First-order logic with incomplete information

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Abstract

We develop first-order logic and some extensions for incomplete information scenarios and consider related complexity issues.

1 Introduction

We define (an extension of) first-order logic for scenarios where the underlying model is not fully known. This is achieved by evaluating a formula with respect to several models simultaneously, not unlike in first-order modal logic. The set (or even a proper class) of models is taken to represent a collection of all possible models. The approach uses some ingredients from Hodges’ team semantics.

We shall not formally define what we mean by incomplete information (or imperfect information for that matter). However, we will not directly investigate any variant of quantifier independence as in IF-logic (which is sometimes referred to as first-order logic with imperfect information).

To demonstrate the defined framework from a technical perspective we also provide a complexity (of satisfiability) result that can be easily extended to further similar systems not formally studied here.

2 First-order logic with incomplete information

Let $\tau$ be a relational signature. Let $F(\tau)$ be the smallest set such that the following conditions hold.

1. For any $R \in \tau$, $Rx_1...x_k \in F(\tau)$. Here $x_1, ..., x_k$ are arbitrary variables (with possible repetitions) from a fixed countably infinite set VAR of first-order variable symbols. $R$ is a $k$-ary relation symbol.
2. $x = y \in F(\tau)$ for all $x, y \in$ VAR.
3. If $\varphi, \varphi' \in F(\tau)$, then $(\varphi \land \varphi') \in F(\tau)$.
4. If $\varphi \in F(\tau)$, then $\neg \varphi \in F(\tau)$.
5. If $x \in$ VAR and $\varphi \in F(\tau)$, then $\exists x \varphi \in F(\tau)$.
The above defines the exact syntactic version of first-order logic we shall consider here.

The semantics of (this version of) first-order logic is here defined with respect to \( \tau \)-interpretation classes; a \( \tau \)-interpretation is a pair \((\mathcal{M}, f)\) where \(\mathcal{M}\) is a \(\tau\)-model and \(f\) a finite function that maps a finite set of variable symbols into the domain of \(\mathcal{M}\). A \(\tau\)-interpretation class is a set (or a class) of \(\tau\)-interpretations with the functions \(f\) having the same domain. From now on we will only consider \(\tau\)-interpretation classes that are sets and call these classes model sets; we acknowledge that a pair \((\mathcal{M}, f)\) is more than a model due to the function \(f\), and indeed such pairs \((\mathcal{M}, f)\) are often called interpretations (while \(f\) is an assignment). Having acknowledged this issue, we shall not dwell on it any more, and we shall even call pairs \((\mathcal{M}, f)\) models. We note that a model set could also be called a model team or even an unknown model (in singular indeed).

A choice function for a model set \(\mathcal{M}\) is a function that maps each model \((\mathcal{M}, f)\) in \(\mathcal{M}\) to some element \(a\) in the domain of \(\mathcal{M}\). Recall that \(h[a/b]\) denotes the function \(h\) modified or extended so that \(b\) maps to \(a\). If \(F\) is a choice function, we let \(\mathcal{M}[F/x]\) denote the class

\[
\{ (\mathcal{M}, f[F(\mathcal{M}, f)/x]) \mid (\mathcal{M}, f) \in \mathcal{M} \}.
\]

We let \(\mathcal{M}[\top/x]\) denote the class

\[
\{ (\mathcal{M}, f[b/x]) \mid (\mathcal{M}, f) \in \mathcal{M} \text{ and } b \in \text{Dom}(\mathcal{M}) \}.
\]

The common domain of a model set \(\mathcal{M}\) is the (possibly empty) intersection of the domains of the models in \(\mathcal{M}\). If \(A\) is any subset (including the empty set) of the common domain of \(\mathcal{M}\), we let \(\mathcal{M}[A/x]\) denote the class

\[
\{ (\mathcal{M}, f[b/x]) \mid (\mathcal{M}, f) \in \mathcal{M} \text{ and } b \in A \}.
\]

Recall that a constant function is a function that maps each input to the same element. Thus a constant choice function for a model set \(\mathcal{M}\) is a choice function that maps each model to the same element in the intersection of the domains of the models in \(\mathcal{M}\). (The empty function is not a constant choice function for any other than the empty model set.) Let \(\mathcal{M}\) be a \(\tau\)-interpretation class, i.e., a model set. The semantics of first-order logic (with incomplete information) is defined as follows.
Technically this logic (first-order logic with incomplete information) adds very little to standard first-order logic: the semantics has simply been lifted to the level of sets of models (or sets of pairs $(\mathfrak{M}, f)$), as the following Proposition shows. However, conceptually the difference with standard first-order logic approach is clear, and further meaningful divergence can be expected to arise in the study of extensions of this base formalism.

The following proposition is easy to prove.

