DECOMPOSITION OF A LÉVY PROCESS SAMPLE PATH INTO BROWNIAN AND JUMP PARTS

JORGE GONZÁLEZ CÁZARES AND JEVGENIJS IVANOVS

Abstract. We introduce a simple procedure for recovering paths of the Brownian and jump components from high-frequency observations of a Lévy process. It relies on reordering of independently sampled normal increments and thus is fully non-parametric with no tuning parameters. The speed of convergence depends on the small-jump activity and is given in terms of the Blumenthal-Getoor index. Among the main reasons underlying the method are small time predominance of the Brownian component, exchangeable structures, and fast convergence of normal empirical quantile functions. Extensions to Itô semimartingales and to the multidimensional case are discussed. Numerical illustrations and examples are also provided.

1. Introduction

Consider a Lévy process $X$ on $[0,1]$ and the decomposition

$$X_t = Y_t + \sigma W_t, \quad t \in [0,1],$$

where $\sigma \geq 0$ and $W$ is a standard Brownian motion independent of the Lévy process $Y$ having no Brownian component. In this work we assume that $\sigma > 0$ and provide a method to recover $W$, and thus also $Y$, from a given sample path of $X$ or rather its high-frequency observations $(X_{i/n})_{i=0,...,n}$ as $n \to \infty$. More precisely, we recover the path of the bridge $(W_t - W_1 t)_{t \in [0,1]}$ and the path of the drifted process $(Y_t + \sigma W_1 t)_{t \in [0,1]}$. Importantly, our method does not rely on the knowledge of the law of $X$ apart from the parameter $\sigma$, and the latter can be efficiently estimated from the given high-frequency observations [1, 8].

Apart from its intrinsic interest, our result may be useful in a variety of applied areas. Oftentimes $\sigma W$ is interpreted as noise, see e.g. [3, 5, 14, 16, 17], and thus the proposed procedure yields the signal $Y$ up to an unknown linear drift. Furthermore, various statistical procedures may benefit from pre-separation of the Brownian part. According to [15, §5] ‘coexistence of the Gaussian part and the jump part makes the parametric estimation problem much more difficult and cumbersome’ and the common strategy then is to use thresholding. As was exemplified in [20] through simulations, a naive choice of the threshold may severely deteriorate estimation performance. Thus our procedure may be used to avoid the difficult practical problem of threshold selection. Furthermore, it can be employed as an alternative to [12, 13] to detect the presence of jumps.

Our method amounts to the following simple algorithm:

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1. simulate an independent standard Brownian motion $W'$ on the grid $(i/n)_{i=1,...,n}$,

2. reorder the increments of $W'$ according to the order of the increments of $X$.

The resultant skeleton $W^{(n)}$ then satisfies:

$$(W_t - W_t^{(n)})_{t \in [0,1]} \overset{P}{\rightarrow} ((W_1 - W_1') t)_{t \in [0,1]}$$

in supremum norm. In words, we recover the Brownian evolution up to some linear drift. In fact, we have a more general result in Theorem 1 establishing the speed of convergence, see also Figure 2 for a numerical illustration. Figure 1 illustrates the algorithm in the case $\sigma = 1$ and $Y$ being a variance gamma process. In addition, we remove the random drift in the approximation $X - W^{(n)}$ of $Y$ by matching the endpoints. This is not possible in practice, and it is done here to make the assessment of signal recovery easier.

Let us provide some intuition. On the small scale the overwhelming number of the increments of $X$ are close to those of $\sigma W$. By self-similarity the scaled increments of $W$ are i.i.d. standard normals, and the respective empirical quantile function exhibits fast convergence due to light tails of the normal distribution. On an intuitive level this explains that ordering the increments of $W'$ according to the increments of $W$ or $X$ may produce a well-coupled process. Nonetheless, the result may still look surprising even for the purely Brownian case. In some sense, a
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The path of the Brownian bridge is determined by the ordering of its infinitesimal increments. Interestingly, in the case of no Brownian component ($\sigma = 0$) our procedure results in a standard Brownian motion independent of the original process $X$, see Proposition 2.

2. THE MAIN RESULT: CONVERGENCE SPEED AND LIMIT LAWS

Denote the Lévy triplet (see [19, §2, Def. 8.2]) of $X$ by $(\gamma, \sigma^2, \Pi(dx))$ and write $\Pi(x) = \Pi(\mathbb{R} \setminus (-x, x))$ for any $x > 0$. The quality of decomposition of the path of $X$ crucially depends on the activity of small jumps. Therefore, we define two indices $0 \leq \beta_* \leq \beta^* \leq 2$ capturing some main characteristics:

$$\beta^* = \inf \left\{ p \geq 0 : \int_{(-1,1)} |x|^p \Pi(dx) < \infty \right\},$$

$$\beta_* = \inf \left\{ p \geq 0 : \liminf_{x \downarrow 0} x^p \Pi(x) = 0 \right\}.$$

The index $\beta^*$ is known as the Blumenthal-Getoor index, whereas $\beta_*$ reminds the Pruitt’s index [18] and, in fact, the latter must be in-between $\beta_*$ and $\beta^*$. Importantly, $\beta_* = \beta^*$ under some weak regularity assumptions, such as regular variation of $\Pi(x)$ at 0 with some index $-\alpha$. In the latter case

$$\beta^* = \beta_* = \alpha,$$

which readily follows from the standard theory [2, §1].

