Weight distributions of six families of 3-weight binary linear codes

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Abstract First, we correct a previous erroneous result about the exponential sum $\sum_{x \in \mathbb{F}_2} \chi_1(ax^{x^2+1} + bx)$. Second, we construct several families of binary linear codes of 3-weight and determine their weight distributions based on the results of the exponential sum. Most of the codes we construct can be used in secret sharing schemes.

Keywords Weight distribution · Binary linear code · Exponential sum

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1 Introduction

Throughout the paper, let $p$ be a prime and $\mathbb{F}_q$ be the finite field with $q = p^e$ elements and $\mathbb{F}_q^*$ be the multiplicative group. Fixed $g$ as a generator of $\mathbb{F}_q^*$. If $e$ is even, suppose $e = 2m$. Let $\alpha$ be a positive integer. Set $d = \gcd(e, \alpha)$ and $\gamma = g^{2d+1}$. Denote by $\langle \gamma \rangle$ the subgroup generated by $\gamma$.

Denote by $\mathbb{F}_p^n$ the vector space of all $n$-tuples over $\mathbb{F}_p$. For $x \in \mathbb{F}_p^n$, the Hamming weight $wt(x)$ is defined to be the number of nonzero coordinates of $x$. This research is supported by National Natural Science Foundation of China (No. 11701001) and Anhui Provincial Natural Science Foundation (No. 1908085MA02).

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For two vectors $x, y \in \mathbb{F}_p^n$, the Hamming distance $\delta(x, y)$ between them is defined as the weight $wt(x - y)$. If $C$ is a $k$-dimensional subspace of $\mathbb{F}_p^n$, then $C$ is called an $[n, k, \delta]$ $p$-ary linear code with minimum distance $\delta$ [16]. The vector in $C$ is called a codeword.

Let $A_i$ indicate the number of codewords with weight $i$ of $C$. The weight enumerator of $C$ is defined as the polynomial $1 + A_1 z + A_2 z^2 + \cdots + A_n z^n$. The sequence $1, A_1, \cdots, A_n$ is called the weight distribution of the code $C$. The weight distribution of linear codes is an significant research topic in coding theory. It can give the minimum distance of the code, hence the error correcting capability. It is well-known that the weight distributions of codes allow the computation of the error probability of error detection and correction with respect to some algorithms [18]. In recent years, many researchers focus on the calculation of weight distributions of linear codes [14,15,25,29].

We call a code $C$ is a $t$-weight code if the number of nonzero elements in the sequence $A_1, \cdots, A_n$ equals $t$. Linear codes with a few weights can be used in combinatorial designs [26], secret sharing [28], association schemes [2], authentication codes [13] and strongly regular graphs [3]. There have been many studies about linear codes with a few weights recently [17,19,20,21,22,24].

The remaining paper is arranged as follows. Sect. 2 presents an exponential sum in characteristic 2 together with its useful results and introduces a generalized method of constructing linear code by defining sets. Sect. 3 constructs binary linear codes with three weights and determines their weight distributions. Sect. 4 summarizes the paper.

2 Preliminaries

Here and after, let $p = 2$. For a positive integer $t$ dividing $e$, let $\text{Tr}_t$ be the trace function from $\mathbb{F}_q$ to $\mathbb{F}_{2^t}$. Namely, for each $x \in \mathbb{F}_q$,

$$\text{Tr}_t(x) = x + x^{2^t} + \cdots + x^{2^{t(e-1)}}.$$
Denote simply by $\text{Tr}$ the *absolute trace function* $\text{Tr}_1$. The *canonical additive character* $\chi_1$ over $\mathbb{F}_q$ is presented by

$$\chi_1(x) = \exp(\pi i \text{Tr}(x))$$

for all $x \in \mathbb{F}_q$. See [23] for more information about trace functions and additive characters over finite fields.

### 2.1 An exponential sum

For any $a \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$, an exponential sum $S_\alpha(a, b)$ is defined as follows.

$$S_\alpha(a, b) = \sum_{x \in \mathbb{F}_q} \chi_1 \left(ax^{2^{\alpha}+1} + bx\right).$$

The values of $S_\alpha(a, b)$ in odd characteristic ($p$ odd) were determined explicitly in [6,7,8]. On the other hand, the values in even characteristic were solved in [5,9].

However, it must be noted that the second part of the results of Theorem 5.3 in [9] (Also see Lemma 4 as below) is false. We have corrected it by proving Lemma 7.

Some results on $S_\alpha(a, b)$ are given in the lemmas as below. They play important roles in solving the parameters of the codes in the paper.

**Lemma 1. ([9, Theorem 4.1])** When $e/d$ is odd, we have

$$\sum_{x \in \mathbb{F}_q} \chi_1 \left(ax^{2^{\alpha}+1}\right) = 0$$

for each $a \in \mathbb{F}_q^*$.

