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The volume of an infinitesimally flexible polyhedron is a multiple root of its volume polynomial

1. In [1] among the many open problems there is one under the number 6 in which the following property of polyhedra is announced as a suggestion:

Theorem. Algebraic volume of an infinitesimally non-rigid polyhedron is a multiple root of its volume polynomial.

Here we present a sketch of a proof of this assertion in three-dimensional space. Because any flexible polyhedron is infinitesimally (inf.) bendable one then as a corollary of our theorem we can affirm that the volume of any flexible polyhedron is a multiple root of its volume polynomial.

2. Let \( P \) be a simplicial orientable inf. bendable polyhedron in \( \mathbb{R}^3 \) of any combinatorial structure \( K \). Let points \( M_i(x_i, y_i, z_i), 1 \leq i \leq n \), be its vertices. Inf. bendability (i.e., flexibility, non-rigidity) of the polyhedron \( P \) means that there are vectors \( Z_i = \{\xi_i, \eta_i, \zeta_i\} \) attached to vertices \( M_i \) satisfying the equations

\[
(x_i - x_j)(\xi_i - \xi_j) + (y_i - y_j)(\eta_i - \eta_j) + (z_i - z_j)(\zeta_i - \zeta_j) = 0, \tag{1}
\]

written for all edges \( (i, j) \in E \) of \( P \) (where by \( E \) we note the set of all edges of \( P \), and indices correspond to numbers \( i \) and \( j \) of end vertices of edges). Under the deformation

\[
(x_i, y_i, z_i) \rightarrow (x_i + \varepsilon\xi_i, y_i + \varepsilon\eta_i, z_i + \varepsilon\zeta_i) \tag{2}
\]

the length \( l_{ij} \) of the edge \( (i, j) \) changes to an inf. small \( o(\varepsilon) \), \( \varepsilon \rightarrow 0 \).

The vectors \( Z_i \) are found as a solution of homogenous linear system of \( |E| \) equations for \( 3n \) unknowns (in reality for \( 3n - 6 \) unknowns because 6 unknowns can be fixed by a motion of \( P \) as a solid). The number of edges is \( |E| = 3n + 6g - 6 \) where \( g \) is topological genus of \( P \). So for the existence of a nontrivial solution of our system it is necessary and sufficient that the rank of the main matrix of system be less then the number of unknowns equal to \( 3n - 6 \). So in the case of inf. bendability of \( P \) the coordinates of its vertices should satisfy to one or more polynomial equations of the form

\[
det D(x) = 0 \quad (g = 0), \quad det D_1(x) = 0, ..., det D_s(x) = 0, \quad s = C_{3n-6}^{3n+6g-6}. \tag{3}
\]

It is important to remark that if an inf. small deformation is trivial that is it an initial velocity vector of a motion of \( P \) as a solid body then the distances between all vertices of \( P \) are changed by the order \( o(\varepsilon), \varepsilon \rightarrow 0 \). With an additional condition the inverse statement is true too:

Lemma 1. If a polyhedron is not a situated on a plane then for any non-trivial inf. deformation there exists a small of nonzero length diagonal whose length changes to the exact order \( O(\varepsilon), \varepsilon \rightarrow 0 \).

Remarks. 1) A small diagonal is the one between two vertices of two faces with a common edge. In some cases this diagonal is in reality an edge. The
lemma is equivalent to the affirmation that there is a dihedral angle changing as an exact $O(\varepsilon)$, $\varepsilon \to 0$.

2) If all vertices of $P$ are situated on a plane then can be that under a nontrivial inf. bending the lengths of all diagonals are changing to the order $o(\varepsilon)$, $\varepsilon \to 0$.

3. Let’s continue the proof. Recall that a volume polynomial of a polyhedron $P$ is any polynomial of the form

$$Q(V, l) = V^{2N} + \sum_{i=1}^{N} a_i(l)V^{2N-2i},$$

where coefficients $a_i(l)$ are some polynomials too in the set $l$ of squares of lengths of edges of $P$ such that after the substitution in (4) instead the volume $V$ and the squares of lengths of edges their expressions in coordinates of vertices $x = (x_1, y_1, z_1, ..., z_n)$ the value $Q(V(x), l(x))$ becomes identically zero relatively to all coordinates.

In [1] one can find a proof of existence theorem for such a polynomial with a detailed description of the background history.

Let’s consider a new polyhedron $P_\varepsilon$ with vertices coordinates $(x_i + \varepsilon \xi_i, y_i + \varepsilon \eta_i, z_i + \varepsilon \zeta_i)$ and with the same combinatorial structure $K$. For the squares of lengths of its edges we have $l_{ij}^2(\varepsilon) = l_{ij}^2 + \varepsilon^2 L_{ij}^2$, where $L_{ij}^2 = (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2$. Renumber all edges by the index $k$, $1 \leq k \leq |E|$ and compose for $P_\varepsilon$ its volume polynomial:

$$Q(V, l, \varepsilon) = V^{2N}(\varepsilon) + \sum_{i=1}^{N} a_i(l_\varepsilon)V^{2N-2i}(\varepsilon) = 0, \forall \varepsilon.$$  

Evidently for $\varepsilon = 0$ this polynomial becomes a volume polynomial for the initial polyhedron with $V_0 = V(0)$. The derivation of (5) by $\varepsilon$ gives

$$Q'_{V}V'_\varepsilon + 2\varepsilon \sum_{i=1}^{N} \left( \frac{\partial a_i(l_\varepsilon)}{\partial l_k} L_{ik}^2 \right) V^{2N-2i} = 0.$$ 

If $\lim_{\varepsilon \to 0} V'_\varepsilon \neq 0$ then one has $Q'_{V}(V_0) = 0$, so the multiplicity of the root $V = V_0$ is proven.

