Inventory Control with Modulated Demand and a Partially Observed Modulation Process

Satya S. Malladi · Alan L. Erera · Chelsea C. White III

Abstract We consider a periodic review inventory control problem having an underlying modulation process that affects demand and that is partially observed by the uncensored demand process and a novel additional observation data (AOD) process. We present an attainability condition, AC, that guarantees the existence of an optimal myopic base stock policy if the reorder cost $K = 0$ and the existence of an optimal $(s,S)$ policy if $K > 0$, where both policies depend on the belief function of the modulation process. Assuming AC holds, we show that (i) when $K = 0$, the value of the optimal base stock level is constant within regions of the belief space and that each region can be described by two linear inequalities and (ii) when $K > 0$, the values of $s$ and $S$ and upper and lower bounds on these values are constant within regions of the belief space and that these regions can be described by a finite set of linear inequalities. A heuristic and bounds for the $K = 0$ case are presented when AC does not hold. Special cases of this inventory control problem include problems considered in the Markov-modulated demand and Bayesian updating literatures.

Keywords partially observed · nonstationary demand · myopic optimality · base stock policy · POMDP

1 Introduction

We consider a periodic review, data driven inventory control problem over finite and infinite planning horizons with instantaneous replenishment. We assume that

S.S. Malladi
Kantar Analytics Practice, India
Tel.: +91 79896 25293
E-mail: sarvanisaty@gmail.com

A.L. Erera
School of Industrial and Systems Engineering, Georgia Institute of Technology, USA

C.C. White III
School of Industrial and Systems Engineering, Georgia Institute of Technology, USA
there are several interconnected processes: the completely observed inventory process that keeps track of the inventory level, the uncensored demand process, the action process that represents replenishment decisions, the underlying modulation process that affects demand, and the additional observation data (AOD) process that together with the demand process partially observes the modulation process. The inventory, demand, and action processes are common to inventory control problems.

When completely observed by the demand and AOD processes, the modulation process models the case where demand is Markov-modulated. When the modulation process is only observed by the demand process and is assumed static, then the model conforms to a model considered by the Bayesian updating literature. Thus, the model presented in this paper generalizes models found in both the Markov modulated literature and the Bayesian updating literature and hence serves as a bridge between these two major research directions.

Additionally, the new model can consider scenarios that have not been considered thus far in the literature. For example, assume the modulation process is partially observed by both the demand and AOD processes and is dynamic, which is consistent with reality when the modulation process models the macro economy and the observations of the AOD process are housing starts, consumer spending, and other indicators that partially observe the state of the macro economy.

Noting that improving demand forecasting is of major on-going interest to industry, the modulation and AOD processes can serve as a modeling basis for improved data-driven demand forecasting using data that includes past demand data. The modulation process can represent an unknown static parameter or index of the demand process and dynamic exogenous and partially observed factors, such as the weather, seasonal effects, and the underlying economy. The AOD process can model observations of the modulation process other than demand; e.g., weather, season, day of the week, and macro-economic indicators, and is motivated by the fact that supply chains are becoming increasingly data driven to support improved real-time supply chain control. For example, consider the demand for housing construction materials. This demand depends directly on the number of new housing starts. Demand for new housing is influenced by the state of the economy, and the state of the economy can be inferred from various macro-economic indicators, such as the rate of growth of the Gross Domestic Product, interest rates, and measures of consumer confidence. Historical economic data can be used to determine the parameter values of the modulation and AOD processes.

We now outline and present the contributions found in the remainder of this paper. In Section 2, we model this inventory control problem as a partially observed Markov decision process (POMDP). We show that for the single period problem, there exists an optimal base stock policy, the value of the optimal base stock level is dependent on the belief function but constant within regions of the belief space, and these regions can be described by a finite set of linear inequalities. We then present a generalization of the Veinott attainability condition, AC.

In Section 3, we assume that reorder cost is zero (\( K = 0 \)) and that AC holds. Our first result extends the single period result in Section 2 to the countable period case and shows that the single period optimal base stock policy is myopic and hence identical for all decision epochs. We also show that use of this optimal policy and AC guarantees that the current base stock level is always at least as great as the current inventory level. We define a partial order on the belief space
and present conditions that guarantee the base stock level is non-decreasing with regard to the partial order.

In Section 4, we assume \( K = 0 \) and that AC does not hold. Preliminary analysis shows that there is an optimal base stock policy for this case; however, this base stock policy is not myopic and is hence stage dependent. This complication identifies an interesting future research challenge and motivates the search for a good, easily computed, and easily implemented heuristic. The heuristic we chose to investigate is the easy to determine and easy to implement optimal policy determined in Section 3. We determine a lower bound on the optimal value function and use the policy determined in Section 3 to generate an upper bound on the optimal value function. We then present an upper bound on the difference between the upper and lower bounds on the optimal value function. We also show that the upper bound on the optimal cost function is piecewise linear in the belief function for the finite horizon case but may not be continuous. Hence, and counter intuitively, improved observation quality of the modulation process may not result in improved systems performance using this heuristic.

In an extensive numerical study, we show that AC holds for at least 63\% of the instances considered. Further, when this condition is violated, we show that the difference between the proposed upper and lower bounds on the optimal value function is less than 0.6\% on average and that the heuristic generates an average percentage optimality gap of 0.51\%. This study supports the claims that (i) AC often holds and (ii) when AC does not hold, the optimal base stock policy determined in Section 3 may be an excellent heuristic.

In Section 5, we consider the \( K > 0 \) case and assume throughout that AC holds. We show that there exists an optimal \((s,S)\) policy and determine upper and lower bounds on \( s \) and \( S \) for the finite and infinite horizon cases, where each bound and the values of \( s \) and \( S \) are dependent on the belief function of the modulation process. Each of these bounds and the values of \( s \) and \( S \) are shown to be constant within regions of the belief space described by a finite number of linear inequalities. An outline of an approach for determining an optimal \((s,S)\) policy and the resultant expected cost function for the finite horizon case are presented in the e-companion.

Conclusions are presented in Section 6.

1.1 Literature Review

This survey is organized around various assumptions made in the literature regarding the modulation process, with emphasis on how the model considered in this paper generalizes much of the Markov-modulated demand and Bayesian updating inventory control literatures. After a brief overview of the general inventory and POMDP literatures, we review the literature concerned with the \( K = 0 \) case, followed by the \( K > 0 \) case.

Inventory control has been studied extensively over six decades; see [28, 21, 42, 54, 50, 10, 11, 23, 30], and [2] for detailed surveys. We also survey several nonparametric approaches. We model the inventory control problem considered in this paper as a POMDP; see [60] and [61] for the POMDP foundational results on which our results are based.
1.1.1 The $K = 0$ case

Assume the modulation process is completely observed and static. This special case of the problem we consider in this paper, for which assumption AC always holds, was first considered by Arrow et al. [2], and Karlin [32], and Karlin [36], various extensions of which are detailed in surveys by Graves et al. [28], Khouja [42], Petruzzi and Dada [54], and Qin et al. [56].

A special case of the problem considered in this paper is obtained by restricting the modulation process to be completely observed and nonstationary with known demand distributions for each period. This case was first considered by Karlin [37, 38], Iglehart and Karlin [33], and Veinott [64, 65]. A base stock policy dependent on the completely observed state of the modulation process (current demand distribution) was proved to be optimal in [37] and [38]. We will later see that this result is an implication of the generalized result presented in this paper. Iglehart and Karlin [33] developed computational approaches for determining the base stock level. Zipkin [72] extended these results to the average cost criterion and to cyclic costs. Veinott [65] proved the existence of an optimal myopic base stock policy when the base stock level at the next decision epoch is guaranteed to exceed the current inventory position after satisfying demand (i.e., the attainability assumption) for independent and correlated nonstationary demands across time periods, respectively. Veinott [65] also provided sufficient conditions for this assumption. We generalize Veinott’s attainability assumption for the problem considered in this paper (AC) and also provide sufficient conditions for this generalized attainability condition. Morton [50] studied an inventory system with the additional option of disposal of inventory at a cost and nonstationary demands in each period. Lovejoy [47] modeled explicit dependence of a generalized demand process on a completely observed modulation process with exogenous parameters and demand history and derived an upper bound on the optimal cost for scenarios such as Markov modulation and additive and multiplicative demand shocks. Song and Zipkin [62] modeled the modulation process as a completely observed underlying “state-of-the-world” in a continuous time framework similar to [33], with a Markov-modulated Poisson demand process. A “state-of-the-world” dependent base stock policy was proved to be optimal. An optimal myopic policy was shown to exist when the attainability of the next period’s base stock level is guaranteed. Sethi and Cheng [59] extended the results of [62] to a discrete time system and obtained analogous results.

Dvoretzky et al. [22], Scarf [57], and Murray and Silver [51] analyzed the case where the modulation process is static, partially observed by the demand process, completely unobserved by the AOD process, and represents unknown parameters of a single stationary distribution. While Scarf [53] proved the optimality of a statistic-dependent base stock policy, Murray and Silver [51] extended the results to determine a Bayesian update on unknown parameters. Azoury and Miller [6] and Azoury [5] extended these results (53) to other distributions and compared this method with non-Bayesian mixture methods. Lovejoy [46] proved the optimality of a myopic base stock policy for “parameter adaptive models” of demand, and Lariviere and Porteus [44] dealt with an unknown stationary distribution of demand partially observed by a scale parameter and a shape parameter. Kamath and Pakkala [34] presented a study of the Bayesian updating mechanism with and without nonstationarity. We note that when the modulation process is static, al-
though partially observed, the attainability assumption always holds. We remark that the problem instances of the problem we consider in this paper, for which AC is sometimes violated, always involve a dynamic modulation process.

Partial observability of demand outcomes, which is different from partial observability of modulation process, results from limitations on the accuracy of inventory book-keeping (in [12] and [52]), and censoring (in [11, 13, 32, 51, 40, 71, 70]). Bensoussan et al. [11, 13] treated Markovian modulation of demand as a special case. Their problem formulation differs from our framework in that their demand process (not the modulation process) is partially observed (censored). Ding et al. [22] presented an analysis of optimal policies for the Bayesian newsvendor problem with and without censoring.

For the case where the modulation process is partially observed by the demand process, completely unobserved by the AOD process, and dynamic, Treharne and Sox [63] proved the existence of an optimal state-dependent base stock policy for an un-capacitated inventory system. Arifoglu and Ozekici [1] proved the optimality of inflated state-dependent base stock policies for capacitated production systems under Markov-modulated demand and supply processes (extending [23]). Bayraktar and Ludkovski [9] studied a completely unobserved Markov-modulated Poisson demand process in a continuous-review inventory system with reorder cost and lost sales (censoring).

Treharne and Sox [63] and Arifoglu and Ozekici [1], however, did not prove the existence of an optimal myopic state-dependent base stock policy, which we prove in this paper, assuming AC holds. Further, we show that the belief space can be partitioned into subsets by a finite set of linear inequalities and that the base stock level is constant within each of these subsets. Such regions have also been observed in the numerical example provided in [9]; however, no explanation is given for such behavior. The linear partition of the belief space we present provides an easily computed approach to determine an optimal base stock level for any given belief vector.

