Critical mass of bacterial populations and critical temperature of self-gravitating Brownian particles in two dimensions

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Abstract

We show that the critical mass $M_c = 8\pi$ of bacterial populations in two dimensions in the chemotactic problem is the counterpart of the critical temperature $T_c = GMm/4k_B$ of self-gravitating Brownian particles in two-dimensional gravity. We obtain these critical values by using the Virial theorem or by considering stationary solutions of the Keller-Segel model and Smoluchowski-Poisson system. We also consider the case of one dimensional systems and develop the connection with the Burgers equation. Finally, we discuss the evolution of the system as a function of $M$ or $T$ in bounded and unbounded domains in dimensions $d = 1, 2$ and $3$ and show the specificities of each dimension. This paper aims to point out the numerous analogies between bacterial populations, self-gravitating Brownian particles and, occasionally, two-dimensional vortices.

Keywords: Chemotaxis, two-dimensional gravity, self-gravitating Brownian particles, nonlinear meanfield Fokker-Planck equations, Burgers equation, two-dimensional turbulence

1 Introduction

In many fields of physics, astrophysics and biology, one is confronted with the description of the evolution of a system of particles which self-consistently attract each other over large distances. One difficulty and richness of the problem arises from the long-range nature of the potential of interaction [1]. This is the case, for example, in biology in relation with the process of chemotaxis [2]. The chemotactic aggregation of bacterial populations (like *Escherichia coli*) or amoebae (like *Dictyostelium discoideum*) is usually studied in the framework of the Keller-Segel model [3] which describes the collective motion of organisms that are attracted by a chemical substance (pheromone) that they produce themselves. The Keller-Segel (KS) model involves a drift-diffusion equation describing the evolution of the concentration of the bacteria in the gradient of concentration of the secreted chemical. In the simplest formulation, the concentration of the chemical is related to the concentration of bacteria by a Poisson equation (this is valid in a limit of large diffusivity of the chemical and for sufficiently large concentrations) [4]. These
equations have been studied by applied mathematicians who obtained rigorous results for the existence and unicity of the solutions and for the conditions of blow-up, modeling chemotactic collapse, in different dimensions of space [5]. In particular, in $d = 2$, there exists a critical mass $M_c$ (independent on the size of the domain) above which the system collapses and forms a Dirac peak. Alternatively, for $M < M_c$, the system spreads to infinity in an unbounded domain or tends to a stationary state in a bounded domain.

Gravity is another example of long-range attractive potential of interaction. In a series of papers, Chavanis & Sire [6, 7, 8, 9, 10, 11, 12, 13, 14] have studied a model of self-gravitating Brownian particles in various dimensions of space. In statistical mechanics, this model is associated with the canonical ensemble in which the temperature is fixed. A lot of analytical results have been obtained and an almost complete description of the system, for all the phases of the dynamics (pre-collapse and post-collapse), has been given in the overdamped limit of the model. In that limit, the evolution of the density of the self-gravitating Brownian gas is governed by the Smoluchowski-Poisson (SP) system. The Smoluchowski equation is a drift-diffusion equation of a Fokker-Planck type. For self-gravitating particles, the gravitational potential inducing the drift is produced by the density of particles through the Newton-Poisson equation. It turns out that the Smoluchowski-Poisson system is isomorphic to the simplified version of the Keller-Segel model of chemotaxis provided that the parameters are suitably re-interpreted, as discussed in [11]. In particular, for the 2D self-gravitating Brownian gas, there exists a critical temperature $T_c$ (independent on the size of the domain) below which the system collapses and forms a Dirac peak. This is the counterpart of the critical mass of bacterial populations. For $T > T_c$, the system evaporates in an unbounded domain or tends to a statistical equilibrium state in a bounded domain.

The object of this paper is to emphasize the parallel between these two systems. In Secs. 2 and 3, we use the Virial theorem to derive the critical mass of bacterial populations and the critical temperature of self-gravitating Brownian particles in two dimensions. In Sec. 4 we show that these critical values can also be obtained by considering stationary solutions of the Keller-Segel model and Smoluchowski-Poisson system. In Sec. 5 we consider the one dimensional problem and point out the connection between the Smoluchowski-Poisson system (or the Keller-Segel model) and the Burgers equation. We use this analogy to provide the general solution of these equations in $d = 1$ in bounded and unbounded domains. Finally, in Sec. 6 we provide a summary of the results obtained by Chavanis & Sire [6, 7, 8, 9, 10, 11, 12, 13, 14] for self-gravitating Brownian particles and adapt them to the context of chemotaxis to clearly show the link between these two problems.

2 The Keller-Segel model

The dynamical evolution of biological populations like bacteria, amoebae, cells... that are attracted by a substance that they emit themselves, is often described by the Keller-Segel (KS) model [3]. In its simplest form, it can be written as

\begin{equation}
\frac{\partial \rho}{\partial t} = D \Delta \rho - \chi \nabla \cdot (\rho \nabla c), \tag{1}
\end{equation}

\begin{equation}
\Delta c = -\lambda \rho. \tag{2}
\end{equation}

Equation (1) is a drift-diffusion equation for the cell density $\rho(\mathbf{r}, t)$. The diffusion term takes into account the erratic motion of the cells (like in Brownian theory) and the drift term with $\chi > 0$ takes into account the chemotactic attraction (one could also consider the case $\chi <$
where the secreted substance is a noxious substance, like a poison, so that chemotaxis is repulsive). It is directed along the gradient of concentration \( c(r, t) \) of the secreted chemical.

In the simplest formulation\(^4\), the production of the chemical by the cells is described by a Poisson equation \( (2) \). This is valid in a limit of high diffusivity of the chemical and for sufficiently large concentrations (see Appendix \( A \)). The Keller-Segel model \( (1)-(2) \) must be supplemented by appropriate boundary conditions. A first physical boundary condition is that the current \( J = D \nabla \rho - \chi \rho \nabla c \) is parallel to the boundary of the domain (or vanishes at infinity in an unbounded domain) so that the total mass is conserved. On the other hand, when dealing with the reduced Keller-Segel model \( (1)-(2) \), one usually assumes that \( c(r, t) \) is the solution of the Poisson equation \( (2) \) in an infinite domain with the usual gauge condition. This yields \( c(r, t) = \frac{\lambda}{(d-2)S_d} \int \rho(r', t) |r - r'|^{-(d-2)} d r' \) (in \( d \neq 2 \)) or \( c(r, t) = \frac{\lambda}{2\pi} \int \rho(r', t) \ln |r - r'| d r' \) (in \( d = 2 \)) whenever the domain containing the bacteria is finite (box) or infinite (here \( S_d \) denotes the surface of a unit sphere in \( d \)-dimensions). A physical discussion of the boundary conditions and of the limitations of the reduced Keller-Segel model \( (1)-(2) \) is provided in Appendix \( A \).

We now proceed in deriving an exact relation that is similar to the Virial theorem in astrophysics (see Sec. 3). As discussed in \( [13] \), it is convenient to take the origin of the system of coordinates at the center of mass which is a fixed quantity. Multiplying Eq. \( (1) \) by \( x_i x_j \) and integrating over the entire domain, we get

\[
\int \frac{\partial \rho}{\partial t} x_i x_j \, dr = \int x_i x_j \frac{\partial}{\partial x_k} \left( D \frac{\partial \rho}{\partial x_k} - \chi \rho \frac{\partial c}{\partial x_k} \right) \, dr.
\]

Introducing the “tensor of inertia”

\[
I_{ij} = \int \rho x_i x_j \, dr,
\]

and integrating the second term by parts twice, we find that

\[
\frac{1}{2} \frac{dI_{ij}}{dt} = D M \delta_{ij} + \chi W_{ij},
\]

where \( M = \int \rho \, dr \) is the total mass of cells and \( W_{ij} = W_{ji} \) is the “potential energy tensor” \( [13] \):

\[
W_{ij} = \int \rho \, x_i \frac{\partial c}{\partial x_j} \, dr.
\]

If the system is confined within a box, we have to account for boundary terms. The first integration by parts in Eq. \( (3) \) yields a residual term

\[
\int \nabla \cdot [x_i x_j (D \nabla \rho - \chi \rho \nabla c)] \, dr = \oint x_i x_j (D \nabla \rho - \chi \rho \nabla c) \cdot d S,
\]

where \( d S \) is the surface element normal to the frontier of the confining box. By virtue of the conservation of mass, the diffusion current in Eq. \( (1) \) is always perpendicular to the surface vector and consequently the term \( (7) \) vanishes. The second integration by parts yields

\[
- \int \frac{\partial}{\partial x_k} \left[ D \rho (x_j \delta_{ki} + x_i \delta_{kj}) \right] \, dr = - \oint D \rho (x_j \delta_{ki} + x_i \delta_{kj}) \, d S_k.
\]

