Lattice Structure For Random Matching Markets

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Abstract

In this paper we study the lattice structure of the random matching set, that is, all lotteries of stable matchings. We define the l.u.b. (least upper bound) and the g.l.b. (greatest lower bound) for both sides of the matching market, and we prove the that with these binary operations the set of random matchings has two dual lattices.

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1 Introduction

Matchings have been studied for several decades now, beginning with Gale and Shapley’s pioneering work (Gale and Shapley, 1962). They introduce the notion of stability and provide an algorithm for finding stable matchings. Since then, a considerable amount of work was carried out on both the theory and applications of stable matchings.

A matching is stable if all agents have acceptable partners and there are no pair of agents, one of each side of the market, for instance, a firm $f$ and worker $w$ that are not currently matched, but where $w$ would like to work for $f$, and $f$ would like to hire $w$. Unfortunately, the set of stable matchings may be empty. Substitutability is the weakest condition that has so far been imposed on agents’ preferences under which the existence of stable matchings is guaranteed. An agent has substitutable preferences if it wants to continue being a partner with agents from the other side of the market even if other agents become unavailable (see Kelso Jr and Crawford, 1982; Roth, 1984, for more detail).

One of the most important results in the matching literature is that the set of stable matchings has a lattice structure. This fact is important for several reasons, it shows that even if agents of the same side of the market compete for agents of the other side,
this conflict is attenuated since, on the set of stable matchings, agents on the same side of the market have a coincidence of interests. Another reason is that many algorithms are based on this lattice structure of the set of stable matchings, for example, algorithms that yield stable matching in centralized markets. The lattice structure of the set of stable matchings is introduced by Knuth (1997) for the marriage market. Given two stable matchings he defines the l.u.b. for men, by matching to each man with the best of the two women matched, and the g.l.b. for men, by matching to each man the less preferred between the two women matched; these are usually called the “pointing functions” relative to a partial order. Roth (1985) shows that these binary operations (l.u.b. and g.l.b.) used in Knuth does not work in the more general many-to-many and many-to-one matching markets introduced respectively by Kelso Jr and Crawford (1982) and Roth (1984) even though substitutable preferences. For a specific many-to-one matching market, so-called the college admission problem, Roth and Sotomayor (1990) present a natural extension of Knuth’s result for q-responsiveness preferences. Martínez et al. (2001) further extended the results proved in Roth and Sotomayor (1990). They identified a weaker condition than responsiveness, called q-separability, and proposed two natural binary operations that give a lattice structure to the set of stable matchings in a model of many-to-one matching with substitutable and q-separable preferences. Such binary operations were similar in spirit to Knuth’s ones. Pepa Risma (2015) generalizes the result of Martínez et al. (2001) by showing that their binary operations work well in a many to one matching market where the preferences of the agents satisfy substitutability and a condition called law of aggregate demand, which is less restrictive than q-separable. Her paper is contextualized in many-to-one matching markets with contracts. Manasero (2019) extend the result in Pepa Risma (2015) to a many-to-many marching market, where one side has a substitutable preference and satisfying law of aggregate demand, and the other side satisfies q-responsiveness. Alkan (2002) consider a market with multiple partners on both sides. Preferences were given by rather general path-independent choice functions that do not necessarily respect any ordering on individuals and satisfies a property called cardinal monotonicity. He shows that the set of stable matchings in any two-sided market with path-independent cardinal monotone choice functions has a lattice structure under the common preferences of all agents on any side of the market. Li (2014) proved that the set of stable matchings in a many-to-many matching market under substitutable and cardinal monotone preferences has a lattice structure. One distinction between Li (2014) and Alkan (2002) lies in the conditions on preferences: Li (2014) works with the complete preference ordering, whereas Alkan (2002) works with the incomplete revealed preference ordering of agents. All of these previously mentioned papers have a common point, they all have a natural definition of the binary operations via pointing functions.

In an another direction, there is an extensive literature that proves that the set of sta-
ble matching has a lattice structure (see Adachi, 2000; Fleiner, 2003; Echenique and Oviedo, 2004; Hatfield and Milgrom, 2005; Ostrovsky, 2008; Wu and Roth, 2018, among others). All of this papers have an unnatural and difficult way to compute the binary operations, using fixed points methods.

In this paper, we study lotteries of stable matchings, here called random matchings. Random matchings are very useful to consider for at least two reasons. Firstly, the randomization allows for a much richer space of possible outcomes and may be essential to achieve fairness property such as anonymity and (ex-ante) equal-treatment-of-equals. Secondly, the framework of random matchings also helps to reason about fractional matchings that capture time-sharing arrangements, (see Roth et al., 1993; Teo and Sethuraman, 1998; Bäiou and Balinski, 2000; Doğan and Yıldız, 2016; Neme and Oviedo, 2019a,b, among others). Roth et al. (1993) define a stable fractional matching to be a not necessarily integer solution of a linear program. They characterize the stable matchings of the marriage market as the extreme points of the convex polytope generated by the linear constraints. Then, in the marriage market, stable fractional matchings and random matchings are equivalent. However, in more general markets, the convex polytope may have fractional extreme points. This means that this equivalence does not holds, (see Bäiou and Balinski, 2000; Neme and Oviedo, 2019b, among others).

Roth et al. (1993), define two binary operations for random matchings of the marriage market. They prove that a set of stable fractional matchings, doted with a partial order (first-order stochastic dominance) has a lattice structure. Neme and Oviedo (2019a) characterize the set of strongly stable fractional matchings of the marriage market. Since the set of strongly stable fractional matching is a subset of the stable fractional matchings, they prove that the set of strongly stable fractional matchings doted by the same partial order has a lattice structure (is a Sub-lattice of the set of stable fractional matchings). The binary operation defined in Roth et al. (1993) can not be extended to a more general market, for instance, the college admission problem with q-responsiveness preferences. As far as we know, these two papers (Roth et al., 1993; Neme and Oviedo, 2019a) are the only ones that study the lattice structure of random matchings.

Our contribution is that the set of random matchings for a very general framework has a lattice structure. That is, as long as the set of (deterministic) stable matchings has a lattice structure, where the binary operations are defined via a pointing function, the set of random matching has a lattice structure. First we show that any random matching can be represented by a unique lottery over stable matchings fulfilling a decreasing property. Then, given two random matchings, we present an algorithm that allows us to represent both lotteries with the same number of terms in the sum. Moreover, the new way to represent each random matching has a very important property: both lotteries have the same scalars term to term. Then, we define a partial order for random
matchings that relies on both this new way to represent each random matching and the partial order of the stable matchings. We prove that this partial order, when the market coincides with the marriage market, is equivalence to the first-order stochastic dominance defined in Roth et al. (1993). We also define the binary operations and prove that the set of random matchings has a dual lattice structure. When the scalar of the lottery are all rational numbers, we show that there is a very easy way to compute the l.u.b. and the g.l.b. with no need of the algorithm that changes the representation of the random matchings.

The paper is organized as follows: In Section 2, we introduce the matching market and preliminary results. In Section 3, first, we prove that there is a unique way to represent a random matching with a decreasing property. Then, we also present an algorithm that changes the representation of two random matchings so that they can be compared and binary operations can be computed. In Section 4, is presented the main result of the paper: The random matching set has a dual lattice structure. Section 5 has two subsections, in the first one, we prove the equivalence between our partial order and the first-order stochastic dominance contextualized in the special case of the marriage market. In the second subsection, we show how to compute the l.u.b. and the g.l.b. for rational random matchings (i.e. random matchings where all scalars of the lottery are rational numbers). Section 6 contains conclusions. Finally, Section 7 is an Appendix with technical results.

