Abstract

We consider detection and localization of an abrupt break in the covariance structure of high-dimensional random data. The paper proposes a novel testing procedure for this problem. Due to its nature, the approach requires a properly chosen critical level. In this regard we propose a purely data-driven calibration scheme. The approach can be straightforwardly employed in online setting and is essentially multiscale allowing for a trade-off between sensitivity and change-point localization (in online setting, the delay of detection). The description of the algorithm is followed by a formal theoretical study justifying the proposed calibration scheme under mild assumption and providing guaranties for break detection. All the theoretical results are obtained in a high-dimensional setting (dimensionality $p \gg n$). The results are supported by a simulation study inspired by real-world financial data.

1 Introduction

The analysis of high dimensional time series is crucial for many fields including neuroimaging and financial engineering. There one often has to deal with processes involving abrupt structural breaks which necessitates a corresponding adaptation of the model and/or the strategy. Structural break analysis
comprises determining if an abrupt change is present in the given sample and if so, estimating the change-point, namely the moment in time when it takes place. In literature both problems may be referred to as change-point or break detection. In this study we will be using terms break detection and change-point localization respectively in order to distinguish between them. The majority of approaches consider only a univariate process [13, 1]. However, in recent years the interest for multi-dimensional approaches has increased. Most of them cover the case of fixed dimension [19, 18, 2, 24, 25]. Some approaches [10, 17, 11] feature high-dimensional theoretical guarantees but only the case of dimensionality polynomially growing in sample size is covered. The case of exponential growth has not been considered so far.

In order to detect a break, a test statistic is usually computed for each point $t$ (e.g. [19]). The break is detected if the maximum of these values exceeds a certain threshold. A proper choice of the latter may be a tricky issue. The classical approach to the problem is based on the asymptotic behavior of the statistic [13, 1, 2, 17, 3, 25]. As an alternative, permutation [17, 19] or parametric bootstrap may be used [17]. Clearly, it seems attractive to choose the threshold in a solely data-driven way employing bootstrap as it is suggested in the recent paper by [10], but a rigorous bootstrap validation is still an open question.

In the current study we are interested in a particular kind of a break – an abrupt transformation in the covariance matrix – which is motivated by applications to finance and neuroimaging. In finance the dynamics of the covariance structure of a high-dimensional process modeling return rates is crucial for a proper asset allocation in a portfolio [12, 4, 14, 20]. Analogously, break analysis in covariance structure of data in functional Magnetic Resonance Imaging is particularly important for the research on neural diseases as well as in context of brain development with emphasis on characterization of the re-configuration of the brain during learning [3, 23, 15].

One approach allowing for the change-point localization is developed in [18], the corresponding significance testing problem is considered in [2]. However, neither of these papers addresses the high-dimensional case.

A widely used break detection approach (named CUSUM) [11, 2, 17] suggests to compute a statistic at a point $t$ as a distance of estimators of some parameter of the underlying distributions obtained using all the data before and after that point. This technique requires the whole sample to be known in advance, which prevents it from being used in online setting. In order to overcome this drawback we propose the following augmentation: choose a
window size \( n \in \mathbb{N} \) and compute parameter estimators using only \( n \) points before and \( n \) points after the central point \( t \) (see Section 2.1 for formal definition). Window size \( n \) is an important parameter and its choice is case-specific (see Section 4 for theoretical treatment of this issue). Using small window results in high variability and low sensitivity, while large window implies higher uncertainty in change-point localization yielding the issue of a proper choice of window size. The multiscale nature of the proposed method enables us to incorporate the advantages of narrower and wider windows by considering multiple window sizes at once in order for wider windows to provide higher sensitivity while narrower ones improve change-point localization.

The contribution of our study is the development of a novel structural break analysis approach which is

- high-dimensional, allowing for up to exponential growth of the dimensionality with the window size
- suitable for online setting
- multiscale, attaining trade-off between break detection sensitivity and change-point localization accuracy
- using a fully data-driven calibration scheme rigorously justified under mild assumptions
- featuring formal sensitivity guaranties in high-dimensional setting

We consider the following setup. Let \( X_1, \ldots, X_N \in \mathbb{R}^p \) denote a sample of independent zero-mean vectors. In online setting the sample size is not fixed in advance. The goal is to test the hypothesis

\[
H_0 := \{ \forall i : \text{Var}[X_i] = \text{Var}[X_{i+1}] \} \tag{1}
\]

versus the alternative suggesting the existence of a break:

\[
H_1 := \{ \exists \tau : \text{Var}[X_\tau] \neq \text{Var}[X_{\tau+1}] \} \tag{2}
\]

and localize the change-point \( \tau \) as precisely as possible or (in online setting) to detect a break as soon as possible.

To this end we define a family of test statistics in Section 2.1 which is followed by Section 2.2 describing a data-driven (bootstrap) calibration scheme. Section 3 presents and discusses a theoretical result justifying the
bootstrap scheme while Section 4 presents a sensitivity result providing a lower bound for a window size $n$ necessary to detect a break of a given extent and hence bounding the uncertainty of the change-point localization (or the delay of detection in online setting). Finally, Section 5 presents a simulation study inspired by real-world financial data supporting the theoretical findings and demonstrating superiority of our approach to a recent one.

2 Proposed approach

The first part of this Section formally defines the test statistics while the second part concentrates on the calibration scheme. Informally, the test statistics may be defined as follows. Provided that the break may happen only at point $t$, one could estimate the covariance matrix using $n$ data-points to the left of $t$, estimate it again using $n$ data-points to the right of it and use the norm of their difference as a test statistic $B_n(t)$. Yet, in practice one does not usually possess such knowledge, therefore we propose to maximize these statistics over all possible locations $t$ yielding $B_n$. Finally, in order to attain a trade-off between break detection sensitivity and change-point localization accuracy we propose a multiscale approach considering multiple window sizes $n \in \mathbb{N}$ and multiple respective test statistics $\{B_n\}_{n \in \mathbb{N}}$ at once.

2.1 Definition of the test statistics

Now we present a formal definition of the test statistic. In order to detect a break we consider a set of window sizes $\mathcal{N} \subset \mathbb{N}$. Denote the size of the widest window as $n_+$ and of the narrowest as $n_-$. Given a sample of length $N$ for each window size $n \in \mathcal{N}$ define a set of central points $T_n := \{n + 1, n + 2, \ldots, N - n + 1\}$. Next, for all $n \in \mathcal{N}$ define a set of indices which belong to the window on the left side from the central point $t \in T_n$ as $I_{l,n}(t) := \{t - n, t - n + 1, \ldots, t - 1\}$ and correspondingly for the window on the right side define $I_{r,n}(t) := \{t, t + 1, \ldots, t + n - 1\}$. Denote the sum of numbers of central points for all window sizes $n \in \mathcal{N}$ as

$$T := \sum_{n \in \mathcal{N}} |T_n|. \quad (3)$$

For each window size $n \in \mathcal{N}$ and each central point $t \in T_n$ define a pair of estimators of covariance matrix as
\[
\hat{\Sigma}_n^l(t) := \frac{1}{n} \sum_{i \in I_n(t)} X_i X_i^T \quad \text{and} \quad \hat{\Sigma}_n^r(t) := \frac{1}{n} \sum_{i \in I_n(t)} X_i X_i^T.
\]

(4)

Let some subset of indices \( I_s \subseteq 1..N \) of size \( s \) (possibly, \( s = N \)) be chosen. Define a scaling diagonal matrix

\[
S = diag(\sigma_{1,1}, \sigma_{1,2}...\sigma_{p,p-1}, \sigma_{p,p})
\]

where the elements \( \sigma_{j,k} \) are standard deviations of corresponding elements of \( X_i X_i^T \) averaged over \( I_s \):

\[
\sigma_{j,k}^2 := \frac{1}{s} \sum_{i \in I_s} \text{Var}[(X_i X_i^T)_{jk}] .
\]

(5)

In practice the matrix \( S \) is usually unknown, hence we propose to plug-in empirical estimators \( \hat{\sigma}_{j,k} \).

For each window size \( n \in \mathbb{N} \) and central point \( t \in T_n \) we define a test statistic \( B_n(t) \)

\[
B_n(t) := \left\| \sqrt{\frac{n}{2}} S^{-1}(\hat{\Sigma}_n^l(t) - \hat{\Sigma}_n^r(t)) \right\|_{\infty}.
\]

(6)

Here and below we write \( \hat{A} \) for a vector composed of stacked columns of matrix \( A \) and use \( \| \cdot \|_{\infty} \) to denote the sup norm. Finally, the family of test statistics \( \{B_n\}_{n \in \mathbb{N}} \) is obtained via maximization over the central points:

\[
B_n := \max_{t \in T_n} B_n(t).
\]

(7)

### 2.2 Decision rule and bootstrap calibration scheme

Our approach rejects \( H_0 \) in favor of \( H_1 \) if at least one of statistics \( B_n \) exceeds a corresponding threshold \( x_n^b(\alpha) \) or formally if \( \exists n \in \mathbb{N} : B_n > x_n^b(\alpha) \).

In order to choose thresholds \( x_n^b(\alpha) \) the following bootstrap scheme is proposed. Define vectors \( \hat{Z}_i \) for \( i \in I_s \) as

\[
\hat{Z}_i := X_i X_i^T - \frac{1}{s} \sum_{i \in I_s} X_i X_i^T.
\]

(8)
Elements $Z^i$ for $i \in 1..N$ of bootstrap sample are proposed to be drawn with replacement from the set $\bigcup_{i \in I_s} \{ \hat{Z}_i, -\hat{Z}_i \}$. Denote the measure which $Z^i$ are distributed with respect to as $P^\flat$.

