A Line Source In Minkowski For The de Sitter Spacetime Scalar Green’s Function:
Massless Minimally Coupled Case

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Abstract

We show how, for certain classes of curved spacetimes, one might obtain its retarded or advanced minimally
coupled massless scalar Green’s function by using the corresponding Green’s functions in the higher dimensional
Minkowski spacetime where it is embedded. Analogous statements hold for certain classes of curved Riemannian
spaces, with positive definite metrics, which may be embedded in higher dimensional Euclidean spaces. The
general formula is applied to \((d \geq 2)\)-dimensional de Sitter spacetime, and the scalar Green’s function is demonstrated to be sourced by a line emanating infinitesimally close to the origin of the ambient \((d + 1)\)-dimensional
Minkowski spacetime and piercing orthogonally through the de Sitter hyperboloids of all finite sizes. This method
does not require solving the de Sitter wave equation directly. Only the zero mode solution to an ordinary differential equation, the “wave equation” perpendicular to the hyperboloid – followed by a one dimensional integral – needs to be evaluated. A topological obstruction to the general construction is also discussed by utilizing it to derive a generalized Green’s function of the Laplacian on the \((d \geq 2)\)-dimensional sphere.
I. MOTIVATION

From understanding correlations of metric fluctuations imprinted on the cosmic microwave photon background and other quantum phenomenon in a non-trivial gravitational field; to extracting the gravitational waveforms generated from the motion of compact bodies around (super-)massive black holes; to describing the propagation of gravitational and electromagnetic signals over cosmological length scales; curved spacetime Green’s functions are central to the quantitative analysis of the dynamics of our classical and quantum field theories. In most cases, the Green’s functions are solved directly by examining the wave equations in the particular curved metric of interest. Motivated by the increasing number of cosmological and gravitational applications, we wish to in this paper, initiate a search for alternate methods to compute them. We hope this will lead to novel representations of Green’s functions in curved space(time)s, and lend new insights into the physical problems themselves.

A number of 4-dimensional curved spacetimes of physical significance have known embeddings in higher dimensional flat spacetimes. For instance, (anti-)de Sitter spacetimes are hyperboloids. The general Friedmann-Lemaître-Robertson-Walker universe has an embedding that was known as early as in §V.D of [1]; while the Schwarzschild black hole’s complete embedding was found in [2], which built upon earlier work in [3]. Moreover, the flat spacetime Green’s functions are also known explicitly for all dimensions, and perhaps some use can be found for them in this context. In particular, since the retarded or advanced Green’s function $G[x, x']$ is nothing but the field seen by an observer at $x$, produced by a spacetime point source at $x'$ on the curved manifold; if this curved spacetime could be embedded in a higher dimensional Minkowski, one cannot help but wonder if a family of higher dimensional beings residing in the latter could somehow concoct an appropriate source to fool the observer that she is sensing a $\delta$-function source in her own curved world.

In this work we delineate how this thought experiment can indeed be performed at least for minimally coupled massless scalar fields propagating in de Sitter spacetime, in dimensions greater or equal to 2. Using its embedding as a hyperboloid in Minkowski of one higher dimension, we will show that the source is a line starting very close to the Minkowski origin, and runs perpendicularly through the de Sitter hyperboloids of all finite sizes. The spacetime location where the line intersects the de Sitter hyperboloid of the observer’s world is interpreted by her as the source of the de Sitter scalar Green’s function. From the perspective of the higher dimensional beings, however, scalar waves are being produced by every point on this line. In this picture one may therefore apply flat spacetime geometric intuition to understand the propagation properties of massless scalar signals in de Sitter
space. If this same thought experiment could ever be conducted for the Schwarzschild black hole – which is embeddable in 6-dimensional Minkowski, where massless fields propagate solely on the light cone\(^1\) – this could lend a different perspective on the propagation of gravitational waves generated by compact bodies orbiting around it, and perhaps provide alternate ways of addressing the associated self-force problem induced by the tail effect.

In section [II], we will derive the general construction, leading up to the formula in eq. (9). Following that we will review the Green’s functions of the Minkowski wave operator and the Euclidean Laplacian in all dimensions, highlighting how one may use the embedding viewpoint to obtain them from the \(d = 3\) Minkowski case. In section [IV] we will then apply the method to obtain the de Sitter spacetime minimally coupled massless scalar Green’s function, and in section [V] we discuss a topological obstruction to the general construction. There, we will propose a generalized Green’s function for the Laplacian on the \(d\)-sphere. We summarize our findings and lay out possible future directions in section [VI]. In appendix [A] we derive the geodesic distance between any two points in de Sitter spacetime; in appendix [B] we discuss the recursion relation between the metric on the \(d\)- and \((d - 1)\)-sphere, and use it to write down the (reduced) Laplacian on the \(d\)-sphere.

II. GENERALITIES

We shall first begin in \((d+n)\)-dimensional Minkowski spacetime endowed with Cartesian coordinates \(\{X^A|A = 0, 1, 2, \ldots, d + n - 1\}\). Its metric is

\[
ds^2 = \eta_{AB}dX^A dX^B \equiv dX \cdot dX,
\]

\[
\eta_{AB} \equiv \text{diag}[1, -1, \ldots, -1].
\]

Suppose we found a set of coordinates \(\{x^\mu, y^A|\mu = 0, 1, 2, \ldots, d-1; A = 1, 2, 3, \ldots, n\}\) that transforms the \((d+n)\)-dimensional Minkowski metric in (1), via

\[
X^A \rightarrow X^A[x, y]
\]

\[
dX^A \rightarrow \frac{\partial X^A[x, y]}{\partial x^\alpha}dx^\alpha + \frac{\partial X^A[x, y]}{\partial y^B}dy^B,
\]

into the form

\[
ds^2 = P^2[y]g_{\mu\nu}[x]dx^\mu dx^\nu + g_{AB}[y]dy^A dy^B,
\]

\(^1\) Massless fields in even dimensional Minkowski travel strictly on the light cone; whereas in odd dimensions they develop a tail, and travel both on and within the light cone; see equations (38) and (39).
where \( \bar{g}_{\mu\nu} \) does not depend on \( y \) while \( P \) and \( g_{AB}^0 \) do not depend on \( x \). Suppose further that the curved spacetime metric \( g_{\mu\nu}[x] \) of interest is realized on the surface \( y = y_0 \), i.e.

\[
g_{\mu\nu}[x] = P^2[y_0] \bar{g}_{\mu\nu}[x],
\]

with its wave operator \( \Box \) acting on a scalar \( \psi \) as

\[
\Box \psi = \frac{1}{P^2[y_0] \sqrt{|g|}} \partial_\mu \left( \sqrt{|\bar{g}|} \bar{g}^{\mu\nu} \partial_\nu \psi \right) \equiv \frac{1}{P^2[y_0]} \Box \psi,
\]

where \( \sqrt{|g|} \) is the square root of the absolute value of the determinant of \( \bar{g}_{\mu\nu} \) and \( \bar{g}^{\mu\nu} \) is its inverse. Then the Green’s function \( G_d[x, x'] \) – whose solution is the central goal of this paper – obeying the equation

\[
\Box_x G_d[x, x'] = \Box_{x'} G_d[x, x'] = \frac{\delta^{(d)}[x - x']}{|g[x]g[x']|^{1/4}} = \frac{\delta^{(d)}[x - x']}{P^d[y_0] |\bar{g}[x]|^{1/4}}
\]

can be deduced from its \((d + n)\)-dimensional flat spacetime counterpart \( G_{d+n}[X - X'] \), through the general formula

\[
G_d[x, x'] = \int d^n y' \sqrt{|g^{-1}[y']|} \frac{P^{d-2}[y']}{P^{d-2}[y_0]} N_0 \sum_{\sigma_{(0+)}} \psi_{\sigma_{(0+)}}^+[y'] G_{d+n}[X[x, y_0] - X'[x', y']].
\]

The \( G_{d+n}[X - X'] \) themselves, which can be found in equations (37) through (39) below, obey the equations

\[
\eta^{AB} \partial_A \partial_B G_{d+n}[X - X'] = \eta^{AB} \partial_A \partial_B G_{d+n}[X - X'] = \delta^{(d+n)}[X - X'],
\]

with the unprimed indices denoting derivatives with respect to \( X \) and the primed indices derivatives with respect to \( X' \). The \(|g^{-1}[y']|\) in eq. (8) is the absolute value of the determinant of the metric \( g^{AB}_{0}[y'] \); the \( \{ \psi_{\sigma_{(0+)}} \} \) is the collection of all regular and mutually orthogonal solutions to the zero mode equation

\[
D_y \psi_{\sigma_{(0+)}}[y] \equiv \partial_A \left( \frac{P^d \sqrt{|g^{-1}|} (g^{A\perp B}) \partial_B \psi_{\sigma_{(0+)}}}{P^{d-2} \sqrt{|g|}} \right) = 0;
\]

\( N_0 \) is the total number of such zero mode solutions; and \((g^{A\perp B})^A B\) is the inverse of \( g_{AB}^0 \). By orthogonality we mean \((\psi_1, \psi_2) = 0\), if we define the inner product to be

\[
(\psi_1, \psi_2) \equiv \int d^n y P^{d-2}[y] \sqrt{|g^{A\perp B} |} \psi_{\sigma_{(0+)}}^+[y] \psi_{\sigma_{(0+)}}[y].
\]
Notice that a $y$-independent constant is always a solution to eq. (11); we are, however, allowing for the possibility that there could be other non-trivial yet regular zero mode solutions.

