Some comments on Gravitational Entropy and the Inverse Mean Curvature Flow

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Abstract

The Geroch-Wald-Jang-Huisken-Ilmanen approach to the positive energy problem may be extended to give a negative lower bound for the mass of asymptotically Anti-de-Sitter spacetimes containing horizons with exotic topologies having ends or infinities of the form $\Sigma \times \mathbb{R}$, in terms of the cosmological constant. We also show how the method gives a lower bound for the mass of time-symmetric initial data sets for black holes with vectors and scalars in terms of the mass, $|Z(Q, P)|$ of the double extreme black hole with the same charges. I also give a lower bound for the area of an apparent horizon, and hence a lower bound for the entropy in terms of the same function $|Z(Q, P)|$. This shows that the so-called attractor behaviour extends beyond the static spherically symmetric case, and underscores the general importance of the function $|Z(Q, P)|$. There are hints that higher dimensional generalizations may involve the Yamabe conjectures.

1 Introduction

Recently Huisken and Ilmanen have made mathematically rigorous an old idea of Geroch’s for proving the positive mass theorem using the inverse mean curvature flow. It was realized by Jang and Wald soon after Geroch’s suggestion that if the method could be made to work it would yield a lower bound for the mass of an asymptotically flat spacetime in terms of the area of any apparent horizon. The inverse mean curvature flow is not in general smooth and has jumps. Nevertheless Huisken and Ilmanen are able to show that despite the jumps the basic idea of Geroch goes through because the functional that he introduced is monotonic through a jump.
In this note I want to point out that Geroch’s formal argument may be extended to cover exotic black holes with non-trivial topology which occur in theories with a negative cosmological term and to cover black holes with scalar and abelian vector fields. That is locally the flow will be monotonic. It seems reasonable to believe that the rigorous methods of Huisken and Ilmanen can then be extended to this setting to give a genuine proof. Some of the results below have been known to me for many years but in the absence of a proper proof for the existence of the Geroch flow I hesitated to publish them. In addition they were of limited cosmological or astrophysical interest. Publication seems more appropriate now that a proper proof is at hand and in the light of the prominent role that these black holes play in current attempts to derive black hole entropy in terms of microstates and in applications to conformally invariant gauge theories via the AdS/CFT correspondence. It is precisely rigorous bounds on the classical entropy of initial data sets which could substantiate some of the ideas currently labelled with the name “holography”

Asymptotically flat black holes with scalar $\phi$ and vector fields carrying electric charges $Q$ and magnetic charges $P$ arise in supergravity and supergravity theories. Their properties are governed by a function $V(\phi, Q, P)$ quadratic in charges and depending on the manifold in which the scalars take their values. We define a function $Z(Q, P)$ which is the value of $\sqrt{V(\phi, Q, P)}$ at its critical point. The bounds we obtain on the entropy and mass may be expressed entirely in terms of $Z(Q, P)$. In supersymmetric theories $Z(Q, P)$ is related to the central charges of the theory but it may be defined more generally. Previous work has largely been in the context of static spherically symmetric solutions (see [8, 9] and references therein). The main point being made here is that $V(\phi, Q, P)$ and $Z(Q, P)$ retain their importance when considering time dependent and non-spherically symmetric situations. This is completely consistent with their microscopic interpretation in terms of states of D-branes. Moreover it provides a remarkable link between thermodynamic ideas and global differential geometry which may well extend much further. Some links between entropy, complexity, and gravitational action in the context of hyperbolic geometry have already been made [11, 12].

