AN OPTIMAL BOUND ON THE NUMBER OF MOVES FOR THE OPEN MANCALA

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ABSTRACT. We prove the optimal bound for the maximal number of moves required to reach a periodic configuration of the open mancala (also called open owari or with other names), inspired by a popular African game.

The proof relies on the notion of $s$-monotonicity, which is a stronger version of the mere monotonicity.

1. INTRODUCTION

The mancala is a family of traditional African games that can be played in many versions and under many different names. It basically comprises a circular list of holes containing zero or more seeds. The basic move consists of selecting a nonempty hole, taking all its seeds and sowing them counterclockwise in the subsequent holes, one seed per hole.

We shall idealize the game by taking an infinite sequence of holes and always selecting the leftmost nonempty hole, sowing to the right. The nonempty holes are supposed to be all consecutive. In this way after each move the leftmost nonempty hole always advances of one position. This is the so called open mancala (see [Bon1], where the game is called open owari).

Although configurations with an infinite number of seeds can theoretically be considered (either an infinite number of seeds in some holes or an infinite number of nonempty holes, provided that there is a leftmost nonempty one), we shall only focus on configurations with a finite total number of seeds and where the nonempty holes are contiguous. Such configurations are described by a finite sequence of positive integers $\lambda_i$, $i = 1, \ldots, \ell$, indicating the number of seeds contained in the set of nonempty holes, starting from the leftmost one. It is clear that the sequence of moves becomes periodic in a finite time, and periodic configurations have been completely classified: they are the so-called (marching groups and augmented marching groups (see [Bon1, Bru, Bon2]), introduced in Definitions 2.8 and 2.9.

In 1983 Martin Gardner brought attention to a card game called Bulgarian Solitaire in his well-known column of Scientific American [Gar], and mentioned a conjecture about the maximal length of a sequence of moves before the onset of periodicity, in the case the number of cards is $n = k(k+1)/2$. The conjecture was proved independently a few years later by [Igu] and [Eti]. In 1998, Griggs & Ho [GrHo] revised the result and proposed a new conjecture on the optimal bound for a generic $n$. The game described by Gardner is isomorphic to the monotone version of the open mancala, where $\lambda_i \geq \lambda_j$ for $i \leq j$.

We deal here with the mancala game (not the monotone variant), for which very few results are known. The main result of the paper is the proof of the optimal bound for the maximal number of moves before reaching periodicity. In Section 3 we prove the lower bound by producing, for any number $n$ of seeds, a configuration that takes exactly the conjectured number of maximal moves. In Section 4 we introduce the main tool in proving the upper bound, that is $s$-monotonicity (see
In Section 5, which is the main part of the paper, we prove that such a number cannot be exceeded (Theorem 5.9).

2. Notations and setting

We denote by \( \mathbb{N} \) the set of nonnegative integers and by \( \mathbb{N}^* \) the set of strictly positive integers. By \( T_k, k \in \mathbb{N} \), we denote the \( k \)-th triangular number

\[
T_k = \sum_{i=0}^{k} i = \frac{k(k+1)}{2},
\]

where in particular \( T_0 = 0 \).

**Definition 2.1.** A configuration \( \lambda \) is a sequence of nonnegative numbers \( \lambda_i, i \geq 1 \), that is \( \lambda: \mathbb{N}^* \rightarrow \mathbb{N} \). The support \( \text{supp}(\lambda) \) of a configuration \( \lambda \) is the set of indices \( \{ i \in \mathbb{N}^*: \lambda_i > 0 \} \).

**Definition 2.2.** A mancala configuration \( \lambda \) is a configuration having a connected support of the form \( \{1, \ldots, \ell\} \) for some \( \ell = \text{len}(\lambda) \in \mathbb{N} \). The index \( \text{len}(\lambda) \) is called the length of the configuration. In the special case of the zero configuration, we shall conventionally define its length to be zero. We shall denote by \( \Lambda \) the set of all mancala configurations.

**Definition 2.3.** If \( \lambda \in \Lambda \), its mass is the number

\[
|\lambda| = \sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\text{len}(\lambda)} \lambda_i.
\]

We denote by \( \Lambda_n \) the set of all mancala configurations of mass \( n \).

The mancala game, in our setting, is a discrete dynamical system associated with a function \( M : \Lambda \rightarrow \Lambda \) mapping the space of mancala configurations \( \Lambda \) in itself.

**Definition 2.4** (Sowing). The sowing of a mancala configuration \( \lambda \) is an operation on \( \lambda \) that results in a new configuration \( \mu = M(\lambda) \) defined as follows:

\[
\mu_i = \begin{cases} 
\lambda_{i+1} + 1 & \text{if } 1 \leq i \leq \lambda_1, \\
\lambda_{i+1} & \text{if } i > \lambda_1.
\end{cases}
\]

Conventionally, \( M \) maps the empty configuration into itself.

It is clear that \( M \) preserves the mass of a configuration, so that it can be restricted to \( \Lambda_n \).

It is convenient to define the (right) shift operator \( \mathcal{E} \), that acts on generic sequences simply by a change in the indices.

**Definition 2.5** (Shift operator). If \( \lambda : \mathbb{N}^* \rightarrow \mathbb{N} \) is a sequence, the sequence \( \mathcal{E}(\lambda) \) is defined by

\[
\mathcal{E}(\lambda)_i = \begin{cases} 
0 & \text{if } i = 1, \\
\lambda_{i-1} & \text{if } i > 1.
\end{cases}
\]

Clearly, the result of the shift operator is never a mancala configuration (with the exception of the empty configuration).

Both \( M \) and \( \mathcal{E} \) can be iterated, the symbol \( M^k \) (resp. \( \mathcal{E}^k \)) denoting the result of \( k \) repeated applications of \( M \) (resp. of \( \mathcal{E} \)). The shift operator has a left inverse \( \mathcal{E}^{-1} \), defined by \( \mathcal{E}^{-1}(\lambda)_1 = \lambda_{i+1} \) and satisfying \( \mathcal{E}^{-1}(\mathcal{E}(\lambda)) = \lambda \) for any configuration \( \lambda \).
Definition 2.6 (Partial ordering and sum). We say that $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all $i \geq 1$. Moreover we say that $\lambda < \mu$ if $\lambda < \mu$ and $\lambda \neq \mu$. If $\lambda, \mu : \mathbb{N}^* \to \mathbb{Z}$ are two integer sequences (in particular if any of them is a configuration in $\Lambda$), we define the sum and difference $\lambda \pm \mu$ componentwise: $(\lambda \pm \mu)_i = \lambda_i \pm \mu_i$.

Remark 2.1 (Comparison). It is easy to check that if $\lambda, \mu \in \Lambda$ and $\lambda \leq \mu$, then $\mathcal{M}(\lambda) \leq \mathcal{M}(\mu)$.

Definition 2.7. A configuration is called monotone if $\lambda_i \geq \lambda_j$ whenever $i \leq j$.

Remark 2.2 (Monotone mancala). If $\lambda$ is a monotone configuration, then so it is $\mathcal{M}(\lambda)$. Moreover, if $\lambda \in \Lambda$ has length $\ell$, then $\mathcal{M}^{\ell-1}(\lambda)$ is monotone.

The following special configurations play an important role.

Definition 2.8 (Marching group). For a given $k \in \mathbb{N}$ we define the special mancala configuration $\text{mg}_k^k$ (marching group of order $k$) with mass $T_k$ and length $k$ as follows:

$$\text{mg}_k^k = \begin{cases} k - i + 1 & \text{if } i \leq k, \\ 0 & \text{otherwise}. \end{cases}$$

It is readily seen that $\mathcal{M}(\text{mg}_k^k) = \text{mg}_k^k$, i.e. the marching groups are fixed points for $\mathcal{M}$. Moreover they are the only fixed points.

Definition 2.9 (Augmented marching group). A mancala configuration $\lambda$ such that $\text{mg}_k^k \leq \lambda < \text{mg}_{k+1}^{k+1}$ for some $k \in \mathbb{N}$ is called an augmented marching group of order $k$. It satisfies $T_k \leq |\lambda| < T_{k+1}$. These are periodic configurations, in the sense that $\mathcal{M}^{k+1}(\lambda) = \lambda$. Moreover, they are the only periodic configurations [Bouh] Theorem 1].

Notice that a marching group is a particular case of an augmented marching group. The period of an augmented marching group of mass $n = T_k + r$ with $1 \leq r \leq k$ is a divisor of $k + 1$ larger than 1. It can be a proper divisor depending on the actual configuration.

A remarkable fact about the open mancala is that every configuration becomes an augmented marching group, and hence periodic, after a finite number of moves. The paper is devoted to find an optimal bound on such a number of moves which depends only on the mass of the initial configuration.

Definition 2.10 (Depth and diameter). The depth of a configuration $\lambda$ is its distance from the periodic configuration, i.e. the number of moves needed to reach periodicity.

For $n \geq 0$ we call the depth of $\Lambda_n$, denoted by $D(n)$, the maximal depth of all configurations $\lambda$ with $|\lambda| = n$.

The diameter of a configuration is the number of moves before the first repetition. It is equal to its depth plus the length of the period to which we arrive minus one. The maximal diameter of all configurations of mass $n$ is called the diameter of $\Lambda_n$.