**Justification of eq. (9)** In the ensuing discourse it is useful to recall that the Green’s function of some hermitian operator $\mathcal{O}$ can be expressed as a mode expansion in terms of its eigenvectors \{\ket{\lambda}\} and eigenvalues \{\lambda\}. (We are exploiting Dirac’s bra-ket notation here.) That is, if we have diagonalized $\mathcal{O}$ and solved all the

$$\mathcal{O}\ket{\lambda} = \lambda\ket{\lambda},$$  \tag{13}$$
the Green’s function equation

$$\mathcal{O}_{\xi} G[\xi,\xi'] = \mathcal{O}_{\xi'} G[\xi,\xi'] = \delta_{\xi,\xi'},$$  \tag{14}$$
can in turn be solved with

$$G[\xi,\xi'] = \sum_{\lambda} \frac{\langle \xi|\lambda\rangle\langle \lambda|\xi'\rangle}{\lambda}.$$
\tag{15}

because of the completeness relation

$$\sum_{\lambda} \langle \xi|\lambda\rangle\langle \lambda|\xi'\rangle = \delta_{\xi,\xi'}.$$
\tag{16}

For instance, the Green’s function in $(d+n)D$ flat spacetime admits the mode sum expansion

$$\tilde{G}_{d+n}[X - X'] = \sum_{\text{eigenvalues}} \frac{\langle X| - k^2 \rangle \langle -k^2|X'\rangle}{-k^2}$$

$$= \int \frac{d^{d+n}k}{(2\pi)^{d+n}} \exp[-ik\mathcal{A}X^{\mathcal{A}}] \exp[+ik\mathcal{B}X^{\mathcal{B}}] \exp[-\mathcal{E}\mathcal{D}k_c k_D].$$
\tag{17}

The plane waves $\exp[\pm ik\mathcal{A}X^{\mathcal{A}}]$ are eigenvectors of the scalar wave operator $\eta^{\mathcal{A}\mathcal{B}}\partial_{\mathcal{A}}\partial_{\mathcal{B}}$, and their common eigenvalue is $-\eta^{\mathcal{E}\mathcal{D}}k_c k_D$. If the flat metric in $(d+n)D$ takes the form in eq. (5), then a direct calculation reveals its associated scalar wave operator acting on a scalar $\psi$ is

$$\eta^{\mathcal{A}\mathcal{B}}\partial_{\mathcal{A}}\partial_{\mathcal{B}}\psi = \frac{1}{P^2[y]} \left( P^2[y_0] \Box_x \psi + D_y \psi \right)$$
\tag{18}

$$= \frac{1}{P^2[y]} \left( \Box_x \psi + D_y \psi \right),$$
\tag{19}

where $\Box_x$ is the scalar wave operator with respect to $g_{\mu\nu}[x]$ in eq. (6) (see eq. (7)) and $D$ has already been introduced in eq. (11). That the two terms inside the parenthesis in eq. (18) is in a separated form,
i.e. $\Box_x$ acts only on the variables $x$ and $\mathcal{D}_y$ only on the $y$, suggests that we should seek to diagonalize the operators $\Box$ and $\mathcal{D}$. Again employing Dirac’s notation – the relevant eigenvector equations are

\[
\Box_x \langle x | \lambda, \sigma(\lambda) \rangle = P^2[y_0]|\Box_x \langle x | \lambda, \sigma(\lambda) \rangle = \lambda \langle x | \lambda, \sigma(\lambda) \rangle \tag{20}
\]
\[
\mathcal{D}_y \langle y | \lambda^\perp, \sigma(\lambda^\perp) \rangle = \lambda^\perp \langle y | \lambda^\perp, \sigma(\lambda^\perp) \rangle \tag{21}
\]

Here, $\{\lambda\}$ are the eigenvalues of $\Box = P^2[y_0]\Box$ and $\{\lambda^\perp\}$ are those of $\mathcal{D}$. The $\sigma(\lambda)$ and $\sigma(\lambda^\perp)$ account for degeneracy: they label all the eigenvectors with the common eigenvalues $\lambda$ and $\lambda^\perp$, respectively.

Now, the operator $\Box$ is hermitian with respect to the inner product

\[
(\psi_1, \psi_2)_{\Box} = \int d^d x \sqrt{|g(x)|} \psi_1^\dagger(x) \psi_2(x), \tag{22}
\]

in that

\[
(\Box \psi_1, \psi_2)_{\Box} = (\psi_1, \Box \psi_2)_{\Box}, \tag{23}
\]

as long as the $\psi$s fall off quickly enough so that integration-by-parts do not yield surface terms. And with similar boundary conditions, the operator $\mathcal{D}$ defined in (11) is a self-adjoint operator with respect to eq. (12):

\[
(\mathcal{D} \psi_1, \psi_2) = (\psi_1, \mathcal{D} \psi_2). \tag{24}
\]

From the integration measure $P^{d-2} \sqrt{|g^\perp|}$ and $\sqrt{|g|}$ in equations (12) and (22) respectively, we can infer that the completeness relations obeyed by the orthonormal eigenvectors are

\[
\sum_\lambda \sum_{\sigma(\lambda)} \langle x | \lambda, \sigma(\lambda) \rangle \langle \lambda, \sigma(\lambda) | x' \rangle = \frac{\delta^{(d)}[x - x']}{|g(x)g(x')|^{1/4}} \tag{25}
\]
\[
\sum_{\lambda^\perp} \sum_{\sigma(\lambda^\perp)} \langle y | \lambda^\perp, \sigma(\lambda^\perp) \rangle \langle \lambda^\perp, \sigma(\lambda^\perp) | y' \rangle = \frac{\delta^{(4)}[y - y']}{P^{d-2}[y] P^{d-2}[y'] |g^\perp[y]g^\perp[y']|^{1/4}}. \tag{26}
\]

At this point, we assert that the Green’s function of $\eta^3 \partial_3 \partial_{3'}$, which we have earlier written explicitly in terms of its Fourier expansion in eq. (17), can also be expanded in terms of the eigenvectors of $\Box$ and $\mathcal{D}$, namely

\[
\tilde{G}_{d+n} [X - X'] = \sum_\lambda \sum_{\sigma(\lambda)} \sum_{\lambda^\perp} \sum_{\sigma(\lambda^\perp)} \frac{\langle x | \lambda, \sigma(\lambda) \rangle \langle y | \lambda^\perp, \sigma(\lambda^\perp) \rangle \cdot \langle \lambda, \sigma(\lambda) | x' \rangle \langle \lambda^\perp, \sigma(\lambda^\perp) | y' \rangle}{\lambda + \lambda^\perp}. \tag{27}
\]

Applying $\eta^3 \partial_3 \partial_3$ or $\eta^3 \partial_3 \partial_{3'}$ on $\tilde{G}_{d+n} [X - X']$ using eq. (19), followed by the eigenvector and completeness relations (20), (21), (25), and (26), allow us to verify that the proposed mode expansion
in eq. (27) satisfies eq. (10) written in the coordinates \((x, y)\) and \((x', y')\),

\[
\eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} \bar{G}_{d+n} \left[ X - X' \right] = \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} \bar{G}_{d+n} \left[ X - X' \right] = \delta^{(d)}[x - x'] \frac{\delta^{(n)}|y - y'|}{|\bar{g}[x]\bar{g}[x']|^{1/4} P^2_{y} P^2_{y'} |y^\perp| |y'^\perp|^{1/4}}.
\]

(28)

(We have used the form of the metric in eq. (5) to take its determinant.) Identifying

\[
\psi^\dagger \sigma_{(0^\perp)}[y] = \left< y | \lambda^\perp = 0, \sigma_{(0^\perp)} \right>
\]

and inserting the mode expansion of eq. (27) into the general formula in eq. (9), one would find that, by the orthogonality of the eigenvectors, only the \(\lambda^\perp = 0\) terms in the \(\sum_{\lambda^\perp}\) would survive, leaving

\[
G_d[x, x'] = \frac{1}{P^{d-2}[y_0]} \sum_{\lambda} \sum_{\sigma_{(\lambda)}} \left< x | \lambda, \sigma_{(\lambda)} \right> \left< \lambda, \sigma_{(\lambda)} | x' \right>.
\]

(30)

Acting \(\Box\) on both sides of eq. (30), invoking the eigenvector equation (20) and completeness relation eq. (25) once again, we therefore conclude the general formula indeed solves the desired equation (8).

Let us remark that, despite the apparent asymmetric treatment of the variables \((x, y)\) versus \((x', y')\) in eq. (9) – we are integrating with respect to \(y'\) and not \(y\) – our derivation here shows that the end result, eq. (30), really is a solution with respect to both \(x\) and \(x'\). Moreover, because the plane waves in the mode expansion of eq. (17) are bounded and regular everywhere, it is therefore reasonable to demand that, if the mode expansion in eq. (27) is its equivalent, the zero mode solution of eq. (11) must be regular everywhere. Finally, notice that little use was actually made of the fact that the metric in eq. (5) has a zero Riemann curvature tensor. This implies, if a lower dimensional space(time) \(P^2[y_0]\bar{g}_{\mu\nu}[x]\) can be embedded, via eq. (5), in a higher dimensional curved space(time) whose scalar wave operator has a known Green’s function \(\bar{G}\), then formula (9) should still be applicable.

**A check** A more pragmatic route to check that eq. (9) works, if one is given an explicit embedding that takes the form in eq. (5), is to apply \(\Box\) on the solution in eq. (9).

\[
\Box x G_d[x, x'] = \int d^n y' \sqrt{|y^\perp[y']|} P^{d-2}[y'] 1 \sum_{\sigma_{(0^\perp)}} \frac{\psi_{(0^\perp)}[y']}{N_0} \Box x \bar{G}_{d+n} \left[ X[x, y] - X'[x', y'] \right].
\]

(31)

For now, we have left the \(y\) unevaluated, i.e. replaced \(y_0\) with \(y\). Let us re-express \(\Box\) in terms of

\(^2\text{The only modification required of eq. (9) is that, since translation invariance is no longer present in a general curved spacetime, }\bar{G}[X - X'] \text{ will now be } \bar{G}[X, X'], \text{ i.e. it is no longer a function of the difference } X - X'.\)
\( \eta^{AB} \partial_A \partial_B \) and \( D \), using eq. (19). Upon recalling eq. (28),

\[
\Box_x G_d[x, x'] = \frac{\delta^{(d)}[x - x']}{P^{d-2}[y] |g[x']g[x']|^{1/4}} - \frac{1}{N_0 P^{d-2}[y]} \sum_{\sigma(0+)} \frac{1}{\psi_{(0+)}^*} \mathcal{D}_y \left( \int d^n y' \sqrt{|g[y']|} P^{d-2}[y'] \bar{\psi}_{(0+)}[y'] G_{d+n} \left[ X[x, y] - X'[x', y'] \right] \right).
\]

If there is only one zero mode and if it is just a constant, as it would be for the cases considered in this paper, eq. (32) simplifies to

\[
\frac{1}{P^2[y]} \Box_x G_d[x, x'] = \frac{\delta^{(d)}[x - x']}{P^d[y] |g[x']g[x']|^{1/4}} - \frac{1}{P^d[y]} \mathcal{D}_y G_d
\]

with

\[
G_d \equiv \int d^n y' \sqrt{|g[y']|} P^{d-2}[y'] G_{d+n} \left[ X[x, y] - X'[x', y'] \right].
\]

When \( y = y_0 \), eq. (33) is the same wave equation in eq. (8) if and only if the integral \( G_d \) is itself a zero mode solution of eq. (11), i.e. iff it is annihilated by the \( D \). Therefore, for a given metric and its embedding in Minkowski spacetime, one need only to write down the corresponding \( G_d \) in eq. (34).