These general thermodynamic ideas extend to all dimensions. However at present the inverse mean curvature flow techniques are restricted to 3+1 dimensional physics. In the last section of this paper I indicate the difficulties one encounters. It appears that overcoming them may involve the Yamabe conjectures, a subject which has also been applied to gravitational entropy in a cosmological context [13, 14]. In fact the calculations in [13, 14] involved the entropy of the matter in a self consistent solution of the Einstein equations resulting in an Einstein static universe, $ESU_4 \equiv \mathbb{R} \times S^3$. Since $ESU$ is both the universal cover of the conformal compactification of four-dimensional Minkowski spacetime $E^{3,1}$ and the conformal boundary of five-dimensional Anti-De-Sitter spacetime $AdS_5$, it is not inconceivable that the AdS/CFT correspondence may entail the Yamabe conjectures in some fundamental way.
2 Time Symmetric Initial data sets

We shall consider three-dimensional initial data sets \( \{ \Sigma, g_{ij} \} \) for general relativity with a cosmological term. The arguments will also go through if we merely demand that the initial data set is maximal (i.e. if the trace \( K = g^{ij}K_{ij} \) of the second fundamental form \( K_{ij} \) vanishes) but since this is an entirely straightforward generalization which introduces no new features, I shall not say more about it here.

The Ricci scalar \( R \) of the initial metric is constrained to satisfy

\[
R = 2\Lambda + 16\pi T_{00}
\]  

(1)

where \( T_{00} \) is the energy density of the matter and \( \Lambda \) is the cosmological constant.

A basic model example is given by

\[
ds^2 = dr^2 \Delta + r^2 d\omega^2
\]

(2)

where \( d\omega^2 \) is the metric of constant Guassian curvature = \( k = 0, \pm 1 \) and

\[
\Delta = k - \frac{2M}{r} - \frac{\Lambda}{3} r^2.
\]

(3)

This three-metric is just that induced on the constant time hypersurfaces of the Kottler or Schwarzschild-De-Sitter static spacetime

\[
ds^2 = -\Delta dt^2 + g_{ij}dx^i dx^j.
\]

(4)

We are mainly interested in the cases when the two-surfaces of constant \( r \) are closed and orientable with genus \( g \). Thus if \( k = 1 \) we have two-spheres with \( g = 0 \), if \( k = 0 \) we have tori with \( g = 1 \) and if \( k = -1 \) we have \( g \geq 2 \). We may view this last case as \( H^2/\Gamma \) where \( H^2 \) is two-dimensional hyperbolic space and \( \Gamma \) is a suitable discrete subgroup of its isometry group \( S(2, 1) \).

The interesting new cases occur when \( \Lambda < 0 \). We then let \( \Lambda = -\frac{3}{a^2} \) and thus

\[
\Delta = \frac{r^2}{a^2} + k - \frac{2M}{r}.
\]

(5)

If \( M = 0 \) we have Anti-de-Sitter spacetime, \( AdS_4 \), or a quotient of it by a discrete group \( \Gamma \). We may think of \( AdS_4 \) as a quadric in \( E^{3,2} \):

\[
(X^0)^2 + (X^4)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = -\frac{3}{\Lambda}.
\]

(6)

The isometry group of \( AdS_4 \) is \( SO(3, 2) \). The three (locally) static forms of the metric correspond to the three different types of one-parameter subgroups of \( SO(2, 1) \subset SO(3, 2) \) acting say on the coordinates \( (X^0, X^4, X^3) \).

Thus the globally static case \( k = 1 \) corresponds to rotations in the \( X^0, X^4 \) 2-plane. The case \( k = -1 \) corresponds to boosts in the \( X^0, X^3 \), and the discrete group \( \Gamma \subset SO(2, 1) \) acting in the \( X^4, X^1, X^2 \) 3-plane. We have a killing horizon and with respect to this Killing field \( AdS_4 \) has temperature \( \frac{1}{2\pi} \sqrt{-\frac{3}{\Lambda}} \).
The case $k = 0$ corresponds to null rotations and there is a degenerate Killing horizon. If one identifies the horizon to get a torus the identifications are also null rotations. However these identifications do not act freely on the horizon and introduce orbifold singularities there.

