Vibration control of a piezoelectric cantilever smart beam by $\mathcal{L}_1$ adaptive control system

A. Ebrahimi-Tirtash, S. Mohajerin, M. R. Zakerzadeh and M. A. Nojoomian

School of Mechanical Engineering, University of Tehran, Tehran, Iran; School of Aerospace Engineering, Amirkabir University of Technology, Tehran, Iran

ABSTRACT
In this paper, the modelling and design of an $\mathcal{L}_1$ state feedback control system applied for vibration control of a smart piezoelectric flexible Euler–Bernoulli cantilever beam is presented. For a Single-Input Single-Output (SISO) case by retaining the first two dominant vibratory modes, the dynamics of the system is presented. Classical $\mathcal{L}_1$ adaptive control law, with time-varying parameters and in presence of disturbance, is employed to suppress the vibration of the beam. Three Piezoelectric patches, two as actuators and the other as a sensor, are bonded to the structure at the support of the beam and along the length of the beam. The beam structure is modelled in the state space form using the concept of piezoelectric theory, the Euler–Bernoulli beam theory and the Finite Element technique. Also, for comparing the performance of the proposed controller, PID and LQR control systems are applied. Simulation results, first for constant parameters and then for time-varying parameters and disturbance, represent the better performance of the proposed control system in comparison to the other mentioned controllers.

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1. Introduction
$\mathcal{L}_1$ adaptive control is capable to manipulate extensive classes of uncertain systems that include parametric and time-varying uncertainties, unmodelled dynamics and nonlinearities (Cao & Hovakimyan, 2006). In particular, by employing a low-pass filter in the adaptation laws, the $\mathcal{L}_1$ adaptive controller provide not only stability of the system (Cao & Hovakimyan, 2007) but also acceptable transient performance (Cao & Hovakimyan, 2008). The idea of the $\mathcal{L}_1$ adaptive-based controller is to guarantee a desired tracking to the reference command, while rejecting the disturbance and compensating for the system uncertainties. The major advantage of $\mathcal{L}_1$ adaptive control amid other adaptive control methods is that it detaches robustness and performance. This controller benefits a state predictor like indirect adaptive controllers; however, the control input is procured after the estimated control signal is filtered. The employment of the filter for producing the control signal renders the controller fulfill fast parameter adaptation and returns measurable performance bounds which make it distinct from the traditional adaptive control systems.

$\mathcal{L}_1$ adaptive control is apparent to have considerable capacities for application in aerospace systems (Dobrokhodov et al., 2013) due to the unmodelled dynamics and uncertainties of the system these systems. It has also applied in attitude control of satellites, UAVs and hover attitude control (Capello et al., 2013; Lee & Singh, 2014) because of its simple structure and minimum computation control effort. In the literature, the performance of $\mathcal{L}_1$ adaptive controller was also obtained from the analysis of an exoskeleton and applied on it (Rifai et al., 2006), shape memory alloy actuated flexible beam (Malinga & Buckner, 2015), cryogenic wind tunnel (Zhu et al., 2018).

Today, thin-walled structures are progressively applied in various fields due to their low weight. However, they have many negative effects, e.g. low damping and high vibration sensibility. Making the structure or beam smart employing piezoelectric patches is a solution to control the oscillation actively. Because of the dimension of the thickness, 1-dimensional FE models based on Euler–Bernoulli beam theory (Tzou & Chai, 2007) and Timoshenko beam theory (Narayanan & Balamurugan, 2003) are much more effective and common compared to 2D and 3D FE models. There have been several approaches for control of a piezoelectric smart beam. Nonetheless, most of these controllers presuppose accurate knowledge of the dynamic model. An overview of recent advances of the modelling and active vibration control of
a smart beam is explained in Ros et al. (2015). Concerning control algorithms for oscillation suppression of smart beams and structures, most of the published papers employed conventional linear control methods, e.g. negative velocity feedback control (Sun & Huang, 2001), traditional PID (Kumar et al., 2014; Zhang et al., 2015), fractional-order PID (Xu et al., 2018), LQR and LQG optimal control (Bruant et al., 2001; Zhang et al., 2010), $H_\infty$ robust control (Sahin & Aridogan, 2011), or nonlinear control methods such as sliding mode control (Itik et al., 2005) and fuzzy controller (Moradi et al., 2014), or MIMO algorithm (Zhu et al., 2012). Most of these papers merely contemplated the free vibration case which has no steady-state error. However, to control the vibration of a large dimensional system with distributed parameters, it is more preferable to design methods based on a small number of modes that use limited numbers of sensors and actuators. If high-frequency dynamics are neglected by modal truncation, the excitation of the residual modes may produce instability in the closed-loop system. Therefore, these residual modes should also be considered in the control design (Ha et al., 2018). Moreover, these systems usually have parametric uncertainties and may be disturbed by external disturbances. Thus, active vibration control in flexible structures should be robust to parametric uncertainties in both dynamics and external disturbances. Variable structure control (VSC) is interesting for nonlinear control of smart beams due to its simplicity and robustness to parametric uncertainties and external disturbances (Bai et al., 2011). Although VSC has many advantages, pure VSC has some disadvantages one of which is that the actuator has to cope with the control actions that could produce premature wear and tear.

