Some approximation results by Bernstein-Kantorovich operators based on $(p, q)$-integers

M. Mursaleen, Khursheed J. Ansari and Asif Khan
Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
mursaleenm@gmail.com; ansari.jkhursheed@gmail.com; asifjnu07@gmail.com

Abstract

In this paper, a new analogue of Bernstein-Kantorovich operators has been introduced and we call it as $(p, q)$-Bernstein-Kantorovich operators. Approximation properties based on Korovkin’s type approximation theorem and some direct theorems for $(p, q)$-Bernstein-Kantorovich operators has been studied. Furthermore, comparisons and some illustrative graphics for the convergence of operators to a function are presented.

Keywords and phrases: $(p, q)$-Bernstein-Kantorovich operators; $q$-Bernstein-Kantorovich operator; modulus of continuity; positive linear operator; Korovkin type approximation theorem.

AMS Subject Classifications (2010): 41A10, 41A25, 41A36, 40A30

1 Introduction and preliminaries

In last two decades, the applications of $q$-calculus emerged as a new area in the field of approximation theory. The rapid development of $q$-calculus has led to the discovery of new generalizations of Bernstein polynomials involving $q$-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş [14] introduced the first $q$-analogue of Bernstein operators [5] and investigated its approximating and shape-preserving properties. Another $q$-generalization of the classical Bernstein polynomials is due to Phillips [21]. Several generalizations of well-known positive linear operators based on $q$-integers were introduced and their approximation properties have been studied by several authors. For instance, $q$-Bleimann, Butzer and Hahn operators in [3]; $q$-analogue of Szász–Kantorovich operators in [15]; $q$-analogue of Stancu-Beta operators in [4]; and $q$-Lagrange polynomials in [19] were defined and their approximation properties were investigated.

Recently Mursaleen et al applied $(p, q)$-calculus in approximation theory and introduced the first $(p, q)$ analogue of Bernstein operators in [17]. and also they introduced and investigated some approximation properties and $(p, q)$-analogue of Bernstein-Stancu Operators [18].

Dalmanoglu [6] has given the Bernstein-Kantorovich [10] operators using $q$-calculus which is given as

$$K_{n,q}(f; x) = [n+1]_q \sum_{k=0}^{n} p_{n,k}(q; x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t) dq t, \quad x \in [0,1], \quad (2.1)$$

$$p_{n,k}(q; x) := \binom{n}{k} x^k \prod_{s=0}^{n-k-1} (1 - q^s x).$$
where $K_{n,q} : C[0,1] \to C[0,1]$ are defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$.

Details on the $q$-calculus can be found in [9] and for the applications of $q$-calculus in approximation theory, one can refer [1].

In this paper, we introduce a new generalization of $q$-Bernstein-Kantorovich operators as $(p, q)$-Bernstein-Kantorovich operators. We study the approximation properties based on Korovkin’s type approximation theorem and also establish some direct theorems. Further, we show comparisons and some illustrative graphics for the convergence of operators to a function.

Let us recall certain notations of $(p, q)$-calculus.

The $(p, q)$-integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \quad 0 < q < p \leq 1.$$ 

The $(p, q)$-Binomial expansion is

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} \binom{n}{k}_{p,q} a^{n-k} b^k x^{n-k} y^k$$

$$[(x + y)^n_{p,q} := (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y).$$

Also, the $(p, q)$-binomial coefficients are defined by

$$\binom{n}{k}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

and the definite integral of the function $f$ is defined by

$$\int_0^a f(x) d_{p,q} x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{p^k+1} f \left( \frac{p^k}{q^k+1} a \right), \quad \text{when} \quad \left| \frac{p}{q} \right| < 1,$$

and

$$\int_0^a f(x) d_{p,q} x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{q^k+1} f \left( \frac{q^k}{p^k+1} a \right), \quad \text{when} \quad \left| \frac{p}{q} \right| > 1.$$

Details on $(p, q)$-calculus can be found in [8, 12, 13, 22, 23].

