MORE CONSTRUCTIONS OF NEAR OPTIMAL CODEBOOKS
ASSOCIATED WITH BINARY SEQUENCES

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Abstract. An \((N, K)\) codebook \(C\) is a collection of unit norm vectors in a
\(K\)-dimensional vectors space. In applications of codebooks such as CDMA,
those vectors in a codebook should have a small maximum magnitude of inner
products, denoted by \(I_{\max}(C)\), between any pair of distinct code vectors. Since
the famous Welch bound is a lower bound on \(I_{\max}(C)\), it is desired to construct
codebooks meeting the Welch bound strictly or asymptotically. Recently, N.
Y. Yu presents a method for constructing codebooks associated with a binary
sequence from a \(\Phi\)-transform matrix. Using this method, he discovers new
classes of codebooks with nontrivial bounds on the maximum inner products
from Fourier and Hadamard matrices. We construct more near optimal code-
books by Yu’s idea. We first provide more choices of binary sequences. We
also show more choices of the \(\Phi\)-transform matrices. Therefore, we can present
more codebooks \(C\) with nontrivial bounds on their \(I_{\max}(C)\). Our work can
serve as a complement of Yu’s work.

1. Introduction

Let \(K\) be a positive integer and \(\mathbb{C}^K\) be the complex space of column vectors
of dimension \(K\). An \((N, K)\) codebook \(C\) is a set of complex column vectors \(v_i =
(c_0, c_1, \cdots, c_{K-1})^T, i = 1, \cdots, N\) of unit norm, i.e., \(v_i^*v_i = \sum_{j=0}^{K-1} |c_j|^2 = 1\) for
every \(v_i \in C\), where \(v_i^*\) means the conjugate transpose of \(v_i\). Denote by \(I_{\max}(C)\) the
maximum magnitude of inner products between a pair of distinct code vectors in \(C\).

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We need a codebook $C$ with $J_{\text{max}}(C)$ as small as possible for applications. However, Welch [24] provides the following well-known lower bound:

$$J_{\text{max}}(C) = \max_{1 \leq k \neq l \leq N} |v_k^* v_l| \geq \frac{N - K}{K(N - 1)}.$$  

The equality holds if and only if for all pairs of $(k, l)$ with $k \neq l$

$$|v_k^* v_l| = \frac{\sqrt{N - K}}{K(N - 1)}.$$  

In this case, $C$ is called a maximum-Welch-bound-equality (abbreviated by MWBE) codebook [22]. MWBE codebooks have been of interest in many areas, e.g., communications [22], combinatorial designs [6,7,26], signal processing [25], packing [3], coding theory [4] and quantum computing [25]. However, constructing MWBE codebooks in an analytic way is extremely difficult as indicated in [22]. The following codebooks are the known classes of MWBE codebooks except nearly trivial cases, i.e., partial Fourier and Hadamard codebooks with $K = N$ or $N - 1$.

- $(N, K)$ MWBE codebooks based on conference matrices [3,23], where $N = 2K = 2^{d+1}$ for a positive integer $d$ or $N = 2K = p^d + 1$ for an odd prime $p$ and a positive integer $d$.
- $(N, K)$ MWBE codebooks based on $(N, K, \lambda)$ cyclic difference sets (DS) [6,7,26], or $(N, K, \lambda)$ almost difference sets (ADS) in finite fields or Abelian groups [6,7].
- $(N, K)$ MWBE codebooks or equiangular tight frames based on $(2, k, v)$-Steiner systems [10].

Since the known classes MWBE exist only for few choices of the parameters $N$ and $K$, many researchers have tried to find the so-called near-optimal codebooks, i.e., codebooks $C$ with $J_{\text{max}}(C)$ nearly achieves the Welch bound. As an extension of the MWBE codebooks based on DS, various types of near-optimal codebooks based on almost (or relative) DS and/or cyclotomic classes are proposed, see [6–8,28–31] etc. Recently, near-optimal codebooks constructed from binary row selection sequences (e.g., binary Sidelnikov sequences) are proposed in [14,27].

Let’s recall the key idea of [27]:

**Yu’s Construction:** Let $\Phi$ be a $L \times N$ matrix with entries $\phi_{i,n}$, $0 \leq i \leq L - 1$ and $0 \leq n \leq N - 1$. Let $\hat{a} = (a_0, \cdots, a_{L-1})$ be a binary sequence of length $L$ and Hamming weight $K$ and $D = \{d_0, \cdots, d_{K-1}\}$ be its support. Consider a $K \times N$ partial matrix of $\Phi$ by selecting $K$ rows from $\Phi$, which row indices are from $D$. With the scaling factor $\frac{1}{\sqrt{K}}$, the $n$-th column vector of the partial matrix is given by

$$c_n = \frac{1}{\sqrt{K}} (\phi_{d_0,n}, \phi_{d_1,n}, \cdots, \phi_{d_{K-1},n})^T, 0 \leq n \leq N - 1.$$  

Then the needed codebook $C_{\Phi}(\hat{a})$ is given by

$$C_{\Phi}(\hat{a}) = \{c_0, c_1, \cdots, c_{N-1}\}.$$  

In this construction, Yu [27] has some constraints on the matrix $\Phi$ as follows:

Property 1: Each entry of $\Phi$ has unit norm.

Property 2: No two distinct column vectors in $\Phi$ are identical. Moreover, for fixed $0 \leq i \leq L - 1$ and for all $n_1, n_2 \in \mathbb{Z}_N$ (the ring of integers modulo $N$), there
exists exactly one \( n \in \mathbb{Z}_N \) such that
\begin{equation}
\phi_{i,n_1}\phi_{i,n_2} = \phi_{i,n},
\end{equation}
where \( \bar{\phi}_{i,n} \) is the complex conjugate of \( \phi_{i,n} \).

Property 3: In (1), \( n = 0 \) if and only if \( n_1 = n_2 \).

Property 4: If \( n \neq 0 \), then \( \sum_{i=0}^{L-1} \phi_{i,n} = 0 \).

Let the \( \Phi \)-transform of \( \tilde{a} \) be
\[
\hat{a} = (\hat{a}_0, \cdots, \hat{a}_{N-1})
\]
with
\[
\hat{a}_n = \sum_{i=0}^{L-1} (-1)^{\alpha_i} \phi_{i,n}, \quad 0 \leq n \leq N - 1.
\]

Yu gets the following result:

**Lemma 1.1** ([27]). In the \((N,K)\) codebook \( \mathcal{C}_\Phi(\tilde{a}) \) constructed above, we have
\[
I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) = \frac{1}{2K} \cdot \max_{1 \leq n \leq N-1} |\hat{a}_n|
\]
where \( \hat{a}_n \) is the \( n \)-th element of \( \Phi \)-transform of \( \tilde{a} \).

