A STUDY OF $k$-DIPATH COLOURINGS OF ORIENTED GRAPHS

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Keywords: Directed Graph, Oriented Graph, Graph Colouring, Graph Homomorphism

Abstract. We examine $t$-colourings of oriented graphs in which, for a fixed integer $k \geq 1$, vertices joined by a directed path of length at most $k$ must be assigned different colours. A homomorphism model that extends the ideas of Sherk for the case $k = 2$ is described. Dichotomy theorems for the complexity of the problem of deciding, for fixed $k$ and $t$, whether there exists such a $t$-colouring are proved.

1. Introduction

Recall that an oriented graph is a digraph obtained from a simple, undirected graph by giving each edge one of its two possible orientations. Recall, also, that if $G$ and $H$ are oriented graphs, then a homomorphism of $G$ to $H$ is a function $\phi$ from the vertices of $G$ to the vertices of $H$ such that $\phi(x)\phi(y) \in E(H)$ whenever $xy \in E(G)$. If $G$ and $H$ are oriented graphs such that there is a homomorphism $\phi$ of $G$ to $H$, then we write $\phi : G \rightarrow H$, or $G \rightarrow H$ if the name of the function $\phi$ is not important.

Let $k$ and $t$ be a positive integers, and let $G$ be an oriented graph. Chen and Wang [3] defined a $k$-dipath $t$-colouring of $G$ to be an assignment of $t$ colours to the vertices of $G$ so that any two vertices joined by a directed path of length at most $k$ are assigned different colours. A 1-dipath $t$-colouring of an oriented graph $G$ is a colouring of the underlying undirected graph of $G$. See Figure 1 for an example of a 3-dipath 4-colouring of an oriented graph. The $k$-dipath chromatic number of $G$, denoted by $\chi_{kd}(G)$, is the smallest positive integer $t$ such that there exists a $k$-dipath $t$-colouring of $G$. Chen and Wang showed that any orientation of a Halin graph has 2-dipath chromatic number at most 7, and there are infinitely many such graphs $G$ with $\chi_{2d}(G) = 7$ [3].

For a positive integer $t$, an oriented $t$-colouring of an oriented graph $G$ is a homomorphism of $G$ to some oriented graph on $t$ vertices. Oriented colourings were first introduced in 1994 [4], and have been a topic of considerable interest in the literature since then; see the recent survey by Sopena [16], and also [2], [5], [17] for related topics. Since oriented graphs have no directed cycles of length two, the definition implies that any two vertices of $G$ joined by a directed path of length at most two are assigned different colours (i.e. they have different
images) in an oriented colouring of $G$. The 2-dipath chromatic number is of interest, in part, because it gives a lower bound for the oriented chromatic number $\chi_o(G)$ – the smallest positive integer $t$ such that $G$ admits an oriented $t$-colouring. It follows from the definition that any oriented colouring of $G$ is a 2-dipath colouring of $G$; hence $\chi_{2d}(G) \leq \chi_o(G)$.

In her Master’s thesis [20] (more recently published as [10]), Sherk (née Young) gives a homomorphism model for 2-dipath $t$-colouring. For each positive integer $t$, she describes an oriented graph $G_t$ with the property that an oriented graph $G$ has a 2-dipath $t$-colouring if and only if there is a homomorphism of $G$ to $G_t$. As is common with such theorems, it is possible to use the homomorphism to $G_t$ to find a 2-dipath $t$-colouring of $G$: the colour assigned to a vertex is determined by its image (but is not equal to it). The existence of this model implies an upper bound for the oriented chromatic number as a function of the 2-dipath chromatic number. It also leads to a proof that deciding whether a given oriented graph has a 2-dipath $t$-colouring is Polynomial if the fixed integer $t \leq 2$, and NP-complete if $t \geq 3$.

A natural question is whether Sherk’s results can be generalized to $k$-dipath $t$-colouring. We seek a model similar to hers, where the homomorphism to the target oriented graph can be used to find a $k$-dipath $t$-colouring of the given oriented graph. In particular, vertices of the given oriented graph $G$ with the same image should be assigned the same colour. We suggest that such model will likely exist only for colouring oriented graphs with no directed cycles of length $k$ or less. Consider the case of 3-dipath $t$-colouring, where $t \geq 3$. Suppose there exists a digraph $H_{3,t}$ with the property that an oriented graph $G$ has a 3-dipath $t$-colouring if and only if there is a homomorphism of $G$ to $H_{3,t}$. The digraph $H_{3,t}$ has no loops, otherwise all vertices of $G$ can be assigned the same colour. By definition, the directed 3-cycle has a 3-dipath 3-colouring: assign each vertex a different colour. Thus, there is a homomorphism of the directed 3-cycle to $H_{3,t}$. Consequently, $H_{3,t}$ has a directed 3-cycle. But, now there is a homomorphism of a directed path of length three to $H_{3,t}$ in which the two end vertices have the same image. Since the ends of a directed path of length 3 must be assigned different colours in a 3-dipath $t$-colouring, this model will not have the desired property. Similar
considerations apply to $k$-dipath $t$-colouring for all pairs of positive integers $k$ and $t$ with $t \geq k$. Hence, a homomorphism model of the type we seek will not exist if the oriented graphs being coloured can have directed cycles of length $k$ or less. Finally, we note that Sherk’s homomorphism model for 2-dipath $t$-colouring is for oriented graphs. These have no directed cycles of length two or less.