**Lemma 2. ([9, Theorem 4.2])** Let $b \in \mathbb{F}_q^*$ and suppose $e/d$ is odd. Then

$$S_\alpha(a, b) = S_\alpha \left(1, bc^{-1}\right),$$

where $c \in \mathbb{F}_q^*$ is the unique element satisfying $c^{2^{\alpha}+1} = a$. Further we have

$$S_\alpha(1, b) = \begin{cases} 0, & \text{if } \text{Tr}_d(b) \neq 1, \\ \pm 2^{-\frac{d-1}{2}}, & \text{if } \text{Tr}_d(b) = 1. \end{cases}$$
Lemma 3. ([9], Theorem 5.2) Let \( e/d \) be even so that \( e = 2m \) for some integer \( m \). Then

\[
S_{\alpha}(a, 0) = \begin{cases} 
(-1)^\frac{e}{d} 2^m, & \text{if } a \neq g^{t(2^d+1)} \text{ for any integer } t, \\
(-1)^\frac{e}{d} 2^{m+d}, & \text{if } a = g^{t(2^d+1)} \text{ for some integer } t,
\end{cases}
\]

where \( g \) is a generator of \( \mathbb{F}_q^* \).

Lemma 4. ([9], Theorem 5.3) Let \( b \in \mathbb{F}_q^* \) and suppose \( e/d \) is even so that \( e = 2m \) for some integer \( m \). Let \( f(x) = a^{2^\alpha}x^{2^\alpha} + ax \in \mathbb{F}_q[x] \). There are two cases.

1. If \( a \neq g^{t(2^d+1)} \) for any integer \( t \), then \( f \) is a permutation polynomial of \( \mathbb{F}_q \).

Let \( x_0 \) be the unique element satisfying \( f(x) = b^{2^\alpha} \). Then

\[
S_{\alpha}(a, b) = (-1)^\frac{e}{d} 2^m \chi_1 \left( ax_0^{2^\alpha+1} \right).
\]

2. If \( a = g^{t(2^d+1)} \) for some integer \( t \), then \( S_{\alpha}(a, b) = 0 \) unless the equation \( f(x) = b^{2^\alpha} \) is solvable. If this equation is solvable, with solution \( x_0 \) say, then

\[
S_{\alpha}(a, b) = \begin{cases} 
(-1)^\frac{e}{d} 2^{m+d} \chi_1 \left( ax_0^{2^\alpha+1} \right), & \text{if } \Tr_d(a) = 0, \\
(-1)^\frac{e}{d} 2^m \chi_1 \left( ax_0^{2^\alpha+1} \right), & \text{if } \Tr_d(a) \neq 0.
\end{cases}
\]

The result of Part 2 of Lemma 4 is false. The corresponding correct result is shown in Lemma 7. To prove Lemma 7, we need the following Lemma 5 and Lemma 6.

Lemma 5 ([9, Lemma 4.2]) Denote by \( \chi_1 \) the canonical additive character of \( \mathbb{F}_q \) with \( q = p^e, p \) any prime. Let \( a \in \mathbb{F}_q \) be arbitrary and let \( d \) be some integer dividing \( e \). Then

\[
\sum_{x \in \mathbb{F}_q} \chi_1(ax) = \begin{cases} 
p^d, & \text{if } \Tr_d(a) = 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Lemma 6 ([9, Lemma 2.1]) Let \( d = \gcd(e, \alpha) \). Then

\[
\gcd(2^\alpha + 1, 2^e - 1) = \begin{cases} 
1, & \text{if } \frac{e}{d} \text{ is odd}, \\
2^d - 1, & \text{if } \frac{e}{d} \text{ is even}.
\end{cases}
\]

Lemma 7. Let \( b \in \mathbb{F}_q^* \) and suppose \( e/h \) is even so that \( e = 2m \) for some integer \( m \). Let \( f(x) = a^{2^\alpha}x^{2^\alpha} + ax \in \mathbb{F}_q[x] \). If \( a = g^{t(2^d+1)} \) for some integer \( t \),
then $S_\alpha(a, b) = 0$ unless the equation $f(x) = b^{2^\alpha}$ is solvable. If this equation is solvable, with solution $x_0$ say, then

$$S_\alpha(a, b) = -(-1)^\frac{m}{2} 2^m \alpha \chi_1 \left( a x_0^{2^\alpha + 1} \right).$$

**Proof.** Just as in the proof of Theorem 5.3 in [9], we have

$$S_\alpha(a, b)S_\alpha(a, 0) = \sum_{x \in \mathbb{F}_q} \left( \chi_1 \left( a x^{2^\alpha + 1} + b x \right) \sum_{x \in \mathbb{F}_q} \chi_1 \left( y^{2^\alpha} (f(x) + b^{2^\alpha}) \right) \right).$$

If $f(x) = b^{2^\alpha}$ has no solutions in $\mathbb{F}_q$, then the inner sum is zero, and so is $S_\alpha(a, b)$. Otherwise, overall, there are $2^{2d}$ solutions given by $x = x_0 + c$, where $f(x_0) = b^{2^\alpha}, f(\beta) = 0$ and $c \in \mathbb{F}_{2^{2d}}$. So we have

$$S_\alpha(a, b)S_\alpha(a, 0) = q \sum_{c \in \mathbb{F}_{2^{2d}}} \chi_1 \left( a(x_0 + \beta c)^{2^\alpha + 1} + b(x_0 + \beta c) \right).$$

After calculation, we get $\text{Tr} \left( a x_0^{2^\alpha + 1} + b x_0 \right) = \text{Tr} \left( a x_0^{2^\alpha + 1} \right)$ and

$$\text{Tr} \left( a(x_0 + \beta c)^{2^\alpha + 1} + b(x_0 + \beta c) \right) = \text{Tr} \left( a x_0^{2^\alpha + 1} + b x_0 \right) + \text{Tr} \left( a(\beta c)^{2^\alpha + 1} \right).$$

