Hill-type formula for Hamiltonian system with Lagrangian boundary conditions

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Abstract

In this paper, we build up Hill-type formula for linear Hamiltonian systems with Lagrangian boundary conditions, which include standard Neumann, Dirichlet boundary conditions. Such a kind of boundary conditions comes from the brake symmetry periodic orbits in \(n\)-body problem naturally. The Hill-type formula connects the infinite determinant of the Hessian of the action functional with the determinant of matrices which depend on the monodromy matrix and boundary conditions. Consequently, we derive the Krein-type trace formula and give nontrivial estimation for the eigenvalue problem. Combined with the Maslov-type index theory, we give some new stability criteria for the brake symmetry periodic solutions of Hamiltonian systems. As an application, we study the linear stability of elliptic relative equilibria in planar 3-body problem.

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1 Introduction

In the present paper, we will study the eigenvalue problems of Hamiltonian systems with Lagrangian boundary conditions. Let \(J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\), the standard symplectic structure \(\omega(x,y)\) on \(\mathbb{R}^{2n}\) is defined by

\[
\omega(x,y) = \langle Jx, y \rangle.
\]

A Lagrangian subspace \(V\) of \((\mathbb{R}^{2n}, \omega)\) is an isotropic subspace of dimension \(n\), that is, for any \(x, y \in V\), \(\omega(x,y) = 0\). Denote by \(\text{Lag}(2n)\) the set of Lagrangian subspaces of \(\mathbb{R}^{2n}\). For \(V_0, V_1 \in \text{Lag}(2n)\), the eigenvalue

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problem of linear Hamiltonian system with Lagrangian boundary condition is to find \( \lambda \in \mathbb{C} \) satisfying that the nontrivial solutions of the following system exist,

\[
\dot{z}(t) = J (B(t) + \lambda D(t)) z(t),
\]

\[
x(0) \in V_0, \quad x(T) \in V_1,
\]

where \( B, D \in C([0, T]; S(2n)) \), the set of continuous paths of symmetric matrices. \( B \) and \( D \) can be considered as bounded operators on \( \mathcal{H} := L^2([0, T]; \mathbb{C}^{2n}) \), defined by \( (Bx)(t) = B(t)x(t) \) and \( (Dx)(t) = D(t)x(t) \).

Unlike the periodic boundary conditions, the Lagrangian boundary conditions are separated boundary conditions which include the Dirichlet, Neumann. However, periodic boundary conditions are closely related to Lagrangian boundary conditions. In fact, we will see that, for a periodic solution with brake symmetry, it is natural to obtain the Lagrangian boundary condition. Readers are referred to [10] and references therein to find the background of brake symmetry in \( n \)-body problem.

Denote by \( A|_E = -J \frac{d}{dt} \) which is densely defined on \( \mathcal{H} \) with the domain \( E \). Obviously, the property of \( A|_E \) depends on its domain \( E \) heavily. The choice of \( E \) is based on the boundary condition. For instance, when we consider the Lagrangian boundary problem \( (1.1)-(1.2) \), the domain \( E_{V_0,V_1} \) is defined as

\[
E_{V_0,V_1} = \left\{ z(t) \in W^{1,2}([0, T]; \mathbb{C}^{2n}) \mid z(0) \in V_0, \ z(T) \in V_1 \right\}.
\]

It is well known that for \( \lambda \in \rho(A|_E) \), the resolvent \( (A|_E - \lambda)^{-1} \) is not a trace class operator, but a Hilbert-Schmidt operator. Generally, in the case that \( A|_E - B \) is non-degenerate, to simplify the notation, we set

\[
\mathcal{F}(B, D; E) = D (A|_E - B)^{-1},
\]

and without of confusion, we will write

\[
A = A|_{E_{V_0,V_1}}, \quad \mathcal{F}(B, D)(= \mathcal{F}) = \mathcal{F}(B, D; E_{V_0,V_1}).
\]

Similar to the resolvent \( (A|_E - \lambda)^{-1} \), in general \( \mathcal{F}(B, D; E) \) is not a trace class operator, but a Hilbert-Schmidt operator. It follows that the Fredholm determinant \( \det(I - \mathcal{F}(B, D; E)) \) is not well-defined, instead we will use the definition of conditional Fredholm determinant, which was introduced in [14]. Some details will be recalled in Section 2.1. It is worth to be pointed out that there is another way to define the infinite dimensional determinant, which is defined by zeta function [26, 6].

The first study of Hill formula was given by G.Hill in [8] when he studied the motion of of lunar perigee, the strict mathematical proof of Hill’s formula was given by H. Poincaré [25]. There are many efforts on Hill-type formula were done, such as [2, 3, 4, 14, 15, 16] and references therein. To state Hill-type formula, we need some notations. Suppose \( \Lambda \in \text{Lag}(2n) \), a Lagrangian frame for \( \Lambda \) is a linear map \( Z : \mathbb{R}^n \to \mathbb{R}^{2n} \) whose image is \( \Lambda \). It is easy to see that the frame is of the form \( Z = \begin{pmatrix} X & Y \end{pmatrix} \), where \( X, Y \) are \( n \times n \) matrices and satisfied

\[
X^T Y = Y^T X.
\]

Denote \( \gamma_{\Lambda}(t) \) the fundamental solution of \( (1.1) \), that is \( \dot{\gamma}_{\Lambda}(t) = J (B + \lambda D) \gamma_{\Lambda}(t) \) with \( \gamma_{\Lambda}(0) = I_{2n} \). Let

\[
\text{Sp}(2n) := \{ M \in \text{GL}(\mathbb{R}^{2n}) \mid M^T J M = J \}
\]

be the symplectic group, it is well known that \( \gamma(t) \in \text{Sp}(2n) \). Let \( Z_0, Z_1 \) be frames of \( \Lambda_0, \Lambda_1 \). Obviously, \( \gamma_{\Lambda}(T) Z_0 \) are frames of \( \gamma_{\Lambda}(T) \Lambda_0 \) and \( \gamma_{\Lambda}(T) Z_0, Z_1 \) are \( 2n \times 2n \) matrices. We have the following Hill-type formula for Hamiltonian system \( (1.1)-(1.2) \).
Theorem 1.1. Assume that $A - B$ is non-degenerate, then
\[ \det(I - F(B, D)) = \det(\gamma_1(T)^{-1} Z_0, Z_1) = \det(\gamma_0(T)^{-1} Z_0, Z_1), \]
where the left side is the conditional Fredholm determinant, and the right side is independent on the choice of the frames $Z_0, Z_1$.

Remark 1.2. In [16], Hu and Wang obtained Hill-type formula for Sturm-Liouville system with separated boundary conditions. It should be pointed out that, there is essential difference between [16, Theorem 1.1] and Theorem 1.1. More precisely, the differential operators in [16, Theorem 1.1] has trace class resolvent, and hence the classical Fredholm determinant can be defined. Therefore, in [16] the techniques from complex analysis can be used, however such techniques do not work well here. Theorem 1.1 will be proved by using the theory of integral operators.

As mentioned above, by brake symmetry, solutions of Lagrangian boundary problem is closely related to that of $S$-periodic boundary problem, where $S \in \text{Sp}(2n) \cap \text{O}(2n)$ (the symplectic orthogonal group). More precisely, we consider the following eigenvalue problem with $S$-periodic conditions
\[ \dot{z}(t) = J(B(t) + \lambda D(t))z(t), \quad z(0) = S z(T). \]

In [14], the following Hill-type formula for $S$-boundary conditions was obtained
\[ \det(I - F(B, D; E_S)) = \det(S \gamma_1(T) - I) \cdot \det(S \gamma_0(T) - I)^{-1}, \]
where
\[ E_S = \{ z \in W^{1,2}([0, T], \mathbb{R}^{2n}) | z(0) = S z(T) \}. \]

Suppose there exists $N \in \text{O}(2n)$ such that $N^2 = I_{2n}, NJ = -JN$ and $NS^T = SN$. Then we define
\[ g : E \to E, \quad z(t) \mapsto N z(T - t) \]
which generates a $\mathbb{Z}_2$ group action on $E_S$. Obviously, $A|_{E_S} g = g A|_{E_S}$. Suppose
\[ NB(T - t) = B(t)N, \quad ND(T - t) = D(t)N, \]
than $Bg = gB$. Therefore
\[ (A|_{E_S} - B - \lambda D)g = g (A|_{E_S} - B - \lambda D), \quad \text{for} \quad \lambda \in \mathbb{R}. \]

Let $V^+(SN), V^+(N)$ and $V^-(SN), V^-(N)$ be the positive and negative definite subspaces of $SN$ and $N$ respectively, then $V^+(SN)$ and $V^+(N)$ are all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$.

Let
\[ E^+_S = \{ z \in E_S, gz = \pm z \}, \]
which are isomorphic to
\[ E^+_S = \{ z \in W^{1,2}([0, T/2], \mathbb{C}^{2n}), z(0) \in V^+(SN), z(T/2) \in V^+(N) \}. \]

We have the following decomposition formula, which build the relationship between the Hill-type formula of $S$-periodic boundary problem and that of Lagrangian boundary problem.
**Theorem 1.3.** Under the above conditions, we have

\[
\det(I - \mathcal{F}(B, D; E_S)) = \det(I - \mathcal{F}(B, D; E_S^1)) \cdot \det(I - \mathcal{F}(B, D; E_S^2)).
\]

By the similar idea to [13], using \(\lambda \mathcal{F}(B, D)\) instead of \(\mathcal{F}(B, D)\), and taking Taylor expansion on \(\lambda\) for both sides of Hill-type formula [13], we have the trace formula. Trace formula is a powerful tool in studying the eigenvalue problem, and hence it is very useful to study the stability problem. The trace formula was first established by Krein in 1950's [23, 24]. Recently, Hu and Wang give the generalization to the Sturm-Liouville system with general separated boundary conditions [14]. In this paper, we will build up the Krein-type trace formula for Hamiltonian systems with Lagrangian boundary condition. Please refer subsection 3.2 for the detail.

The motivation of Krein’s trace formula was to study the stability problem of periodic orbits. Based on trace formula for \(S\)-periodic orbits and Maslov index theory [19], we give some new stability criteria and apply it to study the stability of \(n\)-body problem [13]. As continuous study, we use the trace formula for Hamiltonian system with Lagrangian boundary conditions to obtain stability criteria for the brake symmetry periodic orbits. Consequently, we give some applications on the study of planar 3-body problem.

It is well known that a planar central configuration of the \(n\)-body problem gives rise to solutions where each particle moves on a specific Keplerian orbit while the totality of the particles move on a homographic motion. Follows Meyer and Schmidt [20], we call this solution **elliptic relative equilibria** (ERE for short) if the Keplerian orbit is elliptic. For \(n = 3\), it is well known that there are only two kind of central configurations, the Lagrangian equilateral triangular central configuration and the Euler collinear central configurations. We call the corresponding ERE are elliptic Lagrangian orbits and elliptic Euler orbits because they are first discovered by Lagrange [17] and Euler [7].

