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FRAGMENTED DEFORMATIONS OF PRIMITIVE MULTIPLE CURVES

JEAN–MARC DRÉZET

RESUME. A primitive multiple curve is a Cohen-Macaulay irreducible projective curve $Y$ that can be locally embedded in a smooth surface, and such that $Y_{red}$ is smooth.

This paper studies the deformations of $Y$ to curves with smooth irreducible components, when the number of components is maximal (it is then the multiplicity $n$ of $Y$).

We are particularly interested in deformations to $n$ disjoint smooth irreducible components, which are called fragmented deformations. We describe them completely. We give also a characterization of primitive multiple curves having a fragmented deformation.

SUMMARY

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1. INTRODUCTION

A primitive multiple curve is an algebraic variety $Y$ over $\mathbb{C}$ which is Cohen-Macaulay, such that the induced reduced variety $C = Y_{red}$ is a smooth projective irreducible curve, and that every closed point of $Y$ has a neighborhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [1]. The simplest examples are infinitesimal neighborhoods of projective smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces, cf. [2], theorem 7.1).

Let $Y$ be a primitive multiple curve with associated reduced curve $C$, and suppose that $Y \neq C$. Let $\mathcal{I}_C$ be the ideal sheaf of $C$ in $Y$. The multiplicity of $Y$ is the smallest integer $n$ such that $\mathcal{I}_C^n = 0$. We have then a filtration

$$C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$$
where $C_i$ is the subscheme corresponding to the ideal sheaf $I_C^i$ and is a primitive multiple curve of multiplicity $i$. The sheaf $L = I_C/I_C^2$ is a line bundle on $C$, called the line bundle on $C$ associated to $Y$.

The deformations of double (i.e. of multiplicity 2) primitive multiple curves (also called ribbons) to smooth projective curves have been studied in [11]. In this paper we are interested in deformations of primitive multiple curves $Y = C_n$ of any multiplicity $n \geq 2$ to reduced curves having exactly $n$ components which are smooth ($n$ is the maximal number of components of deformations of $Y$). In this case the number of intersection points of two components is exactly $-\deg(L)$. We give some results in the general case (no assumption on $\deg(L)$) and treat more precisely the case $\deg(L) = 0$, i.e. deformations of $Y$ to curves having exactly $n$ disjoint irreducible components.

1.1. Motivation – Let $\pi: C \to S$ be a flat projective morphism of algebraic varieties, $P$ a closed point of $S$ such that $\pi^{-1}(P) \simeq Y$, $O_C(1)$ a very ample line bundle on $C$ and $P$ a polynomial in one variable with rational coefficients. Let

$$\tau: M_{O_C(1)}(P) \longrightarrow S$$

be the corresponding relative moduli space of semi-stable sheaves (parametrizing the semi-stables sheaves on the fibers of $\pi$ with Hilbert polynomial $P$ with respect to the restriction of $O_C(1)$, cf. [15]).

We suppose first that there exists a closed point $s \in S$ such that $C_s$ is a smooth projective irreducible curve. Then in general $\tau$ is not flat (some other examples on non flat relative moduli spaces are given in [13]). The reason is that the generic structure of torsion free sheaves on $Y$ is more complicated than on smooth curves, and some of these sheaves cannot be deformed to sheaves on the smooth fibers of $\pi$.

A coherent sheaf on a smooth algebraic variety $X$ is locally free on some nonempty open subset of $X$. This is not true on $Y$. But a coherent sheaf $E$ on $Y$ is quasi locally free on some nonempty open subset of $Y$, i.e. on this open subset, $E$ is locally isomorphic to a sheaf of the form $\bigoplus_{1 \leq i \leq n} m_i O_{C_i}$, the sequence of non negative integers $(m_1, \ldots, m_n)$ being uniquely determined (cf. [3], [6]). It is not hard to see that if $E$ can be extended to a coherent sheaf on $C$, flat on $S$, then $R(E) = \sum_{i=1}^n i.m_i$ must be a multiple of $n$. For example, it is impossible to deform the stable sheaf $O_{C_s}$ on $Y$ in sheaves on the smooth fibers, if $1 \leq i < n$.

Now suppose that all the fibers $\pi^{-1}(s)$, $s \neq P$, are reduced with exactly $n$ smooth components. I conjecture that (with suitable hypotheses) a torsion free coherent sheaf on $Y$ can be extended to a coherent sheaf on $C$, flat on $S$, using the fact that we allow coherent sheaves of the reducible fibers $C_s$ that have not the same rank on all the components. This would be a step in the study of the flatness of $\tau$. For example (for suitable $\pi$), there exists a coherent sheaf $E$ on $C$, flat on $S$, such that $E_P = O_C$, and that for $s \neq P$, $E_s$ is the structural sheaf of an irreducible component of $C_s$.

Moduli spaces of sheaves on reducible curves have been studied in [16], [17], [18].
1.2. Maximal reducible deformations – Let $(S, P)$ be the germ of a smooth curve. Let $Y$ be a primitive multiple curve of multiplicity $n \geq 2$ and $k > 0$ an integer. Let $\pi : C \to S$ be a flat morphism, where $C$ is a reduced algebraic variety, such that

- for every closed point $s \in S$ such that $s \neq P$, the fiber $C_s$ has $k$ irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber $C_P$ is isomorphic to $Y$.

We show that by making a change of variable, i.e. by considering a suitable germ $(S', P')$ and a non constant morphism $\tau : S' \to S$, and replacing $\pi$ with $\tau^*C \to S'$, we can suppose that $C$ has exactly $k$ irreducible components, inducing on every fiber $C_s$, $s \neq P$ the $k$ irreducible components of $C_s$. In this case $\pi$ is called a reducible deformation of $Y$ of length $k$.

We show that $k \leq n$. We say that $\pi$ (or $C$) is a maximal reducible deformation of $Y$ if $k = n$.

Suppose that $\pi$ is a maximal reducible deformation of $Y$. We show that if $C'$ is the union of $i > 0$ irreducible components of $C$, and $\pi' : C' \to S$ is the restriction of $\pi$, then $\pi'^{-1}(P) \simeq C_i$, and $\pi'$ is a maximal reducible deformation of $C_i$. Let $s \in S\{P\}$. We prove that the irreducible components of $C_s$ have the same genus as $C$. Moreover, if $D_1, D_2$ are distinct irreducible components of $C_s$, then $D_1 \cap D_2$ consists of $-\deg(L)$ points.

1.3. Fragmented deformations (definition) – Let $Y$ be a primitive multiple curve of multiplicity $n \geq 2$ and $\pi : C \to S$ a maximal reducible deformation of $Y$. We call it a fragmented deformation of $Y$ if $\deg(L) = 0$, i.e. if for every $s \in S\{P\}$, $C_s$ is the disjoint union of $n$ smooth curves. In this case $C$ has $n$ irreducible components $C_1, \ldots, C_n$ which are smooth surfaces.

The variety $C$ appears as a particular case of a gluing of $C_1, \ldots, C_n$ along $C$ (cf. 4.1.5). We prove (proposition 4.1.6) that such a gluing $D$ is a fragmented deformation of a primitive multiple curve if and only if every closed point in $C$ has a neighborhood in $D$ that can be embedded in a smooth variety of dimension 3. The simplest gluing is the trivial or initial gluing $A$. An open subset $U$ of $A$ (and $C$) is given by open subsets $U_1, \ldots, U_n$ of $C_1, \ldots, C_n$ respectively, having the same intersection with $C$, and

$$O_A(U) = \{(\alpha_1, \ldots, \alpha_n) \in O_{C_1}(U \cap C_1) \times \cdots \times O_{C_n}(U \cap C_n); \quad \alpha_1|C = \cdots = \alpha_n|C\},$$

and $O_C(U)$ appears as a subalgebra of $O_A(U)$, hence we have a canonical morphism $A \to C$.

We can view elements of $O_C(U)$ as $n$-tuples $(\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in O_{C_i}(U \cap C_i)$. In particular we can write $\pi = (\pi_1, \ldots, \pi_n)$.

1.4. A simple analogy – Consider $n$ copies of $\mathbb{C}$ glued at 0. Two extreme examples appear: the trivial gluing $A_0$ (the set of coordinate lines in $\mathbb{C}^n$), and a set $C_0$ of $n$ lines in $\mathbb{C}^2$. We can easily construct a bijective morphism $\Psi : A_0 \to C_0$ sending each coordinate line to a line in the plane...
But the two schemes are of course not isomorphic: the maximal ideal of 0 in $A_0$ needs $n$ generators, but 2 are enough for the maximal ideal of 0 in $C_0$.

Let $\pi_{C_0} : C_0 \to C$ be a morphism sending each component linearly onto $C$, and $\pi_{A_0} = \pi_{C_0} \circ \Psi : A_0 \to C$. The difference of $A_0$ and $C_0$ can be also seen by using the fibers of 0: we have

$$\pi_{C_0}^{-1}(0) \simeq \text{spec}(C[t]/(t^n)) \quad \text{and} \quad \pi_{A_0}^{-1}(0) \simeq \text{spec}(C[t_1,\ldots,t_{n-1}]/(t_1,\ldots,t_{n-1})^2).$$

Let $D$ be a general gluing of $n$ copies of $C$ at 0, such that there exists a morphism $\pi : D \to C$ inducing the identity on each copy of $C$. It is easy to see that we have $\pi^{-1}(0) \simeq \text{spec}(C[t]/(t^n))$ if and only if some neighborhood of 0 in $D$ can be embedded in a smooth surface.

1.5. Fragmented deformations (main properties) – Let $\pi : C \to S$ be a fragmented deformation of $Y = C_n$. Let $I \subset \{1,\ldots,n\}$ be a proper subset, $I^c$ its complement, and $C_I \subset C$ the subscheme union of the $C_i, i \in I$. We prove (theorem 4.3.7) that the ideal sheaf $I_{C_I}$ of $C_I$ is isomorphic to $\mathcal{O}_{C_{I^c}}$.

In particular, the ideal sheaf $I_{C_i}$ of $C_i$ is generated by a single regular function on $C$. We show that we can find such a generator such that for $1 \leq j \leq n$, $j \neq i$, its $j$-th coordinate can be written as $\alpha_j \pi_j^{p_{ij}}$, with $p_{ij} > 0$ and $\alpha_j \in H^0(\mathcal{O}_S)$ such that $\alpha_j(P) \neq 0$. If $1 \leq j \leq n$ and $j \neq i$, we can then obtain a generator that can be written as

$$u_{ij} = (u_1,\ldots,u_m),$$

with

$$u_1 = 0, \quad u_m = \alpha_{ij}^{(m)} \pi_m^{p_{im}} \quad \text{for} \ m \neq i, \quad \alpha_{ij}^{(j)} = 1.$$
The constants \( a_{ij}^{(m)} = \alpha_{ij}^{(m)} \in \mathbb{C} \) have interesting properties (propositions \([4.5.2, 4.4.6]\)). Let 
\( p_i = 0 \) for \( 1 \leq i \leq n \). The symmetric matrix \((p_{ij})_{1 \leq i,j \leq n}\) is called the \emph{spectrum} of \( \pi \) (or \( C \)).

It follows also from the fact that \( \mathcal{I}_C = (u_{ij}) \) that \( Y \) is a \emph{simple} primitive multiple curve, i.e. the ideal sheaf of \( C \) in \( Y = C_n \) is isomorphic to \( \mathcal{O}_{C_{n-1}} \). Conversely, we show in theorem \( 4.7.1 \) that if \( Y \) is a simple primitive multiple curve, then there exists a fragmented deformation of \( Y \).

We give in \([4.4, 4.5]\) a way to construct fragmented deformations by induction on \( n \). This is used later to prove statements on fragmented deformations by induction on \( n \).

1.6. \emph{n-stars and structure of fragmented deformations} – An \emph{n-star} of \((S, P)\) is a gluing \( S \) of \( n \) copies \( S_1, \ldots, S_n \) of \( S \) at \( P \), together with a morphism \( \tau : S \to S \) which is an identity on each \( S_i \). All the \( n \)-stars have the same underlying Zariski topological space \( S(n) \).

An \emph{n-star} is called \emph{oblate} if some neighborhood of \( P \) can be embedded in a smooth surface. This is the case if and only \( \tau^{-1}(0) \simeq \text{spec}(\mathbb{C}[t])/(t^n) \).

Oblate \( n \)-stars are analogous to fragmented deformations but simpler. We provide a way to build oblate \( n \)-stars by induction on \( n \).

Let \( \pi : C \to S \) be a fragmented deformation of \( Y = C_n \). We associate to it an oblate \( n \)-star \( S \) of \( S \). Let \( C^\text{top} \) be the Zariski topological space of \( C \). We have an obvious continuous map \( \tilde{\pi} : C^\text{top} \to S(n) \). For every open subset \( U \) of \( S(n) \), \( \mathcal{O}_S(U) \) is the set of \( (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_C(\tilde{\pi}^{-1}(U)) \) such that \( \alpha_i \in \mathcal{O}_{S_i}(U \cap S_i) \) for \( 1 \leq i \leq n \). We obtain also a canonical morphism \( \Pi : C \to S \). We prove (theorem \(6.5.2\)) that \( \Pi \) is flat. Hence it is a flat family of smooth curves, with \( \Pi^{-1}(P) = C \). The converse is also true, i.e. starting from an oblate \( n \)-star of \( S \) and a flat family of smooth curves parametrized by it, we obtain a fragmented deformation of a multiple primitive curve of multiplicity \( n \).

1.7. \emph{Fragmented deformations of double curves} – Let \( Y = C_2 \) be a primitive double curve, \( C \) its associated smooth curve, \( \pi : C \to S \) a fragmented deformation of \( Y \), of spectrum \( \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} \), and \( C_1, C_2 \) the irreducible components of \( C \). For \( i = 1, 2 \), \( q > 0 \), let \( C_i^q \) be the infinitesimal neighborhood of order \( q \) of \( C \) in \( C_i \) (defined by the ideal sheaf \( (\pi_i^q) \)). It is a primitive multiple curve of multiplicity \( q \).

It follows from \([4.3.5]\) that \( C_1^p \) and \( C_2^p \) are isomorphic, and \( C_1^{p+1}, C_2^{p+1} \) are two extensions of \( C_1^p \) in primitive multiple curves of multiplicity \( p + 1 \). According to \([4]\) these extensions are parametrized by an affine space with associated vector space \( H^1(C, T_C) \) (where \( T_C \) is the tangent bundle of \( C \)). Let \( w \in H^1(C, T_C) \) be the vector from \( C_1^{p+1} \) to \( C_2^{p+1} \).

Similarly, the primitive double curves with associated smooth curve \( C \) such that \( \mathcal{I}_C \simeq \mathcal{O}_C \) are parametrized by \( \mathbb{P}(H^1(C, T_C)) \cup \{0\} \) (cf. \([2, 4]\)).

We prove in theorem \(6.0.5\) that the point of \( \mathbb{P}(H^1(C, T_C)) \cup \{0\} \) corresponding to \( C_2 \) is \( \mathbb{C}w \).

1.8. \emph{Notation}: Let \( X \) be an algebraic variety and \( Y \subset X \) a closed subvariety. We will denote by \( \mathcal{I}_{Y,X} \) (or \( \mathcal{I}_Y \) if there is no risk of confusion) the ideal sheaf of \( Y \) in \( X \).
2. Preliminaries

2.1. Local embeddings in smooth varieties

2.1.1. Proposition: Let $X$ be an algebraic variety, $x$ a closed point of $X$ and $n$ a positive integer. Then the three following properties are equivalent:

(i) There exist a neighborhood $U$ of $x$ and an embedding $U \subset Z$ in a smooth variety of dimension $n$.

(ii) The $\mathcal{O}_{X,x}$-module $m_{X,x}$ (maximal ideal of $x$) can be generated by $n$ elements.

(iii) We have $\dim\mathbb{C}(m_{X,x}/m_{X,x}^2) \leq n$.

Proof. It is obvious that (i) implies (ii), and (ii),(iii) are equivalent according to Nakayama’s lemma. It remains to prove that (iii) implies (i).

Suppose that (iii) is true. There exist an integer $N$ and an embedding $X \subset \mathbb{P}_N$. Let $\mathcal{I}_X$ be the ideal sheaf of $X$ in $\mathbb{P}_N$. Let $p$ be the biggest integer such that there exists $f_1, \ldots, f_p \in \mathcal{I}_{X,x}$ whose images in the $\mathbb{C}$-vector space $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$ are linearly independent. Then we have

$$\mathcal{I}_{X,x} \subset (f_1, \ldots, f_p) + m_{\mathbb{P}_N,x}^2.$$ 

In fact, let $f \in \mathcal{I}_{X,x}$. Since $p$ is maximal, the image of $f$ in $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$ is a linear combination of those of $f_1, \ldots, f_n$. Hence we can write

$$f = \sum_{i=1}^{p} \lambda_i f_i + g, \quad \text{with} \quad \lambda_i \in \mathbb{C}, \ g \in m_{\mathbb{P}_N,x}^2,$$

and our assertion is proved. It follows that we have a surjective morphism

$$\alpha : \mathcal{O}_{X,x}/m_{X,x}^2 \twoheadrightarrow \mathcal{O}_{\mathbb{P}_N,x}/((f_1, \ldots, f_p) + m_{\mathbb{P}_N,x}^2).$$

We have

$$\dim\mathbb{C}(\mathcal{O}_{X,x}/m_{X,x}^2) \leq n + 1, \quad \dim\mathbb{C}(\mathcal{O}_{\mathbb{P}_N,x}/((f_1, \ldots, f_p) + m_{\mathbb{P}_N,x}^2)) = N - p + 1.$$

Hence $N - p + 1 \leq n + 1$, i.e. $p \geq N - n$. We can take for $Z$ a neighborhood of $x$ in the subvariety of $\mathbb{P}_N$ defined by $f_1, \ldots, f_{N-n}$, which is smooth at $x$. \hfill \Box

2.2. Flat families of coherent sheaves

Let $(S, P)$ be the germ of a smooth curve and $t \in \mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $\pi : X \rightarrow S$ be a flat morphism. If $\mathcal{E}$ is a coherent sheaf on $X$, $\mathcal{E}$ is flat on $S$ at $x \in \pi^{-1}(P)$ if and only if the multiplication by $t : \mathcal{E}_x \rightarrow \mathcal{E}_x$ is injective. In particular the multiplication by $t : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$ is injective.

2.2.1. Lemma: Let $\mathcal{E}$ be a coherent sheaf on $X$ flat on $S$. Then, for every open subset $U$ of $X$, the restriction $\mathcal{E}(U) \rightarrow \mathcal{E}(U \setminus \pi^{-1}(P))$ is injective.

Proof. Let $s \in \mathcal{E}(U)$ whose restriction to $U \setminus \pi^{-1}(P)$ vanishes. We must show that $s = 0$. By covering $U$ with smaller open subsets we can suppose that $U$ is affine: $U = \text{spec}(A)$. Hence $U \setminus \pi^{-1}(P) = \text{spec}(A_1)$. Let $M = \mathcal{E}(U)$, it is an $A$-module. We have $\mathcal{E}_U = \widetilde{M}$ and $\mathcal{E}(U \setminus \pi^{-1}(P)) = M_I$. Hence if the restriction of $s$ to $U \setminus \pi^{-1}(P)$ vanishes, there exists an integer
n > 0 such that \( t^n s = 0 \). Since the multiplication by \( t \) is injective (because \( E \) is flat on \( S \)), we have \( s = 0 \).

Let \( E \) be a coherent sheaf on \( X \) flat on \( S \). Let \( F \subset E_{x}(\pi^{-1}(P)) \) be a subsheaf. For every open subset \( U \) of \( X \) we denote by \( F(U) \) the subset of \( F(X) \) of elements that can be extended to sections of \( E \) on \( U \). If \( V \subset U \) is an open subset, the restriction \( F(U) \rightarrow F(V) \) induces a morphism \( F(U) \rightarrow F(V) \).

2.2.2. Proposition: \( F \) is a subsheaf of \( E \), and \( E/F \) is flat on \( S \).

Proof. To prove the first assertion, we must show that if \( U \) is an open subset of \( X \) and \( (U_{i})_{i \in I} \) is an open cover of \( U \), then

(i) If \( s \in F(U) \) is such that for every \( i \) we have \( s|_{U_{i}} = 0 \), then \( s = 0 \).
(ii) For every \( i \in I \) let \( s_{i} \in F(U_{i}) \). Then if for all \( i, j \) we have \( s_{i|U_{ij}} = s_{j|U_{ij}} \), then there exists \( s \in F(U) \) such that for every \( i \in I \) we have \( s|_{U_{i}} = s_{i} \).

