Regularity of a parabolic system involving curl

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Abstract
This note presents a regularity result with proof for an initial-boundary value problem of a linear parabolic system involving curl of the unknown vector field, subjected to the boundary condition of prescribing the tangential component of the solution.

Keywords Parabolic curl system · Regularity · Schauder estimate · Maxwell system

Mathematics Subject Classification 35Q60 · 35K51 · 35K65 · 35Q61

1 Introduction
We are interested in the regularity theory of linear parabolic systems involving curl. We believe that the regularity results are well-known to the experts. However it is difficult to find the statements with complete proofs in the literature. Therefore we wish to write out the conclusions with proofs, for convenience reference. We wish to start our program with the equation of the following form

\[
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + a \nabla \times \mathbf{u} + B \nabla \times \mathbf{u} + c \mathbf{u} &= \mathbf{f}, \\
\text{div } \mathbf{u} &= 0, \\
\mathbf{u}_T &= 0, \\
\mathbf{u}(0, x) &= \mathbf{u}^0,
\end{aligned}
\]

(1)

where \( a, c \) are scalar functions, \( B \) is a matrix-valued function, \( Q_T = (0, T] \times \Omega \) with \( \Omega \) being a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \), \( S_T = (0, T] \times \partial \Omega \). We denote \( \nabla \times \mathbf{u} \equiv \nabla \times \mathbf{u} \), and denote by \( \mathbf{u}_T \) the tangential component of \( \mathbf{u} \) at boundary
\( \partial \Omega \), namely \( \mathbf{u}_T = (\mathbf{v} \times \mathbf{u}) \times \mathbf{v} \), where \( \mathbf{v} \) is the unit outer normal vector of \( \partial \Omega \). In this paper, we use \( M(3) \) to denote the set of all \( 3 \times 3 \) matrices, and let
\[
C^{k+a}_{\mathcal{I}_0}(\overline{\Omega}, \text{div } 0) = \left\{ \mathbf{w} \in C^{k+a}(\overline{\Omega}, \mathbb{R}^3) : \text{div } \mathbf{w} = 0 \text{ in } \Omega, \ \mathbf{w}_T = 0 \text{ on } \partial \Omega \right\}.
\]
Note that the boundary condition in (1) is to prescribe the tangential component of the solution, and it makes (1) significantly different to the usual parabolic equation with Dirichlet boundary condition which prescribes the full trace. The regularity of weak solutions of (1) will be used in [4] to establish existence and regularity of weak solutions of the time-dependent model of Meissner states of superconductors.

**Theorem 1.1** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^{3+a} \) boundary, \( Q_T = (0, T] \times \Omega \). Assume that
\[
0 < \alpha < 1, \quad a, c \in C^{a,a/2}(\overline{Q_T}), \quad a(t, x) \geq a_0 > 0, \\
\mathcal{B} \in C^{a,a/2}(\overline{Q_T}, M(3)), \quad \mathbf{f} \in C^{a,a/2}(\overline{Q_T}, \mathbb{R}^3), \quad \mathbf{u}^0 \in C^{2+a}_{\mathcal{I}_0}(\overline{\Omega}, \text{div } 0),
\]
and
\[
a(0, x)[\text{curl } \mathbf{u}^0(x)]_T + [\mathcal{B}(0, x)\text{curl } \mathbf{u}^0(x)]_T = [\mathbf{f}(0, x)]_T, \quad x \in \partial \Omega. \tag{2}
\]
If \( \mathbf{u} \) is a weak solution of (1) on \( Q_T \), then \( \mathbf{u} \in C^{2+a,1+a/2}(\overline{Q_T}, \mathbb{R}^3) \) and
\[
\| \mathbf{u} \|_{C^{2+a,1+a/2}(\overline{Q_T})} \leq C \left\{ \| \mathbf{f} \|_{C^{a,a/2}(Q_T)} + \| \mathbf{u}^0 \|_{C^{2+a}\overline{\Omega}} \right\},
\]
where \( C \) depends only on \( \Omega, T, \alpha \) and the \( C^{a,a/2}(\overline{Q_T}) \) norm of \( a, b, \mathcal{B} \).

In (2) we use \([\cdot]_T\) to denote the tangential component of the enclosed vector. Let us mention that the assumption \( \mathbf{u}^0 \in C^{2+a}_{\mathcal{I}_0}(\overline{\Omega}, \text{div } 0) \) implies that \( \mathbf{u}^0_T = 0 \) for \( x \in \partial \Omega \), which is consistent with the boundary condition \( \mathbf{u}_T = 0 \). This together with the assumption (2) consists of the compatibility condition for the problem (1).

### 2 Estimates near flat boundary

#### 2.1 \( W^{2,1,q} \)-estimates

We consider regularity of weak solutions of (1), where \( a, c \) are scalar functions and \( \mathcal{B} \) is a matrix-valued function. Let \( \mathbf{u} \) be a weak solution of (1). Then \( \mathbf{u} \in L^2(0, T; H^1(\Omega, \mathbb{R}^3)) \). To get higher regularity of the solutions, one may first use the difference method to show that \( \mathbf{u} \in L^2(0, T; H^2(\Omega, \mathbb{R}^3)) \) and \( \partial_t \mathbf{u} \in L^2(0, T; L^2(\Omega, \mathbb{R}^3)) \), then show \( \mathbf{u} \) is of \( C^{2+a,1+a/2} \). Here we use the different approach. We shall start with a weak solution \( \mathbf{u} \in L^2(0, T; H^1(\Omega, \mathbb{R}^3)) \) and show directly \( \mathbf{u} \) is of \( C^{2+a,1+a/2} \).
We shall derive the a priori estimates for smooth functions. Then the regularity of weak solutions follows from the estimates.

By considering cut-off, we only need to examine regularity near boundary. We start with a flat boundary. Denote by $B^+_R$ the upper half ball with center at the origin and radius $R$, and

$$\Sigma_R = \left\{ x = (x_1, x_2, 0) : |x| < R \right\}.$$  

Let

$$Q_{R,T} = (0, T) \times B^+_R, \quad \Gamma_{R,T} = (0, T) \times \Sigma_R.$$  

With the divergence-free condition, (1) can be written in the following form

$$\begin{cases}
\frac{\partial u}{\partial t} - a \Delta u + B \text{curl} u + cu = f, & (t, x) \in Q_{R,T}, \\
u_T = 0, & (t, x) \in \Gamma_{R,T}, \\
u(0, x) = u^0, & x \in B^+_R.
\end{cases} \tag{3}$$

As mentioned in the introduction, the boundary condition in (3) is to prescribe the tangential component, but not the full trace, of the solution. As such, the regularity of (3) is not a direct consequence of the regularity theory of the classical initial-Dirichlet boundary problem of parabolic equations.

The compatibility condition (2) can be written as

$$-a(0, x)[\Delta u^0(x)]_T + [B(0, x)\text{curl} u^0(x)]_T = [f(0, x)]_T, \quad x \in \partial\Omega. \tag{4}$$

Note that $\Gamma_{R,T}$ is the flat part of the parabolic boundary of $Q_{R,T}$. We shall establish the following local estimate:

**Lemma 2.1** Assume that

$$a, c \in C^{\alpha, \alpha/2}(Q_{R_0, T}), \quad a(t, x) \geq a_0 > 0, \quad 1 < q < \infty,$$

$$B \in C^{\alpha, \alpha/2}(Q_{R_0, T}, M(3)), \quad f \in L^q(Q_{R_0, T}, \mathbb{R}^3),$$

$$u^0 \in W^{2, q}(B^+_R, \mathbb{R}^3), \quad \text{div} u^0 = 0 \text{ in } B_{R_0}, \quad u^0_T = 0 \text{ on } \Sigma_{R_0},$$

and assume (2) holds. If $u$ is a weak solution of (3) on $Q_{R_0, T}$, then for any $0 < R < R_0$ we have $u \in W^{2, d}(Q_{R, T}, \mathbb{R}^3)$ and

$$\|u\|_{W^{2, 1, d}(Q_{R, T})} \leq C \left\{ \|f\|_{L^q(Q_{R_0, T})} + \|\nabla x u\|_{L^2(Q_{R_0, T})} + \|u\|_{L^2(Q_{R_0, T})} + \|u^0\|_{W^{2, q}(B^+_R)} \right\},$$

where $C$ depends only on $R_0, R, T, B, a, c, q$.

