Enhançons, Fuzzy Spheres and Multi–Monopoles

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Abstract

We study the “enhançon”, a spherical hypersurface apparently made of D–branes, which arises in string theory studies of large $N$ $SU(N)$ pure gauge theories with eight supercharges. When the gauge theory is 2+1 dimensional, the enhançon is an $S^2$. A relation to charge $N$ BPS multi–monopoles is exploited to uncover many of its detailed properties. It is simply a spherical slice through an Atiyah–Hitchin–like submanifold of the charge $N$ BPS monopole moduli space. In the form of Nahm data, it is built from the $N$ dimensional irreducible representation of $SU(2)$. In this sense the enhançon is a non–commutative sphere, reminiscent of the spherical “dielectric” branes of Myers.
1 Origins of the Enhançon

Consider compactifying ten dimensional type II string theory on the four dimensional K3 surface of volume $V$. This gives a six dimensional theory with $\mathcal{N}=2$ supersymmetry; in other words, sixteen supercharges. Consider further wrapping $N$ D($p+4$)–branes on the $K3$. Then there is an effective $p$–dimensional extended object in the six dimensions. It is in fact a BPS solution, and there are eight supercharges preserved by this situation. Furthermore, there is an $SU(N)$ pure gauge theory with eight supercharges on the $(p+1)$–dimensional world volume of the BPS soliton. This soliton has a description as a bound state of the wrapped brane and a negatively charged D$p$–brane\[1\]. We shall refer to this as the D($p+4$)–D$p^*$ system, where the asterisk ($^*$) is to remind us that this is not an ordinary D$p$ brane, since that would be an instanton.

Let us focus on $p = 2$, hence studying type IIA. The supergravity theory contains twenty–four $U(1)$’s coming from the various R-R potentials in the theory. Of these, twenty–two come from wrapping the two–form on the 19+3 two–cycles of $K3$. The remaining two are special $U(1)$’s for our purposes: One of them arises from wrapping the five–form entirely on $K3$, while the final one is simply the plain one–form already present in the uncompactified theory.

In fact, the BPS soliton is actually a monopole of one of the six dimensional $U(1)$’s. It is obvious which $U(1)$ this is; the diagonal combination of the two special ones we mentioned above. Actually, we can simply ignore the 2 spatial directions in which the soliton is extended and see that the monopole sector (recall that it is also coupled to gravity) is nothing more than the usual problem of monopoles\[2, 3\] in a 3+1 dimensional gauge theory with an adjoint Higgs. The first order “Bogomolnyi” equations\[4\] are:

$$B_i \equiv \frac{1}{2} \epsilon_{ijk} F_{jk} = D_i H ,$$

with

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]; \quad D_i H = \partial_i H + [A_i, H] ,$$

(1.1)

with gauge invariance ($g(x) \in SU(2)$):

$$A_i \rightarrow g^{-1} A_i g + g^{-1} \partial_i g; \quad H \rightarrow g^{-1} H g .$$

(1.2)

Static, finite energy monopole solutions satisfy

$$\|H(x)\| \equiv \frac{1}{2} \text{Tr} [H^* H] \rightarrow H \quad \text{as} \quad r \rightarrow \infty ,$$

(1.3)
where $\mathbf{x} = (x_1, x_2, x_3)$ and $r^2 = x_1^2 + x_2^2 + x_3^2$. The $SU(2)$ is spontaneously broken to $U(1)$, by the Higgs vacuum expectation value (“vev”) $H$, whose magnetic charge the monopoles carry. For orientation, and for later use, the explicitly known fields of the one monopole solution is:

$$H(r) = \frac{1}{r} \left( \coth r - \frac{1}{r} \right) i \sigma_i x_i ; \quad A_i(r) = \frac{1}{r} \left( \frac{1}{\sinh r} - \frac{1}{r} \right) i \epsilon_{ijk} \sigma_j x_k ,$$

where $\mathbf{x} = (0, 0, r)$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(1.5)

Note that it is spherically symmetric and has been normalized (for later use) such that $H \to 1 - r^{-1} + \ldots$, as $r \to \infty$, with unit magnetic charge.

Where did the $SU(2)$ come from? When the $K3$’s volume reaches the value $V_* \equiv (2\pi \sqrt{\alpha'})^4$, our $U(1)$ is enhanced to $SU(2)$. This is a stringy phenomenon which has no description in supergravity, since (for example) the $W$–bosons for this $SU(2)$ are made of wrapped D4–branes.

This is an interesting system to use to study the large $N$ limit of the $\mathcal{N}=2$ $SU(N)$ gauge theory along the lines of recent ideas such as those in ref.[6]. A large number of these D6–D2* objects give a supergravity solution, which might be expected to encode (at least) some of the large $N$ physics. Interestingly, the supergravity solution which one naively writes down suffers from a naked singularity known as a “repulson”[7] which is unphysical, and incompatible with the physics of the gauge theory. One expects that there should be a sensible supergravity solution, valid for $g_s N$ large, where $g_s$ is the string coupling.

In fact[1] the repulson is not present, since it represents supergravity’s best attempt to construct a solution with the correct asymptotic charges. In the solution (not displayed here since we will not need it; see ref.[3]), the volume of the $K3$, set to $V$ asymptotically, actually decreases as one approaches the core of the configuration. At the centre, the $K3$ radius is zero, and this is the singularity.