As mentioned before, we consider a standard Brownian motion $W'$ independent of $X$ and thus of $W$. For each integer $n \in \mathbb{N}$ and function $f : [0, 1] \to \mathbb{R}$, we adopt the notation $\Delta_n f = f(i/n) - f((i-1)/n)$ for $i = 1, \ldots, n$. Let $\pi$ be the (random) permutation of the indices $1, \ldots, n$ such that the ordering of $\Delta_n \pi \cdot X$ coincides with that of $\Delta_n \cdot X$. In other words, if $s$ is a permutation such that $\Delta_n s \cdot X$ is an increasing sequence then also $\Delta_n \pi(s(i)) \cdot W'$ is an increasing sequence. Such permutation $\pi$ is a.s. unique since there are a.s. no ties in either sequence. Finally, we take the corresponding partial sum process

$$W_t^{(n)} = \sum_{i \leq nt} \Delta_n \pi(i) W', \quad t \in [0, 1],$$

which is a Brownian random walk at the jump times converging to a standard Brownian motion (one may keep the bridges between the discretization points so that this is a standard Brownian motion). Note, however, that the joint process $(W_t^{(n)}), i = 1, \ldots, n$, is not a random walk, but the increments are still exchangeable. This is easy to see from the following representation: first order the increments of $W$ and of $W'$ and then apply the same independent random permutation to both, see also [10, Prop. 1.8]. Next we state the main result.

Theorem 1. For $p \in (\beta^*, 2]$ (and $p = 2$ when $\beta^* = 2$) it holds that

$$n^{(2-p)/4} \sup_{t \in [0,1]} |W_t - W_t^{(n)} - (W_1 - W_1^{(n)})t| \xrightarrow{P} 0.$$

Moreover, this convergence fails for $p \in [0, \beta_*)$.

It is noted that the final statement of Theorem 1 implies that with some positive probability the quantity on the left hand side of (2) becomes arbitrarily large for some large $n$. Thus we establish the exact convergence rate in the logarithmic
sense when $\beta_* = \beta^*$ and, in particular, this rate is $n^{-(2-\alpha)/4}$ in the regularly varying case (1).

Note that Theorem 1 also implies the convergence of the bivariate approximation:

$$\left(W^{(n)}_t - W_1^{(n)}t, X_t - \sigma(W^{(n)}_t - W_1^{(n)}t)\right) \overset{\mathbb{P}}{\to} \left(W_t - W_1t, Y_t + \sigma W_1t\right)$$

in supremum norm with exactly the same rate. The problem of identifying varying case (1).

$\beta$ sense when

**Remark 1.** It suffices to let $W$ be a Brownian motion independent of $Y$ under some probability measure $Q$ dominating $P$, that is, $\mathbb{P} \ll Q$. Indeed, the limit in Theorem 1 holds under $Q$ and thus, under $\mathbb{P}$. For example, by Girsanov’s theorem, we may consider $W_t = B_t + \int_0^t b(s, B_s)ds, t \in [0,1]$, where $b$ is a bounded and measurable function and $B$ is a Brownian motion under $\mathbb{P}$ (see [11, §5.6]).

**Remark 2.** Further generalizations are possible. Namely, the convergence in (2) is guaranteed for any process $Y$ (possibly dependent on the Brownian motion $W$) such that the increments of the bivariate process $(W; Y)$ are exchangeable and $Y$ satisfies:

$$n^{2-p/2} \mathbb{E} \left( (\Delta^n Y)^2 \wedge \frac{\log n}{n} \right) \to 0.$$ 

It would be interesting to understand if exchangeability can be replaced by some other structural assumption. One way is to ensure that (6) below is sufficient because of inherent reorderings.

**Remark 3.** Our method readily applies in multivariate setting, where $W$ is an $\mathbb{R}^d$-valued Lévy process, $W$ is an independent standard $d$-dimensional Brownian motion and $\sigma$ is a $d \times d$ matrix (a fixed square-root of the known variance matrix $\Sigma$). Indeed, by applying the decomposition procedure to every element of $X$ we may recover the entire path of the bridge $(\sigma(W_t - tW_1))_{t \in [0,1]}$. In a degenerate case when the rank of $\sigma$ is smaller than $d$, we may use appropriate projections onto reals to reduce the number of required one-dimensional reconstructions to the given rank. Finally, generalizations discussed above still apply, see Remark 1 in particular.

### 2.1. Further results

We have a more precise result in the case when $Y$ is a piecewise constant process, which includes compound Poisson process. Note that one may always add a linear drift to $Y$ since this does not affect $W^{(n)}$. The following result is stated for the discrete skeleton, since it may fail otherwise. The reason is that the maximal deviation of $W$ from its discrete skeleton is of the same order:

$$\sup_{t \in [0,1]} |W_t - W_{\lfloor tn \rfloor/n}| = \Theta_\mathbb{P}(\sqrt{\log n/n}).$$

Throughout, the space $\mathcal{D}[0,1]$ is endowed with the standard Skorohod $J_1$-topology.

**Proposition 1.** Let $(Y_t)_{t \in [0,1]}$ be a piecewise constant process independent of $W$ with jumps $J_1, \ldots, J_N$ at times $T_1, \ldots, T_N$. Then the limit

$$\sqrt{\frac{n}{2 \log n}} \left(W_{\lfloor tn \rfloor/n} - W_1^{(n)}t - (W_1 - W_1^{(n)})t\right) \overset{\mathbb{P}}{\to} \sum_{i=1}^N \text{sign}(J_i)(t - 1_{(T_i \leq t)})$$

holds in $\mathcal{D}[0,1]$. 
In the purely Brownian case ($Y = 0$) the effective rate is $\sqrt{\log \log n}/n$, see Appendix B.

Our final result covers the case $\sigma = 0$. That is, we apply our procedure for a Lévy process without Gaussian part: $X = Y$. Interestingly, our ‘coupling’ results, in the limit, in a Brownian motion independent of $X$. We believe that the following result is true even in the highest activity case $\beta^* = 2$, but its proof seems to require a much more careful analysis.

**Proposition 2.** For $\sigma = 0$ and $\beta^* < 2$ we have the distributional convergence

$$(X_t, W_t^{(n)}) \overset{d}{\to} (X_t, B_t)$$

in $D[0, 1]$, where $B$ is a standard Brownian motion independent of $X$.

