SMOOTH PERFECTNESS FOR THE GROUP OF DIFFEOMORPHISMS

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Abstract. Given a result of Herman, we provide a new elementary proof of the fact that the connected component of the group of compactly supported diffeomorphisms is perfect and hence simple. Moreover, we show that every diffeomorphism $g$, which is sufficiently close to the identity, can be represented as a product of four commutators, $g = [h_1, k_1] \circ \cdots \circ [h_4, k_4]$, where the factors $h_i$ and $k_i$ can be chosen to depend smoothly on $g$.

1. Introduction and statement of the result

Let $M$ be a smooth manifold of dimension $n$. By $\text{Diff}^\infty_c(M)$ we will denote the group of compactly supported diffeomorphisms of $M$. This is a regular Lie group in the sense of Kriegl–Michor, modelled on the convenient vector space $\mathcal{X}_c(M)$ of compactly supported smooth vector fields on $M$, see [9, Section 43.1]. A curve $c: \mathbb{R} \to \text{Diff}^\infty_c(M)$ is smooth iff the associated map $\hat{c}: \mathbb{R} \times M \to M$ is smooth and its support satisfies the following condition: for every compact interval $I \subseteq \mathbb{R}$ there exists a compact set $C \subseteq M$ so that $\text{supp}(c(t)) \subseteq C$, for all $t \in I$, see [9, Section 42.5]. If $M$ is compact, this smooth structure coincides with the well known Fréchet–Lie group structure on $\text{Diff}^\infty_c(M)$.

Given smooth complete vector fields $X_1, \ldots, X_N$ on $M$, we consider the map

$$K: \text{Diff}^\infty_c(M)^N \to \text{Diff}^\infty_c(M),$$

$$(1) \quad K(g_1, \ldots, g_N) := [g_1, \exp(X_1)] \circ \cdots \circ [g_N, \exp(X_N)].$$

Here $\exp(X)$ denotes the flow of a complete vector field $X$ at time 1, and $[k, h] := k \circ h \circ k^{-1} \circ h^{-1}$ denotes the commutator of two diffeomorphisms $k$ and $h$. It is readily checked that $K$ is smooth. Indeed, one only has to observe that $K$ maps smooth curves to smooth curves in view of the characterization of smooth curves from the previous paragraph, cf. [9 Section 27.2]. Writing $\text{id}$ for the identity in $\text{Diff}^\infty_c(M)$, we clearly have $K(\text{id}, \ldots, \text{id}) = \text{id}$. 

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A smooth local right inverse at the identity for $K$ consists of a $C^\infty$-open neighborhood $U$ of the identity in $\Diff^\infty_c(M)$ together with a smooth map
$$\sigma = (\sigma_1, \ldots, \sigma_N): U \to \Diff^\infty_c(M)^N$$
so that $\sigma(id) = (id, \ldots, id)$ and $K \circ \sigma = \id_U$. More explicitly, we require that each $\sigma_i: U \to \Diff^\infty_c(M)$ is smooth with $\sigma_i(id) = \id$ and, for all $g \in U$,
$$g = [\sigma_1(g), \exp(X_1)] \circ \cdots \circ [\sigma_N(g), \exp(X_N)].$$

The $C^\infty$-topology [1] Section 4] is the final topology with respect to all smooth curves, its open subsets are the natural domains for locally defined smooth maps between infinite dimensional manifolds [1] Section 27]. For compact $M$ the $C^\infty$-topology on $\Diff^\infty(M)$ coincides with the Whitney $C^\infty$-topology, cf. [9] Theorem 4.11(1)]. In general the $C^\infty$-topology on $\Diff^\infty_c(M)$ is strictly finer than the one induced from the Whitney $C^\infty$-topology, cf. [9] Section 4.26]. The latter coincides with the inductive limit topology $\lim_K \Diff^\infty_{K}(M)$ where $K$ runs through all compact subsets of $M$, see [9] Section 41.13].

The aim of this paper is to establish the following two results.

**Theorem 1.** Suppose $M$ is a smooth manifold of dimension $n \geq 2$. Then there exist four smooth complete vector fields $X_1, \ldots, X_4$ on $M$ so that the map $K$, see [11], admits a smooth local right inverse at the identity, $N = 4$. Moreover, the vector fields $X_i$ may be chosen arbitrarily close to zero with respect to the strong Whitney $C^0$-topology. If $M$ admits a proper (circle valued) Morse function whose critical points all have index $0$ or $n$, then the same statement remains true with three vector fields.

Particularly, on the manifolds $M = \mathbb{R}^n, S^n, T^n, n \geq 2$, or the total space of a compact smooth fiber bundle $M \to S^1$, three commutators are sufficient. Circle-valued Morse theory was initiated by Novikov [14], see also Pajitnov’s monograph [15] and the references therein. At the expense of more commutators, it is possible to gain further control on the vector fields. More precisely, we have:

**Theorem 2.** Suppose $M$ is a smooth manifold of dimension $n \geq 2$ and set $N := 6(n+1)$. Then there exist smooth complete vector fields $X_1, \ldots, X_N$ on $M$ so that the map $K$, see [11], admits a smooth local right inverse at the identity. Moreover, the vector fields $X_i$ may be chosen arbitrarily close to zero with respect to the strong Whitney $C^\infty$-topology.

Either of the two theorems implies that $\Diff^\infty_c(M)_o$, the connected component of the identity, is a perfect group. This was already proved by Epstein [4] using ideas of Mather [10] [11] who dealt with the $C^r$-case, $1 \leq r < \infty$, $r \neq n + 1$. The Epstein–Mather proof is based on a sophisticated construction, and uses the Schauder–Tychonov fixed point theorem. The existence of a presentation $g = [h_1, k_1] \circ \cdots \circ [h_N, k_N]$ is guarantied, but without any further control on the factors $h_i$ and $k_i$. Rough estimates on the number of necessary factors are well known, too. In view of a result due to Tsuboi [19] Theorem 8.1], one can assume $N = 4^n(n+1)$, provided $g$ is sufficiently close to the identity. More refined estimates for certain classes of manifolds can be found in [20] and [21]. That the factors $h_i$ and $k_i$ can be chosen to depend smoothly on $g$ seems to be folklore as well.

Theorem [11] or [2] actually implies that the universal covering of $\Diff^\infty_c(M)_o$ is a perfect group. This result is known, too, see [18] [12]. Thurston’s proof is based on a result of Herman for the torus [6] [7].
Our proof rests on Herman’s result, too, but is otherwise elementary and different from Thurston’s approach. In fact we only need Herman’s result in dimension 1, which is a more structured situation, see also [22].

Note that the perfectness of $\text{Diff}_c^\infty(M)$ implies that this group is simple, see [3]. The methods used in [3] are elementary and actually work for a rather large class of homeomorphism groups.

