On $\ell_p$-norm Computation over Multiple-Access Channels

(In Invited Paper)

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Abstract

This paper addresses some aspects of the general problem of information transfer and distributed function computation in wireless networks. Many applications of wireless technology foresee networks of autonomous devices executing tasks that can be posed as distributed function computation. In today’s wireless networks, the tasks of communication and (distributed) computation are performed separately, although an efficient network operation calls for approaches in which the information transfer is dynamically adapted to time-varying computation objectives. Thus, wireless communications and function computation must be tightly coupled and it is shown in this paper that information theory may play a crucial role in the design of efficient computation-aware wireless communication and networking strategies. This is explained in more detail by considering the problem of computing $\ell_p$-norms over multiple access channels.

I. INTRODUCTION

Future wireless networks are envisioned to consist of a massive large of communication devices that perform network tasks autonomously. A main enabler for this vision is the ability of the network to extract the useful information from a huge amount of data distributed over the different nodes. In many scenarios, the network reveals its true purpose-centric character, and the individual transmission of every collected sensor value to some sink node can be circumvented. Consider for example an environment monitoring system, where the main objective is to make predictions with respect to a small set of state variables that are facilitated by the laws of physics. In particular, the superposition property of electromagnetic waves provides certain computation capabilities inherently. In turn, this property can drive the convergence between pure transmission of waveforms and performing arithmetic operations on network variables. In this regard, References [1], [2] showed that channel collisions can be exploited through a generalized Computation over Multiple-Access Channels (CoMAC) framework. This framework subsumes techniques and mechanisms for coding and transmission to compute functions at a designated sink node using the superposition property of the multiple-access channel. In [2], [3], the authors showed that natural characteristics allow to compute functions contained in the space of nomographic functions. Continuing the analysis of computable functions within this framework, this work analyzes the computation of $\ell_p$ norms. Computing $\ell_p$ norms is of high practical relevance for many applications, as it allows to compute the number of non-zero elements for $p \to 0$ (and various proxies for $0 < p \leq 1$) or the maximum value for $p \to \infty$ (see Fig. 1). The main contribution of this work is a unified analysis for CoMAC of $\ell_p$ using short sequences and a fixed energy detector at the receiver.

A. Notation

Scalars, vectors and matrices are denoted by lowercase, bold lowercase and bold uppercase letters, respectively. $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}_+$ and $\mathbb{N}$ denote the sets of real, nonnegative real, nonnegative integer and natural numbers. 0, 1 and $I_K$ denote the vector of all zeros, all ones and the identity matrix of size $K \times K$. $\text{tr} \{ \cdot \}$, $\text{vec} \{ \cdot \}$ and $\text{unvec} \{ \cdot \}$ denote the trace of a matrix, the vectorization of a matrix obtained by stacking it’s columns and the inverse vectorization operation. $\mathcal{N}(\cdot, \cdot)$ and $\mathbb{E} \{ \cdot \}$ describe the normal distribution and the expectation operator. $(\cdot)^T$ and $\otimes$ denote transposition and kronecker product, respectively.

II. COMPUTING $\ell_p$-NORMS OVER MULTIPLE-ACCESS CHANNELS

Consider a wireless sensor network consisting of one designated sink node and $K \in \mathbb{N}$ nodes that monitor a physical quantity by sensor values $[x_1, \ldots, x_K]^T := x \in \mathbb{R}^K$. The objective of the network is to compute functions of the form

$$f(x) = \|x\|^p = \sum_{k=1}^{K} |x_k|^p$$

(1)
at the sink node for given values of $p > 0$ (see also Fig. 1 for illustration) Precisely, the class of functions to be computed (referred to as desired functions in what follows) in accordance with [2] is given by the $\ell_p$-(pseudo)-norm to the power of
In order to compute such non-linear functions, we assume as in [2] that each node is equipped with a pre-processing unit \( \varphi : \mathbb{R} \to \mathbb{R}_+ \) and transmit their pre-processed values simultaneously to the sink node using transmit sequences \( S = [s_1, \ldots, s_K] \in \mathbb{R}^{M \times K} \) (see Fig. 2). For ease of exposition, we assume in the following that

(i) the sensor nodes have perfect channel knowledge so that the effect of the communication channel can be perfectly equalized,

(ii) the nodes can be synchronized on frame and symbol level and

(iii) the receiver side detector is restricted to pure energy detection. \(^2\)

Moreover, we treat the problem in the real-valued domain and point out that the analysis for the complex domain follows along similar lines.

