Regular automorphisms and Calogero-Moser families
Cédric Bonnafé

To cite this version:
Cédric Bonnafé. Regular automorphisms and Calogero-Moser families. 2022. hal-03565132

HAL Id: hal-03565132
https://hal.science/hal-03565132
Preprint submitted on 10 Feb 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
REGULAR AUTOMORPHISMS AND CALOGERO-MOSER FAMILIES

CÉDRIC BONNAFÉ

Abstract. — We study the subvariety of fixed points of an automorphism of a Calogero-Moser space induced by a regular element of finite order of the normalizer of the associated complex reflection group \( W \). We determine some of (and conjecturally all) the \( \mathbb{C}^* \)-fixed points of its unique irreducible component of maximal dimension in terms of the character table of \( W \). This is inspired by the mysterious relations between the geometry of Calogero-Moser spaces and unipotent representations of finite reductive groups, which is the theme of another paper [Bon3].

Let \( V \) be a finite dimensional vector space and let \( W \) be a finite subgroup of \( \text{GL}_\mathbb{C}(V) \) generated by reflections. To some parameter \( k \), Etingof and Ginzburg [EtGi] have associated a normal irreducible affine complex variety \( \mathcal{X}_k = \mathcal{X}_k(V, W) \) called a (generalized) Calogero-Moser space. If \( \tau \) is an element of finite order of the normalizer of \( W \) in \( \text{GL}_\mathbb{C}(V) \) stabilizing the parameter \( k \), it induces an automorphism of \( \mathcal{X}_k \).

We denote by \( V_\text{reg} \) the open subset of \( V \) on which \( W \) acts freely, and we assume that \( V_{\text{reg}}^\tau \neq \emptyset \) (then \( \tau \) is called regular). In this case, there exists a unique irreducible component \( (\mathcal{X}_k^\tau)_{\text{max}} \) of \( \mathcal{X}_k^\tau \) of maximal dimension (as it will be explained in Section 2). Recall that \( \mathcal{X}_k \) is endowed with a \( \mathbb{C}^* \)-action and that we have a surjective map \( \text{Irr}(W) \to \mathcal{X}_k^\tau \) defined by Gordon [Gor] (induced by the action of the center of a rational Cherednik algebra on baby Verma modules) whose fibers are called the Calogero-Moser \( k \)-families of \( W \). If \( p \in \mathcal{X}_k^\tau \), we denote by \( \mathcal{F}_p \) its associated Calogero-Moser \( k \)-family. It is a natural question to wonder which \( \mathbb{C}^* \)-fixed points of \( \mathcal{X}_k^\tau \) belong to \( (\mathcal{X}_k^\tau)_{\text{max}} \). The aim of this note is to provide a partial answer in terms of the character table of \( W \):

**Theorem A.** — Assume that \( V_{\text{reg}}^\tau \neq \emptyset \). Let \( p \in \mathcal{X}_k^\tau \) be such that \( \tau(p) = p \). If \( \sum_{\chi \in \text{Irr}(W)} |\tilde{\chi}(\tau)|^2 \neq 0 \), then \( p \in (\mathcal{X}_k^\tau)_{\text{max}} \).

In this statement, if \( \chi \) is a \( \tau \)-stable irreducible character of \( W \), we denote by \( \tilde{\chi} \) an extension of \( \chi \) to the finite group \( W(\tau) \) (note that \( |\tilde{\chi}(\tau)|^2 \) does not depend on the choice of \( \tilde{\chi} \)). Our proof of Theorem A makes an extensive use of the Gaudin operators introduced in [BoRo, §8.3,B]. This result is also inspired by the theory of unipotent representations of finite reductive groups and some conjectures of Broué-Michel [BrMi] on the cohomology of Deligne-Lusztig varieties associated with regular elements in the sense of Springer [Spr] and

The author is partly supported by the ANR: Projects No ANR-16-CE40-0010-01 (GeRepMod) and ANR-18-CE40-0024-02 (CATORE).
by [BMM2, Rem. 4.21] (this will also be discussed in [Bon3]). If we believe in this analogy, we can conjecture that the converse of Theorem A holds:

**Conjecture B.** Assume that \( V_{\text{reg}}^* \neq \emptyset \). Let \( p \in \mathcal{X}^G_k \) be such that \( \tau(p) = p \). Then \( p \in (\mathcal{X}^G_k)_{\text{max}} \) if and only if \( \sum_{\chi \in \mathcal{X}_p} |\chi(\tau)|^2 \neq 0 \).

**General notation.** Throughout this paper, we will abbreviate \( \otimes \mathbb{C} \) as \( \otimes \) and all varieties will be algebraic, complex, quasi-projective and reduced. If \( \mathcal{X} \) is an affine variety, we denote by \( \mathbb{C}[\mathcal{X}] \) its coordinate ring.

If \( X \) is a subset of a vector space \( V \) (or of its dual \( V^* \)), and if \( \Gamma \) is a subgroup of \( \text{GL}_V \), we denote by \( \Gamma_X \) the pointwise stabilizer of \( X \). If moreover \( \Gamma \) is finite, we will identify \( (V^*)^\Gamma \) and \( (V^*)^\Gamma \).

### 1. Set-up

#### Hypothesis and notation.** We fix in this paper a finite dimensional complex vector space \( V \) and a finite subgroup \( W \) of \( \text{GL}_V \). We set

\[
\text{Ref}(W) = \{ s \in W \mid \text{codim}_V V^s = 1 \}
\]

and we assume throughout this paper that

\[
W = \langle \text{Ref}(W) \rangle,
\]

i.e. that \( W \) is a complex reflection group.

**1.A. About \( W \).** We set \( \varepsilon : W \to \mathbb{C}^\times \), \( w \mapsto \det(w) \). We identify \( \mathbb{C}[V] \) (resp. \( \mathbb{C}[V^*] \)) with the symmetric algebra \( S(V^*) \) (resp. \( S(V) \)).

