GRAPHICAL COMBINATORICS AND A DISTRIBUTIVE LAW FOR MODULAR OPERADS

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Abstract. This work presents a detailed analysis of the combinatorics of modular operads. These are operad-like structures that admit a contraction operation as well as an operadic multiplication. Their combinatorics are governed by graphs that admit cycles, and are known for their complexity. In 2011, Joyal and Kock introduced a powerful graphical formalism for modular operads. This paper extends that work. A monad for modular operads is constructed and a corresponding nerve theorem is proved, using Weber’s abstract nerve theory, in the terms originally stated by Joyal and Kock. This is achieved using a distributive law that sheds new light on the combinatorics of modular operads.

Introduction

Modular operads, introduced in [18] to study moduli spaces of Riemann surfaces, are a “‘higher genus’ analogue of operads . . . in which graphs replace trees in the definition.” [18, Abstract].

Roughly speaking, modular operads are \( N \)-graded objects \( P = \{P(n)\}_{n \in \mathbb{N}} \) that, alongside an operadic multiplication (or composition) \( \circ: P(n) \times P(m) \to P(m + n - 2) \) for \( m, n \geq 1 \), admit a contraction operation \( \zeta: P(n) \to P(n - 2) \), \( n \geq 2 \). For example, as in Figure 1, we may multiply two oriented surfaces by gluing them along chosen boundary components, or contract a single surface by gluing together two distinct boundary components.

![Figure 1](image.png)

FIGURE 1. Gluing (multiplication) and self-gluing (contraction) of surfaces along boundary components. Moduli of geometric structures – such as Riemann surfaces – provide many examples of modular operads.

This work considers a notion of modular operads due to Joyal and Kock [23] that incorporates a broad compass of related structures, including modular operads in the original sense of [18] (see Example 1.26) and their coloured counterparts [19], but also wheeled properads [20, 43] (see Example 1.29). More generally, compact closed categories [27] provide examples of modular operads [37] (see Example 1.27). These are closely related to circuit algebras that are used in the study of finite-type knot invariants [1, 14] (see Example 1.28). As such, modular operads have applications across a range of disciplines.

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1Joyal and Kock used the term ‘compact symmetric multicategories (CSMs)’ in [29] to refer to what are here called ‘modular operads’. Indeed, I adopted their terminology in [36] and in a previous version of this paper.
However, the combinatorics of modular operads are complex. In modular operads equipped with a multiplicative unit, contracting this unit leads to an exceptional ‘loop’, that can obstruct the proof of general results. This paper undertakes a detailed investigation into the graphical combinatorics of modular operads, and provides a new understanding of these loops.

In [23], which forms the inspiration for this work, Joyal and Kock construct modular operads as algebras for an endofunctor on a category $\mathbf{GS}$ of coloured collections called ‘graphical species’. Their machinery is significant in its simplicity. It relies only on minimal data and basic categorical constructions, that lend it considerable formal and expressive power.

However, the presence of exceptional loops means that their modular operad endofunctor does not extend to a monad on $\mathbf{GS}$. As a consequence, it does not lead to a precise description of the relationship between modular operads and their graphical combinatorics. (See Section 6 for details.)

This paper contains proofs of the following statements that first appeared in [23] (and were proved – by similar, though slightly less general methods than those presented here – in my PhD thesis [36]):

**Theorem 0.1** (Monad existence Theorem 7.46). The category $\mathbf{MO}$ of modular operads is isomorphic to the Eilenberg-Moore category of algebras for a monad $\mathcal{O}$ on the category $\mathbf{GS}$ of graphical species.

In particular, $\mathcal{O}$ is the algebraically free monad [24] on the endofunctor of $\mathbf{GS}$. Theorem 0.2 – the ‘nerve theorem’ – characterises modular operads in terms of presheaves on a category $\Xi$ of graphs.

**Theorem 0.2** (Nerve Theorem 8.2). The category $\mathbf{MO}$ has a full subcategory $\Xi$ whose objects are graphs. The induced (nerve) functor $N$ from $\mathbf{MO}$ to the category $\text{psh}(\Xi)$ of presheaves on $\Xi$ is fully faithful.

There is a canonical (restriction) functor $R^* : \text{psh}(\Xi) \to \mathbf{GS}$, and the essential image of $N$ consists of precisely those presheaves $P : \Xi^{\text{op}} \to \text{Set}$ that satisfy the so-called ‘Segal condition’:

$P$ is in the essential image of $N$ if and only if it is completely determined by the graphical species $R^* P$.

An obvious motivation for establishing such results is provided by the study of weak, or $(\infty, 1)$-modular operads, by weakening the Segal condition of Theorem 0.2. To this end, Hackney, Robertson and Yau have also recently proved versions of Theorems 0.1 and 0.2, by different methods, and used them to obtain a model of $(\infty, 1)$-modular operads that are characterised in terms of a weak Segal condition [21, 22]. A number of potential applications of such structures are discussed in the introduction to [21].

The aim of this work is to prove Theorems 0.1 and 0.2 in the manner originally proposed by [23] – using the abstract nerve machinery introduced by Weber [42, 6] (see Section 2) – and to use these proofs as a route to a full understanding of the underlying combinatorics, and the contraction of multiplicative units in particular. This method places strict requirements on the relationship between the modular operad monad $\mathcal{O}$ and the graphical category $\Xi$. In fact, to apply the results of [6], the category $\Xi$ must – in a sense that will be made precise in Section 2 – arise naturally from the definition of $\mathcal{O}$.

Neither the construction of the monad $\mathcal{O}$ for modular operads, nor the proof of Theorem 0.2 is entirely straightforward. First, the method of [23], which is closely related to analogous constructions for operads (Examples 5.1, 6.1, c.f. [20, 28, 33, 35]) does not lead to a well-defined monad. Second, as a consequence of the contracted units, the desired monad, once obtained, does not satisfy the conditions for applying the machinery of [6]. To prove Theorems 0.1 and 0.2 it is therefore necessary to break the problem into smaller pieces, thereby rendering the graphical combinatorics of modular operads completely explicit.

Since the obstruction to obtaining a monad in [23] arises from the combination of the modular operadic contraction operation and the multiplicative units (see Section 6), the approach of this work is to first treat these structures separately – via a monad $T$ on $\mathbf{GS}$ whose algebras are non-unital modular operads, and a monad $\mathcal{D}$ on $\mathbf{GS}$ that adjoins distinguished ‘unit’ elements – and then combine them, using the theory of distributive laws [4].
Theorem 0.1 is then a corollary of:

**Theorem 0.3** (Proposition 7.39 & Theorem 7.46). *There is a distributive law $\lambda$ for $T$ over $D$ such that the resulting composite monad $DT$ on $GS$ is precisely the modular operad monad $O$ of Theorem 0.1.*

The graphical category $\Xi$, used to define the modular operadic nerve, arises canonically via the unique fully faithful–bijective on objects factorisation of a functor used in the construction of $O$. Therefore, if the monad $O$ satisfies certain formal conditions – if it ‘has arities’ (see [6]) – then Theorem 0.2 follows from [6, Section 1].

Though the monad $O$ on $GS$ does not have arities, the distributive law in Theorem 0.3 implies that there is a monad $T^*$, on the category $GS^*$ of $D$-algebras, whose algebras are modular operads. Moreover, Theorem 0.2 follows from:

**Lemma 0.4** (Lemma 8.11). *The monad $T^*$ on $GS^*$ has arities, and hence satisfies the conditions of [6, Theorem 1.10].*

I conclude this introduction by briefly mentioning three (related) benefits of this abstract approach.

In the first place, the results obtained by this method provide a clear overview of how modular operads fit into the wider framework of operadic structures, and how other general results may be modified to this setting. For example, by Lemma 0.4, $T^*$ and $\Xi$ satisfy the Assumptions 7.9 of [10], which leads to a suitable notion of weak modular operad via the following corollary:

**Corollary 0.5** (Corollary 8.14). *There is a model structure on the category of presheaves in simplicial sets on $\Xi$. The fibrant objects are precisely those presheaves that satisfy a weak Segal condition.*

Second, since this work makes the combinatorics of modular operads – including the tricky bits – completely explicit, it provides a clear road map for working with and extending the theory.

One fruitful direction for extending this work is to use iterated distributive laws [11] to generalise constructions presented here. In [38], an iterated distributive law is used to construct circuit operads – modular operads with an extra product operation, closely related to small compact closed categories – as algebras for a composite monad on $GS$ (Example 1.28). Once again, the distributive laws play an important role in describing the corresponding nerve. The approach of [11, Section 3] may also be used to construct higher (or $(n,k)$-) modular operads. This can be used to give a modular operadic description of extended cobordism categories.

Finally, the complexities of the combinatorics of contractions can provide new insights into the structures they are intended to model. In current work, also together with L. Bonatto, S. Chettih, A. Linton, M. Robertson, N. Wahl, I am using these ideas to explore singular curves in the compactification of moduli spaces of algebraic curves. (See also Example 1.26, and c.f. [2] for the genus 0 case.)

This work owes its existence to the ideas of A. Joyal and J. Kock and I thank Joachim for taking time to speak with me about it. P. Hackney, M. Robertson and D. Yau’s work has been an invaluable resource. Conversations with Marcy have been particularly helpful. I gratefully acknowledge the anonymous reviewer whose insights have not only improved the paper, but also increased my appreciation of the mathematics.

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Remark 0.6. The following errors appear in the published version [39] and are corrected here:

In [39] Section 4.1, graph embeddings (Definition 4.6) were mistakenly identified with graph monomorphisms ([39], Proposition 4.8) and the terminology of ‘monomorphisms’ was used throughout the paper. The incorrect [39] Proposition 4.8 – which served only to establish terminology in [39] – has been deleted and [39] Lemma 4.7 has been replaced with Definition 4.6 of graph ‘embeddings’, and Lemma 4.7. The examples in Section 4.1 have also been modified accordingly. The terminology of graph embeddings is due to [21], Section 1.3, and replaces the incorrect use of the term (graphical) ‘monomorphism’ in [39].

On [39] page 61, there is a sentence that begins “But then $el(\circ) \cong el(i)$, and hence $S(\circ) \cong S(i)$ …”. This should simply read “But this would imply that $S(\circ) \cong S(i)$ …”, and the rest of Section 6 is unchanged.

Overview and key points. The opening two sections provide context and background for the rest of the work. An axiomatic definition of modular operads is given in Section 1. Section 2 gives a brief review of Weber’s abstract nerve theory, that provides a framework for the later sections. Both these introductory sections include a number of examples to motivate the constructions that follow.

Section 3 is a detailed introduction to the (Feynman) graphs of [23], and Section 4 focuses on their étale morphisms. The monad $T$ for non-unital modular operads is constructed in Section 5.

Section 6 acts as a short intermezzo in which the appearance of exceptional loops in the theory, and why they are problematic in the construction of [23], is explained.

The construction of the monad $O$ for modular operads happens in Section 7. This is the longest and most important section of the work, and contains most of the new contributions. Finally, Section 8 contains the proof of the Nerve Theorem 0.2 as well as a short discussion on weak modular operads.

There have been many other approaches to the issue of loops, some of which are mentioned in Remarks 6.6 and 6.7. But the graphical construction presented in this paper is unique, as far as I am aware, in that it does not incorporate some version of the exceptional loop into the graphical calculus, in order to model contractions of units. (See Remark 6.8.)

In other approaches, the contraction of units is described by adjoining a formal colimit of a diagram of graphs, resulting in the exceptional loop object (see Example 3.16). By contrast, we will see in Section 7 that the definition of modular operads (Definition 1.24) implies that the contracted units are, in fact, described in terms of a formal limit of the very same diagram. This is illustrated in Figure 2.

Moreover, this construction leads to a graphical description of the unit contraction, not by an exceptional loop, but as the singularity of a ‘double cone’ of wheel-shaped graphs (see Section 7.4 and Figure 25).

1. Definitions and examples

The goal of this section is to give an axiomatic definition of modular operads (Definition 1.24), and to provide some motivating examples. As mentioned in the introduction, the term ‘modular operad’ refers here to what are called ‘compact symmetric multicategories (CSMs)’ in [23].

1.1. Graphical species. After establishing some basic notional conventions, we discuss Joyal and Kock’s graphical species [23] that generalise various notions of coloured collection used in the study of operads.

Let $\mathbf{Set}$ be the category of sets and all morphisms between them. A presheaf on a category $C$ is a functor $P: C^{op} \to \mathbf{Set}$. The corresponding functor category is denoted $\text{psh}(C)$. 
Remark 1.2. all presheaves $c$ by restriction. Conversely, a $\Sigma$-presheaf $P$ of elements of $P$ -- where $c$ is an object of $C$ and $x \in P(c)$. Morphisms $(c,x) \to (d,y)$ in $\text{el}_C(P)$ are given by morphisms $f \in C(c,d)$ such that $P(f)(y) = x$.

If a presheaf $P$, on an essentially small category $C$, is of the form $C(-,c)$, for some $c \in C$, then $\text{el}_C(P)$ is the slice category $C/c$ whose objects are pairs $(d,f)$ where $f \in C(d,c)$, and morphisms $(d,f) \to (d',f')$ are given by commuting triangles in $C$:

$$
\begin{array}{ccc}
d & \overset{g}{\rightarrow} & d' \\
\downarrow{f} & & \downarrow{f'} \\
c. & \overset{c}{\leftarrow} & c.
\end{array}
$$

Given a functor $\iota : D \to C$, let $\iota^*C(-,c)$, $d \mapsto C(\iota(d),c)$ be the induced pullback on presheaves. For all $c \in C$, the slice category of $D$ over $c$ is defined by $D/c \overset{\text{def}}{=} \text{el}_D(\iota^*C(-,c))$. (This involves a small abuse of notation, and $D/c$ is more accurately denoted by $\iota/c$.)

In particular, the Yoneda embedding $C \to \text{psh}(C)$ induces a canonical isomorphism $\text{el}_C(P) \cong C/P$ for all presheaves $P$ on $C$, and these categories will be identified in this work.

The groupoid of finite sets and bijections is denoted by $\mathcal{B}$. For $n \in \mathbb{N}$, the set $\{1, \ldots, n\}$ is denoted by $\n$. So $\emptyset = \emptyset$ is the empty set.

Remark 1.2. Let $\Sigma \subset \mathcal{B}$ denote the skeletal subgroupoid on the objects $\n$, for $n \in \mathbb{N}$. A presheaf $P : \mathcal{B}^{\text{op}} \to \text{Set}$ on $\mathcal{B}$, also called a (monochrome or single-sorted) species [24], determines a presheaf on $\Sigma$ by restriction. Conversely, a $\Sigma$-presheaf $Q$ may always be extended to a $\mathcal{B}$-presheaf $Q_{\mathcal{B}}$, by setting

$$
Q_{\mathcal{B}}(X) \overset{\text{def}}{=} \lim_{(n,f) \in \Sigma/X} Q(n)
$$

for all $n \in \mathbb{N}$.

Graphical species, defined in [23] Section 4, are a coloured or multi-sorted version of species.

Let the category $\mathcal{B}^\|$ be obtained from $\mathcal{B}$ by adjoining a distinguished object $\|$ that satisfies

- $\mathcal{B}^\|(\|,\|) = \{\text{id, } \tau\}$ with $\tau^2 = \text{id}$,
- for each finite set $X$ and each element $x \in X$, there is a morphism $ch_x \in \mathcal{B}^\|(\|,X)$ that ‘chooses’ $x$, and $\mathcal{B}^\|(\|,X) = \{ch_x \mid ch_x \circ \tau\}_{x \in X}$,
- $\mathcal{B}^\|(X,Y) = \mathcal{B}(X,Y)$ for all finite sets $X$ and $Y$, and morphisms are equivariant with respect to the action of $\mathcal{B}$. That is, $ch_{f(x)} = f \circ ch_x \in \mathcal{B}^\|(\|,Y)$ for all $x \in X$ and all bijections $f : X \xrightarrow{\cong} Y$. 

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (a) at (0,0) {$\bullet$};
\node (b) at (2,0) {$z$};
\node (c) at (2,-1) {$\text{flip edge}$};
\node (d) at (4,-1) {$\text{id}$};
\node (e) at (6,-1) {$\text{Formal colimit: glue endpoints of edge to form a loop object.}$};
\node (f) at (2,-3) {$\text{Formal limit: pick out midpoint of edge.}$};
\draw[->] (a) -- (b);
\draw[->] (b) -- (c);
\draw[->] (c) -- (d);
\draw[->] (d) -- (e);
\end{tikzpicture}
\caption{An edge graph with no vertices may be flipped or left unchanged. The exceptional loop that ‘glues the edge ends together’ arises as the formal colimit of these endomorphisms. In Section 7 the graph category of Sections 3-5 (and 23) is enlarged to include the morphism $z$ that ‘picks out the midpoint’ of the edge graph with no vertices.}
\end{figure}
Definition 1.3. A graphical species is a presheaf $S : B^S \rightarrow \text{Set}$. The element category of a graphical species $S$ is denoted by $\text{el}(S) \equiv \text{el}_{B^S}(S)$, and the category of graphical species by $GS \equiv \text{psh}(B^S)$. Hence, a graphical species $S$ is described by a species $(S_X)_{X \in B}$, and a set $S_\emptyset$ with involution $S_\tau : S_\emptyset \rightarrow S_\emptyset$, together with, for each finite set $X$, and $x \in X$ a $B$-equivariant projection $S(ch_x) : S_X \rightarrow S_\emptyset$.

Definition 1.4. Given a graphical species $S$, the pair $(S_\emptyset, S_\tau)$ is called the (involutive) palette of $S$ and elements $c \in S_\emptyset$ are colours of $S$. If $S_\emptyset$ is trivial then $S$ is a monochrome graphical species.

For each element $c = (c_x)_{x \in X} \in S_\emptyset^X$, the $c$-(coloured) arity $S_\Lc$ is the fibre above $c \in S_\emptyset^X$ of the map $(S(ch_x))_{x \in X} : S_X \rightarrow S_\emptyset^X$.

Remark 1.5. The involution $\tau$ on $\emptyset$ is responsible for much of the heavy lifting in the constructions that follow. Initially however, its role may seem obscure. I mention two key features here. First, the involution provides the expressive power necessary to describe composition rules involving colours, such as particle spin, that may have an orientation, or direction. (Directed graphical species are discussed in Example 1.10.)

The second is more fundamental. As will be explained in Example 3.20, $B^S$ embeds in a certain category of graphs. Under this embedding, the distinguished object $\emptyset$ is represented as the exceptional edge with no vertices, and the involution $\tau$ as the ‘flip’ map that swaps its ends (see Figure 2). This enables us to encode formal compositions in graphical species – described in terms of graphs – as categorical limits, and thereby derive the results of this paper by purely abstract methods. For example, the involution underlies a well-defined notion of graph nesting, or substitution, in terms of diagram colimits, without the need to specify extra data (see Sections 5 and 6 and compare with, e.g. [43, 20]).

Example 1.6. The terminal graphical species $K$ has trivial palette and $K_X = \{\ast\}$ for all finite sets $X$.

Definition 1.7. A morphism $\gamma \in GS(S, S')$ is palette-preserving if its component $\gamma_\emptyset$ at $\emptyset$ is the identity on $S_\emptyset$. For a fixed palette $(C, \omega)$, $GS^{(C, \omega)}$ is the subcategory of $GS$ on the $(C, \omega)$-coloured graphical species and palette-preserving morphisms.

Example 1.8. For any palette $(C, \omega)$, the terminal $(C, \omega)$-coloured graphical species $K^{(C, \omega)}$ in $GS^{(C, \omega)}$ is described by $K_X^{(C, \omega)} = C^X$ with $K_\emptyset^{(C, \omega)} = \{\ast\}$ for all finite sets $X$ and all $\emptyset \in C^X$.

In particular, let $\sigma_{D_1}$ be the unique non-identity involution on the set $D_1 = \{\text{in, out}\}$. A monochrome directed graphical species is a graphical species with palette $(D_1, \sigma_{D_1})$. The terminal monochrome directed graphical species is denoted by $D_1 = K^{(D_1, \sigma_{D_1})}$. See also Example 1.10.

Remark 1.9. In the graphical representation of the category $B^S$, mentioned in Remark 1.5, a finite set $X$ is represented by a corolla or star graph $C_X$ with legs in bijection with $X$ (Figure 3 left side).

An element $\phi \in S_X$ of a graphical species $S$ is represented as a labelling or decoration of the unique vertex of $C_X$, and a colouring of the legs of $C_X$ by $S_\emptyset$ according to $S(ch_x)$ for $x \in X$ (Figure 3 right side).

Example 1.10. The graphical species $D_1$ was defined in Example 1.8. For each finite set $X$, $D_1|_X = \{\text{in, out}\}^X$ is the set of partitions $X = X_{\text{in}} \sqcup X_{\text{out}}$ of $X$ into input and output sets, with blockwise action of the partition-preserving isomorphisms in $\text{el}(D_1)$.

In other words, $\text{el}(D_1)$ is equivalent to the category $(B \times B^{op})^\emptyset$, obtained from $B \times B^{op}$ by adjoining a distinguished object $\emptyset$ (see Figure 4a) with trivial endomorphism group, and – for all pairs $(X, Y)$ of finite sets – input morphisms $i_x : \emptyset \rightarrow (X, Y)$ for all $x \in X$, and output morphisms $o_y : \emptyset \rightarrow (X, Y)$ for all $y \in Y$, that are compatible with the action of $B \times B^{op}$ (see Figure 4d)).
Figure 3. Graphical species may be represented graphically: \( \phi \in S_X \) is represented as a \( X \)-corolla \( C_X \) with vertex decorated by \( \phi \) and \( x \)-leg coloured by \( c_x = S(ch_x) \).

The objects \((X, Y)\) of \((B \times B^{op})_x^x\) may be represented, as in Figure 4(b), as directed corollas and the distinguished object \((\downarrow)\) as a directed exceptional edge (Figure 4(a)). If \( Y = \{\ast\} \) is a singleton, then \((X, \{\ast\})\) describes a rooted corolla as in Figure 4(c).

Hence \( GS/\mathfrak{D}i \) is equivalent to the category \( \text{psh}((B \times B^{op})_x^x) \) of directed graphical species. The subcategory \( GS(\mathfrak{D}i, \sigma \mathfrak{D}i)/\mathfrak{D}i \) of monochrome directed graphical species is equivalent to \( \text{psh}(B \times B^{op}) \).

Figure 4. (a) The directed exceptional edge \((\downarrow)\); (b) the pair \((X, Y)\) \( \in (B \times B^{op})_x^x \) describes a directed corolla; (c) \((X, \{\ast\})\) describes a rooted corolla \( t_X \); (d) input and output morphisms in \((B \times B^{op})_x^x\).

A \textit{PROP} \cite{30} is a strict symmetric monoidal category \((E, +, 0)\) whose objects are natural numbers and whose monoidal product + is addition on objects. More generally, for any set \( \mathfrak{D} \), a \( \mathfrak{D} \)-coloured \textit{PROP} \((E^{\mathfrak{D}}, \oplus, \emptyset)\) is a strict symmetric monoidal category whose monoid of objects is freely generated by \( \mathfrak{D} \). By Remark 1.2 this is equivalently a presheaf \( P^{\mathfrak{D}} \) on \((B \times B^{op})_x^x\) with \( P^{\mathfrak{D}}(\downarrow) = \mathfrak{D} \) and,

\[
P^{\mathfrak{D}}(X; Y) = \lim_{(m, f) \in E/X \atop (m, g) \in E/Y} \prod_{(c, d) \in \mathfrak{D}^m \times \mathfrak{D}^n} E^{\mathfrak{D}}(c, d)
\]

together with composition and monoidal product maps, and an injection \( P^{\mathfrak{D}}(\downarrow) \hookrightarrow P^{\mathfrak{D}}(1; 1) \) that induces the identities for composition. In particular, PROPs may be described in terms of graphical species.

1.2. Multiplication and contraction on graphical species. Intuitively, a multiplication \( \cdot \) on a graphical species \( S \) is a rule for combining (gluing) distinct elements of \( S \) along pairs of legs (called ‘ports’) with dual colouring as in Figure 5.

The notation ‘\( \rightharpoonup \)’ denotes a partial map of sets. So \( f: A \to B \) is given by a subset \( A' \subset A \) and a function \( A' \to B \).

Definition 1.11. A multiplication \( \cdot \) on a graphical species \( S \) is given by a family of partial maps

\[
\phi^{X,Y}_{x,y} : S_{X,\Pi(x)} \times S_{Y,\Pi(y)} \to S_{X,\Pi(y)},
\]
defined (for all $X, Y$ and $x, y$) whenever $\phi \in S_{XII(x)}$, $\psi \in S_{YII(y)}$ satisfy $S(ch_x)(\phi) = S(ch_y \circ \tau)(\psi)$.

The multiplication $\circ$ satisfies the following conditions:

(m1) (Commutativity axiom.)
Wherever $\phi^{X, Y}_{x, y}$ is defined,
$$\psi \circ^{Y, X}_{y, x} \phi = \phi \circ^{X, Y}_{x, y} \psi$$

(m2) (Equivariance axiom.)
For all bijections $\hat{\sigma} : X \xrightarrow{\cong} W$ and $\hat{\rho} : Y \xrightarrow{\cong} Z$ that extend to bijections $\sigma : X \amalg \{x\} \xrightarrow{\cong} W \amalg \{w\}$ and $\rho : Y \amalg \{y\} \xrightarrow{\cong} Z \amalg \{z\}$,
$$S(\hat{\sigma} \sqcup \hat{\rho})(\phi \circ_{w, z}^{W, Z} \psi) = S(\sigma)(\phi) \circ_{x, y}^{X, Y} S(\rho)(\psi),$$

(where $\hat{\sigma} \sqcup \hat{\rho} : X \amalg Y \xrightarrow{\cong} W \amalg Z$ is the block permutation).

A unit for the multiplication $\circ$ is a map $\epsilon : S_{\emptyset} \to S_2$, $c \mapsto \epsilon_c$ such that, for all $X$ and all $\phi \in S_{XII(x)}$ with $S(ch_x) = c \in S_{\emptyset}$,
$$\phi \circ_{x, 2}^{1, X} \epsilon_c = \epsilon_c \circ_{2, x}^{1, X} \phi = \phi.$$

A multiplication $\circ$ is called unital if it has a unit $\epsilon$. In this case $\epsilon_c$ is a $c$-coloured unit for $\circ$.

If $(\circ, \epsilon : S \to S_2)$ is a unital multiplication on a $(\mathcal{C}, \omega)$-coloured graphical species $S$, then $\epsilon_c \in S_{(c, \omega c)}$ for all $c \in \mathcal{C}$. Let $\sigma_2 \in \mathbf{B}^\emptyset(2, 2)$ be the unique non-identity endomorphism.

**Lemma 1.13.** If $\circ$ admits a unit $\epsilon : S \to S_2$, it is unique. Moreover, $\epsilon$ is compatible with the involutions $\omega = S_\tau$ on $\mathcal{C}$ and $S(\sigma_2)$ in that

$$\epsilon \circ \omega = S(\sigma_2) \circ \epsilon : \mathcal{C} \to S_2.$$

**Proof.** If $\epsilon : S_{\emptyset} \to S_2$ is a unit for $\circ$ then, by definition $S(\sigma_2)\epsilon_c = (S(\sigma_2)\epsilon_c) \circ_{1, 2}^{2, (1)} \epsilon_{\omega c}$ for all $c \in \mathcal{C}$. By equivariance $(S(\sigma_2)\epsilon_c) \circ_{1, 2}^{2, (1)} \epsilon_{\omega c} = \epsilon_c \circ_{2, 1}^{2, (1)} \epsilon_{\omega c}$, so
$$S(\sigma_2)\epsilon_c = (S(\sigma_2)\epsilon_c) \circ_{1, 2}^{2, (1)} \epsilon_{\omega c} = \epsilon_c \circ_{2, 2}^{1, (1)} \epsilon_{\omega c} = \epsilon_{\omega c},$$

whereby the second statement is proved.

Now, let $\lambda : \mathcal{C} \to S_2$, $c \mapsto \lambda_c$ be another unit for $\circ$. Then, for all $c \in \mathcal{C}$,
$$\epsilon_c = \epsilon_c \circ_{1, 2}^{2, (1)} \lambda_c = (S(\sigma_2)\epsilon_c) \circ_{2, 2}^{1, (1)} \lambda_c = \epsilon_{\omega c} \circ_{2, 2}^{1, (1)} \lambda_c = \lambda_c.$$

Hence multiplicative units are unique. $\square$

**Remark 1.15.** Equivalently, a multiplication $\circ$ on $(\mathcal{C}, \omega)$-coloured graphical species $S$ is a family of maps

$$- \circ_{c, d}^e : S_{(e, c)} \times S_{(d, \omega c)} \to S_{(ed)}, \quad c \in \mathcal{C}, \ e \in \mathcal{C}^X, \ d \in \mathcal{C}^Y,$$

Both (1.12) and (1.16) are used in what follows. Where the context is clear, the superscripts may be dropped altogether.
As one would expect, a multiplication \( \circ \) on a graphical species \( S \) is called ‘associative’ if the result of several consecutive multiplications does not depend on their order. This is stated precisely in condition (M1) of Definition 1.24 and visualised in the figure therein.

**Example 1.17.** A graphical species \( O \) equipped with a unital, associative multiplication \((\circ, e)\) is a cyclic operad in the sense of [16]. When the involution is trivial, these are the *entries-only* cyclic operads of [13] (see there for a comparison with cyclic operads as introduced in [17]).

Some advantages of the involutive, graphical species approach to cyclic operads are discussed in [16] and [22, Introduction].

**Example 1.18.** Operads (see e.g. [7]) admit a description as graphical species with unital multiplication:

Recall, from Examples 1.8 and 1.10, the graphical species \( Di \), and the category \((B \times B^{op})^+ \simeq \text{el}(Di)\) whose objects are either the exceptional directed edge \((\downarrow)\), or pairs \((X, Y)\) of finite sets.

If \( Y \cong \{\ast\} \) is a singleton, then \((X, \{\ast\})\) is called a rooted corolla, and denoted by \( t_X \) (Figure 4(c)). Let \( B^+ \subset (B \times B^{op})^+ \) be the full subcategory on \((\downarrow)\) and all rooted corollas \( t_X \).

Presheaves \( O : B^{op} \to \text{Set} \) are described by a set \( \mathfrak{D} = O(\downarrow) \) and sets \( O((c; d), \xi) \), defined for all \( d \in \mathfrak{D} \) and \( \xi \in \mathfrak{D}^X \) (for all \( X \)), and such that the action of \( B \) on \( O \) induces isomorphisms \( O((c_x)_x; d) \cong O((c_{f(x)})_{x}; d) \) for all \( f : X \xrightarrow{\cong} Y \). Hence, a \( \mathfrak{D} \)-coloured operad is a \( B^+ \) presheaf \( O \), together with an operadic composition, and a \( d \)-coloured unit \( 1_d \in O(d; d) \) for each \( d \in \mathfrak{D} \).

The graphical species \( RC \subset Di \) is given by \( \text{el}(RC) \xrightarrow{\cong} B^+ \) under the restriction of the equivalence \( \text{el}(Di) \cong (B \times B^{op})^+ \). So, \( RC_{\emptyset} = Di_{\emptyset} = \{\text{in, out}\} \), \( RC_0 = \emptyset \), and \( RC_{x} \) consists of those \( \phi \in Di_{x} \) such that \( Di(ch_x)(\phi) = \text{(in)} \) for all \( x \in X \), and \( Di(ch_x)(\phi) = \text{(out)} \).

Clearly, \( RC \) inherits the trivial unital multiplication from \( Di \). Moreover, a presheaf \( O : B^{op} \to \text{Set} \) has the structure of an operad precisely when the corresponding graphical species \( O^{\text{GS}} \in \text{GS/RC} \) is equipped with an associative unital multiplication. Hence, the category \( \text{Op} \) of (symmetric) operads is equivalent to the category whose objects are objects of \( \text{GS/RC} \) with an associative unital multiplication, and whose morphisms are morphisms in \( \text{GS/RC} \) that preserve the multiplication and units.

**Remark 1.19.** Examples 1.17 and 1.18 highlight the expressive power of graphical species. The involution \( \tau \) on \( \natural \) means that (undirected) cyclic operads and (directed) operads may be expressed in terms of presheaves on the same underlying category. (See also Examples 1.10 and 1.29)

Intuitively, a contraction \( \zeta \) on a graphical species \( S \) may be thought of as a rule for ‘self-gluing’ single elements of \( S \) along pairs of ports with dual colouring (Figure 6). The presence of a contraction operation enables modular operads to encode algebraic structures – such as those involving trace – that ordinary operads cannot [33, 34].

![Figure 6. Contraction](image-url)
Definition 1.20. A contraction $\zeta$ on $S$ is given by a family of partial maps

$$\zeta_{x,y}^X : S_{X \cup \{x,y\}} \to S_X$$

defined for all finite sets $X$ and all $\phi \in S_{X \cup \{x,y\}}$, such that $S(ch_x)(\phi) = S(ch_y \circ \tau)(\phi)$, and equivariant with respect to the action of $B$ on $S$: If $\sigma : X \cup \{x,y\} \overset{\cong}{\to} Z \cup \{w,z\}$ extends the bijection $\hat{\sigma} : X \overset{\cong}{\to} Z$ by $\sigma(x) = w, \sigma(y) = z$, then for any $\phi \in S_{Z \cup \{w,z\}}$, we have

$$S(\hat{\sigma}) (\zeta_{w,z}^Z(\phi)) = \zeta_{x,y}^X (S(\sigma)(\phi)).$$

If $\zeta$ is a contraction on $S$, then by, equivariance, $\zeta_{x,y}^X(\phi) = \zeta_{y,x}^X(\phi)$ wherever defined.

Remark 1.22. A contraction $\zeta$ on a $\mathcal{C}, \omega$-coloured graphical species $S$ is equivalently a family of maps

$$\zeta_c^c : S_{(c,c,\omega_c)} \to S_c$$

for $c \in \mathcal{C}$, and $c \in \mathcal{C}^X$. Depending on context, both $\zeta_c^c$ (and even $\zeta_c$) and (1.21) will be used.

Let $S$ be a $\mathcal{C}, \omega$-coloured graphical species equipped with a unital multiplication $(\circ, \epsilon)$ and contraction $\zeta$. By Lemma 1.13 there is a contracted unit map

$$o \overset{\text{def}}{=} \zeta \epsilon : \mathcal{C} \to S_0,$$

satisfying $\zeta_c(\epsilon_c) = \zeta_{\omega_c}(\epsilon_{\omega_c})$ for all $c \in \mathcal{C}$.

As will be explained in Sections 6 and 7 the contracted units $o : S_0 \to S_0$ present the main challenge for describing the combinatorics of modular operads.

1.3. Modular operads: definition and examples. Modular operads are graphical species with multiplication and contraction operations that satisfy the nicest possible (mutual) coherence axioms.

Definition 1.24. A modular operad is a graphical species $S$, with palette $(\mathcal{C}, \omega)$, say, together with a unital multiplication $(\circ, \epsilon)$, and a contraction $\zeta$, that together satisfy the following four coherence axioms governing their composition:

(M1) Multiplication is associative.