If the coordinate(s) \( y \) can be re-scaled away before the integral is even evaluated, so that \( G_d \) becomes independent of \( y \), then the solution of the Green’s function wave equation (8) is assured. We will, in fact, use these observations in the forthcoming sections as a check that our calculations are solving the proper wave equations.

**Causal structure**

We have just seen that the Green’s function in the \( d \) dimensional curved spacetime, given in eq. (9), is the flat spacetime Green’s function in \((d + n)\) dimensions projected along the zero modes of \( D \) in eq. (11). More geometrically, however, we can read eq. (9) as saying that the observer on the curved spacetime is really getting bathed by scalar particles produced from an (appropriately weighted) \( n \)-dimensional membrane in the embedding Minkowski, intersecting perpendicularly with the observer’s world.

Analogous statements hold, if we replace in this section all instances of “Minkowski” and “flat spacetime” with “Euclidean space,” \( \eta_{AB} \) with \( \delta_{AB} \) (the Kronecker delta), and “curved spacetime” for “curved Riemannian spaces” (with positive definite metrics). However, a subtlety that is present for curved spacetimes embedded in Minkowski but not for that of curved spaces in Euclidean, is the notation of casual influence: when cause precede effect (or effect precede cause) in the ambient Minkowski spacetime – as recorded, say, by the higher dimensional beings – under what circumstances do cause precede effect (effect precede cause) in the curved spacetime itself? In the same vein – when observer
and source are within each other’s light cone according to the Minkowski beings, does the same hold from the curved spacetime perspective too? We merely raise these issues here in the abstract, but will address it in some detail for de Sitter spacetime in section [IV].

III. FLAT SPACE(TIME)S

Because the flat space(time) Green’s functions lie at the core of the embedding calculations below, in this section, we record their explicit expressions in all relevant dimensions $d$. Both the Minkowski Green’s functions and their Euclidean counterparts in various dimensions are intimately connected to each other [4] – we will illuminate this using the embedding viewpoint and eq. (9).

Minkowski First define

$$\bar{\sigma} \equiv \frac{1}{2} (X - X')^2 \equiv \frac{1}{2} \eta_{AB} (X - X')^A (X - X')^B.$$ (35)

$\bar{\sigma}$, known as Synge’s world function, is half the square of the geodesic distance between $X$ and $X'$ in the $(d + n)$-dimensional flat spacetime. Also define the step function

$$\Theta[z] = 1, \quad z \geq 0$$

$$= 0, \quad z < 0.$$ (36)

The retarded $\bar{G}_d^+[X - X']$ and advanced $\bar{G}_d^-[X - X']$ Green’s functions are

$$\bar{G}_d^\pm[X - X'] = \Theta[\pm(X^0 - X'^0)]\bar{G}_d[\bar{\sigma}].$$ (37)

For even $d \geq 2$, the symmetric Green’s function $\bar{G}_d[\bar{\sigma}]$ is

$$\bar{G}_d[\bar{\sigma}] = \frac{1}{2(2\pi)^{d/2}} \left( \frac{\partial}{\partial \bar{\sigma}} \right)^{d/2} \Theta[\bar{\sigma}].$$ (38)

For odd $d \geq 3$, it is instead

$$\bar{G}_d[\bar{\sigma}] = \frac{1}{\sqrt{2}(2\pi)^{d/2}} \left( \frac{\partial}{\partial \bar{\sigma}} \right)^{d/2} \frac{\Theta[\bar{\sigma}]}{\sqrt{\bar{\sigma}}}. \quad (39)$$

The embedding perspective we are exploring here has in fact been exploited in [4] to obtain the recursion relations in equations (38) and (39), relating $\bar{G}_d$ and $\bar{G}_{d+2}$. We will give a brief review of it here, by phrasing it as an application of the general formula eq. (9).

\footnote{The $\mathcal{G}$ here is not to be confused with the $\mathcal{G}$ in eq. (34), though in the later sections we shall see that they are related.}
Let the \((d+1)\)D flat metric be \(\eta_{\mu\nu}\) and the \((d+2)\)D metric be \(\eta_{AB}\), so that
\[
\eta_{AB} dX^A dX^B = \eta_{\mu\nu} dX^\mu dX^\nu - \left( dX^{d+1} \right)^2.
\]
Comparison against eq. (5) tells us \(P = 1\) and \(g^\perp_{AB} dy^A dy^B = - \left( dX^{d+1} \right)^2\). The zero mode equation in eq. (11) is
\[
\left( \frac{d}{dX^{d+1}} \right)^2 \psi_0 = 0,
\]
whose general solution is
\[
\psi_0 = C_0 + C_1 X^{d+1}, \quad C_{0,1} = \text{constant}.
\]
Since the plane waves \(e^{i k \cdot X}\) are regular and bounded everywhere \(k\) and \(X\) are real, we have to set \(C_1 = 0\), for \(X^{d+1} \to \pm \infty\) grows indefinitely. There is no need to determine \(C_0\) because it cancels out in the formula (9), which now hands us
\[
\bar{G}_{d+1}^\pm [X - X'] = \int_{-\infty}^{\infty} d \left( X^{d+1} - X'^{d+1} \right) \bar{G}_{d+2}^\pm [X - X'].
\]
The same relation must hold if we now shift \(d \to d - 1\). This means
\[
\bar{G}_{d}^\pm [X - X'] = \int_{-\infty}^{\infty} d \left( X^{d+1} - X'^{d+1} \right) \int_{-\infty}^{\infty} d \left( X^{d} - X'^{d} \right) \bar{G}_{d+2}^\pm [X - X'].
\]
Next we re-write the Green’s function so that it depends on only two variables, the time elapsed between observation and emission \(T \equiv X^0 - X'^0\) and the Euclidean distance between observer and source \(R \equiv |\vec{X} - \vec{X}'|\). Using cylindrical symmetry on the \((X^d, X^{d+1})\) plane and rotation symmetry of the \((X^1, \ldots, X^{d-1})\) volume, we may convert eq. (44) into
\[
\bar{G}_{d}^\pm [T, R] = 2\pi \int_{R}^{\infty} dR' R' \bar{G}_{d+2}^\pm [T, R'].
\]
Upon differentiating both sides with respect to \(R\),
\[
-\frac{1}{2\pi R} \frac{\partial}{\partial R} \bar{G}_{d}^\pm [T, R] = G_{d+2}^\pm [T, R].
\]
Global Poincaré symmetry informs us that, in fact, the Green’s function depends solely on the object \(\bar{\sigma}\) in eq. (35); the \(\Theta[\pm T]\) in (37) is merely the instruction to ignore half of the light cone of \(x'\). Because \((-2\pi R)^{-1} \partial \bar{\sigma} / \partial R) \partial / \partial \bar{\sigma} = (2\pi)^{-1} \partial / \partial \bar{\sigma}\), by induction on the number of dimensions \(d\), the equivalence between eq. (46) and equations (38) and (39) follows once the \(d = 2, 3\) cases have been verified.
**Euclidean** The flat Euclidean space Green’s function for all spatial dimensions $d \geq 1$ reads

\[
\bar{G}^{(E)}_d \left[ \vec{X} - \vec{X}' \right] = -\frac{\Gamma \left[ \frac{d}{2} - 1 \right]}{4\pi^{d/2} \left| \vec{X} - \vec{X}' \right|^{d-2}}. \tag{47}
\]

($\Gamma$ is the Gamma function.) This formula is valid even for $d = 2$, where the Green’s function is proportional to $\ln \left| \vec{X} - \vec{X}' \right|$, if one first sets $d = 2 - \epsilon$ and proceed to expand in powers of $|\epsilon| \ll 1$ up to $O(\epsilon^0)$. (Dimensional regularization acts as a long distance regulator here.) For $d = 1$, $|X - X'|$ is to be read as the absolute value of the difference between the coordinates of the observer and source.

In this embedding framework, we see that all the Minkowski and Euclidean Green’s functions follow from the $d = 3$ flat spacetime case via integration and differentiation. Once a concrete expression is gotten for $\bar{G}^{\pm}_3 [X - X']$, say by evaluating its Fourier integral representation in eq. (17), then $\bar{G}^{\pm}_2 [X - X']$ can be obtained by integrating it once, using eq. (43). All even dimensional Green’s functions then follow from $\bar{G}^{\pm}_2 [X - X']$ by applying the differential recursion relation in eq. (46) repeatedly; and all odd dimensions from $\bar{G}^{\pm}_3 [X - X']$.

