Arithmetic quotients of the Bruhat-Tits building
for projective general linear group in positive
characteristic

Satoshi Kondo
Seidai Yasuda

Author address:

Corresponding author: Satoshi Kondo, Middle East Technical University, Northern Cyprus Campus, Kalkanli, Guzelyurt, Mersin 10, Turkey; Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa 277-8583, Japan
E-mail address: satoshi.kondo@gmail.com

Seidai Yasuda, Department of Mathematics, Faculty of Science, Hokkaido University Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
E-mail address: sese@math.sci.hokudai.ac.jp
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Abstract

Let \( d \geq 1 \). We study a subspace of the space of automorphic forms of \( \text{GL}_d \) over a global field of positive characteristic (or, a function field of a curve over a finite field). We fix a place \( \infty \) of \( F \), and we consider the subspace \( \mathcal{A}_{\text{St}} \) consisting of automorphic forms such that the local component at \( \infty \) of the associated automorphic representation is the Steinberg representation (to be made precise in the text).

We have two results.

One theorem (Theorem 16) describes the constituents of \( \mathcal{A}_{\text{St}} \) as automorphic representations and gives a multiplicity one type statement.

For the other theorem (Theorem 12), we construct, using the geometry of the Bruhat-Tits building, an analogue of modular symbols in \( \mathcal{A}_{\text{St}} \) integrally (that is, in the space of \( \mathbb{Z} \)-valued automorphic forms). We show that the quotient is finite and give a bound on the exponent of this quotient.

2010 Mathematics Subject Classification. Primary 11F67 11F70 Secondary 20E42.

Key words and phrases. modular symbol, automorphic representation, positive characteristic, Bruhat-Tits building.

the first author was partially supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.
Acknowledgments

During this research, the first author was supported as a Twenty-First Century COE Kyoto Mathematics Fellow and was partially supported by JSPS KAKENHI Grant number JP17740016, JP21654002, JP25800005 and by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. The second author was partially supported by JSPS KAKENHI Grant number JP16244120, JP21540013, JP24540018, and JP15H03610.

The second author would like to thank Fumiharu Kato for letting him know the works \cite{Wer1}, \cite{Wer2} of Werner.

We thank Professor Kazuya Kato for always raising important questions.
Introduction

1.1. Modular forms and automorphic forms

Automorphic forms are fundamental objects of study in number theory. We obtain some very basic results concerning automorphic forms (satisfying some condition at a fixed prime) for GL$_d$ (where $d$ is a positive integer) over the function field of a curve over a finite field (or a global field of positive characteristic).

Automorphic forms are defined for a reductive algebraic group $G$ over a global field $F$ (either a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q(t)$ for a finite field $\mathbb{F}_q$ of $q$ elements). They are functions on the adele points $G(\mathbb{A}_F)$ which are invariant by the translation by $G(F)$, satisfying certain conditions at some place(s) of $F$. In practice, for characteristic 0 fields $F$, they are realized as functions on some quotient of a symmetric space, satisfying certain (real analytic) conditions. For example, (elliptic) modular forms are automorphic forms for GL$_2$ over the rationals $\mathbb{Q}$. They are functions on the upper half plane (or, equally, on the quotient SL$_2(\mathbb{R})/SO_2(\mathbb{R})$), and certain conditions at the real place amounts to the modular condition of some weight.

In this article, we study certain automorphic forms for GL$_d$ for $d \geq 1$ over a global field of positive characteristic. We fix a place $\infty$ of $F$. The automorphic forms that we consider are functions on GL$_d(\mathbb{A}_F)$, which are invariant under GL$_d(F)$, satisfying a condition at the place $\infty$. The slogan here is that we consider “automorphic forms whose associated automorphic representation is the Steinberg representation at $\infty$”. This will be made precise later. As an analogue of the symmetric space in positive characteristic, there are the Bruhat-Tits buildings. We use the Bruhat-Tits building of PGL$_d$ over $F_\infty$, where $F_\infty$ is the local field (completion) at the place $\infty$. It is a simplicial complex of dimension $d-1$ whose set of simplices are quotients of PGL$_d(F_\infty)$. For example, the set of zero simplices is isomorphic to GL$_d(F_\infty)/$GL$_d(O_\infty)$ where $O_\infty \subset F_\infty$ is the ring of integers, and the set of $(d-1)$-dimensional simplices is isomorphic to GL$_d(F_\infty)/I$ where $I \subset$ GL$_d(O_\infty)$ is the Iwahori subgroup consisting of those matrices that are congruent to an upper triangular matrix modulo the maximal ideal of $O_\infty$. There are many interpretations of the simplices (e.g. in terms of $O_\infty$-lattices, of norms, and, for quotients, of vector bundles over the proper smooth curve $C$ whose function field is $F$) which we will use.

For the dictionary between our function field setup and the setup for modular forms, see the table in Section 1.10.

1.2. Classical modular symbols

In the study of modular forms, especially of weight 2 (automorphic forms for GL$_2$ over $\mathbb{Q}$), one useful tool is modular symbols, as invented by Shimura and Eichler
1. INTRODUCTION

and developed by Manin [Ma1]. To introduce them, recall the following geometric setup for modular forms. We consider some arithmetic subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$. Main examples include congruence subgroups such as $\Gamma_1(N)$ for a positive integer $N$. Then modular forms appear as 1-forms on the analytic space $\Gamma \backslash \mathcal{H}$ or on $\Gamma \backslash \mathcal{H}^*$ where $\mathcal{H}$ is the upper half space and $\mathcal{H}^* = \mathcal{H} \cup \overline{\mathbb{P}^1}(\mathbb{Q})$. The quotients have algebraic models defined over some number field, and the $\Gamma \backslash \mathcal{H}^*$ is a smooth compactification of $\Gamma \backslash \mathcal{H}$. The set $\mathbb{P}^1(\mathbb{Q})$ or the quotient $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$ is called the set of cusps. The Eichler-Shimura isomorphism states that 1-forms on $\Gamma \backslash \mathcal{H}$ of $\Gamma \backslash \mathcal{H}$ give elements in the relative homology $H_1(\Gamma \backslash \mathcal{H}^*, \{\text{cusps}\}; \mathbb{C})$. The relation with modular forms is given by integration, integrating a 1-form (a modular form) on $\Gamma \backslash \mathcal{H}$. One main theorem is that the modular symbols generate the dual symbols, which are amenable to computation.

A modular symbol in this geometric setup is a path from $a$ to $b$ in $\mathcal{H}^*$ for cusps $a, b$, or its class in the relative homology $H_1(\Gamma \backslash \mathcal{H}^*, \{\text{cusps}\}; \mathbb{C})$. The relation with modular forms is given by integration, integrating a 1-form (a modular form) on the path from $a$ to $b$. We may regard modular symbols as elements in the dual of cusp forms. One main theorem is that the modular symbols generate the dual space of cusp forms. This property enables us to study cusp forms using modular symbols, which are amenable to computation.

We refer to Manin’s fairly recent introductory article [Ma2] for applications in Iwasawa theory.

1.3. Higher dimensional modular symbols of Ash and Rudolph

Ash and Rudolph consider higher dimensional modular symbols in [As-Ru]. The automorphic forms they treat are for $\text{GL}_d$ over the rationals $\mathbb{Q}$, and they are functions on the symmetric space $X = \text{SL}_d(\mathbb{R})/\text{SO}_d(\mathbb{R})$. They use the Borel-Serre bordification (a partial compactification) $\overline{X}$ of $X$ (as a generalization of $\mathcal{H}^*$). Let $\Gamma$ be a torsion free arithmetic subgroup and set $M = \Gamma \backslash X$ and $\overline{M} = \Gamma \backslash \overline{X}$. Then their modular symbols are elements of the relative homology group $H_{d-1}(M, \overline{M}; \mathbb{Z})$. By Poincaré duality, this group is dual to $H^N(M; \mathbb{Z}) = H^N(\Gamma; \mathbb{Z})$ where $N = d(d-1)/2$, which is closely related to the space of cusp forms.

To construct elements in the relative homology, they introduce universal modular symbols as elements of the homology groups $H_{d-2}(T_d, \mathbb{Z})$ of the Tits building $T_d$ (the simplicial complex associated with the poset of flags in $\mathbb{Q}^d$). (The Solomon-Tits theorem says that $T_d$ is homotopy equivalent to the bouquet of $(d-2)$-spheres.) This $T_d$ is homotopy equivalent to $X \setminus X$ and the projection map to the quotient give elements in $H_{d-1}(M, \overline{M}; \mathbb{Z})$.

Given an ordered basis $q_1, \ldots, q_d$ of $\mathbb{Q}^d$, they construct a map from a $(d-2)$-sphere to $T_d$. They call the homology class of this sphere a universal modular symbol and denote it by $[q_1, \ldots, q_d]$. It is known that universal modular symbols generate the homology group $H_{d-2}(T_d, \mathbb{Z})$.

Their main theorem is that the ordered bases $q_1, \ldots, q_d$ such that the matrix $(q_1, \ldots, q_d)$ regarded as a matrix lying in $\text{SL}_d$ (not $\text{GL}_d$) span the homology group $H_{d-1}(M, \overline{M}; \mathbb{Z})$ when $\Gamma$ is torsion free. If $\Gamma$ is not torsion free, there always exists a torsion free subgroup $\Gamma' \subset \Gamma$ of finite index. Their result in this case is that the homology group divided by the space of modular symbols is killed by the index $[\Gamma : \Gamma']$. 
They also give an algorithm in expressing a general modular symbol in terms of such "unimodular" modular symbols. We do not consider the analogue of this theorem. One reason is that their proof make use of the fact that \( \mathbb{Z} \) is a Euclidean domain. The analogue of \( \mathbb{Z} \), namely \( A \), in our setup need not be a Euclidean domain.

1.4. Our result on modular symbols

We take \( F \) to be a global field of positive characteristic and fix a place \( \infty \) as above.

The space of automorphic forms we consider can be defined as follows. Let \( C_C = \text{Hom}(GL_d(F)\backslash GL_d(A), \mathbb{C}) \) be the space of \( \mathbb{C} \)-valued functions on the adele points that are invariant by the action of \( GL_d(F) \). We let \( C_C^\infty = \text{U}_K C_C^K \) where \( K \) runs over the open compact subgroups of \( GL_d(A) \) where the superscript \( (\cdot)^K \) means the invariants. This is the space of smooth vectors. Let \( St_{d,C}^I \) denote the Steinberg representation of \( GL_d(F_\infty) \) and \( I \subset GL_d(O_\infty) \) be the Iwahori subgroup. It is known that the Iwahori fixed part \( St_{d,C}^I \) is one dimensional. Take a non-zero vector \( v \in St_{d,C}^I \). Then we define

\[
\mathcal{A}_{St_{d,C}} = \text{Image}(\text{Hom}_{GL_d(F_\infty)}(\text{St}_{d,C}, C_C^\infty) \to C_C^\infty)
\]

where the arrow is the evaluation at \( v \). This is the space of automorphic forms with Steinberg at infinity, as mentioned in Section 1.1, and is the main object of our study.

We use the Bruhat-Tits building \( BT_* \) of \( \text{PGL}_d(F_\infty) \) as an analogue of the symmetric space \( SL_d(\mathbb{R})/SO_d(\mathbb{R}) \). We set \( X_{K,*} = GL_d(F)\backslash GL_d(\mathbb{A}_\infty) \times BT_* / K \) for the ring of finite adeles \( \mathbb{A}_\infty \) and for a compact open subgroup \( K \subset GL_d(\mathbb{A}_\infty) \). (This is the analogue of the double coset description of the \( \mathbb{C} \)-valued points of a Shimura variety.) We show that there is a canonical isomorphism

\[
\lim_{\overleftarrow{K}} H_{d-1}^{BM}(X_{K,*}, \mathbb{C}) \cong \mathcal{A}_{St_{d,C}}
\]

and study the geometry of \( X_{K,*} \).

Let us assume for simplicity that \( X_{K,*} = \Gamma \backslash BT_* \) for some arithmetic subgroup \( \Gamma \subset GL_d(F) \). (In general, it is a finite disjoint union of such quotients. This is analogous to that \( \prod_i \Gamma_i \backslash H \), where \( \Gamma_i \) are some arithmetic groups, is isomorphic to the \( \mathbb{C} \)-valued points of a modular curve.)

Let us give the definition of our modular symbols. Our definition of modular symbols is analogous to the paths in \( H^* \). By the definition of Bruhat-Tits buildings, \( BT_* \) is the union of subsimplicial complexes called apartments. The apartments are indexed by the set of \( F_\infty \)-bases of \( F^{\otimes d} \). Let \( A_{q_1, \ldots, q_d,*} \subset BT_* \) denote the apartment corresponding to the basis \( q_1, \ldots, q_d \). We have a map

\[
A_{q_1, \ldots, q_d,*} \subset BT_* \to \Gamma \backslash BT_*
\]

For an arithmetic subgroup \( \Gamma \) and an \( F \)-basis (that is, a basis of \( F^{\otimes d} \) regarded as a basis of \( F^{\otimes d}_\infty \)), this map is locally finite (in the sense that the inverse image of a simplex is a finite set). The image of the fundamental class of the apartment \( H_{d-1}^{BM}(A_{q_1, \ldots, q_d,*}, \mathbb{Z}) \) in \( H_{d-1}^{BM}(\Gamma \backslash BT_* , \mathbb{Z}) \) is well defined in this case, where \( H^{BM} \) means the Borel-Moore homology. Our modular symbols are defined to be the elements of this form as \( q_1, \ldots, q_d \) runs over all \( F \)-bases.
Our modular symbols are a priori different from the modular symbols coming from universal modular symbols (the analogue of those of Ash and Rudolph). We prove that they actually coincide.

Our main theorem computes a bound of the index of the subgroup generated by modular symbols inside \( H_{d-1}^{BM}(\Gamma \setminus \mathcal{B} T_\bullet, \mathbb{Z}) \). We have a uniform bound which is independent of the choice of \( \Gamma \). The prime-to-\( p \) (the characteristic of \( F \)) part depends only on the base field, and divides the order of \( \text{GL}_d(\mathbb{F}_{q'}) \) for some explicitly given \( q' \). The exponent of the \( p \)-part is given explicitly in terms of \( d \).

### 1.5. Outline of proof for universal modular symbols

In Chapter 11, we prove our main theorem on modular symbols but for universal modular symbols. We give an outline of the proof in this section.

1.5.1. Let \( \Gamma \subset \text{GL}_d(F) \) be an arithmetic subgroup. We construct (as described in Section 1.4 above) modular symbols in the Borel-Moore homology \( H^{BM}_d(\Gamma \setminus \mathcal{B} T_\bullet, \mathbb{Z}) \) as the (fundamental) classes of the apartments corresponding to \( F \)-bases. The goal is to describe the size of cokernel of the (injective) map (the \( \mathbb{Z} \)-module of) modular symbols to the Borel-Moore homology of \( \Gamma \setminus \mathcal{B} T_\bullet \).

To achieve this goal, we use the universal modular symbols of Ash and Rudolph. That is, we construct a map

\[
H_{d-2}(T_d) \to H_{d-1}^{BM}(\Gamma \setminus \mathcal{B} T_\bullet)
\]

from the homology of the Tits building \( T_d \). This homology group is the space generated by universal modular symbols. We compute the cokernel of this map. Then we show that the image coincides with the space of (our) modular symbols.

1.5.2. The analogue of the map

\[
H_{d-2}(T_d) \to H_{d-1}^{BM}(\Gamma \setminus \mathcal{B} T_\bullet)
\]

appears in Ash and Rudolph as described briefly in Section 1.3, however, the construction is different. We believe our method is simpler and might apply to their case as well. There is a remark in Section 1.9.

First, we express the Borel-Moore homology as an inverse limit:

\[
H_{d-1}^{BM}(\Gamma \setminus \mathcal{B} T_\bullet) \cong \lim_{\alpha \to 0} H_{d-1}(\Gamma \setminus \mathcal{B} T_\bullet, \Gamma \setminus \mathcal{B} T_\bullet^{(\alpha)}).
\]

Here, \( \alpha \) runs over positive real numbers, and \( \mathcal{B} T_\bullet^{(\alpha)} \) is a subsimplicial complex consisting of “more unstable than \( \alpha \)” vector bundles.

1.5.3. The following is the key sequence in our proof:

\[
H_{d-2}(T_d) \cong H_{d-1}(\mathcal{B} T_\bullet, \mathcal{B} T_\bullet^{(\alpha)}) \to H_0(\Gamma, H_{d-1}(\mathcal{B} T_\bullet, \mathcal{B} T_\bullet^{(\alpha)})) \cong H_{d-1}^{BM}(\mathcal{B} T_\bullet, \mathcal{B} T_\bullet^{(\alpha)}) \to H_{d-1}(\Gamma \setminus \mathcal{B} T_\bullet, \Gamma \setminus \mathcal{B} T_\bullet^{(\alpha)}).
\]

The second map is the canonical surjection to coinvariants. Let us describe the other three maps.
1.5.4. The first map (isomorphism) is obtained by following the method of Grayson’s (see [Gr Cor 4.2]). Let us explain this in this section.

Recall that each 0-simplex of $\Gamma/BT_\bullet$ can be interpreted as a locally free sheaf (vector bundle) on the curve $C$ whose function field is $F$. It can be seen from the work of Grayson that the semi-stable ones lie in the “middle”, whereas the unstable ones are closer to the “boundary”. The picture to have in mind is in Serre’s book (near [Se p.106, II.2, Thm 9]), where the quotient of the building of dimension 1 (tree) is discussed in detail. Only the finite graph in the middle consist of semi-stable ones and the halflines, after a few steps, consist of unstable ones only.

We define a subsimplicial complexes $BT_\bullet^{(\alpha)}$ by using the Harder-Narasimhan function which measures how unstable a vector bundle is. In particular, all the simplices of $BT_\bullet^{(\alpha)}$ for sufficiently large $\alpha$ correspond to unstable ones. (If $d = 2$, $BT_\bullet^{(\alpha)}$ consists of the halflines which become shorter as $\alpha$ grows bigger.)

Recall on the other hand that a simplex of the Tits building $T_d$ corresponds to a flag in $F^d$. Suppose that a 0-simplex of $BT_\bullet^{(\alpha)}$ corresponds to an unstable vector bundle. Then there is a nontrivial Harder-Narasimhan filtration, and by taking the generic fiber, we obtain a filtration or a flag of $F^\oplus d$. This is how the two spaces $BT_\bullet^{(\alpha)}$ and $T_d$ are related. Using Grayson’s method, we see that they are homotopy equivalent. Then, using that $BT_\bullet$ is contractible, we obtain the first isomorphism.

1.5.5. Let us look at the third map which is an isomorphism. There is a Lyndon-Hochshild-Serre spectral sequence (see Section 11.4) for a pair of spaces converging to the equivariant homology:

$$E^2_{p,q} = H_p(\Gamma, H_q(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z})) \Rightarrow H_{p+q}(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z}).$$

We can compute the $E_2$-page. The Solomon-Tits theorem says that the homotopy type of $T_d$ is a bouquet of $(d - 2)$-spheres. This means that many of the $E_2$-terms are zero and the relevant terms are $E^2_{0,t}$, which are the homology groups of the relative space $(BT_\bullet, BT_\bullet^{(\alpha)})$. Hence we obtain the third map.

1.5.6. For the fourth map, we use the following spectral sequence

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, \chi_\sigma) \Rightarrow H_{p+q}(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z})$$

where $\Sigma_p$ is the set of $p$-simplices of $BT_\bullet \setminus BT_\bullet^{(\alpha)}$, $\Gamma_\sigma$ is the stabilizer group, and $\chi_\sigma$ is the representation associated with orientation.

Because it is a first quadrant spectral sequence and the $E_1$-terms $E^1_{p,t}$ vanish for $t > d$, we obtain the fourth map as the composition of (injective) differential maps. Thus the cokernel of the composition is equal to the cokernel of the fourth map. To compute it, we bound the order of the $E^1$ terms, or the order of the stabilizer groups.

1.5.7. Torsion in arithmetic subgroups. It is common in characteristic zero case (see e.g. Borel-Serre, Ash-Rudolph) to assume that the arithmetic subgroup is without torsion. In that case, all the stabilizer groups are trivial since finite. Then the sought cokernel turns out to be trivial. The general case is reduced to the torsion free case because given an arithmetic subgroup there always exists a torsion free arithmetic subgroup of finite index. The exponent of the cokernel in this case is killed by this index.
However, in positive characteristic, an arithmetic subgroup always has a non-trivial $p$-torsion subgroup so we do not expect to do as well as in the characteristic zero case. Instead, we will see that given an arithmetic subgroup $\Gamma$, there always exists an arithmetic subgroup $\Gamma' \subset \Gamma$ of finite index which is $p'$-torsion free (each torsion element has order prime to $p$).

We can bound the sought cokernel in the $p'$-torsion free case in terms of some power of $p$ depending only on the dimension $d$ by inspecting the shape of $p$-torsion subgroups of an arithmetic subgroup. We can also bound the index $[\Gamma : \Gamma']$ in terms of $d$ and $q$. We arrive at the bound of the cokernel in general in terms of $d$, $p$, and $q$.

1.6. Outline for comparing modular symbols

After computing the cokernel for the image of the universal modular symbols, there remains the task of comparing the image with the space of our modular symbols.

The strategy is summarized in the following diagram (in Section 12.3):

We first take an ordered basis $v_1, \ldots, v_d$ of $F^{\oplus d}$. We then have an embedding $\psi_{v_1, \ldots, v_d} : A_* \to BT_*$. (The image of this was denoted $A_{v_1, \ldots, v_d}$ earlier.) This gives the top arrow as the pushforward by this embedding composed with the projection map $BT_* \to \Gamma \backslash BT_*$. We know (show) that the Borel-Moore homology $H_{d-1}^{BM}(A_*)$ of an apartment is isomorphic to $\mathbb{Z}$ (the fundamental class is a generator). The image of $1 \in \mathbb{Z}$ by this pushforward is the definition of our modular symbol.

The image of $1 \in \mathbb{Z}$ at the right bottom via the horizontal map followed by the left vertical map is the definition of the Ash-Rudolph modular symbol (or the modular symbol coming from the universal modular symbol).
To prove that the two modular symbols coincide, we add the right vertical column and show that the diagram commutes. The commutativity of the squares except for the last one is not difficult.

We follow Grayson [Gr] for the construction of the map (5). We need to look at the proof (which uses [Qu]) and see that it comes from a zigzag of morphisms of posets. We interpret the map (8) in a similar manner in terms of posets and then the commutativity of the last square follows.

1.7. On the structure of the space of automorphic forms

We study automorphic forms $\mathcal{A}_{\text{St},\mathbb{C}}$ with Steinberg at infinity and the intersection $\mathcal{A}_{\text{St,cusp},\mathbb{C}}$ with the cusp forms.

Using known results, we verify (Proposition 15) that $\mathcal{A}_{\text{St,cusp},\mathbb{C}}$ is the direct sum of irreducible cuspidal automorphic representations that are cuspidal and whose local representation at infinity is the Steinberg representation. We also obtain (Theorem 16) a similar description for the space $\mathcal{A}_{\text{St},\mathbb{C}}$. While the space of $L^2$ automorphic forms is well studied, not all subrepresentations of $\mathcal{A}_{\text{St},\mathbb{C}}$ are $L^2$ and the $L^2$ methods do not apply.

Let us give some ideas and the outline of proof. We first show that the space of automorphic forms is isomorphic to the Borel-Moore homology of (a finite union of) the quotient $X_K,\bullet$ of the Bruhat-Tits building by an arithmetic subgroup. We take the dual and work with the cohomology with compact support.

Now, if there existed some good compactification $\overline{X}$ of $X = X_K,\bullet$ then we would be looking at an exact sequence

$$H^{d-2}(\overline{X} \setminus X) \rightarrow H^{d-1}_c(X) \rightarrow H^{d-1}(\overline{X}).$$

(The analogy in the case of modular forms is $X = Y_0(N), \overline{X} = X_0(N), \overline{X} \setminus X = \{\text{cusps}\},$ and $d = 2$. Then $H^1(\overline{X})$ is (roughly) the space of cusp forms, and the remaining task is to compute $H^0(\overline{X} \setminus X)$ as an automorphic representation.) While the compactifications given in [Fu-Ka-Sh1] might be helpful, we do not use them.

There are two steps. The first step is to regard the cohomology as the limit as the boundary becomes smaller (again using the spaces $BT_\alpha^\circ$). The second step is to express the limit as induced representation using a covering spectral sequence.

1.8. Remarks

We give miscellaneous remarks.

1.8.1. Automorphic forms in positive characteristic. We study automorphic forms in positive characteristic. Because there are no archimedean primes, there is no difficulty coming from complex or real analysis. In terms of automorphic forms, this means that all the smooth automorphic forms are admissible automorphic forms (we refer to Cogdell’s Lectures 3,4 in [Co-Kl-Mu] for the definitions). Then the theory of automorphic representations is equivalent to the (simpler) theory of representations of Hecke algebras. For example, decomposition into irreducible automorphic representations is simpler in positive characteristic.

This also means that it is meaningful to consider the automorphic forms with values in $\mathbb{Z}$, and this gives one natural $\mathbb{Z}$-structure ($\mathbb{Z}$-submodule which spans the space of ($\mathbb{C}$-valued) automorphic forms). In the modular form case, there is the cohomology with coefficients in $\mathbb{Z}$ of the topological space $\Gamma \setminus \mathcal{H}$, which spans the
space of modular forms. However, this does not have an interpretation as \( \mathbb{Z} \)-valued functions.

One simplification occurs. The Borel-Moore homology, or the relative homology, in the Ash-Rudolph case is not readily related to the space of automorphic forms, but its Poincaré dual is. The pairing, in terms of automorphic forms, involves integration and hence periods of automorphic forms. For example, in the modular form case, the pairing involves values such as \( \int_0^\infty f(z) \, dz \) where \( f \) is a modular form.

In our case, the Borel-Moore homology is equal to the space of automorphic forms, hence our results are stated directly in terms of automorphic forms, with no reference to any pairing. However, in our application [Ko-Ya2], we do take a pairing of modular symbols and cusp forms, much analogous to the integration. The second author does not know what the natural formulation is.

1.8.2. The space \( A_{st, C} \) of automorphic forms with Steinberg at infinity arises as the étale cohomology of Drinfeld modular varieties. Drinfeld modular varieties may be regarded as a Shimura variety for \( \text{PGL}_d \) over \( F \). (Shimura varieties are defined only in characteristic zero. Note also that there is no Shimura variety for \( \text{GL}_d \) if \( d \geq 3 \).) Thus by studying the cohomology, Laumon [Lau1], [Lau2] obtains a result in the Langlands program for \( \text{GL}_d \) over a global field in positive characteristic.

One drawback is that, via this method, one can treat only those automorphic representations whose local representation is the Steinberg representation at the fixed prime infinity.

1.8.3. Our old preprint. This article may be regarded as a revised version of our preprint [Ko-Ya1]. The main result of loc.cit. is that the space of \( \mathbb{Q} \)-valued cusp forms with Steinberg at infinity is contained in the space generated by modular symbols tensored with \( \mathbb{Q} \). The result in this article implies that.

We took a very different approach there. In [Ko-Ya1], we used the Werner compactification [Wer2], [Wer1] and used the duality twice. One key observation there was that the Werner compactification was best suited when studying the group cohomology of arithmetic subgroups (not the homology of the quotient spaces), which in turn is closely related to the space of cusp forms. We no longer use this observation here.

1.9. Comparing with the Ash-Rudolph method

Let us compare our method with that of Ash and Rudolph.

1.9.1. Ash and Rudolph [As-Ru p.245] use the following sequence:

\[
H_{d-2}(T_d) \cong H_{d-2}(\partial \bar{X}) \cong H_{d-1}(\bar{X}, \partial \bar{X}) \xrightarrow{\pi_*} H_{d-1}(M, \partial M).
\]

Here \( X = \text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R}) \), \( \bar{X} \) is the Borel-Serre bordification, \( M = \Gamma \setminus \bar{X} \) is a manifold with boundary for some arithmetic subgroup \( \Gamma \), and \( M \setminus \partial M = \Gamma \setminus X \). The map \( \pi_* \) is the canonical projection, which is shown to be surjective if \( \Gamma \) is torsion free. In the proof of their result [As-Ru Prop 3.2], however, they use Poincaré duality and show \( H_c^N(\bar{X}) \rightarrow H^N(M) \) is injective where \( N = d(d-1)/2 \). Then they translate these groups into group cohomology of \( \Gamma \) using that the action is free and the contractibility of spaces \( X \) and \( \bar{X} \). They apply the Borel-Serre duality (of group cohomology) to prove the claimed injectivity.
1.9.2. Let us point out some differences with our method.

There is a difficulty in using a (partial) compactification of $\mathcal{B}T_\bullet$. There exists a compactification of Borel-Serre of $\mathcal{B}T_\bullet$ (which is in the same spirit as the Borel-Serre bordification of $X$). (There are also other compactifications. See [Fu-Ka-Sh1] for overview.) One difference is that an arithmetic subgroup does not act freely on (the boundary of) the compactification, even when restricted to some subgroup. Thus there is some difficulty in connecting the homology groups of spaces to group cohomology (as was done by Ash and Rudolph). We also did not find the analogue of the Borel-Serre duality in the literature. There may also be some difficulty arising from the fact that an arithmetic subgroup is usually not torsion free.

These considerations suggest that their method may not apply directly to our case. On the other hand, we believe that our method applies to the case treated by Ash and Rudolph. Our method is more straightforward in that we do not use the two dualities.

1.10. Dictionary

The following table is a dictionary between our function field setup (the right column) and the classical modular forms setup (the left column). See also Sections 1.1.

| base field | $\mathbb{Q}$ | $F$ (a global field in positive characteristic; a function field of a curve over a finite field) |
| place | the real place $\infty$ | a fixed place $\infty$ |
| integers | $\mathbb{Z}$ | $\mathcal{A}$ (integral at all but $\infty$) |
| completion | $\mathbb{R}$ | $F_\infty$, ($\mathcal{O}_\infty$ the ring of integers) |
| rank, dimension | $d = 2$ | $d \geq 1$ |
| algebraic group | $\text{SL}_2$ over $\mathbb{Q}$ | $\text{PGL}_d$ over $F$ |
| symmetric space | $\mathcal{H} = \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$ the upper half plane (real manifold) | $\mathcal{B}T_\bullet$, $\mathcal{B}T_0 = \text{PGL}_d(F_\infty)/\text{PGL}_d(\mathcal{O}_\infty)$ the Bruhat-Tits building (simplicial complex) |
| modular symbol | geodesic from $a$ to $b$ (a, $b \in \mathbb{P}^1(\mathbb{Q})$) | apartment corresponding to a basis of $F^{d \oplus d}$ |
| arithmetic subgroup | $\Gamma \subset \text{SL}_d(\mathbb{Z})$ | $\Gamma \subset \text{GL}_d(F)$ |
| automorphic form | modular form of weight 2 level $\Gamma$ | $\Gamma$-invariant harmonic function (cochain) on $\mathcal{B}T_{d-1}$ |
| automorphic representation | $\text{Of GL}_2(\mathbb{A}_\mathbb{Q})$, discrete series (for weight 2) at $\infty$ | $\text{Of GL}_d(\mathbb{A}_F)$, Steinberg at $\infty$ |
| homology | $H_1^{\text{BM}}(\Gamma \backslash \mathcal{H}, \mathbb{Z})$ | $H_{d-1}^{\text{BM}}(\Gamma \backslash \mathcal{B}T_\bullet, \mathbb{Z})$ |
| cusp | $\mathbb{P}^1(\mathbb{Q})$ | Several choices |
| (partial) compactification | $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ | Several choices |
CHAPTER 2

Simplicial complexes and their (co)homology

The material of this chapter (except for the remark in Section 2.7) already appeared in Sections 3 and 5 of [Ko-Ya2]. We collected them for the convenience of readers.

We define generalized simplicial complexes in Section 2.1, define their (co)homology theories (homology, cohomology, Borel-Moore homology, cohomology with compact support), and mention the universal coefficient theorem and geometric realization.

Later, we will consider quotients of the Bruhat-Tits building by arithmetic subgroups (to be defined). While the Bruhat-Tits building is canonically a simplicial complex, the arithmetic quotient is in general not a simplicial complex. This issue was addressed by Prasad in a paper of Harder [Har2], p.140, Bemerkung]. We introduce this notion because the quotients are naturally (generalized) simplicial complexes. An example of a generalized simplicial complex which is not a strict simplicial complex consists of two vertices with two edges between the vertices.

When defining the (co)homology theories of simplicial complexes, usually we fix an orientation of each simplex and then construct suitable complexes of abelian groups and compute the (co)homology groups. We give a slightly different complex, where we do not fix a choice of orientation of each simplex. This is because the Bruhat-Tits building is not canonically oriented (even though they look like they are canonically oriented).

We end this chapter with a remark in Section 2.7.