2 Preliminaries

This paper is contextualized in a many-to-many matching marker. There are two disjoint set of agents, the set of firms $F$ and the set workers $W$. Each firm has antisymmetric, transitive and complete preference relation over the set of all subsets of $W$ that induce a partial order over the set of stable matchings denoted by $\geq_F$. In the same way, each worker has an antisymmetric, transitive and complete preference relation over the set of all subsets of $F$ that induce a partial order over the set of stable matchings denoted by $\geq_W$. We will denote by $P$ the preference profile for all agents, firms and workers. The matching market will be denoted by $(F, W, P)$. Each preference list of the preference profile fulfill different properties, the firms fulfill substitutability and law of aggregated demand, and the workers fulfill q-responsiveness.\footnote{There are several papers that the reader could check for the basic definitions of the matching market considered in this paper, (see Kelso Jr and Crawford, 1982; Blair, 1988; Roth, 1984; Martínez et al., 2001; Pepa Risma, 2015; Roth and Sotomayor, 1990; Li, 2014). In Manasero (2019) the reader can find all definition since the matching market in that paper is the same matching market that is considered in this paper.}

The set of stable matchings of the matching market $(F, W, P)$ has a lattice structure.
Manasero (2019) present the binary operations using the pointing functions. Given two stable matchings $\mu_1$ and $\mu_2$, the l.u.b. for firms is denoted by $\mu_1 \lor_F \mu_2$ and the g.l.b. for firms is denoted by $\mu_1 \land_F \mu_2$. Similarly, the l.u.b. for workers is denoted by $\mu_1 \lor_W \mu_2$ and the g.l.b. for the workers is denoted by $\mu_1 \land_W \mu_2$.

To define random matchings, first we need to define an incidence vector as follows:

**Definition 1** Given a stable matching $\mu$, a vector $x^{\mu} \in \{0, 1\}^{|F| \times |W|}$ is an incidence vector when $x^{\mu}_{i,j} = 1$ if and only if $j \in \mu(i)$ and $x^{\mu}_{i,j} = 0$ otherwise.

Denote by $S(P)$ to be the set of stable matchings. Now we formally define a random matching as follows.

**Definition 2** We say that $x$ is a random matching, if it is a lottery over stable matchings. That is,

$$x = \sum_{\nu \in S(P)} \lambda_{\nu} x^{\nu}$$

where $0 < \lambda_{\nu} \leq 1$, $\sum_{\mu \in S(P)} \lambda_{\mu} = 1$, and $\mu \in S(P)$.

3 New representations for random matchings

In this section, we present two important results that we use in the next section to prove that the set of random matchings has a lattice structure. First, we prove that given a random matching, there is a unique way to represent it with a decreasing property: each stable matching of the lottery fulfills that $\mu_\ell \geq_F \mu_{\ell+1}$. Since this property is independent of the structure of the preferences, this result can be stated for the other side of the market, i.e. on the eyes of all workers. Also, in this section given two random matchings, we present an algorithm that changes the representation of each lottery with a very particular property. This new way to represent each lottery, allows us to define the binary operations (l.u.b and g.l.b.) for random matchings in a very intuitive and easy way.

Given a random matching $x$, we denote by $\text{supp}(x)$ as follows:

$$\text{supp}(x) = \{(i, j) : x_{i,j} > 0\}.$$ 

Let $x$ be a random matching. We denote the following sets by:

- $A = \left\{ \nu \in S(P), \text{ such that } x = \sum_{\nu \in S(P)} \lambda_{\nu} x^{\nu}; 0 < \lambda_{\nu} \leq 1, \sum_{\nu \in S(P)} \lambda_{\nu} = 1 \right\}.$
- $A^\lor = \left\{ \forall_{\nu \in T} \nu, \text{ such that } T \subseteq A \right\}.$
Let \( x \) be a random matching. Then, there is a unique set of stable matchings that fulfill a decreasing property in the sense of the firm’s preferences. Formally,

**Theorem 1**  
Let \( x \) be a random matching. Then, there are a unique set of stable matchings \( \{\mu_1, \ldots, \mu_l\} \) and a unique set of scalars \( \{\alpha_1, \ldots, \alpha_l\} \) such that, \( x = \sum_{\ell=1}^{l} \alpha_{\ell} \mu^{\ell} \) with \( 0 < \alpha_{\ell} < 1, \sum_{\ell=1}^{l} \alpha_{\ell} = 1 \) for \( \ell = 1, \ldots, I \) and \( \mu^{\ell} \succeq_F \mu^{\ell+1} \) for \( \ell = 1, \ldots, I - 1 \).

**Proof.** Let \( x \) be a random matching. Let \( \mu^{1} \) and \( \alpha_1 \) defined as before.

Let \( x^{1} = x - \frac{\alpha_1}{1-\alpha_1} x^{\mu^{1}}. \)

If \( x^{1} \) is the incidence vector of a stable matching \(^2\), then \( x = \alpha_1 \mu^{1} + (1 - \alpha_1) x^{1} \) with \( 0 < \alpha_1 < 1 \) and \( \mu^{1} \succeq_F x^{1} \).

If not, let \( \mu^{2} = \bigvee_{v \in B_2} \) and \( \alpha_2 = \min \{ x_{i,j}^{1} \text{ such that } (i,j) \in \text{supp}(x^{\mu^{2}}) \} \).

By definition, we have that \( C_2 = \{ v \in B_1, \text{ such that there is a pair } (i,j) \in \text{supp}(x^{v}) \text{ where } x_{i,j}^{1} = \alpha_2 \) and \( x_{i,j}^{\mu^{1}} = 1 \} \), and \( B_3 = B_2 \setminus C_2 \). Notice that, since \( \mu^{2} \in C_2 \), then \( C_2 \neq \emptyset \). Hence, \( B_3 \subset B_2 \).

Let \( x^{2} = x - \frac{\alpha_2}{1-\alpha_2} x^{\mu^{2}}. \)

If \( x^{2} \) is the incidence vector of a stable matching, then \( x = \alpha_1 \mu^{1} + (1 - \alpha_1)\alpha_2 \mu^{2} + (1 - \alpha_1)(1 - \alpha_2)v \), with \( 0 < \alpha_1 < 1, 0 < \alpha_2 < 1 \) and \( \mu^{1} \succeq_F \mu^{2} \geq_F v \).

If not, we continue this process, until ends by the finiteness of the set of stable matchings, and the strictly embedding of the sets \( B_{l} \) (i.e \( B_{l} \subset B_{l+1} \)), (see Lemma 3 in the Appendix).

Denote by \( \beta_{1} = \alpha_1, \beta_{2} = (1 - \alpha_1)\alpha_2, \beta_{3} = (1 - \alpha_1)(1 - \alpha_2)\alpha_3, \ldots \) and so on.

Mathematical induction proves that \( \sum_{l} \beta_{l} = 1 \), for more detail see Lemma 4 in the Appendix.

Then, we represent \( x \) as a lottery of stable matchings such that, \( \mu^{\ell} \succeq_F \mu^{\ell+1} \).

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\(^2\)That is, if there is \( v \in S(P) \) such that \( x^{1} = x^{v} \).
Uniqueness: Assume that the random matching \( x \) has two different representations:

\[
x = \sum_{h=1}^{H} \alpha_h v^h = \sum_{k=1}^{K} \beta_k v^k
\]

with \( 0 < \alpha_h \leq 1 \) for \( h = 1, \ldots, H \); \( 0 < \beta_k \leq 1 \) for \( k = 1, \ldots, K \); \( \sum_{h=1}^{H} \alpha_h = 1 \); and \( \sum_{k=1}^{K} \beta_k = 1 \).

