On torsors under elliptic curves and Serre’s pro-algebraic structures

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Abstract

Let \( O_K \) be a complete discrete valuation ring with algebraically closed residue field of positive characteristic and field of fractions \( K \). Let \( X_K \) be a torsor under an elliptic curve \( A_K \) over \( K \), \( X \) the proper minimal regular model of \( X_K \) over \( S := \text{Spec}(O_K) \), and \( J \) the identity component of the Néron model of \( \text{Pic}^0_{X_K/K} \). We study the canonical morphism \( q: \text{Pic}^0_{X/K} \to J \) which extends the natural isomorphism on generic fibres. We show that \( q \) is pro-algebraic in nature with a construction that recalls Serre’s work on local class field theory. Furthermore, we interpret our results in relation to Shafarevich’s duality theory for torsors under abelian varieties.

Keywords: Elliptic fibrations, models of curves, Shafarevich pairing, abelian varieties, Picard functor, pro-algebraic groups.

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Introduction

This paper concerns some local studies of torsors under an elliptic curve, or more generally, under an abelian variety. In the following, let \( O_K \) be a complete discrete valuation ring with field of fractions \( K \) and algebraically closed residue field \( k \) of positive characteristic \( p > 0 \). Let \( \pi \in O_K \) be a uniformizer of \( O_K \). Let \( S := \text{Spec}(O_K) \), denote by \( s \) its closed point and by \( i: \text{Spec}(k) \to S \) the usual closed immersion. Let \( A_K \) be an elliptic curve over \( K \) and \( X_K \) a torsor under \( A_K \) of order \( d \). Then \( X_K \) is a smooth projective \( K \)-curve whose jacobian is canonically isomorphic to \( A_K' \), the dual of \( A_K \) (see Lemma 2.1.1 (i)). Let \( X \) be the proper minimal regular model of \( X_K \) over \( S \). In general \( X \) is not cohomologically flat in dimension 0 over \( S \) (i.e., the canonical morphism \( k \to H^0(X_s, O_{X_s}) \) is not an isomorphism), and the relative Picard functor \( \text{Pic}^0_{X/K} \) is not representable, not even by an algebraic space. Nevertheless, the functor \( \text{Pic}^0_{X/S} \) is not very far from being representable. For example, let \( J \) denote the identity component of the Néron model of \( J_K := \text{Pic}^0_{X_K/K} \) over \( S \). In [21], 2.3.1, (see § 1.1 for a brief summary) it is shown that there exists an epimorphism of fppf-sheaves

\[
q: \text{Pic}^0_{X/S} \to J
\]
that extends the natural isomorphism on generic fibres. This morphism plays a very important role in a recent work of Liu, Lorenzini and Raynaud where, by considering the induced map Lie(q) between the Lie algebras of Pic^0_{X/S} and J, together with a result of T. Saito, the authors prove a beautiful result about the geometry of the scheme X, namely that the Kodaira type of the special fibre X_s of X is exactly d times the Kodaira type of the special fibre of the minimal regular S-model of the elliptic curve J_K ([16], Theorem 6.6).

One of the aims of this paper is to study the morphism q in order to reveal other interesting properties. More precisely, consider the surjective map induced by q on the S-sections (see the end of §1.1 for the surjectivity of q):

\[ q = q(S) : \text{Pic}^0(X) = \text{Pic}^0_{X/S}(S) \rightarrow J(S). \]

Since the gcd of the multiplicities of the irreducible components of X_s is d (see Lemma 2.1.3), one finds that \( D := \frac{1}{d}X_s \) is a well-defined effective divisor of X, whose sheaf of ideals \( I := \mathcal{O}_X(-D) \subset \mathcal{O}_X \) is invertible of order d, and generates the kernel of q. With the help of Greenberg realization functors, one can show that the morphism q is in fact pro-algebraic in nature, and we get a short exact sequence of pro-algebraic groups over k (see §1.2 for a review on pro-algebraic groups and Greenberg realizations):

\( 0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \text{Pic}^0(X) \xrightarrow{q} J(S) \rightarrow 0, \)

where the second map is given by sending \( \bar{1} \in \mathbb{Z}/d\mathbb{Z} \) to \( \mathcal{O}_X(D) \in \text{Pic}^0(X)(k) = \text{Pic}^0(X) \) (see Corollary 3.4.4 and [57]).

One of the main results of this paper shows that the morphism q in (1) can be thought as an analogue of the norm map studied by Serre in his work on local class field theory [25]. Let us first briefly review Serre’s results. Let \( L/K \) be a finite Galois extension of K with Galois group \( \Gamma_{L/K} \), and let \( U_L \) (respectively \( U_K \)) be the group of units of the valuation ring \( \mathcal{O}_L \) of \( L \) (respectively \( \mathcal{O}_K \) of \( K \)). Since the Brauer group \( Br(K) \) is trivial ([5], 8.1, p. 203), the usual norm map

\( N_{L/K} : U_L \rightarrow U_K \)

is surjective. By using the Greenberg realization functors, one can show that the morphism \( N_{L/K} \) is pro-algebraic in nature. This means that \( N_{L/K} \) is the morphism on k-rational points induced by a morphism of pro-algebraic groups:

\( N_{L/K} : U_L \rightarrow U_K. \)

On the other hand, the two pro-algebraic groups in (1) are naturally filtered: for each \( n \geq 1 \), one can define a pro-algebraic subgroup \( U^n_K \) of \( U_K \) whose group of k-rational points is given by the group \( U^n_K \) of n-units in \( K \):

\[ U^n_K(k) = U^n_K := \ker \left( U_K \rightarrow (\mathcal{O}_K/\pi^n\mathcal{O}_K)^\times \right). \]

Hence \( U_K \) has the following filtration by pro-algebraic subgroups:

\[ \ldots \subset U^{n+1}_K \subset U^n_K \subset \ldots \subset U^1_K \subset U^0_K := U_K. \]

Similarly, \( U_L \) has the following filtration

\[ \ldots \subset U^{n+1}_L \subset U^n_L \subset \ldots \subset U^1_L \subset U^0_L := U_L. \]
In [25], 3.4, Serre proved that these two filtrations are in fact compatible with respect to the norm map \( \psi \). More precisely, for each \( n \in \mathbb{Z}_{\geq 0} \), the map \( N_{L/K} \) in [1] sends \( U_{L}^{n} \) onto \( U_{K}^{n} \), where \( \psi = \psi_{L/K} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \) is the Herbrand function attached to the extension \( L/K \) and used to define the upper numbering of the ramification filtration of the Galois group \( \Gamma_{L/K} \).

In the situation of the present article, the two pro-algebraic groups \( P^{0}(X) \) and \( J(S) \) are also naturally filtered: for each \( n \geq 1 \), we define in § 3.1 a pro-algebraic subgroup \( P^{n}(S) \) of \( P^{0}(X) \) (respectively \( J^{[n]}(S) \) of \( J(S) \)) whose group of \( k \)-rational points is given by

\[
P^{n}(S)(k) \cong \ker (P^{0}(X) \to P^{0}(X_n)) , \quad \text{resp.} \quad J^{[n]}(S)(k) \cong \ker (J(S) \to J(S_n))
\]

where \( X_n \) is the closed subscheme of \( X \) defined by the ideal sheaf \( I^n \), and \( S_n := \text{Spec}(O_K/\pi^n O_K) \). The family of pro-algebraic subgroups \( \{ P^{n}(S) : n \geq 1 \} \) (respectively \( \{ J^{[n]}(S) : n \geq 1 \} \) gives a decreasing filtration of \( P^{0}(X) \) (respectively \( J(S) \)). In order to compare these two filtrations by means of the morphism \( \psi \) in [2], we define at the beginning of § 2.2 a numerical function \( \psi \), whose value at a positive integer \( n \) is the smallest integer \( m \geq 1 \) such that the \( O_K \)-module \( H^1(X_m, O_{X_m}) \) is of length \( n \). The function \( \psi \) appears to be an analogue of Serre’s Herbrand function mentioned above and was introduced in [25]. With this notation, the first main result of this paper can be stated as follows

**Theorem** (See § 3.4.3 and 3.4.4). The morphism \( \psi \) in [2] maps \( P^{[\psi(n)]}(S) \) onto \( J^{[n]}(S) \).

The proof of the above theorem requires a careful analysis of the length of the torsion part of the group \( H^1(X, O_X) \). This length is completely determined in [2.3.1 and 2.3.2].

A second goal of this paper is to study the short exact sequence [2] in the framework of the duality theorems for abelian varieties. By using the short exact sequence [2], we get, for each torsor \( X_K \) of order \( d \), an element of the group \( \text{Ext}^1(J(S), \mathbb{Z}/d\mathbb{Z}) \) of extensions in the category of Serre pro-algebraic groups. More generally, considering torsors of order dividing \( d \) we can actually define a natural map of sets (see § 5 first paragraph, for details):

\[
\Phi_d : dH_0(J, K) \to \text{Ext}^1(J(S), \mathbb{Z}/d\mathbb{Z}).
\]

This construction is analogous to the one used in [25] to relate the Galois group \( \Gamma_K^{ab} \) of the maximal abelian extension of the field \( K \) with the fundamental group of the pro-algebraic group \( U_K \): namely, let \( L/K \) be a finite abelian extension with Galois group \( \Gamma_{L/K} \), and let \( V_L \) be the kernel of the norm map \( N_{L/K} \) in [1]. One then has the following short exact sequence of pro-algebraic groups

\[
0 \to V_L \to U_L \xrightarrow{\alpha_{L/K}} U_K \to 0.
\]

The above sequence provides a homomorphism \( \pi_1(U_K) \to \pi_0(V_L) \) between the fundamental group of the pro-algebraic group \( U_K \) and the group of connected components of \( V_L \). Moreover, there is a canonical isomorphism \( \tau : \pi_0(V_L) \to \Gamma_{L/K} \) (cf. [25], 2.3). Now, the push-out of the sequence (6) via the composition of \( \tau \) with the canonical homomorphism \( \tau : \pi_1(U_K) \to \Gamma_{L/K} \) provides an element of \( \text{Ext}^1(U_K, \Gamma_{L/K}) \), hence a homomorphism \( \pi_1(U_K) \to \Gamma_{L/K} \), where \( \Gamma_{L/K} \) coincides with its component group because it is a finite group. By passing to the limit on \( L \), Serre obtained a homomorphism \( \theta : \pi_1(U_K) \to \Gamma_K^{ab} \). There exists then a homomorphism

\[
\theta^* : H^1(K, \mathbb{Z}/d\mathbb{Z}) \xrightarrow{\sim} \text{Hom}(\Gamma_K^{ab}, \mathbb{Z}/d\mathbb{Z}) \xrightarrow{\sim} \text{Hom}(\pi_1(U_K), \mathbb{Z}/d\mathbb{Z}) \leftarrow \text{Ext}^1(U_K, \mathbb{Z}/d\mathbb{Z}),
\]

which is in fact an isomorphism (cf. [25], 4.1). From this fact Serre deduced the main result of [25], namely that \( \theta \) is an isomorphism, and thus provided a “geometric” characterization of \( \Gamma_K^{ab} \).
Now, our construction of the morphism $\Phi_d$ is an analogue of Serre’s construction. Hence it makes sense to ask if the map $\Phi_d$ is an isomorphism too. To answer this question, we then come to the second main result of this paper, which gives a new construction of Shafarevich’s pairing using the relative Picard functor. Since this discussion also holds for abelian varieties of higher dimension, let us, more generally, consider an abelian $K$-variety $A_K$, and a torsor $X_K$ under $A_K$ of order $d$. Let $A^0$ be the identity component of the Néron model of the dual abelian variety and $\text{Gr}(A^0)$ its perfect Greenberg realization (see § 4.2). Assume $K$ of mixed characteristic. It is known (cf. [1], 8.2.3) that it is still possible to associate with $X_K$ an extension of the pro-algebraic group $\text{Gr}(A^0)$ by $\mathbb{Z}/d\mathbb{Z}$ hence, by push-out, an extension of $\text{Gr}(A^0)$ by $\mathbb{Q}/\mathbb{Z}$. Now, using the canonical isomorphisms $\text{Ext}^1(\text{Gr}(A^0), \mathbb{Q}/\mathbb{Z}) \cong \text{Ext}^1(\text{Gr}(A'), \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z})$ (cf. [21], 5.4), the above association provides an isomorphism

\begin{equation}
\text{H}^1_{fl}(K, A_K) \cong \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z}) \quad (\text{Shafarevich duality})
\end{equation}

(cf. [1], 8.3.6). In general, the isomorphism in (8) was proved by Shafarevich for the prime-to-$p$ parts (27, p. 96), and by Bester and the first author, respectively in [1], 7.1, and [2], Theorem 3, for $K$ of equal positive characteristic; there are also related results of Vvedenskii on elliptic curves (29). In the fourth section of the paper, after recalling the construction of Shafarevich’s duality [8] in the case of mixed characteristic (see § 4.2), we slightly modify Bégueri’s construction using rigidificators to make it work in any characteristic. We get in this way a morphism $\Xi: \text{H}^1_{\text{fl}}(K, A_K) \to \text{Ext}^1(\text{Gr}(A^0), \mathbb{Q}/\mathbb{Z})$ (see Proposition 4.3.3). Then we construct a morphism $\Xi'$ via the relative Picard functor (see Theorem 4.4.1) and we show that $\Xi'$ always coincides with the modified Bégueri construction $\Xi$ and thus with Shafarevich’s duality (8) for $K$ of characteristic 0. In the characteristic $p$ case, the morphism $\Xi'$ coincides with (8) on the prime-to-$p$ parts (see Proposition 4.5.1). The analogous result for the $p$-parts, although expected, is still open.

In the fifth section of the paper, using the canonical isomorphism $A^0 \cong J$ induced by the one in Lemma 2.1.1 (i), as a direct corollary of the previous study of Shafarevich’s pairing, we show that the map $\Phi_d$ in (5), only defined for $A_K$ an elliptic curve, is an injective morphism of groups (see Corollary 5.3.3). Furthermore if $d$ is prime to $p$, or with no restriction on $d$ in the mixed characteristic case, $\Phi_d$ can be identified with the restriction of (8) to the $d$-torsion subgroups via the isomorphism $\text{Ext}^1(\text{Gr}(A^0), \mathbb{Z}/d\mathbb{Z}) \cong \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Z}/d\mathbb{Z})$. In particular $\Phi_d$ is an isomorphism. This is done by showing that, in the case of elliptic curves, the homomorphism $\pi_1(\text{Gr}(A')) \to \mathbb{Z}/d\mathbb{Z}$ corresponding to the short exact sequence (2) coincides with the homomorphism associated with $X_K$ via the morphism $\Xi'$.

This paper arises from the confluence and the comparison of the results contained in the preprints [3] and [28]. Section 2 presents detailed proofs of results contained in the unpublished manuscript [22].

1 The Picard functor and the Greenberg functor

In this section we recall well-known results on Picard functors and Greenberg functors. Let $S := \text{Spec}(\mathcal{O}_K)$ be the spectrum of a discrete valuation ring $\mathcal{O}_K$, $K$ the fraction field of $\mathcal{O}_K$, and $k$ the residue field. Let $\text{Sch}/S$ denote the category of $S$-schemes and $\text{Ab}$ the category of abelian groups. By a curve over $S$ we will mean an $S$-scheme $X \to S$ whose geometric fibres are pure of dimension 1.
1.1 The Picard functor

Let \( f : X \to S \) be a proper morphism of schemes. Let

\[
\text{Pic}_{X/S} : \text{Sch}/S \to \text{Ab}
\]

denote the relative Picard functor of \( X \) over \( S \), i.e., the fppf sheaf associated with the presheaf \( S' \mapsto \text{Pic}(X \times_S S') \). It is also the sheaf for the étale topology associated with the presheaf \( S' \mapsto \text{Pic}(X \times_S S') \) \cite{21}, 1.2).

Suppose furthermore that \( f \) is proper and flat. In general the functor \( \text{Pic}_{X/S} \) is not representable. It is representable by an algebraic space if and only if \( X/S \) is cohomologically flat in dimension 0, i.e., if the formation of the direct image \( f_* \mathcal{O}_X \) commutes with base change. Even when \( \text{Pic}_{X/S} \) is not representable, it has a nice presentation by algebraic \( S \)-spaces. To see this fact, we recall the notion of rigidificator. Given a morphism of schemes \( S' \to S \) and a morphism of \( S \)-schemes \( \alpha : Y \to X, \beta : Y' \to X' \) will denote the base change of \( \alpha \) along \( S' \to S \).

**Definition 1.1.1** (\cite{5} 8.1/5). Let \( i : Y \hookrightarrow X \) be a closed immersion of \( S \)-schemes with \( Y \) finite and flat over \( S \). One says that \( (Y, i) \) is a rigidificator of \( \text{Pic}_{X/S} \) if the following condition holds: for any \( S \)-scheme \( S' \), the map

\[
\Gamma(i') : \Gamma(X', \mathcal{O}_{X'}) \to \Gamma(Y', \mathcal{O}_{Y'})
\]

is injective.

In the sequel let \( f : X \to S \) be proper and flat, and let \( (Y, i) \) be a rigidificator of \( \text{Pic}_{X/S} \); it exists by the hypothesis on \( f \) \cite{21}, proposition 2.2.3 (c)). For any scheme \( S' \) over \( S \), an invertible sheaf on \( X' \) rigidified along \( Y' \), is a pair \((\mathcal{L}, \alpha)\), where \( \mathcal{L} \) is an invertible sheaf on \( X' \) and \( \alpha : \mathcal{O}_{Y'} \cong i'^*\mathcal{L} \) is an isomorphism (i.e., \( \alpha \) is a trivialization of \( i'^*\mathcal{L} \)). An isomorphism between two rigidified invertible sheaves \((\mathcal{L}, \alpha), (\mathcal{M}, \beta)\), on \( X' \) is an isomorphism of \( \mathcal{O}_{X'} \)-modules \( u : \mathcal{L} \to \mathcal{M} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\alpha} & \mathcal{M} \\
\downarrow{i'^*u} & & \downarrow{i'^*\beta} \\
\mathcal{O}_{Y'} & & \mathcal{O}_{Y'}
\end{array}
\]

Let \( (\text{Pic}_{X/S}, Y)(S') \) denote the set of isomorphism classes of invertible sheaves on \( X' \) rigidified along \( Y' \). As \( S' \) varies in the category of \( S \)-schemes (\text{Sch}/S), the association \( S' \mapsto (\text{Pic}_{X/S}, Y)(S') \) defines a functor of abelian groups \( (\text{Pic}_{X/S}, Y) \), called the rigidified Picard functor of \( X/S \) relative to the rigidificator \( Y \). Concerning its representability we have:

**Theorem 1.1.2** \cite{21}, 2.3.1 & 2.3.2. Let \( f : X \to S \) be a proper flat morphism. Then the functor \( (\text{Pic}_{X/S}, Y) \) is representable by an algebraic space over \( S \), locally of finite presentation. Furthermore, if \( \dim(X/S) \leq 1 \), the algebraic space \( (\text{Pic}_{X/S}, Y) \) is formally smooth over \( S \).

One has a canonical morphism of sheaves of groups \( r : (\text{Pic}_{X/S}, Y) \to \text{Pic}_{X/S} \), that forgets the rigidification. Étale locally on \( S' \), any element in \( \text{Pic}_{X/S}(S') \) is represented by an invertible sheaf on \( X' \) such that its pull-back to \( Y' \) is trivial (this is possible since \( Y \) is finite). Hence \( r \) is an epimorphism for the étale topology. To study the kernel of \( r \), let \( V^+_X \) (respectively \( V^+_Y \)) denote the fppf sheaf on \( \text{Sch}/S \), given by \( S' \mapsto \Gamma(X', \mathcal{O}_{X'})^* \) (respectively by \( S' \mapsto \Gamma(Y', \mathcal{O}_{Y'})^* \)). These sheaves are representable by \( S \)-schemes (cf. \cite{21}, 2.4.0). By definition of rigidificator the
natural map $V_X^* \to V_Y^*$ is injective. Let $u: V_Y^* \to (\text{Pic}_{X/S}, Y)$ be the map defined as follows on $S'$-sections, $S'$ an $S$-scheme:

$$a \in V_Y^*(S') = \Gamma(Y_{S'}, \mathcal{O}_{Y_{S'}}^*) \mapsto (\mathcal{O}_{X_{S'}}, a_{a}) \in (\text{Pic}_{X/S}, Y)(S')$$

where $\alpha_{a}: \mathcal{O}_{Y_{S'}} \to \mathcal{O}_{Y_{S'}} = \mathcal{O}_{X_{S'}}|_{YS'}$ is the multiplication by $a$. Clearly $\text{im}(u) \subset \ker(r)$ and thus one obtains a complex of fppf sheaves over $S$:

$$0 \to V_X^* \to V_Y^* \xrightarrow{u} (\text{Pic}_{X/S}, Y) \xrightarrow{r} \text{Pic}_{X/S} \to 0,$$

which is exact for the étale topology ([21], 2.1.2(b), 2.4.1).

We assume for the remainder of the section that the discrete valuation ring $\mathcal{O}_K$ is strictly henselian with algebraically closed residue field. In particular, the Brauer group $\text{Br}(K)$ is trivial. Let $f: X \to S$ be a proper and flat curve with geometrically connected fibres. Denote by $P$ (respectively by $(P, Y)$) the open subfunctor of $\text{Pic}_{X/S}$ (respectively of $(\text{Pic}_{X/S}, Y)$) consisting of invertible sheaves of total degree 0 (respectively of invertible sheaves of total degree 0 rigidified along $Y$). Then $(P, Y)$ is an open algebraic subspace of $(\text{Pic}_{X/S}, Y)$, and $P$ is the schematic closure of $(\text{Pic}_{X/S})_0^K$ in $\text{Pic}_{X/S}$, while $(P, Y)$ is the schematic closure of $(\text{Pic}_{X/S}, Y)_0^K$ in $(\text{Pic}_{X/S}, Y)$. Denote by $E$ the schematic closure of the unit section of $P_K$ in $P$, and define $Q$ as the fppf quotient of $P$ by $E$. It is the biggest separated quotient of $P$. It is representable by a separated group scheme over $S$ ([21], 3.3.1). Denote by $q$ the canonical map

$$q: P \to Q;$$

it is surjective for the fppf topology, and it induces an isomorphism on generic fibres since $E_K$ is the unit section of $P_K = \text{Pic}_{X_K/K}^0$.

**Theorem 1.1.3** ([16], 3.7). Let $f: X \to S$ be a proper and flat curve with $X$ regular and $f_*\mathcal{O}_X = \mathcal{O}_S$. Then the group scheme $Q/S$ is the Néron model of $P_K = \text{Pic}_{X_K/K}^0$.

Suppose then $X$ regular. Denote by $J := Q^0$ the identity component of $Q$, and by $\text{Pic}_{X/S}^0$ the subsheaf of $P \subset \text{Pic}_{X/S}$ of invertible sheaves of degree 0 on each irreducible component of $X$ (for the definition of $F^0$ when $F$ is a sheaf on $S$, see [21], 3.2 (d)). Since the functor $\text{Pic}_{X/S}^0$ has connected fibres, the restriction to $\text{Pic}_{X/S}^0$ of $q$ in ([10]) factors through the identity component $J$ of $Q$. By abuse of notation, the same letter $q$ will denote the induced map:

$$q: \text{Pic}_{X/S}^0 \to J.$$

Finally, since $\text{Br}(K) = 0$, the following map

$$(P, Y)^0(S) = (\text{Pic}_{X/S}, Y)^0(S) \twoheadrightarrow J(S),$$

induced by the morphism $r$ in ([10]) is surjective (see the proof of 9.6/1 of [5]); hence so too is the morphism (again denoted by $q$) induced by ([11]):

$$q = q(S): P^0(S) = \text{Pic}_{X/S}^0(S) \twoheadrightarrow J(S).$$
1.2 Pro-algebraic groups and Greenberg functor

From now on, we suppose that the discrete valuation ring $\mathcal{O}_K$ is complete with algebraically closed residue field $k$. In the following, a pro-algebraic group over $k$ is a pro-object in the category of $k$-group schemes of finite type (see [19], I.4). This notion does not coincide with the notion of pro-algebraic groups introduced by Serre in [24] (§ 2.1 Définition 1), where the author considers the category of $k$-group schemes, but up to purely inseparable morphisms. Since we use both categories, we call the objects of the latter abelian category (see [24], §2.4, Proposition 7) Serre pro-algebraic groups and denote them with bold letters.

Let $G$ be a smooth group scheme of finite type over $S$. The Greenberg functors allow us to construct a pro-algebraic group over $k$, associated with $G$, whose group of $k$-points is $G(S)$. Let us recall the construction. Denote by $W$ the ring of Witt vectors of $k$ and by $\mathbb{W}$ the Witt functor on the category of $k$-algebras $\mathbf{Alg}/k$. Let $n \in \mathbb{Z}_{\geq 1}$ and $\mathcal{O}_{K,n} := \mathcal{O}_K/\pi^n\mathcal{O}_K$. Then $\mathcal{O}_{K,n}$ is canonically a $W$-module of finite length. Let $\mathbb{R}_n$ be its associated Greenberg algebra (see [14], Appendix A), which is by definition the fpqc sheaf on $\mathbf{Alg}/k$ associated with the pre-sheaf: $A \mapsto \mathcal{O}_{K,n} \otimes_W \mathbb{W}(A)$. One defines then $\text{Gr}_n(G)$ as the sheaf on $\mathbf{Alg}/k$ given by $A \mapsto G(\mathbb{R}_n(A))$. It is representable by a smooth $k$-group scheme of finite type ([17], Proposition 7, and [5], Corollary 1). For any $n \geq 2$, the canonical map $\mathcal{O}_{K,n} \to \mathcal{O}_{K,n-1}$ induces a smooth morphism of $k$-group schemes $\alpha_n : \text{Gr}_n(G) \to \text{Gr}_{n-1}(G)$, whose kernel is a connected unipotent $k$-group scheme. Furthermore the canonical map $G(\mathcal{O}_{K,n}) \to G(\mathcal{O}_{K,n})(k)$ is an isomorphism. Thanks to this identification the morphism $\alpha_n(k) : \text{Gr}_n(G)(k) \to \text{Gr}_{n-1}(G)(k)$ is identified with the canonical morphism $G(\mathcal{O}_{K,n}) \to G(\mathcal{O}_{K,n-1})$. The algebraic $k$-groups $\text{Gr}_n(G)$ form then a projective system $\{\text{Gr}_n(G), \alpha_n\}_{n \geq 1}$ of algebraic $k$-groups with smooth transition maps having connected kernels. We will denote the latter projective system by $\text{Gr}(G)$ and call it the Greenberg realization of $G$.

As explained in [17], III § 4, the perfect group schemes $\text{Gr}_n(G)$ associated with the $k$-group schemes $\text{Gr}_n(G)$ are quasi-algebraic groups in the sense of [24], 1.2, and hence the projective system $\text{Gr}(G) := \{\text{Gr}_n(G), \alpha_n\}_{n \geq 1}$ determines a pro-algebraic group in the sense of Serre; we will call it the perfect Greenberg realization of $G$. The group of $k$-rational points of $\text{Gr}(G)$ is $G(S)$. For this reason sometimes $\text{Gr}(G)$ is denoted by $G(S)$.

Observe that $\text{Gr}(G)$ is defined for any smooth $S$-scheme, but it may not be a Serre pro-algebraic group, since its component group may not be profinite: for example, consider as $G$ the lift Néron model of $G_{m,K}$ ([5], 10.1/5); then $\text{Gr}(G)$ is the projective limit of perfect $k$-schemes having component group $\mathbb{Z}$. The functor $\text{Gr}$ is exact on smooth $S$-group schemes (see [4], Lemma 1.1).

Recall that the identity component $G^0$ of a Serre pro-algebraic group $G$ is defined as the smallest closed subgroup $G'$ of $G$ such that $G/G'$ has dimension zero ([24], 5.1/1). In the category of Serre pro-algebraic groups, the component group functor $\pi_0$, which maps $G$ to $G/G^0$, admits a left derived functor $\pi_1$ which is left exact. We list below some well-known facts used in this paper. By simply assuming them, the reader unfamiliar with the theory of Serre pro-algebraic groups will be able to follow the proofs.

(i) If $Y$ is a smooth $S$-group scheme of finite type, then $\text{Gr}(Y^0) \cong \text{Gr}(Y)^0$, $\pi_0(\text{Gr}(Y))$ is isomorphic to the component group of the special fibre $Y_s$, and $\pi_1(\text{Gr}(Y))$ is a profinite group.

(ii) If $P$ is a Serre pro-algebraic group and $P^0$ is its identity component, then $\pi_1(P) \cong \pi_1(P^0)$.

(iii) A complex of smooth $S$-group schemes of finite type $0 \to Y_1 \to Y_2 \to Y_3 \to 0$ which is
exact on \( S \)-sections provides a long exact sequence of profinite groups (cf. [21], 10.2/1)

\[
0 \to \pi_1(\text{Gr}(Y_1)) \to \pi_1(\text{Gr}(Y_2)) \to \pi_1(\text{Gr}(Y_3)) \to \\
\to \pi_0(\text{Gr}(Y_1)) \to \pi_0(\text{Gr}(Y_2)) \to \pi_0(\text{Gr}(Y_3)) \to 0.
\]

Let \( f: Z \to S \) be a proper morphism of schemes such that \( Z \) is of dimension 1 with \( f(Z) = \{s\} \). Note that the morphism \( f \) then factors through some \( S_n := \text{Spec}(\mathcal{O}_{K,n}) \leftrightarrow S \). For instance, when \( X/S \) is a proper flat curve, one can take \( Z = X \times S S_n \) viewed as an \( S \)-scheme. For \( S' \) an \( S \)-scheme, let \( \text{Pic}^0(Z \times S S') \) denote the subgroup of \( \text{Pic}(Z \times S S') \) consisting of line bundles \( L \) on \( Z \times S S' \) such that, for any point \( s' \in S' \) above \( s \in S \), the restriction \( L|_{Z \times S \{s'\}} \) is of degree 0 on each irreducible component of \( Z \times S \{s'\} \). Then we define \( \text{Pic}^0_{Z/S} \) to be the associated fppf sheaf on \( S \) of the following functor

\[
\text{Sch}/S \to \mathfrak{Ab}, \quad S' \mapsto \text{Pic}^0(Z \times S S').
\]

As the map \( f \) is never flat, the Picard functors \( \text{Pic}_{Z/S} \) and \( \text{Pic}^0_{Z/S} \) are not representable. However, as shown by Lipman in [14], the Greenberg realization of the sheaf \( \text{Pic}_{Z/S} \) (respectively of \( \text{Pic}^0_{Z/S} \)) is represented by a smooth \( k \)-group scheme. More precisely, assuming that the morphism \( f \) factors through \( i_n: S_n = \text{Spec}(\mathcal{O}_{K,n}) \leftrightarrow S \), we define \( \text{Gr}(\text{Pic}_{Z/S}) \) (respectively \( \text{Gr}(\text{Pic}^0_{Z/S}) \)) to be the fppf sheaf associated with the presheaf \((\mathbb{1}_4, 1.8)\)

\[
\text{Alg}/k \to \mathfrak{Ab}, \quad A \mapsto \text{Pic}_{Z/S}(\mathbb{R}_n(A)) \quad \text{resp.} \quad A \mapsto \text{Pic}^0_{Z/S}(\mathbb{R}_n(A))
\]

where, as above, \( \mathbb{R}_n \) denotes the Greenberg algebra associated with \( \mathcal{O}_{K,n} \). Note that the definition of \( \text{Gr}(\text{Pic}_{Z/S}) \) does not depend on the choice of the integer \( n \) since \( \text{Gr}(\text{Pic}_{Z/S}) \) is isomorphic to the fppc associated associated with the presheaf \( A \mapsto \text{Pic}(Z \otimes W \mathbb{W}(A)) \) \((\mathbb{1}_4) 1.1 \& 1.8)\); the same proof shows that \( \text{Gr}(\text{Pic}^0_{Z/S}) \) is also independent of \( n \).

