On universality of homogeneous Euler equation

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Abstract

Master character of the multidimensional homogeneous Euler equation is discussed. It is shown that under restrictions to the lower dimensions certain subclasses of its solutions provide us with the solutions of various hydrodynamic type equations. Integrable one dimensional systems in terms of Riemann invariants and its extensions, multidimensional equations describing isoenthalpic and polytropic motions and shallow water type equations are among them.

1 Introduction

Homogeneous Euler equation (also called pressureless Euler equation)

\[ \frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} = 0, \quad i = 1,\ldots,n, \tag{1.1} \]

is one of the basic equations in the theories of fluids, gas and other media at \( n = 3 \) (see e.g. [9, 8, 17]). In spite of the fact that it represents the most simplified version (no pressure, no viscosity etc. [9, 8]) of the full equations, it arises in number of studies in many branches of physics.

Euler equation (1.1) has the remarkable property to be solvable by the straightforward multi-dimensional extension of the classical hodograph equations method [2, 3]. This fact and reductions to lower number of dependent variables has been used to establish the interrelations between equations (1.1) and multi-dimensional Monge-Ampère equations and Bateman equations [3, 4, 5, 10].

In the present paper we will study the restrictions of the \( n \)-dimensional Euler type equation to the lower dimensional spaces \( \mathbb{R}^m \) \((m < n)\). We start with the slightly modified equation (1.1), namely, with the system

\[ \frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} \frac{\beta_k}{\alpha_k} \lambda_k(u) \frac{\partial u_i}{\partial x_k} = 0, \quad i = 1,\ldots,n, \tag{1.2} \]

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where \( \lambda_k(u) = \lambda_k(u_1, \ldots, u_n) \) are arbitrary real-valued functions and \( \alpha_k, \beta_k \) are arbitrary real constants. Solutions of the system \((1.2)\) are provided by the hodograph equations

\[
\alpha_i x_i - \beta_i \lambda_i(u)t + f_i(u) = 0, \quad i = 1, \ldots, n,
\]

where \( f_k(u) = f_k(u_1, \ldots, u_n) \) are arbitrary real-valued functions (in the case \( \lambda_i(u) = u_i \) see 2, 3). Constants \( \alpha_i, \beta_i \) are, obviously, transformable away except the cases when some of them vanish. Exactly such cases are related with restrictions of the system \((1.2)\).

It is shown that restrictions on independent variables \( x_i \), functions \( f_i \) and parameters \( \alpha_i, \beta_i \) in \((1.3)\) gives rise to the various hydrodynamical type systems in the spaces \( \mathbb{R}^m \) \((m \leq n)\). In particular, under the restriction \( x_1 = x_2 = \cdots = x_n \equiv x \) plus certain restrictions on \( g_i \), the hodograph equations \((1.3)\) provides us with the solutions of the system

\[
\frac{\partial u_k}{\partial t} + \lambda_i(u) \frac{\partial u_i}{\partial x} = 0, \quad i = 1, \ldots, n,
\]

that is the classical diagonalized one-dimensional system in terms of Riemann invariants solvable by Tsarev’s generalized hodograph method 16, 1.

Under the restriction to the \((n-1)\)-dimensional subspace given by \( x_n = 0 \), with constraints on functions \( f_i \), the hodograph equations \((1.3)\) provide us with solutions of \((n-1)\)-dimensional hydrodynamical type systems of certain interest. First example is given by equations

\[
\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial v}{\partial x_i}, \quad i = 1, \ldots, n-1,
\]

\[
\frac{\partial v}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial v}{\partial x_k} = 0,
\]

which describes the adiabatic and isoenthalpic motion where \( v = TS \) and \( T \) is temperature and \( S \) is entropy. In the particular case \( f_i = \frac{\partial W}{\partial u_i} \), \( i = 1, \ldots, n \), this system describes the potential motion \( \left(u_i = \frac{\partial \phi}{\partial x_i}\right)\). Second case describes the polytropic motion, namely,

\[
\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i}, \quad i = 1, \ldots, n-1,
\]

\[
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{n-1} \frac{\partial (\rho u_k)}{\partial x_k} = 0,
\]

where the density \( \rho = u_n \) and the pressure \( p = \rho^\gamma \). In this case the functions \( f_i = \frac{\partial W}{\partial u_i} \) and the function \( W \) obeys a determinant type PDE.

Natural two and higher dimensional extensions of the system \((1.4)\) are considered too.

It is noted that solutions of the systems \((1.4)\)-(1.6) and others are obtained in our approach are given by subclasses of solutions of the original system \((1.2)\) which are characterized by the specific choices of functions \( f_i \) and restrictions on the coordinates \( x_1, \ldots, x_n \).

Interrelations between infinite-dimensional Euler equation \((1.2)\) and Burgers and Korteweg-de Vries equations is considered. We have also discussed the phenomenon of the gradient catastrophe for homogeneous Euler equation. It is shown that it first happens at a point on the \( n \)-dimensional hypersurface.

The paper is organized as follows. In section 2 hodograph equations for the generalized homogeneous Euler equations and its one-dimensional reductions are considered. The \((n-1)\)-dimensional
reductions of the homogeneous Euler equations to the \((n-1)\)-dimensional equations describing isoenthalpic motion and its potential version are studied in sections \(3\) and \(4\). Reduction describing polytropic motion is discussed \(5\). Reductions of the three-dimensional Euler equation are considered in section \(6\). Interrelation between infinite-dimensional Euler equation and Burgers and Korteweg-de Vries equations is analyzed in section \(7\). In section \(8\) we discuss the gradient catastrophe for the homogeneous Euler equation.

\section{Generalized homogeneous Euler equation and its one-dimensional reductions}

We start with hodograph equations

\begin{equation}
\alpha_i x_i - \beta_i \lambda_i(u) t + f_i(u) = 0, \quad i = 1, \ldots, n,
\end{equation}

where \(\lambda_i(u)\) and \(f_i(u)\) are arbitrary real valued functions and \(\alpha, \beta_i\) are arbitrary constants. The system \(2.1\) is an obvious extension of the hodograph equations considered in \([2, 3, 4]\).

Differentiating \(2.1\) w.r.t. \(x_k\) and \(t\), one obtains

\begin{equation}
\alpha_i \delta_{ik} + \sum_{l=1}^{n} \frac{\partial g_i}{\partial u_l} \frac{\partial u_l}{\partial x_k} = 0, \quad i, k = 1, \ldots, n.
\end{equation}

and

\begin{equation}
- \beta_i \lambda_i + \sum_{l=1}^{n} \frac{\partial g_i}{\partial u_l} \frac{\partial u_l}{\partial t} = 0, \quad i, k = 1, \ldots, n.
\end{equation}

where \(g_i \equiv \alpha_i x_i - \beta_i \lambda_i(u) t + f_i(u)\).

Relations \(2.2\) and \(2.3\) imply that

\begin{equation}
\frac{\partial u_l}{\partial x_k} = -(A^{-1})_{lk} \alpha_k, \quad i, k = 1, \ldots, n.
\end{equation}

and

\begin{equation}
\frac{\partial u_l}{\partial t} = \sum_{k=1}^{n} (A^{-1})_{lk} \beta_k \lambda_k, \quad l = 1, \ldots, n.
\end{equation}

where the matrix \(A\) has elements

\begin{equation}
A_{lk} = \frac{\partial g_l}{\partial u_k} = -t \frac{\partial \lambda_l}{\partial u_k} + \frac{\partial f_l}{\partial u_k}, \quad l, k = 1, \ldots, n
\end{equation}

and it is assumed that \(\det(A) \neq 0\).