There are many works in the study of linear stability of elliptic Lagrangian orbits and elliptic Euler orbits. Please refer to [9] [13] [20] [21] [22] [29] [27] and references therein for the details. More precisely, let \(e \in (0, 1)\) be the eccentricity of the homothety Keplerian orbits of ERE. The linear stability of elliptic Lagrangian orbits depends on \(e\) and \(\beta \in [0, 9]\) where

\[
\beta = \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{(m_1 + m_2 + m_3)^2}.
\]

Similarly, the elliptic Euler orbits depend \(e\) and \(\delta\), where \(\delta \in [0, 7]\) only depends on masses \(m_1, m_2, m_3\).

In [13], the first nontrivial estimation of the stability region and hyperbolic region of the \(e, \beta\) rectangle \([0, 1) \times [0, 9]\) for elliptic Lagrangian orbits was given. In Section 6, by observing the elliptic Lagrangian orbits with brake symmetry, and we give a better estimation on the stability region by using the trace formula for Hamiltonian system with Lagrangian boundary conditions. Moreover, the first nontrivial estimation of the hyperbolic region for elliptic Euler orbits will be given.

The paper is organized as follows. Section 2 is devoted to preliminaries on conditional Fredholm determinant and conditional trace. In Section 3, we will prove the Hill-type formula by using techniques in integral operators and complex analysis. Based on the Hill-type formula, we get the trace formula. In Section 4 deal with the brake symmetry decomposition for Hill formula. We give some new stability criteria by the trace formula in Section 5. At last, in Section 6, we give new estimation for the stability region of elliptic Lagrangian orbits and estimation for hyperbolic region of elliptic Euler orbits.
2 Preliminaries

In this section, we mainly introduce some fundamental notations and results which will be used later. In Subsection 2.1, we give an overview on conditional Fredholm determinant, details could be found in [14]. In Subsection 2.2, we compute the conditional trace \( F(B, D) \).

2.1 Conditional Fredholm determinant

In this subsection, we will mainly consider the conditional Fredholm determinant. As we have seen, the conditional Fredholm determinants of dynamical systems are our starting point to derive our trace formula.

Let \( J_\infty \) denote the family of compact operators. For \( F \in J_\infty \), let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \) be the singular values of \( A \). In fact, \( \mu_j \) are eigenvalue of \( |F| : = (F^*F)^{1/2} \). For \( p \geq 1 \), we denote \( J_p : = \{ F \in J_\infty \mid \sum \mu_n(F)^p < \infty \} \), and \( \|F\|_p : = (\sum \mu_n(F)^p)^{1/p} \) for \( F \in J_p \). Obviously, \( J_1 \) is the set of trace class operator and \( J_2 \) is the set of Hilbert-Schmidt operators. It is well known that for \( F \in J_1 \), the Fredholm determinant det \((id + F)\) is well defined. However, for \( F \in J_2 \setminus J_1 \), such a Fredholm determinant can not be well-defined. Instead, the regularized determinant

\[
\det_2(id + F) = \det \left( (id + F)e^{-F} \right)
\]

is well-defined since \((id + F)e^{-F} - id \in J_1\). It is known that the regularized determinant has no multiplicative property. Thus we hope to define a kind of conditional Fredholm determinant for \((id + F)\).

The concept of trace finite condition plays an important role in the study of conditional Fredholm determinant. Firstly, we will recall the definition of trace finite condition, which is introduced in [14]. Let \( \{P_k\} \) be a sequence of finite rank projections, such that the following conditions are satisfied,

1. for \( k \leq m \), \( \text{Range}(P_k) \subseteq \text{Range}(P_m) \),
2. \( P_k \) converges to \( id \) in the strong operator topology.

We denote

\[
\mathcal{J}(P_k) : = \{ F \in J_2 \mid \lim_{k \to \infty} Tr(P_kFP_k) \text{ exists and is finite} \}
\]

be the set of operators with trace finite condition respect to \( \{P_k\} \).

It is obvious that \( \mathcal{J}(P_k) \) is linear space and

\[
J_1 \subset J(P_k) \subset J_2.
\]

As been pointed in [15], if \( F \in J(P_k) \), then the conditional Fredholm determinant can be defined,

\[
\det(id + F) = \lim_{k \to \infty} \det(id + P_kFP_k) \\
= \lim_{k \to \infty} \det_2(id + P_kFP_k)e^{Tr(P_kFP_k)} \\
= \det_2(id + F) \lim_{k \to \infty} e^{Tr(P_kFP_k)}. \tag{2.1}
\]
By [15], if \( F \in \mathcal{F}_1 \), then conditional Fredholm determinant \( \det(id + F) \) is the classical Fredholm determinant. Moreover for \( F \in \mathcal{F}_{\infty} \), the function \( \det(id + \alpha F) \) is analytic on \( \alpha \). Correspondingly, in [15] Lemma 3.3, we proved that the conditional Fredholm determinant \( \det \left( (A - B - \alpha D - \nu J)(A + P_0)^{-1} \right) \) is analytic on \( \alpha \). Similarly, it is not hard to see that for \( F \in \mathcal{F}(P_k) \), the function \( \det(id + \alpha F) \), defined by conditional Fredholm determinant, is an entire function.

At the end of this subsection, we will list some fundamental properties of the conditional Fredholm determinant, which were proved in [14].

**Proposition 2.1.**  
(i) If \( F_1, F_2 \in \mathcal{F}(P_k) \), then 
\[
\det((id + F_1)(id + F_2)) = \det(id + F_1)\det(id + F_2).
\]

(ii) \( F \in \mathcal{F}_1 \). Let \( E = E_1 \oplus E_2 \), and \( F_i \in \mathcal{F}(p_{k_i}^{(i)}) \) for \( i = 1, 2 \). Let \( F = F_1 \oplus F_2 \), then \( F \in \mathcal{F}(p_{k_1}^{(1)} \oplus p_{k_2}^{(2)}) \), and 
\[
\det(id + F) = \det(id_{E_1} + F_1)\det(id_{E_2} + F_2),
\]

where \( id_{E_i} \) are identities on \( E_i \), for \( i = 1, 2 \).

### 2.2 Conditional trace for operator with Lagrangian subspace boundary conditions

Suppose \( V_0, V_1 \in \text{Lag}(2n) \). By Changing a symplectic basis, we may assume \( V_0, V_1 \) with Lagrangian frames 
\[
Z_0 = (I_n, 0_n)^T, \quad Z_1 = (C(\theta), S(\theta))^T,
\]
where for \(-\pi/2 < \theta < \pi/2 \),
\[
C(\theta) = \text{diag}(\cos(\theta), \cdots, \cos(\theta_n)), \quad S(\theta) = \text{diag}(\sin(\theta), \cdots, \sin(\theta_n)).
\]

Then \( A \) is a self-adjoint operator on \( \mathcal{H} \) with domain \( E(V_0, V_1) \); moreover, \( A \) has compact resolvent and only has point spectrum. Easy computation shows

\[
\sigma_p(A) = \{ \lambda_{j,k} | \lambda_{j,k} = \theta_j/T + k\pi/T, \ j = 1, 2 \cdots, n, k \in \mathbb{Z} \},
\]

with the corresponding eigenvectors \( e_{j,k} = e^{1\lambda_{j,k}t}e_j = \cos(\lambda_{j,k}t)e_j + \sin(\lambda_{j,k}t)e_{n+j} \), where \( e_j \) is the standard \( j \)-th basis of \( \mathbb{R}^{2n} \). In what follows, let \( P_N \) be the projections from \( \mathcal{H} \) to span \( \{ e_{j,k}; |k| \leq N \} \).

**Proposition 2.2.** (i) For any \( \nu \in \mathbb{C} \) such that \( A - \nu \) is invertible, \( \mathcal{F}(\nu, D) \in \mathcal{F}(P_N) \) and
\[
Tr\mathcal{F}(\nu, D) = \sum_{j=1}^{n} \frac{\cos(\theta_j - T\nu)}{\sin(\theta_j - T\nu)} \int_{0}^{T} (e^{-\nu Jt}De^{\nu Jt}e_j, e_j)dt + \sum_{j=1}^{n} \int_{0}^{T} (e^{-\nu Jt}De^{\nu Jt}e_{n+j}, e_j)dt.
\]

(ii) \( \mathcal{F}(B, D) \in \mathcal{F}(P_N) \) and hence the conditional Fredholm determinant \( \det(I - \mathcal{F}(B, D)) \) is well defined.

Since \( \mathcal{F}(\nu, D) - \mathcal{F}(B, D) = \mathcal{F}(\nu - B)(A - B)^{-1} \in \mathcal{F}_1 \), part ii) follows from part i) easily. The remaining part of this subsection is devote to the proof of part i), which is technical. Readers may skip it on the first glance.

Set \( \hat{D} = \frac{1}{2}(D - JDJ), \hat{D} = \frac{1}{2}(D + JDJ) \), then obviously \( D = \hat{D} + D, \) and \( \hat{D}J = JD, \hat{D}J = -JD. \) Therefore
\[
e^{Jt}\hat{D} = De^{Jt}, \quad e^{Jt}\hat{D} = De^{-Jt}.
\]
Obviously,

\[ \text{Tr} f (v, D) = \text{Tr} F (v, \hat{D}) + \text{Tr} F (v, \check{D}), \]

and \( \text{Tr} F (v, \hat{D}) \) and \( \text{Tr} F (v, \check{D}) \) will be computed separately. Throughout the paper, in order to simplify the notations, the summation \( \sum_k \) always means \( \lim_{N \to \infty} \sum_{|k| \leq N} \).

**Lemma 2.3.** For any \( \nu \in \mathbb{C} \) such that \( A - \nu \) is invertible,

\[ \text{Tr} f (v, \hat{D}) = \sum_{j=1}^{N} \frac{\cos(\theta_j - T \nu)}{\sin(\theta_j - T \nu)} \int_{0}^{T} (\hat{D}e_j, e_j) dt, \]

**Proof.** By the definition

\[
\text{Tr} F (v, \hat{D}) = \frac{1}{T} \sum_{j=1}^{N} \sum_{k} \langle \hat{D} \omega_{\lambda_{j,k}}^{T} e_j, e_{\lambda_{j,k}}^{T} e_j \rangle \frac{1}{\lambda_{j,k} - \nu} \\
= \frac{1}{T} \sum_{j=1}^{N} \int_{0}^{T} (\hat{D}e_j, e_j) dt \sum_{k} \frac{1}{k \pi / T + \theta_j / T - \nu} \\
= \sum_{j=1}^{N} \frac{\cos(\theta_j - T \nu)}{\sin(\theta_j - T \nu)} \int_{0}^{T} (\hat{D}e_j, e_j) dt,
\]

where the third equality is from the following identity [13, Lemma 2.7]

\[
\lim_{N \to \infty} \sum_{|k| \leq N} \frac{1}{\nu + k \pi / T} = T \frac{\cos(T \nu)}{\sin(T \nu)}. \\
\]

The following lemma is needed to compute \( \text{Tr} F (v, \check{D}) \).

