Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables.

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Abstract
We remind known and establish new properties of the Dieudonné and Moore determinants of quaternionic matrices. Using these linear algebraic results we develop a basic theory of plurisubharmonic functions of quaternionic variables.

0 Introduction.
The main point of this paper is that in quaternionic algebra and analysis there exist structures which have analogues over the fields of real and complex numbers, but should reflect different phenomena.

The algebraic part is discussed in Section 1. There we remind the notions of the Moore and Dieudonné determinants of quaternionic matrices. It turns out that (under appropriate normalization) the Dieudonné determinant behaves exactly like the absolute value of the usual determinant of real or complex matrices from all points of view (algebraic and analytic). Let us state some of its properties discussed in more details in Subsection 1.2. Let us denote by $M_n(\mathbb{H})$ the set of all quaternionic $n \times n$ matrices. The Dieudonné determinant $D$ is defined on this set and takes values in non-negative real numbers:

$$D : M_n(\mathbb{H}) \rightarrow \mathbb{R}_{\geq 0}$$
Then one has the following (known) results (see Theorems 1.2.3 and 1.2.4 below and references given at the beginning of Section 1):

**Theorem.**

(i) For any complex $n \times n$- matrix $X$ considered as quaternionic matrix the Dieudonné determinant $D(X)$ is equal to the absolute value of the usual determinant of $X$.

(ii) For any quaternionic matrix $X$

$$D(X^*) = D(X),$$

where $X^*$ denotes the quaternionic conjugate matrix of $X$.

(iii) $D(X \cdot Y) = D(X)D(Y)$.

**Theorem 1.2.5.** Let $A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$ be a quaternionic matrix.

Then

$$D(A) \leq \sum_{i=1}^{n} |a_{1i}|D(M_{1i}).$$

Similar inequalities hold for any other row or column.

(In this theorem $|a|$ denotes the absolute value of a quaternion $a$, and $M_{pq}$ denotes the minor of the matrix $A$ obtained from it by deleting the $p$-th row and $q$-th column).

In a sense, the Dieudonné determinant provides the theory of *absolute value* of determinant. However it is not always sufficient and we lose most of the algebraic properties of the usual determinant. The notion of Moore determinant provides such a theory, but only on the class of quaternionic *hyperhermitian* matrices. Remind that a square quaternionic matrix $A = (a_{ij})$ is called hyperhermitian if its quaternionic conjugate $A^* = A$, or explicitly $a_{ij} = \overline{a_{ji}}$. The Moore determinant denoted by $det$ is defined on the class of all hyperhermitian matrices and takes real values. (The Moore determinant is defined in Subsection 1.1 after Theorem 1.1.8). The important advantage of it with respect to the Dieudonné determinant is that it depends polynomially on the entries of a matrix; it has already all the algebraic and analytic properties of the usual determinant of real symmetric and complex hermitian matrices. Let us state some of them referring for the details to Subsection 1.1 (again, the references are given at the beginning of Section 1).

**Theorem 1.1.9.**
(i) The Moore determinant of any complex hermitian matrix considered as quaternionic hyperhermitian matrix is equal to its usual determinant.

(ii) For any hyperhermitian matrix \( A \) and any quaternionic matrix \( C \)

\[
det(C^*AC) = detA \cdot det(C^*C).
\]

Examples.
(a) Let \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) be a diagonal matrix with real \( \lambda_i \)'s. Then \( A \) is hyperhermitian and the Moore determinant \( detA = \prod_i \lambda_i \).
(b) A general hyperhermitian \( 2 \times 2 \) matrix \( A \) has the form

\[
A = \begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix},
\]

where \( a, b \in \mathbb{R}, q \in \mathbb{H} \). Then its Moore determinant is equal to \( detA = ab - q\bar{q} \).

Next, in terms of the Moore determinant one can prove the generalization of the classical Sylvester criterion of positive definiteness of hyperhermitian matrices (Theorem 1.1.13). In terms of the Moore determinant one can introduce the notion of the mixed discriminant and to prove the analogues of Aleksandrov’s inequalities for mixed discriminants (Theorem 1.1.15 and Corollary 1.1.16).

The (well known) relation between the Dieudonné and Moore determinants is as follows: for any hyperhermitian matrix \( X \)

\[
D(X) = |detX|.
\]

In Section 1 we prove some additional properties of the Dieudonné and Moore determinants; they are used in Section 2.

Note that the Dieudonné determinant was introduced originally by J. Dieudonné in [14] (see also [5] for his theory). It can be defined for arbitrary (non-commutative) field. On more modern language this result can be formulated as a computation of the \( K_1 \)-group of a non-commutative field (see e.g. [12]). Note also that there is a more recent theory of non-commutative determinants (or quasideterminants) due to I. Gelfand and V. Retakh generalizing in certain direction the theory of the Dieudonné determinant. First it was introduced in [20], see also [21], [22], [24] and references therein for further developments and applications. In the recent preprint [23] Gelfand, Retakh, and Wilson have discovered that the formulas for quasideterminants
of quaternionic matrices can be significantly simplified. They also understood the relation between the theory of quasideterminants and the Moore determinant. We would also like to mention a different direction of a development of the quaternionic linear algebra started by D. Joyce [30] and applied by himself to hypercomplex algebraic geometry. We refer also to D. Quillen’s paper [40] for further investigations in that direction. Another attempt to understand the quaternionic linear algebra from the topological point of view was done in [3].

Section 2 of this paper develops the basic theory of plurisubharmonic functions of quaternionic variables on \( H^n \). It uses in essential way the linear algebraic results of Section 1. This theory is parallel to the classical theories of convex functions on \( \mathbb{R}^n \) and plurisubharmonic functions on \( \mathbb{C}^n \).

The formal definition is as follows (for more discussion see Subsection 2.1).

**Definition.** A real valued function

\[
f : H^n \longrightarrow \mathbb{R}
\]

is called quaternionic plurisubharmonic if it is upper semi-continuous and its restriction to any right quaternionic line is subharmonic (in the usual sense). We refer to Subsection 2.1 where we remind the relevant notions.

In this form this definition was suggested by G. Henkin [27]. For the class of continuous plurisubharmonic functions this definition is different but equivalent (by Proposition 2.1.6 below) to the original author’s definition.

**Remark.** On \( H^1 \) the class of plurisubharmonic functions coincides with the class of subharmonic functions. In this case all the results of this paper are reduced to the classical properties of subharmonic functions in \( \mathbb{R}^4 \).

Let us describe the main results on plurisubharmonic functions we prove. We will write a quaternion \( q \) in the usual form

\[
q = t + x \cdot i + y \cdot j + z \cdot k,
\]

where \( t, x, y, z \) are real numbers, and \( i, j, k \) satisfy the usual relations

\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]

Let us introduce the differential operators \( \frac{\partial}{\partial q} \) and \( \frac{\partial}{\partial \bar{q}} \) as follows:

\[
\frac{\partial}{\partial q} f := \frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}, \quad \text{and}
\]

\[
\frac{\partial}{\partial \bar{q}} f := \frac{\partial f}{\partial \bar{t}} - i \frac{\partial f}{\partial \bar{x}} - j \frac{\partial f}{\partial \bar{y}} - k \frac{\partial f}{\partial \bar{z}}.
\]
\[
\frac{\partial}{\partial q} f := \overline{\frac{\partial}{\partial \bar{q}} f} = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} i - \frac{\partial f}{\partial y} j - \frac{\partial f}{\partial z} k.
\]

**Remarks.** (a) The operator $\frac{\partial}{\partial q}$ is called sometimes the Cauchy-Riemann-Moisil-Fueter operator since it was introduced by Moisil in [35] and used by Fueter [17], [18] to define the notion of quaternionic analyticity. For further results on quaternionic analyticity we refer e.g. to [10], [37], [38], [43], and for applications to mathematical physics to [25]. Another used name for this operator is Dirac-Weyl operator. But in fact it was used earlier by J.C. Maxwell in [32], vol. II, pp.570-576, where he has applied the quaternions to electromagnetism.

(b) Note that
\[
\frac{\partial}{\partial ar{q}} = \frac{\partial}{\partial t} + \nabla,
\]
where $\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$. The operator $\nabla$ was first introduced by W.R. Hamilton in [26].

(c) In quaternionic analysis one considers a right version of the operators $\frac{\partial}{\partial q}$ and $\frac{\partial}{\partial \bar{q}}$ which are denoted respectively by $\frac{\partial}{\partial q}$ and $\frac{\partial}{\partial \bar{q}}$. The operator $\frac{\partial}{\partial q}$ is related to $\frac{\partial}{\partial \bar{q}}$ by the same formula as $\frac{\partial}{\partial q}$ is related to $\frac{\partial}{\partial \bar{q}}$, and $\frac{\partial}{\partial \bar{q}}$ is defined as
\[
\frac{\partial}{\partial \bar{q}} f := \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.
\]
For a real valued function $f$ the derivatives $\frac{\partial^2 f}{\partial q_i \partial q_j}$ and $\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}$ are quaternionic conjugate to each other.

First one has a simple

**Proposition 2.1.6.** A real valued twice continuously differentiable function $f$ on the domain $\Omega \subset H^n$ is quaternionic plurisubharmonic if and only if at every point $q \in \Omega$ the matrix $(\frac{\partial^2 f}{\partial q_i \partial q_j})(q)$ is non-negative definite.

Note that the matrix in the statement of proposition is quaternionic hyperhermitian. The more important thing is that in analogy to the real and complex cases one can define for any continuous quaternionic plurisubharmonic function $f$ a non-negative measure $\text{det}(\frac{\partial^2 f}{\partial q_i \partial q_j})(q)$, where $\text{det}$ denotes the Moore determinant (this measure is obviously defined for smooth $f$). We prove the following continuity result.
Theorem 2.1.11. Let \( \{f_N\} \) be sequence of continuous quaternionic plurisubharmonic function in a domain \( \Omega \subset \mathbb{H}^n \). Assume that this sequence converges uniformly on compact subsets to a function \( f \). Then \( f \) is continuous quaternionic plurisubharmonic function. Moreover the sequence of measures \( \det(\frac{\partial^2 f_N}{\partial q_i \partial q_j}) \) weakly converges to the measure \( \det(\frac{\partial^2 f}{\partial q_i \partial q_j}) \).

The proofs of analogous results in real and complex cases can be found in [7], where the exposition of this topic follows the approach of Chern-Levine-Nirenberg [13] and Rauch-Taylor [41]. For the complex case we refer to the classical book by P. Lelong [31]. In generalizations of these results to the quaternionic situation the large part of the difficulties comes from linear algebra since the technique of working with the Moore determinant is not sufficiently developed. For instance there is no formula of decomposition of the Moore determinant in row or column, and thus one should use some more tricky manipulations.

Next we would like to state a result on existence and uniqueness of solution of the Dirichlet problem for quaternionic Monge-Ampère equation (to be defined). In this paper we prove only the uniqueness part; the existence is proved in author’s paper [4].

Definition. An open bounded domain \( \Omega \subset \mathbb{H}^n \) with a smooth boundary \( \partial \Omega \) is called strictly pseudoconvex if for every point \( z_0 \in \partial \Omega \) there exists a neighborhood \( \mathcal{O} \) and a smooth strictly plurisubharmonic function \( h \) on \( \mathcal{O} \) such that \( \Omega \cap \mathcal{O} = \{ h < 0 \} \) and \( \nabla h(z_0) \neq 0 \).

The next result is quaternionic analogue of the results on Dirichlet problem for real and complex Monge-Ampère equations. The real case was solved by Aleksandrov [2], and the complex one by Bedford and Taylor [9].

