A 2-CALABI-YAU REALIZATION OF FINITE-TYPE CLUSTER ALGEBRAS WITH UNIVERSAL COEFFICIENTS

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ABSTRACT. We categorify various finite-type cluster algebras with coefficients using completed orbit categories associated to Frobenius categories. Namely, the Frobenius categories we consider are the categories of finitely generated Gorenstein projective modules over the singular Nakajima category associated to a Dynkin diagram and their standard Frobenius quotients. In particular, we are able to categorify all finite-type skew-symmetric cluster algebras with universal coefficients and finite-type Grassmannian cluster algebras. Along the way, we classify the standard Frobenius models of a certain family of triangulated orbit categories which include all finite-type $n$-cluster categories, for all integers $n \geq 1$.

1. INTRODUCTION

In this article we pursue the representation-theoretic approach to categorification of cluster algebras that has been developed by many authors (see for example [1, 7, 10, 11, 16, 19, 23]). Here, we are able to categorify a new family of skew-symmetric finite-type cluster algebras with geometric coefficients and give a new description of the categories constructed in [23] to categorify finite-type Grassmannian cluster algebras (note that in loc. cit. the authors categorify all Grassmannian cluster algebras). Our main idea is to combine the techniques used to categorify acyclic cluster algebras with those used to categorify geometric cluster algebras.

Cluster categories were introduced in [7] to study coefficient free cluster algebras associated to acyclic quivers using representation theory. They are defined as follows. Let $k$ be an algebraically closed field and $Q$ a finite quiver without oriented cycles. The cluster category $\mathcal{C}_Q$ is defined as the orbit category

$$\mathcal{D}^{b}(\text{mod}(kQ))/\Sigma \circ \tau^{-1},$$

where $\mathcal{D}^{b}(\text{mod}(kQ))$ is the bounded derived category of finitely generated right modules over the path algebra $kQ$ and $\Sigma \circ \tau^{-1}$ is the composition of the suspension functor $\Sigma$ and the inverse of the Auslander-Reiten translation $\tau^{-1}$. Some of the fundamental properties of cluster categories are as follows:

- they are 2-Calabi-Yau triangulated categories (2-CY for short);
- they Krull-Schmidt categories;
- they have a cluster-tilting subcategory.
It was noticed in [8] that Frobenius categories can provide an adequate framework to cate-

ergify cluster algebras with geometric coefficients. This proposal has been successfully

implemented by many authors. For instances we can mention the work of Fu and Keller

[16]; Geiss-Leclerc-Schröer’s catgorification of the multi-homogenus coordinate ring of a

partial flag variety [19]; the categorification of Grassmannian cluster algebras obtained by

Jensen, King and Su in [23]; the recent generalization of [19] and [23] by Demonet and

Iyama [10]; the categorification of cluster algebras associated to ice quivers with potentials

arising from triangulated surfaces carried out by Demonet and Luo in [11], among many

other works.

In this paper we combine the approaches of [7] and [8] to categoryfy cluster algebras

with geometric coefficients using orbit categories associated to Frobenius categories. To

be more precise, let $\mathcal{E}$ be a $k$-linear Frobenius category whose stable category is triangle

equivalent to $\mathcal{D}^b(\text{mod}(kQ))$. Suppose that the autoequivalence $\Sigma \circ \tau^{-1}$ can be lifted to

an exact autoequivalence $E : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$. It follows from previous work of the author of this

note [29] that the completed orbit category $\hat{\mathcal{E}}/\mathcal{E}$ has the structure of a Frobenius category,

whose stable category is triangle equivalent to the cluster category $\mathcal{C}_Q$. It is reasonable to

expect that the Frobenius category $\hat{\mathcal{E}}/\mathcal{E}$ can be used to construct a categorification of a

geometric cluster algebra of type $Q$. The main purpose of this note is to show that this

is indeed the case provided $Q$ is a Dynkin quiver and $\mathcal{E}$ is a standard Frobenius model of

$\mathcal{D}^b(\text{mod}(kQ))$ in the sense of [27], that is, $\mathcal{E}$ is Hom-finite, Krull-Schmidt, and satisfies the

following conditions:

(i) For each indecomposable projective object $P$ of $\mathcal{E}$, the $\mathcal{E}$-module $\text{rad}_\mathcal{E}(?, P)$ and the

$\mathcal{E}^{\text{op}}$-module $\text{rad}_\mathcal{E}(P, ?)$ are finitely generated with simple tops.

(ii) $\mathcal{E}$ is standard in the sense of Ringel [39], i.e. its category of indecomposable objects

is equivalent to the mesh category of its Auslander–Reiten quiver.

So let’s assume from now on that $Q$ is an orientation of a Dynkin diagram $\Delta$ of type $A$, $D$ or

$E$. In general, different Frobenius models will lead to the categorification of different cluster

algebras of type $Q$ with coefficients. An important aspect of our work is that we are able

to identify particular Frobenius models of $\mathcal{D}^b(\text{mod}(kQ))$ which provide a categorification

distinguished cluster algebras such as cluster algebras with universal coefficients and

finite-type Grassmannian cluster algebras (the later happens when $\Delta$ is $D_4$, $E_6$, $E_8$ or of

type $A$).

The standard Frobenius models of $\mathcal{D}^b(\text{mod}(kQ))$ were classified by Keller and Scherotzke

in [27] using a representation-theoretic approach to Nakajima’s quiver varieties. They are

in bijection with certain subsets of the set of vertices of the repetution quiver $\mathbb{Z}Q$ and can be

described as follows. Let $\mathcal{S}$ be the singular Nakajima category (see [27]) associated to $Q$ and

$C \subset \mathbb{Z}Q_0$ an admissible configuration. Let $\mathcal{S}_C$ be the quotient category of $\mathcal{S}$ associated to $C$

constructed in [27] (cf. Definition 2.7). Then the category of finitely generated Gorenstein
projective $S_C$-modules $\text{gpr}(S_C)$ is a standard Frobenius model of $\mathcal{D}^b(\text{mod}(kQ))$. Moreover, every standard Frobenius model of $\mathcal{D}^b(\text{mod}(kQ))$ is equivalent as an exact category to $\text{gpr}(S_C)$ for some admissible configuration $C$. The main result of this paper is as follows:

**Theorem.** Consider $\mathbb{Z}Q_0$ as the set of indecomposable objects of $\mathcal{D}^b(\text{mod}(kQ))$. Let $C$ be an admissible configuration of $\mathbb{Z}Q_0$ invariant under $\Sigma \circ \tau^{-1}$. Then $\Sigma \circ \tau^{-1}$ lifts to an exact automorphism $E : \text{gpr}(S_C) \to \text{gpr}(S_C)$ and the complete orbit category $\text{gpr}(S_C) E$ is a 2-Calabi-Yau realization of a cluster algebra with geometric coefficients of type $\Delta$. Moreover, if $C = \mathbb{Z}Q_0$ (i.e. $S_C = S$) then $\text{gpr}(S) E$ is a 2-Calabi-Yau realization of the cluster algebra with universal coefficients of type $\Delta$.

Cluster algebras with universal coefficients were introduced by Fomin and Zelevinsky in [14] and further investigated by Reading in [34, 35, 36]. These are cluster algebras which are universal with respect to coefficient specialization among cluster algebras associated to a fixed initial quiver. The existence of universal coefficients for finite-type cluster algebras was proved in [14]. In [34], it was shown that universal coefficients always exist if we restrict to the class of geometric cluster algebras and allow them to have an infinite number of coefficients (whose powers can be taken not only in the ring of integer numbers, but in more general rings such as the rational or real numbers). We hope that our approach can be generalized to categorify other cluster algebras with universal coefficients beyond finite-type.

Part of the technical aspects of our construction were carried out in [29], where we studied completed orbit categories associated to Frobenius categories in a more general framework. It is worth pointing out that we consider completed orbit categories rather than usual orbit categories because the Krull-Schmidt property in general is lost when we consider orbit categories, whereas it is always preserved for completed orbit categories.

We can use the insight of [27] to classify the Frobenius models of $C_Q$ satisfying conditions (i) and (ii) using completed orbit categories associated to categories of the form $\text{gpr}(S_C)$. Notice that we shall modify slightly the definition in [27] and admit Frobenius models with infinite-dimensional morphism spaces. Moreover, we can classify the Frobenius models not only of cluster categories but of a larger class of orbit categories associated to $\mathcal{D}^b(\text{mod}(kQ))$. This classification problem was already addressed by Scherotzke in [40, Section 3] using usual orbit categories. The following theorem proved in this note extends Theorem 3.7 of [40] to more general set-ups.

**Theorem.** Let $E : \mathcal{D}^b(\text{mod}(kQ)) \to \mathcal{D}^b(\text{mod}(kQ))$ be a triangle equivalence such that $\mathcal{D}^b(\text{mod}(kQ)) E$ is $\text{Hom}$-finite and equivalent to its triangulated hull (in the sense of [25]). Let $C$ be an admissible configuration invariant under $E$. Suppose moreover that $E$ lifts to an exact autoequivalence $E_\ast : \text{gpr}(S_C) \to \text{gpr}(S_C)$, and that for each indecomposable object $X$ of $\text{gpr}(S_C)$, the group $\text{gpr}(S_C)(X, E^l_\ast(X))$ vanishes for all $l < 0$. Then
(a) The completed orbit category $\gpr(S_C)\sim E_*$ admits the structure of a Frobenius category whose stable category is triangle equivalent to $\mathcal{D}_Q/E$.

(b) The category $\gpr(S_C)\sim E_*$ satisfies conditions (i) and (ii) above.

(c) The map taking $C$ to $\gpr(S_C)$ induces a bijection from the set of $E$-invariant admissible configurations $C \subset \mathbb{Z}Q_0$ onto the set of equivalence classes of Frobenius models $\gpr(S_C)$ of $\mathcal{D}_Q/E$ satisfying (i) and (ii).

The main difference between \cite[Theorem 5.7]{40} and the theorem above is that, by considering completed orbit categories, we are able to consider functors $E : \mathcal{D}^b(\text{mod}(kQ)) \to \mathcal{D}^b(\text{mod}(kQ))$ and admissible configurations $C \subset \mathbb{Z}Q_0$ for which the usual orbit category $\gpr(S_C)/E_*$ is Hom-infinite or fails to be Krull-Schmidt.

This paper is organized as follows. In Section 2 we recall the definition and some of the fundamental properties of Nakajima categories. In Section 3 we state our main results and prove them in Sections 4 and 5. In Section 6 we present some examples.

2. Recollections

In this note we will freely use the basic concepts in the theory of dg categories. The reader can refer to Section 2 of \cite{29} for an account of the results on dg categories and their orbit categories that will be used here. Some of the standard references for this topic are \cite{24} and \cite{12}. Throughout this note all functors between $k$-linear categories are assumed to be $k$-linear.

2.1. Notation. The set of morphisms between two objects $x$ and $y$ of a category $\mathcal{C}$ is denoted by $\mathcal{C}(x, y)$. If $k$ is a field and $\mathcal{E}$ an additive $k$-category, a right $\mathcal{C}$-module is by definition a $k$-linear functor $M : \mathcal{C}^{\text{op}} \to \text{Mod}(k)$, where $\text{Mod}(k)$ is the category of $k$-vector spaces. We let $\text{Mod}(\mathcal{C})$ be the category of all right $\mathcal{C}$-modules, $\text{mod}(\mathcal{C})$ its subcategory of finitely presented modules and $\text{proj}(\mathcal{C})$ the full subcategory of finitely generated projective $\mathcal{C}$-modules. In particular, for each object $x$ of $\mathcal{C}$, we have the finitely generated projective $\mathcal{C}$-module

$$x^\wedge := \mathcal{C}(?, x) : \mathcal{C}^{\text{op}} \to \text{Mod}(k),$$

and the finitely generated injective $\mathcal{C}$-module

$$x^\vee := D(\mathcal{C}(x, ?)) : \mathcal{C}^{\text{op}} \to \text{Mod}(k).$$

Here, $\mathcal{C}(y, z)$ denotes the space of morphisms from $y$ to $z$ in the category $\mathcal{C}$ and $D$ is the duality over the ground field $k$. If the endomorphism ring of every indecomposable object $x$ of $\mathcal{C}$ is local, then each projective $\mathcal{C}$-module is a direct sum of modules of the form $x^\wedge$, therefore we will sometimes refer to $x^\wedge$ (resp. $x^\vee$) as the free (resp. co-free) $\mathcal{C}$-module associated to $x$. Contrary to the notation used for the space of morphisms in $\mathcal{C}$,
the morphism space between two $C$-modules $L$ and $M$ is denoted by $\text{Hom}_C(L, M)$ or simply $\text{Hom}(L, M)$ when there is no risk of confusion.

2.2. Nakajima categories. In this section we recall the definition and some of the properties of the Nakajima categories.

Let $Q$ be a quiver. Let $Q_0$ be its set of vertices and $Q_1$ be its set of arrows. We suppose that $Q$ is finite (both $Q_0$ and $Q_1$ are finite sets) and acyclic ($Q$ has no oriented cycles). The repetition quiver (cf. [38]) $\mathbb{Z}Q$ is the quiver obtained from $Q$ as follows:

- the set of vertices of $\mathbb{Z}Q$ is $\mathbb{Z}Q_0 = Q_0 \times \mathbb{Z}$.
- For each arrow $\alpha : i \rightarrow j$ of $Q$ and each $p \in \mathbb{Z}$, the repetition quiver $\mathbb{Z}Q$ has the arrows $(\alpha, p) : (i, p) \rightarrow (j, p)$ and $\sigma(\alpha, p) : (j, p - 1) \rightarrow (i, p)$.
- $\mathbb{Z}Q$ has no more arrows than those described above.