2.2. Numerical illustration. We conclude this section with numerical illustrations of Theorem 1 and Proposition 1. Suppose $\sigma = 1$, $W$ is a 3-dimensional Bessel process (a Brownian motion under an equivalent probability measure) and $Y$ is an independent strictly $\alpha$-stable process. We consider various values of $\alpha$ and, as the other parameters are less relevant, we fix the skewness parameter at $\beta = 0.5$ and take unit scale. We work with 5 approximation levels $n = 10^3, 10^4, \ldots, 10^7$ and for each scenario we compute the maximal difference between the discretised bridge on a uniform grid of $N = 10^7$ points and its approximation at level $n$:

$$\sup_{i \leq N} \left| W_{i/N} - \frac{i}{N} W_1 - (W_{i/N}^{(n)} - \frac{i}{N} W_1^{(n)}) \right|.$$  

We point out that we do not resample $W'$, i.e. we use the same path for each of the resolution levels $n$. We replicate the procedure 100 times to estimate the expected value of the quantity in (4) and its standard deviation, see Table 1. For the sake of comparison, we also take $n = 1$ giving $W^{(n)} = W'$ so that (4) quantifies the discrepancy between two independent bridges, and the result is $1.202(3333)$. Moreover, Figure 2 provides the log-log plot together with lines corresponding to

| $n$ | $\alpha = 0.2$ | $\alpha = 0.6$ | $\alpha = 1$ | $\alpha = 1.4$ | $\alpha = 1.8$ | $\alpha = 1.99$ |
|-----|----------------|----------------|-------------|-------------|-------------|-------------|
| $10^4$ | 1.1590(0.0430) | 1.1704(0.0438) | 1.1912(0.0598) | 2.2300(0.5534) | 3.2115(0.0825) | 3.3999(0.0786) |
| $10^5$ | 0.0626(0.0175) | 0.0741(0.0199) | 0.0980(0.0241) | 0.1468(0.0415) | 0.2623(0.0715) | 0.3061(0.0793) |
| $10^6$ | 0.0243(0.0086) | 0.0326(0.0083) | 0.0536(0.0137) | 0.1002(0.0306) | 0.2275(0.0649) | 0.2958(0.0793) |
| $10^7$ | 0.0090(0.0033) | 0.0145(0.0036) | 0.0290(0.0079) | 0.0704(0.0221) | 0.2007(0.0551) | 0.2905(0.0811) |
| $10^8$ | 0.0031(0.0017) | 0.0056(0.0018) | 0.0153(0.0043) | 0.0501(0.0160) | 0.1773(0.0511) | 0.2886(0.0821) |

**Table 1.** Means (and standard deviations) of the errors given by (4) when $Y$ is an $\alpha$-stable process

the theoretical rates given by Theorem 1. That is, the lines pass through the given value at $n = 10^5$ and their slopes are given by $-(2 - \alpha)/4$.

Next suppose $\sigma$ and $W$ are as in the previous paragraph but $Y$ is a Poisson process with intensity 3. The Figure 3 below exemplifies the limit established in Proposition 1 above. Note how the signs of the jumps in the limit are opposite to those of $Y$. 


3. Proofs

Consider the process of interest

\[
\left( n^{(2-p)/4} \left( W_t - W^{(n)}_t - (W^{1(n)}_1 - W^{1(n)}_t) \right) \right)_{t \in [0,1]},
\]

with \( p \in (0,2] \) and let \( \xi_{ni} \) be its \( i \)th increment, that is, we apply \( \Delta_{ni}^n \). According to (3) we may restrict our attention to the partial sum process \( \left( \sum_{i \leq tn} \xi_{ni} \right)_{t \in [0,1]} \).

Observe the identities

\[
\xi_{ni} = n^{(2-p)/4} \left( \Delta^n_{i} W - \Delta^n_{\pi(i)} W' - \frac{1}{n} \sum_{j \leq n} (\Delta^n_{j} W - \Delta^n_{j} W') \right)
\]

\[
= n^{-p/4} \left( Z_i - Z'_{\pi(i)} - \frac{1}{n} \sum_{j \leq n} (Z_j - Z'_{j}) \right).
\]
where $Z_1, \ldots, Z_n$ and $Z'_1, \ldots, Z'_n$ are i.i.d. standard normal variables. Recall that $\pi$ is the permutation such that $Z'_{\pi(i)}$ has the same ordering as

$$Z_i + \frac{1}{\sigma} \sqrt{n} \Delta_i^\pi Y.$$  

(5)

Additionally, we define the permutation $\nu$ so that $Z_{\nu(i)}$ is ordered according to (5) and thus the orderings of $Z_{\nu(i)}$ and $Z'_{\pi(i)}$ coincide. Consider the decomposition:

$$\xi_{ni} = \hat{\xi}_{ni} + \tilde{\xi}_{ni} = n^{-p/4} \left( Z_{\nu(i)} - Z'_{\pi(i)} - \frac{1}{n} \sum_{j \leq n} (Z_j - Z'_j) \right) + n^{-p/4} (Z_i - Z_{\nu(i)}),$$

and note that the second term does not depend on $W'$, whereas the first term corresponds essentially to comparing certain order statistics (the order is random and dependent on $X$).

The strategy is to split the analysis of the partial sum process of $\xi_{ni}$ into that of the partial sum processes of $\hat{\xi}_{ni}$ and $\tilde{\xi}_{ni}$. Importantly, $(\hat{\xi}_{ni})_{i=1,\ldots,n}$ and $(\tilde{\xi}_{ni})_{i=1,\ldots,n}$ are both exchangeable. In fact, this is true for any process $Y$ as long as $(\Delta^p_W, \Delta^p_X)$ is exchangeable. Thus the general theory in [10, Thm. 3.13] for exchangeable increment processes is applicable. In this respect, since $\sum_{i \leq n} \hat{\xi}_{ni} = \sum_{i \leq n} \tilde{\xi}_{ni} = 0$, we note that the convergence in probability of the partial sum processes of $\hat{\xi}_{ni}$ and $\tilde{\xi}_{ni}$ to 0 is equivalent to, respectively, the limits

$$\sum_{i \leq n} \hat{\xi}_{ni}^2 \overset{p}{\rightarrow} 0 \quad \text{and} \quad \sum_{i \leq n} \tilde{\xi}_{ni}^2 \overset{p}{\rightarrow} 0.$$  

(6)

Lemma 1 establishes the first limit for any $p > 0$. The second convergence depends on the choice of $p$: it holds for large enough $p$ and fails for sufficiently small $p$.

