On Severi varieties on Hirzebruch surfaces

Ilya Tyomkin

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, USA. e-mail: tyomkin@math.mit.edu

Abstract. In the current paper we prove that any Severi variety on a Hirzebruch surface contains a unique component parameterizing irreducible nodal curves of the given genus in characteristic zero.

1. Introduction

Convention 1. Throughout this paper we work over an algebraically closed field $K$ of characteristic zero, and genus always means geometric genus.

The study of Severi varieties is one of the classical problems in algebraic geometry. Given a smooth projective surface $\Sigma$, a line bundle $\mathcal{L} \in \text{Pic}(\Sigma)$, and an integer $g$, one defines Severi variety $V(\Sigma, \mathcal{L}, g) \subset |\mathcal{L}|$ to be the closure of the locus of nodal curves of genus $g$. Then the subvariety $V^{\text{irr}}(\Sigma, \mathcal{L}, g) \subset |\mathcal{L}|$ parameterizing irreducible curves is of special interest. Originally, these varieties were introduced by Severi (in the plane case) in order to prove the irreducibility of the moduli spaces of curves $\mathcal{M}_g$ in characteristic zero. In Anhang F of his famous book, Vorlesungen über algebraische Geometrie, F. Severi gave a false proof of the irreducibility of $V^{\text{irr}}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d), g)$. And it took more than sixty years, till, in 1986, Harris proved this result [5].

The study of various properties of Severi varieties, in particular their degrees continued, and several formulas were obtained: recursive formulas of Caporaso and Harris for projective plane, and of Vakil for Hirzebruch surfaces, and a non-recursive formula of Mikhalkin

* The author was partially supported by the Postdoctoral fellowship provided by the Clore Foundation.
for toric surfaces. More formulas were obtained by Kontsevich, Ran, Ruan, Tian and others. Although the degrees of Severi varieties have been computed for any projective toric surface, the irreducibility problem is still open in most of the cases.

The goal of the current paper is to give a proof of the irreducibility of \( V^{\text{irr}}(\Sigma, \mathcal{L}, g) \subset |\mathcal{L}| \) on Hirzebruch surfaces. We shall mention that Shevchishin announced the same result, but up to our knowledge his argument is incomplete. We shall also mention that the approach presented in this paper is different from that of Shevchishin [9].

The idea of the proof of the irreducibility of Severi varieties on Hirzebruch surfaces is as follows. Let \( \Sigma \) be a Hirzebruch surface, and let \( \mathcal{L} = \mathcal{O}_{\Sigma}(dL_0 + kF) \) be a line bundle, where \( L_0 \) and \( F \) denote the effective classes generating \( \text{Pic}(\Sigma) \), satisfying \( L_0.F = 1, F^2 = 0, \) and \( L_0^2 = n \). We use the notation \( V_{g,d,k} \) for the Severi variety \( V(\Sigma, \mathcal{L}, g) \).

In these notations the first part of the proof is given by Proposition 5, which states the following: any irreducible component \( V \subseteq V_{d,k,g} \) contains a very reducible nodal curve of a special type. Now, the irreducibility of \( V^{\text{irr}}_{g,d,k} \) follows from Proposition 6, claiming that there exists a unique component containing such a curve, whose generic point corresponds to an irreducible curve.

The proof of Proposition 5 is similar to the plane case proof of Harris presented in [6]. The proof of Proposition 6 is reduced to a combinatorial statement using monodromy-type arguments.

To finish the introduction we shall mention that there are examples of (non-rational) surfaces admitting reducible Severi varieties. Moreover, these Severi varieties can have components of different dimensions. So it is unclear how to characterize the surfaces admitting only irreducible Severi varieties. Nevertheless, we would like to state the following conjecture motivated by our result and Mikhalkin’s work [8]:

**Conjecture 1.** If \( \Sigma \) is a toric surface, \( \mathcal{L} \in \text{Pic} \Sigma \) is an effective class, and \( g \geq 0 \) is a non-negative integer then the Severi variety \( V^{\text{irr}}_{\Sigma, \mathcal{L}, g} \) parameterizing irreducible nodal curves of genus \( g \) in \( |\mathcal{L}| \), that do not contain singular points of \( \Sigma \), is either empty or irreducible.

We shall discuss the proof of this conjecture for rational curves in the last section of the paper, and we hope that the general case can be approached using the rapidly developing methods of tropical geometry combined with the approach presented in the current paper.

Finally, we would like to mention that the case of positive characteristic is still open even for plane curves. It seems that the main missing ingredient is a statement characterizing Severi varieties in terms of their dimensions (similar to Theorem 1). There is a tropical evident that such a statement must exist, however we do not know any algebraic theorem of this type. Nevertheless we suppose that Conjecture 1 is true in arbitrary characteristic.
Acknowledgements. I am very grateful to J. Bernstein, G.-M. Greuel, E. Shustin, and M. Temkin for helpful discussions. Parts of this work were done while the author was a Clore postdoctoral fellow at the Weizmann Institute of Science, a Moore instructor at MIT, and was visiting the Max-Planck-Institut für Mathematik at Bonn. I would like to thank these institutions for their hospitality.

2. Preliminaries

2.1. Deformation theory

In this section we discuss several (basic) facts from the deformation theory of algebraic varieties and algebraic maps. Most of the statements, ideas, and proofs presented here can be found in different sources (see for example [1,2,5,10], and [11] for related topics). However, I decided to write it down here for the completeness of the presentation.

2.1.1. Deformations of maps. Let \( X \) and \( Y \) be smooth algebraic varieties over an algebraically closed field \( K \) of characteristic zero, and let \( f : X \to Y \) be an algebraic map. In this section we discuss the deformation theory of the pair \((X,f)\), namely, we fix \( Y \) and vary \( X \) and \( f \). Denote \( \mathbb{D} = \text{Spec} K[[\epsilon]]/(\epsilon^2) \). We recall that a first-order deformation of \((X,f)\) is a triple

\[-\quad \text{a flat family } \pi : \tilde{X} \to \mathbb{D},\]
\[-\quad \text{a map } F : \tilde{X} \to Y,\]
\[-\quad \text{an isomorphism } \alpha : (\tilde{X}_0, F_0) \to (X,f), \text{ where } \tilde{X}_0 = \tilde{X}/(\epsilon), \text{ and } F_0 = F/(\epsilon).\]

Notation 1. The set of first order deformations of the pair \((X,f)\) modulo isomorphisms is denoted \(\text{Def}^1(X,f)\).

Proposition 1. If \( 0 \to T_X \to f^*T_Y \) is exact, then
\[\text{Def}^1(X,f) \cong H^0(X, N_f),\]
where \( N_f \) denotes the normal sheaf to \( f \), i.e. the cokernel of the map \( df : T_X \to f^*T_Y \).

Proof. First, we choose affine coverings \( Y = \bigcup_{i=1}^n Y_i \) and \( X = \bigcup_{i=1}^n X_i \) such that \( f(X_i) \subset Y_i \) for all \( i \). Now, let \( \xi \in \text{Def}^1(X,f) \) be a first-order deformation. We shall use the following well-known claim

Claim 1. Let \( Z = \text{Spec} A \) be a smooth affine variety over an algebraically closed field \( K \), and let \( Z_\epsilon = \text{Spec} A_\epsilon \) be an infinitesimal extension of \( Z \), i.e. a pair consisting of a flat morphism \( Z_\epsilon \to \mathbb{D} \) together with an isomorphism \( Z_\epsilon/(\epsilon) \cong Z \). Then \( Z_\epsilon \) is isomorphic to the trivial extension, namely \( A_\epsilon \cong A \oplus \epsilon A \).
Due to the claim we can fix trivializations
\[ O_{\tilde{X}}(X_i) \simeq O_{X_i}(X_i) \oplus \epsilon O_{X_i}(X_i). \tag{2.1} \]

Then we obtain the automorphisms
\[ \beta_{ij} : O_{X_{ij}}(X_{ij}) \oplus \epsilon O_{X_{ij}}(X_{ij}) \longrightarrow O_{X_{ij}}(X_{ij}) \oplus \epsilon O_{X_{ij}}(X_{ij}), \]
equal to identity modulo \( \epsilon \). Hence
\[ \beta_{ij}(x + \epsilon y) = x + \epsilon(y + D_{ij}(x)), \]
where \( D_{ij} : O_{X_{ij}}(X_{ij}) \longrightarrow O_{X_{ij}}(X_{ij}) \) are derivations. The maps
\[ F^* : O_Y(Y_i) \longrightarrow O_{X_i}(X_i) \oplus \epsilon O_{X_i}(X_i) \]
are given by \( F^*(x) = f^*(x) + \epsilon D_i(x), \) where
\[ D_i : O_Y(Y_i) \longrightarrow O_{X_i}(X_i) \]
are derivations, and the following equality holds:
\[ D_i - D_j = D_{ij} \circ f^*. \]

Hence the set \( D^\xi = (D_1, ..., D_n) \) defines a global section of the sheaf \( N_f \). It is clear that \( D^\xi \) does not depend on the choice of the trivializations in (2.1). Now, one can easily check that the constructed correspondence provides us with the bijection\(^1\)
\[ \text{Def}^1(X, f) \ni \xi \leftrightarrow D^\xi \in H^0(X, N_f), \]
which in fact is an isomorphism of vector spaces. Moreover, this bijection does not depend on the choice of coverings \( Y = \bigcup_{i=1}^n Y_i \) and \( X = \bigcup_{i=1}^n X_i. \)

\( \square \)

2.1.2. Families of curves on algebraic surfaces. Let \( \Sigma \) be a smooth projective algebraic surface, and let \( \mathcal{L} \) be a line bundle on \( \Sigma \). Consider an irreducible variety \( V \subseteq |\mathcal{L}| \) whose generic element is a reduced curve. The goal of this section is to give a natural upper bound on the dimension of \( V \). Let
\[ \mathcal{L} \leftarrow V \times \Sigma \]

\( \square \)

\(^1\) Here one must use the fact that \( 0 \longrightarrow T_X \longrightarrow f^*T_Y \) is exact.
be the tautological family of curves over $V$, and let $\widetilde{C} \rightarrow C$ be its normalization. Then for almost all $p \in V$ the fiber $\widetilde{C}_p$ is the normalization of the fiber $C_p$.

Let us choose a generic point $0 \in V$. Due to the generic flatness theorem and Proposition 1, we then have a natural map

$$\mu : T_0 V \rightarrow \text{Def}^1(C, f) \cong H^0(C, N_f),$$

where $C = \widetilde{C}_0$ and $f$ is the composition of maps

$$C = \widetilde{C}_0 \rightarrow C \rightarrow \Sigma \times \{0\} = \Sigma.$$

**Proposition 2.**

1. The map $T_0 V \rightarrow H^0(C, N_f)$ is an embedding.

2. $\text{Im}(T_0 V) \cap H^0(C, N^\text{tor}_f) = 0$.

