Bayesian analysis of running holographic Ricci dark energy

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Abstract

Recent astronomical observations strongly support the idea of accelerated expansion of the universe. Several models have been proposed to explain this phenomenon. One of the main challenges in modern cosmology is the comparison of these models. We focused on running holographic Ricci dark energy model and also considered its two versions: one with an additive constant in the vacuum energy density and second one is without an additive constant in the energy density but accounting for possible interaction between dark sectors. The model parameters are constrained using combinations of some of the latest cosmological data such as Hubble parameter data, SNIa(307)+CMB+BAO, SNIa(307)+CMB+BAO+Hubble data, SNIa(580)+CMB+BAO, SNIa(580)+CMB+BAO+Hubble data sets. This paper mainly focused on the comparison of our models with the standard ΛCDM model using Bayesian analysis. We also compare the two versions of our present model using Bayesian analysis. We do not find a strong evidence against the standard model as compared with our model.

1 Introduction

The idea that the cosmological term Λ could be varying (at least slowly) as the universe expands is gaining much attention in the light of the recent cosmological data[1]. A continuously varying vacuum energy which adopts its time dependence from the Hubble parameter $H(t)$ is suggested by quantum field theory in curved space-time[2] and is turn out to be a good choice for dynamical cosmological term [3, 4, 5, 6]. Such a running vacuum energy (RVE) model has been proposed to alleviate the drawbacks of the standard ΛCDM[12, 11], the coincidence problem and the cosmological constant problem, proposed to explain the recent acceleration in the expansion of the universe[7, 8, 9, 10]. The issue, cosmological constant problem, is due to the incredibly small magnitude of the observed value of the cosmological constant compared to its theoretical prediction from quantum field theory[13], while the coincidence of present energy densities of both cosmological constant and dark matter is a coincidence problem. RVE model[14] proposes a cosmological parameter, $\Lambda(H)$, which is decaying as the universe expands and can hopefully solve the previously mentioned issues. Authors in reference[15] have found strong evidence for a slowly varying ‘cosmological constant’ using the latest combined observational data, SNIa+BAO+H(z)+LSS+BBN+CMB, .

Another interesting attempt to explain the decaying dark energy is by applying holographic principle[16, 17], which culminated into an alternative model, the holographic Ricci dark energy(hrde). The hrde has a resemblance in form with the conventional RVE. Energy densities of both the models are combinations of $\dot{H}$ and $H^2$, where over-dot represents a derivative with respect to time. The hrde was initially proposed as a varying dark energy with time dependent equation of state[18]. Due to its similarity in density, it has been considered as an alternative to RVE[19, 20, 21], for convenience let us call it as hrde (running holographic Ricci dark energy). In hrde model an additive constant in the energy density is essential to ensure the transition from a prior decelerated to a late accelerated epoch[19, 20]. In reference[21] we have
shown that such an additive constant in the density is not needed for causing a transition into the late accelerating epoch if one accounts for the interaction between the \( r^2 \)de and the dark matter\(^{22, 23, 24} \). The evolution of the various cosmological parameters in such a model were studied in the light of supernovae, SDSS and recent Planck data \(^{21} \). Taking account of the reasonable performance of \( r^2 \)de in predicting the back ground evolution of the universe, it is worth to contrast it with the standard \( \Lambda \)CDM model in order to assess it’s relative importance. In the present work we make this comparison using the method Bayesian analysis\(^{42, 26, 27, 28} \).

The Bayesian theory of statistics was developed by great mathematicians such as Gauss, Bayes, Laplace, Bernoulli etc. It is possible to assign a probability for a random variable based on repeated measurements. But in cosmological scenario such kind of repeated observations are practically impossible or only rarely possible. Often one have a hypothesis or a theory in cosmology instead of a random variable, the probability of which is to be fixed. Bayesian theory help us in this regard to fix the probability of such hypothesis by using the available data. Following this, it is possible to compare different cosmological models by computing what is known as Bayes factor, which is proportional to the ratio of the probabilities of the models which are to be compared. This method have been used in many works for model comparison\(^{25, 29, 30, 43} \). Here we first compare the \( r^2 \)de having an additive constant in the density with the standard \( \Lambda \)CDM model then compare the \( r^2 \)de devoid of the additive constant, but its interaction with cold dark matter is accounted phenomenologically.