**Theorem 1.2.1** \((\mathbb{1}_4)\). Let \( f: Z \to S \) be as above. Then the functor \( \text{Gr}(\text{Pic}_{Z/S}) \) (respectively \( \text{Gr}(\text{Pic}^0_{Z/S}) \)) is representable by a smooth \( k \)-group scheme (respectively by a smooth and connected \( k \)-group scheme), whose dimension is equal to the length of the \( W \)-module \( H^1(Z, \mathcal{O}_Z) \). Furthermore, the canonical morphisms

\[
\text{Pic}(Z) \to \text{Gr}(\text{Pic}_{Z/S})(k), \quad \text{Pic}^0(Z) \to \text{Gr}(\text{Pic}^0_{Z/S})(k)
\]

are isomorphisms.

**Proof.** The statements for \( \text{Gr}(\text{Pic}_{Z/S}) \) are obtained by combining Theorem 1.2, (1.8), Corollary 8.6 (a) and Theorem 9.1 of \([14]\). To deduce the corresponding statements for \( \text{Gr}(\text{Pic}^0_{Z/S}) \), let \( Z' := Z_{\text{red}} \) be the maximal reduced closed subscheme of \( Z \). The canonical map \( Z' \leftrightarrow Z \) induces a morphism between the Picard functors \( \text{Pic}_{Z/S} \to \text{Pic}_{Z'/S} \), hence also a morphism \( u: \text{Gr}(\text{Pic}_{Z/S}) \to \text{Gr}(\text{Pic}_{Z'/S}) \) between the Greenberg realizations of Picard functors. By Proposition 1.2.2, \( u \) is an epimorphism of smooth \( k \)-group schemes with \( \ker(u) \) a smooth connected unipotent group. On the other hand, by definition, the canonical morphism of sheaves

\[
\text{Pic}^0_{Z/S} \to \text{Pic}_{Z/S} \times_{\text{Pic}_{Z'/S}} \text{Pic}^0_{Z'/S}
\]

is an isomorphism, from whence we deduce a similar isomorphism between the Greenberg realizations:

\[
(13) \quad \text{Gr}(\text{Pic}^0_{Z/S}) \xrightarrow{\sim} \text{Gr}(\text{Pic}_{Z/S}) \times_{\text{Gr}(\text{Pic}_{Z'/S})} \text{Gr}(\text{Pic}^0_{Z'/S}).
\]
Consequently, the canonical morphism \( u^0 : \text{Gr}(\text{Pic}^0_{Z/S}) \to \text{Gr}(\text{Pic}^0_{Z'/S}) \) is an epimorphism of fppf sheaves on \( \text{Spec}(k) \), with kernel isomorphic to \( \ker(u) \). Now, since \( Z' \) is reduced, the canonically map \( Z' \to S \) factors through \( i_1 : \text{Spec}(k) \hookrightarrow S \), and \( \text{Gr}(\text{Pic}^0_{Z'/S}) \cong \text{Pic}^0_{Z'/k} \) (cf. \[14\], p. 29, last paragraph) is representable by a smooth connected \( k \)-group scheme. Therefore, the sheaf \( \text{Gr}(\text{Pic}^0_{Z'/S}) \) is representable and, furthermore, it is isomorphic to \( \text{Gr}(\text{Pic}^0_{Z/S})^0 \) because \( u^0 \) is an epimorphism, with \( \ker(u^0) \cong \ker(u) \) a connected affine \( k \)-group scheme. Finally, consider the following commutative diagram

\[
\begin{array}{c}
\text{Pic}^0(Z) \\
\downarrow \\
\text{Pic}(Z) \times_{\text{Pic}(Z')} \text{Pic}^0(Z') \\
\downarrow \\
\text{Gr}(\text{Pic}_{Z/S})(k) \\
\downarrow \\
\text{Gr}(\text{Pic}_{Z'/S})(k) \\
\end{array}
\]

By the definition of \( \text{Pic}^0 \) and \[14\], both vertical morphisms are bijective. Moreover, by what we have shown at the beginning of the proof and the fact that \( \text{Gr}(\text{Pic}^0_{Z'/S}) \cong \text{Pic}^0_{Z'/k} \); the lower horizontal morphism is also bijective. Therefore, the upper horizontal morphism is bijective. This finishes the proof. \( \square \)

**Proposition 1.2.2.** Let \( \mathcal{J} : Z' \hookrightarrow Z \) be a closed subscheme defined by a nilpotent ideal \( \mathcal{J} \subset \mathcal{O}_Z \). Then the canonical morphism of smooth \( k \)-group schemes

\[ u : \text{Gr}(\text{Pic}_{Z/S}) \to \text{Gr}(\text{Pic}_{Z'/S}) \]

is an epimorphism. Moreover, \( \ker(u) \) is a smooth connected unipotent group.

**Proof.** By Theorem 1.2.1, \( u \) is represented by a morphism of smooth \( k \)-group schemes. Then \( u \) is an epimorphism as soon as it is surjective on \( k \)-rational points. By the previous theorem, we are reduced to showing that the canonical map \( \text{Pic}(Z) \to \text{Pic}(Z') \) is surjective. Consider then the following short exact sequence of abelian sheaves

\[ 0 \to 1 + \mathcal{J} \to \mathcal{O}_Z^* \to \iota_* \mathcal{O}_{Z'}^* \to 0. \]

It provides an exact sequence of cohomology groups

\[ H^1(Z, 1 + \mathcal{J}) \to \text{Pic}(Z) \to \text{Pic}(Z') \to H^2(Z, 1 + \mathcal{J}). \]

Since \( \dim(Z) = 1 \), the \( H^2 \) group on the right vanishes and hence the map \( \text{Pic}(Z) \to \text{Pic}(Z') \) is surjective.

By the proof of Proposition 2.5 in \[14\] (in particular the last paragraph of page 31), \( \ker(u) \) is representable by a connected unipotent group scheme over \( k \). It remains to show that \( \ker(u) \) is smooth. Let \( \mathcal{N} \subset \mathcal{O}_Z \) be the nilpotent radical of \( \mathcal{O}_Z \), and \( Z_n \hookrightarrow Z \) the closed subscheme defined by \( \mathcal{J} \mathcal{N}^n \subset \mathcal{O}_Z \) \((n \geq 0)\). Then we have \( Z_0 = Z' \) and \( Z_m = Z \) for \( m \) an integer sufficiently large. In this way, we obtain an ascending sequence of closed subschemes \( \{Z_n\}_{n \geq 0} \) of \( Z \), and hence a sequence of surjective morphisms of smooth \( k \)-group schemes \( u_n : \text{Gr}(\text{Pic}_{Z_{n+1}/S}) \to \text{Gr}(\text{Pic}_{Z_n/S}) \) \((n \geq 0)\). Therefore, in order to prove that \( \ker(u) \) is smooth, we only need to show that \( \ker(u_n) \) is smooth for all \( n \). Hence, up to replacing \( Z' \hookrightarrow Z \) with \( Z_n \hookrightarrow Z_{n+1} \), we may assume furthermore that \( \mathcal{J} \mathcal{N} = 0 \). Now, the proof of 6.4 in \[14\] shows that \( \ker(u) \) is isomorphic to the kernel of the canonical map \( v : \mathbb{H} \to \mathbb{H}' \), where \( \mathbb{H} \) (respectively \( \mathbb{H}' \)) denotes the associated fppf sheaf of the functors

\[ \text{Alg}/k \to \text{Ab}, \quad A \mapsto H^1(Z_{A/}, \mathcal{O}_{Z_A}), \quad (\text{respectively} \quad A \mapsto H^1(Z'_{A/}, \mathcal{O}_{Z'_A})). \]
with $Z_A := Z \otimes_{\mathcal{O}_K} \mathbb{R}(A)$ and $Z'_A := Z' \otimes_{\mathcal{O}_K} \mathbb{R}(A)$. Both $\mathbb{H}$ and $\mathbb{H}'$ are representable by smooth $k$-group schemes (14. Theorem 1.4 and Corollary 8.4), and $v$ is a surjective morphism of $k$-group schemes (14, Corollary 4.4). To finish the proof, it remains to show that the morphism $v$ is smooth, or equivalently, that the induced map between the Lie algebras $\text{Lie}(v): \text{Lie}(\mathbb{H}) \to \text{Lie}(\mathbb{H}')$ is surjective (14 Prop. 1.1 (e)). By 14, Theorem 8.1, $\text{Lie}(\mathbb{H})$ has a natural grading by $k$-vector subspaces:

$$\text{Lie}(\mathbb{H}) = \bigoplus_{i \geq 0} \text{Lie}^i(\mathbb{H}), \quad \text{with} \quad \text{Lie}^i(\mathbb{H}) \xrightarrow{\sim} \text{im}(\mu_i)^{(-i)},$$

where $\mu_i = \mu_{i,Z}$ is the canonical map $H^1(Z, p^i\mathcal{O}_Z/p^{i+1}\mathcal{O}_Z) \to H^1(Z, \mathcal{O}_Z/p^{i+1}\mathcal{O}_Z)$, and for any $k$-vector space $V$, $V^{(-i)}$ denotes the $k$-vector space with the same underlying abelian group as $V$ and the scalar multiplication $(a, x) \mapsto a^{-1}x$ for any $a \in k, x \in V$. Since the grading is functorial (14, p. 77), in order to showing that $\text{Lie}(v)$ is surjective, we have to prove that the map $\text{im}(\mu_{i,Z}) \to \text{im}(\mu_{i,Z'})$ is surjective. Now, since $\dim(Z) = 1$, the surjective morphism of coherent sheaves $p^i\mathcal{O}_Z/p^{i+1}\mathcal{O}_Z \to \iota_* (p^i\mathcal{O}_{Z'}/p^{i+1}\mathcal{O}_{Z'})$ induces a surjective map

$$H^1(Z, p^i\mathcal{O}_Z/p^{i+1}\mathcal{O}_Z) \longrightarrow H^1(Z', p^i\mathcal{O}_{Z'}/p^{i+1}\mathcal{O}_{Z'}),$$

and this implies the surjectivity of the map $\text{im}(\mu_{i,Z}) \to \text{im}(\mu_{i,Z})$. \hfill \square

## 2 Herbrand functions

Unless explicitly mentioned, the results of this section are directly taken from the unpublished work [23] of Raynaud. Some of them have already appeared in [12]. In this section and the next, let $K$ be a complete discrete valued field with algebraically closed residue field $k$ and ring of integers $\mathcal{O}_K$. Let $A_K$ be an elliptic curve, $X_K$ a torsor under $A_K$ over $K$, and $f: X \to S$ the proper regular minimal model of $X_K/K$ over $S$. Let $X_s = \sum_{i=1}^r n_i C_i$ be the decomposition of the special fibre $X_s$ into the sum of its reduced irreducible components, and denote by $d$ the gcd of the integers $n_i$. Moreover, let $D$ be the divisor of $X$ given by

$$D := \frac{1}{d} X_s = \sum_{i=1}^r \frac{n_i}{d} C_i,$$

and let $\mathcal{I}$ be the (locally principal) ideal sheaf of $D$. The special fibre $X_s$ of $X$ is then defined by the ideal sheaf $\mathcal{I}^d = \pi \mathcal{O}_X \subset \mathcal{O}_X$. For each $n \in \mathbb{Z}_{\geq 1}$, let $\iota_n: X_n \hookrightarrow X$ be the closed immersion defined by the ideal $\mathcal{I}^n \subset \mathcal{O}_X$. For $\mathcal{L}$ an invertible sheaf (respectively for $\Sigma$ a divisor) on $X$ and $Z$ a divisor on $X$, let $\mathcal{L} \cdot Z$ (respectively $\Sigma \cdot Z$) denote the intersection number of $\mathcal{L}$ and $Z$ (respectively of $\Sigma$ and $Z$). Finally, let $\phi(n)$ be the length of the $\mathcal{O}_K$-module $H^1(X_n, \mathcal{O}_{X_n})$. In this section, we will first study the function $n \mapsto \phi(n)$ by means of some numerical invariants attached to $X/S$ (Lemma 2.1.11). Then we will define from $\phi$ two piecewise linear functions $\varphi, \psi$ (see § 2.2 and the figure therein), which are the analogue of the Herbrand functions in Serre’s description of local class field theory (25, § 3) and will be used in the next section.

We recall the following useful result.

**Lemma 2.0.1** ([18], p. 332). Let $Y \to S$ be a proper flat curve such that $Y$ is regular and such that $Y_K/K$ is geometrically irreducible. Let $Y_s = \sum_{i=1}^r n_i E_i$ be the decomposition of the special fibre $Y_s$ into the sum of its reduced irreducible components, and $n$ the gcd of the integers $n_i$. Assume moreover that $\omega_{Y/S} \cdot E_i = 0$ for all $i$, where $\omega_{Y/S}$ denotes the dualizing sheaf of $Y/S$. 

Let finally $Y_1$ be the closed subscheme of $Y$ defined by the ideal sheaf $\mathcal{O}_Y(-\sum_{i=1}^{r} \frac{m_i}{n_i}E_i)$ and $L$ an invertible sheaf over $Y_1$, which is of degree 0 on $E_i$ for all $i$. Then, if $H^0(Y_1, L) \neq 0$, we have $L \cong \mathcal{O}_{Y_1}$ and $H^0(Y_1, \mathcal{O}_{Y_1}) \cong k$.

The curve $Y_1$ on $Y$ is indecomposable of canonical type in the sense of [18], p. 330. The lemma in [18] (p. 332) is stated for an indecomposable curve of canonical type on a smooth proper algebraic surface defined over an algebraically closed field. However, one can check that the proof there works in our setting too.

2.1 Study of the dualizing sheaf

We begin with two classical results. Recall that, given a morphism of $2.1$ Study of the dualizing sheaf

proper algebraic surface defined over an algebraically closed field. However, one can check that lemma in [18] (p. 332) is stated for an indecomposable curve of canonical type on a smooth proper algebraic surface defined over an algebraically closed field. However, one can check that the proof there works in our setting too.

Lemma 2.1.1. Let $A_K$ be an elliptic curve over $K$, and $X_K/K$ a torsor under $A_K$. Let $J_K := \text{Pic}^0_{X_K/K}$ be the jacobian of $X_K/K$.

(i) There is a canonical isomorphism of elliptic curves $\iota: A'_K \sim \to J_K$, where $A'_K$ denotes the dual of $A_K$.

(ii) Let $\sigma: A_K \to J_K$ be the isomorphism obtained by composing the isomorphism $\alpha: A_K \sim \to A'_K$, sending $a \in A_K(K)$ to $\mathcal{O}_{A'_K}(a-o) \in A'_K(K)$. Then the canonical map $\psi: X_K \to \text{Pic}^1_{X_K/K}$ mapping $x$ to $\mathcal{O}_{X_K}(x)$ is equivariant with respect to the isomorphism $\sigma$, where $\text{Pic}^1_{X_K/K}$ is the Picard scheme which classifies the line bundles of degree 1 on $X_K$. In particular, under the identification given by $\sigma$, the (class of the) torsor $X_K$ in $H^1_{fl}(K, A_K)$ corresponds to the (class of the) torsor $\text{Pic}^1_{X_K/K}$ in $H^1_{fl}(K, J_K)$.

Proof. The first fact is proved in [22], XIII, 1.1. For the second fact, in order to verify that the morphism $\psi$ is equivariant with respect to the morphism $\sigma$, by descent, we need only prove the corresponding statement over a separable closure $\bar{K}$ of $K$. Over $\bar{K}$, the isomorphism $\sigma: A_K \to J_K$ is explicitly described by mapping $a \in A_K$ to the class of $\mathcal{O}_{X_K}(a-x_0-x_0)$ with $x_0 \in X_K(\bar{K})$. Hence to show the desired property of (ii), we are reduced to proving the following isomorphism between line bundles on $X_K$: for any $x \in X_K(\bar{K})$, and any $a \in A_K(\bar{K})$, we have $\mathcal{O}_{X_K}(a \cdot x) = \mathcal{O}_{X_K}(a \cdot x_0-x_0) \otimes \mathcal{O}_{X_K}(x)$, or equivalently, $\mathcal{O}_{X_K}(a \cdot x_0-x_0) = \mathcal{O}_{X_K}(a \cdot x-x)$. Indeed, since the isomorphism $\iota_K$ is independent of the choice of $x_0$, we have $\sigma(a) = \iota_K, (a) = [\mathcal{O}_{X_K}(a \cdot x-x)]$.

Remark 2.1.2. In [22], XIII, 1.1, the conclusion in 2.1.1 (i) is proved for any abelian variety $A_K$.

Lemma 2.1.3. Let $d_1$ be the order of (the class of) the torsor $X_K$ in the group $H^1_{fl}(K, J_K)$, $d_2$ the minimal degree of extensions $K'$ of $K$ such that $X_K(K') \neq \emptyset$, and $d_3$ the minimum of the multiplicities of the irreducible components of $X_K$. Then $d_1 = d_2 = d_3 = d$, with $d$ as in [14].

Proof. We will prove this Lemma by showing that $d \leq d_1 \leq d_2 \leq d_3 \leq d$. Let $n \in \mathbb{Z}_{>0}$. By Lemma 2.1.1 (ii), the class $n[X_K] \in H^1(K, A_K) \sim \to H^1(K, J_K)$ can be represented by the irreducible component $\text{Pic}^n_{X_K/S}$ of $\text{Pic}_{X_K/K}$ which classifies the invertible sheaves of degree $n$. Hence, the class $n[X_K] \in H^1(K, A_K)$ is trivial if and only if $\text{Pic}^n_{X_K/K}(K) \neq \emptyset$. On the other
hand, since $\mathcal{O}_K$ is strictly henselian with algebraically closed residue field, we have $\text{Br}(K) = 0$. Hence, $\text{Pic}^n_{X_K/K}(K) = \text{Pic}^n(X_K)$. As a result, $d_1$ is also the minimum of the degrees of the divisors with positive degree on $X_K$. Now, let $\Sigma_K \subset X_K$ be any divisor with positive degree, and let $\Sigma$ be its schematic closure in $X$. Then we have $\deg(\Sigma_K) = \Sigma \cdot X_s = \Sigma \cdot (dD) = d(\Sigma \cdot D)$. Hence $d|\deg(\Sigma_K)$, in particular $d \leq \deg(\Sigma_K)$. As a result, we get $d \leq d_1$. Next, by definition of $d_2$, there exists a closed point of degree $d_2$ on $X_K$, hence a divisor of degree $d_2$ on $X_K$, so we have $d_1 \leq d_2$. Since $\mathcal{O}_K$ is strictly henselian, for each $i$, we can find a positive divisor $\Delta_i$ of $X/S$ of degree $n_i$ (see [15] 9.1/10). In particular, we have $d_2 \leq n_i$ for each $i$, hence we get $d_2 \leq d_3$. Finally, to see that $d_3 \leq d$, note that a suitable combination of $\Delta_i$ gives us a divisor $\Delta'$ of degree $d$ of $X_K$. In general, the divisor $\Delta'$ might not be positive, but since $X_K$ is of arithmetic genus 1, and $d \geq 1$, we have $h^0(X_K, \mathcal{O}_{X_K}(\Delta'_K)) > 0$. Hence there exists a positive divisor $\Delta_K$ of degree $d$ of $X_K$ which is linearly equivalent to $\Delta'_K$. Let $\Delta := \overline{\Delta_K}$ be the schematic closure of $\Delta_K$ in $X$. Then we have

$$d = \deg(\Delta_K) = \Delta \cdot X_s = d(\Delta \cdot D).$$

In particular, $\Delta \cdot D = 1$, and $\Delta \cap D = \{y\}$ consists of only one point, and $D$ is regular at $y$. Let $C_i$ be the irreducible component of $D$ passing through $y$, then $C_i$ is of multiplicity $d$ in $X_s$, whence $d_3 \leq d$. This completes the proof. \hfill $\Box$

Since $S$ is an affine Dedekind scheme and $X$ is regular, by a result of Lichtenbaum ([15] 8.3.16), $X$ is projective over $S$. Let $\omega_{X/S}$ denote the canonical (invertible) sheaf of $f : X \to S$ (see [15], 6.4.7, for the definition and [15], 6.4.32, for the fact that it is isomorphic to the dualizing sheaf). Since $X_K/K$ is smooth projective curve of genus 1 and the formation of the canonical sheaf is compatible with flat base change ([15], 6.4.9(b)), one finds that $(\omega_{X/S})_{X_K} \cong \omega_{X_K/K} \cong \mathcal{O}_{X_K}$ (Example 7.3.35). For any $n \geq 0$, let

$$\omega_n := i_n^*(\omega_{X/S}(X_n)) = i_n^*(\omega_{X/S} \otimes_{\mathcal{O}_{X}} \mathcal{O}_{X}(X_n)).$$

Let $f_n$ denote the composite of $i_n : X_n \hookrightarrow X$ and $f : X \to S$. Then we have

$$(15) \quad \omega_n = i_n^*(\omega_{X/S} \otimes_{\mathcal{O}_{X}} \mathcal{O}_{X}(X_n)) = i_n^*(\omega_{X/S} \otimes_{\mathcal{O}_{X}} \mathcal{T^n}) \cong i_n^*\omega_{X/S} \otimes_{\mathcal{O}_{X_n}} (\mathcal{T^n}/\mathcal{T^{2n}})^v,$$

where $\mathcal{F}^v := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ for any coherent sheaf $\mathcal{F}$ on $X$. The following lemma is well-known.

**Lemma 2.1.4.** (i) We have $f^!\mathcal{O}_S \cong \omega_{X/S}[1]$ and $f_n^!\mathcal{O}_S \cong \omega_n$ respectively in the derived category of quasi-coherent sheaves on $X$ and on $X_n$.

(ii) Let $\mathcal{F}$ be a coherent sheaf on $X_n$. Then we have a canonical isomorphism of $\mathcal{O}_K$-modules

$$\text{Hom}_{\mathcal{O}_K}(H^1(X_n, \mathcal{F}), \mathcal{O}_K/\pi^m \mathcal{O}_K) \xrightarrow{\sim} H^0(X_n, \mathcal{F}^v \otimes \omega_n)$$

for any integer $m \geq n$.

**Proof.** (i). Since the structure morphism $X \to S$ is projective, it factors into a closed immersion $\iota : X \hookrightarrow Y := \mathbb{P}^N_S$ for some $N$, followed by the projection $g : Y \to S$. Observe that by the smoothness and projectivity of $g$ we have $\omega_{Y/S} = \det \Omega_{Y/S}^1 := \Omega_{Y/S}^N$, and $g^!\mathcal{O}_S = \omega_{Y/S}[N]$ in the derived category of quasi-coherent sheaves on $Y$ ([11], Corollary 3.4 p. 383 and proof of Theorem 4.1 p. 389). Let $\mathcal{J} \subset \mathcal{O}_Y$ denote the ideal sheaf defining $\iota$. Then $\iota^*(\mathcal{J}/\mathcal{J}^2)$ is a locally free $\mathcal{O}_X$-module of rank $N - 1$ ([15], Corollary 6.3.8). By [15] 6.4.7, we have

$$(16) \quad \omega_{X/S} \cong \iota^*(\det \Omega_{Y/S}^1) \otimes_{\mathcal{O}_X} \det \left(\iota^*(\mathcal{J}/\mathcal{J}^2)^v\right).$$
Since \( f = g \circ \iota \), from the functoriality of \((-)^!\) with respect to compositions of morphisms of finite type, we have the following isomorphisms in the derived category of quasi-coherent sheaves on \( X \)

\[
f^! \mathcal{O}_S \cong i^!(g^! \mathcal{O}_S) \cong i^!(\omega_{Y/S}[N]) \cong (i^! \omega_{Y/S})[N].
\]

By the computation of \( i^! \) in [11], III, Corollary 7.3, recalling that \( \omega_{X/Y} := \det (i^*(J/J^2))^\vee \) ([11] p. 140–141, Definition (b)), we get

\[
i^! \omega_{Y/S} \cong i^* \omega_{Y/S} \otimes_{\mathcal{O}_X} \det (i^*(J/J^2))^\vee [1 - N] \cong \omega_{X/S}[1 - N],
\]

where we use [10] for the latter isomorphism. We conclude, then, that \( f^! \mathcal{O}_S \cong i^! \omega_{Y/S}[N] \cong \omega_{X/S}[1] \), as asserted. The isomorphism \( f^! \mathcal{O}_S \cong \omega_{n} \) can be derived in exactly the same way, after recalling that \( f_n = f \circ \iota_n : X_n \to X \) a regular immersion of codimension 1.

(ii). The Grothendieck-Serre duality theorem ([11], VII, Corollary 3.4 (c)) implies a canonical isomorphism

\[
(17) \quad \text{RHom}_S(Rf_{n\ast} \mathcal{F}, \mathcal{O}_S) \xrightarrow{\sim} \text{RHom}_{X_n}(\mathcal{F}, f_n^! \mathcal{O}_S).
\]

Now, \( R^i f_{n\ast} \mathcal{F} \) is the Zariski sheaf associated with the presheaf \( U \mapsto H^i(X_n \times_S U, \mathcal{F}|_{X_n \times_S U}) \), for any \( i \in \mathbb{Z} \). Consequently, since \( X_n \) is a scheme of dimension 1, \( R^i f_{n\ast} \mathcal{F} = 0 \) for \( i \neq 0, 1 \). As a result, \( Rf_{n\ast} \mathcal{F} \) has bounded cohomology. Furthermore, \( R^i f_{n\ast} \mathcal{F} \) is the coherent \( \mathcal{O}_S \)-module whose group of global sections is \( H^i(X_n, \mathcal{F}) \); for \( i = 0 \) the fact is evident since \( R^0 f_{n\ast} \mathcal{F} = f_{n\ast} \mathcal{F} \), while for \( i = 1 \) one uses Grothendieck spectral sequence \( H^i(S, R^j f_{n\ast} \mathcal{F}) \Rightarrow H^{i+j}(X_n, \mathcal{F}) \) ([30], 5.8.2) and the vanishing of \( H^i(S, f_{n\ast} \mathcal{F}) \) for \( i > 0 \), due to the affineness of \( S \).

We next compute the \( H^0 \) of the left-hand side complex in (17). Consider the following spectral sequence (see [30] chap. 5, prop. 5.7.6 for the homological version of this spectral sequence, and the extra sign — is due to the fact that \( \text{Hom}_S(-, \mathcal{O}_S) \) is a contravariant functor)

\[
(18) \quad E^{i,j}_2 = \text{Ext}^i_S(R^{-j} f_{n\ast} \mathcal{F}, \mathcal{O}_S) \Rightarrow H^{i+j}(\text{RHom}_S(Rf_{n\ast} \mathcal{F}, \mathcal{O}_S)).
\]

Since \( S \) is the spectrum of a discrete valuation ring, any coherent \( \mathcal{O}_S \)-module \( R^{-j} f_{n\ast} \mathcal{F} \) admits a free resolution of length 1. In particular, \( E^{i,j}_2 = 0 \) for \( i \neq 0, 1 \). Therefore, the differential maps \( d_r : E^{i,j}_r \to E^{i+r,j-r+1}_r \) are all zero for all \( r \geq 2 \). Consequently, \( E^{i,j}_{\infty} = E^{i,j}_2 \), and it is zero whenever \( j \notin \{0, -1\} \) or \( i \notin \{0, 1\} \). The usual six terms short exact sequence associated to the Grothendieck spectral sequence provides then an isomorphism

\[
(19) \quad H^0(\text{RHom}_S(Rf_{n\ast} \mathcal{F}, \mathcal{O}_S)) \xrightarrow{\sim} \text{Ext}^1_S(R^1 f_{n\ast} \mathcal{F}, \mathcal{O}_S)
\]

because \( \text{Hom}_S(f_{n\ast} \mathcal{F}, \mathcal{O}_S) \) is zero, since \( S \) is integral and the stalk of \( f_{n\ast} \mathcal{F} \) at the generic point is trivial.

On the other hand, by (i), we have \( f_1^! \mathcal{O}_S \cong \omega_n \) in the derived category of quasi-coherent sheaves on \( X_n \). As a result, we find

\[
(20) \quad H^0\left(\text{RHom}_{X_n}(\mathcal{F}, f_1^! \mathcal{O}_S)\right) \cong H^0(\text{RHom}_{X_n}(\mathcal{F}, \omega_n)) \cong \text{Hom}_{X_n}(\mathcal{F}, \omega_n) \cong H^0(X_n, \mathcal{F}^\vee \otimes \omega_n).
\]

Now, on applying \( H^0 \) to both sides of (17), using (19), (20), and recalling that the functor \( \Gamma(S, -) \) provides an equivalence between the category of quasi-coherent \( \mathcal{O}_S \)-modules and the category of \( \mathcal{O}_K \)-modules, we get the following isomorphism

\[
\text{Ext}^1_{\mathcal{O}_K}(H^1(X_n, \mathcal{F}), \mathcal{O}_K) \xrightarrow{\sim} H^0(X_n, \mathcal{F}^\vee \otimes \omega_n).
\]
Finally, since the $O_K$-module $H^1(X_n, \mathcal{F})$ is killed by $\pi^n$, we also have

$$\text{Hom}_{O_K}(H^1(X_n, \mathcal{F}), O_K/\pi^mO_K) \cong \text{Ext}^1_{O_K}(H^1(X_n, \mathcal{F}), O_K)$$

for any integer $m \geq n$.

\begin{proof}
\end{proof}

**Lemma 2.1.5** ([23], 3.7.1). For any $i = 1, \ldots, r$, we have $\omega_{X/S} \cdot C_i = 0$.

\begin{proof}
Since $\omega_{X/S}X \cong O_{X_K}$, we have $\omega_{X/S} \cdot X = 0$, i.e., $\sum_{i=1}^r n_i (\omega_{X/S} \cdot C_i) = 0$. In particular, we get the Lemma if $r = 1$. Suppose now $r \geq 2$; since $C_i \cdot X = 0$, we obtain $C_i \cdot C_i < 0$.

If $\omega_{X/S} \cdot C_i < 0$, by [15] Proposition 9.3.10 (a), the divisor $C_i$ is exceptional in the sense of [15] Definition 9.3.1, which gives us a contradiction with the fact that $X/S$ is a minimal regular surface. So $(\omega_{X/S} \cdot C_i) \geq 0$. As a result, $\omega_{X/S} \cdot C_i = 0$ for any $i$.

\begin{proof}
\end{proof}

**Corollary 2.1.6** ([23], 3.7.2). There is a unique integer $n$, $0 \leq n < d$, such that $\omega_{X/S} \cong \mathcal{I}^n$.

\begin{proof}
Since $\omega_{X/S}X \cong O_{X_K}$, $\omega_{X/S} \cong O_X(Y)$, with $Y$ a divisor of $X$ with support contained in $X_K$. Hence, $Y$ is a combination of the components $C_i$. On the other hand, according to the previous Lemma, $Y \cdot C_i = 0$, we have $Y \cdot Y = 0$, hence, $Y$ is a rational multiple of $X_S$ ([5], 9.5/10), i.e., $Y$ is linearly equivalent to $nD$ with $0 \leq n < d$, whence the Corollary follows.

