THE CAUCHY-CROFTON FORMULA AND THE WHITNEY ARC PROPERTY FOR DEFINABLE SETS

ELISA VASQUEZ RIFO

Abstract. We use the Cauchy-Crofton formula to show that every \( \mathbb{Q} \)-bounded definable cell in an O-minimal expansion of a field \( F \supseteq \mathbb{R} \) satisfies the Whitney arc property.

1. Introduction.

The purpose of this paper is to apply the Cauchy-Crofton formula to prove that a cell in an O-minimal expansion of an ordered field \( F \) satisfies an analog of the Whitney arc property. A subset \( A \subseteq \mathbb{R}^n \) satisfies the Whitney arc property if there is a \( K \in \mathbb{R}_{>0} \) such that for all \( x, y \in A \) there is a curve \( \gamma \) in \( A \) joining \( x \) and \( y \) with length(\( \gamma \)) \( \leq K|x - y| \) (Whitney, [7]).

Kurdyka shows in [4] that subanalytic subsets of \( \mathbb{R}^n \) have a stratification such that each strata is built out of Lipschitz functions. A consequence is that each strata satisfies the Whitney arc property. The author’s thesis [6] contains an analogous result for definable sets in an O-minimal expansion of an ordered field, and Pawlucki [5] proves that every definable set has a decomposition into subsets which, after a permutation of the coordinates, are cells built from Lipschitz functions.

By combining the results of this paper with the cell decomposition theorem we obtain a new, conceptually simpler proof that definable sets have decompositions into pieces which satisfy the Whitney arc property; moreover, any cell decomposition will do, and the curves witnessing the Whitney arc property may be chosen so that they vary definably with their endpoints (theorem 5.6). Berarducci and Otero’s work [1] allows one to define the length of a \( \mathbb{Q} \)-bounded definable curve.

Acknowledgements. I would like to thank Patrick Speissegger for suggesting the questions that became the topic of my dissertation [6] and inspired this work.

2. Preliminaries.

2.1. The Berarducci-Otero measure in an o-minimal structure. We fix an O-minimal expansion of an ordered field extension \( F \) of \( \mathbb{R} \). A set \( A \subseteq F^d \) is \( \mathbb{Q} \)-bounded if there is a \( q \in \mathbb{Q} \) such that \( A \subseteq [-q, q]^d \).

We give a description of the real-valued additive measure defined in [1]. This measure is defined on a Boolean algebra of of subsets of \( F^n \) which includes the definable \( \mathbb{Q} \)-bounded subsets of \( F^n \).

Definition 2.1. \( B \subseteq F^d \) is a polyrectangle of dimension \( d \) if \( B \) is a finite union of rectangles \( [q_1, r_1) \times \cdots \times [q_d, r_d) \) with rational coordinates \( q_i, r_i \). The set \( \mathcal{PR}^{(d)}(F) \) is the set of polyrectangles of dimension \( d \) of \( F \). The volume of a rectangle \( [q_1, r_1) \times \cdots \times [q_d, r_d) \) is

\[
\mu([q_1, r_1) \times \cdots \times [q_d, r_d]) = \prod_{i=1}^{d} (r_i - q_i).
\]

If a polyrectangle \( P \) is the disjoint union of rectangles \( R_i, \ i = 1, \ldots, m, \) then \( \mu(P) := \sum_{i=1}^{m} \mu(R_i) \).

Definition 2.2. Let \( A \subseteq F^d \) be a \( \mathbb{Q} \)-bounded set. The outer measure of \( A \) is:

\[
\mu^*(A) := \inf \{ \mu(P) : P \supseteq A, P \in \mathcal{PR}^{(d)}(F) \}.
\]
The inner measure of $A$ is:
\[ \mu_*(A) := \sup \{ \mu(P) : P \subset A, P \in \mathcal{P}R^{(d)}(\mathbb{F}) \}. \]

Here the infimum and supremum are taken in $\mathbb{R}$.

A $\mathbb{Q}$-bounded set $A$ is measurable if $\mu_*(A) = \mu^*(A)$, and in this case the measure of $A$ is defined as $\mu(A) := \mu_*(A) = \mu^*(A)$. One of the main results in [1] is the following

**Theorem 2.3.** Let $A$ be a $\mathbb{Q}$-bounded definable subset of $\mathbb{F}^d$. Then $A$ is measurable. Moreover, if $\dim(A) < d$ then $\mu(A) = 0$.

Let $A \subset \mathbb{F}^d$ be $\mathbb{Q}$-bounded. For $f : \mathbb{F}^d \to \mathbb{F}_{\geq 0}$, we define
\[ \int_A f := \mu([0, f)_A), \]
provided that $[0, f)_A := \{(x, y) : x \in A, 0 \leq y < f(x)\}$ is measurable. For general $f : \mathbb{F}^d \to \mathbb{F}$, we put
\[ \int_A f := \int_A f^+ - \int_A f^-, \]
provided both terms on the right exist, where $f^+$ and $f^-$ are, respectively, the positive and negative part of $f$.

