The coarse Pimsner-Voiculescu sequence

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Abstract

We derive the Pimsner-Voiculescu sequence calculating the $K$-theory of a $C^*$-algebra with $\mathbb{Z}$-action using constructions with equivariant coarse $K$-homology theory. We then investigate to which extend this idea extends to more general equivariant coarse homology theories.

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1 Introduction

If A is a $C^*$-algebra with an action of the group $\mathbb{Z}$, then the Pimsner-Voiculescu sequence \[ PV80, \text{ [Bla98, 10.2.1]} \] is the long exact sequence of $K$-theory groups

\[ K_{*+1}(A \rtimes \mathbb{Z}) \to K_*(\text{Res}^{\mathbb{Z}}(A)) \xrightarrow{1-\zeta_*} K_*(\text{Res}^{\mathbb{Z}}(A)) \to K_*(A \rtimes \mathbb{Z}) , \]  

(1.1)

where $\text{Res}^{\mathbb{Z}}$ denotes the operation of forgetting the $\mathbb{Z}$-action, and $\sigma_*$ is induced by the $\mathbb{Z}$-action on $A$ via functoriality. It is the long exact sequence of homotopy groups associated to a fibre sequence of $K$-theory spectra

\[ \begin{array}{ccc}
K(\text{Res}^{\mathbb{Z}}(A)) & \longrightarrow & K(A \rtimes \mathbb{Z}) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K(\text{Res}^{\mathbb{Z}}(A))
\end{array} \]  

(1.2)

The classical arguments derive such a fibre sequence of spectra from a suitably designed short exact sequence of $C^*$-algebras. Thereby different proofs work with different sequences. They all give the long exact sequence (1.1) of homotopy groups, but the question whether the resulting fibre sequences of spectra are equivalent has not been discussed in the literature so far.

The Baum-Connes conjecture with coefficients for the amenable group $\mathbb{Z}$ is known to hold and asserts that the assembly map

\[ \mu_{\text{Fin}, A}^{K_{\text{asp}}} : \text{KK}^\mathbb{Z}(C_0(\mathbb{R}), A) \to K(A \rtimes \mathbb{Z}) \]  

(1.3)

introduced by Kasparov \[ \text{Kas88} \] (see \[ \text{BELa, 13.9} \] for the spectrum level version) is an equivalence. Accepting to use the Baum-Connes conjecture with coefficients for $\mathbb{Z}$ a fibre sequence of the form (1.2) can be derived in the following two ways.

Observe that $\text{KK}^\mathbb{Z}(C_0(-), A)$ is a locally finite $\mathbb{Z}$-equivariant homology theory. Using the $\mathbb{Z}$-CW-structure of $\mathbb{R}$ with precisely two $\mathbb{Z}$-cells in dimension 0 and 1 and the identification $\text{KK}^\mathbb{Z}(C_0(\mathbb{Z}), A) \simeq K(\text{Res}^\mathbb{Z}(A))$ (use \[ \text{BELa Cor. 1.23} \]) we get a fibre sequence of the form (1.2).

Alternatively, to $A$ one can associate a Davis-Lück type functor $K^\mathbb{Z}_A : \text{ZOrb} \to \text{Sp} \ [\text{Kra20}]$ (see also \[ \text{BELa Def. 16.15} \]) whose values on the $\mathbb{Z}$-orbits $Z$ and $\ast$ are given by

\[ K(\text{Res}^\mathbb{Z}(A)) \simeq K^\mathbb{Z}_A(Z) , \quad K(A \rtimes Z) \simeq K^\mathbb{Z}_A(\ast) . \]  

(1.4)

If we equip $K(\text{Res}^\mathbb{Z}(A))$ with the $\mathbb{Z}$-action induced by functoriality from the $\mathbb{Z}$-action on $A$, and $K^\mathbb{Z}_A(Z)$ with the $\mathbb{Z}$-action induced by functoriality from the $\mathbb{Z}$-action on the orbit $Z$, then the first equivalence in (1.4) is $\mathbb{Z}$-equivariant. The Davis-Lück assembly map (see (1.7) below) turns out to be equivalent to the map

\[ \text{colim}_{\mathbb{Z}} K(A) \to K(A \rtimes \mathbb{Z}) . \]  

(1.5)
Since Kasparov’s assembly map (1.3) and the Davis-Lück assembly map are isomorphic on the level of homotopy groups [Kra20], if one of them is an equivalence, then so is the other. Therefore the Baum-Connes conjecture with coefficients for \( \mathbb{Z} \), stating that Kasparov’s assembly map is an equivalence, also implies that (1.5) is an equivalence. The usual cofibre sequence calculating the coinvariants of a \( \mathbb{Z} \)-object in a stable \( \infty \)-category specializes to

\[
K(\text{Res}^\mathbb{Z}(A)) \xrightarrow{1 - \sigma} K(\text{Res}^\mathbb{Z}(A)) \to K(A \rtimes \mathbb{Z}),
\]

(we write the square (1.2) in this form in order to be able to highlight the map \( 1 - \sigma \)), where \( \sigma \) denotes the action of the generator of \( \mathbb{Z} \) on \( K(\text{Res}^\mathbb{Z}(A)) \). On the level of homotopy groups this cofibre sequence induces the PV-sequence (1.1) including the calculation of the map in the middle.

One outcome of this note is an alternative construction of a square of the form (1.2) in terms of the \( \mathbb{Z} \)-equivariant coarse the \( K \)-homology theory. In particular, in our construction the maps involved in this square acquire a clear geometric interpretation.

The symmetric monoidal category \((\mathbb{Z} \text{BC}, \otimes)\) of \( \mathbb{Z} \)-bornological coarse spaces and the notion of a \( \mathbb{Z} \)-equivariant coarse homology theory have been introduced in [BEKW20b]. Examples of \( \mathbb{Z} \)-bornological coarse spaces are \( \mathbb{Z}_{\text{min,min}} \) and \( \mathbb{Z}_{\text{can,min}} \) given by the group \( \mathbb{Z} \) with the minimal bornology and the minimal or canonical coarse structures. The additional structure of transfers was introduced in [BEKW20a]. For every \( \mathbb{Z} \)-equivariant coarse homology theory \( E^\mathbb{Z} : \mathbb{Z} \text{BC} \to \text{M} \) with transfers and \( \mathbb{Z} \)-bornological coarse space \( X \) in Section 2 we will construct a commutative square

\[
\begin{array}{ccc}
E^\mathbb{Z}(X \otimes \mathbb{Z}_{\text{min,min}}) & \xrightarrow{\epsilon} & E^\mathbb{Z}(X \otimes \mathbb{Z}_{\text{can,min}}) \\
\downarrow & & \downarrow \text{tr} \\
0 & \longrightarrow & E^\mathbb{Z}(X \otimes \mathbb{Z}_{\text{can,min}} \otimes \mathbb{Z}_{\text{min,min}})
\end{array}
\]

(1.6)

which we call the coarse PV-square associated to \( E^\mathbb{Z} \) and \( X \). The morphism \( \epsilon \) in (1.6) is induced by the morphism of bornological coarse spaces \( \mathbb{Z}_{\text{min,min}} \to \mathbb{Z}_{\text{can,min}} \) given by the identity of the underlying sets, and the morphism \( \text{tr} \) is the transfer morphism induced by the coarse covering \( \mathbb{Z}_{\text{min,min}} \to \ast \). The filler of the square is given by a simple argument using the properties of coarse homology theories.

We then study conditions on \( E^\mathbb{Z} \) and \( X \) which imply that the square (1.6) cartesian.

In order to formulate one such condition, for any \( \mathbb{Z} \)-equivariant coarse homology theory \( F^\mathbb{Z} : \mathbb{Z} \text{Orb} \to \text{M} \) we form the functor

\[
HF^\mathbb{Z} : \mathbb{Z} \text{Orb} \to \text{M}, \quad S \mapsto F^\mathbb{Z}(S_{\text{min,max}} \otimes \mathbb{Z}_{\text{can,min}})
\]

and consider the Davis-Lück assembly map

\[
\mu_H^{DL} : \varprojlim \mathbb{Z} \text{Fin} \text{Orb} \xrightarrow{\mu^\mathbb{Z}_{HF}} \colim_{\mathbb{Z} \text{Fin} \text{Orb}} HF^\mathbb{Z} \to HF^\mathbb{Z}(\ast)
\]

(1.7)
for the family $\text{Fin}$ of finite subgroups.

We form the $\mathbb{Z}$-equivariant coarse homology theory $E_{Z,c}^\mathbb{Z} : \mathbb{Z}\text{BC} \to \mathcal{M}$ given by the continuous approximation of the $\mathbb{Z}(X \otimes -) : \mathbb{Z}\text{BC} \to \mathcal{M}$ of $E^\mathbb{Z}$ by $X$.

**Theorem 1.1 (Theorem 3.2).** Assume:

1. $X$ has the minimal bornology and is discrete.
2. $E^\mathbb{Z}$ is strong and strongly additive.

Then the PV-square (2.4) is cartesian if and only if $\mu_{HEX,c}^{DL}$ is an equivalence.

We refer to Section 2 for an explanation and references concerning the additional properties of $\mathbb{Z}$-equivariant coarse homology theories appearing in the above statement. The Condition 1.1.2 on $E^\mathbb{Z}$ is satisfied for many examples of equivariant coarse homology theories, e.g. the coarse topological $K$-theory (1.9) with coefficients in a $C^*$-category with $\mathbb{Z}$-action, or the coarse algebraic $K$-homology associated to an additive category or a left-exact $\infty$-category with $\mathbb{Z}$-action. The first is the main example for the present paper, and we refer to Example 3.17 for detailed references for the second case and to [BCKW] for the third. The condition on $\mu_{HEX,c}^{DL}$ is complicated and not always satisfied, see Example 3.17.

At a first glance the Condition 1.1.1 on $X$ is very restrictive, but using that the corners of the square (1.6) are $\mathbb{Z}$-equivariant coarse homology theories in the variable $X$ one can extend the range of $\mathbb{Z}$-bornological spaces $X$ for which this square is known to be cartesian considerably. Let $Y_0^* : \mathbb{Z}\text{BC} \to \mathcal{Z}\text{Sp}\mathcal{X}$ be the universal $\mathbb{Z}$-equivariant coarse homology theory. Then the property that (1.6) is cartesian only depends on the image $Y_0^*\text{loc}(X) := \ell(Y_0^*(X))$ of $Y_0^*(X)$ under the localization $\ell : \mathcal{Z}\text{Sp}\mathcal{X} \to \mathcal{Z}\text{Sp}\mathcal{X}_{\text{loc}}$ of $\mathcal{Z}\text{Sp}\mathcal{X}$ at the three $\mathbb{Z}$-equivariant coarse homology theories $E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{min,min}})$, $E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{can,min}})$ and $E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{can,min}} \otimes \mathbb{Z}_{\text{min,min}})$ appearing at the corners of (1.6). In Definition 3.14 we introduce $\mathcal{Z}\text{Sp}\mathcal{X}_{\text{loc}}(DL)$ as the localizing subcategory of $\mathcal{Z}\text{Sp}\mathcal{X}_{\text{loc}}$ generated by $Y_0^*\text{loc}(X)$ for all $X$ in $\mathbb{Z}\text{BC}$ which are discrete, have the minimal bornology, and are such that $\mu_{HEX,c}^{DL}$ is an equivalence. Then for $X$ in $\mathbb{Z}\text{BC}$ we have the following consequence of Theorem 1.1.

**Corollary 1.2 (Corollary 3.15).** Assume that $E^\mathbb{Z}$ is strong and strongly additive. If $Y_0^*\text{loc}(X) \in \mathcal{Z}\text{Sp}\mathcal{X}_{\text{loc}}(DL)$, then the PV-square (2.4) is cartesian.

