We show that the nonlinear polarization dynamics of a vertical-cavity surface-emitting laser placed in an external cavity lead to the emission of temporal dissipative solitons. These are vectorial solitons because they appear as localized pulses in the polarized output, but leave the total intensity constant. When the cavity roundtrip time is much longer than the soliton duration, several independent solitons as well as bound states (molecules) may be hosted in the cavity. All these solitons coexist together and with the background solution. The experimental results are well described by a theoretical model that can be reduced to a single delayed equation for the polarization orientation, which allows the vectorial solitons to be interpreted as polarization kinks. A Floquet analysis is used to confirm the mutual independence of the observed solitons.

The field of dissipative solitons (DS) in optical systems has been the subject of intensive research during the past 20 years (see, for example, Akhmediev and Ankiewicz2–4, Ackemann et al.5,6, Descalzi et al.7 and Grelu and Akhmediev8, and references therein). Optical DS are localized light pulses in time or localized beams in space that appear in nonlinear systems kept out of equilibrium by a continuous flow of energy that counteracts the losses5. As a consequence, DS differ substantially from conservative solitons, which originate purely by a compensation between a spreading effect and the nonlinearity.

A first important difference concerns the role of the nonlinearity. Although in the beginning DS were envisioned as weakly modified conservative solitons6, it was later shown that, for strong dissipation, the scenario that leads to DS formation may be more complex. For instance, bright cavity solitons, which appear in the transverse plane of driven resonators7–10, exist even in the presence of defocusing nonlinearities, which favour the spreading effect11. Cavity solitons are cellular patterns generated by fronts that connect different coexisting spatial solutions12 and their existence cannot be reduced to a simple compensation mechanism between nonlinearity and diffraction.

Another fundamental difference is that DS are attractors, that is stable solutions towards which the system evolves spontaneously from a wide set of initial conditions13. This entails that, at variance with their conservative analogues, DS do not rely on a proper seeding of the initial conditions. This renders them extremely interesting for applications and, in particular, for information processing in which DS are used as bit units. For example, recently all-optical buffers based on cavity solitons have been demonstrated both in spatial and in temporal domains3,10,14–16.

Temporal DS have been studied largely in mode-locked lasers5 and, more recently, in Kerr fibre cavities driven by an injected field11; several interesting behaviours have been experimentally demonstrated, as, for example, soliton bound states17, molecules18 repulsive/attacting forces on an extremely long scale19, solitary rain20 and soliton explosion21. When the vectorial degree of freedom of the light is taken into account, as in the case of the Manakov solitons22, more complex behaviours are observed. Antiphase switching between orthogonally linearly polarized states has been observed recently23,24 and interpreted in terms of soliton domain walls, that is localized states that separate domains of emission in orthogonal polarizations24,25. In this paper we demonstrate the existence of a new kind of vectorial soliton. It consists of time-localized rotations of the polarization orientation of the light emitted by a single mode vertical-cavity surface-emitting laser (VCSEL) enclosed in a polarization-sensitive double external cavity. By exploiting the two different times of flight as well as the polarization selectivity of the output, we show that these vectorial solitons can be used to perform different tasks, as highlighted in the following.

In the PSF branch, a polarizing cube (PBS) transmits only the Y component from the cavity for detection, whereas the detection of the X component is obtained using the Faraday rotator (FR) with an exit polarizer (P) transmitting only the X component out of the cavity for detection. The PBS sends the X component into the VCSEL. The PBS sends the X component out of the cavity for detection, whereas the detection of the Y component is obtained using the beam transmitted by BS2. In the XPR branch, the beam is sent to a π/4 Faraday rotator (FR) with an exit polarizer (P) transmitting only the Y component that will be rotated into X. Then, the mirror (M) reflects the light back towards the VCSEL. As a result of the double passage into the FR, Y is finally reinjected into the VCSEL with a polarization axis parallel to X. Neutral density filters (not shown) in both branches allow varying feedback levels.

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cavity, we are able to control the polarization state of the light and generate vectorial DS. When the cavity roundtrip time is much longer than the duration of the vectorial DS, several independent DS and/or bound states (molecules) may be hosted in the cavity where they coexist with the background solution (zero soliton emission). We show that molecules and independent DS can be discriminated experimentally by analysing their noise-induced motion.

