Determination of $U(1)_A$ restoration from pion and $a_0$-meson screening masses: Toward the chiral regime

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We incorporate the effective restoration of $U(1)_A$ symmetry in the 2+1 flavor entanglement Polyakov-loop extended Nambu–Jona-Lasinio (EPNJL) model by introducing a temperature-dependent strength $K(T)$ to the Kobayashi-Maskawa-'t Hooft (KMT) determinant interaction. The coupling constant $K$ is determined from the results of state-of-the-art lattice QCD simulations on pion and $a_0$-meson screening masses. The strength is suppressed in the vicinity of the pseudocritical temperature of chiral transition and hence much faster than the instanton suppression estimated by Pisarski and Yaffe. The EPNJL model shows that the chiral transition is second order at the “light-quark chiral-limit” point where the light quark mass is zero and the strange quark mass is fixed at the physical value. This indicates that there exists a tricritical point. Hence the location is estimated.

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I. INTRODUCTION

Meson masses are important quantities to understand the properties of Quantum Chromodynamics (QCD) vacuum. For example, the difference between pion and sigma meson masses is mainly originated in the spontaneous breaking of chiral symmetry, so that the restoration can be determined from temperature ($T$) dependence of the mass difference. Similar analysis is possible for the effective restoration of $U(1)_A$ symmetry through the difference between pion and $a_0$-meson (Lorentz scalar and isovector meson) masses.

$U(1)_A$ symmetry is explicitly broken by the axial anomaly and the current quark mass. In the effective model, the $U(1)_A$ anomaly is simulated by the Kobayashi-Maskawa-'t Hooft (KMT) determinant interaction [1, 2]. The coupling constant $K$ of the KMT interaction is proportional to the instanton density screened by the medium with finite $T$ [3]. Hence $K$ becomes small as $T$ increases: $K \propto K(T)$. Pisarski and Yaffe estimated the suppression $S(T) = K(T)/K(0)$ for high $T$, say $T \gtrsim 2T_c$ for the pseudocritical temperature $T_c$ of chiral transition, by calculating the Debye-type screening [3]:

$$S(T) = \exp \left[ -\pi^2 \rho^2 T^2 \left( \frac{2}{3} N_c + \frac{1}{3} N_f \right) \right]$$

where $N_c$ ($N_f$) is the number of colors (flavors) and the typical instanton radius $\rho$ is about 1/3 fm, and hence the suppression parameter $b$ is of order $0.7T_c$ for $N_c = N_f = 3$ of our interest [4]; note that 2+1 flavor QCD simulations show $T_c = 154 \pm 9$ MeV [5, 7]. This phenomenon is called “effective restoration of $U(1)_A$ symmetry”, since $U(1)_A$ symmetry is always broken in the current-operator level but effectively restored at higher $T$ in the vacuum expectation value.

Figure 1 shows the current status of knowledge on 2+1 flavor phase diagram for various values of light-quark mass $m_l$ and strange-quark mass $m_s$. QCD shows a first-order phase transition associated with the breaking of chiral ($Z_3$) symmetry at the lower left (upper right) corner [8, 9]. When $m_l$ and $m_s$ are finite, these first-order transitions become second order of 3d Ising ($Z(2)$) universality class, as shown by the solid lines [8, 9]. However, the order of chiral transition is unknown on the vertical line of $m_l = 0$ and $m_s > 0$, and it is considered to be related to the effective $U(1)_A$ restoration. In the two-flavor chiral limit of $(m_l, m_s) = (0, \infty)$ at the the upper left corner, for example, the order may be second order of $O(4)$ class if the effective restoration is not completed at $T = T_c$, because the chiral symmetry becomes $SU_L(2) \times SU_R(2)$ isomorphic to $O(4)$ in the situation and the transition is then expected to be in the 3d $O(4)$ universality class [8, 9]. There are many lattice QCD (LQCD) simulations made so far to clarify the order in the two-flavor chiral limit of $(m_l, m_s) = (0, \infty)$ and the light-quark chiral limit where $m_l$ vanishes with $m_s$ fixed at the physical value, but these are still controversial; see Refs. [10–21] and therein.

Very recently, the effective restoration of $U(1)_A$ symmetry has been investigated by state-of-the-art LQCD simulations on screening masses of pion and $a_0$-meson [22] and susceptibilities for the operators ($\pi^4$ and $\delta^4$) corresponding to pion and $a_0$-meson [23, 24]. The effective restoration of $U(1)_A$ symmetry thus becomes an important current issue.

