Reversibility completes Information Balance in Quantum Measurements

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The measurement in quantum mechanics is fundamentally different from the classical counterpart as it disturbs the measured system. The inevitable disturbance induced by quantum measurements is the key ingredient of the unconditionally secure quantum information processing. Meanwhile, the belief that a quantum measurement is an irreversible process has been revised recently by a non-zero reversibility verified in weak quantum measurements. Therefore, a full quantitative analysis of information balance in quantum measurements, including the reversibility, is essential both from a fundamental and practical point of view but is still lacking. Here we establish a global information balance in general quantum measurements, by completing the full quantitative links between the three information quantities, i.e., information gain, disturbance and reversibility. The reversibility turns out to play a crucial role, filling the gap between the information gain and disturbance, for the balance of the total information. Our work opens a new information-theoretic perspective on quantum measurements as well as finds wider applications in measurement-based quantum information technologies.

Since the earliest days of quantum mechanics when Heisenberg first discussed the γ-ray microscope gedanken experiment [1], the collapse induced by quantum measurements has been at the core of quantum theory. The inevitable disturbance of quantum states by measurements constitutes the fundamental basis of the security in quantum information technologies [2–4]. A quantitative verification of the trade-off between the amount of information gain and disturbance by measurements has been one of the longstanding issues [5–16].

Meanwhile, the common belief that a quantum measurement is an irreversible process has been revised in recent years [17–32]. It turns out that a quantum measurement can be reversed with a non-zero success probability if the measurement apparatus weakly interacts with the system [17, 18]. Such a weak measurement causes a partial collapse so that the input state can be faithfully recovered by a reversing operation. The reversibility has been thus regarded as a fundamental quantity, different from the information gain and disturbance, characterizing a quantum measurement [17–22].

The reversal of quantum measurements has been studied for practical application such as quantum error corrections [23], gate operations [24, 25], and decoherence suppression [26–28], and successfully demonstrated by experiments in e.g. superconducting qubits [29], trapped ion [30] and photonic qubits [26, 28, 31, 32].

In this circumstance, one of the most important questions may be ‘how to verify the balance of information in quantum measurements.’ While, in a unitary evolution, the information conservation seems obvious within the second law of thermodynamics [33–35], the answer is not straightforward for quantum measurements (for example, the approaches based on entropy reduction have suffered from the negative gains [5–7] in spite of recent successful remedies [14, 16]). It becomes further obscure when an observer takes part in by a subsequent reaction or post-selection based on the obtained information (e.g. reversing operation) since treating the overall process as unitary transformations seems implausible [17]. Therefore, verifying the full information balance in quantum measurement and reversal process is of great importance in fundamental quantum physics, but is still lacking. This is also deeply related with measurement-based applications in quantum computations and communications [2, 36], quantum cryptography [37] and quantum controls [38].

In this letter, we establish a global information balance in general quantum measurements. We first introduce the information gain $G$, disturbance $D$, and reversibility $R$ of quantum measurements, defined to be universal (i.e., independent of the input quantum state) and have a clear operational meaning. We then derive the trade-off relations between $G$, $D$ and $R$. We show that the information gain-disturbance ($G$-$D$) relation can be tightened further by the reversibility $R$, beyond the one derived by Banaszek [10]. The relation between the disturbance and reversibility ($D$-$R$) is also derived, which can compensate the information gain-reversibility ($G$-$R$) relation [22].

To our knowledge, this is the first full quantitative links of $G$, $D$ and $R$, which offer a criterion to verify the global information balance. It turns out that the reversibility $R$ plays a vital role, filling the gap between the information gain $G$ and disturbance $D$, as the last piece of the puzzle of the total information balance. Moreover, our result quantitatively strengthens the no-cloning theorem in quantum measurement and reversal process. Since $G$, $D$ and $R$ are directly measurable quantities, the derived trade-off relations are ready to be tested in any physical systems. Our work opens a new perspective to understand the fundamentals of quantum measurements as well as provides a useful tool to explore the information content in measurement-based quantum processors.

Information gain and disturbance—A general quantum measurement can be described by a set of operators $(\hat{M}_r)r = 1, \ldots, N$, satisfying the completeness relation $\sum_{r=1}^{N}\hat{M}_r^\dagger\hat{M}_r = \hat{1}$, where $r$ indicates the measurement outcomes. Assume that the measurement is performed on an arbitrary input state $|\psi\rangle$ prepared in $d$-dimensional Hilbert space. When the measurement outcome is $r$, the input state is changed to a post-measurement state $|\psi_r\rangle = \hat{M}_r|\psi\rangle/\sqrt{p(r, \psi)}$, where $p(r, \psi) = \langle\psi|\hat{M}_r^\dagger\hat{M}_r|\psi\rangle$. The measurement operator can be generally represented by the singular value decomposition
as $\hat{M} = \hat{V}_r\hat{D}_r\hat{W}_r$, with unitary operators $\hat{W}_r$ and $\hat{V}_r$ and a diagonal matrix $\hat{D}_r$. Without loss of generality, we assume that $\hat{W}_0 = \hat{1}$ and the singular values (i.e., diagonal elements) of $\hat{D}_r = \sum_{i=0}^{d-1} \lambda_i^r |i\rangle\langle i| \hat{D}_r$ are defined in decreasing order, $\lambda_0^r \geq \lambda_1^r \geq \ldots \geq \lambda_{d-1}^r \geq 0$. The singular values are satisfying $\sum_r \sum_i (\lambda_i^r)^2 = d$ due to the completeness relation.

Let us consider a general quantum measurement process with a state estimation as illustrated in Fig. 1(a). Assume that, when the outcome is $r$, the input state can be estimated as $|\tilde{\psi}_r\rangle$ with an optimal estimation strategy. The amount of information obtained by quantum measurement is then given as the estimation fidelity, by averaging $|\langle \tilde{\psi}_r | \psi \rangle|^2$ over all input states and the measurement outcomes,

$$\hat{G}(\hat{M}) = \int d\psi \sum_{r=1}^N p(r, \psi) |\langle \tilde{\psi}_r | \psi \rangle|^2. \tag{1}$$

The information gain by a quantum measurement can be then defined as the maximum estimation fidelity (evaluated in Appendix A),

$$\hat{G}_{\text{max}}(\hat{M}) = \frac{1}{d(d+1)} \left[ d + \sum_{r=1}^N (\lambda_0^r)^2 \right], \tag{2}$$

where $\lambda_0^r$ is the largest singular values, which will be denoted in what follows by $\lambda$. It is scaled in the range $1/d \leq \lambda \leq 2/(d+1)$, where the upper bound is reached by a projection measurement and the lower bound is obtained with a unitary operation or a random guess.