Now we are ready to define a bootstrap counterpart $B^\flat_n(t)$ of $B_n(t)$ for all $n \in \mathfrak{N}$ and $t \in T_n$ as

$$B^\flat_n(t) := \left\| \frac{1}{\sqrt{2n}} S^{-1} \left( \sum_{i \in I_s(t)} Z_i^i - \sum_{i \in I_s(t)} Z_i^{-i} \right) \right\|_\infty.$$  \hfill (9)

The counterparts $B^\flat_n$ of $B_n$ for all $n \in \mathfrak{N}$ are naturally defined as

$$B^\flat_n := \max_{t \in T_n} B^\flat_n(t).$$  \hfill (10)

Now for each given $x \in (0, 1)$ we can define quantile functions $z^\flat_n(x)$ such that

$$z^\flat_n(x) := \inf \left\{ z : P^\flat \left\{ B^\flat_n > z \right\} \leq x \right\}.$$  \hfill (11)

Next for a given significance level $\alpha$ we apply multiplicity correction choosing $\alpha^*$ as

$$\alpha^* := \sup \left\{ x : P^\flat \left\{ \exists n \in \mathfrak{N} : B^\flat_n > z^\flat_n(x) \right\} \leq \alpha \right\}$$  \hfill (12)

and finally choose thresholds as $x^\flat_n(\alpha) := z^\flat_n(\alpha^*)$.

**Remark 1.** In most of the cases one may simply choose $I_s = 1...N$ but at the same time it seems appealing to use some sub-sample which a priori does not include a break, if such information is available. On the other hand, the bootstrap justification result (Theorem 1) and sensitivity result (Theorem 2) benefit from larger set $I_s$. The experimental comparison of these options is given in Section 5.

### 2.3 Change-point localization

In order to localize a change-point we have to assume that $I_s \subseteq 1..\tau$. Consider the narrowest window detecting a change-point as $\hat{n}$:

$$\hat{n} := \min \left\{ n \in \mathfrak{N} : B_n > x^\flat_n(\alpha) \right\}$$  \hfill (13)

and the central point where this window detects a break for the first time as

$$\hat{\tau} := \min \left\{ t \in T_{\hat{n}} : B_{\hat{n}}(t) > x^\flat_{\hat{n}}(\alpha) \right\}.$$  \hfill (14)
By construction of the family of the test statistics we conclude (up to the confidence level $\alpha$) that the change-point $\tau$ is localized in the interval

$$[\hat{\tau} - \hat{n}; \hat{\tau} + \hat{n} - 1].$$

(15)

Clearly, if a non-multiscale version of the approach is employed, i.e. $|\mathcal{N}| = \{n\}$, $n = \hat{n}$ and precision of localization (delay of the detection in online setting) equals $n$.

### 3 Bootstrap validity result

This section states and discusses the theoretical result demonstrating validity of the proposed bootstrap scheme i.e.

$$\mathbb{P}\{\forall n \in \mathcal{N} : B_n \leq x_n^b(\alpha)\} \approx 1 - \alpha.$$  

(16)

Our theoretical results require the tails of the underlying distributions to be light. Specifically, we impose Sub-Gaussianity vector condition.

**Assumption 1.**

$$\exists L > 0 : \forall i \in 1..N \sup_{a \in \mathbb{R}^p : \|a\|_2 \leq 1} \mathbb{E}\left[\exp\left(\left(\frac{a^T X_i}{L}\right)^2\right)\right] \leq 2.$$  

(17)

**Theorem 1.** Let Assumption 1 hold and let $X_1, X_2, ..., X_N$ be i.i.d. Allow the parameters $p, |\mathcal{N}|, s, n_-, n_+$ grow with $N$. Further let $N > 2n_+$ and $N > s$ and let the minimal window size $n_-$ and the size $s$ of the set $\mathcal{I}_s$ grow fast enough

$$\frac{|\mathcal{N}| L^4 \log^{19}(pN)}{\min\{n_-, s\}} = o(1).$$  

(18)

Then

$$\left|\mathbb{P}\{\forall n \in \mathcal{N} : B_n \leq x_n^b(\alpha)\} - (1 - \alpha)\right| = o_P(1).$$  

(19)

The formal proof of the theorem can be found in Supplementary Materials Section A along with the finite-sample-size version of the result.
Proof discussion  The proof of the bootstrap validity result mostly relies on the high-dimensional central limit theorems obtained by [9]. That paper also presents bootstrap justification results, yet does not include a comprehensive bootstrap validity statement. The theoretical treatment is complicated by the randomness of \( x_n^\parallel(\alpha) \). Indeed, consider Lemma 6. One cannot trivially obtain result of sort (16) substituting \( \{x_n^\parallel(\alpha)\}_{n \in \mathbb{N}} \) in (82) due to the randomness of \( x_n^\parallel(\alpha) \) and dependence between \( x_n^\parallel(\alpha) \) and \( B_n \). We overcome this by means of so-called “sandwiching” proof technique (see Lemma 4), initially used by [22] and extended by [6]. The authors of [22] had to assume normality and low dimensionality of the data, while in [6] only continuous probability measures \( \mathbb{P} \) and \( \mathbb{P}^0 \) were considered. Our result is free of such limitations.

Online setting  As one can easily see, the theoretical result is stated in off-line setting, when the whole sample of size \( N \) is acquired in advance. In online setting we suggest to control the probability \( \alpha \) to raise a false alarm for at least one central point \( t \) among \( N \) data points (which differs from classical techniques controlling the mean distance between false alarms [21]). Having \( \alpha \) and \( N \) chosen, one should acquire \( s \) data-points (set \( \{X_i\}_{i \in \mathbb{I}_s} \) ) and employ the proposed bootstrap scheme with the bootstrap samples of length \( N \) in order to obtain the critical values. Next, the approach can be naturally applied in online setting and Theorem 1 guarantees the capability of the proposed bootstrap scheme to control the aforementioned probability to raise a false alarm.

4 Sensitivity result

Consider the following setting. Let there be index \( \tau \), such that \( \{X_i\}_{i \leq \tau} \) are i.i.d. and \( \{X_i\}_{i > \tau} \) are i.i.d. as well. Denote covariance matrices \( \Sigma_1 := \mathbb{E} [X_1X_1^T] \) and \( \Sigma_2 := \mathbb{E} [X_{\tau+1}X_{\tau+1}^T] \). Define the break extent \( \Delta \) as

\[
\Delta := ||\Sigma_1 - \Sigma_2||_\infty.
\]  

The question is, how large the window size \( n_+ \) should be in order to reliably reject \( \mathbb{H}_0 \).

**Theorem 2.** *Let Assumption 1 hold and let \( X_1, X_2, ..., X_N \) be i.i.d. Allow the parameters \( p, |\mathfrak{M}|, s, n_-, n_+ \) grow with \( N \) and let the break extent \( \Delta \) decay*
with $N$. Further let $N > 2n_+ \geq 2n_-, N > s, \mathcal{I}_s \subset 1..\tau$ and let the minimal window size $n_-$, the size $s$ of the set $\mathcal{I}_s$ and the maximal window size $n_+$ grow fast enough

\[
\frac{|\mathcal{R}|}{\min \{n_-, s\}} L^4 \log^{19}(pN) = o(1), \quad (21)
\]

\[
\frac{\log(pN)}{n_+ \Delta^2} = o(1). \quad (22)
\]

Then $\mathbb{H}_0$ will be rejected with probability approaching 1.

The formal proof along with the finite-sample-size version is given in Supplementary Materials Section B.

Discussion of sensitivity result  The assumption $\mathcal{I}_s \subseteq 1..\tau$ is only technical. The result may be proven without relying on it by methodologically the same argument.

Obviously, we still cannot explicitly compute the window size sufficient for reliable break detection, since it depends on the underlying distributions. However this result guarantees that the sensitivity of the test does not vanish in high-dimensional setting.

Online setting  Theorem 2 is established in offline setting as well. In online setting it guarantees that the proposed approach can reliably detect a break of an extent not less than $\Delta$ with a delay at most $n_+$ satisfying (22).

Change-point localization guarantees  Theorem 2 implies by construction of statistic $B_n$ that the change-point can be localized with precision up to $n_+$. Hence the bound (22) provides the bound for change-point localization accuracy.

5  Simulation study

5.1 Real-world covariance matrices

We have downloaded stock market quotes for $p = 87$ companies included in S&P 100 with 1-minute intervals for approximately a week ($N = 2211$) using
the API provided by Google Finance\textsuperscript{[1]}. A sample of interest was composed of 1-minute log returns for each of the companies. Our approach with window size $\mathcal{N} = \{30\}$ has detected a break at confidence level $\alpha = 0.05$, while the approach proposed by \cite{19} (referred to as ecp below) has detected nothing. The change-point was localized at the morning of Monday 19 December 2016 (the day when the Electoral College had voted).

Discarding the portion of the data around the estimated change-point we have acquired a pair of data samples which both approaches fail to detect a break in. Denote the realistic covariance matrices estimated on each of these samples as $\Sigma_1$ and $\Sigma_2$.

