Maximizing the number of maximal independent sets of a fixed size

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Abstract

For a fixed graph $G$, a maximal independent set is an independent set that is not a proper subset of any other independent set. P. Erdős, and independently, J. W. Moon and L. Moser (Israel J. Math., 3 (1965): 23–28), and R. E. Miller and D. E. Muller (IBM Res. Rep., (1960): RC-240), determined the maximum number of maximal independent sets in a graph on $n$ vertices, as well as the extremal graphs. In this paper we maximize the number of maximal independent sets of a fixed size for all graphs of order $n$ and determine the extremal graphs. Our result generalizes the classical result.

1 Introduction

Throughout this paper, we consider finite simple connected graphs. Let $G = (V(G), E(G)) = (V, E)$ be such a graph with vertex set $V$ and edge set $E$. Below are some graph theory concepts and notation needed in this paper. Readers are suggested to refer to [1], [2] or [13] for terminologies not specified here.

An independent set (or stable set) of $G$ is a set of pairwise nonadjacent vertices. In order for a set of vertices $U \subseteq V$ to be a maximal independent set (abbr. MIS) of $G$, we require the set $U$ to (1) be independent, and (2) have no strictly super independent set $W$ such that $U \subseteq W \subseteq V$. The set of neighbors of $v \in V$ is denoted by $N(v)$, or if necessary by $N_G(v)$.

A $k$-partite graph is a graph whose graph vertices can be partitioned in to $k$ disjoint sets so that no two vertices within the same set are adjacent. A $k$-partite graph is said to be complete if every pair of graph vertices in the $k$ sets are adjacent. The Turán graph $T_{n,k}$ is the complete $k$-partite graph with $n$ vertices whose partite sets differ in size by at most one [1]. We also define $G_1 + G_2$ as the graph consisting of the disjoint union of two graphs $G_1$ and $G_2$. The Turán graph $T_{n,k}$ and the operation of disjoint union will be useful in the extremal graph structure described in our main theorem (Theorem 1.1).

Given $G$, let $i_t(G)$ be the number of independent sets of size $t$ in $G$ and let $i(G) = \sum_{t \geq 0} i_t(G)$ be the total number of independent sets. While there have been many extremal results on $i(G)$ and $i_t(G)$ over various families of graphs (see e.g. [4] [6] [5] [12] [16]), it makes sense to investigate parallel theories on the MIS’s, the independent sets that are not covered by bigger ones. Let $i_{\text{max}}^t(G)$ be the number of maximal independent sets of size $t$ in $G$ and let $i_{\text{max}}(G) = \sum_{t \geq 0} i_{\text{max}}^t(G)$ be the total number of maximal independent sets. For arbitrary graphs $G$ on $n$ vertices, P. Erdős (see [3]), and independently, Moon and Moser [11], and Miller and Muller [10] determined $i_{\text{max}}^t(G)$ as well as the extremal graphs.

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Nonetheless, studies on the number of maximal independent sets seem to be less adequate (see for instance [3, 9, 11]).

The maximal independence polynomial is defined by

\[ I_{\text{max}}(G; x) := \sum_{U: U \text{ is an MIS of } G} x^{|U|}. \]

By definition, \( i_{k}^{\text{max}}(G) = [x^k] I_{\text{max}}(G; x) \), where by usual convention \([x^k] f(x)\) represents the coefficient of \( x^k \) in the polynomial or series \( f(x) \).

In this note we maximize \( i_{k}^{\text{max}}(G) \) graphs \( G \) on \( n \) vertices.

**Theorem 1.1** Assume \( n = qt + r \), where \( 0 \leq r < t \). For all graphs \( G \) on \( n \) vertices, we have

\[ i_{k}^{\text{max}}(G) \leq q^{t-r}(q+1)^r. \] (1.1)

Furthermore, let \( H = (t-r)K_q + rK_{q+1} \), i.e. disjoint union of \( t \) cliques of the specified orders, then \( H \) is the unique extremal graph.

**Remark 1.2** If \( n \) is a multiple of 3, (1.1) shows that \( i_{k}^{\text{max}}(G) \leq 3^\frac{n}{3} \), and it implies the main theorem in [11] which says that \( i^{\text{max}}(G) \leq 3^\frac{n}{3} \) when 3 divides \( n \). If \( n = 3k - 1 \), (1.1) gives that \( i_{k}^{\text{max}}(G) \leq 2 \cdot 3^{k-1} \). If \( n = 3k + 1 \), (1.1) gives that \( i^{\text{max}}(G) \leq 2^2 \cdot 3^{k-1} \) and \( i_{k}^{\text{max}}(G) \leq 2^2 \cdot 3^{k-1} \). Each case above strengthens a respective case of [11] Theorem 1. (Note that the celebrated result of [11] Theorem 1 says that each extremal graph actually has maximal independent sets of only one certain size.)

