GENERAL RELATIVISTIC ELECTROMAGNETISM AND PARTICLE ACCELERATION IN A PULSAR POLAR CAP

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ABSTRACT

We reconstruct a 3 + 1 formalism of general relativistic electromagnetism and derive the equations of motion of charged particles in a pulsar magnetosphere, taking account of the inclination between the rotation axis and the magnetic axis. Unlike previous works in which space charge is evaluated by assuming the flow velocity to be light speed, we analyze particle motion in the polar cap, finding that gravity significantly changes its dynamics and the condition for acceleration.

Subject headings: acceleration of particles — gravitation — pulsars: general — stars: magnetic fields —

1. INTRODUCTION

The origin of radio emission from pulsars remains a great mystery. One of the likely scenarios is that particles are accelerated along open magnetic field lines and emit γ-rays that subsequently convert into electron-positron pairs under a strong magnetic field. The combination of the primary beam and pair plasma provides the radio emission mechanism (Melrose 2000). The process of particle acceleration has been intensively studied in flat spacetime (Fawley, Arons, & Scharlemann 1977; Arons & Scharlemann 1979; Mestel 1981; Shibata 1991, 1995, 1997). The field-aligned electric field is driven by deviation of the space charge from the Goldreich-Julian charge density (Goldreich & Julian 1969), which is determined by the magnetic field geometry. Therefore, general relativistic (GR) effects on the field geometry are crucial for the formation of the field-aligned electric field and particle acceleration.

Muslimov & Tsygan (1992) initiated the GR analysis of electromagnetic fields around a pulsar; they solved the Maxwell equations with the assumption that the particles move with light speed. Shibata showed, however, that their assumption of constancy of particle speed was not always true (Shibata 1991, 1997): the particle motion and space charge show oscillatory behavior, even on field lines curving toward the rotation axis if the assumed current density is subcritical. Mestel (1996) extended Shibata’s equations of motion to include GR effects, although his analysis is not complete. Recently, Harding & Muslimov (2001) extended the Muslimov-Tsygan analysis to include pair-creation dynamics.

In this paper we derive the basic equations more rigorously and generally: we depart from the Muslimov-Tsygan (and Harding-Muslimov) model by solving the equations of motion for the particles and correct Mestel’s equation and extend it to include the inclination angle between the magnetic axis and the rotation angle. Next we solve the electric field together with the particle motion for the region near the magnetic pole just above the surface. It turns out that the effect of gravity on particle dynamics cannot be ignored, and hence the condition for acceleration is also modified. As pointed out by Arons (1997), a local deficit of space charge takes place on most of the field lines. In particular, particle dynamics on the field lines curving away from the rotation axis (“away field lines”) are changed qualitatively by gravity if the current density is less than the critical value. It is also true that an oscillatory solution exists, even with the GR effect. We find that some oscillatory solutions disappear above a certain height.

This paper is organized as follows: In § 2 we reconstruct a 3 + 1 formalism of GR electromagnetism. In § 3 we derive the equations of motion of charged particles around a magnetized rotating star. In § 4, solving the equations of motion, we analyze particle acceleration in the polar cap. Conclusions and a discussion are given in § 5. We use units where c = 1.

2. 3 + 1 ELECTROMAGNETIC EQUATIONS: GENERAL FORMALISM

A 3 + 1 dimensional decomposition of Maxwell equations was made by Landau & Lifshitz (1975) and Thorne & MacDonald (1982). Landau & Lifshitz derived the 3 + 1 Maxwell equations in a straightforward manner, which we essentially follow. They introduced a metric,1

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -h(dt - g_{\nu\lambda} dx^\nu) dx^\lambda + \gamma_{ij} dx^i dx^j, \]

\[ h \equiv -g_{00}, \quad g_{\mu\nu} \equiv \frac{g_{\mu\nu}}{h}, \quad \gamma^{ij} \equiv g^{ij}, \]

which is formally in a 3 + 1 form. However, it is difficult to understand how the entire four-dimensional spacetime \( (x^\mu, g_{\mu\nu}) \) is decomposed into three-dimensional spatial hypersurfaces \( (x^i, g_{ij}) \), because each hypersurface does not correspond to \( t = \text{const} \).