1.1.2 The \( K > 0 \) case

Scarf [58] and Iglehart [32] proved that there exists an optimal \((s, S)\) policy under finite and infinite horizons, respectively. Iglehart [32] presented the first set of bounds on period-wise reorder points and base stock levels, which were later tightened by Veinott and Wagner [67]. Veinott [63] extended Veinott [65, 64] to the \( K > 0 \) case. More recently, Chen et al. [19] presented sufficiency conditions of divergence and \( K \)-convexity for the optimality of \((s, S)\) policies under time-varying parameters and correlated demand variables modulated by an underlying “state-of-the-world” variable. Our results extended to the \( K > 0 \) case lead to significantly reduced computational effort in determining the optimal policy compared to Bayraktar and Ludkovski [9] when AC holds.

More recently, data-driven and nonparametric approaches for describing demand uncertainty have garnered interest. Related literature includes research presented in the following papers: [16, 20, 27, 52, 51, 16, 43, 7, 14, 8, 45, 26, 59, 49, 24]. Future research may involve a blend of nonparametric approaches with Bayesian approaches, such as the data driven approach presented in this paper.
2 Problem Description and Preliminary Results

We describe the inventory control problem in Section 2.1. We then model the problem as a POMDP and present optimality equations and other standard results in Section 2.2. In Section 2.3 we present results associated with the single period expected cost function that will be useful in later sections and also present the condition AC.

2.1 Problem Definition

We consider an inventory control problem that involves the inventory process \( \{s(t), t = 0, 1, \ldots\} \), the modulation process \( \{\mu(t), t = 0, 1, \ldots\} \), the demand process \( \{d(t), t = 1, 2, \ldots\} \), the additional observation data (AOD) process \( \{z(t), t = 1, 2, \ldots\} \), and the action process \( \{a(t), t = 0, 1, \ldots\} \). These processes are linked by the state dynamics equation \( s(t+1) = f(y(t), d(t+1)) \), where \( y(t) = s(t)+a(t) \), and the given conditional probability \( \Pr(d(t+1), z(t+1), \mu(t+1) | \mu(t)) \). We assume the single period cost accrued between decision epoch \( t \) and \( t+1 \) is \( c(y(t), d(t+1)) \), where \( c(y, d) \) is convex in \( y \) and \( \lim_{|y| \to \infty} c(y, d) = \infty \) for all \( d \). We also assume that \( c(y, d) \) is piecewise linear in \( y \) for all \( d \) and that the facets describing \( c(y, d) \) intersect at integers. We will have particular interest in the case where \( f(y, d) = y - d \), which assumes backlogging, and \( c(y, d) = p(d - y)^+ + h(y - d)^+ \), where \( p \) is the shortage penalty per period for each unit of stockout, \( h \) is the holding cost per period for each unit of excess inventory after demand realization, and \( (g)^+ = \max(g, 0) \).

Without loss of significant generality, this definition of single period cost does not include an ordering cost. It is straightforward to transform an inventory problem with a strictly positive ordering cost into an inventory problem with no ordering cost for a wide variety of cost and dynamic models of inventory position, e.g., \( f(y, d) = y - d \) or \( f(y, d) = (y - d)^+ \) and \( c(s, y, d) = c'(y - s) + p(d - y)^+ + h(y - d)^+ \), where in this case the single period cost accrued between decision epochs is dependent on \( s \) and \( c' \) is the cost per unit ordered.

We assume that the modulation, demand and AOD state spaces are all finite, the inventory process has a countable state space, and the action space is the set of non-negative integers. We assume the action at \( t \) can be selected based on \( s(t), d(t), d(t - 1), \ldots, z(t), z(t - 1), \ldots \), and the prior probability mass vector \( \{\Pr(\mu(0) = \mu_i), \forall i\} \). Thus, the inventory process is completely observed, demand is not censored, and the modulation process is partially observed by the demand and AOD processes. The problem is to determine a policy that minimizes the expected total discounted cost over the infinite horizon, where we let \( \beta \in [0, 1) \) be the discount factor. It is assumed throughout that replenishment is instantaneous.

We remark that the inventory, demand, and action processes are all part of inventory control problems considered in the literature. As indicated in the literature review, the modulation process is also part of the structure of inventory control problems with Markov-modulated demand. The AOD process is intended to provide information about the modulation process, where appropriate, in addition to that provided by the demand process, such as macro-economic data. Throughout we assume demand realization is uncensored and completely revealed. This assumption is in contrast to the censored demand case where only sales data are available to the decision maker.
We note that the conditional probability $\Pr(d(t+1), z(t+1), \mu(t+1) \mid \mu(t))$ is the product of two conditional probabilities:

1. $\Pr(d(t+1), z(t+1) \mid \mu(t+1), \mu(t))$, the demand and AOD probabilities, conditioned on the modulation process
2. $\Pr(\mu(t+1) \mid \mu(t))$, the state transition probabilities for the (Markov-modulated) modulation process.

The Baum-Welch algorithm is typically used to estimate parameters of a POMDP, viz., observation and transition probabilities and initial belief state (see [4] for a review on POMDP training methods).

### 2.2 The POMDP Model and Preliminary Results

This problem can be recast as a partially observed Markov decision problem as follows. Results in [54] and [55] imply that $(s(t), x(t))$ is a sufficient statistic, where $N$ is the number of values the modulation process can take, the belief function $x(t) = \text{row}\{x_1(t), \ldots, x_N(t)\}$, is such that $x_i(t) = \Pr(\mu(t) = \mu_i \mid d(t), \ldots, d(1), z(t), \ldots, z(1), x(0))$, and $x(t) \in X = \{x \in \mathbb{R}^N : x \geq 0 \text{ and } \sum_{i=1}^N x_i = 1\}$. For $g \in \mathbb{R}^N$, let $g^\mathcal{A} = \sum_{n=1}^N g_n$. Thus, the inventory process is completely observed, the modulation process is partially observed through the demand and AOD processes, and the state of the modulation process is characterized by the belief function. Let

$P_{ij}(d, z) = \Pr(d(t+1) = d, z(t+1) = z, \mu(t+1) = j \mid \mu(t) = i)$,

$P(d, z) = \{P_{ij}(d, z)\}$,

$\sigma(d, z, x) = xP(d, z)\mathcal{1} = \sum_{i=1}^N x_i \sum_j P_{ij}(d, z)$,

$\lambda(d, z, x) = \text{row}\{\lambda_1(d, z, x), \ldots, \lambda_N(d, z, x)\} = xP(d, z)/\sigma(d, z, x), \sigma(d, z, x) \neq 0$,

$L(x, y) = E[c(y, d)] = \sum_{d,z} \sigma(d, z, x)c(y, d)$.

Define the operator $H$ as

$$[Hv](x, s) = \min_{y \geq s} \left\{L(x, y) + \beta \sum_{d,z} \sigma(d, z, x)v(\lambda(d, z, x), f(y, d))\right\}. \quad (1)$$

Results in [53] guarantee that there exists a unique cost function $v^*$ such that $v^* = Hv^*$ and that this fixed point is the expected total discounted cost accrued by an optimal policy. We can restrict search for an optimal policy to $t$-invariant functions that select $a(t)$ on the basis of $(s(t), x(t))$, the function $\psi$ such that $\psi(s(t), x(t)) = a(t)$ causing the minimum in (1) to be attained is an optimal policy, and $\lim_{n \to \infty} \|v^* - v_n\| = 0$, where the (finite horizon) cost function $v_{n+1} = Hv_n$ for any given bounded function $v_0$ and $\|\|_\alpha$ is the sup-norm. The function $L(x, y)$ is the expected single period cost, conditioned on belief $x$ and inventory level $y$.

From the perspective of Bayes’ Rule, note that $x = x(t)$ can be thought of as the prior probability mass function of $\mu(t)$, $\sigma(d, z, x)$ is the probability that the demand and AOD processes will have realizations $d = d(t+1)$ and $z = z(t+1)$,
respectively, given $x$, and $x(t+1) = \lambda(d,z,x)$ is the posterior probability mass function of $\mu(t)$, given $d$, $z$, and $x$.

With respect to optimal value function structure, results in [60] guarantee that $v_n(x,s)$ is piecewise linear and concave in $x$ for each fixed $s$ for all finite $n$, assuming $v_0(x,s)$ is also piecewise linear and concave in $x$ for each $s$. In the limit $v^*(x,s)$ may no longer be piecewise linear in $x$ for each $s$; however, concavity will be preserved.

Regarding the value of information, results in [61] and [18] can be used to determine upper and lower bounds on $v_n(x,s)$ for all $n$ and $v^*(x,s)$ as a function of observation quality and that as observation quality improves, optimal expected systems performance will never degrade. This result may not hold for some suboptimal policies, as discussed in Example 3 of Section 4.2.

2.3 $L(x,y)$ Analysis for Backordering Systems

We now examine $L(x,y)$ in more detail for backordering systems where $f(y,d) = y - d$ and $c(y,d) = p(d - y)^+ + h(y - d)^+$. Let $\{d_1, \ldots, d_M\}$ be the set of all possible demand values, where $d_m < d_{m+1}$, for all $m = 1, \ldots, M - 1$. Letting $\sigma(d,x) = \sum_s \sigma(d,z,x)$, define for all $m = 0, \ldots, M$,

$$A_m(x) = h \sum_{k=1}^m \sigma(d_k,x) - p \sum_{k=m+1}^M \sigma(d_k,x),$$

$$B_m(x) = p \sum_{k=m+1}^M d_k \sigma(d_k,x) - h \sum_{k=1}^m d_k \sigma(d_k,x).$$

Note, $A_0(x) = -p$ and $B_0(x) = p \sum_{k=1}^M d_k \sigma(d_k,x)$. Proof of the next result, which provides structure that will prove useful, is straightforward.

**Lemma 1** For all $x \in X$:

(i) $L(x,y) = \begin{cases} 
A_0(x)y + B_0(x) = p \sum_{k=1}^M \sigma(d_k,x)(d_k - y), & y \leq d_1 \\
A_m(x)y + B_m(x), & d_m \leq y \leq d_{m+1}, \\
A_M(x)y + B_M(x) = h \sum_{k=1}^M \sigma(d_k,x)(y - d_k), & d_M \leq y
\end{cases}$,

for all $m = 1, \ldots, M - 1$, $A_{m+1}(x) = A_m(x) + (h+p)\sigma(d_{m+1},x)$, and hence, $A_{m+1}(x) \geq A_m(x)$.

(ii) for all $m = 1, \ldots, M - 1$, $A_{m+1}(x) = A_m(x) + (h+p)\sigma(d_{m+1},x)$, and hence, $A_{m+1}(x) \geq A_m(x)$.

