The $\mathcal{OSP}(2,2|16)$ superconformal theory is free!

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Abstract

The SuperConformal theory in three space-time dimensions with $SO(16)$ 
$R$-symmetry, 128 bosons, and 128 fermions, cannot sustain interactions. 
This result is obtained using both light-cone superspace techniques which 
rely on algebraic consistency, and covariant methods which rely on $SO(16)$ 
Fierz identities which fail to produce the desired algebra.

Keywords: Superspace; Light-cone; Superconformal Theories; Chern-Simons Theories.
1 Introduction

According to W. Nahm\cite{1}, there are two maximally symmetric superconformal theories in three dimensions. The first with $\text{OSp}(2,2|8) \supset \text{SO}(3,2) \times \text{SO}(8)_R$ symmetry, and eight bosons and eight fermions, has been shown by Bagger, Lambert and also Gustavsson (BLG)\cite{2} to be a non-trivial interacting theory.

The second, with $\text{OSp}(2,2|16) \supset \text{SO}(3,2) \times \text{SO}(16)_R$ symmetry, contains 128 bosons and 128 fermions, is very similar to the Bagger-Lambert and Gustavsson theory. The purpose of this letter is to show that it does not sustain interactions.

We arrive at this result in two different ways. Algebraic consistency in light-cone superspace, previously applied to the BLG theory\cite{3}, shows the impossibility to construct dynamical supersymmetry transformations which satisfy all commutation relations of this larger superalgebra.

Covariant methods lead to the same conclusion. $\text{SO}(8)_R$ triality allows the eight bosons and eight fermions of the BLG theory to span the two $\text{SO}(8)_R$ spinor representations. The larger theory then looks similar to the BLG theory (surely at free level); all one has to do is replace $\text{SO}(8)_R$ by $\text{SO}(16)_R$, with its 128 bosons and 128 fermions now spanning its two spinor representations. We start with BLG-like transformations with an auxiliary vector field, but we find that the $\text{SO}(16)$ Fierz identities lead only to trivial closure of these transformations.

2 $N=8$ Light-Cone Superspace

Theories with 128 fermions and bosons are naturally described in a superspace with eight complex Grassmann variables, $\theta^m$ and $\bar{\theta}^m$, $m = 1, \ldots, 8$. On the light-cone $(x^\pm = (x^0 \pm x^3)/\sqrt{2}, \partial^\pm = (\partial^0 \pm \partial^3)/\sqrt{2})$, define chiral derivatives,

$$d^m = -\frac{\partial}{\partial \theta^m} - \frac{i}{\sqrt{2}} \theta^m \partial^+; \quad \bar{d}_n = \frac{\partial}{\partial \bar{\theta}^n} + \frac{i}{\sqrt{2}} \bar{\theta}^n \partial^+;$$

they satisfy

$$\{d^m, \bar{d}_n\} = -i \sqrt{2} \delta^m_n \partial^+ . \quad (1)$$

Introduce the constrained chiral superfield with 256 (128 bosonic, 128 fermionic) degrees of freedom

$$\Phi(y) = \frac{1}{\partial^{+2}} h(y) + i \theta^m \frac{1}{\partial^{+2}} \bar{\psi}_m(y) + i \theta^{mn} \frac{1}{\partial^{+2}} \bar{B}_{mn}(y) - \theta^{mnp} \frac{1}{\partial^{+2}} \bar{\chi}_{mnp}(y) \quad (2)$$

where bar denotes complex conjugation, and
\[ \theta^{a_1 a_2 \ldots a_n} = \frac{1}{n!} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_n}, \quad \tilde{\theta}^{a_1 a_2 \ldots a_n} = \epsilon_{a_1 a_2 \ldots a_n b_1 b_2 \ldots b_{(8-n)}} \theta^{b_1 b_2 \ldots b_{(8-n)}}. \]

The arguments of the fields are the chiral coordinates

\[ y = (x, x^+, y^- \equiv x^- - \frac{i}{\sqrt{2}} \theta^m \tilde{\theta}_m), \]

where \( x \) is the transverse variable. \( \Phi \) and its complex conjugate \( \Phi \) satisfy the chiral constraints

\[ d^m \Phi = 0, \quad \bar{d}_m \Phi = 0, \quad \Phi = \frac{1}{4 \partial^+ \partial^-} d^1 d^2 \ldots d^8 \tilde{\Phi}. \]

The superspace measure is given by

\[ \int d^8 \theta d^8 \bar{\theta} \frac{1}{\partial^+ \partial^-} \Phi \]

The additional operators

\[ q^m = -\frac{\partial}{\partial \theta^m} + \frac{i}{\sqrt{2}} \theta^m \partial^+; \quad \bar{q}_n = \frac{\partial}{\partial \bar{\theta}^n} - \frac{i}{\sqrt{2}} \tilde{\theta}_n \partial^+ \]

satisfy

\[ \{ q^m, \bar{q}_n \} = i \sqrt{2} \delta^m_n \partial^+. \]