Let us mention that some analogues of Theorems 1 and 2 for the homeomorphism groups in the category of topological manifolds have been obtained in [17] by using completely different arguments.

The remaining part of this note is organized as follows: In Section 2 we recall the above mentioned result of Herman and derive a corollary, see Proposition 1, which asserts that the statement of Theorem 2 holds true for the torus, $M = T^n$, with $N = 3$. Using the exponential law we then establish a similar statement for diffeomorphisms on open subsets of $\mathbb{R}^n$, see Proposition 2 in Section 3. This construction allows us to circumvent Thurston’s deformation construction, and at the same time restricts the approach to dimensions $n \geq 2$. In Section 4 we formulate and prove a smooth version of the fragmentation lemma, see Proposition 3, and give a proof of Theorem 2. Finally, in Section 5 we discuss a technique to reduce the number of commutators which will eventually lead to a proof of Theorem 1.

2. HERMAN’S THEOREM REVISITED

Let $T^n := \mathbb{R}^n/\mathbb{Z}^n$ denote the torus. For $\lambda \in T^n$ we let $R_\lambda \in \text{Diff}_c^\infty(T^n)$ denote the corresponding rotation. The main ingredient in the proof of Theorems 1 and 2 is the following result of Herman [7, 6].

**Theorem 3 (Herman).** There exist $\gamma \in T^n$ so that the smooth map

$$T^n \times \text{Diff}^\infty(T^n) \to \text{Diff}^\infty(T^n), \quad (\lambda, g) \mapsto R_\lambda \circ [g, R_\gamma],$$

admits a smooth local right inverse at the identity. Moreover, $\gamma$ may be chosen arbitrarily close to the identity in $T^n$.

Herman’s result is an application of the Nash–Moser inverse function theorem. When inverting the derivative one is quickly led to solve the linear equation $Y = X - (R_\gamma)^*X$ for given $Y \in C^\infty(T^n, \mathbb{R}^n)$. This is accomplished using Fourier transformation. Here one has to choose $\gamma$ sufficiently irrational so that tame estimates on the Sobolev norms of $X$ in terms of the Sobolev norms of $Y$ can be obtained. The corresponding small denominator problem can be solved due to a number theoretic result of Khintchine.

Below we will make use of the following corollary of Herman’s result:

**Proposition 1.** There exist smooth vector fields $X_1, X_2, X_3$ on $T^n$ so that the smooth map $\text{Diff}^\infty(T^n)^3 \to \text{Diff}^\infty(T^n)$,

$$(g_1, g_2, g_3) \mapsto [g_1, \exp(X_1)] \circ [g_2, \exp(X_2)] \circ [g_3, \exp(X_3)],$$

admits a smooth local right inverse at the identity. Moreover, the vector fields $X_i$ may be chosen arbitrarily close to zero with respect to the Whitney $C^\infty$-topology.

**Proof.** This is an immediate consequence of Theorem 3 and the following observation, cf. [7]. The finite dimensional Lie group $\text{PSL}_2(\mathbb{R})$ acts effectively on the circle.
$T^1$ and we have smooth embeddings $T^1 \subseteq \text{PSL}_2(\mathbb{R}) \subseteq \text{Diff}^\infty(T^1)$. Since $\text{PSL}_2(\mathbb{R})$ is a simple Lie group, there exist $Y_1, Y_2 \in \mathfrak{sl}_2(\mathbb{R})$ so that the smooth map

$$\text{PSL}_2(\mathbb{R})^2 \to \text{PSL}_2(\mathbb{R}), \quad (g_1, g_2) \mapsto [g_1, \exp(Y_1)][g_2, \exp(Y_2)],$$

admits a smooth local right inverse at the identity. Moreover, $Y_i$ may be chosen arbitrarily close to 0 in $\mathfrak{sl}_2(\mathbb{R})$. Taking the product of $n$ copies of such local right inverses and using the smooth embeddings $T^n \subseteq \text{PSL}_2(\mathbb{R})^n \subseteq \text{Diff}^\infty(T^n)^n \subseteq \text{Diff}^\infty(T^n)$, the statement follows readily from Theorem [3] with $X_i := (Y_1, \ldots, Y_i) \in \mathfrak{sl}_2(\mathbb{R})^n \subseteq \mathfrak{X}(T^n)^n \subseteq \mathfrak{X}(T^n)$, $i = 1, 2$, and $X_3$ so that $\exp(X_3) = R_y$.

3. The exponential law

If $\mathcal{F}$ is a smooth foliation of $M$ we let $\text{Diff}^\infty(M; \mathcal{F})$ denote the group of compactly supported diffeomorphisms preserving the leaves of $\mathcal{F}$. This is a regular Lie group modelled on the convenient vector space of compactly supported smooth vector fields tangential to $\mathcal{F}$. The group of foliation preserving diffeomorphisms has been studied in [10].

**Lemma 1.** Suppose $M_1$ and $M_2$ are two finite dimensional smooth manifolds and set $M := M_1 \times M_2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the foliations with leaves $M_1 \times \{pt\}$ and $\{pt\} \times M_2$ on $M$, respectively. Then the smooth map

$$F: \text{Diff}^\infty_c(M; \mathcal{F}_1) \times \text{Diff}^\infty_c(M; \mathcal{F}_2) \to \text{Diff}^\infty_c(M), \quad F(g_1, g_2) := g_1 \circ g_2,$$

is a local diffeomorphism at the identity.

**Proof.** We proceed by constructing an inverse. To this end let $\pi: M \to M_2$ denote the canonical projection and consider the smooth map

$$\sigma_2: \text{Diff}^\infty_c(M) \to C^\infty_c(M, M), \quad \sigma_2(g)(x_1, x_2) := (x_1, \pi(g(x_1, x_2))).$$

Since $\text{Diff}^\infty_c(M)$ is $c^\infty$-open in $C^\infty_c(M, M)$, see [9, Section 43.1], the set $\mathcal{U} := \sigma_2^{-1}(\text{Diff}^\infty_c(M))$ is a $c^\infty$-open neighborhood of the identity in $\text{Diff}^\infty_c(M)$. Moreover, $\sigma_2$ restricts to a smooth map $\sigma_2: \mathcal{U} \to \text{Diff}^\infty_c(M; \mathcal{F}_2)$. Note that the diffeomorphism $\sigma_1(g) := g \circ \sigma_2(g)^{-1}$ preserves the leaves of the foliation $\mathcal{F}_1$, and this provides a smooth map $\sigma_1: \mathcal{U} \to \text{Diff}^\infty_c(M; \mathcal{F}_1)$. We thus obtain a smooth map

$$\sigma := (\sigma_1, \sigma_2): \mathcal{U} \to \text{Diff}^\infty_c(M; \mathcal{F}_1) \times \text{Diff}^\infty_c(M; \mathcal{F}_2)$$

such that $F \circ \sigma = \text{id}_\mathcal{U}$ and $\sigma(\text{id}) = (\text{id}, \text{id})$. Since $F$ is injective, we have $\sigma(\mathcal{U}) = F^{-1}(\mathcal{U})$, hence $\mathcal{V} := \sigma(\mathcal{U})$ is $c^\infty$-open too. Clearly, $F$ restricts to a bijection, and hence diffeomorphism, $F: \mathcal{V} \cong \mathcal{U}$, with inverse $\sigma$.