On the other hand, Thorne & MacDonald (1982) adopted the metric used in the Arnowitt-Deser-Misner (ADM)
formalism:
\[
\begin{align*}
\alpha^2 &= -\frac{1}{g_{00}}, \quad \beta_i = g_{0i}, \quad q_{ij} = g_{ij} , \\
\end{align*}
\]
where \(\alpha\) and \(\beta\) are called the lapse function and the shift vector, respectively. With this metric, one can understand the concept of the 3 + 1 decomposition easily, as we describe below. However, they adopted the “congruence” approach and wrote down the equation with coordinate-free language, e.g., the expansion rate of the bundle of the fiducial observers. Although their equations are powerful for some theoretical arguments, one cannot avoid using coordinates to solve them in most cases.

For the purpose of our study, we combine and modify the two methods: we adopt the ADM formalism to decompose spacetime and electromagnetic fields into a 3 + 1 form and write down 3 + 1 electromagnetic equations in a straightforward manner.

First, with the metric in equation (3), we introduce fiducial observers with the 4-velocity
\[
u_\mu = (-\alpha, 0, 0, 0)
\]
and the spacelike hypersurface \(\Sigma\) that is orthogonal to \(u^\mu\) at each time \(t\). Because \(\Sigma(t)\) is characterized by a \(t = \text{const}\) hypersurface, \(x^i\) and \(q_{ij}\) are the 3-coordinates and the 3-metric of \(\Sigma(t)\), respectively. We define the projection tensor, which projects any 4-vector or tensor into \(\Sigma(t)\), as
\[
l_{\mu\nu} = g_{\mu\nu} + u^\mu u^\nu .
\]
Electromagnetic quantities are originally in the 3 + 1 form, i.e., 3-vectors and scalars: the electric field \(E\), magnetic field \(B\), charge density \(\rho\), current density \(\mathbf{J}\), scalar potential \(\phi\), and vector potential \(A^i\). We define those quantities on \(\Sigma(t)\), i.e., those quantities are supposed to be measured by observers with velocity \(u^\mu\). Note that the “ordinary” components of a vector \(\mathbf{P}\) or the contravariant components \(P^\mu\) (Weinberg 1972). If the 3-coordinates \(x^i\) are orthogonal, i.e., \(q_{ij} = \delta_{ij}\), the ordinary components \(P_{(i)}\) are
\[
P_{(i)} = h_{(i)} P^i = \frac{P_i}{h_{(i)}} .
\]
As long as we adopt the 4-velocity from equation (5), which specifies \(\Sigma\), we can convert any 3-vector \(\mathbf{P}\) on \(\Sigma\) into a 4-vector \(P^\mu\) in a natural way:
\[
P^\mu = (0, P^i) .
\]
It is easy to confirm that \(P^\mu\) is the vector on \(\Sigma\): \(P^\mu u_\mu = 0\) and \(P^\mu h_{\mu}^\nu = P^\nu\).

Let us reconstruct the electromagnetic field \(\mathbf{\nabla}\mathbf{\cdot}\mathbf{\nabla}\), the charge-current vector \(\mathbf{\nabla}\mathbf{\cdot}\mathbf{\nabla}\), and the 4-potential \(\mathbf{\nabla}\mathbf{\cdot}\mathbf{\nabla}\) from the quantities defined on \(\Sigma\) and \(u^\mu\) (Thorne & MacDonald 1982):
\[
\begin{align*}
\mathbf{\nabla}\mathbf{\cdot}\mathbf{\nabla} &= u^\mu E^\mu - u^\mu E^\mu + \epsilon^{\mu\nu\lambda\kappa} u_\lambda B_\kappa , \\
\mathbf{\nabla}\mathbf{\cdot}\mathbf{\nabla} &= \rho - \frac{1}{\epsilon_0} \phi u^\mu \partial_\mu \phi + u^\mu A^\mu ,
\end{align*}
\]
where \(\epsilon^{\mu\nu\lambda\kappa}\) is the Levi-Civita tensor with \(\epsilon^{0123} = 1/(-g)^{1/2}\).

