ON NATURAL DEFORMATIONS OF SYMPLECTIC AUTOMORPHISMS OF MANIFOLDS OF $K3^{[n]}$ type

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Abstract. In the present paper we prove that finite symplectic groups of automorphisms of manifolds of $K3^{[n]}$ type can be obtained by deforming natural morphisms arising from $K3$ surfaces if and only if they satisfy a certain numerical condition.

Keywords: Symplectic Automorphisms, $K3^{[n]}$ type, Natural morphism

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1. Introduction

The present paper is devoted to a natural question concerning deformations of automorphisms of hyperkähler manifolds. Roughly speaking, given a $K3$ surface $S$ the group $\text{Aut}(S)$ induces automorphisms of the Hilbert scheme $S^{[n]}$ of $n$ points of $S$. These automorphisms are called natural. Let $X$ be a hyperkähler manifold deformation equivalent to some $S^{[n]}$ and let $G$ be a group of automorphisms of $X$. One can ask whether it is possible to deform $X$ together with $G$ to some $(S^{[n]},G)$, where $G$ is a group of natural automorphisms. In the following we give a positive answer for all finite symplectic automorphism groups whose action on $H^2(X)$ is the natural one and for several different dimensions (cf. Theorem 2.5). We remark that having the natural action on $H^2(X)$ is a necessary condition, since this action is constant under smooth deformations.

There have been several works concerning automorphisms of $K3$ surfaces, we will refer to the foundational work of Nikulin [11], later improved by Mukai [10] in the nonabelian case. By the work of Mukai [10] there are 79 possible finite groups of symplectic automorphisms on $K3$ surfaces and, by a recent classification due to Hashimoto [5], there are 84 different possibilities for their action on $H^2$. Our result holds for all these 84 cases as long as the hypothesis of the global Torelli theorem are satisfied.

In the case of manifolds of $K3^{[n]}$ type the notion of natural morphisms was introduced by Boissière [3] and further analyzed by him and Sarti [4]. In the particular case of symplectic involutions on manifolds of $K3^{[2]}$ type our result is proven in [9].

Notations. If $L$ is a lattice and $G \subset O(L)$ we denote by $T_G(L) := L^G$ the invariant sublattice and by $S_G(L) := T_G(L)^\perp$ the coinvariant sublattice. For $G \subset \text{Aut}(X)$ and $H^2(X,\mathbb{Z})$ endowed with a quadratic form, we denote $T_G(X) := T_G(H^2(X,\mathbb{Z}))$ the invariant sublattice and $S_G(X) := S_G(H^2(X,\mathbb{Z}))$ the coinvariant sublattice. Let $X$ be a hyperkähler manifold and let $G \subset \text{Aut}(X)$. The group $G$ is called symplectic if it acts trivially on $H^{2,0}(X)$, i.e. it preserves the symplectic form. We denote by $\text{Aut}_s(X)$ the subgroup of automorphisms of $X$ preserving the symplectic form. We will call manifolds of $K3^{[n]}$ type all manifold deformation equivalent to the Hilbert scheme of $n$ points on a $K3$ surface.
Preliminaries. In this section we gather some useful results for ease of reference. The reader interested in hyperkähler manifolds can consult [7] and [8] for further references and for a broader treatment of the subject.

A hyperkähler manifold is a simply connected compact Kähler manifold whose $H^{2,0}$ is generated by a symplectic form.

**Theorem 1.1.** Let $X$ be a hyperkähler manifold of dimension $2n$. Then there exists a canonically defined pairing $(, )_X$ on $H^2(X, \mathbb{C})$, the Beauville-Bogomolov pairing, which is a deformation and birational invariant. This form makes $H^2(X, \mathbb{Z})$ a lattice of signature $(3, b_2(X) - 3)$.

For every hyperkähler manifold $X$ and every Kähler class $\omega$ there exists a family of smooth deformations of $X$ over the base $\mathbb{P}^1$. This family is called *twistor family* and denoted $TW_\omega(X)$.

