Calculating statistical distributions from operator relations: the statistical distributions of various intermediate statistics

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Abstract

In this paper, we give a general discussion on the calculation of the statistical distribution from a given operator relation of creation, annihilation, and number operators. Our result shows that as long as the relation between the number operator and the creation and annihilation operators can be expressed as $a^\dagger b = \Lambda (N)$ or $N = \Lambda^{-1} (a^\dagger b)$, where $N$, $a^\dagger$, and $b$ denote the number, creation, and annihilation operators, i.e., $N$ is a function of quadratic product of the creation and annihilation operators, the corresponding statistical distribution is the Gentile distribution, a statistical distribution in which the maximum occupation number is an arbitrary integer. As examples, we discuss the statistical distributions corresponding to various operator relations. In particular, besides Bose-Einstein and Fermi-Dirac cases, we discuss the statistical distributions for various schemes of intermediate statistics, especially various $q$-deformation schemes. Our result shows that the statistical distributions corresponding to various $q$-deformation schemes are various Gentile distributions with different maximum occupation numbers which are determined by the deformation parameter $q$. This result shows that the results given in much literature on the $q$-deformation distribution are inaccurate or incomplete.

1 Introduction

The statistical property of a quantum system is embodied in the operator relation of creation, annihilation, and number operators. When such a relation is given, one can in principle solve the statistical distribution for the system. For example, from $N = a^\dagger a$ and $[a, a^\dagger] = 1$, one can deduce

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the Bose-Einstein distribution, and from $N = a\dagger a$ and $\{a, a\dagger\} = 1$, one can deduce the Fermi-Dirac distribution.

As generalizations of Bose-Einstein and Fermi-Dirac statistics, there are some schemes of intermediate statistics [1, 2, 3, 4]. It has been shown that intermediate-statistics type excitations may exist in many physical systems [5]. The need of intermediate statistics in physics is that there are many composite-particle systems and intermediate-statistics type elementary excitations, e.g., the Cooper pair in the theory of superconductivity, the Fermi gas superfluid [6], the exciton [7], the magnon [8], etc. Concretely, composite bosonic particles consisted of fermions will deviate from Bose-Einstein statistics to a certain extent under some circumstances [9, 10]. In this case, such a system can be viewed as obeying a kind of intermediate statistics, and intermediate statistics can be used as an effective tool for studying such a system.

Different intermediate-statistics schemes correspond to different operator relations. In principle, from the operator relation of a kind of intermediate statistics, one can achieve the corresponding intermediate-statistics distribution. In the following, we will give a general discussion on the calculation of the statistical distribution from a given operator relation.

Let $a\dagger$ and $b$ be creation operator and annihilation operator, and let $N$ be the number operator. Then we must have

$$[N, a\dagger] = a\dagger \quad \text{and} \quad [N, b] = -b. \tag{1}$$

Denoting the eigenstate of the number operator by $|N\rangle$, i.e., $N |N\rangle = N |N\rangle$, we achieve

$$a\dagger |N\rangle = \sqrt{\alpha (N+1)} |N+1\rangle,$$

$$b |N\rangle = \sqrt{\beta (N)} |N-1\rangle, \tag{2}$$

where the coefficients $\alpha (N)$ and $\beta (N)$ are functions of $N$. It should be emphasized that Eq. (2) is a basic relation. From this relation, one can deduce the operator relations among creation, annihilation, and number operators, including the quantization condition. For example, for the Bose case, we have $b = a$, $\alpha (N + 1) = N + 1$, and $\beta (N) = N$; as a result, the bosonic quantization condition reads $[a, a\dagger] = aa\dagger - a\dagger a = 1$ and the relation between $a\dagger$, $a$, and $N$ reads $a\dagger a = N$ or $aa\dagger = N + 1$.

Assume that the relation between the number operator and the creation and annihilation operators is

$$a\dagger b = \Lambda (N), \tag{3}$$

where $\Lambda (N)$ is an analytic function, i.e., $N$ is a function of quadratic product of the creation and
annihilation operators. Then, from Eqs. (1) and (2), we achieve

\[ ba^\dagger = \Lambda (N + 1) . \]  

(4)

The statistical distribution is the ensemble average of the number operator of the \( l \)-th state

\[ \langle N_l \rangle = \frac{1}{\Xi} \text{tr} \left[ e^{-\beta (H - \mu N_{\text{total}})} N_l \right] , \]  

(5)

where \( H = \sum_l N_l \varepsilon_l \) is the Hamiltonian of the system, \( N_{\text{total}} = \sum_l N_l \) is the total number of particles in the system, \( \mu \) is the chemical potential, \( \Xi \) is the grand partition function, \( \varepsilon_l \) is the energy of the \( l \)-th state, and \( \beta = 1/(kT) \). In the following, we will give a general discussion on the derivation of the statistical distribution on the basis of the operator relations among creation, annihilation, and number operators.

As a direct generalization of Bose-Einstein and Fermi-Dirac statistics, Gentile suggested a scheme of intermediate statistics — Gentile statistics in which the maximum occupation number of particles of a quantum state is a finite number \( n \) [1]. Bose-Einstein and Fermi-Dirac statistics become two limit cases of Gentile statistics when \( n \to \infty \) and \( n = 1 \), respectively [11, 12]. In the following, we will show that when the relation between the number operator and the creation and annihilation operators takes the form of Eq. (3), the statistical distribution is always the Gentile distribution with a maximum occupation number determined by the intermediate-statistics parameter.

As examples of the general result, we will first discuss the statistical distributions for Bose-Einstein, Fermi-Dirac, and Gentile cases, and then discuss the statistical distributions for various \( q \)-deformation schemes.

