STRONGLY MINIMAL SETS IN \( j \)-REDUCTS OF DIFFERENTIALLY CLOSED FIELDS

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Abstract. Let \( K := (K; +, \cdot, D) \) be a differentially closed field with constant field \( C \). Let also \( E_j(x, y) \) be the differential equation of the \( j \)-function. We prove a Zilber style classification result for strongly minimal sets in the reduct \( K_{E_j} := (K; +, \cdot, E_j) \) assuming an Existential Closedness (EC) conjecture for \( E_j \). More precisely, assuming EC we show that in \( K_{E_j} \) all strongly minimal sets are geometrically trivial or non-orthogonal to \( C \). The Ax-Schanuel inequality for the \( j \)-function and its adequacy play a crucial role in this classification.

1. Introduction

Understanding the nature of strongly minimal sets in a structure (or theory) is one of the central problems in geometric model theory. It is well known that strongly minimal sets in differentially closed fields satisfy Zilber’s trichotomy, that is, such a set must be either geometrically trivial or non-orthogonal to a Manin kernel \( \mathcal{A} \) (this is the locally modular non-trivial case) or non-orthogonal to the field of constants \( (\mathbb{H}_{nS93}) \). Hrushovski also proved that order 1 strongly minimal sets are either non-orthogonal to the constants or are trivial and \( \aleph_0 \)-categorical \( (\mathbb{H}_{nS95}) \). This (and the lack of counterexamples) led people to believe that every strongly minimal set (of arbitrary order) must be \( \aleph_0 \)-categorical. However, Freitag and Scanlon showed recently that it is not true by proving that the differential equation of the \( j \)-function is strongly minimal and trivial but not \( \aleph_0 \)-categorical \( (\mathbb{F}_{S15}) \). Their proof is based on Pila’s Modular Ax-Lindemann-Weierstrass with Derivatives theorem \( (\mathbb{P}_{113}) \). Later the latter was generalised by Pila and Tsimerman to an Ax-Schanuel theorem for the \( j \)-function incorporating the derivatives of \( j \) \( (\mathbb{P}_{116}) \). In \( (\mathbb{A}_{s16}) \) we showed that if a two-variable differential equation \( E(x, y) \) satisfies an analogue of the Ax-Schanuel theorem for the \( j \)-function then the fibres of \( E \) are strongly minimal and geometrically trivial, thus giving a new proof for the aforementioned theorem of Freitag and Scanlon (note that we actually use Ax-Lindemann-Weierstrass in our proof and not the full Ax-Schanuel).

After that Zilber asked me in a private communication if it is possible to use Ax-Schanuel for the \( j \)-function to classify strongly minimal sets in a “\( j \)-reduct” of a differentially closed field. More precisely, let \( K := (K; +, \cdot, D) \) be a differentially closed field and \( F(y, Dy, D^2y, D^3y) = 0 \) be the differential equation of the \( j \)-function (see Section 4.1). Consider a two-variable equation \( E_j(x, y) \) given by \( F(y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = 0 \) where \( \partial_x = \frac{1}{D} \cdot D \). Then the problem is to classify strongly minimal sets in the reduct \( K_{E_j} := (K; +, \cdot, E_j) \). It turns out that if we assume \( E_j \) satisfies an Existential Closedness statement \( (\mathbb{A}_{s18}) \), which essentially states that if for a system of equations in \( K_{E_j} \) having a solution does not contradict Ax-Schanuel then it does have a solution, then we can prove a dichotomy result for strongly minimal sets in \( K_{E_j} \).

Theorem 1.1. Assume the Existential Closedness conjecture for \( E_j \). Then in \( K_{E_j} \) all strongly minimal sets are geometrically trivial or non-orthogonal to \( C \) (the latter being definable in \( K_{E_j} \)).

The existential closedness conjecture is related to the question of adequacy of the Ax-Schanuel theorem for the \( j \)-function (see \( \mathbb{A}_{s18} \) for details, we also give the necessary preliminaries in Sections 3 and 4). Adequacy means roughly that the Ax-Schanuel inequality governs the geometry of the reduct, hence it is not surprising that it leads to a classification of strongly minimal sets there.

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\( \text{More precisely, it is non-orthogonal to the Manin kernel } A^\# \text{ of a simple abelian variety } A \text{ of } C\text{-trace zero.} \)
We also study strongly minimal sets in a more basic reduct, namely $\mathcal{K}_C := (K; +, \cdot, C)$ where $C$ is the field of constants. Actually, this is the first example that we deal with in this paper. For this reduct we do not have any Ax-Schanuel type statement and we do not need one since it is quite easy to understand definable sets in such a structure. In this case we have the following result.

**Theorem 1.2.** All strongly minimal sets in $\mathcal{K}_C$ are non-orthogonal to $C$.

Most of our observations on pairs of algebraically closed fields are well known and we merely present our approach as a prelude to the aforementioned classification of strongly minimal sets in $j$-reducts.

**Notation and conventions.**

- The length of a tuple $\vec{a}$ will be denoted by $|\vec{a}|$.
- For fields $L \subseteq K$ the algebraic locus (Zariski closure) of a tuple $\vec{a} \in K$ over $L$ will be denoted by $\text{Loc}_L(\vec{a})$ or $\text{Loc}(\vec{a}/L)$.
- Algebraic varieties defined over an (algebraically closed) field $K$ will be identified with the sets of their $K$-rational points.

2. **Pairs of algebraically closed fields**

Model theory of pairs of algebraically closed fields is well studied (see, for example, A vdD16, Kei64). Therefore, most of the results of this section are well known.

Let $\mathcal{K}_C := (K; +, \cdot, C)$ be an algebraically closed field of characteristic 0 with a distinguished algebraically closed subfield $C$ ($C$ is a unary predicate in the language). It is easy to prove that this structure is $\omega$-stable of Morley rank $\omega$. We assume $\mathcal{K}_C$ is sufficiently saturated.

Let $\vec{a} \in K^m$ and $b \in K$.