We can invert these relations as
\[
\frac{d^2 r}{d\tau^2} = \mathbf{\nabla}\mathbf{\cdot}\mathbf{\nabla} - \frac{\gamma^2}{\alpha} (-\partial_0 \alpha + \frac{\alpha}{2} V^i V^j \partial_0 q_{ij} + V \partial_0 \beta) ,
\]
\[
\frac{d^2 r}{d\tau^2} = \gamma \left( -\partial_0 \alpha + \frac{\alpha}{2} V^i V^j \partial_0 q_{ij} + V \partial_0 \beta \right) ,
\]
where \(\gamma = \frac{1}{(1 - V^2)^{1/2}}\), which implies that the scalar \(\gamma\) is nothing but the Lorentz factor. With these quantities and the 3 + 1 electromagnetic quantities, we rewrite the equation of motion (21) as
\[
\frac{d}{d\tau} [\gamma (-\alpha + \beta \cdot V)] = -\frac{\gamma^2}{\alpha} \left( -\partial_0 \alpha + \frac{\alpha}{2} V^i V^j \partial_0 q_{ij} + V \partial_0 \beta \right) ,
\]
\[
\frac{d}{d\tau} (\gamma V_0) = -\frac{\gamma^2}{\alpha} \left( -\partial_0 \alpha + \frac{\alpha}{2} V^i V^j \partial_0 q_{ij} + V \partial_0 \beta \right) ,
\]
(24)
We assume that the spacetime outside a neutron star is stationary and axisymmetric. Because the gravity of charged particles is negligible, the outer spacetime is described by a Kerr metric. We adopt the spherical coordinates \((t, r, \theta, \phi)\) so that \(\theta = 0\) is aligned with the rotation axis. In the slow rotation, for which we take up to the first order of the angular momentum of the star \(L\), the spacetime is described by the metric

\[
\alpha^2 = F(r) \equiv 1 - \frac{\tilde{\omega}^2}{r^2}, \quad \beta^2 = -\omega(\tilde{\omega}) \equiv -\frac{2L}{r^3},
\]

\[
q_{\theta} = \text{diag}(F^{-1}(r), \tilde{r}^2, \tilde{r}^2 \sin^2 \tilde{\theta}),
\]

where \(\tilde{r}\) is the gravitational radius.

We consider a two-step coordinate transformation. First, we introduce the “corotating coordinates” as

\[
\tilde{\phi} \rightarrow \phi - \Omega t,
\]

where \(\Omega\) is the angular velocity of the star (Landau & Lifshitz 1975). The shift vector is, accordingly, transformed as

\[
\tilde{\beta} \rightarrow \beta = \Omega - \omega.
\]

Second, we move on to the “magnetic coordinates” \((t, r, \theta, \varphi)\), where the magnetic dipole axis is aligned with \(\theta = 0\).

The shift vector in equation (25) is, accordingly, transformed into

\[
\beta_0 = 0, \quad \beta_\theta = -\sin \chi \sin \varphi(\Omega - \omega), \quad \beta^r = (\cos \chi \cot \theta \cos \varphi \sin \chi)(\Omega - \omega),
\]

while the lapse function and the spatial metric remain unchanged:

\[
\alpha^2(r) = F'(r), \quad q_{\theta} = \text{diag}(F^{-1}(r), \tilde{r}^2, \tilde{r}^2 \sin^2 \tilde{\theta}).
\]

In this (and any) axisymmetric spacetime, observers with the velocity from equation (5) are called “zero-angular-momentum observers” because of their vanishing angular momentum per unit mass: \(u_{\phi}(\partial / \partial \phi)^{\phi} = 0\) (Thorne & MacDonald 1982).