**Lemma 2.** Suppose $B$ and $T$ are finite dimensional smooth manifolds, assume $T$ compact, and let $\mathcal{F}$ denote the foliation with leaves $\{pt\} \times T$ on $B \times T$. Then the canonical bijection

$$C^\infty_c(B, \text{Diff}^\infty(T)) \xrightarrow{\cong} \text{Diff}^\infty_c(B \times T; \mathcal{F})$$

is an isomorphism of regular Lie groups.

**Proof.** This follows from the exponential law, see [9, Section 42.14].

**Lemma 3.** Let $B$ be a precompact open subset in a finite dimensional smooth manifold $M$. Then there exist compactly supported smooth vector fields $X_1, X_2, X_3$ on $M \times T^n$, tangential to the foliation $\mathcal{F}$ with leaves $\{pt\} \times T^n$, so that the map

$$\text{Diff}^\infty_c(B \times T^n; \mathcal{F})^3 \to \text{Diff}^\infty_c(B \times T^n; \mathcal{F})$$

is an isomorphism of regular Lie groups.
\((g_1, g_2, g_3) \mapsto [g_1, \exp(X_1)] \circ [g_2, \exp(X_2)] \circ [g_3, \exp(X_3)]\)

admits a smooth local right inverse at the identity. Moreover, the vector fields \(X_i\) may be chosen arbitrarily close to zero with respect to the strong Whitney \(C^\infty\)-topology.

**Proof.** According to Proposition \([1]\) there exist smooth vector fields \(Y_1, Y_2, Y_3\) on \(T^n\) so that the map \(\text{Diff}^\infty(T^n)^3 \to \text{Diff}^\infty(T^n)\),

\[(h_1, h_2, h_3) \mapsto [h_1, \exp(Y_1)] \circ [h_2, \exp(Y_2)] \circ [h_3, \exp(Y_3)],\]

admits a smooth local right inverse \(\rho = (\rho_1, \rho_2, \rho_3): U \to \text{Diff}^\infty(T^n)^3\) at the identity. We extend the vector fields \(Y_i\) in a constant manner to smooth vector fields \(Z_i\) on \(M \times T^n\), tangential to \(F\). Multiplying \(Z_i\) with a compactly supported smooth function which equals 1 on \(\bar{B} \times T^n\), we obtain compactly supported smooth vector fields \(X_1, X_2, X_3\) on \(M \times T^n\), tangential to \(F\), so that \(X_i|_B = Z_i|_B\). Moreover, \(X_i\) depends continuously on \(Y_i\) with respect to the Whitney \(C^\infty\)-topologies. Consequently, \(X_i\) may be assumed arbitrarily close to zero with respect to the Whitney \(C^\infty\)-topology on \(M\). Consider the \(c^\infty\)-open neighborhood

\[W := \{ f \in C^\infty_c(B, \text{Diff}^\infty(T^n)) \mid f(B) \subseteq V \}\]

of the identity, and observe that the maps \((\rho_i)_* : W \to C^\infty_c(B, \text{Diff}^\infty(T^n))\) are smooth. Via the isomorphism in Lemma \([2]\) the set \(W\) corresponds to a \(c^\infty\)-open neighborhood \(U\) of the identity in \(\text{Diff}^\infty_c(B \times T^n; F)\), and the maps \((\rho_i)_*\) provide a smooth map \(\sigma = (\sigma_1, \sigma_2, \sigma_3): U \to \text{Diff}^\infty_c(B \times T^n; F)^3\). By construction,

\[g = [\sigma_1(g), \exp(X_1)] \circ [\sigma_2(g), \exp(X_2)] \circ [\sigma_3(g), \exp(X_3)], \quad \text{for all } g \in U.\]

Hence \(\sigma\) is the desired smooth local right inverse. \(\square\)

**Lemma 4.** Suppose \(p \geq 1, q \geq 0, \) set \(n := p + q, \) and let \(F\) denote the foliation with leaves \(\{pt\} \times \mathbb{R}^q\) on \(\mathbb{R}^n\). Moreover, let \(B\) be a precompact open subset in \(\mathbb{R}^n\). Then there exist compactly supported smooth vector fields \(X_1, X_2, X_3\) on \(\mathbb{R}^n\), a \(c^\infty\)-open neighborhood \(U\) of the identity in \(\text{Diff}^\infty_c(B; F)\) and smooth maps \(\sigma_1, \sigma_2, \sigma_3: U \to \text{Diff}^\infty_c(\mathbb{R}^n)\) so that \(\sigma_i(\text{id}) = \text{id}\) and, for all \(g \in U\),

\[g = [\sigma_1(g), \exp(X_1)] \circ [\sigma_2(g), \exp(X_2)] \circ [\sigma_3(g), \exp(X_3)].\]

Moreover, the vector fields \(X_i\) may be chosen arbitrarily close to zero with respect to the strong Whitney \(C^\infty\)-topology on \(\mathbb{R}^n\).

**Proof.** Since \(p \geq 1, \) there exists a smooth embedding \(\varphi: B^p \times T^q \to \mathbb{R}^n\). Set \(U := \varphi(B^p \times T^q)\), where \(B^p\) denotes the open unit ball in \(\mathbb{R}^p\). Clearly, we may assume \(B \subseteq U\). Let \(G\) denote the foliation on \(U\) corresponding to the foliation with leaves \(\{pt\} \times T^q\) on \(B^p \times T^q\), via the embedding \(\varphi\). Furthermore, we may assume \(G|_B = F|_B\). According to Lemma \([3]\) there exist compactly supported smooth vector fields \(X_1, X_2, X_3\) on the image of \(\varphi\) so that the smooth map \(\text{Diff}^\infty_c(U; G)^3 \to \text{Diff}^\infty_c(U; G)\),

\[(g_1, g_2, g_3) \mapsto [g_1, \exp(X_1)] \circ [g_2, \exp(X_2)] \circ [g_3, \exp(X_3)],\]

admits a smooth local right inverse \(\rho = (\rho_1, \rho_2, \rho_3): V \to \text{Diff}^\infty_c(U; G)\) at the identity. We extend the vector fields \(X_i\) by zero to compactly supported smooth vector fields on \(\mathbb{R}^n\), and observe that these extensions may be assumed arbitrarily close to zero with respect to the Whitney \(C^\infty\)-topology on \(\mathbb{R}^n\). Let \(U\) denote the preimage of \(V\) under the canonical inclusion \(\text{Diff}^\infty_c(B; F) \to \text{Diff}^\infty_c(U; G)\). Restricting \(\rho_i, \) we
obtain smooth maps $\sigma_1, \sigma_2, \sigma_3: \mathcal{U} \to \text{Diff}^\infty_c(U; \mathcal{G}) \subseteq \text{Diff}^\infty_c(\mathbb{R}^n)$ with the desired property. \hfill \Box

