Classification of classical and non-local symmetries of fourth-order nonlinear evolution equations *

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Abstract

In this paper, we consider group classification of local and quasi-local symmetries for a general fourth-order evolution equations in one spatial variable. Following the approach developed in \cite{22}, we construct all inequivalent evolution equations belonging to the class under study which admit either semi-simple Lie groups or solvable Lie groups. The obtained lists of invariant equations (up to a local change of variables) contain both the well-known equations and a variety of new ones possessing rich symmetry. Based on the results on the group classification for local symmetries, the group classification for quasi-local symmetries of the equations is also given.

Key words: Group classification, Symmetry group, Local symmetry, Quasi-local symmetry, Semi-simple algebra, Solvable Lie algebra, Fourth-order nonlinear evolution equation.

1. Introduction

The purpose of this article is to classify local and nonlocal symmetries of fourth-order evolution equations of the form

\[ u_t = F(t, x, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, x, u, u_x, u_{xx}, u_{xxx}). \] (1)

Here \( F \) and \( G \) are arbitrary smooth functions and \( F \neq 0 \). Hereafter we adopt notations \( u = u(t, x), \ u_t = \partial u/\partial t, \ u_x = \partial u/\partial x, \ u_{xx} = \partial^2 u/\partial x^2, \) etc.

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This paper is a farther development of the the methods classification approach suggested in [22] and successfully applied to various equations of mathematical and theoretical physics in [3, 4, 11, 12, 13, 22, 23]. It also generalizes the results of [12].

The class partial differential equations of Eq. (1) contains a number of important mathematical physics equations. Among the particular cases of Eq. (1) there are, the Kuramoto-Sivashinsky (KS) equation

\[
    u_t = -u_{xxxx} - u_{xx} - \frac{1}{2}u_x^2,
\]

the extended Fisher-Kolmogorov (eFK) equation

\[
    u_t = -u_{xxxx} + u_{xx} - u^3 + u,
\]

the Swift-Hohenberg (SH) equation

\[
    u_t = -u_{xxxx} - 2u_{xx} - u^3 + (\kappa - 1)u, \quad \kappa \in \mathbb{R}.
\]

It also contains the equation describing surface tension driven thin film flows

\[
    u_t = -(u^3u_{xxx} + f(u, u_x, u_{xx}))_x,
\]

whose appropriate forms have been used to model fluid flows in physical situations such as coating, draining of foams, and the movement of contact lenses [16]. Another interesting particular case is the equation

\[
    u_t = -(f(u)u_{xxx})_x + (g(u)u_x)_x,
\]

which appears in the context of surface dominated motion of thin viscous films and spreading droplets or plasticity (for further details, see, [5] and the references therein). If \( f(u) = u^n, \ g(u) \equiv 0 \), it is the generalized lubrication equation. When \( n = 3 \), it is the capillary-driven flow. Equations with \( n < 3 \) can be used to describe slip models in the vicinity of \( u \to 0 \). Another case of practical importance is \( n = 1 \), which describes the thickness of a thin bridge between two masses of fluid in a Hele-Shaw cell [7].

In [12], we perform preliminary group classification of the equations

\[
    u_t = -u_{xxxx} + G(t, x, u, u_x, u_{xx}, u_{xxx}),
\]

obtained from Eq. (1) by putting \( F = -1 \). Note that the so obtained lists of invariant equations contain equations (5) as particulars cases.

Lie symmetries play the central role in the modern theory of differential equations. One of the primary questions of theory of Lie symmetries is whether a given equation admits nontrivial symmetry. If the answer is positive one can apply the whole wealth of approaches utilizing symmetries to analyze and solve the equation in question. So that group classification of equations (1) is necessary first step in
utilizing any symmetry methods and techniques. In the present paper, we perform preliminary group classification of Eq. (1) and describe all possible forms of the functions $F$ and $G$ such that Eq. (1) admit symmetry groups of the dimension $n \leq 4$.

The history of group classification methods goes back to Lie himself. In [14] he proved that a linear two-dimensional second-order partial differential equation (PDE) can admit at most three-parameter invariance group (apart from the trivial infinite-parameter symmetry group, which is due to linearity). The modern formulation of the problem of group classification of PDEs has been suggested by Ovsyannikov [17]. He introduced a regular method based on the concept of equivalence group (we will refer to it as the Lie-Ovsyannikov method). Ovsyannikov’s approach works at its best when the equivalence group of an equation under study is finite-dimensional and is not very efficient otherwise. So that the equations with arbitrary elements depending on two and more arguments cannot be efficiently handled with the Lie-Ovsyannikov method. To overcome this difficulty, Zhdanov and Lahno developed a different purely algebraic approach enabling to classify classes of PDEs having infinite-dimensional equivalence groups [22].

The method suggested in [22] has been applied to classify broad classes of heat conductivity [4], Schrödinger [24], KdV-type evolution [11], nonlinear wave [13], general second-order quasi-linear evolution [23], third-order nonlinear evolution [3] and fourth-order evolution [12] equations. Here we follow the approach of [22] in order to describe fourth-order nonlinear evolution equations (1) having non-trivial symmetry properties.

Our classification algorithm for local symmetries is a combination of the standard Lie infinitesimal algorithm, equivalence group and the structures of abstract Lie algebra. It consists of three major steps. We give a brief description of the steps involved. Further details can be found in [22].

The first step is to compute the most general symmetry group of (1) together with the classifying equations for $F$ and $G$. In addition, we calculate the maximal local equivalence group admitted by Eq. (1) under consideration. The second step is essentially based on the explicit forms of commutation relations of low dimensional abstract Lie algebras [2, 18, 19]. Using these we construct all inequivalent realizations of symmetry algebras by basis infinitesimal operators admitted by Eq. (1). The third step is inserting the canonical forms of symmetry generators into the classifying equations, solving them, and deriving the invariant equations. Also we need to make sure that the corresponding symmetry algebras are maximal in Lie’s sense.

Since Lie symmetry is not always the best answer to all challenges of the modern theory of nonlinear differential equations, one has always been looking for ways to generalize it. A generalization is to allow for infinitesimals depending on the global behavior of the dependent variable, which is just the way the non-local symmetries arose. The most well studied non-local symmetries are those of linear PDEs [2, 10], while for those of nonlinear PDEs, much less is known. One possible
approach has been suggested by Bluman [6, 8]. He derived non-local symmetries (potential symmetries) admitted by a given differential equation by realizing such symmetries as local symmetries of an associated auxiliary system. There is another way to constructing non-local symmetries which is to apply a non-local transformation to an equation admitting nontrivial Lie symmetries. Then some of Lie symmetries of the initial equation will remain Lie symmetries of the transformed equation, while the others become non-local ones, which are called quasi-local symmetries. The term quasi-local has been introduced independently in [1] and [15] to distinguish non-local symmetries that are equivalent to local ones through non-local transformation. It was noted in [4] that results of the group classification for local symmetries can be utilized to derive quasi-local symmetries of PDEs under study. Zhdanov and Lahno obtained some nontrivial examples of second-order evolution equations with quasi-local symmetries in [21]. recently, Zhdanov suggested a regular group-theoretical approach to the problem of classification of evolution equations that admit quasi-local symmetries [20].

The principal motivation for the present paper is a need for a unified group classification approach enabling to obtain both local/Lie and quasi-local/non-local symmetries of equations of the form (1). This approach is essentially based on the methods developed in the above mentioned papers [22, 20].

The outline of this paper is as follows. In Section 2, we obtain the classifying equations for the functions $F$, $G$ and give the equivalence group of Eq. (1). In Section 3, we provide a detailed description of equations of the form (1) invariant with respect to semi-simple algebras. Section 4 is devoted to the classification of those evolution equations (1) which admit solvable Lie algebras. In Section 5, utilizing the results obtained while classifying for local symmetries, we derive the realizations of Lie algebras leading to equations admitting quasi-local symmetries.

2. Preliminary group analysis of Eq. (1)

In this section we derive the classifying equations for coefficients of ininitesimal operators of symmetry group of Eq. (1). Next, we obtain its most general equivalence group. It is a common knowledge that the Lie transformation group admitted by (1) is generated by the infinitesimal operators of the form

$$V = \tau (t, x, u) \partial_t + \xi (t, x, u) \partial_x + \eta (t, x, u) \partial_u,$$

(6)

where $\tau, \xi, \eta$ are arbitrary, real-valued smooth functions defined in some subspace of the space $V = \mathbb{R}^2 \times \mathbb{R}$ of the independent $\langle t, x \rangle$ and the dependent $\langle u \rangle$ variables.

The vector field (6) generates one-parameter invariance group of (1) iff its coefficients $\tau, \xi, \eta$ satisfy the determining equation (Lie’s invariance criterion)

$$\eta^t - [\tau F_t + \xi F_x + \eta F_u + \eta^x F_{ux} + \eta^{xx} F_{uxx} + \eta^{xxx} F_{uxxx}] u_{xxxx} - \eta^{xxxx} F = 0,$$

$$\tau G_t + \xi G_x + \eta G_u + \eta^x G_{ux} + \eta^{xx} G_{uxx} + \eta^{xxx} G_{uxxx}] u_t = F_{uxxxx} + G = 0,$$
where

\[ \eta^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \]
\[ \eta^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \]
\[ \eta^{xx} = D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi), \]
\[ \eta^{xxx} = D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \]
\[ \eta^{xxxx} = D_x(\eta^{xxx}) - u_{xxxx} D_x(\tau) - u_{xxxxx} D_x(\xi). \]

and the operators \( D_t \) and \( D_x \) denote the total derivatives with respect to the variables \( t \) and \( x \) respectively,

\[ D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{xt} \partial_{u_x} + \ldots, \]
\[ D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \ldots. \]

Equating coefficients of linearly independent terms of invariance condition above to zero yields an over-determined system of linear PDEs. Solving it, we arrive at the following assertion.

**Proposition 1.** The most general symmetry group of (1) is generated by the infinitesimal operators

\[ V = \tau(t) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u, \] (7)
where \( \tau, \xi \) and \( \eta \) are real-valued functions satisfying the classifying equations
\[
(4\xi_u u_x + 4\xi_x - \tau)F - \tau F_t - \xi F_x - \eta F_u + (u_x \xi_x - u_x \eta_x + u_x^2 \xi_u - \eta_x)F_{uu} \\
(-u_{xx} \eta_u - \eta_{xx} + u_x^2 \xi_{uu} - u_x^2 \eta_{uu} - 2u_x \eta_{xu} + 2u_x \xi_x + u_x \xi_{xx} + 2u_x^2 \xi_{uu}) \\
+ 3u_x u_{xxx} \xi_u F_{uu} + (-3u_x^2 \xi_{x,x,u} - \eta_{xxx} + u_x^4 \xi_{uuu} - 3u_x u_{xxx} - 3u_x \eta_{xxx}) \\
+ 3u_{xxx} \xi_x + 3u_x^2 \xi_u + 3u_{xx} \xi_{xx} + 3u_x^3 \xi_{xxx} + u_x \xi_{xxx} + 6u_x^2 u_{xxx} \xi_{uu} - u_x^3 \eta_{uuu} \\
+ 9u_x u_{xxx} \xi_{uu} - u_{xxx} \eta_u + 4u_{x} u_{xxx} \xi_u + 3u_x^2 \xi_{xxx} - 3u_x u_{xxx} \eta_{uu})F_{uuuu} = 0, \\
(-12u_x u_{xxx} \eta_{uuu} - u_x^4 \eta_{uuuu} + u_x^5 \xi_{uuuu} + 4u_x^4 \xi_{uuuu} - 6u_x^2 \eta_{uuuu}) \\
+ 6u_x^3 \xi_{uuuu} - 4u_{xxx} \eta_{uu} - 6u_{xx} \eta_{xxx} - 4u_x \eta_{xxxx} + 12u_x^2 \xi_{xx} - 4u_x^3 \eta_{xxxx} \\
- 3u_x u_{xx} \eta_u + 4u_x^2 \xi_{xx} + 4u_{xx} \xi_{xxx} + 6u_{xxx} \xi_{xx} - \eta_{xxxx} - 6u_x^2 u_{xx} \eta_{uu} \\
+ 16u_x u_{xxx} \xi_{xx} - 4u_x u_{xxx} \eta_{uu} + 10u_x^2 u_{xxx} \xi_{uu} + 15u_x u_x u_{xxx} \xi_{uu} + 10u_x^3 u_{xxx} \xi_{uu} \\
+ 16u_x u_{xxx} \xi_{uu} + 24u_x^2 u_{xxx} \xi_{uu} + 18u_x u_{xxx} \xi_{uu} + (\eta_u - \tau - u_x \xi_u)G \\
- \tau G_t - \xi G_x - \eta G_u + (u_x \xi_x - u_x \eta_x + u_x^2 \xi_u - \eta_x)G_{uu} + (-u_{xx} \eta_u - \eta_{xx}) \\
+ u_x^3 \xi_{uu} - u_x^2 \eta_{uu} - 2u_x \eta_{xu} + 2u_x \xi_x + u_x \xi_{xx} + 2u_x^2 \xi_{xx} + 3u_x u_{xx} \xi_u)G_{uu} \\
(-3u_x^2 \eta_{uu} - \eta_{xxx} + u_x^4 \xi_{uuu} - 3u_x \eta_{xxx} - 3u_x \eta_{xx} + 3u_{xxx} \xi_x + 3u_x^3 \xi_x) \\
+ 3u_x \xi_{xx} + 3u_x^2 \xi_{xuu} + u_x \xi_{xxx} + 6u_x^2 u_{xxx} \xi_{uu} - u_x^3 \eta_{uuu} + 9u_x u_{xxx} \xi_x \\
- u_{xxx} \eta_u + 4u_x u_{xxx} \xi_u + 3u_x^2 \xi_{xxx} - 3u_x u_{xxx} \eta_{uu})G_{uuuu} - u_x \xi_t + \eta = 0.
\]

Here and after the dot over a symbol stands for differentiation with respect to its argument.

If there are no restrictions on \( F \) and \( G \), then (8) should be satisfied identically, which is only possible when the symmetry group is the trivial group of identity transformations. Our goal is finding all specific forms of \( F \) and \( G \) for which Eq. (1) admits non-trivial symmetry groups. The basic idea, as suggested in [22], is to utilize the fact that for arbitrarily fixed \( F \) and \( G \), all admissible vector fields form a Lie algebra. So that we can use the results of classification of abstract low-dimensional Lie algebras to construct all Lie symmetries of the form (7) admitted by (1). The next step is integrating the classifying equations, which yields the explicit form of the functions \( F, G \).

As a first step we need to compute the equivalence group of Eq.(1). To this end we have to construct all possible invertible changes of variables
\[
\bar{t} = T(t, x, u), \quad \bar{x} = X(t, x, u), \quad \bar{u} = U(t, x, u), \quad \frac{D(T, X, U)}{D(t, x, u)} \neq 0,
\]
which do not alter the form of Eq.(1). We present the final result of our calculations without detailed proof.
Proposition 2. The maximal equivalence group of Eq.(11) reads as
\[ \bar{t} = T(t), \quad \bar{x} = X(t, x, u), \quad \bar{u} = U(t, x, u), \] (9)
where \( \dot{T} \neq 0 \) and \( \frac{D(X,U)}{D(t,x,u)} \neq 0 \).

We use equivalence transformation (9) to transform vector field (7) to a simplest possible (canonical) form.

Lemma 1. Within the point transformation (9), vector field (7) is equivalent to one of the following canonical operators
\[ \partial_t, \quad \partial_x. \] (10)

Proof. Transformation (9) converts operator (7) into
\[ \bar{V} = \tau \dot{T} \partial_t + (\tau X_t + \xi X_x + \eta X_u)\partial_x + (\tau U_t + \xi U_x + \eta U_u)\partial_u. \]

The are two cases \( \tau \neq 0 \) and \( \tau = 0 \) to consider.

If \( \tau \neq 0 \), choosing in (9) the function \( T \) to satisfy \( \tau \dot{T} = 1 \) and the functions \( X \) and \( U \) to be any two independent solutions of the equation
\[ \tau Y_t + \xi Y_x + \eta Y_u = 0, \quad Y = Y(t, x, u), \]
leads to the canonical form \( \partial_t \).

Suppose now that \( \tau = 0 \) and \( \xi^2 + \eta^2 \neq 0 \) (otherwise operator (7) vanishes identically), then (7) is transformed to
\[ \bar{V} = (\xi X_x + \eta X_u)\partial_x + (\xi U_x + \eta U_u)\partial_u. \]

If \( \xi \neq 0 \), then choosing in (9) a particular solution of \( \xi X_x + \eta X_u = 1 \) as the function \( X \) and a fundamental solution of \( \xi U_x + \eta U_u = 0 \) as \( U \), transforms (7) into \( \partial_x \).

If \( \xi = 0, \eta \neq 0 \), by making the transformation (9) with \( t = t, \bar{x} = u, \bar{u} = x \), we get the case \( \xi \neq 0 \), which has been already considered.

It is straightforward to verify that there is no transformation (9) which can transform \( \partial_t \) and \( \partial_x \) into another.

Consequently, we have two inequivalent realizations of one-dimensional symmetry algebras \( \partial_t \) and \( \partial_x \). Integrating the classifying equations for each symmetry operator yields the corresponding inequivalent equations of class (11).

In what follows, we use the notation \( A_{k,i} = \langle V_1, V_2, \cdots, V_k \rangle \) to denote a \( k \)-dimensional Lie algebra with basis elements \( V_i \) \((i = 1, 2, \cdots, k) \), the index \( i \) standing for the number of the class to which the given algebra belongs.

Theorem 1. There are two inequivalent equations (11) invariant under one-parameter symmetry groups:
\[ A_1^1 = \langle \partial_t \rangle : \quad u_t = F(x, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(x, u, u_x, u_{xx}, u_{xxx}), \]
\[ A_1^2 = \langle \partial_x \rangle : \quad u_t = F(t, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, u, u_x, u_{xx}, u_{xxx}). \]

Here \( F \) and \( G \) are arbitrary smooth functions. Furthermore, the associated symmetry algebra is maximal in Lie’s sense.
3. Classification of equations invariant under semi-simple Lie algebras

In this section, we classify all the equations of the form (1) admitting semi-simple Lie algebras. The lowest order real semi-simple Lie algebras are isomorphic to one of the following two three-dimensional algebras:

\[ \text{so}(3) : \ [V_1, V_2] = V_3, \ [V_1, V_3] = -V_2, \ [V_2, V_3] = V_1; \]
\[ \text{sl}(2, \mathbb{R}) : \ [V_1, V_2] = 2V_2, \ [V_1, V_3] = -2V_3, \ [V_2, V_3] = V_1. \]

We begin by studying realizations of the algebras \( \text{so}(3) \) within the class of operators (7).

Taking into account of our preliminary classification, we can assume without any loss of generality that one of the basis operators, say \( V_1 \), is reduced to one of the canonical forms \( \partial_t \) and \( \partial_x \).

Let \( V_1 = \partial_t \) and \( V_2, V_3 \) be of the form (7), i.e., \( V_i = \tau_i(t)\partial_t + \xi_i(t, x, u)\partial_x + \eta_i(t, x, u)\partial_u, \ i = 2, 3 \). Using \( [V_1, V_2] = V_3 \) we find that \( \tau_2 = \tau_3 \). Next, it follows from \( [V_2, V_3] = -V_2 \) that \( \tau_3 = -\tau_2 \). With these facts in hand and using the relation \( [V_2, V_3] = V_1 \) we arrive at \( \tau_2\tau_3 - \tau_3\tau_2 = 1 \), which in turn yields equation \( \tau_2^2 + \tau_2^2 = -1 \). This equation has no real solutions, so there are no realizations of \( \text{so}(3) \) with \( V_1 = \partial_t \).

Turn now to the case \( V_1 = \partial_x \). A similar analysis yields a unique realization of \( \text{so}(3) \)

\[ \langle \partial_x, \tan u \sin x\partial_x + \cos x\partial_u, \tan u \cos x\partial_x - \sin x\partial_u \rangle \]

However, this algebra cannot be symmetry algebra of an equation of the form (1).

**Theorem 2.** There are no \( \text{so}(3) \)-invariant equations of the form (1).

In a similar way we analyze possible realizations of the algebra \( \text{sl}(2, \mathbb{R}) \) and get the following assertion.

**Theorem 3.** There are six inequivalent realizations of \( \text{sl}(2, \mathbb{R}) \) by operators (7), which are admitted by Eq.(1). These realizations and the corresponding forms of the functions \( F, G \) are presented below

\[ \text{sl}^1(2, \mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u, \partial_t \rangle : \]

\[ F = x^2 F(\omega_1, \omega_2, \omega_3), \quad G = \frac{G(\omega_1, \omega_2, \omega_3)}{x^2} + \frac{xuu_x - u^2}{x^2}, \]
\[ \omega_1 = xu_x - 2u, \ \omega_2 = xu_{xx} - 2u, \ \omega_3 = xu_{xxx}, \]

\[ \text{sl}^2(2, \mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - tx)\partial_x, \partial_t \rangle : \]

\[ F = \frac{F(u, \omega_1, \omega_2)}{x^3u_x^2}, \]
\[ G = \frac{18x^2u_{xxx} + 87xu_{xx} + 105u_x}{x^6u_x^5} F(u, \omega_1, \omega_2) + \frac{G(u, \omega_1, \omega_2)}{x^3u_x} - \frac{u_x}{4x}, \]
\[ \omega_1 = \frac{x u_{xx} + 3 u_x}{x u_x^2}, \omega_2 = \frac{x^2 u_{xxx} + 9 x u_{xx} + 15 u_x}{x^2 u_x^3}, \]

\[ s_t^3(2, \mathbb{R}) = \{ 2 x \partial_x - u \partial_u, -x^2 \partial_x + x u \partial_u, \partial_x \} : \]

\[ F = \frac{F(t, \omega_1, \omega_2)}{u^8}, \quad G = -8 \frac{2 u_{xxx} u^2 - 9 u_x u_{xx} u + 9 u_x^3}{u^{11}} F(t, \omega_1, \omega_2) + u G(t, \omega_1, \omega_2), \]

\[ \omega_1 = \frac{u_{xxx} - 2 u_x^2}{u^6}, \omega_2 = \frac{u_{xxx} u^2 - 9 u_x u_{xx} u + 12 u_x^3}{u^9}, \]

\[ s_t^4(2, \mathbb{R}) = \{ 2 x \partial_x, -x^2 \partial_x, \partial_x \} : \]

\[ F = \frac{F(t, u, \omega)}{u_x^5}, \quad G = 6 \frac{u_{xx}}{u_x^6} (u_x^2 - u_x u_{xxx}) F(t, u, \omega) + u G(t, u, \omega), \]

\[ \omega = \frac{3 u_{xx}^2 - 2 u_x u_{xxx}}{u_x^4}, \]

\[ s_t^5(2, \mathbb{R}) = \{ 2 x \partial_x - u \partial_u, (\frac{1}{u^4} - x^2) \partial_x + x u \partial_u, \partial_x \} : \]

\[ F = \frac{u^4}{(u^6 + 4 u_x^2)^2} F(t, \omega_1, \omega_2), \]

\[ G = -\frac{u}{4(u^6 + 4 u_x^2)^4} \left[ 146880 u_x^8 + 192 u (641 u^5 - 167 u_{xx}) u_x^6 - 896 u^2 u_{xxx} u_x^5 \right. \]

\[ + 96 u^2 (417 u^{10} - 208 u_{xx} u^5 + 28 u_x^2) u_x^4 + 32 u^3 u_{xxx} (u_x^5 + 20 u_{xx}) u_x^3 \]

\[ + 6 u^3 (855 u^{15} - 1092 u_{xx} u^{10} - 176 u_x^2 u^5 - 160 u_x^3) u_x^2 + 32 u^9 u_{xxx} (2 u^5 + 5 u_{xx}) u_x \]

\[ + 3 u^{14} (77 u^{10} - 162 u_{xx} u^5 + 36 u_x^2)] F(t, \omega_1, \omega_2) + \frac{(u^6 + 4 u_x^2)^{\frac{1}{2}}}{u^2} G(t, \omega_1, \omega_2), \]

\[ \omega_1 = \frac{u^3}{(u^6 + 4 u_x^2)^2} (u^6 - 2 u u_{xx} + 10 u_x^2), \]

\[ \omega_2 = \frac{u^3}{(u^6 + 4 u_x^2)^3} [u_{xxx} u^8 - 9 u_x u_{xx} u^7 + 12 u_x^3 u^6 + 4 u_x (u_x u_{xxx} - 3 a_x^2) u^2 \]

\[ + 36 u_x^3 u_{xx} u - 60 u_x^5], \]

\[ s_t^6(2, \mathbb{R}) = \{ 2 x \partial_x - u \partial_u, -(x^2 + \frac{1}{u^4}) \partial_x + x u \partial_u, \partial_x \} : \]

\[ F = \frac{u^4}{(u^6 - 4 u_x^2)^2} F(t, \omega_1, \omega_2), \]

\[ G = -\frac{u}{4(u^6 - 4 u_x^2)^4} \left[ 146880 u_x^8 - 192 u (641 u^5 + 167 u_{xx}) u_x^6 - 896 u^2 u_{xxx} u_x^5 \right. \]
semi-simple Lie algebras admitted by (1).

