Abundant fission and fusion solutions in the 
(2 + 1)-dimensional generalized 
Calogero–Bogoyavlenskii–Schiff equation

Yuhan Li · Hongli An · Yiyuan Zhang

Received: 27 December 2021 / Accepted: 14 February 2022 / Published online: 10 March 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract Fission and fusion are important phenomena, which have been observed experimentally in many physical areas. In this paper, we study the above two phenomena in the (2 + 1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation. By introducing some new constraint conditions to its \(N\)-solitons, the fission and fusion are obtained. Numerical figures show that the two types of solutions look like the capital letter \(Y\) in spatial structures. Then, by taking a long wave limit approach and complex conjugation restrictions, some hybrid resonance solutions are generated, such as the interaction solutions between the \(L\)-order lumps and \(Q\)-fission (fusion) solitons, as well as hybrid solutions mixed by the \(T\)-order breathers and \(Q\)-fission (fusion) solitons. Dynamical behaviors of these solutions are analyzed theoretically and numerically. The results obtained can be helpful for understanding the fusion and fission phenomena in many physical models, such as the organic membrane, macromolecule material and even-clump DNA, plasmas physics, and so on.

Keywords Hirota bilinear form · \(N\)-soliton solutions · Fission phenomenon · Interaction solutions

1 Introduction

With the development of science and technology, the soliton theory has been widely used in the fields of fluid, optics, chemistry, biology, communications, astrophysics, geophysics, and so on. How to construct exact solutions to the soliton equations has attracted many researchers’ attentions, because with the solutions, people can better understand the phenomena described by the models and further explore other new potential applications. Up to now, many powerful methods have been proposed by mathematicians and physicists to find exact solutions for the soliton equations, such as inverse scattering theory [1], Lie group method [2], variable separation approach [3,4], Darboux and Bäcklund transformation [5–7], Riemann–Hilbert method [8–13], Hirota bilinear method [14,15], bilinear neural network method [16–20], and so on [21–31].

Usually, the interactions between soliton solutions derived by the above methods for nonlinear soliton equations are regarded to be completely elastic because the velocity, shape and amplitude of solitons keep unchanged after the interactions. However, for some nonlinear soliton models, completely inelastic interactions may occur when the wave vectors and velocities of the solitons satisfy some special conditions. For example, at certain time, one soliton may fission to two or
more solitons. Contrarily, at some time, two or more solitons may fusion to one. In the terminology of soliton theory, the above two phenomena are termed soliton fission and soliton fusion, respectively. Interestingly, investigations show that the fission and fusion phenomena have found their applications in many models, such as in organic membrane and macromolecule material [32], in even-clump DNA [33], in Sr–Ba–Ni oxidation crystal and waveguide [34], in nuclear physics, and so on [35]. Therefore, it motivates many scholars to seek fission and fusion solutions in nonlinear differential equations. However, in most work, the authors obtained the fission and fusion solutions by taking a logarithmic transformation with \( f \) in a form of [36–38]:

\[
\begin{align*}
  u &= \alpha \left( \ln f \right)_{x}, \\
  f &= 1 + \sum_{j=1}^{N} e^{k_j x + p_j y + w_j t + \phi_j} .
\end{align*}
\]

What needs to be pointed out is that because of the specialty of the approach, it loses the power to construct the interaction solutions between fission/fusion and other types of waves. Subsequently, Chen et al. considered a general transformation via:

\[
\begin{align*}
  u &= \alpha \left( \ln f \right)_{x}, \\
  f &= \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu_j \eta_j \right) \\
  &\quad + \sum_{j>1} \ln A_{1j} \mu_1 \eta_j + \sum_{j>2} \ln A_{2j} \mu_2 \eta_j ,
\end{align*}
\]

and thereby the interaction solution between a lump and \((N-2)\)-fissionable wave of the Sawada–Kotera equation was obtained [39]. It is a pity that the particular form of the expression (1.2) makes many terms lost compared with the classical \(N\)-soliton solution, which lead to the interaction solutions between fusion and fission waves unobtainable. Recently, Li et al. [40] introduced a typical relation to the parameters involved in the \(N\)-solitons, and they obtained the fusion, fission and some hybrid solutions of the fifth-order KdV system.

Inspired by the work mentioned in [36–40], here we would like to seek fission and fusion solutions in the \((2+1)\)-dimensional generalized Calogero–Bogoyavlenskii–Schiff (gCBS) equation:

\[
\begin{align*}
  u_t + u_{xxx} + 3uu_y + 3u_x v_y + \delta_1 u_y + \delta_2 v_{yy} &= 0, \\
  v_x &= u,
\end{align*}
\]

which was introduced by Bogoyavlenskii [41] and Schiff [42] in two different ways and constitutes a generalization of the \((2+1)\)-dimensional CBS equation

\[
\begin{align*}
  v_{xx} + v_{xxx} x + v_x y + 3v_{xx} v_y + 3v_{xx} v_x &= 0,
\end{align*}
\]

derived from the Korteweg–de Vries equation [43]. Due to the importance and applications of the CBS and gCBS equations, many experts have paid attention to the constructions of their exact solutions [44–48]. Unlike the above work, our plan here is to search the fission, fusion solutions as well as their interactions with some localized waves via an alternative method. We begin our work with a new constraint associated with the \(N\)-solitons. It is shown that under the new constraint, the fission and fusion solutions may be readily constructed. Interestingly, numerical results reveal that the spatial structure of the solutions looks like the capital letter “Y”. After that, we introduce the complex conjugation restrictions and long wave limit to the constraint, and then some interaction solutions are generated, including the hybrid solutions consisting of the fission solitons and fusion solitons, of the fusion/fission solitons and \(T\)-order breathers, as well as \(L\)-order lumps. In order to exhibit the dynamical behaviors that the interaction solutions may possess, we make some theoretical analysis and numerical simulations.