Therefore, the general form of the Virial theorem for the chemotactic problem, taking into account boundary terms, is

\[
\frac{1}{2} \frac{dI_{ij}}{dt} = D M \delta_{ij} + \chi W_{ij} - \frac{1}{2} \oint D \rho (x_j \, d S_i + x_i \, d S_j).
\]
By contracting the indices, we get the scalar Virial theorem

\[
\frac{1}{2} \frac{dI}{dt} = dDM + \chi W_{ii} - D \int \rho \mathbf{r} \cdot d\mathbf{S},
\]

where

\[
I = \int \rho r^2 \, dr,
\]

is the moment of inertia and

\[
W_{ii} = \int \rho \mathbf{r} \cdot \nabla c \, d\mathbf{r},
\]

is the Virial. If the density \( \rho_b \) is uniform on the edge of the box (this is the case at least for a spherically symmetric system), we get

\[
\oint \rho \mathbf{r} \cdot d\mathbf{S} = \rho_b \oint \mathbf{r} \cdot d\mathbf{S} = \rho_b \int \nabla \cdot \mathbf{r} \, d\mathbf{r} = d\rho_b V,
\]

where \( V \) is the volume of the confining box. Thus,

\[
\frac{1}{2} \frac{dI}{dt} = dDM + \chi W_{ii} - dD\rho_b V.
\]

On the other hand, adapting the results of the Appendix of Ref.\[13\] to the present situation, we find for \( d \neq 2 \) that \( W_{ii} = (d-2)W \) where \( W = -\frac{1}{2} \int \rho c d\mathbf{r} \) is the “potential energy”. In that case, we get

\[
\frac{1}{2} \frac{dI}{dt} = dDM + (d-2)\chi W - D \oint \rho \mathbf{r} \cdot d\mathbf{S}, \quad (d \neq 2).
\]

For \( d = 2 \), we find instead that \( W_{ii} = -\lambda M^2/(4\pi) \) and we obtain

\[
\frac{1}{2} \frac{dI}{dt} = 2DM - \frac{\lambda \chi M^2}{4\pi} - D \oint \rho \mathbf{r} \cdot d\mathbf{S} \quad (d = 2).
\]

At equilibrium (\( \dot{I} = 0 \)), the Virial theorem (10) reduces to

\[
dDM + \chi W_{ii} = D \int \rho \mathbf{r} \cdot d\mathbf{S}.
\]

For \( d = 2 \), we have

\[
2DM - \frac{\lambda \chi M^2}{4\pi} = D \oint \rho \mathbf{r} \cdot d\mathbf{S}.
\]

In usual situations, the term in the r.h.s. is positive (this is at least the case for an axisymmetric distribution of particles in a disk where it is equal to \( 2D\rho_b \pi R^2 \)). Since \( \rho \geq 0 \), a sufficient condition is that \( \mathbf{r} \cdot d\mathbf{S} \geq 0 \) on each point of the boundary. This criterion is independent on the distribution itself and only depends on the domain shape. When \( \oint \rho \mathbf{r} \cdot d\mathbf{S} \geq 0 \), the above relation implies that a necessary condition for the existence of steady solutions is that

\[
M \leq M_c = \frac{8\pi D}{\chi \lambda}.
\]

\[1\]There are probably situations where this inequality is violated, in which case we expect that the value of the critical mass will differ from Eq. (19). In such situations, boundary effects should play a prominent role.
If we introduce dimensionless parameters (or take $D = \chi = \lambda = 1$), the critical mass is simply $M_c = 8\pi$. In terms of the critical mass (19), we can rewrite the Virial theorem (16) as

$$\frac{1}{2} \frac{dI}{dt} = \frac{\lambda \chi M}{4\pi} (M_c - M) - D \oint \rho \mathbf{r} \cdot d\mathbf{S} \quad (d = 2).$$

For $M > M_c$, we have $\dot{I} \leq \epsilon < 0$ so that the moment of inertia goes to zero in a finite time. This implies that the system collapses to a Dirac peak at $\mathbf{r} = 0$ (the center of mass) in a finite time. In an unbounded domain, assuming that $\rho$ decreases more rapidly than $r^{-2}$ for $r \to +\infty$, the foregoing relation reduces to

$$\frac{1}{2} \frac{dI}{dt} = \frac{\lambda \chi M}{4\pi} (M_c - M) \quad (d = 2).$$

This relation shows that, in an unbounded domain, there can be stationary solutions ($\dot{I} = 0$) only when $M = M_c$. In that case, the moment of inertia is conserved. For $M < M_c$, the moment of inertia diverges with time $I(t) \to +\infty$ so that the system evaporates. For $M > M_c$ the moment of inertia goes to zero $I(t) \to 0$ in a finite time. Since the case $M = M_c$ lies at the frontier between these two regimes (evaporation or collapse) we expect that, when $M = M_c$, the system evolves either towards a Dirac peak (collapse) or a completely spread profile (evaporation).

The existence of a critical mass for chemotaxis in $d = 2$ has been found by various authors \[15, 16, 17, 18, 11, 20, 19, 21\], using different arguments.

### 3 The Smoluchowski-Poisson system

In the mean field approximation and in a strong friction limit, the dynamics of a one-component self-gravitating Brownian gas is described by the Smoluchowski-Poisson (SP) system

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) \right],$$

$$\Delta \Phi = S_d G \rho.$$  

The Smoluchowski-Poisson system has been studied by Chavanis & Sire \[6, 7, 8, 9, 10, 11, 12, 13, 14\] in different dimensions of space (see also extensions in \[22, 23\] for more general equations of state $p = p(\rho)$). Since the temperature is fixed, the relevant statistical ensemble is the canonical ensemble. The boundary conditions are: (i) the current $\mathbf{J} = \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi$ is parallel to the boundary in a finite domain and vanishes at infinity in an unbounded domain (so that the total mass is conserved). (ii) The gravitational potential is given by $\Phi(\mathbf{r}, t) = -\frac{G}{(d-2)} \int \rho(\mathbf{r}', t) |\mathbf{r} - \mathbf{r}'|^{-(d-2)} d\mathbf{r}'$ (in $d \neq 2$) or $\Phi(\mathbf{r}, t) = -G \int \rho(\mathbf{r}', t) \ln |\mathbf{r} - \mathbf{r}'| d\mathbf{r}'$ (in $d = 2$), using the usual Gauge condition.

It is clear at first sight that the Smoluchowski-Poisson system (22)-(23) is isomorphic to the simplified Keller-Segel model (1)-(2) provided that we make the correspondances

$$D = \frac{k_B T}{\xi m}, \quad \chi = \frac{1}{\xi}, \quad c = -\Phi, \quad \lambda = S_d G.$$

Furthermore, the structure of the solutions depends on a single dimensionless parameter \[7\] which can be written:

$$\eta = \frac{\beta G m}{R^{d-2} \rho} = \frac{\lambda \chi M}{S_d R^{d-2}}.$$
We note, in particular, that the concentration of the chemical $c(r, t)$ in the chemotactic problem is the counterpart of the gravitational potential $\Phi(r, t)$ for self-gravitating Brownian particles (with the opposite sign). Due to this analogy, the results derived for the SP system can be applied to the KS model and vice versa. However, due to the different notations (and also because chemotaxis and gravity are studied by different communities) this connection is not always made. It can be therefore of interest to put the two models in parallel, as we do here.

The Virial theorem for the SP system has been derived in [13] (see also Appendix B). It can be written

$$\frac{1}{2} \xi \frac{dI}{dt} = dNk_B T + (d - 2)W - dPV, \quad (d \neq 2)$$

where $W = \frac{1}{2} \int \rho \Phi d\mathbf{r}$ is the potential energy and we have defined

$$P \equiv \frac{1}{dV} \int p \mathbf{r} \cdot d\mathbf{S},$$

where $p(r) = \rho(r)k_B T/m$ is the local pressure of an isothermal gas. In the case where $p = p_b$ is constant on the boundary, we have $P = p_b$. In dimension $d = 2$, the Virial theorem takes the simple form [13]:

$$\frac{1}{2} \xi \frac{dI}{dt} = 2Nk_B T - \frac{GM^2}{2} - 2PV, \quad (d = 2).$$

At equilibrium ($\dot{I} = 0$), we obtain

$$2Nk_B T - \frac{GM^2}{2} = 2PV,$$

Since $P \geq 0$ in usual circumstances (this is at least the case for an axisymmetric system in a disk where $P = \rho_b k_B T/m$), the above relation implies that a necessary condition for the existence of steady solutions is that

$$T \geq T_c = \frac{GMm}{4k_B}.$$

If we introduce dimensionless parameters (or take $G = M = m = k_B = \xi = 1$), the critical temperature is simply $T_c = 1/4$. In terms of the critical temperature (30), the Virial theorem (28) can be rewritten

$$\frac{1}{2} \xi \frac{dI}{dt} = 2Nk_B(T - T_c) - 2PV.$$

For $T < T_c$, the moment of inertia goes to zero in a finite time so that the system collapses to a Dirac peak at $\mathbf{r} = 0$ in a finite time. At equilibrium ($\dot{I} = 0$), we obtain

$$PV = Nk_B(T - T_c).$$

This is the equation of state of the two-dimensional self-gravitating gas at statistical equilibrium in the thermodynamic limit $N \to +\infty$ with $\eta = GMm/k_B T$ fixed (in that limit, the mean field approximation is exact [24]). This equation of state (and its extension to finite $N$ systems) has been obtained by various authors using different methods [25, 26, 27, 13, 14, 28].