Since the l.u.b. and g.l.b. between two stable matchings is computed by the pointing function, it is straightforward that

\[
\mu^1 = \bigvee_{h=1,\ldots,H} v_h = \bigvee_{k=1,\ldots,K} v_k.
\]

In the same way for each \( \mu^\ell \) of the construction. Then, the lottery is unique. \( \square \)

In the following example, to clarify the exposition of our results, we consider a marriage market instance. The example is taken from Roth and Sotomayor (1990).

Example 1 Let \((M, W, P)\) a marriage market instance where \( M = \{m_1, m_2, m_3, m_4\} \), \( W = \{w_1, w_2, w_3, w_4\} \) and the preferences profile is given by

\[
\begin{align*}
P(m_1) &= w_1, w_2, w_3, w_4 & P(w_1) &= m_4, m_3, m_2, m_1 \\
P(m_1) &= w_2, w_1, w_4, w_3 & P(w_1) &= m_3, m_4, m_1, m_2 \\
P(m_1) &= w_3, w_4, w_1, w_2 & P(w_1) &= m_2, m_1, m_4, m_3 \\
P(m_1) &= w_4, w_3, w_2, w_1 & P(w_1) &= m_1, m_2, m_3, m_4
\end{align*}
\]

Where the set of stable matching and its lattice are the following:

| \( v^1 \) | \( v_{10} \) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) | \( v_7 \) | \( v_8 \) | \( v_9 \) |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| \( v_{1} \) | \( w_1 \) | \( w_2 \) | \( w_3 \) | \( w_4 \) | \( v_{10} \) | \( v_{10} \) | \( v_{10} \) | \( v_{10} \) | \( v_{10} \) | \( v_{10} \) |
| \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) | \( v_{2} \) |
| \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) | \( v_{3} \) |
| \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) | \( v_{4} \) |
| \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) | \( v_{5} \) |
| \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) | \( v_{6} \) |
| \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) | \( v_{7} \) |
| \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) | \( v_{8} \) |
| \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) | \( v_{9} \) |

Let \( x = \frac{1}{6}x^{v^2} + \frac{1}{2}x^{v^3} + \frac{1}{3}x^{v^7} \) be a random matching. Now we will change the representation
of the lottery as in Theorem 1. Notice that

\[ x = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \]

We have that, \( A = \{v_2, v_3, v_7\}, A^\vee = \{v_1, v_2, v_3\} \) and \( A^\wedge = \{v_4, v_7\} \). Then, \( B_1 = \{v_1, v_2, v_3, v_4, v_7\} \). Notice that \( \mu^1 = v_1, \alpha_1 = \frac{1}{6} \) and \( C_1 = \{v_1, v_2\} \).

\[ x^1 = \frac{x - \frac{1}{6}x^{\mu^1}}{1 - \frac{1}{6}} = \begin{pmatrix} 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 2 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]

Now, \( B_2 = B_1 \setminus C_1 = \{v_3, v_4, v_7\} \), \( \mu^2 = v_3, \alpha_2 = \frac{2}{3} \) and \( C_2 = \{v_3\} \). Then,

\[ x^2 = \frac{x - \frac{2}{3}x^{\mu^2}}{1 - \frac{2}{3}} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}. \]

\( B_3 = B_2 \setminus C_2 = \{v_4, v_7\} \), \( \mu^3 = v_4, \alpha_3 = \frac{2}{3} \) and \( C_3 = \{v_4\} \). Then,

\[ x^3 = \frac{x - \frac{1}{3}x^{\mu^3}}{1 - \frac{1}{3}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \]

Notice that \( x^3 = v_7 \). Here, \( B_4 = B_3 \setminus C_3 = \{v_7\} \), \( \mu^4 = v_7, \alpha_4 = 1 \) and \( C_4 = \{v_7\} \). Then, we have that \( B_5 = \emptyset \) and the procedure stops.

Therefore,

\[ x = \frac{1}{6}x^{\mu^1} + (1 - \frac{1}{6})\frac{2}{5}x^{\mu^2} + (1 - \frac{1}{6})(1 - \frac{2}{5})\frac{1}{3}x^{\mu^3} + (1 - \frac{1}{6})(1 - \frac{2}{5})(1 - \frac{1}{3})(1)x^{\mu^4} \]

\[ = \frac{1}{6}x^{\mu^1} + \frac{2}{6}x^{\mu^2} + \frac{1}{6}x^{\mu^3} + \frac{2}{6}x^{\mu^4}. \]

Since \( \mu^1 = v_1, \mu^2 = v_3, \mu^3 = v_4 \) and \( \mu^4 = v_7 \), then \( x \) can be written as:

\[ x = \frac{1}{6}x^{v_1} + \frac{2}{6}x^{v_3} + \frac{1}{6}x^{v_4} + \frac{2}{6}x^{v_7}. \]

As we can see in Figure 1, the stable matchings of the lottery fulfill that \( v_1 \geq_F v_3 \geq_F v_4 \geq_F v_7 \).
From now on, we assume that each random matching is already represented as a lottery over stable matchings in a decreasing way following Theorem 1.

Once we have each random matching represented with the decreasing property, to compare two random matchings and to compute the binary operations we need to change its representation with the following algorithm. This algorithm allows us to change the representation of the two random matchings involve with the following property: both lotteries have the same number of terms of the sum and the same scalars term to term. (Cf. Proposition 1.)

Let \( x \) and \( y \) be two random matchings such that,

\[
x = \sum_{\ell=1}^{l} \alpha_{\ell}^0 \mu_{\ell}^x \quad \text{and} \quad y = \sum_{\ell=1}^{l} \beta_{\ell}^0 \mu_{\ell}^y.
\]

Also denote by

\[
I^0 = \{1, \ldots, I\} \quad \text{and} \quad J^0 = \{1, \ldots, J\}.
\]

**Algorithm:**

**Step 1** Let \( \gamma_1 = \min\{\alpha_1^0, \beta_1^0\} \). Then, for each \( \ell \in I^0, J^0 \) define:

\[
\alpha_{\ell}^1 := \begin{cases} 
\alpha_{\ell}^0 - \gamma_1 & \text{if } \gamma_1 \neq \alpha_{\ell}^0 \text{ and } \ell = 1 \\
\alpha_{\ell}^0 & \text{if } \gamma_1 \neq \alpha_{\ell}^0 \text{ and } \ell > 1 \\
\alpha_{\ell-1}^0 & \text{if } \gamma_1 = \alpha_{\ell}^0 \text{ and } \ell > 1.
\end{cases}
\]

\[
\beta_{\ell}^1 := \begin{cases} 
\beta_{\ell}^0 - \gamma_1 & \text{if } \gamma_1 \neq \beta_{\ell}^0 \text{ and } \ell = 1 \\
\beta_{\ell}^0 & \text{if } \gamma_1 \neq \beta_{\ell}^0 \text{ and } \ell > 1 \\
\beta_{\ell-1}^0 & \text{if } \gamma_1 = \beta_{\ell}^0 \text{ and } \ell > 1.
\end{cases}
\]

If \( \gamma_1 = \alpha_1^0 \), then \( I^1 := I^0 \setminus \max graded \{\ell \in I^0\} \). If \( \gamma_1 \neq \alpha_1^0 \), then \( I^1 := I^0 \).