**Remark 1.** It is important to emphasize that except for the requirement of real-valued entries, there are no additional constraints on the matrix \( S \). This means that transmit sequences are jointly optimized with transmit powers. This stands in contrast to the studies related to CDMA systems, where the spreading sequences are normalized to be of unit norm and transmit powers are defined separately (see for instance [4]).

In the proposed setup depicted in Fig. 2 the received signal \( y \) and the output of the energy detector \( \hat{f} \) are, respectively, given by

\[
\begin{align*}
y &= S\varphi(x) + n \\
\hat{f} &= \|S\varphi(x) + n\|_2^2.
\end{align*}
\]  

(2) (3)

Here, \( \varphi : \mathbb{R}^K \to \mathbb{R}^K \) denotes the concatenated mapping from raw sensor readings \( x \) to transmit symbols \( \varphi(x) \) that is defined to be

\[
\varphi(x) := \begin{bmatrix} |x_1|^p \\
|\ldots| \\
|x_K|^p \end{bmatrix}^T.
\]  

(4)

Fig. 1: \( \ell_p \) (pseudo-)norms for different values of \( p \).

Fig. 2: System structure of the proposed \( \ell_p \)-norm computation scheme.

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1. Here, the prefix _pseudo_ applies to \( p < 1 \), in which case the functions are not a proper norm.

2. For an analysis of different channel state information schemes at the sensor nodes and the effect of coarse frame synchronization we refer the reader to [2]. An analysis of more complex (e.g. affine or nonlinear) receiver-side detectors is beyond the scope of the paper.
Remark 2. In an idealized setting with \( n = 0 \) and \( M \geq K \), the desired function value can be recovered exactly using any \( S \) such that \( S^T S = I_K \). In this case, we have

\[
\hat{f}(x) = \| S \phi(x) \|_2^2 = \phi(x)^T S^T S \phi(x) = \sum_{k=1}^{K} \left( |x_k| \right)^2
\]

However, the case described in Remark 2 is of minor practical relevance, as it excludes the effect of noise and is limited to the case \( K \geq M \), i.e. sequence lengths that exceed the number of network nodes. The more interesting case is a noisy setting with \( M < K \), and we motivate the use of the given pre-processing function and system structure by the identity in (5). We note that in addition to the application for networked computation of \( \ell_p \) norms, the considered setting is also closely related to problems in coding theory and robust dimensionality reduction (see e.g. [5],[6]).

To simplify the subsequent analysis, we fix some \( M < K \) and consider an MSE metric for estimating \( f \) by \( \hat{f} \) using \( S \) and a fixed receiver structure as depicted in Fig. 2. In this case, the MSE can be computed as

\[
J(S) = \mathbb{E} \left[ (f - \hat{f})^2 \right]
\]

\[
= \mathbb{E} \left[ \left( \phi(x)^T (I_K - S^T S) \phi(x) - 2 \phi(x)^T S n - n^T n \right)^2 \right]
\]

\[
= \mathbb{E} \left[ a^2 + b^2 + c^2 - 2ab - 2ac + 2bc \right],
\]

which can be evaluated assuming that the probability distribution functions of \( x \) and \( n \) are given.

