Abstract

We introduce a complex q-Fourier transform as a generalization of the (real) one analyzed in [Milan J. Math. 76 (2008) 307]. By recourse to tempered ultradistributions we show that this complex
plane-generalization overcomes all troubles that afflict its real counterpart.
# 1 Introduction

Nonextensive statistical mechanics (NEXT) [1, 2, 3], a current generalization of the Boltzmann-Gibbs (BG) one, is actively studied in diverse areas of Science. NEXT is based on a nonadditive (though extensive [4]) entropic information measure characterized by the real index $q$ (with $q = 1$ recovering the standard BG entropy). It has been applied to variegated systems such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spinglasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], Fokker-Planck equation’s applications [16], etc.

An idiosyncratic NEXT feature is that can it can be advantageously expressed via $q$-generalizations of standard mathematical concepts [17]. Included are, for instance, the logarithm and exponential functions, addition and multiplication, Fourier transform (FT) and the Central Limit Theorem (CLT) [18]. The $q$-Fourier transform $F_q$ exhibits the nice property of transforming $q$-Gaussians into $q$-Gaussians [18]. Recently, plane waves, and the representation of the Dirac delta into plane waves have been also generalized [20, 21, 23, 24].
We will be concerned here with the fact that a generic analytical expression for the inverse q-FT for arbitrary functions and any value of q does not exist \[24\]. Investigations revolving around this fact and related questions might be relevant for field theory and condensed matter physics, engineering (e.g., image and signal processing), and mathematical areas for which the standard FT and its inverse play important roles. It has been recently shown \[27\] that, in the \(1 < q < 2\) particular case, and for non-negative functions (e.g., probability distributions), it is possible, by using special kinds of information, to obtain a bi-univocal relation between a function and its q-FT.

In this work we focus attention on the fact that, for fixed \(q\), the q-Fourier transform is NOT one-to-one. The \(F_q\)–scenario can be vastly improved, however, by recourse to tempered ultradistributions (TUD) \[19\], that help generalizing such transform to the complex plane. The generalization ameliorates the troubles that (see, for example \[27\]) afflict the real \(F_q\).

Why TUD? Because they solve characterization problems for analytic functions whose boundary values are elements of the spaces of distributions, or, conversely, of finding representations of elements of the quoted spaces of generalized functions by analytic functions. Many papers concern themselves
with the ultradistribution spaces of Sebastiao e Silva [31]. Such spaces are related to the solvability and the regularity problems of partial differential equations. Because of such relation, the study of the structural problems as well as problems of various operations and integral transformations in this setting is interesting in itself. Thus, an analysis of spaces of distributions considered as boundary values of analytic functions having appropriate growth estimates, is of great value. One wishes to deal, in particular, with the Dirac’s integral representation in ultradistribution spaces, with the convolution of tempered ultradistributions and ultradistributions of exponential type (in Quantum Field Theory), and with the integral transforms of tempered ultradistributions, of which the best known is the Fourier complex transformation [19].

2 The Complex q-Fourier Transform and its Inverse

So-called q-exponentials

\[ e_q(x) = [1 + (1 - q)x]^{1/(1-q)}, \]  

(2.1)
are the hallmark of Tsallis’s statistics [1]. They are generalizations of the ordinary exponential functions and coincide with them for \( q = 1 \). We start our considerations by appealing to a complex \( q \)-exponential. More precisely, we speak of \( e_q(ikx) \) for \( 1 < q < 2 \) with \( k \) a real number (see Ref. [23])

\[
e_q(ikx) = [1 + i(1 - q)kx]^{\frac{1}{1 - q}}.
\]

(2.2)

It can be seen that \( e_q(ikx) \) is the cut along the real \( k \)-axis of the tempered ultradistribution (see the Appendix for details)

\[
E_q(ikx) = \{H(x)H[\Im(k)] - H(-x)H[-\Im(k)]\} [1 + i(1 - q)kx]^{\frac{1}{1 - q}},
\]

(2.3)

where \( H(x) \) is the Heaviside’s step function and \( \Im(k) \) the imaginary part of the complex number \( k \).

