Generalizing Darcy’s law for filtration radial flows through porous media

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Abstract. We study the filtration process in porous media and compare the filtration coefficients for two possible geometries of flows: cylindrical and radial ones. Solving balance equations for impurity concentration in the simplest axially-symmetric stationary case, one finds the radial filtration process more effective. Therefore, we restrict our attention to the radial filters with non-homogeneous grain filling. First, we describe the transverse diffusion effect and then suggest the generalization of the Darcy’s filtration law, its dynamical origin being stressed. Using the perturbation method, we find the structure of the Stokes stream function for some particular choices of the porosity.

1 Introduction
We study the hydrodynamics of flow in a porous medium modelling the grain filling in filters [1 - 9]. Using the lattice approximation, we derive the structure of the fluid current and obtain the transverse diffusion coefficient $D$, which proves to be proportional to the diameter $d$ of the grains as constituents of the medium [10 - 14]. However, we begin with some remarks concerning the geometry of flows in porous media.

The objective of the current study was threefold. First, we intended to choose the appropriate geometry for the filtration process, the goal being the improvement of its effectiveness. Beside this, on the base of the lattice approximation, we intended to take into account a very important transverse diffusion effect, which was typical for fluid flows in porous media. Finally, we planned to find some arguments in favour of generalizing the standard Darcy filtration law.

1.1 Comparing filtration coefficients for two flow geometries
Let us first estimate the so-called filtration coefficient, which is given by the ratio of the impurity concentration $n$ for the outgoing flow to that for the incoming one. Taking into account that $n$ satisfies the balance equation of the form

$$\partial_t n + \text{div}(n\vec{u}) = -\beta n,$$

where $\vec{u}$ stands for the velocity of the fluid and $\beta$ – for the absorption coefficient of the porous medium, one gets for the stationary process the relation:

$$\text{div}(n\vec{u}) = -\beta n. \quad (1)$$

For the rude estimation of the solution to (1) in the case of a cylindrical filter, one can approximate the velocity by its constant vertical part $u_0$ and obtain the following solution for the concentration $n(z)$:

$$n(z) = n_0 \exp\left(-\frac{\beta z}{u_0}\right). \quad (2)$$
Consider now a radial filter, with the flow velocity being radial. Taking into account that for the radial flow the velocity appears to be inverse proportional to the radial coordinate $\rho$:

$$u(\rho) = u_0 \frac{\rho_0}{\rho}, \quad \rho \geq \rho_0,$$

where $u_0\rho_0$ stands for the constant of integration, one easily finds the evident form of the stationary balance equation (1):

$$\text{div}(n\tilde{u}) \approx \frac{1}{\rho} \partial_{\rho}(\rho nu) = u_0 \frac{\rho_0}{\rho} \partial_{\rho} n = -\beta n. \quad (3)$$

Its solution has the form

$$n(\rho) = n_0 \exp \left[ \frac{\beta}{2u_0u_0} (\rho_0^2 - \rho^2) \right]. \quad (4)$$

Comparing the formulas (2) and (4), giving the expressions for the filtration coefficient, one can see that for $\rho >> \rho_0$ the radial filter proves to be more effective than the cylindrical one. Taking into account this conclusion, from now on we concentrate on the radial flow in porous media.

2 Materials and Methods

We consider as a material the porous medium in the filter and to model the grain filling we use the lattice approximation method for representing the stationary mass balance equation. In the limiting case of the zero lattice spacing we obtain the differential equation of the second order corresponding to the transverse diffusion. To solve the hydrodynamic equations of motion for axially-symmetric radial flows, we introduce the Stokes stream function. However, for solving the full system of equations we use the perturbation method and exploit the corresponding two small parameters. The first one concerns the ratio of the grain diameter $d$ to the radius $a$ of the filter, the second one being related with the small deformation of the Darcy filtration coefficient.

3 Results

3.1 Lattice approximation for mass balance equation in porous media: transverse diffusion

We use the cylindrical coordinates $\rho, \varphi, z$, with $\varphi$ being the azimuth angle. Let us first consider the discrete variant of the mass conservation equation and number the lattice vertices by the indices $i, j$ (transverse to the flow) and $k$ (along the flow), the corresponding cylindrical coordinates being $\varphi, z$, and $\rho$, respectively. Let us denote the local radial stream of the fluid by

$$G_{ijk} = \Delta S_k u_{ijk},$$

where $u_{ijk}$ stands for the velocity of the flow and $\Delta S_k$ is the area of the gap between the grains. It means that

$$\Delta S_k = \rho_k \Delta \varphi \Delta z S_k,$$

where $S_k$ denotes the porosity of the medium. Therefore, the local conservation law reads
\[ G_{ijk} = r_{k-1} G_{ijk-1} + p_{k-1} (G_{i-1,jk-1} + G_{i+1,jk-1}) + q_{k-1} (G_{ij-1,k-1} + G_{ij+1,k-1}), \]

where the branching coefficients \( r, p, q \) are introduced. Supposing that \( p = 0 \) due to the axial symmetry of the flow, let us write the mass conservation equation

\[ \sum_j G_{jk} = \sum_j G_{jk-1}, \]  

which immediately implies the constraint on the branching coefficients:

\[ r_k + 2q_k = 1. \]  