There are many $\mathbb{Z}$-bornological coarse spaces $X$ which are not necessarily discrete or have the minimal bornology but still satisfy $Y_0^*\text{loc}(X) \in \mathcal{Z}\text{Sp}\mathcal{X}_{\text{loc}}(DL)$. It is even not clear that for such spaces the Davis-Lück assembly map $\mu_{HEX,c}^{DL}$ is an equivalence, but nevertheless the PV-square (2.4) is cartesian.
In order to connect the square (1.6) with the classical PV-sequence (1.2) for a $\mathbb{Z}$-$C^*$-algebra $A$ we take $E^\mathbb{Z} = K\chi^\mathbb{Z}_{\text{Hilb},(A)}$, the coarse algebraic $K$-homology (1.9) with coefficients in the $C^*$-category $\text{Hilb}_c(A)$ of Hilbert $A$-modules and compact operators [BEb], and $X = \ast$. Then $E^\mathbb{Z}$ and $X$ satisfy the assumptions of Theorem 1.1. We further employ the fact that group $\mathbb{Z}$ satisfies the Baum-Connes conjecture with coefficients in order to verify that $\mu^{DL}_{HK,X^\mathbb{Z}}$ is an equivalence. In view of the obvious equivalence $\mu^{DL}_{HK,X^\mathbb{Z}} \simeq \mu^{DL}_{HK,X^\mathbb{Z},,c}$, by Theorem 1.1 the coarse PV-square

$$
\begin{array}{ccc}
K\chi^\mathbb{Z}_{\text{Hilb},(A)}(Z_{\text{min},\text{min}}) & \xrightarrow{\iota} & K\chi^\mathbb{Z}_{\text{Hilb},(A)}(Z_{\text{can},\text{min}}) \\
\downarrow & & \downarrow \text{tr} \\
0 & \rightarrow & K\chi^\mathbb{Z}_{\text{Hilb},(A)}(Z_{\text{can},\text{min}} \otimes Z_{\text{min},\text{min}})
\end{array}
$$

is cartesian. In Proposition 4.5 we explain that this square is equivalent to a square of the form (1.2). We further determine the boundary map explicitly. We thus get a new construction of a Pimsner-Voiculescu sequence.

Note that $\text{Hilb}_c(A)$ for a $\mathbb{Z}$-$C^*$-algebra $A$ is just a particular example of a $C^*$-category with a strict $\mathbb{Z}$-action. More generally, if $C$ is any $C^*$-category with a strict $\mathbb{Z}$-action which admits small orthogonal AV-sums, then we have the $\mathbb{Z}$-equivariant coarse homology theory with transfers

$$K\chi^\mathbb{Z}_C: \mathbb{Z}\text{BC} \rightarrow \text{Sp} \hspace{1cm} (1.9)$$

constructed in [BEb]. For every $X$ in $\mathbb{Z}\text{BC}$, by specializing (1.6), we obtain the coarse PV-square

$$
\begin{array}{ccc}
K\chi^\mathbb{Z}_C(X \otimes Z_{\text{min},\text{min}}) & \xrightarrow{\iota} & K\chi^\mathbb{Z}_C(X \otimes Z_{\text{can},\text{min}}) \\
\downarrow & & \downarrow \text{tr} \\
0 & \rightarrow & K\chi^\mathbb{Z}_C(X \otimes Z_{\text{can},\text{min}} \otimes Z_{\text{min},\text{min}})
\end{array}
$$

(1.10)

If $X$ is discrete and has the minimal bornology, i.e., $X \cong Y_{\text{min},\text{min}}$ for some $\mathbb{Z}$-set $Y$, then by Proposition 1.3 we have an equivalence of $\mathbb{Z}$-equivariant coarse homology theories

$$K\chi^\mathbb{Z}_{C,X,c} \simeq K\chi^\mathbb{Z}_{C,Y} ,$$

where $C_Y$ is a suitably defined $C^*$-category with $\mathbb{Z}$-action depending on $Y$. We can therefore reduce the problem of showing that (1.10) is cartesian to the case of $X = \ast$ at the cost of modifying the coefficient $C^*$-category. Using the Baum-Connes conjecture with coefficients for $\mathbb{Z}$ we can then check that $\mu^{DL}_{HK,X^\mathbb{Z}}$ and hence $\mu^{DL}_{HK,X^\mathbb{Z},,c}$ are equivalences so that $Y_{\text{loc}}(\mathbb{Z}\text{Sp})_\mathbb{Z}\chi_\mathbb{Z}^\mathbb{Z}(DL)$ by Theorem 1.1.

Let $\mathbb{Z}\text{Sp}\chi_\mathbb{Z}^\mathbb{Z}(\text{disc})$ be the localizing subcategory of $\mathbb{Z}\text{Sp}\chi_\mathbb{Z}^\mathbb{Z}$ generated by all $X$ in $\mathbb{Z}\text{BC}$ which are discrete and have the minimal bornology.

**Theorem 1.3** (Theorem 1.2). Assume that $E^\mathbb{Z} = K\chi^\mathbb{Z}_C$. Then

$$\mathbb{Z}\text{Sp}\chi_\mathbb{Z}^\mathbb{Z}(\text{disc}) \subseteq \mathbb{Z}\text{Sp}\chi_\mathbb{Z}^\mathbb{Z}(DL) \hspace{1cm} .$$
For a general $X$ in $\mathbf{ZBC}$ we then apply Corollary 1.2 in order to conclude that (1.10) is cartesian provided $Y_{\text{loc}}^s(X) \in \mathbf{ZSp} \mathfrak{X}_{\text{loc}}(\text{disc})$, see Corollary 4.4.

We now restrict to bornological coarse spaces with trivial $\mathbf{Z}$-action and provide very general conditions on $X$ ensuring that $Y_{\text{loc}}^s(X) \in \mathbf{ZSp} \mathfrak{X}_{\text{loc}}(\text{disc})$. In order to state the result we employ the coarse assembly map

$$\mu_{F,X} : F \mathcal{O}^\infty \mathcal{P}(X) \to F(X) \quad (1.11)$$

for a strong coarse homology theory $F : \mathbf{BC} \to \mathbf{M}$ and a bornological coarse space $X$ which has been introduced in [BE20a, Def. 9.7]. Recall the notion of weakly finite asymptotic dimension [BE20a, Def. 10.3] and bounded geometry [BE20b, Def. 7.77]. In the following we consider the functors $K \mathfrak{X}_C^E(- \otimes \mathbf{Z}_{\text{min},\text{min}})$, $K \mathfrak{X}_C^E(- \otimes \mathbf{Z}_{\text{can},\text{min}})$ and $K \mathfrak{X}_C^E(- \otimes \mathbf{Z}_{\text{can},\text{min}} \otimes \mathbf{Z}_{\text{min},\text{min}})$ from $\mathbf{BC}$ to $\mathbf{Sp}$ as strong non-equivariant coarse homology theories.

**Theorem 1.4** (Theorem 5.1). Assume one of the following:

1. $X$ has weakly finite asymptotic dimension.

2. $X$ has bounded geometry and the three coarse assembly maps $\mu_{K \mathfrak{X}_C^E(- \otimes \mathbf{Z}_{\text{min},\text{min}}),X}$, $\mu_{K \mathfrak{X}_C^E(- \otimes \mathbf{Z}_{\text{can},\text{min}}),X}$ and $\mu_{K \mathfrak{X}_C^E(- \otimes \mathbf{Z}_{\text{can},\text{min}} \otimes \mathbf{Z}_{\text{min},\text{min}}),X}$ are equivalences.

Then $Y_{\text{loc}}^s(X) \in \mathbf{ZSp} \mathfrak{X}_{\text{loc}}(\text{disc})$ and hence (1.10) is cartesian.

It could be true that the coarse PV-square (1.10) is cartesian for all $X$ in $\mathbf{BC}$.

The Condition 1.4.1 on $X$ in implies that the coarse assembly map $\mu_{F,X}$ is an equivalence for any strong coarse homology theory. But for the coarse topological $K$-theory $K \mathfrak{X}_{\text{Hilb,}(\mathcal{C})}$ the coarse assembly map is an equivalence for a much bigger class of bornological coarse spaces, e.g. discrete metric spaces of bounded geometry which admit a coarse embedding into a Hilbert space [Yu00, Thm. 1.1]. We therefore expect that the Assumption 1.4.2 is satisfied for many spaces not having finite asymptotic dimension.

We consider this note as a possibility to demonstrate the usage of results of coarse homotopy homotopy as developed in [BE20b], [BE20a], [BEKW20b], [BEb] and [BELa].

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2 The coarse PV-square

A $\mathbb{Z}$-equivariant $\mathcal{M}$-valued coarse homology theory is a functor

$$E^\mathbb{Z} : \mathcal{ZBC} \to \mathcal{M}$$

from the category $\mathcal{ZBC}$ of $\mathbb{Z}$-bornological coarse spaces to a cocomplete stable $\infty$-category $\mathcal{M}$ such that the functor $E^\mathbb{Z}$ is coarsely invariant, excisive, $u$-continuous and vanishes on flasques [BEKW20b, Def. 3.10]. There exists a universal $\mathbb{Z}$-equivariant coarse homology theory [BEKW20b, Def. 4.9]

$$\text{Yo}^\text{e} : \mathcal{ZBC} \to \mathcal{ZSpX},$$

where $\mathcal{ZSpX}$ is a presentable stable $\infty$-category called the $\infty$-category of coarse motivic spectra.

The category $\mathcal{ZBC}$ admits a symmetric monoidal structure $\otimes$ described in [BEKW20b, Ex. 2.17]. It induces a presentably symmetric monoidal structure on $\mathcal{ZSpX}$ such that functor $\text{Yo}^\text{e}$ refines essentially uniquely to a symmetric monoidal functor [BEKW20b, Sec. 4.3].

If $E^\mathbb{Z} : \mathcal{ZBC} \to \mathcal{M}$ is a $\mathbb{Z}$-equivariant coarse homology theory, then by [BEKW20b, Cor. 4.10] it essentially uniquely factorizes as the composition of $\text{Yo}^\text{e}$ and a colimit-preserving functor $E^\mathbb{Z} : \mathcal{ZSpX} \to \mathcal{M}$ denoted by the same symbol.

If $X$ is an object of $\mathcal{ZBC}$, then the functor $E^\mathbb{Z}(X \otimes -) : \mathcal{ZBC} \to \mathcal{M}$ is again a $\mathbb{Z}$-equivariant coarse homology theory called the twist of $E^\mathbb{Z}$ by $X$.

We will encounter additional conditions and structures on $E$:

1. continuity [BEKW20b] Def. 5.19
2. strongness [BEKW20b] Def. 4.19
3. strong additivity [BEKW20b] Def. 3.12
4. transfers [BEKW20a] Def. 2.53).

For any $\mathbb{Z}$-set $Y$ we can form the objects $Y_{\text{min, max}}$ and $Y_{\text{min, min}}$ in $\mathcal{ZBC}$ obtained by equipping $Y$ with the minimal coarse structure (whose maximal entourage is $\text{diag}(Y)$) and the minimal bornology (consisting of the finite subsets) or the maximal bornology (consisting of all subsets), respectively. The group $\mathbb{Z}$ has a canonical $\mathbb{Z}$-coarse structure generated by the entourages $U_r := \{(n,m) \in \mathbb{Z} \times \mathbb{Z} | |n-m| \leq r\}$ for all $r$ in $\mathbb{N}$. We let $\mathbb{Z}_{\text{can, min}}$ in $\mathcal{ZBC}$ denote the corresponding object.

Let $E^\mathbb{Z} : \mathcal{ZBC} \to \mathcal{M}$ be an equivariant coarse homology theory with transfers. We start
with describing a commutative square
\[
\begin{array}{c}
E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{min}, \text{min}}) \\
\downarrow \text{tr}
\end{array} \quad \xrightarrow{\iota} \quad \begin{array}{c}
E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{can}, \text{min}}) \\
\downarrow \text{tr}
\end{array} \quad (2.1)
\]
of \( M \)-valued \( \mathbb{Z} \)-equivariant coarse homology theories. The map \( \iota \) is induced by the morphism \( \mathbb{Z}_{\text{min}, \text{min}} \to \mathbb{Z}_{\text{can}, \text{min}} \) in \( Z_{\text{BC}} \) given by the identity of the underlying sets. The map \( \text{tr} \) is the transfer along the coarse covering \( \mathbb{Z}_{\text{min}, \text{min}} \to * \).

