Phase-stepping algorithms for synchronous demodulation of nonlinear phase-shifted fringes

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Abstract: Standard phase-stepping algorithms (PSAs) estimate the measuring phase of linear-carrier temporal-fringes with respect to a linear-reference. Linear-carrier fringes are normally obtained using feedback, closed-loop, optical phase-shifting devices. On the other hand, open-loop, phase-shifting devices, usually give fringe patterns with nonlinear phase-shifts. The Fourier spectrum of linear-carrier fringes is composed by Dirac deltas only. In contrast, nonlinear phase-shifted fringes are wideband, spread-spectrum signals. It is well known that using linear-phase reference PSA to demodulate nonlinear phase-shifted fringes, one obtains an spurious-piston. The problem with this spurious-piston, is that it may wrongly be taken as a real optical thickness. Here we mathematically find the origin of this spurious-piston and design nonlinear phase-stepping PSAs to cope with open-loop, nonlinear phase-shifted interferometric fringes. We give a general theory to tailor nonlinear phase-stepping PSAs to demodulate nonlinear phase-shifted wideband fringes.

1. Introduction

Linear-reference phase-shifting algorithms (PSAs) have been used to demodulate linear-carrier temporal fringes since the pioneering work by Bruning et al. [1,2]. To generate linear phase-shifting fringes one normally uses well-calibrated, feedback closed-loop, optical phase-shifters [1,2]. In contrast if one uses open-loop, phase-shifters, one normally obtain wideband, nonlinear-carrier fringes [3-9]. In these cases the PSA must also be wideband to deal with highly nonlinear phase-shifted fringes [3-9]. Hibino et al. indicated that an artifact spurious piston appears in the estimated phase when using a linear-reference PSA to demodulate nonlinear phase-shifted fringes [4-9]. This spurious-piston is a numeric artifact of the linear-reference PSA, which may be wrongly interpreted as physical optical thickness [4-9]. Real optical thickness measuring, is fundamental when testing optical material slabs in semiconductor and display equipment [4-9]. Many systematic errors have been solved in linear-carrier, linear-reference PSAs, such as phase-shift miscalibration, fringe harmonics, experiment vibrations [2-9]. For precision thickness measurement, and nonlinear phase-shifted fringes, several linear-reference PSAs with no spurious-piston have been proposed [4-9]. Recently Kim et al. have pointed-out [9] that this numerical spurious-piston has received little attention because it does not give a waving profiling error (such as detuning or harmonics) when an optical surface is profiled. However, when the central interest is to measure absolute optical thickness of transparent slabs by wavelength-tuning (for example), this numerical-piston translates into errors in thickness [4-9]. This error is given by the product of the demodulated-phase and the synthetic wavelength, which is much greater than the wavelength used [9]. Linear-reference PSAs for demodulating wideband, nonlinear-carrier fringes, have been developed using the Taylor series expansion of the arc-tangent of the phase-error [4-9]. The linear-reference PSA’s coefficients are then calculated to set the first terms of this Taylor expansion to zero [4-9].

In this work we are proposing a different approach for phase demodulating temporal nonlinear-carrier fringes using wideband, synchronous, nonlinear-reference PSA. This is similar to the theory behind chirp-carrier radars [10,11]. In chirp-radars the wideband radio-
frequency (RF) pulse varies quadratically with time. When the RF chirp-pulse bounce back from the radar target, the incoming RF-signal is correlated with a synchronous, local chirp-waveform. In the case of wideband chirp-radar, one is interested in timing the amplitude of the correlation peak between the incoming RF chirp-signal and the chirped local-oscillator. Timing this correlation peak give us the round-trip target distance [10,11].

As Fig. 1 shows, here we are using the same concept of synchronously following the nonlinear phase-shifted carrier fringes using the same nonlinearity phase-shifting as reference.