Identifying now the lattice spacing with the diameter \( d \) of the grain, it can be proved through (5) and (6) that in the continuous limit the following differential equation is valid:

\[ \text{div}(rS \vec{u}) = d \partial_z (S q \partial_z u), \]  

One can rewrite the equation (7) in the form of the stationary conservation law:

\[ \text{div} \vec{j} = 0, \]

where the components of the current \( \vec{j} \) in cylindrical coordinates read:

\[ j_z = S(\rho) u - D(\rho) S(\rho) \partial_z w, \quad j_\rho = r(\rho) S(\rho) w, \]  

and the transverse diffusion coefficient is introduced:

\[ D(\rho) = q(\rho) d(\rho). \]  

Here we use the following denotations for the components of the velocity:

\[ u_\rho = w, \quad u_z = u, \]

the density of the fluid being unity.

Now it is worth - while to stress that the effect of the transverse diffusion in porous media is widely discussed in literature [11, 12].

3.2 The hydrodynamics of the radial flow in porous media: generalizing Darcy's law

To find the profiles of the velocity \( \vec{u} \) and the pressure \( P \), it is necessary to solve also the Euler equation, with the force density \( \vec{f} \) including the gravity acceleration \( \vec{g} \) and the Darcy force

\[ \vec{f}_D = -k_D \vec{u}. \]

In the first approximation, the Darcy coefficient \( k_D \) appears to be constant: \( k_D = k_0 = \text{const} \), but in general, it should be some function of the velocity and the pressure. We suggest a generalization of the Darcy law [7, 15 - 18] by including in \( k_D \) the invariant

\[ I = (\vec{u} \nabla) P \]  

(10)
via the simplest linear form:

\[ k_D = k_a + k \cdot I. \]

The well-known Darcy's law of filtration was found empirically in 1856 by the French engineer H. Darcy [7, 15] who specialized in constructing fountains. According to this law, the velocity of the fluid flow in a porous medium appears to be proportional to the pressure gradient and depends on the structure of grains as constituent elements of the medium. Later various modifications of the Darcy law were suggested, with the correction terms containing some degrees of the velocity, which was considered as a formal small parameter [16]. As an alternative to such an activity, we suggest a dynamical approach based on the structure of the fundamental Euler equation, which contains the velocity and the pressure:

\[(\bar{u} \nabla) \bar{u} + \nabla \bar{P} = \bar{g} - k_p \bar{u}.\] (11)

Some deviations from the standard Darcy law were observed in [20, 21]. We intend to analyze the dependence of the filtration process on the coefficient \(k^*\). First, we write down the mass balance equation (7) and the Euler equation (11) in cylindrical coordinates:

\[(w \partial_{\rho} + u \partial_{z})u + \partial \bar{P} = g - (k_0 + k^* I)u,\] (12)
\[(w \partial_{\rho} + u \partial_{z})w + \partial \bar{P} = - (k_0 + k^* I)w,\] (13)

where the invariant \(I\) reads

\[ I = (w \partial_{\rho} + u \partial_{z}) P. \]

Multiplying the equations (12) and (13) by \(u\) and \(w\) respectively and adding the results, one easily finds for the invariant \(I\) the following expression:

\[ I = \left[1 + k^* (u^2 + w^2)\right]^{1/2} \left[gu - k_0 u^2 - \frac{1}{2} (w \partial_{\rho} + u \partial_{z})(u^2 + w^2)\right]. \] (14)

Using (14), one can eliminate the pressure \(P\) between the equations (12) and (13), the resulting equation for \(u\) and \(w\) being:

\[ \partial_{\rho} \left[(w \partial_{\rho} + u \partial_{z})u + (k_0 + k^* I)u\right] = \partial_{z} \left[(w \partial_{\rho} + u \partial_{z})w + (k_0 + k^* I)w\right]. \] (15)