In an unbounded domain, the Virial theorem (31) reduces to

$$\frac{1}{2} \xi \frac{dI}{dt} = 2Nk_B(T - T_c).$$
This relation shows that, in an unbounded domain, there can be stationary solutions \( \dot{I} = 0 \) only at the critical temperature \( T = T_c \). In that case, the moment of inertia is conserved. For \( T > T_c \), the moment of inertia diverges \( I(t) \rightarrow +\infty \) so that the system evaporates. For \( T < T_c \) the moment of inertia goes to zero \( I(t) \rightarrow 0 \) in a finite time so that the system forms a Dirac peak at \( r = 0 \) in a finite time. Defining the mean-squared radius of the cluster through the relation \( \langle r^2 \rangle = I/M \), we can rewrite Eq. (33) after integration in the form \( \langle r^2 \rangle = 4D(T)t + \langle r^2 \rangle_0 \) where

\[
D(T) = \frac{k_B T}{\xi m}(1 - T_c/T),
\]

is an effective diffusion coefficient. For \( T \gg T_c \) when gravitational effects become negligible, the Smoluchowski equation \( (22) \) reduces to a pure diffusion equation and the diffusion coefficient is given by the Einstein formula \( D(+\infty) = k_B T/\xi m \). However, Eq. (34) shows that the diffusion is less and less effective as temperature decreases and gravitational effects come into play. In particular, the effective diffusion coefficient becomes negative for \( T < T_c \) indicating finite time collapse.

The existence of a critical temperature for self-gravitating systems in \( d = 2 \) has been found by various authors, in different contexts, using different arguments \[29, 30, 31, 32, 7, 28\]. A similar critical temperature (whose value is negative) appears in the statistical mechanics of point vortices in two-dimensional hydrodynamics \[33, 34, 35\].

### 4 The equilibrium density profile in \( d = 2 \)

#### 4.1 Chemotaxis of bacterial populations

The stationary solutions of the KS model \( (1)-(2) \) are such that

\[
\rho = A e^{\chi D c}.
\]

This is similar to the Boltzmann distribution in statistical mechanics provided that we interprete \( T_{eff} = D/\chi \) as an effective temperature and \(-c(r)\) as a potential. This distribution can be obtained by extremizing the functional

\[
F[\rho] = -\frac{1}{2} \int \rho c \, dr + \frac{D}{\chi} \int \rho \ln \rho \, dr,
\]

at fixed mass. This functional is the Lyapunov functional of the KS model. It satisfies \( \dot{F} \leq 0 \) and \( \dot{F} = 0 \) if, and only if, the density is given by Eq. (35). It can also be interpreted as an effective free energy \( F = E - T_{eff}S \) associated with the Boltzmann entropy \[36\]. A steady state of the KS model is an extremum of \( F \) at fixed mass. Furthermore, it is linearly dynamically stable if and only if it is a (local) minimum of this functional \[37\]. The equilibrium state is obtained by substituting Eq. (35) into the Poisson equation (2) yielding

\[
\Delta c = -\lambda A e^{\chi D c}.
\]

This is similar to the Boltzmann-Poisson equation appearing in astrophysics (in the statistical mechanics of stellar systems \[27, 38\] and in isothermal models of stars \[39\]) and in vortex dynamics (in the statistical mechanics of 2D point vortices \[35\]). If we restrict ourselves to axisymmetric solutions, this equation can be solved analytically in \( d = 2 \). The 2D axisymmetric Boltzmann-Poisson equation \[37\] is solved in Appendix C. Here we use a slightly different method working directly on the mass profile.
Considering axisymmetric density profiles, introducing the accumulated mass \( M(r, t) = \int_0^r \rho(r', t)2\pi r'dr' \), using the relation \( \frac{\partial c}{\partial r} = -\frac{\rho M(r,t)}{2\pi r} \) (equivalent to the Gauss theorem) and introducing the variable \( u = r^2 \), it is found that the KS model (1)-(2) is equivalent to the single partial differential equation

\[
\frac{\partial M}{\partial t} = 4Du \frac{\partial^2 M}{\partial u^2} + \frac{\lambda \chi}{\pi} M \frac{\partial M}{\partial u}.
\]

The stationary profiles satisfy

\[
u M'' + \frac{\lambda \chi}{4\pi D} MM' = 0.
\]

Using \( uM'' = (uM')' - M', 2MM' = (M^2)' \) and \( M(0) = 0 \), we obtain after integration the first order differential equation

\[
u M' = M \left( 1 - \frac{\lambda \chi}{8\pi D} M \right).
\]

Since \( M' \geq 0 \), a necessary condition for the existence of steady states is that

\[
M \leq M_c = \frac{8\pi D}{\chi \lambda}.
\]

Solving Eq. (40) when this condition is fulfilled, we find that the equilibrium mass profile is given by

\[
M(r) = \frac{Kr^2}{1 + Kr^2/M_c}.
\]

where \( K \) is a constant of integration determined by the boundary conditions.

(a) In a disk of radius \( R \), the constant \( K \) is determined by the condition \( M(R) = M \). This yields

\[
M(r) = \frac{M}{1 - \frac{M}{M_c} 1 + \frac{M}{M_c} \left( \frac{r}{R} \right)^2}.
\]

The corresponding density profile is

\[
\rho(r) = \frac{1}{\pi R^2} \frac{M}{1 - \frac{M}{M_c} 1 + \frac{M}{M_c} \left( \frac{r}{R} \right)^2}.
\]

Typical density profiles are plotted in Fig. 1 for different values of the ratio \( M/M_c \). The relation between the mass and the central density is

\[
\rho_0 = \frac{M_c M/M_c}{\pi R^2 1 - \frac{M}{M_c}}.
\]

It is plotted in Fig. 2. The central density increases as \( M \to M_c \). For \( M = M_c \), the density profile is a Dirac peak: \( \rho(r) = M_c \delta(r) \).

(b) In an unbounded domain, we see from Eq. (42) that the condition \( M(r) \to M \) for \( r \to +\infty \) requires that \( M = M_c \). Therefore, there exists steady state solutions only for \( M = M_c \).
Figure 1: Density profile (normalized by $M_c/\pi R^2$) as a function of $r/R$ for different values of the total mass $M$. For $M \to M_c$, the density profile tends to a Dirac peak.

Figure 2: Relation between the mass $M/M_c$ and the central density $\rho_0$ (normalized by $M_c/\pi R^2$) in the chemotactic problem. There exists a unique solution for each value of $M \leq M_c$. Since there is no turning point in the series of equilibria $M(\rho_0)$, these steady solutions are stable (minima of free energy $F$ at fixed mass $M$) [7].
They form an infinite family of solutions parameterized by the constant $K$ or equivalently by the central density $\rho_0 = K/\pi$. Their density profile is

$$\rho(r) = \frac{\rho_0}{\left(1 + \frac{\pi \rho_0 r^2}{M_c}\right)^2}. \quad (46)$$

It is suggested in Sec. 2 that, among all the distributions of this family, the only stable stationary solution of the KS model with $M = M_c$ is the Dirac peak corresponding to $\rho_0 \to +\infty$. The dynamical formation of this Dirac peak is shown analytically in [13]. Another possible evolution is an evaporation.

### 4.2 Self-gravitating Brownian particles

We give, for comparison, the equivalent relations for the gravitational problem. The stationary solutions of the Smoluchowski equation (22) are described by the Boltzmann distribution

$$\rho = Ae^{-\beta m \Phi}. \quad (47)$$

This is equivalent to the condition of hydrostatic equilibrium $\nabla p + \rho \nabla \Phi = 0$ for an isothermal equation of state $p(r) = \rho(r) k_B T/m$ [13]. The Lyapunov functional of the SP system is the Boltzmann free energy

$$F[\rho] = \frac{1}{2} \int \rho \Phi \, d\mathbf{r} + k_B T \int \frac{\rho}{m} \ln \frac{\rho}{m} \, d\mathbf{r}. \quad (48)$$

The steady states are obtained by solving the Boltzmann-Poisson equation

$$\Delta \Phi = S_d GAe^{-\beta m \Phi}. \quad (49)$$

They are linearly dynamically stable if, and only if, they are minima of free energy at fixed mass. This is consistent with a condition of thermodynamical stability in the canonical ensemble. In $d = 2$, the SP system is equivalent to a single partial differential equation for the mass profile

$$\xi \frac{\partial M}{\partial t} = 4 Tu \frac{\partial^2 M}{\partial u^2} + 2GM \frac{\partial M}{\partial u}. \quad (50)$$

(a) In a bounded domain, there exists steady states if, and only if,

$$T \geq T_c = \frac{GMm}{4k_B}. \quad (51)$$

The mass profile is given by

$$M(r) = \frac{M}{1 - \frac{r}{R} + \frac{T_c}{T-T_c} \left(\frac{r}{R}\right)^2}, \quad (52)$$

and the corresponding density profile by

$$\rho(r) = \frac{1}{\pi R^2} \frac{M}{1 - \frac{T_c}{T} \left[1 + \frac{T_c}{T-T_c} \left(\frac{r}{R}\right)^2\right]^2}. \quad (53)$$

The relation between the temperature and the central density is

$$\rho_0 = \frac{1}{\pi R^2} \frac{M}{1 - \frac{T}{T_p}}. \quad (54)$$
At $T = T_c$, the density is a Dirac peak $\rho(r) = M\delta(r)$.