Now we analyze the basic equations (15)–(19), (23), and (24) with the metric in equations (29) and (30). First, we suppose that the magnetic field distortion due to the external currents is negligibly small. Then the magnetic field is governed by equations (15) and (18) with \(J = 0\):

\[
\text{div} \mathbf{B} = 0, \quad \text{rot}(\alpha \mathbf{B}) = 0.
\]

We adopt the dipole-like solution of equation (31) (Ginzburg & Ozenoi 1965):

\[
B_{(r)} = \frac{B_0 \sqrt{F}}{f(r)} = B_0 \frac{f(r)}{f(r_*)} \left( \frac{r_*}{r} \right)^3 \cos \theta, \\
B_{(\theta)} = B_0 \frac{\sqrt{F}}{2} \left[ -2f(r) + 3F^{-1}(r) \right] \left( \frac{r_*}{r} \right)^3 \cos \theta, \\
B_{(\varphi)} = B_0 \left( 1 + \frac{r_0}{r} - 3 \frac{r_0}{4r_*} \right) \left( \frac{r_*}{r} \right)^3 \sin \theta, \\
f(r) = -3 \left( \frac{r_*}{r} \right)^3 \ln F(r) + \frac{r_*}{r} \left[ 1 + \frac{r_*}{2r} \right] \approx 1 + \frac{3r_0}{4r},
\]

where \(r = r_*\) denotes the star surface and the approximate expressions are obtained by expanding in powers of \(r_0/r\) or \(r_0/r_*\) (Konno & Kojima 2000).

Equation (16) is satisfied if there exists a potential \(\Phi\) such that

\[
\alpha \mathbf{E} + \beta \times \mathbf{B} = -\nabla \Phi.
\]

Substituting equation (35) into another of Maxwell’s equations (eq. [17]), we obtain

\[
-\text{div} \left( \frac{\nabla \Phi}{\alpha} \right) = 4\pi (\rho - \rho_{\text{GJ}}), \quad 4\pi \rho_{\text{GJ}} = -\text{div} \left( \frac{\beta}{\alpha} \times \mathbf{B} \right),
\]

where \(\rho_{\text{GJ}}\) is the Goldreich-Julian density. Each term is written explicitly as

\[
\text{div} \left( \frac{\nabla \Phi}{\alpha} \right) = \frac{\sqrt{F}}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{\sqrt{F}} \frac{\partial}{\partial r} \left[ \frac{\partial}{\sin \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \theta} \right) + \frac{\partial^2 \Phi}{\partial \varphi^2} \right],
\]

\[
2\pi \rho_{\text{GJ}} = -\alpha \mathbf{B} \cdot \text{rot} \left( \frac{\beta}{\alpha} \right)
\]

\[
= -\frac{B_{(r)}}{\sqrt{F}} \frac{\partial}{\partial r} \left( \sin^2 \theta \frac{\partial \beta^r}{\partial \varphi} + \frac{\partial \beta^\theta}{\partial \varphi} \right)
\]

\[
+ B_{(\varphi)} \left( \sin \theta \frac{\partial}{\partial \theta} + \frac{\partial^2 \beta^r}{\partial r^2} \right).
\]

Let us consider the flow of charged particles. In the case under consideration, the magnetic field dominates over the electric field, where particles are at the lowest Landau level and the inertial drift motion across the magnetic field is negligible. We can therefore assume that particles go along the magnetic field lines:

\[
\mathbf{v} = \kappa \mathbf{B}, \quad \mathbf{J} = \rho \mathbf{V} = \rho_{\text{GJ}} \mathbf{B},
\]

where \(\kappa = V/|\mathbf{B}|\) is a scalar function.