Let us now introduce the stream function \(\Psi\) through the substitution

\[ \rho S r w = -\partial_{z} \Psi, \quad \rho S (u - D \partial_{z} w) = \partial_{\rho} \Psi, \] (16)

the equation (7) being satisfied identically. Taking into account that

\[ u << w, \] (17)

one can use the perturbation method by putting

\[ \Psi = \psi_0 + \psi, \quad u = \bar{u}, \quad w = \bar{w}(\rho) + \bar{w}, \quad \psi_0 \gg \psi, \] (18)

where it is supposed that
\( \rho S \nabla \vec{w} = -\partial_z \psi_0, \quad \partial_z \psi_0 = 0. \) \hspace{1cm} (19)

In view of (19), one finds

\[ \psi_0 = -C_0 \tilde{z}, \quad \vec{w} = C_0 (\rho S r)^{-1}, \] \hspace{1cm} (20)

with \( C_0 \) being the constant of integration.

Due to the definition (16) of the stream function, one gets

\[ \tilde{w} = -(\rho S r)^{-1} \partial_z \psi, \quad u = (\rho S)^{-1} \partial_r \psi + D \partial_z \vec{w}. \] \hspace{1cm} (21)

Inserting (21) into (15), one obtains the final equation for the stream function \( \psi \), which can be linearized with respect to \( u \ll w \) and solved by separating variables:

\[ \psi = R(\rho)Z(z). \] \hspace{1cm} (22)

### 3.3 Structure of the stream function: perturbation approach

Now we suppose that the radial filter in question has the external radius \( a \gg d \), the internal radius \( \rho_0 \) and several plates (layers) of the height \( 2\ell \). Therefore, the boundary condition for the fluid velocity reads:

\[ u(\rho, z = \pm \ell) = 0, \quad -\ell \leq z \leq \ell, \quad \rho_0 \leq \rho \leq a. \] \hspace{1cm} (23)

It is worth while to notice that in our case the following two small parameters can be introduced:

\[ \varepsilon = k^* w^2 \ll 1, \quad \mu = d / a \ll 1. \]

This fact permits one, in the first approximation, to neglect the terms in (15) containing the gravity acceleration \( g \) and the diffusion coefficient \( D \). Taking into account that the asymmetry of \( \tilde{w} \) in \( z \) depends on \( g \), one can suppose that

\[ \tilde{w}(-z) = \tilde{w}(z), \quad \psi(-z) = -\psi(z). \] \hspace{1cm} (24)

In view of the boundary condition (23), the property (24) implies the simplest structure of \( Z(z) \):

\[ Z(z) = \sin (\lambda_n z), \quad \lambda_n = n \pi / \ell, \quad n = 1, 2, 3, ... . \] \hspace{1cm} (25)

Inserting (25), (21) and (22) into (15), one obtains in the first approximation the following equation for the function \( R(\rho) \):

\[ \left[ \frac{\tilde{w}}{\rho S} \left( \frac{R'}{\rho S} \right) + \frac{k_0 R'}{\rho S} \right] = \lambda_n^2 \left[ \frac{\tilde{w} R}{\rho S r} \right] + \lambda_n \left[ \frac{k_0 R}{\rho S r} \right]. \] \hspace{1cm} (26)

The equation (26) can be simplified if one supposes that the function \( r(\rho) \) weakly depends on \( \rho \):

\[ r(\rho) \approx r_0 = \text{const.} \]
Using the denotations:

\[ \lambda_0 = \frac{\lambda_n^2}{\rho}, \quad \sigma_0 = k_0r_0/C_0, \quad p = \rho S \]  

(27)

and introducing the new dependent variable

\[ X(\rho) = (R'/p) - \lambda_0 R / p, \]  

(28)

one derives from (26) the following equation for \( X(\rho) \):

\[ (X / p)' + \sigma_0 X = 0. \]  

(29)

The equation (29) has the evident solution:

\[ X(\rho) = \rho Y_0 \exp \left( -\sigma_0 \rho \right). \]  

(30)

with \( Y_0 \) being the constant of integration. Inserting (30) into (28), one finds the non-homogeneous equation for the function \( R(\rho) \), which can be solved in some particular cases.

