KALAI ORIENTATIONS ON MATROID POLYTOPES

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Abstract. Let \( P \) a polytope and let \( \mathcal{G}(P) \) be the graph of \( P \). Following Gil Kalai, we say that an acyclic orientation \( O \) of \( \mathcal{G}(P) \) is **good** if, for every non-empty face \( F \) of \( P \), the induced graph \( \mathcal{G}(F) \) has exactly one sink. Gil Kalai gave a simple way to tell a simple polytope from the good orientations of its graph. This article is a broader study of “good orientations” (of the graphs) on matroid polytopes.

Dedicated to Michel Las Vergnas on the occasion of his 65th birthday

1. Introduction

Let \( P \) be a simple polytope (see [10] for details on polytopes) and let \( \mathcal{G}(P) \) be its graph (1-skeleton). M. Perles conjectured (see the reference [P] in [7]) and Blind and Mani [2] proved that the graph \( \mathcal{G}(P) \) determines the lattice of faces of \( P \). Kalai [7] gave a short and constructive prove of this result; see also [5, 6] for a discussion and refinement of Kalai’s technique. Kalai’s proof is based on an intrinsical characterization of the “good” orientations of \( \mathcal{G}(P) \) between all the acyclic orientations. Following [7], we say that an acyclic orientation \( O \) of \( \mathcal{G}(P) \) is **good** if for every non-empty face \( F \) of \( P \) the induced graph \( \mathcal{G}(F) \) has exactly one sink, i.e., a vertex of \( \mathcal{G}(F) \) with no lower adjacent vertices. Every linear ordering, \( \{v_1 \prec \cdots \prec v_n\} \), of the vertex set \( V \) of \( \mathcal{G}(M) \) induces an acyclic orientation \( O_\prec \) of the graph, where an edge is directed from its larger end-node to its smaller end-node. The linear ordering \( \{v_1 \prec \cdots \prec v_n\} \) is called **good** if \( O_\prec \) is a good orientation. Each acyclic orientation of an arbitrary graph \( G \) is induced by some linear ordering of its vertices, see [5] Proposition 1.2. Good orderings are in 1-1 correspondence with shelling orderings of the facets of the boundary \( \partial P^\Delta \) of the dual polytope \( P^\Delta \) (see Theorem 2.3 below for a matroidal generalization).

We say that an oriented matroid \( \mathcal{M} \) is a matroid polytope if it is acyclic and all the elements of the ground set \( E(\mathcal{M}) \) are extreme points of \( \mathcal{M} \). The graph \( \mathcal{G}(\mathcal{M}) \) of the matroid polytope \( \mathcal{M} \) is the graph whose vertices [resp. edges] are the faces of rank 1 [resp. 2] of \( \mathcal{M} \). In particular the vertex set of \( \mathcal{G}(\mathcal{M}) \) is the ground set \( E(\mathcal{M}) \). We say that the matroid polytope \( \mathcal{M} \) is **simple** if every vertex of \( \mathcal{G}(\mathcal{M}) \) is incident with exactly rank(\( \mathcal{M} \)) – 1 edges.

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The proof of Gil Kalai [7] remains true if we replace “simple polytope” by “simple matroid polytope” so, the graph $G(\mathcal{M})$ also encloses the face lattice of $\mathcal{M}$. If $F$ is a non-empty face of $\mathcal{M}$ then $F$ also is a matroid polytope (we are identifying $F$ and the restriction of $\mathcal{M}$ to $F$). For more details on oriented matroid theory see Section 2 and [1].

2. K-orderings

Let $\mathcal{M}', \mathcal{M}''$ [resp. $\mathcal{M}$] denotes an acyclic oriented matroid [resp. matroid polytope] of rank $r$ on ground set $E = \{e_1, \ldots, e_n\}$. Let $\mathcal{O} = \mathcal{O}(\mathcal{M}')$ be the class of all the acyclic reorientations of $\mathcal{M}'$. For every linear ordering $E(\mathcal{M})_\prec$ let $\mathcal{O}_\prec$ denotes the orientation of $G(\mathcal{M})$ where $uv$ is a directed edge from $u$ to $v$, if $v \prec u$. Let $\mathcal{G}_\prec(\mathcal{M}) := (G(\mathcal{M}), \mathcal{O}_\prec)$ be the corresponding digraph. If $F$ is a face of $\mathcal{M}$ let $\mathcal{G}_\prec(F)$ be the induced directed subgraph on $F$. Note that an element $e$ is the unique sink of $\mathcal{G}_\prec(\mathcal{M})$ if and only if for every element $e'$ there is a directed path, $e' \prec e$, from $e'$ to $e$.