So we get

$$S_\alpha(a, b)S_\alpha(a, 0) = q \chi_1 \left( a x_0^{2^\alpha + 1} \right) \sum_{c \in \mathbb{F}_{2^{2d}}} \chi_1 \left( a \beta^{2^\alpha + 1} c^{2^\alpha + 1} \right).$$

By Lemma 6, we know $\gcd(2^\alpha + 1, 2^{2d} - 1) = 2^d + 1$. So we have $x^{2^\alpha + 1}$ is a permutation monomial over $\mathbb{F}_{2^{2d}}$. By a change of variable, we get

$$\sum_{c \in \mathbb{F}_{2^{2d}}} \chi_1 \left( a \beta^{2^\alpha + 1} c^{2^\alpha + 1} \right) = \sum_{c \in \mathbb{F}_{2^{2d}}} \chi_1 \left( a \beta^{2^\alpha + 1} \left( c^{2^\alpha + 1} \right)^{2^d + 1} \right) = \sum_{x \in \mathbb{F}_{2^{2d}}} \chi_1 \left( a \beta^{2^\alpha + 1} x^{2^d + 1} \right).$$

It is to see the map $N : \mathbb{F}_{2^{2d}}^* \rightarrow \mathbb{F}_{2^{d}}^*$ defined by $N(x) = x^{2^d + 1}$ is a surjective homomorphism. Therefore, we have

$$\sum_{c \in \mathbb{F}_{2^{2d}}} \chi_1 \left( a \beta^{2^\alpha + 1} c^{2^\alpha + 1} \right) = 1 + (2^d + 1) \sum_{y \in \mathbb{F}_{2^{d}}^*} \chi_1 \left( a \beta^{2^\alpha + 1} y \right).$$

Because $f(\beta) = 0$, we have $(a \beta^{2^\alpha + 1})^{2^\alpha - 1} = 1$. It is well-known that

$$\gcd(2^\alpha - 1, 2^e - 1) = 2^{\gcd(e, \alpha)} - 1 = 2^d - 1.$$
So \((a\beta^{2^n}+1)^{2^d-1} = 1\), which means that \(a\beta^{2^n}+1 \in \mathbb{F}_{2^d}\). Hence \(\text{Tr}_d(a\beta^{2^n}+1) = 0\) since \(e/d\) is even. By Lemma 5, \(\sum_{c \in \mathbb{F}_{2^d}} \chi_1 (a\beta^{2^n}+1c^{2^n}+1) = 2^{2d}\) and
\[
S_\alpha(a,b)S_\alpha(a,0) = q\chi_1 (ax^{2^n}+1) 2^{2d}.
\]
Dividing by \(S_\alpha(a,0)\), the claimed result follows. The proof is finished.

Two examples are given below to test the correctness of Lemma 7.

**Example 1** Let \((\mathbb{F}_q, \alpha, a, b) = (\mathbb{F}_{2^6}, 1, g^3, g^3+g^{33})\). It is easy to see \(f(1) = b^{2^n}\).
By Magma, we have \(\text{Tr}_1(a) = 1 \neq 0\) and
\[
S_\alpha(a,b) = \sum_{x \in \mathbb{F}_q} \chi_1 (g^3x^3 + (g^3 + g^{13})x) = -16.
\]

**Example 2** Let \((\mathbb{F}_q, \alpha, a, b) = (\mathbb{F}_{2^6}, 1, g^9, g^9+g^{36})\). It is easy to see \(f(1) = b^{2^n}\).
By Magma, we have \(\text{Tr}_1(a) = 0\) and
\[
S_\alpha(a,b) = \sum_{x \in \mathbb{F}_q} \chi_1 (g^9x^3 + (g^9 + g^{36})x) = 16.
\]

2.2 Linear codes constructed by defining set

A generic construction of linear codes was proposed by Ding et al. as below (10,12). Let \(D = \{d_1, d_2, \ldots, d_n\}\) be contained in \(\mathbb{F}_{p^e}^*\). A \(p\)-ary linear code \(C_D\) with length \(n\) is defined by

\[
C_D = \{(\text{Tr}(xd_1), \text{Tr}(xd_2), \ldots, \text{Tr}(xd_n)) : x \in \mathbb{F}_{p^e}\}.
\]

Here we call \(D\) the defining set of \(C_D\). Some linear codes with a few weights have been constructed by the method (11,15,27,29,30).

The method to construct linear codes was advanced by Li et al. in (24). Let \(D = \{d_1, d_2, \ldots, d_n\}\) be a subset of \(\mathbb{F}_{p^e}^* \{\{0,0,\ldots,0\}\}\) with a positive integer \(s\). For \(u, v \in \mathbb{F}_{p^e}^*\), let \(u \circ v\) be the inner product of them. Set \(u = (u_1, u_2, \ldots, u_s), v = (v_1, v_2, \ldots, v_s)\). Formally,
\[
u \circ v = u_1v_1 + u_2v_2 + \cdots + u_sv_s.
\]

By extension a \(p\)-ary linear code \(C_D\) with length \(n\) can be defined as follows:

\[
C_D = \{(\text{Tr}(x \circ d_1), \text{Tr}(x \circ d_2), \ldots, \text{Tr}(x \circ d_n)) : x \in \mathbb{F}_{p^e}\}.
\]

We also call \(D\) the defining set. By the generalized method, some classes of linear codes were constructed (1,17).
3 Six families of binary linear codes

In this section, we construct six families of binary linear codes by choosing proper defining sets and determine their parameters using the results of exponential sums.

