An extension of Thomassen’s result on choosability

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Abstract

Thomassen proved that all planar graphs are 5-choosable. Škrekovski strengthened the result by showing that all $K_5$-minor-free graphs are 5-choosable. Dvořák and Postle pointed out that all planar graphs are DP-5-colorable. In this note, we first improve these results by showing that every $K_5$-minor-free or $K_{3,3}$-minor-free graph is DP-5-colorable. In the final section, we further improve these results under the term strictly $f$-degenerate transversal.

1 Introduction

Thomassen \cite{6} proved that all planar graphs are 5-choosable. Škrekovski \cite{9} (see also \cite{3, 11}) extended the result to the class of $K_5$-minor-free graphs. Dvořák and Postle \cite{2} gave a generalization of list coloring, under the name correspondence coloring, which was called DP-coloring by Bernshteyn, Kostochka, and Pron \cite{1}. Let $G$ be a graph and $L$ be a list assignment for $G$. For each vertex $v \in V(G)$, we associate it with a set $L_v = \{v\} \times L(v)$; for each edge $uv \in E(G)$, we associate it with a matching $\mathcal{M}_{uv}$ between $L_u$ and $L_v$. Let $\mathcal{M} = \bigcup_{uv \in E(G)} \mathcal{M}_{uv}$, and we call $\mathcal{M}$ the matching assignment over $L$. The matching assignment $\mathcal{M}$ is called a $k$-matching assignment if $L(v) = \{1, 2, \ldots, k\}$ for every $v \in V(G)$. A cover of $G$ is a graph $H_{L, \mathcal{M}}$ (simply write $H$) meeting two conditions:

- the vertex set of $H$ is the disjoint union of $L_v$ for all $v \in V(G)$; and
- the edge set of $H$ is the matching assignment $\mathcal{M}$.

Let $G$ be a graph and $H$ be a cover of $G$ over a list assignment $L$. An $(L, \mathcal{M})$-coloring of $G$ is an independent set $I$ of $H$ such that $|I \cap L_v| = 1$ for each $v \in V(G)$. A graph $G$ is DP-$k$-colorable if for any list assignment $L(v) \supseteq \{1, 2, \ldots, k\}$ and any matching assignment $\mathcal{M}$, it admits an $(L, \mathcal{M})$-coloring. Note that every DP-$k$-colorable graph is $k$-choosable.

Dvořák and Postle \cite{2} have pointed out that all planar graphs are DP-5-colorable. We improve the result to the following Theorem 1.1, and we also extend the result for planar graphs to the class of $K_{3,3}$-minor-free graphs.

**Theorem 1.1.** All $K_5$-minor-free graphs are DP-5-colorable.

**Theorem 1.2.** All $K_{3,3}$-minor-free graphs are DP-5-colorable.

Let $H$ be a cover of $G$, and let $f$ be a function from $V(H)$ to $\{0, 1, 2, \ldots\}$. A subset $T \subseteq V(H)$ is called a transversal if $|T \cap L_v| = 1$ for each $v \in V(G)$. A transversal $T$ of a cover $H$ is strictly $f$-degenerate if every nonempty subgraph $\Gamma$ in $H[T]$ contains a vertex $x$ with $\deg_{\Gamma}(x) < f(x)$. In other words, all the vertices of $H[T]$ can be ordered as $x_1, x_2, \ldots, x_n$ such that each vertex $x_i$ has less than $f(x_i)$ neighbors on the right.

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hand side. Such an order is an $f$-removing order, and the reverse order $x_n, x_{n-1}, \ldots, x_1$ is an $f$-coloring order.

By definition, a vertex $x$ can never be chosen in a strictly $f$-degenerate transversal if $f(x) = 0$. Hence, we can add some vertices into $L_v$ and define the value of $f$ to be zero on these new vertices, so that all the $L_v$ have the same cardinality. On the other hand, it doesn’t matter what the labels of the vertices are, so we may assume that $L_v = \{v\} \times [s]$, where $s$ is an integer. A cover $H$ together with a function $f$ is called a valued-cover.

In Section 3, we strengthen Theorems 1.1 and 1.2 to Theorem 1.3. In order to demonstrate how Thomassen’s technique in [6] is extended, we first give a proof for Theorem 1.1 in Section 2, and then give one for Theorem 1.3, even though Theorems 1.1 and 1.2 are special cases of Theorem 1.3. For a function $f$, we use $R_f$ to denote the range of $f$.

