THE BROWN MEASURE OF A SUM OF TWO FREE RANDOM VARIABLES, ONE OF WHICH IS TRIANGULAR ELLIPTIC

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ABSTRACT. The triangular elliptic operators are natural extensions of the elliptic deformation of circular operators. We obtain a Brown measure formula for the sum of a triangular elliptic operator \( g_{\alpha,\beta,\gamma} \) with an arbitrary random variable \( x_0 \), which is *-free from \( g_{\alpha,\beta,\gamma} \) with amalgamation over certain unital subalgebra. Let \( c_t \) be a circular operator. We prove that the Brown measure of \( x_0 + g_{\alpha,\beta,\gamma} \) is the push-forward measure of the Brown measure of \( x_0 + c_t \) by an explicitly defined map on \( \mathbb{C} \) for some suitable \( t \). We show that the Brown measure of \( x_0 + c_t \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{C} \) and its density is bounded by \( 1/\pi t \). This work generalizes earlier results on the addition with a circular operator, semicircular operator, or elliptic operator to a larger class of operators. We extend operator-valued subordination functions, due to Biane and Voiculescu, to certain unbounded operators. This allows us to extend our results to unbounded operators.

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1. INTRODUCTION

1.1. Brown measure and its regularization. The limit objects of random matrix models can often be identified as operators in free probability theory. In [11], Brown introduced the Brown measure of operators in the context of semi-finite von Neumann algebras, which is a natural generalization of the eigenvalue counting measure of square matrices with finite dimension. In his breakthrough paper [28], Voiculescu discovered that independent Gaussian random matrices are asymptotically free as the dimension of matrices goes to infinity. The Brown measure of a free random variable is regarded as a candidate for the limit empirical spectral distribution of suitable random matrix models.

Let \( A \) be a von Neumann algebra with a faithful, normal, tracial state \( \phi \). Recall that the Fuglede-Kadison determinant of \( x \in A \) is defined by

\[
\Delta(x) = \exp \left( \int_0^\infty \log t \mu_x(t) \right),
\]

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where \( \mu_{|x|} \) is the spectral measure of \(|x|\) with respect to \( \phi \). It is known \([11]\) that the function \( \lambda \mapsto \log \Delta(x - \lambda I) \) is a subharmonic function whose Riesz measure is the unique probability measure \( \mu_x \) on \( \mathbb{C} \) with the property that

\[
\log \Delta(x - \lambda) = \int_\mathbb{C} \log |z - \lambda|d\mu_x(z), \quad \lambda \in \mathbb{C}.
\]

The measure \( \mu_x \) is called the Brown measure of \( x \), and it can be recovered as the distributional Laplacian of the function \( \log \Delta \) as follows

\[
(1.1) \quad \mu_x = \frac{1}{2\pi} \nabla^2 \log \Delta(x - \lambda) = \frac{2}{\pi} \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda} \log \Delta(x - \lambda).
\]

When \( \mathcal{A} = M_n(\mathbb{C}) \) and \( \phi \) is the normalized trace on \( M_n(\mathbb{C}) \), for \( x \in M_n(\mathbb{C}) \), we have

\[
\mu_x = \frac{1}{n} (\delta_{\lambda_1} + \cdots + \delta_{\lambda_n})
\]

where \( \lambda_1, \cdots, \lambda_n \) are the eigenvalues of \( x \).

Let \( \bar{\mathcal{A}} \) be the set of operators which are affiliated with \( \mathcal{A} \). Denote by \( \log^+(\mathcal{A}) \) the set of operators \( T \in \bar{\mathcal{A}} \) satisfying

\[
\phi(\log^+ |T|) = \int_0^\infty \log^+(t) d\mu_{|T|}(t) < \infty.
\]

It is known \([17]\) that the set \( \log^+(\mathcal{A}) \) is a subspace of \( \bar{\mathcal{A}} \). In particular, for \( T \in \log^+(\mathcal{A}) \) and \( \lambda \in \mathbb{C} \), \( T - \lambda \in \log^+(\mathcal{A}) \). Moreover, the function \( \lambda \mapsto \log \Delta(T - \lambda) \) is again subharmonic in \( \mathbb{C} \). In the noncommutative probability space \( (\mathcal{A}, \phi) \), one can extend the definition of the Brown measure to \( \log^+(\mathcal{A}) \) \([11, 17]\) using the same formula \((1.1)\).