The main result of this paper is the construction of a homomorphism model for $k$-dipath $t$-colouring of oriented graphs with no directed cycles of length $k$ or less. That is, for all positive integers $k$ and $t$ we describe an oriented graph $G_{k,t}$ with the property that an oriented graph $G$, with no directed cycle of length at most $k$, has a $k$-dipath $t$-colouring if and only if $G$ admits a homomorphism to $G_{k,t}$ and, further, the homomorphism can be used to find a $k$-dipath $t$-colouring of $G$.

After presenting this result in Section 3, in Section 4 we determine the complexity of deciding the existence of a $k$-dipath $t$-colouring for all pairs of fixed positive integers $k$ and $t$. When instances are restricted to oriented graphs with no directed cycles of length $k$ or less, it is shown that that this problem is NP-complete whenever $t > k \geq 3$, and Polynomial if $t = k$ or $k \leq 2$. When there are no restrictions, it is shown that this problem is NP-complete whenever $k \geq 3$ and $t \geq 3$, and Polynomial whenever $k \leq 2$ or $t \leq 2$.

2. Preliminaries

In this section we review relevant definitions, the homomorphism model for 2-dipath colourings, and make small improvements to the known upper bounds on the oriented chromatic number of oriented graphs with 2-dipath chromatic number 3 or 4. We also observe some straightforward extensions to $k$-dipath colouring of known results for 2-dipath colouring.

Let $G$ be an oriented graph, and let $x, y \in V(G)$. A directed walk is a sequence of vertices $W = v_0, v_1, v_2, \ldots, v_{\ell-1}, v_{\ell}$, such that $v_i v_{i+1} \in E(G)$ for $i = 0, 1, \ldots, \ell - 1$. The integer $\ell$ is the length of $W$. If $v_0 = x$ and $v_{\ell} = y$, then $W$ is a directed walk from $x$ to $y$. Note that the vertices belonging to $W$ need not be different. If no two vertices of $W$ are the same, then $W$ is a directed path. If all vertices of $W$ are different except $v_0$ and $v_{\ell}$, then $W$ is a directed cycle.

If there is a directed walk from $x$ to $y$, then the vertex $y$ is said to be reachable from $x$. The distance from $x$ to $y$ is defined to be the smallest length of a directed walk from $x$ to $y$, or infinity if no such walk exists. The weak distance between $x$ and $y$, denoted $d_{\text{weak}}(x, y)$, is defined to be the minimum of the distance from $x$ to $y$ and the distance from $y$ to $x$. This parameter is $\infty$ if neither of $x$ and $y$ is reachable from the other. The weak diameter of an oriented graph $G$ is the maximum of the weak distance between any two distinct vertices of $G$. A directed graph is called weakly connected if its weak diameter is finite.

Let $G$ be a directed graph, and $x \in V(G)$. The out-neighbourhood of $x$ is $N^+(x) = \{y : xy \in E(G)\}$, and the in-neighbourhood of $x$ is $N^-(x) = \{y : yx \in E(G)\}$. The vertex $x$ is a source if $N^-(x) = \emptyset$, and is a sink if $N^+(x) = \emptyset$. A universal source is a source such that $N^+(x) = V(G) - \{x\}$, and a universal sink is a sink such that $N^-(x) = V(G) - \{x\}$. More generally, the $\ell$-out-neighbourhood of $x$ is the set of all vertices reachable from $x$ by a directed
walk of length at most $\ell$, and the $\ell$-in-neighbourhood of $x$ is the set of all vertices which can reach $x$ by a directed walk of length at most $\ell$.

The directed girth of an directed graph $H$ is defined to be the minimum length of a directed cycle in $H$, or infinity if $H$ has no directed cycle. Our homomorphism model for $k$-dipath colouring applies only to the family of oriented graphs with directed girth at least $k + 1$.

Let $\mathcal{F}$ be a family of oriented graphs. The oriented chromatic number of $\mathcal{F}$, denoted by $\chi_o(\mathcal{F})$, is the least integer $t$ so that $\chi_o(F) \leq t$ for all $F \in \mathcal{F}$. We say that an oriented graph $H$ is a universal target for $\mathcal{F}$ if $\chi_o(F) \leq |V(H)|$ for all $F \in \mathcal{F}$. If $H$ is a universal target for $\mathcal{F}$, then $\chi_o(\mathcal{F}) \leq \chi_o(O) \leq 7$ [15].

Let $G$ be an oriented graph with directed girth at least $k + 1$. We say $G$ is a $k$-dipath clique if $\chi_{kd}(G) = |V(G)|$. The terminology arises by analogy with undirected graphs, where a clique is a graph for which the chromatic number equals the number of vertices.

Let $G$ be an oriented graph. Define $G^k$ to be the simple graph formed from $G$ as follows:

- $V(G^k) = V(G)$, and
- $E(G^k) = \{uw|0 < d_{\text{weak}}(u, v) \leq k\}$.

It follows from the definitions that there is an equivalence between 2-dipath colourings of $G$ and proper vertex colourings of the simple graph $G^2$ (also, see [10]). This equivalence extends to $k$-dipath colourings of $G$ and proper vertex colourings of $G^k$.

**Observation 2.1.** If $G$ is an oriented graph with directed girth at least $k + 1$, then there is a one-to-one correspondence between $k$-dipath colourings of $G$ and proper colourings of $G^k$.