Clearly if $n = T_k$ the diameter of $\Lambda_n$ equals its depth $D(n)$, whereas if $n = T_k + r$, $1 \leq r \leq k$ the diameter of $\Lambda_n$ is bounded above by $D(n) + k$.[1]

The number $D(n)$ can be seen also as the depth of a graph. Indeed, for a given $n \geq 1$ we can construct the directed graph $\mathcal{G}_n$ having the configurations in $\Lambda_n$ as nodes and an arc from $\lambda$ to $\mu$ whenever $\mu = \mathcal{M}(\lambda)$. The graph $\mathcal{G}_n$ contains exactly $2^{n-1}$ nodes. A cycle of $\mathcal{G}_n$ correspond to sequences of moves that repeat periodically. A configuration is periodic if it belongs to a cycle.

[1] It is also strictly larger than $D(n)$, more precisely a lower bound is given by $D(n) + m - 1$ where $m$ is the smallest divisor of $k + 1$ larger than 1.
In the case \( n \geq 2 \), if we remove all arcs connecting two periodic configurations (arcs that belong to a cycle), the remaining graph \( G_n^o \) is a disjoint union of trees rooted at a periodic configuration, and in each tree all arcs point toward the root. In this setting, \( D(n) \) represents the depth of \( G_n^o \), i.e. the maximal depth of the trees of \( G_n^o \).

2.1. Energy levels interpretations. We can view the mancala configurations and mancala moves in a way reminiscent of the energy levels for the electrons in an atom.

Let us consider the subset \( L \subset \mathbb{N} \times \mathbb{N} \) given by
\[
L = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq j\}.
\]
It is convenient to think of the integers \( i \) and \( j \) as a numbering of square cells instead of coordinates of points, in a way similar to the familiar sea-battle game. Each square cell in \( L \) can either be empty or contain a single seed.

The column index \( i \) corresponds to one of the holes in the mancala game, so that \( \lambda_i \), where \( \lambda \in \Lambda \), is the total number of occupied cells in column \( i \) of \( L \). We shall also associate an energy to a seed positioned in cell \( (i,j) \), given by its second coordinate \( j \). The \( k \)-th level of \( L \) is the set \( \{(1,k), \ldots, (k,k)\} \) of cells having energy \( k \). Finally we let the seeds free to immediately fall down without changing their column, but decreasing their energy (the second coordinate of their current position) provided the new position (and all those in between) are free.

The mancala configuration \( \lambda \in \Lambda \) then corresponds to a positioning of seeds in \( L \) such that \( (i,j) \) is occupied by a seed if and only if \( 1 \leq j \leq \lambda_i \) (see Figure 1).

A mancala move can now be viewed in this setting as follows. Firstly, each seed is moved one position to the left without changing its energy level, while seeds in column 1 are rotated to the last column of that energy level. Namely, a seed at position \( (i,j) \) is moved in the new position \( (i-1,j) \) if \( i > 1 \), \( (j,j) \) if \( i = 1 \). This is just a left rotation of the seeds of each level, so that cells will receive at most one seed. Secondly we let gravity pack all the seeds in each column at the lowest energy possible.

**Figure 1.** The energy interpretation of the mancala configuration 5 3 2 (first drawing, \( t = 0 \)), and some steps of its evolution until 12 mancala moves (last drawing, \( t = 12 \)). The central dot identifies a gap.
It should be noted that if all energy levels up to level \( k \) are completely filled, then a mancala move does not change levels 1 to \( k \) (up to a rearrangement of the seeds), and all the action takes place from the lowest level that is not completely filled (say \( k + 1 \)) up to the highest level that is not completely empty. The lowest active level is the lowest level that is not completely filled.

The overall energy of a configuration (the sum of the individual energy of every seed) does not change after the first part of the move (the rotation within each energy level) whereas it can decrease during the second part (vertical downward shift).

**Definition 2.11 (gap).** We shall call gap an empty cell at the lowest energy level, say \( k \), that is not completely filled. We shall think of a gap as an absent seed, and in this respect we can follow a gap through mancala moves as it rotates left along its energy level.

A gap can be filled when it receives a seed that falls from above after a move, and this can happen only when the gap is in column \( k \). It should be noted that the filling of a gap at the lowest energy level that is not completely filled is permanent, since there is no possibility of further energy decrease for seeds at that energy level. All gaps will be eventually filled, provided there are sufficiently many seeds at energy levels larger than \( k \).

### 3. Lower bound

Finding a good lower bound \( D^*(n) \) for the depth \( D(n) \) requires to devise appropriate configurations and to compute the number of moves required to reach a periodic configuration (which is an augmented marching group).

**3.1. The biaugmented marching group.** Let \( n = T_k + r \) with \( 2 \leq r \leq k + 1 \).

A \( q \)-biaugmented marching group is a mancala configuration \( \lambda \in \Lambda_n \) of the form

\[
\lambda = mg^k + \varepsilon
\]

where \( mg^k \) is the marching group of order \( k \) and \( \varepsilon = (\varepsilon_i) \) is an increment of the form

\[
\begin{align*}
\varepsilon_1 &= 2 \\
\varepsilon_i &= 1 \quad \text{for } 2 \leq i \leq q + 1 \\
\varepsilon_{q+2} &= 0 \\
\varepsilon_i &\in \{0, 1\} \quad \text{for } q + 3 \leq i \leq k + 1
\end{align*}
\]

for some \( 0 \leq q < k \). The second (resp. fourth) row in the definition is void if \( q = 0 \) (resp. if \( q = k - 1 \)). Clearly such a configuration is not an augmented marching group. Appropriate choice of \( q \) and \( \varepsilon \) (compatible with the definition above) allows to obtain all and only the values of \( r \) such that \( 2 \leq r \leq k + 1 \).

**Proposition 3.1.** The depth of a \( q \)-biaugmented marching group is given by \( t = q(k + 2) + 2 \), i.e. it becomes an augmented marching group exactly after \( q(k + 2) + 2 \) moves.

**Proof.** If \( q = 0 \), a direct check shows that after two moves we obtain an augmented marching group. If \( q > 0 \), let us *color* one of the seeds\(^{(2)}\) at position \( i = 1 \); we agree that the colored seed is the last to be sown in the mancala move. Notice that neglecting the colored seed we have a periodic configuration with period (a factor of) \( k + 1 \).

The pile \( \lambda_1 \) contains \( k + 2 \) seeds, so that the colored seed is again at the leftmost position after exactly \( k + 2 \) moves, which exceeds by one the period of the

\(^{(2)}\) The idea of the colored seed is due to Bouchet; indeed, a similar trick was used in [Bou2, Section 3]
augmented marching group obtained neglecting the colored seed. Hence the sequence of increments (zero or one) for \( i > 1 \) at time \( k + 2 \) is shifted to the left of one position, with the colored seed again in the leftmost pile. Then we reach a \((q - 1)\)-biaugmented marching group, and can conclude by induction. □

As an example, Figure 1 shows the energy-level interpretation of a 2-biaugmented marching group with \( k = 3 \) and its evolution. The gray circle in the pictures is the colored seed of the proof.

### 3.2. The biaugmented marching group of the second kind.

Let \( n = T_k + r = T_{k-1} + k + r \) with \( 0 \leq r \leq k - 2 \). A \( q \)-biaugmented marching group of the second kind is a configuration \( \lambda \in \Lambda_n \) of the form \( \lambda = mg^{k-1} + \varepsilon \), where \( mg^{k-1} \) is the marching group of order \( k - 1 \) and \( \varepsilon = (\varepsilon_i) \) is an increment of the form

\[
\begin{align*}
\varepsilon_i &\in \{1, 2\} \quad \text{for } 1 \leq i < k - q - 1 \\
\varepsilon_{k-q-1} &= 2 \\
\varepsilon_i &= 1 \quad \text{for } k - q \leq i < k \\
\varepsilon_k &= 0
\end{align*}
\]

for some \( 0 \leq q < k - 1 \). Again, such a configuration is not an augmented marching group.

**Proposition 3.2.** The depth of a \( q \)-biaugmented marching group of the second kind is given by \( t = (q+1)k \), i.e. it becomes an augmented marching group precisely after \((q + 1)k\) moves.

**Proof.** Let us color red one of the seeds in all piles with increment \( \varepsilon_i = 2 \) and blue another seed in piles with increment \( \varepsilon_i \in \{1, 2\} \). As before, we agree that the colored seeds are the last to be sown in every mancala move, the red seed (if present) being sown after the blue one. We keep on moving as long as no seeds of the same color end up into the same hole and as long as there is no red seed in a hole without a blue one, with the exception of the hole in position \( k + 1 \) that can contain only a single red seed. We track the evolution of the colored seeds. The leftmost hole contains \( k + 1 \) seeds if both colors are present, \( k \) seeds if only the blue seed is present, \( k - 1 \) seeds otherwise, so that the red seeds always move into the position of index \( k + 1 \) and the blue seeds into the position of index \( k \). It turns out that the pattern of red seeds rotates with a period \( k + 2 \) while the pattern of blue seeds rotate with a period of \( k + 1 \); hence the pattern of red seeds slowly slides one position to the right with respect to the pattern of blue seeds every \( k \) moves, and the value of \( q \) decreases by one. The condition about colors stated above is first violated when we move a pile with both colors and then a pile with no colored seeds: after that, we will have a red seed in the \( k \)-th position with no corresponding blue seed, getting an augmented marching group. If \( q = 0 \), this happens exactly after \( k \) moves. Summarizing we have \((q + 1)k\) moves before the periodic configuration is reached. □

### 3.3. Evolution of the Heaviside configuration.

**Definition 3.1** (Heaviside configuration). We define the Heaviside configuration \( h^n \in \Lambda_n \) by

\[
h^n_i = \begin{cases} 
1 & \text{if } 1 \leq i \leq n, \\
0 & \text{if } i > n.
\end{cases}
\]