**Proposition 2.** Suppose $n \geq 2$, and let $B$ denote a precompact open subset of $\mathbb{R}^n$. Then there exist compactly supported smooth vector fields $X_1, \ldots, X_6$ on $\mathbb{R}^n$, a $\mathcal{C}^\infty$-open neighborhood $\mathcal{U}$ of the identity in $\text{Diff}^\infty_c(B)$, and smooth maps $\sigma_1, \ldots, \sigma_6: \mathcal{U} \to \text{Diff}^\infty_c(\mathbb{R}^n)$ so that $\sigma_i(\text{id}) = \text{id}$ and, for all $g \in \mathcal{U}$,

$$g = [\sigma_1(g), \exp(X_1)] \circ \cdots \circ [\sigma_6(g), \exp(X_6)].$$

Moreover, the vector fields $X_i$ may be chosen arbitrarily close to zero with respect to the strong Whitney $C^\infty$-topology.

**Proof.** Fix $p, q \geq 1$ so that $n = p + q$. Without loss of generality we may assume $B = B^p \times B^q$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the foliations with leaves $\mathbb{R}^p \times \{pt\}$ and $\{pt\} \times \mathbb{R}^q$ on $\mathbb{R}^n$, respectively. According to Lemma 1 there exist compactly supported smooth vector fields $X_4, X_5, X_6$ on $\mathbb{R}^n$, a $\mathcal{C}^\infty$-open neighborhood $\mathcal{W}$ of the identity in $\text{Diff}^\infty_c(B; \mathcal{F}_2)$, and smooth maps $\tilde{\sigma}_4, \tilde{\sigma}_5, \tilde{\sigma}_6: \mathcal{W} \to \text{Diff}^\infty_c(\mathbb{R}^n)$ so that $\tilde{\sigma}_i(\text{id}) = \text{id}$ and, for all $g \in \mathcal{W}$,

$$g = [\tilde{\sigma}_4(g), \exp(X_4)] \circ [\tilde{\sigma}_5(g), \exp(X_5)] \circ [\tilde{\sigma}_6(g), \exp(X_6)].$$

Similarly, there exist compactly supported smooth vector fields $X_1, X_2, X_3$ on $\mathbb{R}^n$, a $\mathcal{C}^\infty$-open neighborhood $\mathcal{V}$ of the identity in $\text{Diff}^\infty_c(B; \mathcal{F}_1)$, and smooth maps $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3: \mathcal{V} \to \text{Diff}^\infty_c(\mathbb{R}^n)$ so that $\tilde{\sigma}_i(\text{id}) = \text{id}$ and, for all $g \in \mathcal{V}$,

$$g = [\tilde{\sigma}_1(g), \exp(X_1)] \circ [\tilde{\sigma}_2(g), \exp(X_2)] \circ [\tilde{\sigma}_3(g), \exp(X_3)].$$

In view of Lemma 1 the composition $\text{Diff}^\infty_c(B; \mathcal{F}_1) \times \text{Diff}^\infty_c(B; \mathcal{F}_2) \to \text{Diff}^\infty_c(\mathbb{R}^n)$ admits a smooth local right inverse $\rho = (\rho_1, \rho_2): \mathcal{U} \to \mathcal{V} \times \mathcal{W}$ at the identity. Hence the smooth maps $\sigma_i := \tilde{\sigma}_i \circ \rho_1$, $1 \leq i \leq 3$, and $\sigma_i := \tilde{\sigma}_{i-3} \circ \rho_2$, $4 \leq i \leq 6$, will have the desired property. \hfill \Box

### 4. Smooth fragmentation

Suppose $\mathcal{U} \subseteq M$ is an open subset. Every compactly supported diffeomorphism of $\mathcal{U}$ can be regarded as a compactly supported diffeomorphism of $M$ by extending it identically outside $\mathcal{U}$. The resulting injective homomorphism $\text{Diff}^\infty_c(\mathcal{U}) \to \text{Diff}^\infty_c(M)$ is clearly smooth. Note, however, that a curve in $\text{Diff}^\infty_c(\mathcal{U})$, which is smooth when considered as a curve in $\text{Diff}^\infty_c(M)$, need not be smooth as a curve into $\text{Diff}^\infty_c(\mathcal{U})$. Nevertheless, if there exists a closed subset $A$ of $M$ with $A \subseteq \mathcal{U}$ and if the curve has support contained in $A$, then one can conclude that the curve is smooth in $\text{Diff}^\infty_c(\mathcal{U})$, too. This follows immediately from the characterization of smooth curves given at the beginning of Section 3.1.

The following is a folklore statement which, in one form or the other, can be found all over the literature on diffeomorphism groups [1]. For the reader’s convenience we include a version emphasizing the smoothness of the construction.

**Proposition 3 (Fragmentation).** Let $M$ be a smooth manifold of dimension $n$, and suppose $U_1, \ldots, U_k$ is an open covering of $M$, i.e. $M = U_1 \cup \cdots \cup U_k$. Then the smooth map

$$P: \text{Diff}^\infty_c(U_1) \times \cdots \times \text{Diff}^\infty_c(U_k) \to \text{Diff}^\infty_c(M), \quad P(g_1, \ldots, g_k) := g_1 \circ \cdots \circ g_k,$$

admits a smooth local right inverse at the identity.
Proof. Let \( \pi : TM \to M \) denote the projection of the tangent bundle. Choose a Riemannian metric on \( M \) and let \( \exp \) denote the corresponding exponential map. Choose an open neighborhood of the zero section \( W \subseteq TM \) such that \( (\pi, \exp) : W \to M \times M \) is a diffeomorphism onto an open neighborhood of the diagonal. Fix a linear connection \( \nabla \) on \( TM \), let \( W' \subseteq T^*M \otimes TM \) be an open neighborhood of the zero section, and set
\[
W := \{ X \in \mathfrak{X}_c(M) \mid \text{img}(X) \subseteq W, \text{img}(\nabla X) \subseteq W' \}.
\]
This is a \( C^1 \)-open, hence \( C^\infty \)-open, zero neighborhood in \( \mathfrak{X}_c(M) \). For \( X \in W \) define \( f_X := \exp \circ X \in \mathcal{C}^\infty_c(M, M) \). Choosing \( W \) and \( W' \) sufficiently small, we may assume that every \( f_X, X \in W \), is a diffeomorphism. The map
\[
W \to \mathcal{D}iff^\infty_c(M), \quad X \mapsto f_X
\]
provides a chart of \( \mathcal{D}iff^\infty_c(M) \) centered at the identity. This is the standard way to put a smooth structure on \( \mathcal{D}iff^\infty_c(M) \), see [9, Section 42.1].

