A NEW MODEL OF GROUNDWATER FLOW WITHIN AN UNCONFINED AQUIFER: APPLICATION OF CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

PIERRE AIME FEULEFACK AND JEAN DANIEL DJIDA

African Institute for Mathematical Sciences–Cameroon
Limbe Crystal Gardens, South West Region
P.O. Box 608, Cameroon

ATANGANA ABDON∗

Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of Free Staye
Bloemfontein, 9300, South Africa

(Communicated by Peter E. Kloeden)

Abstract. In this paper, the groundwater flow equation within an unconfined aquifer is modified using the concept of new derivative with fractional order without singular kernel recently proposed by Caputo and Fabrizio. Some properties and applications are given regarding the Caputo-Fabrizio fractional order derivative. The existence and the uniqueness of the solution of the modified groundwater flow equation within an unconfined aquifer is presented, the proof of the existence use the definition of Caputo-Fabrizio integral and the powerful fixed-point Theorem. A detailed analysis on the uniqueness is included. We perform on the numerical analysis on which the Crank-Nicolson scheme is used for discretisation. Then we present in particular the proof of the stability of the method, the proof combine the Fourier and Von Neumann stability analysis. A detailed analysis on the convergence is also achieved.

1. Introduction. The variability of the thickness of an unconfined aquifer with the hydraulic head makes their pumping test non-linear. However the non-linearity associated to an unconfined aquifer makes their analysis more complex than that of the confined aquifer. Therefore, the well-known Theis solution that is considered as one of the fundamental breakthroughs in the development of hydrologic modelling for a constant rate $Q$, should be corrected in order to be used for unconfined aquifers [21]. Wenzel in 1942 notice that the ‘Theis’ solution gave inconsistent estimation of $S_s$ and $K$ attributed to the delayed in the yield of water from storage as water table [8]. Boulton in 1954 suggested that it is theoretically unsound to use the Theis solution for unconfined aquifer flow because it does not account for vertical flow to the pumping well [21]. He proposed a new mechanism for flow towards a full penetrating pumping well under confined aquifer conditions. He extended the...
Theis transient theory to include the effect of delayed yield due to movement of the water table in unconfined aquifer. He proposed solutions reproducing all the three segments of the unconfined aquifer time-drawdown curve. In his formulation of delayed yield, he assume that, as the water table falls, water is released from storage gradually rather than instantaneously as in the free-surface solutions. This approach yields an integro-differential flow equation in term of drawdown $s$ as \[ (1) \]

\[
\left[ \frac{\partial^2 s(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial s(r,t)}{\partial r} \right] = \left[ \frac{S}{T} \frac{\partial s}{\partial t} \right] + \left\{ \gamma S y \int_{0}^{t} \frac{\partial s(r,x)}{\partial x} \exp \left[ -\gamma (t - x) \right] dx \right\}
\]

where the term in square brackets is the instantaneous confined storage, the term in braces is a convolution integral representing storage release gradually since pumping began, due to water table decline. According to Bolton, $\gamma$ represents the time when delayed yield becomes approximately negligible [21]. The term was denoted “delayed index”. With Boulton solution, Prickett in 1965 established an empirical relationship between the delayed index and physical aquifer property and, also proposed a methodology for estimated $S$, $S_y$ and $\gamma$ of unconfined aquifers by analysing pumping tests. Because of the non-physical meaning of the delayed index, the Boulton solution has some misunderstanding to explain the physical mechanism of the the delayed yield process [21]. In 1972 Streltsova brought some comments, but the two solutions end up to be compared equivalent. The delayed yield solutions of Boulton and Streltsova do not take into account the vertical flow within the unconfined aquifer; they cannot be extended to account for partially penetrating pumping and observation wells [18]. Unconfined flow to a well is non-linear in multiple ways, and the application of analytical solutions has required the utilization of advanced mathematical tools. There are still many additional challenges to be addressed related to unconfined aquifer pumping tests [21]. However, since the derivative is involved in the flow equation, some physical observation and fact cannot in any case be well described (or partially described) with more accuracy via the local derivative [2]. New definitions of the concept of derivative with fractional order has been introduced, for instance the Riemann-Liouville Derivative, Caputo derivative just to name a few. However some complaints were made for somewhat complicated mathematical expression of its expression and the consequent complications in solutions of the fractional differential equations. Recently, Caputo and Fabrizio proposed a new definition of fractional derivative given by \[ (2) \]

\[
^{CF}_{a} D_{t}^{\alpha}(f(t)) = \frac{M(\alpha)}{1 - \alpha} \int_{a}^{t} f'(x) \exp \left( \frac{-\alpha(t - x)}{1 - \alpha} \right) dx,
\]

where $M(\alpha)$ is a normalisation function such that $M(0) = M(1) = 1$, which assumes two different representations for the temporal and spatial variable. This work is organised as follows: we present the background of the Caputo-Fabrizio fractional order derivative to connect the unfamiliar potential reader in accordance with the definition and the useful properties. We investigate on the groundwater flow equation within an unconfined aquifer by applying the Caputo-Fabrizio fractional derivative. The existence and uniqueness of the modified equation is proved using the Caputo-Fabrizio fractional integral. We solve the equation analytically using the well known method of separation of variable combining with Laplace transform operator. We present the numerical analysis of the modified groundwater flow equation using Crank-Nicolson scheme for the discretisation of the model for which we perform the stability and the convergence analysis of the method.
2. Caputo-Fabrizio fractional order derivative. The aim of this section is to connect the unfamiliar potential reader in accordance with the definition, the useful properties and applications of the new derivative proposed by Caputo and Fabrizio. This new derivative proposed by M. Caputo and M. Fabrizio generally called Caputo-Fabrizio derivative has nowadays a huge application in many field of sciences and engineering.

2.1. Definition and some properties. Let \( f \in H^1(a, b), \ \alpha \in (0, 1) \) is the order of the derivative and \( a < b \)

**Definition 2.1.** [4]. The well known Caputo fractional time derivative of order \( \alpha \) is given by the following definition,

\[
C_a^D_t^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t f'(x)(t-x)^{-\alpha}dx. \tag{3}
\]

The problem with this definition of fractional order derivative proposed by Caputo is the singularity at the end point of the interval [3].

To avoid the singularity, Caputo and Fabrizio recently introduced a new fractional order derivative without singularity at any point by changing the kernel \((t-x)^{-\alpha}\) with the function \(\exp(-\frac{\alpha(t-x)}{1-\alpha})\) and \(\frac{M(\alpha)}{1-\alpha}\) with \(\frac{M(\alpha)}{1-\alpha}\) and give out a new definition based on the convolution of a first order derivative and the exponential function as below.

**Definition 2.2.** [10] The new definition propose by Caputo and Fabrizio is given as follow for \( f \in H^1(a, b) \) by:

\[
C^F_a D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(x) \exp(-\frac{\alpha(t-x)}{1-\alpha})dx, \tag{4}
\]

where \(M(\alpha)\) is the normalisation function such that \(M(1) = M(0) = 1 \) [3, 5, 10, 20].

The new fractional derivative can also be applied to functions which do not belong to \(H^1(a, b)\). Indeed, the definition (4) can be formulated also for \( f \in L^1(-\infty, b) \) and for any \( \alpha \in (0, 1) \) as

\[
C^F_a D_t^{(\alpha)} f(t) = \alpha \frac{M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(x)) \exp(-\frac{\alpha(t-x)}{1-\alpha})dx. \tag{5}
\]

According to the definition (4), the new fractional derivative is zero when \( f(t) \) is constant, as in the usual Caputo fractional time derivative, that when \( \alpha = 1 \) we obtain the classical first derivative, whereas if \( \alpha = 0 \) , we have from (5) the function \( f(t) \). But, contrary to the the usual Caputo fractional time derivative, the kernel does not have singularity for \( t = x \).

Now, let us put in \( \sigma = \frac{1-\alpha}{\alpha}, \ \sigma \in [0, \infty] \) and \( \alpha \) takes the values \( \alpha = \frac{1}{1+\sigma} \) we have \( \alpha \in (0, 1) \).

By substituting \( \sigma \) in (4) we get

\[
C^F_a D_t^{(\alpha)} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t f'(x) \exp(-\frac{(t-x)}{\sigma})dx. \tag{6}
\]

where \( \sigma \in [0, \infty) \) and \( N(\sigma) \) such that \( N(0) = N(\infty) = 1 \) is the normalisation function such as \(M(\alpha)\).

