Public Channel Cryptography: Chaos Synchronization and Hilbert’s Tenth Problem

Ido Kanter,1 Evi Kopelowitz,1 and Wolfgang Kinzel2

1Department of Physics, Bar-Ilan University, Ramat-Gan, 52900 Israel
2Institute for Theoretical Physics, University of Würzburg, Am Hubland, 97074 Würzburg, Germany

The synchronization process of two mutually delayed coupled deterministic chaotic maps is demonstrated both analytically and numerically. The synchronization is preserved when the mutually transmitted signal is concealed by two commutative private filters that are placed on each end of the communication channel. We demonstrate that when the transmitted signal is a convolution of the truncated time delayed output signals or some powers of the delayed output signals synchronization is still maintained. The task of a passive attacker is mapped onto Hilbert’s tenth problem, solving a set of nonlinear Diophantine equations, which was proven to be in the class of NP-Complete problems. This bridge between two different disciplines, synchronization in nonlinear dynamical processes and the realm of the NPC problems, opens a horizon for a new type of secure public-channel protocols.

Chaotic systems are very unpredictable and two chaotic systems, starting from almost identical initial states, end in completely uncorrelated trajectories. Nevertheless, two chaotic systems which are coupled by some of their internal variables may synchronize to a common identical chaotic motion. Unpredictability or chaos synchronization, of coupled chaotic systems, have attracted a lot of attention, mainly because of the potential to build a secure communication protocol based on artificial chaotic systems or coupled chaotic lasers.

The security of a public-key encryption protocol based on chaos synchronization relies on the fact that two chaotic systems, A and B, synchronize by bi-directional interaction whereas a third unit E, which is only driven by the transmitted signal cannot synchronize. However, it is not obvious that this is possible at all. On one hand, the two mutually coupled chaotic systems influence the dynamics of each other and can accelerate the synchronization by enhancing coherent moves, whereas the unidirectionally coupled system, an attacker, cannot influence the synchronization process. On the other hand, the attacker is allowed to record and to manipulate his recorded signals, without affecting the synchronization process. Note that the two partners, A and B, are not allowed to exchange any secret information; the attacker E knows all the details which A knows about the system of B and vice versa.

For identical partners which synchronize by a bi-directional signal we recently presented a proof that an attacking unit coupled unidirectionally can synchronize as well. The proof is valid for any type of transmitted signals, for instance, a nonlinear function of the time delayed output signals. For non-identical partners which can synchronize, using for instance private commutative filters, it may be difficult for the attacker to synchronize and to reveal the time dependent output signal of the parties, but one cannot exclude efficient advanced software or hardware attacks. A hardware attacker consists of a similar chaotic setup to those of the synchronized chaotic partners, whereas a software attacker is able to mathematically manipulate the recorded signal.

In order to exclude any possible software advanced attack, we map the task of the attacker onto one of the NP-Complete (NPC) problems. The NPC problems are the most difficult problems in NP (non-deterministic polynomial time) and at present, all known deterministic algorithms for NPC problems require running time that is exponential with some tunable parameters of the problem. The main goal of this Letter is to bridge between two different disciplines, synchronization in nonlinear dynamics and the realm of the NPC problems. The establishment of such a bridge proves the lack of any possible efficient software attack, while the mutually coupled chaotic partners are synchronized. Note that the definition of the known NPC problems is static, and here we map a dynamical process onto an NPC problem.

Hilbert’s tenth problem is the tenth on the list of Hilbert’s problems of 1900. Its statement is as follows; given a set of Diophantine equations, polynomials with integer coefficients, finding an integer solution that satisfies the set. The solution of a general set of Diophantine equations is known to be undecidable. However, some subsets of the Diophantine equations are known to be decidable and belong to the class of NPC problems. A class of Hilbert’s tenth problem is to find an integer solution of the following set of Diophantine equations:

\[ D\vec{y} = \vec{\sigma}(z), \]

where \( D \) is an \( m \times n \) matrix of rational constants, \( \vec{y} = (y_1, ..., y_n) \) and \( \vec{\sigma} = (\sigma_1(z), ..., \sigma_m(z)) \) is a column vector. The \( \{\sigma_i(z)\} \) are polynomials with a finite degree greater than one. Finding a non-negative integer solution \( (y_1, ..., y_n, z) \) to the above set was proven to belong to the class of NPC problems. In this Letter we map the task of an attacker in the scenario of two synchronizing chaotic units onto this NPC problem.

We start by defining our synchronization process of two interacting units. Consider two iterated chaotic maps \( x^A \)}
and $x^B$, which are controlled by a general self-feedback function $S_f$ and a general coupling function $S_c$, which are both nonlinear functions of the history $\tau$ steps back

$$
\begin{align*}
x^A_t &= S_f(\vec{x}^A_t) + S_c(\vec{x}^B_t) \\
x^B_t &= S_f(\vec{x}^B_t) + S_c(\vec{x}^A_t)
\end{align*}
$$

(2)

where $\vec{x}_t = (x_{t-1}, ..., x_{t-\tau})$.