For all $b \in \mathcal{C}^X_1, c \in \mathcal{C}^X_2, d \in \mathcal{C}^X_3$ and all $c, d \in \mathcal{C}$, the following square commutes:

$$S_{(b,c)} \times S_{(c,\omega_c,d)} \times S_{(d,\omega_d)} \overset{o_c \times id}{\longrightarrow} S_{(b,c,d)} \times S_{(d,\omega_d)}$$

id $\times o_d$

$$S_{(b,c)} \times S_{(d,\omega_c,d)} \overset{o_c}{\longrightarrow} S_{bcd}.$$
(M2) Order of contraction does not matter.
For all $c \in \mathcal{C}^X$ and $c, d \in \mathcal{C}$, the following square commutes:

$$
\begin{array}{ccc}
S_{(\xi, \omega c, d, \omega d)} & \xrightarrow{\zeta_c} & S_{(\xi, \omega d)} \\
\downarrow \zeta_d & & \downarrow \zeta_d \\
S_{(\xi, \omega c)} & \xrightarrow{\zeta_c} & S_{\xi} \\
\end{array}
$$

(M3) Multiplication and contraction commute.
For all $c \in \mathcal{C}^{X_1}$, $d \in \mathcal{C}^{X_2}$ and $c, d \in \mathcal{C}$, the following square commutes.

$$
\begin{array}{ccc}
S_{(\xi, \omega c, d)} \times S_{(\omega c, \omega d)} & \xrightarrow{\zeta_c \times id} & S_{(\xi, \omega d)} \times S_{(\xi, \omega d)} \\
\downarrow \phi_c \downarrow \phi_c & & \downarrow \phi_c \downarrow \phi_c \\
S_{(\xi, \omega c, \omega d)} & \xrightarrow{\zeta_c} & S_{\xi} \\
\end{array}
$$

(M4) ‘Parallel multiplication’ of pairs.
For all $c \in \mathcal{C}^{X_1}$, $d \in \mathcal{C}^{X_2}$, and $c, d \in \mathcal{C}$, the following square commutes:

$$
\begin{array}{ccc}
S_{(\xi, c, d)} \times S_{(\omega d, \omega c)} & \xrightarrow{\phi_c} & S_{(\xi, d, d)} \times S_{(\xi, \omega d)} \\
\downarrow \phi_c \downarrow \phi_c & & \downarrow \phi_c \downarrow \phi_c \\
S_{(\xi, c, \omega d)} & \xrightarrow{\zeta_c} & S_{\xi} \\
\end{array}
$$

Modular operads form a category $\text{MO}$ whose morphisms are morphisms of the underlying graphical species that preserve multiplication, contraction and multiplicative units.

Informally, the multiplication and contraction operations describe rules for collapsing edges of graphs that represent formal compositions of elements. The coherence axioms (M1)-(M4) say that this is independent of the order in which the edges are collapsed.

**Remark 1.25.** A non-unital modular operad $(S, \diamond, \zeta)$ is a graphical species $S$ equipped with a multiplication $\diamond$ and contraction $\zeta$ satisfying (M1)-(M4), but without the requirement of a multiplicative unit. These form a category $\text{MO}^-$ whose morphisms are morphisms in $\text{GS}$ that preserve the multiplication and contraction operations. Non-unital modular operads are the subject of Section 5.
To provide context and motivation for the constructions that follow, the remainder of this section is devoted to examples.

**Example 1.26. Getzler-Kapranov modular operads.** The monochrome graphical species \( M \) given by \( M_X = \mathbb{N} \) for all \( X \in \mathcal{B} \), admits a unital multiplication \((+,0 \in M_2)\) induced by addition in \( \mathbb{N} \), and a contraction \( t \) induced by the successor operation:

\[
+: M_m \times M_n \to M_{m+n-2}, \quad (g_m, g_n) \mapsto g_m + g_n \quad (m,n \geq 1);
\]

\[
t: M_n \to M_{n-2}, \quad g_n \mapsto g_n + 1 \quad (n \geq 2).
\]

Since a compact oriented surface with boundary is determined, up to homeomorphism, by its genus and number of boundary components, the combinatorics of \( n \) and the spaces \( M \) may encode (moduli spaces) of geometric structures on surfaces. For example, the Deligne-Mumford compactification \( \overline{M}_{g,n} \) of the moduli space of genus \( g \) smooth algebraic curves with \( n \) marked points, may be described, via Belyi’s Theorem, in terms of the space of genus \( g \) Riemann surfaces with \( n \) nodes, and the spaces \( \overline{M}_{g,n} \) form a monochrome modular operad (c.f. [18] Example 6.2).

Getzler and Kapranov originally defined modular operads [18] in terms of the restriction to the stable part \( M^{st} \subset M \) of the graphical species \( M \), bigraded by pairs \((g,n)\) such that \( 2g+n-2 \geq 0 \). So \( M_0^{st} = M_n \) for \( n > 2 \) but \( M_0^{st} = \{2,3,4,\ldots\} \) and \( M_1^{st} = M_2^{st} = \{1,2,3,4,\ldots\} \).

In particular, since \( 0 \notin M_2^{st} \), modular operads in the original sense of [18] are non-unital.

These ideas may be extended to many-coloured cases: for example, one can describe a 2-coloured modular operad for gluing surfaces along open and closed subsets of their boundaries. (See, e.g. [19].)

**Example 1.27. Compact closed categories.** introduced in [25], are symmetric monoidal categories \((C, \otimes, e)\) for which every object \( c \in C \) has a symmetric categorical dual (see [5] [27]): there is an object \( c^* \in C \), and morphisms \( \cap_c : e \to c^* \otimes c \) and \( \cup_c : c \otimes c^* \to e \) such that

\[
(\cup_c \otimes id_c) \circ (id_c \otimes \cap_c) = id_c = (\cap_{c^*} \otimes id_{c^*}) \circ (id_{c^*} \otimes \cup_{c^*}).
\]

Examples of compact closed categories include categories of finite dimensional vector spaces over a given field, or, more generally, finite dimensional projective modules over a commutative ring. Cobordism categories provide other important examples.

There is a canonical monadic adjunction \( \mathbf{MO} \cong \mathbf{Comp}_{inv} \), where \( \mathbf{Comp}_{inv} \) is the category of involutive compact closed categories, whose objects are small compact closed categories \( C \) such that \( c = c^{**} \) for all \( c \in C \). The right adjoint takes an involutive compact closed category \((C, \otimes, e, *)\) with object set \( C_0 \) to a \((C_0, *)\)-coloured modular operad \( S^C \) with coloured arities

\[
S^C(d_1,\ldots,d_n,c_{m_1},\ldots,c_1) = C(c_1 \otimes \cdots \otimes c_m, d_1 \otimes \cdots \otimes d_n).
\]

The modular operad structure on \( S^C \) is induced by composition in \( C \) together with \( \cup \) and \( \cap \). The left adjoint \( \mathbf{MO} \to \mathbf{Comp}_{inv} \) is induced by the free monoid functor on palettes and arities.

These observations underly the proof, in [37], of an ‘operadic’ nerve theorem for compact closed categories in the style of Section 8.
Example 1.28. Circuit algebras—so named because of their resemblance to electronic circuits—are a symmetric version of Jones’s planar algebras, introduced to study finite-type invariants in low-dimensional topology [1, 14].

The category of Set-valued circuit algebras is equivalent to a category CO of circuit operads [35] whose objects are modular operads equipped, via a monadic adjunction MO ⇐ CO, with an extra ‘external product’ operation. Moreover, the adjunction between modular operads and involutive compact closed categories in Example 1.27 factors through the adjunction MO ⇐ CO.

This formal perspective on modular operads, circuit algebras, and compact closed categories leads to interesting questions in a number of directions. For example, we can study the analogous relationships if the definition of modular operads is relaxed by replacing the symmetric action with a braiding, or by considering higher dimensional versions. Related ideas are being explored by Dansco, Halacheva and Robertson in their work on algebraic and categorical structures in low-dimensional topology [15].

Example 1.29. Wheeled properads. Wheeled properads have been studied extensively in [20] and [43]. They describe the connected part (c.f. [41, Introduction]) of wheeled PROPs (i.e. coloured PROPs with a contraction) that have applications in geometry, deformation theory, and other areas [33, 34].

The category WP of (Set-valued) wheeled properads is canonically equivalent to the slice category MO/Di of directed modular operads. This is well-defined since the terminal directed graphical species Di trivially admits the structure of a modular operad (see Example 1.10). An equivalence between wheeled PROPs in linear categories and directed circuit algebras is established in [14].

2. Abstract nerve theorems and distributive laws

The purpose of this largely formal section is to review some basic theory of distributive laws, and provide an overview of Weber’s abstract nerve theory. The simplicial nerve for categories, and the dendroidal nerve for operads provide motivating examples for the latter.

For an overview of monads and their Eilenberg–Moore (EM) categories of algebras, see for example [31, Chapter VI].

2.1. Monads with arities and abstract nerve theory. Given an essentially small category C, a functor F: D → C induces a nerve functor N_D: C → psh(D) by N_D(c)(d) = C(Fd, c) for all c ∈ C, d ∈ D. If N_D is fully faithful, and F and D are suitably nice, then N_D provides a useful tool for studying C.

In the crudest sense, monads with arities are monads whose EM category of algebras may be characterised in terms of a fully faithful nerve, the construction of which is entirely abstract. The aim of this section is to explain, without proofs, the key points of this abstract nerve theory (details may be found in [31, Sections 1-3]). This motivates the framework of this paper, and underlies the proof of the nerve theorem for modular operads, Theorem 8.2 in Section 8.

Recall that every functor admits an (up to isomorphism) unique bo-ff factorisation as a bijective on objects functor followed by a fully faithful functor. For example, if M is a monad on a category C, and CM is the EM category of algebras for M, then the free functor C → CM has bo-ff factorisation C → CM → CM, where CM is the Kleisli category of free M-algebras (see e.g. [31, Section VI.5]).

Hence, for any subcategory D of C, the bo-ff factorisation of the canonical functor D → C ↪ free CM factors through the full subcategory ΘM,D of CM with objects from D.

By construction, the defining functor ΘM,D → CM is fully faithful. It is natural to ask if there are conditions on D and M that ensure that the induced nerve NM,D: CM → psh(ΘM,D) is also fully faithful. This is the motivation for describing monads with arities.
**Definition 2.1.** The essential image \( im^e(F) \) of a functor \( F : E \to C \) is the smallest subcategory of \( C \) that contains the image \( im(F) \) of \( F \) in \( C \) and is closed under isomorphisms in \( C \).

A subcategory \( \iota : D \hookrightarrow C \) is a dense subcategory (and \( \iota \) is a dense functor) if the induced nerve \( N_D : C \to psh(D) \) is full and faithful.

Once again, let \( M = (M, \mu^M, \eta^M) \) be a monad on \( C \). Let \( \iota : D \to C \) be the inclusion of a dense subcategory, and let \( \Theta_{M,D} \) be obtained in the boff factorisation of \( D \to C^M \). There is an induced diagram of functors

\[
\begin{array}{ccc}
\Theta_{M,D} & \xrightarrow{\text{f.f.}} & C^M \\
\text{b.o.} & \downarrow j & \text{free} \\
D^{\text{dense}} & \to & C \\
\end{array}
\]

\[
\begin{array}{ccc}
N_{M,D} & \to & psh(\Theta_{M,D}) \\
\text{forget} & \downarrow & \text{f.f.} \\
N_D & \to & psh(D). \\
\end{array}
\]

where \( j^* \) is the pullback of the bijective on objects functor \( j : D \to \Theta_{M,D} \). The left square of (2.2) commutes by definition, and the right square commutes up to natural isomorphism.

By [29] Proposition 5.1, the inclusion \( \iota : D \to C \) is dense if and only if every object \( c \) of \( C \) is given canonically by the colimit of the functor \( D/c \to C \), \((d,f) \mapsto \iota(d)\).

The monad \( M \) has arities \( D \) if \( N_D \circ M \) takes the canonical colimit cocones \( D/c \) in \( C \) to colimit cocones in \( psh(D) \). In this case, by [12] Section 4, the full inclusion \( \Theta_{M,D} \to C^M \) is dense, and the essential image of the induced fully faithful nerve \( N_{M,D} : C^M \to psh(\Theta_{M,D}) \) is the full subcategory of \( psh(\Theta_{M,D}) \) on those presheaves \( P \) whose restriction \( j^* P \) to \( D \) is in the essential image of \( N_D : C \to psh(D) \).

**Remark 2.3.** The condition that \( M \) has arities \( D \to C \) is sufficient, but not necessary, for the induced nerve \( C^M \to psh(\Theta_{M,D}) \) to be fully faithful.

In fact, by Theorems 8.2 and Remark 8.12, the modular operad monad \( \mathcal{O} \) on the category of graphical species, together with the full dense subcategory \( CGr_{\text{et}} \hookrightarrow GS \) of connected graphs and étale morphisms (described in Section 4), provides an example of a monad that does not have arities, but for which the nerve theorem holds.

Necessary conditions on \( M \) and \( D \hookrightarrow C \), for the induced nerve to be fully faithful are described in [9].

**Example 2.4.** Recall that directed graphs \( G = (s,t : E \rightrightarrows V) \) are presheaves over the small diagram category \( \mathcal{E} \overset{\text{def}}{=} \bullet \rightrightarrows \bullet \), and that the canonical forgetful functor from \( \text{Cat} \) to \( psh(\mathcal{E}) \) – that assigns to a small category \( C \), the directed graph \( G_C \) with vertex set \( V_C \) indexed by objects of \( C \), and edge set \( E_C \) indexed by morphisms of \( C \) – is monadic. So, every directed graph freely generates a small category.

For \( n \in \mathbb{N} \), the finite ordinal \([n]\) may be viewed as a directed linear graph:

\[
[n] = \overset{0}{} \to 1 \to \cdots \to n. \tag{2.5}
\]

The free category on \([n]\) is the \( n \)-simplex \( \Delta(n) \), and \( \Delta \) is the simplex category of simplices \( \Delta(n) \), \( n \in \mathbb{N} \), and functors between them. The category of \( \Delta \)-presheaves, or simplicial sets, is denoted by \( sSet \).

The **classical nerve theorem** states that the induced nerve functor \( N_{\Delta} : \text{Cat} \to sSet \) is fully faithful. Moreover, its essential image consists of precisely those \( P \in sSet \) that satisfy the classical Segal condition, originally formulated in [10]: a simplicial set \( P \) is the nerve of a small category if and only if, for \( n > 1 \), the set \( P_n \) of \( n \)-simplices is isomorphic to the \( n \)-fold fibre product

\[
P_n \cong P_1 \times_{P_0} \cdots \times_{P_1} P_1. \tag{2.6}
\]

The nerve theorem and Segal condition (2.6) may be derived using abstract nerve theory:
Let $\Delta_0 \subseteq \text{psh}(\mathcal{E})$ be the full subcategory on the directed linear graphs $[n]$ whose morphisms $f : [m] \to [n]$ satisfy $f(i + 1) = f(i) + 1$ for all $0 \leq i < m$. In particular, $\mathcal{E}$ embeds in $\Delta_0$ as the full subcategory on the objects $[0]$ and $[1]$, and the full inclusion $\Delta_0 \to \text{psh}(\mathcal{E})$ is precisely the nerve induced by the inclusion $\mathcal{E} \hookrightarrow \Delta_0$. Hence $\mathcal{E}$ is dense in $\Delta_0$. Since $\mathcal{E} \to \Delta_0$ is fully faithful, so is $N_{\Delta_0}$ (by [32, Section VII.2]), so $\Delta_0$ is also dense in $\text{psh}(\mathcal{E})$.

Since $\Delta$ is the category obtained in the bo-ff factorisation of $\Delta_0 \to \text{psh}(\mathcal{E}) \to \text{Cat}$, we consider the following diagram of functors

$$
\begin{array}{ccc}
\Delta & \to & \text{Cat} \\
\mathcal{E} & \downarrow^{j} & \text{free} \\
\Delta_0 & \to & \text{psh}(\mathcal{E})
\end{array}
$$

It is straightforward to prove – using for example [6, Sections 1 & 2] – that the category monad on $\text{psh}(\mathcal{E})$ has arities $\Delta_0$. Hence $N_{\Delta} : \text{Cat} \to \text{sSet}$ is fully faithful, and a simplicial set $P$ is in its essential image if and only if $j^*P$ is in the essential image of $N_{\Delta_0}$. Segal’s condition (2.6) follows from the fact that $\mathcal{E}$ is dense in $\Delta_0$.

**Remark 2.8.** The notion of graph in Example 2.4 is different and, in a suitable sense, dual to the one used in Example 2.10 and in the rest of this paper (from Section 3), where edges function as ‘objects’ and connections between them as ‘morphisms’.

The classical Segal condition (2.6) may be generalised as follows:

As before, let $D \subseteq C$ be a dense subcategory, and, as in Example 2.4, let $C = \text{psh}(\mathcal{E})$ be the category of presheaves on a dense subcategory $\mathcal{E}$ of $D$. So, the dense inclusion $D \to C$ is also full. If $D$ provides arities for a monad $M$ on $\mathcal{C}$, then by [6, Lemma 3.6], a presheaf $P : \Theta_{M,D}^p \to \text{Set}$ is in the essential image of $N_{M,D}$ if and only if

$$
P(jd) = \lim_{e,f \in \mathcal{E}/d} j^*(P)(e) \quad \text{for all } d \in D.
$$

Equation (2.9) is called the Segal condition for the nerve functor $N_{M,D}$.

**Example 2.10.** The category $B^\downarrow$ – whose objects are the directed exceptional edge $(\downarrow)$, and the rooted corollas $t_X$ (for all finite set $X$) – was describe in Example 1.18. Recall that an operad is a presheaf $O$ on $B^\downarrow$, together with a unital composition operation satisfying certain axioms. The forgetful functor $\text{Op} \to \text{psh}(B^\downarrow)$ is monadic, so every presheaf $O$ on $B^\downarrow$ freely generates an operad. Let $M_{\text{Op}}$ be the induced monad.

**Rooted trees** $\mathcal{T}$ are obtained as formal colimits of finite diagrams in $B^\downarrow$ that describe grafting of objects of $B^\downarrow$ root-to-leaf as in Figure 7(b). Let $\Omega_0$ be the category whose objects are such rooted trees $\mathcal{T}$ and whose morphisms $\mathcal{S} \to \mathcal{T}$ are (up to isomorphism) inclusions of rooted trees that preserve vertex valency (as in Figure 7(a)). Then $B^\downarrow \subseteq \Omega_0$ is the full and dense subcategory of rooted trees with zero or one vertex.

Hence, the induced nerve $\Omega_0 \to \text{psh}(B^\downarrow)$ is full and faithful, and $B^\downarrow$ canonically induces a topology on $\Omega_0$ whose sheaves are precisely $B^\downarrow$-presheaves. In particular, $\Omega_0 \to \text{psh}(B^\downarrow)$ is also dense (see e.g. Section 4.4 for comparison), and there is a diagram of functors

$$
\begin{array}{ccc}
\Omega & \to & \text{Op} \\
\mathcal{B}^\downarrow & \downarrow^{j} & \text{free} \\
\Omega_0 & \to & \text{psh}(\mathcal{B}^\downarrow)
\end{array}
$$
in which the left square commutes and the right square commutes up to natural isomorphism. The full subcategory of $\text{Op}$ induced by the bo-ff factorisation of the functor $\Omega_0 \to \text{psh}(B^\perp) \to \text{Op}$ is the dendroidal category $\Omega$ of free operads on rooted trees. This is described in [35], where it was established that the full inclusion $\Omega \hookrightarrow \text{Op}$ is dense, and hence the dendroidal nerve $N\Omega$ is fully faithful.

It is easy to show, e.g. using methods similar to those described in Section 8, that the monad $M\text{Op}$ on $\text{psh}(B^\downarrow)$ has arities $\Omega_0$. Hence, the abstract nerve theory of [6] may also be used to show that the nerve functor $N\Omega : \text{Op} \to \text{psh}(\Omega)$ is fully faithful and its essential image consists of those $\Omega$-presheaves (or dendroidal sets) $O : \Omega^{\text{op}} \to \text{Set}$ that satisfy the dendroidal Segal condition first proved in [12, Corollary 2.6]:

\[(2.12) \quad O(\Sigma) = \lim_{(t,i) \in (B^\downarrow/\Sigma)} j^* O(t) \quad \text{for all symmetric rooted trees } \Sigma.\]

In particular, since $\Delta_0$ is the full subcategory of linear trees in $\Omega_0$, the simplicial nerve theorem for categories is a special case of the dendroidal nerve theorem for operads.

**Definition 2.13.** A pointed endofunctor on a category $C$ is an endofunctor $E$ on $C$ together with a natural transformation $\eta^E : 1_C \Rightarrow E$. An algebra for a pointed endofunctor $(E, \eta^E)$ on $C$ is a pair $(c, \theta)$ of an object $c$ of $C$ and a morphism $\theta \in C(Ec, c)$ such that $\theta \circ \eta^E_c = \text{id}_c \in C(c, c)$.

For example, modular operads are algebras for the pointed endofunctor on $\text{GS}$ described in [23]. However, as discussed in Section 6, the abstract nerve machinery of [6] cannot be modified for algebras of (pointed) endofunctors:

For any monad $M = (M, \mu^M, \eta^M)$ on a category $C$, the EM category $C^M$ of $M$-algebras embeds canonically in the category $C^M$ of algebras for the pointed endofunctor $(M, \eta^M)$. The induced free functor $C \to C^M$, $c \mapsto (Mc, \mu^M_c)$ factors through $C^M$ and depends crucially on the monadic multiplication $\mu^M : M^2 \Rightarrow M$ of $M$.

By contrast, for an arbitrary pointed endofunctor $(E, \eta^E)$ is on $C$, there is, in general, no canonical choice of functor $C \to C^E$.

**2.2. Distributive laws.** With Examples 2.4 and 2.10 in mind, let us return to the case of modular operads. Recall that graphical species are presheaves on the category $B^\perp$ and that modular operads are graphical species equipped with certain operations.

Informally, monads are gadgets that encode, via their algebras, (algebraic) structure on objects of categories. In [23], it is the combination of the contraction structure $\zeta$, and the multiplicative unit structure $\epsilon$ that provides an obstruction to extending the modular operad endofunctor on $\text{GS}$ to a monad (see Section 6). So, one approach to constructing the modular operad monad $\Box$ on $\text{GS}$ could be to find...
monads for the modular operadic multiplication, contraction, and unital structures separately, and then attempt to combine them.

In general, monads do not compose. Given monads \( M = (M, \mu^M, \eta^M) \) and \( M' = (M', \mu^{M'}, \eta^{M'}) \) on a category \( C \), there is no obvious choice of natural transformation \( \mu: (MM')^2 \Rightarrow MM' \) defining a monadic multiplication for the endofunctor \( MM' \) on \( C \).

Observe, however, that any natural transformation \( \lambda: M' M \Rightarrow MM' \) induces a natural transformation

\[
(2.14) \quad \mu \lambda: (MM')^2 \xrightarrow{M\lambda M'} M^2M^2 \xrightarrow{\mu^M \mu^{M'}} MM'.
\]

**Definition 2.15.** A distributive law \( \mathcal{A} \) for \( M \) and \( M' \) is a natural transformation \( \lambda: M' M \Rightarrow MM' \) such that the triple \((MM', \mu, \eta^M \eta^{M'})\) defines a monad \( MMM' \) on \( C \).

A distributive law \( \lambda: M' M \Rightarrow M'M \) determines how the \( M \)-structures and \( M' \)-structures on \( C \) interact to form the structure encoded by the composite monad \( MMM' \).

**Example 2.16.** The category monad on \( \text{psh}(\mathcal{E}) \) (Example 2.4) may be obtained as a composite of the semi-category monad, which governs associative composition, and the reflexive graph monad that adjoins a distinguished loop at each vertex of a graph \( G \in \text{psh}(\mathcal{E}) \). The corresponding distributive law encodes the property that the adjoined loops provide identities for the semi-categorical composition.

(There is also a distributive law in the other direction, but the two structures do not interact in the composite. See also Remark 7.43)

As usual, let \( C^M \) denote the EM category of algebras for a monad \( M \) on \( C \).

By Section 3, given monads \( M, M' \) on \( C \), and a distributive law \( \lambda: M' M \Rightarrow M'M \), there is a commuting square of strict monadic adjunctions:

\[
(2.17)
\]

In Section 4, it is shown that the category \( \text{Gr}_\text{et} \) of connected Feynman graphs and étale morphisms (first defined in [23]) fits into a chain \( B^8 \xhookleftarrow{} \text{Gr}_\text{et} \xhookrightarrow{} \text{GS} \) of fully faithful dense embeddings. And, in Section 7, the modular operad monad \( \mathcal{O} \) on \( \text{GS} \) is constructed as a composite \( \mathcal{D}\mathcal{T} \) of monads \( \mathcal{T} \) (that governs contraction and non-unital multiplication) and \( \mathcal{D} \) (that governs multiplicative units) on \( \text{GS} \).

Hence, by (2.17), there is a monad \( \mathcal{T}_* \) on the EM category \( \text{GS}_* \) of \( \mathcal{D} \)-algebras, such that \( \text{GS}_*^{\mathcal{T}_*} \cong \mathcal{O} \) and a diagram of functors

\[
(2.18)
\]

in which the categories \( B^8, \text{Gr}_* \) and \( \Xi \) are obtained via bo-ff factorisations.

In Section 8, it is shown that \( \mathcal{T}_* \) has arities \( \text{Gr}_* \) (see Section 2), whence it follows that the induced nerve \( N: \mathcal{O} \rightarrow \text{psh}(\Xi) \) is fully faithful and its essential image is characterised in terms of \( B^8 \).
3. Graphs and their morphisms

This section is an introduction to Feynman graphs as defined in [23]. Most of this section and the
next stay close to the original constructions there. Since [23] was just a short note, it contained very few
proofs, and so relevant results are proved in full here. Extensive examples are also given. Where possible,
definitions and examples are presented in a way that builds on Section 1 and highlights similarities with
familiar concepts in basic topology.

This section deals with basic definitions and examples. The following section is devoted to a more
detailed study of the topology of Feynman graphs, in terms of their étale morphisms.

3.1. Graph-like diagrams and Feynman graphs. Roughly speaking, a graph consists of a finite set
of vertices \( V \) and a finite set of connections \( \tilde{E} \), together with an incidence relation: if \( \tilde{E} \) is the set of
orbits of a set \( E \) under an involution \( \tau \), then the incidence is a partial function \( E \to V \) that attaches
connections to vertices. In this paper, all graphs are finite, and may have loops, parallel edges, and loose
ends (ports).

Example 3.1. Section 15 of [3] provides a nice overview of various graph definitions that appear in the
operad literature. The definition that is perhaps most familiar is that found in, for example, [18] and
[8]. There, a graph \( G \) is described by sets \( V \) of vertices and \( E \) of edges, an involution \( \hat{\tau} : E \to E \), and
an incidence function \( \hat{t} : E \to V \). The ports of \( G \) are the fixed points of the involution \( \hat{\tau} \). A formal
exceptional edge graph \( \eta \) is also allowed. Morphisms \( \eta \to G \) are choices \( \{ \ast \} \to E \) of elements of \( E \).

Feynman graphs are defined similarly to the graphs described in Example 3.1, except the involution
on \( E \) must be fixed-point free, while the incidence is allowed to be a partial map \( E \to V \). These subtle
differences make it possible to encode the whole calculus of Feynman graphs in terms of the formal theory
of diagrams in finite sets.

The category of graph-like diagrams is the category \( \text{psh}_f(\mathcal{D}) \) of functors \( \mathcal{D}^{\text{op}} \to \text{Set}_f \), where \( \mathcal{D} \) is the
small category \( \bullet \to \bullet \to \bullet \to \bullet \), and \( \text{Set}_f \) is the category of finite sets and all maps between them.

The initial object in \( \text{psh}_f(\mathcal{D}) \) is the empty graph-like diagram:
\[
\emptyset = \\
\begin{array}{cc}
\circ & 0
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
0
\end{array}

and the terminal object \( \star \) is the trivial diagram of singletons:
\[
\star = \\
\begin{array}{cc}
\circ & 1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}

Feynman graphs, introduced in [23], are graph-like diagrams satisfying extra properties:

Definition 3.2. A Feynman graph is a graph-like diagram
\[
\mathcal{G} = \tau \begin{array}{c}
\circ
\end{array}
\begin{array}{c}
E
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
H
\end{array}
\begin{array}{c}
t
\end{array}
\begin{array}{c}
V
\end{array}

such that \( s : H \to E \) is injective and \( \tau : E \to E \) is an involution without fixed points.

A subgraph \( \mathcal{H} \to \mathcal{G} \) of a Feynman graph \( \mathcal{G} \) is a subdiagram that inherits a Feynman graph structure
from \( \mathcal{G} \).

The full subcategory on graphs in \( \text{psh}_f(\mathcal{D}) \) is denoted by \( \text{Grpsh}_f(\mathcal{D}) \).

Elements of \( V \) are vertices of \( \mathcal{G} \) and elements of \( E \) are called edges of \( \mathcal{G} \). For each edge \( e \), \( \hat{e} \) is the
\( \tau \)-orbit of \( e \), and \( \tilde{E} \) is the set of \( \tau \)-orbits in \( E \). Elements of \( H \) are half-edges of \( \mathcal{G} \). Together with the maps
\( s \) and \( t \), \( H \) encodes a partial map \( E \to V \) describing the incidence for the graph. A half-edge \( h \in H \) may
also be written as the ordered pair \( h = (s(h), t(h)) \).
In general, unless I wish to emphasise a point that is specific to the formalism of Feynman graphs, I will refer to Feynman graphs simply as ‘graphs’.

Remark 3.3. A graph $G$ may be realised geometrically by a one-dimensional space $|G|$ obtained from the discrete space $\{*_{e}\}_{e\in V}$, and, for each $e \in E$, a copy $[0, \frac{1}{2}]_{e}$ of the interval $[0, \frac{1}{2}]$ subject to the identifications $0_{h(h)} \sim *_{t(h)}$ for $h \in H$, and $(\frac{1}{2})_{e} \sim (\frac{1}{2})_{ee}$ for all $e \in E$.

Example 3.4. (See also Figure 8(a).) The graph (i) has edge set $2 = \{1, 2\}$ and no vertices.

\[(i) \overset{\text{def}}{=} \begin{array}{c}
0 \\
1 \\
2
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
0
\end{array}
\]

A stick graph is a graph that is isomorphic to (i).

For any set $X$, $X^\dagger \cong X$ denotes its formal involution.

Example 3.5. (See also Figure 8(b), (c).) The $X$-corolla $C_{X}$ associated to a finite set $X$ has the form

\[C_{X} : \overset{\text{inc}}{\begin{array}{c}
\uparrow \\
\circlearrowleft
\end{array}} \begin{array}{c}
X \sqcup X^\dagger
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
\ast
\end{array}\]

(a) \hspace{2cm} (b) \hspace{2cm} (c)

\begin{array}{c}
1 \\
2
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
x \\
x^\dagger
\end{array}
\begin{array}{c}
1 \\
2
\end{array}
\begin{array}{c}
3
\end{array}

Figure 8. Realisations of (a) the stick graph (i), and the corollas (b) $C_{\{x\}}$ and (c) $C_{3}$.

Definition 3.6. An inner edge of $G$ is an element $e \in E$ such that $\{e, t_e\} \subset \text{im}(s)$. The set $E_{\ast} \subset E$ of inner edges of $G$ is the maximal subset of $\text{im}(s) \subset E$ that is closed under $\tau$, and $\tilde{E}_{\ast}$ is the set of inner $\tau$-orbits $\tilde{e} \in \tilde{E}$ such that $e \in E_{\ast}$.

The set $E_{0} = E \setminus \text{im}(s)$ is the boundary of $G$. Elements $e \in E_{0}$ are ports of $G$.

A stick component of a graph $G$ is a pair $\{e, t_e\}$ of edges of $G$ such that $e$ and $t_e$ are both ports.

Graph morphisms preserve inner edges by definition. The stick graph (i) has $E_0(i) = E(i) = 2$, and, for all finite sets $X$, the $X$-corolla $C_X$ has boundary $E_0(C_X) = X$.

Since $\text{Set}_f$ admits finite (co)limits, so does $\text{psh}_f(D)$, and these are computed pointwise. And, since $\text{Grpsh}_f(D)$ is full in $\text{psh}_f(D)$, (co)limits in $\text{Grpsh}_f(D)$, when they exist, correspond to (co)limits in $\text{psh}_f(D)$.

Example 3.7. The empty graph-like diagram $\varnothing$ is trivially a graph, and is therefore initial in $\text{Grpsh}_f(D)$. However, there is no non-trivial involution on a singleton set, so the terminal diagram $\ast$ in $\text{psh}_f(D)$ is not a graph. Hence, $\text{Grpsh}_f(D)$ is not closed under finite limits in $\text{psh}_f(D)$. (By Examples 3.16 and 3.33 $\text{Grpsh}_f(D)$ is also not closed under finite colimits in $\text{psh}_f(D)$.)

The cocartesian monoidal structure on $\text{Set}_f$ is inherited by $\text{psh}_f(D)$ and $\text{Grpsh}_f(D)$, making these into strict symmetric monoidal categories under pointwise disjoint union $\sqcup$, and with monoidal unit given by the empty graph $\varnothing$.

Example 3.8. Let $X$ and $Y$ be finite sets. The graph $\mathcal{M}_{x_0, y_0}^{X, Y}$, illustrated in Figure 9, has two vertices and one inner edge orbit (highlighted in bold-face in Figure 9). It is obtained from the disjoint union $C_{X \sqcup \{x_0\}} \sqcup C_{Y \sqcup \{y_0\}}$ by identifying the $\tau$-orbits of the ports $x_0$ and $y_0$ according to $x_0 \sim \tau y_0, y_0 \sim \tau x_0$. So,
\( M_{x_0, y_0}^{X, Y} = \tau \big( (X \amalg Y) \amalg (X \amalg Y)^\dagger (x_0, y_0) \big) \xrightarrow{s} ((X \amalg Y)^\dagger (x_0, y_0) \big) \xrightarrow{t} \{v_X, v_Y\}, \)

where \( s \) is the obvious inclusion, and the involution \( \tau \) is described by \( x_0 \leftrightarrow y_0 \) and \( z \leftrightarrow z^\dagger \) for \( z \in X \amalg Y \). The map \( t \) is described by \( t^{-1}(v_X) = X^\dagger \{y_0\} \) and \( t^{-1}(v_Y) = Y^\dagger \{x_0\} \).

In the construction of modular operads, graphs of the form \( M_{x_0, y_0}^{X, Y} \) are used to encode formal multiplications in graphical species.

**Example 3.9.** Formal contractions in graphical species are encoded by graphs of the form \( \mathcal{N}_{x_0, y_0}^{X} \) (see Figure 9): For \( X \) a finite set, the graph \( \mathcal{N}_{x_0, y_0}^{X} \) is the quotient of the corolla \( C_{XII(x_0, y_0)} \) obtained by identifying the \( \tau \)-orbits of the ports \( x_0 \) and \( y_0 \) according to \( x_0 \sim \tau y_0 \) and \( y_0 \sim \tau x_0 \). It has boundary \( E_0 = X \), one inner \( \tau \)-orbit \( \{x_0, y_0\} \) (bold-face in Figure 9), and one vertex \( v \). So,

\[
\mathcal{N}_{x_0, y_0}^{X} = \tau \big( (X \amalg X) \amalg (x_0, y_0) \big) \xrightarrow{s} ((X^\dagger (x_0, y_0) \big) \xrightarrow{t} \{v\}.
\]

![Figure 9](image)

**Figure 9.** Realisations of \( M_{x_0, y_0}^{X, Y} \) and \( \mathcal{N}_{x_0, y_0}^{X} \) for \( X \cong 2, Y \cong 3 \).

Let \( \mathcal{G} \) be a graph with vertex and edge sets \( V \) and \( E \) respectively. For each vertex \( v \), define \( H/v \) as the fibre of \( t \) at \( v \), and let \( E/v \) as \( s(H/v) \subseteq E \).

