HARDY SPACES MEET HARMONIC WEIGHTS

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Abstract. We investigate the Hardy space $H^1_L$ associated with a self-adjoint operator $L$ defined in a general setting in [28]. We assume that there exists an $L$-harmonic non-negative function $h$ such that the semigroup $\exp(-tL)$, after applying the Doob transform related to $h$, satisfies the upper and lower Gaussian estimates. Under this assumption we describe an illuminating characterisation of the Hardy space $H^1_L$ in terms of a simple atomic decomposition associated with the $L$-harmonic function $h$. Our approach also yields a natural characterisation of the $BMO$-type space corresponding to the operator $L$ and dual to $H^1_L$ in the same circumstances.

The applications include surprisingly wide range of operators, such as: Laplace operators with Dirichlet boundary conditions on some domains in $\mathbb{R}^n$, Schrödinger operators with certain potentials, and Bessel operators.

1. Introduction and statement of main results

1.1. Background. The classical notion of Hardy spaces is a mainstream masterpiece in the core of harmonic analysis, see for example [22, 48, 49]. There are several equivalent definitions of the real variable Hardy space $H^1(\mathbb{R}^n)$. For example, $H^1(\mathbb{R}^n)$ can be defined in terms of the maximal function associated with the heat semigroup generated by the Laplace operator $\Delta$ on $\mathbb{R}^n$. Recall that a locally integrable function $f$ on $\mathbb{R}^n$ is said to be in $H^1(\mathbb{R}^n)$ if

$$M_\Delta f(x) = \sup_{t > 0} |e^{t\Delta} f(x)|$$

belongs to $L^1(\mathbb{R}^n)$. If this is the case, then we set

$$\|f\|_{H^1(\mathbb{R}^n)} = \|M_\Delta f\|_{L^1(\mathbb{R}^n)}.$$

The definition above suggests defining Hardy spaces corresponding to a general self-adjoint operator $L$ by simply replacing the standard heat propagator by the semigroup $\exp(-tL)$ in (1.1). Alternatively one can define $H^1_L$ using the square function approach. The theory of Hardy spaces associated with operators has attracted a lot of attention in last decades and has been a very active research topic in harmonic analysis, see for example [1, 3, 4, 8, 13, 20, 28–30, 46, 53] and the references therein. Very systematic and general theory of such Hardy spaces was described in [28]. In a more specific situation, such as some classes of Schrödinger operators, the Hardy spaces $H^1_L$ were studied also by Dziubański and Zienkiewicz, see for example [17, 19, 20].

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In our study we investigate $H^1_L$ in the case, when there exists an $L$-harmonic non-negative function $h$ such that the semigroup $\exp(-tL)$, after applying the Doob transform related to $h$, satisfies the upper and lower Gaussian estimates. In this situation we are able to obtain a natural characterisation of $H^1_L$ in terms of atomic decompositions in which atoms satisfy the cancellation associated with the harmonic function $h$.

Recall that one of the most fundamental aspect of the theory of Hardy spaces is the atomic decomposition theorem obtained by Coifman and Latter, see [9] for $n = 1$ and [36] for $n \geq 2$. It is known that $f \in H^1(\mathbb{R}^n)$ if and only if
\[
f = \sum_{k=1}^{\infty} \lambda_k a_k,
\]
where $\sum_k |\lambda_k| < \infty$ and $a_k$ are classical atoms, i.e. there exist balls $B_k$ such that
\[
supp a_k \subseteq B_k, \quad \|a_k\|_{L^\infty} \leq |B_k|^{-1}, \quad \int_{B_k} a_k(x) \, dx = 0.
\]
Moreover, we can choose $\lambda_k$'s such that
\[
C^{-1} \|f\|_{H^1(\mathbb{R}^n)} \leq \sum_{k=1}^{\infty} |\lambda_k| \leq C \|f\|_{H^1(\mathbb{R}^n)}.
\]
The atomic description of Hardy spaces is particularly useful and it is the primary point of interest of this paper. Our main observation in this study states that under our assumption involving the Doob transform such characterisation remains valid with the cancellation part of condition (1.2) replaced by the relation
\[
\int_{B_k} a_k(x) h(x) \, dx = 0.
\]

Another fundamental aspect of classical theory of Hardy spaces is the duality of $H^1(\mathbb{R}^n)$ and the space of functions of bounded mean oscillation, $BMO(\mathbb{R}^n)$, see [22]. For Hardy and $BMO$ spaces associated with operators such duality was investigated and established in [13]. In the setting which we consider our approach allows us to describe a natural interpretation of such duality.

Recall that in the classical theory the $BMO(\mathbb{R}^n)$ space is defined by the norm
\[
\|f\|_{BMO} = \sup_B |B|^{-1} \int_B |f(x) - f_B| \, dx,
\]
where $f_B = |B|^{-1} \int_B f(x) \, dx$ and the supremum is taken over all balls in $\mathbb{R}^n$. The elements $BMO(\mathbb{R}^n)$ space are defined up to a constant function. It appears that (in a proper sense) $BMO(\mathbb{R}^n)$ is the dual of $H^1(\mathbb{R}^n)$. The new cancellation condition (1.3) suggests that if $h$ is the $L$ harmonic function then the $BMO$ norm associated to $L$ should be defined based on the following expression
\[
\sup_B \inf_c \left( \mu_h(B)^{-1} \int_B |g(x) - c h(x)|^2 h(x) \, d\mu(x) \right)^{1/2} < \infty.
\]
Note that if $h$ is a constant function then in virtue of the John-Nirenberg Inequality the above integral defines the norm equivalent with the classical $BMO$ definition. It is convenient for us to define $BMO$ in terms of the $L^2$ condition, see the proof of Lemma 5.1. Theorem B stated below confirms that the above definition gives a coherent
description of the duality between Hardy and \(BMO\) spaces associated to the operator \(L\) in the considered setting.

The aim of this paper is to study Hardy spaces and their duals for self-adjoint operators defined on spaces of homogeneous type. In particular, we shall study operators related to some harmonic functions in the sense that the heat semigroup kernel, after the Doob transform, satisfies lower and upper Gaussian estimates. Our assumption involving the Doob transform are specific and the resulting theory is not as general as in [28], but it still includes several interesting applications. For example Laplace operators with the Dirichlet boundary conditions which were considered by Auscher, Russ, Chang, Krantz and Stein in [4,8] can be investigated using the proposed framework, see Subsection 6.1 below. Examples also include Schrödinger operators with certain potentials, and Bessel operators. Our result gives a natural and explicit atomic description of Hardy spaces with atoms related to the \(L\)-harmonic function \(h\). Let us mention that in several examples it is possible that there exist two or more different bounded harmonic functions, see for example [7]. We hope it is possible to obtain similar description of the corresponding Hardy and \(BMO\) spaces in the case of several harmonic functions but we intend to investigate such a possibility in a different project.

Our characterization, see Theorems A and B below, is different from the ones studied before, even for well-known classical operators, see e.g. [4,8,18]. In these papers, the atoms that describe Hardy spaces can be divided into two classes: some of them are similar to classical atoms, and some of them do not satisfy cancellation condition (one can think that a function \(|B|^{-1}\chi_B(x)\) is an atom for a proper choice of a ball \(B\), c.f. [24]). Our result gives more homogeneous (and maybe even more natural) description - all the atoms satisfy cancellation condition, but with respect to the harmonic function \(h(x)\). Nevertheless the both descriptions are equivalent, see Section 7 below. A secondary goal of our study is to give a list of examples that satisfy assumptions of Theorems A and B, see Section 6. However, we believe that there are many more operators that fit to our context.

1.2. Assumptions and main results. Let \((X,d,\mu)\) be a metric measure space endowed with a distance \(d\) and a nonnegative Borel doubling measure \(\mu\) on \(X\), c.f. [11,12]. Recall that a measure \(\mu\) satisfies the doubling condition provided that there exists a constant \(C > 0\) such that for all \(x \in X\) and for all \(r > 0\),

\[
\mu(B(x,2r)) \leq C\mu(B(x,r)).
\]

Note that the doubling property implies the following strong homogeneity property,

\[\mu(B(x,\lambda r)) \leq C\lambda^n\mu(B(x,r))\]

for some \(C,n > 0\) uniformly for all \(\lambda \geq 1, r > 0\), and \(x \in X\). In Euclidean space with the Lebesgue measure, the parameter \(n\) corresponds to the dimension of the space, but in our more abstract setting, the optimal \(n\) need not even to be an integer.

Throughout the paper we assume that \(\mu(X) = \infty\). We shall consider operators \(L\), that are always assumed to be self-adjoint, non-negative, and defined on a domain \(\text{Dom}(L) \subseteq L^2(\mu)\). Moreover, we assume that the semigroup \(T_t = \exp(-tL)\) generated
by \( L \) has a nonnegative integral kernel
\[
T_t f(x) = \int_X T_t(x, y) f(y) d\mu(y)
\]
that satisfies the pointwise upper Gaussian estimates, i.e. there exist \( c, C > 0 \), such that
\[
(UG) \quad 0 \leq T_t(x, y) \leq C \mu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{ct} \right), \quad x, y \in X, \ t > 0.
\]

There are several equivalent definitions of Hardy spaces \( H^1_L(X) \) associated with \( L \), see Section 2.1 below. The simplest and most direct is in terms of the maximal operator associated with the heat semigroup generated by \( L \), namely
\[
M_L f(x) = \sup_{t > 0} |e^{-tL} f(x)|
\]
with \( x \in X, f \in L^2(\mu) \). Then we define the space \( H^1_L(X) \) as the completion of the set \( \{ f \in L^2(\mu) : \| M_L f \|_{L^1(\mu)} < \infty \} \) with respect to \( L^1 \)-norm of the maximal function,
\[
\| f \|_{H^1_L(X)} = \| M_L f \|_{L^1(\mu)}.
\]

1.2.1. **Motivation: an atomic decomposition result.** Let us now recall some results from [15]. Assume that we have a space \((X, d, \nu)\) and an operator \( \mathcal{L} \) related to a semigroup \( \mathcal{T}_t = \exp(-t\mathcal{L}) \). Notice, that we have changed the notation: \((X, d, \mu), L, T_t\) are replaced by \((X, d, \nu), \mathcal{L}, \mathcal{T}_t\) (in what follows, the latter will be used for the operators after applying the Doob transform). Following [15], suppose that the semigroup kernel \( \mathcal{T}_t(x, y) \) satisfies lower and upper Gaussian estimates, i.e. there exist \( c_1, c_2, C > 0 \) such that
\[
(ULG) \quad C^{-1} \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{c_1 t} \right) \leq \mathcal{T}_t(x, y) \leq C \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{c_2 t} \right)
\]
for \( x, y \in X \) and \( t > 0 \).

**Proposition 1.1.** [15, Proposition 3] Assume that a semigroup \( \mathcal{T}_t \) satisfies \((ULG)\). Then there exists a function \( \varphi \) such that
\[
(1.6) \quad 0 < c \leq \varphi(x) \leq C
\]
and \( \varphi \) is \( \mathcal{L} \)-harmonic in the sense that for all \( t > 0 \),
\[
(1.7) \quad \mathcal{T}_t \varphi(x) = \varphi(x), \quad \text{a.e.} \ x \in X.
\]

For details we refer the reader to [15, Sec. 2]. Let us notice that \( \varphi \) is unique (up to a constant), see Corollary 2.5. By Liouville’s theorem, the constant functions are the only bounded harmonic functions when \( \mathcal{L} \) is the the Laplace operator \( \Delta \) on \( \mathbb{R}^n \). Following [15], we call a function \( a \) a \((\nu, \varphi)\)-atom if there exists a ball \( B \) such that:
\[
(1.8) \quad \text{supp } a \subseteq B, \quad \| a \|_{\infty} \leq \nu(B)^{-1}, \quad \int a(x) \varphi(x) d\nu(x) = 0.
\]

The atomic Hardy space \( H^1_{at}(\nu, \varphi) \) is defined then in a standard way using \((\nu, \varphi)\)-atoms. It is shown in [15, Theorem 1] that if \( \mathcal{L} \) satisfies \((ULG)\) and an additional geometric continuity assumption, see [15, Theorem 1], then for \( \varphi \) from Proposition 1.1 we have
\[
(1.9) \quad \| f \|_{H^1_L(X)} \simeq \| f \|_{H^1_{at}(\nu, \varphi)}.
\]
Obviously, the assumption \((ULG)\) is quite restrictive. However, there is a more general version \((ULG_h)\) that includes a harmonic function \(h(x)\), which can have bounded values but not separated from zero, or can be even unbounded. Such harmonic functions appear e.g. when studying the Dirichlet Laplacian on a domain above the graph of a bounded \(C^{1,1}\) function on \(\mathbb{R}^n\) or the exterior of a \(C^{1,1}\) compact convex domain in \(\mathbb{R}^n\). Moreover, the same story appears when studying some Schrödinger operators (e.g. \(-\Delta + \gamma |x|^{-2}\) on \(\mathbb{R}^n, n \geq 3, \gamma > 0\), or for some Bessel operator defined on a weighted half-line. We shall discuss the details in Section 6.

1.2.2. Main results. The following assumptions are motivated by the notion of the Doob transform (or \(h\)-transform), see e.g. [26, 27]. Assume that there exists a function \(h : X \to (0, \infty)\) such that:

\((H1)\) : \(h\) is \(L\)-harmonic in the sense that for all \(t > 0\)

\[ T_t h(x) = h(x), \quad \text{a.e. } x \in X. \]

\((H2)\) : The metric-measure space \((X, d, \mu_h)\) is doubling, where \(\mu_{\mathcal{h}}\) is the measure with the density \(h^2(x) d\mu(x)\).

\((H3)\) : There exist \(c_1, c_2, C > 0\) such that for \(x, y \in X\) and \(t > 0\) we have

\((ULG_h)\)

\[
\frac{C^{-1}}{\mu_{\mathcal{h}}(B(x, \sqrt{t}))} \exp \left( -\frac{d(x, y)^2}{c_1 t} \right) \leq T_t(x, y) h(x) h(y) \leq \frac{C}{\mu_{\mathcal{h}}(B(x, \sqrt{t}))} \exp \left( -\frac{d(x, y)^2}{c_2 t} \right).
\]

Let us notice that \((H2)\) and \((H3)\) imply that the action of \(T_t\) on \(h\) is well defined, even if \(h\) is unbounded. Moreover, Proposition 2.3 below says that, in some sense, the assumption \((H1)\) is always true after some mild change of function \(h\). However, we decided to state \((H1)\) as an assumption to emphasize the relation of \(L\)-harmonicity of \(h\) with the estimates \((ULG_h)\).