This integral can be used to define the length of a definable $C^1$ curve $\gamma : (a, b) \to \mathbb{F}^n$, with $(a, b)$ and $\text{Im}(\gamma')$ $\mathbb{Q}$-bounded, by
\[ \text{length}(\gamma) := \int_{(a, b)} |\gamma'(x)|. \]

**2.2. The Cauchy-Crofton Formula.** Let $C_c(M)$ be the collection of compactly supported continuous real valued functions on a manifold $M$. A measure on $M$ is an $\mathbb{R}$-linear mapping $C_c(M) \to \mathbb{R}$ such that: for each compact $K \subset M$, there is a constant $m_K$ such that for every continuous $f$ with compact support contained in $K$,
\[ \int_M f \leq m_K \sup_{x \in M} |f(x)|. \]

Let $M$ be a manifold together with a left action of a Lie group $G$. A measure in $M$ is $G$-invariant if for all $g \in G$ and $f \in C_c(M)$,
\[ \int_M f = \int_M f \circ L_g, \]
where $L_g : M \to M$ denotes the left action by $g$, that is $L_g(m) = g \cdot m$.

Let $G$ be the group of isometries of $\mathbb{R}^n$. Let $AGr_{n-1}(\mathbb{R}^n)$ be the affine Grassmannian of hyperplanes in $\mathbb{R}^n$, that is, $AGr_{n-1}(\mathbb{R}^n)$ is the collection of all affine hyperplanes in $\mathbb{R}^n$. The group $G$ acts on $AGr_{n-1}(\mathbb{R}^n)$ in a canonical way, and $AGr_{n-1}(\mathbb{R}^n)$ has a measure which is invariant under the action by $G$. Furthermore, a $G$-invariant measure in $AGr_{n-1}(\mathbb{R}^n)$ is unique up to a constant factor.

The Cauchy-Crofton formula expresses the length of a compact, embedded curve in $\mathbb{R}^n$ as the average number of points of intersection of the curve with a hyperplane in $\mathbb{R}^n$.

**Theorem 2.4.** Let $G$ be the group of motions of $\mathbb{R}^n$. Then there is a $G$ invariant measure $dL$ on $AGr_{n-1}(\mathbb{R}^n)$ such that for any compact embedded 1-dimensional submanifold $\gamma$ of $\mathbb{R}^n$,
\[ \text{length}(\gamma) = \int_{AGr_{n-1}(\mathbb{R}^n)} |\gamma \cap L| dL, \]
where for $L \in AGr_{n-1}(\mathbb{R}^n)$, $|\gamma \cap L|$ is the number of points of intersection of $\gamma$ with $L$.

See [3] 3.18 for a proof.
3. The length of a definable $\mathbb{Q}$-bounded curve.

We prove in Theorem 3.3 that any definable $\mathbb{Q}$-bounded curve can be reparametrized by a definable, piecewise map with finite first and second derivatives.

An element $x$ of $\mathbb{F}^n$ is finite if it is bounded in magnitude by some natural number, infinite otherwise and infinitesimal if $|x| < r$ for every $r \in \mathbb{R}_{>0}$; in the last case we write $x \approx 0$. For a finite $x \in \mathbb{F}$ the monad of $x$, denoted by $\mu(x)$, consists of all $y \in \mathbb{F}$ with $y - x \approx 0$, we write $y \approx x$ for $y \in \mu(x)$ and we say that $y$ is infinitesimally close to $x$. For finite $x \in \mathbb{F}$ the standard part of $x$ is $st(x) := \sup\{r \in \mathbb{R} : r < x\}$. If $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ is finite, $st(x) := (st(x_1), \ldots, st(x_n))$.

**Lemma 3.1.** Let $f : (a, b) \to (c, d)$ be a definable and twice differentiable function, where $(a, b), (c, d) \subset \mathbb{F}$ are $\mathbb{Q}$-bounded. Then $\overline{f}$ is finite outside a finite union of monads.

**Proof.** For $r \in \mathbb{F}_{>0}$, let $A_r := \{x \in (a, b) : |f'(x)| > 1/r\}$. The family $\{A_r\}_{r \in \mathbb{F}_{>0}}$ is a definable family of sets. Let $r \approx 0$. By the mean value theorem (see (2)), any interval contained in $A_r$ is of infinitesimal length. Since $A_r$ is definable, it follows from $\mathbb{O}$-minimality that $A_r$ is contained in a finite union of monads; let $n_r$ be the minimum number of monads containing $A_r$. By cell decomposition, there is an $N \in \mathbb{N}$ such that for every $r \in \mathbb{F}_{>0}$ the set $A_r$ is a union of at most $N$ disjoint intervals and points. Let $s \approx 0$ be such that

$$n_s = \max_{r \approx 0} n_r,$$

and let $A$ be the finite union of the $n_s$ monads containing $A_s$. For $r < s$, $A_r \subset A_s$ so $A_r \subset A$. For $r > s$, $r \approx 0$, $A_s \subset A_r$ so $n_r$ is at most $n_s$; since $n_s$ is maximal, $n_r = n_s$ and therefore $A_r$ must be contained in $A$. $\overline{f}'$ is finite away from $A$.

