Lyapunov-Type Inequality for a Riemann-Liouville Type Fractional Boundary Value Problem with Robin Boundary Conditions

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Abstract: In this article we establish a Lyapunov-type inequality for two-point Riemann-Liouville fractional boundary value problems associated with well-posed Robin boundary conditions. To illustrate the applicability of established result, we deduce criterion for the nonexistence of real zeros of a Mittag-Leffler function.

Key Words: Riemann-Liouville derivative, boundary value problem, Green’s function, Lyapunov inequality, Mittag-Leffler function

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1. Introduction

In 1907, Lyapunov [32] proved a necessary condition for the existence of a non-trivial solution of Hill’s equation associated with Dirichlet boundary conditions.

Theorem 1.1. [32] If the boundary value problem

\[
\begin{align*}
y''(t) + p(t)y(t) &= 0, \quad a < t < b, \\
y(a) &= 0, \quad y(b) = 0,
\end{align*}
\]

(1.1)

has a nontrivial solution, where \( p : [a, b] \to \mathbb{R} \) is a continuous function, then

\[
\int_a^b |p(s)|\,ds > \frac{4}{(b-a)}.
\]

(1.2)

The Lyapunov inequality (1.2) has several applications in various problems related to differential equations. Due to its importance, the Lyapunov inequality has been generalized in many forms. For more details on Lyapunov-type inequalities and their applications, we refer [9, 36, 37, 41, 43, 44] and the references therein.

On the other hand, many researchers have derived Lyapunov-type inequalities for various classes of fractional boundary value problems in the recent years. For the first time, in 2013, Ferreira [18] generalized Theorem 1.1 to the case where the classical second-order derivative in (1.1) is replaced by an \( \alpha \)-th-order \((1 < \alpha \leq 2)\) Riemann-Liouville type derivative.

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Theorem 1.2. If the fractional boundary value problem
\[
\begin{cases}
D_α^a y(t) + p(t)y(t) = 0, & a < t < b, \\
y(a) = 0, & y(b) = 0,
\end{cases}
\]
has a nontrivial solution, where \( p : [a, b] \to \mathbb{R} \) is a continuous function, then
\[
\int_a^b |p(s)|ds > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{1-\alpha}.
\]
Here \( D_α^a \) denotes the Riemann-Liouville type \( \alpha \)th-order differential operator. In 2014, Ferreira replaced the Riemann-Liouville type derivative in Theorem 1.2 with the Caputo one \( CD_α^a \) and obtained the following Lyapunov-type inequality for the resulting problem:

Theorem 1.3. If the fractional boundary value problem
\[
\begin{cases}
CD_α^a y(t) + p(t)y(t) = 0, & a < t < b, \\
y(a) = 0, & y(b) = 0,
\end{cases}
\]
has a nontrivial solution, where \( p : [a, b] \to \mathbb{R} \) is a continuous function, then
\[
\int_a^b |p(s)|ds > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha - 1)(b-a)]^{\alpha-1}}.
\]

In 2015, Jleli et al. obtained a Lyapunov-type inequality for two-point Caputo fractional boundary value problems associated with Robin boundary conditions. Recently, Ntouyas et al. presented a survey of results on Lyapunov-type inequalities for fractional differential equations associated with a variety of boundary conditions. This article shows a gap in the literature on Lyapunov-type inequalities for two-point Riemann-Liouville fractional boundary value problems associated with Robin boundary conditions.

In 2016, Dhar et al. derived Lyapunov-type inequalities for two-point Riemann-Liouville fractional boundary value problems associated with fractional integral boundary conditions. This article stresses the importance of choosing well-posed boundary conditions for Riemann-Liouville fractional boundary value problems.

Motivated by these developments, in this article, we establish a Lyapunov-type inequality for two-point Riemann-Liouville fractional boundary value problems associated with well-posed Robin boundary conditions.

2. Preliminaries

Throughout, we shall use the following notations, definitions and known results of fractional calculus. Denote the set of all real numbers and complex numbers by \( \mathbb{R} \) and \( \mathbb{C} \), respectively.
**Definition 2.1.** \[ \text{Let } \alpha > 0 \text{ and } a \in \mathbb{R}. \text{ The } \alpha \text{-order Riemann-Liouville fractional integral of a function } y : [a, b] \rightarrow \mathbb{R} \text{ is defined by } \\
I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} y(s) ds, \quad a \leq t \leq b, \quad (2.1) \]

provided the right-hand side exists. For \( \alpha = 0 \), define \( I_a^0 \) to be the identity map.

