ON A CHARACTERIZATION
OF FINITE-DIMENSIONAL VECTOR SPACES

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ABSTRACT. We provide a characterization of the finite dimensionality of vector
spaces in terms of the right-sided invertibility of linear operators on them.

1. INTRODUCTION

In paper [5], found is a characterization of one-dimensional (real or complex) normed
algebras in terms of the bounded linear operators on them, echoing the celebrated
Gelfand-Mazur theorem characterizing complex one-dimensional Banach algebras
(see, e.g., [1–3, 8, 10]).

Here, continuing along this path, we provide a simple characterization of the finite
dimensionality of vector spaces in terms of the right-sided invertibility of linear
operators on them.

2. PRELIMINARIES

As is well-known (see, e.g., [4, 9]), a square matrix $A$ with complex entries is invert-
able iff it is one-sided invertible, i.e., there exists a square matrix $C$ of the same
order as $A$ such that

$$AC = I \text{ (right inverse)} \quad \text{or} \quad CA = I \text{ (left inverse)},$$

where $I$ is the identity matrix of an appropriate size, in which case $C$ is the inverse
of $A$, i.e.,

$$C = A^{-1}.$$

Generally, for a linear operator on a (real or complex) vector space, the existence of
a left inverse implies being invertible. Indeed, let $A: X \to X$ be a linear operator
on a (real or complex) vector space $X$ and a linear operator $C: X \to X$ be its left
inverse, i.e.,

$$CA = I,$$

where $I$ is the identity operator on $X$. Equality (2.1), obviously, implies that

$$\ker A = \{0\},$$

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and hence, there exists an inverse $A^{-1} : R(A) \to X$ for the operator $A$, where $R(A)$ is its range (see, e.g., [7]). Equality (2.1) also implies that the inverse operator $A^{-1}$ is the restriction of $C$ to $R(A)$.

Further, as is easily seen, for a linear operator on a (real or complex) vector space, the existence of a right inverse immediately implies being surjective, which, provided the underlying vector space is finite-dimensional, by the rank-nullity theorem (see, e.g., [6,7]), is equivalent to being injective, and hence, being invertible.

With the underlying space being infinite-dimensional, the arithmetic of infinite cardinals does not allow to conclude, by the rank-nullity theorem, that the surjectivity of a linear operator on the space is equivalent to its injectivity. Hence, in this case the right-sided invertibility for linear operators need not imply invertibility. For instance, on the (real or complex) infinite-dimensional vector space $l_\infty$ of bounded sequences, the left shift linear operator

$$l_\infty \ni x := (x_1, x_2, x_3, \ldots) \mapsto Lx := (x_2, x_3, x_4, \ldots) \in l_\infty$$

is non-invertible since

$$\ker L = \{(x_1, 0, 0, \ldots)\} \neq \{0\}$$

(see, e.g., [6,7]) but has the right shift linear operator

$$l_\infty \ni x := (x_1, x_2, x_3, \ldots) \mapsto Rx := (0, x_1, x_2, \ldots) \in l_\infty$$

as its right inverse, i.e.,

$$LR = I,$$

where $I$ is the identity operator on $l_\infty$.

Not only does the above example give rise to the natural question of whether, when the right-sided invertibility for linear operators on a (real or complex) vector space implies their invertibility, the underlying space is necessarily finite-dimensional, but also serves as an inspiration for proving the “if” part of the subsequent characterization.

3. Characterization

**Theorem 3.1** (Characterization of Finite-Dimensional Vector Spaces).

A (real or complex) vector space $X$ is finite-dimensional iff right-sided invertibility for linear operators on $X$ implies invertibility.

**Proof.**

“Only if” part. Suppose that the vector space $X$ is finite-dimensional with $\dim X = n \ (n \in \mathbb{N})$ and let $B := \{x_1, \ldots, x_n\}$ be an ordered basis for $X$.

For an arbitrary linear operator $A : X \to X$ on $X$, which has a right inverse $C$, i.e.,

$$AC = I,$$

where $I$ is the identity operator on $X$, let $[A]_B$ and $[C]_B$ be the matrix representations of the operators $A$ and $C$ relative to the basis $B$, respectively (see, e.g., [7]).
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Then
\[(3.2) \quad [A]_B [C]_B = I_n,\]
where \(I_n\) is the identity matrix of size \(n\) (see, e.g., [7]).

By the multiplicativity of determinants (see, e.g., [4,9]), equality (3.2) implies that
\[\det ([A]_B) \det ([C]_B) = \det ([A]_B [C]_B) = \det (I_n) = 1.\]

Whence, we conclude that
\[\det ([A]_B) \neq 0,\]
which, by the determinant characterization of invertibility (see, e.g., [4,9]), we infer that the matrix \([A]_B\) is invertible, and hence, so is the operator \(A\) (see, e.g., [6]).

"If" part. Let us prove this part by contrapositive, assuming that the vector space \(X\) is infinite-dimensional. Suppose that \(B := \{x_i\}_{i \in I}\) is a basis for \(X\), where \(I\) is an infinite indexing set, and that \(J := \{i(n)\}_{n \in \mathbb{N}}\) is a countably infinite subset of \(I\).

Let us define a linear operator \(A : X \to X\) on \(X\) as follows:
\[Ax_{i(1)} := 0, \quad Ax_{i(n)} := x_{i(n-1)}, \quad n \geq 2, \quad Ax_i = x_i, \quad i \in I \setminus J,\]
and
\[X \ni x = \sum_{i \in I} c_i x_i \mapsto Ax := \sum_{i \in I} c_i Ax_i,\]
where
\[x = \sum_{i \in I} c_i x_i\]
is the unique basis representation of a vector \(x \in X\) relative to the basis \(B\), in which all but a finite number of the coefficients \(c_i, \ i \in I\), called the coordinates of \(x\) relative to \(B\), are zero (see, e.g., [6,7]).

As is easily seen, \(A\) is a linear operator on \(X\), which is non-invertible since
\[\ker A = \text{span} (\{x_{i(1)}\}) \neq \{0\} .\]

The linear operator \(C : X \to X\) on \(X\) defined as follows:
\[Cx_{i(n)} := x_{i(n+1)}, \quad n \in \mathbb{N}, \quad Cx_i = x_i, \quad i \in I \setminus J,\]
and
\[X \ni x = \sum_{i \in I} c_i x_i \mapsto Cx := \sum_{i \in I} c_i Cx_i,\]
is a right inverse for \(A\) since
\[ACx_{i(n)} := Ax_{i(n+1)} = x_{i(n)}, \quad n \in \mathbb{N}, \quad ACx_i = Ax_i = x_i, \quad i \in I \setminus J.\]

Thus, on a (real or complex) infinite-dimensional vector space, there exists a non-invertible linear operator with a right inverse, which completes the proof of the "if" part, a hence, of the entire statement. \(\square\)
4

MARAT V. MARKIN

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