**Lemma 1.** Let $Z_{(1)} < \cdots < Z_{(n)}$ and $Z'_{(1)} < \cdots < Z'_{(n)}$ be two independent ordered sequences of $n$ standard normal random variables. Then

$$a_n \sum_{i \leq n} \left( Z_{(i)} - Z'_{(i)} - \frac{1}{n} \sum_{j \leq n} (Z_{(j)} - Z'_{(j)}) \right)^2 \overset{p}{\rightarrow} 0$$

whenever $a_n \log n \rightarrow 0$.

**Proof.** Letting $\mu_n$ be the inner sum of the statement we note that

$$\sum_{i \leq n} \left( Z_{(i)} - Z'_{(i)} - \frac{1}{n} \mu_n \right)^2 = \sum_{i \leq n} \left( Z_{(i)} - Z'_{(i)} \right)^2 - \frac{1}{n} \mu_n^2.$$  

Since $\mu_n/\sqrt{n} \sim N(0,2)$ has constant distribution we have $\mu_n^2/(n \log n) \overset{p}{\rightarrow} 0$. Thus it is left to prove that

$$a_n \sum_{i \leq n} \left( Z_{(i)} - Z'_{(i)} \right)^2 = a_n n \| F^{-1}_n - G^{-1}_n \|_2^2 \overset{p}{\rightarrow} 0,$$

where $\| f \|_2^2 = \int_0^1 f^2(x) dx$, and $F^{-1}_n$ and $G^{-1}_n$ are the right-inverses of the empirical distributions of $Z$ and $Z'$, respectively.

Let $\Phi$ denote the standard normal distribution and let $\tilde{F}_n = \sqrt{n}(F^{-1}_n - \Phi^{-1})$ and $\tilde{G}_n = \sqrt{n}(G^{-1}_n - \Phi^{-1})$ be the respective normalised empirical quantile processes. By Minkowski inequality we have

$$n \| F^{-1}_n - G^{-1}_n \|_2^2 = \| \tilde{F}_n - \tilde{G}_n \|_2^2 \leq 2 (\| \tilde{F}_n \|_2^2 + \| \tilde{G}_n \|_2^2).$$
From [4, Thm 4.6(ii)] it can be deduced that \( \| \tilde{\mathcal{F}}_n \|^2_2 / \log \log n \to 1 \). The same is true of \( \tilde{G}_n \), completing the proof. \( \Box \)

It can be shown that \( \log \log n \) is the “right” scale, see Appendix B. The following permutation Lemma (with \( q = 2 \)) is crucial to upper bound \( \sum_{i \leq n} (Z_i - Z_{\nu(i)})^2 \). In this lemma it is more convenient to swap \( \nu \) for \( \nu^{-1} \).

**Lemma 2.** Let \( z_1 \leq \cdots \leq z_n \) be \( n \geq 1 \) ordered real numbers. For arbitrary \( y_i \in \mathbb{R} \) consider a permutation \( \nu \) such that \( z_{\nu^{-1}(i)} + y_{\nu^{-1}(i)} \) is ordered. Then

\[
\sum_{i \leq n} |z_i - z_{\nu(i)}|^q \leq 2^q \sum_{i \leq n} (|y_i|^q \land m^q), \quad q \geq 1,
\]

with \( m = z_n - z_1 \).

**Proof.** Without loss of generality we assume that \( \nu \) has exactly one cycle, since otherwise we just sum over the cycles and increase the respective \( m \) if needed. Moreover, the result is trivial for a cycle of length 1.

It is a basic fact that

\[
|z_i - z_j|^q \leq (|y_i| + |y_j|)^q \leq 2^{q-1} (|y_i|^q + |y_j|^q)
\]

whenever \( i < j \) and \( \nu(i) > \nu(j) \) or \( i > j \) and \( \nu(i) < \nu(j) \), i.e., if the order is flipped. Furthermore, this bound is still true when \( |y_i| \) is replaced by \( |y_i| \land m \).

We call the ordered sequence \( \nu(i), \nu^2(i), \ldots \) the successors of \( i \). For each \( i \) satisfying \( i < \nu(i) \), we define:

\[
b(i) \text{ is the first successor of } i \text{ such that } \nu(b(i)) < \nu(i) \leq b(i),
\]

and note that \( b(i) \) is well defined, see Example 1. From (7) we have the bound:

\[
|z_i - z_{\nu(b(i))}|^q \leq |z_i - z_{\nu(i)}|^q \leq 2^{q-1} (|y_i|^q \land m^q + |y_{b(i)}|^q \land m^q).
\]

The case \( i > \nu(i) \) is analogous but with inequalities reversed in the definition of \( b(i) \). By summing up over all \( i \) we get the upper bound for \( \sum_{i \leq n} |z_i - z_{\nu(i)}|^q \). This bound needs to be reduced since the same \( b \) may appear multiple times.