In LQCD, pole and screening masses are evaluated from the exponential decay of mesonic correlation functions in the temporal and spatial directions, respectively, but for finite $T$ the lattice size is smaller in the temporal direction than in the spatial direction. This makes LQCD simulations less feasible for pole masses than for screening masses. The problem is getting serious as $T$ increases. This is the reason why meson screening masses are calculated in most of LQCD simulations. In fact, as mentioned above, state-of-the-art LQCD calculations were done for meson screening masses with large...
This problem can be solved by introducing the Lorentz-invariant Pauli-Villars (PV) regularization [46].

Even after the unphysical oscillations are removed, heavy numerical calculations are still required to obtain \( \eta_{\xi}(r) \) at large \( r \) [45]. This is the second problem. In the model calculation, the spatial correlation function is obtained first in the momentum representation \( \bar{q} \chi_{\xi}(0, \bar{q}^2) \). Hence we have to make the Fourier transform from \( \chi_{\xi}(0, \bar{q}^2) \) to \( \eta_{\xi}(r) \):

\[
\eta_{\xi}(r) = \frac{1}{4\pi^2ir} \int_{-\infty}^{\infty} dq \bar{q} \chi_{\xi}(0, q^2)e^{iqr}.
\]

The \( \bar{q} \) integration is quite particular at large \( r \), since the integrand consists of a slowly damping function \( \bar{q} \chi_{\xi}(0, \bar{q}^2) \) and a highly oscillating function \( e^{iqr} \). If \( \chi_{\xi}(0, \bar{q}^2) \) has a pole below the cut in the complex \( \bar{q} \) plane, one can easily determine \( M_{\xi,scr} \) from the pole location. In the old formulation of Ref. [45], the condition was not satisfied, since logarithmic cuts appear in the vicinity of the real \( \bar{q} \) axis in addition to physical cuts. Very recently we solved the problem in our previous paper [47], showing that the logarithmic cuts near the real \( \bar{q} \) axis are unphysical and removable. In the new formulation based on the PV regularization, there is no logarithmic cut and a pole appears below physical cuts, as shown later.

In this paper, we incorporate the effective restoration of \( U(1)_A \) symmetry in the 2+1 flavor EPNJL model by introducing a \( T \)-dependent coupling strength \( K(T) \) to the KMT interaction. \( T \) dependence of \( K(T) \) is well determined from state-of-the-art 2+1 flavor LQCD results [22] on pion and \( a_0 \)-meson screening masses. For the derivation of meson screening mass, we extend the previous prescription of Ref. [47] for 2 flavors to 2+1 flavors. The \( K(T) \) determined from the LQCD data is suppressed near \( T_c \) and much faster than the instanton suppression [1]. Using the parameter set, we show that the chiral transition is second order in the light-quark chiral limit. This result indicates that there exists a tricritical point near the “light-quark chiral-limit” point in the \( m_l-m_s \) plane. We then estimate the location.

We recapitulate the EPNJL model and the method of calculating meson screening masses in Sec. II and show the results of numerical calculations in Sec. III, Section IV is devoted to a summary.

\section{Model Setting}

\subsection{EPNJL model}

We start with the 2+1 flavor EPNJL model [38, 39]. The Lagrangian density is

\[
\mathcal{L} = \bar{\psi}(i\gamma_k D^k - \bar{m}_0)\psi + G_\xi(\Phi) \sum_{a=0}^{8} \left[ \left( \bar{\psi} \lambda_a \psi \right)^2 + \left( \bar{\psi} i\gamma_5 \lambda_a \psi \right)^2 \right] - K(T) \left[ \det \bar{\psi}_f(1 + \gamma_5)\psi_f + \det \bar{\psi}_f(1 - \gamma_5)\psi_f \right] - \mathcal{U}(\Phi[A], \bar{\Phi}[A], T)
\]

with quark fields \( \psi = (\psi_u, \psi_d, \psi_s) \) and \( D^\nu = \partial^\nu + iA^\nu \) with \( A^\nu = \delta^\nu_0 g(A^0) a_t a/2 = -\delta^\nu_0 i g(A_1) a_t a/2 \) for the gauge
coupling $g$, where the $\lambda_a$ ($\tau_a$) are the Gell-Mann matrices in flavor (color) space and $A_0 = \sqrt{2/M}$ for the unit matrix $I$ in flavor space. The determinant in (3) is taken in flavor space. For the 2+1 flavor system, the current quark masses $m_0 = \text{diag}(m_u, m_d, m_s)$ satisfy a relation $m_s > m_d \equiv m_u = m_q$.