3.1 Choosing defining sets

Let $\alpha = h$ be a proper divisor of $e$. For $u \in \mathbb{F}_q^*$, $v \in \mathbb{F}_q$, to construct linear codes $C_D$ defined by (2), the defining set $D$ is chosen to be

$$D = D_{(u,v)} = \{(x, y) \in \mathbb{F}_q^2 \setminus \{(0, 0)\} : \text{Tr}(ux^{2^h+1} + vy) = 0\}.$$ 

Let $n = |D_{(u,v)}|$, the length of the linear code $C_{D_{(u,v)}}$. See the lemma below for the length of the code.

**Lemma 8.** Let $u \in \mathbb{F}_q^*$, $v \in \mathbb{F}_q$. Then,

$$n = |D_{(u,v)}| = \begin{cases} \frac{1}{2}q^2 + \frac{1}{2}qS(u, 0) - 1, & \text{if } v = 0, \\ \frac{1}{2}q^2 - 1, & \text{if } v \neq 0. \end{cases}$$

**Proof.** Here and after, $\zeta_2 = -1$. By definition, we have

$$|D_{(u,v)}| = \frac{1}{2} \sum_{x, y \in \mathbb{F}_q} \sum_{z \in \mathbb{F}_2} \zeta_2^{\text{Tr}(z(ux^{2^h+1} + vy))} - 1$$

$$= \frac{1}{2} \sum_{x, y \in \mathbb{F}_q} (1 + \zeta_2^{\text{Tr}(ux^{2^h+1} + vy)}) - 1$$

$$= \frac{1}{2}q^2 + \frac{1}{2} \sum_{x, y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2^h+1} + vy)} - 1$$

$$= \frac{1}{2}q^2 + \frac{1}{2} \sum_{y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(vy)} \sum_{x \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2^h+1})} - 1.$$ 

If $v = 0$, then

$$|D_{(u,0)}| = \frac{1}{2}q^2 + \frac{1}{2} \sum_{x \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2^h+1})} - 1$$

$$= \frac{1}{2}q^2 + \frac{1}{2}qS(u, 0) - 1.$$ 

If $v \neq 0$, then $\sum_{y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(vy)} = 0$. So $|D_{(u,v)}| = \frac{1}{2}q^2 - 1$. We complete the proof.
3.2 Weight distribution of $C_{D(u,v)}$

We shall give the weight distributions of the linear code $C_{D(u,v)}$.
For $(a, b) \in \mathbb{F}_q^2$, let $c_{(a,b)}$ be the corresponding codeword in $C_{D(u,v)}$. Namely,

$$c_{(a,b)} = (\text{Tr}(ax + by))(x,y) \in D(u,v).$$

About the weight $wt(c_{(a,b)})$, we have the following lemma.

**Lemma 9.** Let $(a, b) \in \mathbb{F}_q^2$. There are two cases.

1. If $(a, b) \neq 0$, $v = 0$, then

$$wt(c_{(a,b)}) = \begin{cases} \frac{1}{4}q(q + S(u,0)), & \text{if } b \neq 0, \\ \frac{1}{4}q(q + S(u,0) - S(u,a)), & \text{if } a \neq 0, b = 0. \end{cases}$$

2. If $(a, b) \neq 0$, $v \neq 0$, then

$$wt(c_{(a,b)}) = \begin{cases} \frac{1}{4}q^2, & \text{if } a \neq 0, b = 0, \\ \frac{1}{4}q(q - S(u,a)), & \text{if } b = v, \\ \frac{1}{4}q^2, & \text{if } b \neq v, b \neq 0. \end{cases}$$

**Proof.** Firstly, we set

$$N(a, b) = \{(x, y) \in \mathbb{F}_q^2 : \text{Tr}(ux^{2h+1} + vy) = 0, \text{Tr}(ax + by) = 0\}.$$

$$|N(a, b)| = \frac{1}{4} \sum_{x, y \in \mathbb{F}_q} \sum_{z_1 \in \mathbb{F}_q} \zeta_2^{\text{Tr}(z_1(ux^{2h+1} + vy))} \sum_{z_2 \in \mathbb{F}_q} \zeta_2^{\text{Tr}(z_2(ax + by))}$$

$$= \frac{1}{4} \sum_{x, y \in \mathbb{F}_q} (1 + \zeta_2^{\text{Tr}(ux^{2h+1} + vy)})(1 + \zeta_2^{\text{Tr}(ax + by)})$$

$$= \frac{1}{4}q^2 + \frac{1}{4} \sum_{x, y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2h+1} + vy)} + \frac{1}{4} \sum_{x, y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2h+1} + ax + by + vy)}.$$