Suppose \( i_1, \ldots, i_k \) with \( k > 1 \) are all the indices with

\[
b^* = b(i_1) = \cdots = b(i_k).
\]

Without loss of generality we assume that \( b^* > \nu(b^*) \) and so \( i_j < \nu(i_j) \) for all \( j = 1, \ldots, k \). Moreover, let the numbering be such that the path from \( i_1 \) to \( b^* \) passes through \( i_2, \ldots, i_k \) in this order. Note that \( i_2 < \nu(i_1) \) implies that \( b(i_1) \) occurs before \( i_2 \), a contradiction. Thus we have

\[
i_1 < \nu(i_1) \leq i_2 < \nu(i_2) \leq \cdots < i_k < \nu(i_k) \leq b^*.
\]

But then

\[
|z_{i_1} - z_{\nu(i_1)}|^q + \cdots + |z_{i_k} - z_{\nu(i_k)}|^q \leq |z_{i_1} - b^*|^q,
\]

implying that only one term \( 2^{q-1} (|y_{i_1}|^q \land m^q + |y_{b^*}|^q \land m^q) \) out of \( k \) is necessary. The proof is now complete. \( \Box \)

Note that the constant \( 2^q \) in front of the upper bound can not be reduced in general. For example, let \( q = n = 2 \) and \( z_1 = 0, z_2 = 1, y_1 = 1/2 + \epsilon, y_2 = -y_1 \) with some \( \epsilon > 0 \). Then \( z_1 + y_1 > z_2 + y_2 \) and the bound reads \( 2 \leq 2(1 + 2\epsilon)^2 \).

**Example 1.** Consider the permutation: \( 1 \to 2 \to 4 \to 3 \to 5 \to 1 \). The summary of indices is given below:
Note that the pair \((3, 5)\) was not used in the construction of our bound.

Finally, we need some estimates for the Lévy processes \(Y\).

**Lemma 3.** The following statements hold true for a Lévy process \(Y\) without Brownian component:

(a) For any positive sequence \(a_n \downarrow 0\) satisfying \(a_n\sqrt{n} \to \infty\), we have the limit
\[
P(|Y_{1/n}| > a_n) \to 0
\]
and the following bound for sufficiently large \(n\):
\[
nP(|Y_{1/n}| > a_n) \geq \frac{1}{2} \Pi(2a_n).
\]

(b) For \(p > \beta^*\) as well as \(p = 2\) when \(\beta^* = 2\), we have
\[
n^{1-p/2}E\left(\frac{Y_{1/n}^2 \wedge \log n}{n}\right) \to 0.
\]

(c) If \(\beta^* < 2\) then
\[
\sqrt{n} \log nE(|Y_{1/n}| \wedge 1) \to 0.
\]

The proof is based on standard techniques and is thus deferred to Appendix A.

In the following we say that events \((A_n)_{n \in \mathbb{N}}\) have high probability (for all large \(n\)) if
\[
P(A_n) \to 1.
\]
Clearly, any finite collection of events with high probability jointly have high probability.

**Proof of Theorem 1.** Recall that it is left to consider the quantities in (6). Lemma 1 implies that
\[
\sum_{i \leq n} \hat{\xi}^2_{ni} \to 0
\]
for any \(p > 0\) because the sum can be reordered so that both \(Z\) and \(Z'\) appear in increasing order. Thus is is left to (i) show \(\sum_{i \leq n} \hat{\xi}^2_{ni} \to 0\) for \(p > \beta_s\) (and \(p = 2\) when \(\beta = 2\)), and (ii) to disprove this for \(p \in (0, \beta_s)\).

Part (i). By standard extreme value theory \([6, (3.65)]\) we have
\[
M_n - 2\sqrt{2 \log n} \to 0, \quad \text{where} \quad M_n = \max_{i \leq n} Z_i - \min_{i \leq n} Z_i.
\]
According to (5) and Lemma 2 there is the bound
\[
\sum_{i \leq n} \hat{\xi}^2_{ni} = n^{-p/2} \sum_{i \leq n} (Z_i - Z_{\nu(i)})^2 \leq 4n^{-p/2} \sum_{i \leq n} \left(\frac{n}{\sigma^2} (\Delta_i^n Y)^2 \wedge M_n^2\right).
\]
With high probability \(M_n^2 < 9 \log n\) for all large \(n\). Moreover, by Lemma 3(b),
\[
E\left[n^{-p/2} \sum_{i \leq n} (\Delta_i^n Y)^2 \wedge \log n\right] = n^{1-p/2}E\left(nY_{1/n}^2 \wedge \log n\right) \to 0
\]
whenever \(p > \beta^*\) or \(p = 2 = \beta^*\). Hence we also have \(\sum_{i \leq n} \hat{\xi}^2_{ni} \to 0\) for such a \(p\), proving the first claim.

\[
\begin{array}{c|c|c|c}
   i & \text{direction} & b(i) & \#y_i \text{ in the bound} \\
   \hline
   1 & \rightarrow & 5 & 2 \\
   2 & \rightarrow & 4 & 1 \\
   3 & \rightarrow & 5 & 1 \\
   4 & \leftarrow & 3 & 2 \\
   5 & \leftarrow & 1 & 2 \\
\end{array}
\]
Part (ii). Assume that $p \in (0, \beta_*)$ and recall that $M_n < 3\sqrt{\log n}$ with high probability for all large $n$. Note that $\beta_* > 0$ implies $\Pi(\mathbb{R}) = \infty$, and so $Y$ is not compound Poisson. Let $I$ be the set of indices $i$ such that

$$\sqrt{n}\Delta_n^0 Y_i/\sigma > 6\sqrt{\log n}. \tag{8}$$

The cardinality $N = |I|$ is Binomial$(n, p_n)$ distributed, where $p_n$ satisfies

$$np_n = nP(|Y_{i/n}| > 6a_n/\sigma) \geq \frac{1}{2}\prod_n(c_n), \quad \text{for } a_n = \sqrt{\frac{\log n}{n}},$$

some $c > 0$ and all large $n$, see Lemma 3(a). This implies that $np_n \to \infty$ and so $N = np_n(1 + o_p(1))$. Moreover, Lemma 3(a) shows that $p_n \to 0$ and so $N/n \to 0$.