Quantum algebras (quantum groups), as generalizations of usual Lie algebras, have been discussed widely for many years [13, 14]. Quantum algebras become important in physics since the introduction of \( q \)-deformed harmonic oscillator which provides a bosonic realization of the quantum algebra \( su_q(2) \). Many schemes of \( q \)-oscillator have been constructed, including the \( q \)-deformed Arik-Coon oscillator [15], the \( q \)-deformed Biedenharn-Macfarlane oscillator [16], the parabosonic and parafermionic oscillators [17], the \( q \)-deformed parabosonic and \( q \)-deformed parafermionic quantization schemes [18], the \( q \)-deformed fermionic algebra [19], the Tamm-Dancoff cut-off deformation [18], the two-parameter deformed oscillator [20], etc. Moreover, further studies present some generalized deformed oscillators [21, 22, 23, 24, 25]. Many researches also devoted to the deformed algebras, such as \( su_q(2) \) [16, su_q(1, 1) [26, su_q(1, 1) [20, su_q(N) [28, U_q(gl(2)) [29, 30, U_{p,q}(gl(2)) [31, 32, U_{p,q}(gl(1,1)) [33, \text{q-deformed Lorentz algebra} [34, 35, 36, etc.

Once the operator relation of a \( q \)-deformation scheme is given, the corresponding statistical distribution is determined. In this paper, as examples of the above general result, we discuss the
statistical distributions for various $q$-deformation schemes. We argue that the statistical distributions of various $q$-deformation schemes are the Gentile distributions whose maximum occupation numbers are determined by the $q$-deformation parameter $q$ other than the distributions given in literature; note that Bose-Einstein and Fermi-Dirac distributions are viewed as special cases of the Gentile distribution with maximum occupation numbers $\infty$ and 1 [11]. In other words, our result indicates that the $q$-deformation statistical distributions given in the literature are inaccurate or incomplete. The inaccurate results in literature are obtained by an improper approximation. Concretely, a key step in the derivation of the statistical distribution is to deal with the average value $\langle f(N) \rangle$, where $f(N)$ is a function of $N$. In the literature [37], however, such an average value is approximately taken as $\langle f(N) \rangle \simeq f(\langle N \rangle)$, or, for more details, in the literature the authors take the approximation $\langle q^N \rangle \simeq q^{\langle N \rangle}$ [38, 39, 40, 41, 12, 43, 44, 45, 46, 47, 48, 49, 50]. Nevertheless, the rigorous result should be $\langle N^m \rangle = \langle N \rangle \langle N^{m-1} \rangle - \frac{\partial}{\partial x} \langle N^{m-1} \rangle$ rather than $\langle N^m \rangle = \langle N \rangle^m$, where $x = \beta \varepsilon$ and $\varepsilon$ is the energy of the quantum state. Moreover, in literature there is an alternative way for considering the statistical distributions of various $q$-deformation schemes: replacing the average of number operator $\langle N \rangle$ by the average $\langle a^\dagger a \rangle$ [51, 52, 53, 54, 55, 56, 57]. The reason why taking such a replacement, as stated in the literature [51, 52, 53], is that $\langle N \rangle$ gives a nondeformed Bose-Einstein distribution. Our result shows that the statistical distributions coming from $\langle N \rangle$ are not only the Bose-Einstein statistics; in some cases the statistical distribution is the Gentile distribution and the maximum occupation number is determined by the deformation parameter $q$.

In Section 2 we give a general discussion on the derivation of the statistical distribution from a given operator relation. In Section 3 as examples, we discuss the statistical distributions for various $q$-deformation schemes. The conclusion and outlook are given in Section 4.

## 2 Deducing the statistical distribution from a given operator relation

In this section, we give a general discussion on the calculation of statistical distribution from the operator relation. The statistical distribution of a physical system can be calculated from the basic operator relations [58]. In the following, we will show that the relation between the number operator and the creation and annihilation operators, $\Lambda(N_l) = a^\dagger_l b_l$, determines the statistical distribution. Or, in more details, the statistical distribution is determined by the first two nonnegative integer zeroes of $\Lambda(N_l)$. In this section, we use the subscript $l$ to denote the $l$-th state.
2.1 The statistical distribution

Let $p_0, p_1, p_2, \cdots$ be zeroes of $\Lambda (N_l)$. We first have the following general result:

Let $p_{k_1}$ and $p_{k_2}$ be two zeroes of $\Lambda (N_l)$. Then

$$\langle N_l \rangle = \frac{1}{z^{-1}e^{\beta \varepsilon_l} - 1} - \frac{p_{k_2} - p_{k_1}}{z^{-1}(p_{k_2} - p_{k_1})e^{(p_{k_2} - p_{k_1})\beta \varepsilon_l} - 1} + p_{k_1}$$

(6)

is the statistical distribution corresponding to the operator relation (3), where $z = e^{\beta \mu}$ is the fugacity.

The proof is as follows.

The ensemble average of $\Lambda (N_l)$ can be calculated directly from Eq. (3):

$$\langle \Lambda (N_l) \rangle = \langle a_l^\dagger b_l \rangle = \frac{1}{\Xi} \text{tr} \left[ e^{-\beta (H - \mu N_{\text{total}})} a_l^\dagger b_l \right].$$

(7)

By the relation

$$e^{-\beta (H - \mu N_{\text{total}})} a_l^\dagger = e^{-\beta (\varepsilon_l - \mu)} a_l^\dagger e^{-\beta (H - \mu N_{\text{total}})},$$

(8)

we achieve

$$\langle \Lambda (N_l) \rangle = e^{-\beta (\varepsilon_l - \mu)} \frac{1}{\Xi} \text{tr} \left[ e^{-\beta (H - \mu N_{\text{total}})} b_l a_l^\dagger \right]$$

$$= e^{-\beta (\varepsilon_l - \mu)} \langle \Lambda (N_l + 1) \rangle,$$

(9)

or, equivalently,

$$\frac{\langle \Lambda (N_l) \rangle}{\langle \Lambda (N_l + 1) \rangle} = e^{-\beta (\varepsilon_l - \mu)} = ze^{-x},$$

(10)

where $x = \beta \varepsilon_l$.

