A classical solution of the Standard Model + General Relativity is given by an elliptic function whose periodicity in imaginary time is the origin of cosmological temperature. Nothing beyond the Standard Model is assumed. The solution is a Spin(4)-symmetric universe expanding prior to the electroweak transition. A rapidly oscillating SU(2) gauge field holds the Higgs field to 0 with strength inversely proportional to the scale factor $a$. When $a$ reaches $a_{EW}$ the solution becomes unstable and the electroweak transition begins. $a_{EW}$ is the only free parameter in the solution. The temperature at $a_{EW}$ is $m_H/(6\pi)^{1/2} = 28.8\text{ GeV} = 3.34 \times 10^{14}\text{ K}$ whatever the value of $a_{EW}$.

Consider the classical equations of motion of the Standard Model combined with General Relativity. Assume (1) the only nonzero fields are the space-time metric $g_{\mu\nu}(x)$, the SU(2) gauge field $B_\mu(x) \in \text{su}(2)$, and the Higgs scalar field $\phi(x) \in \mathbb{C}^2$, (2) space is the 3-sphere $S^3$, and (3) the universe is Spin(4)-symmetric. Details of the calculations are shown in the supplemental material [1].

The action (in units of $\hbar$ with $c = 1$) is

$$\frac{1}{\hbar} S = \int \left[ \frac{1}{2\kappa} R + \frac{1}{2g^2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) \right. $$

$$ - D_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} \lambda^2 \left( \phi^a \phi^a - \frac{1}{2} v^2 \right)^2 \right] \sqrt{-g} \, dt \, dx$$  

(1)

Space is the unit 3-sphere $S^3$ identified with SU(2) by

$$\hat{x} \in S^3 \leftrightarrow g_{\hat{x}} = \hat{x}^4 \mathbf{1} + \hat{x}^a i^{-1} \sigma_a \in \text{SU}(2)$$  

(2)

$\sigma_{1,2,3}$ being the Pauli matrices. Points in space-time are $x = (x^0, \hat{x}) = (x^0, g_{\hat{x}})$. Spin(4) = SU(2)$_L \times$ SU(2)$_R$ acts as SO(4) on space-time by

$$x = (x^0, g_{\hat{x}}) \mapsto Ux = (x^0, g_L g_{\hat{x}} g_R^{-1})$$  

(3)

An SO(4)-symmetric metric has the conformally flat form

$$g_{\mu\nu} dx^\mu dx^\nu = a(T)^2 \left( -dT^2 + ds_{S^3}^2 \right)$$  

(4)

d$s_{S^3}^2$ being the metric of the unit 3-sphere.

As a symmetry ansatz for solving the equations of motion, suppose Spin(4) acts on $\phi(x)$ and $B_\mu(x)$ by

$$\phi(x) \mapsto g_L^{-1} \phi(Ux)$$

$$B_\mu(x) dx^\mu \mapsto g_L^{-1} B_\mu(Ux) g_L \, dx(Ux)^\mu$$  

(5)

The Spin(4)-symmetric scalar and gauge fields are

$$\phi(x) = 0 \quad B_\mu(x) dx^\mu = [1 + b(T)] \frac{1}{2} g_\hat{x} g_L^d$$  

(6)

Cosmology is described by the functions $a(T)$ and $b(T)$.
subject to the Wheeler-Dewitt constraint

\[ H = -\epsilon_w^{-4}E_a + E_b = 0 \]

\[ E_a = \frac{1}{2}(\partial_T a)^2 + V_a \quad E_b = \frac{1}{2}(\partial_T b)^2 + V_b \]

(11)

so the zero-mode stability condition is

\[ 0 \leq \frac{3}{4}(1 + b)^2 - \frac{1}{2}a^2\lambda^2 v^2 \]

(19)

b(T) oscillates in the potential \( V_b \) between \( \pm b_{\text{max}} \) where

\[ V_b(b_{\text{max}}) = \frac{1}{2}(b_{\text{max}}^2 - 1)^2 = E_b \]

(20)

Stability of \( \phi = 0 \) at time scales longer than the period of \( b \) oscillation is the time average of condition (19).

\[ 0 \leq \frac{3}{4} - \frac{3}{4}(b^2 - \frac{1}{2}a^2\lambda^2 v^2) \]

(21)

This is satisfied when \( a(T) \) is small. When \( a(T) \) grows large enough that condition (21) is violated, the \( \phi = 0 \) solution becomes unstable. The Higgs field then starts down from the unstable equilibrium at \( \phi = 0 \) towards its vacuum expectation value. This is the beginning of the electroweak transition, at scale factor \( a(T) = a_{\text{ew}} \) where equality holds in condition (21).