The invariant equations listed in Theorem 3 exhaust the list of all corresponding PDEs, provided the functions $F$ and $G$ are arbitrary. Hence it immediately follows that the realizations of the algebra $sl$ admit the following isomorphisms

$$\text{so}(3) \sim su(2) \sim sp(1), \quad sl(2, \mathbb{R}) \sim su(1, 1) \sim so(2, 1) \sim sp(1, \mathbb{R})$$

(see, e.g., [2]). Hence it immediately follows that the realizations of the algebra $sl(2, \mathbb{R})$ exhaust the set of all possible inequivalent realizations of three-dimensional semi-simple Lie algebras admitted by (1).

The next admissible dimension for classical semi-simple Lie algebras is six. There are four non-isomorphic semi-simple Lie algebras over the real number field, namely, $so(4)$, $so^*(4)$, $so(3,1)$, and $so(2,2)$. As $so(4) \sim so(3) \oplus so(3)$, $so^*(4) \sim so(3) \oplus sl(2, \mathbb{R})$, and the algebra $so(3,1)$ contains $so(3)$ as a subalgebra, there are no realizations of these algebras by operator (1). Therefore $so(2,2)$ is the only candidate for six-dimensional semi-simple symmetry algebra of (1).

In view of $so(2,2) \sim sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$, we can choose $so(2,2) = \langle Q_i, K_i \rangle i = 1, 2, 3$, where $\langle Q_1, Q_2, Q_3 \rangle$ and $\langle K_1, K_2, K_3 \rangle$ are all $sl(2, \mathbb{R})$ algebras, and $[Q_i, K_j] = 0$, $(i,j = 1, 2, 3)$. Taking $Q_1, Q_2, Q_3$ to be the basis operators of the realizations of $sl(2, \mathbb{R})$ given in the Theorem 3 and $K_1, K_2, K_3$ be of the general form (1), we establish that realizations of $sl(2, \mathbb{R})$ cannot be extended to a realization of $so(2,2)$.

Hence no equation of form (1) is invariant under six-dimensional semi-simple Lie algebras.

The similar reasoning yields that there are no realizations of eight-dimensional semi-simple Lie algebras $sl(3, \mathbb{R})$, $su(3)$ and $su(2,1)$.

The same assertion holds true for the algebras $A_{n-1}$ ($n > 1$), $D_n$ ($n > 1$), $B_n$ ($n \geq 1$), $C_n$ ($n \geq 1$) and the exceptional semi-simple Lie algebras $G_2, F_4, E_6, E_7, E_8$.

The theorem is proved. $\square$
4. Classification of equations invariant under solvable Lie algebras

Using the concept of compositional series for a solvable algebra we can construct all possible realizations of solvable Lie algebras admitted by Eq.(1) starting from the one-dimensional ones and proceeding to the solvable Lie algebras of the dimension 2, 3, 4, . . . (for more details, see [4]).

In this section, we describe inequivalent equations of the form Eq.(1) which are invariant under solvable Lie algebras of the dimension up to four. Since equations invariant with respect to one-dimensional algebras have already been constructed, we start by analyzing two-dimensional solvable algebras.

4.1. Equations with two-dimensional Lie algebras

There are two non-isomorphic two-dimensional Lie algebras

\[ A_{2,1} : \ [V_1, V_2] = 0, \quad A_{2,2} : \ [V_1, V_2] = V_2. \]

As both \( A_{2,1} \) and \( A_{2,2} \) contain the algebra \( A_1 \), we can assume that the basis operator of the latter is reduced to one of the canonical forms.

We consider in detail the case of \( A_{2,1} \), while for the case of algebra \( A_{2,2} \) we present the final results only.

Let \( V_1 = \partial_t \) and \( V_2 \) be an operator of the most general form (7)

\[ V_2 = \tau(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u. \]

Then the commutation relation implies that \( \dot{\tau} = \dot{\xi} = \dot{\eta} = 0 \). Therefore \( \tau \) is a constant and \( \xi = \xi(x, u), \eta = \eta(x, u) \). So without any loss of generality we can put \( V_2 = \xi(x, u)\partial_x + \eta(x, u)\partial_u \). Before simplifying \( V_2 \) with equivalence transformations (9), we seek those equivalent transformations which preserve the basis operator \( V_1 \)

\[ V_1 \rightarrow \bar{V}_1 = \hat{T}\partial_t + X_t\partial_x + U_t\partial_u = \partial_t. \]

Hence, \( \hat{T} = 1 \) and \( X_t = U_t = 0 \). Consequently, we can take \( T = t, X = X(x, u) \) and \( U = U(x, u) \). Performing this transformation yields

\[ V_2 \rightarrow \bar{V}_2 = (\xi X_x + \eta X_u)\partial_x + (\xi U_x + \eta U_u)\partial_u. \]

If \( \eta = 0 \), we choose \( U = U(u) \) and \( \xi \) satisfying \( \xi X_x = 1 \) thus getting \( V_2 = \partial_x \). Provided \( \eta \neq 0 \), we can take \( X \) as a solution of \( \xi X_x + \eta X_u = 1 \) and select \( U \) to satisfy \( \xi U_x + \eta U_u = 0 \), get \( \partial_x \) again. Thus within the action of equivalence group of Eq.(11), we get the following two-dimensional invariance algebra \( \langle \partial_t, \partial_x \rangle \).

Consider now the case when \( V_1 = \partial_x \) and \( V_2 \) is an operator of the form (7). Inserting \( V_1 \) and \( V_2 \) into the commutation relation yields \( V_2 = \tau(t)\partial_t + \xi(t, u)\partial_x + \eta(t, u)\partial_u \). The equivalence transformation, which leaves \( V_1 \) invariant, reads as

\[ \bar{x} = x + X(t, u), \quad \bar{u} = U(t, u) \]
with $U_u \neq 0$. This transformation reduces $V_2$ to the form

\[ \bar{V}_2 = \tau \dot{T} \partial_t + (\tau X_t + \xi + \eta X_u) \partial_x + (\tau U_t + \eta U_u) \partial_u. \]

We consider the cases $\tau = 0$ and $\tau \neq 0$ separately.

If $\tau = 0$ and $\eta = 0$, then $\bar{V}_2 = \xi \partial_x$. If $\xi_u \neq 0$, choosing $U = \xi$ yields $\bar{u} \partial_x$. If $\xi_u = 0$ and $\xi_t = 0$, then $\bar{V}_2 = \partial_x$. And if $\xi_u = 0$ and $\xi_t \neq 0$, selecting $T = \xi$ yields $\bar{V}_2 = \bar{u} \partial_x$. It is straightforward to verify that the so obtained two-dimensional Lie algebra, $\langle \partial_x, t \partial_x \rangle$, cannot be invariance algebras of an equation of the form (1). Now if $\tau = 0$ and $\eta \neq 0$, we choose $U$ and $X$ satisfying $\eta U_u = 1$ and $\xi + \eta X_u = 0$ and get $\partial_u$.

Turn next to the remaining case $\tau \neq 0$. Choosing of $T$, $X$ and $U$ as the solutions of the equations $\tau \dot{T} = 1$, $\tau X_t + \xi + \eta X_u = 0$ and $\tau U_t + \eta U_u = 0$, respectively, we arrive at the operator $\partial_t$. This yields the already known realization $\langle \partial_x, \partial_t \rangle$.

This completes analysis of the realizations of the algebra $A_{2,1}$. The case of $A_{2,2}$ is treated similarly.

Substituting the so obtained basis operators into the classifying equations and solving the latter yields the corresponding invariant equations.

**Theorem 5.** There exist three Abelian and four non-Abelian two-dimensional symmetry algebras admitted by (1). These algebras and the corresponding invariant equations are given below.

| Algebra | $F$ | $G$ |
|---------|-----|-----|
| $A_{2,1}^{1} \langle \partial_t, \partial_x \rangle$ | $F(u, u_x, u_{xx}, u_{xxx})$ | $G(u, u_x, u_{xx}, u_{xxx})$ |
| $A_{2,1}^{2} \langle \partial_x, \partial_u \rangle$ | $F(t, u_x, u_{xx}, u_{xxx})$ | $G(t, u_x, u_{xx}, u_{xxx})$ |
| $A_{2,1}^{3} \langle \partial_u, u \partial_x \rangle$ | $F(t, x, u_{xx}, u_{xxx})$ | $G(t, x, u_{xx}, u_{xxx})$ |
| $A_{2,2}^{1} \langle -t \partial_t - x \partial_x, \partial_t \rangle$ | $x^3 F(u, xu_x, x^2 u_{xx}, x^3 u_{xxx})$ | $x^{-1} G(u, xu_x, x^2 u_{xx}, x^3 u_{xxx})$ |
| $A_{2,2}^{2} \langle -t \partial_t - x \partial_x, \partial_u \rangle$ | $t^3 F(u, tu_x, t^2 u_{xx}, t^3 u_{xxx})$ | $t^{-1} G(u, tu_x, t^2 u_{xx}, t^3 u_{xxx})$ |
| $A_{2,2}^{3} \langle -x \partial_x + u \partial_u, \partial_x \rangle$ | $u^4 F(t, u_x, u u_x, u^2 u_{xxx})$ | $u G(t, u_x, u u_x, u^2 u_{xxx})$ |
| $A_{2,2}^{4} \langle -x \partial_x, \partial_u \rangle$ | $u^{-3} F(t, u, u^{-2} u_x, u^{-3} u_{xxx})$ | $G(t, u, u^{-2} u_x, u^{-3} u_{xxx})$ |

**4.2. Equations admitting three-dimensional solvable Lie algebras**

We split three-dimensional solvable Lie algebras into two classes. One contains the algebras which are direct sums of lower dimensional algebras, the other includes the remaining algebras. We consider the decomposable and non-decomposable Lie algebras separately.
4.2.1 Three-dimensional decomposable algebras

There exist two non-isomorphic three-dimensional decomposable Lie algebras, \(A_{3,1}\) and \(A_{3,2}\),

\[ A_{3,1} = A_1 \oplus A_1 \oplus A_1, \]

with commutation relations

\[ [V_i, V_j] = 0 \quad (i, j = 1, 2, 3), \]

and

\[ A_{3,2} = A_{2,2} \oplus A_1, \]

with commutation relations

\[ [V_1, V_2] = V_2, \quad [V_1, V_3] = 0, \quad [V_2, V_3] = 0. \]

It is a common knowledge that any three-dimensional solvable Lie algebra contains two-dimensional solvable algebra as a subalgebra. So that to describe all possible realizations of three-dimensional solvable algebras admitted by Eq. (1) it suffices to consider all possible extensions of two dimensional algebras listed in Theorem 5 by vector fields \(V_3\), of the form (7). Then for each of the so obtained realizations we simplify \(V_3\) using equivalence transformations which preserve the operators \(V_1\) and \(V_2\). Having performed these two steps we obtain the following list of invariant equations (1):

\[ A_{3,1} - \text{invariant equations} \]

\[ A_{3,1}^1 = \langle \partial_t, \partial_x, \partial_u \rangle : \]

\[ F = F(u_x, u_{xx}, u_{xxx}), \quad G = G(u_x, u_{xx}, u_{xxx}), \]

\[ A_{3,1}^2 = \langle \partial_t, \partial_u, x\partial_u \rangle : \]

\[ F = F(x, u_{xx}, u_{xxx}), \quad G = G(x, u_{xx}, u_{xxx}), \]

\[ A_{3,1}^3 = \langle \partial_u, x\partial_u, f(t, x)\partial_u \rangle, \quad f_{xx} \neq 0 : \]

\[ F = F(t, x, \omega), \quad G = -\frac{f_{xxx}}{f_{xx}}u_{xx}F(t, x, \omega) + G(t, x, \omega) + \frac{f_t}{f_{xx}}u_{xx}, \]

\[ \omega = u_{xxx} - \frac{f_{xxx}}{f_{xx}}u_{xx}. \]

\[ A_{3,2} - \text{invariant equations} \]

\[ A_{3,2}^1 = \langle -t\partial_t - x\partial_x, \partial_t, \partial_u \rangle : \]

\[ F = x^3F(\omega_1, \omega_2, \omega_3), \quad G = x^{-1}G(\omega_1, \omega_2, \omega_3) \]
\[ \omega_1 = xu_x, \quad \omega_2 = x^2u_{xx}, \quad \omega_3 = x^3u_{xxx}, \]

\[ A_{3,2}^2 = \langle -t\partial_t - u\partial_u, \partial_t, xu\partial_u \rangle : \]
\[ F = u^{-1}e^{x\sigma_1}F(x, \omega_1, \omega_2), \]
\[ G = e^{x\sigma_1}[6\sigma_1^2\sigma_2 - 4\sigma_1\sigma_3 - 3\sigma_1^4]F(x, \omega_1, \omega_2) + G(x, \omega_1, \omega_2), \]
\[ \sigma_1 = u^{-1}u_x, \quad \sigma_2 = u^{-1}u_{xx}, \quad \sigma_3 = u^{-1}u_{xxx}, \]
\[ \omega_1 = \sigma_2 - \sigma_1^2, \quad \omega_2 = \sigma_3 + 2\sigma_1^2 - 3\sigma_1\sigma_2, \]

\[ A_{3,2}^3 = \langle -t\partial_t - x\partial_x, \partial_x, tu\partial_x \rangle : \]
\[ F = tu_x^{-1}F(u, \omega_1, \omega_2), \]
\[ G = -5t^{-1}u_x^{-6}u_{xx}(2u_xu_{xxx} - 3u_{xx}^2)F(u, \omega_1, \omega_2) + u_xG(u, \omega_1, \omega_2) - t^{-1}u, \]
\[ \omega_1 = t^{-1}u_x^{-3}u_{xx}, \quad \omega_2 = t^{-1}u_x^{-5}(u_xu_{xxx} - 3u_{xx}^2), \]

\[ A_{3,2}^4 = \langle -t\partial_t - x\partial_x, \partial_x, \partial_u \rangle : \]
\[ F = t^3F(\omega_1, \omega_2, \omega_3), \quad G = t^{-1}G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = tu_x, \quad \omega_2 = t^2u_{xx}, \quad \omega_3 = t^3u_{xxx}, \]

\[ A_{3,2}^5 = \langle -t\partial_t - x\partial_x, \partial_x, t\partial_t \rangle : \]
\[ F = t^{-1}u_x^{-4}F(u, \omega_1, \omega_2), \quad G = t^{-1}G(u, \omega_1, \omega_2), \]
\[ \omega_1 = u_x^{-2}u_{xx}, \quad \omega_2 = u_x^{-3}u_{xxx}, \]

\[ A_{3,2}^6 = \langle -t\partial_t - x\partial_x, \partial_x, t\partial_t + \partial_u \rangle : \]
\[ F = t^3e^{-4u}F(\omega_1, \omega_2, \omega_3), \quad G = t^{-1}G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = te^{-u}u_x, \quad \omega_2 = t^2e^{-2u}u_{xx}, \quad \omega_3 = t^3e^{-3u}u_{xxx}, \]

\[ A_{3,2}^7 = \langle -x\partial_x - u\partial_u, \partial_x, u\partial_x \rangle : \]
\[ F = u^4u_x^{-4}F(t, \omega_1, \omega_2), \]
\[ G = -5u^4u_x^{-6}u_{xx}(2u_xu_{xxx} - 3u_{xx}^2)F(t, \omega_1, \omega_2) + uu_xG(t, \omega_1, \omega_2), \]
\[ \omega_1 = uu_x^{-3}u_{xx}, \quad \omega_2 = u^2u_x^{-5}(u_xu_{xxx} - 3u_{xx}^2), \]

\[ A_{3,2}^8 = \langle -x\partial_x - u\partial_u, \partial_x, \partial_t \rangle : \]
\[ F = u^4F(\omega_1, \omega_2, \omega_3), \quad G = uG(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = u_x, \quad \omega_2 = uu_x, \quad \omega_3 = u^2u_{xx}, \]

\[ A_{3,2}^9 = \langle -x\partial_x - u\partial_u, \partial_x, tu\partial_u \rangle : \]
\[ F = u^4 u_x^{-4} F(t, \omega_1, \omega_2), \quad G = uG(t, \omega_1, \omega_2) + t^{-1} u \ln |u_x|, \]
\[ \omega_1 = uu_x^{-2} u_x x, \quad \omega_2 = u^2 u_x^{-3} u_x x, \]
\[ A^{10}_{3.2} = \langle -x \partial_x, \partial_x, \partial_t \rangle : \]
\[ F = u_x^{-4} F(t, \omega_1, \omega_2), \quad G = G(t, \omega_1, \omega_2), \]
\[ \omega_1 = u_x^{-2} u_x x, \quad \omega_2 = u_x^{-3} u_x x, \]
\[ A^{11}_{3.2} = \langle -x \partial_x, \partial_x, \partial_u \rangle : \]
\[ F = u_x^{-4} F(u, \omega_1, \omega_2), \quad G = G(u, \omega_1, \omega_2), \]
\[ \omega_1 = u_x^{-2} u_x x, \quad \omega_2 = u_x^{-3} u_x x. \]

### 4.2.2 Three-dimensional non-decomposable algebras

The list of inequivalent three-dimensional non-decomposable Lie algebras consists of the following seven Lie algebras (only nonzero commutation relations are given),

- \[ A_{3.3} : \ [V_2, V_3] = V_1; \]
- \[ A_{3.4} : \ [V_1, V_3] = V_1, \quad [V_2, V_3] = V_1 + V_2; \]
- \[ A_{3.5} : \ [V_1, V_3] = V_1, \quad [V_2, V_3] = V_2; \]
- \[ A_{3.6} : \ [V_1, V_3] = V_1, \quad [V_2, V_3] = -V_2; \]
- \[ A_{3.7} : \ [V_1, V_3] = V_1, \quad [V_2, V_3] = qV_2, \quad (0 < |q| < 1); \]
- \[ A_{3.8} : \ [V_1, V_3] = -V_2, \quad [V_2, V_3] = V_1; \]
- \[ A_{3.9} : \ [V_1, V_3] = qV_1 - V_2, \quad [V_2, V_3] = V_1 + qV_2, \quad (q > 0). \]

All these algebras contain a two-dimensional Abelian ideal as a subalgebra. Thus, we can utilize results of classification of \( A_{2.1} \) invariant equations to construct Eq. (11) which admit non-decomposable three-dimensional solvable Lie algebras. As a result, we arrive at the following invariance equations:

- \[ A_{3.3} \] invariant equations,

\[ A^1_{3.3} = \langle \partial_u, \partial_t, \partial_x + t \partial_u \rangle : \]
\[ F = F(u_x, u_{xx}, u_{xxx}), \quad G = x + G(u_x, u_{xx}, u_{xxx}), \]
\[ A^2_{3.3} = \langle \partial_u, \partial_t, (t + x) \partial_u \rangle : \]
\[ F = F(x, u_{xx}, u_{xxx}), \quad G = u_x + G(x, u_{xx}, u_{xxx}), \]
\[ A^3_{3.3} = \langle \partial_u, \partial_x, t \partial_x + x \partial_u \rangle : \]
\[ F = F(t, u_{xx}, u_{xxx}), \quad G = -\frac{u_x^2}{2} + G(t, u_{xx}, u_{xxx}), \]

\[ A_{3,3}^4 = \{ \partial_u, \partial_x, \partial_t + x\partial_u \} : \]
\[ F = F(u_x - t, u_{xx}, u_{xxx}), \quad G = G(u_x - t, u_{xx}, u_{xxx}), \]

\[ A_{3,3}^5 = \{ \partial_u, x\partial_u, -\partial_x \} : \]
\[ F = F(t, u_{xx}, u_{xxx}), \quad G = G(t, u_{xx}, u_{xxx}), \]

\[ A_{3,3}^6 = \{ \partial_u, x\partial_u, \partial_t - \partial_x \} : \]
\[ F = F(x + t, u_{xx}, u_{xxx}), \quad G = G(x + t, u_{xx}, u_{xxx}), \]

\[ A_{3,3}^7 = \{ x\partial_u, \partial_u, x^2\partial_x + xu\partial_u \} : \]
\[ F = x^8 F(t, \omega_1, \omega_2), \quad G = 4x^6(3u_{xx} + 2u_{xxx})F(t, \omega_1, \omega_2) + xG(t, \omega_1, \omega_2), \]
\[ \omega_1 = x^3u_{xx}, \quad \omega_2 = x^4(3u_{xx} + xu_{xxx}), \]

\[ A_{3,3}^8 = \{ x\partial_u, \partial_u, \partial_t + x^2\partial_x + xu\partial_u \} : \]
\[ F = x^8 F(\omega_1, \omega_2, \omega_3), \quad G = 4x^6(3u_{xx} + 2u_{xxx})F(\omega_1, \omega_2, \omega_3) - xG(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = t + x^{-1}, \quad \omega_2 = x^3u_{xx}, \quad \omega_3 = x^4(3u_{xx} + xu_{xxx}). \]