All in all, the conventional controllers are not able to provide desired performance with minimum control action in different conditions with uncertainties and disturbance, which motivates the use of adaptive-based control system for performance improvement. Therefore, in this paper, due to its robustness properties, $L_1$ control system is developed and employed to act against both the free vibration and the steady-state error for the beam under a disturbance force and uncertainties. It is also guaranteed good transient response as well as improving disturbance rejection without the demand for exhaustive tuning.

This paper addresses the synthesis of an $L_1$ state-feedback controller based (Hovakimyan & Cao, 2010) for deflection control of an aluminum cantilever beam by means of bonded piezoelectric actuator and sensor. Since the controller is full state feedback, all states are obliged to be measured. The controller manipulates applied voltage, which causes strain in the piezo-actuator and as a result, bending the flexible beam to compensate model uncertainties and disturbance. A Finite Element (FEM) model based on Euler–Bernoulli beam theory is constructed. Two cases for simulations are considered, one for the system with unknown constant parameters and the other with time-varying parameters and disturbance. Simulation results for the oscillation control of the smart beam are obtained, and the results are compared with PID and Linear Quadratic Regulator (LQR) control systems. It is observed that the controller damps the oscillation of the beam, even with parameter uncertainties and disturbance force and has better performance in respect to the other mentioned control systems. Using an $L_1$ state-feedback controller based for suppression control of them beam by two actuators and a sensor and considering uncertainties and unmodelled dynamics and disturbances are the main contributions of this paper.

The main contribution of this study is threefold: (1) in comparison with the so called studies $L_1$ adaptive control has the fast adaptation nature which help the closed loop system response in existence of uncertainties or parameters’ variation of system; (2) $L_1$ adaptive control method helps a lot in proposing a practical controller by bounding the input and band-width of the controller effort; (3) $L_1$ adaptive control nature is to lessen the $L_1$ norm of the system’s output which improves the system response.

In what follows, the remaining part of the paper is constructed into five sections. Next section deals with FEM formulation of the piezoelectric smart beam. Then the design of the $L_1$ adaptive controller for the first two vibratory shape modes of the beam is detailed. Afterwards, the simulation results for two case studies is represented and finally, some concluding remarks are drawn in the last Section.

2. Mathematical modelling of the smart beam

Mathematical modelling considers the aluminum cantilever beam with collocated piezoelectric patches. In order to represent the beam dynamics, it is assumed to have two structural degrees of freedoms ($\omega, \theta$) at each nodal point which undergoes translational and rotational displacements, as shown in Figure 1 (Rao & Yap, 2011; Thomson, 2018). In addition, one electrical degree of freedom is considered at each nodal point to be used as the actuator or sensor voltage.

where $\omega_i$ are the transverse displacements of the centroidal axis of the beam (along $z$-axis) and $\theta_i$ are the rotation of the beam cross-section about $y$-axis.

The smart beam is modelled based on Euler–Bernoulli beam theory and is divided into 10 elements as shown in Figure 2, where subscripts ‘$b$’, ‘$p$’ and ‘$a$’ are used for beam element, PZT patch element, actuator and sensor, respectively and parameters $t$, $b$ and $l$ are the thickness, width
and length of each element or piezo patches and $L_b$ is the length of the beam (Kumar et al., 2014).

Each piezoelectric patches are bonded on only one element of the surface of the beam, as shown in Figure 2.

An external disturbing force $F_{Dist}$ is applied by the Actuator2, which exerted at the free end and $F_{Actu}$ is the force generated by the Actuator1 at the fixed end to control the deflection the beam.

### 2.1. Dynamic FEM model of the beam element

In the presented model, the effect of transverse shear forces on the beam deformation is ignored and the smart cantilever beam model is developed using piezoelectric beam elements. We considered one sensor and two actuators in the model dynamics and the remaining beam elements are considered as regular beam elements based on Euler–Bernoulli beam theory assumptions. Finally, all the elements are assembled using Finite Element analysis (Rao & Yap, 2011; Thomson, 2018).

The displacement of the beam along $x, y$ and $z$ axes can be written as:

\[
\begin{align*}
  u(x, y, z, t) & = -z \theta(x, t) = -z \left( \frac{\partial w}{\partial x} \right) \\
  v(x, y, z, t) & = 0 \\
  w(x, y, z, t) & = w(x, y)
\end{align*}
\]

(1)

where $w$ is the transverse displacement of the centroidal axis of the beam (along $z$-axis), $\theta$ and $v$ are the rotation of the beam cross-section and lateral displacement about $y$-axis and $u$ is the axial displacement along the $x$-axis.