Definition 2.1. We say that the linear ordering $\{e_1 < e_2 < \cdots < e_n\}$ of the ground set of a matroid polytope $\mathcal{M}$ is a K-ordering if, for every non-empty face $F$ of $\mathcal{M}$, the directed subgraph $\mathcal{G}_\prec(F)$ has exactly one sink. In particular the element $e_1$ is the unique sink of the digraph $\mathcal{G}_\prec(\mathcal{M})$.

From the “Topological Representation Theorem”, we know that there is a pure regular CW-complex of dimension $r - 1$, with the topology of a PL-sphere and encoding $\mathcal{O}$, see [1 Theorem 5.2.1]. We will denote this CW-complex by $\Delta(\mathcal{O})$, or by $\Delta$ for short, and called it the CW-complex of acyclic reorientations of $\mathcal{M}'$. To every cell $W \in \Delta$ we attach a “sign vector” $\sigma(W) \in \{0, +, -\}^{E(\mathcal{M}')}$ (for precisions see [1]). We will identify $W$ and $\sigma(W)$ and set $W_{(e_i)} = \sigma(W)_{(e_i)}$. The set

$$\text{supp}(W) = \{e_i : W_{(e_i)} \neq 0\}$$

is called the support of $W$. We say that the cell $W'$ is a face of $W$ if the following two conditions hold:

1. $\text{supp}(W') \subset \text{supp}(W)$;
2. For every $e_i \in E$, if $e_i \in \text{supp}(W')$ then we have $W'_{(e_i)} = W_{(e_i)}$.

A cell $P \in \Delta$ has dimension 0 (is a vertex) if, seen as a signed vector, $P$ is a signed cocircuit of $\mathcal{M}$. We will see $\Delta$ as an “abstract” regular cell complex over the set of its vertices. Every cell $W \in \Delta$ is identified with the set of its vertices $V(W)$:

$$W \equiv V(W) := \{P : P \text{ is a vertex of } \Delta \text{ and } P \leq W\}.$$

The facets of $\Delta$ are called the topes and have support equal to $E(\mathcal{M})$. We use the letter $T$ to denote a tope. Every tope $T \in \Delta$ is a PL-ball and its boundary $\partial T$ a PL-sphere. Every tope $T$ fixes one acyclic reorientation $\mathcal{M}'' \in \mathcal{O}(\mathcal{M}')$. To explicit this correspondence we write $T = T(\mathcal{M}'')$. (The opposite tope $-T$ fix the same oriented matroid $\mathcal{M}'$.) If a fixed acyclic oriented matroid $\mathcal{M}'$ is given we set $T(\mathcal{M}') = (+, +, \ldots, +)$. Let $\mathcal{L}(\mathcal{M})$
The following two statements are equivalent:

**Theorem 2.3.** So there is a directed edge \( \rightarrow \) \( (2.3) \)

\( F \) non-empty face unique sink of \( \mathcal{E} \) \( (2.3) \)

**Definition 2.2.** \( T \) of cells of \( \mathcal{E} \).

\( \Xi = 1 \) case, \( \{ \Xi \} \)

\( T \) \( \Xi \) denotes the element of rank \( r - s \) of \( \mathcal{L}(T(\mathcal{M})) \) determined by the following conditions:

\[
\Xi(X)(e_\ell) = \begin{cases} 
0 & \text{if } e_\ell \leq X \\
+ & \text{if } e_\ell \not\leq X.
\end{cases}
\]

The atoms of \( \mathcal{L}(T(\mathcal{M})) \) are the image by \( \Xi \) of the co-atoms of \( \mathcal{L}(\mathcal{M}) \). The set of vertices and facets of CW-complex \( T \), are respectively

\[
\{ \Xi(H) : H \text{ a facet of } \mathcal{M} \} \quad \text{and} \quad \{ \Xi(e) : e \in E(\mathcal{M}) \}.
\]

If \( \mathcal{M} \) is a simple matroid polytope then \( T(\mathcal{M}) \) is an (abstract) simplicial complex of dimension rank(\( \mathcal{M} \)) \( - 1 \). In particular, for every pair \( \{ W, W' \} \) of cells of \( T(\mathcal{M}) \), we have \( V(W \land W') = V(W) \cap V(W') \). The following definition is a particular case of the standard one, see \[13\] \[14\] \[15\] for details. (The equivalence of Conditions (2.2) and \( (2.2) ' \) is left to the reader.)