**A\underline{3.4}– invariant equations,**

\[ A_{3,4}^1 = \{ \partial_u, \partial_t, t\partial_t + \partial_x + (t + u)\partial_u \} : \]
\[ F = e^{-x} F(\omega_1, \omega_2, \omega_3), \quad G = x + G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = e^{-x}u_x, \quad \omega_2 = e^{-x}u_{xx}, \quad \omega_3 = e^{-x}u_{xxx}, \]

\[ A_{3,4}^2 = \{ \partial_u, \partial_t, t\partial_t + (t + u)\partial_u \} : \]
\[ F = u_x^{-1} F(x, \omega_1, \omega_2), \quad G = \ln |u_x| + G(x, \omega_1, \omega_2), \]
\[ \omega_1 = u_x^{-1}u_{xx}, \quad \omega_2 = u_x^{-1}u_{xxx}, \]

\[ A_{3,4}^3 = \{ \partial_u, \partial_x, \partial_t + x\partial_x + (x + u)\partial_u \} : \]
\[ F = e^t F(\omega_1, \omega_2, \omega_3), \quad G = e^tG(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = u_x - t, \quad \omega_2 = e^tu_{xx}, \quad \omega_3 = e^{2t}u_{xxx}, \]

\[ A_{3,4}^4 = \{ \partial_u, \partial_x, x\partial_x + (x + u)\partial_u \} : \]
\[ F = e^{4u_x} F(t, \omega_1, \omega_2), \quad G = e^{4u_x}G(t, \omega_1, \omega_2), \]
\[ \omega_1 = e^{u_x}u_{xx}, \quad \omega_2 = e^{2u_x}u_{xxx}. \]
\[ A_{3.4}^5 = (\partial_u, x \partial_u, -\partial_x + u \partial_x) : \\
F = F(t, \omega_1, \omega_2), \quad G = e^{-x} G(t, \omega_1, \omega_2), \\
\omega_1 = e^x u_x, \quad \omega_2 = e^x u_{xx}, \]

\[ A_{3.4}^6 = (\partial_u, x \partial_u, \partial_t - \partial_x + u \partial_u) : \\
F = F(\omega_1, \omega_2, \omega_3), \quad G = e^t G(\omega_1, \omega_2, \omega_3), \\
\omega_1 = x + t, \quad \omega_2 = e^{-t} u_x, \quad \omega_3 = e^{-t} u_{xx}, \]

\[ A_{3.4}^7 = (x \partial_u, \partial_u, x^2 \partial_x + (1 + x) u \partial_u) : \\
F = x^8 F(t, \omega_1, \omega_2), \quad G = 4x^6(3u_x + 2x u_{xxx}) F(t, \omega_1, \omega_2) + x e^{-\frac{t}{x}} G(t, \omega_1, \omega_2), \\
\omega_1 = x^3 e^\frac{t}{x} u_x, \quad \omega_2 = x^4 e^\frac{t}{x} (3u_x + x u_{xxx}), \]

\[ A_{3.4}^8 = (x \partial_u, \partial_u, \partial_t + x^2 \partial_x + (1 + x) u \partial_u) : \\
F = x^8 F(\omega_1, \omega_2, \omega_3), \quad G = 4x^6(3u_x + 2x u_{xxx}) F(\omega_1, \omega_2, \omega_3) - x e^t G(\omega_1, \omega_2, \omega_3), \\
\omega_1 = t + \frac{1}{x}, \quad \omega_2 = x^3 e^{-t} u_x, \quad \omega_3 = x^4 e^{-t} (3u_x + x u_{xxx}). \]
\[ F = F(x, \omega_1, \omega_2), \quad G = e^t G(x, \omega_1, \omega_2), \]
\[ \omega_1 = e^{-t} u_{xx}, \quad \omega_2 = e^{-t} u_{xxx}. \]

\[ A_{3.6} \text{-- invariant equations,} \]
\[ A^1_{3.6} = \langle \partial_t, \partial_u, t \partial_t - u \partial_u \rangle : \]
\[ F = u_x F(x, u^{-1} x u_{xx}, u^{-1} x u_{xxx}), \quad G = u_x^2 G(x, u^{-1} x u_{xx}, u^{-1} x u_{xxx}), \]
\[ \omega_1 = e^{-x} u_x, \quad \omega_2 = e^{-x} u_{xx}, \omega_3 = e^{-x} u_{xxx}, \]
\[ A^2_{3.6} = \langle \partial_t, \partial_u, t \partial_t + \partial_x - u \partial_u \rangle : \]
\[ F = e^{-t} F(\omega_1, \omega_2, \omega_3), \quad G = e^{-2t} G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = e^t u_x, \quad \omega_2 = e^t u_{xx}, \omega_3 = e^t u_{xxx}, \]
\[ A^3_{3.6} = \langle \partial_x, \partial_u, x \partial_x - u \partial_u \rangle : \]
\[ F = u_x^{-2} F(t, \omega_1, \omega_2), \quad G = u_x^4 G(t, \omega_1, \omega_2), \]
\[ \omega_1 = u_x^{-\frac{3}{2}} u_{xx}, \quad \omega_2 = u_x^{-2} u_{xxx}, \]
\[ A^4_{3.6} = \langle \partial_x, \partial_u, \partial_t + \partial_x - u \partial_u \rangle : \]
\[ F = e^{4t} F(\omega_1, \omega_2, \omega_3), \quad G = e^{-4t} G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = e^{2t} u_x, \quad \omega_2 = e^{3t} u_{xx}, \omega_3 = e^{4t} u_{xxx}, \]
\[ A^5_{3.6} = \langle \partial_x, \partial_u, x \partial_x - u \partial_u \rangle : \]
\[ F = x^4 F(t, \omega_1, \omega_2), \quad G = x^\frac{7}{2} G(t, \omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{3}{2}} u_{xx}, \quad \omega_2 = x^{\frac{5}{2}} u_{xxx}, \]
\[ A^6_{3.6} = \langle \partial_u, x \partial_u, \partial_t + 2x \partial_x + u \partial_u \rangle : \]
\[ F = e^{8t} F(\omega_1, \omega_2, \omega_3), \quad G = e^{t} G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = e^{-2t} x, \quad \omega_2 = e^{3t} u_{xx}, \omega_3 = e^{5t} u_{xxx}. \]

\[ A_{3.7} \text{-- invariant equations,} \]
\[ A^1_{3.7} = \langle \partial_t, \partial_u, t \partial_t + qu \partial_u \rangle : \]
\[ F = u_x^{-\frac{1}{q}} F(x, u_x^{-1} u_{xx}, u_x^{-1} u_{xxx}), \quad F = u_x^{1-\frac{1}{q}} G(x, u_x^{-1} u_{xx}, u_x^{-1} u_{xxx}), \]
\[ A^2_{3.7} = \langle \partial_t, \partial_u, t \partial_t + \partial_x + qu \partial_u \rangle : \]
\[ F = e^{-x}F(\omega_1, \omega_2, \omega_3), \quad G = e^{(q-1)x}G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = e^{-qx}u_x, \quad \omega_2 = e^{-qx}u_{xx}, \quad \omega_3 = e^{-qx}u_{xxx}, \]

\[ A_{3,7}^3 = (\partial_x, \partial_u, x\partial_x + qu\partial_u) : \]
\[ F = u_x^{\frac{1}{q}} F(t, \omega_1, \omega_2), \quad G = u_x^{\frac{1}{q}} G(t, \omega_1, \omega_2), \]
\[ \omega_1 = u_x^{\frac{3}{q}} u_{xx}, \quad \omega_2 = u_x^{\frac{3}{q}} u_{xxx}, \]

\[ A_{3,7}^4 = (\partial_x, \partial_u, \partial_t + x\partial_x + qu\partial_u) : \]
\[ F = e^{qt} F(\omega_1, \omega_2, \omega_3), \quad G = e^{qt} G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = e^{(1-q)t} u_x, \quad \omega_2 = e^{(2-q)t} u_{xx}, \quad \omega_3 = e^{(3-q)t} u_{xxx}, \]

\[ A_{3,7}^5 = (\partial_x, u\partial_x, x\partial_x + (1-q)u\partial_u) : \]
\[ F = x^4 F(t, \omega_1, \omega_2), \quad G = x^{\frac{2}{q-1}} G(t, \omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{3q-2}{q-1}} u_{xx}, \quad \omega_2 = x^{\frac{3q-2}{q-1}} u_{xxx}, \]

\[ A_{3,7}^6 = (\partial_x, u\partial_x, \partial_t + x\partial_x + (1-q)u\partial_u) : \]
\[ F = e^{s(1-q)t} F(\omega_1, \omega_2, \omega_3), \quad G = e^{s} G(\omega_1, \omega_2, \omega_3), \]
\[ \omega_1 = e^{(q-1)t} x, \quad \omega_2 = e^{(1-2q)t} u_{xx}, \quad \omega_3 = e^{(2-3q)t} u_{xxx}. \]

\[ A_{3,8} - \text{invariant equations,} \]

\[ A_{3,8}^1 = (\partial_x, \partial_u, u\partial_x - x\partial_u) : \]
\[ F = (1 + u_x^2)^{-2} F(t, \omega_1, \omega_2), \]
\[ G = -5u_x u_{xx}(1 + u_x^2)^{-4} (2u_x^2 u_{xxx} - 3u_x^2 u_{xx}^2 + 2u_{xxx}^2) F(t, \omega_1, \omega_2) \]
\[ + \frac{2}{3} (1 + u_x^2)^{-2} G(t, \omega_1, \omega_2), \]
\[ \omega_1 = (1 + u_x^2)^{-\frac{3}{2}} u_{xx}, \quad \omega_2 = (1 + u_x^2)^{-\frac{3}{2}} (u_x^2 u_{xxx} - 3u_x^2 u_{xx}^2 + u_{xxx}), \]

\[ A_{3,8}^2 = (\partial_x, \partial_u, \partial_t + u\partial_x - x\partial_u) : \]
\[ F = (1 + u_x^2)^{-2} F(\omega_1, \omega_2, \omega_3), \]
\[ G = \frac{u_{xxx}}{(1 + u_x^2)^{\frac{3}{4}}} [6u_{xx}^2 \arctan^2 u_x + (2u_x^2 u_{xxx} + 12tu_x^2 - 6u_x^2 u_{xx}^2 + 2u_{xxx}) \arctan u_x \]
\[ + 6t^2 u_x^2 - 6tu_x^2 u_{xx} + 2tu_x^2 u_{xxx} + 3u_x^2 u_{xx}^2 + 2tu_{xxx} - 2u_x^3 u_{xxx} - 2u_x u_{xxx}] \]
\[ F(\omega_1, \omega_2, \omega_3) + (1 + u_x^2)^{\frac{1}{2}} G(\omega_1, \omega_2, \omega_3), \]

\[ \omega_1 = \arctan u_x + t, \ \omega_2 = (1 + u_x^2)^{-\frac{3}{4}} u_{xx}, \]

\[ \omega_3 = (1 + u_x^2)^{-3}(u_x^2 u_{xxx} - 3u_x u_{xx}^2 + u_{xxx} + 3u_x^2 \omega_1), \]

\[ A_{3,8}^3 = (\partial_u, x \partial_u, -(x^2 + 1) \partial_x - xu \partial_u) : \]

\[ F = (1 + x^2)^4 F(t, \omega_1, \omega_2), \]

\[ G = 4x(1 + x^2)^2(2x^2 u_{xxx} + 3xu_{xx} + 2u_{xxx}) F(t, \omega_1, \omega_2) + (1 + x^2)^{\frac{1}{4}} G(t, \omega_1, \omega_2), \]

\[ \omega_1 = (1 + x^2)^{\frac{3}{4}} u_{xx}, \ \omega_2 = (1 + x^2)^{\frac{1}{4}} (x^2 u_{xxx} + 3xu_{xx} + u_{xxx}), \]

\[ A_{3,8}^4 = (\partial_u, x \partial_u, \partial_t - (x^2 + 1) \partial_x - xu \partial_u) : \]

\[ F = (1 + x^2)^4 F(\omega_1, \omega_2, \omega_3), \]

\[ G = 4x(1 + x^2)^2 u_{xx}(2x^2 u_{xxx} + 3xu_{xx} + 2u_{xxx}) F(\omega_1, \omega_2, \omega_3) + (1 + x^2)^{\frac{1}{4}} G(\omega_1, \omega_2, \omega_3), \]

\[ \omega_1 = t + \arctan x, \ \omega_2 = (1 + x^2)^{\frac{3}{4}} u_{xx}, \ \omega_3 = (1 + x^2)^{\frac{1}{4}} u_{xxx} - 3\omega_2(\omega_1 - x). \]

\[ A_{3,9} - \text{invariant equations}, \]

\[ A_{3,9}^1 = (\partial_x, \partial_u, (qx + u) \partial_x + (qu - x) \partial_u) : \]

\[ F = (1 + u_x^2)^{-2} e^{-4q \arctan u_x} F(t, \omega_1, \omega_2), \]

\[ G = -5u_x u_{xxx}(1 + u_x^2)^{-4} e^{-4q \arctan u_x} (2u_x^2 u_{xxx} - 3u_x u_{xx}^2 + 2u_{xxx}) F(t, \omega_1, \omega_2) + (1 + u_x^2)^{\frac{1}{8}} e^{-q \arctan u_x} G(t, \omega_1, \omega_2), \]

\[ \omega_1 = (1 + u_x^2)^{-\frac{3}{4}} u_{xx}, \ \omega_2 = (1 + u_x^2)^{-3}(u_x^2 u_{xxx} - 3u_x u_{xx}^2 + u_{xxx}), \]

\[ A_{3,9}^2 = (\partial_x, \partial_u, \partial_t + (qx + u) \partial_x + (qu - x) \partial_u) : \]

\[ F = (1 + u_x^2)^{-2} e^{4qt} \omega_1, \omega_2, \omega_3), \]

\[ G = -5u_x u_{xxx}(1 + u_x^2)^{-4} e^{4qt} (2u_x^2 u_{xxx} - 3u_x u_{xx}^2 + 2u_x u_{xxx} + 3u_x^2) F(\omega_1, \omega_2, \omega_3) + (1 + u_x^2)^{\frac{1}{8}} e^{qt} G(\omega_1, \omega_2, \omega_3), \]

\[ \omega_1 = t + \arctan u_x, \ \omega_2 = (1 + u_x^2)^{-\frac{3}{4}} e^{qt} u_{xx}, \]

\[ \omega_3 = (1 + u_x^2)^{-3} e^{2qt} (u_x^2 u_{xxx} - 3u_x u_{xx}^2 + u_{xxx}), \]

\[ A_{3,9}^3 = (\partial_u, x \partial_u, -(x^2 + 1) \partial_x + (q - x) u \partial_u) : \]

\[ F = (x^2 + 1)^4 F(t, \omega_1, \omega_2), \]
\[G = 4x(x^2 + 1)^2[2(1 + x^2)u_{xxx} + 3xu_{xx}]F(t, \omega_1, \omega_2) + (x^2 + 1)^\frac{3}{2}e^{-q\arctan x}G(t, \omega_1, \omega_2),\]

\[\omega_1 = (x^2 + 1)^\frac{3}{2}e^{q\arctan x}u_{xx},\]

\[\omega_2 = (1 + u_x^2)^\frac{3}{2}e^{q\arctan x}(x^2u_{xxx} + 3xu_{xx} + u_{xxx}),\]

\[A^4_{3,9} = \langle \partial_x, u\partial_x, \partial_t + (q - u)x\partial_x - (u^2 + 1)\partial_u \rangle: \]

\[F = (u^2 + 1)^4u_x^{-4}F(\omega_1, \omega_2, \omega_3),\]

\[G = [-5(u^2 + 1)^4u_x^{-6}u_{xxx}(2u_xu_{xxx} - 3u_{xx}^2) + 8u(u^2 + 1)^3u_x^{-3}u_{xx}(u_{xxx}u_{xxx} - 3u_{xx}^2)\]

\[+ 12(u^2 - 1)(u^2 + 1)^2u_x^{-2}u_{xx}]F(\omega_1, \omega_2, \omega_3) + (u^2 + 1)^\frac{3}{2}e^{q't}u_xG(\omega_1, \omega_2, \omega_3),\]

\[\omega_1 = t + \arctan u, \quad \omega_2 = (u^2 + 1)^\frac{3}{2}e^{-q't}u_x^{-3}u_{xx},\]

\[\omega_3 = (u^2 + 1)^\frac{3}{2}e^{-q't}u_x^{-5}(u_{xxx}u_{xxx} - 3u_{xx}^2) + 3u(u^2 + 1)^\frac{3}{2}e^{-q't}u_x^{-3}u_{xx}.\]

### 4.3 Equations invariant under four-dimensional solvable Lie algebras

Now we perform group classification of Eq. (1) admitting four-dimensional solvable Lie algebras. To this end, we utilize the well known classification of abstract four-dimensional Lie algebras as well as the above obtained results on three-dimensional solvable algebras. For the sake of convenience we treat the cases of decomposable and non-decomposable algebras separately. Note that we skip quite cumbersome calculation details presenting the final results, symmetry algebras and invariant equations, only.

#### 4.3.1 Equations with four-dimensional decomposable algebras

The list of non-isomorphic four-dimensional decomposable Lie algebras contains the following ten algebras:

\[A_{2,2} \oplus A_{2,2} = 2A_{2,2},\]

\[A_{3,1} \oplus A_1 = 4A_1,\]

\[A_{3,2} \oplus A_1 = A_{2,2} \oplus 2A_1,\]

\[A_{3,i} \oplus A_1 (i = 3, 4, \ldots, 9).\]

The exhaustive list of equations (1) admitting one of the above algebras is given in Appendix A.
4.3.2. Equations with four-dimensional non-decomposable algebras

There exist ten non-isomorphic four-dimensional non-decomposable Lie algebras, \( A_{4,i} \) \( (i = 1, 2, \cdots, 10) \):

\[
A_{4,1} : \quad [X_2, X_4] = X_1, \quad [X_3, X_4] = X_2;
\]

\[
A_{4,2} : \quad [X_1, X_4] = qX_1, \quad [X_2, X_4] = X_2, \quad [X_3, X_4] = X_2 + X_3, \quad q \neq 0;
\]

\[
A_{4,3} : \quad [X_1, X_4] = X_1, \quad [X_3, X_4] = X_2;
\]

\[
A_{4,4} : \quad [X_1, X_4] = X_1, \quad [X_2, X_4] = X_1 + X_2, \quad [X_3, X_4] = X_2 + X_3;
\]

\[
A_{4,5} : \quad [X_1, X_4] = X_1, \quad [X_2, X_4] = qX_2, \quad [X_3, X_4] = pX_3,
\]

\[-1 \leq p \leq q \leq 1, \quad pq \neq 0;\]

\[
A_{4,6} : \quad [X_1, X_4] = qX_1, \quad [X_2, X_4] = pX_2 - X_3, \quad [X_3, X_4] = X_2 + pX_3,
\]

\[q \neq 0, \quad p \geq 0;\]

\[
A_{4,7} : \quad [X_2, X_3] = X_1, \quad [X_1, X_4] = 2X_1, \quad [X_2, X_4] = X_2,
\]

\[ [X_3, X_4] = X_2 + X_3; \]

\[
A_{4,8} : \quad [X_2, X_3] = X_1, \quad [X_1, X_4] = (1 + q)X_1, \quad [X_2, X_4] = X_2,
\]

\[ [X_3, X_4] = qX_3, \quad |q| \leq 1; \]

\[
A_{4,9} : \quad [X_2, X_3] = X_1, \quad [X_1, X_4] = 2qX_1, \quad [X_2, X_4] = qX_2 - X_3,
\]

\[ [X_3, X_4] = X_2 + qX_3, \quad q \geq 0; \]

\[
A_{4,10} : \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_2, \quad [X_1, X_4] = -X_2, \quad [X_2, X_4] = X_1.
\]

Each of the above algebras can be decomposed into a semi-direct sum of a three-dimensional ideal \( N \) and a one-dimensional Lie algebra. Analysis of the commutation relations above shows that \( N \) is of the type \( A_{3,1} \) for the algebras \( A_{4,i} \) \( (i = 1, 2, \cdots, 6) \), of the type \( A_{3,3} \) for the algebras \( A_{4,7}, A_{4,8}, A_{4,9} \), and of the type \( A_{3,5} \) for the algebra \( A_{4,10} \). Thus we can utilize the already known realizations of three-dimensional solvable Lie algebras to obtain exhaustive descriptions of the four-dimensional non-decomposable solvable Lie algebras admitted by Eq. (1). The full list of inequivalent symmetry algebras and the corresponding invariant equations is given in Appendix B.

5. Classification of equations admitting quasi-local symmetries

Following the scheme developed in [20] we make use of local (Lie) symmetries of the fourth-order evolution equations (11) to obtain their non-local symmetries.
As established earlier, the most general infinitesimal operator, \( V \), admitted by evolution equation (1) reads as

\[
V = \tau(t) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u
\]

and the maximal equivalence group of Eq. (1) takes the form

\[
\bar{t} = T(t), \quad \bar{x} = X(t, x, u), \quad \bar{u} = U(t, x, u),
\]

where \( \dot{T} \neq 0 \) and \( \frac{D(X,U)}{D(x,u)} \neq 0 \).

If \( \tau = 0 \), we have \( V = \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u \) and there is an equivalence transformation \((t, x, u) \rightarrow (\bar{t}, \bar{x}, \bar{u})\) that reduces \( V \) to the canonical form \( \partial_u \) (we drop the bars). Eq. (1) now becomes

\[
u_t = F(t, x, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, x, u, u_x, u_{xx}, u_{xxx}) \tag{11}
\]

Note that the right-hand side of Eq. (11) does not depend explicitly on \( u \).

Differentiating (11) with respect to \( x \) yields

\[
u_{tx} = Fu_{xxxx} + \left[(F_x + F_{ux}u_x + F_{uxx}u_{xx} + F_{uxxx}u_{xxx})u_{xxx}
+ G_x + G_{ux}u_x + G_{uxx}u_{xx} + G_{uxxx}u_{xxx}\right].
\]

Making the non-local change of variables

\[
\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u_x \tag{12}
\]

and dropping the bars, we finally get

\[
u_t = Fu_{xxxx} + \left[(F_x + F_{ux}u_x + F_{uxx}u_{xx} + F_{uxxx}u_{xxx})u_{xxx}
+ G_x + G_{ux}u_x + G_{uxx}u_{xx} + G_{uxxx}u_{xxx}\right]. \tag{13}
\]

where \( F = F(t, x, u, u_x, u_{xx}) \) and \( G = G(t, x, u, u_x, u_{xx}) \).

Thus the non-local transformation (12) preserves the differential structure of the class of evolution equation (11).

Assume that Eq. (11) admits \( r \)-parameter Lie transformation group

\[
t' = T(t, \bar{\theta}), \quad x' = X(t, x, u, \bar{\theta}), \quad u' = U(t, x, u, \bar{\theta}) \tag{14}
\]

with the vector of group parameters \( \bar{\theta} = (\theta_1, \ldots, \theta_r) \) and \( r \geq 2 \). To obtain the symmetry group of Eq. (13), we need to transform (14) according to (12). Computing the first prolongation of formulas (13), we derive the transformation rule for the first derivative of \( u \)

\[
u'_{x'} = \frac{U_ux_x + U_x}{X_au_x + X_x}.
\]
The transformation group (14) now reads as

\[ t' = T(t, \bar{\theta}), \quad x' = X(t, x, v, \bar{\theta}), \quad u' = \frac{U_v u + U_x}{X_v u + X_x}. \] (15)

with \( v = \partial^{-1} u \) and \( U = U(t, x, v, \bar{\theta}) \). Consequently, Eq. (13) admits symmetry group (15). Provided the right-hand sides of the transformation group (15) depend explicitly on the non-local variable \( v \), then (15) is a quasi-local symmetry of Eq. (13).

Evidently, Eq. (13) admits quasi-local symmetry iff transformation (15) satisfies

\[ X_v \neq 0 \quad \text{or} \quad \frac{\partial}{\partial v} \left( \frac{U_v u + U_x}{X_v u + X_x} \right) \neq 0, \]

or, equivalently

\[ X_v \neq 0 \quad \text{or} \quad X_v = 0, \quad U_{xv}^2 + U_{vv}^2 \neq 0. \]

It is straightforward to express the above constraints in terms of the coefficients of the corresponding infinitesimal operator of group (14) and obtain the following assertion.