Using this observation we generalize a result of Sen [14] for 2-dipath cliques. This result will be used in Section 4.

**Proposition 2.2.** [14] An oriented graph is a 2-dipath clique if and only if it has weak diameter at most 2.

**Proposition 2.3.** Let $k \geq 2$ be an integer. An oriented graph is a $k$-dipath clique if and only if it has weak diameter at most $k$.

**Proof.** Let $G$ be an oriented graph with directed girth at least $k + 1$. We observe that $G^k$ is a complete graph if and only if for each pair of non-adjacent vertices, say $u$ and $v$, there is a directed path of length at most $k$, in some direction, between $u$ and $v$. Equivalently, $G$ has weak diameter at most $k$. \hfill $\Box$

We now review the homomorphism model for 2-dipath colouring [10]. Let $t$ be a positive integer. The oriented graph $\mathcal{G}_t$ is defined as follows.

$V(\mathcal{G}_t) = \{(u_0; u_1, u_2, \ldots, u_t) : u_0 \in \{1, 2, \ldots, t\}, u_i \in \{0, 1\} \text{ if } i \neq u_0, u_{u_0} = \cdot\}$

$E(\mathcal{G}_t) = \{(u_0; u_1, u_2, \ldots, u_t)(x_0; x_1, x_2, \ldots, x_t) : u_{x_0} = 1, x_{u_0} = 0\}$.

It is then proved that an oriented graph $G$ admits a homomorphism to $\mathcal{G}_t$ if and only if $G$ has 2-dipath chromatic number at most $t$. That is, $\mathcal{G}_t$ is a universal target for the
family of oriented graphs with 2-dipath chromatic number at most $t$. Further, suppose $\phi : G \rightarrow G_t$. For each vertex $x$, assigning colour $u_0$ to $x$ if and only if the mapping $\phi$, $\phi(x) = (u_0; u_1, u_2, \ldots, u_t)$, is a 2-dipath $t$-colouring of $G$. Since homomorphisms compose, the oriented chromatic number of $G_t$ is an upper bound for the oriented chromatic number of $G$. The authors of [10] show that $\chi_o(G_t) \leq 2^t - 1$, which gives the following.

**Theorem 2.4.** [10] If $G$ is an oriented graph, then

$$\chi_{2d}(G) \leq \chi_o(G) \leq 2^{\chi_{2d}(G)} - 1.$$  

The topic of universal targets for $k$-dipath colourings is considered in Section 3. Here we offer improvements for the cases $t = 3, 4$ of Theorem 2.4.

**Proposition 2.5.** Let $t \geq 2$ be an integer and let $G$ be an oriented graph with $\chi_{2d} \leq t$.

- If $t \leq 3$, then $\chi_o(G) \leq 5$.
- If $t \leq 4$, then $\chi_o(G) \leq 12$.

**Proof.** Figure 2 shows $G_3$, except for arcs between the source vertices on the left and the sink vertices on the right. By inspection, $G_3$ admits a homomorphism to the tournament of order 5 formed from a copy of a directed 3-cycle together with a universal source vertex and universal sink vertex. This proves the first statement.

Let $H$ be the oriented graph obtained from $G_4$ by deleting all sources and all sinks. Figure 3 gives a mapping of $H$ to the oriented graph of order 10 shown. Thus, this oriented graph, together with a universal source and universal sink vertex, is a homomorphic image of $G_4$. This proves the second statement.

For $k \geq 2$, a $k$-dipath colouring of an oriented graph $G$ is a 2-dipath colouring of $G$. Thus, Theorem 2.4 implies the following result for the $k$-dipath chromatic number.
Figure 3. A homomorphic image of the modified universal target for the family of oriented graphs with $\chi_{2d} \leq 4$.

**Corollary 2.6.** If $G$ is an oriented graph with directed girth at least $k + 1$, then $\chi_o(G) \leq 2^{\chi_{kd}(G)} - 1$.

Note, however, that by Theorem 3.11, $\chi_{kd}$ can always be replaced by $\chi_{2d}$. By contrast, the lower bound for $\chi_o(G)$ given in Theorem 2.4 does not hold if $\chi_{2d}$ is replaced by $\chi_{kd}$. The oriented chromatic number of a directed path on 4 vertices is 3, but its 4-dipath chromatic number is 4.

3. HOMOMORPHISMS AND $k$-DIPATH COLOURING

We begin our study of homomorphisms and $k$-dipath colourings by considering oriented graphs without directed cycles. The transitive tournament on $t$ vertices is denoted by $T_t$.

**Theorem 3.1** ([12]). If $G$ is an acyclic oriented graph, then $G \rightarrow T_t$ if and only if there is no homomorphism of a directed path on $t + 1$ vertices to $G$.

A different way to phrase the condition in the theorem is that if $G \rightarrow T_t$, then $G$ has no directed cycles, and no directed path of length $t$ or more.

**Corollary 3.2.** If $G$ is an acyclic oriented graph and the longest directed path in $G$ has $t$ vertices, then $\chi_{kd}(G) = t$ for all $k \geq t > 1$. 
Proof. Suppose $k \geq t$. Observe that since $G$ has a path on $t$ vertices, we have $\chi_{kd}(G) \geq t$. By Theorem 3.1 there exists a homomorphism $\phi : G \rightarrow T_t$. Since $T_t$ has no directed cycle, any two vertices joined by a directed path must have different images. If the vertices of $T_t$ are regarded as the colours $1, 2, \ldots, t$, then $\phi$ is a $k$-dipath colouring of $G$. Thus $\chi_{kd}(G) \leq t$. □

Since any oriented graph which admits a homomorphism to $T_t$ is acyclic, it follows that $T_t$ is a universal target for $k$-dipath $t$-colouring of acyclic oriented graphs.

