Lower bounds for some decision problems over \( \mathbb{C} \)

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Abstract

Lower bounds for some explicit decision problems over the complex numbers are given.

1 Introduction

This paper is about lower bounds for certain decision problems over \( \mathbb{C} \). (See [3] for the model of computation and for background). In particular, we will provide lower bounds for the complexity of deciding, given \( x \), if \( p^d(x) = 0 \) for some explicit polynomials \( p^d \).

A related problem is to give lower bounds for the evaluation of explicit polynomials. This has been an active subject of research since [6]. See [4] for modern developments and for bibliographical remarks. More recent results appeared in [1] and [2].

Most of those bounds use the Ostrowsky model of computation ([1] page 6): sum and multiplication by an algebraic constant are free, and the complexity of a computation for polynomial \( f(x) \) is the number of non-scalar multiplications, i.e., of multiplications of two polynomials in the variable \( x \). For instance, Horner rule for a degree \( d \) polynomial requires \( d \) non-scalar multiplications.

All those bounds apply trivially to the complexity of evaluating polynomials by a ‘machine over \( \mathbb{C} \)’ as defined in [4], or to the (multiplicative-branching) complexity of a computation tree for evaluating the same polynomial.

Little is known, however, about the application of those bounds to decision problems (Over \( \mathbb{C} \), in the sense of [4], or by a decision tree as in [4], Definition (4.19) page 115. In this definition, each node of a computation tree can perform one algebraic operation or comparison, and therefore a natural measure of complexity is the depth of the tree).

In this paper, only decision problems of the form below will be considered: let \( X \subseteq \mathbb{N} \times \mathbb{C} \), and let \( X_d = \{ x \in \mathbb{C} : (d, x) \in X \} \). Typically, \( d \) is the problem size and \( \#X_d \leq d \). One can think of \( X \) as the disjoint union of the zero-set of a
family of polynomials of degree $\leq d$, where $d \in \mathbb{N}$. The two following forms of a decision problem are natural in this setting:

**Problem 1.** For any fixed $d$, decide whether $x \in X_d$.

**Problem 2.** Decide whether $(d, x) \in X$.

Problem 1 is non-uniform, in the sense that we allow for a different machine over $\mathbb{C}$ or a different decision tree to be used for each value of $d$. However, we want a bound on the running time or on the multiplicative complexity of the tree, as a function of $d$.

Problem 2 is uniform. It is harder than Problem 1, in the sense that it cannot be solved by a decision tree, since $\#X_d$ can be arbitrarily large. It requires a machine over $\mathbb{C}$, that will eventually branch according to the value of $d$.

Lower bounds for Problem 1 are also lower bounds for Problem 2. A trivial, topological lower bound for Problems 1 and 2 when $\#X_d = d$ is $\log_2 d$. Sharper known bounds come from the ‘Canonical Path’ argument, see §2.5: Let $f$ be a univariate polynomial. The complexity of deciding $f(x) = 0$ is bounded below by the minimum of the complexity of evaluating $g(x)$, where $g$ ranges over the non-zero multiples of $f$.

If one assumes some property of $f$ that propagates to its multiple $g$, then one eventually obtains a sharper, non-trivial lower bounds.

In Lemma 1 below, we will give conditions on the roots of $f$ that will provide lower bounds for the evaluation of $g$. Essentially, we will require a subset of the roots to be rapidly growing. This will imply a rapid growth property for the coefficients of $g$. Then, the results of [1, 2] imply a lower bound for the complexity of evaluating $g$. Thus we will be able to construct specific polynomials that are hard to decide in the non-uniform sense, viz.

**Lower bound 1.** The set $X = \{(d, x) \in \mathbb{Z} \times \mathbb{C} : x = 2^{2^i d}, 0 \leq i \leq d, \text{ cannot be solved in time polylog}(d)\}$ in the setting of Problem 1.

**Lower bound 2.** The set $Y = \{(d, x) \in \mathbb{Z} \times \mathbb{C} : p^d(x) = 0, \text{ where } p^d(t) = \sum_{i=0}^d 2^{2^i (d-i)} t^i, \text{ cannot be decided in time polylog}(d)\}$ in the setting of Problem 2.

