Quantizing Open Spin Chains with Variable Length: an example from Giant Gravitons

David Berenstein\textsuperscript{1,2} Diego H. Correa\textsuperscript{2} and Samuel E. Vázquez\textsuperscript{2}

\textsuperscript{1}Department of Physics, University of California at Santa Barbara, CA 93106
\textsuperscript{2}Kavli Institute for Theoretical Physics, University of California at Santa Barbara, CA 93106

We study an XXX open spin chain with variable number of sites, where the variability is introduced only at the boundaries. This model arises naturally in the study of Giant Gravitons in the AdS/CFT correspondence. We show how to quantize the spin chain by mapping its states to a bosonic lattice of finite length with sources and sinks of particles at the boundaries. Using coherent states, we show how the Hamiltonian for the bosonic lattice gives the correct description of semiclassical open strings ending on Giant Gravitons.

PACS numbers: 75.10.Pq, 11.25.Tq
Keywords: Quantum spin chains, AdS/CFT

I. Introduction.

In the last few years many connections have been made relating large $N$ quantum field theories in four dimensions, string theory on negatively curved space-time and quantum spin chain models. These seemingly disparate subjects have been tied together via the AdS/CFT correspondence \cite{1, 2, 3, 4, 5}. In particular, it has become apparent that to understand the field theory beyond the one loop approximation, one needs to deal with the problem of studying spin chain models with varying numbers of sites \cite{6}.

In this article we want to report a new development relating to this collection of subjects for the case of open spin chain models. In particular we will show how one can quantize an XXX spin chain model with variable numbers of sites, where the variability is introduced only at the boundaries. This model makes its natural appearance in the study of string states attached to giant gravitons when seen from the dual CFT point of view, and the variability on the number of sites is obtained already at one loop order. The XXX model has also important applications in condensed matter and statistical physics and here we provide a generalization of the boundary conditions for a finite length chain. For the present paper we will content ourselves with the form of this final answer and present a derivation of the model and a more complete study elsewhere \cite{7}.

We remind the reader the basic geometric properties of giant gravitons \cite{8, 9, 10, 11, 12, 13, 14, 15, 16}. These are D3-branes of $AdS_5 \times S^5$ that wrap an $S^1$ inside the $S^5$, and move in a circle. More specifically, the radius of their orbit in the $S^5$ is determined by their angular momentum $p$, as, $r = R\sqrt{1 - p/N}$. Here $R$ is the radius of the $S^5$ and $N$ is the number of units of five-form flux on the sphere. Hence, the angular momentum is bounded by $p \leq N$, with the equality corresponding to the “maximal” giant graviton. Also, the Hamiltonian we discuss here is associated to the one loop anomalous dimension of certain operators in the dual field theory \cite{13, 14, 15}.

Our main result is that the Hamiltonian of the spin chain model for the variable number of sites that we consider can be transformed to a Hamiltonian for a dual model with a fixed number of sites and non-diagonal boundary conditions. This lets us solve for the ground state of the model and apply the results to the study of the Dirac-Born-Infeld fluctuations of the brane. From the string theory point of view, this Hamiltonian can be understood by a different gauge choice of the Polyakov action than the standard one, and it is a hint that the CFT knows something about the reparametrization invariance of the string.

II. A map from the XXX spin chain model to a system of bosons on a lattice

Let us begin with a description of the configuration space of a finite chain of length $L$ of the XXX model. This is, consider a spin system with $L$ sites, some of which are set to spin up, and some which are set at spin down. We will label the spin down site with a $Y$, and the spin up site by $Z$. A configuration of spins of the system can be mapped into a word on the letters $Y,Z$. For example, we can consider the states as words $Y^L$, or $Y^k Z Y^{L-k-1}$. The first one is the ferromagnetic ground state of the XXX spin chain model, and the second one is a state with one impurity. We can also build multi-impurity states by inserting $Z$ at various locations.