The next result is needed in the proof of Proposition 2.3.2.

**Corollary 2.1.7** ([23], 3.7.3). Suppose that $f: X \to S$ has a section $\sigma$. Let $\mathcal{F}$ be the ideal sheaf defining the closed immersion $\sigma: S \hookrightarrow X$ and set $\omega := \sigma^*(\mathcal{F}/\mathcal{F}^2)$. Then we have a canonical isomorphism $f^*\omega \cong \omega_{X/S}$.

\begin{proof}
By assumption, the torsor $X_K$ has a $K$-rational point; it is then trivial as a torsor under $J_K$. According to Lemma 2.1.3 we have then $d = 1$. As a result, $\omega_{X/S} \cong O_X$ (Corollary 2.1.6), and the canonical morphism $f^*f_\omega \omega_{X/S} \to \omega_{X/S}$ given by adjunction is an isomorphism. On the other hand, we have:

$$O_S \cong (f\sigma)^1O_S \cong \sigma^1(f^1O_S) \cong \sigma^1(\omega_{X/S})[1] \cong \sigma^*\omega_{X/S} \cdot \omega^\vee,$$

where the first isomorphism follows from the fact that $f \circ \sigma = \text{id}_S$, the second one uses the functoriality of $(-)^1$ with respect to compositions of morphisms, the third one is induced from Lemma 2.1.4 (i), and the last one comes from the calculation of $\sigma^1$ ([11], III, Corollary 7.3). Hence $\omega \cong \sigma^*\omega_{X/S} \cong \sigma^*f^*f_\omega \omega_{X/S} \cong f^*\omega_{X/S}$ and we get the following canonical isomorphisms $f^*\omega \cong f^*f_\omega \omega_{X/S} \cong \omega_{X/S}$, as desired.

Let $n \geq 2$ be an integer, and $\mathcal{L}$ an invertible sheaf on $X$, which is of degree 0 on each component of $X_1$. Consider the following short exact sequence of sheaves over $X$

$$0 \to i_{n-1}^*(\mathcal{I}/X_{n-1}) \to i_{ns}O_{X_n} \to i_{1s}O_{X_1} \to 0,$$

where $|X_m|$ stands for the inverse image $i_m^*$. By tensoring by the invertible sheaf $\mathcal{L}^\vee \otimes \omega_{X/S}(X_n) := \mathcal{L}^\vee \otimes \omega_{X/S} \otimes \mathcal{I}^\vee$ and recalling that $\omega_n := i_n^*(\omega_{X/S}(X_n))$, we get the following exact sequence of sheaves over $X$

$$0 \to \mathcal{L}^\vee \otimes i_{n-1s}\omega_{n-1} \to \mathcal{L}^\vee \otimes i_{ns}\omega_n \to \mathcal{L}^\vee \otimes i_{1s}(\omega_n|_{X_1}) \to 0. \tag{21}$$

Since $i_{ns}(\mathcal{L}^\vee|_{X_n} \otimes \omega_n) = i_{ns}(i_n^*\mathcal{L}^\vee \otimes \omega_n) \cong \mathcal{L}^\vee \otimes i_{ns}\omega_n$ and $\mathcal{L}^\vee \otimes i_{1s}(\omega_n|_{X_1}) \cong i_{1s}(\mathcal{L}^\vee|_{X_1} \otimes \omega_n|_{X_1})$ we have the following exact sequence

$$0 \to H^0(X_{n-1}, \mathcal{L}^\vee|_{X_{n-1}} \otimes \omega_{n-1}) \xrightarrow{h_1} H^0(X_n, \mathcal{L}^\vee|_{X_n} \otimes \omega_n) \xrightarrow{h_2} H^0(X_1, \mathcal{L}^\vee|_{X_1} \otimes \omega_n|_{X_1}). \tag{22}$$
As a result, considering the lengths of the finite $\mathcal{O}_K$-modules in (22) and applying Lemma 2.0.1 we get

\[(23) \quad h^0(X_{n-1}, \mathcal{L}^\vee|_{X_{n-1}} \otimes \omega_{n-1}) \leq h^0(X_n, \mathcal{L}^\vee|_{X_n} \otimes \omega_n) \leq h^0(X_{n-1}, \mathcal{L}^\vee|_{X_{n-1}} \otimes \omega_{n-1}) + 1.\]

**Lemma 2.1.8** (23, 3.7.6). For $n$ a positive integer, we have either $\omega_n \cong L|_{X_n}$, and in this case $H^0(X_1, \mathcal{L}^\vee|_{X_1} \otimes \omega_n|_{X_1}) \cong k$ and the sequence (22) is exact, or $\omega_n \not\cong L|_{X_n}$, in which case the canonical morphism $h_1 : H^0(X_{n-1}, \mathcal{L}^\vee|_{X_{n-1}} \otimes \omega_{n-1}) \to H^0(X_n, \mathcal{L}^\vee|_{X_n} \otimes \omega_n)$ is bijective.

**Proof.** Suppose first that $\omega_n \cong L|_{X_n}$, so that $\mathcal{L}^\vee|_{X_n} \otimes \omega_n \cong \mathcal{O}_{X_n}$. Then the map $h_2$ in (22) can be identified (in a non canonical way) with the canonical map

\[(24) \quad H^0(X_1, \mathcal{O}_{X_1}) \rightarrow H^0(X_1, \mathcal{O}_{X_1}).\]

By Lemma 2.0.1 we have $H^0(X_1, \mathcal{O}_{X_1}) \cong k$. Hence, every element of $H^0(X_1, \mathcal{O}_{X_1})$ can be lifted to an element of $H^0(X_n, \mathcal{O}_{X_n})$. As a result, the map (24) is surjective and hence the complex (22) is right exact. Thus we obtain the first assertion. Next, suppose $\omega_n \not\cong L|_{X_n}$. If $(\mathcal{L}^\vee|_{X_n} \otimes \omega_n)|_{X_1} \not\cong \mathcal{O}_{X_1}$ we conclude by (Lemma 2.0.1) that $h_2 = 0$ and hence $h_1$ is an isomorphism. Suppose then $(\mathcal{L}^\vee|_{X_n} \otimes \omega_n)|_{X_1} \cong \mathcal{O}_{X_1}$ and show that still $h_2 = 0$. Let $a \in H^0(X_n, \mathcal{L}^\vee|_{X_n} \otimes \omega_n)$. Suppose that the image of $a$ in $H^0(X_1, \mathcal{L}^\vee|_{X_1} \otimes \omega_n|_{X_1}) \cong k$ (Lemma 2.0.1) is non-zero. Then $a$ is nowhere vanishing because its restriction at any point of $X_1$ (i.e., at any point of $X_n$) is non-zero. Hence the invertible sheaf $\mathcal{L}^\vee|_{X_n} \otimes \omega_n$ is trivial because it admits a nowhere vanishing section. But this contradicts the hypothesis.

Next we investigate the Picard group of $X_n$. Let $n \geq 2$ be an integer. The kernel of the surjective morphism $\text{Pic}(X_n) \rightarrow \text{Pic}(X_{n-1})$ is an $\mathcal{O}_K$-module of finite length killed by $p$. More precisely, consider the closed immersion $i'_{n-1} : X_{n-1} \hookrightarrow X_n$, given by the ideal sheaf $\mathfrak{N} := \mathcal{I}^{n-1}/\mathcal{I}^n \subset \mathcal{O}_{X_n}$. The sheaf $\mathfrak{N}$ is nilpotent (in fact, $\mathfrak{N}^2 = 0$), so we get a short exact sequence of abelian sheaves over $X_n$:

\[0 \rightarrow 1 + \mathfrak{N} \rightarrow \mathcal{O}_{X_n}^* \rightarrow i'_{n-1*} \mathcal{O}_{X_{n-1}}^* \rightarrow 0.\]

Since $\mathfrak{N}^2 = 0$, the morphism $x \mapsto 1 + x$ defines an isomorphism of abelian sheaves

\[(25) \quad \beta : \mathfrak{N} \rightarrow 1 + \mathfrak{N}.\]

Since $X_n$ has dimension 1, the cohomology groups $H^2(X_n, 1 + \mathfrak{N}) \cong H^2(X_n, \mathfrak{N})$ are zero. In this way we get the following long exact sequence

\[(26) \quad H^0(X_{n-1}, \mathcal{O}_{X_{n-1}}^*) \xrightarrow{\partial^*} H^1(X_n, 1 + \mathfrak{N}) \rightarrow \text{Pic}(X_n) \xrightarrow{\alpha} \text{Pic}(X_{n-1}) \rightarrow 0.\]

On the other hand, from the following exact sequence of sheaves over $X_n$:

\[0 \rightarrow \mathfrak{N} \rightarrow \mathcal{O}_{X_n} \rightarrow i'_{n-1*} \mathcal{O}_{X_{n-1}} \rightarrow 0,\]

we obtain a long exact sequence (recall that $H^2(X_n, \mathfrak{N}) = 0$):

\[(27) \quad H^0(X_{n-1}, \mathcal{O}_{X_{n-1}}) \xrightarrow{\partial} H^1(X_n, \mathfrak{N}) \rightarrow H^1(X_n, \mathcal{O}_{X_n}) \xrightarrow{\alpha'} H^1(X_{n-1}, \mathcal{O}_{X_{n-1}}) \rightarrow 0.\]

We have then the following result:

**Lemma 2.1.9** (Dévissage d'Oort, [20], Proposition in § 6). Consider the morphisms $\partial$ and $\partial^*$ given respectively in (27) and (26) above, then one has $\beta(\text{im}(\partial)) = \text{im}(\partial^*)$. 

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As a consequence, \( \ker(\alpha) \cong \coker(\partial) \) (as abelian sheaves). Since \( \mathcal{R} = \mathcal{I}^{n-1}/\mathcal{I}^n \) is an \( \mathcal{O}_K \)-module killed by \( p \), it follows that \( \ker(\alpha) \cong \coker(\partial) \) is an \( \mathcal{O}_K \)-module of finite length killed by \( p \). Hence the order of \( I|_{X_n} \) in \( \text{Pic}(X_n) \) is equal either to the order of \( I|_{X_{n-1}} \), or \( p \) times the order of \( I|_{X_{n-1}} \). Let \( d' \) denote the order of the invertible sheaf \( I|_{X_1} \). Then for any \( n \geq 2 \), the order of \( I|_{X_n} \) is of the form \( d'p^n \). On the other hand, since \( I|_{X_K} \cong \mathcal{O}_K \), and since the invertible sheaf \( I \) is of order \( d \), by Lemma 6.4.4 in [21], for \( m \in \mathbb{Z} \) sufficiently large, the invertible sheaf \( I|_{X_m} \) is of order divisible by \( d \). Moreover, since \( I \) is of order \( d \), the order of \( I|_{X_m} \) divides \( d \). Hence for \( m \gg 0 \), the order of \( I|_{X_m} \) is equal to \( d \). Let \( e \geq 0 \) be the integer such that \( d = d'p^e \). For \( i = 0, \ldots , e \), let \( m_i \) denote the smallest integer \( n \) such that \( I|_{X_n} \) is of order \( d'p^i \).

Let \( \phi(n) := h^1(X_n, \mathcal{O}_{X_n}) \) be the length of the \( \mathcal{O}_K \)-module \( H^1(X_n, \mathcal{O}_{X_n}) \). By Lemma 2.1.4 (ii) (with \( \mathcal{F} = \mathcal{O}_{X_n} \)), \( \phi(n) = h^0(X_n, \omega_n) \). Therefore, according to the first inequality of (23) we have \( \phi(n) \geq \phi(n-1) \). Similarly, by Lemma 2.1.4 again, \( \phi(0) > \phi(n-1) \) if and only if \( h^0(X_n, \omega_n) > h^0(X_{n-1}, \omega_{n-1}) \), hence if and only if \( \omega_n \cong \mathcal{O}_{X_n} \), in which case \( \phi(n) = \phi(n-1) + 1 \) (Lemma 2.1.8). One then gets from Lemma 2.1.9 the following Corollary:

**Corollary 2.1.10.** Let \( n \geq 2 \) be an integer. We have either \( \phi(n) = \phi(n-1) \), in which case the morphism \( \alpha : \text{Pic}(X_n) \to \text{Pic}(X_{n-1}) \) is an isomorphism, or \( \phi(n) = \phi(n-1) + 1 \), in which case \( \ker(\alpha) \) is an \( \mathcal{O}_K \)-module of length 1, and hence a \( k \)-vector space of dimension 1.

**Proof.** If \( \phi(n) = \phi(n-1) \), the morphism \( \alpha' \) in (27) is an isomorphism, and this implies that \( \partial \) in (27) is surjective. Conjecture. When \( \phi(n) = \phi(n-1) \), \( \ker(\alpha') \cong \coker(\partial) \) is an \( \mathcal{O}_K \)-module of length 1. From Lemma 2.1.9 under the identification \( H^1(X_n, \mathcal{R}) \cong H^1(X_n, 1 + \mathcal{R}) \) induced by the isomorphism \( \beta \) in (25), we get that \( \ker(\alpha) \cong \coker(\partial^*) \) is an \( \mathcal{O}_K \)-module of dimension 1. Hence \( \ker(\alpha) \) is a \( k \)-vector space of dimension 1, as desired.

**Lemma 2.1.11** ([23], 3.7.9). 

(i) For \( i = 0, \ldots , e \), the sheaf \( \omega_{m_i} \) is isomorphic to \( \mathcal{O}_{X_{m_i}} \).

(ii) For \( i = 0, \ldots , e - 1 \), there exists an integer \( k_i > 0 \) such that \( m_{i+1} = m_i + k_i d'p^i \).

(iii) The integers \( m \in (m_i, m_{i+1}] \) such that \( \phi(m) = \phi(m-1) + 1 \) are exactly those which can be written as \( m = m_i + hd'p^i \) for some integer \( h \).

**Proof.** (i) Let \( n > 1 \) be an integer such that the order of \( I|_{X_n} \) is different from that of \( I|_{X_{n-1}} \). We then have \( \phi(n) = \phi(n-1) + 1 \), and the canonical map \( \text{Pic}(X_n) \to \text{Pic}(X_{n-1}) \) has a kernel of length 1 (Corollary 21.10). Now use the exact sequence (22) once again. By Lemma 2.1.4 (ii), the injective morphism \( H^0(X_n, \mathcal{R}) \to H^0(X_n, \omega_n) \) has a non-trivial cokernel, and this implies that \( \omega_n \cong \mathcal{O}_{X_n} \) (Lemma 21.18).

For the assertions (ii) and (iii), recall that \( \omega_{m_i} = \omega_{X/S}(m_i D)|_{X_{m_i}} \), and by Corollary 21.6 there exists an integer \( n (1 \leq n \leq d) \) such that \( \omega_{m_i} = \omega_{X/S}(mD) \). Let \( L_m := \omega_{X/S}(mD) \), then \( L_m \cong \mathcal{I}^{n-m} \). But \( \omega_{m_{i+1}} = L_{m+i+1}|_{X_m+i+1} = \mathcal{I}^{n-m_{i+1}}|_{X_m+i+1} \cong \mathcal{O}_{X_{m+i+1}} \). Hence, \( \omega_{m_{i+1}}|_{X_m} = \mathcal{I}^{n-m_i}|_{X_m} \cong \mathcal{O}_{X_m} \) by (i). Since \( \omega_{m_i} = \mathcal{I}^{n-m_i}|_{X_m} \cong \mathcal{O}_{X_m} \), we have \( \mathcal{I}^{n-m_i}|_{X_m} \cong \mathcal{O}_{X_m} \), and thus there exists an integer \( k_i > 0 \) such that \( m_{i+1} = m_i + k_i d'p^i \).

The same argument also gives us that for an integer \( m \) so that \( m_{i+1} > m > m_i \) and \( \phi(m) > \phi(m-1) \), we have \( \omega_m \cong \mathcal{O}_{X_m} \) and there exists an integer \( 0 < h \leq k_i \) verifying \( m = m_i + hd'p^i \). Conversely, let \( m \) be an integer of the form \( m = m_i + hd'p^i \) (for some \( 0 < h \leq k_i \)); we show that \( \phi(m) > \phi(m-1) \).
We may assume that \( m < m_{i+1} \), hence \( \mathcal{I}|_{X_m} \) is of order \( d^p i \). By Lemma 2.1.8 we only need to show that \( \omega_m \cong \mathcal{O}_{X_m} \). But

\[
\omega_m = \mathcal{L}_m|_{X_m} \cong \mathcal{I}|_{\mathcal{I}m} - \mathcal{I}|_{\mathcal{I}m + \mathcal{I}m + \mathcal{I}m - \mathcal{I}m - \mathcal{I}m} \cong \omega_{m_{i+1}}|_{X_m} \cong \mathcal{O}_{X_m},
\]

since \( \omega_{m_{i+1}} \cong \mathcal{O}_{X_{m_{i+1}}} \) and \( \mathcal{I}|_{X_m} \) is of order \( d^p i \). This completes the proof. \( \square \)

2.2 The function \( \psi \)

We now come to the key construction of this section. We define a function \( \varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that the graph of \( \varphi \) is just the upper concave envelope of the set \( \{(n, \phi(n)) \mid n \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{R}^2 \). Then \( \varphi \) is a continuous piecewise linear function with \( \varphi(0) = 0 \), and \( \varphi(1) = 1 \) by Lemma 2.0.1. Since \( X/S \) is proper flat of relative dimension 1, by [9] 7.7.5 (the equivalence of conditions (a’) and (b’) for \( p = -1 \)) and 7.8.4 (the equivalence of conditions (b) and (e) for \( p = -2 \)), we have

\[
H^1(X, \mathcal{O}_X) \otimes \mathcal{O}_K \mathcal{O}_K / \pi^n \mathcal{O}_K \cong H^1(X_{nd}, \mathcal{O}_{X_{nd}}).
\]

Therefore \( \lim_{n \to +\infty} \phi(n d) = +\infty \) since \( H^1(X, \mathcal{O}_X) \otimes \mathcal{O}_K \mathcal{O}_K \cong H^1(X_K, \mathcal{O}_K) \) is of dimension 1 over \( K \). Consequently, the function \( \varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is strictly increasing with unbounded image, hence bijective. Let \( \psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be its inverse. Then \( \psi \) is also continuous and piecewise linear. For any integer \( n \geq 1 \), \( \psi(n) \) is just the smallest integer \( m \geq 1 \) such that \( \phi(m) = n \). If we denote by \( \delta_n \) the order of the invertible sheaf \( \mathcal{I}|_{X_{\psi(n)}} \), then by Lemma 2.1.11 (iii) we have

\[
\psi(n + 1) = \psi(n) + \delta_n,
\]

and for all \( m \in \mathbb{Z} \) such that \( \psi(n) \leq m < \psi(n+1) \), the morphism of groups \( \text{Pic}(X_m) \to \text{Pic}(X_{\psi(n)}) \) is an isomorphism (Corollary 2.1.10). Observe that \( \delta_1 = d' \) with \( d' \) the order of \( \mathcal{I}|_{X_1} \). We call the functions \( \varphi, \psi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1} \) the Herbrand functions; they are exact analogues of the Herbrand functions used by Serre in [25], §3.

To finish this section, we present some corollaries of the previous discussion, which will be useful in the next section. The first follows directly from Lemma 2.1.8 (23), and Lemma 2.1.4 (ii).

**Corollary 2.2.1.** Let \( \mathcal{L} \) be an invertible sheaf on \( X \), of degree 0 on each component of \( X_1 \), and let \( n \) be an integer \( \geq 2 \). Then we have \( h^1(X_{n-1}, \mathcal{L}|_{X_{n-1}}) \leq h^1(X_n, \mathcal{L}|_{X_n}) \leq h^1(X_{n-1}, \mathcal{L}|_{X_{n-1}}) + 1 \). Moreover, \( h^1(X_n, \mathcal{L}|_{X_n}) = h^1(X_{n-1}, \mathcal{L}|_{X_{n-1}}) + 1 \) if and only if \( \mathcal{L}|_{X_n} \cong \omega_n \).
Corollary 2.2.2 ([23], 3.9.1). Let $\mathcal{L}$ be an invertible sheaf on $X$, of degree 0 on each component of $X_1$, and let $n$ be an integer $\geq 1$. Then, if the morphism $i^*_n \colon H^1(X, \mathcal{L}) \to H^1(X_n, \mathcal{L}|_{X_n})$ is not bijective, there exists an integer $m > n$ such that $\mathcal{L}^i|_{X_m} \otimes \omega_m$ is trivial.

Proof. We first remark that $i_n^*$ is surjective, and we have $H^1(X, \mathcal{L}) \cong \lim_{\to \leftarrow} H^1(X_m, \mathcal{L}|_{X_m})$. Moreover, $i_n^*$ is not bijective if and only if there exists $m > n$ such that $H^1(X_n, \mathcal{L}|_{X_n}) \to H^1(X_{m-1}, \mathcal{L}|_{X_{m-1}})$ is not injective. By Lemma 2.1.3, this is equivalent to saying that the injective morphism $H^0(X_{m-1}, \mathcal{L}^i|_{X_{m-1}} \otimes \omega_{m-1}) \to H^0(X_m, \mathcal{L}^i|_{X_m} \otimes \omega_m)$ is not surjective. Hence, there exists an integer $L|\omega_m$ such that $1 < m < n$ and $\mathcal{L}^i|_{X_m} \otimes \omega_m \cong \mathcal{O}_{X_m}$ (Lemma 2.1.8) and one concludes.

For $\mathcal{L}$ an invertible sheaf on $X$ of degree 0 on each component of $X_1$, the $\mathcal{O}_K$-module $H^1(X, \mathcal{L})$ is of infinite length if and only if $\mathcal{L}|_{X_K} \cong \mathcal{O}_K$, that is, if and only if $\mathcal{L}$ is isomorphic to a power of $\mathcal{I} = \mathcal{O}_X(-D)$ (see the proof of Corollary 2.1.6). Hence, when $\mathcal{L}|_{X_K}$ is not isomorphic to $\mathcal{O}_K$, the $\mathcal{O}_K$-module $H^1(X, \mathcal{L})$ is of finite length. Moreover, we have

Corollary 2.2.3 ([23], 3.9.2). Let $\mathcal{L}$ be an invertible sheaf on $X$ of degree 0 on each component of $X_1$, and let $n \geq 1$ be an integer. Suppose that the $\mathcal{O}_K$-module $H^1(X, \mathcal{L})$ is of length $\geq n$. Then $\mathcal{L}|_{X_{\psi(n)}} \cong \mathcal{I}^i|_{X_{\psi(n)}}$ with $i$ a suitable integer.

Proof. We will prove by induction that, under our assumptions and for any $n'$, $1 \leq n' \leq n - 1$, the $\mathcal{O}_K$-module $H^1(X_{\psi(n') - 1}, \mathcal{L}|_{X_{\psi(n') - 1}})$ is of length $n'$. As a result, the canonical map $H^1(X, \mathcal{L}) \to H^1(X_{\psi(n) - 1}, \mathcal{L}|_{X_{\psi(n) - 1}})$ is not bijective (here we define by convention $H^1(X_n, \mathcal{L}|_{X_n}) = 0$). Hence Corollary 2.2.2 provides an integer $m > \psi(n) - 1$ such that $\mathcal{L}|_{X_m} \cong \omega_m \cong \mathcal{I}^{n-m}|_{X_m}$, where $\bar{n}$ is the integer appearing in Corollary 2.1.6. Since $\psi(n) \leq m$, the Corollary follows.

We begin with the case where $n' = n$ (hence $n \geq 2$). By Lemma 2.0.1 either the $\mathcal{O}_K$-module $H^1(X_1, \mathcal{L}|_{X_1})$ is of length 1, which is equivalent to saying that $\mathcal{L}|_{X_1} \cong \mathcal{O}_{X_1}$, or $H^1(X_1, \mathcal{L}|_{X_1})$ is trivial, and, in this case, the natural morphism $H^1(X, \mathcal{L}) \to H^1(X_1, \mathcal{L}|_{X_1})$ is not bijective. By Corollary 2.2.2, there is then an integer $m > 1$, such that $\mathcal{L}|_{X_m} \cong \omega_m = \mathcal{O}_{X/S}(X_m)|_{X_m}$. Hence $\mathcal{L}|_{X_1} \cong \omega_{m}|_{X_1}$ is isomorphic to $\mathcal{I}^{n-m}|_{X_1}$, again applying Corollary 2.1.6. Thus, in both cases, we have $\mathcal{L}|_{X_1} \cong \mathcal{I}^i|_{X_1}$ for a suitable integer $i$. Moreover, for $m'$ an integer such that $1 \leq m' \leq \psi(2) - 1$, the canonical morphism $\text{Pic}^0(X_{m'}) \to \text{Pic}^0(X_1)$ is bijective (see Corollary 2.1.10); hence $\mathcal{L}|_{X_{m'}} \cong \mathcal{I}|_{X_{m'}}$. But $\psi(2) = \psi(1) + \delta_1 = 1 + \delta_1$, and $\mathcal{O}_{X/S}$ is isomorphic to a power of $\mathcal{I}$ (Corollary 2.1.6), there exists a unique integer $m'$ such that $1 \leq m' \leq \psi(2) - 1$, and $\mathcal{L}|_{X_{m'}} \cong \omega_{m'} = (\omega_{X/S} \otimes \mathcal{I}^{m'})|_{X_{m'}}$. Hence, by Lemma 2.2.1 we find that the $\mathcal{O}_K$-module $H^1(X_{\psi(2) - 1}, \mathcal{L}|_{X_{\psi(2) - 1}})$ is of length 1. Suppose now that the above assertion has been verified for $1 \leq n' - 1 < n$ (with $n' < n$). Under the assumptions of the Lemma, the map

$$H^1(X, \mathcal{L}) \to H^1(X_{\psi(n') - 1}, \mathcal{L}|_{X_{\psi(n') - 1}})$$

is not surjective. Hence, there exists an integer $m \geq \psi(n')$, such that $\mathcal{L}|_{X_m} \cong \omega_m$, and so $\mathcal{L}|_{X_{\psi(n')}} \cong \mathcal{I}^i|_{X_{\psi(n')}}$ for $0 \leq i < \delta_{n'}$. On the other hand, since $\psi(n') + 1 = \psi(n') + \delta_{n'}$ (see (28)), there exists a unique integer $m$ such that $\psi(n') \leq m \leq \psi(n' + 1) - 1$, and $\mathcal{L}|_{X_m} \cong \omega_m$, in particular, the $\mathcal{O}_K$-module $H^1(X_{\psi(n' + 1) - 1}, \mathcal{L}|_{X_{\psi(n' + 1) - 1}})$ is of length $n'$. This finishes the induction, and hence also the proof of the Corollary.

2.3 Numerical studies

We maintain the notation introduced in the beginning of this section. So $X_K$ denotes a $K$-torsor under an elliptic curve $A_K$, and $f : X \to S$ denotes its $S$-proper regular minimal model. We consider the $S$-proper minimal regular model $f' : X' \to S$ of the elliptic curve Pic,$^0_{X_K/K}$. As a
result, $f'$ has a canonical section given by $e$, the schematic closure of the identity element of $X'_K$ in $X'$. Its dualizing sheaf is $f^*\omega$ (with $\omega$ defined by the section $e$, see Corollary 2.1.7). The main result of this part is that, by using some numerical invariants of $X/S$, one can then recover the sheaf $\omega_{X/S}$ from the sheaf $\omega$ (Lemma 2.3.1).

According to [16], Theorem 3.8, there exists a morphism of $\mathcal{O}_K$-modules

$$\tau_X : H^1(X, \mathcal{O}_X) \rightarrow H^1(X', \mathcal{O}_{X'})$$

which extends the natural isomorphism over the generic fibre. Moreover, its kernel is the torsion part of $H^1(X, \mathcal{O}_X)$ ([16], 3.1 a)), and the $\mathcal{O}_K$-modules ker$(\tau_X)$ and coker$(\tau_X)$ have the same length. In this section we give an estimate for this length.

By duality, we obtain the following map

$$\tau^\vee_X : H^0(X', \omega_{X'/S}) \cong (H^1(X', \mathcal{O}_{X'}))^\vee \rightarrow (H^1(X, \mathcal{O}_X))^\vee \cong H^0(X, \omega_{X/S}).$$

On the other hand, we have $f'_*\omega_{X'/S} \cong f'_*f^*\omega \cong \omega$ (see Corollary 2.1.7 with $\omega$ defined via the $S$-section of $f'$ associated with the identity element of $X'_K$). Hence we get following canonical map:

$$\tau^\vee_X : \omega(S) \cong H^0(X', \omega_{X'/S}) \rightarrow H^0(X, \omega_{X/S}) = f'_*\omega_{X/S}(S)$$

which is injective, but not an isomorphism in general. Since $S$ is affine, $\tau^\vee_X$ also corresponds to an injective morphism of sheaves, again denoted by $\tau^\vee_X$:

$$(29) \quad \tau^\vee_X : \omega \rightarrow f'_*\omega_{X/S}.$$ 

which gives by adjunction the following non-zero canonical morphism of sheaves on $X$:

$$\tau'^\vee_X : f^*\omega \rightarrow \omega_{X/S}.$$ 

Since $\tau^\vee_X$ is an isomorphism on the generic fibre $X_K$ of $X$, the same holds for $\tau'^\vee_X$. Under the identification given by the restriction of $\tau^\vee_X$ to $X_K$, $f^*(\omega)$ and $\omega_{X/S}$ are naturally identified with two $\mathcal{O}_X$-submodules of $\omega_{X/K} \cong (\omega_{X/S})|_{X_K}$.

**Lemma 2.3.1** ([23], pp. 31–32). We have $\omega_{X/S} = f^*\omega \otimes \mathcal{I}^{-\chi}$ for a suitable non-negative integer $\chi$, as submodule of $\omega_{X/K}$. Moreover, $[\chi/d]$, i.e., the biggest integer $\leq \chi/d$, is the length of the torsion part of $H^1(X, \mathcal{O}_X)$.