5.2 Design of the simulation study, results and discussion

The goal of the current simulation study is to verify that the bootstrap procedure controls first type error rate and evaluate the power of the test and compare it to the power of ecp. Hence we need to generate two types of realistic datasets – with and without a break for power and first type error rate estimation respectively. In order to generate a dataset without a break we independently draw 520 vectors from normal distribution $\mathcal{N}(0, \Sigma_1)$. As for the datasets including a break, they are generated by binding 400 vectors independently drawn from $\mathcal{N}(0, \Sigma_1)$ and 120 vectors independently drawn from $\mathcal{N}(0, \Sigma_2)$.

The results obtained in the simulation study are given in Table\textsuperscript{[1]}\textsuperscript{[1]} One can easily see that the proposed test exhibits proper control of the first type error rate. ecp (being tested in the same setting) has demonstrated proper first type error rate as well, but the power did not exceed 0.1. So, our approach outperforms ecp in all cases apart from $\mathcal{N} = \{7\}$ and $\mathcal{L} = 1..100$.

As expected, the power is higher for larger windows and it may be decreased by adding narrower windows into consideration which is the price to be paid for better change-point localization.

It should be noted that contrary to the intuition expressed in Remark\textsuperscript{[1]} using only a data sub-sample which a priori does not include a break does not necessarily improve the power of the test.

For the case of $\mathcal{L} = 1..100 \subset 1..\tau$ Table\textsuperscript{[1]} also provides mean precision of change-point localization. One can see, that multiscale approach significantly

\textsuperscript{[1]}\url{https://www.google.com/finance}
Table 1: First type error rate and power exhibited by the proposed approach for various choice of set of window sizes $\mathcal{R}$ and sub-set used for bootstrap $\mathcal{I}_s$ at significance level $\alpha = 0.05$. For the case $\mathcal{I}_s \subset 1..\tau$ mean precision of change-point localization is reported as well.

| $\mathcal{R}$ | $\mathcal{I}_s = 1..520$ | $\mathcal{I}_s = 1..100$ |
|---------------|-----------------|-----------------|
|               | I type error rate | power | I type error rate | power | localization |
| {60}          | .02             | 1.00  | .00             | .90   | 60           |
| {30}          | .01             | .90   | .00             | .52   | 30           |
| {15}          | .00             | .76   | .00             | .38   | 15           |
| {7}           | .00             | .34   | .00             | .03   | 7            |
| {60,30}       | .01             | .99   | .00             | .84   | 47.1         |
| {60,30,15}    | .01             | .99   | .00             | .82   | 41.1         |
| {60,30,15,7}  | .01             | .99   | .00             | .78   | 42.0         |
| {30,15}       | .01             | .90   | .00             | .49   | 21.8         |
| {30,15,7}     | .01             | .84   | .00             | .34   | 19.9         |

improves it.

References

[1] Alexander Aue and Lajos Horvath. Structural breaks in time series. *Journal of Time Series Analysis*, 34(1):1–16, 2013.

[2] Alexander Aue, Siegfried Hörmann, Lajos Horváth, and Matthew Reimherr. Break detection in the covariance structure of multivariate time series models. *Ann. Statist.*, 37(6B):4046–4087, 12 2009.

[3] Danielle S. Bassett, Nicholas F. Wymbs, Mason a. Porter, Peter J. Mucha, Jean M. Carlson, and Scott T. Grafton. Dynamic reconfiguration of human brain networks during learning. *Proceedings of the National Academy of Sciences*, 108(18):7641, 2010.

[4] Luc Bauwens, Sébastien Laurent, and Jeroen V K Rombouts. Multivariate GARCH models: a survey. *Journal of Applied Econometrics*, 21(1):79–109, jan 2006.

[5] Gérard Biau, Kevin Bleakley, and David M. Mason. Long signal change-point detection. *Electron. J. Statist.*, 10(2):2097–2123, 2016.
[6] Nazar Buzsun and Valeriy Avanesov. Bootstrap for change point detection. *Manuscript*, 2017.

[7] Victor Chernozhukov, Denis Chetverikov, and Kengo Kato. Comparison and anti-concentration bounds for maxima of gaussian random vectors. Dec 2013.

[8] Victor Chernozhukov, Denis Chetverikov, and Kengo Kato. Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. *Ann. Statist.*, 41(6):2786–2819, 12 2013.

[9] Victor Chernozhukov, Denis Chetverikov, and Kengo Kato. Central limit theorems and bootstrap in high dimensions. Dec 2014.

[10] Haeran Cho. Change-point detection in panel data via double cusum statistic. *Electron. J. Statist.*, 10(2):2000–2038, 2016.

[11] Haeran Cho and Piotr Fryzlewicz. Multiple-change-point detection for high dimensional time series via sparsified binary segmentation. *Journal of the Royal Statistical Society Series B*, 77(2):475–507, 2015.

[12] Mihaela Şerban, Anthony Brockwell, John Lehoczky, and Sanjay Sivastava. Modelling the dynamic dependence structure in multivariate financial time series. *Journal of Time Series Analysis*, 28(5):763–782, 2007.

[13] Miklós Csörgő and Lajos Horváth. *Limit theorems in change-point analysis*. Wiley series in probability and statistics. J. Wiley & Sons, Chichester, New York, 1997.

[14] Robert F. Engle, Victor K. Ng, and Michael Rothschild. Asset pricing with a factor-arch covariance structure. Empirical estimates for treasury bills. *Journal of Econometrics*, 45(1-2):213–237, 1990.

[15] Karl J. Friston. Functional and effective connectivity: A review. *Brain Connectivity*, 1(1):13–36, 2011.

[16] Jana Janková and Sara van de Geer. Confidence intervals for high-dimensional inverse covariance estimation. *Electron. J. Statist.*, 9(1):1205–1229, 2015.
[17] Moritz Jirak. Uniform change point tests in high dimension. *Ann. Statist.*, 43(6):2451–2483, 12 2015.

[18] M. Lavielle and G. Teyssièrè. Detection of multiple change-points in multivariate time series. *Lithuanian Mathematical Journal*, 46(3):287–306, 2006.

[19] David S. Matteson and Nicholas A. James. A nonparametric approach for multiple change point analysis of multivariate data. *Journal of the American Statistical Association*, 109(505):334–345, 2014.

[20] Thomas Mikosch, Søren Johansen, and Eric Zivot. Handbook of Financial Time Series. *Time*, 468(1996):671–693, 2009.

[21] A.N. Shiryaev. *Optimal Stopping Rules*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2007.

[22] V. Spokoiny and N. Willrich. Bootstrap tuning in ordered model selection. *ArXiv e-prints*, July 2015.

[23] Olaf Sporns. *Networks of the brain*. The MIT Press, 2011.

[24] Yao Xie and David Siegmund. Sequential multi-sensor change-point detection. *Ann. Statist.*, 41(2):670–692, 04 2013.

[25] Changliang Zou, Guosheng Yin, Long Feng, and Zhaojun Wang. Nonparametric maximum likelihood approach to multiple change-point problems. *Ann. Statist.*, 42(3):970–1002, 06 2014.
A Proof of bootstrap validity result

Theorem 3. Let Assumption 1 hold and let $X_1, X_2, ..., X_N$ be i.i.d. Moreover, assume that the residual $R < \alpha/2$ where

$$R := (3 + 2 |\mathcal{N}|) \left(2R_B + 2R_{B^\#} + R_{\Sigma}^\# \right),$$  (23)

$$R_{\Sigma}^\# := C \Delta_Y^{1/3} \log^{2/3}(Tp^2),$$  (24)

$\Delta_Y$, $R_B$ and $R_{B^\#}$ are defined in Lemmas 18, 17 and 13 respectively and $C$ is an independent positive constant. Then for all positive $x$, $t$ and $\chi$ it holds that

$$|\mathbb{P} \left\{ \forall n \in \mathcal{N} : B_n \leq x_n^\#(\alpha) \right\} - (1 - \alpha) | \leq R + 2(1 - q),$$  (25)

where

$$q := 1 - p^Z_s(\kappa) - p^{\Omega}_s(t, x) - p^W_s(x) - p^\Sigma(\chi),$$  (26)

probabilities $p^Z_s(\kappa)$, $p^{\Omega}_s(t, x)$, $p^W_s(x)$ and $p^\Sigma(\chi)$ come from Lemmas 12, 15, 14 and 14 respectively and quantiles $\{x_n^\#(\alpha)\}_{n \in \mathcal{N}}$ are yielded by bootstrap procedure described in Section 2.2.

Proof sketch The proof consists of four straightforward steps.

1. Approximate statistics $B_n$ by norms of a high-dimensional Gaussian vector up to the residual $R_B$ using the high dimensional central limits theorem by [9].

2. Similarly, we approximate bootstrap counterparts $B^\#_n$ of the statistics up to the residual $R_{B^\#}$.

3. Prove that the covariance matrix of the Gaussian vector used to approximate $B^\#_n$ in step 2 is concentrated in the ball of radius $\Delta_Y$ centered at its real-world counterpart involved in step 1 and employ the Gaussian comparison result provided by [9] and [8].