Choose a smooth partition of unity \( \lambda_1, \ldots, \lambda_k \) with \( \text{supp}(\lambda_i) \subseteq U_i \), \( 1 \leq i \leq k \), and define
\[
\mathcal{V} := \{ X \in W \mid (\lambda_1 + \cdots + \lambda_i)X \in W \text{ for all } 1 \leq i \leq k \}.
\]
This is a \( C^1 \)-open, hence \( C^\infty \)-open, zero neighborhood in \( \mathfrak{X}_c(M) \), and \( \mathcal{V} \subseteq W \). For \( X \in \mathcal{V} \) and \( 1 \leq i \leq k \) set \( X_i := (\lambda_1 + \cdots + \lambda_i)X \), and note that \( f_{X_i} \in \mathcal{D}iff^\infty_c(M) \). Clearly, \( \text{supp}(f_{X_i}) \subseteq \text{supp}(\lambda_i) \subseteq U_i \). Moreover, for \( 1 < i \leq k \) we have \( X_{i-1} = X_i \) on \( M \setminus \text{supp}(\lambda_i) \), and thus \( f_{X_{i-1}} = f_{X_i} \) on \( M \setminus \text{supp}(\lambda_i) \). We conclude that the diffeomorphism \( (f_{X_{i-1}})^{-1} \circ f_{X_i} \) has compact support contained in \( \text{supp}(\lambda_i) \subseteq U_i \), \( 1 < i \leq k \).

Let \( \mathcal{U} \subseteq \mathcal{D}iff^\infty_c(M) \) denote the \( C^\infty \)-open neighborhood of the identity in \( \mathcal{D}iff^\infty_c(M) \) corresponding to \( \mathcal{V} \) via (2), and define a map
\[
\sigma : \mathcal{U} \to \mathcal{D}iff^\infty_c(U_1) \times \cdots \times \mathcal{D}iff^\infty_c(U_k)
\]
\[
\sigma(f_X) := \left( f_{X_1}, (f_{X_1})^{-1} \circ f_{X_2}, (f_{X_2})^{-1} \circ f_{X_3}, \ldots, (f_{X_{k-1}})^{-1} \circ f_{X_k} \right).
\]
Since the support of \( (f_{X_{i-1}})^{-1} \circ f_{X_i} \) is contained in \( \text{supp}(\lambda_i) \subseteq U_i \), it is clear that \( \sigma \) is a smooth map, cf. the remark at the beginning of this section. Obviously we have \( P(\sigma(f_X)) = f_{X_k} = f_X \) and thus \( P \circ \sigma = \text{id}_\mathcal{U} \). Moreover, it is immediate from the construction that \( \sigma(\text{id}) = \sigma(f_0) = (\text{id}, \ldots, \text{id}) \).

Combining Propositions 2 and 3 permits to show the following result.

**Proposition 4.** Let \( M \) be a smooth manifold of dimension \( n \geq 2 \), set \( N := 6(n+1) \), and suppose \( U \) and \( V \) are \( C^\infty \)-open subsets of \( M \) such that \( \bar{U} \subseteq V \). Then there exist smooth complete vector fields \( X_1, \ldots, X_N \) on \( M \) with \( \text{supp}(X_i) \subseteq V \), a \( C^\infty \)-open neighborhood \( \mathcal{U} \) of the identity in \( \mathcal{D}iff^\infty_c(U) \) and smooth maps \( \sigma_1, \ldots, \sigma_N : \mathcal{U} \to \mathcal{D}iff^\infty_c(V) \) such that \( \sigma_i(\text{id}) = \text{id} \) and, for all \( g \in \mathcal{U} \),
\[
\sigma = [\sigma_1(g), \exp(X_1)] \circ \cdots \circ [\sigma_N(g), \exp(X_N)].
\]
Moreover, the vector fields \( X_i \) may be chosen arbitrarily close to zero with respect to the strong Whitney \( C^\infty \)-topology.

**Proof.** It is well known that there exist open subsets \( U_1, \ldots, U_{n+1} \) of \( M \) so that
\[
\bar{U} \subseteq U_1 \cup \cdots \cup U_{n+1} \subseteq V.
\]
and such that each \( U_i, 1 \leq i \leq n + 1 \), is diffeomorphic to a disjoint union of copies of the open unit ball \( B^n \). Moreover, we may assume that there exist embeddings \( \varphi_i: \bigcup_{\alpha \in A_i} \mathbb{R}^n \to V \), with index sets \( A_i \), so that

\[
\varphi_i \left( \bigcup_{\alpha \in A_i} B^n \right) = U_i, \quad 1 \leq i \leq n + 1.
\]

Applying Proposition 2 to each connected component of \( U_i \), we find complete vector fields \( X_{i,1}, \ldots, X_{i,6} \) on \( M \) with \( \text{supp}(X_{i,j}) \subseteq V \), a \( \infty \)-open neighborhood \( U_i \) of the identity in \( \text{Diff}_c^\infty(U_i) \) and smooth maps \( \sigma_{i,1}, \ldots, \sigma_{i,6}: U_i \to \text{Diff}_c^\infty(V) \) so that \( \sigma_{i,j}(\text{id}) = \text{id} \) and, for all \( g \in U_i \),

\[
g = [\sigma_{i,1}(g), \exp(X_{i,1})] \circ \cdots \circ [\sigma_{i,6}(g), \exp(X_{i,6})].
\]

Moreover, the vector fields \( X_{i,j} \) may be chosen arbitrarily close to zero with respect to the strong Whitney \( C^\infty \)-topology on \( M \). In view of Proposition 3 the map

\[
\text{Diff}_c^\infty(U \cap U_1) \times \cdots \times \text{Diff}_c^\infty(U \cap U_{n+1}) \to \text{Diff}_c^\infty(U),
\]

\[
(g_1, \ldots, g_{n+1}) \mapsto g_1 \circ \cdots \circ g_{n+1}
\]

admits a local smooth right inverse at the identity. Combining this with the \( \sigma_{i,j} \) above, we immediately obtain the statement.

Specializing Proposition 3 to \( U = M \), we obtain Theorem 2.

5. Reducing the Number of Commutators

Proceeding as in \([2]\) permits to reduce the number of commutators considerably, see also \([20]\) and \([21]\).