Define now the set \( \Lambda_{q,\infty} \), defined as

\[
\Lambda_{q,\infty} = \{f(x)/f(x) \in \Lambda_{q,\infty}^+ \land f(x) \in \Lambda_{q,\infty}^-\},
\]

(2.4)

where

\[
\Lambda_{q,\infty}^+ = \left\{ f(x)/f(x) \{1 + i(1 - q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1 - q}} \in \mathcal{L}^1[\mathbb{R}^+] \land \right.
\]

\[
[f(x) \geq 0; 1 \leq q < 2]\}
\]

(2.5)

and

\[
\Lambda_{q,\infty}^- = \left\{ f(x)/f(x) \{1 + i(1 - q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1 - q}} \in \mathcal{L}^1[\mathbb{R}^-] \land \right.
\]

\[
\left. [f(x) \geq 0; 1 \leq q < 2]\}
\]

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\[ f(x) \geq 0; 1 \leq q < 2 \}\).  \(\quad (2.6)\)

With the help of this set and using (2.3) we define our complex Tsallis’ q-Fourier transform (of \(f(x) \in \Lambda_{q,\infty}\)) in the fashion

\[
F(k, q) = \left[ H(q - 1) - H(q - 2) \right] \times \left\{ H[\Im(k)] \int_{0}^{\infty} f(x) \left\{ 1 + i(1 - q)kx[f(x)]^{(q-1)} \right\} \frac{1}{1 - q}, \, dx - \\ H[-\Im(k)] \int_{-\infty}^{0} f(x) \left\{ 1 + i(1 - q)kx[f(x)]^{(q-1)} \right\} \frac{1}{1 - q}, \, dx \right\}. \quad (2.7)
\]

In (2.7) \(q\) is a real variable such that \(1 \leq q < 2\). It is of the essence that the cut along the real axis of this transform is the real Tsallis’ q-Fourier transform given in [18], [23]. Taking into account that for \(q = 1\) the q-Fourier transform is the usual Fourier transform and using the formula for the inversion of the complex Fourier transform immediately yields the inversion formula for (2.7):

\[
f(x) = \frac{1}{2\pi} \oint_{\Gamma} \left[ \lim_{\epsilon \to 0^+} \int_{1}^{2} F(k, q) \delta(q - 1 - \epsilon) \, dq \right] e^{-ikx} \, dk. \quad (2.8)
\]

Eqs. (2.7) and (2.8) solve the problem of inversion of the q-Fourier transform, which is now of the desired one-to-one character (see [25], [26]) for fixed \(q\). Of course, on the real axis we obtain for (2.7) and (2.8):

\[
F(k, q) = \left[ H(q - 1) - H(q - 2) \right] \times
\]
\[
\int_{-\infty}^{\infty} f(x) \{ 1 + i(1 - q)kx[f(x)]^{(q-1)} \}^{\frac{1}{1-q}} \, dx, \quad (2.9)
\]
for the real transform, and
\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \lim_{\epsilon \to 0^+} \int_{1}^{2} F(k, q)\delta(q - 1 - \epsilon) \, dq \right] e^{-ikx} \, dk, \quad (2.10)
\]
for its inverse.

3 A case in which \( F_q \) is not one-to-one

Let us discuss an interesting example and consider Hilhorst’s work [24] to illustrate the unfortunate fact that for fixed \( q \) the q-Fourier transform is not one-to-one. Let \( f(x) \) be given by:

\[
f(x) = \begin{cases} 
\left( \frac{\lambda}{x} \right)^\beta & ; \; x \in [a, b] \; ; \; 0 < a < b \; ; \; \lambda > 0 \\
0 & ; \; x \text{ outside } [a, b]
\end{cases} \quad (3.1)
\]

The corresponding complex q-Fourier transform is:

\[
F(k, q) = [H(q - 1) - H(q - 2)]H[3(k)] \times \lambda^\beta \int_{a}^{b} x^{-\beta} \{ 1 + i(1 - q)k\lambda^{\beta(q-1)}x^{1-\beta(q-1)} \}^{\frac{1}{1-q}} \, dx \quad (3.2)
\]