Since they anticommute with the chiral derivatives,

\[ \{ q^m, \bar{d}_n \} = \{ q^m, d^n \} = 0, \]

their action on the superfield do not affect its chirality. They are used to construct \( SO(16) \) transformations in terms of its \( SU(8) \times U(1) \) subgroup transformations, with parameters \( \omega^m_n \), and \( \omega \):

\[ \delta_{SU(8)} \Phi = i \omega^m_m \left( q^m \bar{q}_m - \frac{1}{8} \delta^m_m q^l \bar{q}_l \right) \frac{1}{\partial^+} \Phi; \]

\[ \delta_{SO(2)} \Phi = \frac{i \omega}{8} \left( q^m \bar{q}_m - \bar{q}_m q^m \right) \frac{1}{\partial^+} \Phi. \]

The coset transformations, with parameters \( \omega^{mn} \), and \( \omega_{mn} \), are given by,

\[ \delta_{\text{coset}} \Phi = i \omega^{mn} \bar{q}_m \bar{q}_n \frac{1}{\partial^+} \Phi; \quad \delta_{\text{coset}} \Phi = i \omega_{mn} q^m q^n \frac{1}{\partial^+} \Phi. \]
3 SuperConformal Structure

In Dirac's light front form \((x^+ = 0)\), the superconformal group generators split as

\[
\text{Conformal Group:}\begin{cases}
\text{Lorentz Group:} & J^+, J^-; J^-
\text{Translations:} & P, P^+, P^-
\text{Dilatation:} & D
\text{Conformal:} & K^+, K^-; K^-
\end{cases}
\]

\[
\text{Supers:}\begin{cases}
\text{Supersymmetry:} & q, \bar{q}; Q, \bar{Q}
\text{Superconformal:} & s, \bar{s}; S, \bar{S}
\end{cases}
\]

with the dynamical generators written in capital calligraphic letters. Note that \(J^+\) and \(K^+\) at \(x^+ = 0\), and \(R\)-symmetries (Eqs. (9-11)) are all kinematical.

With superconformal symmetries, all dynamical operators are obtained by commutation of the dynamical supersymmetry transformations with kinematical operators, so that dynamical supersymmetry transformations determine the full theory.

3.1 Kinematical Transformations

Superconformal kinematical transformations are linear in the fields, even in the interacting theory, starting with

\[
\begin{align*}
\delta P^+ \Phi^a &= -i \partial^+ \Phi^a; \\
\delta P^- \Phi^a &= -i \partial \Phi^a; \\
\delta J^+ \Phi^a &= ix \partial^+ \Phi^a; \\
\delta J^- \Phi^a &= i(\mathcal{A} + \frac{x}{2} \partial + 2) \Phi^a; \\
\delta D \Phi^a &= i(A - \frac{x}{2} \partial) \Phi^a; \\
\delta K^+ \Phi^a &= 2i x A \Phi^a; \\
\delta K^- \Phi^a &= i x^2 \partial^+ \Phi^a.
\end{align*}
\]

(12)

where \(\partial\) is the transverse derivative, and

\[
\mathcal{A} \equiv x^+ \partial^+ - \frac{x}{2} \partial - \frac{1}{2} N + \frac{3}{2}; \quad N \equiv \sum_{m=1}^{8} \left( \partial^m \frac{\partial}{\partial \theta^m} + \bar{\theta}^m \frac{\partial}{\partial \bar{\theta}^m} \right).
\]

(13)

The taxonomic index \(a\) allows for several superfields. The kinematical (spectrum generating) supersymmetries are

\[
\begin{align*}
\delta_q \Phi^a &= \epsilon^m \bar{q}_m \Phi^a; \\
\delta_{\bar{q}} \Phi^a &= \bar{s}_m q^m \Phi^a,
\end{align*}
\]

(14)

and the kinematical superconformal transformations are

\[
\begin{align*}
\delta_s \Phi^a &= -ix \epsilon^m \bar{q}_m \Phi^a; \\
\delta_{\bar{s}} \Phi^a &= ix \bar{s}_m q^m \Phi^a.
\end{align*}
\]

(15)

where \(\epsilon^m\) and \(\bar{s}_m\) are eight anticommuting parameters.
3.2 Dynamical Transformations

The free dynamical supersymmetry transformations (in boldface) are,

$$\delta_{\text{free}}^{\Phi} = \frac{1}{\sqrt{2}} \epsilon^m q_m \frac{\partial}{\partial \Phi} \Phi^a, \quad \delta_{\text{free}}^{\bar{\Phi}} = \frac{1}{\sqrt{2}} \epsilon^m \bar{q}_m \frac{\partial}{\partial \bar{\Phi}} \Phi^a. \quad (16)$$

It is easy to see that they are chiral and satisfy the inside-out constraint.