**Proposition 5.** Let \( M \) be a smooth manifold of dimension \( n \geq 2 \) and put \( N = 6(n + 1) \). Moreover, let \( U \) an open subset of \( M \) and suppose \( \phi \in \text{Diff}^\infty(M) \), not necessarily with compact support, such that the closures of the subsets

\[
U, \phi(U), \phi^2(U), \ldots, \phi^N(U)
\]

are mutually disjoint. Then there exists a smooth complete vector field \( X \) on \( M \), a \( \infty \)-open neighborhood \( U \) of the identity in \( \text{Diff}_c^\infty(U) \), and smooth maps \( g_1, g_2: U \to \text{Diff}_c^\infty(M) \) so that \( g_1(\text{id}) = g_2(\text{id}) = \text{id} \) and, for all \( g \in U \),

\[
g = [g_1(g), \phi] \circ [g_2(g), \exp(X)].
\]

Moreover, the vector field \( X \) may be chosen arbitrarily close to zero in the strong Whitney \( C^\infty \)-topology on \( M \).

**Proof.** Clearly, there exists an open subset \( V \) of \( M \) with \( \bar{U} \subseteq V \) such that the open subsets \( V, \phi(V), \phi^2(V), \ldots, \phi^N(V) \) are mutually disjoint. Define a smooth map \( \psi: \text{Diff}_c^\infty(V)^N \to \text{Diff}_c^\infty(M) \),

\[
\psi(g_1, \ldots, g_N) := \prod_{i=1}^N \phi^{i-1} \circ g_1 \circ \cdots \circ g_N \circ \phi^{-(i-1)}.
\]

If \( g, h \in \text{Diff}_c^\infty(V) \) and \( 0 \leq i \neq j \leq N \), then the diffeomorphisms \( \phi^i \circ g \circ \phi^{-i} \) and \( \phi^j \circ h \circ \phi^{-j} \) commute as their supports are disjoint. Hence

\[
g_1 \circ \cdots \circ g_N = [\psi(g_1, \ldots, g_N), \phi] \circ \prod_{i=1}^N \phi^i \circ g_i \circ \phi^{-i}, \quad (3)
\]
for all $g_1, \ldots, g_N \in \text{Diff}^\infty_c(V)$. According to Proposition 3 there exist smooth complete vector fields $X_1, \ldots, X_N$ on $M$ with $\text{supp}(X_i) \subseteq V$, a $C^\infty$-open neighborhood $U$ of the identity in $\text{Diff}^\infty_c(U)$ and smooth maps $\sigma_1, \ldots, \sigma_N : U \to \text{Diff}^\infty_c(V)$ so that

$$
\sigma_i(id) = \text{id} \quad \text{and, for all } g \in U,
$$

$$
g = [\sigma_1(g), \exp(X_1)] \circ \cdots \circ [\sigma_N(g), \exp(X_N)].
$$

Combining this with (3) we obtain, for all $g \in U$,

$$
g = [\varrho_1(g), \phi] \circ [\varrho_2(g), \exp(X)],
$$

where $\varrho_1, \varrho_2 : U \to \text{Diff}^\infty_c(M)$,

$$
\varrho_1(g) := \psi([\sigma_1(g), \exp(X_1)], \ldots, [\sigma_N(g), \exp(X_N)]),
$$

$$
\varrho_2(g) := \prod_{i=1}^N \phi^i \circ \sigma_i(g) \circ \phi^{-i},
$$

and $X := \sum_{i=1}^N \phi_i^i X_i$, i.e. $\exp(X) = \prod_{i=1}^N \phi_i \circ \exp(X_i) \circ \phi^{-i}$. Since $X$ depends continuously on $X_1, \ldots, X_N$, we may assume $X$ to be arbitrarily close to zero with respect to the strong Whitney $C^\infty$-topology. Clearly, $\varrho_1$ and $\varrho_2$ are smooth, and we have $\varrho_1(id) = \varrho_2(id) = \text{id}$. □

**Lemma 5.** Let $M$ be a smooth manifold of dimension $n$. Then there exists an open covering $M = U_1 \cup U_2 \cup U_3$ and smooth complete vector fields $X_1, X_2, X_3$ on $M$ so that $\exp(X_1)(U_1) \subseteq U_2$, $\exp(X_2)(U_2) \subseteq U_3$, and such that the closures of the sets

$$
U_3, \; \exp(X_3)(U_3), \; \exp(X_3)^2(U_3), \; \ldots
$$

are mutually disjoint. Moreover, the vector fields $X_1, X_2, X_3$ may be chosen arbitrarily close to zero with respect to the strong Whitney $C^0$-topology. If $M$ admits a proper (circle valued) Morse function whose critical points all have index 0 or $n$, then we may, moreover, choose $U_1 = \emptyset$ and $X_1 = 0$.

**Proof.** Fix a proper Morse function $f : M \to \mathbb{R}$, see [3], and let $\mathcal{X}$ denote the set of critical points of $f$. Since $\mathcal{X}$ is a discrete subset of $M$, the properness of $f$ implies that the critical values form a discrete subset of $\mathbb{R}$. We may, moreover, assume that each critical level contains precisely one critical point, see [12]. For notational brevity we will write $\text{ind}(t)$ for the Morse index of the unique critical point corresponding to a critical value $t$ of $f$. For any subset $X \subseteq \mathbb{R}$, we introduce the notation $M_X := f^{-1}(X) \subseteq M$.

Let $U$ be a zero neighborhood in the space of vector fields on $M$ with respect to the strong Whitney $C^0$-topology. A neighborhood basis in this topology is obtained by taking neighborhoods of the zero section in $TM$ and considering vector fields which take values in this subset. We will assume that $U$ is of this form, for a neighborhood of the zero section which is fiberwise convex. Particularly, $U$ is invariant with respect to multiplication by functions whose modulus does not exceed 1. Below we will show that the following assertions hold true:

(a) Suppose $t$ is a regular value of $f$. Then, for sufficiently small $\varepsilon > 0$ and all $\eta > 0$ there exists a vector field $Y_2^{\varepsilon, \eta} \in U$ so that $\text{supp}(Y_2^{\varepsilon, \eta}) \subseteq M_{(t, t + \varepsilon + \eta)}$ and

$$
\exp(Y_2^{\varepsilon, \eta})(M_{(t, t + \varepsilon) \cup M_{(t, t + \eta)}}) \subseteq M_{(t, t + \eta)}.
$$
(b) Suppose \( t \) is a regular value of \( f \). Then, for sufficiently small \( \eta > 0 \) there exists a vector field \( Y^\eta \in \mathcal{U} \) with \( \text{supp}(Y^\eta) \subseteq M_{(t-\eta,t+\eta)} \) such that the closures of the subsets
\[
M_{(t-\eta,t+\eta)}, \ \exp(Y^\eta_1)(M_{(t-\eta,t+\eta)}), \ \exp(Y^\eta_2)(M_{(t-\eta,t+\eta)}), \ldots
\]
are mutually disjoint.