2. Linear and nonlinear phase-shifted interferometric fringes

Let us first show the usual mathematical models for linear and nonlinear phase-shifted interferometric fringes. The model for linear-carrier interferometric fringes is,

$$ I_1(t) = a + b \cos(\varphi + \omega_b t); \quad \varphi \in [0, \pi]; \quad t \in [0, T]. \quad (1) $$

Where $\varphi \in [-\pi, \pi]$ is the measuring phase. On the other hand, nonlinear-carrier fringes are formalized by,

$$ I_2(t) = a + b \cos[\varphi + \omega_b t + \Delta(t)]; \quad \varphi \in (0, \pi]; \quad t \in [0, T]. \quad (2) $$

We are assuming that the nonlinearity $\Delta(t)$ is smooth, and can be determined experimentally [3-9]. Previous papers assume that $\Delta(t)$ can be approximated by few Taylor series terms [3,9]. Here we relax this condition, by requiring only that the derivative of $[\omega_b t + \Delta(t)]$ be bounded within the open interval $(0, \pi)$,

$$ \left[ \omega_b + \frac{d\Delta(t)}{dt} \right] \in (0, \pi); \quad t \in [0, T]. \quad (3) $$

In Fig. 2 we show linear (in blue) and nonlinear (in red) phase-shifted fringes.

- Panel (a) shows in blue, linear phase-shifting, and in red, nonlinear phase-shifting.
- Panel (b) shows linear-carrier fringes. Panel (c) shows nonlinear-carrier fringes.
As we prove in the next sections, the nonlinear phase-shifting $\Delta(i)$ generate a spurious-numerical piston when a linear-reference PSA is used as phase-demodulator [4-9]. Fig.2 shows an example of linear and nonlinear carrier fringes,

### 3. Fourier spectrum for linear and nonlinear phase-shifted fringes

From Eq. (1), linear fringes are single-frequency at $\omega_b$, having a spectrum given by [2],

$$ F\left\{ a + \frac{b}{2} e^{i(\omega + \omega_b)} + \frac{b}{2} e^{-i(\omega + \omega_b)} \right\} = a \delta(\omega) + \frac{b}{2} e^{i\omega} \delta(\omega - \omega_b) + \frac{b}{2} e^{-i\omega} \delta(\omega + \omega_b). \quad (4) $$

Where $F[\cdot]$ is the Fourier transform operator (see Fig. 3(a)). In contrast, highly nonlinear phase-shifted fringes (Eq. (2)) are wideband, and its spectrum may be modeled as,

$$ F\left\{ a + \frac{b}{2} e^{i(\omega + \omega_b)} + \frac{b}{2} e^{-i(\omega - \omega_b)} \right\} = a e^{i\omega} C(\omega) + b e^{i\omega} \delta(-\omega). \quad (5) $$

Where,

$$ C(-\omega) = F\left\{ e^{-i(\omega + \omega_b)} \right\}; \quad C(\omega) = F\left\{ e^{i(\omega + \omega_b)} \right\}. \quad (6) $$

Figure 3 shows schematically the spectrum of linear and wideband nonlinear-carrier fringes. In Eq. (5) the terms $C(-\omega)$ and $C(\omega)$ are wideband spectra.

![Fig. 3. Panel (a) shows the three delta-spectrum of linear-carrier fringes, and in panel (b) the wideband spectrum of nonlinear phase-shifted fringes.](image)

Summarizing, linear-phase carrier fringes have a three delta spectrum (Fig 3(a)); while nonlinear-phase carrier fringes have two spread-spectrum components (Fig. 3(b)).

### 4. Linear and nonlinear reference PSAs

Let us now show the mathematical form of phase-shifting algorithms (PSAs) using linear and nonlinear-reference for demodulating nonlinear-carrier fringes

#### 3.1 Standard linear-reference PSA for demodulating linear-carrier fringes

The general form for standard linear-carrier, linear-reference PSA is [2],

$$ A e^{i\varphi} = \sum_{n=0}^{N-1} \left[ c_n e^{i\omega n} \right] \left[ a + b \cos(\varphi + \omega n) \right]. \quad (7) $$

These are the standard linear-carrier, linear-reference PSAs in use since 1974 [1,2].

#### 3.2 Linear-reference PSA for demodulating nonlinear-carrier fringes

People has proposed linear-reference PSAs to demodulate nonlinear-carrier fringes as [3-9],
Performing the multiplications one obtains,

\[ A_e^{\ell (\varphi + \text{Piston})} = \sum_{n=0}^{N-1} \left[ d_s e^{-i \omega_p n} \right] \left[ a + b \cos \left( \varphi + \omega_p n + \Delta(n) \right) \right]. \] (8)

As we show next, using a linear-reference PSA, we generally obtain a spurious-piston, \( \text{Piston} \neq 0 \) [4-9]. Hibino et al. have proposed linear-reference PSAs to eliminate this spurious piston [4-9]. Here we are proposing an alternative solution, a more natural way (we believe), for disappearing this non-desired, spurious-piston.