**Theorem 6.** Evolution equation (11) can be transformed to Eq. (13) admitting a quasi-local symmetry if (11) admits a Lie symmetry whose infinitesimal generator satisfies one of the following inequalities:

\[ \xi_u \neq 0, \quad (16) \]

\[ \xi_u = 0, \quad \eta_{xu}^2 + \eta_{uu}^2 \neq 0. \quad (17) \]

By making a hodograph transformation

\[ \bar{t} = t, \quad \bar{x} = u, \quad \bar{u} = x, \]

and dropping the bars, the evolution equation

\[ u_t = F(t, u, u_x, u_{xx}, u_{xxx}) u_{xxxx} + G(t, u, u_x, u_{xx}, u_{xxx}) \] (18)

can be transformed to an equation of the form (11). Hence we can get the following assertion.

**Corollary 1.** Equation (18) can be reduced to an equation having a quasi-local symmetry if Eq. (18) admits a Lie symmetry satisfying one of the inequalities

\[ \eta_x \neq 0, \]

or

\[ \eta_x = 0, \quad \xi_{xu}^2 + \xi_{uu}^2 \neq 0. \]

Based on the above reasonings, now we can formulate the algorithm for constructing evolution equations of the form (1) admitting a quasi-local symmetry.
1. Select all invariant equations, whose invariance algebras contain at least one operator of the form \( V = \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u \).

2. For each of these equations, make a suitable local equivalence transformation reducing \( V \) to the canonical form \( \partial_u \), the original equations being transformed to evolution equations of the form (11).

item For each Lie symmetry of invariance algebra admitted by (11) check whether its infinitesimal generator satisfies one of conditions (16), (17) of Theorem 6. This analysis yields the list of evolution equations (11) that can be reduced to those admitting quasi-local symmetries.

item Performing the non-local change of variables (12) transforms Eq. (11) to (13) which has quasi-local symmetries (15).

We processes in this way all the invariant equations obtained in Sections 3, 4 and thus obtain the list of fourth-order evolution equations (11) admitting quasi-local symmetries. Here we give the symmetry algebras of the corresponding invariant equations omitting the expressions for the functions \( F \) and \( G \) equations for brevity.

Semi-simple Lie algebras:

\[ sl^3(2, \mathbb{R}) = \langle 2x \partial_x - u \partial_u, -x^2 \partial_x + xu \partial_u, \partial_x \rangle, \]
\[ sl^4(2, \mathbb{R}) = \langle 2x \partial_x, -x^2 \partial_x, \partial_x \rangle, \]
\[ sl^5(2, \mathbb{R}) = \langle 2x \partial_x - u \partial_u, (\frac{1}{u^4} - x^2) \partial_x + xu \partial_u, \partial_x \rangle, \]
\[ sl^6(2, \mathbb{R}) = \langle 2x \partial_x - u \partial_u, -(x^2 + \frac{1}{u^4}) \partial_x + xu \partial_u, \partial_x \rangle. \]

Three-dimensional solvable Lie algebras:

\[ A_{3,3}^7 = \langle x \partial_u, \partial_u, x^2 \partial_x + xu \partial_u \rangle, \]
\[ A_{3,3}^8 = \langle x \partial_u, \partial_u, \partial_t + x^2 \partial_x + xu \partial_u \rangle, \]
\[ A_{3,4}^7 = \langle x \partial_u, \partial_u, x^2 \partial_x + (1 + x)u \partial_u \rangle, \]
\[ A_{3,4}^8 = \langle x \partial_u, \partial_u, \partial_t + x^2 \partial_x + (1 + x)u \partial_u \rangle, \]
\[ A_{3,8}^1 = \langle \partial_x, \partial_u, u \partial_x - x \partial_u \rangle, \]
\[ A_{3,8}^2 = \langle \partial_x, \partial_u, \partial_t + u \partial_x - x \partial_u \rangle, \]
\[ A_{3,8}^3 = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x - xu \partial_u \rangle. \]
\[
A_{3,8}^{4} = \langle \partial_u, x \partial_u, \partial_t - (x^2 + 1) \partial_x - xu \partial_u \rangle,
A_{3,9}^{4} = \langle \partial_x, \partial_u, (qu + u) \partial_x + (qu - x) \partial_u \rangle,
A_{3,9}^{2} = \langle \partial_x, \partial_u, \partial_t + (qx + u) \partial_x + (qu - x) \partial_u \rangle,
A_{3,9}^{3} = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x + (q - x)u \partial_u \rangle,
A_{3,9}^{4} = \langle \partial_u, xu \partial_x, - (u^2 + 1) \partial_x - (u^2 + 1) \partial_u \rangle.
\]

Four-dimensional solvable Lie algebras:

\[
2A_{2,2}^{1} = \langle -t \partial_t - u \partial_u, \partial_t, \partial_x, u e^x \partial_u \rangle,
2A_{2,2}^{2} = \langle -t \partial_t - x \partial_x, \partial_t, \partial_u, e^x \partial_u \rangle,
2A_{2,2}^{3} = \langle -t \partial_t - u \partial_u, \partial_t, \partial_x, e^x \partial_x + \alpha u e^x \partial_u \rangle, \ \alpha \neq 0,
2A_{2,2}^{11} = \langle -t \partial_t - x \partial_x, \partial_t, \partial_u, e^x \partial_u \rangle,
2A_{2,2}^{12} = \langle -t \partial_t - u \partial_u, \partial_t, \partial_x, \alpha e^x \partial_x + te^x \partial_u \rangle, \ \alpha \neq 0,
2A_{2,2}^{13} = \langle -t \partial_t - x \partial_x, \partial_t, \partial_u, te^u \partial_x \rangle,
2A_{2,2}^{29} = \langle -x \partial_x, \partial_x, \partial_u, te^u \partial_u \rangle,
A_{3,3}^{4} \oplus A_{1} = \langle \partial_x, \partial_u, \partial_t + u \partial_x, \partial_t + \alpha t \partial_x + \alpha \partial_u \rangle, \ \alpha \in \mathbb{R},
A_{3,3}^{5} \oplus A_{1} = \langle \partial_x, u \partial_x, - \partial_u, \partial_t \rangle,
A_{3,4}^{7} \oplus A_{1} = \langle x \partial_u, \partial_u, x^2 \partial_x + xu \partial_u, \partial_t \rangle,
A_{3,4}^{4} \oplus A_{1} = \langle \partial_x, \partial_u, (x + u) \partial_x + u \partial_u, \partial_t \rangle,
A_{3,4}^{7} \oplus A_{1} = \langle u \partial_x, \partial_x, x(1 + u) \partial_x + u^2 \partial_u, u e^{-\frac{1}{2}} \partial_x \rangle,
A_{3,4}^{7} \oplus A_{1} = \langle u \partial_x, \partial_x, x(1 + u) \partial_x + u^2 \partial_u, tue^{-\frac{1}{2}} \partial_x \rangle,
A_{3,4}^{7} \oplus A_{1} = \langle u \partial_x, \partial_x, x(1 + u) \partial_x + u^2 \partial_u, \partial_t \rangle,
A_{3,4}^{8} \oplus A_{1} = \langle u \partial_x, \partial_x, \partial_t + x(1 + u) \partial_x + u^2 \partial_u, u e^t f(t + \frac{1}{u}) \partial_x \rangle, \ f'' \neq 0,
A_{3,8}^{1} \oplus A_{1} = \langle \partial_x, \partial_u, u \partial_x - x \partial_u, \partial_t \rangle,
A_{3,8}^{3} \oplus A_{1} = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x - xu \partial_u, \partial_t \rangle,
A_{3,8}^{3} \oplus A_{1} = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x - xu \partial_u, (x^2 + 1)\frac{1}{2} \partial_u \rangle,
A_{3,8}^{3} \oplus A_{1} = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x - xu \partial_u, t(x^2 + 1)\frac{1}{2} \partial_u \rangle,
A_{3,8}^{4} \oplus A_{1} = \langle \partial_x, u \partial_x, \partial_t - xu \partial_x - (u^2 + 1) \partial_u, (1 + u^2)\frac{1}{2} f(t + \arctan u) \partial_x \rangle, \ f + f'' \neq 0,
\]
\[ A_{3,9}^1 + A_1 = \langle \partial_x, \partial_u, (qx + u)\partial_x + (qu - x)\partial_u, \partial_t \rangle, \]
\[ A_{3,9}^2 + A_1 = \langle \partial_u, x\partial_u, -(x^2 + 1)\partial_x + (q - x)u\partial_u, \partial_t \rangle, \]
\[ A_{3,9}^3 + A_1 = \langle \partial_u, x\partial_u, -(x^2 + 1)\partial_x + (q - x)u\partial_u, (x^2 + 1)\frac{1}{2}e^{-q\arctan x}\partial_u \rangle, \]
\[ A_{3,9}^4 + A_1 = \langle \partial_u, x\partial_u, -(x^2 + 1)\partial_x + (q - x)u\partial_u, t(x^2 + 1)\frac{1}{2}e^{-q\arctan x}\partial_u \rangle, \]
\[ A_{4,1}^1 = \langle \partial_x, \partial_u, \partial_t, x^2\partial_x + (t + xu)\partial_u \rangle, \]
\[ A_{4,2}^1 = \langle \partial_t, \partial_u, \partial_x, qt\partial_t + x\partial_x + (x + u)\partial_u \rangle, \]
\[ A_{4,2}^2 = \langle \partial_t, x\partial_u, \partial_u, qt\partial_t + x^2\partial_x + (x + 1)u\partial_u \rangle, \]
\[ A_{4,3}^1 = \langle \partial_t, \partial_x, \partial_u, t\partial_t + x\partial_u \rangle, \]
\[ A_{4,3}^2 = \langle \partial_t, x\partial_u, \partial_u, t\partial_t + x^2\partial_x + xu\partial_u \rangle, \]
\[ A_{4,4}^1 = \langle \partial_u, \partial_x, \partial_t, t\partial_t + (t + x)\partial_x + (x + u)\partial_u \rangle, \]
\[ A_{4,4}^2 = \langle \partial_x, \partial_u, \partial_t, t\partial_t + x^2\partial_x + [t + (x + 1)u]\partial_u \rangle, \]
\[ A_{4,6}^1 = \langle \partial_t, \partial_x, \partial_u, qt\partial_t + (px + u)\partial_x + (pu - x)\partial_u \rangle, \]
\[ A_{4,6}^2 = \langle \partial_t, x\partial_u, \partial_u, qt\partial_t - (x^2 + 1)\partial_x + (p - x)u\partial_u \rangle, \]
\[ A_{4,7}^1 = \langle \partial_u, \partial_x, \partial_t, x\partial_u, -\partial_t + x\partial_x + 2u\partial_u \rangle, \]
\[ A_{4,7}^2 = \langle \partial_u, \partial_x, \partial_t, x\partial_u, t\partial_t + (t + x)\partial_x + \left(\frac{t^2}{2} + 2u\right)\partial_u \rangle, \]
\[ A_{4,7}^3 = \langle \partial_u, x\partial_u, -\partial_x, x\partial_x + (u - \frac{x^2}{2})\partial_u \rangle, \]
\[ A_{4,7}^4 = \langle \partial_u, x\partial_u, -\partial_x, \partial_t + x\partial_x + (u - \frac{x^2}{2})\partial_u \rangle, \]
\[ A_{4,7}^6 = \langle x\partial_u, \partial_u, x^2\partial_x + xu\partial_u, -x\partial_x + (u - \frac{1}{2x})\partial_u \rangle, \]
\[ A_{4,7}^7 = \langle x\partial_u, \partial_u, x^2\partial_x + xu\partial_u, \partial_t - x\partial_x + (u - \frac{1}{2x})\partial_u \rangle, \]
\[ A_{4,7}^8 = \langle x\partial_u, \partial_u, \partial_t + x^2\partial_x + xu\partial_u, t\partial_t - x\partial_x + (t + u + \frac{t^2}{2})\partial_u \rangle, \]
\[
\begin{align*}
A^3_{4,8} &= \langle \partial_u, \partial_x, t\partial_x + x\partial_u, (1 - q)t\partial_t + x\partial_x + (1 + q)u\partial_u \rangle, \\ A^4_{4,8} &= \langle \partial_u, \partial_x, t\partial_x + x\partial_u, x\partial_x + 2u\partial_u \rangle, \\ A^5_{4,8} &= \langle \partial_u, \partial_x, \partial_t + x\partial_u, qt\partial_t + x\partial_x + (1 + q)u\partial_u \rangle, \\ A^6_{4,8} &= \langle \partial_u, \partial_x, \partial_t + x\partial_u, x\partial_x + u\partial_u \rangle, \\ A^7_{4,8} &= \langle \partial_u, x\partial_u, -\partial_x, -x\partial_x + t\partial_u \rangle, \\ A^8_{4,8} &= \langle \partial_u, x\partial_u, -\partial_x, qx\partial_x + (1 + q)u\partial_u \rangle, \\ A^9_{4,8} &= \langle \partial_u, x\partial_u, -\partial_x, qt\partial_t + x\partial_x + u\partial_u \rangle, \\ A^{10}_{4,8} &= \langle \partial_u, x\partial_u, -\partial_x, x\partial_x + 2u\partial_u \rangle, \\
A^{11}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + qx\partial_x + (1 + q)u\partial_u \rangle, \\
A^{12}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + u\partial_u \rangle, \\
A^{13}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + (1 + q)u\partial_u \rangle, \\
A^{14}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + x\partial_u \rangle, \\
A^{15}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t - qx\partial_x + u\partial_u \rangle, \\
A^{16}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + (tx + u)\partial_u \rangle, \\
A^{17}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + u\partial_u \rangle, \\
A^{18}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + (1 + q)u\partial_u \rangle, \\
A^{19}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + u\partial_u \rangle, \\
A^{20}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + (1 + q)u\partial_u \rangle, \\
A^{21}_{4,8} &= \langle \partial_u, x\partial_u, \partial_t + x\partial_x + u\partial_u \rangle. \\
\end{align*}
\]

Making a suitable local equivalence transformation to reduce basis elements \( V = \xi(t, x, u)\partial_t + \eta(t, x, u)\partial_u \) of each algebra listed above to the canonical form \( \partial_u \), yields evolution equations of the form \( \Box^{\xi} \). Next, differentiating the so obtained equations with respect to \( x \) and replacing \( u_x \) with \( u \) yields an equation of the form \( \Box^{\xi+1} \) that admits a quasi-local symmetry. Here we consider two illustrative
examples.

Example 1. Consider the algebra

\[ sl^4(2, \mathbb{R}) = \{2x\partial_x, -x^2\partial_x, \partial_x\} \]

and make a hodograph transformation

\[ \bar{t} = t, \quad \bar{x} = u, \quad \bar{u} = x, \]

which transforms the original algebra to

\[ \{2u\partial_u, -u^2\partial_u, \partial_u\}. \]

Here we drop the bars. The invariant equation corresponding to the above algebra reads as

\[ u_t = F(t, x, \omega)u_{xxxx} + \frac{3u^3 - 4u_xu_{xx}u_{xxx}}{u_x^2}F(t, x, \omega) + u_xG(t, x, \omega), \]

where \( F, G \) are arbitrary smooth functions and \( \omega = (2u_xu_{xxx} - 3u_x^2)u_x^{-2} \). Differentiating the above equation with respect to \( x \) and replacing \( u_x \) with \( u \) according to (12) we arrive at the evolution equation

\[ u_t = Fu_{xxxx} + (u_{xxx} + \frac{3u^3 - 4u_xu_{xx}u_{xxx}}{u_x^2})(F_x + \sigma F_\omega) \]

\[ - \frac{4u^2(u_{xx}^2 + u_xu_{xxx}) - 13u_xu_{xx}u_x + 6u_x^4}{u_x^3}F + u_xG + uG_x + u\sigma G_\omega, \]

with \( \omega = (2u_xu_{xxx} - 3u_x^2)u_x^{-2} \) and \( \sigma = 2(u^2u_{xxx} - 4uu_xu_{xx} + 3u_x^3)u_x^{-3} \). This equation admits the non-local symmetry transformation group

\[ t' = t, \quad x' = x, \quad u' = \frac{u}{(\theta v + 1)^2}, \]

with the generator \(-2uv\partial_u\), where \( \theta \) is the group parameter and \( v = \partial^{-1}u \).

Example 2. Consider the Lie algebra

\[ 2A_{2,2}^2 = \{-t\partial_t - x\partial_x, \partial_t, \partial_u, e^u\partial_u\} \]

and its corresponding invariant equation

\[ u_t = x^3F(\omega_1, \omega_2)u_{xxxx} - x^3(u_x^4 - 6u_x^2u_{xx} + 4u_xu_{xxx} + 3u_{xx}^2)F(\omega_1, \omega_2) + u_xG(\omega_1, \omega_2), \]

where \( \omega_1 = x(u_x^2 - u_{xx})u_x^{-1}, \quad \omega_2 = x^2(u_x^3 - 3u_xu_{xx} + u_{xxx})u_x^{-1} \) and \( F, G \) are arbitrary smooth functions. Differentiating the above equation with respect to \( x \) and replacing \( u_x \) with \( u \) according to (12) yield the evolution equation

\[ u_t = x^3Fu_{xxxx} + 3x^2Fu_{xxx} + x^3(3u^4 - 6u_x^2u_{xx} + 4u_xu_{xxx} + 3u_{xx}^2)[F\omega_1\sigma_1 \]

\[ + F\omega_2\sigma_2] - x^2[3u^4 + 4xu_x^3u_x - 6(xu_{xx} + 3u_x)u^2 + (-12u_x^2 + 12u_{xx} + 4xu_{xxx})u \]

\[ + 9u_x^2 + 10xu_xu_{xx}]F + u_xG + u\sigma_1G_{\omega_1} + u\sigma_2G_{\omega_2}, \]

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with
\[ \omega_1 = \frac{x(u^2 - u_x)}{u}, \quad \omega_2 = \frac{x^2(u^3 - 3uu_x + u_{xx})}{u}, \]

and
\[ \sigma_1 = \frac{u^3 + xu_xu_x^2 - (xu_{xx} + u_x)u + xu_x^2}{u^2}, \]
\[ \sigma_2 = \frac{x[2u^4 + 2xu_xu_x^3 - 3(xu_{xx} + 2u_x)u^2 + (xu_{xxx} + 2u_{xx})u - xu_xu_{xx}]}{u^2}. \]

The new equation admits the non-local symmetry
\[ t' = t, \quad x' = x, \quad u' = \frac{u}{1 - \theta e^v}, \]

with symmetry generator
\[ e^v u \partial_u. \]

6. Concluding remarks

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Appendix A. Equations admitting decomposable four-dimensional solvable Lie algebras

2$A_{2,2^2}$ invariant equations,

$2A_{2,2}^1$ = invariant equations,

\[ 2A_{2,2}^1 = \langle -t \partial_t - u \partial_u, \partial_t, \partial_x, we^x \partial_u \rangle : \]

\[ F = \frac{e^{\omega x}}{u} F(\omega_1, \omega_2), \]
\[ G = -\frac{u_x e^u}{u^4}[u^3 + (-3u_x + 6u_{xx} + 4u_{xxx})u^2 - 6u_x(u_{xx} + u_x)u + 3u_x^3]F(\omega_1, \omega_2) + e^u G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{(u_{xx} - u_x)u - u_x^2}{u^2}, \quad \omega_2 = \frac{(u_{xxx} - u_x)u^2 - 3u_xu_{xx}u + 2u_x^3}{u^3}, \]

\[ 2A_{2,2}^2 = \langle -t\partial_t - x\partial_x, \partial_t, \partial_u, e^u\partial_u \rangle : \]
\[ F = x^3 F(\omega_1, \omega_2), \quad G = -x^3(u_x^4 - 6u_x^2u_{xx} + 4u_xu_{xxx} + 3u_{xx}^2)F(\omega_1, \omega_2) + u_x G(\omega_1, \omega_2), \]
\[ \omega_1 = xu_x^{-1}(u_x^2 - u_{xx}), \quad \omega_2 = x^2u_x^{-1}(u_x^2 - 3u_xu_{xx} + u_{xxx}), \]

\[ 2A_{2,2}^3 = \langle -t\partial_t - u\partial_u, \partial_t, \partial_x, e^\alpha \partial_x + \alpha u e^\alpha \partial_u \rangle, \quad \alpha \neq 0 : \]
\[ F = \frac{u^{\alpha+3}}{(u_x - \alpha u)^{\alpha+4}} F(\omega_1, \omega_2), \quad G = \frac{u^{\alpha+3}}{(u_x - \alpha u)^{\alpha+4}}[(\alpha^4 - 6\alpha^3 + 11\alpha^2 - 6\alpha)u - 2(2\alpha^3 - 9\alpha^2 + 11\alpha - 3)u_x \]
\[ + (6\alpha^2 - 18\alpha + 11)u_{xx} - (4\alpha - 6)u_{xxx}]F(\omega_1, \omega_2) + \frac{u^\alpha}{(u_x - \alpha u)^\alpha} G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{(u_{xx} - u_x)u - u_x^2}{u^2}, \quad \omega_2 = \frac{(u_{xxx} - u_x)u^2 - 3u_xu_{xx}u + 2u_x^3}{u^3}, \]

\[ 2A_{2,2}^4 = \langle -t\partial_t - x\partial_x, \partial_t, \alpha u \partial_x - u \partial_u, \partial_u \rangle : \]
\[ F = \frac{x^{3-\alpha}}{u^\alpha} F(\omega_1, \omega_2), \quad G = \frac{x^{1-\alpha}}{u^\alpha} G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{xu_x}{u_x}, \quad \omega_2 = \frac{x^2u_{xx}}{u_x}, \]

\[ 2A_{2,2}^5 = \langle -t\partial_t - u\partial_u, \partial_t, xu \partial_x, xu \partial_u \rangle : \]
\[ F = \frac{x^4}{u} e^{\frac{xu}{u}} F(\omega_1, \omega_2), \quad G = -\frac{x^4}{u^4} e^{\frac{xu}{u}} x_x (4u_{xxx}u^2 - 6u_xu_{xx}u + 3u_x^2)F(\omega_1, \omega_2) + e^{\frac{xu}{u}} G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{x^2(u_{xx}u - u_x^2)}{u^2}, \quad \omega_2 = \frac{x^3(u_{xxx}u^2 - 3u_xu_{xx}u + 2u_x^3)}{u^3}, \]