**Definition 2.2.** Let \( \Delta \) be the CW-complex of the acyclic reorientations of \( \mathcal{M} \) and \( T \in \Delta \) be the tope associated to \( \mathcal{M} \). We say that the linear ordering \( \{ \Xi_1 : \Xi_1 \prec \Xi_2 \prec \cdots \prec \Xi_n \} \) is a shelling of the PL-sphere \( \partial T \) and \( \partial T \) is shellable if one of the following equivalent conditions holds:

(2.2) \( \quad \) For every pair of co-atoms \( \Xi_i \prec \Xi_j \) such that \( \Xi_j \cap \Xi_i \neq \emptyset \), there is some facet \( \Xi_\ell \prec \Xi_j \) such that \( \Xi_\ell \cap \Xi_j \) is an abstract simplex of cardinality \( \ell - 2 \) of \( T \) and \( \Xi_i \cap \Xi_j \subseteq \Xi_\ell \cap \Xi_j \);

\((2.2) ' \) \( \quad \) For every pair of vertices \( \{ e_i, e_j \} \), \( 1 \leq i < j \leq n \), on a non-singular face \( F \) of \( \mathcal{M} \), there is some \( \ell < j \) such that \( e_\ell e_\ell \) is a directed edge of the digraph \( \mathcal{G}_\prec(F) \).

**Theorem 2.3.** The following two statements are equivalent:

(2.3) \( \quad \) \( \{ e_1 \prec \cdots \prec e_n \} \) is a K-ordering of \( \mathcal{M} \);

(2.3) \( \quad \) \( \{ \Xi_1 = \Xi(e_1) \prec \Xi_2 \prec \cdots \prec \Xi_n \} \) is a shelling of the PL-sphere \( \partial T(\mathcal{M}) \).

**Proof.** (2.3) \( \Longrightarrow \) (2.3) \( \). As \( i < j \) we know that \( e_j \) is not a sink of \( \mathcal{G}_\prec(F) \). So there is a directed edge \( e_\ell e_\ell \) of \( \mathcal{G}_\prec(F) \) and (2.2) \( ' \) holds.

(2.3) \( \Longrightarrow \) (2.3) \( \). Let \( \{ e_{i_1} \prec \cdots \prec e_{i_w} \} \) be the induced ordering on a non-empty face \( F \) of \( \mathcal{M} \). From Condition (2.2) \( ' \) we know that \( e_{i_1} \) is the unique sink of \( \mathcal{G}_\prec(F) \) and (2.3) \( \) follows.

Let us recall that a linear ordering \( \{ e_1 \prec e_2 \prec \cdots \prec e_n \} \) of the ground set \( E(\mathcal{M}') \) is called a shelling ordering of \( \mathcal{M}' \), if the orientation obtained from \( \mathcal{M}' \) by changing the signs on the initial sets \( E_i := \{ e_1, e_2, \ldots, e_i \} \), \( i = 1, 2, \ldots, n \), is also acyclic. Edmonds and Mandel proved that in this case, \( \{ \Xi_1 = \Xi(e_1) \prec \Xi_2 \prec \cdots \prec \Xi_n \} \) is a shelling of the PL-sphere \( \partial T(A(M)) \), see \[14\] Proposition 4.3.1. Note that if \( F \) is a face of a matroid polytope \( \mathcal{M} \), every shelling ordering of \( \mathcal{M} \) induces a shelling ordering on \( F \). The following
result is a consequence of Theorem 2.3 and the above result of Edmonds and Mandel.

**Corollary 2.4.** Every shelling ordering of a simple matroid polytope is also a K-ordering.

Let \( f_\ell(M) \) be the number of faces of rank \( \ell + 1 \) of \( M \), \(-1 \leq \ell \leq r-1\). By convention set \( f_{-1}(M) = f_{r-1}(M) = 1 \). By analogy with the definition of the \( f \)-vector and the \( h \)-vector of a polytope, we call the vector,

\[
\mathbf{f}(M) := (f_{-1}(M), f_0(M), f_1(M), \ldots, f_{r-2}(M), f_{r-1}(M)),
\]

the \( f \)-vector of the (simple) matroid polytope \( M \) and we call the vector,

\[
\mathbf{h}^*(M) := (h_0^*(M), h_1^*(M), \ldots, h_{r-1}^*(M)),
\]

determined by the formulas

\[
(1) \quad h_\ell^*(M) = \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{r-1-i}{\ell-i} f_{r-1-i}(M), \quad \ell = 0, 1, \ldots, r-1,
\]

the \( h^*- \)vector of \( M \). Note that the \( f \)-vector also can be recovered from the \( h^* \)-vector:

\[
(2) \quad f_\ell(M) = \sum_{i=0}^{r-1-\ell} \binom{r-1-i}{\ell} h_i^*(M), \quad \ell = 0, 1, \ldots, r-1.
\]

(See [10, Section 8.3] for a good survey of \( h \)-vectors of simplicial polytopes and Dehn-Sommerville Equations.) From Euler-Poincaré formula (see [11, Corollary 4.6.11]) we know that

\[
(3) \quad \sum_{i=-1}^{r-1} (-1)^i f_i(M) = 0.
\]

The graph \( G(M) \) is regular of degree \( r-1 \). Fix a K-ordering \( \{e_1 \prec \cdots \prec e_n\} \) of \( M \). Let \( d^+(e) \) [resp. \( d^-(e) \)] denotes the outdegree [resp. indegree] of \( e \in E \). Set

\[
\mathbf{d}^+_\ell(M) := |\{e : e \in E, d^+(e) = \ell\}|
\]

\[
\mathbf{d}^-_\ell(M) := |\{e : e \in E, d^-(e) = \ell\}|
\]

We clearly have that \( \mathbf{d}^+_\ell(M) = \mathbf{d}^-_{r-1-\ell}(M) \).