Since $p_{k_1}$ and $p_{k_2}$ are two zeroes of $\Lambda (N_l)$, $\Lambda (N_l)$ can be expressed as

$$\Lambda (N_l) = (N_l - p_{k_1}) (N_l - p_{k_2}) G (N_l).$$

(11)

Expanding $G (N_l)$ as

$$G (N_l) = \sum_{m=0}^{\infty} c_m N_l^m$$

(12)

and substituting Eqs. (11) and (12) into Eq. (10) gives

$$\sum_{m=0}^{\infty} c_m \langle (N_l - p_{k_1}) (N_l - p_{k_2}) N_l^m \rangle = \sum_{m=0}^{\infty} c_m e^{-\beta (\varepsilon_l - \mu)} \langle (N_l + 1 - p_{k_1}) (N_l + 1 - p_{k_2}) (N_l + 1)^m \rangle.$$  

(13)

Now, we prove that Eq. (6) is a solution of Eq. (13).

First, consider the case of $m = 0$.  

The term with $m = 0$ in Eq. (13) is

$$
\left[1 - e^{-\beta(\epsilon_l - \mu)}\right] \langle N_l^2 \rangle - \left[(p_{k_1} + p_{k_2}) + e^{-\beta(\epsilon_l - \mu)} (2 - p_{k_1} - p_{k_2})\right] \langle N_l \rangle \\
+ \left[p_{k_1}p_{k_2} - e^{-\beta(\epsilon_l - \mu)} (p_{k_1}p_{k_2} - p_{k_1} - p_{k_2} + 1)\right] = 0.
$$

We first prove that

$$
\langle N_l f (N_l) \rangle = \langle f (N_l) \rangle - \frac{\partial}{\partial x} \langle f (N_l) \rangle .
$$

The proof is straightforward:

$$
\langle N_l f (N_l) \rangle = 1 \Xi \text{tr} \left[ \frac{\partial}{\partial x} e^{-\beta(H - \mu N)} f (N_l) \right]
= \left( \frac{1}{\Xi} \frac{\partial \Xi}{\partial x} \right) \frac{1}{\Xi} \text{tr} \left[ e^{-\beta(H - \mu N)} f (N_l) \right] - \frac{\partial}{\partial x} \langle f (N_l) \rangle
= \langle N_l \rangle \langle f (N_l) \rangle - \frac{\partial}{\partial x} \langle f (N_l) \rangle .
$$

This gives

$$
\langle N_l^2 \rangle = \langle N_l \rangle^2 - \frac{\partial \langle N_l \rangle}{\partial x} ,
$$

and Eq. (14) is converted into a differential equation:

$$
\left[1 - e^{-\beta(\epsilon_l - \mu)}\right] \left(\langle N_l \rangle^2 - \frac{\partial \langle N_l \rangle}{\partial x}\right) - \left[(p_{k_1} + p_{k_2}) + e^{-\beta(\epsilon_l - \mu)} (2 - p_{k_1} - p_{k_2})\right] \langle N_l \rangle \\
+ \left[p_{k_1}p_{k_2} - e^{-\beta(\epsilon_l - \mu)} (p_{k_1}p_{k_2} - p_{k_1} - p_{k_2} + 1)\right] = 0.
$$

It can be checked directly that the statistical distribution (6) is a solution of Eq. (18).

Next, consider the case of $m > 0$.

We will prove that if

$$
\frac{\langle F (N_l) \rangle}{\langle F (N_l + 1) \rangle} = ze^{-x} ,
$$

then

$$
\frac{\langle N_l^m F (N_l) \rangle}{\langle (N_l + 1)^m F (N_l + 1) \rangle} = ze^{-x} .
$$

By Eqs. (15) and (19), we have

$$
\langle (N_l + 1) F (N_l + 1) \rangle = ze^x \langle N_l F (N_l) \rangle .
$$

Then

$$
\frac{\langle N_l F (N_l) \rangle}{\langle (N_l + 1) F (N_l + 1) \rangle} = ze^{-x} .
$$

Repeating this procedure proves Eq. (20).

Now, we can prove Eq. (6) directly.

We have shown that the statistical distribution (6) satisfies Eq. (19) with $F (N_l) = (N_l - p_{k_1}) (N_l - p_{k_2})$.

Then, for an arbitrary value of $m$, from Eq. (20), we can see that the distribution (6) is a solution.
\[ \langle (N_l - p_{k_1}) (N_l - p_{k_2}) N_l^m \rangle = e^{-\beta (\varepsilon_l - \mu)} \langle (N_l + 1 - p_{k_1}) (N_l + 1 - p_{k_2}) (N_l + 1)^m \rangle, \quad m = 0, 1, 2, \cdots, \] (23)

and then is a solution of Eq. (13).

This proves the statement that Eq. (13) is the statistical distribution corresponding to the operator relation (3).

This is a general result. For a physical system, we also need to take some physical conditions into account. We will show that only the first two nonnegative integer zeroes of \( \Lambda (N_l) \) contribute to the statistical distribution of a physical system, though when \( p_{k_1} \) and \( p_{k_2} \) are complex, Eq. (6) is still a solution of Eq. (18).