**Proof.** First, we choose a smooth projective irreducible curve $D \subset \Sigma$ intersecting $f(C)$ transversally, such that $h^0(\Sigma, L(-D)) = 0$. Let

$$X \xrightarrow{F} \Sigma \rightarrow D$$

be a first-order deformation of the pair $(C, f)$. We define the new pair $X_D = X \times_{\Sigma} D$ and $F_D : X_D \rightarrow D$.

**Claim 2.** The natural map $X_D \rightarrow D$ is flat.

So we constructed a map

$$\rho : \text{Def}^1(C, f) \rightarrow \text{Def}^1(f(C) \cap D, f_D),$$

where $f_D = F_D/(\epsilon)$.

**Claim 3.** $H^0(C, N^\text{tor}_f) \subset \text{Ker}(\rho)$.

To finish the proof it is enough to show that $\dim(\rho(\mu(T_0 V))) = \dim(V)$. Consider the exact sequence

$$H^0(\Sigma, L(-D)) \rightarrow H^0(\Sigma, L) \rightarrow H^0(D, L \otimes \mathcal{O}_D).$$

The first term is zero, hence the map $\alpha : V \rightarrow |L \otimes \mathcal{O}_D|$ is an embedding. Now

$$T_{\alpha(0)}|L \otimes \mathcal{O}_D| = H^0(f(C) \cap D, \mathcal{O}_{f(C) \cap D}(f(C) \cap D)) = \text{Def}^1(f(C) \cap D, f_D)$$

and $\rho \circ \mu = d\alpha(0)$, which implies $\dim(\rho(\mu(T_0 V))) = \dim(V)$. \qed

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2 Here we use the fact that $\mathbb{K}$ is a field of characteristic zero.
Proof (of Claim 2). First of all we can assume that $D, \Sigma,$ and $C$ are affine. Thus $X$ and $X_D$ are affine as well. Next we shall use the following lemma:

**Lemma 1** ([4], Proposition 6.1). Let $R$ be a commutative ring with identity, and let $M$ be an $R$–module. If $I$ is an ideal of $R$, then $\text{Tor}_1^R(R/I, M) = 0$ if and only if the map $I \otimes_R M \rightarrow M$ is an injection. The module $M$ is flat if and only if this condition is satisfied for every finitely generated ideal $I \subset R$.

In our case $R = \mathbb{K}[\epsilon]/(\epsilon^2)$ and the only non-trivial ideal we have is $I = \mathbb{K}\epsilon$. By the lemma it is enough to show that the map

$$\mathcal{O}_{X_D} \otimes_{\mathcal{O}_D} \mathbb{K}\epsilon \rightarrow \mathcal{O}_{X_D}$$

is an embedding. Since $\mathcal{O}_X$ is flat over $\mathcal{O}_D$, we have an exact sequence

$$0 \rightarrow \mathcal{O}_X \otimes_{\mathcal{O}_D} \mathbb{K}\epsilon \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0.$$

Tensoring this sequence with $\mathcal{O}_D$ over $\mathcal{O}_\Sigma$ we obtain

$$0 \rightarrow \text{Tor}_1^{\mathcal{O}_\Sigma}(\mathcal{O}_D, \mathcal{O}_C) \rightarrow \mathcal{O}_{X_D} \otimes_{\mathcal{O}_D} \mathbb{K}\epsilon \rightarrow \mathcal{O}_{X_D}.$$

It remains to prove that $\text{Tor}_1^{\mathcal{O}_\Sigma}(\mathcal{O}_D, \mathcal{O}_C) = 0$. Let

$$0 \rightarrow \mathcal{O}_\Sigma \rightarrow \mathcal{O}_\Sigma \rightarrow \mathcal{O}_D \rightarrow 0$$

be a free resolution of $\mathcal{O}_D$, where $j \in \mathcal{O}_\Sigma$ is a function defining $D$. Thus

$$\text{Tor}_1^{\mathcal{O}_\Sigma}(\mathcal{O}_D, \mathcal{O}_C) = \text{Ker} \left( \mathcal{O}_C \xrightarrow{f^*(j)} \mathcal{O}_C \right) = 0,$$

since $f(C)$ intersects $D$ transversally. \qed

Proof (of Claim 3). To prove this claim we have to construct the map $\rho$ explicitly. Consider the fibered product diagram

$$\begin{array}{ccc}
  f(C) \cap D & \xrightarrow{f_D} & D \\
  i & & i_D \\
  C & \xrightarrow{f} & \Sigma 
\end{array}$$
The intersection $f(C) \cap D$ is transversal and its points are smooth on both $f(C)$ and $D$. Hence we have the following exact sequences on $f(C) \cap D$:

\[
\begin{array}{c}
0 \rightarrow i^*T_C \xrightarrow{i^*(df)} i^*f^*T_{\Sigma} \xrightarrow{} i^*N_f \xrightarrow{} 0 \\
\| \quad \| \\
0 \rightarrow f_D^*T_D \xrightarrow{df_D} f_D^*i_D^*T_{\Sigma} \xrightarrow{} f_D^*N_{i_D} \xrightarrow{} 0 \\
\| \\
\mathcal{N}_{f_D}
\end{array}
\]

Moreover,

\[i^*(df)(i^*T_C) \oplus df_D(f_D^*T_D) = i^*f^*T_{\Sigma}.
\]

Hence the map $\gamma : \mathcal{N}_{f_D} \rightarrow i^*N_f$ is an isomorphism, and $\rho$ is given by the composition

\[H^0(C, N_f) \xrightarrow{i^*} H^0(f(C) \cap D, i^*N_f) \xrightarrow{\gamma^{-1}} H^0(f(C) \cap D, \mathcal{N}_{f_D}).\]

To finish the proof we note that $df(p) \neq 0$ for any $p \in i(f(C) \cap D)$. Thus $i^*N_f^{tor} = 0$, which implies $H^0(C, N_f^{tor}) \subseteq \text{Ker}(\rho)$. □

**Theorem 1.** Let $\Sigma$ be a smooth projective algebraic surface, $C' \subset \Sigma$ be a smooth curve, and let $L$ be a line bundle on $\Sigma$. Consider a positive dimensional irreducible variety $V \subseteq |L|$ whose generic element $C_0$ is a reduced curve. Assume that $C_0^i.K_{\Sigma} < -1$ for any irreducible component $C_0^i \subseteq C_0$. Then

\[\dim(V) \leq -C_0.K_{\Sigma} + g - 1. \quad (2.2)\]

Furthermore, if the equality holds, and

\[C_0^i.K_{\Sigma} < -3 \quad (2.3)\]

for any singular irreducible component $C_0^i$ of $C_0$, then $V$ has no fixed points, $C_0$ has only nodes as its singularities, and $C_0$ intersects $C'$ transversally.

**Proof.** Let $C$ be the normalization of $C_0$ and let $f : C \rightarrow \Sigma$ be the natural map. Then $\dim(h^0(C, N_f/N_{f}^{tor})) \leq -C_0.K_{\Sigma} + g - 1$ implies (2.2), by Proposition 2. So it is enough to show the analogous inequality for every irreducible component of $C$. Thus we can assume that $C$ is irreducible.
Choose an invertible sheaf $\mathcal{F}$ on $C$ such that the sequence

$$0 \rightarrow \mathcal{N}_f/\mathcal{N}_f^{tor} \rightarrow \mathcal{F} \rightarrow \mathcal{N}_f^{tor} \rightarrow 0$$

is exact (the existence of such $\mathcal{F}$ is completely obvious). Then

$$c_1(\mathcal{F}) = c_1(\mathcal{N}_f) = c_1(f^*T_{C}) - c_1(T_C) = 2g - 2 + c_1(f^*T_{C}) > 2g - 1.$$  

Hence

$$h^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor}) \leq h^0(C, \mathcal{F}) = c_1(\mathcal{F}) + 1 - g = -C_0.K_{\Sigma} + g - 1$$  

by the Riemann-Roch theorem, and the equality holds if and only if $\mathcal{N}_f^{tor} = 0$.

For the second part we note that if the dimension of $V$ equals $-C_0.K_{\Sigma} + g - 1$, then $\mathcal{N}_f^{tor} = 0$, and hence $df \neq 0$ everywhere. So it remains to prove that $C_0$ has no triple points, and that all its double points have two different tangent directions. If $p \in C_0$ is a triple point and $q_1, q_2, q_3 \in C$ are three points mapped to $p$, then any $\xi \in T_0V$ vanishing at $q_1$ and $q_2$, must vanish at $q_3$ as well. However, due to the Riemann-Roch theorem, inequality (2.4), and condition (2.3), there exists $\eta \in H^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor})$ such that $\eta(q_1) = \eta(q_2) \neq \eta(q_3)$. Thus

$$\dim T_0V < h^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor}) = -C_0.K_{\Sigma} + g - 1,$$

which contradicts the equality in (2.2). If $p \in C_0$ is a double point with a unique tangent direction and $q_1, q_2 \in C$ are the two pre-images of $p$, then any $\xi \in T_0V$ vanishing at $q_1$, must also vanish at $q_2$. However, applying Riemann-Roch theorem, inequality (2.4), and condition (2.3), we can find $\eta \in H^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor})$, such that $\eta(q_1) = 0 \neq \eta(q_2)$. Hence

$$\dim T_0V < h^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor}) = -C_0.K_{\Sigma} + g - 1,$$

which is a contradiction.

It remains to prove that if $\dim(V) = -C_0.K_{\Sigma} + g - 1$, then $C_0$ intersects $C'$ transversally. Assume that $\dim(V) = -C_0.K_{\Sigma} + g - 1$. Then the system $V$ has no fixed components; hence $C_0$ does not contain $C'$. If $p$ is either a point of a non-transversal intersection of $C_0 \cap C'$ or a fixed point of $V$, then either $p$ has at least two pre-images $q_1, q_2 \in C$ or any $\xi \in T_0V$ vanishes at $q$, where $q \in C'$ is the unique pre-image of $p$. In the first case any $\xi \in T_0V$ vanishing at $q_1$ must also vanish at $q_2$. So both cases contradict the Riemann-Roch theorem, due to (2.4) and (2.3). □
Lemma 2. Let $(\Sigma, L)$ be a smooth rational surface equipped with a line bundle, and let $L \subset \Sigma$ be a smooth curve. Let $p_1, \ldots, p_r \in L$ be arbitrary points, $k_1, \ldots, k_r$ be non-negative integers, and let
\[ R \subset \left\{ C \in |L| : L \cap C = \sum_{i=1}^{r} k_i p_i \right\} \]
be a non-empty subvariety. Choose a generic curve $C \in R$. If $C$ is reduced and $C_i.(K_{\Sigma} + L) < -1$ for any irreducible component $C_i \subset C$, then
\[ \dim(R) \leq -C.K_{\Sigma} + g(C) - 1 - L.C. \]
Moreover, if the equality holds, and for any irreducible component $C_i$ of $C$ we have $C_i.(K_{\Sigma} + L) < -3$, then $R$ has no fixed points but $p_1, \ldots, p_r$, $C$ has only nodes as its singularities outside of $L$, for any smooth irreducible curve $C'$, not tangent to $L$, a generic curve $D \in R$ intersects $C'$ transversally at any $q \notin \{p_1, \ldots, p_r\}$, and for any $i$, $D$ is a union of smooth branches in a neighborhood of $p_i$, not tangent to $C'$.