**Proof.** Since $f^*\omega$ is invertible and the scheme $X$ is regular, the generically injective morphism $\tau'^\vee_X$ is automatically injective. By tensoring both sides with $\omega_{X/S}^{-1}$, we get an invertible ideal sheaf

$$\mathcal{J} := f^*\omega \otimes \omega_{X/S}^{-1} \rightarrow \mathcal{O}_X.$$ 

The closed subscheme $V(\mathcal{J})$ of $X$ defined by $\mathcal{J}$ has support contained in $X_s$. Moreover, the intersection numbers $V(\mathcal{J}) \cdot C_i = 0$ for any irreducible component $C_i$ of $X_s$ (Lemma 2.1.5). Hence the effective divisor $V(\mathcal{J}) \hookrightarrow X$ is a multiple of $D = X_1 = V(\mathcal{I}) \hookrightarrow X$. So one can find some non-negative integer $\chi \in \mathbb{N}$ such that $\mathcal{J} = f^*\omega \otimes \omega_{X/S}^{-1} = \mathcal{I}^\chi$. In other words, we find the following identification $\omega_{X/S} = f^*\omega \otimes \mathcal{I}^{-\chi}$ as submodules of $\omega_{X/K}$. This proves the first assertion. Under the latter identification, the morphism $\tau'^\vee_X$ is then obtained from the canonical map: $\mathcal{O}_X \hookrightarrow \mathcal{I}^{-\chi}$ after tensoring both sides by $f^*\omega$. Hence the morphism $\tau^\vee_X$ in (29) can now be described as the following composition:

$$\tau^\vee_X : \omega \overset{\sim}{\rightarrow} f_*f^*\omega \cong f_*(f^*\omega \otimes \mathcal{O}_X) \rightarrow f_*(f^*\omega \otimes \mathcal{I}^{-\chi}) = f_*\omega_{X/S},$$

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where the first isomorphism is just the adjunction map, and the third map is induced by the canonical injection $O_X \hookrightarrow I^{-\chi}$. Hence, by using the projection formula
\[ f_*(f^*\omega \otimes I^{-\chi}) \cong \omega \otimes f_*(I^{-\chi}), \]
the morphism $\tau_X^\chi$ is obtained from the canonical map $O_S = f_*O_X \to f_*(I^{-\chi})$ after tensoring by the invertible sheaf $\omega$. Now, if we identify these two sheaves as $O_S$-submodule of $f_K^*O_{X_K}$ (here $f_K$ is the generic fibre of $f$), we have $f_*(I^{-\chi}) = \pi^{-[\chi/d]}O_S \subset O_{\text{Spec}(K)}$. Indeed, if we write $f_*(I^{-\chi}) = \pi^{-r}O_S$ for some non-negative integer $r$, then $r$ is the largest integer $r'$ such that $\pi^{-r'}O_S \subset f_*(I^{-\chi})$. But this last inclusion is equivalent to the inclusion
\[ f^*(\pi^{-r'}O_S) = I^{-dr'} \subset I^{-\chi}, \]
hence is also equivalent to the condition $-dr' \geq -\chi$, namely $r' \leq \chi/d$. The maximum of the possible $r'$ is then given by $r = [\chi/d]$, and thus we obtain $f_*(I^{-\chi}) = \pi^{-[\chi/d]}O_S$. Hence, if we identify $\omega$ and $f_*(\omega_{X/S})$ as $O_S$-submodule of $\omega_K = \omega \otimes O_S O_{\text{Spec}(K)}$, the injection (29) gives us the following equality inside $\omega_K$: $f_*(\omega_{X/S}) = \pi^{-[\chi/d]}\omega$. As a result, we find that $\text{coker}(\tau_X^\chi)$ and the torsion part of $H^1(X, O_X)$ are both of length $[\chi/d]$. \hfill \square

**Proposition 2.3.2** ([23], pp. 31–32). Let $X_K/K$ be a torsor under an elliptic curve $A_K$. Let $\chi \geq 0$ be the integer introduced in Lemma [2.3.7]. Then one has
\[ \chi = (d-1) + k_0(d-d') + k_1(d-d'p) + \ldots + k_{e-1}(d-d'p^{e-1}), \]
where the positive integers $k_i$ were introduced in Lemma [2.1.11] and $d'$ is the order of the invertible sheaf $I|_{X_1}$, with $d = d'p^e$.

**Proof.** By Lemma [2.1.11] we have $\phi(m_e) = 1 + k_0 + \ldots + k_{e-1}$, and, for $n > m_e$, we have $\phi(n) = \phi(n-1) + 1$ if and only if $n - m_e$ is a multiple of $d = d'p^e$. In particular, if we write $m_e = hd - a$ for non-negative integers $h, 0 \leq a < d$, we have $\phi(m_e) = \phi(hd)$. Let $T$ be the torsion subsheaf of $M := R^1f_*O_X$. Consider $L := M/T$, which is free of rank 1 over $S$. We have $R^1f_*(\mathcal{O}_S) = R^1f_*(O_X/\pi^nO_X) \cong M/\pi^nM$. Hence, for $n \geq h$, the length of $M/\pi^nM$, i.e., $\phi(hd)$, increases by 1 with $n$. This means that $T$ is killed by $\pi^h$, and that $\ell(M/\pi^nM) = \ell(T) + \ell(L/\pi^nL)$. In particular, if we take $n = h$, we get
\[ \phi(m_e) = \phi(hd) = \ell(R^1f_*(O_S)) = \ell(M/\pi^hM) = \ell(T) + h. \]

On the other hand, $\omega_{m_e} = I^{-(\chi+m_e)}|_{X_{m_e}}$ is the trivial invertible sheaf (Lemmas [2.1.11] (i) and [2.3.1]), and since $I|_{X_{m_e}}$ is of order $d$, there exists an integer $\alpha$ such that $\chi + m_e = \alpha d$. Hence $\chi = (\alpha - h)d + a$, and we have $\ell(T) = [\chi/d] = \alpha - h$. Thus, (by using the equality (31) and Lemma [2.3.1]), we find that $\ell(T) = [\chi/d] = \alpha - h = \phi(m_e) - h$. Hence $\alpha = \phi(m_e)$, and
\[ \chi = \phi(m_e)d - m_e = (1 + k_0 + \ldots + k_{e-1})d - (1 + k_0d' + \ldots + k_{e-1}d'p^{e-1}) = (d-1) + k_0(d-d') + k_1(d-d'p) + \ldots + k_{e-1}(d-d'p^{e-1}). \]
\hfill \square

**Corollary 2.3.3** ([23], 3.8.2). The following conditions are equivalent:

(i) $X/S$ is cohomologically flat in dimension 0.
(ii) $\chi < d$.

(iii) $e = 0$.

(iv) $T|_{X_1}$ is of order $d$.

Moreover, if these conditions are satisfied, we have $\chi = d - 1$.

Proof. The $S$-scheme $X$ is cohomologically flat in dimension $0$ if and only if $T$, the torsion subsheaf of $R^1f_*\mathcal{O}_X$, is trivial, i.e., if and only if $\ell(T) = [\chi/d] = 0$; the latter assertion is equivalent to saying that $\chi < d$, hence (i)$\iff$(ii). The equivalence between (iii) and (iv) comes from the definition of $e$. If $e > 0$, then by Lemma 2.3.2 (ii) all integers $k_i$ are positive and $\chi \geq d$ by (30). Therefore (ii)$\iff$(iii).

Remark 2.3.4. Further conditions equivalent to the conditions in Corollary 2.3.3 can be found in 3.2.4. By the discussion after Lemma 2.1.9 these conditions are satisfied if $(d, p) = 1$. But note that, as shown in [21] Remark 9.4.3 (d), it is possible for the equivalent conditions of Corollary 2.3.3 to be satisfied when $p$ divides $d$.

3 Filtrations and comparison of the pro-algebraic structures

Recall that $\mathcal{O}_K$ is a complete discrete valuation ring with field of fractions $K$ and algebraically closed residue field $K$. Recall that, as shown in §3.4.5, by the discussion after Lemma 2.1.9 these conditions are satisfied if $(d, p) = 1$. But note that, as shown in [21] Remark 9.4.3 (d), it is possible for the equivalent conditions of Corollary 2.3.3 to be satisfied when $p$ divides $d$.

For this construction, we shall make intensive use of the notion of dilatation of an $S$-scheme. We recall this construction briefly (see [5], §3.2, for more details). Let $H$ be a smooth $S$-scheme of finite type, $W \hookrightarrow H$ a closed subscheme over $k$. Denote by $J$ the ideal sheaf of $W \hookrightarrow H$. Let $Bl_W(H)$ denote the blowing-up of $H$ along the center $W \hookrightarrow H$. Then, by definition, the dilatation of $H$ along the center $W \hookrightarrow H$ is the largest open scheme $H' \subset Bl_W(H)$ such that the ideal $J\mathcal{O}_{H'} \subset \mathcal{O}_{H'}$ is generated by $\pi$. According to [5], 3.2/1, $H'$ is a flat $S$-scheme, satisfying the following universal property: let $Z$ be a flat $S$-scheme and $v: Z \to H$ a morphism of $S$-schemes such that its restriction to special fibres, $v_s: Z_s \to H_s$, factors through $W \to H_s$, then there exists a unique $S$-morphism $v': Z \to H'$ rendering the obvious diagram commutative. Furthermore, if $W$ is a smooth over $k$, then $H'$ is smooth over $S$ ([5], 3.2/3), and if $H$ is an $S$-group scheme, then the same is $H'$ (combine [5], 3.2/1 and 3.2/2).

In order to simplify the presentation, for the remainder of the section we will use the following notation: let $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $n \leq m$ (by convention $n \leq m$ for any $n \in \mathbb{Z}_{\geq 0}$). Denote by $P^{[n,m]}$ the kernel of the canonical morphism of functors $\text{Pic}^0_{X_m/S} \to \text{Pic}^0_{X_n/S}$. Here, we set $X_{\infty} = X$ and $\text{Pic}^0_{X_0/S} = 0$, the final object in the category of abelian fppf-sheaves on $S$. Furthermore let

$$P^{[n]} := P^{[n,\infty]} = \ker(\text{Pic}^0_{X/S} \to \text{Pic}^0_{X_n/S}), \quad P^{[n]} := P^{[0,n]} = \text{Pic}^0_{X_n/S}.$$
In particular, \( P[0] = \text{Pic}^0_{X/S} \). For any integer \( n \geq 1 \), we define by induction a smooth \( S \)-group scheme \( J[n] \) as the dilatation of \( J^{[n-1]} \) along the unit element of the special fibre of \( J_s^{[n-1]} \) (here, \( J[0] := J \)).

**Lemma 3.0.1.** For any \( n \in \mathbb{Z}_{\geq 1} \), we have the following exact sequence:

\[
0 \to J^{[n]}(S) \to J(S) \to J(S_n) \to 0.
\]

**Proof.** The map \( J(S) \to J(S_n) \) is surjective by the smoothness of \( J \). By the universal property of dilatations we get inclusions \( J^{[n]}(S) \subseteq J(S) \) and the exactness of \([32]\) for \( n = 1 \). In order to prove the exactness for \( n > 1 \) we need to work with the local description of dilatations as in \([5], \S\ 3.2\). Let then \( U \subset J \) be an open neighbourhood of the zero section \( 0 := 0_J \) of \( J \), and \( f : U \to Z := \mathbb{A}^n_S = \text{Spec}(\mathcal{O}_K[x_1, \ldots, x_m]) \) an étale morphism of \( S \)-schemes sending \( 0 \in J(S) \) to the zero section \( 0' := 0_Z \) of \( Z \); see \([5], 2.2/11\). Up to shrinking \( U \) we may assume that \( 0_s \in U_s(k) \) is the only point above \( 0'_s \in Z_s(k) = k^m \). Let \( U[n] \) denote the pre-image of \( U \subset J \) via the canonical map \( J^{[n]} \to J \). Then, for \( n \geq 1 \), \( U[n] \) is the dilatation of \( U^{[n-1]} \) along the closed point \( 0_s \in U_s^{[n-1]}(k) \) and \( U[n](S) = J^{[n]}(S) \). Define now inductively \( Z[0] := Z \), and for \( n > 0 \), \( Z[n] \cong \text{Spec}(\mathcal{O}_K[\pi^{-n}x_1, \ldots, \pi^{-n}x_m]) \) the dilatation of \( Z^{[n-1]} \) along the zero section of \( Z_s^{[n-1]} \); on algebras the canonical map \( Z[n] \to Z^{[n-1]} \) sends \( \pi^{-n+1}x_i \) to \( \pi^{-n}x_i := \pi(\pi^{-n}x_i) \). By direct computations, considering \( Z \) as an \( S \)-group scheme via the isomorphism \( Z \cong \mathbb{G}_m^m \), we get an exact sequence

\[
0 \to Z^{[n]}(S) \to Z(S) \to Z(S_n) \to 0
\]

for any \( n \). Moreover, using the étaleness of \( f \), one shows inductively that \( f \) induces a morphism \( f[n] : U[n] \to Z[n] \) which identifies \( U[n] \to U \) with the base change of \( Z[n] \to Z \) along \( f \). In particular, \( f[n] \) is étale, and \( 0_s \in U_s[n] \) is the only point above the zero section of \( Z_s[n] \).

Let \( 0_n \) (respectively \( 0'_n \)) denote the composition of \( S_n \to S \) with the 0 (respectively \( 0'_n \)) section. The morphism \( f \) induces a bijection between \( U[n](S) \) and \( Z[n](S) \) for \( n \geq 1 \). Indeed, for \( n = 1 \), the étaleness of the map \( f \) assures that for any \( \sigma \in Z(S) \) that becomes \( 0' \) modulo \( \pi \) there is a unique \( \sigma' \in U(S) \) that becomes 0 modulo \( \pi \). For higher \( n \) one proceeds by induction recalling that \( 0_1 \in U^{[n-1]}(S_1) \) is the only point above \( 0'_1 \in Z^{[n-1]}(S_1) \), and that the induced morphism \( f^{[n-1]} : U^{[n-1]} \to Z^{[n-1]} \) is étale.

We are now ready to show that \([32]\) is exact. Take \( \tau \in J^{[n]}(S) = U^{[n]}(S) = Z^{[n]}(S) \) and let \( \sigma := f(\tau) \in Z(S) \). Then the reduction of \( \sigma \) modulo \( \pi^n \) is \( 0'_n \) by the exactness of \([34]\). Hence the reduction modulo \( \pi^n \) of \( \tau \) in \( U(S_n) \) must be \( 0_n \in U(S_n) \) since \( U \) is étale over \( Z \), and \( 0_n \) is the only point in \( U_s(k) \) above \( 0'_s \). In particular the reduction of \( \tau \) is \( 0_n \in J(S_n) \) and hence \([32]\) is a complex. Finally, take \( \tau \in J(S) \) whose reduction modulo \( \pi^n \) is \( 0_n \in J(S_n) \). In particular \( \tau \in U(S) \), and \( f(\tau) \in Z(S) \) is contained in the kernel of the natural map \( Z(S) \to Z(S_n) \). Thus, \( f(\tau) \in Z^{[n]}(S) = U^{[n]}(S) = J^{[n]}(S) \), as desired.

By \([3.0.1]\) one then has a diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
J^{[n]}(S) & \to & J^{[n]}(S) & & & & & & & \\
0 & \to & J^{[n-1]}(S) & \to & J(S) & \to & J(S_n) & \to & 0 \\
0 & \to & J^{[n-1]}(S_1) & \overset{\theta_n}{\to} & J(S_n) & \to & J(S_{n-1}) & \to & 0.
\end{array}
\]
3.1 Pro-algebraic structures

Recall that, in this paper, a pro-algebraic group over $k$ is a pro-object in the category of $k$-group schemes of finite type (see §[1.2]). The aim of this subsection is to show that, with the help of Greenberg realization functors, the morphism $q: \Pic^0(X) \to J(S)$ is pro-algebraic in nature.

Let $n \geq 1$ be an integer. Consider $\Gr(P_{[n]})$, the Greenberg realization of the Picard functor $P_{[n]} = \Pic^0_{X/S}$ (Theorem [1.2.1]). The natural morphism of functors $P_{[n+1]} \to P_{[n]}$ induces a morphism of smooth $k$-group schemes $\alpha_n: \Gr(P_{[n+1]}) \to \Gr(P_{[n]})$. Thus we obtain a pro-algebraic group over $k$ (in the sense of §[1.2])

$$\Gr(\Pic^0_{X/S}) := \{(\Gr(P_{[n]}), \alpha_n)\}_{n \geq 1}.$$ 

**Lemma 3.1.1.** The morphism $\alpha_n: \Gr(P_{[n+1]}) \to \Gr(P_{[n]})$ is a smooth and surjective morphism of smooth connected $k$-group schemes. Moreover, either $\alpha_n$ is an isomorphism, in which case we have $\phi(n+1) = \phi(n)$, or $\ker(\alpha_n)$ is a $k$-vector group of dimension 1, in which case we have $\phi(n+1) = \phi(n) + 1$.

**Proof.** By Theorem [1.2.1] (and its proof) $\Gr(P_{[n]})$ is (represented by) a smooth connected $k$-group scheme which is the identity component of $\Gr(\Pic_{X/S})$. Now, by Proposition [2.2.2] the canonical map $\Gr(\Pic_{X_{n+1}/S}) \to \Gr(\Pic_{X_n/S})$ is smooth and surjective with connected unipotent kernel. Hence, so is the restriction of $\alpha_n$ to the identity components. One concludes then by Corollary [2.1.10].

On passing to the projective limit of the perfect group schemes $\Gr(P_{[n]})$ and using the fact that

$$\Pic^0(X) = \lim_{\leftarrow} \Pic^0(X_n) \cong \lim_{\leftarrow} \Gr(P_{[n]})(k),$$

we get a pro-algebraic structure in the sense of Serre on the group $\Pic^0_{X/S}(S) = \Pic^0(X)$. Denote the Serre pro-algebraic group so obtained by

$$\Pic^0(X) := \lim_{\leftarrow} \Gr(P_{[n]}).$$

Similarly, since $P_{[n]}(S) = \ker(\Pic^0_{X/S}(S) \to P_{[n]}(S))$, we find that the group $P_{[n]}(S)$ can also be endowed with a pro-algebraic structure in the sense of Serre, denoted by $P_{[n]}(S)$. Thus we obtain a decreasing filtration of $\Pic^0(X)$ by pro-algebraic subgroups:

$$(35) \quad \ldots \subset P_{[n+1]}(S) \subset P_{[n]}(S) \subset \ldots \subset P_{[1]}(S) \subset P_{[0]}(S) = \Pic^0(X).$$

Secondly, from the $S$-group scheme $J$, we can construct a pro-algebraic group $\{(\Gr_n(J), \beta_n)\}_{n \geq 1}$, where each $\Gr_n(J)$ is smooth, and hence a Serre pro-algebraic algebraic group

$$J(S) := \Gr(J) := \lim_{\leftarrow} \Gr_n(J)$$

whose group of $k$-points is $J(S)$. Moreover, the canonical map $J(S) = \Gr(J)(k) \to \Gr_n(J)(k) = J(S_n)$ is also pro-algebraic in nature, hence its kernel can also be endowed with a pro-algebraic structure. This last pro-algebraic group, according to the short exact sequence ([2.2]), is just the sub-pro-algebraic group $J^{[n]}(S) \subset J(S)$ induced by the canonical map of $S$-group schemes $J^{[n]} \to J$. In this way, we also obtain a decreasing filtration of $J(S)$ by sub-pro-algebraic groups:

$$(36) \quad \ldots \subset J_{[n+1]}(S) \subset J^{[n]}(S) \subset \ldots \subset J_{[1]}(S) \subset J_{[0]}(S) = J(S).$$
On the other hand, for each integer \( n \geq 1 \), the morphism \( q: \text{Pic}^0_X \to J \) induces a morphism of functors \( \text{Pic}^0_{X/S} \times_S S_n = \text{Pic}^0_{X/S} \times S S_n \to J \times_S S_n \); hence a morphism of algebraic \( k \)-groups:

\[
\text{Gr}(\text{Pic}^0_{X/S}) \to \text{Gr}_n(J).
\]

In particular, we obtain a morphism of pro-algebraic groups:

\[
(37) \quad \text{Gr}(\text{Pic}^0_{X/S}) = \{(\text{Gr}(\text{Pic}^n), \alpha_n)\}_{n \geq 1} \to \{(\text{Gr}_n(J), \beta_n)\}_{n \geq 1} = \text{Gr}(J).
\]

In this way, we find that the canonical morphism \( q: \text{Pic}^0(X) \to J(S) \) is the morphism on \( k \)-rational points induced by a morphism of Serre pro-algebraic groups:

\[
(38) \quad q: \text{Pic}^0(X) \to J(S).
\]

In fact, we can be more precise in comparing the two filtrations \( (35) \) and \( (36) \). The main result of this section (see Theorem 3.3.3) says that the above filtrations are compatible via \( q \) and this fact suggests that the morphism \( q \) should be thought as an analogue of the norm map studied by Serre in \( [25] \) § 3.3-3.4. In order to explore the compatibility between the two filtrations, we start by proving a useful result on the length of torsion sheaves.

### 3.2 A result on intersection theory

The results of this section hold for \( \mathcal{O}_K \) any discrete valuation ring; as usual \( S := \text{Spec}(\mathcal{O}_K) \) and \( s \) is the closed point of \( S \). In the following, a coherent sheaf on an integral scheme is called a torsion coherent sheaf if its stalk at the generic point is trivial. Moreover, for a torsion coherent sheaf \( T \) defined over the spectrum of a discrete valuation ring, let \( \ell(T) \) denote the length of \( T \). For \( \alpha \in Z(S) \) an \( S \)-point of a separated \( S \)-scheme \( Z \), let \( \alpha(S) \) be the image of \( S \) by \( \alpha \), together with the reduced subscheme structure.

**Proposition 3.2.1.** Let \( Z \) be a smooth \( S \)-scheme of finite type and \( \xi \) a generic point of its special fibre \( Z_s \). Let \( \alpha: S \to Z \) be a section of \( Z/S \) such that \( \alpha(s) \in \overline{\{\xi\}} \subset Z_s \). Let \( \mathcal{M} \) be a coherent sheaf on \( Z \), whose support \( \text{Supp}(\mathcal{M}) \) is pure of codimension 1 in \( Z \). Suppose that the stalk \( \mathcal{M}_\xi \) of \( \mathcal{M} \) at \( \xi \) is of length \( \ell \) as a torsion \( \mathcal{O}_{Z,\xi} \)-module, and that \( \alpha(S) \nsubseteq \text{Supp}(\mathcal{M}) \).

1. We have \( \ell(\alpha^*\mathcal{M}) \geq \ell \), with equality if and only if the following conditions are satisfied: the support of \( \mathcal{M} \) at \( \alpha(s) \) is contained in \( Z_s \), i.e., \( \overline{\{\xi\}} \subset Z_s \) is the only possible component of \( \text{Supp}(\mathcal{M}) \) containing \( \alpha(s) \), and \( \mathcal{M} \) is Cohen-Macaulay at \( \alpha(s) \).
2. Suppose furthermore that the support of \( \mathcal{M}_K \) on \( Z_K \) is not empty, and let \( H_K \) be the scheme having \( \text{Supp}(\mathcal{M}_K) \) as underlying space with its reduced structure and \( H := \overline{\{\xi\}} \subset Z \) its schematic closure in \( Z \) (which is a relative effective divisor). Suppose moreover that \( \alpha(s) \in H_s \). Let \( \zeta \) be the generic point of an irreducible component of \( H \) passing through \( s \). Then \( \ell(\alpha^*\mathcal{M}) \geq \ell + 1 \). Moreover, if the equality \( \ell(\alpha^*\mathcal{M}) = \ell + 1 \) holds, then (a) \( \mathcal{M} \) is Cohen-Macaulay at \( \alpha(s) \); (b) \( H \) is regular at \( \alpha(s) \), and \( \mathcal{M} \) is of length 1 at \( \zeta \); (c) \( H \) cuts the closed subscheme \( \alpha(S) \to Z \) transversally at \( \alpha(s) \), i.e., the intersection number at \( \alpha(s) \) of the closed subscheme \( \alpha(S) \) (of dimension 1) with \( H \) (of codimension 1 and regular around the point \( \alpha(s) \)) is equal to 1.

Before proving this result consider the following technical Lemma.

**Lemma 3.2.2.** Let \( Z = \text{Spec}(A) \) be a local noetherian regular scheme of dimension 2, and \( \mathcal{M} \) a torsion coherent \( \mathcal{O}_Z \)-module such that \( \text{Supp}(\mathcal{M}) \) is of dimension 1. Let \( H_1, \ldots, H_n \) be the irreducible components of \( \text{Supp}(\mathcal{M}) \), endowed with the reduced subscheme structure. Denote by \( \xi_i \) the generic point of \( H_i \), and by \( \ell_i \) the length of \( \mathcal{M}_{\xi_i} \) as an \( \mathcal{O}_{Z,\xi_i} \)-module. Let finally \( f \in A \) be
an element, which is part of a system of regular parameters of $A$, such that $Z_1 := V(f) \subset Z$ is not contained in $\text{Supp}(M)$. Then $\ell(M/fM) \geq \sum_{i=1}^n \ell_i$, with equality if and only if the following conditions are satisfied: (i) for each $i$, the scheme $H_i$ is regular and cuts $Z_1$ transversally in $Z$; (ii) the $O_Z$-module $M$ is Cohen-Macaulay.

**Proof.** Remark first that a coherent $O_Z$-module $N$ with one dimensional support is Cohen-Macaulay if and only if $N$ has no embedded associated points. Indeed, suppose first that $N$ has no embedded associated points. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be the minimal ideals of the support of $N$, and let $f \in m_A \setminus \bigcup_i \mathfrak{p}_i$ (where $m_A \subset A$ is the maximal ideal). Then multiplication by $f$ provides an injective map ([26], Chapter I, §n and let $f \in \mathfrak{m}_A \setminus \bigcup_i \mathfrak{p}_i$ (where $\mathfrak{m}_A \subset A$ is the maximal ideal). Then multiplication by $f$ provides an injective map ([26], Chapter I, §B Corollary 1 of Proposition 7)

$$N \rightarrow N, \quad n \mapsto f \cdot n.$$ 

Hence, the maximal $M$-sequence of $N$ has at least $1 = \dim(N)$ element, which implies that $N$ is Cohen-Macaulay ([26], §B.1, Definition 1). The converse statement follows from Proposition 13 of §B.2 in [26].

In order to prove the Lemma, we use induction on $n$. Let us begin with the case where $n = 1$. Denote by $\xi = \xi_1$ the generic point of $\text{Supp}(M)_{\text{red}} = H_1 = H$, and by $\ell = \ell_1$ the length of $M$ at $\xi$. Hence, $M_{\xi}$ has a filtration by $O_{Z,\xi}$-submodules:

$$0 = M_{\xi, 0} \subset M_{\xi, 1} \subset \ldots \subset M_{\xi, \ell} = M_{\xi},$$

where the successive quotients are isomorphic to $k(\xi)$. We then define $M_i$ as the inverse image of $M_{\xi, i}$ via the canonical map $M \rightarrow M_{\xi}$, thus obtaining a filtration on $M$:

$$M_{-1} := 0 \subset M_0 \subset M_1 \subset \ldots \subset M_{\ell} = M.$$ 

In general, $M_0 \neq 0$, and it is trivial if and only if $M$ has no embedded associated points. For each $i \geq 0$, let $C_i := M_i / M_{i-1}$, which has no embedded associated point by definition whenever $i \geq 1$. Moreover, $C_{i,\xi} \cong M_{\xi, i} / M_{\xi, i-1}$, hence $C_{0,\xi} = 0$ and for $i \geq 1$, we have $C_{i,\xi} \cong k(\xi)$. In particular, if $i \geq 1$, we have $C_i \neq 0$ with schematic support equal to $H = \text{Supp}(M)_{\text{red}}$. Indeed, we only need to show that the schematic support $\text{Supp}(C_i) = V(\text{Ann}(C_i))$ is reduced. Since $C_{i,\xi} \cong k(\xi)$, $\text{Supp}(C_i)$ is reduced at $\xi$, hence it is generically reduced. Furthermore, since $C_i$ has no embedded associated points, so too is the scheme $\text{Supp}(C_i)$. So $\text{Supp}(C_i)$ is reduced, and hence, equal to $H$ as subscheme of $Z$. On the other hand, for $i \geq 1$, since the $O_Z$-module $C_i$ has no embedded associated points, and $Z_1 = V(f)$ is not contained in $H$, the map “multiplication by $f$”:

$$C_i \rightarrow C_i, \quad x \mapsto f \cdot x$$

is injective for $i \geq 1$. From this, we get a filtration of $M/fM$:

$$0 \subset M_0/fM_0 \subset M_1/fM_1 \subset \ldots \subset M_{\ell}/fM_{\ell} = M/fM,$$

where for each $i \geq 1$, the quotient of $M_i/fM_i$ by $M_{i-1}/fM_{i-1}$ is isomorphic to $C_i/fC_i$ which is non-zero since $C_i \neq 0$. As a result, $\ell(M/fM) \geq \ell$. Moreover, $\ell(M/fM) = \ell$, if and only if the following two conditions are realized: (a) $M_0/fM_0 = 0$ which means $M_0 = 0$ by Nakayama’s lemma; (b) for each $i$ ($1 \leq i \leq \ell$), the $O_Z$-module $C_i/fC_i$ is of length 1 over $O_Z/fO_Z$.

Now, suppose that $\ell(M/fM) = \ell$, or equivalently that the conditions (a) and (b) above are verified. We will prove that $M$ is Cohen-Macaulay, and the schematic support $H$ of $C_i$ is regular and cuts the subscheme $V(f) \hookrightarrow Z$ transversally. In fact, condition (a) implies that the $O_Z$-module $M$ has no embedded associated points, in particular, is Cohen-Macaulay. On the
other hand, suppose that \( \text{Ann}(C_i) = (g) \subset A \) (hence \( H \) is defined by the equation \( g = 0 \) in \( Z \)), and let \( c \in C_i \) be such that \( c \notin fC_i \). Condition (b) together with Nakayama’s Lemma imply that the \( O_Z \)-module \( C_i \) is generated by \( c \). The morphism \( O_Z \to C_i = O_Z c \), defined by \( \lambda \mapsto \lambda c \) is then surjective, with kernel the ideal \( (g) = \text{Ann}(C_i) = \text{Ann}(c) \). Therefore, \( O_Z/(g, f) \cong C_i/fC_i \) is of length 1 over \( O_Z/fO_Z \). Hence \( \text{Supp}(C_i) = \text{Supp}(M)_{\text{red}} = H = V(g) \) is regular and cuts \( V(f) \hookrightarrow Z \) transversally. Conversely, suppose that \( M \) is Cohen-Macaulay and that the scheme \( H \) is regular and cuts \( V(f) \hookrightarrow Z \) transversally. In particular, \( M \) has no embedded associated point, which implies that \( M_0 = 0 \), whence condition (a) holds. Moreover, since \( H = \text{Spec}(A/gA) \) is regular of dimension 1, \( A/gA \) is a principal ideal domain. Therefore, the \( O_H = O_Z/gO_Z \)-module \( C_i \) is free of rank 1. Hence, \( \ell(C_i/fC_i) = \ell(A/(f, g)) = 1 \) since \( H = V(g) \hookrightarrow Z \) cuts \( V(f) \hookrightarrow Z \) transversally. In this way we get condition (b), which completes the proof of the Lemma in the case \( n = 1 \).

Suppose now that the assertion of the lemma has been verified for \( n - 1 \geq 1 \). Let \( M' \subset M \) be the submodule defined as the kernel of the following map

\[
M \to \bigoplus_{i=2}^{n} \iota_i \ast i_i^* M
\]

with \( \iota_i : \text{Spec}(k(\xi_i)) \to Z \) the canonical map, and define \( M'' \) by the following exact sequence:

\[
0 \to M' \to M \to M'' \to 0.
\]

Then \( M'' \) has no embedded associated points (and so is Cohen-Macaulay) and has support \( \cup_{i=2}^{n} H_i \), while \( M' \) has support \( H_1 \). One then has the following exact sequence (since \( M'' \) is Cohen-Macaulay and \( V(f) \notin \text{Supp}(M'') \)):

\[
0 \to M'/fM' \to M/fM \to M''/fM'' \to 0.
\]

Hence, we have \( \ell(M/fM) = \ell(M'/fM') + \ell(M''/fM'') \). Moreover, by the definitions of \( M' \) and \( M'' \), we have \( M'_{\xi_i} \cong M_{\xi_1} \), and \( M''_{\xi_i} \cong M_{\xi_i} \) for \( i = 2, \ldots, n \).