The distribution (6) is a statistical distribution with both maximum occupation number \( p_{k_2} - 1 \) and minimum occupation number \( p_{k_1} \). This is because, by Eq. (6), when \( T \to 0 \), the occupation number of the ground state is \( \langle N_0 \rangle = p_{k_2} - 1 \). The minimum value of Eq. (6) is \( p_{k_1} \); this tells us that \( p_{k_1} \) is the minimum occupation number. In a word, in such a system, a quantum state can be occupied by at most \( p_{k_2} - 1 \) and at least \( p_{k_1} \) particles.

The above result shows that the zeroes of \( \Lambda (N_l) \) correspond to the restriction on the occupation number of a quantum state.

In a physical system, the minimum occupation number must be zero, i.e., \( p_{k_1} = 0 \). This requires that for a physical system, the function \( \Lambda (N_l) \) must have a zero \( N_l = 0 \).

As shown above, \( p_{k_2} - 1 \) corresponds to the maximum occupation number. If we insist on that the maximum occupation number must be a positive integer, then \( p_{k_2} \) should be an integer.

For a physical system, if \( \Lambda (N_l) \) has more than one zero, only the first two nonnegative integer zeroes are physically meaningful: the first zero should be \( p_0 = 0 \), which is the minimum occupation number, and the second zero gives the maximum occupation number. The other zeroes do not contribute to the statistical distribution. The reason is as follows.

\( p_k \) being a zero means that \( \Lambda (p_k) = 0 \). Then, from Eqs. (14) and (15), we have \( b_l a_l^\dagger |p_k - 1\rangle = 0 \) and \( a_l^\dagger b_l |p_k\rangle = 0 \). By Eq. (2), we achieve \( \alpha (p_k) \beta (p_k) = 0 \). A realistic system must have an equilibrium state, so \( |\alpha (p_k)| = |\beta (p_k)| \). Consequently, we have \( \alpha (p_k) = \beta (p_k) = 0 \). This means that starting from the state with no particle, one cannot achieve the state with \( p_k \) particles by repeatedly acting the creation operator.

Now, we can draw our conclusion:

If the relation between the number operator and the creation and annihilation operators can be expressed as \( a^\dagger b = \Lambda (N) \), i.e., \( N \) is a function of quadratic product of the creation and annihilation operators, then the corresponding statistical distribution is determined by the first two nonnegative
integer zeroes. The first nonnegative integer zero determines the minimum occupation number, which should be zero for physical reasons, and the second nonnegative integer zero determines the maximum occupation number. The corresponding statistical distribution is the Gentile statistics distribution \[11\]. Two special cases are that the maximum occupation number equals 1, which recovers the Fermi-Dirac distribution, and that the maximum occupation number tends to \( \infty \), which recovers the Bose-Einstein distribution \[59, 60\].

It should be emphasized that the above conclusion is based on the demand that the maximum occupation number (not the average occupation number) is an integer. If we release this condition, we can obtain other statistical distributions.

2.2 Examples

As examples, we first consider the Bose-Einstein, Fermi-Dirac, and Gentile cases.

The above result shows that if the number operator can be written as a function of the quadratic product of the creation and annihilation operators \( a^\dagger b \), i.e., \( a^\dagger b = \Lambda (N) \), or, \( N = \Lambda^{-1} (a^\dagger b) \), then the corresponding statistical distribution is determined by the zeroes of the function \( \Lambda (N) \) and the distribution must take the form of Eq. (6). This means that, if we insist that the maximum occupation number must be integer, the only physically allowed statistical distributions are Bose-Einstein, Fermi-Dirac, and Gentile statistics (the distribution with a non-zero minimum occupation number has no physical meaning).

(1) The Bose case. For the Bose case,

\[
a^\dagger a = \Lambda (N) = N, \tag{24}
\]

\[
a a^\dagger = \Lambda (N + 1) = N + 1. \tag{25}
\]

The quantization condition reads

\[
[a, a^\dagger] = \Lambda (N + 1) - \Lambda (N) = 1. \tag{26}
\]

The only zero of \( \Lambda (N) \) is \( N = 0 \). This means that in the Bose case, the minimum occupation number is 0, and there is no restriction on the maximum occupation number. The corresponding statistical distribution is the Bose-Einstein distribution.

(2) The Fermi case. For the Fermi case,

\[
a^\dagger a = \Lambda (N) = N (2 - N), \tag{27}
\]

\[
a a^\dagger = \Lambda (N + 1) = (N + 1) (1 - N). \tag{28}
\]

The quantization condition reads

\[
[a, a^\dagger] = \Lambda (N + 1) - \Lambda (N) = 1 - 2N \tag{29}
\]
or
\[
\{a, a^\dagger\} = \Lambda(N+1) + \Lambda(N) = 1 + 2N - 2N^2 = 1.
\] (30)

Note that for the Fermi case, the value of \(N\) can only take 0 or 1, so \(1 + 2N - 2N^2\) always equals 1.

The zeroes of \(\Lambda(N)\) are \(N = 0\) and \(N = 2\). This means that the maximum occupation number in the Fermi case is 1. The corresponding statistical distribution is the Fermi-Dirac distribution.