(b) In an unbounded domain, there exists steady solutions only for $T = T_c$. They are parameterized by the central density $\rho_0 = K/\pi$. The mass and density profiles are

$$M(r) = \frac{Kr^2}{1 + \frac{K}{M}r^2}, \quad \rho(r) = \frac{\rho_0}{(1 + \frac{\pi\rho_0}{M}r^2)^2}.$$  

These solutions have the same value of the free energy $F = -Nk_BT_c[1 + \ln(\pi/N)]$ independent on $\rho_0$. On the other hand, their moment of inertia is infinite (except the solution with $\rho_0 = +\infty$ for which $I = 0$). Since $I$ is conserved at $T = T_c$, they cannot be reached from a generic initial condition with $0 < I < +\infty$. It is suggested in Sec. 3 that the only stable stationary solution of the SP system with $T = T_c$ is the Dirac peak corresponding to $\rho_0 \to +\infty$. The dynamical formation of this Dirac peak is shown analytically in [13]. The system ejects a tiny amount of mass at large distances in order to accommodate for the conservation of $I$. Another possible evolution at $T = T_c$ is an evaporation.

In the above formulae, we can pass from biological notations to gravitational notations by using the relation

$$\frac{T}{T_c} = \frac{M_c}{M},$$  

resulting from the correspondances (24). Therefore, the mass $M$ in chemotaxis plays the role of the inverse temperature $\beta = 1/k_BT$ in gravity. We note also that, in 2D gravity, the condition (51) for the existence of a steady solution above a critical temperature $T_c$ (for a given mass $M$), can be converted into a condition on the mass (for a given temperature $T$)

$$M \leq M_c = \frac{4k_BT}{Gm},$$  

making the link with the biological problem even closer.

5 Analogy with the Burgers equation in $d = 1$

5.1 The case of an infinite domain

In $d = 1$, it is shown in [7] that the Smoluchowski-Poisson system is equivalent to a single differential equation

$$\xi \frac{\partial M}{\partial t} = \frac{k_BT}{m} \frac{\partial^2 M}{\partial x^2} + GM \frac{\partial M}{\partial x},$$  

for the integrated density $M(x,t) = 2\int_0^x \rho(x',t)dx'$. The density and the gravitational potential are related to the mass profile by $\partial_x M = 2\rho(x,t)$ and $\partial_x \Phi = GM(x,t)$. The stationary states of Eq. (58) can be obtained analytically (see Sec. 5.2). In an infinite domain, they are given by

$$M(x) = M \tanh(x/H), \quad \rho(x) = \frac{\rho_0}{\cosh^2(x/H)},$$  

where the length scale and the central density are determined as a function of the temperature by

$$H = \frac{2k_BT}{GMm}, \quad \rho_0 = \frac{M}{2H} = \frac{GM^2m}{4k_BT}.$$  

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There exists a unique steady state solution for each value of the temperature. Introducing
rescaled parameters, or taking equivalently \( \xi = k_B = m = G = M = 1 \), the dynamical equation
(58) takes the form
\[
\frac{\partial M}{\partial t} = T \frac{\partial^2 M}{\partial x^2} + M \frac{\partial M}{\partial x}.
\]
Setting \( v(x, t) = -M(x, t) \) and \( \nu = T \), it becomes equivalent to the one dimensional Burgers
equation
\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2},
\]
which has been studied in connection with hydrodynamical turbulence \[40\] and cosmology \[41\].
In this analogy, the mass plays the role of the velocity \( v = -M \), the density the role of the
velocity increment \( v' = -2 \rho \) and the temperature the role of the viscosity \( \nu = T \). The steady solution of Eq. (62) is
\[
v(x) = - \tanh(x/2\nu).
\]
In 1D turbulence, this solution describes a single shock \[40\] while in the context of self-
gravitating Brownian particles it describes the equilibrium mass profile \[59\] of a one dimensional
isothermal gas \[13\].

The collapse dynamics of the Smoluchowski-Poisson system at \( T = 0 \) has been studied in
\[7,8\] where an analytical solution describing the formation of a Dirac peak has been obtained. The collapse is self-similar and a Dirac peak is formed in the post-collapse regime \[7,8\]. This
Dirac peak is precisely the limiting form of the stationary solution \(59\)-\(60\) at \( T = 0 \). We note
that the solution obtained in \[7,8\] also describes the formation of a singular shock in inviscid 1D
turbulence \( \nu = 0 \) since the equation for the mass profile of self-gravitating Brownian particles
in \( d = 1 \) is isomorphic to the Burgers equation. For \( \nu \neq 0 \), the Burgers equation \(62\) has a
well-known explicit solution. As observed by Hopf \[42\] and Cole \[43\], the change of variables
\( v = -\partial_x \psi \) and \( \psi = 2\nu \ln \theta \) transforms the nonlinear Burgers equation into the heat equation
thereby leading to the explicit solution \[41\]:
\[
\psi(x, t) = 2\nu \ln \left\{ \frac{1}{\sqrt{4\pi t \nu}} \int_{-\infty}^{+\infty} \exp \left[ \frac{1}{2\nu} \left( \psi_0(q) - \frac{(x - q)^2}{2t} \right) \right] dq \right\},
\]
where \( \psi_0(x) = \psi(x, 0) \). Returning to the notations of the initial problem, we note that \( \psi \)
represents the gravitational potential \( \Phi \). This provides the general solution of the SP system
in \( d = 1 \) dimension in an unbounded domain for \( T \neq 0 \). Acedo \[44\] has constructed an explicit
analytical solution of the 1D Smoluchowski-Poisson system from the general formula \(64\). If
we take the limit \( T \to 0 \) and use steepest descent technics, we obtain from Eq. \( 64 \):
\[
\psi(x, t) = \sup_q \left[ \psi_0(q) - \frac{(x - q)^2}{2t} \right], \quad (T \to 0).
\]
A general method to solve this equation is described in \[41\] in the cosmological context. Interestingly, these results can also have applications for the 1D Smoluchowski-Poisson system.

### 5.2 The case of a finite domain

According to Eq. \(58\), the stationary solution of the SP system in \( d = 1 \) satisfies the differential
equation
\[
M'' + \frac{Gm}{k_B T} MM' = 0.
\]
Figure 3: Density profile (normalized by $M/2R$) as a function of $x/R$ for different values of the temperature $\eta = \beta GMmR$. For $T \to 0$, the density profile tends to a Dirac peak.

Integrating this equation once and using $M(0) = 0$ and $M'(0) = 2\rho_0$ to determine the constant of integration, we find that

$$M' + \frac{Gm}{2k_BT}M^2 = 2\rho_0.$$  \hspace{1cm} (67)

This can be rewritten

$$\frac{dM}{a^2 - M^2} = \frac{Gm}{2k_BT}dx, \quad a = \left(\frac{4k_BT\rho_0}{Gm}\right)^{1/2}. \hspace{1cm} (68)$$

This is easily integrated in

$$M(x) = a \tanh \left(\frac{Gma}{2k_BT}x\right), \quad \rho(x) = \frac{\rho_0}{\cosh^2 \left(\frac{Gma}{2k_BT}x\right)}. \hspace{1cm} (69)$$

In an infinite domain, using the fact that $M(x) \to M$ for $x \to +\infty$, we find that $a = M$ and we obtain the relations (59)-(60). In a bounded domain, using the fact that $M(R) = M$ we find that $a$ is solution of

$$M = a \tanh \left(\frac{GmaR}{2k_BT}\right). \hspace{1cm} (70)$$

This equation implicitly determines the central density $\rho_0$ in terms of the temperature $T$. These results can also be obtained by solving the 1D Boltzmann-Poisson equation (see Appendix D) which is equivalent to Eq. (66) [7]. The equilibrium density profile for different values of the temperature is represented in Fig. 3 and the relation between the temperature and the central density is represented in Fig. 4. We have used the notations defined in Appendix D.