In order to obtain a dynamic Finite Element model of the beams, Hamilton’s principle is usually applied, which is given by

\[
\delta \Pi = \int_{t_1}^{t_2} \left( \delta W_p - \delta W_k - \delta W_e \right) dt = 0
\]

\[
W_p = \frac{1}{2} \int_0^{L_b} E I \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx
\]

\[
W_k = \frac{1}{2} \int_0^{L_b} \rho A \left( \frac{\partial w}{\partial t} \right)^2 dx
\]

\[
W_e = \int_0^{L_b} w^T q_{Dist} dx
\]

(2)

where $W_p$, $W_k$ and $W_e$ are total strain energy, total kinetic energy of the beam and total work done due to the external forces, respectively and $q_{Dist}$ represents the disturbed force at the tip of the beam. We assume a cubic polynomial function in the expression for $w$, since there are four nodal variables for the beam element. Therefore, $w$ is approximated as:

\[
w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3
\]

\[
\theta = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2
\]

(3)

The boundary conditions at the ends of the beam element are as:

\[
x = 0 \rightarrow \begin{cases} w(x,t) = w_1 = \alpha_1 \\ w'(x,t) = \frac{\partial w}{\partial x} = \alpha_2 = \theta_1 \end{cases}
\]

\[
x = L_b \rightarrow \begin{cases} w(x,t) = w_2 \\ w'(x,t) = \theta_2 \end{cases}
\]

(4)

where $w_1, \theta_1$ and $w_2, \theta_2$ are the degree of freedom at node 1 and node 2 of the beam element. Application of
Substituting the coefficients \( \alpha_i \) in (3) and writing them in the matrix form, the transverse displacement can be obtained as:

\[
[w(x, t)] = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = [N_t] [q]
\]

where \( N \) gives the shape functions \( N_1(x) \) to \( N_4(x) \) of the beam element as:

\[
[N] = \begin{bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{bmatrix} = \begin{bmatrix} 1 - 3\frac{x^2}{l_b^2} + 2\frac{x^3}{l_b^3} \\ \frac{x^2}{l_b^2} - \frac{x^3}{l_b^3} \\ \frac{3x^2}{l_b^2} - 2\frac{x^3}{l_b^3} \\ -\frac{x^2}{l_b} + \frac{x^3}{l_b^2} \end{bmatrix}
\]

The vector \( q \) is called as the vector of displacements and slopes and is given by

\[
[q] = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}
\]

The first, second spatial derivatives and the time derivative of the displacement function \( w(x, t) \) in (3) are given by:

\[
[w'(x, t)] = \frac{\partial w}{\partial x} = [N'_1(x) N'_2(x) N'_3(x) N'_4(x)] \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = [N_q] [q]
\]

\[
[w''(x, t)] = \frac{\partial w}{\partial x} = [N''_1(x) N''_2(x) N''_3(x) N''_4(x)] \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = [N_q] [q]
\]

It is worth mentioning that, the adhesive used to bond the actuators to the beam is not considered in mass and stiffness of the smart beam element and the temperature effects are contemplated negligible.

### 2.2. Finite element modelling of piezoelectric beam element

Let us consider the beam element represented in Figure 2. Since the piezoelectric patches are very thin and light as compared to the thickness of the beam, the effect of the shear is negligible. Thus, the mass and stiffness matrices of the each piezo patches are the same as (11) with difference in subscript ‘b’ which is substituted by subscript
‘p’ for both actuator and sensor and are not shown for brevity.

The rotational inertia of the piezoelectric layer respect to the neutral axis of the beam, can be obtained as:

\[ l_p = \frac{1}{12} b_p t_p^3 + b_p t_p \left( \frac{t_p + t_b}{2} \right)^2 \]  
\[ t_p = t_a = t_b \quad \text{and} \quad b_p = b_b \]  

(12)

where \( t_b \) and \( t_p \) are the thickness of the piezoelectric and the thickness of the beam, and \( t_p \) is the width of the piezo patch which is considered equal to the width of the beam.

### 2.3. Dynamic equation

The mass and stiffness matrix for the piezoelectric beam element as a collocated pair is given by:

\[ [M] = [M_{b\text{-element}}] + 2[M_{\text{piezo}}]_{\text{actuator}} + [M_{\text{piezo}}]_{\text{sensor}} \]
\[ [K] = [K_{b\text{-element}}] + 2[K_{\text{piezo}}]_{\text{actuator}} + [K_{\text{piezo}}]_{\text{sensor}} \]

(13)

So, the global equation representing the dynamics of the total structure of the beam is obtained by combining all local matrices and vectors.

Also, it is necessary to include the system damping Matrix which is assumed to be linear with respect to the mass and stiffness matrices, so, a proportional damping in the Rayleigh form is considered for the present analysis:

\[ [C] = \rho [M] + \eta [K] \]  

(14)

where \( \rho \) and \( \eta \) are the frictional damping and the structural damping constants, respectively, determined from (Chowdhury & Dasgupta, 2003):

\[ \xi = \frac{\rho}{2\omega_i} + \frac{\eta\omega_i}{2}, \quad i = 1, 2 \]

(15)

where the damping ratio \( \xi \) is considered to be 0.3% for the first two modes and are the natural frequencies of each mode shape.