Combining \(2.1\) and \(2.2\), one gets

\begin{equation}
\frac{\partial u_l}{\partial t} + \sum_{k=1}^{n} \frac{\beta_k}{\alpha_k} \lambda_l \frac{\partial u_l}{\partial x_k} = 0, \quad l = 1, \ldots, n.
\end{equation}

For \(\lambda_k = u_k\), \(\alpha_k = \beta_k = 1\) these calculations has been done in \([2, 3, 4]\).

Relations \(2.2\) and \(2.3\) also imply that

\begin{equation}
\sum_{l=1}^{n} A_{il} \left( \frac{\partial u_l}{\partial t} + \sum_{k=1}^{n} \frac{\beta_k}{\alpha_k} \frac{\partial u_l}{\partial x_k} \right) = 0, \quad i = 1, \ldots, n.
\end{equation}
It is noted that functions $f_i(u)$ are arbitrary one in this construction. They are related to initial data at $t = 0$ via

$$\alpha_i x_i + f_i(u(t = 0)) = 0. \quad (2.9)$$

Hence, hodograph equations (2.1) provide us with the general solutions of the system (2.7).

We would like to note that the system (2.7) and the original Euler equation (1.1), in fact, are equivalent. Indeed it is easy to see that if $u_i$ obey the system (2.7) then $\lambda_k$ obey the system (1.1) and vice versa ($u_i \rightarrow \lambda_i(u)$). It is noted also that the systems (2.7) with different $\lambda_k(u)$ pairwise commute. So one has an infinite hierarchy of equations of the form (2.7).

Now let us consider the simplest reductions of the system (2.7) for which the matrix $A$ is diagonal one, i.e.

$$A_{ik} = -t \frac{\partial \lambda_l}{\partial u_k} + \frac{\partial f_l}{\partial u_k} = 0, \quad l \neq k. \quad (2.10)$$

A way to satisfy this condition is to impose the constraints

$$\frac{\partial \lambda_l}{\partial u_k} = 0, \quad \frac{\partial f_l}{\partial u_k} = 0, \quad l \neq k. \quad (2.11)$$

In this case the relations (2.2) imply that

$$\frac{\partial u_l}{\partial u_k} = 0, \quad l \neq k, \quad (2.12)$$

and the $n$-dimensional system (2.7) (with $\alpha_k = \beta_k = 1$) is decomposed into $n$ decoupled one-dimensional Burgers-Hopf type equations

$$\frac{\partial u_l}{\partial t} + \lambda_l(u) \frac{\partial u_l}{\partial x} = 0, \quad l = 1, \ldots, n. \quad (2.13)$$

Less trivial reduction with the diagonal matrix $A$ arises if one considers the restriction to the one-dimensional subspace given by the condition $x_1 = x_2 = \cdots = x_n \equiv x$. In this reduction, the hodograph system (2.1) assumes the form

$$x - \lambda_i(u) t + f_i(u) = 0, \quad i = 1, \ldots, n, \quad (2.14)$$

and one obtains

$$\frac{\partial u_l}{\partial x} = -\frac{1}{\partial f_l/\partial u_l}, \quad \frac{\partial u_l}{\partial t} = \lambda_l \frac{\partial f_l}{\partial u_l}, \quad l = 1, \ldots, n. \quad (2.15)$$

Hence, one has the system

$$\frac{\partial u_l}{\partial t} + \lambda_l(u) \frac{\partial u_l}{\partial x} = 0, \quad l = 1, \ldots, n. \quad (2.16)$$

The equation (2.14) implies that

$$t = \frac{f_i - f_l}{\lambda_i - \lambda_l}, \quad i \neq l. \quad (2.17)$$

Consequently the condition (2.10) is equivalent to the following one

$$\frac{\partial f_l}{\partial u_k} = \frac{\partial \lambda_l}{\partial u_k} \frac{\lambda_l - \lambda_k}{f_l - f_k}, \quad l \neq k. \quad (2.18)$$
Equations (2.16) represent the well known form of the one-dimensional multi-component hydrodynamic type systems in terms of Riemann invariants (see e.g. [9, 17]). Hodograph equation (2.14) and condition (2.18) are exactly those of the Tsarev generalized hodograph method [16, 11].

So, solutions of the homogeneous Euler equations (2.7), for which functions \( f_l, l = 1, \ldots, n \) in (2.1) are selected according to the condition (2.18), after the restriction to the one-dimensional subspace \( x_1 = \cdots = x_n \) become the solutions of the system (2.16).

It is noted that reduction of the homogeneous Euler equation to the system (2.16) arises also for other one-dimensional restrictions of the \( n \)-dimensional space \( (x_1, \ldots, x_n) \), for instance, given by \( x_2 = x_3 = \cdots = x_n = 0 \). In these cases the characterizations of functions \( f_l \) are quite different from (2.18).

### 3 \((n - 1)\)-dimensional reductions: Jordan system

Here we will consider reductions of the Euler system (2.1) with \( \lambda_k(u) = u_k \) to the \((n - 1)\)-dimensional subspace defined by the restriction \( x_n = 0 \). It is equivalent to require \( \alpha_n = 0 \) in the hodograph equations (2.1). The relation (2.4) implies that \( \frac{\partial u_l}{\partial x_n} = 0, \ l = 1, \ldots, n \) under this restriction, however due to (2.4)

\[
\frac{1}{\alpha_n} \frac{\partial u_l}{\partial x_n} = - (A^{-1})_{ln} \neq 0, \quad l = 1, \ldots, n. \tag{3.1}
\]

Using this relation, one rewrites equation (2.7) as \((\alpha_k = \beta_k = 1, k = 1, \ldots, n - 1, \beta_n = 1)\)

\[
\frac{\partial u_l}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_l}{\partial x_k} - u_n (A^{-1})_{ln} = 0, \quad l = 1, \ldots, n - 1
\]

\[
\frac{\partial u_n}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_n}{\partial x_k} - u_n (A^{-1})_{nn} = 0. \tag{3.2}
\]

Under the requirements

\[
(A^{-1})_{ln} = - \frac{\partial u_n}{\partial x_l}, \quad l = 1, \ldots, n - 1 \tag{3.3}
\]

and

\[
(A^{-1})_{nn} = 0 \tag{3.4}
\]

the system (3.2) assumes the form

\[
\frac{\partial u_l}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_l}{\partial x_k} + u_n \frac{\partial u_n}{\partial x_l} = 0, \quad l = 1, \ldots, n - 1
\]

\[
\frac{\partial u_n}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_n}{\partial x_k} = 0. \tag{3.5}
\]

In terms of variables \( u_i, i = 1, \ldots, n - 1 \) and \( v = u_n^2/2 \), the system looks like

\[
\frac{\partial u_l}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_l}{\partial x_k} + \frac{\partial v}{\partial x_l} = 0, \quad l = 1, \ldots, n - 1
\]

\[
\frac{\partial v}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial v}{\partial x_k} = 0. \tag{3.6}
\]
The system (3.6) represents the \((n - 1)\)-dimensional generalization of the one-dimensional \((n = 2)\) Jordan system introduced in [6].