2.1. Generalized simplicial complexes

2.1.1. Let us recall the notion of (abstract) simplicial complex.

Definition 2.1. A (strict) simplicial complex is a pair \((Y_0, \Delta)\) of a set \(Y_0\) and a set \(\Delta\) of finite subsets of \(Y_0\) which satisfies the following conditions:

- If \(S \in \Delta\) and \(T \subset S\), then \(T \in \Delta\).
- If \(v \in Y_0\), then \(\{v\} \in \Delta\).

In this paper we call a simplicial complex in the sense above a strict simplicial complex, and use the terminology “simplicial complex” in a little broader sense as defined below. We recall that in a strict (abstract) simplicial complex, as recalled above, each simplex is uniquely determined by the set of its vertices.

2.1.2. The definition of (generalized) simplicial complex is as follows. For a set \(S\), let \(\mathcal{P}_{\text{fin}}(S)\) denote the category whose objects are the non-empty finite subsets of \(S\) and whose morphisms are the inclusions.

Definition 2.2. A simplicial complex is a pair \((Y_0, F)\) of a set \(Y_0\) and a presheaf \(F\) of sets on \(\mathcal{P}_{\text{fin}}(Y_0)\) such that \(F(\{\sigma\}) = \{\sigma\}\) holds for every \(\sigma \in Y_0\).
2.1.3. The definition above is too abstract in practice. We now give a working definition which is equivalent to the definition above.

**Definition 2.3.** A (generalized) simplicial complex is a collection \( Y_* = (Y_i)_{i \geq 0} \) of sets indexed by non-negative integers, equipped with the following additional data:

- a subset \( V(\sigma) \subset Y_0 \) with cardinality \( i + 1 \), for each \( i \geq 0 \) and for each \( \sigma \in Y_i \) (we call \( V(\sigma) \) the set of vertices of \( \sigma \)), and
- an element in \( Y_j \), for each \( i \geq j \geq 0 \), for each \( \sigma \in Y_i \), and for each subset \( V' \subset V(\sigma) \) with cardinality \( j + 1 \) (we denote this element in \( Y_j \) by the symbol \( \sigma \times_{V(\sigma)} V' \) and call it the face of \( \sigma \) corresponding to \( V' \))

which satisfy the following conditions:

- For each \( \sigma \in Y_0 \), the equality \( V(\sigma) = \{ \sigma \} \) holds,
- For each \( i \geq 0 \), for each \( \sigma \in Y_i \), and for each non-empty subset \( V' \subset V(\sigma) \), the equality \( V(\sigma \times_{V(\sigma)} V') = V' \) holds.
- For each \( i \geq 0 \) and for each \( \sigma \in Y_i \), the equality \( \sigma \times_{V(\sigma)} V(\sigma) = \sigma \) holds, and
- For each \( i \geq 0 \), for each \( \sigma \in Y_i \), and for each non-empty subsets \( V', V'' \subset V(\sigma) \) with \( V'' \subset V' \), the equality \( \sigma \times_{V(\sigma)} V' \times_{V'} V'' = \sigma \times_{V(\sigma)} V'' \) holds.

Let us call the element of the form \( \sigma \times_{V(\sigma)} V' \) for \( j \) and \( V' \) as above, the \( j \)-dimensional face of \( \sigma \) corresponding to \( V' \). We remark here that the symbol \( \times_{V(\sigma)} \) does not mean a fiber product in any way.

2.1.4. This equivalence of the two definitions is explicitly described as follows. Suppose we are given a simplicial complex \( Y_* \). Then the corresponding \( F \) is the presheaf which associates, to a non-empty finite subset \( V \subset Y_0 \) with cardinality \( i + 1 \), the set of elements \( \sigma \in Y_i \) satisfying \( V(\sigma) = V \).

This alternative definition of a simplicial complex is much closer to the definition of a strict simplicial complex.

2.1.5. Any strict simplicial complex gives a simplicial complex in the sense above in the following way. Let \( (Y_0, \Delta) \) be a strict simplicial complex. We identify \( Y_0 \) with the set of subsets of \( Y_0 \) with cardinality 1. For \( i \geq 1 \) let \( Y_i \) denote the set of the elements in \( \Delta \) which has cardinality \( i + 1 \) as a subset of \( Y_0 \). For \( i \geq 1 \) and for \( \sigma \in Y_i \), we set \( V(\sigma) = \sigma \) regarded as a subset of \( Y_0 \). For a non-empty subset \( V \subset V(\sigma) \), of cardinality \( i' + 1 \), we set \( \sigma \times_{V(\sigma)} V = V \) regarded as an element of \( Y_{i'} \). Then it is easily checked that the collection \( Y_* = (Y_i)_{i \geq 0} \) together with the assignments \( \sigma \mapsto V(\sigma) \) and \( (\sigma, V) \mapsto \sigma \times_{V(\sigma)} V \) forms a simplicial complex.

2.1.6. An example of a (generalized) simplicial complex which is not a strict simplicial complex consists of two vertices with two edges between the two vertices.

2.1.7. The Bruhat-Tits building (for PGL as introduced in Definition 3.2) is a strict simplicial complex. However, its arithmetic quotients are generally merely a (generalized) simplicial complex.

2.1.8. Let \( (Y_0, F) \) and \( (Z_0, G) \) be simplicial complexes.

**Definition 2.4.** A morphism \( (Y_0, F) \to (Z_0, G) \) is a map of sets \( f_0 : Y_0 \to Z_0 \) and a morphism of presheaves \( F \to G \circ P_{\text{fin}}(f_0) \) on \( P_{\text{fin}}(f_0) \). Here \( P_{\text{fin}}(f_0) : P_{\text{fin}}(Y_0) \to P_{\text{fin}}(Z_0) \) is the functor induced by \( f_0 \).
2.1.9. The definition above in terms of the working definition is as follows. Let \( Y_\bullet \) and \( Z_\bullet \) be simplicial complexes. We define a map (morphism) from \( Y_\bullet \) to \( Z_\bullet \) to be a collection \( f = (f_i)_{i \geq 0} \) of maps \( f_i : Y_i \to Z_i \) of sets which satisfies the following conditions:

- for any \( i \geq 0 \) and for any \( \sigma \in Y_i \), the restriction of \( f_0 \) to \( V(\sigma) \) is injective and the image of \( f|_{V(\sigma)} \) is equal to the set \( V(f_i(\sigma)) \), and
- for any \( i \geq j \geq 0 \), for any \( \sigma \in Y_i \), and for any non-empty subset \( V' \subset V(\sigma) \) with cardinality \( j + 1 \) we have \( f_j(\sigma \times_{V(\sigma)} V') = f_i(\sigma) \times_{V(f_i(\sigma))} f_0(V') \).

2.2. Orientation

Usually the (co)homology groups of \( Y_\bullet \) are defined to be the (co)homology groups of a complex \( C_\bullet \) whose component in degree \( i \) is the free abelian group generated by the \( i \)-simplices of \( Y_\bullet \). For a precise definition of the boundary homomorphism of the complex \( C_\bullet \), we need to choose an orientation of each simplex. In this paper we adopt an alternative, equivalent definition of homology groups which does not require any choice of orientations. The latter definition seems a little complicated at first glance, however it will soon turn out to be a better way for describing the (co)homology of the arithmetic quotients Bruhat-Tits building, which seems to have no canonical, good choice of orientations.

2.2.1. We introduce the notion of orientation of a simplex. It will be a \( \{\pm 1\} \)-torsor (where \( \{\pm 1\} \) is the abelian group \( \mathbb{Z}/2\mathbb{Z} \)) associated with each simplex.

Let \( Y_\bullet \) be a simplicial complex and let \( i \geq 0 \) be a non-negative integer. For an \( i \)-simplex \( \sigma \in Y_i \), we let \( T(\sigma) \) denote the set of all bijections from the finite set \( \{1, \ldots, i+1\} \) of cardinality \( i+1 \) to the set \( V(\sigma) \) of vertices of \( \sigma \). The symmetric group \( S_{i+1} \) acts on the set \( \{1, \ldots, i+1\} \) from the left and hence on the set \( T(\sigma) \) from the right. Through this action the set \( T(\sigma) \) is a right \( S_{i+1} \)-torsor.

**Definition 2.5.** We define the set \( O(\sigma) \) of orientations of \( \sigma \) to be the \( \{\pm 1\} \)-torsor \( O(\sigma) = T(\sigma) \times_{S_{i+1}, \text{sgn}} \{\pm 1\} \) which is the push-forward of the \( S_{i+1} \)-torsor \( T(\sigma) \) with respect to the signature character \( \text{sgn} : S_{i+1} \to \{\pm 1\} \).

We give the basic properties which we need in order to consider (co)homology.

2.2.2. When \( i \geq 1 \), the \( \{\pm 1\} \)-torsor \( O(\sigma) \) is isomorphic, as a set, to the quotient \( T(\sigma)/A_{i+1} \) of \( T(\sigma) \) by the action of the alternating group \( A_{i+1} = \text{Ker} \text{sgn} \subset S_{i+1} \). When \( i = 0 \), the \( \{\pm 1\} \)-torsor \( O(\sigma) \) is isomorphic to the product \( O(\sigma) = T(\sigma) \times \{\pm 1\} \), on which the group \( \{\pm 1\} \) acts via its natural action on the second factor.

2.2.3. Let \( i \geq 1 \) and let \( \sigma \in Y_i \). For \( v \in V(\sigma) \) let \( \sigma_v \) denote the \((i-1)\)-simplex \( \sigma_v = \sigma \times_{V(\sigma)} (V(\sigma) \setminus \{v\}) \). There is a canonical map \( s_v : O(\sigma) \to O(\sigma_v) \) of \( \{\pm 1\} \)-torsors defined as follows. Let \( \nu \in O(\sigma) \) and take a lift \( \tilde{\nu} : \{1, \ldots, i+1\} \xrightarrow{\cong} V(\sigma) \) of \( \nu \) in \( T(\sigma) \). Let \( \tilde{\tau}_v : \{1, \ldots, i\} \hookrightarrow \{1, \ldots, i+1\} \) denote the unique order-preserving injection whose image is equal to \( \{1, \ldots, i+1\} \setminus \{\tilde{\nu}^{-1}(v)\} \). It follows from the definition of \( \tilde{\tau}_v \) that the composite \( \tilde{\nu} \circ \tilde{\tau}_v : \{1, \ldots, i\} \to V(\sigma) \) induces a bijection \( \tilde{\nu}_v : \{1, \ldots, i\} \xrightarrow{\cong} V(\sigma) \setminus \{v\} = V(\sigma_v) \). We regard \( \tilde{\nu}_v \) as an element in \( T(\sigma_v) \). We define \( s_v : O(\sigma) \to O(\sigma_v) \) to be the map which sends \( \nu \in O(\sigma) \) to \((-1)^{\tilde{\nu}^{-1}(v)}\) times the class of \( \tilde{\nu}_v \). It is easy to check that the map \( s_v \) is well-defined.
2.2.4. Let \( i \geq 2 \) and \( \sigma \in Y_i \). Let \( v, v' \in V(\sigma) \) with \( v \neq v' \). We have \((\sigma_{v'})_{v} = (\sigma_{v'})_{v'}\). Let us consider the two composites \( s_{v'} \circ s_v : O(\sigma) \to O((\sigma_{v'})_{v'}) \) and \( s_v \circ s_{v'} : O(\sigma) \to O((\sigma_{v'})_{v}) \). It is easy to check that the equality

\[
(2.1) \quad s_{v'} \circ s_v(\nu) = (-1) \cdot s_v \circ s_{v'}(\nu)
\]

holds for every \( \nu \in O(\sigma) \).

2.3. Cohomology and homology

**Definition 2.6.** We say that a simplicial complex \( Y_* \) is locally finite if for any \( i \geq 0 \) and for any \( \tau \in Y_i \), there exist only finitely many \( \sigma \in Y_{i+1} \) such that \( \tau \) is a face of \( \sigma \).

We give the four notions of homology or cohomology for a locally finite (generalized) simplicial complex.

2.3.1. Let \( Y_* \) be a simplicial complex (resp. a locally finite simplicial complex). For an integer \( i \geq 0 \), we set \( Y'_i = \prod_{\nu \in Y_i} O(\sigma) \) and regard it as a \( \{\pm 1\}\)-set. Given an abelian group \( M \), we regard the abelian groups \( \bigoplus_{\nu \in Y_i} M \) and \( \prod_{\nu \in Y_i} M \) as \( \{\pm 1\}\)-modules in such a way that the component at \( \epsilon \cdot \nu \) of \( \epsilon \cdot (m_{\nu}) \) is equal to \( \epsilon m_{\nu} \) for \( \epsilon \in \{\pm 1\} \) and for \( \nu \in Y'_i \).

2.3.2. For \( i \geq 1 \), let \( \partial_{i,\oplus} : \bigoplus_{\nu \in Y_i} M \to \bigoplus_{\nu \in Y'_{i-1}} M \) (resp. \( \partial_{i,\Pi} : \prod_{\nu \in Y_i} M \to \prod_{\nu \in Y'_{i-1}} M \)) denote the homomorphism of abelian groups that sends \( m = (m_{\nu})_{\nu \in Y_i} \) to the element \( \partial_i(m) \) whose coordinate at \( \nu' \in O(\sigma') \subset Y'_{i-1} \) is equal to

\[
(2.2) \quad \partial_i(m)_{\nu'} = \sum_{(v,\sigma,\nu) \in \nu'} m_{\nu'}. 
\]

Here in the sum on the right hand side \( (v,\sigma,\nu) \) runs over the triples of \( \nu \in Y'_0 \setminus V(\sigma') \), an element \( \sigma \in Y_i \), and \( \nu \in O(\sigma) \) which satisfy \( V(\sigma) = V(\sigma') \Pi \{v\} \) and \( s_v(\nu) = \nu' \).

Note that the sum on the right hand side is a finite sum for \( \partial_{i,\oplus} \) by definition. One can see also that the sum is a finite sum in the case of \( \partial_{i,\Pi} \) using the locally finiteness of \( Y_* \).

Each of \( \partial_{i,\oplus} \) and \( \partial_{i,\Pi} \) is a homomorphism of \( \{\pm 1\}\)-modules. Hence it induces a homomorphism \( \tilde{\partial}_{i,\oplus} : (\bigoplus_{\nu \in Y_i} M)_{\{\pm 1\}} \to (\bigoplus_{\nu \in Y'_{i-1}} M)_{\{\pm 1\}} \) (resp. \( \tilde{\partial}_{i,\Pi} : (\prod_{\nu \in Y_i} M)_{\{\pm 1\}} \to (\prod_{\nu \in Y'_{i-1}} M)_{\{\pm 1\}} \)) of abelian groups, where the subscript \( \{\pm 1\} \) means the coinvariants.

2.3.3. It follows from the formula (2.1) and the definition of \( \tilde{\partial}_{i,\oplus} \) and \( \tilde{\partial}_{i,\Pi} \) that the family of abelian groups \( (\bigoplus_{\nu \in Y_i} M)_{\{\pm 1\}} \) (resp. \( (\prod_{\nu \in Y_i} M)_{\{\pm 1\}} \)) indexed by the integer \( i \geq 0 \), together with the homomorphisms \( \tilde{\partial}_{i,\oplus} \) (resp. \( \tilde{\partial}_{i,\Pi} \)) for \( i \geq 1 \), forms a complex of abelian groups. The homology groups of this complex are the homology groups \( H_*(Y_*, M) \) (resp. the Borel-Moore homology groups \( H_*^{BM}(Y_*, M) \)) of \( Y_* \) with coefficients in \( M \).

2.3.4. We note that there is a canonical map \( H_*(Y_*, M) \to H_*^{BM}(Y_*, M) \) from homology to Borel-Moore homology induced by the map of complexes \( ((\bigoplus_{\nu \in Y_i} M)_{\{\pm 1\}})_{i \geq 0} \to ((\prod_{\nu \in Y_i} M)_{\{\pm 1\}})_{i \geq 0} \) given by inclusion at each degree.
2.3.5. The family of abelian groups \( \text{Map}_{\{\pm 1\}}(Y'_i, M) \) (resp. \( \text{Map}_{\text{fin}, \{\pm 1\}}(Y'_i, M) \)) where the subscript \( \text{fin} \) means finite support of the \( \{\pm 1\} \)-equivariant maps from \( Y'_i \) to \( M \) forms a complex of abelian groups in a similar manner. (One uses the locally finiteness of \( Y_i \) for the latter.) The cohomology groups of this complex are the cohomology groups \( H^*(Y_\bullet, M) \) (resp. the cohomology groups with compact support \( H_c^*(Y_\bullet, M) \)) of \( Y_\bullet \) with coefficients in \( M \). There is a canonical map from cohomology with compact support to cohomology.

2.4. Universal coefficient theorem

It follows from the definitions that the following universal coefficient theorem holds.

2.4.1. For a simplicial complex \( Y_\bullet \), there exist canonical short exact sequences

\[
0 \to H_i(Y_\bullet, \mathbb{Z}) \otimes M \to H_i(Y_\bullet, M) \to \text{Tor}_1^\mathbb{Z}(H_{i-1}(Y_\bullet, \mathbb{Z}), M) \to 0
\]

and

\[
0 \to \text{Ext}_1^\mathbb{Z}(H_{i-1}(Y_\bullet, \mathbb{Z}), M) \to H^i(Y_\bullet, M) \to \text{Hom}_\mathbb{Z}(H_i(Y_\bullet, \mathbb{Z}), M) \to 0.
\]

for any abelian group \( M \).

2.4.2. Similarly, for a locally finite simplicial complex \( Y_\bullet \), we have short exact sequences

\[
0 \to \text{Ext}_1^\mathbb{Z}(H_{i+1}^c(Y_\bullet, \mathbb{Z}), M) \to H^c_i(Y_\bullet, M) \to \text{Hom}_\mathbb{Z}(H^c_i(Y_\bullet, \mathbb{Z}), M) \to 0
\]

and

\[
0 \to H^c_i(Y_\bullet, \mathbb{Z}) \otimes M \to H^c_i(Y_\bullet, M) \to \text{Tor}_1^\mathbb{Z}(H^{c+1}_i(Y_\bullet, \mathbb{Z}), M) \to 0
\]

for any abelian group \( M \).

2.5. Some induced maps

Let \( f = (f_i)_{i \geq 0} : Y_\bullet \to Z_\bullet \) be a map of simplicial complexes. For each integer \( i \geq 0 \) and for each abelian group \( M \), the map \( f \) canonically induces homomorphisms \( f_* : H_i(Y_\bullet, M) \to H_i(Z_\bullet, M) \) and \( f^* : H^i(Z_\bullet, M) \to H^i(Y_\bullet, M) \). We say that the map \( f \) is finite if the subset \( f_{i-1}^{-1}(\sigma) \) of \( Y_i \) is finite for any \( i \geq 0 \) and for any \( \sigma \in Z_i \). If \( Y_\bullet \) and \( Z_\bullet \) are locally finite, and if \( f \) is finite, then \( f \) canonically induces the pushforward homomorphism \( f_* : H^c_i(Y_\bullet, M) \to H^c_i(Z_\bullet, M) \) and the pullback homomorphism \( f^* : H^c_i(Z_\bullet, M) \to H^c_i(Y_\bullet, M) \).

2.6. Geometric realization

Let \( Y_\bullet \) be a simplicial complex. We associate a CW complex \( |Y_\bullet| \) which we call the geometric realization of \( Y_\bullet \).

Let \( I(Y_\bullet) \) denote the disjoint union \( \coprod_{i \geq 0} Y_i \). We define a partial order on the set \( I(Y_\bullet) \) by saying that \( \tau \leq \sigma \) if and only if \( \tau \) is a face of \( \sigma \). For \( \sigma \in I(Y_\bullet) \), we let \( \Delta_\sigma \) denote the set of maps \( f : V(\sigma) \to \mathbb{R}_{\geq 0} \) satisfying \( \sum_{v \in V(\sigma)} f(v) = 1 \). We regard \( \Delta_\sigma \) as a topological space whose topology is induced from that of the real vector space \( \text{Map}(V(\sigma), \mathbb{R}) \). If \( \tau \) is a face of \( \sigma \), we regard the space \( \Delta_\tau \) as the closed subspace of \( \Delta_\sigma \) which consists of the maps \( V(\sigma) \to \mathbb{R}_{\geq 0} \) whose support is contained in the subset \( V(\tau) \subset V(\sigma) \). We let \( |Y_\bullet| \) denote the colimit \( \lim_{\sigma \in I(Y_\bullet)} \Delta_\sigma \) in the category of topological spaces and call it the geometric realization of \( Y_\bullet \).
It follows from the definition that the geometric realization $|Y^*|$ has a canonical structure of CW complex.

2.7. **Cellular versus singular**

We give a remark on the use of the term “Borel-Moore homology” in this paragraph. Given a strict simplicial complex, its cohomology, homology and cohomology with compact support (for a locally finite strict simplicial complex) are usually defined as above, and called cellular (co)homology. See for example [Hatc].

On the other hand, there is also the singular (co)homology and with compact support that are defined using the singular (co)chain complex. It is well-known that the cellular (co)homology groups (with compact support) are isomorphic to the singular (co)homology groups (with compact support) of the geometric realization. The same proof applies to the generalized simplicial complexes and gives an isomorphism between the cellular and the singular theories.

For the Borel-Moore homology, we did not find a cellular definition as above, except in Hattori [Hatt] where he does not call it the Borel-Moore homology. He also gives a definition using singular chains and shows that the two homology groups are isomorphic.

There are several definitions of Borel-Moore homology that may be associated to a (strict) simplicial complex. The definition of the Borel-Moore homology for PL manifolds is found in Haefliger [Ha]. There is also a sheaf theoretic definition in Iversen [Iv]. More importantly, there is the general definition which concerns the intersection homology. However, we did not find a statement in the literature and we did not check that the cellular definition in Hattori (same as the one given in this article) is isomorphic to the other Borel-Moore homology theories.
CHAPTER 3

The Bruhat-Tits building and apartments

Let $d \geq 1$ be a positive integer. In this chapter, we recall the definition of the Bruhat-Tits building of $\text{PGL}_d$ over a nonarchimedean local field using lattices and subsimplicial complexes called apartments. (For the general theory of Bruhat-Tits building and apartments, the reader is referred to [Br-Ti] and the book [Ab-Br].) Then we define the fundamental class of an apartment in its $(d-2)$-nd Borel-Moore homology group.

Later, we define modular symbols to be the image of the fundamental classes of apartments associated with $F$-basis ($F$ is a global field) in the Borel-Moore homology of quotients of the Bruhat-Tits building.

3.1. The Bruhat-Tits building of $\text{PGL}_d$

In the following paragraphs, we recall the definition of the Bruhat-Tits building of $\text{PGL}_d$ over a nonarchimedean local field. We recall that it is a strict simplicial complex.

3.1.1. Notation. Let $K$ be a nonarchimedean local field. We let $\mathcal{O} \subset K$ denote the ring of integers. We fix a uniformizer $\varpi \in \mathcal{O}$. Let $d \geq 1$ be an integer. Let $V = K^{\oplus d}$. We regard it as the set of row vectors so that $\text{GL}_d(K)$ acts from the right by multiplication.

3.1.2. The Bruhat-Tits building ($\text{Br-Ti}$) using lattices. We do not recall here the most general definition of the Bruhat-Tits buildings. Let us give a definition using lattices (see also [Gr], §4) first, and then give a more explicit description for later use.

3.1.2.1. An $\mathcal{O}$-lattice in $V$ is a free $\mathcal{O}$-submodule of $V$ of rank $d$. We denote the set of $\mathcal{O}$-lattices in $V$ by $\text{Lat}_{\mathcal{O}}(V)$. We regard the set $\text{Lat}_{\mathcal{O}}(V)$ as a partially ordered set whose elements are ordered by the inclusions of $\mathcal{O}$-lattices.

3.1.2.2. Two $\mathcal{O}$-lattices $L$, $L'$ of $V$ are called homothetic if $L = \varpi^j L'$ for some $j \in \mathbb{Z}$. Let $\overline{\text{Lat}}_{\mathcal{O}}(V)$ denote the set of homothety classes of $\mathcal{O}$-lattices $V$.

We denote by $\text{cl}$ the canonical surjection $\text{cl} : \text{Lat}_{\mathcal{O}}(V) \to \overline{\text{Lat}}_{\mathcal{O}}(V)$.

**Definition 3.1.** We say that a subset $S$ of $\overline{\text{Lat}}_{\mathcal{O}}(V)$ is totally ordered if $\text{cl}^{-1}(S)$ is a totally ordered subset of $\text{Lat}_{\mathcal{O}}(V)$.

3.1.2.3. The pair $(\overline{\text{Lat}}_{\mathcal{O}}(V), \Delta)$ of the set $\overline{\text{Lat}}_{\mathcal{O}}(V)$ and the set $\Delta$ of totally ordered finite nonempty subsets of $\overline{\text{Lat}}_{\mathcal{O}}(V)$ forms a strict simplicial complex.

**Definition 3.2.** The Bruhat-Tits building of $\text{PGL}_d$ over $K$ is the simplicial complex $\mathcal{B}T_\bullet$ associated to this strict simplicial complex.

3.1.3. Explicit description of the building. In the next paragraphs we explicitly describe the simplicial complex $\mathcal{B}T_\bullet$. 

3.1.3.1. For an integer \( i \geq 0 \), let \( \overline{BT}_i \) be the set of sequences \( (L_j)_{j \in \mathbb{Z}} \) of \( \mathcal{O} \)-lattices in \( V \) indexed by \( j \in \mathbb{Z} \) such that \( L_j \supsetneq L_{j+1} \) and \( \varpi L_j = L_{j+1} \) hold for all \( j \in \mathbb{Z} \). In particular, if \( (L_j)_{j \in \mathbb{Z}} \) is an element in \( \overline{BT}_0 \), then \( L_j = \varpi^j L_0 \) for all \( j \in \mathbb{Z} \).

We identify the set \( \overline{BT}_0 \) with the set \( \text{Lat}_\mathcal{O}(V) \) by associating the \( \mathcal{O} \)-lattice \( L \) to an element \( (L_j)_{j \in \mathbb{Z}} \) in the set \( \overline{BT}_0 \). We say that two elements \( (L_j)_{j \in \mathbb{Z}} \) and \( (L'_j)_{j \in \mathbb{Z}} \) in \( \overline{BT}_i \) are equivalent if there exists an integer \( \ell \) satisfying \( L_j = L'_{j+\ell} \) for all \( j \in \mathbb{Z} \). We will see below that the set of the equivalence classes in \( \overline{BT}_i \) is identified with \( BT_i \). For \( i = 0 \), the identification \( \overline{BT}_0 \cong \text{Lat}_\mathcal{O}(V) \) gives an identification \( BT_0 \cong \text{Lat}_\mathcal{O}(V) \).

3.1.3.2. Let \( \sigma \in BT_i \) and take a representative \( (L_j)_{j \in \mathbb{Z}} \) of \( \sigma \). For \( j \in \mathbb{Z} \), let us consider the class \( \text{cl}(L_j) \) in \( \text{Lat}_\mathcal{O}(V) \). Since \( \varpi L_j = L_{j+1} \), we have \( \text{cl}(L_j) = \text{cl}(L_{j+1}) \). Since \( L_j \supsetneq L_k \supsetneq \varpi L_j \) for \( 0 < j < k \leq i \), the elements \( \text{cl}(L_0), \ldots, \text{cl}(L_i) \) in \( \text{Lat}_\mathcal{O}(V) \) are distinct. Hence the subset \( V(\sigma) = \{ \text{cl}(L_j) \mid j \in \mathbb{Z} \} \subset BT_0 \) has cardinality \( i+1 \) and does not depend on the choice of \( (L_j)_{j \in \mathbb{Z}} \). It is easy to check that the map from \( BT_i \) to the set of finite subsets of \( \text{Lat}_\mathcal{O}(V) \) which sends \( \sigma \in BT_i \) to \( V(\sigma) \) is injective and that the set \( \{ V(\sigma) \mid \sigma \in BT_i \} \) is equal to the set of totally ordered subsets of \( \text{Lat}_\mathcal{O}(V) \) with cardinality \( i+1 \). In particular, for any \( j \in \{ 0, \ldots, i \} \) and for any subset \( V' \subset V(\sigma) \) of cardinality \( j+1 \), there exists a unique element in \( BT_j \), which we denote by \( \sigma \times_{V(\sigma)} V' \), such that \( V(\sigma \times_{V(\sigma)} V') = V' \). Thus the collection \( BT_* = \bigcup_{i \geq 0} BT_i \) together with the data \( V(\sigma) \) and \( \sigma \times_{V(\sigma)} V' \) forms a simplicial complex which is canonically isomorphic to the simplicial complex associated to the strict simplicial complex \( (\text{Lat}_\mathcal{O}(V), \Delta) \) which we introduced in the first paragraph of Section 3.1.2. We call the simplicial complex \( BT_* \) the Bruhat-Tits building of \( \text{PGL}_d \) over \( K \).

3.1.3.3. The simplicial complex \( BT_* \) is of dimension at most \( d-1 \), by which we mean that \( BT_i \) is an empty set for \( i > d-1 \). It follows from the fact that \( \overline{BT}_i \) is an empty set for \( i > d-1 \), which we can check as follows. Let \( i > d-1 \) and assume that there exists an element \( (L_j)_{j \in \mathbb{Z}} \) in \( \overline{BT}_i \). Then for \( j = 0, \ldots, i+1 \), the quotient \( L_j/L_{i+1} \) is a subspace of the \( d \)-dimensional \( (\mathcal{O}/\varpi \mathcal{O}) \)-vector space \( L_0/L_{i+1} = L_0/\varpi L_0 \). These subspaces must satisfy \( L_0/L_{i+1} \supsetneq L_1/L_{i+1} \supsetneq \cdots \supsetneq L_{i+1}/L_{i+1} \). It is impossible since \( i+1 > d \).

3.2. Apartments

Here we recall the definition of apartment. It is a simplicial subcomplex of the Bruhat-Tits building. We are interested only in the apartments of \( \text{PGL}_d \) of a nonarchimedean local field and not of other algebraic groups. To describe apartments, we do not use the general theory via root systems but give a simpler, ad hoc treatment, particular to \( \text{PGL}_d \). The readers are referred to [Ab-Br] p. 523, 10.1.7 Example for the general theory.

For example, when \( d = 2 \), it is an infinite sequence of 1-simplices. When \( d = 3 \), it is an \( \mathbb{R}^2 \) tiled by triangles (2-simplices). The geometric realization is homeomorphic to \( \mathbb{R}^{d-1} \).

3.2.1. Set \( A_0 = \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \ldots, 1) \). For two elements \( x = (x_j), y = (y_j) \in \mathbb{Z}^{\oplus d} \), we write \( x \leq y \) when \( x_j \leq y_j \) for all \( 1 \leq j \leq d \). We regard \( \mathbb{Z}^{\oplus d} \) as a partially ordered set with respect to \( \leq \). Let \( \pi : \mathbb{Z}^{\oplus d} \to A_0 \) denote the quotient map.

We say that a subset \( S \) of \( A_0 \) is totally ordered if \( \pi^{-1}(S) \) is a totally ordered subset of \( \mathbb{Z}^{\oplus d} \). The pair \( (A_0, D) \) of the set \( A_0 \) and the set \( D \) of totally ordered
finite nonempty subsets of $A_0$ forms a strict simplicial complex. We denote by $A_* = (A_i)_{i \geq 0}$ the simplicial complex associated to the strict simplicial complex $(A_0, \bigsqcup_{i \geq 0} A_i)$. We note that $A_i$ is an empty set for $i \geq d$, since by definition there is no totally ordered subset of $A_0$ with cardinality larger than $d$.

**3.2.2.** Let $v_1, \ldots, v_d$ be a basis of $V = K^{\oplus d}$. We define a map $\iota_{v_1,\ldots,v_d} : A_* \to \mathcal{B}T_*$ of simplicial complexes as follows.

Let $\overline{\iota}_{v_1,\ldots,v_d} : \mathbb{Z}^{\oplus d} \to \overline{\mathcal{B}}T_0$ denote the map that sends the element $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ to the $O$-lattice $O\omega^{n_1}v_1 \oplus O\omega^{n_2}v_2 \oplus \cdots \oplus O\omega^{n_d}v_d$. The map $\overline{\iota}_{v_1,\ldots,v_d}$ is an order-embedding of partially ordered sets, and induces a map $\iota_{v_1,\ldots,v_d,0} : A_0 \to \overline{\mathcal{B}}T_0$ that makes the diagram

$$
\begin{array}{ccc}
\mathbb{Z}^{\oplus d} & \xrightarrow{\overline{\iota}_{v_1,\ldots,v_d}} & \overline{\mathcal{B}}T_0 \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\iota_{v_1,\ldots,v_d,0}} & \overline{\mathcal{B}}T_0
\end{array}
$$

of sets, where the vertical arrows are the quotient maps, commutative and cartesian.

This implies that the map $\iota_{v_1,\ldots,v_d,0} : A_0 \to \overline{\mathcal{B}}T_0$ sends a totally ordered subset of $A_0$ to a totally ordered subset of $\overline{\mathcal{B}}T_0$. Hence the map $\iota_{v_1,\ldots,v_d,0} : A_0 \to \overline{\mathcal{B}}T_0$ induces a map $\iota_{v_1,\ldots,v_d} : A_* \to \mathcal{B}T_*$ of simplicial complexes.