This ignores rather interesting physics, however. At a finite distance from the putative singularity, the volume of the $K3$ gets to $V=V_*$, so the stringy phenomena—including new massless fields—giving the enhanced $SU(2)$ should have played a role. So the aspects of the supergravity solution near and inside the special radius, called the “enhançon radius”, should not be taken seriously at all, since it ignored this stringy physics.
To a first approximation, the supergravity solution should only be taken as physical down to the enhançon radius \( r_e \). That locus of points, a two–sphere \( S^2 \), is itself called an “enhançon”. It deserves a name, and to be considered as an object in its own right, since D6–D2* objects probing this geometry seem to spread out or smear onto it as they approach it, losing their identity (see ref.[1] and later in this paper). In this way, we see that the enhançon is a hypersurface apparently made of branes which have puffed up into a sphere. There is a natural generalization of this all to situations involving branes of different dimensions, (with the enhançon a sphere of different dimensionality), and including orientifolds. This pertains to \( SU(N) \), \( SO(2N) \), \( SO(2N+1) \) and \( USp(2N) \) gauge theories with eight supercharges in various dimensions[1, 8].

1.1 Overview of This Paper

In the rest of this paper we will uncover many new properties of the enhançon pertaining to the 2+1 dimensional \( SU(N) \) gauge theory. Many of the generic features will have meaning in other dimensions and for other gauge groups. We will obtain detailed information because we can exploit the connection[3, 10, 11] to the classical physics of monopoles. The relevant properties of the enhançon already alluded to so far are reviewed in the next section, and the details that we will need are emphasized, including the perturbative expression for the metric on the spacetime geometry as seen by a probe brane. Section 3 shows that a number of (metric) geometrical details of the enhançon can be learned from the observation that the full non–perturbative spacetime geometry (as seen by the probe in the decoupling limit \( \alpha' \to 0 \)) can be deduced from the Atiyah–Hitchin manifold[4], and appropriate generalisations thereof, which we conjecture to exist as an ADE family. Section 3 focuses on the description of the system of \( N \) monopoles \textit{via} Nahm data[15]. The point is simply that the since the description of the \( N \) coincident D6–D2* branes carrying the \( SU(N) \) is as classical monopoles, we ought to learn more about them by studying the well–established technology for describing multi–monopoles. In this way, we see that there is some essential non–commutativity in the description, and we exploit this in section 5 to show that the enhançon is actually a “fuzzy” or non–commutative sphere[16]. This makes contact with the “dielectric brane” construction of Myers[17], and we discuss the similarities and differences between the two cases.
Figure 1 is a summary of some of the properties of the geometry.

![Figure 1](image)

Figure 1: A summary of the geometry uncovered. There are three spheres shown: The unphysical repulson (innermost) was seen to be removed in ref. [1], and replaced by the enhançon (next). Here, we find that the true non–perturbative enhançon radius is slightly different from this (shown outermost), although the correction is exponentially small at large $N$. The enhançon is a non–commutative sphere. There is smooth matching onto the spherically symmetric supergravity solution at large $N$. The region interior to the enhançon is the core of an $(N−1)$–monopole. There is an unbroken $SU(2)$ there.

Finally, in section 6 we make a tentative but contentful conjecture about a possible stringy dual description of the 2+1 dimensional $SU(N)$ gauge theory at large $N$. It is motivated by the fact that the part of the supergravity geometry which can be written, in the decoupling limit, entirely in terms of gauge theory quantities (after referral to the “frame” of a monopole probe) is a part of monopole moduli space. It is suggested that the Nahm data (a family of $N×N$ matrices depending upon a single coordinate) might be used at large $N$ to describe a matrix string theory with many of the properties needed for a stringy dual. The construction of the string is likely quite analogous to the more familiar matrix strings, but this new string theory inherits rich properties of the monopole physics, since it is built out of Nahm data. Further work is needed on this proposal. We close with an added note on other work.
SU(N) Gauge theory and BPS Monopoles

In fact, the phenomenology of the enhançon in supergravity is consistent with the monopole physics and with the physics of the 2+1 dimensional SU(N) gauge theory at large N. The moduli space of supersymmetric vacua of the theory is parameterized by the vevs of the three adjoint scalars \( \Phi_i \) (\( i = 3, 4, 5 \)) in the vector–multiplet. This is \( 3(N-1) \) dimensional, since they generically live in the Cartan subalgebra of the gauge group when satisfying this condition.

At a generic point on this space, the gauge symmetry is therefore \( U(1)^{N-1} \), (hence the name “Coulomb branch”) and these Abelian gauge fields may be dualized to give \( N-1 \) more scalars. The moduli space is therefore \( 4(N-1) \) dimensional. The complete, quantum corrected moduli space is a smooth hyperkähler manifold, given that there are eight supercharges. In fact [9, 10, 11], the moduli space [21] is that of \( N \) BPS monopoles.

In the probe computation of ref. [1], a single D6–D2* object was used to probe all of the others, in order to investigate how the geometry looks from its point of view. As moving the probe slowly in the background of its siblings is a BPS process, there should be no potential in the effective Lagrangian for this procedure, and only kinetic terms. From these terms may be read the metric of spacetime as seen by the probe [22, 23]. The result of the computation is:

\[
ds^2 = F(r) \left( dr^2 + r^2 d\Omega^2 \right) + F(r)^{-1} (ds/2 - \mu_2 C_\phi d\phi/2)^2 ,
\]

where

\[
F(r) = \frac{Z_6}{2g_s} (\mu_6 V(r) - \mu_2) ,
\]

and \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \) and \( C_\phi = -(r_6/g_s) \cos \theta \). The volume of K3 is \( V(r) = VZ_2(r)/Z_6(r) \), with