**Theorem 1.3.** Assume that $G$ is a $K_5$-minor-free or $K_{3,3}$-minor-free graph, and $(H, f)$ is a valued-cover with $R_f \subseteq \{0, 1, 2\}$. Then $H$ contains a strictly $f$-degenerate transversal.

Assume that $G$ is a plane graph and $C$ is a cycle in it. We will use Int($C$) (resp. Ext($C$)) to denote the subgraph induced by $V(C)$ and the vertices inside (resp. outside) of $C$. The cycle $C$ is a separating cycle of $G$ if both the interior and the exterior of $C$ have at least one vertex.

## 2 DP-5-coloring

A plane triangulation is an embedded plane graph such that each of its faces is bounded by a cycle of length three. A near-triangulation is an embedded plane graph such that each bounded face is bounded by a triangle and the unbounded face (outer face) is bounded by a cycle. An $\ell$-sum of two graphs $G'$ and $G''$ is the graph $G$ such that $G = G' \cup G''$ and $G' \cap G'' = K_\ell$.

The Wagner graph is a 3-regular graph with 8 vertices and 12 edges, see Fig. 1. Note that the Wagner graph is non-planar, thus the Wagner graph cannot be a subgraph of a planar graph.

![Wagner graph](image)

Fig. 1: Wagner graph.

Wagner [10] gave the following characterization of planar graphs in terms of graph minors.

**Theorem 2.1** (Wagner [10]). A graph is planar if and only if it does not contain $K_5$ or $K_{3,3}$ as a minor.

By Wagner’s Theorem, the class of $K_5$-minor-free graphs and the class of $K_{3,3}$-minor-free graphs are two superclasses of planar graphs.

A graph $G$ is maximal $K_5$-minor-free if it does not contain $K_5$ as a minor, but $G + xy$ contains a $K_5$-minor for every pair nonadjacent vertices $x$ and $y$ in $G$. Wagner [10] also gave the following characterization of maximal $K_5$-minor-free graphs.

**Theorem 2.2** (Wagner [10]). Every maximal $K_5$-minor-free graph can be obtained from the Wagner graph and plane triangulations by recursively 2-sums or 3-sums.

The following theorem and its proof are very similar to that in [6], but for completeness we give a complete proof here.

**Theorem 2.3.** Assume that $G$ is a near-triangulation such that the outer face is bounded by a cycle $O = v_1v_2 \ldots v_pv_1$. Let $L$ be a list assignment of $G$ such that $|L(v)| \geq 3$ for each $v \in V(O)$ and $|L(v)| \geq 5$ for each $v \notin V(O)$. If $M$ is a matching assignment for $G$ and $R_0$ is an ($L,M$)-coloring of $G'[\{v_1, v_2\}]$, then $G$ admits an ($L, M$)-coloring such that its restriction on $G'[\{v_1, v_2\}]$ is $R_0$.  

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Theorem 2.5

Proof. The assertion is proved by induction on $|V(G)|$. When $G$ has only three vertices, $G = \emptyset = K_3$ and the assertion is obvious. So we can assume that $|V(G)| \geq 4$ and the assertion is true for smaller graphs. Suppose that $\mathcal{O}$ has a chord $v_iv_j$. It follows that $v_iv_j$ lies in two cycles $C_1$ and $C_2$ of $O + v_iv_j$. Let $v_1v_2$ lie in $C_1$. Applying the induction hypothesis to $\text{Int}(C_1)$, $R_0$ can be extended to an $(L,\mathcal{M})$-coloring of $\text{Int}(C_1)$. After $v_1$ and $v_2$ are colored, it can be further extended to an $(L,\mathcal{M})$-coloring of $\text{Int}(C_2)$. This yields a desired $(L,\mathcal{M})$-coloring of $G$.