3.3 Nonlinear-reference PSA for demodulating nonlinear-carrier fringes

We specifically propose the use of a nonlinear-reference PSA which has the following form,

\[ A_e^{\ell} = \sum_{n=0}^{N-1} \left[ w_n e^{i (\alpha_q n + \Delta(n))} \right] \left[ a + b \cos \left( \varphi + \alpha_q n + \Delta(n) \right) \right]; \quad (w_n \in \mathbb{R}). \] (9)

Note that the nonlinear-reference \( \exp[i(\alpha_q n + \Delta(n))] \) is synchronous with the nonlinear-carrier \( \cos[\varphi + \alpha_q n + \Delta(n)] \); this fact makes the spurious-piston disappear \( \text{Piston}=0 \). The weighting coefficients \( (w_n) \) are chosen to approximate a Hilbert quadrature filter.

5. Spurious-piston using linear phase-shifted reference PSAs

Using a linear-reference PSAs to demodulate nonlinear-carrier fringes (Eq. (8)) one obtains,

\[ A_e^{\ell (\varphi + \text{Piston})} = \sum_{n=0}^{N-1} d_s e^{-i \omega_p n} I_2(n) = \sum_{n=0}^{N-1} \left[ a + b e^{i \varphi} e^{i [\alpha_q n + \Delta(n)]} + b e^{-i \varphi} e^{-i [\alpha_q n + \Delta(n) + \pi]} \right] d_s e^{-i \omega_p n} \] (10)

Performing the indicated multiplications one obtains,

\[ A_e^{\ell (\varphi + \text{Piston})} = a \left[ \sum_{n=0}^{N-1} d_s e^{-i \omega_p n} \right] + b e^{i \varphi} \left[ \sum_{n=0}^{N-1} d_s e^{i \Delta(n)} \right] + b e^{-i \varphi} \left[ \sum_{n=0}^{N-1} d_s e^{-i [2 \alpha_q n + \Delta(n)]} \right] \] (11)

The coefficients \( d_s \) are chosen such that the first and third square-brackets are set to zero as,

\[ \left[ \sum_{n=0}^{N-1} d_s e^{-i \omega_p n} \right] = 0, \quad \text{and} \quad \left[ \sum_{n=0}^{N-1} d_s e^{-i [2 \alpha_q n + \Delta(n)]} \right] = 0. \] (12)

Obtaining the desired analytic signal as,

\[ A_e^{\ell (\varphi + \text{Piston})} = \frac{b}{2} e^{i \varphi} \left[ \sum_{n=0}^{N-1} d_s e^{i \Delta(n)} \right]; \quad \text{Piston} = \arg \left[ \sum_{n=0}^{N-1} d_s e^{i \Delta(n)} \right]. \] (13)

As we see, in general, the spurious-piston is non-zero (\( \text{Piston} \neq 0 \)), and it may give erroneous absolute optical thickness measurements [4-9].

6. No spurious-piston using nonlinear phase-shifted reference PSA

Now using a synchronous (matched-phase) nonlinear reference PSA (Eq. (9)) one gets,

\[ A_e^{\ell} = \sum_{n=0}^{N-1} w_n e^{-i [\alpha_q n + \Delta(n)]} I_2(n) = \sum_{n=0}^{N-1} \left[ a + b e^{i \varphi} e^{i [\alpha_q n + \Delta(n)]} + b e^{-i \varphi} e^{-i [\alpha_q n + \Delta(n) + \pi]} \right] w_n e^{-i [\alpha_q n + \Delta(n)]} \] (14)

Performing the multiplications one obtains,

\[ A_e^{\ell} = a \left[ \sum_{n=0}^{N-1} w_n e^{-i [\alpha_q n + \Delta(n)]} \right] + b e^{i \varphi} \left[ \sum_{n=0}^{N-1} w_n \right] + b e^{-i \varphi} \left[ \sum_{n=0}^{N-1} w_n e^{-i [2 \alpha_q n + \Delta(n)]} \right] \] (15)
As we did before, to obtain just the desired analytic-signal \( \exp[i\varphi] \) one needs,
\[
\left[ \sum_{n=0}^{N-1} w_n e^{-i[n\omega_0 + \Delta(n)]} \right] = 0; \quad \text{and} \quad \left[ \sum_{n=0}^{N-1} w_n e^{-i[2n\omega_0 + \Delta(n)]} \right] = 0. \quad (16)
\]

Obtaining the phase-demodulated signal \( Ae^{i\varphi} \) as,
\[
A_n e^{i\varphi} = \frac{b}{2} e^{i\omega} \sum_{n=0}^{N-1} w_n; \quad (w_n \in \mathbb{R}). \quad (17)
\]
The spurious piston has naturally disappeared (Piston=0) thanks to the use of a synchronous nonlinear-reference \( \exp\{-i[\omega_0 n + \Delta(n)]\} \) in the PSA.