In a more classical computer-science language, we can define the input size of some $(d, x)$ as $\log d$. This means that the integer $d$ is represented in binary notation, while variable $x$ can contain an arbitrary complex number. In that case, ‘time polylog(d) in the setting of Problem 1’ can be refrased as $P_{/\text{poly}}$.

The lower bounds above become now: $X \notin P_{/\text{poly}}$ and $Y \notin P_{/\text{poly}}$.

Non-uniform lower bounds 1 and 2 can be compared to the following easier, uniform lower bound:

**Lower bound 3.** The set $Z = \{(d, x) \in \mathbb{Z} \times \mathbb{C} : q^d(x) = 0, \text{ where } q^d(t) = \sum_{i=0}^d 2^i t^i, \text{ cannot be decided in time polylog}(d)\}$ in the setting of Problem 2.
This means that the set $Z$, where $d$ is represented in binary notation and $x$ is a complex number, does not belong to $\mathcal{P}$ over $\mathbb{C}$.

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2 Background and notations

Definition 1. Let $K \subset L$ be finite algebraic extensions of $\mathbb{Q}$. Let $\nu$ be a valuation in $M_K$. Then we extend the notation $\nu$ to $L$ by:

$$
\nu(x) = \frac{\sum_{\mu} n_{\mu} \mu(x)}{\deg[L : K]}
$$

where the sum ranges over all the valuations $\mu$ of $L$ that are ‘above’ $\nu$, and where $n_{\mu}$ is the ‘local degree’ of $L : K$. The local degree is defined as $n_{\mu} = \deg[L_\mu : K_{\nu}]$, where $K_{\nu}$ is the completion of $K$ under the metric induced by the absolute value $|.|_{\nu}$.

Recall that for $x \in K$, $\deg[L : K] \nu(x) = \sum_{\mu} n_{\mu} \nu(x)$. The case $K = \mathbb{Q}$ is an immediate consequence of Corollary 2 of Theorem 1 in Chapter II, p. 39 of [5].

Definition 2. Let $g$ be a polynomial with algebraic coefficients in some extension $K$ of $\mathbb{Q}$. Let $\nu$ be a valuation in $M_K$. The Newton diagram of $g$ at $\nu$ is the (lower) convex hull of the set $\{(i, \nu(g_i)), i = 0 \cdots d\}$.

The basic property of Newton diagrams used here is the following.

Proposition 1. Suppose that $\zeta_1, \cdots, \zeta_d$ are the roots of a univariate polynomial $g \in K[x]$. Let the roots of $g$ be ordered so that

$$
\nu(\zeta_1) \geq \cdots \geq \nu(\zeta_d)
$$

and let the increasing sequence $i_j$ assume the values $0, d$ and all the values of $i$ where:

$$
\nu(\zeta_i) > \nu(\zeta_{i+1})
$$

Then the sharp corners of the Newton diagram are precisely the points of the form $(i_j, \nu(g_{i_j}))$ for all $j$.

Moreover, the slope of the segment $[(i_{j-1}, \nu(g_{i_{j-1}})), (i_j, \nu(g_{i_j}))]$ is precisely $-\nu(\zeta_{i_j})$.

Proof of Proposition 1. The proof uses the following property of valuations: $\nu(\sum x_i) \geq \min \nu(x_i)$. Furthermore, when that minimum is attained in only one $x_i$, we have equality.
Let $i_{j-1} < k < i_j$. Writing

$$g_{i_{j-1}} = g_{d}\sigma_{d-i_{j-1}}(\zeta_1, \ldots, \zeta_d)$$
$$g_k = g_{d}\sigma_{d-k}(\zeta_1, \ldots, \zeta_d)$$
$$g_{i_j} = g_{d}\sigma_{d-i_j}(\zeta_1, \ldots, \zeta_d)$$

one can pass to the valuation by:

$$\nu(g_{i_{j-1}}) = \nu(g_d) + \nu(\zeta_{i_{j-1}+1}) + \cdots + \nu(\zeta_d)$$
$$\nu(g_k) \geq \nu(g_d) + \nu(\zeta_{k+1}) + \cdots + \nu(\zeta_d)$$
$$\nu(g_{i_j}) = \nu(g_d) + \nu(\zeta_{i_j+1}) + \cdots + \nu(\zeta_d)$$