By effecting the change of variable

\[
y = x^{1-\beta(q-1)}
\]
we obtain for (3.2):

\[
F(k, q) = [H(q - 1) - H(q - 2)]H[\Im(k)] \times \\
\frac{\lambda^\beta}{1 - \beta(q - 1)} \int_{a^{1 - \beta(q - 1)}}^{b^{1 - \beta(q - 1)}} y^{\beta(q - 2)/1 - \beta(q - 1)} \left\{ 1 + i(1 - q)k\lambda^{\beta(q - 1)}y \right\}^{1/1 - q} dy. 
\]

(3.3) can be written equivalently as:

\[
F(k, q) = [H(q - 1) - H(q - 2)]H[\Im(k)] \times \\
\left\{ \left\{ H(q - 1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \times \\
\frac{\lambda^\beta}{1 - \beta(q - 1)} \int_{a^{1 - \beta(q - 1)}}^{b^{1 - \beta(q - 1)}} y^{\beta(q - 2)/1 - \beta(q - 1)} \left\{ 1 + i(1 - q)k\lambda^{\beta(q - 1)}y \right\}^{1/1 - q} dy + \\
\left\{ H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] - H(q - 2) \right\} \times \\
\frac{\lambda^\beta}{\beta(q - 1) - 1} \int_{b^{1 - \beta(q - 1)}}^{\infty} y^{\beta(q - 2)/1 - \beta(q - 1)} \left\{ 1 + i(1 - q)k\lambda^{\beta(q - 1)}y \right\}^{1/1 - q} dy \right\}. 
\]

(3.4)

We appeal now some of the results to be found in Ref. (28) to evaluate

\[
\int_{a^{1 - \beta(q - 1)}}^{\infty} y^{\beta(2-q)/1-\beta(q-1) \left\{ 1 + i(1 - q)k\lambda^{\beta(q - 1)}y \right\}^{1/1 - q} dy = \\
\frac{(q - 1)[1 - \beta(q - 1)]a^{\frac{2 - q}{q - 1}}}{(2-q)(1-q)ik\lambda^{\beta(q - 1)}1 - \beta(q - 1)} \times \\
F \left( \frac{1}{q - 1}, \frac{2 - q}{(q - 1)[1 - \beta(q - 1)]}, \frac{1}{q - 1} + \frac{\beta(2-q)}{1 - \beta(q - 1)} \right); 
\]
\[-\frac{1}{(1-q)ik\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\], \hspace{1cm} (3.5)

and

\[
\int_0^{a^{1-\beta(q-1)}} y^{\beta(q-1) - 1} \left\{ 1 + i(1-q)k\lambda^{\beta(q-1)}y \right\} \frac{1}{\lambda^{(q-1)}} dy =
\]

\[
\frac{\beta(q-1) - 1}{\beta - 1} a^{1-\beta} \times F \left( \frac{1}{\beta(q-1) - 1}, \frac{\beta - 1}{\beta(q-1) - 1}, \frac{\beta q - 2}{\beta(q-1) - 1}; (q-1)ik\lambda^{\beta(q-1)}a^{1-\beta(q-1)} \right), \hspace{1cm} (3.6)
\]

where \(F(a, b, c; z)\) is the hypergeometric function. Thus we obtain for \(F(k, q)\):

\[
F(k, q) = [H(q-1) - H(q-2)]H[\Im(k)] \times
\]

\[
\left\{ \left\{ H(q-1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \times
\]

\[
\frac{(q-1)\lambda^\beta}{(2-q)[(1-q)ik\lambda^\beta]^\frac{1}{q-1}} \times
\]

\[
\left\{ a^{\frac{q-2}{q-1}} F \left( \frac{1}{q-1}, \frac{2-q}{(q-1)[1-\beta(q-1)]}; \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right) -
\]

\[
b^{\frac{q-2}{q-1}} F \left( \frac{1}{q-1}, \frac{2-q}{(q-1)[1-\beta(q-1)]}; \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right) \right\} +
\]

\[
\left\{ H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] - H(q-2) \right\} \frac{\lambda^\beta}{\beta - 1} \times
\]

