Pythagorean triangles within Pythagorean triangles

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July 30, 2010
1 Introduction

Suppose that $CBA$ is a Pythagorean triangle with side-lengths $|AB| = c$, $|CA| = b$, and $|CB| = a$; that is, a right triangle with the right angle at $C$; and with $a, b, c$ being positive integers such that $a^2 + b^2 = c^2$. Then (without loss of generality – $a$ and $b$ may be switched),

$$
\begin{align*}
  a &= d(m^2 - n^2), \\
  b &= d(2mn), \\
  c &= d(m^2 + n^2)
\end{align*}
$$

where $d, m, n$ are positive integers such that

$$
\begin{align*}
  m > n, \ (m,n) = 1, \ and \ m + 1 &\equiv 1 (\text{mod} \ 2)
\end{align*}
$$

**Note:** Throughout this paper, $(X, Y)$ will stand for the greatest common divisor of two integers $X$ and $Y$.

Thus, the condition $(m, n) = 1$ says that $m$ and $n$ are relatively prime, their greatest common divisor is 1. Also, the condition $m + n \equiv 1 (\text{mod} \ 2)$ says that $m$ and $n$ have different parities; one of them is even, the other odd. The formulas in (1), are the well known parametric formulas describing the entire family of Pythagorean triangles or triples.

A derivation of the formulas can be found in references [1] and [2]. For a wealth of historic information on Pythagorean triangles see [2] or [3].

Now, consider a point $P$ on the hypotenuse $AB$, and let $D$ and $E$ be the intersection points of the two lines through $P$ and parallel to $CA$ and $CB$; with the sides $CB$ and $CA$ respectively. Two right triangles are formed; the triangles $BDP$ and $APE$. Let $x$ and $y$ denote the lengths of line segments $DP$ and $PE$ respectively. Also, let $h_1 = |BP|$ and $h_2 = |AP|$. Then,
\[
\begin{align*}
\{ & |DP| = |CE| = x \text{ and } |PE| = |DC| = y. \\
& \text{Thus, } |BD| = |BC| - |DC| = a - y; \\
& \text{and } |AE| = |AC| - |CE| = b - x
\end{align*}
\]

Both right triangles \(BDP\) and \(APE\) are similar to the right triangle of \(CBA\). We have the similarity ratios,

\[
\begin{align*}
\begin{cases}
\frac{x}{b} = \frac{a-y}{a} = \frac{h_1}{c} \\
\frac{y}{a} = \frac{b-x}{b} = \frac{h_2}{c}
\end{cases}
\end{align*}
\]

Since \(a, b, c\) are (positive) integers, it follows, by inspection, from (3i) that if one of \(x, y,\) or \(h_1\) is a rational number, then all three of them must be rational numbers. Hence, either all three \(x, y, h_1\) are rationals, or otherwise, all three of them must be irrational. Likewise, it follows from (3ii) that either all three \(x, y, h_2\) are rational or all three are irrational. Combining these two observations, we infer that

\begin{center}
\textbf{Either all four } \(x, y, h_1, h_2\) \textbf{ are rational numbers or, otherwise, all four of them are irrationals.}
\end{center}

In Section 2, we state three lemmas from number theory. One of them (Euclid’s Lemma) is well known. We offer proofs for the other two.

In Section 3, we prove Theorems 1 and 2; Theorem 2 is a corollary of Theorem 1.

In Section 4, we consider and analyze three special cases. These are the cases when the point \(P\) is the midpoint \(M\) of the hypotenuse \(AB\); when \(P\) is the point \(I\) where the angle bisector of the 90° angle at \(C\) intersects the hypotenuse \(AB\); and when the point \(P\) is the foot \(F\) of the perpendicular from \(C\) to the hypotenuse \(AB\).

Back to Section 3. In Theorem 1 we prove that the two right triangles \(BDP\) and \(PEA\) in Figure 1 are either both Pythagorean or neither of them is a Pythagorean triangle (assuming, of course, that \(BCA\) is a Pythagorean triangle). It then follows, and this is part of Theorem 2, that when the triangle \(BCA\) is a primitive Pythagorean triangle, neither of the triangles,
BDP and PEA are Pythagorean for any position of the point P along the hypotenuse AB.

In Section 5 (Theorem 6), we postulate that given a Pythagorean triangle with side lengths $a = d(m^2 - n^2)$, $b = d(2mn)$, and $c = d(m^2 + n^2)$, where $d, m, n$ are positive integers such that $d \geq 2$, $(m, n) = 1$, $m > n$, and $m + n \equiv 1(\text{mod } 2)$. Then there are exactly $d - 1$ positions of the point P, such that triangles BDP and PEA are both Pythagorean.

In Section 6, we will examine the general question of when, in addition to the two triangles BDP and APE being Pythagorean, the four congruent right triangles (within the rectangle CDPE) CDP, CEP, DCE, and EPD are also Pythagorean. We derive a family of non-primitive Pythagorean triangles CBA with that property.

Note: In addition to the notation $(k, \ell)$ denoting the greatest common divisor of two integers, $k$ and $\ell$, the notation $t|v$, will stand for “The integer $t$ is a divisor of the integer $v$”.