Theorem. Let \( \Omega \) be a strictly pseudoconvex bounded domain in \( \mathbb{H}^n \). Let \( \phi \) be a continuous real valued function on the boundary \( \partial \Omega \). Let \( f \) be a continuous function on the closure \( \overline{\Omega} \), \( f \geq 0 \). Then there exists a unique continuous on \( \overline{\Omega} \) plurisubharmonic function \( u \) such that

\[
\det(\frac{\partial^2 u}{\partial q_i \partial q_j}) = f \quad \text{and} \\
u = \phi \quad \text{on} \quad \partial \Omega.
\]

The uniqueness part in this theorem is an immediate consequence of the following minimum principle which is proved in Subsection 2.2.
Theorem 2.2.1. Let $\Omega$ be a bounded open set in $\mathbb{H}^n$. Let $u$, $v$ be continuous functions on $\bar{\Omega}$ which are plurisubharmonic in $\Omega$. Assume that

$$\text{det} \left( \frac{\partial^2 u}{\partial q_i \partial q_j} \right) \leq \text{det} \left( \frac{\partial^2 v}{\partial q_i \partial q_j} \right) \text{ in } \Omega.$$ 

Then

$$\min \{ u(z) - v(z) | z \in \bar{\Omega} \} = \min \{ u(z) - v(z) | z \in \partial \Omega \}.$$ 

The proof of this theorem closely follows the argument of Bedford and Taylor [9] (Theorem A).

In appendix to this paper we prove the injectivity of Radon transform over quaternionic subspaces in the affine space $\mathbb{H}^n$. Probably this result is not new. It is included here since it was used in the proof of Lemma 2.1.7, and we could not find a reference.

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1 Linear algebra.

In this section we remind the construction and basic properties of the Dieudonné and Moore determinants and investigate further their properties. Part of them will be used in the next sections of this paper. For a survey of quaternionic determinants and references we refer to [6].

First of all remind that over any noncommutative field there exist usual notions of vector spaces over the field (however one should distinguish between left and right ones), their dimension, basis etc. (see e.g. [5]). However there is no construction of quaternionic determinant which would have all the properties of the determinant over commutative field. We are going to discuss this problem in this section. We will discuss only right vector spaces. The case of left ones can be considered similarly. Many results of Section 1 are a folklore. Theorems 1.1.8, 1.1.9, 1.1.4 are not new. We refer for the proofs to [8], [11], [12], [15], [16], [28], [29], [33], [34], [36], [39], [44], [45].

1.1 Hyperhermitian forms and the Moore determinant.

Let $V$ be a right vector space over quaternions.
1.1.1 Definition. A hyperhermitian semilinear form on $V$ is a map $a : V \times V \rightarrow \mathbb{H}$ satisfying the following properties:

(a) $a$ is additive with respect to each argument;
(b) $a(x, y \cdot q) = a(x, y) \cdot q$ for any $x, y \in V$ and any $q \in \mathbb{H}$;
(c) $a(x, y) = a(y, x)$.

1.1.2 Example. Let $V = \mathbb{H}^n$ be the standard coordinate space considered as right vector space over $\mathbb{H}$. Fix a hyperhermitian $n \times n$-matrix $(a_{ij})_{i,j=1}^n$, i.e. $a_{ij} = \bar{a}_{ji}$, where $\bar{x}$ denotes the usual quaternionic conjugation of $x$. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ define

$$A(x, y) = \sum_{i,j} \bar{x}_i a_{ij} y_j$$

(note the order of the terms!). Then $A$ defines hyperhermitian semilinear form on $V$.

In general one has the following standard claims.

1.1.3 Claim. Fix a basis in a finite dimensional right quaternionic vector space $V$. Then there is a natural bijection between hyperhermitian semilinear forms on $V$ and $n \times n$-hyperhermitian matrices.

This bijection is in fact described in previous Example 1.1.2.

1.1.4 Claim. Let $A$ be the matrix of the given hyperhermitian form in the given basis. Let $C$ be transition matrix from this basis to another one. Then the matrix $A'$ of the given form in the new basis is equal

$$A' = C^* AC.$$

1.1.5 Remark. Note that for any hyperhermitian matrix $A$ and for any matrix $C$ the matrix $C^* AC$ is also hyperhermitian. In particular the matrix $C^* C$ is always hyperhermitian.

1.1.6 Definition. A hyperhermitian semilinear form $a$ is called positive definite if $a(x, x) > 0$ for any non-zero vector $x$.

Let us fix on our quaternionic right vector space $V$ a positive definite hyperhermitian form $(\cdot, \cdot)$. The space with fixed such a form will be called hyperhermitian space.
For any quaternionic linear operator \( \phi : V \rightarrow V \) in hyperhermitian space one can define the adjoint operator \( \phi^* : V \rightarrow V \) in the usual way, i.e. \((\phi x, y) = (x, \phi^* y)\) for any \(x, y \in V\). Then if one fixes an orthonormal basis in the space \(V\) then the operator \(\phi\) is selfadjoint if and only if its matrix in this basis is hyperhermitian.

1.1.7 Claim. For any selfadjoint operator in a hyperhermitian space there exists an orthonormal basis such that its matrix in this basis is diagonal and real.

The proof is standard. Now we are going to define the Moore determinant of hyperhermitian matrices. The definition below is different from the original one [36] but equivalent to it.

First note that every hyperhermitian \(n \times n\)-matrix \(A\) defines a hyperhermitian semilinear form on the coordinate space \(\mathbb{H}^n\). It also can be considered as a symmetric bilinear form on \(\mathbb{R}^{4n}\) (which is the realization of \(\mathbb{H}^n\)). Let us denote its \(4n \times 4n\)-matrix by \(\mathbb{R}A\). Let us consider the entries of \(A\) as formal variables (each quaternionic entry corresponds to four commuting real variables). Then \(\det(\mathbb{R}A)\) is a homogeneous polynomial of degree \(4n\) in \(n(2n-1)\) real variables. Let us denote by \(\text{Id}\) the identity matrix. One has the following result.

1.1.8 Theorem. There exists a polynomial \(P\) defined on the space of all hyperhermitian \(n \times n\)-matrices such that for any hyperhermitian \(n \times n\)-matrix \(A\) one has \(\det(\mathbb{R}A) = P^4(A)\) and \(P(\text{Id}) = 1\). \(P\) is defined uniquely by these two properties. Furthermore \(P\) is homogeneous of degree \(n\) and has integer coefficients.

Thus for any hyperhermitian matrix \(A\) the value \(P(A)\) is a real number, and it is called the Moore determinant of the matrix \(A\). The explicit formula for the Moore determinant was given by Moore [36] (see also [6]). From now on the Moore determinant of a matrix \(A\) will be denoted by \(\det A\). This notation should not cause any confusion with the usual determinant of real or complex matrices due to part (i) of the next theorem.

1.1.9 Theorem. (i) The Moore determinant of any complex hermitian matrix considered as quaternionic hyperhermitian matrix is equal to its usual determinant.

(ii) For any hyperhermitian matrix \(A\) and any matrix \(C\)

\[\det(C^*AC) = \det A \cdot \det(C^*C),\]
Example. (a) Let $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$ be a diagonal matrix with real $\lambda_i$'s. Then $A$ is hyperhermitian and the Moore determinant $\det A = \prod_i \lambda_i$.

(b) A general hyperhermitian $2 \times 2$ matrix $A$ has the form

$$A = \begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix},$$

where $a, b \in \mathbb{R}$, $q \in \mathbb{H}$. Then $\det A = ab - q\bar{q}$.

Let us introduce more notation. Let $A$ be any hyperhermitian $n \times n$-matrix. For any non-empty subset $I \subset \{1, \ldots, n\}$ the minor $M_I(A)$ of $A$ which is obtained by deleting the rows and columns with indexes from the set $I$, is clearly hyperhermitian. For $I = \{1, \ldots, n\}$ let $\det M_{\{1, \ldots, n\}} = 1$.

1.1.11 Proposition. For any hyperhermitian $n \times n$-matrix $A$ and any diagonal real matrix $T = \begin{bmatrix} t_1 & 0 \\ & \ddots \\ 0 & t_n \end{bmatrix}$

$$\det(A + T) = \sum_{I \subset \{1, \ldots, n\}} \left( \prod_{i \in I} t_i \right) \cdot \det M_I(A).$$

In particular

$$\det(A + t \cdot Id) = \sum_{I \subset \{1, \ldots, n\}} t^{|I|} \cdot \det M_I(A),$$

where $|I|$ denotes the cardinality of the set $I$.

Remark. Clearly this formula is true for arbitrary $n \times n$-matrix $A$ over a commutative field.

Proof. Fix a hyperhermitian matrix $A$. It is clear that $\det(A + T)$ is a polynomial in $t_1, \ldots, t_n$ of degree $n$. Since

$$A + \begin{bmatrix} t_1 & 0 \\ t_2 & \ddots \\ 0 & \ddots & t_n \end{bmatrix} = A + \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} t_1 & 0 \\ 0 & \ddots \\ 0 & \cdots & t_n \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

10
one can apply induction in $n$ to show that if $A = \begin{bmatrix} a_{11} & \ast \\ \ast & B \end{bmatrix}$, where $a_{11} \in \mathbb{R}$, and $B$ is a hyperhermitian $(n-1) \times (n-1)$- matrix. Set

$$f(t) := \det \left( A + \begin{bmatrix} t & 0 \\ 0 & 0 \\ \ddots & \ddots \\ 0 & 0 \end{bmatrix} \right).$$

It is sufficient to show that $f(t) = \det A + t \cdot \det B$. Clearly $f(0) = \det A$. Let $k$ denote the degree of the polynomial $f$. Using Theorem 1.1.9(ii) one gets

$$f(t) = t^k \cdot \det \begin{bmatrix} t^{-k/2} & 0 \\ 0 & 1 \\ \ddots & \ddots \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 0 \\ \ddots & \ddots \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t^{-k/2} & 1 \\ 0 & 0 \\ \ddots & \ddots \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{t^k} \det \begin{bmatrix} a_{11} t^{-k} \ast + t^{-k+1} & a_{12} t^{-k/2} \ast & \ldots & a_{1n} t^{-k/2} \ast \\ a_{21} t^{-k/2} & \ldots & \ddots & \ast \\ \vdots & \ddots & \ddots & \ddots \\ a_{n1} t^{-k/2} & \ast & \ldots & B \end{bmatrix}. $$

If $k > 1$ then

$$\frac{f(t)}{t^k} \to \det \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = 0$$

when $t \to \infty$. Hence $k = 1$ and

$$\frac{f(t)}{t} \to \det \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} = \det B.$$

Q.E.D.

1.1.12 Lemma. Let $A$ be a non-negative (resp. positive) definite hyperhermitian matrix. Then $\det A \geq 0$ (resp. $\det A > 0$).

Proof. Let us prove it under the assumption that $A$ is positive definite. By Claim 1.1.7 there exists a matrix $C \in Sp(n)$ (i.e. $C^* C = Id$) such that
A = C^* \begin{bmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix} C \quad \text{with } \lambda_i \in \mathbb{R}. \quad \text{Since } A \text{ is positive definite, } \lambda_i > 0 \quad \text{for all } i. \quad \text{By Theorem 1.1.9(ii) } \det A = \det(C^*C) \prod \lambda_i = \prod \lambda_i > 0. \quad \text{Q.E.D.}

The following theorem is a quaternionic generalization of the standard Sylvester criterion.

1.1.13 Theorem (Sylvester criterion). A hyperhermitian \( n \times n \)-matrix \( A \) is positive definite if and only if \( M_{\{i+1,\ldots,n\}}(A) > 0 \) for any \( 0 \leq i \leq n \).