Let $(a, b) \subset \mathbb{F}$. We define $(a, b)_{\mathbb{R}} := (a, b) \cap \mathbb{R}$. For a function $f : (a, b) \to \mathbb{F}^n$ with $\mathbb{Q}$-bounded image we define $\overline{f} : (a, b)_{\mathbb{R}} \to \mathbb{R}$ by $\overline{f}(x) = st(f(x))$. Similarly, if $A \subset \mathbb{F}^m$ and $f : A \to \mathbb{F}^n$ maps finite elements into finite elements, we define $\overline{f} : A_{\mathbb{R}} \to \mathbb{R}^n$ by $\overline{f}(x) = st(f(x))$ where $A_{\mathbb{R}} := A \cap \mathbb{R}^m$.

**Lemma 3.2.** Let $f : (a, b) \to (c, d)$ be a definable and twice differentiable function, where $(a, b), (c, d) \subset \mathbb{F}$ are $\mathbb{Q}$-bounded. Suppose that for $x \not\approx a, b$ both $f'$ and $f''$ are finite. Then $\overline{f}$ is differentiable on the interior of $(a, b)_{\mathbb{R}}$ and for $x \in (a, b)$ with $st(x) \in \text{Int}((a, b)_{\mathbb{R}})$, $st(f'(x)) = \overline{f}'(st(x))$.

**Proof.** We first consider the case where $0 \in (a, b)$ but $0 \not\approx a, b$, $f(0) = 0$, and $f'(0) = 0$. Let $\epsilon \in \mathbb{R}_{>0}$. If $\delta \approx 0$ and $\delta > 0$, then $\left|\frac{f(h)}{h}\right| < \epsilon$ whenever $|h| < \delta$. Otherwise there would be a $\delta > 0$, $\delta \approx 0$ and $h \in (a, b)$, $|h| < \delta$ with

$$|\frac{f(h)}{h}| > \epsilon,$$

thus by the mean value theorem there is an $x$ between $0$ and $h$ such that $|f'(x)| = \left|\frac{f(h)}{h}\right| > \epsilon$, and a $z$ between $0$ and $x$ with $|f''(z)| = \left|\frac{f'(x)}{x}\right|$, but this last fraction is infinite. This shows that the set

$$\{\delta \in \mathbb{F}_{>0} : \text{ for all } h \in (a, b), |h| < \delta \implies \left|\frac{f(h)}{h}\right| < \epsilon/2\}$$

contains all positive infinitesimals. This set is also definable, so by the cell decomposition theorem it is a finite union of intervals and points and therefore it must contain a positive real $\delta$. This shows that $\overline{f}$ is differentiable at $0$ and $\overline{f}'(0) = 0$.

For $x_0$ in $(a, b)$ with $st(x_0) \in \text{Int}((a, b)_{\mathbb{R}})$, consider the function

$$g(x) := f(x_0 - x) - f(x_0) - f'(x_0)x.$$

Since $g(0) = 0$, $g'(0) = 0$ and $0$ is not infinitesimally close to the endpoints of $\text{Dom}(g)$, $\overline{g}$ is differentiable at 0 and $\overline{g}'(0) = 0$. It follows that $\overline{f}$ is differentiable at $st(x_0)$ with derivative $st(f'(x_0))$. \qed
Lemma 3.2 shows that for the restriction with $C_{0} = 0$ and points the intermediate value theorem shows that Lemma 4.1. Let $\gamma : (a, b) \to \mathbb{F}^{n}$ be a definable curve with $Q$-bounded image. Then, there are $a_{0} = a < \cdots < a_{k} = b$ such that each restriction $\gamma|_{(a_{i}, a_{i+1})}$ is either constant or has a reparametrization $\sigma$ with $\sigma'$ finite, $\sigma''(x)$ finite for $x \neq a_{i}, a_{i+1}$, and with $\overline{\sigma}$ an embedded $C^{1}$-curve in $\mathbb{R}^{n}$.

Proof. By the $C^{1}$-cell decomposition theorem $\gamma$ is piecewise $C^{1}$, so without loss of generality we can assume that $\gamma$ is $C^{1}$. Also $\gamma' = 0$ in a finite union of intervals and points, and $\gamma$ is constant on those intervals where $\gamma' = 0$; thus we may assume that $\gamma' \neq 0$. Similarly we can assume that $\gamma$ is injective.