It is useful to consider the regularized function

\[
S(x, \lambda, \varepsilon) = \phi \left( \log \left( (x - \lambda)^* (x - \lambda) + \varepsilon^2 \right) \right).
\]

Then, \( S(x, \lambda, 0) = \lim_{\varepsilon \to 0} S(x, \lambda, \varepsilon) \). The regularized Brown measure is defined as

\[
\mu_x^{(\varepsilon)} = \frac{1}{4\pi} \nabla^2 S(x, \lambda, \varepsilon).
\]

It is known that \( \mu_x^{(\varepsilon)} \to \mu_x \) weakly as \( \varepsilon \to 0 \). It turns out that the regularized Brown measure approximates the Brown measure in a much stronger sense for the operators we are interested in.

### 1.2. Main results

Given two \(*\)-free random variables \( x, y \) in a \( W^*\)-probability space \( (\mathcal{A}, \phi) \), it desirable to find a general method to calculate the Brown measure of \( x + y \). In the special case when \( y \) is an elliptic operator, the third author obtained Brown measure formulas for an arbitrary \( x \) in \([33]\). This generalized previous results of Hall–Ho \([20]\) and Ho–Zhong \([22]\), and introduced a new approach based on the Hermitian reduction method and subordination functions. The purposes of this paper are two-folds. First, we adapt techniques developed in \([5, 33]\) to study the Brown measure of the sum of a larger family of operators \( g_{\alpha, \beta, \gamma} \), called triangular elliptic operators, and another random variable \( x_0 \) that is \(*\)-free from \( g_{\alpha, \beta, \gamma} \) in the sense of operator-valued free probability. Second, we extend our main results to unbounded operator \( x_0 \). Our results for the case \( \alpha = \beta \) extend main results obtained by the third author \([33]\), and generalize main results in \([21]\) obtained by Ho for the special case \( x_0 \) being self-adjoint, and another result by Bordenave–Caputo–Chafaï \([9]\) for unbounded normal operators.
It turns out the family of Brown measures $x_0 + g_{a,b,c}$ can be retrieved from the Brown measure $x_0 + c_t$ by a family of explicitly constructed self-maps of $\mathbb{C}$, where $c_t$ is a circular operator with variance $t$. Hence, it is crucial to understand the Brown measure $\mu_{x_0+c_t}$. We introduce a new approach to study regularity properties for the Brown measure of $x_0 + c_t$ via regularized Brown measure, and obtain a complete description of the Brown measure $\mu_{x_0+c_t}$. We prove that the density of the regularized Brown measure $\mu_{x_0+c_t}$ is bounded by $1/\pi t$. Consequently, we show that $\mu_{x_0+c_t}$ has no singular part and this settles an open question in [33].

Our method is completely different from the PDE method used by Ho [21], where he studied the sum of an unbounded selfadjoint operator and an elliptic operator (corresponding to $\alpha = \beta$). Although our approach is an extension of the method used in [33], the details are substantially more technical and some new ideas are developed to study the distinguished measure $\mu_{x_0+c_t}$. In addition, we extend the subordination functions in operator-valued free probability, due to Biane and Voiculescu, to unbounded operators under some assumptions. The subordination result is of interest by its own and it allows us to extend our results to unbounded operators.

Consider a random matrix $A_N = (a_{ij})_{1 \leq i,j \leq N}$ such that the joint distribution of random variables $(\Re a_{ij}, \Im a_{ij})_{1 \leq i,j \leq N}$ is centered Gaussian with covariance given by

$$
\mathbb{E}a_{ij} = \left\{ \begin{array}{ll}
\frac{N}{2} \delta_{ik} \delta_{jl} & \text{if } i = j, \\
\frac{\alpha + \beta}{2N} \delta_{ik} \delta_{jl} & \text{if } i = j, \\
\frac{\alpha}{2N} \delta_{ik} \delta_{jl} & \text{if } i > j,
\end{array} \right.
$$

where $\alpha, \beta > 0$ and $\gamma \in \mathbb{C}$ such that $|\gamma| \leq \sqrt{\alpha \beta}$. One can show that $A_N$ converges in $*$-moments to some operator $g_{a,b,,\gamma}$, called a triangular elliptic operator, in $\mathcal{A}$.

