On Matrix Product Ansatz for Asymmetric Simple Exclusion Process with Open Boundary in the Singular Case

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Abstract
We study a substitute for the matrix product ansatz for asymmetric simple exclusion process with open boundary in the “singular case” $\alpha\beta = q^N \gamma \delta$, when the standard form of the matrix product ansatz of Derrida et al. (J Phys A 26(7):1493–1517, 1993) does not apply. In our approach, the matrix product ansatz is replaced with a pair of linear functionals on an abstract algebra. One of the functionals, $\varphi_1$, is defined on the entire algebra, and determines stationary probabilities for large systems on $L \geq N + 1$ sites. The other functional, $\varphi_0$, is defined only on a finite-dimensional linear subspace of the algebra, and determines stationary probabilities for small systems on $L < N + 1$ sites. Functional $\varphi_0$ vanishes on non-constant Askey–Wilson polynomials and in non-singular case becomes an orthogonality functional for the Askey–Wilson polynomials.

Keywords Asymmetric simple exclusion process with open boundary · Askey–Wilson polynomials · Matrix product ansatz

1 Introduction and Main Results

The asymmetric simple exclusion process (ASEP) with open boundary on sites $\{1, \ldots, L\}$ is a continuous time Markov chain with state space $\{0, 1\}^L$. Informally, see Fig. 1, particles may arrive at the left boundary at rate $\alpha > 0$ and leave at rate $\gamma \geq 0$. A particle may move to the right at rate 1 or to the left at rate $q < 1$. It may leave at the right boundary at rate $\beta > 0$ or a new particle may arrive there at rate $\delta \geq 0$. At most one particle is allowed at each site. More formal description of the evolution is given as Kolmogorov’s equations (1.1) below.
We are interested in the steady state of the ASEP, so we focus on the stationary distribution of the Markov chain. The standard method relies on Kolmogorov’s prospective equations. Denoting by $P_t(\tau_1, \ldots, \tau_L)$ the probability that Markov chain is in configuration $(\tau_1, \ldots, \tau_L) \in \{0, 1\}^L$ at time $t$, we have

$$
dt P_t(\tau_1, \ldots, \tau_L) = \delta_{\tau_1=1} \left[ \alpha P_t(0, \tau_2, \ldots, \tau_L) - \gamma P_t(1, \tau_2, \ldots, \tau_L) \right]$$

$$+ \delta_{\tau_1=0} \left[ \gamma P_t(1, \tau_2, \ldots, \tau_L) - \alpha P_t(0, \tau_2, \ldots, \tau_L) \right]$$

$$+ \sum_{k=1}^{L-1} \delta_{\tau_k=1, \tau_{k+1}=0} \left[ q P_t(\tau_1, \ldots, \tau_k-1, 0, 1, \tau_k+2, \ldots, \tau_L) - P_t(\tau_1, \ldots, \tau_k-1, 1, 0, \tau_k+2, \ldots, \tau_L) \right]$$

$$+ \delta_{\tau_{L-1}=0} \sum_{k=1}^{L-1} \delta_{\tau_k=0, \tau_{k+1}=1} \left[ P_t(\tau_1, \ldots, \tau_k-1, 1, 0, \tau_k+2, \ldots, \tau_L) - q P_t(\tau_1, \ldots, \tau_k-1, 0, 1, \tau_k+2, \ldots, \tau_L) \right]$$

$$+ \delta_{\tau_L=1} \left[ \beta P_t(\tau_1, \ldots, \tau_{L-1}, 1) - \delta P_t(\tau_1, \ldots, \tau_{L-1}, 0) \right]$$

$$+ \delta_{\tau_L=0} \left[ \delta P_t(\tau_1, \ldots, \tau_{L-1}, 0) - \beta P_t(\tau_1, \ldots, \tau_{L-1}, 1) \right]. \quad (1.1)$$

The stationary distribution $P(\tau_1, \ldots, \tau_L)$ of this Markov chain satisfies

$$
dt P(\tau_1, \ldots, \tau_L) = 0$$

so it solves the system of linear equations on the right hand side of (1.1). An ingenious method of determining the stationary probabilities for all $L$ was introduced by Derrida et al. [12], who consider infinite matrices and vectors that satisfy relations

$$DE - qED = D + E, \quad (1.2)$$

$$\langle W | (\alpha E - \gamma D) \rangle = \langle W |, \quad (1.3)$$

$$\langle \beta D - \delta E | V \rangle = | V \rangle. \quad (1.4)$$

The stationary probabilities are then computed as

$$P(\tau_1, \ldots, \tau_L) = \left\langle W \left| \prod_{j=1}^{L} (\tau_j D + (1 - \tau_j) E) \right| V \right\rangle \langle W | (D + E)^L | V \rangle. \quad (1.5)$$

It has been noted in the literature that the above approach may fail: Essler and Rittenberg [15, p. 3384] point out that matrix representation (1.5) runs into problems when $\alpha \beta = \gamma \delta$. 

Fig. 1 Asymmetric simple exclusion process (ASEP) on $\{1, \ldots, L\}$ with open boundaries, with parameters $\alpha, \beta > 0, \gamma, \delta \geq 0$, and $0 \leq q < 1$. Filled in disks represent occupied sites.
and they point out the importance of a more general condition that \( \alpha \beta - q^n \gamma \delta \neq 0 \) for \( n = 0, 1, \ldots \). We will call this a non-singular case.

The singular case when \( \alpha \beta = q^N \gamma \delta \), is discussed by Mallick and Sandow [27, Appendix A] in the context of finite matrix representations. Of course, this is a singular case for the matrix product ansatz, not for the actual Markov chain. To avoid singularity, Lazarescu [24] presents a perturbative generalization of the matrix product ansatz, which was used in [19] to derive exact current statistics for all values of parameters. Continuity of the ASEP with respect to its parameters is also used to derive recursion for stationary probabilities in [26, proof of Theorem 2.3].

1.1 Solution for the Singular Case

Our goal is to analyze the singular case \( \alpha \beta = q^N \gamma \delta \) directly. We consider an abstract noncommutative algebra \( \mathcal{M} \) with identity \( I \) and two generators \( D, E \) that satisfy relation (1.2). The algebra consists of linear combinations of monomials \( X = D^m E^n \ldots D^{m_k} E^{n_k} \). It turns out that monomials in normal order, \( E^m D^n \), form a basis for \( \mathcal{M} \) as a vector space. We introduce increasing subspaces \( \mathcal{M}_k \) of \( \mathcal{M} \) that are spanned by the monomials in normal order of degree at most \( k \), i.e., \( \mathcal{M}_k \) is the span of \( \{E^m D^n : m+n \leq k\} \). The abstract version of the matrix product ansatz for the singular case uses a pair of linear functionals \( \varphi_0 : \mathcal{M}_N \to \mathbb{C} \) and \( \varphi_1 : \mathcal{M} \to \mathbb{C} \).

\[ \text{Theorem 1} \quad \text{Suppose} \quad \alpha, \beta, \gamma, \delta > 0 \text{ satisfy} \quad \alpha \beta = q^N \gamma \delta \quad \text{for some} \quad N = 0, 1, \ldots \quad \text{Then there exists a pair of linear functionals} \quad \varphi_0 : \mathcal{M}_N \to \mathbb{C} \quad \text{and} \quad \varphi_1 : \mathcal{M} \to \mathbb{C} \quad \text{such that stationary probabilities for the ASEP are} \]

\[ P(\tau_1, \ldots, \tau_L) = \frac{\varphi \left[ \prod_{j=1}^{L} (\tau_j D + (1 - \tau_j) E) \right]}{\varphi (D + E)^L}, \quad (1.6) \]

where \( \varphi = \varphi_0 \) if \( 1 \leq L < N + 1 \) and \( \varphi = \varphi_1 \) if \( L \geq N + 1 \). Furthermore, if \( L = N + 1 \) then the stationary distribution is the product of Bernoulli measures

\[ P(\tau_1, \ldots, \tau_{N+1}) = \prod_{j=1}^{N+1} p_j^{\tau_j} q_j^{1-\tau_j} \]

with \( p_j = \frac{\alpha}{\alpha + \gamma q_j^{-1}} \) and \( q_j = 1 - p_j \).

If \( \alpha, \beta > 0, \gamma, \delta \geq 0 \) are such that \( \alpha \beta \neq q^n \gamma \delta \) for all \( n = 0, 1, \ldots \), then \( \varphi_0 \) is defined on \( \mathcal{M}_\infty = \mathcal{M} \), and (1.6) holds with \( \varphi = \varphi_0 \) for all \( L \).

We remark that part of the conclusion of the theorem is the assertion that the denominators in (1.6) are non-zero for all \( L \). Proposition 3 below determines their signs, which according to Remark 3 may vary also in the non-singular case. The signs determine the direction of the current \( J \) through the bond between adjacent sites, which is defined as \( J = \Pr(\tau_k = 1, \tau_{k+1} = 0) - q \Pr(\tau_k = 0, \tau_{k+1} = 1) \). When \( L \neq N + 1 \), we have \( J = \varphi[(E + \gamma q_j^{-1})/(E + \gamma q_j)] \), so the current is negative for \( 2 \leq L \leq N \), and positive for \( L > N + 1 \). As noted in [2, Sect. 3], the current vanishes for \( L = N + 1 \) due to the detailed balance condition satisfied by the product measure.

The proof of Theorem 1 is given in Sect. 2 and consist of recursive construction of the pair of functionals. In the construction, the left and right eigenvectors in (1.3) and (1.4) are replaced by the left and right invariance requirements:
\[
\varphi \left[ (\alpha E - \gamma D)A \right] = \varphi [A], \\
\varphi \left[ A(\beta D - \delta E) \right] = \varphi [A],
\]
for all \( A \in \mathcal{M} \) when \( \varphi = \varphi_1 \) and for all \( A \in \mathcal{M}_{N-1} \) if \( \varphi = \varphi_0 \). By an adaptation of the argument from [12], functionals that satisfy (1.7) and (1.8) give stationary probabilities, see Theorem 3 for precise statement. Similar modification of (1.3) and (1.4) in the matrix formulation appears in [9, Theorem 5.2]. After the paper was submitted, we learned that the idea of working with an abstract algebra and defining a linear functional by using normal order can be traced back to [10, Sect. 3] who consider periodic ASEPs, so constraints (1.7) and (1.8) do not appear.

In the singular case functional \( \varphi_0 \) is defined on \( \mathcal{N}(\mathcal{N}+1)/2 \)-dimensional space \( \mathcal{M}_N \). However, \( \mathcal{M}_N \) is not an algebra, so this is different from the finite dimensional representations of the matrix algebra which were studied by Essler and Rittenberg [15] and Mallick and Sandow [27]. In Appendix C we present a “matrix model” for all \( \alpha, \beta, \gamma, \delta \) with \( 0 < q < 1 \) that was inspired by Mallick and Sandow [27]. The model reproduces their finite matrix model when the parameters are chosen like in their paper, but cannot be used for general parameters due to lack of associativity.

### 1.2 Relation to Askey–Wilson Polynomials

Reference [33] shows that the stationary distribution of the open ASEP is intimately related to the Askey–Wilson polynomials. Here we extend this relation to cover also the singular case, when the Askey–Wilson polynomials do not have the Jacobi matrix, see discussion below.

In the context of ASEP, the Askey–Wilson polynomials depend on parameter \( q \), and on four real parameters \( a, b, c, d \) which are related to parameters of ASEP by the equations

\[
\alpha = \frac{1 - q}{(1 + c)(1 + d)}, \quad \beta = \frac{1 - q}{(1 + a)(1 + b)}, \quad \gamma = \frac{(1 - q)cd}{(1 + c)(1 + d)}, \quad \delta = -\frac{ab(1 - q)}{(1 + a)(1 + b)},
\]

see [7], [15, (74)], [33], and [27]. In this parametrization, the singularity condition becomes \( abcdq^N = 1 \).