As indicated in [27], a codebook from Yu’s Construction is not new if \( \Phi \) is either the Fourier or the Hadamard matrix and \( \tilde{a} \) is a characteristic sequence of a difference set. If the support of \( \tilde{a} \) forms an ADS, \( \mathcal{C}_\Phi(\tilde{a}) \) may or may not be new, depending on \( \Phi \) and \( \tilde{a} \). Some constructions from ADSs are introduced in [6] and [29,30], but there is still no characterization in this situation. Moreover, when \( \tilde{a} \) is an arbitrary binary sequence or \( \Phi \) is not the character table of an Abelian group (e.g. \( \Phi \) is neither the Fourier matrix nor the Hadamard matrix), no one explicitly describes either whether \( \mathcal{C}_\Phi(\tilde{a}) \) can form a codebook with nontrivially bounded on \( I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) \) or how the upper bound of \( I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) \) can be determined. In other words, for an arbitrary binary sequence \( \tilde{a} \), Yu’s Construction establishes a framework of constructing a codebook \( \mathcal{C}_\Phi(\tilde{a}) \), and Lemma 1.1 then presents a technique for determining the maximum possible value of \( I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) \) with an efficient proof. In this aspect, one may enjoy more freedom of choosing a binary sequence \( \tilde{a} \) for constructing its associated codebook \( \mathcal{C}_\Phi(\tilde{a}) \) and determining \( I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) \). Following this approach, Yu [27] and Hong et al [14] obtain some codebooks with nontrivial bounds on \( I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) \).

Following Yu’s idea, we provide more constructions of near optimal codebooks in this paper. We will provide more choices of binary sequences, namely we will show that one may choose a suitable subset \( D \) of a finite field and then use its characteristic function as the binary sequence \( \tilde{a} \). Using this method, we can provide more codebooks associated with this sequence \( \tilde{a} \). We will determine their \( I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) \) as well. Since the subset \( D \) may not be a DS or ADS in the additive group or multiplicative group of the finite field, the corresponding codebooks may be new. Moreover, note that if one chooses the whole Hadamard matrix or Fourier matrix as the transform matrix, then Yu’s constructions are just the subset construction as in [6,8,26,29,31] etc. Here the so-called subset construction means that one chooses a subset \( D \) in a group and applies the characters of the group to \( D \) to obtain the code vectors. However, if one chooses some special rows and columns of the Hadamard matrix as the transform matrix, the result of \( I_{\text{max}}(\mathcal{C}_\Phi(\tilde{a})) \) may be slightly better than one uses the whole Hadamard matrix as the transform matrix.
The rest of this paper is organized as follows: In Section 2, we provide some notations and preliminaries which are needed in our discussion. We present four classes of near optimal codebooks associated with a binary sequences in Section 3. We also show that one can choose some particular transform matrices neither the Fourier matrices nor the Hadamard matrices and give more classes of near optimal codebooks. In Section 4, we give some conclusion remarks.

2. Notations and preliminaries

We always assume that $p$ is a prime number, $m$ a positive integer, and $q = p^m$. Let $\mathbb{F}_q$ be the finite field of $q$ elements. If $d$ is a positive divisor of $m$, then the trace mapping from $\mathbb{F}_{p^m}$ to $\mathbb{F}_{p^d}$ is defined as

$$\text{tr}_d^m(x) = x + x^{p^d} + x^{p^2d} + \cdots + x^{p^{m-d}}$$

for all $x \in \mathbb{F}_q$.

It is well known that

$$\text{tr}_1^m(x) = \text{tr}_1^d(\text{tr}_d^m(x)) \text{ for all } x \in \mathbb{F}_q,$$

see e.g. [19]. Let $\zeta_p$ be a primitive $p$th root of unity in $\mathbb{C}$. Then for every $a \in \mathbb{F}_q$, the mapping

$$\chi_a(x) = \zeta_p^{\text{tr}_1^m(ax)} \text{ for all } x \in \mathbb{F}_q$$

is an additive character of $\mathbb{F}_q$. All additive characters of $\mathbb{F}_q$ form a group, called the additive character group and denoted by $\hat{\mathbb{F}_q}$, which is isomorphic to the additive group of the finite field $\mathbb{F}_q$. We denote $\chi_1$ simply by $\chi$ in the sequel. We denote by $\mathbb{F}_q^*$ the set of all nonzero elements in $\mathbb{F}_q$. Note that $\mathbb{F}_q^*$ is a cyclic group and each generator is called a primitive element of $\mathbb{F}_q$. Let $\xi$ be a primitive element of $\mathbb{F}_q$.

For any positive integer $n$, let $\zeta_n$ be a primitive $n$th root of unity in $\mathbb{C}$. Then all the multiplicative characters of $\mathbb{F}_q$ are given by $\varphi_i$, $0 \leq i \leq q-2$, defined by

$$\varphi_i(\xi^j) = \zeta_q^{ij}, \text{ for all } 0 \leq j \leq q-2.$$

The group of all multiplicative characters of $\mathbb{F}_q$ is denoted by $\hat{\mathbb{F}_q}$, which is isomorphic to $\mathbb{F}_q^*$. Obviously, $\varphi_0$ is the principal multiplicative character of $\mathbb{F}_q$, if $\varphi^2 = \varphi_0$, then $\varphi$ is called a quadratic character. For more details, see e.g. [19] for details.

Let $f(x)$ be a function from $\mathbb{F}_q$ to $\mathbb{F}_p$. The Hadamard transform of $f(x)$ is defined by

$$W_f(a) = \sum_{x \in \mathbb{F}_q} \zeta_p^{f(x) - \text{tr}_1^m(ax)}, \ a \in \mathbb{F}_q.$$

If $p = 2$, then $\zeta_2 = -1$. Any function $f(x)$ from $\mathbb{F}_{2^m}$ to $\mathbb{F}_2$ is called a Boolean function. A Boolean function $f$ is called bent (resp. semi-bent) if $W_f(a) = \pm 2^{m/2}$ (resp. $W_f(a) \in \{0, \pm 2^{(m+1)/2}\}$) for all $a \in \mathbb{F}_{2^m}$, here $|x|$ is the floor function which is the biggest integer less than or equal to $x$. Of course, a bent function exists only when $m$ is even. If $p > 2$ and $|W_f(a)| = p^{m/2}$ for every $a \in \mathbb{F}_q$, then $f$ is called a generalized bent function. For more details on (semi-)bent functions and generalized bent functions, please refer to [12][13][20][21] etc.