**Lemma 2.1.** The square \((2.1)\) commutes.

**Proof.** We have the following commutative diagram
\[
\begin{array}{cccc}
E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{min}, \text{min}}) & \xrightarrow{\iota} & E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{can}, \text{min}}) & \xrightarrow{\text{tr}} \\
\downarrow \text{tr} & & \downarrow \text{tr} & \\
E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{min}, \text{min}} \otimes \mathbb{Z}_{\text{min}, \text{min}}) & \xrightarrow{\iota \simeq} & E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{can}, \text{min}} \otimes \mathbb{Z}_{\text{min}, \text{min}}) & \xrightarrow{\text{tr} \simeq} \\
\downarrow \partial_{\text{MV}} \simeq 0 & & \downarrow \partial_{\text{MV}} \simeq \Sigma & \\
\Sigma E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{min}, \text{min}}) & \xrightarrow{!} & \Sigma E^\mathbb{Z}(- \otimes \mathbb{Z}_{\text{min}, \text{min}}) & \\
\end{array}
\]
The upper three horizontal maps are all induced from \( \mathbb{Z}_{\text{min}, \text{min}} \to \mathbb{Z}_{\text{can}, \text{min}} \). The map marked by \( ! \) is induced by the equivariant map of \( \mathbb{Z} \)-sets
\[
\begin{array}{c}
X \times \mathbb{Z} \times \mathbb{Z} \to X \times \text{Res}^\mathbb{Z}(\mathbb{Z}_{\text{min}, \text{min}}) \times \mathbb{Z} \ ,
\end{array} \quad (x, n, m) \mapsto (x, n - m, m) .
\]
The morphism \( \partial_{\text{MV}} \) is the Mayer-Vietoris boundary map associated to the decomposition of \( \text{Res}^\mathbb{Z}(\mathbb{Z}) \) into \( (\mathbb{N}, -\mathbb{N}) \). Its left instance vanishes since this decomposition of \( \text{Res}^\mathbb{Z}(\mathbb{Z}_{\text{min}, \text{min}}) \) is coarsely disjoint. Using the Mayer-Vietoris sequence we see that right instance of \( \partial_{\text{MV}} \) is an equivalence since the subsets \( \pm \mathbb{N} \) of \( \text{Res}^\mathbb{Z}(\mathbb{Z}_{\text{can}, \text{min}}) \) are flasque. The commutativity of the upper square encodes the naturality of the transfer, and the commutativity of the lower square reflects the naturality of the Mayer-Vietoris boundary. The middle square arises from applying \( E^\mathbb{Z} \) to a commutative square in \( Z_{\text{BC}} \) and therefore commutes, too.

The commutative diagram \((2.2)\) yields a filler of \((2.1)\). \( \square \)

We now fix \( X \) in \( Z_{\text{BC}} \) and consider the \( \mathbb{Z} \)-equivariant coarse homology theory
\[
E^\mathbb{Z}_X(-) := E^\mathbb{Z}(X \otimes -) : Z_{\text{BC}} \to M \quad (2.3)
\]
obtained from \( E^\mathbb{Z} \) by twisting with \( X \).
Definition 2.2. The commutative square

\[
\begin{array}{ccc}
E_X^Z(\mathbb{Z}_{\text{min,min}}) & \xrightarrow{i} & E_X^Z(\mathbb{Z}_{\text{can,min}}) \\
\downarrow & & \downarrow \tau \\
0 & \rightarrow & E_X^Z(\mathbb{Z}_{\text{can,min}} \otimes \mathbb{Z}_{\text{min,min}})
\end{array}
\]

(2.4)

is called the coarse PV-square associated to \(E^Z\) and \(X\).

3 The coarse PV-sequence

In this section we are interested in conditions on \(E^Z\) and \(X\) ensuring that the coarse PV-square (2.4) is cartesian, i.e. that

\[
E_X^Z(\mathbb{Z}_{\text{min,min}}) \xrightarrow{i} E_X^Z(\mathbb{Z}_{\text{can,min}}) \xrightarrow{\tau} E_X^Z(\mathbb{Z}_{\text{can,min}} \otimes \mathbb{Z}_{\text{min,min}})
\]

(3.1)

is a part of a fibre sequence. In this case it will be called the coarse PV-sequence.

Let \(i: \mathbb{Z}BC_{\text{min}} \rightarrow \mathbb{Z}BC\) denote the inclusion of the full subcategory of \(\mathbb{Z}\)-bornological coarse spaces with the minimal bornology. We let \(i^*\) and \(i!\) denote the operations of restriction and left Kan extension along \(i\). Recall from [BEKW20b, Sec. 5.4] that a coarse homology theory \(F^Z : \mathbb{Z}BC \rightarrow \text{Sp}\) is continuous if the canonical transformation

\[
i!i^* F^Z \rightarrow F^Z
\]

(3.2)

is an equivalence.

Definition 3.1. We call \(F_{\text{c}}^Z := i!i^* F^Z\) the continuous approximation of \(F^Z\).

We apply this construction to the functor \(E^Z_X\) from (2.3) and get a continuous equivariant coarse homology theory \(E_{X,c}^Z\).

The functor \(E_{X,c}^Z\) gives rise to a functor

\[
H E_{X,c}^Z : \text{ZOrb} \rightarrow \text{M}, \quad S \mapsto E_{X,c}^Z(S_{\text{min,max}} \otimes \mathbb{Z}_{\text{can,min}})
\]

from the orbit category of \(\mathbb{Z}\) to \(\text{M}\) and an associated Davis-Lück assembly map \(\mu_{HE_{X,c}^Z}\) given in (1.7). Unfolding the definition, the Davis-Lück assembly map is equivalent to the map

\[
\colim_{BZ} E_{X,c}^Z(\mathbb{Z}_{\text{min,max}} \otimes \mathbb{Z}_{\text{can,min}}) \rightarrow E_{X,c}^Z(\mathbb{Z}_{\text{can,min}})
\]

(3.3)

induced by the projections \(\mathbb{Z}_{\text{min,max}} \rightarrow \ast\), where \(\mathbb{Z}'\) acts on \(\mathbb{Z}_{\text{min,max}}\) by translations.

The main theorem of the present section is:
**Theorem 3.2.** Assume:

1. $X$ has the minimal bornology and is discrete.
2. $E^Z$ is strong and strongly additive.

Then the PV-square (2.4) is cartesian if and only if $\mu^{DL}_{HE,c}$ is an equivalence.

**Proof.** We let $Z'$ be a second copy of $Z$ which acts on $Z_{min,min}$ by translations. Then $Z'$ acts by functoriality on the coarse homology theory $E^Z_X(- \otimes Z_{min,min})$. The coarse covering $Z_{min,min} \to *$ in $Z'$-equivariant, where $Z'$ acts trivially on *. As a consequence, the transfer map $\text{tr}$ for $E^Z_X$ along the coarse coverings $- \otimes Z_{min,min} \to -$ has a factorization

\[
\text{tr} : E^Z_X(-) \xrightarrow{\text{coass}_X} \lim_{Z'} E^Z_X(- \otimes Z_{min,min}) \xrightarrow{\text{ev}} E^Z_X(- \otimes Z_{min,min}).
\]

**Definition 3.3.** For $Z$ in $Z\text{BC}$ we call

\[
\text{coass}_{X,Z} : E^Z_X(Z) \to \lim_{Z'} E^Z_X(Z \otimes Z_{min,min})
\]

the coassembly map for the object $Z$.

In a stable $\infty$-category like $M$ finite limits commute with colimits. Since $\lim_{Z'}$ is a finite limit the functor $\lim_{Z'} E^Z_X(- \otimes Z_{min,min})$ is again a $Z$-equivariant coarse homology theory. We will keep $X$ in the notation since later we will study properties of the coassembly map which may depend on the choice of $X$.

**Proposition 3.4.** Assume that $E^Z$ is strong and the coassembly maps $\text{coass}_{X,Z_{min,min}}$ is an equivalence. Then the following assertions are equivalent:

1. The coassembly map $\text{coass}_{X,Z_{can,min}}$ is an equivalence.
2. The coarse PV-square (2.4) is cartesian.

**Proof.** We consider the commutative diagram

\[
\begin{array}{c}
E^Z_X(Z_{min,min}) \xrightarrow{l} E^Z_X(Z_{can,min}) \\
\downarrow \text{coass}_{X,Z_{min,min}} \quad \downarrow \text{coass}_{X,Z_{can,min}} \\
\lim_{Z'} E^Z_X(Z_{min,min} \otimes Z_{min,min}) \xrightarrow{\text{lim}_{Z'}} \lim_{Z'} E^Z_X(Z_{can,min} \otimes Z_{min,min}) \xrightarrow{\text{ev}} E^Z_X(Z_{can,min} \otimes Z_{min,min})
\end{array}
\]
The vanishing of the composition $\ev \circ \lim_{\mathcal{B}Z'} \iota$ is a consequence of Lemma 2.1. We claim that $\ev$ presents the cofibre of $\lim_{\mathcal{B}Z'} \iota$.

For the moment let us assume the claim. If $\coass_{X,\mathcal{Z}_{\text{can,min}}}$ is an equivalence, then $\tr$ represents the cofibre of $\iota$ and the coarse PV-square (2.4) is cartesian. Vice versa, if the coarse PV-square (2.4) is cartesian, $\coass_{X,\mathcal{Z}_{\text{can,min}}}$ is an equivalence by an application of the Five Lemma.

In order to show the claim we form the commutative diagram

$$
\begin{array}{ccc}
\lim_{\mathcal{B}Z'} E_{X}^{2}(\mathcal{Z}_{\text{min,min}} \otimes \mathcal{Z}_{\text{min,min}}) & \xrightarrow{\ev} & E_{X}^{2}(\mathcal{Z}_{\text{can,min}} \otimes \mathcal{Z}_{\text{min,min}}) \\
\downarrow & & \downarrow \ \\
\lim_{\mathcal{B}Z'} Q & \xrightarrow{0} & \lim_{\mathcal{B}Z'} Q
\end{array}
$$

(3.5)

in $\mathcal{M}$. Here $Q$ is the object of $\mathcal{M}$ defined as the cofibre of the middle map denoted by $\iota$ with the induced action of $\mathcal{Z}'$. The symbol $\sigma$ stands for the action of the generator of $\mathcal{Z}'$. The horizontal sequences are fibre sequences reflecting the usual presentation of the fixed points of a $\mathcal{Z}'$-object in a stable $\infty$-category. The vertical sequences are fibre sequences by construction. We now have the following assertions:

**Lemma 3.5.**

1. The map $E_{X}^{2}(\mathcal{Z}_{\text{min,min}} \otimes \mathcal{Z}_{\text{min,min}}) \rightarrow E_{X}^{2}(\mathcal{Z}_{\text{can,min}} \otimes \mathcal{Z}_{\text{min,min}})$ vanishes.

2. The group $\mathcal{Z}'$ acts trivially on $E_{X}^{2}(\mathcal{Z}_{\text{can,min}} \otimes \mathcal{Z}_{\text{min,min}})$.