(iii) for all $m = 1, \ldots, M - 1$, $B_{m+1}(x) = B_m(x) - (p+h)\sigma(d_{m+1},x)$, and hence, $B_{m+1}(x) \leq B_m(x)$.

(iv) for all $m = 1, \ldots, M$, $A_m(x) \geq A_{m-1}(x) + d_m + B_{m-1}(x) = A_m(x)d_m + B_m(x)$. \(\Box\)

(iii) for all $m = 1, \ldots, M$, $A_m(x) \geq A_{m-1}(x) + d_m + B_{m-1}(x) = A_m(x)d_m + B_m(x)$. \(\Box\)

2.3.1 Base Stock Policy: Linear Partition of Belief Space.

**Lemma 1** Testifies that $L(x,y)$ is piecewise linear and convex in $y$ for all $x \in X$.

Let $s^*(x)$ be the smallest integer that minimizes $L(x,y)$ with respect to $y$. Note that it is sufficient to restrict $s^*(x)$ to the set $\{d_1, \ldots, d_M\}$, i.e., $s^*(x) : x \in \{d_1, \ldots, d_M\}$
Inventory control with Markov-modulation process

Let \( P(d) = \sum_z P(d, z), \forall d. \) For \( m = 1, \ldots, M, \)

\[
X_m = \left\{ x \in X : x \leq \frac{1}{m} \sum_{k=1}^{m-1} P(d_k) \leq \frac{1}{m} \sum_{k=1}^{m} P(d_k) \right\}. \tag{2}
\]

Note that the criterion in (2) can be re-written as:

\[
\sum_{k=1}^{m-1} \sigma(d_k, x) < p/(p + h) \leq \sum_{k=1}^{m} \sigma(d_k, x),
\]

where \( \sigma(d_k, x) \) is the probability of observing demand outcome \( d_k \) when the current belief is \( x. \) This criterion is identical to the newsvendor problem’s criterion for determining the optimal base stock policy with the probability mass function of demand given by \( \sigma(d_k, x), \forall k. \)

Due to the linearity of \( \sigma(d, x) \) in \( x, \) the above criterion results in a linear partition of the belief space. We note that the partition thus obtained is independent of the values of demand and AOD outcomes but depends only on the parameters, \( P_{ij}(d, z), p, \) and \( h. \) We remark that \( X_m \) for all \( m \) can be described by two inequalities linear in \( x, \) which is true irrespective of the values \( N \) and \( M \) take, since \( L(x, y) \) is piecewise linear in \( x \) for fixed \( y. \)

**Example 1** Let \( M = 7, N = 3, h = 1, p = 3, d = [5, 10, 15, 20, 25, 30, 35] \) and

\[
P = \begin{bmatrix}
0.0192 & 0.8744 & 0.1063 \\
0.0437 & 0.4712 & 0.4851 \\
0.4467 & 0.0313 & 0.522
\end{bmatrix},
\]

\[
Q^D = \begin{bmatrix}
0.207 & 0.2321 & 0.0717 & 0.2054 & 0.1519 & 0.0346 & 0.2837 \\
0.2697 & 0.208 & 0.2044 & 0.1942 & 0.0748 & 0.0427 & 0.0062 \\
0.283 & 0.0378 & 0.0429 & 0.0605 & 0.1335 & 0.3001 & 0.3969
\end{bmatrix}
\]

where \( P = \{P_i\} \) and \( P_{ij} = \Pr(\mu(t + 1) = j | \mu(t) = i), \)

\( Q^D = \{q^D_{ij}\}, q^D_{jd} = \Pr(d(t + 1) = d | \mu(t + 1) = j), \) where \( q^D_{jd} \) is independent of \( i. \)

Note that \( P_{ij}(d) = P_{ijd}q^D_{jd} \) is independent of \( z. \) The belief space is given by the triangle with vertices \((1,0,0), (1,1,0), \) and \((0,0,1)\) (described by \( x_1 + x_2 + x_3 = 1, \)

\( x_1 \geq 0, x_2 \geq 0, \) and \( x_3 \geq 0, \)) where modulation state \( n \) indicates a stronger economy than modulation state \( n, \) for all \( n. \) Figure 1 depicts the belief space, \( X, \) overlaid with the partition, \( P_1 \) (derived in Lemma 2). \( P_1 \) divides \( X \) into 4 regions of constant base stock level, viz., \( X_1 \) through \( X_4. \) For any belief vector in \( X_m, \) the optimal order-up-to level for the one period problem is \( d_m. \) Hence, the optimal myopic base stock levels are \( 20, 25, 30, \) and \( 35 \) in \( X_4, X_5, X_6, \) and \( X_7 \) respectively.

If the AOD process is dependent on \( \mu(t + 1) \) (e.g. current state of the economy), and has two outcomes \( \{z_1, z_2\} \) (e.g. real estate price levels) with \( R^Z = \{r^Z_{1z}, r^Z_{2z} = \)
Pr(z(t+1) = z | μ(t+1) = j), then \( P_{ij}(d,z) = P_{ij} q^{d}_{j} r_{z}^{2} \). The regions presented in Figure 1 do not change as (2) does not depend on the values of the outcomes of either the AOD or demand processes. Note \( \lambda(d_{1}, e_{3}) = [0.28, 0.26, 0.46] \) when the AOD process is uninformative. Let \( R^{z} = [1, 0; 0.7, 0.3; 0, 1] \) and hence the AOD process is informative. Then, \( \lambda(d_{1}, z_{1}, e_{3}) = [0.61, 0.39, 0] \) and \( \lambda(d_{1}, z_{2}, e_{3}) = [0, 0.15, 0.85] \). We note that the availability of additional observation data leads to substantially different updated belief functions.

We now present an assumption that would help establish the optimality of the myopic base stock policy in Section 3.

2.4 Definition of the Attainability Condition (AC)

We now present the attainability condition, AC. Let \( s^{*}(x) \) define an order-up-to level \( y \) given a belief state \( x \).

**Attainability Condition (AC)** \( f(s^{*}(x), d) \leq s^{*}(\lambda(d,z,x)) \) for all \( x \in X \), \( d \in \{d_{1}, \ldots, d_{M}\} \), and \( z \).

AC assumes that the amount of inventory after demand is satisfied, which is \( f(s^{*}(x), d) \), never exceeds the order-up-to level at the next decision epoch, which is \( s^{*}(\lambda(d,z,x)) \). This assumption is identical to assumptions made in the inventory literature specialized to the problems considered in [64, 65].

3 When AC holds

3.1 Main Result

We assume throughout this section that AC holds. We now present the main result of this section. Proof of the following result is provided in the e-companion.
**Proposition 1** Assume AC holds, \( L(x, y) \) is piecewise linear in \( y \) for all \( x \in X \), \( s^*(x) \) is the smallest integer that minimizes \( L(x, y) \) with respect to \( y \), and that \( f(y, d) \) is non-decreasing in \( y \) for each \( d \). Then, \( v_n(x, s) = v_n(x, \max\{s^*(x), s\}) \) is non-decreasing and convex in \( s \) for all \( n \) and \( x \). Further, the myopic base stock policy that orders up to \( \max\{s^*(x), s\} \) is an optimal policy.

Thus, when AC is satisfied and recalling that \( s^*(x) \) is determined using the inequalities presented in [2], ordering up to \( \max\{s^*(x), s\} \) at every decision epoch is optimal over finite and infinite horizons.

**3.2 AC Analysis**

Proof of the following preliminary result is straightforward.

**Lemma 3** Assume AC holds, apply the base stock policy “order up to \( \max\{s^*(x), s\} \)”, and assume \( s(t) \leq s^*(x(t)) \). Then, \( s(\tau) \leq s^*(x(\tau)) \) for all \( \tau \geq t \).

Thus, once the inventory level falls at or below the base stock level, AC guarantees that the inventory level will always fall at or below the base stock level at the next decision epoch.

In Example 2, we show that Example 1 satisfies AC, following a preliminary result. Define the binary operator for first order stochastic dominance, \( \preceq \), as follows: for \( x, x' \in X \), \( x \preceq x' \iff \sum_{i=n}^{N} x_i \leq \sum_{i=n}^{N} x'_i \ \forall \ n = 1, \ldots, N \).

**Lemma 4** Assume if \( i \leq i' \), then \( \sum_{k=m}^{M} \sum_{j} P_{ij}(dk_i) \leq \sum_{k=m}^{M} \sum_{j} P_{ij'}(dk_i) \) for all \( m = 1, \ldots, M \). Then, \( x \preceq x' \), implies \( s^*(x) \leq s^*(x') \).

Lemma 4 guarantees that \( s^*(x_N) \geq s^*(x) \) for all \( x \in X \), where \( e_N \in X \) has a 1 as its \( n \)th entry. Let \( \tilde{x}_{d,z} \in X \) be such that \( \tilde{x}_{d,z} \preceq \lambda(d, z, x) \forall x \in X \). A simple linear programming procedure for determining \( \tilde{x}_{d,z} \) is presented in Section A2 of the Appendix. We can now show that the problem presented in Example 1 satisfies AC.

**Example 2** Consider the problem in Example 1. It is straightforward to show that the assumption in Lemma 4 is satisfied. If \( f(s^*(x_N), d) \leq s^*(\tilde{x}_{d,z}) \) for all \( (d, z) \), then AC is satisfied since

\[
f(s^*(x), d) \leq f(s^*(x_N), d) \leq s^*(\tilde{x}_{d,z}) \leq s^*(\lambda(d, z, x)).
\]

Use of the procedure in Section A2 of the Appendix verifies that \( f(s^*(x_N), d) \leq s^*(\tilde{x}_{d,z}) \forall (d, z) \), and hence, AC is satisfied. Figure 2 shows \( \{\lambda(d, z, x) : x \in X\} \) and \( \tilde{x}_{d,z} \), where both \( \tilde{x}_{d,z} \) and \( \lambda(d, z, x) \) are independent of \( z \) (e.g., where \( \Pr(z(t+1) \mid \mu(t+1), \mu(t)) \) is independent of \( \mu(t+1) \) and \( \mu(t) \)). Thus, since Example 1 satisfies AC, ordering up to the myopic base stock level at every decision epoch is optimal over finite and infinite horizons.
We remark that for the case where the modulation process is completely observed and \( f(y,d) \) is non-increasing in \( d \) for all \( y \), AC is equivalent to \( f(s^*(\mu(t)),d(t+1)) \leq s^*(\mu(t+1)) \) for all \( d(t+1) \), and hence \( f(s^*(\mu(t)),d_t) \leq s^*(\mu(t+1)) \). Note that this is equivalent to the attainability assumption presented by Veinott [64, 65] that guarantees the optimality of a myopic base stock policy for the completely observed nonstationary case.

A computational procedure to determine the value function of the POMDP is presented in Section A2 of the Appendix.

4 When AC Does Not Hold

We now consider the case where AC does not hold. Preliminary analysis shows that there is an optimal base stock policy for this case; however, this base stock policy is not myopic and is hence stage dependent. This complication identifies an interesting future research challenge and motivates the search for a good, easily computed and easily implemented heuristic.

