Near – Rings with Generalized Right \(n\)-Derivations

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Abstract

We define a new concept, called "generalized right \(n\)-derivation", in near-ring and obtain new essential results in this field. Moreover we improve this paper with examples that show that the assumptions used are necessary.

Keywords: Generalized right derivations, Generalized right \(n\)-derivations, Prime near-ring, Right derivations, Right \(n\)-derivations.

1. Introduction

A near-ring is defined to be a “set \(N\) with two binary operations \((+\) and \(\cdot\) \) such that (i) \((N,+)\) is a group that is not necessarily abelian; (ii) \((N,\cdot)\) is a semi group; (iii) \(a \cdot (b + c) = a \cdot b + a \cdot c\) for each \(a, b, c \in N\)”. The elements products, such as \(a\) and \(b\) in \(N\), will be \(a \cdot b\), which is specified by \(ab\). \(N\) is zero-symmetric whenever \(0x = 0\), for each \(x \in N\) (\(x0 = 0\) yields from left distributivity). The centre of multiplicative of \(N\) will be represented by \(Z\). For each \(x, y \in N\), \([x,y] = xy - yx\) symbolizes the commutator and \((x,y) = x + y - x - y\) is the additive commutator, while \(x \circ y\) will denote the well-known Jordan product. \(N\) is referred to as prime near-ring in case of \(xNy = \{0\}\), which infers that \(y = 0\) or \(x = 0\). “A non-empty subset \(U\) of \(N\) is named as semigroup left ideal, resp. semigroup right ideal in case of \(NU \subseteq U\) (resp. \(UN \subseteq U\)). But, if \(U\) represents both semigroup right and left ideal, then it will be termed as semigroup ideal”.

For more about near-ring theory and its applications, we make reference to Pilz [1]. In [2], X. Wang defined the derivation as an additive mapping \(d\) from \(N\) into itself which satisfies \(d(xy) = d(x)y + xd(y)\) for each \(x, y \in N\). Later, the derivation concepts generalization have been achieved through various means according to different authors. “Ashraf and Siddiquee well-defined the concepts of \(n\) –derivations, generalized \(n\) –derivations, and \((\sigma,\tau) – n\) – derivation in near ring [3 - 6]”. Also, various properties of such derivations were examined. In 2015, Abdul Rehman and Enaam defined a new concept, called "right \(n\)-derivation", in near-ring and obtained new essential results for researchers in this field [7].

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"An additive mapping \( d \) from \( N \) into itself is said to be right derivation of \( N \) if \( d(xy) = d(x)y + d(y)x \), for each \( x, y \in N \) and \( n \) - additive (i.e. additive in each argument) mapping \( d: N \times N \times \ldots \times N \rightarrow N \) is said to be right \( n \) - derivation of \( N \) if the following equations hold for each \( x_1, x_2, x_n \in N \):

\[
\begin{align*}
\text{for } x_1, x_1', x_2, \ldots, x_n, x_n' \\
\Rightarrow d(x_1, x_2, \ldots, x_n) \Rightarrow d(x_1', x_2, \ldots, x_n) + d(x_1', x_2, \ldots, x_n) x_1 \\
\Rightarrow d(x_1, x_2 x_1', \ldots, x_n) \Rightarrow d(x_1, x_2, \ldots, x_n) x_2 + d(x_1, x_2', \ldots, x_n) x_2 \\
\vdotswithin{\Rightarrow}
\Rightarrow d(x_1, x_2, \ldots, x_n x_n') \Rightarrow d(x_1, x_2, \ldots, x_n) x_n + d(x_1, x_2, \ldots, x_n') x_n
\end{align*}
\]

Motivated by the previous studies, we define here the concepts of generalized right derivation and generalized right \( n \)-derivation in near-ring \( N \). After that, we will give new essential results in this field and generalize some results presented in [7]. Finally, we improve this paper with examples that show that the assumptions used are necessary.

**Note that we will use the abbreviation C.R to refer to the commutative ring.**

**Definition 1.1.** Let \( d \) be a right derivation of \( N \). An additive mapping \( \bar{G} \) from \( N \) into itself is said to be generalized right derivation of \( \bar{G} \) connected with right derivation \( d \) if the following equations hold for each \( x, y \in N \):

\[
\begin{align*}
\bar{G}(x_1, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n) x_1 \\
\Rightarrow \bar{G}(x_1, x_2 x_1', \ldots, x_n) + d(x_1, x_2, \ldots, x_n) x_2 + d(x_1, x_2', \ldots, x_n) x_2 \\
\vdotswithin{\Rightarrow}
\Rightarrow \bar{G}(x_1, x_2, \ldots, x_n x_n') + d(x_1, x_2, \ldots, x_n) x_n + d(x_1, x_2, \ldots, x_n') x_n
\end{align*}
\]