## 2 Construction of Operators

Now, we introduce $(p, q)$-analogue of Bernstein-Stancu operators as

$$K^{(p,q)}_n(f; x) = [n+1]_{p,q} \sum_{k=0}^{n} \frac{(p-q)}{p^k(p-1) - q^k(k-1)} b_{n,k}^{(p,q)}(x) \int_{0}^{[k+1]_{p,q}} f(t) d_{p,q} t, \quad x \in [0,1] \tag{2.1}$$

where

$$b_{n,k}^{(p,q)}(x) = \binom{n}{k}_{p,q} x^k (1-x)^{n-k}_{p,q} = \binom{n}{k}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x).$$

We have the following basic lemmas:

**Lemma 2.1.** For $x \in [0,1], \quad 0 < q < p \leq 1$
(i) \( K_n^{(p,q)}(1; x) = 1 \);
(ii) \( K_n^{(p,q)}(t; x) = \frac{(qx+1-x)^n}{[2p,q(n+1)]_{p,q}} + \frac{(p+1)[n]_{p,q} x}{[2p,q(n+1)]_{p,q}} \); 
(iii) \( K_n^{(p,q)}(t^2; x) = \frac{(qx+1-x)^n}{[3p,q(n+1)]_{p,q}} + \frac{[p^n+2pq+q+1](qx+1-x)^n}{[3p,q(n+1)]_{p,q}} x + \frac{2(p+1)[n]_{p,q} x}{[2p,q(n+1)]_{p,q}} \); 
(iv) \( K_n^{(p,q)}((t - x)^2; x) = \frac{(p^n+1-x)^n}{[3p,q(n+1)]_{p,q}} + \frac{[(p^n+2pq+q+1)(qx+1-x)^n]}{[3p,q(n+1)]_{p,q}} x + \frac{2(p+1)[n]_{p,q} x}{[2p,q(n+1)]_{p,q}} \).

3 Main Results

Let \( C[a, b] \) be the linear space of all real valued continuous functions \( f \) on \([a, b]\) and let \( T \) be a linear operator which maps \( C[a, b] \) into itself. We say that \( T \) is positive if for every non-negative \( f \in C[a, b] \), we have \( T(f, x) \geq 0 \) for all \( x \in [a, b] \).

The classical Korovkin approximation theorem \([2][11][25]\) states as follows: Let \( (T_n) \) be a sequence of positive linear operators from \( C[a, b] \) into \( C[a, b] \). Then \( \lim_n \| T_n(f, x) - f(x) \|_{C[a, b]} = 0 \), for all \( f \in C[a, b] \) if and only if \( \lim_n \| T_n(f_i, x) - f_i(x) \|_{C[a, b]} = 0 \), for \( i = 0, 1, 2 \), where \( f_0(x) = 1 \), \( f_1(x) = x \) and \( f_2(x) = x^2 \).

**Theorem 3.1.** Let \( 0 < q_n < p_n \leq 1 \) such that \( \lim_{n \to \infty} p_n = 1 \) and \( \lim_{n \to \infty} q_n = 1 \). Then for each \( f \in C[0, 1] \), \( K_n^{(p_n,q_n)}(f; x) \) converges uniformly to \( f \) on \([0, 1]\).

Now we will compute the rate of convergence in terms of modulus of continuity.

Let \( f \in C[0, 1] \). The modulus of continuity of \( f \) denoted by \( \omega(f, \delta) \) gives the maximum oscillation of \( f \) in any interval of length not exceeding \( \delta > 0 \) and it is given by the relation

\[
\omega(f, \delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|, \quad x, y \in [0, b].
\]

It is known that \( \lim_{\delta \to 0^+} \omega(f, \delta) = 0 \) for \( f \in C[0, b] \) and for any \( \delta > 0 \) one has

\[
|f(y) - f(x)| \leq \omega(f, \delta) \left( \frac{(y-x)^2}{\delta^2} + 1 \right). \tag{4.4}
\]

**Theorem 3.2.** If \( f \in C[0, 1] \), then

\[
|K_n^{(p,q)}(f; x) - f(x)| \leq 2\omega(f, \delta_n)
\]

takes place, where

\[
\delta_n = \frac{1}{|n + 1|_{p,q}} \left\{ ([2]_{p,q} q^{2n} + [2]_{p,q} (p^2 + 2pq + p + q + 1)q^{n-1}[n]_{p,q} + [2]_{p,q} p(p^2 + p + 1)[n]_{p,q} [n - 1]_{p,q} - 2[3]_{p,q} (p + 1)[n]_{p,q} [n + 1]_{p,q} + [2]_{p,q} [3]_{p,q} [n + 1]^2_{p,q} /[2]_{p,q} [3]_{p,q} \right\}^{1/2}.
\]