Proof. The proof is by induction on $L.C$. If $L.C = 0$, then the lemma follows from Theorem 1, since $L.C_i = 0$ for any irreducible component $C_i \subseteq C$.

Assume now that $k_i > 0$ for some $i$. Without loss of generality $k_1 > 0$. Consider the blow up $\tilde{\Sigma} = Bl_{p_1}(\Sigma)$ with its natural projection $\pi : \tilde{\Sigma} \to \Sigma$. We denote the strict transform of $L$ by $\tilde{L}$. Then $O_{\tilde{\Sigma}}(\tilde{L}) \simeq \pi^*O_{\Sigma}(L) \otimes O_{\tilde{\Sigma}}(-E)$, where $E$ denotes the exceptional divisor $\pi^{-1}(p_1)$. Let $\tilde{C} \subset \tilde{\Sigma}$ be the strict transform of $C$. Then $O_{\tilde{\Sigma}}(\tilde{C}) \simeq \pi^*O_{\Sigma}(C) \otimes O_{\tilde{\Sigma}}(-m_1 E)$, where $m_1 = \text{mult}_{p_1}(C)$. Define $p'_i = E \cap \tilde{L}$ and $p'_i = p_i$ for any $i > 1$, and consider the pullback of (an open dense subset of) $R$
\[ R_1 \subset \left\{ X \in |O_{\tilde{\Sigma}}(\tilde{C})| : g(X) = g(C) \text{ and } X \cap \tilde{L} = \sum_{i=1}^{r} k'_i p'_i \right\}, \]
where $k'_i = m_{p'_i}(\tilde{C}, \tilde{L})$ and $k'_i = k_i$ for all $i > 1$. Since $K_{\tilde{\Sigma}} \equiv \pi^*K_{\Sigma} + E$,
\[ \tilde{C}_i.(K_{\tilde{\Sigma}} + \tilde{L}) = C_i.(K_{\Sigma} + L) < -1 \]
for any irreducible component $\tilde{C}_i \subset \tilde{C}$. Since $\tilde{L}.\tilde{C} = L.C - m_1 < L.C$,
\[ \dim(R) = \dim(R_1) \leq -\tilde{C}.K_{\tilde{\Sigma}} + g(\tilde{C}) - 1 - \tilde{L}.\tilde{C} = -C.K_{\Sigma} + g(C) - 1 - L.C, \]
by the induction hypothesis. Moreover, if the equality holds and for any singular component $C_i$ of $C$ we have $C_i.(K_{\Sigma} + L) < -3$, which
implies $\tilde{C}_i(K_{\Sigma} + \tilde{L}) < -3$, then $\tilde{C}$ has only nodes as its singularities outside of $\tilde{L}$, for any smooth irreducible curve $C'$ not tangent to $\tilde{L}$, a generic curve $D_1 \in R_1$ intersects $C'$ transversally at any $q' \notin \{p'_1, ..., p'_r\}$, and for any $i$, $D_1$ is a union of smooth branches in a neighborhood of $p'_i$ not tangent to $C'$. Thus no germ is tangent to the exceptional divisor $E$, hence $D = f(D_1)$ satisfies the required properties, since $R_1$ has no fixed points but $p'_1, ..., p'_r$. □

2.2. Severi varieties on Hirzebruch surfaces

Let $\Sigma_n = \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be the Hirzebruch surface and let $\pi : \Sigma_n \to \mathbb{P}^1$ be the natural projection. Consider two sections $(1, 0), (0, \sigma) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. They define the maps

$$(\mathbb{P}^1 \setminus Z(\sigma)) \to \Sigma_n.$$ 

We denote the closures of the images of these maps by $L_0$ and $L_\infty$, respectively. It is clear that $L_\infty$ is independent of the choice of $\sigma$. The following facts will be useful

- The Picard group $\text{Pic}(\Sigma_n)$ is a free abelian group generated by the classes $F$ and $L_\infty$, where $F$ denotes the fiber of the projection $\pi$. It is important to mention that $L_0 \equiv nF + L_\infty$.
- The intersection form on $\text{NS}(\Sigma_n) = \text{Pic}(\Sigma_n)$ is given by $F^2 = 0$, $L_\infty^2 = -n$, and $F.L_\infty = 1$.
- Any effective divisor $M \in \text{Div}(\Sigma_n)$ is linearly equivalent to a linear combination of $F$ and $L_\infty$ with non-negative coefficients. Moreover, if $M$ does not contain $L_\infty$, then it is linearly equivalent to a combination of $F$ and $L_0$ with non-negative coefficients.
- The canonical class is

$$K_{\Sigma_n} \equiv -(2L_\infty + (2 + n)F) \equiv -(L_0 + L_\infty + 2F).$$

Now let us define the Severi varieties on $\Sigma_n$.

**Definition 1.** (1) Let

$$\tilde{\Sigma}_n^\delta = \left\{ (p_1, ..., p_\delta) \in \Sigma_n^\delta \mid p_i \neq p_j \text{ for any } i \neq j \right\}$$

be the configuration space of $\delta$ points in $\Sigma_n$. For non-negative integers $d, k, \delta$, we define the decorated Severi variety

$$U_{d,k,\delta} \subset |\mathcal{O}_{\Sigma_n}(dL_0 + kF)| \times \tilde{\Sigma}_n^\delta$$

to be

$$\{(C; p_1, ..., p_\delta) \mid C \text{ is reduced, } L_\infty \not\subset C, \text{ and } p_1, ..., p_\delta \in C \text{ are nodes} \}.$$
(2) Let \( g, d, k \) be non-negative integers. We define the Severi variety \( V_{g,d,k} \subseteq |O_{\Sigma_n}(dL_0 + kF)| \) to be the closure of the locus of reduced nodal curves of genus \( g \) which do not contain \( L_\infty \), and we define \( V^\text{irr}_{g,d,k} \subseteq V_{g,d,k} \) to be the union of the irreducible components whose generic points correspond to irreducible curves.

Next, we establish the basic properties of (decorated) Severi varieties:

**Proposition 3.** (1) Let \( d, k, \delta \) be non-negative integers. Then either \( U_{d,k,\delta} \) is empty or \( U_{d,k,\delta} \) is a smooth equidimensional variety of dimension

\[
\frac{(d + 1)(nd + 2k + 2)}{2} - 1 - \delta = \dim |O_{\Sigma_n}(dL_0 + kF)| - \delta.
\]

(2) Let \( \psi: U_{d,k,\delta} \rightarrow |O_{\Sigma_n}(dL_0 + kF)| \) be the projection to the first factor. Then for \( g = \frac{(d - 1)(nd + 2k - 2)}{2} - \delta \)

\[
V_{g,d,k} = \begin{cases} 
\psi(U_{d,k,\delta}) & \text{if } \delta \geq 0; \\
\emptyset & \text{otherwise.}
\end{cases}
\]

**Proof.** (1) Let us choose arbitrary \((C_0; p_{10}, ..., p_{80}) \in U_{d,k,\delta} \). We can find an open subset \( U \subset \Sigma_n \) isomorphic to \( \mathbb{A}^2 \) and containing all the points \( p_{10}, ..., p_{80} \). Fix a trivialization of \( O_{\Sigma_n}(dL_0 + kF)(U) \simeq \mathbb{K}[x,y] \). Then in a neighborhood of \((C_0; p_{10}, ..., p_{80}) \) \( U_{d,k,\delta} \) is given by the (homogeneous in the coefficients) equations \( f(p_l) = f'_i(p_l) = f'_j(p_l) = 0, 1 \leq l \leq \delta \), where \( f \in H^0(\Sigma_n, O_{\Sigma_n}(dL_0 + kF)) \). We denote \((x_{10}, y_{10}) = p_{10} \) and define \( a_{ij} \) to be the coefficients of \( f(x_{10} + y_{10}), f = \sum a_{ij} x^i y^j \in \mathbb{K}[x,y] \). Let \( f_0(x,y) = \sum \beta_{ij} x^i y^j \) be an equation defining \( C_0 \). Then

\[
d(f(p_l))|_{(f_0,p_{10},...,p_{80})} = da_{100},
\]

\[
d(f'_i(p_l))|_{(f_0,p_{10},...,p_{80})} = da_{110} + 2\beta_{20} dx_i + \beta_{111} dy_i,
\]

and

\[
d(f'_j(p_l))|_{(f_0,p_{10},...,p_{80})} = da_{101} + \beta_{211} dx_i + 2\beta_{102} dy_i.
\]

The points \( p_{10}, ..., p_{80} \) are nodes of \( C_0 \), hence the matrices

\[
\begin{pmatrix}
2\beta_{20} & \beta_{211} \\
\beta_{111} & 2\beta_{102}
\end{pmatrix}
\]

are invertible. So, it remains to prove that \( da_{100} = ... = da_{400} = 0 \) defines a subspace of codimension \( \delta \) in the tangent space to \( |O_{\Sigma_n}(dL_0 + kF)| \) at \( C_0 \). In other words we have to prove that

\[
h^0(C_0, \mathcal{I}(dL_0 + kF)) = h^0(C_0, O_{C_0}(dL_0 + kF)) - \delta,
\]

(2.5)
where $\mathcal{J}$ denotes the ideal of the zero-dimensional reduced subscheme

$$X = \bigcup_{i=1}^\delta p_i \subset C_0.$$  

Thus the following Claim implies (2.5).

**Claim 4.** \( H^1(C_0, \mathcal{J}(C_0)) = 0. \)

We postpone the proof of the claim till the end of the proof of the proposition.