4. Finally, obtain the bootstrap validity result combining the results of steps 1-3.

Proof. Proof of the Theorem consists in applying Lemmas 18, 13 and 6 justifying applicability of sandwiching Lemma 4 on a set of probability not less than $q$ (defined by (26)) which are followed by applying Lemma 1. □
Finite-sample-size bootstrap validity result discussion  The remainder terms $R_B$, $R_{B^*}$ and $R^\pm_\Sigma$ involved in the statement of Theorem 1 are rather complicated. Here we just note that for $p, s, N, n_-, n_+ \to +\infty, N > 2n_+, n_+ \geq n_-$

$$R_B \leq C_1 \left( \frac{L^4 \log^7 (p^2 T n_+)}{n_-} \right)^{1/6},$$  \hspace{1cm} (27)

$$R_{B^*} \leq C_2 \left( \frac{L^4 \log^7 (p^2 T n_+)}{n_-} \right)^{1/6} \log^2 (ps),$$ \hspace{1cm} (28)

$$R^\pm_\Sigma \leq C_3 \left( \frac{L^4 \log^4 (ps)}{s} \right)^{1/6} \log^{2/3} (p^2 T),$$  \hspace{1cm} (29)

while the parameters $\kappa, \chi, t$ are chosen in order to ensure the probability $q$ defined by (26) to be above 0.995, e.g.

$$x = 7.61 + \log (ps),$$  \hspace{1cm} (30)

$$\kappa = 6.91 + \log s,$$ \hspace{1cm} (31)

$$t = 7.61 + 2 \log p,$$ \hspace{1cm} (32)

$$\chi = 6.91.$$ \hspace{1cm} (33)

Here $C_1, C_2, C_3$ are some positive constants independent of $N, p, s, L$. In fact, probability $q$ can be made arbitrarily close to 1 at the cost of worse constants.

It is worth noticing that, unusually, remainder terms $R_B$, $R_{B^*}$ and $R^\pm_\Sigma$ grow with $T$ defined by (3) and hence with the sample size $N$ but the dependence is logarithmic. Indeed, we gain nothing from longer samples since we use only $2n$ data points each time.

**Lemma 1.** Consider a measure $\mathbb{P}$ and a pair of sets $A$ and $B$. Then denoting $p := \mathbb{P} \{ B \}$

$$|\mathbb{P} \{ A \} - \mathbb{P} \{ A \mid B \} | \leq 2(1 - p).$$ \hspace{1cm} (34)
Proof.

\[
|\mathbb{P}\{A\} - \mathbb{P}\{A|B\}| = |\mathbb{P}\{A|B\}\mathbb{P}\{B\} + \mathbb{P}\{A|\overline{B}\}\mathbb{P}\{\overline{B}\} - \mathbb{P}\{A|B\}| \\
= |\mathbb{P}\{A|B\}(p-1) + \mathbb{P}\{A|\overline{B}\}(1-p)| \\
\leq |\mathbb{P}\{A|B\}(p-1)| + |\mathbb{P}\{A|\overline{B}\}(1-p)| \\
\leq 2(1-p).
\]

\[\blacksquare\]

B Proof of the sensitivity result

**Theorem 4.** Let Assumption 1 hold. Also let \(\Delta_Y < 1/2\) and

\[
R_{BP} < \frac{\alpha}{6|\mathcal{M}|},
\]

where \(\Delta_Y\) and \(R_{BP}\) come from Lemmas 12 and 13. Moreover, assume \(\mathcal{I}_s \subseteq 1..\tau\) and \(\tau \geq n_{suff}\), where

\[
n_{suff} := \left(\frac{q||S^{-1}||_\infty - 2\rho + \sqrt{(2\rho - q||S^{-1}||_\infty)^2 - 4\Delta\rho^2}}{\sqrt{2\Delta}}\right)^2,
\]

\[
q = \sqrt{2(1 + \Delta_Y)\log \left(\frac{2N|\mathcal{M}|\rho^2}{\alpha - 3|\mathcal{M}|R_{BP}}\right)},
\]

\[
\rho = \sqrt{2\log p + \chi}.
\]

Let it hold for the widest window that \(n_+ > n_{suff}\). Then with probability at least

\[
1 - p_s^Z(\kappa) - 3p_s^\Sigma(\chi) - p_s^\Omega(t, x) - p_s^W(x)
\]

where \(p_s^Z(\kappa), p_s^\Omega(t, x), p_s^W(x)\) and \(p_s^\Sigma(\chi),\) come from Lemmas 12, 17, 15 and 13 respectively, the hypothesis \(H_0\) will be rejected by the proposed approach at confidence level \(\alpha\).
Discussion of the finite-sample-size sensitivity result The expression (37) and the residual $R_B$ involved in the statement of Theorem 2 are rather complicated. Here we note that for $N, s$ and $p \to +\infty$, for some positive constant $C_4$ independent of $N, s, p$ and $\Delta$ it holds that

$$n_{suff} \leq C_4 \left( 1 + \frac{\log^2(ps)}{\sqrt{s}} \right) \frac{\log (|\mathcal{I}| Np^2)}{\Delta^2}$$  \hspace{1cm} (41)

while the bound (28) for $R_B$ holds as well, and the parameters $x, t$ and $\kappa$ may be chosen as specified by (30), (32) and (31) respectively and $\chi$ may be chosen as $\chi = 7.32$ in order to ensure the probability (40) to be at least 0.99.

As expected, the bound for sufficient window size decreases with the growth of the break extent $\Delta$ and the size of the set $\mathcal{I}$, but increases with dimensionality $p$. It is worth noticing, that the latter dependence is only logarithmic. And again, in the same way as with Theorem 3 the bound increases with the sample size $N$ (only logarithmically) since we use only $2n$ data points.

The assumption $\mathcal{I} \subseteq 1..\tau$ is only technical. The result may be proven without relying on it by methodologically the same argument.

Obviously, we still cannot explicitly compute $n_{suff}$, since it depends on the underlying distributions. However this result guarantees that the sensitivity of the test does not vanish in high-dimensional setting.

Proof of Theorem 4. Consider a pair of centered normal vectors

$$\eta := \left( \eta^1 \eta^2 \ldots \eta^{[n]} \right) \sim \mathcal{N}(0, \Sigma^*_Y),$$  \hspace{1cm} (42)

$$\zeta := \left( \zeta^1 \zeta^2 \ldots \zeta^{[n]} \right) \sim \mathcal{N}(0, \hat{\Sigma}_Y),$$  \hspace{1cm} (43)

$$\Sigma^*_Y := \frac{1}{2n^+} \sum_{j=1}^{2n^+} \text{Var} \left[ Y^n_j \right],$$  \hspace{1cm} (44)

$$\hat{\Sigma}_Y := \frac{1}{2n^+} \sum_{j=1}^{2n^+} \text{Var} \left[ Y^{n^b}_j \right],$$  \hspace{1cm} (45)

where vectors $Y^n_j$ and $Y^{n^b}_j$ are defined in proofs of Lemmas 7 and 11 respectively. Lemma 3 applies here and yields for all positive $q$

$$
\mathbb{P} \{ ||\zeta^{n^+}||_\infty \geq q \} \leq 2 \left| \mathcal{T}_{n^+} \right| p^2 \exp \left( -\frac{q^2}{2 \left| \hat{\Sigma}_Y \right|_\infty} \right),
$$  \hspace{1cm} (46)

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where $\hat{\Sigma}_Y = \text{Var}[\zeta]$ and $|T_{n+}|$ is the number of central points for window of size $n_+$. Applying Lemma 18 on a set of probability at least $1 - p^2(t, x) - p^2(\chi) - p^2(\Sigma)$ yields $\left\|\Sigma^*_Y - \hat{\Sigma}_Y\right\|_{\infty} \leq \Delta_Y$, and hence, due to the fact that $\left\|\Sigma^*_Y\right\|_{\infty} = 1$ by construction,

$$P\left\{\left\|\zeta^{n_+}\right\|_{\infty} \geq q\right\} \leq 2 |T_{n+}| p^2 \exp \left(-\frac{q^2}{2(1 + \Delta_Y)}\right).$$

(47)

Due to Lemma 13 and continuity of Gaussian c.d.f.

$$P^b \left\{B_{n_+}^b \geq \hat{x}_{n_+}^b(\alpha)\right\} \geq \alpha / |\mathfrak{R}| - 2R_B^b$$

(48)

and due to Lemma 13 along with the fact that $|T_{n+}| < N$, choosing $q$ as proposed by equation (38) we ensure that $x_{n_+}^b(\alpha) \leq q$.

Now using Lemma 14 twice for $\hat{\Sigma}_n^l(\tau)$ and $\hat{\Sigma}_n^r(\tau)$ respectively we obtain that with probability at least $1 - 2p^2(\chi)$

$$B_n \geq \sqrt{\frac{n}{2}} \left\|S\right\|_{\infty} (\Delta - 2\delta_n(\chi)).$$

(49)

Finally, we notice that due to definition (37) of $n_{suff}$ and since $n_+ > n_{suff}$

$$B_{n_+} > q$$

and therefore, $H_0$ will be rejected.

Lemma 2. Consider a centered random Gaussian vector $\xi \in \mathbb{R}^p$ with an arbitrary covariance matrix $\Sigma$. For any positive $q$ it holds that

$$P\left\{\max_i \xi_i \geq q\right\} \leq p \exp \left(-\frac{q^2}{2 \left\|\Sigma\right\|_{\infty}}\right).$$

(50)

Proof. By convexity we obtain the following chain of inequalities for any $t$

$$e^{t\xi_i} \leq \mathbb{E} [e^{t\xi_i}] \leq \mathbb{E} [e^{t\Sigma_i}] \leq pe^{t^2\left\|\Sigma\right\|_{\infty}/2}. \quad (51)$$

Chernoff bound yields for any $t$

$$P\left\{\max_i \xi_i \geq q\right\} \leq \frac{pe^{t^2\left\|\Sigma\right\|_{\infty}/2}}{e^{tq}}. \quad (52)$$

Finally, optimization over $t$ yields the claim.
As a trivial corollary, one obtains

**Lemma 3.** Consider a centered random Gaussian vector \( \xi \in \mathbb{R}^p \) with an arbitrary covariance matrix \( \Sigma \). For any positive \( q \) it holds that

\[
P \left\{ \|\xi\|_{\infty} \geq q \right\} \leq 2p \exp \left( -\frac{q^2}{2\|\Sigma\|_{\infty}} \right).\tag{53}
\]

### C Sandwiching lemma

The following lemma is a generalization covering the case of non-continuous probability measures of Lemma 21 of [6].