We shall also study the special configuration \( h^\infty \) with a single seed in all holes \( i \geq 1 \): although \( h^\infty \) is not in \( \Lambda \) since it has unbounded support, the sowing \( M \) is well-defined also for such infinite configurations.
Proposition 3.3. Let \( k \geq 1 \) and \( t = T_k + r \) with \( 0 \leq r \leq k \). Then the configuration after \( t \) moves starting from \( h^\infty \) is given by

\[
\lambda^{(t)}_i = \begin{cases} 
    k - i + 2 & \text{if } 1 \leq i \leq k - r + 1 \\
    k - i + 3 & \text{if } k - r + 1 < i \leq k + 1 \\
    1 & \text{if } i > k + 1.
\end{cases}
\]

Proof. It is easy to check that exactly at every move \( t \) the configuration becomes greater than the marching group \( m \varpi^i \), hence we have the result for \( r = 0 \). In particular, for any \( 0 \leq r \leq k \) the leftmost element of the sequence at time \( t = T_k + r \) is given by \( \lambda^{(t)}_1 = k + 1 \). Hence in the subsequent move we have to add one seed in the holes with indices \( i = 2, \ldots, k + 2 \) and then shift all piles of one position to the left (dropping the leftmost). If \( r < k \) we get

\[
\lambda^{(t+1)}_i = \lambda^{(t)}_{i+1} + 1 \quad \forall i = 1, \ldots, k + 1
\]

and the result follows by induction. If \( r = k \), then \( t + 1 = T_k + k + 1 = T_{k+1} \) and we have to add one seed to the piles with indices \( i = 2, \ldots, k + 2 \), which contain the values \( k + 1, k, k - 1, \ldots \), to begin with. Again the result follows immediately. \( \square \)

Notice that at time \( \bar{t} = T_{k-1} + r \) we move exactly \( k \) seeds, so that we do not touch the piles at position \( i > k \), which come from \( \bar{t} \) shifts to the left. This means that Proposition 3.3 holds for a starting configuration \( h^n \) with \( n = T_k + r \) (we denote it by \( 1^n \)) until time \( \bar{t} \), with truncation to zero for elements with index \( i > k \). Moreover, the number of seeds in the leftmost pile (index \( i = 1 \)) does not change in the moves between \( t = T_{k-1} + 1 \) and \( t = T_k - 1 \).

3.4. The augmented Heaviside sequences. Now we want to analyze the evolution of the augmented sequence obtained from \( h_n \) by adding a seed at the position with index \( m \). We can write the sequence as

\[ h_n + \delta_m \]

where \( \delta_m \) denotes the sequence with a 1 at the position \( m \) and zero elsewhere. The analysis can be conveniently done by using again Bouchet’s trick, see Footnote 2. In this way the evolution of the augmented configuration coincides exactly with the evolution of the non-augmented configuration with the addition of the colored seed, to be positioned appropriately.

At each sowing the colored seed moves one position to the left; when it reaches the leftmost pile (index \( i = 1 \)), at the next move its new position depends on how large the pile is. We however know precisely how the Heaviside sequence evolves, so that we can explicitly predict the position of the colored seed at each time.

We are particularly interested in the case where \( n = T_k + 1 \) and \( m = T_p + 1 \) with \( p \leq k \). In this case, after \( T_p \) moves the colored seed reaches the leftmost pile which, according to Proposition 3.3 will contain also \( p + 1 \) noncolored seeds. This is true even in the special case \( p = k \), where the colored seed is added at the rightmost nonempty pile of the sequence \( h_n \), see discussion after Proposition 3.3. Hence at time \( T_p + 1 \) the colored seed will go into the hole with index \( p + 2 \) and will be again into the leftmost pile at time \( t = T_p + p + 2 = T_{p+1} + 1 \). If \( p + 1 < k \) we can again resort to the Proposition and conclude that the leftmost pile will contain the colored seed and \( p + 2 \) noncolored seeds at that time. This argument can be repeated for \( q \) cycles as long as \( p + q < k \) with the leftmost pile containing \( p + q + 1 \) noncolored seeds at time \( t = T_{p+q} + q \) and we have the complete evolution.

The method of coloring seeds can be also used with more than one seed, provided that we do not place two or more colored seeds in the same hole and as long as the colored seeds do not interact (staying all in different piles). Thanks to the
previous analysis, we indeed have noninteracting colored seeds for $T_k$ moves if we place them in positions with indices subsequent to triangular numbers, and we can exactly predict their position at time $t = T_k + 1$. In particular there will be a colored seed in the leftmost pile if and only if there was a colored seed in position $i = T_k + 1$ (the rightmost pile) of the starting configuration, there will be a colored seed in the second pile if and only if there was a colored seed in position $T_{k-1} + 1$ of the original configuration, and so on. We hence obtain a $q$-biaugmented marching group, where $q$ is the largest integer for which the original configuration had a colored seed at the subsequent of each triangular number $T_j + 1$ for $j = k - q, \ldots, k$. By Proposition 3.1 the biaugmented marching group will become periodic after exactly $q(k + 2) + 2$ further moves.

Our intent being to devise, for a given $n = T_k + r$, a configuration that takes as long as possible to become periodic, it is then mostly convenient to place the colored $r$ seeds in the rightmost positions that correspond to the subsequents of triangular numbers.

**Definition 3.2.** We call augmented Heaviside sequence the configuration of mass $n = T_k + r$, $1 \leq r \leq k + 1$, and length $\ell = T_k + 1$, which is obtained from $h_k$ by adding a seed in the rightmost $r - 1$ positions having index of the form $T_j + 1$ (in particular there will be no added seeds if $r = 1$).

As an example, the augmented Heaviside sequence with mass $n = 19 = T_5 + 4$ is given by

$$1 1 1 1 1 1 2 1 1 1 2 1 1 1 1 2.$$

The preceding arguments give a proof of the following

**Proposition 3.4.** Let $n = T_k + r$ with $1 \leq r \leq k + 1$. Then the corresponding augmented Heaviside sequence becomes an $(r-2)$-biaugmented marching group after $T_k$ moves. In particular, its evolution becomes periodic after exactly $D_1^*(n)$ moves with

$$D_1^*(n) := T_k + (r - 2)(k + 2) + 2. \quad (3.1)$$

The choice $r = k + 1$ is a limit case corresponding to $n = T_{k+1}$, and we have $D_1^*(T_{k+1}) = T_k + (k - 1)(k + 2) + 2 = 3T_k$. Moreover if we write $n = T_{k+1} - s$ with $0 \leq s < k$, then $s = k + 1 - r$ and $D_1^*(n)$ in (3.1) can be equivalently written as $D_1^*(n) = 3T_k - s(k + 2)$.

Direct inspection allows to show that for $r < k + 1$ the particular biaugmented marching group at which we arrive at time $T_k$ evolves into an augmented marching group having period of length exactly $k + 1$ (and not a proper divisor). Hence the diameter of the evolution is $D_1^*(n) + k$.

### 3.5. Truncated augmented Heaviside sequences

We now consider the augmented Heaviside sequence corresponding to $T_{k+1}$: it contains an increment at all the positions corresponding to subsequents of triangular numbers, and the total number of seeds is itself a triangular number. In the rightmost positions, this sequence has a 2 followed by $k - 1$ ones and a final 2. Given $0 \leq r \leq k - 1$, let us remove from this sequence the rightmost $k - r$ piles, obtaining a configuration with mass $T_k + r$ (equivalently, we can obtain the same configuration starting from the augmented Heaviside sequence corresponding to $T_k$ and joining to the right a sequence of $r$ piles with one seed each).

**Definition 3.3.** Let us call truncated augmented Heaviside sequence such a configuration.

As an example, the truncated augmented Heaviside sequence with mass $n = 17 = T_5 + 2$ is given by
Corollary 3.6. A truncated augmented Heaviside sequence with \( n = T_k + r \), \( 0 \leq r \leq k - 2 \), becomes a \((k - r - 2)\)-biaugmented marching group of the second kind after \( T_{k-1} \) moves.

Proof. Simply color the rightmost sequence of ones and track their position during the first \( T_{k-1} \) moves. \( \square \)

Corollary 3.6. A truncated augmented Heaviside sequence with \( n = T_k + r \), \( 0 \leq r \leq k - 2 \), becomes periodic after exactly

\[
D_2^r(n) := T_k + k(k - r - 2) = 3T_{k-1} - rk
\]  

moves.

If \( n = T_k + k \) or \( n = T_k + k - 1 \), the function \( D_2^r(n) \) is not defined by the previous Corollary, and we shall conventionally set it to \(-1\).

As for the augmented Heaviside sequences of Section 3.3, the special biaugmented marching group of the second kind at which we arrive if \( r > 0 \) evolves to an augmented marching group with period exactly \( k+1 \), and again we have a diameter given by \( D_2^r(n) + k \).

3.6. The special “plateau” sequence. In the special case \( n = T_{2r} + r \), \( r \geq 1 \), we need a further type of configuration.

Definition 3.4. For \( n = T_{2r} + r \), \( r \geq 1 \), we define the biaugmented Heaviside sequence in the following way: take the truncated augmented Heaviside sequence for \( n - 1 = T_{2r} + r - 1 \), and then add a third seed in the pile with two seeds at position \( i = T_{2r-1} + 1 \).

As an example, \( n = 24 = T_6 + 3 \) produces the sequence

1 2 1 2 1 1 2 1 1 1 2 1 1 1 3 1 1

Proposition 3.7. The biaugmented Heaviside sequence with \( n = T_{2r} + r \) becomes periodic after exactly

\[
D_3^r(n) := T_{2r} + 2r(r - 1)
\]  

moves.

