GLOBAL WELL-POSEDNESS OF THE 1D DIRAC-KLEIN-GORDON SYSTEM IN SOBOLEV SPACES OF NEGATIVE INDEX

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Abstract. We prove that the Cauchy problem for the Dirac-Klein-Gordon system of equations in 1D is globally well-posed in a range of Sobolev spaces of negative index for the Dirac spinor and positive index for the scalar field. The main ingredient in the proof is the theory of “almost conservation law” and “I-method” introduced by Colliander, Keel, Staffilani, Takaoka and Tao. Our proof also relies on the null structure in the system, and bilinear spacetime estimates of Klainerman-Machedon type.

1. Introduction

We consider the Dirac-Klein-Gordon system (DKG) in one space dimension,
\[
\begin{cases}
-\gamma^0 \partial_t + \gamma^1 \partial_x + M) \psi = \phi \psi, \\
-\square + m^2) \phi = \langle \gamma^0 \psi, \psi \rangle_{C^2}, \\
\end{cases}
\]
with initial data
\[
\psi|_{t=0} = \psi_0 \in H^s, \quad \phi|_{t=0} = \phi_0 \in H^r, \quad \partial_t \phi|_{t=0} = \phi_1 \in H^{r-1}. 
\]
Here \((t,x) \in \mathbb{R}^{1+1}, \psi = \psi(t,x) \in \mathbb{C}^2\) is the Dirac spinor and \(\phi = \phi(t,x)\) is the scalar field which is real-valued; \(M, m > 0\) are constants. Further, \(\langle w, z \rangle_{C^2} = z^*w\) for column vectors \(w, z \in \mathbb{C}^2\), where \(z^*\) is the complex conjugate transpose of \(z\); \(H^s = (1 - \partial_x^2)^{-s/2}L^2(\mathbb{R})\) is the standard Sobolev space of order \(s\), and \(\gamma^0\) and \(\gamma^1\) are the Dirac matrices given by
\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
We remark that with this choice the general requirements for Dirac matrices are verified:
\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}I, \quad (\gamma^0)^* = \gamma^0, \quad (\gamma^1)^* = -\gamma^1
\]
for \(\mu, \nu = 0, 1\), where \((g^{\mu\nu}) = (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})\).

We are interested in studying low regularity global solutions of the DKG system (1) given the initial data (2). Global well-posedness (GWP) of DKG in 1d was first proved by Chadam [4] for data
\[
(\psi_0, \phi_0, \phi_1) \in H^1 \times H^1 \times L^2.
\]
Table 1. GWP for DKG in 1d for data \((\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}\).

|                | \(s\) | \(r\)            |
|----------------|-------|------------------|
| Chadam [4], 1973 | 1     | 1                |
| Bournaveas [2], 2000 | 0     | 1                |
| Fang [9], 2001   | 0     | \((1/2, 1]\)     |
| Bournaveas and Gibbeson [3], 2006 | 0     | \((1/4, 1]\)     |
| Machihara [11], Pecher [13], 2006 | 0     | \((0, 1]\)       |
| Selberg [15], 2007 | \((-1/8, 0]\) | \((-s + \sqrt{s^2 - s}, 1 + s]\) |

This result has been improved over the years in the sense that the regularity requirements on the initial data which ensure global-in-time solutions can be lowered. The earlier known GWP results for DKG in 1d are summarized in Table 1.

It is well known that when \(s \geq 0\), the question of GWP of \((1), (2)\) reduces to the corresponding local question essentially due to the conservation of charge:

\[
\|\psi(t, .)\|_{L^2} = \|\psi_0\|_{L^2}.
\]

However, when \(s < 0\) there is no applicable conservation law. So even if we have a local well-posedness (LWP) result for \(s_0 < s < 0\) for some \(s_0\), it seems that we are stuck when trying to extend this to a global-in-time solution.

The first breakthrough for resolving such problems came from Bourgain [1] who considered the cubic, defocusing nonlinear Schrödinger (NLS) equation in 2d, and proved GWP of NLS below the (conserved) energy norm, i.e., below \(H^1\). The idea behind this method for a PDE is to split the rough initial data (data whose regularity is below the conserved norm; say the \(L^2\) norm from now on) into low and high frequency parts, using a Fourier truncation operator. Consequently, one splits the PDE into two, corresponding to the initial data with low and high frequencies. The data with low frequency becomes smoother, in fact it is in \(L^2\), so by global well-posedness its evolution remains in \(L^2\) for all time.

On the other hand, the difference between the original solution and the evolution of the low frequency data satisfies a modified nonlinear equation evolving the high-frequency part of the initial data. The homogeneous part of this evolution is of course no smoother than the initial data (so it may not be in \(L^2\)), but the inhomogeneous part may be better due to nonlinear smoothing effects. If the nonlinear smoothing brings the inhomogeneous part into \(L^2\), then at the end of the time interval of existence this part can be added to the evolution of the low-frequency data, and the whole process can be iterated. Assuming that sufficiently good a priori estimates are available, this iteration allows one to reach an arbitrarily large existence time, by adjusting the frequency cut-off point of the original initial data. Several authors used Bourgain’s method to prove GWP of dispersive and wave equations with rough data.

Recently, Selberg [15] used Bourgain’s method to prove GWP of 1d-DKG below the charge norm, obtaining the following result (for a comparison with earlier results, see Table 1):
Theorem 1. The DKG system (1) is GWP for data (2) provided
\[-\frac{1}{8} < s < 0, \quad -s + \sqrt{s^2 - s} < r \leq 1 + s.\]

Concerning LWP of 1d-DKG the best result so far, which we state in the next theorem, is due to S. Selberg and the present author [16], building on earlier results by several authors; see [4], [2], [9], [3], [11] and [13].

Theorem 2. The DKG system (1) is LWP for data (2) if
\[s > -\frac{1}{4}, \quad r > 0, \quad |s| \leq r \leq 1 + s.\]

As mentioned earlier, when \( s \geq 0 \) this LWP result can be extended to GWP result essentially due to the presence of conservation of charge. So in view of Theorem 2, we have the following (see also Table 1):

Theorem 3. The DKG system (1) is GWP for data (2) provided
\[s \geq 0, \quad r > 0, \quad |s| \leq r \leq 1 + s.\]

However, in view of Theorems 2, 3 and 1, there is still a gap left between the local and global results known so far. In the present paper, we shall relax the lower bound of \( r \) in Theorem 1. In particular, we fill the following gap left by Theorem 1 (see Figure 1):
\[-\frac{1}{8} < s < 0, \quad s + \sqrt{s^2 - s} < r \leq -s + \sqrt{s^2 - s}\]

We now state our Main theorem.

Theorem 4. The DKG system (1) is GWP for data (2) if (see Figure 1)
\[-\frac{1}{8} < s < 0, \quad s + \sqrt{s^2 - s} < r \leq 1 + s.\]

The technique used here is the theory of “almost conservation law” and “I-method” which was developed by Colliander, Keel, Staffilani, Takaoka and Tao in a series of papers; See for instance [5], [6], [7]. The idea here is to apply a smoothing operator \( I \) to the solution of the PDE. The operator \( I \) is chosen so that it is the identity for low frequencies and an integration operator for high frequencies. The next step is to prove an “almost conservation law” for the smoothed out solution as time passes. Then one hopes that a modified version of LWP Theorem (after \( I \) is introduced) together with the “almost conservation law” will give a GWP result of the PDE for rough data.

In the DKG system, however, there is no conservation law for the field \( \phi \), only for the spinor \( \psi \). Hence, we will not have “almost conservation law” for the \( \phi \) field, which makes the problem harder. To fix this problem we use a product estimate for the Sobolev spaces for the inhomogeneous part of \( \phi \), the “almost conservation law” for the spinor \( \psi \), together with an additional idea used by Selberg [15] of making use of induction argument involving a cascade of free waves.

This paper is organized as follows. In the next section we fix some notation, state definitions, and recall the derivation of the conservation of charge. In Section 3 we shall state some basic linear and bilinear estimates, and prove some null form estimates. In Section 4 we discuss the I-method, state a modified LWP theorem when we introduce the \( I \) operator, state a key Lemma concerning smoothing estimate, and show that a combination of these imply an “almost conservation law” for the
Figure 1. Global well-posedness of DKG holds in the interior of the shaded region. Moreover, we can allow the line $r = 1 + s$ for $-1/8 < s < 0$. The larger region which is contained in the strip $-1/4 < s < 0$ is where Local well-posedness of DKG holds.

2. Preliminaries

2.1. Notation and Definitions. In estimates, $C$ denotes a positive constant which can vary from line to line and may depend on the Sobolev exponents $s$ and $r$ in (2). We use the shorthand $X \lesssim Y$ for $X \leq CY$, and if $C \ll 1$ we use the symbol $\ll$ instead of $\lesssim$. We use the shorthand $X \approx Y$ for $Y \lesssim X \lesssim Y$. Throughout the paper $\varepsilon$ is considered to be a sufficiently small positive number in the sense that $0 < \varepsilon \ll 1$. We also use the notation

$$\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}.$$  

The Fourier transforms in space and space-time are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx, \quad \tilde{u}(\tau, \xi) = \int_{\mathbb{R}^{1+1}} e^{-i(t\tau + x\xi)} u(t, x) \, dt \, dx.$$  

We denote $D = -i\partial_x$, so $\hat{D}u(\xi) = \xi \hat{u}(\xi)$. We also write $D_+ := \partial_t + \partial_x$ and $D_- := \partial_t - \partial_x$, hence $\Box = -D_+ D_-$. 

charge. Here, we also state another key Lemma which is used to control the growth of solution of the Klein-Gordon part of DKG, $\phi$. In Section 5 we put everything from section 4 together and prove our main theorem. In Sections 6 and 7 we prove the two key lemmas stated in section 4. In section 8 we prove the modified LWP theorem.
We use the following spaces of Bourgain-Klainerman-Machedon type: For $a, b \in \mathbb{R}$, define $X_{a,b}^\pm$, $H_{a,b}^\pm$ and $\mathcal{H}_{a,b}^\pm$ to be the completions of $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to the norms

\[
\|u\|_{X_{a,b}^\pm} = \|\langle \xi \rangle^a (\tau \pm |\xi|)^b \tilde{u}(\tau, \xi)\|_{L_{\tau, \xi}^2},
\]
\[
\|u\|_{H_{a,b}^\pm} = \|\langle \xi \rangle^a |\tau| - |\xi|\rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau, \xi}^2},
\]
\[
\|u\|_{\mathcal{H}_{a,b}^\pm} = \|u\|_{H_{a,b}^\pm} + \|\partial_\tau u\|_{H_{a-1,b}^\pm}.
\]

We also need the restrictions to a time slab $S_T = (0, T) \times \mathbb{R}$. The restriction $X_{a,b}^\pm(S_T)$ is a Banach space with norm

\[
\|u\|_{X_{a,b}^\pm(S_T)} = \inf_{\tilde{u} \mid \tilde{u}|_{S_T} = u} \|\tilde{u}\|_{X_{a,b}^\pm}.
\]

The restrictions $H_{a,b}^\pm(S_T)$ and $\mathcal{H}_{a,b}^\pm(S_T)$ are defined in the same way. See [8] for more details about these spaces.