Let $\sigma : \mathbb{Z}Q_1 \rightarrow \mathbb{Z}Q_1$ be the bijection given by

$$
\sigma(\beta) = \begin{cases} 
\sigma(\alpha, p) & \text{if } \beta = (\alpha, p), \\
(\alpha, p - 1) & \text{if } \beta = \sigma(\alpha, p).
\end{cases}
$$

Let $\tau : \mathbb{Z}Q \rightarrow \mathbb{Z}Q$ be the graph automorphism given by the translation by one unit:

$$
\tau(i, p) = (i, p - 1) \quad \text{and} \quad \tau(\beta) = \sigma^2(\beta)
$$

for each vertex $(i, p)$ and each arrow $\beta$ of $\mathbb{Z}Q$.

Let $k$ be a field. Following [17] and [37], we define the mesh category $k(\mathbb{Z}Q)$ to be the quotient of the path category $k\mathbb{Z}Q$ by the ideal generated by the mesh relators, i.e. the $k$-category whose objects are the vertices of $\mathbb{Z}Q$ and whose morphism space from $a$ to $b$ is the space of all $k$-linear combinations of paths from $a$ to $b$ modulo the subspace spanned by all elements $ur_xv$, where $u$ and $v$ are paths and

$$
rs = \sum_{\beta : y \rightarrow x} \beta \sigma(\beta) : \cdots
\begin{array}{c}
\vdots \\
\downarrow \sigma(\beta) \\
\downarrow \beta \\
\downarrow \sigma(\beta) \\
\downarrow \beta \\
\downarrow \sigma(\beta) \\
s \\
\end{array}
\begin{array}{c}
y_1 \\
\vdots \\
\vdots \\
y_s \\
\end{array}
\begin{array}{c}
\sigma(\beta_1) \\
\beta_1 \\
\sigma(\beta_s) \\
\beta_s \\
\end{array}
\begin{array}{c}
x \\
\vdots \\
\vdots \\
x \\
\end{array}
$$

is the mesh relator associated with a vertex $x$ of $\mathbb{Z}Q$. Here the sum runs over all arrows $\beta : y \rightarrow x$ of $\mathbb{Z}Q$. Note that $\tau$ can be thought of as an autoequivalence of $\mathbb{Z}(kQ)$.

**Notation 2.1.** Let $kQ$ be the path algebra of $Q$ and let $\text{mod}(kQ)$ be the category of all finite-dimensional right $kQ$-modules. We adopt the notation used in [27] and denote the bounded derived category of $\text{mod}(kQ)$ by $D_Q$. If $\mathcal{X}$ is an additive category then we write $\text{ind}(\mathcal{X})$ to denote its full subcategory of indecomposable objects.
Theorem 2.2 ([21]). There is a canonical fully faithful functor
\[ H : k(ZQ) \rightarrow \text{ind}(D_Q) \]
rather fully faithful functor taking each vertex \((i,0)\) to the indecomposable projective module associated to the vertex \(i \in Q_0\). It is an equivalence if and only if \(Q\) is a Dynkin quiver (= an orientation of a Dynkin diagram of type ADE).

Definition 2.3. ([27]) The framed quiver \(\tilde{Q}\) associated to \(Q\) is the quiver obtained from \(Q\) by adding, for each vertex \(i \in Q_0\), a new vertex \(i'\) and a new arrow \(i \rightarrow i'\). We consider the repetition quiver \(Z\tilde{Q}\) and call frozen vertices its vertices of the form \((i',n)\), with \(i \in Q_0\) and \(n \in \mathbb{Z}\). The regular Nakajima category \(R\) associated to \(Q\) is the quotient of the path category \(kZ\tilde{Q}\) by the ideal generated by the mesh relators associated to the non-frozen vertices. The singular Nakajima category \(S\) is the full subcategory of \(R\) whose objects are the frozen vertices.

As shown in [21], \(H\) identifies the autoequivalence \(\tau : \mathbb{Z}(kQ) \rightarrow \mathbb{Z}(kQ)\) with the Auslander-Reiten translation of \(D_Q\), which we will also denote by \(\tau\). For Dynkin quivers, the combinatorial descriptions of \(\nu, \Sigma\) and of the image of \(\text{mod}(kQ)\) in \(D_Q\) are given in section 6.5 of [17]. If \(Q\) is Dynkin, let \(\Sigma\) be the unique bijection of the vertices of \(ZQ\) such that \(H(\Sigma x) = \Sigma(H(x))\).

Remark 2.4. We will systematically identify the vertices of \(ZQ\) with the objects of \(\text{ind}(D_Q)\) via the functor \(H\). So when we refer to a vertex of \(ZQ\) as an indecomposable object of \(D_Q\) we will tacitly refer to the object \(H(x)\).

Remark 2.5. Note that \(\Sigma : ZQ_0 \rightarrow ZQ_0\) extends to a bijection \(\mathbb{Z}\tilde{Q}_0 \rightarrow \mathbb{Z}\tilde{Q}_0\) which will be denoted by \(\Sigma\) by a slight abuse of notation. There is also a bijection \(\sigma : \mathbb{Z}\tilde{Q}_0 \rightarrow \mathbb{Z}\tilde{Q}_0\) given by \(\sigma : (i,n) \mapsto (i',n-1)\) and \((i',n) \mapsto (i,n)\) for \(i\) a vertex of \(Q\) and \(n\) an integer.

Example 2.6. Let \(Q\) be the Dynkin quiver \(1 \rightarrow 2\). The quiver \(\tilde{Q}\) is depicted in Figure 1 below. The frozen vertices are represented by small squares \(\square\). In this case the mesh relations imply that \(ba + dc = 0\) in the Nakajima category whereas \(eb \neq 0\).

\[ \begin{array}{ccccccccc}
\square & \rightarrow & \square & \rightarrow & \square & \rightarrow & \square & \rightarrow & \Sigma x & \rightarrow & \square \\
\tau(x) & \rightarrow & x & \rightarrow & a & \rightarrow & b & \rightarrow & \square & \rightarrow & e & \rightarrow & \square
\end{array} \]

**Figure 1.** The quiver of the regular Nakajima category associated to \(A_2\).
Definition 2.7. Let $C$ be a subset of $\mathbb{Z}Q_0$. Denote by $\mathcal{R}_C$ the quotient of $\mathcal{R}$ by the ideal generated by the identities of the frozen vertices not belonging to $\sigma^{-1}(C)$ and by $\mathcal{S}_C$ its full subcategory formed by the vertices in $\sigma^{-1}(C)$. We call $C$ an admissible configuration of $\mathbb{Z}Q_0$ if for each vertex $x \in \mathbb{Z}Q_0$, there is a vertex $c \in C$ such that the space of morphisms from $x$ to $c$ in the mesh category $k(\mathbb{Z}Q)$ does not vanish. Finally, let $\tilde{\mathbb{Z}}Q_C$ be the quiver obtained from $\tilde{\mathbb{Z}}\tilde{Q}$ by delating the set of frozen vertices $\sigma^{-1}(C)$.

2.3. Nakajima categories and derived categories. A Frobenius category is a Quillen-exact category [33] with enough injective objects, enough projective objects and where these two families of objects coincide. By definition, every Frobenius category is endowed with a distinguished class of sequences

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

called conflations (we follow the terminology of [18]). We will call the morphism $L \rightarrow M$ of such a sequence an inflation and the morphism $M \rightarrow N$ a deflation. Let $\mathcal{E}$ be a Frobenius category. The stable category $\mathcal{E}_\ast$ is the quotient of $\mathcal{E}$ by the ideal of morphisms factoring through a projective-injective object. It was shown by Happel [20] that $\mathcal{E}_\ast$ has a canonical structure of triangulated category. Note that it is possible to define the extension functors $\text{Ext}^i_\mathcal{E}$ for $\mathcal{E}$ in the usual way. We have that

$$\text{Ext}^i_\mathcal{E}(L, M) \cong \text{Ext}^i_\mathcal{E}(L, M)$$

for all objects $L$ and $M$ of $\mathcal{E}$ and all integers $i \geq 1$. Now we introduce a class of Frobenius categories that will be very important for our discussion.

Definition 2.8. Let $\mathcal{C}$ be an additive $k$-category. An $\mathcal{C}$-module $M$ is finitely generated Gorenstein projective if there is an acyclic complex

$$P_M : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

of objects in $\text{proj}(\mathcal{C})$ such that $M \cong \text{cok}(P_1 \rightarrow P_0)$ and the complex $\text{Hom}_\mathcal{C}(P_M, P')$ is still acyclic for each module $P'$ in $\text{proj}(\mathcal{C})$. Denote by $\text{gpr}(\mathcal{C})$ the full subcategory of $\text{mod}(\mathcal{C})$ formed by the Gorenstein projective modules. In the situation described above we call $P_M$ a complete projective resolution of $M$.

Every finitely generated projective $\mathcal{C}$-module $P$ lies in $\text{gpr}(\mathcal{C})$ since we may consider the complete projective resolution $\cdots \rightarrow 0 \rightarrow P \rightarrow 0 \rightarrow 0 \rightarrow \cdots$. The following result is a well-known result on Gorenstein projective modules (see [4, Proposition 5.1]).

Lemma 2.9. The category $\text{gpr}(\mathcal{C})$ is an extension closed subcategory of $\text{Mod}(\mathcal{C})$. Moreover, the induced exact structure on $\text{gpr}(\mathcal{C})$ makes it a Frobenius category whose subcategory of projective–injective objects is $\text{proj}(\mathcal{C})$. 7
Remark 2.10. The Gorenstein injective modules are defined analogously, all the results in this note can be naturally adapted to be stated in terms of Gorenstein injective modules.

The following theorem summarizes some of the results of [27] that we shall need.

**Theorem 2.11.** ([27]) Let $Q$ be a Dynkin quiver and $C$ an admissible configuration of $\mathbb{Z}Q$. Then

(i) the restriction functor

$$\text{res} : \text{Mod}(\mathcal{R}_C) \rightarrow \text{Mod}(\mathcal{S}_C)$$

induces an equivalence between the full subcategory of finitely generated projective $\mathcal{R}_C$-modules $\text{proj}(\mathcal{R}_C)$ and the category $\text{gpr}(\mathcal{S}_C)$

$$\text{res} : \text{proj}(\mathcal{R}_C) \sim \text{gpr}(\mathcal{S}_C).$$

In particular, it induces an isomorphism of $\mathbb{Z}\tilde{Q}_C$ onto the Auslander-Reiten quiver of $\text{gpr}(\mathcal{S}_C)$ so that the vertices of $\sigma^{-1}(C)$ correspond to the projective–injective objects;

(ii) there is a $\delta$-functor $\Phi : \text{mod}(\mathcal{S}_C) \rightarrow \mathcal{D}_Q$ defined as the composition

$$\text{mod}(\mathcal{S}_C) \xrightarrow{\Omega} \text{gpr}(\mathcal{S}_C) \xrightarrow{\phi} \mathcal{D}_Q,$$

where $\Omega$ is the syzygy functor and $\phi$ is a distinguished triangle equivalence.

Remark 2.12. In view of part (i) of Theorem 2.11, the indecomposable objects of $\text{gpr}(\mathcal{S}_C)$ are of the form $\text{res}(x^\wedge)$, for a vertex $x$ of $\mathbb{Z}\tilde{Q}_C$. If $y$ is a frozen vertex of $\mathbb{Z}\tilde{Q}_C$, then the projective $\mathcal{S}_C$-module $y^\wedge \in \text{gpr}(\mathcal{S}_C)$ is identified with $\text{res}(y^\wedge)$. For this reason, we denote the indecomposable objects of $\text{gpr}(\mathcal{S}_C)$ by $x^\wedge$, for some vertex $x$ of $\mathbb{Z}\tilde{Q}_C$.

**Assumption 2.13.** From now on we suppose that $Q$ is an orientation of a connected and simply laced Dynkin diagram.

2.4. **Standard Frobenius models of $\mathcal{D}_Q$.** We suppose for the rest of the note that $k$ is algebraically closed. A Frobenius model of $\mathcal{D}_Q$ is a Frobenius category $\mathcal{E}$ together with a triangle equivalence $\mathcal{D}_Q \cong \mathcal{E}$. Theorem 2.11 implies that we can construct Frobenius models of $\mathcal{D}_Q$ from admissible configurations. Indeed, if $C \subset \mathbb{Z}Q_0$ is admissible, then the category $\text{gpr}(\mathcal{S}_C)$ is a Frobenius model of $\mathcal{D}_Q$. Keller and Shardtze proved that the Frobenius categories obtained in this way are precisely the class of Frobenius categories $\mathcal{E}$ satisfying the following conditions:

(P0) $\mathcal{E}$ is $k$-linear, $\text{Ext}$-finite and Krull-Schmidt (see [28]).

(P1) For each indecomposable non-projective object $X$ of $\mathcal{E}$, there is an almost split sequence starting and an almost split sequence ending at $X$.