## 2 The method of Bayesian analysis

In this section we present the basic method of the Bayesian comparison following reference\(^{43, 44} \). The method is based on the famous Bayes theorem, which allow one to evaluate the posterior probability of a given model. Let \( p(M_{r^2 \text{de}}|D,I) \) be the posterior probability of \( M_{r^2 \text{de}} \), the \( r^2 \)de model with Ricci dark energy and dark matter as the cosmic components, such that \( D \), the given set of observational data and \( I \), the back ground information regarding the expansion of the universe are true. According to Bayes theorem it then follows,

\[
p(M_{r^2 \text{de}}|D,I) = \frac{p(M_{r^2 \text{de}}|I)p(D|M_{r^2 \text{de}},I)}{p(D|I)},
\]

(1)

Here \( p(M_{r^2 \text{de}}|I) \) is the prior probability of the model assigned before analyzing the data, given that the back ground information \( I \) is true, \( p(D|M_{r^2 \text{de}},I) \) is the probability for obtaining the data provided the model and the background information are true and is also called the likelihood of the model \( M_{r^2 \text{de}} \). The term \( p(D|I) \) is the probability for obtaining the data if the background information is true, which is the normalization factor. In a similar way the posterior probability for the standard \( \Lambda \)CDM model can be written as,

\[
p(M_{\Lambda \text{CDM}}|D,I) = \frac{p(M_{\Lambda \text{CDM}}|I)p(D|M_{\Lambda \text{CDM}},I)}{p(D|I)},
\]

(2)

where \( p(M_{\Lambda \text{CDM}}|I) \) is the prior probability and \( p(D|M_{\Lambda \text{CDM}},I) \) is the likelihood of the \( M_{\Lambda \text{CDM}} \) respectively. On comparing the posterior probabilities of the two models, we define what is called as the odds ratio in Bayesian model comparison as,

\[
O_{ij} = \frac{p(M_{r^2 \text{de}}|D,I)}{p(M_{\Lambda \text{CDM}}|D,I)} = \frac{p(M_{r^2 \text{de}}|I)p(D|M_{r^2 \text{de}},I)}{p(M_{\Lambda \text{CDM}}|I)p(D|M_{\Lambda \text{CDM}},I)},
\]

(3)

where suffix ‘i’ represents \( M_{rhrde} \) and ‘j’ represents \( M_{Λ CDM} \). The prior probability of either the given model or the standard model is fixed before considering the data and is depending only on the prior information. The prior information is an attempt to quantify the collective wisdom of a researcher in the general field in which the models are being proposed. If this collective information does not prefer one model over the other, the prior probabilities get cancelled out in the above odds ratio, hence we have

\[
O_{ij} = \frac{p(D|M_{rhrde}, I)}{p(D|M_{Λ CDM}, I)} \equiv B_{ij},
\]

where \( B_{ij} \) is now called as the Bayes factor. The likelihood of the model \( M_i \) can be conveniently denoted as \( L(M_i) \) (where \( M_i = M_{rhrde} \)). If \( \alpha, \beta \) are the model parameters of the model, then the likelihood of \( M_i \) can be of the form

\[
L(M_i) = \int d\alpha p(\alpha|M_i) \int d\beta p(\beta|M_i) e^{-\chi^2(\alpha, \beta)/2},
\]

where \( p(\alpha|M_i) \) and \( p(\beta|M_i) \) are the flat prior probabilities of the parameters \( \alpha \) and \( \beta \) respectively and \( \chi^2(\alpha, \beta) \) is the statistical function defined as

\[
\chi^2(\alpha, \beta) = \sum \left[ \frac{A_k - A_k(\alpha, \beta)}{\sigma_k} \right]^2.
\]

Here \( A_k \) are the measured values of the observable, \( A_k(\alpha, \beta) \) are the values of the same observable obtained from the theory and \( \sigma_k \) are the uncertainties in the measurement of the observable. We assume no prior information regarding the parameters except that they are lying in the range \( [\alpha, \alpha + \Delta \alpha] \) and \( [\beta, \beta + \Delta \beta] \) respectively, then \( p(\alpha|M_i) = \frac{1}{\Delta \alpha} \) and \( p(\beta|M_i) = \frac{1}{\Delta \beta} \) are the simplest choice for them. Hence the above equation become

\[
L(M_i) = \frac{1}{\Delta \alpha \Delta \beta} \int_{\alpha-\Delta \alpha}^{\alpha+\Delta \alpha} d\alpha \int_{\beta-\Delta \beta}^{\beta+\Delta \beta} d\beta \exp[-\chi^2(\alpha, \beta)/2].
\]

The range over which the parameters vary can be obtained as follows. In the above integral the exponential term is usually taken to be the likelihood for the combination of the parameters \( \alpha \) and \( \beta \) together i.e.,

\[
L(\alpha, \beta) \equiv e^{-\chi^2(\alpha, \beta)/2}.
\]

The values of the model parameters are to be obtained by constraining the model with the observation data using the process of \( \chi^2 \) minimization. The predicted value of them is the one corresponding to the minimum of \( \chi^2 \), i.e. \( \chi^2_{min} \). A plot of \( L(\alpha, \beta) \) by keeping one of the parameters constant, equal to its predicted equilibrium value, is of Gaussian in shape. For instance, \( \beta \) can be fixed equal to its value corresponding to the \( \chi^2_{min} \) and parameter \( \alpha \) can be varied about its equilibrium value corresponding to \( \chi^2_{min} \). The width of the resulting Gaussian curve can be taken as \( \Delta \alpha \) and the inverse of which is taken to be as the flat prior \( p(\alpha|M_i) \). Similar procedure can be adopted to find \( \Delta \beta \) and the corresponding prior. Having the prior probabilities for the parameters, the marginal likelihood for a parameter, say \( \alpha \) can be written as,

\[
L_\alpha(\alpha) = \frac{1}{\Delta \beta} \int_{\beta-\Delta \beta}^{\beta+\Delta \beta} d\beta \exp[-\chi^2(\alpha, \beta)/2].
\]
Similarly marginal likelihood for $\beta$ can also be obtained. Having all these, the likelihood of the model $L(M_i)$ can be evaluated as per equation (7).