**Proof** Let $u$ be a solution of (3). Write
\[ \mathbf{u} = (u_1, u_2, u_3)', \quad \mathbf{f} = (f_1, f_2, f_3)', \quad B \text{curl} \mathbf{u} = \mathbf{H} = (h_1, h_2, h_3)'. \]

\( u_1, u_2 \) correspond to the tangential component of \( \mathbf{u} \) and \( u_3 \) corresponds to the normal component. Recall the formula (see [2, p.210])

\[
\text{div} \mathbf{u} = \text{div}_t (\pi \mathbf{u}) + 2(\mathbf{v} \cdot \mathbf{u})H(x) + \frac{\partial}{\partial \mathbf{v}}(\mathbf{v} \cdot \mathbf{u}).
\]

In the above \( \pi \mathbf{u} \) denotes the tangential component of \( \mathbf{u} \) on the domain boundary, \( \text{div}_t \) denotes the surface divergence, and \( H(x) \) is the mean curvature of the domain boundary. Applying the above equality on the flat part of the boundary \( \Sigma_R \) where \( H(x) \equiv 0 \) we see that, the boundary condition \( \mathbf{u}_T = 0 \) together with the divergence-free condition \( \text{div} \mathbf{u} = 0 \) implies the Neumann boundary condition for \( u_3 \). In fact, for \( x \in \Sigma_R \) we have

\[ \pi \mathbf{u} = \mathbf{u}_T = (u_1, u_2, 0), \quad \mathbf{v} \cdot \mathbf{u} = u_3. \]

Hence

\[ \frac{\partial u_3}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}(\mathbf{v} \cdot \mathbf{u}) = \text{div} \mathbf{u} - \text{div}_t (\pi \mathbf{u}) - 2(\mathbf{v} \cdot \mathbf{u})H(x). \]

So

\[ u_1 = 0, \quad u_2 = 0, \quad \frac{\partial u_3}{\partial \mathbf{v}} = 0 \text{ on } \Gamma_{R,T}. \]

We can write the equations for \( u_1 \) and \( u_2 \) as follows:

\[
\begin{cases}
\partial_t u_j - a \Delta u_j + cu_j = f_j - h_j, & (t, x) \in Q_{R,T}, \\
u_j = 0, & (t, x) \in \Gamma_{R,T}, \\
u_j(0, x) = u_j^0, & x \in B^+_R,
\end{cases}
\]

\( j = 1, 2 \), and write the equation for \( u_3 \) as follows:

\[
\begin{cases}
\partial_t u_3 - a \Delta u_3 + cu_3 = f_3 - h_3, & (t, x) \in Q_{R,T}, \\
\frac{\partial u_3}{\partial \mathbf{v}} = 0, & (t, x) \in \Gamma_{R,T}, \\
u_3(0, x) = u_3^0, & x \in B^+_R.
\end{cases}
\]

However, since \( \Gamma_{R,T} \) is only a subset of the parabolic boundary of \( Q_{R,T} \), (5) and (6) are not exactly the standard initial-boundary value problem of parabolic equations with Dirichlet or Norman boundary condition. We shall modify \( u_j \)'s to get the standard initial-boundary problems of parabolic equations.

Given \( 0 < R < R_0 \), we can take a domain \( U \) with \( C^{2+\alpha} \) boundary and a smooth function \( \eta \) supported in \( U \) such that
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In fact we can first choose \( \hat{R} \) such that \( R < \hat{R} < R_0 \). Take a smooth cur-off function \( \eta \) such that

\[
\text{spt}(\eta) \subset B_{R}, \quad \eta = 1 \quad \text{in} \quad B_{\hat{R}}, \quad \frac{\partial \eta}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_{\hat{R}}.
\]

(7)

Then we take a domain \( \hat{U} \) with smooth boundary such that

\[
\overline{B_{\hat{R}}} \subset \hat{U} \subset \overline{U} \subset B_{R_0}.
\]

We can choose \( \hat{U} \) such that \( U \equiv \hat{U} \cap B_{R_0}^+ \) has \( C^{2+a} \) boundary. Then \( U \) and \( \eta \) satisfy (7).

Let \( w = \eta u \). Then

\[
w(t, x) = 0 \quad \text{for} \quad 0 < t \leq T, \quad x \in \partial U \setminus \Sigma_{R_1}.
\]

Moreover we actually have \( w(t, x) = 0 \) for \( x \in U \setminus B_{R_1} \), thus

\[
\frac{\partial}{\partial \nu}(\nu \cdot w) = 0 \quad \text{for} \quad x \in \partial U \setminus \Sigma_{R_1}.
\]

If \( x \in \Sigma_{R_1} \), then from (8) we have

\[
\frac{\partial}{\partial \nu}(\nu \cdot w) = \frac{\partial}{\partial \nu}(\eta \nu \cdot u) = \eta \frac{\partial}{\partial \nu}(\nu \cdot u) + (\nu \cdot u) \frac{\partial}{\partial \nu} \eta = 0.
\]

So we have

\[
w_T = 0, \quad \frac{\partial}{\partial \nu}(\nu \cdot w) = 0 \quad \text{if} \quad 0 < t \leq T, \quad x \in \partial U.
\]

(9)

Denote

\[
G_T = (0, T] \times U, \quad L_T = (0, T] \times \partial U.
\]

We see that \( w \) is a weak solution of a modified system on \( G_T \), namely

\[
\begin{cases}
\frac{\partial w}{\partial t} - a \Delta w + B \text{curl } w + cw = F, & \text{div } w = g, \quad (t, x) \in G_T, \\
w_T = 0, & (t, x) \in L_T, \\
w(0, x) = w^0, & x \in U,
\end{cases}
\]

(10)
\[ \mathbf{F} = \eta \mathbf{f} - a \left( \Delta \eta \mathbf{u} + \sum_{j=1}^{3} \partial_j \eta \partial_j \mathbf{u} \right) - \mathcal{B}(\nabla \eta \times \mathbf{u}), \]

\[ g = \nabla \eta \cdot \mathbf{u}, \quad \mathbf{w}^0 = \eta \mathbf{u}^0. \]

Now we write

\[ \mathbf{w} = (w_1, w_2, w_3)', \quad \mathbf{F} = (F_1, F_2, F_3)', \quad \mathcal{B} \text{curl} \mathbf{w} = \mathbf{H} = (H_1, H_2, H_3)'. \]

Then \( w_1, w_2 \) satisfy

\[ \begin{aligned}
\partial_t w_j - a \Delta w_j + c w_j &= F_j - H_j, & (t, x) &\in G_T, \\
w_j &= 0, & (t, x) &\in L_T, \\
w_j(0, x) &= w_j^0, & x &\in U,
\end{aligned} \tag{11} \]

\( j = 1, 2 \), and \( w_3 \) satisfies

\[ \begin{aligned}
\partial_t w_3 - a \Delta w_3 + c w_3 &= F_3 - H_3, & (t, x) &\in G_T, \\
\frac{\partial w_3}{\partial \nu} &= 0, & (t, x) &\in L_T, \\
w_3(0, x) &= w_3^0, & x &\in U.
\end{aligned} \tag{12} \]

From the assumption on \( \mathbf{u}^0 \) and (4) we see that the following compatibility condition for the parabolic Dirichlet problem (11) is satisfied for \( x \in \partial U \) and for \( j = 1, 2 \):

\[ w_j^0(x) = 0, \quad -a(0, x) \Delta w_j^0(x) = F_j(0, x) - H_j(0, x). \]

We can apply the theory of regularity of parabolic equations to get a priori estimates of the solutions \( \mathbf{w} \) in terms of \( F_j \)'s and \( H_j \)'s. However we can not directly get the final estimation by iterating the local estimates on the previously chosen half ball \( B^+_R \). To see this point, recall that the standard iteration processes such as the bootstrap argument require the right hand terms be controlled by the unknowns. In our case \( F_j \)'s can be controlled by \( \nabla \mathbf{u} \), but not by \( \nabla \mathbf{w} \). So we can not improve the regularity on \( F_j \)'s over the whole region \( B^+_R \) by iteration. Nevertheless, we have improved the regularity of \( \mathbf{w} \) in \( B^+_R \), then we get the improved regularity of \( \mathbf{u} \) in \( B^+_R \) with some \( R_1 < R \) where \( \eta = 1 \). Hence we can improve the regularity of \( F_j \)'s over \( B^+_R \). Then we can iterate the above estimation to get the further improved estimates of the solution \( \mathbf{w} \) on \( B^+_R \) in terms of \( F_j \)'s. We iterate this procedure in a finite times to get improved estimates on smaller regions.

Following the above idea, we shall first derive the estimates of the solution \( \mathbf{w} \) of (10) in terms of \( \mathbf{f} \) and \( g \).

(a) First of all, by the Sobolev imbedding in \( \mathbb{R}^3 \) we have

\[ L^2(0, T; H^2(U)) \hookrightarrow L^2(0, T, W^{1,p}(U)), \quad \forall 1 < p < \infty. \]
\[ |\mathbf{H}| \leq C |\text{curl } \mathbf{w}|, \]
\[ |\mathbf{F}| \leq C (|\mathbf{f}| + |\mathbf{u}| + |\nabla \mathbf{u}|). \quad (13) \]

(b) In the following we derive the $L^p$ estimates.

**Step 1.** $L^p$ estimate for $\mathbf{w}_1, \mathbf{w}_2$.

We take the following iteration argument. If $1 < p < \infty$ is such that
\[ \|\text{curl } \mathbf{w}\|_{L^p(G_T)} < \infty, \quad \|\mathbf{F}\|_{L^p(G_T)} < \infty, \quad (14) \]

then
\[ \|\mathbf{H}\|_{L^p(G_T)} < \infty, \quad (15) \]

and we can apply the global $L^p$ estimate for Dirichlet problem of heat equation (see [5, p.176, Theorem 7.17]) to (11) to get, for \( j = 1, 2 \),
\[ \|w_j\|_{W^{2,1,p}(G_T)} \leq C \left\{ \|F_j - H_j\|_{L^p(G_T)} + \|w_j^0\|_{W^{2,p}(U)} \right\} \]
\[ \leq C \left\{ \|\mathbf{f}\|_{L^p(G_T)} + \|\text{curl } \mathbf{w}\|_{L^p(G_T)} + \|w_j^0\|_{W^{2,p}(U)} \right\}, \quad (16) \]

where $C$ depends only on $U, T, a, c, p$.