In superconformal theories, all dynamical information is contained in the dynamical supersymmetry transformations; knowing them suffices to generate all interactions. We set

$$\delta_{\text{int}}^{\Phi} = \delta_{\text{free}}^{\Phi} + \delta_{\text{int}}^{\Phi}, \quad \delta_{\text{int}}^{\bar{\Phi}} = \delta_{\text{free}}^{\bar{\Phi}} + \delta_{\text{int}}^{\bar{\Phi}}, \quad (17)$$

and proceed to determine the form of $\delta_{\text{int}}^{\Phi}$ and its conjugate. Our procedure parallels [3], except for the number of Grassmann variables.

The interacting parts must satisfy chirality, and the inside out constraint

$$d^m \left( \delta_{\text{int}}^{\Phi} \Phi^a \right) = 0, \quad \delta_{\text{int}}^{\Phi} \Phi^a = \frac{d^8}{4 \theta^++} \left( \delta_{\text{int}}^{\Phi} \Phi^a \right)^*, \quad (18)$$

Additional requirements from kinematics are:

(i) Independence from $x^-$ and $x$, since

$$[\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\bar{\Phi}}] \Phi^a = [\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\Phi}] \Phi^a = 0, \quad (19)$$

and

$$[\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\bar{\Phi}}] \Phi^a = [\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\Phi}] \Phi^a = 0. \quad (20)$$

(ii) No transverse derivatives $\partial$, since

$$[\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\bar{\Phi}}] \Phi^a = \frac{i}{\sqrt{2}} \delta_{\text{int}}^{\Phi} \Phi^a, \quad \delta_{\text{int}}^{\Phi} \Phi^a = \frac{i}{\sqrt{2}} \delta_{\text{int}}^{\Phi} \Phi^a, \quad (21)$$

lead to

$$[\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\Phi}] \Phi^a = [\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\Phi}] \Phi^a = 0. \quad (22)$$

(iii) From

$$[\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\Phi}] \Phi^a = -\epsilon_m \epsilon^m \delta_{\text{int}}^{\Phi} \Phi^a, \quad \delta_{\text{int}}^{\Phi} \Phi^a = \epsilon_m \epsilon^m \delta_{\text{int}}^{\Phi} \Phi^a, \quad (23)$$

we deduce,

$$[\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\Phi}] \Phi^a = [\delta_{\text{int}}^{\Phi}, \delta_{\text{int}}^{\Phi}] \Phi^a = 0. \quad (24)$$
(iv) Proper transformations under $J^{+-}$ and $D$ require

\[ [\delta_{J^{+-}}, \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}}] \Phi^a = \frac{i}{2} \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}} \Phi^a, \quad [\delta_{J^{+-}}, \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}}] \Phi^a = \frac{i}{2} \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}} \Phi^a. \quad (25) \]

\[ [\delta_{D}, \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}}] \Phi^a = -\frac{i}{2} \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}} \Phi^a, \quad [\delta_{D}, \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}}] \Phi^a = -\frac{i}{2} \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}} \Phi^a. \quad (26) \]

(v) The correct $U(1)$ $R$-charge,

\[ [\delta_{U(1)}, \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}}] \Phi^a = -\frac{1}{2} \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}} \Phi^a, \quad [\delta_{U(1)}, \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}}] \Phi^a = \frac{1}{2} \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}} \Phi^a. \quad (27) \]

(vi) The sixteen interacting supersymmetries must transform as an $SO(16)$ vector, which, in the $SU(8) \times U(1)$ decomposition, means,

\[ [\delta_{\text{coset}}, \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}}] \Phi^a = 0, \quad [\delta_{\text{coset}}, \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}}] \Phi^a = \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}} \Phi^a, \quad (28) \]

with $\epsilon_m^l = 2\epsilon_{mn}^l \epsilon^n$. Similarly, with $\epsilon^{lm} = 2\omega^{mn} \epsilon_{mn}$,

\[ [\delta_{\text{coset}}, \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}}] \Phi^a = 0, \quad [\delta_{\text{coset}}, \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}}] \Phi^a = \delta_{\epsilon_{\mathcal{Q}}}^{\text{int}} \Phi^a. \quad (29) \]

(vii) Dynamical interacting supersymmetries are cubic powers of superfields.

In three dimensions, Bose fields have mass dimension one-half, so $\Phi$ has half-odd integer dimension, assuming integer power of $\varphi$. In a conformal theory with no dimensionful parameters, the interacting supersymmetry must then be odd powers of superfields. Define

\[ \delta_{\Delta} \Phi^a \equiv \delta_{(J^{+-} - D)} \Phi^a = i \left( x \partial + \frac{1}{2} \right) \Phi^a. \quad (30) \]