The mean operation fidelity of a quantum measurement is calculated by averaging the overlap between the input and post-measurement states $|\langle \psi | \tilde{\psi}_r \rangle|^2$ as

$$\bar{F}(\hat{M}) = \int d\psi \sum_{r=1}^N p(r, \psi) |\langle \psi | \tilde{\psi}_r \rangle|^2. \tag{3}$$

Its maximum can be evaluated as (see Appendix B)

$$\bar{F}_{\text{max}}(\hat{M}) = \frac{1}{d(d+1)} \left[ d + \sum_{r=1}^N \left( \sum_{l=0}^{d-1} \lambda_l^r \right)^2 \right]. \tag{4}$$

denoted in what follows by $\bar{F}$, which is scaled in the range $2/(d+1) \leq \bar{F} \leq 1$. The upper bound is reached by a unitary

(a) Estimation $|\tilde{\psi}_r\rangle$ (b) Estimation $|\tilde{\psi}_r\rangle$

$|\psi\rangle$ $|\tilde{\psi}_r\rangle$ $|\psi\rangle$ $|\tilde{\psi}_r\rangle$ $|\tilde{\psi}_r\rangle$ $|\psi, \hat{M}_r\rangle$

FIG. 1. (a) Quantum measurement $|\hat{M}_r\rangle$ with a state estimation. The input, post-measurement, and estimated states (when the outcome is $r$) are denoted by $|\psi\rangle$, $|\tilde{\psi}_r\rangle$, and $|\tilde{\psi}_r\rangle$, respectively. (b) A reversing operation $|\tilde{\psi}_r\rangle$ selected according to $r$ is subsequently performed, yielding its output state $|\psi, \hat{M}_r\rangle$ for the outcome $l$.

operation, while the lower bound is obtained with a projection measurement. We then define the disturbance for a given quantum measurement as

$$D = 1 - \bar{F}, \tag{5}$$

scaled in the range $0 \leq D \leq (d-1)/(d+1)$.

Note that the definition of $\hat{G}$ and $D(\bar{F})$ are generally valid for arbitrary input states $\rho$, as the values of (1) and (3) with mixed input states are always smaller than $\hat{G}_{\text{max}}$ and $\bar{F}_{\text{max}}$ evaluated in the space of pure states.

Reversibility of quantum measurements—Let us then consider a quantum measurement and reversal process as illustrated in Fig. 1(b). Assume that, when the outcome is $r$, a subsequent measurement $|\hat{R}_r, l\rangle$ is performed aiming to reverse $|\hat{M}_r\rangle$. A reversing operation is a selective process dependent on the result of $|\hat{M}_r\rangle$ and observer’s choice of $|\hat{R}_r, l\rangle$, in which the observer need to post-select the success events (i.e., when the output state is exactly the same with the input). Note that it differs from the recovery (a completely positive trace preserving, CPTP) map [39, 40]. The measurement operator can be represented as $\hat{R}_r = \hat{V}_r\hat{D}_r\hat{W}_r$, with unitary operators $\hat{W}_r$ and $\hat{V}_r$ (without loss of generality we assume $\hat{V}_s = \hat{1}$) and a diagonal matrix $\hat{D}_r = \sum_{l=0}^{d-1} \lambda_l^r |l\rangle\langle l|$. The final output state is given by $|\psi, \hat{M}_r\rangle = \hat{R}_r |\hat{M}_r\rangle |\psi\rangle/\sqrt{p(r, l, \psi)}$, where $p(r, l, \psi) = |\langle \psi | \hat{M}_r |\psi\rangle|^2$.

The mean operation fidelity of the overall process can be then written by

$$\bar{F}(\hat{R}_r |\hat{M}_r\rangle) = \int d\psi \sum_{r=1}^N \sum_{l=1}^M p(r, l, \psi) |\langle \psi, \hat{M}_r |\psi\rangle|^2, \tag{6}$$

$$= \int d\psi \sum_{r=1}^N \sum_{l=1}^M |\langle \psi |\hat{R}_r |\hat{M}_r\rangle |\psi\rangle|^2,$$

which is obviously affected by the uncertainty of the two subsequent measurements, determined by $\hat{W}_r\hat{V}_r$ as $\hat{R}_r |\hat{M}_r\rangle = \hat{D}_r(\hat{W}_r\hat{V}_r) \hat{D}_r$. We are here interested in its maximum value obtained with the minimum uncertainty, $\hat{W}_r\hat{V}_r = \hat{1}$, as (see Appendix C)

$$\bar{F}_{\text{max}}(\hat{R}_r |\hat{M}_r\rangle) = \frac{1}{d(d+1)} \left[ d + \sum_{r=1}^N \sum_{l=1}^M \left( \sum_{l=0}^{d-1} \lambda_l^r \right)^2 \right], \tag{7}$$

the upper bound 1 of which is reached if and only if both $|\hat{M}_r\rangle$ and $|\hat{R}_r, l\rangle$ are unitary operations, and the lower bound $2/(d+1)$ can be obtained when $|\hat{M}_r\rangle$ is a projection measurement regardless of $|\hat{R}_r, l\rangle$.