If \( \gamma_1 = \beta_1^0 \), then \( J^1 := J^0 \setminus \max graded \{\ell \in J^0\} \). If \( \gamma_1 \neq \beta_1^0 \), then \( J^1 := J^0 \).

Define \( \tilde{\mu}_I^x = \mu_I^x \) and \( \tilde{\mu}_I^y = \mu_I^y \).

**Step k**  
If \( |I^{k-1}| > 1 \) or \( |J^{k-1}| > 1 \).

Let \( \gamma_k = \min\{\alpha_k^{k-1}, \beta_k^{k-1}\} \). Then, for each \( \ell \in I^{k-1}, J^{k-1} \) define:

If \( \gamma_k \neq \alpha_k^{k-1} \), then \( I^k := I^{k-1} \) and \( \alpha_{\ell}^k := \begin{cases} 
\alpha_{\ell}^{k-1} - \gamma_k & \text{if } \ell = 1, \\
\alpha_{\ell}^{k-1} & \text{if } \ell > 1.
\end{cases} \)

If \( \gamma_k = \alpha_k^{k-1} \), then \( I^k := I^{k-1} \setminus \max graded \{\ell \in I^{k-1}\} \) and \( \alpha_{\ell-1}^k = \alpha_{\ell}^{k-1} \).

If \( \gamma_k \neq \beta_k^{k-1} \), then \( J^k := J^{k-1} \) and \( \beta_{\ell}^k := \begin{cases} 
\beta_{\ell}^{k-1} - \gamma_k & \text{if } \ell = 1, \\
\beta_{\ell}^{k-1} & \text{if } \ell > 1.
\end{cases} \)

If \( \gamma_k = \beta_k^{k-1} \), then \( J^k := J^{k-1} \setminus \max graded \{\ell \in J^{k-1}\} \) and \( \beta_{\ell-1}^k = \beta_{\ell}^{k-1} \).

Let \( p = |I^0| - |I^{k-1}| \) and \( r = |J^0| - |J^{k-1}| \), then define \( \tilde{\mu}_I^x = \mu_{p+1}^x \) and \( \tilde{\mu}_k^y = \mu_{r+1}^y \).
If \( |I^{k-1}| = 1 \) and \( |J^{k-1}| = 1 \), then \( \gamma_k = \alpha_1^{k-1} = \beta_1^{k-1} \), and define \( \tilde{\mu}_k^x = \mu_1^x \) and \( \tilde{\mu}_k^y = \mu_1^y \), and \( x = \sum_{\ell=1}^k \gamma_\ell \tilde{\mu}_\ell^x \), and \( y = \sum_{\ell=1}^k \gamma_\ell \tilde{\mu}_\ell^y \).

**Lemma 1** The algorithm stops in a finite number of steps. That is, there is a \( \tilde{k} \) such that \( |I^{\tilde{k}-1}| = 1 \) and \( \tilde{\alpha}_1^\tilde{k} = \beta_1^\tilde{k} \).

**Proof.** Note that in each step of the algorithm, we have that \( |I^k| = |I^{k-1}| - 1 \) or \( |J^k| = |J^{k-1}| - 1 \). We also have that in each step \( k \) of the algorithm,

\[
\sum_{\ell \in I^k} \alpha_\ell = \sum_{\ell \in I^{k-1}} \alpha_\ell - \gamma_k \quad \text{and} \quad \sum_{\ell \in J^k} \beta_\ell = \sum_{\ell \in J^{k-1}} \beta_\ell - \gamma_k.
\]

Hence,

\[
\sum_{\ell \in I^k} \alpha_\ell = \sum_{\ell \in I^0} \alpha_\ell - \sum_{t=1}^k \gamma_t = 1 - \sum_{t=1}^k \gamma_t.
\]

Similarly,

\[
\sum_{\ell \in J^k} \beta_\ell = \sum_{\ell \in J^0} \beta_\ell - \sum_{t=1}^k \gamma_t = 1 - \sum_{t=1}^k \gamma_t.
\]

That is, for each \( k \) we have that

\[
\sum_{\ell \in I^k} \alpha_\ell = \sum_{\ell \in J^k} \beta_\ell = 1 - \sum_{t=1}^k \gamma_t. \quad (1)
\]

Since \( |I^0| \) and \( |J^0| \) are finite, \( |I^k| = |I^{k-1}| - 1 \) or \( |J^k| = |J^{k-1}| - 1 \) and (1), we have that there is a \( \tilde{k} \) such that \( |I^{\tilde{k}-1}| = 1 \) and \( \tilde{\alpha}_1^\tilde{k} = \beta_1^\tilde{k} \).

**Proposition 1** The algorithm change the representation of two random matchings. That is, given two random matchings \( x \) and \( y \) such that

\[
x = \sum_{\ell=1}^l \alpha_\ell^0 \mu_\ell^x \quad \text{and} \quad y = \sum_{\ell=1}^l \beta_\ell^0 \mu_\ell^y.
\]

Then there are \( \gamma_\ell, \tilde{\mu}_\ell^x \) and \( \tilde{\mu}_\ell^y \) defined by the algorithm, and \( \tilde{k} \) the last step of the algorithm such that

\[
x = \sum_{\ell=1}^k \gamma_\ell \tilde{\mu}_\ell^x \quad \text{and} \quad y = \sum_{\ell=1}^k \gamma_\ell \tilde{\mu}_\ell^y.
\]

**Proof.** First we prove there is \( k_1 \) such that for \( \alpha_1^0 = \sum_{t=1}^{k_1} \gamma_t \). Since \( \gamma_1 = \min\{ \alpha_1^0, \beta_1^0 \} \), we analyze two cases.
Case 1: $\gamma_1 = a_1^0$. In this case $k_1 = 1$. Then,

$$\gamma_1 \tilde{\mu}_1^x = \gamma_1 \mu_1^x = a_1^0 \mu_1^x.$$  

Notice that $|I^1| = |I^0| - 1$. That is, $1 = p = |I^0| - |I^1|$ and $\tilde{\mu}_2^x = \mu_2^x$. Then, for each $k \geq 2$ we have that $\tilde{\mu}_k^x \neq \mu_k^x$.

Case 2: $\gamma_1 < a_1^0$. In this case we have that $|I^0| = |I^1|$ and $a_1^0 = a_1^0 - \gamma_1$. Then, in the next step $\gamma_2 \leq a_1^1$.

If $\gamma_2 = a_1^1$, then $a_1^0 = \gamma_1 + \gamma_2$ and $|I^0| = |I^1|$. Then, $\tilde{\mu}_2^x = \mu_2^x$ and

$$\gamma_1 \tilde{\mu}_1^x + \gamma_2 \tilde{\mu}_2^x = (\gamma_1 + \gamma_2) \mu_1^x = a_1^0 \mu_1^x.$$  

Notice that $|I^2| = |I^1| - 1$. That is, $1 = p = |I^0| - |I^2|$ and $\tilde{\mu}_3^x = \mu_3^x$. Then, for each $k \geq 3$ we have that $\tilde{\mu}_k^x \neq \mu_k^x$.