2 Fission and fusion solutions of the gCBS equation

With the aid of the Hirota’s bilinear method, the \(N\)-soliton solution of the \((2+1)\)-dimensional gCBS Eq. (1.3) may be readily obtained via

\[
\begin{align*}
  u &= 2 \ln f, \\
  v &= 2 \ln f_x ,
\end{align*}
\]

where the function \(f(x, y, t)\) is given by

\[
\begin{align*}
  f &= \sum_{\mu=0,1} \exp \left( \sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} \mu_i \mu_j \ln A_{ij} \right) ,
\end{align*}
\]

with

\[
\begin{align*}
  \eta_i &= k_i x + p_i y + w_i t + \phi_i , \quad w_i = -p_i (\delta_1 + k_i^2) - \frac{\delta_2 p_i^2}{k_i} , \\
  A_{ij} &= \frac{k_j x + k_j y + w_j t + \phi_j}{(k_j^2 - k_i^2) (k_j x + 2k_j^2 p_j - k_i^2 k_j^2 p_j - \delta_2 (k_j x - k_i p_j))^2} .
\end{align*}
\]

\[ i, j, s = 1, 2, \ldots , N . \]
In the above, the parameters $k_i$, $p_i$, $w_i$ and $\phi_i$ are arbitrary constants related to the amplitude and phase of the $i$-soliton, while $\sum_{\mu=0,1}$ means the summation of all possible combinations of $\mu_j$, $\mu_j = 0, 1$.

In order to construct the fission and fusion solutions of Eq. (1.3), we first give a necessary supplement to the range of $\ln x$: the relation $\exp(\ln x) = 0$ holds if and only if $x = 0$. According to the supplement, one can see that: if all $A_{js} = 0$, all the terms of $\exp(\ln A_{js})$ are removal and so that the expression (2.2) degenerates to a form of (1.1) given in [36], and if $A_{js} = 0$ with $3 \leq j < s \leq N$, then the expression (2.2) reduces to a form of (1.2) established in [39]. In this sense, we conclude that the supplement associated with Eq. (2.2) is more general, which can be used to construct the fission and fusion solutions of the $(2+1)$-dimensional gCBS equation.

**Proposition 1** Assume that the parameters $A_{js}$ described in (2.3) satisfy the following constraint:

$$A_{js} = 0, \quad (1 \leq j < s \leq P, \quad P < j < s \leq N, \quad P + Q = N),$$

(2.4)

which is equivalent to

$$k_i^2[k_i^2 - 3k_i^2k_s + 2k_jk_s^2 \pm \sqrt{k_j(k_j - k_s)^2(k_j^2 - 4k_j^2k_s + 4k_jk_s^2 + 12\delta_2p_j) + 2\delta_2p_j}],$$

(2.5)

then a kind of fission solutions combined by $P$-fission and $Q$-fission solitons can be constructed.

It is necessary to point out that for convenience, we record the solution obtained by Proposition 1 as the fission solution. In fact, the fusion and interaction solution between the fission and fusion are also contained due to the symbols “±” involved in (2.5). Generally, if the symbol “+” corresponds to the fission, then the symbol “-” corresponds to the fusion. In order to distinguish the two physical phenomena, we call the phenomenon fission wherein a soliton is divided into two or more solitons with $y$ changing from minus infinity to positive for a fixed time. In the following, Proposition 1 will be applied to generate the fission and fusion solutions as well as their interaction for the $(2+1)$-dimensional gCBS equation.

Here, we take $P = 2$ and $Q = 2$ in Proposition 1; then the resonance solution is described by

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_x,$$

(2.6)

with

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + A_{13}e^{\eta_1+\eta_3} + A_{14}e^{\eta_1+\eta_2} + A_{23}e^{\eta_2+\eta_3} + A_{24}e^{\eta_2+\eta_4},$$

(2.7)

where the parameters $\eta_j$ and $A_{js}$ are described by (2.3). Just as we point out in the above, the formula (2.6) represents three different types of solutions when appropriate parameters are chosen. The first type is an interaction solution between two different 2-fission waves, seen in Fig. 1. The second type is an interaction between two different 2-fusion waves, seen in Fig. 2. And the third is a solution mixed by a 2-fission and 2-fusion waves, seen in Fig. 3. Interestingly, it is seen from the...
Fig. 2  An interaction between two different 2-fusion solitons of \( u \) determined by Eq. (2.6) with 
\[ k_1 = \frac{1}{2}, k_2 = \frac{1}{2}, k_3 = -1, k_4 = -\frac{3}{2}, \]
\[ p_1 = -\frac{3}{2}, p_2 = \frac{409}{3000} - \frac{9\sqrt{89}}{1000}, p_3 = -\frac{1}{2}, p_4 = -\frac{3}{2} + \frac{3\sqrt{10}}{8}, \]
\[ \delta_1 = -1, \delta_2 = 1. a \ t = -50, b \ t = 0, c \ t = 50 \]

Fig. 3  An interaction between the fission and fusion of \( u \) determined by Eq. (2.6) with 
\[ k_1 = -1, k_2 = -\frac{4}{3}, k_3 = \frac{4}{9}, k_4 = \frac{1}{6}, p_1 = -\frac{2}{3}, \]
\[ p_2 = -\frac{34}{27} + \frac{2\sqrt{97}}{27}, p_3 = \frac{4}{9}, p_4 = \frac{329}{1944} + \frac{5\sqrt{973}}{1944}, \delta_1 = -1, \delta_2 = 1. a \ t = -40, b \ t = 0, c \ t = 40 \]

Figures that the spatial structures of these solutions look like the capital letter \( Y \). Therefore, the fission and fusion solutions obtained can also be called as resonance \( Y \)-type solutions. In order to examine the properties of these interaction solutions to be elastic or inelastic, we take the interaction solution between the fission and fusion waves in Fig. 3 as an example. The illustrative numerical simulations are undertaken in Fig. 4. From this figure, one can see that with time evolution, the amplitudes of each branch of the mixed solution consisting of the fission and fusion keep unchanged for a fixed \( y \) value. Therefore, it is concluded that interactions between the fission and fusion waves are elastic.