**Corollary 3.3.** Let $k$ and $t$ be positive integers. An acyclic oriented graph $G$ has $\chi_{kd}(G) \leq t$ if and only if $G$ admits a homomorphism to $T_t$.

**Corollary 3.4.** Let $k$ and $t$ be positive integers. An acyclic oriented graph $G$ has $\chi_{kd}(G) \leq t$ if and only if $G$ has no directed path on at least $t + 1$ vertices.

Though not a direct analogue, Corollary 3.2 has a similar flavour to the early results on graph colourings of Gallai, Roy, Hasse, and Vitaver (7, 13, 8, 18).

Observe that for $k \geq t$, the $k$-dipath chromatic number of $T_t$ equals $t$. Thus, Corollary 3.2 states that the $k$-dipath chromatic number of $T_t$ is an upper bound on the $k$-dipath chromatic number of any oriented graph which admits a homomorphism to $T_t$. The same statement holds if $T_t$ is replaced by any oriented graph with large enough directed girth.

**Theorem 3.5.** Let $G$ and $H$ be oriented graphs such that $H$ has directed girth at least $k + 1$. If $G \rightarrow H$, then $\chi_{kd}(G) \leq \chi_{kd}(H)$.

Proof. Suppose $\phi : G \rightarrow H$. Let $c$ be a $k$-dipath $t$-colouring of $H$. Let $c' : V(G) \rightarrow \{1, 2, 3, \ldots, t\}$ be defined by $c'(v) = c(\phi(v))$, for all $v \in V(G)$. We claim that this is a $k$-dipath $t$-colouring of $G$.

Let $u, v \in V(G)$. Suppose that there is a directed path from $u$ to $v$ of length $\ell \leq k$. Then there is a directed walk $W$ of length $k$ in $H$ from $\phi(u)$ to $\phi(v)$. Since the directed girth of $H$ is $k + 1$ or more, $W$ is a directed path. Thus, $c(\phi(u)) \neq c(\phi(v))$. Therefore $c'(u) \neq c'(v)$, and $c'$ is a $k$-dipath colouring of $G$. □

We now describe a homomorphism model for $k$-dipath $t$-colouring of oriented graphs with directed girth at least $k + 1$. That is, for all positive integers $k$ and $t$, we will describe an oriented graph $G_{k,t}$ with the property that an oriented graph $G$, with no directed cycle of length at most $k$, has a $k$-dipath $t$-colouring if and only if $G$ admits a homomorphism to $G_{k,t}$ and, further, the homomorphism can be used to find a $k$-dipath $t$-colouring of $G$. The graph $G_{k,t}$ is a universal target for $k$-dipath $t$-colouring. Once this model is in place, Theorem 3.5 could be viewed as a direct consequence of the fact that a composition of homomorphisms is a homomorphism; it arises in the proof for the model, however.

We begin by defining a special set of matrices. Let $k \geq 2$ be an integer. Suppose that an oriented graph $G$, with directed girth at least $k + 1$, admits a $k$-dipath $t$-colouring, $c$. For $x \in V(G)$, given $c(x)$, we want to encode information about the colours of vertices in the $k$-in-neighbourhood of $x$ and $k$-out-neighbourhood of $x$. We define the $k$-dipath colouring matrix of $x$ with respect to $c$ to be the $(2k-1) \times m$ zero-one matrix $A_{x,c}(G)$ with rows indexed by $-(k-1), -(k-2), \ldots, -1, 0, 1, \ldots, (k-2), (k-1)$, columns indexed by $1, 2, 3, \ldots, m$, and
and \((i,j)\)-entry equal to 1 if and only if there exists a vertex \(y \in V(G)\) such that \(c(y) = j\), and

- if \(i \in \{-k-1,-k-2,\ldots,-1\}\), then there is a directed path from \(y\) to \(x\) of length \(i\);
- if \(i \in \{1,2,\ldots,(k-1)\}\), then there is a directed path from \(x\) to \(y\) of length \(i\); and
- if \(i = 0\), then \(x = y\).

When the graph \(G\) is clear from the context, or unimportant in the discussion, the \(k\)-dipath colouring matrix of \(x\) with respect to \(c\) is denoted by \(A_{x,c}\).

We illustrate the definition with an example. Consider the colouring, \(c\), given in Figure 1. Let \(x\) be the unique vertex such that \(c(x) = 3\). The 3-dipath colouring matrix of \(x\) is given by

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
-2 & 0 & 1 & 0 \\
-1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
\end{array}
\]

For example, the value, 1, in the entry \((-1,2)\) arises because there is a vertex \(w\) such that \(c(w) = 2\) and there is a directed path of length 1 from \(w\) to \(x\).