So the equation of motion of the entire structure is obtained as follows:

\[ [M] \{ \ddot{q} \} + [C] \{ \dot{q} \} + [K] \{ q \} = \{ F_{\text{Dist}} \} + \{ F_{\text{Actu}} \} \]

(16)

### 2.4. Piezoelectric constitutive equations

The material behaviour of the piezoelectric actuators can be modelled by the constitutive equations as follows involving two mechanical and two electrical variables

(Periasamy, 2008):

\[ \{ T \} = [c] \{ \varepsilon \} - [e]^T \{ E \} \]
\[ \{ D \} = [e] \{ \varepsilon \} + \{ \chi \} \{ E \} \]

(17)

with \( [e] = [c][d] \), which \( \{ T \}, \{ \varepsilon \}, \{ E \} \) and \( \{ D \} \) are the second Piola–Kirchhoff stress, Green–Lagrange strain, electric field, and electric displacement vectors, respectively. Besides, \([c]\) and \([\chi]\) denote the transformed elastic constitutive matrix and the transformed dielectric constants matrix, \([d]\) and \([e]\) are the transformed piezoelectric strain coefficient matrix and transformed piezoelectric stress coefficient matrix, respectively.

### 2.5. Piezoelectric actuator model

The actuator strain and the applied electric field are derived from the converse piezoelectric equation (Periasamy, 2008):

\[ T_x = E_p d_{31} E_z \]
\[ E_z = \frac{V_a(t)}{t_a} \]

(18)

where \( V_a(t) \) is the applied voltage across the thickness of the actuator and \( t_a \) is the thickness of the actuator layer, as shown in Figure 2. Also, \( E_p \) is the Young’s modulus of elasticity of the piezoelectric patches. The resultant moment \( M_a \) acting on the beam due to the stress is determined by integrating the stress through the structure thickness as:

\[ M_a = E_p d_{31} \bar{z} V_a(t) \]

(19)

where \( \bar{z} \) is the distance between the neutral axis of the beam and the piezoelectric layer. Finally, the applied force by the actuator is obtained as:

\[ \{ F_{\text{Actu}} \} = E_p d_{31} \bar{z} b_p \int_0^{t_p} N_0 V_a(t) \, dx \]

(20)

or can be represented as:

\[ \{ F_{\text{Actu}} \} = \{ P \} V_a(t) \]

(21)

where \( \{ P \} \) is a constant vector depends on the piezoelectric characteristics and its location on the beam.

### 2.6. Piezoelectric sensor model

For calculating the output charge on the sensor surface created by the strains in the beam, direct piezoelectric...
effect is used:
\[
Q(t) = \int_s e_{31} \varepsilon_x \, ds
\]  
(22)

The corresponding generated current is given by:
\[
i(t) = \frac{dQ(t)}{dt}
\]  
(23)

The total charge \(Q(t)\) generated on the sensor surface is the summation of all local charges generated on the sensor layer, so it can be shown as:
\[
i(t) = z e_{31} b \int_0^{b} N_a^T \frac{\partial q}{\partial x} \, dx
\]  
(24)

where \(N_a\) is the second spatial derivative of the mode shape function of the beam and \(z\) is maximum strain and it equals to:
\[
z = \frac{t_b}{2} + t_s
\]

The output voltage of the sensor is obtained as:
\[
V_s(t) = G_S \, i(t)
\]  
(25)

where \(G_S\) is the signal conditioning gain of the device:
\[
V_s(t) = G_S e_{31} b \int_0^{b} N_a^T \frac{\partial q}{\partial x} \, dx
\]  
(26)

which also can be expressed as:
\[
V_s(t) = \{g\}^T \{q\}
\]  
(27)

where \(\{g\}\) is a constant vector.

2.7. State space model of the smart structure

Since the dynamic of FE model has large number of degrees of freedom, a truncated modal matrix \(\{\psi\}\) is introduced to reduce the model order and as a result the computation cost is reduced. So, the nodal displacement vector \(\{q\}\) can be transformed to the reduced vector \(\{\kappa\}\) as (Zhang, 2014):
\[
\{q\} = [\psi] \{\kappa\}
\]  
(28)

Therefore, the new states can be measured from the previous displacement vectors \(\{q\}\) and the truncated Matrix which is defined as the first \(n\) modes shape matrix. By considering the first two bending modes’ vibration of the beam the truncated Matrix is formed by the eigenvectors of the first two modes. Thus, \(\{q\}\) the will be \((20 \times 1)\) displacement vector, \(\{\psi\}\) will be \((20 \times 2)\) truncated Matrix and \(\{\kappa\}\) will be the \((2 \times 1)\) modal coordinate vector.