**Definition 1.2.** If \( d \) is a right \( n \)-derivation of \( N \) and \( \bar{G}: N \times N \times \ldots \times N \rightarrow N \) is an \( n \)-additive mapping on \( N \), then \( \bar{G} \) will be called "generalized right \( n \)-derivation of \( N \) connected with right derivation \( d \) if the following equations hold for each \( x_1, x_1', x_2, \ldots, x_n, x_n' \in N \):

\[
\begin{align*}
\bar{G}(x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n) x_1 + \bar{G}(x_1, x_2, \ldots, x_n) x_1 \\
\Rightarrow \bar{G}(x_1, x_2 x_1', \ldots, x_n) = d(x_1, x_2, \ldots, x_n) x_2 + \bar{G}(x_1, x_2, \ldots, x_n) x_2 \\
\vdotswithin{\Rightarrow}
\Rightarrow \bar{G}(x_1, x_2, \ldots, x_n x_n') = d(x_1, x_2, \ldots, x_n) x_n + \bar{G}(x_1, x_2, \ldots, x_n) x_n
\end{align*}
\]

**Example 1.3.** If \( S \) be a near-ring and zero symmetric then it is obvious that

\[
\mathcal{N} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} x, y, 0 \in S \right\}
\]

is a near-ring with the addition and multiplication of matrices.

Let \( d, \bar{G}: N \rightarrow N \) and \( d_1, \bar{G}_1: N \times N \times \ldots \times N \rightarrow N \) defined by:

\[
\begin{align*}
\bar{G}(\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} x, y, 0 \in S) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} x, y, 0 \in S
\end{align*}
\]

Simply can check that \( \bar{G} \) is a generalized right derivation connected with right derivation \( d \) of \( \mathcal{N} \) and \( \bar{G}_1 \) is a nonzero generalized right \( n \)-derivation connected with right \( n \)-derivation \( d_1 \) of \( \mathcal{N} \).

2. **Preliminaries**

The next lemmas are fundamental to develop the proofs of our work.

**Lemma 2.1 [8].** “Let \( N \) be a near-ring. If there is an element \( z \neq 0 \) of \( Z \) such that \( z + z \in Z \), then \( (\mathcal{N}, +) \) is abelian”.

**Lemma 2.2 [9].** “Let \( N \) be a prime near-ring. If \( z \in \mathcal{Z} \setminus \{0\} \) and \( x \) is an element of \( \mathcal{N} \) such that \( x z \in \mathcal{Z} \) or \( x z \in \mathcal{Z} \), then \( x \in \mathcal{Z} \).

**Lemma 2.3 [9].** “Let \( N \) be a prime near-ring and \( Z \) contains a nonzero semigroup left ideal or nonzero semigroup left ideal, then \( \mathcal{N} \) is a C.R.”

**Lemma 2.4 [6].** “Let \( N \) be a prime near-ring, \( d \) is a nonzero \( n \)-derivation of \( N \) and \( x \in N \). If \( d(N, N, \ldots, N)x = 0 \), then \( x = 0 \).”

**Lemma 2.5 [7].** “Let \( N \) be a prime near-ring and let \( d \neq 0 \) be a right \( n \)-derivation \( d \). If \( d([x, y], x_2, \ldots, x_n) = 0 \) for each \( x, y, \ldots, x_n \in N \), then \( N \) is a C.R.”
Lemma 2.6 “Let \( N \) be a near-ring, then \( N \) is zero symmetric if and only if \( N \) admits a right \( n \) – derivation \( d \).

Proof. If \( N \) is zero symmetric then \( d = 0 \) is a right \( n \)–derivation on \( N \).

Now, if \( N \) admits right derivation \( d \), then \( 0 = d(x_0, x_2, \ldots, x_n) = d(x_0, x_2, \ldots, x_n)0 + d(0, x_2, \ldots, x_n)x = 0x \). Hence, \( N \) is zero symmetric near-ring.

We would like to point out that we will consider \( N \) in the rest of this article as a prime near-ring.

Lemma 2.7 Let \( g \neq 0 \) is a generalized right \( n \)–derivation of \( N \) connected with right \( n \)–derivation \( d \), and \( a \in N \) s.t \( g(N, N, \ldots, N) = \{0\} \), then \( a = 0 \).

Proof. From assumption

\[
g(x_1, x_2, \ldots, x_n)a = 0, \text{ for each } x_1, x_2, \ldots, x_n \in N.
\]

By putting \( ax_1 \) in place of \( x_1 \) in relation (1), we get

\[
0 = g(ax_1, x_2, \ldots, x_n)a
\]

\[
= (d(a, x_2, \ldots, x_n)x_1 + g(x_1, x_2, \ldots, x_n)a)a
\]

\[
= d(a, x_2, \ldots, x_n)x_1a
\]

So, we get \( d(a, x_2, \ldots, x_n)N \) \( a = \{0\} \) for each \( x_2, \ldots, x_n \in N \). It follows that either \( a = 0 \) or \( d(a, x_2, \ldots, x_n) = 0 \) for each \( x_2, \ldots, x_n \in N \).

The following lemmas deduce directly from Lemma 2.7.

Lemma 2.8 [7 : Lemma 2.5]. “If \( d \neq 0 \) is a right \( n \)– derivation of \( N \), and \( a \in N \) s.t \( d(N, N, \ldots, N)a = \{0\} \), then \( a = 0 \).

Lemma 2.9 “If \( g \neq 0 \) is a generalized right derivation of \( N \) connected with right derivation \( d \), and \( a \in N \) s.t \( g(N)a = \{0\} \), then \( a = 0 \).