For an extremely rough estimate of the actual value of \( a_{\text{ew}} \) suppose that the electroweak transition began at redshift \( \approx 10^{10} \) to within some orders of magnitude. Suppose that the present scale factor \( a_0 \) is the Hubble time \( t_H = 4.55 \times 10^{17} \) s again to within some orders of magnitude. Then \( a_{\text{ew}} \) is \( \approx 10^{7} \) s. Combine this in (21) with \( \lambda v = m_H/h \approx (10^{-27} s)^{-1} \) to get \( \langle b^2 \rangle \approx 10^{68} \). So the solutions that might be physically interesting will have \( E_b \) quite a large number, something like \( 10^{68} \) and \( a_{\text{ew}} \) given by equality in (21) is

\[ a_{\text{ew}} = \frac{3}{2} \left( \frac{\langle b^2 \rangle}{\lambda^2 v^2} \right)^{1/2} = \frac{3}{2} \left( \frac{\langle b^2 \rangle}{\lambda^2 v^2} \right)^{1/2} \]

(22)

A complete analysis of the stability condition (17) for all modes of the perturbation \( \phi(x) \) is carried out in [1]. The result is that the zero-mode is the first mode to become unstable as \( a(T) \) increases. So equation (22) remains valid.

The \( \hat{b} \) oscillator is solved by integrating the energy equation (11). Change variables from \( T, b(T) \) to \( z, y(z) \)

\[ T = \epsilon_b z \quad b(T) = b_{\text{max}} y(z) \quad \epsilon_b = (8E_b)^{-1/4} \]

(23)

y oscillates between \( \pm 1 \). The energy equation (11) is

\[ \left( \frac{dy}{dz} \right)^2 = (1 - y^2)(1 - k^2 + k^2y^2) \quad k^2 = 1 + \epsilon_b^2 \]

(24)

solved by the Jacobian elliptic function \( y(z) = \text{cn}(z, k) \) [3, 4]

\[ z = \int_{\text{cn}(z, k)}^{\text{cn}(z, k)} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 + k^2y^2)}} \]

(25)

\( \text{cn}(z, k) \) is doubly periodic in the complex variable \( z \) with real period \( 4K' \) and imaginary period \( 4iK' \)

\[ \frac{z}{\text{cn}(z, k)} = \begin{cases} -2K & \text{if } z = -2K \text{ or } K \text{ or } 2K \text{ or } -K \\ -1 & \text{if } z = -1 \text{ or } 0 \text{ or } 1 \text{ or } -1 \end{cases} \]

(26)
$K$, $K'$ are the complete elliptic integrals of the first kind

$$K(k) = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2+y^2)}}$$  \hspace{1cm} (27)

$$K'(k) = K(k'), \quad k^2 + k'^2 = 1. \text{ Since } \epsilon_b \approx 10^{-34},$$

$$k = 1/\sqrt{2}, \quad b(T) = (2E_b)^{1/4} \text{cn}(\epsilon_b^{-1}T,k)$$

$$b(T + 4\epsilon_b K) = b(T + 4\epsilon_b K)$$  \hspace{1cm} (28)

$K = K' = (1/\sqrt{2}) = \Gamma(1/4)^2 / 4\pi^{1/2} = 1.854075\ldots$

The time average of $cn^2(z,k)$ over a real cycle gives $\langle b^2 \rangle$

$$\langle cn^2 \rangle = \frac{\pi}{2K^2}, \quad \langle b^2 \rangle = (2E_b)^{1/2} \frac{\pi}{2K^2}$$  \hspace{1cm} (29)

Equation (22) now becomes

$$a_{ew} = \frac{(3\pi)^{1/2}(2E_b)^{1/4}}{2K} \frac{h}{m_H} \frac{(6\pi)^{1/2}}{4K\epsilon_b} \frac{h}{m_H}$$  \hspace{1cm} (30)

$E_b$ determines $a_{ew}$ and vice versa, so $a_{ew}$ can be taken as the free parameter of the solution.