Since $\delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}} \Phi^a$ contain no transverse variable, it follows that

\[ [\delta_{\Delta}, \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}}] \Phi^a = \frac{i}{2} (n_{\Phi} - 1) \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}} \Phi^a, \quad (31) \]

where $n_{\Phi}$ is the number of superfields. On the other hand, by matching with the free part, the algebra requires

\[ [\delta_{\Delta}, \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}}] \Phi^a = i \delta_{e^{\epsilon}_{\mathcal{Q}}}^{\text{int}} \Phi^a, \quad (32) \]

which is consistent for $n_{\Phi} = 3$. This theory must contain a tensor with at least four indices, $f^{abcd}$, as in the BLG theory.
The dynamical supersymmetry transformations are written in terms of the basic cubic nested form,

\[ K_\alpha = \frac{1}{\partial^+ A_\alpha} \left( (\partial^{+B_\alpha} \varphi^b) \frac{1}{\partial^+ M_\alpha} (\partial^{+C_\alpha} \varphi^c) (\partial^{+D_\alpha} \varphi^d) \right) \]  

(33)

\[ = \frac{1}{\partial^+ A_\alpha} \left( \partial^{+B_\alpha}, \frac{1}{\partial^+ M_\alpha} (\partial^{+C_\alpha}, \partial^{+D_\alpha}) \right), \]  

(34)

and the coherent state operators [4],

\[ E_\eta = e^{\eta \hat{d}}, \quad E_r = e^{r \hat{\partial}}, \quad E_\epsilon = e^{\epsilon \hat{q}}, \quad E_{\bar{\epsilon}} = e^{\bar{\epsilon} \hat{q}}. \]  

(35)

The Grassmann variables \( \eta^m, \zeta^m \), and \( r, r' \) are dummy variables, and \( \epsilon^m (\bar{\epsilon}_m) \) are the supersymmetry parameters.

Using insertion operators \( U_i, i = 1, 2, 3, 4 \), we define

\[ K_\alpha^{(\eta, \zeta)} = \frac{1}{\partial^+ A_\alpha} \left( E_\eta \partial^{+B_\alpha}, E_{-\eta} \frac{1}{\partial^+ M_\alpha} (E_\zeta \partial^{+C_\alpha}, E_{-\zeta} \partial^{+D_\alpha}) \right) \]  

(36)

\[ = (E_\eta U_1) (E_{-\eta} U_2) (E_\zeta U_3) (E_{-\zeta} U_4) K_\alpha \]  

(37)

and

\[ K_\alpha^{(r, r')} = \frac{1}{\partial^+ A_\alpha} \left( E_r \partial^{+B_\alpha}, E_{-r} \frac{1}{\partial^+ M_\alpha} (E_{r'} \partial^{+C_\alpha}, E_{-r'} \partial^{+D_\alpha}) \right) \]  

(38)

Supersymmetry parameters are introduced through,

\[ K_\alpha^{(\epsilon, \eta, \zeta)} = (E_\epsilon U_1) (E_{-\epsilon} U_2) K_\alpha^{(\eta, \zeta)}, \]  

(39)

\[ = \frac{1}{\partial^+ A_\alpha} \left( E_\epsilon, E_{-\epsilon} \frac{1}{\partial^+ M_\alpha} (E_\zeta \partial^{+C_\alpha}, E_{-\zeta} \partial^{+D_\alpha}) \right), \]  

(40)

In this notation, the dynamical supersymmetry transformations (showing explicitly the superfield taxonomic indices) are of the form,

\[ \delta^{\text{int}} \Phi^a = f^a_{bcd} \sum_\alpha K^{(\epsilon, \eta, \zeta)bcd}_\alpha. \]  

(41)

They are similar to those in the BLG theory [3], except that the indices now run over eight values. Some new features must be noted:

- The correct \( U(1) \) charge now requires

\[ (\eta^m \frac{\partial}{\partial \eta^m} + \zeta^m \frac{\partial}{\partial \zeta^m} - 8) K^{(\epsilon, \eta, \zeta)a}_\alpha = 0, \]  

(42)

so that only terms octal in \( \eta, \zeta \) need to be considered, that is

\[ \zeta^8, \eta \zeta^7, \eta^2 \zeta^6, \eta^3 \zeta^5, \eta^4 \zeta^4, \eta^5 \zeta^3, \eta^6 \zeta^2, \eta^7 \zeta, \eta^8. \]  

(43)
• The proper transformation under $J^{+-}$, together with the $U(1)$ constraint restricts the number of $\partial^\pm$ derivatives to eight,

$$- A_\alpha + B_\alpha - M_\alpha + C_\alpha + D_\alpha = 8. \quad (44)$$

• The correct transformation properties under the coset transformations splits the Ansatz into two types, the even Ansatz where the sum is over $(\eta^8, \eta^6\zeta^2, \eta^4\zeta^4, \eta^2\zeta^6, \zeta^8)$, and odd Ansatz over $(\eta^7, \eta^5\zeta^3, \eta^3\zeta^5, \eta\zeta^7)$, with recursion relations

$$A_\alpha = A_{-\alpha} - 2\alpha, \quad B_\alpha = B_{-\alpha} - 2\alpha, \quad M_\alpha = M_{-\alpha} + 4\alpha,$$

$$C_\alpha = C_{-\alpha} + 2\alpha, \quad D_\alpha = D_{-\alpha} + 2\alpha. \quad (45)$$