(2) The inclusion

$$V_{g,d,k} \subseteq \begin{cases} \psi(U_{d,k,\delta}) & \text{if } \delta \geq 0; \\ \emptyset & \text{otherwise.} \end{cases}$$

is obvious. Let us prove the opposite direction. If \( g \geq \frac{(d-1)(nd+2k-2)}{2} \), then there is nothing to prove. Otherwise let \( U \subseteq U_{d,k,\delta} \) be an irreducible component, and let \((C; p_1, \ldots, p_\delta) \in U\) be a generic point. Then by part (1) and Theorem 1

$$-C.K_{\Sigma_n} + g(C) - 1 \geq \dim \psi(U_{d,k,\delta}) = \dim |O_{\Sigma_n}(dL_0 + kF)| - \delta. \quad (2.6)$$

Hence \( g(C) \geq \frac{(d-1)(nd+2k-2)}{2} - \delta = g_{ar}(C) - \delta \), which is possible only if the equality holds. Thus the equality holds in (2.6) as well, and by Theorem 1 this implies the nodality of the curve \( C \). So

$$\psi(C; p_1, \ldots, p_\delta) \in V_{g,d,k}$$

and we are done. \( \square \)

**Proof (of Claim 4).** Let us denote the irreducible components of \( X = C_0 \) by \( X_i \), and the normalizations of \( X_i \) by \( \tilde{X}_i \). Then the normalization \( \tilde{X} \) of \( X \) is the disjoint union of \( \tilde{X}_i \). Consider the conductor ideal \( \mathcal{J}^{\text{cond}} = \text{Ann}(\oplus \mathcal{O}_{\tilde{X}_i}/\mathcal{O}_{\tilde{X}}) \subseteq \oplus \mathcal{O}_{\tilde{X}_i} = \mathcal{O}_{\tilde{X}} \) and the direct sum of the conductor ideals \( \oplus \mathcal{J}_i^{\text{cond}} = \oplus \text{Ann}(\mathcal{O}_{\tilde{X}_i}/\mathcal{O}_{\mathcal{X}}) \subseteq \mathcal{O}_{\tilde{X}}. \)

Let us notice that \( \mathcal{J}^{\text{cond}} \subseteq \mathcal{O}_{\mathcal{X}} \subseteq \mathcal{O}_{\tilde{X}} \) is an ideal in both algebras \( \mathcal{O}_{\mathcal{X}} \) and \( \mathcal{O}_{\tilde{X}} \), hence it is sufficient to prove that \( H^1(X, \mathcal{J}^{\text{cond}}(X)) = H^1(\tilde{X}, \mathcal{J}^{\text{cond}}(X)) = 0 \), since

$$0 \longrightarrow \mathcal{J}^{\text{cond}}(X) \longrightarrow \mathcal{J}(X)|_X$$

is exact, and the factor is a torsion sheaf.

Consider the exact sequence

$$0 \longrightarrow \mathcal{J}_i^{\text{cond}} \longrightarrow \oplus \mathcal{J}_i^{\text{cond}} \longrightarrow \mathcal{F}_i \longrightarrow 0,$$

where \( \mathcal{F}_i \) are torsion sheaves supported at the preimages of the points of intersection with other irreducible components of \( X \). Let us estimate the degree of \( \mathcal{F}_i \). To do this we can assume that the surface
and the curves are affine. Let \( f_i = 0 \) be an equation of \( X_i \). Then 
\[
\prod_{j \neq i} f_j \cdot \mathcal{J}^\text{cond} \subseteq \mathcal{J}^\text{cond}.
\]
Thus
\[
\deg(\mathcal{J}_i) \leq (X - X_i)X_i = X.X_i - X_i^2.
\]

It is well known that \( \deg(\mathcal{J}_i^\text{cond}) \geq -2\delta(X_i) \) where \( \delta(X_i) \) denotes the total delta invariant of \( X_i \), moreover the equality holds for singular curves on smooth surfaces. Thus
\[
\deg(\mathcal{J}_i^\text{cond}(X)|_{X_i}) = \deg(\mathcal{J}_i^\text{cond}(X)) - \deg(\mathcal{J}_i) \geq \\
X.X_i - 2\delta(X_i) - (X - X_i).X_i = \\
(K_{\Sigma_n} + X_i).X_i - 2\delta(X_i) - K_{\Sigma_n}.X_i = \\
2g_{ar}(X_i) - 2 - 2\delta(X_i) - K_{\Sigma_n}.X_i > 2g(X_i) - 2,
\]
by the adjunction formula, since \( -K_{\Sigma_n}.X_i > 0 \). Applying Riemann-Roch theorem we conclude that \( H^1(X_i, \mathcal{J}^\text{cond}(X)|_{X_i}) = 0 \) for all \( i \), which implies \( H^1(\tilde{X}, \mathcal{J}^\text{cond}(X)) = 0 \).

Corollary 1. If \( V_{g,d,k} \neq \emptyset \) then it has pure dimension \( nd + 2k + 2d + g - 1 \), and for any \( C \in V_{g,d,k} \) having exactly \( \delta = \frac{(d-1)(nd+2k-2)}{2} \) \( g \) nodes, \( V_{g,d,k} \) is smooth at \( C \), and \( T_C(V_{g,d,k}) \simeq H^0(C, \mathcal{J}^\text{cond}(C)) \), where \( \mathcal{J}^\text{cond} \subseteq \mathcal{O}_C \) is the conductor ideal.

Proposition 4. Let \( p_1, ..., p_r \in L_0 \cup L_\infty \) be fixed points, let \( d > 0 \), \( k, k_1, ..., k_r \geq 0 \) be integers, and let
\[
R \subset \left\{ D \in |\mathcal{O}_{\Sigma_n}(dL_0 + kF)| : (L_0 \cup L_\infty) \cap D = \sum_{i=1}^r k_ip_i \right\}
\]
be a non empty subvariety whose generic point \( C \) corresponds to a reduced irreducible curve of genus \( g \). Then
\[
\dim X \leq -(K_{\Sigma_n} + L_0 + L_\infty).(dL_0 + kF) + g - 1.
\]
Moreover, if the equality holds, then \( C \) has only nodes as its singularities outside of \( L_0 \cup L_\infty \), for any smooth irreducible curve \( C' \) not tangent to \( L_0 \cup L_\infty \), a generic curve \( D \in R \) intersects \( C' \) transversally outside of \( \{p_1, ..., p_r\} \), and for any \( i \), \( D \) is a union of smooth branches in a neighborhood of \( p_i \), not tangent to \( C' \).

Proof. \( C.(K_{\Sigma_n} + L_0 + L_\infty) = -2d < -1 \), and if \( C \) is singular then \( d > 1 \), hence \( C.(K_{\Sigma_n} + L_0 + L_\infty) < -3 \). So we can apply Lemma 2 to prove the Proposition. \( \square \)
3. The Result

Theorem 2. Let $g, k, d$ be non-negative integers. If the variety $V_{g,d,k}^{\text{irr}}$ is not empty then it is irreducible.

Proof. Let $L_i$, $1 \leq i \leq d$, be generic curves in the linear system $|O_{\Sigma}(L_0)|$, and let $F_i$, $1 \leq i \leq k$, be generic curves in the linear system $|O_{\Sigma}(F)|$. Define
\[ \Gamma = L_1 \cup \ldots \cup L_d \cup F_1 \cup \ldots \cup F_k \in |O_{\Sigma}(dL_0 + kF)|. \]

Proposition 5. Consider an arbitrary component $V$ of the Severi variety $V_{g,d,k}$. Then $\Gamma \in V$.

Proposition 6. There exists a unique component $V \subset V_{g,d,k}^{\text{irr}}$ containing $\Gamma$.

The theorem now follows. \(\square\)

Remark 1. One can generalize the propositions above to prove the following more general statement: Let $g, k, d, m_1, \ldots, m_r$ be non-negative integers satisfying $\sum m_i = k$. Consider the varieties
\[ V_{g,d,k}(m_1, \ldots, m_r) \subset V_{g,d,k}, \]
parameterizing curves having $r$ points of tangency of orders $m_1, \ldots, m_r$ with $L_\infty$, and
\[ V_{g,d,k}^{\text{irr}}(m_1, \ldots, m_r) = V_{g,d,k}(m_1, \ldots, m_r) \cap V_{g,d,k}^{\text{irr}}. \]
If $V_{g,d,k}^{\text{irr}}(m_1, \ldots, m_r) \neq \emptyset$ then it is irreducible. The generalization is pretty much straightforward, but it makes the presentation more complicated, so we will not write it down in this paper, but rather leave to the interested reader as an exercise.

3.1. Proof of Proposition

$V$ is birational to a product of components of Severi varieties whose generic points correspond to irreducible curves modulo a finite group of symmetries, due to Claim 3 and Theorem 1. Thus, without loss of generality, we can assume that the generic point of $V$ corresponds to an irreducible curve.

Now, to prove the proposition, it is sufficient to show that $V$ contains a nodal curve $E = L \cup E'$ where $L$ is a smooth curve of type $L_0$ and $E' \in V_{g',d',k'}$. If $d = 1$ then $g = 0$, hence $V = V_{0,1,k} = |O_{\Sigma}(L_0 + kF)|$, and there is nothing to prove. So we can assume that $d > 1$. 
Let \( p_1^0, \ldots, p_{nd+k+1}^0 \in L_0 \) and \( p_1^\infty, \ldots, p_k^\infty \in L_\infty \) be generic points. A generic \( C \in V \) intersects \( L_0 \cup L_\infty \) transversally due to Lemma 2 and the locus of curves in \( V \) intersecting \( L_0 \cup L_\infty \) transversally along \( C \cap (L_0 \cup L_\infty) \) has codimension \( |C \cap (L_0 \cup L_\infty)| \) by Proposition 2. Then the locus \( W \) of irreducible curves passing through \( \{p_i^0\}_{i=1}^{nd+k} \cup \{p_j^\infty\}_{j=1}^k \) has pure dimension

\[-(K_{\Sigma_n} + L_0 + L_\infty). (dL_0 + kF) + g - 1;\]

and the locus of curves \( V_{L_0} \subset \overline{W} \) passing through \( p_{nd+k+1}^0 \), i.e. containing \( L_0 \) as a component, has pure dimension

\[-(K_{\Sigma_n} + L_0 + L_\infty). (dL_0 + kF) + g - 2.\]

Consider a map from an irreducible smooth germ curve

\[ j : (T, 0) \rightarrow (\overline{W}, V_{L_0}), \]

such that \( j(0) \in V_{L_0} \) is a generic point and \( j(T^*) = j(T \setminus \{0\}) \subset W \). Then for any \( t \in T^* \), \( C_t \) is a nodal curve of genus \( g \) containing \( p_1^0, \ldots, p_{nd+k+1}^0, p_1^\infty, \ldots, p_k^\infty \), where \( C_t \) denotes the fiber over \( t \) of the corresponding flat family \( \tilde{C} \rightarrow T \). The central fiber \( C_0 \) can be presented as \( C_0 = s_0 L_0 \cup C'_0 \), where \( C'_0 \) is a curve that does not contain \( L_0 \), and \( s_0 \geq 1 \).

**Lemma 3.** The curve \( C'_0 \) does not contain \( L_\infty \), and \( s_0 = 1 \). Moreover, \( C'_0 \) is a nodal curve, and the points of intersection \( C'_0 \cap L_0 \) are smooth points of \( C'_0 \).