**Theorem 2.5.** The integers \( \mathbf{d}^+_\ell(M) \), \( \ell = 0, 1, \ldots, r-1 \), are invariant of the matroid polytope \( M \) (i.e., are independent of the K-ordering) and they are determined by the equalities:

\[
(4) \quad h_\ell^*(M) = \mathbf{d}^+_\ell(M), \quad \ell = 0, 1, \ldots, r-1.
\]
Proof. Note that
\[ f_\ell(M) = r - 1 - \sum_{j=\ell}^{r-1} d_j^- (M) \]
\[ = \sum_{i=0}^{r-1-\ell} \binom{r-1-i}{\ell} d_{r-1-i}^- (M) \]
\[ = \sum_{i=0}^{r-1-\ell} \binom{r-1-i}{\ell} d_{r-1-i}^+ (M). \]
From Equation (2) we conclude that \( d_\ell^+ (M) = h^*_\ell (M). \)

Corollary 2.6. Let \( E_\prec := \{ e_1 \prec e_2 \prec \cdots \prec e_n \} \) be a \( K \)-ordering of \( M \). Then the reverse ordering \( E_\prec^* := \{ e_n \prec^* e_{(n-1)} \prec^* \cdots \prec^* e_1 \} \) is also a \( K \)-ordering of \( M \).

Proof. It is necessary to prove that, for every non-empty face \( F \) of \( M \), the digraph \( G_\prec^* (F) \) has exactly one sink, i.e., \( d_{r-1}^+ (F) = 1 \). From the equalities (4) we know \( d_{r-1}^+ (M) = h^*_{r-1} (M) \). From Euler-Poincaré formula (3) we conclude that
\[ h^*_{r-1} (M) = \sum_{i=0}^{r-1} (-1)^{r-1-i} f_{r-1-i} (M) = \]
\[ = \sum_{j=0}^{r-1-\ell} (-1)^j f_j^+ (M) = f_{-1} (M) = 1. \]
So \( G_\prec^* (M) \) has exactly one sink. As every face \( F \) of \( M \) is a simple matroid polytope the result follows.

3. The Cube

Let \( C^d := \{ x \in \mathbb{R}^d : 0 \leq x_\ell \leq 1, \ \ell = 1, \ldots, d \} \) be the \( d \)-dimensional cube. As the polar of the cube \( C^d \) is the \( d \)-dimensional crosspolytope (a simplicial polytope), it results from Theorem (2.5) above that:
\[ h^*(C^d) = \begin{pmatrix} \binom{d}{0} \\ \binom{d}{1} \\ \vdots \\ \binom{d}{d-1} \\ \binom{d}{d} \end{pmatrix}, \]
i.e., there are exactly \( \binom{d}{\ell} \), \( 0 \leq \ell \leq d \), vertices of \( G(C^d) \) such that \( d^- (e) = d - \ell \).
Let \( B := \{0,1\}^d \) be the set of the vertices of the cube \( C^d \) and for every \( b_i \in B, 1 \leq i \leq 2^d \), set \( e_i = (b_i, 1) \in \mathbb{R}^{d+1} \). The rank \( d + 1 \) cube matroid polytope, \( C^d \), is the oriented matroid determined by the linear dependencies of vectors of \( E := \{ e_i : b_i \in B \} \). The following theorem is closely related to the results presented here. (We present here a slightly different version of the original result.)
Theorem 3.1. Proposition 3.2] Let $C^d$ be the cube matroid polytope of rank at least three. Let $\{e_1 \prec e_2 \prec \cdots \prec e_{2d}\}$ be a linear ordering of $E(C^d)$. Then the following two conditions are equivalent:

\begin{enumerate}
  \item[(3.1.1)] $\{e_1 \prec e_2 \prec \cdots \prec e_{2d}\}$ is a K-ordering of $C^d$;
  \item[(3.1.2)] For every rank three face $F$ of $C^d$, the digraph $G_\prec(F)$ has an unique sink. \hfill \square
\end{enumerate}

In the rank three cube matroid polytope $C^2$, the “K-ordering” and the “shelling orderings” coincide. This result suggest the following problem:

Open Problems 3.2. Is there a simple characterisation of shelling orderings of the cube matroid polytope $C^d$? Are there K-orderings of the cube matroid polytope that are not shelling orderings?

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