Proof. Let us color the extra added third seed in the pile at position \( i = T_{2r-1} + 1 \). By construction, without the colored seed the biaugmented Heaviside sequence becomes a truncated augmented Heaviside sequence with mass \( T_{2r} + r - 1 \), hence by Proposition 3.5, after \( T_{2r-1} \) moves we have an \((r - 1)\)-biaugmented marching group of the second kind with an extra seed (corresponding to an increment of \(+3\) with respect to the marching group) in the leftmost pile. The argument we used to study the biaugmented marching groups can be adapted to the present case, proving that the periodic configuration is reached exactly after the same number of moves as for the biaugmented marching group of the second kind: \((q + 1)k\) with \( q = r - 1 \) and \( k = 2r \). Hence the initial configuration becomes periodic after

\[ T_{2r-1} + 2r^2 = T_{2r-1} + 2r^2 + 2r(r - 1) = T_{2r} + 2r(r - 1). \]  

\( \square \)

Also in this case a direct inspection of the resulting augmented marching group allows to show that its period is exactly \( k+1 \), and again the diameter of the special plateau sequence is \( D_3^r(n) + k \).

The lower bound \( D_3^r(n) \) is only defined for special values of \( n \), of the form \( n = T_{2r} + r = 4T_r \), and we conventionally set it to \(-1\) in all the other cases.

We can now gather the three lower bounds and define

\[
D^*(n) := \max\{D_1^r(n), D_2^r(n), D_3^r(n)\}
\]  

(3.4)
so that the previous results can be summarized as the following:

**Theorem 3.8** (Lower bound).

\[ D(n) \geq D^*(n). \]

**Proof.** Simply gather the results obtained in Propositions 3.4, 3.7 and Corollary 3.6.

A plot of the function \( D^*(n) \) for \( n \leq 45 \) can be seen in Figure 2.

![Figure 2](image)

**Figure 2.** The values of the function \( D^*(n) \) for \( n \leq 45 \). It is \( D^*(45) = 108 \).

**Remark 3.1.** There is an explicit way to express the lower bound \( D^*(n) \). Indeed, writing \( n \geq 1 \) in the form \( n = T_k + r \) with \( k \geq 0 \) and \( 1 \leq r \leq k + 1 \), one can prove that

\[
D^*(n) = \begin{cases} 
D^*_2(n) = T_k + k(k - r - 2) & \text{if } r < k/2 \\
D^*_3(n) = T_{2r} + 2r(r - 1) & \text{if } r = k/2 \\
D^*_1(n) = T_k + (r - 2)(k + 2) + 2 & \text{if } r > k/2.
\end{cases}
\]

Notice that the middle case occurs only when \( k \) is even.

4. **s-Monotonicity**

In this section we introduce one of the main notions of the paper, namely \textit{s-monotonicity}. Firstly we define a sequence which is strictly related to the energy levels interpretation of the mancala game.

**Definition 4.1** (Energy sequence). Given a mancala configuration \( \lambda \in \Lambda \) we define the energy sequence \( e = e(\lambda) : \mathbb{Z} \to \mathbb{N} \) as follows: setting \( m = m(\lambda) = \max\{\lambda_1, \text{len}(\lambda)\} \), we define

\[
e_i = \lambda_i + i - 1 \quad i = 1, \ldots, m
\]

and extend it on \( \mathbb{Z} \) as a periodic function of period \( m \).

Notice that if \( \lambda_1 > \text{len}(\lambda) \) we actually use values of \( \lambda \) outside its support, so that we have \( e_i = i - 1 \) for \( \text{len}(\lambda) < i \leq \lambda_1 \).
Remark 4.1. We have that $\lambda \geq mg^k$ if and only if $\varepsilon(\lambda)_i \geq k$ for all $i \in \mathbb{Z}$.

Definition 4.2 (s-monotonicity). Given $s \geq 1$ and $\lambda \in \Lambda$, we say that $\lambda$ is $s$-monotone if:

(i) $\lambda_1 \leq \text{len}(\lambda) + 1$, and

(ii) its energy sequence $e = \varepsilon(\lambda)$ satisfies the inequality

$$ e_j \leq e_i + 1 \quad \forall i < j \leq i + s $$

(4.2)

(notice that the first requirement $\lambda_1 \leq \text{len}(\lambda) + 1$ is redundant if $s \geq 2$).

It is convenient to introduce also the (void) notion of $0$-monotonicity, satisfied by all mancala configurations.

Definition 4.3 (Increasing plateau). Let $s \geq 0$. An increasing plateau of length $s$ is a subset of $(s + 2)$ consecutive values $e_i, e_{i+1}, \ldots, e_{i+s+1}$ of an energy sequence such that

$$ e_i < e_{i+1} = \cdots = e_{i+s} < e_{i+s+1} $$

if $s \geq 1$

$$ e_i \leq e_{i+1} - 2 $$

if $s = 0$.

It is readily proved that a configuration is $s$-monotone if and only if each increasing plateau has length at least $s$.

Let us give a few examples of $s$-monotone configurations:

- $1 \ 2$ is $0$-monotone but not $1$-monotone
- $1 \ 1 \ 1$ is $1$-monotone but not $2$-monotone
- $4 \ 1 \ 1$ and $2 \ 2 \ 1 \ 1$ are $2$-monotone but not $3$-monotone
- $5 \ 2 \ 2 \ 1$ and $3 \ 3 \ 2 \ 1 \ 1$ are $3$-monotone but not $4$-monotone
- $6 \ 3 \ 3 \ 2 \ 1 \ 1$ and $4 \ 4 \ 3 \ 2 \ 1 \ 1$ are $4$-monotone but not $5$-monotone.

Remark 4.2. If a mancala configuration is $s_2$-monotone, then it is also $s_1$-monotone for any $s_1 < s_2$. Moreover, a configuration is an augmented marching group if and only if it is $s$-monotone for all $s \in \mathbb{N}$.

Remark 4.3. The notions of monotonicity, in the sense of Definition 2.7, and of $1$-monotonicity, in the sense of the previous definition, are slightly different. For instance, the sequence $4 \ 2$ is monotone but not $1$-monotone. However, if a configuration $\lambda$ is monotone, then $\mathcal{M}(\lambda)$ is $1$-monotone. Hence, in view also of next theorem, during the evolution of a configuration there is at most one time when it is monotone but not $1$-monotone.

In passing, we observe that any $1$-monotone sequence has a predecessor that is merely monotone, i.e. a $1$-monotone configuration can be obtained by applying a move to some monotone configurations. Hence the maximal depth/diameter in the family of monotone configurations is exactly one plus the maximal depth/diameter in the family of $1$-monotone configurations.

The following theorem shows that the class of $s$-monotone configurations is closed under mancala moves.

Theorem 4.1. If $\lambda$ is $s$-monotone, then $\mathcal{M}(\lambda)$ is $s$-monotone.

Proof. Let us set $m = \max\{\lambda_1, \text{len}(\lambda)\}$, $\mu = \mathcal{M}(\lambda)$, $\epsilon = \varepsilon(\lambda)$ and $f = \varepsilon(\mu)$.

Clearly any mancala configuration is at least $0$-monotone, hence the set of $0$-monotone configurations is closed under mancala moves. A direct check shows that the property $\lambda_1 \leq \text{len}(\lambda) + 1$ is preserved under a mancala move whenever $\lambda$ is a monotone configuration, indeed we have $\lambda_2 \leq \lambda_1 \leq \text{len}(\mu)$, whence $\mu_1 = \lambda_1 + 1 \leq \text{len}(\mu) + 1$. Consequently the set of $1$-monotone configurations is closed under mancala moves.
Now assume that $s > 1$, hence we have to check only the values of the energy sequence. We divide the proof in five cases.

**Case** $\lambda_1 \geq \text{len}(\lambda)$. We have $m = \text{len}(\mu) = \lambda_1$ and the new period is $m(\mu) = \max\{\mu, \text{len}(\mu)\} = \max\{\lambda_2 + 1, \lambda_1\}$. Since $s > 1$, then $\lambda$ is monotone and $\lambda_2 \leq \lambda_1$, hence we have $m \leq m(\mu) \leq m + 1$ and the new energy sequence $f$ is obtained by a left-shift of $e$ possibly followed by the insertion of a new value. Indeed $f_{i} = \mu_{i} + i - 1 = \lambda_{i+1} + 1 + i - 1 = e_{i+1}$ for $i = 1, \ldots, m - 1$ and $f_{m} = \mu_{m} + m - 1 = m$, since $\mu_{m} = 1$. If $\lambda_2 < \lambda_1$, then $m(\mu) = m$ and the new energy sequence $f$ is exactly a left-shift of $e$, so that $\mu$ is itself $s$-monotone. If $\lambda_2 = \lambda_1$, then $m(\mu) = m + 1$ and the new value to be inserted is $f_{m+1} = \mu_{m+1} + m + 1 - 1 = m$ (being $\mu_{m+1} = 0$) and it is equal to the previous value $f_{m}$. Since inserting a value in the sequence cannot destroy the $s$-monotonicity property if it is equal to the previous (or the next) value, we can conclude that $\mu$ is $s$-monotone.