The reversibility of quantum measurement can be defined as follows: assume that $\hat{R}_r$ for $l = 1, 2,..., s$ are associated with the success events to recover $|\psi\rangle$ s.t.

$$\hat{R}_r |\hat{M}_r |\psi\rangle = \eta_{r,l} |\psi\rangle, \tag{8}$$

with a complex variable $\eta_{r,l}$. The reversibility is defined as the maximum overall success probability,

$$\bar{R}_{\text{max}}(\hat{M}_r) = \max_{|\hat{R}_r\rangle} \int d\psi \sum_{r=1}^N \sum_{l=1}^s |\langle \psi |\hat{R}_r |\hat{M}_r |\psi\rangle|^2, \tag{9}$$
denoted in what follows by $\mathcal{R}$, obtained by averaging the probability of success reversal $\sum_{i=1}^{s} |\langle \phi | \hat{R}_{r,i} | \psi \rangle |^{2} = \sum_{i=1}^{s} |\eta_{r,i}|^{2}$ for each outcome $r$. It can be evaluated (see Appendix D) as
\[
\mathcal{R} = \sum_{r=1}^{N} (\lambda_{r}^{2})^{2},
\]
with the smallest singular values $\lambda_{r}$. Without loss of generality, we can set $s = 1$. The optimal reversing operation is then given by $\hat{R}_{r} = \lambda_{r}^{-1} \hat{D}_{r} \hat{V}_{r}^{*}$, where $\hat{D}_{r} = \sum_{i=0}^{\lambda} (\lambda_{r}^{-1})^{i} |\psi\rangle$ with nonzero $\lambda_{r}$. For example, a measurement, $\hat{M}_{r} = \sqrt{\eta_{r}} |1\rangle |1\rangle + \sqrt{1-\eta_{r}} |0\rangle |0\rangle$ and $\hat{R}_{2} = \sqrt{\eta_{r}} |0\rangle |0\rangle$ when the first and second measurement outcomes are respectively $r = 2$ and $l = 1$. The reversibility here is $\mathcal{R} = 1 - \eta$. Note that the reversibility is scaled $0 \leq \mathcal{R} \leq 1$; a unitary operation is deterministically reversible $\mathcal{R} = 1$, while a projection measurement is irreversible $\mathcal{R} = 0$.

We first derive a useful inequality between the reversibility and the overall operation fidelity as below:

**Lemma 1.** For any quantum measurement $\hat{M}_{r}$ and subsequent reversal $\hat{R}_{r,l}$,
\[
2 + (d - 1)\mathcal{R} \leq (d + 1) \tilde{F}_{\text{max}}(\hat{R}_{r,l} | \hat{M}_{r})
\]
is satisfied.

The equality can be reached if and only if the quantum measurement satisfies $\hat{v}_{i}^{*} \cdot \hat{v}_{j}^{*} = \delta_{ij} |\hat{v}_{i}|^{2}$, $\forall i \neq 1$, where $\hat{v}_{i} = (\lambda_{i}^{-1} |\lambda_{i=1}, \ldots, \lambda_{i=N} \lambda_{N} \rangle)$ for $i = 0, \ldots, d - 1$. The details of the proof of Lemma 1 and its saturation condition are in Appendix E.

**Trade-off relations**—We now have three different quantities, the information gain $\mathcal{G}$, disturbance $\mathcal{D}(\mathcal{F})$, and reversibility $\mathcal{R}$, characterizing a quantum measurement (see Fig. 2). Before presenting our main results, we introduce two trade-off relations derived previously, denoted by $\mathcal{G} - \mathcal{D}$ [10] and $\mathcal{G} - \mathcal{R}$ [22] as below:

(\mathcal{G} - \mathcal{D}: Information gain and Disturbance trade-off) A trade-off relation between the information gain $\mathcal{G}$ and disturbance $\mathcal{D} = 1 - \mathcal{F}$ was derived in Ref. [10] as,
\[
\sqrt{\mathcal{F} - \frac{1}{d + 1}} \leq \sqrt{\mathcal{G} - \frac{1}{d + 1} + \sqrt{(d - 1)\left(\frac{2}{d + 1} - \mathcal{G}\right)}},
\]
quantitatively showing that ‘the more information is obtained by quantum measurement, the more the quantum state is disturbed.’ Numerous relevant fundamentals and practical applications have been studied so far based on (12) [13, 32, 41].

(\mathcal{G} - \mathcal{R}: Information gain and Reversibility trade-off) A trade-off relation between the information gain $\mathcal{G}$ and reversibility $\mathcal{R}$ was derived in Ref. [22] as,
\[
d(d + 1)\mathcal{G} + (d - 1)\mathcal{R} \leq 2d, \quad (13)
\]
which was the first information-theoretic approach introducing the role of the reversibility in quantum measurements. It implies that ‘the more information is obtained by quantum measurement, the less reversible the quantum measurement is.’

We now move to our main results, aiming to complete the total information balance in quantum measurements. Let us first derive a quantitative relation $\mathcal{G} - \mathcal{D} - \mathcal{R}$ as below:

**Theorem 1.** (\mathcal{G} - \mathcal{D} - \mathcal{R}: Tight information gain and Disturbance trade-off with Reversibility) The information gain $\mathcal{G}$, disturbance $\mathcal{D}(\mathcal{F})$, and reversibility $\mathcal{R}$ of quantum measurements satisfy
\[
\sqrt{\mathcal{F} - \frac{1}{d + 1}} \leq \sqrt{\mathcal{G} - \frac{1}{d + 1} + \sqrt{\mathcal{D} \left(\frac{\mathcal{R}}{d(d + 1)}\right)}}, \quad (14)
\]
\[
+ \sqrt{(d - 2)\left(\frac{2}{d + 1} - \mathcal{G} - \mathcal{D} \frac{\mathcal{R}}{d(d + 1)}\right)}.
\]
The details of the proof are in Appendix F. It determines the upper bound of $\mathcal{F}$ or equivalently the lower bound of $\mathcal{D}$ with respect to $\mathcal{G}$ and $\mathcal{R}$. The derived $\mathcal{G} - \mathcal{D} - \mathcal{R}$ relation is fundamentally different from the relation $\mathcal{G} - \mathcal{D}$ as well as $\mathcal{G} - \mathcal{R}$, and provides a tighter bound than $\mathcal{G} - \mathcal{D}$ (or equivalent when $d = 2$). This becomes clear with its saturation condition described below:

($\mathcal{G} - \mathcal{D} - \mathcal{R}$ saturation condition) The $\mathcal{G} - \mathcal{D} - \mathcal{R}$ inequality (14) is saturated if and only if the quantum measurement satisfies following conditions: all $\hat{v}_{i}$ for $i = 0, \ldots, d - 1$ are collinear and $|\hat{v}_{i}| = \ldots = |\hat{v}_{d-2}|$, where $\hat{v}_{i} = (\lambda_{i}^{-1}, \ldots, \lambda_{i}^{N})$. Details are in Appendix F.