(P2) For each indecomposable projective object $P$ of $\mathcal{E}$, the $\mathcal{E}$-module $\text{rad}_\mathcal{E}(?, P)$ and the $\mathcal{E}^{op}$-module $\text{rad}_\mathcal{E}(P, ?)$ are finitely generated with simple tops.
(P3) $\mathcal{E}$ is standard, i.e. its category of indecomposables is equivalent to the mesh category of its Auslander–Reiten quiver (cf. section 2.3, page 63 of [39]).

**Theorem 2.14.** ([27, Corollary 5.25]) The map taking $C$ to $\text{gpr}(S_C)$ induces a bijection from the set of admissible configurations $C \subset \mathbb{Z}Q_0$ onto the set of equivalence classes of Frobenius models $\mathcal{E}$ of $\mathcal{D}_Q$ satisfying $(P0)−(P3)$. The inverse bijection sends a Frobenius model $\mathcal{E}$ to the set $C \subset \mathbb{Z}Q_0$ such that the indecomposable projectives of $\mathcal{E}$ correspond to the vertices $\sigma^{-1}(c), c \in C$, of the Auslander–Reiten quiver of $\mathcal{E}$.

**2.5. Completed orbit categories.** Orbit categories and their completions will be fundamental in what follows. We recall now their definitions and useful properties. Let $\mathcal{C}$ be a $k$-linear category and $F : \mathcal{C} \rightarrow \mathcal{C}$ an automorphism. By definition, the orbit category $\mathcal{C}/F$ has the same objects as $\mathcal{C}$, the set of morphisms from an object $X$ to an object $Y$ is given by

$$\mathcal{C}/F(X,Y) = \bigoplus_{l \in \mathbb{Z}} \mathcal{C}(X,F^l(Y)).$$

The composition of morphisms is given by the formula

$$(f_a) \circ (g_b) = \left( \sum_{a+b=c} F^b(f_a) \circ g_b \right),$$

where $f_a : Y \rightarrow F^a(Z), g_b : X \rightarrow F^b(Y)$ and $a, b \in \mathbb{Z}$. Clearly $\mathcal{C}/F$ is still a $k$-linear category and the canonical projection $p : \mathcal{C} \rightarrow \mathcal{C}/F$ is an additive functor. Suppose that for all objects $X, Y$ of $\mathcal{C}$, the space $\mathcal{C}(X, F^l(Y))$ vanishes for all integers $l \ll 0$. In his case we can define the completed orbit category $\mathcal{C}/F$ as the category whose objects are the same as those of $\mathcal{C}$ and with morphism spaces

$$\mathcal{C}/F(X,Y) = \prod_{l \in \mathbb{Z}} \mathcal{C}(X, F^l(Y)).$$

The vanishing condition imposed on the spaces $\mathcal{C}(X, F^l(Y))$ implies that (2.1) defines a composition of morphisms in $\mathcal{C}/F$. Clearly, the category $\mathcal{C}/F$ is $k$-linear. We still denote the natural projection $\mathcal{C} \rightarrow \mathcal{C}/F$ by $p$.

**Assumption 2.15.** Whenever we refer to the completed orbit category associated to an automorphism $F : \mathcal{C} \rightarrow \mathcal{C}$ we will implicitly assume for all objects $X, Y$ of $\mathcal{C}$, the space $\mathcal{C}(X, F^l(Y))$ vanishes for all integers $l \ll 0$.

**Remark 2.16.** Using a standard procedure, we can replace a category with an autoequivalence by a category with an automorphism (see Section 7 of [3]). So we will consider orbit categories associated to categories equipped with an autoequivalence.

**Remark 2.17.** Each indecomposable object of $\mathcal{C}/F$ is the image of an indecomposable object of $\mathcal{C}$ under $p$. 
Let $\mathcal{B}$ be a dg category endowed with an endomorphism $F: \mathcal{B} \to \mathcal{B}$ inducing an equivalence $H^0(F): H^0(\mathcal{B}) \to H^0(\mathcal{B})$. We define the dg orbit category $\mathcal{B}/F$ as follows: the objects of $\mathcal{B}/F$ are the same as the objects of $\mathcal{B}$. For $X, Y \in \mathcal{B}/F$, we have

\begin{equation} \label{2.3}
\mathcal{B}/F(X, Y) := \colim_p \bigoplus_{n \geq 0} \mathcal{B}(F^n(X), F^p(Y)),
\end{equation}

where the transition maps are given by $F$

\begin{equation*}
\bigoplus_{n \geq 0} \mathcal{B}(F^n(X), F^p(Y)) \xrightarrow{F} \bigoplus_{n \geq 0} \mathcal{B}(F^n(X), F^{p+1}(Y)).
\end{equation*}

**Definition 2.18.** Let $\mathcal{T}$ be a triangulated category endowed with a triangulated equivalence $F: \mathcal{T} \to \mathcal{T}$. Suppose that $\mathcal{T}_{dg}$ is a dg enhancement of $\mathcal{T}$ and that $\tilde{F}: \mathcal{T}_{dg} \to \mathcal{T}_{dg}$ is a dg functor such that $H^0(\tilde{F}) = F$. The triangulated hull of $\mathcal{T}/F$ (with respect to $\tilde{F}$) is the triangulated category $H^0(\text{pretr}(\mathcal{T}_{dg}/\tilde{F}))$, where $\text{pretr}(\mathcal{T}_{dg}/\tilde{F})$ is the pretriangulated hull of $\mathcal{T}_{dg}/\tilde{F}$.

### 3. The main results

This section is devoted to state our main results, their proofs will be given in the subsequent sections. Let $F: \mathcal{D}_Q \to \mathcal{D}_Q$ be a triangle equivalence which is isomorphic to the derived tensor product

\[ \otimes^L_{kQ} M : \mathcal{D}_Q \to \mathcal{D}_Q \]

for some complex $M$ of $kQ$-bimodules. This is not a restriction since all autoequivalences with an "algebraic" construction are of this form (cf. Section 9 of [25]).

We can identify $F$ with an automorphism of the mesh category $k(\mathbb{Z}Q)$ via Happel’s embedding (see Theorem 2.2). A configuration $C \subset \mathbb{Z}Q_0$ is called $F$-invariant if $H^0(\tilde{F}) = F$. The triangulated hull of $\mathcal{T}/F$ (with respect to $\tilde{F}$) is the triangulated category $H^0(\text{pretr}(\mathcal{T}_{dg}/\tilde{F}))$, where $\text{pretr}(\mathcal{T}_{dg}/\tilde{F})$ is the pretriangulated hull of $\mathcal{T}_{dg}/\tilde{F}$.

Let $F^*: \operatorname{Mod}(\mathcal{R}_C) \to \operatorname{Mod}(\mathcal{R}_C)$ be the automorphism on $\operatorname{Mod}(\mathcal{R}_C)$ defined by

\[ F^*(M) = M \circ F^{-1}. \]

In particular, $F^*(x^\wedge) = F(x)^\wedge$ for every vertex $x$ of $\mathbb{Z}Q_C$, so $F^*$ restricts to an exact automorphism $F^*_c : \operatorname{gpr}(\mathcal{S}_C) \to \operatorname{gpr}(\mathcal{S}_C)$.

Consider the dg functor

\[ \tilde{F} := \otimes_{kQ} P^*M : \mathcal{C}^b(\mathcal{Q}) \to \mathcal{C}^b(\mathcal{Q}) \]

where $P^*M$ is a projective resolution of the bimodule $M$. We have that $H^0(\tilde{F}) = F$. 

10
**Theorem 3.1.** Let $F : D_Q \to D_Q$ be a triangle equivalence such that $D_Q/F$ is Hom-finite and equivalent to its triangulated hull. Let $C$ be an admissible configuration invariant under $F$. Suppose moreover that for each indecomposable object $X$ of $\text{gpr}(S_C)$ the group $\text{Hom}_{S_C}(X, F_\ast(X))$ vanishes for all $l < 0$. Then

(i) the completed orbit category $\hat{\text{gpr}}(S_C)/F_\ast$ admits the structure of a Frobenius category whose stable category is triangle equivalent to $D_Q/F$ (see Assumption 2.15);

(ii) the category $\hat{\text{gpr}}(S_C)/F_\ast$ satisfies conditions $(P_0) - (P_3)$ of Section 2.4 and its AR quiver is isomorphic to $\mathbb{Z}\tilde{Q}_C/F$;

(iii) the map taking $C$ to $\hat{\text{gpr}}(S_C)$ induces a bijection from the set of $F$-invariant admissible configurations $C \subset \mathbb{Z}Q_0$ onto the set of equivalence classes of Frobenius models of $D_Q/F$ satisfying $(P0) - (P3)$.

We can see that in the setting of Theorem 3.1, the completed orbit category $\hat{\text{gpr}}(S_C)/F_\ast$ is in general Hom-infinite. Note however that it is always Ext$^1$-finite since $D_Q/F$ is Hom-finite. Keller showed in [25] that $D_Q/F$ is Hom-finite and equivalent to its triangulated hull if the following conditions hold:

(a) for each indecomposable $U$ of $\text{mod}(kQ)$, there are only finitely many $i \in \mathbb{Z}$ such that the object $F^i(U)$ lies in $\text{mod}(kQ)$;

(b) there is an integer $N \geq 0$ such that the $F$-orbit of each indecomposable of $D_Q$ contains an object $\Sigma^nU$, for some $0 \leq n \leq N$ and some indecomposable object $U$ of $\text{mod}(kQ)$.

**Remark 3.2.** The functor $\Sigma^n \circ \tau^{-1}$ satisfies these conditions for all $n \geq 0$ (we are in the Dynkin case, so $n = 0$ is included). These are the functors used to define higher cluster categories, so Theorem 3.1 holds for these kind of categories.

**Theorem 3.3.** Let $\Delta$ be the simply laced Dynkin diagram which underlies the quiver $Q$. Let $F$ be the autoequivalence $\Sigma \circ \tau^{-1} : D_Q \to D_Q$ and $C \subset \mathbb{Z}Q$ be an admissible configuration invariant under $F$.

(i) Then $\hat{\text{gpr}}(S_C)/F_\ast$ is a Frobenius 2-Calabi-Yau realization (in the sense of [16]) of a cluster algebra with geometric coefficients of type $\Delta$.

(ii) If $C = \mathbb{Z}Q$ (i.e. $S_C = S$) then $\hat{\text{gpr}}(S)/F_\ast$ is a 2-Calabi-Yau realization of the cluster algebra with universal coefficients of type $\Delta$ (cf. [14]).

4. Frobenius models of $D_Q/F$

In this section we prove Theorem 3.1.

4.1. The stratifying functor. The functor $\Phi : \text{mod}(S_C) \to D_Q$ of Theorem 2.11 was named in [27] the stratifying functor. It can be used to define a stratification of Nakajima’s graded affine quiver varieties (introduced in [30]) whose strata are parameterized by the
objects of $\text{ind}(\mathcal{D}_Q)$. We will show that $\phi: \text{gpr}(\mathcal{S}_C) \to \mathcal{D}_Q$ is induced by a quasi-functor. The inverse of $\phi: \text{gpr}(\mathcal{S}_C) \to \mathcal{D}_Q$ can be described explicitly as the composition of two functors as follows: on the one hand we consider the path category $kQ$ as a full subcategory of $\mathcal{R}_C$ via the embedding $i \mapsto (i, 0)$. The restriction functor gives a functor $kQ \to \mathcal{S}_C$ taking $x$ to $\text{res}(x^\wedge)$. It gives rise to a $kQ$-$\mathcal{S}_C$-bimodule $X$ given by

$$X(u, x) = \text{Hom}(u^\wedge, \text{res}(x^\wedge)), \text{ for } x \in Q_0 \text{ and } u \in \sigma(C)$$

and therefore a functor

$$L \otimes_{kQ} X : \mathcal{D}_Q \to \mathcal{D}^b(\text{mod}(\mathcal{S}_C)).$$

Notice that for every $kQ$-module $M$ the $\mathcal{S}_C$-module $M \otimes_{kQ} X$ lies in $\text{gpr}(\mathcal{S}_C)$. By definition, the derived category of $\text{gpr}(\mathcal{S}_C)$ (as an exact category) can be identified with a full triangulated subcategory of $\mathcal{D}^b(\text{mod}(\mathcal{S}_C))$. Therefore, we can consider the derived tensor product as a triangulated functor

$$L \otimes_{kQ} X : \mathcal{D}_Q \to \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)).$$

On the other hand, since $\text{gpr}(\mathcal{S}_C)$ is a Frobenius category, there is a canonical triangulated functor

$$\text{can} : \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \to \text{gpr}(\mathcal{S}_C).$$

constructed as follows. Let $\mathcal{P}$ be the full subcategory of $\text{gpr}(\mathcal{S}_C)$ formed by its projective objects and denote by $\mathcal{H}^-(\mathcal{P})$ the homotopy category associated to the category of bounded above complexes with components in $\mathcal{P}$. There is a functor

$$p : \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \to \mathcal{H}^-(\mathcal{P})$$

sending a complex $X$ to a quasi-isomorphic complex $pX \in \mathcal{H}^-(\mathcal{P})$. If $P^\bullet$ is a complex in $\mathcal{H}^-(\mathcal{P})$ and $p \in \mathbb{Z}$ is small enough, then the objects $\Sigma^{-p}(Z^p(P^\bullet)) \in \text{gpr}(\mathcal{S}_C)$ are canonically isomorphic. Moreover, if $P_1^\bullet$ and $P_2^\bullet$ are quasi-isomorphic complexes in $\mathcal{H}^-(\mathcal{P})$ and $p \ll 0$ then we have an isomorphism $\Sigma^{-p}(Z^p(P_1^\bullet)) \cong \Sigma^{-p}(Z^p(P_2^\bullet))$ in $\text{gpr}(\mathcal{S}_C)$. Thus there is a functor

$$t : \mathcal{H}^-(\mathcal{P}) \to \text{gpr}(\mathcal{S}_C)$$

sending a complex $P^\bullet$ to $t(P^\bullet) := \Sigma^{-p}(Z^p(P^\bullet))$ for some $p \ll 0$ (which depends on $P^\bullet$). Then $\text{can} : \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \to \text{gpr}(\mathcal{S}_C)$ is defined as the composition of the functors described above

$$\text{can} = t \circ p.$$

Finally, the functor $\phi^{-1} : \mathcal{D}_Q \to \text{gpr}(\mathcal{S}_C)$ is given by the composition

$$\phi^{-1} : \mathcal{D}_Q \xrightarrow{? \otimes_{kQ} X} \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \xrightarrow{\text{can}} \text{gpr}(\mathcal{S}_C).$$

**Remark 4.1.** The fact that $\phi^{-1}$ is a triangle equivalence is proved in Section 5 of [27].
4.2. A dg lift of $\phi^{-1}$. In this subsection we prove a technical result which we shall need to compare the triangulated structures on the orbit categories that we consider. Namely, we prove that the functor $\phi^{-1}$ admits a dg lift in the sense of [25]. In other words, we prove that $\phi^{-1}$ is the triangulated functor associated to a quasi-functor between pretriangulated dg categories.