The cosmological gauge field $b(T)$ is periodic in imaginary proper time with period $T \sim T + 4i\epsilon_b K$. In comoving time $t$, given by $dt = a(t)dt$ so that the space-time metric is $g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2ds^2_{S^3}$, the imaginary time period is $t \sim t + 4i\epsilon_b / \epsilon_a$. The cosmological gauge field therefore acts as a thermal bath. The period $4K\epsilon_b / \epsilon_a$ in imaginary co-moving time is the inverse temperature,

$$\frac{h}{k_B T_{SU(2)}} = 4K\epsilon_b / \epsilon_a$$  \hspace{1cm} (31)

Using equation (30) for $a_{ew}$, the temperature is

$$k_B T_{SU(2)} = \frac{m_H}{(6\pi)^{1/2}} \frac{a_{ew}}{a} = 28.8 \text{ GeV} \frac{a_{ew}}{a}$$  \hspace{1cm} (32)

At $a = a_{ew}$, the onset of the electroweak transition, the temperature is

$$k_B T_{ew} = 28.8 \text{ GeV} \quad T_{ew} = 3.34 \times 10^{14} \text{ K}$$  \hspace{1cm} (33)

independent of the value of $a_{ew}$.

To solve the $\dot{a}$ oscillator, let

$$\dot{a} = (2E_a)^{1/4}u, \quad \epsilon_a = (2E_a)^{-1/4} \quad k_a^2 = \frac{1}{2} + \frac{1}{4}\epsilon_a^2$$  \hspace{1cm} (34)

The interesting values of $E_a^{1/4} = \epsilon_a E_b^{1/4}$ are numbers $\approx 10^{17}$, so $k_a = 1/\sqrt{2}$. In dimensionless co-moving time $\dot{t} = t/t_1, \quad dt = \dot{t}dt'$, the energy equation (11) is

$$\left( \frac{du}{dt} \right)^2 + \epsilon_a^2u^2 - u^4 = 1$$  \hspace{1cm} (35)

The solution is

$$u = \sqrt{(e^{2t} - 1)} \left(1 - k_a^2 + k_a^2 e^{-2t}\right) = \sqrt{\sinh (2t)}$$  \hspace{1cm} (36)

with the relation to conformal time $T$ given by

$$e^{-t} = \text{cn}(\epsilon_a^{-1}T,k_a)$$  \hspace{1cm} (37)

The scale factor $\dot{a}$ ranges from 0 to $\infty$ as co-moving time $t$ ranges from 0 to $\infty$ and conformal time $T$ ranges from 0 to $\epsilon_a K_a$ where $K_a = K(k_a) = K = 1.854075\ldots$. The time scale of the $b$ oscillations is much shorter than the expansion time scale, $\epsilon_b / \epsilon_a = \epsilon_w / \sqrt{2} = 2.40 \times 10^{-17}$.

Combining (30), (36), and (37) gives

$$a_{ew} = 0.504 (2E_a)^{1/4} \epsilon_a \quad t_{ew} = 0.126$$

$$t_{ew} = 3.21 \times 10^{-11} \text{ s} \quad T_{ew} = 0.501 \epsilon_a$$  \hspace{1cm} (38)

(Here $T_{ew}$ is proper time at $a = a_{ew}$, not temperature.)

The next step will be to find a classical solution for cosmology after the onset of the electroweak transition, for $a > a_{ew}$. The solution will include the $U(1)$ gauge field along with the SU(2) gauge field. The gauge group will be $U(2)$. In $U(2)_L \times U(2)_R$ the subgroup consisting of the $(g_L, g_R)$ with $det g_R = det g_L$ acts on $S^3 \approx SU(2)$ by $g \rightarrow g_L g_R g^{-1}$. The symmetry group of the nonzero Higgs field $\left( \phi \right)$ is the $U(2)$ subgroup $g_L = \left( \begin{array}{ccc} 1 & 0 \\ 0 & det g_R \end{array} \right)$. There should be a classical $U(2)$-symmetric solution describing the descent of the Higgs field to its vacuum expectation value.

Questions that will arise include: how much expansion will take place during the descent? how much spatial anisotropy will be introduced? will the cosmological gauge field violate CP symmetry during the descent?