3. Main Results

Theorem 3.1 If \( g \neq 0 \) is a generalized right \( n \)–derivation of \( N \) associated with right \( n \)–derivation \( d \neq 0, \text{ s.t } g([x, y], x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a C.R.

Proof. By assumption

\[
g([x, y], x_2, \ldots, x_n) = 0, \text{ for each } x, y, x_2, \ldots, x_n \in N.
\]

Replace \( y \) by \( xy \) in (3) to get

\[
g([x, xy], x_2, \ldots, x_n) = 0, \text{ for each } x, y, x_2, \ldots, x_n \in N,
\]

which implies that \( g([x, y], x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \).

Therefore,

\[
d([x, y], x_2, \ldots, x_n)[x, y] + g([x, y], x_2, \ldots, x_n)x = 0, \text{ for each } x, y, x_2, \ldots, x_n \in N.
\]

Using (3) in the previous equation, we get

\[
d([x, y], x_2, \ldots, x_n)[x, y] = 0, \text{ for each } x, y, x_2, \ldots, x_n \in N, \text{ or equivalently}
\]

\[
d(x, x_2, \ldots, x_n)xy = d(x, x_2, \ldots, x_n)y, \text{ for each } x, y, x_2, \ldots, x_n \in N.
\]

Replacing \( y \) by \( yz \) in (4) and using it again, we get

\[
d(x, x_2, \ldots, x_n)[y, z] = 0, \text{ for each } x, y, z, x_2, \ldots, x_n \in N.
\]

Hence, we get

\[
d(x, x_2, \ldots, x_n)N [x, z] = \{0\}, \text{ for each } x, z, x_2, \ldots, x_n \in N.
\]

We arrive at, for any \( xe N \)

\[
either d(x_1, x_2, \ldots, x_n) = 0, \text{ for each } x_2, \ldots, x_n \epsilon N \text{ or } x \epsilon Z
\]

If \( d(x_1, x_2, \ldots, x_n) = 0 \) for each \( x_2, \ldots, x_n \epsilon N \) and for each \( xe N \), then \( d = 0 \) and this contradicts assumption. Therefore, there is \( x_1, x_2, \ldots, x_n \epsilon N \), such that \( d(x_1, x_2, \ldots, x_n) \neq 0 \) and \( x_1 \epsilon Z \).

Since \( x_1 \epsilon Z \), we conclude that \( [x_1, y, z] = x_1, [y, n], \text{ where } y, n \epsilon N \).

By hypothesis we get

\[
g([x_1, y, n], x_2, \ldots, x_n) = 0, \text{ which implies that}
\]

\[
0 = g(x_1, [y, n], x_2, \ldots, x_n)
\]

\[
= d(x_1, x_2, \ldots, x_n)[y, n] + g([y, n], x_2, \ldots, x_n)x_1
\]

\[
= d(x_1, x_2, \ldots, x_n)[y, n], \text{ for each } y, n, x_2, \ldots, x_n \epsilon N.
\]
Therefore,
\[ d(x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n) n \cdot y \text{ for each } y, n, x_1, x_2, \ldots, x_n \in N. \]
Replace \( n \) by \( nt \), where \( t \in N \), in the last equation to get \( d(x_1, x_2, \ldots, x_n) n[y, t] = 0 \) for each \( y, t, n, x_1, x_2, \ldots, x_n \in N \), i.e. \( d(x_1, x_2, \ldots, x_n) N[y, t] = \{0\} \) for each \( y, t, x_2, \ldots, x_n \in N \). But \( d(x_1, x_2, \ldots, x_n) \neq 0 \) and \( N \) is prime, so using Lemma 2.3 implies the required result.

**Corollary 3.2** [7, Theorem 3.11]. Let \( d \neq 0 \) be a right \( n \)-derivation \( d \), s.t. \( d([x, y], x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a C.R.

**Corollary 3.3** Let \( g \neq 0 \) be a generalized right derivation \( N \) connected with the right derivation \( d \neq 0 \) s.t. \( g(x, y) = 0 \) for each \( x, y \in N \), then \( N \) is a C.R.

**Theorem 3.4** If \( g \neq 0 \) be a generalized right \( n \)-derivation \( N \) connected with \( n \)-derivation \( d \neq 0 \) s.t. \( g(N, N, \ldots, N) \subseteq Z \), then \( N \) is a C.R.

**Proof.** Because of \( g \neq 0 \), there exist \( x_1, x_2, \ldots, x_n \in N \) all of them nonzero such that \( g(x_1, x_2, \ldots, x_n) \in Z \{0\} \). We have \( g(x_1 + x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) + g(x_1, x_2, \ldots, x_n) \in Z \), which implies that \( (N, +) \) is an abelian by Lemma 2.1. Therefore, \( d([x, y], x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \) and, finally using Lemma 2.5 complete the proof.

**Corollary 3.5** [7, Theorem 3.1] "If \( d \neq 0 \) be a right \( n \)-derivation \( N \) s.t. \( d(N, N, \ldots, N) \subseteq Z \), then \( N \) is a C.R.