If $v = 0$, then

$$|N(a, b)| = \frac{1}{4}q^2 + \frac{1}{4} \sum_{x, y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2h+1})} + \frac{1}{4} \sum_{x, y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2h+1} + ax + by)}$$

$$= \frac{1}{4}q^2 + \frac{1}{4} \sum_{x \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2h+1})} + \frac{1}{4} \sum_{x \in \mathbb{F}_q} \zeta_2^{\text{Tr}(by)} \sum_{x \in \mathbb{F}_q} \zeta_2^{\text{Tr}(ux^{2h+1} + ax)}$$

$$= \frac{1}{4}q^2 + \frac{1}{4}qS(u,0) + \frac{1}{4}S(u,a) \sum_{y \in \mathbb{F}_q} \zeta_2^{\text{Tr}(by)}.$$

Then we have

$$|N(a, b)| = \begin{cases} \frac{1}{4}q(q + S(u,0)), & \text{if } b \neq 0, \\ \frac{1}{4}q(q + S(u,0) + S(u,a)), & \text{if } a \neq 0, b = 0. \end{cases}$$
Note that \( wt(c(a,b)) = n - |N(a,b)| + 1 \). By Lemma 8, we have
\[
wt(c(a,b)) = \begin{cases} 
\frac{1}{4}q(q + S(u,0)), & \text{if } b \neq 0, \\
\frac{1}{4}q(q + S(u,0) - S(u,a)), & \text{if } a \neq 0, b = 0.
\end{cases}
\]

If \( v \neq 0 \), then
\[
|N(a,b)| = \frac{1}{4}q^2 + \frac{1}{4} \zeta_2 \sum_{x,y \in \mathbb{F}_q} \text{Tr}(ux^{2^{b+1} + ax + vy + by})
\]
\[
= \frac{1}{4}q^2 + \frac{1}{4} \zeta_2 \sum_{y \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q} \text{Tr}(ux^{2^{b+1} + ax})
\]
\[
= \frac{1}{4}q^2 + \frac{1}{4} S(u,a) \zeta_2 \sum_{y \in \mathbb{F}_q} \text{Tr}((v+b)y).
\]

Then we have
\[
|N(a,b)| = \begin{cases} 
\frac{1}{4}q^2, & \text{if } a \neq 0, b = 0, \\
\frac{1}{4}q(q + S(u,a)), & \text{if } b = v, \\
\frac{1}{4}q^2, & \text{if } b \neq v, b \neq 0.
\end{cases}
\]

By Lemma 8, we have
\[
wt(c(a,b)) = \begin{cases} 
\frac{1}{4}q^2, & \text{if } a \neq 0, b = 0, \\
\frac{1}{4}q(q - S(u,a)), & \text{if } b = v, \\
\frac{1}{4}q^2, & \text{if } b \neq v, b \neq 0.
\end{cases}
\]

The proof is finished.

**Case 1.** \( v = 0 \), \( e/h \equiv 1 \) (mod 2).

In this case, we have the following theorem.

**Theorem 1.** Let \( v = 0 \) and \( e/h \equiv 1 \) (mod 2). The code \( CD_{(u,0)} \) is a \([2^{2e-1} - 1, 2e, \delta]\) linear code over \( \mathbb{F}_2 \) with the weight distribution in Table 1, where \( \delta = \frac{1}{4}q(q - 2^{e+h}) \).

| Weight \( w \) | Multiplicity \( A \) |
|---------------|------------------|
| \( \frac{1}{4}q^2 \) | \( q^2 - 2^{e-h} - 1 \) |
| \( \frac{1}{4}q(q - 2^{e+h}) \) | \( 2^{e-h-1} + 2^{e-h-1} \) |
| \( \frac{1}{4}q(q + 2^{e+h}) \) | \( 2^{e-h-1} - 2^{e-h-1} \) |
Proof. Let the notations and symbols be as above. By Lemma 1, Lemma 2 and Lemma 9, the three values of $\text{wt}(c_{(a,b)})$ are $\omega_1, \omega_2, \omega_3$, where

$$\begin{align*}
\omega_1 &= \frac{1}{4}q^2, \\
\omega_2 &= \frac{1}{4}q(q-2^{e-h}) \\
\omega_3 &= \frac{1}{4}q(q+2^{e-h}).
\end{align*}$$

Recall that $A_{\omega_i}$ is the multiplicity of $\omega_i$. By the computation of $|N(a, b)|$ and the first two Pless Power Moment ([16], P. 260), we obtain the system of linear equations as follows:

$$\begin{align*}
A_{\omega_1} &= q(q-1) + q - 2^{e-h} - 1 \\
A_{\omega_2} + A_{\omega_3} &= 2^{e-h} \\
\omega_1 A_{\omega_1} + \omega_2 A_{\omega_2} + \omega_3 A_{\omega_3} &= \frac{1}{2}q^2 \left( \frac{1}{2}q^2 - 1 \right).
\end{align*}$$

Solving the system, we get

$$\begin{align*}
A_{\omega_1} &= q^2 - 2^{e-h} - 1 \\
A_{\omega_2} &= 2^{e-h-1} + \frac{2^{e-h}}{2} \\
A_{\omega_3} &= 2^{e-h-1} - \frac{2^{e-h}}{2}.
\end{align*}$$

Then we get the weight distribution of Table 1. For $(a, b) \in \mathbb{F}_q^2$, $(a, b) \neq (0, 0)$, we can see that $\text{wt}(c_{(a,b)}) > 0$. Hence, the dimension of this code $C_{D(u,0)}$ of Theorem 1 is equal to $2e$. We complete the proof.