We can obtain the decomposed equation of motion by substituting (21) into (16) and multiplying the equation by the transposed modal matrix as:
\[
[\mathbf{M}'] [\dot{\kappa}] + [\mathbf{C}] [\dot{\kappa}] + [\mathbf{K}] [\kappa] = \{\mathbf{F}_{\text{Dist}}\} + \{\mathbf{F}_{\text{Actu}}\}
\]  
(29)

where \([\mathbf{M}']\), \([\mathbf{C}]\) and \([\mathbf{K}]\) denotes the modal mass, damping and stiffness matrices, \(\{\mathbf{F}_{\text{Dist}}\}\) is the modal external force vector and \(\{\mathbf{F}_{\text{Actu}}\}\) is the modal control force vector.

Therefore state vector can be defined as
\[
\{x\} = \{\kappa \ \dot{\kappa}\}
\]  
(30)

Concerning smart structures, the system input is actuation voltages, and the system output can be sensor voltages or displacements.
\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) \\
V_s(t) = Dx(t)
\]  
(31)

Therefore, matrices of the state space equation for the smart beam can be derived as:
\[
\begin{bmatrix}
[\mathbf{A}] &=& \begin{bmatrix} [0] & [I] \\ \mathbf{M}'^{-1} & \mathbf{K}'^{-1} \end{bmatrix} \\
[\mathbf{B}] &=& \begin{bmatrix} [0] \\ \mathbf{M}'^{-1} \{\psi\}^T \{P\} \end{bmatrix} \\
[\mathbf{C}] &=& \begin{bmatrix} \{\psi\} & [0] \end{bmatrix} \\
[\mathbf{D}] &=& \begin{bmatrix} \{g\}^T \{\psi\} & [0] \end{bmatrix}
\end{bmatrix}
\]  
(32)

where, \([\mathbf{A}]\) denotes the system matrix, \([\mathbf{B}]\) is the control matrix, \([\mathbf{C}]\) is the displacement output matrix, \([\mathbf{D}]\) is the sensor output matrix.

3. \(\mathcal{L}_1\) adaptive control

Adaptive control systems have been developed to prevent degradation in the closed-loop behaviour of a controlled system. Indeed, the controller is expected to be adapted to the different kind of uncertainties, disturbances and changes in the system and environment. That is why classical adaptive controllers are employed in such systems. However, it is worth emphasizing that different challenges related to the implementation of a classical adaptive controller exist, such as bad transient behaviour, instability or slow convergence rate and priori knowledge of the system (Rifai et al., 2006). The \(\mathcal{L}_1\) adaptive control theory have guaranteed robustness and transient performance in the presence of fast adaptation, without enforcing persistence of excitation, and without any gain scheduling in the controller parameters. The architecture
of $L_1$ adaptive control system is represented in Figure 3 and it uses four key parts, inspired by Model Reference Adaptive Control (MRAC), that is the controlled system, the state predictor, the adaptation mechanism and the control law including a low-pass filter.

One main difference between $L_1$ adaptive control system and classical MRAC system is the existence of a low-pass filter for $L_1$ that enables the decoupling of control and adaptation. The filter endorses more adaptation rates while remaining the control signal in the bandwidth of the system actuator (Malinga & Buckner, 2015).

The state-feedback type of $L_1$ adaptive control system is considered in this research, therefore all states variables have to be measured, which may not be applicable in real systems in most of the cases.

### 3.1. $L_1$ control problem formulation

In this section, a full state feedback adaptive control system is designed based on the $L_1$ adaptive control theory (Hovakimyan & Cao, 2010), considering a typical class of systems with matched uncertainties introduced as

$$ x(t) = A_p x(t) + b \left( \omega u_p(t) + \theta^T x(t) + \sigma(t) \right), \quad y(t) = c^T x(t), \quad x(0) = x_0 $$

(33)

where $x(t) \in \mathbb{R}^n$ is the state vector which is supposed to be measurable, $u_p(t) \in \mathbb{R}$ is the control input, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ are known constant vectors, $A_p \in \mathbb{R}^{n \times n}$ is a known matrix such that the pair $(A_p, b)$ is controllable, $\theta$ is the vector of time-varying unknown parameters, $\sigma \in \mathbb{R}$ is a time-varying disturbance, while $\omega \in \mathbb{R}$ is unknown constant with known sign, $y(t) \in \mathbb{R}$ is the system’s output. By defining a piecewise-continuous bounded reference $r(t)$, the control goal will be the design of an adaptive state feedback input $u_p(t)$ so that the output $y(t)$ tracks $r(t)$ while the boundedness of the states and the parameters of the system are preserved.

This control input is assumed to compose of two parts defined according to the following:

$$ u_p(t) = u_l(t) + u_a(t) $$

(34)

where $u_l(t) = -K^T x(t)$ is the state feedback control law that lead the system to the desired performance with the lack of uncertainties. This linear controller is designed such that $A_p = A_m + b \omega K^T$ where $A_m \in \mathbb{R}^{n \times n}$ is a known Hurwitz matrix that describes the desired system. The adaptive control law $u_a$ will be described later. Thus, the system in (33) can be written as:

$$ \dot{x}(t) = A_m x(t) + b (\omega u_a(t) + \theta^T x(t) + \sigma(t)), \quad y(t) = c^T x(t), \quad x(0) = x_0 $$

(35)

For the system described in (35), the adaptive controller is made of a control laws, a state predictor and an adaptation law. Chang et al. (2018).