Now we define our atomic Hardy space that will be used to describe \(H^1_L(X)\).

**Definition 1.2.** We call a function \(a\) an \([\mu, h]\)-atom if there exists a ball \(B\) such that:

\[(1.10)\]

\[
\circ \quad \text{supp } a \subseteq B,
\]

\[(1.11)\]

\[
\circ \quad \|a\|_{L^2(h^{-1}, \mu)} \leq \mu_{\mathcal{h}}(B)^{-1/2},
\]

\[(1.12)\]

\[
\circ \quad \int a(x) h(x) d\mu(x) = 0.
\]

Then, by definition, a function \(f\) belongs to the atomic Hardy space \(H^1_{at}[\mu, h]\) if \(f = \sum_k \lambda_k a_k\), where \(a_k\) are \([\mu, h]\)-atoms and \(\sum_k |\lambda_k| < \infty\). Moreover, define

\[
\|f\|_{H^1_{at}[\mu, h]} = \inf \sum_k |\lambda_k|,
\]

where the infimum is taken over all representations of \(f\) as above.

Observe that by (1.10)–(1.11) every \([\mu, h]\)-atom \(a\) satisfies the estimate

\[
\|a\|_{L^1(\mu)} \leq \|a\|_{L^2(h^{-1}, \mu)} \mu_{\mathcal{h}}(B)^{1/2} \leq 1,
\]

so the series \(f = \sum_k \lambda_k a_k\) above converges in \(L^1(\mu)\)-norm and a.e. By a standard argument, \(H^1_{at}[\mu, h]\) is a Banach space.
The main goal of this paper is to provide a natural and simple atomic description of \( H^1_\var{L}(X) \) (in the spirit of (1.8)–(1.9)). Recall that \( A_p(\mu) \) is the Muckenhoupt class, see (3.1) below. Our result can be stated in a following way.

**Theorem A.** Suppose that an operator \( L \), its semigroup \( T_t = \exp(-tL) \), and a function \( h(x) \) satisfy the assumptions (H1)–(H3). There exists \( p_0 \in (1,2] \) such that if \( h^{-1} \in A_{p_0}(\mu,h^2) \), then the spaces \( H^1_\var{L}(X) \) and \( H^1_{at}[\mu,h] \) coincide and

\[
\|f\|_{H^1_\var{L}(X)} \approx \|f\|_{H^1_{at}[\mu,h]}.
\]

We would like to emphasize that in Theorem A the semigroup \( T_t \) does not need to satisfy (ULG). Hence (1.6) is not necessarily valid so it can happen that \( \inf h(x) = 0 \) or \( \sup h(x) = \infty \) (this is the case in many interesting examples). Therefore Theorem A can be seen as a generalization of (1.9) from [15]. However, there is a small cost here, namely we change \( L^\infty \)-type condition on size of atoms into weighted \( L^2 \)-type condition.

Also, note that in [15, Theorem 1] the result requires the following geometric assumption: for every \( x \in X \) the function \( r \mapsto \nu(B(x,r)) \) is a bijection on \((0,\infty)\). Our approach does not require this condition.

Our proof of Theorem A uses strongly the Doob transform, see Subsection 2.2. More precisely, we can introduce a new semigroup \( \hat{T}_t = \exp(-t\hat{L}) \) by (2.5) which acts on \((X, d, \nu), d\nu(x) = h^2(x)d\mu(x)\) and satisfies (ULG) on this changed metric-measure space. Moreover,

\[
H^1_\var{L}(X) \ni f \mapsto h^{-1}f \in H^1_{\hat{L},h^{-1}}(X)
\]

is an isometry between \( H^1_\var{L}(X) \) (related to the measure \( \mu \)) and a weighted Hardy space \( H^1_{\hat{L},h^{-1}}(X) \) (related to the measure \( \nu \)). Therefore, we shall study weighted Hardy spaces for operators \( \mathcal{L} \) satisfying (ULG) in Section 3 below. The proof of (1.9) in [15] uses different methods to the ones used here. In [15] the key step is to use a theorem of Uchiyama [52], which relies on the analysis of grand maximal function. Our approach is based on atomic decompositions for weighted tent spaces, see [38, 42, 44].

The second goal of this paper is to study the space of functions of bounded mean oscillation, the dual of \( H^1_{at}[\mu,h] \). By definition, a function \( g \) is in \( BMO[\mu,h] \) if

\[
\|g\|_{BMO[\mu,h]} := \sup_B \inf_c \left( \mu(B)^{-1} \int_B |g(x) - c\,h(x)|^2 h(x) d\mu(x) \right)^{1/2} < \infty.
\]

In a standard way, elements of \( BMO[\mu,h] \) are classes \( \{g + ch : c \in \mathbb{C} \} \). A natural analogue of the Fefferman-Stein duality result [22] is the following:

**Theorem B.** Suppose that an operator \( L \), its semigroup \( T_t = \exp(-tL) \), and a function \( h(x) \) satisfy the assumptions (H1)–(H3). There exists \( p_0 \in (1,2] \) such that if \( h^{-1} \in A_{p_0}(\mu,h^2) \), then \( BMO[\mu,h] \) is the dual to the Hardy space \( H^1_{at}[\mu,h] \).

The proof of Theorem B and further details are discussed in Section 5. The outline of the remainder of the paper is as follows. In Section 2 we recall some known facts on: Hardy spaces, the Doob transform, Gaussian estimates, and prove some preliminary results. In Section 3 we study the weighted Hardy spaces and the corresponding atomic decompositions. In Sections 4 and 5 we prove our main results, Theorems A and B,
respectively. In Section 6 we provide several examples of operators that satisfy our assumptions.

2. Preliminaries.

We now set notation and some common concepts that will be used throughout the course of the proofs. Let \((X,d,\mu)\) be a metric measure space endowed with a distance \(d\) and a nonnegative Borel doubling measure \(\mu\) on \(X\). The operator \(L\) is related to the semigroup \(T_t\) on the space \((X,d,\mu)\), whereas \(\mathcal{L}\) is related to \(T_t\) on \((X,d,\nu)\). The difference is that we always assume that \(\mathcal{L}\) satisfies \((ULG)\), whereas \(L\) satisfies more general condition \((ULG_h)\). As a consequence \(\varphi(x)\) from Proposition 1.1 is the harmonic function for \(\mathcal{L}\) that is bounded from above and from below. However, the harmonic function \(h\) related to \(L\) is in general unbounded (either from above or from below). Finally, the letters \(c, C\) are positive constants that may change from line to line. The notation \(A \simeq B\) means that \(C^{-1}A \leq B \leq CA\).

2.1. Hardy spaces \(H^1_L(X)\). Let us start with giving a few definitions of the Hardy space \(H^1_L(X)\) adapted to an operator \(L\). At the end all these definitions are the same Hardy space that we shall denote \(H^1_L(X)\). In Subsection 1.1 we already defined \(H^1_L(X) = H^1_{L,max}(X)\) by means of the maximal function. Let us also recall the following Lusin (area) function \(S_Lf\) and Littlewood-Paley function \(G_Lf\) associated to the heat semigroup generated by \(L\)

\[
S_Lf(x) := \left( \int_{d(x,y)<t} \left| t^2 \text{Le}^{-t^2} f(y) \right|^2 \frac{d\mu(y)}{\mu(B(x,t))} \frac{dt}{t} \right)^{1/2}, \quad x \in X,
\]

and

\[
G_Lf(x) := \left( \int_0^\infty \left| t^2 \text{Le}^{-t^2} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in X.
\]

We define the Hardy space \(H^1_{L,s}(X)\) as the completion of \(\{f \in L^2(\mu) : \|S_Lf\|_{L^1(\mu)} < \infty\}\) in \(L^1(\mu)\), see [2, Theorem 4.7], with respect to \(L^1\)-norm of the Lusin (area) function, i.e.

\[
\|f\|_{H^1_{L,s}(X)} = \|S_Lf\|_{L^1(\mu)}.
\]

The space \(H^1_{L,G}(X)\) is defined analogously. Now, we shall discuss another approach to atomic decomposition of \(H^1_L(X)\), which works in a more general context, but gives different (and in some sense more complicated) atoms. At this moment it is enough to make only assumptions from Subsection 1.1. Following [28] let us define an \(L\)-atom \(a\) as follows. Assume that there exists a ball \(B = B(y_0,r) \subseteq X\) and a function \(b \in \text{Dom}(L)\) such that for \(k = 0, 1\) we have:

\[
a = Lb, \quad \text{supp } L^kb \subseteq B, \quad \|(r^2L)^kb\|_{L^2(\mu)} \leq r^2\mu(B)^{-1/2}.
\]

Using \(L\)-atoms, one defines an atomic Hardy space \(H^1_{L,at}(X)\) as in [28, Definition 2.2]. In [28, Theorem 7.1] Hofmann et. al. proved that

\[
\|f\|_{H^1_{L,s}(X)} \simeq \|f\|_{H^1_{L,at}(X)} \leq C \|f\|_{H^1_L(X)}.
\]

Later, in [46, Theorem 1.3], a complementary estimate was proved, namely

\[
\|f\|_{H^1_L(X)} \leq C \|f\|_{H^1_{L,at}(X)}.
\]
Moreover, results from [31, Theorem 1.2] imply that
\[ \|f\|_{H^1_L(S(X))} \simeq \|f\|_{H^1_L(G(X))}. \]
Therefore, all the definitions above lead to the same Hardy space that we shall denote
\[ H^1_L(X) := H^1_{L,\text{max}}(X) = H^1_{L,S}(X) = H^1_{L,G}(X) = H^1_{L,\text{lat}}(X). \]
Let us also mention that \( H^1_L(X) \) has also equivalent norms in terms of non-tangential maximal function and analogues with Poisson semigroup, see [28, 45, 46].

**Remark 2.1.** From [28, Lemma 9.1] it follows that if the semigroup \( T_t = \exp(-tL) \) related to an operator \( L \) is conservative, i.e.,
\[ \int_X T_t(x,y)d\mu(y) = 1, \quad t > 0, \ \ x \in X, \]
then for every \( L \)-atom \( a \) we have \( \int_X a(x)d\mu(x) = 0 \).

### 2.2. Doob transform

In this section we describe one of the most important tools for this paper, i.e. the Doob transform (or \( h \)-transform), see e.g. [26, 27]. Assume that an operator \( L \) related to a metric measure space \((X, d, \mu)\) and a function \( h \) satisfy (H1)–(H3). Notice that here we do not assume that \( h \) is bounded neither from above nor from below. See Section 6 for examples.

On \((X, d)\) define a new measure \( d\nu(x) = h^2(x)d\mu(x) \) and a new kernel
\[ T_t(x, y) = \frac{T_t(x, y)}{h(x)h(y)}. \]
By (H2) the space \((X, d, \nu)\) satisfies the doubling condition. The inequalities from (H3) for \( T_t \) are equivalent to (ULG) for \( T_t \). The Doob transform is a simple multiplication operator
\[ f \mapsto h^{-1}f. \]
Observe that
\[ \|f\|_{L^2(\mu)} = \|h^{-1}f\|_{L^2(\nu)} \]
so the Doob transform is an isometry between these two \( L^2 \) spaces. Moreover, a simple calculation shows that \( T_t \) is a semigroup and its generator \( L \) is also self-adjoint (as an image of \( L \) under isometry). However, the Doob transform is not an isometry between \( L^1 \) spaces but we still have \( \|f\|_{L^1(\mu)} = \|h^{-1}f\|_{L^1(h^{-1}\nu)} \).

Recall now that \( H^1_L(X) \) corresponds to the measure \( \mu \), whereas \( H^1_L(X) \) is defined with respect to \( \nu = \mu h^2 \). A crucial observation in this paper is the following proposition, where \( H^1_{L,G,h^{-1}}(X) \) and \( H^1_{L,\text{max},h^{-1}}(X) \) are weighted Hardy spaces that we define in Section 3 below.

**Proposition 2.2.** Let \( f \in L^1(\mu) \). Then
\[ \|f\|_{H^1_{L,G}(X)} = \|f\|_{H^1_{L,G,h^{-1}}(X)}, \]
\[ \|f\|_{H^1_{L,\text{max}}(X)} = \|f\|_{H^1_{L,\text{max},h^{-1}}(X)}. \]
Proof. It is enough to notice that
\[ \int_X \left( \int_0^\infty \left| t^2 LT_t f(x) \right|^2 \right)^{1/2} d\mu(x) = \int_X \left( \int_0^\infty \left| t^2 \mathcal{L} (h^{-1} f) (x) \right|^2 \right)^{1/2} \frac{d\nu(x)}{h(x)} \]
and
\[ \int_X \sup_{t > 0} \left| \int_X T_t(x, y) f(y) d\mu(y) \right| d\mu(x) = \int_X \sup_{t > 0} \left| \int_X T_t(x, y) \frac{f(y)}{h(y)} d\nu(y) \right| \frac{d\nu(x)}{h(x)}. \]

\[ \square \]

The next statement essentially says that, for the purpose of our discussion here, assumption (H1) is automatically fulfilled provided that assumptions (H2) and (H3) are valid.

**Proposition 2.3.** Assume that for a semigroup \( T_t \) there exists a function \( \tilde{h} \) such that (H2) and (H3) are satisfied. Then, there exist \( C > 0 \) and a function \( \varphi : X \to \mathbb{R} \) such that \( C^{-1} \leq \varphi(x) \leq C \) and for all \( t > 0 \) we have
\[ T_t(\varphi\tilde{h})(x) = \varphi\tilde{h}(x), \quad a.e. \ x \in X. \]

Proof. Assume that \( \tilde{h} \) is such that (H2) and (H3) hold. Then, after the Doob transform the semigroup \( T_t \) satisfies (ULG) and we obtain \( \varphi \) satisfying (1.6) and (1.7), see Subsection 1.2.1. In particular, for \( t > 0 \),
\[ T_t(\varphi\tilde{h})(x) = \tilde{h}(x)T_t\varphi(x) = \tilde{h}(x)\varphi(x). \]
Thus, \( h = \tilde{h}\varphi \) is \( L \)-harmonic and still satisfies (H2)–(H3).

\[ \square \]

### 2.3. Semigroups with two-sided Gaussian bounds.

In this subsection we assume that \( T_t(x, y) \) is a semigroup that satisfy (ULG) on the space \((X, d, \nu)\). Then, there exists a function \( \varphi \), such that (1.6) and (1.7) are satisfied, see Proposition 1.6 and [15, Sec. 2]. It is well known that (ULG) implies certain Hölder regularity in the space variable for \( T_t(x, y) \). For a simple proof see [15, Sec. 4].

