HIGHER CURVED ORBIT SPACES

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Abstract. It is known that the infimum of the sectional curvatures (on the regular part) of orbit spaces of isometric actions on unit spheres in bounded above by 4. We show that the infimum is 1 for “most” actions, and determine the cases in which it is bigger than 1.

1. Introduction

Let \( G \) be a compact Lie group acting non-transitively by isometries on the unit sphere \( S^n \), where \( n \geq 2 \). The orbit space \( X = S^n / G \) is an Alexandrov space of curvature at least 1 and diameter at most \( \pi \). Let \( \kappa_X \) denote the infimum of sectional curvatures of the quotient Riemannian metric in the regular part of \( X \). It was proved in [GL17] that \( \kappa_X \leq 4 \) always holds. Note that \( \kappa_X \) can also be characterized as the largest number \( \kappa \) such that \( X \) is an Alexandrov space of curvature \( \geq \kappa \) (cf. subsection 2.1).

It is apparent from the discussion in [GL17] that \( \kappa_X = 1 \) for “most” representations. This remark motivates the present work. If \( \kappa_X > 1 \), we will say that \( X \) is highly curved; we will also abuse language and say that \( \rho \) is highly curved. Herein we prove the following theorem.

Theorem 1.1. Let \( X = S^n / G \) be the orbit space of an isometric action of a compact Lie group \( G \) on the unit sphere \( S^n \) and assume that \( \dim X \geq 2 \). Then \( X \) is highly curved if and only if:

(i) \( n = 7 \) and \( G^0 = U(2) \) acts on \( S^7 \) as the restriction of the irreducible representation on \( \mathbb{C}^4 \);

or the associated representation of \( G^0 \) is quotient-equivalent to a non-polar (cf. subsection 4.1) sum of two representations of cohomogeneity one; in the latter case, either one of the following cases occur:

(ii) \( X \) is a good Riemannian orbifold of constant curvature 4;

(iii) \( X \) is a a complex weighted projective line (of real dimension two) or a \( \mathbb{Z}_2 \)-quotient thereof;

(iv) \( n = 6 \) and \( G^0 = SU(2) \) acts on \( S^6 \) as the restriction of the representation \( \mathbb{C}^2 \oplus \mathbb{R}^3 \);

(v) \( G^0 = Sp(m) \times U(1) \) and it acts on \( S^{4m-1} \), where \( Sp(m) \) acts diagonally on \( \mathbb{C}^{2m} \oplus \mathbb{C}^{2m} \) and \( U(1) \) acts with weights \( r, s \geq 0 \) where \( r \neq s \).

There are two senses in which the curvature of \( X \) is related to its diameter. First, the more extrinsically curved a \( G \)-orbit is, the closer its focal points are, and thus the sooner a normal geodesic starting there ceases to be minimizing. It is shown in [GSTS] that the infimum over all actions (coming from irreducible
representations, non-transitive on the unit sphere) of the supremum over all orbits of their focal radii is bounded away from zero. Second, and more relevant to this paper, the Bonnet-Myers argument implies that \( \text{diam } X \leq \frac{\pi}{\sqrt{\kappa_X}} \). In [GLLM] it is proved the existence of a (non-explicit) universal positive lower bound for \( \text{diam } X \) in case of a nontransitive action on \( S^n \) with \( n \geq 2 \).

There are families of actions for which \( X \) is a Riemannian orbifold as in (ii) and (iii), see [GL16] for a classification. In case (iii), \( X \) is a bad Riemannian orbifold and \( \rho \) is quotient-equivalent to a circle action or a \( \mathbb{Z}_2 \)-extension thereof; a classification of the representations of maximal-connected groups can be found in [Str94, Table II, Types \( S^2 \) and \( I \)]. The other cases do not yield Riemannian orbifolds. Case (i) is the only example in the list in which the representation is irreducible and reduced (cf. subsection 2.2); incidentally, this representation is not amenable to the general principles developed in this paper and its analysis requires a direct calculation of the curvature involving the Thorpe method, which we take up in subsection 3.6.1. In general, the proof of Theorem 1.1 combines geometric and algebraic arguments, with analyses of special cases and use of representation theory and several classification results. It would be very interesting to find simple geometric reasons why the representations listed in the theorem (and only them) are highly curved.

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2. Preliminaries

2.1. Spherical quotient spaces. Let \( \rho : G \to \text{O}(V) \) be a representation of a compact Lie group on an Euclidean space \( V \). It restricts to an isometric action on the unit sphere \( S(V) \), and all isometric actions of compact Lie groups on unit spheres are obtained in this way. The cohomogeneity of \( \rho \) coincides with \( \dim(S(V)/G) + 1 \). The quotient space \( X = S(V)/G \) is an Alexandrov space stratified by smooth Riemannian manifolds, namely, the projections of the sets of points in \( S(V) \) with conjugate isotropy groups — the connected components of the strata can be equivalently characterized as the connected components of the subsets of \( X \) consisting of points with isometric tangent cones. There is a unique maximal stratum, the set of principal orbits \( X_{\text{reg}} \), which comes equipped with a natural quotient Riemannian metric which makes the projection \( S(V)_{\text{reg}} \to X_{\text{reg}} \) into a Riemannian submersion. Moreover, \( X \) is the completion of the convex open submanifold \( X_{\text{reg}} \), hence Toponogov’s globalization theorem [Pet12] says that \( \kappa_X \) is the largest number \( \kappa \) such that \( X \) is an Alexandrov space of curvature bounded below by \( \kappa \), where \( \kappa_X = \kappa_{\rho} \) is defined as in the introduction.

2.2. Types of equivalence between representations. We say that two representations \( \rho : G \to \text{O}(V) \) and \( \tau : H \to \text{O}(W) \) are quotient-equivalent if they have isometric orbit spaces [GL14]. If, in addition, \( \dim G < \dim H \), we say that \( \rho \) is a reduction of \( \tau \). A representation that admits no reductions is called reduced.

A special case of quotient-equivalence occurs when there is an isometry from \( V \) to \( W \) mapping \( G \)-orbits onto \( H \)-orbits. In this case we say that \( \rho \) and \( \tau \) are orbit-equivalent.

2.3. Local convexity and folding map. For an isometric action of \( G \) on \( S(V) \) with orbit space \( X \), we denote the stratum of \( X \) corresponding to an isotropy group
K by \( X_{(K)} \). Every stratum \( X_{(K)} \) of \( X \) is a (possibly incomplete, disconnected) totally geodesic Riemannian submanifold of \( X \) which is moreover a locally convex subset. It follows that the infimum of the sectional curvatures in \( X_{(K)} \) is also bounded below by \( \kappa_X \).

The set of fixed points of the isotropy group \( K \) of \( G \) is a subspace \( W \) on which the normalizer \( N_G(K) \) acts isometrically. Let \( H := N_G(K)/K \). Then \( H \) acts on \( W \) with trivial principal isotropy groups. The quotient \( Y = S(W)/H \) admits a canonical map \( I_{(K)} : Y \to X \) which is 1-Lipschitz, finite-to-one and length-preserving, and an injective local isometry from an open dense subset of \( Y \) onto \( X_{(K)} \) [GL16]. We will call \( I_{(K)} \) the folding map associated with \( X_{(K)} \). If \( \dim X_{(K)} \geq 2 \), we deduce that \( \kappa_X \leq \kappa_Y \). We have proved (cf. [GL16 §5]):

**Proposition 2.1.** Let \( \rho : G \to O(V) \) be a representation and \( X = S(V)/G \). Assume there is a non-principal stratum \( X_{(K)} \) of dimension \( d \geq 2 \) of \( X \). Then there is another representation \( \tau : H \to O(W) \) such that \( Y = S(W)/H \) has dimension \( d \) and \( \kappa_X \leq \kappa_Y \).

It is known that the folding map associated with the principal stratum \( X_{\text{reg}} \) is a global isometry. The corresponding representation of \( H \) on \( W \) is called the principal reduction of the representation \( \rho \) [Str94].

### 2.4. Rank and strata

We quote from [GL17]:

**Lemma 2.2.** Let a compact Lie group \( G \) of dimension \( g \) and rank \( k \) act by isometries on \( S(V) \). Then:

(i) The smallest dimension of a \( G \)-orbit is at most \( g - k + 1 \).

(ii) If the action has trivial principal isotropy groups, then \( X = S(V)/G \) contains a non-maximal stratum of dimension at least \( k - 2 \).

### 2.5. Index estimates

The following result was proved in [GL17] and gives slightly more than can be directly obtained from O’Neill’s formula. It already shows that “most” representations are not highly curved.

**Lemma 2.3.** Let a compact Lie group \( G \) of dimension \( g \) and rank \( k \) act by isometries on \( S^n \). Let \( \ell \) denote the smallest dimension of an orbit, and let \( m \geq 2 \) denote the dimension of the orbit space \( X = S^n/G \). If \( \kappa > 1 \) then \( \ell \geq m - 1 \); in particular, \( 2g + 2 - k \geq n \).

A very similar reasoning yields the following improved index inequality:

**Lemma 2.4.** Let \( \rho : G \to O(n + 1) \) be a highly curved representation with trivial principal isotropy groups. Let \( m \) be the dimension of \( X \) and \( g = n - m \) be the dimension of \( G \). Assume there exists a regular horizontal geodesic \( \gamma \) in \( S^n \), of length less than \( \pi \), intersecting singular orbits of dimensions \( \ell_1, \ell_2, \ldots, \ell_s \). Then \( g \geq (m - 1) + (g - \ell_1) + (g - \ell_2) + \ldots + (g - \ell_s) \). In particular, if \( s > 1 \) then \( \ell_1 + \ell_2 \geq n - 1 \).