The fractional integral of order \( \alpha \) of \( f \) is defined by

\[
I_t^{(\alpha)} (f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(x)dx, \ t \leq 0. \tag{7}
\]
Looking at this, Jorge Losada and Juan J. Nieto propose the following definition of fractional derivative of order $0 < \alpha < 1$.

$$CF_a D^{(\alpha)}_t f(t) = \frac{1}{1-\alpha} \int_a^t f'(x) \exp(-\frac{\alpha(t-x)}{1-\alpha})dx. \tag{8}$$

The above section 2 allowed us to be familial with the definitions and properties of the Caputo-Fabrizio fractional order derivative and also the definition of Caputo-Fabrizio integral. With these tools in hand we can now investigate further to achieve the objective of this project.

In the next section 3 we shall apply that definition of caputo-Fabizio fractional order derivative and Caputo-Fabrizio integral on the new model of groundwater flow equation within an unconfined aquifer.

3. Application of Caputo-Fabizio fractional order derivative to groundwater flow equation within unconfined aquifer. The main point of this section is to prove the existence and uniqueness of the solution for the new model of groundwater flow equation using Caputo-Fabrizio derivative within an unconfined aquifer.

The modified equation with Caputo-Fabrizio derivative is given by

$$T \left[ \frac{\partial^2 h(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial h(r,t)}{\partial r} \right] = S \frac{CF_0 D^{(\alpha)}_t (h(r,t))}{T} + \chi S_y \int_0^t \frac{\partial h(r,k)}{\partial k} \exp \left[-\chi(t-k)\right] dk. \tag{9}$$

Whenever substituting the Caputo-Fabrizio fractional time derivative by its expression in the sequel equation (9) we have the following

$$T \left[ \frac{\partial^2 h(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial h(r,t)}{\partial r} \right] = S \int_0^t \frac{\partial h(r,k)}{\partial k} \exp \left[-\frac{\alpha(t-k)}{1-\alpha}\right] dk + \chi S_y \int_0^t \frac{\partial h(r,k)}{\partial k} \exp \left[-\chi(t-k)\right] dk,$$

where $\chi$ is an empirical constant. The initial condition are the same as in the case with local derivative.

4. Existence of the solution of the new model of groundwater flow equation within an unconfined aquifer. In this section we use the Fixed-point theorem to prove the existence of the solution for the new model of groundwater flow equation. We can then rewrite Equation (9) as

$$CF_0 D^{(\alpha)}_t (h(r,t)) = \frac{T}{S} \left[ \frac{\partial^2 h(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial h(r,t)}{\partial r} \right] - \frac{\chi S_y}{T} \int_0^t \frac{\partial h(r,k)}{\partial k} \exp \left[-\chi(t-k)\right] dk. \tag{10}$$

We recall that the Caputo-Fabrizio integral is given by

$$CF_0 I^{(\alpha)}_t (f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(k) dk, \quad t \geq 0. \tag{11}$$

We first transform Equation (10) into an integral form multiplying both sides by Caputo-Fabrizio integral and we get
The kernel operator $K$ satisfies the Lipschitz condition and also the contraction if
\[ 0 < \left\{ \lambda_1 \frac{T}{S} + \lambda_2 \frac{1}{a} \right\} + \lambda_3 \sqrt{2\chi \exp(-2\chi T)} \left\| \frac{\chi S_y}{S} \right\| < 1, \quad \lambda_1, \lambda_2, \lambda_3, T_1 > 0. \]

Proof. Let $h_1$ and $h_2$ be two functions in $C^2([a, b] \times [0, T_1], \mathbb{R})$ endowed with the sum of supremum norm of the function and its derivative. That is for $k \geq 0$,
\[ \|f\|_{C^k} = \sum_{0 \leq i \leq k} \|f^{(i)}\|_{\infty}. \]

The associated metric make $C^k$ to be a Banach space. We then have
\[ \|K(r, t, h_2) - K(r, t, h_1)\| = \left\| \frac{T}{S} \left[ \frac{\partial^2 h_2(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial h_2(r, t)}{\partial r} \right] - \frac{\chi S_y}{S} \int_0^t \frac{\partial h_2(r, k)}{\partial k} \exp[-\chi(t - k)] \, dk \right\|, \]
\[ = \left\| \frac{T}{S} \left[ \frac{\partial^2 h_1(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial h_1(r, t)}{\partial r} \right] - \frac{\chi S_y}{S} \int_0^t \frac{\partial h_1(r, k)}{\partial k} \exp[-\chi(t - k)] \, dk \right\|, \]
\[ = \left\| \frac{T}{S} \left[ \frac{\partial^2 h_2(r, t) - h_1(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial(h_2(r, t) - h_1(r, t))}{\partial r} \right] \right\|, \]
\[ - \frac{\chi S_y}{S} \int_0^t \frac{\partial(h_2(r, k) - h_1(r, k))}{\partial k} \exp[-\chi(t - k)] \, dk \right\|. \]

Using the triangle inequality we have
\[ \|K(r, t, h_2) - K(r, t, h_1)\| \leq \left\| \frac{T}{S} \left[ \frac{\partial^2 h_2(r, t) - h_1(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial(h_2(r, t) - h_1(r, t))}{\partial r} \right] \right\|, \]
\[ - \frac{\chi S_y}{S} \int_0^t \frac{\partial(h_2(r, k) - h_1(r, k))}{\partial k} \exp[-\chi(t - k)] \, dk \right\|. \]

Now, in the real world problem, the level is bounded, we consider the only function which satisfy the property
\[ \left\| \frac{\partial h}{\partial r}(r, t) \right\| \leq C. \]

For all $t$ in some interval $[t_0, t_0 + \delta]$ and all $r$ in $[r_0, r_0 + \delta_1]$, where $\delta > 0$, $\delta_1 > 0$ and the constant $C$ is independent of $t$ and $r$. 
Then the derivative operator satisfy the Lipschitz condition. Therefore we can find two positive constants $\lambda_1$ and $\lambda_2$ such that

$$
\|K(r, t, h_2) - K(r, t, h_1)\| \leq \lambda_1^2 \left\| \frac{\mathcal{T}}{S} \right\| \|h_2(r, t) - h_1(r, t)\| + \lambda_2 \left\| \frac{1}{r} \right\| \|h_2(r, t) - h_1(r, t)\|
$$

$$
- \left\| \frac{\chi S_y}{S} \right\| \lambda_3 \left\| \frac{\chi S_y}{S} \right\| \left\| h_2(r, t) - h_1(r, t) \right\| \left( \int_0^t \left| \exp [-\chi(t - k)] \right|^2 dk \right) \right) + \right. 
$$

Integrating the second part of the right hand side term yields

$$
\|K(r, t, h_2) - K(r, t, h_1)\| \leq A \|h_2(r, t) - h_1(r, t)\| + B \|h_2(r, t) - h_1(r, t)\| 
$$

$$
- \lambda_3 \left\| \frac{\chi S_y}{S} \right\| \left\| h_2(r, t) - h_1(r, t) \right\| \left( \int_0^t \left| \exp [-\chi(t - k)] \right|^2 dk \right) \right) + \right. 
$$

where $A = \lambda_1^2 \left\| \frac{\mathcal{T}}{S} \right\|$ and $B = \lambda_2 \left\| \frac{1}{r} \right\|$.

Now, by applying the Cauchy-Schwartz inequality on the second term of the right hand side we have

$$
\|K(k, t, h_2) - K(k, t, h_1)\| \leq A \|h_2(r, t) - h_1(r, t)\| + B \|h_2(r, t) - h_1(r, t)\| 
$$

$$
- \lambda_3 \left\| \frac{\chi S_y}{S} \right\| \left\| h_2(r, t) - h_1(r, t) \right\| \left( \int_0^t \left| \exp [-\chi(t - k)] \right|^2 dk \right) \right) + \right. 
$$

Integrating the second part of the right hand side term yields

$$
\|K(r, t, h_2) - K(r, t, h_1)\| \leq A \|h_2(r, t) - h_1(r, t)\| + B \|h_2(r, t) - h_1(r, t)\| 
$$

$$
- \lambda_3 \sqrt{2\chi \exp(-2\chi T^r)} \left\| \frac{\chi S_y}{S} \right\| \left\| h_2(r, t) - h_1(r, t) \right\| 
$$

where $C = \lambda_3 \sqrt{2\chi \exp(-2\chi T^r)} \left\| \frac{\chi S_y}{S} \right\|$.