**Lemma 2.1.** $\text{MR}(b/\vec{a}) < \omega$ iff $b \in C(\vec{a})^{\text{alg}}$.

**Proof.** If $b$ is transcendental over $C(\vec{a})$ then for any $b' \notin C(\vec{a})^{\text{alg}}$ there is a field automorphism of $K$ fixing $C(\vec{a})$ pointwise and mapping $b$ to $b'$. In particular, it is an automorphism of $\mathcal{K}_C$ and so $\text{tp}(b/\vec{a}) = \text{tp}(b'/\vec{a})$, and this type is the generic type over $\vec{a}$. \hfill $\Box$

Now let $b \in C(\vec{a})^{\text{alg}}$. Then for some polynomial $p$ the equality $p(\vec{a}, \bar{c}, b) = 0$ holds for some finite tuple $\bar{c} \in C^l$. Let $W := \text{Loc}_{\bar{Q}(\bar{a})}(\bar{c}) \subseteq K^l$ be the algebraic locus (Zariski closure) of $\bar{c}$ over $\bar{Q}(\bar{a})$. For every proper subvariety $U \not\subseteq W$ defined over $\bar{a}$ consider the formula

$$\varphi_U(y) = \exists \bar{x}(\bar{x} \in C^l \cap (W \setminus U) \land p(\bar{a}, \bar{x}, y) = 0).$$

Notice that for every $U \not\subseteq W$ the formula $\varphi_U(b)$ holds. Observe also that the set $C^l \cap (W(K) \setminus U(K))$, being a subset of $C^l$, is actually definable with parameters from $C$. This follows from the stable embedding property.

**Proposition 2.2.** The collection of all formulas $\varphi_U(y)$ determines $\text{tp}(b/\vec{a})$.

**Proof.** Assume $b' \models \varphi_U(y)$ for all $U \not\subseteq W$. The collection of formulas

$$\{\bar{x} \in C^l \cap (W \setminus U) \land p(\bar{a}, \bar{x}, b') = 0 : U \not\subseteq W\}$$

(over $\bar{a}, b'$) is finitely satisfiable so it has a realisation $\bar{c}'$. Evidently $\bar{c}'$ is generic in $W$ over $\bar{a}$. Therefore there is an automorphism $\pi$ of $C(\bar{a})$ which fixes $\bar{a}$ pointwise, fixes $C$ setwise and sends $\bar{c}$ to $\bar{c}'$. This automorphism can be extended to an automorphism of $\mathcal{K}_C$ which sends $b$ to $b'$. \hfill $\Box$

**Remark 2.3.** This shows, in particular, that the first-order theory of $\mathcal{K}_C$ is nearly model complete, that is, every formula is equivalent to a Boolean combination of existential formulas.

\footnote{This theory is axiomatised by axiom schemes stating that $K$ is an algebraically closed field and $C$ is an algebraically closed subfield.}
Proof. Let \( K \) be strongly minimal defined by some formula \( \varphi_U \) (a conjunction of formulas of the form (2.1) is again of the same form). Then \( S \subseteq \text{acl}(C \cup \bar{a}) \) and therefore \( S \not\subseteq C \). \( \square \)

Remark 2.5. Let \( S \subseteq K \) be strongly minimal defined by some formula \( \varphi_U \). As we pointed out above \( V := W(K) \setminus U(K) \cap C^d \) is defined over \( C \). So \( V \) can be regarded as a quasi-affine variety over \( C \). Define an equivalence relation \( E \subseteq V \times V \) by
\[
\bar{c}_1 E \bar{c}_2 \text{ iff } \forall y(p(\bar{a}, \bar{c}_1, y) = 0 \leftrightarrow p(\bar{a}, \bar{c}_2, y) = 0).
\]

By the stable embedding property \( E \) is definable in the pure field structure of \( C \). Moreover, there is a natural finite-to-one map from \( S \) to \( V/E \). By elimination of imaginaries in algebraically closed fields \( V/E \) can be regarded as a constructible set in some Cartesian power \( C^k \). The latter must have dimension 1 since \( S \) is strongly minimal. Thus, in the formula \( \varphi_U \) we may assume that the constants live on a curve defined over \( C \). This gives a characterisation of strongly minimal formulas.

3. PreDimensions and Hrushovski constructions

In this section we give the necessary preliminaries on predimensions and Hrushovski constructions. We refer the reader to [Asl17, Asl18] (and the references given there) for details and proofs of the results presented here.

Let \( \mathfrak{L} \) be a countable language and \( \mathfrak{C} \) be a collection of \( \mathfrak{L} \)-structures closed under isomorphism and intersections, that is, if \( A_i \in \mathfrak{C} \), \( i \in I \), are substructures of some \( A \in \mathfrak{C} \) then \( \bigcap_{i \in I} A_i \in \mathfrak{C} \). We will also assume that \( \mathfrak{C} \) has the joint embedding property, i.e. for any \( A, B \in \mathfrak{C} \) there is \( C \in \mathfrak{C} \) such that \( A \) and \( B \) can be embedded into \( C \). Assume further that \( \mathfrak{C} \) contains a smallest structure \( S \in \mathfrak{C} \), that is, \( S \) can be embedded into all structures of \( \mathfrak{C} \).

Definition 3.1. For \( B \subseteq \mathfrak{C} \) and \( X \subseteq B \) the \( \mathfrak{C} \)-closure of \( X \) inside \( B \) (or the \( \mathfrak{C} \)-substructure of \( B \) generated by \( X \)) is the structure\(^3\)
\[
\langle X \rangle_B := \bigcap_{A \subseteq \mathfrak{C}, X \subseteq A \subseteq B} A.
\]
A structure \( A \in \mathfrak{C} \) is finitely generated if \( A = \langle X \rangle_A \) for some finite \( X \subseteq A \). The collection of all finitely generated structures from \( \mathfrak{C} \) will be denoted by \( \mathfrak{C}_{f.g.} \).

Note that in general finitely generated in this sense is different from being finitely generated as a structure. We will assume however that finitely generated structures are countable. Further, a substructure of a finitely generated structure may not be finitely generated but we assume it is the case here.

Since \( S \) is the smallest structure in \( \mathfrak{C} \), it is in fact generated by the empty set, i.e. \( S = \langle \emptyset \rangle \). So, by abuse of notation, we will normally write \( \emptyset \) instead of \( S \).

For \( A, B \in \mathfrak{C} \) by \( A \subseteq_{f.g.} B \) we mean \( A \) is a finitely generated substructure of \( B \). When we have two structures \( A, B \in \mathfrak{C} \) we would like to have a notion of a structure generated by \( A \) and \( B \). However, this cannot be well-defined without embedding \( A \) and \( B \) into a larger \( C \).
Given such a common extension $C$, we will denote $AB_C := \langle A \cup B \rangle_C$. Often we will drop the subscript $C$ meaning that our statement holds for every common extension $C$ (or it is obvious in which common extension we work). This remark is valid also when we write $A \cap B$ which should be understood as the intersection of $A$ and $B$ after identifying them with their images in a common extension.