The temperature $T_{ew} = 3.34 \times 10^{14} \text{ K}$ suggests that $a_{ew}$ is at redshift $\approx 10^{14}$. Considerable expansion will be needed during the electroweak transition. The solution (36) for $a(t)$ has an inflationary epoch $a \sim e^{t/t_1}$ but it does not set in until well after the Spin(4)-symmetric solution becomes unstable at $t_{ew} = 0.126$. The $U(2)$-symmetric solution will need to spend some time in the inflationary neighborhood.

The $U(2)$-symmetric solution will be spatially homogeneous because $U(2)$ takes every point in $S^3$ to every other. But the little group at a point in space is a $U(1)$ that leaves a preferred direction invariant. The $U(2)$-symmetric solution will be anisotropic. The question is how much anisotropy remains at large time.

The cosmological SU(2) gauge field $b(T)$ is $P$-odd and $C$-even so it is $CP$-odd. In the Spin(4)-symmetric solution the oscillations are even in $b \rightarrow -b$. If the $b$ oscillations are critically damped or over-damped in the $U(2)$ solution, then the very last stage of the descent to 0 will break the $b \rightarrow -b$ symmetry and thus break CP.

A second pressing task is to evaluate the corrections to the Spin(4)-symmetric action due to thermal quantum fluctuations of all the Standard Model fields to check whether the classical Spin(4)-symmetric solution is materially affected. The temperature $T_{SU(2)}$ is inversely proportional to $a(t)$ so there is a question how far back before
\( a = a_{\mu\nu} \) can the solution obtain before corrections from unknown high energy physics become significant.

Eventually there will be the question of the origin of the Spin(4) symmetry. Why should the SU(2) gauge bundle be isomorphic to the spin bundle at early time? And why should \( E_b \) be such a large number?

The Spin(4)-symmetric classical cosmology is part of a project to find a classical solution of the Standard Model combined with General Relativity that is analytic in time and that captures the observed macroscopic features of cosmology. The project is an attempt at a top-down, first-principles calculation of a complete macroscopic cosmology within the Standard Model combined with General Relativity. The hope is that the macroscopic classical cosmology can be refined by microscopic corrections into a quantitatively testable theory.

The first part of the project [5] was an SO(4)-symmetric classical solution that captured some qualitative features of late-time cosmology: a big bang followed by decelerating expansion followed by accelerating expansion, driven by purely gravitational dark matter and energy. The hope is that the U(2)-symmetric solution will interpolate between the Spin(4)-symmetric cosmology at early time and the SO(4)-symmetric cosmology at late time.

My particular agenda in this work is to find evidence in support of a speculative fundamental theory of physics in which the laws of physics are produced by the 2d renormalization group [5]. In the pure gravity simplification of this theory, the space-time metric \( g_{\mu\nu} \) is the coupling of a 2d quantum field theory, and the 2d rg drives \( g_{\mu\nu} \) to a solution of \( R_{\mu\nu} = \nabla_{\mu} v_{\nu} + \nabla_{\nu} v_{\mu} \). This is an rg fixed point because the metric changes by \( R_{\mu\nu} \) which is just an infinitesimal reparametrization of space-time. The 2d rg fixed point equation is General Relativity extended by the source term \( \nabla_{\mu} v_{\nu} + \nabla_{\nu} v_{\mu} \). The source term provides dark matter and energy [5]. In the full theory, the classical equations of motion of the Standard Model combined with General Relativity are extended by source terms corresponding to all the gauge symmetries of the theory. The present work started out by investigating the Spin(4)-symmetric equations of motion with such source terms, but the source terms turned out to be identically zero in the solution. So the Spin(4)-symmetric solution is presented here without connection to the 2d rg program of [5]. Later, if a comprehensive macroscopic cosmology with the 2d rg source terms can be derived successfully, then global properties of the 2d rg flow might be looked to for explanation of early-time Spin(4)-symmetry and the magnitude of the gauge field oscillations.

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[1] See the accompanying Supplemental Material consisting of a note “Calculations for "Origin of Cosmological Temperature” and a SageMath notebook performing numerical calculations. The Supplemental Material is also available at https://cocalc.com/share/3c1ab84c375769c3460251d4f2bd43461c1211b5/Supplemental_material/ and at https://www.physics.rutgers.edu/pages/friedan/papers/2020/Origin/Supplemental_material/.

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