Since \( \alpha, \beta > 0 \) and \( \gamma, \delta \geq 0 \), when solving the resulting quadratic equations without loss of generality we can choose \( a, c > 0 \), and then \( b, d \in (-1, 0] \). The explicit expressions are

\[
a = \kappa_+(\beta, \delta), \quad b = \kappa_-(\beta, \delta), \quad c = \kappa_+(\alpha, \gamma), \quad d = \kappa_-(\alpha, \gamma),
\]

where

\[
\kappa_{\pm}(u, v) = \frac{1 - q - u + v \pm \sqrt{(1 - q - u + v)^2 + 4uv}}{2u}.
\]

Recall the \( q \)-hypergeometric function notation

\[
(\alpha_1, \ldots, \alpha_r+1; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, b_2, \ldots, b_r; q)_k} z^k.
\]

Here we use the usual Pochhammer notation:

\[
(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n
\]
and \((a; q)_{n+1} = (1 - aq^n)(a; q)_n\) with \((a; q)_0 = 1\). Later, we will also need the \(q\)-numbers \([n]_q = 1 + q + \cdots + q^{n-1}\) with the convention \([0]_q = 0\), \(q\)-factorials \([n]_q! = [1]_q \cdots [n]_q\) = 
\((1 - q)^{-n}\) \((a; q)_n\) with the convention \([0]_q! = 1\), and the \(q\)-binomial coefficients

\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.
\]

We define the \(n\)th Askey–Wilson polynomial using the \(4\phi_3\)-hypergeometric function, which in the second expression we write more explicitly for all \(x\) rather than for \(x = \cos \psi\).

\[p_n(x; a, b, c, d|q) = a^{-n}(ab, ac, ad; q)_n 4\phi_3\left(\frac{q^{-n}, q^{n-1}abcd, ae^{i\psi}, ae^{-i\psi}}{ab, ac, ad} \mid q \right)
\]

\[= a^{-n}(ab, ac, ad; q)_n \sum_{k=0}^n q^k \frac{(abcdq^{n-1}; q)_k}{(q, ab, ac, ad; q)_k} \prod_{j=0}^{k-1} \left(1 + a^2 q^{2j} - 2axq^j\right).
\]

(1.10)

Although this is not obvious from (1.10), it is known that \(p_n(x; a, b, c, d|q)\) is invariant under permutations of parameters \(a, b, c, d\), and that the polynomial is well defined for all \(a, b, c, d \in \mathbb{C}\). However, in the singular case the degree of the polynomial varies with \(n\) somewhat unexpectedly. It is easy to see from the last expression in (1.10) that if \(abcdq^N = 1\), then for \(0 \leq n \leq N + 1\) the degree of polynomial \(p_n(x; a, b, c, d|q)\) is \(\min\{n, N + 1 - n\}\). In particular, the degrees may decrease and hence there is no three step recursion, or a Jacobi matrix.

The relation of \(\varphi_0\) to Askey–Wilson polynomials is more conveniently expressed using a different pair of generators of algebra \(\mathcal{M}\). Instead of \(\mathbf{E}, \mathbf{D}\), we consider elements \(\mathbf{d}\) and \(\mathbf{e}\) given by

\[
\mathbf{D} = \theta^2 \mathbf{I} + \theta \mathbf{d}, \quad \mathbf{E} = \theta^2 \mathbf{I} + \theta \mathbf{e}, \quad \theta = 1/\sqrt{1-q}.
\]

(1.11)

(Similar transformation was used by several authors, including [33] and [7].)

In this notation, \(\mathcal{M}\) is then an algebra with identity and two generators \(\mathbf{d}, \mathbf{e}\) that satisfy relation

\[
\mathbf{d} \mathbf{e} - q \mathbf{e} \mathbf{d} = \mathbf{I}.
\]

(1.12)

According to Theorem 1, functional \(\varphi_0\) is defined on \(\mathcal{M}_N\) in the singular case, and on all of \(\mathcal{M}\) in the non-singular case. We include non-singular case in the conclusion below by setting \(N = \infty\). The action of \(\varphi_0\) on Askey–Wilson polynomials can now be described as follows.

**Theorem 2** With \(x = \frac{1}{2\theta} (e + d)\), for \(1 \leq n < N + 1\) we have

\[
\varphi_0 \left[p_n(x; a, b, c, d \mid q)\right] = 0.
\]

More generally, for any non-zero \(t \in \mathbb{C}\) let

\[
x_t = \frac{1}{2\theta} \left(\frac{1}{t} e + td\right).
\]

(1.13)

Then

\[
\varphi_0 \left[p_n \left(x_t; at, bt, \frac{c}{t}, \frac{d}{t} \mid q\right)\right] = 0 \quad \text{for} \ 1 \leq n < N + 1.
\]

(1.14)
The proof of Theorem 2 appears in Sect. 3 and is fairly involved. It relies on evaluation of \( \varphi_0 \) on the family of continuous \( q \)-Hermite polynomials, on explicit formula for the connection coefficients between the \( q \)-Hermite polynomials and the Askey–Wilson polynomials which we did not find in the literature, and to complete the proof we need some non-obvious \( q \)-hypergeometric identities. In Appendix B we discuss action of \( \varphi_0 \) and \( \varphi_1 \) on the Askey–Wilson polynomials in the much simpler case of Totally Asymmetric Exclusion process where \( q = 0 \).

### 1.3 Relation to Orthogonality Functional for the Askey–Wilson Polynomials

In the non-singular case when \( q^nabcd \neq 1 \) for all \( n = 0, 1, \ldots \), the Askey–Wilson polynomials \( \{p_n\}_{n=0,1,\ldots} \) are of increasing degrees and satisfy the three step recursion [3, (1.24)]. According to Theorem 1 functional \( \varphi_0 \) is then defined on all of \( M \) and determines stationary probabilities (2.1) for all \( L \geq 0 \). Theorem 2 implies that \( \varphi_0 \) is an orthogonality functional for the Askey–Wilson polynomials, which encodes the relation between ASEP and Askey–Wilson polynomials that was discovered by Uchiyama et al. [33]. In particular, (1.14) corresponds to [33, formula (6.2)] with \( \xi = t \).

Orthogonality can be seen as follows. Theorem 2 says that

\[
\varphi_0[p_n(x; a, b, c, d | q)] = 0
\]

for all \( n \geq 1 \), and it is easy to check, see e.g. [8, Proof of Favard’s theorem], that the latter property together with the three-step recursion for the Askey–Wilson polynomials implies orthogonality:

\[
\varphi_0[p_m(x; a, b, c, d | q)p_n(x; a, b, c, d | q)] = 0
\]

for all \( m \neq n \). This orthogonality relation holds without additional conditions on \( a, b, c, d \) that appear when orthogonality of polynomials \( \{p_n\} \) is considered on the real line [3, Theorem 2.4], or on a complex curve [3, Theorem 2.3]. Since \( \varphi_0[p_n(x; a, b, c, d | q)] \neq 0 \) only for \( n = 0 \), linearization formulas [16] give the value of

\[
\varphi_0\left[p^2_n(x; a, b, c, d | q)\right] = \frac{(ab, ac, ad; q)_n^2}{a^{2n}} \sum_{L=0}^{2n} \frac{q^L (ab, ac, ad; q)_L}{(abcd; q)_L} \times \sum_{j=\max(0,L-n)}^{\min(n,L)} q^{j(j-L)}(q^{-n}, abcdq^{n-1}; q)_j (q, ab, ac, ad; q)_j(q; q)_{L-j} \times \sum_{k=0}^{\min(j, j-L+n)} q^{k}\frac{(q^{-j}, a^2 q^{L+r}; q)_k(q^{-n}, abcdq^{n-1}; q)_k+L-j}{(q)_k(ab, ac, ad; q)_k+L-j},
\]

which may fail to be positive when \( abcd > 1 \).

**Remark 1** After this paper was submitted, we learned about [25] which introduces non-standard truncation condition for the Askey–Wilson polynomials in the singular case \( abcdq^N = 1 \). Their \( q \)-para-Racah polynomials are obtained by taking a limit for special choices of positive parameters \( b, d \) which do not arise from ASEP. Finite dimensional representations of the Askey–Wilson algebra in the singular case are discussed in [1, Sect. 7], [2, p. 15] and [32, Sect. 4].
We begin with two observations from the literature. The first observation is that the proof of Derrida et al. [12] is non-recursive, so it implies that an invariant functional on the finite-dimensional subspace \( M_L \) determines stationary probabilities for ASEP of size \( L \).

**Theorem 3 ([12])** Fix \( L \in \mathbb{N} \). Suppose that \( \varphi \) is a linear functional on \( M_L \) such that \( \varphi \left( (E + D)^L \right) \neq 0 \). If invariance equations (1.7) and (1.8) hold for all \( A \in M_{L-1} \), then the stationary probabilities for the ASEP of length \( L \) are

\[
P(\tau_1, \ldots, \tau_L) = \frac{\varphi \left[ \prod_{j=1}^{L} (\tau_j D + (1 - \tau_j)E) \right]}{\varphi((D + E)^L)}. \tag{2.1}
\]

**Proof** The argument here is the same as the proof in [12, Sect. 11.1] for the matrix version, see also [29, Sect. III]. The important aspect of that proof is that it works with fixed \( L \), i.e., that we do not need to use a recurrence that lowers the value of \( L \) as in [11, formula (8)] or in [26, Theorem 3.2]. We reproduce a version of argument from [12] for completeness and clarity.

For \( L = 1 \) it is easily seen that the stationary distribution is \( P(1) = \frac{\alpha + \delta}{\alpha + \beta + \gamma + \delta} \) with \( P(0) = 1 - P(1) \). On the other hand, Eqs. (1.7) and (1.8) give \( \alpha \varphi[E] - \gamma \varphi[D] = \varphi[I] \) and \( \beta \varphi[D] - \delta \varphi[E] = \varphi[I] \). The solution is:

\[
\varphi[E] = \begin{cases} 
\frac{\beta + \gamma}{\alpha + \gamma} \varphi[I] & \text{if } \alpha \beta \neq \gamma \delta \\
\frac{\beta}{\gamma} \varphi[I] & \text{if } \alpha \beta = \gamma \delta
\end{cases}
\]

\[
\varphi[D] = \begin{cases} 
\frac{\alpha + \delta}{\alpha + \gamma} \varphi[I] & \text{if } \alpha \beta \neq \gamma \delta \\
\frac{\alpha}{\gamma} \varphi[I] & \text{if } \alpha \beta = \gamma \delta
\end{cases}
\]

where we note that \( \varphi[I] = 0 \) when \( \alpha \beta = \gamma \delta \) and in this case we also used the normalization \( \varphi[E + D] = 1 \) to determine the values. In both cases, a calculation shows that

\[
\frac{\varphi[D]}{\varphi[E] + \varphi[D]} = \frac{\alpha + \delta}{\alpha + \beta + \gamma + \delta}
\]

giving the correct value of \( P(1) \).

Suppose that \( L \geq 2 \). Denote by \( p(\tau_1, \ldots, \tau_L) = \varphi \left[ \prod_{j=1}^{L} (\tau_j D + (1 - \tau_j)E) \right] \) the unnormalized probabilities. Since by assumption the denominator in (2.1) is non-zero, it is enough to verify that the right hand side of (1.1) vanishes on \( p(\tau_1, \ldots, \tau_L) \). That is, we want to show that

\[
\begin{align*}
&\left( \delta_{\tau_1=1} - \delta_{\tau_1=0} \right) \left[ \alpha p(0, \tau_2, \ldots, \tau_L) - \gamma p(1, \tau_2, \ldots, \tau_L) \right] \\
&+ \sum_{k=1}^{L-1} \left( \delta_{\tau_k=0, \tau_{k+1}=1} - \delta_{\tau_k=1, \tau_{k+1}=0} \right) \\
&\quad \left[ p(\tau_1, \ldots, \tau_{k-1}, 1, 0, \tau_{k+2}, \ldots, \tau_L) \\
&- q p(\tau_1, \ldots, \tau_{k-1}, 0, 1, \tau_{k+2}, \ldots, \tau_L) \right] \\
&+ \left( \delta_{\tau_L=0} - \delta_{\tau_L=1} \right) \left[ \beta p(\tau_1, \ldots, \tau_{L-1}, 1) - \delta p(\tau_1, \ldots, \tau_{L-1}, 0) \right] = 0.
\end{align*}
\tag{2.2}
\]

Denote

\[
X_k = \prod_{j=1}^{k} (\tau_j D + (1 - \tau_j)E) \quad \text{and} \quad Y_k = \prod_{j=k}^{L} (\tau_j D + (1 - \tau_j)E)
\]

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with the usual convention that empty products are 1. Relation (1.2) implies that
\[ p(\tau_1, \ldots, \tau_k-1, 1, 0, \tau_k+2, \ldots, \tau_L) - q p(\tau_1, \ldots, \tau_k-1, 0, 1, \tau_k+2, \ldots, \tau_L) = \varphi[X_{k-1} (DE - qED)Y_{k+2}] = \varphi[X_{k-1} (D+E)Y_{k+2}]. \]
Noting that
\[ \delta_{\tau_k=0, \tau_{k+1}=1} - \delta_{\tau_k=1, \tau_{k+1}=0} = (1 - \tau_k)\tau_{k+1} - \tau_k (1 - \tau_{k+1}) = \tau_{k+1} - \tau_k, \]
the sum in (2.2) becomes
\[ \sum_{k=1}^{L-1} (\tau_{k+1} - \tau_k) \varphi[X_k (D+E)Y_{k+2}]. \]
Since \( \tau_k, \tau_{k+1} \in \{0, 1\} \), the difference \( \tau_{k+1} - \tau_k \) can take only three values 0, \( \pm 1 \). Considering all four possible cases, we get
\[ (\tau_{k+1} - \tau_k) \varphi[X_{k-1} (D+E)Y_{k+2}] = (\tau_{k+1} - \tau_k) \left( \varphi[X_{k-1} (\tau_{k+1}D + (1 - \tau_{k+1})E)Y_{k+2}] + \varphi[X_{k-1} (\tau_kD + (1 - \tau_k)E)Y_{k+2}] \right) \]
\[ = (\tau_{k+1} - \tau_k) \left( \varphi[X_k Y_{k+2}] + \varphi[X_{k-1} Y_{k+1}] \right) \]
\[ = \varepsilon_k \varphi[X_{k-1} Y_{k+1}] - \varepsilon_{k+1} \varphi[X_k Y_{k+2}], \]
where \( \varepsilon_k = \delta_{\tau_k=1} - \delta_{\tau_k=0} = \pm 1 \). (For the last equality we need to notice that \( X_{k-1} Y_{k+1} = X_k Y_{k+2} \) when \( \tau_k = \tau_{k+1} \).)

