ON THE STRUCTURE OF CALABI-YAU CATEGORIES WITH A CLUSTER TILTING SUBCATEGORY

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Abstract. We prove that for $d \geq 2$, an algebraic $d$-Calabi-Yau triangulated category endowed with a $d$-cluster tilting subcategory is the stable category of a DG category which is perfectly $(d+1)$-Calabi-Yau and carries a non-degenerate $t$-structure whose heart has enough projectives.

Contents

1. Introduction 1
2. Acknowledgments 2
3. Preliminaries 2
4. Embedding 3
5. Determination of the image of $G$ 6
6. Alternative description 10
7. The main theorem 12
Appendix A. Extension of $t$-structures 14
References 17

1. Introduction

In this article, we propose a description of a class of Calabi-Yau categories using the formalism of DG-categories and the notion of ‘stabilization’, as used for the description of triangulated orbit categories in section 7 of [10]. For $d \geq 2$, let $C$ be an algebraic $d$-Calabi-Yau triangulated category endowed with a $d$-cluster tilting subcategory $T$, cf. [18] [13] [14]. Such categories occur for example,

- in the representation-theoretic approach to Fomin-Zelevinsky’s cluster algebras [9], cf. [5] [6] [10] and the references given there,
- in the study of Cohen-Macaulay modules over certain isolated singularities, cf. [12] [18] [11], and the study of non commutative crepant resolutions [28], cf. [12].

From $C$ and $T$ we construct an exact dg category $B$, which is perfectly $(d+1)$-Calabi-Yau, and a non-degenerate aisle $U$, cf. [20], in $H^0(B)$ whose heart has enough projectives. We prove, in theorem 7.1 how to recover the category $C$ from $B$ and $U$ using a general procedure of stabilization defined in section 7. This extends previous results of [19] to a more general framework. It follows from [24] that for $d = 2$, up

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to derived equivalence, the category $\mathcal{B}$ only depends on $\mathcal{C}$ (with its enhancement) and not on the choice of $\mathcal{T}$. In the appendix, we show how to naturally extend a $t$-structure, cf. [2], on the compact objects of a triangulated category to the whole category.

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3. Preliminaries

Let $k$ be a field. Let $\mathcal{E}$ be a $k$-linear Frobenius category with split idempotents. Suppose that its stable category $\mathcal{C} = \mathcal{E}$, with suspension functor $S$, has finite-dimensional Hom-spaces and admits a Serre functor $\Sigma$, see [3]. Let $d \geq 2$ be an integer. We suppose that $\mathcal{C}$ is Calabi-Yau of CY-dimension $d$, i.e. [22] there is an isomorphism of triangle functors

$$S^d \cong \Sigma.$$ 

We fix such an isomorphism once and for all. See section 4 of [18] for several examples of the above situation.

For $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we put

$$\text{Ext}^n(X, Y) = \text{Hom}_\mathcal{C}(X, S^n Y).$$

We suppose that $\mathcal{C}$ is endowed with a $d$-cluster tilting subcategory $\mathcal{T} \subset \mathcal{C}$, i.e.

a) $\mathcal{T}$ is a $k$-linear subcategory,

b) $\mathcal{T}$ is functorially finite in $\mathcal{C}$, i.e. the functors $\text{Hom}_\mathcal{C}(?, X)|_{\mathcal{T}}$ and $\text{Hom}_\mathcal{C}(X, ?)|_{\mathcal{T}}$ are finitely generated for all $X \in \mathcal{C}$,

c) we have $\text{Ext}^i(T, T') = 0$ for all $T, T' \in \mathcal{T}$ and all $0 < i < d$ and

d) if $X \in \mathcal{C}$ satisfies $\text{Ext}^i(T, X) = 0$ for all $0 < i < d$ and all $T \in \mathcal{T}$, then $T$ belongs to $\mathcal{T}$.

Let $\mathcal{M} \subset \mathcal{E}$ be the preimage of $\mathcal{T}$ under the projection functor. In particular, $\mathcal{M}$ contains the subcategory $\mathcal{P}$ of the projective-injective objects in $\mathcal{M}$. Note that $\mathcal{T}$ equals the quotient $\mathcal{M}$ of $\mathcal{M}$ by the ideal of morphisms factoring through a projective-injective.

We dispose of the following commutative square:

$$\begin{array}{ccc}
\mathcal{M} & \subseteq & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{T} & \subseteq & \mathcal{E} = \mathcal{C}.
\end{array}$$

We use the notations of [15]. In particular, for an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ (resp. $\mathcal{C}^-(\mathcal{A})$, $\mathcal{C}^0(\mathcal{A})$, . . .) the category of unbounded (resp. right bounded, resp. bounded, . . .) complexes over $\mathcal{A}$ and by $\mathcal{H}(\mathcal{A})$ (resp. $\mathcal{H}^-(\mathcal{A})$, $\mathcal{H}^0(\mathcal{A})$, . . .) its quotient modulo the ideal of nullhomotopic morphisms. By [21], cf. also [25], the projection functor $\mathcal{E} \to \mathcal{E}$ extends to a canonical triangle functor $\mathcal{H}^b(\mathcal{E})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{E}$. This induces a triangle functor $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{E}$. It is shown in [24] that this functor is a localization functor. Moreover, the projection functor $\mathcal{H}^b(\mathcal{M}) \to$
\( \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \) induces an equivalence from the subcategory \( \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \) of bounded \( \mathcal{E} \)-acyclic complexes with components in \( \mathcal{M} \) onto its kernel. Thus, we have a short exact sequence of triangulated categories

\[
0 \to \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{C} \to 0.
\]

Let \( \mathcal{B} \) be the dg (=differential graded) subcategory of the category \( \mathcal{C}^b(\mathcal{M})_{dg} \) of bounded complexes over \( \mathcal{M} \) whose objects are the \( \mathcal{E} \)-acyclic complexes. We denote by \( G : \mathcal{H}^-\mathcal{M} \to \mathcal{D}(\mathcal{B}^{op})^{op} \) the functor which takes a right bounded complex \( X \) over \( \mathcal{M} \) to the dg module

\[
B \mapsto \text{Hom}^\bullet_{\mathcal{M}}(X, B),
\]

where \( B \) is in \( \mathcal{B} \).

**Remark 3.1.** By construction, the functor \( G \) restricted to \( \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \) establishes an equivalence

\[
G : \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \xrightarrow{\sim} \text{per}(\mathcal{B}^{op})^{op}.
\]

Recall that if \( P \) is a right bounded complex of projectives and \( A \) is an acyclic complex, then each morphism from \( P \) to \( A \) is nullhomotopic. In particular, the complex \( \text{Hom}^\bullet_{\mathcal{M}}(P, A) \) is nullhomotopic for each \( P \) in \( \mathcal{H}^{-}(\mathcal{P}) \). Thus \( G \) takes \( \mathcal{H}^{-}(\mathcal{P}) \) to zero, and induces a well defined functor (still denoted by \( G \))

\[
G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}(\mathcal{B}^{op})^{op}.
\]

### 4. Embedding

**Proposition 4.1.** The functor \( G \) is fully faithful.

For the proof, we need a number of lemmas.

It is well-known that the category \( \mathcal{H}^{-}(\mathcal{E}) \) admits a semiorthogonal decomposition, cf. [4], formed by \( \mathcal{H}^{-}(\mathcal{P}) \) and its right orthogonal \( \mathcal{H}_{\mathcal{E}-ac}(\mathcal{E}) \), the full subcategory of the right bounded \( \mathcal{E} \)-acyclic complexes. For \( X \) in \( \mathcal{H}^{-}(\mathcal{E}) \), we write

\[
pX \to X \to a_pX \to SpX
\]

for the corresponding triangle, where \( pX \) is in \( \mathcal{H}^{-}(\mathcal{P}) \) and \( \mathcal{H}^-_{\mathcal{E}-ac}(\mathcal{E}) \). If \( X \) lies in \( \mathcal{H}^{-}(\mathcal{M}) \), then clearly \( \mathcal{H}^-_{\mathcal{E}-ac}(\mathcal{M}) \) so that we have an induced semiorthogonal decomposition of \( \mathcal{H}^{-}(\mathcal{M}) \).