These constraints further narrow the form of the dynamical supersymmetries. In terms of

$$K_{\alpha(k,8-k)} = \frac{\epsilon^{i_1...i_8}}{k!(8-k)!} \frac{\partial}{\partial \eta^{i_1...i_k}} \frac{\partial}{\partial \zeta^{i_{k+1}...i_8-k}} K^{(\eta,\zeta)}_{\alpha}$$

evaluated at $\eta = \zeta = 0$, we find two linear combinations which satisfy chirality, as well as all kinematical and inside-out constraints, the odd Ansatz,

$$\delta^{\text{odd}} \Phi^a = \sum_{\text{odd}} K^a_\alpha = f_{\text{bcd}} \sum_{\alpha=-2}^{2} (-1)^{\alpha + \frac{1}{2}} K^{\text{bcd}}_{\alpha(4-2\alpha,4+2\alpha)}, \quad (47)$$

and the even Ansatz,

$$\delta^{\text{even}} \Phi^a = \sum_{\text{even}} K^a_\alpha = f_{\text{bcd}} \sum_{\alpha=-2}^{2} (-1)^{\alpha} K^{\text{bcd}}_{\alpha(4-2\alpha,4+2\alpha)}. \quad (48)$$

All kinematic requirements being satisfied, we use these expressions to find the other dynamical transformations. The light-cone Hamiltonian is calculated from,

$$[\delta^{\text{free}}_{\epsilon^Q} + \delta^{\text{int}}_{\epsilon^Q}, \delta^{\text{free}}_{\epsilon^Q} + \delta^{\text{int}}_{\epsilon^Q}] \phi^a = \sqrt{2} \epsilon^m \epsilon^m \delta_{\phi^a} \phi^a.$$

The boost $\delta_{J^-} \Phi^a$ are obtained from the commutator of the Hamiltonian with the transverse conformal transformation,

$$2i \delta_{J^-} \Phi^a = [\delta_{K^+}, \delta_{p^-}] \Phi^a. \quad (50)$$

The first consistency check comes from the commutator of the hamiltonian with the boost which, by conformal symmetry, should vanish,
\[
[ \delta_{\mathcal{P}^-, \delta_{\mathcal{J}^-}} ] \Phi^a = 0 . \tag{51}
\]

We now show that, unlike in the BLG case, it does not vanish. For the odd Ansatz (similar conclusions apply as well to the even Ansatz), and to first order in \( f_{abcd} \), a long calculation shows this commutator to be proportional to

\[
\sum_{\text{odd}} A_\alpha \left( K^{(r'r,1)}_{\alpha} - K^{(1,r'r)}_{\alpha+1} \right) + 2 \sum_{\text{even}} A_\alpha K^{(r,r')}_{\alpha+\frac{1}{2}} - \\
- \sum_{\text{even}} B_\alpha \left( K^{(r'r,1)}_{\alpha+\frac{1}{2}} - K^{(1,r'r)}_{\alpha+\frac{3}{2}} \right) - 2 \sum_{\text{odd}} B_\alpha K^{(r,r')}_{\alpha+1} \tag{52}
\]

where

\[
A_\alpha = (B_\alpha + \alpha - \frac{7}{2}) \partial^+ \mathcal{U}_2 + \left( M_\alpha - C_\alpha - D_\alpha + 5 \right) \partial^+ \mathcal{U}_1, \tag{53}
\]

\[
B_\alpha = \left( C_\alpha - \alpha - 4 \right) \frac{1}{\partial^+} \mathcal{U}_3 - \left( D_\alpha - \alpha - 4 \right) \frac{1}{\partial^+} \mathcal{U}_4 \right) \left( \partial^+ \mathcal{U}_1 \right) \left( \partial^+ \mathcal{U}_2 \right). \tag{54}
\]

In these expressions, the coefficients \((A_\alpha, B_\alpha, M_\alpha, C_\alpha, D_\alpha)\) have yet to be determined.

The choices \((A_{-\frac{1}{2}}, B_{-\frac{3}{2}}, M_{-\frac{3}{2}}, C_{-\frac{5}{2}}, D_{-\frac{3}{2}}) = (5, 5, -6, 1, 1)\) eliminate the largest number of terms, and we focus on those the remaining terms which contain the combination,

\[
Z = \partial^2 \frac{1}{8!} \epsilon_{i_1 \ldots i_8} \bar{d}_{i_1} \ldots \bar{d}_{i_8} . \tag{55}
\]