Thus
\[ \sum_{k=1}^{L-1} (\tau_{k+1} - \tau_k) \varphi[X_k (D+E)Y_{k+2}] = \sum_{k=1}^{L-1} (\varepsilon_k \varphi[X_{k-1} Y_{k+1}] - \varepsilon_{k+1} \varphi[X_k Y_{k+2}]) \]
\[ = \varepsilon_1 \varphi[Y_2] - \varepsilon_L \varphi[X_{L-1}]. \]

By invariance we have
\[ [\alpha p(0, \tau_2, \ldots, \tau_L) - \gamma p(1, \tau_2, \ldots, \tau_L)] = \varphi[(\alpha E - \gamma D)Y_2] = \varphi[Y_2] \]
\[ [\beta p(\tau_1, \ldots, \tau_{L-1}, 1) - \delta p(\tau_1, \ldots, \tau_{L-1}, 0)] = \varphi[X_{L-1} (\beta D - \delta E)] = \varphi[X_{L-1}]. \]

So the left hand side of (2.2) becomes
\[ -\varepsilon_1 \varphi[Y_2] + \varepsilon_1 \varphi[Y_2] - \varepsilon_L \varphi[X_{L-1}] + \varepsilon_L \varphi[X_{L-1}] = 0 \]
proving (2.2). \( \square \)

The second observation is that stationary distribution for ASEP of length \( L = N + 1 \) is given as an explicit product of Bernoulli measures. This fact has been explicitly noted in [14, Sect. 5.2], see also [13, Sect. 4.6.2] and [2, Sect. 3]. The proof consists of verification of detailed balance equations so that individual terms on the right hand side of (1.1) vanish.

**Proposition 1** (Enaud and Derrida [14]) Suppose \( \alpha \beta = q^N \gamma \delta \). If \( L = N + 1 \) then the stationary distribution of the ASEP is the product of Bernoulli measures
\[ P(\tau_1, \ldots, \tau_L) = \prod_{j=1}^{L} p_j^{\tau_j} q_j^{1-\tau_j} \]

with \( p_j = \frac{\alpha}{\alpha + \beta q^{1-r}} \) and \( q_j = 1 - p_j \).

### 2.1 Construction of the Pair of Invariant Functionals

The construction starts with choosing a convenient basis for \( \mathcal{M} \), consisting of monomials in normal order, with all factors \( e \) occurring before \( d \). Such monomials appear in many references, see e.g. Frisch and Bourret [17, p. 368], Bożejko et al. [6, p. 137], Mallick and Sandow [27, p. 4524], or [10, Eq. (19)].

**Proposition 2** Monomials in normal order \( \{ e^m d^n : m, n = 0, 1, \ldots \} \) are a basis of \( \mathcal{M} \) considered as a vector space. In this basis \( \mathcal{M}_k \) is the span of \( \{ e^m d^n : m + n \leq k \} \).

**Proof** It is easy to check by induction that \( q \)-commutation relation (1.12) gives explicit expressions for “swaps” that recursively convert all monomials into linear combinations of monomials in normal order. We have

\[ de^m d^n = q^m e^m d^{n+1} + [m]_q e^{m-1} d^n. \tag{2.3} \]

Similarly, we get

\[ e^m d^n e = q^n e^{m+1} d^n + [n]_q e^{m-1} d^n. \tag{2.4} \]

(Formulas (2.3) and (2.4) hold also for \( m = 0 \) or \( n = 0 \) after omitting the term with \( [0]_q = 0 \).)

The formulas imply that any monomial is a linear combination of monomials in normal order:

\[ d^{n_1} e^{m_1} \cdots d^{n_k} e^{m_k} = q^I e^m d^n + \sum_{i+j \leq m+n-1} a_{i,j} e^i d^j, \tag{2.5} \]

where \( m = m_1 + \cdots + m_k, n = n_1 + \cdots + n_k \) and \( I = \sum_{i=1}^{k} \sum_{j=1}^{i} m_in_j \) is the minimal number of inversions (length) of a permutation that maps \( e^m d^n \) into \( d^{n_1} e^{m_1} \cdots d^{n_k} e^{m_k} \), see e.g. [4]. Compare [27, Appendix A].

Formula (2.5) shows that monomials in normal order span \( \mathcal{M} \). To verify that they are linearly independent we consider a pair of linear mappings (endomorphism) \( D_q \) and \( Z \) acting on polynomials \( C[z] \) which are the \( q \)-derivative and the multiplication mappings:

\[ (D_q p)(z) = \frac{p(z) - p(qz)}{(1 - q)z}, \quad (Z p)(z) = zp(z). \]

The mapping \( d \mapsto D_q \) and \( e \mapsto Z \) extends to homomorphism of algebra \( C(\langle d, e \rangle) \) of polynomials in noncommuting variables \( e, d \) to the algebra \( End(C[z]) \). It is well known that \( D_q Z - qZD_q \) is the identity, so we get an induced homomorphism of algebras

\[ \mathcal{M} = \frac{C(\langle d, e \rangle)}{I} \rightarrow End(C[z]), \]

where \( I \) is the two sided ideal generated by \( de - qed - I \). Therefore, it is enough to prove linear independence of \( \{ Z^m D_q^n \} \).
To prove the latter, consider a finite sum
\[ S = \sum_{m,n \geq 0} a_{m,n} Z^m D^n_q = 0 \]
and suppose that some of the coefficients \( a_{m,n} \) are non-zero. Let \( n_* \geq 0 \) be the smallest value of index \( n \) among the non-zero coefficient \( a_{m,n} \). We note that

\[ Z^m D^n_q(z_{n_*}) = \begin{cases} 0 & \text{if } n > n_*, \\ [n_*]_q z^m & \text{if } n = n_. \end{cases} \]

Therefore, applying \( S \) to the monomial \( z^{n_*} \in \mathbb{C}[z] \) we get

\[ \sum_{m \in M} a_{m,n_*} [n_*]_q z^m = 0, \]

i.e., all \( \{a_{m,n_*} : m \in M\} \) are zero, in contradiction to our choice of \( n_* \). The contradiction shows that all coefficients must be zero, proving linear independence. \( \square \)

Using (1.11) we remark that invariance conditions (1.7) and (1.8) with \( A \in \mathcal{M}_k \) can be written equivalently in our basis of monomials in normal order as

\[ \alpha \varphi[e^{m+1}d^n] - \gamma \varphi[de^m d^n] = \Delta(\gamma - \alpha) \varphi[e^m d^n], \]
\[ -\delta \varphi[e^m d^n e] + \beta \varphi[e^m d^{n+1}] = \Delta(\delta - \beta) \varphi[e^m d^n], \]

where \( m + n \leq k \) and \( \Delta(x) = \theta^{-1} + \theta x \).

### 2.2 Recursive Construction of the Functionals

We define linear functional \( \varphi = \varphi_0 \) or \( \varphi = \varphi_1 \) by assigning its values on all elements of the basis \( \{e^m d^n\} \) and then extending it to \( \mathcal{M}_N \) or \( \mathcal{M} \) by linearity. On the basis, we define \( \varphi \) recursively, extending it from \( \mathcal{M}_k \) to \( \mathcal{M}_{k+1} \) in such a way that the invariance properties (1.7) and (1.8) hold.

#### 2.2.1 Initial Values

We set \( \varphi_0[I] = 1 \). We set

\[ \varphi_1[e^m d^n] = \begin{cases} 0 & \text{if } m + n \leq N, \\ \Pi^{-1} \alpha^n \gamma^m q^{m(m-1)/2} & \text{if } m + n = N + 1, \end{cases} \]

where the normalizing constant \( \Pi = \theta^{N+1} \prod_{j=1}^{N+1} (\alpha + q^{j-1} \gamma) \) is chosen so that \( \varphi_1[(e + d)^{N+1}] = 1/\theta^{N+1} \).

Clearly, \( \varphi_1 \equiv 0 \) on \( \mathcal{M}_N \). We need to check that our initialization of \( \varphi_1 \) has the properties we need for the recursive construction: that invariance conditions hold for \( A \in \mathcal{M}_N \), and that \( \varphi_1 \) determines the stationary measure of ASEP with \( L = N + 1 \).

**Lemma 1** For monomials of degree \( N + 1 \) we have

\[ \varphi_1[D^\tau_1 E^{1-\tau_1} \ldots D^\tau_{N+1} E^{1-\tau_{N+1}}] = \prod_{j=1}^{N+1} p_j^{\tau_j} q_j^{1-\tau_j}, \]

where the weights \( \{p_j\} \) come from stationary product measure in Proposition 1. Furthermore, (1.7) and (1.8) hold for \( A \in \mathcal{M}_N \).
Proof Since $\phi_1$ vanishes on polynomials of lower degree, from (1.11) it is easy to see that

$$\phi_1[D^{\tau_1}e^{1-\tau_1} \ldots D^{\tau_N+1}e^{1-\tau_N+1}] = \theta^{N+1} \phi_1[d^{\tau_1}e^{1-\tau_1} \ldots d^{\tau_N+1}e^{1-\tau_N+1}].$$

So we only need to show that

$$\phi_1[d^{\tau_1}e^{1-\tau_1} \ldots d^{\tau_N+1}e^{1-\tau_N+1}] = \prod_{j=1}^{N+1} p_j^{\tau_j} q_j^{1-\tau_j} / \theta^{N+1}. \quad (2.10)$$

It is easy to see that this formula holds true for $\phi_1[e^m d^{N+1-m}]$. (In fact, this is how we defined $\phi_1[e^m d^{m}]$ when $m+n = N+1$.) All monomials of the form $d^{\tau_1}e^{1-\tau_1} \ldots d^{\tau_N+1}e^{1-\tau_N+1}$ can be obtained from monomials $e^m d^{N+1-m}$ in normal order by applying a finite number of adjacent transpositions, i.e., by swapping pairs of adjacent factors ed or de. (Adjacent transpositions are Coxeter generators for the permutation group, see e.g. [4].) So to complete the proof we check that if formula (2.10) holds for some monomial, then it also holds after we swap the entries at adjacent locations $k, k+1$. Suppose that

$$\theta^{N+1} \phi_1[XeYd] = q_k p_{k+1} \Pi' = \frac{\alpha \gamma q^{k-1}}{(\alpha + q^{k-1} \gamma)(\alpha + q^k \gamma)} \Pi',$$

with $X = d^{\tau_1}e^{1-\tau_1} \ldots d^{\tau_k-1}e^{1-\tau_k-1}$, $Y = e^{1-\tau_k} \ldots d^{\tau_N+1}e^{1-\tau_N+1}$ and $\Pi' = \prod_{j \neq k, k+1} p_j^{\tau_j} q_j^{1-\tau_j}$. Multiplying this by $q$ and replacing $q ed$ by $de - I$, we get

$$\theta^{N+1} \phi_1[XeYd] = \frac{\alpha \gamma q^k}{(\alpha + q^{k-1} \gamma)(\alpha + q^k \gamma)} \Pi' = p_k q_{k+1} \Pi',$$

as $\phi_1$ vanishes on lower degree monomials. So the swap preserves the expression on the right hand side of (2.10). The case when the factors at the adjacent locations are de is handled similarly.

To verify that (1.7) and (1.8) hold for $A \in M_N$ we show that (2.6) and (2.7) hold for $m + n \leq N$. Indeed, both sides are zero if $m + n \leq N - 1$, and if $m + n = N$ then the right hand sides are still zero. By (2.10), the left hand side of (2.6) is

$$\alpha^{n+1} \gamma^{m+1} \left( q^{m(m+1)/2} - q^m q^{m(m-1)/2} \right) / \Pi = 0.$$

The left hand side of (2.7) is

$$\alpha^n \gamma^m q^{m(m-1)/2} (\alpha \beta - q^{m+n} \gamma \delta) / \Pi = 0$$

by singularity assumption.

\[\square\]

2.2.2 Recursive Step for $\varphi = \varphi_0$ or $\varphi_1$

Suppose $\varphi$ is defined on $M_k$ and that invariance conditions hold for $A \in M_{k-1}$. If $m + n = k$ with $1 \leq k < N$ (case of $\varphi_0$) or $k \geq N + 1$ (case of $\varphi_1$). Define

$$\varphi[e^{m+1}d^n] = \frac{1}{(q^N - q^{m+n}) \gamma \delta} \left[ (\beta \Delta(\gamma - \alpha) + \gamma \Delta(\delta - \beta) q^m) \varphi[e^m d^n] \right. \left. + \gamma \delta [n] q^m \varphi[e^m d^{n-1}] + \beta \gamma [n] \varphi[e^{m-1} d^n] \right], \quad (2.11)$$

$$\varphi[e^m d^{n+1}] = \frac{1}{(q^N - q^{m+n}) \gamma \delta} \left[ (\alpha \Delta(\delta - \beta) + \delta \Delta(\gamma - \alpha) q^n) \varphi[e^m d^n] \right].$$
\[ + \alpha \delta [n] q \varphi [e^{m} d^{n-1}] + \gamma q^n [m] q \varphi [e^{m-1} d^n]. \] (2.12)

where \( \Delta(x) = \theta^{-1} + \theta x \) comes from (2.6) and (2.7).