2 Three lemmas from number theory

Lemma 1. (Euclid’s Lemma): Suppose that $a, b, c$, are natural numbers such that $c|ab$ (i.e., $c$ is a divisor of the product $ab$). If $(c, a) = 1$, then $c|b$.

For a proof of this well-known result, the reader may refer to [1] or [2].

Lemma 2. Let $m, n$ be positive integers such that $m > n$, $(m, n) = 1$, and $m + n \equiv 1(\text{mod } 2)$. Then

(i) $(m^2 + n^2, 2mn) = 1$
(ii) $(m^2 + n^2, m^2 - n^2) = 1$
(iii) $(m^2 - n^2, 2mn) = 1$

Proof.

(i) We show that $m^2 + n^2$ and $2mn$ have no prime divisors in common. If, to the contrary, $p$ were a prime divisor of both $m^2 + n^2$ and $2mn$, then $p$ would be odd, since $m^2 + n^2 \equiv 1(\text{mod } 2)$, by virtue of the hypothesis $m + n \equiv 1(\text{mod } 2)$. Thus, $p|2mn$ implies, since $(p, 2) = 1$, that $p|m$.
(by Lemma 1). But \( p \) is a prime, so \( p | mn \) implies that \( p \) must divide at least one of \( m, n \). If \( p | m \), then from \( p | m^2 + n^2 \), it follows that \( p | n^2 \), and so \( p | n \). Thus, \( p | m \) and \( p | n \) contradicting the hypothesis that \( (m, n) = 1 \).

(ii) A similar argument left to the reader (\( p \) must divide the sum of \( m^2 + n^2 \) and \( m^2 - n^2 \), and their difference. Hence, \( p | 2n^2 \) and \( p | 2m^2 \), which since \( p \) is odd, eventually implies \( p | n \) and \( p | m \), a contradiction).

(iii) A similar argument as in (i).

\[ \square \]

3 A theorem and a corollary

**Theorem 1.** Suppose that \( ABC \) is a Pythagorean triangle with the right angle at \( C \); and with the three sidelengths satisfying the formulas in (1), namely \( a = d(m^2 - n^2) \), \( b = d(2mn) \), \( c = d(m^2 + n^2) \), where \( d, m, n \) are positive integers such that \( m > n \), \( (m, n) = 1 \), and \( m + n \equiv 1 \) (mod 2).

Let \( P \) be a point on the hypotenuse \( \overline{AB} \), distinct from \( A \) and \( B \). Furthermore, suppose that \( D \) is the foot of the perpendicular from \( P \) to the side \( \overline{CB} \); and \( E \) the foot of the perpendicular from \( P \) to the side \( \overline{CA} \), as in Figure 1. Then, either both right triangles \( BDP \) and \( APE \) are Pythagorean or neither of them is.

Moreover, if they are both Pythagorean, then the sidelengths \( |BD| = a - y \), \( |DP| = x \), and \( |BP| = h_1 \) of the triangle \( BDP \) satisfy the formulas,

\[ a - y = \delta(m^2 - n^2), \quad x = \delta(2mn), \quad h_1 = \delta(m^2 + n^2). \]

While the sidelengths \( |PE| = y \), \( |EA| = b - x \), \( |PA| = h_2 \) of the triangle \( PEA \) satisfy the formulas

\[ y = (d - \delta)(m^2 - n^2), \quad b - x = (d - \delta)(2mn), \quad h_2 = (d - \delta)(m^2 + n^2) \]

where \( \delta \) is a positive integer such that \( 1 \leq \delta \leq d - 1 \).

**Proof.** Suppose that the triangle \( BDP \) is Pythagorean. We will prove that the triangle \( APE \) must also be Pythagorean; then so must the triangle \( BDP \) be.
Since the triangle $BDP$ is Pythagorean, its three sidelengths, $x$, $a - y$, and $h_1$ (see Figure 1) must be natural numbers. From (3i)

$$x = \frac{b \cdot h_1}{c} \frac{d(2mn)}{d(m^2 + n^2)} \cdot h_1;$$  \hspace{1cm} (4)

$$x = \frac{2mn h_1}{m^2 + n^2}$$

From the conditions $(m, n) = 1$ and $m + n \equiv 1 \pmod{2}$, it follows by Lemma 2(i) that

$$(m^2 + n^2, 2mn) = 1$$ \hspace{1cm} (4i)

Since $x$ is a natural number, equation (4) says that the integer $m^2 + n^2$ must be a divisor of the product $2mn h_1$, which clearly implies, by (4i) and Lemma 1, that $h_1$ must be divisible by $m^2 + n^2$.

$$h_1 = \delta \cdot (m^2 + n^2)$$ \hspace{1cm} (4ii)

for some positive integer $\delta$; and since $h_1$ is the length hypotenuse $\overline{BP}$ (triangle $BDP$), and the point $P$ lies strictly between $A$ and $B$, it is clear that

$$h_1 = \overline{BP} < c = \overline{BA} = d(m^2 + n^2),$$

which together with (4ii) clearly show that

$$1 \leq \delta < d; \text{ or equivalently,}$$

$$1 \leq \delta \leq d - 1$$ \hspace{1cm} (4iii)