The system (3.6) at \(n = 4\) arises also in physics. Indeed, hydrodynamical equations describing adiabatic flow of an ideal fluid are of the form [8, 9]

\[
\frac{\partial u_l}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial u_l}{\partial x_k} + \frac{1}{\rho} \frac{\partial P}{\partial x_l} = 0, \quad l = 1, 2, 3
\]

\[
\frac{\partial S}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial S}{\partial x_k} = 0.
\]

(3.7)

where \(\rho\) is the fluid density, \(P\) stands for pressure and \(S\) is the entropy. The variation of enthalpy \(W\) is given by (see e.g. [9])

\[
\frac{\partial W}{\partial x_i} = T \frac{\partial S}{\partial x_i} + \frac{1}{\rho} \frac{\partial P}{\partial x_i}, \quad i = 1, 2, 3
\]

(3.8)

So for the isoenthalpic motion with constant temperature one has

\[
\frac{1}{\rho} \frac{\partial P}{\partial x_i} = \frac{\partial}{\partial x_i} (T S), \quad i = 1, 2, 3
\]

(3.9)

and, consequently, one immediately concludes that the system (3.6) at \(n = 4\) and \(v = -TS\) describes the adiabatic and isoenthalpic motion of a fluid at constant temperature.

Now let us analyze the conditions (3.3) and (3.4). The relation (2.4) says that

\[
\frac{\partial u_l}{\partial x_l} = -(A^{-1})_{nl}, \quad l = 1, \ldots, n - 1,
\]

(3.10)

and so the condition (3.3) is satisfied if

\[
(A^{-1})_{nl} = (A^{-1})_{ln}, \quad l = 1, \ldots, n - 1.
\]

(3.11)

Thus, equations (2.7) are reducible to (3.6) if the matrix \(A_{lk} = \frac{\partial g_l}{\partial u_k}\) obeys the constraints (3.11), (3.4) or, equivalently

\[
\tilde{A}_{ln} = \tilde{A}_{nl}, \quad l = 1, \ldots, n - 1
\]

\[
\tilde{A}_{nn} = 0,
\]

(3.12)

where \(\tilde{A}\) is the matrix adjugate to \(A\) (i.e. \(A\tilde{A} = \det(A)I_n\)).

Using the known formula for the adjugate matrix \(\tilde{A}\), one can obtain more explicit form of the conditions (3.12). We instead will use an explicit form of the matrix \(A^{-1}\). Indeed, since \(A_{lk} = \frac{\partial u_l}{\partial g_k}\), one has

\[
(A^{-1})_{lk} = \frac{\partial u_l}{\partial g_k}, \quad l, k = 1, \ldots n.
\]

(3.13)

Using (3.13), one rewrites (3.12) as

\[
\frac{\partial u_l}{\partial g_n} = \frac{\partial u_n}{\partial g_l}, \quad l = 1, \ldots, n - 1
\]

\[
\frac{\partial u_n}{\partial g_n} = 0.
\]

(3.14)
Conditions (3.14) imply that

\[ u_l = \frac{\partial}{\partial g_l} \phi(g_1, \ldots, g_n) + A_l(g_1, \ldots, g_{n-1}), \quad l = 1, \ldots, n - 1, \tag{3.15} \]

\[ u_n = \frac{\partial}{\partial g_n} \phi(g_1, \ldots, g_n), \]

where \( \phi(g_1, \ldots, g_n) \) and \( A_l(g_1, \ldots, g_{n-1}) \) are arbitrary functions.

Consider now 1–form \((x_i, i = 1, \ldots, n)\) and \( t \) are fixed

\[ \Omega = \sum_{l=1}^{n} u_l dg_l = d\phi + \sum_{l=1}^{n-1} A_l(g_1, \ldots, g_{n-1}) dg_l \tag{3.16} \]

and perform the Legendre transformation defined by

\[ \sum_{l=1}^{n} g_l du_l = d \left( \sum_{l=1}^{n} u_l g_l \right) - \Omega. \tag{3.17} \]

Due to (3.16) one has

\[ \sum_{l=1}^{n} g_l du_l = dW - \sum_{l=1}^{n-1} A_l(g_1, \ldots, g_{n-1}) dg_l, \tag{3.18} \]

where

\[ W = \left( \sum_{l=1}^{n} u_l g_l \right) - \phi. \tag{3.19} \]

Equation (3.18) rewritten as

\[ \sum_{l=1}^{n} \left( g_l - \frac{\partial W}{\partial u_l} + \sum_{k=1}^{n-1} A_k \frac{\partial g_k}{\partial u_l} \right) du_l = 0, \tag{3.20} \]

implies that

\[ g_l = \frac{\partial W}{\partial u_l} - \sum_{k=1}^{n-1} A_k \frac{\partial g_k}{\partial u_l}, \quad l = 1, \ldots, n. \tag{3.21} \]

The compatibility condition for (3.21) (i.e. \( \partial^2 W/\partial u_l \partial u_k = \partial^2 W/\partial u_k \partial u_l \)) is given by

\[ \frac{\partial g_l}{\partial u_m} - \frac{\partial g_m}{\partial u_l} + \sum_{k,i=1}^{n-1} \left( \frac{\partial A_k}{\partial g_i} - \frac{\partial A_i}{\partial g_k} \right) \frac{\partial g_i}{\partial u_m} \frac{\partial g_k}{\partial u_l} = 0, \quad l, m = 1, \ldots, n. \tag{3.22} \]

Correspondingly for \( f_l \) one has

\[ f_l = \frac{\partial \tilde{W}}{\partial u_l} - \sum_{k=1}^{n-1} \tilde{A}_k \frac{\partial f_k}{\partial u_l}, \quad l = 1, \ldots, n. \tag{3.23} \]

and

\[ \frac{\partial f_l}{\partial u_m} - \frac{\partial f_m}{\partial u_l} + \sum_{k,i=1}^{n-1} \left( \frac{\partial \tilde{A}_k}{\partial g_i} - \frac{\partial \tilde{A}_i}{\partial g_k} \right) \frac{\partial f_i}{\partial u_m} \frac{\partial f_k}{\partial u_l} = 0, \quad l, m = 1, \ldots, n, \tag{3.24} \]

where \( \tilde{W}(u_1, \ldots, u_n) \) and \( \tilde{A}_l(f_1, \ldots, f_{n-1}) \), \( l = 1, \ldots, n - 1 \) are arbitrary functions.