**Lemma 4.1.** The functor \( \mathcal{U} : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{H}^-_{\mathcal{E}-ac}(\mathcal{M}) \) which takes \( X \) to \( a_pX \) is fully faithful.

**Proof.** By the semiorthogonal decomposition of \( \mathcal{H}^{-}(\mathcal{M}) \), the functor \( X \mapsto a_pX \) induces a right adjoint of the localization functor

\[
\mathcal{H}^{-}(\mathcal{M}) \to \mathcal{H}^{-}(\mathcal{M})/\mathcal{H}^{-}(\mathcal{P})
\]
and an equivalence of the quotient category with the right orthogonal $\mathcal{H}_{\mathcal{E}-ac}(\mathcal{M})$.

Moreover, it is easy to see that the canonical functor

$$\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P})$$

taking $X$ to $a_p X$.

**Remark 4.1.** Since the functor $G$ is triangulated and takes $\mathcal{H}^-(\mathcal{P})$ to zero, for $X$ in $\mathcal{H}^b(\mathcal{M})$, the adjunction morphism $X \to a_p X$ yields an isomorphism

$$G(X) \sim G(a_p X) = G(\mathcal{Y}X).$$

Let $\mathcal{D}^{-\mathcal{M}}(\mathcal{M})$ be the full subcategory of the derived category $\mathcal{D}(\mathcal{M})$ formed by the right bounded complexes whose homology modules lie in the subcategory $\text{Mod}^{-\mathcal{M}}$ of $\text{Mod}^{\mathcal{M}}$. The Yoneda functor $\mathcal{M} \to \text{Mod}^{\mathcal{M}}$, $\mathcal{M} \mapsto \mathcal{M}^\wedge$, induces a full imbedding

$$\Psi : \mathcal{H}_{\mathcal{E}-ac}(\mathcal{M}) \hookrightarrow \mathcal{D}^{-\mathcal{M}}(\mathcal{M}).$$

We write $V$ for its essential image. Under $\Psi$, the category $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ is identified with $\text{per}^{-\mathcal{M}}(\mathcal{M})$. Let $\Phi : \mathcal{D}^{-\mathcal{M}}(\mathcal{M}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ be the functor which takes $X$ to the dg module

$$B \mapsto \text{Hom}^\bullet(X_c, \Psi(B)),$$

where $B$ is in $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ and $X_c$ is a cofibrant replacement of $X$ for the projective model structure on $C(\mathcal{M})$. Since for each right bounded complex $\mathcal{M}$ with components in $\mathcal{M}$, the complex $\mathcal{M}^\wedge$ is cofibrant in $C(\mathcal{M})$, it is clear that the functor $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ is isomorphic to the composition $\Phi \circ \Psi \circ \mathcal{Y}$. We dispose of the following commutative diagram

**Lemma 4.2.** Let $Y$ be an object of $\mathcal{D}^{-\mathcal{M}}(\mathcal{M})$. 
a) $Y$ lies in $\text{per}_\mathcal{M}(\mathcal{M})$ iff $H^p(Y)$ is a finitely presented $\mathcal{M}$-module for all $p \in \mathbb{Z}$ and vanishes for all but finitely many $p$.

b) $Y$ lies in $\mathcal{V}$ iff $H^p(Y)$ is a finitely presented $\mathcal{M}$-module for all $p \in \mathbb{Z}$ and vanishes for all $p \gg 0$.

**Proof.**
a) Clearly the condition is necessary. For the converse, suppose first that $Y$ is a finitely presented $\mathcal{M}$-module. Then, as an $\mathcal{M}$-module, $Y$ admits a resolution of length $d + 1$ by finitely generated projective modules by theorem 5.4 b) of [13]. It follows that $Y$ belongs to $\text{per}_\mathcal{M}(\mathcal{M})$. Since $\text{per}_\mathcal{M}(\mathcal{M})$ is triangulated, it also contains all shifts of finitely presented $\mathcal{M}$-modules and all extensions of shifts. This proves the converse.

b) Clearly the condition is necessary. For the converse, we can suppose without loss of generality that $Y^n = 0$, for all $n \geq 1$ and that $Y^n$ belongs to $\text{proj} \mathcal{M}$, for $n \leq 0$. We now construct a sequence

$$
\cdots \to P_n \to \cdots \to P_1 \to P_0
$$

of complexes of finitely generated projective $\mathcal{M}$-modules such that $P_n$ is quasi-isomorphic to $\tau_{\geq -n}Y$ for each $n$ and that, for each $p \in \mathbb{Z}$, the sequence of $\mathcal{M}$-modules $P_n^p$ becomes stationary. By our assumptions, we have $\tau_{\geq 0}Y \sim H^0(Y)$. Since $H^0(Y)$ belongs to $\text{mod} \mathcal{M}$, we know by theorem 5.4 c) of [13] that it belongs to $\text{per}(\mathcal{M})$ as an $\mathcal{M}$-module. We define $P_0$ to be a finite resolution of $H^0(Y)$ by finitely generated $\mathcal{M}$-modules. For the induction step, consider the following truncation triangle associated with $Y$

$$S^{i+1}H^{i-1}(Y) \to \tau_{\geq -i-1}Y \to \tau_{\geq -i}Y \to S^{i+2}H^{i-1}(Y),$$

for $i \geq 0$. By the induction hypothesis, we have constructed $P_0, \ldots, P_i$ and we dispose of a quasi-isomorphism $P_i \sim \tau_{\geq -i}Y$. Let $Q_{i+1}$ be a finite resolution of $S^{i+2}H^{i-1}(Y)$ by finitely presented projective $\mathcal{M}$-modules. We dispose of a morphism $f_i : P_i \to Q_{i+1}$ and we define $P_{i+1}$ as the cylinder of $f_i$. We define $P$ as the limit of the $P_i$ in the category of complexes. We remark that $Y$ is quasi-isomorphic to $P$ and that $P$ belongs to $\mathcal{V}$. This proves the converse.

Let $X$ be in $\mathcal{H}_{\mathcal{E}, \text{ac}}(\mathcal{M})$.

**Remark 4.2.** Lemma [13] shows that the natural $t$-structure of $D(\mathcal{M})$ restricts to a $t$-structure on $\mathcal{V}$. This allows us to express $\Psi(X)$ as

$$\Psi(X) \sim \operatorname{holim}_i \tau_{\geq -i} \Psi(X),$$

where $\tau_{\geq -i} \Psi(X)$ is in $\text{per}_\mathcal{M}(\mathcal{M})$.