\[ 2A_{2,2}^6 = \langle -t\partial_t - x\partial_x, \partial_t, tu \partial_x, tu \partial_x \rangle : \]
\[ F = \frac{1}{t u_x^4} F(u, \omega), \]
\[ G = -5 \frac{2 u_x u_{xx} u_{xxx} - 3 u_{xx}^3}{tu_x^6} F(u, \omega) + \frac{u_{xx}}{tu_x^2} G(u, \omega) - \frac{u}{t}, \]
\[ \omega = \frac{u_x u_{xxx} - 3 u_{xx}^2}{u_x^2 u_{xx}}, \]
\[ 2A_{2,2}^7 = (-t \partial_t - x \partial_x, \partial_x, \alpha t \partial_t + (1 - \alpha) u \partial_u, tu \partial_x), \alpha \neq 1 : \]
\[ F = \frac{u^4}{tu_x^4} F(\omega, \omega_2), \]
\[ G = -5 \frac{4(2u_x u_{xx} u_{xxx} - 3u_{xx}^3)}{tu_x^6} F(\omega_1, \omega_2) + \frac{u^{\alpha - 1} u_x}{tu_x^2} G(\omega_1, \omega_2) - \frac{u}{t}, \]
\[ \omega_1 = \frac{u_{xx} u^{\frac{\alpha - 2}{\alpha - 1}}}{tu_x^3}, \omega_2 = \frac{u_x u_{xxx} - 3u_{xx}^2}{tu_x^5} u^{\frac{\alpha - 3}{\alpha - 1}}, \]
\[ 2A_{2,2}^8 = (-t \partial_t - x \partial_x, \partial_x, \alpha t \partial_x - u \partial_u, \partial_u), \alpha \in \mathbb{R} : \]
\[ F = t^3 F(\omega, \omega_2), \quad G = u_x G(\omega, \omega_2) + \alpha u_x \ln(tu_x), \]
\[ \omega_1 = \frac{tu_{xx}}{u_x}, \omega_2 = \frac{t^2 u_{xxx}}{u_x}, \]
\[ 2A_{2,2}^9 = (-t \partial_t - x \partial_x, \partial_x, t \partial_t - u \partial_u, \partial_u) : \]
\[ F = \frac{1}{t^3 u_{xx}^4} F(\omega_1, \omega_2), \quad G = \frac{1}{t^3 u_{xx}} G(\omega_1, \omega_2), \]
\[ \omega_1 = tu_x, \omega_2 = \frac{u_{xx}}{tu_{xx}^2}, \]
\[ 2A_{2,2}^{10} = (-t \partial_t - x \partial_x, \partial_x, \alpha t \partial_t - u \partial_u, \partial_u), \alpha \neq 0, 1 : \]
\[ F = t^{-\frac{\alpha + 3}{\alpha - 1}} u_x^{-\frac{4\alpha - 3}{\alpha - 1}} F(\omega_1, \omega_2), \quad G = t^{-\frac{\alpha}{\alpha - 1}} u_x^{-\frac{1}{\alpha - 1}} G(\omega_1, \omega_2), \]
\[ \omega_1 = t^{-\frac{1}{\alpha - 1}} u_x^{-\frac{2\alpha - 1}{\alpha - 1}} u_{xx}, \omega_2 = t^{-\frac{\alpha}{\alpha - 1}} u_x^{-\frac{3\alpha - 1}{\alpha - 1}} u_{xxx}, \]
\[ 2A_{2,2}^{11} = (-t \partial_t - x \partial_x, \partial_x, \partial_u, e^u \partial_u) : \]
\[ F = t^3 F(\omega_1, \omega_2), \quad G = -t^3 (u_x^4 - 6u_x^2 u_{xx} + 4u_x u_{xxx} + 3u_{xx}^2) F(\omega_1, \omega_2) + u_x G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{t(u_x^2 - u_{xx})}{u_x}, \omega_2 = \frac{t^2(u_{xx}^3 - 3u_x u_{xxx} + u_{xxxx})}{u_x}, \]
$$2A_{22}^{12} = \langle -t \partial_t - u \partial_u, \partial_u, \partial_x, \alpha e^x \partial_x + te^x \partial_u \rangle, \quad \alpha \neq 0:$$

$$F = \frac{t^3}{(\alpha u_x - t)^4} F(\omega_1, \omega_2),$$

$$G = \frac{t^3}{\alpha (\alpha u_x - t)^4} (6\alpha u_x + 11\alpha u_{xx} + 6\alpha u_{xxx} - 6t) F(\omega_1, \omega_2)$$

$$+ G(\omega_1, \omega_2) + \frac{1}{\alpha} \ln \frac{t}{\alpha u_x - t},$$

$$\omega_1 = \frac{t(\alpha u_x + \alpha u_{xx} - t)}{(\alpha u_x - t)^2}, \quad \omega_2 = \frac{t^2(2\alpha u_x + 3\alpha u_{xx} + \alpha u_{xxx} - 2t)}{(\alpha u_x - t)^3},$$

$$2A_{22}^{13} = \langle -t \partial_t - xu \partial_x, \partial_x, \partial_u, te^u \partial_x \rangle:$$

$$F = \frac{1}{t u_x^2} F(\omega_1, \omega_2),$$

$$G = \frac{26u_x^6 + 35u_x^4 u_{xx} - 10u_x^2 u_{xxx} + 30u_x^2 u_{x}^2 - 10u_x u_{xx} u_{xxx} + 15u_{xxx}^2}{tu_x^6} F(\omega_1, \omega_2)$$

$$+ u_x G(\omega_1, \omega_2) - \frac{1}{t},$$

$$\omega_1 = \frac{u_x^2 + u_{xx}}{tu_x^3}, \quad \omega_2 = \frac{5u_x^4 + 6u_x^2 u_{xx} - u_x u_{xxx} + 3u_{xx}^2}{tu_x^5},$$

$$2A_{22}^{14} = \langle -t \partial_t - xu \partial_x, \partial_x, t \partial_t + \partial_u, e^u \partial_u \rangle:$$

$$F = \frac{e^{4u}}{t^5(u_x^2 - u_{xx})^4} F(\omega_1, \omega_2),$$

$$G = -\frac{e^{4u}(u_x^4 - 6u_x^2 u_{xx} + 4u_x u_{xxx} + 3u_{xx}^2)}{t^5(u_x^2 - u_{xx})^4} F(\omega_1, \omega_2) - \frac{e^{2u}}{t^3(u_x^2 - u_{xx})} G(\omega_1, \omega_2),$$

$$\omega_1 = \frac{tu_x}{e^u}, \quad \omega_2 = \frac{e^u(u_x^3 - 3u_x u_{xx} + u_{xxx})}{t(u_x^2 - u_{xx})^2},$$

$$2A_{22}^{15} = \langle -x \partial_x - u \partial_u, \partial_x, u \partial_u, u \partial_x \rangle:$$

$$F = \frac{u_x^4}{u_x^4} F(t, \omega), \quad G = -\frac{5u_x^4 u_{xx}(2u_x u_{xxx} - 3u_{xx}^2)}{u_x^6} F(t, \omega) - \frac{u_x^2 u_{xx}^2}{u_x^2} G(t, \omega),$$

$$\omega = \frac{u(u_x u_{xxx} - 3u_{xx}^2)}{u_x^2 u_{xx}},$$

$$2A_{22}^{16} = \langle -x \partial_x - u \partial_u, \partial_x, \partial_t + u \partial_u, u \partial_x \rangle:$$
\[ F = \frac{u^4}{u_x^4} F(\omega_1, \omega_2), \quad G = -\frac{5u^4 u_{xx}(2u_x u_{xxx} - 3u_x^2)}{u_x^6} F(\omega_1, \omega_2) - \frac{uu_x}{e^{u_x}} G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{ue^t u_{xx}}{u_x^3}, \quad \omega_2 = \frac{u^2 e^t (u_x u_{xxx} - 3u_x^2)}{u_x^5}. \]

\[ 2A_{22}^{17} = \langle -x \partial_x - u \partial_u, \partial_x, -t \partial_t + \alpha u \partial_x, \partial_t \rangle, \quad \alpha \neq 0 : \]

\[ F = \frac{u^4}{u_x^4} e^{\frac{1}{u_x}} F(\omega_1, \omega_2), \quad G = -\frac{5u^4 u_{xx}(2u_x u_{xxx} - 3u_x^2)}{u_x^6} e^{\frac{1}{u_x}} F(\omega_1, \omega_2) + uu_x e^{\frac{1}{u_x}} G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{uu_{xx}}{u_x^3}, \quad \omega_2 = \frac{u^2 (u_x u_{xxx} - 3u_x^2)}{u_x^5}. \]

\[ 2A_{22}^{18} = \langle -x \partial_x - u \partial_u, \partial_x, -t \partial_t + \alpha u \partial_u, \partial_t \rangle, \quad \alpha \neq 0 : \]

\[ F = u^4 u_x^{-\alpha} F(\omega_1, \omega_2), \quad G = uu_x^{-\alpha} G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{uu_{xx}}{u_x^3}, \quad \omega_2 = \frac{u^2 u_{xxx} - 3u_x^2}{u_x^5}. \]

\[ 2A_{22}^{19} = \langle -x \partial_x - u \partial_u, \partial_x, \partial_t, u e^t \partial_x \rangle : \]

\[ F = \frac{u^4}{u_x^4} F(\omega_1, \omega_2), \quad G = \frac{5u^4 u_{xx}(2u_x u_{xxx} - 3u_x^2)}{u_x^6} F(\omega_1, \omega_2) + uu_x G(\omega_1, \omega_2) - u, \]

\[ \omega_1 = \frac{uu_{xx}}{u_x^3}, \quad \omega_2 = \frac{u^2 (u_x u_{xxx} - 3u_x^2)}{u_x^5}. \]

\[ 2A_{22}^{20} = \langle -x \partial_x - u \partial_u, \partial_x, \partial_t, e^t \partial_t + \alpha u e^t \partial_x \rangle, \quad \alpha \neq 0 : \]

\[ F = \frac{u^4}{u_x^4} e^{\frac{1}{u_x}} F(\omega_1, \omega_2), \quad G = \frac{5u^4 u_{xx}(2u_x u_{xxx} - 3u_x^2)}{u_x^6} e^{\frac{1}{u_x}} F(\omega_1, \omega_2) + uu_x e^{\frac{1}{u_x}} G(\omega_1, \omega_2) - \alpha uu_x, \]

\[ \omega_1 = \frac{uu_{xx}}{u_x^3}, \quad \omega_2 = \frac{u^2 (u_x u_{xxx} - 3u_x^2)}{u_x^5}. \]

\[ 2A_{22}^{21} = \langle -x \partial_x - u \partial_u, \partial_x, \partial_t, u e^t \partial_u \rangle : \]

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\[
F = \frac{u^4}{u_x} F(\omega_1, \omega_2), \quad G = uG(\omega_1, \omega_2) + u \ln u_x,
\]
\[
\omega_1 = \frac{uu_{xx}}{u_x^2}, \quad \omega_2 = \frac{u^2 u_{xxx}}{u_x^3},
\]
2A_{22}^{\alpha} = \langle -x \partial_x - u \partial_u, \partial_x, \partial_t, e^t \partial_t + \alpha u e^t \partial_u \rangle, \quad \alpha \neq 0:
\]
\[
F = \frac{u^4}{u_x^4} F(\omega_1, \omega_2), \quad G = \frac{u}{t} G(\omega_1, \omega_2) + \frac{u}{t} \ln u_x,
\]
\[
\omega_1 = \frac{uu_{xx}}{u_x^2}, \quad \omega_2 = \frac{u^2 u_{xxx}}{u_x^3},
\]
2A_{22}^{\alpha} = \langle -x \partial_x - u \partial_u, \partial_x, t \partial_t, tu \partial_u \rangle:
\]
\[
F = \frac{u^4}{tu_x^4} F(\omega_1, \omega_2), \quad G = \frac{u}{t} G(\omega_1, \omega_2) + \frac{u}{t} \ln u_x,
\]
\[
\omega_1 = \frac{uu_{xx}}{u_x^2}, \quad \omega_2 = \frac{u^2 u_{xxx}}{u_x^3},
\]
2A_{22}^{\alpha} = \langle -x \partial_x, \partial_x, -t \partial_t + \partial_u, \partial_t \rangle:
\]
\[
F = \frac{e^u}{u_x^4} F(\omega_1, \omega_2), \quad G = e^u G(\omega_1, \omega_2),
\]
\[
\omega_1 = \frac{u_{xx}}{u_x^2}, \quad \omega_2 = \frac{u_{xxx}}{u_x^3},
\]
2A_{22}^{\alpha} = \langle -x \partial_x, \partial_x, \partial_t, e^t \partial_t + e^t \partial_u \rangle, \quad \alpha \neq 0:
\]
\[
F = \frac{1}{u_x^4 e^{\alpha u}} F(\omega_1, \omega_2), \quad G = e^{(1-\alpha)u} G(\omega_1, \omega_2) + \frac{1}{\alpha},
\]
\[
\omega_1 = \frac{u_{xx}}{u_x^2}, \quad \omega_2 = \frac{u_{xxx}}{u_x^3},
\]
2A_{22}^{\alpha} = \langle -x \partial_x, \partial_x, \partial_t, e^t \partial_u \rangle:
\]
\[
F = \frac{1}{u_x^4} F(\omega_1, \omega_2), \quad G = G(\omega_1, \omega_2) + u,
\]
\[
\omega_1 = \frac{u_{xx}}{u_x^2}, \quad \omega_2 = \frac{u_{xxx}}{u_x^3},
\]
2A_{22}^{\alpha} = \langle -x \partial_x, \partial_x, -u \partial_u, \partial_u \rangle:
\]
\[
F = \frac{u^4}{u_{xx}^4} F(t, \omega), \quad G = \frac{u_x^2}{u_{xx}} G(t, \omega),
\]
\[ \omega = \frac{u_x u_{xxx}}{u_{xx}^2}, \]

\[ 2A_{2,2}^{28} = \langle -x \partial_x, \partial_x, \partial_t - u \partial_u, \partial_u \rangle : \]

\[ F = \frac{1}{e^{4u}u_x^4} F(\omega_1, \omega_2), \quad G = \frac{1}{e^t} G(\omega_1, \omega_2) + u, \]

\[ \omega_1 = \frac{u_{xx}}{e^t u_x}, \quad \omega_2 = \frac{u_{xxx}}{e^{2t} u_x^3}, \]

\[ 2A_{2,2}^{29} = \langle -x \partial_x, \partial_x, \partial_u, t e^u \partial_u \rangle : \]

\[ F = \frac{u_x^4}{(u_x^2 - u_{xx})^4} F(t, \omega), \]

\[ G = -\frac{u_x^4 (u_x^4 - 6u_x^2 u_{xx} + 4u_x u_{xxx} + 3u_{xx}^2)}{(u_x^2 - u_{xx})^4} F(t, \omega) - \frac{u_x^2}{u_x^2 - u_{xx}} G(t, \omega) \]

\[ + \frac{u_{xx}}{t(u_x^2 - u_{xx})}, \]

\[ \omega = \frac{u_x (u_x^3 - 3u_x u_{xx} + u_{xxx})}{(u_x^2 - u_{xx})^2}. \]

\[ A_{3,1} \oplus A_1 - \text{invariant equations,} \]

\[ A_{3,1}^2 \oplus A_1 = \langle \partial_t, \partial_u, x \partial_u, f(x) \partial_u \rangle, \quad f'' \neq 0 : \]

\[ F = F(x, \omega), \quad G = -\frac{f^{(4)}}{f''} u_{xx} F(x, \omega) + G(x, \omega), \]

\[ \omega = \frac{f'' u_{xxx} - f^{(3)} u_{xx}}{f''}. \]

\[ A_{3,1}^3 \oplus A_1 = \langle \partial_u, x \partial_u, f(t, x) \partial_u, g(t, x) \partial_u \rangle, \quad f_{xx} g_{xxx} - g_{xx} f_{xxx} \neq 0 : \]

\[ F = F(t, x), \]

\[ G = \frac{(f_{xx} u_{xxx} - f_{xxx} u_{xx}) g_{xxx} + (g_{xxx} u_{xx} - g_{xx} u_{xxx}) f_{xxx}}{f_{xxx} g_{xx} - g_{xxx} f_{xx}} F(t, x) + G(t, x) \]

\[ + \frac{(f_t g_{xx} - g_t f_{xx}) u_{xxx} + (g_t f_{xxx} - f_t g_{xxx}) u_{xx}}{f_{xxx} g_{xx} - g_{xxx} f_{xx}} \]

\[ A_{3,2} \oplus A_1 - \text{invariant equations,} \]

\[ A_{3,2}^2 \oplus A_1 = \langle -t \partial_t - u \partial_u, \partial_t, xu \partial_u, u f(x) \partial_u \rangle, \quad f'' \neq 0 : \]

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\[ F = \frac{e^\sigma}{u} F(x, \omega), \]
\[ G = -e^\sigma \left( u_{xx} u^3 - u_x^2 u^2 \right) f^{(4)} + \left[ 4(u_x u_{xxx} + 3u_x^2 u_x u) - 12u_x^2 u_{xx} u + 6u_x^4 \right] f'' \]
\[ \frac{u^4 f''}{F(x, \omega)} \]
\[ + e^\sigma G(x, \omega), \]
\[ \sigma = \frac{x u u_x f'' - x(u_{xx} u - u_x^2) f' + (u_{xx} u - u_x^2) f}{f'' u^2}, \]
\[ \omega = \frac{(-u_{xx} u^2 + u_x^2) f^{(3)} + (u_{xxx} u^2 - 3u_x u_{xx} u + 2u^3)}{u^3 f''}. \]

\[ A^3_{3, 2} \oplus A_1 = \langle -t \partial_t - x \partial_x, \partial_x, tu \partial_x, tf(u) \partial_x \rangle, \quad f'' \neq 0 : \]
\[ F = \frac{1}{tu_x^4} F(u, \omega), \]
\[ G = -u_x \frac{u^4 f^{(4)} + 5(2u_x u_{xxx} - 3u_x^2) f'}{tu_x^5 f''} F(u, \omega) + u_x G(u, \omega) \]
\[ - \frac{u_x (uf' - f)}{tu_x^5 f''} - \frac{u}{t}, \]
\[ \omega = \frac{-u_x^2 u_{xx} f^{(3)} + (u_x u_{xxx} - 3u_x^2) f''}{tu_x^5 f''}. \]

\[ A^3_{3, 2} \oplus A_1 = \langle -t \partial_t - x \partial_x, \partial_x, tu \partial_x, -u \partial_u \rangle : \]
\[ F = \frac{u^4}{tu_x^4} F(\omega_1, \omega_2), \]
\[ G = -\frac{5u^4 u_{xx} (2u_x u_{xxx} - 3u_x^2)}{tu_x^5 f''} F(\omega_1, \omega_2) + uu_x G(\omega_1, \omega_2) - \frac{u}{t}, \]
\[ \omega_1 = \frac{uu_{xx}}{tu_x^3}, \quad \omega_2 = \frac{u^2 (u_x u_{xxx} - 3u_x^2)}{tu_x^5}, \]

\[ A^4_{3, 2} \oplus A_1 = \langle -t \partial_t - x \partial_x, \partial_x, \partial_u, t \partial_t \rangle : \]
\[ F = \frac{1}{tu_x^4} F(\omega_1, \omega_2), \quad G = \frac{1}{t} G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{u_{xx}}{u_x^2}, \quad \omega_2 = \frac{u_{xxx}}{u_x^3}, \]

\[ A^6_{3, 2} \oplus A_1 = \langle -t \partial_t - x \partial_x, \partial_x, t \partial_t + \partial_u, te^{-u} \partial_x \rangle : \]
\[ F = \frac{1}{tu_x^4} F(\omega_1, \omega_2), \]
\[ G = \frac{-26u_x^6 + 35u_x^4u_{xx} + 10u_x^2u_{xxx} - 30u_x^2u_{xx} - 10u_xu_{xx}u_{xxx} + 15u_x^2}{tu_x^6} \]
\[ + \frac{u_x}{lu_x} G(\omega_1, \omega_2) + \frac{1}{t}, \]
\[ \omega_1 = \frac{e^u(u_x^2 - u_{xx})}{tu_x^3}, \quad \omega_2 = \frac{e^u(5u_x^4 - 6u_x^2u_{xx} - u_xu_{xxx} + 3u_x^2)}{tu_x^5} \]

\[ A_{3,2} \oplus A_1 = \langle -x\partial_x - u\partial_u, \partial_x, u\partial_x, \partial_t \rangle : \]
\[ F = \frac{u^4}{u_x^4} F(\omega_1, \omega_2), \]
\[ G = \frac{-5u_x^4u_{xx}(2u_xu_{xxx} - 3u_x^2)}{u_x^6} F(\omega_1, \omega_2) + uu_xG(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{uu_{xx}}{u_x^3}, \quad \omega_2 = \frac{u^2(u_xu_{xxx} - 3u_x^2)}{u_x^5} \]

\[ A_{3,2} \oplus A_1 = \langle -x\partial_x - u\partial_u, \partial_x, \partial_t, u\partial_u \rangle : \]
\[ F = \frac{u^4}{u_x^4} F\left(\frac{uu_{xx}}{u_x^2}, \frac{u^2u_{xxx}}{u_x^3}\right), \quad G = uG\left(\frac{uu_{xx}}{u_x^2}, \frac{u^2u_{xxx}}{u_x^3}\right), \]

\[ A_{3,2} \oplus A_1 = \langle -x\partial_x, \partial_x, \partial_t, \partial_u \rangle : \]
\[ F = \frac{1}{u_x^4} F\left(\frac{uu_{xx}}{u_x^2}, \frac{u_{xxx}}{u_x^3}\right), \quad G = G\left(\frac{uu_{xx}}{u_x^2}, \frac{u_{xxx}}{u_x^3}\right). \]

\[ A_{3,3} \oplus A_1 \text{ - invariant equations}, \]
\[ A_{3,3} \oplus A_1 = \langle \partial_u, \partial_t, (t + x)\partial_u, f(x)\partial_u \rangle, \quad f'' \neq 0 : \]
\[ F = F(x, \omega), \]
\[ G = -\frac{uu_{xx}f^{(4)}}{f''} F(x, \omega) + G(x, \omega) - \frac{f'}{f''} u_{xx} + u_x, \]
\[ \omega = \frac{uu_{xx}f^{(3)} - uu_{xxx}f''}{f''}, \]

\[ A_{3,3} \oplus A_1 = \langle \partial_u, \partial_t, (t + x)\partial_u, \partial_t - \partial_x \rangle : \]
\[ F = F(u_{xx}, u_{xxx}), \quad G = F(u_{xx}, u_{xxx}) + u_x, \]

\[ A_{3,3}^4 \oplus A_1 = \langle \partial_x, \partial_u, \partial_t + u \partial_x, \partial_t + \alpha t \partial_x + \alpha \partial_u \rangle, \quad \alpha \in \mathbb{R} : \]

\[ F = \frac{1}{u_x^4} F(\omega_1, \omega_2), \]

\[ G = -\frac{5u_{xx}(2u_x u_{xxx} - 3u_{xx}^2)}{u_x^6} F(\omega_1, \omega_2) + u_x G(\omega_1, \omega_2) - \alpha u_x + \alpha, \]

\[ \omega_1 = \frac{u_{xx}}{u_x^3}, \quad \omega_2 = \frac{u_x u_{xxx} - 3u_{xx}^2}{u_x^5}, \]

\[ A_{3,3}^5 \oplus A_1 = \langle \partial_x, u \partial_x, -\partial_u, \partial_t \rangle : \]

\[ F = \frac{1}{u_x^4} F(\omega_1, \omega_2), \]

\[ G = -\frac{5u_{xx}(2u_x u_{xxx} - 3u_{xx}^2)}{u_x^6} F(\omega_1, \omega_2) + u_x G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{u_{xx}}{u_x^3}, \quad \omega_2 = \frac{u_x u_{xxx} - 3u_{xx}^2}{u_x^5}, \]

\[ A_{3,3}^6 \oplus A_1 = \langle \partial_x, u \partial_x, \partial_t - \partial_u, f(t + u) \partial_x \rangle, \quad f'' \neq 0 : \]

\[ F = \frac{1}{u_x^4} F(\omega_1, \omega_2), \]

\[ G = \left[ \frac{15u_{xx}^3}{u_x^6} - \frac{10u_{xx} u_{xxx} u_x}{u_x^5} - \frac{u_{xx} f^{(4)}}{u_x^2 f''} \right] F(\omega_1, \omega_2) + u_x G(\omega_1, \omega_2) + \frac{u_{xx} f'}{u_x^2 f''}, \]

\[ \omega_1 = t + u, \quad \omega_2 = \frac{(u_x u_{xxx} - 3u_{xx}^2) f'' - u_x^2 u_{xx} f^{(3)}}{u_x^5 f''}, \]

\[ A_{3,3}^7 \oplus A_1 = \langle x \partial_u, \partial_u, x^2 \partial_x + xu \partial_u, \partial_t \rangle : \]