Now taking $L = A + B + C$ i.e.

$$
L = \lambda_1 \left\| \frac{\mathcal{T}}{S} \right\| + \lambda_2 \left\| \frac{1}{a} \right\| + \lambda_3 \sqrt{2\chi \exp(-2\chi T^r)} \left\| \frac{\chi S_y}{S} \right\|, 
$$

we have

$$
\|K(r, t, h_2) - K(r, t, h_1)\| \leq L \|h_2(r, t) - h_1(r, t)\|. 
$$

(14)

Showing that the kernel operator $K$ satisfy the Lipschitz condition. If also $0 < L < 1$ then $K$ is a contraction. We suppose in the following that $K$ is a contraction (in general true from the given data).

Now let us consider the recursive sequence as follow

$$
h_{n+1}(r, t) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} K(r, t, h_n) + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t K(r, t, h_n) ds. 
$$

The idea is to show that for a given sequence $h_n$ which satisfies our problem, the sequence converges. We claim that $h_n$ converge to the fixed point. To see this we show that the sequence $h_n$ is a Cauchy sequence and hence must converge. Whenever the difference between the two consecutive terms of the sequence is given
A MODEL OF GROUNDWATER FLOW WITHIN AN UNCONFINED AQUIFER

by

\[ h_{n+1}(r, t) - h_n(r, t) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} [K(r, t, h_n) - K(k, t, h_{n-1})] \]
\[ + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t [K(r, t, h_n) - K(r, t, h_{n-1})] ds. \]

If we take the norm on both sides of the above equation we have

\[ \| h_{n+1}(r, t) - h_n(r, t) \| = \left\| \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} [K(r, t, h_n) - K(k, t, h_{n-1})] \right\|
\[ + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \left\| \int_0^t [K(r, t, h_n) - K(r, t, h_{n-1})] ds \right\|. \]

By applying the triangle inequality we get

\[ \| h_{n+1}(r, t) - h_n(r, t) \| \leq \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \| K(r, t, h_n) - K(k, t, h_{n-1}) \|
\[ + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \left\| \int_0^t [K(r, t, h_n) - K(r, t, h_{n-1})] ds \right\|. \]

Now, let us put

\[ H_n(r, t) = h_n(r, t) - h_{n-1}(r, t). \]

Then we have

\[ h_n(r, t) = \sum_{j=0}^n H_j(r, t), \]

with \( H_0(r, t) = h_0 \).

Using the fact that the kernel \( K \) satisfies the Lipschitz condition we get

\[ \| h_{n+1}(r, t) - h_n(r, t) \| \leq \frac{2(1 - \alpha)L}{(2 - \alpha)M(\alpha)} \| h_n(r, t) - h_{n-1}(r, t) \|
\[ + \frac{2\alpha L}{(2 - \alpha)M(\alpha)} \int_0^t \| h_n(r, s) - h_{n-1}(r, s) \| ds. \]

We can then put our expression as follows

\[ \| H_{n+1}(r, t) \| \leq \Lambda \| H_n \| + \Theta \int_0^t \| H_n \| ds, \]

where \( \Lambda = \frac{2(1 - \alpha)L}{(2 - \alpha)M(\alpha)} \) and \( \Theta = \frac{2\alpha L}{(2 - \alpha)M(\alpha)} \).

Using the induction technique with respect to \( n \geq 2 \) we get the following:

\[ \| H_2 \| \leq \Lambda \| H_1 \| + \Theta \int_0^t \| H_1 \| ds \leq \Lambda \| H_1 \| + \Theta T'' \| H_1 \| = (\Lambda + \Theta T_1) \| H_1 \|, \]
\[ \| H_3 \| \leq \Lambda \| H_1 \| + \Theta \int_0^t \| H_1 \| ds \]
\[ \leq \Lambda (\Lambda + \Theta T_1) \| H_1 \| + \Theta \int_0^t (\Lambda + \Theta T_1) \| H_1 \| ds, \]
\[ = (\Lambda + \Theta T_1) \times (\Lambda + \Theta T_1) \| H_1 \| = (\Lambda + \Theta T_1)^2 \| H_1 \|, \]
Uniqueness of the solution of the new model of groundwater flow equation

Suppose that equation (10) subjected to the initial data \( h_0 \) given above has a solution satisfying assumption (13) and that the kernel \( K \) is a contraction. Then the solution is unique.

Proof. Let \( h \) and \( h' \) be two candidate solutions of our initial boundary value problem. Then using Equation (12) we will have

\[
h(r, t) - h'(r, t) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} [K(r, t, h) - K(k, t, h')] + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t [K(k, t, h) - K(k, t, h')] ds.
\]

Applying the norm on the both sides of the above equality we have

\[
\|h(r, t) - h'(r, t)\| = \frac{2(1 - \alpha) L}{(2 - \alpha)M(\alpha)} \|K(r, t, h) - K(r, t, h')\| + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t \|K(k, t, h) - K(k, t, h')\| ds
\]

\[
\leq \frac{2(1 - \alpha) L}{(2 - \alpha)M(\alpha)} \|h - h'\| + \frac{2\alpha L}{(2 - \alpha)M(\alpha)} \int_0^t \|h - h'\| ds.
\]

Finally we have

\[
\|h_n(r, t)\| \leq (\Lambda + \Theta T_1)^{n-1} \|h_1(r, t)\|.
\]

Or

\[
\|h_{n+1}(r, t) - h_n(r, t)\| \leq \left( \frac{2(1 - \alpha) L}{(2 - \alpha)M(\alpha)} + \frac{2\alpha L T_1}{(2 - \alpha)M(\alpha)} \right)^{n-1} \|h_1(r, t) - h(r, 0)\|.
\]

In general, according to the data, the quantity \( C = \left( \frac{2(1 - \alpha) L}{(2 - \alpha)M(\alpha)} + \frac{2\alpha L T_1}{(2 - \alpha)M(\alpha)} \right) \) is between 0 and 1. Using the above result we have for \( m < n \) that

\[
\|h_m(r, t) - h_n(r, t)\| \leq \frac{c_m}{1 - c} \|h_1(r, t) - h(r, 0)\|.
\]

Take \( \epsilon > 0 \) such that \( \frac{c_m}{1 - c} < \epsilon \) and \( m \) large enough, then for \( m, n > N \) we have

\[
\|h_m(r, t) - h_n(r, t)\| < \epsilon.
\]

The sequence \( h_n \) is thus a Cauchy sequence. Since \( C^2 \) with the above mentioned norm is complete then \( h_n \) converges to some limit \( h(r, t) \).

We then use the Banach Fixed Point Theorem to conclude the existence of the solution to our boundary value problem. This completes the proof of the existence of the solution.

5. Uniqueness of the solution of the new model of groundwater flow equation. In this section we need to show whenever the solution exists, that solution is unique.

Proposition 1. Suppose that equation (10) subjected to the initial data \( h_0 \) given above has a solution satisfying assumption (13) and that the kernel \( K \) is a contraction. Then the solution is unique.

Proof. Let \( h \) and \( h' \) be two candidate solutions of our initial boundary value problem. Then using Equation (12) we will have

\[
h(r, t) - h'(r, t) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} [K(r, t, h) - K(k, t, h')] + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t [K(k, t, h) - K(k, t, h')] ds.
\]

Applying the norm on the both sides of the above equality we have

\[
\|h(r, t) - h'(r, t)\| = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \|K(r, t, h) - K(r, t, h')\| + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t \|K(k, t, h) - K(k, t, h')\| ds
\]

\[
\leq \frac{2(1 - \alpha) L}{(2 - \alpha)M(\alpha)} \|h - h'\| + \frac{2\alpha L}{(2 - \alpha)M(\alpha)} \int_0^t \|h - h'\| ds.
\]
We then get
\[ \| h(r,t) - h'(r,t) \| \leq \left( \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)} + \frac{2\alpha L}{(2-\alpha)M(\alpha)} \right) \| h(r,t) - h'(r,t) \|. \]  
(17)

This implies that
\[ \left( 1 - \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)} + \frac{2\alpha L}{(2-\alpha)M(\alpha)} \right) \| h(r,t) - h'(r,t) \| \leq 0. \]  
(18)

Assuming that
\[ \left( 1 - \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)} + \frac{2\alpha L}{(2-\alpha)M(\alpha)} \right) \neq 0, \]
we have
\[ \| h(r,t) - h'(r,t) \| = 0. \]

This implies that
\[ h(r,t) = h'(r,t). \]

We conclude that the solution is unique.