We denote by \( \mathcal{A} \) the set of reflecting hyperplanes of \( W \), namely

\[
\mathcal{A} = \{ V^s \mid s \in \text{Ref}(W) \}.
\]

If \( H \in \mathcal{A} \), we denote by \( \alpha_H \) an element of \( V^* \) such that \( H = \text{Ker}(\alpha_H) \) and by \( \alpha_H^\perp \) an element of \( V \) such that \( V = H \oplus \mathbb{C}\alpha_H^\perp \) and the line \( \mathbb{C}\alpha_H^\perp \) is \( W \)-stable. We set \( e_H = |W_H| \). Note that \( W_H \) is cyclic of order \( e_H \) and that \( \text{Irr}(W_H) = \{ \text{Res}_{W_H}^W \varepsilon^j \mid 0 \leq j \leq e - 1 \} \). We denote by \( e_{H,j} \) the (central) primitive idempotent of \( CW_H \) associated with the character \( \text{Res}_{W_H}^W \varepsilon^{-j} \), namely

\[
e_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \varepsilon(w)w \in CW_H.
\]

If \( \Omega \) is a \( W \)-orbit of reflecting hyperplanes, we write \( e_\Omega \) for the common value of all the \( e_{H,j} \), where \( H \in \Omega \). We denote by \( \mathbb{N} \) the set of pairs \( (\Omega,j) \) where \( \Omega \in \mathcal{A} \) and \( 0 \leq j \leq e_\Omega - 1 \). The vector space of families of complex numbers indexed by \( \mathbb{N} \) will be denoted by \( \mathbb{C}^\mathbb{N} \); elements of \( \mathbb{C}^\mathbb{N} \) will be called parameters. If \( k = (k_{\Omega,j})_{(\Omega,j) \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \), we define \( k_{H,j} \) for all \( H \in \Omega \) and \( j \in \mathbb{Z} \) by \( k_{H,j} = k_{\Omega,j_0} \) where \( \Omega \) is the \( W \)-orbit of \( H \) and \( j_0 \) is the unique element of \( \{0,1,\ldots,e_H - 1\} \) such that \( j \equiv j_0 \pmod{e_H} \).

We denote by \( V_{\text{reg}} \) the set of elements \( \nu \) of \( V \) such that \( W_{\nu} = 1 \). It is an open subset of \( V \) and recall from Steinberg-Serre Theorem [Bro, Theo. 4.7] that

\[
V_{\text{reg}} = V \setminus \bigcup_{H \in \mathcal{A}} H.
\]
1.B. Rational Cherednik algebra at \( t = 0 \). — Let \( k \in \mathbb{C}^\mathbb{N} \). We define the rational Cherednik algebra \( H_k \) (at \( t = 0 \)) to be the quotient of the algebra \( T(V \oplus V^*) \times W \) (the semi-direct product of the tensor algebra \( T(V \oplus V^*) \) with the group \( W \)) by the relations

\[
\begin{align*}
[x, x^\prime] &= [y, y^\prime] = 0, \\
[y, x] &= \sum_{H \in \mathbb{S}} e_H^{-1} \sum_{j=0}^{\infty} e_H (k_{H,j} - k_{H,j+1}) \langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle e_{H,j},
\end{align*}
\]

for all \( x, x^\prime \in V^* \), \( y, y^\prime \in V \). Here \( \langle , \rangle : V \times V^* \to \mathbb{C} \) is the standard pairing. The first commutation relations imply that we have morphisms of algebras \( \mathbb{C}[V] \to H_k \) and \( \mathbb{C}[V^*] \to H_k \). Recall [EtGi, Theo. 1.3] that we have an isomorphism of \( \mathbb{C} \)-vector spaces

\[
\mathbb{C}[V] \otimes \mathbb{C} W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} H_k
\]

induced by multiplication (this is the so-called PBW-decomposition).

**Remark 1.4.** — Let \( (\ell_\Omega)_{\Omega \in \mathbb{S}/W} \) be a family of complex numbers and let \( k' \in \mathbb{C}^\mathbb{N} \) be defined by \( k'_{\Omega,j} = k_{\Omega,j} + \ell_\Omega \). Then \( H_k = H_{k'} \). This means that there is no restriction to generality if we consider for instance only parameters \( k \) such that \( k_{\Omega,0} = 0 \) for all \( \Omega \), or only parameters \( k \) such that \( k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,\epsilon_\Omega - 1} = 0 \) for all \( \Omega \) (as in [BoRo]).

1.C. Calogero-Moser space. — We denote by \( Z_k \) the center of the algebra \( H_k \): it is well-known [EtGi, Theo. 3.3 and Lem. 3.5] that \( Z_k \) is an integral domain, which is integrally closed. Moreover, it contains \( \mathbb{C}[V]^W \) and \( \mathbb{C}[V^*]^W \) as subalgebras [Gor, Prop. 3.6] (so it contains \( P = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W \)), and it is a free \( P \)-module of rank \( |W| \). We denote by \( \mathcal{X}_k \) the affine algebraic variety whose ring of regular functions \( \mathbb{C}[\mathcal{X}_k] \) is \( Z_k \): this is the Calogero-Moser space associated with the datum \( (V, W, k) \). It is irreducible and normal.

We set \( \mathcal{P} = V/W \times V^*/W \), so that \( \mathbb{C} [\mathcal{P}] = P \) and the inclusion \( P \hookrightarrow Z_k \) induces a morphism of varieties

\[
\Upsilon_k : \mathcal{X}_k \to \mathcal{P}
\]

which is finite and flat.

1.D. Calogero-Moser families. — Using the PBW-decomposition, we define a \( \mathbb{C} \)-linear map \( \Omega^{H_k} : H_k \to \mathbb{C} W \) by

\[
\Omega^{H_k}(f w g) = f(0) g(0) w
\]

for all \( f \in \mathbb{C}[V], g \in \mathbb{C}[V^*] \) and \( w \in \mathbb{C} W \). This map is \( W \)-equivariant for the action on both sides by conjugation, so it induces a well-defined \( \mathbb{C} \)-linear map

\[
\Omega^k : Z_k \to Z(\mathbb{C} W).
\]

Recall from [BoRo, Cor. 4.2.11] that \( \Omega^k \) is a morphism of algebras.

Calogero-Moser families were defined by Gordon using his theory of baby Verma modules [Gor, §4.2 and §5.4]. We explain here an equivalent definition given in [BoRo, §7.2]. If \( \chi \in \text{Irr}(W) \), we denote by \( \omega_\chi : \mathbb{C} W \to \mathbb{C} \) its central character (i.e., \( \omega_\chi(z) = \chi(z)/\chi(1) \) is the scalar by which \( z \) acts on an irreducible representation affording the character \( \chi \)). We say that two characters \( \chi \) and \( \chi' \) belong to the same Calogero-Moser \( k \)-family if \( \omega_\chi \circ \Omega^k = \omega_{\chi'} \circ \Omega^k \).
In other words, the map $\omega_\chi \circ \Omega^k : Z_k \to \mathbb{C}$ is a morphism of algebras, so it might be viewed as a point $\varphi_k(\chi)$ of $\mathcal{I}_k$, which is easily checked to be $\mathbb{C}^*$-fixed. This defines a surjective map

$$\varphi_k : \text{Irr}(W) \to \mathcal{I}_k^{\mathbb{C}^*}$$

whose fibers are the Calogero-Moser $k$-families. If $p \in \mathcal{I}_k^{\mathbb{C}^*}$, we denote by $F_p$ the corresponding Calogero-Moser $k$-family.