Equations (3.21) and (3.22) or (3.23) and (3.24) plus the condition (3.4) characterise those solutions of the homogeneous Euler equations which are, at the same time, solutions of the \((n-1)\)-dimensional Jordan system (3.6) or the system (3.7). It is noted that in this case the class of solutions is parametrized by one arbitrary function \( \tilde{W}(u_1, \ldots, u_n) \) (or \( W \)) on \( n \) variables and \((n-1)\) functions \( \tilde{A}_l(f_1, \ldots, f_{n-1}) \) of \( n-1 \) variables.
4 Potential flows

The formulae presented in the previous section are simplified drastically in the particular case when

\[ \frac{\partial A_k}{\partial g_i} = \frac{\partial A_i}{\partial g_k}, \quad i = 1, \ldots, n - 1, \quad (4.1) \]

or, consequently

\[ A_i = \frac{\partial \psi}{\partial g_i}, \quad i = 1, \ldots, n - 1, \quad (4.2) \]

where \( \psi \) is an arbitrary function. So \( g_l = \frac{\partial}{\partial u_l}(W - \psi) \). Equivalently without loss of generality one can put directly \( A_i \equiv 0 \). In this case

\[ g_l = \frac{\partial W}{\partial u_l}, \quad l = 1, \ldots, n, \quad (4.3) \]

and the matrix \( A \) is of the form

\[ A_{lk} = \frac{\partial^2 W}{\partial u_l \partial u_k}, \quad l, k = 1, \ldots, n. \quad (4.4) \]

Then, one has

\[ (A^{-1})_{nn} = \frac{\text{det } B}{\text{det } A}, \quad (4.5) \]

where \( (n - 1) \times (n - 1) \) matrix \( B \) is the algebraic complement to the element \( A_{nn} \), i.e.

\[ B_{lk} = \frac{\partial^2 W}{\partial u_l \partial u_k}, \quad l, k = 1, \ldots, n - 1. \quad (4.6) \]

So, the condition (3.4) assumes the form

\[ \text{det}(B) = 0. \quad (4.7) \]

The form (4.4) of the matrix \( A \) leads to a constraint of the variables \( u_i, i = 1, \ldots, n \). Indeed, since the matrix \( A \) (4.4) is symmetric one, then the matrix \( A^{-1} \) is symmetric too. In such a case the relations (2.4) imply that

\[ \frac{\partial u_l}{\partial x_k} = \frac{\partial u_k}{\partial x_l}, \quad k, l = 1, \ldots, n - 1. \quad (4.8) \]

So

\[ u_i = \frac{\partial \phi}{\partial x_i}, \quad i = 1, \ldots, n - 1, \quad (4.9) \]

where \( \phi(x_1, \ldots, x_{n-1}) \) is some function. Thus in this case the equations (3.6) or (3.7) describe the potential adiabatic isoenthalpic flows.

Due to (4.9) equations (3.6) are equivalent to the following (assuming that all constants of integration vanish)

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \sum_{k=1}^{n-1} \left( \frac{\partial \phi}{\partial x_k} \right)^2 + v = 0, \]

\[ \frac{\partial v}{\partial t} + \sum_{k=1}^{n-1} \frac{\partial \phi}{\partial x_k} \frac{\partial v}{\partial x_k} = 0. \quad (4.10) \]
The first equation (4.10) is well known in hydrodynamics, e.g. for the isoentropic potential motion (see [9, §5,109]).

In our case the elimination of \( v \) from the system (4.10) gives us the following equation for the velocity potential \( \phi \)

\[
\frac{\partial^2 \phi}{\partial t^2} + 2 \sum_{k=1}^{n-1} \frac{\partial \phi}{\partial x_k} \frac{\partial^2 \phi}{\partial x_k \partial t} + \sum_{i,k=1}^{n-1} \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial x_k \partial x_i} = 0. \tag{4.11}
\]

Solutions of this equation provide us the solutions of the system (4.10) via

\[
v = -\frac{\partial \phi}{\partial t} - \frac{1}{2} \sum_{k=1}^{n-1} \left( \frac{\partial \phi}{\partial x_k} \right)^2.
\]

The system (4.10) can be viewed also as the Hamilton-Jacobi equation given by the first of equations (4.10) for the action \( \phi \) with the time-dependent potential \( v(x_1,\ldots,x_{n-1};t) \) obeying the second equation (4.10). So the solutions of the system (4.10) provide us with a solvable \((n-1)\)-dimensional system of classical mechanics.

In the one dimensional case \( n = 2 \) the equation (4.11) is of the form

\[
\frac{\partial^2 \phi}{\partial t^2} + 2 \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x \partial t} + \left( \frac{\partial \phi}{\partial x} \right)^2 = 0. \tag{4.12}
\]

or

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial t} + \left( \frac{\partial \phi}{\partial x} \right)^2 \right) + \frac{\partial}{\partial x} \left( \frac{1}{3} \left( \frac{\partial \phi}{\partial x} \right)^3 \right) = 0. \tag{4.13}
\]

and it is of parabolic type.

Solutions of the system (4.10) and equation (4.11) are provided by hodograph equations (2.1) with function \( g_l \) obeying (4.3) and (4.6), (4.7).

One can obtain the corresponding expression also for the functions \( f_l, l = 1,\ldots,n \). Indeed since \( g_l = x_l - u_l t + f_l \), the 1-form

\[
\tilde{\Omega} = \sum_{l=1}^{n} f_l du_l = -\Omega^* - d \left( \sum_{l=1}^{n} x_l u_l - \frac{t}{2} \sum_{l=1}^{n} u_l^2 \right) \tag{4.14}
\]

is closed due to the fact that \( \Omega^* = dW \). Consequently one has

\[
f_l = \frac{\partial \tilde{W}}{\partial u_l}, \quad l = 1,\ldots,n \tag{4.15}
\]

for some function \( \tilde{W}(u_1,\ldots,u_n) \). In terms of the function \( \tilde{W} \) the condition (4.7) looks like

\[
\det(\tilde{B}) = 0. \tag{4.16}
\]

where

\[
\tilde{B}_{lk} = \frac{\partial^2 \tilde{W}}{\partial u_l \partial u_k} - t \delta_{lk}, \quad l, k = 1,\ldots,n - 1. \tag{4.17}
\]

Thus, for the \((n-1)\)-dimensional Jordan system (3.6) and the system (3.7) and (3.9), hodograph equations (2.1) \((\lambda_i = u_i)\) are the equations \( \frac{\partial W}{\partial u_i} = 0, \quad i = 1,\ldots,n \) defining the critical points of functions \( W = W(u,\bar{x},t) \) of the form \( v = u_n^2/2 \)

\[
W = \sum_{i=1}^{n-1} x_i u_i - t \left( v + \frac{1}{2} \sum_{i=1}^{n-1} u_i^2 \right) + \tilde{W}(u,v) \tag{4.18}
\]
with functions $\tilde{W}$ obeying the constraint (4.10) at the critical points. The potential Jordan system (3.6) describes the dynamics of the critical points of such functions $W$.

In other words, solutions of the potential Jordan system (3.6) or equations (3.7) and (3.9) are those solutions of the homogeneous Euler equation (2.7) which corresponds to the functions $f_i$ in (2.1) being the components of the gradients of functions $W$ of the form (4.18) with obeying the constraints (4.16).

The condition that $\det(\tilde{B})$ belongs to the ideal generated by the functions $\frac{\partial W}{\partial u_i}$, $i = 1, \ldots, n$ is the sufficient one to characterize the above subclasses of solutions. However, more explicit description of the class of the functions $W$ would be, definitely, rather convenient.

For this purpose we first observe that at the critical point $t = \frac{\partial \tilde{W}}{\partial v}$. So the matrix $\tilde{B}$ can be equivalently rewritten as

$$\tilde{B}_{lk} = \frac{\partial^2 \tilde{W}}{\partial u_l \partial u_k} - \delta_{lk} \frac{\partial \tilde{W}}{\partial v}, \quad l, k = 1, \ldots, n - 1.$$ (4.19)

Then since

$$\frac{\partial^2 \tilde{W}}{\partial u_l \partial u_k} - \delta_{lk} \frac{\partial \tilde{W}}{\partial v} = \frac{\partial^2 W}{\partial u_l \partial u_k} - \delta_{lk} \frac{\partial W}{\partial v},$$ (4.20)

the condition (4.16) is equivalent to

$$\det \left( \frac{\partial^2 W}{\partial u_l \partial u_k} - \delta_{lk} \frac{\partial W}{\partial v} \right) = 0.$$ (4.21)

The last step is to extend this condition outside the critical points and to consider (4.21) as the equation defining the function $W$ of the form (4.18). For such functions conditions (4.16) and (4.17) are automatically satisfied at the critical points. Note that the formula (4.18) and (4.21) are natural $(n - 1)$-dimensional extensions of the corresponding formulae $\frac{\partial W}{\partial v} = \frac{\partial^2 W}{\partial u_1^2}$ for the one-dimensional Jordan system [6].