**Definition 3.10.** Edges in the set \( E/v \) are said to be incident on \( v \).

The map \( |\cdot| \colon V \to \mathbb{N}, v \mapsto |v| \) defines \( |H/v| \), the valency of \( v \) and \( V_n \subseteq V \) is the set of \( n \)-valent vertices of \( \mathcal{G} \). A bivalent graph is a graph \( \mathcal{G} \) with \( V = V_2 \).

A vertex \( v \) is bivalent if \( |v| = 2 \). An isolated vertex of \( \mathcal{G} \) is a vertex \( v \in V(\mathcal{G}) \) such that \( |v| = 0 \).

Bivalent and isolated vertices are particularly important in Section 7.

Vertex valency also induces an \( \mathbb{N} \)-grading on the edge set \( E \) (and half-edge set \( H \)) of \( \mathcal{G} \): For \( n \geq 1 \), define \( H_n \) as \( t^{-1}(V_n) \) and \( E_n \) as \( s(H_n) \). Since \( s(H) = E \setminus E_0 = \bigcup_{n \geq 1} E_n \),

\[ E = \bigcup_{n \in \mathbb{N}} E_n. \]

**Example 3.11.** Recall the stick graph (i) from Example 3.4. Since \( H(i) \) is empty, both edges of (i) are ports: \( E(i) = E_0(i) \). The corolla \( C_X \) (Example 3.5) with vertex \( * \) has \( X \cong H/\ast = H^\dagger/\ast \). If \( |X| = n \), then \( |\ast| = n \), so \( V(C_n) = V_n \), and \( E = E_n \amalg E_0 \).

**Example 3.12.** For finite sets \( X \) and \( Y \), the graph \( M_{x_0, y_0}^{X, Y} \) (Example 3.8) has \( E/\nu_X = X^\dagger \{y_0\} \) and \( E/\nu_Y = Y^\dagger \{x_0\} \). If \( X \cong n \) for some \( n \in \mathbb{N} \), then \( \nu_X \subseteq V_{n+1} \), and \( E/\nu_X \subseteq E_{n+1} \).

The graph \( \mathcal{N}_{x_0, y_0}^{X} \) (Example 3.9) has \( E/\nu = X^\dagger \{x_0, y_0\} \cong H \), so \( V = V_{n+2} \) when \( X \cong n \).
Since $\text{Grpsh}_i(D)$ is full in the diagram category $\text{psh}_i(D)$, morphisms $f \in \text{Grpsh}_i(D)(G, G')$ are commuting diagrams in $\text{Set}_i$ of the form
\begin{equation}
\begin{array}{cccccc}
G & E & \tau & E & H & t \\
\downarrow f & \downarrow f_E & \downarrow f_E & \downarrow f_H & \downarrow f_V \\
G' & E' & \tau' & E' & H' & t' \\
\end{array}
\end{equation}

\textbf{Lemma 3.14.} For any morphism $f = (f_E, f_H, f_V) \in \text{Grpsh}_i(D)(G, G')$, the map $f_H$ is completely determined by $f_E$. Moreover if $G$ has no isolated vertices, then $f_E$ also determines $f_V$, and hence $f$.

If $G$ has no stick components or isolated vertices, then $f$ is completely determined by $f_H$.

(A directed version of Lemma 3.14 appeared as [28, Proposition 1.1.11].)

\textbf{Proof.} The map $f_H : H \to H'$ given by $h \mapsto (s')^{-1}f_Es(h)$ is well defined since $s$ is injective. If $G$ has no isolated vertices, then, for each $v \in V$, $H/v$ is non-empty and the map $f_V : V \to V'$ given by $v \mapsto t'(s')^{-1}f_Es(h)$ does not depend on the choice of $h \in H/v$.

If $G$ has no stick components then, for each $e \in E$, there is an $h \in H$ such that $e = s(h)$ or $e = \tau s(h)$, and the last statement of the lemma follows from the first. □

\textbf{Example 3.15.} For any graph $G$ with edge set $E$, $\text{Grpsh}_i(D)((i, G)) \cong E$. The morphism $1 \mapsto e \in E$ in $\text{Grpsh}_i(D)((i, G))$ that chooses an edge $e$ is denoted $ch_e$, or $ch_e^E$.

\textbf{Example 3.16.} The stick graph $(i)$ has endomorphisms $ch_1 = id$ and $ch_2 = \tau$ in $\text{Grpsh}_i(D)$. The coequaliser of $id, \tau : (i) \rightrightarrows (i)$ in the category $\text{psh}_i(D)$ of graph-like diagrams is the exceptional loop $\bigcirc$:

$$
\bigcirc \overset{\text{def}}{=} \begin{array}{ccc}
& 0 & 0 \\
\Uparrow & \mathrm{id} & \Downarrow \\
0 & 1 & 0 \\
\end{array}
$$

Clearly $\bigcirc$ is not a graph since a singleton set does not admit a non-trivial involution. Hence $\text{Grpsh}_i(D)$ does not admit all finite colimits. This example is the subject of Section 6.

\textbf{Definition 3.17.} A morphism $f \in \text{Grpsh}_i(D)(G, G')$ is \textit{locally injective} if, for all $v \in V$, the induced map $f_v : E/v \to E'/f(v)$ is injective, and locally surjective if $f_v : E/v \to E'/f(v)$ is surjective for all $v \in V$.

Locally bijective morphisms are called \textit{étale}.

Local bijections are preserved under composition, so \textit{étale} morphisms form a subcategory, $\text{Gr}_{et}$ of $\text{Grpsh}_i(D)$. This is the subject of Section 4.

\textbf{Example 3.18.} The following display illustrates the two morphisms $f_a$ and $f_b$ in $\text{Grpsh}_i(D)$ described by the commuting diagrams (a) and (b) below. Both morphisms are locally injective, and (b) is surjective, and also locally surjective, hence étale. By Lemma 3.14 both $f_a$ and $f_b$ are completely determined by the image of $E(C_2)$.

\textbf{(a)}

$$
\begin{array}{c}
\begin{array}{cc}
|C_2| & |G| \\
\end{array}
\begin{array}{cc}
\downarrow e_2 & \downarrow f_a \\
\end{array}
\end{array}
$$

\textbf{(b)}

$$
\begin{array}{c}
\begin{array}{cc}
|C_2| & |W| \\
\end{array}
\begin{array}{cc}
\downarrow e_2 & \downarrow f_b \\
\end{array}
\end{array}
$$

In each example, the horizontal maps are the obvious projections, and the columns in the edge sets represent the orbits of the involution.
Example 3.19. Recall Examples 3.8 and 3.9 above. For finite sets $X$ and $Y$, the canonical morphisms $C_X \longrightarrow M_{x_0,y_0}^{XY} \longrightarrow CY$ and $C_X \longrightarrow N_{x_0,y_0}^X$ are locally injective, but not locally surjective.

The canonical morphisms $C_{X\cup\{x_0\}} \longrightarrow M_{x_0,y_0}^{XY} \leftarrow C_{Y\cup\{y_0\}}$ are locally injective and locally surjective (hence étale), but neither is surjective. However, the canonical morphism $C_{X\cup\{x_0\}} \rightarrow N_{x_0,y_0}^X$ is étale and surjective. (See Example 3.18(b) for the case $X = \emptyset$.)

Example 3.20. The assignment $X \mapsto C_X$ describes a full embedding of $\text{Set}_f$ into $\text{Grpsh}(D)$. Since $\text{Grpsh}_f(D)((i, i) \cong B^\emptyset(\emptyset, \emptyset)$ canonically, and any morphism in $\text{Grpsh}_f(D)$ with domain $(i)$ is étale, it follows that $B^\emptyset$ embeds in $\text{Grpsh}(D)$ as the subcategory of étale morphisms between the corollas $C_X (X \in \text{Set}_f)$, and $(i)$.

Remark 3.21. By Example 3.20, $B^\emptyset$ will henceforth also be viewed as a subcategory of $\text{Grpsh}_f(D)$. The choice of notation for objects – $(i)$ or $\emptyset$, $X$ or $C_X$ – will depend on the context. The same notation will be used for morphisms in $B^\emptyset$ and their image in $\text{Grpsh}(D)$. So $ch_x \in B^\emptyset(\emptyset, X)$ may also be written as $ch_x \in \text{Grpsh}_f(D)((i, C_X)$, and $f \in B^\emptyset(X, Y)$ also describes an étale morphism $f \in \text{Grpsh}_f(D)(C_X, C_Y)$.

Example 3.22. For all finite sets $X$ and $Y$, the diagram

$$(3.23) \quad C_{X\cup\{x_0\}} \leftarrow ch_{x_0} \longrightarrow (i) \longrightarrow ch_{y_0} \circ \tau \longrightarrow C_{Y\cup\{y_0\}}$$

is in the image of the inclusion $B^\emptyset \hookrightarrow \text{Grpsh}_f(D)$. It has colimit $M_{x_0,y_0}^{XY}$ in $\text{Grpsh}_f(D)$.

The graph $N_{x_0,y_0}^X$ is the colimit in $\text{Grpsh}_f(D)$ of the diagram of parallel morphisms

$$(3.24) \quad ch_{x_0}, \ ch_{y_0} \circ \tau: (i) \rightrightarrows C_{X\cup\{x_0\}}$$

in the image of $B^\emptyset$ in $\text{Grpsh}_f(D)$. (See also Example 5.12 and Figure 15.)

As will be shown in Section 4, all graphs may be constructed canonically as colimits of diagrams in the image of $B^\emptyset \subset \text{Grpsh}_f(D)$.

3.2. Connected components of graphs. A graph is connected if it cannot be written as a disjoint union of non-empty graphs. Precisely:

Definition 3.25. A non-empty graph-like diagram $\mathcal{G}$ is connected if, for each $f \in \text{psh}(D)(\mathcal{G}, \star \amalg \star)$, the pullback of $f$ along the inclusion $\star \amalg \star \rightrightarrows \star \amalg \star$ is either the empty graph-like diagram $\emptyset$ or $\mathcal{G}$ itself. A graph $\mathcal{G}$ is connected if it is connected as a graph-like diagram.

A (connected) component of a graph $\mathcal{G}$ is a maximal connected subdiagram of $\mathcal{G}$.
By Definition 3.2 a subdiagram $\mathcal{H} \hookrightarrow \mathcal{G}$ is a subgraph precisely when $E(\mathcal{H}) \subseteq E$ is closed under $\tau: E \to E$. Hence:

**Lemma 3.26.** A connected component of a graph $\mathcal{G}$ inherits a subgraph structure from $\mathcal{G}$. If $\mathcal{H} \hookrightarrow \mathcal{G}$ is a subgraph of $\mathcal{G}$, then so is its complement $\mathcal{G} \setminus \mathcal{H}$.

Therefore, every graph is the disjoint union of its connected components.

**Remark 3.27.** A graph $\mathcal{G}$ is connected if and only if its realisation $|\mathcal{G}|$ (Remark 3.3) is a connected space.

**Example 3.28.** Following the terminology of [28], a shrub $\mathcal{S}$ is a graph that is isomorphic to a disjoint union of stick graphs. Hence a shrub $\mathcal{S} = \mathcal{S}(J)$ is determined by a set $J$ (of edges) equipped with a fixed-point free involution $\tau_J: J \rightarrow J$. A morphism in $\text{Grpsh}(\mathcal{D})$ whose domain is a shrub is trivially étale.

Given any graph $\mathcal{G}$, the shrub $\mathcal{S}(E)$ determined by $(E, \tau)$ is canonically a subgraph of $\mathcal{G}$. Components of $\mathcal{S}(E)$ are of the form

$$\{(e_{\bar{e}}) \in E \mid \{e, \tau e\} \to 0 \to 0,$$

for each $\bar{e} \in \tilde{E}$. The inclusion $\{e, \tau e\} \hookrightarrow E$ induces a subgraph inclusion $\iota_{\bar{e}}: \{(\bar{e})\} \hookrightarrow \mathcal{G}$ called the essential morphism at $\bar{e}$ (for $\mathcal{G}$).

Recall from Definition 3.6 that a stick component of $\mathcal{G}$ is a $\tau$-orbit $\{e, \tau e\}$ in the boundary $E_0$ of $\mathcal{G}$. In particular, $\{e, \tau e\}$ is a stick component if and only if $\iota_{\bar{e}}$ is a connected component of $\mathcal{G}$.

**Example 3.29.** Recall that, for each $v \in V$, $E_v = s(t^{-1}(v))$ is the set of edges incident on $v$. Let $v = (E_v)^t$ denote its formal involution. Then the corolla $C_v$ is given by

$$C_v = \bigcup_{(\bar{e}) \in E_v} (E_v)^t \xleftarrow{k} H_v \xrightarrow{t} \{v\}.$$

The inclusion $E_v \hookrightarrow E$ induces a morphism $\iota_v^G$ or $\iota_v: C_v \to \mathcal{G}$ called the essential morphism at $v$ for $\mathcal{G}$. If there exists an edge $e$ such that both $e$ and $\tau e$ are incident on $v$, then $\iota_v$ is not injective on edges.

If $E_v$ is empty – so $C_v$ is an isolated vertex – then $C_v \hookrightarrow \mathcal{G}$ is a connected component of $\mathcal{G}$.

**Example 3.30.** For $k \geq 0$, the line graph $\mathcal{L}^k$ (illustrated in Figure 10) is the connected bivalent graph with boundary $E_0 = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, and

- ordered set of edges $E(\mathcal{L}^k) = (l_j)_{j=0}^{1} \mathcal{L}^k$ where $l_0 = 1_{\mathcal{L}^k} \in E_0$ and $l_{2k+1} = 2_{\mathcal{L}^k} \in E_0$, and the involution is given by $\tau(l_{2i}) = l_{2i+1}$, for $0 \leq i \leq k$,

- ordered set of $k$ vertices $V(\mathcal{L}^k) = (v_i)_{i=1}^k$, such that $E(v_i) = \{l_{2i-1}, l_{2i}\}$ for $1 \leq i \leq k$.

So, $\mathcal{L}^k$ is described by a diagram of the form

$$\bigcup_{1} 2 \overset{2(k)}{\longrightarrow} 2(k) \xrightarrow{\mathcal{L}^k} k.$$

**Example 3.31.** The wheel graph $\mathcal{W} = \mathcal{W}^1$ with one vertex is the graph

(3.32) $$\mathcal{W} \overset{\text{def}}{=} \bigcup_{1} \{a, \tau a\} \xrightarrow{\{a, \tau a\}} \{\ast\}$$

obtained as the coequaliser in $\text{Grpsh}(\mathcal{D})$ of the morphisms $ch_1, ch_2 \circ \tau: (i) \rightarrow C_2$ (see Example 3.18(b)).

More generally, for $m \geq 1$, the wheel graph $\mathcal{W}^m$ (illustrated in Figure 10) is the connected bivalent graph obtained as the coequaliser in $\text{Grpsh}(\mathcal{D})$ of the morphisms $ch_{1, \mathcal{L}^m}, ch_{2, \mathcal{L}^m} \circ \tau: (i) \rightarrow \mathcal{L}^m$. So $\mathcal{W}^m$ has empty boundary and

- $2m$ cyclically ordered edges $E(\mathcal{W}^m) = (a_j)_{j=1}^{2m}$, such that the involution satisfies $\tau a_{2i} = a_1$ and $\tau(a_{2i}) = a_{2i+1}$ for $1 \leq i < m$,

- $m$ cyclically ordered vertices $V(\mathcal{W}^m) = (v_i)_{i=1}^m$, that $E_{v_i} = \{a_{2i-1}, a_{2i}\}$ for $1 \leq i \leq m$. 
So $\mathcal{W}^m$ is described by a diagram of the form $\bigcirc 2(m) \leftarrow 2(m) \rightarrow m$.

In Proposition 4.22 it will be shown that a connected bivalent graph is isomorphic to $L^k$ or $\mathcal{W}^m$ for some $k \geq 0$ or $m \geq 1$.

**Example 3.33.** The wheel graph $\mathcal{W}$ with one vertex is weakly terminal in $\text{Grpsh}_f(D)$: Since $\overline{\mathcal{E}}(\mathcal{W}) \cong V(\mathcal{W}) \cong \{\ast\}$, by Lemma 3.14 morphisms in $\text{Grpsh}_f(D)(\mathcal{G}, \mathcal{W})$ are in canonical bijection with projections in $\text{Grpsh}_f(D)(S(E), i)$. Hence, for all graphs $\mathcal{G}$, there are precisely $2|\overline{\mathcal{E}}| \geq 1$ morphisms $\mathcal{G} \rightarrow \mathcal{W}$ in $\text{Grpsh}_f(D)$ and every diagram in $\text{Grpsh}_f(D)$ forms a cocone over $\mathcal{W}$.

In particular,

$$\text{Grpsh}_f(D)(\mathcal{W}, \mathcal{W}) = \{id_{\mathcal{W}}, \tau_{\mathcal{W}}\} \cong \text{Grpsh}_f(D)(i, \mathcal{W}) \cong \text{Grpsh}_f(D)(i, i).$$

The morphisms $id_{\mathcal{W}}, \tau_{\mathcal{W}} : \mathcal{W} \Rightarrow \mathcal{W}$ do not admit a coequaliser in $\text{Grpsh}_f(D)$ since their coequaliser in $psh_f(D)$ is the terminal diagram $\star$, which is not a graph.

**Example 3.33** leads to another characterisation of connectedness:

**Proposition 3.34.** The following are equivalent:

1. A graph $\mathcal{G}$ is connected;
2. $\mathcal{G}$ is non-empty and, for every morphism $f \in \text{Grpsh}_f(D)(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$, the pullback in $psh_f(D)$ of $f$ along the inclusion $inc_1 : \mathcal{W} \hookrightarrow \mathcal{W} \amalg \mathcal{W}$ is either the empty graph $\varnothing$ or isomorphic to $\mathcal{G}$ itself;
3. for every finite disjoint union of graphs $\amalg_{i=1}^k \mathcal{H}_i$,

$$\text{Grpsh}_f(D)(\mathcal{G}, \amalg_{i=1}^k \mathcal{H}_i) \cong \prod_{i=1}^k \text{Grpsh}_f(D)(\mathcal{G}, \mathcal{H}_i).$$

**Proof.** (1) ⇔ (2): Since $\mathcal{W}$ is weakly terminal, any morphism $f \in \text{psh}_f(D)(\mathcal{G}, \star \amalg \star)$ factors as a morphism $\tilde{f} \in \text{Grpsh}_f(D)(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$ followed by the componentwise projection $\mathcal{W} \amalg \mathcal{W} \rightarrow \star \amalg \star$ in $\text{psh}_f(D)$.

(1) ⇒ (3): For any finite disjoint union of graphs $\amalg_{i=1}^k \mathcal{H}_i$, and $1 \leq j \leq k$, let $p_j \in \text{psh}_f(D)(\amalg_{i=1}^k \mathcal{H}_i, \star \amalg \star)$ be the morphism that projects $\mathcal{H}_j$ onto the first summand, and $\amalg_{i \neq j} \mathcal{H}_i$ onto the second summand. Then, for any graph $\mathcal{G}$ and any $f \in \text{Grpsh}_f(D)(\mathcal{G}, \amalg_{i=1}^k \mathcal{H}_i)$, the diagram

$$\begin{array}{ccc}
P_j \downarrow & \rightarrow & \mathcal{G} \downarrow f \\
\mathcal{H}_j \downarrow inc_j & \rightarrow & \amalg_{i=1}^k \mathcal{H}_i \\
\downarrow p_j & & \downarrow \mathcal{H}_j \downarrow inc_1 \leftarrow \star \amalg \star \\
\star \downarrow inc_1 & & \star \amalg \star 
\end{array}$$
where the top square is a pullback, commutes in $\text{psh}_I(D)$. Since the lower square is a pullback by construction, so is the outer rectangle.

In particular, if $\mathcal{G}$ is connected, then $\mathcal{P}_j$ is either empty or isomorphic to $\mathcal{G}$ itself. But this implies that there is some unique $1 \leq j \leq k$ such that $f$ factors through the inclusion $\text{inc}_j \in \text{Grpsh}_I(D)(\mathcal{H}_j, \bigsqcup_{i=1}^k \mathcal{H}_i)$. In other words, $\text{Grpsh}_I(D)(\mathcal{G}, \bigsqcup_{i=1}^k \mathcal{H}_i) \cong \bigsqcup_{i=1}^k \text{Grpsh}_I(D)(\mathcal{G}, \mathcal{H}_i)$.

(3) $\Rightarrow$ (2): If $\mathcal{G}$ satisfies condition (3), then $\text{Grpsh}_I(D)(\mathcal{G}, \mathcal{W} \sqcup \mathcal{W}) \cong \text{Grpsh}_I(D)(\mathcal{G}, \mathcal{W}) \sqcup \text{Grpsh}_I(D)(\mathcal{G}, \mathcal{W})$. So, taking $\bigsqcup_{i=1}^k \mathcal{H}_i = \mathcal{W} \sqcup \mathcal{W}$ in [3.36], we have $\mathcal{P}_j = \emptyset$ or $\mathcal{P}_j \cong \mathcal{G}$ for $j = 1, 2$. \hfill $\square$

3.3. Paths and cycles. Paths and cycles in a graph $\mathcal{G}$ may be defined using line and wheel graphs (Examples 3.30 and 3.31).

**Definition 3.37.** For any graph $\mathcal{G}$, a morphism $p \in \text{Grpsh}_I(D)(\mathcal{L}_k, \mathcal{G})$ is called a path of length $k$ in $\mathcal{G}$. Given any pair $x_1, x_2 \in E \sqcup V$, $x_1$ and $x_2$ are connected by a path $p \in \text{Grpsh}_I(D)(\mathcal{L}_k, \mathcal{G})$ if \{x$_1$, x$_2$\} $\subset$ im(p). A non-empty graph $\mathcal{G}$ is path connected if each pair of distinct elements $x_1, x_2 \in E \sqcup V$ is connected by a path in $\mathcal{G}$.

**Example 3.38.** The isolated vertex $\mathcal{C}_0$ is trivially path connected. Since $\text{Grpsh}_I(D)(\mathcal{L}_k, \mathcal{C}_1)$ is non-empty only when $k = 0$ or $k = 1$, the unique path $\mathcal{L}_1 = \mathcal{C}_2 \to \mathcal{C}_1$ is the only path that connects the unique vertex $v$ of $\mathcal{C}_1$ with an edge $e \in E(\mathcal{C}_1)$.

**Corollary 3.39** (Corollary to Proposition 3.34). A graph $\mathcal{G}$ is connected if and only if it is path connected.

**Proof.** A morphism $f: \mathcal{G} \to \mathcal{W}_1 \sqcup \mathcal{W}_2$ that does not factor through an inclusion $\mathcal{W} \to \mathcal{W}_1 \sqcup \mathcal{W}_2$ exists if and only if there are distinct $x_1, x_2 \in E \sqcup V$ such that $f(x_1) \in \mathcal{W}_1$ and $f(x_2) \in \mathcal{W}_2$. By Proposition 3.34, since $\mathcal{L}_k$ is connected for all $k$, this is the case if and only if there is no $p \in \text{Grpsh}_I(D)(\mathcal{L}_k, \mathcal{G})$ connecting $x_1$ and $x_2$. \hfill $\square$

**Lemma 3.40.** Let $\mathcal{G}$ be a connected graph. For any pair $(e_1, e_2)$ of edges of $\mathcal{G}$, there is a locally injective path connecting $e_1$ and $e_2$ in $\mathcal{G}$.

**Proof.** For all edges $e$ of $\mathcal{G}$, $\text{ch}_e: (\mathcal{L}_e) = \mathcal{L}^0 \to \mathcal{G}$ describes an injective path connecting $e$ and $\tau e$.

So, let $e_1$ and $e_2 \neq \tau e_1$ be distinct edges of a connected graph $\mathcal{G}$. By Corollary 3.39 there is a path $p \in \text{Grpsh}_I(D)(\mathcal{L}_k, \mathcal{G})$ connecting $e_1$ and $e_2$ in $\mathcal{G}$. Moreover, we may assume, without loss of generality, that, for $i = 1, 2$, $p(i_{\mathcal{L}_e}) \in \{e_1, \tau e_1\}$: if not, we may replace $p$ with a path $p \circ \iota$ -- where $\iota: \mathcal{L}_k' \to \mathcal{L}_k$ ($1 \leq k' < k$) is injective -- for which this holds.

If $p$ is not locally injective then there is some $1 \leq j \leq k$, such that $p(l_{2j-1}) = p(l_{2j}) \in E(\mathcal{G})$.

In this case, if $j = 1$, then $p$ may be replaced by a path $p_1: \mathcal{L}_k^{i-1} \to \mathcal{G}$ obtained by precomposing $p$ with the etale inclusion $\mathcal{L}_k^{i-1} \hookrightarrow \mathcal{L}_k$, $l_i' \mapsto l_{i+2}$, $0 \leq i \leq 2k - 1$.

If $1 < j < k$, then $p(l_{2j-1}) = p(l_{2j})$ implies that $p(v_{j-1}) = p(v_{j+1})$. Therefore, $p$ may be replaced with a path $p_j: \mathcal{L}_k^{i-2}$ of length $k - 2$ given by

$$p_j(l_i') = \begin{cases} p(l_i) & \text{for } 0 \leq i \leq 2j - 3, \\ p(l_{i+4}) & \text{for } 2j - 2 \leq i \leq 2k - 3. \end{cases}$$

Finally, if $j = k$, then replace $p$ with the path $p_k: \mathcal{L}_k^{k-1} \to \mathcal{G}$ obtained by precomposing $p$ with the inclusion $\mathcal{L}_k^{k-1} \hookrightarrow \mathcal{L}_k$, $l_i' \mapsto l_i$, $1 \leq i \leq k - 1$.

By iterating this process (always starting with the lowest value of $j$ for which the path $p$ is not injective at $v_j$), we obtain a unique, locally injective path $p_I$ connecting $e_1$ and $e_2$. \hfill $\square$
Morphisms from wheel graphs $W^m$ describe the higher genus structure of graphs (see Remark 3.43).

**Definition 3.41.** A cycle in $G$ is a morphism $c \in \text{Grpsh}_t(D)(W^m, G)$ for some $m \geq 1$.

A connected graph $G$ is simply connected if it has no locally injective cycles.

A cycle $c: W^m \to G$ is trivial if there is a simply connected graph $H$ such that $c$ factors through $H$.

It is straightforward, using the cyclic ordering on the edges of each $W^m$, to verify that a graph $G$ is simply connected if and only if its geometric realisation $|G|$ is.

**Example 3.42.** For all finite sets $X$, the corolla $C_X$ is trivially simply connected since it has no inner edges and therefore does not admit any cycles.

Since the edge sets of the line graphs $L^k$ are totally ordered for all $k$, there can be no locally injective morphism $W^m \to L^k$. Hence $L^k$ is simply connected. However, for all $k \geq 1$, there are morphisms $\text{Grpsh}_t(D)(W^{2k}, L^{k+1})$ that are surjective on vertices and inner edges. For example, let $V(W^{2k}) = (w_i)_{i=1}^{2k}$ and $V(L^{k+1}) = (v_j)_{j=1}^{k+1}$ be canonically ordered as in Examples 3.30 and 3.31. Then the assignment $w_i \mapsto v_i$ for $1 \leq i \leq 2k + 1$, and $w_{k+1-j} \mapsto v_{k+1-j}$ for $1 \leq j < k$ induces a morphism $q: W^{2k} \to L^{k+1}$ that flattens $W^{2k}$ (Figure 11(a)). This fails to be a local injection at $w_1$ and $w_k$.

More generally, by flattening $W^2$ as above, we see that, for any graph $G$, the set $E_*$ of inner edges of $G$ is non-empty if and only if $\text{Grpsh}_t(D)(W^2, G)$ is (see Figure 11(b)).

![Figure 11](image)

**Figure 11.** (a) A morphism $q: W^4 \to L^3$ in $\text{Grpsh}_t(D)$ that flattens $W^4$. (b) If a graph $G$ has an inner edge, then $\text{Grpsh}_t(D)(W^2, G)$ is non-empty.

**Remark 3.43.** For any graph $G$, we may define an equivalence relation of path homotopy on paths in $G$. Two paths in $G$ are homotopic if applying the proof of Lemma 3.40 to each leads to the same locally injective path $p_I$ in $G$. When $E_* \neq \emptyset$, this relation extends to an equivalence relation on cycles in $G$. If $G$ is also connected, the set of equivalence classes of cycles has a canonical group structure that is isomorphic to the fundamental group $\pi_1(|G|)$ of the geometric realisation of $G$.

The fundamental group construction can be extended, using Proposition 7.17, to all graphs $G$ without isolated vertices. These ideas are not developed in the current work.

4. The étale site of graphs

Recall that $\text{Gr}_{et} \subset \text{Grpsh}_t(D)$ is the bijective on objects subcategory of graphs and étale morphisms and that, by Example 3.20, there is a canonical categorical embedding $\mathbf{B}^g \hookrightarrow \text{Grpsh}_t(D)$ whose image consists of the exceptional graph $(\cdot)$, the corollas $C_X$, and the étale (locally bijective) morphisms between them.

The goal of this section is to describe $\text{Gr}_{et}$ – and its full subcategory $\text{CGr}_{et}$ on the connected graphs – in detail, and establish the chain

$$
\mathbf{B}^g \longleftarrow \text{CGr}_{et} \hookrightarrow \text{GS}
$$

of dense fully faithful categorical embeddings discussed in Section 2.

The following is immediate from Definition 3.17 and the universal property of pullbacks of sets:
Proposition 4.1. A morphism \( f \in \text{Grpsh}_\text{fr}(\mathcal{D}) (\mathcal{G}, \mathcal{G}') \) is étale if and only if the right square in the defining diagram (3.13) is a pullback of finite sets.

Example 4.2. For any graph \( \mathcal{G} \) and each edge \( e \) of \( \mathcal{G} \), the essential morphism \( \iota_e : (\{e\}) \to \mathcal{G} \) (Example 3.28) is trivially étale. For each vertex \( v \) of \( \mathcal{G} \), the essential morphism \( \iota_v : \mathcal{C}_v \to \mathcal{G} \) (Example 3.29) is also étale.

Indeed, a morphism \( f \in \text{Grpsh}_\text{fr}(\mathcal{D}) (\mathcal{G}, \mathcal{G}') \) is étale if and only if \( f \) induces an isomorphism \( \mathcal{C}_v \xrightarrow{\cong} \mathcal{C}_{f(v)} \) for all \( v \in V(\mathcal{G}) \).

Example 4.3. As discussed in Example 3.42, there are no étale morphism \( \mathcal{W}^m \to \mathcal{L}^k \), for any \( k \geq 0, m \geq 1 \).

All étale morphisms between line graphs are pointwise injective, and for \( k, n \in \mathbb{N} \),
\[
\text{Gr}_{\text{et}}(\mathcal{L}^k, \mathcal{L}^n) \cong \begin{cases} 
2(n-k+1) & n \geq k \\
\emptyset & n < k.
\end{cases}
\]

For \( m \geq 1 \), a morphism \( f \in \text{Gr}_{\text{et}}(\mathcal{L}^k, \mathcal{W}^m) \) is pointwise injective precisely when \( k < m \). For all \( k \geq 0 \), \( f \) is fixed by \( f(1_{\mathcal{L}^k}) \in E(\mathcal{W}^m) \). Hence, \( \text{Gr}_{\text{et}}(\mathcal{L}^k, \mathcal{W}^m) \cong E(\mathcal{W}^m) \cong 2(\mathbb{N}) \).

Étale morphisms between wheel graphs are surjective and for \( l, m \geq 1 \)
\[
\text{Gr}_{\text{et}}(\mathcal{W}^l, \mathcal{W}^m) \cong \begin{cases} 
2(l(m-1)) & \text{if } \frac{l}{m} \in \mathbb{N} \\
\emptyset & \text{otherwise}.
\end{cases}
\]

4.1. **Pullbacks and embeddings in \( \text{Grpsh}_\text{fr}(\mathcal{D}) \).** As local isomorphisms, étale morphisms of graphs have similar properties to local homeomorphisms of topological spaces.

**Lemma 4.4.** The graph categories \( \text{Grpsh}_\text{fr}(\mathcal{D}) \) and \( \text{Gr}_{\text{et}} \) admit pullbacks. Moreover, étale morphisms are preserved under pullbacks in \( \text{Grpsh}_\text{fr}(\mathcal{D}) \).

**Proof.** The pullback \( \mathcal{P} = (E, H, V, S, k, \mathcal{Z}) \) of morphisms \( f_1 \in \text{Grpsh}_\text{fr}(\mathcal{D})(\mathcal{G}_1, \mathcal{G}) \) and \( f_2 \in \text{Grpsh}_\text{fr}(\mathcal{D})(\mathcal{G}_2, \mathcal{G}) \) exists in the presheaf category \( \text{psh}_\text{fr}(\mathcal{D}) \). Moreover, since pullbacks in \( \text{psh}_\text{fr}(\mathcal{D}) \) are computed pointwise, \( \mathcal{Z} \) is a fixed-point free involution, and \( \mathcal{S} \) is injective. So, \( \mathcal{P} \) is a graph, and \( \text{Grpsh}_\text{fr}(\mathcal{D}) \) admits pullbacks.

Étale morphisms pull back to étale morphisms since limits commute with limits, and therefore, by symmetry, \( \text{Gr}_{\text{et}} \) admits pullbacks. \( \square \)

**Definition 4.5.** For any morphism \( f \in \text{Grpsh}_\text{fr}(\mathcal{D})(\mathcal{H}, \mathcal{G}) \), not necessarily étale, and any morphism \( w : \mathcal{G}' \to \mathcal{G} \), the preimage \( f^{-1}(\mathcal{G}') \to \mathcal{G}' \) of \( \mathcal{G}' \) under \( f \) is defined by the pullback
\[
\begin{array}{ccc}
\mathcal{f}^{-1}(\mathcal{G}') & \xrightarrow{f} & \mathcal{H} \\
\downarrow & & \downarrow f \\
\mathcal{G} & \xrightarrow{w} & \mathcal{G}'.
\end{array}
\]

In particular, by Lemma 4.4 if \( f : \mathcal{H} \to \mathcal{G} \) is étale, then so is the preimage \( f^{-1}(\mathcal{G}') \to \mathcal{G} \).

Observe that any (possibly empty) graph \( \mathcal{H} \) has the form \( \mathcal{H}' \amalg \mathcal{S} \) where \( \mathcal{H}' \) is a graph without stick components and \( \mathcal{S} \) is a shrub.

**Definition 4.6.** A morphism \( f \in \text{Grpsh}_\text{fr}(\mathcal{D})(\mathcal{H}, \mathcal{G}) \) (with \( \mathcal{H} = \mathcal{H}' \amalg \mathcal{S} \) as above) is called an embedding if the following three conditions hold:

(i) the images \( f(\mathcal{H}') \) and \( f(\mathcal{S}) \) are disjoint in \( \mathcal{G} \);
(ii) the restriction of \( f \) to \( \mathcal{S} \) is injective;
(iii) \( f \) is injective on \( V(\mathcal{H}) \) and \( H(\mathcal{H}) \) (but not necessarily on \( E(\mathcal{H}) \)).