Example 3 Let $(q, h, u, v) = (2^5, 1, 1, 0)$. Then, the corresponding code $C_{D(u,0)}$ has parameters $[511, 10, 192]$, weight enumerator $1 + 10x^{192} + 1007x^{256} + 6x^{320}$.

Case 2. $v = 0, u \notin \langle \gamma \rangle$, $e/h \equiv 0 \pmod{2}$.

In this case, we have the following theorem.

Theorem 2. Let $v = 0, u \notin \langle \gamma \rangle$ and $e/h \equiv 0 \pmod{2}$. The code $C_{D(u,0)}$ is an $[n, 2e]$ linear code over $\mathbb{F}_2$ with the weight distribution in Table 2, where $n = \frac{1}{2}q(q + (-1)^{m}2^m) - 1$.

| Weight w       | Multiplicity A |
|----------------|----------------|
| 0              | 1              |
| $\frac{1}{2}q(q + (-1)^{m}2^m)$ | $q(q-1)$ |
| $\frac{1}{2}q^2$ | $\frac{1}{2}q(q+(-1)^{m}2^{m-1}) - 1$ |
| $\frac{1}{2}q(q + (-1)^{m}2^{m+1})$ | $\frac{1}{2}q(q-(-1)^{m}2^{m-1})$ |
Proof. Let the notations and symbols be as above. By Lemma 3, Lemma 4 and Lemma 9, the three values of wt\((c_{(a, b)})\) are \(\omega_1, \omega_2, \omega_3\), where

\[
\begin{align*}
\omega_1 &= \frac{1}{4}q(q + (-1)^{\frac{m}{2}} 2^m), \\
\omega_2 &= \frac{1}{4}q^2, \\
\omega_3 &= \frac{1}{4}q(q + (-1)^{\frac{m}{2}} 2^{m+1}).
\end{align*}
\]

By the computation of \(|N(a, b)|\) and the first two Pless Power Moment (\[16\], P. 260), we obtain the system of linear equations as follows:

\[
\begin{align*}
A_\omega_1 &= q(q - 1) \\
A_\omega_2 + A_\omega_3 &= q - 1 \\
\omega_1A_\omega_1 + \omega_2A_\omega_2 + \omega_3A_\omega_3 &= \frac{1}{2}q^2n.
\end{align*}
\]

Solving the system, we get

\[
\begin{align*}
A_\omega_1 &= q(q - 1) \\
A_\omega_2 &= \frac{1}{4}q + (-1)^{\frac{m}{2}} 2^{m-1} - 1 \\
A_\omega_3 &= \frac{1}{2}q - (-1)^{\frac{m}{2}} 2^{m-1}.
\end{align*}
\]

Then we get the weight distribution of Table 2. For \((a, b) \in \mathbb{F}_q^2, (a, b) \neq (0, 0)\), we can see that \(wt(c_{(a, b)}) > 0\). Hence, the dimension of this code \(C_{D_{(a, b)}}\) of Theorem 2 is equal to \(2e\). We complete the proof.

Example 4. Let \((q, h, u, v) = (2^6, 1, g, 0)\). Then the corresponding code \(C_{D_{(g, 0)}}\) has parameters \([1791, 12, 768]\), weight enumerator \(1 + 36x^{768} + 4032x^{896} + 27x^{1024}\).

Case 3. \(v = 0, u \in \langle \gamma \rangle, e/h \equiv 0 \pmod{2}\).

In this case, we have the following theorem.

Theorem 3. Let \(v = 0, u \in \langle \gamma \rangle\) and \(e/h \equiv 0 \pmod{2}\). The code \(C_{D_{(v, 0)}}\) is an \([n, 2e]\) linear code over \(\mathbb{F}_2\) with the weight distribution in Table 3, where \(n = \frac{1}{4}q(q - (-1)^{\frac{m}{2}} 2^{m+h}) - 1\).

Table 3 The weight distribution of the codes of Theorem 3.
Proof. Let the notations and symbols be as above. By Lemma 3, Lemma 7 and Lemma 9, the three values of $wt(c_{(a,b)})$ are $\omega_1, \omega_2, \omega_3$, where

$$
\begin{align*}
\omega_1 &= \frac{1}{4} q(q - (-1)^{\frac{n}{2}} 2^{m+h}), \\
\omega_2 &= \frac{1}{4} q^2, \\
\omega_3 &= \frac{1}{4} q(q - (-1)^{\frac{n}{2}} 2^{m+h+1}).
\end{align*}
$$

Recall that $f(x) = a^2 x^{2h} + ax \in \mathbb{F}_q[x]$. In this case, the number of solutions of $f = 0$ in $\mathbb{F}_q$ is $2^{2h}$. So we have that $\dim(\ker f) = 2h$ and $\dim(f(\mathbb{F}_q)) = e - 2h$. By the computation of $|N(a, b)|$ and the first two Pless Power Moment ([16], P. 260), we obtain the system of linear equations as follows:

$$
\begin{align*}
A_\omega_1 &= q(q - 1) + q - 2e - 2h \\
A_\omega_2 + A_\omega_3 &= 2e - 2h - 1 \\
A_\omega_1 A_\omega_2 + A_\omega_1 A_\omega_3 + A_\omega_2 A_\omega_3 &= \frac{1}{2} q^2 n.
\end{align*}
$$

Solving the system, we get

$$
\begin{align*}
A_\omega_1 &= q^2 - 2e - 2h \\
A_\omega_2 &= 2e - 2h - 1 - (-1)^{\frac{n}{2}} 2^{m-h-1} - 1 \\
A_\omega_3 &= 2e - 2h + (-1)^{\frac{n}{2}} 2^{m-h-1}.
\end{align*}
$$

Then we get the weight distribution of Table 3. For $(a, b) \in \mathbb{F}_q^2$, $(a, b) \neq (0, 0)$, we can see that $wt(c_{(a,b)}) > 0$. Hence, the dimension of this code $C_{D(a,b)}$ of Theorem 2 is equal to $2e$. We complete the proof.