Let $Gr_{1}(\mathbb{F}^{n})$ be the Grassmannian of 1-dimensional subspaces of $\mathbb{F}^{n}$. Then $Gr_{1}(\mathbb{F}^{n})$ is the disjoint union of the definable sets

$$A_{i} := \{ l \in Gr_{1}(\mathbb{F}^{n}) : l = \langle v \rangle, \ |v_{i}| \geq |v_{j}| \text{ for } j \geq i \},$$

Let $\phi : (a, b) \to Gr_{1}(\mathbb{F}^{n})$ be the Gauss map of $\gamma$, that is $\phi(t) = (\gamma'(t))$. The sets $\phi^{-1}(A_{i})$ are definable, and therefore are a union of intervals and points. Suppose that $I$ is one of these intervals and let $J := \gamma_{I}(I)$. Since $\gamma'_{I} \neq 0$ on $I$, $J$ contains an interval; and since $J$ is a finite union of intervals and points the intermediate value theorem shows that $J$ is a single interval. Moreover, $J$ is $Q$-bounded. We define $\sigma_{I} : J \to \mathbb{F}^{n}$ as $\sigma_{I} := \gamma_{I} \circ \gamma_{I}^{-1}$. Then $\sigma_{I}$ is a $C^{1}$ function since $\gamma_{I}^{-1}$ is invertible with $C^{1}$ inverse. Moreover, $\sigma_{I}'$ is finite: for $x \in J$, $\sigma_{I}'(x)$ generates the line $\langle \gamma'(\gamma_{I}^{-1}(x)) \rangle \in A_{i}$ thus $(\sigma_{I})'_{j}(x) \geq (\sigma_{I})'_{j}(x)$, but $(\sigma_{I})'_{j}(x) = 1$.

By the cell decomposition theorem and Lemma 3.1, there are points $b_{0}, \ldots, b_{k}$ such that $J = \langle b_{0}, b_{k} \rangle$, $\sigma_{I}'$ and $\sigma_{I}''$ exist on $\langle b_{i}, b_{i+1} \rangle$ and are finite except possibly on the monads of $b_{i}$ and $b_{i+1}$. Lemma 3.2 shows that for the restriction $\sigma$ of $\sigma_{I}$ to one of this subintervals $\overline{\sigma}, \overline{\sigma}'$ and $\overline{\sigma}''$ are differentiable and $\overline{\sigma}' = st(\sigma')$, $\overline{\sigma}'' = st(\sigma'')$. Since $st(\sigma') = \overline{\sigma}'$ it follows that $\overline{\sigma}$ is twice differentiable. Finally, $(\overline{\sigma})_{i}(t) = t$, therefore for $(c, d) \subseteq \text{Dom}(\overline{\sigma})$ we have

$$\overline{\sigma}(c, d) = \{ x \in \mathbb{R}^{n} : c < x_{i} < d \} \cap \text{Im}(\overline{\sigma})$$

showing that $\overline{\sigma}$ is an open map and therefore an embedding.

□

4. The Cauchy-Crofton formula for the Berarducci-Otero length

We prove that for a $Q$-bounded, injective, definable curve $\gamma$ in $\mathbb{F}^{n}$, the length of $\gamma$ is the average number of points of intersection of $\gamma$ with an affine hyperplane defined over $\mathbb{R}$. The proof is a reduction to the standard Cauchy-Crofton formula for the length of a curve in $\mathbb{R}^{n}$. The main point is that the number of points of intersection of $\gamma$ with a hyperplane $L$ defined over the reals is the same as the number of points of intersection of the standard part of $\gamma$, namely $\overline{\gamma}$, with the real points of $L$, as long as $L$ is not tangent to the curve $\overline{\gamma}$.

Lemma 4.1. Let $f : B \to [0, q]$ be a definable function, where $B \subset \mathbb{F}^{n}$ is a $Q$-bounded box and $q \in Q$, then $\overline{f}$ is Riemann integrable, and

$$\int_{B} f = \int_{B_{\mathbb{R}}} \overline{f}.$$ 

Proof. Let $P$ be a polyrectangle. If $P \supset [0, f)$, then $P \supset [0, \overline{f})$. Thus, $\mu^{*}([0, f)) \geq \mu^{*}([0, \overline{f}))$. Also, if $P \subseteq [0, f)$ then $P \subseteq [0, \overline{f})$. Thus $\mu_{*}([0, f)) \leq \mu_{*}([0, \overline{f}))$. Since $[0, f)$ is $\mu$-measurable we get

$$\mu^{*}([0, \overline{f})) \leq \mu([0, f)) \leq \mu([0, \overline{f})).$$

But $\mu^{*}([0, \overline{f})) \geq \mu_{*}([0, \overline{f}))$, thus $[0, \overline{f})$ is $\mu$-measurable, so $\overline{f}$ is Riemann integrable. Moreover,

$$\int_{B_{\mathbb{R}}} \overline{f} = \mu([0, f)) = \int_{B} f.$$ 

□
Let $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ be a definable curve with $\mathbb{Q}$-bounded image $\gamma'$ finite and $\gamma''(x)$ finite for $x \neq 0, 1$. We will assume that $\overline{\gamma}$ is an embedded $C^1$ curve in $\mathbb{R}^n$.