**Proof.** Assume that \( C'_0 = s_0 L_0 \cup C''_0 \cup s_\infty L_\infty \), where \( C''_0 \) contains neither \( L_0 \) nor \( L_\infty \) as its components. Then \( C''_0 = (d - s_0 - s_\infty) L_0 + (k + ns_\infty) F \). After proceeding with an appropriate base change and replacing the family \( \mathcal{C} \) by its normalization, we can consider a semistable model \( \tilde{C} \rightarrow C \rightarrow T \) of the family \( \mathcal{C} \), whose total space is smooth, generic fiber is also smooth and has genus \( g \), and its central fiber is a nodal curve. Let \( \tilde{C}_0 \) be the central fiber of the semistable family, and let \( f : \tilde{C}_0 \rightarrow C_0 \) be the natural map. Then \( \tilde{C}_0 = A_0 \cup B \cup A_\infty \), where \( A_i \subset f^{-1}(L_i) \) are the unions of the connected components of \( \tilde{C}_0 \) mapped surjectively onto \( L_i \), \( B \) is the union of all other components, and the following equality holds\(^3\)

\[ p_a(A_0) + p_a(B) + p_a(A_\infty) - 2 + |A_0 \cap B| + |A_\infty \cap B| = g. \]

Next step is to estimate the degrees of freedom of \( C''_0 \). Let \( D \subset C''_0 \) be any irreducible component with reduced structure, and let \( B_D \) be

\(^3\) if \( s_\infty = 0 \), i.e. \( A_\infty = \emptyset \), then \( p_a(A_\infty) := 1 \).
any irreducible component of $B$ mapped surjectively on $D$. Then, by Proposition 4, $D$ vary in a family of dimension at most

$$-D.(K_{\Sigma_n} + L_0 + L_\infty) + g(D) - 1 + l_D,$$

where $l_D$ is the number of points of $D \cap (L_0 \cup L_\infty)$ distinct from $\{p_i^j\}$.

Case 1: $D \neq F$. Since $-D.(K_{\Sigma_n} + L_0 + L_\infty) = 2D.F > 0$, $D$ vary in a family of dimension at most

$$-B_D.f^*(K_{\Sigma_n} + L_0 + L_\infty) + g(B_D) - 1 + l_{B_D},$$

where $l_{B_D}$ is the number of points mapped onto $(L_0 \cup L_\infty) \setminus \{p_i^j\}$. Moreover, the equality holds if and only if $B_D \xrightarrow{\sim} D$ is the normalization map, $l_{B_D} = l_{D}$, $D$ is nodal away from $L_0 \cup L_\infty$, and all its branches are smooth at the points of intersection with $L_0 \cup L_\infty$.

Case 2: $D = F$. In this case $D$ also varies in a family of dimension at most

$$-B_D.f^*(K_{\Sigma_n} + L_0 + L_\infty) + g(B_D) - 1 + l_{B_D}.$$

Since $p_i^j \in L_i$ are general then $1 \leq l_D \leq 2$. If $l_D = 1$ then there is a point $q \in B_D$ mapped onto one of $\{p_i^j\}$, such that $q$ is a smooth point of $B$. Thus the pullback of $L_i$ to $B$ is reduced at this point hence $B_D \xrightarrow{\sim} D$ is an isomorphism.

The only points of $B$ that are mapped onto $L_i \setminus \{p_i^j\}$ are $A_i \cap B$, hence, using the analysis above and the fact that every connected component of $B$ must intersect $A_0 \cup A_\infty$, we conclude that $C''_0$ varies in a family of dimension

$$\dim \leq -f^*(K_{\Sigma_n} + L_0 + L_\infty).B + p_a(B) - 1 +$$

$$|A_0 \cap B| + |A_\infty \cap B| =$$

$$-(K_{\Sigma_n} + L_0 + L_\infty).C_t + g - 2 +$$

$$(K_{\Sigma_n} + L_0 + L_\infty).(s_0L_0 + s_\infty L_\infty) - p_a(A_0) - p_a(A_\infty) + 3.$$ 

Since $p_a(A_i) \geq 1 - s_i$, 

$$\dim \leq -(K_{\Sigma_n} + L_0 + L_\infty).C_t + g - 2 - (s_0 + s_\infty) + 1.$$ 

On the other hand $C''_0$ must vary in a family of dimension at least $-C_t.(K_{\Sigma_n} + L_0 + L_\infty) + g - 2$, hence $s_0 = 1$, $s_\infty = 0$, and all the inequalities above are equalities. Furthermore,

1. Over any irreducible component of $C''_0$ there is a unique irreducible component of $B$ mapped surjectively onto this component.

\[\text{footnote}{\text{4 Since } \tilde{C}_t \text{ is irreducible, thus } \tilde{C}_0 \text{ is connected.}}\]
On Severi varieties on Hirzebruch surfaces

- $A_0$ is a tree whose root $A_0^R \simeq \mathbb{P}^1$ mapped isomorphically onto $L_0$. Any connected component of $A_0 \setminus A_0^R$ intersects at most one connected component of $B$ and at exactly one point.
- $C''_0$ is reduced.
- Any two irreducible components of $C''_0$ intersect transversally.
- $f(A_0 \cap B)$ is a set of generic points of $L_0$, in particular it is disjoint from $p_1^0, \ldots, p_{nd+k}$.
- $C''_0 = C_0'$ has geometric genus $g + 1 - |A_0 \cap B|$.
- $C'_0$ intersects $L_0$ transversally outside of $f(A_0 \cap B)$.

Now we can describe $C''_0$ explicitly. $C''_0$ is a reduced nodal curve intersecting $L_\infty$ transversally, and it is smooth at the points of intersection with $L_0$. And, finally, if $p \in A_0 \cap B$ then in a neighborhood of $f(p)$ the delta invariant of $C_t$ is equal to the local delta invariant of $C_0$ minus one for all sufficiently small values of $t$, hence if $m$ denotes the order of tangency of $C'_0$ and $L_0$ at $f(p)$ then $C_t$ has $m - 1$ nodes in a small neighborhood of $f(p)$. \hfill \Box

To complete the proof we must show that $V$ contains nodal equigeneric deformations of $C_0$, since any such deformation must be of the form $E = L \cup E'$, where $L \equiv L_0$ and $E' \equiv C'_0$.

Let us denote the points of intersection of $L_0$ with $C'_0$ other than $p_1^0, \ldots, p_{nd+k}$ by $q_1, \ldots, q_r$, and the orders of the tangency by $m_1, \ldots, m_r$ respectively. Thus $\{q_1, \ldots, q_r\} = f(A_0 \cap B)$. Consider now the product $\mathcal{V} = \prod_{i=1}^r \mathcal{V}_i$ of the versal deformations of the tacnodes of orders $m_1, \ldots, m_r$, and consider the natural map

$$\psi : (V, C_0) \longrightarrow \mathcal{V}.$$  

Then $\psi(V) \subset \prod_{i=1}^r \mathcal{V}_i(m_i - 1)$, where $\mathcal{V}_i(h)$ denotes the closure of the locus of deformations having $h$ nodes.

Claim 5 ([3] Lemma 2.8). $\mathcal{V}_i$ are smooth, irreducible of dimension $2m_i - 1$, and for $m_i - 1 \leq h \leq m_i$, $\mathcal{V}_i(h)$ are irreducible of dimension $2m_i - 1 - h$.

We denote the nodes of $C_0$ different from $q_1, \ldots, q_r$ by $o_1, \ldots, o_\delta$. Consider the component $U$ of the decorated Severi variety containing $(C_0; o_1, \ldots, o_\delta)$. Thus $U$ is smooth by Proposition [3] (1). Now, let

$$\phi : (U, (C_0; o_1, \ldots, o_\delta)) \longrightarrow (\mathcal{V}, 0)$$  

be the natural map.

Claim 6. The map $d\phi : T_{(C_0; o_1, \ldots, o_\delta)}U \longrightarrow T_0 \mathcal{V}$ is surjective.

We postpone the proof of the Claim, and first finish the proof of Proposition [3]. Since $U$ and $V$ are smooth, Claim [3] implies that
the central fiber $\phi^{-1}(0)$ is smooth at $(C_0; o_1, ..., o_6)$, and since the subvariety $\prod_{i=1}^r V_i(m_i - 1) \subset V$ is irreducible

$$U' = \phi^{-1} \left( \prod_{i=1}^r V_i(m_i - 1) \right)$$

is also irreducible. Another conclusion of Claim 4 is the surjectivity of $\phi$. Thus the generic point of $U'$ corresponds to a nodal equigeneric deformations of $(C_0; o_1, ..., o_6)$, since $\prod_{i=1}^r V_i(m_i - 1) \subset \prod_{i=1}^r V_i(m_i)$. Consider the natural projection $U' \rightarrow V_{g,d,k}$. Its image intersects $V$, and, since $U'$ is irreducible, it belongs to $V$. Thus $V$ contains nodal equigeneric deformations of $C_0$, and we are done. □

Proof (of Claim 6). To prove the claim, one must interpret the tangent spaces and the differential map in cohomological terms. Following the proof of Proposition 3, one can see that

$$T_{(C_0; o_1, ..., o_6)} U \simeq H^0(C_0, \mathcal{J}(dL_0 + kF)),$$

where $\mathcal{J}$ denotes the ideal sheaf of the zero dimensional scheme $\bigcup_{i=1}^r o_i \subset C_0$. The tangent space to $V$ is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}^{es}(q_i)$, where $\mathcal{O}^{es} = \mathcal{O}_{C_0}/I^{es}$, and $I^{es}$ denotes the equisingularity ideal of $C_0$. We define $X^{es} = \text{Spec} \mathcal{O}^{es}$. Since $A_{2m-1}$ is a simple singularity for any $m \geq 1$ the equisingularity ideal $I^{es}$ is generated locally at a singular point by the partial derivatives of the defining equation of the curve. In these notations the map $d\phi$ is given by the natural restriction map $H^0(C_0, \mathcal{J}(dL_0 + kF)) \rightarrow \bigoplus_{i=1}^r \mathcal{O}^{es}(q_i)$, associated to the short exact sequence

$$0 \rightarrow I^{es}(dL_0 + kF) \rightarrow \mathcal{J}(dL_0 + kF) \rightarrow \bigoplus_{i=1}^r \mathcal{O}^{es}(q_i) \rightarrow 0.$$

To prove that $d\phi$ is surjective it is sufficient to show that

$$H^1(C_0, I^{es}(dL_0 + kF)) = 0. \quad (3.1)$$

It is also a necessary condition since $H^1(C_0, \mathcal{J}(dL_0 + kF)) = 0$ by Claim 4. Consider the short exact sequence of sheaves

$$0 \rightarrow I^{es}_{X^{es}, L_0} \xrightarrow{L_0} I^{es}(L_0) \rightarrow I^{es}_{X^{es} \cap L_0/L_0}(L_0) \rightarrow 0.$$

Thus to prove (3.1), it is sufficient to show that

$$H^1(C_0, I^{es}_{X^{es}, L_0}((d-1)L_0 + kF)) = 0 \quad (3.2)$$

and

$$H^1(L_0, I^{es}_{X^{es} \cap L_0/L_0}(dL_0 + kF)) = 0. \quad (3.3)$$

Let $p$ be any point in the support of $X^{es}$. If $p \notin L_0$ then

$$I^{es}_{X^{es}, L_0, p} = I^{es}_p = \mathcal{J}^{cond}_p$$

and $I^{es}_{X^{es} \cap L_0/L_0, p} = 0$. 
where $J^{cond}$ denotes the conductor ideal of $C_0$. If $p \in L_0$ then consider a local system of coordinates $x, y$ at $p \in \Sigma_n$ such that $L_0$ is given by $y = 0$, and $C_0'$ is given by $y = x^m$. In these notations

$$I_{X^{es}:L_0,L_0}^{\delta} = yO_{C_0,p} + x^{m-1}O_{C_0,p} \supset yO_{C_0,p} + x^mO_{C_0,p} = J_{p}^{cond}$$

and

$$I_{X^{es}\cap L_0/L_0}^{\delta} = x^mO_{L_0,p}.$$ 

Thus $I_{X^{es}\cap L_0/L_0}^{\delta}(dL_0 + kF)$ is a line bundle of degree $C_0,L_0 - C_0',L_0 = n > 0$, hence $H^1(L_0, I_{X^{es}\cap L_0/L_0}^{\delta}(dL_0 + kF)) = 0$ by the Riemann-Roch theorem, which proves (3.3).

