ENTIRE SOLUTIONS IN NONLOCAL MONOSTABLE EQUATIONS: ASYMMETRIC CASE

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Abstract. This paper is concerned with entire solutions of the monostable equation with nonlocal dispersal, i.e., \( u_t = J * u - u + f(u) \). Here the kernel \( J \) is asymmetric. Unlike symmetric cases, this equation lacks symmetry between the nonincreasing and nondecreasing traveling wave solutions. We first give a relationship between the critical speeds \( c^* \) and \( \hat{c}^* \), where \( c^* \) and \( \hat{c}^* \) are the minimal speeds of the nonincreasing and nondecreasing traveling wave solutions, respectively. Then we establish the existence and qualitative properties of entire solutions by combining two traveling wave solutions coming from both ends of real axis and some spatially independent solutions. Furthermore, when the kernel \( J \) is symmetric, we prove that the entire solutions are 5-dimensional, 4-dimensional, and 3-dimensional manifolds, respectively.

1. Introduction. We are concerned with the following nonlocal dispersal equation on \( \mathbb{R} \):

\[
    u_t(x,t) = (J * u)(x,t) - u(x,t) + f(u(x,t)),
\]

with the nonlocal diffusion operator \( (J * u)(x,t) - u(x,t) = \int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]dy \). The kernel \( J \) satisfies

\( (J1): \) \( J \in C^1(\mathbb{R}) \), \( J(x) \geq 0 \), \( \int_{\mathbb{R}} J(x)dx = 1 \), \( J'(x) \in L^1(\mathbb{R}) \), \( \exists \lambda > 0 \) such that \( \int_{\mathbb{R}} J(x)e^{\lambda|x|}dx < +\infty \), and \( \exists a < 0 < b \) such that \( J(a) > 0 \) and \( J(b) > 0 \).

The function \( f \) is monostable type nonlinearity and satisfies

\( (F1): \) \( f \in C^2(\mathbb{R}) \), \( f(0) = f(1) = 0 \), \( f(s) > 0 \) and \( f'(s) \leq f'(0) < 1 \) for \( s \in (0,1) \).

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The condition (J1) implies that the kernel $J$ may be asymmetric. During the past ten years, nonlocal equations as (1) have been introduced and extensively studied to analyze the long range effects of the dispersal, see [2, 9, 10, 11, 13, 20, 21, 24, 32] and references therein. In particular, there has been significant progress in the study of traveling wave solutions for nonlocal equations with symmetric and asymmetric kernel $J$; see Bates [2], Bates et al. [3], Carr and Chmaj [4], Chen [7], Coville and Dupaigne [13], Pan et al. [31], Schumacher [32] and Weinberger [39] for the symmetric case, and Coville [11], Coville et al. [12], Sun et al. [34] and Yagisita [42, 43] for the asymmetric case.

Although the traveling wave solution is a key object characterizing the dynamics of nonlocal dispersal equations, such as (1), it is not enough to understand the whole dynamics. In fact, traveling wave solutions are special examples of the so-called entire solutions, which are defined in the whole space and for all time $t \in \mathbb{R}$. From the viewpoint of biology, entire solutions can model new spreading and invasion behavior of the epidemic and species, respectively; see [28, 30, 45]. Moreover, entire solutions can help us for the mathematical understanding of transient dynamics and the structures of the global attractors. However, the global attractors are rather complicated. Some new types of entire solutions other than traveling wave solutions have been established for various evolution equations with spatially homogeneous environment; see e.g. [8, 18, 19, 27, 29, 37, 41] for reaction-diffusion equations with and without delays, [38] for delayed lattice differential equations with global interaction, [25] for reaction-advection-diffusion equations, [30, 36, 40, 45] etc. for reaction-diffusion or discrete model systems.

Recently, Li et al. [26] and Sun et al. [33] constructed new types of entire solutions for symmetric nonlocal equations with monostable and bistable nonlinearity by combining two traveling wave solutions coming from both ends of real axis and some spatially independent solutions. And Dong et al. [16], Li et al. [28] and Zhang et al. [45] further considered the entire solutions for symmetric nonlocal systems. However, the issue of the existence of entire solutions for nonlocal equation (1) is still open when $J$ is asymmetric. As for entire solutions, it is natural to ask what is the difference between symmetric equations and asymmetric equations.

In fact, there is a close relationship between the nonlocal equation (1) and a local version. Let $J(x) = \frac{1}{2} P(\frac{x}{\varepsilon})$ with $\varepsilon > 0$, where $P(x)$ is a general mollification function with support $x \in [-1, 1]$. If $u(x)$ is smooth, then the Taylor’s formula implies that

$$(J \ast u)(x) - u(x) = \int_{\mathbb{R}} \frac{1}{\varepsilon} P\left(\frac{x - y}{\varepsilon}\right) [u(y) - u(x)] dy$$

$$= \int_{\mathbb{R}} P(-z)[u(x + \varepsilon z) - u(x)] dz$$

$$= \varepsilon^2 \alpha u''(x) + \varepsilon \beta u'(x) + o(\varepsilon^2) \text{ as } \varepsilon \rightarrow 0,$$

where $\alpha = \frac{1}{2} \int_{\mathbb{R}} P(-z)z^2 dz$, $\beta = \int_{\mathbb{R}} P(-z)z dz$. Thus there is a formal analogy between $J \ast u - u$ and $\varepsilon^2 \alpha u'' + \varepsilon \beta u'$ (see [12]). When $J$ is symmetric, it is clear that $\beta = 0$, then (1) can be viewed as an approximation of the classical Laplace diffusion equation

$$u_t = Au'' + f(u).$$

Therefore, equation (1) indeed shares many properties of equation (3). For instance, both of them have maximum principle, and stationary solutions are constants [17].
Especially, the results on the existence and some related properties of traveling wave solutions of equation (1) are similar to those of the reaction diffusion equation (3); see [3, 7] for bistable nonlinearity, [4, 13, 31, 32] and the references therein for monostable nonlinearity. However, for general asymmetric kernel $J$, we see from (2) that a better analogy than (3) for (1) is the following elliptic equation:

$$u_t = Au'' + Bu' + f(u).$$

(4)

Thus, there is an essential difference between symmetric and asymmetric equations, which makes the two types of equations have many different dynamical properties. For instance, Coville et al.[12] and Sun et al.[34] have showed that the minimal speed $c^*$ of asymmetric equation (1) may be nonpositive (The minimal speed $c^* > 0$ for symmetric equation (1)). Additionally, asymmetric equations lack symmetry between decreasing and increasing traveling wave solutions. Therefore, resolving the issue of entire solutions of asymmetric equation (1) represents a main contribution of our current study.

Very recently, Zhang et al. [44] established the front-like entire solution of the asymmetric equation (1) with ignition nonlinearity and obtained its qualitative properties by classifying the sign and size of the wave speeds. In this paper, one of our goals is to study entire solutions of asymmetric equation (1) with monostable nonlinearity. Recall that under the conditions (J1) and (F1) (Since $f$ is of class $C^2$ here, the assumption (F2) in [34] holds naturally), Sun et al. [34] have shown that, there exists a minimal speed $c^* \in \mathbb{R}$ such that for any $c > c^*$, there exists a nonincreasing traveling wave solution $\phi_c(x - ct)$ satisfying

$$J * \phi_c - \phi_c + c\phi_c' + f(\phi_c) = 0,$$

$$\phi_c(-\infty) = 1, \lim_{z \to +\infty} \phi_c(z)e^{\lambda(c)z} = 1,$$

where $\lambda(c)$ is the smallest positive root to $c\lambda = \int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 + f'(0)$ and $c^*$ is defined by (19). When $c > c^*$ and $c \neq 0$, there are $\phi'_c < 0$ in $\mathbb{R}$ and $\lim_{z \to +\infty} \phi'_c(z)e^{\lambda(c)z} = -\lambda(c)$. And a similar argument shows that there exists a minimal speed $\hat{c}^* \in \mathbb{R}$ such that for any $\hat{c} > \hat{c}^*$, there exists a nondecreasing traveling wave solution $\hat{\phi}_c(x + \hat{c}t)$ satisfying

$$J * \hat{\phi}_c - \hat{\phi}_c + \hat{c}\phi'_c + f(\hat{\phi}_c) = 0,$$

$$\lim_{z \to -\infty} \hat{\phi}_c(z)e^{-\mu(c)z} = 1, \hat{\phi}_c(+\infty) = 1,$$

where $\mu(\hat{c})$ is the smallest positive root to $\hat{c}\mu = \int_{\mathbb{R}} J(z)e^{-\mu z}dz - 1 + f'(0)$ and $\hat{c}^*$ is defined by (19). When $\hat{c} > \hat{c}^*$ and $\hat{c} \neq 0$, there are $\hat{\phi}'_c > 0$ in $\mathbb{R}$ and $\lim_{z \to -\infty} \hat{\phi}'_c(z)e^{-\mu(c)z} = \mu(\hat{c})$. In this paper, we always normalize the traveling wave solutions $\phi_c(x - ct)$ and $\hat{\phi}_c(x + \hat{c}t)$ so that $\phi_c(0) = \frac{1}{2}$ and $\hat{\phi}_c(0) = \frac{1}{2}$. Then, for each $c > c^*$ and $\hat{c} > \hat{c}^*$, we set

$$\alpha_c = \lim_{z \to +\infty} \phi_c(z)e^{\lambda(c)z}, \beta_{\hat{c}} = \lim_{z \to -\infty} \hat{\phi}_c(z)e^{-\mu(\hat{c})z}.$$  

(7)

In addition, define $A_c, B_{\hat{c}} > 0$ for each $c > c^*$ and $\hat{c} > \hat{c}^*$ by

$$A_c = \inf \{A > 0 : A \geq \phi_c(z)e^{\lambda(c)z} \text{ for any } z \in \mathbb{R} \}$$

and

$$B_{\hat{c}} = \inf \{B > 0 : B \geq \hat{\phi}_c(z)e^{-\mu(\hat{c})z} \text{ for any } z \in \mathbb{R} \}.$$  

(8)

(9)

It is easy to see that $A_c \geq \alpha_c, B_{\hat{c}} \geq \beta_{\hat{c}}$. 

\[\int \]
In order to establish the entire solutions of (1), it is necessary to study the property of the minimal speeds \( c^* \) and \( \hat{c}^* \).

