9] takes the role of \( S_G(q) \) in this case, the product side being a way of writing the characters of the primary fields in these CFTs [10]. The sum for the character of the smallest dimension field is of the form (1.1) if one takes \((2 - I_n)^{-1}\) instead of the inverse Cartan matrix, where \( I_n \) is the “generalized incidence matrix” for the twisted affine Lie algebra \( A_{2n}^{(2)} \) introduced in [15] (it differs from that of the Lie algebra \( A_n \) only by a 1 on the last entry of the diagonal, i.e. it is the incidence matrix of the tadpole graph \( A_{2n}/\mathbb{Z}_2 \)). As a generalization of an observation made in [1], we note that this sum can be obtained from the one in \( S_{D_{n+2}}(q) \) by avoiding the summation over \( m_{n+1} \) and \( m_{n+2} \).
(i.e. setting both to zero) due to the fact that the upper-left $n \times n$ minor of $C_{D_{n+2}}^{-1}$ is precisely $(2 - I_n)^{-1}$. Eqs. (4.1)–(4.5) then give the effective central charges $\tilde{\ell} = \frac{2n}{2n + 3}$ of these theories, cf. [13][20]. Our discussion in sect. 3 extends in a straightforward way to give a quasi-particle interpretation of the models $M(2, 2n + 3)$ (cf. [1] for the case $n=1$).

5. Discussion

In [1] the fermionic quasi-particle representation (1.2) of the branching functions was found by studying the spectrum of the antiferromagnetic 3-state Potts chain, realized as an RSOS model based on the Dynkin diagram of $D_4$. This $D_4$ model is obtained by an orbifold construction [25][26] from the $A_5$ RSOS model, and the antiferromagnetic spectrum corresponds to the boundary of the I/II regime. It is expected that the fermionic representation of the branching functions given above for $A_n$ can be obtained from a similar study of the $A_{n+2}$ RSOS models at the boundary of the I/II regime. Such a study would extend the computations of Klümper and Pearce [27] and would provide explicit results for the other characters not considered in this paper. We remark, however, that the analogous study of the ferromagnetic region of the $Z_{n+1}$ model of Fateev and Zamolodchikov [28], which has the identical $Z_{n+1}$ parafermionic branching functions as the I/II boundary of the $A_{n+2}$ RSOS models, gives a different fermionic representation [29] because the quasi-particle spectra of the lattice models are quite different.

The $ADE$ algebras that appear in the fermionic sum formulas presented here are the same as the algebras relevant for the integrable massive scattering theories discussed in [13], which describe certain perturbations of the corresponding $ADE$-related coset CFTs. For example, the energy perturbation of the critical Ising model, the well known Ising field theory [30], is related to $A_1$, and the magnetic perturbation of the Ising model describes a scattering theory with 8 particles, as discovered by A. Zamolodchikov [31]. Not only do the numbers of particles in the massive scattering theories agree with those in the quasi-particle picture, also the charge assignments for the quasi-particles in sect. 3 are consistent with those of the particles in the scattering theories. Furthermore, we note the similarity of (3.9), (3.10) with the UV limit (large rapidities) of the Bethe Ansatz equations for the diagonal scattering theories discussed in [13].

We therefore strongly suspect that different fermionic representations of the same branching functions are closely related to various integrable perturbations of the relevant
CFT. It is also interesting to see whether there is any relation between the massless quasi-particle interpretations suggested by our results and the massless $S$-matrix theories which were associated with certain CFTs in \cite{32} (where different $S$-matrices for a given CFT were also thought of as corresponding to different integrable perturbations of that CFT). We believe that characters of $\left(\frac{G^{(1)}}{G^{(2)}}\right)_1 \times \left(\frac{G^{(1)}_1}{G^{(2)}_2}\right)$ not discussed in this paper also have fermionic representations, as in the $A_n$ case discussed in \cite{13}.

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$$E L/2\pi v$$ | $d_{\hat{E}}$ | 2-particle | 4-particle | 6-particle
--- | --- | --- | --- | ---
0 | 1 | 1 | | |
1 | 0 | | | |
2 | 1 | 1 | | |
3 | 1 | 1 | | |
4 | 2 | 2 | | |
5 | 2 | 2 | | |
6 | 3 | 3 | | |
7 | 3 | 3 | | |
8 | 5 | 4 | 1 | |
9 | 5 | 4 | 1 | |
10 | 7 | 5 | 2 | |
11 | 8 | 5 | 3 | |
12 | 11 | 6 | 5 | |
13 | 12 | 6 | 6 | |
14 | 16 | 7 | 9 | |
15 | 18 | 7 | 11 | |
16 | 23 | 8 | 15 | |
17 | 26 | 8 | 18 | |
18 | 33 | 9 | 23 | 1 |
19 | 37 | 9 | 27 | 1 |
20 | 46 | 10 | 34 | 2 |
21 | 52 | 10 | 39 | 3 |
22 | 63 | 11 | 47 | 5 |
23 | 72 | 11 | 54 | 7 |
24 | 87 | 12 | 64 | 11 |
25 | 98 | 12 | 72 | 14 |
26 | 117 | 13 | 84 | 20 |
27 | 133 | 13 | 94 | 26 |
28 | 157 | 14 | 108 | 35 |
29 | 178 | 14 | 120 | 44 |
30 | 209 | 15 | 136 | 58 |