**Definition 3.2.** A *predimension* on $\mathcal{C}_{f.g.}$ is a function $\delta : \mathcal{C}_{f.g.} \to \mathbb{Z}$ with the following properties:

P1 $\delta(\emptyset) = 0$,

P2 If $A, B \in \mathcal{C}_{f.g.}$ with $A \cong B$ then $\delta(A) = \delta(B)$,

P3 (Submodularity) For all $A, B \in \mathcal{C}_{f.g.}$ and $C \in \mathcal{C}$ with $A, B \subseteq C$ we have

\[
\delta(AB) + \delta(A \cap B) \leq \delta(A) + \delta(B).
\]

If, in addition, such a function is monotonic, i.e. $A \subseteq B \Rightarrow \delta(A) \leq \delta(B)$, and hence takes on only non-negative values, then $\delta$ is called a *dimension*.

Given a predimension $\delta$, for a finite subset $X \subseteq_{f.m} A \in \mathcal{C}$ one defines

\[
\delta_A(X) := \delta(\langle X \rangle_A).
\]

**Definition 3.3.** For $A, B \in \mathcal{C}_{f.g.}$, the *relative predimension* of $A$ over $B$ is defined as $\delta(B/A) := \delta(AB) - \delta(A)$. This depends on a common extension of $A$ and $B$, so we work in such a common extension without explicitly mentioning it. When $A \subseteq B$ we work in $B$ and define $\delta(B/A) = \delta(B) - \delta(A)$.

In the next definition $B$ is the ambient structure that we work in.

**Definition 3.4.** Let $A \subseteq B \in \mathcal{C}$. We say $A$ is strong (or self-sufficient) in $B$, denoted $A \leq B$, if for all $X \subseteq_{f.g.} B$ we have $\delta(X \cap A) \leq \delta(X)$. One also says $B$ is a strong extension of $A$. An embedding $A \hookrightarrow B$ is strong if the image of $A$ is strong in $B$.

For $M \in \mathcal{C}$ and a finite set $\bar{a} \subseteq M$ we say $\bar{a}$ is strong in $M$ if $\langle \bar{a} \rangle \leq M$.

**Definition 3.5.** For $B \in \mathcal{C}$ and $X \subseteq B$ we define the self-sufficient closure (or strong closure) of $X$ in $B$ by

\[
[X]_B := \bigcap_{A \in \mathcal{C}, X \subseteq A \subseteq B} A.
\]

An arbitrary (finite or infinite) intersection of strong substructures is strong. It follows from this that $[X]_B \leq B$. Note also that $\leq$ is transitive.

From now on we assume $\delta(A) \geq 0$ for all $A \in \mathcal{C}_{f.g.}$. In other words $\emptyset$ is strong in all structures of $\mathcal{C}$.

**Lemma 3.6.** If $B \in \mathcal{C}$ and $X \subseteq_{f.g.} B$ then

- $[X]_B$ is finitely generated, and
- $\delta([X]_B) = \min\{\delta(Y) : X \subseteq Y \subseteq_{f.g.} B\}$.

A predimension gives rise to a dimension in the following way.

**Definition 3.7.** For $X \subseteq_{f.g.} B$ define

\[
d_B(X) := \min\{\delta(Y) : X \subseteq Y \subseteq_{f.g.} B\} = \delta([X]_B).
\]

For $X \subseteq_{f.m} B$ set $d_B(X) := d_B(\langle X \rangle_B)$.

It is easy to verify that $d$ is a dimension function and therefore we have a natural pregeometry associated with $d$. More precisely, we define $\mathrm{cl}_B : \mathcal{P}(B) \to \mathcal{P}(B)$ (the latter is the power set of $B$) by

\[
\mathrm{cl}_B(X) = \{b \in B : d_B(b/X) = 0\}.
\]

Then $(B, \mathrm{cl}_B)$ is a pregeometry and $d_B$ is its dimension function.
Self-sufficient embeddings can be defined in terms of \( d \). Indeed, if \( A \subseteq B \) then \( A \leq B \) if and only if for any \( X \subseteq_{f_{\text{fin}}} A \) one has \( d_A(X) = d_B(X) \).

Now we formulate conditions under which one can carry out an amalgamation-with-predimension construction. Let \( C \) be as above and let \( \delta \) be a non-negative predimension on \( C_{f,g} \).

**Definition 3.8.** The class \( C \) is called a strong amalgamation class if the following conditions hold.

1. Every \( A \in C_{f,g} \) has at most countably many finitely generated strong extensions up to isomorphism.
2. \( C \) is closed under unions of countable strong chains \( A_0 \subseteq A_1 \subseteq \ldots \).
3. SAP \( C_{f,g} \) has the strong amalgamation property, that is, for all \( A_0, A_1, A_2 \in C_{f,g} \) with \( A_0 \subseteq A_i, i = 1, 2 \), there is \( B \in C_{f,g} \) such that \( A_1 \) and \( A_2 \) are strongly embedded into \( B \) and the corresponding diagram commutes.

The following is a standard theorem.

**Theorem 3.9 (Amalgamation theorem).** If \( C \) is a strong amalgamation class then there is a unique (up to isomorphism) countable structure \( U \in C \) with the following properties.

- **U1** \( U \) is universal with respect to strong embeddings, i.e. every countable \( A \in C \) can be strongly embedded into \( U \).
- **U2** \( U \) is saturated with respect to strong embeddings, i.e. for every \( A, B \in C_{f,g} \) with strong embeddings \( A \to U \) and \( A \to B \) there is a strong embedding of \( B \) into \( U \) over \( A \).

Furthermore, \( U \) is homogeneous with respect to strong substructures, that is, any isomorphism between finitely generated strong substructures of \( U \) can be extended to an automorphism of \( U \).

This \( U \) is called the (strong) Fraïssé limit or the Fraïssé-Hrushovski limit of \( C_{f,g} \). It has a natural pregeometry associated with the predimension function as described above. Note that U2 is normally known as the richness property in the literature and it implies U1.

4. **The \( j \)-function**

Now we study the differential equation of the \( j \)-function. The first three subsections are preliminary. The reader is referred to [AsH18] for details.

4.1. **Modular polynomials and the Ax-Schanuel theorem.** The function \( j \) is a modular function of weight 0 for the modular group \( \text{SL}_2(\mathbb{Z}) \), which is defined and analytic on the upper half-plane \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \).