Now if we pass to the the Kottler solution when $M \neq 0$ we find that if $k \geq 0$ then $M > 0$ guarantees the existence of a regular apparent horizon of area $A = 4\pi r_H^2$ where $r_H$ is the outermost root of $\Delta(r_H) = 0$. In fact

$$2M = \frac{r_H^3}{a^2} + kr_H. \tag{7}$$

If $k = 0$ and $M = 0$ then the spatial section $\Sigma$ is complete and has a cusp at at $r = 0$ which is an infinite spatial distance. The static Kottler spacetime has a degenerate Killing horizon at the cusp in this case. If $k = -1$ we have

$$2M = \frac{r_H^3}{a^2} + kr_H. \tag{8}$$

Thus $M$ can be negative but not too negative. If

$$M > \frac{1}{\sqrt{27a^2}} \tag{9}$$

then we have a regular apparent horizon of area $A = 4\pi|1 - g|r_H^2$. The case of equality

$$M = \frac{1}{\sqrt{27a^2}} \tag{10}$$

gives a cusp at

$$r = \frac{1}{\sqrt{3a^2}}. \tag{11}$$

These observations suggest a general lower bound, valid for all values of $k$ of the form

$$2M \geq \frac{r_H^3}{a^2} + kr_H. \tag{12}$$

We shall shortly argue that precisely this lower bound is obtainable form the inverse mean curvature flow. In particular this seems to imply that the negative mass topologically non-trivial black hole are classically stable. Moreover the degenerate zero temperature limiting solution should be quantum mechanically stable even though it is not a BPS state, that is, it has no Killing spinors and hence admits no supersymmetry.

3 The Geroch flow

The argument initiated by Geroch [1], extended by Wald and Jang [2] and completed by Ilmanen and Huisken [4, 5] goes roughly as follows. One considers
a family of level sets $S_s$ which are smoothly immersed two-surfaces each with metric $h_{ab}$, second fundamental form $F_{ab}p_{ab}$ of area $A(s)$ mean curvature $p = h^{ab}p_{ab}$ and Gaussian curvature $K_G$ evolving in $s$ according to the inverse mean curvature flow equation so the velocity $v_n$ of the surface $S_s$ along its normal $n$ is given by

$$v_n = \frac{1}{p}. \quad (13)$$

It follows that

$$\frac{dA}{ds} = A. \quad (14)$$

One associates with each surface the function:

$$f(s) = \int_{S_s} (4K_G - p^2 - \frac{4\Lambda}{3})dA \quad (15)$$

and finds that

$$\frac{d(fA^{\frac{1}{2}})}{ds} \geq A^{\frac{1}{2}} \int_{S_s} 16\pi T_{00}. \quad (16)$$

Equality is possible if and only if the trace free part of the extrinsic curvature of the surfaces $S(s)$ vanishes and $p$ is constant on the surfaces. Now start the flow from the outermost apparent horizon (which of course may not be connected) on which $p = 0$. One assumes that the flow reaches infinity near which the metric behaves like the model example. If the surfaces tending to $r = \text{constant}$ surfaces, one finds that

$$\lim fA^{\frac{1}{2}} = 64\pi^{\frac{2}{3}} M. \quad (17)$$

If the matter energy density $T_{00}$ is non-negative one may integrate the inequality to get

$$64\pi^{\frac{2}{3}} M \geq fA^{\frac{1}{2}}_{H}. \quad (18)$$

One now evaluates $fA^{\frac{1}{2}}$ on the apparent horizon which in this case will have vanishing mean curvature, $p = 0$ to obtain the desired inequality.

Actually the term in $f$ involving the cosmological constant was first introduced in [6] and the results of [5] may need extending to cover this case. I shall assume in what follows that that is possible.