**Remark 2** If \( \alpha \beta - q^n \gamma \delta \neq 0 \) for all \( n \), we define \( \varphi _0 \) on \( \mathcal{M} \), replacing the above recursion with

\[
\begin{align*}
\varphi _0 [e^{m+1} d^n] &= \frac{1}{\alpha \beta - q^{m+n} \gamma \delta} \left[ (\beta \Delta (\gamma - \alpha) + \gamma \Delta (\delta - \beta) q^m) \varphi [e^{m} d^n] \\
& \quad + \gamma \delta [n] q^m \varphi [e^{m} d^{n-1}] + \beta \gamma [m] q \varphi [e^{m-1} d^n] \right], \\
\varphi _0 [e^{m} d^{n+1}] &= \frac{1}{\alpha \beta - q^{m+n} \gamma \delta} \left[ (\alpha \Delta (\delta - \beta) + \delta \Delta (\gamma - \alpha) q^n) \varphi [e^{m} d^n] \\
& \quad + \alpha \delta [n] q \varphi [e^{m} d^{n-1}] + \gamma q^n [m] q \varphi [e^{m-1} d^n] \right].
\end{align*}
\] (2.13) (2.14)

We need to make sure that this expression is well defined.

**Lemma 2** Fix \( k \neq N \). Suppose \( m' + n' = k + 1 \). Then \( \varphi [e^{m'} d'] \) is well defined: both formulas give the same answer when \( (m', n') \) can be represented as \( (m', n') = (m + 1, n) \) and as \( (m', n') = (m, n + 1) \).

**Proof** We proceed by contradiction. Suppose that \( m, n \) is a pair of smallest degree \( m + n \) where consistency fails. This means that (2.6) and (2.7) still hold for all pairs of lower degree but the solution (2.12) with \( m \) replaced by \( m + 1 \) and \( n \) replaced by \( n - 1 \) does not match the solution in (2.11). We show that this cannot be true by verifying that the numerators are the same,

\[
\begin{align*}
\left( \beta \Delta (\gamma - \alpha) + \gamma \Delta (\delta - \beta) q^m \right) \varphi [e^{m} d^n] \\
& + \gamma \delta [n] q^m \varphi [e^{m} d^{n-1}] + \beta \gamma [m] q \varphi [e^{m-1} d^n] \\
= & \left( \alpha \Delta (\delta - \beta) + \delta \Delta (\gamma - \alpha) q^n \right) \varphi [e^{m+1} d^{n-1}] \\
& + \alpha \delta [n-1] q \varphi [e^{m+1} d^{n-2}] + \gamma q^n [m+1] q \varphi [e^{m+1} d^{n-1}] .
\end{align*}
\] (2.15)

(Formally, the term with the factor \( [n - 1] q \) should be omitted when \( n = 1 \).) The difference between the left hand side and the right hand side of (2.15) is

\[
\Delta (\gamma - \alpha) \left( \beta \varphi [e^{m} d^n] - \delta q^n \varphi [e^{m+1} d^{n-1}] \right) \\
+ \Delta (\delta - \beta) \left( \gamma q^m \varphi [e^{m} d^n] - \alpha \varphi [e^{m+1} d^{n-1}] \right) \\
+ \left( \gamma q^n [m] q \varphi [e^{m+1} d^{n-2}] - \delta q^n [m+1] q \varphi [e^{m+1} d^{n-1}] \right) .
\]

Since \( q^n [n] q = q^n [n-1] q + q^n q^{n-1} \) and \( q^{n-1} [m+1] q = q^{n-1} [m] q + q^m q^{n-1} \), canceling the terms with factor \( q^n q^{n-1} \) we rewrite the above as

\[
\Delta (\gamma - \alpha) \left( \beta \varphi [e^{m} d^n] - \delta q^n \varphi [e^{m+1} d^{n-1}] \right) \\
+ \Delta (\delta - \beta) \left( \gamma q^m \varphi [e^{m} d^n] - \alpha \varphi [e^{m+1} d^{n-1}] \right) .
\]
+ \gamma[n-1]q (\gamma q^n \varphi[e^{m+1}d^n] - \alpha \varphi[e^{m+1}d^{n-1}])
+ \gamma[m]q (\beta \varphi[e^{m-1}d^n] - \delta q^n \varphi[e^{m}d^{n-1}]).

We now use (2.3) and (2.4). We get
\[
\Delta(\gamma - \alpha) \left( \beta \varphi[e^{m}d^n] - \delta \varphi[e^{m+1}d^{n-1}] \right) + \Delta(\gamma - \alpha) \delta[n-1]q \varphi[e^{m}d^{n-2}]
+ \Delta(\delta - \beta) \left( \gamma \varphi[de^{m}d^{n-1}] - \alpha \varphi[e^{m+1}d^{n-1}] \right) - \Delta(\delta - \beta) \gamma[m]q \varphi[e^{m+1}d^{n-1}]
+ \delta[n-1]q \left( \gamma \varphi[de^{m+1}d^{n-2}] - \alpha \varphi[e^{m+1}d^{n-2}] \right) - \gamma \delta[n-1]q \gamma[m]q \varphi[e^{m+1}d^{n-2}]
+ \gamma[m]q \left( \beta \varphi[e^{m-1}d^n] - \delta \varphi[e^{m+1}d^{n-1}] \right) + \gamma[m]q \delta[n-1]q \varphi[e^{m+1}d^{n-2}].
\]

After canceling \gamma \delta[m]q [n-1]q \varphi[e^{m+1}d^{n-2}] we regroup the expression into the sum \( S_1 + S_2 + S_3 \) with
\[
S_1 = \Delta(\gamma - \alpha) \left( \beta \varphi[e^{m}d^n] - \delta \varphi[e^{m}d^{n-1}] \right)
- \Delta(\delta - \beta) \left( \alpha \varphi[e^{m+1}d^{n-1}] - \gamma \varphi[de^{m}d^{n-1}] \right),
\]
\[
S_2 = \delta[n-1]q \left[ \Delta(\gamma - \alpha) \varphi[e^{m}d^{n-2}] - \left( \alpha \varphi[e^{m+1}d^{n-2}] - \gamma \varphi[de^{m}d^{n-2}] \right) \right],
\]
\[
S_3 = \gamma[m]q \left[ \left( \beta \varphi[e^{m-1}d^n] - \delta \varphi[e^{m+1}d^{n-1}] \right) - \Delta(\delta - \beta) \varphi[e^{m+1}d^{n-1}] \right].
\]

From (2.6) and (2.7) we see that \( S_1, S_2, S_3 \) are zero, proving (2.15). □

Formulas (2.11) and (2.12) extend \varphi from \( \mathcal{M}_k \) to \( \mathcal{M}_{k+1} \).

**Lemma 3** Invariance conditions (1.7) and (1.8) hold for \( A \in \mathcal{M}_k \).

**Proof** We verify (2.6) and (2.7) with \( m + n \leq k \). By inductive assumption (2.6) and (2.7) hold when \( m + n < k \), so we only need to consider \( m + n = k \).

Using “swap identities” (2.3) and (2.4) we rewrite these relations as
\[
\alpha \varphi[e^{m+1}d^n] - q^n \gamma \varphi[e^{m}d^{n+1}] = \Delta(\gamma - \alpha) \varphi[e^{m}d^n] + \gamma[m]q \varphi[e^{m-1}d^n] \quad (2.16)
\]
and
\[
- q^n \delta \varphi[e^{m+1}d^n] + \beta \varphi[e^{m}d^{n+1}] = \Delta(\delta - \beta) \varphi[e^{m}d^n] + \delta[n]q \varphi[e^{m}d^{n-1}] \quad (2.17)
\]
with the solution given in (2.11) and (2.12). By linearity this establishes invariance conditions for all \( A \in \mathcal{M}_k \). □

### 2.3 Signs of \( \varphi \) on Monomials

To verify that \( \varphi[(E + D)^k] \neq 0 \), we will need the following version of a formula discussed in [27, Appendix A].

**Lemma 4** If \( X = E^{m_1} \ldots D^{n_k} E^{m_k} D^{n_k} \) is a monomial of degree \( m + n \) with \( m = m_1 + \cdots + m_k \), \( n = n_1 + \cdots + n_k \), then there exist non-negative integers \( b_j, c_j \) and monomials \( Y_j, Z_j \) of degree \( m + n \) such that

\[ \square \] Springer
Both proofs are similar and consist of showing that for the same degree $L$ the same degree $L$ holds for $ϕ_1$. We begin with the recursive proof for functional $ϕ_1$ and $L$ the same degree $L$.

**Proof**

Denote $S = E + D$. Suppose that formulas hold for $X$ with $k \geq 0$ factors. Then for $n = n_{k+1}$ and $m = m_0$ by repeated applications of (1.2) we get

$$D^nE = qD^{n-1}ED + D^{n-1}S = q^2D^{n-2}ED^2 + D^{n-2}SD + D^{n-1}S = \ldots$$

$$= q^nED^n + \sum_{j=1}^{n-1} D^{n-j-1}SD^j$$

and

$$DE^m = qEDE^{m-1} + SE^{m-1} = q^2E^2DE^{m-2} + ESE^{m-2} + SE^{m-1} = \ldots$$

$$= q^mE^mD + \sum_{j=0}^{m-1} E^jSE^{m-1-j}.$$  

Clearly, $D^{n-1-j}SD^j = D^{n-j-1}ED^j + D^n$ is the sum of monomials of degree $n$ and $E/SE^{m-1-j} = E^m + E^jD^{m-1-j}$ is the sum of monomials of degree $m$. We now multiply (2.19) by $XE^{mk+1}$ from the left and use the induction assumption. Similarly, we multiply (2.20) by $D^{mk}X$ from the right and use the induction assumption. This establishes (2.18) by induction.

**Proposition 3**

If $αβ = q^nγδ$ then

1. $(-1)^Lφ_0[(E + D)^L] > 0$ for $L = 0, \ldots, N$,
2. $φ_1[(E + D)^L] > 0$ for $L \geq N + 1$.

**Remark 3**

An inspection of our argument shows that in the non-singular case with $αβ \neq q^nγδ$ for all $n$, we have $φ_0[(E + D)^L] \neq 0$ for all $L$. More precisely, define $M = \min\{n \geq 0 : αβ > q^nγδ\}$, with $M = 0$ when $αβ > γδ$. Then

1. $(-1)^Lφ_0[(E + D)^L] > 0$ for $0 \leq L \leq M$,
2. $(-1)^Mφ_0[(E + D)^L] > 0$ for $L \geq M + 1$.

In particular, the current $J = φ_0[(E + D)^L]/φ_0[(E + D)^L]$ undergoes reversal as the system size increases: $J < 0$ for $1 \leq L \leq M$ and $J > 0$ for $L \geq M + 1$.

**Proof**

Both proofs are similar and consist of showing that for $φ = φ_0$ and for $φ = φ_1$ the value $φ[X]$ on a monomial $X = E^{m_1}D^{n_1} \ldots E^{m_k}D^{n_k}$ is real, and that for all monomials $X$ of the same degree $L = m + n$ with $m = m_1 + \cdots + m_k$, $n = n_1 + \cdots + n_k$, the sign of $φ[X]$ is the same. We begin with the recursive proof for functional $φ = φ_0$ where the signs alternate with $L$. Then we will indicate how to modify the proof for $φ = φ_1$ where the signs are all positive.

For $L = 0$ we have $(-1)^Lφ[X] = 1 > 0$ by the initialization of $φ_0$. Suppose that $(-1)^Lφ[X] > 0$ holds for all monomials $X = E^{m_1}D^{n_1} \ldots E^{m_k}D^{n_k}$ with $m = m_1 + \cdots + m_k = m, n = n_1 + \cdots + n_k$ of degree $L = m + n < N$.

A monomial $Y$ of degree $L + 1$ arises from a monomial $X$ of degree $L$ in one of the following ways: $Y = EX, Y = XD, Y = DX$, or $Y = XE$. Our goal is to show that in each of these cases $φ[Y]$ is a real number of the opposite sign than $φ[X]$.

Cases $Y = EX$ and $Y = XD$ are handled together, and are needed for the other two cases. From (1.7) and (1.8) applied with $A = X$ we get

$$αφ[EX] − γφ[DX] = φ[X]$$

and

$$−δφ[XE] + βφ[XD] = φ[X].$$
Applying (2.18) to $DX$ and to $XE$ we get
\[
\begin{align*}
\alpha \varphi[EX] - q^n \gamma \varphi[XD] &= d_1, \\
-q^n \delta \varphi[EX] + \beta \varphi[XD] &= d_2,
\end{align*}
\]
where by inductive assumption $d_1 = \varphi[X] + \gamma \sum_j c_j \varphi[Z_j]$ is the sum of non-zero real numbers of the same sign $(-1)^L$, and similarly $d_2$ is real and has the sign $(-1)^L$. The solution of this system is

\[
\varphi[EX] = \begin{vmatrix} d_1 - q^n \gamma \\ d_2 \\ \alpha - q^n \gamma \\ -q^n \delta \end{vmatrix} \quad \text{and} \quad \varphi[XD] = \begin{vmatrix} \alpha & d_1 \gamma \\ -q^n \delta & d_2 \\ \alpha & -q^n \gamma \\ -q^n \delta & \beta \end{vmatrix}.
\]

(2.21)

Since the numerators have sign $(-1)^L$ and the denominator $\alpha \beta - q^L \gamma \delta = \gamma \delta (q^N - q^L) < 0$, this establishes the conclusion for all monomials $Y = E^{m_1+1}D^{n_1} \ldots E^{m_k}D^{n_k}$ and $Y = E^{m_1}D^{n_1} \ldots E^{m_k}D^{n_k+1}$ of degree $m + n + 1 = L + 1$.