Example 5 Let $(q, h, u, v) = (2^6, 1, 1, 0)$. Then, the corresponding code $C_{D(1,0)}$ has parameters $[2559, 12, 1024]$, weight enumerator $1 + 9x^{1024} + 4080x^{1280} + 6x^{1536}$.

Example 6 Let $(q, h, u, v) = (2^6, 1, g^3, 0)$. Then the corresponding code $C_{D(g^3,0)}$ has parameters $[2559, 12, 1024]$, weight enumerator $1 + 9x^{1024} + 4080x^{1280} + 6x^{1536}$.

Case 4 $v \neq 0$, $e/h \equiv 1 \pmod{2}$.

Theorem 4. Let $v \neq 0$, $e/h \equiv 1 \pmod{2}$. The code $C_{D(v,v)}$ is a $[\frac{1}{2} q^2 - 1, 2e]$ linear code over $\mathbb{F}_2$ with the weight distribution in Table 4.

Proof. Let the notations and symbols be as above. By Lemma 2 and Lemma 9, for a nonzero codeword $c_{(a,b)}$, its weight $wt(c_{(a,b)})$ has three values
Table 4 The weight distribution of the codes of Theorem 4.

| Weight w     | Multiplicity A |
|--------------|----------------|
| 0            | 1              |
| $\frac{1}{2}q^2$ | $q^2 - 2^{e-h} - 1$ |
| $\frac{1}{2}q(q - 2^{e-h})$ | $2^{e-h} - 1 + 2^{e-h}$ |
| $\frac{1}{2}q(q + 2^{e-h})$ | $2^{e-h} - 1 - 2^{e-h}$ |

$\omega_1, \omega_2, \omega_3$, where

$$
\begin{align*}
\omega_1 &= \frac{1}{4}q^2, \\
\omega_2 &= \frac{1}{4}q(q - 2^{e-h}), \\
\omega_3 &= \frac{1}{4}q(q + 2^{e-h}).
\end{align*}
$$

By the computation of $\vert N(a,b) \vert$ and the first two Pless Power Moment ([10], P. 260), it is easy to get equations of the multiplicity $A_{\omega_i}$. They are listed as follows.

$$
\begin{align*}
A_{\omega_1} &= q(q - 1) + q - 2^{e-h} - 1 \\
A_{\omega_2} + A_{\omega_3} &= 2^{e-h} \\
\omega_1 A_{\omega_1} + \omega_2 A_{\omega_2} + \omega_3 A_{\omega_3} &= \frac{1}{2}q^2(\frac{1}{2}q^2 - 1).
\end{align*}
$$

Solving the system, we get

$$
\begin{align*}
A_{\omega_1} &= q^2 - 2^{e-h} - 1 \\
A_{\omega_2} &= 2^{e-h} \\
A_{\omega_3} &= 2^{e-h} - 2^{e-h}.
\end{align*}
$$

Then we get the weight distribution of Table 4. The dimension of this code $C_{D(u,v)}$ follows from that $wt(c(a,b)) > 0$ for each $((a, b)) \neq (0,0)$. The proof is finished.

**Remark.** Theorem 1 and Theorem 4 have different proving processes, but the codes in them have the same parameters and weight distributions.

**Example 7** Let $(q, h, u, v) = (2^5, 1, 1, 1)$. Then, the corresponding code $C_{D(1,1)}$ has parameters $[511, 10, 192]$, weight enumerator $1 + 10x^{192} + 1007x^{256} + 6x^{320}$.

**Case 5** $v \neq 0, u \not\in \langle \gamma \rangle$, $e/h \equiv 0 \pmod{2}$.

**Theorem 5.** Let $v \neq 0, u \not\in \langle \gamma \rangle$, $e/h \equiv 0 \pmod{2}$. The code $C_{D(u,v)}$ is a $[\frac{1}{2}q^2 - 1, 2e]$ linear code over $F_2$ with the weight distribution in Table 5.