This follows easily from lemma 2.2.1.

Now we prove that \( E/F \) is flat on \( S \). Let \( x \in \pi^{-1}(P) \) and \( u \in (E/F)_{x} \) such that \( tu = 0 \). We must show that \( u = 0 \). Let \( v \in E_{x} \) over \( u \). Then we have \( tv \in F_{x} \). Let \( U \) be a neighborhood of \( x \) such that \( tv \) comes from \( w \in F(U) \). This means that \( w|_{U \cap \pi^{-1}(P)} \in F(U 
\pi^{-1}(P)) \). Since \( t \) is invertible on \( U \cap \pi^{-1}(P) \) we can write \( w = tw' \), with \( w' \in F(U 
\pi^{-1}(P)) \). We have then \( w' = v \) on \( U \cap \pi^{-1}(P) \). Hence \( v \in F_{x} \) and \( u = 0 \).

2.3. Primitive multiple curves

(cf. [1], [2], [3], [4], [5], [7], [8], [10]).

Let \( C \) be a smooth connected projective curve. A multiple curve with support \( C \) is a Cohen-Macaulay scheme \( Y \) such that \( Y_{red} = C \).

Let \( n \) be the smallest integer such that \( Y = C^{(n-1)} \), \( C^{(k-1)} \) being the \( k \)-th infinitesimal neighborhood of \( C \), i.e. \( \mathcal{I}_{C^{(k-1)}} = \mathcal{I}_{C}^{k} \). We have a filtration \( C = C_{1} \subset C_{2} \subset \cdots \subset C_{n} = Y \) where \( C_{i} \) is the biggest Cohen-Macaulay subscheme contained in \( Y \cap C^{(i-1)} \). We call \( n \) the multiplicity of \( Y \).

We say that \( Y \) is primitive if, for every closed point \( x \) of \( C \), there exists a smooth surface \( S \), containing a neighborhood of \( x \) in \( Y \) as a locally closed subvariety. In this case, \( L = \mathcal{I}_{C}/\mathcal{I}_{C_{2}} \) is a line bundle on \( C \) and we have \( \mathcal{I}_{C_{j}} = \mathcal{I}_{C}^{j} \), \( \mathcal{I}_{C_{j}}/\mathcal{I}_{C_{j+1}} = L^{j} \) for \( 1 \leq j < n \). We call \( L \) the line bundle on \( C \) associated to \( Y \). Let \( P \in C \). Then there exist elements \( y, t \) of \( m_{S,P} \) (the maximal ideal of \( \mathcal{O}_{S,P} \)) whose images in \( m_{S,P}/m_{S,P}^{2} \) form a basis, and such that for \( 1 \leq i < n \) we have \( \mathcal{I}_{C_{i},P} = (y^{i}) \).

The simplest case is when \( Y \) is contained in a smooth surface \( S \). Suppose that \( Y \) has multiplicity \( n \). Let \( \tilde{P} \in C \) and \( f \in \mathcal{O}_{S,P} \) a local equation of \( C \). Then we have \( \mathcal{I}_{C_{i},P} = (f^{i}) \) for \( 1 < j \leq n \), in particular \( I_{Y,P} = (f^{n}) \), and \( L = \mathcal{O}_{C}(-C) \).
We will write \( \mathcal{O}_n = \mathcal{O}_{C_n} \) and we will see \( \mathcal{O}_i \) as a coherent sheaf on \( C_n \) with schematic support \( C_i \) if \( 1 \leq i < n \).

If \( \mathcal{E} \) is a coherent sheaf on \( Y \) one defines its \textit{generalized rank} \( R(\mathcal{E}) \) and \textit{generalized degree} \( \text{Deg}(\mathcal{E}) \) (cf. [6], 3-): take any filtration of \( \mathcal{E} 
abla_0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E} \) by subsheaves such that \( \mathcal{E}_i / \mathcal{E}_{i-1} \) is concentrated on \( C_i \) if \( 1 \leq i < n \).

If \( \mathcal{E} \) is a coherent sheaf on \( Y \) one defines its \textit{generalized rank} \( R(\mathcal{E}) \) and \textit{generalized degree} \( \text{Deg}(\mathcal{E}) \).

Let \( \mathcal{O}_Y(1) \) be a very ample line bundle on \( Y \). Then the Hilbert polynomial of \( \mathcal{E} \) is
\[
P_{\mathcal{E}}(m) = R(\mathcal{E}) \deg(\mathcal{O}_C(1)) m + \text{Deg}(\mathcal{E}) + R(\mathcal{E})(1-g)
\]
(where \( g \) is the genus of \( C \)).

We deduce from proposition 2.1.1:

\textbf{2.3.1. Proposition:} \textit{Let \( Y \) be a multiple curve with support \( C \). Then \( Y \) is a primitive multiple curve if and only if \( I_C / I_C^2 \) is zero, or a line bundle on \( C \).}

\textbf{2.3.2. Parametrization of double curves} - In the case of double curves, D. Bayer and D. Eisenbud have obtained in [2] the following classification: if \( Y \) is of multiplicity 2, we have an exact sequence of vector bundles on \( C \)
\[
0 \longrightarrow L \longrightarrow \Omega_{Y|C} \longrightarrow \omega_C \longrightarrow 0
\]
which is split if and only if \( Y \) is the \textit{trivial curve}, i.e. the second infinitesimal neighborhood of \( C \), embedded by the zero section in the dual bundle \( L^* \), seen as a surface. If \( Y \) is not trivial, it is completely determined by the line of \( \text{Ext}^1_{\mathcal{O}_C}(\omega_C, L) \) induced by the preceding exact sequence. The non trivial primitive curves of multiplicity 2 and of associated line bundle \( L \) are therefore parametrized by the projective space \( \mathbb{P}(\text{Ext}^1_{\mathcal{O}_C}(\omega_C, L)) \).

\textbf{2.4. Simple primitive multiple curves}

Let \( C \) be a smooth projective irreducible curve, \( n \geq 2 \) an integer and \( C_n \) a primitive multiple curve of multiplicity \( n \) and associated reduced curve \( C \). Then the ideal sheaf \( \mathcal{I}_C \) of \( C \) in \( C_n \) is a line bundle on \( C_{n-1} \).

We say that \( C_n \) is \textit{simple} if \( \mathcal{I}_C \simeq \mathcal{O}_{n-1} \).

In this case the line bundle on \( C \) associated to \( C_n \) is \( \mathcal{O}_C \). The following result is proved in [8] (théorème 1.2.1):

\textbf{2.4.1. Theorem:} \textit{Suppose that \( C_n \) is simple. Then there exists a flat family of smooth projective curves \( \tau : C \rightarrow \mathbb{C} \) such that \( \tau^{-1}(0) \simeq C \) and that \( C_n \) is isomorphic to the \( n \)-th infinitesimal neighborhood of \( C \) in \( C \).}
3. Reducible reduced deformations of primitive multiples curves

3.1. Connected Components

Let \((S, P)\) be the germ of a smooth curve and \(t \in \mathcal{O}_{S,P}\) a generator of the maximal ideal. Let \(n > 0\) be an integer and \(Y = C_n\) a projective primitive multiple curve of multiplicity \(n\).

Let \(k > 0\) be an integer. Let \(\pi : C \to S\) be a flat morphism, where \(C\) is a reduced algebraic variety, such that

- For every closed point \(s \in S\) such that \(s \neq P\), the fiber \(C_s\) has \(k\) irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber \(C_P\) is isomorphic to \(C_n\).

It is easy to see that the irreducible components of \(C\) are reduced surfaces.

Let \(Z\) be the open subset of \(C \setminus C_P\) of points \(z\) belonging to only one irreducible component of \(C_{\pi(z)}\). Then the restriction of \(\pi : Z \to S \setminus \{P\}\) is a smooth morphism. For every \(s \in S \setminus \{P\}\), let \(C_s = C_s \cap Z\). It is the open subset of smooth points of \(C_s\).

Let \(z \in Z\) and \(s = \pi(z)\). There exist a neighborhood (for the Euclidean topology) \(U\) of \(s\), isomorphic to \(\mathbb{C}\), and a neighborhood \(V\) of \(z\) such that \(V \cong \mathbb{C}^2\), \(\pi(V) = U\), the restriction of \(\pi : V \to U\) being the projection \(\mathbb{C}^2 \to \mathbb{C}\) on the first factor. We deduce easily from the following facts:

- let \(s \in S \setminus \{P\}\) and \(C_1\) an irreducible component of \(C_s\). Let \(z_1, z_2 \in C_1 \cap Z\). Then there exist neighborhoods (in \(Z\), for the Euclidean topology) \(U_1, U_2\) of \(z_1, z_2\) respectively, such that if \(y_1 \in U_1, y_2 \in U_2\) are such that \(\pi(y_1) = \pi(y_2)\), then \(y_1\) and \(y_2\) belong to the same irreducible component of \(C_{\pi(y_1)}\).
- for every continuous map \(\sigma : [0, 1] \to S \setminus \{P\}\) and every \(z \in Z\) such that \(\sigma(0) = \pi(z)\) there exists a lifting of \(\sigma\), \(\sigma' : [0, 1] \to Z\) such that \(\sigma'(0) = z\). Moreover, if \(\sigma'' : [0, 1] \to Z\) is another lifting of \(\sigma\) such that \(\sigma''(0) = z\), then \(\sigma'(1)\) and \(\sigma''(1)\) are in the same irreducible component of \(C_{\sigma(1)}\). More generally, if we only impose that \(\sigma''(0)\) is in the same irreducible component of \(C_{\sigma(0)}\) as \(z\), then \(\sigma'(1)\) and \(\sigma''(1)\) are in the same irreducible component of \(C_{\sigma(1)}\).

3.1.1. Lemma: Let \(\sigma_0, \sigma_1 : [0, 1] \to S \setminus \{P\}\) be two continuous maps such that \(\sigma_0(0) = \sigma_1(0), s = \sigma_0(1) = \sigma_1(1)\). Suppose that they are homotopic. Let \(\sigma_0', \sigma_1'\) be liftings \([0, 1] \to Z\) of \(\sigma_0, \sigma_1\) respectively, such that \(\sigma_0'(0) = \sigma_1'(0)\). Then \(\sigma_0'(1)\) and \(\sigma_1'(1)\) belong to the same irreducible component of \(C_{\sigma(1)}\).

Proof. Let

\[
\Psi : [0, 1] \times [0, 1] \to S \setminus \{P\}
\]

be an homotopy:

\[
\Psi(0, t) = \sigma_0(t), \quad \Psi(1, t) = \sigma_1(t), \quad \Psi(t, 0) = \sigma_0(0), \quad \Psi(t, 1) = \sigma_0(1)
\]

for \(0 \leq t \leq 1\). For every \(u \in [0, 1]\) and \(\epsilon > 0\) let \(I_{u, \epsilon} = [u - \epsilon, u + \epsilon] \cap [0, 1]\). By using the local structure of \(\pi|_Z\) for the Euclidean topology it is easy to see that for every \(u \in [0, 1]\), there exists
an $\epsilon > 0$ such that the restriction of $\Psi$
\[
I_{u, \epsilon} \times [0, 1] \longrightarrow S \setminus \{P\}
\]
can be lifted to a morphism
\[
\Psi' : I_{u, \epsilon} \times [0, 1] \longrightarrow Z
\]
such that $\Psi'(t, 0) = \sigma'_0(0)$ for every $t \in I_{u, \epsilon}$. It follows that if $I_{u, \epsilon} = [a_{u, \epsilon}, b_{u, \epsilon}]$, then $\Psi'(a_{u, \epsilon}, 1)$ and $\Psi'(b_{u, \epsilon}, 1)$ are in the same irreducible component of $C'_{\sigma_0(1)}$. We have just to cover $[0, 1]$ with a finite number of intervals $I_{u, \epsilon}$ to obtain the result. \hfill $\square$

Let $s \in S \setminus \{P\}$, $D_1, \ldots, D_k$ be the irreducible components of $C'_s$ and $x_i \in D_i$ for $1 \leq i \leq k$. Let $\sigma$ be a loop of $S \setminus \{P\}$ with origin $s$, defining a generator of $\pi_1(S \setminus \{P\})$. Let $i$ be an integer such that $1 \leq i \leq k$. The liftings $\sigma' : [0, 1] \to Z$ of $\sigma$ such that $\sigma'(0) = x_i$ end up at a component $D_j$ which does not depend on $x_i$. Hence we can write
\[
j = \alpha_C(i).
\]

3.1.2. Lemma: $\alpha_C$ is a permutation of $\{1, \ldots, k\}$.

Proof. Suppose that $i \neq j$ and $\alpha_C(i) = \alpha_C(j)$. By inverting the paths we find liftings of paths from $D_{\alpha_C(i)}$ to $D_i$ and $D_j$. This contradicts lemma \[3.1.1\] \hfill $\square$

Let $p > 0$ be an integer such that $\alpha^p_C = I_{\{1, \ldots, k\}}$. Let $t$ be a generator of the maximal ideal of $O_{S, P}$, $K$ the field of rational functions on $S$ and $K' = K(t^{1/p})$. Let $S'$ be the germ of the curve corresponding to $K'$, $\theta : S' \to S$ canonical the morphism and $P'$ the unique point of $\theta^{-1}(P)$. Let $D = \theta^*(C)$. We have therefore a cartesian diagram
\[
\begin{tikzcd}
D \ar[r, \rho] \ar[d, \Theta] & S' \ar[d, \theta] \\
C \ar[r, \pi] & S
\end{tikzcd}
\]
where $\rho$ is flat, and for every $s' \in S'$, $\Theta$ induces an isomorphism $D_{s'} \simeq C_{\theta(s')}$. We have
\[
\alpha_D = I_{\{1, \ldots, k\}}.
\]
Let $Z' \subset D$ be the complement of the union of $\rho^{-1}(P')$ and of the singular points of the curves $D_{s'}$, $s' \neq P'$ (hence $Z' = \Theta^{-1}(Z)$).

3.1.3. Proposition: The open subset $Z'$ has exactly $k$ irreducible components $Z'_1, \ldots, Z'_k$. Let $\overline{Z}'_1, \ldots, \overline{Z}'_k$ be their closures in $D$. Then for every $s' \in S' \setminus \{P'\}$, the $\overline{Z}_i' \cap D_{s'}$, $1 \leq i \leq k$, are the irreducible components of $D_{s'}$ minus the intersection points with the other components, and the $\overline{Z}_i' \cap D_{s'}$ are the irreducible components of $D_{s'}$.

3.1.4. Definition: Let $k > 0$ be an integer. A reducible deformation of length $k$ of $C_n$ is a flat morphism $\pi : C \to S$, where $C$ is a reduced algebraic variety, such that
- For every closed point $s \in S$, $s \neq P$, the fiber $C_s$ has $k$ irreducible components, which are smooth and transverse, and any three of these components have no common point.
3.2. Maximal reducible deformations

Let \((S, P)\) be the germ of a smooth curve and \(t \in \mathcal{O}_{S,P}\) a generator of the maximal ideal. Let \(n > 0\) be an integer and \(Y = C_n\) a projective primitive multiple curve of multiplicity \(n\), with underlying smooth curve \(C\). We denote by \(g\) the genus of \(C\) and \(L\) the line bundle on \(C\) associated to \(C_n\).

Let \(\pi : \mathcal{C} \to S\) be a reducible deformation of length \(k\) of \(C_n\). Let \(Z_1, \ldots, Z_k\) be the closed subvarieties of \(\pi^{-1}(S\backslash\{P\})\) such that for every \(s \in S\backslash\{P\}\), \(Z_{s1}, \ldots, Z_{sk}\) are the irreducible components of \(\mathcal{C}_s\) (cf. prop. 3.1.3). For \(1 \leq i \leq k\), we denote by \(J_i\) the ideal sheaf of \(Z_1 \cup \cdots \cup Z_i\) in \(\pi^{-1}(S\backslash\{P\})\). This sheaf is flat on \(S\backslash\{P\}\), and we have

\[
0 = J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset \mathcal{O}_{\pi^{-1}(S\backslash\{P\})}.
\]

The quotients \(\mathcal{O}_{\pi^{-1}(S\backslash\{P\})}/J_1, J_i/J_{i+1}, 1 \leq i < k\), are also flat on \(S\backslash\{P\}\). We obtain the filtration of sheaves on \(\mathcal{C}\)

\[
0 = \mathcal{J}_k \subset \mathcal{J}_{k-1} \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{O}_{\mathcal{C}}.
\]

(cf. 2.2). According to proposition 2.2.2, the quotients \(\mathcal{O}_{\mathcal{C}}/\mathcal{J}_1, \mathcal{J}_i/\mathcal{J}_{i+1}, 1 \leq i < n\), are flat on \(S\). We have \(\mathcal{O}_{\pi^{-1}(S\backslash\{P\})}/J_1 = \mathcal{O}_{Z_1}\). We denote by \(X_i\) the closed subvariety of \(\mathcal{C}\) corresponding to the ideal sheaf \(\mathcal{J}_i\).

Similarly we consider the ideal sheaf \(\mathcal{J}_i'\) of \(Z_{i+1} \cup \cdots \cup Z_n\) on \(\pi^{-1}(S\backslash\{P\})\), the associated ideal sheaf \(\mathcal{J}_i'\) on \(\mathcal{C}\) and the corresponding subvariety \(X_i'\).

3.2.1. Proposition: We have \(k \leq n\).

Proof. Let \(\mathcal{E}_0 = \mathcal{O}_{\mathcal{C}}/\mathcal{J}_1\) and \(\mathcal{E}_i = \mathcal{J}_i/\mathcal{J}_{i+1}\) for \(1 \leq i < n\). The sheaves \(\mathcal{E}_iP\) are not concentrated on a finite number of points. To see this we use a very ample line bundle \(\mathcal{O}(1)\) on \(\mathcal{C}\). The Hilbert polynomial of \(\mathcal{E}_iP\) is the same as that of \(\mathcal{E}_{is}, s \neq P\), hence it is not constant. So we have \(R(\mathcal{E}_i) \geq 1\) \((R(\mathcal{E}_i)\) is the generalized rank of \(\mathcal{E}_i\), cf. 2.3), and since

\[
n = R(\mathcal{O}_{C_n}) = \sum_{i=0}^{k} R(\mathcal{E}_iP),
\]

we have \(k \leq n\). \(\square\)

3.2.2. Definition: We say that \(\pi\) (or \(\mathcal{C}\)) is a maximal reducible deformation of \(C_n\) if \(k = n\).

3.2.3. Theorem: Suppose that \(\mathcal{C}\) is a maximal reducible deformation of \(C_n\). Then we have, for \(1 \leq i < n\)

\[
\mathcal{J}_{iP} = \mathcal{I}_{C_i, C_n}
\]

and \(X_i\) is a maximal reducible deformation of \(C_i\).
Proof. Let $\mathcal{O}_C(1)$ be a very ample line bundle on $C$.

Let $Q$ be a closed point of $C$. Let $z \in \mathcal{O}_{n,Q}$ be an equation of $C$ and $x \in \mathcal{O}_{n,Q}$ over a generator of the maximal ideal of $Q$ in $\mathcal{O}_{C,Q}$. Let $z, x \in \mathcal{O}_{C,Q}$ be over $z, x$ respectively. The maximal ideal of $\mathcal{O}_{n,Q}$ is $(x, z)$. The maximal ideal of $\mathcal{O}_{C,Q}$ is generated by $z, x, t$. It follows from proposition 2.1.1 that there exist a neighborhood $U$ of $Q$ in $C$ and an embedding $j : U \to \mathbb{P}_3$. We can assume that the restriction of $j$ to $\mathbb{Z}_1 \cap U$ is induced by the morphism $\phi : \mathbb{C}[X, Z, T] \to \mathcal{O}_{\mathbb{Z}_1, Q}$ of $\mathbb{C}$-algebras which associates $x, z, t$ to $X, Z, T$ respectively.

Since $C$ is reduced, $U$ is an open subset of a reduced hypersurface of $\mathbb{P}_3$ having $n$ irreducible components, corresponding to $\mathbb{Z}_1, \ldots, \mathbb{Z}_n$. It is then clear that $X_i$, being the smallest subscheme of $C$ containing $\mathbb{Z}_1 \cap C, \ldots, \mathbb{Z}_i \cap C$, is the union in $U$ of the first $i$ hypersurface components.