We remark that although \( f(s^*(x),d) \leq s^*(\lambda(d,z,x)) \) may not hold for all \( (d,z,x) \), this inequality holds for all \( (d,z) \) for at least a subset of \( X \), which supports the conjecture that an optimal policy when AC holds may be a good sub-optimal policy when AC does not hold. For example, let \( X_m \subseteq X \) be such that for \( x' \in X_m \), \( s^*(x') \leq s^*(x) \) \( \forall x \in X \). Then, \( f(s^*(x),d) \leq s^*(\lambda(d,z,x)) \) for all \( (d,z) \) for all \( x \in X_m \).

In this section, we address the question: how good is the policy “order up to \( \max\{s^*(x),s\} \)”, which is optimal when AC holds, as a heuristic when AC does not hold? We begin addressing this question by determining a lower bound \( v^L \) on the optimal value function. We use the “order up to \( \max\{s^*(x),s\} \)” policy to generate an upper bound \( v^U \) on the optimal value function. We then present an upper bound on the difference \( v^U - v^L \). Numerical results support the claim that this heuristic is near-optimal for the broad class of problems considered in the numerical analysis. We also show that the upper bound on the optimal cost function is piecewise linear in the belief function for the finite horizon case but may not be continuous. Hence, and counter intuitively, improved observation quality
of the modulation process may not result in improved systems performance using this heuristic.

4.1 A Lower Bound, $v^L$

We now present a lower bound on $v_n(x, s)$. Let

$$[H^L v](x, s) = L(x, s^*(x)) + \beta \sum_{d, z} \sigma(d, z, x)v(\lambda(d, z, x), s^*(\lambda(d, z, x)))$$

$v^L_{n+1} = H^L v^L_n$, $v^L_0 = 0$, and let $v^L$ be the fixed point of $H^L$, which we note is independent of $s$.

**Proposition 2** For all $x, s$, and $n$, $v^L_n(x, s^*(x)) \leq v_n(x, s)$.

The proof follows from the fact that the controller always brings the inventory to $s^*(x)$, which is not feasible when the inventory is higher than $s^*(x)$. We remark that $v^L_n(x)$ can be computed as was $v_n(x, s)$ for $s \leq s^*(x)$, in Section 3. A tighter lower bound, dependent on $s$ for $s > s^*(x)$, would replace $L(x, s^*(x))$ with $L(x, \max\{s^*(x), s\})$ in the definition of the operator $H^L$. However, since such a definition of $H^L$ would complicate later analysis, we have chosen not to use this tighter lower bound in the development of results in the sections to follow.

4.2 An Upper Bound, $v^U$

Let

$$[H^U v](x, s) = L(x, \max\{s^*(x), s\}) + \beta \sum_{d, z} \sigma(d, z, x)v(\lambda(d, z, x), f(\max\{s^*(x), s\}, d)),$$

$v^U_{n+1} = H^U v^U_n$, $v^U_0 = 0$, and let $v^U$ be the fixed point of $H^U$. We remark that $v^U$ is the expected cost to be accrued by the “order-up-to max\{s^*(x), s\}” policy, which is feasible but may not be optimal when AC is not satisfied, and hence represents an upper bound on the optimal cost function. It is straightforward to prove the following structural result.

**Proposition 3** For all $n$ and $x$, $v^U_n(x, s) = v^U_n(x, s^*(x))$ for $s \leq s^*(x)$, and $v^U_n(x, s)$ is non-decreasing and convex in $s$.

We prove the following result in the e-companion of this paper.

**Lemma 5** For each $n \geq 1$, there is a partition $\mathcal{P}_n$ of $X$ that is defined by a finite set of linear inequalities such that on each element of this partition $v^U_n$ is linear in $x$. Further, $\mathcal{P}_{n+1}$ is at least as fine as $\mathcal{P}_n$ (i.e., if $S \in \mathcal{P}_{n+1}$, then there is an $S' \in \mathcal{P}_n$ such that $S \subseteq S'$).

Thus, $v^U_n(x, s)$ is piecewise linear in $x$ for each $s$. Note that $\mathcal{P}_1$ is defined in Section 2.3.1 However, Example 3 shows that $v^U_n(x, s)$ may be discontinuous and hence not concave in $x$ for each $s$. Thus, according to White and Harrington [48], it may not be true that the improved observation accuracy will improve the performance of the “order up to max\{s^*(x), s\}” policy if AC is not satisfied.
Example 3 Assume $f(y, d) = y - d$, $c(y, d) = p(d - y)^{+} + h(y - d)^{+}$, and $\beta = 0.9$.

Let $N = 2$, $M = 10$, $h = 1$, $p = 2$, 

$$
P = \begin{bmatrix} 0.4670 & 0.5330 \\ 0.4103 & 0.5897 \end{bmatrix},
$$

$$
Q = \begin{bmatrix} 0.1747 & 0.1716 & 0.1417 & 0.1153 & 0.1095 & 0.0993 & 0.0712 & 0.0658 & 0.0368 & 0.0142 \\ 0.0115 & 0.0278 & 0.0537 & 0.0611 & 0.1012 & 0.1176 & 0.1215 & 0.1612 & 0.1667 & 0.1777 \end{bmatrix},
$$

and $d = [0 \ 1 \ 2 \ 3 \ 4 \ 8 \ 12 \ 17 \ 18 \ 19]$. Then, $\min_{x} s^{*}(x) = 12$ and $\max_{x} s^{*}(x) = 17$.

In Example 3, AC does not hold. Figure 3 presents $v_{U}^{2}(x, s)$ and $v_{L}^{2}(x, s)$ for this example. We note the discontinuity in the expected cost function for two periods, $v_{U}^{2}$, obtained by implementing the myopic base stock policy when AC does not hold.

![Fig. 3 $v_{U}^{2}(x, s)$ and $v_{L}^{2}(x, s)$ at $s = 13$.](image)

We remark that although $v_{U}^{n}(x, s)$ is piecewise linear in $x$ for all $s$ and $n$, in the limit as $n$ approaches $\infty$, we may lose piecewise linearity. Thus, although implementing the policy “order up to $\max\{s^{*}(x), s\}$” is straightforward, determining $v^{U}$, or for that matter $v_{U}^{n}$ for large $n$, is computationally demanding. For this reason, we seek an easily computable upper bound on $v_{U} - v_{L}$ in the next section.

4.3 An Upper Bound on $v_{U} - v_{L}$

Let $\Delta = \max_{(x, s)} \left\{ L(x, \max\{s^{*}(x), s\}) - L(x, s^{*}(x)) \right\}$, and consider the following preliminary result.

**Proposition 4** Assume the policy “order up to $\max\{s^{*}(x), s\}$” is applied and $\min\{s^{*}(x) : x \in X\} - d_{M} \leq s(t) \leq \max\{s^{*}(x) : x \in X\} - d_{1}$. Then, for all $\tau \geq t$, 

$$
\min\{s^{*}(x) : x \in X\} - d_{M} \leq s(\tau) \leq \max\{s^{*}(x) : x \in X\} - d_{1}.
$$

The next result follows directly from the definition of $\Delta$ and the result above.
Inventory control with Markov-modulation process

Proposition 5 Assume the policy ‘order up to \( \max\{s^*(x), s\} \)’ is applied and \( \min\{s^*(x) : x \in X\} - d_M \leq s(0) \leq \max\{s^*(x) : x \in X\} - d_1 \). Then,

\[
\Delta = \max_x \{ L(x, \max\{s^*(x) : x \in X\} - d_1) - L(x, s^*(x)) \}.
\]

Thus, assuming \( d_M \) is finite, \( \Delta \) is finite.

We now present a study of the performance of the myopic policy when \( d > d_{lM} \) is a linear program (LP). When \( d \) does not satisfy the attainability condition for all \( x \) in that element, clearly, \( \Delta = 0 \) when \( \max\{s^*(x) : x \in X\} - d_1 \leq \min\{s^*(x) : x \in X\} \). The quantity \( \Delta \) may be computed as follows by solving the LP defined in (3) for each \( m \in M \):

\[
\Delta = \max_{m \in M} \max_x \{ L(x, \max\{d_m, d_{lM} - d_1\}) - L(x, d_m) : x \in X_m \},
\]

where \( M = \{ m : d_\tilde{m} \leq d_m \leq d_{lM} - d_1 \} \) and \( d_\tilde{m} = \min\{s^*(x) : x \in X \} \).

4.4 Computational Analysis of Myopic Policy for Backordering Systems

We now present a study of the performance of the myopic policy when \( \Delta > 0 \) and hence AC is violated. We consider a set of 216 test instances of backordering systems with \( T = 100 \) decision epochs that were randomly generated by the procedure described in Appendix Section A3. Of the 216 instances, \( \Delta > 0 \) for 81 instances, approximately 37% of the instances. For each instance, on 10,000 sampled trajectories, we implement the myopic policy to compute a sample average cost \( \bar{c}_{lF}(x, s) \) and the lower bound policy of resetting the inventory position to the optimal base stock level \( s^*(x) \) of the single period problem (even when doing so requires an infeasible negative reorder quantity) to compute a sample average lower bound cost \( \bar{c}_{lF}(x) \). The algorithms for computing \( \bar{c}_{lF}(x, s) \) and \( \bar{c}_{lF}(x) \) (Algorithms 1 and 2) are presented in Appendix Section A5.
Δ_T over a finite horizon with T decision epochs using the approach outlined in Section 4.3.

We analyze two quantities: the average observed gap \( \tilde{\delta}_T(x, s) = \tilde{v}^{f^*_T}(x, s) - \tilde{v}^{L}(x) \) as a fraction of the maximum possible expected difference \( \Delta_T \) and as a fraction of the sample average cost of the lower bound policy, where the latter quantity represents a sample average percentage gap approximation.

Table 1 presents the results for the 81 instances in this test set where \( \Delta > 0 \). The myopic policy generates an average percentage optimality gap of 0.51%. Not unexpectedly, we note that the average observed gap is significantly smaller than the a priori upper bound \( \Delta_T \) across this instance set. These results appear to be largely independent of the backorder penalty cost \( p \). The average percentage gaps do tend to grow slightly with the number of modulation states \( N \) and the number of possible demand outcomes \( M \). In summary, however, the quality of the myopic policy, which is optimal when AC holds, performs very well on the test instances considered in this numerical study, even when \( \Delta > 0 \).

| \( N \) | \( \tilde{\delta}_T(x, s)/\Delta_T \) | \( \tilde{\delta}_T(x, s)/\tilde{v}^L(x) \) | \( M \) | \( \tilde{\delta}_T(x, s)/\Delta_T \) | \( \tilde{\delta}_T(x, s)/\tilde{v}^L(x) \) |
|---|---|---|---|---|---|
| 2 | 3.55% | 0.50% | 3 | 1.65% | 0.41% |
| 3 | 2.15% | 0.51% | 4 | 3.06% | 0.45% |
| Overall | 2.65% | 0.51% | 5 | 3.02% | 0.60% |

Table 1 Performance of the myopic policy on subset of test instances for which AC is violated.