Knowing the likelihood of the models $M_i$ and $M_j$, the comparison between them can be performed by estimating Bayes factor $B_{ij}$, which is the ratio of likelihood of the two models,

$$B_{ij} = \frac{L(M_i)}{L(M_j)}.$$  \hspace{1cm} (10)

Following the conventional Jeffrey’s scale of reference in Bayesian analysis[46], If the Bayes factor, $B_{ij} < 1$, then the model $M_i$ is not significant as compared to $M_j$. If $1 < B_{ij} < 3$, there is evidence against $M_j$ when compared with $M_i$ but it is not worth more than a bare mention. For $3 < B_{ij} < 20$, the evidence of $M_i$ against $M_j$ is not strong but definite. If $20 < B_{ij} < 150$, the evidence is strong and on the other hand if $B_{ij} > 150$, evidence against $M_j$ is very strong[45, 47, 48].

3 Model Comparison

In this section we employ the method of Bayesian comparison to contrast the significance of rhrde over the standard $\Lambda$CDM model. We consider two versions of rhrde model. The first one characterized by a dark energy density with an additive constant, which causes the transition into the late acceleration. While the second one, doesn’t have that additive constant in the dark energy density and on the other hand the late acceleration is then caused by the non-gravitational interaction between the dark sectors which is introduced in a phenomenological way. We briefly describe both these two models before entering the Bayesian analysis.

3.1 rhrde Models

Model 1 - Ricci dark energy with a bare cosmological constant.

The rhrde is characterized by the energy density with a bare cosmological constant, $\Lambda_0$, of the form [19],

$$\rho_\Lambda(H, \dot{H}) = 3\beta M_p^2 (\dot{H} + 2H^2) + M_p^2 \Lambda_0,$$  \hspace{1cm} (11)

where $M_p^2 = \frac{1}{8\pi G}$ is the reduced Planck mass, $\beta$ is the model parameter, $H$ is the Hubble parameter and $\dot{H}$ is its derivative with cosmic time. This is running in the sense that, it’s equations of state is fixed, $\omega = -1$ while the density is varying as the universe evolves. The conservation law followed by the major cosmic components is,

$$\dot{\rho}_m + \dot{\rho}_r + \dot{\rho}_\Lambda + 3H(\rho_m + \frac{4}{3}\rho_r) = 0,$$  \hspace{1cm} (12)

where $\rho_m, \rho_r$ are the matter and radiation densities respectively and the over dot represents their derivatives with respect to cosmic time. From the Friedman equation the Hubble parameter, $h = H/H_0$ takes the form[20],

$$h^2 = \frac{\Omega_{m0} e^{-3\xi_m x} + \Omega_{r0} e^{-4x} + \frac{\Lambda_0}{3(1-2\beta)H_0^2}}{\xi_m}.$$  \hspace{1cm} (13)
where $\Omega_{m0} = \frac{\rho_{m0}}{3H_0^2}$, $\Omega_{r0} = \frac{\rho_{r0}}{3H_0^2}$, $H_0$ is the present value of the Hubble parameter and $\xi_m = \frac{(1-2\beta)}{(1-3\beta)}$.

The variable $x = \ln a$. The first term in Eqn (13) is corresponds to matter density, second term represent the energy density of radiation and the third term corresponds to the bare cosmological constant. This equation implies an upper limit on the model parameter, $\beta < \frac{1}{2}$.

In the limit $a \to \infty$ the Hubble parameter becomes, $h^2 \to \frac{\Lambda_0}{(1-2\beta)}H_0^2$, a constant which corresponds to an accelerating universe dominated by the bare cosmological constant.