### 2.6. Enlarging group actions

We consider the situation in which the \( G \)-action on \( S(V) \) is the restriction of the action of a compact Lie group \( H \) that contains \( G \) as a closed subgroup. We will need the following extension of the results in [GL17 §2.3]. Recall that polar representations [Dad85] are exactly those whose orbit space has constant curvature 1 [GL15 Introdl.].
Proposition 2.5.  
(a) Suppose an orthogonal representation \( \rho : G \to O(V) \) is the restriction of another representation \( \tau : H \to O(V) \), where \( G \) is a closed subgroup of \( H \). If the cohomogeneity of \( \tau \) is at least 3, then \( \kappa_\rho \leq \kappa_\tau \).

(b) The following classes of representations \( \rho : G \to O(V) \) are not highly curved:

(i) Representations \( \rho \) as in (a), where \( \tau \) is polar and has cohomogeneity at least 3.

(ii) Tensor products, where \( G = G_1 \times G_2 \), \( V = V_1 \otimes V_2 \) and \( \dim V_i \geq 3 \) for \( i = 1, 2 \) \((F = \mathbb{R}, \mathbb{C}, \mathbb{H})\).

(iii) Direct sums \( V = V_1 \oplus V_2 \), where the G-action on \( V_2 \) has cohomogeneity at least 2.

(iv) Direct sums \( V = V_1 \oplus \cdots \oplus V_n \), where \( n > 2 \).

Proof. See [GL17] for (a), (i) and (ii). For (iii), we take \( H = SO(V_1) \times G_2 \), where \( G_2 = G/\ker \rho_2 \), and note that \( S(V)/H \) is the suspension (or spherical cone) over \( S(V_2)/G_2 \). Since \( \dim S(V_2)/G_2 \geq 1 \), the suspension over a non-constant geodesic in \( S(V_2)/G_2 \) is a convex totally geodesic surface in \( S(V)/H \) which is locally isometric to the unit sphere \([BB01], \S 4.3.3\), hence \( \kappa_\tau = 1 \) and we can apply (a). Finally, the case (iv) is reduced to the previous case simply by writing \( V = V_1 \oplus (V_2 + \cdots + V_n) \) and noting that the cohomogeneity of \( G \) on \( V_2 + \cdots + V_n \) is bigger than one. \( \square \)

3. Some interesting examples

In this section we show that a few specific representations are (are not) highly curved. These results are part of the proof of Theorem [1.1]

3.1. The curvature of complex weighted projective lines. Let \( U(1) \) act on \( \mathbb{C} \oplus \mathbb{C} \) with parameters \( (a, b) \), namely, \( \xi \cdot (z, w) = (\xi^a z, \xi^b w) \) for \( \xi \in U(1) \subset \mathbb{C} \) and \( z, w \in \mathbb{C} \). We assume that \( a \) and \( b \) are co-prime, positive integers, and \( a \geq b \). The map

\[
F : (0, \pi/2) \times (0, 2\pi) \to S^3, \quad F(r, \theta) = (\cos r, e^{i\theta} \sin r)
\]

meets all principal orbits, so the orbital metric in the principal stratum of \( X = S^3/\mathbb{C} \) can be easily computed in terms of \( F \) to give

\[
g = dr^2 + \frac{1}{4 \cos^2 r + b^2} \frac{a^2 \sin^2 2r}{\sin^2 r} d\theta^2.
\]

This is a rotationally symmetric metric, whose Gaussian curvature is given by

\[
K(r) = \frac{3a^4 + 26a^2b^2 + 3b^4 + 4(a^4 - b^4) \cos 2r + (a^2 - b^2)^2 \cos 4r}{2(a^2 + b^2 + (a^2 - b^2) \cos 2r)^2}.
\]

We have

\[
K'(r) = \frac{48a^2b^2(a^2 - b^2) \sin 2r}{(a^2 + b^2 + (a^2 - b^2) \cos 2r)^3} > 0,
\]

so that

\[
K_{\text{inf}} = K(0^+) = 1 + \frac{b^2}{a^2}, \quad K_{\text{sup}} = K(\frac{\pi}{2}^-) = 1 + \frac{a^2}{b^2},
\]

and hence \( 1 < \kappa_X < 4 \), unless \( a = b = 1 \) in which case \( X \) is a 2-sphere of constant curvature 4. In any case, \( X \) is highly curved.
3.2. The representation \((\text{SU}(2), \mathbb{C}^2 \oplus \mathbb{R}^3)\). In terms of quaternions, this representation is \((\text{Sp}(1), \mathbb{H} \oplus \mathbb{H})\) where \(q \cdot (x, y) = (qx, qyq^{-1})\). The only non-principal orbit in \(S^6(1)\) corresponds to \(x = 0\). The map

\[
\mathbb{H} \oplus \mathbb{H} \to \mathbb{H} \cong \mathbb{R}^3, \quad (x, y) \mapsto x^{-1}yx
\]

is well-defined and constant on principal orbits, applies the regular part of \(S^6(1)\) onto the interior of the closed ball \(B^3(1)\), and a neighborhood of the singular orbit to a neighborhood of the boundary \(\partial B^3(1) = S^2(1)\). It follows that \(X\) is topologically a 3-sphere. A section of the above map over the regular set is

\[
B^3(1) \subset \mathbb{R}^3 \to S^6(1) \subset \mathbb{H} \oplus \mathbb{H}, \quad v \mapsto (\sqrt{1 - ||v||^2}, v)
\]

which, in spherical coordinates, is written

\[
(r, \theta, \varphi) \in (0, \pi/2) \times (0, 2\pi) \times (0, \pi) \mapsto (\cos r, \sin r \sin \theta \sin \varphi, \sin r \sin \theta \cos \varphi, \sin r \cos \varphi) \in S^6(1) \subset \mathbb{R}^7.
\]

One easily computes the inner products of the horizontal components of \(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\) to obtain the orbital metric coefficients:

\[
g = dr^2 + \frac{\sin^2 2r}{10 - 6 \cos 2r} (d\varphi^2 + \sin^2 \varphi d\theta^2).
\]

This is a warped product \(X_{reg} = [0, \pi/2) \times f S^2(1)\) where \(f(r) = \frac{1}{2} \sqrt{\sin 2r / (\cos^2 r + 4 \sin^2 r)}\) is the coefficient of the metric associated to the complex weighted projective line of weights \((1, 2)\). The sectional curvatures of such a warped product are well known [Pet06 § 3.2.3], namely, they all lie between \(-f''/f\) and \(1 - f'/f f^{-2}\). We have \(\text{Im}(-f''/f) = (7/4, 13)\) and \(\text{Im}(1 - f'/f f^{-2}) = [9, +\infty)\), so \(\kappa_X = 7/4\) and \(X\) is highly curved.

3.3. The representation \((\text{SO}(3), \mathbb{R}^7)\). This representation is induced from the real form \(V\) of \((\text{SU}(2), \text{Sym}^6(\mathbb{C}^2))\) given by

\[
\text{span}_\mathbb{R} \{ e_1^1 + e_2^1, i(e_1^1 - e_2^1), e_1^2e_2 - e_1e_2^2, i(e_1^2e_2 + e_1e_2^2), e_1^1e_2^2 + e_1^2e_2^1, e_1e_2^2 + e_1^2e_2, i(e_1^2e_2^1 - e_1^1e_2^2), i e_1^3e_2 \}.
\]

There is exactly one singular orbit, namely, that through \(p = ie_1^1e_2^3\), whose isotropy group is the (diagonal) maximal torus (circle). It is easy to find \(g = \left( \begin{array}{cc} \alpha & -\beta \\ \beta & \alpha \end{array} \right) \in \text{SU}(2)\) such that \(q = gp \neq -p\) is orthogonal to \(T_p(Gp) = \text{span}_\mathbb{R} \{ e_1^1e_2^3 + e_1^3e_2^1, i(e_1^1e_2^1 - e_1^2e_2) \}\), e.g. any \(\alpha, \beta\) with \(|\alpha|^2 = \frac{1}{2}(1 \pm \frac{1}{\sqrt{5}}), |\beta|^2 = 1 - |\alpha|^2\) will do. It follows that there is a regular horizontal geodesic of length smaller than \(\pi\) that meets \(Gp\) in two points, namely, a minimizing geodesic segment between \(p\) and \(q\). Since \(\ell_1 = \ell_2 = 2\) and \(n = 6\), Lemma [2,3] implies that this representation is not highly curved.

3.4. The representation \((\text{Sp}(1) \times \text{Sp}(1), \mathbb{H}^3 \otimes_\mathbb{H} \mathbb{H})\). We will show that this representation is not highly curved. We consider a double quotient

\[
Y = S^{11}/H \quad Z = S^{11}/K = \mathbb{H}P^2
\]

\[
X = S^{11}/G
\]
where: $H$ is $\text{Sp}(1)$ acting on the left, which we view as the representation of quaternionic type ($\text{SU}(2), \text{Sym}^3(\mathbb{C}^2))$; $K$ is $\text{Sp}(1)$ acting on the right; and $G = H \times K$.