By applying the induction hypothesis, we get

\[
\ell(M/fM) = \ell(M'/fM') + \ell(M''/fM'') \geq \ell(M'_{\xi_1}) + \sum_{i=2}^{n} \ell(M''_{\xi_i}) = \sum_{i=1}^{n} \ell_i,
\]

with equality if and only if \( \ell(M'/fM') = \ell_1 \), and \( \ell(M''/fM'') = \sum_{i=2}^{n} \ell_i \). In other words, equality holds if and only if (a) \( M', M'' \) are Cohen-Macaulay, and (b) the subschemes \( H_i \) are regular cutting \( Z_1 \) transversally in \( Z \). Since \( M'' \) is already Cohen-Macaulay, condition (a) is equivalent to saying that \( M \) is Cohen-Macaulay. This completes the proof of the Lemma.

**Proof of (3.2.1)** Since \( \alpha : S \to Z \) is a section of \( Z/S \), there exist elements \( f_1, \ldots, f_d \) of \( O_{Z,x} \) which generate, together with \( \pi \), the maximal ideal of \( O_{Z,x} \) and \( \alpha(S) = V(f_1, \ldots, f_d) \hookrightarrow Z \). Up to replacing \( Z \) by its localization at \( x \), we may assume that \( Z \) is the spectrum of a regular local ring of dimension \( d + 1 \). In particular \( Z_0 = \{ \xi \} \) is regular and irreducible. We will prove the Proposition by induction on \( d \). The case \( d = 0 \), i.e., \( S = Z \), is trivial. We start illustrating the case \( d = 1 \). In this case, \( Z \) is a 2-dimensional local regular scheme, with \( M \) a torsion coherent module on \( Z \). When \( \ell = 0 \), the conclusion of (1) is clear since in this situation, we always have
\[ \ell(\alpha^* M) \geq \ell = 0, \text{ and an equality means that } x \notin \text{Supp}(M), \text{ or equivalently, } M_s = 0. \]

In fact, here we have \( M = 0 \) since \( x \in Z \) is the only closed point of the local scheme \( Z \). To finish the proof of (1) for \( d = 1 \), we may assume that \( \ell \geq 1 \). Since \( \xi \in \text{Supp}(M) \), the closed subscheme \( Z_s = \overline{\{\xi\}} \subseteq \text{Supp}(M) \) is one of the irreducible components of \( \text{Supp}(M) \) in \( Z \). Now by applying Lemma 3.2.2, we get \( \ell(\alpha^* M) \geq \ell \). The equality holds if and only if \( Z_s \) is the only component of \( Z \), \( Z_s \) cuts \( \alpha(S) \) transversally in \( Z \) and \( M \) is Cohen-Macaulay at \( x \). We now consider assertion (2). By assumption, \( \text{Supp}(M) \) is the union of the one dimensional subscheme \( H \) with, possibly, \( Z_s \) if \( \ell > 0 \); hence \( \ell(\alpha^* M) \geq \ell + 1 \). If \( \ell(\alpha^* M) = \ell + 1 \), on applying Lemma 3.2.2 once again, we see that \( M \) is Cohen-Macaulay at \( x \), the subscheme \( H \subseteq Z \) is irreducible, regular and cuts \( \alpha(S) \) transversally at \( x \). Moreover, \( M \) must have length 1 at the generic point of \( H \). This proves (2), and hence the Proposition, for \( d = 1 \).

For the general case, denote by \( Z_1 \hookrightarrow Z \) the closed subscheme defined by the ideal \((f_1)\), by \( \xi_1 \) the generic point of \( Z_{1,s} \) and by \( M_1 \) the pull-back of \( M \) to \( Z_1 \). Then \( Z_1 \) is a regular local scheme of dimension \( d \), which is not contained in the support of \( M \). In particular, \( M_1 \) is again a torsion coherent sheaf on \( Z_1 \). The morphism \( \alpha: S \to Z \) factors through \( Z_1 \to Z \), and we denote by \( \alpha_1: S \to Z_1 \) the morphism obtained in this way. In particular, \( \alpha^* M \cong \alpha_1^* M_1 \) and \( \alpha(S) \not\subseteq \text{Supp}(M_1) \). Hence, in order to prove the first assertion of (1), we only need to verify that the module \( M_1 \) is of length \( \ell \) at \( \xi_1 \) and then apply the induction hypothesis. To see the inequality \( \ell(M_{1,\xi_1}) \geq \ell \), since

\[ M_{1,\xi_1} \cong M_{\xi_1}/f_1 M_{\xi_1}, \]

we are reduced to showing that the restriction of the torsion sheaf \( \widetilde{M} := M|_{\text{Spec}(O_{Z,\xi_1})} \) to the subscheme \( \text{Spec}(O_{Z,\xi_1}) \to \text{Spec}(O_{Z,\xi_1}) =: \widetilde{Z} \) is of length \( \ell \). By definition, \( \xi \in Z_s \) is contained in the special fibre \( Z_s \) of \( \widetilde{Z} \), and \( M \) has length \( \ell \) at \( \xi \in Z_s \). Hence, we need only apply Lemma 3.2.2 to the two dimensional regular local scheme \( \widetilde{Z} \) to get the conclusion. We can also summarize the previous arguments by the following relations:

\[ \ell(\alpha^* M) = \ell(\alpha_1^* M_1) \geq \ell(M_{1,\xi_1}) = \ell(\overline{\widetilde{M}}/f_1 \overline{\widetilde{M}}) \geq \ell(\overline{\widetilde{M}}) = \ell(M_{\xi_1}) = \ell. \]

Next, we examine the condition \( \ell(\alpha^* M) = \ell \). By (39), we have \( \ell(\alpha^* M) = \ell \) if and only if

(a) \( \ell(M/f_1 M) = \ell(\overline{\widetilde{M}}) = \ell \), and

(b) \( \ell(M_{1,\xi_1}) = \ell(\alpha_1^* M_1) \).

Consider the conditions (a'): \( M \) is Cohen-Macaulay having support contained in \( Z_s \) at \( \xi_1 \) and (b'): \( M_1 \) is Cohen-Macaulay, with support contained in \( Z_{1,s} \) at \( x \). On applying the induction hypothesis to the torsion module \( M_1 \) on the \( d \)-dimensional scheme \( Z_1 \), one checks immediately that condition (b) is equivalent to condition (b'). Furthermore, on applying the induction hypothesis to the torsion module \( \widetilde{M} = M|_{\text{Spec}(O_{Z,\xi_1})} \) on the 2-dimensional local scheme \( \widetilde{Z} = \text{Spec}(O_{Z,\xi_1}) \), and since \( \widetilde{Z} \) is the localization of \( Z \) at the point \( \xi_1 \in Z \), we find that conditions (a) and (a') are equivalent. Hence

\[ \ell(\alpha^* M) = \ell \] if and only if (a') and (b') hold.

Now, we proceed with the proof of the second part of (1). Suppose first \( \ell(\alpha^* M) = \ell \), or equivalently, that the previous conditions (a') and (b') hold, and prove that \( M \) is Cohen-Macaulay with support contained in \( Z_s \) at \( x \). We first claim that the multiplication by \( f_1 \) on \( M \) provides an injective map. To see this fact, let \( M' \) be the submodule of \( M \) formed by the elements killed by a power of \( f_1 \), and let \( M'' = M/M' \). By definition, \( \text{Supp}(M') \subseteq Z_1 \cap \text{Supp}(M) \), which is hence of codimension at least 2. By definition of \( M'' \), the multiplication by \( f_1 \) on \( M'' \) is an injective map, hence the canonical map

\[ M'/f_1 M' \to M/f_1 M = M_1 \]

is injective. On one hand, condition (a') above implies that \( \xi_1 \notin \text{Supp}(M') \) (since \( \text{Supp}(M') \subseteq V(f_1) \cap Z_s = Z_{1,s} \subseteq Z_s \) and \( M \) is Cohen-Macaulay at \( \xi_1 \) by (a')), hence \( \xi_1 \notin \text{Supp}(M'/f_1 M') \).
In particular, $\text{Supp}(\mathcal{M}'/f_1\mathcal{M}') \cap Z_{1,s} \subseteq Z_{1,s}$. On the other hand, condition (b') and the injection \[(40)\] imply that the support of $\mathcal{M}'/f_1\mathcal{M}'$ is contained in $Z_{1,s}$. Hence $\mathcal{M}'/f_1\mathcal{M}' = 0$, and then $\mathcal{M}' = 0$ by Nakayama’s Lemma. As a result, the multiplication by $f_1$ on $\mathcal{M}$ is injective. Moreover the quotient sheaf $\mathcal{M}_1 = \mathcal{M}/f_1\mathcal{M}$ is Cohen-Macaulay of dimension $d - 1 = \dim(Z_1) - 1$ by (b'), hence also $\mathcal{M}$ is Cohen-Macaulay. To see that the support of $\mathcal{M}$ is contained in $Z_s$, suppose $\text{Supp}(\mathcal{M})$ contains a component $\Gamma$ different from $Z_s$ at the point $x$. Since we have shown that $\mathcal{M}$ is Cohen-Macaulay at $x$, it follows that $\Gamma$ is also of codimension 1. By condition (b) above, $\mathcal{M}_1$ has support contained in $Z_{1,s}$ at $x$. So $\Gamma \cap Z_1 \subseteq Z_{1,s}$ which is in fact an equality of sets for reasons of dimension. As a result, one finds that $\xi_1 \in \Gamma$, which means that $\text{Supp}(\mathcal{M})$ has at least two components (of codimension 1) at $\xi_1$, but this is impossible because of the condition (a'). This proves that $\mathcal{M}$ is Cohen-Macaulay with support contained in $Z_s$ at $x$. Conversely, suppose $\mathcal{M}$ is Cohen-Macaulay with support contained in $Z_s$ at $x$. We must prove that $\ell(\alpha^* \mathcal{M}) = \ell$. First of all, this condition implies in particular that $\mathcal{M}$ is Cohen-Macaulay with support contained in $Z_s$ at $\xi_1$, namely condition (a') holds. To complete the proof of (1), we only need to show that condition (b') also holds. It is clear that the support of $\mathcal{M}_1$ is contained in $Z_{1,s}$, so we need only verify that $\mathcal{M}_1$ is Cohen-Macaulay. Since $Z_{1,s} = \text{Supp}(\mathcal{M}_1)$ has dimension equal to $\dim(\mathcal{M}) - 1$, $\mathcal{M}_1$ is also Cohen-Macaulay \((26), \text{Chapter IV \S B.2, Proposition 14)\}. This finishes the proof of (1).

To finish the proof of (2), since this is a local question for the étale topology on $S$, we may assume that $S$ is strictly local, in particular the residue field is an infinite set. This implies that the residue field $k(x)$ of $Z$ at $x$ is also infinite. Since $k(x)$ is an infinite field, up to replacing $f_1$ by $f_1 + \lambda f_2$, for a suitable $\lambda \in \mathcal{O}_{Z,x}$, we may assume that $Z_{1,s} \nsubseteq H_s$, so that $H_{1,s} = H_s \cap Z_{1,s} \hookrightarrow Z_{1,s}$ is of codimension 1 in $Z_{1,s}$ (where $H_1 := H \cap Z_1$).

As we have seen in \((39)\), $\mathcal{M}_1$ has length $\geq \ell$ at $\xi_1$. Since $x \in H_{1,s}$, on applying the induction hypothesis to $Z_1$, we find that
\[(41)\]
$$\ell(\alpha^* \mathcal{M}) = \ell(\alpha^* \mathcal{M}_1) \geq \ell(\mathcal{M}_1, \xi_1) + 1 \geq \ell + 1.$$  

This proves the first assertion in (2). From now on, we suppose that $\ell(\alpha^* \mathcal{M}) = \ell + 1$. According to \((11)\), we get $\ell(\alpha^* \mathcal{M}) = \ell(\alpha^* \mathcal{M}_1) = \ell + 1$, and $\mathcal{M}_1$ is of length $\ell$ at $\xi_1$. By the induction hypothesis, we have (i) $H_{1,\text{red}}$ is irreducible and regular, and moreover $\alpha_1(S)$ cuts $H_1$ transversally in $Z_1$ at $x$; (ii) $\mathcal{M}_1$ is Cohen-Macaulay at $x$ in $Z_1$, and if we denote by $\zeta_1 \in H_1$ the generic point of $H_1$, then $\mathcal{M}_1$ is of length 1 at $\zeta_1$. Denote by $Z'$ the localization of $Z$ at $\zeta_1$, and by $\mathcal{M}'$ the inverse image of $\mathcal{M}$ by the canonical morphism $Z' \to Z$. Then, $\mathcal{M}'/f_1\mathcal{M}'$ is of length 1 over $\mathcal{O}_{Z'}/f_1\mathcal{O}_{Z'}$. Hence, on applying Lemma \((4,22)\) to the torsion module $\mathcal{M}|_{\text{Spec}(\mathcal{O}_{Z,\zeta_1})}$ on the two dimension regular local scheme $\text{Spec}(\mathcal{O}_{Z,\zeta_1})$ we get that $H$ is regular at $\zeta_1$, and it cuts $Z_1$ transversally at $\zeta_1$. Moreover, $\mathcal{M}$ is Cohen-Macaulay with support contained in $H$ at $\zeta_1$, and $\mathcal{M}$ is of length 1 at the generic point $\zeta$ of $H$. Using now the fact that $H_1 = H \cap Z_1$ is irreducible, we find that $H$ itself must be irreducible, since otherwise, $H$ would have at least two components at $\zeta_1$. Therefore, $H_1$ is generically reduced. But $H_1$ is a divisor inside a regular scheme $Z_1$, hence $H_1$ is Cohen-Macaulay and thus $H_1$ is reduced. By assertion (i), we find that $H$ is irreducible and regular, cutting $Z_1$ transversally at $x$ inside $Z$. Now we need only verify that $\mathcal{M}$ is Cohen-Macaulay on $Z$. By (ii), we need only show that $\mathcal{M}$ has no embedded associated points. Denote by $\mathcal{N}'$ the biggest quotient without embedded associated points of $\mathcal{M}$, and denote by $\mathcal{N}$ the $\mathcal{O}_Z$-submodule defined by the following exact sequence
$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N}' \longrightarrow 0;$$
we have the short exact sequence
$$0 \longrightarrow \mathcal{N}/f_1\mathcal{N} \longrightarrow \mathcal{M}/f_1\mathcal{M} \longrightarrow \mathcal{N}'/f_1\mathcal{N}' \longrightarrow 0.$$
According to Nakayama’s Lemma, to complete the proof, we need only show that $\mathcal{N}/f_1\mathcal{N} = 0$. By definition, $\text{Supp}(\mathcal{N}') = \text{Supp}(\mathcal{M})$, and $\ell(\mathcal{N}_\xi') = \ell(\mathcal{M}_\xi) = \ell$. Hence, according to the first part of (2) (which has already been proved), we have $\ell(\alpha^*\mathcal{N}') \geq \ell + 1$. On the other hand, there is a surjection $\alpha^*\mathcal{M} \to \alpha^*\mathcal{N}'$, whence we have $\ell + 1 = \ell(\alpha^*\mathcal{M}) \geq \ell(\alpha^*\mathcal{N}')$. As a result, we have $\ell(\alpha^*\mathcal{N}') = \ell + 1$. Hence the $O_{Z^r}$-module $\mathcal{N}'$ again satisfies the assumptions of the assertion (2) of this Proposition: On applying what was proved a few lines above to the torsion sheaf $\mathcal{N}'$ in place of $\mathcal{M}$, we get $\ell\left(\left(\mathcal{N}'/f_1\mathcal{N}'\right)_\xi\right) = 1$, and $\ell\left(\left(\mathcal{N}'/f_1\mathcal{N}'\right)_\xi\right) = \ell$. Hence $(\mathcal{N}/f_1\mathcal{N})_\xi_1 = (\mathcal{N}/f_1\mathcal{N})_\xi = 0$. As a result, the support of $\mathcal{N}/f_1\mathcal{N}$ is of dimension $< d - 2$. But we have seen that $\mathcal{M}_1 = \mathcal{M}/f_1\mathcal{M}$ is Cohen-Macaulay with support $H_1 \cup Z_1$, of dimension $d - 2$, hence we must have $\mathcal{N}/f_1\mathcal{N} = 0$. This completes the proof. □

3.3 Preliminaries on the comparison between the pro-algebraic structures

The aim of this section is to show that the canonical map of sheaves $q: \text{Pic}^0_X/S \to J$ in (11) induces, for each $n \geq 1$, a morphism of smooth algebraic $k$-groups

\[(42) \quad q_n: \text{Gr}(\text{Pic}^0_{\psi(n)}) = \text{Gr}(\text{Pic}^0_{X_{\psi(n)/S}}) \to \text{Gr}_n(J),\]

and that the maps $q_n$ are compatible in the evident way.

Let $n \geq 1$ be an integer. We have seen in § 3.1 that there exists a morphism of sheaves $\text{Pic}^0_{X_{\text{nd}/S} \times_S S_n} \to J \times_S S_n$. In this way we get a morphism of smooth algebraic $k$-groups

\[(43) \quad q'_n: \text{Gr}(\text{Pic}^0_{\text{nd}}) \to \text{Gr}_n(J).\]

Since $\psi(n) \leq nd$ (cf. § 2.2), there is a canonical morphism of smooth algebraic $k$-groups

$$\text{Gr}(\text{Pic}^0_{\text{nd}}) \to \text{Gr}(\text{Pic}^0_{\psi(n)}).$$

So, in order to prove the existence of $q_n$ as above, it suffices to verify that the morphism $q'_n$ in (43) factors as follows:

$$\begin{array}{ccc}
\text{Gr}(\text{Pic}^0_{\text{nd}}) & \xrightarrow{q'_n} & \text{Gr}_n(J) \\
{\downarrow q_n} & & {\downarrow q_n} \\
\text{Gr}(\text{Pic}^0_{\psi(n)}) & & \\
\end{array}$$

On the other hand, since the morphism of algebraic $k$-groups $\text{Gr}(\text{Pic}^0_{\text{nd}}) \to \text{Gr}(\text{Pic}^0_{\psi(n)})$ has smooth kernel (see Lemma 3.1.1) and $k$ is algebraically closed, we need only check the factorization on the level of $k$-points. Since the maps $\text{Pic}^0(X) \to \text{Pic}^0(X_{\psi(n)}) = \text{Gr}(\text{Pic}^0_{\psi(n)})(k)$ are surjective, the verification is reduced to proving the existence of the following factorization:

$$\begin{array}{ccc}
\text{Pic}^0(X) & \xrightarrow{q} & \text{Pic}^0(X_{\psi(n)}) \\
{\downarrow q} & & {\downarrow q_n} \\
J(S) & \xrightarrow{\psi} & J(S_n) \\
\end{array}$$

where we again denote by $q_n$ the map induced by (42) on $k$-points. With the help of the rigidified Picard functor we will establish this factorization via induction on $n.$
3.4 Comparison of the pro-algebraic structures

In the following discussion we fix a rigidificator \( Y \hookrightarrow X \) of the relative Picard functor \( \text{Pic}_{X/S} \), and for simplicity, we denote by \( G := (\text{Pic}_{X/S},Y)^0 \) the identity component of the rigidified Picard functor of \( X/S \) along \( Y/S \). As usual \( J \) denotes the identity component of the Néron model of \( \text{Pic}^0_{X/K} \) over \( S \). According to Proposition 3.2 in [10], \( G \) is representable by a smooth separated \( S \)-group scheme. Consider the canonical morphism of \( S \)-group schemes \( r: G = (\text{Pic}_{X/S},Y)^0 \rightarrow \text{Pic}^0_{X/S} \) (recalled in §41), which is surjective for the étale topology. Let \( H \) be the schematic closure of \( \ker(r_K) \subset G_K \) in \( G \). It is a flat \( S \)-group scheme of finite type with smooth generic fibre, which is also the kernel of the canonical morphism \( \theta: G \rightarrow J \) (composition of \( r: G \rightarrow \text{Pic}^0_{X/S} \) and the epimorphism \( q: \text{Pic}^0_{X/S} \rightarrow J \); see for example [21], 4.1, for the fact that \( \ker(\theta) = H \).

Since \( S \) is strictly local, the morphism \( r \) induces a surjective map (still denoted by \( r \)) between the \( S \)-sections:

\[
r: G(S) \rightarrow \text{Pic}^0(X).
\]

Let \( \mathfrak{L} \) be the (rigidified) Poincaré sheaf on \( X \times G \). For each \( n \in \mathbb{Z}_{\geq 1} \), we denote by

\[
r_n : G(S) \xrightarrow{r} \text{Pic}^0(X) \xrightarrow{\psi_n} \text{Pic}^0(X_{\psi(n)})
\]

the composition of maps which sends \( \varepsilon \in G(S) \) to \( \mathcal{L}_{\varepsilon} \vert_{X_{\psi(n)}} \in \text{Pic}^0(X_{\psi(n)}) \), where \( \mathcal{L}_{\varepsilon} \) is the sheaf \( (\text{id}_X \times \varepsilon)^*\mathfrak{L} \). These maps are all surjective since \( \mathcal{O}_K \) is strictly henselian.

Let \( p_G : X \times_S G \rightarrow G \) be the projection onto the second factor, and consider the object \( Rp_{G,*}\mathfrak{L} \) in the derived category of \( \mathcal{O}_S \)-modules. It is well known that this complex is quasi-isomorphic to a perfect complex of perfect amplitude contained in \([0,1]\), i.e., locally for the Zariski topology on \( G \), \( Rp_{G,*}\mathfrak{L} \) can be represented by a complex

\[
\ldots \xrightarrow{\mathcal{F}^0} u \xrightarrow{\mathcal{F}^1} \ldots
\]

with \( \mathcal{F}^i \) (\( i = 0,1 \)) locally free \( \mathcal{O}_G \)-modules of the same rank ([9], 6.10.5). The cokernel \( \mathcal{M} \) of \( u \) gives the \( \mathcal{O}_G \)-module \( R^1p_{G,*}\mathfrak{L} \), and for any section \( \varepsilon : S \rightarrow G \) of \( G/S \), the pull-back \( \varepsilon^*\mathcal{M} \) is given by the cohomology group \( H^1(X,\varepsilon \mathcal{L}) \). On the other hand, for \( L \) an invertible sheaf of degree 0 on \( X_K \), \( H^1(X_K,L) \neq 0 \) if and only if \( L \cong \mathcal{O}_{X_K} \). Therefore, the morphism \( u \) above is injective and \( \text{det}(u) \neq 0 \). Hence \( \mathcal{M} \) is a torsion \( \mathcal{O}_G \)-module which admits a resolution of length 1 by locally free \( \mathcal{O}_G \)-modules. In particular, \( \mathcal{M} \) is Cohen-Macaulay, with support \( \text{Supp}(\mathcal{M}) \subset G \) purely of codimension 1 satisfying the inclusion relations (of sets) \( H \subset \text{Supp}(\mathcal{M}) \subset H \cup G_s \).

**Lemma 3.4.1.** Let notation be as above. Then \( \text{Supp}(\mathcal{M}) = H \) as sets.

**Proof.** Let \( \xi \) be the generic point of \( G_s \), and \( \ell \) the length of the \( \mathcal{O}_{G,\xi} \)-module \( \mathcal{M}_\xi \). We will first prove by contradiction that \( \ell = 0 \). Suppose then \( \ell \geq 1 \). Let \( \varepsilon \in G(S) \) be a section of \( G \), and let \( \mathcal{L}_\varepsilon := (\text{id}_X \times \varepsilon)^*\mathfrak{L} \). According to Proposition 3.21, the \( \mathcal{O}_K \)-module \( H^1(X,\mathcal{L}_\varepsilon) \cong \varepsilon^*\mathcal{M} \) is of length at least \( \ell \geq 1 \). By Corollary 2.2.3, this is equivalent to saying that \( \mathcal{L}_\varepsilon \vert_{X_1} \cong \mathcal{I}^i \vert_{X_1} \) with \( i \) a suitable integer. This last fact implies that the surjective homomorphism

\[
r_1 : G(S) \rightarrow \text{Pic}^0(X_1), \ \varepsilon \mapsto \mathcal{L}_\varepsilon \vert_{X_1}
\]

has finite image. However, this produces a contradiction since \( k \) is algebraically closed and so \( \text{Pic}^0(X_1) \cong \text{Pic}^0_{X_1/k}(k) \) is an infinite group. Therefore \( \mathcal{M}_\xi = 0 \). In particular, \( \xi \notin \text{Supp}(\mathcal{M}) \), which completes the proof of the Lemma since \( \text{Supp}(\mathcal{M}) \subset G \) is purely of codimension 1. \( \square \)
Let us begin the comparison of the two filtrations defined in § 3.1 at the level \( n = 1 \). Since \( X_1/S \) can be defined over the closed point \( s \) of \( S \), its Picard functor \( P_{[1]} = \text{Pic}^0_{X_1/S} \) can also be defined over \( s \). Hence, by adjunction, the morphism of functors \( r_1: G \to P_{[1]} \) corresponds to a morphism of algebraic groups over the closed points \( s \) of \( S \):

\[
r_{1,s}: G_s \to P_{[1],s} = \text{Pic}^0_{X_1/k},
\]

which renders the following diagram commutative

\[
\begin{array}{ccc}
G & \xrightarrow{\text{Pic}^0_{X/S}} & P_{[1]} = i_sP_{[1],s} \\
\downarrow & & \downarrow_{1 \circ r_{1,s}} \\
i_sG_s & & 
\end{array}
\]

Let \( x \in G_s(k) \) be a closed point, \( \varepsilon \in G(S) \) a section lifting \( x \) and put \( \mathcal{L}_\varepsilon = (\text{id}_X \times \varepsilon)^*\mathcal{L} \). This is a rigidified invertible sheaf on \( X \). Since \( \text{Supp}(\mathcal{M}) = H \) (Lemma 3.4.1), one finds that \( x = \varepsilon(s) \in H_s(k) \) if and only if the \( \mathcal{O}_K \)-module \( \varepsilon^*\mathcal{M} = H^1(X, \mathcal{L}_\varepsilon) \) is of length \( \geq 1 \). Moreover, in view of Lemma 2.2.3 this last condition is equivalent to saying that \( \mathcal{L}_{\varepsilon|X_1} \cong \mathcal{I}|_{X_1} \) for a suitable integer \( i \). In particular, the image \( r_{1,s}(H_s(k)) \) is a finite set of \( P_{[1],s}(k) \), and the kernel of \( r_{1,s} \) is contained in \( H_s \). Let \( Z \) be the schematic closure of the set of those points \( x \in G_s(k) \) which admit a lifting \( \varepsilon \in G(S) \) such that \( \mathcal{L}_{\varepsilon|X_1} \cong \mathcal{O}_{X_1} \). By continuity of \( r_{1,s} \), the subgroup scheme \( Z \subseteq G_s \) is a union of irreducible components of \( H_{s,\text{red}} \). Denote by \( G_{[1]} \to G \) the dilatation of \( G \) with center \( Z \subseteq G_s \). By definition of \( Z \), we have an exact sequence of smooth \( k \)-group schemes

\[
0 \to Z \to G_s \xrightarrow{r_{1,s}} \text{Pic}^0_{X_1/k} \to 0.
\]

Moreover, according to the universal property of dilatations ([5], 3.2/1), we have the following short exact sequence of abstract groups:

\[
0 \to G_{[1]}(S) \to G(S) \to \text{Pic}^0(X_1) \to 0.
\]

Consider now the morphism \( \theta: G \to J \). Denote by \( G_{[1]}' \) the dilatation of \( G \) with center \( H_{s,\text{red}} = \ker(\theta)_{s,\text{red}} \subseteq G_s \). Since \( H_s \) is the kernel of the canonical map \( \theta_s: G_s \to J_s \), the universal property of dilatations implies that the following sequence is exact

\[
0 \to G_{[1]}'(S) \to G(S) \to J(S_1) \to 0.
\]

Since \( Z \subseteq H_{s,\text{red}} \) is an open subgroup, \( G_{[1]} \) is an open subgroup of \( G_{[1]}' \). From the exact sequences ([15], [46]), we obtain a morphism of groups \( q_1: \text{Pic}^0(X_1) \to J(S_1) \) which makes the external square commute:

\[
\begin{array}{ccc}
G(S) & \xrightarrow{\text{Pic}^0(X)} & \text{Pic}^0(X_1) \\
\downarrow & & \downarrow q \\
G(S) & \xrightarrow{\theta} & J(S) \\
\downarrow & & \downarrow q_1 \\
\end{array}
\]

Moreover, from the fact that \( q \circ r = \theta \) and the surjectivity of \( r \), the square on the right also commutes. The morphism of abstract groups \( q_1 \) is surjective (since all the other maps are), and has kernel generated by \( \mathcal{I}|_{X_1} \in \text{Pic}(X_1) \) (see Corollary 2.2.3). Note that the diagram above fits
also into a bigger commutative diagram

(47) \[
\begin{array}{ccc}
G^{[1]}(S) & \rightarrow & G[S] \\
\downarrow \theta^{[1]} & & \downarrow \theta \\
P^{[1]}(S) & \rightarrow & \Pic^0(X)
\end{array}
\]

On the level of pro-algebraic groups we have then shown that the morphism \( q \) in (37) induces a map

(48) \[ q_1 : \Gr(\Pic^0_{X_{\psi(1)/S}}) \rightarrow \Gr_1(J) \]

because, as we noted in \( \S \, 3.3 \), it is sufficient to check the factorization on \( k \)-points.

In order to proceed with the comparison of the filtrations for higher \( n \), let us denote by \( M^{[1]} \) (respectively by \( M^{[1]'} \)) the inverse image of \( M \) over \( G^{[1]} \) (respectively over \( G^{[1]'} \)) via the morphism \( G^{[1]} \rightarrow G \) (respectively via the morphism \( G^{[1]'} \rightarrow G \)). Let \( H^{[1]} \) (respectively \( H^{[1]'} \)) be the schematic closure of \( H_K \leftrightarrow G^{[1]}_K = G_K \) in \( G^{[1]} \) (respectively in \( G^{[1]'} \)). Then \( M^{[1]} \) (respectively \( M^{[1]'} \)) is a coherent torsion sheaf with support in \( H^{[1]} \cup G^{[1]}_K \) (respectively in \( H^{[1]'} \cup G^{[1]'}_K \)), which admits a resolution of length 1 by locally free \( \mathcal{O}_{G^{[1]}/S} \)-modules (respectively \( \mathcal{O}_{G^{[1]'}/S} \)-modules). In particular, since the schemes \( G^{[1]} \) and \( G^{[1]'} \) are regular, \( M^{[1]} \) and \( M^{[1]'} \) are Cohen-Macaulay as modules. On the other hand, by the universal property of dilatations, the composed morphism \( G^{[1]} \rightarrow G \rightarrow J \) (respectively \( G^{[1]'} \rightarrow G \rightarrow J \)) factors through \( J^{[1]} \rightarrow J \). We denote by \( \theta^{[1]} : G^{[1]} \rightarrow J^{[1]} \) (respectively by \( \theta^{[1]'} : G^{[1]'} \rightarrow J^{[1]} \)) the morphism obtained in this way.