Model 2 - Interacting Ricci dark energy

The dark energy density in this case having the same form as in equation (11) except the last additive term on the right hand side. In addition, the interaction between the dark sectors is taken care off by a phenomenological term. In reference[21] we analyzed the background evolution of the universe in this model by taking into account of the interaction between the rhrde and the dark matter by the phenomenological term, $Q = 3bH\rho_m$, where $b$ is the coupling constant and $\rho_m$ is density of dark matter. Using the Friedmann equation,

$$3H^2 = \rho_m + \rho_{hrde},$$ \hfill(14)

and the conservation laws,

$$\dot{\rho}_{hrde} + 3H(\rho_{hrde} + P_{hrde}) = -Q,$$
$$\dot{\rho}_m + 3H(\rho_m + P_m) = Q,$$ \hfill(15)

one can obtain the Hubble parameter as[21],

$$h^2 = \frac{\Omega_{m0}}{1-b}e^{-3(1-b)x} - \frac{1}{3}\left(\frac{2\Omega_{hrde0}}{\beta} + 3\Omega_{m0} - 4\right)e^{-3x} + \left(\frac{2\Omega_{hrde0}}{3\beta} - \frac{b}{1-b}\Omega_{m0} - \frac{1}{3}\right),$$ \hfill(16)

where $h = H/H_0$, $\Omega_{m0} = \frac{\rho_{m0}}{3H_0^2}$, $\Omega_{hrde0} = \frac{\rho_{hrde0}}{3H_0^2}$, with $\rho_{m0}$ and $\rho_{hrde0}$ as the present value of the dark matter and dark energy densities respectively and $H_0$ is present value of the Hubble parameter. The variable, $x = \ln a$, where $a$ is the scale factor of expansion. The Hubble parameter have the expected asymptotic properties. As $x \to -\infty$(equivalently $a \to 0$) the first two terms in equation (16) dominates which implies a prior deceleration phase. As $x \to +\infty$ (equivalently $a \to \infty$) Hubble parameter tends to a constant which corresponds to the end de Sitter phase.

### 3.2 Data analysis for parameters

The model parameters $\beta$ of model 1 and $\beta b$ of model 2 are evaluated using the $\chi^2$ minimization technique. For the computation we have used supernova 580 data from the SCP "Union2.1" SN Ia compilation[32], $H(z)$ data[33], CMBR data from Planck2013[39, 40] and BAO (Baryon Accoustic Oscillation) data from Sloan Digital Sky Survey(SDSS)[37, 38]. We repeat the computation by replacing 580 supernovae data set with the 307 data from Union compilation[31]. The 580 data set consists of more data points from the low redshift observations. It has been noted that interference with the background in obtaining the low redshift data is relatively high[35] and also Kolmogorv-Smirnov analysis[36] have shown that 307 data set
have relatively larger, i.e. 85% probability of being originated from a common distribution. The theoretical distance modulus for a redshift \( z_i \) is,

\[
\mu_t(\beta, b, H_0, z_i) = 5 \log_{10} \left[ \frac{d_L(\beta, b, H_0, z_i)}{\text{Mpc}} \right] + 25,
\]

where \( d_L \) is its luminosity distance. The \( \mu_t \) calculated for a given redshift is to be compared with the observational data for the same redshift for obtaining \( \chi^2 \) using equation(6), in which \( A_k \) is replaced with the distance modulus \( \mu \).

We have used Hubble parameter data \(^{33}\), contains 38 samples in the red shift range \( 0.07 \leq z \leq 2.36 \). The \( \chi^2 \) has been obtained using equation (6), in which we replace \( A \) with \( H \). In using Cosmic Microwave Background (CMB) data\(^{34}\), the shift parameter \( R \), is taken as the observable instead of \( A \) in equation(6), defined as,

\[
R = \sqrt{\Omega_m} \int_{0}^{z_2} \frac{dz}{h(z)}.
\]

Here \( z_2 \) is the red shift at the last scattering surface. From Planck 2013 data, \( z_2 = 1090.41 \) and \( R = 1.7499 \pm 0.0080 \)\(^{39, 40}\). For Baryon Acoustic Oscillation (BAO) data, the observable used in equation(6) is the acoustic parameter \( A \),

\[
A = \frac{\sqrt{\Omega_m}}{h(z_1) \frac{1}{2}} \left( \frac{1}{z_1} \int_{0}^{z_1} \frac{dz}{h(z)} \right)^{\frac{3}{2}}.
\]

Here \( z_1 = 0.35 \) is the redshift where the signature of the peak acoustic oscillations has been measured\(^{38}\). According to the SDSS data the observational value of the acoustic parameter for flat universe corresponding to the same redshift is, \( A = 0.484 \pm 0.016 \)\(^{41}\).

We first done the \( \chi^2 \) minization with Hubble parameter data alone for both the models. We obtained the parameters for the data combinations SNIa(307)+BAO+CMB, SNIa(307)+BAO+CMB+Hubble data, SNIa(580)+BAO+CMB and SNIa(580)+BAO+CMB+Hubble data. For instance the total \( \chi^2 \) corresponding to Model 2 for the full data set is \( \chi^2(\beta, b) = \chi^2(\beta, b)_{SNIa} + \chi^2(\beta, b)_{CMB} + \chi^2(\beta, b)_{BAO} + \chi^2(\beta, b)_{Hubble} \). By minimizing \( \chi^2 \) function, the parameters for both models are extracted and are summarized in table1 for model 1 and in table2 for model 2. For both models we have fixed the value of the present Hubble parameter at \( H_0 = 69 \text{km}s^{-1}\text{Mpc}^{-1} \) as per the Planck data\(^{39}\). Regarding model 2 we have estimated \( \beta \) and \( b \) and by using the corresponding two parameter contour plane we have estimated the correction factor also as shown in table2. For model 1, the value of the parameter \( \beta \) is found to be almost the same for all the combinations of the data set, but with slight changes from the third decimal place onwards. The same is the case for model 2 regarding parameter \( \beta \). In the case of model 2, the interaction parameter \( b \) is around 0.0470 with Hubble parameter data, but the value of which has been ten times smaller with combinations of data (see table2). Yet another point to be noted is the strength of the parameter \( \beta \), which around 0.01 for model 1 but in the case of model 2, where interaction is being taken care explicitly, this parameter has gone up approximately by ten times and is around 0.46. So in a sense with the phenomenological interaction the strength of dark energy has enhanced substantially.
4 Bayesian analysis for model 1