For every $\varphi \in \hat{\mathbb{F}_q^*}$, the Gaussian sum is defined by

$$G_q(\varphi, \chi) = \sum_{x \in \mathbb{F}_q^*} \varphi(x)\chi(x).$$

Note that $G_q(\varphi_0, \chi) = -1$ and $|G_q(\varphi, \chi)| = \sqrt{q}$ for all nontrivial $\varphi$. We refer to [19] for some basic properties of Gaussian sums.
Let $q = 2^m$. Consider a character sum 

$$C_m^{(k)}(1, \gamma) = \sum_{x \in \mathbb{F}_q} \chi(x^{2^k + 1} + \lambda x),$$

where $k$ is a positive integer with $\gcd(k, m) = 1$. Lahtonen, McGuire, and Ward [17] give the following result:

$$(3) \quad C_m^{(k)}(1, 1) = \left(\frac{2}{m}\right) 2^{(m+1)/2} = \begin{cases} 2^{(m+1)/2}, & \text{if } m \equiv \pm 1 \pmod{8}; \\ -2^{(m+1)/2}, & \text{if } m \equiv \pm 3 \pmod{8}, \end{cases}$$

where $\left(\frac{2}{m}\right)$ is the Jacobi symbol. From [3], there exists $h_1, h_2 \in \mathbb{F}_q$ such that 

$$\chi(h_1^{2^k + 1} + h_1) = 1 \quad \text{and} \quad \chi(h_2^{2^k + 1} + h_2) = -1$$

simultaneously. Also by (3), one can obtain the following result (see [3]).

**Lemma 2.1.** Suppose that $m$ is odd and $\gcd(k, m) = 1$, then

1. $C_m^{(k)}(1, \gamma) = 0$ if and only if $tr_1^m(\gamma) = 0$;
2. if $tr_1^m(\gamma) = 1$, then there is an $h \in \mathbb{F}_{2^m}$ such that $\gamma = h^{2^k} + h^{2^m-k} + 1$ and 

$$(4) \quad C_m^{(k)}(1, \gamma) = \chi(h^{2^k + 1} + h)C_m^{(k)}(1, 1) = \chi(h^{2^k + 1} + h) \left(\frac{2}{m}\right) 2^{(m+1)/2}.$$ 

As a consequence, one has 

$$|C_m^{(k)}(a, b)| \leq \sqrt{2q}.$$ 

For $a, b \in \mathbb{F}_{p^m}$, the Kloosterman sum $K_m(a, b)$ is defined by

$$(5) \quad K_m(a, b) = \sum_{x \in \mathbb{F}_{p^m}} \chi(ax + bx^{-1}),$$

here and after, $0^{-1}$ is defined as 0. Note that $K_m(a, b) = K_m(1, ab) = K_m(ab, 1)$ for $ab \neq 0$, and thus we denote $K_m(1, a)$ by $K_m(a)$ for convenience. It is well known that for every $a, b \in \mathbb{F}_q$,

$$|K_m(a, b)| \leq 2\sqrt{q},$$

see Chapter 5 of [19] and [16] for details.

3. **Near-optimal codebooks from binary sequences defined by a subset of a finite field**

Since the support of any subset $D$ of a finite field $\mathbb{F}_q$ is a binary sequence, we can use it to construct codebooks by Yu’s Construction. In this section, we will present four constructions. One of them is a partial Fourier codebook and two of them are partial Hadamard codebooks, while in the final construction, we consider the support of the Cartesian product of two sets constructed from finite fields. Our first construction is a partial Fourier codebook associated with a binary sequence defined by a subset of a finite field.

**Theorem 3.1** (Construction 1). Let $q = p^m$ with $m$ a positive integer and $\mathbb{F}_q$ be the finite field of $q$ elements. Let $\lambda \in \mathbb{F}_q$ such that $\lambda^p - \lambda = 1$. Define the set 

$$D = \{x \in \mathbb{F}_q^* : \text{tr}(x) = \text{tr}(\lambda x) = 0\}.$$ 

Let $\alpha$ be a primitive element of $\mathbb{F}_q$. Define $\tilde{a} = (a_0, a_1, \cdots, a_{q-2})$ to be the binary sequence of period $N = q - 1$ which is the characteristic sequence of $D$, i.e., $a_i = 1$ if $\alpha^i \in D$ and 0 otherwise. Let $\Phi$ be the $N \times N$ IDFT matrix with entries of $\phi_{i,n} = e^{2\pi i n \frac{n}{N}}$ for $0 \leq i, n \leq N - 1$. Let $\{d_0, d_1, \cdots, d_{K-1}\}$ be the support of $\tilde{a}$ and 

$$c_n = \frac{1}{\sqrt{K}}(\phi_{d_0,n}, \phi_{d_1,n}, \cdots, \phi_{d_{K-1},n})^T, 0 \leq n \leq N - 1.$$
The codebook $C_\Phi(\tilde{a})$ is given by

$$C_\Phi(\tilde{a}) = \{c_0, c_1, \cdots, c_{N-1}\}.$$ 

Then $C_\Phi(\tilde{a})$ is an $(N, K)$ codebook with $N = q-1$ and $K = q/p^2 - 1$. Moreover, we have

$$I_{\max}(C_\Phi(\tilde{a})) = \frac{p^2 - 1}{\sqrt{q} - p^2/\sqrt{q}}.$$ 

Proof. First of all, we compute the cardinality of $D$. It is obvious that

$$|D| = \frac{1}{p^2} \sum_{x \in \mathbb{F}_q^*} \sum_{i=0}^{p-1} \chi^i(x)(\sum_{j=0}^{p-1} \chi^j(\lambda x)) = \frac{1}{p^2} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{x \in \mathbb{F}_q^*} \chi((i + j\lambda)x) = \frac{q}{p^2} - 1.$$ 

For every $n, 1 \leq n \leq N - 1$, one has

$$\hat{a}_n = \sum_{i=0}^{N-1} (-1)^i \phi_{i,n} = \sum_{i=0}^{N-1} (1 - 2a_i) \phi_{i,n} = -2 \sum_{i=0}^{N-1} a_i \phi_{i,n}.$$ 

Since

$$\sum_{i=0}^{N-1} a_i \phi_{i,n} = \frac{1}{p^2} \sum_{k=0}^{N-1} \sum_{i=0}^{p-1} \chi^i(\alpha^k)(\sum_{j=0}^{p-1} \chi^j(\lambda \alpha^k)) e^{2\pi \sqrt{-1} k n/p},$$

we have

$$\sum_{i=0}^{N-1} \sum_{j=0}^{p-1} \sum_{x \in \mathbb{F}_q^*} \chi((i + j\lambda)x) \psi^n(x) = \frac{1}{p^2} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{x \in \mathbb{F}_q^*} \chi(i + j\lambda)x \psi^n(x).$$

where $\psi(x) = e^{\frac{2\pi \sqrt{-1} \log_2(x)}{n}}$ is a multiplicative character of $\mathbb{F}_q$ and $\log_2(x)$ is the index of $x$ in the base of the primitive element $\alpha$, i.e., $x = \alpha^{\log_2(x)}$. Thus by [10, Theorem 5.12], we have

$$(7) \quad \sum_{i=0}^{N-1} a_i \phi_{i,n} = \frac{1}{p^2} \sum_{0 \leq i, j \leq p-1 \atop i+j \neq 0} \tilde{\psi}^n(i + j\lambda)G_q(\psi^n, \chi).$$