**Lemma 3.4.2.** (i) Let \( \xi_1 \) be a generic point of \( G^{[1]'}_s \), then the \( \mathcal{O}_{G^{[1]'}}_{\xi_1} \)-module \( M^{[1]'}_{\xi_1} \) is of length 1.

(ii) The scheme \( H \) is normal.

(iii) The morphism \( \theta^{[1]} : G^{[1]} \rightarrow J^{[1]} \) induces a surjection \( G^{[1]}(S) \rightarrow J^{[1]}(S) \). In particular, \( \theta^{[1]} \) is a faithfully flat morphism of \( S \)-group schemes, with \( \ker(\theta^{[1]}) = H^{[1]} \).

**Proof.** Observe first that we have a commutative diagram with exact rows, where the first row is (45):

\[
\begin{array}{ccc}
0 & \rightarrow & G^{[1]}(S) \\
\downarrow & & \downarrow r_2 \\
0 & \rightarrow & P^{[1,\psi(2)]}(S)
\end{array}
\]

The morphism \( G^{[1]}(S) \rightarrow P^{[1,\psi(2)]}(S) \) is surjective, since the map \( r_2 \) is surjective. Moreover, by Corollary 2.1.10, the group \( P^{[1,\psi(2)]}(S) \) is an \( \mathcal{O}_K \)-module of length 1. Hence, it is an infinite group. Therefore, the composed morphism

\[ G^{[1]}(S) \rightarrow G(S) \rightarrow P_{[\psi(2)]}(S) = \Pic^0(\mathcal{X}_{\psi(2)}) \]
has infinite image. Hence, the composed morphism
\[(49) \quad G^{[1]'}(S) \rightarrow G(S) \rightarrow P_{[\psi(2)]}(S) = \text{Pic}^0(X_{\psi(2)})\]
also has infinite image because \(G^{[1]}\) is an open subgroup of \(G^{[1]'}\).

Next, we consider the map of functors \(r'_2 : G^{[1]'} \rightarrow P_{[\psi(2)]} = \text{Pic}^0_{X_{\psi(2)}/S}\) obtained as the composition of \(G^{[1]'} \rightarrow G\) with \(r_2 : G \rightarrow P_{[\psi(2)]}\). This map induces a morphism of pro-algebraic groups over \(k\), again denoted by \(r'_2\):
\[r'_2 : \text{Gr}(G^{[1]'}) \rightarrow \text{Gr}(P_{[\psi(2)]}).\]

We claim that this morphism factors through \(G^{[1]'}_s = \text{Gr}(G^{[1]'})\):
\[(50) \quad \text{Gr}(G^{[1]'}) \xrightarrow{r'_2} \text{Gr}(P_{[\psi(2)]}) \xrightarrow{r_{2,s}} G^{[1]'}_s \]

Let \(K\) be the kernel of the morphism \(\text{Gr}(G^{[1]'}) \rightarrow G^{[1]'}_s\); it is pro-smooth and connected. Let \(\varepsilon \in G^{[1]'}(S)\) be such that \(\varepsilon(s) \in G^{[1]'}_s\) is the unit element. In particular, the support of the torsion module \(M^{[1]'}\) at \(\varepsilon(s)\) has two irreducible components, which implies, according to Proposition \[3.2.1\] (1), that the \(O_K\)-module \(\varepsilon^*M^{[1]'} = H^1(X, L_s)\) has length at least 2 (here, \(L_s = (\text{id} \times \varepsilon)^* (\mathcal{L}|_{X \times G^{[1]'}})\)). In particular, by Corollary \[2.2.3\], the restriction \(L_s|_{X_{\psi(2)}}\) is the power of \(T|_{X_{\psi(2)}}\). Since the invertible sheaf \(T\) is finite order and since \(k\) is algebraically closed, this implies that the induced map \(K \rightarrow \text{Gr}(P_{[\psi(2)]})\) has finite image, in particular, it is trivial since \(K\) is pro-smooth and connected. This fact ensures the existence of the factorization in (50).

In order to prove (i), let us denote by \(\ell_1\) the length of the \(O_{G^{[1]'}}\)-module \(M^{[1]'}_{\xi_1}\). By definition of \(G^{[1]'}\), we have \(\ell_1 \geq 1\). Suppose \(\ell_1 \geq 2\). Let \(\varepsilon \in G^{[1]'}(S)\) be a section of \(G^{[1]'}\) such that \(\varepsilon(s) \in \{\xi_1\} \subset G^{[1]'}_s\) and denote by \(L_\varepsilon\) the associated rigidified invertible sheaf on \(X\). According to Proposition \[3.2.1\] (1), the \(O_K\)-module \(\varepsilon^*M^{[1]'} \cong H^1(X, L_s)\) has length at least \(\ell_1 \geq 2\). Hence, by Corollary \[2.2.3\], we have \(L_\varepsilon|_{X_{\psi(2)}} \cong T^j|_{X_{\psi(2)}}\) for a suitable integer \(i\). Thus, \(r'_{2,s}(\xi_1 \cup \xi_1) \subset \text{Gr}(P_{[\psi(2)]})\) consists of a single element because \(r'_{2,s}\) is continuous and the set \(\{T^j : j \in \mathbb{Z}\}\) is finite. Using the fact that \(r'_{2,s}\) is a morphism of groups and that \(\{\xi_1\}\) is an irreducible component of \(G^{[1]'}_s\), we deduce that the morphism \(r'_{2,s}\) and hence the map \(r'_2 : G^{[1]'}(S) \rightarrow P_{[\psi(2)]}(S) = \text{Pic}^0(X_{\psi(2)})\) has finite image. This contradicts the assertion on the infinity of the image of (49) proved above. Hence \(\ell_1 = 1\), and this concludes the proof of (i).

Assertion (ii) is just a corollary of (i). In fact, for \(Y_1\) an irreducible component of \(H_{s}\), let \(\xi_1\) denote the generic point of \(G^{[1]'}\) lying above \(Y_1\). Let \(x' \in \{\xi_1\} \subset G^{[1]'}_s\) be a closed point not contained in \(H^{[1]'}_s\), and \(x' : S \rightarrow G^{[1]'}\) a section lifting \(x'\), which also gives a section \(\varepsilon : S \rightarrow G\) by composition with \(G^{[1]'} \rightarrow G\). Let \(x = \varepsilon(s) \in H_s\). Since \(x' \notin H^{[1]'}_s\), and \(\ell(M^{[1]'}_{\xi_1}) = 1\) by assertion (i) of this Lemma, the \(O_K\)-module \(\varepsilon^*M^{[1]'} = \varepsilon^*M\) is of length 1 (see Proposition \[3.2.1\] (1)). According to Proposition \[3.2.1\] (2), this last condition implies that \(H\) is regular at \(x\). Hence \(H\) is regular at the generic point of the irreducible component \(Y_1\) of \(H_{s}\) because \(Y_1\) contains \(x\). Since this can be done for any generic point of \(H_{s}\), one finds that \(H\) is normal by using Serre's criterion of normality (recall that the generic fibre \(H_K\) of \(H\) is regular, and the scheme \(H\), being a divisor of a regular scheme, is Cohen-Macaulay).
For (iii), recall that the composed morphism $G(S) \to \Pic^0(X) \to J(S)$ is surjective (see §111). Since $G[S]''$ is the dilatation of $G$ along $H_{s,\text{red}}$, the surjectivity of the last map implies that the map $G[S]'(S) \to J[S](S)$ is also surjective. Since $G[S] \subset G[S]'$ is an open subgroup, with non-empty special fibre, and the abstract group $G[S]'(S)/G[S](S)$ is a finite group, according to [5], 9.2/6, the morphism $\theta[S]$ also induces a surjection $G[S](S) \to J[S](S)$. In particular, using the fact that the two $S$-group schemes $G[S]$ and $J[S]$ are smooth, we find that the morphism $\theta[S]$ is faithfully flat. Hence $\ker(\theta[S]) = H[S]$ since both $\ker(\theta[S])$ and $H[S]$ are flat closed subgroup schemes of $G[S]$.

By abuse of notation, let us denote by $r_2$ both the following composition of morphisms

$$r_2: G[S] \longrightarrow G \longrightarrow P_{[\psi(2)]}$$

and the induced morphism of pro-algebraic groups over $k$: $\text{Gr}(G[S]) \to \text{Gr}(P_{[\psi(2)]})$. As we have seen in the proof of Lemma 3.4.2, diagram (50), (since $G[S] \subset G[S]'$ is an open subgroup), the map $r_2$ factors through the canonical surjection $\text{Gr}(G[S]) \to G[S]$: $$\begin{array}{ccc}
\text{Gr}(G[S]) & \longrightarrow & \text{Gr}(P_{[\psi(2)]}) \\
\downarrow & & \downarrow \\
G[S] & \longrightarrow & G[S] \\
\end{array}$$

Next, define $Z_1 := \ker(r_{2,s})_{\text{red}} \to G[S]$. Then the same argument used for $Z$ in (111), in the case $n = 1$, implies that $Z_1$ is a union of connected components of $H_{s,\text{red}}$.

Now we use constructions similar to those used in the comparison at the first level. Let $G^2$ (respectively $G^2'$) be the dilatation of $G[S]$ along the closed smooth subgroup $Z_1$ of $G[S]$ (respectively along $H_{s,\text{red}} \to G[S]$), and let $\alpha^1$ be the composed morphism $G^2 \to G[S] \to G$. According to [5], 3.2/3, $G^2$ is a smooth $S$-group scheme, and we have an exact sequence:

$$0 \longrightarrow G^2(S) \longrightarrow G^1(S) \longrightarrow P^{[1,\psi(2)]}(S) \longrightarrow 0.$$

On the other hand, since $H_{s}^{[1]}$ is the kernel of the morphism $G^{[1]} \to J^{[1]}$, according to the universal property of dilatations, we have an exact sequence of abstract groups

$$0 \longrightarrow G^{[2]}(S) \longrightarrow G^{[1]}(S) \longrightarrow J^{[1]}(S) \longrightarrow 0.$$ 

Since $Z_1 \subset H_{s,\text{red}}^{[1]}$ is an open subgroup scheme, $G^{[2]} \subset G^{[2]'}$ is an open subgroup. Hence we obtain a morphism $\alpha: P^{[1,\psi(2)]}(S) \to J^{[1]}(S)$ which makes the following diagram

$$\begin{array}{ccc}
G^{[1]}(S) & \longrightarrow & P^{[1]}(S) \\
\downarrow & & \downarrow \\
J^{[1]}(S) & \longrightarrow & J^{[1]}(S) \\
\end{array}$$
Finally, we summarize our results in the following theorem:

\[ J \overset{\text{K}}{\rightarrow} X \]

Let \( J \) be a K-torsor under an elliptic curve and \( X \) its S-proper regular minimal model. Let \( J \) be the identity component of the Néron model over \( S \) of the jacobian \( \text{Pic}^0_X \). Then for any integer \( n \geq 1 \), the surjective morphism of fppf-sheaves \( q \): \( \text{Pic}^0_X \rightarrow J \) in \( \text{Gr} \) induces a morphism of smooth k-group schemes

\[ q_n: \text{Gr}(\text{Pic}^0_{X_{/\psi(n)}}) \rightarrow \text{Gr}_n(J) \]

making the obvious diagram commute. Moreover, the morphism \( q_n \) defined above is an isogeny of algebraic k-groups, and the group of k-points of the kernel of \( q_n \) is given by

\[ \ker(q_n)(k) = \{ \mathcal{I}_{|X_{\psi(n)}} : i \in \mathbb{Z} \} \subset \text{Gr}(\text{Pic}_{\psi(n)})(k) \cong \text{Pic}^0(X_{\psi(n)}) . \]

**Corollary 3.4.4.** The morphism \( q \): \( \text{Pic}^0(X) \rightarrow J(S) \) in \( \text{Gr} \) maps \( \text{Pic}^0_{\psi(n)}(S) \) onto \( J^n(S) \), thus inducing an isogeny of connected quasi-algebraic groups

\[ q_n: \text{Gr}(\text{Pic}_{\psi(n)}) \rightarrow \text{Gr}_n(J) \]

whose kernel is generated by the element \( \mathcal{I}_{|X_{\psi(n)}} \in \text{Gr}(\text{Pic}_{\psi(n)})(k) = \text{Pic}^0(X_{\psi(n)}) \). Furthermore \( q \) is an epimorphism in the abelian category of Serre pro-algebraic groups (in particular it is surjective on k-sections), and the kernel of \( q \) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \).
Proof. According to the previous theorem, we have \( q(P^{[\psi(n)]}(S)) \subset J^{[n]}(S) \), hence \( q \) induces the isogeny \( q_n \) with properties as stated in the corollary. In particular, being a projective limit of surjective morphisms of quasi-algebraic groups, \( q \) is an epimorphism in the category of Serre pro-algebraic groups. In order to finish the proof, we need only establish that the last inclusion is in fact an equality. To see this, we consider the quotient of \( J^{[n]}(S) \) by \( q(P^{[\psi(n)]}(S)) \). By applying the snake Lemma to the following diagram with exact rows

\[
\begin{array}{c}
0 \rightarrow P^{[\psi(n)]}(S) \rightarrow \text{Pic}^0(X) \rightarrow \text{Gr}(P^{[\psi(n)]}) \rightarrow 0 \\
0 \rightarrow J^{[n]}(S) \rightarrow J(S) \rightarrow \text{Gr}_n(J) \rightarrow 0
\end{array}
\]

we find that the quotient \( J^{[n]}(S) \) by \( P^{[\psi(n)]}(S) \) is a finite pro-algebraic group. Since \( J^{[n]}/S \) has connected fibres, the pro-algebraic group \( J^{[n]}(S) \) is connected, hence the cokernel of the left vertical arrow is necessarily trivial. Thus \( q(P^{[\psi(n)]}(S)) = J^{[n]}(S) \). The kernel of \( q \) is cyclic of order \( d \) because the kernel of any \( q_n \) is a constant finite group and the kernel of \( q \) in \([12]\) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \) and is generated by \( \mathcal{I} \) (\([21] \), Théorème 6.4.1 (3)).

\[\square\]

**Corollary 3.4.5.** Let \( X_K/K \) be a torsor under an elliptic curve, and \( X/S \) its proper regular minimal model. The following conditions are equivalent:

(i) The \( S \)-scheme \( X/S \) is cohomologically flat in dimension 0.

(ii) The Picard functor \( \text{Pic}^0_{X/S} \) is representable, and the canonical map \( \text{Pic}^0_{X/S} \rightarrow J \) is étale.

(iii) The extension of Serre pro-algebraic groups associated to \( X/S \) deduced from \([3.4.4]\) lies in the subgroup \( \text{Ext}^1(\text{Gr}_1(J), \mathbb{Z}/d\mathbb{Z}) \subset \text{Ext}^1(J(S), \mathbb{Z}/d\mathbb{Z}) \).

Proof. The equivalence between (i) and (ii) follows from Proposition 5.2 of \([21]\) and from Corollary \([2.3.3]\). To see (i)\( \Leftrightarrow \) (iii), suppose first that \( X/S \) is cohomologically flat in dimension 0, namely \( \mathcal{I}|_{X_1} \) is of order \( d \) (Corollary \([2.3.3]\)); we then have the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \text{Pic}^0(X) \rightarrow q J(S) \rightarrow 0 \\
0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \text{Gr}(P^{[1]}) \rightarrow q_1 \text{Gr}_1(J) \rightarrow 0
\end{array}
\]

In particular, we get (iii). Conversely, if condition (iii) holds, the morphism \( q \) induces an isomorphism between \( q^{-1}(J^{[1]}(S)) \) and \( J^{[1]}(S) = \ker(J(S) \rightarrow \text{Gr}_1(J)) \). In particular, Serre pro-algebraic group \( q^{-1}(J^{[1]}(S)) \) is connected. On the other hand, \( q^{-1}(J^{[1]}(S)) \) contains the subgroup \( \text{Gr}(P^{[1]}) \) of index \( d/d' \) with \( d' \) the order of \( \mathcal{I}|_{X_1} \). As a result, we find \( q^{-1}(J^{[1]}(S)) = \text{Gr}(P^{[1]}) \) by the connectedness of \( J^{[1]}(S) \). Therefore, \( d = d' \), and \( \mathcal{I}|_{X_1} \) is of order \( d \), hence \( X/S \) is cohomologically flat in dimension 0 (Corollary \([2.3.3]\)).

\[\square\]

We record the following fact, which will not be used in the rest of this paper.

**Remark 3.4.6.** For each integer \( n \geq 1 \), let \( N^{[n]} \) be the kernel of the morphism of \( S \)-group schemes \( \theta^{[n]} : G^{[n]} \rightarrow J^{[n]} \) obtained inductively by a sequence of dilatations of \( \theta : G \rightarrow J \) in the proof of Theorem \([3.4.3]\) (and its omitted induction steps). The proof of Theorem \([3.4.3]\) (especially of Lemma \([3.4.2]\) (ii)) shows that the scheme \( N^{[n]} \) is normal. Moreover, one can verify that the scheme \( N^{[n]} \) is smooth over \( S \) for sufficiently large \( n \).

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4 Shafarevich’s pairing

Let $A_K$ be an abelian variety over $K$. We construct in this section via the rigidified Picard functor a homomorphism $\Xi': H^1(K, A_K) \to \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z})$ which coincides with the restriction of $[5]$ to the $n$-parts when $n \in \mathbb{Z}_{>0}$ is prime to $p$, and more generally, for all positive integers $n$ in the mixed characteristic case (Theorem [4.11]). In Section 5 we will use these constructions to study the morphism $\Phi_d$ in [5]. All group schemes we will work with are assumed to be commutative.

4.1 The component group of a torus

One of the key facts in the construction of Shafarevich’s duality is the pro-algebraic structure of the cohomology group $H^1(K, \mu_n)$, where $\mu_n$ denotes the finite subgroup scheme of $n$-th roots of unity in the multiplicative group $\mathbb{G}_{m,K}$. We first recall this construction. Observe that the Néron model $T$ over $S = \text{Spec}(\mathcal{O}_K)$ of a torus $T_K$ is locally of finite type over $S$, but, in general, not of finite type over $S$. It is of finite type over $S$ if and only if $T_K$ does not contain split tori (cf. [5], 10.2/1). Let $\Lambda_K$ denote the character group of $T_K$.

**Lemma 4.1.1.** There exists a functorial construction that associates to a finite multiplicative group scheme $F_K$ a Serre pro-algebraic group $H^1(K, F_K)$ whose group of $k$-rational points is isomorphic to $H^1_{fl}(K, F_K)$.

**Proof.** (Cf. [1], 4.3.) Let $f: T_{1,K} \to T_{2,K}$ be an isogeny of tori with kernel $F_K$. Let $\Lambda_{i,K}$ (respectively $\Lambda_{i}$) be the character group (respectively the Néron model) of $T_{i,K}$, $i = 1, 2$. Let $T_{1,K}^{(d)}$ denote the torus (déployé) whose character group is the constant free group $\Lambda_{1,K}(K)$, and similarly for $T_{2,K}^{(d)}$. They are split tori with isomorphic component groups, say $\mathbb{Z}^r$. Furthermore the isogeny $f$ induces an isogeny $f^{(d)}: T_{1,K}^{(d)} \to T_{2,K}^{(d)}$ that is injective on component groups. The torus $T_{1,K}'$, defined as the kernel of the quotient map $T_{1,K} \to T_{2,K}^{(d)}$, admits a Néron model of finite type over $S$ because its character group is $\Lambda_{1,K}' \cong \Lambda_{1,K}/\Lambda_{1,K}(K)$. Since $K$ is a $(C_1)$-field, tori are cohomologically trivial. Hence there is an isomorphism $T_{1,K}'(K)/T_{1,K}'(K) \sim T_{1,K}'(K)$ and the complexes of component groups

$$\pi_0(T_{i}'^{(d)}) \to \pi_0(T_{i}) \to \pi_0(T_{i}'^{(d)}) \to 0,$$

are exact. One deduces from this fact that the kernel and the cokernel of the homomorphism $\pi_0(T_{1}) \to \pi_0(T_{2})$ are finite groups.

The identity components of the Néron models $T_i$ are smooth group schemes of finite type ([5], 10.1). Hence their perfect Greenberg realizations are Serre pro-algebraic groups. Let $P$ denote the cokernel of the map $\text{Gr}(T_{1}) \to \text{Gr}(T_{2})$. Now, the cokernel $H^1(K, F_K)$ of the map $\text{Gr}(T_{1}) \to \text{Gr}(T_{2})$ is an extension of the finite group $\pi_0(T_{2})/\pi_0(T_{1})$ by the quotient of $P$ by a finite constant group, hence it is a Serre pro-algebraic group. By construction the group of $k$-points of $H^1(K, F_K)$ is $H^1_{fl}(K, F_K)$. Furthermore, the pro-algebraic group $H^1(K, F_K)$ does not depend on the isogeny $f$ ([1], 4.3 (b)).

For the functoriality, consider a morphism of finite multiplicative group schemes $g: F_K \to F'_K$ and let $f: T_{1,K} \to T_{2,K}$ be an isogeny of tori with kernel $F_K$. Then $F'_K$ embeds in $T_{1,K}' := T_{1,K}/\ker(g)$ and we have an isogeny of tori $f': T_{1,K}' \to T_{2,K}$ with kernel $F'_K$. Since $f$ factors through $f'$, we get a morphism $H^1(K, F_K) \to H^1(K, F'_K)$.

For our later work we will also need the following result:
Lemma 4.1.2. Let \( 0 \to T_K \to G_K \to A_K \to 0 \) be an extension of an abelian variety \( A_K \) by a torus \( T_K \). Let \( A, G, T \), be the Néron models of \( A_K, G_K, T_K \), respectively, over \( S \). Then the above sequence induces a homomorphism of profinite groups \( \pi_1(\Gr(A)) \to \pi_0(\Gr(T)) \) where the index \( \text{tor} \) indicates the torsion subgroup.

Proof. The sequence of Néron models of the above extension is exact on \( S \)-sections; indeed, by the universal property of Néron models, \( T_K(K) = T(S), G_K(K) = G(S), A_K(K) = A(S) \), and \( T_K \) is cohomologically trivial. Hence the sequence of Néron models provides an extension of perfect \( k \)-schemes \( 0 \to \Gr(T) \to \Gr(G) \to \Gr(A) \to 0 \). If \( T \) is of finite type, \( \pi_0(\Gr(T)) = \pi_0(T) = \pi_0(\Gr(T)) \) is finite. Furthermore the above sequence is a sequence of Serre pro-algebraic groups, and the desired map comes from the long exact sequence of the \( \pi_i \)'s (see \[4.2\] (iii)).

Suppose that \( T \) is locally of finite type. Since \( k \) is algebraically closed, \( \pi_0(T) = \pi_0(T) \oplus \pi_0(T)_{\text{ft}} \) with \( \pi_0(T)_{\text{ft}} \) torsion-free. Let \( T^{\text{ft}} \) be the maximal subgroup of \( T \) whose component group is finite. In particular, \( T^{\text{ft}} \) contains the identity component \( T^0 \) and \( \pi_0(T^{\text{ft}}) \cong \pi_0(T) \). Then the sequence we started with extends to a sequence \( 0 \to T^{\text{ft}} \to G \to A^{\text{ft}} \to 0 \) that is exact on \( S \)-sections because all extensions of \( A^{\text{ft}} \) by \( T/T^{\text{ft}} \) are trivial \((\[10\], §5.7, 5.5)). Then we have an exact sequence of Serre pro-algebraic groups \( 0 \to \Gr(T^{\text{ft}}) \to \Gr(G) \to \Gr(A^{\text{ft}}) \to 0 \). By the long exact sequence of the \( \pi_i \)'s we get then a map \( \pi_1(\Gr(A^{\text{ft}})) \to \pi_0(\Gr(T^{\text{ft}})) \). The conclusion follows using the canonical isomorphisms \( \pi_1(\Gr(A^{\text{ft}})) \cong \pi_1(\Gr(A)) \) and \( \pi_0(\Gr(T^{\text{ft}})) \cong \pi_0(T) \).

In the case \( F_K = \mu_n \) we can describe explicitly the component group of \( H^1(K, F_K) \).

Lemma 4.1.3. The component group of \( H^1(K, \mu_n) \) is canonically isomorphic to \( \mathbb{Z}/n\mathbb{Z} \).

Proof. Using the Kummer sequence \( 0 \to \mu_n \to G_{m,K} \to G_{m,K} \to 0 \), we get that \( H^1(K, \mu_n) \) is the cokernel of the \( n \)-multiplication on \( \Gr(G) \) with \( G \) the Néron model of \( G_{m,K} \) over \( S \). By the right exactness of the functor \( \pi_0 \), we then get the isomorphisms \( \pi_0(H^1(K, \mu_n)) \cong \pi_0(\Gr(G))/n\pi_0(\Gr(G)) \cong \mathbb{Z}/n\mathbb{Z} \), since \( \pi_0(\Gr(G)) \) is canonically isomorphic to \( \pi_0(G_s) \), and hence to \( \mathbb{Z} \) (see \[1.2\] (i)).

4.2 Bégueri’s construction

In this section we assume that \( K \) has characteristic 0. Given \( K \)-schemes \( Z_K \) and \( U_K \), let \( Z_{U_K} \) denote the fibred product \( Z_K \times_K U_K \), viewed as a scheme over \( U_K \). Furthermore, we will identify any commutative \( K \)-group scheme with the corresponding fppf sheaf so that Hom and Ext groups or sheaves are always meant for the fppf topology.

Let \( X_K \) be a \( K \)-torsor under \( A_K \) and let \( n \) be a positive integer such that \( n[X_K] \) is trivial in \( H^1(K, A_K) \). Since the order of \( X_K \) is defined as the order of \( [X_K] \), it is the minimum among such integers. The element \( [X_K] \) in \( H^1(K, A_K) \) corresponds to an extension of group schemes over \( K \)

\[
0 \to A_K \longrightarrow B_K \longrightarrow Z \longrightarrow 0,
\]

so that the fibre at \( 1 \in Z \) is isomorphic to \( X_K \) \((\[10\], VII, §1.4)). Since the class of \( X_K \) in \( H^1(K, A_K) \) is \( n \)-torsion, the exact sequence \( (52) \) is the pull-back along \( Z \to \mathbb{Z}/n\mathbb{Z} \) of \( a \), not unique, extension

\[
0 \to A_K \longrightarrow E_K \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0
\]

and \( X_K \) is isomorphic to the fibre of \( E_K \) at \( 1 \in \mathbb{Z}/n\mathbb{Z} \). Let

\[
0 \to A_K \longrightarrow E_K \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0
\]
be the sequence of \( n \)-torsion subgroups. Consider also the exact sequence
\[(55) \quad 0 \to \mu_n \to V_{n,E_K}^* \to \text{Ext}^1(E_K, \mathbb{G}_m)_{n,E_K} \to \text{Ext}^1(E_K, \mathbb{G}_m) \cong A'_K \to 0\]
(cf. [1], 2.3.2) where \( V_{n,E_K}^* \) denotes the torus \( \mathfrak{R}_{n,E_K/K}(\mathbb{G}_m, nE_K) \) representing the Weil restriction functor that associates to a \( K \)-scheme \( S' \) the group \( \mathbb{G}_m, K(S' \times_K nE_K) \). The isomorphism on the right is due to the vanishing of the sheaf \( \text{Ext}^1(K, \mathbb{G}_m) \) for \( i = 1, 2 \); to prove this fact recall that \( n: \mathbb{G}_m \to \mathbb{G}_m \) is an epimorphism for the fpf topology and that \( \text{Ext}^1(Z, \mathbb{G}_m) = 0 \) for \( i > 0 \). Observe that
\[\mu_n \cong \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \cong \text{Hom}(E_K, \mathbb{G}_m).\]
The second map in (55) sends a homomorphism \( f: E_K \to \mathbb{G}_m, K \) to its restriction to \( nE_K \); while the third arrow sends \( g \in \mathbb{G}_m, K(nE_K) \) to (the isomorphism class of) the trivial extension endowed with the section \( g \), and the map \( \tau_E \) forgets the rigidification along \( nE_K \).

We now describe Bégueri’s construction of Shafarevich’s duality following [1]. Let \( F_K \) be a finite \( K \)-group scheme and \( F_K^D \) its Cartier dual. There is a short exact sequence (cf. [1], 2.2.1)
\[(56) \quad 0 \to F_K^D \to V_{F_K}^* \to \text{Ext}^1(F_K, \mathbb{G}_m)_{F_K} \to 0,\]
where the second map forgets the group structure and the third map associates to each \( f \in \mathbb{G}_m, K(F_K) \) the trivial extension endowed with the rigidification induced by \( f \). We also recall the following exact sequence (cf. [1], 2.3.1)
\[(57) \quad 0 \to V_{n,A_K}^* \to \text{Ext}^1(A_K, \mathbb{G}_m)_{n,A_K} \to A'_K \to 0.\]

In [1], 8.2.2, Bégueri first constructs a map
\[\Gamma: H^1(K, nA_K) \to \text{Ext}^1(\mathbb{Gr}(A'), H^1(K, \mu_n))\]
as follows: any element in \( H^1(K, nA_K) \) corresponds to a sequence \( \eta_n \) as in (54). Consider now the diagram
\[
\begin{array}{ccccccccc}
\mu_n & \downarrow & \mu_n & \downarrow & 0 \\
V_{\mathbb{Z}/n\mathbb{Z}}^* & \downarrow v_1 & V_{n,E_K}^* & \downarrow v_2 & V_{n,A_K}^* \\
\text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)_{\mathbb{Z}/n\mathbb{Z}} & \downarrow \tau_E & \text{Ext}^1(E_K, \mathbb{G}_m)_{n,E_K} & \downarrow \tau_A & \text{Ext}^1(A_K, \mathbb{G}_m)_{n,A_K} \\
0 & \downarrow & A'_K & \downarrow & A'_K \\
\end{array}
\]
where the rows are complexes and the vertical sequences are those in (56), for \( F_K = \mathbb{Z}/n\mathbb{Z}, \) (55), (57), respectively. Since \( K \) has characteristic 0, the second row consists of tori, while the third row consists of semi-abelian varieties. Hence they all admit Néron models, locally of finite type over \( S \). On passing to the perfection of the Greenberg realization of the Néron models and considering the cokernels of the maps induced by \( v_1, v_2, v_3 \), one gets a complex of Serre pro-algebraic groups (cf. Lemma [1.1.1])
\[(58) \quad 0 \to H^1(K, \mu_n) \to \text{Ext}^1(E_K, \mathbb{G}_m) \to \text{Gr}(A') \to 0;\]
this is indeed an exact sequence because on $k$-points it induces the exact sequence

$$0 \longrightarrow \text{Ext}^1(Z/nZ, \mathbb{G}_m) \longrightarrow \text{Ext}^1(E_K, \mathbb{G}_m) \longrightarrow \text{Ext}^1(A_K, \mathbb{G}_m) \longrightarrow 0,$$

where the group on the left is isomorphic to $H^1_n(K, \mu_n)$ and the one on the right is isomorphic to $A'(O_K)$. We have thus associated with (58) an extension of $\text{Gr}(A')$ by $H^1(K, \mu_n)$: this is the image of (51) via $\Gamma$.