We will now calculate the likelihood of the model. One of the basic input for this is the prior probability for the parameter $\beta$. For obtaining that, we plotted $\exp(-\chi^2(\beta)/2)$ with $\beta$ and is of Gaussian shape as shown in figure[1]. The width of the variation is taken as $\Delta\beta$, called the prior range of the parameter.

The prior ranges for $\beta$ are found to be: $-0.0665 < \beta < 0.0763$ for Hubble data, $-0.007 < \beta < 0.0342$ for SN(307)+BAO+CMB, $0.0038 < \beta < 0.0435$ for SN(307)+BAO+CMB+Hubble data, $0.0018 < \beta < 0.03155$ for SN(580)+CMB+BAO, $0.00053 < \beta < 0.0295$ for SN(580)+CMB+BAO+Hubble data sets. The likelihood for the model 1 can then be evaluated by using equation (7) as,

$$L(\beta) = \frac{1}{\Delta\beta} \int_{\beta}^{\beta+\Delta\beta} d\beta L_i(\beta).$$  \hspace{1cm} (20)

For obtaining the Bayesian evidence of the model, we have to obtain the likelihood of the standard $\Lambda CDM$ for the same data set. The standard model is two component one with dark matter ($\Omega_m = \frac{\rho_m}{3H^2}$) and cosmological constant $\Lambda$ ($\Omega_\Lambda = \frac{\rho_\Lambda}{3H^2}$) as the constituents. First we obtain the prior ranges of the parameters $\Omega_m$ and $\Omega_\Lambda$ using the relation (8) by replacing $\alpha$ and $\beta$ with $\Omega_m$ and $\Omega_\Lambda$ respectively and are found to be $0.2437 < \Omega_m < 0.3056$ & $0.6363 < \Omega_\Lambda < 0.8712$ for Hubble data, $0.255 < \Omega_m < 0.31$ & $0.695 < \Omega_\Lambda < 0.7802$ for SN(307)+CMB+BAO, $0.2538 < \Omega_m < 0.2935$ & $0.7057 < \Omega_\Lambda < 0.7953$ for SN(307)+CMB+BAO+Hubble, $0.2639 < \Omega_m < 0.3038$ & $0.6669 < \Omega_\Lambda < 0.7225$ for SN(580)+CMB+BAO and $0.2556 < \Omega_m < 0.2883$ & $0.6602 < \Omega_\Lambda < 0.7161$ for the last combination SN(580)+CMB+BAO+Hubble of the data sets. Using corresponding prior probabilities we obtained $L(M_i)$, the likelihood of the standard $\Lambda CDM$ model using the relation (7) by replacing $\alpha$ and $\beta$ with $\Omega_m$ and $\Omega_\Lambda$ respectively for each data sets. The Bayes factor for the model 1 in comparison to the standard $\Lambda CDM$ model can then be evaluated using the expression (10) and are summarized in the last row of table[1].
Table 1: Parameter value, prior and Bayes factor for different data sets of model 1.

| Parameters | Hubbledata | SNIa(307)+CMB+BAO | SNIa(307)+CMB+BAO+Hubbledata | SNIa(580)+CMB+BAO+Hubbledata | SNIa(580)+CMB+BAO+Hubbledata |
|------------|------------|--------------------|------------------------------|-----------------------------|------------------------------|
| $\beta$    | 0.0125     | 0.0145             | 0.013                        | 0.0172                      | 0.015                        |
| $\Delta \beta$ | 0.14       | 0.0412             | 0.0397                       | 0.0287                      | 0.0289                       |
| Bayes factor | 1.09       | 1.45               | 1.88                         | 1.80                         | 2.6                          |

From table it is seen that the Bayes factor for all the data combinations are in the range $1 < B_{ij} < 3$. According to Jeffrey’s scale this indicate that the model 1 have evidence against the standard model but that is just worth for a bare mention.