**Example 1.** Let $X$ be a hyperkähler manifold of $K3^{[n]}$ type. Then $H^2(X, \mathbb{Z})$ endowed with its Beauville-Bogomolov pairing is isomorphic to the lattice $L_n := H^2(K3, \mathbb{Z}) \oplus (2 - 2n)$.

If $X$ is hyperkähler we call a marking of $X$ any isometry between $H^2(X, \mathbb{Z})$ and a lattice $M$. There exists a moduli space of marked hyperkähler manifolds with $H^2(X, \mathbb{Z}) \cong M$ and we denote it by $M_M$.

We will often consider the induced action of $\text{Aut}(X)$ on $O(H^2(X, \mathbb{Z}))$ for a manifold $X$ of $K3^{[n]}$ type. For a general hyperkähler manifold this map might not be injective but in our case it is:

**Lemma 1.2.** Let $X$ be a manifold of $K3^{[n]}$ type. Then the map
\[
\nu(X) : \text{Aut}(X) \to O(H^2(X, \mathbb{Z}))
\]
is injective.

**Proof.** By [6, Theorem 2.1] the kernel of $\nu(X)$ is invariant under smooth deformations. Beauville [1, Lemma 3] proved that, if $S$ is a $K3$ surface with no non-trivial automorphisms, then $\text{Aut}(S^{[n]}) = \{Id\}$, therefore $\{Id\} = \text{Ker}(\nu(S^{[n]})) = \text{Ker}(\nu(X))$. □

The following is a very important theorem which is essential in the proof of our main result. The only truly restrictive hypothesis of Theorem 2.3 is one of the hypotheses of the following:

**Theorem 1.3** (Global Torelli, Verbitsky, Markman and Huybrechts). Let $X$ and $Y$ be two hyperkähler manifolds of $K3^{[n]}$ type and let $n - 1$ be a prime power. Suppose $\psi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is an isometry preserving the Hodge structure. Then there exists a birational map $\phi : X \dasharrow Y$.

Let $M$ be a lattice of signature $(3, r)$. We define $\Omega_M = \mathbb{P}(\{x \in M \otimes \mathbb{C} \mid x^2 = 0, (x, \mathcal{T}) > 0\})$ as the period domain for the lattice $M$. It is an open subset of a quadric hypersurface inside $\mathbb{P}(M \otimes \mathbb{C})$.

In the particular case where $M \cong H^2(X, \mathbb{Z})$ for some hyperkähler manifold $X$, there exists a natural map, the period map $\mathcal{P}$, between the moduli space $\mathcal{M}_M$ and the period domain $\Omega_M$.

Moreover, when Theorem 1.3 holds, two marked manifolds having the same period are birational.
The images of twistor families in \( \mathcal{M}_M \) through the period map are called twistor lines. A fundamental property of period domains is that they are connected by twistor lines (see [8, Proposition 3.7] or [2]).

2. DEFORMATIONS OF PAIRS

**Definition 2.1.** Let \( X \) be a manifold and let \( G \subset Aut(X) \). A \( G \) deformation of \( X \) (or a deformation of the pair \((X,G)\)) consists of the following data:

- A flat family \( \mathcal{X} \to B \), \( B \) connected and \( \mathcal{X} \) smooth, and a distinguished point \( 0 \in B \) such that \( \mathcal{X}_0 \cong X \).
- A faithful action of the group \( G \) on \( \mathcal{X} \) inducing fibrewise faithful actions of \( G \).

Two pairs \((X,G)\) and \((Y,H)\) are deformation equivalent if \((Y,H)\) lies in a \( G \) deformation of \( X \).