If $\gamma_2 < a_1^1$, then repeat this procedure until find $k_1$ such that $\gamma_{k_1} = a_1^{k_1-1}$. Then $a_1^0 = \sum_{i=1}^{k_1} \gamma_i$ and $|I^0| = |I^1| = \ldots = |I^{k_1}|$. Then, $\tilde{\mu}_1^x = \mu_1^x$ for $t = 1, \ldots, k_1$ and

$$\sum_{i=1}^{k_1} \gamma_i \tilde{\mu}_i^x = \sum_{i=1}^{k_1} \gamma_i \mu_i^x = a_1^0 \mu_1^x.$$  

Notice that $|I^{k_1}| = |I^{k_1-1}| - 1$. That is, $1 = p = |I^0| - |I^{k_1}|$ and $\tilde{\mu}_{k_1+1}^x = \mu_{k_1+1}^x$. Then, for each $k \geq k_1 + 1$ we have that $\tilde{\mu}_k^x \neq \mu_k^x$.

Once we find $k_1$, we have to repeat this procedure with each $a_1^0$ for $\ell \geq 2$.

The case for $\beta$ is similar.  

Example 1: Continue. Let $x = \frac{1}{6}x^1 + \frac{2}{6}x^2 + \frac{4}{6}x^3 + \frac{3}{6}x^4 + \frac{2}{6}x^5$ and $y = \frac{1}{4}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4$. Notice that both random matching are represented as in Theorem 1. We will use the algorithm to change the representation of both lotteries. Let $I^0 = \{1, 2, 3, 4\}$ and $J^0 = \{1, 2, 3\}$.

Step 1 $\gamma_1 = \min\{\frac{1}{3}, \frac{1}{4}\} = \frac{1}{6}$. Then, define:

$$
\begin{array}{c|c|c}
\alpha_1^1 & \frac{1}{6} & \beta_1^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \\
\alpha_2^1 & \frac{1}{6} & \beta_2^1 = \frac{1}{2} \\
\alpha_3^1 & \frac{2}{6} & \beta_3^1 = \frac{1}{4}
\end{array}
$$

Then, we have that $I^1 = \{1, 2, 3\}, J^1 = \{1, 2, 3\}, \tilde{\mu}_1^x = \nu_1$ and $\tilde{\mu}_1^y = \nu_2$.

Step 2 $\gamma_2 = \min\{\frac{2}{3}, \frac{1}{12}\} = \frac{1}{12}$. Then, define:

$$
\begin{array}{c|c|c}
\alpha_1^2 & \frac{2}{6} - \frac{1}{12} = \frac{1}{12} & \beta_1^2 = \frac{1}{7} \\
\alpha_2^2 & \frac{1}{6} & \beta_2^2 = \frac{1}{7} \\
\alpha_3^2 & \frac{2}{6} & \beta_3^2 = \frac{1}{4}
\end{array}
$$

Then, we have that $I^2 = \{1, 2, 3\}, J^2 = \{1, 2\}, \tilde{\mu}_2^x = \nu_3$ and $\tilde{\mu}_2^y = \nu_2$.  

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Step 3 \( \gamma_3 = \min \{ \frac{1}{4}, \frac{1}{4} \} = \frac{1}{4} \). Then, define:
\[
\begin{aligned}
\alpha_3^1 &= \frac{1}{4} \\
\alpha_3^2 &= \frac{2}{8} \\
\beta_3^1 &= \frac{3}{4} - \frac{1}{4} = \frac{1}{4} \\
\beta_3^2 &= \frac{1}{4}
\end{aligned}
\]
Then, we have that \( I^3 = \{ 1, 2 \} \), \( J^3 = \{ 1, 2 \} \), \( \tilde{\mu}_3^x = \nu_3 \) and \( \tilde{\mu}_3^y = \nu_4 \).

Step 4 \( \gamma_4 = \min \{ \frac{1}{6}, \frac{1}{4} \} = \frac{1}{6} \). Then, define:
\[
\begin{aligned}
\alpha_4^1 &= \frac{2}{6} \\
\beta_4^1 &= \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \\
\beta_4^2 &= \frac{1}{4}
\end{aligned}
\]
Then, we have that \( I^4 = \{ 1 \} \), \( J^4 = \{ 1, 2 \} \), \( \tilde{\mu}_4^x = \nu_4 \) and \( \tilde{\mu}_4^y = \nu_4 \).

Step 5 \( \gamma_5 = \min \{ \frac{2}{6}, \frac{1}{12} \} = \frac{1}{12} \). Then, define:
\[
\begin{aligned}
\alpha_5^1 &= \frac{2}{6} - \frac{1}{12} = \frac{1}{4} \\
\beta_5^1 &= \frac{1}{4}
\end{aligned}
\]
Then, we have that \( I^5 = \{ 1 \} \), \( J^5 = \{ 1 \} \), \( \tilde{\mu}_5^x = \nu_7 \) and \( \tilde{\mu}_5^y = \nu_4 \).

Step 6 \( \gamma_6 = \min \{ \frac{1}{4}, \frac{1}{4} \} = \frac{1}{4} \).

Then, we have that \( I^6 = J^6 = \emptyset \), \( \tilde{\mu}_6^x = \nu_7 \) and \( \tilde{\mu}_6^y = \nu_5 \).

Therefore, we can represent the random matchings \( x \) and \( y \) as follows:
\[
\begin{aligned}
x &= \frac{1}{6} x^{v_1} + \frac{1}{12} x^{v_3} + \frac{1}{4} x^{v_3} + \frac{1}{4} x^{v_4} + \frac{1}{12} x^{v_7} + \frac{1}{4} x^{v_7}, \\
y &= \frac{1}{6} x^{v_2} + \frac{1}{12} x^{v_2} + \frac{1}{4} x^{v_4} + \frac{1}{4} x^{v_4} + \frac{1}{12} x^{v_4} + \frac{1}{4} x^{v_5}.
\end{aligned}
\]
Observe that \( x \) and \( y \) have six terms in each representation. Moreover, both lotteries have the same scalar term to term.

## 4 Main result

In this section, we first define a partial order. Next, using the partial order and Proposition 1, we define the binary operation for random matchings. The main result of this paper is that with the partial order and these binary operations, the set of random matchings has a dual lattice structure.

From now on, each random matching will be represented as in Proposition 1. That is, given \( x \) and \( y \), we write
\[
x = \sum_{\ell=1}^{\tilde{k}} \gamma_{\ell} \tilde{\mu}_{\ell}^x \quad \text{and} \quad y = \sum_{\ell=1}^{\tilde{k}} \gamma_{\ell} \tilde{\mu}_{\ell}^y
\]
such that, \( 0 < \gamma_{\ell} \leq 1; \sum_{\ell=1}^{\tilde{k}} \gamma_{\ell} = 1; \tilde{\mu}_{\ell}^x \in S(P), \tilde{\mu}_{\ell}^x \geq_F \tilde{\mu}_{\ell+1}^x \) and \( \tilde{\mu}_{\ell}^y \geq_F \tilde{\mu}_{\ell+1}^y \) for \( \ell = 1, \ldots, \tilde{k} - 1 \).

Now we define when \( x \) dominates \( y \) for one side of the market.
Definition 3 We will say that \( x \) dominates \( y \) for all firms, \((x \succeq_F y)\), if and only if \( \tilde{\mu}^x_{\ell} \succeq_F \tilde{\mu}^y_{\ell} \) for each \( \ell = 1, \ldots, \tilde{k} \). Analogously, we define \( x \succeq_W y \) for all workers.

Once each lottery is represented as in Proposition 1, the partial order over random matchings is very natural. Now, the reader can see that the following proof relies on the partial order of (deterministic) stable matchings.

Lemma 2 The relations \( \succeq_F \) and \( \succeq_W \) are partial orders. Also, \( \succeq_F \) and \( \succeq_W \) are dual.