(3) The Gentile case. We can also consider another kind of intermediate statistics — Gentile statistics. In Gentile statistics, a quantum state can be occupied by at most \(n\) particles; \(n \rightarrow \infty\) and 1 recover Bose-Einstein statistics [11, 12] and Fermi-Dirac statistics, respectively. For Gentile statistics, two operator realizations can be constructed as follows:

One is
\[
a^\dagger a = \Lambda(N) = \frac{\sin \frac{N\pi}{n+1}}{\sin \frac{\pi}{n}}.
\] (31)
\[
a a^\dagger = \Lambda(N + 1) = \frac{\sin \frac{(N+1)\pi}{n+1}}{\sin \frac{\pi}{n+1}}.
\] (32)

The quantization condition reads
\[
a a^\dagger - e^{i\frac{2\pi}{n+1}} a^\dagger a = e^{-i\frac{N\pi}{n+1}}
\]
or
\[
a a^\dagger - \cos \frac{\pi}{n+1} a^\dagger a = \cos \frac{N\pi}{n+1}.
\] (33)

The other is [59, 60]
\[
a^\dagger b = \Lambda(N) = \frac{1 - e^{i2\pi \frac{N}{n+1}}}{1 - e^{i2\pi \frac{n+1}{n+1}}},
\] (34)
\[
ba^\dagger = \Lambda(N + 1) = \frac{1 - e^{i2\pi \frac{n+1}{n+1}}}{1 - e^{i2\pi \frac{n+1}{n+1}}}.
\] (35)

Then the quantization condition can be constructed as
\[
[b, a^\dagger] = e^{i2\pi \frac{N}{n+1}}
\] (36)
or
\[
ba^\dagger - e^{i2\pi \frac{n+1}{n+1}} a^\dagger b = 1.
\] (37)

The first two zeroes of \(\Lambda(N)\) are \(N = 0\) and \(N = n + 1\). This means that the maximum occupation number is \(n\) and the corresponding statistical distribution is the Gentile distribution:
\[
\langle N_l \rangle = \frac{1}{z^{-1}e^{\beta\varepsilon_l} - 1} - \frac{n + 1}{z^{-(n+1)}e^{(n+1)\beta\varepsilon_l} - 1}.
\] (38)
As a physical realization of Gentile statistics, Ref. [8] shows that the elementary excitation of the Heisenberg magnetic system — the magnon which is the quantized spin waves — obeys Gentile statistics with a maximum occupation number \( n = 2S \) rather than Bose–Einstein statistics, where \( S \) is the spin quantum number. In the conventional treatment of a magnetic system, one uses the Holstein–Primakoff realization. The Holstein–Primakoff realization is a bosonic realization with an additional putting-in-by-hand restriction on the occupation number. Nevertheless, instead of the bosonic Holstein–Primakoff realization, Ref. [8] constructs a Gentile-type operator realization, in which there is no additional constraint. By comparing with the experimental data, in Ref. [8], one can see that the Gentile realization is more accurate than that of the bosonic realization. In other words, for a magnetic system, a bosonic realization with a restriction on the occupation number is indeed an approximation of the Gentile scheme.

3 The statistical distributions corresponding to various \( q \)-deformation schemes

3.1 The \( q \)-deformation distributions

In this section, as the examples of the above general result, we discuss the statistical distributions for various \( q \)-deformation schemes.

There are many schemes of \( q \)-deformation [61, 62], and there are many discussions on the relation between the operator realization and the corresponding statistical distribution. However, our result will show that the statistical distributions given in literature are inaccurate or incomplete. Our result shows that the \( q \)-deformation statistical distributions are the Gentile distribution with a maximum occupation number which is determined by the deformation parameter \( q \) (here we regard Bose-Einstein and Fermi-Dirac statistics as special cases of Gentile statistics with maximum occupation numbers \( \infty \) and 1). That is to say, different \( q \)-deformation schemes correspond to different kinds of Gentile statistics whose maximum occupation numbers lie on the deformation parameter \( q \).

\[ a^\dagger a = \Lambda (N) = \frac{q^N - q^{-N}}{q - q^{-1}}. \]

In this scheme [16, 28],

\[ aa^\dagger = \Lambda (N + 1) = \frac{q^{N+1} - q^{-(N+1)}}{q - q^{-1}}. \] (39)

The quantization condition reads

\[ aa^\dagger - qa^\dagger a = \Lambda (N + 1) - q\Lambda (N) = q^{-N}. \] (40)

The statistical distribution is determined by the zeroes of \( \Lambda (N) \), given by \( (q^N - q^{-N}) / (q - q^{-1}) = 0 \).
There are two possible cases: $q = e^{i \frac{2\pi}{l}}$ and $q \neq e^{i \frac{2\pi}{l}}$, where $l$ is an arbitrary positive integer, $1 \leq k \leq 2l - 1$, and $k$ and $2l$ are relatively prime.

For $q = e^{i \frac{2\pi}{l}}$, there are many zeroes: $N = 0$ and $N$ equals any integral multiple of $2l$. $N = 0$ is the minimum occupation number, and $2l - 1$ is the maximum occupation number. This is just Gentile statistics with the maximum occupation number $2l - 1$, i.e., the distribution \[38\] with $n = 2l - 1$. In this case, the relation between maximum occupation number and $q$ is

$$n = \frac{2\pi k}{\ln q} - 1 \quad \text{with} \quad q = e^{i \frac{2\pi}{l}}. \quad (41)$$

For $q \neq e^{i \frac{2\pi}{l}}$, the only zero of $\Lambda(N)$ is $N = 0$. This means that the minimum occupation number is 0 and there is no restriction on the maximum occupation number. This is just the Bose-Einstein distribution, the Gentile distribution with an infinite maximum occupation number.

(2) $a^\dagger a = \Lambda(N) = \frac{N}{q - p^{-N}}$. In this scheme, the quantization condition is $aa^\dagger - qa^\dagger a = 1$ [15] and

$$[a, a^\dagger] = q^N. \quad (42)$$

Since $\Lambda(N)$ must be real for any value of $N$, $q$ should be a real number. In this scheme, $0 < q < 1$. Therefore, the only zero is $N = 0$ and the corresponding statistical distribution is the Bose-Einstein distribution.