Let $N'$ be the analogue of $N$, but with 6 replaced by 3 in (8). From the definition of $\nu$ (see also (5)) and the above bound on $M_n$, we conclude that all $Z_{\nu(i)}, i \in I$, must be among the $N'$ largest or among the $N'$ smallest values of $Z_i$ with high probability. As with $N$, we see that $N'/n \to 0$ and thus $Z_i(N') \to \infty$ and $Z_i(n-N') \to +\infty$. The corresponding $Z_i, i \in I$, however, are chosen independently of $Y$ so by the law of large numbers, $[N/2]$ of their moduli $|Z_i|$ must be bounded above by $\Phi^{-1}(4/5)$ with high probability for all large $n$. Finally, we get the following bound with high probability for all sufficiently large $n$:

$$\sum_{i \leq n}(Z_i - Z_{\nu(i)})^2 \geq 3N \geq \prod_n(c_n).$$

Choose $q \in (p, \beta_*)$ and note that necessarily $x^q \prod(x) \to \infty$ as $x \downarrow 0$. Thus for some $c_1 > 0$ and all large $n$ we have the bound

$$n^{-p/2}\prod(c_n) \geq c_1n^{-p/2}n^{1/2}(\log n)^{-q/2} \to \infty.$$ 

This shows that

$$\sum_{i \leq n}\hat{\epsilon}_i^2 = n^{-p/2}\sum_{i \leq n}(Z_i - Z_{\nu(i)})^2 \overset{P}{\to} \infty,$$

instead of convergence to 0. The proof is now complete. \hfill \Box

**Lemma 4.** Let $(Z_1, \ldots, Z_n)$ be exchangeable and independent of $(Z'_1, \ldots, Z'_n)$. Given $1 \leq i_1 < \cdots < i_k \leq n$ and $y_1, \ldots, y_k \in \mathbb{R}$ define

$$\tilde{Z}_i = Z_i + \sum_{u=1}^k y_u 1_{\{i = i_u\}}, \quad i = 1, \ldots, n.$$

Assume there are no ties a.s. and let $\nu$ and $\pi$ be permutations such that the orderings of $(\tilde{Z}_i), (Z_{\nu(i)})$ and $(Z'_{\pi(i)})$ coincide. Then the sequence $((\tilde{Z}_i, Z_{\nu(i)}, Z'_{\pi(i)}))_{i \notin \{i_1, \ldots, i_k\}}$ of length $n - k$ is exchangeable.

**Proof.** We may assume that the sequence $(Z'_i)$ is constant and $(Z_i)$ is the result of uniformly permuting constant numbers. Let $s$ be such that $\tilde{Z}_{s(1)} < \cdots < \tilde{Z}_{s(n)}$. The permutation $s^{-1}$ maps $s(i_u)$ to $i_u$ for $u = 1, \ldots, k$ and is otherwise independently and uniformly distributed. The sequences $(Z_{\nu(i)})$ and $(Z'_{\pi(i)})$ are obtained by sorting $(Z_i)$ and $(Z'_i)$ in increasing order and then permuting according to $s^{-1}$. We conclude that the law of the sequence $((\tilde{Z}_i, Z_{\nu(i)}, Z'_{\pi(i)}))_{i \notin \{i_1, \ldots, i_k\}}$ is invariant under uniform permutations of $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$, completing the proof. \hfill \Box
Proof of Proposition 1. As in the Proof of Theorem 1, we consider the increments \( \xi_{ni} \) but with the scaling \( \sqrt{n/(2 \log n)} \). Let \( Z^{(i)} \) be the corresponding order statistics and recall that, as \( n \to \infty \),

\[
Z^{(n)} - \sqrt{2 \log n} \overset{p}{\to} 0 \quad \text{and} \quad \Delta_n := \max_{i < n} (Z^{(i+1)} - Z^{(i)}) \overset{p}{\to} 0.
\]

First, we focus on the partial sum process corresponding to

\[\hat{\xi}_{ni} = (2 \log n)^{-1/2}(Z_i - Z_{(i)}).\]

We will work on the event \( \{N_\pm = k_\pm\} \) for \( k = k_+ + k_- \geq 1 \), where \( N_\pm \) is the number of positive/negative jumps \( J \); the case of no jumps is trivial. Now the following is true for large \( n \) with probability arbitrarily close to 1. The indices \( [T_j n] \) must be different (the set of such is denoted by \( I \)), every \( \sqrt{n} |J|/\sigma \) must be larger than \( Z^{(n)} - Z^{(1)} \), and the latter is smaller than \( 3 \sqrt{\log n} \). Hence for each \( i = [n T_j] \in I \) the quantity \( Z_i + \sqrt{n} \Delta_n Y/\sigma \) must be among \( k_+ \) largest if the corresponding \( J_j > 0 \) or \( k_- \) smallest if \( J_j < 0 \). Thus

\[(2 \log n)^{-1/2} Z_{(i)} \overset{p}{\to} \pm 1\]

according to the sign of the respective jump \( J_j \). But \( Z_i \) do not depend on the choice of indices \( i \) and so

\[\hat{\xi}_{ni} \overset{p}{\to} - \text{sign}(J_j), \quad i/n = [T_j n]/n \overset{p}{\to} T_j,\]

where \( j \) is the corresponding jump index. It is thus left to show that the partial sum process of \( \hat{\xi}_{ni} \) with \( i \in I \) excluded converges in probability to \((k_+ - k_-)t \) in supremum norm. But the vector \( \hat{\xi}_{ni}, i \notin I \) is also exchangeable, see Lemma 4, and so according to [10, Thm. 3.1.3] it is sufficient to show that

\[
\sum_{i \notin I} \hat{\xi}_{ni} \overset{p}{\to} k_+ - k_- \quad \text{and} \quad \sum_{i \notin I} \hat{\xi}_{ni}^2 \overset{p}{\to} 0.
\]