Since Euclidean space (with the Kronecker delta $\delta_{ij}$ as its metric) can be viewed as a $d$ dimensional space embedded in $(d + 1)$ dimensional Minkowski, with the analog of eq. (5) reading

\[
-ds^2 = \delta_{ij}dX^i dX^j - (dX^0)^2, \quad P = 1, \quad g_{AB}dy^A dy^B = - (dX^0)^2, \tag{48}
\]

that means the Euclidean Green’s function in eq. (47) can be viewed as the massless minimally coupled scalar field generated by a line source in $(d + 1)$D Minkowski spacetime, i.e. a static point source $\propto \delta^{(d)}(\vec{X} - \vec{X}')$, sweeping out a timelike worldline. Upon solving the zero mode equation in eq. (11), which again yields a constant for the regular solution, formula (9) now allows us to see that – once the general $\bar{G}^{\pm}_{d+1} [X - X']$ is known, its Euclidean counterpart can be worked out with a single integration\footnote{The relations in equations (43), (44), and (49) can also be understood by replacing the Green’s functions occurring within the integrals on the right hand side with their Fourier representation in eq. (17).}

\[
\bar{G}^{(E)}_d \left[ \vec{X} - \vec{X}' \right] = \int_{-\infty}^{\infty} d \left( X^0 - X'^0 \right) \bar{G}^{\pm}_{d+1} [X - X']. \tag{49}
\]

**IV. DE SITTER SPACETIME**

De Sitter spacetime in $d$ dimensions is defined as the following hyperboloid embedded in $(d + 1)$ dimensional Minkowski:

\[
-\eta_{\alpha\beta} X^\alpha X^\beta \equiv -X^2 = \frac{1}{H^2}, \quad H > 0. \tag{50}
\]
$H$ is some fixed Hubble parameter. Let $\tau \in \mathbb{R}$, $\rho \geq 0$, $\{\theta^i\}$ be the $(d-1)$ angular coordinates on a $(d-1)$-sphere, and $\hat{n}[\vec{\theta}]$ be the unit radial (spatial) vector. For the most part we will employ hyperbolic coordinates to describe an arbitrary point in Minkowski spacetime outside the light cone of its origin $0^\Lambda$,

$$X^\Lambda[\rho, \tau, \vec{\theta}] = \rho \left( \sinh[\tau], \cosh[\tau] \hat{n}[\vec{\theta}] \right).$$

The Minkowski metric, for $X^2 < 0$, now reads

$$ds^2 = \rho^2 \left( d\tau^2 - \cosh^2[\tau] d\Omega_{d-1}^2 \right) - d\rho^2,$$

– with $d\Omega_{d-1}^2$ being the metric on the $(d-1)$D sphere – such that the induced metric on the $\rho = 1/H$ surface is de Sitter spacetime

$$g^{(dS)}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{H^2} \left( d\tau^2 - \cosh^2[\tau] d\Omega_{d-1}^2 \right).$$

These coordinates, which cover the whole of de Sitter spacetime, are also known as the “closed slicing” coordinates because constant time surfaces describe a closed sphere: $(g^{(dS)}_{\mu\nu} dx^\mu dx^\nu)_{\tau \text{ fixed}} \propto d\Omega_{d-1}^2$.

Our main results will be expressed in terms of the object

$$Z[x,x'] \equiv \left( H^2 X \left[ \rho, x \right] \cdot X' \left[ \rho', x' \right] \right) \bigg|_{\rho = \rho' = H^{-1}}.$$  

With the choice of coordinates in eq. (51), for instance, we have

$$Z \left[ \tau, \vec{\theta}; \tau', \vec{\theta}' \right] = \sinh[\tau] \sinh[\tau'] - \cosh[\tau] \cosh[\tau'] \vec{n} \cdot \vec{n}'. $$

In appendix [A], we show that the square of the geodesic distance between $x$ and $x'$ in de Sitter is

$$L[x,x'] = \left( \frac{1}{H} \cosh^{-1} \left[ - Z[x,x'] \right] \right)^2.$$  

In particular, $x$ and $x'$ lie precisely on or within each other’s null cones, $L[x,x'] \geq 0$, and are thus causally connected, when and only when

$$Z[x,x'] \leq -1.$$  

**Integral representation**  

With these preliminaries aside, we may now apply the general formula (9) to eq. (52). First we identify

$$P[\rho] = \rho, \quad g^{AB}_A dy^A dy^B = -d\rho^2.$$  

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The zero mode equation is

\[ D_\rho \psi_0 = -\frac{1}{\rho^{d-2}} \frac{d}{d\rho} \left( \rho^d \frac{d\psi_0}{d\rho} \right) = 0, \quad (59) \]

and its general solution is

\[ \psi_0[\rho] = C_1 \rho^{1-d} + C_2, \quad C_{1,2} = \text{constant}. \quad (60) \]

The regular solution is achieved with \( C_1 = 0 \), since \( \rho^{1-d} \) blows up as \( \rho \to 0 \), for \( d > 1 \). There is no need to determine \( C_2 \) since it cancels out in eq. (9).

The integral representation of the minimally coupled massless scalar Green’s function in de Sitter spacetime is therefore

\[ G_d[x, x'] = \int_0^\infty d\rho' \left( H\rho' \right)^{d-2} \tilde{G}_{d+1} \left[ X[\rho = H^{-1}, x] - X'[\rho', x'] \right], \quad (61) \]

with the parametrization in eq. (51), and the flat spacetime Green’s functions \( \tilde{G}_{d+1} \) from equations (37) through (39). As already advertised, the higher dimensional beings have set up a line source which begins from one end infinitesimally close to the Minkowski origin 0, and penetrates perpendicularly the de Sitter hyperboloids of all finite sizes. (One can check directly, from eq. (51), that the vector \( \partial_{\rho'} X'^{\alpha} \) is orthogonal to \( \partial_{\tau'} X'^{\alpha} \) and \( \{\partial_{\rho'}, X'^{\alpha}\} \).) The observer constrained to live on the \( d \) dimensional hyperboloid at \( \rho = 1/H \) has been deceived to think she sees a \( \delta \)-function point source at \( (\rho' = 1/H, x') \).

Causal structure for closed slicing

Before we compute eq. (61), we need to first tackle an issue we have already raised in section (II). If a family of observers living in de Sitter spacetime held clocks synchronized to read \( \tau \) occurring in the parametrization of eq. (51), and if cause preceded effect (or effect preceded cause) according to the higher dimensional beings in Minkowski, then does cause necessarily precede effect (effect precede cause) from these de Sitter observers’ viewpoint? In Minkowski spacetime itself, for instance, whether cause or effect comes first is controlled by the \( \Theta[\pm (X^0 - X'^0)] \) in eq. (37). We now answer the above question in the affirmative, and demonstrate that, the retarded \( G^+_d \) (advanced \( G^-_d \)) Green’s function in the closed slicing coordinates of de Sitter, is obtained simply from its retarded (advanced) counterpart in Minkowski:

\[ G_d^{(\text{Closed}[\pm])}[x, x'] = \int_0^\infty d\rho' \left( H\rho' \right)^{d-2} \Theta[\pm (X^0 - X'^0)] \tilde{G}_{d+1} \left[ X[\rho, \tau, \tilde{n}] - X'[\rho', \tau', \tilde{n}'] \right] \]

\[ = \Theta[\pm (\tau - \tau')] H^{d-2} G_d[x, x'], \quad (62) \]

where \( \tilde{G}_{d+1} \) and \( G_d \) are both symmetric (unordered in time); the former can be found in equations (38) and (39), and the latter is the analog of eq. (34):

\[ G_d[x, x'] \equiv \int_0^\infty d\rho' \rho'^{d-2} \tilde{G}_{d+1} \left[ X[\rho, \tau, \tilde{n}] - X'[\rho', \tau', \tilde{n}'] \right]. \quad (64) \]
We have left \( \rho \) unevaluated so that we can exercise the check discussed around equations (32) through (34). The key point here is that, for a given observer location \((\rho = 1/H, \tau, \hat{n})\) and some de Sitter point source\(^5\) position \((\rho' = 1/H, \tau', \hat{n}')\),

\[
\Theta \left[ \pm (X^0[\rho, \tau, \hat{n}] - X^0[\rho', \tau', \hat{n}']) \right] = \Theta \left[ \pm (\tau - \tau') \right]
\]

(65)

for all \( \rho' \) such that \((X - X')^2 \geq 0\). That the observer at \(X\) and the source at \(X'\) has to be within each other’s light cone (in the Minkowski sense), is imposed directly by the flat Green’s functions in eq. (62). (See the occurrence of \(\Theta[\pm(X^0 - X'^0)]\Theta[\bar{\sigma}]\) in equations (37) through (39).) With the parametrization in eq. (51), this translates to

\[
(X - X')^2 = -(\rho^2 + \rho'^2 + 2\rho\rho'Z) \geq 0,
\]

(66)

which, in turn, holds when and only when\(^6\)

\[
-Z - \sqrt{Z^2 - 1} \leq \frac{\rho'}{\rho} \leq -Z + \sqrt{Z^2 - 1}, \quad \text{and} \quad Z \leq 1.
\]

(67)

Notice we have recovered the causality condition \(Z[x, x'] \leq -1\) in eq. (57); it is apparently encoded in the integral representation eq. (61). Fig. 1 illustrates the conditions imposed by eq. (67).

We further consider

\[
V^2 \equiv \left( \frac{X}{\sqrt{-X^2}} - \frac{X'}{\sqrt{-X'^2}} \right)^2 = \frac{-2X \cdot X' - 2\rho\rho'}{\rho'}. \quad (68)
\]

(Of course, \(-X^2 = \rho^2\) and \(-X'^2 = \rho'^2\), but we want to emphasize the global Lorentz invariance of the object \(V^2\).) Since \((X - X')^2 \geq 0\) is equivalent to \(-2X \cdot X' \geq \rho^2 + \rho'^2\), we have

\[
V^2 \geq \frac{(\rho - \rho')^2}{\rho\rho'} \geq 0.
\]

(69)

Let us invoke the fact that the sign of the time component of a timelike or null vector \(U\) in Minkowski is a Lorentz invariant, i.e.

\[
\text{sgn} \left[ U^0 \right] = \text{sgn} \left[ L^0_{\text{Mink}} U^0 \right], \quad \forall U^2 \geq 0,
\]

(70)

\(^{5}\) Which is not to be confused with the (related) line source in Minkowski – at the risk of over-verbosity, we reiterate that the point source on the de Sitter hyperboloid is but one point on the line source in Minkowski.