This terminology is due to Hackney, Robertson and Yau [21, Section 1.3].
Lemma 4.7. An embedding \( f: \mathcal{H} \rightarrow \mathcal{G} \) is either pointwise injective or there exists a pair of ports \( e_1, e_2 \in E_0(\mathcal{H}) \) such that

- \( \tau_\mathcal{H} e_1, \tau_\mathcal{H} e_2 \in s(\mathcal{H}) \), and hence \( e_2 \neq \tau_\mathcal{H} e_1 \) (where \( \tau_\mathcal{H} \) is the involution on \( E(\mathcal{H}) \)),
- \( \tau_\mathcal{G} f(e_2) = f(e_1) \in E_*(\mathcal{G}) \) so \( \{ f(e_1), f(e_2) \} \) forms a \( \tau_\mathcal{G} \)-orbit of inner edges of \( \mathcal{G} \).

If \( e_1, e_2 \in E_0(\mathcal{H}) \) and \( \tau_\mathcal{G} f(e_2) = f(e_1) \in E_*(\mathcal{G}) \), then \( f \) is said to glue \( e_1 \) and \( e_2 \) in \( \mathcal{G} \).

Proof. Let \( f \in \text{Grpsh}_0(\mathcal{D})(\mathcal{G}, \mathcal{H}) \) and assume that \( e \) and \( e' \) are edges of \( \mathcal{H} \) such that \( f(e) = f(e') \). If \( f \) is an embedding, then either \( e \) or \( e' \) is a port, since otherwise \( e = s(h) \) and \( e' = s(h') \), so \( f(h) = f(h') \). Moreover, since \( f(\tau e) = f(\tau e') \), either \( \tau e \) or \( \tau e' \) is a port by the same argument.

Assume therefore, that \( e \) is a port. If \( \tau e \) is a port, then \( e \) and \( \tau e \) define a stick component of \( \mathcal{H} \) and so \( f \) violates either condition (i) or condition (ii) of Definition 4.6.

So, if \( f \) is an embedding, then either \( e, \tau e' \in E_0 \) and \( \tau e, e' \in E_1 \), or \( e, \tau e' \in E_0 \) and \( \tau e, e' \in E_1 \). In particular \( f(e) = f(e') \) and \( f(\tau e) = f(\tau e') \) are inner edges of \( \mathcal{G} \).

\( \square \)

Remark 4.8. Monomorphisms in \( \text{Grpsh}_0(\mathcal{D}) \) are pointwise injective morphisms and hence embeddings. If \( f: \mathcal{H} \rightarrow \mathcal{G} \) is an embedding such that \( e_1, e_2 \in E_0(\mathcal{H}) \) are ports and \( f(e_1) = f(\tau e_2) \in E_*(\mathcal{G}) \), then

\[ f \circ ch_{e_1} = f \circ ch_{\tau e_2} : (i) \rightarrow \mathcal{G} \]

and hence \( f \) is not a monomorphism in \( \text{Grpsh}_0(\mathcal{D}) \).

Example 4.9. Let \( \mathcal{W} \) be the wheel graph with vertex \( v \in V(\mathcal{W}) \). The essential morphism \( \iota_v: \mathcal{C}_v \rightarrow \mathcal{W} \) is a pointwise surjective embedding. In fact, for all \( k \geq 1 \), the canonical morphism \( \mathcal{L}^k \rightarrow \mathcal{W}^k \) (Example 3.31) is an epimorphic embedding that is not a monomorphism.

For all finite sets \( X \) and \( Y \), the canonical étale morphisms \( \mathcal{C}_X \coprod \{ x_0 \} \coprod \mathcal{C}_Y \coprod \{ y_0 \} \rightarrow \mathcal{M}_{x_0,y_0}^{X,Y} \) and \( \mathcal{C}_X \coprod \{ x_0 \} \coprod \mathcal{C}_Y \coprod \{ y_0 \} \rightarrow \mathcal{N}_{x_0,y_0}^X \) are epimorphic embeddings but not monomorphisms.

4.2 Graph neighbourhoods and the essential category \( \text{es}(\mathcal{G}) \). A family of morphisms \( \mathfrak{U} = \{ f_i \in \text{Gr}_{\text{et}}(\mathcal{G}_i, \mathcal{G}) \}_{i \in I} \) is jointly surjective on \( \mathcal{G} \) if \( \mathcal{G} = \bigcup_{i \in I} \text{im}(f_i) \). By Lemma 4.4 \( \text{Gr}_{\text{et}} \) admits pullbacks, and jointly surjective families of étale morphisms \( \{ f_i \in \text{Gr}_{\text{et}}(\mathcal{G}_i, \mathcal{G}) \}_{i \in I} \) define the covers at \( \mathcal{G} \) for a canonical étale topology \( J \) on \( \text{Gr}_{\text{et}} \). Sheaves for this topology are those presheaves \( P: \text{Gr}_{\text{et}}^{\text{op}} \rightarrow \text{Set} \) such that \( P(\mathcal{G}) \cong \lim_{f_i \in \mathfrak{U}} P(\mathcal{G}_i) \) for all graphs \( \mathcal{G} \), and all covers \( \mathfrak{U} = \{ f_i \in \text{Gr}_{\text{et}}(\mathcal{G}_i, \mathcal{G}) \}_{i \in I} \) at \( \mathcal{G} \).

As will be shown in Proposition 1.25 the category \( \text{sh}(\text{Gr}_{\text{et}}, J) \) of sheaves for the étale site \( (\text{Gr}_{\text{et}}, J) \) is canonically equivalent to the category \( \mathcal{GS} \) of graphical species (Definition 1.3).

As motivation for this result, let us first establish more properties of étale morphisms.

Definition 4.10. A neighbourhood of an embedding \( w: \mathcal{G}' \rightarrow \mathcal{G} \) is an étale embedding \( u: \mathcal{U} \rightarrow \mathcal{G} \) such that \( w = u \circ \bar{w}: \mathcal{G}' \rightarrow \mathcal{U} \rightarrow \mathcal{G} \), for some embedding \( \bar{w}: \mathcal{G}' \rightarrow \mathcal{U} \).

A neighbourhood \( (\mathcal{U}, u) \) of \( w: \mathcal{G}' \rightarrow \mathcal{G} \) is minimal if every other neighbourhood \( (\mathcal{U}', u') \) of \( w: \mathcal{G}' \rightarrow \mathcal{G} \) is also a neighbourhood of \( (\mathcal{U}, u) \).

Since vertices \( v \) of \( \mathcal{G} \) correspond to subgraphs \( v: \mathcal{C}_0 \rightarrow \mathcal{G} \), and edges \( e \) of \( \mathcal{G} \) are in bijection with subgraphs \( ch_e: (i) \rightarrow \mathcal{G} \), we may also refer to neighbourhoods of vertices and edges. Moreover, since \( u: \mathcal{U} \rightarrow \mathcal{G} \) is a neighbourhood of \( e \in E \) if and only if it is a neighbourhood of \( \iota_{\bar{e}}: (\bar{e}) \rightarrow \mathcal{G} \), there is no loss of generality in referring to neighbourhoods of \( \tau \)-orbits \( \bar{e} \in \bar{E} \).
Let $\mathcal{S}(E_*) = \bigsqcup_{\tilde{E} \in \mathcal{E}^*} \{\tilde{e}\}$ be the shrub on the inner edges of a graph $\mathcal{G}$. Given any subgraph $\mathcal{I} \hookrightarrow \mathcal{S}(E_*)$, there is a graph $\mathcal{G}_I^\natural$ and a canonical surjective embedding $i_\mathcal{I}^\natural : \mathcal{G}_I^\natural \to \mathcal{G}$ (see Figure 12):

\[
\begin{align*}
\mathcal{G}_I^\natural &\quad \xrightarrow{E \amalg (E(\mathcal{I}))^\dagger} \quad (E(\mathcal{I}))^\dagger \quad \xleftarrow{\tau} \quad E \quad \xleftarrow{s} \quad H \quad \xrightarrow{t} \quad V \\
i_\mathcal{I}^\natural &\quad \downarrow \quad \tau \quad \downarrow \quad \downarrow \quad \downarrow \\
\mathcal{G} &\quad E \quad \xleftarrow{\tau} \quad E \quad \xleftarrow{s} \quad H \quad \xrightarrow{t} \quad V,
\end{align*}
\]

where

- $(E(\mathcal{I}))^\dagger$ is the formal involution $e \mapsto e^\dagger$ of the set $E(\mathcal{I})$ of edges of $\mathcal{I}$,
- the involution $\tau_\mathcal{I}$ on $E \amalg (E(\mathcal{I}))^\dagger$ is defined by
  \[
e \mapsto \begin{cases}
  \tau e, & e \in E \setminus E(\mathcal{I}) \\
e^\dagger, & e \in E(\mathcal{I}),
\end{cases}
\]
- the surjection $E \amalg (E(\mathcal{I}))^\dagger \to E$ is the identity on $E$ and $e^\dagger \mapsto \tau e, e \in E(\mathcal{I})$.

So, $\mathcal{G}_I^\natural$ has inner edges $E_*(\mathcal{G}_I^\natural) = E_* \setminus E(\mathcal{I})$, and boundary $E_0(\mathcal{G}_I^\natural) = E_0 \amalg (E(\mathcal{I}))^\dagger$.

Informally, $\mathcal{G}_I^\natural$ is the graph obtained from $\mathcal{G}$ by ‘breaking the edges’ of $\mathcal{I}$, as in Figure 12.

For $\tilde{e} \in \tilde{E}(\mathcal{I})$, the essential morphism $i_{\tilde{e}} : \{\tilde{e}\} \to \mathcal{G}$ (Example 3.28) factors in two ways through $\mathcal{G}_I^\natural$:

\[
\begin{align*}
(i_{\tilde{e}}) &\quad \xrightarrow{(e,\tau e) \mapsto (e,e^\dagger)} \quad \mathcal{G}_I^\natural \quad \xrightarrow{i_\mathcal{I}^\natural} \quad \mathcal{G}.
\end{align*}
\]

Hence there exist parallel morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_I^\natural$, and a coequaliser diagram in $\text{Grpsh}_h(D)$:

\[
\begin{align*}
\mathcal{I} &\quad \xrightarrow{=} \quad \mathcal{G}_I^\natural \quad \xrightarrow{i_\mathcal{I}^\natural} \quad \mathcal{G}.
\end{align*}
\]

(The choice of morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_I^\natural$ in (4.13) is not unique – there are $2^{|\tilde{E}(\mathcal{I})|}$ pairs – but it is unique up to isomorphism.)

For each $\mathcal{I} \subset \mathcal{S}(E_*)$, the set of components of $\mathcal{I} \amalg \mathcal{G}_I^\natural$, together with the canonical embeddings to $\mathcal{G}$, define an étale cover at $\mathcal{G}$.

The collection $\{(\mathcal{G}_I^\natural, i_\mathcal{I}^\natural)\}_{\mathcal{I} \subset \mathcal{S}(E_*)} \subset \text{Gr}_{\text{et}}/\mathcal{G}$ inherits a poset structure from the poset of subgraphs of $\mathcal{S}(E_*)$, and the graph $\mathcal{G}_\mathcal{S}(E_*)$ with no inner edges is initial in this poset. Moreover, any surjective embedding $\mathcal{G}' \to \mathcal{G}$ factors as $\mathcal{G}' \xrightarrow{\mathcal{G}_s} \mathcal{G}_I^\natural \xrightarrow{i_\mathcal{I}^\natural} \mathcal{G}$ for some unique $\mathcal{I} \subset \mathcal{S}(E_*)$. Hence, we have proved the following:
Lemma 4.14. A neighbourhood $(U, u)$ of an embedding $w \in \text{Grpsh}_I(D)(G', G)$ is minimal if and only if $E_\delta(U) = E_\delta(G')$ and the embedding $G' \to U$ induces a surjection on connected components.

The essential morphisms $i_e : (i_e) \to G$ and $i_v : C_v \to G$ describe minimal neighbourhoods of each edge $e$ and vertex $v$ of $G$.

When $G$ has no stick components, one readily checks that $G_\delta^e = \coprod_{v \in V} C_v$. In particular, for any graph $G$, there is a canonical choice of essential cover $E G$ at $G$ by the essential morphisms $i_e$ and $i_v$.

Definition 4.15. Let $G$ be a graph. The essential category $\text{es}(G)$ of $G$ is the full subcategory of $\text{Gr}_{\text{et}}/G$ on the essential embeddings $i_e : (i_e) \to G$, $e \in \tilde{E}$, and $i_v : C_v \to G$, $v \in V$.

By definition, $\text{es}(G)$ has no non-trivial isomorphisms. Hence, there is a canonical bijection $h = (e, v) \mapsto (\delta_h : i_e \to i_v)$ between half-edges $h$ of $G$, and non-identity morphisms $\delta$ in $\text{es}(G)$.

Lemma 4.16. Each graph $G$ is canonically the colimit of the forgetful functor $\text{es}(G) \to \text{Gr}_{\text{et}}$.

A presheaf $P \in \text{psh}(\text{Gr}_{\text{et}})$ is a sheaf for the étale site $(\text{Gr}_{\text{et}}, J)$ if and only if for all $G$,

\begin{equation}
(4.17) \quad P(G) \cong \lim_{(\mathcal{C}, b) \in \text{es}(G)} P(C).
\end{equation}

Proof. If $e \in E_0$ is a port of $G$, then there is at most one non-trivial morphism $\delta_h = \delta_{(\tau_e, v)}$ with domain $i_e$ in $\text{es}(G)$. In this case, $C_v$ is the colimit of the diagram $i_e \xleftarrow{\delta_h} C_v$. The first statement then follows from (4.12) and (4.13). The second statement is immediate since, by Lemma 4.14, the essential cover $E G$ refines every étale cover $U$ of $G$.

4.3 Boundary-preserving étale morphisms. In general, morphisms $f \in \text{Gr}_{\text{et}}(G', G)$ do not satisfy $f(E_0') \subset E_0$. Those that do are componentwise surjective graphical covering morphisms in the sense of Proposition 4.18 below. In particular, embeddings $f \in \text{Gr}_{\text{et}}(G', G)$ such that $f(E_0') = E_0$ are componentwise isomorphisms.

Proposition 4.18. For any étale morphism $f \in \text{Gr}_{\text{et}}(G', G)$, $f(E_0') \subset E_0$ if and only if there exists an étale cover $\mathcal{U} = \{U_i, u_i\}_{i \in I}$ of $G$, such that, for all $i$, $f^{-1}(U_i)$ is isomorphic to a disjoint union of $k(U_i, f) \in \mathbb{N}$ copies of $U_i$.

In this case, $k(U_i, f) = k_f \in \mathbb{N}$ is constant on connected components of $G$.

Proof. If $u : U \to G$ is an étale embedding for which there is a $k \in \mathbb{N}$ such that $f^{-1}(U) \cong k(U)$, then also $f^{-1}(V) \cong k(V)$ for all embeddings $V \to U$. So, we may assume, without loss of generality, that $\mathcal{U} = E G$ is the essential cover of $G$.

Observe first that, if $f(E_0') \not\subset E_0$, there exists a vertex $v$ of $G$ and a port $e'$ of $G'$ such that $f(e') \in E/v$. Hence $(e') \not\subseteq C_v$ is a connected component of $f^{-1}(C_v)$.

For the converse, let $v \in V$ be a vertex of $G$. Since $f$ is étale, $C_v \cong C_v'$ for all $v' \in V'$ such that $f(v') = v$. By the universal property of pullbacks, the canonical embedding $\coprod_{v' : f(v') = v} C_{v'} \to G'$ factors through $f^{-1}(C_v) \to G'$, and therefore

\[ f^{-1}(C_v) \cong \left( \coprod_{v' : f(v') = v} C_{v'} \right) \llcorner S, \quad \text{for some shrub } S. \]

By construction, a connected component of $S$ must be of the form $i_e : (i_e) \to G'$ for some port $e'$ of $G'$ satisfying $f(e') = e \in E/v$. But $f(E_0') \subset E_0$ by assumption, so there is no such port. Hence $S$ is the empty graph, and so $f^{-1}(C_v) \cong \coprod_{v' : f(v') = v} C_{v'}$, whereby $k(C_v, f) = |f^{-1}(v)| \in \mathbb{N}$. 

It is immediate that \( f^{-1}(i_\mathcal{E}) \cong \bigsqcup_{e' \in E', f(e') = e} \mathcal{E}(i_{e'}) \) for all \( e \in E_u \). So \( k(i_\mathcal{E}, f) = |f^{-1}(e)| \in \mathbb{N} \), and the first statement of the proposition is proved.

By condition (3) of Proposition 3.34, it is sufficient to verify the second part of the proposition componentwise on \( \mathcal{E} \). Therefore, let \( f \in G_{\text{et}}(\mathcal{G}', \mathcal{G}) \) satisfy \( f(E_0) \subset E_0 \), and assume, without loss of generality, that \( \mathcal{G} \) is connected.

If \( \mathcal{G} \cong (i) \) is a stick, there is nothing to prove. Otherwise, for any half-edge \( h = (e, v) \) of \( \mathcal{G} \), if \( e' \in f^{-1}(e) \), then \( e' = s(h') \) for some half-edge \( h' = (e', v') \in f^{-1}(h) \) of \( \mathcal{G}' \). Hence, the following diagram commutes:

\[
\begin{array}{ccc}
\prod_{e' \in f^{-1}(e)} \mathcal{E}(i_{e'}) & \xrightarrow{\prod_{e' \in f^{-1}(e)} \delta_{(e', v')}} & \prod_{e' \in f^{-1}(e)} \mathcal{C}_{v'} \\
\downarrow & & \downarrow \\
\mathcal{E}(i_{e}) & \xrightarrow{\delta_h} & \mathcal{C}_v \\
\end{array}
\]

The first part of the proof implies that both squares are pullbacks. So, if \( f^{-1}(\mathcal{C}_v) \) is isomorphic to \( k_v = k(\mathcal{C}_v, \mathcal{E}_v) \) copies of \( \mathcal{C}_v \), then \( f^{-1}(i_{\mathcal{E}}) \cong k_v(i_{\mathcal{E}}) \) for all \( e \in E_u \). Hence \( \mathcal{C}_v \mapsto k_v \) extends to a functor \( k_\mathcal{E} \) from \( \mathcal{E}(\mathcal{G}) \) to the discrete category \( \mathbb{N} \). Since \( \mathcal{G} \) is connected, so is \( \mathcal{E}(\mathcal{G}) \), and therefore \( k_\mathcal{E} \) is constant. \( \square \)

**Definition 4.19.** A morphism \( f \in G_{\text{et}}(\mathcal{G}', \mathcal{G}) \) is called boundary-preserving if it restricts to an isomorphism \( f_{E_0}: E_0 \xrightarrow{\cong} E'_0 \).

The following is an immediate corollary of Proposition 4.18.

**Corollary 4.20.** If \( \mathcal{G} \) is connected, and \( \mathcal{G}' \) is non-empty, then an étale morphism \( f \in G_{\text{et}}(\mathcal{G}', \mathcal{G}) \) such that \( f(E'_0) \subset E_0 \) is surjective. If \( \mathcal{G}' \) is also connected and its boundary \( E'_0 \) is non-empty, then \( f \) is boundary-preserving if and only if it is an isomorphism.

**Remark 4.21.** The condition that \( E'_0 \) is non-empty is necessary in the statement of Corollary 4.20. For example, for any \( m > 1 \), each of the two étale morphisms \( W^m \rightarrow W \) (Example 4.3) is trivially boundary-preserving, but certainly not an isomorphism.

Recall from Example 3.30 that the line graph \( L^k \) has totally ordered edge set \( E(L^k) = \{l_j\}_{j=0}^{2k+1} \) with ports \( l_0 = 1_{L^k} \) and \( l_{2k+1} = 2_{L^k} \). For each vertex \( w_i \in V(L^k) \), \( E/w_i = \{l_{2i-1}, l_{2i}\} \).

**Proposition 4.22.** Let \( \mathcal{G} \) be a connected graph with only bivalent vertices. Then \( \mathcal{G} = L^k \) or \( \mathcal{G} = W^m \) for some \( k \geq 0 \) or \( m \geq 1 \).

**Proof.** Since \( \mathcal{G} \) is bivalent, every embedding \( L^k \rightarrow \mathcal{G} \) from a line graph is étale.

The result holds trivially if \( \mathcal{G} \cong 0 \) is a stick graph. Otherwise, if \( V = V_2 \) is non-empty, then, for each \( v \in V \), a choice of isomorphism \( L^1 \xrightarrow{\cong} \mathcal{C}_v \) describes an embedding \( L^1 \rightarrow \mathcal{G} \). Since \( \mathcal{G} \) is finite, there is a maximum \( M \geq 1 \) such that there exists an embedding \( f: L^M \rightarrow \mathcal{G} \).

Let \( f \in G_{\text{et}}(L^M, \mathcal{G}) \) be such a map. By Lemma 4.7, \( f \) is either injective on edges, or \( f(2L^M) = \tau f(1L^M) \subset E_0 \). Let \( e_1 = f(1L^M) \), and \( e_2 = f(2L^M) \). If \( e_j \) is not a port of \( \mathcal{G} \) for some \( j = 1, 2 \), then \( e_j \in E/\mathcal{G} \), hence for some vertex \( v \in V_2 \).

If \( f \) is injective on edges, then \( v \) is not in the image of \( f \). But then, since \( \mathcal{G} \) is bivalent, this means that \( f \) factors through an embedding \( L^M \rightarrow L^{M+1} \), contradicting maximality of \( M \). Therefore, \( \{e_1, e_2\} \subset E_0 \) so \( f \) is surjective and boundary-preserving, whence \( L^M \cong \mathcal{G} \) by Corollary 4.20.

Otherwise, if \( f \) is not injective on edges, then, by Lemma 4.7, it must be the case that \( f(l_0) = f(l_{2M}) \). Therefore, \( f \) factors through a cycle \( f: L^M \rightarrow W^M \xrightarrow{\hat{1}_\mathcal{G}} \mathcal{G} \). Since \( f \) is an étale embedding, so is \( \hat{f}: W^M \rightarrow \mathcal{G} \). Hence, \( f \) is boundary-preserving and \( \mathcal{G} \cong W^M \) by Corollary 4.20. \( \square \)
Étale morphisms of simply connected graphs are either subgraph inclusions or isomorphisms. (This is why the combinatorics of cyclic operads are much simpler than those of modular operads.)

**Corollary 4.23.** Let $\mathcal{G}$ be simply connected. If $f \in \text{Grpsh}^e(D)(\mathcal{G}', \mathcal{G})$ is locally injective, then $f$ is pointwise injective on connected components of $\mathcal{G}'$. Hence, if $\mathcal{G}'$ is connected, it is simply connected.

It follows that any étale morphism of simply connected graphs is a pointwise injection.

**Proof.** We may assume, without loss of generality, that $\mathcal{G}'$ and $\mathcal{G}$ are connected. Since the result holds trivially when either $\mathcal{G}'$ or $\mathcal{G}$ is an isolated vertex, assume further that both graphs have non-empty edge sets.

Let $f: \mathcal{G}' \to \mathcal{G}$ be a local injection. For any locally injective path $p: \mathcal{L}^k \to \mathcal{G}'$, the path $f \circ p: \mathcal{L}^k \to \mathcal{G}$ is locally injective in $\mathcal{G}$. If $f \circ p$ is not pointwise injective, then either $f \circ p$ factors through a locally injective cycle in $\mathcal{G}$ – and hence $\mathcal{G}$ is not simply connected – or there are $1 \leq i < j \leq k$ such that $f \circ p(v_i) = f \circ p(v_j) \in V(\mathcal{G})$.

So, let $1 \leq i < j \leq k$ be such that $f \circ p(v_i) = f \circ p(v_j) \in V(\mathcal{G})$. We may assume, moreover, that if $i \leq i' < j' \leq j$ also satisfy $f \circ p(v_{i'}) = f \circ p(v_{j'})$, then $(i', j') = (i, j)$.

Let $L = j - i$. Then there is a cycle $c: \mathcal{W}^L \to \mathcal{G}$ described by $a_{2L-1} = f \circ p(l_{j-1}), c(a_{2L}) = f \circ p(l_{2i})$, and $c(a_{2s}) = f \circ p(l_{2(s+i)})$ (and hence $c(a_{2s-1}) = f \circ p(l_{2(s+i)-1})$) for $1 \leq s < L$. In particular, for $1 \leq s < L$, $c$ is injective at the vertex $w_s$ since $f \circ p$ is locally injective. And $c(a_{2L}) = f \circ p(l_{2i}) \neq f \circ p(l_{2j-1}) = c(a_{2L-1})$ since $f \circ p$ is locally injective, so $c$ is injective at $w_L$. Therefore $c$ is locally injective, and $\mathcal{G}$ is not simply connected.

Hence, if $f: \mathcal{G}' \to \mathcal{G}$ is a local injection from a connected graph $\mathcal{G}'$ to a simply connected graph $\mathcal{G}$, then $f$ is pointwise injective, so $\mathcal{G}'$ is also simply connected.

The final statement is immediate since étale morphisms are locally injective by definition. □

Proposition 4.32 gives analogous results for directed acyclic graphs (Definition 4.13).

### 4.4. Étale sheaves on $\text{Gr}_{\text{et}}$. Recall that graphical species are presheaves on the category $\text{B}^\text{es}$, and that there is a full inclusion $\Phi: \text{B}^\text{es} \hookrightarrow \text{Gr}_{\text{et}}$. To prove that $\Phi$ induces an equivalence $\text{GS} \simeq \text{sh}(\text{Gr}_{\text{et}}, J)$ between graphical species and sheaves for the étale topology on $\text{CG}_{\text{Gr}_{\text{et}}}$, first observe:

**Lemma 4.24.** The inclusion $\Phi: \text{B}^\text{es} \hookrightarrow \text{Gr}_{\text{et}}$ is dense.

**Proof.** It is easy to check that any connected graph without inner edges is isomorphic to $\sqsubseteq$ or $\mathcal{C}_X$ for some finite set $X$, and therefore the essential image $\text{im}^{\text{es}}(\Phi)$ of $\text{B}^\text{es}$ in $\text{Gr}_{\text{et}}$ is the full subcategory of connected graphs with no inner edges. Moreover, it follows immediately from the definition of $\text{es}(\mathcal{G})$ (Definition 4.15) that the canonical inclusion $\text{es}(\mathcal{G}) \hookrightarrow \text{im}^{\text{es}}(\Phi)/\mathcal{G}$ is full and essentially surjective on objects, and hence an equivalence of categories.

Therefore $\text{es}(\mathcal{G}) \simeq \text{B}^\text{es}/\mathcal{G}$, and the lemma follows from Lemma 4.16. □

In particular, $\text{Gr}_{\text{et}}$ is a full subcategory of $\text{GS}$ under the induced nerve functor $\Upsilon^\text{def} = N_{\text{Gr}_{\text{et}}}: \text{Gr}_{\text{et}} \to \text{GS}$, and I will write $\mathcal{G}$, rather than $\Upsilon \mathcal{G}$, where there is no risk of confusion. The category $\text{el}(\Upsilon \mathcal{G}) = \text{B}^\text{es}/\mathcal{G}$, whose objects are elements of $\mathcal{G}$, will be denoted by $\text{el}(\mathcal{G})$.

Let $J_C$ be the restriction to $\text{CG}_{\text{Gr}_{\text{et}}}$ of the topology $J$ on $\text{Gr}_{\text{et}}$.

**Proposition 4.25.** There is a canonical equivalence of categories $\text{sh}(\text{Gr}_{\text{et}}, J) \simeq \text{GS}$, and hence also an equivalence $\text{sh}(\text{CG}_{\text{Gr}_{\text{et}}}, J_C) \simeq \text{GS}$. 
Proof. This is straightforward from the definitions and Lemma 4.24. Namely, the inclusion \( \Phi : B^S \to Gr_{et} \) induces an essential geometric morphism between the presheaf categories \( \text{psh}(B^S) = GS \) and \( \text{psh}(Gr_{et}) \). The right adjoint to the pullback \( \Phi^* : \text{psh}(Gr_{et}) \to GS \) is given by

\[
(4.26) \quad \Phi_* : GS = \text{psh}(B^S) \to \text{psh}(Gr_{et}), \quad S \mapsto (G \mapsto \lim_{(C,b) \in \text{el}(G)} S(C)).
\]

Since \( \Phi \) is fully faithful, so is \( \Phi_* \) (e.g. by [32, Section VII.2]).

By Lemmas 4.16 and 4.24, a presheaf \( P \) on \( Gr_{et} \) is a sheaf for the canonical étale topology \( J \) on \( Gr_{et} \) if and only if, for all graphs \( G \),

\[
(4.27) \quad P(G) \cong \lim_{(C,b) \in \text{el}(G)} P(C).
\]

Hence \( sh(Gr_{et}, J) \cong GS \). Moreover, for all \( J \)-sheaves \( P \), and all graphs \( G \), \( P(G) \) is computed componentwise on \( G \), whence \( sh(CGr_{et}, J) \cong GS \) and the proposition is proved. \( \square \)

I will use the same notation to denote a graphical species \( S \) and the corresponding sheaf on \( (Gr_{et}, J) \). So, for any graph \( G \), \( S(G) \) is the natural limit

\[
\lim_{(C,b) \in \text{el}(G)} S(C).
\]

**Definition 4.28.** An \( S \)-structured graph \( (G, \alpha) \) is a graph \( G \) together with an element \( \alpha \in S(G) \) (or \( \alpha \in GS(G, S) \)). The category of \( S \)-structured graphs is denoted by \( Gr_{et}/S \), and \( CGr_{et}/S \) is the subcategory of connected \( S \)-structured graphs.

**4.5. Directed graphs.** By way of example, and to provide extra context, this section ends with a discussion of directed graphs.

Let \( Di \) be the terminal directed graphical species from Examples 1.8 and 1.10. For any graph \( G \), a \( Di \)-structure \( \xi \in Di(G) \) is precisely a partition \( E = E_{in} \coprod E_{out} \), where \( e \in E_{in} \) if and only if \( (e_\tau) \in E_{out} \). So, \( \tau \) induces bijections \( E_{in} \cong \tilde{E} \cong E_{out} \), and an object \( (G, \xi) \) of \( Gr_{et}/Di \) – called an orientation on \( G \) – is given by a diagram of finite sets

\[
(4.29) \quad \tilde{E} \xleftarrow{\tilde{s}_{in}} H_{in} \xrightarrow{t_{in}} V \xleftarrow{t_{out}} H_{out} \xrightarrow{\tilde{s}_{out}} \tilde{E},
\]

where the maps \( \tilde{s}_{in}, \tilde{s}_{out}, t_{in}, t_{out} \) denote the appropriate (quotients of) restrictions of \( s : H \to E \), respectively \( t : H \to V \). Then morphisms in \( Gr_{et}/Di \) are quadruples of finite set maps making the obvious diagrams commute, and such that the outer left and right squares are pullbacks. In particular, \( Gr_{et}/Di \) is the category of directed graphs and étale morphisms used in [28, Section 1.5] to prove a nerve theorem for properads in the style of [6].

**Example 4.30.** The line graphs \( L^k \) with \( E(L^k) = \{ l_i \}_{i=0}^{2k+1} \) admit a distinguished choice of orientation \( \theta_L^k \in Di(L^k) \) given by

\[
\theta_L^k : E(L^k) \to \{ \text{in, out} \}, \quad l_{2i} \mapsto \text{(in)} \text{ and } l_{2i+1} \mapsto \text{(out)} \text{ for } 0 \leq i \leq k.
\]

For \( m \geq 1 \), the canonical morphism \( L^m \to W^m \) induces an orientation \( \theta_{W^m} \) (with \( a_{2j} \mapsto \text{(in)} \)) on the wheel graph \( W^m \).

**Definition 4.31.** A directed path of length \( k \) in \( (G, \xi) \) is a path \( p : L^k \to G \) in \( G \) such that, for all \( l \in E(L^k) \),

\[
Di(ch_i)(\theta_L^k) = Di(ch_{\text{out}}(l))(\xi) \in \{ \text{in, out} \}.
\]

A directed cycle of length \( m \) in \( (G, \xi) \) is a cycle \( c : W^m \to W^m \to G \) in \( G \) such that the induced morphism \( L^m \to W^m \to G \) is a directed path.

A directed acyclic graph (DAG) is a directed graph \( (G, \xi) \) without directed cycles.
It follows immediately from the definitions that any directed path or cycle in a directed graph \((G, \xi)\) is locally injective. Hence, if \((G, \xi)\) admits a directed cycle, \(G\) is not simply connected. The converse is not true.

The following directed version of Corollary 4.23 is not necessary for the constructions of this paper, so I leave its proof as an exercise for the interested reader:

**Proposition 4.32.** For all étale morphisms \(f : (G', \xi') \to (G, \xi)\) between connected DAGs, the underlying morphism \(f : G' \to G\) is an étale embedding.

Moreover, if \((G, \xi)\) is a DAG and the set of morphisms \((G', \xi') \to (G, \xi)\) in \(Gr_{et}/Di\) is non-empty, then \((G', \xi')\) is a DAG. Hence, any morphism to a DAG in \(Gr_{et}/Di\) is pointwise injective on connected components.

A consequence of Proposition 4.32 is that the combinatorics of properads, which are governed by DAGs, are much simpler than those of wheeled properads or modular operads.

5. **Non-unital modular operads**

The goal of the current section is to construct a monad \(T = (T, \mu^T, \eta^T)\) on \(\mathcal{GS}\) whose EM category of algebras \(\mathcal{GS}^T\) is isomorphic to the category \(\mathcal{MO}^-\) of non-unital modular operads (Remark 1.25).

To provide context for this section, consider the following example:

**Example 5.1.** Recall, from Example 1.18, the category \(B^\downarrow\), whose objects are finite sets \(X\), viewed as rooted corollas \(t_X\), and the directed exceptional edge \((\downarrow)\).

The operad endofunctor \(M_{Op}\) on \(psh(B^\downarrow)\) from Example 2.10 is described in detail in [7, Section 3]. It takes a presheaf \(O : B^{\downarrow\text{op}} \to \text{Set}\) to the presheaf \(M_{Op}O\) on \(B^\downarrow\) with \(M_{Op}O(\downarrow) = O(\downarrow)\), and such that elements of each \(M_{Op}O(t_X)\) are formal operadic compositions (i.e. root-to-leaf graftings of decorated corollas as in Figure 7(b)) of elements of \(O\). In other words, they are represented by rooted trees \(\Sigma \in \Omega\), whose leaves are bijectively labelled by \(X\), together with a decoration of the vertices of \(\Sigma\) by elements of \(O\) (according to valency), that also determines a colouring of edges of \(\Sigma\) by \(O(\downarrow)\).

The monadic unit \(\eta^M_{Op}\) is induced by the inclusion of rooted corollas, or trees with one vertex, in \(\Omega\). So, \(\eta^M_{Op}(\phi) = (t_X, \phi)\) for all \(\phi \in O(t_X)\) (Figure 13, left side). Applying the monad twice describes a nesting of \(O\)-decorated trees, and the multiplication \(\mu^M_{Op}\) for \(M_{Op}\) is induced by erasing the inner nesting (the blue circles in the right hand side of Figure 13).

If \((O, h)\) is an algebra for \(M_{Op}\), then \(h\) describes a rule for collapsing the inner edges of each \(O\)-decorated tree, according to the axioms of operadic composition.

Just as the operad endofunctor \(M_{Op}\) takes a \(B^\downarrow\)-presheaf \(O\) to trees decorated by \(O\), the non-unital modular operad endofunctor \(T\) on \(\mathcal{GS}\) takes a graphical species \(S\) to the graphical species \(TS\) whose elements are formal multiplications and contractions in \(S\), represented by \(S\)-structured connected graphs.
Example 5.4. For \( k \geq 0 \), the line graph \( \mathcal{L}^k \), with \( E_0(\mathcal{L}^k) = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\} \), \( k \geq 0 \) is labelled by \( 1_{\mathcal{L}^k} \mapsto 1 \in 2 \) and therefore has the structure of a \( 2 \)-graph when \( k \geq 1 \). However, \( \mathcal{L}^0 = (i) \) has empty vertex set and is therefore not an (admissible) \( 2 \)-graph.