Let \( \text{GL}_2^+(\mathbb{Q}) \) be the subgroup of \( \text{GL}_2(\mathbb{Q}) \) consisting of matrices with positive determinant (this group acts on the upper half-plane). For \( g \in \text{GL}_2^+(\mathbb{Q}) \) we let \( N(g) \) be the determinant of \( g \) scaled so that it has relatively prime integral entries. For each positive integer \( N \) there is an irreducible polynomial \( \Phi_N(X,Y) \in \mathbb{Z}[X,Y] \) such that whenever \( g \in \text{GL}_2^+(\mathbb{Q}) \) with \( N = N(g) \), the function \( \Phi_N(j(z), j(gz)) \) is identically zero. Conversely, if \( \Phi_N(j(x), j(y)) = 0 \) for some \( x, y \in \mathbb{H} \) then \( y = gx \) for some \( g \in \text{GL}_2^+(\mathbb{Q}) \) with \( N = N(g) \). The polynomials \( \Phi_N \) are called modular polynomials. It is well known that \( \Phi_1(X,Y) = X-Y \) and all the other modular polynomials are symmetric. Two elements \( w_1, w_2 \in \mathbb{C} \) are called modularly independent if they do not satisfy any modular relation \( \Phi_N(w_1, w_2) = 0 \). This definition makes sense for arbitrary fields (of characteristic zero) as the modular polynomials have integer coefficients.

The \( j \)-function satisfies an order 3 algebraic differential equation over \( \mathbb{Q} \), and none of lower order (i.e. its differential rank over \( \mathbb{C} \) is 3). Namely, \( F(j, j', j'', j''') = 0 \) where

\[
F(y_0, y_1, y_2, y_3) = y_3 \frac{y_2}{y_1} - \frac{3}{2} \left( \frac{y_2}{y_1} \right)^2 + \frac{y_0^2 - 1968y_0 + 265420}{2y_0^3(y_0 - 1728)^2} \cdot y_1^2.
\]

Thus

\[
F(y, y', y'', y''') = Sy + R(y)(y')^2,
\]
where $S$ denotes the *Schwarzian derivative* defined by $Sy = \frac{y'''}{y'} - \frac{3}{2} \left( \frac{y''}{y'} \right)^2$ and $R(y) = \frac{y^2 - 19689 + 2654208}{2y - 2}$. 

Here $'$ denotes the derivative of a complex function. Below when we work in an abstract differential field we will always denote its derivation by $D$ and for an element $a$ in that field $a', a'', \ldots$ will be some other elements and not the derivatives of $a$.

Let $(K; +, \cdot, D, 0, 1)$ be a differential field with constant field $C$ and $F(y, Dy, D^2 y, D^3 y) = 0$ be the differential equation of the $j$-function. Consider its two-variable version\footnote{Recall that for a non-constant $x$ we define $\partial_x : y \mapsto \frac{Dy}{Dt}$}.

\begin{equation}
(4.3) \quad f(x, y) := F(y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = 0.
\end{equation}

**Theorem 4.1** (Ax-Schanuel for $j$, [PT16]). Let $(K; +, \cdot, D, 0, 1)$ be a differential field and let $z_i, j_i \in K \setminus C$, $i = 1, \ldots , n$, be such that

$f(z_i, j_i) = 0$.

If $j_i$’s are pairwise modularly independent then

\begin{equation}
(4.4) \quad \text{td}_C C(z_i, \partial z_i, j_i, \partial^2 z_i, j_i : 1 \leq i \leq n) \geq 3n + 1.
\end{equation}

**Corollary 4.2** (Ax-Schanuel without derivatives). If $z_i, j_i$ are non-constant elements in a differential field $K$ with $f(z_i, j_i) = 0$, then

$\text{td}_C C(\bar{z}, \bar{j}) \geq n + 1,$

unless for some $N, i, k$ we have $\Phi_N(j_i, j_k) = 0$.

4.2. The predimension. We consider a binary predicate $E_j(x, y)$ which will be interpreted in a differential field as the set of solutions of the equation $f(x, y) = 0$. This equation excludes the possibility of $x$ or $y$ being a constant. However, if we multiply $f(x, y)$ by a common denominator and make it a differential polynomial then $x$ and $y$ would be allowed to be constants as well. So we add $C^2$ to $E_j$, i.e. any pair of constants is in $E_j$. Further, let $E_j^\pm$ be the set of all $E_j$-points with no constant coordinate.

**Definition 4.3.** The theory $T^0_j$ consists of the following first-order statements about a structure $K$ in the language $\mathfrak{L}_j := \{ +, \cdot, E_j, 0, 1 \}$.

A1 $K$ is an algebraically closed field of characteristic 0.

A2 $C := C_K = \{ c \in K : E_j(0, c) \}$ is an algebraically closed subfield. Further, $C^2 \subseteq E_j(K)$ and if $(z, j) \in E_j(K)$ and one of $z, j$ is constant then both of them are constants.

A3 If $(z, j) \in E_j$ then for any $g \in \text{SL}_2(C)$, $(gz, j) \in E_j$. Conversely, if for some $j$ we have $(z_1, j), (z_2, j) \in E_j$ then $z_2 = gz_1$ for some $g \in \text{SL}_2(C)$.

A4 If $(z, j_1) \in E_j$ and $\Phi_N(j_1, j_2) = 0$ for some $j_2$ and some modular polynomial $\Phi_N(X, Y)$ then $(z, j_2) \in E_j$.

AS If $(z, j_1) \in E_j$, $i = 1, \ldots , n$, with

$\text{td}_C C(\bar{z}, \bar{j}) \leq n,$

then $\Phi_N(j_i, j_k) = 0$ for some $N$ and some $1 \leq i < k \leq n$, or $j_i \in C$ for some $i$.

**Definition 4.4.** An $E_j$-field is a model of $T^0_j$. If $K$ is an $E_j$-field, then a tuple $(\bar{z}, \bar{j}) \in K^{2n}$ is called an $E_j$-point if $(z_i, j_i) \in E_j(K)$ for each $i = 1, \ldots , n$. By abuse of notation, we let $E_j(K)$ denote the set of all $E_j$-points in $K^{2n}$ for any natural number $n$.

It is easy to see that reducts of differential fields to the language $\mathfrak{L}_j$ are $E_j$-fields.