Lemma 1. Assume that \( x \) and \( n \) are independent and distributed as \( x \sim \mathcal{N}(0, \sigma_n^2 I_K) \) and \( n \sim \mathcal{N}(0, \sigma_n^2 I_M) \). Under this assumption, the expectation in (6) decomposes and we have

\[
\mathbb{E} [a^2] = \text{tr} \left\{ M(S^T S \otimes S^T S) \right\} - 2 \text{tr} \left\{ M(I \otimes S^T S) \right\} + \text{tr} \left\{ M \right\}
\]

\[
\mathbb{E} [b^2] = 4 \sigma_n^2 \text{tr} \left\{ CS^T S \right\}
\]

\[
\mathbb{E} [c^2] = \text{tr} \left\{ N \right\}
\]

\[
\mathbb{E} [ac] = M \sigma_n^2 \text{tr} \left\{ C(I - S^T S) \right\}
\]

\[
\mathbb{E} [ab] = \mathbb{E} [bc] = 0,
\]

with

\[
C = \mathbb{E} \left[ \phi(x)^T \phi(x)^T \right]
\]

\[
M = \mathbb{E} \left[ \text{vec} \left\{ \phi(x)^T \right\} \text{vec} \left\{ \phi(x)^T \right\}^T \right]
\]

\[
N = \mathbb{E} \left[ \text{vec} \left\{ nn^T \right\} \text{vec} \left\{ nn^T \right\}^T \right],
\]

Proof: The result for \( \mathbb{E} [a^2] \) follows from

\[
\phi(x)^T A \phi(x) \phi(x)^T B \phi(x) = \text{tr} \left\{ \phi(x)^T A \phi(x) \phi(x)^T B \right\}
\]

\[
= \text{tr} \left\{ \phi(x)^T A \phi(x) \phi(x)^T B \right\} = \text{tr} \left\{ \phi(x)^T A \phi(x) \phi(x)^T B \right\}
\]

\[
= \text{tr} \left\{ (B^\top \otimes A) \text{vec} \left\{ \phi(x)^T \right\} \text{vec} \left\{ \phi(x)^T \right\}^T \right\}
\]

using

\[
\text{tr} \left\{ ABC \right\} = \text{tr} \left\{ CAB \right\} = \text{tr} \left\{ BCA \right\}
\]

\[
\text{tr} \left\{ A^T B \right\} = \text{vec} \left\{ A \right\}^T \text{vec} \left\{ B \right\}
\]

\[
\text{vec} \left\{ ABC \right\} = \left( C^T \otimes A \right) \text{vec} \left\{ B \right\}
\]

\[
(A \otimes B)(C \otimes D) = AC \otimes BD,
\]

and performing the expectation w.r.t. the random variable \( x \). The term \( \mathbb{E} [c^2] \) follows similarly with \( n \) as random variable. \( \mathbb{E} [b^2], \mathbb{E} [ac], \mathbb{E} [ab] \) and \( \mathbb{E} [bc] \) follow from independency of \( x \) and \( n \) and the zero mean assumption on \( n \), respectively. ■
Moreover, we point out the following without a proof.

Lemma 2. The MSE function $J : \mathbb{R}^{M \times K} \to \mathbb{R}_+$ defined by (6) attains a minimum on $\mathbb{R}^{M \times K}$.

To compute the matrices $C$, $M$ and $N$, we note, that all matrix entries are equal to monomials with either non-negative rational exponents $\alpha \in \mathbb{R}^+_{\text{r}}$ for $C$ and $M$, or non-negative integer exponents $\beta \in \mathbb{Z}^+_M$ for $N$. Thus, the entries of $C$ and $M$ can be computed by extending the derivation of central absolute moments in [7] to the monomial case:

\[
Q(\alpha) = \mathbb{E} \left[ \prod_{k=1}^K |x_k|^\alpha_k \right] = \int_{\mathbb{R}^K} |x_1|^\alpha_1 \cdots |x_K|^\alpha_K p_x(x) dx = \frac{(2\sigma^2)^{\sum_{k=1}^K \alpha_k}}{\sqrt{\pi^K}} \prod_{k=1}^K \Gamma \left( \frac{\alpha_k + 1}{2} \right). \tag{11}
\]

Similarly, the entries of $N$ can be obtained by extending the derivation of central moments in [7]:

\[
I(\beta) = \mathbb{E} \left[ \prod_{m=1}^M n^\beta_m \right] = \int_{\mathbb{R}^M} n_1^{\beta_1} \cdots n_M^{\beta_M} p_n(n) dn = \begin{cases} 0 & \text{if some } \beta_m \text{ is odd} \\ \frac{(2\sigma^2)^{\sum_{m=1}^M \beta_m}}{\sqrt{\pi^M}} \prod_{m=1}^M \Gamma \left( \frac{\beta_m + 1}{2} \right) & \text{if all } \beta_m \text{ are even.} \end{cases} \tag{12}
\]

Now we are in a position to state our optimization problem.

