Fixed point subalgebras of Weil algebras: from geometric to algebraic questions

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Abstract The paper is a survey of some results about Weil algebras applicable in differential geometry, especially in some classification questions on bundles of generalized velocities and contact elements. Mainly, a number of claims concerning the form of subalgebras of fixed points of various Weil algebras are demonstrated.

Keywords Local algebra, Weil algebra, automorphism, fixed point subalgebra, natural operator.

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1 Introduction

Motivated by algebraic geometry, André Weil suggested the treatment of infinitesimal objects as homomorphisms from algebras of smooth functions into some real finite-dimensional commutative algebra with unit in 1950’s. In fact, he follows a certain idea of Sophus Lie: so-called $A$-near points (defined by Weil in [9]) represent ‘parametrized infinitesimal submanifolds’. More precisely, let $M$ be a smooth manifold and let $\mathcal{C}^\infty(M, \mathbb{R})$ be its ring of smooth functions into $\mathbb{R}$: $A$-near points of $M$ are defined as $\mathbb{R}$-algebra homomorphism $\mathcal{C}^\infty(M, \mathbb{R}) \rightarrow A$, where $A$ is a certain local $\mathbb{R}$-algebra $A$ (precisely defined below) now called the Weil algebra. This can be...
regarded as the first notable occurrence of local \( \mathbb{R} \)-algebras in differential geometry. New concepts, such as Weil algebras, Weil functors, Weil bundles were introduced and they are widely studied, even to this day, because of their considerable generality. In a modern categorical approach to differential geometry, if we interpret geometric objects as bundle functors, then natural transformations represent a number of geometric constructions. In this context, finding a bijection between natural transformations of two Weil functors \( T^A, T^B \) (generalizing well-known functors of higher order velocities and, of course, the tangent functor as the first of them) and corresponding morphisms of Weil algebras \( A \) and \( B \), has fundamental importance. The theory of natural bundles and operators, including methods for finding natural operators, is very well presented in the monographical work Natural Operations in Differential Geometry [1] (Ivan Kolář, Peter Michor and Jan Slovák, 1993). This paper has the character of a survey: it provides an introduction to Weil algebras and some selected problems which are geometrically motivated and were studied by the author and his collaborators from the algebraic point of view.

2 Starting points: product preserving functors

Let \( F : \text{Mf} \rightarrow \text{FM} \) be a bundle functor from the category \( \text{Mf} \) of manifolds (having smooth manifolds as objects and smooth maps as morphisms) to the category \( \text{FM} \) of fibered manifolds (and fibered manifold morphisms). For example, such a functor is the tangent functor \( T \). For two manifolds \( M_1, M_2 \) we denote the standard projection onto the \( i \)-th factor by \( p_i : M_1 \times M_2 \rightarrow M_i \), where \( i = 1,2 \). \( F \) is called **product preserving** if the mapping

\[
(F(p_1), F(p_2)) : F(M_1 \times M_2) \rightarrow F(M_1) \times F(M_2)
\]

is a diffeomorphism for all manifolds \( M_1, M_2 \). For a product preserving bundle functor we shall always identify \( F(M_1 \times M_2) \) with \( F(M_1) \times F(M_2) \) by the diffeomorphism from the definition. The tangent functor \( T \) is product preserving. Another example of a product preserving functor is the functor \( T^r_k \) of \( k \)-dimensional \( r \)-th order velocities with \( T^1_1 = T \). Further, we obtain a product preserving functor by arbitrary (finite) iterations of product preserving functors.

If we denote by \( \text{WA} \) the category of Weil algebras (the exact definition of Weil algebra is postponed to the next section) and Weil algebra homomorphisms, then the problem of classification of all product preserving functors was solved in works of Kainz and Michor, Luciano and Eck in the 1980’s and reads as follows (see [1]):