Note that by (4iii), we must have $d \geq 2$. Going back to (4i) and using (4ii) we get

$$x = (2mn)\delta$$ \hspace{1cm} (4iv)

and so, by (4iv), (3i), (4ii), and (1), we further obtain

$$a - y = \delta(m^2 - n^2); \hspace{0.5cm} y = a - \delta(m^2 - n^2);$$

$$y = d(m^2 - n^2) - \delta(m^2 - n^2) = (d - \delta)(m^2 - n^2)$$ \hspace{1cm} (4v)
By using (1), (3i), and (4v) we also get
\[ b - x = (d - \delta)(2mn) \]
and
\[ h_2 = (d - \delta)(m^2 + n^2). \]

The proof is complete. \( \square \)

**Theorem 2.** Let \( CBA \) be a Pythagorean triangle, with the 90 degree angle at \( C \). Also, let \( |CB| = a, |CA| = b \) and \( |BA| = c \), be the three side-lengths so that \( a = d(m^2 - n^2), b = d(2mn), c = d(m^2 + n^2) \) where \( m, n, d \) are positive integers such that \( m > n, (m, n) = 1, \) and \( m + n \equiv 1 (\text{mod} 2) \).

Let \( P \) be a point on the hypotenuse \( |BA| \) and strictly between the endpoints \( B \) and \( A \).

Let \( D \) and \( E \) be the feet of the perpendiculars from the point \( P \) to the sides \( CB \) and \( CA \) respectively.

(i) If \( d = 1 \) i.e., if the Pythagorean triangle \( CBA \) is primitive, then neither of the right triangles \( PDB \) and \( PEA \) is Pythagorean.

(ii) If \( d = 2 \), and the point \( P \) is coincident with the midpoint \( M \) of the hypotenuse \( BA \), then both triangles \( PDB \) and \( PEA \) are Pythagorean. Otherwise, if \( P \neq M \), neither of these two triangles is Pythagorean.

(iii) If \( d = 3 \), and the point \( P \) is such that \( \frac{|PB|}{|PA|} = \frac{1}{3} \) or \( \frac{2}{3} \), then both triangles \( PDB \) and \( PEA \) are Pythagorean. Otherwise, if \( \frac{|PB|}{|PA|} \neq \frac{1}{3}, \frac{2}{3} \), then neither of these triangles are Pythagorean.

**Proof.** (i) If \( d = 1 \), then neither of the two right triangles, \( BDP \) and \( PEA \) can be Pythagorean since according to Theorem 1, the natural number \( \delta \) must satisfy \( 1 \leq \delta \leq d - 1 \), which is impossible when \( d = 1 \).

(ii) Suppose that \( d = 2 \).

If the point \( P \) coincides with the midpoint \( M \) of the hypotenuse \( BA \), then each of the triangles \( BDP \) and \( PEA \) is half the size of the triangle \( CBA \). So, by inspection,
which proves that both triangles $BDP$ and $PEA$ are (in fact primitive) Pythagorean triangles. Conversely, if both triangles are Pythagorean, then by Theorem 1, it follows that $1 \leq \delta \leq d - 1 = 2 - 1 = 1$, $1 \leq \delta \leq 1$, $\delta = 1$ which establishes that each of the triangles is half the size of triangle of $CBA$; which implies that $P$ is the midpoint of $BA$. 

(iii) Assume that $d = 3$.

Suppose $\frac{|PB|}{|PA|} = \frac{1}{3}$ or $\frac{2}{3}$. If $\frac{|PB|}{|PA|} = \frac{1}{3}$, then the triangle $BDP$ is $\frac{1}{3}$ the size of triangle $CBA$ and the triangle $PEA$ is $\frac{2}{3}$ the size of $CBA$. We have,

$$|BD| = \frac{a}{2} = \frac{2(m^2 - n^2)}{2} = m^2 - n^2,$$

$$|DP| = \frac{b}{2} = \frac{2(2mn)}{2} = 2mn,$$

$$|BP| = \frac{c}{2} = \frac{2(m^2 + n^2)}{2} = m^2 + n^2,$$

and $|PE| = \frac{2a}{3} = 2(m^2 - n^2)$, $|EA| = \frac{2b}{3} = 2(2mn)$, $|PA| = \frac{2c}{3} = 2(m^2 + n^2)$. It is clear that both triangles $BDE$ and $PEA$ are Pythagorean.

The argument for the case $\frac{|PB|}{|PA|} = \frac{2}{3}$ is similar (we omit the details).

Now, the converse. Assume that both triangles, $BDP$ and $PEA$, are Pythagorean. Then by Theorem 1 we must have,
\[ 1 \leq \delta \leq d - 1 = 3 - 1 = 2; \quad \delta = 1 \text{ or } 2. \]

Using the formulas for the sidelengths (of triangles \( BDP \) and \( PEA \)), found in Theorem 1 we easily see that \( \frac{|PA|}{|PB|} = \frac{1}{3} \), if \( \delta = 1 \). While \( \frac{|PA|}{|PB|} = \frac{2}{3} \) if \( \delta = 2 \). The proof is complete.

\[ \square \]

4 Three special cases

A. Case 1: When the point \( P \) is the midpoint \( M \) of the hypotenuse \( BA \)

By inspection, it is clear that all six right triangles \( BDP, PEA, CDP, EPD, DCE, \) and \( PEC \) are all congruent and each of them is half the size of triangle \( BCA \). Clearly then, by (1), these six triangles will be Pythagorean if and only if the integer \( d \) in (1) is even.