Since $j(\mathbb{Z}_1)$ is a hypersurface, the kernel of $\phi$ is a principal ideal generated by the equation $F$ of the image of $\mathbb{Z}_1$.

Recall that $\mathcal{O}_n = \mathcal{O}_{C_n} = (\mathcal{O}_C)_p$. We have $R(\mathcal{O}_n/\mathcal{J}_1, p) = 1$ according to [1]. Hence there exists a nonempty open subset $V$ of $C_n$ such that $(\mathcal{O}_n/\mathcal{J}_1, p)|_V$ is a line bundle on $V \cap C$. It follows that the projection $\mathcal{O}_n \to \mathcal{O}_C$ vanishes on $\mathcal{J}_1, p|_V$. Since $\mathcal{O}_C$ is torsion free this projection vanishes everywhere on $\mathcal{J}_1$, i.e. $\mathcal{J}_1, p \subseteq \mathcal{I}_{C,C_n}$, with equality on $V$.

The sheaf $\mathcal{E}_0 = \mathcal{O}_C/\mathcal{J}_1$ is the structural sheaf of $\mathbb{Z}_1$, and the projection $\mathbb{Z}_1 \to S$ is a flat morphism. For every $s \in S \setminus \{P\}$, $(\mathcal{Z}_1)_s$ is a smooth curve. The fiber $(\mathcal{Z}_1)_p$ consists of $C$ and a finite number of embedded points. There exist flat families of curves whose general fiber is smooth and the special fiber consists of an integral curve and some embedded points (cf. [12], III, Example 9.8.4). We will show that this cannot happen in our case, i.e. we have $\mathcal{J}_1, p = \mathcal{I}_{C,C_n}$.

Let $m = (X, Z, T) \in \mathbb{C}[X, Z, T]$, and $m_{\mathbb{Z}_1}$ the maximal ideal of $\mathcal{O}_{\mathbb{Z}_1, Q}$. The ideal of $(\mathbb{Z}_1)_p$ in $\mathcal{O}_{n,Q}$ contains $z^q$ and $x^p z$ (for suitable minimal integers $p \geq 0, q > 0$), with $p > 0$ if and only if $Q$ is an embedded point. Hence the ideal of $\mathbb{Z}_1$ in $\mathcal{O}_{C,Q}$ contains elements of type $x^p z - t \alpha, z^q - t \beta$, with $\alpha, \beta \in \mathcal{O}_{C,Q}$.

Let $\mathcal{O}_{\mathbb{Z}_1, Q}$ be the completion of $\mathcal{O}_{\mathbb{Z}_1, Q}$ with respect to $m_{\mathbb{Z}_1}$ and

$$\hat{\phi} : \mathbb{C}((X, Z, T)) \to \mathcal{O}_{\mathbb{Z}_1, Q}$$

the morphism deduced from $\phi$. We can also see $\mathcal{O}_{\mathbb{Z}_1, Q}$ as the completion with respect to $(X, Z, T)$ of $\mathcal{O}_{\mathbb{Z}_1, Q}$ seen as a $\mathbb{C}[X, Z, T]$-module. It follows that $\ker(\hat{\phi}) = (F)$ (cf. [9], lemma 7.15). Note that $\hat{\phi}$ is surjective (this is why we use completions). Let $\alpha, \beta \in \mathbb{C}((X, Y, Z))$ be such that $\hat{\phi}(\alpha) = \alpha, \hat{\phi}(\beta) = \beta$. So we have

$$X^p Z - T \alpha, Z^q - T \beta \in \ker(\hat{\phi}).$$

Hence there exist $A, B \in \mathbb{C}((X, Z, T))$ such that $X^p Z - T \alpha = AF, Z^q - T \beta = BF$. We can write in an unique way

$$A = A_0 + TA_1, \ B = B_0 + TB_1, \ F = F_0 + TF_1,$$

with $A_0, B_0, F_0 \in \mathbb{C}((X, Z))$ and $A_1, B_1, F_1 \in \mathbb{C}((X, Z, T))$, and we have

$$A_0 F_0 = X^p Z, \ B_0 F_0 = Z^q.$$


Since \( F \) is not invertible, it follows that \( F_0 \) is of the form \( F_0 = cZ \), with \( c \in \mathbb{C}((X, Z, T)) \) invertible. So we have \( F = cZ + TF_1 \). It follows that \( z \in (t) \) in \( O_{Z(t)}Q \). This implies that this is also true in \( O_{Z(t)}Q \); in fact the assertion in \( O_{Z(t)}Q \) implies that

\[
z \in \bigcap_{n \geq 0} ((t) + m_{Z(t)}^n)
\]

in \( O_{Z(t)}Q \), and the latter is equal to \( (t) \) according to [14], vol. II, chap. VIII, theorem 9. Hence \( z \in (t) \) in \( O_{Z(t)}Q \), i.e. \( p = 0 \) and \( Q \) is not an embedded point. So there are no embedded points. This implies that \( \overline{J}_{1P} = \mathcal{O}_{C,C_n} \). Similarly, if \( I_j \) denotes the ideal sheaf of \( Z_j \) for \( 1 \leq j \leq n \), we have \( I_{j,P} = \mathcal{O}_{C,C_n} \). Since the restriction of \( \pi : Z_j \to S \) is flat, the curves \( E_j,s, s \neq P \), have the same genus as \( C \), and the same Hilbert polynomial with respect to \( \mathcal{O}_C(1) \).

Now we show that \( X'_1 \) is a maximal reducible deformation of \( C_{n-1} \). We need only to show that \( X'_{1,P} = C_{n-1} \). As we have seen, for \( 2 \leq j \leq n \), a local equation of \( Z_j \) at any point \( Q \in C \) induces a generator \( u_j \) of \( \mathcal{O}_{C,C_n,Q} \). Hence \( u = \prod_{2 \leq j \leq n} u_j \) is a generator of \( \mathcal{O}_{C_n-1,C_n,Q} \). But \( u = 0 \) on \( X'_1 \). It follows that \( X'_{1,P} \subset C_{n-1} \). But the Hilbert polynomial of \( \mathcal{O}_{C_{n-1}} \) is the same as that of the structural sheaves of the fibers of the flat morphism \( X'_1 \to S \) over \( s \neq P \), hence the same as \( \mathcal{O}_X'_{1,P} \). Hence \( X'_{1,P} = C_{n-1} \).

The theorem [3.2.3] is then easily proved by induction on \( n \).

3.2.4. Corollary: Let \( s \in S \setminus \{P\} \) and \( D_1, D_2 \) be two irreducible components of \( C_s \). Then \( D_1 \) is of genus \( g \) and \( D_1 \cap D_2 \) consists of \( -\deg(L) \) points.

Proof. According to theorem [3.2.3], there exists a flat family of smooth curves \( C \) parametrized by \( S \) such that \( C_P = C \) and \( C_s = D_1 \). So the genus of \( D_1 \) is equal to that of \( C \).

Let us prove the second assertion. Again according to theorem [3.2.3] we can suppose that \( n = 2 \). We have then \( \chi(C_s) = \chi(C_2) = 2\chi(C) + \deg(L) \). Let \( x_1, \ldots, x_N \) be the intersection points of \( D_1 \) and \( D_2 \). We have an exact sequence

\[
0 \to \mathcal{O}_{D_2}(-x_1 - \cdots - x_N) \to \mathcal{O}_{C_s} \to \mathcal{O}_{D_1} \to 0.
\]

Whence \( \chi(\mathcal{O}_{C_s}) = \chi(D_1) + \chi(D_2) - N = 2\chi(\mathcal{O}_C) - N \) (according to the first assertion). Whence \( N = -\deg(L) \). \( \square \)

3.2.5. It follows from the previous results that if \( \pi : C \to S \) is a maximal reducible deformation of \( C_n \), then we have

(i) \( \deg(L) \leq 0 \).

(ii) \( C \) has exactly \( n \) irreducible components \( C_1, \ldots, C_n \).

(iii) For \( 1 \leq i \leq n \), the restriction of \( \pi, \pi_i : C_i \to S \) is a flat morphism , and \( \pi_i^{-1}(P) = C_i \).

(iv) For every nonempty subset \( I \subset \{1, \ldots, n\} \), let \( C_I \) be the union of the \( C_i \) such that \( i \in I \), and \( m \) the number of elements of \( I \). Then the restriction of \( \pi, \pi_I : C_I \to S \) is a maximal reducible deformation of \( C_m \).

The following is immediate, and shows that we need only to consider maximal reducible deformations parametrized by a neighborhood of 0 in \( \mathbb{C} \):
3.2.6. Proposition: Let $t \in \mathcal{O}_S(P)$ be a generator of the maximal ideal, and $\pi : C \to S$ a maximal reducible deformation of $C_n$. Let $S' \subset S$ be an open neighborhood of $P$ where $t$ is defined and $C' = \pi^{-1}(U)$, $V = t(U)$. Then $\pi' = t \circ \pi : C' \to V$ is a maximal reducible deformation of $C_n$.

4. Fragmented deformations of primitive multiple curves

The fragmented deformations of primitive multiple curves are particular cases of reducible deformations.

In this chapter $(S, P)$ denotes the germ of a smooth curve. Let $t \in \mathcal{O}_{S,P}$ be a generator of the maximal ideal of $P$. We can suppose that $t$ is defined on the whole of $S$, and that the ideal sheaf of $P$ in $S$ is generated by $t$.

4.1. Fragmented deformations and gluing

Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity $n$.

4.1.1. Definition: Let $k > 0$ be an integer. A general fragmented deformation of length $k$ of $C_n$ is a flat morphism $\pi : C \to S$ such that for every point $s \neq P$ of $S$, the fiber $C_s$ is a disjoint union of $k$ projective smooth irreducible curves, and such that $C_P$ is isomorphic to $C_n$.

We have then $k \leq n$. If $k = n$ we say that $\pi$ (or $C$) is a general maximal fragmented deformation of $C_n$. We suppose in the sequel that it is the case.

The line bundle on $C$ associated to $C_n$ is $\mathcal{O}_C$ (by proposition 3.2.4).

Let $p > 0$ be an integer. Let $K$ be the field of rational functions on $S$ and $K' = K(t^{1/p})$. Let $S'$ be the germ of curve corresponding to $K'$, $\theta : S' \to S$ the canonical morphism and $P'$ the unique point of $\theta^{-1}(P)$. Let $\mathcal{D} = \theta^*(\mathcal{C})$. So we have a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\rho} & S' \\
\downarrow \theta & & \downarrow \theta \\
\mathcal{C} & \xrightarrow{\pi} & S
\end{array}
$$

where $\rho$ is flat, and for every $s' \in S'$, $\Theta$ induces an isomorphism $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$.

4.1.2. Proposition: For a suitable choice of $p$, $\mathcal{D}$ has exactly $n$ irreducible components $\mathcal{D}_1, \ldots, \mathcal{D}_n$, and for every point $s \neq P$ of $S'$, $\mathcal{D}_{1s}, \ldots, \mathcal{D}_{ns}$ are the irreducible components of $\mathcal{D}_s$, for $1 \leq i \leq n$ the restriction of $\rho : \mathcal{D}_{is} \to S'$ is flat, and $\mathcal{D}_{P'} = C_n$.

(See proposition 3.1.3)

4.1.3. Definition: A fragmented deformation of $C_n$ is a general maximal fragmented deformation of length $n$ of $C_n$ having $n$ irreducible components.
We suppose in the sequel that $C$ is a fragmented deformation of $C_n$, union of $n$ irreducible components $C_1, \ldots, C_n$.

4.1.4. Proposition: Let $I \subset \{1, \ldots, n\}$ be a nonempty subset having $m$ elements. Let $C_I = \bigcup_{i \in I} C_i$. Then the restriction of $\pi$, $C_I \to S$, is flat, and the fiber $C_{IP}$ is canonically isomorphic to $C_m$.

(See 3.2.5)

In particular there exists a filtration of ideal sheaves

$$0 \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_{n-1} \subset \mathcal{O}_C$$

such that for $1 \leq i < n$ and $s \in S \setminus \{P\}$, $\mathcal{I}_{is}$ is the ideal sheaf of $\bigcup_{j=i}^{n} C_j$, and that $\mathcal{I}_{IP}$ is that of $C_{n-i}$.

4.1.5. Definition: For $1 \leq i \leq n$, let $\pi_i : C_i \to S$ be a flat family of smooth projective irreducible curves, with a fixed isomorphism $\pi_i^{-1}(P) \simeq C$. A gluing of $C_1, \ldots, C_n$ along $C$ is an algebraic variety $D$ such that

- for $1 \leq i \leq n$, $C_i$ is isomorphic to a closed subvariety of $D$, also denoted by $C_i$, and $D$ is the union of these subvarieties.
- $\bigsqcup_{1 \leq i \leq n}(C_i \setminus C)$ is an open subset of $D$.
- There exists a morphism $\pi : D \to S$ inducing $\pi_i$ on $C_i$, for $1 \leq i \leq n$.
- The subvarieties $C = \pi_i^{-1}(P)$ of $C_i$ coincide in $D$.

For example the previous fragmented deformation $C$ of $C_n$ is a gluing of $C_1, \ldots, C_n$ along $C$.

All the gluings of $C_1, \ldots, C_n$ along $C$ have the same underlying Zariski topological space.

Let $\mathcal{A}$ be the initial gluing of the $C_i$ along $C$. It is an algebraic variety whose points are the same as those of $C$, i.e.

$$\left(\bigsqcup_{i=1}^{n} C_i\right)/\sim,$$

where $\sim$ is the equivalence relation: if $x \in C_i$ and $y \in C_j$, $x \sim y$ if and only if $x = y$, or if $x \in C_{iP} \simeq C$, $y \in C_{jP} \simeq C$ and $x = y$ in $C$. The structural sheaf is defined by: for every open subset $U$ of $\mathcal{A}$

$$\mathcal{O}_\mathcal{A}(U) = \{(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{C_1}(U \cap C_1) \times \cdots \times \mathcal{O}_{C_n}(U \cap C_n); \alpha_1|C = \cdots = \alpha_n|C\}.$$ 

For every gluing $D$ of $C_1, \ldots, C_n$, we have an obvious dominant morphism $\mathcal{A} \to D$. If follows that the sheaf of rings $\mathcal{O}_D$ can be seen as a subsheaf of $\mathcal{O}_\mathcal{A}$.

The fiber $D = \mathcal{A}_P$ is not a primitive multiple curve (if $n > 2$): if $\mathcal{I}_{C,D}$ denotes the ideal sheaf of $C$ in $D$ we have $\mathcal{I}_{C,D}^2 = 0$, and $\mathcal{I}_{C,D} \simeq \mathcal{O}_C \otimes \mathbb{C}^{n-1}$.

4.1.6. Proposition: Let $D$ be a gluing of $C_1, \ldots, C_n$. Then $\pi^{-1}(P)$ is a primitive multiple curve if and only if for every closed point $x$ of $C$, there exists a neighborhood of $x$ in $D$ that can be embedded in a smooth variety of dimension 3.
Proof. Suppose that $\pi^{-1}(P)$ is a primitive multiple curve. Then $I_C/(I_C^2 + (\pi))$ is a principal module at $x$: suppose that the image of $u \in m_{D,x}$ is a generator. The module $m_{D,x}/I_C$ is also principal (since it is the maximal ideal of $x$ in $C$): suppose that the image of $v \in m_{D,x}$ is a generator. Then the images of $u, v, \pi$ generate $m_{D,x}/m_{D,x}^2$, so according to proposition 2.1.1 we can locally embed $D$ in a smooth variety of dimension 3.

Conversely, suppose that a neighborhood of $x \in C$ in $D$ is embedded in a smooth variety $Z$ of dimension 3. The proof of the fact that $\pi^{-1}(P)$ is Cohen-Macaulay is similar to that of theorem 3.2.3. We can suppose that $\pi$ is defined on $Z$. We have $\pi|_C = \pi \notin m_{Z,x}^2$, so $\pi \notin m_{Z,x}^2$. It follows that the surface of $Z$ defined by $\pi$ is smooth at $x$, and that we can locally embed $\pi^{-1}(P)$ in a smooth surface. Hence $\pi^{-1}(P)$ is a primitive multiple curve. \hfill \Box

### 4.2. Fragmented deformations of length 2

Let $\pi : C \to S$ be a fragmented deformation of $C_2$. So $C$ has two irreducible components $C_1, C_2$. Let $\mathcal{A}$ be the initial gluing of $C_1$ and $C_2$ along $C$. For every open subset $U$ of $C$, $U$ is also an open subset of $\mathcal{A}$ and $\mathcal{O}_C(U)$ is a sub-algebra of $\mathcal{O}_A(U)$. For $i = 1, 2$, let $\pi_i : C_i \to S$ be the restriction of $\pi$. We will also denote $t \circ \pi$ by $\pi$, and $t \circ \pi_i$ by $\pi_i$. So we have $\pi = (\pi_1, \pi_2) \in \mathcal{O}_C(C)$.

Let $\mathcal{I}_C$ be the ideal sheaf of $C$ in $C$. Since $C_2 = \pi^{-1}(P)$ we have $\mathcal{I}_C \subset \langle ((\pi_1, \pi_2)) \rangle$.

Let $m > 0$ be an integer, $x \in C$, $\alpha_1 \in \mathcal{O}_{C_1,x}$, $\alpha_2 \in \mathcal{O}_{C_2,x}$. We denote by $[\alpha_1]_m$ (resp. $[\alpha_2]_m$) the image of $\alpha_1$ (resp. $\alpha_2$) in $\mathcal{O}_{C_1,x}/(\pi_1^m)$ (resp. $\mathcal{O}_{C_2,x}/(\pi_2^m)$).

#### 4.2.1. Proposition: 1 – There exists an unique integer $p > 0$ such that $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ is generated by the image of $(\pi_1^p, 0)$.

2 – The image of $(0, \pi_2^p)$ generates $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$.

3 – For every $x \in C$, $\alpha \in \mathcal{O}_{C_1,x}$ and $\beta \in \mathcal{O}_{C_2,x}$, we have $(\alpha \pi_1^p, 0) \in \mathcal{O}_{C,x}$ and $(0, \beta \pi_2^p) \in \mathcal{O}_{C,x}$.

Proof. Let $x \in C$ and $u = (\pi_1 \alpha, \pi_2 \beta)$ whose image is a generator of $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ at $x$ ($\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ is a locally free sheaf of rank 1 of $\mathcal{O}_C$-modules). Let $\beta_0 \in \mathcal{O}_{C_1,x}$ be such that $(\beta_0, \beta) \in \mathcal{O}_{C,x}$. Then the image of

$$u = (\pi_1, \pi_2)(\beta_0, \beta) = (\pi_1 (\alpha - \beta_0), 0)$$

is also a generator of $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ at $x$. We can write it $(\pi_1^p \lambda, 0)$, where $\lambda$ is not a multiple of $\pi_1$.

Now we show that $p$ is the smallest integer $q$ such that $\langle (\pi_1, \pi_2) \rangle_x$ contains the image of an element of the form $(\pi_1^q \mu, 0)$, with $\mu$ not divisible by $\pi_1$. We can write

$$(\pi_1^q \mu, 0) = (u_1, u_2)(\pi_1^p \lambda, 0) + (v_1, v_2)(\pi_1, \pi_2)$$

with $(u_1, u_2), (v_1, v_2) \in \mathcal{O}_{C,x}$. So we have $v_2 = 0$, hence $(v_1, v_2) \in \mathcal{I}_{C,x}$. So we can write $(v_1, v_2)$ as the sum of a multiple of $(\pi_1^p \lambda, 0)$ and a multiple of $(\pi_1, \pi_2)$. Finally we obtain $(\pi_1^q \mu, 0)$ as

$$(\pi_1^q \mu, 0) = (u_{12}, u_{22})(\pi_1^p \lambda, 0) + (v_{11}, 0)(\pi_1, \pi_2)^2.$$ 

In the same way we see that $(\pi_1^q \mu, 0)$ can be written as

$$(\pi_1^q \mu, 0) = (u_{1p}, u_{2p})(\pi_1^p \lambda, 0) + (v_{1p}, 0)(\pi_1, \pi_2)^p,$$

which implies immediately that $q \geq p$. 