**Lemma 2.4.** For \( f \in C[0, T] \), we have

\[
\int_{0}^{T} f(t) \sum_{k} \frac{\cos(2(k \pi / T + \theta_j / T - \nu) t)}{k \pi / T + \theta_j / T - \nu} dt = T \int_{0}^{T} f(t) dt \frac{\cos(\theta_j - v T)}{\sin(\theta_j - v T)} , \\
\int_{0}^{T} f(t) \sum_{k} \frac{\sin(2(k \pi / T + \theta_j / T - \nu) t)}{k \pi / T + \theta_j / T - \nu} dt = T \int_{0}^{T} f(t) dt.
\]

**Proof.** For \( f \in C[0, T] \), set \( F(t) = \int_{0}^{t} f(s) ds \). Notice that

\[
\sum_{|k| \leq N} \frac{\cos(2(\lambda_{j,k} - \nu) t)}{\lambda_{j,k} - \nu} = \frac{\cos(2(\theta_j / T - \nu) t)}{\theta_j / T - \nu} + \sum_{k=1}^{N} \left( \frac{\cos(2(\lambda_{j,k} - \nu) t)}{\lambda_{j,k} - \nu} + \frac{\cos(2(\lambda_{j,-k} - \nu) t)}{\lambda_{j,-k} - \nu} \right),
\]

which converges uniformly. By integration by part, we have

\[
\int_{0}^{T} f \cdot \sum_{k} \frac{\cos(2(\lambda_{j,k} - \nu) t)}{\lambda_{j,k} - \nu} dt = F(T) \sum_{k} \frac{\cos(2(\lambda_{j,k} - \nu) T)}{\lambda_{j,k} - \nu} + 2 \int_{0}^{T} F(t) \sum_{k} \sin(2(\lambda_{j,k} - \nu) t) dt.
\]

\[ \square \]
For the first part of the above equality
\[
\sum_k \frac{\cos(2(\lambda_{jk} - \nu)T)}{\lambda_{jk} - \nu} = \sum_k \frac{\cos(2\theta_j - 2\nu T)}{k\pi/T + \theta_j/T - \nu}
= T \cos(2\theta_j - 2\nu T) \frac{1 + \cos(2\theta_j - 2\nu T)}{\sin(2\theta_j - 2\nu T)}.
\]

For the second part of (2.4)
\[
\int_0^T F(t) \sum_k \sin(2(\lambda_{jk} - \nu)t)dt
= \int_0^T F(t) \sin((2\theta_j/T - 2\nu)t) \sum_k \cos \frac{2k\pi t}{T} dt + \int_0^T F(t) \cos((2\theta_j/T - 2\nu)t) \sum_k \sin \frac{2k\pi t}{T} dt
= 2 \int_0^T F(t) \sin((2\theta_j/T - 2\nu)t) \sum_{k \in \mathbb{N}} \cos \frac{2k\pi t}{T} dt + \int_0^T F(t) \sin((2\theta_j/T - 2\nu)t)dt
= \frac{T}{2} F(T) \sin(2\theta_j - 2\nu T),
= \frac{T}{2} \int_0^T f(t)dt \cdot \sin(2\theta_j - 2\nu T),
\]
where the third equality is from the Fejer Theorem for Fourier series. Thus we have
\[
\int_0^T f(t) \sum_k \frac{\cos(2(\lambda_{jk} - \nu)t)}{\lambda_{jk} - \nu}dt
= T \int_0^T f(t)dt \cdot \cos(2\theta_j - 2\nu T) \frac{1 + \cos(2\theta_j - 2\nu T)}{\sin(2\theta_j - 2\nu T)} + \sin(2\theta_j - 2\nu T)
= T \int_0^T f(t)dt \cdot \frac{1 + \cos(2\theta_j - 2\nu T)}{\sin(2\theta_j - 2\nu T)}.
\]

Similarly
\[
\int_0^T f \cdot \sum_k \frac{\sin(2(\lambda_{jk} - \nu)t)}{\lambda_{jk} - \nu} dt
= \int_0^T f(t)dt \sum_k \frac{\sin(2(\lambda_{jk} - \nu)t)}{\lambda_{jk} - \nu} - 2 \int_0^T F(t) \sum_k \cos(2(\lambda_{jk} - \nu)t)dt,
\]
and
\[
\sum_k \frac{\sin(2(\lambda_{jk} - \nu)t)}{\lambda_{jk} - \nu} = \sum_k \frac{\sin(2\theta_j - 2\nu T)}{k\pi/T + \theta_j/T - \nu} = T(1 + \cos(2\theta_j - 2\nu T)).
\]

As above, we have
\[
\int_0^T F(t) \sum_k \cos(2(\lambda_{jk} - \nu)t)dt = \int_0^T F(t) \cos((2\theta_j/T - 2\nu)t) \sum_k \cos \frac{2k\pi t}{T} dt
- \int_0^T F(t) \sin((2\theta_j/T - 2\nu)t) \sum_k \sin \frac{2k\pi t}{T} dt
= \frac{T}{2} (F(0) + F(T) \cos(2\theta_j - 2\nu T))
= \frac{T}{2} \int_0^T f(t) dt \cos(2\theta_j - 2\nu T).
\]
We have
\[ \int_0^T f \cdot \sum_k \frac{\sin(2(\lambda_{jk} - \nu)t)}{\lambda_{jk} - \nu} dt = T \int_0^T f(t) dt. \]

\[ \square \]

**Lemma 2.5.** For any \( \nu \in \mathbb{C} \) such that \( A - \nu \) is invertible, then \( \mathcal{F}(\nu, \tilde{D}) \in \mathcal{J}(P_N) \) and the conditional trace
\[
Tr \mathcal{F}(\nu, \tilde{D}) = \sum_{j=1}^n \frac{\cos(\theta_j - T\nu)}{\sin(\theta_j - T\nu)} \int_0^T (e^{-\nu Jt} \tilde{D} e^{-\nu Jt} e_j, e_j) dt + \sum_{j=1}^n \int_0^T (e^{-\nu Jt} \tilde{D} e^{\nu Jt} e_{n+j}, e_j) dt.
\]

**Proof.** By the definition
\[
Tr \mathcal{F}(\nu, \tilde{D}) = \frac{1}{T} \sum_{j=1}^n \left\langle \tilde{D} e^{\lambda_{jk} Jt} e_j, e_j \right\rangle \frac{1}{\lambda_{jk} - \nu} = \frac{1}{T} \sum_{j=1}^n \sum_k \int_0^T (e^{-\nu Jt} \tilde{D} e^{2(\kappa/T + \theta_j/T - \nu) Jt} \kappa/T + \theta_j/T - \nu e_j, e_j) dt.
\]

Please note that
\[
\int_0^T (e^{-\nu Jt} \tilde{D} e^{2(\kappa/T + \theta_j/T - \nu) Jt} \kappa/T + \theta_j/T - \nu e_j, e_j) dt = \int_0^T (e^{-\nu Jt} \tilde{D} e_j, e_j) \frac{\cos[2(\kappa/T + \theta_j/T - \nu)t]}{\kappa/T + \theta_j/T - \nu} dt + \int_0^T (e^{-\nu Jt} \tilde{D} e_{n+j}, e_j) \frac{\sin[2(\kappa/T + \theta_j/T - \nu)t]}{\kappa/T + \theta_j/T - \nu} dt.
\]

By (2.2, 2.3) and \( e^{-\nu Jt} \tilde{D} = \tilde{D} e^{Jt} \), the right of (2.5) equals to
\[
\sum_{j=1}^n \frac{\cos(\theta_j - T\nu)}{\sin(\theta_j - T\nu)} \int_0^T (e^{-\nu Jt} \tilde{D} e_j, e_j) dt + \sum_{j=1}^n \int_0^T (e^{-\nu Jt} \tilde{D} e^{\nu Jt} e_{n+j}, e_j) dt,
\]

this is end of the proof. \( \square \)

Please note that \( \sum_{j=1}^n \int_0^T (e^{-\nu Jt} \tilde{D} e^{\nu Jt} e_{n+j}, e_j) dt = 0 \). Part i) of Proposition 2.2 comes from Lemma 2.3 and Lemma 2.5 directly.

**3 Hill-type formula for Hamiltonian systems with Lagrangian boundary conditions**

This section is the main part of our paper, we build up the Hill-type formula in Subsection 3.1 and the trace formula is obtained in Subsection 3.2. At last, we discuss the relationship between the the eigenvalue problem for Hamiltonian system with that of Sturm-Liouville systems in Section 3.3.
3.1 Hill-type formula

Let $\gamma_\alpha$ be the fundamental solution of $B + \alpha D$, that is

$$\dot{\gamma}_\alpha(t) = J(B(t) + \alpha D(t))\gamma_\alpha(t), \quad \gamma(0) = I_{2n}.$$ 

Assume $A - B$ is nondegenerate which is obvious equivalent to $V_0 \triangleleft \gamma_0^{-1}(T)V_1$. We will express $F(B, D)$ by integral operator. Let $Q$ be the unique idempotent matrix on $\mathbb{R}^{2n}$ with kernel $V_0$ and image $\gamma_0^{-1}(T)V_1$, in general $Q$ is not orthogonal. The integral kernel of $F(B, D)$ could be given by

$$K_F(t, t') = \begin{cases} 
JD\gamma_0(t)Q\gamma_0^{-1}(t'), & 0 \leq t' < t; \\
-JD\gamma_0(t)(I_{2n} - Q)\gamma_0^{-1}(t'), & t \leq t' < T.
\end{cases}$$

That means

$$F(B, D)f(t) = \int_0^T K_F(t, t')f(t')dt'.$$

In general, the kernel $K_F$ is not continuous but in $L^2$. However, we may define the trace formally from the viewpoint of integral operators

**Definition 3.1.** For $K_F$, we define

$$TrK_F = Tr \int_0^T JD\gamma_0(t)Q\gamma_0^{-1}(t)dt.$$ 

At first, we will show the following lemma.

**Lemma 3.2.** For any $\beta \in [0, 1]$,

$$TrK_F = Tr \int_0^T [\beta JD\gamma_0(t)Q\gamma_0^{-1}(t) - (1 - \beta)JD\gamma_0(t)(I_{2n} - Q)\gamma_0^{-1}(t)]dt.$$ 

**Proof.** Since $D$ is a path of symmetric matrices, it follows that

$$Tr \int_0^T -JDdt = 0,$$

Hence

$$Tr \int_0^T JD\gamma_0(t)Q\gamma_0^{-1}(t)dt = -Tr \int_0^T JD\gamma_0(t)(I_{2n} - Q)\gamma_0^{-1}(t)dt.$$ 

The proof is complete. \(\square\)

To simply the notation, we let

$$Q_d = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}, \quad Q_n = \begin{pmatrix} 0_n & 0_n \\ 0_n & I_n \end{pmatrix}.$$ 

In the following Lemma 3.3 and Corollary 3.4, we will show that the conditional trace defined in Subsection 2.2 coincides with the formally defined trace in Definition 3.1.