(3) $a^\dagger a = \Lambda(N) = \frac{q^N - p^{-N}}{q - p^{-1}}$. In this scheme, the quantization condition is $aa^\dagger - qa^\dagger a = p^{-N}$ and $aa^\dagger - p^{-1}a^\dagger a = q^N$ [20, 63]. Then we have

$$[a, a^\dagger] = \frac{q^N (q - 1) + p^{-N} (1 - p^{-1})}{q - p^{-1}}. \quad (43)$$

When

$$q = |q| e^{i \frac{2\pi}{l}} \quad \text{and} \quad p = |q|^{-1} e^{i \frac{2\pi}{l}}, \quad (44)$$

where $l$ is a positive integer and $k$ and $2l$ are relatively prime, the two minimum zeroes of $\Lambda(N)$ are

$$N = 0 \quad \text{and} \quad N = 2l. \quad (45)$$

Then the corresponding distribution is the Gentile distribution with the maximum occupation number $2l - 1$, the distribution \[38\] with $n = 2l - 1$.

When $q$ and $p$ take other values, there is only one zero $N = 0$, and the distribution is the Bose-Einstein distribution.

That is to say, for a given $q$ and $p$, this scheme corresponds to Gentile statistics with a maximum occupation number determined by the deformation parameters $q$ and $p$.

(4) $a^\dagger a = \Lambda(N) = N (p + 1 - N)$. In the parafermionic scheme, the matrix element of $a^\dagger$ and $a$ is $a_{N,N+1} = a_{N+1,N}^\dagger = \sqrt{(N + 1) (p - N)}$ [17]. The quantization condition then reads

$$[a, a^\dagger] = p - 2N. \quad (46)$$
When \( p \) is a positive integer, \( \Lambda (N) \) has two zeroes

\[
N = 0 \text{ and } N = p + 1.
\]

(47)

The corresponding statistical distribution is the Gentile distribution with the maximum occupation number \( p \), the distribution \( \text{(38)} \) with \( n = p \).

Such a result can be directly seen from the operator realization of parafermionic quantization.

For \( p = 1 \), the operator realization is just the fermionic case: \( \{a, a^\dagger\} = 1 \) \[17\]. This leads to \( a^2 = 0 \), i.e., the maximum occupation number is 1. For \( p = 2 \), the operator realization is: \( a^3 = 0, \) \( aa^1 = 2a, \) and \( aaa^1 + a^1 aa = 2a \) \[17\]. \( a^3 = 0 \) means that the maximum occupation number is 2.

In other words, one can recognize from the operator realization of the parafermionic quantization that such a \( q \)-deformation scheme corresponds to Gentile statistics.

When \( p \) is not a positive integer, \( \Lambda (N) \) has only one zero \( N = 0 \), the distribution is the Bose-Einstein distribution.

\( a^\dagger a = \Lambda (N) = \frac{\sinh(\tau N) \sinh[\tau (p+1-N)]}{\sinh^2(\tau)} \).

In the \( q \)-deformed para-Fermi quantization scheme \[18, 64\], \( a^\dagger a = - [(N - 1 - p)] [N] \), where \( [x] = \frac{q^x - q^{-x}}{q - q^{-1}} \). The commutation relation, by the substitution \( \tau = \ln q \), then reads

\[
[a, a^\dagger] = \frac{\cosh[\tau (1 + p - 2N)] - \cosh[\tau (1 - p + 2N)]}{2 \sinh^2(\tau)}.
\]

(48)

and \( \Lambda (N) \) can be rewritten as \( \frac{\sinh(\tau N) \sinh[\tau (p+1-N)]}{\sinh^2(\tau)} \).

Like that in the above case, when \( p \) is a positive integer, \( \Lambda (N) \) has two zeroes

\[
N = 0 \text{ and } N = p + 1.
\]

(49)

The distribution is the Gentile distribution with the maximum occupation number \( p \). When \( p \) takes another value, \( \Lambda (N) \) has only one zero, the distribution is the Bose-Einstein distribution.

\( a^\dagger a = \Lambda (N) = N \cos^2 \frac{N\pi}{2} + (N + p - 1) \sin^2 \frac{N\pi}{2} \).

In the parabosonic scheme \[17\], for \( p = 1 \), the operator realization is \( [a, a^\dagger] = 1 \) and this is just the Bose case; for \( p = 2 \), the operator realization is \( aaa^1 - a^1 aa = 2a \), etc. The matrix elements of \( a^1 \) and \( a \) are \( a_{N,N+1} = a_{N+1,N}^\dagger = \sqrt{N + p} \) (\( N = \text{even} \)) and \( a_{N,N+1} = a_{N+1,N}^\dagger = \sqrt{1 + N} \) (\( N = \text{odd} \)). Then \( a^1 a = N + p - 1 \) for \( N = \text{odd} \) and \( a^\dagger a = N \) for \( N = \text{even} \). Such results can be equally rewritten in a compact form: \( a_{N,N+1} = a_{N+1,N}^\dagger = \cos^2 \frac{N\pi}{2} \sqrt{N + p} + \sin^2 \frac{N\pi}{2} \sqrt{1 + N} \) and \( a^\dagger a = N \cos^2 \frac{N\pi}{2} + (N + p - 1) \sin^2 \frac{N\pi}{2} \).

Then the quantization condition reads

\[
[a, a^\dagger] = 2 (1 - p) \sin^2 \frac{N\pi}{2} + p.
\]

(50)

In this case, \( \{a, a^\dagger\} = 2N - p \).
When \( p = 1 - \frac{1}{\sin^2(\tau/2)} \), where \( l > 1 \) is an integer, \( \Lambda (N) \) has two zeroes

\[
N = 0 \text{ and } N = l,
\]

the corresponding distribution is the Gentile distribution with the maximum occupation number \( l - 1 \), the distribution \( 38 \) with \( n = l - 1 \).