The state predictor is described by

$$ \dot{\hat{x}}(t) = A_m \hat{x}(t) + b (\hat{\omega} u_a(t) + \hat{\theta}^T x(t) + \hat{\sigma}(t)), \quad \hat{y}(t) = c^T \hat{x}(t), \quad \hat{x}(0) = x_0 $$

(36)

which has the same structure as the system in (35) where, $\hat{x}(t)$ is the state vector of the predictor and $\hat{\omega}(t), \hat{\theta}(t)$ and $\hat{\sigma}(t)$ are the estimations of the unknown parameters $\omega(t), \theta(t)$ and $\sigma(t)$ governed by adaptation law based on the projection operators [39] as follows and ensures the boundedness of the parametric estimates by definition (see [40] for details):

$$ \dot{\hat{\omega}}(t) = -\Gamma_\omega \text{Proj}(\hat{x}^T(t), P\hat{u}_a(t)), \quad \dot{\hat{\theta}}(t) = -\Gamma_\theta \text{Proj}(\hat{x}^T(t), P\hat{b}(t)), \quad \dot{\hat{\sigma}}(t) = -\Gamma_\sigma \text{Proj}(\hat{x}^T(t), P\hat{b}) $$

(37)

where $\hat{x}(t) = \hat{x}(t) - x(t)$ is the prediction error, $\Gamma_i \in \mathbb{R}^+$ are the adaptation gains, $P = P^T > 0$ is the solution of the algebraic Lyapunov equation $A_m P^T + PA_m = -Q$.
any arbitrary symmetric matrix $Q = Q^T > 0$. The adaptation gain is a design parameter and determine the rate at which the controller parameters will converge to the correct parameters. They are chosen so that the command signal remain reasonable. A too high adaptation gain may lead to badly damped behaviour while a too low adaptation gain will lead to an unacceptable slow response. In this study the adaptation gains are tuned manually by trial-and-error. As a result, the control law given by $L_1$ control system is:

$$u_a(s) = C(s) \left( kgr(s) - \hat{\eta}(s) \right)$$

(38)

where $\hat{\eta}(s)$ is the Laplace transformation of the term $\hat{\omega}u_a(t) + \hat{\theta}^\top x(t) + \hat{\sigma}(t)$, $C(s)$ is strictly proper stable filter as $C(s) = \omega k / (s + \omega k)$ with $C(0) = 1$ where $k > 0$ is a designed parameter and the static gain $kg$ is defined as $kg = -1 / (c^\top A_m^{-1} b)$.

Moreover, the following definition is defined:

$$L = \max_{\theta \in \Theta} \| \theta \|_1, \quad \| . \|_1 \text{ is } L_1 \text{- norm of } \theta,$$

(39)

$$H(s) = (sI - A_m)^{-1} b$$

$$G(s) = (1 - C(s)) H(s)$$

(40)

which means that $L$ is the maximal bound set on the parameter $\theta$ and it is supposed that $H(s)$, $G(s)$ and $C(s)$ are bounded-input and bounded-output stable transfer functions.

The $L_1$ adaptive control described by Equations (36), (37) and (38) is subject to the $L_1$-norm condition as follows:

$$\|G(s)\|_{L_1} L < 1$$

(41)

and it guarantees that the system is bounded-input and bounded-state stable with regard to the initial condition and reference signal.

**Lemma 3.1:** If $k$ verify the $L_1$-norm condition in (41), the closed-loop reference system in following is BIBS stable with respect to $r(t)$ and $x_0$.

$$x_{ref}(t) = A_m x_{ref}(t) + b(\omega u_{ref}(t) + \hat{\theta}^\top x_{ref}(t) + \sigma(t)),$$

$$x{ref}(0) = 0$$

$$u_{ref}(s) = \frac{C(s)}{\omega} \left( kgr(s) - \eta_{ref}(s) \right)$$

$$y_{ref}(t) = c^\top x_{ref}(t).$$

(42)

The proof of the Lemma is described in Hovakimyan and Cao (2010) which is ignored here in favour of brevity.

### 4. Simulation results

In this section, the simulation results is represented, to prove the performance of the proposed $L_1$ adaptive control system.

As described before, the first two dominant vibratory mode shapes of a clamped-free beam with piezoelectric patches bounded on the both surfaces at the cantilevered end of the beam as shown in Figure 4 are considered in the system state space model. The output of the system will be determined from the superposition of the both vibratory mode displacement according to the output matrix.

According to the second section a linear piezoelectric coupled FEM model is constructed based on the Euler–Bernoulli theory.