\[
Z_2 = 1 + \frac{r_2}{r} , \quad r_2 = -\frac{(2\pi)^4 g_s N_\alpha'^{5/2}}{2V} ,
\]

\[
Z_6 = 1 + \frac{r_6}{r} , \quad r_6 = \frac{g_s N_\alpha'^{1/2}}{2} ,
\]

the harmonic functions appearing in the supergravity solution, which we do not display here. The basic D6– and D2–brane charges are [24] \( \mu_6 = (2\pi)^{-6}\alpha'^{-7/2} \) and \( \mu_2 = (2\pi)^{-2}\alpha'^{-3/2} \), respectively.
Notice that the metric (2.3) is singular where the monopole’s mass per unit volume, \( \tau = (\mu_6 V(r) - \mu_2) / g_s \) vanishes, which is at

\[
V(r) = \mu_2 / \mu_6 = (2\pi \sqrt{\alpha'})^4 \equiv V_* .
\]

This happens at the “enhançon” radius

\[
r_e = \frac{2V}{V - V_*} |r_2| .
\]

This is consistent with the fact that a monopole’s mass is set by the value of the Higgs, while its size is inversely proportional to it. So as \( \mu \) approaches zero at \( r_e \), a monopole probe becomes smeared out as it merges into all the other monopoles at the core. The departure from a sharp description as a heavy point–like object —the smearing— is signaled in the kinetic energy’s divergence.

Since the monopoles cannot go to \( r < r_e \) in the supergravity geometry in a supersymmetric way\[1\] consistent with the gauge dynamics and common sense, it is sensible to conclude that there is simply new geometry and physics in that region, as anticipated in the supergravity discussion of the previous section. Perhaps we can learn more about the enhançon by a closer study of the gauge theory, and hence the monopole physics.

The coupling of the \( SU(N) \) gauge theory is given by

\[
g_{YM}^2 = (2\pi)^4 g_s \alpha' \sqrt{\alpha'} V^{-1} .
\]

To isolate the gauge theory, it is prudent to focus on the limit where we attempt a decoupling limit by sending \( \alpha' \rightarrow 0 \), holding the coupling and \( U=r/\alpha' \) finite\[3\]. In this case, the metric becomes

\[
d s^2 = f(U) \left( dU^2 + U^2 d\Omega^2 \right) + f(U)^{-1} \left( d\sigma - \frac{N}{8\pi^2} A_\phi d\phi \right)^2 ,
\]

where

\[
f(U) = \frac{1}{8\pi^2 g_{YM}^2} \left( 1 - \frac{\lambda}{U} \right) ,
\]

the \( U(1) \) monopole potential is \( A_\phi = \pm 1 - \cos \theta \), and \( \sigma = s\alpha' / 2 \). This metric is meaningful only for \( U \geq \lambda \). It is the Euclidean Taub–NUT metric, with a negative mass. It is a hyperKähler manifold, because \( \nabla f = \nabla \times A \), where \( A = (N/8\pi^2) A_\phi d\phi \).
It is intriguing to note\cite{1} (and we shall try to exploit this more fully later) that only the
gauge theory quantities $U$ (a characteristic energy scale) and $\lambda \equiv g_{YM}^2 N$ (the 't Hooft coupling)
survive the limit, while all other details of the supergravity have disappeared. The enhançon is at $U=\lambda$. A crucial point which can be read off from this geometry is that the enhançon appears as the one–loop correction to the gauge coupling, representing the Landau pole. There
are instanton corrections to this (and hence to the manifold), smoothing out the singular nature at the enhançon. The expectation was expressed in ref.\cite{1} that this manifold is would be thereby
corrected to an Atiyah–Hitchin\cite{14}–like manifold, and we shall see that this is true presently.

From the point of view of the monopole description, this manifold should be related to the
metric on the moduli space of monopoles. It is clearly a submanifold of the full $4N−4$ dimension-
sional metric on what is known as the “strongly centered” moduli space of $N$ BPS monopoles\footnote{“Strongly centred” means that we have the relative moduli space, where the overall center of mass and overall phase of the monopoles are not included.}

### 3 The Role of the Atiyah–Hitchin Manifold

Precisely which submanifold we have here should be of interest to us. First observe that we
can change variables in our probe metric (2.12) by absorbing a factor of $\lambda / 2 = g_{YM}^2 N / 2$ into
the radial variable $U$, defining $\rho = 2U / \lambda$. Further absorb $\psi = \sigma 8\pi^2 / N$ and gauge transform to
$A_\phi = −\cos \theta$. Then we get:

$$ds^2 = g_{YM}^2 N^2 / 32\pi^2 ds^2_{TN}, \quad \text{with}$$

$$ds^2_{TN} = \left(1 - \frac{2}{\rho}\right) \left(d\rho^2 + \rho^2 d\Omega^2\right) + 4 \left(1 - \frac{2}{\rho}\right)^{-1} \left(d\psi + \cos \theta d\phi\right)^2.$$ 

The latter is precisely the form of the Taub–NUT metric that one gets by expanding the
Atiyah–Hitchin metric in large $\rho$ and neglecting the exponential corrections.