It follows from the description above that $J^{cond} \subset I_{X^{es}:L_0}$. Thus $H^1(C_0, J^{cond}((d - 1)L_0 + kF)) = 0$ implies (3.2). Consider the normalization $\nu : C_0 \longrightarrow C_0$. It is sufficient to show that

$$H^1(C, J^{cond}((d - 1)L_0 + kF)|_C) = 0, \quad (3.4)$$

for any irreducible component $C \subset \tilde{C}_0$. Since $\nu(C) \neq L_\infty$ by Lemma 4 thus $(2F + L_\infty)\nu(C) > 0$ and $J^{cond}((d - 1)L_0 + kF)$ restricted to $C$ is a line bundle of degree

$$\nu(C).C_0' - \deg(J^{cond}|_C) = \nu(C)^2 - \nu(C).L_0 - 2\delta(\nu(C)) = 2g(C) - 2 - (K_{\Sigma_n} + L_0)\nu(C) = 2g(C) - 2 + (2F + L_\infty)\nu(C) > 2g(C) - 2,$$

where $\delta(\nu(C))$ denotes the number of nodes of $\nu(C)$ (which is equal to the total delta invariant of $\nu(C)$), thus (3.4) follows from the Riemann-Roch theorem. □

3.2. Proof of Proposition 4

We start with some combinatorics.

**Definition 2.** (1) An ordered subset $\mu \subseteq \Gamma^{sing}$ consisting of $r$ nodes is called an $r$-marking on the curve $\Gamma$.

(2) An $r$-marking $\mu$ is called irreducible if and only if the curve $\Gamma \setminus \mu$ is connected.

(3) We define $D$-moves on the set of $r$-markings as follows: let $D, D' \subseteq \Gamma$ be two different irreducible components, let $q, q' \in D \cap D'$ be two nodes, and let $\mu = \{p_1, ..., p_r\}$ be an $r$-marking. Then

$$D_{q,q'}(\mu) = \begin{cases} \{p_1, ..., p_i+1, q, p_{i+1}, ..., p_r\} & \text{if } q \notin \mu \text{ and } q' = p_i, \\ \{p_1, ..., p_i, q', p_{i+1}, ..., p_r\} & \text{if } q = p_i \text{ and } q' \notin \mu, \\ \{p_{\tau_j(1)}, ..., p_{\tau_j(r)}\} & \text{if } q = p_i \text{ and } q' = p_j, \\ \mu & \text{otherwise}, \end{cases}$$
where \( \tau_{ij} \in \mathcal{S}_r \) denotes the elementary transposition \( \tau_{ij} = (i,j) \).

(4) Assume that \( n > 0 \). We define \( T \)-moves on the set of \( r \)-markings as follows: let \( D, D', D'' \subset \Gamma \) be three different irreducible components, and let \( q \in D' \cap D'', q' \in D \cap D'' \), \( q'' \in D \cap D' \) be three nodes, and let \( \mu = \{p_1, \ldots, p_r\} \) be an \( r \)-marking. Then if \( q' \notin \mu \) we define

\[
T_{q,q',q''}(\mu) = \begin{cases}
\{p_1, \ldots, p_{i-1}, q'', p_{i+1}, \ldots, p_r\} & \text{if } q = p_i; q'' \notin \mu, \\
\{p_1, \ldots, p_{i-1}, q, p_{i+1}, \ldots, p_r\} & \text{if } q'' = p_i; q \notin \mu, \\
\{p_{\tau_{ij}(1)}, \ldots, p_{\tau_{ij}(r)}\} & \text{if } q = p_i, q'' = p_j, \\
\mu & \text{if } q, q'' \notin \mu,
\end{cases}
\]

otherwise we define \( T_{q,q',q''}(\mu) = \mu \).

(5) Assume that \( n = 0 \). We define \( Q^h \)-moves as \( Q^h \)-moves as follows: let \( \mu = \{p_1, \ldots, p_r\} \) be an \( r \)-marking, let \( X, X', Y, Y' \subset \Gamma \) be four different irreducible components satisfying \( X \equiv X' \equiv L \), \( Y \equiv Y' \equiv F \), and let \( q \in X \cap Y \), \( q' \in X \cap Y' \), \( q'' \in X' \cap Y' \) be nodes. If \( q', q'' \notin \mu \) we define

\[
Q^h_{q,q',q''}(\mu) = \begin{cases}
\{p_1, \ldots, p_{i-1}, q', p_{i+1}, \ldots, p_r\} & \text{if } q = p_i \text{ and } q' \notin \mu, \\
\{p_1, \ldots, p_{i-1}, q, p_{i+1}, \ldots, p_r\} & \text{if } q' = p_i \text{ and } q \notin \mu, \\
\{p_{\tau_{ij}(1)}, \ldots, p_{\tau_{ij}(r)}\} & \text{if } q = p_i, q' = p_j,
\end{cases}
\]

otherwise, we define \( Q^h_{q,q',q''}(\mu) = \mu \). If \( q', q'' \notin \mu \), we define

\[
Q^v_{q,q',q''}(\mu) = \begin{cases}
\{p_1, \ldots, p_{i-1}, q'', p_{i+1}, \ldots, p_r\} & \text{if } q = p_i \text{ and } q'' \notin \mu, \\
\{p_1, \ldots, p_{i-1}, q, p_{i+1}, \ldots, p_r\} & \text{if } q'' = p_i \text{ and } q \notin \mu, \\
\{p_{\tau_{ij}(1)}, \ldots, p_{\tau_{ij}(r)}\} & \text{if } q = p_i, q'' = p_j,
\end{cases}
\]

otherwise, we define \( Q^v_{q,q',q''}(\mu) = \mu \).

(6) Two \( r \)-markings, \( \mu \) and \( \mu' \), are called equivalent if and only if one can be obtained from another by a sequence of \( T \)-moves, \( Q^h \)-moves, and \( D \)-moves.

**Notation 2.** Let \( \mu = \{p_1, \ldots, p_r\} \) be any \( r \)-marking and let \( C, C' \subset \Gamma \) be two different components. The following notation will be useful:

\[
\mu_{C,C'} = |\mu \cap C \cap C'|, \quad \mu_C = |\mu \cap C|, \quad \text{and } \mu_i = p_i.
\]

**Claim 7.** Let \( r > 0 \) be such an integer, that the set of irreducible \( r \)-markings on the curve \( \Gamma \) is not empty. Then for any pair of distinct irreducible components \( C, C' \subset \Gamma \) and for any \( q \in C \cap C' \), there exist irreducible \( r \)-markings \( \mu \) and \( \mu' \) such that \( q \notin \mu \) and \( q \notin \mu' \).

**Proof.** Obvious. \( \Box \)
From now on we will assume that $n > 0$. The remaining case, $n = 0$, is much easier, and the proof in this case can be obtained via the same lines as in the case we consider. Thus, we leave it to the reader.

**Lemma 4.** If $n > 0$ then any two irreducible $r$–markings on the curve $\Gamma$ are equivalent.

**Proof.** It is enough to prove the lemma for the case $r = k(d - 1) + n\frac{d(d-1)}{2} - (d - 1)$, since any irreducible $r$–marking $\mu$ can be extended to an irreducible marking $\mu^{ex}$ of order $k(d - 1) + n\frac{d(d-1)}{2} - (d - 1)$, and for any $D$ or $T$ move $M$ on the extended marking the natural forgetful map takes $M(\mu^{ex})$ to $M(\mu)$.

Let us prove the lemma by induction on $d + k$. If either $d = 2 = k + 2$ or $d \leq 1$, then the lemma is obvious. Assume that the statement is true for all $d + k \leq m$ and let us prove it for $d + k = m + 1$. We can assume that $m > 2$ and $d \geq 2$. Let $\mu$ be an irreducible $r$–marking.

**Step 1:** The goal of this step is to prove that there exists a marking $\mu' \sim \mu$ such that

$\mu'_{C,C'} = \begin{cases} C.C' - 1 & \text{if } L_d \in \{C, C'\}, \\ C.C' & \text{otherwise.} \end{cases}$ (3.5)

Choose a component $D \subset \Gamma$ in the following way: if $k \neq 0$ then $D = F_k$, otherwise $D = L_1$. Then any irreducible component $D' \subset \Gamma$ different from $D$ and satisfying $D.D' > 0$ belongs to the linear system $|O_{\Sigma_n}(L_0)|$. Moreover, there are at least two such components. Now, let us choose $\tilde{\mu} \sim \mu$, such that

$\tilde{\mu}_D = \max_{\mu' \sim \mu}\{\mu'_D\}$.

Then $\tilde{\mu}_{|\Gamma\setminus D}$ is an irreducible marking on $\Gamma\setminus D$, since otherwise, due to the choice of $D$, we would be able to find two distinct irreducible components $D', D'' \subset \Gamma$ different from $D$ such that $\tilde{\mu}_{D,D'} < D.D'$, $\tilde{\mu}_{D,D''} < D,D''$, and $\tilde{\mu}_{D,D'} = D',D'' > 0$. Hence, there would exist $q'' \in D \cap D'$, $q' \in D \cap D''$, and $q \in D' \cap D''$, such that $q', q'' \notin \mu$ and $q = \tilde{\mu}_i$ for some $i$. Thus $\tilde{\mu}_D < \mu'_D$, where $\mu' = T_{q', q''}(\tilde{\mu}) \sim \mu$, which would be a contradiction.