(c) Suppose \( t \) is a critical value of \( f \) and \( 0 < \text{ind}(t) < n \). Then, for sufficiently small \( \varepsilon > 0 \) and all \( \eta > 0 \), there exists an open covering \( M_{(t-\varepsilon,t+\varepsilon)} = V_1^\varepsilon \cup V_2^\varepsilon \) and vector fields \( Y_1^{\varepsilon,\eta}, Y_2^{\varepsilon,\eta} \in \mathcal{U} \) such that
\[
\exp(Y_1^{\varepsilon,\eta})(V_1^\varepsilon) \subseteq V_2^\varepsilon, \quad \text{supp}(Y_1^{\varepsilon,\eta}) \subseteq M_{(t-\varepsilon,t+\varepsilon+\eta)}, \quad \text{supp}(Y_2^{\varepsilon,\eta}) \subseteq M_{(t-\varepsilon,t+\varepsilon+\eta)}, \quad \text{and} \quad \exp(Y_2^{\varepsilon,\eta})(V_2^\varepsilon) \subseteq M_{(t-\varepsilon,t-\varepsilon+\eta)}.
\]

(d) Suppose \( t \) is a critical value of \( f \) with \( \text{ind}(t) = 0 \), i.e. a local maximum. Then, for sufficiently small \( \varepsilon > 0 \) and all \( \eta > 0 \), there exists a vector field \( Y_2^{\varepsilon,\eta} \in \mathcal{U} \) such that \( \text{supp}(Y_2^{\varepsilon,\eta}) \subseteq M_{(t-\varepsilon,t+\varepsilon+\eta)} \) and
\[
\exp(Y_2^{\varepsilon,\eta})(M_{(t-\varepsilon,t+\varepsilon)}) \subseteq M_{(t-\varepsilon,t-\varepsilon+\eta)}.
\]

(e) Suppose \( t \) is a critical value of \( f \) with \( \text{ind}(t) = 0 \), i.e. a local minimum. Then, for sufficiently small \( \varepsilon > 0 \) and all \( \eta > 0 \), there exists a vector field \( Y_2^{\varepsilon,\eta} \in \mathcal{U} \) such that \( \text{supp}(Y_2^{\varepsilon,\eta}) \subseteq M_{(t-\varepsilon,t-\varepsilon+\eta)} \) and
\[
\exp(Y_2^{\varepsilon,\eta})(M_{(t-\varepsilon,t+\varepsilon)}) \subseteq M_{(t+\varepsilon-\varepsilon,t+\varepsilon)}.
\]

We postpone the proof of these statements and first indicate how the lemma can be derived from them. For each critical value \( t \) of \( f \) we choose \( \varepsilon_t > 0 \) and \( \eta_t > 0 \) as in (c), (d) or (e), respectively, depending on \( \text{ind}(t) \). Shrinking \( \varepsilon_t \) and \( \eta_t \) we may assume that the intervals \( \{t - \varepsilon_t - \eta_t, t + \varepsilon_t + \eta_t\} \) are mutually disjoint. We let
\[
U_1 := \bigcup_{0 < \text{ind}(t) < n} V_1^{\varepsilon_t}, \quad X_1 := \sum_{0 < \text{ind}(t) < n} Y_1^{\varepsilon_t,\eta_t}, \quad U_2 := \bigcup_{0 < \text{ind}(t) < n} V_2^{\varepsilon_t}, \quad X_2 := \sum_{0 < \text{ind}(t) < n} Y_2^{\varepsilon_t,\eta_t},
\]
where \( t \) runs through all critical values of \( f \) corresponding to critical points of index different from 0 and \( n \). Moreover, \( V_1^{\varepsilon_t}, V_2^{\varepsilon_t}, Y_1^{\varepsilon_t,\eta_t}, Y_2^{\varepsilon_t,\eta_t}, Y_2^{\varepsilon_t,\eta_t} \), denote the open subsets and vector fields constructed in (f), respectively. Furthermore, we set
\[
U''_2 := U_2 \cup \bigcup_{\text{ind}(t) = 0,n} M(t-\varepsilon_t, t+\varepsilon_t), \quad X''_2 := X_2 + \sum_{\text{ind}(t) = 0,n} Y_2^{\varepsilon_t,\eta_t},
\]
where \( t \) runs through all critical values of \( f \) corresponding to critical points of index 0 or \( n \), and the vector fields \( Y_2^{\varepsilon_t,\eta_t} \) are the ones constructed in (d) or (e), respectively. By construction we have \( X_1, X_2'' \in \mathcal{U}, \exp(X_1)(U_1) \subseteq U''_2, U_1 \cup U''_2 = M(t-\varepsilon_t, t+\varepsilon_t), \) where the latter union is over all critical values, and
\[
\exp(X_2')(U_2') \subseteq \bigcup_{\text{ind}(t) > 0} M(t-\varepsilon_t, t-\varepsilon_t+\eta_t) \sqcup \bigcup_{\text{ind}(t) = 0} M(t+\varepsilon_t, t-\varepsilon_t+\eta_t).
\]
Note that we are still free to shrink all \( \eta_t \) without affecting these properties. Using (a) and (b) it is now clear how to complete the construction, the open sets \( U_2 \) and \( U_3 \) can be chosen to be of the form \( U_2 = U''_2 \cup M_I \) and \( U_3 = M_J \) where \( I \) and \( J \) are disjoint unions of suitably chosen intervals.
It thus remains to verify assertions (2) through (6) above. If \( t \) is a regular value of \( f \), then there exists an open interval \( I \) containing \( t \) and a diffeomorphisms \( M_I \cong M_t \times I \) intertwining the map \( f: M_I \to I \) with the standard projection \( M_I \times I \to I \), see [13]. In order to prove statements (2) and (6) it thus suffices to write down appropriate vector fields on \( I \) which is straightforward.

Now suppose \( t \) is a critical value of \( f \) with corresponding critical point \( y \) of index \( k \). Choose a Morse chart \([13]\) centered at \( y \), i.e. \( (x_1, \ldots, x_n): W \to \mathbb{R}^n \) are local coordinates such that

\[
f|_W = f(y) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2
\]
on the open neighborhood \( W \) of \( y \). We fix a Riemannian metric \( g \) on \( M \) such that its restriction to \( W \) is the standard Euclidean metric, i.e.