When \( p \) takes another value, \( \Lambda (N) \) has only one zero \( N = 0 \), the distribution is the Bose-Einstein distribution.

\[
(7) \quad a^\dagger a = \Lambda (N) = \frac{\sinh(\tau N) \cosh[\tau(N+2N_0-1)]}{\sinh \tau \cosh \tau} \left[ \cos^2 \frac{N\pi}{2} + \frac{\sinh[\tau(N+2N_0-1)] \cosh(\tau N)}{\sinh \tau \cosh \tau} \sin^2 \frac{N\pi}{2} \right].
\]

In the \( q \)-deformed Wigner quantization scheme \( 18, 64 \), the matrix elements of \( a^\dagger \) and \( a \) are \( a_{N,N+1} = a_{N+1,N} = \sqrt{\left( N + 2N_0 \right) (N + 1)} \) (\( N \) is even) and \( a_{N,N+1} = a_{N+1,N} = \sqrt{\left( N + 2N_0 \right) (N + 1)} \) (\( N \) is odd), where \( \{ x \} = q^{x-q^{-x}} \). Then we have \( a^\dagger a = [N - 1 + 2N_0] \{ N \} \) (\( N \) is odd) and \( a^\dagger a = \{ N - 1 + 2N_0 \} [N] \) (\( N \) is even). Substituting \( \tau = \ln q \) and rewriting the above results in a compact form, we have \( a_{N,N+1} = a_{N+1,N} = \sqrt{\{ N + 2N_0 \} \{ N + 1 \}} \sin \frac{N\pi}{2} + \sqrt{\left[ N + 2N_0 \right] \{ N + 1 \} \cos^2 \frac{N\pi}{2} \}
\]

and \( a^\dagger a = \frac{\sinh(\tau N) \cosh[\tau(N+2N_0-1)]}{\sinh \tau \cosh \tau} \cos^2 \frac{N\pi}{2} + \frac{\sinh[\tau(N+2N_0-1)] \cosh(\tau N)}{\sinh \tau \cosh \tau} \sin^2 \frac{N\pi}{2} \). The commutation relation then reads

\[
[a, a^\dagger] = \frac{\cosh [2\tau (N + N_0)]}{\cosh \tau} - 2 \cos (N\pi) \frac{\sinh [\tau (1 - 2N_0)]}{\sinh (2\tau)}.
\]

In this case, \( \{ a, a^\dagger \} = \frac{\sinh[2\tau (N + N_0)]}{\sinh \tau} \).

When \( N_0 > 0 \), \( \Lambda (N) \) has only one zero \( N = 0 \), the distribution is the Bose-Einstein distribution.

\[
(8) \quad a^\dagger a = \Lambda (N) = \sin^2 \frac{N\pi}{2}. \quad \text{In this scheme } 65,
\]

\[
[a, a^\dagger] = (-1)^N.
\]

In this case, \( \{ a, a^\dagger \} = 1 \). Note that this scheme is just the scheme \( 2 \) with \( q = -1 \), i.e., \( a^\dagger a = \frac{q^{N-1}}{q^{-1}} \bigg |_{q=-1} \).

The two minimum zeroes of \( \Lambda (N) \) are \( N = 0 \) and \( 2 \). This means that the maximum occupation number is 1, and then the corresponding statistical distribution is just the Fermi-Dirac distribution. This is because, from Eq. \( 29 \), in the Fermi case we have \( [a, a^\dagger] = 1 - 2N \). Moreover, in the Fermi case, the value of \( N \) can take on only 0 and 1, while in Eq. \( 65 \), the values of \( (-1)^N \) are only +1 and -1.

\[
(9) \quad a^\dagger a = \Lambda (N) = q^{N-1} \sin^2 \frac{N\pi}{2}. \quad \text{In this scheme } 19,
\]

\[
[a, a^\dagger] = (q^N - q^{N-1}) + (-1)^N (q^N + q^{N-1}).
\]

Like the above case, the two minimum zeroes of \( \Lambda (N) \) are \( N = 0 \) and \( 2 \), and the distribution is the Fermi-Dirac distribution.
It can be seen that the statistical distribution of case (8) and case (9) are the same. This is because the type of statistical distribution is only determined by the zero of \( \Lambda (N) \). So long as the zeroes of \( \Lambda (N) \) are the same, the statistical distributions are the same.

\[ (10) \quad a^\dagger a = \Lambda (N) = \frac{1 - (-q)^N}{1 + q}. \]

In this scheme [66], the operator realization is \( aa^\dagger + qa^\dagger a = 1 \).

We then have

\[ [a, a^\dagger] = (-q)^N. \tag{55} \]

When \( q = 1 \), the two minimum zeroes of \( \Lambda (N) \) are \( N = 0 \) and \( 2 \), and the distribution is the Fermi-Dirac distribution.

When \( q \neq 1 \), there is only one zero \( N = 0 \); the distribution is the Bose-Einstein distribution.

\[ (11) \quad a^\dagger a = \Lambda (N) = N^n. \]

In this scheme [21], the operator realization is \( (aa^\dagger)^n - (a^\dagger a)^n = 1 \).

We then have

\[ [a, a^\dagger] = (N + 1)^n - N^n. \tag{56} \]

For any value of \( n \), there is only one zero; the distribution is the Bose-Einstein distribution.

### 3.2 Comment on the literature result of the \( q \)-deformation statistical distribution

In literature, there are many discussions on the statistical distributions for various \( q \)-deformation schemes, especially the scheme \( aa^\dagger - qa^\dagger a = q^{-N} \) with \( a^\dagger a = \Lambda (N) = (q^N - q^{-N}) / (q - q^{-1}) \).