3. The map $\lim_{\mathcal{B}Z'} \iota$ is split injective.

These facts imply that the maps marked by 0 in the diagram (3.5) vanish. From Lemma 3.6 below applied to the web (3.5) we then get a commutative triangle

$$
\begin{array}{ccc}
\lim_{\mathcal{B}Z'} E_{X}^{2}(\mathcal{Z}_{\text{can,min}} \otimes \mathcal{Z}_{\text{min,min}}) & \xrightarrow{\ev} & \lim_{\mathcal{B}Z'} Q \\
\downarrow & & \downarrow \\
\lim_{\mathcal{B}Z'} Q & \xrightarrow{\sim} & Q
\end{array}
$$

(3.6)

which solves our task.
We consider a web of fibre sequences

\[
\begin{array}{cccc}
A & \rightarrow & B & \rightarrow C \\
\downarrow a & & \downarrow 0 & \downarrow 0 \\
E & \rightarrow & F & \rightarrow G \\
\downarrow l & & \downarrow 0 & \downarrow 0 \\
H & \rightarrow & I & \rightarrow J \\
\downarrow 0 & & \downarrow 0 & \downarrow 0 \\
\Sigma A & \rightarrow & \Sigma B & \rightarrow \Sigma C
\end{array}
\]

in some stable \(\infty\)-category where the maps marked by 0 vanish.

**Lemma 3.6.** There exists a commutative triangle

\[
\begin{array}{ccc}
H & \rightarrow & E \\
\downarrow u & & \downarrow j \\
F & \downarrow \leftarrow & I
\end{array}
\]

*Proof.* The map \(u\) is obtained from the universal property of \(H\) as the cofibre of \(a\) together with the fact that \(j \circ a \simeq 0\) witnessed be the upper left square. We have \(j \simeq u \circ l\).

The potential inverse \(v: F \rightarrow H\) of \(u\) is obtained from the universal property of \(H\) as the fibre of \(I \rightarrow J\) together with the fact that \(d \circ i \simeq 0\) witnessed by the middle right square. We have \(i \simeq m \circ v\).

We claim that \(u\) and \(v\) are inverse to each other equivalences in the homotopy category of \(\mathbf{M}\). To this end we must consider equivalences between maps (we say homotopies) as 2-categorical data. We first observe that \(i \circ u = m\). The map \(u\) is homotopy class of the pair of the map \(j\) together with the homotopy \(\alpha: j \circ a \Rightarrow 0\). Similarly the map \(m\) is the homotopy class of the map \(i \circ j\) together with the homotopy \(\beta: i \circ j \circ a \Rightarrow 0\). The commutativity of the left middle square expresses the fact that \(i \circ \alpha = \beta\). We conclude that \(i \circ u = m\).

We now calculate \(i \circ u \circ v \circ j = m \circ v \circ j = i \circ j\). Since \(i\) is a monomorphism and \(j\) is an epimorphism we conclude that \(u \circ v = \text{id}_F\).

We now show that \(v \circ u = \text{id}_H\). The map \(v\) is the homotopy class of a pair of the map \(i\) and the zero homotopy \(\sigma\) of \(d \circ i\). The map \(l\) is similarly a homotopy class of the pair of the map \(i \circ j\) and the zero homotopy \(\kappa\) of \(d \circ i \circ j\). In this picture the commutativity of the left middle square expresses the fact that \(j^* \sigma = \kappa\). Altogether we conclude that \(j^* v = l\). We now calculate that \(v \circ u \circ l = v \circ j = l\). Since \(l\) is an epimorphism we conclude that \(v \circ u = \text{id}_H\).
**Proof of Lemma 3.5.** Assertion 1 is provided by the two lower squares of (2.2).

We now show Assertion 2. We use the isomorphism

\[
Z_{\text{can,min}} \otimes Z_{\text{min,min}} \rightarrow \text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}}, \quad (m, n) \mapsto (m - n, n)
\]  

(3.9)

in \(Z_{\text{BC}}\). The \(Z'\)-action on \(\text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}}\) is given by \((k, (m, n)) \mapsto (m - k, n + k)\). For given \(k\) in \(Z'\) we can decompose this into the map \(c_k : (m, n) \mapsto (m - k, n)\) and \(a_k : (m, n) \mapsto (m, n + k)\). The map \(c_k\) is close to the identity. By the coarse invariance of coarse homology theories the induced action of \(k\) on \(E^Z_X(\text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}})\) is therefore the equivalent map induced by \(a_k\). The morphism

\[
\text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}} \rightarrow \text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}}
\]

\[
\text{Res}^Z(Z_{\text{can,min}})
\]

between coarse coverings induces the commutative diagram

\[
E^Z_X(\text{Res}^Z(Z_{\text{can,min}})) \quad \xrightarrow{\approx} \quad \text{tr} \quad \xrightarrow{\approx} \quad E^Z_X(\text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}})
\]

\[
E^Z_X(\text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}}) \quad \xrightarrow{a_k} \quad E^Z_X(\text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}})
\]

This shows that the map induced by \(a_k\) is also equivalent to the identity.

We finally show Assertion 3. We use the geometric cone functor \(O^\infty : Z_{\text{UBC}} \rightarrow Z_{\text{BC}}\) which sends a \(Z\)-uniform bornological coarse space \(Y\) to the \(Z\)-bornological coarse space given by \(Z\)-set \(\mathbb{R} \times Y\) with the bornology generated by the subsets \([-n, n] \times B\) for all bounded subsets \(B\) of \(Y\) and \(n\) in \(\mathbb{N}\), and the hybrid coarse structure (this is \(O(Y)_-\) in the notation from [BEKW20b, Sec. 9]). By [BEKW20b, Prop. 9.31] we have a natural cone fibre sequence in \(Z_{\text{Sp}}X\) involving the cone boundary \(\partial^{\text{cone}} : Yo^s(O^\infty(-)) \rightarrow \Sigma Yo^s(F(-))\) which is induced from the transformation \(d^{\text{cone}} : O^\infty(-) \rightarrow \mathbb{R} \otimes F(-)\) between \(Z_{\text{BC}}\)-valued functors and the suspension equivalence \(Yo^s(\mathbb{R} \otimes -) \simeq \Sigma Yo^s(-)\), where \(F : Z_{\text{UBC}} \rightarrow Z_{\text{BC}}\) is the forgetful functor.

We consider \(\mathbb{R}\) as an object in \(Z_{\text{UBC}}\) with the action of \(Z\) by translations and the metric structures. We furthermore consider the invariant subset \(Z\) of \(\mathbb{R}\) with the induced structures. Note that \(F(Z) \cong Z_{\text{can,min}}\) and that the inclusion \(Z_{\text{can,min}} \rightarrow F(\mathbb{R})\) is a coarse equivalence. We let \(Z_{\text{disc}}\) in \(Z_{\text{UBC}}\) denote \(Z\) with the coarse structure replaced by the discrete one. Then the identity map of underlying sets is a coarsification morphism \(Z_{\text{disc}} \rightarrow Z\) in \(Z_{\text{UBC}}\). We have \(F(Z_{\text{disc}}) \cong Z_{\text{min,min}}\), and by [BEKW20b, Prop. 9.33] the induced map \(Yo^s(O^\infty(Z_{\text{disc}})) \rightarrow Yo^s(O^\infty(Z))\) is an equivalence. We consider the following
We now show Assertion 2. It suffices to show that the underlying map is exists the identification $Z$. In particular, if we decompose $R$ and $Z$, in order to conclude that coassembly map coass $X$, order to deal with $X$, since coass of $Z$, homology theories the fact that coass $X, Z$, is excisive and homotopy invariant by [BEKW20b, Prop. 9.36 & 9.38]. In particular, if we decompose $R$ into the $Z$-invariant subsets $\bigcup_{n \in \mathbb{N}} [n, 1/2 + n]$ and $\bigcup_{n \in \mathbb{N}} [n + 1/2, n + 1]$ and use the homotopy equivalences $Z \to \bigcup_{n \in \mathbb{N}} [n, 1/2 + n]$, $n \mapsto n$ and $Z \to \bigcup_{n \in \mathbb{N}} [n + 1/2, n + 1]$, $n \mapsto n + 1$ in $ZUBC$, we get a Mayer-Vietoris sequence

$$ Yo^*(O^\infty(Z)) \oplus Yo^*(O^\infty(Z)) \to Yo^*(O^\infty(Z)) \oplus Yo^*(O^\infty(Z)) \to Yo^*(O^\infty(R)) \cdot (3.12) $$

Since coass $X, O^\infty(R)$ is already known to be an equivalence we can use the Five Lemma in order to conclude that coass $X, O^\infty(R)$ is an equivalence, too.

We now show Assertion 2. It suffices to show that the underlying maps

$$ \partial^\text{cone}_{Zdisc} : E^Z_X(O^\infty(Z_{disc}) \otimes Z_{min,min}) \to \Sigma E^Z_X(Z_{min,min} \otimes Z_{min,min}) $$

The upper two horizontal maps are induced by $Z_{disc} \to Z \to R$. At the lower right corner we used the identification $E^Z_X(Z_{can,min} \otimes Z_{min,min}) \cong E^Z_X(F(R) \otimes Z_{min,min})$ induced by the coarse equivalence $Z_{can,min} \to F(R)$.

It is clear that the following assertions imply Assertion 3.5.3.

**Lemma 3.7.**

1. $\text{coass}_{X,O^\infty(Z_{disc})}$ and $\text{coass}_{X,O^\infty(R)}$ are equivalences.

2. $\lim_{Zdisc} \partial^\text{cone}_{Zdisc}$ and $\lim_{Z} \partial^\text{cone}_{R}$ are equivalences.

3. The map $i$ is split injective.

**Proof.** We start with Assertion 1. By [BEKW20b, Prop. 9.35] we have an equivalence $\partial^\text{cone}_{Zdisc} : Yo^*(O^\infty(Z_{disc})) \cong \Sigma Yo^*(Z_{min,min})$. Hence $\text{coass}_{X,O^\infty(Z_{disc})}$ is equivalent to the suspension of $\text{coass}_{X,Z_{min,min}}$ which is an equivalence by assumption.
and 
\[ \partial^\text{cone}_R : E^Z_X(\mathcal{O}^\infty(\mathbb{R}) \otimes \mathbb{Z}_{\text{min,min}}) \rightarrow \Sigma E^Z_X(\mathcal{F}(\mathbb{R}) \otimes \mathbb{Z}_{\text{min,min}}) \]
are equivalences. As already observed in the previous step, \( \partial^\text{cone}_{Z_{\text{disc}}} \) is an equivalence. We now discuss the case of \( \partial^\text{cone}_{\text{R}} \). We use the isomorphisms
\[ \mathcal{O}^\infty(\mathbb{R}) \otimes \mathbb{Z}_{\text{min,min}} \rightarrow \mathcal{O}^\infty(\text{Res}^Z(\mathbb{R})) \otimes \mathbb{Z}_{\text{min,min}} ; \quad (t, x, n) \mapsto t, x - n, n \]
and
\[ \mathcal{F}(\mathbb{R}) \otimes \mathbb{Z}_{\text{min,min}} \rightarrow \text{Res}^Z(\mathcal{F}(\mathbb{R})) \otimes \mathbb{Z}_{\text{min,min}} ; \quad (x, n) \mapsto (x - n, n) \]
in \( \mathbb{Z} \mathcal{B} \mathcal{C} \). It therefore suffices to show that
\[ \partial^\text{cone}_{\text{Res}^Z(\mathbb{R})} : E^Z_X(\mathcal{O}^\infty(\text{Res}^Z(\mathbb{R})) \otimes \mathbb{Z}_{\text{min,min}}) \rightarrow \Sigma E^Z_X(\mathcal{F}(\text{Res}^Z(\mathbb{R})) \otimes \mathbb{Z}_{\text{min,min}}) \]
is an equivalence. By [BE20a, Prop. 7.12] the uniform bornological coarse space \( \text{Res}^Z(\mathbb{R}) \) is coarsifying. Since \( E^Z \) is assumed to be strong also \( E^Z_X(- \otimes \mathbb{Z}_{\text{min,min}}) \) is a strong non-equivalent coarse homology theory. We can now conclude that \( \partial^\text{cone}_{\text{Res}^Z(\mathbb{R})} \) is equivalent to coarse assembly map \( \mu_{E^Z_X(- \otimes \mathbb{Z}_{\text{min,min}}, \mathcal{F}(\text{Res}^Z(\mathbb{R})) \otimes \mathbb{Z}_{\text{min,min}})} \) from [BE20a, Def. 9.7]. Since \( \mathcal{F}(\text{Res}^Z(\mathbb{R})) \) (i.e. the bornological coarse space \( \mathbb{R} \) with the metric structures) has weakly finite asymptotic dimension the coarse assembly map \( \mu_{E^Z_X(- \otimes \mathbb{Z}_{\text{min,min}}, \mathcal{F}(\text{Res}^Z(\mathbb{R})) \otimes \mathbb{Z}_{\text{min,min}})} \) is an equivalence by [BE20a, Thm. 10.4].