Next, we shall prove that

$\tilde{\mu}_D = D.(\Gamma - D) - 1$. (3.6)

Consider two cases: $k > 0$ and $k = 0$.

**Case 1:** $k > 0$. In this case $D = F_k$, and

$\tilde{\mu}_D \geq r - \left( (k - 1)(d - 1) + n\frac{d(d-1)}{2} - (d - 1) \right)$,
since $\tilde{\mu}_{\Gamma\setminus D}$ is an irreducible marking on $\Gamma\setminus D$. However,
\[
r - \left( (k-1)(d-1) + n\frac{d(d-1)}{2} - (d-1) \right) = d - 1 = D.(\Gamma - D) - 1,
\]
and we are done.

Case 2: $k = 0$. In this case $D = L_1$, and
\[
\tilde{\mu}_D \geq r - \left( n\frac{(d-1)(d-2)}{2} - (d-2) \right),
\]
since $\tilde{\mu}_{\Gamma\setminus D}$ is irreducible. However,
\[
r - \left( n\frac{(d-1)(d-2)}{2} - (d-2) \right) = n(d-1) - 1 = D.(\Gamma - D) - 1.
\]
Thus, $\tilde{\mu}_D = D.(\Gamma - D) - 1$ in both cases.

Now we can complete the proof of the first step. By (3.5) there
exists a unique component $D_1 \subset \Gamma$ satisfying $\tilde{\mu}_{D,D_1} = D.D_1 - 1$, and
$\tilde{\mu}_{D_1,D'} = D.D'$ for all $D' \subset \Gamma$ different from $D_1$. By the induction
hypothesis it remains to prove that there exists $\mu' \sim \tilde{\mu}$ such that
$\mu_{D_1,L_d} = D.L_d - 1$ and $\mu'_D = \tilde{\mu}_D$. If $D_1 = L_d$ then there is nothing
to prove. Otherwise, by the induction assumption and Claim 7 we can
find an irreducible marking $\tilde{\mu} \sim \tilde{\mu}$ with the following properties:
$\tilde{\mu}_{D,D'} = \tilde{\mu}_{D,D'}$ for all $D' \subset \Gamma$, and $\mu_{D_1,L_d} < D_1.L_d$. Then there exist
$q'' \in D \cap D_1, q \in D \cap L_d$, and $q' \in D_1 \cap L_d$ such that $q', q'' \notin \tilde{\mu}$, and
$q = \tilde{\mu}_i$ for some $i$. Thus $\mu' = T_{q,q',q''}(\tilde{\mu})$ satisfies the required
condition.

Step 2: The goal of this step is to prove that any two markings
satisfying (3.5) are equivalent. Let $\mu, \mu'$ be two such markings. Applying
several $D$-moves we can find a marking $\mu'' \sim \mu'$ such that $\mu''$
differs from $\mu$ only by the order of marked points. Namely, there
exists $\tau \in \mathcal{G}_r$ such that $\mu''_i = \mu_{\tau(i)}$ for all $i$. We use the notation $\tau(\mu)$
for such $\mu''$. It remains to prove that $\mu \sim \tau(\mu)$. Without loss of generality
we can assume that $\tau = \tau_{ij}$ is a simple transposition; moreover we
can assume that $i = 1$ and $j = 2$. Let $D_1, D_2, D_3, D_4$ be components
such that $\mu_1 \in D_1 \cap D_2$ and $\mu_2 \in D_3 \cap D_4$. If $\{D_1, D_2\} = \{D_3, D_4\}$,
then we apply $D_{\mu_1,\mu_2}$ to finish the proof. So we can assume that
$\mu_1 \neq D_3, D_4$ and $D_2 \neq D_3$.
If $\mu_1 \in L_d \cap D'$ for some $D'$, then $D' \equiv L_0$ due to (3.5). Thus
there exists a component $D'' \neq D', L_d$ such that $D''D' > 0$. Let $\mu_i \in
D'' \cap D'$ be any node, and let $q \in L_d \cap D''$ be the node not belonging
to $\mu$. Then $\mu \sim T_{\mu_1,q,\mu_1}(\mu)$, and hence we can reduce the statement
to the case when $\mu_1 \notin L_d$. Applying the same argument to $\mu_2$ we get
one of the following: either $\mu_i = \mu_1$, and then $\mu \sim T_{\mu_2,q,\mu_1}(\mu) = \mu''$,
or $\mu_1, \mu_2 \notin L_d$, and thus that $D_i \neq L_d$ for all $i$. Now we shall consider
several cases:
Case 1: $D_2 = D_4 = L_0$. Let $q_1 \in L_d \cap D_1$, $q_2 \in L_d \cap D_2$, and $q_3 \in L_d \cap D_3$ be the nodes not belonging to $\mu$. Then

$$\tau_{12}(\mu) = T_{\mu_2,q_3,q_2}(T_{\mu_1,q_1,q_2}(T_{\mu_2,q_3,q_2}(\mu))) \sim \mu.$$ 

Case 2: $D_2 = D_4 = F$. In this case $D_1 \equiv D_3 \equiv L_0$. Let $\mu_i \in D_1 \cap D_3$ be any node. Then $\tau_{12} = \tau_{1i} \circ \tau_{2} \circ \tau_{1i}$, and thus we reduce to the previous case. So $\tau_{12}(\mu) \sim \mu$.

Case 3: $\{D_1, D_2\} \cap \{D_3, D_4\} = \emptyset$. Without loss of generality $D_1 \equiv D_3 \equiv L_0$. Let $\mu_i \in D_1 \cap D_3$ be any node. Then $\tau_{12} = \tau_{1i} \circ \tau_{2} \circ \tau_{1i}$, so $\tau_{12}(\mu) \sim \mu$, by the first case, and we are done, since cases 1, 2, 3 cover all the possibilities. □

Lemma 5. Let $X = X_1 \cup X_2 \cup \ldots \cup X_r$ be a nodal curve. Assume that $X_2, \ldots, X_r$ are generic curves of types $L_0$ and $F$, and assume that $X_1$ is a generic nodal rational curve whose type is one of $L_0$, $L_0 + F$, or $2L_0$. Consider the variety $W(X, L) \subset (X^{\text{smooth}})^s \times |L|$ given by

$$\{(p_1, \ldots, p_s; \xi) : p_i \neq p_j \text{ for all } i \neq j, \text{ and } \xi(p_1) = \ldots = \xi(p_s) = 0\},$$

where $s = \deg(\mathcal{L} \otimes \mathcal{O}_X)$ and either $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(L_0)$ or $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(F)$. Let $(p_1, \ldots, p_s; \xi) \in W(X, L)$ be an arbitrary point, and let $W \subseteq W(X, L)$ be the irreducible component containing $(p_1, \ldots, p_s; \xi)$. Assume that $n > 0$ and there exist $a \neq b$ satisfying $p_a, p_b \in X_1$. Then

$$(p_{\tau_{ab}(1)}, \ldots, p_{\tau_{ab}(s)}; \xi) \in W,$$

where $\tau_{ab} \in \mathfrak{S}_s$ denotes the elementary transposition $\tau_{ab} = (a \ b)$.

Proof. If $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(F)$, we can assume that $X_1 \equiv 2L_0$. Consider the variety $W(X_1, \mathcal{L})$, which is irreducible since $X_1$ is irreducible and the projection to the first factor $W(X_1, \mathcal{L}) \longrightarrow X_1$ is dominant and has irreducible fibers. The natural forgetful map $f : W \longrightarrow W(X_1, \mathcal{L})$ is dominant, and since $F.X_i \leq 1$ for all $i > 1$ it is also one-to-one. This implies the statement.

Assume now that $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(L_0)$. It is easy to see that $W(X, \mathcal{L})$ is smooth. Let $(x_1, \ldots, x_s; \eta) \in W(X, \mathcal{L})$ be any point with the following properties: $x_i \in X^{\text{smooth}}$ for all $i$, $x_a = x_b$, $\eta(x_a) = d(\eta|_X)(x_a) = 0$, and $x_i \neq x_j$ for all $\{i, j\} \neq \{a, b\}$. Then $(x_1, \ldots, x_s; \eta)$ is a smooth point of $W(X, \mathcal{L})$. Thus to prove the lemma it suffices to show that $W$ contains a point with such properties. Consider the forgetful map $f : \overline{W} \longrightarrow (X^{\text{smooth}})^2$ given by $f(x_1, \ldots, x_s; \eta) = (x_a, x_b)$. This map is surjective, since $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(L_0)$ and $n > 0$, and it is clear that a generic $\alpha \in f^{-1}(\Delta_{X_1})$ satisfies the properties mentioned above. □
Proof (of Proposition 6). We start with the following remark: let $U \subset U_{d,k,\delta}$ be an irreducible component containing $(\Gamma; p_1, ..., p_\delta)$. We define a $\delta$–marking $\mu_{\Gamma; p_1, ..., p_\delta}$ on the curve $\Gamma$ in the following way:

$$\mu_{\Gamma; p_1, ..., p_\delta} = \{p_1, ..., p_\delta\}.$$ 

Then the generic curve $C \in \psi(U)$ is irreducible if and only if the marking $\mu_{\Gamma; p_1, ..., p_\delta}$ is irreducible. The collection of all $\delta$–markings corresponding to $U$ is denoted by

$$\mathcal{M}(U) = \{\mu_{\Gamma; p_1, ..., p_\delta} \mid (\Gamma; p_1, ..., p_\delta) \in U\}.$$ 

Thus Proposition would be proven once we show that $\mathcal{M}(U)$ is closed under $T$-moves and under $D$-moves. This is what we proceed to proving now under the assumption $n > 0$.

Step 1: First, we shall prove that $\mathcal{M}(U)$ is closed under $T$-moves. We choose an arbitrary marking $\mu = \mu_{\Gamma; p_1, ..., p_\delta} \in \mathcal{M}(U)$ and label the rest of the nodes of $\Gamma$ by $p_{\delta+1}, ..., p_{\delta'}$, where $\delta' = dk + \frac{d(d-1)}{2}$. Let $D, D', D'' \subset \Gamma$ be three different irreducible components, and let $q \in D' \cap D''$, $q' \in D \cap D'$, $q'' \in D \cap D'$ be three nodes. If $q' \notin \mu$ then $T_{q,q'}(\mu) = \mu \in \mathcal{M}(U)$ and we are done. So we can assume that $q' \notin \mu$. If $q,q'' \notin \mu$ then again $T_{q,q',q''}(\mu) = \mu \in \mathcal{M}(U)$ and we are done. So without loss of generality we can assume that $q \in \mu$. We shall show that $T_{q,q',q''}(\mu) \in \mathcal{M}(U)$.

Consider the irreducible component $U' \subset U_{d,k,\delta'-1}$ containing the pointed curve $(\Gamma; p_1, ..., p_{\delta'-1})$, and let $f : U' \longrightarrow U$ be the natural forgetful map. It is sufficient to prove that

$$(\Gamma; p_i, p_{i+1}, ..., p_{\delta'-1}) \in U',$$

since $T_{q,q',q''}(\mu) = \mu_{f((\Gamma; p_i, p_{i+1}, ..., p_{\delta'-1}))}$.