2 Proof

For completeness, we work with the complementary graph, and count cliques instead of independent sets. That is, we show that for all graphs \( G \) on \( n = qt + r \) vertices, where \( 0 \leq r < t \), the number of \( t \)-maximal cliques in \( G \) is no more than \( f(n, t) := q^{t-r}(q+1)^r \). Furthermore, the Turán graph \( T_{n,t} = K_{q, \ldots, q, q+1, \ldots, q+t} \) is the unique extremal graph. We achieve this by induction on \( n + t \). Keep in mind that the proposed extremal value \( f(n, t) = q^{t-r}(q+1)^r \) strictly increases with \( n + t \).

**Proof.** Since the cases that \( n < t \) or \( t = 1 \) are trivial, without loss of generality, we assume that \( n \geq t \geq 2 \).

**Case 1.** \( r > 0 \). We start with the case that is more convenient to phrase and Case 2 will be easier to understand.

**Subcase 1a.** \( r > 0 \) and \( \delta(G) \geq n - q \). Let \( v_1, v_2, \ldots, v_t \) be arbitrarily selected. Note that

\[ |V - \bigcap_{i=1}^{t} N(v_i)| = |\bigcup_{i=1}^{t} (V - N(v_i))| \leq \sum_{i=1}^{t} |V - N(v_i)| \leq qt < qt + r = n = |V|. \]

Hence \( \bigcap_{i=1}^{t} N(v_i) \neq \emptyset \), i.e., every \( t \) vertices in \( G \) has a common neighbor. Thus any maximal clique is larger than \( K_t \), so that \( G \) has no \( t \)-maximal cliques, implying that this subcase need not be considered in order to maximize \( i_{k}^{\text{max}}(G) \).

**Subcase 1b.** \( r > 0 \) and \( \delta(G) \leq n - q - 1 \). Choose a vertex \( v \) such that \( d(v) = \delta(G) \leq n - q - 1 \).

Let \( A \) be the set of \( t \)-maximal cliques in \( G \) which contains \( v \), and \( B \) be the set of \( t \)-maximal cliques in \( G \) which does not contain \( v \).

Every \( t \)-maximal clique in \( B \) is a \( t \)-maximal clique of \( G - \{v\} \). By induction hypothesis, as \( n - 1 = qt + r - 1 \), \( |B| \leq q^{t-r+1}(q+1)^{r-1} \).

To calculate \( |A| \), consider the subgraph of \( G \) induced by the neighbors of \( v \). Every \( t \)-maximal clique in \( A \) corresponds to a \((t-1)\)-maximal clique of \( G[N(v)] \). As \( |V(G[N(v)])| \leq n - q - 1 = qt + r - q - 1 = q(t-1) + r - 1 \), inductively, \( |A| \leq q^{(t-1)-(r-1)}(q+1)^{r-1} = q^{t-r}(q+1)^{r-1} \).

Therefore, the total number of \( t \)-maximal cliques in \( G \) is bounded by

\[ |B| + |A| \leq q^{t-r+1}(q+1)^{r-1} + q^{t-r}(q+1)^{r-1} = q^{t-r}(q+1)^r = f(n, t). \]
Theorem 1.1. David Galvin and Yufei Zhao. The number of independent sets in a graph with small maximum degree.

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Béla Bollobás. The above equality holds if and only if the following conditions are simultaneously met.

i). $G - v$ is a $t$-cliques extremal graph of order $n - 1$. Inductively, this requires $G - v = T_{n-1,t} = K_{q,\ldots,q,t+1}$ with $t-r+1$ partite sets of size $q$ and $r-1$ partite sets of size $q+1$.

ii). $d(v) = n - q - 1 = qt + r - q - 1 = q(t - r) + (q + 1)(r - 1)$ and $G[N(v)]$ is a $(t - 1)$-cliques extremal graph of order $n - q - 1$. This requires that $G[N(v)] = T_{n-q-1,t-1} = K_{q,\ldots,q,t+1}$ with $t-r$ partite sets of size $q$ and $r-1$ partite sets of size $q+1$.

Putting i) and ii) together, clearly, the neighbors of $v$ are precisely $t-r$ of the total $t-r+1$ partite sets of size $q$ and $r-1$ partite sets of size $q+1$ in $G - v$. Thus $G = T_{n,t}$ is the unique extremal graph in this case.

Case 2. $r = 0$, so that $n = qt$. The case $r = 0$ is similar to Case 1, with only slight differences in the calculation.