**Lemma 3.3.** For the case $B = \nu I_{2n}$,

$$TrF(\nu, D) = TrK_F.$$
From [13], we have
\[ P := (Z_0, \gamma^{-1}_0(T)Z_1) = \begin{pmatrix} I_n & C(\theta - \nu T) \\ 0_n & S(\theta - \nu T) \end{pmatrix}. \]
Hence
\[ Q = P Q_0 P^{-1} = \begin{pmatrix} 0_n & C(\theta - \nu T) S(\theta - \nu T)^{-1} \\ 0_n & I_n \end{pmatrix}. \]
By definition,
\[ TrK_F = Tr \int_0^T JD\gamma_0(t)Q\gamma^{-1}_0(t)dt \]
\[ = Tr \int_0^T e^{-\nu Jt} D e^{\nu Jt} Q dt \]
\[ = \sum_{j=1}^n \cos(\theta - \nu T) \int_0^T (e^{-\nu Jt} D e^{\nu Jt} e_j, e_j) dt + \sum_{j=1}^n \int_0^T (e^{-\nu Jt} D e^{\nu Jt} e_{n+j}, e_j) dt. \]
Combining with Corollary [27.2] we have the result. \[ \square \]

**Corollary 3.4.** For any \( B \) such that \( A - B \) is invertible, we have
\[ TrF(B, D) = TrK_F. \]

**Proof.** Please note that \( F(B, D) - F(v, D) \in \mathcal{F}. \) We have, \( \mathcal{K}_{F(B, D)} - \mathcal{K}_{F(v, D)} \) is the integral kernel of a trace class operator. By [5, P.244, Theorem 2.1], the kernel \( \mathcal{K}_{F(B, D)} - \mathcal{K}_{F(v, D)} \) is continuous, and hence by [5, P.70, Theorem 8.1],
\[ Tr(F(B, D) - F(v, D)) = Tr \int_0^T (\mathcal{K}_{F(B, D)} - \mathcal{K}_{F(v, D)})(s, s) ds = Tr(\mathcal{K}_{F(B, D)} - \mathcal{K}_{F(v, D)}). \]
The result is from Lemma [3.3]. \[ \square \]

Let \( M(\alpha) = (\gamma_0(T)Z_0, Z_1) \). In the remaining part of this section, we will let
\[ f(\alpha) = \det M(\alpha) \det M^{-1}(0). \] (3.1)
Direct computation shows that
\[ \gamma^{-1}_0(T)M(\alpha) = (\gamma^{-1}_0(T)\gamma_0(T)Z_0, \gamma^{-1}_0(T)Z_1) \]
\[ = P + ((\gamma^{-1}_0(T)\gamma_0(T) - I_{2n})Z_0, 0_{2n \times n}) \]
\[ = P(I_{2n} + P^{-1}(\gamma^{-1}_0(T)\gamma_0(T) - I_{2n})Q_d). \]
From [13], we have \( \frac{d}{d\alpha}(\gamma^{-1}_0(T)\gamma_0(T) - I_{2n}) \big|_{\alpha=0} = \int_0^T \gamma^{-1}_0(t)JD\gamma_0(t)dt. \) Then
\[ f'(0) = TrP^{-1} \int_0^T \gamma^{-1}_0(t)JD\gamma_0(t) dt Q_d \]
\[ = Tr \int_0^T \gamma^{-1}_0(t)JD\gamma_0(t) dt (I - Q), \]
where the second equality from the fact that \( I - Q = PQ_dP^{-1}, PQ_d = Q_d. \) By Lemma [3.2] we have
Lemma 3.5. For any $B$ such that $A - B$ is invertible,

$$\text{Tr} \mathcal{F}(B, D) = \text{Tr} \mathcal{K}_\mathcal{F} = -f'(0).$$

From the above discussion, we have

Theorem 3.6. For any $B$ such that $A - B$ is invertible,

$$\det(I - \alpha \mathcal{F}(B, D)) = f(\alpha). \quad (3.2)$$

Proof. Let $g(\alpha) = \det(I - \alpha \mathcal{F}(B, D))$, then $f$ and $g$ are analytic functions on $\mathbb{C}$ with same zero points and $f(0) = g(0) = 1$. We will show that for $f(\alpha) \neq 0$,

$$g'(\alpha)g^{-1}(\alpha) = f'(\alpha)f^{-1}(\alpha), \quad (3.3)$$

which implies (3.2). For any $\alpha_0$ with $f(\alpha_0) \neq 0$, we have

$$g(\alpha) = \det(I - (\alpha - \alpha_0)\mathcal{F}(B - \alpha_0D, D))g(\alpha_0),$$

so $g'(\alpha_0)g^{-1}(\alpha_0) = -\text{Tr} \mathcal{F}(B - \alpha D, D)$. On the other hand,

$$f(\alpha) = \det M(\alpha) \det M^{-1}(\alpha_0)f(\alpha_0).$$

Using $B + \alpha D$ instead of $B$ in Corollary 3.4 we get

$$-\text{Tr} \mathcal{F}(B - \alpha D, D) = \frac{d}{d\alpha} \det M(\alpha) \det M^{-1}(\alpha_0)|_{\alpha = \alpha_0} = f'(\alpha)f^{-1}(\alpha),$$

and hence the equality (3.3) is proved. □

Remark 3.7. The Hill-type formula (3.2) shows that $f(\alpha)$ is independent on the choice of the frames of $V_0$ and $V_1$. In fact, we can get this by easy computation of (3.1).

3.2 Krein-type trace formula

Since $\mathcal{F} = \mathcal{F}(B, D) \in \mathcal{F}(P_N)$, we have

$$\det(I - \alpha \mathcal{F}) = \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \alpha^m \text{Tr}(F^m)\right) \quad (3.4)$$

Next, we will give the expansion of $f(\alpha)$. Recall that $P = (Z_0, \gamma_0^{-1}(T)Z_1)$, $Q_d = (Z_0, 0_{2n \times n})$. Noting that $M(0) = \gamma_0(T)(Z_0, \gamma_0^{-1}(T)Z_1) = \gamma_0(T)P$, and $\det(\gamma_0(T)) = 1$, we have

$$f(\alpha) = \det\left(P(I_{2n} + P^{-1}(\gamma_0^{-1}(T)\gamma_0(T) - I_{2n})Q_d)\right) \det(P^{-1})$$

$$= \det\left(I_{2n} + P^{-1}(\gamma_0^{-1}(T)\gamma_0(T) - I_{2n})Q_d\right).$$

Let $\hat{\gamma}_0(T) = \gamma_0^{-1}(T)\gamma_0(T)$, by [13] Section 2.2,

$$\hat{\gamma}_0(T) - I_{2n} = \sum_{j=1}^{\infty} \alpha^j M_j,$$
where for $\hat{D}(t) = \gamma_0^T(t)D(t)\gamma_0(t)$,

$$M_j = \int_0^T J\hat{D}(t_1) \int_0^{t_1} J\hat{D}(t_2) \cdots \int_0^{t_{j-1}} J\hat{D}(t_j) dt_j \cdots dt_2 dt_1.$$  

Let $G_j = P^{-1}M_jQ_d$, we have that

$$f(\alpha) = \det(I_{2n} + \sum_{j=1}^{\infty} \alpha^j G_j).$$

Since $f(\alpha)$ vanishes nowhere near 0, we can write $f(\alpha) = \epsilon^g(\alpha)$, then by [13, Formula 2.6], we have

$$g^{(m)}(0)/m! = \sum_{j=1}^{m} \frac{(-1)^k+1}{k} \left( \sum_{j_1+\cdots+j_k=m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right). \tag{3.5}$$

Combining (3.4) and (3.5) with Theorem 3.6, we have the following trace formula.

**Theorem 3.8.** With the above notations, we have that

$$\text{Tr}(F^m) = m \sum_{k=1}^{m} \frac{(-1)^k}{k} \left( \sum_{j_1+\cdots+j_k=m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right).$$

For $m = 1, 2$, the trace formula is simple

$$\text{Tr}F = -\text{Tr}(G_1), \quad \text{Tr}(F^2) = \text{Tr}(G_1^2) - 2\text{Tr}(G_2). \tag{3.6}$$

Since for $m \geq 2$, $F(B, D)$ is trace class operator, we have

$$\text{Tr}(F^m) = \sum_j \lambda_j^{-m},$$

where each $\lambda_j$ appears as many times as its multiplicity.

### 3.3 Relation with the eigenvalue problem of Sturm-Liouville systems

In [14], the Hill-type formula and Krein-type trace formula were given for Sturm-Liouville system. In this subsection, we will study the relationship between the formulas for general Hamiltonian systems and that for Sturm-Liouville systems. When the Hamiltonian system comes from the Legendre transformation of Sturm-Liouville systems, the operator $F(B, D) \in \mathcal{F}_1$, then

$$\det(I - F) = \prod_j (1 - \lambda_j^{-1}), \tag{3.7}$$

$$\text{Tr}(F) = \sum_j \lambda_j^{-1}, \tag{3.8}$$

where $\lambda_j$ are eigenvalues the corresponding Sturm-Liouville systems:

$$-(P\dot{y} + Qy) + Q^T \dot{y} + (R + \lambda R_1)y = 0, \tag{3.9}$$

where $Q$ is a continuous path of $n \times n$ matrices, and $P, R, R_1$ are continuous paths of $n \times n$ symmetric matrices on $[0, T]$. Instead of Legendre convexity condition, we assume that for any $t \in [0, T]$, $P(t)$ is invertible. The
boundary condition given in follows: let \( \lambda_0, \lambda_1 \in \text{Lag}(2n) \), which are phase spaces with standard symplectic structure. Set \( x = P \dot{y} + Q y, z = (x, y)^T \), and the boundary condition is given by

\[
\begin{align*}
    z(0) &\in \Lambda_0, \quad z(T) \in \Lambda_1.
\end{align*}
\]

By the standard Legendre transformation, the linear system (3.9) with the boundary conditions (3.10) corresponds to the linear Hamiltonian system,

\[
\dot{z} = JB_d(t)z, \quad z(0) \in \Lambda_0, \quad z(T) \in \Lambda_1,
\]

with

\[
B_d(t) = \begin{pmatrix}
    P^{-1}(t) & -P^{-1}Q(t) \\
    -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t) - R(t) - \bar{\lambda} R_1(t)
\end{pmatrix}.
\]