**Theorem 1.1.** Assume that (J1) and (F1) hold. Then \( c^* + \hat{c}^* \geq 0 \).

This theorem implies that \( c^* \) and \( \hat{c}^* \) can not be negative at the same time, which is also mentioned in [12]. Especially, for any \( c > c^* \) and any \( \hat{c} > \hat{c}^* \), there must be \( c + \hat{c} > 0 \). In this paper, we only consider traveling wave solutions with non-zero speeds. According to Theorem 1.1, for any \( c > c^* \), \( \hat{c} > \hat{c}^* \), \( c \neq 0 \) and \( \hat{c} \neq 0 \), there must be \((c, \hat{c}) \in C_{ij}\) for some \((i, j)\) with \(i, j = 1, 2, 3\), where

| \( c^* \) | \( \hat{c}^* > 0 \) | \( \hat{c}^* = 0 \) | \( \hat{c}^* < 0 \) |
|---|---|---|---|
| \( c^* = 0 \) | \( C_{11} \) | \( C_{12} \) | \( C_{13} = C_{13}^1 \cup C_{13}^2 \) |
| \( c^* \neq 0 \) | \( C_{31} = C_{31}^1 \cup C_{31}^2 \) | \( C_{32} = C_{32}^1 \cup C_{32}^2 \) | \( \) |

**Table 1. Region of \((c, \hat{c})\)**

\[
\begin{align*}
C_{11} &= (c^*, +\infty) \times (\hat{c}^*, +\infty), & C_{12} &= (c^*, +\infty) \times (0, +\infty), \\
C_{21} &= (0, +\infty) \times (\hat{c}^*, +\infty), & C_{22} &= (0, +\infty) \times (0, +\infty), \\
C_{13} &= (c^*, +\infty) \times (0, +\infty), & C_{13}^2 &= (c^*, +\infty) \times (\hat{c}^*, 0), \\
C_{23} &= (0, +\infty) \times (0, +\infty), & C_{23}^2 &= (0, +\infty) \times (\hat{c}^*, 0), \\
C_{31}^1 &= (0, +\infty) \times (\hat{c}^*, +\infty), & C_{31}^2 &= (c^*, 0) \times (\hat{c}^*, +\infty), \\
C_{32}^1 &= (0, +\infty) \times (0, +\infty), & C_{32}^2 &= (\hat{c}^*, 0) \times (0, +\infty).
\end{align*}
\]

In summary, the signs of traveling wave speeds can be classified as the following three types:

(i): \( c > 0 \) and \( \hat{c} > 0 \), when \((c, \hat{c}) \in C_{++} : = \bigcup_{i, j = 1, 2} C_{ij} \bigcup_{i = 1, 2} C_{13}^1 \bigcup_{j = 1, 2} C_{13}^2;\)

(ii): \( c > 0 \) and \( \hat{c} < 0 \), when \((c, \hat{c}) \in C_{+-} : = \bigcup_{i = 1, 2} C_{33}^2;\)

(iii): \( c < 0 \), \( \hat{c} > 0 \), when \((c, \hat{c}) \in C_{-+} : = \bigcup_{j = 1, 2} C_{33}^2;\)

Moreover, for any \((x, t) \in \mathbb{R}^2\) and \(A, a \in \mathbb{R}\), we denote the regions \(S_{A,a}^i, i = 1, \ldots, 9\), by

\[
\begin{align*}
S_{A,a}^1 &= (-\infty, A] \times (-\infty, a], & S_{A,a}^3 &= (-\infty, A] \times [-a, +\infty), \\
S_{A,a}^3 &= [-A, +\infty) \times (-\infty, a], & S_{A,a}^4 &= [-A, +\infty) \times [-a, +\infty), \\
S_{A,a}^5 &= \mathbb{R} \times (-\infty, a], & S_{A,a}^6 &= \mathbb{R} \times [-a, +\infty), \\
S_{A,a}^7 &= [-A, +\infty) \times (-\infty, a], & S_{A,a}^8 &= [-A, +\infty) \times [-a, a], \\
S_{A,a}^9 &= (-\infty, A] \times [-a, a], & S_{A,a}^10 &= [-A, A] \times [-a, a].
\end{align*}
\]

Our main results are stated as follows.

**Theorem 1.2.** Assume that (J1) and (F1) hold. Then for any \( c > c^* \), \( \hat{c} > \hat{c}^* \) and \((c, \hat{c}) \in C_{++}, h_1, h_2 \in \mathbb{R}, k > 0, \) and \( \chi_1, \chi_2, \chi_3 \in (0, 1) \) with \( \chi_1 + \chi_2 + \chi_3 \geq 2 \), there exists an entire solution \( u(x, t) := u_p(x, t) \) with \( p := p_{\chi_1, \chi_2, \chi_3} = \chi_1c, \chi_2c, \chi_1h_1, \chi_2h_2, \chi_3k \) of (1) such that
Corollary 1.3. Assume that (J1) and (F1) hold. Then for any $c > c^*$, $\hat{c} > \hat{e}^*$ and $(c, \hat{c}) \in C_{++}$, $h_1, h_2 \in \mathbb{R}$, $k > 0$, and $\chi_1, \chi_2, \chi_3 \in \{0, 1\}$ with $\chi_1 + \chi_2 + \chi_3 \geq 2$, the entire solution $u(x, t) := u_p(x, t)$ in Theorem 1.2 also satisfies the following statements:

(i): For every $x \in \mathbb{R}$, there exist $D(x) > C(x) > 0$ such that
\[
C(x)e^{\rho(c, \hat{c})t} \leq u(x, t) \leq D(x)e^{\rho(c, \hat{c})t}
\]
for all \( t \ll -1 \), where
\[
\nu(c, \hat{c}) = \begin{cases} 
\min\{c\lambda(c), \hat{c}\mu(\hat{c}), f'(0)\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 1), \\
\min\{c\lambda(c), \hat{c}\mu(\hat{c})\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 0), \\
\min\{\hat{c}\mu(\hat{c}), f'(0)\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (0, 1, 1), \\
\min\{c\lambda(c), f'(0)\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 0, 1). 
\end{cases}
\]

(ii): When \( \chi_1 = 1 \), the function \( u(x, t) \) is decreasing with respect to \( h_1 \) and converges to 1 as \( h_1 \to -\infty \) uniformly for \((x, t) \in S_{A,a}^3\). However, when \( \chi_2 = 1 \)
(or \( \chi_3 = 1 \)), \( u(x, t) \) is increasing with respect to \( h_2 \) (or \( k \)) and converges to 1 as \( h_2 \to +\infty \) (or \( k \to +\infty \)) uniformly for \((x, t) \in S_{A,a}^4\) (or \((x, t) \in S_{A,a}^5\)).

(iii): Moreover, according to the assumption \( \chi_1, \chi_2, \chi_3 \in \{0, 1\} \) with \( \chi_1 + \chi_2 + \chi_3 \geq 2 \), we further denote the entire solution \( u(x, t) = u_p(x, t) \) of (1) by
\[
u_p(x, t) := \begin{cases} 
u_{p_0}(x, t), & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 1), \\
\nu_{p_1}(x, t), & \text{if } (\chi_1, \chi_2, \chi_3) = (0, 1, 1), \\
\nu_{p_2}(x, t), & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 0, 1), \\
\nu_{p_3}(x, t), & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 0),
\end{cases}
\]
with
\[
u_{p_0}(x, t) \to \begin{cases} u_{p_1}(x, t) \text{ as } h_1 \to +\infty \text{ in } \mathcal{T} \text{ and uniformly on } (x, t) \in S_{A,a}^3, \\
u_{p_2}(x, t) \text{ as } h_2 \to -\infty \text{ in } \mathcal{T} \text{ and uniformly on } (x, t) \in S_{A,a}^5, \\
u_{p_3}(x, t) \text{ as } k \to 0^+ \text{ in } \mathcal{T} \text{ and uniformly on } (x, t) \in S_{A,a}^5, \\
\phi_c(x - ct + h_1) \text{ as } h_2 \to -\infty, k \to 0^+ \text{ in } \mathcal{T} \text{ and uniformly on } (x, t) \in S_{A,a}^5, \\
\phi_c(x + \hat{c}t + h_2) \text{ as } h_2 \to -\infty, h_2 \to -\infty \text{ in } \mathcal{T} \text{ and uniformly on } (x, t) \in S_{A,a}^7, \\
\xi(t) \text{ as } h_1 \to +\infty, h_2 \to +\infty \text{ in } \mathcal{T} \text{ and uniformly on } (x, t) \in S_{A,a}^7, \\
\hat{\phi}_c(x + \hat{c}t + h_2) \text{ as } k \to 0^+ \text{ in } \mathcal{T} \text{ and uniformly on } (x, t) \in S_{A,a}^5, \\
x_{A,a}^3, \\
x_{A,a}^5, \\
x_{A,a}^3, \\
x_{A,a}^3, \\
x_{A,a}^3.
\]

Here, we say that a sequence of functions \( u_p(x, t) \) converges to a function \( u_{p_0}(x, t) \) in the sense of topology \( \mathcal{T} \), if for any compact set \( S \subset \mathbb{R}^2 \), the functions \( u_p(x, t) \) and \( \frac{d}{dt}u_p(x, t) \) converge uniformly in \((x, t) \in S\) to \( u_{p_0}(x, t) \) and \( \frac{d}{dt}u_{p_0}(x, t) \) as \( p \to p_0 \).

**Theorem 1.4.** Assume that (J1) and (F1) hold. Then for any \( c > c^* \), \( \hat{c} > \hat{c}^* \) and \((c, \hat{c}) \in C_{+, -}, h_1, h_2 \in \mathbb{R}, k > 0, \) and \( \chi_1, \chi_2, \chi_3 \in \{0, 1\} \) with \( \chi_1 + \chi_2 + \chi_3 \geq 2 \), there exists an entire solution \( u^-(x, t) = u_p^-(x, t) \) of (1) such that:
(i): When $\chi_1 = \chi_3 = 1$ and $\chi_2 = 0$, all assertions about $u_{(c,0,h_1,0,k)}(x,t)$ in Theorem 1.2 and Corollary 1.3 are true for $u_{(c,0,h_1,0,k)}(x,t)$.

(ii): When $\chi_1 = \chi_2 = \chi_3 = 1$, or $\chi_2 = \chi_3 = 1$ and $\chi_1 = 0$, or $\chi_1 = \chi_2 = 1$ and $\chi_3 = 0$, the assertions (i) and (ii) in Theorem 1.2 are true for $u^-(x,t)$ as for $u(x,t)$.

(iii): When $\chi_1 = \chi_2 = 1$ and $\chi_3 = 0$, the entire solutions $u^-(x,t)$ have the results:

(a): For arbitrarily given number $M > 0$,

$$\lim_{t \to +\infty} \sup_{x \in (-\infty,M]} |u^-(x,t) - \phi_c(x-ct+h_1)| = 0. \quad (11)$$

(b): Especially, for $c > -\hat{c}$,

$$\lim_{t \to -\infty} \left\{ \sup_{x \leq gt} |u^-(x,t) - \phi_c(x-ct+h_1)| + \sup_{x \geq gt} |u^-(x,t) - \hat{\phi}_c(x+\hat{c}t+h_2)| \right\} = 0 \quad (12)$$

with $-\hat{c} < g < c$, and

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |u^-(x,t) - 1| = 0. \quad (13)$$

In fact, when $\chi_1 = \chi_2 = \chi_3 = 1$, or $\chi_2 = \chi_3 = 1$ and $\chi_1 = 0$, or $\chi_1 = \chi_2 = 1$ and $\chi_3 = 0$, the assertions (ii) and (iii) in Corollary 1.3 are true for $u^-(x,t)$ as for $u(x,t)$, where $S^1_{A,a}$ is replaced by $S^3_{A,a}$ in (ii), $S^1_{A,a}$ is replaced by $S^3_{A,a}$ as $h_2 \to -\infty$, $S^1_{A,a}$ is replaced by $S^3_{A,a}$ as $h_2 \to -\infty$ and $k \to 0^+$, and $S^7_{A,a}$ is replaced by $S^9_{A,a}$ as $h_1 \to +\infty$ and $h_2 \to -\infty$ in (iii).