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\[
\left\{ a^{1-\beta} F\left(\frac{1}{q-1}, \frac{\beta-1}{\beta(q-1)-1}, \frac{\beta q-2}{\beta(q-1)-1}; (q-1)ik\lambda^{\beta(q-1)} a^{1-\beta(q-1)}\right) - b^{1-\beta} F\left(\frac{1}{q-1}, \frac{\beta-1}{\beta(q-1)-1}, \frac{\beta q-2}{\beta(q-1)-1}; (q-1)ik\lambda^{\beta(q-1)} b^{1-\beta(q-1)}\right) \right\}. \tag{3.7}
\]

From (3.7) we appreciate a crucial fact: our \( F(k, q) \) does depend on \( a \) and \( b \) (remember that we have shown in section 2 that \( F(k, q) \) is one to one). That the mentioned dependence is of the essence will become evident right now.

If we fix \( q \) and select \( \beta = \frac{1}{(q-1)} \) we immediately reproduce the result obtained by Hilhorst. In fact, in this case we obtain for (3.7):

\[
F(k, q) = \lambda^{\frac{q-1}{2-q}} H[3(k)] [H(q-1) - H(q-2)] \times 
\left[ a^{\frac{q-2}{q-1}} F\left(\frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)ik\lambda\right) - b^{\frac{q-2}{q-1}} F\left(\frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)ik\lambda\right) \right] \tag{3.8}
\]

According to ref. \[29\]:

\[ F(-a, b, b, -z) = (1 + z)^a. \]

Then, (3.8) simplifies to:

\[
F(k, q) = \lambda^{\frac{q-1}{2-q}} H[3(k)] [H(q-1) - H(q-2)].
\]
\[
\left( \frac{q-a^{-1}}{b^{q-1}} \right) \left[ 1 + (1 - q)ik\lambda \right]^{\frac{1}{1-q}}.
\] (3.9)

If we use the value of \( \lambda \) given by Hilhorst, then
\[
\lambda = \left[ \frac{q-1}{2-q} \left( \frac{2-q}{a^{q-1}} - \frac{2-q}{b^{q-1}} \right) \right]^{1-q}.
\]

We now obtain
\[
F(k, q) = H[\Im(k)] [H(q - 1) - H(q - 2)] [1 + (1 - q)ik\lambda]^{\frac{1}{1-q}}.
\] (3.10)

We see according to (3.10) that in this case \( F(k, q) \) is independent of \( a \) and \( b \), i.e., the same for all legitimate pairs \( a-b \), and, as consequence, not one-to-one for fixed \( q \). On the real axis (3.10) takes de form
\[
F(k, q) = [H(q - 1) - H(q - 2)] [1 + (1 - q)i(k + i0)\lambda]^{\frac{1}{1-q}} =
\]
\[
[H(q - 1) - H(q - 2)] [1 + (1 - q)ik\lambda]^{\frac{1}{1-q}},
\] (3.11)

which is the result obtained in Ref. (24).

4 Another example

As a second example we evaluate the q-Fourier transform of the Heaviside function
\[
f(x) = H(x).
\]
In this case,
\[ F(k, q) = H[\Im(k)] \int_0^\infty [1 + (1 - q)ikx]^{\frac{1}{1-q}} \, dx. \] (4.12)

Using the result given in [30] we have
\[ F(k, q) = H[\Im(k)] \frac{\Gamma\left(\frac{2-q}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)} [(1 - q)ik]^{-1}, \] (4.13)

and, finally,
\[ F(k, q) = \frac{i}{2 - q} \frac{H[\Im(k)]}{k}. \] (4.14)

In the same way, if we select:
\[ f(x) = H(-x), \]
we obtain the result:
\[ F(k, q) = \frac{i}{2 - q} \frac{H[-\Im(k)]}{k}. \] (4.15)

Taking into account that \( H(x) + H(-x) = 1 \) we have, for
\[ f(x) = 1, \]
the expression
\[ F(k, q) = \frac{i}{2 - q} \frac{1}{k} = \frac{2\pi}{2 - q} \delta(k), \] (4.16)
which is the formula obtained by us in Ref. [23].

