A recursive normalizing one-step reduction strategy for the distributive lambda calculus

Anton Salikhmetov

May 2, 2014

Abstract

We positively answer the question A.1.6 in [2]: “Is there a recursive normalizing one-step reduction strategy for micro \( \lambda \)-calculus?” Micro \( \lambda \)-calculus refers to an implementation of the \( \lambda \)-calculus due to [1], implementing \( \beta \)-reduction by means of “micro steps” recursively distributing a \( \beta \)-redex \((\lambda x.M)N\) over its body \( M \).

1 Inner spine strategy

First, we provide “micro \( \lambda \)-calculus” with a more systematic name 1.

Definition 1. Distributive reduction is defined as

\[
\beta_d = \beta^i_d \cup \beta^c_d \cup \beta^l_d \cup \beta^a_d,
\]

where

\[
\begin{align*}
\beta^i_d &= \{(\lambda x.M, M) \mid M \in A\}, \\
\beta^c_d &= \{(\lambda x.y.M, y) \mid M \in A\}, \\
\beta^l_d &= \{((\lambda x.\lambda y.M)N, \lambda y.(\lambda x.M)N) \mid M, N \in A\}, \\
\beta^a_d &= \{((\lambda x.M)N, (\lambda x.M)P((\lambda x.N)P)) \mid M, N, P \in A\}.
\end{align*}
\]

Additionally, we denote the following binary relations:

\[
\begin{align*}
(M, N) \in \beta_d &\Rightarrow \forall C[ \quad : C[M] \rightarrow_d C[N], \\
(M, N) \in \beta^i_d &\Rightarrow \forall C[ \quad : C[M] \rightarrow_i C[N], \\
(M, N) \in \beta^c_d &\Rightarrow \forall C[ \quad : C[M] \rightarrow_c C[N], \\
(M, N) \in \beta^l_d &\Rightarrow \forall C[ \quad : C[M] \rightarrow_l C[N], \\
(M, N) \in \beta^a_d &\Rightarrow \forall C[ \quad : C[M] \rightarrow_a C[N].
\end{align*}
\]

Proposition 1. Any \( \beta_d \)-redex is a \( \beta \)-redex, and vice versa.

1Vincent van Oostrom suggested the following constructions a little differently in private correspondence [3]. We took the liberty to change some notations hopefully simplifying the proofs of the two core propositions.
Proof. The proposition directly follows from the definition of $\beta_d$. □

Normalisation of our strategy answering the previously open question, relies on the one hand on normalization of spine reductions for the ordinary $\lambda$-calculus, and on the other hand on termination of pure distribution steps, as encountered in the $\lambda$-calculus with explicit substitutions $\lambda x$.

**Definition 2.** Inner spine strategy contracts the innermost redex among spine redexes (see the definition 4.7 in [5]).

### 1.1 Correctness of distributive reduction

A term is a distributive redex if and only if it is a $\beta$-redex, hence distributive and $\beta$-normal forms coincide. In turn, the spine redexes with respect to distributive reduction coincide with those for ordinary $\beta$-reduction. If $M$ distributively rewrites to $M'$, then in general $M$ need not $\beta$-rewrite to $M'$, but $M$ and $M'$ are $\beta$-convertible.

**Proposition 2.** $M \rightarrow_d N \Rightarrow M =_\beta N$.

**Proof.** Let us consider each subset of $\beta_d$.

1. If $M \rightarrow_i N$, then for some $P$

   $$M \equiv (\lambda x.x) P \land N \equiv P,$$

   but then

   $$(\lambda x.x) P \rightarrow_\beta x[x := P] \equiv P.$$

2. If $M \rightarrow_c N$, then for some $P$

   $$M \equiv (\lambda x.y) P \land N \equiv y,$$

   but then

   $$(\lambda x.y) P \rightarrow_\beta y[x := P] \equiv y.$$

3. If $M \rightarrow_l N$, then for some $P, Q$

   $$M \equiv (\lambda x.\lambda y.P) Q \land N \equiv \lambda y.(\lambda x.P) Q,$$

   but then

   $$(\lambda x.\lambda y.P) Q \rightarrow_\beta (\lambda y.P)[x := Q] \equiv \lambda y.P[x := Q] \leftarrow_\beta \lambda y.(\lambda x.P) Q.$$