Note that $Y$ is a Riemannian orbifold (a quaternionic weighted projective space). View the representation space of $H$ as

$$\text{span}_\mathbb{C}\{e_1^5, e_1^4 e_2, e_1^3 e_2^2, e_1^2 e_2^3, e_1 e_2^4, e_2^5\}$$

and take $p = e_1^5$. The isotropy group $H_p$ is the cyclic group $\mathbb{Z}_5$, say with a generator $h = \text{diag}(e^{i\omega}, e^{-i\omega})$, where $\omega = 2\pi/5$, the tangent space

$$\mathfrak{h} \cdot p = \text{span}_\mathbb{R}\{ie_1^5, e_1^4 e_2, ie_1^4 e_2\}$$

and the normal space

$$N_p(Hp) = \text{span}_\mathbb{C}\{e_1^3 e_2^2, e_1^2 e_2^3, e_1 e_2^4, e_2^5\}.$$ 

It follows that $h$ acts on $\mathfrak{h} \cdot p$ is id$_{\mathbb{R}} \oplus R_{i\omega}$, where $R_{i\omega}$ denotes a rotation of angle $\theta$ on an oriented 2-plane. Similarly, the action of $h$ on $N_p(Hp)$ is id$_{\mathbb{R}^2} \oplus R_{i\omega} \oplus R_{i\omega} \oplus R_{i\omega}$. Since the O’Neill tensor $A^H_p : A^2 N_p(Hp) \to \mathfrak{h} \cdot p$ is $H_p$-equivariant and $4 \pm 2 \neq 0$, 3 mod 5, we find that $w_1 = e_1^3 e_2^2$ and $w_2 = e_1 e_2^4$ satisfy $A^H_p(w_1 \wedge w_2) = 0$, that is, the 2-plane $\sigma$ spanned by $w_1$, $w_2$ projects to a 2-plane of sectional curvature 1 in $Y$.

The quaternionic structure on $\text{Sym}^3(\mathbb{C}^2)$ is induced from that of $\mathbb{C}^2$. Since the latter maps $e_1$ to $e_2$ and $e_2$ to $-e_1$, the former maps $w_1 = e_1^3 e_2^2$ to $-e_1 e_2^4$ and $w_2 = e_1 e_2^4$ to $e_1^4 e_2$, so $\sigma$ is a totally real plane and maps to a 2-plane of sectional curvature 1 in $Z$. Equivalently, the O’Neill tensor of $S^{11} \to \mathbb{H}P^2$ vanishes on $w_1 \wedge w_2$.

Let $x$ be the projection of $p$ to $X$. This is an isolated singular point in $X$, but $p$ is an exceptional point of the $H$-action and a regular point of the $K$-action, so we have continuity at $p$ of the O’Neill tensors of Riemannian submersions to $Y$ and $Z$. It follows that there is a sequence of $G$-regular points $p_n \to p$ and 2-planes $\sigma_n$ tangent to $S^{11}$ at $p_n$ projecting to 2-planes in $X$ with sectional curvature $\to 1$. Hence $\kappa_X = 1$.

### 3.5. The representation ($\text{SU}(2), \mathbb{H}^2$).

We will prove that this representation is not highly curved. Let $G = \text{SU}(2)$. We view this representation as the cubic symmetric power $V = \text{Sym}^3(\mathbb{C}^{2*})$. Namely, write an arbitrary element $g \in \text{SU}(2)$ as

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. This exhibits the matrix representation of the standard action of $\text{SU}(2)$ on $\mathbb{C}^2$ with respect to the canonical basis $\{e_1, e_2\}$. Let $\{u, v\}$ be the dual basis of $\mathbb{C}^{2*}$. The action of $g$ on $\mathbb{C}^{2*}$ is represented by the matrix complex-conjugate to $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ with respect to this basis. Now an orthonormal basis of $V$ is given by

$$\begin{pmatrix} u^3 & uv^2 & v^3 & u^2 v \\ \sqrt{6} & \sqrt{2} & \sqrt{6} & \sqrt{2} \end{pmatrix}.$$
We have chosen the order in the basis in view of the quaternionic structure below.

In this basis, the action of $g$ on $V$ is represented by the matrix

\[
\begin{pmatrix}
\bar{\alpha}^3 & \sqrt{3}\bar{\alpha}\beta^2 & -\beta^3 & -\sqrt{3}\bar{\alpha}^2\beta \\
\sqrt{3}\bar{\alpha}\beta^2 & \alpha(|\alpha|^2 - 2|\beta|^2) & -\sqrt{3}\bar{\alpha}^2\beta & \beta(2|\alpha|^2 - |\beta|^2) \\
\beta^3 & \sqrt{3}\alpha^2\bar{\beta} & \bar{\alpha}^3 & \sqrt{3}\alpha\bar{\beta}^2 \\
\sqrt{3}\alpha^2\bar{\beta} & \beta(|\beta|^2 - 2|\alpha|^2) & \sqrt{3}\alpha\bar{\beta}^2 & \bar{\alpha}(|\alpha|^2 - 2|\beta|^2)
\end{pmatrix}
\]

On the level of Lie algebras, consider the basis

\[
i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}
\]

of $\mathfrak{su}(2)$. We have $[i, j] = 2k$ and cyclic permutations. These matrices, viewed as elements of $\mathfrak{g}$, operate on $V$ as

\[
i_L = \begin{pmatrix} -3i & i \\ 3i & -i \end{pmatrix}, \quad j_L = \begin{pmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix},
\]

and

\[
k_L = \begin{pmatrix} 0 & i\sqrt{3} \\ i\sqrt{3} & 2i \end{pmatrix}.
\]

Beware that $i_L j_L \neq k_L$, etc.

3.5.1. Quaternionic structure. The matrix (3.2) has the form

\[
\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}.
\]

If we identify $V \cong \mathbb{C}^4$ with $\mathbb{H}^2$ via the map

\[
x_1 \frac{u^3}{\sqrt{6}} + x_2 \frac{uv^2}{\sqrt{2}} + y_1 \frac{v^3}{\sqrt{6}} + y_2 \frac{u^2v}{\sqrt{2}} \mapsto (x_1 + jy_1, x_2 + jy_2)
\]

then the representation is given by left multiplication by the quaternionic matrix $A + jB$. In particular, the normalizer of the representation of $G$ on $V$ contains another copy $G'$ of $\text{SU}(2)$ acting on the right. The action of $q \in \text{Sp}(1) \cong G'$ on $V$ is given by right multiplication of $\mathbb{H}^2$ by $q^{-1}$. It is not complex linear.

We describe the action of the basis elements (3.3) of $g'$ as follows:

\[
i_R(x_1 + jy_1, x_2 + jy_2) = ((-ix_1) + j(-iy_1), (-ix_2) + j(-iy_2)),
\]

\[
 j_R(x_1 + jy_1, x_2 + jy_2) = (\bar{y}_1 - j\bar{x}_1, \bar{y}_2 - j\bar{x}_2),
\]

\[
k_R(x_1 + jy_1, x_2 + jy_2) = ((-i\bar{y}_1) + j(i\bar{x}_1), (-i\bar{y}_2) + j(i\bar{x}_2)).
\]
3.5.2. **Cohomogeneity one.** Since $G'$ normalizes $G$, it acts on $X = S(V)/G$. Indeed the group generated by $G$ and $G'$ is the full normalizer $K = \text{SO}(4)$ of $G$ in $\text{O}(V) = \text{O}(8)$. The representation of $K$ on $V$ is the isotropy representation of the symmetric space $G_2/\text{SO}(4)$, of rank 2. It follows that $X/G'$ is one-dimensional and in fact isometric to the interval $[0, \pi/6]$. Now the action of $G'$ on $X$ has cohomogeneity one. A $K$-horizontal geodesic is given by

$$\gamma(t) = \cos \frac{tv^3}{\sqrt{6}} + \sin \frac{tu^2v}{\sqrt{2}}.$$  

It suffices to compute the sectional curvatures of $X$ along the projection of $\gamma$.

3.5.3. **Natural frames.** For future reference, we compute:

- $i_R\gamma(t) = -\cos \frac{tv^3}{\sqrt{6}} - \sin \frac{tu^2v}{\sqrt{2}}$
- $j_R\gamma(t) = -\cos \frac{tv^3}{\sqrt{6}} - \sin \frac{tu^2v}{\sqrt{2}}$
- $k_R\gamma(t) = \cos \frac{tv^3}{\sqrt{6}} + \sin \frac{tu^2v}{\sqrt{2}}$
- $i_L\gamma(t) = -3\cos \frac{tv^3}{\sqrt{6}} + \sin \frac{tu^2v}{\sqrt{2}}$
- $j_L\gamma(t) = \sqrt{3} \sin \frac{tv^3}{\sqrt{6}} + (-2 \sin t + \sqrt{3} \cos t) \frac{tu^2v}{\sqrt{2}}$
- $k_L\gamma(t) = \sqrt{3} \sin \frac{tv^3}{\sqrt{6}} + (2 \sin t + \sqrt{3} \cos t) \frac{tu^2v}{\sqrt{2}}$.

It is useful to note that there is an orthogonal decomposition

$$T_pS^7 = \langle \gamma'(t) \rangle \oplus \langle i_Rp, i_LP \rangle \oplus \langle j_Rp, j_LP \rangle \oplus \langle k_Rp, k_LP \rangle$$

where $p = \gamma(t)$.