5 Reorder Cost Case

We now consider the case where there is a reorder cost \( K \geq 0 \). The following results combine the ideas presented for the \( K = 0 \) case with straightforward extensions of earlier results in the literature. Let the operator \( H^K \) be defined as:

\[
[H^K](x, s) = \min_{y \geq s} \left\{ K \xi(y - s) + [Gv](x, y) \right\},
\]

where \( \xi(k) = 0 \) if \( k = 0 \) and \( \xi(k) = 1 \) if \( k \neq 0 \) and

\[
[Gv](x, y) = L(x, y) + \beta \sum_{d, z} \sigma(d, z, x)v(\lambda(d, z, x), f(y, d)).
\]

We note that when \( K = 0 \), \( H^K = H \), as defined in Section 2.2.

We now assume that \( K > 0 \). Our objective is to present conditions under which \((s, S)\) policies exist and how such policies can be computed.
5.1 K-convexity and Optimal (s, S) Policies

We now present our first result following a key definition: the real-valued function $g$ is K-convex if for any $s \leq s'$,

$$g(\lambda s + (1 - \lambda)s') \leq \lambda g(s) + (1 - \lambda)(g(s') + K), \text{ for all } \lambda \in [0, 1].$$

Proof of the following result is a direct extension of results in [58] and elsewhere.

**Proposition 7** Assume: (i) $v(x, s)$ is K-convex in $s$ for all $x$, (ii) $f(y, d)$ is non-decreasing and convex in $y$ for all $d$, and (iii) $c(y, d)$ is convex in $y$ and $\lim_{|y| \to \infty} c(y, d) \to \infty$ for all $d$. Then,

1. $[Gv](x, y)$ is K-convex in $y$ for all $x$,
2. $[H^K v](x, s)$ is K-convex in $s$ for all $x$, and
3. $[H^K v](x, s) = \begin{cases} K + [Gv](x, S^*(x, v)) & s \leq s^*(x, v) \\ [Gv](x, s) & \text{otherwise,} \end{cases}$

where: $S^*(x, v)$ is the smallest integer minimizing $[Gv](x, y)$ with respect to $y$, and $s^*(x, v)$ is the smallest integer such that $[Gv](x, s^*(x, v)) \leq K + [Gv](x, S^*(x, v))$.

Thus, the fact that $v(x, s)$ is K-convex and non-decreasing in $s$ for all $x$ leads to the existence of an optimal policy that is of $(s, S)$ form: if the inventory drops below $s$, then order up to $S$; otherwise, do not replenish.

5.2 Bounds on $s_n$ and $S_n$

Let $v_0 = 0, v_{n+1} = H^K v_n$ for all $n \geq 0$, and $G_n(x, y) = [Gv_n](x, y)$. Let $S_n(x)$ be the smallest integer such that $G_n(x, S_n(x)) \leq G_n(x, y)$ for all $y$, and let $s_n(x)$ be the smallest integer such that $G_n(x, s_n(x)) \leq K + G_n(x, S_n(x))$. Following [67], we now define four real-valued functions that represent bounds on the set $\{(s_n(x), S_n(x)) : n \geq 0\}$. Let the values $\underline{s}(x), \overline{s}(x), \underline{S}(x)$, and $\overline{S}(x)$ be the smallest integers such that:

$$L(x, \underline{s}(x)) \leq L(x, y) \forall y \quad (5)$$

$$L(x, \overline{s}(x)) \leq K + L(x, \overline{S}(x)) \quad (6)$$

$$\beta K + L(x, \underline{S}(x)) \leq L(x, \overline{S}(x)), \quad \overline{S}(x) \geq \underline{s}(x) \quad (7)$$

$$L(x, \overline{s}(x)) \leq L(x, \overline{S}(x)) + (1 - \beta)K, \quad (8)$$

where, from earlier results, $\underline{S}(x)$ can be restricted to the set $\{d_1, \ldots, d_M\}$ and where $\overline{s}$ is identical to the functions $s^*$ and $S_0$. We remark that the convexity of $L(x, y)$ in $y$ for all $x$ insures that for each $x$, $\underline{s}(x) \leq \overline{s}(x) \leq \overline{S}(x) \leq \underline{S}(x)$.

5.3 A Partition based on $(\underline{s}, \overline{s}, \underline{S}, \overline{S})$

Extending results in [67], we now show that for all $x$ and $n$, $\underline{s}(x) \leq s_n(x) \leq \overline{s}(x)$ and $\underline{S}(x) \leq S_n(x) \leq \overline{S}(x)$ and that for the infinite horizon discounted case, $\underline{s}(x) \leq s^*(x) \leq \overline{s}(x)$ and $\underline{S}(x) \leq S^*(x) \leq \overline{S}(x)$, where $(s^*, S^*)$ represents an $(s, S)$ belief-dependent optimal policy. Proof is presented in the e-companion of this paper.
Assume $f(y, d) = y - d$ and $c(y, d) = p(d - y)^+ + h(y - d)^+$ and recall from Lemma 2 that $S^s(x) = d_m$ if $x$ satisfies

$$x \sum_{k=1}^{m-1} P(d_k) \leq \frac{p}{\mu + K} \leq x \sum_{k=1}^{m} P(d_k).$$

(9)

Given $S(x) = d_m$, let $s(x) = d_i$, $\pi(x) = d_j$, and $\overline{s}(x) = d_n$ satisfy

$$A_i(x)d_i + B_i(x) \leq K + A_m(x)d_m + B_m(x)$$

(10)

$$A_j(x)d_j + B_j(x) \leq (1 - \beta)K + A_m(x)d_m + B_m(x)$$

(11)

$$\beta K + A_m(x)d_m + B_m(x) \leq A_n(x)d_n + B_n(x).$$

(12)

Let $\overline{X}(\underline{x}, \pi, \overline{s})$ be the set of all $x \in X$ such that $\underline{x} = d_i$, $\pi = d_j$, $\overline{s} = d_m$, and $\overline{s} = d_n$ are the smallest integers satisfying the five linear inequalities in Eqs. 9 - 12. Note that the set of all $\overline{X}(\underline{x}, \pi, \overline{s})$ such that $\overline{X}(\underline{x}, \pi, \overline{s})$ is non-null is a partition of $X$.

Example 4 Consider Example 1 with reorder cost, $K = 5$. Each region in the triangle is described by $(i, j, m, n)$ where, $(\underline{x}, \pi, \overline{s}, \overline{S}) = (d(i), d(j), d(m), d(n))$. For example, the region labeled as $(4, 4, 5, > 7)$ in Figure 4 has $(\underline{x}, \pi, \overline{s}, \overline{S}) = (20, 20, 25, 36)$. This implies that $s^s(x) = d_4 = 20, \forall x \in (4, 4, 5, > 7)$. The search interval for $S^s(x)$ is also significantly restricted to the demand outcomes between $\underline{S}$ and $\overline{S}$, making the computation very easy. We remark that it is possible $\overline{S} > d_m$, as indicated (by $> 7$) in Figure 4. The corresponding $\overline{S}$ is 36 in $X_5$ and $X_6$ and it equals 38 in $X_7$.

A description of the determination of the sets $\Gamma_n(s)$, where $v_n(x, s) = \min\{x \gamma : \gamma \in \Gamma_n(s)\}$, can be found in the e-companion of this paper.
6 Conclusions

We have presented and analyzed an inventory control problem having a modulation process that affects demand and that is partially observed by the demand and AOD processes. Assuming AC holds and the reorder cost $K = 0$, we generalized results found throughout the literature that there exists an optimal policy that is a myopic base stock policy. We also developed a simple, easily implemented description of the optimal myopic base stock levels, as a function of the belief function. When AC is violated and $K = 0$, we examined the the base stock policy that is optimal when AC holds as a suboptimal policy yielding an upper bound. We presented a lower bound on the optimal expected cost function and a bound on the difference between the upper and lower bounds. A numerical study indicated that the bound on the difference between these two bounds can be quite small, indicating that even when AC is violated, the optimal base stock policy for the case where AC is not violated may be quite good.

When $K > 0$, we showed that there exists optimal $(s, S)$ policies, dependent on the belief function, and determined upper and lower bounds on $s$ and $S$ for the finite and infinite horizon cases, where each bound is dependent on the belief function of the modulation process. We showed that each of these bounds and the values of $s$ and $S$ for the finite and infinite horizon cases are constant within regions of the belief space and that these regions can be described by a finite number of linear inequalities. We outlined an approach for determining an optimal $(s, S)$ policy and the resultant expected cost function for the finite horizon case.

Future work may focus on extending the approach of formulation and solution presented for the current problem to other problems with similar structure. Another direction of potential interest would be the development of heuristic approaches for the case when AC does not hold, as pursued by Malladi et al. [48] for a distributed production system with mobile production capacity.