### 4 Charged Black Holes

If the energy density takes a particular form then stronger inequalities may be obtained. Thus in Einstein-Maxwell theory the model metric is the Reissner-Nordstrom-De-Sitter for which

$$\Delta = k - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{Z^2}{r^2}. \quad (19)$$

where

$$Z^2 = Q^2 + P^2. \quad (20)$$
and \( P \) and \( Q \) are the electric and magnetic charges. One expects
\[
2M \geq \frac{r_H^3}{a^2} + kr_H + \frac{Z^2}{r_H^2}.
\] (21)
To obtain this one follows Jang \[3\] and notes that
\[
T_{00} = \frac{1}{8\pi}(E^2 + B^2),
\] (22)
where \( E \) and \( B \) are the electric and magnetic fields. They are divergence free with respect to the metric \( g_{ij} \) on \( \Sigma \):
\[
\nabla \cdot E = 0 = \nabla \cdot B.
\] (23)
The electric charge \( Q \) inside a level set is is given by
\[
Q = \frac{1}{4\pi} \int_{S_s} E_n dA
\] (24)
where \( E_n \) is the normal component of the electric field \( E \) and there is a similar expression for the magnetic charge \( P \).

Now
\[
T_{00} \geq \frac{1}{8\pi}(E_n^2 + B_n^2),
\] (25)
and use of the Schwartz Inequality gives:
\[
\int_{S_s} 16\pi T_{00} \geq 32\pi^2 \frac{Z^2}{A(s)}.
\] (26)
One now integrates the inequality to get the desired result.

5 Entropy and Attractors

In this section we shall extend the argument above to the case of gravity coupled to scalars and vectors. In this way we shall vindicate the claim made in \[8\] that the mass of a black hole with given charges is never less than the double extreme hole with the same charges. For more details about attractors, double extreme holes etc and references to the earlier literature the reader is directed to \[8, 9\].

Consider a theory with matter lagrangian
\[
L = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{16\pi} e^{-2\phi} F^2.
\] (27)
In general the scalar \( \phi \) will take a particular limiting value \( \phi^\infty \) at infinity and will vary over the interior. For static extreme holes it must take a particular value, called in the sequel \( \phi^{\text{frozen}} \) on the horizon. Static extreme black holes for which \( \phi \) takes this value everywhere are said to be double extreme and the scalar is said to be frozen. They have the same geometry as the extreme Reissner-Nordstrom
black hole. The mass of any regular static black hole with give charges \((Q, P)\) is never smaller than the value it takes \(|Z(q, p)| = \sqrt{2|QP|}\) for the double extreme hole with those charges. Since the area of the horizon of a double extreme hole is \(A = 2\pi|PQ|\) One expects that this should be a general lower bound for any time symmetric initial data. In fact we shall show that

\[
2M \geq r_H^3 + r_H + \frac{Z^2}{r_H^2}.
\]  

(28)

A simple calculation reveals that in the time symmetric case

\[
T_{00} = \frac{1}{2}(\nabla \phi)^2 + \frac{1}{8\pi}e^{-2\phi}(E^2 + B^2).
\]  

(29)

However now

\[
\nabla \cdot B = 0 = \nabla \cdot D.
\]  

(30)

where the electric induction \(D\) is given by

\[
D = e^{-2\phi}E.
\]  

(31)

Moreover now

\[
Q = \frac{1}{4\pi} \int_{S_s} D_n dA
\]  

(32)

where \(D_n\) is the normal component of the electric induction \(D\). The expression for the magnetic charge remains unchanged.

An application of the Schwartz inequality now gives

\[
\int_{S_s} 16\pi T_{00} \geq \frac{1}{A^2(s)} \int_{S_s} (Q^2 e^{2\phi} + P^2 e^{-2\phi}) dA.
\]  

(33)

Let’s define:

\[
V(\phi, Q, P) = (Q^2 e^{2\phi} + P + 2e^{-2\phi}).
\]  

(34)

At fixed charges \((Q, P)\), both assumed to be non-vanishing, the function \(V(\phi, Q, P)\) attains its least value \(Z^2(Q, P) = 2|QP|\) at the so-called frozen value \(\phi = \phi^{\text{frozen}}\) given by

\[
\phi^{\text{frozen}} = \frac{1}{2} \ln |P/Q|.
\]  

(35)

Thus

\[
\int_{S_s} 16\pi T_{00} \geq 32\pi^2 \frac{Z^2(Q, P)}{A(s)}.
\]  

(36)

The result now follows as in the Resissner-Nordstrom case.