**Lemma 4.3.** We dispose of the following isomorphism

$$\Phi(\Psi(X)) = \Phi(\operatorname{holim}_i \tau_{\geq -i} \Psi(X)) \sim \operatorname{holim}_i \Phi(\tau_{\geq -i} \Psi(X)).$$

**Proof.** It is enough to show that the canonical morphism induces a quasi-isomorphism when evaluated at any object $B$ of $\mathcal{B}$. We have

$$\Phi(\operatorname{holim}_i \tau_{\geq -i} \Psi(X))(B) = \operatorname{Hom}^\bullet(\operatorname{holim}_i \tau_{\geq -i} \Psi(X), B),$$

but since $B$ is a bounded complex, for each $n \in \mathbb{Z}$, the sequence

$$i \mapsto \operatorname{Hom}^n(\tau_{\geq -i} \Psi(X), B)$$
stabilizes as \( i \) goes to infinity. This implies that

\[
\text{Hom}^\bullet(\text{holim}_i \tau_{\geq -i} \Psi(X), B) \xrightarrow{\sim} \text{holim}_i \Phi(\tau_{\geq -i} \Psi(X))(B).
\]

\[\check{\square}\]

**Lemma 4.4.** The functor \( \Phi \) restricted to the category \( \mathcal{V} \) is fully faithful.

**Proof.** Let \( X, Y \) be in \( \mathcal{H}^{-}_{\text{ac}}(\mathcal{M}) \). The following are canonically isomorphic:

\[
\begin{align*}
\text{Hom}_{\mathcal{D}(B^{\text{op}})^{\text{op}}}(\Phi \Psi X, \Phi \Psi Y) \\
\text{Hom}_{\mathcal{D}(B^{\text{op}})}(\Phi \Psi Y, \Phi \Psi X) \\
(4.1) \quad \text{Hom}_{\mathcal{D}(B^{\text{op}})}(\text{holim}_i \Phi \tau_{\geq -i} \Psi Y, \text{holim}_j \Phi \tau_{\geq -j} \Psi X) \\
\text{holim}_i \text{holim}_j \text{Hom}_{\mathcal{D}(B^{\text{op}})}(\Phi \tau_{\geq -i} \Psi Y, \Phi \tau_{\geq -j} \Psi X) \\
(4.2) \quad \text{holim}_i \text{holim}_{\text{per}_M(\mathcal{M})}(\tau_{\geq -j} \Psi X, \tau_{\geq -i} \Psi Y) \\
\text{holim}_i \text{holim}_{\text{per}_M(\mathcal{M})}(\tau_{\geq -j} \Psi X, \tau_{\geq -i} \Psi Y) \\
(4.3) \quad \text{holim}_{\text{per}_M(\mathcal{M})}(\tau_{\geq -j} \Psi X, \tau_{\geq -i} \Psi Y) \\
\text{Hom}_{\mathcal{V}^{\text{op}}}(\Phi(X), \Phi(Y)).
\end{align*}
\]

Here (4.1) is by the lemma 4.3 seen in \( \mathcal{D}(B^{\text{op}}) \), (4.2) is by the fact that \( \Phi \tau_{\geq -i} \Psi Y \) is compact and (4.3) is by the fact that \( \tau_{\geq -i} \Psi Y \) is bounded.

\[\check{\square}\]

It is clear now that lemmas 4.1, 4.3 and 4.4 imply the proposition 4.1.

5. **Determination of the image of \( G \)**

Let \( L_{\rho} : \mathcal{D}^{-}(\underline{\mathcal{M}}) \to \mathcal{D}^{-}(\mathcal{M}) \) be the restriction functor induced by the projection functor \( \mathcal{M} \to \underline{\mathcal{M}} \). \( L_{\rho} \) admits a left adjoint \( L : \mathcal{D}^{-}(\underline{\mathcal{M}}) \to \mathcal{D}^{-}(\mathcal{M}) \) which takes \( Y \) to \( Y \otimes_{\underline{\mathcal{M}}} \mathcal{M} \). Let \( \mathcal{B}^{-} \) be the dg subcategory of \( \mathcal{C}^{-}(\text{Mod} \mathcal{M})_{\text{dg}} \) formed by the objects of \( \mathcal{D}^{-}(\mathcal{M}) \) that are in the essential image of the restriction of \( \Psi \) to \( \mathcal{H}^{-}_{\text{ac}}(\mathcal{M}) \). Let \( \mathcal{B}' \) be the DG quotient, cf. [8], of \( \mathcal{B}^{-} \) by its quasi-isomorphisms. It is clear that the dg categories \( \mathcal{B}' \) and \( \mathcal{B}^{-} \) are quasi-equivalent, cf. [47], and that the natural dg functor \( \mathcal{M} \to \mathcal{C}^{-}(\text{Mod} \mathcal{M})_{\text{dg}} \) factors through \( \mathcal{B}' \). Let \( R' : \mathcal{D}(B^{\text{op}})^{\text{op}} \to \mathcal{D}(\underline{\mathcal{M}}^{\text{op}})^{\text{op}} \) be the restriction functor induced by the dg functor \( \underline{\mathcal{M}} \to \mathcal{B}' \). Let \( \Phi' : \mathcal{D}(\underline{\mathcal{M}}) \to \mathcal{D}(B^{\text{op}})^{\text{op}} \) be the functor which takes \( X \) to the dg module

\[B' \mapsto \text{Hom}^\bullet(X_c, B'),\]

where \( B' \) is in \( \mathcal{B}' \) and \( X_c \) is a cofibrant replacement of \( X \) for the projective model structure on \( \mathcal{C}(\text{Mod} \mathcal{M}) \). Finally let \( \Gamma : \mathcal{D}(\underline{\mathcal{M}}) \to \mathcal{D}(\underline{\mathcal{M}}^{\text{op}})^{\text{op}} \) be the functor that sends \( Y \) to

\[M \mapsto \text{Hom}^\bullet(Y_c, \underline{\mathcal{M}}(?, M)),\]

where \( Y_c \) is a cofibrant replacement of \( Y \) for the projective model structure on \( \mathcal{C}(\text{Mod} \underline{\mathcal{M}}) \) and \( M \) is in \( \underline{\mathcal{M}} \).
We dispose of the following diagram:

\[
\begin{array}{ccc}
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \xrightarrow{\tau} & \mathcal{H}_{\leq \infty}(\mathcal{M}) \\
\downarrow{\scriptstyle L} & & \downarrow{\scriptstyle R'} \\
\mathcal{D}^-(\mathcal{M}) & \xrightarrow{\Phi} & \mathcal{D}(\mathcal{M}^{\text{op}})^{\text{op}} \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \mathcal{M}} \\
\mathcal{D}^-(-) & \xrightarrow{\Phi'} & \mathcal{D}(\mathcal{B}^{\text{op}})^{\text{op}} \\
\end{array}
\]

\[
\rho \rightsquigarrow \phi \leftarrow \star \xrightarrow{\star} \mathcal{B}
\]

\[
\begin{array}{ccc}
\mathcal{D}^-(\mathcal{M}) & \xrightarrow{\Phi'} & \mathcal{D}(\mathcal{B}^{\text{op}})^{\text{op}} & \mathcal{B}' \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \mathcal{M}} & \downarrow{\scriptstyle \mathcal{M}} \\
\mathcal{D}^-(\mathcal{M}) & \xrightarrow{\Phi} & \mathcal{D}(\mathcal{B}^{\text{op}})^{\text{op}} & \mathcal{B}' \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{D}^-(\mathcal{M}) & \xrightarrow{\Phi} & \mathcal{D}(\mathcal{B}^{\text{op}})^{\text{op}} \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \mathcal{M}} \\
\mathcal{D}^-(\mathcal{M}) & \xrightarrow{\Phi'} & \mathcal{D}(\mathcal{M}^{\text{op}})^{\text{op}} \\
\end{array}
\]

Lemma 5.1. The following square

\[
\begin{array}{ccc}
\mathcal{D}^-(\mathcal{M}) & \xrightarrow{\Phi} & \mathcal{D}(\mathcal{B}^{\text{op}})^{\text{op}} \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \mathcal{M}} \\
\mathcal{D}^-(\mathcal{M}) & \xrightarrow{\Phi'} & \mathcal{D}(\mathcal{M}^{\text{op}})^{\text{op}} \\
\end{array}
\]

is commutative.

Proof. By definition \((R' \circ \Phi')(X)(M)\) equals \(\text{Hom}^\ast(X, \mathcal{M}(?, M))\). Since \(\mathcal{M}(?, M)\) identifies with \(L_p M^\wedge\) and by adjunction, we have

\[
\text{Hom}^\ast(X, \mathcal{M}(?, M)) \xrightarrow{\sim} \text{Hom}^\ast(X, L_p M^\wedge) \xrightarrow{\sim} \text{Hom}^\ast((LX)_c, \mathcal{M}(?, M))
\]

where the last member equals \((\tau \circ L)(X)(M)\).