\[
g|_W = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n.
\]
The gradient vector field \( X \) := \( -\text{grad}(f) \) thus has a simple linear form on \( W \),

\[
X|_W = 2x_1 \partial_1 + \cdots + 2x_k \partial_k - 2x_{k+1} \partial_{k+1} - \cdots - 2x_n \partial_n,
\]
where \( \partial_i = \frac{\partial}{\partial x_i}, 1 \leq i \leq n \), denote the associated coordinate vector fields. Set

\[
V_1^\varepsilon := M_{(t-\varepsilon, t+\varepsilon)} \cap \{ x \in W : x_1^2 + \cdots + x_k^2 < \varepsilon/2 \}
\]
and

\[
V_2^\varepsilon := M_{(t-\varepsilon, t+\varepsilon)} \setminus \{ x \in W : x_1^2 + \cdots + x_k^2 \leq \varepsilon/3 \}.
\]
Clearly, \( M_{(t-\varepsilon, t+\varepsilon)} = V_1^\varepsilon \cup V_2^\varepsilon \), and \( V_2^\varepsilon \) is open. For sufficiently small \( \varepsilon > 0 \) the set \( V_2^\varepsilon \)
will be open too. By construction, the flow lines of \( X \) starting at points in \( V_2^\varepsilon \) remain bounded away from the critical point \( y \) by a positive distance. For sufficiently small \( \varepsilon > 0 \), we thus have \( \exp(X)(V_2^\varepsilon) \subseteq M_{(-\infty, t-\varepsilon)} \). Multiplying \( X \) with an appropriate function, we get a vector field \( Y_{2,\eta}^\varepsilon \in \mathcal{U} \) such that \( \text{supp}(Y_{2,\eta}^\varepsilon) \subseteq M_{(t-\varepsilon, t+\varepsilon+\eta)} \), and

\[
\exp(Y_{2,\eta}^\varepsilon)(V_2^\varepsilon) \subseteq M_{(t-\varepsilon, t-\varepsilon+\eta)}.
\]
Moreover, if \( k \neq 0 \), then after possibly shrinking \( \varepsilon \), it is straightforward to write down a vector field \( Y_{1,\eta}^\varepsilon \in \mathcal{U} \), supported in \( W \), such that \( \exp(Y_{1,\eta}^\varepsilon)(V_1^\varepsilon) \subseteq V_2^\varepsilon \) and \( \text{supp}(Y_1^\varepsilon) \subseteq M_{(t-\varepsilon, t+\varepsilon+\eta)} \). This shows (6).

To see (5), i.e. the case \( k = n \), note first that \( M_{(t-2\varepsilon, t+2\varepsilon)} \cap W \) is a connected component of \( M_{(t-2\varepsilon, t+2\varepsilon)} \), for sufficiently small \( \varepsilon > 0 \). On \( M_{(t-\varepsilon, t+\varepsilon+\eta)} \setminus W \) we construct the vector field \( Y_{1,\eta}^\varepsilon \) as above. On \( M_{(t-2\varepsilon, t+2\varepsilon)} \cap W \) it is straightforward to write down a vector field \( Y_{1,\eta}^\varepsilon \) with the desired properties explicitly.

Finally, statement (4) follows from (5) by considering \(-f\).

We are now in a position to complete the proof of Theorem [11]. Fix an open covering \( M = U_1 \cup U_2 \cup U_3 \) and smooth complete vector fields \( X_1, X_2, X_3 \) as in Lemma [5] above. According to Proposition [8] there exists a \( c^\infty \)-open neighborhood \( \mathcal{V} \) of the identity in \( \text{Diff}^\infty_c(M) \) and smooth maps \( \tilde{\sigma}_i: \mathcal{V} \to \text{Diff}^\infty_c(U_i) \) such that \( \tilde{\sigma}_i(\text{id}) = \text{id} \) and, for all \( g \in \mathcal{V} \),

\[
g = \tilde{\sigma}_1(g) \circ \tilde{\sigma}_2(g) \circ \tilde{\sigma}_3(g).
\]
A trivial computation shows that for all \( g \in \mathcal{V} \) we have

\[
g = [\sigma_1(g), \exp(X_1)] \circ [\sigma_2(g), \exp(X_2)] \circ \phi(g),
\]
where \( \sigma_1(g) := \tilde{\sigma}_1(g) \) and
\[
\sigma_2(g) := \exp(X_1) \circ \sigma_1(g) \circ \exp(X_1)^{-1} \circ \tilde{\sigma}_2(g),
\]
\[
\phi(g) := \exp(X_2) \circ \sigma_2(g) \circ \exp(X_2)^{-1} \circ \tilde{\sigma}_3(g).
\]
Clearly, these expressions define smooth maps \( \sigma_1 : \mathcal{V} \to \text{Diff}^\infty_c(U_1) \subseteq \text{Diff}^\infty(M) \), \( \sigma_2 : \mathcal{V} \to \text{Diff}^\infty_c(U_2) \subseteq \text{Diff}^\infty_c(M) \), and \( \phi : \mathcal{V} \to \text{Diff}^\infty_c(U_3) \) such that \( \sigma_1(\text{id}) = \sigma_2(\text{id}) = \phi(\text{id}) = \text{id} \). In view of Proposition 5 and Lemma 5 there exists a smooth complete vector field \( X_4 \) on \( M \), a \( \text{C}^\infty \)-open neighborhood \( \mathcal{W} \) of the identity in \( \text{Diff}^\infty_c(U_3) \), and smooth maps \( \varrho_1, \varrho_2 : \mathcal{W} \to \text{Diff}^\infty_c(M) \) so that \( \varrho_1(\text{id}) = \varrho_2(\text{id}) = \text{id} \), and
\[
h = [\varrho_1(h), \exp(X_3)] \circ [\varrho_2(h), \exp(X_4)],
\]
for all \( h \in \mathcal{W} \). Note that \( \mathcal{U} := \phi^{-1}(\mathcal{W}) \) is a \( \text{C}^\infty \)-open neighborhood of the identity in \( \text{Diff}^\infty_c(M) \). Defining smooth maps \( \sigma_3, \sigma_4 : \mathcal{U} \to \text{Diff}^\infty_c(M) \) by \( \sigma_3(g) := \varrho_1(\phi(g)) \) and \( \sigma_4(g) = \varrho_2(\phi(g)) \), we obtain \( \sigma_3(\text{id}) = \sigma_4(\text{id}) = \text{id} \) and \( g = [\sigma_3(g), \exp(X_1)] \circ [\sigma_2(g), \exp(X_2)] \circ [\sigma_3(g), \exp(X_3)] \circ [\sigma_4(g), \exp(X_4)] \), for all \( g \in \mathcal{U} \). It follows immediately from the corresponding assertions in Proposition 5 and Lemma 5 that the vector fields \( X_1, \ldots, X_4 \) may be chosen arbitrarily close to zero in the strong Whitney \( C^0 \)-topology. If \( M \) admits a (circle) valued proper Morse function whose critical points all have index 0 or \( n \), then we may assume \( X_1 = 0 \), see Lemma 5. Whence, in this case,
\[
g = [\sigma_2(g), \exp(X_2)] \circ [\sigma_3(g), \exp(X_3)] \circ [\sigma_4(g), \exp(X_4)],
\]
for all \( g \in \mathcal{U} \). This completes the proof of Theorem 1.

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