The homomorphism

$$\psi_n: H^1_n(K, A_K) \longrightarrow \text{Ext}^1(\text{Gr}(A^0), Z/nZ)$$

in [1], 8.2.3, is then obtained by applying first $\Gamma$, then the pull-back along $\text{Gr}(A^0) \to \text{Gr}(A')$ and, finally, the push-out along $H^1(K, \mu_n) \to \pi_0(H^1(K, \mu_n)) \cong \mathbb{Z}/n\mathbb{Z}$ (cf. Lemma 4.1.3). Let

$$\psi_n(\eta_n): 0 \longrightarrow Z/nZ \longrightarrow W(X_K) \longrightarrow \text{Gr}(A^0) \longrightarrow 0$$

denote the image of (54) via $\psi_n$. Recall now that (cf. [24], 5.4)

$$\text{Ext}^1(\text{Gr}(A^0), \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Hom}(\pi_1(\text{Gr}(A^0)), \mathbb{Q}/\mathbb{Z}) \leftarrow \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z}).$$

In terms of homomorphisms of profinite groups, the extension (59) then corresponds to a map

$$u^r = u^r_{X_K}: \pi_1(\text{Gr}(A')) \longrightarrow \pi_0(H^1(K, \mu_n)) \cong \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

deduced from (58) (or equivalently, from the pull-back of (58) along $\text{Gr}(A^0) \to \text{Gr}(A')$) via the long exact sequence of $\pi_i$’s.

**Theorem 4.2.1** ([1], 8.2.3, 8.3.6). (i) The extension class $\psi_n(\eta_n)$ in (59) depends only on the class of the sequence (52), i.e., on $[X_K]$; furthermore, $\psi_n$ factors through an isomorphism $\Psi_n: nH^1_n(K, A_K) \sim \text{Ext}^1(\text{Gr}(A^0), Z/nZ)$.

(ii) Let $n', n \in \mathbb{Z}_{>0}$, with $n | n'$; then the following diagram

$$\begin{array}{ccc}
\psi_n & & \psi_{n'} \\
\downarrow & & \downarrow \\
\text{Ext}^1(\text{Gr}(A^0), Z/nZ) & \longrightarrow & \text{Ext}^1(\text{Gr}(A^0), Z/n'Z)
\end{array}$$

commute, where the upper horizontal arrow is the usual inclusion of torsion subgroups of $H^1_n(K, A_K)$ and the lower horizontal arrow is the push-out along the inclusion $Z/nZ \to Z/n'Z$ (in $\mathbb{Q}/\mathbb{Z}$).

(iii) Passing to the limit on $n$, the homomorphisms $\Psi_n$ provide an isomorphism $H^1_n(K, A_K) \sim \text{Ext}^1(\text{Gr}(A^0), \mathbb{Q}/\mathbb{Z})$ and hence Shafarevich duality $H^1_n(K, A_K) \sim \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z})$ in [6] via the isomorphisms (60).

Hence we can deduce that

**Corollary 4.2.2.** Shafarevich’s duality $H^1_n(K, A_K) \sim \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z})$ in [6] maps the class of the torsor $X_K$ to the homomorphism $u^r_{X_K}$ in (61).
We give now an alternative construction of the map $\psi_n$ that will be useful for further applications. The kernel of $\tau_E$ in (55) is a torus, which, for brevity, will be denoted by $T^\tau_K$. Let $T^\tau$ be its Néron model over $S$. We have an exact sequence

$$0 \rightarrow T^\tau_K \rightarrow G^\tau_K := \text{Ext}^1(E_K, \mathbb{G}_m)_n \rightarrow A'_K \rightarrow 0$$

which extends to a sequence of Néron models

$$(62) \quad 0 \rightarrow T^\tau \rightarrow G^\tau \rightarrow A' \rightarrow 0$$

which is exact on $S$-sections because $T^\tau_K$ is cohomologically trivial. On applying the perfection of the Greenberg functor we get an exact sequence

$$(63) \quad 0 \rightarrow \text{Gr}(T^\tau) \rightarrow \text{Gr}(G^\tau) \rightarrow \text{Gr}(A') \rightarrow 0$$

where the first two groups are not Serre pro-algebraic groups in general, because they are projective limits of perfect schemes not necessarily of finite type. Let $j_\ast V^*_n E_K$ denote the Néron model over $S$ of the torus $V^*_n E_K$. Since the map $V^*_n E_K \rightarrow T^\tau_K$, deduced from (55), is an isogeny with kernel $\mu_n$, we have an exact sequence

$$\text{Gr}(j_\ast V^*_n E_K) \rightarrow \text{Gr}(T^\tau) \rightarrow H^1(K, \mu_n) \rightarrow 0,$$

as explained in the proof of Lemma 4.1.1. Now take the push-out of (63) along $h^\tau$; by construction, the resulting exact sequence is the one in (58), i.e., the image of $[X_K]$ via Shafarevich’s duality.

$$\text{Gr}(T^\tau) \rightarrow U \rightarrow \text{Gr}(A^0) \rightarrow 0,$$

and then the push-out of (64) along the composition of maps

$$\text{Gr}(T^\tau) \rightarrow H^1(K, \mu_n) \rightarrow \pi_0(H^1(K, \mu_n)) \cong \mathbb{Z}/n\mathbb{Z},$$

one gets the extension $\psi_n(\eta_n)$ in (59), i.e., the image of $[X_K]$ via Shafarevich’s duality.

Thanks to this new description of Shafarevich’s map, we can characterize the map $u^\tau$ in (61) as follows: let $T^{\tau,\text{ft}}$ be the maximal subgroup scheme of finite type in $T^\tau$ and consider the sequence, exact on $S$-sections,

$$0 \rightarrow T^{\tau,\text{ft}} \rightarrow G' \rightarrow A^0 \rightarrow 0,$$

and obtained from (62), as explained in the proof of Lemma 4.1.2. Then $\psi_n(\eta_n)$, i.e., the push-out of (64) along $h^\tau$ is isomorphic to the push-out of

$$(65) \quad 0 \rightarrow \text{Gr}(T^{\tau,\text{ft}}) \rightarrow \text{Gr}(G') \rightarrow \text{Gr}(A^0) \rightarrow 0$$

along the composition of maps $h^{\tau,\text{ft}}: \text{Gr}(T^{\tau,\text{ft}}) \rightarrow \text{Gr}(T^\tau) \rightarrow H^1(K, \mu_n)$. Hence

$$(66) \quad u^\tau_{X_K} = \pi_0(h^{\tau,\text{ft}}) \circ u^{\tau,\text{ft}},$$

where the homomorphism $u^{\tau,\text{ft}}: \pi_1(\text{Gr}(A')) \rightarrow \pi_0(\text{Gr}(T^{\tau,\text{ft}}))$ is deduced from (65) via the long exact sequence of the $\pi_i$’s.
4.3 An alternative construction using rigidificators

Let $X_K$ be a torus under an abelian variety $A_K$. We will see in this section how the homomorphism $u^*$ in (61) (and in (66)) can be constructed using a rigidificator $x_K$ of the relative Picard functor $Pic_{X_K/K}$ (21, 2.1.1). Observe that any closed point of $X_K$ provides a rigidificator of $Pic_{X_K/K}$.

**Lemma 4.3.1.** Let $X_K$ be a torus under an abelian variety $A_K$, of order $d$. Let $d'$ be the separable index of $X_K$, i.e., the greatest common divisor of the degrees of its finite separable splitting extensions. Then $d|d'$. If $A_K$ is an elliptic curve, then $d = d'$ and the index is indeed the degree of a minimal separable splitting extension.

**Proof.** For the first assertion see [13], comments at pp. 663-664 and Proposition 5. For the latter assertion on elliptic curves see [5], Thm. 9.2. □

**Remark 4.3.2.** Let $x_K = \text{Spec}(K')$ with $K'/K$ a finite separable extension of degree $n$. Then the torus $V^*_x := \mathcal{R}_{K'/K}(\mathbb{G}_{m,K'})$ has component group isomorphic to $\mathbb{Z}$ and the closed immersion $\mathbb{G}_{m,K} \to V^*_x$ ([5], p. 197 last lines), that is the inclusion $K^* \subset K''$ on $K$-sections, induces the $n$-multiplication $n: \mathbb{Z} \to \mathbb{Z}$ on component groups of Néron models over $S$.

The main idea here is to use in (55) a rigidificator $x_K$ of Pic$_{X_K/K}$ in place of $nE_K$. The advantage is that the new construction works even for $K$ of positive characteristic; in the latter case we choose $x_K$ étale so that $V^*_x := \mathcal{R}_{x_K/K(\mathbb{G}_{m,xK})}$ is still a torus.

Observe that a rigidificator $x_K$ is a closed subscheme of $E_K$ and the homomorphisms

$$\text{Hom}(E_K, \mathbb{G}_m) \to V^*_x$$

is still a closed immersion. Indeed any homomorphism $f: E_K \to \mathbb{G}_m$ factors through $\rho: E_K \to \mathbb{Z}/n\mathbb{Z}$ and if $f|_{x_K} = 0$ then $f|_{X_K} = 0$ because $x_K$ is a rigidificator. However $X_K$ is the fibre at 1 of $\rho$ and hence also $f = 0$. We then have an exact sequence

$$0 \to \mu_n \to V^*_x \xrightarrow{\text{Ext}^1(E_K, \mathbb{G}_m)_{x_K}} A'_K \to 0 \quad (67)$$

after recalling the isomorphisms $\mu_n \cong \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \sim \text{Hom}(E_K, \mathbb{G}_m)$. More generally, for any finite étale subscheme $Z_K$ of $E_K$ which satisfies the following property

(*) the canonical map $\text{Hom}(E_K, \mathbb{G}_m) \to V^*_Z$ is a closed immersion,

we can construct an exact sequence as in (67).

Let $T_K$ denote the torus $T^*_K := V^*_x/\mu_n$. The sequence (67) induces an exact sequence

$$0 \to T_K \to \text{Ext}^1(E_K, \mathbb{G}_m)_{x_K} \to A'_K \to 0, \quad (68)$$

and hence a sequence which is exact on $S$-sections (see proof of Lemma 4.1.2)

$$0 \to T^\text{ft} \to G'' \to A^0 \to 0, \quad (69)$$

where $T^\text{ft}$ is the maximal subgroup of finite type over $S$ of the Néron model $T$ of $T_K$. Now consider the cokernel

$$\text{Gr}(j_*V^*_x) \xrightarrow{\phi^*} \text{Gr}(T) \xrightarrow{h} H^1(K, \mu_n) \to 0 \quad (70)$$

of the homomorphism between the perfect Greenberg realizations of the Néron models of $V^*_x$ and $T_K$; by Lemma 4.1.1 it is a Serre pro-algebraic group whose group of $k$-points is $H^1_n(K, \mu_n)$. 42
In order to provide a more useful description of the map $u^T$ in (61), consider the perfect Greenberg realization of (69)

$$(71) \quad 0 \rightarrow \text{Gr}(T^{\text{ft}}) \rightarrow \text{Gr}(G'') \rightarrow \text{Gr}(A^0) \rightarrow 0,$$

and then its push-out along the composition of maps

$$(72) \quad h^{\text{ft}}: \text{Gr}(T^{\text{ft}}) \rightarrow \text{Gr}(T) \xrightarrow{h} \text{H}^1(K, \mu_n).$$

We obtain an exact sequence

$$\zeta: \quad 0 \rightarrow \text{H}^1(K, \mu_n) \rightarrow W' \rightarrow \text{Gr}(A^0) \rightarrow 0$$

and hence a homomorphism

$$u_{X_K} = u: \pi_1(\text{Gr}(A')) \rightarrow \pi_0(\text{H}^1(K, \mu_n)) \cong \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

such that

$$(73) \quad u = \pi_0(h^{\text{ft}}) \circ u^T,$$

where $u^{\text{ft}}: \pi_1(\text{Gr}(A')) \rightarrow \pi_0(\text{Gr}(T^{\text{ft}})) \cong \pi_0(\text{H}^1(K, \mu_n))$ is deduced from the long exact sequence of the $\pi_i$'s of (71).

**Proposition 4.3.3.** The map $\Xi: \text{H}^1(K, A_K) \rightarrow \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z})$, with $[X_K] \mapsto u_{X_K}$, is a group homomorphism. If $\text{char}(K) = 0$ the homomorphism $u_{X_K}$ in (73) coincides with the homomorphism $u_{X_K}^\tau$ in (61). In particular, the homomorphism $\Xi$ is Shafarevich's duality in (8).

**Proof.** We start by showing that, once $X_K$ has been fixed, the construction of $u: \pi_1(\text{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$ in (73) does not depend on the choices of $x_K$, $n$ and $\eta \in \text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A_K)$ above $X_K$.

First we see that $u$ does not depend on the étale finite closed subscheme $x_K$ of $E_K$ satisfying (*). Let $x_K \subseteq y_K$ be two étale subschemes of $E_K$ satisfying (*). Let $T^y_K, h^y, h^{\text{ft}, y}, u^x, u^y$ denote, respectively, the torus in (68), the maps in (70), (72) and (73) for $x_K$, and similarly for $y_K$. The canonical morphism of tori $T^y_K \rightarrow T^x_K$ induces a morphism $\beta: T^{y,\text{ft}} \rightarrow T^{x,\text{ft}}$ between the maximal subgroups of finite type of the Néron models over $S$. Denote by $\beta': \text{Gr}(T^{y,\text{ft}}) \rightarrow \text{Gr}(T^{x,\text{ft}})$ the corresponding map on perfect Greenberg realizations. One then has $\beta' \circ h^{x,\text{ft}} = h^{y,\text{ft}}$ and $u_0(h^{y,\text{ft}}) = \pi_0(h^{x,\text{ft}}) \circ u_0(\beta')$. Furthermore the sequence (71) for $x_K$ is the push-out along $\beta'$ of the sequence (71) for $y_K$. Hence $u^{x,\text{ft}} = u_0(\beta') \circ u^{y,\text{ft}}$. We conclude then that

$$(74) \quad u^x = u_0(h^{x,\text{ft}}) \circ u^{x,\text{ft}} = u_0(h^{x,\text{ft}}) \circ u_0(\beta') \circ u^{y,\text{ft}} = u_0(h^{y,\text{ft}}) \circ u^{y,\text{ft}} = u^y.$$

Let now $n, \hat{n}$ be positive integers such that $n[X_K] = 0$ and $n|\hat{n}$. We can consider the pull-back $\hat{\eta}$ of $\eta$ in (53) along the projection $\mathbb{Z}/\hat{n}\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$. If we proceed with $\hat{\eta}$ as we have done for $\eta$, we get a map $\hat{u}: \pi_1(\text{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$. Observe that the 2-fold extension (67) for $\hat{\eta}$ is the push-out of $\mu_n \rightarrow \mu_{\hat{n}}$ of (67) and that the map $u_0(\text{H}^1(K, \mu_n)) \rightarrow \pi_0(\text{H}^1(K, \mu_{\hat{n}}))$ is the inclusion $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/\hat{n}\mathbb{Z}$. It is now immediate to check that the maps $\hat{u}$ and $u$ coincide.

We have thus obtained a map

$$(75) \quad \text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A_K) \rightarrow \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z}), \quad \eta \mapsto u.$$
To check that this map is indeed a homomorphism, observe that it is functorial in $A_K$. Furthermore we could repeat the construction with any finite constant group $F_K$ in place of $\mathbb{Z}/n\mathbb{Z}$ obtaining in this way a map

$$\text{Ext}^1(F_K, A_K) \rightarrow \text{Hom}(\pi_1(\text{Gr}(A')), \pi_0(\text{H}^1(K, F^D_K)))$$

with $F^D_K$ the Cartier dual of $F_K$. This construction is functorial in $F_K$. The functoriality results are sufficient to conclude that the map in (75) is a homomorphism, because the Baer sum of two extensions as in (53) is found by first taking the direct sum of the two extensions, then applying the push-out along the multiplication of $A'_K$ and finally applying the pull-back along the diagonal map $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

Suppose now that $n$ and $x_K$ are fixed. We show that the map $u$ is trivial if $X_K$ is trivial, i.e., the map in (53) factors through $H^1_{d}(K, A_K) \cong \text{Ext}^1(\mathbb{Z}, A_K)$. Suppose that $X_K$ is trivial and choose a $K$-point $x_K$ of $X_K$. In particular, $V_{x_K}^* = \mathbb{G}_{m, K}$, $T_K \cong \mathbb{G}_{m, K}$ and $\pi_0(T) \cong \mathbb{Z}$. Hence $T^\pi = \mathbb{G}_{m, \mathbb{O}_K}$, the homomorphism $u^\pi: \pi_1(\text{Gr}(A')) \rightarrow \pi_0(\text{Gr}(T^\pi)) = 0$ is the zero map and $u = 0$.

Suppose now that $\text{char}(K) = 0$. To see that the homomorphism $[X_K] \mapsto u_{X_K}$ is Shafarevich’s duality, it is sufficient to check that the homomorphisms $u^\pi$ in (61) and $u$ in (73) coincide (see Corollary 4.2.2). Consider then a finite separable extension $K'/K$ splitting (54) and a point $x_K = \text{Spec}(K')$ of $E_K$ above $1$. In particular $x_K$ is a rigidificator of $\text{Pic}_{X_K/K}$. Set $y_K = \pi E_K$. Then, $u^\pi$ coincides with the map $u^\tau$ in (66) and one can repeat the arguments used in (71) to showing that $u^\tau = u^\pi = u^\pi$.

**Remark 4.3.4.** The original construction by Bégueri works only for $K$ of characteristic zero because in the case of positive characteristic the scheme $V_{\alpha E_K}^*$ (and hence $T^\pi_K$) need not be a torus; in particular it might not admit a Néron model over $S$. The construction via rigidifiers describes in this section works in any characteristic. For $\text{char}(K) = p$ it is not clear that the homomorphism $\Xi$ in Proposition 4.3.3 is Shafarevich’s duality in (8) (see also [2], Theorem 3). We will see in Proposition 4.5.4 that this is the case on the prime-to-$p$ parts.

### 4.4 A construction via the Picard functor

Let $A_K$ be an abelian variety. In this section we present a third possible construction of a homomorphism $H^1(K, A_K) \rightarrow \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z})$, this one making use of the relative Picard functor. We will see in Theorem 4.3.1 that the new construction always coincides with the one in Proposition 4.3.3 and hence with Shafarevich’s duality in the characteristic 0 case.

Let $X_K$ be a torsor under $A_K$ and $x_K = \text{Spec}(K')$ a closed point of $X_K$ with $K'/K$ a finite separable extension of degree $n$; it exists by the smoothness of $X_K$, and $n[X_K] = 0$ by Lemma 4.3.1. No assumption on the characteristic of $K$ is made.

Consider the exact sequence (cf. [22], 2.4.1)

(76) \[ 0 \rightarrow V_{x_K}^* \rightarrow V_{x_K}^* \rightarrow (\text{Pic}_{X_K/K}, x_K)^0 \rightarrow A'_K \rightarrow 0 \]

where we use that $A'_K \simeq \text{Pic}^0_{X_K/K}$ (Remark 2.1.2). Observe that $V_{x_K}^* := \mathcal{R}_{X_K/K}(\mathbb{G}_{m,X_K}) \cong \mathbb{G}_{m, K}$ (22, 2.4.3), $V_{x_K}^*$ is a torus and hence so too is $N_K := \text{Pic}_{x_K/K}/\mathbb{G}_{m, K}$. Let $N$ denote its Néron model. Observe that it follows from Remark 1.3.2 that the component group of $N$ is cyclic of order $n$, hence its perfect Greenberg realization is a Serre pro-algebraic group.

We proceed as in the previous section, first by passing to Néron models and then applying the perfect Greenberg realization to the sequence

(77) \[ 0 \rightarrow N_K \rightarrow (\text{Pic}_{X_K/K}, x_K)^0 \rightarrow A'_K \rightarrow 0, \]
so that we obtain an exact sequence of Serre pro-algebraic groups
\[ 0 \to \text{Gr}(\mathcal{N}) \to \text{Gr}(j_*(\text{Pic}_{X_K/K}, x_K)^0) \xrightarrow{h} \text{Gr}(A') \to 0, \]
and hence a homomorphism
\[ v = v_{X_K} : \pi_1(\text{Gr}(A')) \to \pi_0(\text{Gr}(\mathcal{N})) \cong \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}. \]

In order to compare this construction with the (modified) Bégueri construction of the previous section, i.e., in order to compare the maps \( u \) in (73) and \( v \) in (78), we consider the following diagram
\[ \begin{array}{ccccccccc}
0 & \to & T_K & \to & \text{Ext}^1(E_K, \mathbb{G}_m)_{x_K} & \to & A'_{K, 0} & \to & 0 \\
0 & \to & N_K & \to & (\text{Pic}_{X_K/K}, x_K)^0 & \xrightarrow{h_K} & A'_{K, 0} & \to & 0 \\
\end{array} \]

where the upper sequence is (78), the lower one is (77), \( t_K : T_K := V_{x_K}^*/\mu_n \to V_{x_K}^*/V_{x_K}^0 \to N_K \) is induced by the identity on \( V_{x_K}^* \), and \( f_K \) associates to a \( \mathbb{G}_m \)-extension \( L_K \) of \( E_K \) endowed with a \( x_K \)-section \( \sigma \) its restriction (as torsor) to \( X_K \) endowed with the trivialization along \( x_K \) induced by \( \sigma \). The morphism \( t_K \) is surjective and its kernel is \( V_{x_K}^*/\mu_n \cong \mathbb{G}_{m,K}/\mu_n \cong \mathbb{G}_{m,K} \).

Consider now the induced diagram on Néron models (with exact rows when restricting to \( S \)-sections)
\[ \begin{array}{ccccccccc}
0 & \to & T^{ft} & \to & G'' & \to & A'^{0} & \to & 0 \\
0 & \to & N & \to & j_*(\text{Pic}_{X_K/K}, x_K)^0 & \xrightarrow{f} & A' & \to & 0 \\
\end{array} \]

where the first row is (69). Here \( j_* H_K \) is just a notation for the Néron model of \( H_K \) when it exists. The homomorphism \( u \) in (73) is the composition of the homomorphism \( u^{ft} : \pi_1(\text{Gr}(A'^{0})) \to \pi_0(\text{Gr}(T^{ft})) \) (deduced from the upper sequence) with the homomorphism \( \pi_0(h^{ft}) : \pi_0(T^{ft}) = \pi_0(\text{Gr}(T^{ft})) \to \pi_0(H^1(K, \mu_n)) \),

where \( h^{ft} \) was introduced in (72). It now follows form the above diagram that the map \( v : \pi_1(\text{Gr}(A')) \to \pi_0(\text{Gr}(\mathcal{N})) \) in (78), obtained from the lower exact sequence, satisfies
\[ v = \pi_0(t^{ft}) \circ u^{ft}. \]

We are going to check that \( u \) and \( v \) coincide up to sign, by showing that, up to canonical identifications we have \( \pi_0(h^{ft}) = -\pi_0(t^{ft}) \). To see this fact, consider the following diagram with exact rows and columns
\[ \begin{array}{ccccccccc}
0 & \to & \mu_n & \to & V_{X_K}^* \cong \mathbb{G}_{m,K} & \xrightarrow{n} & \mathbb{G}_{m,K} & \to & 0 \\
0 & \to & \mu_n & \to & V_{x_K}^* & \to & T_K & \to & 0 \\
\end{array} \]
where the middle horizontal sequence is deduced from (67) while the middle vertical sequence comes from (76). Consider the induced diagram of component groups of Néron models

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{n} & \mathbb{Z} \\
\pi_0(j_*V^*_K) & \xrightarrow{} & \pi_0(T) \\
\pi_0(N) & \xrightarrow{} & \pi_0(N) \\
\pi_0(T^ft) & = & \pi_0(T)_{tor}
\end{array}
\]

where \( \iota \) is the inclusion map and the vertical sequences are left exact because \( \mathbb{Z} \) is torsion free (cf. [10], VIII 5.5). We complete the diagram by inserting the cokernels of the horizontal maps

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z} \\
0 & \rightarrow & \pi_0(j_*V^*_K) \\
0 & \rightarrow & \pi_0(N) \\
0 & \rightarrow & \pi_0(T^ft)
\end{array}
\]

where \( \pi_0(h) \circ \iota = \pi_0(h^ft) \) (see (72)) and \( \pi_0(t) \circ \iota = \pi_0(t^ft) \) (see (80)). By Remark 4.3.2, \( \pi_0(j_*V^*_K) \) is isomorphic to \( \mathbb{Z} \), \( \pi_0(N) \) is isomorphic to \( \mathbb{Z}/n\mathbb{Z} \) and, under these identifications, the left vertical sequence coincides with the upper horizontal sequence. More precisely, we have

\[
\pi_0(N) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \pi_0(H^1(K, \mu_n))
\]

where the first isomorphism maps the image of the class of a uniformizer \( \pi' \in K^* = V^*_K(K) \) to the class of 1, while the second isomorphism maps the class of 1 to the image of the cohomology class corresponding to a uniformizer \( \pi \in K^* = \mathbb{G}_{m,K}(K) \). Furthermore, the middle vertical sequence splits as does the middle horizontal sequence, and we may identify \( \pi_0(T) \) with \( \mathbb{Z}/\mathbb{Z}/n\mathbb{Z} \) so that \( \pi_0(h)(a \oplus b) = a - b \) and \( \pi_0(t)(a \oplus b) = b \) with \( b \) the class of \( b \in \mathbb{Z} \) modulo \( n\mathbb{Z} \). Let \( \sigma \) be the section of \( \pi_0(t) \) mapping \( b \) to \( 0 \oplus b \). Then \( \pi_0(h) \circ \sigma = -id_{\mathbb{Z}/n\mathbb{Z}} \). Furthermore \( \sigma \circ \pi_0(t) \circ \iota = \iota \) because \( \sigma \circ \pi_0(t) \circ \iota - \iota \) factors through \( \mathbb{Z} \) and thus is trivial because \( \pi_0(T^ft) \) is torsion. Hence

\[
\pi_0(h^ft) = \pi_0(h) \circ \iota = \pi_0(h) \circ \sigma \circ \pi_0(t) \circ \iota = -t \circ \iota = -\pi_0(t^ft),
\]

and thanks to (81) and (73), we get \( v = \pi_0(t^ft) \circ u^ft = -\pi_0(h^ft) \circ u^ft = -u \). We can then state the main result which is an immediate consequence of what we have just proved and Proposition 4.3.3.

**Theorem 4.4.1.** Let \( A_K \) be an abelian variety over \( K \). The homomorphism

\[
\Xi' : H^1(K, A_K) \longrightarrow \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Q}/\mathbb{Z})
\]

mapping the class \([X_K]\) of the torsor \( X_K \) to the homomorphism \(-v_{X_K} : \pi_1(\text{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z} \) (see (78)) coincides with the homomorphism \( \Xi \) in Proposition 4.3.3 mapping \([X_K]\) to the homomorphism \( u_{X_K} \) in (73). If furthermore the characteristic of \( K \) is zero, then \(-v_{X_K} = u_{X_K} \) coincides with Bégueris’s homomorphism \( u_{X_K} \) in (61) and hence \( \Xi' \) is Shafarevich’s duality in (8).
4.5 Comparison on the prime-to-$p$ parts

We have seen in Theorem 4.4.1 that the homomorphisms $\Xi$ and $\Xi'$ always coincide, and in the mixed characteristic case, that they coincide with Shafarevich’s duality in (5); in particular they are isomorphisms. For $K$ of characteristic $p$, it is not clear in general either that they are isomorphisms or that they correspond to Shafarevich’s duality in (5) (see also [2], Theorem 3). However, we have a partial result on the prime-to-$p$ parts where Shafarevich’s duality is quite easy to describe.

We recall here what Shafarevich’s duality looks like on the prime-to-$p$ parts. Let $n = l^r$ be a positive integer, prime to $p$, and large enough to kill the $l$-primary parts of the component groups of $A_K$ and $A_K'$. Consider the perfect cup product pairing

$$
\langle \ , \ \rangle : \mathrm{H}^1(K, nA_K) \times nA_K'(K) \rightarrow \mathrm{H}^1(K, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}
$$

on the (étale or flat) cohomology groups of the $n$-torsion subgroups of $A_K$ and $A_K'$. Given an extension $\eta_n$ as in (5) (which corresponds to the torsor $X_K$) and a point $a \in nA_K'(K)$, then $\langle \eta_n, a \rangle$ is the class of the pull-back along $a: \mathbb{Z} \rightarrow nA_K'$ of the Cartier dual of $\eta_n$,

$$
n_n^D : 0 \rightarrow \mu_n \rightarrow nE_K^D \rightarrow nA_K' \rightarrow 0,
$$

and it corresponds to the image of $a$ along the boundary map $\partial : nA_K'(K) \rightarrow \mathrm{H}^1(K, \mu_n)$. Furthermore, if $nA^0$ denotes the quasi-finite subgroup of $n$-torsion sections of $A^0$, we have

$$
\pi_1(\text{Gr}(A'))/n\pi_1(\text{Gr}(A')) \sim nA^0(\mathcal{O}_K) \sim n^2A'(\mathcal{O}_K)/nA'(\mathcal{O}_K),
$$

$$
\mathrm{H}^1(K, n^2A)/\mathrm{H}^1(K, nA_K) \sim \mathrm{H}^1(K, A_K)
$$

and hence a perfect pairing

$$
(82) \quad n\mathrm{H}^1(K, A_K) \times \pi_1(\text{Gr}(A'))/n\pi_1(\text{Gr}(A')) \rightarrow \mathrm{H}^1(K, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.
$$

Now, using the isomorphism

$$
\text{Hom}(\pi_1(\text{Gr}(A'))/n\pi_1(\text{Gr}(A')), \mathbb{Z}/n\mathbb{Z}) \sim \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Z}/n\mathbb{Z}),
$$

the pairing in (82) provides an isomorphism

$$
(83) \quad n\mathrm{H}^1(K, A_K) \sim \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Z}/n\mathbb{Z}),
$$

that is the restriction of (5) to the $n$-parts (cf. [2] § 1).

Now, we will show that (83) coincides with the restriction to the $n$-parts of the homomorphism $\Xi$ in Proposition 4.3.3. The map $u_{X_K}^\tau : \pi_1(\text{Gr}(A')) \rightarrow \pi_0(\mathrm{H}^1(K, \mu_n)) \cong \mathrm{H}^1(K, \mu_n)$ in (61) can also be viewed as the composition

$$
(84) \quad \pi_1(\text{Gr}(A')) \xrightarrow{\delta} nA'(\mathcal{O}_K) \sim nA_K'(K) \xrightarrow{\partial} \mathrm{H}^1(K, \mu_n) \cong \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}
$$

where the first map is deduced from the exact sequence

$$
0 \rightarrow nA_K' \rightarrow A_K' \xrightarrow{n} A_K' \rightarrow 0
$$

on passing to Néron models. More precisely we have

$$
0 \rightarrow nA' \rightarrow A' \xrightarrow{n} nA' \rightarrow 0
$$

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where \( nA' \) is a subgroup scheme of \( A' \) that contains \( A'^0 \). In particular, on applying the perfect Greenberg realization functor, we get a homomorphism

\[
\delta: \pi_1(\text{Gr}(A')) \to nA'(\mathcal{O}_K)
\]

via the canonical isomorphisms \( \pi_1(\text{Gr}(nA')) \to \pi_1(\text{Gr}(A')) \), \( nA'(\mathcal{O}_K) \cong \text{Gr}(nA') \to \pi_0(nA') \).