### 3.3.1 The case of constant porosity \( S = S_0 = \text{const} \)

In this case \( p = S_0 \rho \) and from (30) one gets

\[ X = \rho Y_0 \exp \left( -\gamma (\rho^2 - \rho_0^2) \right), \quad \gamma = \sigma_0 S_0 / 2. \]  

(31)

Inserting (31) into (28), one finds the following equation for \( R(\rho) \):

\[ \rho^{-1}(\rho^{-1}R')' - \lambda_0 \rho^{-2} R = S_0^2 Y_0 \exp \left[ -\gamma (\rho^2 - \rho_0^2) \right]. \]  

(32)

Taking into account that the homogeneous part of the equation (32) admits the solution

\[ R_0 = \rho K_1(k_n \rho), \quad k_n^2 = \lambda_0 = \lambda_n^2 / r_0, \]

where \( K_1 \) is the Macdonald function of the first order (or the modified Bessel function) [19], one can represent the radial function \( R(\rho) = R_n(\rho) \) in the form:

\[ R_n(\rho) = \rho K_1(k_n \rho) U_n(\rho). \]  

(33)

Using the substitution (33) and taking into account the equation for \( K_n(z) \):

\[ K_n'' + z^{-1} K_n' = (1 + n^2 z^{-2}) K_n, \]

one can deduce from (32) the corresponding equation for the function \( U_n(\rho) \):

\[ (\rho K_1^2 U_n') = Y_0 S_0^2 \rho^2 K_1 \exp \left[ -\gamma (\rho^2 - \rho_0^2) \right]. \]  

(34)
It should be noted that solution to (34) can be found by simple double integration. Finally, the resulting stream function takes the form
\[
\psi(\rho, z) = \sum_{n=1}^{\infty} U_n(\rho) \rho K_1(k_n \rho) \sin(n \pi z),
\]
(35)
and the velocity components can be easily found through (21).

3.3.2 The case of the porosity \( S = S_1 / \rho \), \( S_1 = \text{const} \)

In this case \( p = \rho S = S_1 \) and the radial function \( R(\rho) \) satisfies the equation
\[
R^\prime - \lambda_0 R = Y_0 \exp[-\sigma_0 S_1 (\rho - \rho_0)].
\]
(36)
The corresponding substitution reads
\[
R_n(\rho) = U_n(\rho) \exp(-k_n \rho),
\]
and the equation for \( U_n(\rho) \) takes the simple form:
\[
\left[U_n^\prime \exp(-2k_n \rho)\right] = Y_0 \exp(-k_n \rho) \exp[-\sigma_0 S_1 (\rho - \rho_0)].
\]

3.3.3 The case of the porosity \( S = S_2 / \rho^2 \), \( S_2 = \text{const} \)

In this case \( p = \rho S = S_2 / \rho \) and the equation (29) has the following solution:
\[
X(\rho) = X_0(\rho_0 / \rho)^{\sigma+1}, \quad \sigma = \sigma_0 S_2, \quad X_0 = \text{const}.
\]
(37)
Inserting (37) into (28), one finds the equation for \( R_n(\rho) \):
\[
\rho R_n^\prime + R_n^\prime - k_n^2 \rho R_n = S_2 X_0(\rho_0 / \rho)^{\sigma+1}.
\]
(38)
The equation (38) can be solved through the substitution
\[
R_n = U_n(\xi) K_0(\xi), \quad \xi = k_n \rho,
\]
(39)
where \( K_0 \) stands for the Macdonald function of the zero order and \( U_n(\xi) \) satisfies the equation
\[
(\xi K_0^2 U_n) = X_0 k_n^\sigma K_0(\rho_0 / \xi)^{\sigma+1}.
\]
(40)
The solution \( U_n \) to the equation (40) can be found by simple double integration.

4 Discussion

For analyzing the structure of the stationary axially-symmetric mass balance equation we used the lattice approximation method and introduced the porosity \( S \) and the branching coefficients \( r, p, q \). In
the continuous limit we obtained the differential equation of the second order describing the transverse diffusion effect, the corresponding diffusion coefficient (9) being proportional to the vertical branching coefficient $q$ and the diameter of the grain $d$.

The next step concerned generalizing the Darcy force in the Euler equation through including the natural invariant (10) into the Darcy coefficient. The special structure of the Darcy force permits one to exclude the pressure from the Euler equation and obtain the main equation (15) for the velocity components. The equation (15) can be solved via the perturbation method, if one neglects the vertical velocity component. The structure of the corresponding Stokes stream function can be found by separating variables. In the first approximation, if one neglects the terms containing the gravity acceleration, the Stokes function has the following remarkable property: it appears to be an odd function with respect to the vertical coordinate $z$. It should be noticed that the Stokes function can be found analytically or expressed through the Macdonald functions for some special choices of the porosity.

Conclusions
1. We studied the hydrodynamics of radial flows in non-homogeneous porous media and showed that the equations of motion should be modified to take into account the effect of the transverse diffusion and that of generalizing the Darcy’s law.
2. Due to introducing the Stokes stream function the special perturbation approach was elaborated for solving the modified equations.
3. For some particular choices of the porosity structure $S(\rho)$ the analytical solutions for the stream function were obtained.

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