### 2.2. Rewriting DKG and Conservation of charge.

To see the symmetry in the DKG system, we shall rewrite (1) as follows: Let

\[
\psi = \begin{pmatrix} u \\ v \end{pmatrix}
\]

for $u, v \in \mathbb{C}$. Then we calculate

\[
(\gamma^0 \partial_t + \gamma^1 \partial_x)\psi = \begin{pmatrix} v_t - v_x \\ u_t + u_x \end{pmatrix}
\]

and

\[
\langle \gamma^0 \psi, \psi \rangle_{C^2} = \bar{u} v + u \bar{v} = 2 \text{Re}(u\bar{v}).
\]

Using this information, we rewrite (1) as

\[
\begin{align*}
i(u_t + u_x) &= Mv - \phi v, \\
i(v_t - v_x) &= Mu - \phi u, \\
\Box \phi &= m^2 \phi - 2 \text{Re}(u\bar{v}),
\end{align*}
\]

with the initial data (2) transformed to

\[
\begin{align*}
u(0) &= u_0 \in H^s, \\
v(0) &= v_0 \in H^s, \\
\phi(0) &= \phi_0 \in H^r, \\
\partial_t \phi(0) &= \phi_1 \in H^{r-1}.
\end{align*}
\]

We shall then work with the Cauchy problem (3), (4) in the rest of the paper.

To motivate the derivation of the “almost conservation law”, we first recall the proof of the conservation of $L^2$-norm of the solution to the Dirac part of the equation (3), using integration by parts. To do this we first assume $u, v$ to be smooth functions that decay at spatial infinity. For general well posed solutions of (3) where $s \geq 0$, the conservation of charge will follow by a density argument.

Multiplying the first and second equations in (3) by $-\bar{u}\bar{v}$ and $-\bar{u}\bar{v}$, respectively, we get

\[
\begin{align*}
\bar{u} u_t + \bar{u} u_x &= -i M \bar{u} v + i \phi \bar{u} v, \\
\bar{v} v_t - \bar{v} v_x &= -i M u \bar{v} + i \phi u \bar{v}.
\end{align*}
\]

Adding these two we obtain

\[
\bar{u} u_t + \bar{v} v_t + \bar{u} u_x - \bar{v} v_x = 2i(-M + \phi) \text{Re}(u\bar{v}).
\]
We now take the real part of this equation to get
\[ \text{Re}(\overline{\mu}u_t) + \text{Re}(\overline{\nu}v_t) + \text{Re}(\overline{\mu}u_x) - \text{Re}(\overline{v}v_x) = 0. \]

Using the identity \((\overline{\mu}u)_t = \overline{\mu}u_t + \overline{\mu}_t u = 2 \text{Re}(\overline{\mu}u_t)\) (and the same identity if we take partial derivative in \(x\)), we have
\[ ((|u|^2)_t + (|v|^2)_t + (|u|^2)_x - (|v|^2)_x) = 0. \]

We get after integrating in \(x\)
\[ \frac{d}{dt} \left( \|u(t)\|^2_{L^2} + \|v(t)\|^2_{L^2} \right) = 0, \]
which implies the conservation charge:
\[ \|u(t)\|^2_{L^2} + \|v(t)\|^2_{L^2} = \|u_0\|^2_{L^2} + \|v_0\|^2_{L^2}. \tag{5} \]

3. Linear and bilinear estimates

The representation formula in Fourier space for the inhomogeneous Dirac Cauchy problem
\[
\begin{cases}
  iD_\pm w_\pm = M w_\pm + F_\pm(t, x), \\
  w_\pm(0, x) = f_\pm(x),
\end{cases} \tag{6}
\]
is given by
\[
\widehat{w_\pm}(t)(\xi) = e^{-i(M \pm \xi)(t)} f_\pm(\xi) + \int_0^t e^{-i(M \pm \xi)(t-t')} F_\pm(t', \xi) dt'. \tag{7}
\]
Similarly, the representation formula in Fourier space for the inhomogeneous Klein-Gordon Cauchy problem
\[
\begin{cases}
  \Box z = m^2 z + F(t, x), \\
  z(0, x) = f(x), \quad \partial_t z(0, x) = g(x),
\end{cases} \tag{8}
\]
is given by
\[
\widehat{z}(t)(\xi) = \cos(t(\xi)_m) \widehat{f}(\xi) + \frac{\sin(t(\xi)_m)}{\langle \xi \rangle_m} \widehat{g}(\xi) + \int_0^t \frac{\sin((t-t')(\xi)_m)}{\langle \xi \rangle_m} \widehat{F}(t')(\xi) dt', \tag{9}
\]
where \(\langle \xi \rangle_m = \sqrt{m^2 + |\xi|^2}\).

3.1. Linear estimates. Throughout the paper, we use the notation
\[ \|z[t]\|_{H^n} \equiv \|z(t)\|_{H^n} + \|\partial_t z(t)\|_{H^{n-1}}. \]

From the solution formulas (7) and (9) we deduce the following energy estimates for the solution of Cauchy problems (6) and (8), respectively:
\[
\|w_\pm(t)\|_{H^n} \leq \|f_\pm\|_{H^n} + \int_0^t \|F_\pm(t')\|_{H^n} dt', \tag{10}
\]
\[
\|z[t]\|_{H^n} \leq C \left( \|f\|_{H^n} + \|g\|_{H^{n-1}} + \int_0^t \|F(t')\|_{H^{n-1}} dt' \right), \tag{11}
\]

1 for some \(C > 0\) and for all \(t > 0\).

The estimates we present in the following two lemmas are a priori estimates for the solutions of the massive Dirac and Klein-Gordon Cauchy problems, and they

\[ \text{If we set } m = 0 \text{ in (8), then the constant } C \text{ in the energy estimate (11) will depend on } t. \]
are crucial for the reduction of the local existence problem to bilinear estimates. These estimates are variants of the estimates in [8, Lemma 5, Lemma 6], i.e., when $M = m = 0$.

**Lemma 1.** Let $1/2 < b \leq 1$, $a \in \mathbb{R}$, $0 < T \leq 1$ and $0 \leq \delta \leq 1 - b$. Then for all data $F_\pm \in X^{a,b-1+\delta}(S_T)$ and $f_\pm \in H^a$, we have the following estimate for the solution (7) of the Dirac Cauchy problem (6):

$$\|u_\pm\|_{X^a_b(S_T)} \leq C \left(\|f_\pm\|_{H^a} + T^\delta \|F_\pm\|_{X^{a,b-1+\delta}(S_T)}\right),$$

where $C$ depends only on $b$.

**Proof.** Define the space $X^{a,\theta}_{M,\pm}$ with a norm

$$\|u\|_{X^{a,\theta}_{M,\pm}} = \|\langle \xi \rangle^a (\tau \pm \xi M)^\theta \tilde{u}(\tau, \xi)\|_{L^2_{\tau, \xi}},$$

where $\xi_M = \xi + M$. In view of [8, Lemma 5] the estimate (12) holds if we replace the space $X^{a,\theta}_{\pm}$ by $X^{a,\theta}_{M,\pm}$. So, to complete the proof it suffices to show

$$X^{a,\theta}_\pm = X^{a,\theta}_{M,\pm}.$$

This reduces to proving

$$\langle \tau \pm \xi \rangle \approx \langle \tau \pm \xi_M \rangle.$$  \hspace{1cm} (13)

But this follows from

$$\langle \tau \pm \xi \rangle \approx 1 + |\tau \pm \xi| \leq 1 + |\tau \pm \xi_M| + M \lesssim \langle \tau \pm \xi_M \rangle,$$

and conversely,

$$\langle \tau \pm \xi_M \rangle \approx 1 + |\tau \pm \xi_M| \leq 1 + |\tau \pm \xi| + M \lesssim \langle \tau \pm \xi \rangle.$$  \hspace{1cm} \qed

**Lemma 2.** Let $1/2 < b \leq 1$, $a \in \mathbb{R}$, $0 < T \leq 1$ and $0 \leq \delta \leq 1 - b$. Then for all data $F \in H^{a-1,b-1+\delta}(S_T)$, $f \in H^a$ and $g \in H^{a-1}$, we have the following estimate for the solution (9) of the Klein-Gordon Cauchy problem (8):

$$\|z\|_{\mathcal{H}^{a,b}(S_T)} \leq C \left(\|f\|_{H^a} + \|g\|_{H^{a-1}} + T^{\delta/2} \|F\|_{H^{a-1,b-1+\delta}(S_T)}\right),$$

where $C$ depends only on $b$.

**Proof.** Define the space $H^{a,\theta}_m$ with a norm

$$\|u\|_{H^{a,\theta}_m} = \|\langle \xi \rangle^a (|\tau| - \langle \xi \rangle)_m^\theta \tilde{u}(\tau, \xi)\|_{L^2_{\tau, \xi}},$$

and the space $\mathcal{H}^{a,\theta}_m$ with a norm

$$\|u\|_{\mathcal{H}^{a,\theta}_m} = \|u\|_{H^{a,\theta}_m} + \|\partial_\tau u\|_{H^{a-1,\theta}_m}.$$

So, in view of [14, Theorem 12] the estimate (14) holds if we replace the spaces $H^{a,\theta}_m$ and $\mathcal{H}^{a,\theta}_m$ by $H^{a,\theta}_m$ and $\mathcal{H}^{a,\theta}_m$, respectively. Hence, to complete the proof it suffices to show

$$H^{a,\theta}_m = H^{a,\theta}_m.$$

This reduces to proving

$$\langle |\tau| - |\xi| \rangle \approx \langle |\tau| - \langle \xi \rangle_m \rangle.$$  \hspace{1cm} (15)
Assume \( \tau \geq 0 \). Then
\[
\langle -\tau + |\xi| \rangle \approx 1 + |\tau + |\xi|| \leq 1 + |\tau + \langle \xi \rangle_m + \langle \xi \rangle_m - |\xi|
\leq 1 + m + |\tau + \langle \xi \rangle_m|
\lesssim \langle -\tau + \langle \xi \rangle_m \rangle.
\]

Conversely,
\[
\langle -\tau + \langle \xi \rangle_m \rangle \approx 1 + |\tau + \langle \xi \rangle_m|
\leq 1 + |\tau + |\xi|| + \langle \xi \rangle_m - |\xi|
= 1 + m + |\tau + |\xi|| \lesssim \langle -\tau + |\xi| \rangle.
\]

Similarly, it can be shown that the estimate (15) holds true for \( \tau < 0 \). This completes the proof of the Theorem. \( \square \)

We shall need the fact that if \( b > 1/2 \), then
\[
\|u(t)\|_{H^b} \leq C \|u\|_{H^{b,b}(S_T)} \leq C \|u\|_{X^{b,b}(S_T)} \quad \text{for } 0 \leq t \leq T,
\]
where \( C \) depends only on \( b \). The following estimate will also be needed in the last section (see [12] for the proof):
\[
\|u\|_{X^{0,\alpha}(S_T)} \leq C T^{1/2 - 2\varepsilon} \|u\|_{X^{0,1/2 - \varepsilon}(S_T)},
\]
for all \( \varepsilon > 0 \) sufficiently small, and \( 0 < T \leq 1 \).

**Lemma 3.** Let \( 2 \leq q \leq \infty \) and \( \varepsilon > 0 \) be sufficiently small. Then
\[
\|u\|_{H^{0,-1/2+1/q-\varepsilon}} \lesssim \|u\|_{L^q_t L^2_x}
\]
where \( \frac{1}{q} + \frac{1}{q'} = 1 \).