**Notation 4.2.** Let $C \subset \mathbb{Z}Q_0$ be an admissible configuration. For the rest of the paper we let $\mathcal{E}_C$ be the Frobenius category $\text{gpr}(S_C)$. In the rest of this section we will drop the subindex $C$ and for simplicity just write $\mathcal{E}$. We let $\mathcal{P}$ be the full subcategory of $\mathcal{E}$ formed by its projective objects.

Let $C^b(\text{proj}(kQ))_{dg}$ be the dg category of bounded complexes of projective $kQ$-modules and $\mathcal{A}c(\mathcal{P})_{dg}$ be the dg category of acyclic complexes of objects of $\mathcal{P}$. The dg categories $C^b(\text{proj}(kQ))_{dg}$ and $\mathcal{A}c(\mathcal{P})_{dg}$ are dg enhancements of the categories $DQ$ and $\mathcal{E}$, respectively. The dg quotient

$$D^b(\mathcal{E})_{dg} := C^b(\mathcal{E})_{dg} / \mathcal{A}c^b(\mathcal{E})_{dg}$$

is a dg enhancement of $D^b(\mathcal{E})$. We will show that the equivalence $\phi^{-1}$ is the triangulated functor associated to a quasi-functor $C^b(\text{proj}(kQ))_{dg} \rightarrow \mathcal{A}c(\mathcal{P})_{dg}$. To prove that the functor $\text{can} : D^b(\mathcal{E}) \rightarrow \mathcal{E}$ is algebraic we need the following.

**Definition 4.3.** Let $\text{dgcat}_k$ denote the category of small dg categories over $k$. It admits the structure of a model category whose weak equivalences are the quasi-equivalences (cf. [41]). Let $\text{Hqe}$ denote the associated homotopy category. Let $\mathcal{C}$ be a small dg category and $\mathcal{B}$ a full dg subcategory of $\mathcal{C}$. We say that a morphism $G : \mathcal{C} \rightarrow \mathcal{C}'$ in $\text{Hqe}$ annihilates $\mathcal{B}$ if its associated functor $H^0(G) : H^0(\mathcal{C}) \rightarrow H^0(\mathcal{C}')$ takes all objects of $\mathcal{B}$ to zero objects.

**Theorem 4.4.** ([24, 42]) There is a morphism $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{B}$ of $\text{Hqe}$ which annihilates $\mathcal{B}$ and is universal among the morphisms annihilating $\mathcal{B}$. Moreover, the morphism $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{B}$ induces an equivalence between the category of quasi-functors $\mathcal{C}/\mathcal{B} \rightarrow \mathcal{C}'$ into the category of quasi-functors $\mathcal{C} \rightarrow \mathcal{C}'$ whose associated functor annihilates $\mathcal{B}$.

**Proposition 4.5.** There is a quasi-functor

$$\tilde{\phi}^{-1} : C^b(\text{proj}(kQ))_{dg} \rightarrow \mathcal{A}c(\mathcal{P})_{dg}$$

such that $H^0(\tilde{\phi}^{-1}) \cong \phi^{-1}$.

**Proof.** Recall that $\phi^{-1}$ is defined as the composition

$$\phi^{-1} : DQ \xrightarrow{\gamma^b_{S_1QX}} D^b(\mathcal{E}) \xrightarrow{\text{can}} \mathcal{E}.$$
It is enough to prove that these two functors admit a dg lift. Let $P^\bullet X$ be a projective resolution of $X$ as a $kQ\cdot S_C$-bimodule. We can compose the dg functor

$$-\otimes_{kQ} P^\bullet X : C^b(\text{proj}(kQ))_{dg} \to C^b(E)_{dg}$$

with the canonical dg functor $C^b(E)_{dg} \to D^b(E)_{dg}$ to obtain a dg lift of $? \otimes_{kQ} X : D_Q \to D^b(E)$). Recall that every module $M \in E$ is of the form $M \cong Z^0(P_M)$, for some complex $P_M \in Ac(P)$. Moreover, since projective resolutions and injective coresolutions of objects in $E$ can be chosen functorially, there is a faithful functor $i : E \to Ac(P)$. Therefore, we can consider $C^b(E)$ as a subcategory of the category $Bi(P)$ of double complexes with components in $P$ (see for instance Sign Trick 1.2.5 of [43]). Let $\text{Tot}(B)$ be the completed total complex associated to a double complex $B \in Bi(P)$. By construction, we have that $\text{can}(M) \cong H^0(\text{Tot}(i(M)))$.

The composition $\text{Tot} \circ i : C^b(E) \to Ac(P)$ defines a dg functor $\text{Tot} \circ i : C^b(E)_{dg} \to Ac(P)_{dg}$ which annihilates $Ac^b(E)_{dg}$. In light of Theorem 4.4, there is a quasi-functor

$$\overline{\text{can}} : D^b(E)_{dg} \to Ac(P)_{dg}$$

whose associated triangle functor is $\text{can} : D^b(E) \to E$. Finally, we can define $\phi^{-1}$ as the quasi-functor associated to the composition of dg functors

$$C^b(\text{proj}(kQ))_{dg} \xrightarrow{\otimes_{kQ} P^\bullet X} C^b(E)_{dg} \xrightarrow{\text{can}} Ac(P)_{dg}.$$ 

\[\square\]

4.3. **Proof of Theorem 3.1.** Recall that we are given an exact autoequivalence $F_* : E \rightarrow E$ extending a triangle functor $F : D_Q \to D_Q$ such that $D_Q/F$ is Hom-finite and triangulated with respect to $\tilde{F}$ (see (3.1)). Since $F_*$ is exact, it restricts to an equivalence $F_* : P \to P$ and therefore it induces a triangulated functor $F_* : E \to E$. Let $\tilde{F}_* : Ac(P)_{dg} \to Ac(P)_{dg}$ be the dg functor defined as $F_*$ componentwise. Note that $\tilde{F}_*$ it induces $F_*$ in homology, i.e. $H^0(\tilde{F}_*) = F_*$. 

**Lemma 4.6.** Under the above assumptions, the dg category $Ac(P)_{dg}/\tilde{F}_*$ is quasi-equivalent to $C^b(\text{proj} kQ)_{dg}/\tilde{F}$. In particular, the triangulated hull of $E/\tilde{F}_*$ is triangle equivalent to the triangulated hull of $D_Q/F$.

**Proof.** Let $\tilde{\phi}$ be an inverse of the quasi-functor $\tilde{\phi}^{-1}$. In view of the universal property of the triangulated hull (see Section 9.4 of [25]) it is enough to show that the diagram
is triangulated and triangle equivalent to \( D \).

**Proof of Theorem 3.1.** (i) By Lemma 4.6 the category \( \mathcal{E}/F_* \) is triangulated and triangle equivalent to \( D_Q/F \). By Theorem 42 of [29], \( \mathcal{E}/F_* \) admits the structure of a Frobenius category whose stable category is triangle equivalent to \( D_Q/F \).

(ii) It is clear that \( \mathcal{E}/F_* \) is \( k \)-linear. By Theorem 42 of [29], we know that it is Krull-Schmidt (see also Lemma 35 of loc. cit.). It is \( Ext^1 \)-finite since \( D_Q/F \) is \( Hom \)-finite. So \( \mathcal{E}/F_* \) satisfies condition (P0).

We know that the canonical projection \( p : D_Q \to D_Q/F \) is exact. Then by the proof of Proposition 1.3 of [7] we have that \( D_Q/F \) satisfies condition (P1) and that its AR quiver is \( \mathbb{Z}Q/F \). Therefore, \( \mathcal{E}/F_* \) satisfies (P1) as well.

Let \( p(z^\wedge) \) be an indecomposable projective object of \( \mathcal{E}/F_* \) (See Remark 2.12 and Remark 2.17). So \( z \) is some frozen vertex of \( \mathbb{Z}Q_C \). Let \( f \) be the morphism in \( \mathcal{E} \) corresponding to the arrow \( z \to \sigma^{-1}(z) \) in \( \mathbb{Z}Q_C \) under the Yoneda embedding \( \mathcal{R}_C \to \mathcal{E} \). By Theorem 2.14, \( f \) is an irreducible morphism. We claim that the image of \( f \) under the canonical projection \( p : \mathcal{E} \to \mathcal{E}/F_* \) is an irreducible morphism. Let \( p(x^\wedge) \in \mathcal{E}/F_* \) be an indecomposable object which is not isomorphic to \( p(z^\wedge) \). Let \( h : p(z^\wedge) \to p(x^\wedge) \) be a non-zero morphism in \( \mathcal{E}/F_* \). Then \( h = (h_i) \), with \( h_i : z^\wedge \to F^i_*(x^\wedge) \). Since \( p(z^\wedge) \) is not isomorphic to \( p(x^\wedge) \), we have that \( F^i(x) \neq z \) for all \( i \in \mathbb{Z} \). Since \( f \) is irreducible, we have that \( h_i = g_i \circ f \) for some \( g_i : \sigma^{-1}(z)^\wedge \to F^i_*(x^\wedge) \). Then \( h = (g_i) \circ p(f) \). This shows that \( p(f) \) is left almost split and that \( p(f) \) is the only irreducible morphism with source \( p(z^\wedge) \). Similarly, let \( g \) be the irreducible morphism in \( \mathcal{E} \) corresponding to the arrow \( \sigma(z) \to z \) in \( \mathbb{Z}Q_C \). As before we conclude that \( p(g) \) is the only irreducible morphism of \( \mathcal{E}/F_* \) ending on \( \sigma(z)^\wedge \). This shows that the map \( p(x^\wedge) \to x \) induces an isomorphism between the AR quiver of \( \mathcal{E}/F_* \) and the quiver \( \mathbb{Z}Q_C/F \). Moreover, the frozen vertices of \( \mathbb{Z}Q_C \) correspond to the projective-injective objects of \( \mathcal{E}/F_* \).

To complete the proof of part (ii) it only remains to show that \( \mathcal{E}/F_* \) is standard. We have to check that there are no relations other that those induced by the mesh relations associated to non-projective indecomposable objects. Let \( r \in \mathcal{E}/F_*(X,Y) \) be a relation in \( \mathcal{E}/F_* \). So \( r \) is a zero morphism that can be expressed as a finite sum non-zero morphism,
that is
\[ r = \sum_{i=1}^{k} f_i \]
where \( f_i = (f_{ij})_{j \in \mathbb{Z}} \in \mathcal{E}/F_*(X,Y) \) is a non-zero morphism for each \( i \) and \( f_{ij} \in \mathcal{E}(X,F^j(Y)) \).
In particular, we have that \( r_j := f_{ij} + \cdots + f_{kj} = 0 \) for all \( j \in \mathbb{Z} \), i.e. \( f_{ij} + \cdots + f_{kj} \) is a relation in \( \mathcal{E} \). Since \( \mathcal{E} \) is standard we have that \( r_{ij} \) is induced by the non-frozen mesh relations of its AR quiver. Since \( p : \mathcal{E} \to \mathcal{E}/F_* \) is exact, it follows that the same holds for \( \mathcal{E}/F_* \).

(iii) It only remains to check that \( C \) is admissible. Let \( \pi(x^\wedge) \) be an inflation, where \( I \) is an injective object of \( \mathcal{E} \) and \( f_i : x^\wedge \to F^i(I) \) is a morphism of \( \mathcal{E} \). In particular, there is a path \( p \) from \( x \) to \( \sigma^{-1}(c) \) for some \( c \in C \). To finish the proof we can proceed exactly as in the proof of Theorem 2.14 given [27].

5. Categorification

In this section we prove Theorem 3.3. First, we recall the notion of cluster algebra with universal coefficients. We assume that the reader has some familiarity with cluster algebras, for general background on this theory we refer to the articles [13, 14].