For all finite sets \( X \), there is a canonical functor \( X:\mathbb{CGr}_\text{iso} \rightarrow \mathbb{CGr}_\text{et} \rightarrow \mathbb{G} \S \). A graphical species \( S \) defines a presheaf on \( X:\mathbb{CGr}_\text{iso} \) with \( S(\mathcal{X}) = S(\mathcal{G}) \) for \( \mathcal{X} = (\mathcal{G}, \rho) \). Objects of the corresponding element category \( X:\mathbb{CGr}_\text{iso}/S \) are called \( S \)-structured \( X \)-graphs.

We can now define the non-unital modular operad endofunctor \( T \) on \( \mathbb{G} \S \), that takes a graphical species \( S \) to equivalence classes of \( S \)-structured graphs.

For all graphical species \( S \), let \( TS \) be defined on objects by

\[
TS_X = \colim_{X \in X:\mathbb{CGr}_\text{iso}} S(\mathcal{X}) \quad \text{for all finite sets } X.
\]

(5.5)

Let \( \text{Aut}_X(\mathcal{X}) \overset{\text{def}}{=} X:\mathbb{CGr}_\text{iso}(\mathcal{X}, \mathcal{X}) \) be the automorphism group of an \( X \)-graph \( \mathcal{X} \). If \( g, g' \in X:\mathbb{CGr}_\text{iso}(\mathcal{X}, \mathcal{X}') \) are parallel \( X \)-isomorphisms, then there are \( \sigma \in \text{Aut}_X(\mathcal{X}) \) and \( \sigma' \in \text{Aut}_X(\mathcal{X}') \) such that \( g' = \sigma' \sigma g \).

Therefore, there is a completely canonical (independent of \( g \in X:\mathbb{CGr}_\text{iso}(\mathcal{X}, \mathcal{X}') \)) choice of natural (in \( \mathcal{X} \)) isomorphism

\[
\frac{S(\mathcal{X})}{\text{Aut}_X(\mathcal{X})} \overset{\cong}{\longrightarrow} \frac{S(\mathcal{X}')}{\text{Aut}_X(\mathcal{X}')}, \quad [\alpha] \mapsto [g(\alpha)], \text{ for } \alpha \in S(\mathcal{X}).
\]

(5.6)

It follows from (5.6), that

\[
TS_X = \coprod_{[\mathcal{X}] \in \pi_0(X:\mathbb{CGr}_\text{iso})} \frac{S(\mathcal{X})}{\text{Aut}_X(\mathcal{X})}
\]

(5.7)

where \([\mathcal{X}] \in \pi_0(X:\mathbb{CGr}_\text{iso})\) is the connected component of \( \mathcal{X} \) in \( X:\mathbb{CGr}_\text{iso} \).

Hence, elements of \( TS_X \) may be viewed as isomorphism classes of \( S \)-structured \( X \)-graphs, and two \( S \)-structured \( X \)-graphs \((\mathcal{X}, \alpha)\) and \((\mathcal{X}', \alpha')\) represent the same class \([\mathcal{X}, \alpha] \in TS_X\) precisely when there is an isomorphism \( g \in X:\mathbb{CGr}_\text{iso}(\mathcal{X}, \mathcal{X}') \) such that \( S(g)(\alpha') = \alpha \).

Since bijections \( f : X \overset{\cong}{\rightarrow} Y \) of finite sets induce isomorphisms \( X:\mathbb{CGr}_\text{iso}/S \overset{\cong}{\rightarrow} Y:\mathbb{CGr}_\text{iso}/S \), the action of \( TS \) on isomorphisms in \( \mathbb{B}^S \) is the obvious one.

The projections \( TS(\mathcal{G}) : TS_X \rightarrow TS_\mathcal{G} = S_\mathcal{G} \) are induced by the projections \( X:\mathbb{CGr}_\text{iso}/S \rightarrow S_\mathcal{G} \) given by \( (\mathcal{X}, \alpha) \mapsto S(\mathcal{X})/(\alpha) \), where \( \mathcal{X} \in \mathbb{CGr}_\text{et}(\mathcal{G}) \) is the map \( ch_{\rho^{-1}(x)} \) defined by \( 1 \mapsto \rho^{-1}(x) \in E_0(\mathcal{G}) \).
This is well-defined since, if \((\mathcal{X}, \alpha)\) and \((\mathcal{X}', \alpha')\) represent the same element of \(TS_X\), then there is an \(X\)-isomorphism \(g: \mathcal{X} \to \mathcal{X}'\) such that \(S(g)(\alpha') = \alpha \in S_X\) and hence
\[
S(ch_{\mathcal{X}}^X)(\alpha') = S(ch_{\mathcal{X}}^X) \circ S(g)(\alpha') = S(ch_{\mathcal{X}}^X)(\alpha).
\]

So \(TS\) describes a graphical species. Moreover, it is clear from the definition that the assignment \(S \mapsto TS\) extends to an endofunctor \(T\) on \(GS\), with unit \(\eta^T: \text{id}_{GS} \Rightarrow T\) given by the canonical maps \(S_X \xrightarrow{\cong} S(\mathcal{C}_X) \to TS_X\) for all \(X\).

5.2. **Gluing constructions.** A monadic multiplication \(\mu^T\) for the pointed endofunctor \((T, \eta^T)\) will be defined in terms of colimits of a certain class of diagrams in \(\text{Gr}_{\text{et}}\). However, since \(\text{Gr}_{\text{et}}\) does not admit general colimits (see Examples \[3.16]\ and \[3.33]\), a small amount of preparation is necessary.

Let \(S\) be a graphical species and \(Y\) a finite set. Since, elements of \(TS_Y\) are represented by \(S\)-structured \(Y\)-graphs, it follows that, for all finite sets \(X\), elements of \(T^2S_X\) are represented by \(X\)-graphs \(\mathcal{X}\) that are decorated by \(S\)-structured graphs. In other words, each \((\mathcal{X}, \beta) \in T^2S_X\) is represented by a functor
\[
eq\]
\[
\text{el}((\mathcal{X}, b)) \mapsto (\mathcal{C}_X, b), \quad \text{where} \ S(b)(\beta) \in TS_X b
\]
such that
\[
\text{el}(\beta)(\mathcal{C}_X, b) \times S(b)(\beta) \ni (\mathcal{C}_X, c) \mapsto (\mathcal{C}_X, c) \in S_X b
\]
for all morphisms in \(el(\mathcal{X})\) of the form
\[
(i) \quad \xrightarrow{ch_{\mathcal{X}}} \quad \mathcal{C}_X b
\]
\[
\xleftarrow{ch_{\mathcal{C}}} \quad b \quad \xrightarrow{\mathcal{X}}
\]

Then, as in the operad case (Example \[5.1\] Figure \[13\]), we would like to think of the monad multiplication as forgetting the vertices of the original graph \(\mathcal{X}\), to obtain an element of \(TS_X\) (Figure \[14\]).

Graphs of graphs are functors that encode this idea:

**Definition 5.8.** Let \(\mathcal{G}\) be a graph. A \(\mathcal{G}\)-shaped graph of graphs is a functor \(\Gamma: el(\mathcal{G}) \to \text{Gr}_{\text{et}}\) such that
\[
\Gamma(ch_{\mathcal{C}}) = (i) \quad \text{for all} \ (i, ch_{\mathcal{C}}) \in el(\mathcal{G}),
\]
\[
E_0(\Gamma(b)) = X_b \quad \text{for all} \ (\mathcal{C}_X, b) \in el(\mathcal{G}),
\]
and, for all \((\mathcal{C}_X, b) \in el(\mathcal{G})\) and all \(x_b \in X_b\),
\[
\Gamma(ch_{\mathcal{X}}) = ch_{\mathcal{X}}^{(b)} \in \text{Gr}_{\text{et}}(i, \Gamma(b)).
\]

A \(\mathcal{G}\)-shaped graph of graphs \(\Gamma: el(\mathcal{G}) \to \text{Gr}_{\text{et}}\) is non-degenerate if, for all \(v \in V\), \(\Gamma(\iota_v)\) has no stick components. Otherwise, \(\Gamma\) is degenerate.

Informally, a non-degenerate \(\mathcal{G}\)-shaped graph of graphs is a rule for substituting graphs into vertices of \(\mathcal{G}\) as in Figure \[14\]. However, this intuitive description of a graph of graphs in terms of graph insertion does not always apply in the degenerate case (see Sections \[6\] and \[7\]).

By Lemma \[4.16\] every graph \(\mathcal{G}\) is the colimit of the (non-degenerate) identity \(\mathcal{G}\)-shaped graph of graphs \(\mathcal{I}^G\) given by the forgetful functor \(el(\mathcal{G}) \to \text{Gr}_{\text{et}}, (\mathcal{C}, b) \mapsto \mathcal{C}\). It follows from Section \[4.2\] that, if \(\mathcal{G}\) has no stick components, this is equivalent to the statement that \(\mathcal{G}\) is the coequaliser of the canonical diagram
\[
S(E_\bullet) \xrightarrow{\coprod_{v \in V} C_v} \coprod_{v \in V} C_v \xrightarrow{\coprod(\iota_v)} \mathcal{G}.
\]
Graphical combinatorics and a distributive law for modular operads

Figure 14. A $G$-shaped graph of graphs $\Gamma$ describes graph substitution: a graph $G_v$ and bijection $E_0(G_v) \cong (E/v) \downarrow$ is assigned to each vertex $v$ of $G$. When $\Gamma$ is non-degenerate, taking its colimit in $Gr_{et}$ corresponds to erasing the inner (blue) nesting.

To prove that all non-degenerate graphs of graphs admit a colimit in $Gr_{et}$, we generalise this observation using a modification of [28, Section 1.5.1], where gluing data for directed graphs were described. A directed graph version of Lemma 5.11 appears as [28, Proposition 1.5.2].

Definition 5.10. Let $S = \bigsqcup_{i \in I} (p_i)$ be a shrub, and let $G$ be a (not necessarily connected) graph without stick components. A pair of parallel morphisms $\delta_1, \delta_2: S \Rightarrow G$ such that

- $\delta_1, \delta_2$ are injective and have disjoint images in $G$; and
- for all $i \in I$, $\delta_1(1_i)$ and $\delta_2(2_i)$ are ports of $G$,

is called a gluing datum in $Gr_{et}$.

Lemma 5.11. Gluing data admit coequalisers in $Gr_{et}$.

Proof. Let $G$ be a graph without stick components and let $\delta_1, \delta_2: S \Rightarrow G$ be a gluing datum with coequaliser $p: G \to \overline{G} = (E, \overline{H}, \overline{V}, \overline{s}, \overline{t}, \overline{\tau})$ in the category $psh_i(D)$ of graph-like diagrams.

Since $\delta_1$ and $\delta_2$ are injective and have disjoint images, the induced map $\overline{\tau}: \overline{E} \to \overline{E}$ is a fixed-point free involution. Moreover $H = \overline{H}$ since, if half-edges $h$ and $h'$ of $G$ are identified in $\overline{H}$, then there is an edge $l \in E(S)$ such that $\delta_1(l) = s(h)$ and $\delta_2(l) = s(h')$. This contradicts the conditions of Definition 5.10 since $G$ has no stick components. Likewise, edges $e, e' \in E(G)$ are identified in $\overline{E}$ if and only if there is an $l \in E(S)$ such that $\delta_1(l) = e$ and $\delta_2(l) = e'$ (or vice versa). Since $G$ has no stick components, and $\delta_1, \delta_2$ have disjoint images, we may assume that $e$ and $\tau e'$ are ports and $e', \tau e \in s(H)$. Therefore, $\overline{\sigma}: \overline{H} \to \overline{E}$ is injective, and $\overline{G}$ is a graph.

In particular, $\overline{V} = V$ since $\overline{H} = H$ and $S$ is a shrub. It follows that $p: G \to \overline{G}$ is an étale embedding, and the lemma is proved.

Example 5.12. The graphs $M_{x,y}^X$ and $N_{x,y}^X$ (Examples [3.8, 3.9, 3.22]) are coequalisers of gluing data [3.23, 3.24]:

\[
(ch_x, ch_y \circ \tau: (i) \Rightarrow (C_{XH(x)} \amalg C_{YH(y)})) \to M_{x,y}^X \quad (ch_x, ch_y \circ \tau: (i) \Rightarrow C_{XH(x,y)}) \to N_{x,y}^X.
\]

This is visualised in Figure 15.

Proposition 5.13. A non-degenerate $G$-shaped graph of graphs $\Gamma: el(G) \to Gr_{et}$ admits a colimit $\Gamma(G)$ in $Gr_{et}$.
then the induced map \(\text{Corollary 5.18.}\) for now. However, the non-degeneracy condition simplifies the proof of Proposition 5.13, and is all that is needed and the morphism \(\text{et}\) on ports. For each \(\mathbf{e} \in \mathfrak{e}(\mathcal{G})\), we may apply this is a gluing datum since \(\mathfrak{e}(\mathcal{G})\) is non-degenerate. Therefore (5.14) has a colimit \(\mathfrak{G}\) on \(\mathfrak{e}(\mathcal{G})\). Hence, by the universal property of colimits, \(\mathfrak{G}\) is a graph. Moreover, since \(\mathfrak{d}(\mathcal{G}) \neq (i)\) is connected, for any port \(e \in \mathfrak{e}_0(\mathcal{G})\), there is a unique half-edge \((\tau e, w) \in H(\mathcal{G})\) and the morphism \(\mathfrak{d}(\delta_{(\tau e, w)}): \mathfrak{d}(\tau e) \to \mathfrak{d}(\tau w)\) induces an inclusion

\[
(\tau e) \hookrightarrow \mathfrak{d}(\tau w) \hookrightarrow \prod_{v \in V} \mathfrak{d}(\tau v).
\]

The inclusions (5.15) and (5.16) describe a functor from \(\mathfrak{d}(\mathcal{G})\) to the diagram (5.14), and hence a cocone of \(\mathfrak{G}\) above \(\mathfrak{d}(\mathcal{G})\).

Conversely, \(\mathfrak{G}\) has a colimit \(\mathfrak{d}(\mathcal{G})\) in the category \(\text{psh}(\mathfrak{D})\) of graph-shaped diagrams and the cocone of \(\mathfrak{d}(\mathcal{G})\) factors through (5.14). Hence, by the universal property of colimits, \(\mathfrak{d}(\mathcal{G}) = \mathfrak{G}\) is a graph. It is the colimit of \(\mathfrak{G}\) in \(\text{CGr}_{\text{et}}\) since \(\mathfrak{G}\) is the coequaliser of (5.14) in \(\text{CGr}_{\text{et}}\).

Remark 5.17. In fact, as will follow from Proposition 7.17, all graphs of graphs admit a colimit in \(\text{CGr}_{\text{et}}\). However, the non-degeneracy condition simplifies the proof of Proposition 5.13 and is all that is needed for now.

Corollary 5.18. If \(\mathcal{G}\) is a graph, and \(\mathfrak{G}\) is a non-degenerate \(\mathcal{G}\)-shaped graph of graphs with colimit \(\mathfrak{d}(\mathcal{G})\), then the induced map \(E(\mathcal{G}) \to E(\mathfrak{G})\) on edges is injective, and restricts to the identity \(E_0(\mathcal{G}) \to E_0(\mathfrak{d}(\mathcal{G}))\) on ports. For each \((C, b) \in \mathfrak{e}(\mathcal{G})\), the universal map \(\mathfrak{d}(b) \to \mathfrak{d}(\mathcal{G})\) is an étale embedding. In particular,

\[
E(\mathfrak{d}(\mathcal{G})) \cong E(\mathcal{G}) \amalg \prod_{v \in V} E_*(\mathfrak{d}(\tau v)).
\]

Proof. The final statement follows directly from the first two.

By the proof of Proposition 5.13, only the inner edges of \(\mathcal{G}\), and, for all \((C, b) \in \mathfrak{e}(\mathcal{G})\), the \(\tau\)-orbits of ports of \(\mathfrak{d}(b)\) are involved in forming the colimit \(\mathfrak{d}(\mathcal{G})\) of \(\mathfrak{G}\). Hence \(\mathfrak{G}\) induces a strict inclusion

\[
\text{et}\text{of}_E \hookrightarrow \prod_{e \in E} \mathfrak{d}(\tau e) \hookrightarrow \mathfrak{d}(\mathcal{G}).
\]

**Figure 15.** Construction of \(\mathfrak{M}_{x,y}^X\) and \(\mathfrak{N}_{x,y}^X\) as coequalisers of gluing data.
that restricts to an identity $E_0(\mathcal{G}) = E_0(\Gamma(\mathcal{G}))$ of ports. The second part is immediate. □

The following corollary was proved, for directed graphs, in [28 Lemma 1.5.12].

**Corollary 5.19.** Let $\Gamma$ be a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\Gamma(\mathcal{G})$ in $\text{Gr}_{\text{et}}$. If $\Gamma(C, b)$ is connected for each $(C, b) \in \text{el}(\mathcal{G})$, then $\Gamma(\mathcal{G})$ is a connected graph if and only if $\mathcal{G}$ is.

**Proof.** A $(i)$-shaped graph of graphs is isomorphic to the identity functor $(i) \mapsto (i)$ with colimit $(i)$.

So, assume that $\mathcal{G}$ has no stick components and let $\Gamma : \text{el}(\mathcal{G}) \to \text{CGr}_{\text{et}}$ be a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\Gamma(\mathcal{G})$.

A morphism $\gamma \in \text{psh}(\mathcal{D})(\Gamma(\mathcal{G}), \star II \star)$ is equivalently described by a commuting diagram in $\text{psh}(\mathcal{D})$:

\[
\begin{array}{ccc}
S(E_\bullet) & \longrightarrow & \prod_{v \in V(\mathcal{G})} \Gamma(\iota_v) \\
\end{array}
\]

Let $\Gamma(\iota_v)$ be connected for each $v \in V$. Then each map $\Gamma(\iota_v) \to \star II \star$ is constant, and morphisms $\prod_{v \in V} \Gamma(\iota_v) \to \star II \star$ are in bijection with morphisms $\prod_{v \in V} \mathcal{C}_v \to \star II \star$.

So, $\text{psh}(\mathcal{D})(\mathcal{G}, \star II \star) \cong \text{psh}(\mathcal{D})(\Gamma(\mathcal{G}), \star II \star)$, and it follows from Proposition 3.34 that $\Gamma(\mathcal{G})$ is connected if and only if $\mathcal{G}$ is. □

Let $\mathcal{G}$ be a graph and $S$ any graphical species. As usual, let $K$ be the terminal graphical species.

**Definition 5.21.** A (non-degenerate) $\mathcal{G}$-shaped graph of $S$-structured graphs is a functor $\Gamma_S : \text{el}(\mathcal{G}) \to \text{Gr}_{\text{et}}/S$ such that the functor $\Gamma : \text{el}(\mathcal{G}) \to \text{Gr}_{\text{et}}$ induced by the unique morphism $S \to K$

\[
\begin{array}{ccc}
\text{el}(\mathcal{G}) & \xrightarrow{\Gamma_S} & \text{Gr}_{\text{et}}/S \\
\end{array}
\]

is a (non-degenerate) $\mathcal{G}$-shaped graph of graphs.

For a connected graph $\mathcal{G}$, objects of the category $(\text{CGr}_{\text{et}}/S)^{(\mathcal{G})}$ are non-degenerate $\mathcal{G}$-shaped graphs of $S$-structured graphs $\Gamma_S : \text{el}(\mathcal{G}) \to \text{Gr}_{\text{et}}/S$, and morphisms are natural transformations.

**Lemma 5.23.** For $\mathcal{G}$ connected, $\mathcal{G} \not\cong \mathcal{C}_0$, two $\mathcal{G}$-shaped graphs of (connected) $S$-structured graphs $\Gamma^1_S, \Gamma^2_S$ are in the same connected component of $(\text{CGr}_{\text{et}}/S)^{(\mathcal{G})}$ if and only if, for all $(C_{X_b}, b) \in \text{el}(\mathcal{G})$, $\Gamma^1_S(b)$ and $\Gamma^2_S(b)$ are in the same connected component of $X_b\text{-CGr}_{\text{et}}/S$.

In particular, if $\Gamma^1_S$ and $\Gamma^2_S$ are in the same connected component of $(\text{CGr}_{\text{et}}/S)^{(\mathcal{G})}$, then $\Gamma^1_S$ and $\Gamma^2_S$ have isomorphic colimits in $\text{CGr}_{\text{et}}/S$.

**Proof.** Let $\mathcal{G} \not\cong \mathcal{C}_0$ be connected. So, if $(C_{X_b}, b) \in \text{el}(\mathcal{G})$, then $X_b \neq \emptyset$. Given a morphism $\phi : \Gamma^1_S \Rightarrow \Gamma^2_S$ in $(\text{CGr}_{\text{et}}/S)^{(\mathcal{G})}$, its component $\phi(b)$ at $b$ is, by definition, a boundary-preserving morphism in $\text{CGr}_{\text{et}}/S$, and hence by Corollary 1.20 an $X_b$-isomorphism in $X_b\text{-CGr}_{\text{et}}/S$.

The converse is immediate, as is the final statement. □

### 5.3. Multiplication for the monad $T$.

The aim of this section is to describe the multiplication $\mu^T : T^2 \Rightarrow T$ in terms of colimits of graphs of graphs.

From now on, all graphs will be connected, unless explicitly stated otherwise.

Let $X$ be a finite set and $\mathcal{X} = (\mathcal{G}, \rho)$ an $X$-graph. If $\Gamma : \text{el}(\mathcal{X}) \to \text{CGr}_{\text{et}}$ is a non-degenerate $\mathcal{X}$-shaped graph of graphs, then its colimit $\Gamma(\mathcal{X}) = \text{colim}_{\text{el}(\mathcal{X})} \Gamma$ exists by Proposition 5.13 and, by Corollary 5.18 it inherits the $X$-labelling $\rho$ of $\mathcal{X}$.

Given a graphical species $S$ and finite set $X$, elements $[X, \beta]$ of $T^2S_X$ are represented by pairs $(\mathcal{X}, \Gamma_S)$ where $\mathcal{X}$ is an $X$-graph, and $\Gamma_S$ is an $X$-shaped graphs of connected $S$-structured graphs. The colimit of
\( \Gamma_S \) in \( \text{CGr}_{\text{et}}/S \) is given by a pair \((\Gamma(\mathcal{X}), \alpha)\), where \( \Gamma(\mathcal{X}) \) is the colimit of the underlying \( \mathcal{X} \)-shaped graph of graphs \( \Gamma \) defined as in (5.22) and \( \alpha \in S(\Gamma(\mathcal{X})) \).

For \( j = 1, 2 \), let \((\mathcal{X}^j, \Gamma^j_S) : \text{el}(\mathcal{X}^j) \to \text{CGr}_{\text{et}}/S \) represent the same element \([\mathcal{X}, \beta] \) of \( T^2S_X \). Then, by definition of \( T \), \( \mathcal{X}^1 \cong \mathcal{X}^2 \) in \( \text{X-CGr}_{\text{iso}} \) and, by Lemma 5.23

\[
(5.24) \quad \text{colim}_{\text{el}(\mathcal{X})} \Gamma_S^1 \cong \text{colim}_{\text{el}(\mathcal{X}^2)} \Gamma_S^2 \in \text{X-CGr}_{\text{iso}}/S.
\]

A multiplication \( \mu^T : T^2 \Rightarrow T \) for \((T, \eta^T)\) will be induced by the (by (5.24) well-defined) assignments:

\[
(5.25) \quad [\mathcal{X}, \beta] \mapsto \Gamma(\mathcal{X}), \alpha]
\]

To see that (5.25) extends to a morphism \( \mu^T : T^2S \to TS \) of graphical species, let \([\mathcal{X}, \beta] \in T^2S_X \) be represented by an \( \mathcal{X} \)-shaped graph of \( S \)-structured graphs \( \Gamma_S : \text{el}(\mathcal{X}) \to \text{CGr}_{\text{et}}/S \) with colimit \( \Gamma_S(\mathcal{X}) = (\Gamma(\mathcal{X}), \alpha) \) in \( \text{X-CGr}_{\text{iso}}/S \).

By Corollary 5.18, there is a canonical inclusion \( E(\mathcal{X}) \hookrightarrow E(\Gamma(\mathcal{X})) \) of edge sets, and for each \( e \in E(\mathcal{X}) \),

\[
S(ch_e^{\Gamma(\mathcal{X})})(\alpha) = S(ch_e^X)(\beta) \in S(i).
\]

Hence, for all \( x \in X \), there is a commuting diagram of sets

\[
\begin{array}{ccc}
T^2S_X & \xrightarrow{\mu^T S_X} & TS_X \\
\downarrow \text{TS(ch_x)} & & \downarrow \text{TS(ch}_x) \\
T^2S(i) = TS(i).
\end{array}
\]

Naturality of \( \mu^T S \) in \( S \) is immediate from the definition and, by a straightforward modification of [28, Section 2.2], it may be shown that \( T = (T, \mu^T, \eta^T) \) satisfies the two axioms for a monad.

**Remark 5.26.** For all graphical species \( S \), \( \mu^T S \) and \( \eta^T S \) are palette-preserving morphisms in \( \text{GS} \). So \( T \) restricts to a monad \( T^{(\mathcal{E}, \omega)} : \text{GS}^{(\mathcal{E}, \omega)} \) for all \((\mathcal{E}, \omega)\). If \( A \) is a \((\mathcal{E}, \omega)\)-coloured graphical species and \( h \in \text{GS}(TA, A) \), then \((A, h)\) is a \( T \)-algebra if and only if it is a \( T^{(\mathcal{E}, \omega)} \)-algebra.

**Example 5.27.** If \( K \) is the terminal graphical species, then \( \text{CGr}_{\text{et}}/K \cong \text{CGr}_{\text{et}} \) and hence elements of \( TK \) are boundary-preserving isomorphism classes of graphs in \( \text{CGr}_{\text{et}} \). The unique morphism \( ! \in \text{GS}(TK, K) \) makes \( K \) into an algebra for \( T \). Likewise, for any palette \((\mathcal{E}, \omega)\), the terminal \((\mathcal{E}, \omega)\)-coloured graphical species \( K^{(\mathcal{E}, \omega)} \) is a \( T \)-algebra together with the unique palette-preserving morphism \( !(\mathcal{E}, \omega) : TK^{(\mathcal{E}, \omega)} \to K^{(\mathcal{E}, \omega)} \).

### 5.4. \( T \)-algebras are non-unital modular operads.

Having constructed the monad \( T \), it remains to prove that \( T \)-algebras are non-unital modular operads.

**Lemma 5.28.** A \( T \)-algebra \((A, h)\) admits a multiplication \( h \circ \) and contraction \( h \zeta \), that are natural with respect to morphisms in \( \text{GS}^T \).

**Proof.** Let \( X \) and \( Y \) be finite sets and let \( M_{X, Y}^X \) be the \( X \) \( Y \)-graph (described in Examples 3.8 and 5.12) obtained by gluing the corollas \( C_{X \cup \{x\}} \) and \( C_{Y \cup \{y\}} \) along ports \( x \) and \( y \).

Let \( S \) be a \((\mathcal{E}, \omega)\)-coloured graphical species. For \( c \in \mathcal{E}^X \), \( d \in \mathcal{E}^Y \) and \( c \in \mathcal{E} \), let \( M_c(\phi, \psi) \) be the element of \( S(M_{x, y}^{X, Y}) \) determined by an ordered pair \((\phi, \psi) \in S(\mathcal{E}^c) \times S(\mathcal{E}^\omega) \). The canonical map \( S(M_{x, y}^{X, Y}) \to TS_{X \cup Y} \) is injective unless \( X = Y = \emptyset \), in which case \([M_c(\phi_1, \psi_1)] = [M_c(\phi_2, \psi_2)] \) when \((\phi_2, \psi_2) = (\psi_1, \phi_1) \in S(\{x\}) \times S(\{y\}) \).

If \((A, h)\) is a \((\mathcal{E}, \omega)\)-coloured \( T \)-algebra, then the maps given by the compositions

\[
h \circ : S(\mathcal{E}^c) \times S(\mathcal{E}^\omega) \xrightarrow{[M(\cdot)]} TS_{cd} \xrightarrow{h} S_{cd}
\]
are $\mathcal{B}^3$-equivariant by construction and hence induce a multiplication on $A$ (see Figure 16).

Recall similarly that, for any finite set $X$, $N_{x,y}^X$ is the $X$-graph (described in Examples 3.9 and 5.12) obtained by gluing the ports $x$ and $y$ of $C_{x,y}$. For $\mathcal{C} \in \mathcal{C}^X$ and $c \in \mathcal{C}$, let $N^c_{x,y}(\phi) \in S(N_{x,y}^X)$ be the element determined by $\phi \in S(\mathcal{C}(c,\omega_c)) \subset S_{x,y}(c)$. Hence, to prove (M1), it suffices to show that, for all $\phi \in \mathcal{C}(c,\omega_c)$, $N^c_{x,y}(\phi) \in S(N_{x,y}^X)$ is the element determined by $\phi \in S(\mathcal{C}(c,\omega_c)) \subset S_{x,y}(c)$.

The only non-trivial boundary-preserving automorphism of $N_{x,y}^X$ is the permutation $\sigma_{x,y} \in \text{Aut}(X \amalg \{x,y\})$ that fixes $X$ and switches $x$ and $y$. So, $[N^c_{x,y}(\phi)] = [N^c_{x,y}(\psi)]$ in $TS^X$ if and only if $\phi = \psi$ or $S(\sigma_{x,y})(\phi) = \psi$.

If $(A,h)$ is a $(\mathcal{C},\omega)$-coloured algebra for $T$, then the maps given by the compositions

$$h\zeta: A(\mathcal{C},\omega) \xrightarrow{[N^A(c)]} TA_c \xrightarrow{h} A_c$$

are $\mathcal{B}^3$-equivariant and induce a contraction $h\zeta$ on $A$ (see Figure 16). Naturality of $h\circ$ and $h\zeta$ is immediate from the construction.

![Figure 16](image-url)

**Figure 16.** If $(A,h)$ is a $T$-algebra, $h$ induces a multiplication and contraction on $A$.

We are now able to show that algebras for the monad $T$ on $G^\mathcal{S}$ are precisely non-unital modular operads.

**Proposition 5.29.** There is a canonical isomorphism of categories $G^\mathcal{T} \cong M^\mathcal{O}$. 

**Proof.** A $T$-algebra structure $h: TA \to A$ equips a graphical species $A$ with a multiplication $\circ = h\circ$, and contraction $\zeta = h\zeta$ as in Lemma 5.28. We must show that $(A,\circ,\zeta)$ satisfies conditions (M1)-(M4) of Definition 1.24.

The proof is based on the observation that (up to its boundary $E_0$) any connected graph with two inner edge orbits has one of the forms illustrated in Figures 17-20 and each of these relates to one of the conditions (M1)-(M4).

Condition (M1) is illustrated in Figure 17. Let $\phi_1 \in A(c,e), \phi_2 \in A(\mathcal{C}(c,\omega_c))$ and $\phi_3 \in A(\mathcal{C}(c,\omega_d))$. By Lemma 5.28 and the monad algebra axioms,

$$(\phi_1 \circ_3 \phi_2) \circ_3 \phi_3 = h [\mathcal{M}^A_d (h(\phi_1,\phi_2),h\eta^\mathcal{T} A(\phi_3))] = h [\mathcal{M}^A_d (h(\phi_1,\phi_2),h\eta^\mathcal{T} A(\phi_3))] = h [\mathcal{M}^A_d (h(\phi_1,\phi_2),h\eta^\mathcal{T} A(\phi_3))]$$

and, likewise

$$\phi_1 \circ (\phi_2 \circ_3 \phi_3) = h [\mathcal{M}^A_d (\phi_1,\phi_2,\phi_3)]$$

Hence, to prove (M1), it suffices to show that, for all $\phi_1, \phi_2, \phi_3$ as above,

$$\mu^\mathcal{T} [\mathcal{M}^A_d (\phi_1,\phi_2),\eta^\mathcal{T} A(\phi_3)] = \mu^\mathcal{T} [\mathcal{M}^A_d (\phi_1,\phi_2),\eta^\mathcal{T} A(\phi_3)]$$

By Example 5.12 and since colimits commute, this follows from:

$$\text{coeq} \mathcal{CG}_{\text{an}} (ch_y, ch_z \circ \tau: (i) \Rightarrow \mathcal{M}_{w,x}^{X_1,\Pi X_2 (y)} \amalg \mathcal{C}_{X_1 \amalg X_2 (z)}) = \text{coeq} \mathcal{CG}_{\text{an}} (ch_w, ch_x \circ \tau: (i) \Rightarrow \mathcal{C}_{X_1 \amalg w} \amalg \mathcal{M}_{w,x}^{X_{1 \amalg X_2 (y)},X_2})$$
The coherence conditions (M2)-(M4) all follow in the same way from the defining axioms of monad algebras. Figures 17-20 illustrate each condition.

Figure 17. Coherence condition (M1) Applying $\mu^T A : T^2 A \to A$ amounts to erasing inner nesting.

Figure 18. Coherence condition (M2)

The induced assignment $(A, h) \mapsto (A, \circ, \zeta)$ clearly extends to a functor $\text{GS}^T \to \text{MO}^{-}$.

The proof of the converse closely resembles that of [18, Theorem 3.7]. Namely, let $(S, \circ, \zeta)$ be a non-unital modular operads. We construct a morphism $h \in \text{GS}(TS, S)$ by successively using $\circ$ and $\zeta$ to collapse inner edge orbits of $S$-structured $X$-graphs $(X, \alpha)$, resulting in a finite sequence of $S$-structured $X$-graphs that terminates in an $S$-structured corolla $(C_X, \phi)$.

As usual, let $X$ be a finite set and let $(X, \alpha)$ be a representative of $[X, \alpha] \in TS_X$.

If $X$ has no inner edges, then $X = C_X$, and so $[X, \alpha] = \eta^T S(\phi)$ for some $\phi \in S_X$. In this case, define

\[(5.30)\] $h[X, \alpha] \overset{\text{def}}{=} \phi \in S_X$.

Otherwise, let $X$ have vertex set $V$, edge set $E$, and let $\bar{e} \in \bar{E}_e$ be the orbit of a pair $e, \tau e$ of inner edges of $X$. Write $t(e) \overset{\text{def}}{=} ts^{-1}(e)$ for the vertex $v$ with $e \in E_v$. 
There are two possibilities: either \( t(e) = t(\tau e) \) or \( t(e) \neq t(\tau e) \).