**Remark 1.** This Lemma also holds if we replace \( H^{0,-1/2+1/q-\varepsilon} \) by \( X^{0,-1/2+1/q-\varepsilon}_\pm \), simply because \( H^{0,\alpha} \hookrightarrow X^{0,\alpha}_\pm \) for any \( \alpha \leq 0 \).

**Proof of lemma 3.** By duality, the estimate is equivalent to
\[
\|u\|_{L^q_t L^2_x} \lesssim \|u\|_{H^{0,1/2-1/q+\varepsilon}}.
\]
By Sobolev embedding in \( t \)
\[
\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{H^{0,1/2+\varepsilon}}.
\]
Interpolating this with
\[
\|u\|_{L^2_t L^2_x} = \|u\|_{L^2_t L^2_x}
\]
gives
\[
\|u\|_{L^q_t L^2_x} \lesssim \|u\|_{H^{0,b}}
\]
where
\[
\frac{1}{q} = \frac{\theta}{\infty} + \frac{1 - \theta}{2}, \quad b = \theta(1/2 + \varepsilon)
\]
for \( \theta \in [0, 1] \). Thus \( b = \frac{1}{2} - \frac{1}{q} + \varepsilon(1 - \frac{1}{2}) < \frac{1}{2} - \frac{1}{q} + \varepsilon \), and hence (18) follows. This concludes the proof of the Lemma. \( \square \)
3.2. Bilinear estimates. We shall need the standard product estimate for the Sobolev spaces $H^s$, which reads as follows:

**Lemma 4.** Suppose $a_1, a_2, a_3 \in \mathbb{R}$. Then

$$\|fg\|_{H^{-a_3}} \lesssim \|f\|_{H^{a_1}} \|g\|_{H^{a_2}}.$$  \hspace{1cm} (19)

provided

$$a_1 + a_2 + a_3 > 1/2,$$
$$a_1 + a_2 \geq 0, \quad a_1 + a_3 \geq 0, \quad a_2 + a_3 \geq 0.$$  \hspace{1cm} (20)

The following estimate is just the analogue of Lemma 4 for the wave-Sobolev space $H^{s,b}$.

**Lemma 5.** \cite{14, 16]. Suppose $a_1, a_2, a_3 \in \mathbb{R}$ satisfy (20). Let $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma > \frac{1}{2}$. Then

$$\|wz\|_{H^{-a_3-\gamma}} \lesssim \|w\|_{H^{s_1,a_1}} \|z\|_{H^{s_2,a_2}}.$$  \hspace{1cm} (21)

The following comparison estimate between elliptic and hyperbolic weights proved in [16] will be needed in the proof of Lemma 7 below. This estimate is used to identify null structure in bilinear estimates.

**Lemma 6.** Denote

$$\Gamma = |\tau| - |\xi|, \quad \Theta_+ = \lambda + \eta, \quad \Sigma_- = \tau - \lambda - (\xi - \eta).$$

Then

$$\min(|\eta|, |\xi - \eta|) \lesssim \max(|\Gamma|, |\Theta_+|, |\Sigma_-|).$$

We now prove the following null form estimates. We remark that the null structure of DKG in 1d is reflected in the difference of signs in the r.h.s. of the estimate (22), and the difference of signs in the r.h.s. and l.h.s. of the estimates (23) and (24) below; for equal signs the estimates would fail.

**Lemma 7.** Let $b = \frac{1}{2} + \varepsilon$ for sufficiently small $\varepsilon > 0$. The bilinear estimates

$$\|wz\|_{H^{-s_1+b-1}} \lesssim \|w\|_{X^{s_2,b}} \|z\|_{X^{s_3,b}},$$  \hspace{1cm} (22)

$$\|wz\|_{X^{-s_3+b-1}} \lesssim \|w\|_{H^{s_1+b}} \|z\|_{X^{s_2,b}},$$  \hspace{1cm} (23)

$$\|wz\|_{X^{-s_3+b-1}} \lesssim \|w\|_{H^{s_1+b}} \|z\|_{X^{s_2,b}},$$  \hspace{1cm} (24)

hold provided

$$s_1 + s_2 + s_3 > \varepsilon,$$
$$s_2 + s_3 \geq -1/2 + \varepsilon,$$
$$s_1 + s_2 \geq 0, \quad s_1 + s_3 \geq 0.$$  \hspace{1cm} (25)

**Remark 2.** The bilinear estimates (22)–(24) will still hold if we replace $z$ in the l.h.s. of the inequalities in these estimates by $\overline{z}$. We also note that these bilinear estimates will imply the corresponding estimates where the spaces are restricted in time (refer \cite{8} for the detail).

**Proof of Lemma 7.** We only prove (22) and (23), since (24) will follow from (23) by symmetry. We first prove (22).

Set

$$F_+(\lambda, \eta) = \langle \eta \rangle^{s_{2}} (\lambda + \eta)^{b} |\overline{w}(\lambda, \eta)|,$$
$$G_-(\lambda, \eta) = \langle \eta \rangle^{s_{3}} (\lambda - \eta)^{b} |\overline{z}(\lambda, \eta)|.$$
Then (22) is equivalent to

\[ J \lesssim \|F_+\|_{L^2} \|G_-\|_{L^2} \]

where

\[ J := \left\| \int_{\mathbb{R}^{1+1}} F_+(\lambda, \eta) G_-((\tau - \lambda, \xi - \eta)) d\lambda d\eta \right\|_{L^2_{\xi}} \]

where \( \Gamma, \Theta_+ \) and \( \Sigma_- \) are defined as in Lemma 6.

By symmetry, we may assume \( |\eta| \leq |\xi - \eta| \). If \( \max(|\Gamma|, |\Theta_+|, |\Sigma_-|) = |\Gamma| \), then in view of Lemma 6 the estimate for \( J \) reduces to (21) with exponents \((a_1, a_2, a_3) = (s_2 + 1 - b, s_3, s_1)\), \((\alpha, \beta, \gamma) = (b, b, 0)\). If \( \max(|\Gamma|, |\Theta_+|, |\Sigma_-|) = |\Theta_+| \) or \( |\Sigma_-| \), then the estimate for \( J \) reduces to (21) with exponents \((a_1, a_2, a_3) = (s_2 + b, s_3, s_1)\), \((\alpha, \beta, \gamma) = (0, b, 1 - b) \) or \((b, 0, 1 - b) \).

Then the conditions on \((a_1, a_2, a_3)\), (20), will be satisfied (for all the cases above) as long as (25) holds.

Next, we prove (23). By duality, proving the estimate (23) is equivalent to proving

\[ \|wz\|_{H^{-s_1,-b}} \lesssim \|w\|_{X^{s_2,b}} \|z\|_{X^{s_3,1-b}} \],

where \( w, z \) are \( \mathbb{C} \)-valued functions. Define \( F_+ \) as before, and redefine \( G_- \) as

\[ G_-(\lambda, \eta) = \langle \eta \rangle^{s_3} (\lambda - \eta)^{-b} |\tilde{z}(\lambda, \eta)|. \]

Then (26) is equivalent to

\[ L \lesssim \|F_+\|_{L^2} \|G_-\|_{L^2} \]

where

\[ L := \left\| \int_{\mathbb{R}^{1+1}} F_+(\lambda, \eta) G_-(\tau - \lambda, \xi - \eta) d\lambda d\eta \right\|_{L^2_{\xi}} \]

We use the same argument as in the estimate for \( J \) above. In view of Lemma 6 we can add \( 1 - b \) to the exponent of either the weight \( \langle \eta \rangle \) or \( \langle \xi - \eta \rangle \), at the cost of giving up one of the hyperbolic weights \( (\Gamma), (\Theta_+) \) or \( (\Sigma_-) \). Then we apply Lemma 5. In fact, we can reduce the estimate for \( L \) to (21) with exponents \((a_1, a_2, a_3) = (s_2 + 1 - b, s_3, s_1) \) or \((s_2, s_3 + 1 - b, s_1) \). Then the condition (20) is satisfied, since we assume (25).

\[ \square \]

4. I-METHOD AND ALMOST CONSERVATION LAW

Let \( s < 0 \) and \( N > 1 \) be fixed. Define the Fourier multiplier operator

\[ \hat{I}f(\xi) = q(\xi) \hat{f}(\xi), \quad q(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s}|\xi|^s, & |\xi| \geq 2N, \end{cases} \]

(27)

with \( q \) even, smooth and monotone.

Observe that on low frequencies \( \{\xi : |\xi| < N\} \), \( I \) is the identity operator. The operator \( I \) commutes with differential operators. We also have the following properties: For \( a, b \in \mathbb{R} \),

\begin{align*}
|If\|_{H^a} & \lesssim \|f\|_{H^a}, \quad |Iw|\|_{H^{a,b}} \lesssim \|w\|_{H^{a,b}}, \quad (28) \\
|f|\|_{H^a} & \lesssim |If|\|_{L^2} \lesssim N^{-s} \|f\|_{H^a}, \quad (29) \\
|f|\|_{H^a} & \lesssim \|T^2f\|_{H^{a-2s}} \lesssim N^{-2s} \|f\|_{H^a}, \quad (30)
\end{align*}

and if \( \text{supp} \, \tilde{z}(t, \cdot) \subset \{\xi : |\xi| \geq N\} \), we have

\[ \|I^{-1}z\|_{H^{a,b}} \lesssim N^s \|z\|_{H^{a-s,b}}. \]
which in turn implies
\[ \| I_z \|_{H^{a,b}} \lesssim N^s \| I^2_z \|_{H^{a-s,b}}. \]  

(31)

Let \( (s, r) \) be such that \(-\frac{1}{6} < s < 0 \) and \(-s \leq r < \frac{1}{2} + 2s\). Then from the modified LWP theorem which we state in the next section, there exists a \( \Delta T > 0 \) depending on
\[ \| Iu_0 \|_{L^2} + \| Iv_0 \|_{L^2} + \| I^2 \phi_0 \|_{H^{r-2s}} + \| I^2 \phi_1 \|_{H^{r-2s-1}}, \]
such that (3), (4) has solution for times \( 0 \leq t \leq \Delta T \). Of course, (3), (4) has solution for \( (s, r) \) in a larger region as in Theorem 2, but now we reprove the Theorem in the above restricted region with a different time of existence of solution.