Then the continuity equation (19) implies (Muslimov & Tsygan 1992; Melset 1996)

\[
0 = \text{div}(\alpha \mathbf{J}) = \text{div}(\alpha \rho_{\text{GJ}} \mathbf{B}) = \mathbf{B} \cdot \nabla (\alpha \rho_{\text{GJ}}),
\]

which reads

\[
\alpha \rho_{\text{GJ}} = \frac{\sqrt{F} J}{B} = \text{const on magnetic field lines}.
\]
Equations (23) and (24) give the motion of charged particles. If we assume $B_0 = E_0 = 0$, equations (35) and (39) give $V_0 = \partial_x \Phi = 0$. Using the relation
\[
E \cdot V = -\frac{\mathbf{V}\Phi}{\alpha} \cdot V = -\frac{1}{\alpha} V^i \partial_i \Phi,
\]
we rewrite one of the equations of motion (eq. [23]) as
\[
\frac{d}{dr}(\sqrt{F\gamma}) = \frac{e}{m} \frac{1}{F} \frac{d\Phi}{dr}.
\] (42)
we rewrite one of the equations of motion (eq. [23]) as
\[
\frac{d}{dr}(\sqrt{F\gamma}) = \frac{e}{m} \frac{1}{F} \frac{d\Phi}{dr}.
\] (43)
The other equation (eq. [24]) need not be solved because the spatial trajectory is determined by equation (39).

Finally, let us consider boundary conditions. We assume that the star crust is a perfect conductor, and hence particles on the surface do not suffer a Lorentz force. Equations (24), (35), (39), and (42) reduce this ideal-MHD condition to $\mathbf{V}\Phi = 0$. We also assume that there exist closed magnetic lines, where the ideal-MHD condition holds. Thus, open field lines in the polar region have another boundary, defined by the "last open field lines," $\theta = \theta_0(r)$. Therefore, our boundary conditions are $\Phi = \mathbf{V}\Phi = 0$ on $r = r_s$ and on $\theta = \theta_0(r)$.

4. PARTICLE ACCELERATION IN THE POLAR CAP

We are interested in the "polar cap" region, $\theta < \theta_0(r)$, where particle acceleration may occur. For simplicity, we restrict ourselves to the region near the magnetic pole just above the surface, where the following approximations hold: (1) $\theta_0(r) \approx \text{const}$, (2) $\theta < 1$ (we take its first order), and (3) $d/dr$ along the particle trajectory $\approx \partial/\partial r$. Therefore, we can apply our equations below only to the region elongated less than the polar cap radius: $r - r_s < r_0$.

As for the nonradial direction, we expand $\Phi$ as
\[
\Phi = \sum_{l,m} \bar{\Phi}(r) Y_{lm}(\theta, \varphi).
\] (44)
We only take the mode of the polar cap scale, $l \approx \pi/\theta_0$.