It is noted that if we take one of \( P \) and \( Q \) to be zero, then we can obtain a pure fission or fusion solution when \( N = 2 \), whose dynamical behaviors are shown in Fig. 5a–b. Except that, when \( N = 3 \) and appropriate parameters are selected, we can also derive a new kind of solution which fuses first and then splits rapidly, seen in Fig. 5c. To the best of our knowledge, the solution fusing first and then splitting rapidly fast together with the interaction solutions in Figs. 1, 2 and 3 constructed by Proposition 1 has not been discussed before.

3 Hybrid solutions between the \( L \)-order lumps and \( Q \)-fission (fusion) solitons

In this section, we shall show that when the constraint condition introduced in Proposition 1 is incorporated with the long wave limit approach, a kind of hybrid solution consisting of the \( L \)-order lumps and \( Q \)-fission (or fusion) solitons can be obtained.

**Proposition 2** The hybrid solution combined by the \( L \)-order lumps and \( Q \)-fission (or fusion) waves can be constructed if the following restrictions are satisfied:
Abundant fission and fusion solutions in the gCBS equation

Fig. 4  Time evolution of the interaction solution between the fission and fusion waves of $u$ given by (2.6). a $u(x, y = -100, t = -40)$, b $u(x, y = -100, t = 0)$, c $u(x, y = -100, t = 40)$, d $u(x, y = 150, t = -40)$, e $u(x, y = 150, t = 0)$, f $u(x, y = 150, t = 40)$

Fig. 5  Dynamical behavior of three different types of solutions of the gCBS equation (2.6) with $Q = 0$. a A fission solution with $k_1 = \frac{1}{3}, k_2 = -\frac{1}{3}, \delta_1 = -1, \delta_2 = 1, p_1 = \frac{1}{2}, p_2 = \frac{71}{216} - \frac{5\sqrt{265}}{216}$. b A fusion solution with $k_1 = \frac{1}{3}, k_2 = -\frac{1}{3}, p_1 = \frac{2}{5}, p_2 = \frac{599}{1350} + \frac{17\sqrt{1834}}{1350}$, $p_3 = \frac{107741}{182250} - \frac{1147\sqrt{1834}}{182250}$. c A solution that fuses first and then splits with $k_1 = \frac{1}{3}, k_2 = -\frac{1}{3}, k_3 = -\frac{37}{35} + \frac{2\sqrt{7333}}{35}, p_1 = \frac{2}{5}, p_2 = \frac{599}{1350} + \frac{17\sqrt{1834}}{1350}$, $p_3 = \frac{107741}{182250} - \frac{1147\sqrt{1834}}{182250}$, $\delta_1 = -1, \delta_2 = 1$
While the trajectories of the L-order lump waves are governed by the parameters \((K_{2i-1}, K_{2i}, \ldots, K_{2L-1})\), the central coordinates of the lumps before and after the interactions with the \(Q\)-fission (or fusion) waves are described by

\[
x_{\text{before}} = \frac{P_{2i-1}P_{2i}t}{K_{2i-1}K_{2i}} + \sum_{s=L+1}^{N} h_{\text{before}}(\lambda_s)k_s, \\
y_{\text{before}} = \frac{(K_{2i-1}P_{2i} + K_{2i}P_{2i-1})t}{K_{2i-1}K_{2i}} - t + \sum_{s=L+1}^{N} h_{\text{before}}(\lambda_s)\xi_s, \\
x_{\text{after}} = \frac{P_{2i-1}P_{2i}t}{K_{2i-1}K_{2i}} + \sum_{s=L+1}^{N} h_{\text{after}}(\lambda_s)k_s, \\
y_{\text{after}} = \frac{(K_{2i-1}P_{2i} + K_{2i}P_{2i-1})t}{K_{2i-1}K_{2i}} - t + \sum_{s=L+1}^{N} h_{\text{after}}(\lambda_s)\xi_s, \\
\]

with

\[
\lambda_s = -\frac{p_s(k_s^3 - k_s + p_s)}{k_s} - \frac{P_{2i-1}P_{2i}k_s}{K_{2i-1}K_{2i}} + \frac{(K_{2i-1}P_{2i} + K_{2i}P_{2i-1})p_s}{K_{2i-1}K_{2i}} - p_s \neq 0, \\
\kappa_s = \frac{P_{2i-1}b_{2i,s} - P_{2i}b_{2i-1,s}}{K_{2i-1}K_{2i}}, \\
\xi_s = \frac{-K_{2i-1}b_{2i,s} + K_{2i}b_{2i-1,s}}{K_{2i-1}P_{2i} - K_{2i}P_{2i-1}}, \\
h_{\text{before}}(x) = \begin{cases} 1, x < 0 \\ 0, x \geq 1 \end{cases}, \\
h_{\text{after}}(x) = \begin{cases} 0, x < 0 \\ 1, x > 1 \end{cases}
\]

The height of the lump waves of \(u\) is given by

\[
h = \frac{2(K_{2i-1}P_{2i} - K_{2i}P_{2i-1})^2}{3K_{2i-1}K_{2i}(K_{2i-1}P_{2i} + K_{2i}P_{2i-1})},
\]

which keeps unchanged before and after the interactions.

According to Proposition 2, when \(L = 1, Q = 2\) and \(N = 4\), the hybrid solution mixed by a lump and 2-fission (fusion) waves can be written as

\[
u = 2(\ln f)_{xx}, \quad f = 2(\ln f)_{x},
\]

with

\[
f = b_{12} + \theta_2 \\
+ e^{\eta_1}(\theta_1 \theta_2 + b_{12} + \theta_1b_{23} + \theta_2b_{13} + b_{13}b_{23}) \\
+ e^{\eta_2}(\theta_1 \theta_2 + b_{12} + \theta_1b_{24} + \theta_2b_{14} + b_{14}b_{24}),
\]

where

\[
\theta_1 = K_1x + P_1y - P_1(\frac{P_1}{k_1} + \delta_1)t, \\
b_{12} = \frac{6k_1^2k_2^2(K_1P_2 + k_2P_1)}{\delta_2(K_1P_2 - K_2P_1)^2}, \\
b_{13} = -\frac{6k_1^2k_3^2(K_3P_3 + k_3P_2)}{K_2^2(2k_3^3P_3 - \delta_2P_3^2) + K_3P_3K_3(k_3^3 + \delta_2P_3^2) - \delta_2P_3^2k_3^2}, \\
(s = 1, 2, \quad j = 3, 4).
\]

