A note on the continuity of Oseledets subspaces
for fiber-bunched cocycles

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Abstract

We prove that restricted to the subset of fiber-bunched elements of the space of $GL(2, \mathbb{R})$-valued cocycles Oseledets subspaces vary continuously, in measure, with respect to the cocycle.

1 Introduction

In its simple form, a linear cocycle is just an invertible dynamical system $f : M \rightarrow M$ and a matrix-valued map $A : M \rightarrow GL(d, \mathbb{R})$. Sometimes one calls linear cocycle (over $f$ generated by $A$), instead, the sequence $\{A^n\}_{n \in \mathbb{Z}}$ defined by

$$A^n(x) = \begin{cases} 
A(f^{n-1}(x)) \cdots A(f(x))A(x) & \text{if } n > 0 \\
Id & \text{if } n = 0 \\
A(f^n(x))^{-1} \cdots A(f^{-1}(x))^{-1} & \text{if } n < 0
\end{cases}$$

for all $x \in M$.

A special class of cocycles is given when the base dynamics $f$ is hyperbolic and the dynamics induced by $A$ on the projective space is dominated by the dynamics of $f$. That is, the rates of contraction and expansion of the cocycle $A$ along an orbit are smaller than the rates of contraction and expansion of $f$. Such a cocycle is called fiber-bunched (see Section 2 for the precise definitions).

Many aspects of fiber-bunched cocycles are rather well understood. For instance, it is known that their cohomology classes are completely characterized by the information on periodic points [2, 8], generically they have simple Lyapunov spectrum [5, 9] and in the case when $d = 2$, Lyapunov exponents are continuous as functions of the cocycle [3]. In this short note, still in the context of fiber-bunched cocycles, we address the problem of continuity of the Oseledets subspaces. More precisely, we prove that restricted to the subset of fiber-bunched elements of the space of $GL(2, \mathbb{R})$-valued cocycles Oseledets subspaces vary continuously, in measure, with respect to the cocycle. The proof of this result relies on ideas from [3] and [4]. In a different context a similar statement was recently gotten by [6].

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2 Definitions and statements

Let \((M, d)\) be a compact metric space and \(f : M \to M\) be a homeomorphism. Given any \(x \in M\) and \(\varepsilon > 0\), we define the local \textit{stable} and \textit{unstable sets} of \(x\) with respect to \(f\) by

\[
W^s(x) := \{ y \in M : d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \geq 0 \},
\]

\[
W^u(x) := \{ y \in M : d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \leq 0 \},
\]

respectively.

Following [1], we say that a homeomorphism \(f : M \to M\) is \textit{hyperbolic with local product structure} (or just \textit{hyperbolic} for short) whenever there exist constants \(C_1, \varepsilon, \tau > 0\) and \(\lambda \in (0, 1)\) such that the following conditions are satisfied:

- \(d(f^n(y_1), f^n(y_2)) \leq C_1 \lambda^nd(y_1, y_2), \forall x \in M, \forall y_1, y_2 \in W^s(x), \forall n \geq 0;\)
- \(d(f^{-n}(y_1), f^{-n}(y_2)) \leq C_1 \lambda^{-n}d(y_1, y_2), \forall x \in M, \forall y_1, y_2 \in W^u(x), \forall n \geq 0;\)
- \(d(x, y) \leq \tau, \text{ then } W^s(x) \text{ and } W^u(y) \text{ intersect in a unique point which is denoted by } [x, y] \text{ and depends continuously on } x \text{ and } y. \text{ This property is called local product structure}.\)

Fix such an hyperbolic homeomorphism and let \(A : M \to GL(d, \mathbb{R})\) be a \(r\)-Hölder continuous map. This means that there exists \(C_2 > 0\) such that

\[
\| A(x) - A(y) \| \leq C_2 d(x, y)^r \text{ for any } x, y \in M.
\]

Let us denote by \(H^r(M)\) the space of such \(r\)-Hölder continuous maps. We endow this space with the \(r\)-Hölder topology which is generated by norm

\[
\| A \|_r := \sup_{x \in M} \| A(x) \| + \sup_{x \neq y} \frac{\| A(x) - A(y) \|}{d(x, y)^r}.
\]

We say that the cocycle generated by \(A\) satisfies the \textit{fiber bunching condition} or that the cocycle is \textit{fiber-bunched} if there exists \(C_3 > 0\) and \(\theta < 1\) such that

\[
\| A^n(x)\| \| A^n(x)^{-1} \| \lambda^{nr} \leq C_3 \theta^n
\]

for every \(x \in M\) and \(n \geq 0\) where \(\lambda\) is the constant given in the definition of hyperbolic homeomorphism.