In the EPNJL model, the coupling strength $G_s(\Phi)$ of the scalar-type four-quark interaction depends on the Polyakov loop $\Phi$ and its Hermitian conjugate $\bar{\Phi}$ as

$$G_s(\Phi) = G_s(0) \left[ 1 - \alpha_1 \Phi \bar{\Phi} - \alpha_2 (\Phi^3 + \bar{\Phi}^3) \right]. \tag{5}$$

This entanglement coupling is charge-conjugation and $Z_3$ symmetric. When $\alpha_1 = \alpha_2 = 0$, the EPNJL model is reduced to the PNJL model. We set $\alpha_2 = 0$ for simplicity, since the $\alpha_2$ term yields the same effect as the $\alpha_1$ term in the present analysis. As shown later in Sec. III the value of $\alpha_1$ is determined from LQCD data on pion and $\sigma_0$-meson screening masses; the resulting value is $\alpha_1 = 1.0$.

For $T$ dependence of $K(T)$, we assume the following form:

$$K(T) = \begin{cases} K(0) & (T < T_1) \\ K(0)e^{-(T-T_1)^2/\beta} & (T \geq T_1) \end{cases}. \tag{6}$$

For high $T$ satisfying $T \gg T_1$, the form (6) is reduced to (11). As shown later in Sec. III the values of $T_1$ and $\beta$ are well determined from LQCD data on pion and $\sigma_0$-meson screening masses; the resulting values are $T_1 = 0.79 T_\text{c} = 121$ MeV and $\beta = 0.23 T_\text{c} = 36$ MeV.

In the EPNJL model, the time component of $A_\mu$ is treated as a homogeneous and static background field, which is governed by the Polyakov-loop potential $U$. In the Polyakov gauge, $\Phi$ and $\bar{\Phi}$ are obtained by

$$\Phi = \frac{1}{3} \text{tr}_c(L), \quad \bar{\Phi} = \frac{1}{3} \text{tr}_c(L^*) \tag{7}$$

with $L = \exp[iA_4/T] = \exp[i\text{diag}(A_1^0, A_2^0, A_3^0)/T]$ for real variables $A_i^0$ satisfying $A_4^0 + A_2^0 + A_3^0 = 0$. For zero quark chemical potential where $\Phi = \bar{\Phi}$, one can set $A_3^0 = 0$ and determine the others as $A_4^0 = -A_1^0 = \cos^{-1}(3\Phi - 1)/2$.

We use the logarithm-type Polyakov-loop potential of Ref. [33] as $U$. The parameter set in $U$ has been determined from LQCD data at finite $T$ in the pure gauge limit. The potential has a parameter $T_0$ and yields a first-order deconfinement phase transition at $T = T_0$. The parameter used to be set to $T_0 = 270$ MeV, since LQCD data show the phase transition at $T = 270$ MeV in the pure gauge limit. In full QCD with dynamical quarks, however, the PNJL model with this value of $T_0$ is found not to explain LQCD results. Nowadays, $T_0$ is then rescaled to reproduce the LQCD results. In the present case, we take $T_0 = 180$ MeV so that the EPNJL model can reproduce LQCD results for the pseudocritical temperature $T_c^{\text{deconf}} = 165$ MeV in the EPNJL model and $170 \pm 7$ MeV in LQCD [48].

Making the mean field approximation (MFA) to (3) leads to the linearized Lagrangian density

$$\mathcal{L}^{\text{MFA}} = \bar{\psi}S^{-1}\psi - U_M - U(\Phi[A], \bar{\Phi}[A], T) \tag{8}$$

with the quark propagator

$$S = (i\gamma_\nu \partial^\nu - i\gamma_0 A_4 - \tilde{M})^{-1}, \tag{9}$$

where $\tilde{M} = \text{diag}(M_u, M_d, M_s)$ with

$$M_u = m_u - 4G_u(\Phi)\sigma_u + 2K(T)\sigma_d\sigma_s,$$
$$M_d = m_d - 4G_u(\Phi)\sigma_d + 2K(T)\sigma_u\sigma_s,$$
$$M_s = m_s - 4G_u(\Phi)\sigma_s + 2K(T)\sigma_u\sigma_d,$$

and $\sigma_f$ means the chiral condensate $\langle \bar{\psi}_f \psi_f \rangle$ for flavor $f$. The mesonic potential $U_M$ is

$$U_M = 2G_u(\Phi)(\sigma_u^2 + \sigma_d^2 + \sigma_s^2) - 4K(T)\sigma_u\sigma_d\sigma_s.$$