**Lemma 4.** Consider a normal multivariate vector \( \eta \) with a deterministic covariance matrix and a normal multivariate vector \( \zeta \) with a possibly random covariance matrix such that

\[
\sup_{\{x_n\}_{n \in \mathbb{R}}} \left| P \left\{ \forall n \in \mathbb{N} : B_n \leq x_n \right\} - P \left\{ \forall n \in \mathbb{N} : \|\eta_n\|_{\infty} \leq x_n \right\} \right| \leq R_B,\tag{54}
\]

\[
\sup_{\{x_n\}_{n \in \mathbb{R}}} \left| P^b \left\{ \forall n \in \mathbb{N} : B_n^b \leq x_n \right\} - P^b \left\{ \forall n \in \mathbb{N} : \|\zeta_n\|_{\infty} \leq x_n \right\} \right| \leq R_B^b,\tag{55}
\]

\[
\sup_{\{x_n\}_{n \in \mathbb{R}}} \left| P \left\{ \forall n \in \mathbb{N} : B_n \leq x_n \right\} - P^b \left\{ \forall n \in \mathbb{N} : B_n^b \leq x_n \right\} \right| \leq R.\tag{56}
\]

where \( \eta_n \) and \( \zeta_n \) are sub-vectors of \( \eta \) and \( \zeta \) respectively. Then

\[
\left| P \left\{ \forall n \in \mathbb{N} : B_n \leq x_n^b(\alpha) \right\} - (1 - \alpha) \right| \leq (3 + 2|\mathbb{N}|)(R + R_B + R_B^b).\tag{57}
\]

**Proof.** Let us introduce some notation. Denote multivariate cumulative distribution function of \( B_n, B_n^b, \|\eta_n\|_{\infty}, \|\zeta_n\|_{\infty} \) as \( P, P^b, \mathcal{N}, \mathcal{N}^b : \mathbb{R}^{\mathbb{N}} \rightarrow [0, 1] \) respectively. Define sets for all \( \delta \in [0, \alpha] \)

\[
\mathcal{Z}_+(\delta) := \{ z : \mathcal{N}(z) \geq 1 - \alpha - \delta \}, \tag{58}
\]

\[
\mathcal{Z}_-(\delta) := \{ z : \mathcal{N}(z) \leq 1 - \alpha + \delta \}. \tag{59}
\]
and their boundaries

\[ \partial Z_+(\delta) := \{ z : N(z) = 1 - \alpha - \delta \}, \quad (60) \]

\[ \partial Z_-(\delta) := \{ z : N(z) = 1 - \alpha + \delta \}. \quad (61) \]

Consider \( \delta = R + R_B + R_{BP} \) and sets \( Z_+ = Z_+(\delta), Z_- = Z_-(\delta), \partial Z_+ = \partial Z_+(\delta), \partial Z_- = \partial Z_-(\delta) \) Define a set of thresholds satisfying confidence level

\[ Z^\flat := \{ z : P^\flat(z) \geq 1 - \alpha \& \forall z_1 < z : P^\flat(z_1) < 1 - \alpha \} \quad (62) \]

here and below comparison of vectors should be understood element-wise. Notice that due to continuity of multivariate normal distribution \( \forall z^\flat \in Z^\flat \) and assumption (55)

\[ |P^\flat(z^\flat) - (1 - \alpha)| \leq R_{BP}. \quad (63) \]

Now for all \( z_- \in \partial Z_- \) and for all \( z^\flat \in Z^\flat \) it holds that

\[ P^\flat(z_-) \leq P(z_-) + R \leq N(z_-) + R + R_B \leq 1 - \alpha - R_{BP} \leq P^\flat(z^\flat) \quad (64) \]

where we have consequently used (55), (54), (60) and (63). In the same way one obtains for all \( z_+ \in \partial Z_+ \) and for all \( z^\flat \in Z^\flat \)

\[ P^\flat(z_+) \geq P^\flat(z^\flat) \quad (65) \]

which implies that \( Z^\flat \subset Z_- \cap Z_+ \).

Now denote quantile functions of \( ||\eta_n||_\infty \) as \( z^N : [0, 1] \rightarrow \mathbb{R}^{[9]} \):

\[ \forall n \in \mathcal{N} : \mathbb{P} \{ ||\eta_n||_\infty \geq z^N_n(x) \} = x. \quad (66) \]

In exactly the same way define quantile functions \( z^{N^\flat} : [0, 1] \rightarrow \mathbb{R}^{[9]} \) of \( ||\zeta_n||_\infty \). Clearly for all \( x \in [0, 1] \),

\[ z^N(x + \delta) \leq z^\flat(x) \leq z^N(x - \delta) \quad (67) \]
and hence
\[ z^\flat(\alpha^*) \leq z^N(\alpha^* - \delta) \leq z^\flat(\alpha^* - 2\delta), \tag{68} \]
\[ 1 - \alpha \leq P^b(z^N(\alpha^* - \delta)) \leq P^b(z^\flat(\alpha^* - 2\delta)). \tag{69} \]

Using Taylor expansion with Lagrange remainder term we obtain for some
0 \leq \kappa \leq 2\delta
\[
\mathcal{N}^\flat \left( z^\flat(\alpha^* - 2\delta) \right) \leq \mathcal{N}^\flat \left( z^N(\alpha^* - 2\delta) \right) + \delta
= \mathcal{N}^\flat \left( z^N(\alpha^*) \right) + \sum_{n \in \mathbb{N}} \partial_{z^N} \mathcal{N}^\flat(z^N(\alpha^*)) \partial_{z^N} \mathcal{N}^\flat(\alpha^*) \kappa + \delta
\leq 1 - \alpha + \sum_{n \in \mathbb{N}} \partial_{z^N} \mathcal{N}^\flat(z^N(\alpha^*)) \partial_{z^N} \mathcal{N}^\flat(\alpha^*) \kappa + 3\delta. \tag{70} \]

Next successively using \[ \mathcal{N} \] and the fact that quantile function is an inverse function of c.d.f. we obtain
\[
\mathcal{N}^\flat \left( z^\flat(\alpha^* - 2\delta) \right) \leq 1 - \alpha + 3\delta + 2\delta |\mathcal{N}|. \tag{71} \]

and therefore
\[
1 - \alpha \leq P^b \left( z^\flat(\alpha^* - 2\delta) \right) \leq 1 - \alpha + \delta \left( 3 + 2 |\mathcal{N}| \right), \tag{72} \]
\[
1 - \alpha \leq P^b \left( z^N(\alpha^* - \delta) \right) \leq 1 - \alpha + \delta \left( 3 + 2 |\mathcal{N}| \right). \tag{73} \]

In the same way one obtains
\[
1 - \alpha - \delta \left( 3 + 2 |\mathcal{N}| \right) \leq P^b \left( z^N(\alpha^* + \delta) \right) \leq 1 - \alpha. \tag{74} \]

Next, by the argument used in the beginning of the proof we obtain
\[
z^N(\alpha^* + \delta), z^N(\alpha^* - \delta) \in \mathcal{Z}_-(\delta \left( 3 + 2 |\mathcal{N}| \right)) \cap \mathcal{Z}_+(\delta \left( 3 + 2 |\mathcal{N}| \right)). \tag{75} \]

As a final ingredient, we need to choose deterministic \( \alpha^+ \) and \( \alpha^- \) such that
(which is possible due to continuity)
\[
\mathcal{N}(z^N(\alpha^- + \delta)) = 1 - \alpha - \delta \left( 3 + 2 |\mathcal{N}| \right), \tag{76} \]
\[
\mathcal{N}(z^N(\alpha^+ - \delta)) = 1 - \alpha + \delta \left( 3 + 2 |\mathcal{N}| \right) \tag{77} \]
}

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so $\alpha^- \leq \alpha^* \leq \alpha^+$ and hence by monotonicity

$$z^N(\alpha^- + \delta) \leq z^N(\alpha^* + \delta) \leq z^*(\alpha^*) \leq z^N(\alpha^* - \delta) \leq z^N(\alpha^+ - \delta)$$  \hspace{1cm} (78)

and finally

$$1 - \alpha - \delta (3 + 2 |\mathcal{F}|) \leq P(z^N(\alpha^- + \delta)) \\
\leq P(z^*(\alpha^*)) \\
\leq P(z^N(\alpha^+ - \delta)) \\
\leq 1 - \alpha + \delta (3 + 2 |\mathcal{F}|).$$  \hspace{1cm} (79)

**Lemma 5.** Consider a random variable $\xi$ and an event $A$ defined on the same probability space. Let c.d.f. $P\{\xi \leq x\}$ and $P\{\xi \leq x & A\}$ be differentiable. Then

$$\frac{\partial_x P\{\xi \leq x & A\}}{\partial_x P\{\xi \leq x\}} \leq 1$$  \hspace{1cm} (80)

**Proof.** Really denoting the complement of set $A$ as $\overline{A}$ we obtain,

$$\frac{\partial_x P\{\xi \leq x & A\}}{\partial_x P\{\xi \leq x\}} = \frac{\partial_x P\{\xi \leq x & A\}}{\partial_x (P\{\xi \leq x & A\} + P\{\xi \leq x & \overline{A}\})} \\
= \frac{\partial_x P\{\xi \leq x & A\}}{\partial_x P\{\xi \leq x & A\} + \partial_x P\{\xi \leq x & \overline{A}\}} \\
= \frac{1}{1 + \frac{\partial_x P\{\xi \leq x & \overline{A}\}}{\partial_x P\{\xi \leq x & A\}}}$$  \hspace{1cm} (81)

Using the fact that derivative of c.d.f. is non-negative we finalize the proof.