Here we choose \(K_1 = K_2 = \frac{7}{4} - \frac{6i}{7}, \quad P_1 = P_2 = 1, \quad k_3 = \frac{1}{2}, \quad k_4 = \frac{3}{4}, \quad P_3 = 1, \quad P_4 = \frac{9}{8}, \quad \delta_1 = -1\) and \(\delta_2 = 1\), a special interaction solution between a lump and fission waves is obtained, whose behaviors are depicted in Fig. 6. From the figure, one can see that the lump wave propagates along the trajectory \(y = \frac{4}{3}x - \frac{25033248}{2688085} \) (marked by red in Fig. 6) before it collides with the fission waves at \(t = 0\). After the collision, it changes the trajectory to \(y = \frac{4}{3}x\) (marked by black in Fig. 6). When we select \(p_3 = \frac{1}{2}, \quad p_4 = \frac{7}{9}\) and keep other parameters unchanged, the solution becomes an interaction solution between a lump and fusion waves. During the
interactions of the lump with the fusion wave, its trajectory changes from \( y = \frac{4}{5}x - \frac{40663520956}{50586174665} \) (marked by red in Fig. 7) to \( y = \frac{4}{5}x \) (marked by black in Fig. 7). By observing Figs. 6 and 7, one can find that, before and after the interactions, the shape, velocity and amplitude of the lump keep unchanged; however, only the trajectory of the lump is shifted, which coincides with the theoretical analysis made in Proposition 2. It is noted that the shift of the trajectory of the lump is a very important phenomenon, which may be used to explain some related phenomena in the ocean.

Making analogous analysis to Proposition 2. On taking \( L = Q = 2 \) while \( N = 6 \) and setting the parameters to satisfy the following conditions

\[
\begin{align*}
k_1 &= k_2^* = K_1^* = K_2^* = K_3 = K_4, \\
p_1 &= p_2 = P_1 = P_2, \\
\epsilon &\to 0, \quad \phi_1 = \phi_2 = \pi i, \\
A_{34} &= A_{56} = 0,
\end{align*}
\]

(3.10)

then we can obtain the interaction solution between the 2-order lumps and 2-fission waves as well as the interaction solution between the 2-order lumps and 2-fusion waves. The time evolution behaviors of the two solutions are exhibited in Figs. 8 and 9.

Moreover, if setting \( L = 1, N = 6 \), and requiring the parameters to satisfy

\[
\begin{align*}
k_1 &= k_2^* = K_1^* = K_2^* = K_3 = K_4, \\
p_1 &= p_2 = P_1 = P_2, \\
\epsilon &\to 0, \quad \phi_1 = \phi_2 = \pi i, \quad A_{34} = A_{56} = 0,
\end{align*}
\]

(3.11)

then we can derive three different types of interaction solutions: (1) a hybrid solution mixed by two different fission waves and a lump, seen in Fig. 10; (2) a hybrid solution combined by two different fusion waves and a lump, seen in Fig. 11; (3) a hybrid solution mixed by a fission, a fusion and a lump, seen in Fig. 12.

4 Hybrid solutions between the \( T \)-order breathers and \( Q \)-fission (fusion) solitons

In this section, we shall show that when we combine the complex conjugation restrictions with Proposition 1, the hybrid solutions between the \( T \)-order breathers and \( Q \)-fission (fusion) solitons can be obtained.
Fig. 7 Interaction behaviors between a lump and a 2-fusion solitons of $u$ determined by Eq. (3.7) with $K_1 = K_2^* = \frac{2}{5} - \frac{6i}{5}$, $P_1 = P_2 = 1$, $k_3 = \frac{1}{2}$, $k_4 = \frac{1}{2}$, $p_3 = \frac{7}{8}$, $p_4 = \frac{7}{8}$, $\delta_1 = -1$, $\delta_2 = 1$. The red line is $y = \frac{4}{5}x - \frac{406635209568}{50586174665}$, and the black is $y = \frac{4}{5}x$. a $t = -45$; b $t = -30$; c $t = -15$; d $t = 0$; e $t = 15$; f: $t = 30$

Fig. 8 Interaction behaviors between two lumps and a 2-fission solitons of $u$ determined by Eq. (3.7) with $K_1 = K_2^* = -\frac{4}{5} - \frac{8i}{5}$, $K_3 = K_4^* = \frac{216}{337} - \frac{384i}{337}$, $P_1 = P_2 = P_3 = P_4 = 1$, $k_5 = \frac{1}{2}$, $k_6 = \frac{3}{7}$, $p_5 = \frac{7}{5}$, $p_6 = \frac{35}{32} - \frac{3\sqrt{7}}{32}$, $\delta_1 = -1$, $\delta_2 = 1$. a $t = -50$; b $t = 0$; c $t = 50$

Proposition 3 Assume that the parameters satisfy Proposition 1 and the following complex conjugation conditions:

$$\eta_{2i-1} = \eta_{2j}^*, \quad A_{sj} = 0, \quad (1 \leq i \leq T, \quad 2T < s < j \leq N, \quad N = 2T + Q),$$

then a kind of hybrid solutions consisting of the $T$-order breather and $Q$-fission (fusion) waves can be constructed.