The assertion (iii) of Theorem 1.4 implies that, when $c > 0$, $\hat{c} < 0$, the entire solution behaves as two traveling wave solutions $\phi(x-ct+h_1)$ and $\hat{\phi}_c(x+\hat{c}t+h_2)$ moving in the same direction as $t \to -\infty$. And the decreasing wave solution $\phi(x-ct+h_1)$ move faster than the increasing one $\hat{\phi}_c(x+\hat{c}t+h_2)$ since $c > -\hat{c}$. At last, the entire solution $u^-(x,t)$ tends to 1 as $t \to +\infty$.

Remark 1.5. For any $c > c^*$, $\hat{c} > \hat{c}^*$ and $(c,\hat{c}) \in C_{-\infty}$, the equation (1) also has an entire solution $u^+(x,t) = u^+_p(x,t)$ similar to $u^-(x,t) = u^-_p(x,t)$ described in Theorem 1.4.

So far, we only constructed entire solutions of (1) for $c > c^*$ and $\hat{c} > \hat{c}^*$. Indeed, Coville [12] also guaranteed the existence of monotone traveling wave solutions with the critical speed $c = c^*$ ($c^* \neq 0$) under conditions (J1) and (F1). If we further assume that $f$ satisfies

(F2): $f''(u) \leq 0$ for $u \in [0,1]$,

then the existence of entire solutions of (1) combining the traveling wave solutions with critical speeds $c = c^*$ and/or $\hat{c} = \hat{c}^*$ can also be established.

Theorem 1.6. Assume that (J1) and (F1)-(F2) hold. When $c^* \hat{c}^* \neq 0$, for any $c \geq c^*$, $\hat{c} \geq \hat{c}^*$ and $cc \neq 0$, $h_1, h_2 \in \mathbb{R}$, $k > 0$ and $\chi_1, \chi_2, \chi_3 \in \{0,1\}$ with $\chi_1 + \chi_2 + \chi_3 \geq 2$, (1) admits an entire solution $v(x,t) := v_p(x,t)$ such that

$$\max\{\chi_1 \phi_c(x-ct+h_1), \chi_2 \hat{\phi}_c(x+\hat{c}t+h_2), \chi_3 \xi(t)\} \leq v(x,t) \leq \bar{v}(x,t)$$

for any $(x,t) \in \mathbb{R}^2$, where

$$\bar{v}(x,t) := \min\{1, \chi_1 \phi_c(x-ct+h_1) + \chi_2 \hat{\phi}_c(x+\hat{c}t+h_2) + \chi_3 \xi(t)\}.$$


Recall that in [26], we have obtained the existence and relative properties of entire solutions for symmetric equation (1) with monostable nonlinearity, that is, $J$ satisfies

\[(J2): \quad J \in C^1(\mathbb{R}), \quad J(x) = J(-x) \geq 0, \quad \int_{\mathbb{R}} J(y)dy = 1 \quad \text{and} \quad J \text{ is compactly supported},\]

and $f$ satisfies (F1). Obviously, our Theorem 1.2 and Corollary 1.3 in this paper can completely cover the previous results in [26]. However, the property (iii) in Theorem 1.4 can not occur when $J$ is symmetric. In fact, when $J$ is symmetric, $\hat{c}^* = c^* > 0$ and $\phi_c(-x - ct)$ is a nondecreasing traveling wave solution of (1) if $\phi_c(x - ct)$ is a nonincreasing one. Thus, $\phi_c(-x - ct) = \phi_c(x + \hat{c}t)$ for any $\hat{c} > \hat{c}^*$ by $\phi_c(0) = \frac{1}{\pi}, \quad \hat{\phi}_c(0) = \frac{1}{\pi}$ and the uniqueness of traveling wave solutions.

However, in [26] we did not consider the uniqueness and continuous dependence of the entire solutions on parameters $c, \hat{c}, h_i(i = 1, 2)$ and $k$. We will devote to this topic in this paper. Unfortunately, the method we used here depends on the symmetry of kernel $J$. Therefore, we just consider the uniqueness and continuous dependence of the entire solutions of (1) with symmetric $J$. The result can be stated as follows.

**Theorem 1.7.** Assume that $(J2)$ and $(F1)$ hold. Let $w(x, t)$ be the entire solution of (1) established in [26]. Then $w(x, t)$ depends continuously on $(c, h, h_1, h_2, k) \in (c^*, +\infty)^2 \times \mathbb{R}^2 \times (0, +\infty)$ in $T$. Moreover, $w(x, t)$ is unique up to translation.

The greatest difficulty in proving the continuous dependency of the entire solution $w(x, t)$ is that the mathematical expression of the solution of Cauchy problem of (1) is too abstract, since the kernel $J$ is abstract. In this paper, we get over it by means of Fourier transform.

The rest of the paper is organized as follows. In Section 2, we give the existence of the solutions for Cauchy problem of (1) and a comparison theorem which is essential in getting the entire solutions we desired. Sections 3, 4 and 5 are devoted to the proofs of Theorems 1.1, 1.2, 1.4 and 1.6, respectively. In the last section, we prove the continuous dependence on parameters and the uniqueness of entire solutions obtained in [26], and end this paper with an important remark.

2. Preliminaries. In this section, we will make some preparations for getting our main results later. Since the main theorems are proved by the aid of solution sequence of the Cauchy problems starting at times $-n$ with suitable initial values, we first consider the following Cauchy problems of (1):

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = (J * u)(x, t) - u(x, t) + f(u(x, t)), & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]  

(14)

**Definition 2.1.** A function $\bar{u}(x, t)$ is called a supersolution of (1) on $(x, t) \in \mathbb{R} \times [\tau, T)$, if $\bar{u}(x, t) \in C^{0,1}([\tau, T], \mathbb{R})$ and satisfies

\[
\frac{\partial}{\partial t} \bar{u}(x, t) \geq (J * \bar{u})(x, t) - \bar{u}(x, t) + f(\bar{u}(x, t)), \quad \forall (x, t) \in \mathbb{R} \times [\tau, T).
\]  

(15)

Furthermore, if for any $\tau < T$, $\bar{u}$ is a supersolution of (1) on $(x, t) \in \mathbb{R} \times [\tau, T)$, then $\bar{u}$ is called a supersolution of (1) on $(x, t) \in \mathbb{R} \times (-\infty, T)$. Similarly, a subsolution $\underline{u}(x, t)$ can be defined by reversing the inequality (15).
Theorem 2.2. Assume that $u_1(x,t),u_2(x,t) \in C^1([0,T],L^\infty(\mathbb{R}))$ and $u_1(x,t),
abla_x u_2(x,t)$ satisfy
\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - (J * u_1 - u_1) - f(u_1) &\geq \frac{\partial u_2}{\partial t} - (J * u_2 - u_2) - f(u_2), \quad (x,t) \in \mathbb{R} \times (0,T],
\end{aligned}
\]
\[u_1(x,0) \geq u_2(x,0), \quad x \in \mathbb{R},\]
where $0 < T \in \mathbb{R}$, $f \in C^1(\mathbb{R})$ and $J$ satisfies (J1). Then $u_1(x,t) \geq u_2(x,t)$ on $\mathbb{R} \times [0,T]$. Moreover, if $u_1(x,t)$ and $u_2(x,t)$ are bounded and uniformly continuous functions on $\mathbb{R} \times [0,T]$ and $u_1(x,t) \neq u_2(x,t)$ on $\mathbb{R} \times [0,T]$, then $u_1(x,t) > u_2(x,t)$ on $\mathbb{R} \times (0,T]$.

Lemma 2.3. Assume that (J1) and (F1) hold. Then
\[(i): \text{for any } u_0(x) \in C(\mathbb{R},[0,1]), (14) \text{ admits a unique solution } u(x,t;u_0) \in C(\mathbb{R} \times [0,\infty),[0,1]).
\[(ii): \text{for any pair of supersolution } \bar{u}(x,t) \text{ and subsolution } \underline{u}(x,t) \text{ of (1) on } \mathbb{R} \times [0,\infty) \text{ with } \underline{u}(x,0) \leq \bar{u}(x,0) \text{ and } 0 \leq \underline{u}(x,t), \bar{u}(x,t) \leq 1 \text{ for } (x,t) \in \mathbb{R} \times [0,\infty), \text{ there holds } 0 \leq \underline{u}(x,t) \leq \bar{u}(x,t) \leq 1 \text{ for all } (x,t) \in \mathbb{R} \times [0,\infty).
\]

Lemma 2.4. Let $J$ satisfy (J1). Assume that $u_1$ and $u_2$ are continuous functions on $\mathbb{R} \times [0,\infty)$ such that $u_1 \geq 0$ and $0 \leq u_2 \leq 1$ on $\mathbb{R} \times [0,\infty)$, $u_2 \leq u_1$ on $\mathbb{R} \times \{0\}$, and
\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - (J * u_1 - u_1) - f'(0)u_1 &\geq \frac{\partial u_2}{\partial t} - (J * u_2 - u_2) - f'(0)u_2 \tag{16}
\end{aligned}
\]
on $(x,t) \in \mathbb{R} \times (0,\infty)$ and $f'(0) > 0$. Then $\min\{1,u_1\} \geq u_2$ on $\mathbb{R} \times [0,\infty)$.

Under the condition that $\exists \ a < 0 < b$ such that $J(a) > 0$ and $J(b) > 0$ in (J1), the proofs of Theorem 2.2, Lemmas 2.3 and 2.4 are constant and standard, thus we do not repeat them here, one can refer to [15], [6, Lemma 2.4], [14, Theorem 1.5.1] and [26, Theorem 2.4 and Lemma 2.3].