**Case 1:** \( t(e) = v_1 \) and \( t(\tau e) = v_2 \) are distinct vertices of \( \mathcal{X} \).

Let \( \mathcal{X}_{/e} \) be the graph obtained from \( \mathcal{X} \) by removing the \( \tau \)-orbit \( \{e, \tau e\} \) and identifying \( v_1 \) and \( v_2 \) to a vertex \( \overline{\tau} \in V/(v_1 \sim v_2) \):

\[
\mathcal{X}_{/e} \overset{\text{def}}{=} \tau \bigcap (E \setminus \{e, \tau e\}) \xrightarrow{\tau} \left(V - \{e, \tau e\}, \{e, \tau e\} \right) \leftarrow \tau \xrightarrow{\tau} V/(v_1 \sim v_2) .
\]

(Here \( \tau \) is the composition of \( t : H \to V \) with the quotient \( V \to V/(v_1 \sim v_2) \).) So, \( \mathcal{X} \) is the colimit of the non-degenerate \( \mathcal{X}_{/e} \)-shaped graph of graphs \( \text{el}(\mathcal{X}_{/e}) \to \text{CG}_{\text{et}} \) given by

\[
(C, b) \mapsto \begin{cases} 
\mathcal{M}^X_{X_1 X_2} & \text{if } (C, b) = (C_{X_1 U X_2}, \overline{b}) \text{ is a neighbourhood of } \overline{v} \in V(\mathcal{X}_{/e}), \\
C & \text{if } (C, b) \text{ is not a neighbourhood of } \overline{v}.
\end{cases}
\]

In particular, if \( (C, b) \in \text{el}(\mathcal{X}_{/e}) \) is not a neighbourhood of \( \overline{v} \), then it describes an element \( (C, b^X) \in \text{el}(\mathcal{X}) \).
For $i = 1, 2$, let $(C_{X_i, U(x_i)}, b'_{i}) \in \mathfrak{el}(X)$ be minimal neighbourhoods of $v_i$ in $X$ such that $b'_1(x_1) = \tau e$ and $b'_2(x_2) = e$. And let $\phi_i \overset{\text{def}}{=} S(b'_i)(\alpha) \in S(C_{X_i, U(x_i)})$. Then there is an $S$-structure $\alpha_{/\bar{e}}$ on $X_{/\bar{e}}$ defined, for all $(C, b) \in \mathfrak{el}(X_{/\bar{e}})$, by

$$S(b)(\alpha_{/\bar{e}}) = \begin{cases} \phi_1 \circ_{x_1, x_2} \phi_2 & \text{if } (C, b) = (C_{X_1, U(x_2)}, \bar{b}), \\ S(b)^V(\alpha) & \text{if } (C, b) \text{ is not a neighbourhood of } \bar{v}. \end{cases}$$

**Case 2:** $t(e) = t(\tau e) = v \in V$.

In this case, the graph $X_{/\bar{e}}$ obtained from $X$ by collapsing $\{e, \tau e\}$ has the form

$$X_{/\bar{e}} \overset{\text{def}}{=} \tau \bigcap (E \setminus \{e, \tau e\}) \xrightarrow{s} (H \setminus s^{-1}\{e, \tau e\}) \xrightarrow{\tau} V,$$

and $X$ is the colimit of the non-degenerate $X_{/\bar{e}}$-shaped graph of graphs $\mathfrak{el}(X_{/\bar{e}}) \to \mathcal{Gr}_{et}$:

$$(C, b) \mapsto \begin{cases} \mathcal{N}^{X_{/\bar{e}}}_{x, y} & \text{if } (C, b) = (C_{X, \bar{b}}) \text{ is a neighbourhood of } v, \\ C & \text{if } (C, b) \text{ is not a neighbourhood of } v. \end{cases}$$

As before, if $(C, b) \in \mathfrak{el}(X_{/\bar{e}})$ is not a neighbourhood of $v$, then it describes an element $(C, b^V) \in \mathfrak{el}(X)$.

Now, let $(C_{X_i, U(x_i)}, b'_i) \in \mathfrak{el}(X)$ be the neighbourhood of $v$ in $X$ such that $b'(x) = \tau e$ and $b'(y) = e$. Let $\phi \overset{\text{def}}{=} S(b')(\alpha) \in S_{X, U(x_i)}. Then there is an $S$-structure $\alpha_{/\bar{e}}$ on $X_{/\bar{e}}$ defined, for all $(C, b) \in \mathfrak{el}(X_{/\bar{e}})$, by

$$S(b)(\alpha_{/\bar{e}}) = \begin{cases} \zeta_{x, y}(\phi) & \text{if } (C, b) = (C_{X, \bar{b}}), \\ S(b)^V(\alpha) & \text{if } (C, b) \text{ is not a neighbourhood of } v. \end{cases}$$

It follows that an ordering $(\bar{e}_1, \ldots, \bar{e}_N)$ of the set $\bar{E}_*$ of inner $\tau$-orbits of $X$ defines a terminating sequence of $S$-structured $X$-graphs:

$$(X, \alpha) \mapsto (X_{/\bar{e}_1}, \alpha_{/\bar{e}_1}) \mapsto ((X_{/\bar{e}_1})_{/\bar{e}_2}, (\alpha_{/\bar{e}_1})_{/\bar{e}_2}) \mapsto \cdots \mapsto (((X_{/\bar{e}_1}) \cdots)_{/\bar{e}_N}, (\alpha_{/\bar{e}_1}) \cdots)_{/\bar{e}_N}).$$

Since $((X_{/\bar{e}_1}) \cdots)_{/\bar{e}_N} = C_X$ has no inner edges, there is a $\phi(X, \alpha) \in S_X$ such that

$$(\alpha_{/\bar{e}_1} \cdots)_{/\bar{e}_N} = \eta^\tau S(\phi(X, \alpha)) \in TS_X.$$

The coherence conditions (M1)-(M4) are equivalent to the statement that $\phi(X, \alpha) \in S_X$ so obtained is independent of the choice of ordering of $\bar{E}_*$. Moreover, by construction, the assignment $(X, \alpha) \mapsto \phi(X, \alpha)$ is equivariant with respect to morphisms in $X \mathcal{Gr}_{et}/S$, and so also independent of the choice of representative $[X, \alpha] \in TS_X$. Hence it extends to a morphism $h : TS \to S, [X, \alpha] \mapsto \phi(X, \alpha)$ in $\mathcal{Gs}$.

To complete the proof of the proposition, it remains to establish that $h$ satisfies the monad algebra axioms for $\mathcal{T}$. Compatibility of $h$ with $\eta^\tau$ is immediate from equation (5.30). Compatibility of $h$ with $\mu^\tau$ follows since the coherence conditions (M1)-(M4) ensure that $h[X, \alpha]$ is independent of the order of collapse of the inner edges of $X$.

So $(S, \circ, \zeta)$ defines a $T$-algebra $(S, h)$, and this assignment extends in the obvious way to a functor $\mathbf{MO}^- \to \mathbf{GS}^\mathcal{T}$ that, by Lemma 5.28 and (M1)-(M4), is inverse to the functor $\mathbf{GS}^\mathcal{T} \to \mathbf{MO}^-$ defined above.

\[ \square \]

6. The problem of loops

Before constructing the (unital) modular operad monad in Section 4, let us first pause to discuss the obstruction to obtaining a monadic multiplication for the modular operad endofunctor in the construction outlined in [23].
**Example 6.1.** In Example 5.1 I sketched the idea behind the construction of the symmetric operad monad $M_{O_p}$ on $psh(\mathcal{B}^\uparrow)$, whose underlying endofunctor takes a $\mathcal{D}$-coloured presheaf $O$ to the $\mathcal{D}$-coloured presheaf of formal operadic compositions in $O$, encoded as $O$-decorated rooted trees. However, I did not describe how the units for the operadic composition are obtained.

Unlike $T$ on $\text{GS}$, the definition of $M_{O_p}$ on $psh(\mathcal{B}^\uparrow)$ allows **degenerate substitution** of the exceptional directed tree ($\downarrow$) into the vertex of the rooted corolla $t_1$ with one leaf (Figure 21). Grafting an exceptional directed edge ($\downarrow$) onto the leaf or root of any tree $T$ leaves $T$ unchanged (see Figure 7(b)). So, if $(O, \theta)$ is a $\mathcal{D}$-coloured algebra for $M_{O_p}$, and hence a $\mathcal{D}$ coloured operad, then the elements $\theta(\downarrow, d) \in O(\downarrow)$ provide the $\mathcal{D}$-coloured units for the operadic composition.

![Figure 21](image)

**Figure 21.** The combinatorics of the operadic unit are represented graphically by the degenerate substitution of the exceptional tree into $t_1$. Applying the monad multiplication $\mu_{M_{O_p}}$ to nested trees in $M_{O_p}^2 O$ deletes vertices decorated by elements of $O(\downarrow)$. (See also Figure 13)

The endofunctor $T^{\text{op}}: \text{GS} \to \text{GS}$ defined in [23] Section 5, whose algebras are modular operads, is obtained by a slight modification of the non-unital modular operad endofunctor $T$, to allow degenerate substitutions analogous to those in Example 6.1.

For a finite set $X$, let $X$-$\text{CGr}_{\text{iso}}^{\text{el}}$ be the groupoid obtained from $X$-$\text{CGr}_{\text{iso}}$ by dropping the condition that $X$-graphs must have non-empty vertex set. So,

$$X$-$\text{CGr}_{\text{iso}}^{\text{el}} = X$-$\text{CGr}_{\text{iso}} \quad \text{for} \quad X \neq 2 \quad \text{and} \quad 2$-$\text{CGr}_{\text{iso}}^{\text{el}} \cong 2$-$\text{CGr}_{\text{iso}} \coprod \{(i, id), (i, \tau)\}.$$

The endofunctor $T^{\text{op}}: \text{GS} \to \text{GS}$ is defined pointwise by

$$T^{\text{op}}S_\emptyset \overset{\text{def}}{=} S_\emptyset,$$

$$T^{\text{op}}S_X \overset{\text{def}}{=} \text{colim}_{X \in X$-$\text{CGr}_{\text{iso}}^{\text{el}}} S(X) \quad \text{for all finite sets} \ X,$$

together with the obvious extension of $T$ on morphisms in $\mathcal{B}^{\text{op}}$.

Since $T S \subset T^{\text{op}}S$ (for all $S$), $\eta^{\text{op}}$ induces a unit $\eta^{\text{op}}$ for the endofunctor $T^{\text{op}}$ (see Definition 2.13).

**Proposition 6.2.** Algebras for the pointed endofunctor $(T^{\text{op}}, \eta^{\text{op}})$ on $\text{GS}$ are modular operads.

**Proof.** Since $T \subset T^{\text{op}}$, algebras for $T^{\text{op}}$ have the structure of non-unital modular operads by Proposition [23] If $(A, h)$ is an algebra for $(T^{\text{op}}, \eta^{\text{op}})$, then each $c \in A_\emptyset$ defines an element $(i, c) \in T^{\text{op}}A_2$, and $h(i, c) \in A_2$ provides a $c$-coloured unit for the induced multiplication. \qed

However $(T^{\text{op}}, \eta^{\text{op}})$ cannot be extended to a monad on $\text{GS}$:

For all graphical species $S$, an element of $T^{\text{op}}S_X$ is represented by an $X$-graph $\mathcal{X}$ and a (possibly degenerate) $X$-shaped graph of $S$-structured connected graphs $\Gamma_S: \text{el}(\mathcal{X}) \to \text{CGr}_{\text{el}}/S$. In particular, if $T^{\text{op}}$ admits a monad multiplication $\mu^{\text{op}}: T^{\text{op}}S \Rightarrow T^{\text{op}}S$, then $\mu^{\text{op}}S$ restricts to $\mu^{\text{op}}S$ on $T S$. But this cannot be well-defined, as the following example shows:
Example 6.3. As usual, let $\mathcal{W}$ be the wheel graph with one vertex $v$ and edges $\{a, \tau a\}$. Its category of elements $\text{el}(\mathcal{W})$ has skeletal subcategory

$$\text{el}(\mathcal{W}) \to \mathcal{C}_2 \xleftarrow{\text{ch}_{1}^{c_2}} \mathcal{W} \xleftarrow{\text{ch}_{1}^{c_2 \circ \tau}} \mathcal{C}_2 \xleftarrow{(1_{c_2} \mapsto a)} \mathcal{W}$$

So, if $S$ is a $(\mathcal{C}, \omega)$-coloured graphical species and $c \in \mathcal{C}$, then there is a (degenerate) $\mathcal{W}$-shaped graph of $S$-structured graphs $\Lambda_{S,c}$ given by

$$\text{el}(\mathcal{W}) \xleftarrow{\Lambda_{S,c}} \mathcal{C}_2 \xleftarrow{\Lambda_{S,c}(\text{ch}_{a})} \mathcal{C}_2 \xleftarrow{\Lambda_{S,c}(1_{c_2} \mapsto a)} \mathcal{W}$$

In particular, $\Lambda_{S,c}(\text{ch}_{a}) = \text{id}_{(i,c)} = \Lambda_{S,c}(\text{ch}_{1}^{c_2} \circ \tau)$ and hence $\Lambda_{S,c}$ has colimit $\text{id}_{(i,c)} : (i,c) \to (i,c)$ in $\text{CGr}_{et}/S$.

We observe immediately that $E_0(i) \neq E_0(\mathcal{W})$ so Corollary 5.18 does not hold for $\Lambda_{S,c}$.

Moreover, if $\Lambda_{S,\omega c}$ is the $\mathcal{W}$-shaped graph of $S$-structured graphs given by

$$(i, \text{ch}_{a} \mapsto (i, \omega c)) \text{ and } (\mathcal{C}_2, (1_{c_2} \mapsto a)) \mapsto (i, \omega c),$$

and if $\tau_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$ is the unique non-trivial (but trivially boundary-preserving) automorphism, then $\Lambda_{S,\omega c}(C, b) = \Lambda_{S,c}(C, \tau_{\mathcal{W}} \circ b)$ for all $(C, b) \in \text{el}(\mathcal{W})$.

Hence $\Lambda_{S,c}$ and $\Lambda_{S,\omega c}$ represent the same element of $T^{\mu_2}S_0$. But $\Lambda_{S,c}$ has colimit $(i,c)$ in $\text{CGr}_{et}/S$ while $\Lambda_{S,\omega c}$ has colimit $(i,\omega c) \in \text{CGr}_{et}/S$, and these are distinct if $c \neq \omega c$.

As Example 6.3 shows, taking colimits in $X-\text{CGr}_{iso}^\mu$ of degenerate graphs of $S$-structured graphs does not always lead to a well-defined class of $S$-structured graphs, let alone one in the correct arity. The issue arises because the coequaliser in $\text{psh}_D$ of the parallel morphisms $\text{id}_{(i)}, \tau : (i) \rightrightarrows (i)$ is the exceptional loop $\bigcirc$, which is not a graph (Example 3.16).

An obvious first attempt at resolving the problem outlined in Example 6.3 is to extend $\mu^{\top}S$ to a well-defined multiplication $\mu^{\top}S : T^{\mu_2}S \to T^{\mu_2}S$, is to enlarge $\text{CGr}_{et}$ to include the exceptional loop $\bigcirc$:

$$\text{CGr}_{et}^\bigcirc$$

Let $\text{CGr}_{et}^\bigcirc$ be the category of fully generalised Feynman graphs and étale morphisms, obtained from $\text{CGr}_{et}$ by adding the object $\bigcirc$ and a unique morphism $(i) \to \bigcirc$.

By definition, $\bigcirc$ is the formal coequaliser of the diagram $\text{id}, \tau : (i) \rightrightarrows (i)$ in $B^\bigcirc \subset \text{CGr}_{et}$. So, we may define its category of elements $\text{el}(\bigcirc) \overset{\text{def}}{=} B^\bigcirc/\bigcirc$, and thereby extend any graphical species $S : B^\bigcirc_{op} \to \text{Set}$ to a presheaf on $\text{CGr}_{et}^\bigcirc$ according to $G \mapsto \text{lim}_{(i) \in G} S$. But this would imply that $S(\bigcirc) \cong S(i)$ for all graphical species $S$.

It follows that $\text{CGr}_{et}^\bigcirc$ does not embed densely or fully in $\text{GS}$. In particular, there is no monad $M$ on $\text{GS}$ with arities $\text{CGr}_{et}^\bigcirc$ (see Section 2).

In particular, let $X-\text{Gr}_{iso}^\bigcirc$ be the groupoid of fully generalised connected $X$-graphs defined by

$$X-\text{Gr}_{iso}^\bigcirc \overset{\text{def}}{=} X-\text{CGr}_{iso}^\mu$$ for $X \neq 0$ and $0-\text{Gr}_{iso}^\bigcirc \overset{\text{def}}{=} 0-\text{CGr}_{iso}^\mu \amalg \{\bigcirc\}$,
and let the endofunctor $T^\circ : GS \to GS$, such that $T^\circ S_X = T^{\text{ad}} S_X$ for $X \neq 0$, be given by

$$
\begin{align*}
T^\circ S & \overset{\text{def}}{=} S, \\
T^\circ S_X & \overset{\text{def}}{=} \lim_{(G, \rho) \in X \cdot Gr}{S(G)}.
\end{align*}
$$

Then, the $W$-shaped graphs of graphs $\Lambda_{S,c}$ and $\Lambda_{S,\omega c}$ described in Example 6.3 represent the same element $[W, \beta] \in T^\circ \mathbb{S}^0$. But, $S(\emptyset) \cong S(\emptyset) = S$ and so

$$
[\emptyset, c] \neq [\emptyset, \omega c] \in T^\circ \mathbb{S}^0 \quad \text{whenever} \quad c \neq \omega c \in S.
$$

It follows that $\mu^T$ cannot be extended to a multiplication $\mu^\circ: T^\circ^2 \Rightarrow T^\circ$.

Indeed, this is not surprising: For all $c \in S$, the contraction $\zeta_c: S_{c, \omega c} \to S_0$ factors through the quotient $\tilde{S}_2$ of $S_2$ under the action of $\text{Aut}(2)$. Hence, $\zeta(\phi)$ loses data relative to $\phi \in S_{c, \omega c}$, and the morphism $(i) \to \emptyset$ in $\text{CG}^{\circ}$ – that collapses a two-element set to a point – would seem to be in the wrong direction!

Remark 6.6. In the graphical formalism of [18, 8] described in Example 3.1 (as well as in, for example [20, 33, 43]), where graph ports are defined to be the fixed points of edge involution, the graph substitution is not defined in terms of a functorial construction, but by ‘removing neighbourhoods of vertices and gluing in graphs’. Therefore, the exceptional loop arises from substitution as in Figure 22.

![Figure 22. Constructing an exceptional loop by removing a vertex and substituting the stick graph.](image)

Remark 6.7. Hackney, Robertson and Yau [22, Definition 1.1] are able to construct the modular operad monad on $GS$, within the framework of Feynman graphs, by including extra boundary data in their definition of graphs. For them, a graph is a pair $(G, \partial(G))$ of a Feynman graph $G$, and subset $\partial(G) \subset E_0$ of ports, that satisfies certain conditions. In their formalism, $(i, 2)$ and $(i, 0)$ are different graphs: $(i, 2)$ is the stick, and $(i, 0)$ plays the role of the exceptional loop $\emptyset$.

However, this approach does not result in a dense functor from the graph category to $GS$. And, though they construct a fully faithful nerve for modular operads in terms of a dense subcategory $U \hookrightarrow \text{MO}$ of graphs, the inclusion is not fully faithful, and so $U$ does not fully describe the graphical combinatorics of modular operads.

The combinatorics of contracted units are examined more closely in Section 7 where the problem of loops discussed in this section will be resolved by adjoining a map that acts as a formal equaliser, rather than a coequaliser, of $\text{id}, \tau: (i) \rightrightarrows (i)$ (see also Figures 2 and 25).

Remark 6.8. To my knowledge, the construction that I present in Section 7 is unique among graphical descriptions of unital modular operads (or wheeled prop(erad)s), in that all others include some version of the exceptional loop as a graph. (See e.g. [33, 34, 20, 43, 3].)
7. Modular operads with unit

Proposition 5.29 identifies the category of non-unital modular operads with the EM category of algebras for the monad $\mathbb{T}$ on $\mathcal{GS}$. The goal of this section is to extend this in order to obtain (unital) modular operads. Some potential obstacles have been discussed in Section 6, where it was also explained why the ‘obvious’ modification of the operad monad (Examples 6.1 and 5.1) does not work for unital modular operads.

This section begins by returning to the definition of modular operads in Definition 1.24 and looking in more detail at the combinatorics of (contracted) units. This combinatorial information can be encoded in a monad $\mathbb{D} = (D, \mu^D, \eta^D)$ on $\mathcal{GS}$.

Once $\mathbb{D}$ is defined, it is a small step to obtaining the distributive law $\lambda: TD \Rightarrow DT$, whose construction provides us with an explicit description of the modular operad monad $\mathbb{D}T$ in terms of equivalence classes of graphs structured by graphical species. Moreover, as discussed in Section 7.5, the construction of $\mathbb{D}T$ is such that it is always possible to work with nice (non-degenerate) representatives of these classes, thereby avoiding the problem of loops described in Section 6.

7.1. Pointed graphical species. By definition, if $(S, \circ, \zeta, \epsilon)$ is a modular operad, then the unit $\epsilon: S_\emptyset \to S_2$ is an injective map such that

\[(7.1) \quad \epsilon \circ S_\tau = S(\sigma_2) \circ \epsilon.\]

The key observation is that the combination of a unit and a contraction implies that, as well as the unit elements in arity 2 provided by $\epsilon: S_\emptyset \to S_2$, modular operads also have distinguished elements in arity 0. Namely, as in [1.23], there is a contracted unit map $o = \zeta \circ \epsilon: S_\emptyset \to S_0$ that satisfies

\[(7.2) \quad o = o \circ S_\tau: S_\emptyset \to S_0.\]

Definition 7.3. Objects of the category $\mathcal{GS}_\ast$ of pointed graphical species are triples $S_\ast = (S, \epsilon, o)$ (or $(S, \epsilon^S, o^S)$) where $S$ is a graphical species and $\epsilon: S_\emptyset \to S_2$, and $o: S_\emptyset \to S_0$ are maps satisfying conditions (7.1) and (7.2) above. Morphisms in $\mathcal{GS}_\ast$ are morphisms in $\mathcal{GS}$ that preserve the additional structure.

Example 7.4. For any palette $(\mathcal{C}, \omega)$, the terminal $(\mathcal{C}, \omega)$-coloured graphical species $K^{(\mathcal{C}, \omega)}$ is trivially pointed and hence terminal in the category of $(\mathcal{C}, \omega)$-coloured pointed graphical species and palette-preserving morphisms.

The category $\mathcal{GS}_\ast$ is also a presheaf category: Let $\mathcal{B}^{\mathcal{S}_\ast}$ be the category obtained from $\mathcal{B}^{\mathcal{S}}$ by formally adjoining morphisms $u: 2 \to \emptyset$ and $z: 0 \to \emptyset$, subject to the relations

- $u \circ ch_1 = \text{id} \in \mathcal{B}^{\mathcal{S}}(\emptyset, \emptyset)$ and $u \circ ch_2 = \tau \in \mathcal{B}^{\mathcal{S}}(\emptyset, \emptyset)$,
- $\tau \circ u = u \circ \sigma_2 \in \mathcal{B}^{\mathcal{S}}(2, \emptyset)$,
- $z = \tau \circ z \in \mathcal{B}^{\mathcal{S}}(0, \emptyset)$.

Lemma 7.5. The following are equivalent:

1. $S_\ast$ is a presheaf on $\mathcal{B}^{\mathcal{S}_\ast}$ that restricts to a graphical species $S$ on $\mathcal{B}^{\mathcal{S}}$,
2. $(S, \epsilon, o)$, with $\epsilon = S_\ast(u)$ and $o = S_\ast(z)$, is a pointed graphical species.

Proof. It is easy to check directly that $\mathcal{B}^{\mathcal{S}_\ast}$ is completely described by

- $\mathcal{B}^{\mathcal{S}}(\emptyset, \emptyset) = \mathcal{B}^{\mathcal{S}}(\emptyset, \emptyset)$ and $\mathcal{B}^{\mathcal{S}}(Y, X) = \mathcal{B}^{\mathcal{S}}(Y, X)$ whenever $Y \not\rightarrow 0$ and $Y \not\rightarrow 2$,
- $\mathcal{B}^{\mathcal{S}}(0, \emptyset) = \{z\}$, and $\mathcal{B}^{\mathcal{S}}(0, X) = \mathcal{B}^{\mathcal{S}}(0, X) \amalg \{ch_x \circ z\}_{x \in X}$,
- $\mathcal{B}^{\mathcal{S}}(2, \emptyset) = \{u, \tau \circ u\}$, and $\mathcal{B}^{\mathcal{S}}(2, X) = \mathcal{B}^{\mathcal{S}}(0, X) \amalg \{ch_x \circ u, ch_x \circ \tau \circ u\}_{x \in X}$ for all finite sets $X$.\]
and the lemma follows immediately. □

By Lemma 7.5 a pointed graphical species \((S,e,o)\) may also be denoted by \(S_+\), and these forms will be used interchangeably. The category of elements of a pointed graphical species \(S_+\) will be denoted by \(\text{el}(S_+) \overset{\text{def}}{=} \text{el}_{B^\#}(S_+)\).

**Lemma 7.6.** The forgetful functor \(GS \to GS_{+,\text{el}}\) is strictly monadic: it has a left adjoint \(GS \to GS_{+,\text{el}}\), and \(GS_+\) is the EM category of algebras for the induced monad \(\mathbb{D} = (D, \mu^D, \eta^D)\) on \(GS\).

**Proof.** The left adjoint \((·)^+\) to the forgetful functor \(GS_+ \to GS\) takes a graphical species \(S\) to its left Kan extension \(S^+\) along the inclusion \((B^\#)^{\text{op}} \to (B^\#)^{\text{op}}\). This does nothing more than formally adjoin elements \(\{c^+_e\}_{c \in S_0}\) to \(S_2\) and \(\{o^+_e\}_{e \in \mathbb{S}_1}\) to \(S_0\) according to the combinatorics of contracted units (7.1). (7.2). Hence, \(S^+\) is described by \((DS,e^+,o^+) = (DS\epsilon^{DS},o^{DS})\) where \(DS_2 = S_2 \sqcup \{c^+_e\}_{c \in S_1}\), \(DS_0 = S_0 \sqcup \{o^+_e\}_{e \in \mathbb{S}_1}\), and \(DS_X = S_X\) for \(X \neq 2, X \neq 0\).

The monadic unit \(\eta^D\) is provided by the inclusion \(S \to DS\), and the multiplication \(\mu^D\) is induced by the canonical projections \(D^2S_2 \to DS_2\). □

### 7.2. Pointed graphs

Let \(\text{CGr}_+\) be the category obtained in the bo-ff factorisation of \((\Upsilon^{-})^+ : \text{CGr}_{+\text{et}} \hookrightarrow GS \to GS_{+,\text{el}}\), so that the following diagram commutes:

\[
\begin{array}{ccc}
\text{B}^\# & \xrightarrow{\text{dense}} & \text{CGr}_+ \\
\uparrow \text{b.o.} & & \uparrow \text{b.o.} \\
\text{B}^\# & \xrightarrow{(\cdot)^+} & \text{GS}_+
\end{array}
\]

\[
\begin{array}{ccc}
\text{B}^\# & \xleftarrow{\text{f.f.}} & \text{CGr}_{+\text{et}} \\
\downarrow \text{f.f.} & & \downarrow \text{f.f.} \\
\text{B}^\# & \xleftarrow{\Upsilon} & \text{GS}
\end{array}
\]

The inclusion \(\text{B}^\# \to \text{CGr}_+\) is fully faithful (by uniqueness of bo-ff factorisation), and also dense, since the induced nerve \(\Upsilon_+ : \text{CGr}_+ \to \text{GS}_+\) is fully faithful by construction.

Let \(\mathcal{G} \in \text{CGr}_{+\text{et}}\) be a graph. By Lemma 7.5 for each edge \(e \in E, e^+_u = c\circ u \in \text{CGr}_+(\mathcal{C}_2, \mathcal{G})\) is the \(c\)-coloured unit for \(\Upsilon_+\mathcal{G}\), and the corresponding contracting unit is given by \(o^+_z = c \circ z \in \text{CGr}_+(\mathcal{C}_0, \mathcal{G})\).

Since the functor \(\Upsilon_+\) embeds \(\text{CGr}_+\) as a full subcategory of \(\text{GS}_+\), I will denote \(\Upsilon_+\mathcal{G} \in \text{GS}_+\) simply by \(\mathcal{G}\) where there is no risk of confusion. In particular, the element category \(\text{el}_+(\mathcal{G})\) is denoted by \(\text{el}_+(\mathcal{G})\) and called the category of pointed elements of a graph \(\mathcal{G}\).

For all pointed graphical species \(S_+\), the forgetful functor \(GS_+ \to GS\) induces injective-on-objects inclusions \(\text{el}(S) \to \text{el}_+(S_+)\).

Recall [31, Section IX.3] that a functor \(\Phi : C \to D\) is final if the slice category \(d/\Phi = \text{def} \Phi^\text{op}/d\) is non-empty and connected for all \(d \in D\), and that this is the case if and only if, for any functor \(\Phi : D \to E\) such that \(\text{colim}_C(\Phi \circ \Psi)\) exists in \(E\), \(\text{colim}_D\Phi\) also exists in \(E\) and the two colimits agree.

**Lemma 7.8.** For all graphs \(\mathcal{G}\), the inclusion \(\text{el}_+(\mathcal{G}) \hookrightarrow \text{el}_+(\mathcal{G})\) is final. Therefore, \(\text{B}^\#\) is dense in \(\text{CGr}_+\) and, for all pointed graphical species \(S_+ = (S,e,o)\),

\[
S(\mathcal{G}) = \text{lim}_{(\mathcal{C},b) \in \text{el}_+(\mathcal{G})} S(\mathcal{C}) = \text{lim}_{(\mathcal{C}',b') \in \text{el}_+(\mathcal{G})} S_+(\mathcal{C}').
\]

**Proof.** By definition, \(\text{el}_+(\mathcal{G})\) is obtained from \(\text{el}(\mathcal{G})\) by adjoining, for each \(e \in E\), the objects \((2, c\circ u)\) and \((0, c\circ z)\) and the morphisms

\[
(2, c\circ u) \xrightarrow{u} (2, c\circ z) \xleftarrow{z} (0, c\circ z).
\]

Hence, for all \((\mathcal{C}, b) \in \text{el}_+(\mathcal{G})\), the slice category \(b/\text{el}(\mathcal{G})\) is connected and non-empty. □
By Lemma 7.8, a morphism \( f \in \text{CGr}_a(\mathcal{G}, \mathcal{G}') \) is described by a functor \( \text{el}(\mathcal{G}) \to \text{el}_a(\mathcal{G}') \) such that, for each \((\mathcal{C}, b) \in \text{el}(\mathcal{G})\), \((\mathcal{C}, b) \mapsto (\mathcal{C}, f(b))\), and there is a commuting diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{g_f(b)} & \mathcal{C}' \\
\downarrow_{f(b)} & & \downarrow_{b'} \\
\mathcal{G}' & & \\
\end{array}
\]

where \( g_f(b) \in B^3(\mathcal{C}, \mathcal{C}') \) and \((\mathcal{C}', b') \in \text{el}(\mathcal{G}')\) is an (unpointed) element of \( \mathcal{G}'\).

**Example 7.9.** (Compare Example 6.3) A surprising consequence of the definitions is that the morphism set \( \text{CGr}_a(\mathcal{W}, i) \) is non-empty. There are two morphisms \( \kappa, \tau \circ \kappa \in \text{CGr}_a(\mathcal{W}, i)\):

\[
\begin{array}{c}
\mathcal{W} \\
\downarrow_{ch_u} \\
\downarrow_{ch_1} \\
\downarrow_{ch_2 \circ \tau} \\
\downarrow_{Ch} \\
\mathcal{C}_2 \\
\end{array}
\]

Hence, \( \text{CGr}_a(\mathcal{W}, i) \cong \text{CGr}_a(\mathcal{W}, \mathcal{W}) \). In particular, for all graphs \( \mathcal{G} \neq \mathcal{W}, \)

\[
\text{CGr}_a(\mathcal{W}, \mathcal{G}) \cong E(\mathcal{G}) \text{ by } ch_e \circ \kappa \mapsto e.
\]

These morphisms play a crucial role in the proof of the nerve theorem, Theorem 8.2.

Now, let \( W \subset V_2 \) be a subset of bivalent vertices of a connected graph \( \mathcal{G} \).

**Definition 7.12.** A vertex deletion functor (for \( W \)) is a \( \mathcal{G} \)-shaped graph of graphs \( \Lambda^\mathcal{G}_W : \text{el}(\mathcal{G}) \to \text{CGr}_a \) such that for \((\mathcal{C}_X, b) \in \text{el}(\mathcal{G})\),

\[
\Lambda^\mathcal{G}_W (b) = \begin{cases} 
(i) & \text{if } (\mathcal{C}_X, b) \text{ is a neighbourhood of } v \in W, \\
\mathcal{C}_X & \text{otherwise}. 
\end{cases}
\]

If \( \Lambda^\mathcal{G}_W \) admits a colimit \( \mathcal{G}_\downarrow_W \) in \( \text{CGr}_a \), then the induced morphism \( \text{del}_W \in \text{CGr}_a(\mathcal{G}, \mathcal{G}_\downarrow_W) \) is called the vertex deletion morphism corresponding to \( W \).

Note, in particular, that a vertex deletion functor \( \Lambda^\mathcal{G}_W \) is non-degenerate if and only if \( W = \emptyset \) in which case \( \Lambda^\mathcal{G}_W \) is the identity graph of graphs \( \mathcal{I}^\mathcal{G} : (\mathcal{C}, b) \to \mathcal{C} \) (Section 5.2).

Moreover, if \( W = W_1 \sqcup W_2 \) and \( \text{del}_W = \text{del}_{W_1}^W : \mathcal{G} \to \mathcal{G}_{\downarrow W} \) exists in \( \text{CGr}_a \), then so do

\[
\text{del}_{W_1}^W : \mathcal{G} \to \mathcal{G}_{\downarrow W_1} \quad \text{and} \quad \text{del}_{W_2}^W : \mathcal{G}_{\downarrow W_1} \to (\mathcal{G}_{\downarrow W_1}) \setminus W_2 = \mathcal{G}_{\downarrow W}^W
\]

and \( \text{del}_W = \text{del}_{W_1}^W \circ \text{del}_{W_2}^W \).
Example 7.13. For \( \mathcal{G} = \mathcal{C}_2 \) and \( W = V = \{ * \} \), \( \Lambda^G_W \) is the constant functor induced by the cocone of \( \text{el}(\mathcal{C}_2) \) over \( (i) \) in \( \text{CGr}_* \):

\[
\begin{array}{ccc}
\text{el}(i) & \xrightarrow{\text{ch}_1} & \mathcal{C}_2 \xleftarrow{\text{ch}_2 \circ \tau} (i) \\
\text{id}(i) & \xrightarrow{\text{id}(i)} & \text{id}(i)
\end{array}
\]

(7.14)

So, \( \Lambda^G_W \) has colimit \( (i) \) in \( \text{CGr}_* \) and \( \text{del}_W = u \in \text{CGr}_*(\mathcal{C}_2, i) \).

In fact, for all \( k \geq 0 \), if \( \mathcal{G} = \mathcal{L}^k \) and \( W = V \), then \( \Lambda^G_W \) is also the constant functor to \( \mathcal{G}_W = (i) \), and \( u^k \text{ def} = \text{del}_W : \mathcal{L}^k \rightarrow \text{CGr}_* \) is induced by the \( \text{CGr}_* \)-cocone under \( \text{el}(\mathcal{L}^k) \):

\[
\begin{array}{ccc}
\text{el}(i) & \xrightarrow{\text{ch}_1} & \mathcal{C}_2 \xleftarrow{\text{ch}_2 \circ \tau} (i) \\
\text{id}(i) & \xrightarrow{\text{id}(i)} & \text{id}(i)
\end{array}
\]

(7.15)

(\text{So } u^i = u : \mathcal{C}_2 \rightarrow (i) \text{ and } u^0 \text{ is just the identity on } (i).)