**Corollary 3.6** If \( g \neq 0 \) is a generalized right derivation connected with right derivation \( d \neq 0 \) of \( N \) s.t. \( g(N) \subseteq Z \), then \( N \) is a C.R.

**Theorem 3.7** If \( g_1 \) and \( g_2 \) are nonzero generalized right \( n \)-derivations connected with nonzero right \( n \)-derivations \( d_1 \) and \( d_2 \), respectively s.t. \( \{g_1(N, N, \ldots, N), g_2(N, N, \ldots, N)\} = \{0\} \), then \( N \) is a C.R.

**Proof.** If \( z \) and \( z + z \) both commute with \( g_2(N, N, \ldots, N) \), hence, for each \( x_1, x_2, \ldots, x_n \in N \), we have
\[
(z + z) g_2 (x_1, x_2, \ldots, x_n) = g_2 (x_1, x_2, \ldots, x_n) (z + z) \tag{7}
\]
and
\[
(z + z) g_2 (x_1, x_2, \ldots, x_n) = g_2 (x_1, x_2, \ldots, x_n) (z + z) \tag{8}
\]
Substituting \( x_1 + x_1 \) instead of \( x_1 \) in (8), we get
\[
(z + z) g_2 (x_1 + x_1', x_2, \ldots, x_n) = g_2 (x_1 + x_1', x_2, \ldots, x_n) (z + z) \text{ for each } x_1, x_1', x_2, \ldots, x_n \in N. \tag{9}
\]
From (7) and (8), the previous equation can be reduced to
\[
(z + z) g_2 (x_1 + x_1' - x_1, x_2, \ldots, x_n) = 0 \text{ for each } x_1, x_1', x_2, \ldots, x_n \in N. \tag{10}
\]
Put \( z = g_1(y_1, y_2, \ldots, y_n) \) to get
\[
g_1(y_1, y_2, \ldots, y_n) g_2 (x_1, x_2, \ldots, x_n) = 0 \text{ for each } x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N. \tag{11}
\]
Use Lemma 2.6 to find that
\[
g_2 (x_1, x_1') (x_2, \ldots, x_n) = 0 \text{ for each } x_1, x_1', x_2, \ldots, x_n \in N. \tag{12}
\]
For each \( w \in N \), \( w(x_1 + x_1' - x_1 - x_1') = w(x_1 + x_1' - x_1 - x_1') =wx_1 + wx_1' - wx_1 - wx_1' = (wx_1 + wx_1') \) which is again an additive commutator, we put \( w(x_1, x_1') \) instead of \( (x_1, x_1') \) in (9) to get
\[
g_2 (w(x_1, x_1'), x_2, \ldots, x_n) = 0 \text{ for each } w, x_1', x_2, \ldots, x_n \in N. \tag{13}
\]
For each \( w \in N \), \( w(x_1 + x_1' - x_1 - x_1') = w(x_1 + x_1' - x_1 - x_1') =wx_1 + wx_1' - wx_1 - wx_1' = (wx_1 + wx_1') \) which is again an additive commutator, we put \( w(x_1, x_1') \) instead of \( (x_1, x_1') \) in (9) to get
\[
g_2 (w(x_1, x_1'), x_2, \ldots, x_n) = 0 \text{ for each } w, x_1, x_2, \ldots, x_n \in N. \tag{14}
\]
Hence, \( (N, +) \) is an abelian group. Therefore, \( d_1 ([x, y], x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \) and, using Lemma 2.5, we finally obtain that \( N \) is a C.R.

**Corollary 3.8** If \( d_1 \) and \( d_2 \) are nonzero right \( n \)-derivations of \( N \), s.t. \( [d_1 (N, N, \ldots, N), d_2 (N, N, \ldots, N)] = \{0\} \), then \( N \) is a C.R.

**Corollary 3.9** If \( g_1 \) and \( g_2 \) are nonzero generalized right derivations of \( N \) connected with the nonzero right derivations \( d_1, d_2 \), respectively s.t. \( \{g_1(N), g_2(N)\} = \{0\} \), then \( N \) is a C.R.

**Theorem 3.10** If \( g_1 \) and \( g_2 \) are nonzero right generalized \( n \)-derivations of \( N \) connected with the nonzero right \( n \)-derivations \( d_1 \) and \( d_2 \), respectively s.t. \( \{g_1(N), g_2(N)\} = \{0\} \), then \( N \) is a C.R.

**Proof.** From hypothesis,
\[
g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1, y_2, \ldots, y_n) = 0 \tag{15}
\]
for each \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N \).

Substitute \( y_1 + y_1' \) in place of \( y_1 \) in (10) to get
\[
g_1(x_1, x_2, \ldots, x_n) g_2(y_1 + y_1', y_1', y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1 + y_1', y_1', y_2, \ldots, y_n) = 0 \tag{16}
\]

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for each $x_1, x_2, \ldots, x_n, y_1, y_1', y_1, y_2, \ldots, y_n \in N$.

Therefore,
$$g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) + g_1(x_1, x_2, \ldots, x_n) g_2(y_1', y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1, y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1', y_2, \ldots, y_n) = 0$$
for each $x_1, x_2, \ldots, x_n, y_1, y_1', y_1, y_2, \ldots, y_n \in N$.