With the above idea, we can now investigate without a doubt on finding the analytical solution for the new model of groundwater flow equation within an unconfined aquifer using the Caputo-Fabrizio-Derivative.

6. Analytical solution for the new model of groundwater flow. We emphasize in this section the well-known method called so far the method of separation of variable which allows us to move from a partial differential equation to an ordinary differential equation that we can may be easily solve via the known methods.

Recall that the new model of groundwater with Caputo-Fabrizio within an unconfined aquifer is given by the following equation
\[ T \left[ \frac{\partial^2 h(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial h(r,t)}{\partial r} \right] = S \frac{M(\alpha)}{1-\alpha} \int_0^t \frac{\partial h(r,x)}{\partial x} \exp \left[ -\alpha \frac{(t-x)}{1-\alpha} \right] dx 
+ \chi S \gamma \int_0^t \frac{\partial h(r,k)}{\partial k} \exp \left[ -\chi (t-k) \right] dk, \]  
(19)

with initial condition \( h(r,0) = h_0 \) and the boundary condition \( \lim_{r \to \infty} h(r,t) = 0 \) and \( Q = 2\pi T \partial_r h(r_b, t) \), \( Q \) is the discharge rate or the rate at which water is being taken out of the aquifer; \( r_b \) is the ratio of the borehole.

Using idea of separation of variable, let us set
\[ h(r,t) = U(r)V(t) \]

By simple derivation we have
\[ \frac{\partial h(r,t)}{\partial r} = U'(r)V(t), \]  
(20)
\[ \frac{\partial^2 h(r,t)}{\partial r^2} = U''(r)V(t), \]  
(21)
\[ \frac{\partial h(r,t)}{\partial t} = U(r)V'(t). \]  
(22)

By substituting Equations (20), (21) and (22) into (19), we have the following ordinary differential equation
\[
U''(r)V(t) + \frac{1}{r} U'(r)V(t) = S \frac{M(\alpha)}{T} \int_0^t U(r)V'(x) \exp \left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx + \frac{\chi Sy}{T} \int_0^t U(r)V''(k) \exp [-\chi(t-x)] dk. \tag{23}
\]

Dividing both sides by \(U(r)\) and after by \(V(t)\) this yields the following equation

\[
\frac{U''(r)}{U(r)} + \frac{1}{r} \frac{U'(r)}{U(r)} = S \frac{M(\alpha)}{T} \int_0^t \frac{V'(x)}{V(x)} \exp \left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx + \frac{\chi Sy}{T} \int_0^t \frac{V''(k)}{V(k)} \exp [-\chi(t-x)] dk. \tag{24}
\]

By separating the variables we have

\[
U''(r) + \frac{1}{r} U'(r) = -\lambda^2 U(r), \tag{25}
\]

and

\[
M_1 \int_0^t V'(x) \exp \left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx + M_2 \int_0^t V'(k) \exp [-\chi(t-x)] dk = -\lambda^2 V(t), \tag{26}
\]

with

\[
M_1 = S \frac{M(\alpha)}{T} \quad \text{and} \quad M_2 = \frac{\chi Sy}{T}.
\]

Where the two expressions have been setted to be constant that we named by \(-\lambda^2\) because they are function of the independent variable \(r \) and \(t\), and the only way that these two can be equal each other is if they are both constant. The square on \(\lambda\) is to avoid the cases of trivial solutions.

Now we need to solve Equations \((25)\) and \((26)\). For Equation \((25)\) we have an ordinary differential equation with non-constant coefficient. This equation can be solved by applying the Frobenius method [2].

In general this equation is known as Bessel equation form, that the exact solution is given on the form

\[
U(r) = AJ_0(r\lambda) + BK_0(r\lambda),
\]

where \(J_0\) is the the Bessel function of the first kind of zero order and \(K_0\) the modified kind. their general form is given by

\[
J_\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \beta + 1)} \left(\frac{z}{2}\right)^{2n+\beta} \quad \text{and} \quad K_\beta(z) = \frac{\pi}{2} (i)^{-\beta} \left(\frac{J_{-\beta}(z) - J_\beta(z)}{\sin(\beta\pi)}\right).
\tag{27}
\]

Using the boundary condition, we obtain \(B = 0\), then the solution is reduced to

\[
U(r) = AJ_0(r\lambda) = A \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + 1)} \left(\frac{r\lambda}{2}\right)^{2n}. \tag{28}
\]

Now we use the well known Laplace transform to solve the second part \((26)\). We have

\[
M_1 \int_0^t V'(x) \exp \left[-\frac{\alpha(t-x)}{1-\alpha}\right] dx + M_2 \int_0^t V'(x) \exp [-\chi(t-x)] dx = -\lambda^2 V(t).
\]
Applying Laplace Transform in both sides we have

\[ M_1 \left[ \mathcal{L}(V'(x)) \mathcal{L} \left( \exp \left( -\frac{\alpha x}{1-\alpha} \right) \right) \right] + M_2 \left[ \mathcal{L}(V'(x)) \mathcal{L}(\exp(-\chi x)) \right] = -\lambda^2 \mathcal{L}(V(t)). \]

(29)

But from the definition of the Laplace transform we have

\[ \mathcal{L}(V'(x)) = sV(s) - V(0), \]

(30)

\[ \mathcal{L}(\exp(-xC\alpha)) = \frac{1}{s + C\alpha}, \quad C\alpha = \frac{\alpha}{1-\alpha}, \]

(31)

\[ \mathcal{L}(\exp(-\chi x)) = \frac{1}{s + \chi}. \]

(32)

If we introduce (30), (31) and (32) into equation (29) we have the following equation

\[ M_1 \left[ \frac{sV(s)}{s + C\alpha} - \frac{V(0)}{s + C\alpha} \right] + M_2 \left[ \frac{sV(s)}{s + \chi} - \frac{V(0)}{s + \chi} \right] = -\lambda^2 V(s), \]

(33)

\[ \Rightarrow V(s) = \frac{\frac{M_1}{s + C\alpha} + \frac{M_2}{s + \chi}}{\frac{M_1}{s + C\alpha} + \frac{M_2}{s + \chi} + \lambda^2} V(0), \]

(34)

To get the exact solution we take the inverse Laplace transform and get

\[ V_n(\lambda t) = \mathcal{L}^{-1}(V(s)). \]

Therefore the general solution of the equation governing the flow in an unconfined aquifer is given by

\[ h(r, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n \frac{(-1)^m}{m! \Gamma(m+1)} \left( \frac{\lambda r}{2} \right)^{2m} V_n(\lambda t). \]

7. Numerical analysis of the groundwater flow within unconfined aquifer.

In this chapter, we solve the new model of groundwater flow within an unconfined aquifer numerically using the Crank-Nicolson method.

For some positive integer \( N \) the grid sizes in time are defined by

\[ \tau = \frac{T}{N}. \]

The grid points in time interval \([0, T]\) are labelled by

\[ t_k = k\tau, \quad k = 1, 2, \ldots, N. \]

Also for some positive integer \( M \) the grid sizes in space are defined by

\[ h = \frac{L}{M}. \]

The grid points in time interval \([0, L]\) are labelled by

\[ r_j = j\tau, \quad j = 1, 2, \ldots, M. \]

We first introduce the Crank-Nicolson scheme for the discretisation of the first and second order space derivative as follows.

\[ \frac{\partial h(r, t)}{\partial r} = \frac{1}{2} \left[ \left( \frac{h(r_{j+1}, t_{k+1}) - h(r_{j+1}, t_{k+1})}{2h} \right) + \left( \frac{h(r_{j+1}, t_{k}) - h(r_{j-1}, t_{k})}{2h} \right) \right] + O(h), \]
\[
\frac{\partial^2 h(r,t)}{\partial r^2} = \frac{1}{2} \left[ \left( \frac{h(r_{j+1}, t_{k+1}) - 2h(r_j, t_{k+1}) + h(r_{j-1}, t_{k+1})}{h^2} \right) + \left( \frac{h(r_{j+1}, t_k) - 2h(r_j, t_k) + h(r_{j-1}, t_k)}{h^2} \right) \right] + O(h^2),
\]

\[ h(r,t) = \frac{1}{2} (h(r_j, t_{k+1}) + h(r_j, t_k)). \]