The vectorial nature of the reported DS is intimately related to the nonlinear polarization response of the VCSEL. As a result of their axial symmetry, these devices lack anisotropies strong enough to pin the polarization orientation26–28, and competition between orthogonal linearly polarized states is likely to occur29. Polarization dynamics do not involve a strong exchange of energy between the light and the active medium. As a consequence, the characteristic timescales of vectorial solitons will not be limited by the gain recovery time, which is promising in view of high-speed photonic applications.

From the theoretical point of view, it is important noting that the nonlinearity in our system is concentrated on a single point (the VCSEL) rather than being distributed along the propagation direction, as in fibre resonators. For this reason, we model our DS by delay differential equations (DDEs) instead of partial differential equations (PDEs) as, for instance, the prototypical cubic–quintic Ginzburg–Landau equation of DS. In fact, DDEs possess the same complexity as PDEs because they both correspond to dynamical systems of infinite dimensionality. Moreover, conceptual links between PDEs and DDEs do exist. It was revealed that a delayed system close to an Andronov–Hopf bifurcation can be described via a diffusive Ginzburg–Landau equation30. More recently, a mapping between a spatially extended laser cavity and an ensemble of coupled delay algebraic equations has been developed31,32. In this paper, we show how temporal localization in DDE systems can be assessed by analysing Floquet exponents. Finally, we reduce the full model to a single delayed equation for the polarization orientation that allows us to interpret the vectorial solitons as polarization kinks and antikinks.

**Results**

The experimental set-up is schematically shown in Fig. 1 and more details are given in the Supplementary Methods. A single-transverse mode VCSEL is coupled to a double external cavity that selects one of the linearly polarized states of the VCSEL (Y, say) and feeds it back twice into the VCSEL: once into the same polarization and once into the orthogonal one. The first arm provides polarization selective feedback (PSF) with a time delay $\tau_1$, whereas the second arm re-injects Y into the orthogonal polarization orientation with time delay $\tau_2$. Such crossed-polarization reinjection (XPR) induces cross-gain modulation of the two linear polarization components28, and thus enhances their competition and leads to polarization dynamics33,34. This double cavity was shown to promote passive mode-locking without a saturable absorber35, although here it is operated in a different regime.

For properly chosen parameters (see the Supplementary Methods), the polarization-resolved outputs of the VCSEL exhibit a train of pulses separated by a time $\tau_1$, as shown in Fig. 2a. In the X-polarization component, the pulses are upwards over a low-intensity background, whereas they are downwards from a high-intensity level in Y. The pulse duration is 50 ps (full-width at half-maximum (FWHM)), fully resolved by the bandwidth of our set-up (33 GHz). These polarization pulses appear anticorrelated and the corresponding total intensity time trace is almost constant, which thus reveals the vectorial character of this dynamics. We interpret these polarization pulses as vectorial DS which travel back and forth in the external cavity and are regenerated at each roundtrip when interact with the VCSEL.

DSs are required to maintain their shape throughout the number of roundtrips covered, although inevitable noise sources (namely, spontaneous emission in the laser, detector shot noise, and mechanical and thermal stability of the experimental set-up) can blur this ideal picture, as shown in Leo et al.33. We assessed the self-similarity of the pulses by performing a statistical analysis of a 100 μs long time series spanning over $18 \times 10^3$ pulses. The distribution of the
maxima of the pulses exhibits a Gaussian shape with a standard deviation of 10% of the average peak intensity. By using the peak of each pulse as a time reference, we superposed all the waveforms obtained with the full bandwidth of the set-up in Fig. 2b. This reveals that, regardless of the peak intensity fluctuations, the shape of the pulse remains robust and stable, and thus supports our interpretation of the pulses in terms of vectorial DS.