Let $C$ be an algebraically closed field with $\text{td}(C/\mathbb{Q}) = \aleph_0$ and let $\mathfrak{C}$ consist of all $E_j$-fields $K$ with $C_K = C$. Note that $C$ is an $E_j$-field with $E_j(C) = C^2$ and it is the smallest structure in $\mathfrak{C}$. From now on, by an $E_j$-field we understand a member of $\mathfrak{C}$. Note that for some $X \subseteq A \in \mathfrak{C}$ we have $\langle X \rangle_A = C(X)^{\text{alg}}$ (with the induced structure from $A$) and $\mathfrak{C}_{f,g}$ consists of those $E_j$-fields that have finite transcendence degree over $C$. 
**Definition 4.5.** For $A \subseteq B \in \mathfrak{C}_{f,g}$, an $E_j$-basis of $B$ over $A$ is an $E_j$-point $\vec{b} = (\vec{z}, \vec{j})$ from $B$ of maximal length satisfying the following conditions:

- $j_i$ and $j_k$ are modularly independent for all $i \neq k$,
- $(z_i, j_i) \notin A^2$ for each $i$.

We let $\sigma(B/A)$ be the length of $\vec{j}$ in an $E_j$-basis of $B$ over $A$. When $A = C$ we write $\sigma(B)$ for $\sigma(B/C)$. It is easy to see that for $A \subseteq B \in \mathfrak{C}_{f,g}$, one has $\sigma(B/A) = \sigma(B) - \sigma(A)$. Further, for $A \in \mathfrak{C}_{f,g}$, define the predimension by

$$\delta(A) := \dim pr_{C}(A) - \sigma(A).$$

Note that the Ax-Schanuel inequality for $j$ implies that $\sigma$ is finite for finitely generated structures. Moreover, for $A, B \subseteq D \in \mathfrak{C}_{f,g}$, the inequality

$$\sigma(AB) \geq \sigma(A) + \sigma(B) - \sigma(A \cap B)$$

holds. Hence $\delta$ is submodular (so it is a predimension) and the Ax-Schanuel inequality states exactly that $\delta(A) \geq 0$ for all $A \in \mathfrak{C}_{f,g}$, with equality holding if and only if $A = C$. The dimension associated with $\delta$ will be denoted by $d$.

Observe also that for $A \subseteq B \in \mathfrak{C}_{f,g}$,

$$\delta(B/A) = \delta(B) - \delta(A) = \dim(B/A) - \sigma(B/A).$$

The class $\mathfrak{C}$ has the strong amalgamation property and satisfies all conditions of the amalgamation theorem and hence there is a strong Fraïssé limit $U$.

**Conjecture 4.6.** (Asl18). Let $K$ be a countable saturated differentially closed field. Then its reduct $K_{E_j}$ is isomorphic to $U$.

In the terminology of Asl18 this conjecture states that the Ax-Schanuel inequality for the differential equation of the $j$-function is (strongly) adequate. In the next section we will formulate an algebraic equivalent of this conjecture.

### 4.3. Existential closedness

Now we describe the Existential Closedness axiom scheme which, along with the above axioms, gives a complete axiomatisation of $\Th(U)$. We will also need it for our classification of strongly minimal sets in $K_{E_j}$.

**Definition 4.7.** Let $n$ be a positive integer, $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$. Denote $\vec{i} = (i_1, \ldots, i_k)$ and define the projection map $\text{pr}_{\vec{i}} : K^n \to K^k$ by

$$\text{pr}_{\vec{i}} : (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_k}).$$

Further, define (by abuse of notation) $\text{pr}_{\vec{i}} : K^{2n} \to K^{2k}$ by

$$\text{pr}_{\vec{i}} : (\vec{x}, \vec{y}) \mapsto (\text{pr}_{\vec{i}} \vec{x}, \text{pr}_{\vec{i}} \vec{y}).$$

It will be clear from the context in which sense $\text{pr}_{\vec{i}}$ should be understood (mostly in the second sense).

**Definition 4.8.** Let $K$ be an algebraically closed field. An irreducible algebraic variety $V \subseteq K^{2n}$ is normal if and only if for any $0 < k \leq n$ and any $1 \leq i_1 < \ldots < i_k \leq n$ we have $\dim \text{pr}_{\vec{i}}V \geq k$. We say $V$ is strongly normal if the strict inequality $\dim \text{pr}_{\vec{i}}V > k$ holds.

**Lemma 4.9.** If $A \subseteq B \in \mathfrak{C}_{f,g}$ and $\vec{b}$ is an $E_j$-basis of $B$ over $A$ then $\text{Loc}_A(\vec{b})$ is normal over $A$.

EC For each normal variety $V \subseteq K^{2n}$ the intersection $E_j(K) \cap V(K)$ is non-empty.
SEC For each normal variety $V \subseteq K^{2n}$ defined over a finite tuple $\vec{a} \subseteq K$, the intersection $E_j(K) \cap V(K)$ contains a point generic in $V$ over $\vec{a}$.

EC and SEC stand for Existential Closedness and Strong Existential Closedness respectively.

**Proposition 4.10.** The strong Fraïssé limit $U$ satisfies SEC (and hence EC).

In fact, all $\aleph_0$-saturated models of $T^{q+}\text{EC}$ satisfy SEC.
Conjecture 4.11. \( (E_j\text{-reducts of } f) \) differentially closed fields satisfy EC.

This is equivalent to Conjecture \( \mathbb{E}_j \). It states that if for a system of equations in \( K_{E_j} \), having a solution does not contradict Ax-Schanuel, then there is a solution. We will refer to both conjectures as the Existential Closedness conjecture or, briefly, EC conjecture.

4.4. Types in \( K_{E_j} \). In this and the next sections we assume the EC conjecture.

Lemma 4.12. Let \( \bar{a} \subseteq K \). If \((\bar{u}, \bar{v})\) is an \( E_j \)-basis of \([\bar{a}]\) then the latter is generated by \( \bar{a}, \bar{u}, \bar{v} \).

Proof. Let \( A = C(\bar{a}, \bar{u}, \bar{v})^{alg} \subseteq [\bar{a}] \). Then \( \text{td}([\bar{a}]/C) \geq \text{td}(A/C) \) and \( \sigma([\bar{a}]) = [\bar{v}] = \sigma(A) \). Hence \( \delta(A) \leq \delta([\bar{a}]) \) and so \( \delta(A) = \delta([\bar{a}]) \). Therefore \( \text{td}([\bar{a}]/C) = \text{td}(A/C) \) and \( A = [\bar{a}] \). \( \Box \)