Moreover, let \( n \) denote a positive integer and assume \( n - 1 < \alpha \leq n \). The \( \alpha \)-order Riemann-Liouville fractional derivative is defined as

\[
D_a^\alpha y(t) = D^n I_a^{n-\alpha} y(t), \quad a \leq t \leq b, \quad (2.2)
\]

where \( D^n \) denotes the classical \( n \)-th order derivative, if the right-hand side exists.

**Definition 2.2.** \[ \text{We denote by } L(a, b) \text{ the space of Lebesgue measurable functions } y : [a, b] \rightarrow \mathbb{R} \text{ for which } \\
\|y\|_L = \int_a^b |y(t)| dt < \infty. \]

**Definition 2.3.** \[ \text{We denote by } C[a, b] \text{ the space of continuous functions } y : [a, b] \rightarrow \mathbb{R} \text{ with the norm } \\
\|y\|_C = \max_{t \in [a, b]} |y(t)|. \]

**Definition 2.4.** \[ \text{Let } 0 \leq \gamma < 1, \text{ and } y : (a, b) \rightarrow \mathbb{R} \text{ and define } y_\gamma(t) = t^\gamma y(t), \quad t \in [a, b]. \text{ We denote by } C_\gamma[a, b] \text{ the weighted space of functions } y \text{ such that } \\
y_\gamma \in [a, b], \text{ and } \\
\|y\|_{C_\gamma} = \max_{t \in [a, b]} |(t - a)^\gamma y(t)|. \]

**Lemma 2.1.** \[ \text{If } \alpha \geq 0 \text{ and } \beta > 0, \text{ then } \\
I_a^\alpha (t - a)^{\beta - 1} = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (t - a)^{\beta + \alpha - 1}, \quad \\
D_a^\alpha (t - a)^{\beta - 1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (t - a)^{\beta - \alpha - 1}. \]

**Lemma 2.2.** \[ \text{Let } \alpha > \beta > 0 \text{ and } y \in C[a, b]. \text{ Then, } \\
D_a^\beta I_a^\alpha y(t) = I_a^{\alpha - \beta} y(t), \quad t \in [a, b]. \]

**Lemma 2.3.** \[ \text{Let } \alpha > 0 \text{ and } n \text{ be a positive integer such that } n - 1 < \alpha \leq n. \text{ If } y \in C(a, b) \cap L(a, b), \text{ then the unique solution of the fractional differential equation } \\
D_a^\alpha y(t) = 0, \quad a < t < b, \quad \text{is} \\
y(t) = C_1 (t - a)^{\alpha - 1} + C_2 (t - a)^{\alpha - 2} + \cdots + C_n (t - a)^{\alpha - n}, \text{ where } C_i \in \mathbb{R}, \ i = 1, 2, \ldots, n. \]

**Lemma 2.4.** \[ \text{Let } \alpha > 0 \text{ and } n \text{ be a positive integer such that } n - 1 < \alpha \leq n. \text{ If } y \in C(a, b) \cap L(a, b), \text{ then } \\
I_a^\alpha D_a^\beta y(t) = y(t) + C_1 (t - a)^{\alpha - 1} + C_2 (t - a)^{\alpha - 2} + \cdots + C_n (t - a)^{\alpha - n}, \text{ for some } C_i \in \mathbb{R}, \ i = 1, 2, \ldots, n. \]
3. Robin Boundary Value Problem

In this section, we obtain a Lyapunov-type inequality for a Robin boundary value problem using the properties of the corresponding Green’s function.