**Proof.** Let the notations and symbols be as above. By Lemma 4 and Lemma 9, for a nonzero codeword $c(a,b)$, its weight $wt(c(a,b))$ has three values
Table 5 The weight distribution of the codes of Theorem 5.

| Weight $w$       | Multiplicity $A$ |
|------------------|------------------|
| 0                | 1                |
| $\frac{q^2}{2}$  | $q^2 - q - 1$    |
| $\frac{q(q - 2^m)}{4}$ | $\frac{q + 2^{m-1}}{2}$ |
| $\frac{q(q + 2^m)}{4}$ | $\frac{q - 2^{m-1}}{2}$ |

$\omega_1, \omega_2, \omega_3$, where

$$
\begin{align*}
\omega_1 &= \frac{1}{2}q^2, \\
\omega_2 &= \frac{1}{2}q(q - 2^m), \\
\omega_3 &= \frac{1}{2}q(q + 2^m).
\end{align*}
$$

By the computation of $|N(a, b)|$ and the first two Pless Power Moment ([16], P. 260), it is easy to get equations of the multiplicity $A_{\omega_i}$ of $\omega_i$. They are listed as follows.

$$
\begin{align*}
A_{\omega_1} &= q(q - 1) - 1, \\
A_{\omega_2} + A_{\omega_3} &= q, \\
\omega_1 A_{\omega_1} + \omega_2 A_{\omega_2} + \omega_3 A_{\omega_3} &= \frac{1}{2}q^2\left(\frac{1}{2}q^2 - 1\right).
\end{align*}
$$

Solving the system, we get

$$
\begin{align*}
A_{\omega_1} &= q^2 - q - 1, \\
A_{\omega_2} &= \frac{1}{2}q + 2^{m-1}, \\
A_{\omega_3} &= \frac{1}{2}q - 2^{m-1}.
\end{align*}
$$

Then we get the weight distribution of Table 5. The dimension of this code $C_{D(u, v)}$ follows from that $wt(c_{(a,b)}) > 0$ for each $((a, b) \neq (0, 0))$. The proof is finished.

**Example 8** Let $(q, h, u, v) = (2^6, 1, g, 1)$. Then, the corresponding code $C_{D(u, v)}$ has parameters $[2047, 12, 896]$, weight enumerator $1 + 36x^{896} + 4031x^{1024} + 28x^{1152}$.

**Case 6** $v \neq 0, u \in \langle \gamma \rangle$, $e/h \equiv 0 \pmod{2}$.

**Theorem 6.** Let $v \neq 0, u \in \langle \gamma \rangle$ and $e/h \equiv 0 \pmod{2}$. The code $C_{D(u, v)}$ is a $[\frac{1}{2}q^2 - 1, 2e]$ linear code over $\mathbb{F}_2$ with the weight distribution in Table 6.

**Proof.** Let the notations and symbols be as above. By Lemma 7 and Lemma 9, for a nonzero codeword $c_{(a,b)}$, its weight $wt(c_{(a,b)})$ has three values $\omega_1, \omega_2, \omega_3$, where

$$
\begin{align*}
\omega_1 &= \frac{1}{2}q^2, \\
\omega_2 &= \frac{1}{2}q(q - 2^{m+h}), \\
\omega_3 &= \frac{1}{2}q(q + 2^{m+h}).
\end{align*}
$$
By the computation of \(|N(a, b)|\) and the first two Pless Power Moment ([10], P. 260), it is easy to get equations about the multiplicity \(A_{\omega_i}\) of \(\omega_i\). They are listed as follows.

\[
\begin{align*}
A_{\omega_1} & = q^2 - 1 - 2^{e-2h} \\
A_{\omega_2} + A_{\omega_3} & = 2^{e-2h} \\
\omega_1 A_{\omega_1} + \omega_2 A_{\omega_2} + \omega_3 A_{\omega_3} & = \frac{1}{2} q^2 (\frac{1}{2} q^2 - 1).
\end{align*}
\]

Solving the system, we get

\[
\begin{align*}
A_{\omega_1} & = q^2 - 1 - 2^{e-2h} \\
A_{\omega_2} & = 2^{e-2h-1} + 2^{m-h-1} \\
A_{\omega_3} & = 2^{e-2h-1} - 2^{m-h-1}.
\end{align*}
\]

Then we get the weight distribution of Table 6. The dimension of this code \(C_{D_{(w,v)}}\) follows from that \(wt(c_{(a,b)}) > 0\) for each \((a, b) \neq (0, 0)\). The proof is finished.

**Example 9** Let \((q, h, u, v) = (2^6, 1, 1, 1)\). Then, the corresponding code \(C_{D_{(1,1)}}\) has parameters [2047, 12, 768], weight enumerator \(1 + 10x^{768} + 4079x^{1024} + 6x^{1280}\).

The above examples have been verified by Magma.

### 4 Concluding Remarks

We correct a wrong result of an exponential sum and construct six families of binary linear codes by the generalised method of defining set. Using the exponential sum theory, we determine the weight distributions of the codes. It is shown that the presented linear codes have three nonzero weights. By Magma, some concrete examples are given to verify the correctness of their corresponding results.
Denote by $w_{\min}$ and $w_{\max}$ the minimum and maximum nonzero weight of the linear code $C_{D(u,v)}$, respectively. If the code $C_{D(u,v)}$ satisfies one of the following five conditions, then it can be easily checked that

$$\frac{w_{\min}}{w_{\max}} > \frac{1}{2}.$$

1. In Theorems 1 or Theorems 4, and $e > h + 1$.
2. In Theorems 2, and $e > 3 - (-1)^{\frac{m}{2}}$.
3. In Theorems 3, and $m > h + 1 + (-1)^{\frac{m}{2}}$.
4. In Theorems 5, and $m > 1$.
5. In Theorems 6, and $m > h + 1$.

By the results in [28], most of the codes $C_{D(u,v)}$ in the paper are suitable for constructing secret sharing schemes with interesting properties.

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