Now, we observe using the Fundamental Theorem of Calculus that
\[ \| Iu(\Delta T) \|_{L^2}^2 + \| Iv(\Delta T) \|_{L^2}^2 = \| Iu_0 \|_{L^2}^2 + \| Iv_0 \|_{L^2}^2 + R_1(\Delta T) + R_2(\Delta T), \]
where
\[ R_1(T) = \int_0^T \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau, \]
\[ R_2(T) = \int_0^T \frac{d}{d\tau}(Iv(\tau), Iv(\tau))d\tau, \]
and \((.,.)\) denotes the scalar product in \( L^2 \). By the first equation in (3),
\[ R_1(T) = \int_0^T \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau \]
\[ = 2 \text{Re} \int_0^T \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau \]
\[ = 2 \text{Re} \int_0^T (I [-iMv + i\phi v - u_x] (\tau), Iu(\tau))d\tau \]
\[ = 2 \text{Re} \int_0^T (iM Iu(\tau), Iu(\tau))d\tau + 2 \text{Re} \int_0^T (iI(\phi v)(\tau), Iu(\tau))d\tau \]
\[ + 2 \text{Re} \int_0^T (-Iu_x(\tau), Iu(\tau))d\tau. \]

But the third term is zero. Indeed,
\[ 2 \text{Re} \int_0^T (-Iu_x(\tau), Iu(\tau))d\tau = -2 \text{Re} \int_0^T \int_R \overline{Iu_x(\tau)} Iu(\tau) dxd\tau \]
\[ = - \int_0^T \int_R \left( \overline{Iu(\tau)} Iu(\tau) \right)_x dxd\tau = 0. \]

Hence
\[ R_1(\Delta T) = 2 \text{Re} \int_0^{\Delta T} \int_R -iM Iu(\tau) Iu(\tau) dxd\tau + 2 \text{Re} \int_0^{\Delta T} \int_R iI(\phi v)(\tau) Iu(\tau) dxd\tau. \]

Similarly, by the second equation in (3)
\[ R_2(\Delta T) = 2 \text{Re} \int_0^{\Delta T} \int_R -iM Iv(\tau) Iv(\tau) dxd\tau + 2 \text{Re} \int_0^{\Delta T} \int_R iI(\phi u)(\tau) Iv(\tau) dxd\tau. \]
We therefore get
\[ R(\Delta T) = R_1(\Delta T) + R_2(\Delta T) \]
\[ = 2 \operatorname{Re} \int_0^{\Delta T} \int_{\mathbb{R}} -2Mi \operatorname{Re}(Iu(\tau)\overline{Iv(\tau)})dxd\tau \]
\[ + 2 \operatorname{Re} \int_0^{\Delta T} \int_{\mathbb{R}} iI(\phi u)(\tau)\overline{Iv(\tau)}dxd\tau + 2 \operatorname{Re} \int_0^{\Delta T} \int_{\mathbb{R}} iI(\phi v)(\tau)\overline{Iu(\tau)}dxd\tau \]
\[ = 2 \operatorname{Re} \int_0^{\Delta T} \int_{\mathbb{R}} iI(\phi)(\tau)\overline{Iv(\tau)}dxd\tau + 2 \operatorname{Re} \int_0^{\Delta T} \int_{\mathbb{R}} iI(\phi)(\tau)\overline{Iu(\tau)}dxd\tau. \]

Now, observe that
\[ -iI\phi Iu\overline{v} - iI\phi Iv\overline{u} = -2i\phi \operatorname{Re}(Iu\overline{v}). \]

Using this identity and the fact that \( I\phi \) is real-valued (recall that the multiplier \( q \) is assumed to be even), we obtain
\[ 2 \operatorname{Re} \left[ \int_0^{\Delta T} \int_{\mathbb{R}} -iI\phi(\tau)Iu(\tau)\overline{v(\tau)} + \int_0^{\Delta T} \int_{\mathbb{R}} -iI\phi(\tau)Iv(\tau)\overline{u(\tau)}dxd\tau \right] = 0. \]

We can therefore add this term to \( R(\Delta T) \) for free. We remark that adding this term to \( R(\Delta T) \) gives us a cancellation on the dangerous interaction in frequencies, and this makes it possible for proving some smoothing estimates. This in turn enables us to get the desired almost conservation law (see below for the details).

We can now write
\[ R(\Delta T) = 2 \operatorname{Re} \int_0^{\Delta T} \int_{\mathbb{R}} i\{I(\phi u) - I\phi Iv\}(\tau)\overline{Iv(\tau)}dxd\tau \]
\[ + 2 \operatorname{Re} \int_0^{\Delta T} \int_{\mathbb{R}} i\{I(\phi v) - I\phi Iv\}(\tau)\overline{Iu(\tau)}dxd\tau. \]

We therefore conclude
\[ \|Iu(\Delta T)\|_{L^2}^2 + \|Iv(\Delta T)\|_{L^2}^2 = \|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 + R(\Delta T). \] (32)

The quantity that could make \( \|Iu(\Delta T)\|_{L^2}^2 + \|Iv(\Delta T)\|_{L^2}^2 \) too large in the future is \( R(\Delta T) \). The idea is then to use bilinear estimates to show that locally in time \( R(\Delta T) \) is small. By Plancherel and Cauchy-Schwarz, we obtain
\[ |R(\Delta T)| \lesssim \|I(\phi u) - I\phi Iv\|_{X_{T}^{0,-b}(S_{\Delta T})} \|Iv\|_{X_{T}^{0,b}(S_{\Delta T})} \]
\[ + \|I(\phi v) - I\phi Iv\|_{X_{T}^{0,-b}(S_{\Delta T})} \|Iu\|_{X_{T}^{0,b}(S_{\Delta T})}, \] (33)
for \( b \in \mathbb{R} \).

We denote
\[ Q_I(f,g) = I(fg) - If \cdot Ig. \]

**Lemma 8.** (Smoothing estimate). Suppose
\[ -1/3 < s < 0, \quad -s < r \leq 1 + 2s. \] (34)

Let \( b = \frac{1}{2} + \varepsilon \) for sufficiently small \( \varepsilon > 0 \) depending on \( s, r \). Then
\[ \|Q_I(\phi, u)\|_{X_{T}^{0,-b}(S_{\Delta T})} \leq CN^{r-\varepsilon+2s+2\varepsilon} \|I^2\phi\|_{H_{T}^{r-2,\varepsilon}(S_{\Delta T})} \|Iu\|_{X_{T}^{0,b}(S_{\Delta T})}, \] (35)
\[ \|Q_I(\phi, v)\|_{X_{T}^{0,-b}(S_{\Delta T})} \leq CN^{r-\varepsilon+2s+2\varepsilon} \|I^2\phi\|_{H_{T}^{r-2,\varepsilon}(S_{\Delta T})} \|Iv\|_{X_{T}^{0,b}(S_{\Delta T})}, \] (36)
where \( C \) depends on \( s, r, \varepsilon, \) but not \( N \) or \( \Delta T \).
In order to apply the $I$-method, we need a variant of Theorem 2, which we call a modified LWP Theorem for the $I$-modified equation

\[
\begin{cases}
    iD_+(Iu) = MIu - I(\phi v), \\
    iD_-(Iv) = MIv - I(\phi u), \\
    \Box(I^2\phi) = m^2I^2\phi - 2I^2(\text{Re}(uv)),
\end{cases}
\]

which is obtained from (3) by applying $I$. The corresponding $I$-initial data obtained from (4) are

\[
\begin{cases}
    Iu(0) = Iu_0 \in L^2, & Iv(0) = Iv_0 \in L^2, \\
    I^2\phi(0) = I^2\phi_0 \in H^{r-2s}, & \partial_t I^2\phi(0) = I^2\phi_1 \in H^{r-2s-1}.
\end{cases}
\]

Combining (32), (33), (35) and (36) we obtain, for $s, r$ and $\varepsilon$ as in Lemma 8,

\[
\begin{align*}
    \|Iu(\Delta T)||^2_{L^2} + \|Iv(\Delta T)||^2_{L^2} & \\
    & \leq \|Iu_0||^2_{L^2} + \|Iv_0||^2_{L^2} + C N^{-r+2s+\varepsilon} \|I^2\phi\||_{H^{r-2s}(S_{\Delta T})} \|Iu||_{X^{0,b}(S_{\Delta T})} \|Iv||_{X^{0,b}(S_{\Delta T})},
\end{align*}
\]

where $C$ depends on $s$, $r$ and $\varepsilon$, but not $N$ or $\Delta T$.

In view of (29) and (30), we have

\[
\begin{align*}
    \|Iu_0||^2_{L^2} + \|Iv_0||^2_{L^2} & \leq AN^{-s}, \\
    \|I^2\phi_0||_{H^{r-2s}} + \|I^2\phi_1||_{H^{r-2s-1}} & \leq BN^{-2s},
\end{align*}
\]

for some $A, B > 0$. Here, $A$ depends on $\|u_0||^2_{L^2} + \|v_0||^2_{L^2}$ whereas $B$ depends on $\|\phi_0||_{H^r} + \|\phi_1||_{H^{r-1}}$.

We now state the modified LWP theorem which will be proved in the last section.

**Theorem 5.** Suppose

\[
-\frac{1}{6} < s < 0, \quad -s \leq r < \frac{1}{2} + 2s,
\]

Let $b = \frac{1}{2} + \varepsilon$ for sufficiently small $\varepsilon > 0$ depending on $s, r$. Assume also that $A$ and $B$ in (40) are such that

\[
C(B + A^2)(N^{-2\varepsilon} + N^{-r+2\varepsilon}) \leq 1.
\]

Then there exists

\[
\Delta T \approx N^{(s-\varepsilon)/(r-2s-2\varepsilon)}
\]

such that (3), (4) has a unique solution

\[
(u, v, \phi) \in X^{b,b}_+(S_{\Delta T}) \times X^{b,b}_-(S_{\Delta T}) \times H^{r,b}(S_{\Delta T})
\]

on the time interval $0 \leq t \leq \Delta T$. Moreover,

\[
\begin{align*}
    \|Iu||_{X^{0,b}(S_{\Delta T})} + \|Iv||_{X^{0,b}(S_{\Delta T})} & \leq CN^{-s}, \\
    \|I^2\phi||_{H^{r-2s,b}(S_{\Delta T})} & \leq C(B + A^2)N^{-2s},
\end{align*}
\]

where $C$ depends on $s$, $r$ and $\varepsilon$, but not $N$ or $\Delta T$.

Combining (39), (44) and (45) we conclude the following almost conservation law:
Corollary 1. Let \( s, r, \Delta T, \varepsilon, A, B, u \) and \( v \) be as in Theorem 5. Then
\[
\| I_u(\Delta T) \|_{L^2}^2 + \| I_v(\Delta T) \|_{L^2}^2 \leq \| I_u0 \|_{L^2}^2 + \| I_v0 \|_{L^2}^2 + C(B + A^2)A^2 N^{-r-2s+2\varepsilon}. \tag{46}
\]

As a consequence of this Corollary and (40), we obtain
\[
\| I_u(\Delta T) \|_{L^2}^2 + \| I_v(\Delta T) \|_{L^2}^2 \leq A^2 N^{-2s} + C(B + A^2)A^2 N^{-r-2s+2\varepsilon}. \tag{47}
\]

We also need to control the growth of \( I^2 \phi \). To do so, we first split \( \phi \) into its homogeneous and inhomogeneous parts. Let \( \phi^{(0)} \) be solution of the homogeneous Klein-Gordon Cauchy problem
\[
\begin{cases}
(\square - m^2) \phi^{(0)} = 0 \\
\phi^{(0)}(0) = \phi_0, \quad \partial_t \phi^{(0)}(0) = \phi_1.
\end{cases}
\tag{48}
\]
Then we write
\[
\phi = \phi^{(0)} + \Phi,
\]
where
\[
\Phi = (\square - m^2)^{-1} (-2(\text{Re}(u\overline{v}))). \tag{49}
\]
Here \( (\square - m^2)^{-1} F \) denotes the solution of \( (\square - m^2) w = F \) with vanishing initial data.

The solution of the homogeneous Cauchy problem (48) in Fourier space is given by
\[
\hat{\phi}^{(0)}(t)(\xi) = \cos(t\xi_m)\hat{\phi}_0(\xi) + \frac{\sin(t\xi_m)}{\xi_m}\hat{\phi}_1(\xi). \tag{50}
\]
Then by the energy estimate we have
\[
\left\| I^2 \phi^{(0)}[t] \right\|_{H^{r-2s}} \leq C\left( \| I^2 \phi_0 \|_{H^{r-2s}} + \| I^2 \phi_1 \|_{H^{r-2s}} \right), \tag{51}
\]
for some \( C > 0 \) and for all \( t \geq 0 \).