4. If $M \rightarrow_a N$, then for some $P, Q, R$

   $$M \equiv (\lambda x.P Q) R \land N \equiv (\lambda x.P) R ((\lambda x.Q) R),$$

   but then

   $$(\lambda x.P Q) R \rightarrow_\beta (P Q)[x := R] \equiv P[x := R] Q[x := R] \leftarrow_\beta (\lambda x.P) R ((\lambda x.Q) R),$$

Since we have traversed $\beta^i_d, \beta^3_d, \beta^l_d, \beta^a_d$, the proposition also stands for $\beta_d$. □
1.2 Useful definitions

**Definition 3.** Full $\beta$-development $M^\bullet$ of a term $M$ is the term obtained by $\beta$-contracting all redexes of $M$.

**Definition 4.** A step is called destructive if the redex contracted is of shape

$$((\lambda x.(\lambda y.P)Q))R,$$

that is, in case of distribution of $N$ over an application $(\lambda y.M_1)M_2$ which itself is a redex.

Our strategy relies on the observation that distributive reduction is preserved when projecting every term to its full $\beta$-development, as long as the steps of the former are not $\beta$-destructive. Non-destructive steps will be mapped to $\beta$-reduction sequences by $\bullet$.

Instead of proving this general fact, we note inner spine steps are non-destructive by innerness, and show that each such inner spine step is mapped to at most a single $\beta$-reduction step by $\bullet$. Moreover, in case a distributive inner spine step is mapped to an empty step by $\bullet$, i.e. if it is erased, then that step did not create a redex, hence it is a purely distributive step. This can be expressed formally by mapping the step to an $x$-step in Bloo and Rose’s $\lambda$-calculus with explicit substitutions $\lambda x$ [6], via the following operation.

**Definition 5.** Explicification $M^\diamond$ of a term $M$ is obtained by replacing each of its redexes

$$(\lambda x.P)Q$$

by the redex $P(x := Q)$ in the $\lambda$-calculus with explicit substitutions $\lambda x$.

1.3 Proof of normalizing property

**Proposition 3.** If $M \rightarrow_d N$, then either the step $M^\bullet \rightarrow_\beta N^\bullet$ contracts a spine redex, or $M^\bullet \equiv N^\bullet$ and $M^\circ \rightarrow_x N^\circ$.

**Proof.** Let us consider each of the possible cases.

1. If an inner spine step $M \rightarrow N$ is due to $M \rightarrow_d N$, then $M^\bullet \equiv N^\bullet$ and $M^\circ \rightarrow_x N^\circ$ immediately follow from the proposition about correctness of distributive reduction.

2. If the inner spine step $M P \rightarrow N P$ is due to $M \rightarrow_d N$, then relying on (1) having been proved, let us consider the following three possible options.

   (a) $M P$ is a $\beta$-redex. Then $M \equiv \lambda x. M'$ and $N \equiv \lambda x. N'$ for some $M'$ and $N'$, and $M' \rightarrow_\beta N'$, hence either

   $$\frac{(M P)^\bullet \equiv M'^\bullet[x := P^\bullet] \rightarrow_\beta N'^\bullet[x := P^\bullet] \equiv (N P)^\bullet}{\text{is a spine step}},$$

   or

   $$\frac{(M P)^\bullet \equiv M'^\bullet[x := P^\bullet] \equiv N'^\bullet[x := P^\bullet] \equiv (N P)^\bullet}{\text{and}}$$

   is a spine step, or

   $$\frac{(M P)^\circ \equiv M'^\circ(x := P^\circ) \rightarrow_x N'^\circ(x := P^\circ) \equiv (N P)^\circ}{\text{and}}.$$


(b) If $M P$ is not, but $N P$ is a $\beta$-redex. Then $N \equiv \lambda x. N'$, and either $M \to_i N$ and $M \equiv (\lambda x. x) N$, or $M \to_i N$ and for some $M'$, $N''$

$$
M \equiv (\lambda xy. M') N'';
N' \equiv (\lambda x . M') N''.
$$

The case of $\beta_i$ is trivial, while for $\beta_d$ we have

$$
(M P)^* \equiv ((\lambda xy. M') N'')^* P^* \equiv
(\lambda y. M') [x := N''^*] P^* \equiv (\lambda y . M'^* [x := N''^*]) P^*
$$

and

$$
(N P)^* \equiv ((\lambda y . N') P)^* \equiv
(\lambda y . (\lambda x . M') N'')^* P^* \equiv M'^* [x := N''^*] [y := P^*],
$$

then $(M P)^* \to_\beta (N P)^*$, hence the proposition stands since a head redex is a spine redex.