Making the path integral over quark fields, one can get the thermodynamic potential (per unit volume) as

$$\Omega = U_M + U - 2 \sum_{f=u,d,s} \int \frac{d^3p}{(2\pi)^3} [3E_{p,f}$$
$$+ \frac{1}{\beta} \ln [1 + 3(\Phi + \Phi e^{-\beta E_{p,f}}) e^{-\beta E_{p,f}} + e^{-3\beta E_{p,f}}]$$
$$+ \frac{1}{\beta} \ln [1 + 3(\Phi + \Phi e^{-\beta E_{p,f}}) e^{-\beta E_{p,f}} + e^{-3\beta E_{p,f}}]] \tag{10}$$

with $E_{p,f} = \sqrt{p^2 + M_f^2}$ and $\beta = 1/T$. We determine the mean-field variables $(X = \sigma_1, \sigma_3, \Phi, \bar{\Phi})$ from the stationary conditions:

$$\frac{\partial \Omega}{\partial X} = 0, \tag{11}$$

where isospin symmetry is assumed for the light-quark sector, i.e., $\sigma_1 \equiv \sigma_3 \equiv \sigma_u = \sigma_d$.

On the right-hand side of (10), the first term (vacuum term) in the momentum integral diverges. We then use the PV regularization [45, 46]. In the scheme, the integral $I(M_f, q)$ is regularized as

$$I^{\text{reg}}(M_f, q) = \sum_{\alpha=0}^{2} C_\alpha I(M_f, q), \tag{12}$$

where $M_{f,0} = M_f$ and the $M_{f,\alpha}$ ($\alpha \geq 1$) mean masses of auxiliary particles. The parameters $M_{f,\alpha}$ and $C_\alpha$ should satisfy the condition $\sum_{\alpha=0}^{2} C_\alpha = \sum_{\alpha=0}^{2} C_\alpha M_{f,\alpha}^2 = 0$. We then assume $(C_0, C_1, C_2) = (1, 1, -2)$ and $(M_{f,1}^2, M_{f,2}^2) = (M_f^2 + 2A^2, M_f^2 + A^2)$. We keep the parameter $A$ finite even after the subtraction (12), since the present model is non-renormalizable. The parameters are taken from Ref. [49] and they are $m_u = 6.2$ MeV, $m_d = 175.0$ MeV, $G_u(0)A^2 = 2.35$ and $K(0)A^5 = 27.8$ for $A = 795$ MeV. This parameter set reproduces mesonic observables at vacuum, i.e., the pion and kaon decay constants $(f_\pi = 92$ MeV and $f_K = 105$ MeV) and their masses $(M_\pi = 141$ MeV and $M_K = 512$ MeV) and the $\eta'$-meson mass $(M_{\eta'} = 920$ MeV). In the present work, we analyze LQCD results of Ref. [22] for pion and $\sigma_0$-meson screening masses. In the LQCD simulation, the pion mass $M_\pi(0)$ at vacuum ($T = 0$) is $175$ MeV and a bit heavier than the experimental value 138 MeV. We then change $m_0$ to 9.9 MeV in the EPNJL model in order to reproduce $M_\pi = 175$ MeV.
B. Meson pole mass

We derive the equations for pion and $a_0$-meson pole masses, following Ref. [34, 50]. The current corresponding to a meson of type $\xi$ is 

$$J_\xi(x) = \bar{\psi}(x) \Gamma_\xi \psi(x) - \langle \bar{\psi}(x) \Gamma_\xi \psi(x) \rangle,$$

(13)

where $\Gamma_\pi = \gamma_5 \lambda_3$ for $\pi$ meson and $\Gamma_{a_0} = \lambda_3$ for $a_0$-meson. We denote the Fourier transform of the mesonic correlation function $\eta_{\xi \xi}(x) \equiv \langle 0 | T \left( J_\xi(x) J_\xi^\dagger(0) \right) | 0 \rangle$ by $\chi_{\xi \xi}(q^2)$ as

$$\chi_{\xi \xi}(q^2) = \chi_{\xi \xi}(q_0^2, q^2) = i \int d^4xe^{iq\cdot x} \eta_{\xi \xi}(x),$$

(14)

where $q = (q_0, q)$ and $T$ stands for the time-ordered product. Using the random-phase (ring) approximation, one can obtain the Schwinger-Dyson equation

$$\chi_{\xi \xi} = \Pi_{\xi \xi} + 2 \sum_{\xi' \xi''} \Pi_{\xi' \xi''} G_{\xi' \xi''} \chi_{\xi'' \xi'}$$

(15)

for $\chi_{\xi \xi}$, where $G_{\xi' \xi''}$ is an effective four-quark interaction and $\Pi_{\xi' \xi''}$ is the one-loop polarization function defined by

$$\Pi_{\xi \xi'}(q^2) = (-i) \int \frac{d^4p}{(2\pi)^4} Tr \left( \Gamma_\xi S(p' + q) \Gamma_{\xi'} S(p') \right)$$