Since we only need to look at the sums, we may permute the indices arbitrarily. In this paragraph we assume that \( Z_i \) is an increasing sequence, and that the elements of \( I \) are given by \( i_1 < \cdots < i_k \). For \( i > i_k \) we have \( \nu(i) = i - k_+ \), for \( i < i_k \) we have \( \nu(i) = i + k_- \) and between any two \( i_j \) and \( i_{j+1} \), the permutation \( \nu \) displaces every index a fixed amount bounded by \( k \). Furthermore, the indices \( i_j \) are chosen uniformly at random (and then sorted), implying \((Z_{i_k} - Z_{i_1})/\sqrt{2 \log n} \overset{p}{\to} 0\). Thus

\[
\sum_{i \notin I, i < i_k} \hat{\xi}_{ni} \overset{p}{\to} k Z_{i_k} - Z_{i_1} \overset{p}{\to} 0,
\]

\[
\sum_{i < i_k} \sum_{j \leq k_-} \frac{Z_j - Z_{i_k + j - 1}}{\sqrt{2 \log n}} \overset{p}{\to} -k_- \quad \text{and} \quad \sum_{i > i_k} \sum_{j \leq k_+} \frac{Z_{n-j} - Z_{i_k - j + 1}}{\sqrt{2 \log n}} \overset{p}{\to} k_+,
\]

which yield the first limit in (9). A simple induction on \( k \) shows that the bound

\[
\sum_{i \notin I} \left| Z_i - Z_{(i)} \right| \leq k (Z_n - Z_1)
\]

holds, establishing the second limit in (9):

\[
\sum_{i \notin I} \hat{\xi}_{ni}^2 \leq \frac{\Delta_n}{2 \log n} \sum_{i \notin I} \left| Z_i - Z_{(i)} \right| \leq k \Delta_n \frac{Z_n - Z_1}{2 \log n} \overset{p}{\to} 0.
\]

It remains to show that the partial sums of \( \hat{\xi}_{ni} \) vanish in probability. Observe that the sequence \( \hat{\xi}_{ni} \) need not be exchangeable. Nevertheless, we may condition on the number of jumps and note that \((\hat{\xi}_{ni})_{i \notin I} \) (of length \( n - k \)) is exchangeable.
Indeed, we need only apply Lemma 4 after conditioning on the ordered values of $Z$ and $Z'$. Now $\sum_{i \notin I} \tilde{\xi}_{ni}^2 \leq \sum_{i \leq n} \tilde{\xi}_{mi}^2 \overset{p}{\to} 0$ according to Lemma 1. Moreover,

$$\sum_{i \notin I} \tilde{\xi}_{ni} = -\sum_{i \in I} \tilde{\xi}_{ni} \overset{p}{\to} 0,$$

because for $i \in I$, both $Z_{\nu(i)}$ and $Z'_{\pi(i)}$ become $\pm \sqrt{2 \log n + o_p(1)}$ (with the same sign) and hence $\tilde{\xi}_{ni} \overset{p}{\to} 0$. This yields

$$\sum_{i \notin I} \tilde{\xi}_{ni} \overset{p}{\to} 0, \quad \sum_{i \in I} \tilde{\xi}_{ni} \overset{p}{\to} 0,$$

and the proof is complete.

Proof of Proposition 2. Note that the bivariate increments $\xi_{ni} = (\Delta^n X, \Delta^n W(n))$ are exchangeable. Moreover, the partial sums of the first coordinate corresponds to the process $X$ observed on the grid $1/n, \ldots, 1$, and those of the second coordinate correspond to some Brownian motion (dependent on $X$) observed on the same grid. Now we apply [10, Thm 3.13] to each coordinate separately, and then jointly. It is only required to show that the cross-variation vanishes:

$$\sum_{i \leq n} (\Delta^n X)(\Delta^n W(n)) \overset{p}{\to} 0.$$

Recall that $\max_{i \leq n} |\Delta^n W'| = O_p(\sqrt{\log n/n})$, and hence we are done in the case when $X$ has bounded variation on compacts. In general, by [8, Thm 2.3], it is sufficient to show that

$$\sqrt{n \log n} E(|X_{1/n}| \wedge 1) \to 0,$$

Lemma 3(c) completes the proof.

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APPENDIX A. SOME ESTIMATES FOR LÉVY PROCESSES

This section is devoted to the proofs of the three statements in Lemma 3. Recall that $(\gamma, 0, \Pi)$ is the Lévy triplet of $Y$ having no Brownian part. For any $x \in (0, 1]$ define the standard quantities:

$$m(x) = \gamma - \int_{|y| < 1} y\Pi(dy), \quad v(x) = \int_{|y| \leq x} y^2\Pi(dy).$$

We also let

$$Y_t = m(x)t + J^{x,1}_t + J^{x,2}_t$$

be the Lévy -Itô decomposition of $Y$, where $J^{x,1}_t$ is the martingale corresponding to the compensated jumps of $Y$ of magnitude less or equal than $x$ and $J^{x,2}_t$ is driftless compound Poisson process containing all jumps of $X$ of magnitude larger than $x$. In particular, $E[(J^{x,1}_t)^2] = v(x)t$. Finally, we consider the integrals

$$I_q = \int_{[-1,1]} |x|^q\Pi(dx), \quad q \geq 0.$$
and recall the following useful lemma (see, e.g., [7, Lem. 9]).

**Lemma 5.** If \( I_q < \infty \) for some \( q \in [0, 2] \), then for any \( x \in (0, 1) \), we have

\[
\Pi(x) \leq \Pi(1) + I_q x^{-q}, \quad |m(x)| \leq |\gamma| + I_q x^{-(q-1)^+}, \quad v(x) \leq I_q x^{q-2}/q.
\]

First, we establish some estimates on the truncated moments.

**Lemma 6.** For any \( p \in (0, 2) \), \( K > 0 \), \( t > 0 \) and \( x \in (0, 1) \), we have

\[
\mathbb{E}(|Y_t|^p \wedge K) \leq (m(x))^2 t^2 + v(x)t^{p/2} + K \Pi(x)t,
\]

\[
\mathbb{P}(|Y_t| \geq K) \leq (m(x))^2 t^2 + v(x)t/K^2 + \Pi(x)t.
\]

**Proof.** Fix \( t > 0 \) and define the event \( A = \bigcap_{s \leq t} \{ J_{s-}^x > 0 \} \) of not observing any jump from \( J_{s-}^x \) on the time interval \([0, t]\). Clearly \( 1 - \mathbb{P}(A) = 1 - e^{-\Pi(x)t} \leq \Pi(x)t \).