5 Bayesian analysis for model 2

In model 2 $\beta$ and $b$ are the parameters. We first describe the calculation process using the data set SNIa(307)+CMB+BAO. The prior probabilities of these parameters are needed for calculating likelihood. The prior for the model parameter $\beta$ is obtained plotting $e^{-\chi^2(\beta)/2}$ with $\beta$, keeping $b$ and $H_0$ ($H_0 = 69 \text{ km s}^{-1} \text{ Mpc}^{-1}$) as constants. The width of the resulting Gaussian, as shown in figure (2), is taken as $\Delta \beta$.

![Figure 2: The plot gives prior range for the model parameter $\beta$ using SNIa(307)+CMB+BAO.](image)

and is found to be $\Delta \beta = 0.0200$. The marginal likelihood for the interaction term $b$ can then be obtained by performing the integral over $\beta$ as in relation (3) by replacing $\alpha$ with $b$. The resulting function is then plotted against $b$ gives a Gaussian shape, see figure (3), which in turn implies a range for $b$ as $-0.01 < b < 0.03$ which is the actual range over which the parameter $b$ varies.

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A similar procedure is adopted for finding the contributing range of the parameter $\beta$. The prior probability of $b$ is obtained from the plot of $exp(-\chi^2(b)/2)$ with $b$ (figure 4), implies $\Delta b = 0.0303$, which is then used to integrate out $b$. 

The marginal likelihood for the parameter $\beta$ thus evaluated and is found that it contribute in the range $0.44 < \beta < 0.48$, see figure 5. In short we have obtained the likelihood of the model parameters by starting with some flat probability.
Figure 5: The plot shows the marginal likelihood of the model parameter $\beta$ for SNIa(307)+CMB+BAO data sets.

The likelihood of the present model $\mathcal{L}(M_i)$ can then be obtained by using equation (7) by using $b$ instead of $\alpha$. The ranges are correspondingly $0.4534 < \beta < 0.487$ & $-0.01993 < b < 0.1191$ for Hubble data, $0.455 < \beta < 0.475$ & $-0.0083 < b < 0.022$ for SNIa(307)+CMB+BAO, $0.454 < \beta < 0.4726$ & $-0.00915 < b < 0.02363$ for SNIa(307)+CMB+BAO+Hubble data set, $0.4525 < \beta < 0.4727$ & $-0.01296 < b < 0.0204$ for SNIa(580)+CMB+BAO data sets and $0.4548 < \beta < 0.4726$ & $-0.0087 < b < 0.02339$ for SNIa(580)+CMB+BAO+Hubble data sets. The Bayes factor for the model 2 corresponding to each data sets was evaluated using the relation (10) and the results are summarized in Table 2.

| Parameters | Hubble data | SNIa(307)+CMB+BAO | SNIa(580)+CMB+BAO | SNIa(307)+CMB+BAO+Hubble data | SNIa(580)+CMB+BAO+Hubble data |
|-----------|-------------|-------------------|-------------------|-----------------------------|-----------------------------|
| $\beta$   | 0.4694$^{+0.007}_{-0.006}$ | 0.4645$^{+0.005}_{-0.004}$ | 0.4642$^{+0.004}_{-0.004}$ | 0.4634$^{+0.004}_{-0.003}$ | 0.4632$^{+0.004}_{-0.003}$ |
| $b$       | 0.0470$^{+0.03}_{-0.06}$ | 0.0075$^{+0.01}_{-0.01}$ | 0.0076$^{+0.006}_{-0.006}$ | 0.0078$^{+0.006}_{-0.002}$ | 0.0077$^{+0.007}_{-0.008}$ |
| $\Delta\beta$ | 0.0336 | 0.02 | 0.0176 | 0.0172 | 0.0178 |
| $\Delta b$ | 0.139 | 0.0303 | 0.0341 | 0.0327 | 0.0322 |
| Bayes factor | 1.9 | 1.6 | 1.72 | 2.9 | 3.2 |

The Bayes factor of model 2 corresponding to the Hubble parameter data, SNIa(307)+CMB+BAO data, SNIa(580)+CMB+BAO data and SNIa(307)+CMB+BAO+Hubble data are all lie in the range $1 < B_{ij} < 3.2$. According Jeffrey’s scale these values indicating that, the model is not worth more than a bare mention than the standard $\Lambda$CDM model. But for the data set SNIa(580)+CMB+BAO+Hubble, the Bayes factor is just above three which indicate that the evidence for model 2 with this full data set is significant but not strong. The difference in the Bayes factors corresponding the data sets SNIa(307)+CMB+BAO+Hubble...
and (580)+CMB+BAO+Hubble is not too large, but for the former data set it below three and for later set it is above three. If you relay on 580 SNIa, which consists of more number of data in the low redshift region, then it seems that model 2 is slightly superior over ΛCDM model. But as noted by some, if one take account of the relatively large background interference in obtaining the low redshift data, then 307 SNIa is comparatively reliable, which leads to the fact that the model 2 is not so worth against the standard model.