Their contributions to the commutator reduce to,

\[
\left[ \mathcal{U}_3 Z_1 - \mathcal{U}_4 g_1 Z_3 + (\mathcal{U}_3 - \mathcal{U}_4) (g_2 - t g_1) Z_3 \frac{1}{\partial^+} (\partial^+ \mathcal{U}_1) (\partial^+ \mathcal{U}_2) \right] K_{-\frac{1}{2}}, \tag{56}
\]

where

\[
(\frac{1}{\partial^+} \mathcal{U}_3) \equiv \mathcal{U}_i, \quad Z \mathcal{U}_i \equiv Z_i; \quad s \equiv (\frac{1}{\partial^+} \mathcal{U}_2) (\partial^+ \mathcal{U}_3), \quad t \equiv (\frac{1}{\partial^+} \mathcal{U}_1) (\partial^+ \mathcal{U}_4), \tag{57}
\]

with

\[
g_1 = 32s^7(1 + t)^2 - 32s^6(1 + t)^3 + 6s^5(1 + t)^4, \tag{58}
\]

\[
g_2 = 32s^8(1 + t)^2 - 48s^7(1 + t)^3 + 18s^6(1 + t)^4 - s^5(1 + t)^5. \tag{59}
\]

Further simplifications reduce Eq. (58) to,
\[
\frac{1}{\partial^+ \sigma} \left( \partial^{+8} X, [32 \partial^{+2} (\partial^{+4}, ) - 48 \partial^{+3} (\partial^{+3}, ) + \\
+ 18 \partial^{+4} (\partial^{+2}, ) - \partial^{+5} (\partial^{+}, )] \right) + O(\partial^{+7} X). \tag{60}
\]

Since these clearly do not vanish, we conclude that an interacting theory based on \(OSp(2,2|16)\) symmetry does not exist. In the next section, we arrive at the same conclusions using the familiar covariant description.

## 4 Covariant Formulation

The covariant form of the BLG theory begins with the supersymmetry transformations,

\[
\begin{align*}
\delta X^I_a & = i \bar{\epsilon} \Gamma^I \Psi_a \\
\delta \Psi_a & = D_\mu X_b^I \gamma^I \epsilon - \frac{1}{6} X_c^I X_d^K f^{bcd}_{\ abc} \Gamma^{IJ} \Gamma \epsilon \\
\delta \tilde{A}_d^b & = i \bar{\epsilon} \gamma_\mu \Gamma^I \Psi_d \epsilon^{cd}_{\ ab}, \tag{61}
\end{align*}
\]

where \(I\) is the \(SO(8)\) vector index. The supersymmetry parameters, \(\epsilon_{\alpha A}\), span one \(SO(8)\) spinor representation, where \(\alpha = 0, 1, 2\) and \(A = 1, 2, \ldots, 8\). The fermions \(\Psi_{\dot{\alpha} A a}\), where \(\dot{\alpha} = 1, \ldots, 8\), span the other \(SO(8)\) spinor representation. The Dirac matrices are \((\Gamma^I)_{AA}\), and \(\gamma_\alpha\), with \(\gamma^{012} \epsilon = \epsilon\). The covariant derivatives are given by

\[
D_\mu X^I_a = \partial_\mu X^I_a + \tilde{A}_{b a} \partial_\mu \tilde{X}_b^I, \quad D_\mu \Psi_{\dot{\alpha} a} = \partial_\mu \Psi_{\dot{\alpha} a} + \tilde{A}_{b a} \Psi_{\dot{\alpha} b}, \tag{62}
\]

where the vector field \(\tilde{A}_{b a}\), with the dimension of mass, is an auxiliary field (canonical boson fields have mass dimension one-half). Two supersymmetry transformations generate, as expected, a translation and a gauge transformation,

\[
[\delta_1, \delta_2] X^I_a = v^a \partial_\mu X^I_a + \Lambda^b_{a b} X^I_b, \tag{63}
\]

where the translation and gauge parameters are

\[
v_\mu = \tau_2 \gamma_\mu \epsilon_1, \quad \Lambda^a_{b b} = v_\mu \tilde{A}_{\mu b} + f^{\alpha}_{\ bcd} (\tau_2 \Gamma^I \epsilon_1) X_c^I X_d^J. \tag{64}
\]

This equation, which relies heavily on the \(SO(8)\) Fierz identities, shows the dependence of the gauge parameters on the bosonic coordinates.

The possible generalization of this algebra to \(SO(16)\) begins with an alternative description based on \(SO(8)\) triality. The vector index \(I\) is simply replaced by a spinor index \(A\); the bosons are labelled by \(X_{A a}\), and the supersymmetry parameters transform as \(SO(8)\) vectors, \(\epsilon^I_\alpha\). This leads to the new transformation rules,
\[ \delta X_{Aa} = \epsilon^I (\Gamma^I)_{\dot{A}\dot{a}} \Psi_{\dot{A}a} \]
\[ \delta \Psi_{\dot{A}a} = D_\mu X_{Aa} \gamma^\mu (\Gamma^I)_{\dot{A}A} \epsilon^I + \frac{1}{6} f^{bcd} a X_{Ab} X_{Bc} X_{Cd} (\Gamma^I)_{\dot{A}A} (\Gamma^I)_{\dot{B}B} \epsilon^I \]
\[ \delta \tilde{A}_{\mu}^b = f^{cde} a \epsilon^I \gamma_\mu (\Gamma^I)_{\dot{A}A} X_{Bc}. \] (65)

As expected, these commutators have the BLG structure, and close on translations and gauge transformations\cite{5}.