**Proposition 2.4.** [15, Corollary 14] Assume that the semigroup kernel \( T_t(x, y) \) satisfies (ULG) and \( \varphi \) is the related \( \mathcal{L} \)-harmonic function. There exist \( C, c, \delta > 0 \) such that
\[ (2.6) \quad \frac{T_t(x, y)}{\varphi(x)\varphi(y)} \leq C \left( \frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{ct} \right) \]
whenever \( d(y, y_0) < \sqrt{t} \).

Let us remark that \( \varphi(x) \simeq C \), so we could skip \( \varphi(x) \) in (2.6). However, we need to divide by \( \varphi(y) \) and \( \varphi(y_0) \) to get Hölder-type inequality. Let us notice that Proposition 2.4 implies the following corollary.

**Corollary 2.5.** If \( T_t(x, y) \) satisfies (ULG), then \( \varphi \) is (up to a constant) the unique bounded harmonic function.
Proof. Let \( \tilde{\varphi} \) be such that \( T_t \tilde{\varphi}(x) = \tilde{\varphi}(x) \) and \(|\tilde{\varphi}(x)| \leq C\). From (2.6) for \( \sqrt{t} > d(y_1, y_2) \) we have
\[
\left| \frac{\tilde{\varphi}(y_1) - \tilde{\varphi}(y_2)}{\varphi(y_1) - \varphi(y_2)} \right| = \left| \frac{T_t \tilde{\varphi}(y_1)}{\varphi(y_1)} - \frac{T_t \tilde{\varphi}(y_2)}{\varphi(y_2)} \right|
\leq \int_X \left| \frac{T_t(y_1, x)}{\varphi(y_1)} - \frac{T_t(y_2, x)}{\varphi(y_2)} \right| \tilde{\varphi}(x) d\nu(x)
\leq C \|	ilde{\varphi}\|_\infty \left( \frac{d(y_1, y_2)}{\sqrt{t}} \right)^\delta \int_X \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{c t} \right) d\nu(x)
\leq C \|	ilde{\varphi}\|_\infty \left( \frac{d(y_1, y_2)}{\sqrt{t}} \right)^\delta.
\]
Taking \( t \to \infty \) we arrive at \( \tilde{\varphi}/\varphi \equiv C \). This completes the proof of Corollary 2.5. □

Let us state another consequence of Proposition 2.4 that we shall use in Section 3.

**Proposition 2.6.** Assume that \( T_t(x, y) \) and \( \delta, c > 0 \) are as in Proposition 2.4 and that \( K_t(x, y) \) is the kernel of the operator \( t\mathcal{L} \exp(-t\mathcal{L}) \). For \( d(y, y_0) < \sqrt{t} \) we have
\[
\left| \frac{K_t(x, y)}{\varphi(x)} - \frac{K_t(x, y_0)}{\varphi(y_0)} \right| \leq C \left( \frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{2c t} \right).
\]

Proof. By the self-improvement property of Gaussian estimates we have that
\[
|K_t(x, z)| \leq C \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, z)^2}{c t} \right),
\]
see e.g. [25] or [43, Theorem 4.]. Observe that
\[
K_t(x, y) = 2 \int_X K_{t/2}(x, z) T_{t/2}(z, y) d\nu(z).
\]
Next, by (2.6),
\[
\left| \frac{K_t(x, y)}{\varphi(x)} - \frac{K_t(x, y_0)}{\varphi(y_0)} \right| \leq C \int_X \left| \frac{T_{t/2}(z, y)}{\varphi(y)} - \frac{T_{t/2}(z, y_0)}{\varphi(y_0)} \right| d\nu(z)
\leq C \left( \frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \nu(B(x, \sqrt{t}))^{-1} \int_X \nu(B(z, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, z)^2 + d(z, y)^2}{c t} \right) d\nu(z)
\leq C \left( \frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{4c t} \right) \int_X \nu(B(z, \sqrt{t}))^{-1} e^{-\frac{d(x, y)^2}{4c t}} d\nu(z)
\leq C \left( \frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{4c t} \right).
\]
The proof of Proposition 2.6 is complete. □

3. **Weighted Hardy spaces**

The theory of weighted Hardy spaces in \( \mathbb{R}^n \) was studied in [23, 51]. In the more general context of spaces of homogeneous type the reader is referred to [31, 38, 45, 46, 53] and references therein.
3.1. Muckenhoupt weights. Recall that a non-negative function \( w \) defined on \( X \) is called a weight if it is locally integrable. We denote by \( \mu_w(A) = \int_A w(x) \, d\mu(x) \) the weighted measure, and by \( \|f\|_{L^p_w(\mu)} = (\int_X |f(x)|^p w(x) \, d\mu(x))^{1/p} \) the weighted \( L^p \)-norm.

We say that \( w \) is in the Muckenhoupt class \( A_p(\mu) \), \( p > 1 \), if there is a constant \( C \) such that
\[
\left( \frac{1}{\mu(B)} \int_B w(x) \, d\mu(x) \right) \left( \frac{1}{\mu(B)} \int_B w(x)^{-1/(p-1)}(x) \, d\mu(x) \right)^{p-1} \leq C
\]
holds for every ball \( B \subset X \). The class \( A_1 \) is defined replacing (3.1) by
\[
\|w^{-1} \chi_B\|_\infty \left( \frac{1}{\mu(B)} \int_B w(x) \, d\mu(x) \right) \leq C,
\]
where \( \chi_B \) is the characterization function of the ball \( B \). The class \( A_\infty(\mu) \) is defined as the union of the \( A_p(\mu) \) classes for \( 1 \leq p < \infty \), i.e., \( A_\infty(\mu) = \bigcup_{p \geq 1} A_p(\mu) \). In the sequel, we shall use the following standard properties of \( A_p(\mu) \) weights. For details we refer the reader to [23, 48, 51].

**Lemma 3.1.** (i) If \( p > 1 \) and \( w \in A_p(\mu) \), then there exists \( \varepsilon > 0 \) such that \( w \in A_{p-\varepsilon}(\mu) \).

(ii) Assume that \( p \geq 1 \), \( w \in A_p(\mu) \). There exists \( C > 0 \) such that for a ball \( B \) and a measurable set \( E \subseteq B \) we have
\[
\left( \frac{\mu(E)}{\mu(B)} \right)^p \leq C \frac{\mu_w(E)}{\mu_w(B)}.
\]

3.2. Weighted Hardy spaces. In this section the weight \( w \) belongs to \( A_\infty(\mu) \) and \( L, T_t \) are as in Subsection 1.1. Recall that the Lusin (area) function \( S_L f \) and Littlewood-Paley function \( G_L f \) are given by (2.1) and (2.2), respectively. We define \( H_{L,S,w}^1(X) \) as the completion of the set \( \{ f \in L^2(\mu) : S_L f \in L^1_w(\mu) \} \) in \( L^1_w(\mu) \), with respect to the norm
\[
\| f \|_{H_{L,S,w}^1(X)} = \| S_L f \|_{L^1_w(\mu)}.
\]
The space \( H_{L,G,w}^1(X) \) are defined analogously. There are several results on the weighted Hardy spaces \( H_{L,w}^1(X) \). In [31, Theorem 1.2] it is proved that for \( w \in A_\infty(\mu) \),
\[
H_{L,S,w}^1(X) = H_{L,G,w}^1(X).
\]

In [38, 44] the authors proved a weighted version of (2.4). Suppose that \( M \in \mathbb{N} \) and \( w \in A_p, 1 \leq p \leq 2 \), we say that a function \( a \in L^2(\mu) \) is called an \( (L, M, w) \)-atom if there exists a ball \( B = B(y_0, r) \) in \( X \) and a function \( b \) such that: \( b \in \text{Dom}(L^M) \) and for \( k = 0, 1, \ldots, M \) we have
\[
a = L^M b, \quad \text{supp } L^k b \subseteq B, \quad \| (r^2 L)^k b \|_{L^2_w(\mu)} \leq r^{2M} \mu_w(B)^{-1/2},
\]
c.f. (2.3) for non-weighted \( L \)-atoms.

**Definition 3.2.** Suppose that \( M \in \mathbb{N} \) and \( w \in A_p(\mu), 1 \leq p \leq 2 \). A function \( f \) belongs to \( H_{L,at,M,w}^1(X) \) if \( f = \sum_k \lambda_k a_k \), where \( a_k \) are \( (L, M, w) \)-atoms, \( \sum_k |\lambda_k| < \infty \) and the series converges in \( L^2(\mu) \). Define
\[
\| f \|_{H_{L,at,M,w}^1(X)} = \inf \sum_k |\lambda_k|,
\]
where $f \in H^1_{L,at,M,w}(X)$ and $f$ is decomposed as above. Then $H^1_{L,at,M,w}(X)$ is defined as a completion of $H^1_{L,at,M,w}(X)$ in the norm $\| \cdot \|_{H^1_{L,at,M,w}(X)}$.

Note that in Definition 3.2, we define $H^1_{L,at,M,w}(X)$ to be the normed space obtained by $L^2(\mu)$ convergence. This approach to the definition of adapted $H^1$ space was used in [28–30, 38, 44]. For further discussion see [2, p. 879 and Rem. 3.15], [30, Def. 3.4 and Theorem 3.5], and Section 8. The following result was proved in [38] in the case $X = \mathbb{R}^n$ and in [44, Theorem 1.10] when $X$ is a space of homogeneous type.

**Theorem 3.3.** Assume that $(X,d,\mu)$ is a doubling metric-measure space and $T_t = \exp(-tL)$ is a semigroup satisfying (UG). Assume that $w \in A_p(\mu)$, $1 \leq p \leq 2$, and $M \in \mathbb{N}$, $M > (p-1)n/2$, where $n$ is as in (1.4). Then

$$\|f\|_{H^1_{L,S,w}(X)} \simeq \|f\|_{H^1_{L,at,M,w}(X)}.$$  

Consequently, one may write $H^1_{L,at,w}(X)$ in place of $H^1_{L,at,M,w}(X)$ when $w \in A_p(\mu)$, $1 \leq p \leq 2$, and $M > (p-1)n/2$ as these spaces are all equivalent. Having in mind (3.2) and Theorem 3.3 we write

$$H^1_{L,w}(X) := H^1_{L,S,w}(X) = H^1_{L,G,w}(X) = H^1_{L,at,w}(X) := H^1_{L,at,M,w}(X)$$

for $M > (p-1)n/2$.

Next, for an operator $L$ related to a metric measure space $(X,d,\mu)$ and a function $h(x)$ satisfy (H1)–(H3), and we consider the semigroup $T_t$ corresponding to the measure $d\nu(x) = h^2(x)d\mu(x)$, as in Subsection 2.2. By (H2) the space $(X,d,\nu)$ satisfies the doubling condition. The inequalities (ULG$_h$) for $T_t$ are equivalent to (ULG) for $T_t$. Recall that, $T_t$ is a semigroup and its generator $\mathcal{L}$ is also self-adjoint, see Section 2.2. As in the above notation corresponding to the operator $L$ the spaces $H^1_{L,S,w}(X)$, $H^1_{L,G,w}(X)$, and $H^1_{L,at,w}(X)$ related to $\mathcal{L}$ are defined analogously and all these weighted Hardy spaces coincide, i.e.

$$H^1_{L,w}(X) = H^1_{L,S,w}(X) = H^1_{L,G,w}(X) = H^1_{L,at,w}(X) := H^1_{L,at,M,w}(X)$$

when $w \in A_p$, $1 < p \leq 2$ and $M \in \mathbb{N}$, $M > (p-1)n/2$.

### 3.3. Alternative atomic characterization with cancellation condition

We shall prove atomic decompositions for $H^1_{L,w}(X)$ with natural and simple atoms related to $\nu$, $w$, and $\mathcal{L}$-harmonic function $\varphi$. Recall that the existence of $\varphi$ satisfying (1.6) follows from (ULG), see [15, Sec. 2].

**Definition 3.4.** We call a function $a$ a $[\nu, \varphi, w]$-atom if there exists a ball $B$ such that:

- $\text{supp } a \subseteq B$,
- $\|a\|_{L^2(\nu)} \leq \nu_w(B)^{-1/2}$,
- $\int_B a(x)\varphi(x)d\nu(x) = 0$.

Then, by definition, a function $f$ belongs to the atomic Hardy space $H^1_{at}[\nu, \varphi, w]$ if $f = \sum k \lambda_k a_k$, where $a_k$ are $[\nu, \varphi, w]$-atoms, $\sum k |\lambda_k| < \infty$, and the series converges in
Moreover, for such representations the expression

\[ \|f\|_{H^1_w([\nu, \varphi, w])} = \inf \sum_k |\lambda_k| \]

defines a norm.

Observe that in Definition 3.4 the atoms satisfy \( \|a\|_{L^1_w(\nu)} \leq 1 \), so the series \( \sum_k \lambda_k a_k \) converges in \( L^1_w(\nu) \)-norm and a.e. Moreover, the space \( H^1_w([\nu, \varphi, w]) \) is a Banach space.

The main result of this section is the following theorem. It states atomic characterization of the Hardy space \( H^1_{L^1_w}(X) \). Later, we shall deduce Theorem A from Theorem 3.5 by using Doob’s transform.

**Theorem 3.5.** Assume that \( L \) satisfies (ULG) and \( \varphi \) is the associated bounded \( L \)-harmonic function, see (1.6) and (1.7). Let \( p_0 = (n + \delta) / n \), where \( \delta \) is from Proposition 2.6 and \( n \) is as in (1.4). If \( w \) is a weight in \( A_{p_0}(\nu) \), then

\[ H^1_{L^1_w}(X) = H^1_{at}[\nu, \varphi, w]. \]

**Proof.** Let \( \varphi \) be the harmonic function for \( L \), \( C^{-1} \leq \varphi(x) \leq C \), see Subsection 2.3.

**Proof of** \( H^1_{L^1_w}(X) \subseteq H^1_{at}[\nu, \varphi, w] \). Assume that \( f \in H^1_{L^1_w}(X) = H^1_{at,1,w}(X) \), see Subsection 3.2. For \( p_0 = 1 + \delta / n \) we have that \( (p_0 - 1)n / 2 = \delta / 2 < 1 \) so we can take \( M = 1 \) in Theorem 3.3. We can assume that \( f \) is in a dense subspace \( H^1_{at,1,w}(X) \), so that we have \( \lambda_k \) and \( (L, 1, w) \)-atoms \( a_k \) as in Definition 3.2 such that

\[ f = \sum_k \lambda_k a_k \]

(convergence in \( L^2(\nu) \) and in \( L^1_w(\nu) \) and a.e.) and

\[ \|f\|_{H^1_{L^1_w}(X)} \simeq \sum_k |\lambda_k|. \]

Observe that \( (L, 1, w) \)-atoms satisfy localization and size condition of Definition 3.4, so to prove that \( a_k \) are \([\nu, \varphi, w]\)-atoms we only need to show that

\[ \int_B a(x) \varphi(x) \, d\nu(x) = 0 \]

for \( a = a_k \). This will be enough since then \( \|f\|_{H^1_{at}[\nu, \varphi, w]} \leq C \|f\|_{H^1_{L^1_w}(X)} \) on a dense subset of \( H^1_{L^1_w}(X) \).