To handle the case $Y = DX$, we use already established information about the sign of monomial $\varphi[EX]$. Using (1.7), we see that the sign of $\gamma \varphi[DX] = \alpha \varphi[EX] - \varphi[X]$ is $(-1)^{L+1}$, and similarly (1.8) determines the sign of $\delta \varphi[XE] = \beta \varphi[XD] - \varphi[X]$ as $(-1)^{L+1}$.

The proof for $\varphi = \varphi_1$ is similar, starting with formula (2.9) which establishes positivity for $L = N + 1$. We then use (2.21) to prove that $\varphi_1[EX] > 0$ and $\varphi_1[XD] > 0$, noting that in the case of $\varphi_1$ we have $d_1, d_2 > 0$ and that the denominator $\alpha \beta - q^L \gamma \delta = \gamma \delta (q^N - q^L) > 0$ as $L \geq N + 1$. Finally, applying $\varphi_1$ to (2.18) we see that $\varphi_1[DX] > 0$ and $\varphi_1[XE] > 0$. □

**Proof** (Conclusion of proof of Theorem 1) Functional $\varphi_0$ satisfies invariance conditions (1.8) and (1.7), and $\varphi_0[(E + D)^L] \neq 0$ for $L \leq N$ by Proposition 3. Therefore, by Theorem 3 we get (1.6) for $L \leq N$. In the non-singular case, by Remark 2 functional $\varphi_0$ is defined on $M$ and by Remark 3 we have $\varphi_0[(E + D)^L] \neq 0$ for all $L$, so Theorem 3 applies.

Functional $\varphi_1$ satisfies invariance conditions (1.8) and (1.7) by Lemma 1 and construction. Proposition 3 states that $\varphi_1[(E + D)^L] > 0$ for $L \geq N + 1$. Therefore, by Theorem 3 we get (1.6) for all $L \geq N + 2$. Proposition 1 gives the stationary distribution for $L = N + 1$, and Lemma 1 shows that this case also arises from (1.6). □

### 3 Proof of Theorem 2

Denote $\varphi_{k,n} = \varphi[e^{k}d^n]$, where $\varphi$ is either $\varphi_0$ or $\varphi_1$. (The latter is needed only for the second part of Theorem 4.) We first rewrite (2.13) and (2.14) using Askey–Wilson parameters (1.9). After a calculation we get

\[
\begin{align*}
\varphi_{m+1,n} &= \frac{1}{1 - abcdqm+n} \left( \theta (c + d - cd(a + b)q^m) \varphi_{m,n} - cd[m]q \varphi_{m-1,n} + abcdq^n[n]q \varphi_{m,n-1} \right), \\
\varphi_{m,n+1} &= \frac{1}{1 - abcdqm+n} \left( \theta (a + b - ab(c + d)q^n) \varphi_{m,n} - ab[n]q \varphi_{m,n-1} + abcdq^{m}[m]q \varphi_{m-1,n} \right).
\end{align*}
\]

(3.1) (3.2)
Our proof relies heavily on monic continuous $q$-Hermite polynomials defined by the three step recurrence
\[ xH_n(x) = H_{n+1}(x) + [n]_q H_{n-1}(x) \] (3.3)
with initial values $H_0(x) = 1$ and $H_{-1}(x) = 0$. These polynomials are convenient because when evaluated at $e + d$ they have explicit expansion in the basis of monomials in normal order.

Somewhat more generally, for $t \in \mathbb{C}$ we consider polynomials $H_n(x; t)$ defined by the three step recurrence
\[ xH_n(x; t) = H_{n+1}(x; t) + [n]_q H_{n-1}(x; t) \] (3.4)
with initial values $H_0(x; t) = 1$ and $H_{-1}(x; t) = 0$. For $t \neq 0$ these two families of polynomials are related by a simple formula $H_n(x; t^2) = t^n H_n(x/t)$.

The following version of [6, Corollary 2.8] follows from (3.4).

**Lemma 5**
\[ H_n(te + d; t) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q t^k e^k d^{n-k}. \]

**Proof** Since $H_0(te + d; t) = 1$ and $H_1(te + d; t) = te^0 + e^0d$, we only need to verify that the right hand side of the formula satisfies recursion (3.4). That is, we have to show that
\[
(te + d) \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q t^k e^k d^{n-k} - t[n]_q \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_q t^k e^k d^{n-k} = \sum_{k=0}^{n+1} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q t^k e^k d^{n+1-k}.
\]

Using (2.3), the left hand side is
\[
\sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q t^{k+1} e^{k+1} d^{n-k} + \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q t^k e^k d^{n-k} - t[n]_q \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_q t^k e^k d^{n-1-k}
\]
\[
= t^{n+1} e^{n+1} + \sum_{k=0}^{n-1} \left[ \begin{array}{c} n \\ k \end{array} \right]_q t^{k+1} e^{k+1} d^{n-k} + \sum_{k=1}^{n} q^k \left[ \begin{array}{c} n \\ k \end{array} \right]_q t^k e^k d^{n+1-k} + d^{N+1}
\]
\[
+ [n]_q \sum_{k=1}^{n} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] q \frac{n}{k} t^k e^{k-1} d^{n-k} - [n]_q \sum_{k=1}^{n} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] q t^k e^{k-1} d^{n-k}
\]
\[
= t^{n+1} e^{n+1} + \sum_{k=1}^{n} \left( \left[ \begin{array}{c} n \\ k-1 \end{array} \right] + q^k \left[ \begin{array}{c} n \\ k \end{array} \right] \right) t^k e^k d^{n+1-k} + d^{N+1} + 0
\]
\[
= \sum_{k=0}^{n+1} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q t^k e^k d^{n+1-k},
\]
as
\[
\left[ \begin{array}{c} n \\ k-1 \end{array} \right] + q^k \left[ \begin{array}{c} n \\ k \end{array} \right] = \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q.
\]
\[\square\]
We now introduce two sequences of functions:

\[ G_n(t) := \varphi_0 \left[ H_n(te + d; t) \right], \]

where \( 0 \leq n < N + 1 \) (we include here non-singular case by allowing \( N = \infty \)), and

\[ F_n(t) := \varphi_1 \left[ H_{n+N}(te + d; t) \right], \quad n \geq 1. \]

It turns out that these sequences satisfy similar recursions.

**Theorem 4** For \( 0 \leq n < N \) we have

\[
G_{n+1}(t) = \frac{\theta}{1 - abcdq^n} \left( (a + b)(1 - tcd)G_n(qt) + (c + d)(t - q^nab)G_n(t) \right)
- \theta^2 \frac{1 - q^n}{1 - abcdq^n} \left( ab(1 - tcd)G_{n-1}(qt) + tcd(t - abq^n)G_{n-1}(t) \right)
\]

(3.6)

with \( G_0(t) = 1 \) and \( G_{-1}(t) = 0 \).

For \( n \geq 1 \) we have

\[
F_{n+1}(t) = \frac{\theta}{1 - q^n} \left( (a + b)(1 - tcd)F_n(qt) + (c + d)(t - q^{n+N}ab)F_n(t) \right)
- \theta^2 \frac{1 - q^{n+N}}{1 - q^n} \left( ab(1 - tcd)F_{n-1}(qt) + tcd(t - abq^{n+N})F_{n-1}(t) \right)
\]

(3.7)

with

\[
F_1(t) = \frac{1}{\theta^{N+1}} \prod_{j=0}^{N} \frac{1}{1 - cdq^j} = \frac{(tcq; q)_{N+1}}{\theta^{N+1}(cdq; q)_{N+1}}
\]

and \( F_0(t) = 0 \).

**Proof** Using the identity \( \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q = q^k \left[ \begin{array}{c} n \\ k \end{array} \right]_q + \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_q \) we write

\[
G_{n+1}(t) = \sum_{k=0}^{n+1} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q \varphi_{k,n+1-k}t^k
= \varphi_{0,n+1} + \sum_{k=1}^{n} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q \varphi_{k,n+1-k}t^k + \varphi_{n+1,0}t^{n+1}
\]

\[
= \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^k \varphi_{k,n+1-k} + t \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \varphi_{k+1,n-k}t^k = A + B \text{ (say)}.
\]

Applying (3.2) to expression \( A \) we get
We have coefficients, relating Wilson polynomials. We will start with the following two explicit formulas for the connection

\[ \text{Proposition 4} \]

\[ \text{For } a, b \in \mathbb{C}, \text{ the connection coefficients in the expansion} \]

\[ p_n(x; 0, 0, 0|q) = \sum_{k=0}^{n} c_{n,k} p_k(x; a, b, 0|q) \quad (3.8) \]

\[ \text{are} \]

\[ c_{n,k} = \sum_{\ell=k}^{n} \binom{n}{k} \binom{\ell}{k} q^{n-\ell} b^{\ell-k}. \quad (3.9) \]
If \( a \neq 0 \), the connection coefficients in the expansion
\[
p_n(x; 0, 0, 0, 0|q) = \sum_{\ell=0}^{n} e_{n,\ell}(a, b, c, d) p_{\ell}(x; a, b, c, d|q)
\]
are
\[
e_{n,\ell}(a, b, c, d) = \sum_{k=\ell}^{n} \binom{n}{k} k^{\ell-\ell} q_{(\ell-k)}(\ell \mid a b q; q)_{k-\ell} 2\phi_2\left( \begin{array}{c}
a^{\ell-k}, \ acq^\ell, \ adq^\ell \\
0, \ abc dq^2\ell
\end{array} \middle| q; q \right).
\]

**Proof** Since (3.8) holds trivially when \( a = b = 0 \), by symmetry of \( p_k(x; a, b, 0, 0|q) \) in parameters \( a, b \), we can assume \( a \neq 0 \). From (A.3) we see that
\[
p_n(x; a, 0, 0, 0|q) = \sum_{k=0}^{n} C_{n,k} p_k(x; a, b, 0, 0|q),
\]
where
\[
C_{n,k} = \frac{q^{k(n-k)}}{a^{n-k}} \binom{n}{k} 2\phi_1\left( \begin{array}{c}
q^{k-n}, \ abq^k \\
0
\end{array} \middle| q; q \right) = \frac{q^{k(n-k)}}{a^{n-k}} \binom{n}{k} (abq^k)^{n-k}
\]
(we used formula (A.2).) In particular (3.10) is valid also for \( a = 0 \). Setting \( a = 0 \) in (3.10), using symmetry again, and renaming \( b \) as \( a \) we get
\[
p_n(x; 0, 0, 0, 0|q) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} p_k(x; a, 0, 0, 0|q).
\]
Combining (3.11) with (3.10) proves that
\[
p_n(x; 0, 0, 0, 0|q) = \sum_{k=0}^{n} c_{n,k} p_k(x; a, b, 0, 0|q),
\]
where \( c_{n,k} \) is given by (3.9). This formula holds for all \( a, b \).

Next we prove the second connection formula for \( a \neq 0 \). From (A.3) follows that the coefficient \( C'_{n,\ell} \) in the expansion
\[
p_k(x; a, b, 0, 0|q) = \sum_{\ell=0}^{n} C'_{n,\ell} p_{\ell}(x; a, b, c, d|q)
\]
is equal to
\[
\binom{k}{\ell} q^{\ell(\ell-k)}(abq^\ell; q)_{k-\ell} 2\phi_2\left( \begin{array}{c}
q^{\ell-k}, \ acq^\ell, \ adq^\ell \\
0, \ abc dq^2\ell
\end{array} \middle| q; q \right).
\]
This ends the proof, since \( e_{n,\ell}(a, b, c, d) = \sum_{k=\ell}^{n} c_{n,k} C'_{n,\ell} \). \( \square \)

Suppose that the degrees of polynomials \( p_k \) are \( k = 0, 1, \ldots, n \). (Recall that this fails for large \( n \) if \( q^nabcd = 1 \) for some \( N = 0, 1, \ldots \) Denote by \( \{a_{n,k}(a, b, c, d)\} \) the coefficients in the expansion
\[
H_n(x) = \sum_{k=0}^{n} a_{n,k}(a, b, c, d) p_k\left( \frac{x}{2\theta} ; a, b, c, d \middle| q \right),
\]
(3.12)
where \( H_n(x) = H_n(x; 1) \) is given by (3.3).

We will need explicit formula for the coefficient \( A_n(a, b, c, d) := a_{n, 0}(a, b, c, d) \). Since \( a_{n, k}(a, b, c, d) \) are invariant under permutations of \( a, b, c, d \), without loss of generality we assume \( a \neq 0 \). This is enough for our purposes, as we have \( a, c > 0 \) for the parameters arising from ASEP.