Let $(C; x_1, ..., x_{\delta'-1}) \in U'$ be a generic element. Then $C$ has a unique component $C_2$ of type $D + D'$ among its $d+k-1$ irreducible components. Moreover, there exists another irreducible component $C_1$ such that $x_1, x_1 \in C_1 \cap C_2$. We denote $C_{\text{fix}} = \bigcup_{i=2}^{d+k-1} C_i$, where $C_1, ..., C_{d+k-1}$ are the irreducible components of $C$.

Consider the locus $U'' \subset U'$ of pointed curves $(C'; x_1', ..., x_{\delta'-1})$ with the following property: $C' = C'_1 \cup C_{\text{fix}}$, where $C'_1 \cong C_1$ is generic. Let $U'' \subset U'$ be the irreducible component containing the pointed curve $(C; x_1, ..., x_{\delta'-1})$, and let

$$\phi : U'' \longrightarrow W(C_{\text{fix}}; \mathcal{O}_{C_{\text{fix}}}(C_1))$$

be the natural forgetful map (cf. Lemma 5). Then the image of $\phi$ is dominant in an irreducible component of $W(C_{\text{fix}}; \mathcal{O}_{C_{\text{fix}}}(C_1))$, and $\phi$ is one-to-one. Hence

$$(\Gamma; p_i, p_{i+1}, ..., p_{\delta'-1}) \in U'' \subset U'$$
by Lemma 5.

**Step 2:** The goal of this step is to prove that $\mathcal{M}(U)$ is closed under D-moves. We choose an arbitrary marking $\mu = \mu_{\Gamma, p_1, \ldots, p_\delta} \in \mathcal{M}(U)$ and label the rest of the nodes of $\Gamma$ by $p_{k+1}, \ldots, p_{\delta'}$, where $\delta' = dk + n\frac{d(d-1)}{2}$. Let $D, D' \subset \Gamma$ be two different irreducible components, and let $q, q' \in D \cap D'$ be two nodes. If $q, q' \notin \mu$ then $D_{q, q'}(\mu) = \mu \in \mathcal{M}(U)$ and we are done. Thus, without loss of generality, we can assume that $q \in \mu$. We shall show that $D_{q, q'}(\mu) \in \mathcal{M}(U)$.

Consider the irreducible component $U' \subset U_{d, k, \delta}$ containing the pointed curve $(\Gamma; p_1, \ldots, p_{\delta})$, and let $f : U' \rightarrow \tilde{U}$ be the natural forgetful map. Then

$$D_{q, q'}(\mu) = \mu_{\Gamma, p_{\tau_{ij}(1)}(\Gamma), \ldots, p_{\tau_{ij}(\delta')}},$$

where $q = p_i$, $q' = p_j$, and $\tau_{ij} \in \mathcal{S}_{\delta'}$ denotes the elementary transposition $\tau_{ij} = (i, j)$. Hence it is sufficient to prove that

$$(\Gamma; p_{\tau_{ij}(1)}, \ldots, p_{\tau_{ij}(\delta')}) \in U'.$$

Let $\Gamma_1, \ldots, \Gamma_{d+k}$ be the irreducible components of $\Gamma$. We can assume that $D = \Gamma_1$, and we denote $\Gamma^{fix} = \bigcup_{l=2}^{d+k} \Gamma_l$.

Consider the locus $U'' \subset U'$ of pointed curves $(C; x_1, \ldots, x_{\delta'})$ with the following property: $C = C_1 \cup \Gamma^{fix}$, where $C_1 \equiv \Gamma_1$ is generic. Let $U''' \subset U''$ be the irreducible component containing $(\Gamma; p_1, \ldots, p_{\delta'})$, and let

$$\phi : U''' \rightarrow W(\Gamma^{fix}, O_{\Gamma^{fix}}(\Gamma_1))$$

be the natural forgetful map (cf. Lemma 5). Then the image of $\phi$ is dominant in an irreducible component of $W(\Gamma^{fix}, O_{\Gamma^{fix}}(\Gamma_1))$, and $\phi$ is one-to-one. Hence

$$(\Gamma; p_{\tau_{ij}(1)}, \ldots, p_{\tau_{ij}(\delta')}) \in U''' \subset U'$$

by Lemma 5. □

4. Rational curves on toric surfaces

The goal of this section is to prove Conjecture for the case of rational curves.

**Proposition 7.** Let $\Sigma = \text{Tor}(\Delta)$ be a toric surface assigned to an integral polygon $\Delta \subset \mathbb{R}^2$, and let $L \in \text{Pic}(\Sigma)$ be an effective class. Consider variety $V$ parameterizing all irreducible nodal rational curves in the linear system $|L|$ belonging to the smooth locus of $\Sigma$. Then $V$ is either empty or irreducible.
Proof. We can resolve the singularities of \( \Sigma \) by a sequence of blow-ups of the singular zero-dimensional orbits. Since \( V \) parameterizes curves that do not contain singularities of \( \Sigma \), the variety \( V \) parameterizes also irreducible rational curves in the pull back of \( L \) to the disingularization of \( \Sigma \). Thus to prove the Proposition, it is sufficient to consider only the case of smooth surface \( \Sigma \). So let us assume that \( \Sigma \) is smooth.

Let \( (a_i, b_i), 1 \leq i \leq n, \) be the primitive integral vectors parallel to the sides of the \( n \)-gon \( \Delta \) oriented counterclockwise. We define \( (a_{n+1}, b_{n+1}) = (a_1, b_1) \). Then \( \{(a_i, b_i), (a_{i+1}, b_{i+1})\} \) is a basis of the integral lattice for any \( 1 \leq i \leq n \), since \( \Sigma \) is smooth.

Now let \( C \in V \) be a generic element, and let \( \phi : \mathbb{P}^1 \rightarrow \Sigma \) be a parameterization of \( C \). If \( C \) coincides with one of the boundary components then \( V \) is a point and we are done. Thus we can assume that \( C \) intersects the boundary divisor at a finite number of points, moreover there are at least two such points, since no chart isomorphic to \( \mathbb{K}^2 \) can contain a complete curve. Thus the first chern class of the normal bundle \( \mathcal{N}_\phi \) is non-negative, and, since \( C \) is nodal, \( \mathcal{N}_\phi \) is a line bundle. So, we can conclude that first \( V \) is equidimensional, and, second, no irreducible component of \( V \) has a fixed point, in particular \( C \) contains no zero-dimensional orbits.

For any \( 1 \leq i \leq n \) we define \( \{c_{ij}\} = \phi^{-1}(L_i) \subset \mathbb{P}^1 \), where \( L_i \subset \Sigma \) is the one-dimensional orbit corresponding to \( (a_i, b_i) \). We can assume that \( \phi(\infty) \in (\mathbb{K}^*)^2 \). The restriction of \( \phi \) to \( \mathbb{A}^1 \setminus \{c_{ij}\} \) is given by two invertible functions \( x(t), y(t) \in \mathbb{K}[t, (t - c_{ij})^{-1}] \), hence

\[
x(t) = \alpha \prod (t - c_{ij})^{m_{ij}}, \quad y(t) = \beta \prod (t - c_{ij})^{n_{ij}}.
\]

Let \( 1 \leq i \leq n \) be any index such that \( k_i = |\phi^{-1}(L_i)| > 0 \). Consider the affine plane \( \text{Spec} \mathbb{K}[x^{a_i}y^{b_i}, x^{a_{i+1}}y^{b_{i+1}}] \subset \Sigma \). In this chart the line \( L_i \) is given by \( x^{a_{i+1}}y^{b_{i+1}} = 0 \) and \( x^{a_i}y^{b_i} \neq 0 \), hence \( a_{i+1}m_{ij} + b_{i+1}n_{ij} = k_{ij} \) and \( a_im_{ij} + b_{ni} = 0 \) for all \( j \), where \( k_{ij} > 0 \) denotes the order of \( \phi^*(L_i) \) at \( c_{ij} \). Since \( \{(a_i, b_i), (a_{i+1}, b_{i+1})\} \) is a (positive) basis of the integral lattice, we can conclude that \( n_{ij} = k_{ij}a_i \) and \( m_{ij} = -k_{ij}b_i \) for all \( i \) and \( j \). Thus

\[
x(t) = \alpha \prod (t - c_{ij})^{-k_{ij}a_i}, \quad y(t) = \beta \prod (t - c_{ij})^{k_{ij}a_i}.
\]

We shall mention that \( \text{deg} x(t) = \text{deg} y(t) = 0 \) since \( \phi(\infty) \in (\mathbb{K}^*)^2 \).

Next we would like to show that \( k_{ij} = 1 \) for all \( i \) and \( j \). If this is not the case, then without loss of generality we can assume that \( k_{11} > 1 \). Thus the locus of rational curves (in the same linear system) that admit the following parameterization

\[
x(t) = \alpha(t - c_{11})^{-(k_{11}-1)b_1}(t - c_{11}^{n_1})^{-b_1} \prod_{(i, j) \neq (1, 1)} (t - c_{ij})^{-k_{ij}b_i},
\]
\[ y(t) = \beta(t - c'_{11})^{(k_{11} - 1)a_1} (t - c''_{11})^{a_1} \prod_{(i,j) \neq (1,1)} (t - c_{ij})^{k_{ij}a_i}, \]

has dimension greater than \( \dim V \), which is a contradiction.

Now we see that \( V \) contains an open dense subset isomorphic to an open subset of

\[ (\mathbb{K}^*)^2 \times \prod_{i=1}^n \text{Sym}^k \mathbb{P}^1 \]

modulo the automorphisms of \( \mathbb{P}^1 \). Thus \( V \) contains an irreducible dense open subset, hence \( V \) is irreducible. \( \square \)

Finally we would like to explain why the assumption that a generic \( C \in V \) does not contain singularities of \( \Sigma \), is necessary. Consider the toric surface \( \Sigma \) assigned to the triangle \{ (0,0), (0,2), (4,0) \}, and let \( \mathcal{L} \) be the tautological line bundle on \( \Sigma \). This surface has unique singular point. Then the locus of irreducible rational curves in \( |\mathcal{L}| \) consists of two irreducible components of dimension 7. The first component was described in the proposition. To see the second component let us consider the desingularization of \( \Sigma \), which is isomorphic to the Hirzebruch surface \( \Sigma_2 \) assigned to the trapezia \{ (0,0), (0,1), (2,1), (4,0) \}. Then the tautological linear system on \( \Sigma_2 \) has dimension 7 and the projection of this system to \( \Sigma \) defines a hyperplane in \( |\mathcal{L}| \) consisting of curves passing through the node of \( \Sigma \). This hyperplane is the second component of the Severi variety.

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