Now we give the rate of convergence of the operators \( K_n^{(p,q)} \) in terms of the elements of the usual Lipschitz class \( Lip_M(\alpha) \).
Let \( f \in C[0,1], M > 0 \) and \( 0 < \alpha \leq 1 \). We recall that \( f \) belongs to the class \( \text{Lip}_M(\alpha) \) if the inequality
\[
|f(t) - f(x)| \leq M|t - x|^{\alpha} \quad (t, x \in [0,1])
\]
is satisfied.

**Theorem 3.3.** Let \( 0 < q < p \leq 1 \). Then for each \( f \in \text{Lip}_M(\alpha) \) we have
\[
|K^{(p,q)}_n (f; x) - f(x)| \leq M\delta_n(x)
\]
where
\[
\delta_n(x) = \sqrt{K^{(p,q)}_n((t-x)^2; x)}.
\]

Next we prove the local approximation property for the operators \( K^{(p,q)}_n \). The Peetre’s \( K \)-functional is defined by
\[
K_2(f, \delta) = \inf\{ \|f - g\| + \delta\|g''\| : g \in W^2\},
\]
where
\[
W^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}.
\]

By [7], there exists a positive constant \( C > 0 \) such that \( K_2(f, \delta) \leq Cw_2(f, \delta^\frac{1}{2}), \delta > 0 \); where the second order modulus of continuity is given by
\[
w_2(f, \delta^\frac{1}{2}) = \sup_{0 < h \leq \delta^\frac{1}{2}} \sup_{x \in [0,1]} |f(x + 2h) - 2f(x + h) + f(x)|.
\]

Also for \( f \in [0,1] \) the usual modulus of continuity is given by
\[
w(f, \delta) = \sup_{0 < h \leq \delta^\frac{1}{2}} \sup_{x \in [0,1]} |f(x + h) - f(x)|.
\]

**Theorem 3.4.** Let \( f \in [0,1] \) and \( 0 < q < p \leq 1 \). Then for all \( n \in \mathbb{N} \), there exists an absolute constant \( C > 0 \) such that
\[
|K^{(p,q)}_n (f; x) - f(x)| \leq Cw_2(f, \delta_n(x)),
\]
where
\[
\delta_n(x) = \sqrt{K^{(p,q)}_n((t-x)^2; x)}.
\]

### 4 Future work:

Recently, we are working on the following operators:
(1) On \((p, q)\)-analogue of Bleimann Butzer and Hahn operators (under preparation).
(2) Approximation by \((p, q)\)-Lorentz polynomials on a Compact disk (under preparation).
(3) A Note On \((p,q)\)-Analogue Of Bernstein-Schurer Operators (submitted).
5 Graphical analysis

With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators to the function \( f(x) = 1 + \sin(7x) \).

From figure (1), we can observe that as the value the \( q \) increases keeping \( p = 1 \), \((p, q)\)-Bernstein-Kantorovich operators given by (2.1) becomes \(q\)-Bernstein-Kantorovich operators which converges towards the function.

\[
 f(x) = 1 + \sin(7x)
\]

**Figure 1:**

In comparison to figure (1), as the value the \( n \) increases, operators given by (2.1) converges towards the function which is shown in figure (2).

**Figure 2:**

Similarly for different values of parameters \( p, q, n \) convergence of operators to the function is shown in figure 3, 4, 5, and 6.
Approximation of a function by $(p,q)$--Bernstein Kantorovich operators

Figure 3:

Approximation of a function by $(p,q)$--Bernstein Kantorovich operators

Figure 4:
Approximation of a function by $(p,q)$–Bernstein Kantorovich operators

For $q=.3$, $p=.5$

For $q=.5$, $p=.8$

For $q=.9$, $p=1$

function

Figure 5:

Approximation of a function by $(p,q)$–Bernstein Kantorovich operators

For $q=.3$, $p=.5$

For $q=.5$, $p=.8$

For $q=.9$, $p=1$

function

Figure 6:

References

[1] A. Aral, V. Gupta and R.P. Agarwal, Applications of $q$-Calculus in Operator Theory, Springer Science+Business Media New York, 2013.