3.5.4. **The Weyl group.** The singular points of the $K$-action on $X$ are $p_1 = \gamma(0)$ and $p_2 = \gamma(\pi/6)$. Their isotropy groups are given by

- $K_{p_1} = \langle (e^{i\theta}, e^{-3i\theta}), (j, j) \rangle$,
- $K_{p_2} = \langle (e^{i\theta}, e^{j\theta}), (i, i) \rangle$.

Note that

$$K_{\text{princ}} = \{ \pm(1, 1), \pm(i, i), \pm(j, j), \pm(k, k) \}.$$

The reflections at $p_1$, $p_2$ are given by

$$w_1 = e^{i\pi/4}(1, -1), \quad w_2 = e^{j\pi/4}(1, 1).$$

Let $w = w_1w_2$. Then $w$ maps $\gamma(t)$ to $\gamma(t - \pi/3)$ and acts by conjugation on $K_{\text{princ}}$ by cyclically permuting $(i, i), (j, j), (k, k)$.
3.5.5. The O'Neill tensor. The vertical space at \( p = \gamma(t) \in S(V) \) is spanned by \( i_{LP}, j_{LP}, k_{LP} \); this is an orthogonal frame. Also \( i_{RP}, j_{RP}, k_{RP} \) is an orthogonal frame. The only nonzero inner products between vectors in the two sets are:

\[
i_0(t) := \langle i_L \gamma(t), i_R \gamma(t) \rangle = 4 \cos^2 t - 1,
\]

\[
j_0(t) := \langle j_L \gamma(t), j_R \gamma(t) \rangle = \langle i_L \gamma(t + \pi/3), i_R \gamma(t + \pi/3) \rangle
\]

and

\[
k_0(t) := \langle k_L \gamma(t), k_R \gamma(t) \rangle = \langle i_L \gamma(t + 2\pi/3), i_R \gamma(t + 2\pi/3) \rangle,
\]

using the action of the Weyl group element \( w \). Moreover

\[
||i_{RP}||^2 = ||j_{RP}||^2 = ||k_{RP}||^2 = 1,
\]

\[
||i_L \gamma(t)||^2 = 1 + 8 \cos^2 t,
\]

\[
||j_L \gamma(t)||^2 = ||i_L \gamma(t + \pi/3)||^2
\]

and

\[
||k_L \gamma(t)||^2 = ||i_L \gamma(t + 2\pi/3)||^2.
\]

Denote by \( i_{R}^h(t) \) the vector field along \( \gamma \) in \( S^7 \) given by the horizontal projection of \( i_R \gamma(t) \). Then

\[
i_{R}^h(t) = i_R \gamma(t) - I_0(t)i_L \gamma(t)
\]

where \( I_0(t) = i_0(t)/||i_L \gamma(t)||^2 \); put also \( J(t) := ||i_{R}^h(t)|| \). Define similarly \( j_{R}^h, k_{R}^h \), \( J_0, K_0, J, K \). A natural horizontal orthonormal frame along \( \gamma \) is now given by \( \gamma' = \partial/\partial t, i_{R}^h/I, j_{R}^h/J, k_{R}^h/K \).

We use O'Neill’s formula to show that there is a value of \( t \) for which the plane spanned by \( i_{R}^h, j_{R}^h, k_{R}^h \) projects to a plane of curvature 1 in \( X \). In fact, equivariantly extend \( i_{R}^h, j_{R}^h, k_{R}^h \) to vector fields in \( S^7 \), denote the Levi-Civitá connection of \( S^7 \) by \( \nabla \) and the O’Neill tensor of \( S^7 \to X \) by \( A \). Then

\[
A_{i_{R}^h}j_{R}^h = \langle \nabla_{i_{R}^h}j_{R}^h, k_L \rangle \frac{k_L}{||k_L||^2},
\]

where

\[
\langle \nabla_{i_{R}^h}j_{R}^h, k_L \rangle = -\langle j_{R}^h, \nabla_{i_{R}^h}k_L \rangle
\]

\[
= -\langle j_{R}^h, k_L(i_{R}^h) \rangle \quad \text{(since \( k_L \) is a linear vector field)}
\]

\[
= k_0(t) + J_0(t)j_0(t) + J_0(t)i_0(t) - I_0(t)j_0(t)(k_{L}i_L \gamma(t), j_L \gamma(t)).
\]

The sectional curvature is

\[
K_X(i_{R}^h \wedge j_{R}^h)|_t = 1 + \frac{3||A_{i_{R}^h}j_{R}^h||_t^2}{J(t)^2 J(t)}
\]

\[
= 1 - 27(-2\sqrt{3} - \sqrt{3}\cos 2t + \sin 2t + 4 \sin 4t)^2
\]

where

\[
P(t) = (5 + 4 \cos 2t)(5 - 2 \cos 2t + 2\sqrt{3}\sin 2t)
\]

\[
(-10 + 2 \cos 2t - 5 \cos 4t + 4 \cos 6t + 2\sqrt{3}\sin 2t + 5\sqrt{3}\sin 4t).
\]

It is easy to prove that there is \( t_0 \in (0, \pi/6) \) such that

\[
-2\sqrt{3} - \sqrt{3}\cos 2t + \sin 2t + 4 \sin 4t = 0,
\]

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which shows the existence of planes with curvature 1. Indeed \( t_0 = \frac{\pi}{3} - \frac{1}{2} \arccos \frac{1}{4} \approx 0.38814 \).

3.6. The representation \((U(2), \mathbb{C}^4)\). This representation is an enlargement of that in subsection 3.5; we retain the notation therein. View \( H = U(2) \) as the group generated by \( G \cong SU(2) \) and a circle subgroup of \( G' \cong SU(2) \), and denote the orbit space \( S^7/H \) by \( Y \). We will show that \( Y \) is highly curved.

3.6.1. Description of all 2-planes in \( X = S^7/SU(2) \) with sectional curvature 1. We use ideas of [Tho71] in dimension 4. Let \( x_t \in X \) be the projection of \( \gamma(t) \). The curvature operator \( R_t : \Lambda^2 T_x X \to \Lambda^2 T_x X \) is self-adjoint, and its matrix with respect to the orthonormal basis

\[
\begin{bmatrix}
\frac{\partial}{\partial t} \wedge \frac{j_1}{T}, & \frac{\partial}{\partial t} \wedge \frac{j_2}{I}, & \frac{\partial}{\partial t} \wedge \frac{j_3}{J}, & \frac{k_1}{K} \wedge \frac{j_1}{I}, & \frac{k_2}{K} \wedge \frac{j_2}{I}, & \frac{k_3}{K} \wedge \frac{j_3}{I}
\end{bmatrix}
\]

is

\[
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
= \begin{pmatrix}
A & B \\
B & C
\end{pmatrix}
\]

where \( a_i, b_i, c_i \) are smooth functions of \( t \in (0, \pi/6) \). The diagonal elements of \( A \) and \( C \) are sectional curvatures and, by the Bianchi identity, the trace of \( B \) is zero. Note that \( c_3 \) was computed in the previous section, and the other functions are computed similarly; their explicit values are listed in the appendix. In view of the action of the Weyl group (subsection 3.5.3), these functions satisfy

\[
a_2(t) = a_1 \left(t + \frac{\pi}{3}\right), \ a_3(t) = a_1 \left(t + \frac{2\pi}{3}\right),
\]

\[
b_2(t) = b_1 \left(t + \frac{\pi}{3}\right), \ b_3(t) = b_1 \left(t + \frac{2\pi}{3}\right),
\]

\[
c_2(t) = c_1 \left(t + \frac{\pi}{3}\right), \ c_3(t) = c_1 \left(t + \frac{2\pi}{3}\right).
\]

Let \( Z(\tilde{R}_t) \), where \( \tilde{R}_t = R_t - I \), denote the subset of the Grassmannian \( G_t := \text{Gr}_2(T_x X) \) consisting of points where the sectional curvature is 1. We will show
that \( \ker(\tilde{R}_t - \mu(t)*) \cap G_t \) is non-empty, where * denotes the Hodge star operator on \( \Lambda^2 T_{x_t}X \) and \( \mu(t) \) is a certain smooth function. It will follow from the remark after Theorem 2.1 and Theorem 4.1 in [Tho71] that \( Z(\tilde{R}_t) = \ker(\tilde{R}_t - \mu(t)*) \cap G_t \).