Proof. The necessity follows from Lemma 1.1.12. Let us prove sufficiency by induction in \( n \). For \( n = 1 \) the statement is trivial. Assume \( n > 1 \). Let

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & & & \\
  \vdots & & & \\
  a_{n1} & & & B
\end{bmatrix}.
\]

Consider the matrix

\[
U = \begin{bmatrix}
  1 & -a_{12}/a_{11} & -a_{13}/a_{11} & \cdots & -a_{1n}/a_{11} \\
  0 & 1 & 0 & \cdots & 0 \\
  & & \cdots & \cdots & \cdots \\
  0 & 0 & 0 & \cdots & 1
\end{bmatrix}.
\]

Then the matrix \( A' := U^*AU \) has the form

\[
A' = \begin{bmatrix}
  a_{11} & 0 & \cdots & 0 \\
  0 & & & B' \\
  \vdots & & & \\
  0 & & &
\end{bmatrix},
\]

where \( B' \) is a hyperhermitian matrix. Moreover for any \( 1 \leq i \leq n \) one has

\[
det M_{\{i+1,\ldots,n\}}(A') = det M_{\{i+1,\ldots,n\}}(A).
\]

Indeed let us check it for \( i = n \) (for \( i < n \) the proof will be the same since the matrix \( U \) is triangular). Namely let us show that \( det A' = det A \). By Theorem 1.1.9 (ii) \( det A' = det A \cdot det(U^*U) \). However using Theorem 1.1.8 and unipotence of \( U \) it is easy to see that \( det(U^*U) = 1 \). Hence the matrix \( B' \)
is positive definite by the induction assumption. Then $A'$ is positive definite, and hence $A$ is as well. Q.E.D.

Let us define now the mixed discriminant of hyperhermitian matrices in analogy with the case of real symmetric matrices \[1\].

**1.1.14 Definition.** Let $A_1, \ldots, A_n$ be hyperhermitian $n \times n$-matrices. Consider the homogeneous polynomial in real variables $\lambda_1, \ldots, \lambda_n$ of degree $n$ equal to $\det(\lambda_1 A_1 + \cdots + \lambda_n A_n)$. The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ divided by $n!$ is called the *mixed discriminant* of the matrices $A_1, \ldots, A_n$, and it is denoted by $(A_1, \ldots, A_n)$.

Note that the mixed discriminant is symmetric with respect to all variables, and linear with respect to each of them, i.e.

\[(\lambda A_1' + \mu A_n', A_2, \ldots, A_n) = \lambda \cdot (A_1', A_2, \ldots, A_n) + \mu \cdot (A_n', A_2, \ldots, A_n)\]

for any real $\lambda, \mu$. Note also that $(A, \ldots, A) = \det A$. We will prove the following generalization of Aleksandrov’s inequalities for mixed discriminants \[1\] (though the proof will be very close to the original one).

**1.1.15 Theorem.** (i) The mixed discriminant of positive (resp. non-negative) definite matrices is positive (resp. non-negative).

(ii) Fix positive definite hyperhermitian $n \times n$-matrices $A_1, \ldots, A_{n-2}$. On the real linear space of hyperhermitian $n \times n$-matrices consider the bilinear form

\[B(X, Y) := (X, Y, A_1, \ldots, A_{n-2}).\]

Then $B$ is non-degenerate quadratic form, and its signature has one plus and the rest are minuses.

**1.1.16 Corollary.** Let $A_1, \ldots, A_{n-1}$ be positive definite hyperhermitian $n \times n$-matrices. Then for any hyperhermitian matrix $X$

\[(A_1, \ldots, A_{n-1}, X)^2 \geq (A_1, \ldots, A_{n-1}, A_{n-1}) \cdot (A_1, \ldots, A_{n-2}, X, X), \quad (1)\]

and the equality is satisfied if and only if the matrix $X$ is proportional to $A_{n-1}$.

**Proof** of Corollary 1.1.16. By Theorem 1.1.15 (i) we get

\[(A_1, \ldots, A_{n-1}, A_{n-1}) > 0.\]
Let
\[ \lambda = \frac{(A_1, \ldots, A_{n-1}, X)}{(A_1, \ldots, A_{n-1}, A_n)}. \]

Let \( X' = X - \lambda A_{n-1} \). Then clearly \((A_1, \ldots, A_{n-1}, X') = 0. \) In the notation of Theorem 1.1.15 it means that \( B(A_{n-1}, A_{n-1}) > 0 \) and \( B(A_{n-1}, X') = 0. \) But the form \( B \) has just one plus. Hence \( B(X', X') \leq 0, \) and the equality is satisfied if and only if \( X' = 0. \) Developing \( B(X', X') \) one gets inequality (1).

The equality case follows as well. Q.E.D.

Proof of Theorem 1.1.15. (1) Let us prove the first part using induction in \( n. \) The case \( n = 1 \) is trivial. Assume that \( n > 1. \) Let \( A_1, \ldots, A_n \) be positive definite hyperhermitian matrices. By Claim 1.1.7 and Theorem 1.1.9 (ii) we can assume that the matrix \( A_n \) is diagonal, i.e. \( A_n = \begin{bmatrix} t_1 & 0 \\ 0 & t_n \end{bmatrix}, \) and \( t_i \)'s are positive. By Proposition 1.1.11

\[ \det(\lambda_1 A_1 + \cdots + \lambda_{n-1} A_{n-1} + \lambda_n A_n) = \sum_{I \subset \{1, \ldots, n\}} (\prod_{i \in I} \lambda_n t_i) \cdot \det M_I(\lambda_1 A_1 + \cdots + \lambda_{n-1} A_{n-1}). \]

Since all the diagonal minors of positive definite matrix are positive definite and since \( t_i > 0 \) the assumption of induction implies the statement.

(2) Let us prove the second part of the theorem, i.e. that \( B \) is non-degenerate. First let us prove it for \( n = 2. \) Assume \( X_0 \) belongs to the kernel of \( B, \) i.e. \( B(X, X_0) = 0 \) for every \( X. \) One can assume that \( X_0 \) is diagonal: \( X_0 = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}. \) For any \( X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \) with real \( x_1, x_2 \) one has \( 2(X, X_0) = t_1 x_2 + t_2 x_1 = 0. \) Hence \( t_1 = t_2 = 0. \) Thus the form \( B \) is non-degenerate. Now, clearly \( B(Id, Id) = 1 > 0. \) Assume that \( X \neq 0 \) is orthogonal to \( Id \) with respect to \( B, \) i.e. \( B(X, Id) = 0. \) It remains to show that \( B(X, X) < 0. \) By Claim 1.1.7 we can assume that \( X \) is diagonal, \( X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}. \) Then \( 2B(X, Id) = x_1 + x_2 = 0. \) But \( B(X, X) = x_1 x_2 = -x_1^2 < 0. \)

Let us assume that \( n > 2. \) Assume also that the theorem is true for matrices of size at most \( n - 1. \) Let us prove first that the form \( B \) is non-degenerate. Assume that \( X_0 \) belongs to the kernel of \( B. \) Since \( A_{n-2} \) is positive
definite, by Claim 1.1.7 one can assume that the matrix $A_{n-2}$ is equal to $Id$ and $X_0$ is diagonal. For $1 \leq i \leq n$ and for $n \times n$-matrices $C_1, \ldots, C_{n-1}$ let us denote by $(C_1, \ldots, C_{n-1})_i$ the mixed discriminant of $(n-1) \times (n-1)$-matrices obtained from $C_j$’s by deleting the $i$-th row and the $i$-th column.

Let $T = \begin{bmatrix} t_1 & 0 \\ \vdots & \ddots \\ 0 & t_n \end{bmatrix}$. Using Proposition 1.1.11 one can easily see that

$$0 = c \cdot (A_1, \ldots, A_{n-2}, T, X_0) = \sum_{i=1}^{n} t_i (A_1, \ldots, A_{n-2}, X_0)_i,$$  

(2)

where $c > 0$ is a normalizing constant. Hence $(A_1, \ldots, A_{n-2}, X_0)_i = 0$ for all $i$. By the induction assumption and Corollary 1.1.16 (which is also satisfied for matrices of size $n-1$)

$$(A_1, \ldots, A_{n-3}, X_0, X_0)_i \leq 0 \text{ for } i = 1, \ldots, n$$  

(3)

with equalities if and only if the matrix $X_0$ vanishes. Since $A_{n-2} = Id$ and $X_0$ belongs to the kernel of $B$ the equality analogous to (2) implies that

$$0 = c \cdot (A_1, \ldots, A_{n-3}, A_{n-2}, X_0, X_0) = \sum_{i=1}^{n} (A_1, \ldots, A_{n-3}, X_0, X_0)_i.$$  

By inequalities (3) one gets that $(A_1, \ldots, A_{n-3}, X_0, X_0)_i = 0$ for all $i$. Hence $X_0$ vanishes by the induction hypothesis. This proves that the form $B$ is non-degenerate.

It remains to compute the signature of $B$. Remind that $B$ depends on positive definite matrices $A_1, \ldots, A_{n-2}$. The space of positive definite matrices is connected (indeed if $A$ and $B$ are positive definite then $tA + (1-t)B$ is positive definite for $0 \leq t \leq 1$). The signature of a family of non-degenerate quadratic forms cannot jump. Hence it is constant. Thus we can assume that $A_1 = \cdots = A_{n-2} = Id$. As in the case $n = 2$ it is sufficient to check that if $X \neq 0$ satisfies $B(X, Id) = 0$ then $B(X, X) < 0$. Again we can assume that $X$ is diagonal,

$$X = \begin{bmatrix} x_1 & 0 \\ \vdots & \ddots \\ 0 & x_n \end{bmatrix}.$$  

15
The condition $B(X, Id) = 0$ means that $\sum_{i=1}^{n} x_i = 0$. Also it is easy to see that

$$\kappa \cdot B(X, X) = 2 \sum_{i<j} x_i x_j,$$

where $\kappa$ is a positive normalization constant. But

$$2 \sum_{i<j} x_i x_j = (\sum_i x_i)^2 - \sum_i x_i^2 = -\sum_i x_i^2 < 0.$$ 

The theorem is proved. Q.E.D.

We will need also the following result.

1.1.17 Theorem. (i) The function $X \mapsto \log(\det X)$ is concave on the cone of positive definite hyperhermitian matrices, namely if $A, B \geq 0$ and $0 \leq t \leq 1$ then

$$\log(\det(tA + (1-t)B)) \geq t \log(\det A) + (1-t) \log(\det B).$$

(ii) The function $X \mapsto (\det X)^{\frac{1}{n}}$ is concave on the cone of the positive definite hyperhermitian matrices.

(iii) If $A, B \geq 0$ then

$$\det(A + B) \geq \det A + \det B.$$ 

Proof. Note that we may assume that $A = I$ and $B$ is real diagonal. Both results follow from the (known) real case. Q.E.D.

1.2 Dieudonné determinant.

We will remind the construction of the Dieudonné determinant referring for the details and proofs to [5]. Also we will prove some properties of it which will be used in the subsequent sections of the paper. Intuitively the Dieudonné determinant of an arbitrary quaternionic matrix has almost the same algebraic and analytic properties as the absolute value of the usual determinant of real or complex matrices. First let us discuss purely algebraic construction.