The dynamics within an external cavity can be usefully described in terms of space–time diagrams in which the time trace is folded over itself at intervals \( \tau_r \), so that the roundtrip number becomes the pseudo-time-discrete variable and the pseudo-space variable correspond to the timing of the vectorial DS modulo \( \tau_r \) (Giacomelli and Politi and Arecchi et al.). This representation pictures the soliton position \( \Delta_n \) within the external cavity as a function of the roundtrip number. When applied to the X polarization trace of Fig. 2a, this reveals that \( \Delta_n \) fluctuates noticeably over a typical timescale of the order of 100 cycles, which suggests that the noise present in the system acts on the soliton as a Langevin force over a free particle, leading to a Brownian-like motion of the soliton within the external cavity as a function of the roundtrips covered. Indeed, the analysis of the time series for \( \Delta_n \) indicates that this variable is described by a first-order autoregressive model, \( \Delta_n = \Delta_{n-1} + R_n \) with \( R_n \), a random term that is distributed as a Gaussian of zero mean and standard deviation \( \sigma_R = 11 \) ps, as shown in Fig. 2d. Moreover, the autocorrelation of \( R_n \) (Fig. 2e) falls off in a few roundtrips, which indicates that the sequence of \( \Delta_n \) corresponds to the regular sampling of a one-dimensional stochastic process \( d_t \Delta_n = \xi(t) \) where \( \xi(t) \) is a Gaussian, slightly coloured noise. The analogy between unidimensional Brownian motion of a free particle and the drift of the soliton position can be traced back to the temporal translational invariance of our autonomous dynamical system.

If \( \tau_r \) is large enough compared to the size of the single soliton, several DS can be hosted within the external cavity and their interaction may be studied. Then, the noise-induced motion of the DS becomes a tool that enables us to discriminate between independent DS and bound states. In Fig. 3a–c, we show the space–time diagrams for different ensembles of coexisting DS. These different realizations have been obtained for the same parameter values; the system may evolve from one situation towards another in response to mechanical perturbations or to parameter changes, in the latter case displaying a high degree of hysteresis. In general, once one of these situations is obtained it persists for a time that corresponds to the environmental stability of the laboratory (a few minutes). Figure 3a shows two DS whose noise-induced trajectories are uncorrelated, and thus evidences their independence, and Fig. 3b depicts a molecule of three DS (3-DS molecule); although the evolution of the ensemble is stochastic, the separation between the DS remains constant at 480 ps, which corresponds to \( \Delta t = \tau_1 - \tau_2 \) (Supplementary Section A and Supplementary Fig. 3a,b). We depict in Fig. 3c the binding and unbinding between two molecules, each one formed by two DS. At roundtrip \( 1 \times 10^4 \), the two molecules approach and form a bound state with four peaks, only visible if one considers the presence of periodic boundary conditions (see Giacomelli and Politi and Arecchi et al. for more details), that afterwards unbind and eventually bind again. Figure 3a–c shows a small sample of a large variety of coexisting situations each one different in terms of DS number (zero included) and arrangement. For \( \tau_r = 2.9 \) ns, molecules composed of a larger number of DS can be observed (Supplementary Section A and Supplementary Fig. 2a), the largest one being an 8-DS molecule that fills the entire roundtrip and forms a soliton ‘crystal’. Increasing the size of the external cavity allows the placement of a higher number of independent DS and larger molecules (Supplementary Section A and Supplementary Fig. 2b,c). In Fig. 3d we show how different vectorial DS may nucleate spontaneously from the continuous wave (CW) solution, that is, from a situation with zero solitons. Although in this case nucleation occurs in an uncontrolled manner, Fig. 3d suggests that these DS might be addressed individually and used as bits for information processing, as in Leo et al.

Theoretical analysis and discussion
To explain the experimental results, we use the spin-flip model, suitably modified to incorporate the effects of gain saturation, PSF and XPR. The choice of parameter values was guided by the experimental situation (see the Supplementary Methods). Figure 4a
as well as pulse widths of the order of the pulse at a distance data can be appreciated in Fig. 2c as a dark shadow that follows Fig. 4b). The presence of this inverted kink in the experimental et al Javaloyes fringence, the PSF and the XPR rates as well as the two delays (see whose orientation is neither of them), the emission consists of a quasilinearly polarized mode coexists with a CW solution, and between pulses (or in the absence | VOL 9 | JULY 2015 | www.nature.com/naturephotonics
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DOI: 10.1038/NPHOTON.2015.92 shows a dynamical state for the single soliton case in good agreement with the experiment: the intensity of each linearly polarized component displays localized antiphase pulsations separated by $t_f$ and followed by a small inverted kink after a time $\Delta \tau$ (see Fig. 4b). The presence of this inverted kink in the experimental data can be appreciated in Fig. 2c as a dark shadow that follows the pulse at a distance $\Delta \tau$. We predict an almost 100% antiphase as well as pulse widths of the order of $\sim 35$ ps, in excellent qualitative agreement with the experimental results. Importantly, such a regime coexists with a CW solution, and between pulses (or in the absence of them), the emission consists of a quasilinearly polarized mode whose orientation is neither $X$ nor $Y$. The polarization orientation is governed by a complex interplay between the dichroism, the birefringence, the PSF and the XPR rates as well as the two delays (see Javaloyes et al. for details). Typically, the suppression ratio between the $Y$ and $X$ CW intensities can be varied between 20 and 5.