References

1. Arifoğlu K, Ozekici S (2010) Optimal policies for inventory systems with finite capacity and partially observed Markov-modulated demand and supply processes. European Journal of Operations Research 204(3):421–438
2. Arrow KJ, Harris T, Marschak J (1951) Optimal inventory policy. Econometrica 19(3):252–272
3. Arrow KJ, Scarf H, Karlin S (1958) Studies in the Mathematical Theory of Inventory and Production. Stanford Press, Stanford
4. Atrash A, Pineau J (2010) A Bayesian method for learning POMDP observation parameters for robot interaction management systems. In: In The POMDP Practitioners Workshop
5. Azoury KS (1985) Bayes solution to dynamic inventory models under unknown demand distribution. Management Science 31(9):1150–1160
6. Azoury KS, Miller BL (1984) A comparison of the optimal ordering levels of Bayesian and non-Bayesian inventory models. Management Science 30(8):993–1003
7. Ban GY (2020) Confidence intervals for data-driven inventory policies with demand censoring. Operations Research 68(2):309–326, DOI 10.1287/opre.2019.1883
8. Ban GY, Rudin C (2019) The big data newsvendor: Practical insights from machine learning. Operations Research 67(1):90–108, DOI 10.1287/opre.2018.1757
9. Bayraktar E, Ludkovski M (2010) Inventory management with partially observed nonstationary demand. Annals of Operations Research 176(1):7–39
10. Bellman R (1958) Review. Management Science 5(1):139–141
11. Bensoussan A, Cakanyildirim M, Sethi SP (2007) A multiperiod newsvendor problem with partially observed demand. Mathematics of Operations Research 32(2):322–344
12. Bensoussan A, Cakanyildirim M, Sethi SP (2007) Partially observed inventory systems: The case of zero-balance walk. SIAM Journal on Control and Optimization 46(1):176–209
13. Bensoussan A, Cakanyildirim M, Minjarez-Sosa JA, Royal A, Sethi SP (2008) Inventory problems with partially observed demands and lost sales. Journal of Optimization Theory and Applications 136(3):321–340
14. Bertsimas D, Kallus N (2020) From predictive to prescriptive analytics. Management Science 66(3):1025–1044, DOI 10.1287/mnsc.2018.3253
15. Besbes O, Muharremoglu A (2013) On implications of demand censoring in the newsvendor problem. Management Science 59(6):1407–1424
16. Bookbinder JH, Lordahl AE (1989) Estimation of inventory re-order levels using the bootstrap statistical procedure. IIE Transactions 21(4):302–312
17. Chang Y, Erera AL, White CC (2015) A leader-follower partially observed, multiobjective Markov game. Annals of Operations Research 235(1):103–128
18. Chang Y, Erera AL, White CC (2015) Value of information for a leader-follower partially observed Markov game. Annals of Operations Research 235(1):129–153
19. Chen LG, Robinson LW, Roundy RO, Zhang RQ (2015) Technical note- New sufficient conditions for (s, S) policies to be optimal in systems with multiple uncertainties. Operations Research 63(1):186–197
20. Cheung WC, Simchi-Levi D (2019) Sampling-based approximation schemes for capacitated stochastic inventory control models. Mathematics of Operations Research 44(2):668–692, DOI 10.1287/moor.2018.0940
21. Choi TM (ed) (2014) Handbook of Newsvendor Problems: Models, Extensions and Applications. Springer, New York
22. Ding X, Puterman ML, Bisi A (2002) The censored newsvendor and the optimal acquisition of information. Operations Research 50(3):517–527
23. Dvoretzky A, Keifer J, Wolfowitz J (1952) The inventory problem: III case of unknown distributions of demand. Econometrica 20(3):450–466
24. Ferreira KJ, Lee BHA, Simchi-Levi D (2016) Analytics for an online retailer: Demand forecasting and price optimization. Manufacturing & Service Operations Management 18(1):69–88
25. Gallego G, Hu H (2004) Optimal policies for production/inventory systems with finite capacity and Markov-modulated demand and supply processes. Annals of Operations Research 126(1):21–41
26. Gallego G, Moon I (1993) The distribution free newsboy problem: Review and extensions. The Journal of the Operational Research Society 44(8):825–834
27. Godfrey GA, Powell WB (2001) An adaptive, distribution-free algorithm for the newsvendor problem with censored demands, with applications to inventory and distribution. Management Science 47(8):1101–1112
28. Graves SC, Rinnooy Kan AHG, Zipkin PH (1993) Logistics of Production and Inventory, Handbooks in Operations Research and Management Science, vol 4. Elsevier
29. Huh WT, Rusmevichientong P (2009) A nonparametric asymptotic analysis of inventory planning with censored demand. Mathematics of Operations Research 34(1):103–123
30. Huh WT, Janakiraman G, Muckstadt JA, Rusmevichientong P (2009) An adaptive algorithm for finding the optimal base-stock policy in lost sales inventory systems with censored demand. Mathematics of Operations Research 34(2):397–416, DOI 10.1287/moor.1080.0367
31. Huh WT, Levi R, Rusmevichientong P, Orlin JB (2011) Adaptive data-driven inventory control with censored demand based on Kaplan-Meier estimator. Operations Research 59(4):929–941
32. Iglehart D (1963) Optimality of (s, S) policies in the infinite horizon dynamic inventory problem. Management Science 9(2):259–267
33. Iglehart D, Karlin S (1962) Optimal policy for dynamic inventory process with nonstationary stochastic demands, Stanford, CA: Stanford University Press, chap 8
34. Kamath R, Pakkala TPM (2002) A Bayesian approach to a dynamic inventory model under an unknown demand distribution. Computers and Operations Research 29(2002):403–422
35. Karlin S (1958) One-Stage Inventory Models with Uncertainty, Stanford Press, Stanford
36. Karlin S (1958) Optimal Inventory Policy for the Arrow-Harris-Marschak Dynamic Model, Stanford Press, Stanford
37. Karlin S (1959) Dynamic inventory policy with varying stochastic demands. Management Science 6(3):231–258
38. Karlin S (1959) Optimal policy for dynamic inventory process with stochastic demands subject to seasonal variations. Journal of the Society of Industrial and Applied Mathematics 8(4):611–629
39. Katehakis MN, Smit LC (2012) On computing optimal (q,r) replenishment policies under quantity discounts. Annals of Operations Research 200(1):279–298
40. Katehakis MN, Melamed B, Shi JJ (2015) Optimal replenishment rate for inventory systems with compound poisson demands and lost sales: a direct treatment of time-average cost. Annals of Operations Research DOI 10.1007/s10479-015-1998-y
41. Katehakis MN, Melamed B, Shi JJ (2016) Cash-flow based dynamic inventory management. Production and Operations Management 25(9):1558–1575, DOI 10.1111/poms.12571
42. Khouja M (1999) The single-period (news-vendor) problem: literature review and suggestions for future research. Omega 27(5):537–553
43. Klabjan D, Simch-Levi D, Song M (2013) Robust stochastic lot-sizing by means of histograms. Production and Operations Management 22(3):691–710
44. Lariviere M, Porteus E (1999) Stalking information: Bayesian inventory management with unobserved lost sales. Management Science 45(3):346–363
45. Levi R, Perakis G, Uichanco J (2015) The data-driven newsvendor problem: New bounds and insights. Operations Research 63(6):1294–1306
46. Lovejoy WS (1990) Myopic policies for some inventory models with uncertain demand distributions. Management Science 36(6):724–738
47. Lovejoy WS (1992) Stopped myopic policies in some inventory models with generalized demand processes. Management Science 38(5):688–707
48. Malladi SS, Erera AL, White III CC (2021) Managing mobile production-inventory systems influenced by a modulation process. Annals of Operations Research accepted.
49. Mamani H, Nassiri S, Wagner MR (2017) Closed-form solutions for robust inventory management. Management Science 63(5):1625–1643, DOI 10.1287/mnsc.2015.2391
50. Morton TE (1978) The nonstationary infinite horizon inventory problem. Management Science 24(14):1474–1482
51. Murray GR Jr, Silver EA (1966) A Bayesian analysis of the style goods inventory problem. Management Science 12(11):785–797
52. Ortiz OL, Erera AL, White CC (2013) State observation accuracy and finite-memory policy performance. Operational Research Letters 41(5):477–481
53. Perakis G, Roels G (2008) Regret in the newsvendor model with partial information. Operations Research 56(1):188–203
54. Petruzzi NC, Dada M (1999) Pricing and the newsvendor problem: A review with extensions. Operations Research 47(2):183–194
55. Puterman ML (1994) Markov Decision Processes: Discrete Stochastic Dynamic Programming. John Wiley & Sons, Inc., Hoboken, New Jersey
56. Qin Y, Wang R, Vakharia AJ, Chen Y, Seref MM (2011) The newsvendor problem: Review and directions for future research. European Journal of Operational Research 213(2):361–374
57. Scarf H (1959) Bayes solutions of the statistical inventory problem. Annals of Mathematical Statistics 30(2):490–508
58. Scarf H (1960) The optimality of \((S, s)\) policies in the dynamic inventory problem. In: Arrow K, Karlin S, Suppes P (eds) Mathematical Methods in the Social Sciences, chap 13
59. Sethi SP, Cheng F (1997) Optimality of \((s, S)\) policies in inventory models with Markovian demand. Operations Research 45(6):931–939
60. Smallwood RD, Sondik EJ (1973) The optimal control of partially observable Markov processes over a finite horizon. Operations Research 21(5):1071–1088
61. Sondik EJ (1978) The optimal control of partially observable Markov processes over the infinite horizon: Discounted costs. Operations Research 26(2):282–304
62. Song JS, Zipkin P (1993) Inventory control in a fluctuating demand environment. Operations Research 41(2):282–304
63. Treharne JT, Sox CR (2002) Adaptive inventory control for nonstationary demand and partial information. Management Science 48(5):607–624
64. Veinott AF Jr (1965) Optimal policy for a multi-product, dynamic, nonstationary inventory problem. Management Science 12(3):206–222
65. Veinott AF Jr (1965) Optimal policy in a dynamic, single product, nonstationary inventory model with several demand classes. Operations Research 13(5):761–778
66. Veinott AF Jr (1966) On the optimality of \((s, S)\) inventory policies: New conditions and a new proof*. SIAM Journal on Applied Mathematics 14(5):1067–
67. Veinott AF Jr, Wagner HM (1965) Computing optimal (s, S) inventory policies. Management Science 11(5):525–552
68. White CC, Harrington D (1980) Application of Jensen’s inequality to adaptive suboptimal design. Journal of Optimization Theory and Applications 32(1):89–99
69. Xin L, Goldberg DA (2015) Distributionally robust inventory control when demand is a martingale. arXiv preprint arXiv:151109437
70. Yuan H, Luo Q, Shi C (2021) Marrying stochastic gradient descent with bandits: Learning algorithms for inventory systems with fixed costs. Management Science online, DOI 10.1287/mnsc.2020.3799
71. Zhang H, Chao X, Shi C (2020) Closing the gap: A learning algorithm for lost-sales inventory systems with lead times. Management Science 66(5):1962–1980, DOI 10.1287/mnsc.2019.3288
72. Zipkin P (1989) Critical number policies for inventory models with periodic data. Management Science 35(1):71–80
Appendix

A1 Proof of Result in Section 2

Proof (Proof of Lemma 2.) If \( s^*(x) = d_m \), then

\[
A_{m-1}(x)d_{m-1} + B_{m-1}(x) > A_m(x)d_m + B_m(x),
\]

\[
A_{m+1}(x)d_{m+1} + B_{m+1}(x) \geq A_m(x)d_m + B_m(x),
\]

which leads to the result.

A2 Proofs of Results in Section 3

Proof (Proof of Proposition 1.) By induction. Letting \( v_0 = 0 \), we note that

\[
v_1(s, x) = \min_{y \geq s} L(x, y) = L(x, \max\{s^*(x), s\})
\]

for all \( x \) and \( L(x, \max\{s^*(x), s\}) \) is non-decreasing and convex in \( s \). Thus, the result holds true for \( n = 1 \) (and, trivially for \( n = 0 \)). Assume the result holds for \( n \). Then, for \( s \leq s^*(x) \),

\[
v_{n+1}(x, s) \leq L(x, s^*(x)) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), f(s^*(x), d))
\]

\[
= L(x, s^*(x)) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), s^*(\lambda(d, z, x))) \quad \text{(using Section 2.4)}.
\]

Also, \( v_{n+1}(x, s) \geq \min_{y \geq s} L(x, y) + \beta \sum_{d,z} \sigma(d, z, x)\min_y v_n(\lambda(d, z, x), f(y, d)) \)

\[
= L(x, s^*(x)) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), s^*(\lambda(d, z, x)))
\]

\[
= L(x, s^*(x)) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), f(s^*(x), d)).
\]

Thus, for \( s \leq s^*(x) \),

\[
v_{n+1}(x, s) = L(x, s^*(x)) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), f(s^*(x), d)),
\]

and \( v_{n+1}(x, s) = v_{n+1}(x, s^*(x)) \).