**Theorem 3.1.** Let $1 < \alpha \leq 2$ and $h : [a, b] \to \mathbb{R}$. The fractional boundary value problem

$$
\begin{cases}
D^\alpha_a y(t) + h(t) = 0, & a < t < b, \\
I^{2-\alpha}_a y(t) - D^{\alpha-1}_a y(a) = 0, & y(b) + D^{\alpha-1}_a y(b) = 0,
\end{cases}
$$

(3.1)

has the unique solution

$$
y(t) = \int_a^b G(t, s)h(s)ds,
$$

(3.2)

where

$$
G(t, s) = \begin{cases}
\left(\frac{t-a}{A}\right)^{\alpha-1} \Gamma(\alpha) + 1, & a < t \leq s \leq b, \\
\left(\frac{s-a}{A}\right)^{\alpha-1} \Gamma(\alpha) + 1 - \left(\frac{t-s}{A}\right)^{\alpha-1} \Gamma(\alpha), & a < s \leq t \leq b.
\end{cases}
$$

(3.3)

**Proof.** Applying $I^\alpha_a$ on both sides of (3.1) and using Lemma 2.4, we have

$$
y(t) = -I^\alpha_a h(t) + C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2},
$$

(3.4)

for some $C_1, C_2 \in \mathbb{R}$. Applying $I^{2-\alpha}_a$ on both sides of (3.4) and using Lemmas 2.1 - 2.2 we get

$$
I^{2-\alpha}_a y(t) = -I^2_a h(t) + C_1 \Gamma(\alpha)(t-a) + C_2 \Gamma(\alpha - 1).
$$

(3.5)

Applying $D^{\alpha-1}_a$ on both sides of (3.4) and using Lemmas 2.1 - 2.2 we get

$$
D^{\alpha-1}_a y(t) = -I^1_a h(t) + C_1 \Gamma(\alpha).
$$

(3.6)

Using $I^{2-\alpha}_a y(a) - D^{\alpha-1}_a y(a) = 0$ in (3.5) and (3.6), we get

$$
-C_1 (\alpha - 1) + C_2 = 0.
$$

(3.7)

Using $y(b) + D^{\alpha-1}_a y(b) = 0$ in (3.4) and (3.3), we get

$$
C_1 [(b-a)^{\alpha-1} + \Gamma(\alpha)] + C_2 (b-a)^{\alpha-2} = I^\alpha_a h(b) + I^1_a h(b).
$$

(3.8)

Solving (3.7) and (3.8) for $C_1$ and $C_2$, we have

$$
C_1 = \frac{1}{A} \int_a^b \left(\frac{b-s}{\Gamma(\alpha)} + 1\right)h(s)ds,
$$

and

$$
C_2 = (\alpha - 1)C_1,
$$

where

$$
A = (b-a)^{\alpha-1} + (\alpha - 1)(b-a)^{\alpha-2} + \Gamma(\alpha).
$$
Substituting $C_1$ and $C_2$ in (3.1), the unique solution of (3.1) is

$$y(t) = -\frac{1}{\Gamma(a)} \int_a^t (t-s)^{a-1} h(s) ds + \frac{(t-a)^{a-1}}{A} \int_a^b \left( \frac{(b-s)^{a-1}}{\Gamma(a)} + 1 \right) h(s) ds$$

$$+ \frac{(\alpha-1)(t-a)^{a-2}}{A} \int_a^b \left( \frac{(b-s)^{a-1}}{\Gamma(a)} + 1 \right) h(s) ds$$

$$= \int_a^t \left[ \frac{(t-a)^{a-1} + (\alpha-1)(t-a)^{a-2}}{A} \right] \left( \frac{(b-s)^{a-1}}{\Gamma(a)} + 1 \right) h(s) ds$$

$$+ \int_t^b \left[ \frac{(t-a)^{a-1} + (\alpha-1)(t-a)^{a-2}}{A} \right] \left( \frac{(b-s)^{a-1}}{\Gamma(a)} + 1 \right) h(s) ds$$

$$= \int_a^b G(t, s) h(s) ds.$$

Hence the proof. □

Now, we prove that this Green’s function is positive and obtain an upper bound for the Green’s function and its integral.