---

**Figure 4.** Cantilevered beam with bonded piezoelectric patches.
State space model of the beam with a sensor and actuators at the clamped end described by (32) is obtained by:

\[
A = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
-578.99 & 1.23 \times 10^{-8} \\
-1.25 \times 10^{-8} & -22023.13 \\
1 & 0 \\
0 & 1 \\
-0.14 & -4.34 \times 10^{-13} \\
-4.35 \times 10^{-13} & -0.89
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0.0001 \\
0.0007
\end{bmatrix}
\]

Table 1. Principal characteristics of the Aluminum beam.

| Parameter (symbol) | Value (unit) |
|--------------------|--------------|
| Length \((l_b)\) | 0.5 (m) |
| Length of each element \((l_b)\) | 0.05 (m) |
| Width \((b_b)\) | 0.05 (m) |
| Thickness \((t_b)\) | 0.001 (m) |
| Young’s modulus \((E_b)\) | 70 (Gpa) |
| Density \((\rho_b)\) | 2700 (kg/m³) |
| Poisson’s ratio \((\nu_b)\) | 0.3 |

Table 2. Principal characteristics of the piezoelectric.

| Parameter (symbol) | Value (unit) |
|--------------------|--------------|
| Length \((l_p)\) | 0.05 (m) |
| Thickness \((t_p)\) | 0.0005 (m) |
| Width \((b_p)\) | 0.05 (m) |
| Young’s modulus \((E_p)\) | 63 (Gpa) |
| Density \((\rho_p)\) | 7600 (kg/m³) |
| Poisson’s ratio \((\nu_p)\) | 0.3 |
| Transformed piezoelectric strain coefficient \((d_{31} = d_{32})\) | \(2.3 \times 10^{-12} (m/V)\) |
| Transformed dielectric constant \((\chi_{33})\) | \(1.5 \times 10^{-12} (F/m)\) |

The smart beam is made up of aluminum as the host structure and the three PZT patches as a sensor and actuators. The Actuator2 acts for the disturbance generation while, the Actuator1 one generates the bending moment as a result of input control voltage. The material characteristic of the beam and PZT patches are represented in Tables 1 and 2, respectively.

Two case studies are considered in this research. First, the \(L_1\) adaptive controller is implemented with constant unknown parameters and is compared with conventional PID and LQR control systems by considering initial displacement of the tip of the beam equals to 50 mm. Then in the second case, to consider parametric uncertainty and ignored high frequency dynamics and disturbance, the extended architecture of \(L_1\) controller with unknown time-varying parameters and also the disturbance applies on the beam.

**Case study I:**

In this case, (33) is reduced to \(\dot{x}(t) = A_p x(t) + b(u_p(t) + \theta^T x(t))\) such that, \(\theta(t)\) is a vector of constant unknown parameters with nominal amount of \([100; -120; 50; 400]\) and \(A_p, b\) and \(c\) are according to (33). The pair \((A_p, b)\) is also controllable.

\(A_p\) is a minimum-phase system with a right-half plane zero and the other in the left-half plane and two complex conjugate pairs of poles near the origin that correspond to slowly decaying components. The frequency responses of the system by considering sensor voltage and beam tip displacement as the output are represented in Figure 5. It shows that the first two natural frequencies of the cantilever smart beam are 3.83 and 23.62 Hz.

![Bode Diagram](image-url)

**Figure 5.** Bode diagrams of the system with sensor voltage as the output and the beam tip displacement as the output.
The desired model is selected as

\[
A_m = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-593.68 & 3267.54 & 0.12 & -1.35 \\
-101.54 & 558.82 & 1.82 & -10.22
\end{bmatrix}
\]

which it means that its poles are placed in \([-1.5, -1.6, -2, -3]\) that are more stable poles with respect to the plant model poles and therefore gives the desired performance. The adaptation rates \(\Gamma_i\) are considered as 1000, 500, 100, \(k_g\) can be computed as per \(k_g = -1 / (c^T A_m^{-1} b)\) and using the low-pass filter design procedure described in the previous section, it is derived as \(C(s) = 150/(s + 150)\) by considering \(\omega = 150\) and \(k = 1\) in the low-pass filter equation and the smart beam starts from an initial condition and shall be suppressed into its free condition.

To compare, the LQR optimal control system is developed from state space model of the system, which is described as

\[
\dot{x} = Ax + Bu \\
y = Cx \\
z = Gx
\]

\[\text{Figure 6.}\] The dynamic behaviour of the smart beam with unknown constant parameters and under the initial condition: (a) Displacement, (b) Sensor Voltage.
where $y$ is measured output corresponds to the states that are available for control and $z$ is controlled output that we would like to make as small as possible and in the shortest amount of time.