**Lemma 4.2.** Let $f : AGr_{n-1}(\mathbb{F}^n) \rightarrow \mathbb{F}$ be the function

$$f(L) = \begin{cases} \lvert \gamma \cap L \rvert & \text{if } \lvert \gamma \cap L \rvert \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_0 := f|_{AGr_{n-1}(\mathbb{R}^n)}$, and let $g : AGr_{n-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be the function

$$g(L) = \begin{cases} \lvert \overline{\gamma} \cap L \rvert & \text{if } \lvert \overline{\gamma} \cap L \rvert \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{AGr_{n-1}(\mathbb{R}^n)} g = \int_{AGr_{n-1}(\mathbb{R}^n)} f_0.$$

**Proof.** Let $L \in AGr_{n-1}(\mathbb{F}^n) \subset AGr_{n-1}(\mathbb{R}^n)$, and denote by $L_{\mathbb{R}}$ the set of $\mathbb{R}$-points of $L$. Suppose that $L_{\mathbb{R}}$ intersects $\gamma$ transversely and let $p \in \gamma \cap L$. Then there are $t_0 < t_1$ such that $\gamma_{[t_0, t_1]} \cap L = \{p\}$ and $\overline{\gamma(t_0), \gamma(t_1)}$ lie on opposite sides of $L$. Then $\gamma(t_0), \gamma(t_1)$ must lie on opposite sides of $L$, so there is a $t \in (t_0, t_1)$ such that $\gamma(t) \in L$. Since $t \approx st(t)$ and $\gamma'$ is finite, $\gamma(st(t)) \approx \gamma(t)$. Thus $st(\gamma(st(t))) = st(\gamma(t)) \in st(L) = L_{\mathbb{R}}$, i.e. $\overline{\gamma(st(t))} \subset L$. But $L$ intersects $\gamma$ only at $p$ when the parameter runs in $[t_0, t_1]$ and $st(t) \in [t_0, t_1]$ so $\gamma(st(t)) = p$, in particular $\gamma(t) \approx p$. It follows that $g(L) \leq f(L)$ whenever $L$ is transverse to $\overline{\gamma}$.

On the other hand, if $f(L) > g(L)$ then there are two infinitesimally close points of $\gamma$ in $L$, that is, there are $\gamma(t_0), \gamma(t_1) \in L$ with $\gamma(t_0) \approx \gamma(t_1)$ and say $t_0 < t_1$. Since $\gamma$ is injective, we have $t_0 \approx t_1$. Assume $t_0 \neq 0, 1$. By Lemma 3.2 for all $s, t \approx t_0$, and $i = 1, \ldots, n$, we have $\gamma_i'(s) \approx \gamma_i'(t)$. By the mean value theorem, there are $u_1, \ldots, u_n \in (t_0, t_1)$ such that $\gamma_i'(u_i)(t_1 - t_0) = \gamma_i(t_1) - \gamma_i(t_0)$, therefore for all $t \approx t_0$,

$$\gamma'(t) \approx \frac{1}{t_1 - t_0} (\gamma(t_1) - \gamma(t_0)).$$

This means that $st(\gamma'(t))$ is parallel to $st(\frac{1}{t_1 - t_0} (\gamma(t_1) - \gamma(t_0)))$, in other words, if $l$ is the secant line through $\gamma(t_0), \gamma(t_1)$, we must have $st(l)$ tangent to $\gamma$ at $st(t_0)$. In particular, we have that $L$ is tangent to $\overline{\gamma}$ at some point of $\gamma$. Thus $f(L) \leq g(L)$ whenever $L$ is transverse to $\overline{\gamma}$ and not infinitesimally close to $\gamma(0), \gamma(1)$. We have shown that $f|_{AGr_{n-1}(\mathbb{R}^n)}$ and $g$ agree almost everywhere, thus the conclusion follows.

**Corollary 4.3.** Let $\gamma : [0, 1] \rightarrow \mathbb{F}^n$ be a definable curve with $\mathbb{Q}$-bounded image, $\gamma'$ finite and $\gamma''(x)$ finite for $x \neq 0, 1$. Suppose that $\overline{\gamma}$ is an embedded $C^1$ curve in $\mathbb{R}^n$. Then,

$$\text{length}(\gamma) = \int_{AGr_{n-1}(\mathbb{R}^n)} \lvert \gamma \cap L \rvert dL.$$

**Proof.** $\overline{\gamma}$ is an embedded curve, so by the Cauchy-Crofton formula

$$\text{length}(\overline{\gamma}) = \int_{AGr_{n-1}(\mathbb{R}^n)} \lvert \overline{\gamma} \cap L \rvert dL.$$

By Lemma 4.1, length($\gamma$) = length($\overline{\gamma}$), so the corollary follows immediately from lemma 4.2.