**Theorem 3.2.** The Green’s function $G(t, s)$ for (3.1) satisfies $G(t, s) > 0$ for $(t, s) \in (a, b] \times (a, b]$.

**Proof.** For $a < t \leq s \leq b$,

$$G(t, s) = \left( \frac{(t-a)^{a-1} + (\alpha-1)(t-a)^{a-2}}{A} \right) \left( \frac{(b-s)^{a-1}}{\Gamma(a)} + 1 \right) > 0.$$

Now, suppose $a < s \leq t \leq b$. Consider

$$G(t, s) = \left( \frac{(t-a)^{a-1} + (\alpha-1)(t-a)^{a-2}}{A} \right) \left( \frac{(b-s)^{a-1}}{\Gamma(a)} + 1 \right) - \frac{(t-s)^{a-1}}{\Gamma(a)}$$

$$= \frac{1}{A \Gamma(a)} \left[ (t-a)^{a-1}(b-s)^{a-1} + (\alpha-1)(t-a)^{a-2}(b-s)^{a-1} \right]$$

$$+ \Gamma(a)(t-a)^{a-1} + (\alpha-1)\Gamma(a)(t-a)^{a-2} - \frac{(t-s)^{a-1}}{\Gamma(a)}$$

$$= \frac{1}{A \Gamma(a)} \left( [(t-a)^{a-1}(b-s)^{a-1} - (b-a)^{a-1}(t-s)^{a-1}] \right.$$  

$$+ (\alpha-1)[(t-a)^{a-2}(b-s)^{a-1} - (b-a)^{a-2}(t-s)^{a-1}]$$  

$$+ \Gamma(a)[(t-a)^{a-1} - (t-s)^{a-1}] + (\alpha-1)\Gamma(a)(t-a)^{a-2} \right)$$

$$= \frac{1}{A \Gamma(a)} [E + B + C + D]. \quad (3.9)$$

Clearly, $A \Gamma(a) > 0$. Consider

$$(t-a)(b-s) - (b-a)(t-s) = (s-a)(b-t) \geq 0,$$
implies
\[ E = (t - a)^{\alpha - 1}(b - s)^{\alpha - 1} - (b - a)^{\alpha - 1}(t - s)^{\alpha - 1} \geq 0. \tag{3.10} \]

Since
\[ a < s \leq t \leq b, \]
we have
\[ (t - a)^{\alpha - 2} \geq (b - a)^{\alpha - 2}, \quad (b - s)^{\alpha - 1} \geq (t - s)^{\alpha - 1} \quad \text{and} \quad (t - a)^{\alpha - 1} > (t - s)^{\alpha - 1}, \]
implies
\[ B = (\alpha - 1)[(t - a)^{\alpha - 2}(b - s)^{\alpha - 1} - (b - a)^{\alpha - 2}(t - s)^{\alpha - 1}] \geq (\alpha - 1)(b - a)^{\alpha - 2}[(b - s)^{\alpha - 1} - (t - s)^{\alpha - 1}] \geq 0, \tag{3.11} \]
and
\[ C = \Gamma(\alpha)[(t - a)^{\alpha - 1} - (t - s)^{\alpha - 1}] > 0. \tag{3.12} \]
Clearly,
\[ D = (\alpha - 1)\Gamma(\alpha)(t - a)^{\alpha - 2} > 0. \tag{3.13} \]

Using (3.10) - (3.13) in (3.9), we have \( G(t, s) > 0 \). Hence the proof. \( \square \)

**Theorem 3.3.** For the Green’s function \( G(t, s) \) defined in (3.3),
\[ \max_{(t, s) \in (a, b) \times (a, b)} G(t, s) = G(t, t) \]
and
\[ \max_{t \in [a, b]} (t - a)^{2-\alpha} G(t, t) = \left( \frac{b - a + \alpha - 1}{A} \right) \left( \frac{(b - a)^{\alpha - 1}}{\Gamma(\alpha)} + 1 \right). \]