Theorem 3. Let \( BCA \) be a Pythagorean triangle with the 90 degree angle at \( C \) and \( |CB| = a = d(m^2 - n^2), |CA| = b = d(2mn), |BA| = c = d(m^2 + n^2) \), where \( d, m, n \) are positive integers such that \( m > n, (m, n) = 1 \) and \( m + n \equiv 1(\text{mod} \ 2) \).

Let \( M \) be the midpoint of the hypotenuse \( BA \) and \( D, E \) the feet of the perpendiculars from \( M \) to the sides \( CB \) and \( CA \) respectively (so \( D \) and \( E \) are the midpoints of \( CB \) and \( CA \)). Then, the six right angles, \( BDM, MEA, CDM, EMD, DCE, \) and \( MEC \) are congruent and have sidelengths as follows:

- length of horizontal side \( = \frac{d}{2}(2mn) = dmn \)
- length of vertical side \( = \frac{d(m^2 - n^2)}{2} \)
- length of hypotenuse \( = \frac{d(m^2 + n^2)}{2} \)
If \( d \) is an even natural number, then the above six triangles are Pythagorean; otherwise, if \( d \) is odd, they are non-Pythagorean.

B. Case 2: When the point \( P \) is the foot \( I \) of the angle bisector of the 90° angle at \( C \)

Using the notation of Theorem 1, we have
\[
\overline{BD} = a - y, \quad \overline{DI} = x,
\]
\[
\overline{BP} = h_1, \quad \overline{EA} = b - x,
\]
\[
\overline{EI} = y, \text{ and } \overline{IA} = h_2.
\]

Clearly, we have \( x = y \) in this case. Note that the four congruent isosceles right triangles \( DCI, IEC, DCE, DIE \) cannot be Pythagorean (no Pythagorean triangle is isosceles).

By Theorem 1, the two right triangles \( BDI \) and \( IEA \) are either both Pythagorean or neither of them are. If they are both Pythagorean, then by Theorem 1 we have, in particular, \( x = \delta (2mn) \) and \( y = (d - \delta)(m^2 - n^2) \) with \( m, n, d, \delta \) being positive integers such that \( m > n, (m, n) = 1, m + n \equiv 1 \pmod{2} \) and \( 1 \leq \delta \leq d - 1 \) (and so \( d \geq 2 \)).

Since \( x = y \), we must have
\[
\delta (2mn) = (d - \delta)(m^2 - n^2) \quad (5)
\]

By Lemma 2(iii), we know that \((m^2 - n^2, 2mn) = 1\). So, by Lemma 1 and (5) it follows that \( 2mn|d - \delta \) and \( m^2 - n^2|\delta \) which, in turn, leads to (when we go back to (5))
\[
\begin{align*}
\delta &= K \cdot (m^2 - n^2), \\
\delta - d &= K \cdot (2mn), \\
\text{for some positive integer } K. \\
\end{align*}
\]

(5i)

Note that clearly, from (5i), \(1 \leq \delta \leq d-1\). In fact, the smallest possible value of \(d\) is 7; obtained for \(K = 1\) and \(m = 2, n = 1\). Moreover, \(1 \leq \delta \leq d - 4\) since the smallest possible value of \(K \cdot (2mn)\) is 4.

Using (5i) and Theorem 1, one can compute in terms of \(m, n, K\). The other four sidelengths of the triangles \(BDI\) and \(IEA\). Also, by (5i) we get \(x = \delta (2mn) = K(2mn)(m^2 - n^2) = y\). We now state the following theorem.

**Theorem 4.** Let \(CBA\) be a Pythagorean triangle with the 90° angle at \(C\) and sidelengths given by

\[
\begin{align*}
|CB| &= a = d(m^2 - n^2), \\
|CA| &= b = d(2mn), \\
|BA| &= c = d(m^2 + n^2),
\end{align*}
\]

where \(d, m, n\) are positive integers such that, \(m > n\), \(m + n \equiv 1(\text{mod } 2)\), and \((m, n) = 1\). Let \(I\) be the foot of the perpendicular of the angle bisector (of the 90° angle at \(C\)) to the hypotenuse \(BA\).