5.1. Universal coefficients for finite-type quivers. Let \( n \) be a positive integer, \([1,n] := \{1,\ldots,n\} \) and \( T^n \) be the \( n \)-regular tree. To construct a rank \( n \) cluster algebra with coefficients on a semifield \( \mathbb{P} \) one needs to specify the following initial information:

- a quiver \( Q \) without cycles of length 1 or 2 such that \( Q_0 = [1,n] := \{1,\ldots,n\} \);
- an \( n \)-tuple of coefficients \( y = (y_1,\ldots,y_n) \in \mathbb{P}^n \).

We always assume that the initial cluster is \((x_1,\ldots,x_n)\). The cluster algebra associated to this data will be denoted by \( \mathcal{A}_{\mathbb{P}}(Q,y) \) and is a subalgebra of \( \mathbb{P}[x_1^{\pm 1},\ldots,x_n^{\pm 1}] \). The set of cluster variables of \( \mathcal{A}_{\mathbb{P}}(Q,y) \) will be denoted by \( \{x_{i,t}\} \) where \( i \in [1,n] \) and \( t \in T \). Similarly, its set of coefficients will be denoted by \( \{y_{i,t}\} \).

**Definition 5.1.** An **ice quiver** consist of a pair \((Q,f)\), where \( Q \) is a quiver together with a distinguished subset \( f \subset Q_0 \) whose elements are called frozen vertices such that there are no arrows between any two vertices of \( f \). When we refer to ice quivers we will assume that \( Q_0 \setminus f = [1,n] \) and \( f = [n+1,n+m] \).

**Remark 5.2.** Cluster algebras with coefficients in a tropical semifield are called geometric cluster algebra. Note that a frozen quiver determines the initial information of a geometric cluster algebra. Therefore, we will systematically define geometric cluster algebras using ice quivers.

**Definition 5.3.** Let \( \mathcal{A} = \mathcal{A}_{\mathbb{P}}(Q,y) \) and \( \overline{\mathcal{A}} = \mathcal{A}_{\overline{\mathbb{P}}}(Q,y') \) be cluster algebras with sets of cluster variables \( (x_{i,t}) \) and \( (\overline{x}_{i,t}) \), respectively. We say that \( \overline{\mathcal{A}} \) is obtained from \( \mathcal{A} \) by a **coefficient specialization** if there is a (unique) homomorphism of multiplicative groups \( \varphi : \mathbb{P} \to \overline{\mathbb{P}} \) such that 

\[ \varphi(y_i) = \overline{y}_i \]
$\mathbb{P} \to \mathbb{P}'$ that extends to a (unique) ring homomorphism $\varphi: \mathcal{A} \to \overline{\mathcal{A}}$ such that $\varphi(x_{i,t}) = \overline{x}_{i,t}$ for all $i$ and all $t$. We call $\varphi$ a coefficient specialization.

**Remark 5.4.** By Proposition 12.2 of [14] we know that $\varphi$ is a coefficient specialization if and only if $\varphi(y_{i,t}) = \overline{y}_{i,t}$ and $\varphi(y_{i,t} \oplus 1) = \overline{y}_{i,t} \oplus 1$ for all $y_{i,t}$. Here $t \mapsto (\overline{y}_t, B_t)$ (resp. $t \mapsto (y_t, B_t)$) is the underlying $Y$-pattern for $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$).

**Definition 5.5.** We say that a cluster algebra $\mathcal{A}_\mathbb{P}(Q, y)$ has universal coefficients if every cluster algebra of the form $\mathcal{A}_{\mathbb{P}'}(Q, y')$ is obtained from $\mathcal{A}_\mathbb{P}(Q, y)$ by a (unique) coefficient specialization.

The existence of a cluster algebra with universal coefficients is not clear. If it exists, then it can be regarded as an invariant of the mutation class of a quiver (these cluster algebras do not depend on the choice of initial seed). In [14], the authors constructed cluster algebras with universal coefficients for any cluster-finite quiver (i.e. a quiver mutation equivalent to a Dynkin quiver). In a series of papers [34, 35, 36], Reading studied the existence of universal coefficients for cluster algebras with geometric coefficients beyond finite-type. For the convenience of the reader, we recall the construction of finite-type cluster algebras with universal coefficient worked out in [14].

**Definition 5.6.** Let $\Delta$ be a simply-laced Dynkin and $\Phi^\Delta$ its root system. We fix a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ and denote by $\Phi_{\geq -1}$ the set of almost positive roots, i.e. the union of the set of positive roots $\Phi_{>0}$ and the set of negative simple roots. If $\alpha \in \Phi^\Delta$, then $[\alpha : \alpha_i]$ denotes the multiplicity of $\alpha_i$ in $\alpha$. Let $\mathbb{P}_\Delta^{univ} := \text{Trop}(p_\alpha : \alpha \in \Phi_{\geq -1})$ be the tropical semifield whose generators $p_\alpha$ are labeled by the set $\Phi_{\geq -1}$.

Recall that every bipartite orientation $Q$ of $\Delta$ is endowed with a function $\varepsilon: Q_0 \to \{+, -\}$ that associates a sign to each vertex:

$$\varepsilon(i) = \begin{cases} + & \text{if } i \text{ is a source,} \\ - & \text{if } i \text{ is a sink.} \end{cases}$$

**Theorem 5.7.** ([14, Theorem 12.4]) Let $Q$ be a bipartite orientation of a simply laced Dynkin diagram $\Delta$. Let $\mathcal{A}_Q^{univ}$ be the cluster algebra with coefficients defined over $\mathbb{P}_\Delta^{univ}$ whose initial quiver is $Q$ and whose $n$-tuple of initial coefficients is $(y_1, \ldots, y_n)$ where

$$y_j = \prod_{\alpha \in \Phi_{\geq -1}} p_\alpha^{\varepsilon(j)[\alpha : \alpha_j]}.$$  

Then $\mathcal{A}_Q^{univ}$ has universal coefficients.

**Remark 5.8.** Since $\mathcal{A}_Q^{univ}$ turns out to be a geometric cluster algebra it can be completely described by an ice quiver. The equation (5.1) allows us to describe at once an ice quiver determining $\mathcal{A}_Q^{univ}$.
5.2. Frobenius 2-CY realizations. Throughout this subsection we will assume that all categories are Krull-Schmidt. We refer the reader to [28] for background on Krull-Schmidt categories. Let $\mathcal{E}$ be a Frobenius category. We say that $\mathcal{E}$ is stably 2-Calabi-Yau (stably 2-CY for short) if its stable category $\mathcal{E}$ is $\text{Hom}$-finite and 2-CY as a triangulated category. This means that for every pair of objects $X$ and $Y$ of $\mathcal{E}$ there is a bifunctorial isomorphism

$$\text{Hom}_E(X,Y) \cong D\text{Hom}_E(Y,\Sigma^2(X)),$$

where $\Sigma$ is the suspension functor of $\mathcal{E}$. Note that every stably 2-CY Frobenius category is $\text{Ext}^1$-finite (i.e. for every pair of objects $X$ and $Y$ of $\mathcal{E}$ the dimension of $\text{Ext}^1_E(X,Y)$ is finite).

**Definition 5.9.** Suppose that $\mathcal{C}$ is either an $\text{Ext}^1$-finite Frobenius category or a $\text{Hom}$-finite triangulated category. A cluster-tilting subcategory of $\mathcal{C}$ is a full additive subcategory $T \subset \mathcal{C}$ which is stable under taking direct factors and such that

(i) for each object $X$ of $\mathcal{C}$ the functors $\mathcal{C}(X,?)|_T : T \to \text{Mod}(k)$ and $\mathcal{C}(?,X) : T^{op} \to \text{Mod}(k)$ are finitely generated;

(ii) an object $X$ of $\mathcal{E}$ belongs to $T$ if and only if we have $\text{Ext}^1_E(T,X) = 0$ for all objects $T$ of $\mathcal{T}$.

An object $T$ is called cluster-tilting if it is basic (its indecomposable summands are pairwise distinct) and $\text{add}(T)$ is a cluster-tilting subcategory. Equivalently, $T$ is cluster-tilting if and only if it is rigid and each object $X$ satisfying $\text{Ext}^1_E(T,X) = 0$ belongs to $\text{add}(T)$.

**Remark 5.10.** We will identify a basic cluster-tilting object $T$ with the cluster-tilting subcategory $\text{add}(T)$ and occasionally refer to $T$ as a cluster-tilting subcategory.

The following result is straightforward and its proof is left to the reader.

**Lemma 5.11.** Let $\mathcal{E}$ be an $\text{Ext}^1$-finite Frobenius category. Then the natural projection $\mathcal{E} \to \mathcal{E}$ induces a bijection between the cluster-tilting subcategories of $\mathcal{E}$ and the cluster-tilting subcategories of $\mathcal{E}$. 

The quiver of an additive subcategory $T \subset \mathcal{C}$ is denoted by $Q_T$. By definition, the set of vertices of $Q_T$ is the set of isomorphism classes of indecomposable objects in $T$. The arrows from the class of an object $X$ to the class of an object $Y$ corresponds to a basis of the space of irreducible maps $\text{rad}(X,Y)/\text{rad}^2(X,Y)$, where $\text{rad}(,)$ denotes the radical in $\mathcal{T}$, i.e. the ideal of all non-isomorphisms from $X$ to $Y$. The quiver of a subcategory $T$ of $\mathcal{C}$ can be thought of as an ice quiver by considering the vertices corresponding to the indecomposable projective objects of $\mathcal{C}$ that lie in $T$ to be frozen and neglect the arrows between them. We will write $Q_T^{fr}$ for the resulting ice quiver. We say that $\mathcal{C}$ has no loops (resp. 2-cycles) if the ice quiver of every cluster-tilting subcategory has no loops (resp. 2-cycles).
Remark 5.12. Let $T$ be a basic object of $\mathcal{C}$ and let $\mathcal{C}(T) = \text{add}(T)$. Consider the basic algebra $A = \text{End}_C(T)$. If $\mathcal{C}$ is Hom-infinite then in general $A$ will be infinite-dimensional. So it is not immediate that $A$ can be expressed as the quotient of a path algebra by an admissible ideal. Nevertheless, the fact that $\mathcal{C}$ is Krull-Schmidt implies that $A$ is a semiperfect ring [28, Corollary 4.4]. By definition, this means that every finitely generated right $A$-module has a projective cover. An equivalent condition is that every simple right $A$-module is of the form $eA/e \text{rad}(A)$, where $e \in A$ is an idempotent (for more background on semi-perfect rings see [2]). Therefore, we can associate a quiver to $A$ computing the dimension of $\text{Ext}^1(S,S')$ for every pair of simple $A$-modules $S$ and $S'$. Then the quiver of $A$ will be equal to $Q_T$.

Definition 5.13 ([8]). Let $\mathcal{E}$ be an $\text{Ext}^1$-finite Frobenius category whose stable category is 2-CY. The cluster-tilting subcategories of $\mathcal{E}$ determine a cluster structure on $\mathcal{E}$ if the following conditions hold:

0) There is at least one cluster-tilting subcategory in $\mathcal{E}$.

1) For each cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{E}$ and each non-projective indecomposable object $X$ of $\mathcal{T}$, there is a non-projective indecomposable object $X^*$ of $\mathcal{E}$, unique up to isomorphism and not isomorphic to $X$, such that the additive subcategory $\mathcal{T}' = \mu_X(\mathcal{T})$ of $\mathcal{E}$ whose set of indecomposable objects

$$\text{ind}(\mathcal{T}') = \text{ind}(\mathcal{T}) \setminus \{X\} \cup \{X^*\}$$

is a cluster-tilting subcategory.

2) In the situation of 1), there are conflations

$$0 \to X^* \xrightarrow{g} E \xrightarrow{h} X \to 0 \quad \text{and} \quad 0 \to X \xrightarrow{s} E' \xrightarrow{t} X^* \to 0,$$

where $h$ and $t$ are minimal right $\text{add}(\mathcal{T} \setminus \{X\})$-approximations and $g$ and $s$ are minimal left $\text{add}(\mathcal{T} \setminus \{X\})$-approximations. We call these sequences the exchange conflations associated to $\mathcal{T}$ and $\mathcal{T}'$.

3) $\mathcal{E}$ has no loops or 2-cycles.

4) For each cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{E}$ and each non-projective indecomposable object $X$ of $\mathcal{T}$ we have that $Q^{fr}_{\mu_X(\mathcal{T})}$ and $\mu_X(Q^{fr}_T)$ are related by quiver mutation, i.e. $Q^{fr}_{\mu_X(\mathcal{T})} = \mu_X(Q^{fr}_T)$. We say that $\mathcal{E}$ (resp.)

Theorem 5.14. ([8, Theorem II.1.6]) Let $\mathcal{E}$ be a stably 2-CY Frobenius category with some cluster-tilting subcategory. If $\mathcal{E}$ has no loops or 2-cycles, then the cluster-tilting subcategories determine a cluster structure for $\mathcal{E}$.

Definition 5.15. [16] A Frobenius category $\mathcal{E}$ is a 2-Calabi-Yau realization of the cluster algebra associated to an ice quiver $(Q,f)$ if $\mathcal{E}$ has a cluster structure and has a cluster-tilting object $T$, such that the ice quiver of $\text{add}(T)$ is $(Q,f)$.

5.3. The quiver of a cluster-tilting subcategory. We return to the context of Nakajima categories and assume that $Q$ is a Dynkin quiver. For the rest of the paper $F$ will denote
the autoequivalence $\Sigma \circ \tau^{-1} : \mathcal{D}_Q \to \mathcal{D}_Q$. Moreover, we let $C \subset ZQ$ be an $F$-invariant configuration, and denote by $\mathcal{E}$ the Frobenius category $\text{gpr}(S_C)$ and by $F_* : \mathcal{E} \to \mathcal{E}$ the exact functor extending $F$.