In addition to the characteristics of a single DS, we reproduce the coexistence of multiple independent DS (see Fig. 4c). Additional DS can be written at arbitrary positions without perturbing the already present localized structures. This is obtained by inducing slips in the feedback phase (see the Supplementary Methods). The numerical addressing of individual DS discloses their application potential for information processing and the polarization dynamics timescale allows addressing rates of the order of several tens of gigabytes per second to be achieved, as shown in Supplementary Section E.

Soliton molecules can be reproduced as well, as shown in Fig. 4d. We attribute the existence of a specific binding distance to the small kink generated in the wake of the vectorial soliton by the replica impinging the VCSEL a second time after a time interval $\Delta \tau$, in agreement with the experimental results shown in Fig. 2c. We noticed the existence of several exponentially decreasing replicas at times $2\Delta \tau$, $3\Delta \tau$ and so on, that generate a weak binding force as well as additional equilibrium distances for the molecules (see Supplementary Section E).

The vectorial DS can be considered as periodic solutions of a high-dimensional dynamical system (see Supplementary Section B), which allows their linear stability to be studied using Floquet theory. As a proof of their mutual independence, we evaluated the Floquet multipliers for both independent DS and bound states. The most significant results from our analysis are depicted in Fig. 5. In the case of three independent DS, three quasidegenerate multipliers exist in the vicinity of $\mu = 1$. We analysed the eigenvectors associated with these neutral modes and found that they correspond to relative translations of each soliton, and thus define temporal localization for delayed systems. Interestingly, the analysis of the bound states yielded a single multiplier $\mu = 1$, which confirms that the ensemble moves as a single entity. Finally, the presence of many weakly damped oscillatory modes (Fig. 5) explains the variations of height and the sensitivity of the system to noise.

A description based on the Stokes parameters for polarized light allows us to interpret the antiphase dynamics as rotations along the equator of the Poincaré sphere, and thus unveil the vectorial character of the DS. We represent in Fig. 6a the temporal trace that corresponds to a single structure. It is found that the orbit proceeds essentially along the equator of the sphere ($|S_E| \lesssim 0.1$). The system starts from the stable quasilinearly polarized state represented by a green circle and performs an almost complete rotation to reach the blue circle. The pulse corresponds to this first rotation. As these two polarizations are degenerate in a representation based solely on the intensity dynamics of $X$ and $Y$, one would think they had reached the initial point. Yet, it is only after receiving the second delayed perturbation—with an additional time delay

Figure 4 | Theoretical temporal traces. a, Single vectorial DS created by perturbing the phase of the PSF (see the Supplementary Methods). b, Zoom on the pulse detail: the antiphase dip is followed by a small inverted kink after a time delay $\Delta \tau$, which corresponds to the reinsertion of $Y$ into $X$ after a longer time $t_f$. Other exponentially decreasing replica of this secondary kink (not visible) follow at time intervals $\Delta \tau$. c, Folded space–time representation: the shadow following the pulse corresponds to the inverted dip at $\Delta \tau$. In c, the second soliton was created far from the first one to yield two independent objects, as evidenced by their uncorrelated motion. Conversely, if the second soliton is nucleated at some precise closer distance, one obtains a bound state (d) in which the two solitons exhibit correlated motions. The period of the solutions is slightly superior to $t_f$, such secular drift being a result of the finite response time of the VCSEL.

Figure 5 | Floquet multipliers $\mu$. The values of $\mu$ are represented by black circles for the single soliton solution. All the multipliers have a modulus smaller than unity, which indicates stability of the solution. A zoom around $\mu = 1$ reveals a single multiplier, as expected for a periodic solution in an autonomous dynamical system. For the solution with three independent DS, we represent, for clarity, only the rightmost multipliers (red crosses). There are three quasidegenerate Floquet multipliers that have no equivalent in the case of the single soliton solution. This demonstrates the existence of three degenerate neutral modes that correspond to independent translational motions of each soliton. That the three multipliers are not exactly equal to unity is a consequence of the ultraweak residual interactions between distant DS. Such residual interaction implies that—strictly speaking—multiple DS are not totally independent for any finite value of the delay.
The polarization angle performs a full cycle, but remains essentially close to the equatorial plane as indicated by the weakness of the Δη = 1, and the ±45° emission corresponds to S2 = ±1. Single orbit unfolded in time. Here, two plateaux exist that correspond to the antiphase dip followed by the small inverted kink replica after a time Δτ. A stable antikink was generated by inverting the phase of the XPR, that is, setting β → −β. The small kinks in b and c around t = 0.7τ correspond to the secondary replica at time 2Δτ.