We note also that one gets the same system (3.6) considering the restrictions to the $(n - 1)$-dimensional subspaces defined by conditions $x_i = x_k$ with fixed $i$ and $k$. In these cases one has characterizations of the functions $f_i$ similar to those considered above.

Finally we note that the subclass of the hodograph equations (2.1) with $g_i = \frac{\partial W}{\partial u_i}$, $i = 1, \ldots, n$ and, hence, $\lambda_i = \frac{\partial F}{\partial u_i}$, $i = 1, \ldots, n$ where $W$ and $F$ are some functions give us solutions of the potential reduction $u_i = \frac{\partial \phi}{\partial x_i}$, $i = 1, \ldots, n$ of the homogeneous Euler equation (2.7). In this case equation (2.7) is equivalent to the following ($\alpha_i = \beta_i = 1$, $i = 1, \ldots, n$)

$$\frac{\partial \phi}{\partial t} + F \left( \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n} \right) = 0.$$ (4.22)

### 5 $(n - 1)$-dimensional reductions: Polytropic gas

Now we will consider equations (5.2) with the constraints

$$(A^{-1})_{ln} = -u_n^a \frac{\partial u_n}{\partial x_l}, \quad l = 1, \ldots, n - 1,$$ (5.1)

and

$$(A^{-1})_{nn} = -\sum_{k=1}^{n-1} \frac{\partial u_k}{\partial x_k},$$ (5.2)
where $a$ is an arbitrary real number. Under these constraints equations (3.2) assume the form

$$
\frac{\partial u_l}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_l}{\partial x_k} = -\frac{1}{a + 2} \frac{\partial}{\partial x_l} \left( u_n^{a+2} \right), \quad l = 1, \ldots, n - 1,
$$

(5.3)

For $a = -1$ this system represents the shallow water equation in $(n - 1)$--dimension with $u_n$ being the fluid height $h$. For arbitrary $a$ it describes the polytropic motion with pressure $p = \frac{1}{a+3} \rho^{a+3}$ and the density $\rho = u_n$ (see e.g. [9, 17]).

The formula (2.4) implies that

$$
\frac{\partial u_n}{\partial x_l} = -(A^{-1})_n l, \quad l = 1, \ldots, n - 1,
$$

(5.4)

and

$$
\sum_{k=1}^{n-1} \frac{\partial u_k}{\partial x_k} = -\sum_{k=1}^{n-1} (A^{-1})_{kk}.
$$

(5.5)

So, the constraint (5.1) and (5.2) are equivalent to the following

$$(A^{-1})_{ln} = u_n^a (A^{-1})_nl, \quad l = 1, \ldots, n - 1$$

$$(A^{-1})_{nn} = \sum_{k=1}^{n-1} (A^{-1})_{kk}.$$

(5.6)

Using (3.13), one rewrites these constraints as

$$
\frac{\partial u_l}{\partial g_n} = u_n^a \frac{\partial u_n}{\partial g_l} = \frac{\partial}{\partial g_l} \left( u_n^{a+1} \right), \quad l = 1, \ldots, n - 1
$$

(5.7)

and

$$
\frac{\partial u_n}{\partial g_n} = \sum_{k=1}^{n-1} \frac{\partial u_k}{\partial g_k}.
$$

The first condition (5.7) imply that

$$
\frac{1}{a + 1} u_n^{a+1} = \frac{\partial \phi}{\partial g_n},
$$

(5.8)

where $\phi(g_1, \ldots, g_n)$ and $B_l(g_1, \ldots, g_{n-1})$ are arbitrary functions. On can find the corresponding formulae and constraints for $g_l$ in a way similar to that described in section 3. Here we will consider the simplest case $B_l(g_1, \ldots, g_{n-1}) = 0$, $l = 1, \ldots, n - 1$. In this case one has

$$
g_l = \frac{\partial W}{\partial u_l}, \quad l = 1, \ldots, n - 1,
$$

$$
g_n = u_n^{-a} \frac{\partial W}{\partial u_n},
$$

(5.9)

for some function $W$. 

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So the matrix $A$ is of the form

$$A = \begin{pmatrix} B & V_1 \\ V_2 & -u_n^{-a} \frac{\partial \tilde{W}}{\partial u_n} \end{pmatrix},$$  \hspace{1cm} (5.10)

where $B$ is an $(n-1) \times (n-1)$ matrix with elements $\frac{\partial^2 W}{\partial u_k \partial u_l}$, $V_1$ is a column with $(n-1)$ elements $u_n^{-a} \frac{\partial^2 W}{\partial u_n \partial u_l}$, $V_2$ is a row with $(n-1)$ elements $u_n^{-a} \frac{\partial^2 W}{\partial u_n \partial u_l}$. Hence, the second condition \((5.6)\) assumes the form

$$\det B - \sum_{k=1}^{n-1} \det C_k = 0,$$  \hspace{1cm} (5.11)

where $C_k$ are algebraic complements of the elements $A_{kk}$.

So, the solutions of the system \((5.3)\) describing polytropic motion are those solutions of the homogeneous $n$–dimensional Euler equation which correspond to the choice \((5.9)\) of functions $g_l$ with $W$ obeying the constraint \((5.11)\) on the manifold $g_l = 0, i = 1, \ldots n$.

Analogously to the previous section one can show that the functions $f_l$ are given by

$$f_l = \frac{\partial \tilde{W}}{\partial u_l}, \quad l = 1, \ldots, n - 1, \quad (5.12)$$

$$f_n = u_n^{-a} \frac{\partial \tilde{W}}{\partial u_n},$$

for some function $\tilde{W}$. So the function $W$ is of the form

$$W = \sum_{i=1}^{n-1} x_i u_i - t \left( \frac{1}{2} \sum_{k=1}^{n-1} u_k^2 + \frac{1}{a+2} u_n^{a+2} \right) + \tilde{W}.$$  \hspace{1cm} (5.13)

Since

$$\frac{\partial^2 W}{\partial u_k \partial u_l} = -t \delta_{lk} + \frac{\partial^2 \tilde{W}}{\partial u_k \partial u_l}, \quad l, k = 1 \ldots, n - 1 \quad (5.14)$$

and

$$t = u_n^{-1-a} \frac{\partial \tilde{W}}{\partial u_n},$$  \hspace{1cm} (5.15)

the condition \((5.11)\) is equivalent to

$$\det \left( \frac{\partial^2 \tilde{W}}{\partial u_l \partial u_k} - u_n^{-1-a} \delta_{lk} \frac{\partial \tilde{W}}{\partial u_n} \right) - \sum_{k=1}^{n-1} \det \tilde{C}_k = 0,$$  \hspace{1cm} (5.16)

where $\tilde{C}_k$ are principal minors of the matrix

$$\tilde{A}_{lk} = \frac{\partial^2 \tilde{W}}{\partial u_l \partial u_k} - u_n^{-1-a} \delta_{lk} \frac{\partial \tilde{W}}{\partial u_n}.$$  \hspace{1cm} (5.17)

Finally using the relation

$$\frac{\partial^2 W}{\partial u_l \partial u_k} - u_n^{-1-a} \delta_{lk} \frac{\partial W}{\partial u_n} = \frac{\partial^2 \tilde{W}}{\partial u_l \partial u_k} - u_n^{-1-a} \delta_{lk} \frac{\partial \tilde{W}}{\partial u_n}, \quad l, k = 1, \ldots, n - 1, \quad (5.18)$$
one can extend the equation (5.16) outside the critical points \( \frac{\partial W}{\partial u_i} = 0 \) to obtain the equation characterizing the function \( W \), i.e.

\[
\det \left( \frac{\partial^2 W}{\partial u_l \partial u_k} - u_n^{-1-a} \delta_{lk} \frac{\partial W}{\partial u_n} \right) - \sum_{k=1}^{n-1} \det C_k = 0 ,
\]

where \( C_k \) are principal minors of the matrix (5.17) with the substitution \( \tilde{W} \to W \).