[2] F. Altomare, M. Campiti, Korovkin type approximation theory and its applications, de Gruyter Stud. Math. 17, Berlin, 1994.

[3] A. Aral, O. Doğru, Bleimann Butzer and Hahn operators based on $q$-integers, J. Inequal. Appl., (2007) 1-12. Art. ID 79410.

[4] A. Aral, V. Gupta, On $q$-analogue of Stancu-beta operators, Appl. Math. Letters, 25 (2012) 67–71.

[5] S.N. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités, Comm. Soc. Math. Kharkow (2), 13 (1912-1913) 1-2.
[6] O. Dalmanoglu, Approximation by Kantorovich type \( q \)-Bernstein operators, in Proceedings of the 12th WSEAS International Conference on Applied Mathematics, Cairo, Egypt (2007), pp. 113-117.

[7] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.

[8] M.N. Hounkonnou, J. Désiré, B. Kyemba, \( R(p,q) \)-calculus: differentiation and integration, SUT Journal of Mathematics, Vol. 49, No. 2 (2013), 145-167.

[9] V. Kac, P. Cheung, Quantum Calculus, Springer-Verlag New York, 2002.

[10] L. V. Kantorovich, Sur certains développements suivant les polynômes de la forme de S. Bernstein, I, II, C. R. Acad. URSS, (1930), 563-568, 595-600.

[11] P. P. Korovkin, Linear operators and approximation theory, Hindustan Publishing Corporation, Delhi, 1960.

[12] J. Katriel, M. Kibler, Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers, J. Phys. A: Math. Gen. 25 (1992) 2683-2691. Printed in the UK.

[13] R. Jagannathan, K. S. Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, Proceedings of the International Conference on Number Theory and Mathematical Physics, 20-21 December 2005.

[14] A. Lupaş, A \( q \)-analogue of the Bernstein operator, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, 9(1987) 85-92.

[15] N.I. Mahmudov and V. Gupta, On certain \( q \)-analogue of Szász Kantorovich operators, J. Appl. Math. Comput., 37 (2011) 407-419.

[16] M. Mursaleen, Asif Khan, Generalized \( q \)-Bernstein-Schurer operators and some approximation theorems, Journal of Function Spaces and Applications, Volume 2013, Article ID 719834, 7 pages http://dx.doi.org/10.1155/2013/719834

[17] M. Mursaleen, Khursheed J. Ansari, Asif Khan, On \((p,q)\)-analogue of Bernstein Operators, (submitted).

[18] M. Mursaleen, Khursheed J. Ansari, Asif Khan, Some Approximation Results by \((p,q)\)-analogue of Bernstein-Stancu Operators, (accepted) Appl. Math. Comput., 10.1016/j.amc.2015.03.135.

[19] M. Mursaleen, A. Khan, H.M. Srivastava and K.S. Nisar, Operators constructed by means of \( q \)-Lagrange polynomials and \( A \)-statistical approximation, Appl. Math. Comput., 219 (2013) 6911-6918.

[20] G. Nowak, Approximation properties for generalized \( q \)-Bernstein polynomials, Math. Anal., pp. 1-10, Sep. 2009.

[21] G.M. Phillips, Bernstein polynomials based on the \( q \)-integers, The heritage of P.L.Chebyshev, Ann. Numer. Math., 4 (1997) 511-518.

[22] V. Sahai, S. Yadav, Representations of two parameter quantum algebras and \( p,q \)-special functions, J. Math. Anal. Appl. 335 (2007) 268-279.

[23] P. N. Sadjang On the fundamental theorem of \((p,q)\)-calculus and some \((p,q)\)-Taylor formulas, arXiv:1309.3934v1 [math.QA], 2013.

[24] D.D. Stancu, Approximation of functions by a new class of linear polynomials operators, Rev. Roum. Math. Pures et Appl. 13 (8)(1968) 1173-1194.

[25] F. Schurer, Linear positive operators in approximation theory, Math. Inst. Techn. Univ. Delft Report, 1962.