**Proposition 5**

\[
A_n(a, b, c, d) = \theta^n \sum_{k=0}^{n} c_{n, k} \frac{(ab; q)_k}{a_k} \phi_2 \left( q^{-k}, ac, ad \mid q; q \right),
\]

with \( c_{n, k} \) given by (3.9).

**Proof** By comparing the three step recursions, it is clear that \( H_n(x) = \theta^n p_n \left( \frac{x}{2\theta}; 0, 0, 0, 0 \right) \).

Hence, by Proposition 4, \( A_n(a, b, c, d) = \theta^n \epsilon_n(0)(a, b, c, d) \).

It turns out that \( A_n(a, b, c, d) \) is related to the moment of the \( n \)th \( q \)-Hermite polynomial introduced in (3.5).

**Proposition 6** For \( 0 \leq n < N, a, c > 0 \) and \( t \neq 0 \) we have

\[
t^n G_n(1/t^2) = A_n(at, bt, c/t, d/t).
\]

For the proof, we need to rewrite both sides of this equation.

For the next lemma, we write \( G_n(z) \) as \( G_n(z; a, b, c, d) \) with explicitly written Askey–Wilson parameters. In this notation, Proposition 6 says

\[
t^n G_n(1/t^2; a, b, c, d) = A_n(at, bt, c/t, d/t),
\]

which is the same as \( t^n G_n(1/t^2; a/t, b/t, ct, dt) = A_n(a, b, c, d) \).

**Lemma 6** Expression

\[
B_n(a, b, c, d) := (abcd; q)_n G_n(t^2; at, bt, c/t, d/t) / \theta^n n!
\]

(3.13)

does not depend on \( t \) and satisfies the following recursion for \( 0 \leq n < N \):

\[
B_{n+1}(a, b, c, d) = (a + b)(1 - cd)q^{n/2} B_n(a/\sqrt{q}, b/\sqrt{q}, c/\sqrt{q}, d/\sqrt{q}) + (c + d)(1 - q^n ab) B_n(a, b, c, d)
- (1 - q^n)(1 - abcd q^{n-1}) \left( ab(1 - cd)q^{(n-1)/2} B_{n-1}(a/\sqrt{q}, b/\sqrt{q}, c/\sqrt{q}, d/\sqrt{q})
+ cd(1 - abq^n) B_{n-1}(a, b, c, d) \right)
\]

(3.14)

with the initial value \( B_0(a, b, c, d) = 1 \), and \( B_{-1}(a, b, c, d) = 0 \).

**Proof** Denote by \( \tilde{G}_n(t^2; a, b, c, d) \) the right hand side of (3.13). Inserting this expression into (3.6) we get recursion

\[
\tilde{G}_{n+1}(t^2; a, b, c, d) = (a + b)(1 - cd)\tilde{G}_n(q t^2; a, b, c, d)
+ (c + d)(1 - q^n ab)\tilde{G}_n(t^2; a, b, c, d)
- (1 - q^n)(1 - abcd q^{n-1}) \left( ab(1 - cd)\tilde{G}_{n-1}(q t^2; a, b, c, d)
+ cd(1 - abq^n)\tilde{G}_{n-1}(t^2; a, b, c, d) \right)
\]

(3.15)
with the coefficients that do not depend on $t$. Since the initial condition $\widetilde{G}_{-1} = 0$ and $\widetilde{G}_0 = 1$ does not depend on $t$, therefore the solution of the recursion does not depend on $t$. We check this by induction, assuming that this assertion holds for $\widetilde{G}_0, \ldots, \widetilde{G}_n$. Denoting $i = t\sqrt{q}$ we have

$$\widetilde{G}_n(q i^2; a, b, c, d) = (abcd; q)_n \frac{G_n(q i^2; at, bt, c/t, d/t)}{\theta^n}$$

$$= q^{n/2} (abcd; q)_n \frac{G_n(i^2; a/\sqrt{q}, b/\sqrt{q}, \sqrt{q}c/i, \sqrt{q}d/i)}{\theta^{n/2}}$$

$$= q^{n/2} B_n(a/\sqrt{q}, b/\sqrt{q}, c/\sqrt{q}, d/\sqrt{q}).$$

Thus (3.15) shows that $\widetilde{G}_{n+1}(t^2; a, b, c, d)$ does not depend on $t$, and recursion (3.14) follows.

Next we rewrite the right hand side of the equation in Proposition 6. Denote

$$\widetilde{A}_n(a, b, c, d) = (abcd; q)_n A_n(a, b, c, d)/\theta^n$$

$$= (abcd; q)_n \sum_{k=0}^n c_{n,k} (ab; q)_k \frac{(a/b)^{3\phi_2} (q^{-k}, ac, ad; q)}{0, abcd}.$$ 

We rewrite this as

$$\widetilde{A}_n(a, b, c, d) = (abcd; q)_n \sum_{k=0}^n (ab; q)_k c_{n,k} \beta_k$$

with

$$\beta_k(a, b, c, d) = \frac{1}{a^k} 3\phi_2 \left( q^{-k}, ad, ac; q \right) = \frac{1}{a^k} \sum_{j=0}^k \frac{(q^{-k}, ad, ac; q)_j}{(q, abcd; q)_j} q^j.$$

In order to prove Proposition 6 it is enough to show that $\widetilde{A}_n(a, b, c, d) = B_n(a, b, c, d)$. Since both expressions are $1$ when $n = 0$, we only need to verify that $\widetilde{A}_n(a, b, c, d)$ satisfies recursion (3.14). To accomplish this goal, we need auxiliary recursions for the coefficients $c_{n,k}$ and $\beta_k$.

**Lemma 7** With the usual convention that $c_{n,k} = 0$ if $k > n$ or $k < 0$, for all $n \geq 0$ and all $k$, we have

$$c_{n+1,k} = c_{n,k-1} + q^k (a + b) c_{n,k} - q^k (1 - q^n) ab \cdot c_{n-1,k}. \quad (3.16)$$

Furthermore, for $n \geq 1$ and $0 \leq k \leq n$ we have

$$(1 - q^{k+1}) c_{n,k+1} = (1 - q^n) c_{n-1,k}. \quad (3.17)$$

**Proof** Let $h_n(x) = p_n(x; 0, 0, 0, 0|q)$ and $Q_n(x) = p_n(x; a, b, 0, 0|q)$. Then (3.8) is

$$h_n(x) = \sum_{k=0}^n c_{n,k} Q_k(x), \quad n \geq 0.$$ 

Comparing the three step recursions

$$2 x h_n(x) = h_{n+1}(x) + (1 - q^n) h_{n-1}(x)$$

$$\Box$$ Springer
and
\[
2x Q_n(x) = Q_{n+1}(x) + q^n(a + b) Q_n(x) + (1 - q^n)(1 - q^{n-1}ab) Q_{n-1}(x), \quad (3.18)
\]
see, e.g., [22, (3.8.4)], we get
\[
c_{n+1,k} = c_{n,k-1} + q^k(a + b)c_{n,k} \\
+ (1 - q^{k+1})(1 - q^k ab)c_{n,k+1} - (1 - q^n)c_{n-1,k}. \quad (3.19)
\]
Indeed, expanding both sides of \(2x h_n(x) = h_{n+1}(x) + (1 - q^n) h_{n-1}(x)\) and applying (3.18) to
the expansion on left hand side we get
\[
\sum_{k=0}^{n} c_{n,k} \left( Q_{k+1}(x) + q^k(a + b) Q_k(x) + (1 - q^k)(1 - q^{k-1}ab) Q_{k-1}(x) \right)
= \sum_{k=0}^{n+1} c_{n+1,k} Q_k(x) + (1 - q^n) \sum_{k=0}^{n-1} c_{n-1,k} Q_k(x).
\]
The formula follows by comparing the coefficients at \(Q_k(x)\).

Since \(c_{n,k} = c_{n,k}(a, b)\) is a homogeneous polynomial of degree \(n - k\) in variables \(a\) and \(b\), we can separate the components of recursion (3.19) into the pair of recursions. The
terms of degree \(n - k - 1\) give (3.17). The terms of degree \(n + 1 - k\) give \(c_{n+1,k} = c_{n,k-1} + q^k(a + b)c_{n,k} - (1 - q^{k+1})q^k ab \cdot c_{n,k+1}\), which gives (3.16) after using (3.17).

**Corollary 1**

\[
(ab; q)_k c_{n+1,k} - (1 - q^n ab)(ab; q)_{k-1} c_{n,k-1} = (a + b) \left( \frac{ab}{q}; q \right)_k q^k c_{n,k} \\
- (1 - q^n)ab \left( \frac{ab}{q}; q \right)_k q^k c_{n-1,k}.
\]

**Proof** It is enough to prove that
\[
(1 - abq^{k-1})c_{n+1,k} = (a + b) \left( 1 - \frac{ab}{q} \right) q^k c_{n,k} + (1 - q^n ab)c_{n,k-1} \\
- (1 - q^n)ab \left( 1 - \frac{ab}{q} \right) q^k c_{n-1,k}.
\]
Since \(c_{n,k}\) is a homogeneous polynomial of degree \(n - k\) in variables \(a\) and \(b\) this is equivalent to a pair of identities
\[
c_{n+1,k} = q^k(a + b)c_{n,k} + c_{n,k-1} - q^k(1 - q^n)ab \cdot c_{n-1,k}, \quad (3.20)
\]
which is (3.16), and
\[
- abq^{k-1}c_{n+1,k} = - ab(a + b)q^{k-1}c_{n,k} - q^n ab \cdot c_{n,k-1} \\
+ (1 - q^n)a^2 b^2 q^{k-1} c_{n-1,k}.
\]
To prove (3.21) it is enough to verify that
\[
q^k c_{n+1,k} = q^k(a + b)c_{n,k} + q^{n+1} c_{n,k-1} - q^k(1 - q^n)ab \cdot c_{n-1,k}.
\]
To do this, we subtract this expression from (3.20) and use (3.17).
We also need the following recursion which was discovered by Mathematica package qZeil [28], but for which we have a standard proof.

**Lemma 8** For $0 \leq n < N$, $a \neq 0$ and $b, c, d \in \mathbb{C}$ we have

\[
(1 - abcdq^n)\beta_{n+1}(a, b, c, d) = (c + d - cd(a + b)q^n)\beta_n(a, b, c, d) - cd(1 - q^n)\beta_{n-1}(a, b, c, d). \tag{3.22}
\]

The initial condition for this recursion is $\beta_0 = 1, \beta_{-1} = 0$.

**Proof** For $a \neq 0$, consider the Al-Salam–Chihara polynomials

\[
\tilde{Q}_n(x; a, b) = \frac{a^n}{(ab; q)_n}p_n(x; a, b, 0, 0|q) = 3\phi_2\left(q^{-n}, ae^{i\psi}, ae^{-i\psi} \mid q; q\right), \tag{3.23}
\]

where $x = \cos \psi$. The three step recursion for polynomials $\tilde{Q}_n(x)$ is

\[
2x\tilde{Q}_n(x; a, b) = a^{-1}(1 - abq^n)\tilde{Q}_{n+1}(x; a, b) + (a + b)q^n\tilde{Q}_n(x; a, b)
+ a(1 - q^n)\tilde{Q}_{n-1}(x; a, b) \tag{3.24}
\]

with $\tilde{Q}_0(x; a, b) = 1$ and $\tilde{Q}_{-1}(x; a, b) = 0$. (This is a version of (3.18) under different normalization.) For $c, d > 0$ let $x_s = \frac{1}{2}\left(\sqrt{\frac{c}{a}} + \sqrt{\frac{d}{c}}\right)$. It is easy to see that

\[
\tilde{Q}_n\left(x_s; a\sqrt{cd}, b\sqrt{cd}\right) = 3\phi_2\left(q^{-n}, ac, ad \mid 0, abcd; q\right) = a^n\beta_n(a, b, c, d).
\]

Indeed, to extend polynomial $\tilde{Q}_n(x)$ from $x = \cos \psi \in [-1, 1]$ to $x > 1$ we replace $e^{\pm i\psi}$ in (3.23) by $\sqrt{c/d}$ and $\sqrt{d/c}$ at $x = x_s$.

Recursion (3.24) implies that

\[
\left(\sqrt{\frac{c}{a}} + \sqrt{\frac{d}{c}}\right)a^n\beta_n = \frac{1}{a\sqrt{cd}}(1 - abcdq^n)a^{n+1}\beta_{n+1} + (a\sqrt{cd} + b\sqrt{cd})q^n a^n\beta_n
+ a\sqrt{cd}(1 - q^n)a^{n-1}\beta_{n-1}.
\]

This implies (3.22) for $a \neq 0$ and $c, d > 0$. We now use the fact that $\beta_n(a, b, c, d)$ is a rational function of $a, b, c, d$, with the denominator that has factors $a^k$ and $1 - abcdq^k$, $0 \leq k \leq n < N$. Thus recursion (3.22) extends to all $a, b, c, d$ within the domain of $\beta_n(a, b, c, d)$.

**Proof** (Proof of Proposition 6) We will show that

\[
\tilde{A}_n(a, b, c, d) := (abcd; q)_n\sum_{k=0}^{n}(ab; q)_k c_{n,k}\beta_k
\]

satisfies recursion (3.14).