\(^{6}\) To understand these statements, note that the inverted parabola on the left hand side of eq. (66) has no real roots, with respect to the variable \(\rho'/\rho\), for \(|Z| < 1\). For \(Z \leq -1\), the region where \((X - X')^2\) is positive lies between its two real roots. (The \(Z > +1\) region gives negative \(\rho'/\rho\), and is irrelevant for our purposes here.) Also observe that \(\rho'/\rho = 0\), corresponding to one end of the line source, yields no solution to eq. (66) because the light cone of \(0^3\) in Minkowski does not intersect the hyperboloid of any finite size \(\rho\).
FIG. 1: The light gray region is the range of \( Z[x, x'] \) and \( \rho' \) contributing to the signal at the observer’s location on the de Sitter hyperboloid (i.e. at \( X[\rho = H^{-1}, x] \)), as described by the symmetric Green’s function of eq. (64). The dimensionless \( Z[x, x'] \) is defined in eq. (54). For a fixed \( Z \), the range of \( \rho' \) lying within the light gray region quantifies which segment of the line source, parametrized in eq. (51), produced the scalar waves seen by the observer at \( x \). Here, we are expressing \( \rho' \) in units of \( 1/H \), the size of the de Sitter hyperboloid.

where \( L^\mathfrak{a}_{\mathfrak{b}} \) is an arbitrary Lorentz boost. This informs us that

\[
\Theta \left[ \pm(X^0 - X'^0) \right] \quad \text{and} \quad \Theta[\pm(\tau - \tau')] = \Theta \left[ \pm(\sinh[\tau] - \sinh[\tau']) \right] = \Theta \left[ \pm \left( \frac{X^0}{\sqrt{-X^2}} - \frac{X'^0}{\sqrt{-X'^2}} \right) \right]
\]

(71)

(72)

are Lorentz scalars whenever \( (X - X')^2 \geq 0 \). The statement in eq. (65) is proven, therefore, if we can set \( X'^0 \) to zero in some global Lorentz frame, everywhere along the line source that is on or within the light cone of the observer. Let us choose to boost along only the \( \mathbf{n}' \) direction, so that

\[
L^\mathfrak{a}_{\mathfrak{b}} = \begin{bmatrix}
\cosh[\tau'] & -\sinh[\tau'] \\
-\sinh[\tau'] & \cosh[\tau']
\end{bmatrix}, \quad \mathfrak{a}, \mathfrak{b} \in \{0, \mathbf{n}'\}.
\]

(73)
Then, applying it to eq. (51),

\[ X^0 \rightarrow L^0_{21} X^0 = 0, \quad \forall \rho' \geq 0. \quad (74) \]

We have framed our derivation of the retarded and advanced de Sitter Green’s functions in terms of the closed slicing coordinates in eq. (51). It is important to note that, however, because the object \( \Theta[\pm(\tau - \tau')] \) is written in terms of the time coordinates in a particular “slicing” of de Sitter, it is not a coordinate scalar. The practical strategy we will adopt here is to focus on the symmetric Green’s function in eq. (64). As we will shortly demonstrate, it depends solely on \( Z \) and therefore can be viewed in a manner independent of the parametrization chosen on de Sitter itself. Only when one does pick a particular set of coordinates – for example, the flat slicing in eq. (83) below – do we then multiply it by \( \Theta[\pm(t - t')] \) (where \( t \) and \( t' \) are the time coordinates), to get the retarded (+) and advanced (−) Green’s functions.

**Technicalities**

We now turn to performing the integral in eq. (64). First rescale the integration variable \( \rho'' \equiv \rho'/\rho \), which effectively sets \( \rho = 1 \), so that now \( \bar{\sigma} = -(1/2)(\rho''^2 + 1 + 2\rho''Z) \). Because the flat spacetime symmetric Green’s functions depend only on \( \bar{\sigma} \), we may proceed to replace in equations (38) and (39)

\[ \frac{\partial}{\partial \bar{\sigma}} \rightarrow -\frac{1}{\rho''} \frac{\partial}{\partial Z}. \quad (75) \]

The limits of \( \rho'' \) integration are determined by the causal condition \( (X - X')^2 \geq 0 \) (from the \( \Theta[\bar{\sigma}] \) in in equations (38) and (39)); the result has already been found in eq. (67). For \( Z > -1 \) the integral is to be set to zero. At this point, the even and odd \( d \) integrals are respectively

\[ G_{\text{even}} d[x, x'] = \frac{1}{(2\pi)^{d/2}} \left( -\frac{\partial}{\partial Z} \right)^{d-1} \left( \Theta[-Z - 1] \int_{-Z - \sqrt{Z^2 - 1}}^{Z + \sqrt{Z^2 - 1}} \frac{d\rho'' \rho''^{d+2}}{\rho''^2 + 1 + 2\rho''Z} \right) \]  

(76)

and

\[ G_{\text{odd}} d[x, x'] = \frac{1}{2(2\pi)^{d/2}} \left( -\frac{\partial}{\partial Z} \right)^{d-1} \left( \Theta[-Z - 1] \int_{-Z - \sqrt{Z^2 - 1}}^{Z + \sqrt{Z^2 - 1}} d\rho'' \rho''^{d+1} \right). \]  

(77)

Notice equations (76) and (77) have both become independent of \( \rho \), the size of the observer’s de Sitter hyperboloid. Referring back to the discussion around eq. (34), we may conclude these are indeed the (symmetric) de Sitter Green’s functions.

Since the integral in eq. (77) for odd \( d \) involves merely a power of \( \rho'' \), the remaining challenge is that of the even \( d \) case in eq. (76). We massage it by changing variables to \( u \equiv \rho'' + Z \), followed by
\[
\sin \varphi \equiv \sqrt{Z^2 - 1} u, \text{ and thereby bringing us to }
\]
\[
\int_{-Z+\sqrt{Z^2-1}}^{-Z-\sqrt{Z^2-1}} \frac{d\rho''}{\sqrt{-\rho''(\rho''+1+2\rho''Z)}} = \int_0^\pi d\varphi \left( (-Z) + \sqrt{(-Z)^2 - 1 \cos \varphi} \right)^{d-2}.
\]

A glance at 8.913.3 of [5] tells us this is \(\pi\) times the Legendre polynomial of degree \((d-2)/2\), \(P_{d-2}^{(2)}[-Z]\).

**Results**

To sum, the retarded \(G_d^+\) and advanced \(G_d^-\) Green’s functions of the minimally coupled massless scalar field, in the closed slicing coordinates of eq. (51), are given by \(G_d^{(\text{Closed})\pm}[x, x'] = \Theta[\pm(\tau - \tau')] \mathcal{H}^{d-2} G_d[x, x']\). The symmetric Green’s functions \(G_d\), in turn, are expressed in terms of \(Z[x, x']\) defined in eq. (54),

\[
G_{\text{even}} d[x, x'] = \frac{\pi}{(2\pi)^{d/2}} \left( -\frac{\partial}{\partial Z} \right)^{\frac{d-1}{2}} \left( \Theta[-Z-1] P_{d-2}^{(2)} [-Z] \right),
\]

\[
G_{\text{odd}} d[x, x'] = \frac{1}{(d-1)(2\pi)^{\frac{d-1}{2}} \left( -\frac{\partial}{\partial Z} \right)^{\frac{d-1}{2}}}
\]

\[
\times \left( \Theta[-Z-1] \left\{ (-Z + \sqrt{Z^2 - 1})^{\frac{d-1}{2}} - (-Z - \sqrt{Z^2 - 1})^{\frac{d-1}{2}} \right\} \right).
\]

The \(P_{d-2}^{(2)}[-Z]\) is the Legendre polynomial and the terms inside the curly brackets in eq. (80) can be re-expressed in terms of \(\sqrt{Z^2 - 1} U_{d-1}^{(2)}[-Z]\), with \(U_{d-1}^{(2)}[-Z]\) being Chebyshev’s polynomial of the second kind – see 8.940 of [5].

The presence of \(\Theta[-Z-1]\) in equations (79) and (80) is necessary to impose the \(Z[x, x'] \leq -1\) causality condition, that \(x\) and \(x'\) lie within each other’s light cone (in the de Sitter sense). Furthermore – again from the de Sitter perspective – terms that contain \((-\partial Z) \Theta[-Z-1] = \delta[Z+1]\) and higher derivatives of the \(\delta\)-function describe propagation of signals on the light cone, whereas the terms containing \(\Theta[-Z-1]\), namely

\[
G^{(\text{Tail})}_{\text{even}} d[x, x'] = \frac{\pi \Theta[-Z-1]}{(2\pi)^{d/2}} \frac{(d-2)!}{2^{\frac{d-2}{2}} \left( \frac{d-2}{2} \right)!},
\]

\[
G^{(\text{Tail})}_{\text{odd}} d[x, x'] = \frac{\Theta[-Z-1]}{(d-1)(2\pi)^{\frac{d-1}{2}} \left( -\frac{\partial}{\partial Z} \right)^{\frac{d-1}{2}}}
\]

\[
\times \left\{ (-Z + \sqrt{Z^2 - 1})^{\frac{d-1}{2}} - (-Z - \sqrt{Z^2 - 1})^{\frac{d-1}{2}} \right\},
\]

describe the tail effect. Despite its zero mass, a portion of the scalar field propagates inside the light cone of the source.\(^7\)

**Four dimensions**

There is mounting observational evidence that our universe underwent a period of accelerated expansion during the very earliest moments of its existence, and is currently

\(^7\) In eq. (81), we have used \(d^m P_m[z]/dz^m = (2m)!/(2^m m!)\), for integers \(m = 0, 1, 2, 3, \ldots\); this follows from the Rodrigues formula for the Legendre polynomials. The tail of the de Sitter Green’s function in even dimensions is therefore a constant.
entering a similar “dark energy” dominated phase. To zeroth order, the metric of such a spacetime is de Sitter. The same observational evidence indicates that, what is most relevant, however, is not the closed slicing parametrization we chose in eq. (51), where constant time surfaces of de Sitter are closed spheres – but rather the flat slicing parametrization, where constant time surfaces are spatially flat and have infinite volume. This choice of coordinates is given by the formulas

\begin{align*}
X^0 &= \frac{1}{2\eta} \left( \eta^2 - \vec{x}^2 - \frac{1}{H^2} \right), \\
X^4 &= \frac{1}{2\eta} \left( -\eta^2 + \vec{x}^2 - \frac{1}{H^2} \right), \\
X^1 &= \frac{x^1}{H\eta}, \\
X^2 &= \frac{x^2}{H\eta}, \\
X^3 &= \frac{x^3}{H\eta},
\end{align*}

which in turn give us the following form of the de Sitter metric:

\begin{equation}
\begin{aligned}
g_{\mu\nu}^{(dS)} d\mathbf{x}\mu d\mathbf{x}\nu &= d\eta^2 - d\vec{x}^2 \left( \frac{1}{H^2} \right).
\end{aligned}
\end{equation}

Here, \(-\infty < \eta < 0\) and \(\vec{x}^2 \equiv \sum_{i=1}^{3} (x^i)^2\). In these coordinates,

\begin{equation}
-\frac{1}{Z} = \frac{1}{2\eta\eta'} \left( \eta - \eta' \right)^2 - \left( \vec{x} - \vec{x}' \right)^2.
\end{equation}

By setting \(-Z[\delta[Z+1] = \delta[Z+1],\) the symmetric Green’s function in eq. (79) takes the form

\begin{equation}
\begin{aligned}
G_4[x,x'] &= \frac{1}{4\pi} (\delta[Z+1] + \Theta[-Z-1]), \\
&= \frac{1}{4\pi} (\eta\eta'\delta[S] + \Theta[S]), \\
&S \equiv \frac{1}{2} \left( (\eta - \eta')^2 - (\vec{x} - \vec{x}')^2 \right),
\end{aligned}
\end{equation}

To obtain the retarded and advanced Green’s functions, we multiply by \(\Theta[\pm(\eta - \eta')]\).

\begin{equation}
G_4^{[\pm\delta]}[x,x'] = \Theta[\pm(\eta - \eta')]H^2G_4[x,x'].
\end{equation}

These results are consistent with, for instance, the discussion in [9].