We have chosen as simple example but it is clear that it extends to include the general class of theories with any number of abelian vector and scalar fields coupled to gravity with an action which is quadratic in the vectors.
6 The second variation

To conclude, we turn to the information about the area of a stable minimal 2-surface in a time symmetric slice which comes from the second variation of the area, i.e. the Hessian of the area functional, and demanding that it be positive. The method is standard. Its application in the electrovac case goes back at least to [16]. For a one parameter variations $S_t = S + h n^i$ by an amount $h$ along the normals $n^i$ one has:

$$\frac{d^2A}{dt^2} = -\int_S h^2 \left[ 3 R_{ij} n^i n^j - 2 \sigma^2 \right] + \int_S |\nabla h|^2$$ \hspace{1cm} (37)

where the integral is taken over the minimal surface $S$ and $\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab}$ is the magnitude of the trace free part of the second fundamental form $\sigma_{ab}$ of the minimal surface $S$. The Gauss-Codazzi equation gives

$$2 K_G = R - 2 R_{ij} n^i n^j - 2 \sigma^2.$$ \hspace{1cm} (38)

Therefore:

$$\frac{d^2A}{dt^2} = \int \left[ |\nabla h|^2 + K_G h^2 - \frac{1}{2} R h^2 - \sigma^2 h^2 \right].$$ \hspace{1cm} (39)

If $h$ is taken to be constant over the surface and we use the Gauss-Bonnet theorem then we must have:

$$4\pi (1 - g) - \Lambda A - \int_S 8\pi T_{00} > 0,$$ \hspace{1cm} (40)

where the integral is taken over the minimal surface $S$.

If the cosmological constant is positive and the positive energy condition holds we see a stable minimal surface must have spherical topology, an old result. Moreover it’s area must exceed $4\pi$. If the cosmological constant is positive there will be a cosmological horizon. This is not stable since its area decreases if it is moved uniformly outward or inward. If that is the only negative mode for the second variation one says that the surface is of index one. Thus the second variation should be positive if

$$\int_S h = 0.$$ \hspace{1cm} (41)

A spectral argument explained to me by S.T. Yau gives the the following lower bound for a surface of genus $g$

$$\int_S \left[ |\nabla h|^2 + K_G h^2 \right] / \int_S h^2 \leq 4\pi (3 - g) / A.$$ \hspace{1cm} (42)

Thus if $g = 0$ we get:

$$A_C \leq \frac{12\pi}{\Lambda}.$$ \hspace{1cm} (43)
This upper bound also follows from Geroch’s method. If the inverse mean curvature flow starts from a black hole horizon of area $A_H$ and reaches a cosmological horizon of area $A_C$, one obtains:

$$\sqrt{A_H} \left( \frac{12}{A} - A_H \right) \leq \sqrt{A_C} \left( \frac{12}{A} - A_C \right).$$  \hspace{1cm} (44)

(see [15] for a recent discussion of related results for positive cosmological constant).

If the cosmological constant is negative and $k = -1$ and the positive energy condition holds then the second variation yields a lower bound:

$$A > \frac{4\pi(g - 1)}{|\Lambda|}.$$  \hspace{1cm} (45)

Thus there is a universal topology dependent lower bound for the entropy of a topological black hole.

If we are dealing with scalars and vectors, as in the previous section we know that:

$$\int_S T_{00} \geq 2\pi^2 Z^2 (Q, P) A.$$  \hspace{1cm} (46)

This allows a strengthening of the bounds above. Consider the case of black holes with spherical topology. If the cosmological constant is zero then we have a lower bound for the entropy with fixed charges, valid for any time-symmetric initial data set:

$$A > 4\pi Z^2.$$  \hspace{1cm} (47)

Again, this substantiates and extends to the non-spherically symmetric case the claims made in [8].