It is easy to check that the map $\iota_{v_1,\ldots,v_d} : A_i \to \mathcal{B}T_i$ is injective for every $i \geq 0$. We define a simplicial subcomplex $A_{v_1,\ldots,v_d,*}$ of $\mathcal{B}T_*$ to be the image of the map $\iota_{v_1,\ldots,v_d,i}$ so that $A_{v_1,\ldots,v_d,i}$ is the image of the map $\iota_{v_1,\ldots,v_d,i}$ for each $i \geq 0$. We call the subcomplex $A_{v_1,\ldots,v_d,*}$ of $\mathcal{B}T_*$ the apartment in $\mathcal{B}T_*$ corresponding to the basis $v_1, \ldots, v_d$. Since the map $\iota_{v_1,\ldots,v_d,i}$ is injective for every $i \geq 0$, the map $\iota_{v_1,\ldots,v_d}$ induces an isomorphism $A_* \xrightarrow{\cong} A_{v_1,\ldots,v_d,*}$ of simplicial complexes.

**3.3. the fundamental class**

We introduce a special element $\beta$ in the group $H_{d-1}^{BM}(A_*, \mathbb{Z})$, which is an analogue of the fundamental class.

**3.3.1.** Let $\sigma \in A_i$ and let $V(\sigma) \subset A_0$ denote the set of vertices of $\sigma$. By definition, $V(\sigma)$ consists of exactly $i + 1$ elements and the inverse image $\widetilde{V}(\sigma) = \pi^{-1}(V(\sigma))$ is a totally ordered subset of $\mathbb{Z}^d$.

**Lemma 1.** As a totally ordered set, $\widetilde{V}(\sigma)$ is isomorphic to $\mathbb{Z}$.

**Proof.** Let $\Sigma : \mathbb{Z}^{\oplus d} \to \mathbb{Z}$ denote the map that sends $(n_1, \ldots, n_d)$ to $n_1 + \cdots + n_d$. Then the composite of the inclusion $\widetilde{V}(\sigma) \hookrightarrow \mathbb{Z}^{\oplus d}$ with $\Sigma$ is order-preserving and injective since $\widetilde{V}(\sigma)$ is totally ordered and for any $x, y \in \mathbb{Z}^{\oplus d}$, the relations $x \leq y$ and $\Sigma(x) < \Sigma(y)$ implies $x < y$. Since $\widetilde{V}(\sigma)$ is closed under the addition of $\pm(1, \ldots, 1)$, it follows that $\widetilde{V}(\sigma)$ is isomorphic, as a totally ordered set, to a subset of $\mathbb{Z}$ which is unbounded both from below and from above. This shows that $\widetilde{V}(\sigma)$ is isomorphic to $\mathbb{Z}$. $\square$

Let $x \in V(\sigma)$ and let us choose a lift $\tilde{x} \in \widetilde{V}(\sigma)$ of $x$. Lemma 1 implies that there exists a maximum element $\tilde{x}'$ of $\widetilde{V}(\sigma)$ satisfying $\tilde{x}' < \tilde{x}$. We set $e(\sigma, x) = \tilde{x} - \tilde{x}' \in \mathbb{Z}^d$. Then $e(\sigma, x)$ does not depend on the choice of the lift $\tilde{x}$ and we have $e(\sigma, x) > 0$ and $\sum_{x \in V(\sigma)} e(\sigma, x) = (1, \ldots, 1)$. 
3.3.2. Now let us assume that \( \sigma \in A_{d-1} \). Then the set \( \{ e(\sigma, x) \mid x \in V(\sigma) \} \) is equal to the set \( \{ e_1, \ldots, e_d \} \) where \( e_1, \ldots, e_d \) denotes the standard \( \mathbb{Z} \)-basis of \( \mathbb{Z}^d \).

This implies that, for any \( i \in \{ 1, \ldots, d \} \), there exists a unique element \( x_i \in V(\sigma) \) satisfying \( e(\sigma, x_i) = e_i \). The map \( \hat{\sigma} : \{ 1, \ldots, d \} \to V(\sigma) \) that sends \( i \) to \( x_i \) is bijective. Hence the map \( \hat{\sigma} \) defines an element of \( T(\sigma) \) (see Section 2.2.1 for the definition). We denote by \( [\hat{\sigma}] \) the class of \( [\hat{\sigma}] \) in \( O(\sigma) \). We let \( \hat{\beta} \) denote the element \( \hat{\beta} = (\hat{\beta}_\nu)_{\nu \in A'_{d-1}} \) in \( \prod_{\nu \in A'_{d-1}} \mathbb{Z} \) where \( \hat{\beta}_\nu = 1 \) if \( \nu = [\sigma] \) for some \( \sigma \in A_{d-1} \) and \( \hat{\beta}_\nu = 0 \) otherwise. We denote by \( \beta \) the class of \( \hat{\beta} \) in \( (\prod_{\nu \in A'_{d-1}} \mathbb{Z})\{\pm 1\} \).

3.3.3. Recall that we defined in Section 2.3 a chain complex which computes the Borel-Moore homology of \( A_\bullet \).

**Proposition 2.** The element \( \beta \in (\prod_{\nu \in A'_{d-1}} \mathbb{Z})\{\pm 1\} \) is a \((d-1)\)-cycle in the chain complex which computes the Borel-Moore homology of \( A_\bullet \).

**Proof.** The assertion is clear for \( d = 1 \) since the \((d-2)\)-nd component of the complex is zero. Suppose that \( d \geq 2 \). Let \( \tau \) be an element in \( A_{d-2} \).

Since \( \sum_{x \in V(\tau)} e(\tau, x) = (1, \ldots, 1) \), there exists a unique vertex \( y \in V(\tau) \) such that \( e(\tau, x) \) belongs to \( \{ e_1, \ldots, e_d \} \) for \( x \neq y \) and \( e(\tau, y) \) is equal to the sum of two distinct elements of \( \{ e_1, \ldots, e_d \} \). Let us write \( e(\tau, y) = e_j + e_{j'} \). Let us choose a lift \( \tilde{y} \in \tilde{V}(\tau) \) of \( y \) and set \( \tilde{y}' = \tilde{y} - e(\tau, y) \). Then \( \tilde{y}' \in \tilde{V}(\tau) \) and there are exactly two elements in \( \mathbb{Z}^d \) which is larger than \( \tilde{y}' \) and which is smaller than \( \tilde{y} \), namely \( \tilde{y} - e_j \) and \( \tilde{y} - e_{j'} \). We set \( y_1 = \pi(\tilde{y} - e_j) \) and \( y_2 = \pi(\tilde{y} - e_{j'}) \). For \( i = 1, 2 \), let \( \sigma_i \in A_{d-1} \) denote the unique element satisfying \( V(\sigma_i) = V(\tau) \cup \{ y_i \} \). It is easily checked that the set of the elements in \( A_{d-1} \) which has \( \tau \) as a face is equal to \( \{ \sigma_1, \sigma_2 \} \). Let \( \iota : V(\sigma_1) \cong V(\sigma_2) \) denote the bijection such that \( \iota(x) = x \) for any \( x \in V(\tau) \) and \( \iota(y_1) = y_2 \). Then the composite \( \iota \circ \hat{\sigma}_1 \) is equal to the composite \( \hat{\sigma}_2 \circ (jj') \), where \((jj')\) denotes the transposition of \( j \) and \( j' \). Since the signature of \((jj')\) is equal to \(-1\), it follows that the component in \( (\prod_{\nu \in O(\tau)} \mathbb{Z})\{\pm 1\} \) of the image of \( \beta \) under the boundary map \( (\prod_{\nu \in A'_{d-2}} \mathbb{Z})\{\pm 1\} \to (\prod_{\nu' \in A'_{d-1}} \mathbb{Z})\{\pm 1\} \) is equal to zero. This proves the claim. \( \square \)

**Definition 3.3.** We refer to the class in \( H^\text{BM}_{d-1}(A_\bullet, \mathbb{Z}) \) defined by the \((d-1)\)-cycle \( \beta \) as the fundamental class of the apartment.
Let \( d \geq 1 \) and \( F \) be a global field of positive characteristic. We give the definition of our main object of study, an arithmetic subgroup \( \Gamma \subset \text{GL}_d(F) \).

Fixing a place \( \infty \) of \( F \) and denoting by \( F_\infty \) the completion at \( \infty \) of \( F \), arithmetic subgroups act on the Bruhat-Tits building \( \mathcal{B} \Gamma \) of \( \text{PGL}_d(F_\infty) \).

We are interested in the \((d-1)\)-st Borel-Moore homology of the quotient \( \Gamma \backslash \mathcal{B} \Gamma \). This is an analogue of \( \Gamma \backslash \mathcal{H} \) where \( \mathcal{H} \) is the upper half space, \( \Gamma \) is a congruence subgroup of \( \text{SL}_d(\mathbb{Z}) \). The first cohomology group is known to be related to the space of elliptic modular forms.

In Section 4.4, we define modular symbols in the Borel-Moore homology group. Recall that a building is a union of subsimplicial complexes called apartments. They are indexed by bases of \( F_\infty^d \), but we restrict to those coming from bases of \( F^d \) (we say \( F \)-basis). A modular symbol is the image of the fundamental class of an apartment associated with an \( F \)-basis.

We state our main result in Section 4.5. It computes a bound on the index of the subgroup generated by modular symbols in the Borel-Moore homology of the quotient.

Recall that one property of an arithmetic subgroup \( \Gamma \) of \( \text{GL}_d(\mathbb{Q}) \) is that there exists a subgroup of finite index \( \Gamma' \subset \Gamma \) such that \( \Gamma \) is torsion free. In positive characteristic of characteristic \( p \), this is not expected to hold true. Instead, there always exist a \( p \)-torsion subgroup (i.e., the torsion subgroup is a \( p \)-group) of finite index. We compute a bound on the index in Section 4.2.

4.1. Arithmetic subgroups

We define arithmetic subgroups. The main examples of arithmetic subgroups are the congruence subgroups. We verify the basic properties (see (1)-(5) below) in Section 4.1.3. We then see that an arithmetic subgroup acts on the Bruhat-Tits building with finite stabilizer groups and the map from an apartment to a quotient is a locally finite map (so the pushforward map of Borel-Moore homology groups is defined).

4.1.1. Let us give the setup. (This is common when considering Drinfeld modules.) We let \( F \) denote a global field of positive characteristic \( p > 0 \). Let \( C \) be a proper smooth curve over a finite field whose function field is \( F \). Let \( \infty \) be a place of \( F \) and let \( K = F_\infty \) denote the local field at \( \infty \). We let \( A = H^0(C \setminus \{\infty\}, \mathcal{O}_C) \). Here we identified a closed point of \( C \) and a place of \( F \). We write \( \hat{A} = \lim_{\leftarrow} A/I \), where the limit is taken over the nonzero ideals of \( A \). We let \( A^\infty = \hat{A} \otimes_A F \) denote the ring of finite adeles.

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Definition 4.1. A subgroup $\Gamma \subset \text{GL}_d(K)$ is called arithmetic subgroup if there exists a compact open subgroup $K^\infty \subset \text{GL}_d(\hat{A}^\infty)$ such that $\Gamma = \text{GL}_d(F) \cap K^\infty \subset \text{GL}_d(K)$.

4.1.2. Congruence subgroups. Let $I \subset A$ be a nonzero ideal. Set $\Gamma_I = \ker[\text{GL}_d(A) \to \text{GL}_d(A/I)]$ where the map is the canonical map induced by the projection $A \to A/I$. Then $\Gamma_I$ is an arithmetic subgroup because we can take $K_\infty$ to be $\ker[\text{GL}_d(\hat{A}) \to \text{GL}_d(\hat{A}/I\hat{A})]$. These are also known as congruence subgroups.

4.1.3. Let $\Gamma$ be an arithmetic subgroup. Then $\Gamma \cap \text{SL}_d(F) = \Gamma \cap \text{SL}_d(K)$ is a subgroup of $\Gamma$ of finite index, and is an $S$-arithmetic subgroup of $\text{SL}_d$ over $F$ for $S = \{\infty\}$ in the paper of Harder $\text{[Har1]}$.

4.1.4. Let $\Gamma \subset \text{GL}_d(K)$ be a subgroup. We consider the following Conditions (1) to (5) on $\Gamma$.

(1) $\Gamma \subset \text{GL}_d(K)$ is a discrete subgroup,
(2) $\{\det(\gamma) | \gamma \in \Gamma\} \subset O_\infty^\times$ where $O_\infty$ is the ring of integers of $K$,
(3) $\Gamma \cap Z(\text{GL}_d(K))$ is finite.

Let $A_* = A_{v_1,\ldots,v_d}$ denote the apartment corresponding to a basis $v_1,\ldots,v_d \in K^{\oplus d}$ (defined in Section 3.2.2).

(4) For any apartment $A_* = A_{v_1,\ldots,v_d}$ with $v_1,\ldots,v_d \in F^{\oplus d}$, the composition $A_* \hookrightarrow B\mathcal{T}_* \to \Gamma \backslash B\mathcal{T}_*$ is quasi-finite, that is, the inverse image of any simplex by this map is a finite set.

(5) (Harder) The cohomology group $H^{d-1}(\Gamma, \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space.

We verify below that the conditions above are satisfied for any arithmetic subgroup $\Gamma$.

4.1.5. These properties are used in the following way. The condition (2) implies that each element in the stabilizer group of a simplex fixes the vertices of the simplex. Under the condition (1), the condition (3) implies that the stabilizer of a simplex is finite. This implies that the $\mathbb{Q}$-coefficient group homology of $\Gamma$ and the homology of $\Gamma \backslash B\mathcal{T}_*$ are isomorphic. The condition (4) will be used to define a class in Borel-Moore homology of $\Gamma \backslash B\mathcal{T}_*$ starting from an apartment (Section 4.4). We do not use Condition (5) in this form but we record it here because it is related to the finite dimensionality of the space of cusp forms of fixed level.

4.1.6. For the rest of this subsection, we give a proof that these Conditions hold true.

Proposition 3. For an arithmetic subgroup $\Gamma$, Conditions (1)-(5) of Section 4.1.4 hold.

Proof. This follows from Lemmas below. $\square$

Lemma 4. For an arithmetic subgroup $\Gamma$, Conditions (1), (2), and (3) hold.

Proof. Condition (1) holds trivially. We note that there exists an element $g \in \text{GL}_d(\hat{A}^\infty)$ such that $gK^\infty g^{-1} \subset \text{GL}_d(\hat{A})$. Since $\det(\gamma) \in F^\times \cap \hat{A}^\times \subset O_\infty^\times$ for $\gamma \in \Gamma$, (2) holds. Because $F^\times \cap \text{GL}_d(\hat{A})$ is finite, (3) holds. $\square$

Lemma 5. Let $\Gamma$ be an arithmetic subgroup. Then (4) holds.
Proof. We show that the inverse image of each simplex of $\Gamma \setminus B_\Gamma^+$ under the map in (4) is finite. For $0 \leq i \leq d - 1$, the set of $i$-dimensional simplices $B_\Gamma^+$ is identified (see Section 4.4.3 for the identification) with the coset $GL_d(K)/\overline{K}_\infty$ for an open subgroup $\overline{K}_\infty \subset GL_d(K)$ which contains $K\times \overline{K}_\infty$ as a subgroup of finite index for some compact open subgroup $\overline{K}_\infty \subset GL_d(K)$.

Let $T \subset GL_d$ denote the diagonal maximal torus. The set of simplices of $A_\bullet$ of fixed dimension is identified with the image of the map

$$\prod_{w \in S_d} gwT(K) \to GL_d(K)/\overline{K}_\infty$$

for some $g \in GL_d(F)$.

Since $S_d$ is a finite group, it then suffices to show that for any $w \in S_d$, the map

$$\text{Image}[gwT(K) \to GL_d(K)/\overline{K}_\infty] \to \Gamma \setminus GL_d(K)/\overline{K}_\infty$$

is quasi-finite. The inverse image under the last map of the image of $gwT(K)$ is isomorphic to the set

$$\{\gamma \in \Gamma | \gamma gwT \in gwT(K)\overline{K}_\infty\} = \Gamma \cap gwT(K)\overline{K}_\infty(gwT)^{-1} = \Gamma \cap (gwT(K)t\overline{K}_\infty t^{-1}(gw)^{-1}.
$$

Hence, if we let $g' = gw$ and $K'_\infty = t\overline{K}_\infty t^{-1}$, this set equals

$$\Gamma \cap g'(K)K'_\infty g'^{-1} = GL_d(F) \cap (K\times K'_\infty g'^{-1}) = g'(GL_d(F) \cap (g'^{-1}\overline{K}_\infty g'' \cap T(K)K'_\infty g'').$$

The finiteness of this set is proved in the following lemma.

Lemma 6. For any compact open subgroup $K \subset GL_d(k)$, the set $GL_d(F) \cap T(K)K$ is finite.

Proof. Let $U = T(K) \cap \overline{K}$. Then $T(O_\infty) \supset U$ and is of finite index. Note that there exist a non-zero ideal $I \subset A$ and an integer $N$ such that $K \subset I^{-1}\varpi^K_\infty N\text{Mat}_d(A) \times \text{Mat}_d(O_\infty)$ where $\varpi^K_\infty$ is a uniformizer in $O_\infty$.

Let $\alpha : T(K)/U \to T(K)/T(O_\infty) \cong \mathbb{Z}^d$ be the (quasi-finite) map induced by the inclusion $U \subset T(O_\infty)$. For $h \in T(K)$, we write $(h_1, \ldots, h_d) = \alpha(h)$. Then for $i = 1, \ldots, d$, the $i$-th row of $hK$ is contained in $(I^{-1}A \times \varpi^K_\infty N\varpi^K_\infty h_iO_\infty)^\oplus$. Hence, for sufficiently large $h_i$, the intersection $hK \cap GL_d(F)$ is empty. We then have, for sufficiently large $N'$,

$$(4.1) \quad GL_d(F) \cap T(K)K = \prod_{h \in T(K)/U, h_1, \ldots, h_d \leq N'} GL_d(F) \cap hK.$$

The adelic norm of the determinant of an element in $GL_d(F)$ is 1, while that of an element in $hK$ is $|\det h|_\infty = \sum_{i=1}^d h_i$. So (4.1) equals

$$\prod_{h \in T(K)/U, h_1, \leq N', \sum h_i = 0} GL_d(F) \cap hK.$$

The index set of the disjoint union above is finite since $\alpha$ is quasi-finite, and $GL_d(F) \cap hK$ is finite since $GL_d(F)$ is discrete and $hK$ is compact. The claim follows.

Lemma 7. Let $\Gamma$ be an arithmetic subgroup. Then (5) holds.

Proof. This follows from [Har1] p.136, Satz 2].
4.2. \(p^r\)-torsion free subgroups

For an arithmetic subgroup (or congruence subgroup) \(\Gamma\) of \(\text{SL}_d(\mathbb{Z})\), there always exists a torsion free subgroup \(\Gamma' \subset \Gamma\) of finite index. In our positive characteristic (of characteristic \(p\)) setup, an arithmetic subgroup always contains some \(p\)-torsion. We therefore define an arithmetic subgroup to be \(p^r\)-torsion free if any torsion element has a \(p\)-power order. A similar fact that any arithmetic subgroup contains a \(p^r\)-torsion free subgroup of finite index holds true.

4.2.1. Let \(p > 0\) denote the characteristic of \(F\).

**Definition 4.2.** We say that a subgroup \(\Gamma \subset \text{GL}_d(K)\) is \(p^r\)-torsion free if any element of \(\Gamma\) of finite order is of order a power of \(p\).

**Lemma 8.** Let \(\Gamma \subset \text{GL}_d(F)\) be an arithmetic subgroup and let \(v \neq \infty\) be a place of \(F\). Let \(F_v\) denote the completion of \(F\) at \(v\), and \(O_v \subset F_v\) its ring of integers. Suppose that there exists an element \(g_v \in \text{GL}_d(F_v)\) such that the image of \(\Gamma\) in \(\text{GL}_d(F_v)\) is contained in \(g_v(I_d + \varphi_v M_d(O_v))g_v^{-1}\), where \(I_d\) is the identity \(d\)-by-\(d\) matrix, \(\varphi_v\) is the maximal ideal of \(O_v\), and \(M_d(O_v)\) denotes the ring of \(d\)-by-\(d\) matrices with coefficients in \(O_v\). Then \(\Gamma\) is \(p^r\)-torsion free.

**Proof.** It suffices to show that any matrix \(h \in I_d + \varphi_v M_d(O_v)\) of finite order is of order a power of \(p\). Let us fix an algebraic closure \(\overline{F}_v\) of \(F_v\) and let \(\overline{O}_v\) denote its ring of integers. Then \(\overline{O}_v\) is a valuation ring. Let \(h \in I_d + \varphi_v M_d(O_v)\) be an element of finite order. Then any eigenvalue \(\alpha\) of \(h\) in \(\overline{F}_v\) belongs to \(\overline{O}_v\) and is congruent to 1 modulo the maximal ideal of \(\overline{O}_v\). Since \(h\) is of finite order, \(\alpha\) is a root of unity. This implies \(\alpha = 1\). Hence \((h - I_d)^N = 0\) for sufficiently large \(N\). Choose a power \(q\) of \(p\) satisfying \(q \geq N\). Then \(h^q - I_d = (h - I_d)^q = 0\). This shows that the order of \(h\) is a power of \(p\).

**Corollary 9.** Let \(\Gamma \subset \text{GL}_d(F)\) be an arithmetic subgroup and let \(v \neq \infty\) be a place of \(F\). Suppose that, as a subgroup of \(\text{GL}_d(F_v)\), the group \(\Gamma\) is contained in a pro-\(p\) open compact subgroup \(\mathbb{K}_v\) of \(\text{GL}_d(F_v)\). Then \(\Gamma\) is \(p^r\)-torsion free.

**Proof.** Let us choose an open subgroup \(\mathbb{K}'_v\) of \(\mathbb{K}_v \cap (I_d + \varphi_v M_d(O_v))\) such that \(\mathbb{K}'_v\) is a normal subgroup of \(\mathbb{K}_v\). Then by Lemma\(\Box\) \(\Gamma' = \Gamma \cap \mathbb{K}'_v\) is \(p^r\)-torsion free. Since \(\Gamma'\) is equal to the kernel of the composite \(\Gamma \hookrightarrow \mathbb{K}_v \to \mathbb{K}_v / \mathbb{K}'_v\) and \(\mathbb{K}_v / \mathbb{K}'_v\) is a finite \(p\)-group, \(\Gamma'\) a normal subgroup of \(\Gamma\) and the quotient \(\Gamma / \Gamma'\) is a finite \(p\)-group. It follows that \(\Gamma\) is \(p^r\)-torsion free.

**Corollary 10.** Let \(I \subset A\) be a nonzero ideal. Then \(\Gamma_I\) (see Section\(\Box\)) is \(p^r\)-torsion free.

**Proof.** Let \(v\) be a prime which divides \(I\). Then \(\Gamma\) is contained in the compact open subgroup \(I_d + \varphi_v M_d(O_v)\) where \(I_d\) is the identity matrix. Hence the claim follows from Corollary\(\Box\).

4.2.2. As the compact open subgroups

\[\ker[\text{GL}_d(\hat{A}) \to \text{GL}_d(\hat{A}/J\hat{A})]\]

as \(J\) runs over the ideals form a fundamental system of neighborhoods of the identity matrix \(I_d\), an arithmetic subgroup \(\Gamma\) contains some congruence subgroup \(\Gamma_I\) and it is of finite index in \(\Gamma\).
4.3. Arithmetic quotients of the Bruhat-Tits building

Let us define simplicial complex $\Gamma \backslash \mathcal{B}T_\bullet$ for an arithmetic subgroup $\Gamma$ and check that the canonical quotient map is well defined.

4.3.1. We need a lemma.

**Lemma 11.** Let $i \geq 0$ be an integer, let $\sigma \in \mathcal{B}T_i$ and let $v, v' \in V(\sigma)$ be two vertices with $v \neq v'$. Suppose that an element $g \in \text{GL}_d(K)$ satisfies $|\det g|_\infty = 1$. Then we have $gv = v'$.

**Proof.** Let $\bar{\sigma}$ be an element $(L_j)_{j \in \mathbb{Z}}$ in $\mathcal{B}T_i$ such that the class of $\bar{\sigma}$ in $\mathcal{B}T_i$ is equal to $\sigma$. There exist two integers $j, j' \in \mathbb{Z}$ such that $v, v'$ is the class of $L_j, L_{j'}$, respectively. Assume that $gv = v'$. Then there exists an integer $k \in \mathbb{Z}$ such that $L_jg^{-1} = \omega_k^L L_{j'} = L_{j' + (i + 1)k}$. Let us fix a Haar measure $d\mu$ of the $K$-vector space $V_\infty = K^{\oplus d}$. As is well-known, the push-forward of $d\mu$ with respect to the automorphism $V_\infty \to V_\infty$ given by the right multiplication by $\gamma$ is equal to $|\det \gamma|_\infty d\mu$. Since $|\det g|_\infty = 1$, it follows from the equality $L_jg^{-1} = L_{j' + (i + 1)k}$ that the two $O_\infty$-lattices $L_j$ and $L_{j' + (i + 1)k}$ have the same volume with respect to $d\mu$. Hence we have $j = j' + (i + 1)k$, which implies $L_j = \omega_k^L L_{j'}$. It follows that the class of $L_j$ in $\mathcal{B}T_0$ is equal to the class of $L_{j'}$, which contradicts the assumption $v \neq v'$.

4.3.2. quotients and the canonical maps. Let $\Gamma \subset \text{GL}_d(K)$ be an arithmetic subgroup. It follows from Lemma [11] (using Condition (2) of Section 4.1.4) that for each $i \geq 0$ and for each $\sigma \in \mathcal{B}T_i$, the image of $V(\sigma)$ under the surjection $\mathcal{B}T_0 \to \Gamma \backslash \mathcal{B}T_0$ is a subset of $\Gamma \backslash \mathcal{B}T_0$ with cardinality $i + 1$. We denote this subset by $V(\text{cl}(\sigma))$, since it is easily checked that it depends only on the class $\text{cl}(\sigma)$ of $\sigma$ in $\Gamma \backslash \mathcal{B}T_i$. Thus the collection $\Gamma \backslash \mathcal{B}T_\bullet = (\Gamma \backslash \mathcal{B}T_i)_{i \geq 0}$ has a canonical structure of a simplicial complex such that the collection of the canonical surjection $\mathcal{B}T_i \to \Gamma \backslash \mathcal{B}T_i$ is a map of simplicial complexes $\mathcal{B}T_\bullet \to \Gamma \backslash \mathcal{B}T_\bullet$.

4.4. Definition of modular symbols

We define modular symbols here.

4.4.1. Notice that apartments are defined for any basis of $F_\infty^{\oplus d}$. However, the ones of interest in number theory are those associated with the $F$-bases (i.e., a basis of $F_\infty^{\oplus d}$ regarded as a basis of $F_\infty^{\oplus d}$).

4.4.2. Let $v_1, \ldots, v_d$ be an $F$-basis (that is, a basis of $F_\infty^{\oplus d}$ regarded as a basis of $F_\infty^{\oplus d}$). We consider the composite

$$A_\bullet \xrightarrow{\beta_{v_1, \ldots, v_d}} \mathcal{B}T_\bullet \to \Gamma \backslash \mathcal{B}T_\bullet.$$ (4.2)

Condition (4) implies that the map (4.2) is a finite map of simplicial complexes in the sense of Section 2.5. It follows that the map (4.2) induces a homomorphism

$$H_{d-1}^\text{BM}(A_\bullet, \mathbb{Z}) \to H_{d-1}^\text{BM}(\Gamma \backslash \mathcal{B}T_\bullet, \mathbb{Z}).$$

We let $\beta_{v_1, \ldots, v_d} \in H_{d-1}^\text{BM}(\Gamma \backslash \mathcal{B}T_\bullet, \mathbb{Z})$ denote the image under this homomorphism of the element $\beta \in H_{d-1}^\text{BM}(A_\bullet, \mathbb{Z})$ introduced in Section 2.5. We call this the class of the apartment $A_{v_1, \ldots, v_d}$. 

**Definition 4.3.** We let $\text{MS}(\Gamma)_{\mathbb{Z}} \subset H_{d-1}^\text{BM}(\Gamma \backslash \mathcal{B}T_\bullet, \mathbb{Z})$ denote the submodule generated by the classes $\beta_{v_1, \ldots, v_d}$ as $v_1, \ldots, v_d$ runs over the set of ordered $F$-bases.
4.5. Statement of Main Theorem

We are ready to state our theorem. The proof begins in Chapter 11 and ends in Chapter 12.

**Theorem 12.** Let $\Gamma \subset \text{GL}_d(K)$ be an arithmetic subgroup. We write $\text{MS}(\Gamma)_\mathbb{Z} \subset H^{BM}_{d-1}(\Gamma \backslash BT_\bullet, \mathbb{Z})$ for the submodule generated by the classes of apartments associated to $F$-bases.

1. We have
   \[ H^{BM}_{d-1}(\Gamma \backslash BT_\bullet, \mathbb{Q}) = \text{MS}(\Gamma)_\mathbb{Z} \otimes \mathbb{Q} \]

2. Suppose that $\Gamma$ is $p'$-torsion free. Set
   \[ c(d) = \frac{(d-2)}{2} \left(1 + \frac{(d-1)(d-2)}{2}\right) \]
   Then
   \[ p^{c(d)} H^{BM}_{d-1}(\Gamma \backslash BT_\bullet, \mathbb{Z}) \subset \text{MS}(\Gamma)_\mathbb{Z}. \]

3. Let $v \neq \infty$ be a prime of $F$, and let $F_v$ denote the completion of $F$ at $v$. Let $\mathbb{K}_v$ be a pro-$p$ open compact subgroup of $\text{GL}_d(F_v)$. Let us consider the intersection $\Gamma' = \Gamma \cap \mathbb{K}_v$ in $\text{GL}_d(F_v)$. Then
   \[ p^{c(d)[\Gamma : \Gamma']} H^{BM}_{d-1}(\Gamma \backslash BT_\bullet, \mathbb{Z}) \subset \text{MS}(\Gamma)_\mathbb{Z}. \]

4. Let $v_0 \neq \infty$ be a prime of $F$ such that the cardinality $q_0$ of the residue field $\kappa(v_0)$ at $v_0$ is smallest among those at the primes $v \neq \infty$. Set
   \[ N(d) = \prod_{i=1}^{d}(q_0 - 1). \]
   Then
   \[ p^{c(d)} N(d) H^{BM}_{d-1}(\Gamma \backslash BT_\bullet, \mathbb{Z}) \subset \text{MS}(\Gamma)_\mathbb{Z}. \]

5. Suppose that $d = 2$. Then
   \[ H^{BM}_{1}(\Gamma \backslash BT_\bullet, \mathbb{Z}) = \text{MS}(\Gamma)_\mathbb{Z}. \]
CHAPTER 5

Automorphic Forms with Steinberg at infinity

Let $d \leq 1$ and $F$ be a global field. An automorphic form for $\text{GL}_d$ over $F$ is a $\mathbb{C}$-valued function on $\text{GL}_d(F) \backslash \text{GL}_d(A_F)$, where $A_F$ is the ring of adeles, satisfying some more conditions. Instead of studying all automorphic forms, we study certain subset consisting of those satisfying a certain condition at a fixed place $\infty$ of $F$. We call them automorphic forms with Steinberg at infinity.

In Section 5.1, we recall the basics and some results on automorphic forms from Laumon’s book [Lau2]. The reader is also referred to [Bo-Ja] and Cogdell’s lectures [Co-Ki-Mu].

In Section 5.2, we give the definition of the automorphic forms of interest to us. These are the automorphic forms that appear when studying the cohomology of Drinfeld modular varieties.

We can apply known results to determine (Proposition 15) the direct sum decomposition, as automorphic representation, of the space of cusp forms that are Steinberg at infinity. For the space of all automorphic forms that are Steinberg at infinity, we have Theorem 16. This does not seem to follow from previously known results. We remark that this space is not contained in the space of square integrable automorphic forms. The proof of this theorem is given in Chapter 7.

5.1. Automorphic forms with values in $\mathbb{C}$

Let $F$ be the global field in positive characteristic. We fix a place $\infty$ of $F$. We write $A$ for the ring of adeles of $F$.