**Proposition 2.1.** Let $\varphi$ be an FO-formula. Then we have

- $\mathcal{M} \models^+ \varphi$ if and only if $(\mathfrak{M}, f) \models_{\text{FO}} \varphi$ for all $(\mathfrak{M}, f) \in \mathcal{M}$,
- $\mathcal{M} \models^- \varphi$ if and only if $(\mathfrak{M}, f) \not\models_{\text{FO}} \varphi$ for all $(\mathfrak{M}, f) \in \mathcal{M}$.

**Corollary 2.2.** Let $\varphi$ be an FO-formula. Then

- $\{(\mathfrak{M}, f)\} \models^+ \varphi$ if and only if $(\mathfrak{M}, f) \models_{\text{FO}} \varphi$,
- $\{(\mathfrak{M}, f)\} \models^- \varphi$ if and only if $(\mathfrak{M}, f) \not\models_{\text{FO}} \varphi$.

We then extend the above defined syntax for first-order logic by a formula construction rule $\varphi \mapsto Cx\varphi$. We call the resulting language $L^*_{C}$. We let $L^*_{C}$ be the fragment of $L^*_C$ where $Cx$ is not allowed in the scope of negation operators.

We extend the semantics based on model sets as follows, where by a constant choice function we mean a choice function that sends all inputs to the same (existing) element.

$$\mathcal{M} \models^+ Cx\varphi \quad \text{iff} \quad \mathcal{M}[F/x] \models^+ \varphi \text{ for some constant choice function } F \text{ for } \mathcal{M}$$

The reading of the operator $Cx$ could be something in the lines of there existing a common $x$, or perhaps a shared or constant $x$, or even known or constructible $x$. The above suffices for $L^*_C$. To define a (possible) semantics
for $L^*_C$, we give the following clause, where $M$ denotes the common domain of $\mathcal{M}$.

$$\mathcal{M} \models Cx\varphi \quad \text{iff} \quad \mathcal{M}[M/x] \models \varphi$$

We shall discuss $L^*_C$ somewhat little as it is somewhat harder to interpret intuitively than $L_C$.

Let us say that two formulae $\varphi, \varphi' \in L^*_C$ are existential variants if $\varphi$ can be obtained from $\varphi'$ by replacing some (possibly none) of the quantifiers $\exists x$ by $Cx$ and some (possibly none) of the quantifiers $Cx$ by $\exists x$. The following is easy to prove (cf. Corollary 2.2).

**Proposition 2.3.** Let $\varphi$ be an FO-formula and assume $\varphi' \in L^*_C$ is an existential variant of $\varphi$. Then

- $\{(\mathcal{M}, f)\} \models^+ \varphi'$ iff $(\mathcal{M}, f) \models_{\text{FO}} \varphi$, 
- $\{(\mathcal{M}, f)\} \models^- \varphi'$ iff $(\mathcal{M}, f) \not\models_{\text{FO}} \varphi$.

It would be interesting and relatively easy to extend in a natural way the first-order part\(^1\) of the above framework to involve generalized quantifiers (following [12]). Another option would be to consider operators that give a Turing-complete formalism (following [11] or even [13]), possibly following a direct game-theoretic approach rather than the team semantics flavoured one given above. This would lead to formalisms for parallelism and distributed computation when used with model sets as opposed to models. However, while these generalizations can be done such that the resulting formalisms are easily seen natural, the formalism here that uses $Cx$ is harder to interpret especially if we allow for $Cx$ in the scope of negations. If we use the semantics in formulae with $Cx$ occurrences, then disjunction together with $Cx$ can become peculiar\(^2\). Indeed, consider the model set $\mathcal{M}$ with two disjoint models and nothing else. Now $\mathcal{M}$ satisfies $\neg (\neg \varphi' \land Cx(x = x)) \lor C(x = x)$ while not satisfying $Cx(x = x)$. Thus the reading of $\lor$ indeed is should be “there are two cases such that $\varphi$ and $\psi$” or even “the possibilities split into two cases such that in the first case $\varphi$ and in the second case $\psi$.” (Note that dependence logic requires a similar reading of $\lor$ to be fully natural.)

This is not an unnatural reading especially if one is attempting to unify semantics and proofs, thereby relating $\lor$ with the proof by cases protocol. Adopting the perspective that a model set is (intuitively) a single fixed but unknown object (for example any group from a collection of groups that extend a particular single group\(^4\)) is very natural and in such a framework it is natural to make statements about splitting into cases. (\(^The \) (unknown)

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\(^1\)The part without operators $Cx$.

\(^2\)Note that $\varphi \lor \psi$ simply means $\neg (\neg \varphi \land \neg \psi)$ here.

\(^3\)See [5] for similar considerations.