The results of this and previous sections strongly support the idea that

$$S = \pi Z^2$$  \hspace{1cm} (48)

is the irreducible amount of entropy of a black hole with fixed charges $(Q, P)$. This irreducible amount of entropy is only attained for extreme black holes. Any additional gravitational fields can only increase it. This is consistent with the microscopic derivations from D-brane theory but still leaves open the question of the origin of the extra contribution to the entropy in the case of non-extreme states. Microscopically this may be due to excited states of D-branes but the extrapolation to macroscopic geometric configurations of classical initial data in the non-BPS situations is not all all obvious.

### 7 The Horowitz-Myers conjecture

One motivation for this present work was to see if one might use the inverse mean curvature flow to tackle an interesting conjecture of Horowitz and Myers [10]. This was made in the context of the $AdS_5/CFT$ correspondence which
concerns’s 4+1 dimensional spacetimes. Thus the mean curvature flow is not immediately applicable. However there is a version in \( p + 1 \) dimensions for all \( p \), including \( p = 3 \). Unfortunately it turns out that the mean curvature flow is not applicable in that case either. To see why, we note that the Horowitz-Myer’s example is obtained by interchanging the role of time with one of the torus coordinates in the \( k = 0 \) Kottler metrics:

\[
ds^2 = -\frac{v^2}{a^2} dt^2 + \frac{dr^2}{\left(\frac{v^2}{a^2} - \frac{r_0^4}{a^2 r} \right)} + \left(\frac{r^2}{a^2} - \frac{r_0^4}{a^2 r} \right) dx^2 + \frac{r^2}{a^2} dy^2. \tag{49}
\]

Regularity demands that we identify \( x \) with period \( L_x = \frac{4\pi a^2}{3r_0} \). The period of the coordinate \( y \), call it \( L_y \) is arbitrary. The initial data set thus has topology \( \mathbb{R}^2 \times S^1 \). Note that the spacetime has no event horizon. Horowitz and Myers’ show that the ADM mass (obtained by comparing with the case \( r_0 = 0 \)) is negative and equals

\[-\frac{4\pi^2 L_y a^2}{27 L_x^2}. \tag{50}\]

They give evidence to support the conjecture that this is a lower bound for the ADM mass of such data sets, specified by \( L_x \) and \( L_y \). As in the case of the negative mass \( k = 1 \) black holes this suggestion is striking because the lower bound is attained for a non-BPS state.

The difficulty with applying the inverse mean curvature flow is that because the spacetime has no event horizon, the initial data set need not have, and indeed doesn’t have, an apparent horizon. Thus there is no natural 2-surface from which to start the flow. There is a minimal geodesic of length \( r_0 L_y \) at \( r = r_0 \) but this is no good because if we consider a small tube of radius \( r \) surrounding the geodesic the function \( f \) would tend to minus infinity like \( r^{-1} \) as we shrunk the tube onto the geodesic. Since the area of the tube shrinks as \( r \) the product \( \sqrt{A f} \) diverges like \( r^{-\frac{1}{2}} \). The monotonicity property thus seems to give nothing interesting. On the other had if we were to start the flow from a very small sphere we would, assuming that the flow reaches infinity, merely obtain the result that the quantity \( r_0 \) is positive.

8 Cosmic Censorship versus Bogomol’nyi

If \( \Lambda < 0 \) and \( k = 1 \) we obtain the Cosmic Censorship lower bound:

\[ M \geq r_H + \frac{r_H^2}{a^2} + \frac{Z^2}{r_H}. \tag{51} \]

By extremizing with respect to \( r_H \) we find, after some elementary algebra and manipulation of surds, the following interesting the lower bound for \( M \) in terms of \( |Z| \) and

\[ M \geq \sqrt{\frac{2}{3}} |Z| \left[ \sqrt{1 + \sqrt{1 - 4\Lambda Z^2}} + \frac{1}{\sqrt{1 + \sqrt{1 - 4\Lambda Z^2}}} \right]. \tag{52} \]
Of course in the limit that \( \Lambda \to 0 \) we recover the familiar Bogomol’nyi bound:

\[
M \geq |Z|, \tag{53}
\]

but it is clear that the Cosmic Censorship lower bound \([52]\) is strictly greater than the Bogomolnyi lower bound \([53]\) if \( \Lambda < 0 \).