Let $\gamma : [0, 1] \rightarrow \mathbb{F}^n$ be a definable, injective curve with $\mathbb{Q}$-bounded image. Suppose that $0 = a_0 < \cdots < a_k = 1$ is a partition of $[0, 1]$ such that:

Each restriction $\gamma_{[a_i, a_{i+1})}$, $i = 0, \ldots, k - 1$, has a reparametrization $\alpha_i$ with $\alpha'_i$ finite, $\alpha''_i(x)$ finite for $x \neq 0, 1$, and $\overline{\alpha}_i$ an embedded $C^1$ curve in $\mathbb{R}^n$ (theorem 3.3) guarantees that such a partition
always exists). Then
\[
\sum_{i=0}^{k-1} \int_{\text{Gr}_{n-1}(\mathbb{R}^n)} |\alpha_i \cap L| dL = \int_{\text{Gr}_{n-1}(\mathbb{R}^n)} |\gamma \cap L| dL.
\]
Therefore we can define
\[
\text{length}(\gamma) := \sum_{i=0}^{k-1} \text{length}(\alpha_i),
\]
and this is independent of the partition and reparametrization chosen. We thus have:

**Corollary 4.4.** *(Cauchy-Crofton formula for the Berarducci-Otero length)* Let \( \gamma : [0, 1] \to \mathbb{F}^m \) be a definable, injective curve with \( \mathbb{Q} \)-bounded image. Then,
\[
\text{length}(\gamma) = \int_{\text{Gr}_{n-1}(\mathbb{R}^n)} |\gamma \cap L| dL.
\]

5. LENGTH IN DEFINABLE FAMILIES OF CURVES

We now prove that there is a bound on the lengths of the curves in a \( \mathbb{Q} \)-bounded definable family. We conclude by using this result to prove that cells have the Whitney arc property.

**Definition 5.1.** Let \( A \subset \mathbb{F}^n, B \subset \mathbb{F}^m \) be definable sets. Let \( \lambda \subset A \times ([0, 1] \times B) \subset \mathbb{F}^n \times \mathbb{F}^{1+m} \) be a definable set such that for every \( x \in A \), the fiber over \( x \)
\[
\lambda_x := \{ y \in [0, 1] \times B : (x, y) \in \lambda \}
\]
is a curve \( \lambda_x : [0, 1] \to B \). We view \( \lambda \) as describing the family of curves \( \{\lambda_x\}_{x \in A} \). Such a family is a definable family of curves (in \( B \), parametrized by \( A \)).

**Definition 5.2.** A definable \( \mathbb{Q} \)-bounded set \( A \subset \mathbb{F}^n \) satisfies the Whitney arc property if there is a number \( K \in \mathbb{R}_{\geq 0} \) such that for every \( x, y \in A \) there is a definable curve \( \gamma : [0, 1] \to A \) with \( \gamma(0) = a, \gamma(1) = b \) and \( \text{length}(\gamma) \leq K |x - y| \).

**Proposition 5.3.** If \( A \subset \mathbb{F}^m \) is definable and definably connected, then there is a definable family ofinjective curves \( \lambda \subset A^2 \times ([0, 1] \times A) \) such that for every \( a, a' \in A \), \( \lambda_{(a,a')}((0) = a, \lambda_{(a,a')}((1) = a' \), and \( \lambda_{(a,a')} \) is piecewise \( C^1 \).

**Proof.** We use induction on \( m \). The case \( m = 1 \) is trivial. For \( m > 1 \), assume first that \( A \) is a cell. By induction, we may assume that \( A \) is an open cell in \( \mathbb{F}^m \), for, if \( A \) itself is not open, then \( A \) is the graph of a function \( g : U \to \mathbb{F}^l, U \subset \mathbb{F}^{m-1} \), and we may lift the paths in \( U \) to paths in \( A \) by using \( g \). Let \( C \) be the projection of \( A \) into \( \mathbb{F}^{m-1} \) so that \( A = (f, g)_C \) for some definable functions \( f, g \) on \( C \). By induction there is a definable family of curves \( \Lambda \) in \( C \) with the required property. Assume that \( f, g \) take values in \( \mathbb{F} \) (the other cases are handled similarly). Let \( (y, r), (z, s) \in A \) with \( y, z \in C \). We first connect \((y, r) \) to \((y, (f(y) + g(y))/2) \) by a vertical path in \( A \). The path \( \Lambda_{(y,z)} \) in \( C \) connecting \( y \) and \( z \) lifts to the path
\[
t \to (\Lambda_{(y,z)}(t), (f(\Lambda_{(y,z)}(t)) + g(\Lambda_{(y,z)}(t)))/2)
\]
connecting \((y, (f(y) + g(y))/2) \) to \((z, (f(z) + g(z))/2) \). The last point can be connected to \((z, s) \) by a vertical path in \( A \). Concatenating these three paths, we get a path \( \lambda_{(y,r),(z,s)} \) in \( A \) connecting \((y, r) \) and \((z, s) \). The collection of these paths constitutes the required definable family.