Do the two mutually coupled chaotic maps synchronize under such circumstances? The positive answer is demonstrated below for the simplest chaotic maps, the Bernoulli map.$^2$ The dynamics of the two mutually coupled units $x^A_t$ and $x^B_t$ can be analyzed analytically and is given by

$$
\begin{align*}
x_{t+1}^A &= (1-\varepsilon)f(x_{t}^A) + \varepsilon[\kappa f(x_{t-1}^A) + (1-\kappa)R^A(x_{t}^B)] \\
x_{t+1}^B &= (1-\varepsilon)f(x_{t}^B) + \varepsilon[\kappa f(x_{t-1}^B) + (1-\kappa)R^B(x_{t}^A)]
\end{align*}
$$

(3)

where $f(x) = (ax) \mod 1$, and a Bernoulli map is chaotic for $a > 1$. The parameter $\varepsilon$ indicates the weight of the delayed terms, $\kappa$ stands for the strength of the self-coupling term, and $R^A, B(x_{t}^A, B)$ are the received signals of each partner. Note that $[0,1]$ is the allowed range for $\varepsilon$ and $\kappa$. For the simple case of $R^A, B(x_{t}^A, B) = f^A, B(x_{t}^A, B)$, a linear expansion of the distance $d_t = x_{t+1}^A - x_{t+1}^B$ leads to $d_t = (1-\varepsilon)d_t + \varepsilon a(2\kappa-1)d_{t-1}$.$^{17, 18}$ By assuming that the distance converges/diverges exponentially in time, $d_t \propto d^t$, we find that the largest conditional Lyapunov exponent is negative and synchronization is achieved for $(a-1)/2(\varepsilon) < \kappa < (2\varepsilon + 1 - a)/2\varepsilon$ as is depicted in figure (1a).

In order to map the task of an attacker on this synchronization process to the presented NPC problem, we have to include the following four adjustments to the system: (a) private commutative filters, (b) transmission of integer signals, (c) additional nonlinear terms to the transmitted signal and (d) periods of cutoffs in communication. Our next goal is to explain each one of these adjustments and to show that synchronization is still maintained when applying all of the adjustments simultaneously, and finally to show that the task of the attacker is mapped onto the NPC problem, eq. (1).

The first adjustment is extending the configuration, equation (2), to the case of non-identical units $x^A$ and $x^B$. Both units are using different functions (filters) $g_A$ and $g_B$, and the two transmitted signals are $S_c(\vec{x}_t^A)$ and $S_c(\vec{x}_t^B)$, see figure 2. These functions are private, only $x^A$ knows $g_A$ and $x^B$ knows $g_B$. The coupling functions $S_c(\vec{x}_t^B)$, $S_c(\vec{x}_t^A)$ are simply the received signals which are $g_A(g_B(\vec{x}_t^B))$ and $g_B(g_A(\vec{x}_t^A))$, respectively. In order to preserve synchronization as a fixed point of the dynamics we only use filters that commute, $g_A(g_B(\vec{x})) = g_B(g_A(\vec{x}))$. Since an attacker does not know the filters he cannot use them for his hardware attack.
We now turn to discuss the complexity of a unidirectional listener. To avoid any software attack or any other advanced attacks we now map the task of the attacker to the NPC problem, eq. (1). Assuming a synchronization state $\tilde{x} = \tilde{x}^P = \tilde{x}_t$. In one time step, the transmitted signals on both directions, $T_{t+1}^{A,B}$, consist of $3N-2$ unknown variables: $\{K_{A,B}^\nu, f(x_0), ..., f(x_{t-N+1})\}$. On the next time step, two new equations emerge: $T_{t+1}^{A,B}$. These keys are normalized and $C_t > 0$ it is possible that the received signal is greater than one, however in practice it does not affect the synchronization process, and alternatively one can apply mod 1 again on the received signal. Both methods give the same regime of synchronization.

For the case of clipped keys and output signals simulations with $m_0 = 32$ indicate that the regime in the phase space where synchronization exists is only slightly affected by the quantization of the keys and the transmitted signals. A typical result for different values of $m$ is depicted in figure 1(a).

The last adjustment ((d)) of our setup is the implementation of dynamical filters. For $N_1$ steps the partners are using the above mentioned prescription. For the next $N_2$ steps no communication between the partners occurs, and each partner is updating his states following his own history of continuous signals with $\kappa = 1$ in eq. (3). After each period of silence, $N_2$, each partner is selecting a new set of private filters, and in addition, they select the nonlinear contribution to the transmitted signal to be a function of the signal at a time step, $N_0$, belonging to the previous silence period [19].

Simulations indicate that while the synchronization time and phase space are affected by the nonlinear additional term in eq. (1) and by the silence periods, $t_{\text{synch}}$ still scales linearly with $\tau$ as depicted in figure 3(b), and synchronization is achieved in a non-negligible fraction of the phase space. For instance, synchronization for $\rho_t = 2, 3, 4, 5$ with equal probability, $C_t \in [0, 0.1]$, $N = 10$, $N_1 = 20$, $N_2 = 20$ and $N_0 = 40(t \mod 40) - 5$ is depicted in figure 3(b).