Now (14) is true for $p = 2$ by the assumption, hence by (16) we have
\[ w_j \in W^{2,1,2}(G_T), \quad j = 1, 2. \]

Then by Sobolev imbedding (see [3, p.26, Theorem 3.14 (i)] with
\[ p_1 \equiv q = \frac{(n + 2)p}{n + 2 - p} = \frac{5p}{3} = \frac{10}{3} \]
when $p = 2$ we see that $|\nabla_x w_j| \in L^{p_1}(G_T)$ with $p_1 = 10/3$, and
\[ \|\nabla_x w_j\|_{L^{p_1}(G_T)} \leq C \|w_j\|_{W^{2,1,2}(G_T)} \leq C \left\{ \|\mathbf{f}\|_{L^3(G_T)} + \|\text{curl } \mathbf{w}\|_{L^3(G_T)} + \|w_j^0\|_{W^{2,2}(U)} \right\} \]
for $j = 1, 2$, where $C$ depends only on $U, T, a, c, p_1$.

**Step 2.** $L^p$ estimate for $\mathbf{w}_3$.

If $1 < p < \infty$ is such that (14) holds hence (15) is true, then we can apply the global $L^p$ estimate for Neumann problem of heat equation to (12) to get
\[ \|w_3\|_{W^{2,1,p}(G_T)} \leq C \left\{ \|F_3 - H_3\|_{L^p(G_T)} + \|w_3^0\|_{W^{2,p}(U)} \right\} \]
\[ \leq C \left\{ \|\mathbf{f}\|_{L^p(G_T)} + \|\text{curl } \mathbf{w}\|_{L^p(G_T)} + \|w_3^0\|_{W^{2,p}(U)} \right\}, \quad (17) \]

where $C$ depends only on $U, T, a, c, p$. 

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Now (14) is true for \( p = 2 \) by the assumption, hence by (17) \( w_3 \in W^{2,1,2}(G_T) \). Then from the Sobolev imbedding (see [3, p.26, Theorem 3.14 (i)] with \( p = 2 \) we see that \( |\nabla_x w_3| \in L^{p_1}(G_T) \) with \( p_1 = 10/3 \), and
\[
\|\nabla_x w_3\|_{L^{p_1}(G_T)} \leq C\|w_3\|_{W^{2,1,2}(G_T)},
\]
where \( C \) depends only on \( U, T, a, c, p_1 \).

Combining the results for \( w_1, w_2, w_3 \) we get
\[
\|\nabla_x w\|_{L^{p_1}(G_T)} \leq \|w\|_{W^{2,1,2}(G_T)} \leq \|w\|_{W^{2,1,2}(G_T)} \leq C\bigg\{\|F\|_{L^2(G_T)} + \|\text{curl } w\|_{L^2(G_T)} + \|w^0\|_{W^{2,2}(U)}\bigg\},
\]
where \( C \) depends only on \( U, T, a, c, p_1 \). It follows that (14) is true with \( p \) replaced by \( p_1 = 10/3 \).

Step 3. By the choice of the cut-off function \( \eta \) we have \( \eta = 1 \) on \( \overline{B^+_R} \), so \( w = u \) on \( B^+_R \). From steps 1 and 2 we see that
\[
u \in W^{2,1,2}(Q_{R,T}, \mathbb{R}^3),
\]
and
\[
\|u\|_{L^3(Q_{R,T})} \leq \|u\|_{W^{2,1,2}(Q_{R,T})} \leq \|w\|_{W^{2,1,2}(G_T)} \leq C\bigg\{\|F\|_{L^2(G_T)} + \|\text{curl } w\|_{L^2(G_T)} + \|w^0\|_{W^{2,2}(U)}\bigg\},
\]
where \( C \) depends only on \( R, R_0, T, U, \eta, a, c, p_1 \). We can construct the domain \( U \) in (7) and the cut-off function \( \eta \) which depend only on \( R \) and \( R_0 \). Then in the above inequality the constant \( C \) depends only on \( R, R_0, T, a, c \). Then we have
\[
\|\text{curl } w\|_{L^2(G_T)} \leq C\bigg\{\|u\|_{L^2(G_T)} + \|\nabla_x u\|_{L^2(G_T)}\bigg\},
\]
\[
\|w^0\|_{W^{2,2}(U)} \leq C\|u^0\|_{W^{2,2}(B^+_R)}
\]
where \( C \) depends only on \( R, R_0 \). From (13) and (19) we see that, for \( p_1 = 10/3 \),
\[
\|F\|_{L^{p_1}(Q_{R,T})} \leq C\bigg\{\|f\|_{L^{p_1}(Q_{R,T})} + \|u\|_{L^{p_1}(Q_{R,T})} + \|\nabla u\|_{L^{p_1}(Q_{R,T})}\bigg\}
\]
\[
\leq C\bigg\{\|f\|_{L^{p_1}(G_T)} + \|F\|_{L^2(G_T)} + \|\text{curl } w\|_{L^2(G_T)} + \|w^0\|_{W^{2,2}(U)}\bigg\}
\]
\[
\leq C\bigg\{\|f\|_{L^{p_1}(G_T)} + \|u\|_{L^2(G_T)} + \|\nabla_x u\|_{L^2(G_T)} + \|\text{curl } w\|_{L^2(G_T)} + \|w^0\|_{W^{2,2}(U)}\bigg\}
\]
\[
\leq C\bigg\{\|f\|_{L^{p_1}(Q_{R_0,T})} + \|u\|_{L^2(Q_{R_0,T})} + \|\nabla_x u\|_{L^2(Q_{R_0,T})} + \|u^0\|_{W^{2,2}(B^+_R)}\bigg\},
\]
where \( C \) depends only on \( R, R_0, T, a, c, p_1 \).

Step 4. Let \( 0 < R_2 < R_1 < R_0 \) be given. We change \( R \) to \( R_1 \) in the above argument to get that \( u \in W^{2,1,2}(Q_{R_1,T}) \) with

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\[ \|u\|_{L^p(Q_{R_1}, T)} \leq \|u\|_{W^{2,1}(Q_{R_1}, T)} \leq C \left\{ \|f\|_{L^p(Q_{R_0}, T)} + \|u\|_{L^2(Q_{R_0}, T)} + \|\nabla_x u\|_{L^2(Q_{R_0}, T)} + \|u^0\|_{W^{2,1}(B_{R_0}^+)} \right\}, \]

where \( p_1 = 10/3 \) and \( C \) depends only on \( R_1, R_0, T, a, c, p_1 \). Then

\[ \|F\|_{L^p(Q_{R_1}, T)} \leq C \left\{ \|f\|_{L^p(Q_{R_0}, T)} + \|u\|_{L^2(Q_{R_0}, T)} + \|\nabla_x u\|_{L^2(Q_{R_0}, T)} + \|u^0\|_{W^{2,1}(B_{R_0}^+)} \right\}, \]

where \( C \) depends only on \( R_1, R_0, T, a, c, p_1 \). From (18) and (19) we have

\[ \|H\|_{L^p(Q_{R_1}, T)} \leq C \|\text{curl } w\|_{L^p(Q_{R_1}, T)} \leq C \left\{ \|f\|_{L^p(Q_{R_0}, T)} + \|u\|_{L^2(Q_{R_0}, T)} + \|\nabla_x u\|_{L^2(Q_{R_0}, T)} + \|u^0\|_{W^{2,1}(B_{R_0}^+)} \right\}, \]

where \( C \) depends only on \( R_1, R_0, T, a, c, p_1 \).

Then we can repeat the above argument with \( R \) and \( R_0 \) replaced by \( R_2 \) and \( R_1 \) to get \( W^{2,1,p_1} \) estimate on \( \Omega_{R_2}, T \). More precisely, we can take a domain \( U_2 \) with \( C^{2+a} \) boundary and a smooth function \( \eta_2 \) supported in \( U_2 \) such that

\[ B_{R_2}^+ \subset U_2 \subset B_{R_1}^+, \quad \eta_2(x) = 1 \text{ if } x \in B_{R_2}^+, \quad \eta_2(x) = 0 \text{ if } x \in B_{R_2}^+ \setminus U_2. \]

Set \( G_T^2 = (0, T) \times U_2 \). Then the conditions (14) and (15) hold on \( G_T^2 \) with \( p \) replaced by \( p_1 \). As in steps 1 and 2, we apply the \( L^p \) estimates of heat equation to get (16) and (17) on \( G_T^2 \) with \( p \) replaced by \( p_1 \); then we get (19) with \( R \) and \( R_0 \) replaced by \( R_2 \) and \( R_1 \). So we have now

\[ \|u\|_{L^p(Q_{R_2}, T)} \leq \|u\|_{W^{2,1,p_1}(Q_{R_2}, T)} \leq C \left\{ \|f\|_{L^p(Q_{R_1}, T)} + \|u\|_{L^p(Q_{R_1}, T)} + \|\nabla_x u\|_{L^p(Q_{R_1}, T)} + \|u^0\|_{W^{2,1,p_1}(B_{R_1}^+)} \right\}, \]

where

\[ p_2 = \frac{(n + 2)p_1}{n + 2 - p_1} = \frac{5p_1}{5 - p_1} = 10 \]

and \( C \) depends only on \( R_1, R_0, T, a, c, p_2 \). It further follows that

\[ F \in L^{p_2}(Q_{R_2}, \mathbb{R}^3), \quad H \in L^{p_2}(Q_{R_2}, \mathbb{R}^3). \]