According to Proposition 3, when $N = 4$, the interaction solution composed of the 1-order breather and
Abundant fission and fusion solutions in the gCBS equation

Fig. 9 Interaction behaviors between two lumps and a 2-fusion solitons of $u$ determined by Eq. (3.7) with $K_1 = K_2^* = -\frac{4}{5} - \frac{8i}{5}$, $K_3 = K_4^* = \frac{216}{257} - \frac{184i}{257}$, $P_1 = P_2 = P_3 = P_4 = 1$, $k_5 = \frac{1}{2}$, $k_6 = \frac{3}{2}$, $p_5 = \frac{2}{7}$, $p_6 = \frac{35}{72} + \frac{3\sqrt{77}}{72}$, $\delta_1 = -1$, $\delta_2 = 1$. a) $t = -10$; b) $t = 0$; c) $t = 10$

Fig. 10 Interaction behaviors between a lump and two 2-fission solitons of $u$ determined by Eq. (3.7) with $K_1 = K_2^* = -\frac{36}{25} - \frac{48i}{25}$, $P_1 = P_2 = 1$, $k_3 = 1$, $k_4 = \frac{5}{2}$, $k_5 = 1$, $k_6 = \frac{1}{2}$, $p_3 = \frac{1}{7}$, $p_4 = \frac{55}{64} - \frac{5\sqrt{33}}{64}$, $p_5 = \frac{3}{7}$, $p_6 = \frac{3}{2}$, $\phi_3 = 10$, $\phi_4 = 0$, $\phi_5 = 40$, $\phi_6 = 30$, $\delta_1 = -1$, $\delta_2 = 1$. a) $t = -30$; b) $t = 0$; c) $t = 30$

Fig. 11 Interaction behaviors between a lump and two 2-fusion solitons of $u$ determined by Eq. (3.7) with $K_1 = K_2^* = -\frac{36}{25} - \frac{48i}{25}$, $P_1 = P_2 = 1$, $k_3 = 1$, $k_4 = \frac{5}{2}$, $k_5 = -\frac{1}{2}$, $k_6 = -1$, $p_3 = \frac{1}{7}$, $p_4 = \frac{224}{1125} - \frac{2\sqrt{207}}{1125}$, $p_5 = -\frac{1}{2}$, $p_6 = -\frac{11}{8} + \frac{\sqrt{87}}{8}$, $\phi_3 = \phi_4 = \phi_6 = -10$, $\phi_5 = -50$, $\delta_1 = -1$, $\delta_2 = 1$. a) $t = -50$; b) $t = 0$; c) $t = 50$
fission (fusion) waves can be derived from (2.1), in which the function \( f \) is given by

\[
f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + A_{12}e^{\eta_1 + \eta_2} + A_{13}e^{\eta_1 + \eta_3} + A_{14}e^{\eta_1 + \eta_4} + A_{23}e^{\eta_2 + \eta_3} + A_{24}e^{\eta_2 + \eta_4} + A_{12}A_{13}A_{23}e^{\eta_1 + \eta_2 + \eta_3} + A_{12}A_{14}A_{24}e^{\eta_1 + \eta_2 + \eta_4}.
\]  

(4.2)

In the above, \( \eta_1 = \eta_2^s, \eta_3, \eta_4 \) and \( A_{js} \) (1 \( \leq j < s \leq 4, j = 1, 2 \)) are determined by (2.3). Here, we choose the parameters as \( k_1 = k_2^s = -\frac{1}{2} + \frac{i}{8}, k_3 = \frac{3}{4}, k_4 = -\frac{7}{10}, k_5 = \frac{4}{7}, k_6 = \frac{1}{6} \), \( p_3 = -\frac{1}{2}, p_4 = -\frac{763}{1000} + 7\sqrt{1287}/1000, p_5 = \frac{4}{7}, p_6 = \frac{329}{1000} + 5\sqrt{577}/1000, \phi_3 = \phi_4 = -30, \phi_5 = -5, \phi_6 = 3, \delta_1 = -1, \delta_2 = 1 \).

(a) t = -40, (b) t = 0, (c) t = 60

Fig. 12 Interaction behaviors among one lump, a 2-fusion and a 2-fission solitons of \( u \) determined by Eq. (3.7) with \( K_1 = K_2^s = -\frac{36}{125} - \frac{48i}{125}, P_1 = P_2 = 1, k_3 = -\frac{1}{2}, k_4 = -\frac{7}{10}, k_5 = \frac{4}{7}, k_6 = \frac{1}{6} \), \( p_3 = -\frac{1}{2}, p_4 = -\frac{763}{1000} + 7\sqrt{1287}/1000, p_5 = \frac{4}{7}, p_6 = \frac{329}{1000} + 5\sqrt{577}/1000, \phi_3 = \phi_4 = -30, \phi_5 = -5, \phi_6 = 3, \delta_1 = -1, \delta_2 = 1 \). a an interaction between a breather and a 2-fission solitons with \( p_4 = \frac{35}{32} - \frac{17\sqrt{17}}{32} \), b an interaction between a breather and a 2-fusion with \( p_4 = \frac{35}{32} + \frac{17\sqrt{17}}{32} \), \( \phi_1 = \phi_2^s = -10, \phi_3 = \phi_4 = 0, \delta_1 = -1 \) and \( \delta_2 = 1 \), a typical solution mixed by the 1-order breather and 2-fission waves is exhibited in Fig. 13a. When we only change the value of \( p_4 \) into \( p_4 = \frac{35}{32} + \frac{17\sqrt{17}}{32} \), then the solution becomes to an interaction between the 1-order breather and 2-fusion waves, whose behaviors are shown in Fig. 13b. From these two figures, one can easily see that the breathers involved in the interaction solution are periodic and localized on the \( xoy \)-plane.

Similarly, when we take \( N = 6, T = 2 \) and set two pairs of parameters to satisfy the complex conjugation conditions via

\[
\eta_1 = \eta_2^s, \eta_3 = \eta_4^s, A_{56} = 0.
\]  

(4.3)
Abundant fission and fusion solutions in the gCBS equation 2499

Fig. 14 Dynamical behaviors of the interaction solution between the 2-order breathers and fission waves with $k_1 = k_2^* = -\frac{1}{16} + \frac{i}{12}$, $k_3 = k_4^* = \frac{1}{12} - \frac{i}{8}$, $k_5 = 1$, $k_6 = \frac{1}{2}$, $p_1 = p_2^* = -\frac{1}{10} - \frac{i}{8}$.