In the general case, since \( A \) is definably connected, we can write it as the union of cells \( C_1, \ldots, C_k \), where for \( i < k \) either \( C_i \) intersects the closure of \( C_{i+1} \), or \( C_{i+1} \) intersects the closure of \( C_i \) (see Theorem 3.27). By definable choice (see Chapter 6, (1.2)), we can definably pick an element \( e(C_i, C_{i+1}) \) in \( C_i \cap \mathbb{F}^{i+1} \) (if \( C_i \cap \mathbb{F}^{i+1} \neq \emptyset \)) and a definable curve \( \gamma_i : [0, e] \to C_{i+1} \) such that \( \lim_{t \to 0} \gamma_i(t) = e(C_i, C_{i+1}) \). Combining this with the fact that the result was already proved for cells we get the desired family \( \lambda \). □
Lemma 5.4. Let $\lambda \subset A \times ([0, 1] \times B) \subset \mathbb{F}^n \times \mathbb{F}^{m+1}$ be a definable and $\mathbb{Q}$-bounded family of injective curves. Then there is a $K \in \mathbb{R}_{>0}$ such that for any $x \in A$, length$(\lambda_x) \leq K$.

Proof. Let $\lambda \subset A \times ([0, 1] \times B) \subset \mathbb{F}^n \times \mathbb{F}^{m+1}$ be a definable family of curves. By Proposition 3.6, there is a natural number $N$ such that for any affine $(m - 1)$-plane $L \subset \mathbb{F}^m$ and $x \in A$, if $L \cap \lambda_x$ is finite then it contains at most $N$ points. Let

$$A := \bigcup_{x \in A} \lambda_x([0, 1]).$$

Take $x \in A$, then by Corollary 4.4,

$$\text{length}(\lambda_x) = \int_{A Gr_{m-1}(\mathbb{F}^n)} |\lambda_x| dL \leq \int_{L \cap \lambda_x \neq \emptyset} N dL = N \int_{L \cap \Lambda \neq \emptyset} dL.$$

The last integral is finite since $\lambda$ is $\mathbb{Q}$-bounded, thus

$$K := N \int_{L \cap \Lambda \neq \emptyset} dL$$

is the required constant. \hfill $\square$

Corollary 5.5. If $A \subset \mathbb{F}^n$ is definable, $\mathbb{Q}$-bounded, and definably connected, then there is a definable family of injective curves $\lambda \subset A^2 \times ([0, 1] \times A)$ and $K \in \mathbb{R}_{>0}$ such that for any pair of points $x, y \in A$, $\lambda(x, y)$ is a piecewise $C^1$ curve in $A$ joining $x$ and $y$ with length$(\lambda(x, y)) \leq K$.

For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0, \infty)^n$. The $\alpha$- box centered at $a$ is the open box

$$B(a, \alpha) := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \in (a_i - \alpha_i, a_i + \alpha_i) \right\}.$$

In what follows we use the max norm in $\mathbb{F}^n$, that is for $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$, $|x| := \max\{|x_i| : i = 1, \ldots, n\}$.

Theorem 5.6. Let $A \subset \mathbb{F}^n$ be a cell. If $A$ is $\mathbb{Q}$-bounded, then there is a $K \in \mathbb{R}_{>0}$ and a definable family of injective curves $\gamma \subset A^2 \times [0, 1] \times A$ such that for $x, y \in A$, $\gamma_x(y)(0) = x$, $\gamma_x(y)(1) = y$, and length$(\gamma(x, y)) \leq K|x - y|$. In particular $A$ has the Whitney arc property.

Proof. For $\lambda \subset \mathbb{F}^n$ and $a \in \mathbb{F}^n$ let $f_{a, \lambda} : \mathbb{F}^n \rightarrow \mathbb{F}^n$ be the dilation about $a$, that is $f_{a, \lambda}(x) = \lambda(x - a) + a$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0, 1)^n$. For every $a \in A$ and $\delta \in (0, 1) \subset \mathbb{F}$, the set $B(a, \delta) \cap A$ is a cell and therefore is definably connected. For $a \in A$ and $\delta \in (0, 1)$ define

$$B_{\delta, a} := f_{a, 1/\delta}(B(a, \delta) \cap A).$$

This set is $\mathbb{Q}$-bounded. Also, $B_{\delta, a}$ is definably connected since $B(a, \delta) \cap A$ is. By Proposition 5.3 there is a definable family of curves $\lambda^{\delta, a} \subset B_{\delta, a}^2 \times ([0, 1] \times B_{\delta, a})$ such that for $b, b' \in B_{\delta, a}$, $\lambda^{\delta, a}_{b, b'}$ is piecewise $C^1$, $\lambda^{\delta, a}_{b, b'}(0) = b$ and $\lambda^{\delta, a}_{b, b'}(1) = b'$. Consider

$$\lambda := \left\{ (\delta, a, x, y, \epsilon, z) \in (0, 1) \times A \times (\mathbb{F}^n)^2 \times ([0, 1] \times \mathbb{F}^n) : (x, y, \epsilon, z) \in \lambda^{\delta, a}_{b, b'} \right\}.$$