We first note that

\[
c_{n,k}(a/\sqrt{q}, b/\sqrt{q}) = q^{(k-n)/2}c_{n,k}(a, b)
\]

and

\[
\beta_k(a/\sqrt{q}, b/\sqrt{q}, c/\sqrt{q}, d/\sqrt{q}) = q^{k/2}\beta_k(a, b, c, d).
\]

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We therefore want to show that
\[
\frac{\tilde{A}_{n+1}(a, b, c, d)}{(abcd; q)_n} = (a + b)(1 - cd) \sum_{k=0}^{n} \left( \frac{ab}{q} \right)_k q^k c_{n,k} \beta_k
\]
\[+ (c + d)(1 - q^n ab) \sum_{k=0}^{n} (ab; q)_k c_{n,k} \beta_k
\]
\[- (1 - q^n)ab(1 - cd) \sum_{k=0}^{n-1} \left( \frac{ab}{q} \right)_k q^k c_{n-1,k} \beta_k
\]
\[- (1 - q^n)cd(1 - abq^n) \sum_{k=0}^{n} (ab; q)_k c_{n-1,k} \beta_k.
\]

We will be working with the right hand side of this equation. The sum of the first and the third term is equal to
\[
(1 - cd) \sum_{k=0}^{n} \left[ (a + b) \left( \frac{ab}{q} \right)_k q^k c_{n,k} - (1 - q^n)ab \left( \frac{ab}{q} \right)_k q^k c_{n-1,k} \right] \beta_k.
\]
By Corollary 1 this is equal
\[
(1 - cd) \sum_{k=0}^{n} (ab; q)_k c_{n+1,k} \beta_k - (1 - cd)(1 - abq^n) \sum_{k=0}^{n} (ab; q)_{k-1} c_{n,k-1} \beta_k
\]
\[= (1 - abcdq^n) \sum_{k=0}^{n} (ab; q)_k c_{n+1,k} \beta_k - cd(1 - abq^n) \sum_{k=0}^{n} (ab; q)_k c_{n+1,k} \beta_k
\]
\[- (1 - cd)(1 - abq^n) \sum_{k=0}^{n} (ab; q)_{k-1} c_{n,k-1} \beta_k,
\]
since \((1 - cd) = (1 - abcdq^n) - cd(1 - abq^n)\).

It follows that what we want to show is
\[
\frac{\tilde{A}_{n+1}(a, b, c, d)}{(abcd; q)_n} = (1 - abcdq^n) \sum_{k=0}^{n} (ab; q)_k c_{n+1,k} \beta_k + (1 - abq^n)S,
\]
where
\[
S = S_1 - S_2 - S_3 - S_4
\]
\[= (c + d) \sum_{k=0}^{n} (ab; q)_k c_{n,k} \beta_k - (1 - q^n)cd \sum_{k=0}^{n} (ab; q)_k c_{n-1,k} \beta_k
\]
\[- cd \sum_{k=0}^{n} (ab; q)_k c_{n+1,k} \beta_k - (1 - cd) \sum_{k=0}^{n} (ab; q)_{k-1} c_{n,k-1} \beta_k.
\]

We will finish the proof by showing that \(S\) is equal to \((1 - abcdq^n) (ab; q)_n c_{n+1,n+1} \beta_{n+1} \cdot \)
By Lemma 7

\[ S_3 = cd \sum_{k=0}^{n} (ab; q)_k c_{n+1,k} \beta_k = S'_3 + S''_3 - S''''_3 \]

\[ S'_3 = cd(a + b) \sum_{k=0}^{n} (ab; q)_k q^k c_{n,k} \beta_k + cd \sum_{k=0}^{n} (ab; q)_k c_{n,k-1} \beta_k \]

\[ S''_3 = - cd \sum_{k=0}^{n} (ab; q)_k q^k (1 - q^n) ab \cdot c_{n-1,k} \beta_k . \]

Since \( cd (ab; q)_k = cd (ab; q)_{k-1} - abcdq^{k-1} (ab; q)_{k-1} = -(1 - cd) (ab; q)_{k-1} + (1 - abcdq^{k-1}) (ab; q)_{k-1} \) we see that

\[ S''_3 = -(1 - cd) \sum_{k=0}^{n} (ab; q)_{k-1} c_{n,k-1} \beta_k \]

\[ I + \sum_{k=0}^{n} (1 - abcdq^{k-1}) (ab; q)_{k-1} c_{n,k-1} \beta_k = -S_4 + I. \]

Writing \( abq^k = -(1 - abq^k) + 1 \) we can rewrite \( S''_3 \) as

\[ S''''_3 = -cd \sum_{k=0}^{n} (ab; q)_{k+1} (1 - q^n) c_{n-1,k} \beta_k + (1 - q^n) cd \sum_{k=0}^{n} (ab; q)_k \beta_k \]

\[ J \]

\[ = -cd \sum_{k=0}^{n} (ab; q)_{k+1} (1 - q^{k+1}) c_{n,k+1} \beta_k \]

\[ S_2 \]

\[ + (1 - q^n) cd \sum_{k=0}^{n} (ab; q)_k c_{n-1,k} \beta_k = -J + S_2. \]

Combining all the expressions together we obtain

\[ S = (S_1 - S'_3 - J) - I. \]

The first expression is equal

\[ S_1 - S'_3 - J \]

\[ = \sum_{k=0}^{n} (ab; q)_k c_{n,k} \left[ c + d - cd(a + b)q^k \right] \beta_k - cd \sum_{k=0}^{n} (ab; q)_k (1 - q^k) c_{n,k} \beta_{k-1} \]

\[ \Box \]
This shows that \( \psi \). Suppose that polynomial \( a \) holds true, due to the assumption on the degree of \( \psi \).

This ends the proof, as \( c_{n+1,n+1} = c_{n,n} = 1 \).

**Proof** (Proof of Theorem 2) The proof does not use explicitly singularity condition \( q^N abcd = 1 \), except for the constraints that it implies on the domain of \( \varphi_0 \) and on the degrees of the polynomials \( \{ p_k : k = 1, \ldots, N \} \).

For \( n = 1 \) this is a calculation, which is also covered by the induction step. Suppose that \( p_k \) is of degree \( k \) and

\[
\varphi_0 [p_k(x_t; at, bt, c/t, d/t|q)] = 0 \quad \text{for } k = 1, \ldots, n.
\]

Suppose that polynomial \( p_{n+1} \) is of degree \( n + 1 \). Then, recalling (1.13), we have

\[
H_{n+1}(e/t^2 + d; 1/t^2) = H_{n+1}(2\theta x_t/t; 1/t^2) = \frac{1}{t_{n+1}} H_{n+1}(2\theta x_t)
\]

\[
= \frac{1}{t_{n+1}} \sum_{k=0}^{n+1} a_{n+1,k}(at, bt, c/t, d/t) p_k(x_t; at, bt, c/t, d/t|q)
\]

by (3.12). Since \( p_0 = 1 \), by inductive assumption we have

\[
\varphi_0 [H_{n+1}(e/t^2 + d; 1/t^2)] = \frac{1}{t_{n+1}} a_{n+1,0}(at, bt, c/t, d/t)
\]

\[
+ \frac{1}{t_{n+1}} a_{n+1,0} \varphi_0 [p_{n+1} (x_t; at, bt, c/t, d/t|q)].
\]

This shows that \( \varphi_0[p_{n+1}(x_t; at, bt, c/t, d/t|q)] = 0 \), provided that \( a_{n+1,n+1} \neq 0 \), which holds true due to the assumption on the degree of \( p_{n+1} \), and provided that

\[
a_{n+1,0}(at, bt, c/t, d/t) = t^{n+1} G_{n+1}(1/t^2),
\]

which holds true by Proposition 6.

Since the degree of polynomial \( p_n \) is \( n \) for \( n \leq \lfloor (N+1)/2 \rfloor \), this establishes the conclusion such \( n \). For \( n > \lfloor (N+1)/2 \rfloor \), polynomial \( p_n \) is a constant multiple of polynomial \( p_{N+1-n} \), so the conclusion also holds.

**4 Conclusions**

In this paper we construct a functional \( \varphi_0 \), or a pair of functionals \( \varphi_0, \varphi_1 \), on an abstract algebra that give stationary probabilities for an ASEP of length \( L \) with arbitrary parameters.
Formula (2.1) for the probabilities extends the celebrated matrix product ansatz [12] to the singular case with $\alpha \beta = q^N \gamma \delta$. Our approach avoids an associativity pitfall that may arise in matrix product models. In Appendix C we exhibit an example of such a matrix model that satisfies the usual conditions (1.2) (1.3) (1.4), yet it cannot be used to compute stationary probabilities.

While verifying that our functionals give non-zero answers for un-normalized probabilities, we noted an interesting phenomenon of current reversal as the system size $L$ increases when $\alpha \beta < \gamma \delta$ and $0 < q < 1$.

In the non-singular case, we prove that functional $\varphi_0$ may serve as an orthogonality functional for the Askey–Wilson polynomials with fairly general parameters. Part of this connection persists in the singular case $\alpha \beta = q^N \gamma \delta$ when the degrees of the first $N$ Askey–Wilson polynomials do not exceed $(N + 1)/2$. In Appendix B we give explicit formulas for the (formal) Cauchy–Stieltjes transforms of both functionals when $q = 0$.

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A Auxiliary Identities

Here we collect $q$-hypergeometric formulas used in this paper. Cauchy’s $q$-binomial formula is

$$(x; q)_n = \sum_{k=0}^{n} \binom{n}{k} q^{\frac{k(k-1)}{2}} x^k.$$  \hspace{1cm} (A.1)

Heine’s summation formula [18, (1.5.3)] reads

$$2\phi_1 \left( q^{-n}, \frac{b}{c} \middle| \frac{q}{q} \right) = \frac{(c/b; q)_n b^n}{(c; q)_n}.$$  \hspace{1cm} (A.2)

We also need the connection coefficients of the Askey–Wilson polynomials.

**Theorem 5** ([3]) If $a_4 \neq 0$ then

$$p_n(x; b_1, b_2, b_3, a_4|q) = \sum_{k=0}^{n} c_{n,k} p_k(x; a_1, a_2, a_3, a_4|q),$$

where

$$c_{n,k} = (b_1 b_2 b_3 a_4; q)_k \frac{q^{k(n-k)}(q; q)_n (b_1 a_4 q^k, b_2 a_4 q^k, b_3 a_4 q^k; q)_{n-k}}{a_4^{n-k}(q; q)_{n-k} (q, a_1 a_2 a_3 a_4 q^{k-1}; q)_k} \times 5\phi_4 \left( q^{k-n}, b_1 b_2 b_3 a_4 q^{n-k}, a_1 a_4 q^k, a_2 a_4 q^k, a_3 a_4 q^k, b_1 a_4 q^k, b_2 a_4 q^k, b_3 a_4 q^k, a_1 a_2 a_3 a_4 q_2^k \middle| q; q \right).$$  \hspace{1cm} (A.3)
B Totally Asymmetric Case

Our recursions simplify when \( q = 0 \), i.e., the case of Totally Asymmetric Exclusion Process. Then the conclusion of Theorem 2 can be derived more directly, and there is also additional information about \( \varphi_1 \) in the singular case \( abcd = 1 \).