**Lemma 5.16.** The canonical projection $p : \mathcal{D}_Q \to C_Q$ induces a bijection between the cluster-tilting subcategories of $\mathcal{D}_Q$ and the cluster-tilting subcategories of $C_Q$.

**Proof.** This is a direct consequence of [7, Proposition 2.2] where cluster-tilting subcategories are called Ext-configurations. Explicitly, if $T$ is a basic cluster-tilting object of $C_Q$, then under this bijection the cluster-tilting subcategory $\text{add}(T)$ corresponds to the cluster-tilting subcategory $T = \text{add}(F^j(T) \mid j \in \mathbb{Z})$. Moreover, $\text{add}(T)$ is canonically isomorphic to $T/F$. $\square$

**Remark 5.17.** Lemma 5.11 and Lemma 5.16 together tell us that the cluster-tilting subcategories of $\mathcal{D}_Q$, $C_Q$, $\mathcal{E}$ and $\widehat{\mathcal{E}}/F_*$ are in bijection. Moreover, any cluster-tilting subcategory of $\mathcal{E}/F_*$ is of the form $\tilde{T}/F_*$ for some cluster-tilting subcategory $\tilde{T}$ of $\mathcal{E}$. From now on we fix such a subcategory $\tilde{T}$ and denote its stable category by $\mathcal{T}$. The diagram of Figure 2 is helpful to keep track of the notation we will use. We let $\mathcal{T}/F = \text{add}(T)$, where $T = T_1 \oplus \cdots \oplus T_n$ is a basic cluster-tilting object of $C_Q$ and the $T_i$’s are its indecomposable summands. Moreover, if $P_1, \ldots, P_r$ are the indecomposable projective objects of $\mathcal{E}/F_*$ and $\tilde{T} = T \oplus P_1 \oplus \cdots \oplus P_r$, then $\tilde{T}/F_*$ is equivalent to $\text{add}(\tilde{T})$. We would like to compute the quiver of $\tilde{T}/F_*$. In view of Remark 5.12 it is enough to compute an explicit (co)resolution of the simple $\tilde{T}/F_*$-modules. Moreover, these simple modules are one-dimensional, supported on a single indecomposable object of $\tilde{T}/F_*$. 

![Figure 2. Bijection between cluster-tilting subcategories.](image)

We consider the quotient category $\mathcal{D}_Q/\mathcal{T}$ and let $x^{\wedge}_{D/\mathcal{T}} = \mathcal{D}_Q/\mathcal{T}(?, x)$ be the $\mathcal{D}_Q/\mathcal{T}$-module associated to an object $x$ of $\mathcal{D}_Q$. Note that if $x$ lies in $\mathcal{T}$ then $x^{\wedge}_{D/\mathcal{T}}$ is the zero module. The module $x^{\vee}_{D/\mathcal{T}}$ is defined analogously. Both $x^{\wedge}_{D/\mathcal{T}}$ and $x^{\vee}_{D/\mathcal{T}}$ are in a natural way $\mathcal{R}_C$-modules. Consider the finitely generated projective $\mathcal{R}_C$-module
\[ P_{x,T} := \bigoplus_{\sigma(y) \in C} \mathcal{D}_Q/T(y,x) \otimes \sigma(y)^\wedge. \]

The multiplicity of \( \sigma(y)^\wedge \) in \( P_{x,T} \) equals the dimension of the vector space \( \mathcal{D}_Q/T(y,x) \). In a completely analogous way, we define the finitely cogenerated injective \( \mathcal{R}_C \)-module

\[ I_{x,T} := \prod_{\sigma^{-1}(y) \in C} D\mathcal{D}_Q/T(x,y) \otimes \sigma^{-1}(y)^\vee. \]

The modules of the form \( x^\wedge_{\mathcal{D}/T} \) and \( x^\vee_{\mathcal{D}/T} \) will be very helpful to compute the resolutions of the simple \( \tilde{T} \)-modules. Note that in practice these modules can be computed quite explicitly.

**Definition 5.18.** ([26]) Let \( T = \bigoplus_{i=1}^n T_i \) be the decomposition of \( T \) as a direct sum of indecomposable objects. For any object \( X \) of \( \mathcal{C}_Q \) there is a triangle

\[ T^X_1 \to T^X_0 \to X \to \Sigma T^X_1 \]

where \( T^X_0 = \bigoplus_{i=1}^n T^x_i^{a_i} \) and \( T^X_1 = \bigoplus_{i=1}^n T^x_i^{b_i} \). With this notation, the **index of** \( X \) **with respect to** \( T \) **is the integer vector**

\[ \text{ind}_T(X) = (a_1 - b_1, \ldots, a_n - b_n). \]

**Remark 5.19.** Even though the triangle in (5.2) is not unique, it can be proved that the index is well defined. Moreover, it was proved in [26] that in any such triangle the morphism \( T^X_0 \to X \) is a left \( T \)-approximation. When this morphism is a minimal left \( T \)-approximation we will call the triangle minimal.

**Notation 5.20.** (i) Let \( z = z_1 \oplus \cdots \oplus z_s \) be the decomposition of an object \( z \in \mathcal{D}_Q \) as a direct sum of indecomposable objects (see Remark 2.4). The projective \( \mathcal{R}_C \)-module \( z_1^\wedge \oplus \cdots \oplus z_s^\wedge \) will be denoted by \( z^\wedge \). We proceed analogously for injective modules.

(ii) The Yoneda functor induces an equivalence between \( \mathcal{R}_C \) and \( \text{ind}(\mathcal{E}) \). So any \( \mathcal{R}_C \)-module can be extended to an \( \mathcal{E} \)-module. We denote the extension of \( x^\wedge \) by \( x^\wedge_\tilde{T} \). The restriction of \( x^\wedge_\tilde{T} \) to \( \tilde{T} \) will be denoted by \( x^\wedge_T \). We proceed analogously for injective modules.

Moreover, the \( \mathcal{R}_C \)-modules of the form \( P_{x,T} \) and \( I_{x,T} \) can be first extended to \( \mathcal{E} \) and then restricted to \( \tilde{T} \). The finitely generated projective \( \tilde{T} \)-modules obtained in this way will be denoted by \( P_{x,\tilde{T}} \) and \( I_{x,\tilde{T}} \), respectively.

**Lemma 5.21.** Let \( x \in \mathbb{Z}_Q0 \) be a non-frozen vertex considered as an indecomposable object of \( \mathcal{D}_Q \) (see Remark 2.4). Let \( T^x_i \to T^x_0 \to x \to \Sigma T^x_i \) be a triangle in \( \mathcal{D}_Q \) such that \( T^x_0 \) and \( T^x_i \) are objects of \( T \), and \( T^x_0 \to x \) is a minimal left \( T \)-approximation. Then there is a minimal projective resolution of \( \mathcal{R}_C \)-modules

\[ 0 \to T^x_i \to P_{x,T} \oplus T^x_0 \to x^\wedge \to x^\wedge_{\mathcal{D}/T} \to 0 \]
and a minimal injective coresolution of $\mathcal{R}_C$-modules

\begin{equation}
0 \to x_{D/T}^\vee \to x^\vee \to I_{x,T} \oplus \Sigma T_0^\sigma \to \Sigma T_1^\sigma \to 0.
\end{equation}

Proof. We can lift the triangle $T_1^x \to T_0^x \to x \to \Sigma T_1^x$ to a conflation in $0 \to T_0^x \to P \oplus T_0^\sigma \to x \to 0 \in \mathcal{E}$, where $P$ is a projective object and $P \oplus T_0^\sigma \to x$ is a minimal left $\tilde{T}$-approximation. Applying the left exact functor $(\cdot)^\vee : \mathcal{R}_C \to \text{mod}(\mathcal{R}_C)$ to this conflation we obtain the exact sequence

$$0 \to T_1^x \to P^\vee \oplus T_0^\sigma \to x^\vee \to M \to 0.$$ 

By definition, $M(y)$ corresponds to all morphisms $y \to x$ that cannot be factorized through the minimal left $\tilde{T}$-approximation $P \oplus T_0^\sigma \to x$. That is, all morphism from $y \to x$ in $\mathcal{E}/\tilde{T} \cong D_Q/T$. This shows that $M$ is isomorphic to $x_{D/T}^\vee$. Therefore, we have obtained a projective resolution of $x_{D/T}^\vee$ which is minimal because $T_0^\sigma \to x$ is a minimal left $T$-approximation. It only remains to describe $P^\vee$ explicitly. We know that $P^\vee$ is a direct sum of indecomposable projective $\mathcal{R}_C$-modules. To calculate the multiplicity of $\sigma(y)^\vee$ in $P^\vee$ it is enough to calculate the dimension of $\text{Ext}^1_{\mathcal{R}_C}(x_{D/T}^\vee, S_{\sigma(y)})$, where $S_{\sigma(y)}$ is the simple $\mathcal{R}_C$-module supported in $\sigma(y)$. We do this calculating the value of the derived functor

$$\text{RHom}(x_{D/T}^\vee, S_{\sigma(y)}) = \text{RHom}(x_{D/T}^\vee, \sigma(y)^\vee \to y^\vee) = (0 \to DD/T(y,x)).$$

We conclude that the multiplicity of $\sigma(y)^\vee$ in $P$ is equal to the dimension of $D/T(y,x)$. Which implies that $P^\vee \cong P_{x,T}$. To obtain the injective coresolution we proceed in an analogous way, we apply the right exact functor $(\cdot)^\vee = D(\mathcal{R}_C(\cdot, \cdot)) : \mathcal{R}_C \to \text{mod}(\mathcal{R}_C)$ to a short exact sequence $0 \to x \to I \oplus \Sigma T_0^\sigma \to \Sigma T_1^x \to 0$ lifting the triangle $T_0^x \to x \to \Sigma T_1^x \to \Sigma T_0^\sigma$. We obtain an exact sequence

$$0 \to x_{D/T}^\vee \to x^\vee \to I^\vee \oplus \Sigma T_0^\sigma \vee \to \Sigma T_1^x \vee \to 0.$$ 

As before we can deduce that $I$ is isomorphic to $I_{x,T}$. \hfill $\square$

**Corollary 5.22.** Let $x$ be a vertex in $C$ and $T_1^x \to T_0^x \to x \to \Sigma T_1^x$ be a triangle in $D_Q$ such that $T_0^x$ and $T_1^x$ are objects of $\mathcal{T}$, and $T_0^\sigma \to x$ is a minimal $T$-approximation. Then the minimal projective resolution of the simple $\mathcal{T}$-module $S_{\sigma^{-1}}$ supported on $\sigma^{-1}(x)^\vee$ is given by

$$0 \to T_1^x \to P_{x,T}^\vee \oplus T_0^\sigma \to \sigma^{-1}(x)^\vee \to S_{\sigma^{-1}} \to 0,$$

and its minimal injective coresolution is given by

\begin{equation}
0 \to S_{\sigma^{-1}} \to x_{D/T}^\vee \to I_{x,T} \oplus \Sigma T_0^\vee \to \Sigma T_1^\vee \to 0.
\end{equation}
Proof. From the description of the AR-quiver of \( \mathcal{E} \), we know that the simple \( \mathcal{E} \)-module \( S_{\sigma^{-1}(x)^\wedge} \) supported on the indecomposable object \( \sigma^{-1}(x)^\wedge \) admits a projective resolution

\[
0 \to x_{\mathcal{E}}^\wedge \to \sigma^{-1}(x)^\wedge_{\mathcal{E}} \to S_{\sigma^{-1}(x)^\wedge} \to 0
\]

Applying the restriction functor \( \text{res} : \text{Mod}(\mathcal{E}) \to \text{Mod}(\tilde{T}) \) to this sequence we obtain the exact sequence of \( \tilde{T} \)-modules

\[
0 \to x_{\tilde{T}}^\wedge \to \sigma_{\tilde{T}}(x)^\wedge \to S_{\sigma^{-1}(x)^\wedge} \to 0.
\]

Note that the resolution of \( R_C (5.3) \) induces an exact sequence of \( \tilde{T} \)-modules

\[
0 \to T_{i,\tilde{T}}^x \to P_{x,\tilde{T}} \oplus T_{0,\tilde{T}}^x \to x_{\tilde{T}}^\wedge \to 0.
\]

We can splice these last two sequences to obtain the minimal projective resolution. The minimal injective coresolution is obtained analogously.

To obtain a (co)resolution of the \( \tilde{T} \)-modules associated to non-frozen vertices we need to recall the notion of mutation of cluster-tilting objects in \( C_Q \) (cf. [22]). Let \( T \) be a basic cluster-tilting object of \( C_Q \) and \( T_i \) one of its indecomposable summands. Then the exchange triangles associated to \( T \) and \( T_i \) are triangles of the form

\[
T_i^* \xrightarrow{g} B \xrightarrow{h} T_i \to \Sigma T_i^* \quad \text{and} \quad T_i \xrightarrow{s} B' \xrightarrow{t} T_i^* \to \Sigma T_i,
\]

where \( h \) and \( t \) are minimal right \( \text{add}(\tilde{T} \setminus \{T_i\}) \)-approximations and \( g \) and \( s \) are minimal left \( \text{add}(\tilde{T} \setminus \{T_i\}) \)-approximations. Changing the objects appearing in these triangles by isomorphic objects if necessary, we can assume that these sequences are induced by triangles in \( D_Q \) of the form

\[
T_i^* \xrightarrow{g} B \xrightarrow{h} T_i \to \Sigma T_i^* \quad \text{and} \quad F^{-1}(T_i) \xrightarrow{F^{-1}} B' \xrightarrow{F^{-1}} T_i^* \to \Sigma F^{-1}(T_i),
\]

respectively. We lift these triangles to conflations in \( \mathcal{E} \) of the form

\[
0 \to T_i^* \to E \to T_i \to 0 \quad \text{and} \quad 0 \to F^{-1}(T_i) \to E' \to T_i^* \to 0,
\]

respectively.