**Proposition 2.** [7] Let \( f(t) \) be function in \( C^2([a,b]) \) and let the order of the fractional derivative be \( 0 < \alpha < 1 \), then the first-order approximation of the Caputo-Fabrizio derivative at a point \( t_n = n\tau \), with \( \tau \) the step, is given

\[
\frac{M(\alpha)}{\alpha} \sum_{k=1}^{n} \left( \frac{h(r_j, t_{k+1}) - h(r_j, t_k)}{\tau} + O(\tau) \right) d_k,~\Delta t,
\]

where

\[
d_{k,\tau} = \exp \left[ -\frac{\alpha\tau}{1-\alpha}(n-k) \right] - \exp \left[ -\frac{\alpha\tau}{1-\alpha}(n-k+1) \right].
\]

**Proof.** Let us set

\[
I_1 = \frac{M(\alpha)}{1-\alpha} \int_0^t \frac{\partial h(r,x)}{\partial x} \exp \left[ -\frac{\alpha(t-x)}{1-\alpha} \right] dx.
\]

\[
I_1 = \frac{M(\alpha)}{1-\alpha} \sum_{k=1}^{n} \int_{(k-1)\tau}^{k\tau} \left( \frac{h(r_j, t_{k+1}) - h(r_j, t_k)}{\tau} + O(\tau) \right) \exp \left[ -\frac{\alpha (n\tau - x)}{1-\alpha} \right] dx
\]

integrating the term with integral we have

\[
= \frac{M(\alpha)}{1-\alpha} \sum_{k=1}^{n} \left( \frac{h(r_j, t_{k+1}) - h(r_j, t_k)}{\tau} + O(\tau) \right) \times
\]

\[
\left\{ \frac{1-\alpha}{\alpha} \left( \exp \left[ -\frac{\alpha\tau}{1-\alpha}(n-k+1)\right] - \exp \left[ -\frac{\alpha\tau}{1-\alpha}(n-k)\right] \right) \right\}
\]

\[
= \frac{M(\alpha)}{\alpha} \sum_{k=1}^{n} \left( \frac{h(r_j, t_{k+1}) - h(r_j, t_k)}{\tau} + O(\tau) \right) \times
\]

\[
\left( \exp \left[ -\frac{\alpha\tau}{1-\alpha}(n-k) \right] - \exp \left[ -\frac{\alpha\tau}{1-\alpha}(n-k+1) \right] \right).
\]

This complete the proof. \( \square \)

We now apply the above proposition on

\[
I_2 = \chi S_y \int_0^t \frac{\partial h(r,\tau)}{\partial \xi} \exp \left[ -\chi(t-\xi) \right] d\xi,
\]

and get

\[
I_2 = \chi S_y \sum_{l=1}^{n} \int_{(l-1)\tau}^{l\tau} \left( \frac{h(r_j, t_{l+1}) - h(r_j, t_l)}{\tau} + O(\tau) \right) \exp \left[ -\chi(n\tau - \xi) \right] d\xi
\]

\[
\Rightarrow I_2 = S_y \sum_{l=1}^{n} \left( \frac{h(r_j, t_{l+1}) - h(r_j, t_l)}{\tau} + O(\tau) \right) \sigma_{l,\tau},
\]
with
\[ \sigma_{l,\tau} = \exp[-\chi \tau (n-l)] - \exp[-\chi \tau (n-l+1)]. \]

By putting all those tools together in Equation (19) and using the simplification \( h(r_j, t_k) = h^k_j \) the numerical approximation of the groundwater flow within an unconfined aquifer with Caputo-Fabrizio derivative through the Crank-Nicolson scheme is given as follow:

\[
T \frac{1}{2} \left[ \frac{h^k_{j+1} - 2h^k_j + h^k_{j-1}}{h^2} + \frac{h^k_{j+1} - 2h^k_j + h^k_{j-1}}{h^2} \right] \\
+ \frac{1}{2r_j} \left[ \frac{h^k_{j+1} - h^k_{j-1}}{2h} + \frac{h^k_{j+1} - h^k_{j-1}}{2h} \right] \\
= \frac{SM(\alpha)}{\alpha} \sum_{l=1}^{k} \left( \frac{h^l_{j+1} - h^l_j}{\tau} \right) d_{l,\tau} + S_y \sum_{l=1}^{k} \left( \frac{h^l_{j+1} - h^l_j}{\tau} \right) \sigma_{l,\tau}. \tag{35}
\]

8. Stability analysis of the Crank-Nicolson scheme. In this section, we analyse the stability conditions of the Crank-Nicolson scheme used to solve the modified equation of groundwater flow within an unconfined aquifer using Caputo-Fabrizio fractional order derivative. We will use the approach known as Von Neumann stability analysis [27].

Let \( \eta^k_j = h^k_j - \bar{h}^k_j \), where \( \bar{h}^k_j \) is the approximate solution at the point \((r_j, t_k)\), with \( j = 1, 2, \cdots, M-1 \) and \( k = 1, 2, \cdots, N \), and commonly in addition \( \eta^k_j = [\eta^k_1, \eta^k_2, \cdots, \eta^k_{M-1}] \)

Let Consider the function
\[ \eta^k(x) = \begin{cases} 
\eta^k_j & \text{if } r_j < x \leq r_j + \frac{h}{2}, j = 1, 2, \cdots, M - 1, \\
0 & \text{if } L - \frac{h}{2} < x \leq L.
\end{cases} \]

Then we can expand the function \( \eta^k \) in the Fourier form series as [6]
\[
\eta^k(x) = \sum_{\nu=-\infty}^{\infty} \delta_\nu(v) \exp \left( \frac{2i\pi \nu x}{L} \right), \quad \text{with} \quad \delta_\nu(x) = \frac{1}{L} \int_0^L \rho^\nu(x) \exp \left( \frac{2i\pi \nu x}{L} \right) dx.
\tag{36}
\]

With the above definition we have the following identity established by Chen et al. [11],
\[ \|p^2\|_2^2 = \sum_{\nu=-\infty}^{\infty} \|\delta_\nu(v)\|. \]

In Equation (35), all the parameters included are positive (the storativity coefficient \( S \), the transmisivity \( T \), the specific yield \( S_y \)), also we have \( 0 < \alpha < 1 \). This will may be help for further investigation.

Now, we apply Von Neumann stability analysis. To do this we assume that the solution \( \eta^k_j \) can be represented as [27]
\[ \eta^k_j = \delta(k) \exp(i\mu jh) \] with \( i = \sqrt{-1} \) and \( \mu \) a number, \tag{37}
By replacing the above Expression (37) in Equation (35) we have the following

\[ T \left\{ \frac{1}{2} \left( \frac{\eta^{k+1}_{j+1} - 2\eta^k_{j+1} + \eta^k_{j-1}}{2h^2} \right) + \frac{1}{2r_j} \left( \frac{\eta^{k+1}_{j+1} - \eta^k_{j+1}}{2h} \right) \right\} 
+ \frac{1}{2r_j} \left( \frac{\eta^{k+1}_{j+1} - \eta^k_{j+1}}{2h} \right) \right\} 
= \frac{SM(\alpha)}{\alpha} \sum_{l=1}^{k} \left( \frac{\eta^{l+1}_{j} - \eta^l_{j}}{\tau} \right) d_{l,\tau} + \frac{S_y}{\tau} \sum_{l=1}^{k} \left( \frac{\eta^{l+1}_{j} - \eta^l_{j}}{\tau} \right) \sigma_{l,\tau} . \]

The above equation yields the following

\[ \frac{T}{4h^2} \left( \eta^{k+1}_{j+1} - 2\eta^k_{j+1} + \eta^k_{j-1} \right) + \frac{1}{4r_j h} \left( \eta^{k+1}_{j+1} - \eta^k_{j+1} + \eta^k_{j-1} \right) \]

\[ = \frac{SM(\alpha)}{\alpha} \sum_{l=1}^{k} \left( \frac{\eta^{l+1}_{j} - \eta^l_{j}}{\tau} \right) d_{l,\tau} + \frac{S_y}{\tau} \sum_{l=1}^{k} \left( \frac{\eta^{l+1}_{j} - \eta^l_{j}}{\tau} \right) \sigma_{l,\tau} . \quad (38) \]

We can also put it in this form

\[ [(\eta^{k+1}_{j+1} + \eta^k_{j-1}) + (\eta^{k+1}_{j+1} + \eta^k_{j-1}) - 2\eta^k_{j+1} - 2\eta^k_{j}] + \]

\[ \lambda_j \left[(\eta^{k+1}_{j+1} - \eta^k_{j-1}) + (\eta^{k+1}_{j-1} - \eta^k_{j+1}) - \Delta \sum l \left( \eta^{l+1}_{j} - \eta^l_{j} \right) \Theta_{l,\tau} \right] . \quad (39) \]

With

\[ \Delta = \left( \frac{SM(\alpha)}{\alpha \tau} + \frac{S_y}{\tau \tau} \right) , \quad \Theta_{l,\tau} = (d_{l,\tau} + \sigma_{l,\tau}) \quad \text{and} \quad \lambda_j = \frac{h}{2r_j} \].