Let $F$ be an infinite field, not necessarily commutative. Let $M_n(F)$ denote the ring of $n \times n$-matrices with coefficients in $F$. Let $GL_n(F)$ denote the group of invertible $n \times n$-matrices. By an elementary matrix one calls a matrix which has units on the diagonal and at most one non-zero element
out of the diagonal. Let $E_n$ denote the subgroup of $GL_n(F)$ generated by all elementary matrices. Set also $F_{ab}^* := F^*/[F^*, F^*]$ the abelinization of the multiplicative group of $F$ (here $F^*$ denotes the multiplicative group of $F$, and $[F^*, F^*]$ denotes its commutator subgroup).

1.2.1 Theorem (Dieudonné). Let $n \geq 2$. The group $E_n$ is normal subgroup of $GL_n(F)$. For the quotient-group $GL_n(F)/E_n$ there exists a natural isomorphism $D : GL_n(F)/E_n \rightarrow F_{ab}^*$.

This isomorphism $D$ is uniquely defined by the property that for any invertible diagonal matrix $X = \begin{bmatrix} x_1 & 0 \\ \vdots & \ddots \\ 0 & x_n \end{bmatrix}$, $D(X) = \prod_i x_i \text{ mod } [F^*, F^*]$.

1.2.2 Definition (Dieudonné determinant). The Dieudonné determinant is a map

\[ D : M_n(F) \rightarrow F_{ab}^* \cup \{0\} \]

defined as follows: if $X$ is an invertible matrix then $D(X)$ is as in Theorem 1.2.1; if $X$ is not invertible then $D(X) := 0$.

Note also that it is convenient to define the Dieudonné determinant of elements of $F$, i.e. $1 \times 1$-matrices, as $D(0) = 0$ and for $x \neq 0$ as $D(x) := x \text{ mod } [F^*, F^*]$.

Let us state some basic general properties of the Dieudonné determinant. For the proofs we again refer to [D].

1.2.3 Theorem. (i) $D(Id) = 1$.

(ii) For $X, Y \in M_n(F)$

\[ D(XY) = D(X)D(Y). \]

(iii) For any block-matrix $A = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ with $X, Y$ being square matrices

\[ D(A) = D(X)D(Y). \]

(iv) If one interchanges two rows or two columns of the matrix then the Dieudonné determinant is multiplied by $-1 \text{ mod } [F^*, F^*]$. 

17
Now let us consider in more details the case of quaternionic field $F = \mathbb{H}$. The commutator subgroup $[\mathbb{H}^*, \mathbb{H}^*]$ coincides with the subgroup of quaternions of absolute value 1. Thus we can identify $\mathbb{H}^*_a b$ with the multiplicative group $\mathbb{R}_{>0}$ by

$$q \mod [\mathbb{H}^*, \mathbb{H}^*] \leftrightarrow |q| := \sqrt{q\bar{q}}.$$

So in the quaternionic case the Dieudonné determinant maps

$$D : M_n(\mathbb{H}) \rightarrow \mathbb{R}_{\geq 0}.$$ 

In the rest of the paper we will denote by $D(X)$ the Dieudonné determinant of a quaternionic matrix $X$, and by $\det(X)$ the Moore determinant of a hyperhermitian matrix $X$.

1.2.4 Theorem. (i) For any complex $n \times n$-matrix $X$ considered as quaternionic matrix the Dieudonné determinant $D(X)$ is equal to the absolute value of the usual determinant of $X$.

(ii) Let $X$ be a quaternionic hyperhermitian $n \times n$-matrix. Then its Dieudonné determinant $D(X)$ is equal to the absolute value of its Moore determinant $|\det(X)|$.

(iii) For any $X$

$$D(X^*) = D(X),$$

where $X^*$ denotes the transposed and quaternionic conjugate matrices respectively.

For any $n \times n$-matrix $X$ and any subsets $I, J \subset \{1, \ldots, n\}$ let us denote by $M_{I,J}(X)$ the matrix obtained from $X$ by deleting the rows with indexes in $I$ and columns with indexes in $J$. The following result is a weakened version of usual formula of the decomposition of the determinant with respect to a row. Note that this result is satisfied for the absolute value of complex matrices.

1.2.5 Theorem. Let $A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$ be a quaternionic matrix. Then

$$D(A) \leq \sum_{i=1}^{n} |a_{1i}| D(M_{1i}).$$

\[1\text{Added in Sept 2024: in the contrary to the commutative case, the equality } D(X^t) = D(X) \text{ is not satisfied in general, where } X^t \text{ is the transposed matrix of } X; \text{ see Section 4.2 in } C.-Y. \text{ Lin and C.-F. Yu, Dieudonné's determinants and structure of general linear groups over division rings revisited, Bull. Inst. Math. Acad. Sin. (N.S.) 16 (2021), no. 1, 21–47.}\]
Similar inequalities hold for any other row or column.

**Proof.** From Theorem 1.2.3 it follows that

\[
D \left( \begin{bmatrix} a & 0 & \ldots & 0 \\ * & \vdots & & B \\ * & & \ddots & \ast \\ * & & & \ast \end{bmatrix} \right) = |a|D(B).
\]

Hence to prove the statement it is sufficient to show that the Dieudonné determinant is subadditive with respect to the first row; namely if the matrices \( A, A', A'' \) are such that the first row of \( A \) is the sum of first rows of \( A' \) and \( A'' \) and all the other rows are the same, then \( D(A) \leq D(A') + D(A'') \).

But the Dieudonné determinant has the following property over arbitrary (non-commutative) field \( F \) ([5], Thm. 4.5):

\[
D(A) \subset D(A') + D(A''),
\]

where the inclusion and addition are understood in the sense of conjugacy classes modulo \([F^*, F^*]\). But under our identification of \( \mathfrak{H}_{ab} \) with \( \mathbb{R}_{>0} \) the last inclusion implies the desired inequality \( D(A) \leq D(A') + D(A'') \). Q.E.D.

The next two propositions will be used in the sequel. It will be convenient to introduce the following notation. Set \( M'_{IJ}(A) := M_{\{1, \ldots, n\} - I, \{1, \ldots, n\} - J}(A) \), i.e. it denotes the minor which stays on the intersection of the rows with indexes from \( I \) and columns with indexes from \( J \).

1.2.6 Proposition. Let \( A \) be hyperhermitian non-negative definite \( n \times n \)-matrix. Fix an integer \( k, 1 \leq k \leq n \) and two subsets \( I, J \subset \{1, \ldots, n\} \) of cardinality \( k \). Then

\[
2D(M'_{IJ}(A)) \leq D(M'_{II}(A)) + D(M'_{JJ}(A)).
\]

**Proof.** For simplicity of the notation and without loss of generality we may assume that \( I \cup J = \{1, \ldots, n\} \), \( I = \{1, \ldots, k\} \), and \( J = \{n - k + 1, \ldots, n\} \).

First let us reduce to the case \( I \cap J = \emptyset \). We have

\[
A = \begin{bmatrix} * & * & \ast \\ * & M'_{I\cap J, I\cap J}(A) & \ast \\ * & \ast & \ast \end{bmatrix}.
\]
For generic matrix $A$ the (hyperhermitian) minor $M'_{I\cap J,I\cap J}(A)$ is invertible. Then by Claim 1.1.7 one can choose an invertible matrix $U_0$ such that $U_0^* M'_{I\cap J,I\cap J}(A) U_0 = Id$. Let $U = \begin{bmatrix} Id & 0 & 0 \\ 0 & U_0 & 0 \\ 0 & 0 & Id \end{bmatrix}$. Consider matrix $A_1 := U^* A U$. Clearly $D(M'_{I\cap J}(A_1)) = D(M'_{I\cap J}(A)) D(U)^2$, and similarly for $M'_{I,J}$ and $M'_{J,J}$. Hence replacing $A$ by $A_1$ we may assume that $M'_{I\cap J,I\cap J}(A) = Id$. Thus $A$ has the form

$$A = \begin{bmatrix} \ast & X & \ast \\ X^* & Id & Y^* \\ \ast & Y & \ast \end{bmatrix}.$$ 

Set $V = \begin{bmatrix} Id & -X & 0 \\ 0 & Id & 0 \\ 0 & -Y & Id \end{bmatrix}$. Consider

$$A_2 := V A V^* = \begin{bmatrix} P & 0 & R \\ 0 & Id & 0 \\ R^* & 0 & Q \end{bmatrix}.$$ 

Here $P$ and $Q$ are hyperhermitian matrices. Then $A_2$ has the same Dieudonné determinants of the minors $M'_{I\cap J}, M'_{I,J}, M'_{J,J}$ as $A$. Hence we may replace $A$ by $A_2$, and we will denote it by the same letter $A$. Then $M'_{I\cap J}(A) = \begin{bmatrix} P & 0 \\ 0 & Id \end{bmatrix}$, $M'_{I,J}(A) = \begin{bmatrix} 0 & R \\ Id & 0 \end{bmatrix}$, $M'_{J,J}(A) = \begin{bmatrix} Id & 0 \\ 0 & Q \end{bmatrix}$. So one has to show that

$$2D(R) \leq D(P) + D(Q).$$

This inequality is the statement of the proposition for the matrix $\tilde{A} := \begin{bmatrix} P & R \\ R^* & Q \end{bmatrix}$ which is also hyperhermitian and positive definite since $A$ is.

Replacing $\tilde{A}$ by the matrix $\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \tilde{A} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}^*$ with $U_1, U_2 \in Sp(k)$ one can assume that the matrices $P$ and $Q$ are diagonal.

Fix now some $U, V \in Sp(k)$ (the choice of them will be clear later). Let

$$T := \begin{bmatrix} P^{1/2} U P^{-1/2} & 0 \\ 0 & Q^{1/2} V Q^{-1/2} \end{bmatrix}.$$ 

Then

$$T \tilde{A} T^* = \begin{bmatrix} P & R \\ R^*_1 & Q \end{bmatrix},$$
where \( R_1 = P^{1/2}U(P^{-1/2}RQ^{-1/2})V^*Q^{1/2} \). Note that \( D(R_1) = D(R) \). Since \( P \) and \( Q \) are diagonal, by a choice of \( U, V \in Sp(k) \) one can make the matrix \( R_1 \) diagonal.

Finally we are reduced to the hyperhermitian non-negative definite matrix

\[
A = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & \lambda_k & \nu_k \\
\bar{\nu}_1 & 0 & \mu_1 \\
\vdots & \ddots & \ddots \\
0 & \bar{\nu}_k & 0 & \mu_k
\end{bmatrix}
\]

We have to show that

\[
2 \prod_{i=1}^{k} |\nu_i| \leq \prod_{i=1}^{k} |\lambda_i| + \prod_{i=1}^{k} |\mu_i|.
\]

Consider the \( 2 \times 2 \)-matrix

\[
\begin{bmatrix}
\lambda_i & \nu_i \\
\bar{\nu}_i & \mu_i
\end{bmatrix}
\]

which is clearly non-negative definite.

Take a vector \( \left( \begin{array}{c} 1 \\ t \cdot q \end{array} \right) \) for any \( t \in \mathbb{R} \) and any quaternion \( q \) of norm 1. Applying that matrix to this vector we get

\[
\lambda_i + t^2 \mu_i + 2t \text{Re}(\nu_i q) \geq 0.
\]

Hence \( |\nu_i| \leq \sqrt{\lambda_i \mu_i} \). Then

\[
2 \prod_{i=1}^{k} |\nu_i| \leq 2 \prod_{i=1}^{k} |\lambda_i| \cdot \prod_{i=1}^{k} |\mu_i| \leq \prod_{i=1}^{k} |\lambda_i| + \prod_{i=1}^{k} |\mu_i|.
\]

Q.E.D.