**Table 1:** Total, $d_{\hat{E}}$, and number of 2-, 4-, and 6-particle states contributing to the $A_1$ representation of the character $q^{1/48}^{(3)}\chi_{1,1}$ of the critical Ising model.
| \( \hat{E} L/2\pi v \) | \( d_{\hat{E}} \) | 1-particle | 2-particle | 3-particle |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | 0 | | | |
| 2 | 1 | 1 | | |
| 3 | 1 | 1 | | |
| 4 | 2 | 2 | | |
| 5 | 2 | 2 | | |
| 6 | 3 | 3 | | |
| 7 | 3 | 3 | | |
| 8 | 5 | 4 | 1 | |
| 9 | 5 | 4 | 1 | |
| 10 | 7 | 4 | 3 | |
| 11 | 8 | 4 | 4 | |
| 12 | 11 | 5 | 6 | |
| 13 | 12 | 5 | 7 | |
| 14 | 16 | 6 | 10 | |
| 15 | 18 | 6 | 12 | |
| 16 | 23 | 6 | 17 | |
| 17 | 26 | 6 | 20 | |
| 18 | 33 | 6 | 26 | 1 |
| 19 | 37 | 6 | 30 | 1 |
| 20 | 46 | 7 | 36 | 3 |
| 21 | 52 | 7 | 40 | 5 |
| 22 | 63 | 7 | 48 | 8 |
| 23 | 72 | 7 | 54 | 11 |
| 24 | 87 | 7 | 64 | 16 |
| 25 | 98 | 7 | 71 | 20 |
| 26 | 117 | 7 | 82 | 28 |
| 27 | 133 | 7 | 90 | 36 |
| 28 | 157 | 7 | 102 | 48 |
| 29 | 178 | 7 | 111 | 60 |
| 30 | 209 | 8 | 123 | 78 |

**Table 2:** Total, \( d_{\hat{E}} \), and number of 1-, 2-, and 3-particle states contributing to the \( E_8 \) representation of the character \( q^{1/48} \chi_{1,1}^{(3)} \) of the critical Ising model.
| $\hat{E} L/2\pi v$ | $d_{\hat{E}}$ | $m$ | $d_{\hat{E},m}$ | $P_{\text{min}}(m) L/2\pi$ |
|-----------------|-------------|-----|----------------|-------------------|
| 2               | 1           | $(1,0,0,0,0,0,0,0)$ | 1               | $(2,0,0,0,0,0,0,0)$ |
| 3               | 1           | $(1,0,0,0,0,0,0,0)$ | 1               | $(2,0,0,0,0,0,0,0)$ |
| 4               | 2           | $(1,0,0,0,0,0,0,0)$ | 1               | $(2,0,0,0,0,0,0,0)$ |
|                 |             | $(0,1,0,0,0,0,0,0)$ | 1               | $(0,4,0,0,0,0,0,0)$ |
| 5               | 2           | .... |                | ....             |
| 6               | 3           | .... |                | ....             |
|                 |             | $(0,0,1,0,0,0,0,0)$ | 1               | $(0,0,6,0,0,0,0,0)$ |
| 7               | 3           | .... |                | ....             |
| 8               | 5           | .... |                | ....             |
|                 |             | $(0,0,0,1,0,0,0,0)$ | 1               | $(0,0,0,8,0,0,0,0)$ |
|                 |             | $(2,0,0,0,0,0,0,0)$ | 1               | $(\frac{7}{2},0,0,0,0,0,0,0)$ |
| 9               | 5           | .... |                | ....             |
| 10              | 7           | .... |                | ....             |
|                 |             | $(2,0,0,0,0,0,0,0)$ | 2               | $(\frac{7}{2},0,0,0,0,0,0,0)$ |
|                 |             | $(1,1,0,0,0,0,0,0)$ | 1               | $(4,6,0,0,0,0,0,0)$ |
| 11              | 8           | .... |                | ....             |
|                 |             | $(2,0,0,0,0,0,0,0)$ | 2               | $(\frac{7}{2},0,0,0,0,0,0,0)$ |
|                 |             | $(1,1,0,0,0,0,0,0)$ | 2               | $(4,6,0,0,0,0,0,0)$ |
| 12              | 11          | .... |                | ....             |
|                 |             | $(2,0,0,0,0,0,0,0)$ | 3               | $(\frac{7}{2},0,0,0,0,0,0,0)$ |
|                 |             | $(1,1,0,0,0,0,0,0)$ | 3               | $(4,6,0,0,0,0,0,0)$ |
|                 |             | $(0,0,0,0,1,0,0,0)$ | 1               | $(0,0,0,0,12,0,0,0)$ |
| 13              | 12          | .... |                | ....             |
|                 |             | $(2,0,0,0,0,0,0,0)$ | 3               | $(\frac{7}{2},0,0,0,0,0,0,0)$ |
|                 |             | $(1,1,0,0,0,0,0,0)$ | 4               | $(4,6,0,0,0,0,0,0)$ |
| 14              | 16          | .... |                | ....             |
|                 |             | $(2,0,0,0,0,0,0,0)$ | 4               | $(\frac{7}{2},0,0,0,0,0,0,0)$ |
|                 |             | $(1,1,0,0,0,0,0,0)$ | 5               | $(4,6,0,0,0,0,0,0)$ |
|                 |             | $(0,0,0,0,0,1,0,0)$ | 1               | $(0,0,0,0,0,14,0,0)$ |
|                 |             | $(1,0,1,0,0,0,0,0)$ | 1               | $(5,0,9,0,0,0,0,0)$ |

**Table 3:** More details about the structure of the $E_8$ representation of table 2, namely the occupation number vectors $m$ contributing for a given energy gap $\hat{E}$, the degeneracy $d_{\hat{E},m}$ at given $\hat{E}, m$, and the minimal momenta $P_{\text{min}}(m)$ assembled into a vector $P_{\text{min}}(m)$, with $P_{\text{min}}(m) \equiv 0$ if $m_a = 0$. For $\hat{E} > 4$ we only show $m$ with $d_{\hat{E},m} > 1$ or those that have not appeared for smaller $\hat{E}$. 