So we can assume that $\mathcal{O}$ has no chord. Let $v_1, u_1, u_2, \ldots, u_m, v_{p_1}$ be the neighbors of $v_p$ in a natural cyclic order around $v_p$. Since all the bounded faces of $G$ are bounded by triangles and $\mathcal{O}$ has no chord, $P = v_1u_1u_2 \ldots u_mv_{p_1}$ is a path and $\mathcal{O}' = P \cup (\mathcal{O} - v_p)$ is a cycle. Let $j$ and $\ell$ be two distinct elements in $(L(v_p))$ which do not conflict with the color of $v_1$ under the matching $\mathcal{M}_{v_1v_p}$. Now define $L'(v) = L(v)$ for every $v \notin \{u_1, u_2, \ldots, u_m, v_p\}$, for $1 \leq i \leq m$, define $L'(u_i)$ from $L(u_i)$ by deleting the neighbors of $j, \ell \in L(v_p)$ under the matching $\mathcal{M}_{v_1u_i}$. It is easy to check that $|L'(v)| \geq 3$ for all $v \in \mathcal{O}'$ and $|L'(v)| \geq 5$ for all $V(G) - \{v_p\} - V(\mathcal{O}')$. Applying the induction hypothesis to $\mathcal{O}'$ and its interior and the new list $L'$, we have an $(L',\mathcal{M})$-coloring for $G - v_p$. There is at least one color in $\{j, \ell\} \subset L(v_p)$ which do not conflict with the color of $v_{p_1}$ under $\mathcal{M}_{v_{p_1}v_p}$, so we can assign it to the vertex $v_p$. This completes the proof.

Theorem 2.4. Assume that $G$ is a maximal $K_5$-minor-free graph. If $K$ is a subgraph of $G$ isomorphic to $K_2$ or $K_3$, then every DP-5-coloring $\varphi$ of $K$ can be extended to a DP-5-coloring of $G$.

Proof. Suppose to the contrary that $G$ is a counterexample with $|V(G)|$ as small as possible.

Assume that $G$ is a plane triangulation and $K$ is a separating 3-cycle of $G$. Note that $\text{Int}(K)$ and $\text{Ext}(K)$ are both plane triangulations and maximal $K_5$-minor-free graphs. By minimality, every DP-5-coloring $\varphi$ of $K$ can be extended to a DP-5-coloring $\varphi_1$ of $\text{Int}(K)$ and a DP-5-coloring $\varphi_2$ of $\text{Ext}(K)$. Combining $\varphi_1$ and $\varphi_2$ yields a DP-5-coloring of $G$, a contradiction.

Assume that $G$ is a plane triangulation and $K = [x_1x_2x_3]$ bounds a 3-face. Note that $G$ has at least four vertices. We can redraw the plane triangulation such that $K$ is the boundary of the outer face. Note that $G - x_3$ is a near-triangulation. Since $x_3$ on $K$ is precolored, every uncolored vertex incident with the outer face of $G - x_3$ has at least four admissible colors other than $\varphi(x_3)$. Applying Theorem 2.3 to $G - x_3$, we obtain a DP-5-coloring of $G$ whose restriction on $K$ is the precoloring $\varphi$.

Assume that $G$ is a plane triangulation and $K = y_1y_2$. We can further assume that $y_1y_2$ is incident with a 3-face $[y_1y_2y_3]$. Clearly, the precoloring of $K$ can be extended to a DP-5-coloring of $G[y_1, y_2, y_3]$ and we can reduce the problem to the previous case.

If $G$ is the Wagner graph, then we can greedily extend the precoloring of $K$ to a DP-5-coloring of $G$ since $G$ is 3-regular.

By Theorem 2.2, we can assume that $G$ is a 2-sum or 3-sum of two maximal $K_5$-minor-free graphs $G_1$ and $G_2$ with $K \subset G_1$. By minimality, the precoloring $\varphi$ of $K$ can be extended to a DP-5-coloring $\varphi_1$ of $G_1$. By minimality once again, we can extended the restriction of $\varphi_1$ on $G_1 \cap G_2$ to $G_2$. This yields a DP-5-coloring of $G$ whose restriction on $K$ is the precoloring $\varphi$.

Now, we can easily prove Theorem 1.1.

Theorem 1.1. All $K_5$-minor-free graphs are DP-5-colorable.

Proof. Since every $K_5$-minor-free graph is a spanning subgraph of a maximal $K_5$-minor-free graph, it suffices to prove the result for maximal $K_5$-minor-free graphs. We can first color two adjacent vertices in $G$, and extend the coloring to the whole graph according to Theorem 2.4.

Wagner [10] also gave a characterization of maximal $K_{3,3}$-minor-free graphs by 2-sums.

Theorem 2.5 (Wagner [10]). Every maximal $K_{3,3}$-minor-free graph can be obtained from the complete graph $K_5$ and plane triangulations by recursively 2-sums.
Since the proof of the following result is similar to that in Theorem 2.4, we leave it as an exercise to the readers.

**Theorem 2.6.** Assume that $G$ is a maximal $K_{3,3}$-minor-free graph. If $K$ is a subgraph of $G$ isomorphic to $K_2$, then every DP-$5$-coloring of $K$ can be extended to a DP-$5$-coloring of $G$.