1.2.7 Proposition. Let \( A = (a_{ij}), B \) be \( n \times n \)-hyperhermitian matrices. Then the mixed discriminant satisfies

\[
|\text{det}(A, B, \ldots, B)| \leq c_n \cdot \max_{i,j} |a_{ij}| \cdot \left( \sum_{|I|,|J|=n-1} D(M'_{IJ}(B)) \right),
\]

where \( c_n \) is a constant depending on \( n \) only.

Proof. Since \( \text{det}(A, B, \ldots, B) \) is linear in \( A \) it is sufficient to prove the inequality in the following two cases:
1) \( A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0 \\
\end{bmatrix} \); 2) \( A = \begin{bmatrix} 0 & q & \bar{q} & 0 \\
\bar{q} & 0 & 0 & \cdots \\
0 & \cdots & 0 & 0 \\
\end{bmatrix} \).

The first case follows from Proposition 1.1.11. Let us consider the second case. Replacing \( A \) by the matrix

\[
\begin{bmatrix}
\frac{|q|}{|q|} & 0 \\
1 & \ddots & 0 \\
0 & \ddots & \ddots \\
0 & \cdots & 1 & 0 \\
\end{bmatrix} \begin{bmatrix}
0 & |q| & 0 & \cdots \\
|q| & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

we can assume that \( A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 \\
\end{bmatrix} \). Let \( B = \begin{bmatrix} P & R^* \\
R & Q \end{bmatrix} \). Here \( P \) and \( Q \) are hyperhermitian matrices of sizes \( 2 \times 2 \) and \( (n-2) \times (n-2) \) respectively.

**Claim.**

\[
\det(A, B, \ldots, B) \leq M'_{\{2,3,\ldots,n\},\{1,3,\ldots,n\}}(B) + M'_{\{1,3,\ldots,n\},\{2,3,\ldots,n\}}(B).
\]

It remains to prove this claim. We may also assume that \( Q \) is invertible. Set \( S := \begin{bmatrix} \text{Id} & -RQ^{-1} \\
0 & \text{Id} \end{bmatrix} \). Consider

\[
B_1 := SBS^* = \begin{bmatrix}
* & 0 \\
0 & * \\
\end{bmatrix}
\]

Note also that \( S^*AS = A \). It is easy to see that

\[
M'_{\{2,3,\ldots,n\},\{1,3,\ldots,n\}}(B_1) = M'_{\{2,3,\ldots,n\},\{1,3,\ldots,n\}}(B) \quad \text{and} \quad M'_{\{1,3,\ldots,n\},\{2,3,\ldots,n\}}(B_1) = M'_{\{1,3,\ldots,n\},\{2,3,\ldots,n\}}(B).
\]

Hence it is sufficient to prove the claim under assumption \( R = 0 \), i.e. \( B = \begin{bmatrix} P & 0 \\
0 & Q \end{bmatrix} \). Then clearly

\[
\det(A, B, \ldots, B) = \det \left( \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix}, P \right) \cdot \det Q.
\]
If $P = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ then

$$|\det \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P \right)| = Re(b_{12}) \leq |b_{11}| + |b_{22}|,$$

where the last inequality follows from Proposition 1.2.6. Proposition 1.2.7 is proved. Q.E.D.

From Propositions 1.2.6 and 1.2.7 one can easily deduce

1.2.8 Proposition. Let $A = (a_{ij})$ be a hyperhermitian matrix and $B_1, \ldots, B_{n-1}$ be non-negative definite hyperhermitian matrices. Then

$$c_n \cdot \max_{i,j} |a_{ij}| \cdot \sum_{|I| = n-1} \sum_{1 \leq i_1, \ldots, i_{n-1} \leq n-1} |\det(A, B_1, \ldots, B_{n-1})| \leq \det(M'_{H}(B_{i_1}), \ldots, M'_{H}(B_{i_{n-1}})),$$

where $c_n$ is a constant depending on $n$ only.

2 Plurisubharmonic functions of quaternionic variables.

In this part we will develop a basic theory of plurisubharmonic functions of quaternionic variables.

2.1 Main notions.

First let us remind few standard notions. Below $\Omega$ will denote an open domain. As usual we will denote by $C^k(\Omega)$ the class of $k$ times continuously differentiable functions on $\Omega$, and by $C^k_0(\Omega)$ the class of $k$ times continuously differentiable functions on $\Omega$ with compact support. We will also denote by $L^\infty(\Omega)$ (resp. $L^\infty_{loc}(\Omega)$) the class of bounded (resp. locally bounded) measurable functions on $\Omega$.

2.1.1 Definition. A real valued function $f : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}$ is called subharmonic if

(a) $f$ is upper semi-continuous, i.e. $f(x_0) \geq \limsup_{x \to x_0} f(x)$ for any $x_0 \in \Omega$;
(b) \( f(x_0) \leq \int_{S(x_0, r)} f(x) \, d\sigma \) for any point \( x_0 \) and for any sufficiently small \( r > 0 \). Here \( S(x_0, r) \) denotes the sphere of radius \( r \) with center at \( x_0 \), and \( \sigma \) is the Lebesgue measure on it normalized by one.

### 2.1.2 Definition

A real valued continuous function

\[
f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}
\]

is called **convex** if its restriction to any (real) line is subharmonic.

### 2.1.3 Definition

A real valued function

\[
f : \Omega \subset \mathbb{C}^n \longrightarrow \mathbb{R}
\]

is called **plurisubharmonic** if it is upper semi-continuous and its restriction to any complex line is subharmonic.

Now let us introduce a new definition.

### 2.1.4 Definition

A real valued function

\[
f : \Omega \subset \mathbb{H}^n \longrightarrow \mathbb{R}
\]

is called **quaternionic plurisubharmonic** if it is upper semi-continuous and its restriction to any right quaternionic line is subharmonic.

It is easy to see that any (quaternionic) plurisubharmonic function is subharmonic.

### 2.1.5 Example

1) Any convex function on \( \mathbb{H}^n \) is quaternionic plurisubharmonic.

2) Fix on \( \mathbb{H}^n \) one of the complex structures compatible with the quaternionic structure; say, let us fix \( i \). Let \( f \) be a plurisubharmonic function with respect to this complex structure in the sense of Definition 2.1.3. It is easy to see that \( f \) is plurisubharmonic in the quaternionic sense.

Let \( q \) be a quaternionic coordinate,

\[
quaternion{t} = t + ix + jy + kz,
\]

where \( t, x, y, z \) are real numbers. Consider the following operators defined on the class of smooth \( \mathbb{H} \)-valued functions of the variable \( q \in \mathbb{H} \):

\[
\frac{\partial}{\partial q} f := \frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}, \text{ and}
\]

24
\[ \frac{\partial}{\partial q} f := \frac{\overline{\partial}}{\partial \overline{q}} f = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} i - \frac{\partial f}{\partial y} j - \frac{\partial f}{\partial z} k. \]

Note that \( \frac{\partial}{\partial q} \) is called sometimes Cauchy-Riemann-Moisil-Fueter operator, and sometimes Dirac-Weyl operator (see the introduction). It is easy to see that \( \frac{\partial}{\partial q} \) and \( \frac{\partial}{\partial \overline{q}} \) commute, and if \( f \) is a real valued function then

\[ \frac{\partial}{\partial q} \frac{\partial}{\partial \overline{q}} f = \Delta f = \left( \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) f. \]

For any real valued \( C^2 \)-smooth function \( f \) the matrix \( (\frac{\partial^2 f}{\partial \overline{q}_i \partial q_j})^n_{i,j=1} \) is obviously hyperhermitian. For brevity we will use the following notation:

\[ \text{det}(f_1, \ldots, f_n) := \text{det} \left( (\frac{\partial^2 f_1}{\partial \overline{q}_i \partial q_j}), \ldots, (\frac{\partial^2 f_n}{\partial \overline{q}_i \partial q_j}) \right), \]

where \( \text{det} \) denotes the mixed discriminant of hyperhermitian matrices (see Definition 1.1.14). Note also that the operators \( \frac{\partial}{\partial q} \) and \( \frac{\partial}{\partial \overline{q}} \) commute. One can easily check the following identities.

**Claim.** (i) Let \( f : \mathbb{H}^n \rightarrow \mathbb{H} \) be a smooth function. Then for any \( \mathbb{H} \)-linear transformation \( A \) of \( \mathbb{H}^n \) (as right \( \mathbb{H} \)-vector space) one has the identities

\[ \left( \frac{\partial^2 f(Aq)}{\partial \overline{q}_i \partial q_j} \right) = A^* \left( \frac{\partial^2 f}{\partial \overline{q}_i \partial q_j}(Aq) \right) A. \]

(ii) If, in addition, \( f \) is real valued then for any \( \mathbb{H} \)-linear transformation \( A \) of \( \mathbb{H}^n \) and any quaternion \( a \) with \( |a| = 1 \)

\[ \left( \frac{\partial^2 f(A(q \cdot a))}{\partial \overline{q}_i \partial q_j} \right) = A^* \left( \frac{\partial^2 f}{\partial \overline{q}_i \partial q_j}(A(q \cdot a)) \right) A. \]

**2.1.6 Proposition.** A real valued twice continuously differentiable function \( f \) on the domain \( \Omega \subset \mathbb{H}^n \) is quaternionic plurisubharmonic if and only if at every point \( q \in \Omega \) the matrix \( (\frac{\partial^2 f}{\partial \overline{q}_i \partial q_j})(q) \) is non-negative definite.

The proof of this proposition is straightforward. The following lemma will be useful in the sequel.

**2.1.7 Lemma.** Let \( f_0, f_1, \ldots, f_n \) be real valued compactly supported sufficiently smooth functions on \( \mathbb{H}^n \). The \( (n+1) \)-linear functional

\[ L(f_0, f_1, \ldots, f_n) := \int_{\mathbb{H}^n} f_0(q) \cdot \text{det}(f_1, \ldots, f_n)(q) dq \]

is symmetric with respect to all \( f_0, f_1, \ldots, f_n \).
Proof. Note that $L$ is symmetric with respect to the last $n$ arguments. Thus it is sufficient to check that

$$L(f_0, f_1, f_2, \ldots, f_n) = L(f_1, f_0, f_2, \ldots, f_n)$$

(4)

for any smooth compactly supported functions $f_0, f_1, \ldots, f_n$. Both sides of (4) make sense if $f_0$ is a generalized function. Since linear combinations of delta-functions of points $\delta_q$ are dense in the space of all the generalized functions it is sufficient to prove (4) for $f_0 = \delta_0$, namely

$$\det(f_1, \ldots, f_n)|_{q=0} = \int_{\mathbb{R}^n} f_1(q) \det(\delta_0, f_2, \ldots, f_n).$$

(5)

Clearly the right hand side in equation (5) depends only on derivatives at 0 of $f_1, \ldots, f_n$ up to order 2. Consider the terms of the Taylor series of $f_1$ at 0:

$$f_1(q) = g(q) + h(q) + O(|q|^3),$$

where $g$ is a polynomial of degree one, and $h$ is a quadratic term. So it is sufficient to prove the following two statements:

Case 1.

$$L(h, \delta_0, f_2, \ldots, f_n) = \det(h, f_2, \ldots, f_n)|_{q=0}$$

(6)

for any smooth compactly supported function $h$ which is equal to a homogeneous polynomial of degree 2 in a neighborhood of 0, and for any smooth compactly supported functions $f_2, \ldots, f_n$.

Case 2.

$$L(g, \delta_0, f_2, \ldots, f_n) = 0$$

(7)

for any smooth compactly supported function $g$ which is equal to a polynomial of degree 1 in a neighborhood of 0, and for any smooth compactly supported functions $f_2, \ldots, f_n$.