Let \( \mu_1^\pm(t, e) = b_i(t) \pm \sqrt{(a_i(t) - 1)(c_i(t) - 1)} \) for \( i = 1, 2, 3 \). Take \( \mu = \mu_1^- \) and put \( \alpha_i(t) = a_i(t) - 1, \beta_i(t) = b_i(t) - \mu(t), \gamma_i(t) = c_i(t) - 1 \). Using the explicit formulae in the appendix, one checks that

\[
((b_1 - b_2)^2 - \alpha_1 \gamma_1 - \alpha_2 \gamma_2)^2 = 4\alpha_1 \alpha_2 \gamma_1 \gamma_2
\]

and

\[
(b_1 - b_2)^2 - \alpha_1 \gamma_1 - \alpha_2 \gamma_2 > 0, \quad b_1 > b_2.
\]

It follows that

\[
\mu_1^-(t) = \mu_1^+(t) \quad \text{for all } t.
\]

Similarly, one checks that

\[
\mu_1^-(t) = \begin{cases} 
\mu_1^+(t) & \text{for } t \leq t_0, \\
\mu_1^-(t_0) & \text{for } t \geq t_0.
\end{cases}
\]

It immediately follows that \( \ker(\tilde{R}_t - \mu(t)*) \) is 3-dimensional and spanned by

\[
-\beta_1(t) \frac{\partial}{\partial t} \wedge \frac{i h}{\mathcal{I}} + \alpha_1(t) \frac{\partial}{\partial t} \wedge \frac{k h}{2} = -\beta_2(t) \frac{\partial}{\partial t} \wedge \frac{j h}{\mathcal{J}} + \alpha_2(t) \frac{k h}{2} \wedge \frac{j h}{\mathcal{J}}
\]

and

\[
-\beta_3(t) \frac{\partial}{\partial t} \wedge \frac{k h}{2} + \alpha_3(t) \frac{j h}{\mathcal{J}} \wedge \frac{j h}{\mathcal{J}}.
\]

Let \( r_1, r_2, r_3 \) be the corresponding coordinates on \( \ker(\tilde{R}_t - \mu(t)*) \). The Plücker and normalization relations defining \( \ker(\tilde{R}_t - \mu(t)*) \cap G_t \) now are

\[
\sum_{i=1}^3 r_i^2 \alpha_i \beta_i = 0, \quad \sum_{i=1}^3 r_i^2 (\alpha_i^2 + \beta_i^2) = 1.
\]

Solving these relations yields:

\[\text{(3.6)}\]

\[
r_1 = \begin{cases} 
\pm \frac{1}{\sqrt{A_1(t)}} \cosh \theta & \text{if } t < t_0 \\
\theta & \text{if } t = t_0 \\
\frac{1}{\sqrt{-A_1(t)}} \sinh \theta & \text{if } t > t_0,
\end{cases}
\]

\[
r_2 = \begin{cases} 
\frac{1}{\sqrt{-A_2(t)}} \sinh \theta & \text{if } t < t_0 \\
\pm \left( \frac{\alpha_1(t_0) \beta_1(t_0)}{\alpha_2(t_0) \beta_2(t_0)} \right)^{1/2} \theta & \text{if } t = t_0 \\
\pm \frac{1}{\sqrt{-A_2(t)}} \cosh \theta & \text{if } t > t_0
\end{cases}
\]

and

\[
r_3^2 = \frac{1 - r_1^2 (\alpha_1^2(t) + \beta_1^2(t)) - r_2^2 (\alpha_2^2(t) + \beta_2^2(t))}{\alpha_3^2(t) + \beta_3^2(t)},
\]

where

\[
A_1 = \alpha_1^2 + \beta_1^2 - \frac{\alpha_1 \beta_1}{\alpha_3 \beta_3} (\alpha_3^2 + \beta_3^2), \quad A_2 = \alpha_2^2 + \beta_2^2 - \frac{\alpha_2 \beta_2}{\alpha_3 \beta_3} (\alpha_3^2 + \beta_3^2).
\]

We deduce that for each \( t \in (0, \pi/6) \) there is a one-parameter family of 2-planes in \( T_{x_t}X \) with sectional curvature 1.
For some nonzero $\xi \in \mathfrak{h}/\mathfrak{g} = \mathfrak{u}(2)/\mathfrak{su}(2)$, the induced $\xi_R$ is a unit vertical vector field of $X \to Y$. Let now $u$, $v$ be linearly independent tangent vectors to $S^7$, horizontal with respect to $S^7 \to Y$. Then
\begin{equation}
\langle \xi_R(u), v \rangle
\end{equation}
is a component of the O’Neill tensor of $S^7 \to Y$, evaluated at $u \wedge v$, which is complementary to the O’Neill tensor of $S^7 \to X$; however, note that $\xi_R$ is not orthogonal to the vertical distribution of $S^7 \to Y$ (spanned by $i_L$, $j_L$, $k_L$). The inner product $\langle \cdot, \cdot \rangle$ is also a component of the O’Neill tensor of $S^7 \to CP^3$, where the circle action for this Hopf action is infinitesimally generated by $\xi_R$. Let $\sigma$ be the 2-plane tangent to $X$ which is the projection of $u \wedge v$, and assume that $\sigma$ has sectional curvature 1. Then $\sigma$ projects to a 2-plane of curvature 1 in $Y$ if and only if $u \wedge v$ projects to a totally real 2-plane in $CP^3$; in fact, $\xi_R$ induces the complex structure of $CP^3$.

Our method to prove that $\kappa^Y > 1$ is to show that no 2-plane in $X$ with sectional curvature 1, horizontal with respect to $X \to Y$, can correspond to a totally real 2-plane in $CP^3$. We first show it suffices to consider 2-planes in $X$ with sectional curvature 1 along the projection of the geodesic $\gamma$, as long as we take into account also the non-horizontal planes with respect to $X \to Y$. In fact, let $\sigma$ be a 2-plane in $T_xX$ with sectional curvature 1 and horizontal with respect to $X \to Y$, where $x \in X$ projects to a regular point of $Y$. There is $g \in G'$ such that $gx = x_t$ for some $t \in (0, \pi/6)$. Now $g_*\sigma$ is a 2-plane in $T_{gx}X$ with sectional curvature 1 and it is horizontal with respect to $g_*\xi_R = (Ad_g\xi)_R$ which is in general different from $\xi_R$, but
\begin{equation}
\langle \xi_R(u), v \rangle = \langle (Ad_g\xi)_R(gu), gv \rangle;
\end{equation}
note that in principle $Ad_g\xi$ can be parallel to any element of $g'$. Conversely, given a 2-plane $\sigma$ in $T_xX$ with sectional curvature 1 for some $t \in (0, \pi/6)$, represented by $u \wedge v$ where $u, v$ are vectors tangent to $S^7$, horizontal with respect to $S^7 \to X$, we observe that $\frac{d}{dt}x_t$ does not belong to $\sigma$ (since the $\alpha_i$ are positive on $(0, \pi/6)$). Therefore there is a unique, up to sign, unit vector field $n_R$ on $S^7$, which is normal to $u$, $v$, $\gamma'(t)$, where $n \in g'$ (cf. (3.8)). We choose $g \in G'$ such that $Ad_g n = \xi$ so that $\xi_R(g\gamma(t)) = g_*(n_R\gamma(t))$ is normal to $gu \wedge gv$, which represents the 2-plane $g\sigma$ in $T_{gx}X$ with sectional curvature 1 and horizontal with respect to $X \to Y$.

Next we apply the method. Let $\sigma$ be a 2-plane in $T_xX$ with sectional curvature 1. Let
\begin{equation}
u = u_0 \frac{\partial}{\partial t} + u_1 \frac{j_R}{J} + u_2 \frac{j_R}{J} + u_3 \frac{k_R}{K} \end{equation}
and
\[ v = v_0 \frac{\partial}{\partial t} + v_1 \frac{j^i}{I} + v_2 \frac{j^j}{J} + v_3 \frac{k^h}{K} \]
be tangent vectors to \( S^7 \) such that \( u \wedge v \) projects to \( \sigma \); let \( (\sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{23}, \sigma_{31}, \sigma_{12}) \)
be the coordinates of \( \sigma \) in the basis \( (3.4) \), so that \( \sigma_{01} = u_0v_1 - u_1v_0 \) etc. The unit normal vector is induced by the following non-zero element of \( g' \):
\[ (3.8) \quad n = (JK\sigma_{23} i + KJ\sigma_{31} j + IJ\sigma_{12} k)/\sqrt{J^2K^2\sigma_{23}^2 + K^2I^2\sigma_{31}^2 + I^2J^2\sigma_{12}^2}. \]

We can now compute:
\[ (3.9) \quad \langle n_R(u), v \rangle = r_1^2 \alpha_1(t)(\alpha_1(t)E(t) - \beta_1(t)F(t)) \]
\[ + r_2^2 \alpha_2(t)(\alpha_2(t)E(t + \pi/3) - \beta_2(t)F(t + \pi/3)) \]
\[ + r_3^2 \alpha_3(t)(\alpha_3(t)E(t + 2\pi/3) - \beta_3(t)F(t + 2\pi/3)), \]
where
\[ E(t) = 1 - J_0(t)j_0(t) - K_0(t)k_0(t) - J_0(t)K_0(t)i_0(t) \]
and
\[ F(t) = 2J(t)K(t)I(t)\sin 2t/I(t). \]

Explicit formulae for \( E \) and \( F \) are given in the appendix.