(c) Neither of $M P$ and $N P$ is a $\beta$-redex, then $(N P)^* \equiv N^* P^*$, $(N P)^\circ \equiv N^\circ P^\circ$, and the proposition stands.

3. If an inner spine step $P M \to P N$ is due to $M \to_d N$, then relying on (1) having been proved let us note that $P M$ cannot be a redex. Therefore, $P N$ is not a redex either. But then again either

$$(P M)^* \equiv P^* M^* \to_\beta P^* N^* \equiv (P N)^*$$

is a spine step, or

$$(P M)^* \equiv P^* M^* \equiv P^* N^* \equiv (P N)^*$$

and

$$(P M)^\circ \equiv P^\circ M^\circ \to_x P^\circ N^\circ \equiv (P N)^\circ.$$

4. If an inner spine step $\lambda x . M \to \lambda x . N$ is due to $M \to_d N$, then relying on (1) having been proved we immediately get that either

$$(\lambda x . M)^* \equiv \lambda x . M^* \to_\beta \lambda x . N^* \equiv (\lambda x . N)^*$$

is a spine step, or

$$(\lambda x . M)^* \equiv \lambda x . M^* \equiv \lambda x . N^* \equiv (\lambda x . N)^*$$

and

$$(\lambda x . M)^\circ \equiv \lambda x . M^\circ \to_x \lambda x . N^\circ \equiv (\lambda x . N)^\circ.$$

Thereby, the possible cases have been treated thoroughly. \qed

**Proposition 4.** Inner spine strategy is normalizing.
Proof. By the previous proposition, an infinite distributive reduction from some term \( M \) having a normal form \( \hat{M} \), would give rise to an infinite spine \( \beta \)-reduction from \( M^* \), unless from some moment \( N \) on in the distributive reduction all further terms are mapped to \( N^* \). But then by the same proposition, the infinite distributive reduction from \( N \) would give rise to an infinite \( x \)-reduction from \( N^\circ \).

Infinite spine \( \beta \)-reductions are impossible from \( M^* \) since \( M \) and \( M^* \) are \( \beta \)-convertible, hence have the same \( \beta \)-normal form \( \hat{M} \), and spine strategies are needed strategies, hence normalising [5].

In turn, infinite \( x \)-reductions are impossible since \( x \)-reduction (the substitution rules) is known to be terminating for the \( \lambda x \)-calculus [6]. \( \square \)

The essence of our strategy is to avoid destruction of redexes. In particular, the inner spine strategy avoids that distribution of the outer redex in \((\lambda x.(\lambda y.P)Q)R\) destroys the inner one, thereby blocking Klop’s counterexample to preservation of strong normalisation for distributive reduction.

References

[1] G. Révész. Axioms for the theory of lambda-conversion. SIAM Journal on Computing, 14(2): 373–382, May 1985.

[2] J. W. Klop. Term rewriting systems. Notes prepared for the seminar on Reduction Machines. Organized by C. Böhm, Ustica, September 1985.

[3] V. van Oostrom. The inner spine strategy is normalising for distributive \( \lambda \)-calculus. Private correspondence.

[4] K. H. Rose. Explicit Substitution — Tutorial and Survey. BRICS LS-96-3, 1996.

[5] H. P. Barendregt, J. R. Kennaway, J. W. Klop, and M. R. Sleep. Needed reduction and spine strategies for the lambda calculus. Information and Computation, 75(3): 191–231, December 1987.

[6] C. J. Bloo. Preservation of Termination for Explicit Substitution. PhD thesis, Technische Universiteit Eindhoven, October 2, 1997.