Let now be $V_2'(0) = 0$. Algebraic volume of a polyhedron is defined as the sum of oriented volumes of tetrahedra with a common vertex and the bases on oriented faces of the polyhedron. Let this common vertex be taken as a vertex of $P$ and choose this vertex as the origin for coordinate system. Then the volume of a tetrahedron with vertices $M_i, M_j, M_k$ is given by the formula

$$V_{ijk}(\varepsilon) = \frac{1}{6} \det \begin{pmatrix} x_i + \varepsilon \xi_i & y_i + \varepsilon \eta_i & z_i + \varepsilon \zeta_i \\ x_j + \varepsilon \xi_j & y_j + \varepsilon \eta_j & z_j + \varepsilon \zeta_j \\ x_k + \varepsilon \xi_k & y_k + \varepsilon \eta_k & z_k + \varepsilon \zeta_k \end{pmatrix}.$$ 

By calculating these determinants for all faces for the total volume we have a presentation:

$$V(P_\varepsilon) = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_3.$$  

2
By the supposition \( V'_1(0) = 0 \), so \( V_1 = 0 \). If \( V_3 \neq 0 \) then two isometric polyhedra \( P_\varepsilon \) and \( P_{-\varepsilon} \) have different volumes that is the polynomial \(^{\mathfrak{5}}\) has two different roots tending under \( \varepsilon \to 0 \) to the same root \( V_0 \) of the initial volume polynomial. Thus the volume of our \( P \) is a multiple root of its volume polynomial.

Now we consider the case \( V_1 = V_3 = 0 \) so \( V(P_\varepsilon) \) is

\[
V(P_\varepsilon) = V_0 + \varepsilon^2 V_2.
\]

We suppose also that \( V_0 \neq 0 \) (because the root \( V = 0 \) is multiple already). Let \( Q'_V(0) \neq 0 \). Then the equation \(^{\mathfrak{5}}\) determinates \( V \) as an analytical implicit function \( V = V(l) \) of \( |E| \) variables \( l_{ij}^2 = l_{ij}^2 + \varepsilon^2 L_{ij}^2 \) as independent arguments in some full neighborhood of values of edge lengths of \( P \) which are not related with coordinates of vertices (recall that in general only some collections of non-negative numbers can be presented as squares of lengths of a polyhedron).

Accordingly \(^{\mathfrak{2}}\) any small diagonal \( d \) satisfies an polynomial equation of the form

\[
D(l, V, d) = A_0(l, V)d^{2K} + A_1(l, V)d^{2K-2} + \ldots + A_K(l, V) = 0,
\]

where coefficients \( A_i, 1 \leq i \leq K \) are some polynomials too in squares of edge lengths and square of the polyhedron’s volume which not all are identically zero. For polyhedra \( P_\varepsilon \) the coefficients \( A_i \) in (ref\(^{\mathfrak{6}}\)) are presentable as follows

\[
A_i = a_{i0} + ai_1\varepsilon^2 + \ldots + a_{im}\varepsilon^{2n},
\]

and not all coefficients \( a_{ij} \) are zero.

**Lemma 2.** Any small diagonal of polyhedra \( P_\varepsilon \) is represented in the form \( d = d_0 + o(\varepsilon), d_0 \neq 0 \).

In the proof of lemma one should to distinguish two cases 1) there is at least one coefficient \( a_{i0} \neq 0 \) and 2) all coefficients \( a_{i0} = 0 \). In the first case we consider for \( D \) from \(^{\mathfrak{6}}\) its derivative \( D'_d(l, V, d) \). If the derivative is not zero then \( d \) is expressed from \(^{\mathfrak{6}}\) as an implicit function and its Taylor expansion consists only of powers of \( \varepsilon^{2m} \). If this derivative is zero then \( d \) satisfies a similar equation of 2 powers less (after the cancellation by \( d \neq 0 \)) and one can continue the same considerations and finally we arrive either to a case with the possibility to present \( d \) by a Taylor expansion with even powers of \( \varepsilon \) or to a biquadratic equation. In the case 2) we should reduce all the coefficients by the maximal common degree \( \varepsilon^{2m} \) and we arrive to the case 1).

Now we note that by lemma 1 there exists at least one small diagonal of the form \( d = d_0 + a\varepsilon, a \neq 0 \), which is in contradiction with lemma 2. So the supposition \( Q'_V \neq 0 \) is not true.

Let’s remark that a seeming theorem should be true for inf. bendable polyhedra in any space \( R^n, n > 3 \) because by \(^{\mathfrak{3}}\) for them there exist volume polynomials too, but for the moment we don’t have a needed affirmation about the existence of equations for small diagonals similar to \(^{\mathfrak{9}}\). It would be interesting also to find an algebraic and geometrical interpretation for the multiplicity order of the volume as a root of a volume equation.
References

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[3] A.A. Gaifullin Generalization of Sabitov’s theorem to polyhedra of arbitrary dimensions. Discr. and Comput. Geometry, 52:2 (2014), 195-220.