Proposition 1. Let $\hat{f}$ be given by (3), and let $x \sim N(0, \sigma^2_x I_K)$ and $n \sim N(0, \sigma^2_n I_M)$ be independent. Then, the optimal sequences for computing $f = \|x\|^p_p$ given the receiver structure in Fig. 2 w.r.t. an MSE metric can be obtained as a solution to the problem

\[
\min_{S \in \mathbb{R}^{M \times K}} \text{tr} \left\{ M(S^T S \otimes S^T S) \right\} - 2\text{tr} \left\{ M(I \otimes S^T S) \right\} \\
+ \text{tr} \left\{ M \right\} + 4\sigma^2_n \text{tr} \left\{ C S^T S \right\} \\
+ \text{tr} \left\{ N \right\} - 2M\sigma^2_n \text{tr} \left\{ C (I - S^T S) \right\} \tag{13}
\]

\[
= \min_{S \in \mathbb{R}^{M \times K}} J(S). 
\]

Proof: The proof follows directly by combining Lemma 2, Lemma 1 and (6).

To the best of our knowledge, a closed-form solution to this problem is not known. Consequently we are going to approach the problem by numerical methods.

III. FIRST-ORDER OPTIMIZATION OF TRANSMIT SEQUENCES

In this section, we develop a simple gradient descent algorithm to optimize the cost function $J$ over the unconstrained input domain $\mathbb{R}^{M \times K}$. Unfortunately, the problem of Proposition 1 (Problem (13)) is not convex and there is no guarantee that the algorithm converges to a global minimum. The problem of designing an algorithm with global convergence is left as an open problem for future research.

Instead, we make use of the well-known gradient descent iteration to optimize the MSE function $J$:

\[
S^{(t+1)} = S^{(t)} - \mu^{(t)} \nabla_S J(S^{(t)}), \tag{14}
\]

where a suitable step-size that guarantees a non-increasing sequence of objective values $J(S^{(t)})$ can be obtained using the Armijo criterion [3, 9]

\[
J \left( S^{(t)} - \mu^{(t)} \nabla_S J(S^{(t)}) \right) \leq J(S^{(t)}) - c\mu^{(t)} \| \nabla_S J(S^{(t)}) \|^2_F. \tag{15}
\]

In fact, it can be shown that a sufficiently small constant step size would guarantee a non-increasing sequence of objective values as well. Since the objective function is bounded below (it is greater than zero), we can conclude that the sequence $J(S^{(t)})$ must converge under (15). This fact will be used for termination condition.

To obtain an analytic expression for $\nabla_S J(S)$ we refer to the analytic expression in Lemma 1 and (6), and simple standard formulas (e.g. [10]) yield

\[
\nabla_S \mathbb{E} \left[ s^2 \right] = 4\sigma_n^2 (SC^T + SC) \\
\nabla_S \mathbb{E} \left[ sc \right] = -M\sigma_n^2 (SC^T + SC) \\
\n\nabla_S \mathbb{E} \left[ s^2 \right] = \nabla_S \mathbb{E} \left[ c^2 \right] = 0. \tag{16}
\]
It remains to compute \( \nabla S \mathbb{E}[a^2] \). To this end, we introduce the following lemma:

**Lemma 3.** Let \( A \in \mathbb{R}^{K^2 \times K^2} \) and \( \mathcal{R}(A) \) denote a permutation of \( A \) given by

\[
\mathcal{R}(A) = [\text{vec} \{ A_{1,1} \}, \ldots, \text{vec} \{ A_{K,1} \}, \text{vec} \{ A_{1,2} \}, \ldots, \text{vec} \{ A_{K,K} \}]^T
\]

where \( A_{i,j} \in \mathbb{R}^{K \times K}, \ i, j \in \{1, \ldots, K\} \) denote the \( K \times K \) block matrix partitioning of \( A \). If \( \mathcal{R}(A) \) has a singular value decomposition

\[
\mathcal{R}(A) = \sum_{k=1}^{K^2} \sigma_k u_k v_k^T,
\]

where \( \sigma_k, u_k \) and \( v_k \) are the corresponding singular values and singular vectors, then the matrices \( U_k = \text{unvec} \{ u_k \} \) and \( V_k = \text{unvec} \{ v_k \} \) form a decomposition

\[
A = \sum_{k=1}^{K^2} \sigma_k U_k \otimes V_k.
\]