We finally show Assertion 3. We consider the Mayer-Vietoris sequence (3.12). Using the intersection with \( \mathbb{Z} \) of decomposition of \( \mathbb{R} \) into the subsets \( \bigcup_{n \in \mathbb{N}} [n, 1/2 + n] \) and \( \bigcup_{n \in \mathbb{N}} [n + 1/2, n + 1] \) we get an analogous Mayer-Vietoris sequence for \( Y_0^s(\mathcal{O}^\infty(\mathbb{Z}_{\text{disc}})) \). The inclusion \( \mathbb{Z}_{\text{disc}} \rightarrow \mathbb{R} \) induces the map of Mayer-Vietoris sequences (since \( \mathcal{O}^\infty \) is invariant under coarsification we can omit the subscript disc)

\[ \xymatrix{ Y_0^s(\mathcal{O}^\infty(\mathbb{Z})) \ar@{-->}[dr]_{x \oplus y \mapsto x} & Y_0^s(\mathcal{O}^\infty(\mathbb{Z})) \oplus Y_0^s(\mathcal{O}^\infty(\mathbb{Z})) \ar@{-->}[dr]_{\alpha \oplus \beta} \ar[r]^e & Y_0^s(\mathcal{O}^\infty(\mathbb{Z})) \ar@{-->}[dr]_{\alpha \oplus \beta} \ar[l]_{(a,b) \mapsto a+b} \ar[l]_{x \mapsto x \oplus 0} \ar[l]_{x \mapsto x \oplus \sigma(x)} \ar[l]_{x \mapsto x \oplus \sigma(x) \oplus y \oplus \sigma(y) + q(y)} & Y_0^s(\mathcal{O}^\infty(\mathbb{Z})) \oplus Y_0^s(\mathcal{O}^\infty(\mathbb{Z})) \ar[l]_{x \mapsto x \oplus \sigma(x) \oplus y \oplus \sigma(y) + q(y)} \ar[l]_{x \mapsto x \oplus \sigma(x) \oplus y \oplus \sigma(y) + q(y)} \ar[r] & Y_0^s(\mathcal{O}^\infty(\mathbb{R})) \ar[l]_{x \mapsto x \oplus \sigma(x) \oplus y \oplus \sigma(y) + q(y)} \ar[l]_{x \mapsto x \oplus \sigma(x) \oplus y \oplus \sigma(y) + q(y)} } \]

where \( \sigma \) indicates the map induced by action of the generator of \( \mathbb{Z} \). The existence of the split indicated by the dashed arrow implies that \( e \) is an epimorphism. The left vertical arrow has a left-inverse (indicated by the left dotted arrow) such that corresponding left square commutes. It induces a map \( \beta \) as indicated. We have \( \beta \circ \alpha \circ e \simeq e \) and hence \( \beta \circ \alpha \simeq \text{id}_{Y_0^s(\mathcal{O}^\infty(\mathbb{Z}))} \). Applying \( E^Z_X \) we obtain the desired left-inverse \( E^Z_X(\beta) \) of \( i \simeq E^Z_X(\alpha) \).

This completes the proof of Proposition 3.7.

\[ \square \]

**Proposition 3.8.** If \( E^Z \) is strongly additive and \( X \) is discrete, then \( \text{coass}_{X,\mathbb{Z}_{\text{min,min}}} \) is an equivalence.
Proof. We first calculate the target of the coassembly map explicitly.

\[
\lim_{BZ'} E^Z(X \otimes Z_{\text{min,min}} \otimes Z_{\text{min,min}}) \overset{3.9}{=} \lim_{BZ'} \prod_{\text{Res}^Z(Z)} E^Z(X \otimes Z_{\text{min,min}}) \\
\overset{\text{ev}_0}{=} E^Z(X \otimes Z_{\text{min,min}})
\]

(3.13)

For the equivalence marked by ! we use the isomorphism

\[
X \otimes \text{Res}(Z_{\text{min,min}}) \otimes Z_{\text{min,min}} \cong \bigfreeunion_{Z_{\text{min,min}}} X \otimes Z_{\text{min,min}}
\]

(see [BEKW20b, Ex. 2.16] for the free union) in \(\mathsf{ZBC}\). At this point it is important that \(X\) is discrete. The equivalence ! then follows from the assumption that \(E^Z\) is strongly additive. The group \(Z'\) acts freely transitively on the index set \(\text{Res}^Z(Z)\) by translations, and also on \(Z_{\text{min,min}}\). The equivalence \(\text{ev}_0\) is the projection onto the factor with index 0 in \(\text{Res}^Z(Z)\). Using [BEKW20a, Lem. 2.59], or more concretely [BEKW20a, (2.22)], we see that the composition of the transfer with (3.13) is equivalent to the identity. Hence (3.13) is an inverse equivalence for \(\text{coass}_{X,Z_{\text{can,min}}}\).

We recall the fibre sequence of functors

\[
\Sigma^{-1} F^\infty \xrightarrow{\beta} F^0 \to F
\]

(3.14)

from \(\mathsf{ZBC}\) to \(\mathsf{ZSp}^X\) introduced in [BEKW20b, Def. 11.9 & (11.2)], where \(\beta\) is called the motivic forget-control map and \(F^0 \cong \text{Yo}^a\). The motivic forget control map induces a the forget control map

\[
\gamma_{E^X} : E^Z_X(F^\infty(Z_{\text{can,min}})) \to \Sigma E^Z_X(F^0(Z_{\text{can,min}})) \cong \Sigma E^Z_X(Z_{\text{can,min}}).
\]

(3.15)

**Proposition 3.9.** Assume:

1. \(X\) has the minimal bornology and is discrete.

2. \(E^Z\) is strong and strongly additive.

Then \(\text{coass}_{X,Z_{\text{can,min}}}\) is an equivalence if and only if \(\gamma_{E^X}\) is an equivalence.

Proof. The following commutative square

\[
\begin{array}{ccc}
\Sigma^{-1} E^Z_X(F^\infty(Z_{\text{can,min}})) & \xrightarrow{\gamma_{E^X}} & E^Z_X(Z_{\text{can,min}}) \\
\downarrow \text{coass}_{X,F^\infty(Z_{\text{can,min}})} & & \downarrow \text{coass}_{X,Z_{\text{can,min}}} \\
\lim_{BZ'} \Sigma^{-1} E^Z_X(F^\infty(Z_{\text{can,min}}) \otimes Z_{\text{min,min}}) & \xrightarrow{\lim_{BZ'} \delta} & \lim_{BZ'} \Sigma^{-1} E^Z_X(Z_{\text{can,min}} \otimes Z_{\text{min,min}})
\end{array}
\]

(3.16)
is at the heart of the proof of the split injectivity of the Davis-Lück assembly map for CP-functors given in [BEKW20a]. In the present paper we reverse the flow of information and use it in order to deduce properties of the coassembly map \(\text{coass}_{X,Z_{\text{can,min}}}\). The map \(\delta\) in (3.16) is also induced by the forget control map \(\gamma_{E^Z_X}\) in (3.14).

Let \(P_U(Z_{\text{can,min}})\) in \(\mathbb{Z}_{\text{UBC}}\) be the Rips complex of \(Z_{\text{can,min}}\) of size \(U\), where \(U\) is any invariant entourage of \(Z_{\text{can,min}}\). By definition we have

\[
F^\infty(Z_{\text{can,min}}) \simeq \colim_{U \in \mathcal{Z}_{\text{can,min}}} \text{Yo}^*(\mathcal{O}^\infty(P_U(Z_{\text{can,min}}))) .
\]

If \(U_1 = \{(n,m) \mid |n-m| \leq 1\}\), then we have a natural identification \(\mathbb{R} \cong P_{U_1}(Z_{\text{can,min}})\).

We then observe that the maps \(\mathbb{R} \to P_{U_r}(Z_{\text{can,min}})\) are homotopy equivalences in \(\mathbb{Z}_{\text{UBC}}\) for all \(r \in [1, \infty)\). This implies the equivalence \(\text{Yo}^*(\mathcal{O}^\infty(\mathbb{R})) \cong F^\infty(Z_{\text{can,min}})\). The proof of Lemma 3.7.1 actually shows that \(\text{coass}_{X,\mathcal{O}^\infty(\mathbb{R})}\) is an equivalence provided \(\text{coass}_{X,Z_{\text{min,min}}}\) is an equivalence. But this is the case by Proposition 3.8. We conclude that \(\text{coass}_{X,F^\infty(Z_{\text{can,min}})}\) is an equivalence. Note that we used here the assumptions that \(E^Z\) is strongly additive and that \(X\) is discrete.

In order to show that \(\lim_{BZ^\infty} \delta\) is an equivalence it suffices to show that the underlying map \(\delta\) is one. The isomorphism (3.9) induces the vertical equivalences in the commutative square

\[
\begin{array}{ccc}
\Sigma^{-1}E^Z_X(F^\infty(Z_{\text{can,min}}) \otimes Z_{\text{min,min}}) & \xrightarrow{\delta} & \Sigma^{-1}E^Z_X(Z_{\text{can,min}} \otimes Z_{\text{min,min}}) \\
\cong & & \cong \\
\Sigma^{-1}E^Z_X(F^\infty(\text{Res}^Z(Z_{\text{can,min}})) \otimes Z_{\text{min,min}}) & \xrightarrow{\delta'} & \Sigma^{-1}E^Z_X(\text{Res}^Z(Z_{\text{can,min}}) \otimes Z_{\text{min,min}})
\end{array}
\]

The map \(\delta'\) is the coarse Baum-Connes assembly map from [BE20a, Def. 9.7] for the non-equivariant coarse homology theory \(E^Z_X(- \otimes Z_{\text{min,min}})\) which is strong since \(E^Z\) was assumed to be strong. Since \(\text{Res}^Z(Z_{\text{can,min}})\) has finite asymptotic dimension this coarse Baum-Connes assembly map is an equivalence by [BE20a, Thm. 10.4]. We conclude that the morphism \(\lim_{BZ^\infty} \delta\) in (3.16) is an equivalence.

We now have shown that the morphisms \(\lim_{BZ^\infty} \delta\) and \(\text{coass}_{X,F^\infty(Z_{\text{can,min}})}\) in (3.16) are equivalences. Hence \(\text{coass}_{X,Z_{\text{can,min}}}\) is an equivalence if and only of \(\gamma_{E^Z_X}\) is an equivalence.

\[\square\]

Proposition 3.10. The Davis-Lück assembly map \(\mu_{HE^Z_X,c}^{DL}\) is equivalent to the forget control map \(\gamma_{E^Z_X}\) in (3.15).

Proof. Recall that \(E^Z_X,c\) is the continuous approximation of \(E^Z_X\), see (3.2).