Lemma 5.2. The functor \(L\) reflects isomorphisms.

Proof. Since \(L\) is a triangulated functor, it is enough to show that if \(L(Y) = 0\), then \(Y = 0\). Let \(Y\) be in \(\mathcal{D}^-(\mathcal{M})\) such that \(L(Y) = 0\). We can suppose, without loss of generality, that \(H^p(Y) = 0\) for all \(p > 0\). Let us show that \(H^0(Y) = 0\). Indeed, since \(H^0(Y)\) is an \(\mathcal{M}\)-module, we have \(H^0(Y) \cong L^0 H^0(Y)\), where \(L^0 : \text{Mod} \mathcal{M} \to \text{Mod} \mathcal{M}\) is the left adjoint of the inclusion \(\text{Mod} \mathcal{M} \to \text{Mod} \mathcal{M}\). Since \(H^0(Y)\) vanishes in degrees \(p > 0\), we have

\[
L^0 H^0(Y) = H^0(LY)
\]

By induction, one concludes that \(H^p(Y) = 0\) for all \(p \leq 0\).

Proposition 5.1. An object \(Y\) of \(\mathcal{D}^-(\mathcal{M})\) lies in the essential image of the functor \(\Psi : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}^-(\mathcal{M})\) iff \(\tau_{\geq n} Y\) is in \(\text{per} \mathcal{M}(\mathcal{M})\), for all \(n \in \mathbb{Z}\) and \(L(Y)\) belongs to \(\text{per} \mathcal{M}(\mathcal{M})\).

Proof. Let \(X\) be in \(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\). By lemma 4.2, \(\tau_{\geq n} \Psi(X)\) is in \(\text{per} \mathcal{M}(\mathcal{M})\), for all \(n \in \mathbb{Z}\). Since \(X\) is a bounded complex, there exists an \(s \leq 0\) such that for all\( m < s\) the \(m\)-components of \(\Psi(X)\) are in \(\mathcal{P}\), which implies that \(L \Psi(X)\) belongs to \(\text{per} \mathcal{M}(\mathcal{M})\).

Conversely, suppose that \(Y\) is an object of \(\mathcal{D}^-(\mathcal{M})\) which satisfies the conditions. By lemma 1.2, \(Y\) belongs to \(\mathcal{V}\). Thus we have \(Y = \Psi(Y')\) for some \(Y'\) in \(\mathcal{H}_{\leq \infty}(\mathcal{M})\). We now consider \(Y'\) as an object of \(\mathcal{H}^-(\mathcal{M})\) and also write \(\Psi\) for the functor \(\mathcal{H}^-(\mathcal{M}) \to \mathcal{D}^-(\mathcal{M})\) induced by the Yoneda functor. We can express \(Y'\) as

\[
Y' \simeq \lim_i \text{hocolim} \sigma_{\geq i} Y',
\]
where the $\sigma_{\geq -i}$ are the naive truncations. By our assumptions on $Y'$, $\sigma_{\geq -i}Y'$ belongs to $H^b(M)/H^b(P)$, for all $i \in \mathbb{Z}$. The functors $\Psi$ and $L$ clearly commute with the naive truncations $\sigma_{\geq -i}$ and so we have

$$L(Y) = L(\Psi Y') \xrightarrow{\sim} \operatorname{hocolim}_i L(\sigma_{\geq -i} \Psi Y') \sim \operatorname{hocolim}_i \sigma_{\geq -i} L(\Psi Y').$$

By our hypotheses, $L(Y)$ belongs to $\operatorname{per}(M)$ and so there exists an $m \gg 0$ such that

$$L(Y) = L(\Psi Y') \xrightarrow{\sim} \sigma_{\geq -m} L(\Psi Y') = L(\sigma_{\geq -m} \Psi Y').$$

By lemma 5.2, the inclusion

$$\Psi(\sigma_{\geq -m} Y') = \sigma_{\geq -m} \Psi Y' \longrightarrow \Psi(Y') = Y$$

is an isomorphism. But since $\sigma_{\geq -m} Y'$ belongs to $H^b(M)/H^b(P)$, $Y$ identifies with $\Psi(\sigma_{\geq -m} Y')$. √

**Remark 5.1.** It is clear that if $X$ belongs to $\operatorname{per}(M)$, then $\Gamma(X)$ belongs to $\operatorname{per}(M^{\text{op}})^{\text{op}}$. We also have the following partial converse.

**Lemma 5.3.** Let $X$ be in $D_{\text{mod}}^-(M)$ such that $\Gamma(X)$ belongs to $\operatorname{per}(M^{\text{op}})^{\text{op}}$. Then $X$ is in $\operatorname{per}(M)$.

**Proof.** By lemma 4.2 b) we can suppose, without loss of generality, that $X$ is a right bounded complex with finitely generated projective components. Applying $\Gamma$, we get a perfect complex $\Gamma(X)$. In particular $\Gamma(X)$ is homotopic to zero in high degrees. But since $\Gamma$ is an equivalence $\operatorname{proj}M \sim \longrightarrow (\operatorname{proj}M^{\text{op}})^{\text{op}}$, it follows that $X$ is already homotopic to zero in high degrees. √

**Lemma 5.4.** The natural left aisle on $\operatorname{per}(M^{\text{op}})$ satisfies the conditions of proposition A.1 b).

**Proof.** Clearly the natural left aisle $U$ in $\operatorname{per}(M)^{\text{op}}$ is non-degenerate. We need to show that for each $C \in \operatorname{per}(M)^{\text{op}}$, there is an integer $N$ such that $\operatorname{Hom}(C, S^N U) = 0$ for each $U \in U$. We dispose of the following isomorphism

$$\operatorname{Hom}_{\operatorname{per}(M)^{\text{op}}}(C, S^N U) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{per}(M)}(S^{-N} U^{\text{op}}, C),$$

where $U^{\text{op}}$ denotes the natural right aisle on $\operatorname{per}(M)$. Since by theorem 5.4 c) of [18] an $\mathcal{M}$-module admits a projective resolution of length $d + 1$ as an $\mathcal{M}$-module and $C$ is a bounded complex, we conclude that for $N \gg 0$

$$\operatorname{Hom}_{\operatorname{per}(M)}(S^{-N} U^{\text{op}}, C) = 0.$$

This proves the lemma. √

We denote by $\tau_{\leq n}$ and $\tau_{\geq n}$, $n \in \mathbb{Z}$, the associated truncation functors on $D(B^{\text{op}})^{\text{op}}$.

**Lemma 5.5.** The functor $\Phi : D^-(M) \rightarrow D(B^{\text{op}})^{\text{op}}$ restricted to the category $V$ is exact with respect to the given $t$-structures.
The functor $\Phi$ is triangulated and so we dispose of the triangle $\tau$ the truncation triangle $\tau^\per M_{\tau}$ where $\Phi(\tau)$ is in $\tau \Phi^\per M_{\tau}$, which implies that $\Phi(X)$ belongs to $\tau \Phi^\per M_{\tau}$. The following have the same classes of objects:

\begin{align}
\mathcal{D}(\mathcal{B}_{\geq 0})^\op & \quad \text{and} \quad \mathcal{D}(\mathcal{B}_{> 0})^\op \\
(\per(\mathcal{B}_{\leq 0})^\op) & \quad \text{and} \quad (\per(\mathcal{B}_{< 0})^\op)_{> 0},
\end{align}

where in (5.1) we consider the right orthogonal in $\mathcal{D}(\mathcal{B}_{\geq 0})^\op$ and in (5.2) we consider the left orthogonal in $\mathcal{D}(\mathcal{B}_{> 0})^\op$. These isomorphisms show us that $\Phi(X)$ belongs to $\tau \Phi^\per M_{\tau}$. Let $\Phi(\tau)$ be in $\tau \Phi^\per M_{\tau}$.