Now let us assume that neither the first site nor the last site is allowed to be a $Z$. In effect, this boundary condition arises from the giant gravitons. We can consider all words in $Y,Z$ as being generically of the following form:

$$Y Z^{n_1} Y Z^{n_2} Y \ldots Y Z^{n_k} Y,$$

where $L = k + 1 + \sum_i n_i$ and the $n_i$ are non-negative integers (notice there are $k+1$ different $Y$’s in the expression). The spin $s_z$ of the system relative to the ferromagnetic ground state is $s_z = s_z^0 = \sum_i n_i$. We can map this state to a state of a lattice with $k$ sites where at each site $i$ there is a boson with occupation number $n_i$. The total occupation number is then $\sum_i n_i$. This duality map does not preserve the length of the spin chain. If we consider the set of all states of a fixed length $L$ for the XXX model, this will be mapped to a collection of states of different length $k$ on our bosonic lattice depending on the total
spin of the configuration. Similarly, the set of all boson configurations of length 
k gets mapped to a collection of arbitrarily large spin chain configurations of the XXX
model, depending on the total occupation number.

Now let us consider the standard ferromagnetic XXX
spin chain Hamiltonian. This Hamiltonian preserves the
number of $Z$. This means that the Hamiltonian acting
on the states of the bosonic chain does not mix lengths
of the spin chain, and can also be understood as a lattice
model which preserves the boson number. This turns out
to be a nearest neighbor Hamiltonian as well.

To derive the Hamiltonian for the boson chain, we first
define the oscillator-like operators, $\hat{a}_i$, $\hat{a}^\dagger_i$ acting at each
site with the property $\hat{a}_i |n_i\rangle = |n_i - 1\rangle$, $\hat{a}_i |0\rangle = 0$ and
$\hat{a}^\dagger_i |n_i\rangle = |n_i + 1\rangle$ (These are shift operators for an infinite
basis). It follows that these operators describe a free
Fock space for a single species (e.g. $[16]$ and references
therein). They obey the so-called “Cuntz” algebra for a
single species $[17]$,

$$\hat{a}\hat{a}^\dagger = I, \quad \hat{a}^\dagger\hat{a} = I - |0\rangle\langle 0| . \quad (2)$$

The number operator $\hat{n}_i$ can also be built from these
operators as in $[16]$. Note that operators at different
sites are commuting, so “particles” filling the sites respect
bosonic statistics. This algebra can also be considered
as the $q \to 0$ limit of the deformed harmonic oscillator
algebra $\hat{a}\hat{a}^\dagger - qa^\dagger a = 1$.

The Hamiltonian takes a particularly simple form,

$$H = 2\lambda \sum_{i=1}^{L} \hat{a}_i^\dagger \hat{a}_i - \lambda \sum_{i=1}^{L-1} (\hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_{i+1}^\dagger \hat{a}_i) , \quad (3)$$

where we have chosen the ferromagnetic ground state to
have zero energy, and for the giant gravitons $\lambda = \frac{2\pi}{L}$
with $g_s$ the string coupling $[4][9]$. The first term tells us
that there is a finite amount of energy in each oscillator,
which is $2\lambda$ if the bosonic oscillator is occupied and
zero otherwise (in the XXX model this is twice the en-
ergy for a domain wall between a region of $Z$ followed
by one of $Y$). The second term is interpreted as a hopping
term for bosons to move between sites, so that the energy
is reduced with bosons that are not localized. Clearly,
if we consider the collection of all possible spin chain
lengths for both the XXX model and our “Cuntz oscil-
lator” model, we obtain a complete identification of the
two dynamical systems. In the above, we have assumed
fairly standard boundary conditions for the spin chain (it
preserves the total spin), although for the $Z$ in the XXX
model we have some sort of Dirichlet boundary condition,
as we are forbidding the $Z$ to be at the edge of the word.
This boundary condition is integrable and the model is
solvable by Bethe Ansatz techniques $[18]$. Similar limits in
$q$-boson hopping models have been studied for the case
of periodic boundary conditions $[19]$.