Elaborating on the predictions of this reduced model, we decreased Δτ to obtain a single uninterrupted cycle and we were able to find DS as short at 25 ps, both in the full and the simplified model (Supplementary Section E). Also, exploiting the symmetry property (Φ, β) → (−Φ, −β), we deduced the existence of topological antikink solutions, which are depicted in Fig. 6c. The kink and the antikink cannot be separated in our experimental measurements. More complicated kinks and antikinks that do not correspond to entire rotations also exist and will be the topic of further studies.

In summary, we have demonstrated the existence of vectorial DS and molecules in a VCSEL enclosed in a double external cavity. A simple theoretical model is found to reproduce the main experimental evidences. The independence of DS has been proved experimentally by studying their noise-induced motion and theoretically by analysing the Floquet multipliers. We have numerically shown that these DS can be individually addressed by perturbing the feedback cavity phase, and thus reveal their application potential as bits for information processing and data buffering. Using Stokes parameters we interpreted the observed solitons as rotations along the equator of the Poincaré sphere. Finally, a multiple timescale analysis allowed us to reduce the dynamics to a single delayed equation for the polarization orientation. In this context, we interpreted our DS as topological kinks or antikinks similar to those of the Sine–Gordon equation.

Methods
Methods and any associated references are available in the online version of the paper.

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Author contributions
M.M. performed the experimental characterization under the supervision of M.G. and assisted by S. Barland. M.M., S.B. and M.G. performed the statistical analysis of the experimental data. J.J. developed the theoretical and the numerical analysis and wrote the manuscript together with S.B. and M.G. All the authors participated in the interpretation of the results.

Additional information
Supplementary information is available in the online version of the paper. Reprints and permissions information is available online at www.nature.com/reprints. Correspondence and requests for materials should be addressed to M.G.

Competing financial interests
The authors declare no competing financial interests.
Methods

Set-up and VCSEL details. Two different lasers from ULM-photonics lasing at 850 nm (ULM.850-PMTNS46FOP) were used. They were single transverse mode lasers with a suppression ratio larger than 10 dB at the rated power of 1 mW. Both emit linearly polarized light and exhibit a birefringence around 7.5 GHz. Their low dichroism and birefringence helps to minimize the effects of PSF and XPR. Their threshold is around 0.5 mA with the bluest polarization mode, which we call Y, appearing at threshold. Both devices exhibit a circular dichroism and polarization switching can be implemented by injecting an external linearly polarized beam of low power (<100 nW). It is important that the linear polarization component in our case). The DS are obtained by pumping currents of 2.6 mM < J < 3.1 mA. In this experimental analysis we explored τ_i values that ranged from 1.3 ns up to 10.8 ns. The first value is the lowest limit possible with our set-up, whereas the second one is the maximum value explored. The detection set-up consists of two 8 GHz d.c. coupled detectors combined with a 33 GHz scope (Tektronix DPO73304D) as well as a 35 GHz a.c. coupled photodetector to resolve the details of the pulse shape. For this purpose, the data can be acquired in an oversampling mode with an acquisition point every 200 fs, which allows us to exploit fully the analogical bandwidth of our set-up. The combined rise time of our photodetector/oscilloscope couple is t_{rise} ≈ 14 ps. Here, we assumed that the rise time can be estimated as t_{rise} = 0.35/BW, with BW the bandwidth in hertz, and that the combined rise-time is t_r = (t_{rise} + t_{pd})^{1/2}, being t_{pd} the rise-time of the photodetector.