(16)

with $p' = (p_0 + iA_4, p)$, where the trace $Tr$ is taken in flavor, Dirac and color spaces. Here the quark propagator $S(q)$ in momentum space is diagonal in flavor space: $S(q) = \text{diag}(S_u, S_d, S_s)$. For $\xi = \pi$ and $a_0$, furthermore, $G_{\xi \xi'}$ and $\Pi_{\xi \xi'}$ are diagonal ($G_{\xi \xi'} = G_{\xi} \delta_{\xi \xi'}$, $\Pi_{\xi \xi'} = \Pi_{\xi} \delta_{\xi \xi'}$), because we impose isospin symmetry for the light-quark sector and employ the random-phase approximation. One can then easily solve the Schwinger-Dyson equation for $\xi = \pi$ and $a_0$:

$$\chi_{\xi \xi} = \Pi_{\xi \xi} \left( \frac{1}{1 - 2G_{\xi} \Pi_{\xi \xi}} \right)$$

(17)

with the effective couplings $G_\pi$ and $G_{a_0}$ defined by

$$G_{a_0} = G_\pi(\Phi) + \frac{1}{2} K(T) \sigma_s,$$

(18)

$$G_\pi = G_\pi(\Phi) - \frac{1}{2} K(T) \sigma_s.$$  

As for $T = 0$, $\Pi_\pi$ and $\Pi_{a_0}$ have the following explicit forms:

$$\Pi_{a_0} = \int \frac{d^4k}{(2\pi)^4} \left( \gamma_\mu (p' + q)^\mu + M_f \right) \left( \gamma_\mu (p + q)^\mu + M_f \right)$$

$$\times \left( \left( p' + q \right)^2 - M_f^2 \right) \left( p + q \right)^2 - M_f^2$$

$$= 4i[I_1 + I_2 - q^2 - 4M_f^2 I_3],$$

(20)

$$\Pi_\pi = \int \frac{d^4k}{(2\pi)^4} \left( \gamma_\mu (p' + q)^\mu + M_f \right) \left( \gamma_\mu (p + q)^\mu + M_f \right)$$

$$\times \left( \left( p' + q \right)^2 - M_f^2 \right) \left( p + q \right)^2 - M_f^2$$

$$= 4i[I_1 + I_2 - q^2 I_3],$$

(21)

and

$$I_1 = \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c,d} \left[ \frac{1}{p^2 - M} \right],$$

(22)

$$I_2 = \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c,d} \left[ \frac{1}{(p' + q)^2 - M} \right],$$

(23)

$$I_3 = \int \frac{d^4p}{(2\pi)^4} \text{tr}_{c,d} \left[ \frac{1}{(p' + q)^2 - M^2} \right].$$

(24)

where $\text{tr}_{c,d}$ ($\text{tr}_c$) means the trace in color and Dirac space, respectively, and $M = M_u = M_d$. For finite $T$, the corresponding equations are obtained by the replacement

$$p_0 \rightarrow i\omega_n = i(2n + 1)\pi T,$$

$$\int \frac{d^4p}{(2\pi)^4} \rightarrow iT \int_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}.$$

(25)

The meson pole mass $M_{\xi \text{pole}}$ is a pole of $\chi_{\xi \xi}(q_0^2, q^2)$ in the complex $q_0$ plane. Taking the rest frame $q = (q_0, 0)$ for convenience, one can get the equation for $M_{\xi \text{pole}}$ as

$$[1 - 2G_{\xi} \Pi_{\xi}(q_0^2, 0)] \bigg|_{q_0 = M_{\xi \text{pole}}, i\xi} = 0,$$

(26)

where $\Gamma$ is the decay width to $q\bar{q}$ continuum. The method of calculating meson pole masses is well established in the PNJL model [34, 50].

C. Meson screening mass

We derive the equations for pion and $a_0$-meson screening masses, following Ref. [47]. This is an extension of the method of Ref. [47] for 2 flavors to 2+1 flavors.