By using (10) again in the preceding equation, we get
$$g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) + g_1(x_1, x_2, \ldots, x_n) g_2(y_1', y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1, y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1', y_2, \ldots, y_n) = 0$$
for each $x_1, x_2, \ldots, x_n, y_1, y_1', y_1, y_2, \ldots, y_n \in N$.

Which means that
$$g_1(1, x_2, \ldots, x_n) g_2(y_1, y_1', y_2, \ldots, y_n) = 0 \quad \text{for each } x_1, x_2, \ldots, x_n, y_1, y_1', y_2, \ldots, y_n \in N.$$ By Lemma 2.6, we obtain $g_2(y_1, y_1', y_2, \ldots, y_n) = 0$ for each $y_1, y_1', y_2, \ldots, y_n \in N$. Now, by putting $w(y_1, y_1', y_2, \ldots, y_n)$ where $w \in N$, in the previous equation, we get $g_2(w(y_1, y_1'), y_2, \ldots, y_n) = 0$ for each $y_1, y_1', y_2, \ldots, y_n \in N$. So, we have $d_2(w, y_2, \ldots, y_n)(y_1, y_1') = 0$. By using Lemma 2.8, we conclude that $(y_1, y_1') = 0$ for each $y_1, y_1' \in N$ i.e., $(N, +)$ is an abelian group. Therefore, $d_1([x, y], x_2, \ldots, x_n) = 0$ for each $x, y, x_2, \ldots, x_n \in N$ and, using Lemma 2.5, we finally obtain that $N$ is a C.R.

**Corollary 3.11** If $d_1$ and $d_2$ are right n-derivations of $N$ s.t $d_1(x_1, x_2, \ldots, x_n) d_2(y_1, y_2, \ldots, y_n) + d_2(x_1, x_2, \ldots, x_n) d_1(y_1, y_2, \ldots, y_n) = 0$ for each $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$, then $N$ is a C.R.

**Corollary 3.12** If $g_1$ and $g_2$ are nonzero generalized right derivations of $N$ connected with the nonzero right derivations $d_1$ and $d_2$, respectively s.t $g_1(x) g_2(y) + g_2(x') g_1(y') = 0$ for each $x, y \in N$, then $N$ is a C.R.

**Theorem 3.13** If $g_1$ is a nonzero generalized right n-derivation of $N$ connected with the nonzero right n-derivation $d_1$, and $g_2$ is a nonzero generalized n-derivation of $N$ associated with the nonzero n-derivation $d_2$.

(i) If $g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1, y_2, \ldots, y_n) = 0$ for each $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$ then $N$ is a C.R.

(ii) If $g_2(x_1, x_2, \ldots, x_n) g_1(y_1, y_2, \ldots, y_n) + g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) = 0$ for each $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$ then $N$ is a C.R.

**Proof.**

(i) By hypothesis,}
$$g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1, y_2, \ldots, y_n) = 0 \quad \text{for each } x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N. \quad (11)$$

Substituting $y_1 + y_1'$, where $y_1' \in N$, for $y_1$ in (11), we get
$$g_1(x_1, x_2, \ldots, x_n) g_2(y_1 + y_1', y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1 + y_1', y_2, \ldots, y_n) = 0 \quad \text{for each } x_1, x_2, \ldots, x_n, y_1, y_1', y_2, \ldots, y_n \in N, x_1,$$

So, we have
$$g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) + g_1(x_1, x_2, \ldots, x_n) g_2(y_1', y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1, y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_1(y_1', y_2, \ldots, y_n) = 0 \quad \text{for each } x_1, x_2, \ldots, x_n, y_1, y_1', y_2, \ldots, y_n \in N.$$

Using (11) in the previous equation implies that
$$g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_2, \ldots, y_n) + g_1(x_1, x_2, \ldots, x_n) g_2(y_1', y_2, \ldots, y_n) + g_2(x_1, x_2, \ldots, x_n) g_2(-y_1, y_2, \ldots, y_n) + g_1(x_1, x_2, \ldots, x_n) g_2(-y_1', y_2, \ldots, y_n) = 0 \quad \text{for each } x_1, x_2, \ldots, x_n, y_1, y_1', y_2, \ldots, y_n \in N,$$

which means that
$$g_1(x_1, x_2, \ldots, x_n) g_2(y_1, y_1', y_2, \ldots, y_n) = 0 \quad \text{for each } x_1, x_2, \ldots, x_n, y_1, y_1', y_2, \ldots, y_n \in N.$$

Now, using Lemma 2.7, we conclude that
$$g_2(y_1, y_1', y_2, \ldots, y_n) = 0 \quad \text{for each } y_1, y_1', y_2, \ldots, y_n \in N \quad (12)$$

Now putting $w(y_1, y_1')$ instead of $(y_1, y_1')$, in (12) and usw it again to get $d_2(w, y_2, \ldots, y_n)(y_1, y_1') = 0$ for each w, $y_1, y_1', y_2, \ldots, y_n \in N$, use Lemma 2.4 lastly to conclude that $(y_1, y_1') = 0$ for each $y_1, y_1' \in N$. Thus, $(N, +)$ is an abelian, hence $d_1([x, y], x_2, \ldots, x_n) = 0$ for each $x, y, x_2, \ldots, x_n \in N$, and finally, we obtain that $N$ is a C.R by Lemma 2.5.