The above assumptions simplify equations (36) to
\[
-\frac{\sqrt{F}}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) + \frac{l(l+1)}{Fr^2} \Phi = 4\pi \left( \frac{J}{V} - \rho_{\text{GR}} \right),
\] (45)
\[
4\pi \rho_{\text{GR}} = \frac{2B_0\Omega}{\sqrt{F}} \left[ \left( 1 - \frac{\omega}{\Omega} \right) \cos \chi + \theta Q \cos \varphi \sin \chi \right],
\] (46)
\[
Q = \frac{r_0 g}{2r} - \frac{3\omega}{2\Omega} + \frac{r g}{4r^2} + \frac{3}{2\Omega} \left( 1 - \frac{3r g}{2r} + \omega \right)
\]
\[
\approx \frac{3}{2} - \frac{11r g}{2r} + \frac{3}{4\Omega},
\] (47)
where $B \equiv \frac{B_0}{\Omega} = \Omega/2 + O(\theta^2)$. Equation (43) describes the time evolution of the particles; however, if the particles make a stationary flow, we can read their spatial configuration by replacing the time $t$ with the spatial variable $r$:
\[
\frac{d}{dr}(\sqrt{F\gamma}) = \frac{e}{m} \frac{1}{F} \frac{d\Phi}{dr}.
\] (48)
Using normalized variables (Shibata 1997),
\[
j = -\frac{2\sqrt{F}(s)\Omega(s)}{\Omega B(s)} = \text{const}, \quad \phi(s) = \frac{e}{m} \Phi(s),
\]
\[
s = \sqrt{\frac{2\Omega B_0 e}{mc^2}},
\] (49)
\[
j(s) = -\frac{2\sqrt{F}(s)\rho_{\text{GR}}(s)}{\Omega B(s)}
\]
\[
= \left[ 1 - \frac{\omega(s)}{\Omega} \right] \cos \chi + \theta(s)Q(s) \cos \varphi \sin \chi,
\] (50)
we rewrite equations (45) and (48) as
\[
\frac{\sqrt{F}}{s^2} \frac{d}{ds} \left( s^2 \frac{d\phi}{ds} \right) = \frac{l(l+1)}{Fr^2} \phi = \frac{B}{B_0 \sqrt{F}} \left( \frac{j}{V} - \bar{j} \right),
\] (51)
\[
\frac{d}{ds} \left( \sqrt{F}\gamma \right) = \frac{1}{d\phi} \left( \frac{1}{F} \frac{d\phi}{ds} \right)
\] (52)
The particle trajectory is given by integrating $\frac{d\theta}{dr} = \frac{B}{B_0}$, resulting in
\[
\theta(s) \approx \theta_s \frac{s}{s_k} \quad \text{for } s < 1,
\] (53)
where an asterisk denotes a value at the star surface $r = r_s$ for any variable. In the following analysis we fix some of the parameters: $r_s = 10$ km, $r_g = 2$ km, $\Omega = 2\pi/(0.1)$ s, $\omega_0/\Omega = 0.1$, $\theta_0 = (\Omega r_g/c)^{1/2} \approx 0.046$ (where $\pi/\theta_0 \approx 69$), $\chi = 30^\circ$, and $\gamma_s = 1.0001$.

We can argue for equation (51) on the analogy of particle motion by regarding $\phi$ as the position and $s$ as the time. At a given point (or time) $j > j(s)$, the "force term" (right-hand side) of equation (51) is always positive; $\phi$ and $\gamma$ grow increasingly. If $j < j(s)$, on the other hand, the force term can be either positive or negative, depending on $V$, and hence oscillating behavior is expected. If we set $r_g = \omega = 0$ (no gravity) and $\chi = 0$ (no inclination), we reproduce Shibata's result, $\bar{j} = 1$ (Shibata 1997). The expression of $\bar{j}$ in equation (51) indicates that the effects of both gravity and inclination reduce $\bar{j}$, or equivalently, the critical value of $j$.

Numerical solutions of equations (51) and (52) in Figures 1–3 verify the above arguments. It is apparent that gravity reduces the critical value of $j$: a monotonic increase of $\gamma$ appears with lower values of $j$ than in the non-GR case.

Let us consider the dynamics for $j \approx j_s$ carefully. To see the behavior analytically, we expand $\bar{j}$ with $\xi \equiv s/s_k - 1$ up to the first order, resulting in
\[
\bar{j}(\xi) = \bar{j}_0(1 + C\xi), \quad C \approx 3 \left( \frac{\omega_0}{\Omega} + \frac{\theta_0}{4} \cos \varphi \tan \chi \right).
\] (54)
This implies that $\bar{j}$ increases or decreases according to the sign of $C$. For the particles on the magnetic axis ($\theta = 0$), $\bar{j}$ is an increasing function because of the term $\omega_0/\Omega$ in the GR case, while it is constant in the non-GR case. This gravitational effect is illustrated in Figure 1: contrary to the non-GR case (top), in the GR case (bottom) the oscillation for $j < j_s$ does not continue to infinity, but the particle velocity