If $j = 0$, then

$$\sum_{i=1}^{p-1} \tilde{\psi}^n(i)G_q(\psi^n, \chi) = \begin{cases} (p-1)G_q(\psi^n, \chi), & \text{if } (p-1)|n; \\ 0, & \text{otherwise.} \end{cases}$$

The remaining sum of $(7)$ becomes

$$\sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \tilde{\psi}^n(i + j\lambda)G_q(\psi^n, \chi) = \sum_{j=1}^{p-1} \tilde{\psi}^n(j) \sum_{i=0}^{p-1} \tilde{\psi}^n(i + \lambda)G_q(\psi^n, \chi) = (p-1)(\sum_{i=0}^{p-1} \tilde{\psi}^{np}(\lambda))G_q(\psi^n, \chi).$$
Combining together, we have
\[
\sum_{i=0}^{N-1} a_i \phi_{i,n} = \frac{1}{p^2} (p - 1) (1 + \sum_{i=0}^{p-1} \psi^{np^i} (\lambda)) G_q (\psi^n, \chi).
\]
If \(\psi^n (\lambda) = 1\) (i.e., the multiplicative order of \(\lambda\) divides \(n\)), then \(\psi^{np^i} (\lambda) = 1\) for all \(0 \leq i \leq p - 1\). So, the maximal value for \(\sum_{i=0}^{N-1} a_i \phi_{i,n}\) is
\[
\left| \sum_{i=0}^{N-1} a_i \phi_{i,n} \right| = \frac{p^2 - 1}{p^2} \left| G_q (\psi^n, \chi) \right| = \frac{(p^2 - 1) \sqrt{q}}{p^2}.
\]
By Lemma 1.1, we have
\[
I_{\text{max}} (C_\Phi (\tilde{\alpha})) = \frac{1}{2K} \cdot \max_{1 \leq n \leq N-1} \left| \tilde{\alpha}_n \right| = \frac{1}{K} \max_{1 \leq n \leq N-1} \left| \sum_{i=0}^{N-1} a_i \psi_{i,n} \right| = \frac{p^2 - 1}{\sqrt{q} - p^2/\sqrt{q}}.
\]
This completes the proof. \(\square\)

**Remark 1.**
1. **Comparison to the Welch bound:** In the above construction, the corresponding Welch bound is \(I_{\text{Welch}} = \sqrt{\frac{(p^2 - 1) q - p^2}{q(q - 2)}}\) and
   \[I_{\text{max}} (C_\Phi (\tilde{\alpha})) - I_{\text{Welch}} \to 0 \quad (\text{as } q \to \infty),\]
   and
   \[1 < \frac{I_{\text{max}} (C_\Phi (\tilde{\alpha}))}{I_{\text{Welch}}} \to \sqrt{p^2 - 1} \quad (\text{as } q \to \infty).\]
2. Let \(p = 2\). We have \(I_{\text{max}} (C_\Phi (\tilde{\alpha})) = \frac{3}{\sqrt{q} - 4/\sqrt{q}}\) from above, which is the best result in this construction in the sense \(\frac{I_{\text{max}} (C_\Phi (\tilde{\alpha}))}{I_{\text{Welch}}} \to \sqrt{\frac{3}{2}}\) (as \(q \to \infty\)). In fact, if we take
   \[D = \{ x \in \mathbb{F}_q : \text{tr} (x) = 1 = \text{tr} (\lambda x) \},\]
   then we have \(I_{\text{max}} (C_\Phi (\tilde{\alpha})) = \frac{2}{\sqrt{q}}\) and so \(\frac{I_{\text{max}} (C_\Phi (\tilde{\alpha}))}{I_{\text{Welch}}} \to \frac{2}{\sqrt{3}}\) (as \(q \to \infty\)), which is a little better than the results from Construction 1.
3. The above defining set \(D\) is neither a DS nor an ADS in \(\mathbb{F}_q^*\) in general.
4. Moreover, such kind of \(D\) is easy to be defined, we hope that there are some more optimal or near-optimal codebooks along them.
5. If we define
   \[C_H (\tilde{\alpha}) = \{ \epsilon_n : n = -\infty, \text{ or } 0 \leq n \leq q - 2 \} \cup \{ \epsilon_1, \ldots, \epsilon_K \},\]
   then \(C_H (\tilde{\alpha})\) is an \((N', K')\) codebook, where
   \[N' = q - 1 + q/p^2 - 1, K' = K = q/p^2 - 1\]
   and
   \[
   I_{\text{max}} (C_\Phi (\tilde{\alpha})) = \max \left\{ \frac{p^2 - 1}{\sqrt{q} - p^2/\sqrt{q}}, \frac{1}{\sqrt{K}} \right\} = \frac{p^2 - 1}{\sqrt{q} - p^2/\sqrt{q}}.
   \]
   \[I_{\text{Welch}} = \sqrt{\frac{q - 1}{(q/p^2 - 1)(q + q/p^2 - 3)}}.\]
Thus when \(q \to \infty\), one has
\[
I_{\text{max}} (C_\Phi (\tilde{\alpha})) - I_{\text{Welch}} \to 0
\]
and

\[ 1 < I_{max}(C_\Phi(\tilde{a}))/I_{Welch} = \left(1 - \frac{1}{p^2}\right)^2 \frac{\sqrt{q/(q-1)}}{1 - q/p^2} \rightarrow \sqrt{1 + p^2(1 - 1/p^2)}. \]

Our second construction is a partial Hadamard codebook associated with a characteristic sequence of a subset of a finite field. Let’s recall first the Hadamard matrix as follows.

**Definition 3.2.** Let \( q = p^m \) and \( \alpha \) be a primitive element of \( \mathbb{F}_q \). Then the \( q \times q \) Hadamard matrix \( H = [h_{i,j}] \) is defined by

\[
h_{i,j} = \begin{cases} 1, & i = 0 \text{ or } j = 0; \\ \zeta_p^{\text{tr} \mathbb{F}_p(q^{i+j-2})}, & \text{otherwise.} \end{cases}
\]

Recall that, if \( q = 2^m \) and \( f(x) \) is a bent function, let \( D = \{ x \in \mathbb{F}_q : f(x) = 0 \} \). Then \( D \) is a DS in \( \mathbb{F}_q \) which can be used to construct a MWBE codebook [22]. Thus in what follows, we consider the case of non-bent functions. Firstly, if \( f(x) \) is a balanced semi-bent function, then we have the following result. Note that if \( f(x) \) is semi-bent but is not balanced, then there exists an \( a \in \mathbb{F}_q \) such that \( f(x) + \text{tr}(ax) \) is a balanced semi-bent function.