As mentioned in Sec. B it is not easy to make the Fourier transform from $\chi_{\xi \xi}(0, q^2)$ to $\eta_{\xi \xi}(r)$ particularly at large $r$. When the direct integration on the real $\tilde{q}$ axis is difficult, one can consider a contour integral in the complex $\tilde{q}$ plane by using the Cauchy’s integral theorem. However, $\chi_{\xi \xi}(0, \tilde{q}^2)$ has logarithmic cuts in the vicinity of the real $\tilde{q}$ axis [45], and it is reported in Ref. [45] that heavy numerical calculations are necessary for evaluating the cut effects. In our previous work [47], we showed that these logarithmic cuts are unphysical and removable. Actually, we have no logarithmic cut, when analytic continuation is made for the $I_3(q)$ after $p$ integration. Namely, the Matsubara summation over $n$ should be taken after the $p$ integration in (25). We then express $I_3$ as
an infinite series of analytic functions:

\[ F_3^{\text{eff}}(0, \tilde{q}^2) = iT \sum_{n=0}^{\infty} \sum_{\alpha} 2C_\alpha \]

\[ \times \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + M_{j,n,\alpha}^2} \frac{1}{p + q^2 + M_{j,n,\alpha}^2} \]

\[ = \frac{iT}{2\pi^2} \sum_{j,n,\alpha} C_\alpha \int_0^1 dx \int_0^\infty dk \frac{k^2}{(k^2 + (x-x^2)q^2 + M_{j,n,\alpha}^2)^2} \]

\[ = \frac{iT}{4\pi q} \sum_{j,n,\alpha} C_\alpha \sin^{-1} \left( \frac{\tilde{q}}{\sqrt{q^2 + M_{j,n,\alpha}^2}} \right) \tag{27} \]

with

\[ M_{j,n,\alpha}(T) = \sqrt{M_{\alpha}^2 + ((2n+1)\pi T + A_{\alpha}^{(3)})^2}, \tag{28} \]

where \( M_\alpha = M_{\text{vac},\alpha} = M_{\text{dir},\alpha} \). We have numerically confirmed that the convergence of \( n \)-summation is quite fast in (27). Each term of \( F_3^{\text{eff}}(0, \tilde{q}^2) \) has two physical cuts on the imaginary axis, one is an upward vertical line starting from \( \tilde{q} = 2iT M_{j,n,\alpha} \), and the other is a downward vertical line from \( \tilde{q} = -2iT M_{j,n,\alpha} \). The lowest branch point is \( \tilde{q} = 2iT M_{j=1,n=0,\alpha=0} \). Hence 2M\( j=1,n=0,\alpha=0 \) is regarded as "threshold mass" in the sense that the meson screening-mass spectrum becomes continuous above the point.

We can obtain the meson screening mass \( M_{\xi,\text{scr}} \) as a pole of \( \chi_{\xi,\xi}(0, \tilde{q}^2) \),

\[ [1 - 2G_\xi \Pi_\xi(0, \tilde{q}^2)]_{\tilde{q}=iM_{\xi,\text{scr}}} = 0. \tag{29} \]

If the pole at \( \tilde{q} = iM_{\xi,\text{scr}} \) is well isolated from the cut, i.e., \( M_{\xi,\text{scr}} < 2M_{j=1,n=0,\alpha=0} \), one can determine the screening mass from the pole location without making the \( \tilde{q} \) integral. In the high-\( T \) limit, the condition tends to \( M_{\xi,\text{scr}} < 2\pi T \).

### III. NUMERICAL RESULTS

The EPNJL model has three adjustable parameters, \( \alpha_1 \) in the entanglement coupling \( G_{\text{e}}(\Phi) \) and \( T_1 \) in the KMT interaction \( K(T) \). These parameters can be clearly determined from LQCD data [22] for pion and \( a_0 \)-meson screening masses, \( M_{\pi,\text{scr}} \) and \( M_{a_0,\text{scr}} \), as shown below.

Figure 2 shows \( T \) dependence of \( M_{\pi,\text{scr}} \) and \( M_{a_0,\text{scr}} \). Best fitting is obtained when \( \alpha_1 = 1.0, T_1 = 0.79T_c = 121 \) MeV and \( b = 0.23T_c = 36 \) MeV. Actually, the EPNJL results (solid and dot-dash lines) with this parameter set well account for LQCD data [22] for both \( M_{\pi,\text{scr}} \) and \( M_{a_0,\text{scr}} \). The mass difference \( \Delta M_{\pi,\text{scr}} = M_{a_0,\text{scr}}(T) - M_{\pi,\text{scr}}(T) \) is sensitive to \( K(T) \) because of (18) and (19), and hence the values of \( b \) and \( T_1 \) were well determined from \( \Delta M_{\pi,\text{scr}}(T) \). In the region \( T \gtrsim 180 \) MeV where \( \Delta M_{\pi,\text{scr}}(T) \) is tiny, the slope of the solid or dot-dash line is sensitive to \( \alpha_1 \), as shown later in Fig. 5. Hence the value of \( \alpha_1 \) was also well determined from the slope. \( T \) dependence of \( M_{\pi,\text{scr}} \) and \( M_{a_0,\text{scr}} \) is thus a good quantity to determine the parameter set of effective models.