This is a definable and $\mathbb{Q}$-bounded family of curves. Thus by Lemma 5.4 there is a $K_1 \in \mathbb{R}_{>0}$ such that for every $\delta \in (0, 1)$, $a \in A$, and $b, b' \in B_{\delta, a}$, length$(\lambda^{\delta, a}_{b, b'}) \leq K_1$.

Similarly, by Corollary 5.3 there is a definable family of curves $\Lambda \subset A^2 \times ([0, 1] \times A)$ and $K_2 \in \mathbb{R}_{>0}$ such that for each $x, y \in A$, $\Lambda_{x, y} : [0, 1] \rightarrow A$ is a piecewise $C^1$ curve in $A$ joining $x$ and $y$, and length$(\Lambda_{x, y}) \leq K_2$.

Now let $x, y$ be distinct points in $A$, and assume that $|x - y| < \min\{\alpha_j/3\}$. Let

$$\delta := \frac{3|x - y|}{\min\{\alpha_j\}}, \quad y' := f_{x, 1/\delta}(y) = \frac{1}{\delta}(y - x) + x.$$
Then $\delta \in (0,1)$ and for any $j$, 

$$|x_j - y_j| < \frac{3}{2} \frac{\alpha_j}{\min\{\alpha_j\}} |x - y| = \frac{\delta \alpha_j}{2}.$$ 

Thus, $y \in B(x, \delta \alpha)$, that is, $y' \in B_{\delta,x}$. Consider the curve $\lambda_{x,y}^{\delta,x}$ in $B_{\delta,x}$, joining $x$ and $y'$, and let $\gamma_{x,y} : [0,1] \to \mathbb{R}^n$ be defined by 

$$\gamma_{x,y}(t) := f_{x,\delta}(\lambda_{x,y}^{\delta,x}(t)) = \delta (\lambda_{x,y}^{\delta,x}(t) - x) + x.$$ 

Then $\gamma_{x,y}(t)$ is a curve in $A$ joining $x$ and $y$, and moreover, 

$$\text{length}(\gamma_{x,y}) = \delta \text{length} \left( \lambda_{x,y}^{\delta,x} \right) \leq \delta K_1 = \frac{3K_1}{\min\{\alpha_j\}} |x - y|.$$ 

Now assume that $|x - y| \geq \frac{1}{3} \min\{\alpha_j\}$, and let $\gamma_{x,y} := \Lambda_{x,y}$. Then $\gamma$ is a curve in $A$ joining $x$ and $y$ and 

$$\text{length}(\gamma_{x,y}) \leq K_2 \frac{K_2}{|x - y|} |x - y| \leq \frac{3K_2}{\min\{\alpha_j\}} |x - y|.$$ 

The collection of curves $\gamma_{x,y}$ constitutes the required definable family. 

$$K := \frac{3 \max\{K_1, K_2\}}{\min\{\alpha_j\}},$$ 

is the required constant. \qed 

As an immediate consequence we have: 

**Corollary 5.7.** Let $A \subset \mathbb{R}^n$ be a definable and $\mathbb{Q}$-bounded set. Then any cell in a cell decomposition of $A$ satisfies the Whitney arc property. 

**References**

[1] A. Berarducci, M. Otero. An additive measure in o-minimal expansions of fields. The Quarterly Journal of Mathematics 55 (2004), no. 4, 411–419. 

[2] L. van den Dries, *Tame Topology and o-minimal Structures*, no. 248 in LMS Lecture Note Series, Cambridge University Press, 1998. 

[3] R. Howard. The kinematic formula in Riemannian homogeneous spaces. Memoirs of the American Mathematical Society, no. 509. 

[4] K. Kurdyka. On a subanalytic stratification satisfying a Whitney property with exponent 1. Real algebraic geometry proceedings (Rennes, 1991), 316–322, Lecture Notes in Math., 1524, Springer, Berlin, 1992. 

[5] W. Pawlucki. Lipschitz Cell Decomposition in O-Minimal Structures. I. RAAG preprint server 2007. http://www.maths.manchester.ac.uk/raag/ 

[6] E. Vasquez Rifo, Geometric partitions of definable sets and the Cauchy-Crofton formula, Ph.D. thesis, University of Wisconsin-Madison, Madison, WI 53704, August 2006. 

[7] H. Whitney. Functions differentiable on the boundaries of regions. Ann. of Math. (2) 35 (1934), no. 3, 482–485. 

Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, 
E-mail address: evasquez@math.umn.edu