In the simplest case \( n = 2 \) all above formulae become rather compact. For arbitrary \( a \) the function \( W \) is of the form

\[
W = xu - t \left( \frac{1}{2} u^2 + \frac{1}{a + 2} v^{a+2} \right) + \tilde{W}
\]

while the equation (5.19) becomes

\[
\frac{\partial^2 W}{\partial u^2} - v^{-a} \frac{\partial^2 W}{\partial v^2} + av^{-1-a} \frac{\partial W}{\partial v} = 0 .
\]

For \( a = -1 \) this equation is quite similar to that associated with is isentropic motion of fluid (see e.g. [9], §105). In the one dimensional case the system (5.3) is diagonalizable to the following

\[
\frac{\partial \Gamma_{\pm}}{\partial t} = \lambda_{\pm} \frac{\partial \Gamma_{\pm}}{\partial x}
\]

with the Riemann invariants \( \Gamma_{\pm} \) and the characteristics velocities \( \lambda_{\pm} \) given by [17]

\[
\Gamma_{\pm} = -\frac{a + 2}{2} u \pm v^{\frac{a+2}{2}} , \quad \lambda_{\pm} = -u \pm v^{\frac{a+2}{2}} .
\]

In terms of the Riemann invariants, the equation (5.21) becomes the classical Euler-Poisson-Darboux equation

\[
(\Gamma_+ - \Gamma_-) \frac{\partial^2 W}{\partial u \partial v} = -\frac{a}{a + 2} \left( \frac{\partial W}{\partial \Gamma_+} - \frac{\partial W}{\partial \Gamma_-} \right) .
\]

For the classical shallow water equation \( a = -1 \) equation (5.24) coincides with that studied in [7].

We see that equations characterizing functions \( W \) are nonlinear for multi-dimensional systems (3.6) and (5.3), in contrast to the one dimensional situation with the linear equation \( \frac{\partial W}{\partial v} = \frac{\partial^2 W}{\partial u^2} \) and equation (5.24). Such situation seems to be typical in applications of the hodograph equation to multi-dimensional PDEs [11, 12], except, of course, the master homogeneous Euler equation (2.7).

We note also that one can study in a similar manner the dimensional reductions of the generalised equation (2.7) with arbitrary function \( \lambda_i(u) \).

6 Reductions of the three-dimensional Euler equations

In this section we consider two particular examples of the Euler equation in three dimensions.

First, let us start with the two-dimensional restriction of the hodograph equation given by

\[
(x_1 = x_2 = x, x_3 = y),
\]

\[
x - \lambda_1 t + f_1 = 0 ,
\]

\[
x - \lambda_2 t + f_2 = 0 ,
\]

\[
y - \lambda_3 t + f_3 = 0 .
\]

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Differentiation of (6.1) w.r.t. $x, y$ and $t$ gives

$$\frac{\partial u_l}{\partial x} = -(A^{-1})_{l1} - (A^{-1})_{l2},$$
$$\frac{\partial u_l}{\partial y} = -(A^{-1})_{l3},$$
$$\frac{\partial u_l}{\partial t} = \sum_{k=1}^{3} (A^{-1})_{lk} \lambda_k, \quad l = 1, 2, 3. \tag{6.2}$$

Combining expressions (6.2), one obtains

$$\frac{\partial u_1}{\partial t} + \lambda_1 \frac{\partial u_1}{\partial x} + \lambda_3 \frac{\partial u_1}{\partial y} = (\lambda_2 - \lambda_1)(A^{-1})_{12},$$
$$\frac{\partial u_2}{\partial t} + \lambda_2 \frac{\partial u_2}{\partial x} + \lambda_3 \frac{\partial u_2}{\partial y} = (\lambda_1 - \lambda_2)(A^{-1})_{21}, \tag{6.3}$$
$$\frac{\partial u_3}{\partial t} + \lambda_3 \frac{\partial u_3}{\partial x} + \lambda_1 \frac{\partial u_3}{\partial y} = (\lambda_2 - \lambda_1)(A^{-1})_{32}.$$

Imposing the constraint

$$(A^{-1})_{12} = (A^{-1})_{21} = (A^{-1})_{32} = 0, \tag{6.4}$$

one gets the system

$$\frac{\partial u_1}{\partial t} + \lambda_1 \frac{\partial u_1}{\partial x} + \lambda_3 \frac{\partial u_1}{\partial y} = 0, \tag{6.5}$$
$$\frac{\partial u_2}{\partial t} + \lambda_2 \frac{\partial u_2}{\partial x} + \lambda_3 \frac{\partial u_2}{\partial y} = 0,$$
$$\frac{\partial u_3}{\partial t} + \lambda_3 \frac{\partial u_3}{\partial x} + \lambda_1 \frac{\partial u_3}{\partial y} = 0,$$

which is two dimensional extension of the one-dimensional system for Riemann invariant $u_1$ and $u_2$.

Constraints (6.4) are equivalent to the following three equations for three functions $g_1, g_2, g_3$

$$\frac{\partial g_1}{\partial u_3} \frac{\partial g_3}{\partial u_2} - \frac{\partial g_1}{\partial u_2} \frac{\partial g_3}{\partial u_3} = 0$$
$$\frac{\partial g_2}{\partial u_3} \frac{\partial g_3}{\partial u_1} - \frac{\partial g_2}{\partial u_1} \frac{\partial g_3}{\partial u_3} = 0$$
$$\frac{\partial g_1}{\partial u_2} \frac{\partial g_3}{\partial u_1} - \frac{\partial g_1}{\partial u_1} \frac{\partial g_3}{\partial u_2} = 0 \tag{6.6}$$

In terms of the functions $f_i, i = 1, 2, 3$ obeying equations (6.6) with the substitution $(t = (f_1 - f_2)/(\lambda_1 - \lambda_2))$

$$\frac{\partial g_l}{\partial u_k} = \frac{\partial f_l}{\partial u_k} - \frac{f_1 - f_2}{\lambda_1 - \lambda_2} \frac{\partial \lambda_l}{\partial u_k}, \quad l, k = 1, 2, 3. \tag{6.7}$$

So any solution of the three-dimensional homogeneous Euler equation, constructed using the functions $f_i, i = 1, 2, 3$ obeying equations (6.6), is a solution of the two-dimensional system (6.5).