The first interesting remark is that, to some extent, all symplectic automorphism groups of a hyperkähler manifold can be deformed:

**Remark 1.** Let \( X \) be a hyperkähler manifold such that \( G \subset Aut_s(X) \) and \( |G| < \infty \). Let \( \omega \) be a \( G \) invariant Kähler class. Then \( TW_\omega(X) \) is a \( G \) deformation of \( X \) over \( \mathbb{P}^1 \).

There is also a notion of local universal \( G \) deformation, for a proof of its existence we refer to [9].

**Lemma 2.2.** Let \( X \) be a manifold of \( K3^{[n]} \) type and let \( G \subset Aut_s(X) \). Then there exists a universal local \( G \) deformation of \( X \) sitting inside \( Def(X) \). It is locally given by the \( G \)-invariant part of \( H^1(T_X) \) and it is of dimension \( \text{rank}(T_G(X)) - 2 \). Moreover two birational manifolds with isomorphic actions of \( G \) on cohomology have intersecting local \( G \)-deformations.

**Proof.** Let \( X \) be birational to \( Y \) and let the action of \( G \) on \( H^2(Y) \) coincide with the action of \( G \) on \( H^2(X) \) induced by the birational transformation between \( X \) and \( Y \). Let us take a representative \( U \) of \( Def(X) \) and let \( x \) be a very general point inside \( U^G \), which is a representative of the local \( G \) deformations of \( X \) and \( Y \). Let \( \mathcal{Y}_x \) and \( \mathcal{X}_x \) be the two hyperkähler manifolds corresponding to \( x \) on \( U^G \). We have \( Pic(\mathcal{Y}_x) = Pic(\mathcal{X}_x) = S_G(X) \) and \( \mathcal{Y}_x \) is birational to \( \mathcal{X}_x \). However any \( G \) invariant Kähler class on \( \mathcal{Y}_x \) is orthogonal to \( Pic(\mathcal{Y}_x) \) and therefore also to the set of effective curves on \( \mathcal{Y}_x \), which is therefore empty. Thus the Kähler cone of \( \mathcal{Y}_x \) coincides with the positive cone and \( \mathcal{Y}_x = \mathcal{X}_x \).

We remark that the local \( G \) deformations around two birational manifolds might not meet for a nonsymplectic group \( G \).

**Definition 2.3.** Let \( S \) be a K3 surface and let \( G \subset Aut_s(S) \) be a group of symplectic automorphisms on \( S \). \( G \) induces a group of symplectic morphisms on \( S^{[n]} \) which we still denote as \( G \). We call the pair \((S^{[n]},G)\) a natural pair, following [3]. We call standard any pair \((X,H)\) deformation equivalent to a natural pair.

A natural question is asking under which condition a pair \((X,G)\) is standard. In the rest of the paper we make the following assumption and we prove that it is equivalent to \((X,G)\) being standard.
Definition 2.4. Let $X$ be a manifold of $K3^{[n]}$ type and let $G \subset \text{Aut}_s(X)$. The group $G$ is numerically standard if the following holds

- $S_G(X) \cong S_H(S)$,
- $T_G(X) \cong T_H(S) \oplus (t)$,
- $t^2 = -2(n-1)$, $(t, H^2(X, \mathbb{Z})) = 2(n-1)\mathbb{Z}.$

For some $K3$ surface $S$ and some $H \subset \text{Aut}_s(S)$ such that $H \cong G$.

Notice that for a standard pair $(X, G)$ the group $G$ is numerically standard, since by [4] a natural pair is numerically standard. Now the main result of the paper can be explicitly stated:

Theorem 2.5. Let $X$ be a manifold of $K3^{[n]}$ type and let $n-1$ be a prime power. Let $G \subset \text{Aut}_s(X)$ be a finite group of numerically standard automorphisms. Then $(X, G)$ is a standard pair.

In this section we prove Theorem 2.5 using some properties of a particular period domain defined by the action of a finite group $G$ of symplectic automorphisms of a manifold $X$ of $K3^{[n]}$ type.