\textbf{van Vleck determinant} In 4D, the light cone part of the scalar Green’s function takes the generic form – see, for example, §14.2 of [7] –

\begin{equation}
\Theta \left[ \pm(x^0 - x'^0) \right] \frac{\sqrt{\Delta[x,x']}}{4\pi} \delta \left[ L[x,x']/2 \right],
\end{equation}

where \(\Delta[x,x']\) is known as the van Vleck determinant. It describes properties of null congruences in the spacetime under consideration. \((L[x,x'],\) already introduced earlier, is the square of the geodesic distance between \(x'\) and \(x\).) From \(L[x,x'] = (\cosh^{-1}[-Z]/H)^2\) in appendix A, by writing \(\delta [L[x,x']/2] = \delta[Z+1]/(1/2)\partial L/\partial Z\), and comparing equations (86) and (88) with (89), we may therefore infer the van Vleck determinant of 4D de Sitter spacetime to be

\begin{equation}
\Delta[x,x'] = \frac{(\cosh^{-1}[-Z])^2}{Z^2 - 1}.
\end{equation}
V. TOPOLOGICAL OBSTRUCTIONS AND GREEN’S FUNCTIONS ON d-SPHERE

Having derived the de Sitter Green’s functions from the Minkowski ones, let us now ask if there are potential obstructions to our general construction. In this section, we shall restrict our attention to Riemannian spaces with positive definite metrics. Recall that the Gauss-Stokes’ theorem tells us the integral of the divergence of the gradient of a scalar field \( \varphi \) over some volume, is equal to the flux of this gradient field over the boundary of the same volume. In a closed volume such the \( d \)-dimensional sphere, i.e. where there is no boundary, the integral of such a total divergence has to vanish identically. In physical terms, topology forbids a net positive charge on a \( d \)-sphere without a compensating negative charge: the total integral of the charge density \( J \equiv \Box \varphi \) has to be zero.

Now, a \( d \)-sphere can be defined by the following embedding in \((d + 1)D\) Euclidean space:

\[
\sum_{\mathfrak{a}=1}^{d+1} (X^\mathfrak{a})^2 \equiv \vec{X}^2 = R^2, \quad R > 0.
\] (91)

The \((d + 1)D\) Euclidean metric in spherical coordinates is

\[
d\bar{s}^2 = r^2 d\Omega_d^2 + dr^2,
\] (92)

where \( d\Omega_d^2 \) is the metric on the \( d \)-sphere. This is of the form in eq. \([5]\), with the identifications

\[
P[r] = r, \quad g^{AB}_d dy^A dy^B = dr^2.
\] (93)

An arbitrary point in the \((d + 1)D\) Euclidean space has coordinates

\[
\vec{X}[r, \hat{n}] = r\hat{n}[\vec{\theta}],
\] (94)

where \( \hat{n}[\vec{\theta}] \) is the unit radial vector parametrized by \( d \) angular coordinates \( \vec{\theta} \).

Let us press on with the application of the general formula in eq. \([9]\), to see how it will break down. The zero mode equation is

\[
D_r \psi_0 = \frac{1}{r^{d-2}} \frac{d}{dr} \left( r^d \frac{d\psi_0}{dr} \right) = 0,
\] (95)

whose general solution is

\[
\psi_0[r] = C_1 r^{1-d} + C_2,
\] (96)

prompting us to set \( C_1 = 0 \) to maintain regularity as \( r \to 0 \). The general formula in eq. \([9]\) becomes

\[
G_d^{(E^+)}[\hat{n}, \hat{n}'; r] = \int_0^\infty dr' \left( \frac{r'}{R} \right)^{d-2} G_{d+1}^{(E)} \left[ \vec{X}[r, \hat{n}] - \vec{X}'[r', \hat{n}'] \right]
\] (97)

\[
\equiv -\frac{\Gamma \left[ \frac{d-1}{2} \right]}{4\pi \frac{d+1}{2} R^{d-2}} G_d^{(E^+)}[\hat{n}, \hat{n}'; r]
\] (98)
where \( \tilde{G}^{(E)}_{d+4} \) can be found in eq. \([47]\); and we have extracted the analog of eq. \([34]\) for evaluation

\[
\tilde{G}^{(E+)}_d[\hat{n}, \hat{n}'; r] = \int_0^\infty \frac{dr' r^{d-2}}{|r^2 + r'^2 - 2rr' \hat{n} \cdot \hat{n}'|^{d-1/2}}.
\]  

We will eventually set \( r = R \), but just as in the previous section, we will leave it unevaluated for now. At first sight, one may be tempted to re-scale \( r'' \equiv r'/r \) in eq. \([99]\), and conclude that the resulting integral is independent of \( r \). However, the situation is more subtle because eq. \([99]\) is “logarithmically divergent”:

\[
\lim_{r_{IR} \to \infty} \int_0^{r_{IR}} \frac{dr' r'^{d-2}}{|r^2 + r'^2 - 2rr' \hat{n} \cdot \hat{n}'|^{d-1/2}} = \int_0^{r_{IR}} \frac{dr' r'^{d-2}}{r^{d-1}} = \ln[r_{IR}].
\]  

From the following indefinite integrals, with integers \( m = 0, 1, 2, 3, \ldots \),

\[
\int z^{2m} (az^2 + bz + c)^{-1+(2m)/2} \, dz = \frac{(-)^m}{(1/2)_m} \partial_a^m \left( a^{-1/2} \ln \left[ 2\sqrt{a} \sqrt{az^2 + bz + c} + 2az + b \right] \right)
\]  

\[
\int z^{1+2m} (az^2 + bz + c)^{-1-m} \, dz = \frac{(-)^m}{(1)_m} \partial_a^m \left( \frac{1}{2a} \ln \left[ az^2 + bz + c \right] - \frac{2b}{\sqrt{4ac - b^2}} \tan^{-1} \left[ \frac{2az + b}{\sqrt{4ac - b^2}} \right] \right),
\]  

where \( (\beta)_m \equiv \beta(\beta+1)(\beta+2) \ldots (\beta+(m-1)) \) is the Pochhammer symbol – we may in fact understand \([99]\) to mean, for even \( d \geq 2 \),

\[
\lim_{r_{IR} \to \infty} \int_0^{r_{IR}} \frac{dr' r'^{d-2}}{(r^2 + r'^2 - 2rr' \hat{n} \cdot \hat{n}')^{d-1/2}} = -\ln[r/r_{IR}] + \ln[2] + \frac{(-)^{d/2}}{(1/2)^{d-2}} \partial_a^\frac{d-2}{2} \left( \frac{\ln[a] - \ln[\sqrt{a} \cdot \hat{n} \cdot \hat{n}]}{\sqrt{a}} \right) \bigg|_{a=1}
\]  

and, for odd \( d \geq 3 \),

\[
\lim_{r_{IR} \to \infty} \int_0^{r_{IR}} \frac{dr' r'^{d-2}}{(r^2 + r'^2 - 2rr' \hat{n} \cdot \hat{n}')^{d-1/2}} = -\ln[r/r_{IR}] + \frac{(-)^{d/2}}{(1)^{d-3}} \partial_a^\frac{d-3}{2} \left( \frac{\ln[a]}{2} + \frac{\hat{n} \cdot \hat{n}' \pi}{\sqrt{a - (\hat{n} \cdot \hat{n}')^2}} \tan^{-1} \left[ \frac{\hat{n} \cdot \hat{n}'}{\sqrt{a - (\hat{n} \cdot \hat{n}')^2}} \right] \right) \bigg|_{a=1}.
\]  

Notice the lower and upper (divergent) limits of eq. \([99]\) have conspired to yield a dependence on \( r \) through \( \ln[r/r_{IR}] \); if the integral had converged, as was the case for the de Sitter calculation, \( r \) could

\[
\frac{(-)^m}{(\beta)_m} \partial_a^m \left( \frac{1}{a^\beta} \right) \bigg|_{a=1} = 1, \quad \beta \neq 0.
\]  

The result

\[
\frac{(-)^m}{(\beta)_m} \partial_a^m \left( \frac{1}{a^\beta} \right) \bigg|_{a=1} = 1, \quad \beta \neq 0.
\]  

is useful in evaluating some of the derivatives occurring in the intermediate steps.
have been re-scaled away. The situation is akin to that in quantum field theory, where the need to tame otherwise divergent calculations forces one to introduce extra dimension-ful scales in the problem.

Recalling the discussion around eq. (34), our integral representation in eq. (99) is therefore not a valid solution to the Green’s function of the Laplacian on the $d$-sphere, because $\mathcal{D}_r \ln[r/r_{1R}] \neq 0$. Of course no such solution should exist, for the topological reasons we have already mentioned. But the constraint that equal amounts of positive and negative charge need to exist on the $d$-sphere suggests that we can construct a modified Green’s function sourced by one positive point charge located at $\hat{n}_+$ on the sphere and one negative point charge at $\hat{n}_-$, i.e.

$$R^{d-2} \square_{\mathbb{S}^{d-1}} G^{(E)\pm}_d[\hat{n} \cdot \hat{n}_+, \hat{n} \cdot \hat{n}_-; R] = \delta^{(d)}[\hat{n} - \hat{n}_+] - \delta^{(d)}[\hat{n} - \hat{n}_-],$$

(106)

where $\delta^{(d)}[\hat{n} - \hat{n}']$ is shorthand for the appropriate $\delta$-function measure. For instance, on the 2-sphere, we have $\delta^{(2)}[\hat{n} - \hat{n}'] = \delta[\cos \theta - \cos \theta'] \delta[\phi - \phi']$.