We seek a generalization of this algebra where the 128 bosonic coordinates span one \(SO(16)\) spinor representation, and the fermionic coordinates the other. We label the first spinor by \(A = 1, \ldots, 128\), the second by \(\dot{A} = 1, \ldots, 128\). The vector index \(I\) runs over sixteen values.

We posit the supersymmetry transformations of the bosons,
\[ \delta X_{Aa} = \epsilon^I (\Gamma^I)_{\dot{A}\dot{a}} \Psi_{\dot{A}a}. \] (66)

The fermions’ supersymmetry transformations contain the same covariant derivatives with the auxiliary one-form,
\[ \delta \Psi_{\dot{A}a} = D_\mu X_{Aa} \gamma^\mu (\Gamma^I)_{\dot{A}A} \epsilon^I + \delta' \Psi_{\dot{A}a}, \] (67)
augmented by \(\delta'\), the most general transformation allowed by \(SO(16)\).

This supersymmetry variation transforms as \textbf{1920} vector-spinor representation, and from,
\[ (128 \times 128 \times 128) = 326144 + 13312 + 1920. \] (68)
we see that \(\delta' \Psi_{\dot{A}a}\) contains cubic antisymmetric products of bosons.

Bi-spinors transform as antisymmetric forms,
\[ 128 \times 128 = [1 + 1820 + 6435]_{\text{sym}} + [8008 + 120]_{\text{anti}}. \] (69)
where \textbf{6435} is the self-dual eight form. Only the two- and six-forms appear in their antisymmetric product. For any two spinors, \(Y_A\) and \(Z_B\), we thus have
\[ 128 Y_A Z_B = (1)_{AB} (YZ) - \frac{1}{2} (\Gamma^{(2)})_{AB} (YT^{(2)} Z) + \frac{1}{4!} (\Gamma^{(4)})_{AB} (YT^{(4)} Z) - \frac{1}{6!} (\Gamma^{(6)})_{AB} (YT^{(6)} Z) + \frac{1}{2 \cdot 8!} (\Gamma^{(8)})_{AB} (YT^{(8)} Z), \] (70)
from which all Fierz identities are derived. The antisymmetric product of two bosons comes in two covariant expressions,
\[ X_{Ab} (\Gamma^{l_1 l_2 l_3 l_4 l_5 l_6})_{AB} X_{Bc} = (X_b \Gamma^{(6)} X_c), \quad X_{Ab} (\Gamma^{l_1 l_2})_{AB} X_{Bc} = (X_b \Gamma^{(2)} X_c). \] (71)
In contrast, $SO(8)$ has only one antisymmetric covariant expression, $X_b \Gamma^{(2)} X_c$. Hence the most general form of the supersymmetry variation is

$$
\delta' \Psi_{\cal A} = f_{abcd} \ell^I \left[ k_1 (\Gamma^J X_b)(X_c \Gamma^{IJ} X_d) + k_2 (\Gamma^{(2)} X_b)(X_c \Gamma^{(2)} X_d) \\
+ k_3 (\Gamma^{(5)} X_b)(X_c \Gamma^{(5)} X_d) + k_4 (\Gamma^{(6)} X_b)(X_c \Gamma^{(6)} X_d) \right] \tag{72}
$$

Assuming antisymmetry of $f_{abcd}$, direct evaluation of the commutator of two supersymmetries yields,

$$
[ \delta_1, \delta_2 ] X_{\cal A \alpha} = (\epsilon^I c_1) D_{\alpha} X_{\cal A \alpha} + \\
+ f_{abcd} \epsilon^{IJ} \left[ k_1 (\Gamma^I \Gamma^K X_b)_{\cal A}(X_c \Gamma^{JK} X_d) + k_2 (\Gamma^I \Gamma^{(2)} X_b)_{\cal A}(X_c \Gamma^{(2)} X_d) + \\
+ k_3 (\Gamma^I \Gamma^{(5)} X_b)_{\cal A}(X_c \Gamma^{(5)} X_d) + k_4 (\Gamma^I \Gamma^{(6)} X_b)_{\cal A}(X_c \Gamma^{(6)} X_d) \right], \tag{73}
$$

where $\epsilon^{IJ} = \epsilon_2^I \epsilon_1^J - (I \leftrightarrow J)$. We use,

$$
\Gamma^I \Gamma^K = \Gamma^{IK} + \delta^{IK}, \tag{74}
$$

and apply the Fierz transformations to obtain

$$
(\Gamma^{(5)} X_b)_{\cal A}(X_c \Gamma^{(5)} X_d) = \sum_{n=0,2,4} c_n (\Gamma^{(2n)} X_b)_{\cal A}(X_c \Gamma^{(5)} X_d), \tag{75}
$$