3. Proof of Theorem 1.1.

Proof. Define
\[
\rho(\lambda) = \frac{1}{\lambda} \left\{ \int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 + f'(0) \right\}, \tag{17}
\]
\[
\hat{\rho}(\lambda) = \frac{1}{\lambda} \left\{ \int_{\mathbb{R}} J(-z)e^{\lambda z}dz - 1 + f'(0) \right\}, \tag{18}
\]
Then
\[
c^* = \inf_{\lambda > 0} \rho(\lambda), \quad \hat{c}^* = \inf_{\lambda > 0} \hat{\rho}(\lambda). \tag{19}
\]
Set
\[
v(\lambda) = \lambda \rho(\lambda) = \int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 + f'(0), \tag{20}
\]
\[
\hat{v}(\lambda) = \lambda \hat{\rho}(\lambda) = \int_{\mathbb{R}} J(-z)e^{\lambda z}dz - 1 + f'(0). \tag{21}
\]
Since \( J \in C^4(\mathbb{R}) \) and \( \exists a < 0 < b \) such that \( J(a) > 0 \) and \( J(b) > 0 \), we have

\[
\lim_{\lambda \to +\infty} \rho(\lambda) = \lim_{\lambda \to +\infty} \int_{\mathbb{R}} zJ(z)e^{\lambda z}dz = \lim_{\lambda \to +\infty} \left\{ \int_{0}^{+\infty} zJ(z)e^{\lambda z}dz - \int_{0}^{+\infty} zJ(-z)e^{-\lambda z}dz \right\} = \lim_{\lambda \to +\infty} \left\{ \int_{0}^{+\infty} z \left[ J(z)e^{\lambda z} - J(-z)e^{-\lambda z} \right]dz \right\} = +\infty.
\]

Similarly,

\[
\lim_{\lambda \to +\infty} \dot{\rho}(\lambda) = +\infty.
\]

Moreover, \( v(0) = \dot{v}(0) = f'(0) > 0 \), which implies that

\[
\lim_{\lambda \to 0^+} \rho(\lambda) = +\infty, \quad \lim_{\lambda \to 0^+} \dot{\rho}(\lambda) = +\infty.
\]

Thus, there exist \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) such that

\[
\inf_{\lambda > 0} \rho(\lambda) = \rho(\lambda_1), \quad \inf_{\lambda > 0} \dot{\rho}(\lambda) = \dot{\rho}(\lambda_2).
\]

By a simple calculation, we have

\[
\rho'(\lambda_1) = 0 \iff v(\lambda_1) = \lambda_1v'(\lambda_1),
\]

\[
\dot{\rho}'(\lambda_2) = 0 \iff \dot{v}(\lambda_2) = \lambda_2\dot{v}'(\lambda_2).
\]

Then

\[
c^* + \dot{c}^* = \rho(\lambda_1) + \dot{\rho}(\lambda_2) = \frac{v(\lambda_1)}{\lambda_1} + \frac{\dot{v}(\lambda_2)}{\lambda_2} = v'(\lambda_1) + \dot{v}'(\lambda_2)
\]

\[
= \int_{\mathbb{R}} zJ(z)e^{\lambda_1 z}dz + \int_{\mathbb{R}} zJ(-z)e^{\lambda_2 z}dz
\]

\[
= \int_{0}^{+\infty} zJ(z) e^{\lambda_1 z}dz + \int_{0}^{+\infty} zJ(-z) e^{\lambda_2 z}dz + \int_{0}^{+\infty} zJ(z) e^{-\lambda_2 z}dz + \int_{0}^{+\infty} zJ(-z) e^{-\lambda_1 z}dz \geq 0.
\]

\[
\Box
\]

4. **Proofs of Theorems 1.2 and 1.4.** In this section, if there are no another statement, we always assume that \( J \) satisfies (J1) and \( f \) satisfies (F1). Assume \( c > c^*, \dot{c} > \dot{c}^* \) and \( c \cdot \dot{c} \neq 0 \). Let \( h_1, h_2 \in \mathbb{R}, k > 0 \) be any given numbers, choose a positive integer \( n_0 \) such that \( ke^{-f'(0)n_0} < 1 \), and let \( \chi_1, \chi_2, \chi_3 \in \{0, 1\} \) with \( \chi_1 + \chi_2 + \chi_3 \geq 2 \). Then for any \( n \geq n_0 \), we denote

\[
\varphi^n(x) := \max\{\chi_1\phi_c(x + cn + h_1), \chi_2\tilde{\phi}_c(x - \tilde{c}n + h_2), \chi_3ke^{-f'(0)n}\}, \quad x \in \mathbb{R}.
\]

Let \( u_n(x, t) \) be the unique solution of the following Cauchy problem

\[
\begin{cases}
(u_n)_t = J * u_n - u_n + f(u_n), & x \in \mathbb{R}, t > -n, \\
u_n(x, -n) = u_{n,0}(x) := \varphi^n(x), & x \in \mathbb{R}.
\end{cases}
\tag{22}
\]

Then by the comparison principle, we have

\[
0 \leq u_n(x, t) \leq 1, \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq -n.
\tag{23}
\]
Next, we show some priori estimates uniformly in \( n \) of \( u_n(x,t) \), which allow us to take the limit as \( n \to +\infty \). Moreover, some properties fulfilled by the functions \( u_n(x,t) \) will hold well for the limit function \( u(x,t) \).

**Lemma 4.1.** There exists a positive constant \( C_1 \), which is independent of \( x,t,n \) and \((\chi_1\hat{c},\chi_2\hat{c},\chi_1h_1,\chi_2h_2,\chi_3k)\) such that for all \( n \in \mathbb{N}, t \geq -n + 1 \) and \( x \in \mathbb{R} \),

\[
|u_n(x,t)|, \ |(u_n)_{tt}(x,t)| \leq C_1. \tag{24}
\]

In addition, if there exists \( C_2 \) independent of \( n,x \) and \((\chi_1\hat{c},\chi_2\hat{c},\chi_1h_1,\chi_2h_2,\chi_3k)\) such that

\[
|u_{n,0}(x_1) - u_{n,0}(x_2)| \leq C_2|x_1 - x_2| \text{ for any } n \in \mathbb{N} \text{ and } x_1, x_2 \in \mathbb{R}. \tag{25}
\]

Then there exist positive constants \( M' \) and \( M'' \), which are independent of \( x,t,n \) and \((\chi_1\hat{c},\chi_2\hat{c},\chi_1h_1,\chi_2h_2,\chi_3k)\), such that the solutions \( u_n(x,t) \) of (22) satisfy

\[
\left| u_n(x + \eta, t) - u_n(x, t) \right| \leq M'\eta, \tag{26}
\]

\[
\left| \frac{\partial u_n}{\partial t}(x + \eta, t) - \frac{\partial u_n}{\partial t}(x, t) \right| \leq M''\eta, \tag{27}
\]

for any \( x \in \mathbb{R}, t > -n \) and \( \eta > 0 \).

**Proof.** Since the functions \( u_n \) are uniformly bounded and \( f \) is of \( C^2 \), it is easy to show that (24) holds.

Moreover, in view of \( J' \in L^1(\mathbb{R}) \), there exists \( L > 0 \) such that

\[
\int_{\mathbb{R}} |J(x + \eta) - J(x)|\,dx = \eta \int_{\mathbb{R}} \left| \int_0^1 J'(x + \theta\eta)\,d\theta \right|\,dx
\]

\[
\leq \eta \int_{\mathbb{R}} \int_0^1 |J'(x + \theta\eta)|\,d\theta\,dx \leq L\eta \text{ for any } \eta > 0.
\]

Thus, the rest of the proof is similar to that of [26, Proposition 2.5]. \( \Box \)

**Lemma 4.2.** When \( \chi_3 = 1 \) and \( \chi_1 + \chi_2 \geq 1 \), there exists \( \epsilon_* \in (0,1) \) and a continuous function \( v: [0,\epsilon_*] \to \mathbb{R} \) such that

\[
\min \left\{ \epsilon, \ k \epsilon f'(0)^t (1 - v(\epsilon)\epsilon) \right\} \leq \xi_n(t) \leq u_n(x,t), \ n \geq n_0, t > -n, \epsilon \in [0,\epsilon_*], \tag{28}
\]

where \( \xi_n(t) \) is a solution of the ordinary differential equation

\[
\xi_n'(t) = f(\xi_n(t)), \quad \xi_n(-n) = ke^{-f'(0)n}.
\]

**Proof.** By the comparison principle, we have that for any \( t \geq -n \) and any \( x \in \mathbb{R} \)

\[
0 < \max \left\{ \chi_1\phi_c(x - ct + h_1), \xi_n(t), \chi_2\phi_c(x + et + h_2) \right\} \leq u_n(x,t) < 1. \tag{29}
\]

We next prove that

\[
\min \left\{ \epsilon, \ k \epsilon f'(0)^t (1 - v(\epsilon)\epsilon) \right\} \leq \xi_n(t) \leq k \epsilon f'(0)^t.
\]

The remainder of the proof is similar to that of [26, Proposition 2.6] and thus omitted. \( \Box \)

The following lemma gives an upper bound of the functions \( u_n(x,t) \), which is independent of \( n \).
Lemma 4.3. For any couple \((x, t) \in \mathbb{R} \times [-n, +\infty)\), the functions \(u_n(x, t)\) satisfy
\[ u_n(x, t) \leq \min\{1, \Pi_1(x, t), \Pi_2(x, t), \Pi_3(x, t)\}, \]
where \(\Pi_i(x, t), i = 1, 2, 3\) are defined in Theorem 1.2.

Proof. We will only prove \(u_n(x, t) \leq \Pi_1(x, t)\) for all \((x, t) \in \mathbb{R} \times [-n, +\infty)\), since the proofs of other inequalities are similar. Without loss of generality, we assume \(\chi_1 = 1\) and set \(v_n(x, t) = u_n(x, t) - \phi_c(x - ct + h_1)\). Then we will compare \(v_n(x, t)\) with the solution of a linear equation. Obviously, \(0 \leq v_n(x, t) \leq 1\) for \((x, t) \in \mathbb{R} \times [-n, +\infty)\). Since \(\phi_c(x - ct + h_1)\) is a solution of (1) and \(f'(s) \leq f'(0)\) for any \(s \in [0, 1]\), a direct computation shows that
\[(v_n)_t = J * v_n - v_n + f(u_n) - f(\phi_c(x - ct + h_1)) \leq J * v_n + f'(0)v_n.\]
Take
\[w_n(x, t) := \chi_2 B_\xi e^{\mu(\xi)(x + ct + h_2)} + \chi_3 k e^{f'(0)t}, \quad x \in \mathbb{R}, t \geq -n.\]
According to the definition of \(\mu(\xi)\), we have
\[
\frac{\partial w_n}{\partial t} - (J * w_n - w_n + f'(0)w_n)
= \chi_2 B_\xi \mu(\xi)e^{\mu(\xi)(x + ct + h_2)} + \chi_3 k e^{f'(0)t}
\begin{aligned}
&- \left( \int_{\mathbb{R}} J(x - y)w_n(y, t)dy - w_n(x, t) + f'(0)w_n(x, t) \right) \\
&= \chi_2 B_\xi e^{\mu(\xi)(x + ct + h_2)} \left( \hat{\xi} \mu(\hat{\xi}) - \int_{\mathbb{R}} J(z)e^{-\mu(\xi)z}dz + 1 - f'(0) \right)
= 0,
\end{aligned}
\]
which implies that \(w_n(x, t)\) is a solution of the following Cauchy problem
\[
\begin{cases}
\frac{\partial w_n}{\partial t} = J * w_n - w_n + f'(0)w_n, \\
w_n(x, -n) = \chi_2 B_\xi e^{\mu(\xi)(x - \hat{\xi}n + h_2)} + \chi_3 k e^{-f'(0)n}
\end{cases}
\]
Note that \(\hat{\phi}_c(\xi) \leq B_\xi e^{\mu(\xi)\xi}\) for all \(\xi \in \mathbb{R}\). Thus we have
\[
v_n(x, -n) = \max\{\phi_c(x + cn + h_1), \chi_2 \hat{\phi}_c(x - \hat{\xi}n + h_2), \chi_3 ke^{-f'(0)n}\} - \phi_c(x + cn + h_1)
\leq \chi_2 \hat{\phi}_c(x - \hat{\xi}n + h_2) + \chi_3 ke^{-f'(0)n}
\leq \chi_2 B_\xi e^{\mu(\xi)(x - \hat{\xi}n + h_2)} + \chi_3 ke^{-f'(0)n}
= w_n(x, -n).
\]
It then follows from Lemma 2.4 that \(v_n(x, t) \leq w_n(x, t)\) for all \((x, t) \in \mathbb{R} \times [-n, +\infty),\) that is \(u_n(x, t) \leq \Pi_1(x, t)\). The proof is complete. \(\square\)