**Theorem 1.2.** All $K_{3,3}$-minor-free graphs are DP-$5$-colorable.

**Proof.** Since each $K_{3,3}$-minor-free graph is a spanning subgraph of a maximal $K_{3,3}$-minor-free graph, it suffices to show the result for maximal $K_{3,3}$-minor-free graphs. We can first color two adjacent vertices in $G$, and further extend the precoloring to the whole graph according to Theorem 2.6.

3 \textbf{Strictly $f$-degenerate transversal}

In this section, we extend the results on DP-$5$-coloring to particular strictly $f$-degenerate transversal. The following two lemmas were presented by Nakprasit and Nakprasit [5, Lemma 2.3] with a different term.

For a vertex subset $K$ of $V(G)$, or a subgraph $K$ of $G$, we use $H_K$ to denote the cover restricted on $K$, i.e., $H_K := H[\bigcup_{v \in K} L_v]$.

**Lemma 3.1.** Assume that $G$ is a graph and $K$ is a subgraph of $G$. Let $(H, f)$ be a valued cover, and $T$ be a transversal of $H_K$ such that $H[T]$ has no edges and $f(x) = 1$ for each $x \in T$. If $T$ can be extended to a strictly $f$-degenerate transversal $T'$ of $H$, then there exists an $f$-removing order of $T'$ such that the vertices in $T$ are on the rightest of the order.

**Proof.** Let $S'$ be an $f$-removing order of $T'$. Since $f(x) = 1$ for each $x \in T$, every vertex in $T$ has no neighbor on the right of the order $S'$, so we can move all the vertices in $T$ to the rightest of the order. In other words, we can delete all the vertices in $T$ from the order $S'$ and put the vertices in $T$ on the right side of all the other vertices of $S'$. Observe that the resulting order satisfies the desired condition.

**Lemma 3.2.** Assume that $G = G_1 \cup G_2$, $V(G_1 \cap G_2) = K$ and $G_1$ is an induced subgraph of $G$. Let $(H, f)$ be a valued cover of $G$, and $H_i$ be the restriction of $H$ on $G_i$ for $i \in \{1, 2\}$. If $R$ is a strictly $f$-degenerate transversal of $H_1$, and $R \cap H_K$ can be extended to a strictly $f^*$-degenerate transversal $R^*$ of $H^*$, where $H^*$ is obtained from $H_2$ by deleting all the edges in $H_K$, and $f^*$ is obtained from $f$ by defining $f^*(x) = 1$ for each $x \in R \cap H_K$, then $R \cup R^*$ must be a strictly $f$-degenerate transversal of $H$.

**Proof.** It suffices to give an $f$-removing order of $H[R \cup R^*]$. By Lemma 3.1, there exists an $f^*$-removing order of $R^*$ such that the vertices in $R \cap H_K$ are on the rightest of the order. Then we list all the vertices of $R^* \setminus (R \cap H_K)$ according to the $f^*$-removing order and then list the vertices of $R$ according to an $f$-removing order. It is easy to check that the resulting order is an $f$-removing order for $H[R \cup R^*]$.

We first extend Theorem 2.3 to the following result. Note that Theorem 3.1 was first proved in [5, Theorem 1.6], but the following proof is a little bit different from that one.

**Theorem 3.1.** Assume that $G$ is a near-triangulation such that the outer face is bounded by a cycle $O = v_1v_2 \ldots v_p v_1$. Let $(H, f)$ be a valued cover of $G$ with $R_f \subseteq \{0, 1, 2\}$ such that

\[ f(v, 1) + \cdots + f(v, s) \geq 3 \text{ for every } v \in V(O) \]  

and

\[ f(v, 1) + \cdots + f(v, s) \geq 5 \text{ for every } v \notin V(O). \]

If $R_0$ is a strictly $f$-degenerate transversal of $H[L_{v_1} \cup L_{v_2}]$, then $R_0$ can be extended to a strictly $f$-degenerate transversal of $H$. 

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Proof. We prove the assertion by induction on \(|V(G)|\). When \(G\) has exactly three vertices, \(G = \mathcal{O} = K_3\) and the assertion is obvious. Then \(|V(G)| \geq 4\) and the assertion is true for smaller graphs. Suppose that \(\mathcal{O}\) has a chord \(uv\). It follows that \(uv\) lies in two cycles \(C_1\) and \(C_2\) of \(\mathcal{O} + uv\) with \(v_1 v_2\) in \(C_1\). Let \(G_1 := \text{Int}(C_1)\) and \(G_2 := \text{Int}(C_2)\). Applying the induction hypothesis to \(G_1, R_0\) can be extended to a strictly \(f\)-degenerate transversal \(R\) of \(H_1\), and then \(R \cap H[L_u \cup L_w]\) can be extended to a strictly \(f^*\)-degenerate transversal \(R^*\) of \(H^*\) as in Lemma 3.2. Therefore, \(R^* \cup R\) is a desired strictly \(f\)-degenerate transversal of \(H\).