Also, let \(D\) and \(E\) be the feet of the perpendiculars from the point \(I\) to the sides \(CB\) and \(CA\) respectively. Then, the two right triangles \(BDI\) and \(IEA\) are both Pythagorean precisely when (i.e., if and only if), \(d = K \cdot (m^2 - n^2 + 2mn)\) for some integer \(K\). If \(d = K \cdot (m^2 - n^2 + 2mn)\), then the sidelengths of triangle \(BDI\) are given by \(|DI| = x = K \cdot (2mn)(m^2 - n^2)\), \(h_1 = |BI| = K(m^2 - n^2)(m^2 + n^2) = K(m^4 - n^4)\), and \(|BD| = a - y = K \cdot (m^2 - n^2)^2\) and the sidelengths of triangle \(IEA\) are given by

\[
\begin{align*}
|IE| &= y = K(2mn)(m^2 - n^2), \\
|EA| &= b - x = K(2mn)(2mn) = K \cdot (2mn)^2,
\end{align*}
\]
and \( h_2 = |TA| = K \cdot (2mn)(m^2 + n^2). \)

If the integer \( d \) is not divisible by \( m^2 - n^2 + 2mn \), then neither of the triangles, \( BDI \) and \( IEA \), is Pythagorean.

C. Case 3: When the point \( P \) is the foot \( F \) of the perpendicular from the vertex \( C \) to the hypotenuse \( BA \)

In this part, instead of using Theorem 1, we will first compute the sidelengths of the triangles \( BDF, FEA \), and the four congruent triangles \( FDC, DFE, DCE, \) and \( CFE \) in terms of (the sidelengths) \( a, b, c \). After that we will implement the formulas in (1) in order to express the above sidelengths in terms of the integers \( d, m, n \).

After that we will implement Lemma 2 to be able to draw the conclusions which will lead to Theorem 5. Note that since \( F \) is the foot of the perpendicular from \( C \) to the hypotenuse \( BA \), the aforementioned six right triangles are all similar to the triangle \( CBA \). Let \( \omega \) and \( \varphi \) be the degree measures of the angles \( \angle CBA \) and \( \angle CAB \) respectively (see Figure 3).

We have (and we set)
Furthermore,

\[ \sin \omega = \frac{y}{h} = \cos \varphi = \frac{b}{c} \text{ and } \cos \omega = \frac{h}{b} = \frac{a}{c}; \]

and thus \( h = \frac{ab}{c} \), which implies \( y = h \cdot \cos \varphi = h \cdot \frac{b}{c} = \frac{ab}{c} \cdot \frac{b}{c} = \frac{ab^2}{c^2}. \) So, \( a - y = a - \frac{ab^2}{c^2} = \frac{a(c^2 - b^2)}{c^2} \) (since \( c^2 = a^2 + b^2 \)) \( \frac{a}{c} \cdot \frac{a^2}{c^2} = \frac{a^3}{c^2}. \) So, \( a - y = \frac{a^3}{c^2}. \)

Next we calculate the lengths \( x \) and \( b - x \). We have \( \tan \omega = \cot \varphi = \frac{y}{x}; \)

\[ \cot \omega = \tan \varphi = \frac{x}{y}, \text{ and } \tan \varphi = \frac{a}{b} \text{ which gives } \frac{x}{y} = \frac{a}{b}, x = \frac{a}{b} \cdot y. \]

Since \( y = \frac{ab^2}{c^2} \) (see above), we obtain \( x = \frac{a}{b} \cdot \frac{ab^2}{c^2} = \frac{ba^2}{c^2}. \) From this we get \( b - x = b - \frac{ba^2}{c^2} = \frac{b(c^2 - a^2)}{c^2} = \frac{b \cdot b^2}{c^2} = \frac{b^3}{c^2}, \) since \( c^2 - a^2 = b^2. \)

Also, \( \sin \omega = \frac{x}{h_1}; h_1 = \frac{1}{\sin \omega} \cdot x = \frac{c}{b} \cdot \frac{ba^2}{c^2} = \frac{a^2}{c}. \) Similarly, we have \( \sin \varphi = \frac{y}{h_2}; h_2 = \frac{1}{\sin \varphi} \cdot y = \frac{c}{a} \cdot \frac{ab^2}{c^2} = \frac{b^2}{c}. \) We summarize these lengths as follows:

**Sidelengths of triangle BDF**

\[
\begin{align*}
|BD| &= a - y = \frac{a^3}{c^2}, \quad |DF| = x = \frac{ba^2}{c^2}, \quad |BF| = h_1 = \frac{a^2}{c} \\
|BC| &= a, \quad |CA| = b, \quad |BA| = c \\
|DC| &= FE = y, \quad |BD| = a - y \\
|BF| &= h_1, \quad FA = h_2, \quad CF = |DE| = h
\end{align*}
\]
Sidelengths of triangle FEA

\[
\left( |FE| = y = \frac{ab^2}{c^2}, \ |EA| = b - x = \frac{b^3}{c^2}, \ |FA| = h_2 = \frac{b^2}{c} \right) \quad (6\text{ii})
\]

Sidelengths of the four congruent triangles FDC, DFE, DCE, CFE

\[
\left( |DC| = |FE| = y = \frac{ab^2}{c^2}, \ |DF| = |CE| \\
= x = \frac{ba^2}{c^2}, \ |CF| = |DE| = \frac{ab}{c} = h \right) \quad (6\text{iii})
\]

Next, we combine the length formulas in (6i), (6ii), and (6iii) with the formulas in (1), since CBA is a Pythagorean triangle, to obtain the following.