$p_3 = p_4^* = -\frac{1}{8} - \frac{i}{6}$, $p_5 = \frac{3}{4}$, $p_6 = \frac{3}{8}$, $\phi_1 = \phi_2 = \phi_3 = \phi_4 = -2$, $\phi_5 = \phi_6 = 0$, $\delta_1 = -1$, $\delta_2 = 1$. a $t = -30$, b $t = 0$, c $t = 35$

Fig. 15 Dynamical behaviors of the interaction solution between the 2-order breathers and fusion waves with $k_1 = k_2^* = -\frac{1}{16} + \frac{i}{12}$, $k_3 = k_4^* = \frac{1}{12} - \frac{i}{8}$, $k_5 = 1$, $k_6 = \frac{1}{2}$, $p_1 = p_2^* = -\frac{1}{10} - \frac{i}{8}$.

$p_3 = p_4^* = -\frac{1}{8} - \frac{i}{6}$, $p_5 = \frac{1}{3}$, $p_6 = \frac{7}{8}$, $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 2$, $\phi_5 = \phi_6 = 0$, $\delta_1 = -1$, $\delta_2 = 1$. a $t = -20$, b $t = 0$, c $t = 20$

Based on the 6-solitons expression given in (2.1), we can derive the interaction solutions between the 2-order breathers and 2-fission (fusion) waves, whose dynamical behaviors are shown in Figs. 14 and 15. Moreover, if we set $N = 6$, $T = 1$ and only require a pair of parameters to satisfy the complex conjugation conditions via

$$\eta_1 = \eta_2^*, \quad A_{34} = A_{56} = 0,$$

then the hybrid solution between a 1-order breather, a fission and a fusion can be constructed, whose behaviors are displayed in Fig. 16.

It is known that the breather, as a kind of localized periodic wave, has been found applications in optics, biophysics and condensed matter physics [49]. Here, we have shown that the localized periodic wave can interact with the fission (fusion) to form an interaction solution. Based on the important applications of the breather and fission (fusion) in many physical fields, we hope that the interaction solutions obtained can provide experts some new insights for investigations of some related problems.

**Remark 1** We would like to point out that the interaction solutions with fission/fusion phenomena we derive in this paper is different to what were given by Hossen et al. in Ref. [50]. Following their method [50], we can also obtain similar results which are shown in Appendix I.
Fission and fusion are very important physical phenomena, which have been observed in many fields, such as organic membrane, biophysics, plasma physics, nuclear physics and life science. In this paper, we investigate the fission and fusion phenomena in the $(2+1)$-dimensional generalized CBS equation by introducing a more general constraint to the $N$-solitons (seen in Proposition 1). Numerical simulations show that such solutions look like the capital letter $Y$ from the spatial structure. In addition, we obtain some interaction solutions, such as the mixed solutions by a lump and fission (fusion) waves, by a lump, a fission and fusion, by 2-order lumps and fission (fusion) waves, by a breather and fission (fusion) waves, by 2-order breathers and fission (fusion) waves. Dynamical behaviors of these solutions are discussed numerically and theoretically, which indicate that the selections of the parameters have great impact on the solutions. The method given in the paper is effective, which can be applied to investigate the fission and fusion phenomena in other physical models. However, there still exist many interesting problems that need further considerations. For example, how to use the fusion and fission solutions to explain any physical phenomena in the related areas? Can such type solutions be obtained by other methods, such as the bilinear neural network method? Except the solutions given here, does other kind of interesting solutions exist? Based on the importance and applications of fission and fusion phenomena, all these solutions are deserved deep investigations.

Acknowledgements We would like to express our sincere thanks to the referees for their valuable comments and suggestions.

Funding This work is supported by the National Natural Science Foundation of China under Grant No. 1177 5116 and Jiangsu Qinglan high-level talent Project.

Data availability statement We declare that the data supporting the findings of this study are available within the article.

Declarations

Conflict statement We declare we have no conflict of interests.

Appendix I: On the interaction solutions with fission phenomena

In this appendix, we shall show that by adopting a "rational-cosh-cos"-type test function as given in Ref. [50], the interaction solutions with fission phenomena of the $(2+1)$-dimensional generalized CBS equation can be obtained.

For this purpose, the function $f$ is described in the form of

$$f = g^2 + h^2 + a_7 + p \cosh(\xi_1) + q \cos(\xi_2),$$  \hspace{1cm} (A.1)

where

$$g = a_1x + a_2y + a_3t, \hspace{0.5cm} h = a_4x + a_5y + a_6t,$$

$$\xi_1 = m_1x + m_2y + m_3t, \hspace{0.5cm} \xi_2 = k_1x + k_2y + k_3t.$$  \hspace{1cm} (A.2)

In the above, $p, q, k_j (j = 1, 2, 3)$ and $a_j (j = 1, 2, \cdots, 7)$ are real constants to be determined, while
Abundant fission and fusion solutions in the gCBS equation 2501

Fig. 17 The profile of the interaction solution \( v \) determined by (A.4) with \( a_1 = 2, a_4 = 3, a_5 = 2.5, a_7 = 1.8, m_1 = 0.8, k_1 = 2, \delta_1 = 2, p = 0, t = 0. \) a) \( q = 0, \) b) \( q = 8, \) c) \( q = 20 \)

Fig. 18 The profile of the interaction solution \( v \) determined by (A.4) with \( a_1 = 2, a_4 = 3, a_5 = 2.5, a_7 = 1.8, m_1 = 0.8, k_1 = 2, \delta_1 = 2, p = 0.08, t = 0. \) a) \( q = 0, \) b) \( q = 10, \) c) \( q = 20 \)

\( m_j (j = 1, 2, 3) \) are real or pure imaginary parameters. Substituting Eq. (A.1) along with Eq. (2.1) into Eq. (1.3) and collecting all the coefficients of \( x, y, t, \cosh(\xi_1), \sinh(\xi_1), \cos(\xi_2) \) and \( \sin(\xi_2) \), then we get a set of algebraic equations in \( a_j, k_j, m_j, p, q, \delta_1 \) and \( \delta_2 \). After solving these algebraic equations, we get the following relations of the parameters:

\[
\begin{align*}
    a_2 &= -\frac{a_4 a_5}{a_1}, \\
    a_3 &= \frac{a_1 a_5}{a_1}, \\
    a_6 &= -\delta_1 a_5, \\
    k_2 &= 0, \\
    k_3 &= -\delta_1 k_1, \\
    m_2 &= 0, \\
    m_3 &= -\delta_1 m_1, \\
    \delta_2 &= 0.
\end{align*}
\] (A.3)