To prove (3.3), we follow the argument similar to [28, Lemma 9.1]. Recall that \( \varphi \) is bounded and \( T_t \) are uniformly bounded on \( L^p(\nu) \), \( 1 \leq p \leq \infty \). By the functional calculus we have

\[ (I + \mathcal{L})^{-1} = \int_0^{\infty} e^{-t} T_t dt \]

and, by (1.7),

\[ (I + \mathcal{L})^{-1} \varphi(x) = \varphi(x), \quad \text{for a.e. } x. \]
Using (3.4) twice,

\[
\int_B a(x) \varphi(x) \, d\nu(x) = \int_B a(x)(I + \mathcal{L})^{-1} \varphi(x) \, d\nu(x)
= \int_B (I + \mathcal{L})^{-1} \mathcal{L} b(x) \varphi(x) \, d\nu(x)
= \int_B (I + \mathcal{L})^{-1} (I + \mathcal{L}) b(x) \varphi(x) \, d\nu(x) - \int_B (I + \mathcal{L})^{-1} b(x) \varphi(x) \, d\nu(x)
= \int_B b(x) \varphi(x) \, d\nu(x) - \int_B b(x)(I + \mathcal{L})^{-1} \varphi(x) \, d\nu(x)
= 0.
\]

Let us notice that in the calculations above, we use that \( a \) and \( b \) have compact supports, \( \varphi \) is bounded, \( \mathcal{T}_t \) has the upper Gaussian estimates, and

\[
\|a\|_{L^1(\nu)} \leq \|a\|_{L^2(\nu)} \left( \int_B w^{-1} \right)^{1/2} \leq \|a\|_{L^2(\nu)} \frac{\nu(B)}{\nu_w(B)^{1/2}} < \infty.
\]

Here we have also used \( A_2(\nu) \) condition for \( w \). The same estimate holds for \( b \).

**Proof of** \( H_1^1[\nu, \varphi, w] \subseteq H_{\mathcal{T}_t}^1(X) \). First, let us show that for every \([\nu, \varphi, w]\)-atom \( a \), there is a constant \( C \) independent of \( a \) such that

\[
\|G_\mathcal{L} a\|_{L^1(\nu)} \leq C,
\]

where \( G_\mathcal{L} f(x) = \left( \int_0^\infty |t^2 \mathcal{L}_t f(x)|^2 \frac{dt}{t} \right)^{1/2} \).

Let \( a \) be a \([\nu, \varphi, w]\)-atom, so that \( \text{supp} \, a \subseteq B = B(y_0, r) \). Then, since \( w \in A_2(\nu) \) and \( G_\mathcal{L} \) is bounded on \( L^2_w(\nu) \),

\[
\|G_\mathcal{L} a\|_{L^1(2B, \nu)} \leq \nu_w(2B)^{1/2} \|G_\mathcal{L} a\|_{L^2_w(\nu)} \leq \nu_w(2B)^{1/2} \|a\|_{L^2_w(\nu)} \leq C.
\]

Let \( x \not\in 2B \) and \( y \in B \). Then, \( d(x, y) \simeq d(x, y_0) > r \). Let \( \mathcal{K}_{t^2} = t^2 \mathcal{T}_t \) be as in Proposition 2.6.

\[
G_\mathcal{L} a(x)^2 = \int_0^\infty \left| \int_X \mathcal{K}_{t^2}(x, y) a(y) \, d\nu(y) \right|^2 \frac{dt}{t}
= \int_r^\infty + \int_0^r =: E_1 + E_2.
\]

Observe that

\[
\nu(B(x, t))^{-1} = \frac{\nu(B(x, d(x, y_0)))}{\nu(B(x, t))} \nu(B(x, d(x, y_0)))^{-1}
\leq C \left( 1 + \frac{d(x, y_0)}{t} \right)^n \nu(B(x, d(x, y_0)))^{-1},
\]
where \( n > 0 \) is the doubling dimension, see (1.4). In \( E_1 \) we have \( t \leq r < d(x, y_0) \). Let \( \delta > 0 \) be as in (2.7). Using (2.8) and (3.7),

\[
E_1 \leq C \int_0^r \left( \int_B \nu(B(x, t))^{-1} \exp \left( -\frac{d(x, y)^2}{ct^2} \right) |a(y)| d\nu(y) \right)^2 \frac{dt}{t} \\
\leq C \nu(B(x, d(x, y_0)))^{-2} \|a\|_{L^1(\nu)}^2 \int_0^r \left( \frac{d(x, y_0)}{t} \right)^{2n} \exp \left( -\frac{d(x, y_0)^2}{ct^2} \right) \frac{dt}{t} \\
\leq C \nu(B(x, d(x, y_0)))^{-2} \|a\|_{L^1(\nu)}^2 \int_0^r \left( \frac{d(x, y_0)}{t} \right)^{-2\delta} \frac{dt}{t} \\
\leq C \frac{r^{2\delta}}{d(x, y_0)^{2\delta}} \nu(B(y_0, d(x, y_0)))^{-2} \|a\|_{L^1(\nu)}^2.
\]

Recall that \( \int a(x) \varphi(x) d\mu(x) = 0 \). For \( E_2 \) we note that \( d(y, y_0) < r \), so (2.7) and (3.7) yield

\[
E_2 \leq C \int_r^\infty \left| \int_B \left( \frac{K_{2\nu}(x, y)}{\varphi(y)} - \frac{K_{2\nu}(x, y_0)}{\varphi(y_0)} \right) a(y) \varphi(y) d\nu(y) \right|^2 \frac{dt}{t} \\
\leq C \int_r^\infty \left( \frac{r}{t} \right)^{2\delta} \nu(B(x, t))^{-2} \left[ \int_B \exp \left( -\frac{d(x, y)^2}{ct^2} \right) |a(y)| d\nu(y) \right]^2 \frac{dt}{t} \\
\leq C \nu^{2\delta}(B(x, d(x, y_0)))^{-2} \|a\|_{L^1(\nu)}^2 \int_0^\infty \left( 1 + \frac{d(x, y_0)}{t} \right)^{2n} \exp \left( -\frac{d(x, y_0)^2}{ct^2} \right) \frac{dt}{t^{1+2\delta}} \\
\leq C \frac{r^{2\delta}}{d(x, y_0)^{2\delta}} \nu(B(x, d(x, y_0)))^{-2} \|a\|_{L^1(\nu)}^2.
\]

Notice that \( \delta \leq 1 \) and \( n \geq 1 \) so \( p_0 = 1 + n/\delta \leq 2 \) and \( w \in A_2(\nu) \). By the Cauchy-Schwarz inequality we have

\[
\|a\|_{L^1(\nu)} \leq \|a\|_{L^2(\nu)} \left( \int_B w^{-1}(x) d\nu(x) \right)^{1/2} \leq C \frac{\nu(B)}{\nu_w(B)}.
\]

Summarizing the estimates above we arrive at

\[
G_a(x) \leq C \frac{r^\delta}{d(x, y_0)^\delta} \nu(B(x, d(x, y_0)))^{-1} \nu(B) \frac{\nu(B)}{\nu_w(B)}.
\]
Denote $S_j(B) = 2^{j+1}B \setminus 2^jB$. If $x \in S_j(B)$ then $\nu(B(x, d(x, y_0))) \simeq \nu(2^jB)$ and

$$
\|G_La\|_{L^1\nu(B)} = \sum_{j \geq 1} \int_{S_j(B)} G_La(x)w(x)\nu(x)
\leq \nu(B) \sum_{j \geq 1} \int_{S_j(B)} (2^j\nu)^{-\delta} w(x)\nu(x)
\leq \nu(B) \sum_{j \geq 1} 2^{-j\delta} \nu_w(2^jB) / \nu(2^jB)
\leq C \sum_{j \geq 1} 2^{-j\delta} \left( \frac{\nu(2^jB)}{\nu(B)} \right)^{p_1-1}
\leq C \sum_{j \geq 1} 2^{-j(\delta-n(p_1-1))} \leq C,
$$

whence (3.6) follows. Here we have used the doubling condition and Lemma 3.1(ii) for $w \in A_{p_1}(\nu)$, where $p_1 < p_0 = 1 + \delta/n$. Recall that $p_1 < p_0$ can be chosen by the self-improvement property of $A_{p_0}(\nu)$, see Lemma 3.1(i).

Now let $f \in H^1_{\nu}(\nu, \varphi, w)$ and there is a sequence $(\lambda_j)_j$ in $\ell^1$ and a sequence $(a_j)_j$ of $[\nu, \varphi, w]$-atoms such that $\sum_j \lambda_j a_j$ converges to $f$ in $L^1_{\nu}(\nu)$ with $\sum_j |\lambda_j| \leq 2\|f\|_{H^1_{\nu}[\nu, \varphi, w]}$.

So by (3.6) we have

$$
\left\| \sum_{j=1}^k \lambda_j a_j - \sum_{j=k+1}^l \lambda_j a_j \right\|_{H^1_{\nu}(X)} \leq \sum_{j=k+1}^l |\lambda_j| \|G_La_j\|_{L^1\nu(B)} \leq C \sum_{j=k+1}^l |\lambda_j|
$$

whenever $l > k > 0$. Then there exists $g \in H^1_{\nu}(X)$ such that $\sum_j \lambda_j a_j$ converges to $g$ in $H^1_{\nu}(X)$. By Theorem 8.2 from the Appendix we have that $H^1_{\nu}(X) = H^1_{\nu}(\nu, \varphi, w) \subseteq L^1(\nu, \varphi, w)$ and we have that $g \in H^1_{\nu}(X) \subseteq L^1_{\nu}(X)$.

Therefore, $f = g \in H^1_{\nu}(X)$ with

$$
\|f\|_{H^1_{\nu}(X)} \leq C \lim_{k \to \infty} \sum_{j=1}^k |\lambda_j| \|G_La_j\|_{L^1\nu(B)} \leq C \sum_{j=1}^\infty |\lambda_j| \leq C \|f\|_{H^1_{\nu}[\nu, \varphi, w]}
$$

so $H^1_{\nu}[\nu, \varphi, w] \subseteq H^1_{\nu}(X)$. The proof of Theorem 3.5 is complete.

**Remark 3.6.** Under the assumptions of Theorem 3.5 there exists $C > 0$ such that for every $[\nu, \varphi, w]$-atom $a$ we have

$$
\|M_La\|_{L^1\nu(B)} \leq C,
$$

where $C > 0$ is independent of $a$.

**Proof.** Denote $M_Lf(x) = \sup_{t > 0} |T_tf(x)|$. Similarly as before, it is enough to show $\|M_La\|_{L^1\nu(B)} \leq C$ with $C$ independent of $a$.

Let $a$ be a $[\nu, \varphi, w]$-atom, so that supp $a \subseteq B = B(y_0, r)$. Then, since $w \in A_2(\nu)$ and $M_L$ is bounded on $L^p_w(\nu),$

$$
\|M_La\|_{L^p_w(2B, \nu)} \leq \nu_w(2B)^{1/2} \|M_La\|_{L^1\nu(B)} \leq C\nu_w(2B)^{1/2} \|a\|_{L^1\nu} \leq C.
$$
Now, let \( x \not\in 2B \), so that \( d(x, y) \leq d(x, y_0) \geq r \) for \( y \in B \). We have

\[
M_L a(x) \leq \sup_{t > r^2} |T_t a(x)| + \sup_{t \leq r^2} |T_t a(x)| =: E_1 + E_2.
\]

Note that in \( E_1 \) we have \( t > r^2 \geq d(y, y_0)^2 \), so we can use (2.6). Hence, by Definition 3.4,

\[
E_1 = \sup_{t > r^2} \int_X \left( \frac{T_t(x, y)}{\varphi(y)} - \frac{T_t(x, y_0)}{\varphi(y_0)} \right) a(y) \varphi(y) d\nu(y) = C \sup_{t > r^2} \int_X \left( \frac{d(y, y_0)}{\sqrt{t}} \right)^\delta \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y_0)^2}{c^2 t} \right) |a(y)| d\nu(y)
\]

\[
\leq C r^\delta \|a\|_{L^1(\nu)} \sup_{t \leq r^2} \left( t^{-\delta/2} \nu(B(x, \sqrt{t}))^{-1} e^{-\frac{d(x, y_0)^2}{c^2 t}} \right) .
\]

For \( E_2 \) we use (ULG) for \( T_t(x, y) \) getting

\[
E_2 \leq C \sup_{t \leq r^2} \int_X \nu(B(x, \sqrt{t}))^{-1} \exp \left( -\frac{d(x, y)^2}{c^2 t} \right) |a(y)| d\nu(y)
\]

\[
\leq C r^\delta \|a\|_{L^1(\nu)} \sup_{t \leq r^2} \left( t^{-\delta/2} \nu(B(x, \sqrt{t}))^{-1} e^{-\frac{d(x, y_0)^2}{c^2 t}} \right) .
\]

By joining these estimates we arrive at

\[
E_1 + E_2 \leq C r^\delta \|a\|_{L^1(\nu)} \nu(B(x, d(x, y_0)))^{-1} \sup_{t > 0} \left( t^{-\delta/2} \nu(B(x, d(x, y_0))) e^{-\frac{d(x, y_0)^2}{c^2 t}} \right)
\]

\[
\leq C r^\delta \|a\|_{L^1(\nu)} \nu(B(x, d(x, y_0)))^{-1} \sup_{t > 0} \left( t^{-\delta/2} \left( 1 + \frac{d(x, y_0)}{\sqrt{t}} \right)^n e^{-\frac{d(x, y_0)^2}{c^2 t}} \right)
\]

\[
\leq C \left( \frac{r}{d(x, y_0)} \right)^\delta \nu(B(y_0, d(x, y_0)))^{-1} \|a\|_{L^1(\nu)} .
\]

The rest of the proof goes exactly as for the Littlewood-Paley operator \( G_L \), see (3.8).

\[ \square \]

Remark 3.7. Under the additional assumption \( w \in A_1(\nu) \) it is possible to strengthen Remark 3.6 and prove that \( H^1_{ad}[\nu, \varphi, w] \subseteq H^1_{L, max, w}(X) \). The proof is based on weak-type boundedness of \( M_L \) on the space \( L^1_w(\nu) \), see for example [28, Lemma 4.3].