5.1.1. The contents of this section is summarized in the following two diagrams. We define each object and explain the inclusions:

\[
\begin{align*}
C_{\mathbb{C}, \infty}^\infty &\subset C_{\mathbb{C}}^\infty &\subset & C_{\mathbb{C}} &\subset & \mathcal{A}_{\mathbb{C}, \infty}^\infty &\subset & \mathcal{A}_{\mathbb{C}}^\infty \\
\cup &\cup & & & & & & \\
C_{\mathbb{C}, \text{cusp}}^\infty &\subset & C_{\mathbb{C}}^\infty &\subset & \mathcal{A}_{\mathbb{C}, \text{cusp}}^\infty &\subset & \mathcal{A}_{\mathbb{C}}^\infty
\end{align*}
\]

5.1.2. We set

\[C_{\mathbb{C}} = \text{Hom}(\text{GL}_d(F) \backslash \text{GL}_d(A), \mathbb{C}).\]

This is a $\text{GL}_d(A)$-module, where an element $g$ acts as $(gf)(x) = f(xg)$. We set

\[C_{\mathbb{C}}^\infty = \bigcup_k C_{\mathbb{C}}^k\]

where $C_{\mathbb{C}}^k$ denotes the invariants, and $K \subset \text{GL}_d(A)$ runs over compact open subgroups. This is the space of smooth vectors. This is a $\text{GL}_d(A)$-submodule of $C_{\mathbb{C}}$. 27
Let $\chi_{C,\infty}$ denote the abelian group of smooth complex characters of $Z(F) \backslash Z(A)$ \cite[p.4]{Lau2}. For $\chi \in \chi_{C,\infty}$, we denote by
\[ C_{C,\infty}^\infty = C_{\chi}(\text{GL}_d(F) \backslash \text{GL}_d(A), \mathbb{C}) \subset \mathbb{C}_C^\infty \]
the $\mathbb{C}$-vector subspace of the functions $\varphi \in C_{C}^\infty$ such that
\[ \varphi(zm) = \chi(z)\varphi(m) \]
for $z \in Z(A), m \in \text{GL}_d(A)$ \cite[p.9, 9.1.9]{Lau2}.

5.1.3. We let $A_C \subset C_C^\infty$ denote the space of automorphic forms. We use the definitions from \cite[p.2, 9.1]{Lau2}. Then we set \cite[p.12, 9.1.14]{Lau2}
\[ A_{C,\chi} = A_C \cap C_{C,\chi}^\infty \]
for $\chi \in \chi_{C,\infty}$, we set
\[ A_{C,\chi} \subset A_{C,\chi} \cap A_{C,\chi} \]

5.1.4. Let $\chi \in \chi_C^\infty$ be a unitary character. We let $A_{C,\chi}^2 \subset A_{C,\chi}$ denote the $\mathbb{C}$-vector subspace of square-integrable automorphic forms \cite[p.24, 9.3]{Lau2}. Lemma 9.3.3 of \cite{Lau2} says
\[ A_{C,\chi} \subset A_{C,\chi}^2. \]

5.2. Automorphic forms with Steinberg at infinity

5.2.1. Let $\text{Std}_{d,C}$ denote the Steinberg representation. It is an admissible representation of $\text{GL}_d(F_\infty)$. Later we will also use the $\mathbb{Q}$-vector space version $\text{Std}_{d,Q}$.

5.2.1.1. Let $\mathcal{I} \subset \text{GL}_d(O_\infty)$ denote the Iwahori subgroup. It is known that the Iwahori fixed part $\text{Std}_{d,C}$ of the Steinberg representation is a one dimensional vector space. Take a nonzero element $v \in \text{Std}_{d,C}$. For a $\text{GL}_d(F_\infty)$-module $M$, we set $M_{\text{Std}}$ to be the image by the evaluation at $v$:
\[ M_{\text{Std}} = \text{Image}[\text{Hom}_{\text{GL}_d(F_\infty)}(\text{Std}_{d,C}, M) \to M] \subset M \]
By the one-dimensionality, $M_{\text{Std}}$ does not depend on the choice of $v$.

We have the following lemma.

**Lemma 13.** We have
(1) $C_{C,\text{Std}}^\infty = A_{C,\text{Std}},$
(2) $C_{C,\text{cusp},\text{Std}}^\infty = A_{C,\text{cusp},\text{Std}}.$

**Proof.** (Proof of (1)) The $\text{GL}_d(F_\infty)$-representation generated by a vector of $C_{C,\text{Std}}^\infty$ is the Steinberg representation. Since the Steinberg representation is an admissible representation, by definition of automorphic form (see, for example, \cite[Prop. 4.5, p.196]{Bo-Ja}), the vector is an automorphic form. $\square$
5.2.1.2. (See [Lau2] p.35) Let \( \chi \in \chi^\infty_C \) and let us assume that \( \chi \) is trivial on \( \mathbb{F}_\infty^\times \). Then \( \chi \) is automatically of finite order (since \( \mathbb{F}_\infty^\times \) is compact) therefore unitary. Let \( f \in \mathcal{A}_{\text{cusp,St}} \) and let \( \chi \) be its central quasi-character. Then \( \chi(F_\infty) = 1 \) because \( \mathbb{F}_\infty^\times \) acts trivially for the Steinberg representation. Therefore, as seen above, \( \chi \) is unitary. We write \( \mathcal{A}_{\text{cusp,St},\chi} \subset \mathcal{A}_{\text{cusp,St}} \) for the subspace consisting of those cusp forms whose unitary central character is \( \chi \).

5.2.1.3. Let \( \mathcal{A}_{\text{cusp,St},\chi} \subset \mathcal{A}_{\text{cusp,St}} \) denote the discrete spectrum (see [Lau2] 9.3.4, p.25). For a quasi-character \( \chi \in \chi^\infty_C \) such that \( \chi(F_\infty) = 1 \) (hence \( \chi \) is unitary), Corollary 9.5.6 of [Lau2] p.36 implies \( \mathcal{A}_{\text{cusp,St},\chi} = \mathcal{A}_{\text{cusp,St}} \).

5.2.1.4. Let \( \mathcal{A}_{\text{cusp,St}} \) denote the vector space of compactly supported functions. Then Harder’s theorem (see [Lau2] Theorem 9.2.6, p.16) implies that \( \mathcal{A}_{\text{cusp,St}} \subset \mathcal{A}_{\text{disc,St}} \). The elements of \( \mathcal{A}_{\text{disc,St}} \) are automorphic and square-integrable hence \( \mathcal{A}_{\text{disc,St}} \subset \mathcal{A}_{\text{disc}} \). Now, for a unitary character \( \chi \), let us write \( \mathcal{A}_{\text{cusp,St},\chi} = \mathcal{A}_{\text{cusp,St}} \cap \mathcal{A}_{\text{disc,St}} \).

**Proof.** (Proof of (2)) From the discussions above, we obtain

\[
\mathcal{A}_{\text{cusp,St},\chi} \subset \mathcal{A}_{\text{cusp,St}} \subset \mathcal{A}_{\text{disc,St}} = \mathcal{A}_{\text{cusp,St},\chi}.
\]

This proves (2). \( \square \)

5.3. Admissibility

The admissibility follows from a result of Harder.

**Proposition 14.** The vector spaces \( \mathcal{A}_{\text{cusp,St}} \) and \( \mathcal{A}_{\text{St}} \) are admissible representations of \( \text{GL}_d(A) \).

**Proof.** Because of the condition at \( \infty \), we can use the result of Harder [Bo-Ja] p.198, Proposition 5.2] to see that \( \mathcal{A}_{\text{St}} \) is admissible. Then the claim follows for the subspace \( \mathcal{A}_{\text{cusp,St}} \). \( \square \)

5.4. Decompositions

5.4.1. There are well-known theorems on the structure of the space of cusp forms, which in turn imply the following.

**Proposition 15.** As a representation of \( \text{GL}_d(A^\infty) \),

\[
\mathcal{A}_{\text{cusp,St}} \cong \bigoplus_{\pi} \pi^\infty,
\]

where \( \pi = \pi^\infty \otimes \pi_\infty \) runs over (the isomorphism classes of) the irreducible cuspidal automorphic representations of \( \text{GL}_d(A) \) such that \( \pi_\infty \) is isomorphic to the Steinberg representation of \( \text{GL}_d(F_\infty) \).

**Proof.** The theorems to use are [Lau2] Theorem 9.2.14, p.22] due to Gelfand and Piatetski-Shapiro, and the multiplicity one theorem [Lau2] Remark 9.2.15, p.22] due to Shalika. \( \square \)
5.4.2. We prove the following structure theorem for $A_{\mathcal{C,St}}$ in Chapter 7.

**Theorem 16.** Let $\pi = \pi^\infty \otimes \pi_\infty$ be an irreducible smooth representation of $GL_d(\mathbb{A})$ such that $\pi^\infty$ appears as a subquotient of $A_{\mathcal{C,St}}$. Then there exist an integer $r \geq 1$, a partition $d = d_1 + \cdots + d_r$ of $d$, and irreducible cuspidal automorphic representations $\pi_i$ of $GL_{d_i}(\mathbb{A})$ for $i = 1, \ldots, d$ which satisfy the following properties:

- For each $i$ with $0 \leq i \leq r$, the component $\pi_{i,\infty}$ at $\infty$ of $\pi_i$ is isomorphic to the Steinberg representation of $GL_{d_i}(F_\infty)$.
- Let us write $\pi_i = \pi_i^\infty \otimes \pi_i,\infty$. Let $P \subset GL_d$ denote the standard parabolic subgroup corresponding to the partition $d = d_1 + \cdots + d_r$. Then $\pi^\infty$ is isomorphic to a subquotient of the unnormalized parabolic induction $\text{Ind}_{P(\mathbb{A}^\infty)}^{GL_d(\mathbb{A}^\infty)} \pi_1^\infty \otimes \cdots \otimes \pi_r^\infty$.

Moreover for any subquotient $H$ of $A_{\mathcal{C,St}}$ which is of finite length as a representation of $GL_d(\mathbb{A}^\infty)$, the multiplicity of $\pi$ in $H$ is at most one.
CHAPTER 6

Double cosets for automorphic forms

We introduce here the main arithmetic object of study $X_{\mathbb{K},\bullet}$ for a compact open subgroup $\mathbb{K} \subset \text{GL}_d(\mathbb{A}^\infty)$. It is a (generalized) simplicial complex. This is an analogue of the $\mathbb{C}$-valued points of a Shimura variety as a double coset.

In Section 6.1.1 we define a (generalized) simplicial complex in the form of double coset. We show (Proposition 23) that (the limit of) the Borel-Moore homology is isomorphic to the space of our automorphic forms and the homology is isomorphic to the subspace of cusp forms. This gives a precise relation between the Borel-Moore homology/homology of the geometry and the space of automorphic forms. In particular, we see how our modular symbols are related to automorphic forms. Later, we study this geometry to prove Theorem 16.

6.1. Simplicial complexes for automorphic forms

We give the definition of some double coset $X_{\mathbb{K},\bullet}$ associated with a compact open subgroup $\mathbb{K}$.

6.1.1. For an open compact subgroup $\mathbb{K} \subset \text{GL}_d(\mathbb{A}^\infty)$, let $\tilde{X}_{\text{GL}_d,\mathbb{K},\bullet}$ denote the disjoint union $\tilde{X}_{\text{GL}_d,\mathbb{K},\bullet} = (\text{GL}_d(\mathbb{A}^\infty)/\mathbb{K}) \times \mathcal{B}T\bullet$ of copies of the Bruhat-Tits building $\mathcal{B}T\bullet$ indexed by $\text{GL}_d(\mathbb{A}^\infty)/\mathbb{K}$. We often omit the subscript $\text{GL}_d$ on $\tilde{X}_{\text{GL}_d,\mathbb{K},\bullet}$ when there is no fear of confusion. The group $\text{GL}_d(\mathbb{A}) = \text{GL}_d(\mathbb{A}^\infty) \times \text{GL}_d(F_\infty)$ acts on the simplicial complex $\tilde{X}_{\mathbb{K},\bullet}$ from the left. $X_{\mathbb{K},\bullet} = \text{GL}_d(F)\backslash\tilde{X}_{\mathbb{K},\bullet}$ is the subgroup $\text{GL}_d(F) \subset \text{GL}_d(\mathbb{A})$.

For $0 \leq i \leq d - 1$, we let $X_{\mathbb{K},\bullet,i} = X_{\text{GL}_d,\mathbb{K},\bullet,i}$ denote the quotient $X_{\mathbb{K},\bullet,i} = \text{GL}_d(F)\backslash\tilde{X}_{\mathbb{K},\bullet,i}$. We let $X_{\mathbb{K},\bullet} = \text{GL}_d(F)\backslash\tilde{X}_{\mathbb{K},\bullet} = \text{GL}_d(F)\backslash(\text{GL}_d(\mathbb{A}^\infty) \times \mathcal{B}T\bullet)/\mathbb{K}$.

6.1.2. The non-adelic description is as follows. We set $J_{\mathbb{K}} = \text{GL}_d(F)\backslash\text{GL}_d(\mathbb{A}^\infty)/\mathbb{K}$. For each $j \in J_{\mathbb{K}}$, we choose an element $g_j \in \text{GL}_d(\mathbb{A}^\infty)$ in the double coset $j$ and set $\Gamma_j = \text{GL}_d(F) \cap g_j\mathbb{K}g_j^{-1}$. Then the set $X_{\mathbb{K},i}$ is isomorphic to the disjoint union $\prod_{j \in J_{\mathbb{K}}} \Gamma_j\backslash\mathcal{B}T\bullet$. For each $j$, the group $\Gamma_j \subset \text{GL}_d(F)$ is an arithmetic subgroup as defined in Section 4.1.1. It follows that the tuple $X_{\mathbb{K},\bullet} = (X_{\mathbb{K},i})_{0 \leq i \leq d-1}$ forms a simplicial complex which is isomorphic to the disjoint union $\prod_{j \in J_{\mathbb{K}}} \Gamma_j\backslash\mathcal{B}T\bullet$.

6.1.3. Since the simplicial complex $\tilde{X}_{\text{GL}_d,\mathbb{K},\bullet}$ is locally finite, it follows that the simplicial complex $X_{\mathbb{K},\bullet}$ is locally finite. Hence, as in Section 2.3 for an abelian group $M$, we may consider the cohomology groups with compact support $H^*_c(X_{\mathbb{K},\bullet}, M)$ and the Borel-Moore homology groups $H^*_{\text{BM}}(X_{\mathbb{K},\bullet}, M)$. Of the simplicial complex $X_{\mathbb{K},\bullet}$. 
6.1.4. Since the simplicial complex $X_{K,\bullet}$ has no $i$-simplex for $i \geq d$ as was remarked in Section 3.1.3.3 it follows that the $(d-1)$-st homology and Borel-Moore homology are isomorphic to the group of chains. This implies that the map

$$H_{d-1}(X_{K,\bullet}, M) \to H_{d-1}^{BM}(X_{K,\bullet}, M)$$

is injective for any abelian group $M$. We regard $H_{d-1}(X_{K,\bullet}, M)$ as a subgroup of $H_{d-1}^{BM}(X_{K,\bullet}, M)$ via this map.

6.2. Pull-back maps for homology groups

We study some functoriality for two open compact subgroups; then we are able to define the inductive limit of the Borel-Moore homology/homology groups.

6.2.1. Let $\mathbb{K}, \mathbb{K}' \subset GL_d(\mathbb{A}^\infty)$ be open compact subgroups with $\mathbb{K}' \subset \mathbb{K}$. We denote by $f_{K,K'}$ the natural projection map $X_{K',i} \to X_{K,i}$. Since $\mathbb{K}'$ is a subgroup of $\mathbb{K}$ of finite index, it follows that for any $i$ with $0 \leq i \leq d-1$ and for any $i$-simplex $\sigma \in X_{K,i}$, the inverse image of $\sigma$ under the map $f_{K,K'}$ is a finite set. Let $i$ be an integer with $0 \leq i \leq d-1$ and let $\sigma' \in X_{K',i}$. Let $\sigma$ denote the image of $\sigma'$ under the map $f_{K,K'}$. Let us choose an $i$-simplex $\tilde{\sigma}'$ of $\tilde{X}_{K',\bullet}$ which is sent to $\sigma'$ under the projection map $\tilde{X}_{K',\bullet} \to X_{K',\bullet}$. Let $\tilde{\sigma}$ denote the image of $\tilde{\sigma}'$ under the map $\tilde{X}_{K',i} \to \tilde{X}_{K,i}$. We let

$$\Gamma_{\tilde{\sigma}} = \{ \gamma \in GL_d(F) \mid \gamma \tilde{\sigma}' = \tilde{\sigma} \}$$

and

$$\Gamma_{\tilde{\sigma}} = \{ \gamma \in GL_d(F) \mid \gamma \tilde{\sigma} = \tilde{\sigma} \}$$

denote the stabilizer group of $\tilde{\sigma}'$ and $\tilde{\sigma}$, respectively.

Lemma 17. Let the notation be as above.

1. The group $\Gamma_{\tilde{\sigma}}$ is a finite group and the group $\Gamma_{\tilde{\sigma}'}$ is a subgroup of $\Gamma_{\tilde{\sigma}}$.

2. The isomorphism class of the group $\Gamma_{\tilde{\sigma}'}$ (resp. $\Gamma_{\tilde{\sigma}}$) depends only on $\sigma'$ (resp. $\sigma$) and does not depends on the choice of $\tilde{\sigma}'$.

Proof. For the case of a 0-dimensional simplex of (1), see $\text{[GR]}$ Proof of Theorem 0.8]. In the stabilizer group for a higher dimensional simplex, the subgroup of elements that fix each vertex is of finite index. This proves (1). The claim (2) can be checked easily.

Definition 6.1. The lemma above shows in particular that the index $[\Gamma_{\tilde{\sigma}} : \Gamma_{\tilde{\sigma}'}]$ is finite and depends only on $\sigma'$ and $f_{K,K'}$. We denote this index by $e_{K,K'}(\sigma')$ and call it the ramification index of $f_{K,K'}$ at $\sigma'$.

6.2.2. Let $M$ be an abelian group. Let $i$ be an integer with $0 \leq i \leq d$. We set $X'_{K,i} = \prod_{\sigma \in X_{K,i}} O(\sigma)$. The map $f_{K,K'} : X_{K',\bullet} \to X_{K,\bullet}$ induces a map $X'_{K,i} \to X'_{K,i}$ which we denote also by $f_{K,K'}$.

6.2.2.1. Let $m = (m_{\nu}^{i})_{\nu \in X_{K,i}'}$ be an element of the $\{\pm 1\}$-module $\prod_{\nu \in X_{K,i}'} M$. We define the element $f_{K,K'}(m)$ in $\prod_{\nu \in X_{K,i}'} M$ to be

$$f_{K,K'}(m) = (m_{\nu}^{i})_{\nu \in X_{K,i}'}$$

where for $\nu' \in O(\sigma') \subset X'_{K,i}$, the element $m'_{\nu'} \in M$ is given by $m'_{\nu'} = e_{K,K'}(\sigma') m f_{K,K'}(\nu')$.

The following lemma can be checked easily.

Lemma 18. Let the notation be as above.
6.4. The Steinberg representation and harmonic cochains

(1) The map \( f_{k^i,k}^*: \prod_{\nu \in X_{k,i}^I} M \rightarrow \prod_{\nu' \in X_{k,i}^I} M \) is a homomorphism of \( \{\pm 1\}\)-modules.

(2) The map \( f_{k^i,k}^*: \prod_{\nu \in X_{k,i}^I} M \rightarrow \prod_{\nu' \in X_{k,i}^I} M \) sends an element in the subgroup \( \bigoplus_{\nu \in X_{k,i}^I} M \subset \prod_{\nu \in X_{k,i}^I} M \) to an element in \( \bigoplus_{\nu' \in X_{k,i}^I} M \).

(3) For \( 1 \leq i \leq d - 1 \), the diagrams

\[
\begin{array}{ccc}
\prod_{\nu \in X_{k,i}^I} M & \xrightarrow{\delta_i \cdot \Pi} & \prod_{\nu' \in X_{k,i}^I} M \\
\downarrow f_{k^i,k}^* & & \downarrow f_{k^i,k}^* \\
\prod_{\nu' \in X_{k,i}^I} M & \xrightarrow{\delta_i \cdot \Pi} & \prod_{\nu' \in X_{k,i}^I} M \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\bigoplus_{\nu \in X_{k,i}^I} M & \xrightarrow{\delta_i \cdot \Pi} & \bigoplus_{\nu' \in X_{k,i}^I} M \\
\downarrow f_{k^i,k}^* & & \downarrow f_{k^i,k}^* \\
\bigoplus_{\nu' \in X_{k,i}^I} M & \xrightarrow{\delta_i \cdot \Pi} & \bigoplus_{\nu' \in X_{k,i}^I} M \\
\end{array}
\]

are commutative.

6.3. Limits

6.3.1. For an abelian group \( M \), we set

\[
H_\ast(X_{\lim,\bullet}, M) = H_\ast(X_{\text{GL}_d,\lim,\bullet}, M) = \lim_{K} H_\ast(X_{K,\bullet}, M)
\]

and

\[
H_{BM}^\ast(X_{\lim,\bullet}, M) = H_{BM}^\ast(X_{\text{GL}_d,\lim,\bullet}, M) = \lim_{K} H_{BM}^\ast(X_{K,\bullet}, M).
\]

Here the transition maps in the inductive limits are given by \( f_{k^i,k}^* \).

6.3.2. For \( g \in \text{GL}_d(A^\infty) \), we let \( \tilde{\xi}_g : X_{K,\bullet} \xrightarrow{\sim} \tilde{X}_{g^{-1}Kg,\bullet} \) denote the isomorphism of simplicial complexes induced by the isomorphism \( \text{GL}_d(A^\infty)/K \xrightarrow{\sim} \text{GL}_d(A^\infty)/g^{-1}Kg \) that sends a coset \( hK \) to the coset \( hg^{-1}Kg \) and by the identity on \( BT_{\bullet} \). The isomorphism \( \tilde{\xi}_g \) induces an isomorphism \( \xi_g : X_{K,\bullet} \xrightarrow{\sim} X_{g^{-1}Kg,\bullet} \) of simplicial complexes. For two elements \( g, g' \in \text{GL}_d(A^\infty) \), we have \( \xi_{gg'} = \xi_{g'} \circ \xi_g \).

6.3.3. The isomorphisms \( \xi_g \) for \( g \in \text{GL}_d(A^\infty) \) give rise to a smooth action (i.e., the stabilizer of each vector is a compact open subgroup) of the group \( \text{GL}_d(A^\infty) \) on these inductive limits. If \( M \) is a torsion free abelian group, then for each compact open subgroup \( K \subset \text{GL}_d(A^\infty) \), the homomorphism \( H_\ast(X_{K,\bullet}, M) \rightarrow H_\ast(X_{\lim,\bullet}, M) \) is injective and its image is equal to the \( K \)-invariant part \( H_\ast(X_{\lim,\bullet}, M)^K \) of \( H_\ast(X_{\lim,\bullet}, M) \). Similar statement holds for \( H_{BM}^\ast \).

6.4. the Steinberg representation and harmonic cochains

In this section, we consider coefficients of \( \mathbb{Q} \)-vector spaces. We show how the limit of the Borel-Moore homology/homology corresponds to a sub \( \text{GL}_d(A^\infty) \) representation of the space of automorphic forms.
6.4.1. Let $\text{St}_{d,\mathbb{C}}$ denote the Steinberg representation as defined, for example, in [Lau] p.193. It is defined with coefficients in $\mathbb{C}$, but it can also be defined with coefficients in $\mathbb{Q}$ in a similar manner. We let $\text{St}_{d,\mathbb{Q}}$ denote the corresponding representation.

6.4.2.

**Definition 6.2.** A harmonic cochain with values in a $\mathbb{Q}$-vector space $M$ is defined as an element of $\text{Hom}(H^{-1}_c(\mathcal{B}^\bullet, \mathbb{Q}), M)$.

**Lemma 19.** For a $\mathbb{Q}$-vector space $M$, there is a canonical, $\text{GL}_d(F_\infty)$-equivariant isomorphism between the module of $M$-valued harmonic $(d-1)$-cochains and the module $\text{Hom}_\mathbb{Q}(\text{St}_{d,\mathbb{Q}}, M)$.

**Proof.** It is shown in [Bo] 6.2,6.4 that $\text{St}_{d,\mathbb{C}}$ is canonically isomorphic to $H^{-1}_c(\mathcal{B}^\bullet, \mathbb{C})$ as a representation of $\text{GL}_d(F_\infty)$. One can check that this map is defined over $\mathbb{Q}$. This proves the claim. \hfill \Box

6.4.3. We let $\mathcal{B}^\bullet_j,*$ denote the quotient $\mathcal{B}^\bullet_j/F_\infty$. This set is identified with the set of pairs $(\sigma, v)$ with $\sigma \in \mathcal{B}^\bullet_j$, $v \in \mathcal{B}^\bullet_0$ a vertex of $\sigma$, which we call a pointed j-simplex. Here the element $(L_i)_{i \in \mathbb{Z}}$ mod $K^\times$ of $\mathcal{B}^\bullet_j/K^\times$ corresponds to the pair $((L_i)_{i \in \mathbb{Z}}, L_0)$ via this identification.

6.4.4. We identify the set $\tilde{\mathcal{B}}^\bullet_0$ with the coset $\text{GL}_d(K)/\text{GL}_d(\mathcal{O})$ by associating to an element $g \in \text{GL}_d(K)/\text{GL}_d(\mathcal{O})$ the lattice $\mathcal{O}_V g^{-1}$. Let $I = \{(a_{ij}) \in \text{GL}_d(\mathcal{O}) \mid a_{ij} \text{ mod } \mathcal{O} = 0 \text{ if } i > j\}$ be the Iwahori subgroup. Similarly, we identify the set $\tilde{\mathcal{B}}^\bullet_{d-1}$ with the coset $\text{GL}_d(K)/I$ by associating to an element $g \in \text{GL}_d(K)/I$ the chain of lattices $(L_i)_{i \in \mathbb{Z}}$ characterized by $L_i = \mathcal{O}_V \Pi_i g^{-1}$ for $i = 0,\ldots,d$. Here, for $i = 0,\ldots,d$, we let $\Pi_i$ denote the diagonal $d \times d$ matrix $\Pi_i = \text{diag}(\varpi, \ldots, \varpi, 1, \ldots, 1)$ with $\varpi$ appearing $i$ times and $1$ appearing $d - i$ times.

6.4.5. Let $\mathbb{K} \subset \text{GL}_d(\mathbb{A})$ be an open compact subgroup. Let $M$ be a $\mathbb{Q}$-vector space. Let $C^\mathbb{K}(M)$ denote the $(\mathbb{Q}$-vector space of locally constant $M$-valued functions on $\text{GL}_d(F)\backslash \text{GL}_d(\mathbb{A})/(\mathbb{K} \times F_\infty^\times)$. Let $C^\mathbb{K}(M) \subset C^\mathbb{K}(M)$ denote the subspace of compactly supported functions.

**Lemma 20.**

1. There is a canonical isomorphism 
   \[ H^{-1}_{d-1}(X_{\mathbb{K}^\bullet}, M) \cong \text{Hom}_{\text{GL}_d(F_\infty)}(\text{St}_{d,\mathbb{Q}}, C^\mathbb{K}(M)), \]
   where $C^\mathbb{K}(M)$ denotes the space of locally constant $M$-valued functions on $\text{GL}_d(F)\backslash \text{GL}_d(\mathbb{A})/(\mathbb{K} \times F_\infty^\times)$.

2. Let $v \in \text{St}_{d,\mathbb{Q}}$ be a non-zero Iwahori-spherical vector. Then the image of the evaluation map 
   \[ \text{Hom}_{\text{GL}_d(F_\infty)}(\text{St}_{d,\mathbb{Q}}, C^\mathbb{K}(M)) \]
   \[ \to \text{Map}(\text{GL}_d(F)\backslash \text{GL}_d(\mathbb{A})/(\mathbb{K} \times F_\infty^\times), M) \]
   at $v$ is identified with the image of the map 
   \[ H^{-1}_{d-1}(X_{\mathbb{K}^\bullet}, M) \to \text{Map}(\text{GL}_d(F)\backslash (\text{GL}_d(\mathbb{A}_\infty)/(\mathbb{K} \times \mathcal{B}^\bullet_{d-1})), M) \]
   \[ \cong \text{Map}(\text{GL}_d(F)\backslash \text{GL}_d(\mathbb{A})/(\mathbb{K} \times F_\infty^\times), M). \]
Proof. For a $\mathbb{C}$-vector space $M$, (1) is proved in [Ko-Ya2] Section 5.2.3, and (2) is [Ko-Ya2] Corollary 5.7. The proofs and the argument in loc. cit. work for a $\mathbb{Q}$-vector space $M$ as well.

Corollary 21. Under the isomorphism in (1), the subspace $H_{d-1}(X_{K,\bullet}, M) \subset H^B_{d-1}(X_{K,\bullet}, M)$ corresponds to the subspace $\text{Hom}_{\text{GL}_d(F)}(\text{St}_d, \mathbb{Q}, C^K_c(M)) \subset \text{Hom}_{\text{GL}_d(F)}(\text{St}_d, \mathbb{Q}, C^K_c(M))$.

Proof. This follows from Lemma 20 (2) and the definition of the homology group $H_{d-1}(X_{K,\bullet}, M)$.

6.4.6. The proof of the following lemma is straightforward and is left to the reader.

Lemma 22. Let the notation be as above.

(1) Suppose that $K'$ is a normal subgroup of $K$. Then the homomorphism $f_{K',K}^*$ induces an isomorphism $H^B_{d-1}(X_{K,\bullet}, M) \cong H^B_{d-1}(X_{K',\bullet}, M)^{K/K'}$ and a similar statement holds for $H_d$.

(2) Let $M$ be a $\mathbb{Q}$-vector space. Then the diagrams

$$
\begin{array}{ccc}
H^B_{d-1}(X_{K,\bullet}, M) & \cong & \text{Hom}_{\text{GL}_d(F)}(\text{St}_d, \mathbb{Q}, C^K_c(M)) \\
\downarrow f_{K',K}^* & & \downarrow \\
H^B_{d-1}(X_{K',\bullet}, M) & \cong & \text{Hom}_{\text{GL}_d(F)}(\text{St}_d, \mathbb{Q}, C^{K'}_c(M))
\end{array}
$$

and

$$
\begin{array}{ccc}
H_{d-1}(X_{K,\bullet}, M) & \cong & \text{Hom}_{\text{GL}_d(F)}(\text{St}_d, \mathbb{Q}, C^K_c(M)) \\
\downarrow f_{K',K}^* & & \downarrow \\
H_{d-1}(X_{K',\bullet}, M) & \cong & \text{Hom}_{\text{GL}_d(F)}(\text{St}_d, \mathbb{Q}, C^{K'}_c(M))
\end{array}
$$

are commutative. Here the horizontal arrows are the isomorphisms given in Lemma 20 and Corollary 21 and the right vertical arrows are the map induced by the quotient map $\text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A})/(K \times F^\infty) \to \text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A})/(K \times F^\infty)$.

Corollary 23. We have isomorphisms

$$
H_{d-1}(X_{\text{lim}, \mathbb{C}}) \cong \mathcal{A}_{\mathbb{C}, \text{cusp, St}}$$

$$
H^B_{d-1}(X_{\text{lim}, \mathbb{C}}) \cong \mathcal{A}_{\mathbb{C}, \text{St}}
$$

Proof. This follows from the previous lemma and the definitions.
CHAPTER 7

Proof of Theorem [16]

We give a proof of Theorem [16]

In Section 7.1 we introduce some terminology for locally free $O_C$-modules of rank $d$ and then describe the sets of simplices of $X_{K,•}$ in terms of chains of locally free $O_C$-modules of rank $d$. In Section 7.2 we follow Section 4 of [Gr] and define certain subsimplicial complexes of $X_{K,•}$.

7.1. Chains of locally free $O_C$-modules

7.1.1. Let $\eta : \text{Spec } F \to C$ denote the generic point of $C$.

**Definition 7.1.** For each $g \in \text{GL}_d(\hat{A}^\infty)$ and an $O_\infty$-lattice $L_\infty \subset O_\infty^{\oplus d}$, we denote by $\mathcal{F}[g, L_\infty]$ the $O_C$-submodule of $\eta_* F^{\oplus d}$ characterized by the following properties:

- $\mathcal{F}[g, L_\infty]$ is a locally free $O_C$-module of rank $d$.
- $\Gamma(\text{Spec } A, \mathcal{F}[g, L_\infty])$ is equal to the $A$-submodule $\tilde{A}^{\oplus d}g^{-1} \cap F^{\oplus d}$ of $F^{\oplus d}$.
- Let $i_\infty$ denote the morphism $\text{Spec } O_\infty \to C$. Then $\Gamma(\text{Spec } O_\infty, i_\infty^* \mathcal{F}[g, L_\infty])$ is equal to the $O_\infty$-submodule $L_\infty$ of $F_\infty^{\oplus d} = \Gamma(\text{Spec } O_\infty, i_\infty^* \eta_* F^{\oplus d})$.

7.1.2. Let $\mathcal{F}$ be a locally free $O_C$-modules of rank $d$. Let $I \subset A$ be a non-zero ideal. We regard the $A$-module $A/I$ as a coherent $O_C$-module of finite length.

**Definition 7.2.** A level $I$-structure on $\mathcal{F}$ is a surjective homomorphism $\mathcal{F} \to (A/I)^{\oplus d}$ of $O_C$-modules.