**Proof.** We first prove that $\Phi(V_{\leq 0}) \subset \mathcal{D}(\mathcal{B}_{\geq 0})^\op$. Let $X$ be in $V_{\leq 0}$. We need to show that $\Phi(X)$ belongs to $\mathcal{D}(\mathcal{B}_{\geq 0})^\op$. The following have the same classes of objects:

\begin{align}
\mathcal{D}(\mathcal{B}_{\geq 0})^\op & \quad \text{and} \quad \mathcal{D}(\mathcal{B}_{> 0})^\op \\
(\per(\mathcal{B}_{\leq 0})^\op) & \quad \text{and} \quad (\per(\mathcal{B}_{< 0})^\op)_{> 0},
\end{align}

for all $P \in \per \mathcal{M}(\mathcal{M})_{> 0}$. Now, by lemma [lemma 4.4] the functor $\Phi$ is fully faithful and so

\[ \text{Hom}_{\mathcal{D}(\mathcal{B}_{\geq 0})^\op}(\Phi(X), \Phi(P)) = 0, \]

which implies that $\Phi(\tau) \Phi^\per M_{\tau}$ belongs to $\tau \Phi^\per M_{\tau}$. Let us now consider $X$ in $V$. We dispose of the truncation triangle $\tau_{\leq 0} X \to X \to \tau_{> 0} X \to S \tau_{\leq 0} X$.

The functor $\Phi$ is triangulated and so we dispose of the triangle

\[ \Phi \tau_{\leq 0} X \to X \to \Phi \tau_{> 0} X \to S \Phi \tau_{\leq 0} X, \]

where $\Phi \tau_{\leq 0} X$ belongs to $\mathcal{D}(\mathcal{B}_{\geq 0})^\op$. Since $\Phi$ induces an equivalence between $\per \mathcal{M}(\mathcal{M})$ and $\per(\mathcal{B}_{\geq 0})^\op$ and $\text{Hom}(P, \tau_{> 0} X) = 0$, for all $P$ in $V_{\leq 0}$, we conclude that $\Phi \tau_{> 0} X$ belongs to $\mathcal{D}(\mathcal{B}_{> 0})^\op$. This implies the lemma.

**Definition 5.1.** Let $\mathcal{D}(\mathcal{B}_{> 0})_f$ denote the full triangulated subcategory of $\mathcal{D}(\mathcal{B}_{> 0})^\op$ formed by the objects $Y$ such that $\tau_{\geq -n} Y$ in $\per(\mathcal{B}_{> 0})^\op$, for all $n \in \mathbb{Z}$, and $R(Y)$ belongs to $\per \mathcal{M}(\mathcal{M})^\op$.

**Proposition 5.2.** An object $Y$ of $\mathcal{D}(\mathcal{B}_{> 0})^\op$ lies in the essential image of the functor $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}(\mathcal{B}_{> 0})^\op$ iff it belongs to $\mathcal{D}(\mathcal{B}_{> 0})_f$.

**Proof.** Let $X$ be in $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$. It is clear that the $\tau_{\geq -n} G(X)$ are in $\per(\mathcal{B}_{> 0})^\op$ for all $n \in \mathbb{Z}$. By proposition [5.1] we know that $L \Psi Y(X)$ belongs to $\per \mathcal{M}(\mathcal{M})$. By lemma [lemma 4.4] and remark [6.1] we conclude that $R \Phi(X)$ belongs to $\per \mathcal{M}(\mathcal{M})^\op$. Let now $Y$ be in $\mathcal{D}(\mathcal{B}_{> 0})^\op$. We can express it, by the dual of lemma [lemma 3.2] as the homotopy limit of the following diagram

\[ \cdots \to \tau_{\leq -n+1} Y \to \tau_{\leq -n} Y \to \tau_{\geq -n} Y \to \cdots, \]

where $\tau_{\geq -n} Y$ belongs to $\per(\mathcal{B}_{> 0})^\op$, for all $n \in \mathbb{Z}$. But since $\Phi$ induces an equivalence between $\per \mathcal{M}(\mathcal{M})$ and $\per(\mathcal{B}_{> 0})^\op$, this last diagram corresponds to a diagram

\[ \cdots \to M_{-n-1} \to M_{-n} \to M_{-n+1} \to \cdots. \]
in $\text{per}_{\mathcal{M}}(\mathcal{M})$. Let $p \in \mathbb{Z}$. The relations among the truncation functors imply that the image of the above diagram under each homology functor $H^p$, $p \in \mathbb{Z}$, is stationary as $n$ goes to $+\infty$. This implies that

$$H^p \lim_{n \to \infty} M_{-n} \xrightarrow{\sim} \lim_{n \to \infty} H^p M_{-n} \cong H^p M_j,$$

for all $j < p$. We dispose of the following commutative diagram

$$\begin{array}{ccc}
\text{holim}_{n} M_{-n} & \xrightarrow{\sim} & \text{holim}_{n} \tau_{\geq -i} M_{-n} \\
\downarrow \sim & & \downarrow \sim \\
\tau_{\geq -i} \text{holim}_{n} M_{-n} & \text{holim}_{n} \tau_{\geq -i} M_{-n} & \cong M_{-i}
\end{array}$$

which implies that

$$\tau_{\geq -i} \text{holim}_{n} M_{-n} \xrightarrow{\sim} M_{-i},$$

for all $i \in \mathbb{Z}$. Since $\text{holim}_{n} M_{-n}$ belongs to $\mathcal{V}$, lemma 6.1 allows us to conclude that $\Phi(\text{holim}_{n} M_{-n}) \cong Y$. We now show that $\text{holim}_{n} M_{-n}$ satisfies the conditions of proposition 5.1. We know that $\tau_{> -i} \text{holim}_{n} M_{-n}$ belongs to $\text{per}_{\mathcal{M}}(\mathcal{M})$, for all $i \in \mathbb{Z}$. By lemma 5.1, $(\Gamma \circ L)(\text{holim}_{n} M_{-n})$ identifies with $R(Y)$, which is in $\text{per}(\mathcal{M}^{op})^{op}$. Since $\text{holim}_{n} M_{-n}$ belongs to $\mathcal{V}$, its homologies lie in $\text{mod}_{\mathcal{M}}$ and so we are in the conditions of lemma 5.1 which implies that $L(\text{holim}_{n} M_{-n})$ belongs to $\text{per}_{\mathcal{M}}(\mathcal{M})$. This finishes the proof.

\section{Alternative Description}

In this section, we present another characterization of the image of $G$, which was identified as $\mathcal{D}(\mathcal{B}^{op})^{op}$ in proposition 5.2. Let $M$ denote an object of $\mathcal{M}$ and also the naturally associated complex in $\mathcal{H}(\mathcal{M})$. Since the category $\mathcal{H}^{b}(\mathcal{M})/\mathcal{H}^{b}(\mathcal{P})$ is generated by the objects $M \in \mathcal{M}$ and the functor $G$ is fully faithful, we remark that $\mathcal{D}(\mathcal{B}^{op})^{op}$ equals the triangulated subcategory of $\mathcal{D}(\mathcal{B}^{op})^{op}$ generated by the objects $G(M)$, $M \in \mathcal{M}$. The rest of this section is concerned with the problem of characterizing the objects $G(M)$, $M \in \mathcal{M}$. We denote by $P_M$ the projective $\mathcal{M}$-module $\mathcal{M}(?, M)$ associated with $M \in \mathcal{M}$ and by $X_M$ the image of $M$ under $\Psi \circ \Upsilon$.