Assume \( s \geq s^*(x) \). Note

\[
v_{n+1}(x, s) \leq L(x, s) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), f(s, d)).
\]

Also, \( v_{n+1}(x, s) \geq \min_{y \geq s} L(x, y) + \beta \sum_{d,z} \sigma(d, z, x)\min_{y \geq s} v_n(\lambda(d, z, x), f(y, d)) \)

\[
= L(x, s) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), f(s, d)),
\]
and hence for \( s \geq s^*(x) \), 
\[
v_{n+1}(x, s) = L(x, s) + \beta \sum_{d,z} \sigma(d, z, x)v_n(\lambda(d, z, x), f(s, d))
\]
and is non-decreasing and convex in \( s \).

\textbf{Proof (Proof of Lemma 4.)} It is sufficient to show that if \( y \leq y' \) and \( x \preceq x' \), then,
\[
L(x, y) - L(x, y') \leq L(x', y) - L(x', y'),
\]
which follows from the assumptions and [55, Lemma 4.7.2].

Ideally, we would want to select \( \hat{x}^{d,z} \) so that \( s^*(x') \leq s^*(\hat{x}^{d,z}) \) for all \( x' \) such that \( x' \preceq \lambda(d, z, x) \) \( \forall x \in X \), for all \( (d,z) \), which would strengthen Lemma 3 as much as possible. We construct such an \( \hat{x}^{d,z} \) after the following preliminary result.

\textbf{Lemma 6} The set \( \{ \lambda(d, z, x) : x \in X \} = \left\{ \sum_{i} \xi_i \lambda(d, z, e_i) : \xi_i \geq 0 \forall i, \sum_i \xi_i = 1 \right\} \).

We remark that if \( x \preceq x' \) and \( x \preceq x'' \), then \( x \preceq \alpha x' + (1-\alpha)x'' \) for all \( \alpha \in [0,1] \). Thus, if \( \hat{x}^{d,z} \) is such that \( \hat{x}^{d,z} \preceq \lambda(d, z, x) \) for all \( i \), then \( \hat{x}^{d,z} \) is such that \( \hat{x}^{d,z} \preceq x' \) for all \( x' \in \{ \lambda(d, z, x) : x \in X \} \).

\textbf{Construction of } \( \hat{x}^{d,z} \)

We now construct \( \hat{x}^{d,z} \). Let
\[
\hat{x}^{d,z}_N = \min \{ \lambda_N(d, z, e_i), i = 1, \ldots, N \}
\]
\[
\hat{x}^{d,z}_n = \min \left\{ \sum_{k=n}^N \lambda_k(d, z, e_i), i = 1, \ldots, N \right\} - \sum_{k=n+1}^N \hat{x}^{d,z}_k, \quad n = N - 1, \ldots, 2
\]
\[
\hat{x}^{d,z}_1 = 1 - \sum_{k=2}^N \hat{x}^{d,z}_k.
\]

By construction, \( \hat{x}^{d,z} \preceq \lambda(d, z, x) \) \( \forall x \in X \). We now show that \( \hat{x}^{d,z} \in X \) and that \( s^*(x') \leq s^*(\hat{x}^{d,z}) \) for all \( x' \in X \) such that \( x' \preceq \lambda(d, z, x) \) \( \forall x \in X \).

\textbf{Lemma 7} (i) \( \hat{x}^{d,z} \in X \). (ii) For any \( x' \preceq \lambda(d, z, x) \) \( \forall x \in X \), \( s^*(x') \leq s^*(\hat{x}^{d,z}) \).

\textbf{Proof (Proof of Lemma 7.)} We have the following:

(i) Clearly, \( 0 \leq \hat{x}^{d,z}_N \leq 1 \) and \( \sum_{n=1}^N \hat{x}^{d,z}_n = 1 \). It is sufficient to show \( 0 \leq \hat{x}^{d,z}_n \), \( n = N - 1, \ldots, 1 \). Note
\[
\sum_{k=n+1}^N \hat{x}^{d,z}_k = \min_{1 \leq i \leq N} \left\{ \sum_{k=n+1}^N \lambda_k(d, z, e_i) \right\} \leq \sum_{k=n+1}^N \lambda_k(d, z, e_i) \leq \sum_{k=n}^N \lambda_k(d, z, e_i), \quad \forall i.
\]

Thus, \( \sum_{k=n+1}^N \hat{x}^{d,z}_k \leq \min_{1 \leq i \leq N} \left\{ \sum_{k=n}^N \lambda_k(d, z, e_i) \right\} = \sum_{k=n}^N \hat{x}^{d,z}_k \), and hence \( \hat{x}^{d,z}_n \geq 0 \).
Computing the Expected Cost Function, $v_n$

We now present a procedure for computing $v_n(s, x)$. We only consider the case where $s = s^*(x)$ due to Proposition 1 and Lemma 3. For notational simplicity, we assume that $\Pr\{z(t+1) \mid \mu(t+1), \mu(t)\}$ is independent of $\mu(t+1)$ and $\mu(t)$. Extension to the more general case is straightforward.

Assume $v_0 = 0$, let $n = 1$, and recall $v_1(x, s^*(x)) = L(x, s^*(x))$. Note $L(x, y) = x\overline{y}$, where $\overline{y} = \sum_{d,z} P(d, z) 1 c(y, d)$. Let $\Gamma_1 = \{\overline{y}\}$, and note that if $c(y, d) = p(d-y)^+ + h(y-d)^+$, it is sufficient to consider only $y \in \{d_1, \ldots, d_M\}$. Then, $v_1(x, s^*(x)) = \min \{x\overline{y} : \gamma \in \Gamma_1\}$. Assume there is a finite set $\Gamma_n$ such that $v_n(x, s^*(x)) = \min \{x\overline{\gamma} : \gamma \in \Gamma_n\}$. Then,

$$v_{n+1}(x, s^*(x)) = L(x, s^*(x)) + \beta \sum_{m=1}^M \sigma(d_m, x) v_n(\lambda(d_m, x), f(s^*(x), d_m))$$

$$= \min \{x\overline{\gamma} : \gamma \in \Gamma_1\} + \beta \sum_{m=1}^M \sigma(d_m, x) v_n(\lambda(d_m, x), s^*(\lambda(d_m, x)))$$

$$= \min \{x\overline{\gamma} : \gamma \in \Gamma_1\} + \beta \sum_{m=1}^M \sigma(d_m, x) \min \{\lambda(d_m, x) : \gamma \in \Gamma_n\}$$

$$= \min_{\overline{\gamma}} \gamma_1 \ldots \gamma_M \left\{x\overline{\gamma} + \beta \sum_{m=1}^M \sigma(d_m, x) \lambda(d_m, x)\gamma_m \right\}$$

$$= \min_{\overline{\gamma}} \gamma_1 \ldots \gamma_M \left\{x\overline{\gamma} + \beta \sum_{m=1}^M P(d_m)\gamma_m \right\}$$

Thus, $\Gamma_{n+1}$ is the set of all $\gamma$ such that $\gamma = \overline{\gamma} + \beta \sum_{m=1}^M P(d_m)\gamma_m$, where $\overline{\gamma} \in \Gamma_1$ and $\gamma_m \in \Gamma_n$ for all $m = 1, \ldots, M$, and for all $n$, $v_n(x, s^*(x))$ is piecewise linear and concave in $x$.

Let $|\Gamma|$ be the cardinality of the set $\Gamma$. Then, $|\Gamma_{n+1}| = |\Gamma_1||\Gamma_n|^M$, where $|\Gamma_1| \leq M$, and hence the cardinality of $\Gamma_n$ can grow rapidly. Many of the vectors in the sets $\Gamma_n$ are redundant and can be eliminated, reducing both computational and storage burdens. An exhaustive literature study of elimination procedures and other solution methods for solving POMDPs can be found in [17].

A3 Proofs of Results in Section 4

Proof (Proof of Lemma 3) Assume $f(y, d) = y - d$ and $c(y, d) = p(d-y)^+ + h(y-d)^+$, recall that elements of $P_1$ are sets of the form $\{x \in X : s^*(x) = d_m\}$ for
all \(d_m\) such that \(\min_x s^*(x) \leq d_m \leq \max_x s^*(x)\). Further recall that \(\{x \in X : s^*(x) = d_m\}\) is the set of all \(x\) such that
\[
\sum_{k=1}^{m-1} \sigma(d_k, x) < p/(p + h) \leq \sum_{k=1}^{m} \sigma(d_k, x),
\]
or equivalently,
\[
x \sum_{k=1}^{m-1} P(d_k) \leq p/(p + h) \leq x \sum_{k=1}^{m} P(d_k),
\]
which represents two linear inequalities. Further, for \(x \in \{x \in X : s^*(x) = d_m\}\),
\[
v^U_l(x, s) = A_l(x)d_l + B_l(x)
\]
for \(l = \max \{s^*(x), s\}\), where we note
\[
A_l(x)d_l + B_l(x) = x \left[ h \sum_{k=1}^{l} (d_l - d_k) P(d_k) + p \sum_{k=l+1}^{M} (d_k - d_l) P(d_k) \right],
\]
where \(A_l(x)\) and \(B_l(x)\) are defined in Section 2.3. Thus, on each element of \(P_1\),
\(v^U_l\) is linear in \(x\) for each \(s\) and each element of \(P_1\) is described by a finite number of linear inequalities.

Let \((x, s)\) be such that \(d_l = \max \{s^*(x), s\} \leq d_{l+1}\) for all \(x\) in an element \(\{x \in X : s^*(x) = d_m\}\). Further, let \(d_l(d, z) \leq \max \{s^*(\lambda(d, z, x)), \max \{s^*(x), s\} - d\} \leq d_{l(d, z) + 1}\) for all \(x\) in an element \(\{x \in X : s^*(\lambda(d, z, x)) = d_{m(d)}\}\), which is the set of all \(x\) such that:
\[
\lambda(d, z, x) \sum_{k=1}^{m(d)-1} P(d_k) \leq p/(p + h) \leq \lambda(d, z, x) \sum_{k=1}^{m(d)} P(d_k),
\]
or equivalently, for all \(x\) such that \(\sigma(d, x) \neq 0,
\[
x \sum_{k=1}^{m(d)-1} P(d_k) \leq (p/(p + h))x \sum_{k=1}^{m(d)} P(d_k) \leq x \sum_{k=1}^{m(d)} P(d_k),
\]
where we assume \(m\) and \(m(d)\) for all \(d\) have been chosen so that the finite set of linear inequalities describes a non-null subset of \(X\). We note that for such a subset,
\[
v^U_{m+1}(x, s) = A_l(x)d_l + B_l(x)
\]
\[
+ \beta \sum_{d, z} \sigma(d, z, x) [A_l(d, z)(\lambda(d, z, x))d_l(d, z) + B_l(d, z)(\lambda(d, z, x))]
\]
\[
= x \left[ h \sum_{k=1}^{l} (d_l - d_k) P(d_k) + p \sum_{k=l+1}^{M} (d_k - d_l) P(d_k) \right]
\]
\[
+ \beta \sum_{d} \left[ h \sum_{z} \sum_{k=1}^{l(d, z)} (d_l(d, z) - d_k) P(d, z) P(d_k) \right]
\]
\[
+ p \sum_{z} \sum_{k=l(d, z) + 1}^{N} (d_l(d, z) - d_l(d, z)) P(d, z) P(d_k) \right].
\]
The resulting partition $\mathcal{P}_2$ is at least as fine as $\mathcal{P}_1$ and each element in $\mathcal{P}_2$ is described by a finite set of linear inequalities. We have shown that on each element in $\mathcal{P}_2$, $v^D_s(x, s)$ is linear in $x$ for each $s$. A straightforward induction argument shows how $v^D_s(x, s)$ may be discontinuous in $x$ for fixed $s$.