Denote by $S_{\mathcal{G} - \mathcal{D} - \mathcal{R}}$ and $S_{\mathcal{G} - \mathcal{D}}$ the sets of quantum measurements saturating $\mathcal{G} - \mathcal{D} - \mathcal{R}$ and $\mathcal{G} - \mathcal{D}$ (see [10] for the condition), respectively. Notably, $S_{\mathcal{G} - \mathcal{D} - \mathcal{R}} \supset S_{\mathcal{G} - \mathcal{D}}$, i.e., quantum measurements saturating $\mathcal{G} - \mathcal{D}$ also saturate $\mathcal{G} - \mathcal{D} - \mathcal{R}$, but the converse is not always true. It indicates that the information balance of more general quantum measurements can be tightly characterized by $\mathcal{G} - \mathcal{D} - \mathcal{R}$ than $\mathcal{G} - \mathcal{D}$. It is also analytically shown that the right hand side of the inequality (14) is always lower than or equal to the right hand side of the inequality (12) (see Appendix G), which guarantees that $\mathcal{G} - \mathcal{D} - \mathcal{R}$ tightens $\mathcal{G} - \mathcal{D}$.

We then introduce another useful inequality as below:

**Lemma 2.** The overall operation fidelity of quantum measurement $\hat{M}_{r}$ and reversal $\hat{R}_{r,l}$ is upper bounded by the operation fidelity of $\hat{M}_{r}$, i.e.,
\[
\tilde{F}_{\text{max}}(\hat{R}_{r,l} | \hat{M}_{r}) \leq \tilde{F}_{\text{max}}(\hat{M}_{r}).
\]
The proof is in Appendix H. It shows that the disturbance in quantum measurements never decreases by any subsequent measurement reversal. By the definition of the optimal reversing operation, the inequality is valid for any subsequent measurement \( \{ M_{i,j} \} \) after \( \{ M_{i} \} \) s.t. \( \mathcal{F}_{\text{max}}(M_{i,j} | M_{i}) \leq \mathcal{F}_{\text{max}}(R_{i,j} | M_{i}) \). This is intuitively plausible by the second law of thermodynamics. It indicates the non-increasing of the average overlap of the output state with the input by reversing operations, so that it differs from but may be fundamentally related with the data processing inequality [34]. The equality holds for unitary operations or projection measurements.

We can then derive a trade-off relation between the disturbance and reversibility as below:

**Theorem 2.** \((D\cup R)\): Disturbance and Reversibility trade-off

The disturbance \( D \) and reversibility \( R \) of quantum measurements satisfy

\[
(d - 1)R + (d + 1)D \leq d - 1. \tag{16}
\]

**Proof.** From Lemma 1 and Lemma 2, \((d - 1)R \leq (d + 1)\mathcal{F}_{\text{max}}(R_{i,j} | M_{i}) - 2 \leq (d + 1)\mathcal{F} - 2\). By \( D = 1 - \mathcal{F} \), we can derive the inequality (16). \( \square \)

It compensates other trade-off relations, determining the upper bound of \( R \) by \( D \). It implies that ‘the more disturbance induced by a quantum measurement, the less reversible the measurement is’. The equality holds for projection measurements or unitary operations. Now, we have full quantitative links between the information gain, disturbance, and reversibility of quantum measurements as illustrated in Fig. 2.

**Information balance**—A quantum measurement can be optimized to extract more information \( G \) from the input state. Such an optimization inevitably increases the disturbance \( D \) (i.e., decreases \( \mathcal{F} \)) from the relation \( G \cup D \). The operation fidelity \( \mathcal{F} \) can be interpreted here as the remaining information about the input state in the post-measurement state. Meanwhile, there exist a hidden part of information, by which the input state can be recovered by a subsequent reversing operation at the output state, the amount of which is associated with \( R \). It turns out that \( R \) fills the gap between \( G \) and \( D \), and tightens further beyond \( G \cup D \). The upper bounds of \( G \cup D \cup R \) are determined by the trade-off relations as plotted in Fig. 3, showing that the three quantities are balanced within the total quantum information.

Consider some examples of quantum measurements:

1. First, assume that a projection measurement \( \hat{P} = |i\rangle\langle i| \) is performed on arbitrary \( d \)-dimensional quantum states. It allows one to obtain the maximum information \( G = 2/(d + 1) \). Therefore, neither of any information remains at the post-measurement state \( \mathcal{F} = 2/(d + 1) \) nor is recoverable \( R = 0 \). It is straightforward to see that these quantities saturate all the trade-off relations.

2. Consider a weak quantum measurement with operators \( M_1 = \sqrt{p} |1\rangle\langle 1| + |2\rangle\langle 2| \) and \( M_2 = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| \), performed on an arbitrary quantum state \( |\psi\rangle \) in 3-dimensional Hilbert space. While the input state \( |\psi\rangle \) is completely collapsed on \( |1\rangle \) when the outcome \( r = 1 \), it is partially collapsed when \( r = 2 \) so that the measurement is reversible (for \( p < 1 \)). The optimal reversing operators are \( R_{1,1} = \sqrt{1 - p}|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| \) and \( R_{2,2} = \sqrt{p}|0\rangle\langle 0| + |2\rangle\langle 2| \). The amount of obtained information is \( G = (4 + p)/12 \), while the remaining and reversible ones are \( \mathcal{F} = (2 + \sqrt{1 - p})/3 \) and \( R = 1 - p \), respectively. These quantities satisfy all the trade-off relations (but do not saturate any of them).

3. Consider a quantum measurement \( \hat{M}_i = \sqrt{p}|i\rangle\langle i| + \sqrt{(1 - p)/2}(|i\rangle\langle i| - |0\rangle\langle 0|) \), for \( i = 0, 1, 2 \) for \( 1/3 \leq p \leq 1 \). It becomes a projection when \( p = 1 \) and a unitary operator when \( p = 1/3 \). It causes a partial collapse of the input state and can be reversed for \( 1/3 < p < 1 \) (its optimal reversing operators are given in Appendix J). For this, we can obtain \( G = (1 + p)/4 \), \( \mathcal{F} = (3 - p + 2\sqrt{2p(1 - p)})/4 \), and \( R = 3(1 - p)/2 \), which are saturating \( G \cup D \cup R \) relations.