Proof of Theorem 1.2. (i) Note that \(\phi_c(x - ct)\) and \(\hat{\phi}_c(x + \hat{c}t)\) are monotone traveling wave solutions of (1) which satisfy (5) and (6), respectively. Then for any \(c > c^*, \hat{c} > \hat{c}^*\) and \(c\hat{c} \neq 0\), we have \(|\phi'_c| \leq \frac{2 + M_3}{|c\hat{c}|}\) and \(|\hat{\phi}'_c| \leq \frac{2 + M_3}{|c\hat{c}|}\), where \(M_3 = \max s \in [0, 1] f(s)\), which make (25) of Lemma 4.1 hold. Thus the solutions \(u_n(x, t)\) of (22) are globally Lipschitz in \(x\). Following (24) and Lemma 4.1, the Arzela-Ascoli Theorem and the diagonal extraction imply that there exists a subsequence \(\{u_{n_i}\}_{i \in N}\) of \(\{u_n\}_{n \in N}\) such that \(u_{n_i}(x, t)\) converge uniformly to a function \(u(x, t)\) in \(T\). From the equation satisfied by \(u_n(x, t)\), we know that the limit function \(u(x, t)\) is an entire solution of
Proof of Corollary 1.3. (i) We only prove the case of \((f_1, \chi_2, \chi_3) = (1, 1, 1)\), since the proofs of other cases are similar. According to (10), we have

\[
\max\{c_1(x - ct + h_1), \hat{c}(x + \hat{c}t + h_2), \chi(t)\} \leq u(x, t) \leq A_{\varepsilon} e^{-\lambda\varepsilon(x - ct + h_1)} + B_{\varepsilon} e^{\mu\varepsilon(x + \hat{c}t + h_2)} + \chi(t)
\]

(31)
for \((x, t) \in \mathbb{R}^2\). Note that for any fixed \(x \in \mathbb{R}, h_i \in \mathbb{R}(i = 1, 2)\) and \(k > 0\), there are

\[
\lim_{t \to -\infty} \phi_\varepsilon(x - ct + h_1)e^{\lambda(t)(x - ct + h_1)} = \alpha_\varepsilon,
\]
\[
\lim_{t \to -\infty} \hat{\phi}_\varepsilon(x + ct + h_2)e^{-\mu(t)(x + ct + h_2)} = \beta_\varepsilon.
\]

Then the assertion (i) follows from (31).

(iii) We only prove the case that \(u_{p_0}(x, t)\) converges to \(u_{p_2}(x, t)\) as \(h_2 \to -\infty\) in \(T\) and uniformly on \((x, t) \in S^1_{A,a}\), since the other cases can be shown similarly. When \((\chi_1, \chi_2, \chi_3) = (1, 1, 1)\), we denote \(\varphi^n(x)\) by \(\varphi^n_{p_0}\) and \(u_n(x, t)\) by \(u^n_{p_0}(x, t)\), respectively. And when \((\chi_1, \chi_2, \chi_3) = (1, 0, 1)\), we denote \(\varphi^n(x)\) by \(\varphi^n_{p_2}\) and \(u_n(x, t)\) by \(u^n_{p_2}(x, t)\), respectively. Let

\[
w^n(x, t) = u^n_{p_0}(x, t) - u^n_{p_2}(x, t), \quad (x, t) \in \mathbb{R} \times [-n, +\infty),
\]
then \(0 \leq w^n(x, t) \leq 1\) for all \((x, t) \in \mathbb{R} \times (-n, +\infty)\). Since \(f'(u) \leq f'(0)\) for \(u \in [0, 1]\), we have

\[
\frac{\partial w^n(x, t)}{\partial t} = (J * w^n)(x, t) - w^n(x, t) + f(u^n_{p_0}(x, t)) - f(u^n_{p_2}(x, t))
\]
\[
= (J * w^n)(x, t) - w^n(x, t) + f'(u^n_{p_2}(x, t) + \theta u^n(x, t))w^n(x, t)
\]
\[
\leq (J * w^n)(x, t) - w^n(x, t) + f'(0)w^n(x, t)
\]

for any \(x \in \mathbb{R}, t > -n\), where \(\theta \in (0, 1)\).

Define the function \(\hat{w}(x, t) = B_\varepsilon e^{\mu(t)(x + \hat{c}t + h_2)}\) for \((x, t) \in \mathbb{R}^2\). Note that \(\hat{c}\mu(\hat{c}) = \int_\mathbb{R} f(z)e^{-\mu(\hat{c})z} - 1 + f'(0)\), direct computation shows that

\[
\frac{\partial \hat{w}}{\partial t} = J * \hat{w} - \hat{w} + f'(0)\hat{w}
\]

Moreover,

\[
w^n(x, -n) = u^n_{p_0}(x, -n) - u^n_{p_2}(x, -n) \leq \hat{\phi}_\varepsilon(x - \hat{c}n + h_2)
\]
\[
\leq B_\varepsilon e^{\mu(t)(x - \hat{c}n + h_2)} = \hat{w}(x, -n).
\]

It then follows from Lemma 2.4 that

\[
0 \leq w^n(x, t) = u^n_{p_0}(x, t) - u^n_{p_2}(x, t) \leq B_\varepsilon e^{\mu(t)(x + \hat{c}t + h_2)}
\]

for all \((x, t) \in \mathbb{R} \times [-n, +\infty)\). Since \(\lim_{n \to +\infty} u^n_{p_2}(x, t) = u_{p_0}(x, t)(i = 0, 1)\), we get

\[
0 \leq u_{p_0}(x, t) - u_{p_2}(x, t) \leq B_\varepsilon e^{\mu(t)(x + \hat{c}t + h_2)}
\]

for all \((x, t) \in \mathbb{R}^2\), which implies that \(u_{p_0}(x, t)\) converges to \(u_{p_2}(x, t)\) as \(h_2 \to -\infty\) uniformly on \((x, t) \in S^1_{A,a}\) since \(\hat{c} > 0\). And since \(f\) is of class \(C^2\), the conclusion is obvious. (ii) can be proved by the same argument as that in [26, Theorem 1.1], thus we omit the details. This completes the proof. □

Proof of Theorem 1.4. Note that \(c > 0\) and \(\hat{c} < 0\). Since \(c + \hat{c} > 0\), we have \(c > -\hat{c}\). Let \(h_1, h_2 \in \mathbb{R}, k > 0\) be any given numbers. And set \(\chi_1, \chi_2, \chi_3 \in \{0, 1\}\) with \(\chi_1 + \chi_2 + \chi_3 \geq 2\). For any \(n \in \mathbb{N}\), let \(u^-_n(x, t) = u^-_{n,p}(x, t)\) be the unique solution of the Cauchy problem

\[
\begin{aligned}
(u^-_n)_t &= J * u^-_n - u^-_n + f(u^-_n), \quad x \in \mathbb{R}, \quad t > -n, \\
u^-_n(x, -n) &= u^-_{n,0}(x) := \varphi^n(x), \quad x \in \mathbb{R}.
\end{aligned}
\]

(32)

According to the pervious discussions, Lemmas 4.1-4.3 still hold for \(u^-_n(x, t)\). Since the assertions (i) and (ii) are obvious, we only prove (a) and (b) of (iii).
(a) Assume $\chi_1 = \chi_2 = 1$ and $\chi_3 = 0$. Since $u^-(x,t)$ satisfy
\begin{equation}
\max \left\{ \phi_c(x - ct + h_1), \hat{\phi}_c(x + \hat{c}t + h_2) \right\} \leq u^-(x,t) \leq 
\min \left\{ 1, \phi_c(x - ct + h_1) + B_c e^{\mu(\hat{c})(x + \hat{c}t + h_2)}, A_c e^{-\lambda(c)(x - ct + h_1) + \hat{\phi}_c(x + \hat{c}t + h_2)} \right\}
\end{equation}
for any $(x,t) \in \mathbb{R}^2$, we have
\[ 0 \leq u^-(x,t) - \phi_c(x - ct + h_1) \leq B_c e^{\mu(\hat{c})(x + \hat{c}t + h_2)} \]
and
\[ 0 \leq u^-(x,t) - \hat{\phi}_c(x + \hat{c}t + h_2) \leq A_c e^{-\lambda(c)(x - ct + h_1)} \]
for any $(x,t) \in \mathbb{R}^2$. In view of $\hat{c} < 0$, the result (a) is obvious.

(b) Assume $c > \hat{c}$. If there exists a function $x(t)$ for $t \ll -1$ such that for any fixed $h_1$ and $h_2$,
\begin{equation}
x(t) + \hat{c}t + h_2 \to -\infty \quad \text{and} \quad x(t) - ct + h_1 \to +\infty
\end{equation}
as $t \to -\infty$, thus $x + \hat{c}t + h_2 \to -\infty$ for $x \leq x(t)$ and $x - ct + h_1 \to +\infty$ for $x \geq x(t)$ as $t \to -\infty$. It then follows that
\[ B_c e^{\mu(\hat{c})(x + \hat{c}t + h_2)} \to 0 \quad \text{for} \quad x \leq x(t) \]
and
\[ A_c e^{-\lambda(c)(x - ct + h_1)} \to 0 \quad \text{for} \quad x \geq x(t) \]
as $t \to -\infty$, which implies (12) holds. Now we show that there exists $x(t)$ satisfying (34). In fact, according to (34) and $c > \hat{c} > 0$, we just assume $x(t) = gt$ with $-\hat{c} < g < c$, which makes (34) hold. And we obtain (12). Furthermore, according to (33), there is
\begin{equation}
\max \left\{ \phi_c(x - ct + h_1), \hat{\phi}_c(x + \hat{c}t + h_2) \right\} \leq u^-(x,t) \leq 1.
\end{equation}
Still set $x(t) = gt$ with $-\hat{c} < g < c$. We have $\phi_c(x - ct + h_1) \to 1$ for $x \leq gt$ since $x - ct + h_1 \to -\infty$ and $\hat{\phi}_c(x + \hat{c}t + h_2) \to 1$ for $x \geq gt$ since $x + \hat{c}t + h_2 \to +\infty$ as $t \to +\infty$. And (13) is a straightforward consequence of (35). The proof is complete. \hfill \Box

5. Proof of Theorem 1.6. Let $v_n(x,t)$ be the unique solution of (22) with $u_n(x,t)$ replaced by $v_n(x,t)$, we first give an upper bound of $v_n(x,t)$.