\textbf{Lemma 6.1.} \textit{We dispose of the following isomorphism}

$$\text{Hom}_{\mathcal{D}(\mathcal{M})}(X_M, Y) \xleftarrow{\sim} \text{Hom}_{\text{mod}_{\mathcal{M}}}(P_M, H^0(Y)),$$

\textit{for all $Y \in \mathcal{D}(\mathcal{M})$.}

\textit{Proof.} Clearly $X_M$ belongs to $\mathcal{D}_{\mathcal{M}}(\mathcal{M})_{\leq 0}$ and is of the form

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where $P_n \in \mathcal{P}$, $n \geq 0$. Now Yoneda’s lemma and the fact that $H^m(Y)(P_n) = 0$, for all $m \in \mathbb{Z}$, $n \geq 0$, imply the lemma.

\textit{Remark 6.1.} Since the functor $\Phi$ restricted to $\mathcal{V}$ is fully faithful and exact, we have

$$\text{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(G(M), \Phi(Y)) \xleftarrow{\sim} \text{Hom}_{\text{per}(\mathcal{B}^{op})^{op}}(\Phi(P_M), H^0(\Phi(Y))),$$

\textit{for all $Y \in \mathcal{V}$.}
We now characterize the objects $G(M) = \Phi(X_M)$, $M \in \mathcal{M}$, in the triangulated category $\mathcal{D}(\mathcal{B}^{op})$. More precisely, we give a description of the functor

$$R_M := \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?, \Phi(X_M)) : \mathcal{D}(\mathcal{B}^{op})^{op} \to \text{Mod} k$$

using an idea of M. Van den Bergh, cf. lemma 2.13 of [7]. Consider the following functor

$$F_M := \text{Hom}_{\text{per}(\mathcal{B}^{op})}(H^0(?), \Phi(P_M)) : \text{per}(\mathcal{B}^{op}) \to \text{mod} k.$$

Remark 6.2. Remark 6.1 shows that the functor $R_M$ when restricted to $\text{per}(\mathcal{B}^{op})$ coincides with $F_M$.

Let $DF_M$ be the composition of $F_M$ with the duality functor $D = \text{Hom}(?, k)$. Note that $DF_M$ is homological.

Lemma 6.2. We dispose of the following isomorphism of functors on $\text{per}(\mathcal{B}^{op})$

$$DF_M \cong \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), ?[d + 1]).$$

Proof. The following functors are canonically isomorphic to $D\Phi$

- (6.1) $D\text{Hom}_{\text{per}(\mathcal{B}^{op})}(H^0(?), \Phi(P_M))$
- (6.2) $D\text{Hom}_{\text{per}(\mathcal{B}^{op})}(\Phi H^0(?), \Phi(P_M))$
- (6.3) $D\text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi X_M, ?)$
- (6.4) $\text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?[-d - 1], X_M)$
- (6.5) $\text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(?)[-d - 1], \Phi(X_M))$
- (6.6) $\text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), ?[d + 1])$

Step (6.1) follows from the fact that $\Phi$ is exact. Step (6.2) follows from the fact that $\Phi$ is fully faithful and we are considering the opposite category. Step (6.3) is a consequence of lemma 6.1. Step (6.4) follows from the $(d + 1)$-Calabi-Yau property and remark 4.2. Step (6.5) is a consequence of $\Phi$ being fully faithful and step (6.6) is a consequence of working in the opposite category. Since the functor $\Phi^{op}$ establishes an equivalence between $\text{per}(\mathcal{M})^{op}$ and $\text{per}(\mathcal{B}^{op})$ the lemma is proven.

Now, we consider the left Kan extension $E_M$ of $DF_M$ along the inclusion $\text{per}(\mathcal{B}^{op}) \hookrightarrow \mathcal{D}(\mathcal{B}^{op})$. We dispose of the following commutative triangle:

$$\begin{array}{ccc}
\text{per}(\mathcal{B}^{op}) & \overset{DF_M}{\longrightarrow} & \text{mod} k \\
\downarrow & & \downarrow \\
\mathcal{D}(\mathcal{B}^{op}) & \overset{E_M}{\longrightarrow} & .
\end{array}$$

The functor $E_M$ is homological and preserves coproducts and so $DE_M$ is cohomological and transforms coproducts into products. Since $\mathcal{D}(\mathcal{B}^{op})$ is a compactly generated triangulated category, the Brown representability theorem, cf. [23], implies that there is a $Z_M \in \mathcal{D}(\mathcal{B}^{op})$ such that

$$DE_M \cong \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?, Z_M).$$

Remark 6.3. Since the duality functor $D$ establishes and anti-equivalence in mod $k$, the functor $DE_M$ restricted to $\text{per}(\mathcal{B}^{op})$ is isomorphic to $F_M$. 

Theorem 6.1. We dispose of an isomorphism
\[ G(M) \sim \to Z_M. \]

Proof. We now construct a morphism of functors from \( R_M \) to \( DE_M \). Since \( R_M \) is representable, by Yoneda’s lemma it is enough to construct an element in \( DE_M(\Phi(X_M)) \).

Let \( C \) be the category \( \text{per}(B^{op}) \downarrow \Phi(X_M) \), whose objects are the morphisms \( Y' \to \Phi(X_M) \) and let \( C' \) be the category \( X_M \downarrow \text{per}(M) \), whose objects are the morphisms \( X_M \to X' \). The following are canonically isomorphic:

\[ \begin{align*}
DE_M(\Phi(X_M)) & \overset{\text{colim}}{\longrightarrow} \text{colim} \ C \text{Hom}_D(B^{op})(\Phi(X_M), Y') \\
& \overset{\text{colim}}{\longrightarrow} \text{colim} \ C' \text{Hom}_D(M)(X', [-d-1], X_M) \\
& \overset{\text{lim}}{\longrightarrow} \text{lim} \ i \text{Hom}_D(-M)(\tau \geq -i X_M, X_M) \\
& \overset{\text{lim}}{\longrightarrow} \text{lim} \ i \text{Hom}_D(-M)(\tau \geq -i X_M, \tau \geq -i X_M)
\end{align*} \]

Step (6.7) is a consequence of the definition of the left Kan extension and lemma 6.2. Step (6.8) is obtained by considering the opposite category. Step (6.9) follows from the fact that the system \( (\tau \geq -i X_M)_{i \in \mathbb{Z}} \) forms a cofinal system for the index system of the colimit. Step (6.10) follows from the \((d+1)\)-Calabi-Yau property.

Now, the image of the identity by the canonical morphism
\[ \text{Hom}_D(M)(X_M, X_M) \longrightarrow \text{lim} \ i \text{Hom}_D(M)(X_M, \tau \geq -i X_M), \]

give us an element of \( (DE_M)(\Phi(X_M)) \) and so a morphism of functors from \( R_M \) to \( DE_M \). We remark that this morphism is an isomorphism when evaluated at the objects of \( \text{per}(B^{op}) \). Since both functors \( R_M \) and \( DE_M \) are cohomological, transform coproducts into products and \( D(B^{op}) \) is compactly generated, we conclude that we dispose of an isomorphism
\[ G(M) \sim \to Z_M. \]

\[ \checkmark \]

7. The main theorem

Consider the following commutative square as in section 3

\[ \begin{array}{ccc}
\mathcal{M} & \overset{\mathcal{C}}{\longrightarrow} & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{T} & \overset{\mathcal{C}}{\longrightarrow} & \mathcal{E} = \mathcal{C}.
\end{array} \]

In the previous sections we have constructed, from the above data, a dg category \( B \) and a left aisle \( U \subset H^0(B) \), see 24, satisfying the following conditions:

- \( B \) is an exact dg category over \( k \) such that \( H^0(B) \) has finite-dimensional Hom-spaces and is Calabi-Yau of CY-dimension \( d + 1 \),
- \( U \subset H^0(B) \) is a non-degenerate left aisle such that:
  - for all \( B \in \mathcal{B} \), there is an integer \( N \) such that \( \text{Hom}_{H^0(B)}(B, S^NU) = 0 \) for each \( U \in U \),
- the heart $\mathcal{H}$ of the $t$-structure on $H^0(B)$ associated with $\mathcal{U}$ has enough projectives.