13
Conclusions

Using tempered ultradistributions we have introduced a complex q-Fourier transform $F(k, q)$ which exhibits nice properties and is one-to-one.

This solves a serious flaw of the original $F_q$—definition, i.e., not being of the essential one-to-one nature, as illustrated in detail in Section 3.

In this work we have shown that if we eliminate the requirement that $q$ be fixed and let it float in its proper interval $[1, 2)$, the complex generalization of the $F_q$—definition restores the one-to-one character.
5 Appendix: Tempered Ultradistributions and Distributions of Exponential Type

For the benefit of the reader we give a brief summary of the main properties of distributions of exponential type and tempered ultradistributions.

Notations. The notations are almost textually taken from Ref. [32]. Let \( \mathbb{R}^n \) (resp. \( \mathbb{C}^n \)) be the real (resp. complex) n-dimensional space whose points are denoted by \( x = (x_1, x_2, \ldots, x_n) \) (resp \( z = (z_1, z_2, \ldots, z_n) \)). We shall use the notations:

(a) \( x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \) ; \( \alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \)

(b) \( x \geq 0 \) means \( x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \)

(c) \( x \cdot y = \sum_{j=1}^{n} x_j y_j \)

(d) \( |x| = \sum_{j=1}^{n} |x_j| \)

Let \( \mathbb{N}^n \) be the set of n-tuples of natural numbers. If \( p \in \mathbb{N}^n \), then \( p = (p_1, p_2, \ldots, p_n) \), and \( p_j \) is a natural number, \( 1 \leq j \leq n \). \( p + q \) stands for \( (p_1 + q_1, p_2 + q_2, \ldots, p_n + q_n) \) and \( p \geq q \) means \( p_1 \geq q_1, p_2 \geq q_2, \ldots, p_n \geq q_n \). \( x^p \) entails \( x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n} \). We shall denote by \( |p| = \sum_{j=1}^{n} p_j \) and call \( D^p \) the differential operator \( \partial^{p_1 + p_2 + \ldots + p_n} / \partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_n^{p_n} \)

For any natural \( k \) we define \( x^k = x_1^k x_2^k \ldots x_n^k \) and \( \partial^k / \partial x^k = \partial^{nk} / \partial x_1^k \partial x_2^k \ldots \partial x_n^k \)
The space $\mathcal{H}$ of test functions such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any $p$ and $q$, being defined [see Ref. (32)] by means of the countably set of norms

$$\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p, x} e^{p|x|}|D^q\hat{\phi}(x)|, \quad p = 0, 1, 2, ... \quad (5.1)$$

The space of continuous linear functionals defined on $\mathcal{H}$ is the space $\Lambda_\infty$ of the distributions of the exponential type given by (ref.[32]).

$$T = \frac{\partial^k}{\partial x^k} [e^{k|x|} f(x)] \quad (5.2)$$

where $k$ is an integer such that $k \geq 0$ and $f(x)$ is a bounded continuous function. In addition we have $\mathcal{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \Lambda_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (ref.[33]).

The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{\phi}(x)} e^{iz\cdot x} \, dx \quad (5.3)$$

According to ref.[32], $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call $\mathcal{H}$ the set of all such functions.

$$\mathcal{H} = \mathcal{F} \{\mathcal{H}\} \quad (5.4)$$

The topology in $\mathcal{H}$ is defined by the countable set of semi-norms:

$$\|\phi\|_k = \sup_{z \in V_k} |z^k|z\phi(z)|, \quad (5.5)$$
where \( V_k = \{ z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : \text{Im}z_j \leq k, 1 \leq j \leq n \} \)

The dual of \( \mathcal{H} \) is the space \( \mathcal{U} \) of tempered ultradistributions [see Ref. (32)]. In other words, a tempered ultradistribution is a continuous linear functional defined on the space \( \mathcal{H} \) of entire functions rapidly decreasing on straight lines parallel to the real axis. Moreover, we have \( \mathcal{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{U} \).