Let us consider Case 1. If we write down the formula for $L(h, \delta_0, f_2, \ldots, f_n)$ as a polynomial in $\frac{\partial^2 f_k}{\partial t_i \partial t_j}$, $\frac{\partial^2 f_k}{\partial t_i \partial x_j}$ etc. and in $\frac{\partial^2 \delta_0}{\partial t_i \partial t_j}$, $\frac{\partial^2 \delta_0}{\partial t_i \partial x_j}$ etc. then we see that the derivatives of $\delta_0$ enter at each monomial only once because of linearity of $L$ with respect to each argument. For example consider a monomial containing $\frac{\partial^2 \delta_0}{\partial t_i \partial t_j}$. Let it be $\int_{\mathbb{R}^n} h \cdot \frac{\partial^2 \delta_0}{\partial t_i \partial t_j} \cdot \partial^2 f_2 \cdot \ldots \cdot \partial^2 f_n$, where $\partial^2 f_k$ denotes certain partial derivative of order 2 of $f_k$. But

$$\int_{\mathbb{R}^n} h \cdot \frac{\partial^2 \delta_0}{\partial t_i \partial t_j} \cdot \partial^2 f_2 \cdot \ldots \cdot \partial^2 f_n = \frac{\partial^2}{\partial t_i \partial t_j} (h \cdot \partial^2 f_2 \cdot \ldots \cdot \partial^2 f_n)|_{q=0} = \ldots$$

26
\[
\frac{\partial^2 h}{\partial t_i \partial t_j}(0) \cdot \partial^2 f_2(0) \cdots \partial^2 f_n(0),
\]
where the last equality is satisfied since the first derivatives of \( h \) at 0 vanish. Thus in each monomial the term \( h \cdot \frac{\partial^2 \delta_0}{\partial t_i \partial t_j} \) is just replaced by \( \frac{\partial^2 h}{\partial t_i \partial t_j}(0) \). Hence the final expression is \( \det(h, f_2, \ldots, f_n)\big|_{q=0} \). This proves the first case.

Let us prove Case 2. It is convenient to prove a more general statement.

**Claim.** Let \( U \) be a fixed neighborhood of the origin 0. Let \( g \) be any smooth compactly supported function which is equal to a polynomial of degree 1 inside \( U \). Let \( f_1 \) be a generalized function with support contained in \( U \). Let \( f_2, \ldots, f_n \) be smooth compactly supported functions.

Then
\[
\int_{\mathbb{H}^n} g \det(f_1, f_2, \ldots, f_n) = 0.
\]

The proof of the claim will be by induction in \( n \). If \( n = 1 \) then using selfadjointness of the Laplacian one gets:

\[
\int_{\mathbb{H}} g \Delta f_1 = \int_{\mathbb{H}} \Delta g \cdot f_1 = \int_U \Delta g \cdot f_1 = 0.
\]

Assume that \( n > 1 \). It is well known (see Appendix) that the linear combinations of delta-functions of quaternionic hyperplanes are dense in the space of all generalized functions (this fact is equivalent to the injectivity of the Radon transform with respect to quaternionic hyperplanes). Hence it is sufficient to prove the claim for \( f_1 = \delta_L \), where \( L \) is the hyperplane \( \{ q_1 = 0 \} \).

Since \( \delta_L \) is invariant with respect to translations in directions \( q_2, \ldots, q_n \) then \( \frac{\partial^2 \delta_L}{\partial q_i \partial q_j} = 0 \) unless \( i = j = 1 \). Thus

\[
\left( \frac{\partial^2 \delta_L}{\partial q_i \partial q_j} \right) = \begin{bmatrix}
\Delta_1 \delta_L & 0 & \cdots & 0 \\
0 & \ddots & & \\
\vdots & & 0 & \\
0 & & & 0
\end{bmatrix},
\]

where \( \Delta_1 \) denotes the Laplacian with respect to the first coordinate: \( \Delta_1 = \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z_1^2} \). Using Proposition 1.1.11 it is easy to see that

\[
c \cdot \det(\delta_L, f_2, \ldots, f_n) = \Delta_1 \delta_L \cdot \det(B_2, \ldots, B_n),
\]

27
where $c$ is a positive normalizing constant, and $B_k$ denotes the $(n-1) \times (n-1)$-matrix $(\det \frac{\partial^2 f_k}{\partial q_i \partial q_j})_{i,j=2}^n$. Then

$$
\int_{\mathbb{H}^n} g \cdot \det(\delta_L, f_2, \ldots, f_n) = \int_{\mathbb{H}^n} g \cdot \Delta_1 \delta_L \cdot \det(B_2, \ldots, B_n).
$$

Clearly the last expression depends only on the 2-jets of $g, f_2, \ldots, f_n$ in the direction $q_1$. Thus we may assume that the functions $f_k$ are of the form

$$
f_k(q_1, q_2, \ldots, q_n) = p_k(q_1) \cdot f'_k(q_2, \ldots, q_n),
$$

where $p_k(q_1)$ are polynomials (of degree at most 2) depending only on $t_1, x_1, y_1, z_1$, and $f'_k$ are smooth compactly supported functions depending only on $q_2, \ldots, q_n$.

Since $\deg g \leq 1$ we may assume (by linearity) that either $g(q_1, q_2, \ldots, q_n) = g(q_1)$ or $g(q_1, q_2, \ldots, q_n) = g(q_2, \ldots, q_n)$. In the first case

$$
\int_{\mathbb{H}^n} g \cdot \Delta_1 \delta_L \cdot \det(B_2, \ldots, B_n) =
$$

$$
\Delta_1 (p_2(q_1) \cdots p_n(q_1)) |_{q_1=0} \cdot \int_{L} \det(B'_2, \ldots, B'_n),
$$

where $B'_k$ denotes the matrix $(\frac{\partial^2 f'_k}{\partial q_i \partial q_j})_{i,j=2}^n$. The last integral vanishes by the induction assumption.

Now consider the second case $g(q_1, q_2, \ldots, q_n) = g(q_2, \ldots, q_n)$. We have

$$
\int_{\mathbb{H}^n} g \cdot \Delta_1 \delta_L \cdot \det(B_2, \ldots, B_n) = \Delta_1 (p_2 \cdots p_n) |_{q_1=0} \int_{L} g \cdot \det(B_2, \ldots, B_n).
$$

Again the last expression vanishes by the induction assumption. Thus our claim, and hence Proposition 2.1.6, are proved. Q.E.D.

The next result is again a quaternionic analogue of the corresponding property of convex functions and complex plurisubharmonic functions. We adopt the arguments of Chern-Levine-Nirenberg [13] and Rauch-Taylor [41] (see also [7]).

2.1.8 Proposition. Let $\Omega \subset \mathbb{H}^n$ be an open domain. Assume that a sequence $\{f_N\}$ of twice continuously differentiable quaternionic plurisubharmonic functions converges uniformly on compact subsets to a twice continuously differentiable function $f$. Then $f$ is also quaternionic plurisubharmonic, and for every continuous function $\phi$ with compact support in $\Omega$

$$
\int_{\Omega} \phi \cdot \det(\frac{\partial^2 f_N}{\partial q_i \partial q_j}) \to \int_{\Omega} \phi \cdot \det(\frac{\partial^2 f}{\partial q_i \partial q_j}) \text{ as } N \to \infty.
$$
We will need a lemma. But first let us introduce a notation. For subsets \( I, J \subset \{1, \ldots, n\} \) and a function \( g \) let us denote by \( M'_{IJ}(g) \) the matrix which stays on the intersection of rows with indexes from \( I \) and columns with indexes from \( J \) in the matrix \( \det(\frac{\partial^2 g}{\partial q_i \partial q_j})_{i,j=1}^n \). Also for a set \( U \) and a function \( g \) defined on it let us denote by \( ||g||_{L^\infty(U)} := \sup_{q \in U} |g(q)| \), and by \( ||g||_{C^k(U)} \) the maximum of \( L^\infty(U) \)-norms of all partial derivatives of \( g \) up to order \( k \). Below we will denote for brevity \( \det(\frac{\partial^2 g}{\partial q_i \partial q_j})_{i,j=1}^n \) by \( \det(g) \).

2.1.9 Lemma. Let \( I, J \) be subsets of \( \{1, \ldots, n\} \) of cardinality \( k \). Let \( f \in L^\infty_{\text{loc}}(\Omega) \), and let \( g \) be a twice continuously differentiable quaternionic plurisubharmonic function on a domain \( \Omega \subset \mathbb{H}^n \). Let \( K \) be a compact subset of \( \Omega \), and let \( U \) be a compact neighborhood of \( K \) in \( \Omega \). Then

\[
| \int_K f \cdot D(M'_{IJ}(g)) | \leq C(U) ||f||_{L^\infty(\Omega)} ||g||_{L^\infty(U)}^k,
\]

where \( C(U) \) is a constant depending on \( U \) only.

Proof. Since \( g \) is plurisubharmonic , Proposition 1.2.6 implies the estimate \( D(M'_{IJ}(g)) \leq D(M'_{II}(g)) + D(M'_{JJ}(g)) \). Hence

\[
| \int_K f \cdot D(M'_{IJ}(g)) | \leq ||f||_{L^\infty(\Omega)} \cdot \int_K (D(M'_{II}(g)) + D(M'_{JJ}(g))).
\]

It remains to prove that for any subset \( I \subset \{1, \ldots, n\} \) of cardinality \( k \)

\[
\int_K \det(M'_{II}(g)) \leq C(U) ||g||_{L^\infty(U)}^k.
\]

Let us prove this inequality by induction in \( k \). For \( k = 0 \) the statement is trivial. Assume that \( k > 0 \). Let us fix a non-negative function \( \gamma \in C^\infty_0(\Omega) \) such that \( \gamma|_K \equiv 1 \) and \( \gamma \) vanishes on \( \Omega - \Omega \). Then using Lemma 2.1.7

\[
\int_K \det M'_{II}(g) \leq \int_{\mathbb{H}^n} \gamma \cdot \det M'_{II}(g) = \int_{\mathbb{H}^n} g \cdot \det_I(\gamma, g, \ldots, g),
\]

where \( \det_I \) denotes the mixed discriminant of matrices of order \(|I|\). By Proposition 1.2.7 the last expression is at most

\[
||g||_{L^\infty(U)} \cdot ||\gamma||_{C^2(U)} \sum S,T \int_U D_{ST}(g),
\]

29
where the sum extends over all subsets $S, T$ of $I$ of cardinality $k - 1$. Again by Proposition 1.2.6

$$\int_U D_{ST}(g) \leq \int_U (\det_S(g) + \det_T(g)).$$

Now the estimate follows by the assumption of induction. Q.E.D.

Now let us prove Proposition 2.1.8. First let us show that the limit $f$ is plurisubharmonic. This is obvious since the restriction of $f$ to any quaternionic line is subharmonic as the uniform limit of subharmonic functions.

Let us prove the second part of Proposition 2.1.8. Let $K := \text{supp}\phi$. Fix $\varepsilon > 0$, and a compact neighborhood $U$ of $K$. Let us choose a function $\psi \in C^\infty_0(\Omega)$ such that $||\phi - \psi||_{L^\infty(U)} \leq \varepsilon$. We have

$$|\int_K \psi (\det(f_N) - \det(f))| \leq C(U) ||f_N||_{L^\infty(U)} ||\psi||_{C^2(\Omega)}$$

where the last inequality follows from Lemma 2.1.9. Thus it is sufficient to prove that

$$\int_\Omega \psi \cdot \det(f_N) \text{ tends to } \int_\Omega \psi \cdot \det(f) \text{ as } N \to \infty.$$
by Proposition 1.2.8 (here we have used the fact that the functions \( f_N \) and \( f \) are plurisubharmonic). Now let us estimate the last expression.

\[
\int_{\text{supp}\psi} \det_I(f_N, \ldots, f_N, f, \ldots, f) \leq \int_{\text{supp}\psi} \det_I(f + f_N, \ldots, f + f_N) \leq C'\|f + f_N\|_{L_\infty}^{n-1},
\]

where the last inequality holds by Lemma 2.1.9. Hence the expression (8) tends to 0 as \( N \to \infty \). This proves Proposition 2.1.8. Q.E.D.