9 Higher dimensions

As originally formulated the Geroch technique works only for three spatial dimensional initial data sets. However there is an analogue of the positive mass theorem in all higher dimensions. This has been proved by Schoen and Yau and by Witten. A reformulation of Geroch’s idea designed to see how this might work in higher dimensions (shown to me by Douglas Eardley in 1980) goes as follows: One considers a foliation of an \( n \)-dimensional manifold of the form:

\[
ds^2 = e^{2\psi}dr^2 + r^2h_{ab}(x,r)dx^adx^b, \tag{54}
\]

where

\[
\frac{\partial}{\partial r}\det h_{ab} = 0, \tag{55}
\]

Thus the trace of the second fundamental form is

\[
p = \frac{n-1}{r}. \tag{56}
\]

Let

\[
q_{ab} = \frac{1}{2}e^{-\psi}\frac{\partial}{\partial r}h_{ab} \tag{57}
\]

so that

\[
q_{ab} = \frac{1}{r^2}(p_{ab} - \frac{1}{n-1}h_{ab}p) \tag{58}
\]

The Ricci scalar is given by

\[
n^R = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}[(n-1)r^{n-2}(1-e^{-2\psi})] - \frac{2}{r^2}e^{-\psi}D_aD_\psi - q_{ab}q^{ab} + \frac{1}{r^2}[n^{-1}R - (n-1)(n-2)]. \tag{59}
\]

where \( n^{-1}R \) is the scalar curvature of the the level sets \( r = \text{constant} \).

One may now multiply this equation by \( r^{n-1}\sqrt{\det h_{ab}}dr^{n-1}x \) and integrate to get the relevant identities. One assumes that the outermost apparent horizon is located at \( r = r_H \) on which \( \psi = 0 \). If the metric is asymptotically flat, it is convenient to normalize such that

\[
\int_{S_r} \sqrt{\det h_{ab}}d^{n-1}x = \text{vol}(S^{n-1}), \tag{60}
\]

where \( \text{vol}(S^{n-1}) \) is the volume of a standard round sphere of unit radius. Then at large distances

\[
\psi \sim \frac{a}{2r^{n-2}} \tag{61}
\]
where the total mass $M$ is given by

$$M = \frac{(n-1)\text{vol}(S^{n-1})}{16\pi} a$$

We get the integral inequality

$$a \geq r_H^{n-2} + \int F r^{n-3} \, dr$$

where

$$F(r) = \int_{S_r} \sqrt{\det h_{ab}} d^{n-1}x [r^{n-1} R - (n-2)(n-1)].$$

To make the theorems work we need this to be non-positive. If $n = 3$ this follows from the Gauss-Bonnet theorem. For $n > 3$ the situation is less clear. The integrand vanishes for a round sphere and this fact leads to some obvious positivity properties if we assume that the levels sets are the orbits of an $SO(n)$ isometry group. If the metric is not $SO(n)$-invariant however we have, without further information, little control of the integrand. The quantity $F$ is of course related to the Yamabe constant. The fact that this has already arisen in the context of entropy in cosmology [13, 14] is striking.

Note that to derive inequalities from the second variation we needed to use the Gauss-Bonnet theorem. In higher dimensions what appears in the second variation is the quantity

$$\int_S R$$

where $R$ is the Ricci scalar of the apparent horizon. If the comological constant is non-negative this must be postive. In fact the Ricci scalar should probably be point wise positive. This would place some topological restrictions on the topology of the apparent horizon.

Thus it seems that the only real evidence for a bound to the area of apparent horizons comes from the higher dimensional versions of the collapsing shell calculations [17].

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