Let \(A_{k,t}\) denote the set of all possible \(k\)-dipath colouring matrices over all \(k\)-dipath \(t\)-coloured oriented graphs with directed girth at least \(k + 1\). We note that \(A_{k,t}\) is necessarily finite, as each member is a \(2k-1 \times t\) matrix with entries from \(\{0,1\}\). Since vertices at weak distance at most \(k\) must be assigned different colours, each element of \(A_{k,t}\) is a matrix that satisfies several conditions. Suppose \(A\) is the \(k\)-dipath colouring matrix of \(x\) with respect to \(c\). If the colour of \(x\) is \(j\), then no vertex at weak distance at most \(k\) from \(x\) can be assigned colour \(j\). Thus, the entry of \(A\) in column \(j\) and row 0 equals 1, and all other entries in column \(j\) are zero. If there is a directed path of length \(i < k\) from \(y\) to \(x\), then no vertex \(w\) for which there is a directed path of length \(k-i\) from \(x\) to \(w\) can have the same colour as \(y\). If the colour of \(y\) is \(j\), then the entry of \(A\) in column \(j\) and row \(-i\) equals 1, and the entries of \(A\) in column \(j\) and rows 0, 1, \ldots, \(k-i\) are all equal to zero. Note that no assertion can be made about the entries of \(A\) in column \(j\) and rows \(-(i-1), -(i-2), \ldots, -1\) because there is no guarantee that a vertex \(v\) for which there is a directed path of length at most \(i-1\) to \(x\) lies on the directed path from \(y\) to \(x\). Similar considerations apply to vertices \(w\) for which there is a directed path of length \(i < k\) from \(x\) to \(w\). These conditions are formalized succinctly as follows.

**Observation 3.6.** Let \(A = [a_{ij}] \in A_{k,t}\). Suppose that \(a_{ij} = 1\). Then,

- if \(i = 0\), then \(a_{\ell j} = 0\) for all \(\ell \neq i\);
- if \(i < 0\), then \(a_{\ell j} = 0\), \(0 \leq \ell \leq k - i\); and
- if \(i > 0\), then \(a_{-\ell j} = 0\), \(0 \leq \ell \leq k - i\).
We now construct an oriented graph, $G_{k,t}$, which is a universal target for the family of $k$-dipath $t$-colourable oriented graphs of directed girth at least $k + 1$. The vertex set $V(G_{k,t}) = A_{k,t}$. We describe the edge set informally at first, and then formally. Suppose the matrices $A = A_{x,c}$ and $B = B_{y,c}'$ are vertices of $G_{k,t}$. Let $c(x) = m_1$ and $c'(y) = m_2$. Then $AB \in E(G)$ if

1. $A$ encodes the fact that $x$ has an out-neighbour of colour $m_2$;
2. $B$ encodes the fact that $y$ has an in-neighbour of colour $m_1$;
3. if $A$ encodes the fact that a vertex of colour $m_3$ is joined to $x$ by a directed path of length $i < k$, then $B$ must encode the fact that a vertex of colour $m_3$ is joined to $y$ by a directed path of length $i + 1$;
4. if $B$ encodes the fact that $y$ is joined to a vertex of colour $m_3$ by a directed path of length $i < k$, then $A$ must encode the fact that $x$ is joined to a vertex of colour $m_3$ by a directed path of length $i + 1$.

We now formally define the edge set of $G_{k,t}$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be vertices of $G_{k,t}$ (i.e. matrices in $A_{k,t}$). Then $AB \in E(G_{k,t})$ if, whenever $a_{0m_1} = b_{0m_2} = 1$, the following conditions all hold:

1. $a_{1m_2} = 1$;
2. $b_{-1m_1} = 1$;
3. if $0 < i < k - 1$ and $a_{-im_3} = 1$, then $b_{-(i-1)m_3} = 1$; and
4. if $0 < i < k - 1$ and $b_{im_3} = 1$, then $b_{(i+1)m_3} = 1$.

We now establish some properties of $G_{k,t}$.

**Lemma 3.7.** The digraph $G_{k,t}$ has directed girth at least $k + 1$. In particular, it is an oriented graph.

**Proof.** Let $A_1, A_2, \ldots, A_\ell, A_1$ be a directed cycle in $G_{k,t}$. Suppose $\ell \leq k$, and $A_1$ has a 1 in entry $(0, m_1)$. This implies $A_2$ has a 1 in entry $(-1, m_1)$ and a 1 in entry $((\ell - 1), m_1)$, contrary to Observation 3.6. □

**Lemma 3.8.** For integers $t \geq 2$ and $k \geq 2$ the $k$-dipath chromatic number of $G_{k,t}$ is at most $t$.

**Proof.** Consider the colouring, $c$, given by $c(A) = m_1$, where $m_1$ the is unique column of the matrix of $A$ for which the entry $(0, m_1)$ of $A$ is a 1. We claim that $c$ is a $k$-dipath colouring of $G_{k,t}$. Let $A_1, A_2, \ldots, A_\ell$ be a directed path of length $\ell \leq k$ in $G_{k,t}$. If there exists a pair of indices $i$ and $j$ such that $1 \leq i < j \leq k$ and $c(A_i) = c(A_j)$, then $A_{i+1}$ has a 1 in entry $(-1, c(A_i))$ and a 1 in entry $((j - (i + 1)), c(A_i))$, contrary to Observation 3.6. □

**Theorem 3.9.** Let $t \geq 2$ and $k \geq 2$ be integers. If $G$ is an oriented graph with directed girth at least $k + 1$, then $\chi_{kd}(G) \leq t$ if and only if $G \rightarrow G_{k,t}$. 9
Theorem 3.11. Let $G$ be an oriented graph with directed girth at least $k + 1$. If $G \to \mathcal{G}_{k,t}$, then by Lemmas 3.7 and 3.8 and by Theorem 3.5, $\chi_{kd}(G) \leq t$.