\[
\begin{align*}
\frac{a - y}{(m^2 + n^2)^2} & = \frac{d \cdot (m^2 - n^2)^3}{(m^2 + n^2)^2}, \quad \frac{x}{(m^2 + n^2)^2} = \frac{d \cdot (m^2 - n^2)^2 \cdot (2mn)}{(m^2 + n^2)^2} \\
y & = \frac{d \cdot (m^2 - n^2) \cdot (2mn)^2}{(m^2 + n^2)^2}, \quad b - x = \frac{d \cdot (2mn)^3}{(m^2 + n^2)^2} \\
h_1 & = \frac{d \cdot (m^2 - n^2)^2}{m^2 + n^2}, \quad h_2 = \frac{d \cdot (2mn)^2}{m^2 + n^2} \\
h & = \frac{d \cdot (2mn) \cdot (m^2 - n^2)}{m^2 + n^2} \quad (7)
\end{align*}
\]

The following lemma from number theory is well-known and comes in handy.

**Lemma 3.** Let \(i_1, i_2, i_3, e_1, e_2, e_3\) be positive integers such that \((i_1, i_2) = 1 = (i_1, i_3)\). Then,

(a) \((i_1^{e_1}, i_2^{e_2}) = 1\)
(b) \((i_1^{e_1}, i_2^{e_2}, i_3^{e_3}) = 1\)
It follows from Lemmas 2 and 3 that

\[
\begin{cases}
(m^2 + n^2)^2, (m^2 - n^2)^3 = 1, \\
(m^2 + n^2)^2, (2mn)^2 = 1 \\
(m^2 + n^2, (m^2 - n^2)^2) = 1, \\
(m^2 + n^2, (2mn)^2) = 1 \\
(m^2 + n^2, (2mn) \cdot (m^2 - n^2)) = 1 \\
((m^2 + n^2)^2, (m^2 - n^2)^2 \cdot (2mn)) = 1 \\
((m^2 + n^2)^2, (m^2 - n^2) \cdot (2mn)^2) = 1
\end{cases}
\]  

(8)

A careful look at formulas (7) and the coprimeness conditions in (8), in conjunction with Lemma 1, reveals that either all six triangles, \(BDF, FEA, FDC, DFE, DCE,\) and \(CFE\) are Pythagorean; or none of them are.

They are all Pythagorean precisely (i.e., if and only if) the integer \(d\) is divisible by \((m^2 + n^2)^2\), i.e., when

\[
\begin{cases}
d = K \cdot (m^2 + n^2)^2 \\
\text{for some positive integer } K
\end{cases}
\]

(9)

This is precisely when all seven numbers \(y, a - y, x, b - x, h_1, h_2,\) and \(h\) are integers. When (9) holds true, we can compute, via (7) and (6i), (6ii), and (6iii) all the sidelengths in terms of the integers \(m, n,\) and \(K\).

We have the following theorem.

**Theorem 5.** Let \(CBA\) be a Pythagorean triangle, with the 90-degree angle at \(C\). With \(|CB| = a = d(m^2 - n^2), |CA| = b = d(2mn), |BA| = |
\[ d(m^2 + n^2) = c, \text{ where } d, m, n \text{ are positive integers such that } m > n, \ (m, n) = 1, \text{ and } m + n \equiv 1(\text{mod}2). \]

Also, let \( F \) be the foot of the perpendicular from the vertex \( C \) to the hypotenuse \( \overline{BA} \). Then, the six similar triangles \( BDF, FEA, \) (and the four congruent ones) \( FDC, DFE, DCE, CFE \) are either all Pythagorean or none of them are. They are all Pythagorean precisely when (i.e., if and only if) \( d = K \cdot (m^2 + n^2)^2 \), for some positive integer \( K \). When \( d \) satisfies the said condition, the sidelengths of the above six triangles are given by the following formulas.

For triangle \( BDF \)

\[ |BD| = a - y = K \cdot (m^2 - n^2), \quad |DF| = x = K \cdot (m^2 - n^2)^2 \cdot (2mn), \text{ and } h_1 = K \cdot (m^2 + n^2) \cdot (m^2 - n^2)^2 \]

For triangle \( FEA \):

\[ |FE| = y = K \cdot (m^2 - n^2) \cdot (2mn)^2, \quad |EA| = b - x = K \cdot (2mn)^3, \text{ and } h_2 = K \cdot (m^2 + n^2) \cdot (2mn)^2. \]

For the four congruent triangles \( FDC, DFE, DCE, CFE \):

\[ |DC| = |FE| = y = K \cdot (m^2 - n^2) \cdot (2mn)^2, \]

\[ |DF| = |CE| = x = K \cdot (2mn) \cdot (m^2 - n^2)^2, \]

and

\[ h = |CF| = |DE| = K \cdot (2mn) \cdot (m^2 - n^2) \cdot (m^2 + n^2) = K \cdot (2mn) (m^4 - n^4). \]

**Numerical Examples**

If we take \( K = 1 \) and \( mn \leq 4 \), then \( K = 1 \) and \( m = 2, n = 1 \); or \( K = 1 \) and \( m = 4, n = 1 \).