**Theorem 3.3** (Construction 2). Let \( q = 2^m \) with \( m \) a positive integer and let \( f(x) \) be a balanced semi-bent function. Define the set

\[
D = \{ x \in \mathbb{F}_q : f(x) = 0 \}.
\]

Let \( \alpha \) be a primitive element of \( \mathbb{F}_q \). Define \( \tilde{a} = (a_{-\infty}, a_0, a_1, \ldots, a_{q-2}) \) to be the binary sequence of period \( q \) which is the characteristic sequence of \( D \), i.e., \( a_{-\infty} = 1 \) and \( a_i = 1 \) if \( a^i \in D \) and 0 otherwise. Let \( \Phi \) be the \( q \times q \) partial Hadamard matrix as defined in Definition 3.2. Let \( \{d_0, d_1, \ldots, d_{K-1}\} \) be the support of \( \tilde{a} \) and

\[
c_n = \frac{1}{\sqrt{K}} (h_{d_0,n}, h_{d_1,n}, \ldots, h_{d_{K-1},n})^T, \text{ for } n = -\infty, 0, \ldots, q - 2.
\]

The codebook \( C_H(\tilde{a}) \) is given by

\[
C_H(\tilde{a}) = \{c_{-\infty}, c_0, \ldots, c_{q-2}\}.
\]

Then \( C_H(\tilde{a}) \) is an \( (q, q/2) \) codebook and

\[
I_{max}(C_H(\tilde{a})) = 2 \left( \frac{m+2}{2} \right)^m.
\]

**Proof.** First of all, since \( f(x) \) is balanced we have

\[
K = |D| = \frac{q}{2}.
\]

For finding \( I_{max}(C_H(\tilde{a})) \), note that we consider \( \sum_i a_i h_{i,j} \) with \( j \neq -\infty \) because \( I_{max}(C_H(\tilde{a})) \) is the maximal value of inner products of all pairs of distinct vectors in \( C_H(\tilde{a}) \) and \( \tilde{h}_{i,j} h_{i,j} = h_{i,-\infty} = 1 \) for \( i = -\infty, 0, \ldots, q - 2 \) if and only if \( j_1 = j_2 = -\infty \). Now, it is easy to see that

\[
\sum_i a_i h_{i,j} = \frac{1}{2} \sum_{x \in \mathbb{F}_q} (1 + (-1)^{\text{tr}(f(x))}) \chi(\beta x) = \frac{1}{2} \sum_{x \in \mathbb{F}_q} (1 + \chi(f(x))) \chi(\beta x)
\]
with $\beta = \alpha^{j-2}$. So,

\[
\sum_i a_i h_{i,j} = \frac{1}{2} W_f(\beta).
\]

By the definition of semi-bent, we have

\[
\max_{\beta \in \mathbb{F}_q^*} \left| \sum_i a_i h_{i,j} \right| = 2^{\left\lfloor \frac{m+2}{2} \right\rfloor - 1}.
\]

Therefore, Lemma 1.1 implies

\[
I_{\max}(\mathcal{C}_H(\tilde{a})) = \frac{2^{\left\lfloor \frac{m+2}{2} \right\rfloor - 1}}{q/2} = 2^{\left\lfloor \frac{m+2}{2} \right\rfloor - m}.
\]

This completes the proof. \hfill \Box

**Remark 2.** (1) Since $N = q$ and $K = q/2$ in this construction, the corresponding Welch bound is

\[
I_{Welch} = \sqrt{\frac{q - q/2}{q/2 \cdot (q - 1)}} = \frac{1}{\sqrt{2^m - 1}}.
\]

Thus,

\[
I_{\max}(\mathcal{C}_H(\tilde{a}))-I_{Welch} \to 0,
\]

and

\[
1 < \frac{I_{\max}(\mathcal{C}_H(\tilde{a})))}{I_{Welch}} = \sqrt{2^m - 1} \times 2^{\left\lfloor \frac{m+2}{2} \right\rfloor - m} \to \begin{cases} \sqrt{2}, & m \text{ is odd;} \\ 2, & m \text{ is even}, \end{cases}
\]
as $m \to \infty$. Therefore, when $m$ is odd, the ratio $I_{\max}(\mathcal{C}_H(\tilde{a})) / I_{Welch}$ is better than the ratio that when $m$ is even.

(2) Some constructions of quadratic semi-bent functions have been obtained in the literature, note that a quadratic function over $\mathbb{F}_q (q = 2^m)$ is the function of the form $f(x) = \sum_{i=0}^{m-1} c_i x^{2^i} \in \mathbb{F}_q[x]$. We just give a list of the known families of quadratic semi-bent functions on $\mathbb{F}_{2^m}$ with $m$ is odd:

- $f(x) = \text{tr}(x^{2^i+1}), \gcd(k, m) = 1$, [11].
- $f(x) = \frac{1}{2} \text{tr}(x^{2^i+1})$, [1].
- $f(x) = \text{tr}(x^{2^i+1} + x^{2^j+1} + x^{2^k+1}), 1 \leq i < j < k < (m-1)/2, i+j = k, \gcd(i, m) = \gcd(j, m) = \gcd(k, m) = 1$, [2].
- $f(x) = \frac{1}{2} \text{tr}(x^{2^i+1} + x^{2^m+1}) \in \mathbb{F}_2[x], \gcd(c(x), x^m+1) = x+1$, where $c(x) = \sum_{i=1}^{m-1} c_i x^{m-i}$, [15].
- $f(x) = \text{tr}(x^{2^i+1} + x^{2^j+1}), \gcd(m, i+j) = \gcd(m, i-j) = 1$, [15].
- $f(x) = \frac{1}{2} \text{tr}(x^{1+2^{k+i}}), \gcd(2k + rd, m) = 1$, [15].
- $f(x) = \sum_{i=1}^{(q-1)/2} \text{tr}(x^{1+2^m}) + \text{tr}(x^{1+2^n}), n = pq, 3 \not| p, p, q$, odd and $\gcd(p, q) = 1$, [9].

(3) When $m$ is even, there are plenty constructions of semi-bent functions, including some trace functions of monomials of Niho type, see [20] and the references therein.

(4) If $m$ is odd, we define

\[
\mathcal{C}_H(\tilde{a}) = \{ c_n : n = -\infty, \text{or } 0 \leq n \leq q-2 \text{ and } n \text{ even} \} \cup \{ e_1, \ldots, e_K \},
\]
where $e_i$ is the $K$-dimensional vector with the $i$-th component 1 and 0 elsewhere. Then $I_{\text{max}}(C_H(\tilde{a}))$ and the Welch bound remain invariant. However, the transformation matrix and the coodbook are different.

Using the inverse function and Kloosterman sums, one can obtain the following construction.