Definition 2.6. Let $M$ be a lattice of signature $(3, r)$ and let $G \subset \text{O}(M)$. We call $\Omega_{G,M}$ the set of points $\omega$ in the period domain $\Omega_M$ such that $\omega \in T_G(M) \otimes \mathbb{C}$.

Definition 2.7. Let $\mathcal{M}_n := \mathcal{M}_{L_n}$ be the moduli space of marked manifolds of $K3^{[n]}$ type and let $G \subset \text{Aut}_s(X)$ for some marked $(X, f) \in \mathcal{M}_n$. Let us denote with $G$ the group of symmetries induced by $G$ on the lattice $L_n$ and let $\Omega_{G,n} := \Omega_{G,L_n}$ be as above. Then we define $\mathcal{M}_{G,n} \subset \mathcal{M}_n$ as the counterimage through the period map of $\Omega_{G,n}$.

By the following remark the set $\mathcal{M}_{G,n}$ is the set of marked pairs $(X, f)$ such that $f^{-1}(S_G(L_n)) \subset \text{Pic}(X)$ for an appropriate marking $f$ and $\Omega_{G,n}$ is just the period domain $\Omega_{T_G(L_n)}$.

Remark 2. Let $X$ be a hyperkähler manifold and let $G \subset \text{Aut}_s(X)$ be a finite group. Then $T_G(X)$ contains $T(X)$ and $S_G(X) \subset \text{Pic}(X)$. Moreover $T_G(X)$ has signature $(3, r)$ for some $r \geq 0$. A proof of this fact can be found in [1, Proposition 6].

This means that, through a chain of twistor families, we can connect any marked point $(X, f) \in \mathcal{M}_{G,n}$ with $G \subset \text{Aut}_s(X)$ numerically standard to a marked point $(Y, g)$ that has the same period of a natural pair $(S^{[n]}, G)$ for an appropriate marking $f'$ of $S^{[n]}$. Since by Remark 1 twistor families are $G$ deformations, we have that $(X, G)$ and $(Y, G)$ are deformation equivalent.

Proof of Theorem 2.5. Let $X$ be a manifold of $K3^{[n]}$ type and let $n-1$ be a prime power. Let $G \subset \text{Aut}_s(X)$ be a finite numerically standard group of symplectic automorphisms. Since $\Omega_{G,n}$ is connected by twistor lines, $(X, G)$ is deformation equivalent to $(Y, G)$ and $\mathcal{P}(Y, f) = \mathcal{P}(S^{[n]}, f') \in \Omega_{G,n}$. Here $S$ is a $K3$ surface with $G \subset \text{Aut}_s(S)$ and $\text{Pic}(S) = S_G(S)$, i.e. the very general $K3$ surface with $G \subset \text{Aut}_s(S)$. By Theorem 1.3 there is a birational map $\phi$ between $Y$ and $S^{[n]}$ which gives an induced action of $G$ on $S^{[n]}$ (possibly nonregular). Let us denote by $H$ the group induced on $S^{[n]}$ by $\phi$ and let us keep calling $G$ the group induced by the automorphisms of $S$. We obtain our claim by proving that $H = G$ (as actions on $S^{[n]}$), since in that case $(Y, G)$ and $(S^{[n]}, H)$ would be deformation equivalent through their local universal $G$-deformations.
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Notice that, by the assumption on the numerical standardness, the actions of $G$ and $H$ already coincide on $H^2(S^{[n]}, \mathbb{Z})$. Let now $g \in G$ and let $h$ be the element of $H$ such that $g^* = h^* in H^2(S^{[n]}, \mathbb{Z})$. Let $r$ be the order of $g$. Then $g \circ h^{r-1}$ induces the identity on $H^2(S^{[n]}, \mathbb{Z})$. Therefore, by Lemma 1.2, $g^{-1}h^{r-1}$, which implies $G = H$ as group of automorphisms of $S^{[n]}$.

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