**Proof:** The proof follows directly from Corollary 2.2 in [11].

**Remark 3.** In our case \( M \) has the additional property that \( \mathcal{R}(M) = M = M^T \), which is stated here without proof. Hence, the SVD in Lemma 3 can be replaced by an EVD. Consequently, the matrix \( M \) can be decomposed as

\[
M = \sum_{k=1}^{K^2} M_k \otimes M_k,
\]

with \( M_k := \sqrt{\sigma_k} U_k \equiv \sqrt{\sigma_k} V_k \).

Now we are in a position to obtain an analytic expression for \( \nabla \mathbb{E}[a^2] \).

**Proposition 2.** Let \( \mathbb{E}[a^2] \) be given according to Lemma 1 and \( M \) according to Lemma 3 and Remark 3. Then, \( \nabla \mathbb{E}[a^2] \) is given by

\[
\nabla S \mathbb{E}[a^2] = 2 \sum_{k=1}^{K^2} \text{tr} \{ M_k S^T S \} \cdot (SM_k + SM_k^T) - 2 \sum_{k=1}^{K^2} \text{tr} \{ M_k \} \cdot (SM_k + SM_k^T).
\]

**Proof:** Define \( \nabla S \mathbb{E}[a^2] := \Delta^{(1)} - 2\Delta^{(2)} + \Delta^{(3)} \) with

\[
\Delta^{(1)} := \nabla S \text{tr} \{ M(S^T S \otimes S^T S) \}
\]

\[
\Delta^{(2)} := \nabla S \text{tr} \{ M(I \otimes S^T S) \}
\]

\[
\Delta^{(3)} := \nabla S \text{tr} \{ M \}.
\]

Using Lemma 3, Remark 3 and trace derivatives (e.g. [10]) we obtain

\[
\Delta^{(1)} = \nabla S \sum_{k=1}^{K^2} \text{tr} \{ (M_k \otimes M_k) \cdot (S^T S \otimes S^T S) \}
\]

\[
= \nabla S \sum_{k=1}^{K^2} \text{tr} \{ M_k S^T S \} \text{tr} \{ M_k S^T S \}
\]

\[
= 2 \sum_{k=1}^{K^2} \text{tr} \{ M_k S^T S \} \cdot (SM_k + SM_k^T),
\]
\[ \Delta^{(2)} = \nabla_S \sum_{k=1}^{K^2} \text{tr} \left\{ (M_k \otimes M_k) \cdot (1 \otimes S^T S) \right\} \]  
\[ = \nabla_S \sum_{k=1}^{K^2} \text{tr} \{M_k\} \cdot \text{tr} \{M_k S^T S\} \]  
\[ = \sum_{k=1}^{K^2} \text{tr} \{M_k\} \cdot (SM_k^T + SM_k), \]

which completes the proof.

Using Proposition 2 and 16, the overall gradient of \( J(S) \) w.r.t. \( S \) is given by

\[ \nabla_S J(S) = \nabla_S \mathbb{E}[a^2] + \nabla_S \mathbb{E}[b^2] - 2\nabla_S \mathbb{E}[ac] \]
\[ = \Delta^{(1)} - 2\Delta^{(2)} + (2M + 4)\sigma_n^2 (SC^T + SC). \]

In practice, the sum involved in the computation of \( \Delta^{(1)} \) and \( \Delta^{(2)} \) can be truncated depending on the decay of singular values \( \sigma_k \) to reduce the computational burden. The resulting gradient descent algorithm is described in Alg. 1.