**D** **Similarity of joint distributions of** $\{B_n\}_{n \in \mathbb{N}}$ **and** $\{B^p_n\}_{n \in \mathbb{N}}$

**Lemma 6.** Let Assumption \[18\] hold and $\Delta_Y < 1/2$ where $\Delta_Y$ comes from Lemma \[18\]. Also let $X_1, X_2, \ldots X_N$ be i.i.d. Then for all positive $x, t$ and $\chi$ on a set of probability at least $1 - p^2_x(\kappa) - p^3_x(t, x) - p^4_x(x) - p^5(\chi)$
\[
\sup_{\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}} \left| P \left\{ \forall n \in \mathbb{N} : B_n \leq x_n \right\} - P^b \left\{ \forall n \in \mathbb{N} : B_n^b \leq x_n \right\} \right| \leq R
\] (82)

where
\[
R := R_B + R_{B^b} + R_{\Sigma}^\pm
\] (83)

\[
R_{\Sigma}^\pm = C \Delta_Y^{1/3} \log^{2/3} (Tp^2)
\] (84)

\(p_s^Z(\kappa), \ p_s^O(t, x), \ p_s^{W}(x)\) and \(p^\Sigma(\chi)\), come from Lemmas 12, 17, 15 and 14 respectively, \(R_B\) and \(R_{B^b}\) are defined in Lemmas 7 and 13 respectively and \(C\) is an independent constant.

**Proof.** Consider a pair of normal vectors \(\eta\) and \(\zeta\)
\[
\eta := \begin{pmatrix} \eta^1 & \eta^2 & \cdots & \eta^{[n]} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_Y^*),
\] (85)

\[
\zeta := \begin{pmatrix} \zeta^1 & \zeta^2 & \cdots & \zeta^{[n]} \end{pmatrix} \sim \mathcal{N}(0, \hat{\Sigma}_Y),
\] (86)

\[
\Sigma_Y^* := \frac{1}{2n^*} \sum_{j=1}^{2n^*} \text{Var} \left[ Y^n_j \right],
\] (87)

\[
\hat{\Sigma}_Y := \frac{1}{2n^*} \sum_{j=1}^{2n^*} \text{Var} \left[ Y^{n^b}_j \right],
\] (88)

where vectors \(Y\) and \(Y^b\) are defined in proofs of Lemmas 7 and 13 respectively. Applying Lemma 21 along with Lemma 19 yields
\[
\sup_{A \in A^{se}} \left| P \{ \eta \in A \} - P \{ \zeta \in A \} \right| \leq C \Delta_Y^{1/3} \log^{2/3} (Tp^2) \] (89)

and the fact that \(\forall k \in 1..p : (\text{Var}[\zeta])_{kk} = 1\) provides independence of the constant \(C\). Here \(A^{se}\) denotes a set of hyperrectangles in the sense of Definition 4 and clearly for all \(\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}\) the set \(\{\forall n \in \mathbb{N} : B_n < x_n\}\) is a hyperrectangle. Subsequently applying Lemmas 7 and 13 we finalize the proof. \(\Box\)

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E Gaussian approximation result for $B_n$

**Lemma 7.** Let Assumption 1 hold. Then

$$\sup_{\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}} \left| \mathbb{P} \{ \forall n \in \mathbb{N} : B_n \leq x_n \} - \mathbb{P} \{ \forall n \in \mathbb{N} : ||\eta^n||_{\infty} \leq x_n \} \right| \leq R_B \quad (90)$$

Where

$$\begin{pmatrix} \eta^1 & \eta^2 & \ldots & \eta^{[N]} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_Y^*), \quad (91)$$

$$\Sigma_Y^* := \frac{1}{2n_+} \sum_{j=1}^{2n_+} \text{Var} [Y_j^n], \quad (92)$$

$$R_B := C_B \left( F \log^7 (2p^2Tn_+) \right)^{1/6}, \quad (93)$$

$$F := \frac{1}{2n_-} \left( \beta \log 2 \sqrt{\frac{2}{\sqrt{2} - 1}} \right)^2 \sqrt{\frac{1}{2n_+} \left( \frac{n_+}{n_-} \right)^{1/3}} \left( ||S^{-1}||_{\infty} M_3 \right)^2 \sqrt{\frac{1}{2n_+ n_-} (||S^{-1}||_{\infty} M_4)^2} \quad (94)$$

with $\gamma$ defined by (102), $\beta$ by (103), and $Y$ along with its sub-matrices $Y^n$ by (99) and (97). Also, $M_3^3$ and $M_4^4$ stand for the third and the fourth maximal centered moments of $(X_1)_k(X_1)_l$ and $C_B$ is an independent constant.

**Proof.** First, we define for all $i \in 1..n$

$$Z_i := S^{-1} \left( X_iX_i^T - \Sigma^* \right) \quad (95)$$

and notice that

$$B_n(t) := \left\| \frac{1}{\sqrt{2n_+}} \left( \sum_{i \in T_n(t)} Z_i - \sum_{i \in T_n(t)} Z_i \right) \right\|_{\infty}. \quad (96)$$

Next, consider a matrix $Y_n$ with $2n_+$ columns
Clearly, columns of the matrix are independent and

\[ B_n = \frac{1}{\sqrt{2n_+}} \sum_{l=0}^{2n_+} (Y^n)_l \]  

Next, we define a block matrix composed of \( Y_n \) matrices:

\[
Y := \begin{pmatrix}
Y^1 \\
Y^2 \\
... \\
Y^{[\mathcal{N}]} \\
\end{pmatrix}.
\]

Again, vectors \( Y_l \) are independent and for all \( \{x_n\}_{n \in \mathcal{N}} \subset \mathbb{R} \) the set

\[ \{ \forall n \in \mathcal{N} : B_n \leq x_n \} \]

is a hyperrectangle in the sense of Definition 1.

The rest of the proof consists in applying Lemma 20. Denote

\[
G_{n_+} = \sqrt{\frac{n_+}{n_-}} \left( \beta \log 2 \vee \frac{\sqrt{2}}{\sqrt{2} - 1} \gamma \right) \vee \left( \frac{n_+}{n_-} \right)^{1/6} M_3 \vee \left( \frac{n_+}{n_-} \right)^{1/4} M_4.
\]
In the same way as in Lemma 10 one shows that the assumptions of Lemma 8 hold for components of $Z_i$ with
\[ \gamma := L^2 \|S^{-1}\|_\infty, \]  \tag{102} \]
\[ \beta := L^2 \|S^{-1}\|_\infty \|\Sigma^*\|_\infty. \]  \tag{103} \]
Therefore, condition \((160)\) holds with $G_{n+}$ defined by equation \((101)\). In order to see that condition \((2)\) is fulfilled with $b = 1$ notice that
\[ \frac{1}{2n+} \sum_{j=1}^{n+} \mathbb{E} \left[ (Y^n_{ij})^2 \right] \geq \min_j \text{Var} [(Z_1)_j] = 1. \]  \tag{104} \]
Next, observe that for any $k$-th component $Z_{ik}$ of $Z_i$ and a central point $t$ (both determined by $j$):
\[ \frac{1}{2n+} \sum_{j=1}^{2n+} \mathbb{E} \left[ |Y^n_{ij}|^3 \right] = \frac{1}{2n+} \sum_{i \in \mathcal{I}_n(t) \cup \mathcal{I}_n^c(t)} \mathbb{E} \left[ \left( \sqrt{\frac{n+}{n}} |Z_{ik}| \right)^3 \right] = \frac{1}{2n+} \sum_{i \in \mathcal{I}_n(t) \cup \mathcal{I}_n^c(t)} \left( \frac{n+}{n} \right)^{3/2} \mathbb{E} \left[ |Z_{ik}|^3 \right] \]
\[ = \frac{2n}{2n+} \left( \frac{n+}{n} \right)^{3/2} \mathbb{E} \left[ |Z_{ik}|^3 \right] \]
\[ = \sqrt{\frac{n+}{n}} \mathbb{E} \left[ |Z_{ik}|^3 \right] \]
\[ \leq \sqrt{\frac{n+}{n-}} \left( \|S^{-1}\|_\infty M_3 \right)^3. \]  \tag{105} \]
In the same way:
\[ \frac{1}{2n+} \sum_{i=1}^N \mathbb{E} \left[ |Y^n_{ij}|^4 \right] \leq \frac{n+}{n-} \left( \|S^{-1}\|_\infty M_4 \right)^4. \]  \tag{106} \]
Therefore, condition \((3)\) holds with $B_{n+}$, so Lemma 20 applies here and provides us with the claimed bound. Moreover, $C_B$ depends only on $b = 1$ which implies that the constant $C_B$ is independent. \qed
Lemma 8. Consider a random variable $\xi$. Suppose $\forall x \geq 0$ the following bound holds:

$$\Pr \{ |\xi| \geq \gamma x + \beta \} \leq e^{-x}. \quad (107)$$

Then

$$\mathbb{E} \left[ \exp \left( \frac{|\xi|}{G} \right) \right] \leq 2 \quad (108)$$

for

$$G = \beta \log 2 \lor \frac{\sqrt{2}}{\sqrt{2} - \gamma}. \quad (109)$$

Proof. Integration by parts yields

$$\mathbb{E} \left[ \exp \left( \frac{|\xi|}{G} \right) \right] \leq \exp \left( \frac{\beta}{G} \right) + \frac{\gamma}{G} \int_{0}^{+\infty} \exp \left( \frac{\gamma x + \beta}{G} \right) e^{-x} dx. \quad (110)$$

$$\int_{0}^{+\infty} \exp \left( \frac{\gamma x + \beta}{G} \right) e^{-x} dx = \frac{G}{G - \gamma} \exp \left( \frac{\beta}{G} \right). \quad (111)$$

$$\mathbb{E} \left[ \exp \left( \frac{|\xi|}{G} \right) \right] \leq \frac{G}{G - \gamma} \exp \left( \frac{\beta}{G} \right) \leq 2. \quad (112)$$

Using the same technique the following lemma, which may be of use in order to bound the moments $M_3^3$ and $M_4^4$, can be proven.