Proof.

We will prove that \( \succeq_F \) is a partial order. The proof for \( x \succeq_W y \) is analogous.

Transitivity: If \( x \succeq_F y \) and \( y \succeq_F z \), then \( x \succeq_F z \). We have that \( \tilde{\mu}^x_{\ell} \succeq_F \tilde{\mu}^y_{\ell} \) for each \( \ell = 1, \ldots, \tilde{k} \) and \( \tilde{\mu}^y_{\ell} \succeq_F \tilde{\mu}^z_{\ell} \) for each \( \ell = 1, \ldots, \tilde{k} \). Then, by the transitivity of the partial order \( \succeq_F \), we have that \( \tilde{\mu}^x_{\ell} \succeq_F \tilde{\mu}^z_{\ell} \) for each \( \ell = 1, \ldots, \tilde{k} \). Then, \( x \succeq_F z \).

Reflexivity: \( x \succeq_F x \) for all \( x \). By the reflexivity of the partial order \( \succeq_F \), it is straightforward that \( \tilde{\mu}^x_{\ell} \succeq_F \tilde{\mu}^x_{\ell} \) for each \( \ell = 1, \ldots, \tilde{k} \). Then, \( x \succeq_F x \).

Antisymmetry: If \( x \succeq_F y \) and \( y \succeq_F x \), then \( x = y \). We have that \( \tilde{\mu}^x_{\ell} \succeq_F \tilde{\mu}^y_{\ell} \) and \( \tilde{\mu}^y_{\ell} \succeq_F \tilde{\mu}^x_{\ell} \) for each \( \ell = 1, \ldots, \tilde{k} \). Then, by the antisymmetry of the partial order \( \succeq_F \), we have that \( \tilde{\mu}^x_{\ell} = \tilde{\mu}^y_{\ell} \) for each \( \ell = 1, \ldots, \tilde{k} \). Then, \( x = y \). Therefore, the relation \( \succeq_F \) is a partial order.

Duality: Since the partial orders \( \succeq_F \) and \( \succeq_W \) for stable matchings are dual, then the partial orders \( \succeq_F \) and \( \succeq_W \) are dual.

Let \( \vee_F \) and \( \wedge_F \) be the binary operations \((l.u.b. \text{ and } g.l.b.)\) defined between two (deterministic) stable matching. Now, we extend these binary operations for random matchings. Formally,

Definition 4 Given two random matchings \( x \) and \( y \) such that

\[
\begin{align*}
x &= \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}^x_{\ell} & \text{and} & \quad y &= \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}^y_{\ell}
\end{align*}
\]

where \( 0 < \gamma_\ell \leq 1; \sum_{\ell=1}^{\tilde{k}} \gamma_\ell = 1, \tilde{\mu}^x_{\ell} \in S(P), \tilde{\mu}^y_{\ell} \succeq_F \tilde{\mu}^z_{\ell+1} \) and \( \tilde{\mu}^z_{\ell+1} \succeq_F \tilde{\mu}^y_{\ell} \).

We define the following random matchings:

\[
\begin{align*}
x \vee_F y & := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell (\tilde{\mu}^x_{\ell} \vee_F \tilde{\mu}^y_{\ell}), & x \wedge_F y & := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell (\tilde{\mu}^x_{\ell} \wedge_F \tilde{\mu}^y_{\ell}),
\end{align*}
\]

and

\[
\begin{align*}
x \vee_W y & := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell (\tilde{\mu}^x_{\ell} \vee_W \tilde{\mu}^y_{\ell}), & x \wedge_W y & := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell (\tilde{\mu}^x_{\ell} \wedge_W \tilde{\mu}^y_{\ell}),
\end{align*}
\]
Now, we are in position to prove that these binary operations defined for random matchings are actually the l.u.b. and g.l.b. for each side of the market.

**Theorem 2** Let \( x \) and \( y \) be two random matchings. Then, for \( X \in \{F,W\} \) we have that

\[
x \lor_X y = l.u.b. \prec_X (x, y) \quad \text{and} \quad x \lor_X y = g.l.b. \preceq_X (x, y).
\]

Also,

\[
x \lor_F y = x \lor_X W y \quad \text{and} \quad x \lor_W y = x \lor_X F y.
\]

**Proof.**

- We will prove that \( x \lor_X y = l.u.b. \prec_X (x, y) \).
  
  (i) \( x \lor_X y \succeq_X x \):
  
  Since \( \bar{\mu}_x^y \lor_X \bar{\mu}_x^y \succeq_X \bar{\mu}_x^y \) for each \( \ell = 1, \ldots, k \), then \( x \lor_X y \succeq_X x \).

  (ii) \( x \lor_X y \succeq_X y \):
  
  Since \( \bar{\mu}_x^y \lor_X \bar{\mu}_x^y \succeq_X \bar{\mu}_x^y \) for each \( \ell = 1, \ldots, k \), then \( x \lor_X y \succeq_X y \).

  (iii) If \( z \succeq_X x \) and \( z \succeq_X y \), then \( z \succeq_X x \lor_X y \):
  
  We have that \( \bar{\mu}_x^y \succeq_X \bar{\mu}_x^y \) and \( \bar{\mu}_x^y \succeq_X \bar{\mu}_x^y \) for each \( \ell = 1, \ldots, k \). Since, \( \bar{\mu}_x^y \lor_X \bar{\mu}_x^y \) is the l.u.b. \( \prec_X (\bar{\mu}_x^y, \bar{\mu}_x^y) \), then \( \bar{\mu}_x^y \succeq_X \bar{\mu}_x^y \lor_X \bar{\mu}_x^y \) for each \( \ell = 1, \ldots, k \). Hence, \( z \succeq_X x \lor_X y \).

- The proof for \( x \lor_X y = g.l.b. \preceq_X (x, y) \) is analogous.

- \( x \lor_F y = x \lor_X W y \):
  
  Recall that the lattices of stable matchings are dual, that is given \( \mu_1, \mu_2 \in S(P) \) \( \mu_1 \lor_F \mu_2 = \mu_1 \land_W \mu_2 \). By Definition 4, we have that for \( 0 < \gamma_\ell \leq 1; \sum_{\ell=1}^k \gamma_\ell = 1 \), \( \bar{\mu}_x^y \in S(P) \), \( \bar{\mu}_x^y \succeq_F \bar{\mu}_x^y+1 \) and \( \bar{\mu}_x^y \succeq_F \bar{\mu}_x^y+1 \),

  \[
x \lor_F y = \sum_{\ell=1}^k \gamma_\ell (\bar{\mu}_x^y \lor_F \bar{\mu}_x^y) = \sum_{\ell=1}^k \gamma_\ell (\bar{\mu}_x^y \land_W \bar{\mu}_x^y) = x \lor_X W y.
\]

- The proof for \( x \lor_W y = x \lor_X F y \) is analogous.