We denote \( \gamma_d(t) \) the fundamental solution of (3.11). Let \( Z_0, Z_1 \) be frames of \( \Lambda_0, \Lambda_1 \). To simplify the notation, set \( \mathcal{A} = -\frac{d}{dt}(P \frac{d}{dt} + Q) + Q^T \frac{d}{dt} + R \), which is a self-adjoint operator on \( L^2([0, T], \mathbb{R}^n) \) with domain:

\[
D(\Lambda_0, \Lambda_1) = \{ y \in W^{2,2}([0, T], \mathbb{R}^n), z(0) \in \Lambda_0, z(T) \in \Lambda_1 \}.
\]

We assume \( \mathcal{A} \) is nondegenerate, that is, 0 is not an eigenvalue of (3.9). It is obvious that \( \lambda \) is a nonzero eigenvalue of the systems (3.9) if and only if \(-1/\lambda\) is an eigenvalue of \( R_1 \mathcal{A}^{-1} \). In what follows, the multiplicity of an eigenvalue \( \lambda_j \) means the algebraic multiplicity of \( R_1 \mathcal{A}^{-1} \) at \(-1/\lambda_j\). It was proven in [16] that

\[
\prod_j (1 - \lambda_j^{-1}) = \det(I + R_1 \mathcal{A}^{-1}) = \det(\gamma_1(T)Z_0, Z_1) \cdot \det(\gamma_0(T)Z_0, Z_1)^{-1},
\]

Let \( D = \text{diag}(0_n, R_1) \), then Hill formula (3.2) shows that

\[
\det(I + \mathcal{F}(B_0, D)) = \det(\gamma_1(T)Z_0, Z_1) \cdot \det(\gamma_0(T)Z_0, Z_1)^{-1},
\]

we have

**Corollary 3.9.** Under the above notations

\[
\det(I + \mathcal{F}(B_0, D)) = \det(I + R_1 \mathcal{A}^{-1}),
\]

consequently, \( \sigma(\mathcal{F}(B_0, D)) = \sigma(R_1 \mathcal{A}^{-1}) \) with the same multiplicity.

**Proof.** Please note that (3.14) follows from (3.12) and (3.13) directly. In (3.14), let \( \lambda R_1 \) take place of \( R_1 \), and we have

\[
\det(I + \lambda \mathcal{F}(B_0, D)) = \det(I + \lambda R_1 \mathcal{A}^{-1}).
\]

This shows that \( \sigma(\mathcal{F}(B_0, D)) = \sigma(R_1 \mathcal{A}^{-1}) \). Moreover, by [14] Theorem 3.5, for \( \zeta \in \mathcal{F}(P_N) \), \( \lambda_0 \) is an eigenvalue of \( \zeta \) of algebraic multiplicity \( k \) if and only if \(-\lambda_0^{-1} \) is zero point of \( \det(I + \zeta \mathcal{A}) \) of order \( k \). Since \( \det(I + \lambda \mathcal{F}(B_0, D)) = \det(I + \lambda R_1 \mathcal{A}^{-1}) \), the algebraic multiplicity of \( \lambda_j \in \sigma(\mathcal{F}(B_0, D)) \) is same as that of \( \lambda_j \in \sigma(R_1 \mathcal{A}^{-1}) \). \( \square \)
In general (3.7-3.8) is not right. For example, let \( n = 1, T = 1, B = 0_2, D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and let \( V_0, V_1 \) with frame \((1, 0)^T, (0, 1)^T\) separately. Easy computation shows that \( JD = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \); and \( \gamma(t) = \begin{pmatrix} e^{-at} & 0 \\ 0 & e^{at} \end{pmatrix} \), which implies that the corresponding Hamiltonian system has no eigenvalue. Therefore \( \prod_j (1 - \lambda_j^{-1}) \) should be understood as 1 and \( \sum_j \lambda_j^{-1} = 0 \). On the other hand, from the Hill-type formulas (3.2), we have

\[
\det(I - \mathcal{F}(0_2, D)) = e^{-1} \neq \prod_j (1 - \lambda_j^{-1}), \quad \text{Tr} \mathcal{F}(0_2, D) = 1 \neq \sum_j \lambda_j^{-1}.
\]

4 Brake symmetry decomposition for \( S \)-periodic orbits

In this section, we will deal with the brake symmetry decomposition. The relationship between the conditional Fredholm determinant of \( S \)-periodic solutions with brake symmetry and that of the solution of corresponding Lagrangian boundary problem will be considered. At first, we need the following lemma.

**Lemma 4.1.** Recall that \( \hat{E}_S^\pm \) and \( E_S^\pm \) are defined in (1.4) and (1.5)

\[
\det(I - \mathcal{F}(B, D; \hat{E}_S^\pm)) = \det(I - \mathcal{F}(B, D; E_S^\pm)).
\]

**Proof.** We only prove the equality for \( E_S^+ \). Let \( U^+: \hat{E}_S^+ \to E_S^+ \) be the isomorphic maps, then a function \( f \) is an eigenvector of \( \mathcal{F}(B, D; \hat{E}_S^+) \) corresponding to the eigenvalue \( \lambda \), if and only if \( U^+ f \) is an eigenvector of \( \mathcal{F}(B, D; E_S^+) \) corresponding to the same eigenvalue. By (28),

\[
\det_2(I - \mathcal{F}(B, D; \hat{E}_S^+)) = \det_2(I - \mathcal{F}(B, D; E_S^+)).
\]

Moreover, let \( \hat{P}_N \) be the orthogonal projection on \( \hat{E}_N^+ = \hat{E}_S^+ \cap \oplus_{\nu \leq N} (\ker(A_{E_S^+} - \nu)) \), then the conditional trace

\[
\text{Tr} \mathcal{F}(B, D; \hat{E}_N^+) = \lim_{N \to \infty} \text{Tr} \mathcal{F}(B, D; \hat{E}_S^+|_{E_N^+}).
\]

Similarly, let

\[
E_N^+ = E_S^+ \cap \oplus_{\nu \leq N} (\ker(A_{E_S^+} - \nu)),
\]

and \( P_N \) the orthogonal projection onto \( E_N^+ \). Since \( U^+ A_{E_S^+} = A_{E_S^+} U^+ \), \( U^+ \hat{E}_N^+ = E_N^+ \), we have

\[
\text{Tr} \mathcal{F}(B, D; \hat{E}_N^+|_{E_N^+}) = \text{Tr} \mathcal{F}(B, D; E_S^+|_{E_N^+}).
\]

It follows that the conditional trace

\[
\text{Tr} \mathcal{F}(B, D; \hat{E}_S^+) = \text{Tr} \mathcal{F}(B, D; E_S^+).
\]

By (2.1), we have the desired equality. \( \Box \)

By (1.4), the condition \( (1.4) \) implies that

\[
A|_{E_S^+} - B - \lambda D = (A|_{E_S} - B - \lambda D)|_{E_S^+} \oplus (A|_{E_S} - B - \lambda D)|_{E_S^-}.
\]

We have
Theorem 4.2. Under the above conditions,

\[ \det(I - \mathcal{F}(B, D; E_S)) = \det(I - \mathcal{F}(B, D; E_S^\perp)) \cdot \det(I - \mathcal{F}(B, D; E_S^\perp)). \]

and

\[ Tr\mathcal{F}^k(B, D; E_S) = Tr\mathcal{F}^k(B, D; E_S^\perp) + Tr\mathcal{F}^k(B, D; E_S^\perp). \]

In [14], we built the following Hill-type formula for \( S \)-periodic solutions

\[ \det(I - \mathcal{F}(B, D; S)) = \det(S\gamma_1(T) - I_{2n})/ \det(S\gamma_0(T) - I_{2n}). \]  \hfill (4.1)

In the remaining part of the section, we will show that the right side of (4.1) could also be decomposed under \( \mathbb{Z}_2 \) action. From the brake symmetry, we have

\[ S\gamma_A(T) = SN\gamma_A^{-1}(T/2)N\gamma_A(T/2) \]

By changing basis, we suppose \( N = \text{diag}(I_n, -I_n) \). Since \( S \in \text{Sp}(2n) \cap O(2n) \), then we can assume \( S = \left( \begin{array}{cc} C & -D \\ D & C \end{array} \right) \), where \( C^T D \) is symmetric, and \( C^T C + D^T D = I \). Since \( NS^T = SN \), then we have \( C = C^T \), \( D = D^T \) and \( CD = DC \). So we can choose basis such that both \( C \) and \( D \) are diagonalized, and we may assume \( C = \cos(\theta), D = \sin(\theta) \) with \( \theta = \text{diag}(\theta_1, \cdots, \theta_n) \). We write \( S \) as \( R_\theta := \left( \begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array} \right) \), it is obvious that \( R(\theta/2)^2 = S \). Easy computation shows that

\[ R_{\theta/2}NR_{\theta/2} = N, SN = \left( \begin{array}{cc} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{array} \right). \]

We have

\[ \det(S\gamma_A(T) - I_{2n}) = \det(SN\gamma_A^{-1}(T/2)N\gamma_A(T/2) - I_{2n}) \]
\[ = \det(R_{\theta/2}NR_{\theta/2}R_{\theta/2}^{-1}N\gamma(T/2)R_{\theta/2} - I_{2n}) \]
\[ = \det(N\hat{M}_A^{-1}N\hat{M}_A - I_{2n}), \]

where we set \( \hat{M}_A = \gamma_A(T/2)R_{\theta/2} \). By assume \( \hat{M}_A = \left( \begin{array}{cc} a_A & b_A \\ c_A & d_A \end{array} \right) \), then

\[ \det(S\gamma_A(T) - I_{2n}) = (-1)^n \det(N\hat{M}_A - \hat{M}_A N) \]
\[ = (-1)^n \det\left( \begin{array}{cc} 0 & 2b_A \\ -2c_A & 0 \end{array} \right) = (-1)^n 2^{2n} \det(b_A) \det(c_A). \]

We have

\[ \det(S\gamma_1(T) - I_{2n})/ \det(S\gamma_0(T) - I_{2n}) = \frac{\det(c_1) \det(b_1)}{\det(c_0) \det(b_0)}. \]

Let \( Z_+ = \text{diag}(I_n, 0_n) \) (\( Z_- = \text{diag}(0_n, I_n) \)) be the frame of \( V^+(N)(V^-(N)) \), then a frame \( \hat{Z}_\pm \) of \( V^\pm(SN) \) can be given by \( R_{\theta/2}Z_\pm \). We have

\[ \det(\gamma_A(T/2)\hat{Z}_+, Z_+) = \det(\hat{M}_A Z_+, Z_+) = (-1)^n \det(c_A), \]
and
\[ \det(\gamma_1(T/2)\hat{Z}_+, Z_+) / \det(\gamma_0(T/2)\hat{Z}_+, Z_+) = \det(c_1) / \det(c_0). \]

Similar, we have
\[ \det(\gamma_1(T/2)\hat{Z}_-, Z_-) = \det(\hat{M}Z_-, Z_-) = \det(b_1), \]

and
\[ \det(\gamma_1(T/2)\hat{Z}_-, Z_-) / \det(\gamma_0(T/2)\hat{Z}_-, Z_-) = \det(b_1) / \det(b_0). \]

We get the decomposition formula
\[ \det(S \gamma_1(T) - I_{2n}) / \det(S \gamma_0(T) - I_{2n}) = \frac{\det(\gamma_1(T/2)\hat{Z}_+, Z_+)}{\det(\gamma_0(T/2)\hat{Z}_+, Z_+)} \cdot \frac{\det(\gamma_1(T/2)\hat{Z}_-, Z_-)}{\det(\gamma_0(T/2)\hat{Z}_-, Z_-)}. \]

5 Relation with the relative Morse index and stability criteria

In this section, we will give the relation of conditional Fredholm determinant and relative Morse index, moreover we give some new stability criteria for the symmetry periodic orbits.