When $\alpha = \beta = t$ and $\gamma = 0$, then $A_N$ converges in $*$-moments to the circular operator $c_t$ with variance $t$, which has the same distribution as $\sqrt{t/2}(s_1 + is_2)$, where $\{s_1, s_2\}$ is a semicircular system. Let $\{g_{t_1}, g_{t_2}\}$ be freely independent semicircular operators with mean zero and variances $t_1, t_2$ respectively. Choose $t_1 = (t + |\gamma|)/2, t_2 = (t - |\gamma|)/2$ and $\theta \in [0, 2\pi]$ such that $e^{2i\theta} = \gamma/|\gamma|$. When $\alpha = \beta = t$ and $|\gamma| \leq t$, then $A_N$ converges in $*$-moments to twisted elliptic operator $g_{t,\gamma}$, which has the same distribution as $e^{i\theta}(g_{t_1} + ig_{t_2})$.

Consider the open set

$$
\Xi_t = \left\{ \lambda \in \mathbb{C} : \phi \left[ \frac{(x_0 - \lambda)^* (x_0 - \lambda)}{t} \right] > \frac{1}{t} \right\}.
$$

Fix $\lambda \in \Xi_t$ and let $w = w(0; \lambda, t)$ be a function of $\lambda$ taking positive values such that

$$
\phi \left[ \frac{(x_0 - \lambda)^* (x_0 - \lambda) + w^2}{t} \right] = \frac{1}{t},
$$

and let $w(0; \lambda, t) = 0$ for $\lambda \in \mathbb{C} \setminus \Xi_t$. We denote

$$
\Phi_{t,\gamma}(\lambda) = \lambda + \gamma \cdot p_\lambda^{(0)}(w),
$$

where

$$
p_\lambda^{(0)}(w) = -\phi \left[ (x_0 - \lambda)^* ((x_0 - \lambda)(x_0 - \lambda)^* + w(0; \lambda, t)^2)^{-1} \right].
$$

**Theorem 1.1.** [33] Theorem 4.2 and 5.4] The Brown measure of $x_0 + c_t$ has no atom and it is supported in the closure of $\Xi_t$. It is absolutely continuous with respect to the Lebesgue
measure in $\Xi_t$. Moreover, the density in $\Xi_t$ is strictly positive and can be expressed explicitly in terms of $w(0; \lambda, t)$.

For any $\gamma \in \mathbb{C}$ such that $|\gamma| \leq t$, the Brown measure of $x_0 + g_{t, \gamma}$ is the push-forward measure of the Brown measure of $x_0 + c_t$ under the map $\Phi_{t, \gamma}$.

The push-forward property of a similar type was first observed for some special cases when $g_{t, \gamma}$ is a semicircular element, and self-adjoint operator $x_0$ in [22] by PDE methods. In a follow-up work [20], it was proved for the sum of an imaginary multiple of semicircular elements and a self-adjoint operator $x_0$.

When $\alpha \neq \beta$, one can show [13, 5] that the limit operator $y = g_{\alpha, \beta, \gamma}$ of $A_N$ lives in some operator-valued $W^*$-probability space $(A, E, B)$ where the unital subalgebra $B$ can be identified as $B = L^\infty[0, 1]$ and $E : A \to B$ is the conditional expectation (see Section 4.1). Given $x \in A$, by identifying $E(x) = f$ for some $f \in L^\infty[0, 1]$, the tracial state $\phi : A \to \mathbb{C}$ is determined by

$$
\phi(x) = \phi(E(x)) = \int_0^1 f(s) ds.
$$

Let $x_0 \in \log^+(A)$ be an operator that is $*$-free from the family of operators $\{g_{\alpha, \beta, \gamma}\}_{\gamma \leq \sqrt{\alpha \beta}}$ with amalgamation over $B$ such that $x_0$ is $*$-free from $B$ with respect to the tracial state $\phi$. Our main interest is the Brown measure of $x_0 + g_{\alpha, \beta, \gamma}$.