**Theorem 3.4** (Construction 3). Let $q = p^m$. Define the set
\[ D = \{ x \in \mathbb{F}_q : \text{tr}(x^{q-2}) = 0 \}. \]
Let $\alpha$ be a primitive element of $\mathbb{F}_q$. Define $\tilde{a} = (a_{-\infty}, a_0, a_1, \ldots, a_{q-2})$ to be the binary sequence of period $q$ which is the characteristic sequence of $D$, i.e., $a_{-\infty} = 1$ and $a_i = 1$ if $\alpha^i \in D$ and 0 otherwise. Let $\Phi$ be the $q \times q$ partial Hadamard matrix as defined in Definition 3.2. Let $\{d_0, d_1, \ldots, d_{K-1}\}$ be the support of $\tilde{a}$ and $c_n = \frac{1}{\sqrt{K}}(h_{d_0,n}, h_{d_1,n}, \ldots, h_{d_{K-1},n})^T, n = -\infty, 0, \ldots, q-2$.

The codebook $C_H(\tilde{a})$ is given by
\[ C_H(\tilde{a}) = \{ c_{-\infty}, c_0, \ldots, c_{q-2} \}. \]
Then $C_H(\tilde{a})$ is an $(N, K)$ codebook with $N = q, K = q/p$ and we have
\[ I_{\text{max}}(C_H(\tilde{a})) \leq \frac{2(p-1)}{\sqrt{q}}. \]

**Proof.** At first,
\[ |D| = \frac{1}{p} \sum_{x \in \mathbb{F}_q} \left( \sum_{i=0}^{p-1} \chi(i(x^{q-2})) \right) = \frac{q}{p}. \]
As we have seen in the proof of Construction 2, we consider $\sum_i a_i h_{i,j}$ with $j \neq -\infty$. Now
\[ \sum_{i=0}^{N-1} a_i h_{i,j} = \frac{1}{p} \sum_{x \in \mathbb{F}_q} \left( \sum_{i=0}^{p-1} \chi(i(x^{q-2})) \chi(\beta x) \right) = \frac{1}{p} \sum_{i=1}^{p-1} K_m(\beta, i) \]
with $\beta = \alpha^{j-2}$. So,
\[ \max_{\beta \in \mathbb{F}_q^*} \left| \sum_i a_i h_{i,j} \right| \leq \frac{2(p-1)\sqrt{q}}{p}. \]
Therefore, Lemma 1.1 implies
\[ I_{\text{max}}(C_H(\tilde{a})) \leq \frac{2(p-1)}{\sqrt{q}}. \]
This completes the proof. \qed

**Remark 3.** (1) In the Construction 3, we have $N = q$ and $K = q/p$, and so the corresponding Welch bound is
\[ I_{\text{Welch}} = \sqrt{\frac{q - q/p}{q/p \cdot (q-1)}} = \frac{\sqrt{p-1}}{\sqrt{q-1}}. \]
This implies
\[ 1 < \frac{I_{\text{max}}(C_H(\tilde{a}))}{I_{\text{Welch}}} < 2\sqrt{p-1}. \]
When \( p = 2 \), we have \( 1 < \frac{I_{\text{max}}(C_H(\hat{a}))}{I_{\text{Welch}}} < 2 \) which is the best situation in this construction.

(2) The function \( \text{tr}(x^{-1}) \) cannot be a bent function since a bent function cannot be balanced. The set \( D \) is neither a DS nor an ADS in \( \mathbb{F}_q \).

(3) If we define
\[
C_H(\hat{a}) = \{ c_n : n = -\infty \text{ or } , 0 \leq n \leq q - 2 \text{ and } p|n \} \cup \{ e_1, \cdots , e_K \},
\]
then we have \( N = 2q/p, K = q/p, \) and
\[
I_{\text{max}}(C_H(\hat{a})) \leq \frac{2(p - 1)}{\sqrt{q}}.
\]
The new Welch bound is
\[
I_{\text{Welch}} = \sqrt{\frac{2q/p - q/p}{q/p \cdot (2q/p - 1)}} = \frac{\sqrt{p}}{\sqrt{2q - p}}.
\]
Thus, we also have
\[
I_{\text{max}}(C_H(\hat{a})) - I_{\text{Welch}} \to 0 \text{ and } 1 < \frac{I_{\text{max}}(C_H(\hat{a}))}{I_{\text{Welch}}} < 2\sqrt{2p}.
\]

In our last construction in this section, a codebook is taken from transform matrix of direct product of two finite fields \( A = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \), and the associated sequence is the characteristic sequence \( \hat{a} \) of a subset of \( A \). Assume that the subset is \( D = D_1 \times D_2 \) and its support is \( \{ d_1, d_2, \cdots , d_K \} \) where \( d_i = (d_i^{(1)}, d_i^{(2)}) \in D_1 \times D_2 \). Trivially, if \( D_1, D_2 \) are independent, the codebook is just a direct product of two codebooks. Thus in order to find good codebooks, we will use such binary sequences that combine \( D_1 \) and \( D_2 \) together. We now present our next construction, which is inspired by \([14, 30]\).

**Theorem 3.5** (Construction 4). Let \( q_1 = p_1^{2m_1} \) and \( q_2 = p_2^{2m_2} \) be odd prime powers. Let \( \eta_1 \in \hat{\mathbb{F}}_{p_1}^* \) and \( \eta_2 \in \hat{\mathbb{F}}_{p_2}^* \) be the quadratic characters of \( \mathbb{F}_{p_1}^* \) and \( \mathbb{F}_{p_2}^* \), respectively. Moreover, let \( \alpha_1 \) be a primitive element of \( \mathbb{F}_{q_1} \) and \( \alpha_2 \) be a primitive element of \( \mathbb{F}_{q_2} \).

Define
\[
D = \{(x_1, x_2) \in \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} | \eta_1(\text{tr}_{1}^{m_1}(x_1^{m_1} + 1))\eta_2(\text{tr}_{2}^{m_2}(x_2^{m_2} + 1)) = 1\}.
\]

Denote \( N = q_1q_2 \) and \( \hat{a} = (a_0, a_1, \cdots , a_{N - 1}) \) to be the characteristic sequence of \( D \), i.e., for \( i = (i_1, i_2) \), \( a_i = 1 \) if \( (\alpha_1^{i_1}, \alpha_2^{i_2}) \in D \) and 0 otherwise. Let \( \Phi \) be the \( N \times N \) character table of the additive group of \( \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \) with entries \( \psi_{i,n} = \chi_{\alpha_1^{i_1}}(\alpha_1^{i_1})\chi_{\alpha_2^{i_2}}(\alpha_2^{i_2}) \), for \( i = (i_1, i_2) \) and \( n = (n_1, n_2) \) so that
\[
i_1, n_1 \in \{-\infty, 0, \ldots , q_1 - 2\} \text{ and } i_2, n_2 \in \{-\infty, 0, \ldots , q_2 - 2\},
\]
respectively. Let \( \{d_0, d_1, \cdots , d_{K - 1}\} \) be the support of \( \hat{a} \) and
\[
c_n = \frac{1}{\sqrt{K}}(\psi_{d_0,n}, \phi_{d_1,n}, \cdots , \phi_{d_{K - 1},n})^T.
\]
The codebook \( C_{\Phi}(\hat{a}) \) is given by
\[
C_{\Phi}(\hat{a}) = \{c_0, c_1, \cdots , c_{N - 1}\}.
\]
Then \( C_{\Phi}(\hat{a}) \) is a \((N, K)\) codebook with \( N = q_1q_2 \) and
\[
K = \frac{p_1^{m_1 - 1}p_2^{m_2 - 1}(p_1^{m_1} + 1)(p_2^{m_2} + 1)(p_1 - 1)(p_2 - 1)}{2}.
\]
Moreover, we have
\[
I_{\max}(C_{\Phi}(\alpha)) = \frac{\max\{1 + \sqrt{q}, 1 + \sqrt{q_2}\}}{(\sqrt{q_1} + 1)(\sqrt{q_2} + 1)}.
\]

For proving the theorem, we need two results on character sums from [14].