Suppose $\chi_{kd}(G) \leq t$. Let $c$ be a $k$-dipath colouring of $G$ using $t$ colours. Consider the mapping $\phi : V(G) \to V(\mathcal{G}_{k,t})$, where for all $v \in V(G)$, $\phi(v) = A_v$, the $k$-dipath colouring matrix of $v$ with respect to $c$. Let $uv$ be an arc of $G$. We claim $A_uA_v$ is an arc of $\mathcal{G}_{k,t}$. Suppose $A_u$ has a 1 in entry $(0, m_1)$, and $A_v$ has a 1 in entry $(0, m_2)$ (i.e., $c(u) = m_1$ and $c(v) = m_2$). To show that $A_u$ and $A_v$ are adjacent in $\mathcal{G}_{k,t}$ we must check that the four conditions in the definition are satisfied:

1. Since $c(v) = m_2$, $A_u$ has a 1 in entry $(1, m_2)$.
2. Since $c(u) = m_1$, $A_v$ has a 1 in entry $(-1, m_1)$.
3. Suppose there exists $i > 0$ such that $A_u$ has a 1 in entry $(-i, m_3)$. Since $c$ is a $k$-dipath colouring the entries of $A_u$ in column $m_3$ and rows $0, 1, \ldots, k - i$ are all equal to 0.
4. Suppose there exists $i > 0$ such that $A_v$ has a 1 in entry $(i, m_3)$. Since $c$ is a $k$-dipath colouring, the entries of $A_v$ in column $m_3$ and rows $-(k - i), -(k - i + 1), \ldots, -1, 0$ are all equal to zero.

This proves the claim. Therefore $\phi : G \to \mathcal{G}_{k,t}$ is a homomorphism.

□

Corollary 3.10. For integers $t \geq 2$ and $k \geq 2$, $\chi_{kd}(\mathcal{G}_{k,t}) = t$.

Proof. We show that $\chi_{kd}(\mathcal{G}_{k,t}) \geq t$. The result then follows from Lemma 3.8. Consider the transitive tournament $T_t$. Assigning each vertex a different colour is a $k$-dipath $t$-colouring of $T_t$. Hence, by Theorem 3.9, there is a homomorphism $T_t \to \mathcal{G}_{k,t}$.

Since $T_t$ does not have a $k$-dipath $m$-colouring for $m < t$, by Theorem 3.9, there is no homomorphism $T_t \to \mathcal{G}_{k,m}$. Since a composition of homomorphisms is a homomorphism, it follows that $\mathcal{G}_{k,m} \not\to \mathcal{G}_{k,t}$ when $m < t$. Therefore, $\chi_{kd}(\mathcal{G}_{k,t}) > t - 1$. The result now follows.

□

Our final result of this section shows that the homomorphism model captures three facts. First, if $t < k$, then no oriented graph with directed girth at least $k + 1$ and a $k$-dipath $t$-colouring can have a directed path of length greater than $t$. Any such digraph is $t$-dipath $t$-colourable (as the proof shows). Second, if $t \geq k$, then any $k$-dipath $t$-colouring of an oriented graph is also a $k$-dipath $(t + 1)$-colouring, and there exist oriented graphs with $k$-dipath chromatic number $t + 1$. Third, every $(k + 1)$-dipath $t$-colouring of an oriented graph is a $k$-dipath $t$-colouring.

Theorem 3.11. Let $k \geq 2$ and $t \geq 2$ be integers. Then

1. if $t \leq k$, then $\mathcal{G}_{k,t} \to \mathcal{G}_{t,t}$;
2. $\mathcal{G}_{k,t} \to \mathcal{G}_{k,(t+1)}$ and $\mathcal{G}_{k,(t+1)} \not\to \mathcal{G}_{k,t}$;
3. if $t > k$, then $\mathcal{G}_{(k+1),t} \to \mathcal{G}_{k,t}$ and $\mathcal{G}_{k,t} \not\to \mathcal{G}_{(k+1),t}$.
Proof. We first prove (1). Suppose $t \leq k$. Then no oriented graph with directed girth at least $k + 1$ and a $k$-dipath $t$-colouring has a directed path of length greater than $t$. By Theorem 3.1 there is a homomorphism of $G$ to the transitive tournament $T_t$. In particular, $G_{k,t} \rightarrow T_t$. 

But, by Theorem 3.9, $T_t \rightarrow G_{t,t}$. Therefore, $G_{k,t} \rightarrow G_{t,t}$. 

We now prove (2). It is clear that the subgraph of $G_{k,(t+1)}$ induced by the vertices (colouring matrices) in which every entry in column $t + 1$ is zero (i.e. $k$-dipath $(t + 1)$-colourings in which colour $t + 1$ is never used) is isomorphic to $G_{k,t}$. Thus, $G_{k,t} \rightarrow G_{k,(t+1)}$. On the other hand, the transitive tournament on $t + 1$ vertices has a homomorphism to $G_{k,(t+1)}$ but not to $G_{k,t}$. Therefore, $G_{k,(t+1)} \not\rightarrow G_{k,t}$.