Let $K_\infty^\infty \subset \text{GL}_d(\hat{A})$ be the kernel of the homomorphism $\text{GL}_d(\hat{A}) \to \text{GL}_d(\hat{A}/\hat{I})$. The group $\text{GL}_d(A/I) \cong \text{GL}_d(\hat{A})/K_\infty^\infty$ acts from the left on the set of level $I$-structures on $\mathcal{F}$, via its left action on $(A/I)^{\oplus d}$. (We regard $(A/I)^{\oplus d}$ as an $A$-module of row vectors. The left action of $\text{GL}_d(A/I)$ on $(A/I)^{\oplus d}$ is described as $g \cdot b = bg^{-1}$ for $g \in \text{GL}_d(A/I)$, $b \in (A/I)^{\oplus d}$.)

**Definition 7.3.** For a subgroup $K \subset \text{GL}_d(\hat{A})$ containing $K_\infty^\infty$, a level $K$-structure on $\mathcal{F}$ is a $K/K_\infty^\infty$-orbit of level $I$-structures on $\mathcal{F}$.

For an open subgroup $K \subset \text{GL}_d(\hat{A})$, the set of level $K$-structures on $\mathcal{F}$ does not depend, up to canonical isomorphisms, on the choice of an ideal $I$ with $K_\infty^\infty \subset K$.

7.1.3. Let $K \subset \text{GL}_d(\hat{A})$ be an open subgroup. Let $(g, \sigma)$ be an $i$-simplex of $\tilde{X}_{K,•}$. Take a chain

$$\cdots \supseteq L_{-1} \supseteq L_0 \supseteq L_1 \supseteq \cdots$$

of $O_\infty$-lattices of $F_\infty^{\oplus d}$ which represents $\sigma$. To $(g, \sigma)$ we associate the chain

$$\cdots \supseteq \mathcal{F}[g, L_{-1}] \supseteq \mathcal{F}[g, L_0] \supseteq \mathcal{F}[g, L_1] \supseteq \cdots$$
of $O_C$-submodules of $\eta_*F^{\oplus d}$. Then the set of $i$-simplices in $\tilde{X}_{\mathbf{K}, \bullet}$ is identified with the set of the equivalence classes of chains

$$
\cdots \not\cong F_{-1} \not\cong F_0 \not\cong F_1 \not\cong \cdots
$$

of locally free $O_C$-submodules of rank $d$ of $\eta_*\eta^*O_C^{\oplus d}$ with a level $\mathbf{K}$-structure such that $F_{j-i-1}$ equals the twist $F_j(\infty)$ as an $O_C$-submodule of $\eta_*F^{\oplus d}$ with a level $\mathbf{K}$-structure for every $j \in \mathbb{Z}$.

Two chains $\cdots \not\cong F_{-1} \not\cong F_0 \not\cong F_1 \not\cong \cdots$ and $\cdots \not\cong F_{-1} \not\cong F_0 \not\cong F_1 \not\cong \cdots$ are equivalent if and only if there exists an integer $l$ such that $F_j = F'_{j+l}$ as an $O_C$-submodule of $\eta_*F^{\oplus d}$ with a level structure for every $j \in \mathbb{Z}$.

7.1.4. Let $g \in GL_d(\mathbb{A}^\infty)$ and let $L_\infty$ be an $O_\infty$-lattice of $F^{\oplus d}$. For $\gamma \in GL_d(F)$, the two $O_C$-submodules $F[g, L_\infty]$ and $F[\gamma g, L_\infty]$ are isomorphic as $O_C$-modules. The set of $i$-simplices in $X_{\mathbf{K}, \bullet}$ is identified with the set of the equivalence classes of chains $\cdots \not\cong F_1 \not\cong F_0 \not\cong F_{-1} \not\cong \cdots$ of injective non-isomorphisms of locally free $O_C$-modules of rank $d$ with a level $\mathbf{K}$-structure such that the image of $F_{j+i+1} \to F_j$ equals the image of the canonical injection $F_j(\infty) \to F_j$ for every $j \in \mathbb{Z}$.

Two chains $\cdots \not\cong F_1 \not\cong F_0 \not\cong F_{-1} \not\cong \cdots$ and $\cdots \not\cong F_1 \not\cong F_0 \not\cong F_{-1} \not\cong \cdots$ are equivalent if and only if there exists an integer $l$ and an isomorphism $F_j \cong F'_{j+l}$ of $O_C$-modules with level structures for every $j \in \mathbb{Z}$ such that the diagram

$$
\begin{array}{c}
\cdots \not\cong F_1 \not\cong F_0 \not\cong F_{-1} \not\cong \cdots \\
\cong \quad \cong \\
\cdots \not\cong F'_{l+1} \not\cong F'_l \not\cong F'_{l-1} \not\cong \cdots
\end{array}
$$

is commutative.

7.1.5. the Harder-Narasimhan polygons. We use functions $\Delta_{pF}$ from the theory of Harder-Narasimhan polygons. Let us recall the properties we use below.

Let $F$ be a locally free $O_C$-module of rank $r$. For an $O_C$-submodule $F' \subset F$ (note that $F'$ is automatically locally free), we set $z_F(F') = (\text{rank}(F'), \deg(F')) \in \mathbb{Q}^2$. It is known that there exists a unique convex, piecewise affine, affine on $[i-1, i]$ for $i = 1, \ldots, r$, continuous function $p_F : [0, r] \to \mathbb{R}$ on the interval $[0, r]$ such that the convex hull of the set $\{z_F(F') \mid F' \subset F\}$ in $\mathbb{R}^2$ equals $\{(x, y) \mid 0 \leq x \leq r, y \leq p_F(x)\}$. We define the function $\Delta p_F : \{1, \ldots, r-1\} \to \mathbb{R}$ as $\Delta p_F(i) = 2p_F(i) - p_F(i-1) - p_F(i+1)$. Then $\Delta p_F(i) \geq 0$ for all $i$. We note that for an invertible $O_C$-module $L$, $\Delta p_{F \otimes L}$ equals $\Delta p_F$. The theory of Harder-Narasimhan filtration \cite{Har-Na} implies that, if $i \in \text{Supp}(\Delta p_F) = \{i \mid \Delta p_F(i) > 0\}$, then there exists a unique $O_C$-submodule $F' \subset F$ satisfying $z_F(F') = (i, p_F(i))$. We denote this $O_C$-submodule $F'$ by $F_{(i)}$. The submodule $F_{(i)}$ has the following properties.

- If $i, j \in \text{Supp}(\Delta p_F)$ with $i \leq j$, then $F_{(i)} \subset F_{(j)}$ and $F_{(j)}/F_{(i)}$ is locally free.
- If $i \in \text{Supp}(\Delta p_F)$, then $p_{F_{(i)}(x)} = p_F(x)$ for $x \in [0, i]$ and $p_{F/F_{(i)}}(x-i) = p_F(x) - \deg(F_{(i)})$ for $x \in [i, r]$.

**Lemma 24.** Let $F$ be a locally free $O_C$-module of finite rank, and let $F' \subset F$ be an $O_C$-submodule of the same rank. Then we have $0 \leq p_F(i) - p_{F'}(i) \leq \deg(F) - \deg(F')$ for $i = 1, \ldots, \text{rank}(F) - 1$. 
We have

\[ \Delta \]

Lemma 25. Let \( F \) be a locally free \( \mathcal{O}_C \)-module of rank \( d \). Let \( F' \subset F \) be an \( \mathcal{O}_C \)-submodule of the same rank. Suppose that \( \Delta_{p_F}(i) > \deg(F) - \deg(F') \). Then we have \( F_{(i)} = F_{(i)} \cap F' \).

**Proof.** Immediate from the definition of \( p_F \). \( \square \)

Let \( X \) be a subsimplicial complex of the building but here we also consider subsimplicial complexes of the quotients. On the other hand, Lemma 24 shows that \( \deg(\mathcal{F}) \) of flags in \( \mathcal{F} \) is a submodule of the same rank. Suppose that \( \Delta_D(p) \neq 0 \). \( \Delta \)

It suffices to prove that \( \mathcal{F}_{(i)} \subset \mathcal{F}(i) \). Assume otherwise. Let us consider the short exact sequence

\[ 0 \to \mathcal{F}_{(i)} \cap \mathcal{F}(i) \to \mathcal{F}_{(i)} \to \mathcal{F}_{(i)} / (\mathcal{F}_{(i)} \cap \mathcal{F}(i)) \to 0 \]

Let \( r \) denote the rank of \( \mathcal{F}_{(i)} \cap \mathcal{F}(i) \). By assumption, \( r \) is strictly smaller than \( i \). Hence

\[ \deg(\mathcal{F}_{(i)}) = \deg(\mathcal{F}_{(i)} \cap \mathcal{F}(i)) + \deg((\mathcal{F}_{(i)} / (\mathcal{F}_{(i)} \cap \mathcal{F}(i))) \]

\[ \leq p_F(r) + p_F(\mathcal{F}(i)) \]

\[ \leq p_F(i) - (i - r)(p_F(i) - p_F(i - 1)) + (i - r)(p_F(i + 1) - p_F(i)) \]

\[ = \deg(\mathcal{F}(i)) - (i - r)\Delta_{p_F}(i) \]

\[ < \deg(\mathcal{F}(i)) - (\deg(F) - \deg(F')). \]

On the other hand, Lemma 24 shows that \( \deg(\mathcal{F}(i) \cap \mathcal{F}') \geq \deg(\mathcal{F}(i)) - (\deg(F) - \deg(F')) \). This is a contradiction. \( \square \)

### 7.2. SOME SUBSIMPLICIAL COMPLEXES

Using the functions \( \Delta_{p_F} \) defined above, we introduce some subsimplicial complexes of \( X_{\mathbb{K}_\bullet} \) and of \( \widetilde{X}_{\mathbb{K}_\bullet} \). In [Gr], Grayson considered subsimplicial complexes of the building but here we also consider subsimplicial complexes of the quotients. We introduce three spaces.

**7.2.1.** Given a subset \( D \subset \{1, \ldots, d - 1\} \) and a real number \( \alpha > 0 \), we define the simplicial subcomplex \( X_{\mathbb{K}_\bullet}(\alpha, D) \) of \( X_{\mathbb{K}_\bullet} \) as follows: A simplex of \( X_{\mathbb{K}_\bullet} \) belongs to \( X_{\mathbb{K}_\bullet}(\alpha, D) \) if and only if each of its vertices is represented by a locally free \( \mathcal{O}_C \)-module \( F \) of rank \( d \) with a level \( \mathbb{K} \)-structure such that \( \Delta_{p_F}(i) \geq \alpha \) holds for every \( i \in D \).

Let \( X_{\mathbb{K}_\bullet}(\alpha) \) denote the union \( X_{\mathbb{K}_\bullet}(\alpha) = \bigcup_{D \neq \emptyset} X_{\mathbb{K}_\bullet}(\alpha, D) \).

**7.2.2.** Write \( D = \{i_1, \ldots, i_{r-1}\} \) with \( i_1 < \cdots < i_{r-1} \). Let \( \text{Flag}_D \) denote the set

\[ \text{Flag}_D = \{f = [0 \subset V_1 \subset \cdots \subset V_{r-1} \subset F_{\leq d}] \mid \dim(V_j) = i_j\} \]

of flags in \( F_{\leq d} \).

Let \( \widetilde{X}_{\mathbb{K}_\bullet}(\alpha, D) \) denote the inverse image of \( X_{\mathbb{K}_\bullet}(\alpha, D) \) by the morphism \( \widetilde{X}_{\mathbb{K}_\bullet} \to X_{\mathbb{K}_\bullet} \). For \( f = [0 \subset V_1 \subset \cdots \subset V_{r-1} \subset F_{\leq d}] \in \text{Flag}_D \), let \( \widetilde{X}_{\mathbb{K}_\bullet}(\alpha, D, f) \) denote the simplicial subcomplex of \( \widetilde{X}_{\mathbb{K}_\bullet}(\alpha, D) \) consisting of the simplices in \( \widetilde{X}_{\mathbb{K}_\bullet} \) whose representative \( \mathcal{F}_{(i)} \) satisfies \( \mathcal{F}_{(i)}(\alpha) \geq \mathcal{F}_{(i)} \geq \mathcal{F}_{(i)}(1) \geq \cdots \geq \mathcal{F}_{(i)}(\alpha, D, f) \) for every \( i \in \mathbb{Z}, j = 1, \ldots, r - 1 \). Lemma 24 implies that, for \( \alpha > (d - 1) \deg(\infty) \), \( \widetilde{X}_{\mathbb{K}_\bullet}(\alpha, D) \) is decomposed into a disjoint union \( \widetilde{X}_{\mathbb{K}_\bullet}(\alpha, D) = \bigsqcup_{f \in \text{Flag}_D} \widetilde{X}_{\mathbb{K}_\bullet}(\alpha, D, f) \).
For $g \in \text{GL}_d(\mathbb{A}^\infty)$, we set
\[
\overline{Y}_\mathbb{K}^{(\alpha),\mathcal{D},g} = \overline{X}_\mathbb{K}^{(\alpha),\mathcal{D},f_0} \cap \left( P_{\mathcal{D}}(\mathbb{A}^\infty)g/(g^{-1}P_{\mathcal{D}}(\mathbb{A}^\infty)g) \times \mathcal{B}_T \right)
\]
and $Y^{(\alpha),\mathcal{D},g} = P_{\mathcal{D}}(F)\backslash \overline{Y}_\mathbb{K}^{(\alpha),\mathcal{D},g}$.

If $g = 1$, we may without loss of generality assume that the group $\text{GL}_d(\mathbb{A}^\infty)$ contains representatives of $P_{\mathcal{D}}(\mathbb{A}^\infty)\backslash \text{GL}_d(\mathbb{A}^\infty)$, then we have $\overline{X}_\mathbb{K}^{(\alpha),\mathcal{D},f_0} = \prod_{g \in T} \overline{Y}_\mathbb{K}^{(\alpha),\mathcal{D},g}$.

### 7.3. The finite adele actions

**Lemma 26.** For every $g \in \text{GL}_d(\mathbb{A}^\infty)$ satisfying $g^{-1}\mathbb{K}g \subset \text{GL}_d(\mathbb{A})$, there exists a real number $\beta_g > 0$ such that the isomorphism $\xi_g : X_{\mathbb{K}} \rightarrow X_{g^{-1}\mathbb{K}g}$ sends $X_{\mathbb{K}}$ to $X_{g^{-1}\mathbb{K}g}$ for all $\alpha > \beta_g$, and for all nonempty subset $\mathcal{D} \subset \{1, \ldots, d\}$.

**Proof.** Take two elements $a, b \in \mathbb{A}^\infty \cap \mathbb{A}$ such that both $ag$ and $bg^{-1}$ lie in $\text{GL}_d(\mathbb{A}^\infty) \cap \text{Mat}_d(\mathbb{A})$. Then for any $h \in \text{GL}_d(\mathbb{A}^\infty)$ we have $a\mathbb{A}^\infty d h^{-1} \subset a\mathbb{A}^\infty d h^{-1} \subset b^{-1}\mathbb{A}^\infty d h^{-1}$. This implies that, for any vertex $x \in X_{\mathbb{K}g}$, if we take suitable representatives $F_x, F_{\xi_g(x)}$ of the equivalence classes of locally free $\mathcal{O}_C$-modules corresponding to $x, \xi_g(x)$, then there exists a sequence of injections $F_x(-\text{div}(a)) \rightarrow F_{\xi_g(x)} \rightarrow F_x(\text{div}(b))$. Applying Lemma 24, we see that there exists a positive real number $m_g > 0$ not depending on $x$ such that $|p_{F_x}(i) - p_{F_{\xi_g(x)}}(i)| < m_g$ for all $i$. Hence the claim follows.

**Remark 7.4.** Any open compact subgroup of $\text{GL}_d(\mathbb{A}^\infty)$ is conjugate to an open subgroup of $\text{GL}_d(\mathbb{A})$. The set of open subgroups of $\text{GL}_d(\mathbb{A})$ is cofinal in the inductive system of all open compact subgroups of $\text{GL}_d(\mathbb{A}^\infty)$. Therefore, to prove Theorem 16 we may without loss of generality assume that the group $\mathbb{K}$ is contained in $\text{GL}_d(\mathbb{A})$, and we may replace the inductive limit in the definition of $H_{d-1}(X_{\text{lim}, M})$ and $H_{d-1}(X_{\text{lim}, M})$ with the inductive limit $\prod_{\mathbb{K} \subset \text{GL}_d(\mathbb{A})}$.

From now on until the end of this section, we exclusively deal with the subgroups $\mathbb{K} \subset \text{GL}_d(\mathbb{A}^\infty)$ contained in $\text{GL}_d(\mathbb{A})$. The notation $\lim_{\mathbb{K} \subset \text{GL}_d(\mathbb{A})}$ henceforth means the inductive limit $\lim_{\mathbb{K} \subset \text{GL}_d(\mathbb{A})}$.

Thus the group $\text{GL}_d(\mathbb{A}^\infty)$ acts on $\lim_{\mathbb{K} \subset \text{GL}_d(\mathbb{A})}^* \left( H^*(X_{\mathbb{K}}^{(\alpha),\mathcal{D}}, \mathbb{Q}) \right)$ in such a way that the exact sequence (7.4) is $\text{GL}_d(\mathbb{A}^\infty)$-equivariant.

### 7.3.1. An argument similar to that in the proof of Lemma 26 shows that, for each $g \in \text{GL}_d(\mathbb{A}^\infty)$ satisfying $g^{-1}\mathbb{K}g \subset \text{GL}_d(\mathbb{A})$, there exists a real number $\beta_g > 0$ such that the isomorphism $\tilde{\xi}_g$ sends $X^{(\alpha),\mathcal{D},f}$ to $X^{(\alpha-\beta_g),\mathcal{D},f}$ for $\alpha > \beta_g$ and for any $f \in \text{Flag}_D$.

### 7.3.2. For $\gamma \in \text{GL}_d(F)$, the action of $\gamma$ on $X_{\mathbb{K}}^{(\alpha),\mathcal{D},f}$ bijectively to $X_{\mathbb{K}}^{(\alpha-\gamma),\mathcal{D},f}$. Let $f_0 = [0] \subset F(\mathbb{A}^\infty) \subset F(\mathbb{A}^\infty) \subset \cdots \subset F^{d-1}(\mathbb{A}^\infty) \subset F^{d}(\mathbb{A}^\infty)$ be the standard flag. The group $\text{GL}_d(F)$ acts transitively on $\text{Flag}_D$ and its stabilizer at $f_0$ equals $P_{\mathcal{D}}(F)$. Hence for $\alpha > (d-1)\text{deg}(-1)$, $X_{\mathbb{K}}^{(\alpha),\mathcal{D}}$ is isomorphic to the quotient $P_{\mathcal{D}}(F)\backslash \overline{X}_{\mathbb{K}}^{(\alpha),\mathcal{D},f_0}$. 
For $g \in \text{GL}_d(A^\infty)$, we set
\[
\tilde{Y}^{(\alpha),D}_{g} = \tilde{X}^{(\alpha),D}_{g} \cap (P_0(A^\infty)g/(g^{-1}P_0(A^\infty)g \cap K) \times \mathcal{B}T_\bullet)
\]
and $Y^{(\alpha),D}_{g} = P_0(F)\tilde{Y}^{(\alpha),D}_{g}$. We omit the superscript $g$ on $\tilde{Y}^{(\alpha),D}_{g}$ and $Y^{(\alpha),D}_{g}$ if $g = 1$.

We note that, when $K$, $\alpha$ and $D$ are fixed, $\tilde{Y}^{(\alpha),D}_{g}$ and $Y^{(\alpha),D}_{g}$ depend only on the class $\mathcal{F} = P_0(A^\infty)g$ of $g$ in $P_0(A^\infty)\backslash \text{GL}_d(A^\infty)$. By abuse of notation, we denote $\tilde{Y}^{(\alpha),D}_{g}$ and $Y^{(\alpha),D}_{g}$ by $\tilde{Y}^{(\alpha),D}_{\mathcal{F}}$ and $Y^{(\alpha),D}_{\mathcal{F}}$, respectively. Then we have
\[
\tilde{X}^{(\alpha),D}_{g} = \bigcup_{\mathcal{F} \in P_0(A^\infty)\backslash \text{GL}_d(A^\infty)} \tilde{Y}^{(\alpha),D}_{\mathcal{F}}.
\]
Hence we have
\[
X^{(\alpha),D}_{g} = \bigcup_{\mathcal{F} \in P_0(A^\infty)\backslash \text{GL}_d(A^\infty)} Y^{(\alpha),D}_{\mathcal{F}}
\]
for $\alpha > (d - 1)\deg(\infty)$.

### 7.3.3. We use a covering spectral sequence

\[
E_1^{p,q} = \bigoplus_{g \in \mathcal{D} = p+1} H^q(X^{(\alpha),D}_{g}, \mathbb{Q}) \Rightarrow H^{p+q}(X^{(\alpha)}_{g}, \mathbb{Q})
\]
with respect to the covering $X^{(\alpha)}_{g} = \bigcup_{1 \leq i \leq d-1} X^{(\alpha),\{i\}}_{g}$ of $X^{(\alpha)}_{g}$. For $\alpha' \geq \alpha > 0$, the inclusion $X^{(\alpha),D}_{g} \rightarrow X^{(\alpha'),D}_{g}$ induces a morphism of spectral sequences. Taking the inductive limit, we obtain the spectral sequence
\[
E_1^{p,q} = \bigoplus_{g \in \mathcal{D} = p+1} \lim_{\alpha} H^q(X^{(\alpha),D}_{g}, \mathbb{Q}) \Rightarrow \lim_{\alpha} H^{p+q}(X^{(\alpha)}_{g}, \mathbb{Q}).
\]

For $g \in \text{GL}_d(A^\infty)$ satisfying $g^{-1}K_g \subset \text{GL}_d(A^\infty)$, let $\beta_g$ be as in Lemma 26. Then for $\alpha > \beta_g$ the isomorphism $\xi_g : X^{(\alpha)}_{g} \xrightarrow{\sim} X^{(\alpha-\beta)}_{g}$ induces a homomorphism from the spectral sequence (7.2) for $X^{(\alpha)}_{g}$ to that for $X^{(\alpha-\beta)}_{g}$. Passing to the inductive limit with respect to $\alpha$ and then passing to the inductive limit with respect to $g$, we obtain the left action of the group $\text{GL}_d(A^\infty)$ on the spectral sequence
\[
E_1^{p,q} = \bigoplus_{g \in \mathcal{D} = p+1} \lim_{\alpha} H^q(X^{(\alpha),D}_{g}, \mathbb{Q}) \Rightarrow \lim_{\alpha} H^{p+q}(X^{(\alpha)}_{g}, \mathbb{Q}).
\]

### 7.4. Finiteness and an application

We prove that the complement of the boundary is finite. Then we express the Borel-Moore homology and cohomology with compact support as a limit of relative homology and cohomology respectively.

**Lemma 27.** For any $\alpha > 0$, the set of the simplices in $X^{(\alpha)}_{g}$ not belonging to $X^{(\alpha)}_{g}$ is finite.

**Proof.** Let $\mathcal{P}$ denote the set of continuous, convex functions $p' : [0, d] \rightarrow \mathbb{R}$ with $p'(0) = 0$ such that $p'(i) \in \mathbb{Z}$ and $p'$ is affine on $[i - 1, i]$ for $i = 1, \ldots, d$. It is known that for any $r \geq 1$ and $f \in \mathbb{Z}$, there are only a finite number of isomorphism classes of semi-stable locally free $\mathcal{O}_C$-modules of rank $r$ with degree $f$. Hence the theory of Harder-Narasimhan filtration, for any $p' \in \mathcal{P}$, the set of the isomorphism classes of locally free $\mathcal{O}_C$-modules $\mathcal{F}$ with $p_x = p'$ is finite. Let us give an action of the group $\mathbb{Z}$ on the set $\mathcal{P}$, by setting $(a \cdot p')(x) = p'(x) + a \deg(\infty)x$ for $a \in \mathbb{Z}$ and
7.4.1. Lemma \(\text{(2)}\) implies that \(H^{BM}_{d-1}(X_{K^\bullet}, \mathbb{Q})\) is canonically isomorphic to the projective limit \(\lim_{\alpha \to 0} H^{d-1}(X_{K^\bullet}, X_{K^\bullet}^{(\alpha)}; \mathbb{Q})\) and \(H^{d-1}_{\varepsilon}(X_{K^\bullet}, \mathbb{Q})\) is canonically isomorphic to the inductive limit \(\lim_{\alpha \to 0} H^{d-1}(X_{K^\bullet}, X_{K^\bullet}^{(\alpha)}; \mathbb{Q})\). Thus from the (usual) long exact sequence of relative homology, we have an exact sequence (7.4)

\[
\lim_{\alpha \to 0} H^{d-2}(X_{K^\bullet}^{(\alpha)}; \mathbb{Q}) \to H^{d-1}_{\varepsilon}(X_{K^\bullet}, \mathbb{Q}) \to H^{d-1}(X_{K^\bullet}, \mathbb{Q}) \to \lim_{\alpha \to 0} H^{d-1}(X_{K^\bullet}^{(\alpha)}; \mathbb{Q}).
\]

### 7.5. Some isomorphisms

**Proposition 28.** For \(\alpha' \geq \alpha > (d-1) \deg(\infty)\), the homomorphism \(H^*(X_{K^\bullet}^{(\alpha)}, \mathbb{Q}) \to H^*(X_{K^\bullet}^{(\alpha')}, \mathbb{Q})\) is an isomorphism.

**Lemma 29.** For any \(g \in \text{GL}_d(\mathbb{K}^\infty)\), the simplicial complex \(\tilde{X}_{K^\bullet}^{(\alpha)}; D, F_0 \cap \{g \mathbb{K}\} \times B\mathbb{T}_\bullet\) is non-empty and contractible.

**Proof.** Since \(\tilde{X}_{K^\bullet}^{(\alpha)}; D, F_0 \cap \{g \mathbb{K}\} \times B\mathbb{T}_\bullet\) is isomorphic to \(\tilde{X}_{K^\bullet}^{(\alpha)}; D, F_0 \cap \{(g \mathbb{GL}(\hat{A})) \times B\mathbb{T}_\bullet\}\), we may assume that \(\mathbb{K} = \mathbb{GL}(\hat{A})\). We set \(X = \tilde{X}_{K^\bullet}^{(\alpha)}; D, F_0 \cap \{(g \mathbb{GL}(\hat{A})) \times B\mathbb{T}_\bullet\}\).

We proceed by induction on \(d\), in a manner similar to that in the proof of Theorem 4.1 of [Gr]. Let \(i \in \mathcal{D}\) be the minimal element and set \(d' = d - i\). We define the subset \(\mathcal{D}' \subset \{1, \ldots, d' - 1\}\) as \(\mathcal{D}' = \{i' - i \mid i' \in \mathcal{D}, i' \neq i\}\). We define \(f_0 \in \text{Flag}_D\) as the image of the flag \(f_0 \in F^\oplus\) with respect to the the projection \(F^\oplus \to F^\oplus / F^\oplus \cap \{0\} \cong F^\oplus\). Take an element \(g' \in \text{GL}_d(\mathbb{A}^\infty)\) such that the quotient \(\mathbb{A}^\oplus - g^{-1} / (\mathbb{A}^\oplus - g^{-1} \cap \{0\})\) equals \(A^\oplus - g^{-1}\) as an \(\hat{A}\)-lattice of \(\mathbb{A}^\infty\). We set \(X' = \tilde{X}_{K^\bullet}^{(\alpha)}; \mathcal{D}', f_0 \cap \{(g' \mathbb{GL}(\hat{A})) \times B\mathbb{T}_{GL^\bullet}\}\) if \(\mathcal{D}'\) is non-empty. Otherwise we set \(X' = \tilde{X}_{K^\bullet}^{(\alpha)}; \mathcal{D}', f_0 \cap \{(g' \mathbb{GL}(\hat{A})) \times B\mathbb{T}_{GL^\bullet}\}\).

By induction hypothesis, \(|X'|\) is contractible. There is a canonical morphism \(h : X \to X'\) which sends an \(O_{\mathcal{C}}\)-submodule \(F[g, L_\infty]\) of \(\eta_* F^\oplus\) to the \(O_{\mathcal{C}}\)-submodule \(F[g, L_\infty] / F[g, L_\infty](i)\) of \(\eta_* F^\oplus\). Let \(e : \text{Vert}(X) \to \mathbb{Z}\) and \(e' : \text{Vert}(X') \to \mathbb{Z}\) denote the maps that send a locally free \(O_{\mathcal{C}}\)-module to the integer \([p_F(1) / \deg(\infty)]\). We fix an \(O_{\mathcal{C}}\)-submodule \(F_0\) of \(\eta_* F^\oplus\) whose equivalence class belongs to \(X\). By twisting \(F_0\) by some power of \(O_{\mathcal{C}}(\infty)\) if necessary, we may assume that \(p_F(i) - p_F(i - 1) > \alpha\). We fix a splitting \(F_0 = F_0(i) \oplus F_0''\). This splitting induces an isomorphism \(\varphi : \eta_* F_0'' \cong \eta_* F_0^\oplus\). Let \(h' : X' \to X\) denote the morphism that sends an \(O_{\mathcal{C}}\)-submodule \(F'\) of \(\eta_* F_0^\oplus\) to the \(O_{\mathcal{C}}\)-submodule \(F_0(i) / \varphi^{-1}(F')(\infty)\). For each \(n \in \mathbb{Z}\), define a morphism \(G_n : X \to X\) by sending an \(O_{\mathcal{C}}\)-submodule \(F\) of \(\eta_* F_0^\oplus\) to the \(O_{\mathcal{C}}\)-submodule \(F_0(i) / (n + e(F)) \cong F\) of \(\eta_* F_0^\oplus\). Then the argument in [Gr] p. 85-86 shows that \(f\) and \(h' \circ h \circ f\) are homotopic for any map \(f : Z \to |X|\) from a compact space \(Z\) to \(|X|\). Since the map \(h' \circ h \circ f\) factors through the contractible space \(|X'|\), \(f\) is null-homotopic. Hence \(|X|\) is contractible.
7.6. Proof of Theorem 16

7.6.1. For a subset $D$ of $\{1, \ldots, d-1\}$, we define the algebraic groups $P_D$, $N_D$ and $M_D$ as follows. We write $D = \{i_1, \ldots, i_r\}$, with $i_0 = 0 < i_1 < \cdots < i_{r-1} < i_r = d$ and set $d_j = i_j - i_{j-1}$ for $j = 1, \ldots, r$. We define $P_D$, $N_D$ and $M_D$ as the standard parabolic subgroup of $GL_d$ of type $(d_1, \ldots, d_r)$, the unipotent radical of $P_D$, and the quotient group $P_D/N_D$ respectively. We identify the group $M_D$ with $GL_{d_1} \times \cdots \times GL_{d_r}$.

Let us consider the smooth $GL_d(\mathbb{A})$-module, $\lim\limits_{\mu K} \lim\limits_{\gamma g} H^\ast(X_{\mu K}, D, \mathbb{Q})$. For a fixed $K$, we have

\[
\lim\limits_{\mu K} \lim\limits_{\gamma g} H^\ast(X_{\mu K}, D, \mathbb{Q}) = \prod_{\mu P(f)(\mathbb{A}) \setminus GL_d(\mathbb{A})} \lim\limits_{\gamma g} H^\ast(Y_{\mu K}, D, \mathbb{Q}),
\]

since $H^\ast(Y_{\mu K}, D, \mathbb{Q}) \rightarrow H^\ast(Y_{\mu K}, D, \mathbb{Q})$ is an isomorphism for $\alpha' \geq \alpha > (d-1)\text{deg}(\mu)$. We note that $\lim\limits_{\mu K} \lim\limits_{\gamma g} H^\ast(Y_{\mu K}, D, \mathbb{Q})$ is a smooth $g^{-1}P_D(\mathbb{A})g$-module for any $g \in GL_d(\mathbb{A})$. Via (7.4) we regard $\lim\limits_{\mu K} \lim\limits_{\gamma g} H^\ast(X_{\mu K}, D, \mathbb{Q})$ as a submodule of $\prod_{\mu P(f)(\mathbb{A}) \setminus GL_d(\mathbb{A})} \lim\limits_{\mu K} \lim\limits_{\gamma g} H^\ast(Y_{\mu K}, D, \mathbb{Q})$.