1.E. Other parameters. — Let $\mathcal{C}$ denote the space of maps $\text{Ref}(W) \to \mathbb{C}$ which are constant on conjugacy classes of reflections. The element

$$\sum_{(\Omega,j) \in \mathbb{N}} \sum_{H \in \Omega} (k_{H,j} - k_{H,j+1}) e_{H \varepsilon H,j}$$

of $Z(CW)$ is supported only by reflections, so there exists a unique map $c_k \in \mathcal{C}$ such that

$$\sum_{(\Omega,j) \in \mathbb{N}} \sum_{H \in \Omega} (k_{H,j} - k_{H,j+1}) e_{H \varepsilon H,j} = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) s.$$

Then the map $\mathbb{C}^N \to \mathcal{C}$, $k \mapsto c_k$ is linear and surjective. With this notation, we have

$$[y, x] = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s,$$

for all $y \in V$ and $x \in V^*$. Here, $\alpha_s = \alpha_V^s$ and $\alpha_s^\vee = \alpha_V^{s^\vee}$.

1.F. Actions on the Calogero-Moser space. — The Calogero-Moser space $\mathcal{I}_k$ is endowed with a $\mathbb{C}^*$-action and an action of the stabilizer of $k$ in $N_{\text{GL}_C(V)}(W)$, which are described below.

1.F.1. Grading, $\mathbb{C}^*$-action. — The algebra $T(V \oplus V^*) \rtimes W$ can be $\mathbb{Z}$-graded in such a way that the generators have the following degrees

$$\begin{cases}
\deg(y) = -1 & \text{if } y \in V, \\
\deg(x) = 1 & \text{if } x \in V^*, \\
\deg(w) = 0 & \text{if } w \in W.
\end{cases}$$

This descends to a $\mathbb{Z}$-grading on $H_k$, because the defining relations (1.2) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra $Z_k$ is also $\mathbb{Z}$-graded. So the Calogero-Moser space $\mathcal{I}_k$ inherits a regular $\mathbb{C}^*$-action. Note also that by definition $P = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of $Z_k$. 
1.F.2. Action of the normalizer. — The group $N_{GL(V)}(W)$ acts on the set $\mathcal{V}$ and so on the space of parameters $C^N$. If $\tau \in N_{GL(V)}(W)$, then $\tau$ induces an isomorphism of algebras $H_k \rightarrow H_{\tau(k)}$. So, if $\tau(k) = k$, then it induces an action on the algebra $H_k$ (and so on its center $Z_k$ and on the Calogero-Moser space $\mathcal{I}_k$).

We say that $\tau$ is a regular element of $N_{GL(V)}(W)$ if $V_{reg}^\tau \neq \emptyset$.

**Notation.** From now on, and until the end of this paper, we fix a parameter $k \in C^N$ and a regular element $\tau$ of finite order of $N_{GL(V)}(W)$ such that $\tau(k) = k$.

We denote by $\mathcal{I}_k^\tau$ the variety of fixed points of $\tau$ in $\mathcal{I}_k$, endowed with its reduced structure. All the above constructions are $\tau$-equivariant: for instance, the map $\varphi_k : \text{Irr}(W) \rightarrow \mathcal{I}_k^{C^\tau}$ is $\tau$-equivariant.

Let us recall the following consequence [Spr, Prop. 3.5 and Theo. 4.2] of the above hypothesis:

**Theorem 1.6 (Springer).** — The group $W^\tau$ acts as a reflection group on $V^\tau$ and the natural map $V^\tau/W^\tau \rightarrow (V/W)^\tau$ is an isomorphism of varieties.

**Corollary 1.7.** — The natural map $(V_{reg}^\tau \times V^{*\tau})/W^\tau \rightarrow ((V_{reg} \times V^*)/W)^\tau$ is an isomorphism of varieties.

**Proof.** — Since $W$ acts freely on $V_{reg} \times V^*$, the quotient $(V_{reg} \times V^*)/W$ is smooth. Consequently, the variety of fixed points $((V_{reg} \times V^*)/W)^\tau$ is also smooth. Similarly, $(V_{reg}^\tau \times V^{*\tau})/W^\tau$ is smooth. Since a bijective morphism between smooth varieties is an isomorphism, we only need to show that the above natural map is bijective.

First, if $(v_1, v_1^\tau)$ and $(v_2, v_2^\tau)$ are two elements of $V_{reg}^\tau \times V^{*\tau}$ belonging to the same $W$-orbit, there exists $w \in W$ such that $v_2 = w(v_1)$. Since $v_1$ and $v_2$ are $\tau$-stable, we also have $\tau(w)(v_1) = v_2$, and so $v_1 = w^{-1}\tau(w)(v_1)$. Since $v_1 \in V_{reg}$, this forces $\tau(w) = w$ and the injectivity follows.

Now, if $(v, v^*) \in V_{reg} \times V^*$ is such that its $W$-orbit is $\tau$-stable, then the $W$-orbit of $v$ is $\tau$-stable. So Theorem 1.6 shows that we may assume that $\tau(v) = v$. The hypothesis implies that there exists $w \in W$ such that $\tau(v) = w(v)$ and $\tau(v^*) = w(v^*)$. But $\tau(v) = v \in V_{reg}$, so $w = 1$. In particular, $\tau(v^*) = v^*$, and the surjectivity follows.

2. Irreducible component of maximal dimension

Let $(\mathcal{I}_k)_{reg}$ denote the open subset $\mathcal{I}_k^{-1}(V_{reg}/W \times V^*/W)$. By [EtGi, Prop. 4.11], we have a $C^\infty$-equivariant and $\tau$-equivariant isomorphism

$$\mathcal{I}_k \simeq (V_{reg} \times V^*)/W.$$  

This shows that $(\mathcal{I}_k)_{reg}$ is smooth and so $(\mathcal{I}_k^\tau)_{reg}$ is also smooth. By Corollary 1.7, this implies that

$$(\mathcal{I}_k^\tau)_{reg} \simeq (V_{reg}^\tau \times V^{*\tau})/W^\tau.$$  

In particular it is irreducible. We denote by $(\mathcal{I}_k^\tau)_{max}$ its closure: it is an irreducible closed subvariety of $\mathcal{I}_k^\tau$. 