It is to noted that, model 2 has progressively higher Bayes factor compared to model 1 for the respective data sets except a slight deviation for SNIa(580)+CMB+BAO. So including the interaction via non-gravitational phenomenological term could increase the acceptance of the model. More over Byes factor of model 2 relative to model 1 is found to be $B_{ij} = 1.25$, indicating the slight advantage of model 2.

6 Conclusion

Current cosmological data favors a slowly varying dark energy. A feasible solution to this, is vacuum energy varying as the universe expand, known as running dark energy. The standard running dark energy models proposed, consists of dark energy density depends on the rate of change of Hubble parameter and also the even powers of the parameter. In contrast to this the authors have analyzed holographic Ricci dark energy as a running dark energy. In the present work we have analyzed the significance of this model relative to the standard ΛCDM model using Bayesian statistics.

Bayesian analysis is an effective tool for selecting the most appropriate model based on the current observational data. We have used supernova, Hubble parameter, BAO and CMB data sets to extract the statistical evidence of two versions of running holographic Ricci dark energy model: (1) rhrde with an additive constant in the energy density and (2) rhrde without additive constant in the energy density and its interaction with dark matter is accounted by a phenomenological term, $Q = 3bH\rho_m$. We have tried five combinations of the above data sets: Hubble parameter data, SNIa(307)+CMB+BAO, SNIa(307)+CMB+BAO+Hubble data, SNIa(580)+CMB+BAO, SNIa(580)+CMB+BAO+Hubble data. It is to be noted that the SNIa(580) data consist of more data points in the low red shift region than SNIa(307). Our analysis shows that the Bayes factor for model 1 is between one and three for all the data combinations and is least for the Hubble parameter data alone and highest for SNIa(580)+CMB+BAO+Hubble data. The inference is that model 1 is not worth more than a bare mention against the standard ΛCDM model, even though the background evolution of the universe of this model is almost comparable with that of the ΛCDM model[20].

Model 2, in which the interaction is being accounted with phenomenological term, also predict comparable back ground evolution for the late universe[21]. The results of the Bayesian statistical analysis is summarized in table 2. For Hubble parameter data alone, the Bayes factor is around 1.9 corresponding to which the interaction parameter is evaluated to be around $0.0470^{+0.06}_{-0.03}$ and in standards of Jeffreys the model is not so worth in this data set. For the combinations data set the interaction parameter is found to be almost the same for all of the combinations except the slight change in the fourth decimal place and is an order of magnitude less compared the usage of Hubble parameter data as evident from the table. Bayes factors for the data sets SNIa(307)+CMB+BAO and SNIa(580)+CMB+BAO are between one and two. But when add Hubble parameter data to both these, the Bayes factor corresponding to SNIa(307)+CMB+BAO+Hubble
become near to three, while for SNIa(580)+CMB+BAO+Hubble data is become above three, i.e. 3.2. This shows that the combinations of data statistically favoring model 2 and for SNIa(580)+CMB+BAO+Hubble data the evidence of Model 2 is definite against ΛCDM but at the same time not so strong. We also check the Bayes factor for model 2 with respect to model 1 and is found to be $B_{ij} = 1.25$. This indicates that model 2 is significant compared to model 1.

There are recent analyses which claim that running vacuum models with energy density similar to the form of rhrde is preferred as compared to the concordance ΛCDM in the light of recent data like WMAP9, Planck 2013 and the recent Planck 2015 data. In reference [15] the authors did an overall fit to the running vacuum model with SNIa+BAO+$H(z)$+LSS+BBN+CMB and found that the corresponding class of running vacuum appears significantly more favored than the ΛCDM. A remarkable coincidence of the such vacuum energy models with the rhrde is the appearance of the terms $H^2$ and $\dot{H}$. Other than this, the parameters of these class of models are different. The present analysis is a clear evidence that the rhrde is equally competent as the conventional running vacuum models in supporting the decaying ‘cosmological constant’.