Let $g \in GL_d(\mathbb{A})$. For an open compact subgroup $K \subset GL_d(\mathbb{A})$ satisfying $g^{-1}Kg \subset GL_d(\mathbb{A})$, there exists a real number $\beta_1 > \beta_2$ such that the isomorphism $\tilde{\xi}_{g}$ sends $\tilde{V}_{\alpha}(\mu, D, g')$ to $\tilde{V}_{gKg^{-1}}(\alpha-\beta', D, g') \subset \tilde{X}_{gKg^{-1}}$, for $\alpha > \beta'$, for any $f \in \text{Flag}_D$ and for any $g' \in GL_d(\mathbb{A})$. This induces a morphism $\xi_{g'\cdot K} : Y_{\mu K}, D, g' \rightarrow Y_{g'\cdot K}, D, g'$ of (generalized) simplicial complexes. By varying $\alpha$, we have a homomorphism

\[
\xi_{g'\cdot K} : \lim\limits_{\alpha} H^\ast(Y_{g'Kg^{-1}}, D, g', \mathbb{Q}) \rightarrow \lim\limits_{\alpha} H^\ast(Y_{\mu K}, D, g', \mathbb{Q}).
\]

The homomorphism $\xi_{g'\cdot K}$ is an isomorphism since the homomorphism $\xi_{g'\cdot K}^{-1}$ gives its inverse. By varying $K$, we obtain an isomorphism

\[
\xi_{g'\cdot K} : \lim\limits_{\mu K} \lim\limits_{\alpha} H^\ast(Y_{\mu K}, D, g', \mathbb{Q}) \rightarrow \lim\limits_{\alpha} H^\ast(Y_{g'Kg^{-1}}, D, g', \mathbb{Q}).
\]

Let us choose a complete set $T \subset GL_d(\mathbb{A})$ of representatives such that $1 \in T$. Then $\lim\limits_{\mu K} \lim\limits_{\alpha} H^\ast(X_{\mu K}, D, \mathbb{Q})$ is a submodule of $\prod_{g \in T} \lim\limits_{\mu K} \lim\limits_{\alpha} H^\ast(Y_{\mu K}, D, g, \mathbb{Q})$.

We set

\[
H_{Q}^{D} = \lim\limits_{\mu K} \lim\limits_{\alpha} H^\ast(Y_{\mu K}, D, g, \mathbb{Q}).
\]
The isomorphism $\xi_{g,1}^* : \lim_{\alpha} \lim_{g \in T} H^*(Y_{\mathbb{K},\bullet}, Q) \xrightarrow{\sim} H^*_{Q,\bullet}$ for each $g \in T$ gives an isomorphism
\[
\prod_{g \in T} \lim_{\alpha} \lim_{\mathbb{K}} H^*(Y_{\mathbb{K},\bullet}, Q) \cong \prod_{g \in T} H^*_{Q,\bullet}.
\]
Via this isomorphism we regard $\lim_{\mathbb{K}} \lim_{\alpha} H^*(X_{\mathbb{K},\bullet}, Q)$ as a submodule of $\prod_{g \in T} H^*_{Q,\bullet}$.

Let $g' \in GL_d(\mathbb{A})$ be an arbitrary element. For each $g \in T$, let us write $gg' = h_sg_g$ with $g_g \in T$ and $h_g \in P_D(\mathbb{A})$. Then, with respect to the inclusion $\lim_{\mathbb{K}} \lim_{\alpha} H^*(X_{\mathbb{K},\bullet}, Q) \hookrightarrow \prod_{g \in T} H^*_{Q,\bullet}$, the action of $g'$ on $\lim_{\mathbb{K}} \lim_{\alpha} H^*(X_{\mathbb{K},\bullet}, Q)$ is compatible with the automorphism $\theta(g')$ of $\prod_{g \in T} H^*_{Q,\bullet}$ that sends $(x_g)_{g \in T}$ to $(\xi_{h_g,1}(x_{g}))(g \in T)$. The automorphisms $\theta(g')$ for various $g'$ give an action of $GL_d(\mathbb{A})$ on $\prod_{g \in T} H^*_{Q,\bullet}$ from the left and an element $x \in \prod_{g \in T} H^*_{Q,\bullet}$ belongs to the submodule $\lim_{\mathbb{K}} \lim_{\alpha} H^*(X_{\mathbb{K},\bullet}, Q)$ if and only if $x$ is invariant under some open compact subgroup $\mathbb{K} \subset GL_d(\mathbb{A})$. Observe that the smooth part of $\prod_{g \in T} H^*_{Q,\bullet}$ with respect to the action of $GL_d(\mathbb{A})$ introduced above is equal to the (unnormalized) parabolic induction $Ind_{P_D(\mathbb{A})}^{GL_d(\mathbb{A})} \lim_{\mathbb{K}} \lim_{\alpha} H^*(Y_{\mathbb{K},\bullet}, Q)$. Thus we obtain an isomorphism
\[
\lim_{\mathbb{K}} \lim_{\alpha} H^*(X_{\mathbb{K},\bullet}, Q) \cong Ind_{P_D(\mathbb{A})}^{GL_d(\mathbb{A})} \lim_{\mathbb{K}} \lim_{\alpha} H^*(Y_{\mathbb{K},\bullet}, Q).
\]
It is straightforward to check that this isomorphism is independent of the choice of $T$.

**Proposition 30.** Let the notations be above. Then as a smooth $GL_d(\mathbb{A})$-module, $\lim_{\mathbb{K}} \lim_{\alpha} H^*(X_{\mathbb{K},\bullet}, Q)$ is isomorphic to
\[
\bigotimes_{j=1}^{r} \text{Ind}_{P_D(\mathbb{A})}^{GL_d(\mathbb{A})} \lim_{\mathbb{K}_j \subset GL_d(\mathbb{A})} H^*(X_{GL_d,K_j,\bullet}, Q),
\]
where the group $P_D(\mathbb{A})$ acts on $\bigotimes_{j=1}^{r} \lim_{\mathbb{K}_j \subset GL_d(\mathbb{A})} H^*(X_{GL_d,K_j,\bullet}, Q)$ via the quotient $P_D(\mathbb{A}) \rightarrow M_D(A^\infty) = \prod_j GL_d(\mathbb{A}),$ and $\text{Ind}_{P_D(\mathbb{A})}^{GL_d(\mathbb{A})}$ denotes the parabolic induction unnormalized by the modulus function.

We give a proof in the following subsections.

**7.6.2.** For $j = 1, \ldots, r$, let $\mathbb{K}_j \subset GL_d(\mathbb{A})$ denote the image of $\mathbb{K} \cap P_D(\mathbb{A})$ by the composition $P_D(\mathbb{A}) \rightarrow M_D(A^\infty) \rightarrow GL_d(\mathbb{A})$.

We define the continuous map $\bar{\pi}_{D,j} : \tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),D} \rightarrow |\tilde{X}_{GL_d,K_j,\bullet}|$ of topological spaces in the following way. Let $\sigma$ be an $i$-simplex in $\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),D}$. Take a chain $\cdots \supset F_{-1} \supset F_0 \supset F_1 \supset \cdots$ of $\mathcal{O}_C$-modules representing $\sigma$. For $l \in \mathbb{Z}$ we set $F_{l,j} = F_{l,\iota_j}(F_{l,\iota_j-1})$, which is an $\mathcal{O}_C$-submodule of $\eta_{\alpha} F_{l,\iota_j}^{(\alpha),D}$. We set $S_j = \{ l \in \mathbb{Z} \mid F_{l,j} \neq F_{l+1,j} \}$. Define the map $\psi_j : \mathbb{Z} \rightarrow S_j$ as $\psi_j(l) = \min\{ l' \geq l \mid l' \in S_j \}$. Take an order-preserving bijection $\varphi_j : S_j \xrightarrow{\sim} \mathbb{Z}$. For $l \in \mathbb{Z}$ set $F'_{l,j} = F_{\varphi_j^{-1}(l),j}$. Then the chain $\cdots \supset F'_{-1} \supset F'_{0} \supset F'_{1} \supset \cdots$ defines a simplex $\sigma'$ in $\tilde{X}_{GL_d,K_j,\bullet}$. We define a continuous map $|\sigma| \rightarrow |\sigma'|$ as the affine map sending the vertex of $\sigma$ corresponding to $F_l$ to the vertex of $\sigma'$ corresponding to $F'_{\varphi_j^{-1}(l)}$. Gluing these maps, we obtain
a continuous map \( \tilde{\pi}_{D,j} : |Y_{K,j}^{(\alpha),D}| \to |X_{GL_{d,j},K,j}| \). We set \( \tilde{\pi}_D = (\tilde{\pi}_{D,1}, \ldots, \tilde{\pi}_{D,r}) : |Y_{K,j}^{(\alpha),D}| \to \prod_{j=1}^r |X_{GL_{d,j},K,j}| \). This continuous map descends to the continuous map \( \pi_D : |Y_{K,j}^{(\alpha),D}| \to \prod_{j=1}^r |X_{GL_{d,j},K,j}| \).

7.6.3. If \( g \in P_D(\mathbb{K}^{\infty}) \) and \( g^{-1}Kg \subset GL_d(\hat{A}) \), then the isomorphism \( \xi_g : X_{K,j} \cong X_{g^{-1}Kg} \) sends \( Y_{K,j}^{(\alpha),D} \) inside \( Y_{g^{-1}Kg}^{(\alpha-\beta_g),D} \). If we denote by \( (g_1, \ldots, g_r) \) the image of \( g \) in \( M_D(\mathbb{K}^{\infty}) = \prod_{j=1}^r GL_d(\mathbb{K}^{\infty}) \), then the diagram

\[
\begin{array}{ccc}
|Y_{K,j}^{(\alpha),D}| & \xrightarrow{\xi_g} & |Y_{g^{-1}Kg}^{(\alpha-\beta_g),D}| \\
\pi_D \downarrow & & \pi_D \downarrow \\
\prod_{j=1}^r |X_{GL_{d,j},K,j}| & \xrightarrow{(\xi_{g_1}, \ldots, \xi_{g_r})} & \prod_{j=1}^r |X_{GL_{d,j},g_j^{-1}Kg_j,j}| \\
\end{array}
\]

is commutative.

7.6.4. With the notations as above, suppose that the open compact subgroup \( \mathbb{K} \subset GL_d(\hat{A}) \) has the following property.

(7.7) The homomorphism \( P_D(\mathbb{K}^{\infty}) \cap \mathbb{K} \to \mathbb{K}_1 \times \cdots \times \mathbb{K}_r \) is surjective.

For a simplicial complex \( X \), we set \( I_X = \text{Map}(\pi_0(X), \mathbb{Q}) \), where \( \pi_0(X) \) is the set of the connected components of \( X \). Let us consider the following commutative diagram.

\[
\begin{array}{ccc}
H^*(M_D(F), \text{Map}(\prod_{j=1}^r \pi_0(X_{GL_{d,j},K,j}), \mathbb{Q})) & \longrightarrow & H^*(P_D(F), I_{Y_{K,j}^{(\alpha),D}}) \\
\downarrow & & \downarrow \\
H^*_M(D(F), \prod_{j=1}^r |X_{GL_{d,j},K,j}|, \mathbb{Q}) & \longrightarrow & H^*_P(D(F), |Y_{K,j}^{(\alpha),D}|, \mathbb{Q}) \\
\uparrow \quad (7.8) \quad \uparrow \\
H^*(\prod_{j=1}^r |X_{GL_{d,j},K,j}|, \mathbb{Q}) & \longrightarrow & H^*(|Y_{K,j}^{(\alpha),D}|, \mathbb{Q}).
\end{array}
\]

Here \( H^*_M(D(F)) \) and \( H^*_P(D(F)) \) denote the equivariant cohomology groups.

7.6.5. Proposition 31. All homomorphisms in the above diagram (7.3) are isomorphisms.

Proof. We prove that the upper horizontal arrow and the four vertical arrows are isomorphisms.

First we consider the upper horizontal arrow.

Lemma 32. For \( q \geq 1 \), the group \( H^q(N_D(F), I_{Y_{K,j}^{(\alpha),D}}) \) is zero.

Proof of Lemma 32. For each \( x \in N_D(F) \setminus \pi_0(Y_{K,j}^{(\alpha),D}) \), take a lift \( \bar{x} \in \pi_0(Y_{K,j}^{(\alpha),D}) \) of \( x \) and let \( N_x \subset N_D(F) \) denote the stabilizer of \( \bar{x} \). Then the group \( H^*(N_D(F), I_{Y_{K,j}^{(\alpha),D}}) \) is isomorphic to the direct product

\[
\prod_{x \in N_D(F) \setminus \pi_0(Y_{K,j}^{(\alpha),D})} H^*(N_x, \mathbb{Q}).
\]
We note that the group \( N_D(F) \) is a union \( N_D(F) = \bigcup_i U_i \) of finite subgroups of \( p \)-power order where \( p \) is the characteristic of \( F \). This follows easily from [Ke-Wei p.2, 1.A.2 Lemma] or from [Ke-Wei p.60, 1.L.1 Theorem]. Hence \( N_x = \bigcup_i (U_i \cap N_x) \).

We claim \( H^j(N_x, \mathbb{Q}) = 0 \) for \( j \geq 1 \). Because the projective system of cochain complexes that compute \( H^r(U_i \cap N_x, M) \), where \( M \) is a \( \mathbb{Q} \)-vector space, satisfies the Mittag-Leffler condition, the cohomology \( H^j(N_x, \mathbb{Q}) \) equals the projective limit \( \lim_{\leftarrow} H^j(U_i \cap N_x, \mathbb{Q}) \) (see [Wei p.83, Ex 3.5.2, 3.5.8]). Since \( U_i \cap N_x \) is a finite group, their higher cohomology with \( \mathbb{Q} \) coefficient vanishes. This proves the claim, hence the lemma.

We note that \( \pi_0(\text{GL}_{d_j} A) \) is canonically isomorphic to \( \text{GL}_{d_j}(\mathbb{A}^\infty)/\mathbb{K}_j \) for \( j = 1, \ldots, r \), and Lemma 29 implies that \( I_{Y_{\mathbb{K}_j}} \) is a compact open subgroup of \( \text{GL}_{d_j}(\mathbb{A}^\infty)/\mathbb{K}_j \). Since \( N_D(F) \) is dense in \( N_D(\mathbb{A}^\infty) \), the group \( H^0(N_D(F), I_{Y_{\mathbb{K}_j}}) \) is canonically isomorphic to the group \( \text{Map}(\rho, \mathbb{A}^\infty)/\prod_j \mathbb{K}_j \). Hence the upper horizontal arrow of the diagram (7.8) is an isomorphism.

Next we consider the vertical arrows. Each connected component of \( \tilde{X}_{\text{GL}_{d_j}, g_j^{-1} g_j} \) is contractible since it is isomorphic to the Bruhat-Tits building for \( \text{GL}_{d_j} \). Recall that the simplicial complex \( X_{\text{GL}_{d_j}, g_j^{-1} g_j} \) is the quotient of \( \tilde{X}_{\text{GL}_{d_j}, g_j^{-1} g_j} \) by the action of \( \text{GL}_{d_j} \). For any simplex \( \sigma \) in \( \tilde{X}_{\text{GL}_{d_j}, g_j^{-1} g_j} \), the stabilizer group \( \Gamma_{\sigma} \subset \text{GL}_{d_j} \) of \( \sigma \) is finite, as remarked in Section 4.1.3. Hence the left two vertical arrows in the diagram (7.8) are isomorphisms. Similarly, bijectivity of the two right vertical arrows in the diagram (7.8) follows from Lemma 29. Thus we have a proof of Proposition 30.

PROOF OF PROPOSITION 30 Let us consider the lower horizontal arrow in the diagram (7.8). By Proposition 30 it is an isomorphism. We note that the compact open subgroups \( \mathbb{K} \subset \text{GL}_d(\mathbb{A}) \) with property (7.7) form a cofinal subsystem of the inductive system of all open compact subgroups of \( \text{GL}_d(\mathbb{A}^\infty) \). Therefore, passing to the inductive limits with respect to \( \alpha \) and \( \mathbb{K} \) with property (7.7), we have \( \lim_{\alpha} \lim_{\mathbb{K}} H^*(Y_{\mathbb{K}_j, \mathbb{K}}, \mathbb{Q}) \cong \bigotimes_{j=1}^r \lim_{\alpha} \lim_{\mathbb{K}} H^*(\text{GL}_{d_j}, \mathbb{K}_j, \mathbb{Q}) \) as desired.

7.6.6. Let \( \mathbb{K}, \mathbb{K}' \subset \text{GL}_d(\mathbb{A}^\infty) \) be two compact open subgroups with \( \mathbb{K}' \subset \mathbb{K} \). The pull-back morphism from the cochain complex of \( X_{\mathbb{K}}, \mathbb{K} \) to that of \( X_{\mathbb{K}'}, \mathbb{K}' \) preserves the cochains with finite supports. Thus we have pull-back homomorphisms \( H^*_c(X_{\mathbb{K}}, \mathbb{K}, \mathbb{Q}) \to H^*_c(X_{\mathbb{K}'}, \mathbb{K}', \mathbb{Q}) \) which is compatible with the usual pull-back homomorphism \( H^*(X_{\mathbb{K}}, \mathbb{Q}) \to H^*(X_{\mathbb{K}'}, \mathbb{Q}) \). For an abelian group \( M \), we let \( H^*(X_{\lim}, M) = H^*(X_{\lim}, \mathbb{Q}) \otimes M \) and \( H^*_c(X_{\lim}, M) = H^*_c(X_{\lim}, \mathbb{Q}) \otimes M \). The inductive limits \( \lim_{\mathbb{K}} H^*(X_{\mathbb{K}}, M) \) and \( \lim_{\mathbb{K}} H^*_c(X_{\mathbb{K}}, M) \), respectively. If \( M \) is a \( \mathbb{Q} \)-vector space, then for each compact open subgroup \( \mathbb{K} \subset \text{GL}_d(\mathbb{A}^\infty) \), the homomorphism \( H^*(X_{\mathbb{K}}, M) \to H^*(X_{\lim}, M) \) is injective and its image is equal to the \( \mathbb{K} \)-invariant part \( H^*(X_{\lim}, M)^\mathbb{K} \) of \( H^*(X_{\lim}, M) \). Similar statement holds for \( H^*_c \). It follows from Proposition 14 that the inductive limits \( H^{d-1}(X_{\lim}, \mathbb{Q}) \) and \( H^{d-1}_c(X_{\lim}, \mathbb{Q}) \) are admissible \( \text{GL}_d(\mathbb{A}^\infty) \)-modules, and are isomorphic to the contragradient of \( H_{d-1}(X_{\lim}, \mathbb{Q}) \) and \( H^BM_{d-1}(X_{\lim}, \mathbb{Q}) \), respectively.
7.6.7. Proof of Theorem [16] Since $\text{St}_{d,C}$ is self-contragradient, it follows from the compatibility of the normalized parabolic inductions with taking contragradient that it suffice to prove that any irreducible subquotient of $H^{d-1}_{\text{c}}(X_{\text{lim}}, \bullet, \mathbb{C})$ satisfies the properties in the statement of Theorem [16]. Let $\pi$ be an irreducible subquotient of $H^{d-1}_{\text{c}}(X_{\text{lim}}, \bullet, \mathbb{C})$. Then Proposition [30] combined with the spectral sequence (7.3) shows that there exists a subset $D \subset \{ \pi \}$ isomorphic to a subquotient of $\text{Ind}_{\text{GL}_r}^{\text{GL}_d}(\chi)$ such that $\pi^\infty$ is isomorphic to a subquotient of $\bigotimes_{j=1}^r H^{d_j-1}_{\text{c}}(X_{\text{GL}_{d_j}, F}, \bullet, \mathbb{C})$. Here $r = 2d + 1$, and $d_1, \ldots, d_r \geq 1$ are the integers satisfying $D = \{ d_1, d_1 + d_2, \ldots, d_1 + \cdots + d_{r-1} \}$ and $d_1 + \cdots + d_r = d$. By Proposition [13], $\pi^\infty$ is isomorphic to a subquotient of the non-$\infty$-component of the induced representation from $\text{P}_{d}(\mathbb{A})$ to $\text{GL}_d(\mathbb{A})$ of an irreducible cuspidal automorphic representation $\pi_1 \otimes \cdots \otimes \pi_r$ of $\text{M}_{d}(\mathbb{A})$ whose component at $\infty$ is isomorphic to the tensor product of the Steinberg representations.

It remains to prove the claim of the multiplicity. The Ramanujan-Petersson conjecture proved by Lafforgue [Laf, p.6, Théorème 1] shows that each place $v$ of $F$, the representation $\pi_{i,v}$ is tempered. Hence for almost all places $v$ of $F$, the representation $\pi_v$ of $\text{GL}_d(F_v)$ is unramified and its associated Satake parameters $\alpha_{v,1}, \ldots, \alpha_{v,d}$ have the following property: for each $i$ with $1 \leq i \leq r$, exactly $d_i$ parameters of $\alpha_{v,1}, \ldots, \alpha_{v,d}$ have the complex absolute value $q_v^{a_i/2}$ where $q_v$ denotes the cardinality of the residue field at $v$ and $a_i = \sum_{i < j \leq r} d_j - \sum_{1 \leq j < i} d_j^2$. This shows that the subset $D$ is uniquely determined by $\pi$. It follows from the multiplicity one theorem and the strong multiplicity one theorem that the cuspidal automorphic representation $\pi_1 \otimes \cdots \otimes \pi_r$ of $\text{M}_{d}(\mathbb{A})$ is also uniquely determined by $\pi$.

Hence it suffices to show the following lemma.

**Lemma 33.** The representation $\text{Ind}_{\text{P}_d(F_v)}^{\text{GL}_d(F_v)} \pi_{1,v} \otimes \cdots \otimes \pi_{r,v}$ of $\text{GL}_d(F)$ is of multiplicity free for every place $v$ of $F$.

**Proof.** For $1 \leq i \leq r$, let $\Delta_i$ denote the multiset of segments corresponding to the representation $\pi_{i,v} \otimes \det( )^{a_i/2}$ in the sense of [Ze]. We denote by $\Delta_i^1$ the Zelevinski dual of $\Delta_i$. Let $i_1, i_2$ be integers with $1 \leq i_1 < i_2 \leq r$ and suppose that there exist a segment in $\Delta_i^1$ and a segment in $\Delta_j^1$ which are linked. Since $\pi_{i_1,v}$ and $\pi_{i_2,v}$ are tempered, it follows that $i_2 = i_1 + 1$ and that there exists a character $\chi$ of $F_v^\times$ such that both $\pi_{i_1,v} \otimes \chi$ and $\pi_{i_2,v} \otimes \chi$ are the Steinberg representations. In this case the multiset $\Delta_i^1$ consists of a single segment for $j = 1, 2$ and the unique segment in $\Delta_i^1$ and the unique segment in $\Delta_j^1$ are juxtaposed. Thus the claim is obtained by applying the formula in [Ze, 9.13, Proposition, p.201].

This finishes the proof of Theorem [16].
CHAPTER 8

Universal Modular Symbols

We recall here the definition of universal modular symbols following Ash and Rudolph [As-Ru §2]. Let $F$ be a field and $d \geq 1$. Let $q_1, \ldots, q_d$ be an ordered basis (a basis with the order fixed) of $F^{\oplus d}$. A universal modular symbol $[q_1, \ldots, q_d]$ associated with the ordered basis is then a $(d-2)$-nd homology class of the Tits building $T_{F^{\oplus d}}$ of the algebraic group $\text{SL}_d$ over $F$.

The treatment here may look slightly different from the paper [As-Ru] because we put our emphasis on posets. By definition, the Tits building $T_{F^{\oplus d}}$ is (the classifying space of) the poset of $F$-subspaces of $F^{\oplus d}$. Its barycentric subdivision is then the classifying space of the poset of flags in $F^{\oplus d}$. On the other hand, the boundary of the barycentric subdivision of the standard $(d-1)$-simplex $\Delta_{d-1}$ is the classifying space of the poset of subsets of $\{1, \ldots, d\}$, excluding $\{1, \ldots, d\}$ and $\emptyset$. A choice of an ordered basis will give a morphism from this poset to the poset of flags. This in turn induces a map of homology and the image of the fundamental class of the homology of the boundary of the standard simplex is defined as the universal symbol corresponding to the ordered basis.

We mention some results of [As-Ru] in Section 8.4. Of course, our main aim is to consider the analogue of their Proposition 3.2; this will be covered in Chapters 11.

We use that the universal modular symbols generate $H_{d-2}(T_{F^{\oplus d}})$. Their main result (see Theorem 36) is applicable when our base ring $A$ is a Euclidean domain.

8.1. the Tits building for the special linear groups

Let us recall the definition of the Tits building.

8.1.1. Let $F$ be a field. (We will take $F$ to be our ground global field of positive characteristic for our application.) Let $d \geq 1$ be a positive integer, and $W$ be a $d$-dimensional vector space over $F$. We denote by $Q(W)$ the poset of nonzero proper vector subspaces $0 \subsetneq W_1 \subsetneq W$. Let $\mathcal{P}_{\text{tot}}(Q(W))$ denote the poset of totally ordered finite subsets of $Q(W)$. By definition, the Tits building $T_W$ is the simplicial complex $(Q(W), \mathcal{P}_{\text{tot}}(Q(W)))$ associated with the poset $Q(W)$. In the terminology of [Qu §1], $T_W$ is the classifying space of $Q(W)$.

8.1.2. We use the poset of flags in $W$, which gives rise to the barycentric subdivision of $T_W$. A flag in $W$ is a sequence, for some $1 \leq i \leq d$,

$$0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_i \subsetneq W$$

of $F$-subvector spaces. Let $F(W)$ denote the set of flags in $W$. For two flags $F_1, F_2$, we set $F_1 \leq F_2$ if $F_2$ is a refinement of $F_1$. With this order, $F(W)$ is a poset. We have a canonical identification $F(W) = \mathcal{P}_{\text{tot}}(Q(W))$ of two posets.
We denote by $T'_W$ the simplicial complex $(F(W), \mathcal{P}_{\text{tot}}(F(W)))$ where $\mathcal{P}_{\text{tot}}(F(W))$ is the set of nonempty totally ordered finite subsets of $F(W)$. Then $T'_W$ is the barycentric subdivision of $T_W$. (See proof of Lemma 1.9 in [Gr].)

8.1.3. Let us take an isomorphism $W \cong \mathbb{F}^d$ and regard $W$ as the set of column vectors. Then $\text{GL}_d(F)$ acts on $W$ by multiplication from the left, on the poset $Q(W)$, on $T_W$ and on $T'_W$.

8.1.4. As mentioned in [Fu-Ka-Sh1], p.165, 2.2.3, $F(W)$ is isomorphic as a poset to the set of parabolic subgroups (ordered by inclusion) of any of $\text{GL}_d$, $\text{PGL}_d$, and $\text{SL}_d$ over $F$. This gives the description of $T_W$ as the Tits building of the semisimple algebraic group $\text{SL}_d$ over $F$. (See [As-Ru], p.242, Section 2.)

8.2. the boundary of the first barycentric subdivision of a standard simplex

We now give the simplicial complex which describes the boundary of the first barycentric subdivision of the standard $(d-1)$-simplex. This description is essentially the same as that in [As-Ru], p.243.

8.2.1. Let $B$ be a nonempty finite set. Let $\mathcal{P}'(B)$ denote the set of subsets $J \subset B$ satisfying $J \neq \emptyset, B$. We regard $\mathcal{P}'(B)$ as a partially ordered set with respect to the inclusions.

Let $\mathcal{P}_{\text{tot}}(\mathcal{P}'(B))$ denote the set of non-empty totally ordered subsets of $\mathcal{P}'(B)$. Then the pair $(\mathcal{P}'(B), \mathcal{P}_{\text{tot}}(\mathcal{P}'(B)))$ forms a finite simplicial complex. It is the classifying space of the poset $\mathcal{P}'(B)$.

Let $d = |B|$ denote the cardinality of $B$. Then this simplicial complex is isomorphic to the boundary of the first barycentric subdivision of the standard $(d-1)$-simplex. The set of vertices of this standard $(d-1)$-simplex $\Delta$ is the set $B$.

8.2.2. The following is the picture of the boundary of the first barycentric subdivision of the standard 2-simplex:

8.2.3. an orientation. If $B$ is totally ordered, then the order gives an orientation of $\Delta$. This orientation in turn gives an orientation of the subdivision, and of the boundary of the barycentric subdivision. Let us describe this explicitly.
The number of \((d - 1)\)-dimensional simplices in \((\mathcal{P}'(B), \mathcal{P}_{tot}(\mathcal{P}'(B)))\) is equal to \(d!\). They are the simplices \(\sigma_g\) whose vertices are

\[
\{g(1)\}, \{g(1), g(2)\}, \ldots, \{g(1), g(2), \ldots, g(d - 1)\}
\]

for each \(g \in S_d\) in the \(d\)-th symmetric group.

The orientation of \(\sigma_g\) that we use below (which is the one determined from the orientation of \(\Delta\)) is as follows. If \(g\) is an even permutation, then the orientation is to be given by the increasing order \(\{g(1)\}, \{g(1), g(2)\}, \ldots, \{g(1), g(2), \ldots, g(d - 1)\}\). If \(g\) is odd, then the orientation is the opposite of the order above.

This choice of orientation (or the total order) gives an element (the fundamental class) in

\[
H_{d-2}((\mathcal{P}'(B), \mathcal{P}_{tot}(\mathcal{P}'(B))), \mathbb{Z})
\]

8.3. the universal modular symbols

We define the universal modular symbols of Ash-Rudolph \([\text{As-Ru}]\) Definition 2.1).

8.3.1. Let \(d \geq 1\) be an integer. Let \(B = \{1, \ldots, d\}\) be a totally ordered finite set of cardinality \(d\). Let \(F\) be a field and \(W\) be a \(d\)-dimensional \(F\)-vector space.

Let \(q_1, \ldots, q_d \in W\) be a basis of \(W\).

We define a morphism of posets

\[
\phi = \phi_{q_1, \ldots, q_d} : \mathcal{P}'(B) \to Q(W)
\]

as follows. Let \(1 \leq i \leq d\) and \(J = \{j_1, \ldots, j_i\} \subset B\) belonging to \(\mathcal{P}'(B)\). Then we set

\[
\phi(J) = \langle q_{j_1}, \ldots, q_{j_i} \rangle \subseteq W
\]

to be the vector subspace spanned by \(\{q_{j_1}, \ldots, q_{j_i}\}\).

The morphism of posets induces a morphism of classifying spaces which we also denote by \(\phi = \phi_{q_1, \ldots, q_d}\):

\[
\phi = \phi_{q_1, \ldots, q_d} : ((\mathcal{P}'(B), \mathcal{P}_{tot}(\mathcal{P}'(B)))) \to T_W = (Q(W), \mathcal{P}_{tot}(Q(W))).
\]

8.3.2. We write \([q_1, \ldots, q_d]\) for the image of the fundamental class in

\[
H_{d-2}((F(W), \mathcal{P}_{tot}(F(W))), \mathbb{Z})
\]

by the pushforward by \(\phi_{q_1, \ldots, q_d}\).

For convenience, we set \([q_1, \ldots, q_d]\) = 0 if \(q_1, \ldots, q_d\) do not form a basis.

These elements are called universal modular symbols.

When \(W = F^{\oplus d}\), for \(Q \in \text{GL}_d(F^{\oplus d})\), we write \([Q]\) for the universal modular symbol \([q_1, \ldots, q_d]\) where \(q_i\) is the \(i\)-th row for \(1 \leq i \leq d\).

8.4. Some results of Ash and Rudolph

We record here some results of Ash and Rudolph on universal modular symbols which may or may not hold in our case. The first statements in the case of \(\text{GL}_2\) were given by Manin \([\text{Ma1}]\).
8.4.1. We have a description of a \( \mathbb{Z} \)-basis of the homology.

Proposition 34 (Prop 2.3, p.244, [As-Ru]). Let \( U \) denote the subgroup of \( \text{GL}(W) \) consisting of unipotent upper triangular matrices. Then the symbols \( [Q] \) as \( Q \) runs through \( U \), make up a \( \mathbb{Z} \)-basis of \( H_{d-2}(T_W, \mathbb{Z}) \).

Proof. See loc. cit. \( \square \)

8.4.2. The following proposition gives some basic properties of the universal modular symbols.

Proposition 35 ([As-Ru, Prop 2.2, p.243]). The universal modular symbols enjoy the following properties:
1. It is anti-symmetric.
2. \([aq_1, q_2, \ldots, q_d] = [q_1, \ldots, q_d]\) for any nonzero \( a \in F \).
3. If \( q_1, \ldots, q_{d+1} \) are all nonzero, then
   \[
   \sum_{i=1}^{d+1} (-1)^{i+1} [q_1, \ldots, \hat{q}_i, \ldots, q_{d+1}] = 0.
   \]
4. If \( A \in \text{GL}_d(F) \), then \([AQ] = A \cdot [Q]\), where the dot denotes the action of \( \text{GL}_F(W) \) on the homology of \( T_W \).

Proof. See [As-Ru, Prop 2.2, p.243]. \( \square \)

8.4.3. We mention their main theorem. Here \( T_d \) is the Tits building for subspaces in \( L^n \) with \( L \) the field of fractions of a Euclidean domain \( \Lambda \).

Theorem 36 ([As-Ru, Thm 4.1, p.247]). As \( Q \) runs over \( \text{SL}(d, \Lambda) \), the universal modular symbols \( [Q] \) generate \( H_{d-2}(T_d; \mathbb{Z}) \).

Proof. See [As-Ru, Thm 4.1] \( \square \)

Ash and Rudolph use this theorem for \( \Lambda = \mathbb{Z} \). For our setup, the ring \( A \) is the analogue of \( \mathbb{Z} \) but this is in general not a Euclidean domain. We do not know to what extent this type of statement holds true.
CHAPTER 9

On finite $p$-subgroups of arithmetic subgroups

9.1. Introduction

The result of this chapter will be used in Section 11.5.2.