The other case is that \(\mathcal{O}\) has no chord. Let \(v_1, u_1, u_2, \ldots, u_m, v_{p-1}\) be the neighbors of \(v_p\) in a natural cyclic order around \(v_p\), and let \(U = \{u_1, u_2, \ldots, u_m\}\). Since all the bounded faces of \(G\) are bounded by triangles and \(\mathcal{O}\) has no chord, we have \(P = v_1 u_1 u_2 \ldots u_m v_{p-1}\) is a path and \(\mathcal{O}' = P \cup (\mathcal{O} - v_p)\) is a cycle. For each \(x \in \{v_p\} \times [s]\), let

\[
f'((x) = \begin{cases} 
\max\{0, f(x) - 1\}, & \text{if } x \text{ is adjacent to } R_0 \cap L_{v_1} \text{ under } \mathcal{M}_{v_1,v_p}; \\
\max\{0, f(x) - 1\}, & \text{if } x \in U \times [s] \text{ and } x \text{ is connected to a vertex in } X^*; \\
f(x), & \text{otherwise.}
\end{cases}
\]

Since \(R_0 \cap L_{v_1}\) has at most one neighbor in \(L_{v_p}\), we have \(f'(v_p, 1) + \cdots + f'(v_p, s) \geq 2\). Let

\[X' = \{ x \in \{v_p\} \times [s] : f'(x) > 0 \}.
\]

Case 1. \(|X'| \geq 2\).

Let \(X^*\) be a subset of \(X'\) with \(|X^*| = 2\). A new function \(f^1\) on \(H - L_{v_p}\) is defined as

\[
f^1(x) = \begin{cases} 
\max\{0, f(x) - 1\}, & \text{if } x \in U \times [s] \text{ and } x \text{ is connected to a vertex in } X^*; \\
f(x), & \text{otherwise.}
\end{cases}
\]

It follows that, for each \(u \in \mathcal{O}'\), we have

\[\sum_{x \in L_u} f^1(x) \geq 3.
\]

By induction hypothesis and Lemma 3.1, \((H - L_{v_p}, f^1)\) contains a strictly \(f^1\)-degenerate transversal \(R^1\) with an \(f^1\)-removing order \(S^1\) such that the vertices in \(R_0\) are on the rightest of the order. Let \((v_p, c_p)\) be a vertex in \(X^*\) which is not adjacent to \(R^1 \cap L_{v_{p-1}}\). Therefore, we insert \((v_p, c_p)\) into \(S^1\) such that it is the reciprocal third element to obtain an \(f\)-removing order of a strictly \(f\)-degenerate transversal of \(H\).

Case 2. \(|X'| = 1\).

Without loss of generality, assume that \(X' = \{(v_p, 1)\}\). Since \(f'(v_p, 1) + \cdots + f'(v_p, s) \geq 2\) and \(R_f \subseteq \{0, 1, 2\}\), we have \(f'(v_p, 1) = 2\). Define a function \(f^1\) on \(H - L_{v_p}\) by

\[
f^1(x) = \begin{cases} 
0, & \text{if } x \in U \times [s] \text{ and } x \text{ is adjacent to } (v_p, 1) \text{ in } H; \\
f(x), & \text{otherwise.}
\end{cases}
\]

Note that the range of \(f\) is a subset of \(\{0, 1, 2\}\), for each \(u \in \mathcal{O}'\),

\[\sum_{x \in L_u} f^1(x) \geq 3.
\]

By induction hypothesis, \((H - L_{v_p}, f^1)\) admits a strictly \(f^1\)-degenerate transversal \(R^1\) with an \(f^1\)-removing order \(S^1\) such that the vertices in \(R_0\) are on the rightest of the order. Let \(S\) be a sequence obtained from \(S^1\) by inserting \((v_p, 1)\) into \(S^1\) such that \((v_p, 1)\) is the immediate predecessor of \((v_{p-1}, c_{p-1})\), where \((v_{p-1}, c_{p-1}) \in L_{v_{p-1}} \cap R^1\). Recall that \(f^1(v_p, 1) = 2\), it is not hard to check that \(S\) is an \(f\)-removing order of a strictly \(f\)-degenerate transversal of \(H\).
Instead of proving Theorem 1.3, we prove the following stronger theorem for $K_5$-minor-free graphs, and leave the corresponding result for $K_{3,3}$-minor-free graphs to the readers.