**Proof.** First, we show that for any fixed \( t \in (a, b] \), \( G(t, s) \) increases from \( G(t, a) \) to \( G(t, t) \), and then decreases to \( G(t, b) \). Let \( t \leq s < b \). Consider
\[ \frac{d}{ds} G(t, s) = -\left( \frac{(t - a)^{\alpha - 1} + (\alpha - 1)(t - a)^{\alpha - 2}}{A} \right) \frac{(\alpha - 1)(b - s)^{\alpha - 2}}{\Gamma(\alpha)} < 0, \]
implies $G(t, s)$ is a decreasing function of $s$. Now, suppose $a < s < t$. Consider

$$\frac{d}{ds} G(t, s) = -\left(\frac{(t-a)^{\alpha-1} + (\alpha-1)(t-a)^{\alpha-2}}{\Gamma(\alpha)}\right) \frac{(\alpha-1)(b-s)^{\alpha-2}}{A} \Gamma(\alpha) + \frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}$$

$$= \frac{(\alpha-1)}{A \Gamma(\alpha)} \left[ - (t-a)^{\alpha-1}(b-s)^{\alpha-2} - (\alpha-1)(t-a)^{\alpha-2}(b-s)^{\alpha-2} \right]$$

$$+ \frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}$$

$$= \frac{(\alpha-1)}{A \Gamma(\alpha)} \left( - (t-a)^{\alpha-1}(b-s)^{\alpha-2} + (b-a)^{\alpha-1}(t-s)^{\alpha-2} \right)$$

$$+ (\alpha-1) \left[ - (t-a)^{\alpha-2}(b-s)^{\alpha-2} + (b-a)^{\alpha-2}(t-s)^{\alpha-2} \right]$$

$$+ \Gamma(\alpha)(t-s)^{\alpha-2}$$

$$= \frac{(\alpha-1)}{A \Gamma(\alpha)} [L + M + N]. \quad (3.14)$$

Clearly, $\frac{(\alpha-1)}{A \Gamma(\alpha)} > 0$. Consider

$$(t-a)(b-s) - (b-a)(t-s) = (s-a)(b-t) \geq 0,$$

implies

$$M = (\alpha-1) \left[ - (t-a)^{\alpha-2}(b-s)^{\alpha-2} + (b-a)^{\alpha-2}(t-s)^{\alpha-2} \right] \geq 0. \quad (3.15)$$

Since

$$a < s < t \leq b,$$

we have

$$(t-s)^{\alpha-2} \geq (b-s)^{\alpha-2} \text{ and } (b-a)^{\alpha-1} \geq (t-a)^{\alpha-1},$$

implies

$$L = -(t-a)^{\alpha-1}(b-s)^{\alpha-2} + (b-a)^{\alpha-1}(t-s)^{\alpha-2}$$

$$\geq (b-s)^{\alpha-2} \left[ -(t-a)^{\alpha-1} + (b-a)^{\alpha-1} \right] \geq 0. \quad (3.16)$$

Clearly,

$$N = \Gamma(\alpha)(t-s)^{\alpha-2} > 0. \quad (3.17)$$

Using $(3.15)$ - $(3.17)$ in $(3.14)$, we have $G(t, s) > 0$, implies $G(t, s)$ is an increasing function of $s$. Thus, we have

$$\max_{(t,s)\in[a,b] \times (a,b)} G(t, s) = G(t, t).$$
Clearly,
\[
\max_{t \in [a, b]} (t - a)^{2-\alpha} G(t, t) = \left( \frac{t - a + \alpha - 1}{A} \right) \left( \frac{(b - t)^{\alpha-1}}{\Gamma(\alpha)} + 1 \right) \\
= \left( \frac{b - a + \alpha - 1}{A} \right) \left( \frac{(b - a)^{\alpha-1}}{\Gamma(\alpha)} + 1 \right).
\]

\[\square\]

Theorem 3.4. For the Green’s function \( G(t, s) \) defined in (3.3),
\[
\max_{t \in [a, b]} \int_a^b (t - a)^{2-\alpha} G(t, s) ds = \left( \frac{b - a + \alpha - 1}{A} \right) \left( \frac{(b - a)^{\alpha}}{\Gamma(\alpha + 1)} + (b - a) \right).
\]

Proof. Consider
\[
\int_a^b (t - a)^{2-\alpha} G(t, s) ds \\
= \int_a^t \left[ \left( \frac{t - a + \alpha - 1}{A} \right) \left( \frac{(b - s)^{\alpha-1}}{\Gamma(\alpha)} + 1 \right) - \frac{(t - s)^{\alpha-1}(t - a)^{2-\alpha}}{\Gamma(\alpha)} \right] ds \\
+ \int_t^b \left( \frac{t - a + \alpha - 1}{A} \right) \left( \frac{(b - s)^{\alpha-1}}{\Gamma(\alpha)} + 1 \right) ds \\
= \left( \frac{t - a + \alpha - 1}{A} \right) \left( \frac{(b - a)^{\alpha}}{\Gamma(\alpha + 1)} + (b - a) \right) - \frac{(t - a)^2}{\Gamma(\alpha + 1)}.
\]