III. Non-diagonal boundary conditions

Let us now consider a new version of the Hamilto-
nian in the Cuntz oscillator which corresponds to a non-
diagonal boundary condition (here, we refer to the fact
that the total boson occupation number does not com-
 mute with the Hamiltonian)

$$H = 2\lambda \alpha^2 + 2\lambda \sum_{i=1}^{L} \hat{a}_i^\dagger \hat{a}_i - \lambda \sum_{i=1}^{L-1} (\hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_{i+1}^\dagger \hat{a}_i) + \lambda \alpha (\hat{a}_1^\dagger + \hat{a}_1) + \lambda \alpha (\hat{a}_L^\dagger + \hat{a}_L) , \quad (4)$$

where $\alpha = \sqrt{1 - p/N}$ with $p$ and $N$ integers (they are
arbitrarily large with $p \leq N$, so their fraction $p/N$ is a
general real number between 0 and 1). Notice that when
$p/N \to 1$ the result reduces to the Hamiltonian in the
previous section. Turning $p/N$ away from one produces
a new boundary condition for the spin chain. From the
point of view of the Cuntz oscillator algebra, we have a
source/sink of bosons at each end. If we go back to
the XXX spin chain, here we have a generalization that
adds and subtracts sites to the model, and corresponds
to a spin chain with a dynamical number of sites. This
is the mathematical problem that we discussed in the
introduction. The Hamiltonian we have written above is
the one that can be derived from attaching strings to a
non-maximal giant graviton from the dual CFT $[7]$.

We would like to diagonalize $H$ and find the spectrum
of this string. In the case of the maximal giant graviton
($p/N = 1$) the Hamiltonian is integrable can be diagno-
slized using the Bethe ansatz. For $p \neq N$, we do not know
at this moment how to diagonalize $1$, and wether it is
integrable or not. See however $[20]$ for a similar problem.

We have been able to find the ground state explicitly

$$|\Psi_0\rangle = (1 - \alpha^2)^{L/2} \sum_{n_1, \ldots, n_L = 0}^{\infty} (-\alpha)^{n_1 + \cdots + n_L} |n_1, \ldots, n_L\rangle , \quad (5)$$

and it has energy $E = 0$. The expectation value of the
number operator for the ground state is,

$$\langle \Psi_0 | \hat{n} | \Psi_0 \rangle = \frac{LN}{p} \left( 1 - \frac{p}{N} \right) , \quad (6)$$

which is generically of order $L$, unless $p \ll N$.

Now we want to study the semiclassical limit for these
open strings (we abuse notation here because in the
AdS/CFT correspondence these represent strings at-
tached to a giant graviton). For that we need to take
$L \sim \sqrt{N} \to \infty$ (one also needs to take $\lambda \to \infty$ with
$\lambda/L^2$ fixed, and we also keep $p/N$ fixed). We now need
to consider coherent states of the operators $[21]$, along the
lines of $[21]$. Building coherent states for this algebra is
not trivial. The reason is that the naive coherent states,
$\hat{a}|z\rangle = z|z\rangle$ with $|z\rangle \sim \sum_n z^n|n\rangle$, are not complete. Sev-
eral solutions to this problem has been proposed in the
literature $[22][23][24][25]$. For us this is not of much
concern because we are only interested in the classical
action. However, one can be more formal and define the overcomplete coherent states along the lines of [24] using the general $q$-deformed algebra. Constructing the partition function then goes in the familiar way [24]. Taking the $L \to \infty$ takes us to the classical limit. Therefore we can take $q \to 0$ directly in the classical action.

In the $q \to 0$ limit, the coherent states of [24] reduce to the familiar form, $|z\rangle = \sqrt{1-|z|^2} \sum_{n=0}^\infty z|n\rangle$, with $z \in \mathbb{C}$ and $|z| < 1$. The coherent state for multiple sites can be written as $|z\rangle = |z_1\rangle \otimes \cdots \otimes |z_L\rangle$. If we label the coherent states as $z_i = r_i e^{i\phi_i}$ and take the continuum limit $z_i(t) \to z(t, \sigma)$, we get the following classical action,

$$S = \int dt \left( i \langle z \partial_{\bar{t}} |z\rangle - \langle z|H|z\rangle \right)$$