We note that the system (6.5) does not reduce to the expression (2.16) in the naive one dimensional limit $x = y$. The reason is that the constraint (6.4) represent only the part of the constraint (2.10).
In order to recover this system we combine the expressions (6.2) into another system of equations (equivalent to (6.4)), namely,

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + \lambda_1 \frac{\partial u_1}{\partial x} &= (\lambda_2 - \lambda_1)(A^{-1})_{12} + \lambda_3(A^{-1})_{13}, \\
\frac{\partial u_2}{\partial t} + \lambda_2 \frac{\partial u_2}{\partial x} &= (\lambda_1 - \lambda_2)(A^{-1})_{21} + \lambda_3(A^{-1})_{23}, \\
\frac{\partial u_3}{\partial t} + \lambda_3 \frac{\partial u_3}{\partial x} &= \lambda_1(A^{-1})_{31} + \lambda_2(A^{-1})_{32}.
\end{align*}
\] (6.8)

Now, requiring that \(x = y\) and \((A^{-1})_{lk} = 0, l \neq k, l,k = 1,2,3\), i.e. \(A_{lk} = 0, l \neq k\), one obtains the 3-component system (2.16).

As second example we consider the one-dimensional reduction of the Euler equation with \(\lambda_k = u_k\) and \(\alpha_1 = 1, \alpha_2 = \alpha_3 = 0\) and \(\beta_1 = \beta_2 = 1, \beta_3 = 0\). So we start with the hodograph system

\[
\begin{align*}
x - u_1 t + f_1 &= 0, \\
-t + f_2 &= 0, \\
f_3 &= 0,
\end{align*}
\] (6.9)

where \(x = x_1, x_2 = x_3 = 0\) and we redefine the function \(f_2 \rightarrow u_2 f_2\). Differentiating (6.9) w.r.t. \(x\) and \(t\), we obtain

\[
\begin{align*}
\frac{\partial u_l}{\partial x} &= -(A^{-1})_{l1}, \quad l = 1,2,3, \\
\frac{\partial u_l}{\partial t} &= (A^{-1})_{l1} u_1 + (A^{-1})_{l2}.
\end{align*}
\] (6.10)

Consequently one has the system

\[
\frac{\partial u_l}{\partial t} + u_1 \frac{\partial u_l}{\partial x} = (A^{-1})_{l2}, \quad l = 1,2,3.
\] (6.11)

Now we impose the constraints

\[
(A^{-1})_{12} = -\frac{\partial u_2}{\partial x}, \quad (A^{-1})_{22} = -\frac{\partial u_3}{\partial x}, \quad (A^{-1})_{32} = 0.
\] (6.12)

Due to the relation (6.10), these constraints are equivalent to the following

\[
(A^{-1})_{12} = (A^{-1})_{21}, \quad (A^{-1})_{22} = (A^{-1})_{31}, \quad (A^{-1})_{32} = 0.
\] (6.13)

Using the explicit form of the 3 \(\times\) 3 inverse of the matrix \(A\), one obtains the following system of equations

\[
\begin{align*}
\frac{\partial g_1}{\partial u_3} \frac{\partial g_3}{\partial u_2} - \frac{\partial g_2}{\partial u_3} \frac{\partial g_3}{\partial u_1} + \frac{\partial g_3}{\partial u_3} \left( \frac{\partial g_2}{\partial u_1} - \frac{\partial g_1}{\partial u_2} \right) &= 0, \\
\frac{\partial g_1}{\partial u_1} \frac{\partial g_3}{\partial u_2} - \frac{\partial g_2}{\partial u_1} \frac{\partial g_3}{\partial u_2} + \frac{\partial g_3}{\partial u_1} \left( \frac{\partial g_2}{\partial u_2} - \frac{\partial g_1}{\partial u_3} \right) &= 0, \\
\frac{\partial g_1}{\partial u_2} \frac{\partial g_3}{\partial u_1} - \frac{\partial g_1}{\partial u_1} \frac{\partial g_3}{\partial u_2} &= 0.
\end{align*}
\] (6.14)

Any solution of this system with the substitution

\[
\begin{align*}
\frac{\partial g_1}{\partial u_l} = \frac{\partial f_1}{\partial u_l} - f_2 \delta_{l1}, \quad \frac{\partial g_2}{\partial u_l} = \frac{\partial f_2}{\partial u_l}, \quad \frac{\partial g_3}{\partial u_l} = \frac{\partial f_3}{\partial u_l},
\end{align*}
\] (6.15)
provide us with the functions $f_1, f_2, f_3$ for which three dimensional homogeneous Euler equations is reducible to the system

$$
\begin{align*}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} &= 0, \\
\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} &= 0, \\
\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} &= 0,
\end{align*}
$$

(6.16)

which is the 3-component one-dimensional Jordan system described in [6].

It is not difficult to show that the system (6.14) has a solution for which

$$
\begin{align*}
g_i &= \frac{\partial W}{\partial u_i}, \quad i = 1, 2, 3
\end{align*}
$$

(6.17)

where the function $W$ obeys the equation

$$
\begin{align*}
\frac{\partial W}{\partial u_2} &= \frac{\partial^2 W}{\partial u_1^2}, \quad \frac{\partial W}{\partial u_3} = \frac{\partial^3 W}{\partial u_1^3}.
\end{align*}
$$

(6.18)

Hodograph equations (6.9) represent the critical points equations $\frac{\partial W}{\partial u_i} = 0$, $i = 1, 2, 3$ for the function

$$
W = xu_1 - t \left( \frac{1}{2} u_1^2 + u_2 \right) + W(u_1, u_2, u_3),
$$

(6.19)

which obeys the equations (6.18).

Equations (6.18) and function $W$ (6.19) are exactly those given in the paper [6].

7 Infinite-dimensional Euler equation: reductions to Jordan chain, Burgers and Korteweg-de Vries equations

The above result on the 3-component Jordan system can be extended to the $n$-component case. Indeed, let us consider the hodograph equations for arbitrary $n$ and $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_n = 0$, $\beta_1 = \beta_2 = 1$, $\beta_3 = \cdots = \beta_n = 0$, i.e. the equations ($x = x_1$)

$$
\begin{align*}
x - u_1 t + f_1 &= 0, \\
-t + f_2 &= 0, \\
f_m &= 0, \quad m = 3, \ldots, n
\end{align*}
$$

(7.1)

where, for convenience we redefine the function $f_2 \to u_2 f_2$. Relation (2.4) imply that $\frac{\partial u}{\partial x_k} = 0$, $k = 2, 3, \ldots, n$ and

$$
\begin{align*}
\frac{\partial u}{\partial x} &= -(A^{-1})_{11}, \\
\frac{\partial u}{\partial t} &= (A^{-1})_{11} u_1 + (A^{-1})_{12}.
\end{align*}
$$

(7.2)

Combining (7.2), one gets

$$
\begin{align*}
\frac{\partial u}{\partial t} + u_1 \frac{\partial u}{\partial x} &= (A^{-1})_{12}, \quad l = 1, \ldots, n.
\end{align*}
$$

(7.3)
Imposing the constraint

\[(A^{-1})_{l2} = - \frac{\partial u_{l+1}}{\partial x}, \quad l = 1, \ldots, n-1, \tag{7.4}\]

\[(A^{-1})_{n2} = 0,\]

one obtains the \(n\)-components system

\[\begin{align*}
\frac{\partial u_l}{\partial t} + u_1 \frac{\partial u_l}{\partial x} + \frac{\partial u_{l+1}}{\partial x} &= 0, \quad l = 1, \ldots, n-1, \\
\frac{\partial u_n}{\partial t} + u_1 \frac{\partial u_n}{\partial x} &= 0, \tag{7.5}
\end{align*}\]

that is the \(n\)-component Jordan system introduced in [6].