\(^4\)Groups have a relational representation here since we are considering relational signatures.
group $G$ has property $P$ or $G$ has property $Q$... This is true especially because proofs are often (or almost always) made for a fixed but unknown object or objects. Thus the above semantics works for formalising that kind of thinking. Category theory of course can also be thought to operate this way but here we have a very simple logic that can also directly speak about the internal structure of objects. It is obviously easy to expand the above framework, but we shall leave that for later.

3 Satisfiability and applications

We say that a sentence $\varphi \in L_C$ is satisfiable if there is some nonempty model set $M \models^+ \varphi$. The satisfiability problem for a fragment $F$ of $L_C$ takes a sentence of $F$ as an input and asks whether some nonempty model set satisfies $\varphi$, i.e., whether $M \models^+ \varphi$ for some nonempty model set $M$.

The two-variable fragment of $L_C$ is the set of formulae that use instances of only the two variables $x$ and $y$. We next show a complexity result concerning the two-variable fragment of $L_C$, although it is easy to see that the related argument rather flexibly generalizes to suitable other fragments as well. We discuss two-variable logic for convenience and also as it and its variants (even in the team semantics context) have received a lot of attention in recent years, see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 12, 14].

Proposition 3.1. The satisfiability problem of the two-variable fragment of $L_C$ is $\text{NEXPTIME}$-complete.

Proof. Define the following translation $T$ from $L_C$ into FO, where $D$ is a fresh unary relation symbol (intuitively representing the common domain of a model set).

$$
T(Rx_1...x_k) = Rx_1...x_k \\
T(x = y) = x = y \\
T(\neg \varphi) = \neg T(\varphi) \\
T((\varphi \land \psi)) = (T(\varphi) \land T(\psi)) \\
T(\exists x \varphi) = \exists x T(\varphi) \\
T(Cx \varphi) = \exists x (Dx \land T(\varphi))
$$

We will prove below (in a couple of steps) that a formula $\varphi$ of $L_C$ is satisfied by some nonempty model set iff $T(\varphi)$ is satisfied by some model (in the classical sense). This will conclude the proof of the current proposition as it is well known that the satisfiability problem of two-variable first order logic is $\text{NEXPTIME}$-complete.

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5 Similar statements are omnipresent. Further operators arise for related statements, such as "It is possible that $G$ has property $P$," etcetera. This modality statement obviously seems to say (more or less) that the subset (of the current model set) where $G$ has $P$ is nonempty.
We first note that if $T(\varphi)$ is satisfiable by some model $(\mathfrak{M}, f)$, then we have $\{(\mathfrak{M}, f)\} \models T(\varphi)$ by Corollary 2.2. From here it is very easy to show that $\varphi$ is satisfiable by the same model set by evaluating formulae step by step from outside in (and recalling the syntactic restrictions of $L_C$). Thus it now suffices to show that if some nonempty model set satisfies $\varphi$, then $T(\varphi)$ is satisfied by some model. To prove this, we begin by making the following auxiliary definition.

Let $\mathcal{M}$ be a model set and let $\mathcal{M}_D$ denote the model set obtained from $\mathcal{M}$ by adding a unary predicate $D$ to each model that covers exactly the common domain of $\mathcal{M}$. Recall that we have already fixed $\varphi$.

Claim. $\mathcal{M} \models T(\varphi)$ implies $\mathcal{M}_D \models T(\varphi)$.

The claim is easy to prove by evaluating formulae from outside in using the semantics for model sets.

Assume that $\mathcal{M} \models T(\varphi)$ for some nonempty model set $\mathcal{M}$. Thus $\mathcal{M}_D \models T(\varphi)$ by the claim. Thus $(\mathfrak{M}, f) \models_{\text{FO}} T(\varphi)$ for all $(\mathfrak{M}, f) \in \mathcal{M}_D$ by Proposition 2.1. Therefore (since $\mathcal{M}_D$ is nonempty) we have $(\mathfrak{M}, f) \models_{\text{FO}} T(\varphi)$ for some $(\mathfrak{M}, f) \in \mathcal{M}_D$. This concludes the very easy proof.

Going from perfect to imperfect information is in general extremely easy to justify in several ways, so let us look at more concrete and even rather specific and particular possible applications of model sets.

Ontology-based data access and related querying frameworks obviously offer a natural application for model sets. The work there is rather active, see, e.g., [2, 4] and the references therein. Another obvious and quite different application is distributed computing. One (of many) ways to model a computer network via logic would be to combine the approaches of [9] (which accounts for communication) with [11] (which accounts for the local (Turing-complete) computation). The nodes of [9] would become first-order models, so the domains considered would be model sets (with relations that connect models to other models). See also [10] for some (simple) adaptations of the framework in [9].