FIG. 4: Simulation results for the fraction of the phase space, $(\varepsilon, \kappa)$, where synchronization is achieved for $\tau = 100$, $a = 1.1$, $N = 10$ and $\phi = 2$. (a) With quantized linear filters for $m = 3, 11$ (red-unclipped) (b) With quantization $m = 6$ and the same parameters as for Fig. 3(b).
equations consist of previously unknown variables and one new unknown variable \( f(x_{t+1}) \). Therefore by adding more time steps we are adding more equations than new variables. Actually the number of required equations to decode the keys of length \( N \) is \( 6(N-1) \). Therefore, the number of required iterations is \( 3(N-1) \). In order for a passive attacker to construct the entire signal, he needs to eavesdrop over at least \( 3(N-1) \) successive time steps. His task in such a scenario is therefore to solve a set of nonlinear Diophantine equations\[14, 15\]. The nonlinearity emerges since the attacker does not know neither the attacker is at least NPC, where the number of photons and in principle convolutional filters can be implemented.

We thank Johannes, Kestler, Uri Feige and Aviezri Fraenkel for many fruitful discussions.

[1] H. G. Schuster, W. Just. *Deterministic Chaos*. Wiley VCH, (2005).
[2] A. Pikovsky, M. Rosenblum, J. Kurths. *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge Univ. Press, N.Y. (2001).
[3] L. M. Pecora, T. L. Carroll, Phys. Rev. Lett. 64, 821 (1990).
[4] A. Abel and W. Scharz, Proc. IEEE, 90, 691 (2002).
[5] E. Klein, R. Mislovaty, I. Kanter, W. Kinzel, Phys. Rev. E 72, 016214 (2005).
[6] G. D. VanWiggeren, R. Roy, Science 279, 1198 (1998).
[7] A. Argyris, D. Syrvidis, L. Largier, V. Annonovazzi-Lodi, P. Colet, I. Fischer, J. Garcia-Ojalvo, C. R. Mirasso, L. Pesquera, K. A. Shore, Nature 438, 343 (2005).
[8] E. Klein, N. Gross, E. Kopelowitz, M. Rosenbluh, W. Kinzel, L. Khaykovich, I. Kanter, Phys. Rev. E 74, 046201 (2006).
[9] M. Rosen-Zvi, E. Klein, I. Kanter and W. Kinzel, Phys. Rev. E. 66, 066135 (2002).
[10] A. Klimov, A. Mityagin and A. Shamir, ASIACRYPT 288-298 (2002).
[11] I. Kanter, E. Kopelowitz, W. Kinzel and J. Kestler, arXiv:0712.2712v1.
[12] M. R. Garey and D. S. Johnson, *Computers and Intractability*, W H Freeman Corporation, (1979).
[13] Davis, M. Amer. Math. Monthly 80, 233-269 (1973).
[14] http://mathworld.wolfram.com/DiophantineEquation.html and references therein.
[15] C. H. Papadimitriou *Computational Complexity*, Addison Wesley (1994).
[16] E. M. Gurari and O. H. Ibarra, J. Assoc. Comput. Mach. 26, 567-581 (1979).
[17] J. Kestler, W. Kinzel, I. Kanter, Phys. Rev. E 76, 035202 (2007).
[18] S. Lerpi, G. Giacomelli, A. Politi, F. T. Arecchi, Physica D 70, 235 (1993).
[19] Synchronization under a scenario of silence periods is based on the fact that resynchronization time of two mutually coupled chaotic maps is shorter in comparison to desynchronization time. This inequality was also recently observed in an experiment of two mutually coupled semiconductor lasers, Phys. Rev. Lett. 98, 154101 (2007).
[20] F. Takens, in Dynamical Systems and Turbulence (Warwick 1980), edited by D. A. Rand and L.-S. Young (Springer-Verlag, Berlin, 1980), vol. 898 of Lecture Notes in Mathematics, pp. 366-381.
[21] C. H. Papadimitriou, JACM 28, 4 (1981) proves that this problem is in NP. A proof that this problem is in NPC was offered by Uri Feige based on the following reduction from SAT. For every variable \( z \) in the input SAT formula, introduce two variables in the system of equations, \( z_0 \) for a positive literal corresponding to the variable, \( z_1 \) for a negative literal and add the equation \( z_0 + z_1 = 3 \). This constraint forces exactly one of these literals to have value 1 (interpreted as “true”) and the other to have value 2 (interpreted as false). For every clause \( C_i \) add the equation stating that the sum of the literals in the clause plus a new variable \( c_i \) is equal to twice the number of variables in the clause. This can be satisfied by setting \( c_i \) to be equal to the number of satisfied literals in the clause, which is positive if and only if the formula is satisfiable.
[22] I. Borosh and A. S. Fraenkel, Math. Comp. 20, 107 (1966).