It follows that \( p \) does not depend on \( x \) and that \( \mathcal{I}_C/(\langle \pi_1, \pi_2 \rangle) \) is a subsheaf of \( \langle (\pi_1^0, 0)/(\pi_1^{p+1}, 0) \rangle \cong \mathcal{O}_C \). Since \( \mathcal{I}_C/(\langle \pi_1, \pi_2 \rangle) \) is of degree 0 by (by corollary 3.2.4) it follows that \( \mathcal{I}_C/(\langle \pi_1, \pi_2 \rangle) \cong \langle (\pi_1^0, 0)/(\pi_1^{p+1}, 0) \rangle \), from which we deduce assertion 1- of proposition 4.2.1. The second assertion comes from the fact that \( (0, \pi_2^p) = \pi^p - (\pi_1^0, 0) \).

To prove the third, we use the fact that there exists \( \alpha' \in \mathcal{O}_{C_2,x} \) such that \( (\alpha, \alpha') \in \mathcal{O}_{C,x} \) (because \( C_1 \subset C \)). Hence \( (\pi_1^0, 0)(\alpha, \alpha') = (\pi_1^0\alpha, 0) \in \mathcal{O}_{C,x} \). Similarly, we obtain that \( (0, \pi_2^p\beta) \in \mathcal{O}_{C,x} \). \( \square \)

According to the proof of proposition 4.2.1, for every \( x \in C \), \( p \) is the smallest integer \( q \) such that there exists an element of \( \mathcal{O}_{C,x} \) of the form \( (\pi_1^q\alpha, 0) \) (resp. \( (0, \pi_2^q\alpha) \)), with \( \alpha \in \mathcal{O}_{C_1,x} \) (resp. \( \alpha \in \mathcal{O}_{C_2,x} \)) not vanishing on \( C \).

Let \( x \in C \) and \( \alpha_1 \in \mathcal{O}_{C_1,x} \). Since \( C_1 \subset C \) there exists \( \alpha_2 \in \mathcal{O}_{C_2,x} \) such that \( (\alpha_1, \alpha_2) \in \mathcal{O}_{C,x} \). Let \( \alpha'_2 \in \mathcal{O}_{C_2,x} \) such that \( (\alpha_1, \alpha'_2) \in \mathcal{O}_{C,x} \). We have then \( (0, \alpha_2 - \alpha'_2) \in \mathcal{O}_{C,x} \). So there exists \( \alpha \in \mathcal{O}_{C_2,x} \) such that \( \alpha_2 - \alpha'_2 = \pi_2^p\alpha \). It follows that the image of \( \alpha_2 \) in \( \mathcal{O}_{C_2,x}/(\pi_2^p) \) is uniquely determined. Hence we have:

**4.2.2. Proposition:** There exists a canonical isomorphism
\[
\Phi : C_1^{(p)} \rightarrow C_2^{(p)}
\]
between the infinitesimal neighborhoods of order \( p \) of \( C_1 \) and \( C_2 \) (i.e. \( \mathcal{O}_{C_1^{(p)}} = \mathcal{O}_{C_1}/(\pi_1^p) \)), such that for every \( x \in C \), \( \alpha_1 \in \mathcal{O}_{C_1,x} \) and \( \alpha_2 \in \mathcal{O}_{C_2,x} \), we have \( (\alpha_1, \alpha_2) \in \mathcal{O}_{C,x} \) if and only if \( \Phi_x([\alpha_1]_p) = [\alpha_2]_p \). For every \( \alpha \in \mathcal{O}_{C_1,x} \) we have \( \Phi_x(\alpha)|C = \alpha|C \), and \( \Phi_x(\pi_1) = \pi_2 \).

The simplest case is \( p = 1 \). In this case \( \Phi : C \rightarrow C \) is the identity and \( C = A \) (the initial gluing).

**4.2.3. Converse** - Recall that \( A \) denotes the initial gluing of \( C_1, C_2 \) (cf. 4.1.5). Let \( \Phi : C_1^{(p-1)} \rightarrow C_2^{(p-1)} \) be an isomorphism inducing the identity on \( C \) and such that \( \Phi(\pi_1) = \pi_2 \). We define a subsheaf of algebras \( \mathcal{U}_\Phi \) of \( \mathcal{O}_A \) : \( \mathcal{U}_\Phi = \mathcal{O}_A \) on \( A \setminus C \), and for every point \( x \) of \( C \)
\[
\mathcal{U}_{\Phi,x} = \{ (\alpha_1, \alpha_2) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x} ; \Phi_x([\alpha_1]_p) = [\alpha_2]_p \}.
\]
It is easy to see that \( \mathcal{U}_\Phi \) is the structural sheaf of an algebraic variety \( A_\Phi \), that the inclusion \( \mathcal{U}_\Phi \subset \mathcal{O}_A \) defines a dominant morphism \( A \rightarrow A_\Phi \) inducing an isomorphism between the underlying topological spaces (for the Zariski topology), and that the composed morphisms \( C_i \subset A \rightarrow A_\Phi, \ i = 1, 2 \), are immersions. Moreover, the morphism \( \pi : A \rightarrow S \) factorizes through \( A_\Phi \) :
\[
A \longrightarrow A_\Phi \longrightarrow S
\]
and \( \pi_\Phi : A_\Phi \rightarrow S \) is flat.

For \( 2 \leq i \leq p \), let \( \Phi^{(i)} : C_1^{(i)} \rightarrow C_2^{(i)} \) be the isomorphism induced by \( \Phi \).

**4.2.4. Proposition:** \( \pi_\Phi^{-1}(P) \) is a primitive double curve.
Proof. Let $x$ be a closed point of $C$. We first show that $\mathcal{I}_{C,x}^2 \subset (\pi)$. Let $u = (\pi_1 \alpha, \pi_2 \beta) \in \mathcal{I}_{C,x}$. Let $\beta' \in \mathcal{O}_{C_2,x}$ be such that $\Phi_x([\alpha]_p) = [\beta']_p$. We have then $v = (\alpha, \beta') \in \mathcal{O}_{C,x}$. We have $u - \pi v = (0, \pi_2 (\beta - \beta')) \in \mathcal{O}_{C,x}$. Therefore $[\pi_2 (\beta - \beta')]_p = \Phi_x(0) = 0$. Hence $\pi_2 (\beta - \beta') \in (\pi_2^2)$. We can then write

$$u = \pi v + (0, \pi_2^2 \gamma).$$

Let $u' \in \mathcal{I}_{C,x}$, that can be written as $u' = \pi v' + (0, \pi_2^2 \gamma')$. We have then

$$uu' = \pi (\pi vv' + (0, \pi_2 \gamma') v + (0, \pi_2 \gamma) v' + (0, \pi_2^{2p-1} \gamma \gamma')) \in (\pi).$$

It remains to show that $\mathcal{I}_{C,x}/(\pi) \simeq \mathcal{O}_{C,x}$. We have

$$\mathcal{I}_{C,x} = \{(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x}; \Phi_x([\pi_1 \alpha]_p) = [\pi_2 \beta]_p\} = \{(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x}; \Phi_x([\alpha]_p^{-1}) = [\beta]_p^{-1}\},$$

$$(\pi)_x = \{(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x}; \Phi_x([\alpha]_p) = [\beta]_p\}.$$ 

So if $(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{I}_{C,x}$, we have $w = \Phi_x([\alpha]_p) - [\beta]_p \in (\pi_2^{p-1})_x / (\pi_2^2)_x \simeq \mathcal{O}_{C,x}$. Hence we have a morphism of $\mathcal{O}_{C,x}$-modules

$$\lambda: \mathcal{I}_{C,x} \longrightarrow \mathcal{O}_{C,x}$$

$$\lambda((\pi_1 \alpha, \pi_2 \beta)) \longmapsto w$$

whose kernel is $(\pi)_x$. We have now only to show that $\lambda$ is surjective, which follows from the fact that $\lambda(\pi_1^p, 0) = 1$. 

$\square$

4.3. Spectrum of a fragmented deformation and ideals of sub-deformations

Let $\pi: \mathcal{C} \to S$ be a fragmented deformation of $C_n, C_1, \ldots, C_n$ the irreducible components of $\mathcal{C}$. For $1 \leq i \leq n$, let $\pi_i = \pi|_{\mathcal{C}_i}$. As in 4.2, we denote also $t \circ \pi_i$ by $\pi_i$. Let $I = \{i, j\}$ be a subset of $\{1, \ldots, n\}$, with $i \neq j$. Then $\pi: \mathcal{C}_I \to S$ is a fragmented deformation of $C_2$. According to 4.2 there exists a unique integer $p > 0$ such that $\mathcal{I}_{\mathcal{C}_I} / (\pi)$ is generated by the image of $(\pi_1^p, 0)$ (and also by the image of $(0, \pi_2^p)$). Recall that $p$ is the smallest integer $q$ such that $\mathcal{I}_{\mathcal{C}_I}$ contains a non zero element of the form $(\pi_1^q \lambda, 0)$ (or $(0, \pi_2^q \mu)$), with $\lambda|_C \neq 0$ (resp. $\mu|_C \neq 0$). Let

$$p_{ij} = p_{ji} = p,$$

and $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1 \leq i, j \leq n}$ is called the spectrum of $\mathcal{C}$.

4.3.1. Generators of $(\mathcal{I}_{C}^p + (\pi)) / (\mathcal{I}_{C}^{p+1} + (\pi))$ - Let $i, j \in \{1, \ldots, n\}$ be such that $i \neq j$. Let $x \in \mathcal{C}$. Since $\mathcal{C}_{\{i,j\}} \subset \mathcal{C}$ there exists an element $u_{ij} = (u_m)_{1 \leq m \leq n}$ of $\mathcal{O}_{C,x}$ such that $u_i = 0$ and $u_j = \pi_{ij}^{(p)}$. According to proposition 4.1.4, the image of $u_{ij}$ generates $\mathcal{I}_{C} / (\mathcal{I}_{C}^p + (\pi))$ at $x$.

According to proposition 4.2.1 and the fact that the image of $u_{ij}$ generates $\mathcal{I}_{\mathcal{C}_{\{i,j\}}} / (\mathcal{I}_{\mathcal{C}_{\{i,j\}}}^2 + (\pi))$, for every integer $m$ such that $m \neq i, j$ and that $1 \leq m \leq n$, $u_m$ is of the form $u_m = \alpha^{(m)}_{ij} \pi_{m}^{(p)}$, with $\alpha^{(m)}_{ij} \in \mathcal{O}_{C_m,x}$ invertible. Let $\alpha^{(i)}_{ij} = 0$ and $\alpha^{(j)}_{ij} = 1$.

4.3.2. Proposition: $1 - \alpha^{(m)}_{ij|C}$ is a non zero constant, uniquely determined and independent of $x$. 

2. Let $a_{ij}^{(m)} = a_{ij}^{(m)} \in \mathbb{C}$. Then we have, for all integers $i, j, k, m, q$ such that $1 \leq i, j, k, m, q \leq n$, $i \neq j$, $i \neq k$:

$$a_{ik}^{(m)} a_{ij}^{(q)} = a_{ik}^{(q)} a_{ij}^{(m)}.$$ 

In particular we have $a_{ij}^{(m)} = a_{ik}^{(m)} a_{ij}^{(k)}$ and $a_{ij}^{(m)} a_{im}^{(j)} = 1$.

Proof. Let $u'_{ij}$ have the same properties as $u_{ij}$. Then $v = u'_{ij} - u_{ij} \in I_{C,x}^{2} + (\pi)$. So the image of $v$ in $O_{C,x}$ belongs to $I_{C,x}^{2} + (\pi)$. It follows that the $m$-th component of $v$ is a multiple of $\pi^{m+1}$. Hence $a_{ij}^{(m)}$ is uniquely determined. It follows that when $x$ varies the $a_{ij}^{(m)}$ can be glued together and define a global section of $O_{C}$, which must be a constant. This proves 1-

Now we prove 2-. There exists $u \in O_{C,x}$ such that the $k$-th component of $u$ is $a_{ij}^{(k)}$, and $u$ is invertible. Then the image of $(v_{m}) = \frac{u_{ij}}{u}$ generates $I_{C}/(I_{C}^{2} + (\pi))$, and $v_{k} = 1$. Hence according to 1-, we have $v_{m|C} = a_{ik}^{(m)}$, i.e.

$$\frac{a_{ij}^{(m)}}{a_{ij}^{(k)}} = a_{ik}^{(m)}.$$ 

We have the same equality with $q$ instead of $m$, whence 2- is easily deduced. \hfill $\Box$

Let $p$ be an integer such that $1 \leq p < n$, and $(i_{1}, j_{1}), \ldots, (i_{p}, j_{p}) p$ pairs of distinct integers of $\{1, \ldots, n\}$. Then the image of $\prod_{m=1}^{p} u_{ijm}$ is a generator of $(I_{C}^{2} + (\pi))/(I_{C}^{p+1} + (\pi))$.

Let $I \subset \{1, \ldots, n\}$ be a nonempty subset, distinct from $\{1, \ldots, n\}$. Let $i \in I, j \in I$. Let

$$u_{I,i} = \prod_{j \in I} u_{ji}.$$ 

Recall that $C_{I} = \bigcup_{j \in I} C_{j} \subset C$.

4.3.3. Proposition: The ideal sheaf of $C_{I}$ is generated by $u_{I,i}$ at $x$.

Proof. According to proposition 4.1.6 there exists an embedding of a neighborhood of $x$ in a smooth variety of dimension 3. In this variety each $C_{i}$ is a smooth surface defined by a single equation. The ideal of the union of the $C_{i}$, $i \in I$ is the product of these equations. \hfill $\Box$

4.3.4. Proposition: Let $i, j, k$ be distinct integers such that $1 \leq i, j, k \leq n$. Then if $p_{ij} < p_{jk}$, we have $p_{ik} = p_{ij}$.

Proof. We can come down to the case $n = 3$ by considering $C_{i,j,k}$. We can suppose that $p_{23} \leq p_{12} \leq p_{13}$, and we must show that $p_{23} = p_{12}$. We have $u_{21} = (\pi^{p_{12}}, 0, \alpha_{21}^{(3)} \pi_{3}^{p_{23}})$, $u_{31} = (\pi_{1}^{p_{13}}, \alpha_{31}^{(2)} \pi_{2}^{p_{23}}, 0)$. So

$$u_{31} - \pi^{p_{13}-p_{12}} u_{21} = (0, \alpha_{31}^{(2)} \pi_{2}^{p_{23}}, -\alpha_{21}^{(3)} \pi_{3}^{p_{23}+p_{13}-p_{12}}) \in O_{C,x}.$$ 

Taking the image of this element in $O_{C_{12}x}$, we see that $p_{23} \geq p_{12}$, hence $p_{23} = p_{12}$. \hfill $\Box$
4.3.5. Proposition: 1 - Let $i$, $j$ be distinct integers such that $1 \leq i, j \leq n$. Then we have $I_{C,x} = (u_{ij}) + (\pi)$. 

2 - Let $v = (v_n)_{1 \leq n \leq n} \in I_{C,x}$ such that $v_i$ is a multiple of $\pi_i^p$, with $p > 0$. Then we have $v \in (u_{ij}) + (\pi^p)$.

Proof. Let $N = 1 + \max_{1 \leq k \leq n}(q_i)$, where $q_i = \sum_{j=1}^{n} p_{ij}$. For every integer $j$ such that $1 \leq j \leq n$ we have $(0, \ldots, 0, \pi_j^{q_j}, 0, \ldots, 0) \in \mathcal{O}_C(C)$. Hence $I_{C,x}^N \subset (\pi)$. We will show by induction on $k$ that $I_{C,x} \subset (u_{ij}) + (\pi) + I_{C,x}^k$. Taking $k = N$ we obtain 1-. 

For $k = 1$ it is obvious. Suppose that it is true for $k - 1 \geq 1$. It is enough to prove that $I_{C,x}^{k-1} \subset (u_{ij}) + (\pi) + I_{C,x}^k$. Let $w_1, \ldots, w_{k-1} \in I_{C,x}$. Since the image of $u_{ij}$ generates $I_{C,x}/(I_{C,x}^2 + (\pi))$, we can write $w_p$ as 

$$w_p = \lambda_p u_{ij} + \pi\mu_p + \nu_p,$$

with $\lambda_p, \mu_p \in \mathcal{O}_{C,x}$ and $\nu_p \in I_{C,x}^2$. So we have 

$$w_1 \cdots w_{k-1} = \lambda u_{ij} + \pi\mu + \nu,$$

with $\lambda, \mu \in \mathcal{O}_{C,x}$ and $\nu \in I_{C,x}^{2k-2}$. Since $2k - 2 \geq k$, we have $w_1 \cdots w_{k-1} \in (u_{ij}) + (\pi) + I_{C,x}^k$. This proves 1-. 

We prove 2- by induction on $p$. The case $p = 1$ follows 1-. Suppose that it is true for $p - 1 \geq 1$. So we can write $v$ as 

$$v = \lambda u_{ij} + \pi^{p-1}\mu,$$

with $\lambda, \mu \in \mathcal{O}_{C,x}$. We can write $v_i$ as $v_i = \alpha\pi^p$. So we have $\alpha\pi_i^p = \pi_i^{p-1}\mu_i$, whence $\mu_i = \alpha\pi_i$. Hence $\mu \in I_{C,x}$. According to 1- we can write $\mu$ as $\mu = \theta u_{ij} + \pi\tau$, with $\theta, \tau \in \mathcal{O}_{C,x}$. So 

$$v = (\lambda + \pi^{p-1}\theta)u_{ij} + \pi^p\tau,$$

which proves the result for $p$. 

\[\square\]

4.3.6. The ideal sheaves $I_{C,j}$ - Recall that $I \subset \{1, \ldots, n\}$ is a nonempty subset, distinct from $\{1, \ldots, n\}$. For every subset $J$ of $\{1, \ldots, n\}$, let $J^c = \{1, \ldots, n\}\setminus J$ and $\mathcal{O}_J = \mathcal{O}_{C,j}$. It follows from proposition 4.3.3 that $I_{C,j}$ is a line bundle on $C_{I^c}$.

From now on, we suppose that $S \subset \mathbb{C}$ and $P = 0$ (cf. proposition 3.2.6).

4.3.7. Theorem: We have $I_{C,1} \cong \mathcal{O}_{I^c}$.

Proof. By induction on $n$. If $n = 2$ the result follows from proposition 4.2.1 and the fact that $S \subset \mathbb{C}$. Suppose that it is true for $n - 1 \geq 2$. We will prove that it is true for $n$ by induction on the number of elements $q$ of $I^c$. Suppose first that $q = 1$ and let $i$ be the unique element of $I^c$. Then according to proposition 4.3.3, $I_{C,1}$ is generated by $(0, \ldots, 0, \pi_i^q, 0, \ldots, 0)$, so the result is true in this case. Suppose that it is true if $1 \leq q < k < n$, and that $q = k$. Let $K = \{1, \ldots, n - 1\}$. We can assume that $I \subset K$. 