Moreover, \((\mathcal{I}_k^\tau)_{\text{reg}}\) has dimension \(2 \dim V^\tau\) by Corollary 1.7. So \(\dim \mathcal{I}_k^\tau \geq 2 \dim V^\tau = \dim(\mathcal{I}_k^\tau)_{\text{max}}\). But, on the other hand, \(\Upsilon_k(\mathcal{I}_k^\tau) \subset (V/W)^\tau \times (V^*/W)^\tau\). Since \(\Upsilon_k\) is a finite morphism, we get from Theorem 1.6 that \(\dim \mathcal{I}_k^\tau \leq 2 \dim V^\tau\). Hence

\[
\dim \mathcal{I}_k^\tau = \dim(\mathcal{I}_k^\tau)_{\text{max}} = 2 \dim V^\tau.
\]

This shows that \((\mathcal{I}_k^\tau)_{\text{max}}\) is an irreducible component of maximal dimension of \(\mathcal{I}_k^\tau\).

**Proposition 2.4.** — The closed subvariety \((\mathcal{I}_k^\tau)_{\text{max}}\) of \(\mathcal{I}_k^\tau\) is the unique irreducible component of maximal dimension.

**Proof.** — Let \(\mathcal{X}\) be an irreducible component of \(\mathcal{I}_k^\tau\) of dimension \(2 \dim V^\tau\). Since \(\Upsilon_k\) is finite, the image \(\Upsilon_k(\mathcal{X})\) is closed in \(V/W \times V^*/W\), irreducible of dimension \(2 \dim(V^\tau)\) and contained in \((V/W)^\tau \times (V^*/W)^\tau\). By Theorem 1.6, we get that \(\Upsilon_k(\mathcal{X}) = (V/W)^\tau \times (V^*/W)^\tau\).

Let \(\mathcal{U} = \Upsilon_k^{-1}(V/W \times V^*/W) \cap \mathcal{X}\). Then \(\mathcal{U}\) is a non-empty open subset of \(\mathcal{X}\): since \(\mathcal{X}\) is irreducible, this forces \(\mathcal{U}\) to have dimension \(2 \dim(V^\tau)\). But \(\mathcal{U}\) is contained in \((\mathcal{I}_k^\tau)_{\text{reg}}\) which is irreducible of the same dimension, so the closure of \(\mathcal{U}\) contains \((\mathcal{I}_k^\tau)_{\text{reg}}\). This proves that \(\mathcal{X} = (\mathcal{I}_k^\tau)_{\text{max}}\).

**Corollary 2.5.** — \(\Upsilon_k((\mathcal{I}_k^\tau)_{\text{max}}) = (V/W)^\tau \times (V^*/W)^\tau\).

It is natural to ask which \(\mathbb{C}^\times\)-fixed points of \(\mathcal{I}_k^\tau\) belong to \((\mathcal{I}_k^\tau)_{\text{max}}\). Inspired by the representation theory of finite reductive groups (see [BrMi] and [BMM2, Rem. 4.21]), we propose an answer to this question in terms of the character table of the finite group \(W(\tau)\) (see [Bon3, Ex. 12.9] for some explanations). We first need some notation.

If \(\chi \in \text{Irr}(W)\), we denote by \(E_{\chi}\) a \(CW\)-module affording the character \(\chi\). If moreover \(\chi\) is \(\tau\)-stable, we fix a structure of \(CW(\tau)\)-module on \(E_{\chi}\) extending the structure of \(CW\)-module, and we denote by \(\tilde{\chi}\) its associated irreducible character of \(W(\tau)\). Note that the real number \(|\tilde{\chi}(\tau)|^2\) does not depend on the choice of \(\tilde{\chi}\).

**Conjecture 2.6.** — Recall that \(\tau\) is regular. Let \(p \in \mathcal{I}_k^{\mathbb{C}^\times}\) be such that \(\tau(p) = p\). Then \(p\) belongs to \((\mathcal{I}_k^\tau)_{\text{max}}\) if and only if \(\sum_{\chi \in \tilde{\mathfrak{g}}_p} |\tilde{\chi}(\tau)|^2 \neq 0\).

**Remark 2.7.** — Let \(\mathfrak{F}\) be a \(\tau\)-stable Calogero-Moser family. Then \(\mathfrak{F}\) contains a unique irreducible character \(\chi_{\mathfrak{F}}\) with minimal \(b\)-invariant [BoRo, Theo. 7.4.1], where the \(b\)-invariant of an irreducible character \(\chi\) is the minimal natural number \(j\) such that \(\chi\) occurs in the \(j\)-th symmetric power of the natural representation \(V\) of \(W\). From this characterization, we see that \(\chi_{\mathfrak{F}}\) is \(\tau\)-stable. In particular, any \(\tau\)-stable Calogero-Moser family contains at least one \(\tau\)-stable character. ■

In general, we are only able to prove the “if” part of Conjecture 2.6.

**Theorem 2.8.** — Recall that \(\tau\) is regular. Let \(p \in \mathcal{I}_k^{\mathbb{C}^\times}\) be such that \(\tau(p) = p\). If \(\sum_{\chi \in \tilde{\mathfrak{g}}_p} |\tilde{\chi}(\tau)|^2 \neq 0\), then \(p\) belongs to \((\mathcal{I}_k^\tau)_{\text{max}}\).

The next two sections are devoted to the proof of Theorem 2.8.
3. Verma modules

3.A. Definition. — Recall that $\mathbb{C}[V] \times W$ is a subalgebra of $H_k$ (it is the image of $1 \otimes CW \otimes \mathbb{C}[V]$ by the PBW-decomposition 1.3). If $E$ is a $\mathbb{C}[W]$-module, we denote by $E^\#$ the $(\mathbb{C}[V]^* \rtimes W)$-module extending $E$ by letting any element $f \in \mathbb{C}[V]^*$ acting by multiplication by $f(0)$. If $\chi \in \text{Irr}(W)$, we define an $H_k$-module $\Delta(\chi)$ as follows:

$$\Delta(\chi) = H_k \otimes_{\mathbb{C}[V]^* \rtimes W} E^\chi.$$

Then $\Delta(\chi)$ is called a Verma module of $H_k$ (see [BoRo, §5.4.A]; in this reference, $\Delta(\chi)$ is denoted by $\Delta(E^\chi)$). Let $H_k^{\text{reg}}$ denote the localization of $H_k$ at $\mathfrak{p}_{\text{reg}} = \mathbb{C}[V_{\text{reg}}/W] \otimes \mathbb{C}[V]^*$. By [EtGi, Prop. 4.11], we have an isomorphism $\mathbb{C}[V_{\text{reg}} \times V^*] \rtimes W \simeq H_k^{\text{reg}}$. We denote by $\Delta^{\text{reg}}(\chi)$ the localization of $\Delta(\chi)$ at $H_k^{\text{reg}}$. So, by restriction to $\mathbb{C}[V_{\text{reg}} \times V^*]$, the Verma module $\Delta(\chi)$ might be viewed as a $W$-equivariant coherent sheaf on $V_{\text{reg}} \times W$. We also view $e\Delta(\chi)$ as a coherent sheaf on $I_k$, so that $e\Delta^{\text{reg}}(\chi)$ may be viewed as a coherent sheaf on $(V_{\text{reg}} \times V^*)/W$. If $p \in I_k$ (or if $(v,v^*) \in V_{\text{reg}} \times V^*$), we denote by $e\Delta(\chi)_p$ (respectively $e\Delta^{\text{reg}}(\chi)_p$) the restriction of $e\Delta(\chi)$ (respectively of $e\Delta^{\text{reg}}(\chi)$) at the point $p$ (respectively $W \cdot (v,v^*) \in (V_{\text{reg}} \times V^*)/W \simeq (I_k)_{\text{reg}}$, respectively $(v,v^*)$). It follows from the definition that the support of $e\Delta(\chi)$ is contained in $\Upsilon^{-1}(V/W \times 0)$, and recall that, through the isomorphism $I_k^{\text{reg}} \simeq (V_{\text{reg}} \times V^*)/W$, $\Upsilon^{-1}(V_{\text{reg}}/W \times 0)$ is not necessarily contained in $(V_{\text{reg}} \times \{0\})/W$.

Lemma 3.1. — Let $\chi \in \text{Irr}(W)$ and let $p \in I_k^{\text{att}}$. Then $e\Delta(\chi)_p \neq 0$ if and only if $\chi \in \mathfrak{s}_p$.

Proof. — Let $p_0$ denote the maximal ideal of the algebra $P = \mathbb{C}[\mathfrak{f}]$ consisting of functions which vanish at 0. Then $\Delta(\chi)/p_0\Delta(\chi)$ is a representation of the restricted rational Cherednik algebra $H_k/p_0H_k$ which coincides with the baby Verma module defined by Gordon [Gor, §4.2]. As $I_k^{\text{att}} = \Upsilon^{-1}(0)$, the result follows from the very definition of Calogero-Moser families in terms of baby Verma modules and the fact that it is equivalent to the definition given in §1.D.

3.B. Bialynicki-Birula decomposition. — We denote by $I_k^{\text{att}}$ the attracting set of $I_k$ for the action of $\mathbb{C}^\times$, namely

$$I_k^{\text{att}} = \{ p \in I_k | \lim_{\xi \to 0} \xi p \text{ exists} \}.$$

Recall from [BoRo, Chap. 14] the following facts:

Proposition 3.2. — With the above notation, we have:

(a) The map $\lim : I_k^{\text{att}} \to I_k^{\mathbb{C}^\times}, p \mapsto \lim_{\xi \to 0} \xi p$ is a morphism of varieties.
(b) $I_k^{\text{att}} = \Upsilon^{-1}(V/W \times \{0\})$.
(c) If $\mathcal{J}$ is an irreducible component of $I_k^{\text{att}}$, then $\mathcal{J}$ is $\mathbb{C}^\times$-stable and $\Upsilon_{\text{reg}}(\mathcal{J}) = V/W \times \{0\}$ and $\lim(\mathcal{J})$ is a single point.
(d) If $\chi \in \text{Irr}(W)$, then the support of $e\Delta(\chi)$ is a union of irreducible components of $I_k^{\text{att}}$.
(e) If $\mathcal{J}$ is an irreducible component of $I_k^{\text{att}}$, then there exists $\chi \in \text{Irr}(W)$ such that the support of $e\Delta(\chi)$ contains $\mathcal{J}$. 

We first propose a characterization of points \( p \in \mathcal{I}_k^\times \) which belong to \((\mathcal{I}_k^\times)^{\max}\) in terms of Verma modules.

**Lemma 3.3.** — Let \( p \in \mathcal{I}_k^\times \) and assume that \( \tau(p) = p \). Then \( p \in (\mathcal{I}_k^\times)^{\max} \) if and only if there exist \( \chi \in \mathfrak{g}^k \) and \((v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau} \) such that \( e \Delta(\chi)_{W,(v, v^*)} \neq 0 \).

**Proof.** — Let \((\mathcal{I}_k^\times)^{\text{att}}\) denote the attracting set of \((\mathcal{I}_k^\times)^{\max}\). Then Corollary 2.5 implies that \( \Upsilon_k((\mathcal{I}_k^\times)^{\max}) = (V/W)^\tau \times \{0\} \). Since \( \Upsilon_k \) is a finite morphism, the same arguments used in [BoRo, Chap. 14] to prove the Proposition 3.2 above yields the following statements:

(a) The map \( \lim : (\mathcal{I}_k^\times)^{\text{att}} \longrightarrow (\mathcal{I}_k^\times)^{\max}, p \mapsto \lim_{\kappa \to 0} 5p \) is a morphism of varieties.

(b) \((\mathcal{I}_k^\times)^{\text{att}} = (\mathcal{I}_k^\times)^{\max} \cap \Upsilon_k^{-1}((V/W)^\tau \times \{0\})\).

(c) If \( \mathcal{J} \) is an irreducible component of \((\mathcal{I}_k^\times)^{\text{att}}\), then \( \mathcal{J} \) is \( C^\times \)-stable and \( \Upsilon_k(\mathcal{J}) = (V/W)^\tau \times \{0\} \) and \( \lim(\mathcal{J}) \) is a single point.

Assume that \( p \in (\mathcal{I}_k^\times)^{\max} \). Let \( \mathcal{J} \) be an irreducible component of \((\mathcal{I}_k^\times)^{\max} \cap \lim^{-1}(p) \). Then \( \mathcal{J} \) is contained in an irreducible component \( \mathcal{J}' \) of \((\mathcal{I}_k^\times)^{\text{att}}\). Since \( \lim(\mathcal{J}') \) is a single point by (c), we have \( \lim(\mathcal{J}') = \{p\} \) and \( \mathcal{J} = \mathcal{J}' \). Still by (c), this says that \( \Upsilon_k(\mathcal{J}) = (V/W)^\tau \times \{0\} \). So let \( q \in \mathcal{J} \) be such that \( \Upsilon_k(q) \in (V_{\text{reg}}/W)^\tau \times \{0\} \).