Let \(\mu\) be an ergodic \(f\)-invariant probability measure on \(M\) with local product structure. Roughly speaking, the last property means that \(\mu\) is locally equivalent to the product measure \(\mu^s \times \mu^u\) where \(\mu^s\) and \(\mu^u\) are measures on the local stable and unstable manifolds respectively induced by \(\mu\) via the local product structure of \(f\). Since we are not going to use explicitly this property we just refer to [3] for the precise definition.

It follows from a famous theorem due to Oseledets (see [10]) that for \(\mu\)-almost every point \(x \in M\) there exist numbers \(\lambda_1(x) > \ldots > \lambda_k(x)\), and a direct sum decomposition \(\mathbb{R}^d = E^s_{x,A} \oplus \ldots \oplus E^u_{x,A}\) into vector subspaces such that

\[
A(x)E^s_{x,A} = E^s_{f(x),A} \text{ and } \lambda_i(x) = \lim_{n \to \infty} \frac{1}{n} \log \| A^n(x) v \|
\]
for every non-zero \( v \in E^i_x \) and \( 1 \leq i \leq k \). Moreover, since our measure \( \mu \) is assumed to be ergodic the Lyapunov exponents \( \lambda_i(x) \) are constant on a full \( \mu \)-measure subset of \( M \) as well as the dimensions of the Oseledets subspaces \( E^i_x \). Thus, we will denote by \( \lambda^+(A, \mu) = \lambda_k(x) \) and \( \lambda^-(A, \mu) = \lambda_1(x) \) the extremal Lyapunov exponents and by \( E^s_x = E^s_k \) and \( E^u_x = E^u_1 \) the stable and unstable spaces respectively. It follows by the Sub-Additive Ergodic Theorem of Kingman (see [7] or [10]) that the extremal Lyapunov exponents are also given by

\[
\lambda^+(A, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)\|
\]

(2.1) and

\[
\lambda^-(A, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|(A^n(x))^{-1}\|^{-1}
\]

for \( \mu \) almost every point \( x \in M \). The objective of this note is to understand, for a fixed base dynamics \( f \), how does the map \( A \to E^i_x \) vary in the case when \( d = 2 \), that is, in the case when the cocycle \( A \) takes values in \( GL(2, \mathbb{R}) \).

Let \( d \) be the distance on the projective space \( \mathbb{P}(\mathbb{R}^2) \) defined by the angle between two directions. We say that an element \( A \) of \( H^r(M) \) with \( \lambda^+(A, \mu) > \lambda^-(A, \mu) \) is a continuity point for the Oseledets decomposition with respect to the measure \( \mu \) if the Oseledets subspaces are continuous, in measure, as functions of the cocycle. More precisely, for any sequence \( \{(A_k)_{k \in \mathbb{N}} \} \subset H^r(M) \) converging uniformly with holonomies to \( A \) in the \( r \)-Hölder topology and any \( \varepsilon > 0 \), we have

\[
\mu\left( \left\{ x \in M; \ d(E^s_{A_k}, E^s_A) < \varepsilon \quad \text{and} \quad d(E^u_{A_k}, E^u_A) < \varepsilon \right\} \right) \xrightarrow{k \to \infty} 1.
\]

Thus, our main result is the following

**Theorem 2.1.** If \( A \in H^r(M) \) is a fiber-bunched cocycle with \( \lambda^+(A, \mu) > \lambda^-(A, \mu) \) then it is a continuity point for the Oseledets decomposition with respect to the measure \( \mu \).