Theoretical model. We used the spin-flip model (SFM)\(^\text{rev}\), suitably modified to incorporate the effects of both PSF and XPR. We adopted a mixed description in terms of linearly polarized components of the field, X and Y, where PSF and XPR are easily expressed, and circularly polarized components of the field, E_{\pm} = (X \pm Y)/2^{1/2}, where the SFM is naturally expressed. In this framework, the dynamics of the system is written as:

\[
\begin{align*}
\dot{X} &= (1 + ia)(G E_{\pm} + G E_{\mp})/\sqrt{2} - X - zX + \beta e^{-i\tau} Y(t - \tau_i) \\
\dot{Y} &= (1 + ia)(G E_{\pm} - G E_{\mp})/\sqrt{2} + zY + \beta e^{-i\tau} Y(t - \tau_i) \\
TD_{fi} &= \mu - D_z - G_z (|E_{\pm}|^2) + \frac{\gamma_i}{\gamma} (D_z - D_{fi})
\end{align*}
\]  

where \(a\) is the linewidth enhancement factor, \(D_z\) are the scaled carrier densities in each spin channel and \(G_z = D_z/(1 - e|E_{\pm}|^2)\) is the gain for each circularly polarized field component including gain saturation. The terms \(Y(t - \tau_i)\) and \(Y(t - \tau_i)\) in the evolution equations for the X and Y components describe the effects of PSF and of XPR, which have strengths \(\eta\) and \(\beta\) phases \(\Omega\) and \(\alpha\), and time delays \(\tau_i\) and \(\tau_r\), respectively.

In equations (2)–(4), time has been scaled to the cavity decay rate \(k\), \(T^{-1} = \gamma_{fi}/k\) and \(\eta_i\) represents the scaled carrier and spin-difference decay rates. The bias current normalized to the threshold is \(\mu\). In addition, we defined \(z = (\eta_i + \eta_r)/\eta\), where \(\eta_i\) and \(\eta_r\) describe the linear dichroism and birefringence, respectively, of the cavity.

Finally, we added to the time evolution in equations (2)–(4) independent Langevin sources that describe noise caused by the spontaneous emission with variance \(\xi = 10^{-5}\). We assumed typical values \(\alpha = 2, \eta = 2 \times 10^{-5}, \beta = 0, \gamma_i = 5 \times 10^{-6}, \mu = 10, T = 500\) and \(\eta_i = 6\eta_r\), which (taking \(\kappa^2 = 2\)) correspond to a carrier lifetime \(\tau_{\gamma_i} = 1\) ns, a spin-difference decay rate of \(\tau_{\gamma_i} = 16.6\) ps, a frequency splitting \(\gamma_{fi}/k \approx 8\) GHz and a relaxation oscillation frequency \(\gamma_{r} = (2\pi)^{-1}(2k - 1\eta_{r})k^{1/2} \approx 15\) GHz. The values of the PSF and XPR parameters are \(\eta_i = 0.09\) and \(\beta = 0.06\). The feedback phases were taken as \(\Omega = a = 0\) for the sake of simplicity. However, DS were found for other values of the phases. The time delays in Figs 4a,b and 6 were taken as \(\tau_i = 1.500\) ps and \(\tau_r = 1.600\) and correspond to 3 ns and 3.2 ns, respectively. We took \(\tau = 1.700\) (that is, 3.4 ns) in Fig. 4c,d to visualize better the shadow of the pulse and \(\tau = 1.900\) (that is, 3.6 ns) in Fig. 6b,c to clarify the existence of the plateau.

For completeness, we recall that the normalized Stokes parameters read:

\[
S_0 = |X|^2 + |Y|^2, S_1 = (|X|^2 - |Y|^2)/S_0, S_2 = 2R(XY^*)/S_0, S_3 = -2\Im(XY^*)/S_0
\]

Numerical simulations. Equations (2) and (4) were integrated using a fourth-order Runge-Kutta method with constant step size \(h = 10^{-4}\) complemented by Hermite interpolation of the delayed terms. DS can be generated numerically by starting from a random initial condition around the off solution; during the transient regime, several spikes that correspond to the so-called relaxation oscillations are generated and can result in one or several DS in the asymptotic regime. A more controlled approach is, however, more appropriate to generate molecules. We use as an initial condition the CW mode, that is the solution with zero soliton. From this state, by inverting the feedback phase \(\Omega \rightarrow -\pi\) (with a 10 ps rise time), one can generate polarization slips that will eventually stabilize as vectorial DS after a few tens of roundtrips in the cavity. Notice that setting the phase back to its original value, even within the same roundtrip, generates another soliton. However, if the interval between the two slips is shorter than the soliton size, no structure is created. After the creation of a soliton, additional perturbations can be applied to generate multiple independent solitons, or soliton molecules if the perturbations are properly timed.