4. Consider a quantum measurement, \( \hat{M}_i = \sqrt{p}|i\rangle\langle i| + \sqrt{(1 - p)/3}(|i\rangle\langle i| + |1\rangle\langle 1| + |2\rangle\langle 2|) \), for \( p = 0, 1, 2 \) for \( 2/5 \leq p \leq 1 \), where \( |i\rangle \equiv |i \mod 3 \rangle \) in 3-dimensional Hilbert space. The optimal reversing operation is given in Appendix J. We can obtain \( G = (1 + p)/4 \), \( \mathcal{F} = (3 + \sqrt{2(1 - p)} + (\sqrt{3} + \sqrt{6})\sqrt{p(1 - p)})/6 \), and \( R = 1 - p \). Notably, these saturate \( G \cup D \cup R \) relation, i.e., \( \{ M_i \} \in S_{G \cup D \cup R} \) but not \( G \cup D \), i.e., \( \{ M_i \} \notin S_{G \cup D} \), showing that \( S_{G \cup D \cup R} \supset S_{G \cup D} \).

Finally, we note that there would exist some missing part of the information, accounted by none of \( G \), \( \mathcal{F} \), nor \( R \). Such a missing part may be due to either the non-optimality of the quantum measurement or the ignorance in the estimation. In this context, the optimal quantum measurement can be defined as the measurement saturating \( G \cup D \cup R \) relation without any unaccounted part of the total information. This generalizes further the optimality based on the minimal disturbance with \( G \cup D \) relation [32, 41].

**Remarks**—Our result guarantees that the total information does not increase in quantum measurements and reversals so that it can be interpreted as a quantitative refinement of the no-cloning theorem. In addition, it allows us to generalize the universal cloning machine, which has been analyzed so far as...
either a deterministic process with a lower fidelity [42–47] or a probabilistic one for exact cloning [48, 49]. For example, consider a $1 \rightarrow N + 1$ asymmetric cloning machine combined with a reversing operation, which works as

$$|\psi\rangle \rightarrow \eta_r,\ell |\tilde{\psi}_r\rangle \otimes |\ell\rangle,$$

with a success probability $|\eta_r,\ell|^2$ for outcomes $r$ and $\ell$. It becomes equivalent with the optimal measurement and reversal process when $N \rightarrow \infty$ as illustrated in Fig. 4. Compared to the deterministic version [45–47], our result provides a way to optimize the cloning process further with enhanced fidelities [50–54] by sacrificing the success probability. Its quantitative upper bounds of the performance are determined by the trade-off relations between $G$, $D$, and $R$.

The total information originally contained in the input state is generally transferred by a quantum measurement to the i) obtained information $G$, ii) remaining information (at the post-measurement state) $F$, and iii) reversible information $R$. Our result shows that the three quantities are balanced by the trade-off relations. The reversibility $R$ of quantum measurements turns out to play an important role to complete the total information balance, filling the gap between the information gain $G$ and disturbance $D = 1 - F$. Note that the three quantities are defined to be universal (i.e., independent of the input state) and directly measurable with clear operational meaning, fulfilling a general requirement of the information content in quantum measurements [16]. The derived trade-off relations are thus generally applicable for any quantum measurement process and ready to be tested experimentally in any physical systems.

An important path for further research is to analyze the effects of noise in quantum measurements. The information flow in sequential quantum measurements would be also an interesting next step of research, in which the uncertainty relation between measurements may be crucial. It may be also valuable to translate our result into the context of quantum thermodynamics [55, 56]. We believe that our work not only enhances our understanding on the information-theoretic nature of quantum measurements but also has the potential for wider application in quantum information technologies.

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A. Information Gain

For given \{\hat{M}_r\}, the fidelity between the input and estimated states \(|\psi(\hat{r}\psi)\rangle^2\) can be averaged over all possible input states and measurement outcomes \(r\) as

\[
\bar{G}(\hat{M}_r) = \int d\psi \frac{1}{N} \sum_{s=1}^{N} p(r, \psi)(\overline{\psi}_r, \psi)^2. \tag{A1}
\]

Let us introduce the Schur’s lemma, which can be used for any operator \(\tilde{O}\) in \(d \times d\) Hilbert space;

\[
\int_{G} dg (\tilde{U}_g \otimes \tilde{U}_g) \tilde{O} \left[ \hat{U}_g \otimes \hat{U}_g \right] = \alpha_1 \hat{1} \otimes \hat{1} + \alpha_2 \hat{S}, \tag{A2}
\]

\[
\alpha_1 = \frac{d^2 \text{Tr} [\tilde{O} - \text{Tr} (\hat{S}) \hat{S}]}{d^2 - 1}, \quad \alpha_2 = \frac{d^2 \text{Tr} [\hat{S} \tilde{O}] - d \text{Tr} (\hat{O})}{d^2 - 1},
\]

where \(\hat{U}_g\) is a unitary representation of \(d\)-dimensional unitary group \(G = U(d)\) such that \(\int_{G} dg = 1\) and \(\hat{S}\) is a swap operator defined as \(\hat{S}|i\otimes |j\rangle = |j\otimes |i\rangle\). Then, the Eq. (A1) is written by

\[
\sum_{r=1}^{N} \int d\psi \langle \psi| \hat{M}_r|\hat{M}_r|\overline{\psi}_r\rangle \langle \overline{\psi}_r| \psi\rangle = \frac{1}{(d+1)} \left[ d + \sum_{r=1}^{N} |\overline{\psi}_r\rangle^2 \right], \tag{A3}
\]

where the second term can be written again by \(\langle \overline{\psi}_r| \hat{M}_r|\hat{M}_r|\overline{\psi}_r\rangle = \langle \overline{\psi}_r| \hat{D}_r^j \hat{V}_r \hat{D}_r^i |\overline{\psi}_r\rangle = \sum_i \langle \overline{\psi}_r| \hat{D}_r^j |i\rangle \langle i| \hat{D}_r^i |\overline{\psi}_r\rangle = \sum_i (\mathcal{A}_r^j)^2 |\langle \overline{\psi}_r|\rangle|^2\). Then, the information gain can be defined as its maximum value obtained when the estimate state is \(|\overline{\psi}_r\rangle = |0\rangle\) for outcome \(r\) so that

\[
G_r \equiv \bar{G}_{\text{max}}(\hat{M}_r) = \frac{1}{(d+1)} \left[ d + \sum_{i=1}^{N} (\mathcal{A}_r^j)^2 \right]. \tag{A4}
\]

Note that this is valid for arbitrary input states \(\hat{p}\), since the maximum value is always obtained in the space of pure states.