Clearly,
\[
\max_{t \in [a, b]} \left[ \left( \frac{t - a + \alpha - 1}{A} \right) \left( \frac{(b - a)^{\alpha}}{\Gamma(\alpha + 1)} + (b - a) \right) - \frac{(t - a)^2}{\Gamma(\alpha + 1)} \right] \\
= \left( \frac{b - a + \alpha - 1}{A} \right) \left( \frac{(b - a)^{\alpha}}{\Gamma(\alpha + 1)} + (b - a) \right).
\]

Hence the proof. \[\square\]

We are now able to formulate a Lyapunov-type inequality for the Robin boundary value problem.

Theorem 3.5. If the following fractional boundary value problem
\[
\begin{align*}
D_a^\alpha y(t) + p(t)y(t) &= 0, \quad a < t < b, \\
I_a^{2-\alpha} y(a) - D_a^{\alpha-1} y(a) &= 0, \quad y(b) + D_a^{\alpha-1} y(b) = 0,
\end{align*}
\]
has a nontrivial solution, then
\[
\int_a^b (s - a)^{\alpha+2} |p(s)| ds > \frac{[(b - a)^{\alpha-1} + (\alpha - 1)(b - a)^{\alpha-2} + \Gamma(\alpha)] \Gamma(\alpha)}{[(b - a)^{\alpha-1} + \Gamma(\alpha)](b - a + \alpha - 1)}.
\]

Proof. Let \( \mathfrak{B} = C_{2-\alpha}[a, b] \) be the Banach space of functions \( y \) endowed with norm
\[
\|y\|_{C_{2-\alpha}} = \max_{t \in [a, b]} |(t - a)^{2-\alpha} y(t)|.
\]
It follows from Theorem \ref{Theorem:3.1} that a solution to (3.18) satisfies the equation

\[ y(t) = \int_a^b G(t, s)p(s)y(s)ds. \]

Hence,

\[ \|y\|_{C^2-\alpha} = \max_{t \in [a,b]} \left| (t - a)^{2-\alpha} \int_a^b G(t, s)p(s)\|y(s)\|ds \right| \]

\[ \leq \max_{t \in [a,b]} \left[ \int_a^b (t - a)^{2-\alpha}G(t, s)\|p(s)\|\|y(s)\|ds \right] \]

\[ \leq \|y\|_{C^2-\alpha} \left[ \max_{t \in [a,b]} \int_a^b (t - a)^{2-\alpha}G(t, s)(s - a)^{\alpha-2}|p(s)|ds \right] \]

\[ \leq \|y\|_{C^2-\alpha} \left[ \max_{t \in [a,b]} (t - a)^{2-\alpha}G(t, t) \right] \int_a^b (s - a)^{\alpha-2}|p(s)|ds, \]

or, equivalently,

\[ 1 < \left[ \max_{t \in [a,b]} (t - a)^{2-\alpha}G(t, t) \right] \int_a^b (s - a)^{\alpha-2}|p(s)|ds. \]

An application of Theorem \ref{Theorem:3.3} yields the result. \hfill \square

Consider the one and two-parameter Mittag-Leffler functions \cite{31}

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \] (3.20)

and

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \] (3.21)

where \( z, \beta \in \mathbb{C} \) and \( \Re(\alpha) > 0 \).

We use Theorem \ref{Theorem:3.5} to obtain an interval in which the function of Mittag-Leffler functions (3.20) and (3.21) has no real zeros.