According to proposition 4.3.3. we have, for every \( x \in C \), \( \mathcal{I}_{\mathcal{C}_i,x} \cong \mathcal{O}_{I^c x} \). We have \( \mathcal{I}_K \subset \mathcal{I}_L \), and \( \mathcal{I}_K \cong \mathcal{O}_{(n)} \). We have

\[
\mathcal{I}_L / \mathcal{I}_K = \mathcal{I}_L \mathcal{C}_K
\]

(the ideal sheaf of \( \mathcal{C}_L \) in \( \mathcal{C}_K \)). From the first induction hypothesis we have

\[
\mathcal{I}_L \mathcal{C}_K \cong \mathcal{O}_{(I \cup \{n\})^c}.
\]

So we have an exact sequence of sheaves

\[
0 \rightarrow \mathcal{O}_{(n)} \rightarrow \mathcal{I}_L \rightarrow \mathcal{O}_{I^c \setminus \{n\}} \rightarrow 0.
\]

Now we will compute \( \text{Ext}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \). According to [6], 2.3, we have an exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \rightarrow \text{Ext}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \rightarrow \text{Hom}(\text{Tor}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}).
\]

Since \( \text{Tor}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c}) \) is concentrated on \( \mathcal{C}_{I^c \setminus \{n\}} \), we have

\[
\text{Hom}(\text{Tor}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}) = \{0\}.
\]

So we have

\[
\text{Ext}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) = \text{Ext}^1_{\mathcal{C}_L} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}).
\]

Let \( \mathcal{J} \) denote the ideal sheaf of \( \mathcal{C}_{\{n\}} \) in \( \mathcal{C}_{I^c} \). The ideal sheaf of \( \mathcal{C}_{I^c \setminus \{n\}} \) is generated by \( w = (0, \ldots, 0, \pi_n^m) \), with \( m = \sum_{i \in I^c \setminus \{n\}} p_i \). So we have an exact sequence of sheaves on \( \mathcal{C}_{I^c} \)

\[
0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{I^c} \rightarrow \mathcal{O}_{I^c \setminus \{n\}} \rightarrow 0,
\]

where \( \alpha \) is the multiplication by \( w \). By the induction hypothesis there exists a surjective morphism \( \mathcal{O}_{I^c} \rightarrow \mathcal{J} \), so we get a locally free resolution of \( \mathcal{O}_{I^c \setminus \{n\}} \)

\[
\mathcal{O}_{I^c} \rightarrow \mathcal{O}_{I^c} \rightarrow \mathcal{O}_{I^c \setminus \{n\}} \rightarrow 0,
\]

that can be used to compute \( \mathcal{E}xt^1_{\mathcal{O}_{I^c}} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \). It follows easily that

\[
\mathcal{E}xt^1_{\mathcal{O}_{I^c}} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \cong \mathcal{O}_{\{n\}}/(\pi_n^m).
\]

We have \( \text{Hom}(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) = 0 \), hence

\[
\text{Ext}^1_{\mathcal{O}_{I^c}} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \cong H^0(\mathcal{E}xt^1_{\mathcal{O}_{I^c}} (\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}))
\]

\[
\cong H^0(\mathcal{O}_{\{n\}}/(\pi_n^m))
\]

\[
\cong H^0(\mathcal{O}_{\nu}/(\pi_n^m))
\]

\[
\cong C[\pi_n]/(\pi_n^m).
\]

We will now describe the sheaves \( \mathcal{E} \) such that there exists an exact sequence

\[
(2) \quad 0 \rightarrow \mathcal{O}_{\{n\}} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{I^c \setminus \{n\}} \rightarrow 0.
\]

Let \( \nu \in C[\pi_n]/(\pi_n^m) \) be associated to this exact sequence, and \( \nu \in H^0(\mathcal{O}_S) \) over \( \nu \). Let

\[
\tau : \mathcal{O}_{\{n\}} \rightarrow \mathcal{O}_{\{n\}} \oplus \mathcal{O}_{I^c}
\]

\[
u \mapsto (\nu u, w u)
\]

Then according to the preceding resolution of \( \mathcal{O}_{I^c \setminus \{n\}} \) and the construction of extensions (cf. [5], 4.2), we have \( \mathcal{E} \cong \text{coker}(\tau) \). It is easy to see that if \( \nu = -1 \) then \( \mathcal{E} \cong \mathcal{O}_{I^c} \). If \( \nu \) is invertible,
then we have also \( \mathcal{E} \simeq \mathcal{O}_{\mathcal{I}^c} \), because the corresponding extension can be obtained from the one corresponding to \( \nu = -1 \) by multiplying the left morphism of the exact sequence by \( \nu \).

A similar construction can be done for extensions of \( \mathcal{O}_{\mathcal{I}^c, x} \)-modules (for every \( x \in C \))

\[
0 \to \mathcal{O}_{(n), x} \to V \to \mathcal{O}_{\mathcal{I}^c \setminus \{n\}, x} \to 0.
\]

These extensions are classified by \( \mathcal{O}_{(n), x}/(\pi^n_m) \), and \( \mathcal{O}_{\mathcal{I}^c, x} \) corresponds to \(-1\).

Conversely we consider extensions

\[
0 \to \mathcal{O}_{(n), x} \xrightarrow{\lambda} \mathcal{O}_{\mathcal{I}^c, x} \xrightarrow{\mu} \mathcal{O}_{\mathcal{I}^c \setminus \{n\}, x} \to 0.
\]

Using the facts that \( \text{Hom}(\mathcal{O}_{(n), x}, \mathcal{O}_{\mathcal{I}^c, x}) \) is generated by the multiplication by \( \mathbf{w} \) and \( \text{Hom}(\mathcal{O}_{\mathcal{I}^c, x}, \mathcal{O}_{\mathcal{I}^c \setminus \{n\}, x}) \) by the restriction morphism, it is easy to see that \( \lambda, \mu \) are unique up to multiplication by an invertible element of \( \mathcal{O}_{\mathcal{I}^c, x} \). Hence the elements of \( \text{Ext}^1_{\mathcal{O}_{\mathcal{I}^c, x}}(\mathcal{O}_{\mathcal{I}^c \setminus \{n\}, x}, \mathcal{O}_{(n), x}) \) corresponding to the preceding extensions are exactly the invertible elements of \( \mathcal{O}_{(n), x}/(\pi^n_m) \).

It follows that the extensions \([2]\) where \( \mathcal{E} \) is locally free correspond to invertible elements of \( \mathbb{C}[\pi^n]/(\pi^n_m) \), and we have seen that in this case we have \( \mathcal{E} \simeq \mathcal{O}_{\mathcal{I}^c} \). Hence we have \( \mathcal{I}_{\mathcal{C}_i} \simeq \mathcal{O}_{\mathcal{I}^c} \) and theorem \( 4.3.7 \) is proved.

**4.3.8. Corollary:** The ideal sheaf of \( \mathcal{C}_i \) is globally generated by an element \( \mathbf{u}_i \) such that for every integer \( i \) such that \( 1 \leq i \leq n \) and \( i \notin I \), the \( i \)-th coordinate of \( \mathbf{u}_i \) belongs to \( H^0(\mathcal{O}_S) \).

**4.4. Properties of the fragmented deformations**

We use the notations of \( 4.3 \)

Let \( i \) be an integer such that \( 1 \leq i \leq n \) and \( J_i = \{1, \ldots, n\} \setminus \{i\} \). We denote by \( \mathcal{B} \) the image of \( \mathcal{O}_C \) in \( \prod_{1 \leq j \leq n} \mathcal{O}_{C_j}/(\pi^j_q) \); it is a sheaf of \( \mathbb{C} \)-algebras on \( C \). Let \( \mathcal{B}_i \) be the image of \( \mathcal{O}_{C_{J_i}} \) in \( \prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j}/(\pi^j_q) \); it is also a sheaf of \( \mathbb{C} \)-algebras on \( C \). For every point \( x \in C \) and every \( \alpha = (\alpha_m)_{1 \leq m \leq n} \) in \( \prod_{1 \leq j \leq n} \mathcal{O}_{C_j, x} \), we denote by \( b_i(\alpha) \) its image in \( \prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j, x} \) (obtained by forgetting the \( i \)-th coordinate of \( \alpha \)).

If \( p, k \) are positive integers, with \( k \leq n, x \in C \) and \( \alpha \in \mathcal{O}_{C_k, x} \), let \( [\alpha]_p \) denote the image of \( \alpha \) in \( \mathcal{O}_{C_k, x}/\pi^p_k \).

**4.4.1. Proposition:** There exists a morphism of sheaves of algebras on \( C \)

\[
\Phi_i : \mathcal{B}_i \to \mathcal{O}_C/\pi^q_i
\]

such that for every point \( x \) of \( C \) and all \( (\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{C_{J_i}, x}, \alpha_i \in \mathcal{O}_{C_i, x} \), we have

\[
\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{C, x} \text{ if and only if } \Phi_i(x)(b_i(\alpha)) = [\alpha_i]_q.
\]

**Proof.** Let \( (\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{C_{J_i}, x} \). Since \( C_{J_i} \subset C \), there exists \( \alpha_i \in \mathcal{O}_{C_i, x} \) such that

\[
(\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{C_i, x}.
\]

If \( \alpha_i \in \mathcal{O}_{C_i, x} \) has the same property, we have

\[
(0, \ldots, \alpha_i, 0, \ldots, 0) \in \mathcal{I}_{J_i, x}. \quad \text{So according to proposition } 4.3.3 \text{, we have } [\alpha_i]_q = [\alpha'_i]_q.
\]

Hence we have a well defined morphism of algebras \( \theta_x : \mathcal{O}_{C_{J_i}, x} \to \mathcal{O}_{C_{J_i}}/(\pi^q_i) \) sending

\[
(\alpha_m)_{1 \leq m \leq n, m \neq i} \to [\alpha_i]_q. \quad \text{If } j \in J_i, \text{ we have, according to proposition } 4.3.3 \,
\theta_x(0, \ldots, 0, \pi^q_j, 0, \ldots, 0) = 0. \quad \text{Hence } \theta_x \text{ induces a morphism of algebras } \mathcal{B}_{i, x} \to \mathcal{O}_{C_i, x}/(\pi^q_i). \quad \Box
\]
The morphism $\Phi_i$ has the following properties: for every point $x$ of $C$

(i) For every $\alpha = (\alpha_m)_{1 \leq m \leq n, m \neq i} \in B_i, x$, we have $\Phi_{i,x}(\alpha) = \alpha_{m|i}$ for $1 \leq m \leq n, m \neq i$.

(ii) We have $\Phi_{i,x}((\pi_m)_{1 \leq m \leq n, m \neq i}) = \pi_i$.

(iii) Let $j, k \in \{1, \ldots, n\}$ be such that $i, j, k$ are distinct. Let $v$ be the image of $u_{jk}$ in $B_i$.

Then there exists $\lambda \in O_{C_i,x}$ such that $\Phi_{i,x}(v) = \lambda \pi_j^{p_{ij}}$.

(iv) Let $j$ be an integer such that $1 \leq j \leq n$ and $j \neq i$. Let $v$ be the image of $u_{ij}$ in $B_i, x$.

Then we have $\ker(\Phi_{i,x}) = (v)$.

4.4.2. **Converse** - Let $C'$ be a gluing of $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n$ along $C$, which is a fragmented deformation of a primitive multiple curve of multiplicity $n - 1$. Let $(p_{jk})_{1 \leq j, k \leq n, j, k \neq i}$ be the spectrum of $C'$. Let $p_{ij}$, $1 \leq j \leq n, j \neq i$ be positive integers, and $p_{ii} = 0$. For $1 \leq j \leq n$, let

$$q_j = \sum_{1 \leq k \leq n} p_{kj}.$$ 

Let $B_i$ be the image of $\mathcal{O}_{C'}$ in $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j}/(\pi_j^{q_{ij}})$ and

$$\Phi_i : B_i \rightarrow \mathcal{O}_{C_i}/(\pi_i^{q_i})$$

a morphism of sheaves of algebras on $C$ satisfying properties (i), (ii), (iii) above. Let $A$ be the subsheaf of algebras of $\mathcal{A}$ defined by: $A = \mathcal{A}$ on $\mathcal{A}_{top} \setminus C$, and for every point $x$ of $C$, and every $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \prod_{m=1}^n \mathcal{O}_{C_m, x}$, $\alpha \in A_x$ if and only if $b_i(\alpha) \in B_i$ and $\Phi_i(x)(b_i(\alpha)) = [\alpha_i]_{q_i}$. It is easy to see that $A$ is the structural sheaf of a gluing of $C_1, \ldots, C_n$ along $C$, which is a fragmented deformation of a primitive multiple curve of multiplicity $n$, and that $C' = A_{i-1,i,i+1\ldots,n}$.

We give now some applications of the preceding construction.

4.4.3. **Corollary**: Let $N$ be an integer such that $N \geq \max_{1 \leq i \leq n}(q_i)$. Let $x \in C$, $\beta \in \mathcal{O}_{C_1, x} \times \cdots \mathcal{O}_{C_n, x}$ and $u \in \mathcal{O}_{C,x}$ such that $u_C \neq 0$. Suppose that $[\beta u]_N \in \mathcal{O}_{C,x}/(\pi^N)$. Then we have $[\beta]_N \in \mathcal{O}_{C,x}/(\pi^N)$.

**Proof.** By induction on $n$. It is obvious if $n = 1$. Suppose that the lemma is true for $n - 1$. Let $I = \{1, \ldots, n - 1\}$. So we have $[\beta_{C_1 \times \cdots C_{n-1}}]_N \in \mathcal{O}_{C_1,x}/(\pi_1, \ldots, \pi_{n-1})^N$ by the induction hypothesis. Let $\gamma$ (resp. $v$) be the image of $\beta$ (resp. $u$) in $\mathcal{B}_n$. To show that $[\beta]_N \in \mathcal{O}_{C,x}/(\pi^N)$ it is enough to verify that

$$\Phi_n(\gamma) = [\beta_n]_{q_n}.$$ 

We have $\Phi_n(\gamma) = [\beta_n u_n]_{q_n}$ because $[\beta u]_N \in \mathcal{O}_{C,x}/(\pi^N)$, and $\Phi_n(v) = [u_n]_{q_n}$ because $u \in \mathcal{O}_{C,x}$. So we have

$$\Phi_n(\gamma)[u_n]_{q_n} = \Phi_n(\gamma)\Phi_n(v) = \Phi_n(\gamma v) = [\beta_n u_n]_{q_n} = [\beta_n]_{q_n}[u_n]_{q_n}.$$ 

Since $u_C \neq 0$, $[u_n]_{q_n}$ is not a zero divisor in $\mathcal{O}_{C_n,x}/(\pi_{q_n}^n)$, so we have $\Phi_n(\gamma) = [\beta_n]_{q_n}$. \hfill $\square$

4.4.4. **Corollary**: Let $q = \max_{1 \leq i \leq n}(q_i)$ and $p$ the number of integers $i$ such that $1 \leq i \leq n$ and $q_i = q$. Then we have $p \geq 2$. 

Proof. Suppose that \( q_i = q \). Then we have \( \pi_i^{q-1} \neq 0 \) in \( \mathcal{O}_C/(\pi_i^q) \). Since \( \pi_i = \Phi_i((\pi_m)_{1 \leq m \leq n, m \neq i}) \), we have \( (\pi_m^{q-1})_{1 \leq m \leq n, m \neq i} \neq 0 \) in \( \mathcal{B}_i \). So we cannot have \( q_m < q_i \) for all the \( m \neq i \). \( \square \)

Let \( i \) be an integer such that \( 1 \leq i \leq n \),

\[
\mathcal{H} = \prod_{1 \leq j \leq n} (\pi_j^{q-1})/(\pi_j^q) \simeq \mathcal{O}_C^n \quad \text{(res. } \mathcal{H}_i = \prod_{1 \leq j \leq n, j \neq i} (\pi_j^{q-1})/(\pi_j^q) \simeq \mathcal{O}_C^{n-1}).
\]

It is an ideal sheaf of \( \prod_{1 \leq j \leq n} \mathcal{O}_C/(\pi_j^q) \) (resp. \( \prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_C/(\pi_j^q) \)). Let \( \mathcal{J} = \mathcal{H} \cap \mathcal{B} \) (resp. \( \mathcal{J}_i = \mathcal{H}_i \cap \mathcal{B}_i \)), which is an ideal sheaf of \( \mathcal{B} \) (resp. \( \mathcal{B}_i \)).

4.4.5. Proposition: There exists a unique \( \lambda(\mathcal{C}) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}_n(\mathbb{C}) \) such that for every \( u = (u_j)_{1 \leq j \leq n} \in \mathcal{H} \), we have \( u \in \mathcal{J} \) if and only if \( \lambda_1 u_1 + \cdots + \lambda_n u_n = 0 \). The \( \lambda_i \) are all non zero.

Proof. We have \( (\pi_m)_{1 \leq m \leq n, m \neq i}, \mathcal{J}_i = 0 \). Hence \( \pi_i \Phi_i(\mathcal{J}_i) = 0 \) and \( \Phi_i(\mathcal{J}_i) \subset (\pi_i^{q-1})/(\pi_i^q) \). The restriction of \( \Phi_i \), \( \mathcal{J}_i \rightarrow (\pi_\lambda^{q-1})/(\pi_\lambda^q) \) is a morphism \((n - 1)\mathcal{O}_C \rightarrow \mathcal{O}_C\) of vector bundles on \( C \). From this, the existence of \( (\lambda_1, \ldots, \lambda_n) \) follows from that.

If \( \lambda_i = 0 \), we have \( (0, \ldots, 0, \pi_i^{q-1}, 0, \ldots, 0) \in \mathcal{O}_C(\mathcal{C}) \). This is impossible because according to proposition 4.3.3, \( (0, \ldots, 0, \pi_i^{q-1}, 0, \ldots, 0) \) generates the ideal sheaf of \( \mathcal{C}_{\mathcal{J}_i} \) in \( \mathcal{C} \).

For all distinct integers \( i, j \) such that \( 1 \leq i, j \leq n \), let \( I_{ij} = \{1, \ldots, n\} \setminus \{i, j\} \). Then according to proposition 4.3.3, \( u_{I_{ij}} \) generates the ideal sheaf of \( \mathcal{C}_{I_{ij}} \). We have \( u_{I_{ij}} = (b_k)_{1 \leq k \leq n} \), with \( b_k = 0 \) if \( k \neq i, j \), \( b_i = \pi_i^{a_{ji}} \) and

\[
b_j = \left( \prod_{1 \leq m \leq n, m \neq i, j} a_{mj}^{(j)} \right) \pi_j^{q_j - p_{ij}}. \]

So we have \( \pi_j^{p_{ij}} u_{I_{ij}} \in \mathcal{J}_i \), which gives the equation

\[
\frac{\lambda_i}{\lambda_j} = -\prod_{1 \leq m \leq n, m \neq i, j} a_{mj}^{(j)}. \quad (3)
\]

4.4.6. Proposition: For all distinct integers \( i, j, k \) such that \( 1 \leq i, j, k \leq n \), we have

\[
a_{ki}^{(j)} = -a_{jk}^{(i)} a_{kj}^{(i)} .
\]

Proof. We need only to treat the case \( n = 3 \), and we get the preceding formula by writing that \( \lambda_3 = \frac{\lambda_1}{\lambda_2} \lambda_2 \), and by using (3). \( \square \)

4.4.7. Proposition: Let \( (\alpha_1 \pi_1^{m_1}, \ldots, \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C},x} \), with \( \alpha_1, \ldots, \alpha_n \) invertible. Let \( M = m_1 + \cdots + m_n \). then

\[
\left( \frac{1}{\alpha_1} \pi_1^{M-m_1}, \ldots, \frac{1}{\alpha_n} \pi_n^{M-m_n} \right) \in \mathcal{O}_{\mathcal{C},x} .
\]
Proof. By induction on $n$. It is obvious for $n = 1$. Suppose that it is true for $n - 1 \geq 1$. Let $I = \{1, \ldots, n - 1\}$. Then $(\alpha_1\pi_1, \ldots, \alpha_{n-1}\pi_{n-1}) \in \mathcal{O}_{C_I,x}$. Hence, by the induction hypothesis, we have

$$(\frac{1}{\alpha_1}\pi_1^{M-m_1-m_n}, \ldots, \frac{1}{\alpha_{n-1}}\pi_{n-1}^{M-m_{n-1}-m_n}) \in \mathcal{O}_{C_I,x}.$$ 

So there exists $\gamma \in \mathcal{O}_{C_n,x}$ such that

$$u = \left(\frac{1}{\alpha_1}\pi_1^{M-m_1-m_n}, \ldots, \frac{1}{\alpha_{n-1}}\pi_{n-1}^{M-m_{n-1}-m_n}, \gamma\right) \in \mathcal{O}_{C,x}.$$ 

Multiplying by $(\alpha_1\pi_1^m, \ldots, \alpha_n\pi_n^m)$ we see that $(\pi_1^{M-m_1}, \ldots, \pi_{n-1}^{M-m_{n-1}}, \gamma\alpha_n\pi_n^m) \in \mathcal{O}_{C,x}$. Subtracting $\pi^{M-m}$, we find that $(0, \ldots, 0, \gamma\alpha_n\pi_n^m, \pi_n^{M-m_n}) \in \mathcal{O}_{C,x}$. There exists $\alpha \in \mathcal{O}_{C,x}$ such that the $n$-th coordinate of $\alpha$ is $\alpha_n$, and $\alpha$ is invertible. It follows that

$$v = (0, \ldots, 0, \gamma\pi_n^m - \frac{1}{\alpha_n}\pi_n^{M-m_n}) \in \mathcal{O}_{C,x}.$$ 

Now we have

$$\pi^m u - v = \left(\frac{1}{\alpha_1}\pi_1^{M-m_1}, \ldots, \frac{1}{\alpha_n}\pi_n^{M-m_n}\right) \in \mathcal{O}_{C,x}. \quad \square$$

4.4.8. Corollary: Let $V \subset U$ be open subsets of $C$, and suppose that $U \cap C \neq \emptyset$. Let $\alpha \in \mathcal{O}_C(V)$ and $\beta \in \mathcal{O}_A(U)$ such that $\beta|_V = \alpha$. Then $\beta \in \mathcal{O}_C(U)$.