Step 5. The above iteration is valid for a similar system in dimension \( n \). Fix \( 0 < R < R_0 \) and let

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If $p_k < n + 2$, after having got
\[
\mathbf{u} \in W^{2,1,p_k}(Q_{R_k,T}, \mathbb{R}^n),
\]

we apply the iteration argument with $p = p_{k+1}$ on $Q_{R_k,T}$ to conclude that
\[
\mathbf{u} \in W^{2,1,p_{k+1}}(Q_{R_{k+1},T}, \mathbb{R}^n),
\]

and
\[
\|\mathbf{u}\|_{W^{2,1,p_{k+1}}(Q_{R_{k+1},T})} \leq C \left\{ \|\mathbf{f}\|_{L^{p_{k+1}}(Q_{R_k,T})} + \|\mathbf{u}\|_{L^{p_k}(Q_{R_k,T})} + \|\nabla \mathbf{u}\|_{L^{p_k}(Q_{R_k,T})} + \|\mathbf{u}^0\|_{W^{2,p_{k+1}}(B_{R_k}^*)} \right\} (24)
\]

where $C$ depends only on $n, R, R_0, T, a, c, p_k$.

Note that $0 < R_0 < R_0^* < R_0^* < R_0$ for all $k$ and $p_k + 1 > n + 2$ for some $k$ (when $n = 3$ we have $p_2 > 5$). Hence by Sobolev imbedding we have $\mathbf{u} \in L^q(Q_{R_{k+1},T}, \mathbb{R}^n)$ for all $1 < q < \infty$.

**Step 6.** Now we come back to the case with $n = 3$. For any $1 < q < \infty$, we can choose $U$ and $\eta$ such that (7) and (8) hold. Then we repeat the $L^q$ estimation for $\mathbf{w} = \eta \mathbf{u}$ on $G_T = (0, T) \times U$, which implies that
\[
\mathbf{u} \in W^{2,1,q}(Q_{R,T}, \mathbb{R}^3),
\]

and
\[
\|\mathbf{u}\|_{W^{2,1,q}(Q_{R,T})} \leq \|\mathbf{w}\|_{W^{2,1,q}(G_T)} \leq C \left\{ \|\mathbf{f}\|_{L^q(G_T)} + \|\mathbf{w}\|_{L^q(G_T)} + \|\text{curl}\ \mathbf{w}\|_{L^q(G_T)} + \|\mathbf{w}^0\|_{W^{2,q}(U)} \right\}
\]

where $C$ depends only on $R, R_0, T, a, c, q$. \hfill \square
2.2 \( C^{1+\alpha(1+\alpha)/2} \)-estimates

**Corollary 2.2** Under the assumption of Lemma 2.1, if \( 0 < \alpha < 1 \) and \( 0 < R < R_0 \), the weak solution \( u \) is of \( C^{1+\alpha(1+\alpha)/2} \), and

\[
\|u\|_{C^{1+\alpha(1+\alpha)/2}(\overline{Q}_{R,T})} \leq C \left\{ \|f\|_{L^q(Q_{R_0,T})} + \|u\|_{L^2(Q_{R_0,T})} + \|\nabla u\|_{L^2(Q_{R_0,T})} + \|u^0\|_{W^{2,q}(B^+_R)} \right\},
\]

where \( 5/(1-\alpha) < q < \infty \), and \( C \) depends only on \( R, R_0, T, a, c, \alpha, q \).

**Proof** We apply the Sobolev imbedding given in [3, p.26, Theorem 3.14 (3)] (with \( p = q > 5/(1-\alpha) \)) to conclusion that

\[
u \in C^{1+\alpha(1+\alpha)/2}(\overline{Q}_{R,T}, \mathbb{R}^3)
\]

and

\[
\|u\|_{C^{1+\alpha(1+\alpha)/2}(\overline{Q}_{R,T})} \leq C \|u\|_{W^{2,q}(Q_{R,T})}.
\]

Then the conclusion follows from Lemma 2.1. \( \square \)

2.3 Schauder estimates

**Lemma 2.3** Assume that

\[
a, c \in C^{\alpha,\alpha/2}(\overline{Q}_{R_0,T}), \quad a(t,x) \geq a_0 > 0,
\]

\[
B \in C^{\alpha,\alpha/2}(\overline{Q}_{R_0,T}, M(3)), \quad f \in C^{\alpha,\alpha/2}(\overline{Q}_{R_0,T}, \mathbb{R}^3),
\]

\[
u^0 \in C^{2+\alpha}(\overline{B}^+_{R_0}, \mathbb{R}^3), \quad \text{div} \nu^0 = 0 \text{ in } B_{R_0}^+, \quad \nu^0_T = 0 \text{ on } \Sigma_{R_0},
\]

and assume (2) holds. If \( u \) is a weak solution of (3) on \( Q_{R_0,T} \), then for any \( 0 < R < R_0 \) we have \( u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{R,T}) \) and

\[
\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_{R,T})} \leq C \left\{ \|f\|_{C^{\alpha,\alpha/2}(\overline{Q}_{R_0,T})} + \|u\|_{L^2(Q_{R_0,T})} + \|\nabla u\|_{L^2(Q_{R_0,T})} + \|u^0\|_{C^{2+\alpha}(\overline{B}^+_{R_0})} \right\},
\]

where \( C \) depends only on \( R_0, R, T, \alpha, B, a, c \).
**Proof** In the following we use the fact that if \( u \in C^{1+\alpha, (1+\alpha)/2}(\Omega_{R,T}) \), then \( \nabla \times u \in C^{\alpha,\alpha/2}(\Omega_{R,T}) \).

Take \( R_1 \) depending only on \( R \) and \( R_0 \) such that \( R < R_1 < R_0 \). From Corollary 2.2 we know that \( u \in C^{1+\alpha, (1+\alpha)/2}(\Omega_{R_1,T}, \mathbb{R}^3) \), hence \( \text{curl } u \in C^{\alpha,\alpha/2}(\Omega_{R_1,T}, \mathbb{R}^3) \), thus

\[
B \text{ curl } u \in C^{\alpha,\alpha/2}(\Omega_{R_1,T}, \mathbb{R}^3).
\]

Take a domain \( U \) and a smooth cut-off function \( \eta \) such that (7) is satisfied with \( R_0 \) replaced by \( R_1 \). Set \( G_T = (0, T] \times U \) and \( L_T = (0, T] \times \partial U \). Set \( w = \eta u \). Then \( w \) is a weak solution of (10), where \( F \in C^{\alpha,\alpha/2}(\Omega_T, \mathbb{R}^3) \) because \( \text{curl } F \in C^{\alpha,\alpha/2}(\Omega_T, \mathbb{R}^3) \) and \( \partial_j \in C^{\alpha,\alpha/2}(\Omega_T, \mathbb{R}^3) \).

Again we write \( w = (w_1, w_2, w_3)' \), \( F = (F_1, F_2, F_3)' \), \( B \text{ curl } w = H = (H_1, H_2, H_3)' \). Then \( w_1, w_2 \) are weak solutions of (11) and \( w_3 \) is a weak solution of (12).

Applying the global Schauder estimate for Dirichlet problem (see [5, p.78, Theorem 4.28]) to (11) we have

\[
\|w_j\|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)} \leq C_D \left\{ \|F_j - H_j\|_{C^{\alpha,\alpha/2}(\Omega_T)} + \|w_j^0\|_{C^{2+\alpha}(\Omega)} \right\}, \quad j = 1, 2.
\]

Then applying the global Schauder estimate for Neumann problem [5, p.79, Theorem 4.31] to (12) we have

\[
\|w_3\|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)} \leq C_N \left\{ \|F_3 - H_3\|_{C^{\alpha,\alpha/2}(\Omega_T)} + \|w_3^0\|_{C^{2+\alpha}(\Omega)} \right\},
\]

Therefore \( w \in C^{2+\alpha,1+\alpha/2}(\Omega_T, \mathbb{R}^3) \), and

\[
\|w\|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)} \leq C \left\{ \|F\|_{C^{\alpha,\alpha/2}(\Omega_T)} + \|H\|_{C^{\alpha,\alpha/2}(\Omega_T)} + \|w^0\|_{C^{2+\alpha}(\Omega)} \right\}. \tag{26}
\]

Note that

\[
\|F\|_{C^{\alpha,\alpha/2}(\Omega_T)} \leq C \left\{ \|f\|_{C^{\alpha,\alpha/2}(\Omega_T)} + \|u\|_{C^{\alpha,\alpha/2}(\Omega_T)} + \|\nabla \times u\|_{C^{\alpha,\alpha/2}(\Omega_T)} \right\},
\]

\[
\|H\|_{C^{\alpha,\alpha/2}(\Omega_T)} \leq C \|\text{curl } w\|_{C^{\alpha,\alpha/2}(\Omega_T)} \leq C \|w\|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)}.
\]