Consider the elementary inequality \( |Y_t|^p \wedge K \leq |m(x)| t + J_{t-}^x \|1_A + K1_{A^c} \). Taking expectations and applying Jensen’s inequality we obtain the bound

\[
\mathbb{E}(|Y_t|^p \wedge K) \leq (m(x))^2 t^2 + \mathbb{E}(\{J_{t-}^x \|1\})^{p/2} + K(1 - \mathbb{P}(A)),
\]

because \( \mathbb{E}J_{t-}^x = 0 \). The first inequality readily follows. Using Markov’s inequality we readily get

\[
\mathbb{P}(|Y_t| \geq K) \leq \mathbb{P}(|Y_t| \wedge K \geq K) \leq \mathbb{E}(|Y_t|^2 \wedge K^2)/K^2
\]

and the second result follows from the first with \( p = 2 \). \( \square \)

**Lemma 7.** For any \( \epsilon > 0 \) and \( a_t \downarrow 0 \) satisfying \( a_t/\sqrt{t} \to \infty \) as \( t \downarrow 0 \) we have

\[
\liminf_{t \to 0} \frac{\mathbb{P}(|X_t| > a_t)}{t \Pi(a_t(1 + \epsilon))} \geq 1.
\]

**Proof.** Take \( x = x_t = a_t(1 + \epsilon) \) and consider the event that \( J_{x_t}^x \) has exactly one jump in \([0, t]\), which yields the lower bound

\[
\mathbb{P}(|Y_t| > a_t) \geq t(1 + o(1))\Pi(x_t)\mathbb{P}(|J_{x_t}^x| + m(x_t)|t < a_t \epsilon).
\]

Here we use \( t\Pi(x_t) \to 0 \) which follows from Lemma 5 with \( q = 2 \) and the assumption \( ta_t^2 \to 0 \). Furthermore, \( |m(x_t)|t/a_t \to 0 \) and \( \mathbb{P}(|J_{x_t}^x| > a_t \epsilon/2) \to 0 \) from Markov’s inequality and the fact that \( \mathbb{E}(|J_{x_t}^x|^2)/a_t^2 = tv(x_t)/a_t^2 \to 0 \). \( \square \)

**Proof of Lemma 3.** Part (a). The inequality follows from Lemma 7. The limit is a consequence of the second inequality in Lemma 6 with \( t = 1/n \) and \( K = x = a_n \), Lemma 5 with \( q = 2 \) and the fact that \( a_n \sqrt{n} \to \infty \).

Part (b). From Lemma 6 with \( x_n^2 = n^{-1} \log n \) we have the bound

\[
n^{-p/2} \mathbb{E}(Y_{1/n}^2 \wedge \frac{\log n}{n}) \leq n^{-p/2} m(x_n)^2 + n^{-p/2} v(x_n) + n^{-p/2} \log(n)\Pi(x_n).
\]

Assume that \( \beta^* < 2 \), pick \( q < p \) such that \( I_q < \infty \), and apply Lemma 5. The first term vanishes because \(-p/2 + (q-1)^+ \leq 0\). The second term vanishes because \( 1 - p/2 - (2 - q)/2 < 0 \). The third term vanishes since \(-p/2 + q/2 < 0 \). Finally, if \( \beta^* = 2 \) then taking \( q = p = 2 \), proceeding as in the previous case and using the facts that \( x^2 \Pi(x) \to 0 \) and \( v(x) \to 0 \) as \( x \to 0 \), gives the result.

Part (c). Applying Lemma 6 with \( x = n^{-1/4} \) to \( Y \) gives

\[
n \log(n)\mathbb{E}(|Y_{1/n} \wedge 1|^2) \leq 2n^{-1} \log(n)m\left(n^{-1/4}\right)^2 + 2\log(n)(v(n^{-1/4}) - \sigma^2) + 2n^{-1} \log(n)\Pi(n^{-1/4})^2.
\]
Take \( q \) satisfying \( \beta^* \vee 1 < q < 2 \) and apply Lemma 5 to show that this quantity indeed tends to 0.

\[
\square
\]

**Appendix B. Purely Brownian case**

An interesting problem is to identify the exact rate of convergence of the skeletons in Proposition 1 for the purely Brownian case, that is, when \( Y = 0 \). Here we show that this rate is \( \sqrt{\log \log n/n} \). We can not, however, establish the limit law, nor its existence.

As in §3, consider the variables

\[
\xi_{ni} = \frac{1}{\sqrt{\log \log n}} \left( Z_{\nu(i)} - Z'_{\pi(i)} - \frac{1}{n} \sum_{j \leq n} (Z_j - Z'_j) \right), \quad i = 1, \ldots, n.
\]

For any \( n \in \mathbb{N} \), let \( S^{(n)}_t = \sum_{i \leq tn} \xi_{ni} \) be its cumulative sum process. We clearly have \( S^{(n)}_0 = S^{(n)}_1 = 0 \) and the jumps of \( S^{(n)} \) are exchangeable.

Following the proof of Lemma 1, more specifically, the bounds in terms of the functions \( \tilde{F}_n \) and \( \tilde{G}_n \), we easily deduce that for any \( a > 4 \), the quadratic variation of \( S^{(n)} \) satisfies

\[
P \left( \sum_{i \leq n} \xi_{ni}^2 > a \right) \to 0.
\]

The stated tightness in turn implies, according to [10, Lem. 3.9], that the processes \( S^{(n)} \) are tight (and nonvanishing) in the Skorokhod space \( \mathcal{D}[0,1] \). This establishes the claimed rate of convergence of skeletons.

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*University of Warwick and Aarhus University*