Now, let \( \mathcal{J} \) be an irreducible component of \( \mathcal{I}_k^{\text{att}} \) containing \( \mathcal{J} \). By Proposition 3.2(e), there exists \( \chi \in \text{Irr}(W) \) such that the support of \( e \Delta(\chi) \) contains \( \mathcal{J} \). In particular, \( e \Delta(\chi)_p \neq 0 \) and so \( \chi \in \mathfrak{g}_p \) by Lemma 3.1. But also \( e \Delta(\chi)_q \neq 0 \). Since \( q \in (\mathcal{I}_k^\times)^{\max} \) and \( \Upsilon_k(q) \in V_{\text{reg}}/W \), it follows that there exists \((v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau} \) such that \( e \Delta(\chi)_{W,(v, v^*)} \neq 0 \), as desired.

Conversely, assume that there exist both \( \chi \in \mathfrak{g}_p^k \) and \((v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau} \) such that \( e \Delta(\chi)_{W,(v, v^*)} \neq 0 \). Let \( \mathcal{J} \) be an irreducible component of \( \mathcal{I}_k^{\text{att}} \) contained in the support of \( e \Delta(\chi) \). Then \( p \in \mathcal{J} \) and so \( p = \lim W \cdot (v, v^*) \). Since \( W \cdot (v, v^*) \in (\mathcal{I}_k^\times)^{\max} \) by the definition of \((\mathcal{I}_k^\times)^{\max}\), this implies that \( p \in (\mathcal{I}_k^\times)^{\max} \), as desired. \( \square \)

4. Gaudin algebra

**4.A. Definition.** — We recall here the definition of Gaudin algebra [BoRo, §8.3.B]. First, let \( C[V_{\text{reg}}][W] \) denote the group algebra of \( W \) over the algebra \( C[V_{\text{reg}}] \) (and not the semi-direct product \( C[V_{\text{reg}}] \rtimes W \)). For \( y \in V \), let

\[
\mathcal{D}_y^k = \sum_{s \in \text{Ref}(W)} \epsilon(s) c_k(s) \langle y, \alpha_s \rangle \alpha_s s \in C[V_{\text{reg}}][W].
\]

Now, let \( \text{Gau}_k(W) \) be the sub-\( C[V_{\text{reg}}] \)-algebra of \( C[V_{\text{reg}}][W] \) generated by the \( \mathcal{D}_y^k \)'s (where \( y \) runs over \( V \)): it will be called the Gaudin algebra (with parameter \( k \)) associated with \( W \).

Let \( C(V) \) denote the function field of \( V \) (which is the fraction field of \( C[V] \) or of \( C[V_{\text{reg}}] \)) and let \( C(V) \otimes_{C[V_{\text{reg}}]} \text{Gau}_k(W) \) denote the subalgebra \( C(V) \otimes_{C[V_{\text{reg}}]} \text{Gau}_k(W) \) of the group algebra \( C(V)[W] \). Recall [BoRo, §8.3.B] that

\[
\text{Gau}_k(W) \text{ is a commutative algebra,}
\]

but that \( \text{Gau}_k(W) \) is generally non-split, as shown by the examples treated in [Bon1, §4] and [Lac].
4.B. Generalized eigenspaces. — If \( v \in V_{\text{reg}} \), we denote by \( D_{k,v}^y \) the specialization of \( D_k^y \) at \( v \), namely \( D_{k,v}^y \) is the element of the group algebra \( \mathbb{C}W \) equal to

\[
D_{k,v}^y = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} s.
\]

Now, if \( v^* \in V^* \) and if \( M \) is a \( \mathbb{C}W \)-module, we define \( M_{k,v,v^*}^y \) to be the common generalized eigenspace of the operators \( D_{k,v}^y \) for the eigenvalue \( \langle y, v^* \rangle \), for \( y \) running over \( V \). Namely,

\[
M_{k,v,v^*}^y = \{ m \in M | \forall y \in V, (D_{k,v}^y - \langle y, v^* \rangle \text{Id}_M)^{\dim(M)}(m) = 0 \}.
\]

Then

\[
(4.2) \quad M = \bigoplus_{v^* \in V^*} M_{k,v,v^*}^y,
\]

since \( \text{Gau}_k(W) \) is commutative.

**Lemma 4.3.** — Let \( \chi \in \text{Irr}(W) \) and let \( (v, v^*) \in V_{\text{reg}} \times V^* \). Then the following are equivalent:

1. \( e\Delta(\chi)\mathbb{C}[V_{\text{reg}}]_{(v,v^*)} \neq 0 \).
2. \( \Delta_{\text{reg}}(\chi)_{v,v^*} \neq 0 \).
3. \( E_{k,v,v^*}^\chi \neq 0 \).

**Proof.** — The equivalence between (1) and (2) follows from the Morita equivalence between \( \mathbb{C}[V_{\text{reg}} \times V^*]_W \) and \( \mathbb{C}[V_{\text{reg}} \times V^*] \rtimes W \) proved in [BoRo, Lem. 3.1.8(b)]. Now, as a \( \mathbb{C}[V_{\text{reg}}] \)-module, \( \Delta_{\text{reg}}(\chi) \simeq \mathbb{C}[V_{\text{reg}}] \otimes E^\chi \), and the equivalence between (2) and (3) follows from the computations in [BoRo, §8.3.B].

4.C. Proof of Theorem A (i.e. Theorem 2.8). — Let \( \chi \in \text{Irr}(W) \) be \( \tau \)-stable and such that \( \tilde{\chi}(\tau) \neq 0 \) and let \( v \in V_{\text{reg}}^* \). By Lemmas 3.1 and 4.3, it is sufficient to show that there exists \( v^* \in V^* \) such that \( E_{k,v,v^*}^\chi \neq 0 \).