Lemma 3.11. The forget control map \(\gamma_{E^Z_X}\) is equivalent to \(\gamma_{E^Z_X,c}\).
Proof. The full category $\mathcal{CE}$ of $\mathbb{Z}\text{Sp}X$ of objects $W$ such that $\mathcal{E}_{X,c}^\mathbb{Z}(W) \to \mathcal{E}_X^\mathbb{Z}(W)$ is an equivalence is localizing. It contains the objects $\text{Yo}^*(Y)$ for all $Y$ in $Z\text{BC}$ with the minimal bornology, so in particular $\text{Yo}^*(Z_{\text{can,min}})$ and $\text{Yo}^*(Z_{\text{min,min}})$. The Mayer-Vietoris sequence (3.12) and the equivalence $\text{Yo}^*(O^\infty(\mathbb{Z})) \simeq \Sigma \text{Yo}^*(Z_{\text{min,min}})$ [BEKW20b, Prop. 9.33 & 9.35] imply that $\text{Yo}^*(O^\infty(\mathbb{R}))$ belongs to $\mathcal{CE}$. Finally, in the proof of Proposition 3.9 we have seen that $F^\infty(Z_{\text{can,min}}) \simeq \text{Yo}^*(O^\infty(\mathbb{R}))$. Therefore $F^\infty(Z_{\text{can,min}})$ belongs to $\mathcal{CE}$. The natural transformation $\mathcal{E}_{X,c}^\mathbb{Z} \to \mathcal{E}_X^\mathbb{Z}$ induces a commutative square

$$
\begin{array}{ccc}
\mathcal{E}_{X,c}^\mathbb{Z}(F^\infty(Z_{\text{can,min}})) & \xrightarrow{\gamma_{X,c}^\mathbb{Z}} & \Sigma \mathcal{E}_{X,c}(Z_{\text{can,min}}) \\
\downarrow & & \downarrow \\
\mathcal{E}_X^\mathbb{Z}(F^\infty(Z_{\text{can,min}})) & \xrightarrow{\gamma_{X}^\mathbb{Z}} & \Sigma \mathcal{E}_X(Z_{\text{can,min}})
\end{array}
$$

The observations above imply that the vertical morphisms are equivalences. □

By [BEKW20c, Cor. 8.25] applied to the continuous coarse homology theory $\mathcal{E}_{X,c}^\mathbb{Z}$ the Davis-Lück assembly map $\mu_{H^\mathbb{E}_X}^{DL}$ is equivalent to the forget control map

$$
\mathcal{E}_{X,c}^\mathbb{Z}(F^\infty(Z_{\text{can,min}}) \otimes Z_{\text{max,max}}) \to \Sigma \mathcal{E}_{X,c}(Z_{\text{can,min}} \otimes Z_{\text{max,max}}) \tag{3.18}
$$

induced by the map $\beta$ in (3.14). In the following we explain that the additional factor $Z_{\text{max,max}}$ can be dropped. First of all, since $\mathbb{Z}$ is torsion free, the projection $F^\infty(Z_{\text{can,min}}) \otimes Z_{\text{max,max}} \to F^\infty(Z_{\text{can,min}})$ is an equivalence. Furthermore, the projection $Z_{\text{can,min}} \otimes Z_{\text{max,max}} \to Z_{\text{can,min}}$ is even a coarse equivalence. Thus the map in (3.18) is equivalent to $\gamma_{X,c}^\mathbb{Z}$. By Lemma 3.11 we conclude that $\mu_{H^\mathbb{E}_X}^{DL}$ is equivalent to $\gamma_{X,c}^\mathbb{Z}$. □

The proof of Theorem 3.2 now follows from a combination of the assertions of Propositions 3.4, 3.8, 3.9 and 3.10. □

We call a morphism in $\mathbb{Z}\text{Sp}X$ a local equivalence if is sent to an equivalence by the coarse homology theories $E^\mathbb{Z}(- \otimes Z_{\text{min,min}})$, $E^\mathbb{Z}(- \otimes Z_{\text{can,min}})$, and $E^\mathbb{Z}(- \otimes Z_{\text{can,min}} \otimes Z_{\text{min,min}})$ appearing the coarse PV-square. Note that in view of Remark 3.19 below we could drop the last entry of this list without changing the notion of a local equivalence.

Definition 3.12. We let

$$
\ell : \mathbb{Z}\text{Sp}X \to \mathbb{Z}\text{Sp}X_{\text{loc}}
$$

be the localization at the local equivalences and set $\text{Yo}^*_{\text{loc}} := \ell \circ \text{Yo}^*$. 

Note that the notion of a local equivalence and $\ell : \mathbb{Z}\text{Sp}X \to \mathbb{Z}\text{Sp}X_{\text{loc}}$ depend on the choice of $E^\mathbb{Z}$ though this is not indicated in the notation. The three coarse homology theories listed above have colimit-preserving factorizations $\mathbb{Z}\text{Sp}X_{\text{loc}} \to M$ which will be denoted by the same symbols. The property that the coarse PV-square (2.4) associated to $E^\mathbb{Z}$ and $X$ is cartesian only depends on the class $\text{Yo}^*_{\text{loc}}(X)$. 

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Definition 3.13. We let $\text{PV}_{\mathbb{E}^z}$ denote the full subcategory of $\text{ZSp}X_{\text{loc}}$ of objects $X$ for which the coarse PV-square (2.3) is cartesian.

By construction $\text{PV}_{\mathbb{E}^z}$ is a localizing subcategory of $\text{ZSp}X_{\text{loc}}$.

Definition 3.14. We define $\text{ZSp}X_{\text{loc}}\langle DL \rangle$ as the localizing subcategory generated by $\text{Yo}^*_\text{loc}(Y_{\text{min},\text{min}})$ for all $\mathbb{Z}$-sets $Y$ such that $\mu_{HEZ_{\text{min},\text{min}},c}^{DL}$ is an equivalence.

Theorem 3.2 now has the following immediate consequence.

Corollary 3.15. If $E^\mathbb{Z}$ is strong and strongly additive, then $\text{PV}_{\mathbb{E}^z} \subseteq \text{ZSp}X_{\text{loc}}\langle DL \rangle$.

Proposition 3.16. If $E^\mathbb{Z}$ is strong and strongly additive, then $\text{Yo}^*_\text{loc}(Z_{\text{min},\text{min}}) \in \text{ZSp}X_{\text{loc}}\langle DL \rangle$.

Proof. The map $\delta$ in (3.17) is the forget-control map $\gamma_{E^\mathbb{Z}}$, and it has been shown in the proof of Lemma 3.9 that it is an equivalence. We now apply Proposition 3.10.

Example 3.17. In this example we show that it can happen that $\text{Yo}^*_\text{loc}(\ast) \notin \text{PV}_{\mathbb{E}^z}$. Let $A$ be an additive category with strict $\mathbb{Z}$-action and let $E^\mathbb{Z} := KAX^\mathbb{Z}$ denote the coarse algebraic $K$-homology [BEKW20b, Def. 8.8]. This functor is strong [BEKW20b, Prop. 8.18], continuous [BEKW20b, Prop. 8.17], strongly additive [BEKW20b, Prop. 8.19] and has transfers [BEKW20a, Thm. 1.4]. We have an obvious equivalence $\mu_{HKA_{\mathbb{Z}},c}^{DL} \simeq \mu_{HKA_{\mathbb{Z}}}^{DL}$. The Davis-Lück assembly map $\mu_{HKA_{\mathbb{Z}}}^{DL}$ is known to be split-injective (see e.g. [BEKW20c] through this is not the original reference), and its cofibre can be expressed in terms of so-called nil-terms, see e.g. [LS16]. So $\text{Yo}^*_\text{loc}(\ast) \in \text{PV}_{KA_{\mathbb{Z}}}$ if and only if these nil-terms vanish. If $R$ is a non-regular ring and $A = \text{Mod}(R)$ with the trivial $\mathbb{Z}$-action, then the nil-terms can be non-trivial and hence $\text{Yo}^*_\text{loc}(\ast) \notin \text{PV}_{KA_{\mathbb{Z}}}$. 

We finally calculate the boundary map of the coarse PV-sequence (3.1) explicitly. Let $X$ be in $\mathbb{Z}BC$. Recall that $\mathbb{Z}'$ acts $E^\mathbb{Z}_{X}(Z_{\text{min},\text{min}})$ by functoriality via its action on $Z_{\text{min},\text{min}}$ by translations.

Proposition 3.18. If the coassembly maps $\text{coass}_{X,Z_{\text{min},\text{min}}}$ and $\text{coass}_{X,Z_{\text{can},\text{min}}}$ are equivalences, then the coarse PV-sequence is equivalent to a fibre sequence

$$
\Sigma^{-1}E^\mathbb{Z}_{X}(Z_{\text{can},\text{min}}) \to E^\mathbb{Z}_{X}(Z_{\text{min},\text{min}}) \to E^\mathbb{Z}_{X}(Z_{\text{min},\text{min}})
$$
Proof. If \(\text{coass}_{X, Z_{\min, min}}\) and \(\text{coass}_{X, Z_{\can, min}}\) are equivalences, then in view of Proposition 3.4 and (3.5) the coarse PV-sequence (3.1) is equivalent to a fibre sequence
\[
E^Z_X(Z_{\can, min}) \to E^Z_X(Z_{\can, min} \otimes Z_{\min, min}) \xrightarrow{1-\sigma} E^Z_X(Z_{\can, min} \otimes Z_{\min, min}).
\]
The isomorphism (3.9) yields the first equivalence in the chain of equivalences
\[
E^Z_X(Z_{\can, min} \otimes Z_{\min, min}) \simeq E^Z_X(\text{Res}^Z(Z_{\can, min}) \otimes Z_{\min, min}) \simeq \Sigma E^Z_X(Z_{\min, min}) (3.19)
\]of \(\mathcal{Z}'\)-objects in \(\mathcal{M}\). In order to see the second equivalence note that after applying the isomorphism (3.9) the group \(\mathcal{Z}'\) acts diagonally on \(\text{Res}^Z(Z_{\can, min}) \otimes Z_{\min, min}\). But arguing as in the proof of Lemma 3.5 we can replace the \(\mathcal{Z}'\)-action on the factor \(\text{Res}^Z(Z_{\can, min})\) by the trivial action.

If \(E^Z\) is strong and strongly additive, \(Y_{\loc}^*(X)\) is in \(\mathcal{Z}\text{Sp}_{\loc}(\text{disc})\), and \(\mu_{HE^Z_{X,c}}\) is an equivalence, then the assumptions of Proposition 3.18 are satisfied.

Remark 3.19. Using the equivalence (3.19) we observe that a morphism in \(\mathcal{Z}\text{Sp}_{\loc}(\mathcal{X})\) is a local equivalence if and only it is sent to an equivalence by \(E^Z(\_ \otimes Z_{\min, min})\) and \(E^Z(\_ \otimes Z_{\can, min})\).

4 Topological coarse \(K\)-homology

Let \(\mathcal{C}\) be a \(C^*\)-category with a strict \(\mathbb{Z}\)-action which admits all orthogonal AV-sums [BEa, Def. 7.1]. Then we can consider the spectrum-valued strong coarse homology theory
\[
K\mathcal{X}_C^Z : \mathcal{B}\mathcal{C} \to \mathcal{S}\mathcal{P}
\]
from [BEb, Def. 6.1.2] for the homological functor \(K^*\text{Cat} : C^*\text{Cat}^{\text{nu}} \to \mathcal{S}\mathcal{P}\). In order to simplify the notation, in the present paper we omit the subscript \(c\) appearing in this reference which indicates continuity. The coarse homology theory \(K\mathcal{X}_C^Z\) is continuous [BEb, Thm. 6.3], strong [BEb, Prop. 6.5], strongly additive [BEb, Thm 11.1] and admits transfers by [BEb, Thm 9.7]. So we can take \(K\mathcal{X}_C^Z\) as an example for \(E^Z\) in the preceding sections and define the associated stable \(\infty\)-category \(\mathcal{Z}\text{Sp}_{\loc}\) as in Definition 3.12.

Definition 4.1. We let \(\mathcal{Z}\text{Sp}_{\loc}(\text{disc})\) denote the localizing subcategory of \(\mathcal{Z}\text{Sp}_{\loc}\) generated by the objects \(Y_{\loc}^*(Y_{\min, min})\) for all \(Y\) in \(\mathcal{Z}\text{Set}\).