Subtracting, one obtains:

$$\nu(g_{i_j}) - \nu(g_{i_{j-1}}) = -\nu(\zeta_{i_{j-1}+1}) - \cdots - \nu(\zeta_d) = -(i_{j} - i_{j-1})\nu(\zeta_{i_j})$$
$$\nu(g_{i_j}) - \nu(g_k) \leq -\nu(\zeta_{k+1}) - \cdots - \nu(\zeta_d) \leq -(i_{j} - k)\nu(\zeta_{i_j})$$

This concludes the proof. \[\square\]

3 Uniform lower bounds

We can now prove Lower Bound 3.
Proof of Lower bound \[3\]. The Newton diagram of \(q^d\) at 2 is \(\{(i, 2^i) : 0 \leq i \leq d\}\). (This latest set is convex, since the points lie on the curve \(y = 2^x\) and this curve is convex). Therefore, there is a unique root \(\zeta\) of \(q^d\) that minimizes \(\nu(\zeta)\).

Since \(q^{d-1}_d = (-\sum \zeta_i)q^{d}_d\), where the sum ranges over all the roots, we have:

\[
\nu_2(q^{d-1}_d) = \nu_2(q^{d}_d) + \min \nu_2(\zeta_i) = \nu_2(q^{d}_d) + \nu_2(\zeta)
\]

Replacing by the actual values of the coefficients, one gets:

\[
\nu_2(\zeta) = -2^{d-1}
\] (1)

Now, suppose that there is a machine \(M\) that decides \(q^d(t) = 0\) in time \(\mathrm{polylog}(d)\). One can assume without loss of generality that this machine has no constant but 0 and 1. Let its running time be bounded by \(T = a(\log d)^b\).

Let us fix \(d > 2 + T^2\). We will derive a contradiction.

Let \(g\) be the polynomial defining the canonical path (recall that \(d\) is fixed now, so this is the path followed by generic \(t \in \mathbb{C}\)). It can be computed in time \(\leq T^2\), so we have the following bounds:

\[
\deg g \leq 2T^2
\]

\[
0 \leq \nu_2(g_p) \leq 2T^2
\]

Since \(\zeta\) is also a root of \(g\), there are coefficients \(g_i\) and \(g_j, i \neq j\), such that:

\[
(j - i)\nu_2(\zeta) = \nu_2(g_i) - \nu_2(g_j)
\] (2)

Thus, \(|\nu_2(\zeta)| \leq |\nu_2(g_i)| + |\nu_2(g_i)|\). This implies:

\[
|\nu_2(\zeta)| \leq 2^{1+T^2} < 2^{d-1}
\]

Replacing by equation [3], one obtains \(2^{d-1} < 2^{d-1}\), a contradiction. \(\square\)

4 Non-uniform lower bounds

**Lemma 1.** Let \(g = g(t)\) be a degree \(D\) polynomial with algebraic coefficients. Let \(\nu\) be a (non-archimedian) valuation of \(K = \mathbb{Q}[g_0, \ldots, g_D]\). Let \(\xi_1, \ldots, \xi_D\) be the roots of \(g\), and assume they are ordered in such way that:

\[
\nu(\xi_1) \geq \cdots \geq \nu(\xi_D)
\]

Suppose that there is a subsequence \(\zeta_j = \xi_{i_j+1}, j = 1 \cdots d\), such that the following holds:

1. \(\nu(\zeta_d) \geq 1\)
2. \(\nu(\zeta_j) \geq 2(i_{j+1} - i_j) \nu(\zeta_{j+1})\), for \(0 \leq j \leq d - 1\).
Then $g$ cannot be evaluated in less than
\[ L \geq \sqrt{\frac{d}{28 \log_2 D + 1}} \]
multiplications.