Subcase 2a. $r = 0$ and $\delta(G) \geq n - q + 1$. For any $t$ vertices $v_1, v_2, \ldots, v_t$, as

$|V - \bigcap_{i=1}^{t} N(v_i)| = |\bigcup_{i=1}^{t} (V - N(v_i))| \leq (q - 1)t < |V|,

they must have a common neighbor. Thus $G$ has no $t$-maximal cliques.

Subcase 2b. $r = 0$ and $\delta(G) \leq n - q$. Choose a vertex $v$ such that $d(v) = \delta(G) \leq n - q$.

Define $A$ and $B$ as in Case 1. As $n - 1 = qt - 1 = (q - 1)t + t - 1$, by similar arguments, inductively,

$|B| \leq (q - 1)^{t-1}.

On the other hand, as $|V(G[N(v)])| \leq n - q = q(t - 1)$, by induction, we have $|A| \leq q^{t-1}.

Thus the total number of $t$-maximal cliques in $G$ is limited by

$|B| + |A| \leq (q - 1)^{t-1} + q^{t-1} = q^t = f(n,t).

The above extremal value is achieved if and only if the following conditions are simultaneously met.

i). $G - v$ is a $t$-cliques extremal graph of order $n - 1$. This means $G - v = T_{n-1,t} = K_{q,\ldots,q,t+1}$ with $t - 1$ partite sets of size $q$ and $1$ partite set of size $q - 1$.

ii). $d(v) = n - q = q(t - 1)$ and $G[N(v)]$ is a $(t - 1)$-cliques extremal graph of order $n - q$. This requires that $G[N(v)] = T_{n-q,t-1} = K_{q,\ldots,q}$ with $t - 1$ partite sets of size $q$.

Altogether, it is implied that $G = T_{n,t}$ is the unique extremal graph in Case 2 as well.

\[\square\]

Remark 2.1. Theorem 1.1 says that the Turán graph $T_{n,t} = K_{q,\ldots,q,t+1}$ is the unique extremal graph of the complementary scenario. That is, $T_{n,t}$ has the maximum number of maximal cliques of size $t$. Equivalently, $T_{n,t} = H = (t-r)K_q + rK_{q+1}$, disjoint union of $t$ cliques of most possibly balanced sizes, is the unique extremal graph that has the maximum number of maximal independent sets of size $t$.

References

[1] Béla Bollobás. Modern graph theory, volume 184 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[2] Reinhard Diestel. Graph theory, volume 173 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 2000.

[3] Zoltán Füredi. The number of maximal independent sets in connected graphs. J. Graph Theory, 11(4):463–470, 1987.

[4] David Galvin. Two problems on independent sets in graphs. Discrete Math., 311(20):2105–2112, 2011.

[5] David Galvin and Yufei Zhao. The number of independent sets in a graph with small maximum degree. Graphs Combin., 27(2):177–186, 2011.
[6] Wenying Gan, Po-Shen Loh, and Benny Sudakov. Maximizing the number of independent sets of a fixed size. *Combin. Probab. Comput.*, 24(3):521–527, 2015.

[7] Han Hu, Toufik Mansour, and Chunwei Song. On the maximal independence polynomial of certain graph configurations. *Rocky Mountain J. Math.*, 47(7):2219–2253, 2017.

[8] Jenq-Jong Lin and Min-Jen Jou. The largest number of maximal independent sets in quasi-unicyclic graphs. *Util. Math.*, 111:85–93, 2019.

[9] Min-Sheng Lin. Counting independent sets and maximal independent sets in some subclasses of bipartite graphs. *Discrete Appl. Math.*, 251:236–244, 2018.

[10] R. E. Miller and D. E. D. E. Muller. A problem of maximum consistent subsets. *IBM Res. Rep. RC*-240, J. T. Watson Research Center, Yorktown Heights, NY, 1960.

[11] J. W. Moon and L. Moser. On cliques in graphs. *Israel J. Math.*, 3:23–28, 1965.

[12] Ashwin Sah, Mehtaab Sawhney, David Stoner, and Yufei Zhao. The number of independent sets in an irregular graph. *J. Combin. Theory Ser. B*, 138:172–195, 2019.

[13] Douglas B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.

[14] Herbert S. Wilf. The number of maximal independent sets in a tree. *SIAM J. Algebraic Discrete Methods*, 7(1):125–130, 1986.

[15] David R. Wood. On the number of maximal independent sets in a graph. *Discrete Math. Theor. Comput. Sci.*, 13(3):17–19, 2011.

[16] Yufei Zhao. The number of independent sets in a regular graph. *Combin. Probab. Comput.*, 19(2):315–320, 2010.