**Lemma 3.6 ([14 Lemma 3]).** For \( b \in \mathbb{F}_{p^{2m}} \), \( y \in \mathbb{F}_p^* \), it holds that
\[
\sum_{x \in \mathbb{F}_q^*} \chi(yx^{p^m+1} + bx) = -p^m \chi(-\frac{b^{p^m+1}}{y}) - 1,
\]

where the function \( E(\cdot) \) is defined by \( E(0) = 1 \) and \( E(z) = 0 \) for all \( z \neq 0 \).

We now give the proof of Theorem 3.5.

**Proof.** First of all,
\[
K = \frac{1}{2} \sum_{(x_1, x_2) \in \mathbb{F}_{p_1} \times \mathbb{F}_{p_2}} \left( \eta_1(\text{tr}_{1}^{m_1}(x_1^{m_1+1}))\eta_2(\text{tr}_{1}^{m_2}(x_2^{m_2+1})) \right)
\]
\[
= \frac{p_1^{m_1-1}p_2^{m_2-1}(p_1^{m_1} + 1)(p_2^{m_2} + 1)(p_1 - 1)(p_2 - 1)}{2}.
\]

In the remaining part of this proof, we extend \( \eta_1 \in \hat{\mathbb{F}}_{p_1} \) to \( \mathbb{F}_{p_1} \) by \( \eta_1(0) = 0 \) and \( \eta_2 \in \mathbb{F}_{p_2}^* \) to \( \mathbb{F}_{p_2} \) by \( \eta_2(0) = 0 \), respectively. Now,
\[
\sum_i a_i \psi_{i,n} = \frac{1}{2} \sum_{x_1 \in \mathbb{F}_{p_1}^*} \sum_{x_2 \in \mathbb{F}_{p_2}^*} \left( \eta_1(\text{tr}_{1}^{m_1}(x_1^{m_1+1}))\eta_2(\text{tr}_{1}^{m_2}(x_2^{m_2+1})) + 1 \right) \chi_b(x_1) \chi_b(x_2)
\]
\[
- \frac{1}{2} \sum_{x_1 \in \mathbb{F}_{p_1}^*} \sum_{x_2 \in \mathbb{F}_{p_2}^*} E(\text{tr}_{1}^{m_1}(x_1^{m_1+1})) \chi(b_1x_1) \chi(b_2x_2)
\]
\[
- \frac{1}{2} \sum_{x_1 \in \mathbb{F}_{p_1}^*} \sum_{x_2 \in \mathbb{F}_{p_2}^*} E(\text{tr}_{1}^{m_2}(x_2^{m_2+1})) \chi(b_1x_1) \chi(b_2x_2)
\]
\[
+ \frac{1}{2} \sum_{x_1 \in \mathbb{F}_{p_1}^*} \sum_{x_2 \in \mathbb{F}_{p_2}^*} E(\text{tr}_{1}^{m_1}(x_1^{m_1+1})) E(\text{tr}_{1}^{m_2}(x_2^{m_2+1})) \chi(b_1x_1) \chi(b_2x_2)
\]
\[
= : \sum_i - \sum_2 - \sum_3 + \sum_4.
\]

By [19 Eq.(5.16)], we have
\[
\eta_i(\text{tr}_{1}^{m_1}(x_i^{m_i+1})) = \frac{1}{p_i} \sum_{c_i \in \mathbb{F}_{p_i}} G(\eta_i, \tilde{\chi}_{c_i}) \chi(c_i x_i^{m_i+1}), i = 1, 2.
\]
Combining together, we obtain by Lemma 3.7, and since

\[ \sum \]

Note also that we can also have \( \eta_i \left( \text{tr}_1^{m_i}(x_i^{m_i+1}) \right) = \frac{1}{p_i} \sum_{c_i \in \mathbb{F}_{p_i}} G(\eta_i, \bar{x}_c) \chi(c_i x_i^{m_i+1}).\) Below, we denote by \( T_i \) the number \( \text{tr}_1^{m_i}(b_i^{m_i+1}).\)

**Case (1):** \( b_1 = 0, b_2 \neq 0.\) In this case,

\[
\sum_1 = \frac{q_1 - 1}{2} + \frac{1}{2p_1p_2} \sum_{c_1, c_2} G(\eta_1, \tilde{x}_1)G(\eta_2, \tilde{x}_2)
\]

\[
\sum \chi(c_1 x_1^{m_1+1})\chi(c_2 x_2^{m_2+1})\chi(b_2 x_2)
\]

\[
= \frac{q_1 - 1}{2} - \frac{(p_1^{m_1} - 1)(p_1^{m_1+1})}{2p_1p_2} \sum_{c_1, c_2 \neq 0} G(\eta_1, \tilde{x}_1)G(\eta_2, \tilde{x}_2)
\]

\[
\sum \chi(b_2 x_2 + c_2 x_2^{m_2+1})
\]

\[
= \frac{q_1 - 1}{2}
\]

since

\[
\sum_{c_1 \neq 0} \eta_1(-c_1)G(\eta_1, \chi) = 0.
\]

Meanwhile, we have

\[
\sum_2 = \frac{1}{2} \sum_{x_1 \in \mathbb{F}_{q_1}, x_2 \in \mathbb{F}_{q_2}} E(\text{tr}_1^{m_1}(x_1^{m_1+1}))\chi(b_2 x_2)
\]

\[
= -\frac{1}{2} \sum_{x_1 \in \mathbb{F}_{q_1}} E(\text{tr}_1^{m_1}(x_1^{m_1+1}))
\]

\[
= -\frac{1}{2} (p_1^{m_1} + 1)(p_1^{m_1-1} - 1),
\]

\[
\sum_3 = \frac{1}{2} \sum_{x_1 \in \mathbb{F}_{q_1}, x_2 \in \mathbb{F}_{q_2}} E(\text{tr}_1^{m_2}(x_2^{m_2+1}))\chi(b_2 x_2)
\]

\[
= \begin{cases} 
\frac{1}{2} (q_1 - 1)(-p_2^{m_2} + p_2^{m_2-1} - 1), & \text{if } T_2 = 0; \\
\frac{1}{2} (q_1 - 1)(p_2^{m_2-1} - 1), & \text{if } T_2 \neq 0,
\end{cases}
\]

by Lemma 3.7 and

\[
\sum_4 = \begin{cases} 
\frac{1}{2} (p_1^{m_1} + 1)(p_1^{m_1-1} - 1)^2, & \text{if } T_2 = 0; \\
\frac{1}{2} (p_1^{m_1} + 1)(p_1^{m_1-1} - 1)^2, & \text{if } T_2 \neq 0.
\end{cases}
\]