For any graph \( \mathcal{G} \), a pointwise étale injection \( \iota \in \text{CGr}_{\text{et}}(\mathcal{L}^k, \mathcal{G}) \) describes a subset \( V(\mathcal{L}^k) = W \subset V_2(\mathcal{G}) \) of bivalent vertices of \( \mathcal{G} \). Hence, \( \text{del}_W \in \text{CGr}_*(\mathcal{G}, \mathcal{G}_W) \) exists in \( \text{CGr}_* \) and there is an edge \( e_W = \text{del}_W(\iota(1_{\mathcal{L}^k})) \in E(\mathcal{G}_W) \) so that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{L}^k & \xrightarrow{\iota} & \mathcal{G} \\
\text{id}(W) & \xrightarrow{\text{del}_W} & \mathcal{G}_W
\end{array}
\]

(7.16)

Example 7.16. Let \( * \) be the unique vertex of the wheel graph \( \mathcal{W} \). By Example 7.9 \( \mathcal{W}_\{*, i\} = \text{colim}_{\mathcal{W}} \Lambda^W_{\{*, i\}} \) exists and is isomorphic to \( (i) \) in \( \text{CGr}_{\text{et}} \). (See also Section 6.) The induced morphism \( \text{del}_W \) is precisely \( \kappa : \mathcal{W} \rightarrow (i) \).

For \( m \geq 1 \), let \( W \subset V(\mathcal{W}^m) \) be the image of \( V(\mathcal{L}^{m-1}) \) under an étale morphism \( \iota \in \text{CGr}_{\text{et}}(\mathcal{L}^{m-1}, \mathcal{W}^m) \). So \( V(\mathcal{W}^m) = W \text{ II } \{ *, i \} \), and by (7.15), \( \iota \) induces a vertex deletion morphism \( \text{del}_W \in \text{CGr}_*(\mathcal{W}^m, \mathcal{W}) \).

Therefore \( \kappa^m \text{ def} = \text{del}_W(\mathcal{W}^m) \) is given by the composite \( \kappa^m = \kappa \circ \text{del}_W : \mathcal{W}^m \rightarrow \mathcal{W} \rightarrow (i) \).

In particular, for all \( m \geq 1 \), there are precisely two distinct morphisms, \( \kappa^m \) and \( \tau \circ \kappa^m \), in \( \text{CGr}_*(\mathcal{W}^m, i) \).

Hence, for all graphs \( \mathcal{G} \),

\[ \text{CGr}_*(\mathcal{W}^m, \mathcal{G}) = \text{CGr}_{\text{et}}(\mathcal{W}^m, \mathcal{G}) \text{ II } \{ \text{ch}_e \circ \kappa^m \}_{e \in E(\mathcal{G})} \]

Proposition 7.17. For all graphs \( \mathcal{G} \) and all \( W \subset V_2 \), the colimit \( \mathcal{G}_W \) of \( \Lambda^G_W \) exists in \( \text{CGr}_* \).

Moreover, \( E_0(\mathcal{G}) = E_0(\mathcal{G}_W) \) unless \( \mathcal{G} = \mathcal{W}^m \) and \( W = V \) for some \( m \geq 1 \).

Proof. If \( W \) is empty, then \( \mathcal{G}_W = \mathcal{G} \) and \( \text{del}_W \) is the identity on \( \mathcal{G} \). On the other hand, if \( W = V \) then, by Proposition 4.22 \( \mathcal{G} = \mathcal{L}^k \) or \( \mathcal{G} = \mathcal{W}^m \) for some \( k \geq 0 \) or \( m \geq 1 \), and so \( \mathcal{G}_W = (i) \) by Examples 7.13 and 7.16. For \( \mathcal{G} = \mathcal{L}^k \), the vertex deletion morphism \( u^k : \mathcal{L}^k \rightarrow (i) \) induces a bijection on boundaries, so the proposition is proved when \( W = V \) or \( W = \emptyset \).

Assume therefore, that \( \emptyset \neq W \subset V \) is a proper, non-empty subset of (bivalent) vertices of \( \mathcal{G} \).

Let \( \mathcal{G}^W \subset \mathcal{G} \) be the subgraph with vertices \( V(\mathcal{G}^W) = W \), half-edges \( H(\mathcal{G}^W) = \bigsqcup_{v \in W} H/v \) and whose edge set \( E(\mathcal{G}^W) \) is the \( \tau \)-closure of \( \bigsqcup_{v \in W} E/v \). (See Figure 23)
By Proposition 4.22, $G^W \cong \coprod_{i=1}^{m} \mathcal{L}^{k_i}$ is a disjoint union of line graphs, with $k_i \geq 1$ for all $i$:

\[(7.18) \quad E_0(G^W) = \coprod_{i=1}^{m} (1_{L^{k_i}}, 2_{L^{k_i}}), \quad \text{and}\quad \left( \coprod_{i=1}^{m} (1_{L^{k_i}}) \right) \cap \left( \coprod_{i=1}^{m} (2_{L^{k_i}}) \right) = \emptyset \quad \text{in} \ E(G).\]

The graph $G_{\setminus W}$ is obtained by applying $u^{k_i} : \mathcal{L}^{k_i} \to (i)$ on each component $\mathcal{L}^{k_i} \to G$ in turn. Since $W \neq V$ and components of $G^W$ are disjoint in $G$, this is independent of the order of $\{L^{k_i}\}_{i=1}^{m}$, and hence $\text{del}_{\setminus W} : G \to G_{\setminus W}$ exists in $\text{CGr}_{\sim}$. The construction is summarised in the diagram

\[
\begin{array}{ccc}
G^W & \leftarrow & \coprod_{i=1}^{m} \mathcal{L}^{k_i} \\
\coprod_{i=1}^{m} u^{k_i} & & i \\
\coprod_{i=1}^{m} (i) & \downarrow & \text{del}_{\setminus W} \\
& \coprod_{i} \mathcal{L}^{k_i} \setminus W, & \\
\end{array}
\]

\[
\text{where, for} \ 1 \leq i \leq m, \ e_i = \text{del}_{\setminus W}(i(1_{L^{k_i}})) \in E(G_{\setminus W}).
\]

When $W \neq V$, the graph $G_{\setminus W}$ (see Figure 23) is described explicitly by:

\[
G_{\setminus W} = \tau_W \bigcap_{G} E_{\setminus W} \xrightarrow{s_{\setminus W}} H_{\setminus W} \xrightarrow{t_{\setminus W}} V_{\setminus W},
\]

where

\[
\begin{align*}
V_{\setminus W} &= V \setminus W, \\
H_{\setminus W} &= H \setminus H(G^W) = H \setminus (\bigcup_{v \in W} E/v), \\
E_{\setminus W} &= E \setminus (\bigcup_{v \in W} E/v).
\end{align*}
\]

The maps $s_{\setminus W}, t_{\setminus W}$ are just the restrictions of $s$ and $t$ and the involution $\tau_W : E_{\setminus W} \to E_{\setminus W}$ is given by

\[
\begin{align*}
\tau_W(e) &= \tau e \quad \text{for} \ e \in E \setminus E(G^W), \\
\tau_W(1_{L^{k_i}}) &= 2_{L^{k_i}} \quad \text{for} \ 1 \leq i \leq m.
\end{align*}
\]

By (7.18), this is fixed point free and induces an identity $E_0(G) = E_0(G_{\setminus W})$ on boundaries. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{vertex-deletion-diagram}
\caption{Vertex deletion $\text{del}_{\setminus W} : G \to G_{\setminus W}$, with colours indicating $G^W \subset G$ and $W \subset V_2$, and $\coprod_{i=1}^{3} u^{k_i}(G^W) \subset G_{\setminus W}$.}
\end{figure}

**Definition 7.19.** The similarity category $\text{CGr}_{\sim} \hookrightarrow \text{CGr}_{\ast}$ is the identity on objects subcategory of $\text{CGr}_{\ast}$, whose morphisms are generated under composition by $z : C_0 \to (i)$, the vertex deletion morphisms, and graph isomorphisms. Morphisms in $\text{CGr}_{\sim}$ are called similarity morphisms, and connected components of $\text{CGr}_{\sim}$ are similarity classes. Graphs in the same connected component of $\text{CGr}_{\sim}$ are similar.

**Example 7.20.** Up to isomorphism, the only morphisms in $\text{CGr}_{\ast}$ with codomain $(i)$ are similarity morphisms of the form $z : C_0 \to (i)$, $\kappa^m : W^m \to (i)$ ($m \geq 1$), and $u^k : L^k \to (i)$ ($k \geq 0$).
**Corollary 7.21** (Corollary to Proposition 7.17). The pair \((\text{CGr}_{\text{sim}}, \text{CGr}_{\text{et}})\) of subcategories of \(\text{CGr}_s\) defines a weak factorisation system on \(\text{CGr}_s\).

In particular, if \(E_0(\mathcal{G}) \neq \emptyset\) and \(f \in \text{CGr}_s(\mathcal{G}, \mathcal{G}')\) is boundary-preserving, then \(f = f_{\chi_{W_f}} \circ \text{del}_{W_f}\) where \(f_{\chi_{W_f}} \in \text{CGr}_s(\mathcal{G}_{\backslash W_f}, \mathcal{G}')\) is an isomorphism.

**Proof.** The only non-identity morphisms in \(\text{CGr}_s\) with (co)domain \(C_0\) are of the form \(c_{e'} \circ z = c_{r_{e'}} \circ z : C_0 \rightarrow \mathcal{G}\) for some graph \(\mathcal{G}\) with edge \(e\), and \(z\) has the left lifting property with respect to morphisms in \(\text{CGr}_s\). Moreover, any morphism \(f \in \text{CGr}_s(\mathcal{G}, \mathcal{G}')\) between connected graphs \(\mathcal{G} \not\cong C_0\) and \(\mathcal{G}' \not\cong C_0\) factors uniquely as \(f = f_{\chi_{W_f}} \circ \text{del}_{W_f}\), where \(W_f\) is the set of bivalent vertices \(w\) of \(\mathcal{G}\) such that, if \((C_2, b)\) is a minimal neighbourhood of \(w\), then \(f \circ b = c_{e'} \circ u : C_2 \rightarrow \mathcal{G}'\) for some (necessarily unique) edge \(e'\) of \(\mathcal{G}'\). Hence \((\text{CGr}_{\text{sim}}, \text{CGr}_{\text{et}})\) describes a weak factorisation system on \(\text{CGr}_s\).

The second statement follows immediately from Corollary 4.20. \(\square\)

**Example 7.22.** For all graphical species \(S\) and all graphs \(\mathcal{G}\) with no isolated vertices, by Corollary 7.21

\[(7.23)\]

\[DS(\mathcal{G}) = \prod_{W \subset V_2} S(\mathcal{G} \setminus W).\]

**Example 7.24.** In particular, for all graphs \(\mathcal{G}\) and all \(k \in \mathbb{N}\), \((7.23)\) gives

\[\text{CGr}_{\text{et}}(\mathcal{L}^k, \mathcal{G}) \cong \prod_{j=0}^k \binom{k}{j} \text{CGr}_{\text{et}}(\mathcal{L}^j, \mathcal{G}).\]

By Corollary 7.21, a morphism \(f \in \text{CGr}_s(\mathcal{G}, \mathcal{G}')\) is uniquely characterised by a commuting diagram of the form \((7.25)\), and such that \(f^{-1}(\mathcal{E}') \subset V_0 \sqcup V_2\) is either a single isolated vertex or a (possibly empty) subset of bivalent vertices, and the induced square \(7.26\) is a pullback.

\[(7.25)\]

\[
\begin{array}{cccccc}
E & \xleftarrow{\tau} & E & \xleftarrow{s} & H & \xrightarrow{t} & V \\
\downarrow f_E & & \downarrow f_E & & \downarrow f_H & & \downarrow f_V \\
E' & \xleftarrow{\tau'} & E' & \xleftarrow{s'_{\text{id}}} & H' \sqcup E' & \xrightarrow{t'_{\text{id}}} & V' \sqcup E' \\
\end{array}
\]

\[(7.26)\]

\[
\begin{array}{cccccc}
H & \xrightarrow{t} & (V \setminus V_0) \\
\downarrow f_H & & \downarrow f_V \\
H' \sqcup E' & \xrightarrow{t'_{\text{id}}} & (V \setminus V_0) \sqcup \mathcal{E}' \\
\end{array}
\]

If \(\mathcal{G} \not\cong C_0\), and \(f = f_{\chi_{W_f}} \circ \text{del}_{W_f}\), where \(f_{\chi_{W_f}}\) is a morphism in \(\text{CGr}_{\text{et}}\), then \(W_f = f^{-1}(\mathcal{E}') \subset V_2\).

**Example 7.27.** The morphisms \(z \in \mathbf{B}^3(0, 8) = \text{CGr}_s(C_0, 1)\) and \(u \in \mathbf{B}^3(2, 8) = \text{CGr}_s(C_2, 1)\) are described by commuting diagrams \((7.28)\) and \((7.29)\):

\[(7.28)\]

\[
\begin{array}{ccccc}
C_0 & \xleftarrow{0} & 0 & \xrightarrow{\{\ast\}} \\
\downarrow z & & \downarrow & & \downarrow \\
\{1, 2\} & \xleftarrow{id} & \{1, 2\} & \xrightarrow{q} & \{1\}
\end{array}
\]

\[(7.29)\]

\[
\begin{array}{ccccc}
C_2 & \xleftarrow{\{1^*_{c_2}, 2^*_{c_2}, 3^*_{c_2}\}} & \{1^*_{c_2}, 2^*_{c_2}\} & \xrightarrow{\{\ast\}} \\
\downarrow u & & \downarrow & & \downarrow \\
\{1, 2\} & \xleftarrow{id} & \{1, 2\} & \xrightarrow{q} & \{1\}
\end{array}
\]

The following extension of Lemma 3.14 says that most morphisms in \(\text{CGr}_s\) are completely determined by their action on edges:
Lemma 7.30. If $\mathcal{G} \not\cong \mathcal{C}_0$ and $\mathcal{G}' \not\cong \mathcal{W}$, then $f_E$ is sufficient to define $f \in CGr_*(\mathcal{G}, \mathcal{G}')$.

Proof. Let $v \in V_2$ be a bivalent vertex of $\mathcal{G}$ with incident edges $E/v = \{e_1, e_2\} \subset E_2$. If $f_E(e_1) \neq f_E(e_2)$, then $f_V(v) = t's^{-1}(f_E(e_1)) \in V'$. Otherwise, $f_E(e_1) = f_E(e_2)$. Then, either

$$f_V(v) = q'(f_E(e_1)) = q'(f_E(e_2)) \in \widetilde{E'},$$

or, there is a vertex $v'$ of $\mathcal{G}'$ with $E/v = \{f_E(e_1), f_E(e_2)\}$, in which case $\mathcal{G}' = \mathcal{W}$. \hfill $\square$

Example 7.31. Lemma 7.30 does not hold if $\mathcal{G}' = \mathcal{W}$. For example, there are only two maps of edges $E(\mathcal{W}^2) \to E(\mathcal{W})$ that are compatible with the involution, and these correspond to the two morphisms in $CGr_*(\mathcal{W}^2, \mathcal{W})$. However, there are six distinct morphisms in $CGr_*(\mathcal{W}^2, \mathcal{W})$.

7.3. $S_*$-structured graphs. The étale topology on $CGr_{et}$ extends to a topology on $CGr_*$ whose covers at $\mathcal{G}$ are jointly surjective collections $\mathcal{U} \subset CGr_*/\mathcal{G}$. By Lemma 7.3 a presheaf $P : CGr^{op}_{et} \to Set$ is a sheaf for this topology if and only if, for all graphs $\mathcal{G}$, $P(\mathcal{G}) \cong \lim_{(C,b) \in \mathcal{U}(\mathcal{G})} P(\mathcal{C})$. In particular, there is a canonical equivalence $sh(CGr_*, J_*) \simeq G_*$ (compare Section 4.4).

Let $S_*$ be a pointed graphical species.

Definition 7.32. (Compare Definition 4.28) An $S_*$ structure on a connected graph $\mathcal{G}$ is an element $\alpha \in S_*(\mathcal{G}) \cong G_*(\mathcal{G}, S_*)$. The category of (connected) $S_*$-structured graphs is denoted by $CGr_*/S_*$.

An $S_*$-structured graph $(\mathcal{G}, \alpha)$ is called admissible if $\mathcal{G} \not\cong (\cdot)$ is not a stick graph.

Example 7.33. For all pointed graphical species $S_ = (S, \epsilon, o)$, all $k \geq 0$, $m \geq 1$, the vertex deletion morphisms $u^k \in CGr_*(L^k, i)$ and $\kappa^m \in CGr_*(W^m, i)$ induce injective maps

$$S_*(u^k) : S_5 = S(i) \to S(L^k), \quad \text{and} \quad S_*(\kappa^m) : S_5 \to S(W^m).$$

For each $c \in S_5$, there are $c$-coloured unit structures on $L^k$ and $W^m$ (as pictured in Figure 24):

$$L^k(\epsilon_c) \overset{\text{def}}{=} S_*(u^k)(c) \in S_*(L^k) \quad \text{and} \quad W^m(\epsilon_c) \overset{\text{def}}{=} S_*(\kappa^m)(c) \in S_*(W^m).$$

![Figure 24. The $c$-coloured unit structures $L^k(\epsilon_c)$ and $W^k(\epsilon_c)$.](image)

For any subset $W$ of bivalent vertices of a graph $\mathcal{G}$, and any $S_*$-structure $\alpha \mid W \in S(\mathcal{G} \mid W)$, there is a unique $S_*$-structure $\alpha \in S(\mathcal{G})$ such that $\text{del}_W \in CGr_*/S_*(\alpha, \alpha \mid W)$: If $(G_2, b) \in \text{el}(\mathcal{G})$ is a neighbourhood of $v \in W$, then $\text{del}_W \circ b = ch_v \circ u$ for some $u \in E(\mathcal{G} \mid W)$. Hence $\alpha \in S(\mathcal{G})$ is determined by

$$S_*(b)(\alpha) = S_*(\text{del}_W \circ b)(\alpha \mid W) = S_*(u)(S(ch_v)(\alpha \mid W)) = \epsilon(S(ch_v)(\alpha \mid W)).$$

Definition 7.34. Let $(\mathcal{G}, \alpha)$ be an $S_*$-structured graph. Then

$$W_\alpha \overset{\text{def}}{=} \{v \mid \text{there is a neighbourhood } (C, b) \text{ of } v \text{ such that } S(b)(\alpha) \in \text{im}(\epsilon) \cup \text{im}(o)\} \subset V_0 \amalg V_2,$$

is the set of vertices $\alpha$-decorated by (contracted) units.

An $S_*$-structure $(\mathcal{G}, \alpha)$ is called reduced if $W_\alpha = \emptyset$. 

Let \( G \not\cong C_0 \) and \( W \subset V_2(G) \). There exists an \( S_* \)-structure \( \alpha_W \in S(G \setminus W) \) such that \( \text{del}_W \in \text{CGr}_*(G, G(W)) \) describes a morphism in \( \text{CGr}_*/S_*(G, (G, \alpha), (G \setminus W, \alpha \setminus W)) \) if and only if \( W \subset W_\alpha \). By definition, \( (G \setminus W, \alpha \setminus W) \) is reduced if and only if \( W = W_\alpha \).

### 7.4. Similar structures.

The issues that can arise from trying to build degenerate substitution by the stick graph into the definition of the modular operad monad have been outlined in Section 6. Degenerate substitutions, and therefore exceptional loops, can be avoided if there is a suitable notion of equivalence of \( S_* \)-structured graphs, for which all constructions may be obtained in terms of admissible representatives.

This principle informs the construction of the distributive law \( \lambda: TD \Rightarrow DT \).

By Proposition 7.17 any similarity morphism that is not of the form \( z: C_0 \rightarrow (i) \) or \( \kappa^m: W^m \rightarrow (i) \) preserves boundaries. Let \( G \) be a connected graph with non-empty vertex set. A boundary-preserving similarity morphism \( f \in \text{CGr}_\text{sim}(G, G') \), together with an \( X \)-labelling \( \rho: E_0 \rightarrow X \) of \( G \), induces an \( X \)-labelling on \( G' \).

Recall that \( X-\text{CGr}_\text{iso} \) is the category of (admissible) \( X \)-graphs and boundary-preserving isomorphisms. The category \( X-\text{CGr}_\text{sim} \) is obtained from \( X-\text{CGr}_\text{iso} \) by adjoining all similarity morphisms from objects of \( X-\text{CGr}_\text{iso} \) and, where necessary, their \( X \)-labelled codomains:

- If \( X \not\cong 0, X \not\cong 2 \), then \( X-\text{CGr}_\text{iso} \) is a bijective on objects subcategory of \( X-\text{CGr}_\text{sim} \) whose morphisms are similarity morphisms that preserve the labelling of the ports.
- For \( X = 2 \), \( 2-\text{CGr}_\text{iso} \) is obtained from \( 2-\text{CGr}_\text{iso} \) by adjoining the morphisms \( \text{del}_V \): \( L^k \rightarrow (i) \), and hence also the labelled stick graphs \((i, id)\) and \((i, \tau)\). An admissible \( 2 \)-graph \( X \in 2-\text{CGr}_\text{iso} \subset 2-\text{CGr}_\text{sim} \) is in the same connected component as \((i, id)\) if and only if \( X = (L^k, id_{L^k}) \) for some \( k \geq 1 \). In particular, \( \tau: (i) \rightarrow (i) \) does not induce a morphism in \( 2-\text{CGr}_\text{sim} \).
- When \( X = 0 \), the morphisms \( \text{del}_V \): \( W^m \rightarrow (i) \), and \( z: C_0 \rightarrow (i) \) are not boundary-preserving, and do not induce any labelling on the ports of \((i)\). So, the objects of \( 0-\text{CGr}_\text{sim} \) are the admissible \( 0 \)-graphs in \( 0-\text{CGr}_\text{iso} \), together with \((i)\). In particular, \( W^m, C_0 \) and \((i)\) are in the same connected component of \( 0-\text{CGr}_\text{sim} \), and \((i)\) is terminal in this component.

To simplify notation in what follows, let \( C_0|_V \overset{\text{def}}{=} (i) \) and \( \text{del}_V = z: C_0 \rightarrow (i) \) in \( \text{CGr}_* \).

**Definition 7.35.** For all \( S_* \in \text{GS}_* \), the slice category \( X-\text{CGr}_\text{sim}/S_* \) induced by the canonical functor \( X-\text{CGr}_\text{sim} \rightarrow \text{CGr}_* \rightarrow \text{GS}_* \) is the category of similar \( S_* \)-structured \( X \)-graphs.

Admissible \( S_* \)-structured \( X \)-graphs \( (\mathcal{X}, \alpha) \) are called similar, written \( (\mathcal{X}, \alpha^1) \sim (\mathcal{X}, \alpha^2) \) (or \( \alpha^1 \sim \alpha^2 \)), if they are in the same connected component of \( X-\text{CGr}_\text{sim}/S_* \).

Let \( (\mathcal{X}, \alpha) \) be an admissible \( S_* \)-structured \( X \)-graph such that \( (\mathcal{X}, \alpha) \not\cong (C_0, o_c) \). Then

\[
(\mathcal{X}_\alpha^1, \alpha^1) \overset{\text{def}}{=} (\mathcal{X}|_{W_\alpha}, \alpha|_{W_\alpha})
\]

is reduced and terminal in the connected component of \( (\mathcal{X}, \alpha) \) in \( X-\text{CGr}_\text{sim}/S_* \). It is admissible unless \( (\mathcal{X}, \alpha) = L^k(e_c) \), or \( (\mathcal{X}, \alpha)W^m(e_c) \) for some \( c \in S_k \) and \( k, m \geq 1 \).

If \( (\mathcal{X}_\alpha^1, \alpha^1) \) is not admissible, then it is represented by \((i, c)\) for some \( c \in S_k \). If \( c \not\in WC \), then \((i, c)\) and \((i, \omega_c)\) are in disjoint components of \( 2-\text{CGr}_\text{sim}/S_* \). By contrast, \( z: C_0 \rightarrow (i) \) induces morphisms \( (C_0, o_c) \rightarrow (i, c) \) and \((C_0, o_c) \rightarrow (i, \omega_c) \) in \( \text{CGr}_* \). So, for all \( c \in S_k \), \( \bar{c} \) defines a unique element \([i, c] = [i, \omega_c] \) in \( 0-\text{CGr}_\text{sim}/S_* \).

Moreover, the similarity maps \( \kappa: W \rightarrow (i) \leftarrow C_0: z \in \text{CGr}_* \) induce morphisms of \( S_* \)-structured graphs:

\[
(7.36) \quad W(\epsilon_c) \quad \overset{\text{def}}{=} (i, c) \quad \overset{\text{def}}{=} (C_0, o_c) \quad \overset{\text{def}}{=} (i, \omega_c) \quad W(\epsilon_{\omega_c})
\]

So, \( W(\epsilon_c) \sim (C_0, o_c) \) and there is a double-cone shaped diagram in \( 0-\text{CGr}_\text{sim}/S_* \) (Figure 25).
Lemma 7.37. For all pointed graphical species $S_*$ and all finite sets $X$, there is a canonical bijection

\[(7.38) \quad \text{colim}_{X \in X \cdot \mathcal{Gr}_{\sim}} S_*(X) \cong \pi_0(X \cdot \mathcal{Gr}_{\sim}/S_*).
\]

Proof. Since $X \cdot \mathcal{Gr}_{\sim}/S \subset X \cdot \mathcal{Gr}_{\sim}/S_*$, there is a surjection of connected components $\pi_0(X \cdot \mathcal{Gr}_{\sim}/S) \to \pi_0(X \cdot \mathcal{Gr}_{\sim}/S_*)$. Every component of $X \cdot \mathcal{Gr}_{\sim}/S_*$ has a representative in $X \cdot \mathcal{Gr}_{\sim}/S$, and the result follows from (5.7) and Definition 7.35.

□

7.5. A distributive law for modular operads. Let $S$ be a graphical species and $X$ a finite set. An element of $TDS_X$ is represented by an $X$-graph $\mathcal{X}$, with a $DS$-structure $\alpha \in DS(\mathcal{X}) = S^+(\mathcal{X})$. The idea is to construct a distributive law $\lambda: TD \Rightarrow DT$ such that $\lambda S$ is invariant under similarity morphisms.

Proposition 7.39. There is a distributive law $\lambda: TD \Rightarrow DT$ such that for all graphical species $S$ and finite sets $X$, and all elements $[\mathcal{X}, \alpha], [\mathcal{X}', \alpha']$ of $TDS_X$,

$$\lambda S[\mathcal{X}, \alpha] = \lambda S[\mathcal{X}', \alpha'] \text{ in } DT S_X \text{ if and only if } [\mathcal{X}, \alpha] \sim [\mathcal{X}', \alpha'] \in X \cdot \mathcal{Gr}_{\sim}/S^+.$$ 

Proof. Since the endofunctor $D$ just adjoins elements, there are canonical inclusions $TS \hookrightarrow DTS$ and $TS \twoheadrightarrow TDS$. The natural transformation $\lambda: TD \Rightarrow DT$ will restrict to the identity on $TS$.

For a finite set $X$, elements of $TDS_X$ are represented by $DS$-structured $X$-graphs $(\mathcal{X}, \alpha)$, whereas elements of $DTS_X$ are either of the form $\epsilon_c^{DTS}, \sigma_c^{DTS}$ for $c \in S_3$, or are represented by $S$-structured $X$-graphs $(\mathcal{X}', \alpha')$. Observe also that an object $(\mathcal{X}, \alpha) \in X \cdot \mathcal{Gr}_{\sim}/S^+$ is reduced and admissible if and only if $(\mathcal{X}, \alpha) \in X \cdot \mathcal{Gr}_{\sim}/S$, and hence $[\mathcal{X}, \alpha] \in TS_X$.

Let $(\mathcal{X}, \alpha) \in X \cdot \mathcal{Gr}_{\sim}/DS$. If $X = 2$, and $(\mathcal{X}, \alpha)$ has the form $\mathcal{L}^k(\epsilon_c)$, set

$$\lambda S[\mathcal{X}, \alpha] = \epsilon_c^{DS} \in DT S_2.$$ 

And, if $X = 0$, and $(\mathcal{X}, \alpha) = \mathcal{W}^m(\epsilon_c)$ or $(\mathcal{X}, \alpha) = (\Lambda_0, \alpha_0)$, set

$$\lambda S[\mathcal{X}, \alpha] = \sigma_c^{DTS} \in DT S_0.$$ 

Otherwise, the component of $(\mathcal{X}, \alpha)$ in $X \cdot \mathcal{Gr}_{\sim}/S^+$ has an admissible and reduced (hence terminal) object $(\mathcal{X}_a^\perp, \alpha_a^\perp)$, so we can set

$$\lambda S[\mathcal{X}, \alpha] = [\mathcal{X}_a^\perp, \alpha_a^\perp] \in TS_X \subset DTS_X.$$ 

The assignment so defined clearly extends to a natural transformation $\lambda: TD \Rightarrow DT$.

The verification that $\lambda$ satisfies the four axioms [4] Section 1 for a distributive law is straightforward but unenlightening, so I prove just one here, that the following diagram of natural transformations commutes:
Let \([X, \alpha] \in TD^2 S_X\). The result is immediate when \(X = C_0\). Moreover, all the maps in (7.40) restrict to the identity on \(T S \subset TD^2 S\).

Therefore, we may assume that \([X, \alpha] \notin T S_X\) and that \(X \notin C_0\). For \(j = 1, 2\), define the sets \(W_j\), of vertices decorated by distinguished elements adjoined in the \(j\)th application of \(D\):

\[ W_j \overset{\text{def}}{=} \{ v \mid v \text{ has a minimal neighbourhood } (C, b) \text{ with } D^2 S(b)(\alpha) \in \text{im}(\epsilon^{DS}) \} \subset V_2. \]

Then \(T \mu^D S[X, \alpha] = [X, \alpha^D] \in T D S_X\), where \(\alpha^D \in DS(X)\) is given by

\[ DS(b)(\alpha^D) = \begin{cases} \epsilon^{DS} & \text{if } (C, b) \text{ is a minimal neighbourhood of } v \in W^1 \sqcup W^2, \\ D^2 S(b)(\alpha) \in S(C) & \text{otherwise.} \end{cases} \]

If \(W^1 \sqcup W^2 \neq V\), then (7.40) gives

\[
\begin{array}{ccc}
[X, \alpha] & \xrightarrow{\lambda_{DS}} & [X^\setminus W^2, \alpha^\setminus W^2] \\
& & \xrightarrow{\lambda_{TS}} [X^\setminus (W^1 \sqcup W^2), \alpha^\setminus (W^1 \sqcup W^2)] \\
& & \xrightarrow{\lambda_S} [X^\setminus (W^1 \sqcup W^2), \alpha^\setminus (W^1 \sqcup W^2)] \\
& & \in TS_X
\end{array}
\]

If \(W^1 \sqcup W^2 = V\), then \(T(\mu^D)[X, \alpha]\) has the form \(L^k(\epsilon^{DS})\) or \(W^m(\epsilon^{DS})\) and both paths map to the corresponding (contracted) unit in \(DT S\).

It follows that there is a composite monad \(\mathbb{D}T\) on \(GS\), induced by \(\lambda\). Moreover, by [3] Section 3, \(\lambda: TD \Rightarrow DT\) induces a lift \(T_*\) of \(T\) to \(GS_\ast\), such that the EM categories \(GS^{DT}_\ast\) and \(GS^*_\ast\) are canonically isomorphic. (See also Section 2.2)

**Corollary 7.41.** The monad \(T_* = (T_*, \mu^T_*, \eta^T_*)\) on \(GS_\ast\) is given by

\[ T_* S_h = S_h, \text{ and } T_* S_X = \text{colim}_{X \in X \cdot \text{Grm}} S(X). \]

The unit \(\eta^T_*: 1_{GS} \Rightarrow T_*\) and multiplication \(\mu^T_*: T_*^2 \Rightarrow T_*\) are induced by the unit \(\eta^T: 1_{GS} \Rightarrow T\) and multiplication \(\mu^T: T^2 \Rightarrow T\) for \(T\). In other words, if \([X, \alpha]_*\) denotes the class of \([X, \alpha] \in TS_X\) in \(T_* S_X\), then

\[ \eta^T_* \lbrack \phi \rbrack = \eta^T \lbrack \phi \rbrack \text{ and } \mu^T_* \lbrack X, \beta \rbrack = \mu^T \lbrack X, \beta \rbrack_* \]

**Proof.** Let \(S_\ast = (S, \epsilon, o)\) be a pointed graphical species, and let \(h_\mathbb{D}: DS \rightarrow S\) denote the \(\mathbb{D}\)-algebra structure on \(S\), so \(\epsilon = h_\mathbb{D} (\epsilon^+\) and \(o = h_\mathbb{D} (o^+)\).

Observe first that

\[ (\mu^D \mu^T) \circ (D \lambda T) \circ (D TD \eta^T) = (\mu^D T) \circ (D \lambda) : DT D \Rightarrow DT. \]

So, by [3] Section 3, \(T_* (S_\ast)\) is described by the coequaliser

\[
\begin{array}{ccc}
DT DS & \xrightarrow{D \phi_h} & DT S \\
\downarrow \& & \downarrow \pi \\
D^2 T S.
\end{array}
\]
The occurrence of $\lambda$ means that the lower path of (7.42) identifies similar elements of $TDS \hookrightarrow DTDS$ and, since $\epsilon = h_0(\epsilon^+)\), it follows that similar elements of $TS \subset DTS$ are identified by the quotient $\pi$. Reduced elements of $TS_X \subset DTDS_X \to DTS_X \supset TS_X$ are unchanged under both paths in (7.42). Therefore $T,S_X = \text{colim}_{X \in X-CGr_{\text{sim}}S_*(\mathcal{X})}$ as required. Clearly $[\mathcal{L}^k(\epsilon)]_*$ and $[\mathcal{W}^m(\epsilon)]_* = [C_0, o]_*$ provide (contracted) units for $T,S_*$.  

Let $(\mathcal{X}, \beta)$ represent an element of $T^2S_X$ such that $(\mathcal{X}, \beta) \not\equiv (C_0,(C_0,o_\epsilon))$, $(\mathcal{X}, \beta) \not\equiv (C_0,[\mathcal{W}^m(\epsilon))]$. And let $\text{del}(W) \in X-CGr_{\text{sim}}((\mathcal{X}, \beta),(\mathcal{X}/W, \beta/W))$, so $W \subset W_\alpha$. Then $(C_2, b) \in \text{el}(\mathcal{X})$ is a neighbourhood of $v \in W$ if and only if there is a $k \geq 1$, and a $c \in S_0$ such that $S(b)(\beta) = [\mathcal{L}^k(\epsilon)]_*$ in $TS_2$. In particular, $\mu \circ [\mathcal{X}, \beta]$ and $\mu \circ [\mathcal{X}/W, \beta/W]$ are represented by similar objects of $X-CGr_{\text{sim}}/S_*$.  

Finally, let $(\mathcal{X}, \beta) \in TS(\mathcal{X})$ be similar to $(C_0,(C_0,o_\epsilon))$, and hence also $(\mathcal{X}, \beta) \sim (C_0,[\mathcal{W}^m(\epsilon)]_*)$ in $0-CGr_{\text{sim}}/T,S_*$. Then, either $\mu \circ [\mathcal{X}, \beta] = (C_0,o_\epsilon)$ or $\mu \circ [\mathcal{X}, \beta] = [\mathcal{W}^M(\epsilon)]$ for some $M \geq 1$. It follows that $\mu$ preserves similarity classes of $(C_0,[\mathcal{W}^m(\epsilon)])$ in $TS(C_0) \to T^2S_0$.  