![Graph showing the dependence of \( M_{\pi,\text{scr}} \) and \( M_{a_0,\text{scr}} \) on \( T \).]

The \( K \) thus determined has stronger suppression than the Pisarski-Yaffe instanton estimation (1), since \( S(T) \) at \( T = 2T_c \) is much smaller in the present parameter set than in the Pisarski-Yaffe estimation. As a future work, it is quite interesting to clarify how the present suppression is explained by instantons.

In Fig. 2, the solid and dot-dash lines are lower than the threshold mass \( 2M_{j=1,n=0,\alpha=0} \) (dotted line). This guarantees that the \( M_{\pi,\text{scr}} \) and \( M_{a_0,\text{scr}} \) determined from the pole location in the complex-\( q \) plane agree with those from the exponential decay of \( \gamma_{\xi,\xi}(r) \) at large \( r \).

In the EPNJL model with the present parameter, the chiral susceptibility \( \chi_{\text{li}} \) for light quarks has a peak at \( T = 163 \) MeV, as shown later in Fig. 7(a). This indicates \( T_c = 163 \) MeV. The model result is consistent with LQCD data \( T_c = 154 \pm 9 \) MeV of Refs. 5, 6. For the deconfinement transition, meanwhile, the parameter \( T_0 \) is adjusted to reproduce LQCD data on \( T_{\text{deconf}} \), as already mentioned in Sec. II. In fact, the Polyakov-loop susceptibility \( \chi_{\Phi,\Phi} \) has a peak at \( T = 165 \) MeV in the EPNJL model, as shown in Fig. 7(b). The model result \( T_{\text{deconf}} = 165 \) MeV is consistent with LQCD data \( T_{\text{deconf}} = 170 \pm 7 \) MeV of Ref. 48.

Figure 3 shows \( T \) dependence of the renormalized chiral condensate \( \Delta l_{\text{s, s}} \) defined by

\[ \Delta l_{\text{s, s}} = \frac{\sigma_l(T) - \frac{m_s}{m_l} \sigma_s(T)}{\sigma_l(0) - \frac{m_s}{m_l} \sigma_s(0)}, \tag{30} \]

and the Polyakov loop \( \Phi \). The present EPNJL model well reproduces LQCD data [5] for the magnitude of \( \Delta l_{\text{s, s}} \) in addition to the value of \( T_c \). The present model overestimates LQCD data for the magnitude of \( \Phi \), although it yields a result consistent with LQCD for \( T_{\text{deconf}} \). The overestimation in the magnitude of \( \Phi \) is a famous problem in the PNJL model. Actually, many PNJL calculations have this overestimation.
This is considered to come from the fact that the definition of the Polyakov loop is different between LQCD and the PNJL model \cite{51,52}. In LQCD the definition is $\Phi_{\text{LQCD}} = \langle \text{tr}_c \exp[i \int_0^T d\tau A_4] \rangle / 3$, while in the PNJL model based on the Polyakov gauge and the mean field approximation the definition is $\Phi_{\text{PNJL}} = \text{tr}_c \exp[i \langle A_4 \rangle / T] / 3$, although both are order parameters of $Z_3$ symmetry \cite{51,52}.

Now we investigate effects of $T$-dependent KMT interaction $K(T)$ on $M_{\pi,\text{scr}}$ and $M_{a_0,\text{scr}}$. In Fig. 4 $T$-dependence of $K(T)$ is switched off; namely, results of the EPNJL model with $K(T) = K(0)$ are shown. One can see that $T$-dependence of $K(T)$ reduces the mass difference between $M_{\pi,\text{scr}}$ and $M_{a_0,\text{scr}}$ significantly in a range $150 \lesssim T \lesssim 180$ MeV, comparing Fig. 4 with Fig. 2. At $T = 176$ MeV where first-order chiral and deconfinement transitions take place, $M_{\pi,\text{scr}}$ has a jump while $M_{a_0,\text{scr}}$ has a cusp. Meson screening mass is thus a good indicator for a first-order transition.

Next we analyze effects of the entanglement coupling $G_s(\Phi)$ on $M_{\pi,\text{scr}}$ and $M_{a_0,\text{scr}}$. In Fig. 5 both $T$ dependence of $M_{\pi,\text{scr}}$ and $M_{a_0,\text{scr}}$ are sensitive to the value of $K(T)$ and the entanglement of $G_s(\Phi)$ are switched off; namely, results of the standard PNJL model with constant $K$ are shown. The PNJL model cannot reproduce the LQCD data at all. One can see that the entanglement coupling is significant at $T \gtrsim T_c$, comparing Fig. 5 with Fig. 4. In particular, the slopes of $M_{\pi,\text{scr}}$ and $M_{a_0,\text{scr}}$ at $T \gtrsim 180$ MeV are quite sensitive to the value of $\alpha_1$.