The relation between \( H^1_{ad}[\nu, \varphi, w] \) and \( H^1_{L, max, w}(X) \) stated in Remarks 3.6 and 3.7 are not required in our main argument, so we will not investigate them further here.

4. PROOF OF THEOREM A

To prove Theorem A, we recall that \( T_t = \exp(-tL) \) and \( h \) is the \( L \)-harmonic function for which (ULG\( h \)) holds. As usual, denote

\[
d\nu(x) = h^2(x) d\mu(x),
\]

and

\[
T_t(x, y) = \frac{T_t(x, y)}{h(x)h(y)} .
\]
Notice that (H1) means that \( h(x) \) is harmonic for \( T_t \). As a consequence
\[
\int_X \mathcal{T}_t(x, y) \, d\nu(y) = 1, \quad x \in X,
\]
so \( \varphi \equiv 1 \) is the harmonic function for \( \mathcal{L} \), see Section 2.2.

**Proof of** \( H^1_{L}(X) \subset H^1_{ad}[\mu, h] \). Let \( p_0 = 1 + \delta n^{-1} \), where \( n \) is the dimension on the space of homogeneous type \((X, d, \nu)\) and \( \delta \) is the Hölder exponent for \( \mathcal{T}_t(x, y) \), see (2.6). Assume that \( f \in H^1_{L}(X) \) or, equivalently, \( \tilde{f} := h^{-1} f \in H^1_{L, h^{-1}}(X) \), see Section 2.2. From Theorem 3.5, we have that \( \tilde{f} = \sum \lambda_k \tilde{a}_k \), where \( \sum \lambda_k |\lambda_k| \approx \| \tilde{f} \|_{H^1_{L, h^{-1}}(X)} \) and there exist balls \( \tilde{B}_k \) such that:
\[
\text{supp } \tilde{a}_k \subseteq \tilde{B}_k, \quad \| \tilde{a}_k \|_{L^2_{h^{-1}}(\nu)} \leq \nu_{h^{-1}}(B_k)^{-1/2}, \quad \int \tilde{a}_k(x) \, d\nu(x) = 0.
\]
Then
\[
f = h \tilde{f} = \sum_k \lambda_k a_k,
\]
where \( a_k = h \tilde{a}_k \). Obviously, \( \text{supp } a_k \subseteq \tilde{B}_k \) and \( \int a_k(x) h(x) \, d\mu(x) = \int \tilde{a}_k(x) \, d\nu(x) = 0 \). Moreover,
\[
\| a_k \|_{L^2_{h^{-1}}(\mu)} = \| \tilde{a}_k \|_{L^2_{h^{-1}}(\nu)} \leq \nu_{h^{-1}}(B_k)^{-1/2} = \mu_h(B_k)^{-1/2}
\]
as desired.

**Proof of** \( H^1_{ad}[\mu, h] \subset H^1_{L}(X) \). Note that if \( a \) is an atom as in Theorem A, then for \( \tilde{a} = h^{-1} a \) we have
\[
\int_B \tilde{a}(x) \, d\nu(x) = 0, \quad \| \tilde{a}_k \|_{L^2_{h^{-1}}(\mu)} \leq \nu_{h^{-1}}(B_k)^{-1/2}.
\]
By Proposition 2.2 and Theorem 3.5,
\[
(4.1) \quad \| a \|_{H^1_L(X)} = \| h^{-1} a \|_{H^1_{L, h^{-1}}(X)} \leq C
\]
with a constant \( C \) independent of \( a \). We then follow an argument as in Theorem 3.5 to obtain that \( H^1_{ad}[\mu, h] \subset H^1_{L}(X) \). The proof of Theorem A is complete.

As a consequence of Theorem A, we have the following result.

**Corollary 4.1.** Assume that \( L \) and \( h \) satisfy all the assumptions of Theorem A. If \( f \) belongs to \( H^1_{L}(X) \) and, additionally, \( h \) is in \( L^\infty(X) \), then
\[
(4.2) \quad \int f(x) h(x) \, d\mu(x) = 0.
\]

**Proof.** Let \( T_t \) be the semigroup related to \( L \). By Theorem A we have \( f = \sum_k \lambda_k a_k \), where \( \sum_k |\lambda_k| < \infty \) and \( a_k(x) \) are \([\mu, h] \)-atoms. Define
\[
f_N(x) = \sum_{k=1}^N \lambda_k a_k(x).
\]
Obviously, \( f_N \to f \) in \( L^1(\mu) \) and \( \int_X f_N(x) h(x) \, d\mu(x) = 0 \) since \( f_N \in L^1(\mu) \) and \( h \in L^\infty \). It follows that
\[
\int_X f(x) h(x) \, d\mu(x) = 0.
\]
Remark 4.2. The assumption that \( h \) is bounded is necessary in Corollary 4.1. If \( h \) is unbounded then (4.2) does not need to hold (or even the integral is not well defined). See Section 6 for examples.

5. Proof of Theorem B

We start our discussion with the following lemma.

Lemma 5.1. If \( f \in H_{a\ell}^1[\mu, h] \) and \( g \in \text{BMO}[\mu, h] \), then the pairing \( \langle f, g \rangle \) can be defined and satisfies

\[
|\langle f, g \rangle| \leq C \| f \|_{H_{a\ell}^1[\mu, h]} \| g \|_{\text{BMO}[\mu, h]}.
\]

Proof. Let \( a \) be a \([\mu, h]\)-atom. Obviously, the integral \( \int a(x)g(x)d\mu(x) \) does not depend on \( c \) when \( g = g_1 + ch \). Moreover,

\[
\left| \int_X a(x)g(x)d\mu(x) \right| \leq \int_X |a(x)||g(x) - ch(x)|d\mu(x)
\]

\[
= \int_X |a(x)|h(x)^{-1/2}|g(x) - ch(x)|h(x)^{1/2}d\mu(x)
\]

\[
\leq \|a\|_{L^2_{h^{-1}}(\mu)} \left( \int_B |g(x) - ch(x)|^2h(x)d\mu(x) \right)^{1/2}
\]

\[
\leq C \|g\|_{\text{BMO}[\mu, h]}.
\]

Therefore, \( \int_X \sum_{j=1}^k \lambda_ja_j(x)g(x)d\mu(x) \) is a Cauchy sequence and we define the pairing \( \langle f, g \rangle \) as its limit for an arbitrary \( f \in H_{a\ell}^1[\mu, h] \).

Proof of Theorem B. By Lemma 5.1, it follows that if \( g \in \text{BMO}[\mu, h] \), then

\[
l_g(f) = \langle f, g \rangle
\]

is a linear bounded functional on \( H_{a\ell}^1[\mu, h] \) with norm at most \( C \|g\|_{\text{BMO}[\mu, h]} \).

On the other hand let \( l \) be a linear functional on \( H_{a\ell}^1[\mu, h] \). Without loss of generality we assume that \( \|l\|_{H_{a\ell}^1[\mu, h] \rightarrow \mathbb{C}} \leq 1 \). For fixed \( B \) let us define the Hilbert space

\[
H_B = \left\{ f \in L^2_{h^{-1}}(\mu|_B) : \int_B f(x)h(x)d\mu(x) = 0 \right\}.
\]

Obviously, if \( f \in H_B \), then \( f \in H_{a\ell}^1[\mu, h] \) with

\[
\|f\|_{H_{a\ell}^1[\mu, h]} \leq \mu_h(B)^{1/2} \|f\|_{L^2_{h^{-1}}(\mu)}
\]

Therefore, by the Riesz representation theorem there is \( \tilde{g}_B \) (defined up to \( ch^2(x)\chi_B(x) \)) such that

\[
l_B(f) = \int_B f(x)\tilde{g}_B(x)h^{-1}(x)\ d\mu(x)
\]

and

\[
\|\tilde{g}_B\|_{L^2_{h^{-1}}(\mu)} = \|l_B\|_{H_B \rightarrow \mathbb{C}} \leq \mu_h(B)^{1/2}.
\]

Let \( c_B \) be a constant chosen so that for the function \( \tilde{g}_B = \tilde{g}_B + c_Bh^2 \) we have \( \int_{B_0}\tilde{g}_Bd\mu = 0 \) on some fixed ball \( B_0 \). Take an increasing family \( B_0 \subseteq B_1 \subseteq \ldots \) of balls such that
$\bigcup_{n \in \mathbb{N}} B_n = X$. Since $\tilde{g}_{B_n}$ agrees with $\tilde{g}_{B_{n+1}}$ on $B_n$ we have that $\tilde{g}_{B_n} - \tilde{g}_{B_{n+1}} = c_n h^2$ on $B_n \supseteq B_0$. But the left hand side has zero integral on $B_0$, so $c_n = 0$. Define $g_B = h^{-1}\tilde{g}_B$ and

$$g(x) = h^{-1}(x) \lim_{n \to \infty} \tilde{g}_{B_n}(x).$$

Notice that the limit exists, and $g$ coincides with $g_B$ on a ball $B$. Finally,

$$\left( \frac{1}{\mu_h(B)} \int_B |g(x) - c_B h(x)|^2 h(x) d\mu(x) \right)^{\frac{1}{2}} = \left( \frac{1}{\mu_h(B)} \int_B |\tilde{g}_B(x)h^{-1}(x)|^2 h(x) d\mu(x) \right)^{\frac{1}{2}}$$

$$= \left( \frac{1}{\mu_h(B)} \left\| \tilde{g}_B \right\|_{L^2_B(\mu)}^2 \right)^{\frac{1}{2}} \leq C.$$

This proves that $g \in BMO[\mu, h]$ and $\|g\|_{BMO[\mu, h]} \leq C$. Also, $l(f) = \int_X f(x)g(x) d\mu(x)$ whenever $f$ is a finite combination of atoms. This ends the second part of the proof. □

6. Applications

As an illustration of our results we shall discuss several examples. Our main results, Theorems A and B can be applied to a wide range of operators such as: operators with Dirichlet boundary conditions on some domains in $\mathbb{R}^n$, Schrödinger operators, and Bessel operators.

For further references let us notice here that the assumption $h^{-1} \in A_p(\mu_{h^2})$ from Theorem A is equivalent to

$$\sup_B \frac{h(B)}{h^2(B)} \left( \frac{h^{2+\frac{r-1}{r}}(B)}{h^2(B)} \right)^{r-1} \leq C,$$

where $B$ is a ball, $p > 1$, and $h^q(B) = \mu_{h^q}(B)$ for $q > 0$.

6.1. Dirichlet Laplacian on $\Omega \subset \mathbb{R}^n$. One of the main motivations for the present paper is the description of the Hardy spaces corresponding to the Dirichlet Laplacian. We believe that the applications of our approach which we describe in Theorems 6.1 and 6.3 below provide an illuminating way of understanding the results concerning Dirichlet Laplace operator obtained by Auscher, Russ, Chang, Krantz and Stein in [4,8]. Assume that a domain $\Omega$ (an open and connected subset) in $\mathbb{R}^n$ is given. By $\Delta_\Omega$ we will denote the Laplace operator with the Dirichlet boundary conditions defined on $\Omega$. We shall consider two particular classes of the set $\Omega$ described in Examples 1.1 and 1.2 below.

6.1.1. Example 1.1: The domain above the graph of a bounded $C^{1,1}$ function. Assume that $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ is such that:

$$|\nabla \Gamma(x)| \leq C_1,$$

$$|\nabla \Gamma(x) - \nabla \Gamma(y)| \leq C_2 |x - y|$$

and consider the following domain in $\mathbb{R}^n, n \geq 3$,

$$\Omega = \left\{ x \in \mathbb{R}^n : x_n > \Gamma(x_1, ..., x_{n-1}) \right\},$$

i.e. the region above the graph of a bounded $C^{1,1}$ function $\Gamma$. One of the main applications of our results is the following theorem.
Theorem 6.1. Assume that $\Omega$ is as in (6.4), where $\Gamma$ is bounded and satisfies (6.2)–(6.3). Then there exists a function $h: \Omega \to (0, \infty)$, such that

$$h(x) \simeq \text{dist}(x, \Omega^c)$$

and the Hardy space $H_{\Delta_B}^1(\Omega)$ coincides with $H_{al}[\mu, h]$, where $\mu$ is the Lebesgue measure on $\Omega$.

Theorem 6.1 is a direct consequence of Theorem A, Proposition 2.3, and Lemma 6.2 below. Let us first recall that the estimates on the heat kernel $T_t(x, y)$ for the Dirichlet Laplacian $\Delta_{\Omega,D}$ on $\Omega$ were given in [47]. It was shown there that

$$T_t(x, y) \geq C \left( \frac{\rho(x)\rho(y)}{t} \wedge 1 \right) t^{-n/2} \exp\left(-\frac{|x-y|^2}{c_1 t}\right), \tag{6.5}$$

and

$$T_t(x, y) \leq C \left( \frac{\rho(x)\rho(y)}{t} \wedge 1 \right) t^{-n/2} \exp\left(-\frac{|x-y|^2}{c_2 t}\right), \tag{6.6}$$

uniformly for $x, y \in \Omega$ and $t > 0$. Here $a \wedge b = \min\{a, b\}$ and $\rho(x) = \text{dist}(x, \Omega^c)$ is the distance between $x$ and $\partial \Omega$.

Lemma 6.2. Let $\Omega$ be a domain given by a bounded $C^{1,1}$ function $\Gamma$, see (6.2)–(6.3). Then, the function $\tilde{h} = \rho$ defined on $\Omega$ satisfies (H2)–(H3). Moreover, for $p > 1$ we have $\tilde{h}^{-1} \in A_p(\nu)$, where $d\nu(x) = \tilde{h}^2(x)dx$ on $\Omega$.

Proof. From (6.5) and (6.6), we see that

$$\frac{C^{-1}}{t + \rho(x)\rho(y)} t^{-n/2} \exp\left(-\frac{|x-y|^2}{c_1 t}\right) \leq T_t(x, y) \leq C \frac{\rho(x)\rho(y)}{t} t^{-n/2} \exp\left(-\frac{|x-y|^2}{c_2 t}\right). \tag{6.7}$$

First, we claim that

$$\nu(B(x, r)) \simeq r^n(r + \rho(x))^2. \tag{6.8}$$

To prove the claim observe that for $y \in B(x, r)$ we have $\rho(y) \leq \rho(x) + r$, which immediately gives the upper bound. To see the lower bound recall that $C_1$ is the constant from (6.2) and consider the set

$$S = B(x, r) \cap \{ y \in \mathbb{R}^n : y_n \geq x_n + r/2 + C_1|x_1, \ldots, x_{n-1}| - (y_1, \ldots, y_{n-1})|\}.$$

Observe that $|S| \simeq r^n$ and $S \subseteq \Omega$. Moreover, if $y \in S$ then $\rho(y) \simeq (r + \rho(x))$ and, consequently we get the lower estimate from (6.8).