**Case** $\lambda_1 \leq \text{len}(\lambda) - 3$. We have $m = \text{len}(\lambda)$ and $\text{len}(\mu) = \text{len}(\lambda) - 1$. Since $\lambda_2 \leq \lambda_1$ we have $\mu_1 \leq \lambda_1 + 1 \leq \text{len}(\lambda) - 2 = \text{len}(\mu) - 1$, so that $f_{1} \leq m - 2$ and $m(\mu) = m - 1$. Moreover, $\lambda_1 < m - 1$ implies that $\mu_{m-1} = \lambda_m$, so that $f_{m-1} = \mu_{m-1} + m - 2 = \lambda_m + m - 2 \geq m - 1$. Using the inequalities above we conclude

$$f_{1} < f_{m-1} = f_{0}. \quad (4.3)$$

The energy sequence $f$ is obtained from the energy sequence $e$ with the following operations:

1. translate the values of $e$ to the left of one position $e'_{i} = e_{i+1}$. This has no impact on the $s$-monotonicity;
2. decrease by one the values with indices in the range $\lambda_1 + 1, \ldots, m$;
3. remove the value at index $m$ and extend to $\mathbb{Z}$ with period $m - 1$.

The last two operations can only impact $s$-monotonicity in the case of indices $i \leq 0 < j$ with $j - i \leq s$ (up to addition of multiples of $m - 1$). However using (4.3) we have $f_{j} - f_{i} < (f_{j} - f_{1}) + (f_{0} - f_{1}) \leq 1 + 1$.

**Case** $\lambda_1 = \text{len}(\lambda) - 2$. The argument is similar to the previous, but (4.3) now becomes

$$f_{1} \leq f_{m-1} = f_{0}. \quad (4.4)$$

Moreover the set of indices (within the period $\{1, \ldots, m\}$) where we decrease the energy by one (step 2 above) reduces to $\{m - 1, m\}$ and is further reduced to the single index $m - 1$ after removal of the energy value with index $m$ (step 3 above). Now suppose by contradiction that $f_{j} - f_{i} \geq 2$ with $i \leq 0 < j$, $j - i \leq s$. This is only possible if both $f_{j} - f_{1} = 1$ and $f_{0} - f_{1} = 1$. The latter equation implies $i < 0$ and, in view of the discussion above, it can be rewritten as $1 = f_{m-1} - f_{m-1+i} = e_{m-1} - e_{m+i}$ which contradicts the $s$-monotonicity of $\lambda$.

**Case** $\lambda_1 = \text{len}(\lambda) - 1$ and $\lambda_2 < \lambda_1$. It turns out that $m(\mu) = \text{len}(\lambda) - 1 = m - 1$. Moreover $e_{0} = e_{m} = \lambda_{m} + m - 1 \geq m$ whereas $e_{1} = \lambda_{1} < m$, so that $e_{1} < e_{0}$. The resulting energy $f$ can be obtained by a left shift of $e$ followed by removal of element at index $m$, which is the same as first removing the element at index 1 of $e$ and then performing a left shift. The fact that $e_{0} > e_{1}$ allows to ensure that the removal of the value $e_{1}$ does not impact the $s$-monotonicity, which is then unaffected by the final left shift.

**Case** $\lambda_1 = \text{len}(\lambda) - 1$ and $\lambda_2 = \lambda_1$. It turns out that in this case $m(\mu) = m(\lambda) = m = \text{len}(\lambda)$ and that the energy sequence of $\mu$ is just a left-shift of the energy sequence of $\lambda$: $f_{i} = e_{i+1}$.

These five cases cover all possible situations so that we conclude the proof. □

For particular configurations it is possible to prove that repeated mancala moves actually increase the order of monotonicity.
The following is our first important result for which the distinction between monotonicity and 1-monotonicity is crucial. For instance, the result is false for the sequence $4, 2$, which is monotone but not 1-monotone.

**Lemma 4.2.** Let $\lambda \in \Lambda$, $s \in \mathbb{N}$ and $2 \leq q \in \mathbb{N}$ such that

(i) $\lambda$ is $s$-monotone;

(ii) $\lambda \geq mg^{q-1}$;

(iii) $\text{len}(\lambda) = q - 1$, (i.e. $\lambda_q = 0$).

Then $M^q(\lambda)$ (the configuration after $q$ moves) is $(s + 1)$-monotone.

**Proof.** Denote by $\mu$ the configuration obtained after $q$ moves. Since we apply a number of moves that is larger than the length of the sequence, it follows that $\mu$ is at least 1-monotone (it is already monotone after $q - 2$ moves, hence 1-monotone after $q - 1$ moves by Remark 4.3). This covers the case $s = 0$, and we can suppose $s \geq 1$. In particular we can assume that $q - 1 \leq \lambda_1 \leq q$. \(\square\)

Using the energy level interpretation of the game, we want to track the position after $q$ moves of the seeds with energy larger than $q$. First observe that requirement (ii) implies that all energy levels up to the $(q - 1)$-th are completely filled and that seeds at level $q$ (which is not completely filled) are subject to $q$ rotations to the left and end up at the same initial position. A seed in column $i$ (notice that requirement (iii) implies that $i < q$) at level $q + j$ is also rotated $q$ times to the left, however the $(q + j)$-th energy level has length $q + j$, so that it will end up in column $i + j$ and can possibly decrease its energy (that is, its height) once or more than once. Requirements (i) and (iii) imply that there is no seed at height $q + 1$ in the first $s$ columns, whereas seeds in column $s + 1$ and height at least $q + 1$ get moved to the right after the set of $q$ moves. This allows to conclude that $f_i = 0$ for all $i \leq j, j - i \leq s + 1$, where $f = \varepsilon(\mu)$ is the energy sequence of $\mu$ given by Definition 4.1. If there were no energy decrease, then all increasing plateaus would increase their length by 1 and the $(s + 1)$-monotonicity would follow. The decrease in energy after each move involves all seeds in the final columns and it can be easily checked that it maintains the $(s + 1)$-monotonicity. \(\square\)

**Definition 4.4** ($q$-canonical configurations). Let $2 \leq q \in \mathbb{N}$, $n \geq T_q$ and $\lambda \in \Lambda_n$. We say that $\lambda$ is $q$-canonical if

(1) $\lambda \geq mg^{q-1}$;

(2) $\text{len}(\lambda) = q - 1$ \((\ast)\)

(3) the gap (see Definition 2.11) in the rightmost position (column $q$) in the energy level at height $q$ is the last to be filled after repeated moves. Notice that requirement $n \geq T_q$ implies that the $q$-th energy level will eventually fill up completely.

Due to the central importance of $q$-canonical configurations a few remarks are in order.

**Remark 4.4.** The value $q$ of a $q$-canonical configuration is necessarily the height of the lowest active energy level (the lowest level that is not completely filled). This follows from requirements (1) and (2) of the above definition. Hence a configuration cannot be $q$-canonical with two different values of $q$.

**Remark 4.5.** In view of the comments following Definition 2.11, in particular the fact that filling a gap is permanent, we can indeed define a filling order on the
The configuration $55362$, shown on the left with its energy-level interpretation, is not a 6-canonical configuration. The three gaps at level 6 are filled in the order given by the corresponding numbers. After three mancala moves we obtain the configuration $95322$ (middle picture), which is a 6-canonical configuration. Another 6-canonical configuration is obtained after 6 further moves.

**Remark 4.6.** An augmented marching group cannot be $q$-canonical, since if $\lambda \in \Lambda_n$ is an augmented marching group with $n \geq T_q$ then $\lambda \geq mq^q$. This is incompatible with requirement (2) of Definition 4.4.

**Remark 4.7.** Since the gap in the rightmost position of a $q$-canonical configuration $\lambda$ returns in the same position after every sequence of exactly $q$ moves, it follows that the $q$-th energy level fills up after exactly $rq$ moves, for some $r \in \mathbb{N}^*$. In particular $\mathcal{M}^q(\lambda) \geq mq^q$. Moreover $\mathcal{M}^q(\lambda)$ is itself a $q$-canonical configuration for all $i = 0, \ldots, r - 1$ and there are no other $q$-canonical configurations in between.

As an example, let us consider the configuration $\lambda = 55362$, it satisfies requirements (1) and (2) of Definition 4.4 with $q = 6$ (see Figure 3 for an energy-level interpretation), it has three gaps at the energy level 6 with three seeds at higher level, enough to completely fill the sixth energy level ($|\lambda| = 21 = T_6$). However this configuration is not a 6-canonical configuration because the rightmost gap in column 6 is not the last filled. This can only be seen by letting the configuration evolve and annotating the filling order of the three gaps. In this example only seven mancala moves are sufficient to see that the gap in column 6 is filled first, whereas the gap that is filled last is the one originally in column 3. The configuration $\lambda = 95322$ (obtained from the previous one with three mancala moves) is indeed a 6-canonical configuration. As noted in Remark 4.7, after six mancala moves we have either another 6-canonical configuration or the sixth level becomes completely filled. It turns out that we have a total of four other 6-canonical configurations, one after each sequence of six moves starting from the 6-canonical configuration $95322$.

**Theorem 4.3.** A $q$-canonical configuration cannot be $(q - 1)$-monotone. In other words it is at most $q - 2$-monotone.