Equation (39) is equivalent to

\[ [(\eta^{k+1}_{j+1} + \eta^k_{j-1}) + (\eta^{k+1}_{j+1} + \eta^k_{j-1}) - 2\eta^k_{j+1} - 2\eta^k_{j}] + \]

\[ \lambda_j \left[(\eta^{k+1}_{j+1} - \eta^k_{j-1}) + (\eta^{k+1}_{j-1} - \eta^k_{j+1}) - \Delta \sum l \left( \eta^{l+1}_{j} - \eta^l_{j} \right) \Theta_{l,\tau} \right] \quad (40) \]

Using \( \eta^k_{j} \) as in Expression (37) we have the following

\[ \frac{1}{2} \left[ \delta(k + 1)e^{i\mu jh}(e^{i\mu h} + e^{-i\mu h}) + \delta(k)e^{i\mu jh}(e^{i\mu h} + e^{-i\mu h}) \right] 
- 2\delta(k + 1)e^{i\mu jh} \]

\[ + \lambda_j \left[ \delta(k + 1)e^{i\mu jh}(e^{i\mu h} - e^{-i\mu h}) + \delta(k)e^{i\mu jh}(e^{i\mu h} - e^{-i\mu h}) \right] 
+ \Delta \sum l \left( \delta(l + 1)e^{i\mu jh} - \delta(l)e^{i\mu jh} \right) \Theta_{l,\tau} \]

\[ \quad = \Delta \sum l \left( \delta(l + 1)e^{i\mu jh} - \delta(l)e^{i\mu jh} \right) \Theta_{l,\tau} \quad (41) \]
If we divide both sides of above equation (41) by \( \exp(i\mu jh) \) we have

\[
\begin{align*}
&\left[ \delta(k + 1) \cos(\mu h) + \delta(k) \cos(\mu h) - \delta(k + 1) - \delta(k) \right] \\
&+ i\lambda_j \left[ \delta(k + 1) \sin(\mu h) + \delta(k) \sin(\mu h) \right] - \Lambda_r \Theta_{k,\tau} (\delta(k + 1) - \delta(k)) \\
&= \Lambda_r \sum_{l=1}^{k-1} (\delta(l + 1) - \delta(l)) \Theta_{l,\tau},
\end{align*}
\]

(42)

by simplification we get

\[
\begin{align*}
\frac{1}{\Lambda_r \Theta_{k,\tau}} \left[ -\delta(k + 1)(1 - \cos(\mu h)) - \delta(k)(1 - \cos(\mu h)) \right] \\
+ i\frac{\lambda_j}{\Lambda_r \Theta_{k,\tau}} \left[ \delta(k + 1) \sin(\mu h) + \delta(k) \sin(\mu h) \right] - (\delta(k + 1) - \delta(k)) \\
= \frac{1}{\Theta_{k,\tau}} \sum_{l=1}^{k-1} (\delta(l + 1) - \delta(l)) \Theta_{l,\tau}.
\end{align*}
\]

(43)

and follow with

\[
\begin{align*}
\frac{2}{\Lambda_r \Theta_{k,\tau}} \left[ -\delta(k + 1) \sin^2 \left( \frac{\mu h}{2} \right) - \delta(k) \sin^2 \left( \frac{\mu h}{2} \right) \right] \\
+ i\frac{\lambda_j}{\Lambda_r \Theta_{k,\tau}} \left[ \delta(k + 1) \sin(\mu h) + \delta(k) \sin(\mu h) \right] - (\delta(k + 1) - \delta(k)) \\
= \frac{1}{\Theta_{k,\tau}} \sum_{l=1}^{k-1} (\delta(l + 1) - \delta(l)) \Theta_{l,\tau}.
\end{align*}
\]

(44)

With further simplification we have

\[
\begin{align*}
- \left[ 1 + \frac{2}{\Lambda_r \Theta_{k,\tau}} \sin^2 \left( \frac{\mu h}{2} \right) \right] \delta(1) + \left[ 1 - \frac{2}{\Lambda_r \Theta_{k,\tau}} \sin^2 \left( \frac{\mu h}{2} \right) \right] \delta(k) \\
+ i\frac{\lambda_j}{\Lambda_r \Theta_{k,\tau}} \left[ \delta(k + 1) \sin(\mu h) + \delta(k) \sin(\mu h) \right] \\
= \frac{1}{\Theta_{k,\tau}} \sum_{l=1}^{k-1} (\delta(l + 1) - \delta(l)) \Theta_{l,\tau}.
\end{align*}
\]

(45)

Equation (45) lead to the following system

\[
\begin{align*}
\left[ 1 + \frac{2}{\Lambda_r \Theta_{k,\tau}} \sin^2 \left( \frac{\mu h}{2} \right) \right] \delta(1) &= \left[ 1 - \frac{2}{\Lambda_r \Theta_{k,\tau}} \sin^2 \left( \frac{\mu h}{2} \right) \right] \delta(0) \quad \text{if } k = 0,
\delta(k + 1) \sin(\mu h) &= -\delta(k) \sin(\mu h) \quad \text{for all } k, \text{ since the RHS is a pure real},
\left[ 1 + \frac{2}{\Lambda_r \Theta_{k,\tau}} \sin^2 \left( \frac{\mu h}{2} \right) \right] \delta(k + 1) &= \left[ 1 - \frac{2}{\Lambda_r \Theta_{k,\tau}} \sin^2 \left( \frac{\mu h}{2} \right) \right] \delta(k) \\
- \frac{1}{\Theta_{k,\tau}} \sum_{l=1}^{k-1} (\delta(l + 1) - \delta(l)) \Theta_{l,\tau}
\end{align*}
\]

for \( k = 1, 2 \cdots, N - 1. \)

(46)

We can put Equation (46) in the simple ways as
\[
\begin{cases}
\delta(1) = \left[1 - \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right] \delta(0) \text{ if } k = 0, \\
\delta(k+1) \sin(\mu h) = -\delta(k) \sin(\mu h) \text{ for all } k \text{ since the RHS is a pure real,} \\
\delta(k+1) = \frac{\left[1 - \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right] \delta(k) - \frac{1}{\Theta_{k,r}} \sum_{l=1}^{k-1} (\delta(l) - \delta(l)) \Theta_{l,r}}{\left[1 + \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right]} \\
\text{for } k = 1, 2, \ldots , N - 1.
\end{cases}
\]  

(47)

With the above system in hand, the main idea here is to show that for all \( k = 1, 2, \ldots , N - 1 \), the following theorem is verified.

**Theorem 8.1.** Assuming that \( \delta(k+1) \) is the solution of Equation (47), then for every integer number \( k \) greater or equal to 1, the following inequality is satisfied

\[
\|\delta(k)\| < \|\delta(0)\|.  
\]  

(48)

**Proof.** This proof would be done by applying the well known technique of recursive on the integer number \( k \).

- For \( k = 0 \), we have

\[
\left\| \frac{\delta(1)}{\delta(0)} \right\| = \left\| \frac{\left[1 - \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right] \delta(k) - \frac{1}{\Theta_{k,r}} \sum_{l=1}^{k-1} (\delta(l) - \delta(l)) \Theta_{l,r}}{\left[1 + \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right]} \right\| < 1. 
\]

It is easy to see that the ratio of the above equation is always less than one no matter what the value of the parameters. This algorithm is unconditional stable for the range \( k = 0 \).