Introducing rescaled variables or taking equivalently $\xi = k_BT = m = G = M = R = 1$, the dynamical equation for the mass profile in $d = 1$ is given by Eq. (61) with the boundary
Figure 4: Relation between the normalized inverse temperature $\eta = \beta GMmR$ and the central density $\rho_0$ (normalized by $M/2R$). There exists a unique solution for each value of the temperature. Since there is no turning point in the series of equilibria $\eta(\rho_0)$, these steady solutions are stable (minima of free energy $F$ at fixed mass $M$) [7].

conditions $M(0, t) = 0, M(1, t) = 1$ and $M(x, 0) = M_0(x)$. With the change of variables $v(x, t) = -M(x, t), v = -\partial_x \psi$ and $\psi = 2T \ln \theta$ of Sec. 5.1, we find that the function $\theta(x, t)$ satisfies the diffusion equation

\begin{equation}
\frac{\partial \theta}{\partial t} = T \frac{\partial^2 \theta}{\partial x^2},
\end{equation}

with the boundary conditions

\begin{equation}
\theta'(0, t) = 0, \quad \theta'(1, t) = \frac{1}{2T} \theta(1, t), \quad \theta_0(x) = e^{\frac{1}{2T} \int_0^1 M_0(y) dy}.
\end{equation}

Using the method of separation of variables, we find that the general solution of these equations is

\begin{equation}
\theta(x, t) = a_0 e^{T \lambda_0^2 t} \cosh(\lambda_0 x) + \sum_{n=1}^{+\infty} a_n e^{-T \lambda_n^2 t} \cos(\lambda_n x),
\end{equation}

where $\lambda_0$ is the solution of $\tanh(\lambda_0) = 1/(2T \lambda_0)$ and $\lambda_n$ are the solutions of $\tan(\lambda_n) = -1/(2T \lambda_n)$. The coefficients are determined from the initial condition

\begin{equation}
\theta_0(x) = a_0 \cosh(\lambda_0 x) + \sum_{n=1}^{+\infty} a_n \cos(\lambda_n x).
\end{equation}

Using the relations satisfied by $\lambda_0$ and $\lambda_n$, the following identities can be obtained by standard calculations

\begin{equation}
\int_0^1 \cos(\lambda_n x) \cos(\lambda_m x) dx = \frac{1 + 4T^2 \lambda_n^2 - 2T \lambda_n^\delta_{nm}}{2(1 + 4T^2 \lambda_0^2)} \delta_{nm}.
\end{equation}
\[ \int_0^1 \cosh(\lambda_0 x) \cos(\lambda_n x) \, dx = 0, \]

\[ \int_0^1 \cosh^2(\lambda_0 x) \, dx = \frac{4T^2 \lambda_0^2 - 1 + 2T}{2(4T^2 \lambda_0^2 - 1)}. \]

Then, we find that the coefficients \( a_n \) are given in terms of the initial condition by

\[ a_0 = \frac{2(4T^2 \lambda_0^2 - 1)}{4T^2 \lambda_0^2 - 1 + 2T} \int_0^1 \theta_0(x) \cosh(\lambda_0 x) \, dx, \]

\[ a_n = \frac{2(1 + 4T^2 \lambda_0^2)}{1 + 4T^2 \lambda_n^2 - 2T} \int_0^1 \theta_0(x) \cos(\lambda_n x) \, dx. \]

The foregoing equations provide the general solution of the SP system in \( d = 1 \) in a bounded domain. For \( t \to +\infty \), we find from Eq. (73) that \( M(x, t) \to M(x) = 2T \lambda_0 \tanh(\lambda_0 x) \) with \( 1 = 2T \lambda_0 \tanh(\lambda_0) \) which returns the stationary solution (69)-(70).

### 5.3 Generalization to other dimensions

Note, finally, that the Smoluchowski-Poisson system in \( d \) dimensions is equivalent to the single differential equation [7]:

\[ \xi \frac{\partial M}{\partial t} = \frac{k_B T}{m} \left( \frac{\partial^2 M}{\partial r^2} - \frac{d - 1}{r} \frac{\partial M}{\partial r} \right) + \frac{G}{r^{d-1}} \frac{\partial M}{\partial r}, \]

for the mass profile \( M(r, t) = \int_0^r \rho(r', t) S d r' \). With the change of variables \( x = r^d/d \), the foregoing equation becomes

\[ \xi \frac{\partial M}{\partial t} = \frac{k_B T}{m} \left( dx \right)^{\frac{2(d-1)}{d}} \frac{\partial^2 M}{\partial x^2} + GM \frac{\partial M}{\partial x}. \]

Introducing rescaled variables, or using equivalently \( \xi = k_B = m = G = M = R = 1 \), and setting \( v(x, t) = -M(x, t) \), this can be put in the form of a Burgers equation:

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu(x) \frac{\partial^2 v}{\partial x^2}, \]

with a position dependent viscosity

\[ \nu(x) = T\left( dx \right)^{\frac{2(d-1)}{d}}. \]

The viscosity is a pure power-law scaling like \( 1, x \) and \( x^{4/3} \) in \( d = 1, 2 \) and 3 dimensions respectively. For \( T = 0 \), we obtain the inviscid Burgers equation in any dimension of space.

In conclusion, it is interesting to note that the spherically symmetric Smoluchowski-Poisson system is connected to the 1D Burgers equation which arises in many domains of physics. This increases the interest of the self-gravitating Brownian gas model [9]. We note however that, in our case, \( v(x, t) = -M(x, t) \) is a negative quantity for \( x \geq 0 \) while in 1D turbulence the velocity \( v \) can take positive and negative values.
6 Dynamics of bacterial populations and self-gravitating Brownian particles

The Virial theorem (26) can be used to obtain general results on the dynamics of self-gravitating Brownian particles without solving the equations of motion. However, the explicit resolution of the Smoluchowski-Poisson system, as considered by Chavani & Sire [6, 7, 8, 9, 10, 11, 12, 13, 14], gives more precise information on the evolution of the density profile of the particles. In this section, we provide a summary of these results in the case of self-gravitating Brownian particles and transpose them to the chemotactic problem by using the notations of biology. This should reinforce the link between these two topics that are studied by different communities.

6.1 Self-gravitating Brownian particles

First, consider a two-dimensional self-gravitating Brownian gas enclosed within a circular box of radius \( R \). (a) Regular steady states exist for \( T > T_c \) and their density profile is known analytically, see Eq. (53). They are global minima of the free energy (48) at fixed mass. Using the free energy as a Lyapunov functional, we deduce that the SP system relaxes towards these steady states. The particles are confined by the box since the free energy as a Lyapunov functional, we deduce that the SP system relaxes towards these steady states. The particles are confined by the box since the free energy is a Lyapunov functional, we deduce that the SP system relaxes towards these steady states.

(b) For \( T = T_c \), the steady state is a Dirac peak \( \rho(r) = M \delta(r) \) containing the whole mass so that \( P = 0 \) (i.e. \( \rho(R) = 0 \)). The dynamics of the SP system at \( T = T_c \) has been studied in Sec. IV.B. of [7]. The collapse is self-similar with a scaling function similar to Eq. (53), and the Dirac peak containing the whole mass is formed for \( t \to +\infty \). The central density increases exponentially rapidly as \( \rho_0(t) \sim e^{2\sigma t} \). (c) For \( T < T_c \), there is no steady state and the system undergoes gravitational collapse. The dynamics of the SP system at \( T < T_c \) has been studied in Sec. IV.C. of [7]. For \( T < T_c \) the collapse is not exactly self-similar. In a finite time \( t_{coll} \), the system develops a Dirac peak containing a fraction \( T/T_c \) of the total mass \( M \), surrounded by a halo whose tail decreases as \( \rho \sim r^{-\alpha(t)} \) with \( \alpha(t) \) converging extremely slowly to \( \alpha = 2 \) for \( t \to t_{coll} \), like \( 2 - \alpha(t) = \sqrt{2 \ln \ln \rho_0(t) / \ln \rho_0(t)} \) with \( \rho_0(t) \sim (t_{coll} - t)^{-1} \), so that the evolution looks self-similar. For \( t \to t_{coll} \), the density profile behaves like \( \rho(r, t) \sim (T/T_c)M \delta(r) + \tilde{\rho}(r, t) \) where the evolution of \( \tilde{\rho}(r, t) \) is studied in [7]. From the Virial theorem Eq. (21), we note that \( \dot{I} \leq \epsilon < 0 \) for \( T < T_c \) so that the moment of inertia tends to its minimum value \( I = 0 \) in a finite time \( t_{end} \). This corresponds to the formation of a Dirac peak at \( r = 0 \) containing the whole mass \( M \). Since this final state is different from the structure obtained at \( t = t_{coll} \), this means that the evolution continues in the post-collapse regime \( t_{coll} \leq t \leq t_{end} \). In this post-collapse regime, the Dirac peak formed at \( t = t_{coll} \) accretes the mass of the surrounding halo until all the mass is at \( r = 0 \) at \( t_{end} \). This regime has not yet been fully described.