$$
(\Gamma^{(4)} X_b)_{\cal A}(X_c \Gamma^{(4)} X_d) = \sum_{n=0,2,4} c_n (\Gamma^{(2n)} X_b)_{\cal A}(X_c \Gamma^{(4)} X_d), \tag{76}
$$

and for $k = 2, 6,$

$$
(\Gamma^{(k)} X_b)_{\cal A}(X_c \Gamma^{(k)} X_d) = \sum_{n=0,2,4} c_n (\Gamma^{(2n)} X_b)_{\cal A}(X_c \Gamma^{(k)} X_d), \tag{77}
$$

where $128(c_0, c_2, c_4) = (1, \frac{1}{2}, \frac{1}{24})$ are Fierz coefficients. Use of the identities

$$
\Gamma^{(k)} \Gamma^{(n)} \Gamma^{(k)} = g_{n,k} \Gamma^{(n)}, \tag{78}
$$

$$
\Gamma^{(k)} \Gamma^{(n)} \Gamma^{(k)} = a_{n,k} \Gamma^{(n)} \Gamma^{(k)} + b_{n,k} \{ \Gamma^{(k)} \Gamma^{(n)} \Gamma^{(k)} \}, \tag{79}
$$

$$
\Gamma^{(k)} \Gamma^{(n)} \Gamma^{(k)} + \Gamma^{(k)} \Gamma^{(n)} \Gamma^{(k)} = c_{n,k} \Gamma^{(n)} \Gamma^{(k)} + d_{n,k} \{ \Gamma^{(k)} \Gamma^{(n)} \Gamma^{(k)} \}. \tag{80}
$$
expresses the commutator in terms of four combinations,

\[
\begin{align*}
A_{a}^{IJ} & \equiv f_{a}^{bcd}X_{bA}(X_{c}^{I}X_{d}^{J}), \\
B_{a}^{IJ} & \equiv f_{a}^{bcd}(\Gamma^{(4)}X_{b})_{A}(X_{c}^{I}X_{d}^{J}), \\
C_{a}^{IJ} & \equiv f_{a}^{bcd}(\Gamma^{(4)}X_{b})_{A}(X_{c}^{I}(\Gamma^{IJ}\Gamma^{(4)}X_{d})^{(4)}X_{d}), \\
D_{a}^{IJ} & \equiv f_{a}^{bcd}(\Gamma^{(8)}X_{b})_{A}(X_{c}^{I}(\Gamma^{IJ}\Gamma^{(8)}+\Gamma^{IJ}\Gamma^{(4)}X_{d})^{(8)}X_{d}).
\end{align*}
\] (81)

Not all are independent as they satisfy two equations,

\[
\begin{align*}
c_{2}(a_{4,3}A_{a}^{IJ}+b_{4,3}B_{a}^{IJ})+c_{4}b_{8,3}C_{a}^{IJ} & = -2\frac{16!}{13!}A_{a}^{IJ}, \\
c_{2}(a_{4,7}A_{a}^{IJ}+b_{4,7}B_{a}^{IJ})+c_{4}b_{8,7}C_{a}^{IJ} & = -2\frac{16!}{9!}A_{a}^{IJ}.
\end{align*}
\] (82)(83)

After numerical evaluation of the coefficients \(a_{n,k}, b_{n,k}, c_{n,k},\) and \(d_{n,k},\) we find,

\[
\begin{align*}
C_{a}^{IJ} & = -\frac{3}{4}(416A_{a}^{IJ}-B_{a}^{IJ}), \\
D_{a}^{IJ} & = -2520(3744A_{a}^{IJ}+B_{a}^{IJ}),
\end{align*}
\] (84)

which enables us to express the commutator in terms of \(A_{a}^{IJ}\) and \(B_{a}^{IJ}\). The commutator turns out to be proportional to

\[
[\delta_{1}, \delta_{2}]X_{Aa} = \left(\bar{\gamma}^{J}_{2}\gamma^{\mu}\epsilon_{I}^{J}\right)D_{\mu}X_{a} + \epsilon^{IJ}\frac{1}{768}(k_{1} - 2(k_{2} - 780(k_{3} - 6k_{4}))(1632A_{a}^{IJ} + B_{a}^{IJ}) ,
\]

so that the desired \(A_{a}^{IJ}\) term always comes accompanied with the unwanted \(B_{a}^{IJ}\).

We conclude that the commutator can be written as a gauge transformation, but, unlike the BLG case, its gauge parameter is independent of the structure function \(f_{abcd}\).

5 Conclusions

We have shown by two different techniques that there is no interacting superconformal theory in three dimensions with 128 fermions and bosons, and sixteen supersymmetries. We used the light-cone superspace with eight Grassmann variables, and its constrained chiral superfield, which in four dimensions describes \(\mathcal{N} = 8\) Supergravity.

It is still possible that in six dimensions, there exists an interacting superconformal theory with 256 degrees of freedom. That such a free theory exists had been noted earlier by Hull[6]. Our result seems to suggest that it also does not have an interacting analog.
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