\( \square \)

**Theorem 3** Let \((F,W,P)\) be a matching market and denote by \( RM \) the set of random matchings. Then, \((RM, \succeq_F)\) and \((RM, \succeq_W)\) are dual lattices.
Example 1: Continue. Once we have the random matchings $x$ and $y$ are represented as in Proposition 1, we can compute the for instance $x \cup_F y$ and $x \cap_F y$ as follows:

$$x = \frac{1}{6}x^v + \frac{1}{12}x^{v_1} + \frac{1}{4}x^{v_3} + \frac{1}{6}x^{v_4} + \frac{1}{12}x^{v_7} + \frac{1}{4}x^{v_7},$$

$$y = \frac{1}{6}x^{v_2} + \frac{1}{12}x^{v_2} + \frac{1}{4}x^{v_4} + \frac{1}{6}x^{v_4} + \frac{1}{12}x^{v_5} + \frac{1}{4}x^{v_5}.$$

$$x \cup_F y = \frac{1}{6}x^{v_1 \vee v_2} + \frac{1}{12}x^{v_3 \vee v_4} + \frac{1}{4}x^{v_3 \vee v_4} + \frac{1}{6}x^{v_4 \vee v_4} + \frac{1}{12}x^{v_7 \vee v_4} + \frac{1}{4}x^{v_7 \vee v_4}$$

$$= \frac{1}{6}x^{v_1} + \frac{1}{12}x^{v_2} + \frac{1}{4}x^{v_3} + \frac{1}{6}x^{v_4} + \frac{1}{12}x^{v_5} + \frac{1}{4}x^{v_5},$$

$$x \cap_F y = \frac{1}{6}x^{v_1 \wedge v_2} + \frac{1}{12}x^{v_3 \wedge v_4} + \frac{1}{4}x^{v_3 \wedge v_4} + \frac{1}{6}x^{v_4 \wedge v_4} + \frac{1}{12}x^{v_7 \wedge v_4} + \frac{1}{4}x^{v_7 \wedge v_4}$$

$$= \frac{1}{6}x^{v_2} + \frac{1}{12}x^{v_4} + \frac{1}{4}x^{v_4} + \frac{1}{6}x^{v_4} + \frac{1}{12}x^{v_7} + \frac{1}{4}x^{v_7}.$$

Analogously for $x \cup_W y$ and $x \cap_W y$.

5 Two particular cases

This section analyze two cases. First, the particular case in which all agents have quota equal to one, this is known in the literature as the marriage market. Second, we compute in a very simple way the binary operations between two rational random matchings, that is, each scalar of the lottery is a rational number.

5.1 When the matching market is the marriage market

In this sub-section we prove that in the special case of the marriage market $(M, W, P)$, the partial order defined in Section 4 is equivalent to the first-order stochastic dominance defined in Roth et al. (1993). For the marriage market, Roth et al. (1993) define a stable fractional matching as a not necessarily integer solution of a linear program (see Roth et al., 1993, for more detail). Recall that, for the marriage market, random matchings and stable factional matchings are equivalent. Then, a random matching is a bi-stochastic matrix, where each entry, $x_{m,w}$ is the probability of man $m$ and woman $w$ to be matched.

---

3 The extreme points of the polytope generated by the linear constraints are exactly the stable matchings, (see Theorem 13 in Roth et al., 1993).
Remark 1 Given the random matching $x$, the entry $x_{m,w} = \sum_{k:w=\mu_k^x(m)} \gamma_k$ where $\gamma_k$ is defined by the algorithm. Then,
$$
\sum_{j \geq m,w} x_{m,j} = \sum_{k:w=\mu_k^x(m) \geq m,w} \gamma_k.
$$
In the same way for the random matching $y$.

Now, we present the definition of the partial order used in Roth et al. (1993).

Definition 5 $x$ weakly dominates $y$ in man’s opinion, (here denoted by $x \succeq^* y$) if
$$
\sum_{j \geq m,w} x_{m,j} \geq \sum_{j \geq m,w} y_{m,j}
$$
for each $w \in W$.

We say that $x \succeq^*_M y$ if and only if $x \succeq^*_m y$ for each $m \in M$.

Proposition 2 The partial order $(\succeq_M)$ is equivalent to $\succeq^*_M$.

Proof. To prove that $\succeq_M$ is equivalent to $\succeq^*_M$, we need to prove that for each $m \in M$ and each $w \in W$,
$$
\sum_{j \geq m,w} x_{m,j} \geq \sum_{j \geq m,w} y_{m,j}
$$
if and only if
$$
\sum_{k:w=\mu_k^x(m) \geq m,w} \gamma_k \geq \sum_{k:w=\mu_k^y(m) \geq m,w} \gamma_k.
$$

Assume that for each $m \in M$ and each $w \in W$,
$$
\sum_{j \geq m,w} x_{m,j} \geq \sum_{j \geq m,w} y_{m,j}.
$$
Also assume that there are $m_1 \in M$, $w_1, w_2 \in W$ and $k_1 \in \{1, \ldots, \tilde{k}\}$ such that
$$
w_1 = \mu_{k_1}^x(m_1) < m_1 \mu_{k_1}^y(m_1) = w_2
$$
and for each $k < k_1$ we have that $\mu_k^x(m_1) \geq m_1 \mu_k^y(m_1)$. Then, we have that
$$
\sum_{k:w=\mu_k^x(m) \geq m,w} \gamma_k + \gamma_{k_1} = \sum_{k:w=\mu_k^y(m) \geq m,w} \gamma_k.
$$
Hence, by Remark 1, we have that
$$
\sum_{j \geq m,w} x_{m,j} = \sum_{k:w=\mu_k^x(m) \geq m,w} \gamma_k < \sum_{k:w=\mu_k^y(m) \geq m,w} \gamma_k = \sum_{j \geq m,w} y_{m,j}.
$$
a contradiction.

\[ \sum_{k=1}^{\infty} \gamma_k = \sum_{m \geq m_0} x_{m,j} \geq \sum_{k=1}^{\infty} \gamma_k = \sum_{j \geq m_0} y_{m,j}. \]

Therefore, the partial order \( (\geq_M) \) is equivalent to \( \geq_M^* \).

\[ \square \]

Notice that, since the two partial orders are equivalent, the binary operations defined in Roth et al. (1993) and the ones defined in this paper, Definition 4, are also equivalent. Since \( \nabla_X \) and \( \bar{\nabla}_X \) are defined for a more general market than the ones defined in Roth et al. (1993), we say that \( \nabla_X \) and \( \bar{\nabla}_X \) of Definition 4, generalizes the binary operations in Roth et al. (1993).

5.2 Binary operations for Rational Random Matchings

In this sub-section we compute the g.l.b. and l.u.b. for two rational random matchings. A rational random matching is a random matching where each scalar of the lottery is a rational number.

Let \( x \) and \( y \) be two rational random matchings, that is:

\[ x = \sum_{i \in I} a_i \mu_i^x, \quad (2) \]

such that, \( 0 < a_i \leq 1; \sum_{i \in I} a_i = 1, \mu_i^x \in S(P), a_i \) is a rational number and \( \mu_i^x \geq \mu_{i+1}^x \);

where \( I = \{ i : i = 1, ..., I \} \).

\[ y = \sum_{j \in J} b_j \mu_j^y, \quad (3) \]

such that, \( 0 < b_j \leq 1; \sum_{j \in J} b_j = 1, \mu_j^y \in S(P), b_j \) is a rational number and \( \mu_j^y \geq \mu_{j+1}^y \);

where \( J = \{ j : j = 1, ..., J \} \).

Since \( a_i \) and \( b_j \) are positive rational numbers, we have that for each \( a_i \) there are natural numbers \( a_i, b_i \) such that \( a_i = \frac{a_i}{b_i} \). In the same way, for each \( b_j \) there are natural numbers \( c_j, d_j \) such that \( b_j = \frac{c_j}{d_j} \).