Proof of Lemma 4: We can assume without loss of generality that the ordering
of the $\xi_i$ satisfies:
\[ \cdots \xi_{ij} < \xi_{ij+1} = \xi_j \leq \xi_{ij+2} \cdots \]
For $j \in \{1, \cdots, d-1\}$ we have:
\[ \nu(g_{ij}) - \nu(g_{ij+1}) = \nu(\xi_{ij+1}) + \cdots + \nu(\xi_{ij+1}) \]
Hence, using $\nu(\xi_{ij+1}) > \nu(\xi_d) \geq 1$:
\[ \nu(\zeta_j) \leq \nu(g_{ij}) - \nu(g_{ij+1}) \leq (i_{j+1} - i_j)\nu(\zeta_j) \]
By the same argument, for $j \in \{0, \cdots, d-2\}$:
\[ \nu(\zeta_{j+1}) \leq \nu(g_{ij+1}) - \nu(g_{ij+2}) \leq (i_{j+2} - i_{j+1})\nu(\zeta_{j+1}) \]
Hence,
\[ \frac{\nu(g_{ij}) - \nu(g_{ij+1})}{\nu(g_{ij+1}) - \nu(g_{ij+2})} \geq \frac{\nu(\zeta_j)}{(i_{j+1} - i_j)\nu(\zeta_{j+1})} \geq 2 \]
Set $G_j = \nu(g_{ij})$ for $j = 0, \cdots, d-1$. We know that the $G_j$ are such that
\[ |G_{j+1} - G_j| < \frac{1}{2}|G_j - G_{j-1}|. \]
Hence
\[ \#\{\sum s_j G_j, s_j \in \{0; 1\}\} = 2^d \]
Hence:
\[ \#\{\nu(\prod_{s \in S} g_s), S \subset \{0, \cdots, D\}\} \geq 2^d \]
and hence
\[ \mu(g) = \#\{\sum_{S \subset \{0, \cdots, D\}} \theta_S \prod_{s \in S} g_s, \theta_S \in \{0; 1\}\} \geq 2^{2d} \]
By Lemma 1 in [1] or by Lemma 4 in [2],
\[ \mu(g) \leq 2^{(D+1)^{28L^2}} \]
and hence, taking logs:
\[ (D + 1)^{28L^2} \geq 2^d \]
Taking logs again:

\[ 28L^2 \geq \frac{d}{\log_2 D + 1} \]

and hence:

\[ L \geq \sqrt{\frac{d}{28 \log_2 D + 1}} \]

Note: Lemma 1 in [1] is slightly more general than Lemma 4 in [2]. However, using Lemma 4 in [2] it is possible to replace all the appearances of the number 28 in the statement and proof of Lemma 1 above by the number 21.

Proof of Lower Bound. We see from its Newton diagram that the polynomial \( p \) has distinct roots \( \zeta_1, \ldots, \zeta_d \) with:

\[ \nu_2(\zeta_i) = 2^{d(d-i+1)} - 2^{d(d-i)} = 2^{d(d-i)}(2^d - 1) \]

So we have \( \nu_2(\zeta_d) = 2^d - 1 > 1 \), and

\[ \nu_2(\zeta_i)/\nu_2(\zeta_{i+1}) = 2^d \]

(3)

Assume that there are \( a, b \) such that for each \( d \), there is a machine \( M \) over \( \mathbb{C} \) deciding \( p(t) = 0 \) in time \( T = a(\log d)^b \). Its generic path is defined by a polynomial \( g(t) \) of degree \( \leq 2T \).

Let us fix \( d > 28(T+1)T^2 \). In particular \( d \geq T + 1 \). We are in the conditions of Lemma 1 where \( D = 2T \). From that Lemma, it follows that

\[ T \geq \sqrt{\frac{d}{28 \log_2 2T^2 + 1}} \geq \sqrt{\frac{d}{28(T+1)}} \]

Hence,

\[ 28T^2(T + 1) \geq d \]

contradicting our choice of \( d \).

Equation (3) holds trivially in the proof of Lower bound [1]. The rest of the proof is verbatim the same.

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