A simple way to understand the relative Morse index \( I(A - B, A - B - D) \) is from the viewpoint of spectral flow. For reader’s convenience, we first give a brief review of the spectral flow. The spectral flow was introduced by Atiyah, Patodi and Singer [1] in their study of index theory on manifolds with boundary. Let \( \{A(\theta), \theta \in [0, 1]\} \) be a continuous path of self-adjoint Fredholm operators on a Hilbert space \( \mathcal{H} \). Roughly speaking, the spectral flow of path \( \{A(\theta), \theta \in [0, 1]\} \) counts the net change in the number of negative eigenvalues of \( A(\theta) \) as \( \theta \) goes from 0 to 1, where the enumeration follows from the rule that each negative eigenvalue crossing to the positive axis contributes +1 and each positive eigenvalue crossing to the negative axis contributes −1, and for each crossing, the multiplicity of eigenvalue is counted.

We come back to the Hamiltonian systems, suppose \( B(s, t) \in C([0, 1] \times [0, T], S(2n)) \). For \( s \in [0, 1] \), let \( B_s \in C([0, T], S(2n)) \). For such two operators \( A - B_0 \) and \( A - B_1 \), we can define the relative Morse index via spectral flow. In fact, by [10], we have,
\[ \overline{I}(A - B_0, A - B_1) = -S f([A - B(s), s \in [0, 1]]). \]

We list some fundamental property of the relative Morse index, detail could found in [13]

(I) For \( B_0, B_1, B_2 \), then
\[ \overline{I}(A - B_0, A - B_1) + \overline{I}(A - B_1, A - B_2) = \overline{I}(A - B_0, A - B_2). \]

(II) Let \( D = B_1 - B_0 \), and we can simply let \( B(s) = B_0 + sD \). Let \( \kappa = \{s_0 \in [0, 1], \ker(A - B(s_0)) \neq 0\} \),
\[ \overline{I}(A - B_0, A - B_1) \leq \sum_{s_0 \in \kappa} \nu(A - B(s_0)). \]

(III) if \( D > 0 \), then \( \overline{I}(A - B, A - B - D) \geq 0 \). By careful analysis [10], the crossing form
\[ \overline{I}(A - B, A - B - D) = \sum_{s_0 \in \kappa \cap [0, 1)} \nu(A - B(s_0)). \]
Similarly

\[ \mathcal{I}(A - B, A - B + D) = - \sum_{x_0 \in A \cap (0, 1]} v(A - B(x_0)). \]

(IV) Suppose \( D_1 \leq D \leq D_2 \), then

\[ \mathcal{I}(A - B, A - B - D_1) \leq \mathcal{I}(A - B, A - B - D) \leq \mathcal{I}(A - B, A - B - D_2). \]

Similar with the \( S \)-periodic boundary conditions [14], we have the next theorem

**Theorem 5.1.** Assume \( A - B \) and \( A - B - D \) are non-degenerate, then \( \det(I - \mathcal{F}(B, D)) > 0(< 0) \) if and only if \( \mathcal{I}(A - B, A - B - D) \) is even (odd).

**Proof.** Let \( P_N \) be the orthogonal projections onto \( V_N = \Theta_{|V| \leq N} \ker(A - \gamma) \). Then

\[ \det((A - B)(A + P_0)^{-1}) = \lim_{N \to \infty} \det((A|V_n - P_n BP_n)(A|V_n + P_0)^{-1}). \]

and the sign of \( \det((A - B)(A + P_0)^{-1}) \) is same as the sign of \( \det((A|V_n - P_n BP_n)(A|V_n + P_0)^{-1}) \) for \( N \) large enough. Now by the same argument as [14, Theorem 6.2], we have \( \det((A|V_n - P_n BP_n)(A|V_n + P_0)^{-1}) \) is positive (negative) if and only if the difference of total multiplicity of the negative eigenvalues of \( A|V_n - P_n BP_n \) and \( A|V_n \) is even(odd) for \( N \) is large enough. By the continuousness of the relative Morse index, this is equivalent to that \( \mathcal{I}(A - B, A - B) \) is even(odd), hence sign \( \det((A - B)(A + P_0)^{-1}) = (-1)^{\mathcal{I}(A - B - D)} \). By the same reason, for \( A - B - D \), we also have sign \( \det((A - B - D)(A + P_0)^{-1}) = (-1)^{\mathcal{I}(A - B - D)} \). Since \( \det(I - \mathcal{F}(B, D)) \det((A - B)(A + P_0)^{-1}) = \det((A - B - D)(A + P_0)^{-1}) \), we get

\[ \text{sign } \det(I - \mathcal{F}(B, D)) = (-1)^{\mathcal{I}(A - B - D)} \cdot (-1)^{-\mathcal{I}(A - B)} = (-1)^{\mathcal{I}(A - B, A - B - D)}, \]

it’s easy to see that \( \det(I - \mathcal{F}(B, D)) \) is positive (negative) if and only if \( \mathcal{I}(A - B, A - B - D) \) is even(odd). □

In [13], we had use trace of \( \mathcal{F}_k(B, D) \) to nontrivial estimation of relative Morse index. Although in [13], we deal with the operators with of \( S \)-periodic case, it is totally same for the Lagrangian boundary conditions. The following theorem is from [13]

**Theorem 5.2.** Suppose \( A - B \) is non-degenerate. Suppose that there are \( D_1, D_2 \in \mathcal{B}(2n) \) such that \( D_1 < D < D_2 \), with \( D_1 < 0, D_2 > 0 \), if there exists \( k \in 2\mathbb{N} \), such that \( \text{Tr} \mathcal{F}_k(B + \gamma I, D_j) < 1 \) for \( j = 1, 2 \), then \( A - B - D \) is non-degenerate, and moreover \( \mathcal{I}(A - B, A - B - D) = 0 \).

Connected with the trace formula (3.6), We can give an estimation of relative Morse index by the trace of matrices. As a corollary of Theorem 5.2, We have

**Corollary 5.3.** Suppose \( A - B \) is non-degenerate. Suppose that there are \( D \geq 0 \), \( \text{Tr}(G_1^2) - 2\text{Tr}(G_2) < 1 \), then \( A - B - D \) is non-degenerate, and \( \mathcal{I}(A - B, A - B - D) = 0 \). Similar for the case \( D < 0 \).

Suppose \( x(t) \) is a \( T \)-periodic solution with the fundamental solutions \( \gamma(t) \). \( x \) is called (spectral) stable if \( \sigma(\gamma(T)) \in \mathbb{U} \), is called hyperbolic if \( \sigma(\gamma(T)) \cap \mathbb{U} = \emptyset \). To estimate the stability, we use the Maslov-type index \( i_{\omega}(\gamma) \), which is essentially same as the relative Morse index [19]. Roughly speaking, for a continuous path \( \gamma(t) \in \text{Sp}(2n) \), \( \omega \in \mathbb{U} \), the Maslov-type index \( i_{\omega}(\gamma) \) is defined by the intersection number of \( \gamma \) and \( \text{Sp}_{\omega}(2n) = \{ M \in \text{Sp}(2n) \mid \det(M - \omega I_{2n}) = 0 \} \). Details could be found in [18, 19], some brief review could be found in [11]. From [10] Theorem 2.5 and Lemma 4.5, we have the following proposition.
Proposition 5.4. For $S = \pm I$, we have

\[
I(A|_{E_S}, A|_{E_S} - B) = \begin{cases} 
  i_1(\gamma) + n, & \text{if } S = I_{2n}, \\
  i_{-1}(\gamma) & \text{if } S = -I_{2n}.
\end{cases}
\]

Let $e(M)$ be the total number of eigenvalues of $M$ on $U$, a simple but useful stability criteria is following

\[
e(\gamma(T))/2 \geq |i_{-1}(\gamma) - i_1(\gamma)|.
\] (5.1)

All the above results, for that the relative Morse index equals to Maslov-type index and for the stability criteria, could be proved for any $S$ boundary condition with $S \in \text{Sp}(2n) \cap O(2n)$, and details could be found in [10], [13].

Consider the linear system

\[
\dot{z}(t) = JB_1(t)z(t), \quad z(0) = z(T),
\] (5.2)

where $B_1 = B + D$. We assume (5.2) satisfied the brake symmetry condition with respect to $N$ as given in §4. From [14], for $S = \pm I$, we have decomposition of the relative Morse index

\[
I(A|_{E_S} - B, A|_{E_S} - B - D) = I(A|_{E_S^+} - B, A|_{E_S^+} - B - D) + I(A|_{E_S^-} - B, A|_{E_S^-} - B - D).
\]

We have

Lemma 5.5. Assume $B$ and $D$ satisfy brake symmetry with respect to $N$. Suppose $|i_1(\gamma_0) - i_{-1}(\gamma_0)| = n$, and $I(A|_E - B, A|_E - B - D) = 0$, for $E = E_S^\pm$, $S = \pm I$, then $\gamma_1(T)$ is stable.

Proof. Please note that $I(A|_E - B, A|_E - B - D) = 0$, for $E = E_S^\pm$, $S = \pm I$ implies $i_{\pm}(\gamma_1) = i_{\pm}(\gamma_0)$, then the result from (5.1). \qed

As a corollary, we have

Corollary 5.6. Suppose $|i_{-1}(\gamma_0) - i_1(\gamma_0)| = n$, $D^\pm \leq D \leq D^+$ with $D^+ \geq 0$, $D^- \leq 0$, and $TrF^2(B, D^\pm, E) < 1$ for $E = E_S^\pm$, then $\gamma_1(T)$ is stable.

From Corollary 5.6 and the trace formula, we can give stability criteria for the brake symmetry orbits. This criteria will applied for elliptic Lagrangian orbits, please see the detail in next section.