Now, consider the inhomogeneous part, (49). Since the multiplier \( q \) is assumed to be even, we obtain
\[
I^2 \text{Re}(u\overline{v}) = \text{Re}(I^2(u\overline{v})) = \text{Re}(I(Iu \cdot I\overline{v})) + \text{Re}(IQ_I(u, \overline{v})).
\]
Using this identity, we write
\[
I^2 \Phi = (\square - m^2)^{-1} (-2 \text{Re}(I(Iu \cdot I\overline{v}))) + (\square - m^2)^{-1} (-2 \text{Re}(IQ_I(u, \overline{v}))). \tag{52}
\]
We then prove the following:

Lemma 9. Suppose
\[
-1/4 < s < 0, \quad 0 < r < 1/2 + 2s. \tag{53}
\]
Let \( b = \frac{1}{2} + \varepsilon \) for sufficiently small \( \varepsilon > 0 \) depending on \( s, r, \) and \( \Delta T \) be as in Theorem 5. Then
\[
\| I^2 \Phi[\Delta T] \|_{H^{r-2s}} \leq C\Delta T(\| I_u0 \|_{L^2}^2 + \| I_v0 \|_{L^2}^2)
+ C\Delta T N^{-r+2s+2\varepsilon} \| I^2 \phi \|_{H^{r-2s,b}(S\Delta T)} \| I_u \|_{X^0_{r,b}(S\Delta T)} \| I_v \|_{X^0_{r,b}(S\Delta T)}
+ CN^{-1/2+2\varepsilon} \| I_u \|_{X^0_{r,b}(S\Delta T)} \| I_v \|_{X^0_{r,b}(S\Delta T)}, \tag{54}
\]
where \( C \) depends on \( s, r, \) and \( \varepsilon, \) but not \( N \) or \( \Delta T \).

Then, by (40), (44), (45) and (54) we conclude
Corollary 2. Let $A$, $B$, $\Delta T$ be as in Theorem 5 and $s, r, \varepsilon$ be as in Lemma 9. Then
\[
\|T^2 \hat{\Phi}[\Delta T]\|_{H^{r-2s}} \leq CA^2 \left( \Delta TN^{-2s} + (B + A^2)\Delta T N^{-r-2s+2\varepsilon} + N^{-1/2-2s+2\varepsilon} \right).
\] (55)

By (40) and (51), we also have
\[
\|T^2 \phi^{(0)}[t]\|_{H^{r-2s}} \leq CBN^{-2s},
\] (56)
for some $C > 0$ and for all $t \geq 0$.

5. Proof of Theorem 4

We first remark that by propagation of higher regularity (see Remark 1.4 in [15] for the detail on this argument), it suffices to prove Theorem 4 for $r < 1/2 + 2s$. We therefore fix $s$ and $r$ satisfying
\[
-\frac{1}{8} < s < 0, \quad s + \sqrt{s^2 - s} < r < \frac{1}{2} + 2s.
\] (57)

Observe that this region is contained in the intersection of the regions in (34), (41) and (53), so the statements made in Theorem 5, Lemmas 8 and 9, Corollaries 1 and 2, (47) and (55) hold true for $s, r$ satisfying (57).

Global well-posedness of (3), (4) will follow if we show well-posedness on $[0, T]$ for arbitrary $0 < T < \infty$. We have already shown in Theorem 5 that (3), (4) is well-posed on $[0, \Delta T]$, where the size of $\Delta T$ is given by (43). Now, we divide the interval $[0, T]$ into subintervals of length $\Delta T$. Let $K$ be the number of subintervals, so
\[
K = \frac{T}{\Delta T} \approx N^{(-s+\varepsilon)/(r-2s-2\varepsilon)}.
\] (58)

To reach the given time $T$, we need to advance the solution from $\Delta T$ to $2\Delta T$ etc. up to $K\Delta T$, successively.

We shall use induction argument to show well-posedness of (3), (4) up to time $T$. We denote the solution of (3), (4) on the $n$-th subinterval $[(n-1)\Delta T, n\Delta T]$, where $1 \leq n \leq K$, by $(u_n, v_n, \phi_n)$. Now, consider the DKG system
\[
\begin{align*}
iD_+u_n &= Mu_n - \phi_n v_n, \\
iD_-v_n &= Mv_n - \phi_n u_n, \\
\Box \phi_n &= m^2 \phi_n - 2 \text{Re}(u_n \overline{v_n}).
\end{align*}
\] (59)

The initial data for this system at $t = (n-1)\Delta T$ is specified by the induction scheme
\[
\begin{align*}
u_n((n-1)\Delta T) &= u_{n-1}((n-1)\Delta T) \in H^s, \\
\phi_n((n-1)\Delta T) &= \phi_{n-1}((n-1)\Delta T) \in H^s, \\
\partial_t \phi_n((n-1)\Delta T) &= \partial_t \phi_{n-1}((n-1)\Delta T) \in H^{s-1}.
\end{align*}
\] (60)

The corresponding $I$-system will be
\[
\begin{align*}
iD_+(Iu_n) &= MIu_n - I(\phi_n v_n), \\
iD_-(Iv_n) &= MIv_n - I(\phi_n u_n), \\
\Box (I^2 \phi_n) &= m^2 I^2 \phi_n - 2I^2(\text{Re}(u_n \overline{v_n})).
\end{align*}
\] (61)
with the $I$-initial data at $t = (n - 1)\Delta T$:
\[
\begin{align*}
Iu_{n}(n - 1)\Delta T &= Iu_{n-1}(n - 1)\Delta T \in L^2, \\
IV_{n}(n - 1)\Delta T &= IV_{n-1}(n - 1)\Delta T \in L^2, \\
I^2\phi_{n}(n - 1)\Delta T &= I^2\phi_{n-1}(n - 1)\Delta T \in H^{r-2s}, \\
\partial_t I^2\phi_{n}(n - 1)\Delta T &= \partial_t I^2\phi_{n-1}(n - 1)\Delta T \in H^{r-2s-1}.
\end{align*}
\] (62)

Note that for $n = 1$, this $I$-initial value problem corresponds to (37), (38).

In the following estimates and the rest of this section we shall use the notation
\[S_{n\Delta T} = [(n - 1)\Delta T, n\Delta T] \times \mathbb{R}.
\]
Recall that $(u_n, v_n, \phi_n)$ is a solution of DKG on the $n$th subinterval $[(n-1)\Delta T, n\Delta T]$ for given data at $t = (n - 1)\Delta T$. Then in view of (39) we have
\[
\begin{align*}
\|Iu_n(n\Delta T)\|_{L^2}^2 + \|IV_n(n\Delta T)\|_{L^2}^2 &
\leq \|Iu_n((n - 1)\Delta T)\|_{L^2}^2 + \|IV_n((n - 1)\Delta T)\|_{L^2}^2 \\
&+ CN^{-r+2s+2\varepsilon} \|I^2\phi_n\|_{H^{r-2s, b}(S_{n\Delta T})} \|Iu_n\|_{X^{0, b}_{n\Delta T}} \|IV_n\|_{X^{0, b}_{n\Delta T}}.
\end{align*}
\] (63)

On the other hand, splitting $\phi_n$ into its homogeneous and inhomogeneous parts, $\phi_n = \phi_n^{(0)} + \Phi_n$, we have in view of (51) and (54)
\[
\begin{align*}
\|I^2\Phi_n[n\Delta T]\|_{H^{r-2s}} &
\leq C\Delta T(\|Iu_n(n - 1)\Delta T\|_{L^2}^2 + \|IV_n((n - 1)\Delta T)\|_{L^2}^2) \\
&+ C\Delta TN^{-r+2s+2\varepsilon} \|I^2\phi_n\|_{H^{r-2s, b}(S_{n\Delta T})} \|Iu_n\|_{X^{0, b}_{n\Delta T}} \|IV_n\|_{X^{0, b}_{n\Delta T}} \\
&+ CN^{-1/2+2\varepsilon} \|Iu_n\|_{X^{0, b}_{n\Delta T}} \|IV_n\|_{X^{0, b}_{n\Delta T}},
\end{align*}
\] (64)

and
\[
\sup_{0 \leq t \leq T} \|I^2\phi_n^{(0)}(t)\|_{H^{r-2s}} \leq C \|I^2\phi_n[(n - 1)\Delta T]\|_{H^{r-2s}}.
\] (65)

Our induction hypotheses will be
\[
\begin{align*}
\|Iu_n(n - 1)\Delta T\|_{L^2}^2 + \|IV_n((n - 1)\Delta T)\|_{L^2}^2 &\leq A_n N^{-s}, \\
\|I^2\phi_n(n - 1)\Delta T\|_{H^{r-2s}} &\leq B_n N^{-2s},
\end{align*}
\] (66) (67)

for some $1 \leq n \leq K$, where $A_n$ and $B_n$ are independent of $N$. Again, at the first induction step, $n = 1$, (66) and (67) hold by (40). Now, by Theorem 5 we know that $(u_n, v_n, \phi_n)$ solves (59), (60) on the $n$-th subinterval $[(n - 1)\Delta T, n\Delta T]$ where the size of $\Delta T$ is given by (43), provided that the bootstrap condition
\[
C(B_n + A_n^2)(N^{-2\varepsilon} + N^{-r+2\varepsilon}) \leq 1
\] (68)
is satisfied. Moreover, these solutions satisfy the bound
\[
\begin{align*}
\|Iu_n\|_{X^{0, b}_{n\Delta T}} + \|IV_n\|_{X^{0, b}_{n\Delta T}} &\leq CA_n N^{-s}, \\
\|I^2\phi_n\|_{H^{r-2s, b}(S_{n\Delta T})} &\leq C(B_n + A_n^2)N^{-2s}.
\end{align*}
\] (69) (70)

So, if we can prove that $A_n$ and $B_n$ stay bounded for all $1 \leq n \leq K$, then (68) will be satisfied for all $1 \leq n \leq K$, choosing $\varepsilon$ small enough and $N$ large enough (recall $r > 0$). We can therefore apply Theorem 5 $K$ times, and hence prove well-posedness on $[0, T]$. 
By (69), (70) and the induction hypotheses (66) and (67), the estimates (63) and (64) imply

$$\|Iu_n(n\Delta T)\|^2_{L^2} + \|Iv_n(n\Delta T)\|^2_{L^2} \leq A_n^2 N^{-2s} + C(B_n + A_n^2)A_n^2 N^{-r-2s+2\epsilon}, \quad (71)$$

$$\|I^2\Phi_n[n\Delta T]\|_{H^{r-2s}} \leq CA_n^2 \left( \Delta TN^{-2s} + (B_n + A_n^2)\Delta TN^{-r-2s+2\epsilon} + N^{-1/2-2s+2\epsilon} \right), \quad (72)$$

whereas (65) and (67) imply

$$\sup_{0 \leq t \leq T} \left\| I^2\phi_n^{(0)}[t] \right\|_{H^{r-2s}} \leq CB_n N^{-2s}. \quad (73)$$

By (62) and (71) we obtain

$$\|Iu_{n+1}(n\Delta T)\|^2_{L^2} + \|Iv_{n+1}(n\Delta T)\|^2_{L^2} = \|Iu_n(n\Delta T)\|^2_{L^2} + \|Iv_n(n\Delta T)\|^2_{L^2} \leq A_n^2 N^{-2s} + C(B_n + A_n^2)A_n^2 N^{-r-2s+2\epsilon}.$$