7. Spectral design for nonlinearly phase-shifted reference PSAs

In previous section we gave an algebraic approach for calculating the coefficients \( w_n \in \mathbb{R} \) for nonlinear phase-shifted reference PSAs. Here we develop a more intuitive spectral design. The impulse response of the nonlinear reference PSA (Eq. (9)) is,
\[
h_2(t) = \sum_{n=0}^{N-1} w_n e^{i[n\omega_0 + \Delta(n)]} \delta(t - n); \quad (w_n \in \mathbb{R}). \quad (18)
\]
Then its FTF is \( H_2(\omega) = F[h_2(t)] \),
\[
H_2(\omega) = \sum_{n=0}^{N-1} w_n e^{i[n\omega_0 + \Delta(n)]} \delta(t - n) = \sum_{n=0}^{N-1} w_n e^{i[n\omega_0 + \Delta(n)]} e^{-i\omega_0}. \quad (19)
\]
The coefficients \( w_n \) of \( H_2(\omega) \) are chosen to fulfill with the wideband conditions,
\[
H_2(\omega) = 0 \quad \text{for} \quad \omega \in [-\pi, 0]
\]
\[
H_2(\omega) \neq 0 \quad \text{for} \quad \omega \in (0, \pi). \quad (20)
\]

As Fig. 4(a) shows \( H_2(\omega) \) for a square-window \( (w_n = 1.0) \). We can see that \( H_2(\omega) \) is not zero for \( \omega \in [-\pi, 0] \), obtaining an erroneous phase. One solution to this is to use an apodizing window [12,13]. We used a Gaussian window \( w_n = e^{-[n-0.5(N-1)]^2}; (g < 1.0) \), and its FTF \( H_2(\omega) = F[h_2(t)] \) is shown in Fig. 4(b). Of course other weightings windows \( (w_n \neq 1) \) may be used [12,13].
In Fig. 5 we show the (normalized frequency) harmonic response of the apodized, nonlinear reference PSA.

### 8. Signal-to-noise ratio (SNR) for linear and nonlinear reference PSA

Here we find the SNR [2] of the phase-demodulated nonlinear-carrier fringes corrupted by additive white Gaussian noise (AWGN). The noisy fringes are,

$$I_n(t) = \sum_{n=0}^{N-1} \{ a + b \cos[\phi + \omega_0 t + \Delta(t)] + N(t) \} \delta(t-n).$$  \hspace{1cm} (21)

Where the noise spectral density $S(\omega)$ is flat, and it is given by,

$$S(\omega) = F[R_{xy}(\tau)] = F[E\{N(t)N(t+\tau)\}] = \frac{N_0}{2}; \quad \omega \in [-\pi,\pi].$$  \hspace{1cm} (22)

Being $R_{xy}(\tau) = E\{N(t)N(t+\tau)\}$ the ensemble autocorrelation function of $N(t)$ [2]. The flat noise power-spectrum of $N(t)$ is modified to $(N_0/2) |H_2(\omega)|^2$ [2].

For nonlinear-carrier fringes, and linear-reference PSA, the SNR is given by,

$$\text{SNR}_1 = \frac{\text{Signal Energy}}{\text{Noise Energy}} = \frac{\left( \frac{b}{2} \right)^2 \int_{-\omega_0}^{\omega_0} d\omega \int_{-\Delta}^{\Delta} |H_1(\omega)|^2 d\omega}{\left( \frac{N_0}{2} \right) \int_{-\omega_0}^{\omega_0} |H_2(\omega)|^2 d\omega}.$$  \hspace{1cm} (23)

On the other hand for nonlinear-carrier and reference, the SNR is given by,

$$\text{SNR}_2 = \frac{\text{Signal Energy}}{\text{Noise Energy}} = \frac{\left( \frac{b}{2} \right)^2 \sum_{n=0}^{N-1} \int_{-\omega_0}^{\omega_0} d\omega \int_{-\Delta}^{\Delta} |H_1(\omega)|^2 d\omega}{\left( \frac{N_0}{2} \right) \int_{-\omega_0}^{\omega_0} |H_2(\omega)|^2 d\omega}.$$  \hspace{1cm} (24)