Lemma 5.1. Assume that $J$ and $f$ satisfy the hypotheses of Theorem 1.6. The speeds $c$ and $\hat{c}$ are described as in Theorem 1.6, then for any $h_1, h_2 \in \mathbb{R}, k > 0$, $\chi_1, \chi_2, \chi_3 \in \{0, 1\}$ with $\chi_1 + \chi_2 + \chi_3 \geq 2$, and any couple $(x,t) \in \mathbb{R} \times [-n, +\infty)$, the functions $v_n(x,t)$ satisfy
\[ v_n(x,t) \leq \min \{1, \chi_1 \phi_c(x - ct + h_1) + \chi_2 \hat{\phi}_c(x + \hat{c}t + h_2) + \chi_3 \xi(t)\} \]

Proof: Set $\bar{v}(x,t) := \min \{1, \chi_1 \phi_c(x - ct + h_1) + \chi_2 \hat{\phi}_c(x + \hat{c}t + h_2) + \chi_3 \xi(t)\}$, by comparison principle and Lemma 2.3, we just need to prove that $\bar{v}(x,t)$ is a supersolution of (1) on $(x,t) \in \mathbb{R}^2$.

If $\chi_1 \phi_c + \chi_2 \hat{\phi}_c + \chi_3 \xi \geq 1$, then $\bar{v}(x,t) = 1$ and the conclusion is obvious. Therefore, we assume that $\chi_1 \phi_c + \chi_2 \hat{\phi}_c + \chi_3 \xi < 1$, and just prove $\frac{\partial \bar{v}}{\partial t} \geq J * \bar{v} - \bar{v} + f(\bar{v})$. Note
Lemma 6.1. That
\[
\frac{\partial \tilde{\nu}}{\partial t} = -\chi_1 c \phi'_c + \chi_2 \hat{c} \phi'_c + \chi_3 \xi'
\]
= \chi_1 (J * \phi_c - \phi_c + f(\phi_c)) + \chi_2 (J * \hat{\phi}_c - \hat{\phi}_c + f(\hat{\phi}_c)) + \chi_3 f(\xi),
and
\[
J * \tilde{\nu} - \tilde{\nu} + f(\tilde{\nu}) = \chi_1 (J * \phi_c - \phi_c) + \chi_2 (J * \hat{\phi}_c - \hat{\phi}_c) + f(\chi_1 \phi_c + \chi_2 \hat{\phi}_c + \chi_3 \xi).
\]
Hence, it suffices to show that
\[
f(\chi_1 \phi_c + \chi_2 \hat{\phi}_c + \chi_3 \xi) \leq \chi_1 f(\phi_c) + \chi_2 f(\hat{\phi}_c) + \chi_3 f(\xi).
\]
We just prove the case of \( \chi_1 = \chi_2 = \chi_3 = 1 \) since the proofs of other cases are similar. Using the concavity of the function \( f \), we obtain
\[
\frac{f(\phi_c + \hat{\phi}_c + \xi) - f(\phi_c)}{\phi_c + \xi} \leq \frac{f(\hat{\phi}_c)}{\hat{\phi}_c},
\]
\[
\frac{f(\phi_c + \hat{\phi}_c + \xi) - f(\xi)}{\phi_c + \hat{\phi}_c} \leq \frac{f(\xi)}{\xi},
\]
which implies that
\[
\phi_c f(\phi_c + \hat{\phi}_c + \xi) \leq f(\phi_c)(\phi_c + \hat{\phi}_c + \xi), \quad \hat{\phi}_c f(\phi_c + \hat{\phi}_c + \xi) \leq f(\hat{\phi}_c)(\phi_c + \hat{\phi}_c + \xi),
\]
\[
\xi f(\phi_c + \hat{\phi}_c + \xi) \leq f(\xi)(\phi_c + \hat{\phi}_c + \xi).
\]
It then follows that \( f(\phi_c + \hat{\phi}_c + \xi) \leq f(\phi_c) + f(\hat{\phi}_c) + f(\xi) \). The proof is complete. □

According to Lemma 5.1, the remaining proof of Theorem 1.6 is similar to that of Theorem 1.2, and is omitted.

6. Proof of Theorem 1.7. In this section, we prove Theorem 1.7 under the conditions (J2) and (F1). Since it is a further work of [26], for convenience, we use \( \phi_c(-x - \epsilon t) \) instead of \( \hat{\phi}_c(x + \epsilon t) \) as the nondecreasing traveling wave solutions of equation (1).

Lemma 6.1. The functions \( \phi_c(z) \) are continuous with respect to \( c \in (c^*, +\infty) \) in \( C^1_{loc}(\mathbb{R}) \).

Proof. Note that \( |\phi'_c| \leq \frac{2 + M_4}{c} \) with \( M_3 = \max_{s \in [0,1]} f(s), f \in C^2(\mathbb{R}) \) and \( J \) is compactly supported. By differentiating the equation
\[
-c \phi'_c = J * \phi_c - \phi_c + f(\phi_c)
\]
with respect to \( z \), we get
\[
-c \phi''_c = J * \phi'_c - \phi'_c + f'(\phi_c)\phi'_c.
\]
Thus, \( |\phi''_c| \leq \frac{2 + M_4}{c^2} |\phi'_c| \) with \( M_4 = \max_{s \in [0,1]} f'(s) \). It is easy to see that there exists a constant \( M_5 > 0 \) which is independent of \( x \) and \( c \) such that
\[
|\phi'_c|, |\phi''_c| \leq M_5.
\]
If \( c_i \to c \in (c^*, +\infty) \), then by the unique boundedness of \( |\phi'_c(z)|, |\phi''_c(z)| \) in \( z \in \mathbb{R} \) on \( l \in \mathbb{N} \) and by a diagonal extraction process, there exists a subsequence \( c_{i_l} \) such that \( \phi_{c_{i_l}} \to \phi \) in \( C^1_{loc}(\mathbb{R}) \), where \( \phi \) is a solution of
\[
-c \phi' = J * \phi - \phi + f(\phi) \quad \text{in } \mathbb{R}.
\]
By passing to the limit \( c_i \to c \), the function \( \phi \) is nonincreasing in \( \mathbb{R} \), since \( \phi_{c_i} \) are normalized in 0, it follows that \( \phi(0) = \frac{1}{e} \). Note that \( f \) is positive on \((0, 1)\), this yields that \( \phi(-\infty) = 1 \) and \( \phi(+\infty) = 0 \). Therefore, \( \phi \) is a traveling wave front of (1) with speed \( c \). Following Carr and Chmaj [4] (or [34]), we have \( \phi \equiv \phi_c \). Therefore, the whole sequence \( \phi_{c_i} \to \phi_c \) in \( C^1_{loc}(\mathbb{R}) \) as \( l \to +\infty 
abla \).

**Lemma 6.2.** \( \alpha_c = \lim_{z \to +\infty} \phi_c(z)e^{\lambda_c z} \) is continuous in \( c \in (c^*, +\infty) \).

**Proof.** Fix \( c_0 \in (c^*, +\infty) \) and let \( c_i \to c_0 \) as \( l \to +\infty \) with \( c_i > c^* \) for each \( l \in \mathbb{N} \). Then by Carr and Chmaj [4] (or [34]), for each \( c \in (c^*, +\infty) \), there exists a unique traveling wave front \( \tilde{\phi}_c \) such that \( \tilde{\phi}'_c < 0 \), \( \tilde{\phi}_c(+\infty) = 0 \), \( \tilde{\phi}_c(-\infty) = 1 \), and

\[
\lim_{z \to +\infty} \tilde{\phi}_c(z)e^{\lambda c z} = 1.
\]

We claim that \( \tilde{\phi}_{c_i} \to \tilde{\phi}_{c_0} \) in \( C^1_{loc}(\mathbb{R}) \) as \( l \to +\infty \). Indeed, since \( c_i \to c_0 \) as \( l \to +\infty \), there exists a subsequence \( c_i \) and a function \( \tilde{\phi} \) such that \( \tilde{\phi}_{c_i} \to \tilde{\phi} \) in \( C^1_{loc}(\mathbb{R}) \), where \( \tilde{\phi} \) is nonincreasing and satisfies

\[
-c_0 \tilde{\phi}' = J * \tilde{\phi} - \tilde{\phi} + f(\tilde{\phi}) \quad \text{in } \mathbb{R}.
\]

On the other hand, by [34], there exists two constants \( q > 1 \) and \( \gamma > 1 \), independent of \( c_i \), such that

\[
e^{-\lambda(c_0) z} - q e^{-\gamma \lambda(c_0) z} \leq \tilde{\phi}_{c_i}(z) \leq e^{-\lambda(c_0) z} + q e^{-\gamma \lambda(c_0) z} \quad \text{for any } z \in \mathbb{R}.
\]

Therefore, as \( l \to +\infty \),

\[
e^{-\lambda(c_0) z} - q e^{-\gamma \lambda(c_0) z} \leq \tilde{\phi}(z) \leq e^{-\lambda(c_0) z} + q e^{-\gamma \lambda(c_0) z} \quad \text{for any } z \in \mathbb{R},
\]

which implies that \( \tilde{\phi} \) is not a constant and satisfies \( \lim_{z \to +\infty} \tilde{\phi}(z)e^{\lambda(c_0) z} = 1 \). Then it follows from [34] (or Carr and Chmaj [4]) that \( \tilde{\phi} \equiv \tilde{\phi}_{c_0} \). Consequently, the whole sequence \( \tilde{\phi}_{c_i} \to \tilde{\phi}_{c_0} \) in \( C^1_{loc}(\mathbb{R}) \) as \( l \to +\infty \).

Furthermore, note that the function \( \tilde{\phi}_c = \phi_c \left( \cdot + \frac{\ln \alpha_c}{\lambda(c)} \right) \) is also a solution of

\[
-c \tilde{\phi}'_c = J * \tilde{\phi}_c - \tilde{\phi}_c + f(\tilde{\phi}_c), \quad \tilde{\phi}_c(-\infty) = 1, \quad \tilde{\phi}_c(+\infty) = 0
\]

with \( \lim_{z \to +\infty} \tilde{\phi}_c(z)e^{\lambda c z} = 1 \). Thus, in view of the uniqueness of traveling wave solution in [4] and [34], we have \( \tilde{\phi}_c = \tilde{\phi}_c \), that is \( \tilde{\phi}_c = \phi_c \left( \cdot + \frac{\ln \alpha_c}{\lambda(c)} \right) \).