For this, let \( \mathcal{E} \) denote the set of \( v^* \in V^* \) such that \( E_{k,v,v^*}^\chi \neq 0 \). Then it follows from (4.2) that

\[
(\ast) \quad E^\chi = \bigoplus_{v^* \in \mathcal{E}} E_{k,v,v^*}^\chi.
\]

Since \( \tau(v) = v \), we have

\[
\tau D_{k,v} = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} \tau s \tau^{-1} = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \tau^{-1}(\alpha_s) \rangle}{\langle v, \tau^{-1}(\alpha_s) \rangle} s = D_{k,v}^\tau \tau(y),
\]

Consequently,

\[
\tau E_{k,v,v^*}^\chi = E_{k,v,v^*}^\chi.
\]

But \( \tilde{\chi}(\tau) = \text{Tr}(\tau, E^\chi) \neq 0 \), so \( \tau \) must fix at least one of the generalized eigenspaces in the decomposition \( (\ast) \). In other words, this implies that there exists \( v^* \in \mathcal{E} \) such that \( \tau(v^*) = v^* \), as desired. The proof is complete.
5. Complements

5.A. Conjectures. — The variety $\mathcal{I}_k$ is endowed with a Poisson structure [EtGi, §1] and so the variety of fixed points $\mathcal{I}_k^\tau$ inherits a Poisson structure too, as well as all its irreducible components. Recall from Springer Theorem 1.6 that $W^\tau$ is a reflection group for its action on $V^\tau$, so we can define a set of pairs $\mathcal{N}_\tau$ for the pair $(V^\tau, W^\tau)$ as well as $\mathcal{N}$ has been defined for the pair $(V, W)$ and, for each parameter $l \in \mathbb{C}^{\mathbb{N}^*}$, we can define a Calogero-Moser space $\mathcal{I}_l(V^\tau, W^\tau)$. The following conjecture is a particular case of [Bon2, Conj. B] (see [Bon2] for a discussion about the cases where this conjecture is known to hold):

**Conjecture 5.1.** — Recall that $\tau$ is regular. Then there exists a linear map $\lambda : \mathbb{C}^k \to \mathbb{C}^{\mathbb{N}^*}$ and, for each $k \in \mathbb{C}^k$, a $\mathbb{C}^\times$-equivariant isomorphism of Poisson varieties

$$\iota_k : (\mathcal{I}_k^\tau)_{\text{max}} \; \sim \; \mathcal{I}_{\lambda(k)}(V^\tau, W^\tau).$$

Assume that Conjecture 5.1 holds and keep its notation. Then $\iota_k$ restricts to a map $\iota_k : (\mathcal{I}_k^\tau)_{\text{max}} \; \sim \; \mathcal{I}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$. If $p \in \mathcal{I}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$, we denote by $\tilde{\mathcal{I}}_{\lambda(k)}^{(\tau)}$ the corresponding Calogero-Moser $\lambda(k)$-family of $W^\tau$. The next conjecture, still inspired by the representation theory of finite reductive groups (see again [Bon3, Ex. 12.9] for some explanations), makes Conjecture B more precise:

**Conjecture 5.2.** — Recall that $\tau$ is regular and assume that Conjecture 5.1 holds. If $p \in (\mathcal{I}_k^\tau)^{\mathbb{C}^\times}$, then

$$\sum_{\chi \in \tilde{\mathcal{I}}_{\lambda(k)}^{(\tau)}} |\tilde{\chi}^{(\tau)}(\tau)|^2 = \sum_{\psi \in \tilde{\mathcal{I}}_{\rho(k)}^{(\tau)}} |\psi(1)|^2.$$

Note that this last conjecture is compatible with the fact that

$$\sum_{\chi \in \text{Irr}(W^\tau)} |\tilde{\chi}^{(\tau)}(\tau)|^2 = |W^\tau| = \sum_{\psi \in \text{Irr}(W^\tau)} |\psi(1)|^2,$$

where the first equality follows from the second orthogonality relation for characters.

5.B. Roots of unity. — We consider in this subsection a particular (but very important) case of the general situation studied in this paper. We fix a natural number $d \geq 1$ and a primitive $d$-th root of unity $\zeta_d$. The group of $d$-th roots of unity is denoted by $\mu_d$. An element $w \in W$ is called $\zeta_d$-regular if the element $\zeta_d^{-1}w$ of $\text{N}_{\text{GL}_d}(V)(W)$ is regular. In other words, $w$ is $\zeta_d$-regular if and only if its $\zeta_d$-eigenspace meets $V_{\text{reg}}$. The existence of a $\zeta_d$-regular element is not guaranteed: we say that $d$ is a regular number of $W$ if such an element exists.

**Hypothesis.** We assume in this subsection, and only in this subsection, that $d$ is a regular number of $W$. We denote by $w_d$ a $\zeta_d$-regular element and we also set $\tau_d = \zeta_d^{-1}w_d$, so that $\tau_d$ is a regular element of $\text{N}_{\text{GL}_d}(V)(W)$.

Recall from [Spr] that $w_d$ is uniquely defined up to conjugacy. Note that

$$V^{\tau_d} = \text{Ker}(w_d - \zeta_d \text{Id}_V), \quad W^{\tau_d} = C_W(w_d) \quad \text{and} \quad \mathcal{I}_k^{\tau_d} = \mathcal{I}_k^{\mu_d}.$$
Since $\tau_d$ induces an inner automorphism of $W$, all the irreducible characters are $\tau_d$-stable. Moreover, if $\chi \in \text{Irr}(W)$, then $\tilde{\chi}(\tau_d) = \xi \chi(w_d)$ for some root of unity $\xi$, so $|\tilde{\chi}(\tau_d)|^2 = |\chi(w_d)|^2$. This allows to reformulate both Theorem A and Conjecture B in this case:

**Conjecture 5.4.** — Recall that $d$ is a regular number. Let $p \in \mathcal{I}^C_k$. Then $p$ belongs to $(\mathcal{I}^H_k)_{\text{max}}$ if and only if $\sum_{\gamma \in \overline{\delta}_p} |\chi(\gamma)|^2 \neq 0$.

**Theorem 5.5.** — Recall that $d$ is regular. Let $p \in \mathcal{I}^C_k$ be such that $\sum_{\gamma \in \overline{\delta}_p} |\chi(\gamma)|^2 \neq 0$. Then $p$ belongs to $(\mathcal{I}^H_k)_{\text{max}}$.