References

[1] J. Sola, *Int J Mod Phys A.*, **31** (2016) 1630035.
[2] J. Sola, H. Stefancic, *Phys. Lett. A* **21** (2006) 479.
[3] J. Sola, A. Gomez-Valent, and J. de Cruz Perez, *Ap. J* **811** (2015) L14.
[4] J. Sola, *J. Phys. Conf. Ser.* **453** (2013) 012015.
[5] J. Sola and A. Gomez-Valent, *Int. J. Mod. Phys. D* **24** (2015) 1541003.
[6] J. Sola, A. Gomez-Valent, and J. de Cruz Perez, *Ap. J* **836** (2017) 43.
[7] A. G. Riess et al., *Ap. J.* **116** (1998) 1009.
[8] S. Perlmutter et al., *Ap. J.* **517** (1999) 565.
[9] D. N. Spergel et al., *Astrophys. J. Suppl. Ser.* **148** (2003) 175.
[10] M. Tegmark et al., *Ap. J.* **606** (2004) 702.
[11] N. A. Bahcall, J. P. Ostriker, S. Perlmutter and P.J. Steinhardt, *Science* **284** (1999) 1481-1488.
[12] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int.J.Mod.Phys.D* **15** (2006) 1753-1936,
[13] L. Amendola, and S. Tsujikawa, *Cambridge University Press* (2010).
[14] I. L. Shapiro, J. Sola, *Phys. Lett. B* **475** (2000) 236; I. L. Shapiro, J. Sola, *Nucl. Phys. B Proc. Supp.* **127** (2004) 71.
[15] J. Sola, A. Gomez-Valent, and J. de Cruz Perez, *Ap. J.*, **836** (2017) 43, [arXiv:1602.02103v4](https://arxiv.org/abs/1602.02103)
[16] R. Bousso, Rev. Mod. Phys 74 (2002) 825.
[17] G. t Hooft, Gerard Conf.Proc. C 930308 (1993) 284-296, arXiv:gr-qc/9310026; L. Susskind, J. Math. Phys. 36 (1995) 6377.
[18] P. Praseetha and T. K. Mathew, Int. J. Mod. Phys. D 23 1450024 (2014).
[19] M.B. Lopez and Y.Tavakoli, Phys.Rev.D 87 (2013) 023515.
[20] G. Paxy, T. K. Mathew, Mod. Phys.Lett. A 31 (2016) 1650075.
[21] G. Paxy, V. M. Sheeef, T. K. Mathew, Ind. Nat. J. Mod. Phys. D 28 (2019) 1950060.
[22] S. H. Pereira and J. F. Jesus, Phys.Rev. D 79 (2009) 043517.
[23] S. Som, A. Sil, Astrophys. Space Sci. 352 (2014) 867.
[24] H. Jian-Hua, B. Wang, JCAP 0806 (2008) 010.
[25] B. Santos, N. C. Devi, and J. S. Alcaniz, Phys. Rev. D 95 (2017) 123514.
[26] A. Heavens, Y. Fantaye, E. Sellentin, H. Eggers, Z. Hosenie, S. Kroon, and A. Mootoovaloo, Phys. Rev. Lett. 119 (2017) 101301.
[27] S. Santos da Costa, M. Benetti, and J. Alcaniz, JCAP 1803 (2018) no.03 004.
[28] U. Andrade, C. A. P. Bengal, J. S. Alcaniz, and B. Santos, Phys. Rev. D 97 (2018) 083518.
[29] J.F. Jesus, R. Valentime and F. Andrade-Oliveira, JCAP 09 (2017) 030.
[30] P. Serra, A. Heavens and A. Melchior, Mon. Not. R. Astron. Soc. 379 (2007) 169175.
[31] M. Kowalski et al., Ap.J. 686 (2008) 749.
[32] N. Suzuki et al, Ap. J 746 (2012) 85.
[33] O. Farooq, F. Madiyar, S. Crandall, B. Ratra, Ap. J 835 (2017) 26.
[34] J.R. Bond, G. Efstathiou and M. Tegmark, Mon. Not. R. Astron. Soc. 291 (1997) L33.
[35] B. Schwarzchild, Phys. Today 57 (2004) 19.
[36] F. B. Bianco et al., Ap. J 686 (2011) 749.
[37] D. J. Eisenstein, et al., [SDSS Collaboration] Astrophys. J. 633 (2005) 560 .
[38] M. Tegmark et al., Phys. Rev. D 74 (2006) 123507.
[39] P.A.R. Ade et al.,[Planck collaboration] Astron. Astrophys. 571 (2014) A16.
[40] D.L. Shafer and D. Huterer, Phys. Rev. D 89 (2014) 063510.
[41] C. Blake, E. Kazin, F. Beutler, T. Davis, D. Parkinson et al., Mon. Not.Roy. Astron. Soc. 418 (2011) 1707.
[42] R.E. Kass and A.E. Raftery, *J. Am. Stat. Assoc.* **90** (1995) 773.

[43] M. V. John and J. V. Narlikar, *Phys. Rev. D* **65** (2002) 043506.

[44] M. V. John, *Ap. J* **630** (2005) 667-674.

[45] R. Trotta., *Mon. Not. R. Astron. Soc.* **378** (2007) 72.

[46] H. Jefferys, *Theory of probability(3 ed.) Oxford UK. Oxford university press* (1961).

[47] P. S. Drell, T. J. Loredo, and I. Wasserman, *Ap. J.* **530** (2000) 593.

[48] A. Sasidharan, N.D.J. Mohan, M.V. John et al., *Eur. Phys. J. C*, **78** (2018) 628.

[49] J. Sola, *The Fourteenth Marcel Grossmann Meeting* (2017) 2363-2370; [arXiv:1601.01668](https://arxiv.org/abs/1601.01668) [gr-qc]

[50] J. Sola, A. Gomez-Valent, and J. de Cruz Perez, [arXiv:1606.00450](https://arxiv.org/abs/1606.00450)

[51] Sola, J. de Cruz Perez and A. Gomez-Valent, *EPL* **121** (2018) 39001.