Inserting them into the transformation (2.1), we can get the following solution to Eq. (1.3)

\[
v = -\frac{2a_1^2 (x^2 + a_1^2) x + pm_1 \sinh(m_1(-x + \delta_1 t)) - q k_1 \sin(k_1(-x + \delta_1 t))}{a_1^2 x^2 + a_1^2 (a_1^2 x^2 + a_2^2 (y - \delta_1 t)^2 + \alpha_2) + a_1^2 a_2^2 (y - \delta_1 t)^2 + \alpha_2^2 \cosh(m_1(-x + \delta_1 t)) + qa_1^2 \cos(k_1(-x + \delta_1 t))}.
\] (A.4)

It is found that when different conditions are imposed onto the parameters \( p \) and \( q \), different interaction solutions can be obtained:

1) On setting \( p = 0 \) and \( q = 0 \), the solution \( v \) in (A.4) exhibits a single lump, which has one valley and one peak (seen in Fig. 17a). However, on setting \( p = 0 \) and \( q \neq 0 \), the solution \( v \) represents an interaction solution between a lump and a periodic wave, whose behaviors are displayed in Fig. 17b–c. Comparison shows that there are one peak and one valley in Fig. 17b and with \( q \) gradually increasing it splits into two peaks and two valleys by fission in Fig. 17c. That’s to say, the fission phenomenon occurs in the lump wave.

2) On setting \( p \neq 0 \) and \( q = 0 \), the solution \( v \) in (A.4) displays an interaction solution wherein the lump get into a double kink waves (seen in Fig. 18a), while
when \( p \neq 0 \) and \( q \neq 0 \), the solution \( v \) is shown to be an interaction solution among the lump, double kinks and periodic waves. Inspection reveals that one valley and one peak of the lump in Fig. 18b split into two valleys and two peaks in Fig. 18c with \( q \) gradually increases.

**References**

1. Ablowitz, M.J., Clarkson, P.A.: Soliton, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, New York (1991)
2. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Springer, New York (1989)
3. Lou, S.Y., Lu, J.Z.: Special solutions from the variable separation approach: the Davey-Stewartson equation. J. Phys. A Math. Gen. 29, 4209–4215 (1996)
4. Lou, S.Y., Chen, L.L.: Formal variable separation approach for nonintegrable models. J. Math. Phys. 40, 6491–6500 (1999)
5. Rogers, C., Schief, W.: Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory. Cambridge University Press, New York (2002)
6. Matveev, V.A., Salle, M.A.: Darboux Transformations and Solitons. Springer, Berlin (1991)
7. Liu, Q.P., Manas, M.: Darboux transformations for supersymmetric KP hierarchies. Phys. Lett. B. 485, 293–300 (2000)
8. Trogdon, T., Deconinck, B.: Numerical computation of the finite-genus solutions of the Korteweg-de-Vries equation via Riemann-Hilbert problems. Appl. Math. Lett. 26, 5–9 (2013)
9. Xu, J., Fan, E.G.: Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: Without solitons. J. Differ. Equ. 259, 1098–1148 (2015)
10. Wang, D.S., Guo, B.L., Wang, X.L.: Long-time asymptotics of the focusing Kundu-Eckhaus equation with nonzero boundary conditions. J. Differ. Equ. 266, 5209–5253 (2019)
11. Zhao, P., Fan, E.G.: Finite gap integration of the derivative nonlinear Schrödinger equation: A Riemann-Hilbert method. Physica D 402, 132213 (2020)
12. Tian, S.F.: Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method. J. Differ. Equ. 262, 506–558 (2017)
13. Tian, S.F., Zhang, T.T.: Long-time asymptotic behavior for the Gerdjikov-Ivanov type of derivative nonlinear Schrödinger equation with time-periodic boundary condition. Proc. Am. Math. Soc. 146, 1713–1729 (2018)
14. Hirota, R.: The Direct Method in Soliton Theory. Cambridge University Press, Cambridge (2004)
15. Hu, X.B.: Generalized Hirota’s bilinear equations and their soliton solutions. J. Phys. A: Math. Gen. 26, L465–L471 (1993)
16. Zhang, R.F., Bilige, S.D., Chaolu, T.: Fractal solitons, arbitrary function solutions, exact periodic wave and breathers for a nonlinear partial differential equation by using bilinear neural network method. J. Syst. Sci. Complex. 34, 122–139 (2021)
17. Zhang, R.F., Bilige, S.D., Liu, J.G., Li, M.C.: Bright-dark solitons and interaction phenomenon for p-gBKP equation by using bilinear neural network method. Phys. Scr. 96, 025224 (2021)
18. Zhang, R.F., Li, M.C., Yin, H.M.: Rogue wave solutions and the bright and dark solitons of the (3+1)-dimensional Jimbo-Miwa equation. Nonlinear Dyn. 103, 1071–1079 (2021)
19. Zhang, R.F., Li, M.C., Gan, J.Y., Li, Q., Lan, Z.Z.: Novel trial functions and rogue waves of generalized breaking soliton equation via bilinear neural network method. Chaos, Solitons Fractals 154, 111692 (2022)
20. Gai, L.T., Ma, W.X., Bilige, S.D.: Abundant multilayer network model solutions and bright-dark solitons for a (3 + 1)-dimensional p-gBLMP equation. Nonlinear Dyn. 106, 867–877 (2021)
21. Gai, L.T., Ma, W.X., Li, M.C.: Lump-type solution and breather lump-kink interaction phenomena to a (3 + 1)-dimensional GBK equation based on trilinear form. Nonlinear Dyn. 100, 2715–2727 (2020)
22. Fan, E.G.: Extended tanh-function method and its applications to nonlinear equations. Phys. Lett. A 277, 212–218 (2000)
23. Yan, Z.Y.: New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations. Phys. Lett. A 292, 100–106 (2001)
24. Ma, W.X., You, Y.C.: Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. Trans. Am. Math. Soc. 357, 1753–1778 (2005)
25. Ma, W.X.: Complexiton solutions to integrable equations. Nonlinear Anal. 63, e2461-2471 (2005)
26. Gai, L.T., Ma, W.X., Li, M.C.: Lump-type solutions, rogue wave type solutions and periodic lump-stripe interaction phenomena to a (3 + 1)-dimensional generalized breaking soliton equation. Phys. Lett. A 384, 126178 (2020)
27. Chen, Y., Wang, Q.: A unified rational expansion method to construct a series of explicit exact solutions to nonlinear evolution equations. J. Appl. Math. Comput. 177, 396–409 (2006)
28. Guo, H.D., Xia, T.C., Hu, B.B.: High-order lumps, high-order breathers and hybrid solutions for an extended (3 + 1)-dimensional Jimbo-Miwa equation in fluid dynamics. Nonlinear Dyn. 100, 601–614 (2020)
29. An, H.L., Feng, D.L., Zhu, H.X.: General M-lump, high-order breather and localized interaction solutions to the (2 + 1)-dimensional Sawada-Kotera equation. Nonlinear Dyn. 98, 1275–1286 (2019)
30. Yuan, P.S., Qi, J.X., Li, Z.L., An, H.L.: General M-lumps, T-breathers, and hybrid solutions to (2+1)-dimensional generalized KdK system. Phys. Lett. A 352, 124769 (2021)
31. Liu, Y.Q., Wen, X.Y., Wang, D.S.: Novel interaction phenomena of localized waves in the generalized (3 + 1)-dimensional KP equation. Comput. Math. Appl. 78, 1–19 (2019)
32. Serkin, V.N., Chapela, V.M., Percino, J., Belyaeva, T.L.: Nonlinear tunneling of temporal and spatial optical solitons through organic thin films and polymeric waveguides. Opt. Commun. 192, 237–244 (2001)
33. Hisakado, M.: Breather trapping mechanism in piecewise homogeneous DNA. Phys. Lett. A 227, 87–93 (1997)
Abundant fission and fusion solutions in the gCBS equation