For the case of $N = 2$, the Atiyah–Hitchin manifold is the full non–perturbative result for
the moduli space of the $SU(2)$ gauge theory\footnote{1}. In fact for this manifold, written in these
coordinates, the periodicity of $\psi$ is $2\pi$ (see later) and so the $SU(2)$ isometry of the Taub–NUT
manifold is broken to $SO(3) \equiv SU(2)/\mathbb{Z}_2$ by the exponential corrections. The quotientied
sphere $S^3/\mathbb{Z}_2$ at infinity is an orbit under this.
In unscaled coordinate of the $U(1)$ probe gauge theory, this means that the quantity $4\pi^2\sigma = \tilde{\sigma}$ is the $2\pi$ periodic dual scalar to the photon. This periodicity is independent of $N$. Therefore, for arbitrary $N$, the periodicity of $\psi$ is $4\pi/N$. This allows us to characterise the manifolds we need for all $N$:

They have only local $SO(3)$ action, and globally the isometry is broken and we have only the action of $SU(2)/\mathbb{Z}_N$. The manifolds are therefore asymptotically the negative mass Taub–NUT, and the space at infinity is an $S^3/\mathbb{Z}_N$. We will characterise the form of the exponential corrections for arbitrary $N$ in the next section.

### 3.1 The Non–Perturbative Corrections

So this is our first set of information that we learn by studying the monopole physics: The exponential corrections to our manifold (as seen by the probe) —and hence information about the neighbourhood and interior of the enhançon geometry— are of the same form as those for the Atiyah–Hitchin manifold (with a generalisation we shall characterise shortly). This is remarkably fortuitous, and will teach us more presently. For definiteness, let us display the full Atiyah–Hitchin manifold [14, 25]:

$$ds^2_{AH} = f^2 d\rho^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2,$$

where

$$\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi ;$$
$$\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi ;$$
$$\sigma_3 = d\psi + \cos \theta d\phi ;$$

$$\frac{2bc \, da}{f \, d\rho} = (b - c)^2 - a^2, \text{ and cyclic perms.}; \quad \rho = 2K \left( \frac{\sin \beta}{2} \right), \quad (3.14)$$

and $K(k)$ is the elliptic integral of the first kind:

$$K(k) = \int_0^{\frac{\pi}{2}} \left(1 - k^2 \sin^2 \tau \right)^{\frac{3}{2}} d\tau. \quad (3.15)$$

Also, $k=\sin(\beta/2)$, the “modulus”, runs from 0 to 1, so $\pi \leq \rho \leq \infty$.

The difference between this and negative mass Taub–NUT (3.13) at large $\rho$ is exponential, i.e., of the form $e^{-\rho}$. In the case of $SU(2)$ gauge theory ($N = 2$), this translates (using the
formulae above (3.13) into precisely the right form to be instanton corrections $e^{-U/g^2_{YM}}$, and this has been proven to be the correct interpretation from a number of points of view\textsuperscript{9, 12}.

For $SU(N)$, we expect instantons in the field theory to have essentially the same action, and so this translates into a set of exponential corrections of the form $e^{-N\rho/2}$. So for large $N$ therefore, the corrections to the Taub–NUT manifold are quite small, but the instantons smooth it out on a small enough scale nonetheless. (This smallness of the intanton corrections to $\mathcal{N} = 2$ $SU(N)$ gauge theory moduli space at large $N$ has been noticed in other contexts, e.g. in ref.\textsuperscript{13}.)

So in short, our fully corrected moduli space, which also contains information about the spacetime geometry, is given by a family of manifolds naturally generalising the Atiyah–Hitchin manifold, after rescaling $\rho$, and $\psi$. It would be interesting to characterise these manifolds further. One expects them to be smooth, or at least to contain $4(N-2)$ parameters which allow them to be deformed to a neighbouring smooth manifold for the simple reason that the gauge theory moduli space, not having a Higgs branch to connect to, is expected to be smooth. There is a concern that this is only true for the full $4(N-1)$ dimensional manifold representing the moduli space, and that a restriction to a 4 dimensional submanifold can introduce singularities. This is where the $4(N-2)$ deformation parameters come in, as they represent the frozen moduli of the other monopoles/vacua, now entering as parameters in the reduced theory; intuitively, one expects the process of a single monopole merging with $N$ others to be smooth, or at least smoothenable by moving the others around\textsuperscript{2}.

\subsection*{3.2 The Case of Two Monopoles}

It is now worth reminding ourselves about the physics of the Atiyah–Hitchin manifold, using it as a prototype for our case involving general $N$. The Atiyah–Hitchin manifold\textsuperscript{14} is the metric on the strongly centred moduli space of two BPS monopoles. In fact, the two monopole solution itself (i.e., the gauge and Higgs fields) is not spherically symmetric\textsuperscript{3}. At best, it is axisymmetric, and this is when the two monopoles are coincident. The coordinate $\rho$ represents the asymptotic separation of the monopoles. It really only has this meaning when the monopoles are separated

\footnote{\textsuperscript{2}I am grateful to Gary Gibbons, Juan Maldacena, Robert Myers, and Edward Witten for suggestions and comments on this issue.}

\footnote{\textsuperscript{3}This is generally true for the $N$ monopole solution\textsuperscript{4}, as we shall discuss.}
quite far apart, and then the metric reduces to $ds_{TN}^2$. Closer than this, the monopoles cease to be distinct. Actually, the singularity at $\rho = 2$ in the Taub–NUT metric is completely meaningless, as it is well outside the range of validity of the large $\rho$ expansion used to get that metric. Further proof of this comes from the fact that the monopoles are coincident at $\rho = \pi$. This special (axisymmetric) solution\cite{18, 26} has monopole charge 2, and really has no sensible description in terms of individual monopoles at all. Generically, all we can say (this can be confirmed by a study of the location of the zeros of the Higgs field part of the monopole solution\cite{27}) is that for any $\rho$ the monopoles are spaced symmetrically along an axis, which we can choose to be the $x_3$ axis. The Higgs field has zeros even when the monopoles cease to have any sensible meaning (since they grow large and diffuse), and are often used as a guide to the “location” of the monopoles, despite their finite core size. When $\rho = \pi$, the two Higgs zeros are both at the origin, and this is the coincident case.