(Recall that $A$ is the initial gluing of $C_1, \ldots, C_n$ (cf. [4.1.5])).

Proof. This can be proved easily by induction on $n$, using proposition [4.4.1]. \qed

4.5. Construction of fragmented deformations

Consider a fragmented deformation

$$\pi = \pi^{[n-1]} = (\pi_1, \ldots, \pi_{n-1}) : C^{[n-1]} \rightarrow S$$

of $C_{n-1}$, with $n - 1$ irreducible components $C_1, \ldots, C_{n-1}$. Let $(p^{[n-1]}_{ij})_{1 \leq i, j < n}$ be its spectrum. For $1 \leq i < n$, let $q^{[n-1]}_i = \sum_{1 \leq j < n} p^{[n-1]}_{ij}$. We denote by $I^{[n-1]}_C$ the ideal sheaf of $C$ in $C^{[n-1]}$. Let

$$\lambda(C^{[n-1]}) = (\lambda_1, \ldots, \lambda_{n-1}).$$

Let $p_1, \ldots, p_{n-1,n}$ be positive integers, $q_i = q^{[n-1]}_i + p_{in}$ for $1 \leq i < n$, and $q_n = p_{1n} + \cdots + p_{n-1,n}$. Let $u \in H^0(I^{[n-1]}_C)$ whose image generates $I^{[n-1]}_C/(I^{[n-1]}_C + (\pi))$, of the form

$$u = (\beta_1\pi_1^{p_1}, \ldots, \beta_{n-1}\pi_{n-1,n}^{p_{n-1,n}}),$$

with $\beta_i \in H^0(\mathcal{O}_S)$ invertible for $1 \leq i < n$.

Let $B^{[n-1]}$ be the image of $\mathcal{O}_{C^{[n-1]}}$ in $\mathcal{O}_{C_1}/(\pi_1^{q_1}) \times \cdots \times \mathcal{O}_{C_{n-1}}/(\pi_{n-1}^{q_{n-1}})$. We will also denote by $u$ the image of $u$ in $B^{[n-1]}$. Let $Q = B^{[n-1]}/(u)$, $\rho : B^{[n-1]} \rightarrow Q$ the projection and $\pi_n = \rho(\pi)$.

4.5.1. Proposition: We have $\pi_n^{q_n} = 0$.\footnote{The proof of this proposition is omitted here.}
Proof. According to proposition 4.4.7, we have for every $x \in C$
\[ v = \left( \frac{1}{\beta_1} \pi_1^{q_1-p_1}, \ldots, \frac{1}{\beta_n} \pi_n^{q_n-p_{n-1,n}} \right) \in \mathcal{O}_{C^{[n-1]}_x}. \]
Hence $\pi^{q_n} = v u \in (u)$ in $\mathcal{O}_{C^{[n-1]}_x}$, and $\pi^{n}_n = 0 \,.$

4.5.2. Proposition: 1 – We have $\pi^{q_n-1}_n = 0$ if and only if
\[ \frac{\lambda_1}{\beta_1} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}} = 0. \]
We suppose now that $\frac{\lambda}{\beta_{1,C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1,C}} \neq 0$. Let $x \in C$. Then

2 – For every $\epsilon \in \mathcal{B}_x^{[n-1]}$ such that $\epsilon|_C \neq 0$, we have $\pi^{q_n-1} \epsilon \notin (u)$.

3 – For every $\eta \in \mathcal{B}_x^{[n-1]/(u)}$, and every integer $k$ such that $1 \leq k < q_n$, we have $\pi^k \eta = 0$ if and only if $\eta$ is a multiple of $\pi^{q_n-k}$.

4 – $\mathcal{B}_x^{[n-1]/(u)}$ is a flat $\mathbb{C}[\pi]/(\pi^{q_n})$-module.

Proof. We have $\pi^{q_n-1}_n = 0$ if and only if $(\pi^{q_n-1}_1, \ldots, \pi^{q_n-1}_n) \in (u)$ in $\mathcal{B}_x^{[n-1]}$. We have, in $\mathcal{O}_{C_1} \times \cdots \times \mathcal{O}_{C_{n-1}}$,
\[ (\pi^{q_n-1}_1, \ldots, \pi^{q_n-1}_n) = (\beta_1 \pi_1, \ldots, \beta_{n-1} \pi_n^{q_n-1}), \left( \frac{1}{\beta_1} \pi_1^{q_1-1}, \ldots, \frac{1}{\beta_p} \pi_p^{q_p-1} \right), \]
and $\pi^{q_n-1} = 0$ if and only if there exist $\eta \in \mathcal{O}_{C[n-1],x}$, $a_i \in \mathcal{O}_{C_i,x}$, $1 \leq i < n$, such that
\[ (\pi^{q_n-1}_1, \ldots, \pi^{q_n-1}_n) = \eta u + (a_1 \pi^{q_n-1}_1, \ldots, a_{n-1} \pi^{q_n-1}_n). \]
This equality is equivalent to
\[ (\frac{1}{\beta_1} \pi_1^{q_1-1}, \ldots, \frac{1}{\beta_{n-1}} \pi_{n-1}^{q_{n-1}-1}) - \eta = (a_1 \pi_1^{q_1-1}, \ldots, a_{n-1} \pi_{n-1}^{q_{n-1}-1}). \]
Since for $1 \leq i < n$, we have $(0, \ldots, 0, \pi_i^{q_i-1}, 0, \ldots, 0) \in \mathcal{O}_{C[n-1],x}$, we have $\pi^{q_n-1} = 0$ if and only if
\[ (\frac{1}{\beta_1} \pi_1^{q_1-1}, \ldots, \frac{1}{\beta_{n-1}} \pi_{n-1}^{q_{n-1}-1}) \in \mathcal{O}_{C[n-1],x}. \]
So the result of 1- follows from the definition of $\lambda(C^{[n-1]})$ (cf. prop. 4.4.5), 2- is an easy consequence.

Now we prove 3-, by induction on $k$. Suppose that it is true for $k = 1$, and that $\pi^k \eta = 0$, with $2 \leq k < q_n$. We have $\pi^{k-1} \eta$, $\eta$, $\pi^{q_n} \eta = 0$, so according to the induction hypothesis, $\pi^{q_n} \eta$ is a multiple of $\pi^{q_n-k+1}$. $\pi^{q_n} \eta = \pi^{q_n-k+1} \lambda$. So $\pi^{q_n-k} \eta = 0$. Since 3 is true for $k = 1$, we can write $\eta - \pi^{q_n-k} \lambda = \pi^{q_n-k+1} \epsilon$, i.e. $\eta = \pi^{q_n-k} \lambda + \pi^{q_n-k+1} \epsilon$, and 3- is true for $k$.

II remains to prove 3- for $k = 1$. Suppose that $\pi \eta = 0$ (with $\eta \neq 0$). We can write $\eta$ as $\eta = \pi^m \theta$, where $\theta$ is not a multiple of $\pi$, and $0 \leq m < q_n$. Let $\theta \in \mathcal{B}_x^{[n-1]}$ be over $\theta$. Since $\mathcal{B}_x^{[n-1]} = (u) + (\pi)$ according to proposition 4.3.5, the condition “$\theta$ is not a multiple of $\pi$” is equivalent to $\bar{\theta} \notin \mathcal{B}_x^{[n-1]}$. We have $\pi^{m+1} \bar{\theta} \in (u)$, so according to 2-, we have $m + 1 \geq q_n$, which proves 3- for $k = 1$. The last assertion is an easy consequence of 3-. \qed
4.5.3. Example: Let $N$ be an integer, $s \in H^0(O_S)$ invertible, and $k,l$ integers such that $1 \leq k,l < n$, $k \neq l$. Suppose that for every integer $i$ such that $1 \leq i < n$ and $i \neq k$ we have $N > p^{[n-1]}_k$ and $N \geq q^{[n-1]}_i - q^{[n-1]}_k + p^{[n-1]}_k$. We take $u = u_{kl} = s \pi^N$. We have then $\beta_i = \alpha_{kl}^{(i)}$ if $i \neq k$, and $\beta_k = -s$. The condition $\sum_{1 \leq i < n, i \neq k} \frac{\lambda_i}{\beta_{1|C}} + \frac{\lambda_k}{\beta_{n-1|C}} \neq 0$ is fulfilled if and only if

$$
\sum_{1 \leq i < n, i \neq k} \frac{\lambda_i}{a_{kl}^{(i)}} - \frac{\lambda_k}{s|C} \neq 0.
$$

4.5.4. Construction of fragmented deformations – Suppose that $\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$. From proposition 4.5.2, it is easy to prove that

1. There exists a flat morphism of algebraic varieties $\tau : Y \to \text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$ with a canonical isomorphism of sheaves of $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$-algebras $O_Y \simeq \mathcal{Q}$, such that $\tau^{-1}(*) = C$ (where $*$ is the closed point of $\text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n})))$.
2. There exist a family of smooth curves $C_n$ and a flat morphism $\pi_n : C_n \to S$ extending $\tau$ (recall that $S$ is a germ). Hence $Y$ is the inverse image of the subscheme of $C_n$ corresponding to the ideal sheaf $(\pi_n^{q_n})$. The existence of $C_n$ can be proved using Hilbert schemes of curves in projective spaces. Of course $C_n$ need not be unique.

We obtain a gluing $C$ of $C_1, \ldots, C_n$ by defining the sheaves of algebras $O_C$ (on the Zariski topological space corresponding to the initial gluing $\mathcal{A}$) as in 4.4.2 using for $\Phi_n$ the quotient morphism $B^{[n-1]} \to \mathcal{Q}$. It is easy to see that $\pi^{-1}(P)$ is a primitive multiple curve $C_n$ of multiplicity $n$ extending $C_{n-1}$, hence $C$ is a fragmented deformation of $C_n$.

4.5.5. Remark: 1 – The multiple curve $C_n$ depends on the choice of the family $C_n$ extending the family $Y$ parametrized by $\text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n})$).

2 – The multiple curve $C_{n-1}$ is completely defined by $B^{[n-1]}$, because $(\pi_n^q) \times \cdots (\pi_{n-1}^{q-1}) \subset (\pi)$. But it is not enough to know $B^{[n-1]}$ and $u$ to define $C_n$. In fact we need $O_{C_i}/(\pi_i^{q_i+1})$, $1 \leq i \leq n$.

4.6. Basic elements

We use the notations of 4.3 and 4.4.

Let $m = (m_1, \ldots, m_n)$ be an $n$-tuple of positive integers, and

$$
\Pi^m = (\pi_1^{m_1}) \times \cdots \times (\pi_n^{m_n}).
$$

4.6.1. Definition: Let $x \in C$. An element $u$ of $O_{C,x}$ is called basic at order $m$ if there exist polynomials $P_1, \ldots, P_n \in \mathbb{C}[X]$ such that

$$
u \equiv (P_1(\pi_1), \ldots, P_n(\pi_n)) \quad (\text{mod. } \Pi^m).
$$

If $u = (P_1(\pi_1), \ldots, P_n(\pi_n))$, we say that $u$ is basic.

Let $q = (q_1, \ldots, q_n)$. Then according to corollary 4.4.8, if $u$ is basic at order $q$, then for every $y \in C$, we have $(P_1(\pi_1), \ldots, P_n(\pi_n)) \in O_{C,y}$. So $(P_1(\pi_1), \ldots, P_n(\pi_n))$ is defined on a neighborhood of $C$. 
4.6.2. Lemma: Let \( u, v, w \in \mathcal{O}_{\mathcal{C},x} \) such that \( w = uw \) and \( w \neq 0 \). Suppose that \( u \) and \( w \) are basic at every order. Then \( v \) is basic at every order.

Proof. Let \( N \) be a positive integer such that \( N > 0 \) and \( N = (N, \ldots, N) \). Suppose that \( w \equiv (Q_1(\pi_1), \ldots, Q_n(\pi_n)) \pmod (\pi^N) \), where \( Q_1, \ldots, Q_n \in \mathbb{C}[X] \). Let \( m = (m_1, \ldots, m_n) \) be an \( n \)-tuple of positive integers, and \( v = (v_i)_{1 \leq i \leq n} \). Suppose that

\[
Q_i(\pi_i) \equiv P_i(\pi_i).v_i \pmod (\pi^N)
\]

for \( 1 \leq i \leq n \). We can write \( P_i(X) \) as \( P_i(X) = X^{n_i}R_i(X) \), where \( R_i(X) \in \mathbb{C}[X] \) is such that \( R_i(0) \neq 0 \). Then \( Q_i(X) \) is also divisible by \( X^{n_i} \): \( Q_i(X) = X^{n_i}S_i(X) \), and we have in \( \mathcal{O}_{\mathcal{A},x} \):

\[
S_i(\pi_i) \equiv R_i(\pi_i).v_i \pmod (\pi^N)
\]

for some integer \( N' > 0 \). We can write \( R_i(X) = a_i.(1 - X.T_i(X)) \), with \( a_i \in \mathbb{C}^* \), \( T_i \in \mathbb{C}(X) \). We have then

\[
v_i \equiv \frac{S_i(\pi_i)}{a_i} \sum_{p=1}^{m_i-1} (\pi_i T_i(\pi_i))^p \pmod (\pi^m).
\]

\( \square \)

For \( 1 \leq i \leq n \), let \( u_{(i)} = ((u_{(i)j})_{1 \leq j \leq n} \) be a generator of the ideal sheaf \( \mathcal{I}_{\mathcal{C}_i} \) of \( \mathcal{C}_i \) in \( \mathcal{C} \), such that for \( 1 \leq j \leq n \), \( u_{(i)j} \in \mathbb{C}[[\pi_j]] \) (cf. corollary 4.3.8).

4.6.3. Proposition: Let \( v \in \mathcal{O}_{\mathcal{C},x} \), then \( v \) is basic at every order if and only if for every \( n \)-tuple \( m \) of positive integers, there exist an integer \( q > 0 \) and \( P_1, \ldots, P_q \in \mathbb{C}[X] \) such that

\[
v \equiv \sum_{1 \leq j \leq q} P_j(\pi).u_{(i)}^j \pmod \Pi^m.
\]

Proof. We use the notations of the proof of lemma 4.6.2. Suppose that \( v = (v_j)_{1 \leq j \leq n} \) is basic at every order. Let \( N \) be a positive integer and \( N = (N, \ldots, N) \). We will prove by induction on \( q \geq 0 \) that we can write \( v \) as

\[
v \equiv \sum_{0 \leq j \leq q} P_j(\pi).u_{(i)}^j + \gamma_q u_{(i)}^{q+1} \pmod \Pi^N
\]

with \( P_0, \ldots, P_q \in \mathbb{C}[X] \), and \( \gamma_q \in \mathcal{O}_{\mathcal{C},x} \). This proves proposition 4.6.3 if \( q \) and \( N \) are big enough.

For \( q = 0 \), we have \( v \equiv P(\pi)(\mod \pi^N) \), for some \( P \in \mathbb{C}[X] \), and we can take \( P_0 = P \). Suppose that the result is true for \( q \) and that we have (4). Since \( v - \sum_{1 \leq j \leq q} P_j(\pi).u_{(i)}^j \) is basic at any order, using the same method as in the proof of lemma 4.6.2, we see that \( \gamma_q \) is basic at order \( N' \), where \( N' = (N', \ldots, N') \), for some integer \( N' \gg 0 \). As in the case \( q = 0 \) we have

\[
\gamma_q \equiv P_{q+1}(\pi) + u_{(i)}\gamma_{q+1} + \gamma_{q+2} \pmod \Pi^N,
\]

with \( P_{q+1} \in \mathbb{C}[X] \). Hence

\[
v \equiv \sum_{0 \leq j \leq q+1} P_j(\pi).u_{(i)}^j + \gamma_{q+1} u_{(i)}^{q+2} \pmod \Pi^N.
\]
4.6.4. Proposition: Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{C,x} \) be such that there exists \( P_1, \ldots, P_{n-1} \in \mathbb{C}[X] \) such that, for \( 1 \leq i \leq n-1 \), we have \( \alpha_i \equiv P_i(\pi_i) \pmod{\pi_i^{q_i})} \). Then there exists \( P_n \in \mathbb{C}[X] \) such that \( \alpha_n \equiv P_n(\pi_n) \pmod{\pi_n^{q_n})} \), i.e. \( \alpha \) is a basic element of order \( q \).

Proof: By induction on \( n \). The case \( n = 2 \) is an easy consequence of proposition 4.2.2. Suppose that \( n \geq 3 \) and that the result is true for \( n - 1 \).

By subtracting multiples of \((0, \ldots, 0, \pi_i^{q_i}, 0, \ldots, 0)\) we may assume that for \( 1 \leq i \leq n-1 \), \( \alpha_i \in \mathbb{C}[\pi_i] \). By subtracting a regular function on a neighborhood of \( C \) in \( \mathcal{C} \), and a multiple of \((\pi_1^{q_1}, 0, \ldots, 0)\) we may also assume that \( \alpha_1 = 0 \). The ideal sheaf of \( \mathcal{C}_1 \) is generated by \( u_{(1)} \). We can then write \( \alpha = \beta u_{(1)} \), with \( \beta = (\beta_i)_{1 \leq i \leq n} \in \mathcal{O}_{C,x} \). We have

\[
(\alpha_1, \ldots, \alpha_{n-1}) = (\beta_1, \ldots, \beta_{n-1}), (u_{(1)}), (u_{(1)} u_{(1)n-1})
\]

hence by lemma 4.6.2 \((\beta_1, \ldots, \beta_{n-1})\) is a basic element at any order. By the induction hypothesis, there exists \( Q \in \mathbb{C}[X] \) such that \( \beta_n \equiv Q(\pi_n) \pmod{\pi_n^{q_n}} \). Since \( u_{(1)n} \) is a multiple of \( \pi_n^{p_n} \) (from the definition of \( p_{1n} \)), it follows that \( \alpha_n \equiv u_{(i)n} Q(\pi_n) \pmod{\pi_n^{q_n}} \). \( \square \)

4.7. Simple primitive curves and fragmented deformations

Let \( C_n \) be a primitive multiple curve of multiplicity \( n \) and associated smooth curve \( C \). Let \( \mathcal{I}_C \) be the ideal sheaf of \( C \) in \( C_n \). It is obvious from proposition 4.3.5 1-., that if there exists a fragmented deformation of \( C_n \), then we have \( \mathcal{I}_{C,C_n} \simeq \mathcal{O}_{C,n-1} \), i.e. \( C_n \) is simple (cf. 2.4). Conversely we have

4.7.1. Theorem: Let \( C_n \) be a simple primitive multiple curve of multiplicity \( n \). Then there exists a fragmented deformation of \( C_n \).

Proof: According to theorem 2.4.1 there exists a flat family of smooth projective curves \( \tau : C \rightarrow \mathbb{C} \) such that \( \tau^{-1}(0) \simeq C \) and that \( C_n \) is isomorphic to the \( n \)-th infinitesimal neighborhood of \( C \) in \( \mathcal{C} \). Let \( \rho_n : \mathbb{C} \rightarrow \mathbb{C} \) be the map defined by \( \rho_n(z) = z^n \), and \( \theta = \rho_n \circ \tau : C \rightarrow \mathbb{C} \). It is a flat morphism, \( \theta^{-1}(0) = C_n \), and for every \( z \neq 0 \) in the image of \( \tau \), \( \theta^{-1}(z) \) is a disjoint union of \( n \) smooth irreducible curves. We can then apply the process of proposition 3.1.3 to obtain the desired fragmented deformation: it is \( C \times_{\mathbb{C}} \mathbb{C} \)

\[
\begin{array}{ccc}
\mathcal{C} \times_{\mathbb{C}} \mathbb{C} & \xrightarrow{\pi} & \mathbb{C} \\
\downarrow & & \downarrow \rho_n \\
\mathcal{C} & \xrightarrow{\theta} & \mathbb{C}
\end{array}
\]

\( \square \)

4.7.2. Remark: let \((p_{ij})\) be the spectrum of the fragmented deformation constructed in the proof of theorem 4.7.1. Then it is easy to see that \( p_{ij} = 1 \) for \( 1 \leq i, j \leq n, i \neq j \). If \( x \in C \), then
\[(C \times_C C)_x = O_{C,x} \otimes_{O_{C,x}} O_{C,x},\] and if \(t = I_C \in O_{C,x}\), we have for \(1 \leq k \leq n\)
\[(\pi_1, \ldots, \pi_{k-1}, 0, \pi_{k+1}, \ldots, \pi_n) = \frac{1}{n-1}(1 \otimes t - e^{\frac{2\pi i}{n}}(t \otimes 1)) .\]

5. Stars of a curve

5.1. Definitions

Let \(S\) be a smooth irreducible curve, and \(P \in S\) (we can also take for \((S, P)\) the germ of a smooth curve). Let \(n\) be a positive integer.