B. Maximum Operation Fidelity

The fidelity between the input and output states \(|\langle \psi_1|\psi\rangle|^2\), for a given quantum measurement \(|\hat{M}_r\rangle\), can be averaged over all possible input states and measurement outcomes \(r\) as

\[
\bar{F}(\hat{M}_r) = \int d\psi \sum_{r=1}^{N} p(r, \psi)(\overline{\psi}_r, \psi)^2 \tag{A5}
\]

\[
= \int d\psi \sum_{r=1}^{N} p(r, \psi) \langle \psi| \hat{M}_r|\hat{M}_r|\overline{\psi}_r\rangle \langle \overline{\psi}_r| \psi\rangle / P(r, \psi)
\]

\[
= \int d\psi \sum_{r=1}^{N} \langle \psi| \hat{M}_r|\overline{\psi}_r\rangle^2.
\]

By the Schur’s lemma, we can rewrite it as

\[
\bar{F}(\hat{M}_r) = \frac{1}{d(d+1)} \left[ d + \sum_{r=1}^{N} |\text{Tr} \hat{M}_r|^2 \right]. \tag{A6}
\]

From \(|\text{Tr} \hat{M}_r| = |\sum_i (\hat{D}_r^j \hat{V}_r \hat{D}_r^i |i\rangle| = |\sum_i \mathcal{A}_r^j \langle \hat{V}_r |i\rangle| \leq \sum_i \mathcal{A}_r^j \langle \hat{V}_r |i\rangle| \leq \sum_i \mathcal{A}_r^j \), the maximum fidelity is obtained as

\[
\bar{F} \equiv \bar{F}_{\text{max}}(\hat{M}_r) = \frac{1}{d(d+1)} \left[ d + \sum_{r=1}^{N} (\sum_{i=0}^{d-1} \mathcal{A}_r^j)^2 \right]. \tag{A7}
\]

by which the disturbance can be defined as \(\mathcal{D} = 1 - \bar{F}\).

C. Maximum Operation Fidelity after Reversal

For given \(|\hat{M}_r\rangle\) and \(|\hat{R}_{ij}\rangle\), the operation fidelity is

\[
\bar{F}(\hat{R}_{ij}, \hat{M}_r) = \int d\psi \sum_{r=1}^{N} \sum_{l=1}^{M} \langle \psi| \hat{R}_{ij} \hat{M}_r |\psi\rangle |\langle \hat{R}_{ij} \hat{M}_r |\psi\rangle|^2 \tag{A8}
\]

where the right hand side can be written by

\[
\int d\psi \sum_{r=1}^{N} \sum_{l=1}^{M} \langle \psi| \hat{R}_{ij} \hat{M}_r |\psi\rangle |\langle \hat{R}_{ij} \hat{M}_r |\psi\rangle|^2 = \int d\psi \sum_{r=1}^{N} \sum_{l=1}^{M} \langle \psi| \hat{R}_{ij} \hat{M}_r |\psi\rangle \langle \psi| \hat{R}_{ij} \hat{M}_r |\psi\rangle \tag{A9}
\]

From Eq. (A6), it can be written again by

\[
\bar{F}(\hat{R}_{ij}, \hat{M}_r) = \frac{1}{d(d+1)} \left[ d + \sum_{r=1}^{N} |\text{Tr} \hat{R}_{ij} \hat{M}_r|^2 \right]. \tag{A10}
\]

Since

\[
|\text{Tr} \hat{R}_{ij} \hat{M}_r| = \left| \sum_i (\hat{D}_i^j \hat{V}_i \hat{D}_i^j |i\rangle) \right| \leq \sum_i \mathcal{A}_i^j \langle \hat{V}_i |i\rangle| \leq \sum_i \mathcal{A}_i^j \langle \hat{V}_i |i\rangle|,
\]

the maximum of \(\bar{F}(\hat{R}_{ij}, \hat{M}_r)\) is obtained when \(\hat{W}_i^j = 1\). Therefore,

\[
\bar{F}_{\text{max}}(\hat{R}_{ij}, \hat{M}_r) = \frac{1}{d(d+1)} \left[ d + \sum_{i=1}^{N} \sum_{l=0}^{d-1} |\mathcal{A}_i^j|^2 \right]. \tag{A11}
\]

D. Reversibility

We assume that \(\hat{R}_{ij}\) for \(l = 1, 2, ..., s\) are associated with the success reversal events such that

\[
\hat{R}_{ij} \hat{M}_r |\psi\rangle = \eta_{ij} |\psi\rangle. \tag{A12}
\]
with a complex variable $\eta_{t,i}$. Since $\hat{\mu} - \sum_{t=1}^{d} \hat{\mu}_{t,i} \hat{\mu}_{t,i}$ is positive definite from the completeness relation,

\[
\sup_{|\phi\rangle} \langle \phi | \sum_{i=1}^{d} \hat{\mu}_{t,i} \hat{\mu}_{t,i} | \phi \rangle \leq 1
\]  

(A13)

is satisfied for arbitrary quantum state $|\phi\rangle$. Then,

\[
\sup_{|\phi\rangle} \langle \phi | \sum_{i=1}^{d} \hat{\mu}_{t,i} \hat{\mu}_{t,i} | \phi \rangle = \sup_{|\psi\rangle} \langle \psi | \sum_{i=1}^{d} \hat{\mu}_{t,i} \hat{\mu}_{t,i} | \psi \rangle
\]

\[
= \sup_{|\psi\rangle} \frac{\langle \psi | \sum_{i=1}^{d} \hat{\mu}_{t,i} \hat{\mu}_{t,i} | \psi \rangle}{p(r, \psi)}
\]

\[
= \sum_{i=1}^{d} |p_{r,i}|^2
\]

inf_{|\psi\rangle} p(r, |\psi\rangle).
\]  

(A14)

so that $\sum_{i=1}^{d} |p_{r,i}|^2 \leq \inf_{|\psi\rangle} p(r, |\psi\rangle)$ is satisfied. For an arbitrary input state $|\psi\rangle = \sum_{i=0}^{d-1} a_i |\psi_i\rangle$ where $\sum_{i=0}^{d-1} |a_i|^2 = 1$, $\inf_{|\psi\rangle} p(r, |\psi\rangle)$ is obtained when $a_{r-1} = 1$ and all other $a_i$ are zero, because the singular values of $\hat{\mu}_r$ are assumed to be defined in decreasing order, so that

\[
\sum_{i=1}^{d} |p_{r,i}|^2 \leq \inf_{|\psi\rangle} p(r, |\psi\rangle) = \left(\lambda^{d-1}_{r-1}\right)^2.
\]  