Lemma 9. Under assumptions of Lemma 8

$$\mathbb{E} [ |\xi|^3 ] \leq \beta^3 + 3 \gamma \beta^2 + 6 \beta \gamma^2 + 2 \gamma^3, \quad (113)$$

$$\mathbb{E} [ \xi^4 ] \leq \beta^4 + 4 \gamma \beta^3 + 12 \beta^2 \gamma^2 + 6 \beta \gamma^3 + 24 \gamma^4. \quad (114)$$
Lemma 10. Under Assumption 1 it holds for all $i \in 1..N$ and positive $\kappa$ that

$$\mathbb{P} \{ \forall k \in 1..p : |(Z_i)_k| \leq \|S^{-1}\|_\infty L^2 (\kappa + \log p + \|\Sigma^*\|_\infty) \} \geq 1 - e^{-\kappa}. \quad (115)$$

Proof. According to the definition (95) of $Z_i$ for its arbitrary element $(Z_i)_k$ one obtains for some $l, m \in 1..p$:

$$(Z_i)_k = S_{kk}^{-1} ((X_i)_l(X_i)_m - \Sigma_{lm}^*). \quad (116)$$

By sub-Gaussianity Assumption 1 it holds for all positive $x$ that

$$\mathbb{P} \{ \forall k \in 1..p : |(X_i)_k| \leq x \} \geq 1 - pe^{-x^2/L^2}. \quad (117)$$

Hence

$$\mathbb{P} \{ \forall k \in 1..p : |(Z_i)_k| \leq \|S^{-1}\|_\infty (x^2 + \|\Sigma^*\|_\infty) \} \geq 1 - pe^{-x^2/L^2} \quad (118)$$

and finally a change of variables establishes the claim.

\[\square\]

F Gaussian approximation result for $B^b_n$

Denote

$$\Sigma_Y^* := \frac{1}{2n_+} \sum_{i=1}^{2n_+} \text{Var} [Y_i] \quad (119)$$

$$\hat{\Sigma}_Y := \frac{1}{2n_+} \sum_{i=1}^{2n_+} \text{Var} [Y^b_i] \quad (120)$$

where vectors $Y_j$ and $Y^b_j$ are defined by (97) and (128) respectively.

Lemma 11.

$$\sup_{\{x_n\}_{n \in \mathbb{N} \subset \mathbb{R}}} \left| \mathbb{P}^b \{ \forall n \in \mathbb{N} : B^b_n \leq x_n \} - \mathbb{P}^b \{ \forall n \in \mathbb{N} : \|\zeta_n\|_\infty \leq x_n \} \right| \leq \hat{C}_B \left( F^b \log^7 (2p^2 T n_+) \right)^{1/6} \quad (121)$$
Where $\hat{C}_{B^b}$ depends only on $\min_{k \in 1..p}(\hat{\Sigma}_Y)_{kk}$,

$$
\begin{pmatrix}
\zeta^1 & \zeta^2 & \ldots & \zeta|\mathcal{N}| \n
\end{pmatrix} \sim \mathcal{N}(0, \hat{\Sigma}_Y),
$$

(122)

$$
\hat{\Sigma}_Y := \frac{1}{2n_+} \sum_{j=1}^{2n_+} \text{Var} \left[ Y_{j}^{n_b} \right],
$$

(123)

$$
F^b = \left( \frac{1}{2n_- \log 2} \right)^{1/3} \sqrt{\frac{1}{2n_+ n_-}} \left\| S^{-1} \right\|_\infty^2 (M^b)^2
$$

(124)

$$
M^b = \max_{i \in I_n} \left\| \hat{Z}_i \right\|_\infty.
$$

(125)

**Proof.** This proof is similar to the proof of Lemma 7.

Consider a matrix which is a bootstrap counterpart of $Y^n$

$$(Y^{n_b})^T := \sqrt{\frac{n_+}{n}} \times
$$

$$
\begin{pmatrix}
Z_1^b & O & \ldots & O & -Z_{2n_+1}^b & \ldots \\
Z_2^b & Z_2^b & \ldots & \ldots & \ldots & \ldots \\
\ldots & Z_3^b & \ldots & \ldots & \ldots & \ldots \\
Z_n^b & \ldots & \ldots & \ldots & \ldots & \ldots \\
-Z_{n+1}^b & Z_{n+1}^b & \ldots & \ldots & \ldots & \ldots \\
-Z_{n+2}^b & -Z_{n+2}^b & \ldots & \ldots & \ldots & \ldots \\
\ldots & -Z_{n+3}^b & \ldots & O & \ldots & \ldots \\
-Z_{2n}^b & \ldots & \ldots & Z_{2n, -2n+1}^b & O & \ldots \\
O & -Z_{2n+1}^b & \ldots & Z_{2n, -2n+2}^b & Z_{2n, -2n+2}^b & \ldots \\
O & O & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & -Z_{2n, -1}^b & -Z_{2n, -1}^b & \ldots \\
O & O & \ldots & -Z_{2n}^b & -Z_{2n}^b & \ldots
\end{pmatrix}.
$$

(126)

Clearly, columns of the matrix are independent and

$$
B^b_n = \frac{1}{\sqrt{2n_+}} \sum_{t=0}^{2n_+} (Y^{n_b})_t
$$

(127)

Next, we define a block matrix composed of $Y^{n_b}$ matrices:
\[ Y^\flat := \begin{pmatrix} Y_1^{\flat b} \\ Y_2^{\flat b} \\ \vdots \\ Y_n^{\flat b} \end{pmatrix}. \] (128)

Again, vectors \( Y_i^\flat \) are independent and for all \( \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \) the set

\[ \{ \forall n \in \mathbb{N} : B_n^\flat < x_n \} \] (129)

is a hyperrectangle in the sense of Definition 1. Now notice that

\[
\frac{1}{2n_+} \sum_{j=1}^{2n_+} \mathbb{E} \left[ |Y_{ij}|^3 \right] \leq \sqrt{\frac{n_+}{n_-}} \left| S^{-1} \right| \max_{i \in I_s} ||Z_i||_\infty,
\] (130)

\[
\frac{1}{2n_+} \sum_{j=1}^{2n_+} \mathbb{E} \left[ |Y_{ij}|^4 \right] \leq \frac{n_+}{n_-} \left| S^{-1} \right| \max_{i \in I_s} ||Z_i||_\infty.
\] (131)

And finally apply Lemma 20.

**Lemma 12.** Under Assumption 4 it holds for all positive \( \kappa \) that

\[
\mathbb{P} \{ \forall i \in I_s : ||Z_i||_\infty \leq Z_s(\kappa) \} \geq 1 - p^Z_s(\kappa)
\] (132)

where

\[
Z_s(\kappa) := \left| |S^{-1}| \right| \left( \kappa + \log p + ||\Sigma^*||_\infty \right),
\] (133)

\[
p^Z_s(\kappa) := se^{-\kappa}.
\] (134)

**Proof.** Proof of the Lemma consists in applying Lemma 10 and appropriate multiplicity correction.

**Lemma 13.** Let Assumption 4 hold and \( \Delta_Y < 1/2 \). Then for all positive \( \kappa \) with probability at least \( 1 - p^Z_s(\kappa) \)

\[
\sup_{\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}} \mathbb{P}^b \left\{ \forall n \in \mathbb{N} : B_n^b \leq x_n \right\} - \mathbb{P}^b \left\{ \forall n \in \mathbb{N} : ||Y_{n \ell}^b||_\infty \leq x_n \right\} \leq R_B
\] (135)
Where
\[ R_{B^p} := C_{B^p} \left( \hat{F} \log^7(2p^2Tn+) \right)^{1/6}, \]  
\[ \hat{F} := \left( \frac{1}{2n_- \log^2 2} \lor \frac{1}{2n_+} \left( \frac{n_+}{n_-} \right)^{1/3} \lor \sqrt{\frac{1}{2n_+n_-}} \right) \|S^{-1}\|_\infty^2 (Z_s(\kappa))^2 \]  
and \( \hat{C}_{B^p} \) is an independent constant.