Consider now a two-dimensional self-gravitating Brownian gas in an infinite domain. (a) For \( T > T_c \), the particles have a diffusive motion (evaporation) with an effective diffusion coefficient given by Eq. (34) increasing linearly with the distance \( T - T_c \) to the critical temperature. This evaporation process is self-similar and it has been analytically studied in Secs. IV.B. and IV.C. of [13] for \( T \gg T_c \) and \( T \to T_c^+ \) respectively. (b) For \( T = T_c \), the effective diffusion coefficient vanishes \( D(T_c) = 0 \), so that the moment of inertia is conserved: \( \dot{I} = 0 \). The dynamics of the SP system at \( T = T_c \) has been studied in Sec. D of [13]. There exists an infinite family of steady state solutions (55) parameterized by the central density \( \rho_0 \) but they have \( I = +\infty \) (for \( \rho_0 < +\infty \)) and cannot be reached by the system for generic initial conditions (with \( I < +\infty \)). An analytical dynamical solution has been found in which the system forms a Dirac peak of mass \( M - \epsilon \) for \( t \to +\infty \) and ejects a tiny amount of mass \( \epsilon \ll 1 \) at large distances so as to satisfy the moment of inertia constraint (if \( I \neq 0 \) initially). Note that in an unbounded domain [13], the central density diverges logarithmically with time, \( \rho_0(t) \sim \ln t \), while the...
divergence is exponential in a bounded domain \[7\]. Another possible evolution at \(T = T_c\) is an evaporation. (c) For \(T < T_c\), the effective diffusion coefficient in Eq. \(34\) is negative, implying finite time blow-up. In particular, \(\langle r^2 \rangle = 0\) for \(t_{end} = \langle r^2 \rangle_0/4|D(T)|\) where \(\langle r^2 \rangle_0\) is calculated at \(t = 0\) from the center of mass. This leads to the formation of a Dirac peak \(\rho(r) = M\delta(r)\) at \(r = 0\) containing the whole mass. The solution obtained in \[7\] in a bounded domain probably describes the collapse of the core accurately but, in the absence of a confining box, the collapse is accompanied by an unlimited expansion of the halo. In the post-collapse regime, the Virial theorem indicates that all the matter falls at \(r = 0\) in a finite time.

For comparison, let us describe the situation in \(d = 3\). When the system is confined within a spherical box of radius \(R\), there exists a critical temperature \(k_BT_c = \frac{GMm}{2.52R}\) which depends on the box radius \(R\) (contrary to the case \(d = 2\)). (a) For \(T \geq T_c\), there exists equilibrium states. They have a density contrast \(\rho(0)/\rho(R) \leq 32.1\). These equilibrium states are metastable\(^2\), i.e. local minima of the free energy \[48\] at fixed mass \[6, 7\]. For \(T = T_c\), the equilibrium state is not a Dirac peak (contrary to the case \(d = 2\)). The density profile is regular with a density contrast \(\rho(0)/\rho(R) = 32.1\). (b) For \(0 < T < T_c\), there is no equilibrium state and the system undergoes a self-similar collapse \(\rho(r, t) = \rho_0(t)f(r/r_0(t))\) leading to a finite time singularity at \(t = t_{coll}\). The collapse time behaves with the distance to the critical point as \(t_{coll} = t_*\langle \eta - \eta_* \rangle^{-1/2}\) where \(\eta = \beta GMm/R\) and \(t_* = 0.91767702...\) \[9\]. This pre-collapse can be described analytically in \(d > 2\) and the invariant density profile is known exactly; the density decreases as \(r^{-2}\) at large distances \[6, 7\]. The central density increases with time as \(\rho_0 \sim (t_{coll} - t)^{-1}\) and the core radius decreases as \(r_0(t) \sim (t_{coll} - t)^{1/2}\). Therefore, this pre-collapse regime does not create a core since the mass \(M_0(t) \sim \rho_0 r_0^d\) at \(r = 0\) goes to zero as \(M_0(t) \sim (t_{coll} - t)^{(d-2)/2}\) at \(t = t_{coll}\). However, a Dirac peak is formed in the post-collapse regime for \(t > t_{coll}\) \[8\]. The mass of the Dirac peak increases as \(M_0(t) \sim (t - t_{coll})^{(d-2)/2}\) for \(t \rightarrow t_{coll}^+\) and the residual density obeys a backward dynamical scaling \(\rho(r, t) = \rho_0(t)g(r/r_0(t))\) with \(\rho_0 \sim (t - t_{coll})^{-1}\) and \(r_0 \sim (t - t_{coll})^{1/2}\). The Dirac peak accretes all the mass in an infinite time \(t \rightarrow +\infty\). (c) For \(T = 0\), the system undergoes a self-similar collapse and develops a finite time singularity at \(t = t_{coll}\) which can be studied analytically in any dimension. The invariant density profile is solution of an implicit equation and the density decreases as \(r^{-2d/(d+2)}\) at large distances \[6, 7\]. The central density increases with time as \(\rho_0 \sim (t_{coll} - t)^{-1}\) and the core radius decreases as \(r_0(t) \sim (t_{coll} - t)^{(d+2)/2d}\). Therefore, this pre-collapse regime does not create a core since the mass \(M_0(t) \sim (t_{coll} - t)^{d/2}\) at \(r = 0\) goes to zero at \(t_{coll}\). However, a Dirac peak is formed in the post-collapse regime for \(t > t_{coll}\) \[7, 8\]. The mass of the Dirac peak increases as \(M_0(t) \sim (t - t_{coll})^{d/2}\) for \(t \rightarrow t_{coll}^+\) and the residual density obeys a backward dynamical scaling \(\rho(r, t) = \rho_0(t)g(r/r_0(t))\) with \(\rho_0 \sim (t - t_{coll})^{-1}\) and \(r_0 \sim (t - t_{coll})^{(d+2)/2d}\). At \(T = 0\), the Dirac peak accretes all the mass in a finite time \(t_{end} = 1/d\) \[8\].

For 3D self-gravitating Brownian particles in an unbounded domain, there is no steady state. The system can either collapse (as in a bounded domain) or evaporate. The evaporation process has been treated in Sec. V. of \[13\] and the collapse in \[6, 7\]. The choice between collapse or evaporation depends on a complicated notion of basin of attraction that is function of the initial condition.

Finally, in \(d = 1\) the problem can be mapped on the Burgers equation which has the explicit solution \(64\) in an unbounded domain and \(73\) in a bounded domain.

\(^2\)Since there is no global minimum of free energy for an isothermal self-gravitating gas, these equilibrium states may not always be reached by the system. For some peculiar initial conditions, the system may rather collapse and form a singularity (Dirac peak) with infinite free energy as in the case \(T < T_c\) (see below). This depends on a complicated notion of basin of attraction as illustrated in \[6\]. However, for generic initial conditions with \(T \geq T_c\), the SP system usually converges towards a “gaseous” steady state with smooth density profile (local minimum of free energy).
6.2 Chemotactic aggregation of bacterial populations

The reduced Keller-Segel model (1)-(2) and the Smoluchowski-Poisson system (22)-(23) are isomorphic so we can directly transpose the results obtained for self-gravitating Brownian particles to the biological context. By reformulating these results with notations appropriate to chemotaxis, we aim to facilitate the comparison with the numerous results obtained in chemotaxis by applied mathematicians [5]. Since we are just transposing the results of the previous section to a different context, we shall give less details in their description.

First, consider the two-dimensional axisymmetric Keller-Segel model (1)-(2) in a circular domain of radius $R$. (a) Regular steady states exist for $M < M_c$ and their density profile is known analytically, see Eq. (44). (b) For $M = M_c$, the steady state is a Dirac peak $\rho(r) = M_c \delta(r)$ containing the whole mass. The collapse is self-similar and the Dirac peak is formed for $t \to +\infty$. The central density increases exponentially rapidly as $\rho_0(t) \sim e^{\sqrt{2}t}$. (c) For $M > M_c$, there is no steady state and the system undergoes chemotactic collapse. For $M > M_c$ the collapse is not exactly self-similar. In a finite time $t_{\text{coll}}$, the system develops a Dirac peak of mass $M_c$ surrounded by a halo whose tail decreases as $\rho \sim r^{-\alpha(t)}$ with $\alpha(t)$ converging extremely slowly to $\alpha = 2$ for $t \to t_{\text{coll}}$, so that the evolution looks self-similar. This corresponds to $\rho(r, t) \to M_c \delta(r) + \tilde{\rho}(r, t)$. This non self-similar collapse has been studied in detail in [7]. Some results had been obtained earlier by Herrero & Velazquez [18] but they are different from those of [7]. The reason is not well-understood but it may be that the solutions constructed in [18], while mathematically correct, are unstable. By contrast, the solution of [7] shows a very good agreement with direct numerical simulations of the reduced Keller-Segel model in $d = 2$ (as explained previously, the work [7] is presented for self-gravitating Brownian particles but the results also apply to biological populations). From the Virial theorem Eq. (20), we note that $I \leq \epsilon < 0$ for $M > M_c$ so that the moment of inertia tends to its minimum value $I = 0$ in a finite time $t_{\text{end}}$. This corresponds to the formation of a Dirac peak at $r = 0$ containing the whole mass $M$. Therefore, we expect a post-collapse regime for $t_{\text{coll}} \leq t \leq t_{\text{end}}$ leading ultimately to a Dirac peak $\rho(r) = M \delta(r)$ containing all the mass at $t_{\text{end}}$. To our knowledge, this post-collapse regime has not yet been fully characterized.