**Proof (Proof of Proposition 4.)** It is sufficient to show the result holds for $\tau = t+1$. There are two cases. First, let $s(t) \leq s^*(x(t))$. Then, $s(t+1) = s^*(x(t)) - d(t)$. We note

$$
\min\{s^*(x) : x \in X\} - d_M \leq s^*(x(t)) - d(t) \\
\leq \max\{s^*(x) : x \in X\} - d_1 \quad \text{and hence,} \\
\min\{s^*(x) : x \in X\} - d_M \leq s(t+1) \leq \max\{s^*(x) : x \in X\} - d_1
$$

Second, let $s^*(x(t)) \leq s(t)$. Then, $s(t+1) = s(t) - d(t) \geq s^*(x(t)) - d(t)$. We note

$$
\min\{s^*(x) : x \in X\} - d_M \leq s^*(x(t)) - d(t) \leq s(t) - d(t) \\
\leq \max\{s^*(x) : x \in X\} - d_1, \quad \text{and hence,} \\
\min\{s^*(x) : x \in X\} - d_M \leq s(t+1) \leq \max\{s^*(x) : x \in X\} - d_1.
$$

A4 Design of Instances for Computational Study

We describe the generation of computational instances for Section 4.4. Each instance describes a backordering system with no fixed ordering cost. For each combination of number of modulation states $N \in \{2, 3\}$, number of demand outcomes $M \in \{3, 4, 5\}$, randomly generate $M$ unique ordered integer demand outcomes from $[0, D_1]$ for each $D_1 \in \{20, 100, 250, 500, 750, 1000\}$. Set the lowest demand outcome $d(0) = 0$ (to encourage A1 violation), randomly sample probability transition matrix $P(i, j)$ and probability mass function for each modulation state $\{Q(d, j)\}$ such that the $N$ ordered expected demands $ED_i, i = 1, \ldots, N$ are quite distinct and satisfy:

- $ED_1 \leq 0.5d(M)$ and $ED_2 > 0.5d(M)$ and $ED_2 - ED_1 > 0.25d(M)$ OR $ED_2 - ED_1 > 0.5d(M)$, when $N = 2$ and
- $ED_1 \leq 0.4d(M)$ and $ED_2 > 0.4d(M)$ and $ED_2 \leq 0.7d(M)$ and $ED_3 > 0.7d(M)$ and $ED_2 - ED_1 > 0.2d(M)$ and $ED_3 - ED_2 > 0.2d(M)$, when $N = 3$.

Set the number of decision epochs $T$ to 100 and vary backorder cost per unit per period $p$ as $\{1.5, 2, 3\}$, while keeping the holding cost $h$ at 1.
Algorithm 1 Sample Average Cost of Myopic Policy $\tilde{v}^D_I(x, s)$

1: procedure $\tilde{v}^D_I(x, s)$
2: for $i = 1, \ldots, N$ do
3: 
4: Set trajectory counter $n \leftarrow 1$.
5: while $n < n_{\text{Samples}}$ do
6: 
7: Initialize $(x, s) \leftarrow (e_i, 0)$. Initialize partially observed modulation state $\mu_0 \leftarrow \mu^i$.
8: 
9: Calculate $y = s^*(x)$.
10: Randomly sample the current modulation state $\mu$ with probability $\epsilon(\mu_0)P_{ij}$.
11: Randomly sample the current demand outcome $d^n_t$ with probability $Pr(d_t | \mu)$.
12: Compute $c^n_t = h(y - d^n_t)^+ + p(d^n_t - y)^+ + \lambda(d^n_t, x)$, $s \leftarrow -y - d_t$.
13: Update $\mu_0 \leftarrow \mu$, $x \leftarrow \lambda(d^n_t, x)$, $s \leftarrow y - d_t$.
14: Compute $c^n = \sum_t \beta^{t-1}c^n_t$.
15: $n \leftarrow n + 1$.
16: Compute $\tilde{v}^D_I(e_i, 0) = \sum_n c^n / n_{\text{Samples}}$.

Algorithm 2 Sample Average Cost of Lower Bound Policy $\tilde{v}^D_I(x)$

1: procedure $\tilde{v}^D_I(x)$
2: for $i = 1, \ldots, N$ do
3: 
4: Set trajectory counter $n \leftarrow 1$.
5: while $n < n_{\text{Samples}}$ do
6: 
7: Initialize $x \leftarrow e_i$.
8: 
9: Calculate $y = s^*(x)$.
10: Randomly sample the current demand outcome $d^n_t$ with probability $Pr(d_t | \mu)$.
11: Randomly sample the current demand outcome $d^n_t$ with probability $Pr(d_t | \mu)$.
12: Compute $c^n_t = h(y - d^n_t)^+ + p(d^n_t - y)^+ + \lambda(d^n_t, x)$.
13: $n \leftarrow n + 1$.
14: Compute $\tilde{v}^D_I(e_i) = \sum_n c^n / n_{\text{Samples}}$.

A6 Proofs of Results in Section 5

Proof (Proof of [Proposition 7]) The proof of [Proposition 7] is a direct extension of the results in [38].

Lemma 8 For all $x$ and $n$:

(i) if $s \leq s'$, then $v_n(x, s) \leq v_n(x, s') + K$
(ii) if $y \leq y'$, then $G_n(x, y') - G_n(x, y) \geq L(x, y') - L(x, y) - \beta K$
(iii) if $s \leq s'$, then $v_n(x, s) \geq v_n(x, s')$
(iv) if $y \leq y'$, then $G_n(x, y') - G_n(x, y) \leq L(x, y') - L(x, y) \leq 0$.

Proof (Proof of [Lemma 8])

(i) This result follows from the $K$-convexity of $v_n(x, s)$ in $s$, which is a direct implication of the second item of [Proposition 7].
The proof of Proposition 8 requires four lemmas.

**Lemma 9** For all \( n \) and \( x \), \( S(x) = S_0(x) \leq S_n(x) \).

**Lemma 10** For all \( n \) and \( x \), \( s_n(x) \) can be selected so that \( s_n(x) \leq \tilde{S}(x) \).

**Lemma 11** For all \( n \) and \( x \), \( S_n(x) \) can be selected so that \( S_n(x) \leq \tilde{S}(x) \).

**Lemma 12** For all \( n \) and \( x \), \( s(x) \leq s_n(x) \).

**Proof (Proof of Proposition 8)** The proof of these results follow from the proofs of Lemmas 2 - 5 in [67]. Proof of Proposition 8(a) follows from Lemmas 9 - 12, and Proposition 8(b) follows from (a) and Proposition 7.

Determining \( \Gamma_n(s) \)

As was true for the case \( K = 0 \), when \( K > 0 \), there is a finite set of vectors \( \Gamma_n(s) \) such that \( v_n(x, s) = \min \{ x \gamma : \gamma \in \Gamma_n(s) \} \) for all \( s \). Note that \( \Gamma_0(s) = \{ \emptyset \} \) for all \( s \), where \( \emptyset \) is the column \( N \)-vector having zero in all entries. Given \( \{ \Gamma_n(s) : \forall s \} \), we present an approach for determining \( \{ \Gamma_{n+1}(s) : \forall s \} \). Recalling Section A2, let \( T = \{ \gamma_1, \ldots, \gamma_M \} \) be such that \( \min \_y L(x,y) = \min \{ x \gamma : \gamma \in T \} \). Note

\[
G_n(x,y) = L(x,y) + \beta \sum_{d,z} \sigma(d,z,x)v_n(\lambda(d,z,x),f(y,d)),
\]

for \( y \in \{ d_1, \ldots, d_M \} \). Then,

\[
v_n(\lambda(d,z,x),f(y,d)) = \min \{ \lambda(d,z,x) \gamma : \gamma \in \Gamma_n(f(y,d)) \}.
\]

Let \( \Gamma_n'(y) \) be the set of all vectors of the form

\[
\gamma + \beta \sum_{d,z} P(d,z) \gamma(d,z),
\]

where \( \gamma \in \Gamma', \quad \text{and} \quad \gamma(d,z) \in \Gamma_n(f(y,d)). \) Then, \( G_n(x,y) = \min \{ x \gamma : \gamma \in \Gamma_n'(y) \} \) and

\[
v_n+1(x,s) = \begin{cases} 
K + G_n(x,S_n(x)) & s \leq s_n(x) \\
G_n(x,s) & \text{otherwise},
\end{cases}
\]

where \( S_n(x) \) and \( s_n(x) \) are the smallest integers such that

\[
G_n(x,S_n(x)) \leq G_n(x,y) \quad \forall y,
\]

\[
G_n(x,s_n(x)) \leq K + G_n(x,S_n(x)).
\]
Let $X_n(s', S')$ be the set of all $x \in X$ such that $s_n(x) = s'$ and $S_n(x) = S'$. Thus, if $x \in X_n(s', S')$, then $s'$ and $S'$ are the smallest integers such that

$$G_n(x, S'(x)) \leq G_n(x, y) \forall y.$$  

$$G_n(x, s'(x)) \leq K + G_n(x, S'(x)).$$

Since $G_n(x, y)$ is piecewise linear and convex in $x$ for each $y$, $X_n(s', S')$ is described by a finite set of linear inequalities. We remark that $\{X_n(s', S') : s' \leq S', \text{ and } X_n(s', S') \neq \emptyset\}$ is a partition of $X$. Further, we remark that if $X(s', \underline{S}, \overline{S}) \cap X_n(s', S') \neq \emptyset$, then search for $(s', S')$ can be restricted to $\underline{s} \leq s' \leq \overline{s}$ and $\underline{S} \leq S' \leq \overline{S}$. Let $\Gamma_{n+1}(s) = \{K_{\gamma} + \gamma : \gamma \in \Gamma_n(S')\}$ for all $s \leq s'$, and let $\Gamma_{n+1}(s) = \Gamma_n(s)$ for all $s > s'$. Thus, $v_{n+1}(x, s) = \min\{x \gamma : \gamma \in \Gamma_{n+1}(s)\}$ for all $s$. 