5.1.1. Definition: An \(n\)-star (or more simply, a star) of \((S, P)\) is an algebraic variety \(\mathcal{S}\) such that

1. \(\mathcal{S}\) is the union of \(n\) irreducible components \(S_1, \ldots, S_n\), with fixed isomorphisms \(S_i \simeq S\), \(1 \leq i \leq n\).
2. For \(1 \leq i < j \leq n\), \(S_i \cap S_j\) has only one closed point, namely \(P\).
3. There exists a morphism \(\pi: \mathcal{S} \to S\), such that for \(1 \leq i \leq n\), the restriction \(\pi|_{S_i}: S_i \to S\) is the isomorphism \(S_i \simeq S\) of (i).

All the \(n\)-stars of \((S, P)\) have the same underlying Zariski topological space \(S(n)\) and set of closed points. The latter is \(\bigcup_{1 \leq i \leq n} \hat{S}_i)/\sim\), where \(\hat{S}_i\) is the set of closed points of \(S_i\), and the equivalence relation \(\sim\) is defined by: for \(x \in \hat{S}_i\) and \(y \in \hat{S}_j\), \(x \sim y\) if and only if \(i = j\) and \(x = y\), or \(x = P \in \hat{S}_i\) and \(y = P \in \hat{S}_j\). An open subset of \(\mathcal{S}\) is defined by open subsets \(U_1\) of \(S_1\), \ldots, \(U_n\) of \(S_n\), such that for \(1 \leq i < j \leq n\), we have \(P \in U_i\) if and only if \(P \in U_j\).

The initial star \(\mathcal{S}_0\) of \((S, P)\) is defined as follows: for every open subset \(U\) of \(S(n)\), \(O_{\mathcal{S}_0}(U)\) is the set of \((\alpha_1, \ldots, \alpha_n) \in O_{S_1}(U \cap S_1) \times \cdots O_{S_n}(U \cap S_n)\) such that if \(P \in U\) then \(\alpha_1(P) = \cdots = \alpha_n(P)\).

For every \(n\)-star \(\mathcal{S}\) of \((S, P)\), there is a unique dominant morphism \(\mathcal{S}_0 \to \mathcal{S}\) inducing the identity on each component. So \(O_{\mathcal{S},P}\) is a subring of \(O_{\mathcal{S}_0,P}\).

Note that (iii) is equivalent to

(iii)’ For every \(\alpha \in O_{\mathcal{S},P}\), we have \((\alpha, \ldots, \alpha) \in O_{\mathcal{S},P}\).

5.1.2. Definition: An oblate \(n\)-star (or more simply, an oblate star) of \((S, P)\) is an \(n\)-star \(\mathcal{S}\) such that some neighborhood of \(P\) in \(\mathcal{S}\) can be embedded in a smooth surface.

5.1.3. Proposition: An \(n\)-star \(\mathcal{S}\) is oblate if and only if \(\pi^{-1}(P) \simeq \text{spec}(\mathbb{C}[X]/(X^n))\).

(cf. prop. 4.1.6).

Let \(I \subset \{1, \ldots, n\}\) be a nonempty subset. Let \(\mathcal{S}^{(I)} = \bigcup_{i \in I} S_i \subset \mathcal{S}\). If \(\mathcal{S}\) is oblate then \(\mathcal{S}^{(I)}\) is oblate too.
5.2. Properties of oblate stars

Let \( \mathcal{S} \) be an oblate \( n \)-star of \( S \). Recall that \( t \) denotes a generator of the maximal ideal of \( P \) in \( S \). We will denote this generator on \( S_i \subset \mathcal{S} \) by \( t_i \). We will also denote by \( \pi \) the element \( t \circ \pi \) of the maximal ideal of \( P \) in \( \mathcal{S} \). Let \( I_P \) be the ideal sheaf of \( P \) in \( \mathcal{S} \).

We begin with 2-stars:

5.2.1. Proposition: Suppose that \( n = 2 \). Then

1 – There exists a unique integer \( p > 0 \) such that \( I_{P,P}/(\pi) \) is generated by the image of \( (t_1^0, 0) \).
2 – The image of \( (0, t_2^0) \) is also a generator of \( I_{P,P}/(\pi) \).
3 – \( (0, t_2^0) \) (resp. \( (t_1^0, 0) \)) is a generator of the ideal sheaf of \( S_1 \) (resp. \( S_2 \)) at \( P \).
4 – \( \mathcal{O}_{S(2),P} \) consists of pairs \( (\alpha, \beta) \in \mathcal{O}_{S,P} \times \mathcal{O}_{S,P} \) such that \( \alpha - \beta \in (t^0) \).

(cf. prop. 4.2.1 and 4.2.2).

Now suppose that \( n \geq 2 \). Let \( I = \{i, j\} \subset \{1, \ldots, n\} \), with \( i \neq j \). Then \( S_i \cup S_j \subset \mathcal{S} \) is a 2-star of \( S \). Hence by proposition 5.2.1 there exists a unique integer \( p_{ij} > 0 \) such that \( I_{P,P}/(\pi) \) (on \( S_i \cup S_j \)) is generated by the image of \( (t_{ij}^p, 0) \) (and also by the image of \( (0, t_{ji}^p) \)). Let \( p_{ii} = 0 \). Then the symmetric matrix \( (p_{ij})_{1 \leq i,j \leq n} \) is called the spectrum of \( \mathcal{S} \).

There exists an element \( \nu_{ij} = (\nu_m)_{1 \leq m \leq n} \) such that \( \nu_i = 0 \) and \( \nu_j = t_{ij}^{p_{ij}} \). For every integer \( m \) such that \( 1 \leq m \leq n \), \( m \neq i, j \), there exists an invertible element \( \beta_{ij}^{(m)} \in \mathcal{O}_{S,P} \) such that \( \nu_m = \beta_{ij}^{(m)} t_{ij}^{p_{ij}} \). Let \( \beta_{ij}^{(i)} = 0, \beta_{ij}^{(j)} = 1 \).

5.2.2. Proposition: Let \( b_{ij}^{(m)} = \beta_{ij}^{(m)}(P) \in \mathbb{C} \). Then we have, for all integers \( i,j,k,m,q \) such that \( 1 \leq i, j, k, m, q \leq n \), \( i \neq j, i \neq k \)

\[
\begin{align*}
b_{ij}^{(m)} b_{ij}^{(q)} &= b_{ik}^{(q)} b_{ij}^{(m)}.
\end{align*}
\]

In particular we have \( b_{ij}^{(m)} = b_{ik}^{(m)} b_{ij}^{(k)} \) and \( b_{ij}^{(m)} b_{im}^{(j)} = 1 \).

For all distinct integers \( i, j, k \) such that \( 1 \leq i, j, k \leq n \), we have

\[
\begin{align*}
b_{ki}^{(j)} &= -b_{ik}^{(j)} b_{kj}^{(k)}.
\end{align*}
\]

(cf. prop. 4.3.2 and 4.4.6).

Let \( p \) be an integer such that \( 1 \leq p < n \), and \( (i_1, j_1), \ldots, (i_p, j_p) \) \( p \) pairs of distinct integers of \( \{1, \ldots, n\} \). Then the image of \( \prod_{m=1}^{p} v_{i_m j_m} \) is a generator of \( (I_{P,P}^p + (\pi))/(I_{P,P}^{p+1} + (\pi)) \).

Let \( I \subset \{1, \ldots, n\} \) be a nonempty subset, distinct from \( \{1, \ldots, n\} \). Let \( i \in \{1, \ldots, n\} \setminus I \). Let

\[
\begin{align*}
v_{I,i} = \prod_{j \in I} v_{ji}.
\end{align*}
\]

5.2.3. Proposition: The ideal sheaf of \( \mathcal{S}^{(I)} \) in \( \mathcal{S} \) is generated by \( v_{I,i} \) at \( P \).
(cf. prop. 4.3.3).

Note that if \( I = \{1, \ldots, n\} \setminus \{i\} \) then \( v_{I,i\mid S_j} = 0 \) if \( j \neq i \), and \( v_{I,i\mid S_i} = t_i^q \) with \( q_i = \sum_{1 \leq j \leq n} p_{ij} \).

Let \( i \) be an integer such that \( 1 \leq i \leq n \) and \( J_i = \{1, \ldots, n\} \setminus \{i\} \). Let \( \mathcal{K}_i \) be the image of \( O_\mathcal{S} \) in \( \prod_{1 \leq j \leq n, j \neq i} O_{S_j}/(t_j^q) \). We can view \( \mathcal{K}_i \) as a \( \mathbb{C} \)-algebra. For every \( \alpha = (\alpha_m) \in O_{S_i,P} \), let \( k_i(\alpha) \) be the image of \( \alpha \) in \( \mathcal{K}_i \).

5.2.4. Proposition: There exists a morphism of \( \mathbb{C} \)-algebras
\[
\Psi_i : \mathcal{K}_i \rightarrow O_{S_i,P}/(t_i^q)
\]
such that for every \( (\alpha_m)_{1 \leq m \leq n, m \neq i} \in O_{S(i),P} \), \( \alpha_i \in O_{S_j,P} \), we have \( \alpha = (\alpha_m)_{1 \leq m \leq n} \in O_{S,P} \) if and only if \( \Psi_i(k_i(\alpha)) = [\alpha_i]_{q_i} \).

(cf. prop. 4.4.1).

The morphism \( \Psi_i \) has the following properties:

(i) For every \( (\alpha_m)_{1 \leq m \leq n, m \neq i} \in O_{S(i),P} \), we have \( \Psi_i(\alpha)(P) = \alpha_m(P) \) for \( 1 \leq m \leq n, m \neq i \).
(ii) We have \( \Psi_i((t_m)_{1 \leq m \leq n, m \neq i}) = t_i \).
(iii) Let \( j, k \in \{1, \ldots, n\} \) be such that \( i, j, k \) are distinct. Let \( \mathbf{w} \) be the image of \( v_{jk} \) in \( \mathcal{B}_i \).
Then there exists \( \lambda \in O_{S_i,P}^* \) such that \( \Psi_i(\mathbf{w}) = \lambda t_i^{p_{ij}} \).
(iv) Let \( j \) be an integer such that \( 1 \leq j \leq n \) and \( j \neq i \). Let \( \mathbf{w} \) be the image of \( v_{ij} \) in \( \mathcal{K}_i \).
Then we have \( \ker(\Psi_i) = (\mathbf{w}) \).

5.2.5. Converse – Let \( \mathcal{S}^{[n-1]} \) be a \((n-1)\)-star of \( S \), with components \( S_1, \ldots, S_{n-1} \), of spectrum \((p_{jk})_{1 \leq j,k \leq n-1}\). Let \( p_{nj} = p_{jn}, 1 \leq j < n \) be positive integers, and \( p_{mm} = 0 \). For \( 1 \leq j \leq n \), let \( q_j = \sum_{1 \leq k \leq n} p_{kj} \).

Let \( S_n \) be another copy of \( S \). Let \( \mathcal{K}_n \) be the image of \( O_{\mathcal{S}^{[n-1]}} \) in \( \prod_{1 \leq j \leq n-1} O_{S_j}/(t_j^q) \) and
\[
\Psi_n : \mathcal{K}_n \rightarrow O_{S_n}/(t_n^q)
\]
a morphism of \( \mathbb{C} \)-algebras satisfying properties (i), (ii), (iii) above. Let \( \mathcal{K} \) be the subsheaf of algebras of \( O_{\mathcal{S}_0} \) defined by: \( \mathcal{K} = O_{\mathcal{S}_0} \) on \( \mathcal{S}_0 \setminus \{P\} \), and for every \( \alpha = (\alpha_m)_{1 \leq m \leq n} \in O_{\mathcal{S}_0,P} \), \( \alpha \in \mathcal{K}_P \) if and only if \( \Psi_n(\alpha') = [\alpha_n]_{q_n} \) (where \( \alpha' \) is the image of \( (\alpha_m)_{1 \leq m \leq n-1} \) in \( \mathcal{K}_n \)).

It is easy to see that \( \mathcal{K} \) is the structural sheaf of an oblate \( n \)-star of \( S \).

Let \( \mathcal{H} = \prod_{1 \leq j \leq n} (t_j^{q_{j-1}})/(t_j^q) \cong \mathbb{C}^n \) and \( \mathcal{K} \) be the image of \( O_{\mathcal{S}} \) in \( \prod_{1 \leq j \leq n} O_{S_j}/(t_j^q) \). We can view \( \mathcal{K} \) as a \( \mathbb{C} \)-algebra. Let \( \mathcal{J} = \mathcal{H} \cap \mathcal{K} \).

5.2.6. Proposition: There exists a unique \( \lambda(\mathcal{S}) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}_n(\mathbb{C}) \) such that for every \( u = (u_j)_{1 \leq j \leq n} \in \mathcal{H} \), we have \( u \in \mathcal{J} \) if and only if \( \lambda_1 u_1 + \cdots + \lambda_n u_n = 0 \). The \( \lambda_i \) are all non zero.
We have 

\[ \frac{\lambda_i}{\lambda_j} = - \prod_{1 \leq m \leq n, m \neq i, j} b_{mi}^{(j)}. \]

### 5.3. Construction of oblate stars of a curve

Consider an oblate \((n - 1)\)-star of \(S, S^{[n-1]}\), with \(n - 1\) irreducible components \(S_1, \ldots, S_{n-1}\), copies of \(S\). Let \((p_{ij}^{[n-1]})_{1 \leq i, j \leq n}\) be its spectrum. For \(1 \leq i < n\), let \(q_i^{[n-1]} = \sum_{1 \leq j < n} p_{ij}^{[n-1]}\). We denote by \(I_{\omega_p^{[n-1]}}\) the ideal of \(P\) in \(O_{\omega_p^{[n-1]}, \omega}\). Let \(\lambda(S^{[n-1]}) = (\lambda_1, \ldots, \lambda_{n-1})\).

Let \(p_{1n}, \ldots, p_{n-1,n}\) be positive integers, \(q_i = q_i^{[n-1]} + p_{in}\) for \(1 \leq i < n\), and \(q_n = p_1 + \cdots + p_{n-1,n}\). Let \(u \in I_{\omega_{p_1^{[n-1]}}^{[n-1]}}\) whose image generates \(I_{\omega_p^{[n-1]}}^{[n-1]}/((I_{\omega_p^{[n-1]}}^{[n-1]})^2 + (\pi))\), of the form

\[ u = (\beta_1 t_1^{p_{1n}}, \ldots, \beta_n t_n^{p_{n-1,n}}), \]

with \(\beta_i \in O_{\omega_i, P}\) invertible for \(1 \leq i < n\).

Let \(K^{[n-1]}\) be the image of \(O_{\omega_p^{[n-1]}, \omega}\) in \(O_{\omega_1}/(t_1) \times \cdots \times O_{\omega_{n-1}}/(t_{n-1})\). We will also denote by \(u\) the image of \(u\) in \(K^{[n-1]}\). Let \(Q = K^{[n-1]}/(u), \rho : K^{[n-1]} \to Q\) the projection and \(t_n = \rho(\pi)\).

#### 5.3.1. Proposition: 1

- We have \(t_n^{q_n} = 0\).

- We have \(t_n^{q_n} = 0\) if and only if

\[ \frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} = 0. \]

We suppose now that \(\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0\). Then

- For every \(\epsilon \in K^{[n-1]}\) such that \(\epsilon(P) \neq 0\), we have \(t_n^{q_n} \epsilon \not\in (u)\).  

- For every \(\eta \in K^{[n-1]}/(u)\), and every integer \(k\) such that \(1 \leq k < q_n\), we have \(t_n^k \eta = 0\) if and only if \(\eta\) is a multiple of \(t_n^{q_n-k}\).

- \(K^{[n-1]}/(u)\) is a flat \(\mathbb{C}[t_n]/(t_n^{q_n})\)-module.

(cf. prop. 4.5.2).

#### 5.3.2. Construction of stars of a curve

Suppose that \(\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0\). From proposition 5.3.1 5-- it is easy to prove, using 5.2.5 that there is a unique oblate \(n\)-star \(S\) such that \(S^{[n-1]}\) is the union \(\bigcup_{1 \leq i \leq n-1} S_i\) in \(S\) and \(\Psi_n\) is the quotient map \(K_n = K^{[n-1]} \to Q\).
5.4. Morphisms of stars

Recall that if \( S \) is an oblate \( n \)-star of \( S \), then we have a canonical inclusion of sheaves of algebras (on the underlying topological space \( S(n) \) of \( S \)) \( \mathcal{O}_S \subset \mathcal{O}_{S_0} \).

Let \( S, S' \) be oblate \( n \)-stars of \( S \), with irreducible components \( S_1, \ldots, S_n \), and \( f : S \to S' \) a morphism inducing the identity on all the components. Such a morphism exists if and only if \( \mathcal{O}_S \subset \mathcal{O}_{S'} \), and in this case \( f \) is unique and is induced by the previous inclusion. Let \((p_{ij})\) (resp. \((p'_{ij})\)) be the spectrum of \( S \) (resp. \( S' \)).

5.4.1. Proposition: \( p_{ij} \leq p'_{ij} \) for \( 1 \leq i, j \leq n \). If \( f \) is not the identity morphism then there exist \( i, j \) such that \( p_{ij} < p'_{ij} \).

Proof. Let \( I = \{i, j\} \). Then \( f \) induces a morphism \( S^{(I)} \to S'^{(I)} \). So we have \( \mathcal{O}_{S^{(I)}, p} \subset \mathcal{O}_{S'^{(I)}, p} \).

From proposition 5.2.1, it follows that \( p_{ij} \leq p'_{ij} \).

Suppose now that \( p'_{ij} = p_{ij} \) for \( 1 \leq i, j \leq n \). We must prove that \( S = S' \), i.e. that \( \mathcal{O}_{S', p} = \mathcal{O}_{S, p} \). This is done by induction on \( n \). For \( n = 2 \) it is obvious. Suppose that it is true for \( n - 1 \). Let \( I = \{1, \ldots, n - 1\} \). Then \( f \) induces a morphism \( f_{n-1} : S^{(I)} \to S'^{(I)} \). It follows from the induction hypothesis that \( S^{(I)} = S'^{(I)} \). Since the integers \( q_i \) are the same for \( S \) and \( S' \), the algebras \( K_n \) for \( S \) and \( S' \) (cf. proposition 5.2.4) are also the same. Now let \( \alpha \in \mathcal{O}_{S, p} \), and let \( \beta \in K_n \) be the image of \( \alpha \). Let \( \alpha' \in \mathcal{O}_{S', p} \) be such that its image in \( K_n \) is also \( \beta \). Then \( \alpha - \alpha' \) belongs to the ideal generated by the \((0, \ldots, 0, t_{i'}^0, 0, \ldots, 0), 1 \leq i \leq n \), which is included in \( \mathcal{O}_{S', p} \). Hence \( \alpha \in \mathcal{O}_{S', p} \). \( \square \)

5.4.2. Lemma: Suppose that \( f \) is not the identity morphism. Then there exist an ideal \( \mathcal{I} \subset \mathcal{O}_{S', p} \) and \( u \in \mathcal{I}, v \in \mathcal{O}_{S, p} \) such that

\[
\begin{align*}
   u \otimes v &\neq 0 & & \text{in} & & \mathcal{I} \otimes_{\mathcal{O}_{S', p}} \mathcal{O}_{S, p} \\
   uv &\neq 0 
\end{align*}
\]

and \( uv = 0 \).

Proof. Let \( q_1 = \sum_{i=1}^n p_{1i}, q_1' = \sum_{i=1}^n p'_{1i} \). According to proposition 5.4.1 we can assume that \( q_1 < q_1' \). Let \( u \) be a generator of the ideal of \( S_1 \) in \( \mathcal{O}_{S', p} \) and \( \mathcal{I} = (u) \). Let \( v = (t_{i^0}^q, 0, \ldots, 0) \).