**Example 5.6 (Symmetric group).** — We assume here, and only here, that $W = S_n$ acting on $V = \mathbb{C}^n$ by permutation of the coordinates, for some $n \geq 2$. The canonical basis of $\mathbb{C}^n$ is denoted by $(y_1, \ldots, y_n)$. Then there is a unique orbit of hyperplanes, that we denote by $\Omega$, and $\epsilon_{\Omega} = 2$. To avoid too easy cases, we also assume that $k_{0,0} \neq k_{0,1}$ (so that $\mathcal{I}_k$ is smooth [EtGi, Cor. 1.14]) and that $d \geq 2$. Saying that $d$ is a regular number is equivalent to say that $d$ divides $n$ or $n - 1$. Therefore, we will denote by $j$ the unique element of $\{0, 1\}$ such that $d$ divides $n - j$ and we set $r = (n - j)/d$. Then $w_d$ is the product of $r$ disjoint cycles of length $d$, so one can choose for instance

$$w_d = (1, 2, \ldots, d)(d+1, d+2, \ldots, 2d) \cdots ((r-1)d+1, (r-1)d+2, \ldots, rd).$$

Then $V^\tau_d$ is $r$-dimensional, with basis $(v_1, \ldots, v_r)$ where $v_a = \sum_{b=1}^d \zeta_d^{-b} e_{(a-1)d+b}$ and the group $G_W(w_d) \simeq G(d, 1, r)$ acting “naturally” as a reflection group on $V^\tau_d = \bigoplus_{r=1}^d \mathbb{C} v_a$.

We also need some combinatorics. We denote by $\text{Part}(n)$ (resp. $\text{Part}^d(r)$) the set of partitions of $n$ (resp. of $d$-partitions of $r$). If $\lambda \in \text{Part}(n)$, we denote by $\chi_\lambda$ the irreducible character of $S_n$ (with the convention of [GePf]: for instance $\chi_n = 1$ and $\chi_1^n = \varepsilon$), by $\text{cor}_d(\lambda)$ the $d$-core of $\lambda$, by $\text{quo}_d(\lambda)$ its $d$-quotient. We let $\text{Part}(n, d)$ denote the set of partitions of $n$ whose $d$-core is the unique partition of $j \in \{0, 1\}$. Then the map

$$\text{quo}_d : \text{Part}(n, d) \longrightarrow \text{Part}^d(r)$$

is bijective. Finally, if $\mu \in \text{Part}^d(r)$, we denote by $\chi_\mu$ the associated irreducible character of $G_W(w_d) = G(d, 1, r)$ (with the convention of [GeJa]). It follows from Murnaghan-Nakayama rule that

$$\chi_\lambda(w_d) \neq 0 \text{ if and only if } \lambda \in \text{Part}(n, d),$$

and that

$$\chi_\lambda(w_d) = \pm \chi_{\text{quo}_d(\lambda)}(1)$$

for all $\lambda \in \text{Part}(n, d)$ (see for instance [BMM1, Page 47]).

Now, the smoothness of $\mathcal{I}_k$ implies that the map $\varphi_k : \text{Irr}(S_n) \longrightarrow \mathcal{I}^C_k$ is bijective (so that Calogero-Moser $k$-families of $S_n$ are singleton) and it follows from [BoMa] that Conjecture 5.1 holds (except that we do not know if the isomorphism respects the Poisson structure), so that we have a $\mathbb{C}^\times$-equivariant isomorphism of varieties

$$\iota_k : (\mathcal{I}^H_k)_{\text{max}} \sim \mathcal{I}_k(V^\tau_d, G(d, 1, r))$$
for some explicit $\lambda(k) \in \mathbb{C}^{N_d}$. Moreover, $\mathcal{F}_{\lambda(k)}(V^{r_d}, G(d,1,r))$ is smooth so that the map $\varphi_{\lambda(k)}^{r_d} : \text{Irr}(G(d,1,r)) \rightarrow \mathcal{F}_{\lambda(k)}(V^{r_d}, G(d,1,r))^{C^\times}$ is bijective (that is, Calogero-Moser $\lambda(k)$-families of $G(d,1,r)$ are singleton). Now, by [BoMa], we have that
\begin{equation}
\varphi_k(\chi_\lambda) \in (\mathcal{F}_k^{\mu_d})_{\text{max}} \text{ if and only if } \lambda \in \text{Part}(n,d),
\end{equation}
and that
\begin{equation}
\iota_k(\varphi_k(\chi_\lambda)) = \varphi_{\lambda(k)}^{r_d}(\chi_{\text{quo}_d(\lambda)})
\end{equation}
for all $\lambda \in \text{Part}(n,d)$. Then (5.7), (5.8), (5.9) and (5.10) show that Conjectures 5.4 and 5.2 hold for the symmetric group. ■

References

[Bon1] C. Bonnafé, On the Calogero-Moser space associated with dihedral groups, Ann. Math. Blaise Pascal 25 (2018), 265-298.
[Bon2] C. Bonnafé, Automorphisms and symplectic leaves of Calogero-Moser spaces, preprint (2021), arXiv:2112.12405.
[Bon3] C. Bonnafé, Calogero-Moser spaces vs unipotent representations, preprint (2021), arXiv:2112.13684.
[BoMa] C. Bonnafé & R. Maksimau Fixed points in smooth Calogero-Moser spaces, Ann. Inst. Fourier 71 (2021), 643-678.
[BoRo] C. Bonnafé & R. Rouquier, Cherednik algebras and Calogero-Moser cells, preprint (2017), arXiv:1708.09764.
[Bro] M. Broué, Introduction to complex reflection groups and their braid groups, Lecture Notes in Math. 1988, Springer-Verlag, Berlin, 2010. xii+138 pp.
[BMM1] M. Broué, G. Malle & J. Michel, Generic blocks of finite reductive groups, in Représentations unipotentes génériques et blocs des groupes réductifs finis, Astérisque 212 (1993), 7-92.
[BMM2] M. Broué, G. Malle & J. Michel, Split spetses for primitive reflection groups, Astérisque 359 (2014), vi+146 pp.
[BrMi] M. Broué & J. Michel, Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées, in Finite reductive groups (Luminy, 1994), 73-139, Progr. Math. 141, Birkh"auser Boston, Boston, MA, 1997.
[EtGi] P. Etingof & V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Invent. Math. 147 (2002), 243-348.
[GeJa] M. Geck & N. Jacon, Representations of Hecke algebras at roots of unity, Algebra and Applications 15, Springer-Verlag London, Ltd., London, 2011, xii+401 pp.
[GePf] M. Geck & G. Pfeiffer, Characters of finite Coxeter groups and Iwahori-Hecke algebras, London Mathematical Society Monographs, New Series 21, The Clarendon Press, Oxford University Press, New York, 2000, xvi+446 pp.
[Gor] I. Gordon, Baby Verma modules for rational Cherednik algebras, Bull. London Math. Soc. 35 (2003), 321-336.
[Lac] A. Lacabanne, On a conjecture about cellular characters for the complex reflection group $G(d,1,n)$, Ann. Math. Blaise Pascal 27 (2020), 37-64.
[Spr] T.A. Springer, Regular elements of finite reflection groups, Invent. Math. 25 (1974), 159-198.