The doubling condition (H2) for $(\Omega, \rho^2(x)dx)$ follows from (6.8). Moreover, (H3) is a consequence of (6.7), (6.8) and the estimate

$$\frac{C^{-1}}{\max(\nu(B(x, \sqrt{t})), \nu(B(y, \sqrt{t})))} \leq \frac{t^{-n/2}}{t + \rho(x)\rho(y)} \leq \frac{C}{\min(\nu(B(x, \sqrt{t})), \nu(B(y, \sqrt{t})))}.$$

Similarly to (6.8) we can prove that for $q > 0$ we have

$$h^q(B(x, r)) \simeq r^n(r + h(x))^q, \tag{6.9}$$

where $h^q(B)$ is the measure with the density $h^q(x)dx$ on $\Omega$. Then $h^{-1} \in A_p(\nu)$ for all $p > 1$ follows from (6.9) and (6.1).
6.1.2. Example 1.2: Exterior domain outside bounded convex $C^{1,1}$ set. Assume that $\Omega \subset \mathbb{R}^n$ is the exterior of a $C^{1,1}$ compact convex domain, which means that $\Omega^c$ is convex, bounded, and its boundary is locally a $C^{1,1}$ function, see (6.2)–(6.3).

**Theorem 6.3.** Assume that $\Omega$ is the exterior of a $C^{1,1}$ compact convex domain with boundary that is locally $C^{1,1}$, see (6.2)–(6.3). Then there exists a function $h : \Omega \to (0, \infty)$, such that

$$h(x) \simeq \min(1, \text{dist}(x, \Omega^c))$$

and the Hardy space $H^1_{\Delta_n}(\Omega)$ coincides with $H^1_{\mu}([\mu, h])$, where $\mu$ is the Lebesgue measure on $\Omega$.

Theorem 6.3 is a direct consequence of Theorem A, Proposition 2.3, and Lemma 6.5 below. In [54] the following estimates were proven on the heat kernel $T_t(x,y)$ for the Dirichlet Laplacian:

$$T_t(x,y) \geq C_1 \left( \frac{\rho(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\rho(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-n/2} \exp \left( -\frac{|x-y|^2}{ct} \right), \quad (6.10)$$

and

$$T_t(x,y) \leq C_2 \left( \frac{\rho(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\rho(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-n/2} \exp \left( -\frac{|x-y|^2}{ct} \right), \quad (6.11)$$

uniformly for $x, y \in \Omega$ and $t > 0$, where $\rho(x) = \text{dist}(x, \Omega^c)$. For $x \in \Omega$ define

$$\tilde{h}(x) = \min(1, \rho(x)). \quad (6.12)$$

**Lemma 6.4.** On $\Omega$ denote the measure $\sigma_q$ that has the density $\tilde{h}^q(x)dx$. Then

$$\sigma_q(B(x,r)) \simeq \begin{cases} r^n & \text{if } r \geq 1 \text{ or } \rho(x) \geq 1 \\ r^n(r + \rho(x))^q & \text{if } r \leq 1 \text{ and } \rho(x) \leq 1 \end{cases}.$$ 

In particular, for $\nu := \sigma_2$ we have

$$\nu(B(x,r)) \simeq \begin{cases} r^n & \text{if } r \geq 1 \text{ or } \rho(x) \geq 1 \\ r^n(r + \rho(x))^2 & \text{if } r \leq 1 \text{ and } \rho(x) \leq 1 \end{cases}.$$ 

**Sketch of the proof.** First, observe that since $\Omega^c$ is convex, then

$$|B(x,r)| = |\{y \in \Omega : d(x,y) < r\}| \simeq r^n.$$ 

Moreover, if $\rho(x) \geq 1$ or $r \geq 1$, then on substantial part (i.e. on the set with measure $\simeq r^n$) of the set $\Omega \cap B(x,r)$ the measure $\nu$ is just the Lebesgue measure. In the opposite case, i.e. $r \leq 1$, $\rho(x) \leq 1$ we are close to boundary and $\tilde{h}(y) \simeq \rho(y)$. Then, the lemma follows by considering two cases: $\rho(x) \geq 2r$ and $\rho(x) \leq 2r$. The details are left to the reader.

In particular, for $\nu := \sigma_2$ we have

$$\nu(B(x,r)) \simeq \begin{cases} r^n & \text{if } r \geq 1 \text{ or } \rho(x) \geq 1 \\ r^n(r + \rho(x))^2 & \text{if } r \leq 1 \text{ and } \rho(x) \leq 1 \end{cases}.$$ 

**Lemma 6.5.** The function $\tilde{h}$ from (6.12) satisfies (H2)–(H3). Moreover, if $d\nu(x) = \tilde{h}^2(x)dx$ on $\Omega$ then for any $p > 1$ we have $\tilde{h}^{-1} \in A_p(\nu)$.
Proof. Observe first, that from (6.1) and Lemma 6.4 we have that $\tilde{h}(x)$ satisfies (H2) and the $A_p$ condition. Now, we shall show (ULG) for $\tilde{h}(x)$. The estimates (H3) will follow from (6.10)–(6.11) provided that we prove

$$
\left(\frac{\rho(x) \wedge 1}{\rho(x) \wedge 1}(\rho(y) \wedge 1)\right)^{-n/2} \leq \frac{C}{\min(\nu(B(x, \sqrt{t})), \nu(B(y, \sqrt{t})))},
$$

and the conclusion of Theorem 1. Nevertheless, all these definitions are equivalent with either $\nu(B(x, \sqrt{t}))$ or $\nu(B(y, \sqrt{t}))$. Recall that the estimates on $\nu(B(x, r))$ are given in Lemma 6.4. Obviously, when $t \geq 1$ there is nothing to prove, so let us assume that $t \leq 1$ and denote

$$W = \left(\frac{\rho(x) \wedge 1}{\rho(x) \wedge 1}(\rho(y) \wedge 1)\right)^{-n/2}.$$

By symmetry we shall always consider $x, y$ such that $\rho(x) \leq \rho(y)$. We claim that

$$C^{-1}\nu(B(y, \sqrt{t}))^{-1} \leq W \leq C\nu(B(x, \sqrt{t}))^{-1}.$$

The claim follows by a careful analysis of the cases:

- $\sqrt{t} \leq 1 \leq \rho(x), \rho(y)$,
- $\sqrt{t} \leq \rho(x) \leq 1 \leq \rho(y)$,
- $\sqrt{t} \leq \rho(x) \leq \rho(y) \leq 1$,
- $\rho(x) \leq \sqrt{t} \leq 1 \leq \rho(y)$,
- $\rho(x) \leq \sqrt{t} \leq \rho(y) \leq 1$,
- $\rho(x) \leq \rho(y) \leq \sqrt{t} \leq 1$.

The proof of Lemma 6.5 will be finished when we prove the estimate (6.13) in each case. This follows easily from Lemma 6.4. Here we will only present one case and leave the others to the reader. Assume then that $\rho(x) \leq \sqrt{t} \leq \rho(y) \leq 1$. In this case we have

$$W \simeq \rho(y)^{-1}t^{-(n+1)/2}, \quad \nu(B(y, \sqrt{t}))^{-1} \simeq \rho(y)^{-2}t^{-n/2}, \quad \nu(B(x, \sqrt{t}))^{-1} \simeq t^{-(n+2)/2}.$$

and (6.13) follows. \quad \square

Remark 6.6. When $\Omega$ is the upper-half space $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, it follows by the reflection method, see for example [50, (6) p. 57], that the heat kernel $T_t(x, y)$ related to Dirichlet Laplacian $\Delta_{\mathbb{R}^n_+}$ on $\mathbb{R}^n_+$ satisfies

$$T_t(x, y) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x'-y'|^2}{4t}}\left(e^{-\frac{|x_n-y_n|^2}{4t}} - e^{-\frac{|x_n+y_n|^2}{4t}}\right),$$

for $n \geq 2$. In this case, the function $h(x)$ from Theorem 6.1 equals $x_n$, see [27, p. 6].

Notice that in for $\Omega = \mathbb{R}^n_+$ as in Remark 6.6 the conclusion of Theorem A describes the Hardy space by different atoms to the ones known from [4, 8]. Here all the atoms are as in Definition 1.2 with $h(x) = x_n$, whereas in [4, 8] the atoms were different, see [8, Proposition 1.5] and [4, Theorem 1]. Nevertheless, all these definitions are equivalent that is they give different descriptions of the same space.
Remark 6.7. When $\Omega$ is the space $\mathbb{R}^n \setminus \overline{B}(0,1) = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 > 1\}$, it is known that $h(x) = \log|x|$ if $n = 2$ and $h(x) = 1 - |x|^{-n+2}$ if $n > 2$, see [27, p. 6].

Remark 6.8. Note that the statements of Theorems 6.1 and 6.3 essentially coincide saying that $H^1_{\Delta_n}(\Omega) = H^1_{al}[\mu, h]$, where $h$ is the positive harmonic function equal to zero on the boundary of $\Omega$. Note however, that in Theorem 6.1 the function $h$ is unbounded whereas $h \in L^\infty(\Omega)$ in Theorem 6.3. Hence in these two settings the nature of $H^1_{al}[\mu, h]$ is different in a way described in Corollary 4.1 and Remark 4.2.

6.2. Schrödinger operators. Consider $X = \mathbb{R}^n$ with the Lebesgue measure and the Schrödinger operators

$$L_V = -\Delta + V,$$

where $V \geq 0$ and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Since we assume $V \geq 0$ then by the Feynmann-Kac formula we always have (UG). However, the semigroup kernel can be much smaller than classical heat kernel due to the influence of the potential. The Hardy spaces for $H^1_{L_V}(\mathbb{R}^n)$ were intensively studied, see e.g. [14, 18–20, 28]. It appears that geometric conditions on atoms depend heavily on the dimension $n$ and size of the potential $V$. Let us recall two examples.

6.2.1. Example 2.1: Potentials from a Kato class. Let $n \geq 3$. Then it is known that $T_t = \exp(-tL_V)$ satisfies (ULG) if and only if $V$ is such that

$$\Delta^{-1}V \in L^\infty(\mathbb{R}^n),$$

see [20] for details. Then a harmonic function $\varphi_V$ for $L_V$ is given by the formula

$$\varphi_V(x) = \lim_{t \to \infty} \int_{\mathbb{R}^n} T_t(x,y) \, dy$$

and satisfies (1.6). In this case for $f \in H^1_{L_V}(\mathbb{R}^n)$ we always have $\int f(x)\varphi_V(x)dx = 0$. For details see [15, 20, 21].

6.2.2. Example 2.2: Inverse square potential. Consider the inverse square potential $V(x) = \gamma|x|^{-2}$ on $\mathbb{R}^n$ with $\gamma > 0$ and $n \geq 3$. This operator was studied in several papers, see for example [32, 35, 37, 40]. Specifically, the Hardy space related to this operator was studied in [14, 18]. The space $H^1_{L_V}(\mathbb{R}^n)$ has local character in the sense that atoms are either classical atoms or local atoms of the type $|Q|^{-1}\chi_Q$ for some family of cubes $Q$, see [18]. Obviously, for $f \in H^1_{L_V}(\mathbb{R}^n)$ there cannot be a general cancellation condition like in (4.2) with any nontrivial function $h$.

However, as we shall see, there is also another approach to atomic decompositions for $H^1_{L_V}(\mathbb{R}^n)$. Consider the function

$$h_V(x) = |x|^\tau,$$

where

$$\tau = \frac{\sqrt{(n-2)^2 + 4\gamma} - (n-2)}{2} > 0.$$ 

It appears that $h_V$ is strictly related to the analysis of $T_t = \exp(-tL_V)$, see [32].

Lemma 6.9. The function $h_V$ satisfies (H2)–(H3).
we have the following.\[(6.14)\]
\[\sigma_\beta(B(x,r)) = r^n(|x| + r)^\beta.\]

The doubling condition for \(d\sigma_{2r}(x) = h_\nu(x)^2\,dx\) follows easily from (6.14). The semigroup related to \(L_\nu\) satisfies \((\text{ULG}_n)\), see for example [32, Theorem 1.2]. See also [37,40]. \(\square\)

**Lemma 6.10.** For \(p > 1\) we have \(h^{-1}_\nu \in A_p(\nu)\), where \(d\nu(x) = h_\nu(x)^2\,dx\) on \(\mathbb{R}^n\).

**Proof.** It is enough to verify the condition (6.1) and this follows from (6.14). \(\square\)

**Corollary 6.11.** Let \(n \geq 3\), \(V(x) = \gamma|x|^{-2}, h_\nu(x) = |x|^\tau\), where \(\tau = (\sqrt{(n - 2)^2 + 4\gamma} - (n - 2))/2 > 0\). Then the spaces \(H^1_{L_\nu}(\mathbb{R}^n)\) and \(H^1_{d\nu}[\mu, h_\nu]\) coincide and have equivalent norms. Here \(\mu\) is the Lebesgue measure on \(\mathbb{R}^n\).

6.3. Bessel operators.

For \(\alpha > -1\) and \(\alpha \neq 1\) on \(X = (0, \infty)\) we consider the Euclidean distance and the measure \(d\mu(x) = x^\alpha\,dx\). The Bessel differential operator is given by

\[L_B f(x) = -f''(x) - \frac{\alpha}{x} f'(x), \quad x > 0.\]

Observe that a function \(h\) satisfies \(L_B h = 0\) if

\[h_B(x) = C_1 + C_2 x^{1-\alpha}.\]

6.3.1. **Example 3.1: Dirichlet Laplacian on \((0, \infty)\).** Let us start with very basic example: \(\alpha = 0\), \(X = (0, \infty)\), and \(L_{(0, \infty)} f = -f''\) with Dirichlet boundary condition at \(x = 0\). The semigroup generated by this operator has the integral kernel

\[T_t(x,y) = (4\pi t)^{-1/2} \left(\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right),\]

where \(x,y,t > 0\). Obviously, the space \(H^1_{L_{(0, \infty)}}(X)\) is well studied, see e.g. [4,8] (one could also use the results from [33] and [34] to get atomic and Riesz transform characterizations of \(H^1_{L_{(0, \infty)}}(X)\)). In particular, \(H^1_{L_{(0, \infty)}}(X)\) can be described by atomic decompositions, where atoms are either classical atoms on \((0, \infty)\) or local atoms of the type \(a(x) = |I_m|^{-1} \chi_{I_m}(x), I_m = (2^m/2, 2^{m+1}/2), m \in \mathbb{Z}\).