Note that despite the fact that the solution is axisymmetric, far away from it, in spacetime ($r \to \infty$), the Higgs field is

$$\frac{1}{2} \text{Tr}[H(r)^*H(r)] = H - \frac{Ne_m}{r} + \cdots$$

(3.16)

for (here) $N = 2$, and this form of the Higgs is generally true for all $N$. Here $e_m = 2\pi/e$ is the basic unit of magnetic charge, Dirac fixed in terms of the electric charge $e$.

Actually, the metric components in the neighbourhood of $\rho = \pi$ are\cite{27}:

$$a = 2(\rho - \pi) + O\left((\rho - \pi)^2\right) + \cdots,$$

$$b = \pi + O\left((\rho - \pi)^2\right) + \cdots,$$

$$c = -\pi + O\left((\rho - \pi)^2\right) + \cdots$$

(3.17)

and so the metric appears to be singular there, given that $a \to 0$. In fact, the $S^3$ of $(\psi, \theta, \phi)$ collapses to a two–sphere, $S^2$, there, but this point is actually a coordinate “bolt” singularity. It is the removal of this bolt which requires $\psi$ to be $2\pi$ periodic, a fact which featured in the previous two subsections.

It should be noted that we have only described a simple cover of the moduli space of two monopoles. There is an addition $\mathbb{Z}_2$ which identifies configurations which correspond to each other after an exchange of the (identical) monopoles. In this way the bolt becomes an $\mathbb{RP}_2$
instead of an $S^2$. We will not have such a symmetry here, so our bolt (or generalisation thereof), for $N > 2$ will always be an $S^2$.

### 3.3 Large $N$ and Spacetime Physics

How are we to make sense of the appearance of the Atiyah–Hitchin–like manifolds in our case, and what can we learn about spacetime physics? Well, we have a four dimensional submanifold of the full metric on moduli space, and so most of the parameters ($4N - 8$ of them) have been fixed. In ref.[1] all of the branes were placed at the origin $r = 0$ in the supergravity discussion. So all of the parameters were frozen except the single probe brane’s position and phase. Although in supergravity they are naively at $r = 0$, this is not the case, and they are smeared out into a sphere of radius $r_e \sim N$.

Now, the Atiyah–Hitchin coordinate $\rho$ should have an interpretation as a separation from the center of mass. For the 2–monopole case that would tell us very little about the spacetime geometry, but since we have $N$ monopoles, and $N$ is large, $U = g_{YM}^2 \rho N/2$ is a good radial coordinate for spacetime, as the center of mass is still close to $U = 0$ when only one probe is separated off. This is why at large $N$ our scaled Atiyah–Hitchin–like manifold has a dual meaning as a relative moduli space for monopoles as well as the spacetime geometry seen by the probe. For small $N$, the coordinate $U$ is not as good a guide to the spacetime geometry.

Note that for any $N$, the spacetime Higgs field will asymptotically behave as in equation (3.16). This is why the supergravity solution can be spherically symmetric, as its asymptotically spherically symmetric geometry matches on to this behaviour, while the deviation from spherical symmetry is a detail only visible in terms subleading in large $N$. Taking the expression for the volume $V(r)$ and expanding gives:

$$\frac{V(r)}{V_s} - 1 = \left(\frac{V}{V_s} - 1\right) - \left(\frac{V}{V_s} + 1\right) \frac{g_s \alpha'^{1/2} N}{r} + \cdots,$$

confirming the earlier statement about the relation between the volume and the Higgs field, and fixing $H = (V/V_s - 1)$ and $e_m = (1 + V/V_s) g_s \alpha'^{1/2}/2$. (Later, we will set $H = 1$ and hence $V = 2V_s$, since we are free to choose these parameters at our convenience. Note that the explicit one–monopole solution displayed in eqn.(1.4) is so normalized, and has $e_m$ set to 1.)
Taking seriously the other lessons learned from the two monopole case, the analogue of the bolt sphere at \( \rho = \pi \) is where the probe merges into all the other branes. For small \( N \), the correction from \( \rho = 2 \) to the bolt radius is significant, while for large \( N \), as we have seen, the instanton corrections are small. Inside the bolt radius there is really no meaning to the coordinate \( \rho \) as having anything to do with distinct monopoles. In fact, this is very robust: Scattering two identical monopoles using the Atiyah–Hitchin manifold shows that \( \rho = \pi \) is truly the distance of closest approach: a head–on collision results in a 90° scattering angle at \( \rho = \pi \). The finite core size of the monopoles takes over. We inherit this qualitative behaviour here for arbitrary \( N \).

So in fact we learn that there is indeed a sharp meaning to the sphere of closest approach for the monopoles. It is also where they become massless, and also become indistinct. It is precisely where there occurs a bolt coordinate singularity in the smooth Atiyah–Hitchin manifold. The radii \( \rho < 2 \) (or \( U < g_{YM}^2 N \)) do not have any meaning for the individual monopole probes.

4 The Multi–Monopole from Nahm Data

In brane/supergravity language, we naively placed the branes all at the same place (\( r = 0 \)), at the point of \( SU(N) \) symmetry, the origin of the Coulomb branch. As we know from other examples[20], quantum corrections in the gauge theory alter the structure of that point. In fact, the enhançon is precisely a manifestation of this, since the monopoles are really not at the origin, but smeared into a sphere.