Let \( \bar{a} = (a_1, \ldots, a_n) \in K^m \) be a tuple with \( d(\bar{a}) = k \) and \( b \in K \) with \( d(b/\bar{a}) = 0 \), i.e. \( b \in \text{cl}(\bar{a}) \). This means that \( d(ab) = k \). Pick an \( E_j \)-basis \((\bar{z}, \bar{j})\) of \( B := [\bar{a}, b] \) By Lemma 4.12 \( B = C(\bar{a}, b, \bar{z}, \bar{j})^{alg} \). We claim that \( b \in C(\bar{a}, \bar{z}, \bar{j})^{alg} \). Indeed, if it is not true then

\[
\text{dim} V_{\bar{a}, \bar{z}} = \text{td}(B/C) - \text{td}(\bar{a}/C) = \delta(B) + \sigma(B) - \text{td}(\bar{a}/C) = k + l - \text{td}(\bar{a}/C).
\]

Also, denote \( W := \text{Loc}_{\bar{a}}(\bar{c}) \). For each proper Zariski closed subvariety \( U \subseteq W \), defined over \( \mathbb{Q}(\bar{a}) \), and each positive integer \( N \) consider the formulae

\[
\xi_{U,N}(\bar{c}, \bar{u}, \bar{v}) := \left( \bar{c} \in C(\bar{c}) \cap (W \setminus U) \cap (\bar{u}, \bar{v}) \in V_{\bar{a}, \bar{z}} \cap \cap_{n=1}^{N} \cap_{i \neq r} \Xi_{n}(\bar{u}_i, v_r) \neq 0 \right),
\]

\[
\psi_{U,N}(\bar{c}, \bar{u}, \bar{v}, y) := \xi_{U,N}(\bar{c}, \bar{u}, \bar{v}) \wedge p(\bar{c}, \bar{e}, \bar{u}, \bar{v}, y) = 0,
\]

\[
\varphi_{U,N}(\bar{y}) := \exists \bar{c}, \bar{u}, \bar{v} \psi_{U,N}(\bar{c}, \bar{u}, \bar{v}, y).
\]

Observe that \( \varphi_{U,N} \) is defined over \( \bar{a} \) and \( \varphi_{U,N}(b) \) holds in \( K_{E_j} \).

Proposition 4.13. The formulae \( \varphi_{U,N} \) axiomatise the type \( \text{tp}(b/\bar{a}) \), that is, if for some \( b' \in K \) the formula \( \varphi_{U,N}(b') \) holds for each \( U \subseteq W \) and each \( N > 0 \) then \( \text{tp}(b/\bar{a}) = \text{tp}(b'/\bar{a}) \).

Proof. Consider the type \( q(\bar{c}, \bar{u}, \bar{v}) \) over \( \bar{a}, b' \) consisting of all formulae \( \psi_{U,N}(\bar{c}, \bar{u}, \bar{v}, b') \) for all \( U \subseteq V \) and \( N > 0 \). Then \( q \) is finitely satisfiable and hence there is a realisation of \( q \) in \( K_{E_j} \) which we denote by \( \bar{c}', \bar{z}', \bar{j}' \).

Observe that \( \bar{c}' \) is generic in \( W \) over \( \bar{a} \). Hence \( \bar{c} \) and \( \bar{c}' \) have the same algebraic type over \( \bar{a} \). In particular, \( \text{dim} V_{\bar{a}, \bar{z}'} = \text{dim} V_{\bar{a}, \bar{z}'} \). Further, \( \bar{j}'_1, \ldots, \bar{j}'_l \) are pairwise modularly independent. So if \( B' := C(\bar{a}, \bar{z}', \bar{j}')^{alg} \) then \( \sigma(B') \geq l \). On the other hand

\[
\text{td}(B'/C) = \text{td}(\bar{a}/C) + \text{td}(\bar{z}', \bar{j}'/C(\bar{a})) \leq \text{td}(\bar{a}/C) + \text{dim} V_{\bar{a}, \bar{z}'} = k + l,
\]

where the last equality follows from (4.5). Therefore \( \delta(B') \leq (k + l) - l = k \). However, \( B' \) contains \( \bar{a} \) and since \( \text{td}(\bar{a}) = k, \delta(B') \) cannot be smaller than \( k \). Thus, \( \delta(B') = k \) and \( \sigma(B') = l \) and the inequality in (1.10) must be an equality, i.e. \( \text{td}(\bar{z}', \bar{j}'/C(\bar{a})) = \text{dim} V_{\bar{a}, \bar{z}'} \). This means that \( \bar{z}', \bar{j}' \) is generic in \( V_{\bar{a}, \bar{z}'} \) over \( C(\bar{a}) \). Therefore, there is a field isomorphism \( \pi : B \to B' \) which fixes \( \bar{a} \) pointwise, fixes \( C \) setwise, sends \( \bar{c} \) to \( \bar{c}' \) and sends \( (\bar{z}, \bar{j}) \) to \( (\bar{z}', \bar{j}') \). Since \( \sigma(B') = l \), the tuple \( (\bar{z}', \bar{j}') \) is an \( E_j \)-basis of \( B' \) and \( \pi \) is an isomorphism of \( B \) and \( B' \) as \( E_j \)-fields.

Finally, as \( p(\bar{a}, \bar{e}, \bar{z}, \bar{j}, b) = 0 \) and \( p(\bar{a}, \bar{c}, \bar{z}', \bar{j}', b') = 0 \), we could have chosen \( \pi \) so that \( \pi(b) = \pi(b') \). Now both \( B \) and \( B' \) are strong in \( K_{E_j} \) and the latter is homogeneous with respect to strong substructures, hence \( \pi \) can be extended to an automorphism of \( K_{E_j} \). This shows that \( b \) and \( b' \) have the same type over \( \bar{a} \). \( \Box \)
Remark 4.14. In general, all types in $\mathcal{K}_{E_j}$ are determined by formulas of the above form and their negations. In particular, every formula is equivalent to a Boolean combination of existential formulas in $\mathcal{K}_{E_j}$ and hence its theory is nearly model complete.

**Theorem 4.15.** If $\bar{a} \subseteq K$ then $\text{acl}(C(\bar{a})) = C(\bar{a})^{\text{alg}}$.

**Proof.** It suffices to prove that for $\bar{a} \subseteq K$ we have $\text{acl}(\bar{a}) \subseteq C(\bar{a})^{\text{alg}}$. Assume $b \in \text{acl}(\bar{a})$. Then $d(b/\bar{a}) = 0$ and $\text{tp}(b/\bar{a})$ is determined by existential formulas $\varphi_{U,N}(y)$. Since $b \in \text{acl}(\bar{a})$, some formula $\varphi_{U,N}(y) \in \text{tp}(b/\bar{a})$ has finitely many realisations in $\mathcal{K}_{E_j}$.