Lemma 5.23. Let \( x \) be a vertex of \( \mathbb{Z}Q \) considered as an indecomposable object of \( D_Q \) such that \( p(x) = T_i \). With the notation above, the minimal projective resolution of the simple \( \tilde{T} \)-module \( S_{x^\wedge} \) supported on \( x^\wedge \) is given by

\[
0 \to F^{-1}(x)^\wedge_{\tilde{T}} \to E^\wedge_{\tilde{T}} \to E^\wedge_{\tilde{T}} \to x^\wedge_{\tilde{T}} \to S_{x} \to 0,
\]

the minimal injective coresolution of the simple \( \tilde{T} \)-module \( S_{x^\wedge} \) is given by

\[
0 \to S_x \to x_{\tilde{T}}^\vee \to E_{\tilde{T}}^\vee \to E_{\tilde{T}}^\vee \to F(x)^\vee_{\tilde{T}} \to 0.
\]

Proof. Notice that the exchange conflations remain exact when applying the left exact functor \( ( \quad )^\wedge : \tilde{T} \to \text{mod} \tilde{T} \). We can splice the two resulting sequences to obtain the minimal projective resolution. The coresolution is obtained analogously.
We have built the ground to describe the projective resolutions of the simple \( \hat{T} \tilde{\rightarrow} F \)-modules. By slight abuse of notation, we denote by \( p \) the natural projection \( E \rightarrow \mathcal{E} \tilde{\rightarrow} F \) (we used \( p \) to denote the natural projection \( D_Q \rightarrow C_Q \), but this last projection is induced by \( E \rightarrow \mathcal{E} \tilde{\rightarrow} F \)). Restricting \( p \) to \( \hat{T} \), we obtain a pair of adjoint functors

\[
\begin{array}{ccc}
\mathsf{Mod}(\hat{T}) & \xrightarrow{\pi} & \mathsf{Mod}(\hat{T}/F) \\
p^* & \downarrow & \\
\end{array}
\]

where \( p^* \) is the restriction functor and \( \pi \) its left adjoint. At the level of objects \( p_* \) is defined as \( \pi(M)(p(x)) = \prod_{i \in \mathbb{Z}} M(F^i(x)) \). This makes it clear that \( \pi \) preserves the simple modules. The following lemma was proved in [29] using the fact that \( \pi \) preserves projectives.

**Lemma 5.24.** ([29, Lema 34]) Let \( M \) be a finitely presented \( \hat{T} \)-module.

(i) Let \( L \) be an \( \hat{T} \)-module admitting a resolution \( \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow L \rightarrow 0 \) by finitely generated projective \( \hat{T} \)-modules \( P_i \). Then the complex

\[
\cdots \rightarrow \pi(P_1) \rightarrow \pi(P_0) \rightarrow \pi(L) \rightarrow 0
\]

is a resolution of \( \pi(L) \) by finitely generated projective \( \hat{T}/F \)-modules;

(ii) for each \( \hat{T} \)-module \( L \) admitting a resolution by finitely generated projective \( \hat{T} \)-modules, there are canonical isomorphisms

\[
\mathsf{Ext}_{\hat{T}/F}^i(\pi(L), \pi(M)) \cong \prod_{i \in \mathbb{Z}} \mathsf{Ext}_F^i(L, F_{\pi *}^i(M)).
\]

We can apply \( \pi \) to the (co)resolutions of Corollary 5.22 and Lemma 5.23 to obtain the (co)resolutions of the simple \( \hat{T}/F \)-modules. For simplicity, we denote \( \pi(x_{\hat{T}}^\wedge) \) by \( x_{\hat{T}/F}^\wedge \) and \( \pi(x_{\hat{T}}^\vee) \) by \( x_{\hat{T}/F}^\vee \).

**Corollary 5.25.** Let \( x \) be a vertex in \( C \). Then the minimal projective resolution of the simple \( \hat{T}/F \)-module \( S_{\sigma^{-1}(x)^\wedge} \) associated to the indecomposable object \( p(\sigma^{-1}(x)^\wedge) \) is

\[
0 \rightarrow T_{1,\hat{T}/F}^x \rightarrow T_{0,\hat{T}/F}^x \oplus P_{x,\hat{T}/F} \rightarrow \sigma^{-1}(x)^\wedge \rightarrow S_{\sigma^{-1}(x)^\wedge} \rightarrow 0,
\]

and its minimal injective coreolution is

\[
0 \rightarrow S_{p(\sigma^{-1}(x)^\wedge)} \rightarrow x_{\hat{T}/F}^\vee \rightarrow I_{x,\hat{T}/F} \oplus \Sigma T_0^x \rightarrow \Sigma T_1^x \rightarrow 0.
\]

**Corollary 5.26.** Let \( x \) be a vertex in \( \mathbb{Z}Q \) and suppose that \( x^\wedge \) is an indecomposable object of \( \hat{T} \). Then the minimal projective resolution of the simple \( \hat{T}/F \)-module \( S_{p(x^\wedge)} \) is

\[
0 \rightarrow x_{\hat{T}/F}^\wedge \rightarrow E_{1,\hat{T}/F}^x \rightarrow E_{0,\hat{T}/F}^x \rightarrow x_{\hat{T}/F}^\wedge \rightarrow S_{p(x^\wedge)} \rightarrow 0
\]
and its minimal injective coresolution is

\[ 0 \to S_p(x^\vee) \to x^\vee_{\overline{T} / F_x} \to E^\vee_{\overline{T} / F_x} \to E^\vee_{\overline{T} / F_x} \to x^\vee_{\overline{T} / F_x} \to 0. \]

5.4. A distinguished labeling of \( \text{ind}(E / F_x) \). From now on we assume \( Q \) is a bipartite orientation of a Dynkin diagram \( \Delta \). Let \( W \) be the Weyl group associated to \( \Phi^\Delta \) and \( s_i \in W \) be the reflection associated to the simple root \( \alpha_i \). Following [15] we define two bijections \( \tau_{\pm} : \Phi_{\geq -1} \to \Phi_{\geq -1} \) by the formula

\[
\tau_\varepsilon(\alpha) = \begin{cases} 
\alpha & \text{if } \alpha = -\alpha_j \text{ with } \varepsilon(j) = -\varepsilon; \\
t_\varepsilon(\alpha) & \text{otherwise}
\end{cases}
\]

where

\[
t_\varepsilon = \prod_{\varepsilon(k) = \varepsilon} s_k.
\]

These elements are well defined because the reflections associated to vertices having the same sign mutually commute. It follows from the same fact that each \( t_\varepsilon \) is an involution which makes it evident that each \( \tau_\varepsilon \) is bijective.

**Notation 5.27.** Let \( i \) be a vertex of \( Q \). Denote by \( P(i), I(i) \) and \( S(i) \) the projective, injective and simple module associated to \( i \), respectively. We consider the objects of \( \text{mod}(kQ) \) as objects in \( D_Q \) via the canonical embedding \( \text{mod}(kQ) \to D_Q \) which sends a module to a complex concentrated in degree 0.

**Remark 5.28.** Let \( \text{ind}(kQ) \) denote the set of indecomposable right \( kQ \)-modules and \( \Phi_{>0} \) be the set of positive roots. The function \( \text{ind}(kQ) \to \Phi_{>0} \) given by

\[ M \mapsto \sum_{i=1}^n d_i \alpha_i, \]

where \( \dim(M) = (d_1, \ldots, d_n) \) is the dimension vector of \( M \), is a bijection. We denote by \( M_\alpha \) the indecomposable module corresponding to \( \alpha \in \Phi_{>0} \). It is well known that the set of indecomposable object of \( C_Q \) is the union of the set of indecomposable \( kQ \)-modules and the set \( \{ \Sigma P(i) \}_{i=1}^n \) formed by the indecomposable projective \( kQ \)-modules shifted by \( \Sigma \). Therefore, the bijection \( \text{ind}(kQ) \to \Phi_{>0} \) can be extended to a bijection \( \text{ind}(C_Q) \to \Phi_{\geq -1} \) by sending the object \( \Sigma P(i) = \tau P(i) \) to the negative simple root \( -\alpha_i \). In particular, the suspension functor \( \Sigma = \tau : C_Q \to C_Q \) induces a bijection (which we still call) \( \tau : \Phi_{\geq -1} \to \Phi_{\geq -1} \). Moreover, it can be verified that \( \tau_- \circ \tau_+ = \tau \).

For each vertex \( i \) of \( Q \) let \( C_i : \text{mod}(kQ) \to \text{mod}(k\mu_i(Q)) \) be the Bernstein-Gelfand-Ponomarev reflection functor (cf. [5]). The functor \( C_i \) induces an autoequivalence

\[ C_D : D_Q \to D_Q \]
Corollary 5.30. We know that \( \varepsilon(i) = \varepsilon \). The following is a well known result.

Lemma 5.29. The functors \( C^\varepsilon_D \) satisfy the following properties:

- \( C^\varepsilon_D \circ C^\varepsilon_D \cong \text{Id} \),
- \( \Sigma \circ C^\varepsilon_D \cong C^\varepsilon_D \circ \Sigma \),
- \( C^\varepsilon_D(P(i)) = \Sigma^{-\varepsilon}I(i) \),
- if \( \alpha \) is not a simple root, then \( C^\varepsilon_D(M_\alpha) = M_{r_\alpha(\alpha)} \).

Corollary 5.30. Let \( M_\alpha \) be the indecomposable \( kQ \)-module corresponding to \( \alpha \in \Phi_{>0} \). Let \( i \) be a vertex of \( Q \). If \( i \) is a source then

\[
[\tau_+(\alpha) : \alpha_i] = \dim \text{Ext}^1_{kQ}(M_\alpha, S(i)).
\]

If \( i \) is a sink of \( Q \) then

\[
[\tau_-(\alpha) : \alpha_i] = \dim \text{Ext}^{-1}_{kQ}(M_\alpha, S(i)).
\]

Proof. We know that \( [\tau_\varepsilon(\alpha) : \alpha_i] = \dim \text{Hom}(M_{r_\varepsilon(\alpha)}, I(i)) \). By Lemma 5.29 we have the following isomorphisms

\[
\text{Hom}(M_{r_\varepsilon(\alpha)}, I(i)) \cong \text{Hom}(M_{r_\varepsilon(\alpha)}, \Sigma^\varepsilon C^\varepsilon_D(P(i)))
= \text{Hom}(C^\varepsilon_D(M_{r_\varepsilon(\alpha)}), \Sigma^\varepsilon P(i))
= \text{Hom}(M_{r_\varepsilon(\alpha)}, \Sigma^\varepsilon P(i))
= \text{Hom}(M_\alpha, \Sigma^\varepsilon P(i)).
\]

If \( i \) is a source then \( P(i) = S(i) \). If \( i \) is a sink then \( P(i) = S(i) \). The claim follows. \( \Box \)

Let \( C = \mathbb{Z}Q_0 \) be the maximal admissible configuration (so \( R_C = R \)). We identify the objects of \( C_Q \) with the non-projective objects of \( E/F_\ast \). In particular, the bijection \( \text{ind}(C_Q) \to \Phi_{\geq -1} \) of Remark 5.28 can be used to label the indecomposable non-projective objects of \( E/F_\ast \). Indeed, we denote by \( X_\alpha \) the non-projective indecomposable object of \( E/F_\ast \) corresponding to \( \alpha \in \Phi_{\geq -1} \) under the bijection. We can extend this labeling to the indecomposable projective objects of \( E/F_\ast \) as follows: let \( P \) be an indecomposable projective object of \( E/F_\ast \). Then \( P = p(y^\wedge) \) for some frozen vertex \( y \) of \( \mathbb{Z}Q \). The indecomposable non-projective object \( p(\sigma(y)^\wedge) \) is isomorphic to \( X_\alpha \) for some \( \alpha \in \Phi_{\geq -1} \). We label \( P \) by \( \tau_+(\alpha) \) and write \( P_{\tau_+(\alpha)} \). Note that the indecomposable projective object \( p(\sigma^{-1}(x)^\wedge) \) is labeled by \( \tau_-(\alpha) \) because \( \tau_-(\alpha) = \tau_+ \circ \tau^{-1}(\alpha) \).

Example 5.31. Let \( Q : 1 \to 2 \leftarrow 3 \) be a bipartite quiver of type \( A_3 \). There are two \( \tau \)-orbits in \( \Phi_{\geq -1} \) that can be visualized as follows:

\[
\begin{array}{cccccccc}
-\alpha_1 & \tau_+ & \alpha_1 & \tau & \alpha_1 + \alpha_2 & \tau_+ & \alpha_2 + \alpha_3 & \tau & \alpha_3 & \tau_+ & -\alpha_3 \\
\tau_- & \tau_- & \tau_- & \tau_- & \tau_- & \tau_- & \tau_- & \tau_- & \tau_- & \tau_- & \tau_-
\end{array}
\]
In Figure 3 below we have represented the AR quiver of the category $E^\hat{}/F_\ast$. Its vertices are labeled by the rule described above.