$$= -L \int dt \int_0^1 d\sigma \left[ \frac{r^2 \phi}{1-r^2} + \frac{\lambda}{L^2} (r^2 + r^2 \phi^2) \right]$$

$$- \lambda \int dt \left[ \alpha^2 \sin^2 \phi + (\alpha \cos \phi + r)^2 \right] \bigg|_{\sigma=0}$$

$$- \lambda \int dt \left[ \alpha^2 \sin^2 \phi + (\alpha \cos \phi + r)^2 \right] \bigg|_{\sigma=1} \ ,$$

where the dot and primes are derivatives with respect to $t$ and $\sigma$ respectively. Another approach is to consider our states as a spin $j = 1/2$ representation of the $SL(2)$ algebra. Building the coherent states goes as in [27], and one can recover the same classical action [7] after some field redefinitions.

Ignoring the boundary terms (see below), the equations of motion for the action [7] are

$$\frac{r^2 \dot{\phi}}{1-r^2} + \frac{\lambda}{L^2} (r^2 + r^2 \phi^2) = 0 \ , \tag{8}$$

$$\frac{r^2 \dot{\phi}}{1-r^2} + \frac{\lambda}{L^2} (r \phi^2 - r''') = 0 \ . \tag{9}$$

The classical Hamiltonian for the coherent states is,

$$\langle H \rangle = \frac{\lambda}{L} \int_0^1 d\sigma (r^2 + r^2 \phi^2)$$

$$+ \lambda \left[ \alpha^2 \sin^2 \phi + (\alpha \cos \phi + r)^2 \right] \bigg|_{\sigma=0}$$

$$+ \lambda \left[ \alpha^2 \sin^2 \phi + (\alpha \cos \phi + r)^2 \right] \bigg|_{\sigma=1} \ .$$

The average number of $Z$-s in the open string is,

$$\langle \hat{n} \rangle = \int_0^1 d\sigma \frac{r^2}{1-r^2} \ , \tag{11}$$

and using Eq. [8] we have,

$$\partial_t \langle \hat{n} \rangle = 2\frac{\lambda}{L} \left( 1 - \frac{p}{N} \right) (\phi'|_{\sigma=0} - \phi'|_{\sigma=1}) \ . \tag{12}$$

So in general $\langle \hat{n} \rangle$ is not conserved and therefore the string will oscillate in length. This is the way we measure the length of the spin chain according to the XXX model. Note however that we must ensure that $\langle \hat{n} \rangle$ remains bounded.

The boundary terms in the Hamiltonian [10] can give rise to a large energy. From the point of view of the dual string theory, this means that moving the ends of the open string costs a lot of energy compared to the fluctuations of the bulk of the string. Thus the lowest energy classical configurations will have the boundary terms in [10] set to zero. This gives rise to the following Dirichlet boundary conditions:

$$r|_{\sigma=0,1} = \sqrt{1 - \frac{p}{N}} \ ,$$

$$\phi|_{\sigma=0,1} = \pi \ . \tag{14}$$

In fact, comparing with the results in the literature [8] one can see that the boundary condition for $r$ is just the radius of orbit of the giant graviton (in units of $R$). On the other hand, the boundary condition on $\phi$ implies [12] that, in general, the length of the string is not fixed. The ground state will be such that that $r' = \phi' = 0$. This is a homogeneous string and this corresponds to a massless excitation of the brane. Notice that this matches precisely the form of the ground state we wrote, and this can be considered as a dual derivation of the spectrum of massless excitations of the giant graviton itself [25].

The spacetime interpretation of the boundary conditions [13] and [14] can be made more clear by deriving the action [7] directly as a limit of the Polyakov action in a particular gauge. Since in our case the number of $Y$s in the open string is constant ($= L + 1 \sim L$), we should use a gauge that distributes the angular momentum in $Y$ homogeneously along the string, this gauge has also been considered in [24]. Moreover, the boundary condition for $\phi$ suggests that we work in a frame where the giant graviton is static.