The hypotheses that \( A \) is fiber-bunched and \( \mu \) has local product structure are only used to apply the results about continuity of Lyapunov exponents from [3]. Thus, more generally, if we have a sequence \( \{(A_k)_{k \in \mathbb{N}} \} \subset H^r(M) \) converging uniformly with holonomies to \( A \) as in the main theorem of [3], then

\[
\mu\left( \left\{ x \in M; \ d(E^s_{A_k}, E^s_A) < \varepsilon \quad \text{and} \quad d(E^u_{A_k}, E^u_A) < \varepsilon \right\} \right) \xrightarrow{k \to \infty} 1.
\]

Consequently, our result also applies if we restrict ourselves to the space of locally constant cocycles endowed with the uniform topology.

## 3 Proof of the theorem

Let us consider the projective cocycle \( F_A : M \times \mathbb{P}(\mathbb{R}^2) \to M \times \mathbb{P}(\mathbb{R}^2) \) associated to \( A \) and \( f \) which is given by

\[ F_A(x, v) = (f(x), PA(x)v) \]
where \( PA \) denotes the action of \( A \) on the projective space. We say that an \( F_A \)-invariant measure \( m \) on \( M \times P(\mathbb{R}^2) \) projects to \( \mu \) if \( \pi_* m = \mu \) where \( \pi : M \times P(\mathbb{R}^2) \to M \) is the canonical projection on the first coordinate. Given a non-zero element \( v \in \mathbb{R}^2 \) we are going to use the same notation to denote its equivalence class in \( P(\mathbb{R}^2) \).

Let \( \mathbb{R}^2 = E^s_x \oplus E^u_x \) be the Oseledets decomposition associated to \( A \) at the point \( x \in M \). Consider also

\[
m^s = \int_M \delta_{(x,E^s_x)} \, d\mu(x)
\]

and

\[
m^u = \int_M \delta_{(x,E^u_x)} \, d\mu(x)
\]

which are \( F_A \)-invariant probability measures on \( M \times P(\mathbb{R}^2) \) projecting to \( \mu \). Moreover, by the Birkhoff ergodic theorem and (2.1) we have that

\[
\lambda^- (A,\mu) = \int_{M \times P(\mathbb{R}^2)} \varphi_A(x,v) \, dm^s(x,v)
\]

and

\[
\lambda^+ (A,\mu) = \int_{M \times P(\mathbb{R}^2)} \varphi_A(x,v) \, dm^u(x,v)
\]

where \( \varphi_A : M \times P(\mathbb{R}^2) \to \mathbb{R} \) is given by

\[
\varphi_A(x,v) = \log \frac{\| A(x)v \|}{\| v \|}.
\]

By the (non-uniform) hyperbolicity of \( (A,\mu) \) we have the following.

**Lemma 3.1.** Let \( m \) be a probability measure on \( M \times P(\mathbb{R}^2) \) that projects down to \( \mu \). Then, \( m \) is \( F_A \)-invariant if and only if it is a convex combination of \( m^s \) and \( m^u \) for some \( f \)-invariant functions \( \alpha, \beta : M \to [0,1] \) such that \( \alpha(x) + \beta(x) = 1 \) for every \( x \in M \).

**Proof.** One implication is trivial. For the converse one only has to note that every compact subset of \( P(\mathbb{R}^2) \) disjoint from \( \{E^u, E^s\} \) accumulates on \( E^u \) in the future and on \( E^s \) in the past. \( \square \)

**Proof of Theorem 2.1.** Suppose that \( A \) is a fiber-bunched cocycle such that \( \lambda^+ (A,\mu) > \lambda^- (A,\mu) \). As the subset of fiber-bunched elements of \( H^r(M) \) is open we may assume without loss of generality that \( A_k \) is fiber-bunched for every \( k \in \mathbb{N} \). Moreover, since the Lyapunov exponents depend continuously on the cocycle \( A \) (see Theorem 1.1 from [3]) and \( \lambda^+ (A,\mu) > \lambda^- (A,\mu) \) we may also assume that \( \lambda^+ (A_k,\mu) > \lambda^- (A_k,\mu) \) for every \( k \in \mathbb{N} \). We will prove just the assertion about the unstable spaces, that is, that \( \mu (\{x \in M; d(E^u_{x_k}, E^u_x) < \delta\}) \xrightarrow{k \to \infty} 1 \). The case of the stable spaces is analogous.