One of the crucial points of the present investigation is that the entire physics of the moduli space of the gauge theory is given in terms of the classical BPS monopoles, and so we should look no further than that system in order to learn more about the enhançon, and what it means. So how does one describe a clump of \( N \) monopoles?

Charge \( N \) multi–monopoles generalising the charge 1 BPS solution[5] were constructed by a number of elegant techniques. On the one hand, algebraic techniques[28] were employed by Ward[18], with generalisations[29]; on the other hand, a connection[30] to Bäcklund transformations and inverse scattering techniques was employed in refs.[26, 31]. Since the Bogomolnyi equations (1.1) are related by dimensional reduction to the self–dual equations in four
Euclidean dimensions, there is another elegant description via an extension of the ADHM construction\[15, 32, 33\]. The basic (“covariant” Nahm) equations are:

$$\frac{d\Phi_i}{d\sigma} + [\Phi_0, \Phi_i] = \frac{1}{2}\epsilon_{ijk}[\Phi^j, \Phi^k],$$

(4.19)

where $i, j, k$ run over 1, 2, 3. Here, $\Phi_0(\sigma)$ and $\Phi_i(\sigma)$ are $N \times N$ anti–Hermitian $SU(N)$ matrices, with $\Phi^*_i(\sigma) = -\Phi_i(\sigma)$ and $\Phi_i(\sigma) = -\bar{\Phi}_i(-\sigma)$. The coordinate $\sigma$ has range $-H \leq \sigma \leq H$, where $H$ is the asymptotic value of the Higgs field, which we shall presently set to 1 in much of the rest of this paper. The data appropriate to monopoles arise as solutions to this equation which are regular in the interior of $[-H, H]$, with appropriate boundary conditions at $\sigma = \pm H$. Those boundary conditions require that $\Phi_i$ have simple poles there, and that the residues of those poles (which are of course $N\times N$ matrices) are irreducible representations of $SU(2)$. There is an $SU(N)$ gauge invariance,

$$\Phi_0 \rightarrow G\Phi_0 G^{-1} - \frac{dG}{d\sigma}G^{-1}, \quad \Phi_i \rightarrow G\Phi_i G^{-1},$$

(4.20)

where the $G(\sigma) \in SU(N)$, and are the identity at the ends of the interval. There is a specific construction (also following the ADHM techniques) for converting the solutions of these equations — the “Nahm data” — into expressions for the spacetime fields $A_i(x), H(x)$, which we shall not reproduce here since for $N > 1$, closed forms are not known. Hitchin\[19\] has shown using algebraic methods that this method constructs all of the monopole solutions and indeed that it is equivalent to the aforementioned monopole constructions based on those of Ward\[18, 29\].

The gauge transformations (4.20) can be used to set $\Phi_0$ to zero, giving the standard Nahm equations, but should be left unfixed in order to perform the full hyperKähler quotient\[33, 32\] which constructs the metric on the moduli space of Nahm data. Nakajima\[34\] has shown that the metric thus computed is indeed the monopole moduli space, and it is smooth.

This Nahm system arises naturally in the brane description as the condition on the brane fields for supersymmetric vacua, resulting in a hyperKähler quotient. The $\Phi$ are adjoint scalars in a gauge theory on the brane. The most natural brane system where this arises is probably that of\[32\] $N$ D1–branes stretched perpendicularly between two D3–branes separated by a distance $2H$. The coordinate $\sigma$ is that along the D1–branes, and the $\Phi$ are the positions of the D1–branes inside the D3–branes. The boundary conditions arise by considering the
1+1 dimensional theory on the world–volume as a theory with “impurities” located at $\sigma = \pm H$, which is natural, since the massless 1–3 strings are localized there\cite{36, 37}. These Nahm equations can also be derived in the brane wrapped on $K3$ system we started with here\cite{4}. The asymptotic value of $K3$, $V$, sets the parameter $H$ via $H = (V/\sqrt{\alpha'})$. As the supergravity parameter $r$ runs from $\infty$ to $r_e$ (or more properly as $U$ runs from $\infty$ to $\lambda$) the coordinate $\sigma$ runs from $H$ to zero. Let us set $H = 1$ henceforth.

5 The Enhançon as a Fuzzy Sphere

It is easy to see that the enhançon is itself a fuzzy sphere in spacetime as follows. The D6–D2* system is dual to a system of $N$ D3–branes stretched between a pair of NS5–branes in type IIB string theory. The $SU(N)$ gauge theory is on the flat part of the D3–branes. The Nahm equations above (4.19) have the following meaning: The $\Phi_i(\sigma)$, multiplied by $2\pi\sqrt{\alpha'}$, are coordinates in the $\mathbb{R}^3$ part of the NS5–branes where the D3–branes end. The coordinate $\sigma$ is the coordinate between the NS5–brane. The 5+1 dimensional theory on the NS5–branes is the spontaneously broken 3+1 dimensional $SU(2)$ theory if we ignore the two spatial directions common to both the branes\cite{11}. Translating further, there is a factor of $1/g_s$ in front of the commutator in the Nahm equation. (This is instead of $g_s$, appropriate to the case of D1–branes ending on D3–branes.)

There is a “double trumpet” shape describing the $N$ stretched D3–branes pulling on the NS5–branes, as depicted in ref.\cite{1} and reproduced in figure 2. At the centre of the shape, there is a two–sphere where the fivebranes touch, restoring the $SU(2)$. Crucially, this is only a two–sphere for $N$ large enough, since the radius of the sphere is proportional to $N$, and only for spheres large enough are we far enough away from the details of the interior of the multi–monopole configuration to see an approximately spherically symmetric situation. (Recall that the multi–monopole is not spherically symmetric\cite{4}.)