(A15)

Therefore, the reversibility can be defined as its maximum

\[
\mathcal{R} \equiv \mathcal{R}_{\text{max}}(\hat{\mu}_r) = \max_{|\psi\rangle} \int d\psi \sum_{r=1}^{d} \sum_{i=1}^{d} |\langle \psi | \hat{\mu}_{r,i} | \psi \rangle|^2
\]

\[
= \sum_{r=1}^{d} \left(\lambda^{d-1}_{r-1}\right)^2.
\]  

(A16)

**E. Proof of Lemma 1**

For optimally chosen $\{\hat{\mu}_{t,i}\}$ to reverse $\{\hat{\mu}_r\}$, the singular values are given by $\lambda^{d-1}_{r-1} = \lambda^{d-1}_{r-1}/\lambda^{d-1}_{r-1}$. If we define $\vec{v}_t = (\lambda^{d-1}_{r-1}, \lambda^{d-1}_{r-1}, \ldots, \lambda^{d-1}_{r-1}, \lambda^{d-1}_{r-1})$, the second term of $\mathcal{F}_{\text{max}}(\hat{\mu}_r, \hat{\mu}_r)$ in Eq. (A11)

\[
\sum_{r=1}^{d} \sum_{i=1}^{d} \left( \lambda^{d-1}_{r-1} \lambda^{d-1}_{r-1} \right)^2 = \sum_{i,j} \vec{u}_i \cdot \vec{u}_j
\]

\[
= \sum_{i,j} (\vec{u}_i \cdot \vec{u}_j) + \sum_{i,j} (\vec{u}_j \cdot \vec{u}_i)
\]

\[
= d^2 \sum_{i,j} \vec{u}_i \cdot \vec{u}_j,
\]  

(A17)

and $\sum_{i,j} \vec{u}_i \cdot \vec{u}_j \geq 0$, the equality is reached with a condition $\vec{u}_i \vec{u}_j = \delta_{ij} |\vec{u}|^2$. From the completeness relation of the overall measurements,

\[
\sum_{r=1}^{d} \sum_{i=1}^{d} (\lambda^{d-1}_{r-1})^2 = \sum_{r=1}^{d} \sum_{i=1}^{d} |\vec{u}|^2 = d.
\]

Therefore,

\[
\mathcal{F}_{\text{max}}(\hat{\mu}_r, \hat{\mu}_r) = \frac{1}{d(d+1)} \left[ d + d^2 \mathcal{R}_{\text{max}}(\hat{\mu}_r) + \sum_{i,j} \sum_{1 \leq i \neq j} \vec{u}_i \cdot \vec{u}_j \right]
\]

\[
\geq \frac{1}{d(d+1)} \left[ 2d + (d^2 - d) \mathcal{R}_{\text{max}}(\hat{\mu}_r), \right]
\]

and we can obtain

\[
2 + (d - 1) \mathcal{R}_{\text{max}}(\hat{\mu}_r) \leq (d + 1) \mathcal{F}_{\text{max}}(\hat{\mu}_r, \hat{\mu}_r).
\]  

(A18)

with the equality condition $\vec{u}_i \vec{u}_j = \delta_{ij} |\vec{u}|^2$. □

**F. Proof of Theorem 1**

Let us first define $\vec{v}_i = (\lambda^{d-1}_{i-1}, \ldots, \lambda^{d-1}_{N})$. Then, $g \equiv d(d + 1)/2 - d = \sum_{i,j} (\lambda^{d-1}_{i-1} \lambda^{d-1}_{j})^2 = \sum_{i,j} \vec{v}_i \cdot \vec{v}_j$, and $\mathcal{R} = \sum_{i,j} (\lambda^{d-1}_{i-1})^2 = |\vec{v}_i|^2$. The completeness relation can be written by $\sum_{i,j} (\lambda^{d-1}_{i-1})^2 = \sum_{i,j} |\vec{v}_i|^2 = d$. From the Schwarz inequality,

\[
f \leq \sum_{i,j=0}^{d-1} |\vec{v}_i| |\vec{v}_j| = \left( \sum_{i=0}^{d-1} |\vec{v}_i| \right)^2
\]

\[
= \left( \sqrt{d} + \sqrt{\mathcal{R}} + \sum_{i=1}^{d-2} |\vec{v}_i| \right)^2,
\]  

(A19)

where the equality can be reached where all the vectors $\vec{v}_i$ are collinear. Then, from the inequality of arithmetic and quadratic means,

\[
\sum_{i=1}^{d-2} |\vec{v}_i| \leq \sqrt{(d - 2) \sum_{i=1}^{d-2} |\vec{v}_i|^2},
\]  

(A20)

where the equality can be reached when $|\vec{v}_1| = \cdots = |\vec{v}_{d-2}|$. Here, the right hand side can be rewritten by the completeness relation as $\sqrt{(d - 2)(d - g - \mathcal{R})}$. Therefore, we can obtain

\[
\sqrt{f} \leq \sqrt{d} + \sqrt{d} + \sqrt{(d - 2)(d - g - \mathcal{R})},
\]

(A21)

and equivalently

\[
\sqrt{d(d+1)} \leq \sqrt{d - 1} + \sqrt{\mathcal{R}} \frac{\sqrt{d(d+1)}}{d(d+1)}
\]

\[
+ \sqrt{(d - 2)\left( \frac{2}{d+1} - \mathcal{R} \right)}\frac{\sqrt{d(d+1)}}{d(d+1)}
\]

(A22)
G. $G$-D-R tightens $G$-D

Assume that we have $d-1$ non-negative real numbers $x_i$ where $i = 0, \cdots, d-1$. From the inequality between arithmetic and quadratic mean,

$$\left(\frac{1}{d-1} \sum_{i=1}^{d-1} x_i\right)^2 \leq \frac{1}{d-1} \sum_{i=1}^{d-1} x_i^2,$$

(A23)

where the equality holds if and only if $x_1 = x_2 = \cdots = x_{d-1}$. By letting $x_1 = \sqrt{R}$ and $x_2 = \cdots = x_{d-1} = \sqrt{(d-g-R)/(d-2)}$, we can obtain

$$\sqrt{R} + \sqrt{(d-2)(d-g-R)} \leq \sqrt{(d-1)(d-g)},$$

(A24)

which indicates that $G$-D-R is tighter than $G$-D.