For each \( k \in \mathbb{N} \), let us consider the measure

\[
m_k = \int_M \delta_{(x,E^u_{x_k})} \, d\mu(x)
\]
and let $m^u$ be the measure given by

$$m^u = \int_M \delta_{(x,E_x^{u,A})} d\mu(x).$$

These are $F_{A_k}$ and $F_A$-invariant probability measures on $M \times \mathbb{P}(\mathbb{R}^2)$ respectively, projecting to $\mu$. Moreover, $m_k \xrightarrow{k \to \infty} m^u$. Indeed, let $(m_{kj})_{j \in \mathbb{N}}$ be a convergent subsequence of $(m_k)_{k \in \mathbb{N}}$ and suppose that it converges to $\eta$. Since for each $j \in \mathbb{N}$ the measure $m_{kj}$ is $F_{A_{kj}}$-invariant and projects to $\mu$ it follows that $\eta$ is an $F_A$-invariant measure projecting to $\mu$. Moreover, since

$$\lambda^+(A_{kj},\mu) \xrightarrow{j \to \infty} \lambda^+(A,\mu)$$

once the Lyapunov exponents are continuous as functions of the cocycle (see [3]) and

$$\lambda^+(A_{kj},\mu) = \int_{M \times \mathbb{P}(\mathbb{R}^2)} \varphi_{A_{kj}} dm_{kj} \xrightarrow{j \to \infty} \int_{M \times \mathbb{P}(\mathbb{R}^2)} \varphi_A d\eta$$

we get that

$$\lambda^+(A,\mu) = \int_{M \times \mathbb{P}(\mathbb{R}^2)} \varphi_A d\eta.$$

Thus, invoking Lemma 3.1 and using the fact that $\mu$ is ergodic it follows that $\eta = m^u$. Indeed, otherwise we would have $\eta = \alpha m^s + \beta m^u$ with $\alpha > 0$ and consequently

$$\int_{M \times \mathbb{P}(\mathbb{R}^2)} \varphi_A d\eta = \alpha \lambda^-(A,\mu) + \beta \lambda^+(A,\mu) < \lambda^+(A,\mu).$$

Therefore, $m_k \xrightarrow{k \to \infty} m^u$ as claimed.

Let $g : M \to \mathbb{P}(\mathbb{R}^2)$ be the measurable map given by

$$g(x) = E_x^{u,A}.$$

Note that its graph has full $m^u$-measure. By Lusin’s Theorem, given $\varepsilon > 0$ there exists a compact set $K \subset M$ such that the restriction $g_K$ of $g$ to $K$ is continuous and $\mu(K) > 1 - \varepsilon$. Now, given $\delta > 0$, let $U \subset M \times \mathbb{P}(\mathbb{R}^2)$ be an open neighborhood of the graph of $g_K$ such that

$$U \cap (K \times \mathbb{P}(\mathbb{R}^2)) \subset U_\delta$$

where

$$U_\delta := \{(x,v) \in K \times \mathbb{P}(\mathbb{R}^2); \ d(v,g(x)) < \delta\}.$$

By the choice of the measures $m_k$,

$$m_k(U_{\delta}) = \mu(\{x \in K; \ d(E_x^{u,A_k},E_x^{u,A}) < \delta\}). \quad (3.2)$$

Now, as $m_k \xrightarrow{k \to \infty} m^u$ it follows that $\liminf m_k(U) \geq m^u(U) > 1 - \varepsilon$. On the other hand, as $m_k(K \times \mathbb{P}(\mathbb{R}^2)) = \mu(K) > 1 - \varepsilon$ for every $k \in \mathbb{N}$, it follows that

$$m_k(U_\delta) \geq m_k(U \cap (K \times \mathbb{P}(\mathbb{R}^2))) \geq 1 - 2\varepsilon \quad (3.3)$$

for every $k$ large enough. Thus, combining (3.2) and (3.3), we get that $\mu(\{x \in M; \ d(E_x^{u,A_k},E_x^{u,A}) < \delta\}) \geq 1 - 2\varepsilon$ for every $k$ large enough completing the proof of Theorem 2.1. 

$\square$
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