Consider now the case of 2D systems in an infinite domain. (a) For $M < M_c$, $I(t) \to +\infty$ for $t \to +\infty$ and the system evaporates. This regime has been studied in [13]. (b) For $M = M_c$, $dI/dt = 0$ so that the moment of inertia is conserved. It is shown analytically in [13] that the system forms a Dirac peak for $t \to +\infty$ and ejects a tiny amount of mass at large distances so as to satisfy the moment of inertia constraint. In an unbounded domain [13], the central density diverges logarithmically with time, $\rho_0(t) \sim \ln t$, while the divergence is exponentially fast in a bounded domain [7]. For $M = M_c$, the system can also evaporate. (c) For $M > M_c$, $I(t) = 0$ in a finite time $t_{\text{end}}$ so the system collapses to a Dirac peak containing all the mass in a finite time. The evolution is expected to be similar to the case of a box-confined system studied in [7].

In $d = 3$, in a bounded domain, there exists a critical mass $M_c = 31.7\frac{\rho_0}{\chi}$ which depends on the box radius $R$ (contrary to the case $d = 2$). This is the equivalent of the Emden temperature $k_B T_c = \frac{GMm}{2\pi^2 R}$ in astrophysics. (a) For $M \leq M_c$, there exists metastable equilibrium states, i.e. local minima of the functional $\mathcal{F}$, to which the system is attracted for generic initial conditions. (b) For $M > M_c$, there is no equilibrium state and the system undergoes a self-similar collapse leading to a finite time singularity at $t = t_{\text{coll}}$. The collapse time behaves with the distance to the critical point as $t_{\text{coll}} = t_s(\eta - \eta_c)^{-1/2}$ where $\eta = \lambda M \chi/4\pi DR$ and $t_s = 0.91767702\ldots$. This pre-collapse regime can be described analytically [6][7]. The density profile decreases as $r^{-2}$. The central density increases with time as $\rho_0 \sim (t_{\text{coll}} - t)^{-1}$ and the core radius decreases as $r_0(t) \sim (t_{\text{coll}} - t)^{1/2}$. A Dirac peak is formed in the post-collapse...
regime for \( t > t_{\text{coll}} \). The mass of the Dirac peak increases as \( M_0(t) \sim (t - t_{\text{coll}})^{1/2} \). The Dirac peak accrets all the mass for \( t \to +\infty \) [8]. (c) The peculiar case \( D = 0 \) (no diffusion) where the particles are driven solely by the chemotactic drift is equivalent to the case \( T = 0 \) for self-gravitating Brownian particles described in Sec. [6, 1].

In an unbounded domain in \( d = 3 \), there is no steady state. The system can either collapse (as in a bounded domain) or evaporate. The evaporation process has been treated in [13] and the collapse in [6, 7]. The choice between collapse or evaporation probably depends on a complicated notion of basin of attraction that is function of the initial condition.

Finally, in \( d = 1 \) the problem can be mapped on the Burgers equation which has the explicit solution (64) in an unbounded domain and (73) in a bounded domain.

7 Conclusion

In this paper, we have stressed the analogy between the chemotaxis of bacterial populations and the dynamics of self-gravitating Brownian particles. In particular, in \( d = 2 \) dimensions, we have shown that the critical mass of bacterial populations \( M_c \) is the counterpart of the critical temperature \( T_c \) of self-gravitating brownian particles. These analogies are not well-known because these topics (chemotaxis and gravity) are usually studied by very different communities and the self-gravitating Brownian gas model has been introduced only recently in physics [6]. Yet, we think that the inter-relation between these disciplines is important to develop and the present paper is a step in that direction. In particular, we have obtained the value of the critical mass by using a relation which turns out to be equivalent to the Virial theorem in astrophysics. We think that many other connections can be made between the two disciplines, and this will be considered in future works.

We have also qualitatively discussed the dynamical evolution of a self-gravitating Brownian gas (or a chemotactic system) in different dimensions of space by presenting a synthesis of the results obtained in Chavanis & Sire [6, 7, 8, 9, 10, 11, 12, 13, 14]. This gives a clear picture of the collapse dynamics of a spherically symmetric system to a single cluster. If we come back to the general problem which does not need to be spherically symmetric, we expect that several collapses will take place at different locations of the domain (if sufficiently large). Each collapse will be described by the spherical solution that we have found. But the resulting clusters will themselves have a non-trivial dynamics and will “merge” together so that their number will decrease with time until a single Dirac peak containing the whole mass remains at the end. In that case, the evolution toward the final Dirac peak is progressive. This problem shares some analogies with the dynamics of vortices in 2D decaying turbulence [45, 46] although the equations of motion are of course different. The analogy with 2D turbulence may be interesting to develop. The aggregation of clumps in our gravitational Brownian model [6] could also be studied by exploiting the analogy with the Burgers equation [11]. These are directions of investigation that we plan to explore in the future.

A A remark on the boundary conditions

When we consider the more general Keller-Segel model [3, 4, 5, 11]

\[
\frac{\partial \rho}{\partial t} = D \Delta \rho - \chi \nabla (\rho \nabla c),
\]

\[
\frac{\partial c}{\partial t} = D' \Delta c + a \rho - bc,
\]
the boundary conditions are the Neumann conditions

\begin{equation}
\nabla \rho \cdot \mathbf{n} = \nabla c \cdot \mathbf{n} = 0,
\end{equation}

where \( \mathbf{n} \) is a unit vector normal to the boundary of the box. Equation (85) describes the evolution of the concentration of the chemical. The chemical diffuses with a diffusion coefficient \( D' \), is created by the bacteria at a rate \( a \) and is degraded at a rate \( -b \). With the Neumann boundary conditions, there is no current of particles (bacteria and chemical) across the box so we have \( \rho = c = 0 \) outside the box. From Eqs. (84) and (86), we note that the average concentration of bacteria \( \overline{\rho}(t) = M/V = \overline{\rho}_0 \) is conserved. From Eqs. (85) and (86), we note that the average concentration of chemical satisfies \( d\overline{c}/dt + b\overline{c} = a\overline{\rho} \) so that \( \overline{c}(t) = (a\overline{\rho}_0/b)(1 - e^{-bt}) + \tau_0 e^{-bt} \) tending to \( \overline{c}(+\infty) = a\overline{\rho}_0/b \) at equilibrium. Following J"ager & Luckhaus [4], we set \( a = \lambda D' \) and consider the limit \( D' \to +\infty \) with \( \lambda \sim 1 \). This leads to

\begin{equation}
\frac{\partial \rho}{\partial t} = D\Delta \rho - \chi \nabla (\rho \nabla c),
\end{equation}

\begin{equation}
\Delta c = -\lambda (\rho - \overline{\rho}_0).
\end{equation}

This model is still well-posed mathematically with the Neumann boundary conditions (86). We emphasize that only the gradient of concentration \( \nabla c \) enters in Eqs. (86)-(88). Therefore, the concentration \( c \) itself is un-determined when we make the above-mentioned approximations since it is obtained only within an additive constant. When the density blows up (chemotactic collapse) so that \( \rho(r, t) \gg \overline{\rho}_0 \), it is justified to consider the model (1)-(2) where Eq. (88) is replaced by a Poisson equation. It is only in that case (large diffusivity of the chemical \( D' \to +\infty \) and high concentration \( \rho(r, t) \gg \overline{\rho}_0 \) of the bacteria) that the Keller-Segel model for the chemotaxis becomes equivalent to the Smoluchowski-Poisson system for self-gravitating Brownian particles. Some authors have studied the model (1)-(2) on general grounds, i.e. not necessarily being an approximation valid when \( \rho(r, t) \gg \overline{\rho}_0 \). We stress, however, that this model (1)-(2) is not well-posed mathematically with the Neumann boundary conditions (86). Indeed, integrating Eq. (2) and using the divergence theorem, we have \( \oint \nabla c \cdot \mathbf{n} dS = -\lambda M \neq 0 \), so that the Neumann boundary conditions (86) cannot be satisfied in that case. One way to circumvent this difficulty is to use the boundary conditions defined in Sec. 2. We must keep in mind, however, that these boundary conditions do not determine the physical concentration \( c_{\text{phys}}(r, t) \) but only a “field” \( c(r, t) \) that has the same gradient (this field can take positive or negative values). Furthermore, this field does not satisfy the requirement \( \nabla c \cdot \mathbf{n} = 0 \) so it is not clear how close it is related to the solution of the problem (86)-(88). Finally, although the problem (1)-(2) is well-posed mathematically with the boundary conditions of Sec. 2 we now find that \( c(r, t) \neq 0 \) outside the box. This is a physical problem because there is no reason why the chemical should exit the material box (unless it has porous properties). Of course, the practical solution is to consider the concentration \( c(r, t) \) inside the box only, and ignore that field outside. Another possibility is to impose \( c = 0 \) on the boundary of the domain (Dirichlet) and \( c(r, t) = 0 \) outside. In the case of self-gravitating Brownian particles, the concentration \( -c(r, t) \) is replaced by the gravitational potential \( \Phi(r, t) \). We can have \( \Phi(r, t) \neq 0 \) outside the box enclosing the particles because the gravitational field “traverses” the box (a material body enclosed within a container creates a gravitational force outside this container) so that the boundary conditions defined in Sec. 3 are natural. We conclude therefore that the reduced chemotactic model (1)-(2) is not very well-posed physically compared with the initial problem (84)-(86). By contrast, the Smoluchowski-Poisson system (22)-(23) is rigorous on a physical point of view as the gravitational potential is always solution of the Newton-Poisson equation (23) with the boundary conditions given in Sec. 3.
B  Overdamped Virial theorem for spherically symmetric systems