By using same arguments as in (i), we can proof (ii)
Corollary 3.14 Let \( d_1 \neq 0 \) be a right \( n \)-derivation of \( N \), and \( d_2 \neq 0 \) be an \( n \)-derivation of \( N \).

(i) If \( d_1(x_1, x_2, \ldots, x_n)d_2(y_1, y_2, \ldots, y_n) + d_2(x_1, x_2, \ldots, x_n) d_1(y_1, y_2, \ldots, y_n) = 0 \) for each \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N \), then \( N \) is a C.R.

(ii) If \( d_2(x_1, x_2, \ldots, x_n) d_1(y_1, y_2, \ldots, y_n) + d_1(x_1, x_2, \ldots, x_n) d_2(y_1, y_2, \ldots, y_n) = 0 \) for each \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N \), then \( N \) is a C.R.

Corollary 3.15 Let \( g_1 \) be a nonzero generalized right derivation of \( N \) associated with the nonzero right derivation \( d_1 \), and \( g_2 \) be a nonzero generalized derivation of \( N \) associated with the nonzero derivation \( d_2 \).

(i) If \( g_1(x)d_2(y) + g_2(x)d_1(y) = 0 \) for each \( x, y \in N \), then \( N \) is a C.R.

(ii) If \( g_2(x)d_1(y) + g_1(x)d_2(y) = 0 \) for each \( x, y \in N \) then \( N \) is a C.R.

Theorem 3.16. If \( N \) is two-torsion free, then \( N \) admits no nonzero generalized right \( n \)-derivation associated with the nonzero right derivation \( d \), such that \( g(x \circ y, x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \).

Proof. Assume that
\[
g(x \circ y, x_2, \ldots, x_n) = 0 \quad \text{for each} \quad x, y, x_2, \ldots, x_n \in N. \tag{13}
\]
Put \( x \) in place of \( y \) in (13) to get \( g(x \circ xy, x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \), which implies that \( g(x \circ xy, x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \). It follows that \( d(x, x_2, \ldots, x_n)(x \circ y) + g(x \circ y, x_2, \ldots, x_n)x = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \), use (13) in the last equation to get
\[
d(x, x_2, \ldots, x_n)(x \circ y) = 0 \quad \text{for each} \quad x, y, x_2, \ldots, x_n \in N, \tag{14}
\]
Replacing \( y \) by \( yz \), where \( z \in N \), in (14), we get
\[
d(x, x_2, \ldots, x_n)yz = -d(x, x_2, \ldots, x_n)xyz = d(x, x_2, \ldots, x_n)y(-x)(-z) \quad \text{for each} \quad x, y, z, x_2, \ldots, x_n \in N. \tag{15}
\]
In the last equation, using the fact
\[
-d(x, x_2, \ldots, x_n)yz = d(x, x_2, \ldots, x_n)(-x)(-z) \quad \text{for each} \quad x, y, z, x_2, \ldots, x_n \in N \implies \tag{16}
\]
d implies that \( d(x, x_2, \ldots, x_n)[x, z] = 0 \) for each \( x, y, z, x_2, \ldots, x_n \in N \). Replacing \( x \) by \( -x \) in the previous equation, we get
\[
d(-x, x_2, \ldots, x_n)[x, z] = 0 \quad \text{for each} \quad x, y, z, x_2, \ldots, x_n \in N. \tag{17}
\]
Hence, we get
\[
d(-x, x_2, \ldots, x_n)[x, z] = \{0\} \quad \text{for each} \quad x, z, x_2, \ldots, x_n \in N. \tag{18}
\]
By primeness we find that
For each \( x \in N \), either \( d(-x, x_2, \ldots, x_n) = 0 \) for each \( x_2, \ldots, x_n \in N \) or \( x \in Z \).

Since \( (x, x_2, \ldots, x_n) = -d(-x, x_2, \ldots, x_n) = 0 \), we get:
For each fixed \( x \in N \), either \( d(x, x_2, \ldots, x_n) = 0 \) for each \( x_2, \ldots, x_n \in N \) or \( x \in Z \).

If \( d(x, x_2, \ldots, x_n) = 0 \) for each \( x_2, \ldots, x_n \in N \) and for each \( x \in N \), we get \( d = 0 \), and this contradicts assumption. Therefore, there exist \( x_1, x_2, \ldots, x_n \in N \), all being nonzero, such that \( d(x_1, x_2, \ldots, x_n) \neq 0 \) and \( x_1 \in Z \). Since \( x_1 \in Z \), we conclude that \( (x_1, y \circ z) = x_1(y \circ z), where y, z \in N \) and \( g(x_1 y \circ z, x_2, \ldots, x_n) = 0 \) for each \( x_1, y, z, x_2, \ldots, x_n \in N \).