We now describe our main results. Given $\alpha, \beta > 0$, we denote

$$
\Xi_{\alpha, \beta} = \left\{ \lambda : \phi(|x_0 - \lambda|^2) > \frac{\log \alpha - \log \beta}{\alpha - \beta} \right\}.
$$

For $\lambda \in \Xi_{\alpha, \beta}$, let $s(\lambda)$ be the positive number $s$ determined by

$$
\frac{\log \alpha - \log \beta}{\alpha - \beta} = \phi \left\{ \left( (\lambda - x_0)^*(\lambda - x_0) + s^2 \right)^{-1} \right\},
$$

and for $\lambda \in \mathbb{C} \setminus \Xi_{\alpha, \beta}$, we put $s(\lambda) = 0$. For $\gamma \in \mathbb{C}$ such that $|\gamma| \leq \sqrt{\alpha \beta}$, we define the map $\Phi_{\alpha, \beta, \gamma} : \mathbb{C} \to \mathbb{C}$ by

$$
\Phi_{\alpha, \beta, \gamma}(\lambda) = \lambda + \gamma \phi \left\{ (\lambda - x_0)^* \left( (\lambda - x_0)(\lambda - x_0)^* + s(\lambda)^2 \right)^{-1} \right\}.
$$

**Theorem 1.2** (See Theorem 5.4 and Theorem 5.8). Let $x_0 \in \log^+(A)$ be an operator that is $*$-free from the family of operators $\{g_{\alpha, \beta, \gamma}\}_{\gamma \leq \sqrt{\alpha \beta}}$ with amalgamation over $B$ such that $x_0$ is $*$-free from $B$ with respect to the tracial state $\phi$. Given $\alpha, \beta > 0$ such that $\alpha \neq \beta$, the support of the Brown measure $\mu_{x_0 + g_{\alpha, \beta, 0}}$ of $x_0 + g_{\alpha, \beta, 0}$ is the closure of the set $\Xi_{\alpha, \beta}$.

The Brown measure $\mu_{x_0 + g_{\alpha, \beta, 0}}$ is absolutely continuous and the density is strictly positive in $\Xi_{\alpha, \beta}$. Moreover, the density formula can be expressed in terms of $s(\lambda)$.

Given $\gamma \in \mathbb{C}$ such that $|\gamma| \leq \sqrt{\alpha \beta}$, the Brown measure $\mu_{x_0 + g_{\alpha, \beta, \gamma}}$ is the push-forward measure of $\mu_{x_0 + g_{\alpha, \beta, 0}}$ under the map $\Phi_{\alpha, \beta, \gamma}$. If, in addition, the map $\Phi_{\alpha, \beta, \gamma}$ is non-singular at any $\lambda \in \Xi_{\alpha, \beta}$, then it is also one-to-one in $\Xi_{\alpha, \beta}$.

The above Theorem 1.2 recovers the Brown measure formula obtained in [22, Section 4] and [5, Section 6] as a special case for $x_0 = 0$. See Example 5.9. We note that the map $\Phi_{\alpha, \beta, \gamma}$ could be singular in general. This restriction does not allow us to calculate the Brown measure $\mu_{x_0 + g_{\alpha, \beta, \gamma}}$ directly. We adapt an approach introduced in [33] as follows. We show that the regularized Brown measure $\mu^{(c)}_{x_0 + g_{\alpha, \beta, \gamma}}$ is the pushforward measure of the Brown measure $\mu^{(c)}_{x_0 + g_{\alpha, \beta, 0}}$ by some regularized map $\Phi^{(c)}_{\alpha, \beta, \gamma}$. We then show that the pushforward connection is preserved by passing to the limit. Hence, the following diagram commutes.
In [33], it was left open if the Brown measure of $\mu_{x_0+c_t}$ has any singular continuous part. We show that it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$ and the statement of Theorem 1.1 still holds for unbounded operators.