\[ F = x^8 F(\omega_1, \omega_2), \]

\[ G = 4x^6 (3u_{xx} + 2x u_{xxx}) F(\omega_1, \omega_2) + x G(\omega_1, \omega_2), \]

\[ \omega_1 = x^3 u_{xx}, \quad \omega_2 = x^4 (3u_{xx} + xu_{xxx}). \]

\[ A_{3,4} \oplus A_1 - \text{invariant equations}, \]

\[ A_{3,4}^1 \oplus A_1 = \langle \partial_u, \partial_t, t \partial_t + \partial_x + (t + u) \partial_u, \partial_x \rangle : \]

\[ F = \frac{1}{u_x} F(\omega_1, \omega_2), \quad G = G(\omega_1, \omega_2) + \ln u_x, \]
\[ \omega_1 = \frac{u_{xx}}{u_x}, \quad \omega_2 = \frac{u_{xxx}}{u_x}, \]

\[ A_{3,4}^1 \oplus A_1 = \langle \partial_u, \partial_t, t \partial_t + \partial_x + (t + u) \partial_u, e^x \partial_x \rangle : \]

\[ F = e^{-x} F(\omega_1, \omega_2), \quad G = -e^{-x} u_x F(\omega_1, \omega_2) + G(\omega_1, \omega_2) + x, \]

\[ \omega_1 = e^{-x}(u_x - u_{xx}), \quad \omega_2 = e^{-x}(u_x - u_{xxx}). \]

\[ A_{3,4}^4 \oplus A_1 = \langle \partial_x, \partial_u, (x + u) \partial_x + u \partial_u, \partial_t \rangle : \]

\[ F = \frac{e^{u_x}}{u_x^4} F(\omega_1, \omega_2), \]

\[ G = \frac{5u_{xx}(3u_{xx}^2 - 2u_x u_{xxx})e^{\frac{1}{u_x}}}{u_x^6} F(\omega_1, \omega_2) + u_x e^{\frac{1}{u_x}} G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{u_x e^{\frac{1}{u_x}}}{u_x^3}, \quad \omega_2 = \frac{(u_x u_{xxx} - 3u_{xx}^2)e^{\frac{2}{u_x}}}{u_x^2}, \]

\[ A_{3,4}^5 \oplus A_1 = \langle \partial_x, u \partial_x, x \partial_x - \partial_u, e^{-u} \partial_x \rangle : \]

\[ F = \frac{1}{u_x^4} F(t, \omega), \]

\[ G = -\frac{u_{xx}(u_x^4 + 10u_x u_{xxx} - 15u_{xx}^2)}{u_x^6} F(t, \omega) + u_x e^{-u} G(t, \omega), \]

\[ \omega = \frac{(u_x^2 u_{xxx} + u_x u_{xxx} - 3u_{xx}^2)e^u}{u_x^5}, \]

\[ A_{3,4}^5 \oplus A_1 = \langle \partial_x, u \partial_x, x \partial_x - \partial_u, t e^{-u} \partial_x \rangle : \]

\[ F = \frac{1}{u_x^4} F(t, \omega), \]

\[ G = -\frac{u_{xx}(u_x^4 + 10u_x u_{xxx} - 15u_{xx}^2)}{u_x^6} F(t, \omega) + u_x e^{-u} G(t, \omega) + \frac{u_{xx}}{tu_x^2}, \]

\[ \omega = \frac{(u_x^2 u_{xxx} + u_x u_{xxx} - 3u_{xx}^2)e^u}{u_x^5}, \]

\[ A_{3,4}^5 \oplus A_1 = \langle \partial_x, u \partial_x, x \partial_x - \partial_u, \partial_t \rangle : \]

\[ F = \frac{1}{u_x^4} F(\omega_1, \omega_2), \]

\[ G = -\frac{5u_{xx}(2u_x u_{xxx} - 3u_{xx}^2)}{u_x^6} F(\omega_1, \omega_2) + e^{-u} u_x(\omega_1, \omega_2), \]

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\[ \omega_1 = \frac{e^u u_{xx}}{u_x^3}, \quad \omega_2 = \frac{(u_x u_{xxx} - 3 u_{xx}^2)e^u}{u_x^5}, \]

\[ A_{3,4}^6 \oplus A_1 = \langle \partial_x, u \partial_x, \partial_t + x \partial_x - \partial_u, f(t + u)e^f \partial_x \rangle, \quad f'' \neq 0 : \]

\[ F = \frac{1}{u_x^4} F(\omega_1, \omega_2), \]

\[ G = -\frac{u_{xx} f^{(4)}}{u_x^2 f''} + \frac{5 u_{xx}(2 u_x u_{xxx} - 3 u_{xx}^2)}{u_x^6} F(\omega_1, \omega_2) + u_x e^f G(\omega_1, \omega_2) \]

\[ + \frac{(f + f') u_{xx}}{f'' u_x^2}, \]

\[ \omega_1 = t + u, \quad \omega_2 = \frac{(u_x u_{xxx} - 3 u_{xx}^2)f'' - u_x^2 u_{xx} f^{(3)}}{u_x^5 e^f f''}, \]

\[ A_{3,4}^7 \oplus A_1 = \langle u \partial_x, \partial_x, x(1 + u) \partial_x + u^2 \partial_u, u e^{-\frac{1}{u}} \partial_x \rangle : \]

\[ F = \frac{u^8}{u_x^4} F(t, \omega), \]

\[ G = -\frac{u^4}{u_x^6} \left[ 5 u^4 u_{xx}(2 u_x u_{xxx} - 3 u_{xx}^2) + 8 u^3 u_x^2(3 u_{xx}^2 - u_x u_{xxx}) + u_x^4 u_{xx}(1 - 12 u^2) \right] F(t, \omega) \]

\[ + u u_x e^{-\frac{1}{u}} G(t, \omega), \]

\[ \omega = \frac{u^3}{u_x^5} \left[ u^2(u_x u_{xxx} - 3 u_{xx}^2) - 3 u_x^2 u_{xx} u - u_x^2 u_{xx} \right] e^{\frac{1}{u}}, \]

\[ A_{3,4}^7 \oplus A_1 = \langle u \partial_x, \partial_x, x(1 + u) \partial_x + u^2 \partial_u, t u e^{-\frac{1}{u}} \partial_x \rangle : \]

\[ F = \frac{u^8}{u_x^4} F(t, \omega), \]

\[ G = -\frac{u^4}{u_x^6} \left[ 5 u^4 u_{xx}(2 u_x u_{xxx} - 3 u_{xx}^2) + 8 u^3 u_x^2(3 u_{xx}^2 - u_x u_{xxx}) + u_x^4 u_{xx}(1 - 12 u^2) \right] F(t, \omega) \]

\[ + u u_x e^{-\frac{1}{u}} G(t, \omega) + \frac{u^4 u_{xx}}{t u_x^2}, \]

\[ \omega = \frac{u^3}{u_x^5} \left[ u^2(u_x u_{xxx} - 3 u_{xx}^2) - 3 u_x^2 u_{xx} u - u_x^2 u_{xx} \right] e^{\frac{1}{u}}, \]

\[ A_{3,4}^7 \oplus A_1 = \langle u \partial_x, \partial_x, x(1 + u) \partial_x + u^2 \partial_u, \partial_t \rangle : \]

\[ F = \frac{u^8}{u_x^4} F(\omega_1, \omega_2), \]
\[
G = -\frac{u^4[5u^4u_{xx}(2u_xu_{xxx} - 3u_{xx}^2) + 8u^3u_x^2(3u_{xx}u_x - u_{xxx}) - 12u^2u_x^4u_{xx}]}{u_x^6} F(\omega_1, \omega_2)
\]
\[
+ uu_x e^{-\frac{1}{u}} G(\omega_1, \omega_2),
\]
\[
\omega_1 = \frac{u^3 e^\frac{1}{u} u_{xx}}{u_x^3}, \quad \omega_2 = \frac{u^4 e^\frac{1}{u} [u(u_xu_{xxx} - 3u_{xx}^2) + 3u_x^2u_{xxx}]}{u_x^5},
\]
\[A_{3,4}^8 \oplus A_1 = \langle u \partial_x, \partial_x, \partial_t + x(1 + u) \partial_x + u^2 \partial_u, ue^f(t + \frac{1}{u}) \partial_x \rangle, \quad f'' \neq 0 : \]
\[
F = \frac{u^8}{u_x^4} F(\omega_1, \omega_2),
\]
\[
G = \left\{ \frac{u^6[5u^2u_{xx}(3u_{xx}^2 - 2u_xu_{xxx}) + 8u^2u_x^2(u_xu_{xxx} - 3u_{xx}^2) - 12u_x^4u_{xx}]}{u_x^6}
\right.
\]
\[
- \frac{u^4f^{(4)}u_{xx}}{f''u_x^2} \right\} F(\omega_1, \omega_2) - uu_x e^f G(\omega_1, \omega_2) + \frac{(f' + f)u^4u_{xx}}{f''u_x^2},
\]
\[
\omega_1 = t + \frac{1}{u}, \quad \omega_2 = \frac{u^4[u(u_xu_{xxx} - 3u_{xx}^2) + 3u_x^2u_{xxx}]f''}{u_x^2e^f} + \frac{u^3u_{xx}f^{(3)}}{e^f u_x^3}.
\]

\[A_{3,5} \oplus A_1 - \text{invariant equations,} \]
\[A_{3,5}^1 \oplus A_1 = \langle \partial_u, \partial_t, t \partial_t + u \partial_u, \partial_x \rangle : \]
\[
F = \frac{1}{u_x} F(\frac{u_{xx}}{u_x}, \frac{u_{xxx}}{u_x}), \quad G = G(\frac{u_{xx}}{u_x}, \frac{u_{xxx}}{u_x}),
\]
\[A_{3,5}^2 \oplus A_1 = \langle \partial_u, \partial_t, t \partial_t + \partial_x + u \partial_u, e^x \partial_u \rangle : \]
\[
F = e^{-x} F(\omega_1, \omega_2), \quad G = -e^{-x} u_x F(\omega_1, \omega_2) + G(\omega_1, \omega_2),
\]
\[
\omega_1 = e^{-x}(u_{xx} - u_x), \quad \omega_2 = e^{-x}(u_{xxx} - u_x),
\]
\[A_{3,5}^3 \oplus A_1 = \langle \partial_x, \partial_u, xu_x + u \partial_u, \partial_t \rangle : \]
\[
F = \frac{1}{u_x^4} F(u_x, \frac{u_{xxx}}{u_x^2}), \quad G = \frac{1}{u_x} G(u_x, \frac{u_{xxx}}{u_x^2}),
\]
\[A_{3,5}^4 \oplus A_1 = \langle \partial_x, u \partial_x, xu_x + u \partial_u, \partial_t \rangle : \]
\[
F = \frac{1}{u_x^4} F(u, \omega), \quad G = -\frac{5u_x^2u_{xxx} - 3u_{xx}^2}{u_x^6} F(u, \omega) + \frac{u_{xx}}{u_x^2} G(u, \omega),
\]
\[
\omega = \frac{u_x u_{xxx} - 3u_{xx}^2}{u_x^2 u_{xx}},
\]
\[43\]
\[A^6_{3,5} \oplus A_1 = \langle \partial_x, u \partial_x, t \partial_t - u \partial_u, \partial_x \rangle \quad f'' \neq 0:\]
\[F = \frac{1}{u_x^4} F(u, \omega),\]
\[G = -\left[ \frac{5u_{xx}(2u_x u_{xxx} - 3u_x^2)}{u_x^6} + \frac{u_{xxx} f(4)}{u_x^2 f''} \right] F(u, \omega) + u_x e' G(u, \omega) + \frac{u_{xx} f}{u_x^2 f''},\]
\[\omega = \frac{(3u_{xx} - u_x u_{xxx}) f'' + 2u_x u_{xxx} f(3)}{e^2 u_x^2 f''}.
\]

\[A_{3,6} \oplus A_1 - \text{invariant equations},\]
\[A^1_{3,6} \oplus A_1 = \langle \partial_u, \partial_t, t \partial_t - u \partial_u, \partial_x \rangle:\]
\[F = u_x F(\omega_1, \omega_2), \quad G = u_x^2 G(\omega_1, \omega_2),\]
\[\omega_1 = \frac{u_{xx}}{u_x}, \quad \omega_2 = \frac{u_{xxx}}{u_x},\]
\[A^2_{3,6} \oplus A_1 = \langle \partial_u, \partial_t, t \partial_t - u \partial_u, e^{-x} \partial_u \rangle:\]
\[F = e^{-x} F(\omega_1, \omega_2), \quad G = u_x e^{-x} F(\omega_1, \omega_2) + e^{-2x} G(\omega_1, \omega_2),\]
\[\omega_1 = e^x (u_x + u_{xx}), \quad \omega_2 = e^x (u_x - u_{xxx}),\]
\[A^3_{3,6} \oplus A_1 = \langle \partial_x, \partial_u, x \partial_x - u \partial_u, \partial_t \rangle:\]
\[F = \frac{1}{u_x^4} F(\omega_1, \omega_2), \quad G = \frac{1}{u_x^4} G(\omega_1, \omega_2),\]
\[\omega_1 = \frac{u_{xx}}{u_x^2}, \quad \omega_2 = \frac{u_{xxx}}{u_x^2},\]
\[A^5_{3,6} \oplus A_1 = \langle \partial_x, u \partial_x, x \partial_x + 2u \partial_u, u^4 f(t) \partial_x \rangle:\]
\[F = \frac{u_x^4}{u_x^4} F(t, \omega),\]
\[G = \frac{5u^2 u_{xx}}{4u_x^6} [4u^2 (3u_x^2 - 2u_x u_{xxx}) - 3u_x^4] F(t, \omega) + u_x^2 u_x G(t, \omega) - \frac{4u^2 u_{xx} f'}{u_x^2 f},\]
\[\omega = \frac{u_x^4}{u_x^5} [2u (u_x u_{xxx} - 3u_x^2) + 3u_x^2 u_{xx}],\]
\[A^5_{3,6} \oplus A_1 = \langle \partial_x, u \partial_x, x \partial_x + 2u \partial_u, \partial_t \rangle:\]
\[F = \frac{u_x^4}{u_x^4} F(\omega_1, \omega_2),\]
\[
G = -\frac{5u^4u_{xx}(2u_xu_{xxx} - 3u_{xx}^2)}{u_x^6}F(\omega_1, \omega_2) + u_x^{\frac{1}{2}}u_xG(\omega_1, \omega_2),
\]
\[
\omega_1 = \frac{u_x^{\frac{2}{3}}u_{xx}}{u_x^3}, \quad \omega_2 = \frac{u_x^{\frac{2}{3}}(u_xu_{xxx} - 3u_{xx}^2)}{u_x^5}.
\]

\[A_{3.7} \oplus A_1 - \text{invariant equations},\]

\[A_{3.7}^1 \oplus A_1 = \langle \partial_t, \partial_u, t\partial_t + qu\partial_u, \partial_x \rangle :\]
\[
F = u_x^{\frac{1}{q}}F(\omega_1, \omega_2), \quad G = u_x^{1-\frac{1}{q}}G(\omega_1, \omega_2),
\]
\[
\omega_1 = \frac{u_{xx}}{u_x}, \quad \omega_2 = \frac{u_{xxx}}{u_x},
\]

\[A_{3.7}^2 \oplus A_1 = \langle \partial_t, \partial_u, t\partial_t + \partial_x + qu\partial_u, e^{qx}\partial_u \rangle :\]
\[
F = q^{-3}e^{-x}F(\omega_1, \omega_2), \quad G = -u_xe^{-x}F(\omega_1, \omega_2) + e^{(q-1)x}G(\omega_1, \omega_2),
\]
\[
\omega_1 = e^{-qx}qu_x - u_{xx}, \quad \omega_2 = e^{-qx}q^2u_x - u_{xxx},
\]

\[A_{3.7}^3 \oplus A_1 = \langle \partial_x, \partial_u, x\partial_x + qu\partial_u, \partial_t \rangle :\]
\[
F = u_x^{\frac{4}{q-1}}F(\omega_1, \omega_2), \quad G = u_x^{\frac{4}{q-1}}G(\omega_1, \omega_2),
\]
\[
\omega_1 = \frac{u_{xx}}{u_x}, \quad \omega_2 = \frac{u_{xxx}}{u_x},
\]

\[A_{3.7}^4 \oplus A_1 = \langle \partial_u, x\partial_u, (1 - q)x\partial_x + u\partial_u, x^{\frac{1}{1-q}}\partial_u \rangle :\]
\[
F = x^4F(t, \omega),
\]
\[
G = -\frac{(2q-1)(3q-2)}{(q-1)^2}x^2u_{xx}F(t, \omega) + x^{\frac{1}{1-q}}G(t, \omega),
\]
\[
\omega = x^{\frac{2q-1}{q-1}}[(q-1)xu_{xxx} + (2q-1)u_{xx}],
\]

\[A_{3.7}^5 \oplus A_1 = \langle \partial_u, x\partial_u, (1 - q)x\partial_x + u\partial_u, tx^{\frac{1}{1-q}}\partial_u \rangle :\]
\[
F = x^4F(t, \omega),
\]
\[
G = -\frac{(2q-1)(3q-2)}{(q-1)^2}x^2u_{xx}F(t, \omega) + x^{\frac{1}{1-q}}G(t, \omega) + \frac{(q-1)^2x^2u_{xx}}{qt},
\]
\[
\omega = x^{\frac{2q-1}{q-1}}[(q-1)xu_{xxx} + (2q-1)u_{xx}],
\]

\[A_{3.7}^6 \oplus A_1 = \langle \partial_u, x\partial_u, (1 - q)x\partial_x + u\partial_u, \partial_t \rangle :\]
\[ F = x^4 F(\omega_1, \omega_2), \quad G = x^{\frac{1}{2-q}} G(\omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{2q-1}{q-1}} u_{xx}, \quad \omega_2 = x^{\frac{3q-2}{q-1}} u_{xxx}, \]
\[ A_{3,7}^6 \oplus A_1 = \langle \partial_u, x \partial_u, \partial_t + (1-q)x \partial_x + u \partial_u, x^{\frac{1}{1-q}} \partial_u \rangle : \]
\[ F = e^{4(1-q)t} F(\omega_1, \omega_2), \]
\[ G = -\frac{(2q-1)(3q-2)u_{xx} e^{4(1-q)t}}{(q-1)^2 x^2} F(\omega_1, \omega_2) + e^t G(\omega_1, \omega_2), \]
\[ \omega_1 = e^{(q-1)t} x, \quad \omega_2 = \frac{e^{(2-3q)t}[(q-1)xu_{xxx} + (2q-1)u_{xx}]}{x}, \]
\[ A_{3,7}^6 \oplus A_1 = \langle \partial_u, x \partial_u, \partial_t + (1-q)x \partial_x + u \partial_u, \partial_t + \alpha x^{\frac{1}{1-q}} \partial_u \rangle, \quad \alpha \neq 0 : \]
\[ F = x^4 F(\omega_1, \omega_2), \]
\[ G = -\frac{\alpha (2q-1)(3q-2)x^{\frac{1}{1-q}} [\ln x + (q-1)t]}{(q-1)^2} F(\omega_1, \omega_2) + x^{\frac{1}{1-q}} G(\omega_1, \omega_2), \]
\[ \omega_1 = (q-1)^3 x^{\frac{2q-1}{q-1}} u_{xx} - \alpha q [\ln x + (q-1)t], \]
\[ \omega_2 = (q-1)^4 x^{\frac{3q-2}{q-1}} u_{xxx} + \alpha q (2q-1)[\ln x + (q-1)t]. \]

\[ A_{3,8} \oplus A_1 - \text{invariant equations,} \]
\[ A_{3,8}^1 \oplus A_1 = \langle \partial_x, \partial_u, u \partial_x - x \partial_u, \partial_t \rangle : \]
\[ F = \frac{1}{(1 + u_x^2)^2} F(\omega_1, \omega_2), \]
\[ G = -\frac{u_x u_{xx} (2u_x^2 u_{xxx} - 3u_x u_x^2 + 2u_{xxxx})}{(1 + u_x^2)^4} F(\omega_1, \omega_2) + (1 + u_x^2)^2 G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{u_x}{(1 + u_x^2)^{\frac{3}{2}}}, \quad \omega_2 = \frac{u_x^2 u_{xxx} - 3u_x u_x^2 + u_{xxxx}}{(1 + u_x^2)^3}, \]
\[ A_{3,8}^3 \oplus A_1 = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x - xu \partial_u, \partial_t \rangle : \]
\[ F = (1 + x^2)^4 F(\omega_1, \omega_2), \]
\[ G = 4x(1 + x^2)^2 (2x^2 u_{xxx} + 3xu_{xx} + 2u_{xxxx}) F(\omega_1, \omega_2) + (1 + x^2)^2 G(\omega_1, \omega_2), \]
\[ \omega_1 = (1 + x^2)^{\frac{3}{2}} u_{xx}, \quad \omega_2 = (1 + x^2)^{\frac{5}{2}} (x^2 u_{xxx} + 3xu_{xx} + u_{xxxx}), \]
\[ A_{3,8}^3 \oplus A_1 = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x - xu \partial_u, (x^2 + 1)^{\frac{3}{2}} \partial_u \rangle : \]
\[ F = (1 + x^2)^4 F(t, \omega), \]
\[ G = x(1 + x^2)^2 (8x^3 u_{xxx} + 12x^2 u_{xx} + 8x u_{xxx} + 3u_{xx}) F(t, \omega) + (1 + x^2)^2 G(t, \omega), \]
\[ \omega = (1 + x^2)^2 (x^2 u_{xxx} + 3x u_{xx} + u_{xxx}), \]
\[ A_{3,8}^3 \oplus A_1 = \langle \partial_u, x \partial_x, -(x^2 + 1) \partial_x - xu \partial_u, t(x^2 + 1)^{\frac{1}{2}} \partial_u \rangle : \]
\[ F = (1 + x^2)^4 F(t, \omega), \]
\[ G = (1 + x^2)^2 (8x^3 u_{xxx} + 12x^2 u_{xx} + 8x u_{xxx} + 3u_{xx}) F(t, \omega) + (1 + x^2)^2 G(t, \omega) + \frac{u_{xx}}{t} (1 + x^2)^2, \]
\[ \omega = (1 + x^2)^2 (x^2 u_{xxx} + 3x u_{xx} + u_{xxx}), \]
\[ A_{3,8}^4 \oplus A_1 = \langle \partial_x, u \partial_x, \partial_t - xu \partial_x - (u^2 + 1) \partial_u, (1 + u^2)^{\frac{1}{2}} f(t + \arctan u) \partial_x \rangle f + f'' \neq 0 : \]
\[ F = \frac{(1 + u^2)^4}{u_x^2} F(\omega_1, \omega_2), \]
\[ G = \left\{ \frac{(1 + u^2)^2}{u_x^2} \right\} \left\{ 24u_x^4 u_{xxx} \arctan^2 u - 8u_x^2 (3u_x^2 u_{xx} (u - 2t) + (1 + u^2)(u_x u_{xxx} - 3u_{xx}^2)) \right. \]
\[ \left. \arctan u + 12u_x^4 u_{xx} (u^2 - 2tu + 2t^2) + 8u_x^2 (1 + u^2)(u_x u_{xxx} - 3u_{xx}) (u - t) \right. \]
\[ \left. - 5(1 + u^2)^2 u_{xx} (2u_x u_{xxx} - 3u_{xx}^2) \right\} + \frac{(1 + u^2)^2 u_{xx}}{(f + f'' u_x^2)} \left\{ 3(1 - 8\omega_1^2) f + 8\omega_1 f'' \right. \]
\[ \left. + 2(1 - 12\omega_1^2) f'' + 8\omega_1 f^{(3)} - f^{(4)} \right\} F(\omega_1, \omega_2) + (1 + u^2)^2 u_x G(\omega_1, \omega_2) \]
\[ + \frac{f'(1 + u^2)^2 u_{xx}}{(f + f'' u_x^2)}, \]
\[ \omega_1 = t + \arctan u, \]
\[ \omega_2 = (1 + u^2)^{\frac{3}{2}} \left\{ \frac{u_{xx}}{u_x^3} (3u - \frac{f' + f^{(3)}}{f + f''}) + (1 + u^2) \frac{u_x u_{xxx} - 3u_{xx}^2}{u_x^2} \right\}. \]

\[ A_{3,9} \oplus A_1 - \text{invariant equations}, \]
\[ A_{3,9}^1 \oplus A_1 = \langle \partial_x, \partial_u, (qx + u) \partial_x + (qu - x) \partial_u, \partial_t \rangle : \]
\[ F = \frac{1}{(1 + u_x^2)^2 e^{4q \arctan u_x}} F(\omega_1, \omega_2), \]