\( \mathcal{U} \) can also be characterized in the following way [see Ref. (32)]: let \( \mathcal{A}_\omega \) be the space of all functions \( F(z) \) such that:

(A) \( F(z) \) is analytic for \( \{ z \in \mathbb{C}^n : |\text{Im}(z_1)| > p, |\text{Im}(z_2)| > p, ..., |\text{Im}(z_n)| > p \} \).

(B) \( F(z)/z^p \) is bounded continuous in \( \{ z \in \mathbb{C}^n : |\text{Im}(z_1)| \geq p, |\text{Im}(z_2)| \geq p, ..., |\text{Im}(z_n)| \geq p \} \), where \( p = 0, 1, 2, ... \) depends on \( F(z) \).

Let \( \Pi \) be the set of all \( z \)-dependent pseudo-polynomials, \( z \in \mathbb{C}^n \). Then \( \mathcal{U} \) is the quotient space

\( C) \quad \mathcal{U} = \mathcal{A}_\omega/\Pi \)

By a pseudo-polynomial we understand a function of \( z \) of the form
\[ \sum_{s} z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \] with \( G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \mathcal{A}_\omega \).

Due to these properties it is possible to represent any ultradistribution as
\[ F(\phi) = \langle F(z), \phi(z) \rangle = \oint_{\Gamma} F(z)\phi(z) \, dz \]  
(5.6)

\[ \Gamma = \Gamma_1 \cup \Gamma_2 \cup ... \Gamma_n, \] where the path \( \Gamma_j \) runs parallel to the real axis from \( -\infty \) to \( \infty \) for \( \text{Im}(z_j) > \zeta, \zeta > p \) and back from \( \infty \) to \( -\infty \) for \( \text{Im}(z_j) < -\zeta, -\zeta < -p \). (\( \Gamma \) surrounds all the singularities of \( F(z) \)).

Eq. (5.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of the “Dirac formula” for ultradistributions [see Ref. (31)]

\[ F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)} \, dt \]  
(5.7)

where the “density” \( f(t) \) is such that

\[ \oint_{\Gamma} F(z)\phi(z) \, dz = \int_{-\infty}^{\infty} f(t)\phi(t) \, dt. \]  
(5.8)

While \( F(z) \) is analytic on \( \Gamma \), the density \( f(t) \) is in general singular, so that the r.h.s. of (5.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on \( \Gamma \), \( F(z) \) is bounded by a power of \( z \) \[ 32 \]

\[ |F(z)| \leq C|z|^p, \]  
(5.9)

where \( C \) and \( p \) depend on \( F \).
The representation (5.6) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ does not alter the ultradistribution:

$$\oint_{\Gamma} \{F(z) + P(z)\} \phi(z) \, dz = \oint_{\Gamma} F(z) \phi(z) \, dz + \oint_{\Gamma} P(z) \phi(z) \, dz$$

However,

$$\oint_{\Gamma} P(z) \phi(z) \, dz = 0.$$ 

As $P(z) \phi(z)$ is entire analytic in some of the variables $z_j$ (and rapidly decreasing), we obtain:

$$\oint_{\Gamma} \{F(z) + P(z)\} \phi(z) \, dz = \oint_{\Gamma} F(z) \phi(z) \, dz.$$ (5.10)

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References

[1] C. Tsallis, J. Stat. Phys. 52 (1988) 479.

[2] M. Gell-Mann, C. Tsallis (Eds.), Nonextensive Entropy: Interdisciplinary Applications, Oxford University Press, New York, 2004; C. Tsallis, Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World, Springer, New York, 2009.

[3] A. R. Plastino, A. Plastino, Phys. Lett A 177 (1993) 177.

[4] C. Tsallis, M. Gell-Mann, Y. Sato, Proc. Natl. Acad. Sci. USA 102 (2005) 15377; F. Caruso, C. Tsallis, Phys. Rev. E 78 (2008) 021102.

[5] P. Douglas, S. Bergamini, F. Renzoni, Phys. Rev. Lett. 96 (2006) 110601; G.B. Bagci, U. Tirnakli, Chaos 19 (2009) 033113.

[6] B. Liu, J. Goree, Phys. Rev. Lett. 100 (2008) 055003.