Let now \( X_K \) be a torsor under \( A_K \) of order \( d \) with \( d \) a power of a prime integer \( l \), \( l \neq p \). Let \( n = l' \) be a multiple of \( d \) large enough to kill the \( l \)-primary parts of the component groups of \( A_K \) and \( A'_K \). Fix an extension corresponding to \( nE_K \) above 1 \( \in \mathbb{Z}/n\mathbb{Z} \) in (83). We show that the composition of the maps in (84) coincides with the map \( u \) in (73). This is sufficient to conclude that our construction via rigidifiers (or equivalently via the relative Picard functor) is Shafarevich’s duality on the prime-to-\( p \) parts.

With notation as in (53), observe that the \( n \)-multiplication on \( A_K \) factors through \( E_K \) so that we have a homomorphism \( \gamma: E_K \to A_K \), with kernel \( nE_K \) such that \( \gamma \circ \alpha = n \). Consider the sequence in (67). We have a diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & \mu_n & \to & nE^D_K & \to & \text{Ext}^1(A_K, \mathbb{G}_m)^n & \to & A'_K & \to & 0 \\
\downarrow \mathrlap{=} & & \downarrow \mathrlap{=} & & \downarrow \mathrlap{=} & & \downarrow \gamma^* & & \downarrow \mathrlap{=} & & 0 \\
0 & \to & \mu_n \cong \text{Hom}(E_K, \mathbb{G}_m) & \to & V_{x_K}^{sc} & \to & \text{Ext}^1(E_K, \mathbb{G}_m)_{x_K} & \to & A'_K & \to & 0.
\end{array}
\]

Indeed \( nE^D_K \cong \text{Hom}(E_K, \mathbb{G}_m) \) maps canonically to \( V_{x_K}^{sc} \cong \text{Mor}(x_K, \mathbb{G}_m) \); hence \( nA'_K \) maps to the torus \( T_K := V_{x_K}^{sc}/\mu_n \) in (68). The push-out of the exact sequence \( 0 \to nA'_K \to A'_K \to A'_K \to 0 \) along \( nA'_K \to T_K \) provides the sequence (68) and the homomorphism \( \gamma^* \) sends a \( \mathbb{G}_m \)-extension of \( A_K \) to its pull-back along \( \gamma \) endowed with its canonical trivialization along \( x_K \), induced by the canonical trivialization along \( nE_K \). Moreover, the boundary map \( \partial: nA'_K(K) \to H^1(K, \mu_n) \) (of finite groups) is the composition of \( \nu: nA'_K(K) \to T_K(K) \) with the boundary map \( h: T_K(K) \to H^1(K, \mu_n) \), i.e.,

\[
\partial = h \circ \nu.
\]

Recall furthermore that the kernel of the \( n \)-multiplication on \( A' \) is a quasi-finite group scheme over \( \mathcal{O}_K \) whose finite part is an étale finite group scheme over \( \mathcal{O}_K \) of order prime to \( p \), hence constant, because \( \mathcal{O}_K \) is strictly henselian. On the level of Serre pro-algebraic groups we then have a diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & nA'(\mathcal{O}_K) & \to & \text{Gr}(A') & \to & \text{Gr}(nA') & \to & 0 \\
\downarrow \nu & & \downarrow \alpha^* & & \downarrow \mathrlap{=} & & \downarrow \mathrlap{=} & & 0 \\
0 & \to & \text{Gr}(T) & \to & \text{Gr}(\text{Ext}^1(E_K, \mathbb{G}_m)_{x_K}) & \to & \text{Gr}(A') & \to & 0
\end{array}
\]

Since the vertical map on the left factors through a map \( \nu^{ft}: nA'(\mathcal{O}_K) \to \text{Gr}(T^{ft}) \), the homomorphism \( u^{ft}: \pi_1(\text{Gr}(A')) \to \pi_0(\text{Gr}(T^{ft})) = \pi_0(\text{Gr}(T))_{tor} \) in (73) factors through the map \( \delta: \pi_1(\text{Gr}(A')) \to nA'(\mathcal{O}_K) \) in (85) and hence

\[
u^{ft} \cong \partial \circ \delta \cong h \circ \nu \circ \delta = \pi_0(h^{ft}) \circ \pi_0(\nu^{ft}) \circ \delta = \pi_0(h^{ft}) \circ u^{ft} = u,
\]

i.e., the homomorphism \( u: \pi_1(\text{Gr}(A')) \to \mathbb{Z}/n\mathbb{Z} \) in (73) coincides with that in (84). Thus we have
Proposition 4.5.1. Let $K$ be a complete discrete valued field with algebraically closed residue field and $A_K$ an abelian variety over $K$. Then the homomorphism $\Xi$ in Proposition 4.3.3 coincides with Shafarevich’s duality $H^1_{fl}(K, A_K) \sim \text{Hom}(\pi_1(\text{Gr}(A)'), \mathbb{Q}/\mathbb{Z})$ (see (8) and [2], Theorem 3), on the prime-to-$p$ parts.

The comparison for the $p$-parts in the equal positive characteristic case is still open.

5 Comparison between (5) and (8)

In this last section, we return to the study of torsors under an elliptic curve $A_K$, and we examine the relation between the fundamental short exact sequence (2) of Serre pro-algebraic groups with Shafarevich’s duality of abelian varieties. Let $n \geq 1$ be an integer and $X_K$ a torsor under the elliptic curve $A_K$ of order $d$ dividing $n$. Let $X$ denote the $S$-proper minimal regular model of $X_K$.

Thanks to Corollary 3.4.4 we are provided with a short exact sequence of Serre pro-algebraic groups

$$0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \text{Pic}^0(X) \xrightarrow{q} J(S) \rightarrow 0$$

by sending $\bar{1} \in \mathbb{Z}/d\mathbb{Z}$ to $O_X(D) \in \text{Pic}^0(X)$. If we push out this short exact sequence by the canonical map

$$\mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}, \; \bar{1} \mapsto \frac{n}{d} \cdot \bar{1},$$

we get an element, denoted by $\Phi_n(X_K)$, of the group $\text{Ext}^1(\text{Gr}(J), \mathbb{Z}/n\mathbb{Z})$ of extensions of the pro-algebraic group $\text{Gr}(J) = J(S)$ by the constant group $\mathbb{Z}/n\mathbb{Z}$. Recall that by Lemma 2.1.1(i), there is a canonical isomorphism of elliptic curves $\iota: A'_K \sim J_K$, thus the group $\text{Gr}(J)$ does not depend on the torsor $X_K$. In this way we get the following canonical map of sets:

$$\Phi_n: nH^1_{fl}(K, A_K) \rightarrow \text{Ext}^1(\text{Gr}(J), \mathbb{Z}/n\mathbb{Z}).$$

Motivated by the isomorphism (7), one might ask if this morphism is always an isomorphism. Our strategy in studying this question is to relate the above construction to Shafarevich’s pairing in (8) by using our new construction in §4.4 as an intermediate bridge.

5.1 Some set-theoretical considerations

We begin with the following lemma, which follows from Lemma 2.0.1.

Lemma 5.1.1. The schematic closure $Y$ in $X$ of any closed point $x_K$ of $X_K$ provides a rigidifier of the Picard functor $\text{Pic}_{X/S}$.

Proof. We need only verify the injectivity of the map $H^0(X_s, \mathcal{O}_{X_s}) \rightarrow H^0(Y_s, \mathcal{O}_{Y_s})$ (Corollary 2.2.2 of [21]). More generally, we will prove by induction on $n$ that the canonical morphism $H^0(X_n, \mathcal{O}_{X_n}) \rightarrow H^0(Y_n, \mathcal{O}_{Y_n})$ is injective, where $Y_n := Y \times_X X_n$. Let $Y$ be defined by the ideal sheaf $\mathcal{J}$. Then $Y_n$ is defined by the ideal sheaf $\mathcal{I}_n + \mathcal{J}$. Let us begin with the case $n = 1$: by Lemma 2.0.1 we know that $H^0(X_1, \mathcal{O}_{X_1}) = k$. Let $\varepsilon \in H^0(X_1, \mathcal{O}_{X_1})$; then $\varepsilon$ is a global function on $X_1$ and so is constant. As a result, the image of $\varepsilon$ in $H^0(Y_1, \mathcal{O}_{Y_1})$ is zero if and only
if \( \varepsilon = 0 \) that is, the morphism \( H^0(X, O_{X_1}) \to H^0(Y, O_{Y_1}) \) is injective. In order to complete the induction, consider the following diagram of sheaves over \( X \)

\[
\begin{array}{cccccc}
0 & \xrightarrow{T^n} & T^n & \xrightarrow{\mathcal{O}_X} & 0 \\
\downarrow & & \downarrow & & \\
0 & \xrightarrow{T^n+\mathcal{J}} & 0 & \xrightarrow{0} & 0
\end{array}
\]

where \( H^0(X, \mathcal{O}_X) = H^0(X_m, O_{X_m}) \) and \( H^0(X, \mathcal{O}_X) = H^0(Y_m, O_{Y_m}) \). Hence we need only establish the injectivity of the morphism

\[
H^0 \left( X, \frac{T^n}{T^{n+1}} \right) \to H^0 \left( X, \frac{T^n + \mathcal{J}}{T^{n+1} + \mathcal{J}} \right).
\]

Observe first that \( \frac{T^n}{T^{n+1}} \cong T^n \otimes O_X \frac{\mathcal{O}_X}{T} \). Hence

\[
H^0 \left( X, \frac{T^n}{T^{n+1}} \right) \cong H^0 \left( X, T^n \otimes O_X \frac{\mathcal{O}_X}{T} \right) = H^0 \left( X_1, T^n |_{X_1} \right).
\]

Furthermore, consider the map \( \frac{T^n}{T^{n+1}} \otimes O_X \frac{\mathcal{O}_X}{T} \otimes \mathcal{J} \to \frac{T^n + \mathcal{J}}{T^{n+1} + \mathcal{J}} \) that, on sections, maps \( \bar{a} \otimes \bar{b} \) to \( \bar{a} \bar{b} \). It is well defined and surjective. Since \( I \) is invertible, \( Y \) is integral and \( Y \nsubseteq X_1 \), so our map is also injective. In particular, we find

\[
\frac{T^n + \mathcal{J}}{T^{n+1} + \mathcal{J}} \cong \frac{T^n + \mathcal{J}}{T^{n+1} + \mathcal{J}} \otimes O_X \frac{\mathcal{O}_X}{T} \otimes \mathcal{J} \cong \frac{T^n + \mathcal{J}}{T + \mathcal{J}} \otimes O_X \frac{\mathcal{O}_X}{T + \mathcal{J}}.
\]

Hence

\[
H^0 \left( X, \frac{T^n + \mathcal{J}}{T^{n+1} + \mathcal{J}} \right) \cong H^0 \left( X, T^n \otimes O_X \frac{\mathcal{O}_X}{T + \mathcal{J}} \right) = H^0 \left( Y_1, T^n |_{Y_1} \right).
\]

We are then reduced to proving that the restriction map

\[
H^0 \left( X_1, T^n |_{X_1} \right) \to H^0 \left( Y_1, T^n |_{Y_1} \right)
\]

is injective. Since \( I \) is an invertible sheaf, according to Lemma \ref{lemma:2.0.1} the first group is trivial or its consists of constant functions; hence the result follows.

**Corollary 5.1.2.** Let \( x_K = \text{Spec}(K') \) be a closed point of \( X_K \) with \( K'/K \) a finite separable extension of degree \( d \) (see Lemma \ref{lemma:4.3.1}), and let \( Y := \{x_K\} \subset X \) be the schematic closure of \( x_K \). Then the subscheme \( Y \hookrightarrow X \) of \( X \) is a rigidificator of the Picard functor \( \text{Pic} \). Moreover, the scheme \( Y \) is regular.

**Proof.** The first statement follows directly from Lemma \ref{lemma:5.1.1} Now, \( Y \) is a local scheme since \( O_K \) is complete. Let \( y \) be the closed point of \( Y \). Since \( Y \) is of degree \( d \) over \( S \) we have \( Y \cdot X_s = d \). Furthermore, \( X_s = dD \) as divisor of \( X \), hence \( Y \cdot D = 1 \) and \( Y \) cuts \( D \) transversally at \( y \). Let \( A \) be the local ring of \( X \) at \( y \). Let \( f \) (respectively \( g \)) be a local equation around the point \( y \) which defines \( Y \) (respectively \( D \)). Then the local ring \( A/(f,g) \) is of length 1, hence it is isomorphic to \( k \). As a result, \( (f,g) \) is a system of parameters of the two dimensional regular local ring \( A \). Therefore, \( Y \) and \( D \) both are regular at the point \( y \). Thus the scheme \( Y \) is regular. \( \square \)
In the following, let $Y$ be the rigidificator given in Corollary 5.1.2. We will use notation as in §1.1 and §3. In particular, $G = (\text{Pic}_{X/S}, Y)^0$ is the identity component of the rigidified Picard scheme $(\text{Pic}_{X/S}, Y)$, and we have the following canonical map

$$r : G = (\text{Pic}_{X/S}, Y)^0 \rightarrow \text{Pic}_{X/S}^0$$

that forgets the rigidification. Let $N$ be the kernel of the morphism $r$. In general, this fppf-sheaf $N$ is not representable, but it has representable fibres. Following §3.4, let $H = N_K \rightarrow (\text{Pic}_{X/S}, Y)^0 = G$ denote the schematic closure of $N_K$ in $G$; it is representable by a flat $S$-group scheme of finite type. Then the fppf quotient $G/H$ gives us the identity component $J$ of the $S$-Néron model of the Jacobian $J_K = \text{Pic}_{X_K/K}^0$ of the curve $X_K/K$, and one has the following exact sequence of $S$-group schemes:

$$0 \rightarrow H \rightarrow G \stackrel{\theta}{\rightarrow} J \rightarrow 0.$$

which induces an exact sequence of abstract groups (§1.1):

(90)  
$$0 \rightarrow H(S) \rightarrow G(S) \rightarrow J(S) \rightarrow 0.$$

On the other hand, by definition, we have another exact sequence of sheaves, which is exact for the étale topology (since (9) in §1.1 is exact for the étale topology):

$$0 \rightarrow N \rightarrow G \stackrel{r}{\rightarrow} \text{Pic}_{X/S}^0 \rightarrow 0.$$

Since $S$ is strictly henselian, the latter sequence induces the following short exact sequence of abstract groups:

(91)  
$$0 \rightarrow N(S) \rightarrow G(S) \rightarrow \text{Pic}^0(X) \rightarrow 0.$$

On combining (90) and (91), we get the following commutative diagram of abstract groups with exact rows:

(92)  
$$\begin{array}{ccccccccc}
0 & \rightarrow & H(S) & \rightarrow & G(S) & \rightarrow & J(S) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H(K) & \rightarrow & G(K) & \rightarrow & J(K) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & N(K) & \rightarrow & G(K) & \rightarrow & \text{Pic}_{X/S}^0(K) & \rightarrow & 0 \\
\end{array}$$

where the lower sequence is exact on the right because $N_K$ is a torus, the upper vertical map on the right is surjective ([5], 9.5/2) and the remaining vertical maps are all injective.

5.2 The pro-algebraic nature of diagram (92)

For $n \in \mathbb{Z}_{\geq 1}$, as in §4.2 we put $S_n = \text{Spec}(\mathcal{O}_{K,n}) = \text{Spec}(\mathcal{O}_K/\pi^n\mathcal{O}_K)$ and let $\mathbb{R}_n$ denote the Greenberg algebra associated with $\mathcal{O}_{K,n}$ (Appendix A of [14]). The aim of this subsection is to show, with the help of Greenberg realization functors, that the diagram (92) is pro-algebraic in nature.
First, the sheaf $H$ is representable by an $S$-group scheme separated of finite type. Hence its Greenberg realization $\text{Gr}_n(H)$ is representable by a $k$-scheme of finite type (§1.2) and we have the following short exact sequence:

$$0 \to \text{Gr}_n(H) \to \text{Gr}_n(G) \to \text{Gr}_n(J) \to 0.$$ 

For the right exactness, we need only prove that the map $\text{Gr}_n(G) \to \text{Gr}_n(J)$ induces a surjective map on the groups of $k$-rational points, i.e., that the morphism of group $G(S_n) \to J(S_n)$ is surjective. This last statement follows from the surjectivity of the maps $\theta(S) : G(S) \to J(S)$ (see §1.3 last paragraph) and $J(S) \to J(S_n)$. On passing to the projective limit of the associated perfect group schemes, one obtains an extension of Serre pro-algebraic groups

$$0 \to \text{Gr}(H) \to \text{Gr}(G) \to \text{Gr}(J) \to 0$$

which says that (90) is pro-algebraic in nature.

Next, we consider the fppf sheaf $N$. Let us first remark that for any $k$-algebra $A$, by considering the $\mathcal{O}_{K,n}$-algebra $\mathbb{R}_n(A)$, we have the following exact sequence of groups:

$$0 \to N(\mathbb{R}_n(A)) \to G(\mathbb{R}_n(A)) \to \text{Pic}^0_{X/S}(\mathbb{R}_n(A)).$$

Let $\text{Gr}_n(N)$ be the fppf sheaf associated with the pre-sheaf $A \mapsto N(\mathbb{R}_n(A))$. By taking the associated fppf sheaves, we get the following exact complex of algebraic $k$-groups (where the representability of $\text{Gr}_n(N)$ follows from the representability of the last two functors by smooth $k$-group schemes):

$$(93) \quad 0 \to \text{Gr}_n(N) \to \text{Gr}_n(G) \to \text{Gr}_n(\text{Pic}^0_{X/S}).$$

By taking the $k$-rational points, we get the usual exact sequence

$$0 \to N(S_n) \to G(S_n) \to \text{Pic}^0_{X/S}(S_n) \to 0$$

which is exact on the right since $\mathcal{O}_{K,n}$ is strictly henselian. So the complex (93) is in fact a short exact sequence of algebraic $k$-groups. Now, by taking the projective limit with respect to $n$ in the sequence of perfect group schemes associated with (93) we get a short exact sequence of Serre pro-algebraic groups:

$$0 \to \mathbb{N}(S) \to \text{Gr}(G) \to \text{Pic}^0(X) \to 0.$$ 

Finally, the group scheme $N_K$ is a torus, $\text{Pic}^0_{X/K/K} \cong A'_K$ is an elliptic curve and $G_K = (\text{Pic}^0_{X/S}, \mathcal{Y})_K$ is a semi-abelian variety; hence they all admit Néron models over $S$, which will be denoted by $\mathcal{N}$, $A'$, and $\mathcal{G}$ respectively; in particular they are smooth group schemes over $S$. Moreover, according to Remark 1.3.2 the $S$-group schemes $\mathcal{N}$, $A'$ are of finite type over $S$, and hence the same holds for $G$.

By the Néron mapping property we have the following two canonical maps

$$f_G : G \to \mathcal{G}, \quad \text{and} \quad f_A : J \to A'.$$

As a consequence, the morphisms

$$G(S) \to G(K) = \mathcal{G}(S), \quad \text{and} \quad J(S) \to \text{Pic}^0_{X/S}(K) \cong A'(S)$$

in diagram (92) come from the morphisms of Serre pro-algebraic groups

$$\text{Gr}(G) \to \text{Gr}(\mathcal{G}), \quad \text{and} \quad \text{Gr}(J) \to \text{Gr}(A').$$

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induced by \( f_G, f_A \). This implies the existence of a morphism of pro-algebraic groups

\[
\alpha: H(S) \longrightarrow N(K) = \text{Gr}(\mathcal{N})
\]

which realizes the lower left vertical inclusion in (92). Summarizing, we have that (92) comes from a commutative diagram (with exact rows) of Serre pro-algebraic groups

\[
\begin{array}{cccccc}
0 & \to & N(S) & \to & \text{Gr}(G) & \to & \text{Pic}^0(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H(S) & \to & \text{Gr}(G) & \to & \text{Gr}(J) & \to & 0 \\
\downarrow \alpha & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N(K) & \to & \text{Gr}(G) & \to & \text{Pic}^0_{X/S}(K) \cong \text{Gr}(A') & \to & 0
\end{array}
\]

5.3 Comparison

We deduce from (94) a commutative diagram of profinite groups:

\[
\begin{array}{cccccc}
\pi_1(\text{Gr}(J)) & \overset{\sim}{\longrightarrow} & \pi_1(\text{Gr}(A')) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\pi_0(H(S)) & \overset{\pi_0(\alpha)}{\longrightarrow} & \pi_0(N(K))
\end{array}
\]

The upper arrow is an isomorphism because by Lemma 2.1.1 we have a canonical isomorphism \( A' \sim J \), hence \( \text{Gr}(A') \sim \text{Gr}(J) \).

In order to give an explicit description of the morphism \( \pi_0(\alpha) \), recall first of all that the group of connected components \( \pi_0(N(K)) \cong \pi_0(\mathcal{N}) \) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \) (cf. Remark 4.3.2 and (79)), with identification given by

\[
\beta: \pi_0(N(K)) \longrightarrow \mathbb{Z}/d\mathbb{Z}, \quad \text{class of } \pi' \text{ in } N_K(K) = k(x_K)^*/K^* \mapsto \bar{1}
\]

where \( \pi' \in k(x_K) \) is a uniformizer. Furthermore, the class of \( \pi' \) in \( N_K(K) \), viewed as element of \( G(K) = (\text{Pic}_{X/S}, Y)^0(K) \) (see §1.1.), is the trivial line bundle on \( X_K \) with the rigidification on \( Y_K \) given by the multiplication by \( \pi' \).

Second, the component group of \( H(S) \) is also \( \mathbb{Z}/d\mathbb{Z} \). Indeed our group scheme \( H \) coincides with the one denoted by \( H_1 \) in [16], pp. 18–21. We then have the following exact sequence

\[
V_{Y^*}^+(S) \longrightarrow H(S) \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0,
\]

(loc. cit., Theorem 3.5) where the first map is the natural factorization of \( V_{Y^*}^+ \to N :\to G := (\text{Pic}_{X/S}, Y)^0 \) through \( H \to G \) (since \( V_Y^* \) is flat over \( S \)), and the second map is defined by

\[
\gamma: H(S) \longrightarrow \mathbb{Z}/d\mathbb{Z}, \quad \left( \mathcal{O}_X \left( \frac{m}{d} X_s \right), a \right) \mapsto \bar{m} \in \mathbb{Z}/d\mathbb{Z}.
\]

(see [16], 3.5). Applying the perfect Greenberg functor to the morphisms \( V_X^* \to V_Y^* \to G \) one sees that the map \( \gamma \) is of pro-algebraic nature and we write:

\[
\pi_0(\gamma): \pi_0(H(S)) \overset{\sim}{\longrightarrow} \mathbb{Z}/d\mathbb{Z}
\]
 Lemma 5.3.1. The following diagram
\[
\begin{array}{cccccc}
\pi_0(H(S)) & \xrightarrow{\pi_0(\alpha)} & \pi_0(N(K)) \\
\downarrow{\pi_0(\gamma)} & & \downarrow{\beta} \\
\mathbb{Z}/d\mathbb{Z} & \xrightarrow{1 \mapsto 1} & \mathbb{Z}/d\mathbb{Z}
\end{array}
\]
commutes. In particular, the morphism $\pi_0(\alpha)$ is an isomorphism.

Proof. Recall that $D$ denotes the vertical divisor $\frac{1}{d}X_s$. Since the map $H(S) \to N(K)$ sends $(\mathcal{O}_X(-D), a)$ to its generic fibre, and the generic fibre of $\mathcal{O}_X(-D)$ is trivial, we are reduced to verifying that a rigidification on $Y_K$ can be given by multiplication by the uniformizer $\pi'$, and that this rigidification extends to a rigidification of $\mathcal{O}_X(-D)$ on $Y$.

Now consider $\mathcal{O}_X(-D)$. This gives us an ideal sheaf of $\mathcal{O}_X$. Recall that $Y$ is regular (Corollary 5.1.2), hence $Y = \text{Spec}(R')$ with $R'$ a complete discrete valuation ring whose field of fractions is $K' := k(x_K)$. Next, we claim that the intersection of $Y$ and $D$, viewed as a divisor of $Y$, is defined by the equation $\pi' = 0$. In fact, let $y \in Y$ be its closed point, and consider the local ring $\mathcal{O}_{X,y}$ which is regular of dimension 2. Let $r \in \mathcal{O}_{X,y}$ (respectively $t \in \mathcal{O}_{X,y}$) be a defining equation of $Y$ (respectively of $D$) around $y \in X$. Since $X_s = dD$ as divisor of $X$, we have $(\pi) = (t^d) \subset \mathcal{O}_{X,y}$. By definition, $Y/S$ is of degree $d$; it follows that the intersection number in $X$

$$Y \cdot X_k = \ell(\mathcal{O}_{X,y}/(r, \pi)) = \ell(\mathcal{O}_{X,y}/(r, t^d)) = \ell(R'/(t')^d)$$

is equal to $d$, with $t'$ the image of $t$ in $R' = \mathcal{O}_{X,y}/(r)$. This implies that $t'$ is an uniformizer of $R'$ (and the maximal ideal of $\mathcal{O}_{X,y}$ is generated by $r$ and $t$). Hence $t' = u\pi'$ with $u' \in R'$ a unit. As a result, the intersection $Y \cap D$ is defined by the equation $t' = 0$, or equivalently, by the equation $\pi' = 0$ in $Y$.

So by the claim, we have $\mathcal{O}_Y(-D \cap Y) = (\pi')$. We get in this way a rigidification of $\mathcal{O}_X(-D)$ along $Y$

$$a: \widetilde{R}' = \mathcal{O}_Y \to \mathcal{O}_X(-D)|_Y = (\pi'), \quad 1 \mapsto \pi',$$

with $\widetilde{R}'$ the coherent module associated with $R'$. Now, if we restrict to the generic point, we get

$$a_K: \widetilde{K}' = \mathcal{O}_{Y_K} \to \mathcal{O}_X(-D)|_{Y_K} = (\pi') = \widetilde{K}', \quad 1 \mapsto \pi'.$$

\[
\square
\]

Now, on forgetting the rigidifications, we have the following exact sequence of Serre pro-algebraic groups

\[
\begin{array}{ccccccc}
0 & \xrightarrow{} & H(S) & \xrightarrow{\gamma'} & \text{Gr}(G) & \xrightarrow{} & \text{Gr}(J) & \xrightarrow{} & 0, \\
& & \downarrow{\gamma'} & & \downarrow{\gamma'} & & \downarrow{\gamma'} & & \\
0 & \xrightarrow{} & <\mathcal{O}_X(D)> & \xrightarrow{q} & \text{Pic}^0(X) & \xrightarrow{q} & \text{Gr}(J) & \xrightarrow{} & 0
\end{array}
\]

where $\gamma'$ is given on $k$-rational sections by $(\mathcal{O}_X(\frac{m}{d}D), a) \to \mathcal{O}_X(\frac{m}{d}D)$, and the vertical map in the middle is given by $(\mathcal{L}, a) \mapsto \mathcal{L}$. Hence, if we identify the kernel $<\mathcal{O}_X(D)>$ of $q$ with $\mathbb{Z}/d\mathbb{Z}$ by sending $\mathcal{O}_X(D)$ to $1 \in \mathbb{Z}/d\mathbb{Z}$, we get the extension of pro-algebraic groups:

\[
(97) \quad 0 \to \mathbb{Z}/d\mathbb{Z} \to \text{Pic}^0(X) \to \text{Gr}(J) \to 0,
\]
and this will give us an element of $\text{Ext}^1(\text{Gr}(J), \mathbb{Z}/d\mathbb{Z})$, which is canonically isomorphic to $\text{Ext}^1(\text{Gr}(A'), \mathbb{Z}/d\mathbb{Z})$. By construction the map $\gamma': H(S) \to \mathcal{O}_X(D) \cong \mathbb{Z}/d\mathbb{Z}$ is the composition of $H(S) \to \pi_0(H(S))$ with $\pi_0(\gamma): \pi_0(H(S)) \to \mathbb{Z}/d\mathbb{Z}$. Hence the commutativity of (85) and (87) implies that the extension (97) is the opposite of the extension obtained from lower exact sequence in (94) by taking first the push-out along $N(\mathbb{K}) \to \pi_0(N(\mathbb{K})) \cong \mathbb{Z}/d\mathbb{Z}$ and then the pull-back along the canonical map $\text{Gr}(J) \to \text{Gr}(A')$. We can summarize these facts as follows:

**Proposition 5.3.2.** The extension (97) is the opposite of the extension obtained from the homomorphism (78) via the canonical isomorphisms

$$
\text{Hom}(\pi_1(\text{Gr}(A'), \mathbb{Z}/d\mathbb{Z})) \overset{\sim}{\to} \text{Ext}^1(\text{Gr}(A'), \mathbb{Z}/d\mathbb{Z}) \overset{\sim}{\leftarrow} \text{Ext}^1(\text{Gr}(J), \mathbb{Z}/d\mathbb{Z}).
$$

The minus sign depends on the choice of the isomorphism $\mathcal{O}_X(D) \cong \mathbb{Z}/d\mathbb{Z}$.

**Corollary 5.3.3.** Let $A_K$ be an elliptic curve and $J$ the identity component of the Néron model $A'$ of the dual elliptic curve $A'_K$ over $S$. Let $n \in \mathbb{Z}_{\geq 1}$. The map $\Phi_n: nH^1(K, A_K) \to \text{Ext}^1(\text{Gr}(J), \mathbb{Z}/n\mathbb{Z})$ in (88) is an injective morphism of groups, which is an isomorphism if one of the following conditions is verified:

- The field $K$ has characteristic 0.
- The integer $n$ is prime to $p$.

If one of the above conditions is satisfied, then the composition of $\Phi_n$ with the isomorphism $\text{Ext}^1(\text{Gr}(J), \mathbb{Z}/n\mathbb{Z}) \overset{\sim}{\to} \text{Hom}(\pi_1(\text{Gr}(A')), \mathbb{Z}/n\mathbb{Z})$ coincides with the restriction of Shafarevich’s duality (83) to the $n$-parts.

**Proof.** In view of the previous Proposition and results in §4 (Theorem 4.4.1 and Proposition 4.3.3) only the injectivity requires verification. Since, there is no non-zero morphism from the connected pro-algebraic group $\text{Gr}(J)$ to a constant finite group, the canonical map (88) induces an injective maps between the group of extensions

$$
\text{Ext}^1(\text{Gr}(J), \mathbb{Z}/n'\mathbb{Z}) \longrightarrow \text{Ext}^1(\text{Gr}(J), \mathbb{Z}/n\mathbb{Z})
$$

when $n'|n$. Hence, we only need to show that, for $X_K$ a torsor under $A_K$ of order $d$, the extension (87) is non-zero in $\text{Ext}^1(\text{Gr}(J), \mathbb{Z}/d\mathbb{Z})$ unless $d = 1$. Since the pro-algebraic group $\text{Pic}^0(X)$ is also connected, extension (87) is split if only if $d = 1$, and this fact implies that the torsor $X_K$ is in fact trivial.

**Remark 5.3.4.** The problem of extending the above Corollary to the $p$-parts in the equal characteristic case reduces to showing that $\Xi$ in Proposition 4.3.3 is always an isomorphism, for example, by checking that it coincides with (83) on the $p$-parts too. Although we have partial results in this direction, for example in the case of abelian varieties with totally degenerate reduction, a full answer is yet not at hand.

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