Hence, $\mu \circ [\mathcal{X}, \beta]$ is well-defined and provides the multiplication for $T_*$. Clearly $\eta \circ [\mathcal{X}, \beta]$ is canonically isomorphic to $\eta \circ (\mathcal{X}, \beta)$, and the corollary follows.  

In particular, to compute the image of $[\mathcal{X}, \beta] \in T^2S_*$ under the multiplication $\mu \circ ([\mathcal{X}, \beta] : T^2S_* \to T,S_*)$, it is sufficient to choose a non-degenerate representative of $\beta$ (i.e. a non-degenerate $\mathcal{X}$-shaped graph of $S$-structured graphs) and quotient by similarity at the end.

Remark 7.43. There is also a distributive law $DT \Rightarrow TD$. Algebras for the composite monad $\mathbb{D}T$ are just the cofibred coproducts of algebras for $\mathbb{D}$ and $T$. There is no further relationship between the two structures. (See also Example 2.16)  

7.6. $\mathbb{D}T$-algebras are modular operads. At last we rea are ready to prove the first main theorem – that modular operads are $\mathbb{D}T$-algebras in GS.  

Let $(S, \circ, \zeta, \epsilon)$ be a modular operad. Since $(S, \circ, \zeta)$ is a non-unital modular operad, it is equipped with a $T$-algebra structure $p_T = p_T^\circ : TS \to S$, by Proposition 5.29. Moreover $S_* = (S, \epsilon, \zeta, \epsilon)$ is a pointed graphical species, and hence also a $\mathbb{D}$-algebra.

Lemma 7.44. The defining functor $X-CGr_{\text{sim}}/S \to TS_X \xrightarrow{p_T} S_X$ factors through $\pi_0(X-CGr_{\text{sim}}/S_*)$.  

Proof. Since $(S, \circ, \zeta, \epsilon)$ is a modular operad, $p_T$ satisfies

\begin{equation}
  p_T[M_c(\phi, \epsilon)] = \phi \circ_c \epsilon = \phi = p_T[\eta(\phi)] \text{ wherever defined.}
\end{equation}

In particular, let $\alpha \in S(\mathcal{X})$ for $\mathcal{X} \not\equiv C_0$ and let $W \subset W_\alpha$ be such that $W \not\equiv V(\mathcal{X})$. Then

\begin{equation*}
  p_T[\mathcal{X}, \alpha] = p_T[\mathcal{X}/W, \alpha/W]
\end{equation*}

by the proof of Proposition 5.29.

Therefore, it remains to check that $p_T[C_0, o_\epsilon] = p_T[\mathcal{W}(\epsilon)]$ for all $c \in S_0$. This is immediate since $W \equiv \mathcal{X}_1(\epsilon)$ (Example 3.9), and hence

\begin{equation*}
  p_T[C_0, o_\epsilon] = o_\epsilon \circ \zeta(\epsilon) = p_T[\mathcal{W}(\epsilon)] = p_T[\mathcal{W}^m(\epsilon)]
\end{equation*}

by construction, since $(S, p_T)$ is a $T$-algebra.  

It is now straightforward to prove that $\mathbb{D}T$ is the desired modular operad monad on GS.

Theorem 7.46. The $EM$ category $GS^{\mathbb{D}T}$ of algebras for $\mathbb{D}T$ is canonically isomorphic to MO.
Proof. Let \((A, h)\) be a \(\mathbb{D}T\)-algebra with corresponding \(\mathbb{D}\) and \(T\) structure morphisms
\[
h_\mathbb{D} \overset{\text{def}}{=} h \circ (D\eta^\mathbb{T}A) : DA \to A, \quad \text{and} \quad h_T \overset{\text{def}}{=} h \circ (D\eta^\mathbb{D}A) : TA \to A.
\]
Since \(\eta^\mathbb{D}\) is just an inclusion, \(h_T = h|_{TA} : TA \to A\) is the restriction of \(h\) to \(TA \subset DTA\). By Proposition 5.29, \(A\) is equipped with a multiplication \(\circ = h \circ [\mathcal{M}(\cdot, \cdot)]\) and contraction \(\zeta = h \circ [\mathcal{N}(\cdot)]\), as described in the proof of Lemma 5.28 so that \((A, \circ, \zeta)\) is a non-unital modular operad.

It remains to show that \(\epsilon\) provides a unit for the multiplication \(\circ\). By the monad algebra axioms, there are commuting diagrams in \(\mathbb{G}_S\):

\[
(7.47) \quad A \xrightarrow{\eta^\mathbb{D}\eta^\mathbb{T}A} DTA \quad \quad (7.48) \quad (DT)^2A \xrightarrow{DTA} D^2T^2A \xrightarrow{\mu^\mathbb{D}\mu^\mathbb{T}A} DTA
\]

Let \(X\) be a finite set, \(c \in (A_0)^X\), and let \(\phi \in A_{c^{-}}\). By definition of \(\lambda\),
\[
DTA[M_{c_x}(\eta^\mathbb{T}A(\phi), \epsilon^{DTA}_{c_x})] = [C_X, \eta^\mathbb{T}A(\phi)] = (\eta^\mathbb{T})^2A(\phi)
\]
for all \(x \in X\). So \([M_{c_x}(\eta^\mathbb{T}A(\phi), \epsilon^{DTA}_{c_x})] \hookrightarrow \phi\) under the top-right path in (7.48).

Now, \(\phi \circ_{c_x} \epsilon_{c_x} = h[M_{c_x}(\phi, \epsilon_{c_x})]\) by definition, and \([M_{c_x}(\phi, \epsilon_{c_x})] = DTh[M_{c_x}(\eta^\mathbb{T}(\phi), \epsilon^{DTA}_{c_x})]\) by the monad algebra axioms. Then, since (7.48) commutes,
\[
\phi \circ_{c_x} \epsilon_{c_x} = h[M_{c_x}(\phi, \epsilon_{c_x})] = hDTh[M_{c_x}(\eta^\mathbb{T}(\phi), \epsilon^{DTA}_{c_x})] = \phi,
\]
and \(\epsilon\) is a unit for \(\circ\).

Conversely, a modular operad \((S, \circ, \zeta, \epsilon)\) induces a pointed graphical species \(S_\# = (S, \epsilon, o = \zeta)\). By Proposition 5.29, \((S, \circ, \zeta, \epsilon)\) has a \(T\)-algebra structure \(p_T : TS \to S\) such that
\[
(7.49) \quad \circ = p_T \circ [\mathcal{M}(\cdot, \cdot)] \quad \text{and} \quad \zeta = p_T \circ [\mathcal{N}(\cdot)].
\]
and, for all \(c \in S_\#\) and all \(m \geq 1\), since \(\epsilon\) is a unit for \(\circ\),
\[
(7.50) \quad o_{c} \overset{\text{def}}{=} \zeta_{c} = p_T[W^m(c)] = p_T[W^\infty(c)] = p_T[W^m(c)].
\]

Let \(p : DTS \to S\) be defined by
\[
(7.51) \quad p(\epsilon^{DTS}) = \epsilon : S_\# \to S_2, \quad p(\zeta) = \zeta : S_\# \to S_0, \quad \text{and} \quad p = p_T : TS \to S\text{ on } TS \subset DTS.
\]
Then (7.47) commutes for \((A, h) = (S, p)\) since \((S, p_T)\) is a \(T\)-algebra.

It remains to check that (7.48) commutes for \((A, h) = (S, p)\). This is clear for the adjoined (contracted) units \(\epsilon^{(DT)^2S}\) and \(\epsilon^{DT^2S}\) in \((DT)^2S\). Otherwise, if \([\lambda, \beta] \in TDTS_X\), then exactly one of the following four conditions holds:

(i) \(X = 0\) and \([\lambda, \beta] = [0, o_{c}^{DTS}]\). In this case, it is immediate from the definitions of \(p\) and \(\lambda\) that the image of \([\lambda, \beta]\) under both paths in (7.48) is \(o_{c}\);

(ii) \(X = 0\) and \([\lambda, \beta] = [W^m(\epsilon^{DTS}_{c})]\) for some \(m \geq 1\), and \(c \in S_\#\). The application of \(\lambda TS\) means that the top-right path takes \([W^m(\epsilon^{DTS}_{c})]\) to \(o_{c} \in S_0\).

Since \(p(\epsilon^{DTS}) = \epsilon\), the bottom-left path takes \([W^m(\epsilon^{DTS}_{c})]\) first to \([W^m(\epsilon_{c})] \in TS_0\) and then, by (7.50), to \(o_{c}\).

(iii) \(X = 2\) and \([\lambda, \beta] = [L^k(\epsilon^{DTS}_{c})]\) for some \(k \geq 1\), and \(c \in S_\#\). Once again, the application of \(\lambda TS\) means that the top-right path takes \([L^k(\epsilon^{DTS}_{c})]\) to \(c_{c} \in S_2\).
The bottom-left path takes \( \mathcal{L}^k(\epsilon) \) first to \( \mathcal{L}^k(\epsilon) \) in \( TS_2 \) by applying \( p \) inside and then, by Lemma 7.44,
\[
p[\mathcal{L}^k(\epsilon)] = p_T[\mathcal{L}^k(\epsilon)] = \epsilon \in S_2;
\]
(iv) Otherwise, \( \Lambda T S[X, \beta] = [X^+ \beta, \beta^+] \) where \( (X^+ \beta, \beta^+) \) is the unique reduced and admissible \( S \)-structured graph that is similar to \( (X, \beta) \) in \( X \)-\text{CGr}_{\text{sim}}/S_\ast \), and hence \( [X^+ \beta, \beta^+] \in T^2 S_X \).

So, for the top-right path we have
\[
[X, \beta] \Rightarrow p \circ \mu T S[X^+ \beta, \beta^+] = p_T \circ \mu T S[X^+ \beta, \beta^+] \in S_X.
\]

Since \( (S, p_T) \) is a \( T \)-algebra \( p_T \circ \mu T S[X^+ \beta, \beta^+] = p_T \circ T p T S[X^+ \beta, \beta^+] \) and, by Lemma 7.44 this is precisely \( p \circ T p[X, \beta] \).

Therefore (7.48) commutes, and \((S, p)\) has the structure of a \( \mathbb{D}T \)-algebra.

It is straightforward to verify that the assignment \((S, \circ, \zeta, \epsilon) \mapsto (S, p)\) is natural and that the functors \( \text{MO} \Rightarrow \text{GS}^{\mathbb{D}T} \) so defined are each others’ inverses. \( \square \)

Remark 7.52. In particular, \( \mathbb{D}T \) is the algebraically free monad \([20]\) on the endofunctor \( T^{\text{dis}} \) from Section 6.

8. A NERVE THEOREM FOR MODULAR OPERADS

By Theorem 7.46 and 1, there is a diagram of functors

\[
\begin{array}{ccccccccc}
\Xi & \xrightarrow{f.f.} & \text{MO} & \xrightarrow{N} & \text{psh}(\Xi) \\
& & & & \downarrow j^* \\
\text{B}^\text{b.o.} & \xrightarrow{\text{dense}} & \text{CGr} \subseteq & \xrightarrow{\text{f.f.}} & \text{GS} & \xrightarrow{\text{f.f.}} & \text{psh}(\text{CGr}^\text{et}) \\
& & & & \downarrow \text{forget}^\text{D} \\
\text{B}^\text{b.o.} & \xrightarrow{\text{dense}} & \text{CGr}^\text{et} & \xrightarrow{\text{f.f.}} & \text{GS} & \xrightarrow{\text{f.f.}} & \text{psh}(\text{CGr}^\text{et}) \\
\end{array}
\]

where \( \Xi \) is the category obtained in the bo-ff factorisation of \( \text{CGr}^\text{et} \rightarrow \text{GS} \rightarrow \text{MO} \), and also in the bo-ff factorisation of \( \text{CGr}^\text{b.o.} \rightarrow \text{GS} \rightarrow \text{MO} \).

The goal of this section is to prove the following nerve theorem for modular operads using the abstract machinery described in Section 2.

**Theorem 8.2.** The functor \( N: \text{MO} \rightarrow \text{psh}(\Xi) \) is full and faithful. Its essential image consists of precisely those presheaves \( P \) on \( \Xi \) whose restriction to \( \text{psh}(\Xi) \) are graphical species. In other words,
\[
P(\mathcal{G}) = \lim_{(C, b) \in \mathcal{E}(\mathcal{G})} P(C) \text{ for all graphs } \mathcal{G}.
\]

Remark 8.4. A version of this theorem was stated in \([23]\), and another version was proved, by different methods from those presented here, in \([21]\) Theorem 3.8. In \([20]\), I proved a version of Theorem 8.2 by almost the same methods as presented here, but without the use of the distributive law. In all these versions, the statement of the Segal condition (8.3) is the same.

An overview (following \([6]\) Sections 1-3]1) of monads with arities was given in Section 2. If the monad \( \mathbb{D}T \) on \( \text{GS} \) had arities \( \text{CGr}^\text{et} \), then Theorem 8.2 would follow immediately from \([6]\) Section 1]. Unfortunately, this is not the case. The obstruction, unsurprisingly, relates to the contracted units (see Remark 8.12).

The remainder of this work is devoted to showing instead that \( T \ast \) has arities \( \text{CGr}^\ast \subseteq \text{GS}^\ast \). In this case, the nerve \( N: \text{MO} \rightarrow \text{psh}(\Xi) \) is fully faithful. Moreover, because \( \text{B}^\text{b.o.} \) is dense in \( \text{CGr}^\ast \), the essential image of \( N \) is characterised by the \( \Xi \)-presheaves \( P \) that satisfy the Segal condition (8.3).
By construction, \( \Xi \subset \text{MO} \) is the full subcategory on the modular operads \( \Xi(\mathcal{G}) \), free on connected graphs \( \mathcal{G} \in \text{CGr}_{\text{et}} \). So, the first step is to study these in more detail.

8.1. **The free modular operad on a graph.** Fix a connected graph \( \mathcal{H} = (E, H, V, s, t, \tau) \). To streamline the notation, let \( T\mathcal{H} \overset{\text{def}}{=} T\mathcal{Y}\mathcal{H} \) denote the free non-unital modular operad on \( \mathcal{H} \), and \( T_* \mathcal{H} \overset{\text{def}}{=} T_* \mathcal{Y}_* \mathcal{H} \), the corresponding free unital modular operad on \( \mathcal{H} \).

Of course, \( T_* \mathcal{H}(i) = \{ ch_e \}_{e \in E} = \text{CGr}_*(i, \mathcal{H}) \). Recall that the unit for \( \mathcal{Y}_* \mathcal{H} \) is given by \( ch_e \mapsto \epsilon_e^\mathcal{H} \overset{\text{def}}{=} ch_e \circ u \in \text{CGr}_*(C_2, \mathcal{H}) \), and the contracted unit for \( \mathcal{Y}_* \mathcal{H} \) is given by \( ch_e \mapsto o^\mathcal{H}_e \overset{\text{def}}{=} ch_e \circ z \in \text{CGr}_*(C_0, \mathcal{H}) \).

So, by Proposition 7.39 \( T_* \mathcal{H} \) has units

\[
ch_e \mapsto \epsilon^\mathcal{H}_e \overset{\text{def}}{=} [\eta^\mathcal{H} \epsilon^\mathcal{H}_e], \quad \text{and contracted units}
\]

\[
ch_e \mapsto o^\mathcal{H}_e \overset{\text{def}}{=} [\eta^\mathcal{H} o^\mathcal{H}_e],
\]

and contracted units

\[
ch_e \mapsto o^\mathcal{H}_e \overset{\text{def}}{=} [\eta^\mathcal{H} o^\mathcal{H}_e] = [C_0, ch_e \circ z] = [\mathcal{W}^m, ch_e \circ \kappa^m].
\]

Let \( X \) be a finite set. By Corollary 7.41 elements of \( T_* \mathcal{H}_X \) are represented by pairs \((\mathcal{X}, f)\) where \( \mathcal{X} \) is an admissible \( X \)-graph and \( f \in \text{CGr}_*(\mathcal{X}, \mathcal{H}) \). Pairs \((\mathcal{X}^1, f^1)\) and \((\mathcal{X}^2, f^2)\) represent the same element \([\mathcal{X}, f]_* \in T_* \mathcal{H}_X \) if and only if there is a commuting diagram

\[
\begin{array}{ccc}
\mathcal{X}^1 & \xrightarrow{g^1} & \mathcal{X}_-^1 \leftarrow & g^2 & \mathcal{X}^2 \\
\downarrow f^1 & & & \downarrow f^2 & \\
\mathcal{H} & & & \mathcal{H} & \\
\end{array}
\]

in \( \text{CGr}_* \) such that, for \( j = 1, 2 \), \( g^j \) is a morphism in \( X \cdot \text{CGr}_{\text{sim}} \), and \( f^j : \mathcal{X}^j_+ \to \mathcal{H} \) is an (unpointed) étale morphism in \( \text{CGr}_{\text{et}} \).

Outside the (contracted) units, \( \mathcal{X}_+ \) is admissible. Otherwise \( f^j = ch_e \in \text{CGr}_{\text{et}}(i, \mathcal{H}) \) for some \( e \in E \).

In particular, the following special case of (8.5) commutes in \( \text{CGr}_* \) for all \( e \in E \) and all \( m \geq 1 \):

\[
\begin{array}{ccc}
C_0 & \xrightarrow{z} & (i) \leftarrow & \kappa^m & \mathcal{W}^m \\
\downarrow ch_e \circ z & & & \downarrow ch_e & \downarrow ch_e \circ \kappa^m \\
\mathcal{H} & & & \mathcal{H} & \\
\end{array}
\]

This will be essential in the proof of Theorem 8.2.

8.2. **The category** \( \Xi \). By (8.1), \( \Xi \) is the restriction to \( \text{CGr}_* \) of the Kleisli category of \( T_* \). So, for all pairs \((\mathcal{G}, \mathcal{H})\) of graphs

\[
\Xi(\mathcal{G}, \mathcal{H}) = GS_*(\mathcal{G}, T_* \mathcal{H}) \cong T_* \mathcal{H}(\mathcal{G}).
\]

In particular, for \( \mathcal{G} \cong C_X \) or \( \mathcal{G} \cong (i) \), \( \Xi(\mathcal{G}, \mathcal{H}) \cong T_* \mathcal{H}(\mathcal{G}) \) has been described in Section 8.1.

For the general case, it follows from Section 8.1 that a morphism \( \alpha \in \Xi(\mathcal{G}, \mathcal{H}) \) is represented by a non-degenerate \( \mathcal{G} \)-shaped graph of graphs \( \Gamma \) with colimit \( \Gamma(\mathcal{G}) \), together with a morphism \( f \in \text{CGr}_*(\Gamma(\mathcal{G}), \mathcal{H}) \).

Let \( \mathcal{G} \not\cong C_0 \) and \( \mathcal{H} \) be graphs. For \( i = 1, 2 \), let \( \Gamma^i \) be a non-degenerate \( \mathcal{G} \)-shaped graph of graphs with colimit \( \Gamma^i(\mathcal{G}) \), and let \( f^i \in \text{CGr}_*(\Gamma^i(\mathcal{G}), \mathcal{H}) \). For each \((\mathcal{C}, b) \in \text{el}(\mathcal{G})\), let \( i^i_0 : \Gamma^i(b) \to \Gamma^i(\mathcal{G}) \) denote the defining embedding.
Lemma 8.7. The pairs \((\Gamma^1, f^1), (\Gamma^2, f^2)\) represent the same morphism \(\alpha \in \Xi(\mathcal{G}, \mathcal{H})\) if and only if there is a non-degenerate \(\mathcal{G}\)-shaped graph of graphs \(\Gamma\) with colimit \(\Gamma(\mathcal{G})\), and a morphism \(f \in \text{CGr}_*(\Gamma(\mathcal{G}), \mathcal{H})\) such that there is a commuting diagram

\[
\Gamma(\mathcal{G}) \xrightarrow{f^1} \Gamma(\mathcal{G}) \xleftarrow{f^2} \Gamma(\mathcal{G})
\]

in \(\text{CGr}_*\) where the morphisms in the top row are vertex deletion morphisms.

Proof. If \((\Gamma^1, f^1)\) and \((\Gamma^2, f^2)\) represent the same morphisms \(\alpha \in \Xi(\mathcal{G}, \mathcal{H})\), then, for all \((\mathcal{C}_{X_b}, b) \in \text{el}(\mathcal{G})\), \((\Gamma^1(b), f^1 \circ \iota^1_b)\) and \((\Gamma^2(b), f^2 \circ \iota^2_b)\) are similar in \(X_b, \text{CGr}_{\text{sim}}/\mathcal{T}_\mathcal{G}\) by definition. Therefore, by Section 8.1 since \(\mathcal{G} \not\cong \mathcal{C}_0\), there is an admissible graph \(\Gamma(b)\) and a morphism \(f_b \in \text{CGr}_*(\Gamma(b), \mathcal{H})\) such that the following diagram – in which the horizontal morphisms are vertex deletion morphisms between graphs with non-empty boundaries – commutes in \(\text{CGr}_*\):

\[
\Gamma^1(b) \xrightarrow{f^1} \Gamma(b) \xleftarrow{f_b} \Gamma^2(b)
\]

If \((\Gamma(\mathcal{G}), f)\) is the colimit of the non-degenerate \(\mathcal{G}\)-shaped graph of \(\mathcal{T}\)-structured graphs defined by \((\mathcal{C}_{X_b}, b) \mapsto (\Gamma(b), f_b)\) for all \((\mathcal{C}_{X_b}, b) \in \text{el}(\mathcal{G})\), then (8.8) commutes by construction. The converse follows immediately from the definitions. \(\square\)

Since every graph \(\mathcal{G}\) is trivially the colimit of the identity \(\mathcal{G}\)-shaped graph of graphs \(\mathcal{I}\mathcal{G}^\mathcal{G}: (\mathcal{C}, b) \mapsto \mathcal{C}\) (Section 5.2), the assignment \(f \mapsto [\mathcal{G}, f] \in \Xi(\mathcal{G}, \mathcal{H})\) induces an inclusion of categories \(\text{CGr}_* \hookrightarrow \Xi\).

It follows that there is weak ternary factorisation system on \(\Xi\): Morphisms in \(\Xi\) factor as boundary-preserving morphisms \([\Gamma]: \mathcal{G} \to \Gamma(\mathcal{G})\) represented by non-degenerate graphs of graphs \(\Gamma\), followed by morphisms in \(\text{CGr}_*\), which themselves factor as \((\text{CGr}_{\text{sim}}, \text{CGr}_{\text{et}})\) by Corollary 7.24.

8.3. Factorisation categories. More generally, let \(\text{GS}_{*,T_*}\) be the Kleisli category of \(T_*\) given by \(\text{GS}_{*,T_*}(S_*, S'_*) = \text{GS}_*(S_*, T, S'_*)\) for all \(S_*, S'_* \in \text{GS}_*\). In particular, the graphical category \(\Xi \subset \text{GS}_{*,T_*}\) is the full subcategory whose objects are graphs \(\mathcal{G} \in \text{CGr}_{\text{et}}\).

Let \(S_*\) be a pointed graphical species. Elements of \(T_*, S_X\) correspond to similarity classes of \(S_*\)-structured \(X\)-graphs \((X, \gamma)\). So, for any graph \(\mathcal{G}\), a morphism \(\beta \in \text{GS}_*(\mathcal{G}, T, S_*)\) is represented by a non-degenerate \(\mathcal{G}\)-shaped graph of \(S\)-structured graphs \(\Gamma_S\). The colimit of \(\Gamma_S\) describes an \(S_*\) structured graph \((\Gamma(\mathcal{G}), \alpha), \) where \(\Gamma(\mathcal{G})\) is the colimit of the underlying \(\mathcal{G}\)-shaped graph of graphs \(\Gamma: \text{el}(\mathcal{G}) \to \text{CGr}_{\text{et}}/S \to \text{CGr}_{\text{et}}\), which represents a morphism \([\Gamma] \in \text{GS}_{*,T_*}(\mathcal{T}_*, \mathcal{G}, \mathcal{T}_*, \Gamma(\mathcal{G}))\), as in Section 8.2.

So, let \(S_*\) be a pointed graphical species, \(\mathcal{G}\) a graph, and let \(\beta \in \text{GS}_*(\mathcal{G}, T, S)\). The following definition is from [9] Section 2.4:

Definition 8.9. The factorisation category \(\text{fact}_*(\beta)\) of \(\beta\) is the category whose objects are pairs \((\Gamma, \alpha)\), where \(\Gamma\) is a non-degenerate \(\mathcal{G}\)-shaped graph of graphs with colimit \(\Gamma(\mathcal{G})\) and \(\alpha \in \text{GS}_*(\Gamma(\mathcal{G}), S) \cong S(\Gamma(\mathcal{G}))\) is such that \(\beta\) is given by the composition of morphisms in \(\text{GS}_{*,T_*}\):

\[
\mathcal{G} \xrightarrow{[\Gamma]} \Gamma(\mathcal{G}) \xrightarrow{\alpha} S_*.\]
Morphisms in $\text{fact}_*(\beta)(((\Gamma^1, \alpha^1), (\Gamma^2, \alpha^2))$ are commuting diagrams in $\text{GS}_{*T_*}$.

\[(8.10)\]

\[
\begin{array}{ccc}
\Gamma^1(\mathcal{G}) & \xrightarrow{g} & S_* \\
\Gamma^2(\mathcal{G}) & \xrightarrow{\alpha^1} & S_* \\
\end{array}
\]

such that $g$ is a morphism in $\text{CGr}_* \subset \text{GS}_{*T_*}$.

By [6] Proposition 2.5], the monad $T_*$ has arities $\text{CGr}_*$ if the following lemma holds for all pointed graphical species $S_*$, all graphs $\mathcal{G} \in \text{CGr}_*$ and all $\beta \in \text{GS}_*(\mathcal{G}, T_*, S_*)$:

**Lemma 8.11.** The category $\text{fact}_*(\beta)$ is connected.

**Proof.** This follows easily from the discussion above, and in particular Section 8.1.

Let $S_*$ be a pointed graphical species. For $X$ a finite set, $S_*$-structured $X$-graphs $(\mathcal{X}^1, \alpha^1), (\mathcal{X}^2, \alpha^2)$ represent the same element of $T_*S_X$ if and only if they are similar in $X-\text{CGr}_{\text{sim}}/S_* \cong \text{GS}_*(C_X, T_*, S_*)$. So, by Section 8.2 the lemma holds whenever $\mathcal{G} = ()$ or $\mathcal{G} = C_X$ is a corolla (including $C_0$, by (8.6)).

Now, let $\mathcal{G} \not\cong C_0$ be any connected graph. Elements of $\text{GS}_*(\mathcal{G}, T_*, S) \cong T_*S(\mathcal{G})$ are represented by non-degenerate $\mathcal{G}$-shaped graphs of $S_*$-structured graphs. Since there is no object of the form $(C_0, b)$ in $\text{el}(\mathcal{G})$, two such non-degenerate $S_*$-structured graphs of graphs, $\Gamma^1_{S_*}, \Gamma^2_{S_*}$, represent the same element of $T_*S(\mathcal{G})$ if and only if for all $(C_X, b) \in \text{el}(\mathcal{G})$, $\Gamma^1_{S_*}(C_X, b) \sim \Gamma^2_{S_*}(C_X, b)$ in $X_{S_*}\text{CGr}_{\text{sim}}/S_*$, whereby the colimits $\Gamma^1_{S_*}(\mathcal{G})$ and $\Gamma^2_{S_*}(\mathcal{G})$ are also similar in $\text{CGr}_*/S_*$. Hence, $\text{fact}_*(\beta)$ is connected by Corollary 7.41.

Theorem 8.2 now follows from [6] Sections 1 & 2.

**Proof of Theorem 8.3.** The category $\text{CGr}_*$ is dense in $\text{GS}_*$. By [6] Proposition 2.5], the monad $T_*$ on $\text{GS}_*$ has arities $\text{CGr}_*$ if and only if $\text{fact}_*(\beta)$ is connected for all $S_*$, $\mathcal{G}$ and $\beta \in \text{GS}_*(\mathcal{G}, T_*, S_*)$.

Hence, by Lemma 8.11 $T_*$ has arities $\text{CGr}_* \subset \text{GS}_*$ and the induced nerve functor $N: \text{MO} \to \text{psh}(\Xi)$ is fully faithful by [6] Propositions 1.5 & 1.9.

Moreover, by [6] Theorem 1.10], its essential image is the subcategory of those presheaves on $\Xi$ whose restriction to $\text{CGr}_*$ are in the image of the fully faithful embedding $\text{GS}_* \hookrightarrow \text{psh}(\text{CGr}_*)$.

In other words, a presheaf $P$ on $\Xi$ is in the essential image of $N$ if and only if, for all $\mathcal{G}$, $P(\mathcal{G}) = \lim_{(C, b) \in \text{el}_*(\mathcal{G})} P(C)$. By finality of $\text{el}(\mathcal{G}) \subset \text{el}_*(\mathcal{G})$, this is the case precisely when $P(\mathcal{G}) = \lim_{(C, b) \in \text{el}(\mathcal{G})} P(C)$.

**Remark 8.12.** To see that the modular operad monad $\mathbb{D}T$ on $\text{GS}$ does not have arities, let us use the method of [6] Section 2) to construct its unpointed factorisation categories.

For any graphical species $S$ and graph $\mathcal{G}$, $\text{GS}_{\mathbb{D}T}(\mathcal{Y}\mathcal{G}, S) \cong \text{GS}_*(\mathcal{Y}, \mathcal{G}, T_*, S^*) \cong T_*S^+(\mathcal{G})$ canonically.

So, a morphism $\beta: \mathcal{Y}\mathcal{G} \to S$ in the Kleisli category $\text{GS}_{\mathbb{D}T}$ is represented by a $\mathcal{G}$-shaped graph of graphs $\mathfrak{G}$ with colimit $\Gamma(\mathfrak{G})$, and a $DS$-structure $\alpha \in \text{GS}_*(\Gamma(\mathfrak{G}), DS) \cong DS(\Gamma(\mathfrak{G}))$.

Such pairs $(\mathfrak{G}, \alpha)$ are the objects of the unpointed factorisation category $\text{fact}(\beta)$. Morphisms in $\text{fact}(\beta)(((\Gamma, \alpha), (\Gamma', \alpha'))$ are morphisms in $\text{CGr}_{\text{et}}(\Gamma(\mathfrak{G}), \Gamma'(\mathfrak{G}'))$ making the diagram (8.10) commute.

By [6] Proposition 2.5], $\mathbb{D}T$ has arities $\text{CGr}_{\text{et}}$ if and only if $\text{fact}(\beta)$ is connected for all $S$, $\mathcal{G}$, and $\beta$.

To see that this is not the case, let $S = \mathcal{Y}()$, and so $TS \cong S$. Let $\mathcal{G} = C_0$ and let $\beta = 0 = z: C_0 \to ()$. Then the diagrams $C_0 \to C_0 \xrightarrow{z} ()$, and $C_0 \to W \xrightarrow{z} ()$ describe objects in $\text{fact}(\beta)$. Since there are no
non-trivial morphisms in $\text{CGr}_{\text{et}}$ with domain or codomain $\mathcal{C}_0$, these objects are in disjoint components of $\text{fact}(\beta)$. Therefore, $\text{fact}(\beta)$ is not connected and $\mathbb{D}T$ does not have arities $\text{CGr}_{\text{et}}$.

8.4. Weak modular operads. In [21,22], Hackney, Robertson and Yau have proved a version of Theorem [8.2] in terms of a bijective on objects subcategory $U$ of $\Xi$ that was constructed precisely so as to have a generalised Reedy structure. The inclusion $U \hookrightarrow \text{MO}$ is not fully faithful since the category $U$ does not contain any morphisms in $\text{CGr}_* \hookrightarrow \Xi$ that factor through $z: \mathcal{C}_0 \to (i)$ or $\kappa^m: \to (i), \ m \geq 1$, nor does it contain any morphisms of $\text{CGr}_{\text{et}}$ that are not embeddings. However, by [22, Theorem 3.6], $U$ is dense in $\text{MO}$ and hence induces a fully faithful nerve.

Furthermore, by [21, Theorem 3.8], the category $\text{psh}_{s\text{Set}}(U)$ of $s\text{Set}$-valued presheaves on $U$ admits a cofibrantly generated model structure, obtained by localising the Reedy model structure at the Segal morphisms

$$\lim_{(C,b)\in \text{el}(\mathcal{G})} P(C) \to P(\mathcal{G}),$$

and the fibrant objects for this model structure are those simplicial presheaves on $U$ that satisfy the weak Segal condition

$$P(\mathcal{G}) \simeq \lim_{(C,b)\in \text{el}(\mathcal{G})} P(C), \text{ for all graphs } \mathcal{G} \in U.$$ (8.13)

The method of [21,22] cannot be applied in the current case since there is no (obvious) generalised Reedy structure on $\Xi$. However, in [10], Caviglia and Horel describe a general class of rigification results whereby, given a dense inclusion $D \hookrightarrow C$ satisfying certain conditions, an equivalence is established between $s\text{Set}$-valued presheaves on $D$ that satisfy a weak Segal condition, and $C$ objects internal to $s\text{Set}$ that satisfy the Segal condition on the nose. In [10, Section 7], this result is applied to a certain class of monads with arities. This leads directly to the following corollary of Theorem 8.2.

**Corollary 8.14.** There is a model category structure on the category $\text{psh}_{s\text{Set}}(\Xi)$ of functors $P: \Xi^{op} \to s\text{Set}$ whose fibrant objects are those $P$ that satisfy the weak Segal condition:

$$P(\mathcal{G}) \simeq \lim_{(C,b)\in \text{el}(\mathcal{G})} P(C) \text{ for all graphs } \mathcal{G} \in \text{CGr}_{\text{et}}.$$ (8.15)

**Proof.** The monad $T_\ast$ has arities $\text{CGr}_*$ and $\text{el}(\mathcal{G})$ is connected and essentially small for all connected graphs $\mathcal{G}$. Therefore the assumptions of [10, Assumptions 7.9] are satisfied. By [10, Section 7.5], $\text{MO}$ is equivalent to the category of models in $\text{Set}$ of the limit sketch $L = (\text{CGr}_*, \{(\mathcal{G}/B^{\text{op}})_{\mathcal{G} \in \text{CGr}_*}\})$.

Moreover, there is a Segal model structure on the category of $s\text{Set}$ valued models for $L$ and, by [10, Proposition 7.1], this can be transferred to a model structure on $\text{psh}_{s\text{Set}}(\Xi)$ whose fibrant objects are those presheaves that satisfy the weak Segal condition.

In current work with M. Robertson, we are comparing the existing models for weak modular operads. We expect that there is a direct Quillen equivalence between the model structure on $\text{psh}_{s\text{Set}}(\Xi)$ of Corollary 8.14 and the model structure on $\text{psh}_{s\text{Set}}(U)$ of [21].

**Remark 8.16.** Theorem 8.2 was originally formulated in [23, in terms of the graphical category $\overline{\text{Gr}}$, whose morphisms are described in [23, Section 6]. This is the bijective on objects subcategory of $\Xi$ that does not contain any morphisms in $\text{CGr}_*$ that factor through $z: \mathcal{C}_0 \to (i)$ or $\kappa: W \to (i)$. In particular $\overline{\text{Gr}}$ does not embed fully in $\text{MO}$.

There are bijective on objects inclusions $U \subset \overline{\text{Gr}} \subset \Xi$. Hence, since $\Xi$ and $U$ are both dense in $\text{MO}$, so is $\overline{\text{Gr}}$, and the inclusion yields a fully faithful nerve functor $\text{MO} \to \text{psh}(\overline{\text{Gr}})$ whose essential image satisfies the same Segal condition [8.3]. (See also [22, Theorem 3.6 & Section 4] for more details.)
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