In this Appendix, we show that the overdamped Virial theorem (26) can be established very simply in the case of spherically symmetric systems. In that case, the moment of inertia can be expressed as a function of the mass profile in the form

\[
I(t) = \int_0^R \frac{\partial M}{\partial r}(r, t) r^2 dr = MR^2 - 2 \int_0^R M(r, t) r dr,
\]

where the second equality follows from an integration by parts. Taking the time derivative of Eq. (89), inserting Eq. (80) and using simple integrations by parts, we obtain

\[
\frac{1}{2} \xi \dot{I} = -\frac{k_B T}{m} [RM'(R, t) - dM] - \int_0^R \frac{GM(r, t)}{r^{d-2}} \frac{\partial M}{\partial r}(r, t) dr.
\]

This can be rewritten in the form

\[
\frac{1}{2} \xi \dot{I} = dNk_B T - \int_0^R \frac{GM(r, t)}{r^{d-2}} \frac{\partial M}{\partial r}(r, t) dr - dPV,
\]

where we have used \( P = k_B T \rho(R)/m \) and \( V = \frac{1}{d} S_d R^d \). Noting finally (see the Appendix of [13]) that

\[
W_{\nu} = -\int_0^R \frac{GM(r, t)}{r^{d-2}} \frac{\partial M}{\partial r}(r, t) dr,
\]

we obtain Eq. (26). In particular, for \( d = 2 \) we immediately deduce Eq. (28) from Eq. (91).

C  Explicit solution of the 2D Boltzmann-Poisson equation

In this Appendix, we explicitly solve the Boltzmann-Poisson equation (37) in \( d = 2 \) for axisymmetric solutions. The density (35) can be rewritten

\[
\rho = \rho_0 e^{\frac{\psi}{D}(c-c_0)},
\]

where \( \rho_0 \) and \( c_0 \) are the values of the density at the center of the domain. The Boltzmann-Poisson equation takes the form

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dc}{dr} \right) = -\frac{\lambda \rho_0 e^{\frac{\psi}{D}(c-c_0)}}{D}.
\]

Introducing \( \psi = -\frac{\lambda}{D}(c - c_0) \) and \( \xi = (\lambda \rho_0 / D)^{1/2} r \), we obtain

\[
\frac{1}{\xi} \frac{d}{d\xi} \left( \xi \frac{d\psi}{d\xi} \right) = e^{-\psi},
\]

\[
\psi(0) = \psi'(0) = 0.
\]
With the change of variables \( t = \ln \xi \) and \( \psi = 2 \ln \xi - z \), Eq. (95) can be rewritten
\[
\frac{d^2 z}{dt^2} = -e^z = -\frac{d}{dz}(e^z).
\]
This corresponds to the motion of a particle in a potential \( V(z) = e^z \). Using the initial condition (96) which translates into \( (z \to -\infty, dz/dt = 2) \) for \( t \to -\infty \), the first integral is
\[
\frac{1}{2} \left( \frac{dz}{dt} \right)^2 + e^z = 2.
\]
This first order differential equation is readily integrated yielding
\[
\tanh^{-1} \sqrt{1 - \frac{1}{2} e^z} = t + C.
\]
Returning to original variables, we get
\[
e^{-\psi} = \frac{8\lambda^2}{(1 + \lambda^2 \xi^2)^2},
\]
where \( \lambda \) is a constant of integration related to \( C \). It is determined by \( \psi(0) = 0 \) yielding \( 8\lambda^2 = 1 \) so we finally obtain
\[
e^{-\psi} = \frac{1}{(1 + \frac{1}{8\xi^2})^2}.
\]
Using the Gauss theorem, the total mass is given by
\[
\frac{dc}{dr}(R) = -\frac{\lambda M}{2\pi R}.
\]
Let \( \alpha = (\lambda \chi \rho_0/D)^{1/2}R \) denote the value of \( \xi \) at the edge of the box so that \( \xi = \alpha r/R \). With these notations, the foregoing relation can be rewritten
\[
\alpha \psi'(\alpha) = \frac{\lambda M \chi}{2\pi D}.
\]
Using Eq. (100) and introducing the critical mass (41), we find that this relation is equivalent to
\[
\frac{\alpha^2}{8} = \frac{M}{M_c - M}.
\]
Recalling the definition of \( \alpha \), this equation determines the relation between the mass and the central density according to
\[
\rho_0 = \frac{M_c}{\pi R^2} \frac{M/M_c}{1 - \frac{M}{M_c}}.
\]
Finally, noting that
\[
\rho = \frac{\rho_0}{(1 + \frac{\alpha^2}{8} (\frac{r}{R})^2)^2},
\]
and using Eqs. (104) and (105) we finally obtain Eq. (44). On the other hand, in an infinite domain, the Gauss theorem \( \lim_{\xi \to +\infty} \xi \psi'(\xi) = \frac{\lambda M}{2\pi D} \) and Eq. (101) imply that \( M = M_c = \frac{8\pi D}{\chi \lambda} \). Then, using Eqs. (93) and (101) we obtain Eq. (46).

The two-dimensional Boltzmann-Poisson equation (37) appeared in very different topics: self-gravitating isothermal gaseous cylinders in hydrostatic equilibrium [29, 30], statistical equilibrium states of two-dimensional stellar systems in the microcanonical ensemble [26, 31], statistical equilibrium states of two-dimensional self-gravitating Brownian particles in the canonical ensemble [7], statistical mechanics of point vortices in two-dimensional hydrodynamics [47, 33, 34, 35], chemotaxis of bacterial populations [18].
D Explicit solution of the 1D Boltzmann-Poisson equation

In this Appendix, we explicitly solve the Boltzmann-Poisson equation (37) in $d = 1$. Writing $\rho = \rho_0 e^{-\psi}$ with $\psi = \beta m (\Phi - \Phi_0)$ where $\rho_0$ is the central density and $\Phi_0$ the central potential, and introducing the scaled distance $\xi = (2G\beta m \rho_0)^{1/2} x$, the Boltzmann-Poisson equation in $d = 1$ can be written

$$\frac{d^2 \psi}{d\xi^2} = e^{-\psi},$$

with $\psi(0) = \psi'(0) = 0$. This is similar to the equation of motion of a particle in a potential $V(\psi) = e^{-\psi}$. The first integral is $E = \frac{1}{2}(d\psi/d\xi)^2 + e^{-\psi}$. Using the initial conditions we find that $E = 1$. Therefore, we get

$$\frac{d\psi}{\sqrt{2(1 - e^{-\psi})}} = d\xi.$$

This is integrated into

$$\tanh^{-1}\sqrt{1 - e^{-\psi}} = \frac{1}{\sqrt{2}} \xi,$$

and we finally obtain

$$e^{-\psi} = \frac{1}{\cosh^2(\xi/\sqrt{2})}.$$

In a infinite domain, the Gauss theorem $(d\Phi/dx)(+\infty) = GM$ in scaled variables becomes

$$\lim_{\xi \to +\infty} \psi'(\xi) = M \left(\frac{\beta G m}{2 \rho_0}\right)^{1/2}.$$

Using Eq. (110), this yields Eq. (60)-b. Then, Eq. (110) yields Eq. (59). In a finite domain, we call $\alpha = (2\beta G m \rho_0)^{1/2} R$ the value of the scaled distance $\xi$ at the box radius $R$. Then, we have $\xi = (\alpha/R) x$. The Gauss theorem $(d\Phi/dx)(R) = GM$ in scaled variables becomes

$$\alpha \psi'(\alpha) = \beta GM m R \equiv \eta.$$

Using Eq. (110), we find that

$$\eta = \sqrt{2}\alpha \tanh(\alpha/\sqrt{2}).$$

We also have by definition

$$\frac{2R\rho_0}{M} = \frac{\alpha^2}{\eta}.$$

Therefore, Eqs. (113) and (114) determine the central density in terms of the temperature. Finally, the density and mass profiles can be written

$$\rho(x) = \frac{\rho_0}{\cosh^2(\alpha x/\sqrt{2}R)}, \quad M(x) = \frac{M}{\tanh(\alpha/\sqrt{2}) \tanh(\alpha x/\sqrt{2}R)}.$$

The central density monotonically increases as the temperature decreases (see Fig. 4). At $T = 0$, the density profile is a Dirac peak: $\rho(x) = M \delta(x)$ (see Fig. 3). Noting that $\alpha = \beta G m R a/\sqrt{2}$, these results are equivalent to those of Sec. 5.2. The one-dimensional Boltzmann-Poisson equation (107) appeared in very different topics: highly flattened galactic disks [48], stellar systems stratified in plane parallel layers [49], one dimensional self-gravitating Brownian particles [7], statistical mechanics of two-dimensional turbulence in a shear layer [50].
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