**Proof.** Let $\lambda$ be a $q$-canonical configuration, $q \geq 2$. If $q = 2$, the lowest value possible, $\lambda$ is 2-canonical (Definition 4.4) implies $|\lambda| \geq T_2 = 3$ and (requirement (2)) $\lambda$ reduces to the single number $\lambda_1$. Hence $\lambda_1 = |\lambda| \geq 3$ which is in conflict with requirement (i) of 1-monotonicity (see Definition 4.2). If $q \geq 3$ suppose by contradiction that $\lambda$ is $q - 1$ monotone. Set $e = e(\lambda)$ (the energy sequence), with period $m(\lambda) = \lambda_1 \geq q - 1$. Either $e_1 = \lambda_1 = q - 1$ or $e_q = q - 1$. Energy levels strictly larger than $q$ must also be present, otherwise $\lambda$ would be an augmented marching group, incompatible with $q$-canonicity. Hence there must exist an increasing plateau, and its length cannot be longer than $m(\lambda) - 2$ due to the periodicity of $e$. $q - 1$-monotonicity then implies $\lambda_1 = m(\lambda) \geq q + 1$ which directly contradicts 2-monotonicity due to the presence of an increasing plateau of length 1. \qed
5. Upper bound

5.1. Analyzing the evolution of a configuration. In this section we shall fix \( n \geq T_k \) for some \( k \in \mathbb{N} \) and consider a mancala configuration \( \lambda \in \Lambda_n \). We let \( \lambda \) evolve (applying mancala moves) until it reaches an augmented marching group; we denote by \( \lambda(t) = \mathcal{M}^t(\lambda) \) the result of \( t \in \mathbb{N} \) mancala moves, so that in particular \( \lambda(0) = \lambda \). Let \( \tilde{t} \) be the smallest time \( t \) for which \( \lambda(t) \) is an augmented marching group, in particular \( \lambda(\tilde{t}) \geq mg^k \). We also denote by \( t^* \leq \tilde{t} \) the smallest time \( t \) such that \( \lambda(t) \geq mg^k \).

During the evolution \( \lambda(t) \) will occasionally become a \( q \)-canonical configuration. Specifically we indicate with \( t_1 < t_2 < \cdots < t_s < t^* \) the times before \( t^* \) when we have a \( q \)-canonical configuration for some \( q \in \mathbb{N} \). We call such times critical times. Specifically we have a sequence \( q_1 \leq q_2 \leq \cdots \leq q_s \) such that \( \lambda(t_i) \) is a \( q_i \)-canonical configuration.

Consider two consecutive critical times \( t_i \) and \( t_{i+1} \). Remark 4.7 applied to \( \lambda(t_i) \) tells us that after \( q_i \) moves, we have either another \( q_i \)-canonical configuration, with no other \( q \)-canonical configurations in between, in which case \( t_{i+1} - t_i = q_i \) and \( q_{i+1} = q_i \), or we have just filled the \( q_i \)-th energy level so that we have \( t_{i+1} - t_i = q_i \) and \( q_{i+1} > q_i \). In both cases we have \( t_{i+1} - t_i = q_i \). We can now apply Lemma 4.2 to \( \lambda(t_i) \) with \( q = q_i \) and conclude that the \( s \)-monotonicity of \( \lambda(t_{i+1}) \) increases strictly. Since \( \lambda(t_1) \) is \( 0 \)-monotone, we obtain by induction that \( \lambda(t_{i+1}) \) is \( i \)-monotone and that \( \lambda(t^*) \) is \( s \)-monotone.

It is possible for the set of critical times to be empty. In this case we shall see that the filling up of the energy levels is very fast.

Lemma 5.1. If \( \lambda \geq mg^{s-1} \) and \( |\lambda| \geq T_q \), then there exists \( 0 \leq i < q \) such that either \( \mathcal{M}^i(\lambda) \geq mg^q \) or \( \mathcal{M}^i(\lambda) \) is a \( q \)-canonical configuration. If \( \lambda \) is itself \( q \)-canonical, the result holds true with \( i = q \) (besides the trivial value \( i = 0 \)).

Proof. If the \( q \)-th energy level is full, then there is nothing to prove, otherwise locate the gap that will be filled last in column \( j \). If \( j = q \) we already have a \( q \)-canonical sequence, otherwise we apply \( j \) moves in order to rotate the gap position into column \( q \), now we either have a \( q \)-canonical sequence or we filled the \( q \)-th energy level. The second part of the Lemma easily follows by first performing a single move and then reasoning as in the first part. \( \square \)

Corollary 5.2. If the set of critical times is empty \( (s = 0) \), then one has \( t^* \leq T_{k-1} \), i.e.

\[
\lambda(T_{k-1}) \geq mg^k.
\]

Proof. It can be readily seen by repeated applications of the previous Lemma. \( \square \)

Similarly, the \( q_1 \)-canonical sequence \( \lambda(t_1) \) is reached after at most \( T_{q_1-1} \) moves, i.e. \( t_1 \leq T_{q_1-1} \).

Now let

\[
I = \{ i \in \mathbb{N} : 1 \leq i \leq s - 1, \ q_{i+1} > q_i \},
\]

\[
J = \{ i \in \mathbb{N} : 1 \leq i \leq s - 1, \ q_{i+1} = q_i \}
\]

(\text{notice that } I \cup J = \{1, \ldots, s - 1\}), and decompose

\[
t_s = t_1 + \sum_{i \in I} (t_{i+1} - t_i) + \sum_{j \in J} (t_{j+1} - t_j).
\]

\((5) \lambda(t^*) \) can be itself \( q \)-canonical, however \( \lambda(t^*) \geq mg^k \) would imply \( q > k \), hence \( n \geq T_{k+1} \).

In other words, if we restrict \( T_k \leq n < T_{k+1} \), then \( \lambda(t^*) \) cannot be \( q \)-canonical.
Lemma 5.3. Let \( t \) and again invoke Lemma 5.1 (more than once if \( q > 1 + (q_s - 1) \) moves). In the end we have the estimate

\[
\lambda_s \leq T_{q_s} + |I| + q_s|J| \leq T_{q_s} + q_s(s - 1).
\]  

(5.1)

At each critical time we gain one level of monotonicity, so that \( \lambda(t_s) \) is \((s - 1)\)-monotone. The maximal possible value of \( q_s \) is \( q_s = k \), which gives an upper bound \( k - 2 \) for the \( s \)-monotonicity of \( \lambda(t_s) \) (Theorem 4.3) and hence we have \( s \leq k - 1 \). We get

\[
\lambda_s \leq T_{q_s} + q_s(k - 2).
\]

To obtain the configuration \( \lambda(t^*) \geq mg^k \) we need to apply a move to \( \lambda(t_s) \) and again invoke Lemma 5.1 (more than once if \( q_s < k \) for a total of at most \( 1 + (k - 1) + \cdots + (q_s - 1) \) moves). We notice that the last term in the previous sum is also present in the sum that defines \( T_{q_s} \), so that using (5.1) we arrive at

\[
t^* \leq T_{q_s} + 1 + (q_s - 1) + q_s(s - 1) = T_{q_s} + q_s \leq T_{q_s} + ks.
\]

We summarize the previous analysis in the following lemma.

Lemma 5.3. Let \( n \geq T_k \) and \( \lambda \in \Lambda_n \). Then there exists \( 0 \leq s \leq k - 1 \) such that after at most \( t = T_{q_s} + ks \) moves the resulting configuration \( \lambda(t) \) satisfies:

1. \( \lambda(t) \geq mg^k \);
2. \( \lambda(t) \) is \( s \)-monotone.

If \( n = T_k \) or \( n = T_k + 1 \) then the resulting configuration is an augmented marching group.

Definition 5.1. Given \( 0 < b \leq a \) two integers, we define \( V(a, b) \) by

\[
V(a, b) = \sum_{i \in \mathbb{N}^{a-bb>0}} (a - ib).
\]

The value of \( V \) can be explicitly computed by first expressing \( a = qb + p \) with \( q = \left\lfloor \frac{a}{b} \right\rfloor \) and \( 0 \leq p < b \) (Euclidean division) and then using the formula for the sum of the first natural numbers. We end up with \( V(a, b) = qb + \frac{b}{2}q(q - 1) \). We have the special values

- \( V(a, a) = 0 \), \( V(a, b) = a - b \) if \( a \leq 2b \) (only one term in the above sum),
- \( V(a, b) = 2a - 3b > a - b \) if \( 2b < a \leq 3b \) (two terms in the above sum),
- \( V(a, b) > 2a - 3b \) if \( a > 3b \) (more than two terms in the above sum).

Moreover, if \( b < a \) then \( V \) is strictly increasing with respect to \( a \) and strictly decreasing with respect to \( b \). We can then “invert” \( V \) with respect to \( b \) and define the function

\[
W(a, v) = \max\{b \in \mathbb{N}^+ : b \leq a, \ V(a, b) \geq v\},
\]

which is (weakly) decreasing with respect to \( v \) and (weakly) increasing with respect to \( a \). We have the special values

- \( W(a, v) = a - v \) if \( 1 \leq v \leq \frac{a}{2} + 1 \),
- \( W(a, v) = \left\lceil \frac{2a - v}{3} \right\rceil \), if \( \frac{a}{2} \leq v \leq a + 3 \).

Referring to Figure 4 the value of \( V(a, b) \) can be graphically interpreted as the number of unit squares completely included in the right triangle having base (long cathetus) of length \( a \) and height (short cathetus) of length \( \frac{b}{2} \).

With the same graphical interpretation in mind function \( W(a, v) \) can be viewed as the maximal value of \( b \) such that the right triangle of sides \( a \) and \( \frac{b}{2} \) contains at least \( v \) unit squares.
Figure 4. Five unit squares are completely contained in a right triangle of sides 7 and 7/3, corresponding to $a = 7$ and $b = 3$. In other words $5 = V(7,3)$ (see Definition 5.1).

For $n \geq T_k$, let us consider a $k$-canonical configuration. By Lemma 5.1, after a cycle of $k$ moves we have that the resulting configuration is either itself $k$-canonical or it has just become larger than $mg^k$. Iterating this procedure, we want to relate the number of cycles needed to become larger than $mg^k$ with the degree of monotonicity of the configuration.