H. Proof of Lemma 2

From Eq. (A11), the operation fidelity for the measurement $\{\hat{M}_r\}$ and reversal $\{\hat{R}_r\}$ is given by

$$\bar{F}_{\text{max}}(\hat{R}_{r,i}\hat{M}_r) = \frac{1}{d(d+1)} \left[ d + \sum_{r=1}^{N} \sum_{i=1}^{M} \left( \sum_{i=0}^{d-1} \lambda_i \lambda_i^r \right)^2 \right],$$

where the second term can be written by

$$\sum_{i=1}^{N} \sum_{i=1}^{M} \left( \sum_{i=0}^{d-1} \lambda_i \lambda_i^r \right)^2 = \sum_{i=1}^{N} \sum_{i=1}^{M} \left( \sum_{i=1}^{d-1} \lambda_i \lambda_i^r \right)^2,$$

(A25)

by a vector defined as $\vec{a}_i = (\lambda_i \lambda_i^r \cdots \lambda_i^M)$. Then, by the Schwarz inequality,

$$\sum_{i=1}^{N} \sum_{i=1}^{M} \sum_{i=1}^{d-1} \lambda_i \lambda_i^r \lambda_i \lambda_i^r \lambda_i \lambda_i^r = \sum_{i=1}^{N} \sum_{i=1}^{M} \left( \sum_{i=1}^{d-1} \lambda_i \lambda_i^r \lambda_i \lambda_i^r \lambda_i \lambda_i^r \right)^2,$$

(A26)

by a vector defined as $\vec{a}_i = (\lambda_i^r \lambda_i^{r+1} \cdots \lambda_i^{r+M})$. Then, by the Schwarz inequality,

$$\sum_{i=1}^{N} \sum_{i=1}^{M} \left( \sum_{i=1}^{d-1} \lambda_i \lambda_i^r \lambda_i \lambda_i^r \lambda_i \lambda_i^r \right)^2 = \sum_{i=1}^{N} \sum_{i=1}^{M} \left( \sum_{i=1}^{d-1} \lambda_i \lambda_i^r \lambda_i \lambda_i^r \lambda_i \lambda_i^r \right)^2,$$

(A27)

and, from Eq. (A7),

$$\bar{F}_{\text{max}}(\hat{R}_{r,i}\hat{M}_r) \leq \bar{F}_{\text{max}}(\hat{M}_r).$$

(A28)

\[ \square \]

FIG. A1. Left: Comparison the upper bounds by $G$-D-R and $G$-D relations with the measurement in Appendix I. The bound by $G$-D-R is tighter than the one by $G$-D. Right: The gap is due to the reversibility $R$, which is also in trade-off relation with the information gain $G$.

I. Example of the tighter bound with $G$-D-R

Let us consider a quantum measurement with operators, $\hat{M}_r = \sqrt{p(i)}(i) + \sqrt{(1-p)(3-i)(i+1)(i+1) + \sqrt{p(1-p)}}(i+1)(i+2)(i+1)$ for $i = 0, 1, 2$, in the region $0.458619 \leq p \leq 1$, where the basis is $|j\rangle \equiv |i\rangle \mod 3$ in 3-dimensional Hilbert space. The information gain by this measurement is $G = (1 + p)/4$, the operation fidelity is $\mathcal{F} = 1/4 + \sqrt{\mathcal{R} + \sqrt{(1-p)(3-p) + \sqrt{p(1-p)}}}/4$, and the reversibility is $R = p(1-p)$. These saturate $G$-D-R but not $G$-D. The relation between the information gain and the reversibility can be written exactly as

$$\mathcal{R} = (4G - 1)(2 - 4G).$$

(A29)

Thus, the equality of the $G$-D-R with $d = 3$ is written by

$$\mathcal{F} = \frac{1}{4} + \left( \sqrt{G - \frac{1}{4}} + \sqrt{R} + \sqrt{\frac{G - R}{4}} \right)^2 = \frac{1}{4} + \left( \sqrt{G - \frac{1}{4}} + \sqrt{\frac{(4G - 1)(1 - 4G)}{6}} \right)^2.$$

(A30)

In Fig. A1, we plot the bound and compare with the one by $G$-D given as

$$\mathcal{F} = 1 - G + \sqrt{-1 + 6G - 8G^2}$$

(A31)

for $d = 3$. We can observe that the bound by $G$-D-R is tighter than the one by $G$-D. The gap is due to the reversibility $R$, which is also in trade-off with $G$ as plotted in Fig. A1.

J. Optimal reversing operations for the example (3) and (4)

(3) The quantum measurement with operators, $\hat{M}_r = \sqrt{p(i)}(i) + \sqrt{(1-p)(3-i)(i+1)(i+1)}$, $i = 0, 1, 2$, for $1/3 \leq p < 1$, can be reversed (for each outcome $r = i$) by an optimal reversing
operation with
\[
\hat{R}_{i,1} = \sqrt{\frac{1 - p}{2p}} |i\rangle \langle i| + (\hat{\sigma}_z - |i\rangle \langle i|)
\]
\[
\hat{R}_{i,2} = \sqrt{\frac{3p - 1}{2p}} |i\rangle \langle i|.
\]  
(A32)

Note that it succeeds when the measurement outcome of the reversing operation is \(l = 1\).

(4) The quantum measurement defined by \(\hat{M}_i = \sqrt{p} |i\rangle \langle i| + \sqrt{2(1 - p) / 3} |i + 1\rangle \langle i + 1| + \sqrt{(1 - p) / 3} |i + 2\rangle \langle i + 2|\), \(i = 0, 1, 2\) for \(2/5 \leq p \leq 1\), where \(|i\rangle \equiv |i \mod 3\rangle\) in 3-dimensional Hilbert space can be optimally reversed (for each outcome \(r = i\)) by

\[
\hat{R}_{i,1} = \sqrt{\frac{1 - p}{3p}} |i\rangle \langle i| + \frac{1}{\sqrt{2}} |i + 1\rangle \langle i + 1| + |i + 2\rangle \langle i + 2|.
\]

\[
\hat{R}_{i,2} = \sqrt{\frac{4p - 1}{3p}} |i\rangle \langle i| + \frac{1}{\sqrt{2}} |i + 1\rangle \langle i + 1|.
\]  
(A33)

The reversal succeeds when the outcome is \(l = 1\).