6 Stability of elliptic relative equilibria in planar 3-body problem

In this section, we use the trace formula to study the stability of ERE in planar 3-body problem. A brief introduction for the stability of ERE is given in Subsection 6.1. The applications on studying the stability of the elliptic Lagrangian orbits and elliptic Euler orbits are given in Subsection 6.2 and Subsection 6.3 separately.
6.1 Brief introduction to the stability of elliptic relative equilibria

In 2005, Meyer and Schmidt [20] strongly used the structure of the central configuration for the elliptic relative equilibria and symplectically decomposed the fundamental solution of the orbits into two parts, one of which corresponding to the Keplerian solution and the other is the essential part of the dynamics, needed for studying the stability. For the planner three body problem, the only central configurations is case of the Lagrangian triple and Euler collinear control configurations, which the corresponding ERE is called elliptic Lagrangian solutions and Elliptic Euler orbits. In this case, the essential part can be written in the following form.

Let \( e \) be the eccentricity, \( t \) be the truly anomaly and \( r_e(t) = (1 + e \cos(t))^{-1} \). In the rotating coordinate system and by using the true anomaly as the variable, Meyer and Schmidt [20] gave a very useful form of the essential part

\[
\mathcal{B}_e(t) = \begin{pmatrix}
I_2 & -J \\
J & I_2 - r_e(t)R
\end{pmatrix}, \quad t \in [0, 2\pi], \quad e \in [0, 1),
\]

(6.1)

where \( R \) can be considered as the regularized Hessian of the central configurations. Thus the corresponding Sturm-Liouville system is

\[-\ddot{y} - 2J_2 \dot{y} + r_e(t)Ry = 0.\]

Let \( \gamma_e(t) \) be the fundamental solution of (6.1), that is

\[\gamma_e(t) = J\mathcal{B}_e(t)\gamma_e(t), \quad \gamma_e(0) = I.\]

The ERE is called spectrally stable (or elliptic) if all the eigenvalues of \( \gamma_e(2\pi) \) belong to the unit circle \( \mathbb{U} \), and it is called linearly stable if moreover \( \gamma_e(2\pi) \) is semi-simple. By contrast, the ERE is called hyperbolic if no eigenvalue of \( \gamma_e(2\pi) \) locates on \( \mathbb{U} \), and is called elliptic-hyperbolic if only part of eigenvalues locate on \( \mathbb{U} \).

We assume \( R = \alpha I_2 + \eta N \) for \( \alpha, \eta \geq 0 \) with \( N = \text{diag}(1, -1) \), which include the case of Lagrangian and Euler orbits. Obviously, \( NR = RN \). Denote \( N = \text{diag}(N, -N) \). Direct computation shows that

\[NB_e(T - t) = \mathcal{B}_e(t)N, \quad e \in [0, 1),\]

which means the system admits the brake symmetry. We have the decomposition formula [14]

\[
i_1(\gamma_e) + n = I(A|_{E_1}, A|_{E_1} - \mathcal{B}_e) + I(A|_{E_1}, A|_{E_1} - \mathcal{B}_e),
\]

\[
i_{-1}(\gamma_e) = I(A|_{E_{-1}}, A|_{E_{-1}} - \mathcal{B}_e) + I(A|_{E_{-1}}, A|_{E_{-1}} - \mathcal{B}_e),
\]

and

\[\dim \ker(\gamma_e(2\pi) - 1) = \dim(V^+(N) \cap \gamma_e(\pi)V^+(N)) + \dim(V^-(N) \cap \gamma_e(\pi)V^-(N)).\]

\[\dim \ker(\gamma_e(2\pi) + 1) = \dim(V^-(N) \cap \gamma_e(\pi)V^+(N)) + \dim(V^+(N) \cap \gamma_e(\pi)V^-(N)).\]

Let

\[D_e := \mathcal{B}_e - \mathcal{B}_0 = \text{diag}(0_2, e \cos(t)r_e(t)R),\]

and denote

\[D_e^\pm := (D_e \pm |D_e|)/2,\]

where \(|D_e| = (D_e^2)^{1/2}\). Then, we have \(D_e^+ \geq 0 \) and \(D_e^- \leq 0 \).
Proposition 6.1. For \( E = E_i^+, i = 1, 2 \), if \( Tr F^2(B_0, D_\gamma^+, E) \) < 1 then \( \gamma_1 \) is \( \pm 1 \) non-degenerate and
\[
i_1(\gamma_1) = i_1(\gamma_0), \quad i_{-1}(\gamma_1) = i_{-1}(\gamma_0).
\]

6.2 Stability of elliptic Lagrangian orbits

We will give a new estimation to the left stability region of the Lagrangian orbits. In this case \( \alpha = 3/2 \), \( \eta = \frac{\sqrt{3 - \beta}}{2} \), \( \beta \in [0, 9] \). Please note that \( R \) only depend on \( \beta \) and \( R_\beta > 0 \) for \( \beta \in (0, 9] \). We denote \( \gamma_{\beta,e} \) be the fundamental solutions corresponding to \( 2B_{\beta,e} \). By (55) and (58) in [11], Lemma 4.1, we obtain
\[
i_1(\gamma_{\beta,e}) = 0, \quad \forall (\beta, e) \in [0, 9] \times [0, 1).
\]

Set
\[
D_{\beta,e}(t) = 2B_{\beta,e}(t) - B_{\beta,0}(t) = e \cos(t)r_e(t)K_{\beta},
\]
where \( K_{\beta} = \text{diag}(0, 0, \frac{3+\sqrt{3-\beta}}{2}, \frac{3-\sqrt{3-\beta}}{2}) \), then \( A - B_{\beta,e} = A - B_{\beta,0} - D_{\beta,e} \). Let \( \cos^+(t) = (\cos(t) + |\cos(t)|)/2 \), and denote
\[
K_{\beta}^{\pm} = \cos^+(t)K_{\beta},
\]
then
\[
D_{\beta,e} = er_e(t)K_{\beta}^{\pm}.
\]

We denote
\[
f^+(\beta) = Tr(F^2(B_{\beta,0}, K_{\beta}^{-}; E_{-1}^+)),
\]
which is a positive function. The following theorem holds true.

Theorem 6.2. For \( \beta \in [0, 3/4) \), \( \gamma_{\beta,e}(2\pi) \) is spectrally stable if
\[
0 \leq e < \min\{1 + f^+(\beta)^{\frac{1}{2}}\}^{-1}.
\]

Proof: The inequality \( 0 \leq e < \min\{1 + f^+(\beta)^{\frac{1}{2}}\}^{-1} \) implies that \( Tr(F^2(B_{\beta,0}, e(1 - e)^{-1}K_{\beta}^{-}; E_{-1}^+)) < 1 \), then \( (A - B_{\beta,0} - e(1 - e)^{-1}K_{\beta}^{-})|E_{-1}^+ \) is non-degenerate, hence we have \( I(A|_{E_{-1}^+}, A - B_{\beta,0} - e(1 - e)^{-1}K_{\beta}^{-}|E_{-1}^+) = I(A|_{E_{-1}^+}, A - B_{\beta,0}|E_{-1}^+) \). Moreover we have
\[
i_{-1}(\gamma_{\beta,e}) = I(A|_{E_{-1}^+}, A - B_{\beta,0}|E_{-1}^+) + I(A|_{E_{-1}^+}, A - B_{\beta,e}|E_{-1}^+)
\geq I(A|_{E_{-1}^+}, A - B_{\beta,0} - e(1 - e)^{-1}K_{\beta}^{-}|E_{-1}^+) + I(A|_{E_{-1}^+}, A - B_{\beta,0} - e(1 - e)^{-1}K_{\beta}^{-}|E_{-1}^+)
= I(A|_{E_{-1}^+}, A - B_{\beta,0}|E_{-1}^+) + I(A|_{E_{-1}^+}, A - B_{\beta,e}|E_{-1}^+)
= i_{-1}(\gamma_{\beta,e}) = 2.
\]

So \( e(\gamma_{\beta,e}) \geq 2|i_1(\gamma_{\beta,e}) - i_{-1}(\gamma_{\beta,e})| = 4 \). This complete the proof. \qed
To compute $f^\pm(\beta)$, let \( \{e_j\}_{j=1}^4 \) be the standard basis of \( \mathbb{R}^4 \), then the frames of \( V^+(N) \) and \( V^-(N) \) could be given by \( (e_1, e_4) \) and \( (e_2, e_3) \) separately. Obviously, \( \gamma_{\beta,0}(t) = \exp(JB_{\beta,0}t) \). We first consider \( f^+(\beta) \), in this case, the boundary conditions is given by \( x(0) \in V^-(N) \) and \( x(\pi) \in V^+(N) \). Then we have

\[
P^+ = (e_2, e_3, \exp(-JB_{\beta,0}\pi)e_1, \exp(-JB_{\beta,0}\pi)e_4),
\]

and \( Q^+_d = (e_2, e_3, 0, 0) \). Setting

\[
\hat{K}_\beta(t) = \exp(-JB_{\beta}(t))JK_\beta \exp(JB_{\beta}(t)),
\]

we have

\[
M_1 = \int_0^\pi \cos^{-1}(t)\hat{K}_\beta(t)dt = \int_{\pi/2}^\pi \cos(t)\hat{K}_\beta(t)dt,
\]

and

\[
M_2 = \int_0^\pi \cos^{-1}(t_1)\hat{K}_\beta(t_1)dt_1 \int_0^{t_1} \cos^{-1}(t_2)\hat{K}_\beta(t_2)dt_2
\]

\[
= \int_{\pi/2}^\pi \cos(t_1)\hat{K}_\beta(t_1)dt_1 \int_0^{t_1} \cos(t_2)\hat{K}_\beta(t_2)dt_2.
\]

We have

\[
G_1^+ = (P^+)^{-1}M_1Q^+_d, \quad G_2^+ = (P^+)^{-1}M_2Q^+_d, \quad f^+(\beta) = Tr(G_1^+)^2 - 2Tr(G_2^+).
\]

Similarly, for \( f^-(\beta) \), the boundary conditions is given by \( x(0) \in V^+(N) \) and \( x(\pi) \in V^-(N) \), and

\[
P^- = (e_1, e_4, \exp(-JB_{\beta,0}\pi)e_2, \exp(-JB_{\beta,0}\pi)e_3),
\]

and \( Q^-_d = (e_1, e_4, 0, 0) \). We have

\[
G_1^- = (P^-)^{-1}M_1Q^-_d, \quad G_2^- = (P^-)^{-1}M_2Q^-_d, \quad f^-(\beta) = Tr(G_1^-)^2 - 2Tr(G_2^-).
\]

As some basic computation given in [13], with the help of matlab, we have

![Figure 1: The stable region S.](image-url)
In Figure 1, the points \( O_\approx (0, 0.3483), \ O_\approx (0, 0.5858), \Gamma_\approx = \{(\beta, e)|e = 1/(1 + \sqrt{f^-}(\beta))\}, \ \Gamma_\approx = \{(\beta, e)|e = 1/(1 + \sqrt{f^+}(\beta))\} \).