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2 Eq. (47) is slightly different from the expression from Muslimov & Tsigan (1992) and Harding & Muslimov (2001): $Q = 3r g/2r - 3\omega/2\Omega + \lfloor 3/2 \right\rfloor (1 - 3r g/2r + \omega/2\Omega)$. We believe that there is an error in their calculation. Nevertheless, this difference does not change the results practically; because we have assumed $\theta \leq 1$ in eq. (46), all terms but $3/2$ (the dominant one) in $Q$ (eq. [47]) are negligible.
becomes $V = 0$ at some point. Although the dynamics are independent of $\gamma_*$ in the non-GR case, the ending point $V = 0$ ($\gamma = 1$) depends on $\gamma_*$ in the GR case: it becomes farther for larger $\gamma_*$.

The dynamics of particles off the magnetic axis ($\theta \neq 0$) are more interesting. In the absence of gravity ($\omega_\phi = 0$), the sign of $C$ depends simply on whether particles move toward ($\cos \varphi > 0$) or away from ($\cos \varphi < 0$) the rotation axis, as claimed by Shibata (1997). He showed, for example, that particles are accelerated after oscillation along away field lines if $j$ is slightly less than 1. This behavior is reproduced in Figure 2 (top). Gravity changes this behavior drastically. If we take $\omega_\phi/\Omega = 0.1$, typically, $C$ becomes positive, even on away field lines, unless $\theta_* > 0.4|\sec \varphi \cot \chi > 0.4 \cot \chi$. Unless $\chi$ is large enough, in most of the small-$\theta$ region, $j$ simply increases, and acceleration after oscillation cannot occur. This argument is confirmed by the numerical result in Figure 2 (bottom).

For the particles moving toward the rotation axis, $\cos \varphi$ is positive, and therefore $j$ always increases along the field lines. The results in Figure 3 are also understandable. Among the solutions in Figure 3 (bottom), the solutions with $j = 1.003j_*$ could be realistic because the particle energy becomes so large that electron-positron pairs are created with the finite electric field.

5. CONCLUSIONS AND DISCUSSION

We have reconstructed a 3 + 1 formalism of general relativistic electromagnetism and have derived the equations of motion of charged particles in a pulsar magnetosphere. Our basic equations are the correct and generalized version of those of Muslimov & Tsygan (1992) and of Mestel (1996) in the sense that the equations include arbitrary current...
density, arbitrary velocity that satisfies the equation of motion, and arbitrary inclination angle between the rotation axis and the magnetic axis.

We have solved our equations of motion together with the electric field structure along the magnetic field for the region near the magnetic pole just above the surface in our approximate method. We have found that the effect of gravity on particle dynamics cannot be ignored, and hence the condition for acceleration is also modified. In particular, particle dynamics on the away field lines are changed qualitatively by gravity if the current density is subcritical.

Monotonic increase of $\gamma$ with the boundary condition $E \cdot B = 0$ at the stellar surface and at infinity is found with the GR effect, and the achieved Lorentz factor is significantly larger than in the non-GR case (Muslimov & Tsygan 1992). With subcritical current densities, we find oscillatory solutions even with the GR effect. Since we take the current densities as a free parameter, the local accelerator model can be linked with the global model by adjusting the current density.

We have also found that some oscillatory solutions disappear above a certain height. This may indicate an intermittent flow of charged particles rather than a steady outflow if the imposed current density is smaller than a threshold. This may cause pulse nulling.

As pointed out by Arons (1997), an important GR effect is the local increase of $\gamma$, which appears on most of the field lines, regardless of whether the large-scale field curvature of the magnetic field lines is away or toward. He suggested that this fact can be responsible for the axisymmetric radio emission about the magnetic axis. If the current density is taken as a free parameter, the supercritical current density in the polar annuli can also be responsible for the distribution of the radio-emitting region. We shall extend our analysis to a more global region to construct a pulsar model in a subsequent work.

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