Now let us study \textit{continuous} quaternionic plurisubharmonic functions which are not necessarily smooth. For every continuous plurisubharmonic function \( f \) we will define a non-negative measure such that if \( f \) is smooth it coincides with \( \det(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j}) \). To do it let us observe first of all that any continuous plurisubharmonic function \( f \) on a domain \( \Omega \subset \mathbb{H}^n \) can be approximated by \( C^\infty \)-smooth plurisubharmonic functions uniformly on compact subsets of \( \Omega \). (To see it consider the convolution of \( f \) with the delta-sequence of non-negative \( C^\infty \)-smooth functions. Each such convolution is infinitely smooth and plurisubharmonic). The next theorem is first main result of this section; it provides the definition of the measure \( \det(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j}) \) for any continuous plurisubharmonic function \( f \).

\textbf{2.1.10 Theorem.} Let \( f \) be a continuous quaternionic plurisubharmonic function on a domain \( \Omega \). Let \( \{f_N\} \) be a sequence of twice continuously differentiable plurisubharmonic functions converging to \( f \) uniformly on compact subsets of \( \Omega \). Then \( \det(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j}) \) weakly converges to a non-negative measure on \( \Omega \). This measure depends only on \( f \) and not on the choice of an approximating sequence \( \{f_N\} \).

This measure will be denoted by \( \det(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j}) \).

\textbf{Proof.} By Lemma 2.1.9 one sees that for any compact subset \( K \subset \Omega \) the sequence of measures \( \det(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j})|_K \) is bounded. Thus it is sufficient to show that for any continuous compactly supported function \( \phi \) the sequence \( \int_\Omega \phi \cdot \det(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j}) \) is a Cauchy sequence. Let us fix \( \varepsilon > 0 \), and a function \( \psi \in C_0^\infty(\Omega) \) such that \( ||\phi - \psi||_{C^0(\Omega)} < \varepsilon \). Let us also fix an arbitrary compact subset \( K \subset \Omega \) and a compact neighborhood \( U \) of \( K \) in \( \Omega \). As in the proof of Proposition 2.1.8 we have

\[
| \int_K \psi(\det(f_N) - \det(f_M)) - \int_K \phi(\det(f_N) - \det(f_M)) | \leq \varepsilon.
\]
\[ | \int_K (\psi - \phi) \det(f_N) | + | \int_K (\psi - \phi) \det(f_M) | \leq C(U) (||f_N||^n_{C^0(U)} + ||f_M||^n_{C^0(U)}) \cdot \varepsilon, \]

where the last inequality follows from Lemma 2.1.9. For large \( M \) and \( N \) the last expression can be estimated from above by \( 3C(U)||f||^n_{C^0(U)} \cdot \varepsilon \). Hence it is sufficient to prove that for any function \( \psi \in C_0^\infty(\Omega) \) the sequence \( \int_\Omega \psi \cdot \det((\partial^2 f_N/\partial q_i \partial q_j)) \) is a Cauchy sequence. We have the following estimate exactly as in the inequality (8) (with \( f_M \) instead of \( f \)):

\[ | \int_\Omega \psi \cdot (\det(f_N) - \det(f_M)) | \leq C \cdot ||f_N - f_M||_{C^0(\text{supp}\psi)} \cdot ||\psi||_{C^2(\Omega)} \sum_{i=0}^{n-1} \sum_{|I|=n-1} |I| \int_{\text{supp}\psi} \det_I(f_N, \ldots, f_N, f_M, \ldots, f_M). \]

Again as in the proof of Proposition 2.1.8 we get

\[ \int_{\text{supp}\psi} \det_I(f_N, \ldots, f_N, f_M, \ldots, f_M) \leq C ||f_N + f_M||^{n-1}_{C^0(\text{supp}\psi)} < C'. \]

This proves Theorem 2.1.10. Q.E.D.

The second main result of this section is as follows.

2.1.11 Theorem. Let \( \{f_N\} \) be a sequence of continuous quaternionic plurisubharmonic functions in a domain \( \Omega \subset \mathbb{H}^n \). Assume that this sequence converges uniformly on compact subsets to a function \( f \). Then \( f \) is continuous quaternionic plurisubharmonic function. Moreover the sequence of measures \( \det(\partial^2 f_N/\partial q_i \partial q_j) \) weakly converges to the measure \( \det(\partial^2 f/\partial q_i \partial q_j) \).

Proof. The limit \( f \) is a plurisubharmonic function. Indeed the restriction of \( f \) to any quaternionic line is subharmonic as a uniform limit of subharmonic functions.

Let us prove the second part of the statement. The functions \( f_N \) can be approximated uniformly on compact subsets as good as we wish by smooth plurisubharmonic functions \( g_N \) such that the sequence \( g_N \) will converge uniformly on compact subsets to \( f \). Then the result follows from previous Theorem 2.1.10. Q.E.D.
2.2 The minimum principle.

In this subsection we prove the following minimum principle.

2.2.1 Theorem. Let $\Omega$ be a bounded open set in $\mathbb{H}^n$. Let $u, v$ be continuous functions on $\bar{\Omega}$ which are plurisubharmonic in $\Omega$. Assume that

$$\det\left(\frac{\partial^2 u}{\partial \bar{q}_i \partial q_j}\right) \leq \det\left(\frac{\partial^2 v}{\partial \bar{q}_i \partial q_j}\right) \text{ in } \Omega.$$ 

Then

$$\min\{u(z) - v(z) | z \in \bar{\Omega}\} = \min\{u(z) - v(z) | z \in \partial \Omega\}.$$ 

The exposition follows very closely to Section 3 of [9]. From now on we will denote for brevity the matrix $\frac{\partial^2 u}{\partial \bar{q}_i \partial q_j}$ by $\partial^2 u$.

2.2.2 Proposition. Let $\Omega$ be a bounded domain in $\mathbb{H}^n$ with smooth boundary, and let $u, v \in C^2(\bar{\Omega})$ be psh functions on $\Omega$. If $u = v$ on $\partial \Omega$ and $u \geq v$ in $\Omega$, then

$$\int_{\Omega} \det(\partial^2 u) \leq \int_{\Omega} \det(\partial^2 v).$$

Proof. First we can write $\Omega = \{\rho < 0\}$ with $\rho$ being a smooth function, $\partial \Omega = \{\rho = 0\}$, and $\nabla \rho|_{\partial \Omega} \neq 0$. We have

$$\int_{\Omega} (\det(\partial^2 u) - \det(\partial^2 v)) = \sum_{i=0}^{n-1} \int_{\Omega} \det(u, \ldots, u_i, \overbrace{v, \ldots, v}^{i \text{ times}}, \overbrace{u, \ldots, u}^{n-i-1 \text{ times}}, (u - v)).$$

Let us prove that each summand is non-positive. We will need the following lemma.

2.2.3 Lemma. Let $\beta \in C^2(\bar{\Omega})$, $\beta|_{\partial \Omega} \equiv 0$. Let $u_1, \ldots, u_{n-1} \in C^3(\bar{\Omega})$. Then

$$\int_{\Omega} \det(u_1, \ldots, u_{n-1}, \beta) = -\int_{s \in \partial \Omega} \det(u_1|_{\tilde{T}_s}, \ldots, u_{n-1}|_{\tilde{T}_s}) \frac{\partial \beta}{\partial \nu(s)} ds,$$

where $T_s$ denotes the tangent space to $\partial \Omega$ at $s$, $\tilde{T}_s$ denotes the quaternionic subspace $T_s \cap i \cdot T_s \cap j \cdot T_s \cap k \cdot T_s$, $\nu(s)$ is the inner normal to $\partial \Omega$, and $ds$ is the surface area measure.
Let us continue proving Proposition 2.2.2 assuming this lemma. Since 
\[ u \geq v \]
we can represent 
\[ u - v = \alpha \cdot \rho, \]
where \( \alpha \leq 0 \). Using Lemma 2.2.3 we have
\[
\begin{align*}
\int_{\Omega} \det(u, \ldots, u, v, \ldots, v, u - v) = \\
- \int_{\partial \Omega} \det(u, \ldots, u, v, \ldots, v) \frac{\partial}{\partial \nu(s)} (\alpha \cdot \rho) ds = \\
- \int_{\partial \Omega} \det(u, \ldots, v) \frac{\partial \rho}{\partial \nu(s)} \cdot ds.
\end{align*}
\]
But since \( \alpha \leq 0 \) and \( \frac{\partial \rho}{\partial \nu(s)} \leq 0 \) the last expression is non-positive. Q.E.D.

**Proof** of Lemma 2.2.3. We have
\[
\int_{\Omega} \det(u_1, \ldots, u_{n-1}, \beta) = \int_{\mathbb{H}^n} \chi_{\Omega} \det(u_1, \ldots, u_{n-1}, \beta).
\]
By Lemma 2.1.7 the last expression is symmetric with respect to all the arguments. Hence it is equal to
\[
\int_{\mathbb{H}^n} \beta \det(u_1, \ldots, u_{n-1}, \chi_{\Omega}).
\]
One easily checks the following

**2.2.4 Claim.** \( \frac{\partial}{\partial x_i}(\chi_{\Omega}) \) is a distribution of order zero with support on \( \partial \Omega \).
*This distribution is equal to \(-\frac{\partial}{\partial x_i}|\text{vol}|.\)*

Now let us fix a point \( s_0 \in \partial \Omega \). Let us choose an orthonormal coordinate system \( (q_1, \ldots, q_n) \) in \( \mathbb{H}^n \), \( q_m = t_m + ix_m + jy_m + kz_m \), such that \( \frac{\partial}{\partial x_1} = \nu(s_0) \).

Let \( \xi, \eta \) be translation invariant vector fields, each of them parallel to one of the chosen coordinate axes, and at least one of them is different from \( \frac{\partial}{\partial x_1} \). In the formula for \( \det(u_1, \ldots, u_{n-1}, \chi_{\Omega}) \) consider the term containing \( \xi(\eta(\chi_{\Omega})) \). It is a product of this last term by some smooth function \( F \). Let us consider the integral \( \int_{\mathbb{H}^n} \beta \cdot F \cdot \xi(\eta(\chi_{\Omega})) \). We may assume that at \( s_0 \) \( \eta \in T_{s_0}(\partial \Omega) \). Then
\[
\int_{\mathbb{H}^n} \beta \cdot F \cdot \xi(\eta(\chi_{\Omega})) = - \int_{\partial \Omega} \beta \cdot F \cdot \xi(\nu|\text{vol}) = \int_{\partial \Omega} \xi(\beta \cdot F) \cdot \eta|\text{vol}).
\]
But since $\beta |_{\partial \Omega} \equiv 0$ the last expression is equal to $\int_{\partial \Omega} \xi(\beta) \cdot F \cdot (\eta | \text{vol})$. Note that since $\eta \in T_{s_{0}}(\partial \Omega)$ the expression under the last integral vanishes at the point $s_{0}$. Hence the only summand which remains is

$$\frac{\partial \beta}{\partial \nu(s_{0})} \cdot F \cdot (\nu(s_{0}) | \text{vol}).$$

It is easy to see that in this case

$$F = \text{det}(u_{1}|_{\tilde{T}_{s_{0}}}, \ldots, u_{n-1}|_{\tilde{T}_{s_{0}}}),$$

and $\nu(s_{0}) | \text{vol} = -ds$. This proves the lemma. Q.E.D.