In order to prove that \( \alpha_c \) is continuous in \( c \) at \( c_0 \), it is enough to show that \( \tilde{\phi}_c(0) \) is continuous in \( c \) at \( c_0 \). We argue by contradiction. Assume that \( \tilde{\phi}_{c_0}(0) \to \tilde{\phi}_c(0) \), but \( \alpha_{c_i} \neq \alpha_{c_0} \) for a sequence \( c_i \to c_0 \). Thus without loss of generality, there exists a real \( \varepsilon > 0 \) and a subsequence \( c_{i'} \to c_0 \) such that \( \alpha_{c_{i'}} \leq \alpha_{c_0} - \varepsilon \). Consequently, one has

\[
\tilde{\phi}_{c_{i'}}(0) = \phi_{c_{i'}} \left( \frac{\ln \alpha_{c_{i'}}}{\lambda(c_{i'})} \right) \geq \phi_{c_{i'}} \left( \frac{\ln(\alpha_{c_0} - \varepsilon)}{\lambda(c_{i'})} \right)
\]

since \( \phi_c \) is decreasing. On the other hand, we have

\[
\tilde{\phi}_{c_{i'}}(0) \to \tilde{\phi}_{c_0}(0) = \phi_{c_0} \left( \frac{\ln \alpha_{c_0}}{\lambda(c_0)} \right)
\]

and

\[
\phi_{c_{i'}} \left( \frac{\ln(\alpha_{c_0} - \varepsilon)}{\lambda(c_{i'})} \right) \to \phi_{c_0} \left( \frac{\ln(\alpha_{c_0} - \varepsilon)}{\lambda(c_0)} \right)
\]
since the function $\phi_c(\cdot)$ is continuous in $C^1_{loc}(\mathbb{R})$ with respect to $c$. Consequently,
\[
\phi_{c_0} \left( \frac{\ln \alpha_{c_0}}{\lambda(c_0)} \right) \geq \phi_{c_0} \left( \frac{\ln(\alpha_{c_0} - \varepsilon)}{\lambda(c_0)} \right).
\]
This is impossible because $\phi_{c_0}$ is decreasing. Since $\widetilde{\phi}_c(z)$ is continuous in $c$ at $c_0$ for any $z \in \mathbb{R}$, it is obvious that $\phi_c(0)$ is continuous in $c$ at $c_0$. This completes the proof. 

\textbf{Lemma 6.3.} $A_c$ defined as (8) is continuous in $c \in (c^*, +\infty)$.

\textit{Proof.} Fix $c_0 \in (c^*, +\infty)$ and let $c_l \to c_0$ as $l \to +\infty$ with $c_l > c^*$ for each $l \in \mathbb{N}$. We argue by contradiction. Assume that $A_{c_l} \to A_0 \in \mathbb{R} \cup \{\infty\}$ as $l \to +\infty$ (up to extraction of some subsequence) and $A_0 \neq A_{c_0}$. Since $A_{c_l} \geq e^{\lambda(c_l)} \phi_{c_l}(z)$ for any $z \in \mathbb{R}$, we have $A_0 \geq e^{\lambda(c_0)} \phi_{c_0}(z)$. Therefore, $A_0 > A_{c_0}$.

Fix $b = \min \{\frac{A_{c_0} + A_0}{2}, A_0 + 1\}$. Then there exists $L \in \mathbb{N}$ such that for any $l > L$, $A_{c_l} > b$. On the other hand, since $\alpha_{c_l} \to \alpha_{c_0} \leq A_{c_0}$ and $\lambda(c_l) \to \lambda(c_0)$, there exists a constant $z_0 > 0$, independent of $c_l$, such that $e^{\lambda(c_l)} \phi_{c_l}(z) \leq b$ for any $|z| > z_0$. For $z \in [-z_0, z_0]$, in view of $e^{\lambda(c_0)} \phi_{c_0}(z) \leq A_{c_0}$, $\phi_{c_l}(z) \to \phi_{c_0}(z)$ in $C^1_{loc}(\mathbb{R})$ and the equicontinuity of $e^{\lambda(c_\cdot)}$ on $l$, there exists $L' > L$ such that $e^{\lambda(c_\cdot)} \phi_{c_l}(z) \leq b$ for any $l > L'$ and $z \in [-z_0, z_0]$. Consequently, $e^{\lambda(c_\cdot)} \phi_{c_l}(z) \leq b$ for any $l > L'$ and $z \in \mathbb{R}$, which contradicts to $A_{c_0} > b$ for any $l > L$. The proof is complete. 

\textbf{Proof of Theorem 1.7.} Consider a sequence
\[
(c_1, c_2, h_{11}, h_{21}, h_{12}, h_{22}, k) \in (c^*, +\infty)^2 \times \mathbb{R}^2 \times (0, +\infty).
\]
Set $w_l(x, t) = w_{c_1, c_2, h_{11}, h_{21}, k_1}$ and $w(x, t) = w_{c, c_1, h_{1}, h_{2}, k}(x, t)$. Let $0 < \xi(t) < 1$ is a solution of $\xi(t) = f(\xi)$ in $\mathbb{R}$ and $\xi(t) \sim ke^{f(0)t}$ as $t \to -\infty$.

In view of the a priori estimate (24) and Lemma 4.1, there exists a function $\bar{w}(x, t)$ such that $w_l(x, t) \to \bar{w}(x, t)$ as $l \to +\infty$ (up to extraction of some subsequence) in the sense of $\mathcal{T}$. In particular, the function $\bar{w}(x, t)$ is an entire solution of (1) and also satisfy the estimate (24) and Lemma 4.1. Since the functions $\xi(t)$ are uniformly bounded in $C^2(\mathbb{R})$, we can assume that they converge in $C^1_{loc}(\mathbb{R})$ to a function $\xi(t)$, which is a solution of $\xi' = f(\xi)$ in $\mathbb{R}$ and $\xi(t) \to ke^{f(0)t}$ as $t \to -\infty$.

Then from [26], we obtain that the function $\bar{w}(x, t)$ fulfills the following estimate
\[
\begin{align*}
\max \{ & \phi_c(x - ct + h_1), \xi(t), \phi_c(-x - \hat{c}t + h_2) \} \\
& \leq \bar{w}(x, t) \\
& \leq \min \left\{ 1, \phi_c(x - ct + h_1) + ke^{f(0)t} + A_2e^{-\lambda(h_2)}(x - \hat{c}t + h_2), \right. \\
& \left. A_2e^{-\lambda(h_2)(x - \hat{c}t + h_1)} + ke^{f(0)t} + \phi_c(-x - \hat{c}t + h_2), \right. \\
& \left. \xi(t) + A_2e^{-\lambda(h_2)(x - \hat{c}t + h_1)} + A_2e^{-\lambda(h_2)(x - \hat{c}t + h_2)} \right\}. \\
\end{align*}
\]

Now we prove that $\bar{w}(x, t) \equiv w(x, t) = w_{c, \hat{c}, h_{11}, h_{21}, k}(x, t)$ for any $(x, t) \in \mathbb{R}^2$. Remember that the functions $w_n(x, t)$ converge to the function $w(x, t)$ in the sense of $\mathcal{T}$, where $w_n(x, t)$ are solutions of the Cauchy problems
\[
(w_n)_t = J * w_n - w_n + f(w_n), \quad x \in \mathbb{R}, \quad t > -n,
\]
with the initial conditions
\[
w_n(x, -n) = w_{n, 0}(x) := \max \left\{ \phi_c(x + cn + h_1), ke^{f(0)n}, \phi_c(-x + \hat{c}n + h_2) \right\}.
\]
Let us compare the functions \( \tilde{w}(\cdot, -n) \) to the functions \( w_{n,0}(\cdot) \). Note that functions \( \phi_c(z) \) and \( \phi_e(z) \) are decreasing and approach 1 and 0 as \( z \to -\infty \) and \( z \to +\infty \), respectively. Set

\[
\gamma_n = \left( \frac{f'(0) - c\lambda(c)}{\lambda(c)} - \frac{f'(0) - \hat{c}\lambda(\hat{c})}{\lambda(\hat{c})} \right) n.
\]

Then

\[
\gamma_n + cn + h_1 \to +\infty
\]

and

\[
-\gamma_n + \hat{c} n + h_2 \to +\infty
\]
as \( n \to +\infty \). This implies that

\[
\phi_c(\gamma_n + cn + h_1) \sim \alpha_c e^{-\lambda(c)(\gamma_n + cn + h_1)} = o(e^{-f'(0)n})
\]

and

\[
\phi_e(-\gamma_n + \hat{c}n + h_2) \sim \alpha_e e^{-\lambda(\hat{c})(-\gamma_n + \hat{c}n + h_2)} = o(e^{-f'(0)n})
\]
as \( n \to +\infty \), in view of \( c\lambda(c) > f'(0) \) for any \( c \geq c^* \). Thus, for \( n \) large enough, there exists two real numbers \( y_n < z_n \) such that

\[
y_n < \gamma_n < z_n
\]

and

\[
w_{n,0}(x) = \begin{cases} 
\phi_c(x + cn + h_1), w'_{n,0}(x) < 0, & \text{if } x < y_n, \\
\phi_c(y_n + cn + h_1) = ke^{-f'(0)n}, & \text{if } x = y_n, \\
k e^{-f'(0)n}, w'_{n,0}(x) = 0, & \text{if } y_n < x < z_n, \\
k e^{-f'(0)n}, & \text{if } x = z_n, \\
\phi_e(-x + \hat{c}n + h_2), w'_{n,0}(x) > 0, & \text{if } x > z_n.
\end{cases}
\]

It is easy to see that \( y_n \) and \( z_n \) satisfy

\[
y_n = \frac{f'(0) - c\lambda(c)}{\lambda(c)} n + \frac{\ln \alpha_c - \ln k}{\lambda(c)} - h_1 + o(1),
\]

\[
z_n = -\frac{f'(0) - \hat{c}\lambda(\hat{c})}{\lambda(\hat{c})} n - \frac{\ln \alpha_c - \ln k}{\lambda(\hat{c})} + h_2 + o(1)
\]
as \( n \to +\infty \). In fact, the formula for \( y_n \) comes directly from the equality \( \phi_c(y_n + cn + h_1) = ke^{-f'(0)n} \) and the asymptotic behavior of \( \phi_c \) given by (7), so does \( z_n \).

Notice that \( A_c e^{-\lambda(c)z} \geq \phi_c(z) \) for any \( z \in \mathbb{R} \). By (36) and the definition of \( (y_n, z_n) \), we get

\[
|\tilde{w}(x, -n) - w_{n,0}(x)| \leq \begin{cases} 
ke^{-f'(0)n} + A_c e^{-\lambda(\hat{c})(-x + \hat{c}n + h_2)}, & \text{if } x \leq y_n, \\
\xi(-n) - ke^{-f'(0)n} + A_c e^{-\lambda(c)(x + cn + h_1)} & \text{if } y_n \leq x \leq z_n, \\
+ A_c e^{-\lambda(\hat{c})(-x + \hat{c}n + h_2)}, & \text{if } x \geq z_n,
\end{cases}
\]

On the other hand, denote \( \hat{J}(\xi) \) as the Fourier transform of \( J \). Under the condition (J2), by [23, Theorem 10.6.2] and [35, Theorems 7.1 and 7.4 and Plancherel’s formula], it is obvious that \( \hat{J}(\xi) \) is differentiable, \( \hat{J}(\xi) \in L^1(\mathbb{R}) \) and \( \hat{J}(\xi) \in L^2(\mathbb{R}) \). In addition, because \( J \) is compactly supported, it follows from [5] that \( \hat{J}(\xi) \sim 1 - A\xi^2 + o(|\xi|^2) \) and \( \hat{J}(\xi) \sim -\xi \) as \( \xi \to 0 \), where \( A = -1/2J''(0) > 0 \). According
to [22, Lemmas 2.1 and 2.2] (see also [1]), the fundamental solution of the following Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t} = J * u - u, & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]

is the solution of it with initial value \( u_0 = \delta_0 \), and it can be decomposed as

\[ S(x, t) = e^{-t} \delta_0(x) + K(t), \]

where \( K(t) = \int_{\mathbb{R}} (e^{t(J(z) - 1)} - e^{-t}) e^{i\xi \cdot x} d\xi \) satisfies \( \|K(t)\|_{L^1(\mathbb{R})} \leq 2 \) for any \( t > 0 \). It is easy to get that \( \|S(x, t)\|_{L^1(\mathbb{R})} \leq 3 \) for any \( t > 0 \). Furthermore, by [5, Lemma 2.2 and Remark 2.1],

\[ \| K(t) \|_{L^\infty(\mathbb{R})} \leq c t e^{-\delta t} \| \hat{J} \|_{L^1(\mathbb{R})} + c(1 + t)^{-\frac{1}{2}} \]

for some positive constants \( c, \delta \) and any \( t > 0 \).