Where $(N_0/2) \int |H_1(\omega)|^2 d\omega$, and $(N_0/2) \int |H_2(\omega)|^2 d\omega$ are the total filtered-noise energy. The energy of $A_2 e^{i\phi}$ using the nonlinear-reference PSA is generally higher than the energy of $A e^{i(\phi + \Delta)}$ using a linear-reference PSA, this is because,

$$\left( \frac{b}{2} \right)^2 \sum_{n=0}^{N-1} w_n \int_{-\Delta}^{\Delta} |H_1(\omega)|^2 d\omega \geq \left( \frac{b}{2} \right)^2 \left| \sum_{n=0}^{N-1} w_n e^{-i\Delta(n)} \right|^2.$$  \hspace{1cm} (25)

Assuming that both have about the same bandwidth $|H_1(\omega)| = |H_2(\omega)|$, one obtains,
As conclusion, the SNR is generally higher for a PSA with nonlinear-reference.

9. Example of a 13-steps Gaussian-window nonlinear-reference PSA

Here we are given a computer simulation example of a 13-step nonlinear-reference PSA applied to nonlinear-carrier fringes. The most usual phase-shifted nonlinearity is quadratic, \( \Delta(n) = \varepsilon n^2 \) [3-9]. We start by considering nonlinear-carrier fringes as,

\[
I(t; \varphi) = \sum_{n=0}^{12} \left[ 1 + \cos(\varphi + \omega_n n + \varepsilon n^2) \right] \delta(t - n); \quad (\omega_0 = 0.35\pi, \varepsilon_2 = 0.05\omega_0).
\]

The non-linear phase and the interferometric chirp-waveform is shown in Fig. 6.

The specific form of the 13-steps nonlinear, chirp-reference PSA is given by,

\[
A_2 e^{i\varphi} = \sum_{n=0}^{12} w_n e^{i[\omega_n n + \varepsilon n^2]} I(n; \varphi); \quad w_n = e^{-0.1[\varepsilon - 0.5(n-1)^2]}.
\]

We have assumed no linear detuning. The PSA reference \( \exp\{-i(\omega_0 n + \varepsilon n^2)\} \) is synchronous with the chirp-fringes. The FTF \( (H_2(\omega)) \) of the nonlinear-reference PSA is,

\[
H_2(\omega) = F[h_2(t)] = F\left[ \sum_{n=0}^{12} w_n e^{i[\omega_n n + \varepsilon n^2]} \delta(t - n) \right] = \sum_{n=0}^{12} w_n e^{i[\omega_n n + \varepsilon n^2]} e^{-i\omega_0}.
\]

And its temporal and spectral graphs are shown in Fig. 7.

Next Fig. 8 shows, superimposed, the fringe-data and the chirp-reference PSA spectra.
We evaluate our nonlinear-reference PSA demodulation error by the following formula,

$$\varphi_{\text{Error}} = \varphi - \arg \left[ \sum_{n=0}^{12} w_n e^{i(\omega_0 n + \omega_0 n^2)} I(n; \varphi) \right]; \quad \varphi \in [0, 2\pi].$$

Finally Fig. 9 shows the phase estimation error $\varphi_{\text{Error}}$, for $\varphi \in [0, 2\pi]$.

**Figure 9**

Phase error given by Eq. (32). Note the vertical scale is within [-0.03, 0.03] radians.

Figure 9 shows that the peak phase demodulation error $\varphi_{\text{Error}}$ is about 0.04 radians.

**10. Example of a 13-steps square-window nonlinear-reference PSA**

Here we analyze a square-window nonlinear-reference PSA for the same fringes used in the previous section $\Delta(n) = \varepsilon_n n^2$. Then our square-window PSA is,

$$A_2 x e^{i\varphi_{\text{square}}} = \sum_{n=0}^{12} w_n e^{i(\omega_0 n + \omega_0 n^2)} \left\{ 1 + \cos \left[ \varphi + \omega_0 n + \omega_0 n^2 \right] \right\}; \quad w_n = 1.0.$$  

The spectral graph of the FTF associated to this PSA is shown in Fig. 10.