Now, assuming that for any \( k \) greater than or equal to 1, the inequality is satisfied and let us show that the inequality is also satisfied for range \( k + 1 \), i.e.

\[
\|\delta(k + 1)\| < \|\delta(0)\|. 
\]

We have

\[
\delta(k + 1) = \frac{\left[1 - \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right] \delta(k) - \frac{1}{\Theta_{k,r}} \sum_{l=1}^{k-1} (\delta(l) - \delta(l)) \Theta_{l,r}}{\left[1 + \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right]}.
\]  

(49)

However applying the triangle inequality, we get

\[
\|\delta(k + 1)\| \leq \frac{\left[1 - \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right] \left\| \delta(k) \right\| + \frac{1}{\Theta_{k,r}} \left\| \sum_{l=1}^{k-1} (\delta(l) - \delta(l)) \Theta_{l,r} \right\|}{\left[1 + \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right]},
\]

\[
\leq \frac{\left[1 - \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right] \left\| \delta(k) \right\| + \frac{1}{\Theta_{k,r}} \sum_{l=1}^{k-1} \Theta_{l,r} \left\| \delta(l) - \delta(l) \right\|}{\left[1 + \frac{2}{\pi \tau \hat{\Theta}_{k,r}} \sin^2 \left(\frac{\mu h}{T} \right)\right]},
\]

(50)
Applying the recurrence hypothesis, we have

\[
\|\delta(k+1)\| < \frac{|1 - \frac{2}{\Lambda_r \Theta_{h_r}} \sin^2 \left(\frac{\mu h}{2}\right)| \|\delta(k)\| + \frac{1}{\Theta_{k_r}} \sum_{l=1}^{k-1} \Theta_{l_r} \|\delta(l)\|}{1 + \frac{2}{\Lambda_r \Theta_{h_r}} \sin^2 \left(\frac{\mu h}{2}\right)},
\]

(51)

Applying the recurrence hypothesis, we have

\[
\|\delta(k+1)\| < \frac{\left|1 - \frac{2}{\Lambda_r \Theta_{h_r}} \sin^2 \left(\frac{\mu h}{2}\right) + \frac{1}{\Theta_{k_r}} \sum_{l=1}^{k-1} \Theta_{l_r}\right| \|\delta(0)\|}{1 + \frac{2}{\Lambda_r \Theta_{h_r}} \sin^2 \left(\frac{\mu h}{2}\right)} \|\delta(0)\|,
\]

\[
\|\delta(k+1)\| < \|\delta(0)\|,
\]

Therefore

\[
\frac{\|\delta(k+1)\|}{\|\delta(0)\|} < 1.
\]

Also we have

\[
\delta(k+1) \sin(\mu h) = -\delta(k) \sin(\mu h),
\]

(52)

Taking the norm both side of Equation (52) we have

\[
\|\delta(k+1)\| = \|\delta(k)\|.
\]

Applying the recurrence hypothesis, we have

\[
\|\delta(k+1)\| < \|\delta(0)\|,
\]

(53)

\[
\Rightarrow \frac{\|\delta(k+1)\|}{\|\delta(0)\|} < 1.
\]

(54)

**Conclusion.** With the above inequality in all the cases we can say that the Crank-Nicolson scheme for the new groundwater flow with Caputo-Fabrizio fractional order derivative is unconditionally stable.

9. **Convergence of the Crank-Nicolson scheme.** In this section, we present the convergence of the Crank-Nicolson scheme of the groundwater flow with Caputo-Fabrizio derivative. We assume that our problem has a smooth solution \(h(r_j, t_k)\) at the point \((x_j, t_k)\), \(j = 1, 2, \cdots, M; k = 1, 2, \cdots, N - 1\).

Let \(h_j^k\) be the numerical approximation solution of \(h(r_j, t_k)\). Let us set

\[
\zeta_j^k = h(r_j, t_k) - h_j^k
\]

and

\[
\Psi^k = [0, \zeta_1^k, \zeta_2^k, \cdots, \zeta_M^k]^T.
\]

(55)
\[ \zeta_j^k \text{ satisfies the following equation} \]
\[ ((\zeta_j^{k+1} + \zeta_j^{k+1}) + (c_j^{k+1} + c_j^{k+1}) - 2\zeta_j^{k+1} - 2c_j^k) + \lambda_j \left( [\zeta_j^{k+1} - \zeta_j^{k+1}] + (c_j^{k+1} - c_j^{k+1}) \right) = R_j^{k+1} + \Lambda_r \sum_{l=1}^{k-1} (c_j^{l+1} - c_j^l) \Theta_{l,r}, \tag{56} \]

where \( R_j^{k+1} \) is the truncation error.

From the discretisation of the Crank-Nicolson, we can find in [14] for a smooth function, the detailed error analysis. However, there exists the constants \( C_1, C_2, C_3 \) and \( C_4 \) such that the following equalities hold uniformly on \([0, L]\) and \([0, T]\), and for all \( 0 < \alpha < 1 \).

\[
\frac{\partial h(r,t)}{\partial r} = \frac{1}{2} \left[ \frac{h(r_{j+1}, t_{k+1}) - h(r_{j-1}, t_{k+1})}{2h} + \frac{h(r_{j+1}, t_k) - h(r_{j-1}, t_k)}{2h} \right] + C_1 h, \\
\frac{\partial^2 h(r,t)}{\partial r^2} = \frac{1}{2} \left[ \frac{h(r_{j+1}, t_{k+1}) - 2h(r_j, t_{k+1}) + h(r_{j-1}, t_{k+1})}{h^2} \right] + \frac{h(r_{j+1}, t_k) - 2h(r_j, t_k) + h(r_{j-1}, t_k)}{h^2} \right] + C_2 h^2, \\
\]

\[
\frac{C_F}{\alpha} D_{\alpha}^r (h(r_j, t_k)) = \frac{M(\alpha)}{\alpha} \sum_{l=1}^{k} \left( \frac{h(r_j, t_{l+1}) - h(r_j, t_l)}{\tau} \right) d_{l,r} + C_3 \tau, \tag{57} \]

where

\[
d_{l,r} = \exp \left[ -\frac{\alpha \tau}{1-\alpha} (k-l) \right] - \exp \left[ -\frac{\alpha \tau}{1-\alpha} (k-l+1) \right],
\]

and

\[
\chi S_y \int_0^t \frac{\partial h(r, \xi)}{\partial \xi} \exp \left[ -\chi (t - \xi) \right] d\xi = S_y \sum_{l=1}^{k} \left( \frac{h(r_j, t_{l+1}) - h(r_j, t_l)}{\tau} \right) \sigma_{l,r} + C_4 \tau,
\]

with

\[
\sigma_{l,r} = \exp \left[ -\chi \tau (n-l) \right] - \exp \left[ -\chi \tau (n-l+1) \right].
\]

From the above equalities, and from [7, 12], we have

\[
|R_j^{k+1}| \leq C_0 \tau (\tau + h^2) \quad \text{for} \quad j = 1, 2, \cdots, M; \quad k = 1, 2, \cdots, N; \tag{58}
\]

where \( C_0 \) is a constant taking in account the Caputo-Fabrizio derivative error combining with the spacial error.

We can then prove the following theorem.

**Theorem 9.1.** Suppose that our problem has a smooth solution \( h(r,t) \). Let \( h_j^k \) be the numerical approximation solution obtained by using the crank-Nicolson scheme, then there exists a positive constant \( C \) independent on \( j, k, h \) and \( \tau \) such that

\[
\| \Psi_j^k \|_{\infty} \leq C k \tau (\tau + h^2), \quad k = 1, 2, \ldots, N. \tag{59}
\]

**Proof.** We prove this theorem via the recurrence technique on the natural number \( k \).

For \( k = 0 \) we have

\[
|\zeta_j^1| = \| \Psi^1 \|_{\infty}. \tag{60}
\]
Since $\Psi^k = [0, \zeta_1^k, \zeta_2^k, \ldots, \zeta_M^k]^T$, we then have $\Psi^0 = 0$ and using Equation (57) we have

$$\|\Psi^1\|_\infty = |\zeta_1^1|$$

(61)

$$\leq |\zeta_{j+1}^1| + |\zeta_{j-1}^1| + 2|\zeta_j^1| + \lambda_j (|\zeta_{j+1}^1| + |\zeta_{j-1}^1|) + \Lambda_\tau \Theta_{k,\tau} |\zeta_j^{k+1}|$$

(62)

$$= |R_j^1| \leq C_0 \tau (\tau + h^2).$$

(63)

Take $C = C_0$, then $\|\Psi^1\|_\infty \leq C\tau (\tau + h^2)$. This show that (59) hold for $k = 0$.

Let us secondly assume that

$$\|\Psi^k\|_\infty \leq C\tau (\tau + h^2).$$

(64)

Then, we have

$$\|\Psi^{k+1}\|_\infty = |\zeta_j^{k+1}|$$

$$\leq |R_j^{k+1}| + \lambda_j \sum_{l=1}^{k} (|\zeta_l^{l+1} - \zeta_l^l| \Theta_{l,\tau})$$

$$\leq |R_j^{k+1}| + \Lambda_\tau \sum_{l=1}^{k} (|\zeta_l^{l+1} - \zeta_l^l| \Theta_{l,\tau})$$

$$\leq |R_j^{k+1}| + \lambda_j \sum_{l=1}^{k} (|\zeta_j^{l+1} - \zeta_j^l| \Theta_{l,\tau})$$

$$\leq |R_j^{k+1}| + \Lambda_\tau \sum_{l=1}^{k} (|\zeta_j^l| \Theta_{l,\tau})$$

$$\leq |R_j^{k+1}| + C' |\zeta_j^k|$$

$$\leq C\kappa (\tau + h^2), \text{ with } C = (1 - C')$$

This complete the proof.