Lemma 5.4. Let $\lambda \in \Lambda_n$ with $n = T_k + r$, $r \geq 1$ and suppose that
1. $\lambda$ is a $k$-canonical configuration;
2. $\lambda$ is $s$-monotone for some $s \in \mathbb{N}$.
Let $\sigma \geq 1$ denote the number of cycles of $k$ moves that are required to become larger than $mg^k$ and suppose that $s + \sigma \geq 2$. Then
$$r \leq V(k-1,s + \sigma - 1) - 1.$$ Equivalently, reasoning by contradiction, we have the upper bound
$$\sigma \leq \max \{W(k-1,r+1) + 1 - s, 0\}.$$

Proof. First we analyze the case $\sigma = 1$, so that $s \geq 1$. Since the configuration is $s$-monotone and $k$-canonical, in this case $\lambda_1 \leq k$, hence $e_1, \ldots, e_s \leq k$, $e_{s+1}, \ldots, e_{2s} \leq k + 1$ and so on, where $e$ is the energy sequence $e = \varepsilon(\lambda)$. Then the total number of seeds with energy above $k$ cannot be larger than $V(k-1,s)$. This is obtained by observing that we have a sequence of $q$ plateaus of length $s$ and increasing height from 0 to $q-1$ and a final top plateau of length $p$ and height $q$. This implies that $r \leq V(k-1,s) - 1$ where the "-1" is a consequence of the fact that the $k$-th energy level is not completely full. For a generic value of $\sigma$ we simply perform $\sigma - 1$ cycles of $k$ moves obtaining a configuration with level of monotonicity $s + \sigma - 1$ with still $\lambda_k = 0$ and we can resort to the special case discussed above. □

5.2. Optimal upper bound. The following lemma will be fundamental in proving the optimal upper bound.

Lemma 5.5. Let $\lambda \in \Lambda_n$ with $n = T_k + r$, $0 \leq r \leq k$, and denote by $\sigma_q$ the number of $q$-canonical configurations encountered during the evolution of $\lambda$, with $q = 2, \ldots, k$.

Then the total number $s = \sum_q \sigma_q$ of canonical configurations encountered, and hence the least level of monotonicity of the first configuration that becomes larger than (or equal to) $mg^k$, is bounded by
$$s \leq W(k-1,r+1) + 1.$$ In particular, for $r < \frac{k}{2}$ we have
$$s \leq k - r - 1 \quad (5.2)$$
and for $\frac{k-3}{2} \leq r \leq k$ we have
$$s \leq \left\lfloor \frac{2k-r}{3} \right\rfloor, \quad (5.3)$$
where we used the special values of function $W$ computed in Definition 5.1.
Proof. If \( \sigma_q = 0 \) for all \( q \), then we can resort to Corollary 5.2 and conclude immediately. Otherwise, set \( \sigma_q = \sum_{i\leq q} \sigma_i \) and let \( \bar{q} \) be the largest value of \( q \) such that \( \sigma_q > 0 \). We apply Lemma 5.4 with \( k = \bar{q} \), \( r = n - T_{\bar{q}} \) and \( s = \sum_{i<\bar{q}} \sigma_i = s_{\bar{q}-1} \) and obtain an upper bound for \( \sigma_{\bar{q}} \) that, since \( \sigma_q > 0 \) can be written as

\[
\sigma_{\bar{q}} \leq W(\bar{q} - 1, n - T_{\bar{q}} + 1) + 1 - s_{\bar{q}-1}
\]

or equivalently as

\[
s = s_{\bar{q}} \leq W(\bar{q} - 1, n - T_{\bar{q}} + 1) + 1 \leq W(k - 1, r + 1) + 1
\]

where the last inequality follows from the monotonicity of \( W \) with respect to its arguments.

The evolution of a configuration \( \lambda \in \Lambda_n \), with \( n = T_k + r \) and \( 1 \leq r \leq k \), will be decomposed into two stages:

1. the evolution from the beginning up to the first time \( t \) when \( \lambda(t) \geq mg^k \), where as usual \( \lambda(t) = M^t(\lambda) \);
2. the subsequent evolution until we reach an augmented marching group.

The duration of each of these stages will be estimated from above with bounds that depend upon a few parameters and we shall find a uniform bound valid for all possible feasible choices of such parameters finally obtaining an upper bound for \( D(n) \).

We need some further lemmas.

**Lemma 5.6.** Let \( \lambda \in \Lambda_n \) and let \( k \) be the largest integer such that \( \lambda \geq mg^k \). Write \( n = T_k + r \) and suppose that \( \lambda \) is \( r \)-monotone. Then \( \lambda \) is an augmented marching group.

**Proof.** By contradiction suppose that \( \lambda \) is not an augmented marching group, then the energy sequence has an element with value \( k \) (otherwise \( \lambda \geq mg^{k+1} \)) and an element with value \( k + 2 \) (otherwise \( \lambda \) would be an augmented marching group). This implies the existence of an increasing plateau at level \( k + 1 \), which must have length at least \( r \). An energy level of \( k + 1 \) at position \( i \) is possible without using one of the \( r \) exceeding seeds only if \( i = k + 1 \), however if that position is part of the increasing plateau, then the seed at level \( k + 2 \) must necessarily stay on top of a seed at level \( k + 1 \). In all cases we need at least \( r + 1 \) seeds to build such an increasing plateau, which contradicts the assumptions.

**Lemma 5.7.** Let \( \lambda \in \Lambda_n \) with \( n = T_k + r \), \( 0 \leq r \leq k \), \( \lambda \geq mg^k \) and \( s \)-monotone. If \( M^{t^*}(\lambda) \) becomes an augmented marching group, then

\[
t^* \leq \max\{r - 1 + (r - s - 1)(k + 1), 0\}.
\]

**Proof.** Using the energy level interpretation we identify the gap at level \( k + 1 \) that will be filled last, and call \( i_0 \) its column index.

We consider two cases.

**Case** \( i_0 < r \) or \( i_0 = k + 1 \).

The first term \( r - 1 \) in the estimate is then the maximal number of moves that are required to move that gap into the rightmost \( (k + 1) \)-th column. We observe that starting from that configuration the augmented marching group will be obtained exactly after a number of moves multiple of \( k + 1 \).

After each cycle of \( k + 1 \) moves the level of monotonicity increases at least of one unit (Lemma 4.2), so that after \( r - s - 1 \) cycles of \( k + 1 \) moves we obtain a configuration \( \mu \) that is at least \( (r - 1) \)-monotone.

Now we want to prove that \( (r - 1) \)-monotonicity (and not higher) is actually impossible under the circumstances. This would imply at least \( r \)-monotonicity that in turn implies that we have an augmented marching group thanks to Lemma 5.6.
Suppose then by contradiction that \( \mu \) is \((r-1)\)-monotone but not \( r \)-monotone and hence it is not an augmented marching group. This means that the gap at level \( k+1 \), column \( k+1 \) is still empty. The energy sequence of \( \mu \) has then both elements at level \( k \) and elements at level \( k+2 \), so that there exists at least an increasing plateau at level \( k+1 \) (i.e. a sequence of energy elements as contiguous positions at level \( k+1 \) following an element at level \( k \) and preceding an element at level \( k+2 \)). Such plateau has length at least \( r-1 \) due to the \((r-1)\)-monotonicity. Now recall that we have an excess of \( r \) seeds with energy larger or equal to \( k+1 \) (levels \( k \) and below are completely filled thanks to the assumptions). Column \( k+2 \) is the only one having energy \( k+1 \) with no need to use one of the excess seeds, however we have barely enough seeds to form an increasing plateau of length \( r-1 \) which requires \( r+1 \) seeds that reduce to \( r \) only if we take advantage of the energy step at column \( k+2 \).

Recalling that column \( k+1 \) has energy level \( k \) (the gap is still empty), the only possible disposition of the \( r \) excess seeds is the one with \( r-1 \) of them in columns 1 to \( r-1 \) (energy \( k+1 \)) and the last excess seed itself in column \( r-1 \) (then with energy \( k+2 \)). This configuration becomes an augmented marching group after exactly \( r \) moves, however we know that an augmented marching group is reached after a multiple of \( k+1 \) moves and hence \( r = k+1 \), a contradiction.

Since \((r-1)\)-monotonicity is impossible for \( \mu \), then we have at least \( r \)-monotonicity and hence we have an augmented marching group.

**Case** \( r \leq i_0 < k+1 \).

Let us consider the result of the evolution as soon as it becomes an augmented marching group and call it \( \mu \), the position of the “marked” gap is then column \( k+1 \) and it has just been filled with the last of the \( r \) seeds the were originally at energy levels larger then \( k+1 \). Immediately to the left of this seed there is a contiguous sequence of, say, \( \rho-1 \) seeds, \( 1 \leq \rho \leq r \) at level \( k+1 \), preceded by a gap in column \( k+1-\rho \). Clearly this latter gap (let us call it “gap number 2”) can be traced back to its corresponding position in the initial configuration \( \lambda \), and will originally be empty in \( \lambda \) and will continue to be empty during the whole evolution. It is located in column \( i_0-\rho \) of \( \lambda \), to the left of the marked gap (in column \( i_0 \)).

No seed at energy larger then \( k+1 \) (they are the “active” seeds, the only ones that “move” with respect to the \( k+1 \) level) can ever cross or reach the column containing gap number 2, so that the active seeds originally to the left of gap number 2 or to the right of the marked gap (call this region the “outer region”) can ever interact with the region between gap number 2 and the active gap.

Without loss of generality we can then suppose that the outer region has reached its final configuration already in \( \lambda \). This actually means that there are no active seeds to the right of the marked gap.