**Proof.** The proof consists in subsequently applying Lemmas 11 and 12 ensuring \( M^2 \leq Z_s(\kappa) \) with probability at least \( 1 - p_s^2(\kappa) \), while assumed bound \( \|\Sigma_Y - \hat{\Sigma}_Y\|_\infty \leq \Delta_Y < 1/2 = \min_{1 \leq k \leq p} (\Sigma_Y^*)_{kk} \) implies the existence of a deterministic constant \( C_{B^p} > \hat{C}_{B^p} \).

Denote
\[ W_i := X_iX_i^T, \]  
\[ \Omega^* := E \left[ (W_1 - \overline{\Sigma}) (W_1 - \overline{\Sigma})^T \right], \]  
\[ \hat{\Omega} := E_{I_S} \left[ (W_i - \overline{\Sigma}) (W_i - \overline{\Sigma})^T \right] \]  
where notation \( E_{I_S}[\cdot] \) is used as a shorthand for averaging over \( I_S \), e.g.
\[ E_{I_S}[\xi_i] = \frac{1}{s} \sum_{i \in I_S} \xi_i, \]  
and similarly \( Var_{I_S}[\cdot] \) denotes an empirical covariance matrix computed using the same set, e.g.
\[ Var_{I_S}[\xi_i] = E_{I_S} \left[ \xi_i \xi_i^T \right]. \]  
The results of this section rely on the following lemma which is a trivial corollary of Lemma 6 by [16] providing the concentration result for empirical covariance matrix.
**Lemma 14.** Consider an i.i.d. \( p \)-dimensional sample of length \( s \). Let Assumption \( \mathcal{A} \) hold for some \( L > 0 \). Then for any positive \( \chi \)

\[
\mathbb{P} \left\{ \left\| \sum^* - \mathbb{E}_{I_S} [W_i] \right\|_\infty \geq \delta_s(\chi) \right\} \leq p^\Sigma(\chi) := 2e^{-\chi},
\]

(143)

where

\[
\delta_s(\chi) := 2L^2 \left( \frac{2 \log p + \chi}{s} + \sqrt{\frac{4 \log p + 2\chi}{s}} \right).
\]

(144)

Straightforwardly applying Assumption \( \mathcal{A} \) and a proper multiplicity correction yields the following result.

**Lemma 15.** Under Assumption \( \mathcal{A} \) it holds for all positive \( x \) that

\[
\mathbb{P} \left\{ \forall i \in I_s : \left\| W_i - \sum^* \right\|_\infty \leq \mathcal{W}_s(x) \right\} \geq 1 - p_s^W(x),
\]

(145)

where

\[
\mathcal{W}_s(x) := x^2 + ||\sum^*||_\infty,
\]

(146)

\[
p_s^W(x) := pse^{-x}.
\]

(147)

**Lemma 16.** Under Assumption \( \mathcal{A} \) with probability at least \( 1 - p_s^W(s) - p^\Sigma(\chi) \)

\[
\left\| \text{Var}_{I_S} [W_i - \mathbb{E}_{I_S} [W_i]] - \hat{\Omega} \right\|_\infty \leq 2\mathcal{W}_s(x)\delta_s(\chi) + \delta_s(\chi)^2.
\]

(148)

**Proof.** By the construction of bootstrap procedure and definition \( [95] \)

\[
\text{Var}_{I_S} [W_i - \mathbb{E}_{I_S} [W_i]] = \frac{1}{s} \sum_{i \in I_s} (W_i - \mathbb{E}_{I_S} [W_i]) (W_i - \mathbb{E}_{I_S} [W_i])^T
\]

\[
= \frac{1}{s} \sum_{i \in I_s} (W_i - \sum^* + \sum^* - \mathbb{E}_{I_S} [W_i]) (W_i - \sum^* + \sum^* - \mathbb{E}_{I_S} [W_i])^T
\]

\[
= \hat{\Omega} + \frac{1}{s} \sum_{i \in I_s} (\sum^* - \mathbb{E}_{I_S} [W_i]) (\sum^* - \mathbb{E}_{I_S} [W_i])^T + \frac{2}{s} (W_i - \sum^*) (\sum^* - \mathbb{E}_{I_S} [W_i])
\]

(149)

Applying Lemmas \( [15] \) and \( [14] \) yields the claim. \( \blacksquare \)
Lemma 17. Let Assumption 1 hold for some \( L > 0 \). Then for any positive \( t \) and \( x \)

\[
P\left\{ \left\| \hat{\Sigma}^* - \hat{\Omega} \right\|_\infty \geq \Delta^\Omega_s(t, x) \right\} \leq p^\Omega_s(t, x) \quad (150)
\]

where

\[
\Delta^\Omega_s(t, x) := \frac{2(\mathcal{W}_s(x))^2 t}{3s} \left( 1 + \sqrt{1 + \frac{9\sigma^2_{\hat{\Omega}}}{t(2(\mathcal{W}_s(x))^2)^2}} \right), \quad (151)
\]

\[
p^\Omega_s(t, x) := p^2 e^{-t} + p^W_s(x). \quad (152)
\]

Proof. Consider a random variable

\[
\zeta_{lm} := (W_i - \Sigma^*)_{ij} (W_i - \Sigma^*)_{jm} - \Omega_{ij}^*. \quad (153)
\]

By Lemma 15 we can bound it as \( |\zeta_{lm}| \leq \frac{2(\mathcal{W}_s(x))^2}{3s} \) with probability at least \( 1 - p^\Omega_s(x) \). Due to \( \zeta_{ij} \) being centered Bernstein inequality applies:

\[
P \left\{ \mathbb{E} \left[ \zeta_{lm} \right] \geq \frac{2(\mathcal{W}_s(x))^2 t}{3s} \left( 1 + \sqrt{1 + \frac{9\sigma^2_{\hat{\Omega}}}{t(2(\mathcal{W}_s(x))^2)^2}} \right) \right\} \leq e^{-t}. \quad (154)
\]

Lemma 18. Under Assumption 1 for any positive \( t, x \) and \( \chi \) with probability at least \( 1 - p^\Omega_s(t, x) - p^W_s(x) - p^\Sigma(\chi) \) it holds that

\[
\left\| \text{Var} [Z] - \text{Var} \left[ Z^\Omega \right] \right\|_\infty \leq \Delta_Y := \left\| S^{-1} \right\|_\infty^2 (\Delta^\Omega_s(t, x) + 2\mathcal{W}_s(x)\delta_s(\chi) + \delta_s^2(\chi)). \quad (155)
\]

Proof. Proof consists in applying Lemmas 17 and 16.

Lemma 19. Under Assumption 1 for any positive \( t, x \) and \( \chi \) with probability at least \( 1 - p^\Omega_s(t, x) - p^W_s(x) - p^\Sigma(\chi) \) it holds that

\[
\left\| \Sigma^* - \Sigma^\Omega \right\|_\infty \leq \Delta_Y \quad (156)
\]

where \( \Delta_Y \) comes from Lemma 18.
H General Gaussian approximation result

In this section we briefly describe the result obtained in [9].

Throughout this section consider an independent sample \( x_1, \ldots, x_n \in \mathbb{R}^p \) of centered random variables. Define their Gaussian counterparts \( y_i \sim \mathcal{N}(0, \text{Var}[x_i]) \) and denote their scaled sums as

\[
S^X_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \quad \text{and} \quad S^Y_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} y_i.
\]  

(157)

Definition 1. We call a set \( A \) of a form \( A = \{ w \in \mathbb{R}^p : a_i \leq w_i \leq b_i \ \forall i \in \{1..p\} \} \) a hyperrectangle. A family of all hyperrectangles is denoted as \( A^{\text{re}} \).

Assumption 2. \( \exists b > 0 \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ x_{ij}^2 \right] \geq b \quad \text{for all} \quad j \in \{1..p\}.
\]  

(158)

Assumption 3. \( \exists G_n \geq 1 \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ |x_{ij}|^{2+k} \right] \leq G_n^{2+k} \quad \text{for all} \quad j \in \{1..p\} \quad \text{and} \quad k \in \{1, 2\},
\]  

(159)

\[
\mathbb{E} \left[ \exp \left( \frac{|x_{ij}|}{G_n} \right) \right] \leq 2 \quad \text{for all} \quad j \in \{1..p\} \quad \text{and} \quad i \in \{1..n\}.
\]  

(160)

Lemma 20 (Proposition 2.1 by [9]). Let Assumption 2 hold for some \( b \) and Assumption 3 hold for some \( G_n \). Then

\[
\sup_{A \in A^{\text{re}}} \left| \mathbb{P}\{S^X_n \in A\} - \mathbb{P}\{S^Y_n \in A\} \right| \leq C \left( \frac{G_n^2 \log^7(pn)}{n} \right)^{1/6}
\]  

(161)

and \( C \) depends only on \( b \).
I Gaussian comparison result

By the technique given in the proof of Theorem 4.1 by [9] one obtains the following generalization of the result given in [7].

Lemma 21. Consider a pair of covariance matrices $\Sigma_1$ and $\Sigma_2$ of size $p \times p$ such that

$$||\Sigma_1 - \Sigma_2||_\infty \leq \Delta$$

(162)

and $\forall k : C_1 \geq (\Sigma_1)_{kk} \geq c_1 > 0$. Then for random vectors $\eta \sim \mathcal{N}(0, \Sigma_1)$ and $\zeta \sim \mathcal{N}(0, \Sigma_2)$ it holds that

$$\sup_{A \in A^c} |P\{\eta \in A\} - P\{\zeta \in A\}| \leq C \Delta^{1/3} \log^{2/3} p,$$

(163)

where $C$ is a positive constant which depends only on $C_1$ and $c_1$. 