On the other hand, our results provide a new atomic description of \(H^1_{L_{(0, \infty)}}(X)\). The (unbounded) harmonic function for \(L_{(0, \infty)}\) is simply

\[h(x) = x.\]

Let \(\nu\) be the measure on \((0, \infty)\) with the density \(x^2\,dx\). One can easily check that

\[T_t(x,y) = \frac{T_t(x,y)}{h(x)h(y)} \simeq \nu(B(\sqrt{x}y, \sqrt{t}))^{-1} \exp\left(-\frac{(x-y)^2}{4t}\right)\]

\[\simeq \nu(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{(x-y)^2}{ct}\right),\]

so (H3) holds. It is also easy to verify that (H1)–(H2) hold and \(x^{-1} \in A_p(\nu)\) for every \(p > 1\). As a result of Theorem A we have the following.
Corollary 6.12. If a function $f$ belongs to $H^1_{L([0,\infty),D)}(X)$, then there exist: $\lambda_k, a_k$, and intervals $B_k$, such that $f = \sum_k \lambda_k a_k$, $\sum_k |\lambda_k| \simeq \|f\|_{H^1_{L([0,\infty),D)}(X)}$ and $a_k$ are atoms that satisfy:

- $\text{supp } a_k \subseteq B_k$,
- $\left( \int_{B_k} |a_k(x)|^2 \frac{dx}{x} \right)^{1/2} \leq \left( \int_{B_k} x \, dx \right)^{-1/2}$
- $\int_{B_k} a_k(x) \, dx = 0$.

In other words, an atomic Hardy space with two types of atoms: global (with cancellations) and local (without cancellations) can be described in a different, more uniform way, where all the atoms have cancellation condition, but w.r.t. a different, unbounded harmonic function. In Appendix we provide a sketch of a direct proof of the equality of these two atomic spaces.

6.3.2. Example 3.2: Bessel operator on $(0, \infty)$ with Neumann boundary condition at $x = 0$. Probably the most natural case is to consider $L_{B,(0,\infty),N}$, i.e. the operator $L_B$ with Neumann boundary condition at $x = 0$. If $n := \alpha + 1 \in \mathbb{N}$ then the analysis of $L_{B,(0,\infty),N}$ is equivalent to the analysis of the radial part of the Laplacian $-\Delta$ on $\mathbb{R}^n$. However, $L_{B,(0,\infty),N}$ can be considered also for non-integer $\alpha$’s. For the results on the Hardy spaces related to $L_{B,(0,\infty),N}$ the reader is referred to [5, 16, 41]. Let us only mention briefly, that

$$h_{B,(0,\infty),N}(x) = 1$$

is the harmonic function and all atoms have cancellation of the form: $\int a(x) \, d\mu(x) = 0$. In other words, the Hardy space $H^1_{(L_{B,(0,\infty),N})}(X)$ is the geometric Hardy space in the sense of [12].

6.3.3. Example 3.3: Bessel operator on $(0, \infty)$ with Dirichlet boundary condition at $x = 0$. Now, consider $L_{B,(0,\infty),D}$, i.e. the space with the Dirichlet boundary condition at $x = 0$. This example coincides with Example 3.2 if $\alpha > 1$. However, for $\alpha \in (-1,1)$ the function

$$h_{B,(0,\infty),D}(x) = x^{1-\alpha}$$

is unbounded and harmonic for $L_{B,(0,\infty),D}$. The crucial estimates (H3) follow from [27, Theorem 5.11]. The rest of assumptions of Theorem A is a direct calculation.

6.3.4. Example 3.4: Bessel operator on $(1, \infty)$ with Dirichlet boundary condition at $x = 1$. Let $\alpha > -1$, $\alpha \neq 1$, and $L_{B,(1,\infty),D}$ be the Bessel operator with Dirichlet boundary condition at $x = 1$. In the case $n := \alpha + 1 \in \mathbb{N}$ one can think about Brownian motion on $\mathbb{R}^n \setminus B(0,1)$ killed when entering unit ball. Here

$$h(x) = |1 - x^{1-\alpha}|$$

is the harmonic function. Observe that for $\alpha \in (-1,1)$ the function $h$ is unbounded, whereas for $\alpha > 1$ we have bounded $h$, but $\lim_{x \to 1^+} h(x) = 0$. Similarly as in the previous example, the estimates (H3) follow from [27, Theorem 5.11] and the rest of the assumptions of Theorem A can be verified directly.
7. Equivalence of different atomic decompositions

In some examples our results give a new atomic description of \( H^1_{L,X,D}(X) \) even for the operators, for which another simple atomic description was known before. Let us explain this phenomena a bit in the simple example of Dirichlet Laplacian on \((0,\infty)\), see Subsection 6.3.1.

Let \( X = (0,\infty) \) be a space equipped with the Lebesgue measure and denote \( I_m = (2^m, 2^{m+1}) \), \( m \in \mathbb{Z} \). Consider the Dirichlet Laplacian \( L = -\Delta_{X,D} \) on \( X \). It is known, see [8] and [4], that if \( f \in H^1_{L,X,D}(X) \), then \( f = \sum_k \lambda_k a_k \), where \( \sum_k |\lambda_k| \approx \|f\|_{H^1_{L,X,D}(X)} \) and \( a_k \) are either:

- \( \alpha_1 \)-atoms: classical atoms on \((0,\infty)\), i.e. for \( a \) there exists an interval \( B \) such that:
  
  \[
  \text{supp } a \subseteq B, \quad \|a\|_{L^2(X)} \leq |B|^{-1/2}, \quad \int a(x) \, dx = 0
  \]
  
  or

- \( \alpha_2 \)-atoms: local atoms of the form \( a = |I_m|^{-1} \chi_{I_m} \), \( m \in \mathbb{Z} \).

On the other hand \( h(x) = x \) is \( L \)-harmonic and \( h(0) = 0 \). One can easily prove that the kernel of the semigroup \( \exp(-tL_{X,D}) \) satisfies (ULG\(_h\)) and the measure \( x^2 \, dx \) is doubling on \((0,\infty)\). Moreover, using (6.1), on easily verifies that \( h^{-1} \in A_p(x^2 \, dx) \) for any \( p > 1 \). As a consequence, from Theorem A we deduce that each \( f \in H^1_{L,X,D}(X) \) can be written as \( f = \sum_k \lambda_k b_k \), where \( \sum_k |\lambda_k| \approx \|f\|_{H^1_{L,X,D}(X)} \) and \( b_k \) are

- \( \beta \)-atoms: for \( a \) there exists a ball \( B \) such that:
  
  \[
  \text{supp } b \subseteq B \subseteq I_k, \quad \|b\|_{L^2(\mathbb{R}^d)} \leq \left( \int_B x \, dx \right)^{-1/2}, \quad \int xb(x) \, dx = 0.
  \]

At a first glance it may be surprising, that \( H^1_{L,X,D}(X) \) has these two different atomic decompositions. However, recall that since \( h(x) = x \) is unbounded, we cannot say that \( \int f(x) x \, dx = 0 \) even if \( f = \sum_k \lambda_k b_k \) and \( \int b_k(x) x \, dx = 0 \) for each \( k \), see Corollary 4.1 and Remark 4.2. The purpose of the following lemma is to show that \( \alpha_1 \)-atoms and \( \alpha_2 \)-atoms can be decomposed into \( \beta \)-atoms and vice versa. However, we shall not consider arbitrary atoms, but, for simplicity of the presentation, we shall assume that the support of every atom considered is already contained in some dyadic interval \( I_m \).

Lemma 7.1. There exists \( C > 0 \) such that:

(a) if \( a \) is an \( \alpha_2 \)-atom supported in \( I_m, \ m \in \mathbb{Z} \), then there exist: \( \beta \)-atoms \((b_k)_{k \geq 0}\)
and numbers \((\lambda_k)_{k \geq 0}\) such that \( a = \sum_{k=0}^{\infty} \lambda_k b_k \) and \( \sum_{k=0}^{\infty} |\lambda_k| \leq C \),

(b) if \( a \) is an \( \alpha_1 \)-atom supported in \( B \subseteq I_m, \ m \in \mathbb{Z} \), then there exist: \( N \in \mathbb{N} \),
\( \beta \)-atoms \((b_k)_{k \geq 0}^{N}, \alpha_2 \)-atom \( a_{N+1} \), and numbers \((\lambda_k)_{k=0}^{N+1}\) such that \( a = \sum_{k=0}^{N} \lambda_k b_k + \lambda_{N+1} a_{N+1} \) and \( \sum_{k=0}^{N+1} |\lambda_k| \leq C \).
(c) if $b$ is a $\beta$-atom supported in $B \subseteq I_m$, $m \in \mathbb{Z}$, then there exist: $N \in \mathbb{N}$, $\alpha_1$-atoms $(a_k)^N_{k=0}$, $\alpha_2$-atom $a_{N+1}$, and numbers $(\lambda_k)^{N+1}_{k=0}$ such that $b = \sum_{k=0}^{N+1} \lambda_k a_k$ and
\[
\sum_{k=0}^{N+1} |\lambda_k| \leq C.
\]

Proof. To prove (a) consider $a = |I_m|^{-1} \chi_{I_m}$ for some $m \in \mathbb{Z}$. Write
\[
a(x) = \sum_{k=0}^{\infty} 2^{-k} b_k(x)
\]
where
\[
2^{-k} b_k(x) = \tau_k \chi_{I_{m+k}}(x) - \tau_{k+1} \chi_{I_{m+k+1}}(x), \quad k = 0, 1, \ldots.
\]

Fix $\tau_0 = |I_m|^{-1} = 2^{-m}$. Recursively, we define
\[
\tau_{k+1} = \tau_k \int_{I_{k+1}} x \, dx = \tau_k \left( \frac{(2^{k+1})^2 - (2^k)^2}{2} \right) = \frac{\tau_k}{4}
\]
so that $\int b_k(x) x \, dx = 0$ for $k = 0, 1, \ldots$ and $\tau_k = 2^{-m} 4^{-k}$. Observe that $\text{supp} b_k \subseteq I_{m+k} \cup I_{m+k+1}$ and
\[
\|b_k\|_{L^2} \simeq C 2^k |\tau_k| \left( \int_{I_{m+k} \cup I_{m+k+1}} \frac{dx}{x} \right)^{1/2} \leq C 2^k 2^{-m} 4^{-k} \simeq C 2^{-m-k}
\]
\[
\simeq \left( \int_{I_{m+k} \cup I_{m+k+1}} x \, dx \right)^{-1/2}.
\]

Therefore, $C^{-1} b_k$ are $\beta$-atoms and (a) is proved.

To prove (b) assume that $a$ is an $\alpha_1$-atom supported in $B \subseteq I_m$ with some $m$. Fix a sequence of intervals $B = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_N = I_m$, where $|Q_{k+1}|/|Q_k| \leq 2$ and $2^N \simeq |I_m|/|B|$. Write
\[
a(x) = \sum_{k=0}^{N} \lambda_k b_k(x) + \lambda_{N+1} a_{N+1}(x)
\]
where
\[
\lambda_k = 2^{k-m} |B|, \quad k = 0, \ldots, N + 1,
\]
\[
\lambda_0 b_0(x) = a(x) - \tau_0 \chi_{Q_0}(x),
\]
\[
\lambda_k b_k(x) = \tau_{k-1} \chi_{Q_{k-1}} - \tau_k \chi_{Q_k}(x), \quad k = 1, \ldots, N
\]
\[
\lambda_{N+1} a_{N+1} = \tau_N \chi_{Q_N}
\]
and $\tau_k$’s will be specified later on. Observe first that $\sum_{k=0}^{N+1} |\lambda_k| \simeq 2^{N-m} |B| \simeq C$. Let $y_0$ be the center of $B \subseteq I_m$ and recall that $\int a(x) \, dx = 0$. Choose $\tau_k$ so that $\int x b_k(x) \, dx = 0$ for $k = 0, \ldots, N$. In particular
\[
\tau_0 = \left( \int_B x \, dx \right)^{-1} \int a(x) x \, dx
\]
\[
|\tau_0| = \left( \int_B x \, dx \right)^{-1} \left| \int a(x)(x - y_0) \, dx \right| \leq (2^m |B|)^{-1} |B| \int |a(x)| \, dx \leq 2^{-m}
\]
and, for \( k = 1, \ldots, N \),
\[
\tau_k = \tau_{k-1} \frac{\int_{Q_{k-1}} x \, dx}{\int_{Q_k} x \, dx},
\]
\[
|\tau_k| \leq |\tau_{k-1}|.
\]

What is left is to check that \( a_{N+1}/C \) is an \( \alpha_2 \)-atom and \( a_k/C \) are \( \beta \)-atoms for \( k = 0, \ldots, N \). It is clear that \( a_{N+1} = \lambda_{N+1}^{-1} \tau_N \chi_{I_m} \) and \( |\lambda_{N+1}^{-1} \tau_N| \leq C 2^{N-m} |B| |\tau_0| \leq C |I_m|^{-1} \). Moreover, for \( k = 0, \ldots, N \) we have: \( \supp b_k \subseteq Q_k \cup Q_{k-1} \) \( (Q_{-1} = \emptyset) \), \( \int x b_k(x) \, dx = 0 \) (by the choice of \( \tau_k \)), and \[
\| b_k \|_{L^2(dx/x)} \leq C |\tau_k| \lambda_k^{-1} \left( \int_{Q_k \cup Q_{k-1}} 2^{-m} \, dx \right)^{1/2} \leq 2^{-m} 2^{m-k} |B|^{-1} (2^{k-m} |B|)^{1/2}
\]
\[
\simeq (2^{k+m} |B|)^{-1/2} \simeq \left( \int_{Q_k \cup Q_{k-1}} x \, dx \right)^{-1/2}.
\]
This shows that \( b_k/C \) are \( \beta \)-atoms for \( k = 0, \ldots, N \) and the proof of \( (b) \) is complete. The proof of \( (c) \) is essentially the same as the case of \( (b) \). We leave the details to the interested reader. \( \square \)

8. APPENDIX: EMBEDDING \( H^1_{L,S,w}(X) \subseteq L^1_w(X) \)

In this section we consider a self-adjoint operator \( L \) as in Section 3.2, i.e. \( (X, d, \mu) \) is a doubling metric-measure space, see (1.4), \( \mu(X) = \infty \), and the kernel \( T_t(x, y) \) corresponding to the semigroup \( \exp(-tL) \) satisfies \( (U) \).