The final step is then to observe that in this situation \( \lambda \) can be interpreted as the result of the evolution (with \( k+1-i_0 \) moves) from a configuration \( \bar{\lambda} \) that itself satisfies the hypotheses of this Lemma and the marked gap already in the \( k+1 \) position (in other words we can “steal” a few moves) and we can resort to the first case of the Lemma.

The proof is thus complete. \( \square \)

We now analyze the evolution of \( \lambda \in \Lambda_n, n = T_k+r, 0 \leq r \leq k \). We denote with \( s \in \mathbb{N} \) the number of \( q \)-canonical configurations encountered during the evolution with \( q \leq k \).
Using Lemma 5.8 we obtain the constraint \( s \leq W(k - 1, r + 1) + 1 \), which in particular gives

\[
\begin{align*}
0 & \leq s \leq k - r - 1 & \text{if } r < \frac{k}{2}, \\
0 & \leq s \leq \left\lfloor \frac{2k - r}{3} \right\rfloor & \text{if } \frac{k - 3}{2} \leq r \leq k + 1, \\
0 & \leq s \leq r & \text{if } r = \frac{k}{2}.
\end{align*}
\]

(5.4)

The third estimate is actually just a special case of the second one; the first and second estimates coincide for \( r \) in the common interval \( \frac{k - 3}{2} \leq r < \frac{k}{2} \).

As usual we denote by \( \lambda(t) = M^t(\lambda) \) and we set

- \( t_A \) the smallest time such that \( \lambda(t_A) \geq mg^k \)
- \( t^* \) the smallest time such that \( \lambda(t^*) \) is an augmented marching group.

**Lemma 5.8.** We have the following estimates:

1. \( t_A \leq T_{k-1} + ks \);
2. \( t^* - t_A = 0 \) if \( s \geq r \);
3. \( t^* - t_A \leq r - 1 + (r - s - 1)(k + 1) \) if \( s < r \).

**Proof.** The first point is exactly Lemma 5.3.

Point (2) is simply Lemma 5.6.

Point (3) follows from Lemma 5.7.

We then have a total estimate of \( t^* = t_A + (t^* - t_A) \) as a function of \( s \) that reads as follows:

\[
t^* \leq \begin{cases} 
T_{k-1} + (r - 1)(k + 2) - s & \text{if } s < r, \\
T_{k-1} + ks & \text{if } s \geq r.
\end{cases}
\]

(5.5)

Clearly in the case \( s < r \) it is most convenient to take \( s = 0 \), whereas if \( s \geq r \) the maximum value is achieved by selecting the largest possible value of \( s \), which is dictated by the constraints (5.4).

We recall from Section 3 the definition of \( D^*(n) \) as

\[
D^*(n) = \begin{cases} 
3T_{k-1} - kr & \text{if } r < \frac{k}{2}, \\
3T_{k-1} - k(r - 1) & \text{if } r = \frac{k}{2}, \\
T_{k-1} + (k + 2)(r - 1) & \text{if } r > \frac{k}{2}.
\end{cases}
\]

**Theorem 5.9 (Depth of \( \Lambda_n \)).** Let \( n = T_k + r \) with \( k \geq 2 \) and \( 0 \leq r \leq k \). Then \( D(n) = D^*(n) \). The case \( k \leq 1 \) corresponds to \( n \leq 2 \), easily studied manually.

**Proof.** In view of the results of Section 7 it suffices to prove the upper bound \( D(n) \leq D^*(n) \). Let \( \lambda \in \Lambda_n \). We invoke Lemma 5.3 so that there exists \( s \) satisfying the constraints (5.4) such that after at most \( T_{k-1} + ks \) moves the resulting configuration is \( s \)-monotone and \( \geq mg^k \). Now we enforce Lemma 5.8 and distinguish three cases.

**Case** \( r < \frac{k}{2} \).

If \( s \geq r \), we use the first inequality of (5.4) in (5.5) to conclude

\[
t^* \leq T_{k-1} + k(k - r - 1) = D^*(n).
\]

If \( s < r \), using \( s \geq 0 \), and \( 2r \leq k - 1 \), i.e. \( r \leq k - r - 1 \) we have

\[
t^* \leq T_{k-1} + (r - 1)(k + 2) \leq T_{k-1} + rk + 2r - k - 2 < T_{k-1} + k(k - r - 1) - 2
\]

so that in any case \( t^* \leq D^*(n) \).

**Case** \( r = \frac{k}{2} \).

Again we distinguish the case \( s \geq r \), together with the third inequality of (5.4) to obtain

\[
t^* \leq T_{k-1} + kr = 3T_{k-1} - k(k - r - 1) = D^*(n).
\]
Whereas if \( s < r \), again using \( s \geq 0 \) we have

\[
t^* \leq T_{k-1} + (r - 1)(k + 2) = 3T_{k-1} - k(r - 1) - 2 < D^*(n).
\]

**Case** \( r > \frac{k}{2} \).

We start with the case \( s < r \) and again use \( s \geq 0 \) to get

\[
t^* \leq T_{k-1} + (r - 1)(k + 2) = D^*(n).
\]

Finally, if \( s \geq r \), enforcing the second inequality in (5.4), using \( 2r - k \geq 1 \) we obtain

\[
t^* \leq T_{k-1} + \frac{2k - r}{3} = T_{k-1} + (r - 1)(k + 2) - (2r - k) \left( \frac{2}{3}k + 1 \right) + 2 < D^*(n).
\]

We have covered all the possible cases and the proof is complete. \( \Box \)

The previous Theorem 5.9, and in particular the upper bound

\[D(n) \leq D^*(n)\]

also allows to compute the maximal diameter of \( \Lambda_n \):

**Corollary 5.10** (Diameter of \( \Lambda_n \)). The diameter of \( \Lambda_n \) (see Definition 2.10) is
given by \( D^*(n) + P(n) - 1 \) where \( P(n) \) is the maximal period of an augmented
marching group, i.e. \( P(n) = 1 \) if \( n = T_k \) is a triangular number, otherwise \( P(n) = k \)
if \( n = T_k + r \), with \( 0 < r \leq k \), is not a triangular number.

**Proof.** That the diameter cannot exceed \( D^*(n) + P(n) - 1 \) immediately follows from
Theorem 5.9. The reverse inequality follows from the constructions of Section 3, in
particular we make use of Proposition 3.4, Corollary 3.6, Proposition 3.7 and the
discussion about the period of the resulting augmented marching groups following
each of these results. \( \Box \)

6. Monotone mancala and Bulgarian solitaire

The notion of \( s \)-monotonicity naturally introduces a sort of graduation on the
graph of moves. Theorem 4.1 indeed asserts that the subgraph obtained by considering
only the \( s \)-monotone configurations for some \( s \in \mathbb{N} \) is closed under mancala
moves (no arc leaves this subset). It is then natural to ask what is the depth of
such subgraphs.

More precisely, let us denote by \( \Lambda_{n,s} \) and \( \Lambda_{n,\text{mon}} \), \( n, s \in \mathbb{N} \) the subsets of \( \Lambda_n \)
given by

\[
\Lambda_{n,s} = \{ \lambda \in \Lambda_n : \lambda \text{ is } s\text{-monotone} \}, \quad \Lambda_{n,\text{mon}} = \{ \lambda \in \Lambda_n : \lambda \text{ is monotone} \}.
\]

Similarly, with \( \mathcal{G}_{n,s} \) and \( \mathcal{G}_{n,\text{mon}} \) we denote the subgraphs of \( \mathcal{G}_n \) having nodes in \( \Lambda_{n,s} \)
and \( \Lambda_{n,\text{sub}} \), respectively.

Then the depth of \( \mathcal{G}_{n,s} \), denoted by \( D(n, s) \), is the maximal distance from a node
of \( \mathcal{G}_{n,s} \) to a periodic configurations. In this context, recalling that \( 0 \)-monotonicity is
a void notion, our main Theorem 5.9 gives a value for the depth of \( \mathcal{G}_{n,0} = \mathcal{G}_n \).

**Remark 4.3** relates \( \mathcal{G}_{n,1} \) to \( \mathcal{G}_{n,\text{mon}} \) and in particular relates their respective depth:
add one to \( D(n, 1) \) to obtain the depth of the graph of monotone configurations. This is of special interest in view of

**Remark 6.1.** The monotone mancala graph is isomorphic to the graph of the Bul-
garian solitaire [Eti],[GrHo],[Igu].
The Bulgarian solitaire, also mentioned in the Introduction, is played as follows. The starting position consists of a number of piles, each with a (possibly different) number of cards. Each move consists in removing one card from each pile and collecting the removed cards in a new pile. Piles that become empty are simply neglected and the order of the piles (and of the cards in a pile) is inessential. To a Bulgarian solitaire configuration we can associate a mancala configuration by defining \( \lambda_i \) equal to the number of piles with at least \( i \) cards. In this way we obtain a valid mancala configuration which is clearly monotone. It can be readily shown that this association is one to one and that it is consistent with the respective game rules.

We hence have the following conjecture [GrHo].

**Conjecture 6.1.** The depth of \( G_{n,1} \) is given by

\[
D(n, 1) = \begin{cases} 
2T_{k-1} - rk - 1, & \text{for } 0 \leq r < \left\lfloor \frac{k}{2} \right\rfloor \\
T_{k-1} + r - 1, & \text{for } r = \left\lfloor \frac{k}{2} \right\rfloor \text{ or } \left\lfloor \frac{k}{2} \right\rfloor + 1 \\
2T_k - (k + 1 - r)(k + 2) - 1, & \text{for } \left\lfloor \frac{k}{2} \right\rfloor + 1 < r \leq k + 1.
\end{cases}
\]

In [GrHo] the authors proved this result for some special values of \( n \), while the above expression is proved to be a lower bound.

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