For yet another example, let $\varphi$ and $\psi$ denote your favourite theorems. One can now ask: “Does $\psi$ follow from $\varphi$?” The first answer could be: “Yes, since $\varphi$ is a true theorem, it in fact already follows from an empty set of assumptions.” The next answer could be a bit more interesting: for example, if $\varphi$ and $\psi$ were theorems of arithmetic, one could try to investigate if $\psi$ follows from $\varphi$ as a logical consequence, i.e., even without the axioms of arithmetic. Different approaches to relevance have been widely studied, and the example below is not unrelated to that.

Let $D$ be a deduction system (or some conceptually similar algorithm). Now, for each $n \in \mathbb{N}$, let $\rightarrow_n^D$ denote the connective defined such that $\varphi \rightarrow_n^D \psi$ holds if $\psi$ can be deduced from the premiss $\varphi$ in $n$ deduction steps (applications of deduction rules) in $D$. Here ‘$\varphi \rightarrow_n^D \psi$ holds’ is a metalogical
statement; we could consider closing the underlying logic under $\to^D_n$ and the other connectives, but we shall not do that now. We note that also statements $(\varphi_1, ..., \varphi_k) \to^D_n \psi$, containing several premises, can be introduced.

Statements $\varphi \to^D_n \psi$ capture aspects of relevance. The idea here is that whether $\psi$ follows from $\varphi$, depends on the particular background knowledge and abilities as well as the computational capacity (of an agent, for example). With this interpretation, it is indeed highly contingent whether something follows in $n$ steps from something else. It here depends on the particularities of $D$. Also, how immediately something follows from something else, is a matter of degree. This is the role of the subindex $n$.

This kind of a framework is one example (of many) that can be elaborated in a possibly more interesting way by using an approach to proofs that is directly (indeed, directly) linked to semantics, with connectives corresponding to proof steps. Model sets offer such possibilities in a natural way. It is worth noting that also refutation calculi (rather than proof calculi), and generalizations thereof, fit into the framework well. The related approaches can be based on the dual systems of [12]. That framework obviously offers quite natural possibilities for generalizations of model sets as well.

Multiperspective thought provides another example of immediate applications. Such thought seems to be considered controversial by many. Yet, it is mostly very simple, and it is indeed surprising that it is so often considered problematic. The difficulties in understanding related statements are often due to the assumption of bivalence and the assumption that concepts have fully fixed meanings in contexts where such assumptions are naïve.

Let us consider a very simple formal framework that captures—and thus elucidates—at least some aspects of multiperspective thought.

Consider a model set $\mathcal{U}$ which we shall call the universe. A property is a subset $P \subseteq \mathcal{U}$. (Properties need not be closed under isomorphism.) A weight function is a mapping $w : \text{Pow}(\mathcal{U}) \to S$, where $S$ is some non-empty set and $\text{Pow}$ the power set operator; the set $S$ could be, for example, the set of real numbers $\mathbb{R}$. We call the set $S$ the set of weights and the elements $s \in S$ are obviously called weights. Let $S_m$ denote the set of multisets over $S$, i.e., collections of elements of $S$ that also enable different multiplicities of elements to occur. A function $E : S_m \to V$ is called an evaluation function, where the set $V$ is an arbitrary set of values $v \in V$. For example, for finite $\mathcal{U}$ and with $S = \mathbb{R}$, the function $E$ could be the operator that gives the sum of any collection of inputs.

Now, let $s$ be a one-to-one function from $\text{Pow}(\mathcal{U})$ into a set of state-

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6Indeed, many philosophical debates stem from considering underdetermined concepts determined. Here ‘underdetermined’ could mean ‘not fully defined’ and determined ‘fully defined.’ Alternatively, ‘underdetermined’ could here stand for ‘not specified up to a sufficient extent’ and determined for ‘specified up to a sufficient extent.’

7We allow for infinite multiplicities, but limit the largest possible multiplicity with some sufficiently large cardinal, for example something greater than the power set of $\mathcal{U}$.
ments, so each property $P$ is associated with a statement $s(P)$. The weight of the statement $s(P)$ is $w(P)$ and the value of a set $K$ of statements is $E(\{w(P) \mid s(P) \in K\})$, where the argument set is a multiset of weights of statements.

So, we (or a group of people) can know that different properties hold, i.e., we know the actual model is inside different sets $P \subseteq U$. Each property $P$ contributes a weight. We combine the weights and get different values, depending on which properties are involved. For example, some true properties can contribute a negative number (as a weight) and others a positive one. The full value is the value obtained by considering the multiset of weights of all properties. (For example, the full value could be the sum of all weights.) Now, the full value then, in the end, can be associated with a truth value, if desired. For example, I can state that John is rich as he has a million dollars on his bank account, and Jill can state that John is not rich as he has a debt of two million dollars. Obviously these observations of partial knowledge (i.e., John has money and debt) are simply contributions towards an ultimate picture, and the possible seeming syntactic contrariness of the related partial statements (John is rich and John is not rich) amounts to nothing much at all.

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