Therefore,
\[
0 = g(x_1(y \circ z), x_2, \ldots, x_n)
= d(x_1, x_2, \ldots, x_n)(y \circ z) + g(y \circ z, x_2, \ldots, x_n)x_1
= d(x_1, x_2, \ldots, x_n)(yoz) \quad \text{for each} \quad y, z \in N.
\]
which implies that
\[
d(x_1, x_2, \ldots, x_n)yoz = -d(x_1, x_2, \ldots, x_n)zx for each \quad y, z \in N. \tag{19}
\]
Replace \( z \) by \( t \), where \( t \in N \), in the last equation and use it to get \( d(x_1, x_2, \ldots, x_n)yt = \{0\} \) for each \( y, z, t \in N \). Since \( d(x_1, x_2, \ldots, x_n) \neq 0 \) and \( N \) is prime, we conclude that \( N \) is a C.R in view of Lemma 2.3. Now, return to (13) to get \( 2d(xy, x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \), it follows that \( d(xy, x_2, \ldots, x_n) = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \) by two torsion freeness of \( N \), and this get \( d(x, x_2, \ldots, x_n)y + d(y, x_2, \ldots, x_n)x = 0 \) for each \( x, y, x_2, \ldots, x_n \in N \). If we replace \( x \) by \( xz \), where \( z \in N \), in the previous equation we find that \( d(xz, x_2, \ldots, x_n)y + d(y, x_2, \ldots, x_n)xz = 0 \) for each
\[ x, y, z, x_2, \ldots, x_n \in \mathbb{N}, \text{ it follows that } d(y, x_2, \ldots, x_n)N_x = \{0\} \text{ for each } x, y, x_2, \ldots, x_n \in \mathbb{N}. \text{ Since } d \neq 0, \text{ primeness of } N \text{ forces that } x = 0 \text{ for each } x \in \mathbb{N}: \text{ a contradiction.} \]

**Theorem 3.17** If \( g \neq 0 \) is a generalized n-derivation of \( N \) connected with the right n-derivation \( d \neq 0 \) \( s.t \{ g(x, x_2, \ldots, x_n), y \} \in Z \) for each \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a C.R.

**Proof.** By assumption
\[
[g(x, x_2, \ldots, x_n), y] \in Z \text{ for each } x, y, x_2, \ldots, x_n \in N
\]
Hence
\[
[[g(x, x_2, \ldots, x_n), y], t] = 0 \text{ for each } x, y, t, x_2, \ldots, x_n \in N.
\]
Replacing \( y \) by \([g(x, x_2, \ldots, x_n), y] \) in (17), we get
\[
[[g(x, x_2, \ldots, x_n), [g(x, x_2, \ldots, x_n), y]], t] = 0 \text{ for each } x, y, t, x_2, \ldots, x_n \in N.
\]
In view of (16), equation (18) assures that
\[
[g(x, x_2, \ldots, x_n), y] \in N \text{ for each } x, y, t, x_2, \ldots, x_n \in N
\]
The primeness of \( N \) implies that \([g(x, x_2, \ldots, x_n), y] = 0 \text{ for each } x, y, x_2, \ldots, x_n \in N \) and hence \( g(N, N, \ldots, N) \subseteq Z \). The application of Theorem 3.4 assures that \( N \) is a C.R.

**Corollary 3.18** [7, Theorem 3.15] If \( d \neq 0 \) is a right n-derivation of \( N \), \( s.t \{ d(x, x_2, \ldots, x_n), y \} \in Z \) for each \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a C.R.

**Corollary 3.19** If \( g \neq 0 \) is a generalized right derivation of \( N \) connected with the right derivation \( d \neq 0 \) \( s.t \{ g(x, x_2, \ldots, x_n), y \} \in Z \) for each \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a C.R.

**Theorem 3.20** If \( g \neq 0 \) is a generalized right n-derivation of \( N \) connected with the right n-derivation \( d \neq 0 \) \( s.t \{ g(x, x_2, \ldots, x_n) \circ y \} \in Z \) for each \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a C.R.

**Proof.** By assumption
\[
g(x, x_2, \ldots, x_n) \circ y \in Z \text{ for each } x, y, x_2, \ldots, x_n \in N
\]
(a) \( g(x, x_2, \ldots, x_n) = \) \( g(x, x_2, \ldots, x_n)(-y) \) for each \( x, y, x_2, \ldots, x_n \in N
\]
Substituting \( y \) for \( y \) in (21) and using it again, we obtain
\[
g(x, x_2, \ldots, x_n) = \) \( g(x, x_2, \ldots, x_n)y \)
\]
Using the fact that \(-y \in N \) \( \in Z \) \( \in Z \) \( \in Z \) \( \in Z \) implies that \( g(x, x_2, \ldots, x_n) = \) \( -g(x, x_2, \ldots, x_n)y \) for each \( x, y, x_2, \ldots, x_n \in N
\]
This implies that
\[
g(y(-x, x_2, \ldots, x_n)) = \) \( g(-x, x_2, \ldots, x_n)y \) for each \( x, y, z, x_2, \ldots, x_n \in N
\]
Using equation (23) in (24), we conclude that
\[
y(g(x + x_2, \ldots, x_n) + g(x, x_2, \ldots, x_n)) \in Z \text{ for each } x, y, x_2, \ldots, x_n \in N
\]
For each \( x, y, t, x_2, \ldots, x_n \in N \), we get
\[
ty[g(x + x_2, \ldots, x_n) + g(x + x_2, \ldots, x_n)]
\]
This implies that
\[
g(x + x_2, \ldots, x_n) \in N \in Z \text{ for each } x, y, t, x_2, \ldots, x_n \in N
\]
The primeness of N implies that either $g(x + x, x_2, \ldots, x_n) + g(x + x, x_2, \ldots, x_n) = 0$ and thus $g = 0$ "which is a contradiction" or $N = Z$, hence $g(N, N, \ldots, N) \subseteq Z$ and using Theorem 3.4 assures that $N$ is a C.R.