Figure 6 shows three types of EPNJL calculations for the mass difference $\Delta M_{\text{scr}}(T)$ between pion and $a_0$-meson screening masses. The solid, dot-dash and dotted lines denote results of the EPNJL model, the EPNJL model with $K(T) = K(0)$ and the standard PNJL model with $K(T) = K(0)$, respectively. See Fig. 2 for LQCD data.

Finally we consider the order of chiral transition near the physical point $(m_l, m_s) = (6.2\text{[MeV]}, 175\text{[MeV]})$ in the $m_l$--$m_s$ plane (in the Columbia plot). First $m_l$ is varied from 9.9
to 0 MeV with $m_s$ fixed at 175 MeV.

Figure 7 presents $T$ dependence of the chiral susceptibility $\chi_{ll}$ for light quarks and the Polyakov-loop susceptibility $\overline{\chi}_{\Phi\Phi}$ in three points, “simulation point (S-point)” of $(m_l, m_s) = (9.9$ [MeV], 175 [MeV]), “physical point (P-point)” of $(m_l, m_s) = (6.2$ [MeV], 175 [MeV]) and “light-quark chiral-limit point (C_l point)” of $(m_l, m_s) = (0$ [MeV], 175 [MeV]). In general, $T_c$ and $T^\text{deconf}_c$ determined from peak positions of $\chi_{ll}$ and $\overline{\chi}_{\Phi\Phi}$ depend on $m_l$ and $m_s$. However, as shown in panel (a), the $T_c$ thus determined is 163 MeV at S-point and 160 MeV at P-point, and hence the value little varies between the two points. At C_l point, $\chi_{ll}$ diverges at $T = T_c = 153$ MeV. The chiral transition is thus second order at C_l-point at least in the mean-field level. This result suggests that the effective $U(1)_A$ restoration is not completed at $T = T_c$. This suggestion is supported by LQCD data at S-point in Fig. 6 where $\Delta M_{\text{scr}}(T_c)$ is about a half of $\Delta M_{\text{scr}}(0)$.

As shown in panel (b), $m_l$ dependence of $T^\text{deconf}_c$ is even smaller; namely, $T^\text{deconf}_c = 165$ MeV for S-point and C_l-point and 163 MeV for P-point. At C_l-point, $\overline{\chi}_{\Phi\Phi}$ has a sharp peak at $T = 153$ MeV. It is just a result of the propagation of divergence from $\chi_{ll}$ to $\overline{\chi}_{\Phi\Phi}$, and never means that a second-order deconfinement takes place there.

Next, both $m_l$ and $m_s$ are varied near P-point. Figure 8 shows the value of $\log[\chi_{ll}(T_c)]$ near P-point in the $m_l$-$m_s$ plane. The value is denoted by a change in hue. Three second-order chiral transitions (solid lines) meet at $(m_l, m_s) \approx (0$ [MeV], 127 [MeV]). This is a tricritical point (TCP) of chiral phase transition.

IV. SUMMARY

In conclusion, we present a simple method for calculating meson screening masses in PNJL-like models. This allows us to compare model results with LQCD data on meson screening masses. Meson screening masses are quite useful to determine model parameters. In particular, the mass difference between pion and $a_0$-meson is effective to determine $T$ dependence of the KMT interaction. The EPNJL model with the present parameter set is useful for estimating the order of chiral transition at the light-quark chiral-limit point and the location of the tricritical point, since it is hard to reach the chiral regime directly with LQCD.

Fig. 7: $T$ dependence of (a) chiral susceptibility $\chi_{ll}$ and (b) Polyakov-loop susceptibility $\overline{\chi}_{\Phi\Phi}$ at S-point, P-point and C_l-point. Here $\chi_{ll}$ and $\overline{\chi}_{\Phi\Phi}$ are dimensionless and their definition is the same as in the LQCD formulation. Calculations are done by the EPNJL model with the present parameter set. The dotted, dot-dash and solid lines stand for the results at S-point, P-point and C_l-point, respectively. At C_l-point, $\chi_{ll}$ is divided by 10 and diverges at $T = T_c = 153$ MeV.

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Fig. 8: Order of chiral transition near physical point in the \(m_l - m_s\) plane. The value of \(\log|\chi_{ll}(T_c)|\) is shown by a change in hue. Simulation point, physical point, light-quark chiral-limit point and tricritical point are denoted by S, P, C\(_l\) and TCP. The solid lines stand for second-order chiral transitions.

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