We therefore have

$$A_{n+1}^2 \leq A_n^2 + C(B_n + A_n^2)A_n^2 N^{-r+2\epsilon}. \quad (74)$$

On the other hand, by (62), (72) and (73) we get

$$\left\| I^2\phi_{n+1}[n\Delta T] \right\|_{H^{r-2s}} = \left\| I^2\phi_n[n\Delta T] \right\|_{H^{r-2s}} \leq \left\| I^2\phi_n^{(0)}[n\Delta T] \right\|_{H^{r-2s}} + \left\| I^2\Phi_n[n\Delta T] \right\|_{H^{r-2s}} \leq CB_n N^{-2s} + CA_n^2 \left( \Delta TN^{-2s} + (B_n + A_n^2)\Delta TN^{-r-2s+2\epsilon} + N^{-1/2-2s+2\epsilon} \right).$$

Therefore,

$$B_{n+1} \leq CB_n + CA_n^2 \Delta T + C(B_n + A_n^2)A_n^2 \Delta TN^{-r-2s+2\epsilon} + CA_n^2 N^{-1/2+2\epsilon}. \quad (75)$$

However, the presence of a constant C in front of $B_n$ in the first term of the r.h.s. of this inequality is bad, since then $B_n$ will grow exponentially in $n$; after $n$ induction steps, $B_n \approx C^n$. To fix this problem, we follow [15] to write $\phi_n^{(0)}$ as a cascade of free waves:

$$\phi_n^{(0)} = \phi_1^{(0)} + \phi_2^{(0)} + \cdots + \phi_n^{(0)} + \phi_{n+1}^{(0)},$$

for $n \geq 1$, where

$$\left\{ \begin{array}{l}
\Box - m^2 \phi_n^{(0)} = 0 \\
\phi_n^{(0)}(n\Delta T) = \Phi_n(n\Delta T), \\
\partial_t \phi_n^{(0)}(n\Delta T) = \partial_t \Phi_n(n\Delta T).
\end{array} \right. \quad (76)$$

Now, by energy inequality and (72) we have

$$\left\| I^2\phi_{n+1}^{(0)}[t] \right\|_{H^{r-2s}} \leq CA_n^2 \left( \Delta TN^{-2s} + (B_n + A_n^2)\Delta TN^{-r-2s+2\epsilon} + N^{-1/2-2s+2\epsilon} \right), \quad (77)$$

in the entire time interval $0 \leq t \leq T$.

We now replace the induction hypothesis (67) by the stronger condition

$$\sup_{0 \leq t \leq T} \left\| I^2\phi_n^{(0)}[t] \right\|_{H^{r-2s}} \leq B_n N^{-2s}. \quad (78)$$

Since $\phi_{n+1}^{(0)} = \phi_n^{(0)} + \phi_{n+1}^{(0)}$, we have

$$\left\| I^2\phi_{n+1}^{(0)}[t] \right\|_{H^{r-2s}} \leq \left\| I^2\phi_n^{(0)}[t] \right\|_{H^{r-2s}} + \left\| I^2\phi_{n+1}^{(0)}[t] \right\|_{H^{r-2s}},$$
for all $0 \leq t \leq T$. Then using (78) and (77), we conclude
\begin{equation}
B_{n+1} \leq B_n + CA_n^2 \Delta T + C(B_n + A_n^2)A_n^2 \Delta TN^{-r+2\varepsilon} + CA_n^2 N^{-1/2+2\varepsilon}.
\end{equation}
This estimate will be a replacement for the “bad” estimate (75).

Now, we claim that if $\varepsilon > 0$ is chosen small enough, and then $N$ large enough, depending on $\varepsilon$, then for $1 \leq n \leq K$,
\begin{equation}
A_n \leq \rho \equiv 2A_1, \quad B_n \leq \sigma \equiv 2B_1 + 4CTA_1^2.
\end{equation}
We proceed by induction. Assume that (80) holds for $1 \leq n < k$, for some $k \leq K$.
Then (68) reduces to
\begin{equation}
C(\sigma + \rho^2)(N^{-2\varepsilon} + N^{-r+2\varepsilon}) \leq 1,
\end{equation}
for $n < k$. Since $r > 0$, we can choose $\varepsilon$ very small and $N$ very large to ensure that (81) is satisfied. So by (74) and (79), and the assumption that (80) holds for $n < k$, we get (for $n < k$)
\begin{align*}
A_{n+1}^2 & \leq A_1^2 + nC\sigma^2 N^{-r+2\varepsilon} \\
B_{n+1} & \leq B_1 + n\left[C\rho^2 \Delta T + C\sigma^2 \Delta TN^{-r+2\varepsilon} + C\rho^2 N^{-1/2+2\varepsilon}\right].
\end{align*}
Furthermore, (80) will be satisfied for $A_k$ and $B_k$ provided that
\begin{align*}
(k-1)C\rho^2 N^{-r+2\varepsilon} & \leq 3A_1^2 \\
(k-1) \left(C\rho^2 \Delta T + C\sigma^2 \Delta TN^{-r+2\varepsilon} + C\rho^2 N^{-1/2+2\varepsilon}\right) & \leq B_1 + 4CTA_1^2.
\end{align*}
Now, since $k \leq K = T/(\Delta T) \leq C\eta N^{(-s+\varepsilon)/(r-2s-2\varepsilon)}$, by (43), it suffices to have
\begin{align*}
C\sigma^2 N^{-s+\varepsilon}/(r-2s-2\varepsilon) & \leq 3A_1^2, \\
CT\sigma^2 N^{-r+2\varepsilon} & \leq B_1/2, \\
C\rho^2 N^{(-s+\varepsilon)/(r-2s-2\varepsilon)-1/2+2\varepsilon} & \leq B_1/2, \\
CT\rho^2 & \leq 4CTA_1^2.
\end{align*}
Here, to get the l.h.s. of (85) we used the fact that $(k-1)\Delta T \leq K\Delta T = T$; In fact, (85) holds with equality, since $\rho = 2A$. Since $r > 0$, (83) will be satisfied by choosing first $\varepsilon$ small enough and then $N$ sufficiently large. To satisfy (82) and (84), it suffices to have
\begin{equation}
\frac{-s+\varepsilon}{r-2s-2\varepsilon} - r + 2\varepsilon < 0, \quad \frac{-s+\varepsilon}{r-2s-2\varepsilon} - 1/2 + 2\varepsilon < 0.
\end{equation}
The first condition is equivalent to $r^2 - 2sr + s > \varepsilon(4(r-s) + 1 - 4\varepsilon)$. Choosing $\varepsilon > 0$ very small, this reduces to $r^2 - 2sr + s > 0$, i.e., $r > s + \sqrt{s^2 - s}$, which holds by assumption (57). The second condition in (86) is weaker than the first condition since by assumption (57), $r < 1/2 + 2s$ and $s < 0$.
Thus, (80) holds for $n = 1, \cdots, K$, and hence the proof is complete.

6. PROOF OF LEMMA 8

Taking the Fourier transform in space, we get
\begin{equation}
[Q_f(f, g)]_{\xi} = \int \left[\hat{q}(\xi) - \bar{q}(\eta)\bar{q}(\xi - \eta)\right] \hat{f}(\eta)\hat{g}(\xi - \eta) d\eta.
\end{equation}
Recall that the symbol $q(\xi) = 1$ for $|\xi| < N$. 

We now write \( u = u_1 + u_h, v = v_1 + v_h, \phi = \phi_1 + \phi_h \) with \( \hat{u}_1, \hat{v}, \hat{\phi}_1 \) supported on \( \{ \xi : |\xi| < N \} \) and \( \hat{u}_h, \hat{v}_h, \hat{\phi}_h \) supported on \( \{ \xi : |\xi| \geq N \} \). Since we are considering (weighted) \( L^2 \) norms, we can replace \( \hat{u}, \hat{v} \) and \( \hat{\phi} \) by \( |\hat{u}|, |\hat{v}| \) and \( |\hat{\phi}| \). Assume therefore that \( \hat{u}, \hat{v}, \hat{\phi} \geq 0 \).

We only prove (35) since the proof for (36) is quite similar. The only difference is that to prove (35), we use the product estimate (23), but to prove (36), we use (24). We prove (35) for all possible interactions. As a matter of convenience we skip the time restriction in this section.

6.1. Low/low interaction. Recalling (87), we have

\[
[Q_I(\phi_1, u_1)](\zeta) = \int [q(\xi) - q(\eta)q(\xi - \eta)]\hat{\phi}_1(\eta)\hat{u}_1(\xi - \eta)\,d\eta.
\]

But since \( |\eta|, |\xi - \eta| \ll N \), which in turn implies \( |\xi| < N \), the expression inside the square bracket in the above integral vanishes.

6.2. Low/high interaction. Then

\[
[Q_I(\phi_1, u_h)](\zeta) = \int [q(\xi) - q(\eta)q(\xi - \eta)]\hat{\phi}_1(\eta)\hat{u}_h(\xi - \eta)\,d\eta,
\]

because \( q(\eta) = 1 \) on the support of \( \hat{\phi}_1 \). By the mean value theorem,

\[
|q(\xi) - q(\xi - \eta)| \leq |q'(\zeta)||\eta|,
\]

where \( \zeta \) lies between \( \xi \) and \( \xi - \eta \).

Now, assume \( |\xi - \eta| \gg N \). Then \( |\eta| \ll |\xi - \eta| \), and this implies

\[
|\xi| \approx |\xi - \eta| \approx |\zeta|.
\]

Hence

\[
|q'(\zeta)| \approx N^{-s} |\zeta|^{s-1} \approx N^{-s} |\xi - \eta|^{s-1}.
\]

Next, assume \( |\xi - \eta| \approx N \). If \( |\zeta| < N \), then \( q'(\zeta) = 0 \). If \( |\zeta| > 2N \), then

\[
|q'(\zeta)| \approx N^{-s} |\zeta|^{s-1} \approx N^{-s} |\xi - \eta|^{s-1}.
\]

Finally, assume \( N \leq |\zeta| \leq 2N \). In this case, we define \( q(\xi) = \chi(\xi/N) \) where \( \chi \) is a smooth, even and monotone function defined by

\[
\chi(\sigma) = \begin{cases} 1 & \text{if } 0 \leq \sigma < 1, \\ \sigma^s & \text{if } \sigma > 1. \end{cases}
\]

Then

\[
|q'(\zeta)| \lesssim N^{-1} \lesssim N^s |\xi - \eta|^{s-1}.
\]

We therefore conclude

\[
|q(\xi) - q(\xi - \eta)| \lesssim N^{-s} |\xi - \eta|^{s-1} |\eta|.
\]

Interpolating this with the trivial estimate

\[
|q(\xi) - q(\xi - \eta)| \lesssim N^{-s} |\xi - \eta|^s
\]

we get

\[
|q(\xi) - q(\xi - \eta)| \lesssim N^{-s} |\xi - \eta|^s |\xi - \eta|^{-\theta} |\eta|^\theta,
\]

for \( \theta \in [0, 1] \).
Then
\[
|Q_I(\phi_I, u_h)|^2(\xi) \lesssim \int |\eta|^\theta \hat{\phi}_I(\eta) |\xi - \eta|^{-\theta} N^{-s} |\xi - \eta|^s \hat{u}_h(\xi - \eta) d\eta
\]
\[
\lesssim |D^\theta \hat{\phi}_I \cdot D^{-\theta} \hat{u}_h|^2(\xi).
\]
(88)