Let now $\mathcal{A}$ be a dg category and $\mathcal{W} \subset H^0(\mathcal{A})$ a left aisle satisfying the above conditions. We can consider the following general construction: Let $\mathcal{Q}$ denote the category of projectives of $\mathcal{H}$. We claim that the following inclusion

$$\mathcal{Q} \hookrightarrow \mathcal{H} \hookrightarrow H^0(\mathcal{A}),$$

lifts to a morphism $\mathcal{Q} \xrightarrow{j} \mathcal{A}$ in the homotopy category of small dg categories, cf. [15] [26] [27]. Indeed, recall the following argument from section 7 of [17]: Let $\tilde{\mathcal{Q}}$ be the full dg subcategory of $\mathcal{A}$ whose objects are the same as those of $\mathcal{Q}$. Let $\tau_{\leq 0} \tilde{\mathcal{Q}}$ denote the dg category obtained from $\tilde{\mathcal{Q}}$ by applying the truncation functor $\tau_{\leq 0}$ of complexes to each Hom-space. We dispose of the following diagram in the category of small dg categories

$$\begin{array}{ccc}
\tilde{\mathcal{Q}} & \hookrightarrow & \mathcal{A} \\
\downarrow & & \downarrow \\
\tau_{\leq 0} \tilde{\mathcal{Q}} & \hookrightarrow & H^0(\tilde{\mathcal{Q}}) \\
\downarrow & & \downarrow \\
\mathcal{Q} & \hookrightarrow & H^0(\mathcal{A})
\end{array}$$

Let $X, Y$ be objects of $\mathcal{Q}$. Since $X$ and $Y$ belong to the heart of a $t$-structure in $H^0(\mathcal{A})$, we have

$$\text{Hom}_{H^0(\mathcal{A})}(X, Y[-n]) = 0,$$

for $n \geq 1$. The dg category $\mathcal{A}$ is exact, which implies that

$$H^{-n}\text{Hom}_{\tilde{\mathcal{Q}}}(X, Y) \cong \text{Hom}_{H^0(\mathcal{A})}(X, Y[-n]) = 0,$$

for $n \geq 1$. This shows that the dg functor $\tau_{\leq 0} \tilde{\mathcal{Q}} \to H^0(\tilde{\mathcal{Q}})$ is a quasi-equivalence and so we dispose of a morphism $\mathcal{Q} \xrightarrow{j} \mathcal{A}$ in the homotopy category of small dg categories. We dispose of a triangle functor $j^* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{Q})$ given by restriction. By proposition A.1, the left aisle $\mathcal{W} \subset H^0(\mathcal{A})$ admits a smallest extension to a left aisle $\mathcal{D}(\mathcal{A}^{op})^{op}_{\leq 0}$ on $\mathcal{D}(\mathcal{A}^{op})^{op}$. Let $\mathcal{D}(\mathcal{A}^{op})^{op} \rightarrow \mathcal{D}(\mathcal{A}^{op})^{op}$ denote the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{op})^{op}$ formed by the objects $Y$ such that $\tau_{\geq -n}Y$ is in $\text{per}(\mathcal{A}^{op})^{op}$, for all $n \in \mathbb{Z}$, and $j^*(Y)$ belongs to $\text{per}(\mathcal{Q}^{op})^{op}$.

**Definition 7.1.** The stable category of $\mathcal{A}$ with respect to $\mathcal{W}$ is the triangle quotient

$$\text{stab}(\mathcal{A}, \mathcal{W}) = \mathcal{D}(\mathcal{A}^{op})_{j}^{op}/\text{per}(\mathcal{A}^{op})^{op}.$$

We are now able to formulate the main theorem. Let $\mathcal{B}$ be the dg category and $\mathcal{U} \subset H^0(\mathcal{B})$ the left aisle constructed in sections 1 to 5.

**Theorem 7.1.** The functor $G$ induces an equivalence of categories

$$\tilde{G} : \mathcal{C} \xrightarrow{\sim} \text{stab}(\mathcal{B}, \mathcal{U}).$$
Proof. We dispose of the following commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{\tilde{G}} & \text{stab}(B, \mathcal{U}) \\
\uparrow & & \uparrow \\
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \xrightarrow{G} & D(\mathcal{B}^{op})^{op} \\
\uparrow & & \uparrow \\
\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) & \xrightarrow{\sim} & \text{per}(\mathcal{B}^{op})^{op}.
\end{array}
\]

The functor \(G\) is an equivalence since it is fully faithful by proposition 4.1 and essentially surjective by proposition 5.2. Since we dispose of an equivalence \(\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \xrightarrow{\sim} \text{per}(\mathcal{B}^{op})^{op}\) by construction of \(B\) and the columns of the above diagram are short exact sequences of triangulated categories, the theorem is proved. \(\square\)

Appendix A. Extension of \(t\)-structures

Let \(\mathcal{T}\) be a compactly generated triangulated category with suspension functor \(S\). We denote by \(\mathcal{T}_c\) the full triangulated sub-category of \(\mathcal{T}\) formed by the compact objects, see [23]. We use the terminology of [20]. Let \(\mathcal{U} \subseteq \mathcal{T}_c\) be a left aisle.

**Proposition A.1.**

a) The left aisle \(\mathcal{U}\) admits a smallest extension to a left aisle \(\mathcal{T}_{\leq 0}\) on \(\mathcal{T}\).

b) If \(\mathcal{U} \subseteq \mathcal{T}_c\) is non-degenerate (i.e., \(f : X \to Y\) is invertible iff \(H^p(f)\) is invertible for all \(p \in \mathbb{Z}\)) and for each \(X \in \mathcal{T}_c\), there is an integer \(N\) such that \(\text{Hom}(X, S^N U) = 0\) for each \(U \in \mathcal{U}\), then \(\mathcal{T}_{\leq 0}\) is also non-degenerate.

**Proof.**

a) Let \(\mathcal{T}_{\leq 0}\) be the smallest full subcategory of \(\mathcal{T}\) that contains \(\mathcal{U}\) and is stable under infinite sums and extensions. It is clear that \(\mathcal{T}_{\leq 0}\) is stable by \(S\) since \(\mathcal{U}\) is. We need to show that the inclusion functor \(\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}\) admits a right adjoint. For completeness, we include the following proof, which is a variant of the 'small object argument', cf. also [1]. We dispose of the following recursive procedure. Let \(X = X_0\) be an object in \(\mathcal{T}\). For the initial step consider all morphisms from any object \(P\) in \(\mathcal{U}\) to \(X_0\). This forms a set \(I_0\) since \(\mathcal{T}\) is compactly generated and so we dispose of the following triangle

\[
\prod_{P \in I_0} P \longrightarrow X_0 \longrightarrow X_1 \xrightarrow{\sim} \prod_{P \in I_0} P.
\]

For the induction step consider the above construction with \(X_n, n \geq 1\), in the place of \(X_{n-1}\) and \(I_n\) in the place of \(I_{n-1}\). We dispose of the following diagram

\[
\begin{array}{ccccccc}
X = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \cdots & \longrightarrow & X' \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\prod_{P \in I_0} P & \longrightarrow & \prod_{P \in I_1} P & \longrightarrow & \prod_{P \in I_2} P & \longrightarrow & \prod_{P \in I_3} P & \longrightarrow & X',
\end{array}
\]

where \(X'\) denotes the homotopy colimit of the diagram \((X_i)_{i \in \mathbb{Z}}\). Consider now the following triangle

\[
S^{-1} X' \longrightarrow X'' \longrightarrow X \longrightarrow X',
\]
where the morphism $X \to X'$ is the transfinite composition in our diagram. Let $P$ be in $U$. We remark that since $P$ is compact, $\text{Hom}_T(P, X') = 0$. This also implies, by construction of $\mathcal{T}_{\leq 0}$, that $\text{Hom}_T(R, X') = 0$, for all $R$ in $\mathcal{T}_{\leq 0}$. The long exact sequence obtained by applying the functor $\text{Hom}_T(R, ?)$ to the triangle above shows that

\[ \text{Hom}(R, X'') \to \text{Hom}(R, X) \]