The next result is a slight generalization of Theorem 2.2.1; it is completely parallel to Theorem 3.2 of [9].

**2.2.5 Theorem.** Let $\Omega$ be a bounded open set in $\mathbb{R}^{n}$. Let $v$ be a continuous function on $\bar{\Omega}$ which is psh in $\Omega$. Let $u$ be a locally bounded (not necessarily continuous) psh function on $\Omega$ such that

$$\lim_{\zeta \rightarrow z \in \partial \Omega} (u(\zeta) - v(\zeta)) \geq 0;$$

and

$$\lim_{\varepsilon \rightarrow 0} \text{det}(\partial^{2} u_{\varepsilon}) \leq \text{det}(\partial^{2} v) \text{ in } \Omega,$$

where $u_{\varepsilon} = u \ast \chi_{\varepsilon}$ and $\chi_{\varepsilon}$ is a usual smoothing kernel of psh functions (exactly as in the complex case, see [31], p.45). Then $u \geq v$ in $\Omega$.

**Proof.** Assume that the theorem is false. Then there exists $z_{0} \in \Omega$ such that $u(z_{0}) < v(z_{0})$. Let $\eta_{0} = (v(z_{0}) - u(z_{0}))/2$. Then for all $0 < \eta < \eta_{0}$ the set

$$G(\eta) = \{ z \in \Omega | u(z) + \eta < v(z) \} \ni z_{0}$$

is nonempty, open (since $u - v$ is upper semi-continuous), relatively compact subset of $\Omega$ (because of the first assumption of the theorem).

Let $u_{\varepsilon} = u \ast \chi_{\varepsilon}$, $v_{\varepsilon} = v \ast \chi_{\varepsilon}$ be regularizations of $u, v$ so that $u_{\varepsilon}, v_{\varepsilon}$ are defined on

$$\Omega_{\varepsilon} = \{ z \in \Omega | \text{distance from } z \text{ to } \partial \Omega \text{ exceeds } \varepsilon \},$$

and $u_{\varepsilon} \geq u$, $v_{\varepsilon} \geq v$. Since $v$ is continuous, $v_{\varepsilon} \rightarrow v$ uniformly on compact subsets of $\Omega$. Define

$$G(\eta, \delta) = \{ z \in \Omega | u(z) + \eta < v(z) + \delta |z - z_{0}|^{2} \}.$$
There exists an increasing function $\delta(\eta) > 0$, $0 < \eta < \eta_0$, such that $G(\eta, \delta)$ is nonempty, open, and relatively compact in $\Omega$ for all $0 < \delta \leq \delta(\eta)$. Clearly $z_0 \in G(\eta, \delta)$. Next choose $\varepsilon(\eta, \delta) > 0$ so small that $0 < \varepsilon < \varepsilon(\eta, \delta)$ implies

$$\Omega_\varepsilon \supset G(\eta/2, \delta), \ 0 < \eta < \eta_0, \ 0 < \delta < \delta(\eta/2).$$

For such $\varepsilon, \eta, \delta$ let us define

$$G(\eta, \delta, \varepsilon) = \{ z \in G(\eta/2, \delta) | u(z) + \eta < v_\varepsilon(z) + \delta|z - z_0|^2 \}.$$

If $\varepsilon$ is so small that $|v(z) - v_\varepsilon(z)| \leq \eta/4$ whenever $z \in G(\eta/2, \delta)$ and $\varepsilon < \varepsilon(\eta, \delta)$ then it is easy to see that

$$G(\eta, \delta, \varepsilon) \subset G(3\eta/4, \delta) \subset G(\eta/2).$$

In particular, $G(\eta, \delta, \varepsilon)$ is a relatively compact subset of $\Omega_\varepsilon$, so $v_\varepsilon$ is $C^\infty$ in a neighborhood of the closure of $G(\eta, \delta, \varepsilon)$.

Finally choose $\tau(\eta, \delta, \varepsilon)$ so small that for $\eta, \delta, \varepsilon$ as above and $0 < \tau < \tau(\eta, \delta, \varepsilon)$ we have that

$$G(\eta, \delta, \varepsilon, \tau) := \{ z \in G(\eta/2, \delta) | u_\tau(z) + \eta < v_\varepsilon(z) + \delta|z - z_0|^2 \}$$

is a nonempty, open, relatively compact subset of $\Omega_\varepsilon$. Since $u_\tau \geq u$ we have $G(\eta, \delta, \varepsilon, \tau) \subset G(\eta, \delta, \varepsilon)$, and because $z_0 \in G(\eta, \delta, \varepsilon)$ we have $z_0 \in G(\eta, \delta, \varepsilon, \tau)$ for sufficiently small $\tau$.

We will apply Proposition 2.2.2 with $G(\eta, \delta, \varepsilon, \tau)$ instead of $\Omega$ and the functions defining this set. However in general this domain does not have smooth boundary. But, by Sard’s lemma, the value $\eta$ is a regular value of the $C^\infty$-function $v_\eta(z) + \delta|z - z_0|^2 - u_\tau(z)$ for almost all values of $\eta$. Thus we can take sequence of numbers $\tau_n \rightarrow 0$ and apply Proposition 2.2.2 for almost all values of $\eta$. Consequently we have by Proposition 2.2.2 and Theorem 1.1.17 (iii)

$$\int det(\partial^2 u_\tau) = \int det\partial^2(u_\tau + \eta) \geq \int det\partial^2(v_\varepsilon + \delta|z - z_0|^2) \geq$$

$$\int det\partial^2 v_\varepsilon + \delta^n \int det(\partial^2 |z - z_0|^2) = \int det\partial^2 v_\varepsilon + \delta^n \cdot c_n vol(G(\eta, \delta, \varepsilon, \tau)),$$

where all the integrals are taken over $G(\eta, \delta, \varepsilon, \tau)$, and $c_n$ is a positive constant depending on $n$ only. When $\tau \rightarrow 0$ the open sets $G(\eta, \delta, \varepsilon, \tau)$ increase to
\( G(\eta, \delta, \varepsilon) \). If \( \mu = \lim_{\tau \to 0} \det(\partial^2 u_\tau) \) then we deduce from the last estimate that

\[
\mu(G(\eta, \delta, \varepsilon)) \geq \int_{G(\eta, \delta, \varepsilon)} \det(\partial^2 v_\varepsilon) + c_n \delta^n \text{vol}(G(\eta, \delta, \varepsilon))
\]

for almost all \( 0 < \eta < \eta_0, 0 < \delta < \delta(\eta) \), and \( 0 < \varepsilon < \varepsilon(\eta, \delta) \). Now let \( \varepsilon \to 0 \). The measures \( \det(\partial^2 v_\varepsilon) \) converge weakly to \( \det(\partial^2 v) \) by Theorem 2.1.11. Also \( G(\eta, \delta, \varepsilon) \supset G(\eta, \delta) \). Next

\[
\cap_{\varepsilon > 0} G(\eta, \delta, \varepsilon) \subset K(\eta, \delta) := \{ z \in \Omega | u(z) + \eta \leq v(z) + \delta |z - z_0|^2 \}.
\]

Thus for almost all \( \eta \) we have

\[
\mu(K(\eta, \delta)) \geq \int_{G(\eta, \delta)} \det(\partial^2 v) + c_n \delta^n \cdot \text{vol}(G(\eta, \delta)).
\]

Let us denote \( \nu := \det(\partial^2 v) \). By assumption \( \mu \leq \nu \). Thus we get

\[
\nu(K(\eta, \delta)) \geq \nu(G(\eta, \delta)) + c_n \delta^n \cdot \text{vol}(G(\eta, \delta)).
\]

Also \( G(\eta, \delta) \subset K(\eta, \delta) \subset G(\eta', \delta) \) for \( \eta' < \eta \). Hence

\[
\nu(G(\eta, \delta)) \leq \nu(K(\eta, \delta)) \leq \nu(G(\eta', \delta)) \text{ for } \eta' < \eta.
\]

However \( \eta \to \nu(G(\eta, \delta)) \) is a decreasing function of \( \eta \). Hence at the points of continuity of this function we have

\[
\nu(G(\eta, \delta)) \geq \nu(G(\eta, \delta)) + c_n \delta^n \cdot \text{vol}(G(\eta, \delta)).
\]

But this contradicts to the fact that \( G(\eta, \delta) \) is a nonempty open set. This proves Theorem 2.2.5 (and hence Theorem 2.2.1). Q.E.D.

**Appendix.**

In this appendix we prove that the linear combinations of delta-functions of quaternionic hyperplanes in \( \mathbb{H}^n \) are dense in the space of distributions (this fact was needed in the proof of Lemma 2.1.7). By the Hahn-Banach theorem it is equivalent to the injectivity of the Radon transform over quaternionic hyperplanes. We believe that the injectivity of quaternionic Radon transform
is a well known fact, but we include the proof for completeness, since we could not find a reference.

Let us fix hyperhermitian metric on \( H^n \), i.e. a Euclidean metric such that for any two vectors \( x, y \in H^n \) and any quaternion \( a \) with \( \|a\| = 1 \)

\[
(x \cdot a, y \cdot a) = (x, y).
\]

Let \( f \) be any smooth compactly supported function on \( H^n \). The quaternionic Radon transform of \( f \) is a function on the manifold of all affine quaternionic hyperplanes defined as

\[
Rf(E) = \int_E f(q) dq,
\]

where the integration is with respect to the volume form on \( E \) defined by the metric.

**Proposition.** The quaternionic Radon transform is injective.

**Proof.** We will just present the inversion formula completely analogous to the complex Radon transform (see [19]). Let us fix the origin \( 0 \in H^n \) for convenience. Let us denote by \( A \) the manifold of affine quaternionic hyperplanes in \( H^n \). For any point \( q \in H^n \) let \( P_q \) denote the manifold of quaternionic hyperplanes passing through \( q \). For \( E \in A \) let us denote by \( E^\perp \) the quaternionic line orthogonal to \( E \) and passing through the origin \( 0 \).

Let us define the operator

\[
D : C^\infty(A) \longrightarrow C^\infty(H^n)
\]

as follows. Let \( g \in C^\infty(A) \). Set

\[
Dg(q) := \int_{E \in P_q} (\Delta_{E^\perp})^{2(n-1)}g(E + w)dE,
\]

where \( \Delta_{E^\perp} \) denotes the (4-dimensional) Laplacian with respect to \( w \in E^\perp \), and the integration is with respect the Haar measure on \( P_q \).

**Claim.** For any smooth rapidly decreasing function \( f \) of \( H^n \)

\[
D(Rf) = c \cdot f,
\]

where \( c \) is a non-zero constant.

It is sufficient to check this claim pointwise, say at 0. The operators \( R \) and \( D \) commute with translations and the action of the group \( Sp_n \). Then
\(\mathcal{D}(Rf)(0)\) defines a distribution invariant with respect to the action of \(Sp_n\). Moreover it is easy to check that this distribution is homogeneous of degree \(-4n\) (exactly as the delta-function at 0). It is easy to see that there is at most one dimensional space of \(Sp_n\)-invariant distributions homogeneous of degree \(-4n\). Hence they must be proportional to the delta-function at 0. Thus \(\mathcal{D}(Rf) = c \cdot f\) for some constant \(c\). So see that \(c \neq 0\) it is sufficient to check it by an explicit computation for the function \(f(q) = \exp(-|q|^2/2)\).

Q.E.D.

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