Recall that \( H^1_{L,w}(X) = H^1_{L,S,w}(X) = H^1_{L,at,M,w}(X) \), see Section 3.2 and Theorem 3.3. Our main goal here is to prove that \( H^1_{L,w}(X) \) embeds continuously in \( L^1_w(X) \) as a Banach space. Notice that although the atoms in Definition 3.2 satisfy \( \|a\|_{L^1_w(\mu)} \leq C \) the inclusion \( H^1_{L,w}(X) \subseteq L^1_w(X) \) is not trivial, since in Definition 3.2 the Hardy space is defined by an abstract completion of the space \( \mathbb{H}^1_{L,at,M,w}(X) \) and this space is defined by \( L^2(\mu) \)-convergence of the series \( \sum_k \lambda_k a_k \). Nevertheless, using methods from Auscher, McIntosh and Morris [2] we shall prove in this appendix the following theorem that is needed in the proof of Theorem 3.5.

**Theorem 8.1.** Let \( w \in A_2(\mu) \). Then \( H^1_{L,S,w}(X) \subseteq L^1_w(\mu) \) and
\[
\|f\|_{L^1_w(\mu)} \leq C \|f\|_{H^1_{L,S,w}(X)}
\]
with \( C \) independent of \( f \).

To prove Theorem 8.1 recall the following definition. Assume that \( E_1 \) is a normed space contained in a Banach space \( E_2 \) with \( \|x\|_{E_2} \leq C \|x\|_{E_1} \) for \( x \in E_1 \). We say that \( E_1 \) has a completion in \( E_2 \) if there exists a Banach space \( E_3 \) such that: \( E_1 \subseteq E_3 \subseteq E_2 \), \( E_1 \) is dense in \( E_3 \), \( \|x\|_{E_1} = \|x\|_{E_3} \) for \( x \in E_1 \), and \( \|x\|_{E_2} \leq C \|x\|_{E_3} \) for \( x \in E_3 \). For more comments see [2, p. 870-871]. We shall prove the following.

**Proposition 8.2.** Let \( w \in A_2(\mu) \). Then the completion \( H^1_{L,S,w}(X) \) of \( \{f \in L^2(\mu) : \|S_L f\|_{L^1_w(\mu)} < \infty\} \) in \( L^1_w(\mu) \) exists.
Notice that Theorem 8.1 follows directly from Proposition 8.2 and the fact that the completion of a normed space is unique. The rest of this section is devoted to prove Proposition 8.2.

First, we recall the notion of the weighted tent spaces on $X$ from [44], see also [10, 42]. For a measurable function $F$ defined on $X \times (0, \infty)$ consider

$$
\mathcal{A}F(x) := \left( \int_0^\infty \int_{d(y,x)<t} |F(y,t)|^2 \frac{d\mu(y)}{\mu(B(x,t))} \frac{dt}{t} \right)^{1/2}.
$$

For $p \in [1, \infty)$ and $w \in A_\infty(\mu)$, the tent space $T^p_w(X)$ is defined as the space of measurable functions $F$ on $X \times (0, \infty)$ for which $\mathcal{A}F \in L^p_w(\mu)$. This is equipped with

$$
\|F\|_{T^p_w(X)} := \|\mathcal{A}F\|_{L^p_w(\mu)}.
$$

For simplicity we will often write $T^p(X)$ in place of $T^p_w(X)$ with $w = 1$. The tent space $T^\infty_w(X)$ is the Banach space of all $F$ satisfying

$$
\|F\|_{T^\infty_w(X)} := \sup \sup_{x \in X} \left( \frac{1}{\mu_w(B)} \int_{T(B)} |F(y,t)|^2 \frac{\mu(B(y,t))}{\mu_w(B(y,t))} d\mu(y) \frac{dt}{t} \right)^{1/2},
$$

where $\mathbb{B}(x)$ denotes the set of all balls $B \subseteq X$ with the property that $x \in B$, and the tent $T(B) = \{ (y, t) \in X \times (0, \infty) : d(y, X \setminus B) \geq t \}$. Using a similar argument as in [6, Lemma 2.5], where the case $X = \mathbb{R}^n$ is considered, it can be verified that for $w \in A_\infty(\mu)$, the pairing

$$
\langle F, G \rangle := \int_{X \times (0, \infty)} F(x, t) G(x, t) \frac{d\mu(x)dt}{t}
$$

realizes $T^\infty_w(X)$ as equivalent with the Banach space dual of $T^1_w(X)$.

Recall that a measurable function $A$ on $X \times (0, \infty)$ is said to be a $T^1_w$-atom if there exists a ball $B \subseteq X$ such that $A$ is supported in the tent $T(B)$ and

$$
\|A\|_{T^1_w(X)} \leq \mu_w(B)^{-1/2}.
$$

It has been proved in [44, Theorem 1.8], see also [44, Remark 3.3], that every $F \in T^1_w(X)$ has an atomic decomposition.

**Lemma 8.3.** Let $w \in A_\infty(\mu)$. For every element $F \in T^1_w(X)$ there exist a sequence $\{\lambda_j\}_{j=1}^\infty$ and a sequence of $T^1_w$-atoms $\{A_j\}_{j=1}^\infty$ such that $\sum |\lambda_j| < \infty$ and

$$
F = \sum_j \lambda_j A_j \quad \text{in } T^1_w(X) \quad \text{and a.e. in } X \times (0, \infty).
$$

Moreover,

$$
\sum_j |\lambda_j| \approx \|F\|_{T^1_w(X)},
$$

where the implicit constants depend only on the homogeneous space properties of $X$.

Finally, if $F \in T^1_w(X) \cap T^2(X)$, then the decomposition (8.1) also converges in $T^2(X)$.

Let $K_{\cos(t\sqrt{\lambda})}(x, y)$ denote the integral kernel of the operator $\cos(t\sqrt{\mathcal{L}})$. It is known, see for example [43], that there exists a constant $c_0 > 0$ such that for every $t > 0$,

$$
\text{supp}K_{\cos(t\sqrt{\lambda})} \subset D_t := \{(x, y) \in X \times X : d(x, y) \leq c_0 t\}.
$$
Let $M \geq 1$, and let $\varphi \in C^\infty_0(\mathbb{R})$ be even, supp $\varphi \subset (-c_0^{-1}, c_0^{-1})$ with $\varphi \geq c > 0$ on $(-1/(2c_0), 1/(2c_0))$. Let $\Phi$ denote the Fourier transform of $\varphi$. Set $\Psi(x) := x^{2(M+1)}\Phi(x)$, $x \in \mathbb{R}$. Consider the operator $\pi_{\varphi,L} : T^2(X) \to L^2(\mu)$, given by
\[
\pi_{\varphi,L}(F)(x) := \int_0^\infty \Psi(t\sqrt{L})(F(\cdot, t))(x)\frac{dt}{t},
\]
where the improper integral converges weakly in $L^2(\mu)$. The bound
\[
\|\pi_{\varphi,L}F\|_{L^2(\mu)} \leq C_M \|F\|_{T^2(X)}, \quad M \geq 0,
\]
follows readily by duality and the $L^2(\mu)$ quadratic estimate. We have the following analogue of the well-known argument of [44, Proposition 4.10].

**Lemma 8.4.** Let $w \in A_2(\mu)$. Suppose that $A$ is a $T^1_w$-atom associated to a ball $B \subset X$ (or more precisely, to its tent $T(B)$). Then for every $M \geq 1$, there is a uniform constant $C_M$ such that $C_M^{-1} \pi_{\varphi,L}(A)$ is an $(L, M, w)$-atom with the concentric ball $2B$.

With the above preliminary results, we now start to prove Theorem 8.2 by adapting an argument as in [2, Lemma 3.9 and Theorem 3.10].

**Proof of Proposition 8.2.** Let $w \in A_2(\mu)$ and set $E^1_{L,S,w}(X) = \{f \in L^2(\mu) : \|S_Lf\|_{L^1(\mu)} < \infty\}$.

**Step 1:** $E^1_{L,S,w}(X) \subseteq L^1_w(\mu)$ and $\|f\|_{L^1_w(\mu)} \leq C \|S_Lf\|_{L^1_w(\mu)}$ for $f \in E^1_{L,S,w}(X)$.

Let $f \in E^1_{L,S,w}(X)$ and set
\[
F(\cdot, t) := t^2L e^{-t^2L} f.
\]
We note that $F \in T^2(X) \cap T^1_w(X)$, by the definition of $E^1_{L,S,w}(X)$. Therefore, by Lemma 8.3, we have that
\[
F = \sum_j \lambda_j A_j,
\]
where each $A_j$ is a $T^1_w$-atom, the sum converges in both $T^2(X)$ and $T^1_w(X)$, and
\[
\sum_j |\lambda_j| \leq C \|F\|_{T^2(X)} = C \|S_Lf\|_{L^1_w(\mu)}.
\]
Also, by $L^2$-functional calculus, see [39], and using that $f \in L^2(\mu)$, we have the “Calderón reproducing formula"
\[
f(x) = c_\Psi \int_0^\infty \Psi(t\sqrt{L})(t^2L e^{-t^2L} f)(x)\frac{dt}{t} = c_\Psi \pi_{\varphi,L}(F) = c_\Psi \sum_j \lambda_j \pi_{\varphi,L}(A_j),
\]
where the last sum converges in $L^2(\mu)$ and $E^1_{L,S,w}(X)$. Moreover, by Lemma 8.4, for every $M \geq 1$, we have that up to multiplication by some harmless constant $C_M$, each $a_j := c_\Psi \pi_{\varphi,L}(A_j)$ is an $(L, M, w)$-atom, and so $\|a_j\|_{L^1_w(\mu)} \leq C$ with a constant $C > 0$ independent of $j$. Consequently, $\sum_j \lambda_j \pi_{\varphi,L}(A_j)$ converges to $\tilde{f}$ in $L^1_w(\mu)$. We must have $f = \tilde{f} \in L^1_w(\mu)$ since $L^1_w(\mu)$ and $L^2(\mu)$ are embedded in $L^{1/2}_{loc}(\mu)$ (notice that $w \in A_2(\mu)$ implies $\int_B w^{-1} \, dt < \infty$) and so $\sum_j \lambda_j \pi_{\varphi,L}(A_j)$ converges to $f$ in $L^1_w(\mu)$ with
\[
\|f\|_{L^1_w(\mu)} = \lim_{n \to \infty} \left\| \sum_j \lambda_j \pi_{\varphi,L}(A_j) \right\|_{L^1_w(\mu)} \leq C \sum_j |\lambda_j| \leq C \|S_Lf\|_{L^1_w(\mu)} = C \|f\|_{E^1_{L,S,w}(X)},
\]
and so \( E_{L,S,w}^1(X) \subseteq L_w^1(\mu) \).

**Step 2:** The completion of \( E_{L,S,w}^1(X) \) in \( L_w^1(\mu) \) exists.

We shall use the following proposition that states necessary and sufficient condition for the existence of a completion inside a given Banach space, see [2, Proposition 2.2].

**Lemma 8.5.** Let \( E_1 \) and a normed space and suppose that \( E_1 \subseteq E_2 \) for some Banach space \( E_2 \), so the identity \( I : E_1 \to E_2 \) is complete. It is enough to consider \( E_1 \) for the existence of a completion inside a given Banach space, see [2, Proposition 2.2].

(i) the completion of \( E_1 \) in \( E_2 \) exists;

(ii) for each Cauchy sequence \((x_n)\) in \( E_1 \) that converges to 0 in \( E_2 \), it follows that \((x_n)\) converges to 0 in \( E_1 \).

The proof of the above lemma is a simple functional analysis argument. To complete the proof of Proposition 8.2 we consider \( E_1 = E_{L,S,w}^1(X) \) and \( E_2 = L_w^1(\mu) \). From (ii) of Lemma 8.5 it is enough to consider \((f_n)\), that is a Cauchy sequence in \( E_{L,S,w}^1(X) \) and converges to 0 in \( L_w^1(\mu) \). We claim that \((f_n)\) converges to 0 in \( E_{L,S,w}^1(X) \). For all \( n \) we have that

\[
\|f_n\|_{E_{L,S,w}^1(X)} = \|S_Lf_n\|_{L_w^1(\mu)} = \|t^2Le^{-t^2L}f_n\|_{T_w^1(X)}
\]

and, since \((f_n)\) is Cauchy sequence in \( E_{L,S,w}^1(X) \), there exists \( U \) in \( T_w^1(X) \) such that \( t^2Le^{-t^2L}f_n \) converges to \( U \) in \( T_w^1(X) \).

Denote

\[
\mathcal{M} = \left\{ F \in T^2(X) \cap T_w^\infty(X) : \frac{\pi_L F}{w} \in L^\infty(\mu) \right\},
\]

where \( \pi_L F(x) := \int_0^\infty t^2Le^{-t^2L}F dt/t \). Following the method from [2] we claim that \( \mathcal{M} \) is weak-star dense in \( T_w^\infty(X) \). Indeed, one can show that for \( F \in T_w^\infty(X) \) the functions \( F \cdot \chi_{A_n} \) belongs to \( \mathcal{M} \), where \( A_n = [1/n, n] \times \{ x \in X : d(x, x_0) < n, w^{-1}(x) \leq n \} \), c.f. [2, Remark 3.11 and p.882]. Using the duality, we have that for any \( F \in \mathcal{M} \),

\[
|\langle U, F \rangle| \leq \|U - t^2Le^{-t^2L}f_n\| + \|t^2Le^{-t^2L}f_n\| \leq C \|U - t^2Le^{-t^2L}f_n\|_{T_w^1} \|F\|_{T_w^\infty} + \|f_n\|_{L_w^1(\mu)} \|w^{-1} \cdot \pi_L F\|_{L^\infty(\mu)}.
\]

Moreover, since \( \|w^{-1} \cdot \pi_L F\|_{L^\infty(\mu)} < \infty \) and \( \|F\|_{T_w^\infty} < \infty \), the preceding convergence results imply that

\[
\langle U, F \rangle = 0, \quad \forall F \in \mathcal{M}.
\]

Then, since \( U \in T_w^1(X) \) and \( \mathcal{M} \) is weak-star dense in \( T_w^\infty(X) \), it follows that \( \langle U, F \rangle = 0 \) for all \( F \in T_w^\infty(X) \), hence \( U = 0 \) and \((f_n)\) converges to 0 in \( E_{L,S,w}^1(X) \) as claimed. This proves that the completion \( H_{L,S,w}^1(X) \) of \( E_{L,S,w}^1(X) \) in \( L_w^1(\mu) \) exists. Hence, the proof of Proposition 8.2 is complete.

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