Since \(\frac{\partial u_{l+1}}{\partial x} = -(A^{-1})_{l+1,1}, \quad l = 1, \ldots, n-1\), the constraints (7.4) are equivalent to the following

\[\begin{align*}
(A^{-1})_{l2} &= (A^{-1})_{l+1,1}, \quad l = 1, \ldots, n-1, \\
(A^{-1})_{n2} &= 0, \tag{7.6}
\end{align*}\]

or

\[\begin{align*}
\tilde{A}_{l2} &= \tilde{A}_{l+1,1}, \quad l = 1, \ldots, n-1, \\
\tilde{A}_{n2} &= 0. \tag{7.7}
\end{align*}\]

Using the explicit expression for the elements of the adjugate matrix \(\tilde{A}\), one rewrites the constraints (7.7) as the system of \(n\) differential equations for the functions \(g_i, \quad i = 1, \ldots, n\) or \(f_i, \quad i = 1, \ldots, n\).

It is not difficult to show that this system has a solution for which

\[g_i = \frac{\partial W}{\partial u_i}, \quad f_i = \frac{\partial \tilde{W}}{\partial u_i}, \quad i = 1, \ldots, n \tag{7.8}\]

where the functions \(W\) and \(\tilde{W}\) obey the equations

\[\frac{\partial W}{\partial u_k} = \frac{\partial^k W}{\partial u_1^k}, \quad k = 2, \ldots, n, \tag{7.9}\]

and

\[W = xu_1 - t \left( \frac{1}{2} u_1^2 + u_2 \right) + \tilde{W}(u_1, \ldots, u_n), \tag{7.10}\]

that coincides with those formulae presented in [6]. It is noted that in this one-dimensional reduction the constraint (4.8) is absent.

Now, following [6] one can consider the system (7.5) in the formal limit \(n \to \infty\) and get the infinite Jordan chain which has been discussed in different contexts in [7, 14, 6]. So the Jordan chain represent a particular reduction of the infinite-dimensional Homogeneous Euler equation.

In the paper [15] it was observed that the Jordan chain admits differential reductions to various integrable partial differential equations, for example, to the Burgers equation and Korteweg-de Vries equation. Indeed, if one imposes the constraint

\[u_2 = \frac{\partial u_1}{\partial x}, \tag{7.11}\]
then the first equation \((l = 1)\) in (7.5) becomes the Burgers equation

\[
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial^2 u_1}{\partial u_1^2} = 0,
\tag{7.12}
\]

while the other equations (7.5) with \(l = 2, 3, \ldots\) represent themselves the recursive relations to define \(u_3, u_4, \ldots\).

If one requires that

\[
u_2 = \frac{\partial^2 u_1}{\partial x^2},
\tag{7.13}
\]

then the Jordan chain is reduced to the Korteweg-de Vries equation

\[
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial^3 u_1}{\partial u_1^3} = 0.
\tag{7.14}
\]

Constraints (7.11) and (7.13) can be rewritten in terms of the elements \((A^{-1})_{lk}\). Indeed, the differential consequence of (7.11), namely, \(\frac{\partial u_2}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial u_1}{\partial x} \right)\) after the use of (2.4), assumes the form

\[
(A^{-1})_{21} + \sum_{k=1}^{\infty} \frac{\partial (A^{-1})_{11}}{\partial u_k} (A^{-1})_{k1} = 0.
\tag{7.15}
\]

The differential consequence of (7.13) is equivalent to the following

\[
(A^{-1})_{21} - \sum_{k,l=1}^{\infty} \frac{\partial}{\partial u_l} \left( \frac{\partial (A^{-1})_{kl}}{\partial u_k} (A^{-1})_{k1} \right) (A^{-1})_{l1} = 0.
\tag{7.16}
\]

Though the constraint (7.15), (7.16) are rather cumbersome, one concludes that the solutions of the Burgers and Korteweg-de Vries equations represent particular subclasses of solutions of the infinite-dimensional homogeneous Euler equation.

8 Gradient catastrophe for the homogeneous Euler equations

All the results presented in the previous sections are valid under the assumption that \(\det(A) \neq 0\). If instead

\[
\det(A) \equiv \det \left( \frac{\partial f_l}{\partial u_k} - t \frac{\partial \lambda_l}{\partial u_k} \right) = 0,
\tag{8.1}
\]

then, according to (2.4), (2.5) solutions of the equation (2.7) and other equations exhibit the gradient catastrophe \(\frac{\partial u}{\partial x} \to \infty, \frac{\partial u}{\partial t} \to \infty\).

Let us consider such a situation for the classical homogeneous Euler equation \((\lambda_k = u_k, \alpha_k = \beta_k = 1)\). In this case the equation (8.1) is simplified to

\[
\det(A) \equiv \det \left( \frac{\partial f_l}{\partial u_k} - t \delta_{lk} \right) = 0,
\tag{8.2}
\]

i.e. to the characteristic polynomial equation

\[
t^n + \sum_{k=0}^{n-1} B_k(u) t^k = 0
\tag{8.3}
\]
of the $n \times n$ matrix $\tilde{A}_{lk} = \frac{\partial f_l}{\partial u_k}$. Due to (2.30) the functions $f_l(u)$ are the local inverse of the initial values of $u(t=0,x)$ and, consequently, coefficients $B_k$ depend only on $u_1, \ldots, u_n$ only.

Thus, gradient catastrophe for the homogeneous Euler equation happens in general on the $n-$dimensional hypersurface in $\mathbb{R}^{n+1}$ given by the equation (8.3). If the polynomial (8.3) has no real roots then the gradient catastrophe does not happen for given initial data $u_i(t=0,x)$. Let us assume that equation (8.3) has at least one real root $t_c$. So the gradient catastrophe happens on the hypersurface $S$ given by

$$t_c = \phi(u_1, \ldots, u_n)$$

(8.4)

where $\phi(u)$ is a certain function constructed out from the (local) inverse of the $u(t=0,x)$. Usually discussed first moment of appearance of gradient catastrophe corresponds to the minimum value of $t_c$, i.e. to the situation when

$$\frac{\partial t_c}{\partial u_i} = \frac{\partial}{\partial u_i} \phi(u_1, \ldots, u_n) = 0, \quad i = 1, \ldots, n$$

(8.5)

plus a condition on the second derivatives (for generic catastrophes such condition is given by the classical condition on the Hessian of $\phi$).

For generic initial data the function $\phi$ is generic. Consequently $n$ equations (8.5) has generically a single solution $u_1^c, \ldots, u_n^c$.

Thus, generically, the gradient catastrophe for the homogeneous Euler equation first happens at the time

$$t_{c_{\text{min}}} = \phi(u_1^c, \ldots, u_n^c)$$

(8.6)

at the point $u_1^c, \ldots, u_n^c$ on the hypersurface $S$ (8.4). Then it expands on the whole hypersurface (8.4).

It is noted that for the first time such property of the gradient catastrophe for multi-dimensional equations has been observed in [11, 12].

In more detail the gradient catastrophes for the homogeneous Euler equation, related equations and its regularization will be considered in a separate paper.

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