34. Kip, D., Wesner, M., Herden, C., Shandarov, V.: Interaction of spatial photorefractive solitons in a planar waveguide. Appl. Phys. B. 68, 971–974 (1999)
35. Stoitcheva, G., Ludu, L., Draayer, J.P.: Antisoliton model for fission model. Math. Comput. Simul. 55, 621–625 (2001)
36. Wang, Y.F., Tian, B., Jiang, Y.: Soliton fusion and fission in a generalized variable-coefficient fifth-order Korteweg-de Vries equation in fluids. Appl. Math. Comput. 292, 448–456 (2017)
37. Chen, A.H.: Multi-kink solutions and soliton fission and fusion of Sharma-Tasso-Olver equation. Phys. Lett. A 374, 2340–2345 (2010)
38. Yan, Z.W., Lou, S.Y.: Soliton molecules in Sharma-Tasso-Olver-Burgers equation. Appl. Math. Lett. 104, 106271 (2020)
39. Chen, A.H., Huang, F.F.: Fissionable wave solutions, lump solutions and interactional solutions for the (2 + 1)-dimensional Sawada-Kotera equation, Phys. Scr. 94, 055206 (8pp) (2019)
40. Zhang, Z., Qi, Z.Q., Li, B.: Fusion and fission phenomena for (2 + 1)-dimensional fifth-order KdV system. Appl. Math. Lett. 116, 107004 (2021)
41. Bogoyavlenskii, O.I.: Overturning solitons in new two-dimensional integrable equations. Math. USSR Izvestiya. 34, 245–260 (1990)
42. Schiff, J.: Integrability of Chern-Simons-Higgs Vortex equations and a reduction of the self-dual Yang-Mills equations to three dimensions, Painlevé Trascendents, their Asymptotics and Physical Applications. (1992), pp. 393–405
43. Toda, K., Yu, S.J.: The investigation into the Schwarz-Korteweg-de Vries equation and the Schwarz derivative in (2 + 1) dimensions. J. Math. Phys. 41, 4747–4751 (2000)
44. Chen, S.T., Ma, W.X.: Lump solutions of a generalized Calogero-Bogoyavlenskii-Schiff equation. Comput. Math. Appl. 76, 1680–1685 (2018)
45. Wazwaz, A.M.: The (2+1) and (3+1)-dimensional CBS equations: multiple soliton solutions and multiple singular soliton solutions, Z. Naturforsch. 65a, 173-181 (2010)
46. Bruzon, M.S., Gandarias, M.L., Muriel, C., Ramrez, J., Saez, S., Romero, F.R.: The Calogero-Bogoyavlenskii-Schiff equation in 2 + 1 dimensions. Theor. Math. Phys. 137(1), 1367–1377 (2003)
47. Roshid, H.O.: Multi-soliton of the (2 + 1)-dimensional CBS equation and KdV equation. Comput. Meth. Differ. Equ. 7, 86–95 (2019)
48. Roshid, H.O., Khan, M.H., Wazwaz, A.M.: Lump, multilump, cross kinky-lump and manifold periodic-soliton solutions for the (2 + 1)-D Calogero-Bogoyavlenskii-Schiff equation. Heliyon 6, e03701 (2020)
49. Wang, J.Q., Tian, L.X., Guo, B.L., Zhang, Y.N.: Nonlinear stability of breather solutions to the coupled modified Korteweg-de Vries equations. Commun. Nonlinear Sci. Numer. Simul. 90, 105367 (2020)
50. Hossen, M.B., Roshid, H.O., Ali, M.Z., Rezazadeh, H.: Novel dynamical behaviors of interaction solutions of the (3 + 1)-dimensional generalized B-type Kadomtsev-Petviashvili model. Phys. Scr. 96, 125236 (2021)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.