(a) \( K = 1, \ m = 2, \ n = 1 \). We obtain the following:
\[ d = 1 \cdot (2^2 + 1^2)^2 = 5^2 = 25, \ h = 60, \ h_1 = 45, \ h_2 = 80, \]
\[ y = 48, \ a - y = 75 - 48 = 27, \ x = 36, \ b - x = 100 - 36 = 64, \]
\[ a = 75, \ b = 100, \ c = 125 \]

(b) \( K = 1, \ m = 4, \ n = 1 \). We have the following:
\[ d = 289, \ a - y = 15, \ x = 1800, \ h_1 = 3825, \]
\[ y = 960, \ b - x = 512, \ h_2 = 1088, \ h = 1404 \]
\[ a = 4335, \ b = 2312, \ c = 4913 \]

5 Exactly \((d - 1)\) positions of \( P \)

Given a Pythagorean triangle \( CBA \), as in Figure 1, and with the point \( P \) on the hypotenuse \( BA \), and \( D \) and \( E \) being the perpendicular projections of \( P \) on the sides \( CB \) and \( CA \) respectively. We know from Theorem 1 that either both triangles \( BDP \) and \( PEA \) are Pythagorean, or neither of them are. The integer \( \delta \), as described in Theorem 1 must satisfy \( 1 \leq \delta \leq d - 1 \); which means that \( d \geq 2 \) is a necessary condition. There are \((d - 1)\) choices for \( \delta \). If we subdivide the hypotenuse \( BA \) into \( d \) equal length segments, each segment having length \( m^2 + n^2 \), it is easily seen that for each such position of the point \( P \) both triangles \( BDP \) and \( PEA \) are Pythagorean. There are exactly \((d - 1)\) such positions for the point \( P \) along the hypotenuse \( BA \). These are the points \( P_1, \ldots, P_{d-1} \); so that each of the consecutive line segments \( BP_1, P_1P_2, \ldots, P_{d-1}A \) (exactly \( d \) line segments) has length \( m^2 + n^2 \).

We postulate the following theorem.

**Theorem 6.** Let \( CBA \) be a Pythagorean triangle with the \( 90^\circ \) angle at the vertex \( C \). With sidelengths given by \( |CB| = a = d(m^2 - n^2), \ |CA| = b = d(2mn), \ |BA| = d(m^2 + n^2) \), where \( d, m, n \) are positive integers such that \( d \geq 2, \ m > n, \ (m, n) = 1, \) and \( m + n \equiv (\text{mod } 2) \). Also, let \( P_1, \ldots, P_{d-1} \) be the \((d - 1)\) points on the hypotenuse \( BA \) such that the \( d \) consecutive line segments \( BP_1, P_1P_2, \ldots, P_{d-1}A \) have equal lengths; each having length \( m^2 + n^2 \). Then there are exactly \((d - 1)\) points \( P \) on the hypotenuse \( BA \) such that both triangles \( BDP \) and \( PEA \) are Pythagorean where \( D \) and
are the feet of the perpendiculars from \( P \) to the sides \( CB \) and \( CA \) respectively. These \((d-1)\) points are precisely the points \( P_1, \ldots, P_{d-1} \) described above. Furthermore, each pair of Pythagorean triangles \( BD_iP_i \) and \( P_iEA \) have sidelengths given by
\[
|BD_i| = i \cdot (m^2 - n^2), \quad |D_iP_i| = i(2mn), \quad |BP_i| = i(m^2 + n^2), \quad |EA_i| = (d - i)(2mn), \quad |P_iA| = (d - i)(m^2 + n^2), \quad |P_iE_i| = (d - i)(m^2 - n^2), \quad \text{for } i = 1, \ldots, d - 1; \text{ and where } D_i \text{ and } E_i \text{ are the perpendicular projections of the point } P_i \text{ onto the sides } CB \text{ and } CA \text{ respectively.}
\]

6 Other cases

In this section, we explore the following question. If in addition to the two triangles in Figure 1, \( BDP \) and \( PEA \) being Pythagorean, we require that the four congruent triangles \( DCE, PEC, CDP, EPD \), also be Pythagorean. What are the necessary and sufficient conditions for this to occur?

For these four congruent triangles to be Pythagorean, the integers \( x = |DP| = |CE| \) and \( y = |DC| = |PE| \) must satisfy the condition,
\[
x^2 + y^2 = \text{ perfect square.}
\]

Combining this with Theorem 6 leads to the following theorem.

**Theorem 7.** Let \( CBA \) be a Pythagorean triangle with the 90 degree angle at the vertex \( C \); and with sidelengths, \( a = |CB| = d(m^2 - n^2), \quad b = |CA| = d(2mn), \quad c = |BA| = d(m^2 + n^2) \) where \( d, m, n \) are positive integers such that \( d \geq 2, \quad m > n, \quad (m, n) = 1, \quad \text{and } m + n \equiv 1(\text{mod } 2) \). Let \( P \) be a point on the hypotenuse \( BA \), and \( D \) and \( E \) be the feet of the perpendiculars from the point \( P \) onto the sides \( CB \) and \( CA \) respectively. Also, let \( x = |DP| = |CD|, \quad y = |DC| = |PE| \) so that
\[
a - y = |BD| \quad \text{and} \quad b - x = |EA|.
\]

Then, the two right triangles \( BDP \) and \( PEA \), as well as the four congruent triangles, \( DCE, PEC, CDP, EPD \), are all (six triangles) are Pythagorean if and only if there exist positive integers \( D, M, N \) such that
\[
M > N, \quad (M, N) = 1, \quad M + N \equiv 1(\text{mod } 2)
\]
and with either
The following example shows that there exist nonprimitive Pythagorean triangles such that there is no point $P$ on the hypotenuse $BA$ such that all six triangles $BDP$, $PEA$, $DCE$, $PEC$, $CDP$, $EPD$, are Pythagorean.