We can make this a bit more precise as follows: First note that in the one–monopole case, the solution is spherically symmetric, and the Nahm data is simply the one dimensional representation of $SU(2)$, i.e., all the $f_i(\sigma)$ are equal. Assume that in our case, we are far

\footnote{This follows since these systems are dual to one another\cite{1}. The details will appear in ref.\cite{38}.}
Figure 2: (a) The configuration of D3–branes stretching between NS5–branes. (b) The resulting “double trumpet” shape of the NS5–branes at large $N$. (A separated probe is also shown.) This system has a natural description in terms of the Nahm equations as explained in the text. The enhançon is the place (an $S^2$) where the NS5–branes touch.

away enough from the core that we can borrow this behaviour, making the symmetric choice $\Phi_i(\sigma) = -if(\sigma)\Sigma_i$. In doing this, we connect to the discussion of ref.[39]. There, this was shown to correspond to an infinite trumpet shape representing $N$ D1–branes merging into an orthogonal D3–brane. The required poles at the ends of the Nahm interval correspond to the flaring of the trumpet as it expands into the perpendicular shape. The Nahm equations become:

$$\frac{df}{d\sigma} = -\frac{f^2}{g_s},$$

and the solution we seek here is made by gluing together two copies of the trumpet end to end at the centre of the interval:

$$f(\sigma) = \frac{g_s}{\sigma \pm 1}, \quad \text{for } \sigma \in [0, \pm 1].$$

This gives a shape like that in figure 2, but is only an approximation. The full solution should connect smoothly through the interior of the interval. A cross section at some value of $\sigma$ is a non–commutative, or “fuzzy” sphere[10] of radius given by (remembering to put in the factor of $2\pi\sqrt{\alpha'}$ for dimensions)

$$R^2 = 4\pi^2\alpha' \sum_i \text{Tr}(\Phi_i^2) = 4\pi^2\alpha'(N^2 - 1) f^2(\sigma),$$

(5.23)
There is a minimum value, $f_e \sim g_s$ where the NS5–branes touch at $\sigma = 0$. There, the radius is

$$R_N = 2\pi \sqrt{\alpha' g_s \sqrt{N^2 - 1}} \sim 2\pi \sqrt{\alpha' g_s N},$$

(5.24)

which compares well with the supergravity expression (2.10) for the enhançon radius, which is

$$r_e = g_s N \sqrt{\alpha'} \left( \frac{V}{V_*} - 1 \right)^{-1} \sim \frac{N}{H e}.$$

(5.25)

This particular fuzzy sphere is the enhançon. It is a sensible smooth sphere of non–zero radius at large $N$. Notice also that it has roughly the correct behaviour for the $N$ monopole size in terms of the Higgs vev $H$ and the electric charge $e$.

For small $N (\neq 1)$ it is very non–spherical, while it collapses to zero size in the case $N=1$: The minimum value $f_e$ is zero for a single monopole, and the double trumpet profile pinches off, as can be deduced (see ref.[40]) from a study of the Higgs field (1.4) for the explicitly known one–monopole solution. We plot this in figure 3.

![Figure 3: A plot of a slice of the two NS5–branes’ shape, as deduced from the Higgs field (1.4) for the single monopole. It is the only exactly spherically symmetric case. The double trumpet shape pinches off to zero size in this case.](image)

Returning to large $N$, it is in this sense that we see the connection between the dielectric brane construction of ref.[17] (see also ref.[41]) (where branes puff up into a sphere in the
presence of a background R–R field) and the enhançon\(^5\). Both phenomena can be used as examples of a new mechanism for excising undesirable spacetime singularities\(^[12]\), but the dielectric mechanism is adapted to \(\mathcal{N}=1\) supersymmetry preserving vacua\(^[13]\), while here we have \(\mathcal{N}=2\) (counting in four dimensional units). The connection between the two is simply that there are non–zero commutators for the adjoint scalars (forming \(N\)–dimensional irreducible representations of \(SU(2)\)) which have been shown in one case\(^[17]\) to induce multipole couplings to higher rank R–R fields. This is equivalent to the growth of extra dimensions on the brane.

In the present case, the Nahm equations are the means by which non–zero commutators arise in an \(\mathcal{N}=2\) supersymmetry preserving way. The higher dimensional aspect of the branes is realized in terms of their description as a finite sized multi–monopole configuration. The non–commutative sphere which is the enhançon is a preferred slice through this geometry, forming the effective shell around the \(N\)–monopole core.

6 A Candidate Dual?

One of the goals of a study of the supergravity solution mentioned in the introduction was to see if there is a large \(N\) dual supergravity solution. As pointed out in ref.\(^[1]\), even with the improved understanding of the geometry by recognizing the role of the enhançon, the decoupling/scaling limit \(\alpha'\rightarrow0\) (with \(U = r/\alpha'\) finite) gives a ten dimensional supergravity solution which was not appropriate as a truly decoupled dual theory. One sign of this (among others) is the simple fact that the resulting geometry contained parameters which recalled the original data of the type IIA compactification. In other words, it did not assemble into an expression referring purely to gauge theory quantities, as happens in simpler cases where there is a genuine supergravity dual\(^[6]\).

It would certainly be an excellent situation if there was a large \(N\) dual theory all the same, and a persistent open question is whether there exists such a theory, and whether it is a useful dual, in the sense of being weakly coupled (or at least tractable) when the gauge theory is strongly coupled.