Of course we have \( \alpha_2(t) = \alpha_1(t + \pi/3) \), \( \alpha_3(t) = \alpha_1(t + 2\pi/3) \), \( \gamma_2(t) = \gamma_1(t + \pi) \) and \( \gamma_3(t) = \gamma_1(t + 2\pi) \) for \( 0 < t < \pi/6 \) (cf. \( (3.8) \)), but the situation for the \( \beta_i \)'s is more complicated since it involves \( \mu \). To remedy this situation, we introduce \( \tilde{\beta}_1 \). From the appendix we read
\[ \beta_1(t) = 27 \frac{|1 - 4 \cos 2t|}{\sqrt{(5 + 4 \cos 2t)^3(21 - 20 \cos 2t + 8 \cos 4t)}}. \]
Put
\[ \tilde{\beta}_1(t) = -27 \frac{1 - 4 \cos 2t}{\sqrt{(5 + 4 \cos 2t)^3(21 - 20 \cos 2t + 8 \cos 4t)}}. \]

Then
\[ \tilde{\beta}_1(t) = \beta_1(t) \quad \text{for } 0 < t < \pi/6; \]
\[ \tilde{\beta}_1(t + \pi/3) = -\beta_1(t + \pi/3) = \beta_2(t) \quad \text{for } 0 < t < \pi/6; \]
\[ \tilde{\beta}_1(t + 2\pi/3) = \begin{cases} -\beta_1(t + 2\pi/3) = \beta_3(t) & \text{for } 0 < t \leq t_0, \\ \beta_1(t + 2\pi/3) = \beta_3(t) & \text{for } t_0 < t \leq \pi/6. \end{cases} \]

Now we can rewrite \( (3.9) \) as
\[ (3.10) \quad \langle n_R(u), v \rangle = r_1^2 C_1(t) + r_2^2 C_2(t) + r_3^2 C_3(t) \]
where
\[ C_1(t) = \alpha_1(t) \left[ \alpha_1(t)E(t) - \tilde{\beta}_1(t)F(t) \right] \]
\[ \text{and } C_2(t) = C_1(t + \pi/3), C_3(t) = C_1(t + 2\pi/3) \text{ for } 0 < t < \pi/6. \]
Finally
\[ \alpha_1(t) > 0 \quad \text{for all } t \in \mathbb{R}, \]
and we compute
\[ (3.11) \quad \alpha_1(t)E(t) - \tilde{\beta}_1(t)F(t) = \frac{-54(1 + 2 \cos 4t)^2}{(5 + 4 \cos 2t)^3(21 - 20 \cos 2t + 8 \cos 4t)} \leq 0. \]
for all \( t \in \mathbb{R} \), and on the interval \([0, 5\pi/6]\) it vanishes precisely for \( t = \pi/6, \pi/3, 2\pi/3 \) and \( 5\pi/6 \).

It is clear that the quadratic form (3.10) is nonpositive everywhere and could only non-trivially vanish at the endpoints \( t = 0 \) and \( t = \pi/6 \). On the other hand, \( C_1 \) is bounded away from zero near \( t = 0 \) and \( C_2 \) is bounded away from zero near \( t = \pi/6 \) (cf. (3.11) and Fig. [2]). Moreover (3.6) shows that \( r_1 \to \pm 3/4 \) as \( t \to 0 \) and \( r_2 \to \pm 7/32 \) as \( t \to \pi/6 \). This proves that \( \langle n_R(u), v \rangle \) is bounded away from zero on the set of 2-planes of sectional curvature 1 of \( X \) and finishes the proof that \( \kappa_Y > 1 \).

4. Main result

We are going to finish the proof of Theorem [1.1] in this section. Let \( \rho : G \to O(V) \) be a representation of a compact Lie group and assume that \( X = S(V)/G \) has dimension \( m \geq 2 \).

4.1. Polar case. This is precisely the case in which the orbit space \( X = S(V)/G \) is a good Riemannian orbifold of constant curvature 1 \([GL15, Intro]\). In case of connected groups, these representations are classified and are orbit-equivalent to isotropy representations of symmetric spaces \([Dad85]\).

4.2.Disconnected case. Let \( \rho_0 \) be the restriction of \( \rho \) to the identity component \( G^0 \) of \( G \). Then the projection \( X_0 = S(V)/G^0 \to X = S(V)/G \) is a Riemannian covering over the set of regular points of \( X \). We deduce \( \kappa_{\rho} = \kappa_{\rho_0} \). This shows it suffices to prove the results for representations of connected compact Lie groups \( G = G^0 \).

4.3. Reducible case. Assume that the representation \( \rho : G \to O(V) \) is reducible. We will prove that \( \rho \) is as in cases (ii), (iii), (iv), (v) of Theorem [1.1]. Note that \( X \) has diameter at least \( \pi/2 \) (cf. [GL14]). This already implies \( \kappa_X \leq 4 \). In view of Proposition [2.5], we further know that \( \kappa_X = 1 \) unless \( \rho \) is the sum of two (irreducible) representations of cohomogeneity one. In the latter case, assume \( G \) is connected and \( \rho \) is non-polar. Then either \( \rho \) is listed in Tables 2 and 3 in [GL16, section 6], or \( X \) is not a Riemannian orbifold (i.e. \( \rho \) is not infinitesimally polar). The representations in Table 2 have good Riemannian orbifolds of constant curvature 4 as orbit spaces (case (ii) of Theorem [1.1]), and among those listed in Table 3, the only reduced representation is case 11, which yields a complex weighted projective line as orbit space (case (iii) of Theorem [1.1] discussed in subsection 3.1), and the other representations reduce to a \( \mathbb{Z}_2 \)-extension of case 11. Going through the proof of Proposition 2 in [GL16], we see that in the non-infinitesimally polar case, \( \rho \) must be as in cases (iv) or (v) of Theorem [1.1]. Case (iv) is analyzed in subsection 3.2. In case (v) the group lies in between \( \mathfrak{sp}(m) \) and \( \mathfrak{sp}(m)T^2 \), so Proposition [2.5(a)] and [GL16, Table 1] yield \( \kappa = 4 \).

We hereafter assume \( \rho \) is irreducible.

4.4. The case \( \text{rank}(G) = 1 \). Every representation of \( U(1) \) with cohomogeneity at least 3 is reducible, so we may assume \( G \) is covered by \( SU(2) \). According to Lemma [2.3], the only irreducible, non-polar representations that need to be considered are \( (\text{SO}(3), \mathbb{R}^7) \) and \( (\text{SU}(2), \mathbb{H}^2) \). They were examined respectively in subsections 3.3 and 3.5, and are not highly curved.

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4.5. Initial cases. We begin with \( m = 2 \). Due to [Str94], in this case the classification of irreducible representations of connected groups with cohomogeneity 3 yields good Riemannian orbifolds of constant curvature 4 [Str94 Table II, case III]. Assume now that \( m = 3 \). By the classification of irreducible representations of connected groups of cohomogeneity 4 [GL14, Theorem 1.11], our representation either has a toric reduction (i.e. reduces to a finite extension of a torus action) and hence is not highly curved, or is given by the action of \( U(2) \) on \( \mathbb{C}^4 \), which has already been discussed in subsection 3.5 and is case (i) of Theorem 1.1. Consider now the restriction of a polar representation of \( SO(2) \) into one of the cases (i), (ii), (iii), (iv) or (v), so that \( \dim Y \leq 3 \). By the assumption on the minimality of \( m \), this implies that \( \rho \) is not highly curved, or is given by the action of \( U(2) \) on \( \mathbb{C}^4 \), which has already been discussed in subsection 3.5 and is case (i) of Theorem 1.1. Consider now \( m = 4 \). By the classification of irreducible representations of connected groups of cohomogeneity 5 [GL14, Theorem 1.11], our representation either has a toric reduction or reduces to the action of \( SO(3) \times U(2) \) on \( \mathbb{R}^{12} = \mathbb{R}^3 \otimes \mathbb{R}^4 \). This representation is the restriction of a polar representation of \( SO(3) \times SO(4) \), so it is not highly curved by Proposition 2.5.

4.6. Formulation. We are going to complete the proof of Theorem 1.1 by proving that there exist no irreducible representations \( \rho \) of connected compact Lie groups with \( m \geq 5 \) that are highly curved. Suppose, to the contrary, that there exists such a representation \( \rho : G \to O(V) \). We may assume that \( m \) is minimal among all such examples. We may also assume that for this \( m \), the number \( g = \dim G \) is minimal among all such examples. We fix \( \rho \) throughout the proof.

4.7. Reduction. By the assumption on the minimality of \( g \), the representation \( \rho \) is reduced (cf. subsection 2.2): for any other representation \( \tau : H \to O(W) \) such that \( S(W)/H \) is isometric to \( X \), we have \( \dim(H) \geq g \). In particular, this implies that the action of \( G \) on \( S(V) \) has trivial principal isotropy groups.

4.8. Type of representation. We claim that the normalizer \( N \) of \( \rho(G) \) in \( O(V) \) has \( \rho(G) \) as its identity component. Otherwise, we find a connected subgroup \( H \) of \( O(V) \) containing \( \rho(G) \) with one dimension more. The inclusion \( \tau : H \to O(V) \) is an irreducible representation and an enlargement of \( \rho \). The quotient space \( Y = S(V)/H \) has dimension at least \( m - 1 \geq 4 \), and \( \kappa_r \geq \kappa_\rho \geq 1 \) due to Proposition 2.5.

Note that \( \tau \) and \( \rho \) cannot have the same orbits, for this would contradict the triviality of principal isotropy groups of \( \rho \). In view of the minimality of \( m \), this implies that \( m = 5 \) and \( \dim Y = 4 \), but this is in contradiction with subsection 1.5.

We deduce that \( \rho \) cannot be of quaternionic type, and in case it is of complex type, \( G \) is covered by \( U(1) \times G' \) for a connected compact Lie group \( G' \).