From these and (26) we get

\[
\|w\|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)} \leq C \left\{ \|f\|_{C^{\alpha,\alpha/2}(\Omega_T)} + \|u\|_{C^{1+\alpha,1+\alpha/2}(\Omega_T)} + \|w^0\|_{C^{2+\alpha}(\Omega)} \right\}.
\]

Using the construction of \( G_T \) and Corollary 2.2 with \( R \) replaced by \( R_1 \), and with the index \( q \) determined by \( \alpha \), we have
\[ \|u\|_{C^{1+a,(1+a)/2}(\overline{Q}_T)} \leq \|u\|_{C^{1+a,(1+a)/2}((Q_{R_1}, T)} \]
\[ \leq C \left\{ \|f\|_{L^2(Q_{R_1}, T)} + \|u\|_{L^2(Q_{R_0}, T)} + \|\nabla_x u\|_{L^2(Q_{R_0}, T)} + \|u^0\|_{C^{1+a}((\overline{Q}_{R_0}))} \right\} \]
\[ \leq C \left\{ \|f\|_{C^{a/2}(\overline{Q}_{R_0}, T)} + \|u\|_{L^2(Q_{R_0}, T)} + \|\nabla_x u\|_{L^2(Q_{R_0}, T)} + \|u^0\|_{C^{1+a}((\overline{Q}_{R_0}))} \right\}. \]

where \( C \) depends only on \( R, R_0, T, \alpha, B, a, c \), as \( R_1 \) is determined by \( R \) and \( R_0 \). Hence we get

\[ \|u\|_{C^{1+a,(1+a)/2}(\overline{Q}_{R,T})} \leq C \left\{ \|f\|_{C^{a/2}(\overline{Q}_{R_0}, T)} + \|u\|_{L^2(Q_{R_0}, T)} + \|\nabla_x u\|_{L^2(Q_{R_0}, T)} + \|u^0\|_{C^{1+a}((\overline{Q}_{R_0}))} \right\}. \]

\[ \square \]

3 Estimates near curved boundary

3.1 Computations in local coordinates near boundary

Let us briefly recall the local coordinates near boundary \( \partial \Omega \) determined by a diffeomorphism that straightens a piece of surface, see [6, section 3] and [1, Appendix]. Let us fix a point \( x_0 \in \partial \Omega \), and introduce new variables \( y_1, y_2 \) such that \( \partial \Omega \) can be represented (at least near \( x_0 \)) by \( r = r(y_1, y_2) \), and \( r(0,0) = x_0 \). Here and henceforth we denote \( y = (y_1, y_2) \) and use the notation \( r_j(y) = \partial_j r(y) \), \( r_{ij} = \partial_{y_i} \partial_{y_j} r(y) \), etc. Let

\[ \mathbf{n}(y) = \frac{r_1(y) \times r_2(y)}{|r_1(y) \times r_2(y)|}. \]

We choose \((y_1, y_2)\) in such a way that \( \mathbf{n}(y) \) is the inward normal of \( \partial \Omega \), and that the \( y_1 \)- and \( y_2 \)-curves on \( \partial \Omega \) are the lines of principal curvatures; thus, \( r_1(y) \) and \( r_2(y) \) are orthogonal to each other. Let

\[ g_{ij}(y) = r_i(y) \cdot r_j(y), \quad i, j = 1, 2, \quad g(y) = \det(g_{ij}(y)) = g_{11}(y)g_{22}(y). \]

Let us define a map \( \mathcal{F} \) by

\[ x = \mathcal{F}(y, z) = r(y_1, y_2) + zn(y_1, y_2). \]

\( \mathcal{F} \) is a diffeomorphism from a ball \( B_R(0) \) onto a neighborhood \( \mathcal{U} \) of the point \( x_0 \), and it maps the half ball \( B_R^+(0) \) onto a subdomain \( \mathcal{U} \cap \Omega \), and maps the disc \( \{(y_1, y_2, 0) : y_1^2 + y_2^2 < R^2 \} \) onto a subset \( \Gamma \) of \( \partial \Omega \).

Denote the partial derivative \( \partial_{y_j} \) by \( \partial_j \), \( j = 1, 2 \), and denote \( \partial_z \) by \( \partial_3 \). Let
Given a vector field $\mathbf{B}$ defined on $\overline{\Omega}$, we can represent $\mathbf{B}$ in a neighborhood of $x_0 \in \partial \Omega$ in the new variables $(y, z) \in B^+_R = F^{-1}(\mathcal{U} \cap \Omega)$ as follows:

$$
\tilde{\mathbf{B}}(y, z) = \mathbf{B}(F(y, z)) = \sum_{j=1}^{3} G_{ij}(y, z) b_j(y, z) \partial_j F(y, z),
$$

where

$$
b_j(y, z) = \mathbf{B}(F(y, z)) \cdot \partial_j F(y, z),
$$

$$
\tilde{B}_j(y, z) = \frac{b_j(y, z)}{\sqrt{G_{ij}(y, z)}}.
$$

We compute, at the point $x = F(y, z)$,

$$
curl \mathbf{B}(x) = \sum_{j=1}^{3} \tilde{R}_j(y, z) \mathbf{E}_j(y, z),
$$

$$
div \mathbf{B} = \frac{1}{\sqrt{G}} \left[ \sum_{j=1}^{2} \partial_j \left( \sqrt{G_{ij}} \tilde{B}_j \right) + \partial_3 \left( \sqrt{G_{33}} \tilde{B}_3 \right) \right],
$$

where

$$
\tilde{R}_1(y, z) = \frac{1}{\sqrt{G_{22}G_{33}}} [\partial_2 (\tilde{B}_3 \sqrt{G_{33}}) - \partial_3 (\tilde{B}_2 \sqrt{G_{22}})] = \frac{1}{\sqrt{G_{22}}} [\partial_2 b_3 - \partial_3 b_2],
$$

$$
\tilde{R}_2(y, z) = \frac{1}{\sqrt{G_{33}G_{11}}} [\partial_3 (\tilde{B}_1 \sqrt{G_{11}}) - \partial_1 (\tilde{B}_3 \sqrt{G_{33}})] = \frac{1}{\sqrt{G_{11}}} [\partial_3 b_1 - \partial_1 b_3],
$$

$$
\tilde{R}_3(y, z) = \frac{1}{\sqrt{G_{11}G_{22}}} [\partial_1 (\tilde{B}_2 \sqrt{G_{22}}) - \partial_2 (\tilde{B}_1 \sqrt{G_{11}})] = \frac{1}{\sqrt{G_{11}G_{22}}} [\partial_1 b_2 - \partial_2 b_1].
$$

If we write
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\[ \text{curl}^2 \mathbf{B} = \sum_{j=1}^{3} \tilde{T}_j(y, z) \mathbf{E}_j(y, z), \]

then

\[ \tilde{T}_1(y, z) = \frac{1}{\sqrt{G_{22}G_{33}}} [\partial_2(\sqrt{G_{33}} \tilde{R}_3) - \partial_3(\sqrt{G_{22}} \tilde{R}_2)] \]
\[ = \frac{1}{\sqrt{G_{22}G_{33}}} \partial_2 \left\{ \frac{\sqrt{G_{33}}}{\sqrt{G_{11}G_{22}}} \left[ \partial_1(\tilde{B}_2 \sqrt{G_{22}}) - \partial_2(\tilde{B}_1 \sqrt{G_{11}}) \right] \right\} \]
\[ - \frac{1}{\sqrt{G_{22}G_{33}}} \partial_3 \left\{ \frac{\sqrt{G_{22}}}{\sqrt{G_{33}G_{11}}} \left[ \partial_3(\tilde{B}_1 \sqrt{G_{11}}) - \partial_1(\tilde{B}_3 \sqrt{G_{33}}) \right] \right\}, \]
\[ \tilde{T}_2(y, z) = \frac{1}{\sqrt{G_{33}G_{11}}} [\partial_3(\sqrt{G_{11}} \tilde{R}_1) - \partial_1(\sqrt{G_{33}} \tilde{R}_3)] \]
\[ = \frac{1}{\sqrt{G_{33}G_{11}}} \partial_3 \left\{ \frac{\sqrt{G_{11}}}{\sqrt{G_{22}G_{33}}} \left[ \partial_2(\tilde{B}_3 \sqrt{G_{33}}) - \partial_3(\tilde{B}_2 \sqrt{G_{22}}) \right] \right\} \]
\[ - \frac{1}{\sqrt{G_{33}G_{11}}} \partial_1 \left\{ \frac{\sqrt{G_{33}}}{\sqrt{G_{11}G_{22}}} \left[ \partial_1(\tilde{B}_2 \sqrt{G_{22}}) - \partial_2(\tilde{B}_1 \sqrt{G_{11}}) \right] \right\}, \]
\[ \tilde{T}_3(y, z) = \frac{1}{\sqrt{G_{11}G_{22}}} [\partial_1(\sqrt{G_{22}} \tilde{R}_2) - \partial_2(\sqrt{G_{11}} \tilde{R}_1)] \]
\[ = \frac{1}{\sqrt{G_{11}G_{22}}} \partial_1 \left\{ \frac{\sqrt{G_{22}}}{\sqrt{G_{33}G_{11}}} \left[ \partial_3(\tilde{B}_1 \sqrt{G_{11}}) - \partial_1(\tilde{B}_3 \sqrt{G_{33}}) \right] \right\} \]
\[ - \frac{1}{\sqrt{G_{11}G_{22}}} \partial_2 \left\{ \frac{\sqrt{G_{11}}}{\sqrt{G_{22}G_{33}}} \left[ \partial_2(\tilde{B}_3 \sqrt{G_{33}}) - \partial_3(\tilde{B}_2 \sqrt{G_{22}}) \right] \right\} . \]