We use the above notation. The point $(\bar{z}, \bar{j}) \in V_{\bar{a},\bar{c}}$ is generic over $\bar{a}, \bar{c}$. Observe that $(\bar{z}, \bar{j})$ must contain an $E_j$-basis of $A = C(\bar{a})^{\text{alg}}$. Denote it by $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}}) := ((\bar{z}, \bar{j}) \setminus (\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$, i.e. $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$ consists of all coordinates $(z_{\bar{i}}, j_i)$ of $(\bar{z}, \bar{j})$ for which $(z_{\bar{i}}, j_i) \notin A^2$. In other words, $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$ is an $E_j$-basis of $B = [A(b)]$ over $A$. Let $W$ be an irreducible component over $L := Q(\bar{a}, \bar{c}, \bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})^{\text{alg}}$ of the fibre of $V_{\bar{a},\bar{c}}$ above $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$ containing $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$. Then it is defined over $L$ and $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$ is generic in $W$ over $L$.

Since $\varphi_{U,N}(b)$ holds, in particular we have $p(\bar{a}, \bar{c}, \bar{z}, \bar{j}, b) = 0$. Assume

$$p(\bar{a}, \bar{c}, \bar{z}, \bar{j}, Y) = Y^n + s_{n-1}(\bar{z}_0, \bar{j}_0)Y^{n-1} + \cdots + s_0(\bar{z}_0, \bar{j}_0)$$

where each $s_i(X_1, X_2)$ is a rational functions over $L$. If for all $i$ $s_i(\bar{z}_0, \bar{j}_0) \in L$ then $b \in L \subseteq A$. Otherwise assume without loss of generality that $s_0(\bar{z}_0, \bar{j}_0) \notin L$.

Since $A \subseteq B$, by Lemma 4.9 $W$ is normal. By SEC there is a point $(\bar{z}_1, \bar{j}_1) \in W(K) \cap E_j(K)$ generic over $L(\bar{z}_0, \bar{j}_0)$. If $s_0(\bar{z}_1, \bar{j}_1) = s_0(\bar{z}_0, \bar{j}_0)$ then the function $s_0(X_1, X_2)$ is constant on $W$. On the other hand, $W$ is defined over $L$, so the constant value of $s_0(X_1, X_2)$ must belong to $L$. This is a contradiction, hence $s_0(\bar{z}_1, \bar{j}_1) \neq s_0(\bar{z}_0, \bar{j}_0)$. Now pick a generic point $(\bar{z}_2, \bar{j}_2) \in W(K) \cap E_j(K)$ over $L(\bar{z}_0, \bar{j}_0, \bar{z}_1, \bar{j}_1)$. By the above argument the elements $s_0(\bar{z}_0, \bar{j}_0), s_0(\bar{z}_1, \bar{j}_1), s_0(\bar{z}_2, \bar{j}_2)$ are pairwise distinct. Iterating this process we will construct a sequence $(\bar{z}_i, \bar{j}_i), i = 0, 1, 2, \ldots$ such that for each $i$

$$\mathcal{K}_{E_j} \models \xi_{U,N}(\bar{c}, \bar{z}_i, \bar{z}_i, \bar{j}_i)$$

and $s_0(\bar{z}_i, \bar{j}_i), i = 0, 1, 2, \ldots$ are pairwise distinct. This shows that the formula $\varphi_{U,N}(y)$ has infinitely many realisations (for there are only finitely many monic polynomials of a given degree the roots of which belong to a finite set of elements). This is a contradiction. □

**Corollary 4.16.** For any $\bar{a} \subseteq K$ we have $\text{acl}(\bar{a}) \subseteq [\bar{a}]$.

4.5. **Classification of strongly minimal sets in $\mathcal{K}_{E_j}$.** Recall that we assume the EC conjecture.

**Theorem 4.17.** Let $S \subseteq K$ be a strongly minimal set. Then either $S$ is geometrically trivial or $S \nsubseteq C$.

**Proof.** Assume $S$ is defined over $\bar{a}$ and denote $A := C(\bar{a})^{\text{alg}}$. Pick $b \in S \setminus \text{acl}(A)$ (if such an element does not exist then $S \nsubseteq C$). Denote $B' := [Ab]$ and let $\bar{z}_b \in K$ be such that $E_j(\bar{z}_b, b)$ holds. Now if $B = B(\bar{z}_b)$ (with the induced structure from $\mathcal{K}_{E_j}$) then $\delta(B) = \delta(B') = d(A)$ as $d(b/A) = 0$. Hence $B \subseteq K$. Choose a maximal $E_j$-field $A'$ with $A \subseteq A' \subseteq B$ such that $b \notin \text{acl}(A')$. Since strong minimality of a set and the nature of the geometry of a strongly minimal set do not depend on the choice of the set of parameters over which the set is defined, we may extend $A$ and assume $A' = A$. This means that if $e \in B \setminus A$ then $b \in \text{acl}(Ae)$. In particular, $\text{acl}(A) = A$.

Let $(\bar{z}, \bar{j}) \in B^{2l}$ be an $E_j$-basis of $B$ with $j_l = b$. Further, extending $\bar{a}$ we may assume that $V := \text{Loc}_A(\bar{z}, \bar{j}) \subseteq K^{2l}$ is defined over $\bar{a}$. Then $\text{tp}(b/A)$ is determined by the formulae

$$\chi_N(y) := \exists \bar{u}, \bar{v} \left( (\bar{u}, \bar{v}) \in V \cap E^N_j \land y = u_l \land \bigwedge_{n=1}^N \bigwedge_{i \neq r} \Phi_n(v_i, v_r) \neq 0 \right).$$

Now pick pairwise acl-independent elements $b_1, \ldots, b_l \in S \setminus A$. We will show that $b_i \notin \text{acl}(Ab_1 \ldots b_{i-1})$. Since $S$ is strongly minimal, $\text{tp}(b_i/A) = \text{tp}(b/A)$ for all $i$. By saturatedness
of $K_{E_j}$ for each $i$ there is $(\bar{z}^i, \bar{j}^i) \in V \cap E_j^\times$ such that $\bar{j}^i$ is pairwise modularly independent and $j^i_i = b_i$. Denote $B_i = A(\bar{z}^i, \bar{j}^i)^{\text{alg}}$.