We have \( uv = 0 \). We must prove that \( u \otimes v \neq 0 \). We need only to find an \( \mathcal{O}_{S', p} \)-module \( M \) and a \( \mathcal{O}_{S', p} \)-bilinear map \( \phi : \mathcal{I} \otimes_{\mathcal{O}_{S', p}} \mathcal{O}_{S, p} \to M \) such that \( \phi(u \otimes v) \neq 0 \). We take \( M = \mathcal{O}_{S_1, p}/(t_{i^0}^q) \), which is a quotient of \( \mathcal{O}_{S'} \). It is easy to verify that

\[
\phi : ((\lambda_i)_{1 \leq i \leq n} u, (w_i)_{1 \leq i \leq n}) \mapsto \lambda_1 w_1 \quad (\text{mod } t_{i^0}^q)
\]

is well defined, bilinear, and that \( \phi(u \otimes v) \neq 0 \). \( \square \)

5.4.3. Corollary: Suppose that \( f \) is not the identity morphism. Let \( Y \) be an algebraic variety and \( g : Y \to S \) a morphism such that \( g^* : \mathcal{O}_{S, p} \to \mathcal{O}_{Y, p} \) is injective. Then \( f \circ g : Y \to S' \) is not flat.
Proof. We use the notations of the proof of lemma 5.4.2. We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{S,P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P} \\
\downarrow{\lambda_S} & & \downarrow{\lambda_Y} \\
\mathcal{I} \otimes \mathcal{O}_{S',P} & \xrightarrow{I_{\mathcal{I}} \otimes g^*} & \mathcal{I} \otimes \mathcal{O}_{S',P} \mathcal{O}_{Y,P} \\
\downarrow{\mu_S} & & \downarrow{\mu_Y} \\
\mathcal{O}_{S,P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P}
\end{array}
\]

where \(\lambda_S(\alpha) = u \otimes \alpha\), \(\mu_S(u \otimes \alpha) = u\alpha\), and \(\lambda_Y, \mu_Y\) are defined similarly. It follows that \(\mu_Y(u \otimes g^*v) = 0\). We will show that \(u \otimes g^*v \neq 0\), and this will imply that \(f \circ g\) is not flat. Let \(w = (t_1^{q_1}, 0, \ldots, 0)\). Then we have \(\mathcal{I} \simeq \mathcal{O}_{S',P}/(w)\), and from the exact sequence of \(\mathcal{O}_{S',P}\)-modules \(0 \to (w) \to \mathcal{O}_{S',P} \to \mathcal{I} \to 0\) we deduce that \(\ker(\lambda_Y) = (w).\mathcal{O}_{Y,P}\). Suppose that \(u \otimes g^*v = 0\). Then \(g^*v\) is a multiple of \(w\). \(g^*v = w.a\), for some \(a \in \mathcal{O}_{Y,P}\). But we have \(w = g^*\pi^{q_1-q_1}v\). Hence \(g^*v.(1 - g^*\pi^{q_1-q_1}) = 0\). Since \(1 - g^*\pi^{q_1-q_1}\) is invertible, we have \(g^*v = 0\), which is false since \(g^*\) is injective. Hence \(u \otimes g^*v \neq 0\). \(\square\)

5.5. Structure of ideals

Let \(\mathcal{S}\) be an oblate \(n\)-star of \(\mathcal{S}\).

5.5.1. Proposition: Let \(\mathcal{I} \subset \mathcal{O}_{S,P}\) be a proper ideal. Then

1 - There exists a positive integer \(k\) such that \(k \leq n\) and a filtration by ideals

\[
\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \cdots \subset \mathcal{I}_1 = \mathcal{I}
\]

such that, for \(1 \leq i \leq k\) there exists a positive integer \(j\) such that \(j \leq n\) and an isomorphism \(\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}\) of \(\mathcal{O}_{S,P}\)-modules.

2 - If \(\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}\), then \(\mathcal{I}_{i+1} \subset \mathcal{I}_j\) and \(\mathcal{I}_i \notin \mathcal{I}_j\).

Proof. We prove 1- by induction on \(n\). The case \(n = 1\) is trivial. Suppose that \(n > 1\) and that the result is true for \(n - 1\). Let \(\mathcal{J}_1\) be the ideal sheaf of \(S_1 \subset \mathcal{S}\), and \(\mathcal{S}' = S_2 \cup \cdots \cup S_{n-1} \subset \mathcal{S}\). We can view \(\mathcal{J}_1\) as an ideal of \(\mathcal{O}_{S',P}\). We can suppose that \(\mathcal{I} \notin \mathcal{O}_{S',P}\), i.e. that some element of \(\mathcal{I}\) has a nonzero first coordinate. Let \(m\) be the smallest positive integer such that \(\mathcal{I}\) contains an element \(u\) of the form

\[
u = (t_1^m, \alpha_2, \ldots, \alpha_n).
\]

Then every element \(v\) of \(\mathcal{I}\) can be written as

\[
\lambda \in \mathcal{O}_{S,P}\] and \(v' \in \mathcal{J}_1 \cap \mathcal{I}\), and the first coordinate of \(\lambda\) is uniquely determined. It follows that \(\mathcal{I}/(\mathcal{J}_1 \cap \mathcal{I}) \simeq \mathcal{O}_{S_1,P}\). We can apply the recurrence hypothesis to the ideal \(\mathcal{J}_1 \cap \mathcal{I}\) of \(\mathcal{O}_{S',P}\) and get a filtration of it, from which we deduce the filtration of \(\mathcal{I}\). This proves 1- for \(n\).

Now we prove 2-. Let \(\alpha \in \mathcal{O}_{S,P}\setminus \mathcal{I}_{S_j}\). Let \(u \in \mathcal{I}_i\) be over a generator of \(\mathcal{I}_i/\mathcal{I}_{i+1}\). Then the image of \(\alpha u\) in \(\mathcal{I}_i/\mathcal{I}_{i+1}\) is not zero, i.e. \(\alpha u \notin \mathcal{I}_{i+1}\). Hence \(\alpha \notin \mathcal{I}_{i+1}\), and \(\mathcal{I}_{i+1} \subset \mathcal{I}_{S_j}\). Let
\( v_i = (0, \ldots, 0, t_i^q, 0, \ldots, 0) \in \mathcal{O}_{S,P} \). Then the image of \( v_i u \) in \( \mathcal{I}_i/\mathcal{I}_{i+1} \) is not zero, hence \( u \notin \mathcal{I}_{S_i} \) and \( \mathcal{I}_i \not\subseteq \mathcal{I}_{S_j} \).

\[ \square \]

### 5.6. Star associated to a fragmented deformation

We keep the notations of chapter 4. Let \( n \geq 2 \) be an integer, \( \pi : C \to S \) a fragmented deformation of \( C_n \), and \( C_1, \ldots, C_n \) the irreducible components of \( C \).

Recall that \( S(n) \) is the underlying (Zariski) topological space of any \( n \)-star of \( S \). Let \( C^{\text{top}} \) be the underlying topological space of \( C \). We have an obvious continuous map \( \pi : C^{\text{top}} \to S(n) \). Let \( \mathcal{A}_n \) be the sheaf of algebras on \( S(n) \) defined by: for every open subset \( U \) of \( S(n) \), \( \mathcal{A}_n(U) \) is the algebra of \( (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_C(\pi^{-1}(U)) \) such that \( \alpha_i \in \mathcal{O}_{S_i}(U \cap S_i) \) for \( 1 \leq i \leq n \).

According to corollary 4.4.8 for every \( x \in \mathcal{C} \), \( \mathcal{A}_{n,P} \) is the algebra of \( (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x} \) such that \( \alpha_i \in \mathcal{O}_{S,P} \) for \( 1 \leq i \leq n \).

#### 5.6.1. Proposition: The sheaf \( \mathcal{A}_n \) is the structural sheaf of an oblate \( n \)-star of \( S \).

**Proof.** By induction on \( n \). The case \( n = 1 \) is obvious. Suppose that \( n > 1 \) and that the result is true for \( n-1 \). Let \( C' = C_1 \cup \cdots \cup C_{n-1} \subset \mathcal{C} \), and \( \mathcal{A}_{n-1} \) the corresponding oblate \((n-1)\)-star of \( S \). Let

\[ \Phi_n : \mathcal{B}_n \to \mathcal{O}_{C_n}/(\pi_n^{q_n}) \]

be the morphism of proposition 4.4.1. According to proposition 4.6.4 \( \Phi_n \) induces a morphism

\[ \Psi_n : \mathcal{K}_n \to \mathcal{O}_{S_n,P}/(t_n^{q_n}) \]

By the definitions of \( \mathcal{A}_n \) and \( \Phi_n \), if \( u = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{S_1,P} \times \cdots \times \mathcal{O}_{S_1,P} \), then \( u \in \mathcal{A}_{n,P} \) if and only if \( \Psi_n(u') = v \), where \( u' \) (resp. \( v \)) is the image of \( u \) in \( \mathcal{K}_n \) (resp. \( \mathcal{O}_{S_n,P}/(t_n^{q_n}) \)). The result follows then from 5.2.5.

\[ \square \]

We denote by \( S(\mathcal{C}) \) (or more simply \( S \)) the oblate \( n \)-star corresponding to \( \mathcal{A}_n \), so \( \mathcal{O}_{S(\mathcal{C})} = \mathcal{A}_n \). From the definition of \( \mathcal{A}_n \) we get a canonical morphism

\[ \Pi : \mathcal{C} \to \mathcal{S} \]

such that \( \Pi|_{C_i} = \pi_i : C_i \to S_i \) for \( 1 \leq i \leq n \).

#### 5.6.2. Theorem: The morphism \( \Pi \) is flat.

**Proof.** We need only to prove that \( \Pi \) is flat at any point \( x \) of \( \mathcal{C} \). Let \( \mathcal{I} \subset \mathcal{O}_{S,P} \) be a proper ideal. We have to show that the canonical morphism of \( \mathcal{O}_{S,P} \)-modules

\[ \tau = \tau_\mathcal{I} : \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{S,P}} \mathcal{I} \to \mathcal{O}_{\mathcal{C},x} \]

is injective. According to proposition 5.5.1 there is a filtration by ideals

\[ \{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \cdots \subset \mathcal{I}_1 = \mathcal{I} \]

such that, for \( 1 \leq i \leq k \) there exists a positive integer \( j \) such that \( j \leq n \) and an isomorphism \( \mathcal{I}_i/\mathcal{I}_{i+1} \cong \mathcal{O}_{S_j,P} \) of \( \mathcal{O}_{S,P} \)-modules. We will prove the injectivity of \( \tau \) by induction on \( k \).
Recall that for $1 \leq j \leq n$, $I_{S_j, P} = I_{S_j, S, P}$ is a principal ideal, generated by an element $u_j$ which is also a generator of $I_{C_j, x} = I_{C_j, C, x}$ (cf. corollary 4.3.8 and proposition 5.2.3), and that the only zero coordinate of $u_j$ is the $j$-th.

Suppose that $k = 1$, so $I$ is isomorphic to $O_{S_j, P}$ for some $j$. Let $u$ be a generator of $I$ and $w \in O_{C, x} \otimes O_{S, P} I$, that can be written as $w = v \otimes u$, $v \in O_{C, x}$. Suppose that $\tau(v \otimes u) = vu = 0$. Since $I$ is annihilated by $I_{S_j, P}$, we have $I \subset ((0, \ldots, 0, t_j^{0j}, 0, \ldots, 0))$. Since $vu = 0$, the $j$-th component of $v$ is zero, i.e. $v \in I_{C_j, x}$. Hence $v$ is a multiple of $u_j$: $v = \alpha u_j$. We have then
\[
\begin{align*}
w &= \alpha u_j \otimes u \\
   &= \alpha \otimes u_j u \quad \text{(because $u_j \in O_{S, P}$)} \\
   &= 0 \quad \text{(because $u_j u = 0$)}.
\end{align*}
\]

Hence $\tau$ is injective.

Suppose that the result is true for $k - 1 \geq 1$ and that the filtration of $I$ is of length $k$. According to proposition 5.5.1, we have $I/I_2 \simeq O_{S_j, P}$ for some $j$. Let $u \in I$ be such that its image in $I/I_2$ is a generator, and $w \in O_{C, x} \otimes O_{S, P} I$ such that $\tau(w) = 0$. We can write $w$ as $w = \alpha \otimes v + \beta \otimes u$, with $\alpha, \beta \in O_{C, x}$ and $v \in I_2$. Since $\alpha v + \beta u = 0$, we have $\beta u \in O_{C, x} I_2$, and $O_{C, x} I_2 \subset I_{C_j}$ by proposition 5.5.1, i.e. the $j$-th coordinate of $\beta u$ is zero. By proposition 5.5.1, the $j$-th coordinate of $u$ does not vanish, hence the $j$-th coordinate of $\beta$ is zero, i.e. $\beta \in I_{C_j}$. Hence $\beta$ is a multiple of $u_j$: $\beta = \gamma u_j$. We have then
\[
\beta \otimes u = \gamma u_j \otimes u = \gamma \otimes u_j u,
\]
and $u_j u \in I_2$ (because its image in $I/I_2$ vanishes). It follows that $w$ is the image of an element $w' \in O_{C, x} \otimes O_{S, P} I_2$. We have $\tau_{I_2}(w') = 0$, hence by the induction hypothesis $w' = 0$. It follows that we have also $w = 0$. \qed

5.6.3. Remark: If $S'$ is an oblate $n$-star of $S$, and if $\Pi' : C \to S'$ is a flat morphism compatible with the projections to $S$, then we have $S' = S(C)$ and $\Pi' = \Pi$. This is an easy consequence of corollary 5.4.3.

5.6.4. Converse - Let $\pi : S \to S$ be an oblate $n$-star of $S$. Let $\Pi : C \to S$ be a flat morphism such that for every closed point $s \in S$, $\Pi^{-1}(s)$ is a smooth irreducible projective curve. Let $C = \Pi^{-1}(P)$ and $\tau = \pi \circ \Pi : C \to S$. Then $C_n = \tau^{-1}(P)$ is a primitive multiple curve of multiplicity $n$ and associated smooth curve $C$, and $C$ is a fragmented deformation of $C_n$. This is an easy consequence of proposition 4.1.6.
Let \( \pi : C \to \mathbb{C} \) be a fragmented deformation of length 2. The corresponding double curve \( C_2 \) is \( \pi^{-1}(0) \). Suppose that the spectrum of \( C \) is \( \left( \begin{array}{cc} 0 & p \\ p & 0 \end{array} \right) \). This means that the infinitesimal neighborhoods of order \( p \) of \( C \) in \( C_1 \) and \( C_2 \) are isomorphic, i.e. we have an isomorphism of sheaves of algebras on \( C \)

\[
\Phi : \mathcal{O}_C / (\pi^p_1) \to \mathcal{O}_C / (\pi^p_2),
\]

and for every point \( x \) of \( C \), we have

\[
\mathcal{O}_{C,x} = \{ (\alpha_1, \alpha_2) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x} : \alpha_2 \pmod{\pi^p_2} = \Phi(\alpha_1 \pmod{\pi^p_1}) \}.
\]

Let \( C^p_i \) denote the infinitesimal neighborhood of order \( k \) of \( C \) in \( C_i \), \( i = 1, 2, k > 0 \). It is a primitive multiple curve of multiplicity \( k \) and associated smooth curve \( C \), and we have \( C^p_1 = C^p_2 \). Hence \( C^{p+1}_1 \) and \( C^{p+1}_2 \) appear as extensions of \( C^p \) in primitive multiple curves of multiplicity \( p + 1 \). According to [4] and [3] these extensions are classified by \( H^1(C, T_C) \) (\( T_C \) being the tangent sheaf on \( C \)). More precisely, we say that two such extensions, \( D, D' \) are isomorphic if there exists an isomorphism \( D \simeq D' \) leaving \( C^p \) invariant. Then if \( \mathcal{H} \) is the set of isomorphism classes of such extensions, a bijection \( \lambda : H^1(C, T_C) \to \mathcal{H} \) is defined in [4], such that \( \lambda(0) = C^{p+1}_1 \).

On the other hand, it follows from [2], [4] that the primitive double curves with associated smooth curve \( C \) and associated line bundle \( \mathcal{O}_C \) are classified by \( \mathbb{P}(H^1(C, T_C)) \cup \{0\} \).

### 6.0.5. Theorem:

The point of \( \mathbb{P}(H^1(C, T_C)) \cup \{0\} \) corresponding to \( C_2 \) is \( \mathbb{C}. \lambda^{-1}(C^{p+1}_2) \).

**Proof.** According to [4], there exists an open covering \( (U_i)_{i \in I} \) of \( C \) such that for \( k = 1, 2 \), the open subset of \( C^{p+1}_k \) corresponding to \( U_i \) is isomorphic to \( U_i \times \text{spec}(C[t]/(t^{p+1})) \). Here \( t \) is \( \pi_1 \) on \( C_1 \) and \( \pi_2 \) on \( C_2 \). We obtain then cocycles \( (\theta^{(k)}_{ij})_{i,j \in I} \), where \( \theta^{(k)}_{ij} \) is an automorphism of \( U_{ij} \times \text{spec}(C[t]/(t^{p+1})) \). We can also suppose that \( \omega_{C|U_i} \) is trivial, for every \( i \in I \). Let \( dx_{ij} = dx \) be a generator of \( \omega_C(U_{ij}) \). Since the ideal sheaf of \( C \) in \( C^{p+1}_k \) is the trivial sheaf on \( C^p_k \), we can write, using the notations of [4], \( \theta^{(k)}_{ij} = \phi_{\mu^{(k)}_{ij}} \), with \( \mu^{(k)}_{ij} \in \mathcal{O}_{C}(U_{ij})[t]/(t^p) \), i.e. for every \( \alpha \in \mathcal{O}_{C}(U_i) \), we have, at the level of regular functions

\[
\theta^{(k)}_{ij}(\alpha) = \sum_{m=0}^{p} \frac{1}{m!} (\mu^{(k)}_{ij} t)^m \frac{d^m \alpha}{dx^m},
\]

and \( \theta^{(k)}_{ij}(t) = t \). Since \( C^p_1 = C^p_2 \) we can suppose that \( \mu^{(1)}_{ij} \equiv \mu^{(2)}_{ij} \pmod{t^p} \). Hence \( \tau_{ij} = \mu^{(2)}_{ij} - \mu^{(1)}_{ij} \in (t^{p-1})/(t^p) \simeq \mathcal{O}_C(U_i) \). The family \( (\tau_{ij}) \) is (in some sense) a cocycle representing \( \lambda^{-1}(C^{p+1}_2) \) (cf. [4], [3]).

We have \( (\pi^{p+1}_1) + (\pi^{p+1}_2) \subset (\pi) \) in \( \mathcal{O}_C \). Hence \( C_2 = \pi^{-1}(0) \) is contained in the subscheme \( Z \) of \( \mathcal{C} \) corresponding to the ideal sheaf \( (\pi^{p+1}_1) + (\pi^{p+1}_2) \). We have

\[
\mathcal{O}_Z(U_{ij}) = \{ (\alpha_1, \alpha_2) \in \mathcal{O}_{C_1}(U_{ij})/(t^{p+1}) \times \mathcal{O}_{C_2}(U_{ij})/(t^{p+1}) : \Phi(\alpha_1 \pmod{t^p}) = \alpha_2 \pmod{t^p} \}
\]

\[
= \{ (\alpha_1, \alpha_2) \in \mathcal{O}_{C}(U_{ij})[t]/(t^{p+1}) \times \mathcal{O}_{C}(U_{ij})[t]/(t^{p+1}) : \alpha_1 \equiv \alpha_2 \pmod{t^p} \}.
\]

To obtain \( \mathcal{O}_{C_2}(U_{ij}) \), we have just to quotient by \( \pi = (t, t) \), and we obtain

\[
\mathcal{O}_{C_2}(U_{ij}) = \mathcal{O}_Z(U_{ij})/(t, t) \simeq \mathcal{O}_{C}(U_{ij})[z]/(z^2),
\]
the last isomorphism being

\[(a_0 + a_1 t + \cdots + a_{p-1} t^{p-1} + \alpha t^p, a_0 + a_1 t + \cdots + a_{p-1} t^{p-1} + \beta t^p) \mapsto \alpha_0 + (\beta - \alpha) z.\]

Now we can make explicit the automorphism of \(O_C(U_{ij})[z]/(z^2)\) induced by \(\theta_{ij}\) (these isomorphisms will define the cocycle corresponding to \(C_2\)). It is easy to see that this isomorphism is \(\phi_{rij,1}\), which proves theorem 6.0.5. 

\[\square\]

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