Let \( \mathbf{u} \) be a solution of (1). In the neighbourhood \( \mathcal{U} \) near boundary, we write

\[ \tilde{\mathbf{u}}(t, y, z) = \mathbf{u}(t, \mathcal{F}(y, z)) = \sum_{j=1}^{3} \tilde{u}_j(t, y, z) \mathbf{E}_j(y, z), \]

\[ \text{div} \mathbf{u} = \frac{1}{\sqrt{G}} \left[ \sum_{j=1}^{2} \partial_j \left( \sqrt{G_{jj}} \tilde{u}_j \right) + \partial_3 \left( \sqrt{G_{33}} \tilde{u}_3 \right) \right], \]

(29)

\[ \text{curl} \mathbf{u}(x) = \sum_{j=1}^{3} \tilde{R}_j(t, y, z) \mathbf{E}_j(y, z), \quad \text{curl}^2 \mathbf{u} = \sum_{j=1}^{3} \tilde{T}_j(t, y, z) \mathbf{E}_j(y, z), \]

\[ \mathcal{B} \text{curl} \mathbf{u} = \mathbf{h} = \sum_{j=1}^{3} \tilde{h}_j(t, y, z) \mathbf{E}_j(y, z), \quad \mathbf{f} = \sum_{j=1}^{3} \tilde{f}_j(t, y, z) \mathbf{E}_j(y, z). \]

Recall that \( G_{33} = 1 \). We have
\[ \tilde{T}_1 = \frac{1}{\sqrt{G_{22}G_{33}}} \partial_2 \left\{ \frac{\sqrt{G_{33}}}{\sqrt{G_{11}G_{22}}} \left[ \partial_1 (\tilde{u}_1 \sqrt{G_{22}}) - \partial_2 (\tilde{u}_1 \sqrt{G_{11}}) \right] \right\} \\
- \frac{1}{\sqrt{G_{22}G_{33}}} \partial_3 \left\{ \frac{\sqrt{G_{22}}}{\sqrt{G_{33}G_{11}}} \left[ \partial_2 (\tilde{u}_1 \sqrt{G_{11}}) - \partial_3 (\tilde{u}_3 \sqrt{G_{33}}) \right] \right\} \\
= \frac{1}{G_{22}\sqrt{G_{11}}} \left[ \partial_{12} (\tilde{u}_2 \sqrt{G_{22}}) - \partial_{22} (\tilde{u}_2 \sqrt{G_{11}}) \right] \\
+ \frac{1}{\sqrt{G_{22}G_{33}}} \partial_2 \left\{ \frac{\sqrt{G_{33}}}{\sqrt{G_{11}G_{22}}} \left[ \partial_1 (\tilde{u}_2 \sqrt{G_{22}}) - \partial_2 (\tilde{u}_1 \sqrt{G_{11}}) \right] \right\} \\
- \frac{1}{G_{33}\sqrt{G_{11}}} \left[ \partial_{33} (\tilde{u}_1 \sqrt{G_{11}}) - \partial_{13} (\tilde{u}_3 \sqrt{G_{33}}) \right] \\
- \frac{1}{\sqrt{G_{22}G_{33}}} \partial_3 \left\{ \frac{\sqrt{G_{22}}}{\sqrt{G_{33}G_{11}}} \left[ \partial_3 (\tilde{u}_1 \sqrt{G_{11}}) - \partial_1 (\tilde{u}_3 \sqrt{G_{33}}) \right] \right\} \\
= \frac{1}{G_{22}\sqrt{G_{11}}} \left[ \sqrt{G_{22}} \partial_{12} \tilde{u}_2 + \partial_2 \tilde{u}_2 \partial_1 \sqrt{G_{22}} + \partial_1 (\tilde{u}_2 \partial_2 \sqrt{G_{22}}) \\
- \sqrt{G_{11}} \partial_{22} \tilde{u}_1 - \partial_2 \tilde{u}_1 \partial_2 \sqrt{G_{11}} - \partial_2 (\tilde{u}_1 \partial_2 \sqrt{G_{11}}) \right] \\
+ \frac{1}{\sqrt{G_{22}G_{33}}} \partial_2 \left\{ \frac{\sqrt{G_{33}}}{\sqrt{G_{11}G_{22}}} \left[ \partial_1 (\tilde{u}_2 \sqrt{G_{22}}) - \partial_2 (\tilde{u}_1 \sqrt{G_{11}}) \right] \right\} \\
- \frac{1}{G_{33}\sqrt{G_{11}}} \left[ \sqrt{G_{11}} \partial_{33} \tilde{u}_1 + \partial_3 \tilde{u}_1 \partial_3 \sqrt{G_{11}} + \partial_3 (\tilde{u}_1 \partial_3 \sqrt{G_{11}}) - \partial_{13} \tilde{u}_3 \right] \\
- \frac{1}{\sqrt{G_{22}G_{33}}} \partial_3 \left\{ \frac{\sqrt{G_{22}}}{\sqrt{G_{33}G_{11}}} \left[ \partial_3 (\tilde{u}_1 \sqrt{G_{11}}) - \partial_1 (\tilde{u}_3 \sqrt{G_{33}}) \right] \right\} \\
= -\frac{1}{G_{22}} \partial_{12} \tilde{u}_1 - \partial_{33} \tilde{u}_1 + \frac{1}{\sqrt{G}} \partial_{12} \tilde{u}_2 + \frac{1}{\sqrt{G_{11}}} \partial_{13} \tilde{u}_3 + \phi_1, \]

where
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\[
\phi_1 = \frac{1}{G_{22} \sqrt{G_{11}}} \left[ \partial_2 \ddot{u}_2 \partial_1 \sqrt{G_{22}} + \partial_1 (\ddot{u}_2 \partial_2 \sqrt{G_{22}}) - \partial_2 \ddot{u}_1 \partial_1 \sqrt{G_{11}} - \partial_2 (\ddot{u}_2 \partial_2 \sqrt{G_{11}}) \right]
\]

\[
+ \frac{1}{\sqrt{G_{22}}} \left[ \partial_2 \left( \frac{1}{\sqrt{G_{11} G_{22}}} \right) \left[ \partial_1 (\ddot{u}_2 \sqrt{G_{22}}) - \partial_2 (\ddot{u}_1 \sqrt{G_{11}}) \right] \right]
\]

\[
- \frac{1}{\sqrt{G_{11}}} \left[ \partial_3 \ddot{u}_1 \partial_2 \sqrt{G_{11}} + \partial_2 (\ddot{u}_1 \partial_3 \sqrt{G_{11}}) \right] - \frac{1}{\sqrt{G_{22}}} \partial_3 \left( \frac{\sqrt{G_{22}}}{\sqrt{G_{11}}} \right) \left[ \partial_3 (\ddot{u}_1 \sqrt{G_{11}}) - \partial_3 \ddot{u}_3 \right].
\]

\[
\tilde{T}_2 = \frac{1}{\sqrt{G_{22} G_{11}}} \partial_3 \left\{ \frac{\sqrt{G_{11}}}{\sqrt{G_{22} G_{33}}} \left[ \partial_2 (\ddot{u}_3 \sqrt{G_{33}}) - \partial_3 (\ddot{u}_2 \sqrt{G_{22}}) \right] \right\}
\]

\[
- \frac{1}{\sqrt{G_{22} G_{11}}} \partial_1 \left\{ \frac{\sqrt{G_{33}}}{\sqrt{G_{11} G_{22}}} \left[ \partial_1 (\ddot{u}_2 \sqrt{G_{22}}) - \partial_2 (\ddot{u}_1 \sqrt{G_{11}}) \right] \right\}
\]

\[
= \frac{1}{G_{33} \sqrt{G_{22}}} \left[ \partial_3 \ddot{u}_3 - \sqrt{G_{22}} \partial_3 \ddot{u}_2 - \partial_3 \ddot{u}_2 \partial_3 \sqrt{G_{22}} - \partial_1 (\ddot{u}_2 \partial_3 \sqrt{G_{22}}) \right]
\]

\[
+ \frac{1}{\sqrt{G_{33} G_{11}}} \partial_3 \left( \frac{\sqrt{G_{11}}}{\sqrt{G_{22} G_{33}}} \right) \left[ \partial_2 (\ddot{u}_3 \sqrt{G_{33}}) - \partial_3 (\ddot{u}_2 \sqrt{G_{22}}) \right]
\]

\[
- \frac{1}{\sqrt{G_{33} G_{11}}} \partial_1 \left( \frac{\sqrt{G_{33}}}{\sqrt{G_{11} G_{22}}} \right) \left[ \partial_1 (\ddot{u}_2 \sqrt{G_{22}}) - \partial_2 (\ddot{u}_1 \sqrt{G_{11}}) \right]
\]