We can now state the following theorem.

**Theorem 9.2.** The crank-Nicolson scheme is convergent, and there exists a positive constant $K$ independent of $j, k, h, \tau$ such that

$$|h_j^k - h(r_j, t_k)| \leq K(\tau + h^2), \quad j = 1, 2, \ldots, M - 1 \quad k = 1, 2, \ldots, N.$$  

(64)

**Proof.** To prove this theorem, we use Theorem (9.2). We have proved that

$$\|\Psi^k\|_\infty \leq C\kappa (\tau + h^2), \quad k = 1, 2, \ldots, N.$$  

Since $k \tau \leq T'$ for each $k$, we have

$$\|\Psi^k\|_\infty \leq C T' (\tau + h^2).$$

Also we have $\Psi^k = [0, \zeta_1^k, \zeta_2^k, \ldots, \zeta_M^k]^T$, the above inequality give that

$$|\zeta_j^k| = |h_j^k - h(r_j, t_k)| \leq K(\tau + h^2), \quad j = 1, 2, \ldots, M - 1 \quad k = 1, 2, \ldots, N;$$

(65)
with $K = CT$. This completes the proof of the convergence of Crank-Nicolson scheme.

We have then shown that the technique of Crank-Nicolson apply on the modified groundwater flow within an unconfined aquifer with Caputo-Fabrizio fractional order derivative is stable and converges.

10. **Conclusion.** The aim of this work was to study the groundwater flow equation within an unconfined aquifer for which the classical local derivative has been changed by the new fractional order derivative without singular kernel recently proposed by Michele Caputo and Mauro Fabrizio. We have presented some definitions and properties concerning the Caputo-Fabrizio derivative. We have investigated on the existence and uniqueness of the solution for the modified groundwater flow equation within an unconfined aquifer using the fraction Caputo-Fabrizio order derivative. The proof used the point fixed Theorem and the definition of the Caputo-Fabrizio integral. A detail analysis on the uniqueness has been done. After the proof of the existence and uniqueness, we used the method of separation of variable combining with Laplace transform to proposed the analytical solution. We also investigated on the numerical analysis of the modified groundwater flow equation by using the Crank-Nicolson technique for discretisation. In particular we proved the stability of the method, the proof combined the Fourier analysis method and Von Neumann analysis method. Also a detail analysis has been done as regards the convergence of the Crank-Nicolson method.

**Acknowledgments.** The authors are grateful to the anonymous of the referee for his/her suggestions and comments. The first author would like to thank the second and the third authors for their helpful discussions and advises, in particular to Prof. A. Abdon.

**REFERENCES**

[1] R. T. Alqahtani, Fixed-point theorem for Caputo–Fabrizio fractional Nagumo equation with nonlinear diffusion and convection, in J. Nonlinear Sci. Appl, 9 (2016), 1991–1999.

[2] A. Atangana and B. S. T. Alkahtani, New model of groundwater flowing within a confined aquifer: Application of Caputo-Fabrizio derivative, in Arabian Journal of Geosciences, Springer, 9 (2016), 8pp.

[3] A. Atangana and B. S. T. Alkahtani, Analysis of the Keller–Segel model with a fractional derivative without singular kernel, in Entropy, Multidisciplinary Digital Publishing Institute, 17 (2015), 4439–4453.

[4] A. Atangana and N. Bildik, The use of fractional order derivative to predict the groundwater flow, in Hindawi Publishing Corporation, Mathematical Problems in Engineering, 2013 (2013), Art. ID 543026, 9 pp.

[5] A. Atangana and P. D. Vermeulen, Analytical solutions of a space-time fractional derivative of groundwater flow equation, in Hindawi, 2014 (2014), Art. ID 381753, 11 pp.

[6] A. Atangana and J. F. Botha, A generalized groundwater flow equation using the concept of variable-order derivative, in Boundary Value Problems, Springer, 2013 (2013), 1–11.

[7] A. Atangana and J. J. Nieto, Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel, in Advances in Mechanical Engineering, SAGE Publications 7 (2015), 1687814015613758.

[8] N. S. Boulton, Unsteady radial flow to a pumped well allowing for delayed yield from storage, in Int. Assoc. Sci. Hydrol. Publ, 2 (1954), 472–477.

[9] H. Brezis, Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.

[10] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, in Progr. Fract. Differ. Appl, 1 (2015), 1–13.
A MODEL OF GROUNDWATER FLOW WITHIN AN UNCONFINED AQUIFER

[11] C.-M. Chen, et al, A Fourier method for the fractional diffusion equation describing subdiffusion, in Journal of Computational Physics, 227 (2007), 886–897.
[12] C.-M. Chen, et al, Numerical methods for solving a two-dimensional variable-order anomalous subdiffusion equation, in Mathematics of Computation, 81 (2012), 345–366.
[13] A. Cloot and J. F. Botha, A generalised groundwater flow equation using the concept of non-integer order derivatives, in Water SA, Water Research Commission (WRC), 32 (2007), 1–7.
[14] K. Diethelm, N. J. Ford and A. D. Freed, Detailed error analysis for a fractional Adams method, in Numerical algorithms, Springer, 36 (2004), 31–52.
[15] Eng. Deeb Abdel-Ghafour, Pumping test for groundwater aquifers analysis and evaluation, 2005, available from: https://docplayer.net/11404875-Pumping-test-for-groundwater-aquifers-analysis-and-evaluation-by-eng-deeb-abdel-ghafour.html.
[16] G. Gambolati, Analytic element modeling of groundwater flow, in Eos, Transactions American Geophysical Union, 77 (1995), 103–103.
[17] G. Garven and R. A. Freeze, Theoretical analysis of the role of groundwater flow in the genesis of stratabound ore deposits, in Mathematical and Numerical Model, American Journal of Science, 284 (1984), 1085–1124.
[18] H. M. Haitjema, Analytic element modeling of groundwater flow, in nc San Diego, CA, USA Google Scholar, Academic Press, (1995), 33–75.
[19] L. F. Konikow and D. B. Grove, Derivation of equations describing solute transport in ground water, in US Geological Survey, Water Resources Division, 77 (1977).
[20] J. Losada and J. J. Nieto, Properties of a new fractional derivative without singular kernel, in Progr. Fract. Differ. Appl., 1 (2015), 87–92.
[21] P. K. Mishra and K. L. Kuhlman, Unconfined aquifer flow theory: from Dupuit to present, in Advances in Hydrogeology, Springer, New York, NY (2013), 185–202.
[22] Pollock and W. David, Documentation of computer programs to compute and display pathlines using results from the US Geological Survey modular three-dimensional finite-difference ground-water flow model, in US Geological Survey, 89 (1989).
[23] J. R. Prendergast, R. M. Quinn and J. H. Lawton, The gaps between theory and practice in selecting nature reserves, in Conservation Biology, Wiley Online Library, 13 (1999), 484–492.
[24] S. A. Sauter and C. Schwab, Boundary Element Methods, Springer Series in Computational Mathematics, 39, Springer-Verlag, Berlin, 2011.
[25] C. V. Theis, The relation between the lowering of the Piezometric surface and the rate and duration of discharge of a well using ground-water storage, in Eos, Transactions American Geophysical Union, Wiley Online Library, 16 (1935), 519–524.
[26] G. K. Watugala, Sumudu transform: A new integral transform to solve differential equations and control engineering problems, in Integrated Education, TaylorFrancis, 24 (1993), 35–43.
[27] S. B. Yuste and L. Acedo, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations, in Journal on Numerical Analysis, SIAM, 42 (2005), 1862–1874.
[28] I. S. Zektser, E. Lorne and others, Groundwater Resources of the World: And Their Use, lllP Series on groundwater, 6th edition, Unesco, 2004.

Received July 2017; revised March 2018.

E-mail address: pierre.feulefack@aims-cameroon.org
E-mail address: jeandaniel.djida@aims-cameroon.org
E-mail address: abdonatangana@yahoo.fr