**Example:** Take $d = 5$, $m = 2$, $n = 1$. Then the sidelengths of triangle $CBA$ are $a = 5 \cdot (2^2 - 1^2) = 15$, $b = 5 \cdot (2 \cdot 2 \cdot 1) = 20$, and $c = 5 \cdot (2^2 + 1^2) = 25$. The possible values of the integer $\delta$ are $\delta = 1, 2, \ldots, d - 1 = 1, 2, 3, 4$. Using the formulas $y = \delta (m^2 - n^2) = D \cdot (M^2 - N^2)$ and $x = (d - \delta)(2mn) = D \cdot (M^2 - N^2)$ we have the following.

1. $\delta = 1 :$ $y = 3$, $x = (5 - 1) \cdot 4 = 16$
   and $y^2 + x^2 = 9 + 256 = 265$ not an integer square.

2. $\delta = 2$, $y = 2 \cdot 3 = 6$, $x = (5 - 2) \cdot 4 = 12$
   and $y^2 + x^2 = 36 + 144 = 180$, not a perfect square.

3. $\delta = 3$, $y = 3 \cdot 3 = 9$, $x = (5 - 3) \cdot 4 = 8$
   and $y^2 + x^2 = 81 + 64 = 145$, not an integer square.

4. $\delta = 4$, $y = 4 \cdot 3 = 12$, $x = (5 - 4) \cdot 4 = 4$
   and $y^2 + x^2 = 144 + 16 = 160$, not a perfect square.

There are many ways in which one can use the conditions (10i) or (10ii) of Theorem 7 in order to produce families of Pythagorean triangles such that each member (of those families) has the property that there is a point $P$ on
its hypotenuse such that all six triangles (as described in Theorem 7) are Pythagorean. We produce such a family.

**Family 1:** Consider (10i):

\[
\begin{align*}
y &= \delta (m^2 - n^2) = D (M^2 - N^2) \\
x &= (d - \delta)(2mn) = D(2MN)
\end{align*}
\]  

(10i)

Let $K$ be a positive integer.

Take $D = K \cdot mn (m^2 - n^2)$.

From the second equation in (10i) we obtain

\[d - \delta = K \cdot MN (m^2 - n^2)\]

and from the first equation in (10i) we get

\[\delta = Kmn (M^2 - N^2)\]

Hence, $d = \delta + KMN (m^2 - n^2) = K \cdot [mn (M^2 - N^2) + MN (m^2 - n^2)]$.

Obviously $1 \leq \delta \leq d - 1$ and $d \geq 2$, as required. We have the following.

**Family 1**

Let $m, n, M, N$ be positive integers such that $m > n$, $(m, n) = 1$, $m + n \equiv 1 \pmod{2}$, $M > N$, $(M, N) = 1, M+N \equiv 1 \pmod{2}$. Also, let $K$ be a positive integer and $\delta = Kmn (M^2 - N^2)$, $d = K \cdot [mn (M^2 - N^2) + MN (m^2 - n^2)]$.

Consider the Pythagorean triangle $CBA$ with sidelengths $|CB| = a = d (m^2 - n^2)$, $|CA| = b = d(2mn)$, $|BA| = c = d (m^2 + n^2)$. Let $P$ be the point on the hypotenuse $|BA|$ such that $|BP| = h_1 = \delta (m^2 + n^2)$; and let $D$ and $E$ be the perpendicular projections of $P$ onto the sides $CB$ and $CA$ respectively.

Then all six right triangles $BDP$, $PEA$, $DCE$, $PEC$, $CDP$ and $EPD$ are Pythagorean.

**References**

[1] Rosen, Kenneth H., *Elementary Number Theory and Its Applications*, Fifth edition (2005).
For Pythagorean triangles see pp. 510-515.
For Lemma 1, see page 109.

[2] Sierpinski, W., *Elementary Theory of Numbers*, original edition, Warsaw, Poland (1964).
ISBN: 0-568-52758-3, Elsevier Publishing (1988)
For Pythagorean triangles see pp. 38-42.
For Lemma 1, see page 14.
For Lemma 3, see page 15.

[3] Dickson, L. E., *History of the Theory of Numbers, Vol. II*, AMS Chelsea Publishing, Providence, RI (1992).
(unaltered textual reprint of the original book, first published by Carnegie Institute of Washington in 1919, 1920, and 1923.)
ISBN: 0-8218-1935-6
For Pythagorean triangles see pp. 165-190.