\(^5\)Non–commutativity in the enhançon geometry was suspected in ref.\(^[1]\), and a relation to the dielectric branes was suspected by many.
To this end, let us note again that in the decoupling limit, the part of the supergravity geometry transverse to the branes assembles into purely gauge theory quantities, when it is referred to from the “frame” of a brane probe. We ought to regard this as a clue. As the geometry is (a subspace of) the moduli space of multi–monopoles, this leads one to speculate that there may be something to be gained by focusing one’s attention there.

On general grounds, one might expect that this large $N$ dual theory might be a string theory whose world sheet genus expansion is isomorphic to the $1/N$ expansion in the planar diagrams of the field theory, in the usual manner[44]. Besides this, the putative string theory should encode the structure of (and allow access to) the $4N−4$ dimensional moduli space of the Coulomb branch of vacua. The $SU(2)_R$ symmetry of the gauge theory should be present, as should the phenomenon of the enhançon, etc.

Here is a proposal for such a string theory. Simply take the large $N$ limit of the Nahm data, but looked at in a different way! The $N\times N$ matrices $\Phi_i(\sigma)$, for $(i = 0, 1, 2, 3)$, which satisfy the Nahm equations, may be thought of at large $N$ as giving the coordinates of a string in a four dimensional transverse space. It is a matrix string constructed by letting the matrices explore the full $4N−4$ dimensional moduli space (or possibly a cover of it), which is imposed by the Nahm equations (4.19) and the accompanying gauge invariance (4.20).

This proposal is more easily motivated by analogy with a simpler matrix string construction, that of the ten dimensional case of type IIA. There, we have eight $N\times N$ matrices $X^i(\sigma)$. While they may be thought of the collective coordinates of $N$ D1–branes, and hence parameterising $(\mathbb{R}^8)^N/S_N$, the now–standard route[45, 46] reinterprets them at large $N$ as light cone “string fields” giving a description the shape of a single IIA string in eight transverse dimensions $\mathbb{R}^8$. The “second–quantized” description of the string theory is simply the $(1+1)$–dimensional $U(N)$ gauge theory of the $X^i$.

In fact, the very same structures are present here, but yielding a (with respect) potentially much more interesting string, at least for our purposes, since it contains all of the ingredients to make a dual string for the $(2+1)$–dimensional $SU(N)$ gauge theory. The naive interpretation of the $\Phi_i(\sigma)$’s is as the non–commutative collective coordinates of $N$ D–strings stretched along a finite interval and acting as BPS monopoles. (We are ignoring the two directions common to both types of brane in the setup of section 5.) The proposal here is that at large $N$, the Nahm
data $\Phi_1(\sigma)$ can be thought of as string fields for a single string having a 4 dimensional transverse target space. To obtain the “long string” sectors required to enable a stringy limit, two modifications to our setup can be considered: The first is that the whole system of NS5–branes described in section 3 be placed on a circle, by compactifying the direction in which the branes are separated. The second is that the Donaldson–Nahm $U(N)$ gauge transformations (4.20), which reduce to the identity at $\sigma = \pm 1$, be allowed to include permutations at the ends of the $\sigma$ interval in this periodic system. In this way, we include configurations in the theory corresponding to a D–string winding around a large number of times before terminating on an NS5–brane.

A periodic version of the 1+1 dimensional “impurity” gauges theories of the sort described in refs.[36, 37] should provide the dynamics. To get to strong coupling for the gauge theory, one must further tune the length of the interval (and hence the Higgs vev) and the size of the circle to be small, holding the ratio fixed.

By construction, the string thus defined has many of the properties which we seek for our dual string. It refers correctly to the moduli space of vacua of the gauge theory by using the monopole moduli space in an essential way. Different vacua of the gauge theory correspond to different backgrounds for the string theory.

A tantalizing consequence of this conjecture (which clearly needs more work) for a dual string is that it may provide a dictionary between many of the elegant results about monopole moduli space (such as the scattering of slowly moving monopoles, geodesics representing bound states, etc.), and properties of the dual gauge theory. It will certainly be interesting to pursue this further.

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**Added Note on Other Work**

There is other recent work on the large $N$ limit of monopoles and Nahm data, with connections to brane configurations, and non–commutativity[6] which we ought to mention here. While they appear to be separate avenues of investigation, it should be interesting to learn if there is

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6The author is grateful to Micha Berkooz and Kimyeoug Lee for pointing out refs.[50, 51] after this manuscript first appeared.
anything in those approaches which might help with some of the issues discussed here.

Existing proposals by Fairlie and collaborators[47] concerning a matrix/M–theory interpretation of Nahm equations (in diverse dimensions) at large \( N \) emphasize a connection to the Moyal bracket (see also ref.[48, 49]), to non–commutative geometry, and the physics of membranes. In ref.[50], Lee studies the case of an infinite sheet of BPS monopoles, highlighting its non–commutative description, and a relation to D1–strings stretching between D3–branes.

It is interesting to note that both of these sets of work make connections at large \( N \) to a natural two dimensional non–commutative surface, while in this paper, we have pointed out that the enhançon itself is a non–commutative sphere. Perhaps a firmer connection might be made between these approaches which might lead to a description of the dynamics of the enhançon as a non–commutative membrane via those techniques.

Also, the work of Berkooz[51] studies a 1+1 dimensional impurity model (related to the one we have in mind in this paper) as a DLCQ realization of the 4+1 dimensional \( SU(N) \) Yang–Mills theory found on D4–branes. Features such as the crossover between open and closed string effective descriptions of the gauge theory at large \( N \) are highlighted.

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