Now fix a couple \((x_0, t_0) \in \mathbb{R}^2\). For \(|t_0| < n\), we can compare \( \tilde{w} - w_n \) with a solution of the linear equation

\[ v_t = J * v - v + f'(0)v, \]

which has an initial condition at time \(-n\) as the right-hand side of the inequality of (37). Thus, we have

\[
|\tilde{w}(x_0, t_0) - w_n(x_0, t_0)| \\
\leq e^{f'(0)(t_0+n)} \int_{\mathbb{R}} S(x_0 - y, t_0 + n)v(y, -n)dy \\
= e^{f'(0)(t_0+n)} \int_{-\infty}^{y_n} S(x_0 - y, t_0 + n)\left[ k e^{-f'(0)n} + A_k e^{-\lambda(c)(-y+\epsilon\lambda+\lambda_1)} \right]dy \\
+ e^{f'(0)(t_0+n)} \int_{y_n}^{z_n} S(x_0 - y, t_0 + n)\left[ \lambda(-n) - k e^{-f'(0)n} \right] + A_k e^{-\lambda(c)(y+\epsilon\lambda+\lambda_1)}dy \\
+ e^{f'(0)(t_0+n)} \int_{z_n}^{+\infty} S(x_0 - y, t_0 + n)\left[ k e^{-f'(0)n} + A_k e^{-\lambda(c)(y+\epsilon\lambda+\lambda_1)} \right]dy.
\]

Call \( I, II, \) and \( III \) the three terms in the right-hand side of this last equality, respectively. Consider the first integral \( I \) and write it as \( I = I_1 + I_2 \). With the change of variable \( z = x_0 - y \), we have

\[
I_1 = e^{f'(0)(t_0+n)} \int_{-\infty}^{y_n} S(x_0 - y, t_0 + n)ke^{-f'(0)n}dy \\
= ke^{f'(0)t_0} \int_{-\infty}^{y_n} S(x_0 - y, t_0 + n)dy \\
= ke^{f'(0)t_0} \int_{x_0 - y_n}^{+\infty} S(z, t_0 + n)dz.
\]

Note that

\[ y_n = \frac{f'(0) - c\lambda(c)}{\lambda(c)} n + \frac{\ln \alpha_c - \ln k}{\lambda(c)} - h_1 + o(1) \]

and \( c\lambda(c) > f'(0) \). We have \( y_n \rightarrow -\infty \) as \( n \rightarrow +\infty \). Then

\[ I_1 = ke^{f'(0)t_0} \int_{x_0 - y_n}^{+\infty} S(z, t_0 + n)dz \rightarrow 0 \]
follows from \( \|S(x,t)\|_{L^1(\mathbb{R})} \leq 3 \) for any \( t > 0 \) and \( x_0 - y_n \to +\infty \) as \( n \to +\infty \). On the other hand,

\[
I_2 = e^{f'(0)(t_0 + n)} \int_{-\infty}^{y_n} S(x_0 - y, t_0 + n) A e^{-c(\hat{\lambda}(\hat{\xi}) (y + \hat{\xi}n + h_2))} dy
\]

\[
= A e^{f'(0)t_0} \int_{-\infty}^{y_n} S(x_0 - y, t_0 + n) e^{\hat{\lambda}(\hat{\xi}) (y + \hat{\xi}n + h_2) e^{(f'(0) - \hat{\xi}\hat{\lambda}(\hat{\xi}))}} dy
\]

\[
\to 0,
\]

since \( y_n \to -\infty \) and \( e^{(f'(0) - \hat{\xi}\hat{\lambda}(\hat{\xi}))} \to 0 \) as \( n \to +\infty \). Thus, \( I \to 0 \) as \( n \to +\infty \).

Similarly, we have \( III \to 0 \) as \( n \to +\infty \).

Lastly, the integral \( II \) can be divided into three terms \( II_1, II_2 \) and \( II_3 \) with obvious notation. First of all,

\[
II_1 = e^{f'(0)(t_0 + n)} \int_{y_n}^{z_n} S(x_0 - y, t_0 + n) \xi(-n) - ke^{-f'(0)t} dy
\]

\[
\leq 3e^{f'(0)(t_0 + n)} \xi(-n) - ke^{-f'(0)t} \to 0
\]

as \( n \to +\infty \), according to \( \|S(x,t)\|_{L^1(\mathbb{R})} \leq 3 \) for any \( t > 0 \) and \( \xi(t) \sim ke^{f'(0)t} \) as \( t \to -\infty \). Now, we deal with term \( II_2 \).

\[
II_2 = e^{f'(0)(t_0 + n)} \int_{y_n}^{z_n} S(x_0 - y, t_0 + n) A e^{-c\lambda(c)(y + cn + h_1)} dy
\]

\[
= A e^{f'(0)(t_0 + n)} \int_{y_n}^{z_n} e^{-(t_0 + n) \delta_0(x_0 - y) + K_{t_0 + n}(x_0 - y)} e^{-c\lambda(c)(y + cn + h_1)} dy
\]

\[
\leq A e^{f'(0)(t_0 + n)} \int_{y_n}^{z_n} e^{-(t_0 + n) \delta_0(x_0 - y)} e^{-c\lambda(c)(y + cn + h_1)} dy
\]

\[
+ A e^{f'(0)(t_0 + n)} \int_{y_n}^{z_n} K_{t_0 + n}(x_0 - y) e^{-c\lambda(c)(y + cn + h_1)} dy.
\]

We call \( II_{2,1} \) and \( II_{2,2} \) the two terms on the right-hand side of the last inequality. Thus,

\[
II_{2,1} = A e^{f'(0)(t_0 + n)} e^{-(t_0 + n) e^{-c\lambda(c)(x_0 + cn + h_1)}}
\]

\[
= A e^{f'(0)(t_0 + n)} e^{-c\lambda(c)(x_0 + h_1)}
\]

\[
= A e^{f'(0)(t_0 + n)} e^{-c\lambda(c)(x_0 + h_1)} e^{(\int_{\mathbb{R}} J(z)e^{c\lambda(z)dz})n}
\]

since \( c\lambda(c) = \int_{\mathbb{R}} J(z)e^{c\lambda(z)dz} - 1 + f'(0) \). Obviously,

\[
II_{2,1} \to 0 \quad \text{as} \quad n \to +\infty
\]
due to the fact $-\int_{\mathbb{R}} I(z) e^{\lambda(c)z} dz < 0$. In addition, we have
\[
|I_{2,2}| = A e^{f'(0)(t_0 + n)} \int_{y_n}^{z_n} K_{t_0 + n}(x_0 - y) e^{-\lambda(c)(y + c_n + h_1)} dy \\
\leq A e^{f'(0)(t_0 + n)} \left\| K_{t_0 + n} \right\|_{L^\infty(\mathbb{R})} \int_{y_n}^{z_n} e^{-\lambda(c)(y + c_n + h_1)} dy \\
= A e^{f'(0)(t_0 + n)} e^{-c\lambda(c)n - \lambda(c)h_1} \left\| K_{t_0 + n} \right\|_{L^\infty(\mathbb{R})} \int_{y_n}^{z_n} e^{-\lambda(c)y} dy \\
= \frac{A e}{\lambda(c)} e^{f'(0) - c\lambda(c)n} e^{f'(0)t_0 - \lambda(c)h_1} \left\| K_{t_0 + n} \right\|_{L^\infty(\mathbb{R})} \left[ e^{-\lambda(c)y} - e^{-\lambda(c)z_n} \right] \\
= \frac{A e}{\lambda(c)} e^{f'(0) - c\lambda(c)h_1} \left\| K_{t_0 + n} \right\|_{L^\infty(\mathbb{R})} e^{-\lambda(c)z_n}.
\]
Since
\[
\left\| K_{t_0 + n} \right\|_{L^\infty(\mathbb{R})} \leq c(t_0 + n)e^{-\delta(t_0 + n)} \left\| \hat{f} \right\|_{L^1(\mathbb{R})} + c(1 + t_0 + n)^{-\frac{1}{2}},
\]
we have
\[
\left\| K_{t_0 + n} \right\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad n \to +\infty.
\]
Note that
\[
e^{f'(0) - c\lambda(c)n} e^{-\lambda(c)y} = e^{f'(0) - c\lambda(c)n} e^{c\lambda(c) - f'(0)n + \ln k + \ln \zeta + \lambda(c)h_1 + o(1)} = e^{\ln k - \ln \zeta + \lambda(c)h_1 + o(1)}.
\]
And $f'(0) - c\lambda(c) < 0$ and $z_n \to +\infty$ as $n \to +\infty$. It follows that
\[
I_{2,2} \to 0 \quad \text{as} \quad n \to +\infty.
\]
Therefore,
\[
H_2 \to 0 \quad \text{as} \quad n \to +\infty.
\]
Similarly, $I_3 \to 0$ as $n \to +\infty$.

Eventually, $|\ddot{w}(x_0, t_0) - w_n(x_0, t_0)| \to 0$ as $n \to +\infty$. Note that $w_n(x_0, t_0) \to w(x_0, t_0)$ as $n \to +\infty$ and $(x_0, t_0) \in \mathbb{R}^2$ is arbitrary, we get that $\ddot{w} \equiv w$. Then $w_1 \to w$ as $t \to +\infty$ in the sense of $T$ due to the uniqueness of the limit.

By the same estimates as above, we can show that the entire solution of (1) is unique.

Finally, we end this paper by giving a meaningful remark to demonstrate the differences caused by the decay rates of the traveling wave solutions and the spatially independent solution when $J$ is symmetric and asymmetric.

**Remark 6.4.** When $J$ is symmetric, for any $c, \hat{c} \geq c^* = \hat{c}^*$, we have $c\lambda(c), \hat{c}\lambda(\hat{c}) > f'(0)$. Let $y(t) = \phi_c(x(t) - \hat{c}t + h_1) = \phi_{\hat{c}}(-x(t) - \hat{c}t + h_2)$. Then
\[
y(t) = o(e^{f'(0)t}) \quad \text{as} \quad t \to -\infty,
\]
which implies that $y(t)$ decays faster than $\zeta(t)$ at the points $x(t)$ as $t \to -\infty$.

However, when $J$ is asymmetric, (38) may not hold, which means the function $\zeta(t)$ may not play a part in the construction of entire solutions in Theorems 1.2, 1.4 and 1.6 even if $\chi_3 = 1$. 

\[\top\]
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