\[
= \frac{1}{G_{33} \sqrt{G_{22}}} \left[ \partial_3 \ddot{u}_3 - \sqrt{G_{22}} \partial_3 \ddot{u}_2 - \partial_3 \ddot{u}_2 \partial_3 \sqrt{G_{22}} - \partial_1 (\ddot{u}_2 \partial_3 \sqrt{G_{22}}) \right]
\]

where
\[
\phi_2 = -\frac{1}{\sqrt{G_{22}}} \left[ \partial_3 \tilde{u}_2 \partial_3 \sqrt{G_{22}} + \partial_3 (\tilde{u}_2 \partial_3 \sqrt{G_{22}}) \right] + \frac{1}{\sqrt{G_{11}}} \partial_3 \left( \frac{\sqrt{G_{11}}}{\sqrt{G_{22}}} \right) \left[ \partial_2 \tilde{u}_3 - \partial_3 (\tilde{u}_2 \sqrt{G_{22}}) \right] \\
- \frac{1}{G_{11} \sqrt{G_{22}}} \left[ \partial_1 \tilde{u}_2 \partial_1 \sqrt{G_{22}} + \partial_1 (\tilde{u}_2 \partial_1 \sqrt{G_{22}}) - \partial_2 \tilde{u}_2 \partial_1 \sqrt{G_{11}} - \partial_1 (\tilde{u}_2 \partial_1 \sqrt{G_{11}}) \right] \\
- \frac{1}{\sqrt{G_{11}}} \partial_1 \left( \frac{1}{\sqrt{G}} \right) \left[ \partial_1 (\tilde{u}_2 \sqrt{G_{22}}) - \partial_2 (\tilde{u}_1 \sqrt{G_{11}}) \right].
\]

\[
\tilde{T}_3 = \frac{1}{\sqrt{G_{11} G_{22}}} \partial_1 \left( \frac{\sqrt{G_{22}}}{\sqrt{G_{11} G_{11}}} \right) \left[ \partial_3 (\tilde{u}_1 \sqrt{G_{11}}) - \partial_1 (\tilde{u}_3 \sqrt{G_{33}}) \right] \\
- \frac{1}{\sqrt{G_{11} G_{22}}} \partial_2 \left( \frac{\sqrt{G_{11}}}{\sqrt{G_{22} G_{33}}} \right) \left[ \partial_2 (\tilde{u}_3 \sqrt{G_{33}}) - \partial_3 (\tilde{u}_2 \sqrt{G_{22}}) \right] \\
+ \frac{1}{\sqrt{G_{11} G_{22}}} \partial_2 \left( \frac{\sqrt{G_{11}}}{\sqrt{G_{22} G_{33}}} \right) \left[ \partial_2 (\tilde{u}_3 \sqrt{G_{33}}) - \partial_3 (\tilde{u}_2 \sqrt{G_{22}}) \right] \\
= \frac{1}{G_{11} \sqrt{G_{33}}} \left[ \partial_{13} \tilde{u}_1 \sqrt{G_{11}} + \partial_3 \tilde{u}_3 \sqrt{G_{11}} + \partial_1 (\tilde{u}_1 \partial_3 \sqrt{G_{11}}) - \partial_1 (\tilde{u}_3 \sqrt{G_{33}}) \right] \\
+ \frac{1}{\sqrt{G_{11} G_{22}}} \partial_2 \left( \frac{\sqrt{G_{11}}}{\sqrt{G_{22} G_{33}}} \right) \left[ \partial_3 (\tilde{u}_1 \sqrt{G_{11}}) - \partial_1 (\tilde{u}_3 \sqrt{G_{33}}) \right] \\
- \frac{1}{G_{22} \sqrt{G_{33}}} \left[ \partial_{22} \tilde{u}_3 - \sqrt{G_{22}} \partial_{22} \tilde{u}_3 - \partial_2 \tilde{u}_2 \partial_3 \sqrt{G_{22}} - \partial_3 (\tilde{u}_2 \partial_2 \sqrt{G_{22}}) \right] \\
- \frac{1}{\sqrt{G_{11} G_{22}}} \partial_2 \left( \frac{\sqrt{G_{11}}}{\sqrt{G_{22} G_{33}}} \right) \left[ \partial_2 (\tilde{u}_3 \sqrt{G_{33}}) - \partial_3 (\tilde{u}_2 \sqrt{G_{22}}) \right] \\
= -\frac{1}{G_{11}} \partial_1 \tilde{u}_3 - \frac{1}{G_{22}} \partial_2 \tilde{u}_3 + \frac{1}{\sqrt{G_{11}}} \partial_3 \tilde{u}_3 + \frac{1}{\sqrt{G_{22}}} \partial_2 \tilde{u}_3 + \phi_3,
\]

where

\[
\phi_3 = \frac{1}{G_{11}} \left[ \partial_1 \tilde{u}_1 \partial_3 \sqrt{G_{11}} + \partial_3 (\tilde{u}_1 \partial_1 \sqrt{G_{11}}) \right] + \frac{1}{\sqrt{G}} \partial_1 \left( \frac{\sqrt{G_{22}}}{\sqrt{G_{11}}} \right) \left[ \partial_3 (\tilde{u}_1 \sqrt{G_{11}}) - \partial_1 \tilde{u}_3 \right] \\
+ \frac{1}{G_{22}} \left[ \partial_2 \tilde{u}_2 \partial_3 \sqrt{G_{22}} + \partial_3 (\tilde{u}_2 \partial_2 \sqrt{G_{22}}) \right] - \frac{1}{\sqrt{G}} \partial_2 \left( \frac{\sqrt{G_{11}}}{\sqrt{G_{22}}} \right) \left[ \partial_2 \tilde{u}_3 - \partial_3 (\tilde{u}_2 \sqrt{G_{22}}) \right].
\]
3.2 Proof of Theorem 1.1

Proof We only need to derive regularity near boundary. Let \( x^0 \in \partial \Omega \) and we take a neighbourhood near \( x^0 \), and take a differentiable isomorphism to map the neighbourhood \( U \) of \( x^0 \) to a domain \( U \) with flat boundary \( \Gamma \), and the image of \( x^0 \) locates in the interior of \( \Gamma \).

The equation in (1) can be written as

\[
\partial_t \tilde{u}_j + \tilde{a} \tilde{T}_j + \tilde{c} \tilde{u}_j = \tilde{f}_j - \tilde{h}_j.
\]

Now we simplify the formula by using the condition \( \text{div} \mathbf{u} = 0 \), which gives

\[
0 = \sum_{j=1}^{2} \partial_j \left( \sqrt{\frac{G}{G_{jj}}} \tilde{u}_j \right) + \partial_3 \left( \sqrt{\frac{G}{G_{33}}} \tilde{u}_3 \right)
\]

\[
= \partial_1 (\sqrt{G_{22}} \tilde{u}_1) + \partial_2 (\sqrt{G_{11}} \tilde{u}_2) + \partial_3 (\sqrt{G} \tilde{u}_3)
\]

\[
= \sqrt{G_{22}} \partial_1 \tilde{u}_1 + \sqrt{G_{11}} \partial_2 \tilde{u}_2 + \sqrt{G} \partial_3 \tilde{u}_3 + \tilde{u}_1 \partial_1 \sqrt{G_{22}} + \tilde{u}_2 \partial_2 \sqrt{G_{11}} + \tilde{u}_3 \partial_3 \sqrt{G}.
\]

Hence

\[
\sqrt{G_{22}} \partial_1 \tilde{u}_1 + \sqrt{G_{11}} \partial_2 \tilde{u}_2 + \sqrt{G} \partial_3 \tilde{u}_3 = -\tilde{u}_1 \partial_1 \sqrt{G_{22}} - \tilde{u}_2 \partial_2 \sqrt{G_{11}} - \tilde{u}_3 \partial_3 \sqrt{G}.
\]

Write (31) as

\[
\frac{1}{\sqrt{G}} \partial_2 \tilde{u}_2 + \frac{1}{\sqrt{G_{11}}} \partial_3 \tilde{u}_3 = -\frac{1}{G_{11}} \partial_1 \tilde{u}_1 - \frac{1}{G_{11} \sqrt{G_{22}}} \{ \tilde{u}_1 \partial_1 \sqrt{G_{22}} + \tilde{u}_2 \partial_2 \sqrt{G_{11}} + \tilde{u}_3 \partial_3 \sqrt{G} \}.
\]

Differentiating in \( y_1 \) yields

\[
\frac{1}{\sqrt{G}} \partial_{12} \tilde{u}_2 + \frac{1}{\sqrt{G_{11}}} \partial_{13} \tilde{u}_3 = -\frac{1}{G_{11}} \partial_{11} \tilde{u}_1 + \zeta_1,
\]

where

\[
\zeta_1 = -\partial_1 \tilde{u}_1 \partial_1 \left( \frac{1}{G_{11}} \right) - \partial_2 \tilde{u}_2 \partial_1 \left( \frac{1}{\sqrt{G}} \right) - \partial_3 \tilde{u}_3 \partial_1 \left( \frac{1}{\sqrt{G_{11}}} \right)
\]

\[
- \partial_1 \left\{ \frac{1}{G_{11} \sqrt{G_{22}}} \{ \tilde{u}_1 \partial_1 \sqrt{G_{22}} + \tilde{u}_2 \partial_2 \sqrt{G_{11}} + \tilde{u}_3 \partial_3 \sqrt{G} \} \right\}.
\]