Combining together, we obtain

\[
\sum_i a_i \psi_{i,n} = \begin{cases} 
\frac{1}{2p_1p_2} \sqrt{p_1 q_2} (1 + \sqrt{q_1}) (p_1 - 1)(p_2 - 1), & \text{if } T_2 = 0; \\
\frac{1}{2p_1p_2} \sqrt{p_1 q_2} (1 + \sqrt{q_1}) (p_1 - 1), & \text{if } T_2 \neq 0.
\end{cases}
\]

**Case (2):** \( b_1 \neq 0, b_2 = 0.\) Using similar arguments as in Case (1), we have

\[
\sum_i a_i \psi_{i,n} = \begin{cases} 
\frac{1}{2p_1p_2} \sqrt{q_1 q_2} (1 + \sqrt{q_2}) (p_1 - 1)(p_2 - 1), & \text{if } T_1 = 0; \\
\frac{1}{2p_1p_2} \sqrt{q_1 q_2} (1 + \sqrt{q_2}) (p_2 - 1), & \text{if } T_1 \neq 0.
\end{cases}
\]
Case (3): $b_1b_2 \neq 0$. We have
\[
\sum_{1} \sum_{c_1, c_2 \neq 0} G(\eta_1, \tilde{\chi}_1) G(\eta_2, \tilde{\chi}_2) \\
\times \sum_{x_1, x_2 \neq 0} \chi(c_1 x_1^{m_1 + 1} + b_1 x_1) \chi(c_2 x_2^{m_2 + 1} + b_2 x_2).
\]
By Lemma 3.6 again, one has
\[
\sum_{c \neq 0} G(\eta, \tilde{\chi}_c) \sum_{x \in F_q^2} \chi(c x^{p^m + 1} + bx) = \sum_{c \neq 0} G(\eta, \tilde{\chi}_c) (\beta^{p^m + 1}) \chi(-\frac{b p^{m + 1}}{c})
\]
\[
= -p^m \sum_{c \neq 0} \eta(-c) G(\eta, \chi)(-\frac{b p^{m + 1}}{c})
\]
\[
= -p^m G(\eta, \chi) \sum_{c \neq 0} \eta(c) \chi(\frac{\eta(p^{m + 1})}{\eta(c)})
\]
\[
= \left\{ \begin{array}{ll}
-p^{m + 1} & \text{if } \eta(p^{m + 1}) \neq 0; \\
0 & \text{if } \eta(p^{m + 1}) = 0.
\end{array} \right.
\]
Therefore,
\[
\sum_{1} = \left\{ \begin{array}{ll}
\frac{1}{n} & \text{if } T_1 T_2 \neq 0; \\
0 & \text{otherwise.}
\end{array} \right.
\]
Meanwhile, one has
\[
\sum_{2} = -\frac{1}{2} \left\{ \begin{array}{ll}
-p^{m_1} + p^{m_1 - 1} - 1, & \text{if } T_1 = 0; \\
p^{m_1 - 1} - 1, & \text{if } T_1 \neq 0,
\end{array} \right.
\]
\[
\sum_{3} = -\frac{1}{2} \left\{ \begin{array}{ll}
-p^{m_2} + p^{m_2 - 1} - 1, & \text{if } T_2 = 0; \\
p^{m_2 - 1} - 1, & \text{if } T_2 \neq 0,
\end{array} \right.
\]
and
\[
\sum_{4} = \left\{ \begin{array}{ll}
\frac{1}{4}(p^{m_1 - 1} - 1)(p^{m_2 - 1} - 1), & \text{if } T_1 T_2 \neq 0; \\
\frac{1}{4}(p^{m_1 - 1} + p^{m_1 - 1} - 1)(p^{m_2 - 1} - 1), & \text{if } T_1 = 0, T_2 \neq 0; \\
\frac{1}{4}(p^{m_1 - 1} - 1)(-p^{m_2} + p^{m_2 - 1} - 1), & \text{if } T_1 \neq 0, T_2 = 0; \\
\frac{1}{4}(-p^{m_1} + p^{m_1 - 1} - 1)(-p^{m_2} + p^{m_2 - 1} - 1), & \text{if } T_1 = 0, T_2 = 0.
\end{array} \right.
\]
Comparing with Case (1) and (2), we find that the absolute value of $\sum_i a_i \psi_{i,n}$ in this case is less than those in Case (1) and (2). Thus,
\[
\max_{1 \leq n \leq N-1} \left| \sum_i a_i \psi_{i,n} \right| = \frac{1}{2p_1 p_2} (p_1 - 1)(p_2 - 1) \sqrt{q_1 q_2} \max \{1 + \sqrt{q_1}, 1 + \sqrt{q_2}\}.
\]
The desired results now follows.

\[\square\]

**Remark.** (1) By using Magma, we test the behavior of $I_{\max}(C_F(\bar{a}))/I_{Welch}$ in Construction 4, the ratio increase rapidly when $p_1, p_2$ is fixed and $m_1, m_2$ increase. However, when $m_1, m_2$ is fixed, then the ratio increase slowly with $p_1, p_2$ growing.

(2) Compared with the codebooks in [14 and 30], the codebooks in Construction 4 may have a longer length but the ratio $I_{\max}(C_F(\bar{a}))/I_{Welch}$ are worse than that in the above mentioned two codebooks in general.
4. Concluding remarks

The main contribution of this paper is to provide more constructions of near-optimal codebooks. We show that the support of a suitable subset of a finite field can serve as a binary sequence in Yu’s construction. Then one can get some near-optimal codebooks by using this sequence to select the corresponding rows from the transform matrix, which can be a partial Hadamard matrix or a partial Fourier matrix. We also illustrate the possibility of choosing partial rows (and/or columns) of the Hadamard matrix or Fourier matrix as the transform matrix. Moreover, we show that one can add some words to the codebooks in Yu’s constructions to get more better codebooks in general. Note that a possible way to find new optimal codebooks is to find new transform matrices. Note also that there are many matrices can be served as the transform matrix in Yu’s construction, e.g., some (not all, since they should satisfy the restriction (3) in Yu’s construction) generalized Hadamard matrices, some (not all) jacket matrices [18], etc.

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