Finally, we prove (3). Suppose $t > k$. Since a $(k + 1)$-dipath $t$-colouring is a $k$-dipath $t$-colouring, it follows from Theorem 3.9 that $G_{(k+1),t} \rightarrow G_{k,t}$. To see the second statement, note that a directed cycle of length $k + 1$ has a homomorphism to $G_{k,t}$ but, by Lemma 3.7, no homomorphism to $G_{(k+1),t}$. Therefore, $G_{k,t} \not\rightarrow G_{(k+1),t}$. 

Directed graphs $G$ and $H$ are called homomorphically equivalent if there are homomorphisms $G \rightarrow H$ and $H \rightarrow G$. If $G$ and $H$ are homomorphically equivalent, then a directed graph $G$ admits a homomorphism to $G$ if and only if it admits a homomorphism to $H$. A directed graph is called a core if it is not homomorphically equivalent to any of its proper subgraphs. Every directed graph has a core which is unique up to isomorphism [6, 19].

**Corollary 3.12.** Let $t$ and $k$ be positive integers. If $t \leq k$, then the core of $G_{k,t}$ is $T_t$.

**Proof.** Note that $T_t$ is a core. The existence of the required homomorphism is noted in the proof of statement (1) in Theorem 3.11. 

4. Complexity of $k$-dipath Colourings

In this section we consider the following decision problem.

**$k$-DIPATH $t$-COLOURING**

Instance: an oriented graph, $G$.

Question: does $G$ have a $k$-dipath $t$-colouring?

The dichotomy theorem stated below covers the cases where $k = 2$ [10, 20]. We shall find a similar result for all remaining cases.

**Theorem 4.1.** [10, 20] Let $t \geq 1$ be a fixed integer. If $t \leq 2$, then 2-DIPATH $t$-COLOURING is Polynomial. If $t \geq 3$, then 2-DIPATH $t$-COLOURING is NP-complete.

Given a simple graph $G$, we construct an oriented graph, $H_{k,t}$ ($t > k \geq 3$), such that $\chi_{kd}(H) \leq t$ if and only if $\chi(G) \leq t$. Let $G$ be a simple graph, and let $\tilde{G}$ be an arbitrary acyclic orientation of $G$. Corresponding to each vertex $v \in V(\tilde{G})$ the oriented graph $H_{k,t}$ contains:

1. vertices $v_i$ and $v_o$;
2. a transitive tournament on $t - k + 1$ vertices with source vertex $s_v$ and sink vertex $t_v$.
For each vertex $v$, if $\phi$ corresponding to the arcs of $\tilde{a}_k G$

Suppose now that $k \geq t > k$ and $H$

Then, in $H_k$, it has directed girth at least $k + 1$ and each vertex $v_i$ has out-degree 0.

**Observation 4.2.** $\chi_{kd}(H_{k,t}) \geq t$.

**Proof.** For any vertex $v \in V(G)$, observe that the subgraph of $H_{k,t}$ induced by the $t - k + 1$ vertices of the transitive tournament corresponding to $v$, together with the vertices $x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}, v_i$ is a $k$-dipath clique on $t$ vertices. $\square$

**Observation 4.3.** Let $t > k \geq 3$. If $H_{k,t}$ has a $k$-dipath $t$-colouring, then for every $v \in V(G)$ and every $k$-dipath $t$-colouring, $c$, of $H_{k,t}$, $c(v_o) = c(v_i)$.

**Proof.** For any $v \in V(G)$, observe that the subgraph of $H_{k,t}$ induced by the vertex $v_o$, the $t - k + 1$ vertices of the transitive tournament corresponding to $v$ and the vertices $x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}, v_i$ is a $k$-dipath clique on $t$ vertices. Since the subgraph of $H_{k,t}$ induced by the $t - k + 1$ vertices of the transitive tournament corresponding to $v$, together with the vertices $x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}, v_i$ is also $k$-dipath clique on $t$ vertices, it follows that $c(v_o) = c(v_i)$. $\square$

**Lemma 4.4.** If $G$ is a simple graph and $H_{k,t}$ is constructed from $G$ as above, then, for all $t > k \geq 3$, the graph $G$ is $t$-colourable if and only if $H_{k,t}$ has a $k$-dipath $t$-colouring.

**Proof.** Suppose that $H_{k,t}$ has a $k$-dipath $t$-colouring, $c$. Let $\phi : V(G) \to \{1, 2, 3, \ldots, t\}$ be defined by $\phi(v) = c(v_i)$. We claim that $\phi$ is a proper colouring of $G$. Suppose $ab \in E(\tilde{G})$. Then, in $H_{k,t}$, the vertices $a_o$ and $b_i$ are joined by a directed path of length $k$. Therefore $c(a_o) \neq c(b_i)$. By Observation 4.3, $c(a_o) = c(b_i)$. Therefore, $c(a_i) \neq c(b_i)$, and $\phi(a) \neq \phi(b)$. This proves the claim. Hence $G$ is $t$-colourable.