5.5. **Proof of Theorem** 3.3. We are ready to prove our second main result. Let $C = \mathbb{Z}Q_0$ be the maximal admissible configuration. Throughout this subsection we let $T$ be the object $\bigoplus_{i \in Q_0} X_{-\alpha_i} \in C_Q$. Clearly, $T$ is a cluster-tilting object whose quiver is $Q$. Let $T/F$ be the associated cluster-tilting subcategory of $C_Q$ (see Remark 5.17). Note that the exchange triangles associated to $X_{-\alpha_i}$ and $T$ are of the form

$$X_{\alpha_i} \rightarrow B \rightarrow X_{-\alpha_i} \rightarrow 0$$

and

$$X_{-\alpha_i} \rightarrow B' \rightarrow X_{\alpha_i} \rightarrow \Sigma X_{-\alpha_i}.$$

**Proposition 5.32.** Let $i$ be a vertex of $Q$.

(i) If $i$ is a source then the exchange conflations associated to $X_{-\alpha_i}$ and $\hat{T}/F_\ast$ have the form:

$$0 \rightarrow X_{-\alpha_i} \rightarrow \bigoplus_{i \rightarrow j} X_{-\alpha_j} \oplus P_{-\alpha_i} \rightarrow X_{\alpha_i} \rightarrow 0$$

and

$$0 \rightarrow X_{\alpha_i} \rightarrow \bigoplus_{\alpha \in \Phi_{>0}} P_\alpha^{[\alpha; \alpha_i]} \rightarrow X_{-\alpha_i} \rightarrow 0.$$

(ii) If $i$ is a sink then the exchange conflations associated to $X_{-\alpha_i}$ and $\hat{T}/F_\ast$ have the form:

$$0 \rightarrow X_{-\alpha_i} \rightarrow \bigoplus_{\alpha \in \Phi_{>0}} P_\alpha^{[\alpha; \alpha_i]} \rightarrow X_{\alpha_i} \rightarrow 0.$$
and

\[ 0 \to X_{\alpha_i} \to \bigoplus_{j \to i} X_{-\alpha_j} \oplus P_{-\alpha_i} \to X_{-\alpha_i} \to 0. \]

**Proof.** Let \( i \) be a source of \( Q \) and \( x \in (\mathbb{Z}\tilde{Q})_0 \) be a non-frozen vertex such that

\[ p(x^\land) = X_{-\alpha_i}. \]

In particular,

\[ p(\sigma^{-1}(x)^\land) = P_{\tau(-\alpha_i)} = P_{-\alpha_i} \quad \text{and} \quad p(\tau^{-1}(x)^\land) = X_{\alpha_i}. \]

The mesh relation starting at \( x \) gives a conflation

\[ 0 \to x^\land \to \bigoplus_{x \to y} y^\land \oplus (\sigma^{-1}(x))^\land \to (\tau^{-1}(x))^\land \to 0 \]

in \( \mathcal{E} \), where the sum ranges over the set of arrows of \( \mathbb{Z}\tilde{Q} \) whose tail is \( x \) and whose head is a non-frozen vertex \( y \). By the exactness of \( p \), there is a conflation

(5.14) \[ 0 \to X_{-\alpha_i} \to \bigoplus_{x \to y} p(y^\land) \oplus P_{-\alpha_i} \to X_{\alpha_i} \to 0 \]

in \( \mathcal{E}/F \). By construction, the image of (5.14) in the stable category \( \mathcal{E}/F_* \cong \mathcal{D}_Q/F \) is the image of the mesh relation in \( \mathcal{D}_Q \) starting at \( \Sigma P(i) \). Thus

\[ \bigoplus_{x \to y} p(y^\land) = \bigoplus_{i \to j \in Q_1} \Sigma P(j). \]

Since \( \Sigma P(j) \) corresponds to \(-\alpha_j\) under the bijection \( \text{ind}(C_Q) \to \Phi_{\geq -1} \) the sequence (5.14) identifies with the sequence (5.12).

Since \( i \) is a source, there is a triangle

\[ X_{\alpha_i} \to 0 \to X_{-\alpha_i} \to X_{-\alpha_i} \]

in \( C_Q \). Therefore, in this case, the exact sequence of \( \mathcal{R} \)-modules in (5.3) is given by

\[ 0 \to (\Sigma^{-1}x)^\land \to P_{x,T} \to x^\land \to x_{D/T}^\land \to 0, \]

and gives rise to the conflation

(5.15) \[ 0 \to \Sigma^{-1}x^\land \to P_{x,T} \to x^\land \to 0 \]

in \( \mathcal{E} \). We apply the canonical projection \( p : \mathcal{E} \to \mathcal{E}/F_* \) to this conflation to obtain the conflation

(5.16) \[ 0 \to \Sigma^{-1}X_{-\alpha_i} \to p(P_{x,T}) \to X_{-\alpha_i} \to 0 \]

of \( \mathcal{E}/F_* \). We claim that (5.13) corresponds to (5.16). Indeed: since \( C = \mathbb{Z}Q_0 \) and

\[ \Sigma^{-1}p(x^\land) = \tau^{-1}p(x^\land) \]
in $E\hat{}/F_\ast$, then (5.16) takes the form

$$0 \to X_{\alpha_i} \to \bigoplus_{y \in \mathbb{Z}Q} D(Q(y, x) \otimes p(\sigma(y)^\wedge) \to X_{-\alpha_i} \to 0.$$  

We have seen that $x = \Sigma P(i) = \Sigma S(i)$. We obtain the following isomorphisms

\begin{equation}
\bigoplus_{y \in \mathbb{Z}Q} D(Q(y, x) = \bigoplus_{\alpha \in \Phi_{\geq 0}} \operatorname{Ext}_{kQ}^1(M_\alpha, S(i)).
\end{equation}

(5.17)

We obtain that the multiplicity of $P_\alpha$ in $P_x$ is equal to the dimension of $\operatorname{Ext}_{kQ}^1(M_{\tau_+(\alpha)}, S_i)$. By Corollary 5.30 we know that $\dim \operatorname{Ext}_{kQ}^1(M_{\tau_+(\alpha)}, S_i) = [\alpha, \alpha_i]$, i.e.

$$p(P(x)) = \bigoplus_{\alpha \in \Phi_{\geq 0}} P_{\alpha}^{[\alpha : \alpha_i]}.$$  

The claim follows. We can treat similarly the case where $i$ is a sink.  

**Proof of Theorem 3.3.** (i) Let $C$ be an admissible configuration. Part (i) states that $E/F_\ast$ is a Frobenius 2-CY realization of a cluster algebra with coefficients of type $Q$. This is a direct consequence of Theorem II.1.6 of [8].

(ii) In view of Remark 5.8, it is enough to prove that there is a cluster-tilting object of $E\hat{}/F_\ast$ whose frozen quiver coincides with the initial coefficients defining $A_{Q}^{\text{univ}}$ described in Theorem 5.7. We claim that

$$\tilde{T} = \bigoplus_{i \in Q_0} X_{-\alpha_i} \oplus \bigoplus_{\alpha \in \Phi_{\geq -1}} P_\alpha$$

satisfies this assertion. It is clear that $\tilde{T}$ is a cluster-tilting object since $\bigoplus_{i \in Q_0} X_{-\alpha_i}$ is a cluster-tilting object of $C_Q$. Let $T = \text{add}(T)$. Denote by $i$ the vertex of $Q^F_T$ corresponding to the indecomposable object $X_{-\alpha_i}$, and by $[\alpha]$ the frozen vertex of $Q^F_T$ corresponding to $P_\alpha$, $\alpha \in \Phi_{\geq -1}$. We can describe $Q^F_T$ using the projective resolutions of the simple $E\hat{}/F_\ast$-modules described in Corollary 5.25 and Corollary 5.26. To compute these resolutions it is enough to describe the exchange conflations associated to $\tilde{T}$ given in Proposition 5.32. If $i$ is a source of $Q$ then there are

- $[\alpha : \alpha_i]$ arrows from $[\alpha]$ to $i$ for each $\alpha \in \Phi_{>0}$,
- one arrow from $i$ to $j$ for each arrow $i \to j$ in $Q$,
- one arrow from $i$ to the vertex corresponding to $-\alpha_i$.

This is precisely the frozen quiver encoding by the initial coefficients of $A_{Q}^{\text{univ}}$ described in Theorem 5.7. We obtain a similar description when $i$ is a sink of $Q$. The claim follows.  

□
6. Examples

Example 6.1. Let $Q : 1 \to 2$ be a Dynkin quiver of type $A_2$. Thus, $\varepsilon(1) = +$ and $\varepsilon(2) = -$. By Theorem 5.7, the cluster algebra with universal coefficients of type $A_2$ is defined over the tropical semifield

$$\mathbb{P} = \text{Trop}(p_{-\alpha_1}, p_{-\alpha_2}, p_{\alpha_1}, p_{\alpha_1+\alpha_2}, p_{\alpha_2}).$$

The cluster algebra $\mathcal{A}^\text{univ}_{A_2}$ is completely determined by the ice quiver

```
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$-\alpha_2$};
  \node (4) at (2,1) {$\alpha_2$};
  \node (5) at (1,1) {$\alpha_1 + \alpha_2$};
  \node (6) at (2,2) {$\alpha_1$};

  \draw[->] (1) -- (2);
  \draw[->] (2) -- (3);
  \draw[->] (5) -- (4);
  \draw[->] (5) -- (3);
  \draw[->] (4) -- (6);
  \draw[->] (3) -- (6);
\end{tikzpicture}
```

where the frozen vertex corresponding to the generator $\alpha \in \Phi_{\geq -1}$ is denoted by $\bar{\alpha}$.

Recall that there is a bijection between the cluster variables in $\mathcal{A}^\text{univ}_{A_2}$ and the roots in $\Phi_{\geq -1}$. We let $x_\alpha$ denote the cluster variable associated to $\alpha$. Then the exchange relations in $\mathcal{A}^\text{univ}_{A_2}$ are:

\[
x_{-\alpha_2} x_{\alpha_2} = p_{-\alpha_2} x_{-\alpha_1} + p_{\alpha_2} p_{\alpha_1+\alpha_2},
\]

\[
x_{-\alpha_1} x_{\alpha_1+\alpha_2} = p_{\alpha_1} x_{\alpha_2} + p_{-\alpha_1} p_{\alpha_2},
\]

\[
x_{\alpha_2} x_{\alpha_1} = p_{\alpha_1+\alpha_2} x_{\alpha_1+\alpha_2} + p_{-\alpha_1} p_{-\alpha_2},
\]

\[
x_{\alpha_1+\alpha_2} x_{-\alpha_2} = p_{\alpha_2} x_{\alpha_1} + p_{-\alpha_2} p_{\alpha_1},
\]

\[
x_{\alpha_1} x_{-\alpha_1} = p_{-\alpha_1} x_{-\alpha_2} + p_{\alpha_1} p_{\alpha_1+\alpha_2}.
\]

In this case, the quiver of $\mathcal{E}/F_\alpha$ is the following

```
P_{\alpha_2} \rightarrow X_{\alpha_1+\alpha_2} \rightarrow P_{\alpha_1} \rightarrow X_{-\alpha_1} \rightarrow P_{-\alpha_1} \rightarrow X_{\alpha_1} \rightarrow X_{\alpha_2} \rightarrow P_{\alpha_2}.
```

We can verify that the exchange relations above correspond to the exchange conflations in $\mathcal{E}/F$. For instances, the exchange relation

\[
x_{\alpha_2} x_{\alpha_1} = p_{\alpha_1+\alpha_2} x_{\alpha_1+\alpha_2} + p_{-\alpha_1} p_{-\alpha_2}
\]

corresponds to the non-split short exact sequences

\[
0 \rightarrow X_{\alpha_1} \rightarrow P_{\alpha_1+\alpha_2} \oplus X_{\alpha_1+\alpha_2} \rightarrow X_{\alpha_2} \rightarrow 0.
\]
and

\[ 0 \rightarrow X_{\alpha_2} \rightarrow P_{-\alpha_1} \oplus P_{-\alpha_2} \rightarrow X_{\alpha_1} \rightarrow 0. \]

The ice quiver of \( T = X_{-\alpha_1} \oplus X_{-\alpha_2} \oplus P_{-\alpha_1} \oplus P_{-\alpha_2} \oplus P_{\alpha_1} \oplus P_{\alpha_2} \oplus P_{\alpha_1 + \alpha_2} \) is

\[ \begin{array}{c}
P_{\alpha_2} \\
\uparrow \\
P_{\alpha_1 + \alpha_2} \\
\downarrow \\
P_{\alpha_1} \\
\rightarrow \\
X_{\alpha_2} \\
\leftarrow \\
P_{-\alpha_2} \\
\downarrow \\
X_{-\alpha_1} \\
\rightarrow \\
P_{-\alpha_1}
\end{array} \]

as expected.

Using Theorem 3.1 we can see that the categorification of finite-type Grassmannian cluster algebras introduced in [23] is equivalent to \( \text{gpr}(S_C)/F \), for an appropriate choice of admissible configuration \( C \). This follows from the fact that these categories are standard Frobenius models of a finite-type cluster category.

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