[7] R.G. DeVoe, Phys. Rev. Lett. 102 (2009) 063001.

[8] R.M. Pickup, R. Cywinski, C. Pappas, B. Farago, P. Fouquet, Phys. Rev. Lett. 102 (2009) 097202.

[9] L.F. Burlaga, N.F. Ness, Astrophys. J. 703 (2009) 311.
[10] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra, A. Rapisarda, Phys. Rev. E 75 (2007) 055101(R); B. Bakar, U. Tirnakli, Phys. Rev. E 79 (2009) 040103(R); A. Celikoglu, U. Tirnakli, S.M.D. Queiros, Phys. Rev. E 82 (2010) 021124.

[11] V. Khachatryan, et al., CMS Collaboration, J. High Energy Phys. 1002 (2010) 041; V. Khachatryan, et al., CMS Collaboration, Phys. Rev. Lett. 105 (2010) 022002.

[12] Adare, et al., PHENIX Collaboration, Phys. Rev. D 83 (2011) 052004; M. Shao, L. Yi, Z.B. Tang, H.F. Chen, C. Li, Z.B. Xu, J. Phys. G 37 (8) (2010) 085104.

[13] M.L. Lyra, C. Tsallis, Phys. Rev. Lett. 80 (1998) 53; E.P. Borges, C. Tsallis, G.F.J. Ananos, P.M.C. de Oliveira, Phys. Rev. Lett. 89 (2002) 254103; G.F.J. Ananos, C. Tsallis, Phys. Rev. Lett. 93 (2004) 020601; U. Tirnakli, C. Beck, C. Tsallis, Phys. Rev. E 75 (2007) 040106(R); U. Tirnakli, C. Tsallis, C. Beck, Phys. Rev. E 79 (2009) 056209.

[14] L. Borland, Phys. Rev. Lett. 89 (2002) 098701.

[15] A. R. Plastino, A. Plastino, Phys. Lett A 174 (1993) 834.
[16] A. R. Plastino, A. Plastino, Physica A 222 (1995) 347.

[17] E. P. Borges, Physica A 340 (2004) 95.

[18] S. Umarov, C. Tsallis, S. Steinberg, Milan J. Math. 76 (2008) 307; S. Umarov, C. Tsallis, M. Gell-Mann, S. Steinberg, J. Math. Phys. 51 (2010) 033502.

[19] hep-th, arXiv:hep-th/0309271 (2003).

[20] M. Jauregui, C. Tsallis, J. Math. Phys. 51 (2010) 063304.

[21] A. Chevreuil, A. Plastino, C. Vignat, J. Math. Phys. 51 (2010) 093502.

[22] M. Mamode, J. Math. Phys. 51 (2010) 123509.

[23] A. Plastino and M.C.Rocca: J. Math. Phys 52, 103503 (2011).

[24] H.J.Hilhorst: J. Stat. Mech. P10023 (2010)

[25] M. Jauregui and C. Tsallis: Phys. Lett. A 375, 2085 (2011).

[26] M. Jauregui, C. Tsallis and E.M.F. Curado: arXiv:1108.2690v1.

[27] M. Jauregui, C, Tsallis, Phys. Lett. A 375 (2011) 2085.
[28] L. S. Gradshtein and I. M. Ryzhik: *Table of Integrals, Series, and Products*. Fourth edition, Academic Press (1965) 3.194 1 and 3.194 2 pages 284 and 285.

[29] M. Abramowitz and I. A. Stegun: *Handbook of Mathematical Functions*. National Bureau of Standards. Applied Mathematical Series 55 Tenth Printing (1972), 15.1.8 page 556.

[30] L. S. Gradshtein and I. M. Ryzhik: *Table of Integrals, Series, and Products*. Fourth edition, Academic Press (1965) 3.194 3 page 285.

[31] J. Sebastiao e Silva: Math. Ann. **136**, 38 (1958).

[32] M. Hasumi: Tohoku Math. J. **13**, 94 (1961).

[33] L. Schwartz: *Théorie des distributions*. Hermann, Paris (1966).