Now, choosing \(\theta = r - 2s\) and applying the product estimate (23), we get
\[
\|Q_I(\phi_I, u_h)\|_{X^{0,-b}} \lesssim \|D^{r-2s} \phi_I \cdot D^{-r+s+2s} u_h\|_{X^{0,-b}}
\]
\[
\lesssim \|D^{r-2s} \phi_I\|_{H^{0,b}} \|D^{-r+s+2s} u_h\|_{X^{2r,b}}
\]
\[
\lesssim N^{-r+s+2s} \|\phi_I\|_{H^{-r-2s,b}} \|u_h\|_{X^{0,b}}.
\]

6.3. **High/low interaction.** A calculation similar to the preceding low/high interaction estimate gives
\[
|Q_I(\phi_h, u_t)|^2(\xi) \lesssim |D^{-\theta} \phi_h \cdot D^\theta u_t|^2(\xi).
\]
Take \(\theta = 0\). Applying the product estimate (23) and (31), we get
\[
\|Q_I(\phi_h, u_t)\|_{X^{0,-b}} \lesssim \|I(\phi_h u_h)\|_{X^{0,-b}} + \|I \phi_h \cdot I u_h\|_{X^{0,-b}}.
\]

6.4. **High/high interaction.** Here, we do not take advantage of any cancellation. We instead use the triangle inequality to get
\[
\|Q_I(\phi_h, u_h)\|_{X^{0,-b}} \leq \|I(\phi_h u_h)\|_{X^{0,-b}} + \|I \phi_h \cdot I u_h\|_{X^{0,-b}}.
\]
By (28), the product estimate (23), and (31), we get
\[
\|I(\phi_h u_h)\|_{X^{0,-b}} \lesssim \|\phi_h u_h\|_{X^{0,-b}}
\]
\[
\lesssim \|\phi_h\|_{H^{-r+s+2s,b}} \|u_h\|_{X^{2r,b}}
\]
\[
= \|\phi_h\|_{H^{-r-s+2s,b}} N^s N^{-s} \|u_h\|_{X^{0,b}}
\]
\[
\lesssim N^{-r-s+2s} \|\phi_h\|_{H^{-r-s+2s,b}} N^s \|u_h\|_{X^{0,b}}
\]
\[
= N^{-r+s+2s} \|\phi_h\|_{H^{-r-s+b}} \|u_h\|_{X^{0,b}}
\]
\[
\lesssim N^{-r+s+2s} \|I^2 \phi_h\|_{H^{-r-2s,b}} \|u_h\|_{X^{0,b}}.
\]
and
\[
\|I \phi_h \cdot I u_h\|_{X^{0,-b}} \lesssim \|I \phi_h\|_{H^{2r,b}} \|I u_h\|_{X^{0,b}}
\]
\[
\lesssim N^{-r+s+2s} \|I \phi_h\|_{H^{-r-s,b}} \|I u_h\|_{X^{0,b}}
\]
\[
\lesssim N^{-r+s+2s} \|I^2 \phi_h\|_{H^{-r-2s,b}} \|I u_h\|_{X^{0,b}}.
\]
7. Proof of Lemma 9

First, we estimate the first term in the right hand side of (52). By energy inequality, (28), Lemma 4 and (39) we get (recall that \( r < 1/2 + 2s \))

\[
\left\| \left( \Box - m^2 \right)^{-1} 2 \Re(Iu \cdot Iu) \right\|_{H^{r-2s}_T} \leq C \int_0^T \left\| \Re(I(Iu(t) \cdot Iu(t))) \right\|_{H^{r-2s-1}_T} dt
\]

\[
\leq C \int_0^T \left\| Iu(t) \right\|_{L^2_x} \left\| Iv(t) \right\|_{L^2_x} dt
\]

\[
\leq C \int_0^T \left\| Iu(t) \right\|_{L^2_x}^2 + \left\| Iv(t) \right\|_{L^2_x}^2 dt
\]

\[
\leq C \Delta T \left( \left\| Iu_0 \right\|_{L^2_x}^2 + \left\| Iv_0 \right\|_{L^2_x}^2 \right)
\]

\[
+ C \Delta T N^{-1/2+2\varepsilon} \left\| I^2 \phi \right\|_{H^{r-2s,0}(S_T)} \left\| Iu \right\|_{X^0_+(S_T)} \left\| Iv \right\|_{X^0_-(S_T)}.
\]

Now, we estimate the second term in the right hand side of (52). We claim that

\[
\left\| \left( \Box - m^2 \right)^{-1} 2 \Re(IQ_I(u, \tau)) \right\|_{H^{r-2s}_T} \leq CN^{-1/2+2\varepsilon} \left\| Iu \right\|_{X^0_+(S_T)} \left\| Iv \right\|_{X^0_-(S_T)}.
\]

Assume for the moment that this claim is true. Then a combination of the estimates (52), (89) and (90) proves the Lemma.

It remains to prove the claim, (90). By (16), Lemma 2 and (28)

\[
\left\| \left( \Box - m^2 \right)^{-1} \Re(IQ_I(u, \tau)) \right\|_{H^{r-2s}_T} \leq C \left\| \left( \Box - m^2 \right)^{-1} \Re(IQ_I(u, \tau)) \right\|_{H^{r-2s,b}(S_T)}
\]

\[
\leq C \left\| IQ_I(u, \tau) \right\|_{H^{r-2s-1,b-1}(S_T)}
\]

\[
\leq C \left\| Q_I(u, \tau) \right\|_{H^{r-2s-1,b-1}(S_T)}
\]

Then to estimate \( \left\| Q_I(u, \tau) \right\|_{H^{r-2s-1,b-1}(S_T)} \) we follow a similar argument as in the preceding subsection. As a matter of convenience we skip the time restriction in the rest of the section. The contribution from the low/low frequency interaction, \( Q_I(u_l, v_l) \), vanishes by the same argument as in the low/low frequency case in the preceding section. For the low/high frequency case we use (88) with \( \theta = 0 \) (the high/low frequency case is similar) to get

\[
\left\| Q_I(u_l, \tau_l) \right\|_{X^{0,b}_H} \lesssim \left\| u_l \cdot \tau_l \right\|_{X^{-1/2+2\varepsilon,b}}
\]

\[
\lesssim N^{-1/2+2\varepsilon} \left\| u_l \right\|_{X^0_+} \left\| Iv_h \right\|_{X^0_+}.
\]
To estimate the contribution from high/high interaction, we first use the triangle inequality to get

\[ \| Q_I(u_h, \overline{v}_h) \|_{H^{r-2s-1,b-1}} \leq \| I(u_h \overline{v}_h) \|_{H^{r-2s-1,b-1}} + \| I u_h \cdot I \overline{v}_h \|_{H^{r-2s-1,b-1}}. \]

Then applying (22), we obtain

\[ \| I(u_h \overline{v}_h) \|_{H^{r-2s-1,b-1}} \lesssim \| u_h \overline{v}_h \|_{H^{r-2s-1,b-1}} \]
\[ \lesssim \| u_h \|_{X^{-1/4, b}} \| \overline{v}_h \|_{X^{-1/4+2e, b}} \]
\[ \lesssim N^{-1/4-s} \| u_h \|_{X^{s,b}} N^{-1/4-s+2e} \| \overline{v}_h \|_{X^{s,b}} \]
\[ \lesssim N^{-1/2+2e} \| u_h \|_{X^{s,b}} \| \overline{v}_h \|_{X^{s,b}}, \]

and

\[ \| I u_h \cdot I \overline{v}_h \|_{H^{r-2s-1,b-1}} \lesssim \| I u_h \|_{X^{-1/4, b}} \| I \overline{v}_h \|_{X^{-1/4+2e, b}} \]
\[ \lesssim N^{-1/2+2e} \| I u_h \|_{X^{s,b}} \| I \overline{v}_h \|_{X^{s,b}}. \]

8. Proof of Theorem 5

Assume \( 0 < \Delta T < 1 \). Define

\[ \| I w \|_{X^{0,b}(S_{\Delta T})} = \| I u \|_{X^{0,b}(S_{\Delta T})} + \| I v \|_{X^{0,b}(S_{\Delta T})}, \]
\[ \| I w_0 \|_{L^2} = \| I u_0 \|_{L^2} + \| I v_0 \|_{L^2}. \]

Applying Lemma 1 to the first two equations and Lemma 2 to the third equation of the \( I \)-system (37), we get

\[ \| I u \|_{X^{0,b}(S_{\Delta T})} \leq C \left\{ \| I u_0 \|_{L^2} + \| I(\phi v) \|_{X^{0,b-1}(S_{\Delta T})} \right\}, \]
\[ \| I v \|_{X^{0,b}(S_{\Delta T})} \leq C \left\{ \| I v_0 \|_{L^2} + \| I(\phi u) \|_{X^{0,b-1}(S_{\Delta T})} \right\}, \]
\[ \| I^2 \phi \|_{H^{r-2s,b}(S_{\Delta T})} \leq C \left\{ \| I^2 \phi[0] \|_{H^{r-2s}} + \| I^2(u \overline{\pi}) \|_{H^{r-2s-1,b-1}(S_{\Delta T})} \right\}. \]

Now, we claim the following:

\[ \| I(\phi u) \|_{X^{0,b-1}(S_{\Delta T})} \leq C \Gamma_1 \| I^2 \phi \|_{H^{r-2s,b}(S_{\Delta T})} \| I u \|_{X^{0,b}(S_{\Delta T})}, \]
\[ \| I(\phi v) \|_{X^{0,b-1}(S_{\Delta T})} \leq C \Gamma_1 \| I^2 \phi \|_{H^{r-2s,b}(S_{\Delta T})} \| I v \|_{X^{0,b}(S_{\Delta T})}, \]
\[ \| I^2(u \overline{\pi}) \|_{H^{r-2s-1,b-1}(S_{\Delta T})} \leq C \Gamma_2 \| I u \|_{X^{0,b}(S_{\Delta T})} \| I v \|_{X^{0,b}(S_{\Delta T})}, \]

where

\[ \Gamma_1 = \Gamma_1(N, \Delta T) : = (\Delta T)^{2r-4s-4e} + N^{-r+2s+2e}, \]
\[ \Gamma_2 = \Gamma_2(N, \Delta T) : = (\Delta T)^{1-4e} + N^{-1/2+2e}. \]

Assume for the moment that the claim is true. Then

\[ \| I w \|_{X^{0,b}(S_{\Delta T})} \leq C \| I w_0 \|_{L^2} + C \Gamma_1 \| I^2 \phi \|_{H^{r-2s,b}(S_{\Delta T})} \| I w \|_{X^{0,b}(S_{\Delta T})}, \]
\[ \| I^2 \phi \|_{H^{r-2s,b}(S_{\Delta T})} \leq C \| I^2 \phi[0] \|_{H^{r-2s}} + C \| I w \|_{X^{0,b}(S_{\Delta T})} \]

(91)

(92)

(93)

(94)