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\[ G = -\frac{5}{(1 + u_x^2)^4} \frac{u_x u_{xx} (2u_x^2 u_{xxx} - 3u_x u_{xx}^2 + 2u_{xxx})}{4 e^q \arctan u_x} F(\omega_1, \omega_2) + \frac{(1 + u_x^2)^2}{e^q \arctan u_x} G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{u_x}{(1 + u_x^2)^2} e^q \arctan u_x, \quad \omega_2 = \frac{u_x^2 u_{xxx} - 3u_x u_{xx}^2 + u_{xxx}}{(1 + u_x^2)^3} e^q \arctan u_x, \]

\[ A^3_{3,9} \oplus A_1 = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x + (q - x) u \partial_u, \partial_t \rangle : 
\]

\[ F = (x^2 + 1)^4 F(\omega_1, \omega_2), \]

\[ G = 4(x^2 + 1)^2 (2x^2 u_{xxx} + 3x u_{xx} + 2u_{xxx}) F(\omega_1, \omega_2) + \frac{(x^2 + 1)^2}{e^q \arctan x} G(\omega_1, \omega_2), \]

\[ \omega_1 = (x^2 + 1)^2 e^q \arctan x, \]

\[ \omega_2 = (1 + u_x^2)^2 e^q \arctan x (x^2 u_{xxx} + 3x u_{xx} + u_{xxx}), \]

\[ A^3_{3,9} \oplus A_1 = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x + (q - x) u \partial_u, (x^2 + 1)^2 e^{-q \arctan x} \partial_u \rangle : 
\]

\[ F = (x^2 + 1)^4 F(t, \omega), \]

\[ G = (x^2 + 1)^2 [8x^3 u_{xxx} + 12x^2 u_{xx} + 8x u_{xxx} + (3 - q^2) u_{xx}] F(t, \omega) + \frac{(1 + x^2)^2}{e^q \arctan x} G(t, \omega), \]

\[ \omega = (1 + x^2)^2 (x^2 u_{xxx} + 3x u_{xx} + q u_{xx} + u_{xxx}), \]

\[ A^3_{3,9} \oplus A_1 = \langle \partial_u, x \partial_u, -(x^2 + 1) \partial_x + (q - x) u \partial_u, t(x^2 + 1)^2 e^{-q \arctan x} \partial_u \rangle : 
\]

\[ F = (x^2 + 1)^4 F(t, \omega), \]

\[ G = (x^2 + 1)^2 (8x^3 u_{xxx} + 12x^2 u_{xx} + 8x u_{xxx} + (3 - q^2) u_{xx}) F(t, \omega) + \frac{(1 + x^2)^2}{e^q \arctan x} G(t, \omega) \]

\[ + \frac{(1 + x^2)^2 u_{xx}}{(1 + q^2)t}, \]

\[ \omega = (1 + x^2)^2 (x^2 u_{xxx} + 3x u_{xx} + q u_{xx} + u_{xxx}) e^q \arctan x, \]

\[ A^3_{3,9} \oplus A_1 = \langle \partial_u, u \partial_x, \partial_t + (q - u) x \partial_x - (u^2 + 1) \partial_u, (u^2 + 1)^2 e^{q t} \partial_x \rangle : 
\]

\[ F = \frac{(u^2 + 1)^4}{u_x^4} F(\omega_1, \omega_2), \]

\[ G = (u^2 + 1)^2 [\frac{5}{u_x^6} (2u_x u_{xxx} - 3u_x^2) + \frac{8u(u^2 + 1)}{u_x^4} (u_x u_{xxx} - 3u_x^2) \]

\[ + \frac{3u_x^4}{u_x^2} (4u^2 + 1)] F(\omega_1, \omega_2) + (u^2 + 1)^2 e^{q t} u_x G(\omega_1, \omega_2) + \frac{q(u^2 + 1)^2 u_{xx}}{u_x^2}, \]

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\[ \omega_1 = t + \arctan u, \]
\[ \omega_2 = \left( \frac{u^2 + 1}{u_x^2 e^{qt}} \right) \left[ (u^2 + 1)(u_xu_{xxx} - 3u_{xx}^2) + 3uu_x^2 u_{xx} \right]. \]

\[ A_4^1 \oplus A_1 = \langle \partial_u, u \partial_u, \partial_t + (q - u)x \partial_x - (u^2 + 1) \partial_x, f(t + \arctan u)(u^2 + 1)\frac{1}{2} e^{qt} \partial_x \rangle, \]
\[ f + f'' \neq 0: \]
\[ F = \left( \frac{u^2 + 1}{u_x^4} \right) F(\omega_1, \omega_2), \]
\[ G = \left( \frac{u^2 + 1}{u_x^2} \right) \left[ -\frac{5(u^2 + 1)^2 u_{xx}}{u_x^6} (2u_x u_{xxx} - 3u_{xx}^2) + \frac{8u(u^2 + 1)}{u_x^4} (u_x u_{xxx} - 3u_{xx}^2) \right. \]
\[ + \frac{u_{xx}}{u_x^2} (2(6u^2 + 1) + \frac{f - f^{(4)}}{f + f''})] F(\omega_1, \omega_2) + (u^2 + 1)\frac{1}{2} e^{qt} u_x G(\omega_1, \omega_2) \]
\[ + \frac{(u^2 + 1)^2 u_{xx} (q f + f')}{u_x^2 (f + f'')}, \]
\[ \omega_1 = t + \arctan u, \]
\[ \omega_2 = \left( \frac{u^2 + 1}{u_x^2 e^{qt}} \right) \left[ (u^2 + 1)(u_x u_{xxx} - 3u_{xx}^2) + 3uu_x^2 u_{xx} - \frac{f + f^{(3)}}{f + f''} \right] u_x^2 u_{xx}, \]

Appendix A. Equations admitting non-decomposable four-dimensional solvable Lie algebras

\[ A_{4,1} \text{ -- invariant equations,} \]
\[ A_{4,1}^1 = \langle \partial_u, \partial_x, \partial_t, t \partial_x + x \partial_u \rangle : \]
\[ F = F(u_{xx}, u_{xxx}), \quad G = G(u_{xx}, u_{xxx}) - \frac{u_{xx}^2}{2}, \]
\[ A_{4,1}^2 = \langle \partial_u, x \partial_u, \partial_t, -\partial_x + tx \partial_u \rangle : \]
\[ F = F(u_{xx}, u_{xxx}), \quad G = G(u_{xx}, u_{xxx}) - \frac{x^2}{2}, \]
\[ A_{4,1}^3 = \langle x \partial_u, \partial_u, \partial_t, x^2 \partial_x + (t + xu) \partial_u \rangle : \]
\[ F = x^8 F(\omega_1, \omega_2), \quad G = 4x^2 \left( 2\omega_2 - 3x \omega_1 \right) F(\omega_1, \omega_2) + xG(\omega_1, \omega_2) - \frac{1}{2x}, \]
\[ \omega_1 = x^3 u_{xx}, \quad \omega_2 = x^4 (x u_{xxx} + 3u_{xx}), \]

\[ A_{4,2} - \text{invariant equations}, \]
\[ A_{4,2}^1 = \langle \partial_t, \partial_u, \partial_x, q t \partial_t + x \partial_x + (x + u) \partial_u \rangle : \]
\[ F = e^{(1-q)u_x} F(\omega_1, \omega_2), \quad G = e^{(1-q)u_x} G(\omega_1, \omega_2), \]
\[ \omega_1 = e^{u_x} u_{xx}, \quad \omega_2 = e^{2u_x} u_{xxx}, \]
\[ A_{4,2}^2 = \langle \partial_x, \partial_u, \partial_t, t \partial_t + qx \partial_x + (t + u) \partial_u \rangle \quad q \neq 1 : \]
\[ F = u_x^{\frac{q-1}{q}} F(\omega_1, \omega_2), \quad G = G(\omega_1, \omega_2) - \frac{\ln |u_x|}{q-1}, \]
\[ \omega_1 = u_x^{\frac{1-2q}{q-1}} u_{xx}, \quad \omega_2 = u_x^{\frac{1-3q}{q-1}} u_{xxx}, \]
\[ A_{4,2}^3 = \langle \partial_x, \partial_u, \partial_t, t \partial_t + x \partial_x + (t + u) \partial_u \rangle : \]
\[ F = \frac{1}{u_{xx}^3} F(u_x, u_{xxx}^2 u_{xx}^2), \quad G = G(u_x, \frac{u_{xxx}^2 u_{xx}^2}{u_{xx}^2}) - \ln |u_x|, \]
\[ \omega_1 = e^x u_{xx}, \quad \omega_2 = e^x u_{xxx}, \]
\[ A_{4,2}^4 = \langle \partial_t, \partial_u, x \partial_u, q t \partial_t - \partial_x + u \partial_u \rangle : \]
\[ F = e^{qx} F(\omega_1, \omega_2), \quad G = e^{(q-1)x} G(\omega_1, \omega_2), \]
\[ \omega_1 = e^x u_{xx}, \quad \omega_2 = e^x u_{xxx}, \]
\[ A_{4,2}^5 = \langle \partial_t, x \partial_u, \partial_u, q t \partial_t + x^2 \partial_x + (x + 1) u \partial_u \rangle : \]
\[ F = x^8 e^{\frac{q}{x}} F(\omega_1, \omega_2), \]
\[ G = 4x^6 e^{\frac{q}{x}} (3u_{xx} + 2x u_{xxx}) F(\omega_1, \omega_2) + x e^{\frac{q-1}{x}} G(\omega_1, \omega_2), \]
\[ \omega_1 = x^3 e^{\frac{1}{x}} u_{xx}, \quad \omega_2 = x^4 e^{\frac{1}{x}} (3u_{xx} + xu_{xxx}), \]
\[ A_{4,2}^6 = \langle \partial_u, x \partial_u, \partial_t, t \partial_t + (q - 1)x \partial_x + (tx + qu) \partial_u \rangle \quad q \neq 1 : \]
\[ F = x^{\frac{4q-5}{4q-1}} F(\omega_1, \omega_2), \quad G = xG(\omega_1, \omega_2) + \frac{x \ln |x|}{q-1}, \]
\[ \omega_1 = x^{\frac{q-2}{4q-1}} u_{xx}, \quad \omega_2 = x^{\frac{2q-3}{4q-1}} u_{xxx}, \]
\[ A_{4,2}^7 = \langle \partial_u, x \partial_u, \partial_t, t \partial_t + (tx + u) \partial_u \rangle : \]
\[ F = \frac{1}{u_{xx}} F(x, \frac{u_{xxx}}{u_{xx}}), \quad G = G(x, \frac{u_{xxx}}{u_{xx}}) + x \ln |u_{xx}|, \]
\[ A_{4,2}^8 = \langle x \partial_u, \partial_u, \partial_t, t \partial_t + (1 - q)x \partial_x + (t + u) \partial_u \rangle \quad q \neq 1 : \]
\[ F = x^{4 \omega - 3} F(\omega_1, \omega_2), \quad G = G(\omega_1, \omega_2) - \ln \frac{x}{q - 1}, \]

\[ \omega_1 = x^{\frac{2q - 4}{q - 1}} u_{xx}, \quad \omega_2 = x^{\frac{2q - 2}{q - 1}} u_{xxx}, \]

\[ A_{4.2}^0 = (x \partial_u, \partial_u, \partial_t, t \partial_t + (t + u) \partial_u) : \]

\[ F = \frac{1}{u_{xx}} F(x, \frac{u_{xxx}}{u_{xx}}), \quad G = G(x, \frac{u_{xxx}}{u_{xx}}) + \ln |u_{xx}|. \]

\[ A_{4.3} - \text{invariant equations,} \]

\[ A_{4.3}^1 = (\partial_t, \partial_u, \partial_x, t \partial_t + x \partial_u) : \]

\[ F = e^{-u_x} F(u_{xx}, u_{xxx}), \quad G = e^{-u_x} G(u_{xx}, u_{xxx}), \]

\[ A_{4.3}^2 = (\partial_x, \partial_u, \partial_t, x \partial_x + t \partial_u) : \]

\[ F = \frac{1}{u_x^2} F(\omega_1, \omega_2), \quad G = G(\omega_1, \omega_2) - \ln |u_x|, \]

\[ \omega_1 = \frac{u_{xx}}{u_x^2}, \quad \omega_2 = \frac{u_{xxx}}{u_x^2}, \]

\[ A_{4.3}^3 = (\partial_t, \partial_u, x \partial_u, t \partial_t - \partial_x) : \]

\[ F = e^x F(u_{xx}, u_{xxx}), \quad G = e^x G(u_{xx}, u_{xxx}), \]

\[ A_{4.3}^4 = (\partial_t, x \partial_u, \partial_u, t \partial_t + x^2 \partial_x + xu \partial_u) : \]

\[ F = x^8 e^{\frac{1}{2}} F(\omega_1, \omega_2), \]

\[ G = 4x^6 (3u_{xx} + 2x u_{xxx}) e^{\frac{1}{2}} F(\omega_1, \omega_2) + xe^{\frac{1}{2}} G(\omega_1, \omega_2), \]

\[ \omega_1 = x^3 u_{xx}, \quad \omega_2 = x^4 (3u_{xx} + xu_{xxx}), \]

\[ A_{4.3}^5 = (\partial_u, x \partial_u, \partial_t, x \partial_x + (tx + u) \partial_u) : \]

\[ F = x^4 F(\omega_1, \omega_2), \quad G = x G(\omega_1, \omega_2) + x \ln |x|, \]

\[ \omega_1 = xu_{xx}, \quad \omega_2 = x^2 u_{xxx}, \]

\[ A_{4.3}^6 = (x \partial_u, \partial_u, \partial_t, -x \partial_x + t \partial_u) : \]

\[ F = x^4 F(\omega_1, \omega_2), \quad G = G(\omega_1, \omega_2) - \ln |x|, \]

\[ \omega_1 = x^2 u_{xx}, \quad \omega_2 = x^3 u_{xxx}. \]
$A_{4.4}$— invariant equations,

$A_{4.4}^1 = \langle \partial_u, \partial_x, \partial_t, t \partial_t + (t + x) \partial_x + (x + u) \partial_u \rangle$:

\[
F = e^{3u_x} F(\omega_1, \omega_2), \quad G = G(\omega_1, \omega_2) - \frac{u_x^2}{2},
\]

$\omega_1 = e^{u_x} u_{xx}, \quad \omega_2 = e^{2u_x} u_{xxx},$

$A_{4.4}^2 = \langle \partial_u, x \partial_u, \partial_t, t \partial_t - \partial_x + (tx + u) \partial_u \rangle$:

\[
F = e^x F(\omega_1, \omega_2), \quad G = e^x G(\omega_1, \omega_2) - \frac{x^2}{2},
\]

$\omega_1 = e^x u_{xx}, \quad \omega_2 = e^x u_{xxx},$

$A_{4.4}^3 = \langle x \partial_u, \partial_u, \partial_t, \partial_t + x^2 \partial_x + [t + (x + 1)u] \partial_u \rangle$:

\[
F = x^8 e^{\frac{1}{x}} F(\omega_1, \omega_2),
\]

\[
G = 4x^6 (3u_{xx} + 2xu_{xxx}) e^{\frac{1}{x}} F(\omega_1, \omega_2) + xG(\omega_1, \omega_2) - \frac{1}{2x},
\]

$\omega_1 = x^3 e^{\frac{1}{x}} u_{xx}, \quad \omega_2 = x^4 e^{\frac{1}{x}} (3u_{xx} + xu_{xxx}).$

$A_{4.5}$— invariant equations,

$A_{4.5}^1 = \langle \partial_t, \partial_x, \partial_u, t \partial_t + qx \partial_x + pu \partial_u \rangle, \ p \neq q$:

\[
F = u_{xx}^{\frac{1-q}{p-1}} F(u_x, \frac{u_{xxx}}{u_{xx}^2}), \quad G = u_{xx}^{\frac{1-q}{p-1}} G(u_x, \frac{u_{xxx}}{u_{xx}^2}),
\]

$\omega_1 = u_x^{\frac{p-2q}{p-q}} u_{xx}, \quad \omega_2 = u_x^{\frac{p-3q}{p-q}} u_{xxx},$

$A_{4.5}^2 = \langle \partial_t, \partial_x, \partial_u, t \partial_t + qx \partial_x + qu \partial_u \rangle$:

\[
F = u_{xx}^{\frac{1-q}{p-1}} F(u_x, \frac{u_{xxx}}{u_{xx}^2}), \quad G = u_{xx}^{\frac{1-q}{p-1}} G(u_x, \frac{u_{xxx}}{u_{xx}^2}),
\]

$A_{4.5}^3 = \langle \partial_x, \partial_t, \partial_u, qt \partial_t + x \partial_x + pu \partial_u \rangle, \ p \neq 1$:

\[
F = u_x^{\frac{4-q}{p-1}} F(\omega_1, \omega_2), \quad G = u_x^{\frac{4-q}{p-1}} G(\omega_1, \omega_2),
\]

$\omega_1 = u_x^{\frac{2-q}{p-q}} u_{xx}, \quad \omega_2 = u_x^{\frac{3-q}{p-q}} u_{xxx},$

$A_{4.5}^4 = \langle \partial_x, \partial_t, \partial_u, t \partial_t + x \partial_x + u \partial_u \rangle, \ p \neq 1$:

\[
F = u_x^{-3} F(u_x, \frac{u_{xxx}}{u_{xx}^2}), \quad G = G(u_x, \frac{u_{xxx}}{u_{xx}^2}),
\]

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\[ A^5_{1,5} = \langle \partial_x, \partial_u, \partial_t, pt \partial_t + x \partial_x + qu \partial_u \rangle, \; q \neq 1 : \]
\[ F = u_x^{\frac{4-q}{q-1}} F(\omega_1, \omega_2), \quad G = u_x^{\frac{2-q}{q-1}} G(\omega_1, \omega_2), \]
\[ \omega_1 = u_x^2 u_{xx}, \quad \omega_2 = u_x^{\frac{3-q}{q-1}} u_{xxx}, \]
\[ A^6_{1,5} = \langle \partial_x, \partial_u, \partial_t, pt \partial_t + x \partial_x + u \partial_u \rangle : \]
\[ F = u_x^{p-1} F(\omega_1, \omega_2), \quad G = u_x^{p-1} G(u_x, \frac{u_{xxx}}{u_{xx}}), \]
\[ A^7_{1,5} = \langle \partial_t, \partial_u, x \partial_u, t \partial_t + (q - p) x \partial_x + qu \partial_u \rangle, \; p \neq q : \]
\[ F = x^{\frac{-1+4q-4p}{q-p}} \partial_x F(\omega_1, \omega_2), \quad G = x^{\frac{-1}{q-p}} G(\omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{q-2p}{q-p}} u_{xx}, \omega_2 = x^{\frac{3q-2p}{q-p}} u_{xxx}, \]
\[ A^8_{1,5} = \langle \partial_t, \partial_u, x \partial_u, t \partial_t + qu \partial_u \rangle, \; q \neq 0 : \]
\[ F = u_x^{\frac{q-1}{q}} F(x, \frac{u_{xxx}}{u_{xx}}), \quad G = u_x^{\frac{1}{q}} G(x, \frac{u_{xxx}}{u_{xx}}), \]
\[ A^9_{1,5} = \langle \partial_t, x \partial_u, \partial_u, t \partial_t + (p - q) x \partial_x + pu \partial_u \rangle, \; p \neq q : \]
\[ F = x^{\frac{-1+4q-4p}{q-p}} \partial_x F(\omega_1, \omega_2), \quad G = x^{\frac{-1}{q-p}} G(\omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{q-2p}{q-p}} u_{xx}, \omega_2 = x^{\frac{3q-2p}{q-p}} u_{xxx}, \]
\[ A^{10}_{1,5} = \langle \partial_u, \partial_t, x \partial_u, q t \partial_t + (1 - p) x \partial_x + u \partial_u \rangle, \; p \neq 1 : \]
\[ F = x^{\frac{4q-4p-q}{p-1}} \partial_x F(\omega_1, \omega_2), \quad G = x^{\frac{q-1}{p-1}} G(\omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{2q-1}{p-1}} u_{xx}, \omega_2 = x^{\frac{3q-2}{p-1}} u_{xxx}, \]
\[ A^{11}_{1,5} = \langle \partial_u, \partial_t, x \partial_u, t \partial_t + u \partial_u \rangle : \]
\[ F = \frac{1}{u_x} F(x, \frac{u_{xxx}}{u_{xx}}), \quad G = G(x, \frac{u_{xxx}}{u_{xx}}), \]
\[ A^{12}_{1,5} = \langle \partial_u, x \partial_u, \partial_t, pt \partial_t + (1 - q) x \partial_x + u \partial_u \rangle, \; q \neq 1 : \]
\[ F = x^{\frac{p+4q-4}{q}} \partial_x F(\omega_1, \omega_2), \quad G = x^{\frac{p-1}{q}} G(\omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{2q-1}{q-1}} u_{xx}, \omega_2 = x^{\frac{3q-2}{q-1}} u_{xxx}, \]
\[ A^{13}_{1,5} = \langle \partial_u, x \partial_u, \partial_t, pt \partial_t + u \partial_u \rangle : \]
\[ F = \frac{1}{u_p^x} F(x, \frac{u_{xxx}}{u_{xx}}), \quad G = u_x^{1-p} G(x, \frac{u_{xxx}}{u_{xx}}), \]
\[ A_{4,5}^{14} = \langle x\partial_u, \partial_t, \partial_u, qt\partial_x + (p - 1)x\partial_x + pu\partial_u \rangle, \quad p \neq 1: \]
\[ F = x^{\frac{4p-4}{p-1}}F(\omega_1, \omega_2), \quad G = x^{\frac{p-2}{p-1}}G(\omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{p-2}{p-1}}u_{xx}, \quad \omega_2 = x^{\frac{2p-3}{p-1}}u_{xxx}, \]
\[ A_{4,5}^{15} = \langle x\partial_u, \partial_u, \partial_t, pt\partial_t + (q - 1)x\partial_x + qu\partial_u \rangle, \quad q \neq 1: \]
\[ F = x^{\frac{4q-4}{q-1}}F(\omega_1, \omega_2), \quad G = x^{\frac{q-2}{q-1}}G(\omega_1, \omega_2), \]
\[ \omega_1 = x^{\frac{q-2}{q-1}}u_{xx}, \quad \omega_2 = x^{\frac{2q-3}{q-1}}u_{xxx}. \]