9.1.1. We are interested in the group homology of stabilizer groups of arithmetic subgroup action on the Bruhat-Tits building $BT_*$. These homology groups appear as the $E_1$-terms of the spectral sequence in Section 11.5. In order to obtain the estimate of the exponent, we do not study the differentials of the spectral sequence directly, but we give an estimate of the exponent of the $E_1$-terms, using the fact that group homology of a finite group is killed by the order of the group. In this chapter, we treat the $p$-part.

9.1.2. Let $\Gamma$ be an arithmetic subgroup acting on $BT_*$. Suppose that $\Gamma$ is a pro-$p$ group, where $p$ is the characteristic of $F$. Let $\sigma$ be a simplex of $BT_*$ and let $\Gamma_\sigma$ be the stabilizer of $\sigma$. Then $\Gamma_\sigma$ is a finite $p$-subgroup of $GL_d(F)$. We will bound the exponent of the homology groups of $\Gamma_\sigma$. We do not use the fact that it is a stabilizer group, but merely the fact that it is a $p$-subgroup of $GL_d(F)$.

In Lemma 38 below, it is shown that there exists a flag in $F^{\oplus d}$ which is stabilized by $\Gamma_\sigma$. This in turn gives a filtration of the stabilizer group by normal subgroups such that successive quotients are either trivial or an elementary abelian $p$-group (Corollary 39). Since the group homology of an elementary abelian $p$-group is killed by $p$, we obtain a certain bound of the order of the homology groups by the length of the filtration.

9.2. Lemmas

**Lemma 37.** Let $H$ be a finite $p$-group. Suppose that there exists a decreasing filtration

$$H = H^0 \supset H^1 \supset \cdots \supset H^{\ell - 1} \supset H^\ell = \{1\}$$

of $H$ by normal subgroups of $H$ such that $H^{i-1}/H^i$ is an elementary abelian $p$-group for $i = 1, \ldots, \ell$. Let $R$ be a principal ideal domain and let $\chi : H \to R^\times$ be a character of $H$. Then for any integer $s \geq 1$, the $s$-th homology group $H_s(H, \chi)$ is an abelian group killed by $p^{1+s(\ell-1)}$.

**Proof.** First we prove the claim for $\ell = 1$. In this case $H$ is an elementary abelian $p$-group. Since the claim follows from the Künneth theorem if $\chi$ is trivial, we may assume that $\chi$ is non-trivial. Let $H'$ denote the kernel of $\chi$. Since $H$ is an elementary abelian $p$-group, there exists a cyclic subgroup $H'' \subset H$ of order $p$ such that the map $H' \times H'' \to H$ given by the multiplication in $H$ is an isomorphism of groups. Via this isomorphism, the character $\chi$ is regarded as the external tensor product over $R$ of the trivial character of $H'$ and the restriction $\chi|_{H''}$ of $\chi$ to $H''$. 

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Hence it follows from the splitting of the short exact sequence in the Künneth theorem that \( H_s(H, \chi) \) is killed by \( p \).

We prove the claim for \( \ell > 1 \) by induction on \( \ell \). Let us consider the Hochschild-Serre spectral sequence (cf. [Br p.171, VII, (6.3)])

\[
E^2_{s,t} = H_s(H/H^{\ell-1}, H_t(H^{\ell-1}, \chi)) \implies H_{s+t}(H, \chi).
\]

The claim for the elementary abelian \( H^{\ell-1} \) shows that \( E^2_{s,t} \) is killed by \( p \) for \( t \geq 1 \).

Since the claim is known for \( H/H^{\ell-1} \) by the inductive hypothesis, \( E^2_{s,0} \) is killed by \( p^{1+s(\ell-2)} \). Hence \( E^2_{r,0} \) is killed by \( p^{1+q(\ell-2)} \) and \( E^2_{r,-i} \) are killed by \( p \) for \( i = 1, \ldots, s \) and for any \( r \geq 2 \). Thus the spectral sequence above shows that \( H_s(H, \chi) \) is killed by \( p^{1+s(\ell-1)} \cdot \prod_{i=1}^{s} p = p^{1+s(\ell-1)} \).

**Lemma 38.** Let \( V \) be a non-zero, finite dimensional \( F \)-vector space, and \( H \) a finite \( p \)-group which acts \( F \)-linearly on \( V \) from the right. Then there exists a flag \( 0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{\ell} = V \) in \( V \) by \( F \)-linear subspaces such that \( V_i \cdot H = V_i \) and \( H \) acts trivially on \( V_i/V_{i-1} \) for \( i = 1, \ldots, \ell \).

**Proof.** First we prove that the \( H \)-invariant part \( V^H \) is non-zero by induction on the order \( p^m \) of \( H \). If \( m \leq 1 \), then \( H \) is generated by an element \( h \in H \). Let \( \rho(h) \in \text{GL}_F(V) \) denote the action of \( h \) on \( V \). Since \( h^p = 1 \), we have \( (\rho(h) - \text{id}_V)^p = 0 \). This implies that 1 is an eigenvalue of \( \rho(h) \). Hence \( V \) has a non-zero \( H \)-invariant vector. Suppose that \( m \geq 2 \). Since any nontrivial finite \( p \)-group has a non-trivial center, there exists an element \( h \) of order \( p \) in the center of \( H \). Let \( W \subset V \) denote the subspace of \( h \)-invariant vectors. Then \( W \neq 0 \) and \( H/(\langle h \rangle) \) acts on \( W \). Hence by inductive hypothesis, \( V^H = W^{H/(\langle h \rangle)} \) is non-zero.

Next we prove the claim by induction on \( d = \dim_F V \). The claim is clear when \( d = 1 \) since \( V^H \neq \{0\} \) implies that \( H \) trivially acts on \( V \). Suppose that \( d \geq 2 \). Set \( W = V/V^H \). If \( W = \{0\} \), then \( H \) acts trivially on \( V \) and the claim is clear. Suppose otherwise. Then the action of \( H \) on \( V \) induces a right action of \( H \) on \( W \).

By inductive hypothesis, there exists a flag \( 0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{\ell'} = W \) in \( W \) by \( F \)-linear subspaces such that \( W_i \cdot H = W_i \) and \( H \) acts trivially on \( W_i/W_{i-1} \) for \( i = 1, \ldots, \ell' \). Then the preimage \( V_i \) of \( W_{i-1} \) under the quotient map \( V \to W \) for \( i = 1, \ldots, \ell = \ell' + 1 \) gives a desired flag in \( V \).

**Corollary 39.** Let \( H \subset \text{GL}_d(F) \) be a finite subgroup that is a finite \( p \)-group. Then there exists an integer \( \ell \leq d \) and a decreasing filtration

\[
H = H^0 \supset H^1 \supset \cdots \supset H^{\ell-2} \supset H^{\ell-1} = \{1\}
\]

of \( H \) by normal subgroups of \( H \) such that \( H^{i-1}/H^i \) is either trivial or an elementary abelian \( p \)-group for \( i = 1, \ldots, \ell - 1 \).

**Proof.** It follows from Lemma 38 that there exists a flag \( 0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{\ell} = F^d \) in \( F^d \) by \( F \)-linear subspaces such that \( V_i \cdot H = V_i \) and \( H \) acts trivially on \( V_i/V_{i-1} \) for \( i = 1, \ldots, \ell \). Since \( F^d \) is \( d \)-dimensional, we have \( \ell \leq d \). For \( i = 0, \ldots, \ell - 1 \), set \( H^i = \text{Ker}(H \to \text{GL}_F(V_{i+1})) \subset H \). Then \( H^i \) is a normal subgroup of \( H \) and we have \( H = H^0 \supset H^1 \supset \cdots \supset H^{\ell-1} = \{1\} \). Let \( 0 \leq \ell \leq \ell - 2 \). Then for \( h \in H^i \), the \( F \)-linear map \( V_i+2 \to V_{i+1} \) that sends \( v \in V \) to \( vh - v \) induces an \( F \)-linear map \( \varphi_h : V_{i+2}/V_{i+1} \to V_{i+1} \). Let \( \varphi : H^i \to \text{Hom}_F(V_{i+2}/V_{i+1}, V_{i+1}) \) denote the map that sends \( h \in H^i \) to \( \varphi_h \). Then one can check easily that the map \( \varphi \) is a homomorphism of groups and that \( H^{i+1} \) is equal to the kernel of \( \varphi \). This shows that \( H^i/H^{i+1} \) is an elementary abelian \( p \)-group since it is isomorphic
to a subgroup of $\text{Hom}_F(V_{i+2}/V_{i+1}, V_{i+1})$. Hence the groups $H^0, \ldots, H^{\ell-1}$ have the desired property.

\[\square\]

**Corollary 40.** Let $H \subset \text{GL}_d(F)$ be a finite subgroup that is a finite $p$-group. Let $\chi : H \to \mathbb{Z}^\times$ be a character. Then the homology groups $H_s(H, \chi)$ is killed by $p^{1+s(d-2)}$ for $s \geq 1$.

**Proof.** This follows from Lemma 37 and Corollary 39. \[\square\]
CHAPTER 10

Some spectral sequences

In this section, we mention two spectral sequences that we use in Chapter 11. They are ordinary ones that are found in textbooks such as [Br] or [Wei]. Because there are minor differences, we record them here. The reader may safely skip this chapter.

10.1. Cellular structure on some products

Let $Y_\bullet$ be a simplicial complex and $Y'_\bullet$ a simplicial set. The geometric realizations $|Y_\bullet|$ and $|Y'_\bullet|$ have canonical structures of CW complexes, induced by the triangulations given by the simplices in $Y_\bullet$ and $Y'_\bullet$, respectively. The complex computing the cellular (co)homology groups of the CW complex $|Y_\bullet|$ is identical to the complex in Section 2.3 computing the (co)homology groups of $Y_\bullet$.

Suppose that $|Y_\bullet|$ is locally compact. Then, by [Hatc, Theorem A.6, p.525], the structures of CW complexes on $|Y_\bullet|$ and $|Y'_\bullet|$ naturally induce a structure of CW complex on the direct product $|Y_\bullet| \times |Y'_\bullet|$ of topological spaces, where the cells of dimension $i$ are indexed by the disjoint union

$$\prod_{j=0}^{i} Y_j \times Y'^{\text{nd}}_{i-j}$$

for any integer $i \geq 0$, where $Y^{\text{nd}}_{i-j} \subset Y'_{i-j}$ denotes the set of non-degenerate $(i-j)$-simplices in $Y'_\bullet$, and the closure of any cell whose index is in $Y_j \times Y'^{\text{nd}}_{i-j}$ is isomorphic to the direct product of a $j$-simplex and an $(i-j)$-simplex equipped with a natural structure of CW complex. We call this structure of CW complex on $|Y_\bullet| \times |Y'_\bullet|$ the product cellular structure on $|Y_\bullet| \times |Y'_\bullet|$.

For a CW complex $Z$, we denote by $C^{\text{cell}}(Z, \mathbb{Z})$ the complex computing the cellular homology groups of $Z$. Since the two projections $|Y_\bullet| \times |Y'_\bullet| \rightarrow |Y_\bullet|$ and $|Y_\bullet| \times |Y'_\bullet| \rightarrow |Y'_\bullet|$ are cellular maps, they induce maps $C^{\text{cell}}(|Y_\bullet| \times |Y'_\bullet|, \mathbb{Z}) \rightarrow C_\bullet(|Y_\bullet|, \mathbb{Z})$ and $C^{\text{cell}}(|Y_\bullet| \times |Y'_\bullet|, \mathbb{Z}) \rightarrow C_\bullet(|Y'_\bullet|, \mathbb{Z})$ of complexes. We will later use the terminology in this paragraph when $Y_\bullet = B\Gamma^\bullet$ and $Y'_\bullet = E\Gamma^\bullet$. In this case, we note that $Y_\bullet = B\Gamma^\bullet$ is locally compact.

10.2. Equivariant homology

Let $\Gamma \subset \text{GL}_d(K)$ be an arithmetic subgroup. We define the simplicial set (not a simplicial complex) $E\Gamma^\bullet$ as follows. We define $E\Gamma_n = \Gamma^{n+1}$ to be the $(n+1)$-fold direct product of $\Gamma$ for $n \geq 0$. The set $\Gamma^{n+1}$ is naturally regarded as the set of maps of sets $\text{Map}(\{0, \ldots, n\}, \Gamma)$ and from this one obtains naturally the structure of a simplicial set. We let $|E\Gamma^\bullet|$ denote the geometric realization of $E\Gamma^\bullet$. Then $|E\Gamma^\bullet|$ is contractible. We let $\Gamma$ act diagonally on each $E\Gamma_n$ ($n \geq 0$). The induced action on $|E\Gamma^\bullet|$ is free.
Let $M$ be a topological space on which $\Gamma$ acts. The diagonal action of $\Gamma$ on $M \times |E\Gamma|_i$ is free. We let $H^i_\Gamma(M, B) = H_i(\Gamma \backslash (M \times |E\Gamma|_i), B)$ where $B$ is a coefficient ring, and call it the equivariant homology of $M$ with coefficients in $B$. We also use the relative version, and define equivariant cohomology in a similar manner.

### 10.3. the Lyndon-Hochschild-Serre spectral sequence

Let $Y_\bullet \subset Z_\bullet$ be simplicial complexes with compatible $\Gamma$-action. We have (see [Br] p.172, VII 7) the following spectral sequence.

$$E^2_{p,q} = H_p(\Gamma, H_q(Z_\bullet, Y_\bullet; \mathbb{Z})) \Rightarrow H^q_{\Gamma}(Z_\bullet, Y_\bullet; \mathbb{Z})$$

### 10.4. Another spectral sequence

We use another spectral sequence which also converges to equivariant homology groups. We make a slight change from [Br VII 7] because we use particular cell structures. The reader may skip this section.

#### 10.4.1. Let $\Gamma$ be an arithmetic subgroup. We let $Y_\bullet$ be a simplicial complex with $\Gamma$-action. We consider the product $|Y_\bullet| \times |E\Gamma|_i$ as in Section 10.1.

Since $\Gamma$ acts freely on the set $E\Gamma_i$ for every $i \geq 0$, it also acts freely on the set of $i$-dimensional cells with respect to the product cellular structure on $|Y_\bullet| \times |E\Gamma|_i$, where $\Gamma$ acts diagonally on the product $|Y_\bullet| \times |E\Gamma|_i$.

Hence the product cellular structure on $Y_\bullet \times |E\Gamma|_i$ induces a structure of CW complex on the quotient $\Gamma \backslash |Y_\bullet| \times |E\Gamma|_i$, which enable us to consider the complex $C^\text{cell}_\Gamma(\Gamma \backslash |Y_\bullet| \times |E\Gamma|_i)$ computing the cellular homology groups of $\Gamma \backslash |Y_\bullet| \times |E\Gamma|_i$.

#### 10.4.2. Consider the quotient maps

$$|Y_\bullet| \to \Gamma \backslash |Y_\bullet|$$

and

$$|Y_\bullet| \times |E\Gamma|_i \to \Gamma \backslash |Y_\bullet| \times |E\Gamma|_i.$$ 

These induce morphisms of complexes

$$C^\text{cell}_\Gamma(|Y_\bullet|, \mathbb{Z}) \to C^\text{cell}_\Gamma(\Gamma \backslash |Y_\bullet|, \mathbb{Z})$$

and

$$C^\text{cell}_\Gamma(|Y_\bullet| \times |E\Gamma|_i, \mathbb{Z}) \to C^\text{cell}_\Gamma(\Gamma \backslash |Y_\bullet| \times |E\Gamma|_i, \mathbb{Z}).$$

It is easy to see that the induced maps

$$C^\text{cell}_\Gamma(|Y_\bullet|, \mathbb{Z})_\Gamma \cong C^\text{cell}_\Gamma(\Gamma \backslash |Y_\bullet|, \mathbb{Z})$$

and

$$C^\text{cell}_\Gamma(|Y_\bullet| \times |E\Gamma|_i, \mathbb{Z})_\Gamma \cong C^\text{cell}_\Gamma(\Gamma \backslash |Y_\bullet| \times |E\Gamma|_i, \mathbb{Z}),$$

where the subscript $\Gamma$ denotes $\Gamma$-coinvariants, are isomorphisms.

#### 10.4.3. Since $|E\Gamma|_i$ is contractible, the complex $C^\text{cell}_\Gamma(|E\Gamma|_i, \mathbb{Z})$ of $\mathbb{Z}[\Gamma]$-modules is quasi-isomorphic to $\mathbb{Z}$ with trivial $\Gamma$-action, regarded as a complex concentrated at degree 0.

Since $\Gamma$ acts freely on the set of $i$-dimensional cells in $|E\Gamma|_i$ for every $i \geq 0$, the complex $C^\text{cell}_\Gamma(|E\Gamma|_i, \mathbb{Z})$ gives a $\mathbb{Z}[\Gamma]$-free resolution of $\mathbb{Z}$ with trivial $\Gamma$-action. (We note that $C^\text{cell}_\Gamma(|E\Gamma|_i, \mathbb{Z})$ is not identical to the standard $\mathbb{Z}[\Gamma]$-free resolution of $\mathbb{Z}$, since the degenerate simplices in $E\Gamma$ are removed in $C^\text{cell}_\Gamma(|E\Gamma|_i, \mathbb{Z})$.)
10.4.4. **By construction, we can identify the complex** $C^\bullet_{ullet'}(\vert Y \vert \times |E\Gamma|, \mathbb{Z})$

with the simple complex associated with the double complex

$$C_{\bullet,\bullet'} = C^\bullet_{\bullet'}(|Y|, \mathbb{Z}) \otimes_{\mathbb{Z}} C_{\bullet'}(|E\Gamma|, \mathbb{Z})$$

of $\mathbb{Z}[\Gamma]$-modules (with respect to a suitable sign convention).

10.4.5. **Let** $C_{\bullet,0}$ **denotes the double complex whose entry at the bidegree** $(i,j)$ **is equal to** $C_{i,0}$ **if** $j = 0$, **and is zero otherwise.**

The projection map $|Y| \times |E\Gamma| \to |Y|$ **induces the map of complexes of** $\mathbb{Z}[\Gamma]$-modules:

$$C^\bullet_{\bullet'}(|Y|, \mathbb{Z}) \to C^\bullet_{\bullet'}(|Y|, \mathbb{Z}).$$

This map is induced by the projection map $C_{\bullet,\bullet'} \to C_{\bullet,0}$

of double complexes.

10.4.6. **As we remarked in the previous paragraph, the complex** $C^\bullet_{\bullet'}(|E\Gamma|, \mathbb{Z})$

gives a $\mathbb{Z}[\Gamma]$-free resolution of $\mathbb{Z}$. **Hence for each** $j \geq 0$, **the complex** $C^\bullet_{\bullet'}(|Y|, \mathbb{Z}) \otimes_{\mathbb{Z}} C^\bullet_{\bullet'}(|E\Gamma|, \mathbb{Z})$

gives a $\mathbb{Z}[\Gamma]$-free resolution of the $\mathbb{Z}[\Gamma]$-module $C^\bullet_{\bullet'}(|Y|, \mathbb{Z})$. **Hence the double complex** $(C_{\bullet,\bullet'})^r$

induces a spectral sequence

$$E^1_{s,t} = H_t(\Gamma, C^s_{\bullet'}(|Y|, \mathbb{Z})) \Rightarrow H^F_{s+t}(Y, \mathbb{Z})$$

where

$$H^F_{s+t}(Y, \mathbb{Z}) = H_{s+t}(C_{\bullet,\bullet'}(|Y|, \mathbb{Z})|_{\Gamma = 1} \otimes_{\mathbb{Z}[\Gamma]} C^s_{\bullet'}(|Y|, \mathbb{Z})).$$

By definition, we have

$$E^2_{s,0} = H_s(C^\bullet_{\bullet'}(|Y|, \mathbb{Z})|_{\Gamma = 1} \otimes_{\mathbb{Z}[\Gamma]} C^s_{\bullet'}(|Y|, \mathbb{Z})) \cong H_s(\Gamma \setminus Y, \mathbb{Z}).$$

10.4.7. **For each** $i \geq 0$ **and for each** $\sigma \in \Gamma \setminus Y$, **let us choose a representative** $\tilde{\sigma} \in Y_i$ **of** $\sigma$. **Let** $\Gamma_{\tilde{\sigma}} \subset \Gamma$ **denote the stabilizer of** $\tilde{\sigma}$. **As we mentioned in Section 10.1.3, it follows from the conditions** (1) **and** (3) **that** $\Gamma_{\tilde{\sigma}}$ **is a finite subgroup of** $\Gamma$. **The group** $\Gamma_{\tilde{\sigma}}$ **acts non-trivially on the set** $O(\tilde{\sigma})$ **of orientations of** $\tilde{\sigma}$. **Hence we obtain a character** $\chi_{\tilde{\sigma}} : \Gamma_{\tilde{\sigma}} \to \{\pm 1\}$ **where an element** $\gamma \in \Gamma_{\tilde{\sigma}}$ **is sent to** $-1$ **under** $\chi$ **if and only if** $\gamma$ **acts non-trivially on** $O(\tilde{\sigma})$. **Then it follows from the construction that for each** $i \geq 0$, **the** $\mathbb{Z}[\Gamma]$-module $C^\bullet_{\bullet'}(|Y|, \mathbb{Z})$

is isomorphic to the direct sum

$$C^\bullet_{\bullet'}(|Y|, \mathbb{Z}) \cong \bigoplus_{\sigma \in \Gamma \setminus Y_i} \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_{\tilde{\sigma}}]} \chi_{\tilde{\sigma}}$$

where $\chi_{\tilde{\sigma}} = \mathbb{Z}$ **with the action of** $\Gamma_{\tilde{\sigma}}$ **given by** $\chi_{\tilde{\sigma}}$. **By Shapiro’s lemma, this induces an isomorphism**

$$E^1_{s,t} \cong \bigoplus_{\sigma \in \Gamma \setminus Y_s} H_t(\Gamma_{\tilde{\sigma}}, \chi_{\tilde{\sigma}})$$

of abelian groups.

10.4.8. **Let** $Y_\bullet$ **be as above and let** $Z_\bullet$ **be a subsimplicial complex with** $\Gamma$-action. **Then we obtain the following spectral sequence in a similar manner:**

$$E^1_{s,t} = H_t(\Gamma, C^\bullet_{\bullet'}(|Y|, \mathbb{Z})/C^\bullet_{\bullet'}(|Z|, \mathbb{Z})) \Rightarrow H^F_{s+t}(Y, Z, \mathbb{Z}).$$

The $E^1$ terms are

$$E^1_{s,t} \cong \bigoplus_{\sigma \in \Gamma \setminus Y_s \setminus \Gamma \setminus Z_s} H_t(\Gamma_{\tilde{\sigma}}, \chi_{\tilde{\sigma}}).$$
CHAPTER 11

Proof for universal modular symbols

In this chapter, we prove Propositions 49, 50, 51, 53 and Corollary 52. These are the statements of our main theorem (Theorem 12) with our modular symbols replaced by (the image of) the universal modular symbols. We show that our modular symbols coincide with the universal modular symbols in Chapter 12.

The content of this chapter is outlined in Section 1.5.

11.1. Modular symbols for automorphic forms

Modular symbols are elements of $H_{d-1}(\Gamma \backslash B\mathcal{T}_\bullet, \mathbb{Z})$ for some arithmetic subgroup $\Gamma$. For a possible application to automorphic forms, we may look at all the connected components. Recall the space $A_{K,\text{St},\mathbb{Z}}$. We have a non-adelic description of $X_{K,\bullet}$. We set $J_K = GL_d(F) \backslash GL_d(A_\infty)/K$, take a set $\{g_j\}$ of representatives of $J_K$, and set $\Gamma_j = GL_d(F) \cap g_j K g_j^{-1}$. Then

$$X_{K,\bullet} \cong \bigsqcup_{j \in J_K} \Gamma_j \backslash B\mathcal{T}_\bullet.$$  

For a compact open subgroup $K \subset GL_d(A_F)$, we set

$$\text{MS}(K) = \bigoplus_j \text{MS}(\Gamma_j) \subset \bigoplus_j H_{d-1}(\Gamma_j \backslash B\mathcal{T}_\bullet, \mathbb{Z}) \cong H_{d-1}(X_{K,\bullet}, \mathbb{Z}) = A_{K,\text{St},\mathbb{Z}}.$$  

This is the space of modular symbols for $K$-invariant automorphic forms $A_{K,\text{St},\mathbb{Z}}$.

We are interested in the size of the cokernel of the map $\text{MS}(K) \subset A_{K,\text{St},\mathbb{Z}}$. For this, it suffices to study the cokernel of $\text{MS}(\Gamma) \subset H_{d-1}(\Gamma \backslash B\mathcal{T}_\bullet, \mathbb{Z})$ for an arithmetic group $\Gamma$. (Because each $\Gamma_j$ is an arithmetic subgroup.)

11.2. Connection with the Tits building

Let $\Gamma$ be an arithmetic subgroup. By definition, there exists a compact open subgroup $K \subset GL_d(A)$ such that $\Gamma = K \cap GL_d(F)$.

Recall the non-adelic description from Section 6.1. We set $J_K = GL_d(F) \backslash GL_d(A_\infty)/K$, take a set $\{g_j\}$ of representatives of $J_K$, and set $\Gamma_j = GL_d(F) \cap g_j K g_j^{-1}$. Then

$$X_{K,i} \cong \bigsqcup_{j \in J_K} \Gamma_j \backslash B\mathcal{T}_i.$$  

Let $e \in \Gamma \subset GL_d(A)$ be the identity element. Then $\Gamma = \Gamma_e$. We will identify $\Gamma \backslash B\mathcal{T}_\bullet$ with a connected component $\Gamma_e \backslash B\mathcal{T}_\bullet \subset X_{K,\bullet}$. We also consider $B\mathcal{T}_\bullet$ as a subset $B\mathcal{T}_\bullet \subset X_{K,\bullet}$ by identifying it with $\{e\} \times B\mathcal{T}_\bullet$. We have a map

$$B\mathcal{T}_\bullet \subset X_{K,\bullet} \to X_{K,\bullet}.$$
11. Proof for Universal Modular Symbols

which factors through $\Gamma \backslash B T_\bullet$.

Let $\alpha$ be a positive real number. Let $D \subset \{1, \ldots, d-1\}$ be a subset. Let $f = [0 \subset V_0 \subset \cdots \subset V_{r-1} \subset F_{\overline{d}}] \in \text{Flag}_D$ be a flag. We defined a subsimplex $\tilde{X}^{(\alpha), D, f}$ in Section 7.2. We define

$$BT_{\bullet}^{(\alpha), D, f} = BT_{\bullet} \cap \tilde{X}^{(\alpha), D, f} \subset \tilde{X}^{(\alpha)}_{K_{\bullet}}.$$

We set

$$BT_{\bullet}^{(\alpha)} = \bigcup_{D, f} BT_{\bullet}^{(\alpha), D, f}.$$

Lemma 41. The space $BT_{\bullet}^{(\alpha), D, f_0}$ is non-empty and contractible.

Proof. This is Lemma 29 stated for $BT_{\bullet}$. □

Recall we defined $T_{F_{\overline{d}}}$ to be the Tits building of $\text{SL}_d$ over $F$.

Corollary 42 (see Corollary 4.2 [Gr]). There is a $\Gamma$-equivariant homotopy equivalence

$$|BT_{\bullet}^{(\alpha)}| \sim |T_{F_{\overline{d}}}|.$$

Proof. The proof is the same as that in loc. cit. using Lemma 41 above. □

11.3. the Solomon-Tits theorem

Theorem 43 (Solomon-Tits). If $d \geq 2$, the Tits building $T_{F_{\overline{d}}}$ has the homotopy type of bouquet of $d-2$ spheres.

Proof. See [Gr] Thm 5.1]. □

Lemma 44. We have

$$H_i(BT_\bullet \cdot BT_\bullet^{(\alpha)}; \mathbb{Z}) \cong \begin{cases} H_{d-2}(T_{F_{\overline{d}}}, \mathbb{Z}) & i = d-1, \\ 0 & i \neq d-1. \end{cases}$$

Proof. Since $BT_\bullet$ is contractible (see [Gr] Thm 2.1), the claim follows from Corollary 42 and the Solomon-Tits theorem (Theorem 43). □

11.4. the Lyndon-Hochschild-Serre spectral sequence

We have the following spectral sequence.

$$E^2_{p, q} = H_p(\Gamma, H_q(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z})) \Rightarrow H^\Gamma_{p+q}(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z})$$

By Lemma 44, we have $E^2_{p, q} = 0$ unless $q = d-1$. Hence the spectral sequence collapses at $E^2$ and we have moreover

$$E^2_{0, d-1} \cong E_{d-1}.$$

Composing with the canonical surjection to the coinvariants, we obtain a surjection

$$H_{d-1}(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z}) \rightarrow H_0(\Gamma, H_{d-1}(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z})) \cong H_{d-1}^\Gamma(BT_\bullet, BT_\bullet^{(\alpha)}; \mathbb{Z}).$$

11.5. the second spectral sequence

Let us write $X = BT_\bullet$ and $X' = BT_\bullet^{(\alpha)}$ for short.
11.5. The second spectral sequence

11.5.1. We use the following spectral sequence ([Br §VII 7], see also Section 10.4)

\[ E_1^{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_{\sigma}, \chi_{\sigma}) \Rightarrow H^{\Gamma}_{p+q}(X, X'; Z) \]

where \( \Sigma_p \) is the set of \( p \)-simplices of \( X \setminus X' \). Note that by definition

\[ E_2^{i,0} = H_i(X, X'; Z) \]

for \( 0 \leq i \leq d \).

Because \( E_1^{p,q} = 0 \) for \( p \leq -1 \), we have

\[ E_\infty^{d-1,0} = \cdots = E_d^{d-1,0}. \]

Because \( E_1^{p,q} = 0 \) for \( q \leq -1 \), we have

\[ E_k^{d-1,0} = \text{Ker} \, d^{k-1} \]

where \( d^{k-1} : E_{d-1}^k \to E_{d-1+k,k-1}^k \) is the differential.

Composing with the canonical map \( E_{d-1}^d \to E_{d-1,0}^d \), we obtain a homomorphism

\[ H^{\Gamma}_{d-1}(X, X'; Z) \to H_{d-1}(\Gamma \setminus X, \Gamma \setminus X'; Z) \]

whose image is \( E_{d-1,0}^d \).

11.5.2. Let us give some estimate on the size of the image \( E_{d-1,0}^d \). Let \( e_{p,q}^r \) denote the exponent of \( E_{p,q}^r \). For each \( k \), we have an exact sequence

\[ E_{d-1}^{k+1} \to E_{d-1}^k \xrightarrow{d^{k-1}} E_{d-1+k,k-1}^k \]

hence

\[ e_{d-1+k,k-1}^k E_{d-1,0}^k \subset \text{Ker} \, d^{k-1} = E_{d-1,0}^{k+1} \subset E_{d-1,0}^k \]

Therefore

\[ \prod_k e_{d-1+k,k-1}^k E_{d-1,0}^k \subset E_{d-1,0}^d \subset E_{d-1,0}^2 \]

It is clear that \( e_{p,q}^r \) divides \( e_{p,q}^1 \) for all \( p, q, r \), hence

\[ \prod_k e_{d-1+k,k-1}^1 E_{d-1,0}^2 \subset E_{d-1}^d \subset E_{d-1,0}^2 \]

It follows from Corollary 40 that \( E_{s,t}^1 \) is killed by \( p^{1+(d-2)} \) for \( t \geq 1 \). That is, \( e_{s,t}^1 \) divides \( p^{1+(d-2)} \). We therefore have

\[ p^{e(d)} E_{d-1}^2 \subset E_{d-1}^d \subset E_{d-1}^2 \]

where \( e(d) = (d-2) \left( 1 + \frac{(d-1)(d-2)}{2} \right) \).

We arrive at the following lemma.

**Lemma 45.** Suppose \( K \subset \text{GL}_d(\mathbb{A}^\infty) \) be a pro-\( p \) compact open subgroup. Let \( \Gamma = \text{GL}_d(F) \cap K \). Then

\[ H^{\Gamma}_{d-1}(X, X'; Z) \to H_{d-1}(\Gamma \setminus X, \Gamma \setminus X'; Z) \]

is injective and the cokernel is annihilated by \( p^{e(d)} \).
11.6. Corestriction and transfer

11.6.1. Let \( \Gamma \supset \Gamma' \) be arithmetic subgroups. For a \( \mathbb{Z}[\Gamma] \)-module \( M \), we denote by \( \text{cor}_M : M_{\Gamma'} \to M_{\Gamma} \) the map induced by the identity map of \( M \).