(95)
Using (95), the estimate (94) reduces to
\[
\|Iw\|_{X^{0,b}(S_{\Delta T})} \leq C \|Iw_0\|_{L^2} + CT_1 \|I^2\phi[0]\|_{H^{r-2s}} + \|Iw\|_{X^{0,b}(S_{\Delta T})} + CT_1 \Gamma_1 \|Iw\|_{X^{0,b}(S_{\Delta T})}^3 \\
\leq CAN^{-s} + CBN^{-2s} \Gamma_1 \|Iw\|_{X^{0,b}(S_{\Delta T})} + CT_1 \Gamma_2 \|Iw\|_{X^{0,b}(S_{\Delta T})}^3
\]
(96)
So if
\[
CBN^{-2s} \Gamma_1 (2CAN^{-s}) + CT_1 \Gamma_2 (2CAN^{-s})^3 \leq CAN^{-s},
\]
then it follows by a boot-strap argument (see the Remark below for the detail on this argument) that
\[
\|Iw\|_{X^{0,b}(S_{\Delta T})} \leq 2CAN^{-s}.
\]
Now, if we choose
\[
\Delta T \approx N(s-\varepsilon)/(r-2s-2\varepsilon),
\]
the boot-strap condition (97) reduces to (modifying C)
\[
C(B + A^2) \left(N^{-2\varepsilon} + N^{-r+2\varepsilon}\right) \leq 1.
\]
(100)
On the other hand, by (95) we get (modifying C)
\[
\|I^2\phi\|_{H^{r-2s,b}(S_{\Delta T})} \leq CBN^{-2s} + 4CA^2 N^{-2s} \left(N^{(s-\varepsilon)(1-4\varepsilon)/(r-2s-2\varepsilon)} + N^{-1/2+2\varepsilon}\right).
\]
The second term in the r.h.s. of this inequality can be bounded by \(C(B + A^2)N^{-2s}\) since the quantity in the bracket is very small. So, we obtain
\[
\|I^2\phi\|_{H^{r-2s,b}(S_{\Delta T})} \leq 2C(B + A^2)N^{-2s}.
\]
(101)
Remark 3. The above estimates imply LWP of (3), (4) with time of existence up to \(\Delta T > 0\) given by (99) provided that the condition (100) is satisfied. The boot-strap argument mentioned above can be shown using the standard iteration argument: Set \(u^{(-1)} = v^{(-1)} = 0\), and define for \(n \geq -1\) inductively
\[
\begin{cases}
  iD_+(Iu^{(n+1)}) = MIu^{(n)} - I(\phi^{(n)}u^{(n)}), \\
  iD_-(Iu^{(n+1)}) = MIu^{(n)} - I(\phi^{(n)}u^{(n)}), \\
  Iu^{(n+1)}(0) = Iv_0 \in L^2, \quad Iv^{(n+1)}(0) = Iv_0 \in L^2,
\end{cases}
\]
(102)
where
\[
\Box v^{(n)} = m^2 v^{(n)} - 2 \text{Re}(\mu^{(n)} v^{(n)}),
\]
with the same data as for \(\phi\).
Then, defining \(y_n = \|Iv^n\|_{X^{0,b}}\) for \(n \geq 0\), (96) becomes
\[
y_{n+1} \leq CAN^{-s} + CBN^{-2s} \Gamma_1 y_n + CT_1 \Gamma_2 y_n^3.
\]
By (12) and (40), \(y_0 \leq 2CAN^{-s}\). Now, if (100) holds, we conclude by induction that \(y_n \leq 2CAN^{-s}\) for all \(n \geq 0\). On the other hand, we know from [16] that \((u^{(n)}, v^{(n)}) \to (u, v) \in X^{s,b}_+ \times X^{s,b}_-\) as \(n \to \infty\), which implies \(Iw^{(n)} \to Iw \in X^{0,b}\) as \(n \to \infty\), and hence (98) follows.

It remains to prove the claim; i.e., (91)–(93). The estimates (91) and (92) are symmetrical. Hence we only prove (91) and (93). As in Section 6, we decompose \(u, v, \phi\) into low and high frequencies, and prove the bilinear estimates for all possible interactions.
8.1. Proof of (91). We recall that \( s > -1/6 \), \( -s \leq r < 1/2 + 2s \), \( b = 1/2 + \varepsilon \), and the operator \( I \) is the identity for low frequencies. Note also that low-low interaction yields low frequency output. Then for the low/low interaction, we have by (28) and the product estimate (23)

\[
\|I(\phi_t u_t)\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})} \leq C \|\phi_t u_t\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})}
\]

(103)

On the other hand, by (28), Lemma 3, Hölder in \( t \), (19) and (16), we have

\[
\|I(\phi_t u_t)\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})} \leq C \|\phi_t u_t\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})}
\]

(104)

Then interpolation between (103) and (104), for \( 0 \leq \theta \leq 1 \), gives

\[
\|I(\phi_t u_t)\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})} \leq C(\Delta T)^{(1-2\varepsilon)\theta} \|\phi_t\|_{H_{2r(1-\theta)+(1/2+\varepsilon)\theta}^{2s+b}(S_{\Delta T})} \|u_t\|_{X_{\alpha, \mu}^{0,b}(S_{\Delta T})}.
\]

We take \( \theta = \frac{2r-b-4s}{1-2s} \) (by the hypothesis made on \( s, r \), we then have \( \theta \in [0, 1] \)). This implies \( 2\varepsilon(1-\theta)+(1/2+\varepsilon)\theta = r-2s \) and \((1-2\varepsilon)\theta = 2r-4s-4\varepsilon \). Consequently,

\[
\|I(\phi_t u_t)\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})} \leq C(\Delta T)^{2r-4s-4\varepsilon} \|\phi_t\|_{H^{r-2s+b}(S_{\Delta T})} \|u_t\|_{X_{\alpha, \mu}^{0,b}(S_{\Delta T})}.
\]

(105)

The contribution from low/high can be estimated using (28) and the product estimate (23) as

\[
\|I(\phi_t u_h)\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})} \leq C \|\phi_t u_h\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})}
\]

(106)

\[
\leq C \|\phi_t\|_{H^{r-2s+b}(S_{\Delta T})} \|u_h\|_{X_{\alpha, \mu}^{r+2s-4s-2\varepsilon+b}(S_{\Delta T})}
\]

\[
\leq C N^{-r+2s+2\varepsilon} \|\phi_t\|_{H^{r-2s+b}(S_{\Delta T})} \|u_h\|_{X_{\alpha, \mu}^{0,b}(S_{\Delta T})}.
\]

The contribution from high/low can be estimated using (28), the product estimate (23), and (31) as

\[
\|I(\phi_t u_l)\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})} \leq C \|\phi_t u_l\|_{X_{\alpha, \mu}^{0,-1}(S_{\Delta T})}
\]

(107)

\[
\leq C \|\phi_t\|_{H^{r-2s+b}(S_{\Delta T})} \|u_l\|_{X_{\alpha, \mu}^{0,b}(S_{\Delta T})}
\]

\[
\leq C N^{-r+s+2\varepsilon} \|\phi_t\|_{H^{r-s+b}(S_{\Delta T})} \|u_l\|_{X_{\alpha, \mu}^{0,b}(S_{\Delta T})}
\]

\[
\leq C N^{-r+2s+2\varepsilon} \|I^2 \phi_t\|_{H^{r-2s+b}(S_{\Delta T})} \|u_l\|_{X_{\alpha, \mu}^{0,b}(S_{\Delta T})}.
\]
Similarly, we estimate the high/high interaction using (28), the product estimate (23), and (31) as

\[ 
\|I(\phi_h u_h)\|_{X^0,b-1(S\Delta T)} \leq C \|\phi_h u_l\|_{X^0,b-1(S\Delta T)} \\
\leq C \|\phi_h\|_{H_r^{r+s},b(S\Delta T)} \|u_h\|_{X^0,b(S\Delta T)} \\
= C \|\phi_h\|_{H_r^{r+s},b(S\Delta T)} N^s N^{-s} \|u_h\|_{X^0,b(S\Delta T)} \\
\leq C N^{-r+s+2\varepsilon} \|I\phi_h\|_{H_r^{r-s},b(S\Delta T)} \|Iu_h\|_{X^0,b(S\Delta T)} \\
\leq C N^{-r+2s+2\varepsilon} \|I^2 \phi_h\|_{H_r^{r-s},b(S\Delta T)} \|Iu_h\|_{X^0,b(S\Delta T)} .
\]

(108)

Therefore, (91) follows from the estimates (105)–(108).

8.2. Proof of (93). We recall that \( s > -1/6 \), \( s < r < 1/2 + 2s \) and \( b = 1/2 + \varepsilon \). Noting that \( I \) is the identity for low frequencies, we have by (28), Lemma 5 and (17)

\[ 
\|I^2(u_l^N)\|_{H_r^{-2s+1,b-1}(S\Delta T)} \leq C \|u_l^N\|_{H_r^{-2s-1/2+\varepsilon}(S\Delta T)} \\
\leq C \|u_l\|_{X^0,b(S\Delta T)} \|v_l\|_{X^0,b(S\Delta T)} \\
\leq C (\Delta T)^{1-4\varepsilon} \|u_l\|_{X^{0,\varepsilon/2}(S\Delta T)} \|v_l\|_{X^{0,\varepsilon/2}(S\Delta T)} \\
\leq C (\Delta T)^{1-4\varepsilon} \|u_l\|_{X^0,b(S\Delta T)} \|v_l\|_{X^0,b(S\Delta T)} .
\]

(109)

The contribution from low/high interaction is estimated using (28) and the product estimate (22) as

\[ 
\|I^2(u_l^N)\|_{H_r^{-2s+1,b-1}(S\Delta T)} \leq C \|u_l^N\|_{H_r^{-1/2,b-1}(S\Delta T)} \\
\leq C \|u_l\|_{X^0,b(S\Delta T)} \|v_l\|_{X^{-1/2+2\varepsilon},b(S\Delta T)} \\
= C \|u_l\|_{X^0,b(S\Delta T)} \|v_l\|_{X^{-1/2+\varepsilon},b(S\Delta T)} \\
\leq C \|u_l\|_{X^0,b(S\Delta T)} N^{-1/2-2\varepsilon} \|v_l\|_{X^0,b(S\Delta T)} \\
= C N^{-1/2+2\varepsilon} \|u_l\|_{X^0,b(S\Delta T)} \|v_l\|_{X^0,b(S\Delta T)} .
\]

(110)

By symmetry

\[ 
\|I^2(u_l^N)\|_{H_r^{-2s+1,b-1}(S\Delta T)} \leq C N^{-1/2+2\varepsilon} \|u_l\|_{X^0,b(S\Delta T)} \|v_l\|_{X^0,b(S\Delta T)} .
\]

(111)

Finally, for the high/high interaction we obtain using (28) and the product estimate (22)

\[ 
\|I^2(u_l^N)\|_{H_r^{-2s+1,b-1}(S\Delta T)} \leq C \|u_l^N\|_{H_r^{-1/2,b-1}(S\Delta T)} \\
\leq C \|u_l\|_{X^{0,1/4+2\varepsilon},b(S\Delta T)} \|v_l\|_{X^{0,1/4},b(S\Delta T)} \\
\leq C N^{-1/4+\varepsilon} \|u_l\|_{X^{0,b}(S\Delta T)} N^{-1/4-\varepsilon} \|v_l\|_{X^{0,b}(S\Delta T)} \\
= C N^{-1/2+2\varepsilon} \|Iu_l\|_{X^0,b(S\Delta T)} \|Iv_l\|_{X^0,b(S\Delta T)} .
\]

(112)

We therefore conclude that (93) follows from the estimates (109)–(112).
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