Write (31) as

\[
\frac{1}{\sqrt{G}} \partial_1 \tilde{u}_1 + \frac{1}{\sqrt{G_{22}}} \partial_3 \tilde{u}_3 = -\frac{1}{G_{22}} \partial_2 \tilde{u}_2 - \frac{1}{G_{22} \sqrt{G_{11}}} \{ \tilde{u}_1 \partial_1 \sqrt{G_{22}} + \tilde{u}_2 \partial_2 \sqrt{G_{11}} + \tilde{u}_3 \partial_3 \sqrt{G} \}.
\]

Differentiating in \( y_2 \) yields
\[ \frac{1}{\sqrt{G}} \partial_{12} \bar{u}_1 + \frac{1}{\sqrt{G_{22}}} \partial_{23} \bar{u}_3 = - \frac{1}{G_{22}} \partial_{22} \bar{u}_2 + \zeta_2, \tag{33} \]

where

\[ \zeta_2 = - \partial_1 \bar{u}_1 \partial_2 \left( \frac{1}{\sqrt{G}} \right) - \partial_2 \bar{u}_2 \partial_2 \left( \frac{1}{G_{22}} \right) - \partial_3 \bar{u}_3 \partial_2 \left( \frac{1}{\sqrt{G_{22}}} \right) \]

\[ - \partial_2 \left\{ \frac{1}{G_{22} \sqrt{G_{11}}} \left[ \bar{u}_1 \partial_1 \sqrt{G_{22}} + \bar{u}_2 \partial_2 \sqrt{G_{11}} + \bar{u}_3 \partial_3 \sqrt{G} \right] \right\}. \]

Write (31) as

\[ \frac{1}{\sqrt{G_{11}}} \partial_{13} \bar{u}_1 + \frac{1}{\sqrt{G_{22}}} \partial_{23} \bar{u}_2 = - \partial_3 \bar{u}_3 + \frac{1}{G} \left[ \bar{u}_1 \partial_1 \sqrt{G_{22}} + \bar{u}_2 \partial_2 \sqrt{G_{11}} + \bar{u}_3 \partial_3 \sqrt{G} \right]. \]

Differentiating in \( z \) yields

\[ \frac{1}{\sqrt{G_{11}}} \partial_{13} \bar{u}_1 + \frac{1}{\sqrt{G_{22}}} \partial_{23} \bar{u}_2 = - \partial_3 \bar{u}_3 + \zeta_3, \tag{34} \]

where

\[ \zeta_3 = - \partial_1 \bar{u}_1 \partial_3 \left( \frac{1}{\sqrt{G_{11}}} \right) - \partial_2 \bar{u}_2 \partial_3 \left( \frac{1}{G_{22}} \right) - \partial_3 \left\{ \frac{1}{G} \left[ \bar{u}_1 \partial_1 \sqrt{G_{22}} + \bar{u}_2 \partial_2 \sqrt{G_{11}} + \bar{u}_3 \partial_3 \sqrt{G} \right] \right\}. \]

Plugging (32), (33), (34) into the equalities of \( \tilde{T}_1, \tilde{T}_2, \tilde{T}_3 \) respectively, we get

\[ \tilde{T}_1 = - \frac{1}{G_{11}} \partial_{11} \bar{u}_1 - \frac{1}{G_{22}} \partial_{22} \bar{u}_1 - \partial_3 \bar{u}_1 + \zeta_1 + \phi_1, \]

\[ \tilde{T}_2 = - \frac{1}{G_{11}} \partial_{11} \bar{u}_2 - \frac{1}{G_{22}} \partial_{22} \bar{u}_2 - \partial_3 \bar{u}_2 + \zeta_2 + \phi_2, \]

\[ \tilde{T}_3 = - \frac{1}{G_{11}} \partial_{11} \bar{u}_3 - \frac{1}{G_{22}} \partial_{22} \bar{u}_3 - \partial_3 \bar{u}_3 + \zeta_3 + \phi_3. \]

On \( \Gamma \) we have \( \bar{u}_1 = \bar{u}_2 = 0 \). Then from (31) we see that

\[ \partial_3 \bar{u}_3 + H \bar{u}_3 = 0, \]

where \( H = \frac{\partial_3 \sqrt{G}}{\sqrt{G}} \). So we find that \( \bar{u}_1, \bar{u}_2 \) satisfy

\[
\begin{align*}
\partial_t \bar{u}_j - \frac{\bar{a}}{G_{11}} \partial_{11} \bar{u}_j - \frac{\bar{a}}{G_{22}} \partial_{22} \bar{u}_j - \bar{a} \partial_3 \bar{u}_j + \bar{c} \bar{u}_j &= F_j & \text{in } (0, T) \times U, \\
\bar{u}_j &= 0 & \text{on } (0, T) \times \Gamma, \\
\bar{u}_j &= \bar{u}_j^0 & \text{in } U, 
\end{align*}
\]

and \( \bar{u}_3 \) satisfies

\[
\begin{align*}
\frac{\partial t}{\partial_3} \bar{u}_3 - \frac{\bar{a}}{G_{11}} \partial_{11} \bar{u}_3 - \frac{\bar{a}}{G_{22}} \partial_{22} \bar{u}_3 - \bar{a} \partial_3 \bar{u}_3 + \bar{c} \bar{u}_3 &= \bar{F}_j & \text{in } (0, T) \times U, \\
\bar{F}_j &= 0 & \text{on } (0, T) \times \Gamma, \\
\bar{u}_3 &= \bar{u}_3^0 & \text{in } U, 
\end{align*}
\]
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\[
\begin{aligned}
&\partial_t \tilde{u}_3 - \frac{\tilde{a}}{G_{11}} \partial_{11} \tilde{u}_3 - \frac{\tilde{a}}{G_{22}} \partial_{22} \tilde{u}_3 - \tilde{a} \partial_3 \tilde{u}_3 + \tilde{c} \tilde{u}_3 = F_3 \quad \text{in } (0, T) \times U, \\
&\partial_3 \tilde{u}_3 + H \tilde{u}_3 = 0 \quad \text{on } (0, T) \times \Gamma, \\
&\tilde{u}_3 = \tilde{u}_3^0 \quad \text{in } U,
\end{aligned}
\]

where

\[
F_j = \tilde{f}_j - \tilde{h}_j - \tilde{a}(\zeta_j + \phi_j), \quad j = 1, 2, 3.
\]

Note that \(\tilde{h}_j, \zeta_j, \phi_j\) contain derivatives of \(\tilde{u}_j\) up to the first order, and contain terms involving \(G_{jj}\) and their derivatives up to order 2. Hence if \(\partial \Omega\) is of \(C^{3+\epsilon}\), then those terms are determined by \(\tilde{u}_j\) and their first order derivatives, with coefficients that are \(C^\alpha\) in \(y_1, y_2, z\).

Note that the boundary condition for \(\tilde{u}_3\) can be changed to a homogeneous Neumann boundary condition if we consider a new function

\[
\tilde{u}_3 = e^{\int_0^t Hdx} \tilde{u}_3.
\]

Although the boundary conditions for \(\tilde{u}_j\) are satisfied only on \(\Gamma\), a part of the boundary of \(U\), we can multiply \(\tilde{u}_j\) by a smooth cut-off function such that \(\tilde{u}_j\) satisfies the same type boundary condition on \(\partial U \setminus \Gamma\). So we can repeat the proof of Lemmas 2.1 and 2.3, to derive the Schauder estimates for \(\tilde{u}_j, j = 1, 2, 3\), in \((0, T) \times B_{R'}^+\). Using the diffeomorphism \(F\) we obtain the Schauder estimates of \(u\) in \((0, T) \times \mathcal{V}\), where \(\mathcal{V}\) is a neighborhood of \(\mathcal{U}\) which contains the point \(x^0\).

Then the conclusion of Theorem 1.1 follows by covering a tubular neighborhood of \(\partial \Omega\) with a finite number of domains as above, and using interpolation. \(\square\)

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**Declarations**

**Conflict of interest** The author declares that he has no conflict of interest.

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**References**

1. Bates, P., Pan, X.B.: On a problem related to vortex nucleation of 3 dimensional superconductors. Commun. Math. Phys. 276, 571–610 (2007)
2. Dautray, R., Lions, J.-L.: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 3. Springer, New York (1990)
3. Hu, B.: Blow-up Theories for Semilinear Parabolic Equations. Lecture Notes in Math, vol. 2018. Springer, Berlin (2011)
4. Kang, K. K., and Pan, X. B.: On a quasilinear parabolic curl system motivated by time evolution of Meissner states of superconductors. SIAM J. Math. Anal. (To appear)
5. Lieberman, G.M.: Second Order Parabolic Differential Equations. World Scientific Publishing Co., Inc, River Edge (1996).
6. Pan, X.B.: Surface superconductivity in 3 dimensions. Trans. Am. Math. Soc. 356, 3899–3937 (2004)

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