This modified solution is nothing but the difference between two of the “Green’s functions” in eq. (97), with one $\hat{n}' \rightarrow \hat{n}_+$ and the other $\hat{n}' \rightarrow \hat{n}_-$. That this difference has to solve eq. (106), is because all the pieces independent of $\hat{n}$ and $\hat{n}'$ in equations (104) and (105) cancel out. In particular, the cancellation of the $\ln[r/r_{1R}]$ simultaneously cures the logarithmic divergence and yields a $r$-independent answer: if one now applies $\square$ to this difference, the analog of eq. (33) is

$$R^{d-2} \square_{\mathbb{S}^{d-1}} G^{(E)\pm}_d[\hat{n} \cdot \hat{n}_+, \hat{n} \cdot \hat{n}_-; R] = \delta^{(d)}[\hat{n} - \hat{n}_+] - \delta^{(d)}[\hat{n} - \hat{n}_-]$$

(107)

$$+ \frac{\Gamma \left[ \frac{d-1}{2} \right]}{4\pi^{d-1}} \mathcal{D}_r \left( G^{(E)\pm}_d[\hat{n}, \hat{n}_+; r] - G^{(E)\pm}_d[\hat{n}, \hat{n}_-; r] \right) \bigg|_{r=R},$$

i.e. the second line is zero.

**Results** We have thus proposed the following modified Green’s function of the Laplacian on the $d$-sphere of radius $R$, embedded in $(d+1)$D Euclidean space, obeying eq. (106). Its integral representation is

$$G^{(E)\pm}_d[\hat{n} \cdot \hat{n}_+, \hat{n} \cdot \hat{n}_-; R] = \int_0^\infty dr' \left( \frac{r'}{R} \right)^{d-2} \left( G^{(E)}_{d+1} \left[ \hat{X}[R, \hat{n}] - \hat{X}'[r', \hat{n}_+] \right] - G^{(E)}_{d+1} \left[ \hat{X}[R, \hat{n}] - \hat{X}'[r', \hat{n}_-] \right] \right),$$

(108)

with $G^{(E)}_{d+1}$ in eq. (47), and the $X$ and $X'$ parametrization of eq. (94). The integrals work out to yield, for $d \geq 2$,

$$G^{(E)\pm}_{\text{even } d} \left[ \hat{n} \cdot \hat{n}_+, \hat{n} \cdot \hat{n}_-; R \right] = \frac{\Gamma \left[ \frac{d+1}{2} \right]}{4\pi^{d+1} R^{d-2}} \left( - \right)^\frac{d-2}{2} \left( \frac{\partial}{\partial \alpha} \right)^\frac{d-2}{2} \left( \frac{1}{\sqrt{\alpha}} \ln \left[ \frac{\sqrt{\alpha} - \hat{n} \cdot \hat{n}_+}{\sqrt{\alpha} - \hat{n} \cdot \hat{n}_-} \right] \right)_{\alpha=1},$$

(109)
\[ G_{\text{odd } d}^{(E|\pm)} [\hat{n} \cdot \hat{n}_+, \hat{n} \cdot \hat{n}_-; R] = -\frac{\Gamma \left[ \frac{d-1}{2} \right]}{4\pi^{\frac{d+1}{2}} R^{d-2}} \left( \frac{(-)^{d-3}}{2} \left( \frac{\partial}{\partial d} \right) \right)^{\frac{d-3}{2}} \left. \right|_{a=1} \times \left( \frac{\hat{n} \cdot \hat{n}_+}{a \sqrt{a - (\hat{n} \cdot \hat{n}_+)^2}} \left( \frac{\pi}{2} + \tan^{-1} \left[ \frac{\hat{n} \cdot \hat{n}_+}{\sqrt{a - (\hat{n} \cdot \hat{n}_+)^2}} \right] \right) \right) \]  
\[ \quad - \frac{\hat{n} \cdot \hat{n}_-}{a \sqrt{a - (\hat{n} \cdot \hat{n}_-)^2}} \left( \frac{\pi}{2} + \tan^{-1} \left[ \frac{\hat{n} \cdot \hat{n}_-}{\sqrt{a - (\hat{n} \cdot \hat{n}_-)^2}} \right] \right), \]  

\( \text{(110)} \)

A more symmetric version of these proposals, in that there is only one independent source vector \( \hat{n}' \) and that the Green’s function is invariant under the exchange \( \hat{n} \leftrightarrow \hat{n}' \), is achieved by setting \( \hat{n}_+ = -\hat{n}_- \equiv \hat{n}' \). (This means the positive and negative charges now lie on each other’s antipodal points.)

\[ G_{\text{even } d}^{(E|S)} [\hat{n} \cdot \hat{n}'; R] = -\frac{\Gamma \left[ \frac{d-1}{2} \right]}{4\pi^{\frac{d+1}{2}} R^{d-2}} \left( \frac{(-)^{d-2}}{1(2)\frac{d-2}{2}} \left( \frac{\partial}{\partial d} \right) \right)^{\frac{d-2}{2}} \left. \left( \frac{1}{\sqrt{a}} \ln \left[ \frac{\sqrt{a} + \hat{n} \cdot \hat{n}'}{\sqrt{a} - \hat{n} \cdot \hat{n}'} \right] \right) \right|_{a=1}, \]  
\[ \text{(111)} \]

\[ G_{\text{odd } d}^{(E|S)} [\hat{n} \cdot \hat{n}'; R] = -\frac{\Gamma \left[ \frac{d-1}{2} \right]}{4\pi^{\frac{d+1}{2}} R^{d-2}} \left( \frac{(-)^{d-2}}{1(2)\frac{d-2}{2}} \left( \frac{\partial}{\partial d} \right) \right)^{\frac{d-2}{2}} \left. \left( \frac{\hat{n} \cdot \hat{n}'}{a \sqrt{a - (\hat{n} \cdot \hat{n}_-)^2}} \right) \right|_{a=1}. \]  
\[ \text{(112)} \]

**Checks**

By direct differentiation, we will now check that \( \Box G_{\text{d}}^{(E|\pm)} = 0 \) almost everywhere. Because the results in equations \( \text{(109)} \) and \( \text{(110)} \) can be expressed as a difference of two identical expressions, except one is a function of the sole variable \( \cos \theta_+ \equiv \hat{n} \cdot \hat{n}_+ \) and the other of the sole variable \( \cos \theta_- \equiv \hat{n} \cdot \hat{n}_- \), this means we may instead check that \( \Box \) acting on the portion of \( G_{\text{d}}^{(E|\pm)} \) that depends only on \( \cos \theta_+ \) (or only on \( \cos \theta_- \)) gives a non-zero pure number. \(^9\) In appendix \( \text{(B)} \) we show that the Laplacian \( \Box \) acting on a scalar function \( \psi \) that depends on the angular coordinates \( \hat{\theta} \) only through the dot product \( \cos \theta_{\pm} \) is

\[ \Box \psi [\hat{n}[\hat{\theta}] \cdot \hat{n}_{\pm}] = \frac{1}{\sin^{d-1}[\theta_{\pm}]} \frac{\partial}{\partial \theta_{\pm}} \left( \sin^{d-1}[\theta_{\pm}] \frac{\partial}{\partial \theta_{\pm}} \psi[\cos \theta_{\pm}] \right). \]  
\[ \text{(113)} \]

Since the reduced Laplacian in eq. \( \text{(113)} \) now depends only on \( d \) and on a single angle \( \theta_{\pm} \), it can be implemented readily on a computer \( \text{(12)} \) – we have ran this check for \( d = 2 \) through \( 40 \).

Let us further specialize to \( d = 2, 3 \),

\[ G_{2}^{(E|\pm)} [\hat{n} \cdot \hat{n}'; R] = -\frac{1}{4\pi} \left( \ln [1 - \cos \theta_-] - \ln [1 - \cos \theta_+] \right), \]  
\[ \text{(114)} \]

\(^9\) For even \( d \), we may replace in eq. \( \text{(109)} \), \( \ln \left[ \frac{\sqrt{a} + \hat{n} \cdot \hat{n}'}{\sqrt{a} - \hat{n} \cdot \hat{n}_-} \right] \to \ln [\sqrt{a} - \cos \theta_-] - \ln [\sqrt{a} - \cos \theta_+]. \)

\(^{10}\) That it gives a non-zero pure number (as opposed to exactly zero) corroborates the topology based argument that we cannot interpret the \( \cos \theta_+ \) dependent portion of \( G_{\text{d}}^{(E|\pm)} \) as the field generated by a single positive charge and the \( \cos \theta_- \) dependent portion as that by a negative charge; it is only when their difference is taken, that \( G_{\text{d}}^{(E|\pm)} \) is then annihilated by the Laplacian \( \Box \) almost everywhere.
and observe that the generalized Green’s functions in equations (114) and (115) are regular everywhere except when the observer is right on top of one of the two sources at \( \hat{n}_\pm \). For \( d = 2 \), this follows from the fact that the logs in (114) are finite unless one of the cosines is unity. For \( d = 3 \), \( \cot \theta \pm \) blows up at \( \theta \pm = 0, \pi \) but because of the factor \( (\pi - \theta \pm) \), we have \( \lim_{\theta \pm \to \pi} (\pi - \theta \pm) \cot \theta \pm = -1 \).

That there are only two points on the sphere where the field is unbounded, at least for \( d = 2, 3 \), supports our claim that equations (109) and (110) are the fields generated by two (and only two) charges of opposite signs.

VI. SUMMARY AND FUTURE DIRECTIONS

In this work, we have derived the formula in eq. (9) to compute the minimally coupled massless scalar Green’s function in a curved space(time) \( g_{\mu\nu}[x] = P^2[y = y_0]g_{\mu\nu}[x] \) using the corresponding Green’s function in flat space(time), if the former is embeddable in the latter via eq. (5). We have used it to work out the scalar Green’s functions in de Sitter spacetime, and the results can be found in equations (61), (79), (80), and (86) through (88). We then turned to the case of the Laplacian on the \( d \)-sphere, where we know no solution to the field generated by a single point source should exist because of topology. Equation (9) broke down by acquiring a divergence that depended logarithmically on the cutoff introduced to regulate the answer, which in turn yielded a term that did not satisfy the zero mode equation (11). The remedy we proposed was to solve for the field generated by one positive and one negative point charge, and this generalized Green’s functions for the \( d \)-sphere can be found in equations (108) through (112).