**Remark 6.3.** compare with the result in [13].

In Figure 2, the points \( O_\approx = (0, 0.3483), \ O_\approx = (0, 0.3333), \Gamma_\approx \) is given in [13]. From this picture, it easy to know that we can get a better estimation of the stability region by using the trace formulas in this paper. The reason is that

\[
Tr(F^2(\mathcal{B}_\beta, 0, e(1 - e)^{-1}K_\beta^-; E^-_1)) = Tr(F^2(\mathcal{B}_\beta, 0, e(1 - e)^{-1}K_\beta^-; E^-_1)) + Tr(F^2(\mathcal{B}_\beta, 0, e(1 - e)^{-1}K_\beta^-; E^-_1)) + Tr(F^2(\mathcal{B}_\beta, 0, e(1 - e)^{-1}K_\beta^-; E^-_1)).
\]

In [13], we need to estimate \( Tr(F^2(\mathcal{B}_\beta, 0, e(1 - e)^{-1}K_\beta^-; E^-_1)) < 1 \), but in this paper we only need to estimate \( Tr(F^2(\mathcal{B}_\beta, 0, e(1 - e)^{-1}K_\beta^-; E^-_1)) < 1 \) and \( Tr(F^2(\mathcal{B}_\beta, 0, e(1 - e)^{-1}K_\beta^-; E^-_1)) < 1 \). Obviously this condition is weaker, hence we can get a better result.

**Remark 6.4.** In [9], there exist two \(-1\)-degenerate curves on the stability bifurcation diagram of Lagrangian solution. This two curves corresponding to spaces \( E^+_1 \) and \( E^-_1 \), hence curves \( \Gamma_+ \) and \( \Gamma_+ \) give a lower bound of this two \(-1\) degenerate curves respectively.

### 6.3 Elliptic-Hyperbolic region of elliptic Euler orbits

The Euler orbits have been studied in [21], [29], [12], in this case, \( R = \text{diag}(-\delta, 2\delta + 3) \), where \( \delta \in [0, 7] \) only depends on mass \( m_1, m_2, m_3 \). Please refer to Appendix A of [21] for the details. Although there is no physical meaning for \( \delta > 7 \), we will assume \( \delta \geq 0 \) to make the mathematical theory complete.

Let \( \gamma_{\delta, e} \) be the fundamental solutions of \( \mathcal{B}_{\delta, e}(t) \) which is given by (6.1), that is \( \gamma_{\delta, e} = J_2\mathcal{B}_{\delta, e}(t)\gamma_{\delta, e}, t \in [0, 2\pi], \gamma_{\delta, e}(0) = I_4 \). The stability problem can be studied via the Maslov-type index [29], then we first review their results. For any \( j \in \mathbb{N} \), there exists 1-degenerate curves \( \Gamma_j = Gr(\varphi_j(e)) \), and we also let
\[ \Gamma_0 = Gr(\varphi_0(e)) \text{ with } \varphi_0(e) = 0. \] Then \( \gamma_{\delta,e} \) only degenerates at \( \cup_{j=1}^{\infty} \Gamma_j \) and \( \dim \ker(\gamma_{\delta,e}(2\pi) - I_4) = 2 \) if \( (\delta, e) \in \cup_{j=1}^{\infty} \Gamma_j \). The Maslov-type index satisfies

\[
i_1(\gamma_{\delta,e}) = 2j + 3, \quad \text{if} \quad \varphi_j(e) < \delta \leq \varphi_{j+1}(e), \quad j \in \mathbb{N} \cup \{0\}.
\]

Similarly, for \( \forall j \in \mathbb{N} \), there exists pair \( -1 \)-degenerate curves \( \Upsilon^+ = Gr(\psi^+_j(e)) \).

Let \( \psi^+_j(e) = \min\{\psi^+_j(e), \psi^-_j(e)\} \) and \( \psi^+_j(e) = \max\{\psi^+_j(e), \psi^-_j(e)\} \). Moreover, we set \( \psi^0 = \psi^{-0} = 0 \), then for \( k \in \mathbb{N} \) we have

\[
i_{-1}(\gamma_{\delta,e}) = \begin{cases} 2j, & \text{if } \delta \in (\psi^j_{j-1}, \psi^j_j), \\ 2j + 1, & \text{if } \delta \in (\psi^j_j, \psi^j_{j+1}). \end{cases}
\]

Direct computation shows that \( \psi^+_j(0) = \psi^-_j(0) \), but it is not clear if, for \( e > 0 \), there exist other intersection points. There is a monotonicity property for Maslov-type index, that is for \( \omega \in \mathbb{U} \)

\[
i_\omega(\gamma_{\delta_1,e}) \leq i_\omega(\gamma_{\delta_2,e}), \quad \text{if} \quad \delta_1 \leq \delta_2.
\]

For any \( e \in [0,1) \), the \( \pm 1 \) degenerate curves satisfies

\[
0 < \psi_1^+(e) < \psi_1^-(e) < \psi_2^+(e) < \ldots < \psi_j^+(e) < \psi_j^-(e) < \ldots,
\]

and for \( j \in \mathbb{N} \)

\[
\varphi_j(0) = \frac{j - 3 + \sqrt{9j^4 - 14j^2 + 9}}{4}, \quad \psi_j^+(0) = \psi_j^-(0) = \frac{(j + \frac{1}{2})^2 - 3 + \sqrt{9(j + \frac{1}{2})^4 - 14(j + \frac{1}{2})^2 + 9}}{4}.
\]

Moreover for the region between the \( \pm 1 \)-degenerate curves, \( \gamma_{\delta,e}(2\pi) \) is elliptic-hyperbolic and for the region between the pairs of \( -1 \)-degenerate curves \( \gamma_{\delta,e}(2\pi) \) is hyperbolic.

We always set \( \psi^k_+ \) to be the degenerate curve in the sense that \( V^-(N) \cap \gamma_{\delta,e}(2\pi)V^+(N) \) nontrivial and similarly \( \psi^-_k \) to be the degenerate curve in the sense that \( V^+(N) \cap \gamma_{\delta,e}(2\pi)V^-(N) \) nontrivial.

Set

\[
D_{\delta,e}(t) = B_{\delta,e}(t) - B_{0,0}(t) = e \cos(t)r_\varepsilon(t)K_\delta,
\]

where \( K_\delta = \text{diag}(0, 0, -\delta, 2\delta + 3) \). For \( t \in [0, \pi] \), we set \( K^+_\delta(t) = \text{diag}(0, 0, -\cos(t)\delta, \cos(t)(2\delta + 3)) \) and \( K^-_\delta(t) = \text{diag}(0, 0, -\cos(t)\delta, \cos(t)(2\delta + 3)) \). then

\[
D^\pm_{\delta,e} = er_\varepsilon(t)K^\pm_\delta.
\]

For \( E = E^\pm_1 \), let

\[
g^\pm_1(\delta) = TrF^2(B_{0,0}, K^+_\delta, E^\pm_1), \quad g^\pm_2(\delta) = TrF^2(B_{0,0}, K^-_\delta, E^\pm_1).
\]

**Theorem 6.5.** For \( \delta \in [0, \psi_1^+(0)) \), \( \gamma_{\delta,e}(2\pi) \) is elliptic-hyperbolic, if

\[
0 < e < \min\{1 + g^+_0(\delta)^2\}^{-1}\}
\]

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Proof. The inequality $0 < e < \min\{(1 + g_1^+(\delta)^{1/2})^{-1}\}$ implies that $Tr(F^2(B_{\delta,0} + \frac{e}{1-e}K_\delta^+; E_{E_1}^\pm)) < 1$, then $(A - B_{\delta,0} - \frac{e}{1-e}K_\delta^+)|_{E_{E_1}^\pm}$ is non-degenerate, hence we have $I(A|_{E_{E_1}^\pm}, A - B_{\delta,0} - \frac{e}{1-e}K_\delta^+|_{E_{E_1}^\pm}) = I(A|_{E_{E_1}^\pm}, A - B_{\delta,0}|_{E_{E_1}^\pm})$. Then

$$i_{-1}(\gamma_{\delta,e}) = I(A|_{E_{E_1}^\pm}, A - B_{\delta,0}|_{E_{E_1}^\pm}) + I(A|_{E_{E_1}^\pm}, A - B_{\delta,0}|_{E_{E_1}^\pm}),$$

$$\leq I(A|_{E_{E_1}^\pm}, A - B_{\delta,0} - \frac{e}{1-e}K_\delta^+|_{E_{E_1}^\pm}) + I(A|_{E_{E_1}^\pm}, A - B_{\delta,0} - \frac{e}{1-e}K_\delta^+|_{E_{E_1}^\pm}),$$

$$= I(A|_{E_{E_1}^\pm}, A - B_{\delta,0}|_{E_{E_1}^\pm}) + I(A|_{E_{E_1}^\pm}, A - B_{\delta,0}|_{E_{E_1}^\pm}),$$

$$= i_{-1}(\gamma_{\delta,0}) = 2.$$

So $i_{-1}(\gamma_{\delta,e})$ does not increase and it implies that the region $\{(\delta, e)| 0 < e < \min\{(1 + g_1^+(\delta)^{1/2})^{-1}\}\}$ is between curves $Gr(\psi_1^0(e))$ and $Gr(\psi_1^\beta(e))$, from [29], we know it’s elliptic-hyperbolic between this two curve. □

**Theorem 6.6.** For $\delta \in (\psi_j^0(0), \psi_{j+1}^0(0))$, $j \in \mathbb{N}$, $\gamma_{\delta,e}(2\pi)$ is elliptic-hyperbolic, if

$$0 < e < \min\{(1 + g_1^+(\delta)^{1/2})^{-1}, (1 + g_2^+(\delta)^{1/2})^{-1}\}$$

Proof. For $0 < e < \min\{(1 + g_1^+(\delta)^{1/2})^{-1}\}$, like the proof of Theorem 6.1, we get $i_{-1}(\gamma_{\delta,e})$ does not increase. For $0 < e < \min\{(1 + g_2^+(\delta)^{1/2})^{-1}\}$, like the proof of Theorem 6.5, we get index $i_{-1}(\gamma_{\delta,e})$ does not decreasing. So the region $\{(\delta, e)| 0 < e < \min\{1/(1 + g_1^+(\delta)), 1/(1 + g_2^+(\delta))\}\}$ must between curves $Gr(\psi_1^j(e))$ and $Gr(\psi_{j+1}^j(e))$, from [29], we know it’s elliptic-hyperbolic between this two curve. □

For Lagrangian solution, we have given the deals in computing the function $f^\pm(\beta)$. By the same way, we also can compute $g_1^+(\delta)$ and $g_2^+(\delta)$. With the help of matlab, we have the estimation of the elliptic-hyperbolic (EH) region of Euler solution.

![Figure 3: The EH region for the Euler solution with $\delta \in [0, \psi_1^0(0))$.](image)
In Figure 3, the points \( J_+ \approx (0, 0.3483), J_- \approx (0, 0.5858), \psi_- = \{(\beta, e)|e = 1/(1 + \sqrt{g_1^-(\beta)})\}, \psi_+ = \{(\beta, e)|e = 1/(1 + \sqrt{g_1^+(\beta)})\} \).

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