Suppose now that $G$ has a $t$-colouring, $\phi$. We construct a $k$-dipath $t$-colouring, $c$, of $H_{k,t}$. For each vertex $v \in V(G)$, set $c(v_o) = c(v_i) = \phi(v)$. We claim that $c$ can be completed to a $k$-dipath $t$-colouring of $H_{k,t}$. Since, for every $v \in V(G)$, if $\phi(v_o) = i$, then vertices of the transitive tournament corresponding to $v$ together with the vertices $x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}$ can be assigned colours from the set $\{1, 2, 3, \ldots, t\} \setminus \{i\}$ so that the resulting partial colouring has the property that no two vertices at weak distance at most $k$ are assigned the same colour. To complete the $k$-dipath $t$-colouring, we must assign colours to the vertices of $H_{k,t}$ corresponding to the arcs of $\tilde{G}$. Since each vertex $v_i$ has out-degree 0, for every $ab \in E(\tilde{G})$, if $\phi(a_o) = i$ and $\phi(b_i) = j$, then vertices $a_{b_1}, a_{b_2}, a_{b_3}, \ldots, a_{b_{k-1}}$ can be assigned colours from
the set \( \{1, 2, 3, \ldots, t\} \setminus \{i, j\} \) so that the result is a \( k \)-dipath \( t \)-colouring of \( H_{k,t} \). This proves the claim. \( \square \)

**Theorem 4.5.** Let \( t \) and \( k \geq 3 \) be fixed positive integers. When restricted to instances of directed girth at least \( k+1 \), the decision problem \( k \)-DIPATH \( t \)-COLOURING is NP-complete if \( t > k \), and Polynomial if \( t \leq k \).

*Proof.* The problem is clearly in NP. If \( t > k \) then NP-completeness follows from Lemma 4.4. Suppose \( t \leq k \). By Corollary 3.12, an oriented graph \( G \) with directed girth at least \( k+1 \) has a \( k \)-dipath \( t \)-colouring if and only if it admits a homomorphism to the transitive tournament \( T_t \). Homomorphism to \( T_t \) can be checked in polynomial time [1]. The result now follows. \( \square \)

We now show that if the girth restriction is removed, then the dichotomy changes.

**Theorem 4.6.** If \( k > 2 \) and \( t > 2 \), then \( k \)-DIPATH \( t \)-COLOURING is NP-complete. If \( k \leq 2 \) or \( t \leq 2 \), then \( k \)-DIPATH \( t \)-COLOURING is Polynomial.

*Proof.* If \( t > k > 2 \), the result follows from Theorem 4.5. If \( k = 2 \), the result follows from Theorem 4.1. Hence assume \( k \geq 3 \) and \( t \leq k \). Observe that if \( t < k \), then no oriented graph with a \( k \)-dipath \( t \)-colouring can have a directed path of length greater than \( t \). Thus, we may further assume that \( t = k \geq 3 \).

We have previously noted that the problem belongs to NP. The transformation is from \( t \)-colouring. Suppose an instance of \( t \)-colouring, a simple, undirected graph \( G \) is given. We will replace each edge of \( G \) by the oriented graph which we now construct.

Let \( C \) be the directed cycle of length \( t \) with vertex sequence \( v_0, v_1, \ldots, v_{t-1}, v_0 \). Add four new vertices \( x_0, x_1, y_0, y_1 \), and arcs to make the directed paths \( x_0, x_1, v_1 \) and \( y_0, y_1, v_2 \). This completes the construction of \( F \). Observe that \( x_0 \) is joined to every vertex of \( C \) except \( v_0 \) by a directed path of length at most \( t \). Every vertex of \( C \) is assigned a different colour in a \( k \)-dipath \( t \)-colouring of \( F \). Hence, the vertices \( x_0 \) and \( v_0 \) must be assigned the same colour. Similarly, the vertices \( y_0 \) and \( v_1 \) must be assigned the same colour. In particular, \( x_0 \) and \( y_0 \) must be assigned different colours. Furthermore, any assignment of two different colours to \( x_0 \) and \( y_0 \) can be extended to a \( k \)-dipath \( t \)-colouring of \( F \).

Construct an oriented graph \( G' \) from \( G \) as follows. Replace each edge \( xy \) of \( G \) with a copy \( F_{xy} \) of \( F \) by identifying \( x_0 \) with \( x \) and \( y_0 \) with \( y \). The construction can clearly be carried out in polynomial time. Observe that each vertex in \( V(G) \cap V(G') \) has in-degree 0.

We claim that \( G \) is \( t \)-colourable if and only if \( G' \) has a \( k \)-dipath \( t \)-colouring. Suppose that a \( t \)-colouring of \( G \) is given. For each copy of \( F \) in \( G' \), this assignment gives different colours to \( x_0 \) and \( y_0 \). By our earlier observation, this assignment can be extended to a \( k \)-dipath \( t \)-colouring of \( F \). Since each vertex in \( V(G) \cap V(G') \) has in-degree 0, there is no directed path of positive length joining vertices in different copies of \( F \), this results in a \( k \)-dipath \( t \)-colouring of \( G' \). Now suppose a \( k \)-dipath \( t \)-colouring of \( G' \) is given. By our observation on colourings of \( F \), any two vertices which are adjacent in \( G \) are assigned different colours. Hence restricting this colouring to \( V(G) \) gives a \( t \)-colouring of \( G \). This completes the proof. \( \square \)
The problem of deciding whether the \( k \)-th power of a graph \( G \) is \( t \)-colourable is known to be NP-complete \[11\, 9\]. The results above imply that, if \( t > k \), the problem of deciding whether the underlying graph of the \( k \)-th power of an oriented graph \( G \) (i.e. two vertices are adjacent if and only if they are at weak distance at most \( k \)) is \( t \)-colourable is NP-complete, even when restricted to powers of oriented graphs with directed girth at least \( k + 1 \), and also that, if \( t = k \geq 3 \), the problem of deciding whether the \( k \)-th power of an oriented graph \( G \) is \( t \)-colourable is NP-complete.

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