4.9. Consequences. We already know that \( \rho \) is not polar and \( \text{rank}(G) \geq 2 \). Proposition 2.1 and Lemma 2.2 together with the choice of \( m \) imply that: either \( X \) does not contain strata of dimensions between 2 and \( m - 1 \), whence the rank of \( G \) is at most 3; or \( X \) does contain a stratum \( X_{(K)} \) of dimension in that range whose associated folding map \( I_{(K)} \) is defined on the orbit space of a representation falling into one of the cases (i), (ii), (iii), (iv) or (v), so that \( \dim Y \leq 5 \) and thus the rank of \( G \) is at most 7.

In view of subsection 4.8 and Proposition 2.5 there only remain the following possibilities:

(a) \( \rho \) is an irreducible representation of real type a simple Lie group \( G' \);
(b) \( G = U(1) \times G' \) and \( \rho = \theta \otimes \rho' \), where \( \theta \) is the representation of \( U(1) \) on \( \mathbb{C} \) and \( \rho' \) is an irreducible representation of complex type of a simple Lie group \( G' \);
(c) $G = \text{Sp}(1) \times G'$ and $\rho = \phi \otimes \rho'$, where $\phi$ is the representation of $\text{Sp}(1)$ on $\mathbb{H}$ and $\rho'$ is an irreducible representation of quaternionic type of a simple Lie group $G'$.

(d) $G = G_1 \times G_2$ and $\rho = \rho_1 \otimes \rho_2$, where $V = \mathbb{F}^2 \otimes_F V_2$, $\dim_F V_2 \geq 2$ and $\mathbb{F}$ can be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

4.10. Kollros' tables and first three cases.

4.10.1. Case (a). In view of Lemma 2.3

\begin{equation}
\text{dim } V \leq 2 \text{dim } G + 3 - \text{rk } G
\end{equation}

and

\begin{equation}
\text{dim } V \leq 2 \text{dim } G + 2.
\end{equation}

Using (4.13) and [Kol02, Lemma 2.6], we deduce that $\rho$ must be one of: ($\text{SO}(7), \Lambda^3 \mathbb{R}^7$), ($\text{Spin}(15), \mathbb{R}^{128}$) (half-spin), ($\text{SO}(8), \Lambda^3 \mathbb{R}^8$), ($G_2, \mathbb{S}^3 \mathbb{R}^7$). The second and fourth representations admit enlargements to $\text{Spin}(16)$ (spin representation) and $\text{SO}(7)$, respectively, which are polar representations, hence they are not highly curved. The third representation fails to satisfy (4.12). The first representation ($\text{SO}(7), \Lambda^3 \mathbb{R}^7$) admits an isotropy group $K = G_p \cong \text{SO}(2)^3$ which is a maximal torus of $G$ (say, $p = a e_1 \wedge e_2 \wedge e_3 + b e_3 \wedge e_4 \wedge e_5$ for generic coefficients $a, b$). Now the fixed point set $W$ of $K$ is 3-dimensional and $H = N_G(K)/K$ is finite, so $Y = S(W)/H$ has constant curvature 1 and the existence of the folding map $I_{(K)} : Y \to X$ implies $\kappa_X = 1$.

4.10.2. Case (b). In view of Lemma 2.3

\begin{equation}
\text{dim } G' + 7 \leq \text{dim } G' + 2 + m = \text{dim } V \leq 2 \text{dim } G' + 4 - \text{rk } G'.
\end{equation}

In case $\text{rk } G' = 1$ we may assume $G' = \text{SU}(2)$ and then (4.14) gives a contradiction. In case $\text{rk } G' \geq 2$, (4.14) gives $\text{dim } V \leq 2 \text{dim } G' + 2$ and we can use [Kol03, Proposition] to deduce that $\rho = (\text{U}(7), \Lambda^3 \mathbb{C}^7)$. This representation is not highly curved because it can be enlarged to ($\text{SU}(8), \Lambda^4 \mathbb{C}^8$), which is a polar representation. Indeed, $\Lambda^3 \mathbb{C}^7$ can be viewed as an $\text{U}(7)$-invariant real form of $\Lambda^4 \mathbb{C}^8$ via

$$x \in \Lambda^3 \mathbb{C}^7 \mapsto \frac{1}{2}(x \wedge e_8 + \epsilon(x \wedge e_8)) \in \Lambda^4 \mathbb{C}^8$$

where $\epsilon$ is the Hodge star operator followed by complex conjugation (see also [Yam p.882]).

4.10.3. Case (c). In view of Lemma 2.3

\begin{equation}
\text{dim } V \leq 2 \text{dim } G' + 8 - \text{rk } G'.
\end{equation}

In case $\text{rk } G' = 1$, we may assume $G = \text{Sp}(1) \times \text{Sp}(1)$ and then (4.15) gives $V = \mathbb{H}^3 \otimes_\mathbb{H} \mathbb{H}$. This representation is covered by subsection 3.4.

In case $\text{rk } G' \geq 2$, (4.15) gives $\text{dim } V \leq 2 \text{dim } G' + 6$ and we can use [Kol03, Proposition] to deduce that $\rho = (\text{Sp}(1) \times \text{Spin}(11), \mathbb{H} \otimes_\mathbb{H} \mathbb{H}^{10})$ or $(\text{Sp}(1) \times \text{Spin}(13), \mathbb{H} \otimes_\mathbb{H} \mathbb{H}^{32})$. These representations are not highly curved because they can be respectively enlarged to $(\text{Sp}(1) \times \text{Spin}(12), \mathbb{H} \otimes_\mathbb{H} \mathbb{H}^{16})$ and $(\text{Spin}(16), \mathbb{R}^{128})$, which are polar.
4.11. The remaining case. In view of subsection 4.10 there remains only to
classify the case of irreducible representations \( \rho \) that can be decomposed as a
tensor product \( \rho_1 \otimes \rho_2 \) where \( V = \mathbb{F}^2 \otimes_{\mathbb{F}} V_2, G = G_1 \times G_2, s := \dim_{\mathbb{F}} V_2 \geq 2 \) and \( \mathbb{F} \) can be \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \).

Consider a pure tensor \( v = v_1 \otimes v_2 \). Consider the enlargement to the polar
representation of cohomogeneity 2 of a compact connected Lie group \( H \). Consider a
geodesic \( \gamma \) starting at \( v \) in a certain \( H \)-horizontal direction. It is then automatically
\( G \)-horizontal, and we may choose it to contain \( G \)-regular points. Since the quotient
\( S(V)/H \) is an interval of length \( \pi/4 \), the point \( \gamma(\pi/2) \) is again on the same \( H \)-orbit,
hence it is again a pure tensor.

4.11.1. The case \( V = \mathbb{R}^2 \otimes_{\mathbb{R}} V_2 \). Here \( G_1 = \text{SO}(2) \) and \( V_2 \) is a representation of
real type (since \( V \) is irreducible).

**Proposition 4.1.** Assume that the action of \( G_2 \) on \( S(V_2) \) has singular orbits. Then
\( \kappa_{\rho} = 1 \).

**Proof.** We have \( n = 2s - 1 \). Since \( \rho_2 \) has singular orbits, for any \( v_2 \) in \( S(V_2) \), the
\( G_2 \)-orbit through \( v_2 \) has dimension at most \( s - 2 \). Moreover, there is a point \( w_2 \)
such that the orbit \( G_2 \cdot w_2 \) has dimension at most \( s - 3 \). Thus a regular horizontal
geodesic \( \gamma \) as above that starts at the pure tensor \( w_1 \otimes w_2 \), starts at an orbit of
dimension at most \( s - 2 \) and intersects at time \( \pi/2 \) an orbit of dimension at most
\( s - 1 \). Since \( (s-2) + (s-1) < 2s-2 = n-1 \), we obtain \( \kappa_{\rho} = 1 \) from Lemma 2.24. \( \Box \)

It remains to consider the possibility that \( \rho_2 \) has no singular orbits.

In case \( \rho_2 \) is a representation of cohomogeneity one of real type, we deduce from the
classification that \( \rho \) is either polar or has cohomogeneity three, contrary to our
assumptions.

If \( G_2 \) acts non-transitively and without singular orbits on \( S(V_2) \), then it is a
(non-Abelian) group of rank 1, but all representations of real type of \( \text{SO}(3) \) admit
singular orbits, so this case cannot occur.

4.11.2. The case \( V = \mathbb{H}^2 \otimes_{\mathbb{H}} V_2 \). We have \( \rho(G) \subset \text{Sp}(2) \otimes \text{Sp}(V_2) \), where \( V_2 \) is
complex irreducible of quaternionic type and \( s := \dim_{\mathbb{H}} V_2 \geq 2 \). We may also
assume \( G_1 \) and \( G_2 \) are simple, for otherwise we could rearrange the factors of \( V \)
and fall into case of a real tensor product.

Since there does not exist a representation non-equivalent but orbit-equivalent
to \( \text{Sp}(2) \times \text{Sp}(s) \), if \( G_1 \neq \text{Sp}(2) \) then we may enlarge \( \rho \) to a representation \( \hat{\rho} \) of
\( G_1 \times \text{Sp}(s) \) and still have \( \hat{\rho} \) of cohomogeneity at least 3. Due to Proposition 2.9
it suffices to check that \( \hat{\rho} \) is not highly curved. Indeed this representation is an
enlargement of the doubling of the vector representation of \( \text{Sp}(s) \). The latter has
cohomogeneity 6. Since the action of \( G_1 \times \text{Sp}(s) \) is clearly not orbit-equivalent to
that of \( \hat{\rho} \), its orbit space has smaller dimension, and then we already know
that \( \kappa_{\hat{\rho}} = 1 \).