**Corollary 3.21 [7, Theorem 3.17].** Let $d$ be nonzero right $n$-derivation of $N$. If $d(x, x_2, \ldots, x_n) \circ y \in Z$ for each $x, y, x_2, \ldots, x_n \in N$, then $N$ is a C.R.

**Corollary 3.22.** Let $g$ be generalized right derivation of $N$ associated with the nonzero right derivation $d$. If $g(x) \circ y \in Z$ for each $x, y \in N$, then $N$ is a C.R.

Primeness assumption is necessary in our results and the following example will show that:

**Example 3.23.** Let $S$ be a zero-symmetric and two-torsion free near-ring. It is obvious that

\[ M = \left( \begin{array}{ccc} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), x, y, 0 \in S \]

is zero symmetric near-ring " not prime" with addition and multiplication of matrices.

Define $d_1, g_1, d_2, g_2 : M \times M \times \ldots \times M \rightarrow M$ such that

\[
\begin{align*}
&d_1 \left( \begin{array}{ccc} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \ldots, \begin{array}{ccc} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \begin{array}{ccc} 0 & x_1x_2 \ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \\
g_1 \left( \begin{array}{ccc} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \ldots, \begin{array}{ccc} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \begin{array}{ccc} 0 & y_1y_2 \ldots y_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \\
d_2 \left( \begin{array}{ccc} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \ldots, \begin{array}{ccc} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \begin{array}{ccc} 0 & 0 & x_1x_2 \ldots x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \\
g_2 \left( \begin{array}{ccc} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \ldots, \begin{array}{ccc} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \begin{array}{ccc} 0 & 0 & y_1y_2 \ldots y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \end{align*}
\]

It is easy to check that $g_1$ and $g_2$ are nonzero right generalized $n$-derivations of $M$ associated with the right $n$-derivations $d_1, d_2$, respectively and

(i) Let $A \in M$, $A = \left( \begin{array}{ccc} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$ such that $x, y \neq 0$, then we can see that $g_1(M, M, \ldots, M)A = 0$. But $A \neq 0$ and

(ii) $g_1(M, M, \ldots, M) \subseteq Z$;

(iii) $[g_1(M, M, \ldots, M), g_2(M, M, \ldots, M)] = \{0\}$;

(iv) $g_1(A_1, A_2, \ldots, A_n) + g_2(A_1, A_2, \ldots, A_n) = g_1(B_1, B_2, \ldots, B_n) = 0$ for each $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n \in M$;

(v) $A_1g_1(B_1, B_2, \ldots, B_n) = g_1(A_1, A_2, \ldots, A_n)B_1$ for each $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n \in M$;

(vi) $g_1([A, B], A_2, \ldots, A_n) = 0$ for each $A, B, A_2, \ldots, A_n \in M$;

(vii) $g_1(A \circ B, A_2, \ldots, A_n) = 0$ for each $A, B, A_2, \ldots, A_n \in M$.

But $M$ is not a C.R.

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**References**

1. G. Pilz, 1983. *Near-Rings. (Second Edition)*, North Holland /American Elsevier, Amsterdam.
2. Wang, X. 1994. Derivations in prime near-rings, *Proc. Am. Math. Soc*. 121: 361–366.
3. Ashraf A, Siddeeque M. A. 2013. On permuting $n$-derivations in near-rings. *Commun. Kor. Math. Soc*. 28(4): 697–707.
4. Ashraf A, Siddeeque M. A. 2013. On \((\sigma,\tau)\)-n-derivations in near-rings. *Asian-European Journal of Mathematics*. 6(4): 1-14.

5. Ashraf A, Siddeeque MA. 2014. On generalized n-derivations in near-rings. *Palestine Journal of Mathematics*. 3(1): 468-480.

6. Ashraf A, Sideeque M. A. and Parveen N. 2015. On semigroup ideals and n-derivations in near-rings. *Science Direct Journal of Taibah University for Science*. (9): 126–132.

7. Majeed A. H and Adhab E. F. 2016. Right n-derivations in prime Near–Rings. *Journal Al-Qadisiyah /Pure Sciences*. 2(3): 31-41.

8. Bell H. E and Mason G. 1987. On derivations in near-rings, Near-rings and Near-fields, ed. G. Betsch. (North-Holland/American Elsevier, Amsterdam). 137: 31–35.

9. Bell HE. 1997. On Derivations in Near-Rings II. Near-rings, Near-fields and k-loops. *Kluwer Academic Publishers. Dordrecht*. 426: 191–197.