To do this we follow [30] and write the Polyakov action in momentum space,

$$S_p = \sqrt{\lambda Y M} \int dt \int_0^\pi \frac{d\sigma}{2\pi} L \ , \tag{15}$$

where,

$$L = p_\mu \partial_\mu x^\mu + \frac{1}{2} A^{-1} \left[ G^{\mu\nu} p_\mu p_\nu + G_{\mu\nu} \partial_\mu x^\nu \partial_\nu x^\nu \right]$$

$$+ BA^{-1} p_\mu \partial_\mu x^\mu \ . \tag{16}$$

Here we defined $\lambda Y M = g_y^2 Y M N = R^4/\alpha'^2$, and $A, B$ play the role of Lagrange multipliers implementing the constraints.

Now we write the metric of $\mathbb{R} \times S^5$ as,

$$ds^2 = -dt^2 + |X|^2 + |Y|^2 + |Z|^2 \ , \tag{17}$$

where $|X|^2 + |Y|^2 + |Z|^2 = 1$. The giant graviton is orbiting in the $Z$ direction with $Z = \sqrt{1-p/N} e^{it}$ and
wraps the remaining $S^3$. We put our string at $X = 0$ and define the coordinates,

$$Z = re^{i(t-\phi)}, \quad Y = \pm \sqrt{1-r^2}e^{i\phi}, \quad \text{(18)}$$

for which the giant graviton is static at $r = \sqrt{1-p/N}$ and $\phi$ constant. The metric becomes,

$$ds^2 = -(1-r^2)dt^2 + 2r^2dtd\phi + \frac{1}{1-r^2}dr^2 + r^2d\phi^2 + (1-r^2)d\phi^2. \quad \text{(19)}$$

The momentum in $\phi$ is conserved and is given by,

$$L = \sqrt{\lambda_{YM}} \int_0^\tau \frac{d\sigma}{2\pi} p_\phi \equiv \sqrt{\lambda_{YM}} \mathcal{J}. \quad \text{(20)}$$

We choose a gauge in which angular momentum $p_\phi$ is homogeneously distributed and $\tau$ coincides with the global time in the metric,

$$t = \tau, \quad p_\phi = 2\mathcal{J} = \text{const.} \quad \text{(21)}$$

We then implement the constraints that follow from varying $A$ and $B$ in \[25\] directly in the action as done in \[15\]. Then, using the equations of motion of $p_\tau$ and $p_\phi$, the action is written in terms of the fields $r$ and $\phi$ and their derivatives. Finally, we take the limit $\mathcal{J} \to \infty$ and assume that the time derivatives are of the order $\partial_\sigma x^\mu \sim 1/\mathcal{J}^2$. To order $\mathcal{O}(1/\mathcal{J}^2)$,

$$S_p \approx -L \int_0^\pi \frac{dt d\sigma}{\pi} \left[ \frac{r^2}{1-r^2} \frac{d^2\phi}{d\sigma^2} + 1 \frac{8}{8\mathcal{J}} (r^2 + r^2 \phi'^2) \right]. \quad \text{(22)}$$

Then, rescaling $\sigma \to \pi \sigma$ and using $\lambda_{YM}/(8\pi^2) = \lambda$, we get the action \[4\] (without the boundary terms).

Therefore, we see that the fields $r$ and $\phi$ of the coherent states are the spacetime coordinates for an open string attached to a giant graviton in a coordinate system for which the brane is static. Furthermore, we see how the particular world-sheet gauge is encoded in the CFT side: we have chosen to label our states in such a way so that the $Y$s are distributed homogeneously along the operator. This has very strong implications in the AdS/CFT correspondence, because we are seeing explicitly the reparametrization invariance of the string worldsheet: the gauge that makes the calculation more natural is different than the one considered in other semiclassical setups \[20, 21\].

S. E. Vázquez would like to thank L. Balents for discussions. D.B. work was supported in part by a DOE OJI award, under grant DE-FG02-91ER40618 and NSF under grant No. PHY99-07949. D.H.C. work was supported in part by NSF under grant No. PHY99-07949 and by Fundación Antorchas. S.E.V. work was supported by an NSF graduate fellowship.