Denote by \( e \) to be the least common multiple (lcm) of all denominators \( b_i, d_j \) for each \( i \in I \) and for each \( j \in J \). That is,

\[ e = \text{lcm}(b_1, \ldots, b_{l}, d_1, \ldots, d_{h}). \]

Then, we can write \( a_i = \frac{a_i}{b_i} = \frac{a_i \cdot d}{e} \) and \( b_i = \frac{c_j}{d_j} = \frac{c_j \cdot d}{e} \) for each \( i \in I \) and for each \( j \in J \). This way, we can write all the scalar \( a \) and \( b_j \) with the same denominator.

Denote by \( \gamma_k = \frac{1}{e} \) and define
Let $x$ and $y$ be two rational random matchings (i.e. each

$$
\tilde{X}_k := \begin{cases} 
\mu_1^x & \text{for } k = 1, \ldots, \frac{a_1}{b_1}e \\
\mu_2^x & \text{for } k = \frac{a_1}{b_1}e + 1, \ldots, \frac{a_2}{b_2}e \\
\vdots & \\
\mu_k^x & \text{for } k = \frac{a_{k-1}}{b_{k-1}}e + 1, \ldots, \frac{a_k}{b_k}e \\
\end{cases}
$$

Then, we have that

$$
x = \sum_{i=1}^{l} \alpha_i \mu_i^x = \sum_{j=1}^{l} \frac{a_i}{b_j} \mu_i^x = \sum_{k=1}^{e} \frac{1}{e} \tilde{X}_k.
$$

In the same way, we have that

$$
y = \sum_{j=1}^{h} \beta_j \mu_j^y = \sum_{i=1}^{h} \frac{c_j}{d_i} \mu_j^y = \sum_{k=1}^{\epsilon} \frac{1}{e} \tilde{Y}_k.
$$

Then, it is easy to compute the $x \vee_F y$ and $x \lor_F y$. Similarly for $x \vee_W y$ and $x \lor_W y$.

Now we formally state the theorem.

**Theorem 4** Let $x$ and $y$ be two rational random matchings (i.e. each $\alpha$ and each $\beta$ from (2) and (3) are rational numbers). Then, for $X \in \{F, W\}$ we have that

$$
x \vee_X y = \sum_{k=1}^{e} \frac{1}{e} (\tilde{X}_k \lor_X \tilde{Y}_k) \quad \text{and} \quad x \lor_X y = \sum_{k=1}^{e} \frac{1}{e} (\tilde{X}_k \lor_X \tilde{Y}_k).
$$

**Example 1: Continue.** Let $x$ and $y$ be the random matchings represented as in Theorem 1,

$$
x = \frac{1}{6} x^{v_1} + \frac{1}{12} x^{v_3} + \frac{1}{4} x^{v_4} + \frac{1}{6} x^{v_7} + \frac{1}{12} x^{v_{17}},
$$

$$
y = \frac{1}{6} x^{v_2} + \frac{1}{12} x^{v_4} + \frac{1}{4} x^{v_4} + \frac{1}{12} x^{v_4} + \frac{1}{4} x^{v_{15}}.
$$

Let $e = \text{lcm}(6, 4, 12) = 12$. Then, the random matchings $x$ and $y$ are equivalent to:

$$
x = \frac{1}{12} x^{v_1} + \frac{1}{12} x^{v_1} + \frac{1}{12} x^{v_3} + \frac{1}{12} x^{v_3} + \frac{1}{12} x^{v_3} + \frac{1}{12} x^{v_3} + \frac{1}{12} x^{v_4} + \frac{1}{12} x^{v_4}
$$

$$
+ \frac{1}{12} x^{v_7} + \frac{1}{12} x^{v_7} + \frac{1}{12} x^{v_7} + \frac{1}{12} x^{v_{17}},
$$

$$
y = \frac{1}{12} x^{v_2} + \frac{1}{12} x^{v_2} + \frac{1}{12} x^{v_2} + \frac{1}{12} x^{v_4} + \frac{1}{12} x^{v_4} + \frac{1}{12} x^{v_4} + \frac{1}{12} x^{v_4} + \frac{1}{12} x^{v_4}.$$
Then,
\[
x \vee_{F} y = \frac{1}{12}x^{v_{1}} + \frac{1}{12}x^{v_{3}} + \frac{1}{12}x^{v_{5}} + \frac{1}{12}x^{v_{7}}.
\]

Analogously for \( x \vee_{W} y \), \( x \vee_{W} y \) and \( x \vee_{W} y \).

6 Conclusions

In this paper we prove an important result that involves two very studied topics of matching literature: random matchings and lattice structure. We prove that the random matching set has a dual lattice structure for a very general matching marker such as: a many-to-many matching market with substitutable preferences that also fulfill law of aggregated demand on one side, and q-responsiveness preferences on the other side. This market is the most general matching market that is known that the binary operations between two stable matchings, \( l.u.b \) and \( g.l.b \), are computed via a pointing function.

7 Appendix

Lemma 3 There exists \( v \in A \), such that \( v \in C_{1} \)

Proof. Let \((i', j') \in \text{supp}(x)\) such that \( x_{i', j'} = \alpha_{1} \). Then, by definition of \( \alpha_{1} \), we have that \( x_{i', j'}^{\mu_{1}} = 1 \). Since \( \mu_{1} \) is computed via pointing function, then we have that there is \( \bar{v} \in A \) such that \( x_{i', j'}^{\bar{v}} = 1 \). Therefore, \( \bar{v} \in C_{1} \) \( \square \)

Lemma 4 Given \( \beta_{l} \) defined in the proof of Theorem 1, then \( \sum_{l=1}^{l} \beta_{l} = 1 \).

Proof. Let \( \beta_{l} \) defined as: \( \beta_{1} = \alpha_{1} \), \( \beta_{2} = (1 - \alpha_{1})\alpha_{2} \), \( \beta_{3} = (1 - \alpha_{1})(1 - \alpha_{2})\alpha_{3} \), ... We will prove that \( \sum_{l=1}^{l} \beta_{l} = 1 \) by mathematical induction over the steps that it takes for \( x^{l} \) to be an incidence vector of a stable matching in the proof of Theorem 1

- If \( x^{1} \in S(P) \). Then,

\[
\beta_{1} = \alpha_{1}, \quad \beta_{2} = (1 - \alpha_{1})
\]

\[
\sum_{l=1}^{l=1} \beta_{l} = \alpha_{1} + (1 - \alpha_{1}) = 1
\]
• If \( x_n \in S(P) \). Then,

\[
\beta_1 = \alpha_1, \quad \beta_2 = (1 - \alpha_1)\alpha_2, \quad \beta_3 = (1 - \alpha_1)(1 - \alpha_2)\alpha_3, \ldots,
\]

\[
\beta_{n-1} = (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n-1})\alpha_n, \quad \beta_n = (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n-1})(1 - \alpha_n)
\]

\[
\sum_{l=1}^{n} \beta_l = \alpha_1 + (1 - \alpha_1)\alpha_2 + \ldots + (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n-1})(1 - \alpha_n) = 1
\]

• If \( x_{n+1} \in S(P) \). Then we will prove that \( \sum_{l=1}^{n+1} \beta_l = 1 \).

\[
\sum_{l=1}^{n+1} \beta_l = \alpha_1 + (1 - \alpha_1)\alpha_2 + \ldots + (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n-1})(1 - \alpha_n) = \alpha_1 + (1 - \alpha_1)\alpha_2 + \ldots + \left[(1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n-1})(1 - \alpha_n) \right] \frac{\alpha_{n+1} + (1 - \alpha_{n+1})}{=1} = \sum_{l=1}^{n} \beta_l = 1
\]

\[\square\]

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