Considering all states can be measured, an optimized control gain can be procured by minimizing a cost function. The cost function can be described as the sum of the energy of the system input and the controlled output, which is expressed as

$$J_{LQR} = \int_0^\infty [z(t)^T Q z(t) + \rho u(t)^T R u(t)] \, dt$$  \hspace{1cm} (44)

where $Q$ and $R$ are symmetric positive-definite matrices which are approximated by Bryson's rule (Hespanha, 2018) as $Q_{ii} = \frac{1}{\max(|z_i|)}$, $R_{ii} = \frac{1}{\max(|u_i|)}$, and $\rho$ a positive constant to set up a compromise between the two contradictory objectives, diminishing the energy of the controlled output or the energy of control signal. The manipulated vector of the full state feedback LQR optimal control can be described as $u = -Kx$ where the control gain is given by $K = R^{-1}B^TP$ and the symmetric positive definite matrix $P$ is the solution of the Algebraic Riccati

Equation as follows:

$$A^T P + PA + Q - PB R^{-1} B^T P = 0$$  \hspace{1cm} (45)

Assuming sensor voltage as the measured output and displacement of the beam’s tip as the controlled output and also setting the maximum sensor voltage and actuator voltage to be 10 and 300 V the weighting matrices $Q$ and $R$ are obtained as Table 3 by using the procedure Bryson’s rule. In addition, the PID parameters are also demonstrated in Table 3.

The results are represented in Figures 6 and 7 in which the benefits of $\mathcal{L}_1$ controller can be observed in comparison with the PID and LQR control system. As in Figure 6(a) the tuned PID controller can only reduce the amplitude of the oscillation and cannot suppress the system in a reasonable time period, while the $\mathcal{L}_1$ in less than 5 sec is capable to control the oscillation with a desirable performance. Consequently, the sensor voltage produced by the $\mathcal{L}_1$ controller is reduced in a short period of time (Figure 6(b)). It is because of fast-adaptation characteristic of $\mathcal{L}_1$ adaptive controller, which the parameters converge faster and consequently the deflection of the beam is stabilized faster.

Figure 7 depicts the control effort of the control systems for the first case study, which shows better control action by the $\mathcal{L}_1$ adaptive rather than PID and LQR control systems.

**Case study II**:

In the second case study, all the situation is same as the previous case except for system formulation that (33) is employed in which $\theta$ is considered an unmatched component as a periodic signal with functionality as $\theta_i(t) = \phi_i \sin(10t + 0.1)$ which belongs to the compact

### Table 3. Parameters of LQR and PID Control Systems for the case study I.

| Parameters | Quantity |
|------------|----------|
| LQR Control | $Q$ | 0.01 |
| | $R$ | $5 \times 10^{-4}$ |
| | $\rho$ | 1 |
| PID Control | $P$ | 1 |
| | $I$ | 10 |
| | $D$ | 0.29 |

**Figure 7.** Control Signal for the first case study of the system with constant unknown parameters.
Figure 8. The dynamic behaviour of the smart beam with time-varying parameters and disturbance: (a) Displacement, (b) Sensor Voltage.

set [120,160] and, $\omega$ is an unknown constant $\sigma(t)$ is the step disturbance generated by the top piezoelectric actuator. Figure 8(a,b) represent the outputs of the system with time-varying sinusoidal parameter and step disturbance starts at the beginning up to the time 2 s. with the amplitude of 10 Volts when $L_1$ adaptive control is employed. The figure shows the effectiveness of the $L_1$ adaptive control even with unknown time-varying parameter in comparison with PID and LQR control system. It also depicts that the $L_1$ adaptive control is capable to cancel the time-varying disturbance quickly. It is because of fast adaptation of the unknown parameters by introducing high adaptation gain. It is worth mentioning that for employing LQR and PID control strategies Equation (33) is changed into the

| Parameters | Quantity  |
|------------|-----------|
| $Q$ LQR control | 0.005     |
| $R$ LQR control | $1 \times 10^{-4}$ |
| $\rho$ LQR control | 1         |
| $P$ PID control  | $5 \times 10^3$ |
| $I$ PID control  | $3 \times 10^2$ |
| $D$ PID control  | 10        |
Figure 9. Control effort for the system with time-varying parameters and disturbance.

following and the control parameters are demonstrated in Table 4.

\[
\begin{align*}
\dot{x}(t) &= (A_p - b \theta^T) x(t) + b \omega u_p(t) + b \sigma(t), \\
y(t) &= c^T x(t)
\end{align*}
\]

(46)

The control effort for the second case study represented in Figure 9 also shows this fact that despite time-varying parameters and disturbance, the actuation voltage is limited to 300 Volts and it decreases as the deflection of the beam damps.

5. Conclusion

Present paper deals with mathematical design and simulation of active vibration control of a piezoelectric smart beam via $\mathcal{L}_1$ adaptive control system by considering first two vibratory mode shapes of the structure. The control system consists of a state predictor and the filtered estimated control signal is synthesized using deflection of the beam as the feedback. Two controllers are designed for the two scenarios and various responses are obtained for each scenario. In the first case where the system is considered with unknown constant parameters, the $\mathcal{L}_1$ adaptive controller is compared with LQR and PID control schemes and the results represent the effectiveness of the $\mathcal{L}_1$ adaptive controller design. In the second case, the system extended into the time-varying one which is undergone a step disturbance. It is seen that, the $\mathcal{L}_1$ adaptive controller manages to give a robust response and a good performance in existence of uncertainties and disturbances, without any retuning, and without persistence of excitation.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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