For \( q = 0 \), Reference [3] relates Askey–Wilson polynomials \( p_n \) to the Chebyshev polynomials \( U_n \) of second kind. Denote by \( s_j(a, b, c, d) \) the \( j \)th symmetric function, i.e. \( s_1 = a + b + c + d, \ s_2 = ab + ac + ad + bc + bd + cd, \ s_3 = abc + abd + acd + bcd, \ s_4 = abcd. \) Then with \( U \) we have

\[
p_0 = U_0 \\
p_1 = (1 - s_4)U_1 + (s_3 - s_1)U_0 \\
p_2 = U_2 - s_1U_1 + (s_2 - s_4)U_0 \\
p_n = U_n - s_1U_{n-1} + s_2U_{n-2} - s_3U_{n-3} + s_4U_{n-4} \quad \text{for } n \geq 3.
\]

Recall that \( G_n(1) = \varphi_0[H_n(e + d)] = \varphi_0[U_n(x)]. \) So in the non-singular case the conclusion of Theorem 2 follows from the following relations between \( G_n(1) \).

\[
(1 - s_4)G_1(1) + (s_3 - s_1)G_0(1) = 0, \tag{B.1}
\]

\[
G_2(1) - s_1G_1(1) + (s_2 - s_4)G_0(1) = 0, \tag{B.2}
\]

\[
G_n(1) - s_1G_{n-1}(1) + s_2G_{n-2}(1) - s_3G_{n-3}(1) + s_4G_{n-4}(1) = 0, \quad n \geq 3. \tag{B.3}
\]

These relations can be established by analyzing explicit solutions of recursion (3.6). We first determine the initial (irregular) solutions

\[
G_1(t) = \frac{(c + d)(t - ab) + (a + b)(1 - cdt)}{1 - abcd}
\]

and

\[
G_2(t) = t(c + d)G_1(t) + \frac{(a + b)(1 - cdt)(a + b - ab(c + d)) - ab(1 - cdt) - cdt^2}{1 - abcd}
\]

which we use with \( t = 1 \) to verify (B.1) and (B.2). Next, we use (3.6) with \( t = 0 \) and \( n \geq 1 \) to determine \( \alpha_n = G_n(0) \) from the recursion of order 2,

\[
\alpha_{n+1}(0) = (a + b)\alpha_n - ab\alpha_{n-1}. \tag{B.4}
\]

Since in our setting arising from ASEP parameters \( b \leq 0 < a \) are not equal, the general solution is

\[
\alpha_n = C_1a^n + C_2b^n.
\]

The constants \( C_1, C_2 \) are determined from the initial values of \( G_0(0) = 1 \) and \( G_1(0) = \frac{a + b - ab(c + d)}{1 - abcd} \). We get

\[
\alpha_n = \frac{(1 - bc)(1 - bd)}{(a - b)(1 - abcd)}a^{n+1} + \frac{(1 - ac)(1 - ad)}{(b - a)(1 - abcd)}b^{n+1}.
\]

Next we solve the recursion for \( \alpha_n = G_n(1) \). This is now a non-homogeneous recursion

\[
\alpha_{n+1} = (1 - cdt)((a + b)\alpha_n - ab\alpha_{n-1}) + (c + d)\alpha_n - cdt\alpha_{n-1},
\]

which we simplify using (B.4) into

\[
\alpha_{n+1} = (1 - cdt)\alpha_{n+1} + (c + d)\alpha_n - cdt\alpha_{n-1}.
\]
Since $d \leq 0 < c$, the general solution of this recursion is

$$G_n(1) = z_n = B_1 a^{n+3} + B_2 b^{n+3} + K_1 c^{n+3} + K_2 d^{n+3}, \quad n \geq 0$$

where

$$B_1 = \frac{(1 - bc)(1 - bd)(1 - cd)}{(a - b)(a - c)(a - d)(1 - abcd)}, \quad B_2 = \frac{(1 - ac)(1 - ad)(1 - cd)}{(b - a)(b - c)(b - d)(1 - abcd)}$$

come from the undetermined coefficients method and

$$K_1 = \frac{(1 - ab)(1 - ad)(1 - bd)}{(c - a)(c - b)(c - d)(1 - abcd)}, \quad K_2 = \frac{(1 - ab)(1 - ac)(1 - bc)}{(d - a)(d - b)(d - c)(1 - abcd)}$$

come from matching the initial values. It turns out that the explicit values of the constants are only needed for verification of the initial equations, as Eq. (B.3) holds for any linear combination of $a^n, b^n, c^n, d^n$.

Proceeding in similar way we can also derive a version of Theorem 2 that relates functional $\varphi_1$ to Askey–Wilson polynomials. We have

$$F_0(t) = 0, \quad F_1(t) = \frac{1 - cdt}{1 - cd}.$$ 

The recursion for $\alpha_n = F_n(0)$ is (B.4), so using the above initial values we get the solution

$$F_n(0) = \frac{a^n - b^n}{(a - b)(1 - cd)}, \quad n \geq 0.$$ 

The recursion for $F_n(1)$ is

$$F_{n+1}(1) = (c + d)F_n(1) - cd F_{n-1}(1) + \frac{a^n - b^n}{a - b}, \quad n \geq 1.$$ 

Here the constants are simpler and a calculation gives

$$F_n(1) = \frac{a^{n+2}}{(a - b)(a - c)(a - d)} + \frac{b^{n+2}}{(b - a)(b - c)(b - d)} + \frac{c^{n+2}}{(c - a)(c - b)(c - d)} + \frac{d^{n+2}}{(d - a)(d - b)(d - c)}, \quad n \geq 0.$$ 

Noting that in the singular case $p_1$ is a constant, we have $\varphi_1[p_n(x)] = 0$ for all $n = 0, 1, \ldots$

Motivated by the generating function $\sum_{n=0}^{\infty} H_n(x)z^n = 1/(1 + z^2 - xz)$, let denote by $\varphi[(1 + z^2 - (e + d)z)^{-1}]$ the power series $\sum_{n=0}^{\infty} \varphi[H_n(e + d)]z^n$. We can now summarize the above formulas more concisely.

**Proposition 7** If $abcd \neq 1$ then for $|z|$ small enough

$$\varphi_0[(1 + z^2 - (e + d)z)^{-1}] = 1 + z^2abcd + \frac{zabcd(a + b + c + d - (1/a + 1/b + 1/c + 1/d))}{(1 - az)(1 - bz)(1 - cz)(1 - dz)}.$$

If $abcd = 1$ then for $|z|$ small enough

$$\varphi_1[(1 + z^2 - (e + d)z)^{-1}] = \frac{z}{(1 - az)(1 - bz)(1 - cz)(1 - dz)}.$$

The first expression matches the formula from [30, Theorem 4.1] who computed the integral of $1/(1 + z^2 - xz)$ with respect to the Askey–Wilson measure with $q = 0$ under the assumptions which in our setting boil down to $ac \leq 1$ and $abcd < 1$. 

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C A Matrix Model

According to Mallick and Sandow [27] stationary probabilities for ASEP with large \( L \) can be computed from a finite matrix model when the parameters satisfy condition \( q^m ac = 1 \) for some \( m \geq 0 \). Here we present a version of this model, together with a caution about a subtle issue that may affect some infinite matrix models.

Recalling that in (1.9) we chose \( a > 0 \), for \( q > 0 \) we consider two infinite matrices

\[
E = \theta^2 \begin{bmatrix}
1 + \frac{1}{a} & 0 & 0 & \cdots & 0 & \cdots \\
1 & 1 + \frac{1}{aq} & 0 & \cdots \\
0 & 1 & \ddots & & \ddots & \ddots \\
& \vdots & \ddots & \ddots & \ddots & \\
0 & 0 & \cdots & 0 & 1 & 1 + \frac{1}{aq^{n-1}} \\
& \vdots & & \cdots & \ddots & \\
& & & & & & \ddots
\end{bmatrix}
\]

\[
D = \theta^2 \begin{bmatrix}
1 + a & 0 & 0 & \cdots \\
0 & 1 + aq & 0 & \cdots \\
& \vdots & \ddots & \ddots & \\
0 & 0 & \cdots & 1 + aq^{n-1} \\
& \vdots & & \cdots & \\
& & & & & \ddots
\end{bmatrix}
\]

It is straightforward to verify that identity (1.2) is satisfied. Conditions (1.3) and (1.4) become recursions for the components of the vectors \([W] = [w_1, w_2, \ldots] \) and \([V] = [v_1, v_2, \ldots]^T \).

In parametrization (1.9), conditions (1.3) and (1.4) become (C.5) and (C.6), and the resulting recursions are

\[
\frac{1}{aq^{k-1}} w_k + w_{k+1} = (c + d)w_k - acdq^{k-1}w_k,
\]

\[
a b \left( v_{k-1} + \frac{1}{aq^{k-1}} v_k \right) = (a + b)v_k - aq^{k-1}v_k.
\]

With \( w_1 = v_1 = 1 \), the solutions are explicit

\[
w_n = \prod_{k=1}^{n-1} \left( c + d - acdq^{k-1} - \frac{1}{aq^{k-1}} \right) = \frac{(ac, ad; q)_{n-1}}{(-a)^{n-1} q^{(n-1)(n-2)/2}},
\]

\[
v_n = \frac{a^{n-1} b^{n-1}}{\prod_{k=1}^{n-1} (a(1 - q^k) + b(1 - 1/q^k))} = \frac{(-a)^{n-1} q^{(n-1)/2}}{(q, qa/b; q)_{n-1}}.
\]

We therefore get explicit formula

\[
\langle W | I | V \rangle = \sum_{k=1}^{\infty} v_k w_k = \sum_{k=1}^{\infty} q^{k-1} \frac{(ac, ad; q)_{k-1}}{(q, qa/b; q)_{k-1}} = _2\phi_1 \left( \frac{ac, ad}{qa/b} \left| q \right. \right).
\]
valid for $0 < q < 1$. Somewhat more generally, since $d$ in (1.11) becomes a diagonal matrix with the sequence $\{\theta a q^{k-1}\}$ on the diagonal, we get

$$\langle W|d^L|V\rangle = a^L \theta^L 2\phi_1 \left( \begin{array}{c} ac, \ a d \\ qa/b \end{array} \right) \langle q; q^{L+1}\rangle.$$  \hfill (C.4)

(We will use this formula for $L = 0, 1$ in Sect. C.1).

We now consider the case when parameters $a, c$ are such that $acq^m = 1$ for some integer $m \geq 0$. In this case the infinite series terminate as Formula (C.2) gives $w_n = 0$ for all $n \geq m + 2$. Since each monomial $X$ is a lower-triangular matrix, in this case components $v_k$ with $k \geq m + 2$ do not enter the calculation of $\langle W|X|V\rangle$, so we can truncate $e, d, I$ to their $m+1$ by $m+1$ upper left corners, recovering a version of the finite matrix model from [27].

Using (A.2) one can show that

$$2\phi_1 \left( \begin{array}{c} q^{-m}, \ a d \\ qa/b \end{array} \right) \langle q; q\rangle = \frac{(bdq^{-m}; q)_m}{(bc; q)_m}.$$

Thus, in agreement with findings in Mallick and Sandow [27],

$$\langle W|I|V\rangle = \frac{(bdq^{-m}; q)_m}{(bc; q)_m} \varphi_0(X)$$

vanishes if and only if $bd \in \{q, q^2, \ldots, q^m\}$, i.e., in the singular case when $q^Nabcd = 1$ for some $N = 0, \ldots, m - 1$. One would expect that in this case the matrix model should be related to functional $\varphi_1$ by a simple renormalization but we have not verified the details.

**Remark 4** From the reviewer report we learned that [23] and [20] relate the finite-dimensional representations from Mallick and Sandow [27] to convex combinations of Bernoulli shock measures with $m$ shocks. It would be interesting to see how this is reflected in the structure of functional $\varphi$.

In the non-singular case (but still with $q^mac = 1$) the relation is straightforward. Due to shared recursion and initialization at $I$, it is clear that functional $\varphi_0$ is indeed related to the matrix model by

$$\langle W|X|V\rangle = \frac{(bdq^{-m}; q)_m}{(bc; q)_m} \varphi_0(X).$$

A natural question then arises how the functionals $\varphi_0$, or $\varphi_1$, are related to this matrix model for more general parameters $a, b, c, d$. The surprising answer is that there is no such relation, as we explain next.

**C. 1 A Caution About Matrix Models**

It is known, [5,21], but perhaps this is not appreciated enough, that multiplication of infinite matrices may fail to be associative for other reasons than divergence. And precisely this difficulty afflicts the above matrix model when $acq^n \neq 1$ for all $n$. To see the source of the difficulty, we rewrite (1.3) and (1.4) as

$$\langle W|e = \theta(c + d)\langle W| - cd\langle W|d, \hfill (C.5)$$

$$ab|e|V\rangle = \theta(a + b)|V\rangle - d|V\rangle. \hfill (C.6)$$
To indicate clearly the order of matrix multiplications, let’s denote vector $\langle W | e$ by $\langle \tilde{W} |$ and vector $e | V \rangle$ by $| \tilde{V} \rangle$. Using (C.5) and (C.6), we could compute the product $\langle W | e | V \rangle$ of three matrices either as $\langle \tilde{W} | V \rangle$, or as $\langle W | \tilde{V} \rangle$. From the first calculation we get

$$\langle \tilde{W} | V \rangle = \theta (c + d) 2 \phi_1 \left( \frac{q^{-m} \cdot ad}{qa/b} | q; q^2 \right) - acd \theta 2 \phi_1 \left( \frac{q^{-m} \cdot ad}{qa/b} | q; q^2 \right).$$

where we used (C.4) with $L = 0$ and 1 on the right hand side. The second calculation gives a different answer

$$ab \langle W | \tilde{V} \rangle = \theta (a + b) 2 \phi_1 \left( \frac{ac \cdot ad}{qa/b} | q; q^2 \right) - a \theta 2 \phi_1 \left( \frac{ac \cdot ad}{qa/b} | q; q^2 \right).$$

In fact, we have

$$\langle \tilde{W} | V \rangle = \theta \sum_{k=1}^{\infty} \left( \frac{1}{aq^{k-1}} w_k + w_{k+1} \right) v_k$$

$$\langle W | \tilde{V} \rangle = \theta \sum_{k=1}^{\infty} w_k \left( v_{k-1} + \frac{1}{aq^{k-1}} v_k \right) \text{ with } v_{-1} = 0.$$ 

So from (C.2) and (C.3) we get

$$\langle \tilde{W} | V \rangle - \langle W | \tilde{V} \rangle = \lim_{n \to \infty} \sum_{k=1}^{n} (w_{k+1} v_k - w_k v_{k-1}) = \lim_{n \to \infty} w_{n+1} v_n = - \frac{\theta (ac \cdot ad; q)_{\infty}}{a (q, qa/b; q)_{\infty}}.$$ 

This shows that in general multiplication of matrices $\langle W |, e$ and $| V \rangle$ is not associative. Since $d \leq 0$, the two answers match only when $q^m ac = 1$ for some $m$, i.e., in the terminating case. This is precisely the case considered by [27], and of course multiplication of finite dimensional matrices is associative.

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