Let $X''_{n-1}, n \geq 1$, be an object as in the following triangle

\[ X = X_0 \to X_n \to X''_{n-1} \to S(X) \]

A recursive application of the octahedron axiom implies that $X''_{n-1}$ belongs to $S(\mathcal{T}_{\leq 0})$, for all $n \geq 1$. We dispose of the isomorphism

\[ \text{hocolim}_{n} X''_{n-1} \to S(X'') \]

Since $\text{hocolim}_{n} X''_{n-1}$ belongs to $S(\mathcal{T}_{\leq 0})$, we conclude that $X''$ belongs to $\mathcal{T}_{\leq 0}$. This shows that the functor that sends $X$ to $X''$ is the right adjoint of the inclusion functor $\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}$. This proves that $\mathcal{T}_{\leq 0}$ is a left aisle on $\mathcal{T}$. We now show that the $t$-structure associated to $\mathcal{T}_{\leq 0}$, cf. [20], extends, from $\mathcal{T}_c$ to $\mathcal{T}$, the one associated with $U$. Let $X$ be in $\mathcal{T}_c$. We dispose of the following truncation triangle associated with $U$

\[ X_U \to X \to X^U_{\perp} \to SX_U \]

Clearly $X_U$ belongs to $\mathcal{T}_{\leq 0}$. We remark that $U^\perp = \mathcal{T}_{\leq 0}^\perp$, and so $X^U_{\perp}$ belongs to $\mathcal{T}^\perp_{>0} := \mathcal{T}^\perp_{\leq 0}$.

We now show that $\mathcal{T}_{\leq 0}$ is the smallest extension of the left aisle $U$. Let $V$ be an aisle containing $U$. The inclusion functor $V \hookrightarrow \mathcal{T}$ commutes with sums, because it admits a right adjoint. Since $V$ is stable under extensions and suspensions, it contains $\mathcal{T}_{\leq 0}$.

b) Let $X$ be in $\mathcal{T}$. We need to show that $X = 0$ iff $H^p(X) = 0$ for all $p \in \mathbb{Z}$. Clearly the condition is necessary. For the converse, suppose that $H^p(X) = 0$ for all $p \in \mathbb{Z}$. Let $n$ be an integer. Consider the following truncation triangle

\[ H^{n+1}(X) \to \tau_{>n}X \to \tau_{>n+1}X \to SH^{n+1}(X) \]

Since $H^{n+1}(X) = 0$ we conclude that

\[ \tau_{>n}X \in \bigcap_{m \in \mathbb{Z}} \mathcal{T}_{>m} \]

for all $n \in \mathbb{Z}$. Now, let $C$ be a compact object of $\mathcal{T}$. We know that there is a $k \in \mathbb{Z}$ such that $C \in \mathcal{T}_{\leq k}$. This implies that

\[ \text{Hom}_T(C, \tau_{>n}X) = 0 \]

for all $n \in \mathbb{Z}$, since $\tau_{>n}X$ belongs to $(\mathcal{T}_{\leq k})^\perp$. The category $\mathcal{T}$ is compactly generated and so we conclude that $\tau_{>n}X = 0$, for all $n \in \mathbb{Z}$. The following truncation triangle

\[ \tau_{\leq n}X \to X \to \tau_{>n}X \to S\tau_{\leq n}X \]

implies that $\tau_{\leq n}X$ is isomorphic to $X$ for all $n \in \mathbb{Z}$. This can be rephrased as saying that

\[ X \in \bigcap_{n \in \mathbb{N}} \mathcal{T}_{\leq -n} \]
Now by our hypothesis there is an integer $N$ such that
\[ \text{Hom}_T(C, \mathcal{U}_{\leq -N}) = 0. \]
Since $C$ is compact and by construction of $\mathcal{T}_{\leq -N}$, we have
\[ \text{Hom}_T(C, \mathcal{T}_{\leq -N}) = 0. \]
This implies that $\text{Hom}_T(C, X) = 0$, for all compact objects $C$ of $\mathcal{T}$. Since $\mathcal{T}$ is compactly generated, we conclude that $X = 0$. This proves the converse.

\[ \text{Lemma A.1.} \]
Let $(Y_p)_{p \in \mathbb{Z}}$ be in $\mathcal{T}$. We dispose of the following isomorphism
\[ H^n(\coprod_p Y_p) \xrightarrow{\sim} \coprod_p H^n(Y_p), \]
for all $n \in \mathbb{Z}$.

\[ \text{Proof.} \] By definition $H^n := \tau_{\geq n} \tau_{\leq n}, n \in \mathbb{Z}$. Since $\tau_{\geq n}$ admits a right adjoint, it is enough to show that $\tau_{\leq n}$ commute with infinite sums. We consider the following triangle
\[ \prod_p \tau_{\leq n} Y_p \to \prod_p Y_p \to \prod_p \tau_{> n} Y_p \to S(\prod_p \tau_{\leq n} Y_p). \]
Here $\prod_p \tau_{\leq n} Y_p$ belongs to $\mathcal{T}_{\leq n}$ since $\mathcal{T}_{\leq n}$ is stable under infinite sums. Let $P$ be an object of $S^n \mathcal{U}$. Since $P$ is compact, we have
\[ \text{Hom}_T(P, \prod_p \tau_{> n} Y_p) \xleftarrow{\sim} \prod_p \text{Hom}_T(P, \tau_{> n} Y_p) = 0. \]
Since $\mathcal{T}_{\leq n}$ is generated by $S^n \mathcal{U}$, $\prod_p \tau_{> n} Y_p$ belongs to $\mathcal{T}_{\geq n}$. Since the truncation triangle of $\prod_p Y_p$ is unique, this implies the following isomorphism
\[ \prod_p \tau_{\leq n} Y_p \xrightarrow{\sim} \tau_{\leq n}(\prod_p Y_p). \]
This proves the lemma.

\[ \text{Proposition A.2.} \]
Let $X$ be an object of $\mathcal{T}$. Suppose that we are in the conditions of proposition A.1 b). We dispose of the following isomorphism
\[ \text{hocolim}_i \tau_{\leq i} X \xrightarrow{\sim} X. \]

\[ \text{Proof.} \] We need only show that
\[ H^n(\text{hocolim}_i \tau_{\leq i} X) \xrightarrow{\sim} H^n(X), \]
for all $n \in \mathbb{Z}$. We dispose of the following triangle, cf. [23],
\[ \prod_p \tau_{\leq p} X \to \prod_q \tau_{\leq q} X \to \text{hocolim}_i \tau_{\leq i} X \to S(\prod_p \tau_{\leq p} X). \]
Since the functor $H^n$ is homological, for all $n \in \mathbb{Z}$ and it commutes with infinite sums by lemma A.1, we obtain a long exact sequence
\[ \cdots \to \prod_p H^n(\tau_{\leq p} X) \to \prod_q H^n(\tau_{\leq q} X) \to H^n(\text{hocolim}_i \tau_{\leq i} X) \to \prod_p H^n(S(\tau_{\leq p} X) \to \prod_q H^n S(\tau_{\leq q} X) \to \cdots \]
We remark that the morphism \( \prod_p \text{H}^n(S_{\tau \leq p}X) \to \prod_q \text{H}^n(S_{\tau \leq q}X) \) is a split monomorphism and so we obtain
\[
\text{H}^n(X) = \text{colim}_i \text{H}^n(S_{\tau \leq i}X) \xrightarrow{\sim} \text{H}^n(\text{hocolim}_i S_{\tau \leq i}X).
\]

\[\square\]

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