We close by touching on a number of directions we wish to pursue further.

Shifting the wave operator by an additive constant ought to correspond to shifting the eigenvalues \{\lambda\} in eq. (15) by the same constant. This probably means our arguments in section (II) can be extended to solve the Green’s function of \( \Box + m^2 + \xi R \) in constant curvature geometries such as de Sitter. (\( R \) is the Ricci scalar and \( \xi \) is a constant.) Also, the topology of the \( d \)-sphere is no longer an obstacle once a non-zero mass term is introduced: the charge density \( J \equiv (\Box + m^2)\varphi \) does not have to integrate to zero. It may be instructive to solve the Green’s function of \( \Box + m^2 \) on a sphere using the embedding method, and then take the \( m \to 0 \) limit to study exactly what goes bad.

In section (IV) we discussed how the notion of causal influence in the \((d + 1)D\) Minkowski, as encoded in the integral representation of eq. (61), is compatible with that on the \( d \)-dimensional de
Sitter hyperboloid itself. It would deepen our understanding of the embedding paradigm and its range of applicability, if this discussion on causal influence can be made in the abstract, or, at the very least, examined for enough examples that general lessons can be extracted. It is also of physical significance to understand whether eq. (9) can be extended to the case of higher rank tensors, including that of fermions. An important case is that of the linearized wave equations resulting from perturbing Einstein’s field equations about some fixed geometry. (Note that it is often technically advantageous to first perform a scalar-vector-tensor decomposition of these equations before proceeding.)

Finally, we wonder if there are other types of embeddings than the one in eq. (5) that would allow us to solve for the curved space(time) Green’s functions from flat ones. For instance, in [8] (building upon earlier work in [9]-[11]), a perturbation theory was devised to solve for the scalar, vector and tensor Green’s functions in the metric \( g^{(d)}_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \), using the corresponding Green’s functions in \( \eta_{\mu\nu} \). On the other hand, we have seen in section (III) that the \( d \)-dimensional flat spacetime Green’s functions is sourced by a line extending into one higher spatial dimension, i.e. parallel to the \( x^d \)-axis. Does this embedding picture hold up once non-zero perturbations \( h_{\mu\nu} \) are allowed? If so, is it possible to phrase the perturbation theory of [8] in terms of perturbation theory of how \( g^{(d)}_{\mu\nu} \) can be embedded in \( \eta^{(d+1)}_{\alpha\beta} \), as well as what the line source would deform into?

VII. ACKNOWLEDGMENTS

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Appendix A: Geodesic Distance in de Sitter Spacetime

In this section we wish to show that the square of the geodesic distance between two points \( x \) and \( x' \) in de Sitter spacetime is given by

\[
\left( \frac{1}{H} \cosh^{-1} \left[ -Z[x, x'] \right] \right)^2,
\]

where

\[
Z \equiv H^2 X \cdot X' \equiv H^2 \eta_{\alpha\beta} X^\alpha X'^\beta
\]
evaluated on the de Sitter hyperboloid,

\[ \begin{align*}
X &= X[\rho = 1/H, x], \\
X' &= X'[\rho' = 1/H, x'].
\end{align*} \tag{A3} \]

In the spirit of the embedding approach we have adopted in this paper, we will use the method of Lagrange multipliers. That is, we will add to the action whose variation yields the geodesic equation in Minkowski, a constraint that the motion must take place on the de Sitter hyperboloid:

\[ L \equiv \int_0^1 d\lambda \left( \dot{\xi} \cdot \ddot{\xi} + s^2 (\xi \cdot \xi + 1) \right). \tag{A4} \]

Note that eq. (A4) evaluated on a geodesic yields the square of the geodesic distance. (See §3 of [7].) We have set \( \frac{1}{H} \to 1 \), and the dot denotes derivative with respect to the affine parameter \( \lambda \): \( \dot{\xi} \equiv d\xi/d\lambda \). The boundary conditions are that \( \xi \) begins at \( X' \) and ends at \( X \).

\[ \begin{align*}
\xi[0] &= X', \\
\xi[1] &= X, \\
X \cdot X &= X' \cdot X' = -1.
\end{align*} \tag{A5} \]

Varying eq. (A4) with respect to \( \xi \) yields

\[ \ddot{\xi} = s^2 \xi. \tag{A6} \]

The solution satisfying equations (A5) and (A6) is

\[ \xi[\lambda] = \frac{X - \cosh[s] X'}{\sinh[s]} \sinh[s\lambda] + X' \cosh[s\lambda]. \tag{A7} \]

Varying eq. (A4) with respect to \( s \) yields the statement that \( \xi^2 = -1 \). Since this needs to hold at any \( \lambda \in [0, 1] \), we can choose to evaluate the statement at \( \lambda = 1/2 \); using the constraints \( X^2 = X'^2 = -1 \), we deduce

\[ \xi^2[\lambda = 1/2] = \frac{X \cdot X' - 1}{\cosh[s] + 1} = -1, \tag{A8} \]

thereby informing us that \( \cosh[s] = -X \cdot X' \). We may now perform the integral occurring in \( L \), assuming \( \xi \) is a geodesic (i.e. dropping the constraint term) to obtain

\[ L = -\frac{s}{\sinh^2[s]} \left( s + \cosh[s] \sinh[s] + X \cdot X' (\sinh[s] + s \cosh[s]) \right). \tag{A9} \]

If we then replace \( X \cdot X' \to -\cosh[s] \) and put back the \( 1/H \), we recover \( L = (s/H)^2 \), the result in eq. (A1). For consistency, we have also checked that

\[ \frac{1}{4} g^{\mu\nu}[x] \frac{\partial L}{\partial x^\mu} \frac{\partial L}{\partial x^\nu} = \frac{1}{4} g^{\mu\nu}[x'] \frac{\partial L}{\partial x'^\mu} \frac{\partial L}{\partial x'^\nu} = L, \tag{A10} \]

using the 4D spatially flat results in equations (84) and (85).\footnote{See, for example, equations 3.5 and 3.6 of [7]; the \( \sigma \) there is our \( L/2 \).}
Appendix B: Spherical Coordinates On d-Sphere

Let \( \hat{n}' \) be some fixed unit radial vector in \((d+1)\)D Euclidean space. The main goal of this section is to show that: on the \(d\)-sphere, the Laplacian acting on a scalar field \( \psi \) that depends on the \(d\) angular coordinates \( \vec{\theta} \) only through the dot product \( \cos \theta' \equiv \hat{n}[\hat{\theta}] \cdot \hat{n}' \) is

\[
\frac{1}{\sqrt{\Omega_d}} \sum_{i,j=1}^{d} \frac{\partial}{\partial \theta_i} \left( \sqrt{\Omega_d} (\Omega_d)^{ij} \frac{\partial}{\partial \theta_j} \psi \left[ \hat{n} \cdot \hat{n}' \right] \right) = \frac{1}{\sin^{d-1}[\theta']} \frac{\partial}{\partial \theta'} \left( \sin^{d-1}[\theta'] \frac{\partial}{\partial \theta'} \psi \left[ \cos \theta' \right] \right). \tag{B1}
\]

Here, \( \sqrt{\Omega_d} \) is the square root of the determinant of the metric on the \(d\)-sphere and \((\Omega_d)^{ij}\) is the inverse of the same metric.

Recursion relation

By spherical symmetry we may set \( \hat{n}' \) to lie parallel to the \((d+1)\)-axis of the Euclidean space where the \(d\)-sphere is embedded; thus \( \theta' = \theta^d \) is really the \(d\)th angle on the \(d\)-sphere. Let us now derive a recursion relation between the metric on the \(d\)- and the \((d-1)\)-sphere. The reduced Laplacian result in eq. (B1) will then follow automatically once we know how to calculate \( \sqrt{\Omega_d} \) and the \((d,d)\) component of the inverse metric.

Let the unit radial vector orthogonal to the \((d+1)\)-axis be \( \hat{n}_\perp \). It does not depend on \( \theta' \) but is necessarily a function of the rest of the \((d-1)\) angles \( \{\theta^1, \theta^2, \ldots, \theta^{d-1}\} \). In terms of these objects, we may therefore write an arbitrary unit radial vector in the \((d+1)\)D Euclidean space as

\[
\hat{n}[\hat{\theta}] = \cos[\theta'] \hat{n}' + \sin[\theta'] \hat{n}_\perp, \quad \hat{n}' \cdot \hat{n}_\perp = 0. \tag{B2}
\]

The coefficient of \((d\theta^i)^2\), for \(i = 1, 2, \ldots, d\), in the \(d\)-sphere metric can be computed by differentiating the unit vector in eq. (B2) with respect to \( \theta^i \) and then dotting it with itself. For example, \((\partial \hat{n}/\partial \theta^d)^2 \equiv (\partial \hat{n}/\partial \theta^d)^2 = 1\), and the metric on the \((d-1)\)-sphere is

\[
d\Omega^2_{d-1} \equiv \sum_{i,j=1}^{d-1} (\Omega_{d-1})^{ij} d\theta^i d\theta^j = \sum_{i,j=1}^{d-1} \frac{\partial \hat{n}_\perp}{\partial \theta^i} : \frac{\partial \hat{n}_\perp}{\partial \theta^j} d\theta^i d\theta^j. \tag{B3}
\]

From these considerations we can deduce the recursion relation between the metric on the \(d\)-sphere and that of the \((d-1)\)-sphere.

\[
d\Omega^2_d = (d\theta')^2 + \sin^2[\theta'] d\Omega^2_{d-1}. \tag{B4}
\]

as well as the corresponding relation between the square root of their determinants

\[
\sqrt{\Omega_d} = \sin^{d-1}[\theta'] \sqrt{\Omega_{d-1}}. \tag{B5}
\]

Equations (B4) and (B5) imply the result in eq. (B1).
Notice eq. (B4) is of the form in eq. (5). This suggests the possibility that once the $d = 2, 3$ results are known, by generalizing the construction in section (II) to allow for one positive and one negative source (instead of a single positive one), the results in section (V) may be reached by recursion from the $d = 2, 3$ ones, very much like how it was done in section (III) for flat space(time)s.

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