We define the transfer map \( \text{tr}_M : M_{\Gamma} \to M_{\Gamma'} \) (cf. [Bru, III, 9. (B), p. 81]) by

\[
\text{tr}(m) = \sum_{g \in \Gamma' \setminus \Gamma} gm^\Gamma
\]

where \( m^\Gamma \) and \( m^{\Gamma'} \) denote the class of \( m \in M \) in the coinvariants \( M_{\Gamma} \) and \( M_{\Gamma'} \) respectively. By definition, the composite \( \text{cor}_M \circ \text{tr}_M \) is equal to the map given by the multiplication by \( [\Gamma : \Gamma'] \).

11.6.2. Let \( M_{2,\bullet} \to M_{1,\bullet} \) be a morphism of complexes of \( \mathbb{Z}[\Gamma] \)-modules. Since \( \text{cor}_M \) and \( \text{tr}_M \) are functorial in \( M \), we have a commutative diagram

\[
\begin{array}{ccc}
(M_{2,\bullet})_{\Gamma} & \xrightarrow{\text{tr}_{M_{2,\bullet}}} & (M_{2,\bullet})_{\Gamma'} \\
\downarrow & & \downarrow \\
(M_{1,\bullet})_{\Gamma} & \xrightarrow{\text{tr}_{M_{1,\bullet}}} & (M_{1,\bullet})_{\Gamma'}
\end{array}
\]

of complexes of abelian groups.

11.6.3. Let \( Y_\bullet \supset Z_\bullet \) be simplicial complexes with (compatible) \( \Gamma \)-actions. Set \( M_{1,\bullet} = C^\text{cell}(Y_\bullet, Z_\bullet)/C^\text{cell}(Z_\bullet, Z_\bullet) \) and \( M_{2,\bullet} = C^\text{cell}(Y_\bullet \times |E\Gamma_\bullet|, Z_\bullet \times |E\Gamma_\bullet|) \).

The quotient maps of spaces give a morphism \( M_{2,\bullet} \to M_{1,\bullet} \) of complexes.

By taking the \((d-1)\)-st homology, we obtain the following commutative diagram of abelian groups:

\[
\begin{array}{ccc}
H^\Gamma_{d-1}(Y_\bullet, Z_\bullet; \mathbb{Z}) & \xrightarrow{\beta_\Gamma} & H^\Gamma_{d-1}(Y_\bullet, Z_\bullet; \mathbb{Z}) \\
\downarrow \beta_\Gamma & & \downarrow \beta_\Gamma \\
H_{d-1}(\Gamma \setminus Y_\bullet, \Gamma \setminus Z_\bullet; \mathbb{Z}) & \xrightarrow{\beta_\Gamma} & H_{d-1}(\Gamma \setminus Y_\bullet, \Gamma \setminus Z_\bullet; \mathbb{Z})
\end{array}
\]

Here \( \beta_\Gamma \) and \( \beta_{\Gamma'} \) are the maps induced by the quotient maps for the groups \( \Gamma \) and \( \Gamma' \) respectively.

By taking the cokernels of the vertical arrows, we obtain the maps

\[ \text{Coker} \beta_\Gamma \xrightarrow{\alpha_1} \text{Coker} \beta_{\Gamma'} \xrightarrow{\alpha_2} \text{Coker} \beta_\Gamma. \]

By construction, the composite \( \alpha_2 \circ \alpha_1 \) is equal to the map given by the multiplication by \( [\Gamma : \Gamma'] \).

11.6.4. Corollary 46. Let \( v \neq \infty \) be a prime of \( F \), and let \( F_v \) denote the completion of \( F \) at \( v \). Let \( \mathbb{K}_v \) be a pro-\( p \) open compact subgroup of \( \text{GL}_d(F_v) \).

Let us consider the intersection \( \Gamma' = \Gamma \cap \mathbb{K}_v \) in \( \text{GL}_d(F_v) \). Then

\[ H^\Gamma_{d-1}(X, X'; \mathbb{Z}) \to H_{d-1}(\Gamma \setminus X, \Gamma \setminus X'; \mathbb{Z}) \]

is injective and the cokernel is annihilated by \( p^{e(d)}[\Gamma : \Gamma'] \).

Proof. It follows from Lemma 45 that \( \text{Coker} \beta_\Gamma \) is killed by \( p^{e(d)} \). Hence by the discussion above, \( \text{Coker} \beta_\Gamma \) is killed by \( p^{e(d)}[\Gamma : \Gamma'] \). This implies the claim. \( \square \)
11.6.5. We prove the analogue of Theorem [124].

**Corollary 47.** Let $v_0 \neq \infty$ be a prime of $F$ such that the cardinality $q_0$ of the residue field $\kappa(v_0)$ at $v_0$ is smallest among those at the primes $v \neq \infty$. Set $N(d) = \prod_{i=1}^{d}(q_0^i - 1)$.

Then the cokernel of the injective map \[ H^\Gamma_{d-1}(X,X';\mathbb{Z}) \to H_{d-1}(\Gamma\setminus X, \Gamma\setminus X';\mathbb{Z}) \]
is annihilated by $p^{(d)}N(d)$.

**Proof.** Let $\alpha$ be a positive real number. We use the discussion above with $Y_\bullet = X_\bullet$ and $Z_\bullet = X_\bullet^{(\alpha)}$.

Since $\Gamma$ is an arithmetic subgroup, $\Gamma$ is contained, as a subgroup of $\text{GL}_d(F_{v_0})$, in a compact open subgroup of $\text{GL}_d(F_{v_0})$. Let $O_{v_0}$ denote the ring of integers in $F_{v_0}$. Since any maximal compact subgroup of $\text{GL}_d(F_{v_0})$ is conjugate of $\text{GL}_d(O_{v_0})$, there exists $g \in \text{GL}_d(F_{v_0})$ such that $g^{-1}\Gamma g$ is contained in $\text{GL}_d(O_{v_0})$. Let $\kappa(v_0)$ denote the residue field at $v_0$ and choose a $p$-Sylow subgroup $P$ of $\text{GL}_d(\kappa(v_0))$. Let $\Gamma'$ denote the inverse image of $P$ under the composite \[ \Gamma \xrightarrow{f_1} \text{GL}_d(O_{v_0}) \xrightarrow{f_2} \text{GL}_d(\kappa(v_0)), \]
where $f_1$ is the map that sends $\gamma \in \Gamma$ to $g^{-1}\gamma g$ and $f_2$ is the map induced by the ring homomorphism $O_{v_0} \to \kappa(v_0)$. Since $gf_2^{-1}(P)g^{-1}$ is a pro-$p$ compact open subgroup of $\text{GL}_d(F_{v_0})$, it follows from Corollary [46] that the cokernel of the map $\beta$ above is killed by $p^{(d)}[\Gamma : \Gamma']$. Since $[\Gamma : \Gamma']$ divides $[\text{GL}_d(\kappa(v_0)) : P] = N(d)$, the claim follows. \[ \square \]

11.7. the Mittag-Leffler condition

Let $\alpha \in \mathbb{Z}$ be a positive integer. We write $X = \mathcal{B}T_\bullet$ and $X^{(\alpha)} = \mathcal{B}T^{(\alpha)}_\bullet$ for short. Let \[ A^{(\alpha)} = \text{Image}(H_{d-1}(X,X^{(\alpha)}) \to H_0(\Gamma, H_{d-1}(X,X^{(\alpha)})) \to H_{d-1}(\Gamma\setminus X, \Gamma\setminus X^{(\alpha)})) \]

For $\alpha < \alpha'$, the diagram \[ \begin{array}{ccc}
H_{d-1}(X,X^{(\alpha)}) & \longrightarrow & H_{d-1}(\Gamma\setminus X, \Gamma\setminus X^{(\alpha)}) \\
\downarrow & & \downarrow \\
H_{d-1}(X,X^{(\alpha')}) & \longrightarrow & H_{d-1}(\Gamma\setminus X, \Gamma\setminus X^{(\alpha')})
\end{array} \]
is commutative (the maps are those induced by the inclusion map $X^{(\alpha')} \subset X^{(\alpha)}$). Since the left vertical map is an isomorphism (both are isomorphic to $H_{d-2}(T_{F@d}, \mathbb{Z})$), we obtain that $A^{(\alpha)}$ surjects to $A^{(\alpha')}$. Hence the projective system \[ \{A^{(\alpha)}\}_\alpha \]
satisfies the Mittag-Leffler condition.

We set \[ C^{(\alpha)} = \text{Coker}(H_{d-1}(X,X^{(\alpha)}) \to H_0(\Gamma, H_{d-1}(X,X^{(\alpha)})) \to H_{d-1}(\Gamma\setminus X, \Gamma\setminus X^{(\alpha)})) \]
so that we have an exact sequence \[ 0 \to A^{(\alpha)} \to H_{d-1}(\Gamma\setminus X, \Gamma\setminus X^{(\alpha)}) \to C^{(\alpha)} \to 0 \]
for each $\alpha$. Note that each $C^{(\alpha)}$ is a finite abelian group.
Because the Mittag-Leffler condition is satisfied, the following sequence
\[ 0 \to \lim_{\alpha} A^{(\alpha)} \to \lim_{\alpha} H_{d-1}(\Gamma \setminus X, \Gamma \setminus X^{(\alpha)}) \to \lim_{\alpha} C^{(\alpha)} \to 0 \]
is also exact (see [Wei] p.83).

It follows that the cokernel of the map
\[ H_{d-2}(T_{F \otimes \epsilon}, \mathbb{Z}) \to \lim_{\alpha} H_{d-1}(\Gamma \setminus \mathcal{B}_T \cdot, \Gamma \setminus \mathcal{B}_T^{(\alpha)}; \mathbb{Z}) \]
is annihilated by the least common multiple of the exponents of \( C^{(\alpha)} \). We turn the corollaries above into the following propositions.

**11.8. Proof**

11.8.1.

**Lemma 48.** We have
\[ H_{BM}^{d-1}(\Gamma \setminus \mathcal{B}_T \cdot, \mathbb{Z}) \cong \lim_{\alpha} H_{d-1}(\Gamma \setminus \mathcal{B}_T \cdot, \Gamma \setminus \mathcal{B}_T^{(\alpha)}; \mathbb{Z}). \]

**Proof.** There is a canonical map \( H_{BM}^{d-1}(\Gamma \setminus \mathcal{B}_T \cdot, \mathbb{Z}) \to H_{d-1}(\Gamma \setminus \mathcal{B}_T \cdot, \Gamma \setminus \mathcal{B}_T^{(\alpha)}; \mathbb{Z}) \) which assemble to give a map to the limit. Then the claim follows from Lemma 27.

11.8.2. We consider the composite of the map (11.1) and the map in Lemma 48:
\[ H_{d-2}(T_{F \otimes \epsilon}, \mathbb{Z}) \to H_{BM}^{d-1}(\Gamma \setminus \mathcal{B}_T \cdot, \mathbb{Z}). \]
Let us write \( \text{MS}(\Gamma)_{AR} \) for the image of the map. By Proposition 34, this is the submodule generated by the images of the universal modular symbols. We write \( AR \) to mean Ash-Rudolph modular symbols.

11.8.3. Let us prove the following proposition, which will imply Theorem 12(2) later.

**Proposition 49.** Suppose \( \mathbb{K} \subset \text{GL}_d(\mathbb{A}^\infty) \) be a pro-\( p \) compact open subgroup. Let \( \Gamma = \text{GL}_d(F) \cap \mathbb{K} \). Then
\[ p^{e(d)} H_{d-1}^{BM}(\Gamma \setminus \mathcal{B}_T \cdot, \mathbb{Z}) \subset \text{MS}(\Gamma)_{AR}. \]

**Proof.** This follows from Lemmas 45, 44, the surjection at the end of Section 11.3 and the discussion in Section 11.7 on the inverse limit.

11.8.4. Let us prove the following proposition, which will imply Theorem 12(3) later.

**Proposition 50.** Let \( v \neq \infty \) be a prime of \( F \), and let \( F_v \) denote the completion of \( F \) at \( v \). Let \( \mathbb{K}_v \) be a pro-\( p \) open compact subgroup of \( \text{GL}_d(F_v) \). Let us consider the intersection \( \Gamma' = \Gamma \cap \mathbb{K}_v \) in \( \text{GL}_d(F_v) \). Then
\[ p^{e(d)} |\Gamma : \Gamma'| H_{d-1}^{BM}(\Gamma \setminus \mathcal{B}_T \cdot, \mathbb{Z}) \subset \text{MS}(\Gamma)_{AR}. \]

**Proof.** This follows from Corollary 46, Lemma 48 and the discussion in Section 11.7 on the inverse limit.
11.9. THE CASE $d = 2$

11.8.5. We prove the analogue of Theorem 12(4).

**Proposition 51.** Let $v_0 \neq \infty$ be a prime of $F$ such that the cardinality $q_0$ of the residue field $\kappa(v_0)$ at $v_0$ is smallest among those at the primes $v \neq \infty$. Set $N(d) = \prod_{i=1}^{d}(q_i^{0} - 1)$. Then

$$p^{r(d)}N(d)H_{d-1}^{BM}(\Gamma \backslash BT\bullet, \mathbb{Z}) \subset MS(\Gamma)_{AR}$$

**Proof.** This follows from Corollary 47, Lemma 48 and the discussion in Section 11.7 on the inverse limit. □

**Corollary 52.** We have

$$H_{d-1}^{BM}(\Gamma \backslash BT\bullet, \mathbb{Q}) = MS(\Gamma)_{AR} \otimes \mathbb{Q}$$

**Proof.** This follows immediately from the proposition. □

11.9. the case $d = 2$

When $d = 2$, we do not need the computation of the homology groups of the stabilizer groups. Hence we obtain the best result that the exponent is 1.

**Proposition 53.** When $d = 2$, we have

$$H_{1}^{BM}(\Gamma \backslash BT\bullet, \mathbb{Z}) = MS(\Gamma)_{AR}$$

**Proof.** When $d = 2$, in the argument given in Section 11.5.2 we only need to look at one edge map in the spectral sequence. It is easy to see that the edge map is surjective. Hence the claim follows. □
Comparison of modular symbols

We defined two kinds of modular symbols. Our modular symbols come from fundamental classes of apartments, whereas the modular symbols of Ash-Rudolph (or those coming from the universal modular symbols) originate in the homology of the Tits building. In this chapter, we show that these two definitions coincide. We refer to Section 1.6 for the outline.

We have a proof of our main theorem (Theorem 12).

12.1. Statement of comparison proposition

The goal of this chapter is to prove the following proposition.

Let \( q_1, \ldots, q_d \) be an ordered basis of \( F^{\oplus d} \). In Section 1.4 we defined (our) modular symbol in the Borel-Moore homology of an arithmetic quotient. Let us write \( [q_1, \ldots, q_d]_A \) for it.

In Section 8.3 we defined universal modular symbol \( [q_1, \ldots, q_d] \) in the homology of the Tits building.

In Section 11.8.2 we defined the modular symbol of Ash-Rudolph to be the image of the universal modular symbol in the Borel-Moore homology of an arithmetic quotient. Let us write \( [q_1, \ldots, q_d]_{AR} \) for it.

The main task of this chapter is to show that they coincide.

Proposition 54. Let the notation be as above. We have

\[ [q_1, \ldots, q_d]_A = [q_1, \ldots, q_d]_{AR}. \]

In particular, we have \( \text{MS}(\Gamma) = \text{MS}(\Gamma)_{AR}. \)

12.2. Proof of Theorem 12

Using this proposition, we are now able to prove our main theorem.

Proof. (Proof of Theorem 12)

Using Proposition 54 the statements follow from Propositions 49, 50, 51, 53 and Corollary 52.

\( \Box \)

12.3. Outline

12.3.1. Let \( \alpha \) be a sufficiently large real number. We defined a subsimplicial complex \( BT_\bullet^{(\alpha)} \subset BT_\bullet \) in Section 11.2.

Given an ordered basis \( q_1, \ldots, q_d \), we defined an injective map \( \phi = \phi_{q_1, \ldots, q_d} : A_\bullet \rightarrow BT_\bullet \) in Section 3.2.2. We regard \( A_\bullet \) as a subsimplicial complex via \( \phi \).

We set \( A_\bullet^{(\alpha)} = A_\bullet \cap BT_\bullet^{(\alpha)} \).
12.3.2. The $S^{d-2}$ is the $(d - 2)$-sphere.

We use the notation from Section 8.1. We let $W = F^{d}$ and write $T_d = T_W$. The notation $\mathcal{P}'(B)$ appeared in Section 8.2. We use $|X|$ to denote the geometric realization of a poset $X$. (See below for precise definition.)

12.3.3. To prove the proposition, we look at the following diagram which appeared in Section 1.6. All the coefficients are $\mathbb{Z}$. The map (1) is from Lemma 48. The map (2) is the limit of the pushforward by the canonical projection for each $\alpha$. The map (3) is the canonical projection to one of the entries in the limit. It is an isomorphism because the maps (4)(5) are isomorphisms. The map (5) is from Corollary 12. The map (4) is the boundary map in the long exact sequence for a pair of spaces. It is an isomorphism because $BT_\bullet$ is contractible.

The map (6) is the canonical projection. The map (7) is the boundary map, which is an isomorphism because $A_\bullet$ is contractible. The map (8) will be constructed below.

The map (9) is the pushforward by the map induced by the morphism of posets $\mathcal{P}'(B) \to Q(W)$ associated with the ordered basis.

The map (10) appeared in Section 4.4.

The inclusions $A_\bullet \subset BT_\bullet$ and $A_{\alpha}^{\bullet} \subset BT_{\alpha}^{\bullet}$ give the rest of the horizontal maps.

The commutativity of the rectangles except the bottom one (with (5)(8)(9)) is easy to see.

12.3.4. Recall the definition of $[q_1, \ldots, q_d]_{AR} \in H_d^{BM}(\Gamma \setminus BT_\bullet)$ from Section 11.8.2. We have the universal modular symbol $[q_1, \ldots, q_d]$ in $H_{d-2}(T_d)$ of the left bottom corner, and $[q_1, \ldots, q_d]_{AR}$ is the image via the left vertical maps of $[q_1, \ldots, q_d]$.
By construction, $|P'(B)|$ is canonically homotopy equivalent to $S^{d-2}$. By the definition (Section 8.3) of universal modular symbol, $[q_1, \ldots, q_d]$ is the image of the fundamental class of $H_{d-1}(|P'(B)|)$ via the map (9).

Our modular symbol $[q_1, \ldots, q_d]$, is the image by (10) of the fundamental class of the apartment (Section 4.4).

Thus to prove the proposition, the strategy is to construct the map (8) such that the bottom square is commutative.

12.4. Quillen’s lemma

We put our emphasis on posets in what follows. For some terminology and results, we refer to [Qu] p.102–103

12.4.1. For a poset $W$, we defined the classifying space as the simplicial complex $(W, P_{\text{tot}}(W))$ where $P_{\text{tot}}(W)$ is the set of finite totally ordered subsets of $W$. We let $|W|$ denote the geometric realization of the classifying space. We call $W$ contractible if $|W|$ is contractible. We say that a morphism of posets $W_1 \to W_2$ is a homotopy equivalence if the induced map $|W_1| \to |W_2|$ of geometric realizations is a homotopy equivalence.

12.4.2. We use the following lemma of Quillen.

LEMMA 55 (Cor 1.8 [Qu]). Let $X, Y$ be posets and $Z \subset X \times Y$ be a closed subset (i.e., $z' \leq z \in Z$ implies $z' \in Z$).

Let $p_1 : Z \to X, p_2 : Z \to Y$ be the maps induced by the projections. We set $Z_x = \{y \in Y \mid (x, y) \in Z\}$ and $Z_y = \{x \in X \mid (x, y) \in Z\}$. If $Z_x$ (resp. $Z_y$) contractible for all $x \in X$ (resp. all $y \in Y$) then $p_1$ (resp. $p_2$) are homotopy equivalences.

12.5. A key diagram of posets

12.5.1. For a simplicial complex $U_\bullet$, we define $(U_\bullet)$ to be the poset of simplicies of $U_\bullet$. The classifying space of $(U_\bullet)$ is the barycentric subdivision of $U_\bullet$. (See proof of Lemma 1.9 of [Gr].)

12.5.2. We construct the following commutative diagram of posets:

$$
\begin{tikzcd}
(BT_{(\alpha)}^\bullet) \arrow[r, h_1] & (A(B)^{(\alpha)}_\bullet) \\
U \arrow[d, f_1] \arrow[r, h_2] & T \arrow[d, g_1] \\
Q(\mathbb{F}^{\oplus d}) \arrow[r, h_3] & P'(B)
\end{tikzcd}
$$

(12.2)

We will see that the maps $f_1$, $f_2$, $g_1$, $g_2$ are homotopy equivalences. Upon taking the geometric realizations, the diagram gives rise to the bottom square of the diagram (12.1).

12.6. The left column

We construct the left column of the key diagram. This is a slight generalization of the construction in Grayson [Gr].
12.6.1. Let $d \geq 1$ and $\mathcal{K} = \text{GL}_d(\mathbb{A})$. We defined $X_{\mathcal{K},\bullet}$ and $X_{\mathcal{K},\bullet}$. The nonadic description is $X_{\mathcal{K},\bullet} = \prod_{\mathcal{J} \in \mathcal{K}} \mathcal{J} \setminus \mathcal{B}_T\bullet$.

We consider the maps $\phi_j : \mathcal{B}_T\bullet \to \mathcal{J}_j \setminus \mathcal{B}_T\bullet$ for $j \in J_{\mathcal{K},\bullet}$. We set $\mathcal{B}_T\bullet(\alpha,j) = \phi_j^{-1}(X_{\mathcal{K},\bullet}^{(\alpha)})$. We note that this does not depend on the choice of $\mathcal{K}$. We defined $\mathcal{B}_T\bullet(\alpha)$ earlier. We have $\mathcal{B}_T\bullet(\alpha) = \mathcal{B}_T\bullet(\alpha,e)$ where $e$ is the identity element.

12.6.2. Let us write $F_d = Q(\mathcal{F}^{\otimes d})$ for short. We define a map

$$S_j : \mathcal{B}_T0 \to F_d \cup \{0\}$$

as follows. We have the map $\phi_{j,0} : \mathcal{B}_T0 \to X_{\mathcal{K},0}$ of 0-simplices of $\phi$. Let $\sigma \in \mathcal{B}_T0$. Then by Section 12.1, $\phi_{j,0}(\sigma) \in X_{\mathcal{K},0}$ is represented by a chain of locally free $\mathcal{O}_{\mathcal{K}}$-submodule of rank $d$ of $\eta_\ast \eta^\ast \mathcal{O}_{\mathcal{K}}^{\otimes d}$. Take one of the sheaves, say $E$, in the chain and consider its Harder-Narasimhan filtration

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_r = E \subset \eta_\ast \mathcal{F}^{\otimes d}$$

The restriction to the generic fiber Spec $F \subset C$ gives a flag of $\mathcal{F}^{\otimes d}$. We let $S_j(\sigma)$ denote the corresponding element in $F_d \cup \{0\}$.

12.6.3. For sufficiently large $\alpha$, the 0-simplices of $\mathcal{B}_T(\alpha)$ consist of unstable ones only. Thus the map $S$ gives $S_j : \mathcal{B}_T(\alpha) \to F_d$.

Let $f \in F_d$ be a flag. We let $Z_{j,\alpha} \subset \mathcal{B}_T(\alpha)$ be the subsimplicial complex consisting of simplices $\sigma$ such that

$$S(\sigma) \geq f$$

for all vertices $v$ of $\sigma$.

**Lemma 56.** $Z_{j,\bullet}^{(\alpha)}$ is contractible.

**Proof.** (We use the proof of Lemma 29 almost word-for-word.) We set $X = Z_{j,\bullet}^{(\alpha)}$. We proceed by induction on $d$. Let $f$ be the flag

$$0 \subset V_1 \subset \cdots \subset V_{r-1} \subset \mathcal{F}^{\otimes d}$$

with dim $V_j = i_j$ for $1 \leq j \leq r-1$ so that $D = \{i_1, \ldots, i_{r-1}\}$ with $i_1 < \cdots < i_{r-1}$.

Set $d' = d - i_1$ and $D' = \{i' - i_1, i' \in D \setminus \{i_1\}\} \subset \{1, \ldots, d' - 1\}$.

We define $f_1 \in \text{Flag}_{D'}$ as the image of the flag $f_1$ with respect to the projection

$$\mathcal{F}^{\otimes d} \to \mathcal{F}^{\otimes d}/V_1$$

We take an isomorphism $\mathcal{F}^{\otimes d}/V_1 \cong \mathcal{F}^{\otimes (d-i_1)}$ so that $f_1$ corresponds to the standard flag $f'_0$:

$$0 \subset \mathcal{F}^{\otimes (i_2-i_1)} \subset \mathcal{F}^{\otimes (i_3-i_1)} \subset \cdots \subset \mathcal{F}^{\otimes (i_{r-1}-i_1)} \subset \mathcal{F}^{\otimes (d-i_1)}$$

Take a representative $g_j \in \text{GL}_d(\mathbb{A})$ of $j \in J_{\mathcal{K}}$. Consider the map that sends an $\mathcal{O}_{\mathcal{K}}$-submodule $\mathcal{F}[g_j, L_{\infty}] \subset \eta, \mathcal{F}^{\otimes d}$ to the $\mathcal{O}_{\mathcal{K}}$-submodule $\mathcal{F}[g_j, L_{\infty}]/\mathcal{F}[g_j, L_{\infty}]_{(i_1)} \subset \eta, \mathcal{F}^{\otimes d'}$. There exists $j' \in J_{\text{Gl}_{d-1}(\mathbb{A})}$ such that the above map gives a morphism $h : X \to X'$ where we set $X' = Z_{j',\bullet}^{(\alpha)}$ if $D' \neq \emptyset$, and $X' = \mathcal{B}_T\mathcal{G}_{d-1}\bullet$, if $D' = \emptyset$. By inductive hypothesis, $|X'|$ is contractible.

Let $\epsilon : \text{Vert}(X) \to Z$ and $\epsilon' : \text{Vert}(X') \to Z$ denote the maps that send a locally free $\mathcal{O}_{\mathcal{K}}$-module $\mathcal{F}$ to the integer $[p_\mathcal{F}(1)/\det(\mathbb{Z})]$.

We fix an $\mathcal{O}_{\mathcal{K}}$-submodule $\mathcal{F}_0$ of $\eta, \mathcal{F}^{\otimes d}$ whose equivalence class belongs to $X$. By twisting $\mathcal{F}_0$ by some power of $\mathcal{O}_{\mathcal{K}}(\infty)$ if necessary, we may assume that $p_{\mathcal{F}_0}(i_1)$—
Lemma 57. For sufficiently large $\alpha$, the poset $U_\sigma$ is contractible.

Proof. We actually prove that if $\alpha > 3 \deg(\infty)$ then $U_\sigma$ has a maximal element, hence contractible. Let us show that $\bigcap_v S(v)$ is a maximal element, where the intersection is taken by regarding each $S_j(v)$ as a subset of $Q(F^{\oplus d})$. The maximality is clear. We need to show that $\bigcap_v S_j(v)$ is nonempty.

For a locally free $\mathcal{O}_C$-submodule $G \subset \eta_* F^{\oplus d}$ of rank $d$ and a vector subspace $B \subset F^{\oplus d}$ we denote by $G \cap \eta_* B$ the intersection of $G$ and $\eta_* B$ in $\eta_* F^{\oplus d}$. Let us choose a vertex $v_0$ of $\sigma$. Let $F_0 \subset \eta_* F^{\oplus d}$ be a locally free $\mathcal{O}_C$-submodule of rank $d$ that represents $v_0$. Since $v_0$ belongs to $BT^{(\alpha,j)}$, there exists a nonzero vector subspace $W_0 \subset F^{\oplus d}$ such that $F_0 \cap \eta_* W_0$ is a part of the Harder-Narasimhan filtration of $F_0$ and the minimal slope of the successive quotients of the Harder-Narasimhan filtration of $F_0 \cap \eta_* W_0$ is greater than the sum of $3 \deg(\infty)$ and the maximal slope of the successive quotients of the Harder-Narasimhan filtration of $F_0/(F_0 \cap \eta_* W_0)$. Let $v$ be a vertex of $\sigma$. Let us choose a locally free $\mathcal{O}_C$-submodule $F_1 \subset \eta_* F^{\oplus d}$ of rank $d$ that represents a vertex of $\sigma$ in such a way that $F_0 \subset F_1 \subset F_0(\infty)$. Then for any vector subspace $W' \subset F^{\oplus d}$, we have $F_0 \cap \eta_* W' \subset F_1 \cap \eta_* W' \subset (F_0 \cap \eta_* W')(\infty)$. Hence we have

$$0 \leq \deg(F_1 \cap \eta_* W') - \deg(F_0 \cap \eta_* W') \leq \dim_F W'.$$

This implies that $F_1 \cap \eta_* W_0$ is a part of the Harder-Narasimhan filtration of $F_1$. Hence $\bigcap_v S_j(v)$ contains $W_0$. In particular $\bigcap_v S_j(v)$ is nonempty. This proves the contractibility.

This implies that $f_2$ is a homotopy equivalence by Lemma 55.
12.6.6. Set
\[ U_f = \{ \sigma \in (BT^\bullet_\alpha) | (\sigma, f) \in U \} = \{ \sigma \in (BT^\bullet_\alpha) | S(v) \geq f \text{ for all vertices } v \text{ of } \sigma \}. \]

This is \( Z_{f, \bullet}^{(j, \alpha)} \), which is contractible. Hence by Lemma 55, \( f_2 \) is a homotopy equivalence.

12.7. The right column

12.7.1. Now, take an ordered basis \( q_1, \ldots, q_d \) of \( F^\oplus \).
We define \( h_3 : P_{\text{tot}}(P'(B)) \to F(F^\oplus) \) to be the map that appeared in Section 8.3.

12.7.2. We have a map (from Section 3.2.2)
\[ \varphi_{q_1, \ldots, q_d} : A(B)_\bullet = A_\bullet \to BT_\bullet. \]
We identify \( A(B)_\bullet \) and \( A_\bullet \) as subspaces of \( BT_\bullet \) via this map.
We set
\[ A^{(\alpha)}_\bullet = BT^{(\alpha)}_\bullet \cap A_\bullet \]
and define
\[ h_1 : (A^{(\alpha)}_\bullet) \to (BT^{(\alpha)}_\bullet) \]
to be the morphism of posets induced by the inclusion \( A^{(\alpha)}_\bullet \subset BT^{(\alpha)}_\bullet \) of simplicial complexes.

12.7.3. Note that the image of the restriction map
\[ S|_{A^{(\alpha)}_0} : A^{(\alpha)}_0 \to F_d \]
is contained in \( P'(B) \), which is regarded as a subset of \( F_d \) via \( h_3 \).

12.7.4. We set
\[ T = \{ (\beta, \gamma) \in (A^{(\alpha)}_\bullet) \times P'(B) | S(v) \geq \gamma \text{ for all vertices } v \text{ of } \beta \}. \]
It is a closed subset of \( (A^{(\alpha)}_\bullet) \times P'(B) \).

12.7.5. We define each of the maps
\[ g_1 : T \subset (A^{(\alpha)}_\bullet) \times P'(B) \to (A^{(\alpha)}_\bullet) \]
\[ g_2 : T \subset (A^{(\alpha)}_\bullet) \times P'(B) \to P'(B) \]
as the inclusion followed by the projection.
12.7.6. Let
\[ T_f = \{ \sigma \in (A^d) \mid (\sigma, f) \in T \} = \{ \sigma \in (A^d) \mid S(v) \geq f \text{ for all vertices } v \text{ of } \sigma \} \].

**Lemma 58.** \( T_f \) is contractible.

**Proof.** We use the setup in the proof of Lemma 56. We proceed by induction on \( d \). We let \( A^d \subset X \). Take \( d', i_1, D' \) as before. We set \( Y' = A^d \subset X' \) if \( D' \neq \emptyset \), \( Y', i_1 \subset X' \) if \( D' = \emptyset \). The inclusions are those corresponding to the ordered basis \( q_{i_1+1}, \ldots, q_d \) of \( F^{d-d(i_1)}/(F^{i_1} \oplus \{0\}^{d-d(i_1)}) \).

By the inductive hypothesis, \(|Y'|\) is contractible. Notice that the restriction of the map \( h : X \to X' \) to \( Y \) has its image inside \( Y' \). We can also check that the image of the restriction of \( h' \) to \( Y' \) is in \( Y \). Then the same argument as in the proof of Lemma 56 proves that \(|Y'|\) is contractible. \( \square \)

This implies \( g_1 \) is a homotopy equivalence.

12.7.7. Let
\[ T_{\sigma} = \{ f \in T_d \mid (\sigma, f) \in T \} = \{ f \in T_d \mid S(v) \geq f \text{ for all vertices } v \text{ of } \sigma \} \].

This is contractible because it has a maximal element. This implies \( g_2 \) is a homotopy equivalence.

12.8. Proof of Proposition 54

The map (5) in the diagram is defined to be the isomorphism of homology groups induced by \( f_1 \circ f_2^{-1} \). The map (8) is defined to be the isomorphism of homology groups induced by \( g_1 \circ g_2^{-1} \).

By the commutativity of the diagram (12.2), we have the commutativity of the bottom square of the diagram (12.1). This proves Proposition 54.
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