**Theorem 3.2.** Assume that $G$ is a $K_5$-minor-free graph, and $(H, f)$ is a valued-cover with $R_f \subseteq \{0, 1, 2\}$. If $K$ is a subgraph isomorphic to $K_2$ or $K_3$, and $f(v, 1) + \cdots + f(v, s) \geq 5$ for each $v \in V(G)$, then every strictly $f$-degenerate transversal of $H_K$ can be extended to a strictly $f$-degenerate transversal of $H$.

**Proof.** Suppose to the contrary that $(G, H, f, R_0)$ is a counterexample with $|V(G)|$ as small as possible, where $R_0$ is a strictly $f$-degenerate transversal of $H_K$. Similar to the previous results, we only need to consider the case that $G$ is a maximal $K_5$-minor-free graph.

Assume that $G$ is a plane triangulation and $K$ is a separating triangle of $G$. Note that $\text{Ext}(K)$ and $\text{Int}(K)$ are both plane triangulations and maximal $K_5$-minor-free graphs. By minimality and Lemma 3.2, $R_0$ can be extended to a strictly $f$-degenerate transversal of $H$.

Assume that $G$ is a plane triangulation and $K = [x_1x_2x_3]$ bounds a 3-face. We can redraw the plane triangulation such that $K$ bounds the outer face. Let $(x_3, c_3)$ be in $R_0$, define a function $f'$ on $H - L_{x_3}$ by

$$f'(x) = \begin{cases} 0, & \text{if } x \in \{u\} \times \{s\} \text{ with } u \notin \{x_1, x_2\} \text{ and } x \text{ is connected to } (x_3, c_3) \text{ in } H; \\ f(x), & \text{otherwise.} \end{cases}$$

Note that the graph $G - x_3$ is a near-triangulation. Since the range of $f$ is a subset of $\{0, 1, 2\}$, we have that, for each $w$ on the outer face of $G - x_3$,

$$\sum_{x \in \{w\} \times \{s\}} f'(x) \geq 3.$$

By Theorem 3.1, $R_0 \setminus \{(x_3, c_3)\}$ can be extended to a strictly $f'$-degenerate transversal of $H \setminus L_{x_3}$ with an $f'$-removing order $S'$ such that the two vertices in $R_0 \setminus \{(x_3, c_3)\}$ are on the rightest of the order. According to an $f$-removing order of $R_0$, we can insert $(x_3, c_3)$ into $S'$ such that the three vertices in $R_0$ are the three rightest elements in the order to obtain an $f$-removing order of a strictly $f$-degenerate transversal of $H$.

Assume that $G$ is a plane triangulation and $K = x_1x_2$. We may assume that $x_1x_2$ is incident with a 3-face $[x_1x_2x_3]$. Clearly, $R_0$ can be extended to a strictly $f$-degenerate transversal of $H_{[x_1, x_2, x_3]}$, and we can reduce the problem to the previous case.

If $G$ is the Wagner graph, then we can greedily extend $R_0$ to a strictly $f$-degenerate transversal of $H$ since $G$ is 3-regular.

By Theorem 2.2, assume that $G$ is a 2-sum or 3-sum of two maximal $K_5$-minor-free graphs $G_1$ and $G_2$ with $K \subset G_1$. By minimality and Lemma 3.2, $R_0$ can be extended to a strictly $f$-degenerate transversal of $H$.

In Theorems 3.1 and 3.2, there is a restriction on $f$, i.e., the range of $f$ is a subset of $\{0, 1, 2\}$. If the restriction can be dropped, the results can imply two theorems due to Thomassen. Thomassen proved that every planar graph can be partitioned into a 3-degenerate graph and an independent set [8], and every planar graph can be partitioned into a 2-degenerate graph and a forest [7]. So the second author and some others made the following conjecture in [4].

**Conjecture.** Assume that $G$ is a planar graph and $(H, f)$ is a positive-valued cover. If $s \geq 2$ and $f(v, 1) + \cdots + f(v, s) \geq 5$ for each $v \in V(G)$, then $H$ admits a strictly $f$-degenerate transversal.

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