Otherwise \( G_1 = \text{Sp}(2) \) and the group \( G_2 \) has rank at most 5. If \( g_2 \) and \( k_2 \) denote
the dimension and rank of \( G_2 \), resp., Lemma 2.3 yields \( \dim_{\mathbb{H}} V_2 = 4s \leq g_2 + \frac{21-k_2}{2} \leq g_2 + 10 \). Referring to [CP05] Table, p. 71], we deduce that \( \rho_2 \) must be one of

\[ (\text{Sp}(1), \mathbb{H}^2), (\text{Spin}(11), \mathbb{H}^{16}), (\text{Sp}(1), \mathbb{H}^3), (SU(6), \mathbb{C}^6), (\text{Sp}(3), \mathbb{H}^3 \mathbb{C}), (\text{Sp}(3), \mathbb{H}^3 \mathbb{C}). \]

In the first case, \( \rho \) is a representation of cohomogeneity 3, which is not highly curved.
The second representation does not satisfy \( 4s \leq g_2 + \frac{21-k_2}{2} \). In order to deal with
the third representation, note that the maximal dimension of a $\text{Sp}(2) \times \text{Sp}(1)$-orbit through a pure tensor in $\mathbb{H}^2 \otimes \mathbb{H}^2$ is $7 + 3 = 10$, so we find a regular horizontal geodesic of length $\pi/2$ which meets two orbits of dimension at most $10$. Since $10 + 10 < 22 = 23 − 1$, we obtain $\kappa = 1$ from Lemma [2.3]

To rule out the last two representations, one can use the following proposition.

**Proposition 4.2.** Assume that the action of $G_2$ on the quaternionic projective space $\mathbb{HP}^{s−1}$ has an orbit of codimension at least $8$. Then $\kappa_\rho = 1$.

**Proof.** We have $n = 8s − 1$. The dimension of the orbit through any pure tensor $v_1 \otimes v_2$ is at most $7 + t$, where $t$ is the maximal dimension of the $G_2$-orbits on $\mathbb{HP}^{s−1}$. Thus the dimension of the $G$-orbit through $v_1 \otimes v_2$ is at most $7 + (4s − 5) = 4s + 2$. Under the standing assumptions, we find a regular horizontal geodesic $\gamma$ of length $\pi/2$ which meets an orbit of dimension at most $7 + (4s − 12) = 4s − 5$ and an orbit of dimension at most $4s + 2$. Since $(4s + 2) + (4s − 5) = 8s − 3 < n − 1$, we obtain $\kappa_\rho = 1$ from Lemma [2.3] \[ \square \]

Note that the action of $G_2$ on $\mathbb{HP}^{s−1}$ has an orbit of codimension at least $8$ if and only if the lift to an irreducible representation $\tilde{\rho}_2$ of $\text{Sp}(1) \times G_2$ on $\mathbb{H}^n$ has an orbit of codimension at least $9$. The remaining two cases for $\rho_2$ yield for $\tilde{\rho}_2$ the isorpy representations of the symmetric spaces $E_6/(\text{SU}(6) \cdot \text{SU}(2))$ and $F_4/(\text{Sp}(3) \cdot \text{Sp}(1))$, whose restricted root systems have Coxeter type $F_4$. The worst case for us is the second one, in which all multiplicities are $1$. Corresponding to a subsystem of type $B_3$, we find a singular orbit of $\tilde{\rho}_2$ of codimension $4 + 9 \cdot 1 = 13 ≥ 9$, so Proposition [4.2] applies.

4.11.3. The case $V = \mathbb{C}^2 \otimes_\mathbb{C} V_2$. We have $G_1 = \text{U}(2)$, $\rho(G) \subset \text{U}(2) \otimes \text{SU}(V_2)$ and $s := \dim_\mathbb{C} V_2 ≥ 2$. We may assume that $G_2$ has no circle factor and that $\rho_2$ is irreducible and of complex type.

Similar to Proposition [4.2] one proves:

**Proposition 4.3.** Assume that the action of $G_2$ on the complex projective space $\mathbb{CP}^{s−1}$ has an orbit of codimension at least $4$. Then $\kappa_\rho = 1$.

Owing to Proposition [4.3] it remains only to discuss the case in which the action of $G_2$ on $\mathbb{CP}^{s−1}$ has all orbits of codimension at most $3$. Under this assumption, that action lifts to an irreducible representation $\tilde{\rho}_2$ of $\text{U}(1) \times G_2$ on $\mathbb{C}^s$ all of whose nonzero orbits have codimension at most $4$ and hence $\tilde{\rho}_2$ has cohomogeneity at most $3$.

If the cohomogeneity of $\tilde{\rho}_2$ is one or two, then this is a polar representation whose restriction to the non $\text{U}(1)$-factor remains irreducible. Going through the classification, we see that $\tilde{\rho}_2$ is one of the isotropy representations of the symmetric spaces:

- $\text{SU}(s + 1)/\text{U}(s)$, $\text{SU}(2 + \frac{s}{2})/\text{SU}(2) \times \text{U}(\frac{s}{2})$ $(s > 2)$,
- $\text{SO}(10)/\text{U}(5)$, $E_6/(\text{U}(1) \cdot \text{Spin}(10))$.

In the first case, $\rho$ is a polar representation so it is not highly curved. In the other cases, the restricted root system of the symmetric space has Coxeter type $B_3$ with multiplicities $(2, s − 3)$, $(4, 5)$ and $(9, 6)$, so we find an orbit of $\tilde{\rho}_2$ of codimension $2 + s − 3 = s − 1 > 4$, $2 + 5 = 7 > 4$, $2 + 9 = 11 > 4$. This remark rules out all cases.
If the cohomogeneity of \( \hat{\rho}_2 \) is 3, recall that \( \rho_2 \) is irreducible of complex type and \( \text{rk} G_2 \leq 5 \), so from the classification [HL71, Str94] we get that \( \hat{\rho}_2 \) is one of the isotropy representations of the symmetric spaces: \( \text{Sp}(3)/U(3) \), \( \text{SO}(12)/U(6) \), \( \text{SU}(6)/S(U(3) \times U(3)) \) or \( \text{SU}(7)/S(U(3) \times U(4)) \). All symmetric spaces have Coxeter type \( B_3 \) and the worst case for us is \( \text{Sp}(3)/U(3) \) in which all multiplicities are 1. In this case, corresponding to a subsystem of type \( B_2 \), we find a singular orbit of \( \hat{\rho}_2 \) of codimension \( 3 + 4 \cdot 1 = 7 > 4 \), which cannot be. This finishes the proof of the theorem.

5. Appendix

\[
a_1[t] := 1 + \frac{27}{(5 + 4\cos[2t])^2}
\]

\[
a_2[t] := 1 + \frac{27}{(-5 + 2\cos[2t] + 2\sqrt{3}\sin[2t])^2}
\]

\[
a_3[t] := 1 + \frac{27}{(5 - 2\cos[2t] + 2\sqrt{3}\sin[2t])^2}
\]

\[
c_1[t] := 1 + \frac{27}{(5 + 2\cos[2t] + 2\sqrt{3}\sin[2t])^2}
\]

\[
c_2[t] := 1 + \frac{27}{(5 - 2\cos[2t] + 2\sqrt{3}\sin[2t])^2}
\]

\[
c_3[t] := 1 - \frac{27}{(5 + 2\cos[2t] + 2\sqrt{3}\sin[2t])^2}
\]

\[
b_1[t] := -\frac{648(2 - 10\cos[2t] + 2\cos[4t] - 5\cos[6t] + 2\cos[8t])\sin[2t]}{(5 + 4\cos[2t])^2(21 - 20\cos[2t] + 8\cos[4t])^2 \sqrt{\sin[6t]^2 / (55 + 16\cos[6t])}}
\]

\[
b_2[t] := -\frac{648(1 + 2\cos[4t])\sin[2t]}{(5 + 4\cos[2t])^2(21 - 20\cos[2t] + 8\cos[4t])^2 \sqrt{\sin[6t]^2 / (55 + 16\cos[6t])}}
\]

\[
b_3[t] := (324(1 + 2\cos[4t])\sin[2t] (5\cos[2t] - 2\cos[4t] + \sqrt{3}(5\sin[2t] + 2\sin[4t]))) / (5 + 4\cos[2t])^2(21 - 20\cos[2t] + 8\cos[4t])^2 \sqrt{\sin[6t]^2 / (55 + 16\cos[6t])}
\]
\[
\begin{align*}
\mu[t] &:= -27 \sqrt{\frac{(1 - 4\cos[2t])^2}{64(2 - 10\cos[2t] + 2\cos[4t] - 5\cos[6t] + 2\cos[8t])\sin[2t]}} \\
E[t] &:= \frac{(-1 + 2\cos[2t])(1 + 2\cos[2t])^2}{21 - 20\cos[2t] + 8\cos[4t]} \\
F[t] &:= \frac{(1 + 2\cos[2t])\sqrt{\frac{(5 + 4\cos[2t])(1 + 2\cos[4t])^2\cos[2t]^2\sin[2t]}{21 - 20\cos[2t] + 8\cos[4t]}}}{5 + 4\cos[2t]}
\end{align*}
\]

Figure 2. Graphs of...

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