**A4.6— Invariant Equations,**

\[ A_{4,6}^1 = \langle \partial_t, \partial_x, \partial_u, qt\partial_t + (px + u)\partial_x + (pu - x)\partial_u \rangle: \]
\[ F = e^{(q-4p)\arctan u_x} \frac{(u^2 + 1)^2}{u_x^2} F(\omega_1, \omega_2), \]
\[ G = -\frac{5u_xu_{xx}(2u_x^2u_{xxx} - 3u_xu_{xx}^2 + 2u_{xxx})e^{(q-4p)\arctan u_x}}{(u_x^2 + 1)^4} F(\omega_1, \omega_2) \]
\[ + (u_x^2 + 1)^\frac{3}{2}e^{(q-p)\arctan u_x} G(\omega_1, \omega_2), \]
\[ \omega_1 = \frac{u_{xx}}{(u_x^2 + 1)^{\frac{3}{2}}} e^{-p\arctan u_x}, \quad \omega_2 = \frac{u_x^2u_{xxx} - 3u_xu_{xx}^2 + u_{xxx}}{(u_x^2 + 1)^3} e^{-2p\arctan u_x}, \]
\[ A_{4,6}^2 = \langle \partial_t, \partial_x, \partial_u, x\partial_t - (x^2 + 1)\partial_x + (p - x)u\partial_u \rangle: \]
\[ F = (x^2 + 1)^4 e^{q\arctan x} F(\omega_1, \omega_2), \]
\[ G = 4x(x^2 + 1)^2(2x^2u_{xxx} + 3xu_{xx} + 2u_{xxx})e^{q\arctan x} F(\omega_1, \omega_2) \]
\[ + (x^2 + 1)^\frac{5}{2}e^{(q-p)\arctan x} G(\omega_1, \omega_2), \]
\[ \omega_1 = (x^2 + 1)^\frac{3}{2} e^{p\arctan x} u_{xx}, \quad \omega_2 = (x^2 + 1)^\frac{3}{2} e^{p\arctan x} (x^2u_{xxx} + 3xu_{xx} + u_{xxx}). \]

**A4.7— Invariant Equations,**

\[ A_{4,7}^1 = \langle \partial_u, \partial_x, \partial_t, \partial_x + x\partial_u, -\partial_t + x\partial_x + 2u\partial_u \rangle: \]
\[ F = e^{-4t}F(u_{xx}, e^{-t}u_{xxx}), \quad G = e^{-2t}G(u_{xx}, e^{-t}u_{xxx}) - \frac{u_x^2}{2}, \]
\[ A_{4,7}^2 = \langle \partial_u, \partial_x, \partial_t + x\partial_u, t\partial_t + (t + x)\partial_x + \left(\frac{t^2}{2} + 2u\right)\partial_u \rangle: \]
\[ F = (t - u_x)^3 F(u_{xx}, \omega), \quad G = (t - u_x)G(u_{xx}, \omega) + (t - u_x) \ln |t - u_x|, \]
\[ \omega = (t - u_x)u_{xxx}, \]
\[ A_{4,7}^3 = \langle \partial_u, x \partial_u, -\partial_x, x \partial_x + (u - \frac{x^2}{2}) \partial_u \rangle : \]
\[ F = \frac{1}{(u_{xx} + 1)^4} F(t, \omega), \quad G = \frac{1}{u_{xx} + 1} G(t, \omega), \]
\[ \omega = \frac{u_{xxx}}{(u_{xx} + 1)^2}, \]
\[ A_{4,7}^4 = \langle \partial_u, x \partial_u, -\partial_x, \partial_t + x \partial_x + (u - \frac{x^2}{2}) \partial_u \rangle : \]
\[ F = e^{4t} F(\omega_1, \omega_2), \quad G = e^t G(\omega_1, \omega_2), \]
\[ \omega_1 = e^t (u_{xx} + 1), \omega_2 = e^{2t} u_{xxx}, \]
\[ A_{4,7}^5 = \langle \partial_u, x \partial_u, \partial_t - \partial_x, t \partial_t + x \partial_x + \left( \frac{t^2}{2} + tx + 2u \right) \partial_u \rangle : \]
\[ F = (t + x)^3 F(u_{xx}, \omega), \quad G = (t + x)G(u_{xx}, \omega) + (t + x) \ln |t + x|, \]
\[ \omega = (t + x)u_{xxx}, \]
\[ A_{4,7}^6 = \langle x \partial_u, \partial_u, x^2 \partial_x + xu \partial_u, -x \partial_x + (u - \frac{1}{2x}) \partial_u \rangle : \]
\[ F = x^8 e^{-4x^3u_{xx}} F(t, \omega), \]
\[ G = 4x^6 e^{-4x^3u_{xx}} (2xu_{xxx} + 3u_{xx}) F(t, \omega) + xe^{-2x^3u_{xx}} G(t, \omega), \]
\[ \omega = x^4 e^{-x^3u_{xx}} (xu_{xxx} + 3u_{xx}), \]
\[ A_{4,7}^7 = \langle x \partial_u, \partial_u, x^2 \partial_x + xu \partial_u, \partial_t - x \partial_x + (u - \frac{1}{2x}) \partial_u \rangle : \]
\[ F = x^8 e^{4t} F(\omega_1, \omega_2), \]
\[ G = 4x^6 e^{4t} (2xu_{xxx} + 3u_{xx}) F(\omega_1, \omega_2) + xe^{2t} G(\omega_1, \omega_2), \]
\[ \omega_1 = x^3 u_{xx} + t, \quad \omega_2 = x^4 e^t (xu_{xxx} + 3u_{xx}), \]
\[ A_{4,7}^8 = \langle x \partial_u, \partial_u, \partial_t + x^2 \partial_x + xu \partial_u, t \partial_t - x \partial_x + (t + u + \frac{t^2x}{2}) \partial_u \rangle : \]
\[ F = x^5 (1 + tx)^3 F(\omega_1, \omega_2), \]
\[ G = 4x^5 t (1 + tx)(2xu_{xxx} + 3u_{xx})t + 2u_{xxx}] F(\omega_1, \omega_2) + (1 + tx) G(\omega_1, \omega_2) \]
\[ + (1 + tx) \ln \frac{1 + tx}{x}, \]
\[ \omega_1 = x^3 ux, \quad \omega_2 = x^4 [(xuxxx + 3uxx)t + uxxx]. \]

**A\(_{4,8}\) - invariant equations,**

\[ A_{4,8}^1 = \langle \partial_u, \partial_t, \partial_x + t\partial_u, t\partial_t + qx\partial_x + (q + 1)u\partial_u \rangle : \]
\[ F = u_x^{q-1} F(\omega_1, \omega_2), \quad G = u_x^q G(\omega_1, \omega_2) + x, \]
\[ \omega_1 = u_x^{q-1} u_{xx}, \quad \omega_2 = u_x^{2q-1} u_{xxx}, \]

\[ A_{4,8}^2 = \langle \partial_u, \partial_t, (t + x)\partial_u, t\partial_t + x\partial_x + (q + 1)u\partial_u \rangle : \]
\[ F = x^3 F(\omega_1, \omega_2), \quad G = x^q G(\omega_1, \omega_2) + u_x, \]
\[ \omega_1 = x^{1-q} u_{xx}, \quad \omega_2 = x^{2-q} u_{xxx}, \]

\[ A_{4,8}^3 = \langle \partial_u, \partial_x, t\partial_x + x\partial_u, (1 - q)t\partial_t + x\partial_x + (1 + q)u\partial_u \rangle, \quad q \neq 1 : \]
\[ F = t^{\frac{3+q}{1-q}} F(\omega_1, \omega_2), \quad G = t^{\frac{2q}{1-q}} G(\omega_1, \omega_2) - \frac{u_x^2}{2}, \]
\[ \omega_1 = tu_{xx}, \quad \omega_2 = t^{\frac{3+q}{q}} u_{xxx}, \]

\[ A_{4,8}^4 = \langle \partial_u, \partial_x, t\partial_x + x\partial_u, x\partial_x + 2u\partial_u \rangle, \quad q \neq 1 : \]
\[ F = \frac{1}{u_{xx}^q} F(t, u_{xx}), \quad G = \frac{1}{u_{xx}^{2q}} G(t, u_{xx}) - \frac{u_x^2}{2}, \]

\[ A_{4,8}^5 = \langle \partial_u, \partial_x, \partial_t + x\partial_u, qt\partial_t + x\partial_x + (1 + q)u\partial_u \rangle, \quad q \neq 0 : \]
\[ F = (t - u_x)^{\frac{1+q}{q}} F(\omega_1, \omega_2), \quad G = (t - u_x)^{\frac{1}{q}} G(\omega_1, \omega_2), \]
\[ \omega_1 = (t - u_x)^{\frac{1-q}{q}} u_{xx}, \quad \omega_2 = (t - u_x)^{\frac{2-q}{q}} u_{xxx}, \]

\[ A_{4,8}^6 = \langle \partial_u, \partial_x, \partial_t + x\partial_u, x\partial_x + u\partial_u \rangle : \]
\[ F = \frac{1}{u_{xx}^q} F(\omega_1, \omega_2), \quad G = \frac{1}{u_{xx}^{2q}} G(\omega_1, \omega_2), \]
\[ \omega_1 = t - u_x, \quad \omega_2 = \frac{u_{xxx}}{u_{xx}^2}, \]

\[ A_{4,8}^7 = \langle \partial_u, x\partial_u, -\partial_x, -x\partial_x + t\partial_u \rangle : \]
\[ F = \frac{1}{u_{xx}^q} F(t, \omega), \quad G = G(t, \omega) + \frac{\ln |u_{xx}|}{2}, \]
\[ \omega = \frac{u_{xxx}}{u_x^2}, \]

\[ A_{4,8}^8 = \langle \partial_u, x \partial_u, -\partial_x, qx \partial_x + (1 + q)u \partial_u \rangle, \quad q \neq 1 : \]

\[ F = \frac{u_{xx}^{q-1}}{u_x} F(t, \omega), \quad G = u_{xx}^{q+1} G(t, \omega), \]

\[ \omega = \frac{1}{u_x^2} u_{xxx}, \]

\[ A_{4,8}^9 = \langle \partial_u, x \partial_u, -\partial_x, x \partial_x + 2u \partial_u \rangle : \]

\[ F = \frac{1}{u_x^4} F(t, u_{xx}), \quad G = \frac{1}{u_x^2} G(t, u_{xx}), \]

\[ A_{4,8}^{10} = \langle \partial_u, x \partial_u, -\partial_x, \partial_t + qx \partial_x + (1 + q)u \partial_u \rangle : \]

\[ F = e^{(1+q)t} F(\omega_1, \omega_2), \quad G = e^{q\omega} G(\omega_1, \omega_2), \]

\[ \omega_1 = e^{(q-1)t} u_{xx}, \quad \omega_2 = e^{(2q-1)t} u_{xxx}, \]

\[ A_{4,8}^{11} = \langle \partial_u, x \partial_u, \partial_t - \partial_x, qt \partial_t + (1 + q)x \partial_x + qu \partial_u \rangle, \quad q \neq 0 : \]

\[ F = (t + x)^2 F(\omega_1, \omega_2), \quad G = (t + x)^{\frac{1}{q}} G(\omega_1, \omega_2), \]

\[ \omega_1 = (t + x)^{\frac{q-1}{q}} u_{xx}, \quad \omega_2 = (t + x)^{\frac{2q-1}{q}} u_{xxx}, \]

\[ A_{4,8}^{12} = \langle \partial_u, x \partial_u, \partial_t - \partial_x, x \partial_x \rangle : \]

\[ F = F(t + x, \frac{u_{xxx}}{u_{xx}}), \quad G = u_{xx} G(t + x, \frac{u_{xxx}}{u_{xx}}), \]

\[ A_{4,8}^{13} = \langle \partial_u, x \partial_u, \partial_t - \partial_x, \alpha \partial_t + x \partial_x \rangle, \quad \alpha \neq 0 : \]

\[ F = F(\omega_1, \omega_2), \quad G = e^{\frac{\omega_1}{\alpha}} G(\omega_1, \omega_2), \]

\[ \omega_1 = e^{-\frac{\omega_1}{\alpha}} u_{xx}, \quad \omega_2 = e^{-\frac{\omega_1}{\alpha}} u_{xxx}, \]

\[ A_{4,8}^{14} = \langle x \partial_u, \partial_u, x^2 \partial_x + xu \partial_u, -qx \partial_x + u \partial_u \rangle, \quad q \neq 1 : \]

\[ F = x^{\frac{q+1}{q-1}} u_{xx}^{\frac{q}{q-1}} F(t, \omega), \]

\[ G = 4x^{\frac{q+1}{q-1}} \frac{u_{xx}^{1+q}}{4q} (3u_{xx} + 2xu_{xxx}) F(t, \omega) + x^{\frac{q+1}{q-1}} u_{xx}^{\frac{1+q}{q}} G(t, \omega), \]

\[ \omega = x^{\frac{q+1}{q-1}} u_{xx}^{\frac{q-1}{q}} (3u_{xx} + xu_{xxx}), \]

\[ A_{4,8}^{15} = \langle x \partial_u, \partial_u, x^2 \partial_x + xu \partial_u, -x \partial_x + u \partial_u \rangle : \]

\[ F = \frac{F(t, x^3u_{xx})}{x^8(3u_{xx} + xu_{xxx})}, \]
\[
G = \frac{3u_{xx} + 2xu_{xxx}}{x^6(3u_{xx} + xu_{xxx})^4}F(t, x^3u_{xx}) + \frac{G(t, x^3u_{xx})}{x^7(3u_{xx} + xu_{xxx})^2},
\]

\[A_{4.8}^{16} = \langle x\partial_u, \partial_u, x^2\partial_x + xu\partial_u, x\partial_x + (tx + u)\partial_u \rangle:
\]

\[
F = \frac{x^2}{u_{xx}}F(t, \omega),
\]

\[
G = \frac{4(3u_{xx} + 2xu_{xxx})}{u_{xx}^2}F(t, \omega) + xG(t, \omega) + \frac{x}{2} \ln |x^3u_{xx}|,
\]

\[\omega = \frac{3u_{xx} + xu_{xxx}}{x^4u_{xx}}.
\]

\[A_{4.8}^{17} = \langle x\partial_u, \partial_u, x^2\partial_x + xu\partial_u, \partial_t - qx\partial_x + u\partial_u \rangle:
\]

\[
F = x^8e^{qt}F(\omega_1, \omega_2),
\]

\[
G = 4x^6e^{qt}(3u_{xx} + 2xu_{xxx})F(\omega_1, \omega_2) + xe^{(1+q)t}G(\omega_1, \omega_2),
\]

\[\omega_1 = x^3e^{(q-1)t}u_{xx}, \quad \omega_2 = x^4e^{(2q-1)t}(3u_{xx} + xu_{xxx}),
\]

\[A_{4.8}^{18} = \langle u\partial_x, \partial_x, \partial_t + xu\partial_x + u^2\partial_u, qt\partial_t + x\partial_x - qu\partial_u \rangle, \quad q \neq 0:
\]

\[
F = \frac{u^5(1 + tu)^3}{u_{xx}^3}F(\omega_1, \omega_2),
\]

\[
G = \frac{u^3(1 + tu)^3}{u_{xx}^3}[-5(2u_xu_{xxx} - 3u_{xx}^2)u_{xx}u^2 + 8u_xu_{xxx} - 3u_{xx}^2]u_{xx}^2u
\]

\[
+ 8u_x^4u_{xx}]F(\omega_1, \omega_2) - u^{1-\frac{3}{4}}(1 + tu)^{\frac{1}{4}}u_xG(\omega_1, \omega_2),
\]

\[\omega_1 = \frac{u^{2+\frac{3}{4}}(1 + tu)^{1-\frac{3}{4}}}{u_{xx}},
\]

\[\omega_2 = \frac{u^{6-\frac{3}{4}}(1 + tu)^{\frac{1}{4}-2}}{u_x^5}[(u_xu_{xxx} - 3u_{xx}^2)u + 3u_{xx}^2u_{xx}],
\]

\[A_{4.8}^{19} = \langle u\partial_x, \partial_x, \partial_t + xu\partial_x + u^2\partial_u, x\partial_x \rangle:
\]

\[
F = x^8F(\omega_1, \omega_2),
\]

\[
G = 4x^6(3u_{xx} + 2xu_{xxx})F(\omega_1, \omega_2) + x^4u_{xx}G(\omega_1, \omega_2),
\]

\[\omega_1 = t + \frac{1}{x} \quad \omega_2 = \frac{x(3u_{xx} + xu_{xxx})}{u_{xx}}.
\]
\(A_{4.9}\) - invariant equations,

\[A_{4.9}^1 = \langle \partial_u, \partial_x, t \partial_x + x \partial_u, -(1 + t^2) \partial_t + (q - t)x \partial_x + \left(-\frac{x^2}{2} + 2qu\right) \partial_u \rangle : \]

\[F = e^{-4q \arctan t} (1 + t^2) F(\omega_1, \omega_2), \quad G = \frac{e^{-2q \arctan t}}{1 + t^2} G(\omega_1, \omega_2) - \frac{u_x^2}{2}, \]

\[\omega_1 = (1 + t^2)u_{xx} - t, \quad \omega_2 = e^{-q \arctan t} (1 + t^2)^3 u_{xxx}. \]

\(A_{4.10}\) - invariant equations,

\[A_{4.10}^1 = \langle \partial_x, \partial_u, x \partial_x + u \partial_u, u \partial_x - x \partial_u \rangle : \]

\[F = \frac{(1 + u_x^2)^4}{u_{xx}^4} F(t, \omega), \]

\[G = \left[15 \frac{u_x^2}{u_{xx}} (1 + u_x^2)^2 - 10 \frac{u_x u_{xxx}}{u_{xx}^3} (1 + u_x^2)^3 \right] F(t, \omega) + \frac{(1 + u_x^2)^2}{u_{xx}^2} G(t, \omega), \]

\[\omega = \frac{u_{xxx}}{u_{xx}^2} (1 + u_x^2) - 3u_x, \]

\[A_{4.10}^2 = \langle \partial_x, \partial_u, x \partial_x + u \partial_u, \partial_t + u \partial_x - x \partial_u \rangle : \]

\[F = \frac{(1 + u_x^2)^4}{u_{xx}^4} F(\omega_1, \omega_2), \]

\[G = \left[15 \frac{u_x(1 + u_x^2)}{u_{xx}} (u_x - 2\omega_1) - 10 \frac{u_{xxx}(1 + u_x^2)^3}{u_{xx}^3} (u_x - \omega_1) \right] F(\omega_1, \omega_2) \]

\[+ \frac{(1 + u_x^2)^2}{u_{xx}^2} G(\omega_1, \omega_2), \]

\[\omega_1 = t + \arctan u_x, \quad \omega_2 = \frac{u_{xxx}}{u_{xx}^2}(1 + u_x^2) + 3(\omega_1 - u_x), \]

\[A_{4.10}^3 = \langle \partial_x, \partial_u, \partial_t + x \partial_x + u \partial_u, u \partial_x - x \partial_u \rangle : \]

\[F = \frac{e^{ut}}{(1 + u_x^2)^2} F(\omega_1, \omega_2), \]

\[G = -5 \frac{u_x u_{xx}}{(1 + u_x^2)^4} [2u_{xxx}(1 + u_x^2) - 3u_x u_{xxx}^2] e^{ut} F(\omega_1, \omega_2) + (1 + u_x^2)^4 e^{ut} G(\omega_1, \omega_2), \]

\[\omega_1 = \frac{u_{xx}}{(1 + u_x^2)^3} e^t, \quad \omega_2 = \frac{u_{xxx}(1 + u_x^2) - 3u_x u_{xx}^2 e^{2t}}{(1 + u_x^2)^3}, \]

\[A_{4.10}^4 = \langle \partial_x, \partial_u, \partial_t + x \partial_x + u \partial_u, \partial_t + u \partial_x - x \partial_u \rangle : \]
\[ F = \frac{e^{4(t + \arctan u_x)}}{(1 + u_x^2)^2} F(\omega_1, \omega_2), \]

\[ G = -5 \frac{u_x u_{xx}}{(1 + u_x^2)^4} [2 u_{xxx}(1 + u_x^2) - 3 u_x u_{xx}^2] e^{3t + 4 \arctan u_x} F(\omega_1, \omega_2) \]

\[ + (1 + u_x^2)^{\frac{3}{2}} e^{\arctan u_x} G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{u_{xx}}{(1 + u_x^2)^{\frac{3}{2}}} e^{t + \arctan u_x}, \quad \omega_2 = \frac{u_{xxx}(1 + u_x^2) - 3 u_x u_{xx}^2}{(1 + u_x^2)^3} e^{2(t + \arctan u_x)}, \]

\[ A_{4,10}^5 = \langle \partial_u, x \partial_u, u \partial_u, -(1 + x^2) \partial_x - xu \partial_u \rangle : \]

\[ F = (1 + x^2)^4 F(t, \omega), \]
\[ G = 4x(1 + x^2)^2 [2(1 + x^2) u_{xxx} + 3x u_{xx}] F(t, \omega) + (1 + x^2)^2 u_{xx} G(t, \omega), \]

\[ \omega = \frac{u_{xxx}}{u_{xx}} (1 + x^2) + 3x, \]

\[ A_{4,10}^6 = \langle \partial_u, x \partial_u, u \partial_u, \partial_t - (1 + x^2) \partial_x - xu \partial_u \rangle : \]

\[ F = (1 + x^2)^4 F(\omega_1, \omega_2), \]

\[ G = -8(1 + x^2)^2 \int_t^\omega [\tan^2(\omega_1 - s) + 1][3(x - \tan(\omega_1 - s)) u_{xx} + (1 + x^2) u_{xxx}] ds \]

\[ F(\omega_1, \omega_2) + (1 + x^2)^2 u_{xx} G(\omega_1, \omega_2), \]

\[ \omega_1 = t + \arctan x, \quad \omega_2 = \frac{u_{xxx}}{u_{xx}} (1 + x^2) + 3(x - \omega_1), \]

\[ A_{4,10}^7 = \langle \partial_u, x \partial_u, \partial_t + u \partial_u, -(1 + x^2) \partial_x - xu \partial_u \rangle : \]

\[ F = (1 + x^2)^4 F(\omega_1, \omega_2), \]

\[ G = 4x(1 + x^2)^2 [2(1 + x^2) u_{xxx} + 3x u_{xx}] F(\omega_1, \omega_2) + (1 + x^2)^\frac{3}{2} e^t G(\omega_1, \omega_2), \]

\[ \omega_1 = (1 + x^2)^\frac{3}{2} e^{-t} u_{xx}; \quad \omega_2 = (1 + x^2)^\frac{3}{2} e^{-t} [(1 + x^2) u_{xxx} + 3x u_{xx}], \]

\[ A_{4,10}^8 = \langle \partial_u, x \partial_u, \partial_t + u \partial_u, \partial_t - (1 + x^2) \partial_x - xu \partial_u \rangle : \]

\[ F = (1 + x^2)^4 F(\omega_1, \omega_2), \]

\[ G = 4x(1 + x^2)^2 [2(1 + x^2) u_{xxx} + 3x u_{xx}] F(\omega_1, \omega_2) + (1 + x^2)^\frac{3}{2} e^{t + \arctan x} G(\omega_1, \omega_2), \]

\[ \omega_1 = \frac{(1 + x^2)^\frac{3}{2}}{e^{t + \arctan x} u_{xx}}, \quad \omega_2 = \frac{(1 + x^2)^\frac{3}{2}}{e^{t + \arctan x} [(1 + x^2) u_{xxx} + 3x u_{xx}]} \]