**Figure 10**

Spectral response (FTF) for the square-window, nonlinear-reference PSA. This square window cannot be used because it has large response in the origin and the left side of the fringe spectrum. This FTF is a bad approximation of a one-sided Hilbert quadrature filter.
As Fig. 11 shows, the DC background of the fringes is not fully filtered-out, and also large energy from the unwanted conjugate-signal leaks into the desired analytic signal.

![Square Window Phase Error](image)

Fig. 11. The blue trace shows the phase-error for the 13-step, square-window, nonlinear-reference PSA. For comparison, the red trace is the phase error corresponding to the Gaussian window seen in previous section. Note that the vertical scale is now [-0.1,0.1] radians.

Figure 11 shows in the blue trace the phase-error of the square-window nonlinear-reference PSA. We summarize this section by remarking the fact that synchronously following the nonlinear-carrier variations of the fringes is not enough. One must also apply an apodizing weighting window to the nonlinear-reference PSA [12,13].

11. Discussion of the proposed nonlinear-reference PSA theory

Before our general summary, we want to make a clear wrap-up of our contribution. The basic theory presented herein is completely general. The only restriction is that the nonlinear-carrier fringe spectrum be bandlimited. For the readers’ convenience we rewrite the few equations required. The nonlinear-carrier fringes are modeled as,

\[ I_2(t) = a + b \cos(\varphi + \omega_0 t + \Delta(t)) \quad t \in [0,T]; \quad \omega_0 \in (0,\pi) . \]  

(32)

Where the only restriction about the phase-shifted variation \([\omega_0 t + \Delta(t)]\) is,

\[ \left[ \omega_0 + \frac{d \Delta(t)}{dt} \right] \in (0,\pi) ; \quad t \in [0,T] . \]  

(33)

This restriction on \(\Delta(t)\) is more general to previous efforts which assumed that \(\Delta(t)\) should be expansible as a Taylor series [3-9]. We then proposed our nonlinear-reference PSA as,

\[ A_2 e^{i\varphi} = \sum_{n=0}^{N-1} w_n e^{i[\omega_n + \Delta(n)]} I_2(n); \quad (w_n \in \mathbb{R}). \]  

(34)

The window \((w_n)\) shape the FTF of this PSA, and it may be taken from a wide set of functions [12,13]. This analytic signal has no spurious piston. The FTF of this PSA is,

\[ H_2(\omega) = F \left[ h_2(t) \right] = F \left\{ \sum_{n=0}^{N-1} w_n e^{i[\omega_n + \Delta(n)]} \delta(t-n) \right\} = \sum_{n=0}^{N-1} w_n e^{i[\omega_n + \Delta(n)]} e^{-in\omega} . \]  

(35)

The coefficients \((w_n)\) shape this FTF to cover the right hand-side spectrum of the fringes as,

\[ H_2(\omega) = 0 \quad \text{for} \quad \omega \in [-\pi,0] \]

\[ H_2(\omega) \neq 0 \quad \text{for} \quad \omega \in (0,\pi) . \]  

(36)

These five equations succinctly show our theory for designing nonlinear-reference PSAs for phase-demodulating nonlinear-carrier fringes without spurious-piston.
12. Summary

Here we have given a frequency transfer function (FTF) approach for designing nonlinear phase-shifting algorithms (PSAs) applied to demodulate nonlinear phase-shifted fringes. The estimated nonlinear phase-step variations of the fringes [4-9], constitute also our nonlinear phase-step reference for the PSA. We then find the real-valued PSA coefficients ($w_n$) that shapes the FTF spectrum of the PSA. The spectral shape of the nonlinear reference PSA smoothly approximate a Hilbert quadrature filter. As such, the spectral FTF shaping must render almost zero the left side (including zero) of the fringes spectrum.

Phase-demodulation of nonlinear phase-shifted fringes using linear-reference PSAs has been studied before [3-9]. As mentioned, in general, linear-reference PSAs obtain an artifact, numeric piston when demodulating nonlinear-carrier fringes [4-9]. This spurious-piston may render the phase measurement of absolute optical thickness erroneous [4-9]. Hibino et al have eliminated this spurious-piston by imposing some conditions on the coefficients of linear-reference PSAs [4-9]. Here we have seen that using a nonlinear-reference PSA synchronous with the nonlinear-carrier fringes, the spurious piston disappears in a more natural way (we believe). We think that our nonlinear-phase reference PSA approach shed new light for understanding temporal phase-demodulation of broadband interferometric fringes.

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