It is clear that $\dim V = \text{td}_A(\bar{z}, \bar{j}) = \text{td}(B/A) = \delta(B/A) + \sigma(B/A)$. Therefore

$$\delta(B_i) = \text{td}(B_i/C) - \sigma(B_i) \leq \dim V + \text{td}(A/C) - l = \delta(B/A) + \sigma(B/A) + \delta(A) - l = \delta(B) = d(A),$$

and so $B_i \leq K$ and $(\bar{z}^i, \bar{j}^i)$ is an $E_j$-basis of $B_i$. We can conclude now that $[Ab_i] \subseteq B_i$, hence $\text{acl}(Ab_i) \subseteq B_i$. Moreover, as in the previous section there is an automorphism of $K_{E_j}$ over $A$ that maps $B$ onto $B_i$ (and maps $(\bar{z}, \bar{j})$ to $(\bar{z}^i, \bar{j}^i)$). In particular, for every $e \in B_i \setminus A$ we have $b_i \in \text{acl}(Ae)$.

We claim that $\bar{j}^i$ and $\bar{j}^m_k$ are modularly independent unless $(i, r) = (m, k)$ or $j^i_r, j^m_k \in A$. Assume for contradiction that for some $i \neq m$ the elements $j^i_r$ and $j^m_k$ are modularly dependent and $j^i_r \notin A$. Then $b_i \in \text{acl}(Aj^i_r) = \text{acl}(A j^m_k) \subseteq B_m$. Hence $b_m \in \text{acl}(Ab_i)$ which is a contradiction, for we assumed $b_i$'s are pairwise $\text{acl}$-independent. This shows in particular that $(t \geq 2) A \subseteq K$ as otherwise we would have $b \in [A]$ and $S \subseteq \text{acl}(Ab)$ in which case $S \not< C$.

Now let $\bar{B}_k := B_1 \ldots B_k$ be the $E_j$-subfield of $K_{E_j}$ generated by $B_1, \ldots, B_k$ where $k \leq t$. The above argument shows that

$$\sigma(\bar{B}_k/A) = \sum_{i=1}^k \sigma(B_i/A) = k \cdot \sigma(B/A).$$

By submodularity of $\delta$ we have

$$\delta(\bar{B}_k) \leq \delta(\bar{B}_{k-1}) + \delta(B_k) - \delta(\bar{B}_{k-1} \cap B_k)$$

for each $k$. Since $\delta(\bar{B}_{k-1} \cap B_k) \geq d(A)$, we can show by induction that $\delta(\bar{B}_k) = d(A)$ and $\bar{B}_k \leq K$. Thus,

$$td_C(\bar{B}_k) = \delta(\bar{B}_k) + \sigma(\bar{B}_k) = d(A) + \sigma(\bar{B}_k). \tag{4.7}$$

On the other hand, using submodularity of $\text{td}$ and $-\sigma$ we get by induction

$$td_C(\bar{B}_k) \leq td_C(\bar{B}_{k-1}) + td_C(B_k) - td_C(\bar{B}_{k-1} \cap B_k) = d(A) + \sigma(\bar{B}_{k-1}) + d(A) + \sigma(B_k) - \delta(\bar{B}_{k-1} \cap B_k) - \sigma(\bar{B}_{k-1} \cap B_k) \leq d(A) + \sigma(\bar{B}_k),$$

where $\delta(\bar{B}_{k-1} \cap B_k) \geq d(A)$ for $A \subseteq \bar{B}_{k-1} \cap B_k$. In fact we must have equalities everywhere in the above inequality due to (4.7). In particular,

$$\sigma((\bar{B}_{k-1} \cap B_k)/A) = \sigma(\bar{B}_{k-1}/A) + \sigma(B_k/A) - \sigma(\bar{B}_k/A) = 0.$$

So

$$\text{td}((\bar{B}_{k-1} \cap B_k)/A) = \delta((\bar{B}_{k-1} \cap B_k)/A) + \sigma((\bar{B}_{k-1} \cap B_k)/A) = 0.$$

This implies that $\bar{B}_{k-1} \cap B_k = A$. In particular, $b_i \notin \bar{B}_{i-1}$. On the other hand, $\text{acl}(Ab_1 \ldots b_{i-1}) \subseteq [Ab_1 \ldots b_{i-1}] \subseteq \bar{B}_{i-1}$. Thus, $b_i \notin \text{acl}(Ab_1 \ldots b_{i-1})$ as required. \hfill $\square$

We can also prove that some sets are strongly minimal. Let $A := C(\bar{a})^{\text{alg}} \leq K$. Assume $V \subseteq K^2$ is an algebraic curve defined over $A$, i.e. $\dim V = 1$. Consider the formula

$$\chi(y) := \exists \bar{u}, \bar{v} \ ((\bar{u}, \bar{v}) \in V \cap E_j^\times \land p(\bar{a}, \bar{u}, \bar{v}, y) = 0),$$

where $p$ is some irreducible algebraic polynomial.

**Proposition 4.18.** If $S := \chi(K_{E_j})$ is infinite then $S$ is strongly minimal.
Proof. We need to show that over any set of parameters all non-algebraic elements in $S$ realise the same type. By the stable embedding property we may choose all extra parameters from the set $S$ itself. Assume $e, e', b_1, \ldots, b_t \in S$ with $e, e' \notin A(\bar{b})^{\text{alg}}$. We will show that $\text{tp}(e/\text{A}(\bar{b})) = \text{tp}(e'/\text{A}(\bar{b}))$.

Choose existential witnesses $(z, j), (z', j'), (z_i, j_i) \in V(K) \cap E_j^*(K)$ for $\chi(e), \chi(e')$ and $\chi(b_i)$ respectively. Since $e \notin A(\bar{b})^{\text{alg}}$ and $p(\bar{a}, z, j, e) = 0$ and $\dim V = 1$, the point $(z, j)$ is generic in $V$ over $A(\bar{b})$. Similarly $(z', j')$ is generic in $V$. So $(z, j)$ and $(z', j')$ have the same algebraic type over $A(\bar{b})$. On the other hand, $\delta(\bar{b}/A) \leq 0$, therefore $\delta(\bar{b}/A) = 0$. Thus $\delta(e/\text{A}(\bar{b}) = \delta(e'/\text{A}(\bar{b}))$ and $(z, j)$ and $(z', j')$ form $E_j^*$-bases of $A(\bar{b}, e)^{\text{alg}}$ and $A(\bar{b}, e')^{\text{alg}}$ over $A(\bar{b})^{\text{alg}}$ respectively. Hence, as in the proof of Proposition 4.13, $e$ can be mapped to $e'$ by an automorphism of $K_{E_j}$ over $A(\bar{b})$. □

Remark 4.19. When $A$ is not strong in $K$ we may actually work over $[A]$ since strong minimality of a set does not depend on the choice of the set of parameters over which the set is defined. Hence the assumption $A \leq K$ does not restrict generality.

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