Recall the Definition 3.14 of \(\mathcal{Z}\text{Sp}_{\loc}(DL)\).

Theorem 4.2. We have \(\mathcal{Z}\text{Sp}_{\loc}(\text{disc}) \subseteq \mathcal{Z}\text{Sp}_{\loc}(DL)\).
Proof. The main input is that the amenable group $\mathbb{Z}$ satisfies the Baum-Connes conjecture with coefficients. Using the main result of \cite{Kra20}, see also \cite[Thm. 1.7]{BELa}, on the level of homotopy groups the Davis-Lück assembly map $\mu_{HKX_C}$ from \eqref{eq:assembly} is isomorphic to the Baum-Connes assembly map \eqref{eq:baum-connes} with coefficients in the free $\mathbb{Z}$-$C^*$-algebra $A(I(C))$ generated by $C$. The latter was introduced in \cite[Def. 3.7]{Joa03}. Consequently, we know that $\mu_{HKX_C}^{DL}$ is an equivalence.

Let $Y$ be a $\mathbb{Z}$-set. We form the $C^*$-category with $\mathbb{Z}$-action $V_C^Z(Y_{\min,\min} \otimes \mathbb{Z}_{\min,\min})$ by specializing \cite[Def. 4.19.2]{BELa} (note that in the present paper we use a different notation). The $\mathbb{Z}$-action is induced by functoriality from the right action on $\mathbb{Z}_{\min,\min}$. As explained in \cite[Sec. 4]{BELa} the objects of $V_C^Z(Y_{\min,\min} \otimes \mathbb{Z}_{\min,\min})$ are triples $(C, \rho, \mu)$, where $(C, \rho)$ is a $G$-object in the multiplier category $MC$ of $C$ (see \cite[Def. 3.1]{BELa}), and $\mu$ is an invariant finitely additive measure on $Y_{\min,\min} \otimes \mathbb{Z}_{\min,\min}$ with values in multiplier projections on $C$ such that $\mu(\{(y, n)\}) \in C$ for every point $(y, n) \in Y \times \mathbb{Z}$. The morphisms of $V_C^Z(Y_{\min,\min} \otimes \mathbb{Z}_{\min,\min})$ are the invariant and $\text{diag}(Y \times \mathbb{Z})$-controlled multiplier morphisms.

We need an AV-sum completion $C_Y$ of $V_C^Z(Y_{\min,\min} \otimes \mathbb{Z}_{\min,\min})$, see \cite[Def. 7.1]{BELa} for the notion of an AV-sum. It turns out to be useful to work with an explicit model for $C_Y$ given as follows. The objects of $C_Y$ are again triples $(C, \rho, \mu)$ as above, but we replace the condition $\mu(\{(y, n)\}) \in C$ (which prevents the existence of infinite AV-sums in $V_C^Z(Y_{\min,\min} \otimes \mathbb{Z}_{\min,\min})$) by the more general condition that the images of $\mu(\{(y, n)\})$ for all $(y, n)$ in $Y \times \mathbb{Z}$ are isomorphic to AV-sums of families of unital objects of $C$ (see \cite[Def. 2.14]{BELa}). Morphisms $A: (C, \rho, \mu) \to (C', \rho', \mu')$ in this category are invariant $\text{diag}(X \times \mathbb{Z})$-controlled morphisms $A: C \to C'$ in $MC$ such that $A\mu(\{(y, n)\}) \in C$ for all $(y, n)$ in $Y \times \mathbb{Z}$. Note that $C_Y$ contains $V_C^Z(Y_{\min,\min} \otimes \mathbb{Z}_{\min,\min})$ as the full subcategory of unital objects.

We let $K\mathcal{A}_{C_Y}^{\mathbb{Z},Y_{\min,\min},c}$ be the continuous approximation (see Definition \ref{def:approximation}) of the coarse homology theory $K\mathcal{A}_{C}^{\mathbb{Z},C_{\min,\min}}$.

**Proposition 4.3.** We have an equivalence $K\mathcal{A}_{C_Y}^{\mathbb{Z},Y_{\min,\min},c} \simeq K\mathcal{A}_{C}^{\mathbb{Z},Y_{\min,\min}}$ of $\mathbb{Z}$-equivariant coarse homology theories.

**Proof.** We consider the inclusion of groups $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $n \mapsto (n, 0)$. We let $\mathbb{Z} \times \mathbb{Z}$ act on $C$ via the projection onto the first factor. Identifying $\text{Ind}_Z^{\mathbb{Z} \times \mathbb{Z}}(-) \simeq - \otimes \mathbb{Z}_{\min,\min}$ we have the induction equivalence \cite[10.5.1]{BELa}

\[ K\mathcal{A}_{C,Y_{\min,\min}}^{\mathbb{Z}, (-)} \cong K\mathcal{A}_{C}^{\mathbb{Z},X_{\min,\min}}(Y_{\min,\min} \otimes (-) \otimes \mathbb{Z}_{\min,\min}) . \] (4.1)

On the r.h.s. the first copy of $\mathbb{Z}$ acts diagonally on $Y \times \mathbb{Z} \times (-)$, while the second copy only acts on $\mathbb{Z}_{\min,\min}$. We have a natural isomorphism

\[ Y \times (-) \times \mathbb{Z} \cong Y \times \mathbb{Z} \times (-) , \quad (x, -, n) \mapsto (x, n, n^{-1} -) \]
of functors from $\mathcal{ZBC}$ to $(\mathcal{Z} \times \mathcal{Z})\mathcal{BC}$. On the target the first factor of $\mathcal{Z} \times \mathcal{Z}$ only acts on $Y \times \mathcal{Z}$, while the second factor now acts diagonally on $\mathcal{Z} \times (-)$. We get an equivalence

$$K\mathcal{X}_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(Y_{min,min} \otimes (-) \otimes Z_{min,min}) \simeq K\mathcal{X}_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(Y_{min,min} \otimes Z_{min,min} \otimes (-)) \quad (4.2)$$

By definition [BEK85, Def. 6.1] of the coarse $K$-homology functor we have an equivalence

$$K\mathcal{X}_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(Y_{min,min} \otimes Z_{min,min} \otimes (-)) \simeq K^{\ast}Cat(V_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(Y_{min,min} \otimes Z_{min,min} \otimes (-))) \quad (4.2)$$

We now construct a natural transformation

$$V_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(Y_{min,min} \otimes Z_{min,min} \otimes (-)) \rightarrow V_{\mathcal{C}}^{\mathcal{Z}}(Z) \quad (4.3)$$

where $\mathcal{Z}$ runs over $\mathcal{ZBC}_{min}$. Let $(\mathcal{C}, \rho, \mu)$ be an object $V_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(Y_{min,min} \otimes Z_{min,min} \otimes (-))$. The functor $(4.3)$ sends this object to an object $((\mathcal{C}, \rho_{\mathcal{Z} \times \mathcal{Z}},{\text{pr}}_{\mathcal{Z} \times \mathcal{Z}}), \sigma, \kappa)$. We first observe that $(\mathcal{C}, \rho_{\mathcal{Z} \times \mathcal{Z}},{\text{pr}}_{\mathcal{Z} \times \mathcal{Z}})$ is an object of $\mathcal{C}_{\mathcal{Y}}$. We define the measure $\kappa$ by $\kappa(\{z\}) = \mu(\mathcal{Y} \times \mathcal{Z} \times \{z\})$. Note that $\kappa(\{z\})$ is an endomorphism of $(\mathcal{C}, \rho_{\mathcal{Z} \times \mathcal{Z}},{\text{pr}}_{\mathcal{Z} \times \mathcal{Z}})$ in $\mathcal{C}_{\mathcal{Y}}$. We further set $\sigma := \rho_{1 \times \mathcal{Z}}$. Using that $\mathcal{Z}$ has the minimal bornology we then observe that $((\mathcal{C}, \rho_{\mathcal{Z} \times \mathcal{Z}},{\text{pr}}_{\mathcal{Z} \times \mathcal{Z}}), \sigma, \kappa)$ indeed belongs to $V_{\mathcal{C}}^{\mathcal{Z}}(\mathcal{Z})$.

On morphisms the functor $(4.3)$ is given by the identity. Naturality in $\mathcal{Z}$ is obvious. The transformation identifies invariant and controlled morphisms on both sides. We conclude that the transformation is fully faithful.

We finally observe that it is essentially surjective. Let $((\mathcal{C}, \tilde{\rho}, \tilde{\mu}), \sigma, \kappa)$ be an object of $V_{\mathcal{C}}^{\mathcal{Z}}(\mathcal{Z})$. We define $\mu$ such that $\mu(\{(y, n, z)\}) = \tilde{\mu}(\{(y, n)\})\kappa(z)$ and $\rho$ such that by $\rho_{(n, m)} = \tilde{\rho}_{n} \sigma_{m}$. Then $(\mathcal{C}, \rho, \mu)$ is a preimage in $V_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(Y_{min,min} \otimes Z_{min,min} \otimes (-))$.

Applying $K^{\ast}Cat$ to the natural equivalence $(4.3)$ to we get a natural equivalence

$$K^{\ast}Cat(V_{\mathcal{C}}^{\mathcal{Z} \times \mathcal{Z}}(X_{min,min} \otimes Z_{min,min} \otimes (-))) \simeq K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}(Z) \quad (4.4)$$

for $Z$ in $\mathcal{ZBC}_{min}$. Composing $(4.1)$, $(4.2)$ and $(4.4)$ we get the equivalence

$$K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}(Y_{min,min}) \simeq K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}} (4.5)$$

of functors on $\mathcal{ZBC}_{min}$. The desired equivalence is now given by

$$K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}(Y_{min,min}) \simeq i_{\mathcal{Y}}^{\ast}K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}(Y_{min,min}) \simeq i_{\mathcal{Y}}^{\ast}K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}} \simeq K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}$$

where the last equivalence is an equivalence since $K\mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}$ is already continuous. □

We can now finish the proof of Theorem [4.2]. We have equivalences of morphisms

$$\mu^{-DL}_{H K \mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}} (Y_{min,min})} \simeq \gamma_{K \mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}} (Y_{min,min})} \simeq \mu^{DL}_{H K \mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}} \quad (4.2)$$

At the beginning of this proof we have seen that $\mu^{DL}_{H K \mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}}$ is an equivalence. We conclude that $\mu^{DL}_{H K \mathcal{X}_{\mathcal{C}_{\mathcal{Y}}}^{\mathcal{Z}}}$ is an equivalence, too. It follows that $\mathcal{ZSp} \mathcal{X}_{\text{loc}}(DL)$ contains $Y_{\text{loc}}^e(Y_{min,min})$ for all $\mathcal{Z}$-sets $Y$, and this implies the assertion of the theorem. □
Corollary 3.15 now implies:

**Corollary 4.4.** We have $\mathbb{Z}Sp\chi_{\text{disc}} \subseteq PV_{KX}\mathbb{Z}$.

Let $A$ be a unital $C^*$-algebra with an action of $\mathbb{Z}$. We consider the $C^*$-category $\operatorname{Hilb}_c(A)$ of Hilbert $A$-modules and compact operators which has the induced $\mathbb{Z}$-action [BEa, Ex. 2.9]. Then the square (1.8) is cartesian by Corollary 4.4. We let $Z'$ denote the group which acts by automorphisms on $\operatorname{Res}^Z(A)$ (via the identification of $Z'$ with the original group $Z$) and on $Z_{\min,\min}$ by translations. We let $\sigma$ denote the action by functoriality of the generator of $Z'$ on various derived objects.

**Proposition 4.5.** We have equivalences

1. $K\chi^Z_{\operatorname{Hilb}_c(A)}(Z_{\min,\min}) \simeq K(\operatorname{Res}^Z(A))$
2. $K\chi^Z_{\operatorname{Hilb}_c(A)}(Z_{\text{can, min}}) \simeq K(A \rtimes \mathbb{Z})$
3. $K\chi^Z_{\operatorname{Hilb}_c(A)}(Z_{\text{can, min}} \otimes Z_{\min,\min}) \simeq \Sigma K(\operatorname{Res}^Z(A))$.