However, all these results on the \( q \)-deformation statistical distribution are inaccurate or incomplete.

When calculating the statistical distribution from a given operator relation, one may encounter a problem of the calculation of \( \langle f (N) \rangle \), the average of a function of the number operator \( N \). In some literature, the authors use an approximation

\[ \langle f (N) \rangle = f (\langle N \rangle), \tag{57} \]

i.e., approximately replace the average of the function of the number operator \( N \) by the function of the average of the number operator \( \langle N \rangle \). Concretely, for example, in Ref. [37], the authors use the approximation \( q^{\langle N \rangle} \simeq \langle q^N \rangle \). This gives \( \langle f (N) \rangle \simeq \langle f (\langle N \rangle) \rangle = \langle a^\dagger a \rangle \). This result has been used as the basis of some other works by many authors, e.g., [38, 39, 40, 41, 42, 43, 44]. Moreover, such a treatment also appears in other literature [45, 46, 47, 48, 49, 50]. In fact, some authors had noticed that the distribution obtained by such a way is not properly the distribution of the average occupation number [43]. The statistical distribution they obtained is of course not the Gentile distribution. Nevertheless, as analyzed above, the statistical distributions corresponding to such \( q \)-deformation schemes are Gentile distributions with various maximum occupation numbers determined by the deformation parameter \( q \).
In literature, there is also an alternative way for considering the $q$-deformation statistical distribution [54, 55, 56, 57]: replacing the number operator $N$ by $a^\dagger a$. In other words, in this treatment it is $\Lambda (N)$, rather than $N$, that plays the role of the number operator; for example, in the $q$-deformation scheme $aa^\dagger - qa^\dagger a = q^{-N}$, $[N] = (q^N - q^{-N}) / (q - q^{-1})$ plays the role of number operator rather than $N$ itself. The reason why such a replacement is adopted is that, as the authors state, if still using $N$ as the number operator, the corresponding statistical distribution is a nondeformed Bose-Einstein distribution [51, 52, 53]. This statement is incomplete. According to our above result, taking the scheme $aa^\dagger - qa^\dagger a = q^{-N}$ as an example, we can see that even if starting from $N$ rather than $a^\dagger a$ to construct the statistical distribution, there is still a deformed distribution when $q = e^{\frac{i\pi}{2l}}$: the Gentile distribution with the maximum occupation number $2l - 1$.

4 Conclusions and outlook

In this paper, we give a general discussion on the calculation of the statistical distribution from a given operator relation. Our result shows that as long as the relation between the number operator and creation and annihilation operators can be expressed as $a^\dagger b = \Lambda (N)$, i.e., $N$ is a function of quadratic product of the creation and annihilation operators, the corresponding statistical distribution is determined by the first two nonnegative integer zeroes of the function $\Lambda (N)$. The statistical distribution cannot be anything but the Gentile distribution, including the two limit cases of the Gentile distribution —- Bose-Einstein and Fermi-Dirac distributions.

On the basis of the general result of the relation between the operator relation and the statistical distribution, we systematically discuss the statistical distributions of various $q$-deformation schemes. We point out that the statistical distributions corresponding to all $q$-deformation schemes are Gentile distributions with various maximum occupation numbers which are determined by the deformation parameter $q$.

Our result shows that for the statistical distribution of the $q$-deformation scheme, there are two inaccurate results in the literature: (1) replacing $\langle f (N) \rangle$ by $f (\langle N \rangle)$ in the calculation of the statistical distribution, and (2) considering that the statistical average $\langle N_i \rangle$ gives only the Bose-Einstein distribution.

As an outlook, we would like to suggest several topics worth to be considered in further studies.

The operator relations considered in the present paper are only of the form $a^\dagger b = \Lambda (N)$ or $N = \Lambda^{-1} (a^\dagger b)$. Nevertheless, our result also implies that the operator relations corresponding to many intermediate statistical distributions are not the Gentile distribution, e.g., the Haldane-Wu statistics, and cannot be expressed as $a^\dagger b = \Lambda (N)$. Therefore, we also need to consider the
intermediate statistics corresponding to more general operator relations.

The statistical distributions corresponding to some $q$-deformation schemes, as shown in the present paper, are Gentile distributions with different maximum occupation numbers. In further studies, we can consider more general cases. Concretely, Gentile statistics is a special case of generalized statistics introduced in Ref. [67]. In the generalized statistics, the maximum occupation numbers of different quantum states can take on different values. Starting from the generalized statistics, one can construct a system with both bosonic states and fermionic states. For example, Ref. [67] constructs an exactly solvable phase transition model which shows that a system with one bosonic ground state and fermionic excited states can display Bose-Einstein-condensation type phase transition. By generalizing the result of the present paper, we can build a set of operator relations for the generalized statistics.

Recent researches show that there are some physical systems, such as a Heisenberg magnetic system, obey Gentile statistics rather than Bose-Einstein or Fermi-Dirac statistics [8]. As shown in the present paper, many $q$-deformation operator schemes correspond to Gentile statistics with a maximum occupation number determined by $q$. This inspires us to apply various $q$-deformation operator schemes to such physical systems.

The idea of $q$-analog has wide-ranging applications in many areas, e.g., in statistical mechanics [68, 69] and in probability theory [70, 71]. In physics, an important $q$-analog is the quantum algebra. The applications of quantum algebras cover many physical fields, such as nuclear physics [62, 72], gravity [73, 74], noncommutative space-time [75], supersymmetric Yang-Mills [76], string theory [77], quantum entanglement [78], and statistical physics [79, 80, 81]. The bridge between operator relation and statistical distribution allows us to study such systems more deeply.

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