Data: initial iterate \( S^{(0)} \), desired norm \( p \)
Result: optimized matrix \( S \in \mathbb{R}^{M \times K} \)
Initialization: compute \( C, M, M_k, N \);
while \( J(S^{(t)}) - J(S^{(t-1)}) \geq \varepsilon J(S^{(t)}) \) and \( t \leq T \) do
\[ \text{find } \mu^{(t)} \text{ that satisfies (15);} \]
\[ S^{(t+1)} = S^{(t)} - \mu^{(t)} \nabla_S J(S); \]
end

Algorithm 1: Gradient descent algorithm for optimizing sequence matrices.

IV. NUMERICAL RESULTS

To evaluate the performance of the proposed first-order optimization scheme we simulate a network consisting of \( K \in \{6, 16\} \) nodes and sequence lengths \( M \in \{3, 6\} \) to compute the desired function \( f = ||x||_p^p \) for \( p \in [10^{-3}, 4] \). The signal and noise powers are set to \( \sigma_x^2 = 1 \) and \( \sigma_n^2 \in \{0.01, 0.1\} \), the number of Monte Carlo iterations is \( 10^5 \) and the gradient descent optimization is carried out using relative threshold \( \varepsilon = 10^{-5} \), Armijo parameter \( c = 0.5 \) and a maximum of \( T = 10^5 \) gradient descent iterations. For comparison, we choose equiangular tight frames (ETFs), which are known to meet both Welch Bound and Maximum Welch Bound with equality and are good candidate solutions for many applications in communications and coding (see e.g. [5]). For the simulations, we use scaled versions of the \( 3 \times 6 \) and \( 6 \times 16 \) ETFs from [12, p. 78f], where the scaling factor is obtained by line-search to optimize (6) (denoted by WBE). The result is fed as initial iterate into the gradient descent optimization algorithm from Alg. 1 (denoted by ALG 1). The results are depicted in Fig. 3 and 4. According to our simulation results, we can achieve considerable performance gains over ETFs for \( p \) (strictly) between \( 10^{-3} \) and 4. On the other hand, the performance gains for the case \( p = 4 \) and noise level \( \sigma_n^2 = 10^{-3} \) as well as the case \( p = 10^{-3} \) and noise level \( \sigma_n^2 = 0.1 \) are rather moderate.

Remark 4. It is important to emphasize that the transmit powers (norms) of the compared sequences are allowed to be different in our setting. However, higher transmit powers do not necessarily result in a lower estimation error due to the fixed energy detector at the receiver (see Fig. 2). In fact, our simulation results show, that optimized sequences can even have lower total/maximum transmit power in some cases.

V. CONCLUSION

In this paper, we studied the problem of computing \( \ell_p \) - norms over the wireless channel using a previously proposed scheme in an idealized setting comprising perfect channel equalization and node synchronization. Assuming a simple energy detection scheme at a designated sink node and scalar pre-processing units at the transmitter nodes we optimize sequences for the best MSE performance. For the case of Gaussian priors on signal and noise, we give a unified error-analysis for the resulting
Fig. 3: Simulation results for $\sigma_x^2 = 1$ and $\sigma_n^2 = 10^{-3}$. Monte Carlo results are shown in solid, analytical results from [13] in dashed linestyle.

Fig. 4: Simulation results for $\sigma_x^2 = 1$ and $\sigma_n^2 = 0.1$. Monte Carlo results are shown in solid, analytical results from [13] in dashed linestyle.

MSE as a function of $p$ and the deployed transmit sequences. By using a simple gradient descent scheme, we showed that (Maximum) Welch Bound Equality Sequences, which are a good candidate solution for network tasks involving interference avoidance, can be outperformed in terms of an MSE criterion, though the performance gains in the investigated small-scale network are rather moderate. An interesting direction to further improve the MSE performance is the use of more complex receiver structures as well as fixed-rank manifold based optimization methods as outlined in [9]. Promising applications of the outlined computation scheme involve measuring the sparsity of sensor values in a network or the maximum sensor value. However, for measuring the sparsity of the sensor values, the Gaussian signal prior poses a limitation in the sense that typical realizations are not sparse. Using a more accurate model for sparse signals by sparse processes or compressible distributions constitutes an interesting task, however, the required analysis seems to be much more complicated.

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