Furthermore, the coarse PV-sequence is equivalent to a fibre sequence

$$
\Sigma^{-1}K(A \rtimes \mathbb{Z}) \rightarrow K(\operatorname{Res}^Z(A)) \xrightarrow{1-\sigma} K(\operatorname{Res}^Z(A)).
$$

**Proof.** In order to prepare the argument for the second assertion we will actually show that there is an equivalence

$$K\chi^Z_{\operatorname{Hilb}_c(A)}(Z_{\min,\min}) \simeq K(\operatorname{Res}^Z(A))$$

of spectra with an action of $Z'$. Using that $Z_{\min,\min} \cong \operatorname{Ind}_1^Z(*)$, by [BEb, Cor. 10.5.2] have an equivalence

$$K\chi^Z_{\operatorname{Hilb}_c(A)}(Z_{\min,\min}) \xrightarrow{\sim} K\chi^Z_{\operatorname{Res}^Z(\operatorname{Hilb}_c(A))}(*) .$$

(4.6)

Unfolding [BEb, Def. 4.19] we get an isomorphism $V_{\operatorname{Res}^Z(\operatorname{Hilb}_c(A))}(*) \cong \operatorname{Res}^Z(\operatorname{Hilb}_c(A))^u$, where $(-)^u$ denotes the operation of taking the full subcategory of unital objects. We therefore get an equivalence

$$K\chi^Z_{\operatorname{Res}^Z(\operatorname{Hilb}_c(A))}(*) \simeq K^{C^*}\operatorname{Cat}(\operatorname{Res}^Z(\operatorname{Hilb}_c(A))^u) .$$

(4.7)

We consider $A$ as a $C^*$-category $A$ with a single object. We then have a fully faithful functor $\psi : A \rightarrow \operatorname{Res}^Z(\operatorname{Hilb}_c(A))^u$ which sends the unique object of $A$ to the Hilbert $A$-module given by $A$ with the right $A$-multiplication and the scalar product $\langle a, a' \rangle := a^* a'$. Since $A$ is assumed to be unital the latter object of $\operatorname{Hilb}_c(A)$ is indeed unital. The functor
A \rightarrow \text{Res}^Z(\text{Hilb}_c(A)^u) is a Morita equivalence [BEa, 18.15]. Since $K^{\ast}\text{Cat}$ is Morita invariant and extends the $K$-theory functor for $C^\ast$-algebras we get the equivalence \[ K(\text{Res}^Z(A)) \cong K^{\ast}\text{Cat}(A) \cong K^{\ast}\text{Cat}(\text{Res}^Z(\text{Hilb}_c(A)^u)). \] (4.8)

The equivalence in Assertion [1] is the composition of the equivalences (4.6), (4.7) and (4.8).

We let $\sigma$ denote the generator of $\mathbb{Z}^\prime$. Then for $n$ in $\mathbb{Z}_{\min,\min}$ we have $\sigma(n) = n + 1$. We will show that the composition $\phi \circ (4.6)$ and the equivalence (4.8) preserve the action of $\sigma$ up to equivalence. Using the explicit description of the equivalence (4.6) given in the proof of [BEb, Prop. 10.1] the equivalence $\phi \circ (4.6)$ is induced by the functor $\phi : V_{\text{Hilb}_c(A)}(\mathbb{Z}_{\min,\min}) \rightarrow V_{\text{Res}^Z(\text{Hilb}_c(A))^\ast}(\ast) \cong \text{Res}^Z(\text{Hilb}_c(A))^u$.

The functor $\phi$ sends the object $(C, \rho, \mu)$ of $V_{\text{Hilb}_c(A)}(\mathbb{Z}_{\min,\min})$ to the submodule $C(0) := \mu(\{0\})C$ in $\text{Hilb}_c(A)^u$. In contrast to the general case considered in [BEb, Prop. 10.1] here we can take this preferred image of the projection $\mu(\{0\})$. The functor $\phi$ furthermore sends a morphism $B : (C, \rho, \mu) \rightarrow (C', \rho', \mu')$ in $V_{\text{Hilb}_c(A)}(\mathbb{Z}_{\min,\min})$ to $\mu'(\{0\})B\mu(\{0\}) : C(0) \rightarrow C'(0)$.

Note that $\phi(\sigma(C, \rho, \mu)) = \phi((C, \rho, \sigma_*\mu)) = (\sigma_*\mu)(\{0\})C = \mu(\{-1\})C =: C(-1)$.

Using the invariance of $\mu$ we see that the unitary multiplier $\rho_\sigma$ restricts to a unitary multiplier $\rho_{\sigma, -1, 0} : C(-1) \rightarrow \sigma C(0)$. We can therefore define a natural unitary isomorphism $v : \phi \circ \sigma \rightarrow \sigma \circ \phi$ such that its evaluation at $(C, \rho, \mu)$ is given by $\rho_{\sigma, -1, 0}$. Since $K^{\ast}\text{Cat}$ sends unitarily equivalent functors to equivalent maps [BEa, Lem. 17.11] we conclude that $\phi \circ (4.6)$ preserves the action of $\sigma$ up to equivalence.

In order to show that (4.8) commutes with $\sigma$ up to equivalence we construct a natural unitary isomorphism $w : \psi \circ \sigma \rightarrow \sigma \circ \psi$. The evaluation of $w$ at the unique object of $A$ is the unitary isomorphism $\sigma^{-} : A \rightarrow \sigma A$ of Hilbert $A$-modules, where $\sigma A$ is the vector space $A$ with the new Hilbert $A$-module described in [BEa, Ex. 2.10], and $a \mapsto \sigma a$ is the action of $\mathbb{Z}^\prime$ on $A$.

Assertion [2] follows from [BEb, Prop. 2.8.3] applied to $X = \ast$ and $G = \mathbb{Z}$.

Assertion [3] follows from Assertion [1] and the equivalence in (3.19).

Finally, the last assertion is a consequence of Proposition 3.18 and the observation that the equivalence in Assertion [1] preserves the $\sigma$-actions up to equivalence. \[ \square \]
5 What is in $\mathbb{Z}Sp\mathcal{X}_\text{loc}(\text{disc})$

In this section we again consider the example $E^Z = K\mathcal{X}^Z_C$. By Corollary 4.4 we have $\mathbb{Z}Sp\mathcal{X}_\text{loc}(\text{disc}) \subseteq PV_{K\mathcal{X}^Z_C}$. Hence we know that the coarse PV-square (1.10) is cartesian for discrete $X$. In the present section we show that $\mathbb{Z}Sp\mathcal{X}_\text{loc}(\text{disc})$ contains the motives of many non-discrete $\mathbb{Z}$-bornological coarse spaces. The main result is the following theorem.

**Theorem 5.1.** Assume one of the following:

1. $X$ has weakly finite asymptotic dimension.
2. $X$ has bounded geometry and the coarse Baum-Connes assembly maps $\mu_{K\mathcal{X}^Z_C(- \otimes \mathbb{Z}\text{min,min}),X}$ and $\mu_{K\mathcal{X}^Z_C(- \otimes \mathbb{Z}\text{can,min}),X}$ are equivalences.

Then $Yo^*_\text{loc}(X) \in \mathbb{Z}Sp\mathcal{X}_\text{loc}(\text{disc})$ and the coarse PV-square (1.10) is cartesian.

**Proof.** By Corollary 4.4 the first part of the assertion implies the second.

Considering a bornological coarse space as a $\mathbb{Z}$-bornological coarse space we get a functor $\text{Res}_\mathbb{Z} : \text{BC} \to \mathbb{Z}\text{BC}$. A morphism in $\mathbb{Z}\mathcal{X}$ is called a local equivalence if it is sent to an equivalence by the functors $K\mathcal{X}^Z_C(- \otimes \mathbb{Z}\text{min,min})$ and $K\mathcal{X}^Z_C(- \otimes \mathbb{Z}\text{can,min})$ which are considered as non-equivariant homology theories. As in the $\mathbb{Z}$-equivariant case we let $\ell : \mathbb{Z}\mathcal{X} \to \mathbb{Z}Sp\mathcal{X}_\text{loc}$ denote the localization at the local equivalences. We get a commutative diagram

If $X$ in $\mathbb{Z}\mathcal{X}$ satisfies Assumption 1.4.1 then $Yo^*_\text{loc}(X) \in \mathbb{Z}\mathcal{X}(\text{disc})$ by [BE20b, Thm 5.59]. This immediately implies that $\text{Res}_\mathbb{Z}(Yo^*_\text{loc}(X)) \in \mathbb{Z}Sp\mathcal{X}_\text{loc}(\text{disc})$.

We now assume that $X$ satisfies Assumption 1.4.2. Since the assertion of the theorem only depends on the coarse equivalence class of $X$ we can assume that $X$ has strongly bounded geometry [BE20b, Def. 7.75]. Then for every coarse entourage $U$ of $X$ the Rips complex $P_U(X)$ [BE20a, Ex. 2.6] is a finite-dimensional simplicial complex. The spherical path metric induces a bornological coarse structure on the Rips complex such that $X \to P_U(X)$ is an equivalence of bornological coarse spaces. Recall from [BE20a, Def. 9.7] that the...
universal coarse assembly map is induced by the morphism
\[ \mu_\mathcal{Y}_0^s, X : \colim_{U \in C_X} \mathcal{Y}_0^s(\mathcal{O}^\infty(P_U(X))) \to \Sigma \mathcal{Y}_0^s(X) \] (5.1)
derived from the cone sequence. For any non-equivariant strong homology theory \( F : \mathcal{BC} \to \mathcal{M} \) the coarse assembly map \( \mu_{F,X} \) from (1.11) is then given by \( \mu_{F,X} \simeq F(\mu_\mathcal{Y}_0^s, X) \).

The Assumption [1,4,12] together with Remark [3,19] imply that \( \mu_\mathcal{Y}_0^s, X \) is a local equivalence. Hence applying \( \ell \) to (5.1) we get the equivalence
\[ \mu_\mathcal{Y}_0^s_{\text{loc}}, X : \colim_{U \in C_X} \mathcal{Y}_0^s_{\text{loc}}(\mathcal{O}^\infty(P_U(X))) \to \Sigma \mathcal{Y}_0^s_{\text{loc}}(X) \] (5.2)
in \( \text{Sp}_{\mathcal{X}_{\text{loc}}} \). The functor \( \mathcal{Y}_0^s_{\text{loc}} \circ \mathcal{O}^\infty \) is homotopy invariant and excisive for closed decompositions of uniform bornological coarse spaces. Furthermore if \( Y \) is a set, then
\[ \mathcal{Y}_0^s_{\text{loc}}(\mathcal{O}^\infty(Y_{\text{min,min,disc}})) \simeq \Sigma \mathcal{Y}_0^s_{\text{loc}}(Y_{\text{min,min}}) \in \text{Sp}_{\mathcal{X}_{\text{loc}}}(\text{disc}) \, . \]
Using that \( \text{Sp}_{\mathcal{X}_{\text{loc}}}(\text{disc}) \) is a thick subcategory of \( \text{Sp}_{\mathcal{X}_{\text{loc}}} \) and \( P_U(X) \) is finite-dimensional we can now use a finite induction by the skeleta of \( X \) in order to conclude that
\[ \mathcal{Y}_0^s_{\text{loc}}(\mathcal{O}^\infty(P_U(X))) \in \text{Sp}_{\mathcal{X}_{\text{loc}}}(\text{disc}) \, . \]
Since \( \text{Sp}_{\mathcal{X}_{\text{loc}}}(\text{disc}) \) even localizing (5.2) finally implies that
\[ \mathcal{Y}_0^s_{\text{loc}}(X) \in \text{Sp}_{\mathcal{X}_{\text{loc}}}(\text{disc}) \, . \]

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