**Theorem 3.6.** Let \( 1 < \alpha \leq 2 \). Then, the function \( E_\alpha(x) + (1 - x)E_{\alpha,\alpha}(x) + E_{\alpha,\alpha-1}(x) \) has no real zeros for

\[ |x| \leq \frac{(\alpha - 1)(\alpha + \Gamma(\alpha))\Gamma(\alpha)}{\alpha(1 + \Gamma(\alpha))}. \]

**Proof.** Let \( a = 0, b = 1 \) and consider the fractional boundary value problem

\[ \begin{cases}
D_0^\alpha y(t) + \lambda y(t) = 0, & 0 < t < 1, \\
I_0^{\alpha-\alpha} y(0) - D_0^{\alpha-1} y(0) = 0, & y(1) + D_0^{\alpha-1} y(1) = 0.
\end{cases} \] (3.22)

By Corollary 5.1 of \cite{31}, the general solution of the fractional differential equation

\[ D_0^\alpha y(t) + \lambda y(t) = 0 \]

is given by

\[ y(t) = c_1 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + c_2 t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t \in (0, 1]. \] (3.23)
Denote by
\[ g(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) = t^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-\lambda)^n t^{\alpha n}}{\Gamma(\alpha n + \alpha)}. \]

Then
\[ g'(t) = t^{\alpha-2} \sum_{n=0}^{\infty} \frac{(-\lambda)^n t^{\alpha n}}{\Gamma(\alpha n + \alpha - 1)} = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha). \]

Note that
\[ I_0^{\alpha-\alpha} g(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha)} I_0^{2-\alpha} \ell^{\alpha n+\alpha-1} \] (3.24)
\[ = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha)} \frac{\Gamma(\alpha n + \alpha)}{\Gamma(\alpha n + \alpha + 1)} \ell^{\alpha n} \]
\[ = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha + 1)} \ell^{\alpha n} = tE_{\alpha,2}(-\lambda t^\alpha), \]

\[ D_0^{\alpha-1} g(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha - 1)} D_0^{\alpha-1} \ell^{\alpha n+\alpha-1} \] (3.25)
\[ = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha - 1)} \frac{\Gamma(\alpha n + \alpha)}{\Gamma(\alpha n + 1)} \ell^{\alpha n} \]
\[ = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + 1)} \ell^{\alpha n} = E_{\alpha}(-\lambda t^\alpha), \]

\[ I_0^{\alpha-\alpha} g'(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha - 1)} I_0^{2-\alpha} \ell^{\alpha n+\alpha-2} \]
\[ = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha - 1)} \frac{\Gamma(\alpha n + \alpha - 1)}{\Gamma(\alpha n + 1)} \ell^{\alpha n} \]
\[ = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + 1)} \ell^{\alpha n} = E_{\alpha}(-\lambda t^\alpha), \] (3.26)

and

\[ D_0^{\alpha-1} g'(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha - 1)} D_0^{\alpha-1} \ell^{\alpha n+\alpha-2} \]
\[ = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha - 1)} \frac{\Gamma(\alpha n + \alpha - 1)}{\Gamma(\alpha n)} \ell^{\alpha n-1} \]
\[ = -\lambda \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n + \alpha)} \ell^{\alpha n+\alpha-1} = -\lambda g(t). \]
Also, note that
$$I_0^{-\alpha}g(0) = 0, \quad D_0^{\alpha-1}g(0) = 1, \quad I_0^{2-\alpha}g'(0) = 1, \quad D_0^{\alpha-1}g'(0) = 0. \quad (3.28)$$

Using $I_0^{2-\alpha}g(0) - D_0^{\alpha-1}g(0) = 0$, we get $c_1 = c_2$. Using $y(1) + D_0^{\alpha-1}y(1) = 0$, we get that the eigenvalues $\lambda \in \mathbb{R}$ of (3.22) are the solutions of
$$E_\alpha(-\lambda) + (1 - \lambda)E_{\alpha,\alpha}(-\lambda) + E_{\alpha,\alpha-1}(-\lambda) = 0, \quad (3.29)$$

and the corresponding eigenfunctions are given by
$$y(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t \in (0, 1]. \quad (3.30)$$

By Theorem 3.5, if a real eigenvalue $\lambda$ of (3.22) exists, i.e. $\lambda$ is a zero of (3.29), then
$$|\lambda| > \frac{(\alpha - 1)(\alpha + \Gamma(\alpha))\Gamma(\alpha)}{\alpha(1 + \Gamma(\alpha))}.$$

Hence the proof. \hfill \Box

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