**Theorem 1.3** (See Theorem 7.12). The Brown measure $\mu_{x_0+c_t}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$ and the result in Theorem 1.1 holds for any $x_0 \in \log^+ (A)$ that is *-free from $\{c_t, g_t, \gamma\}$. Moreover, the density function $\rho_{t,x_0}$ of the Brown measure has an upper bound

$$\rho_{t,x_0}(\lambda) \leq \frac{1}{\pi t}.$$

In [9], Bordenave, Caputo and Chafaï investigated the spectrum of the Markov generator of random walk on a randomly weighted oriented graph. They identified the limit distribution as the Brown measure of $a + c$, where $a$ is a Gaussian distributed normal operator and $c$ is the standard circular operator, freely independent from $a$. Hence, Theorem 1.3 implies that $\mu_{a+c}$ is absolutely continuous and its density is bounded by $1/\pi$. Our main results also provide potential applications to unify the deformed i.i.d. random matrix model, the deformed Wigner random matrix model, and the deformed elliptic random matrix model.

We hope to study this question somewhere else.

By comparing precise formulas in Theorem 1.1 with Theorem 1.2, we can view the main results in [33] as the special case of our results when $\alpha = \beta = t$, where $\log \frac{\alpha}{\beta}$ is regarded as $\frac{1}{t}$.

**Corollary 1.4.** Under the same assumption as Theorem 1.2, given $\alpha, \beta > 0$, set $t = \frac{\alpha - \beta}{\log \alpha - \log \beta}$. Suppose $x_0$ is *-free from $\{c_t, g_t, \gamma\}$ in $(A, \phi)$. Then,

1. the Brown measure of $x_0 + g_{a,0}$ is the same as the Brown measure of $x_0 + c_t$. In particular, it is absolutely continuous in $\mathbb{C}$ and its density is bounded by $1/\pi t$;
2. the push-forward map $\Phi_{t,\gamma}$ between Brown measures $\mu_{x_0+g_{a,0}}$ and $\mu_{x_0+g_{a,\gamma}}$ is the same as the push-forward map $\Phi_{t,\gamma}$ between Brown measures $\mu_{x_0+c_t}$ and $\mu_{x_0+g_t,\gamma}$.

The paper has six more sections. We review some results from free probability theory in Section 2. We study the subordination functions in Section 3 and regularized Brown measure. We calculate the Brown measure formulas and prove the push-forward property in Section 4. In Section 5, we study some further properties of the push-forward map. In Section 6, we study the Brown measure of an addition with an elliptic operator.

2. Preliminaries

2.1. Free probability and subordination functions. An operator-valued $W^*$-probability space $(A, \mathcal{E}, \mathcal{B})$ consists of a von Neumann algebra $A$, a unital *-subalgebra $\mathcal{B} \subset A$, and a conditional expectation $\mathcal{E} : A \to \mathcal{B}$. Thus, $\mathcal{E}$ is a unital linear positive map satisfying:

1. $\mathcal{E}(b) = b$ for all $b \in \mathcal{B}$, and
2. $\mathcal{E}(b_1 x b_2) = b_1 \mathcal{E}(x) b_2$ for all $x \in A, b_1, b_2 \in \mathcal{B}$. Let
(\mathcal{A}_i)_{i \in I}$ be a family of subalgebras $\mathcal{B} \subset \mathcal{A}_i \subset \mathcal{A}$. We say that $(\mathcal{A}_i)_{i \in I}$ are free with amalgamation over $\mathcal{B}$ with respect to the conditional expectation $\mathbb{E}$ (or free with amalgamation in $(\mathcal{A}, \mathbb{E}, \mathcal{B})$) if for every $n \in \mathbb{N}$,

$$
\mathbb{E}(x_1 x_2 \cdots x_n) = 0
$$

whenever $x_j \in \mathcal{A}_{i_j}$ ($j = 1, 2, \cdots, n$) for some indices $i_1, i_2, \cdots, i_n \in I$ such that $i_1 \neq i_2, i_2 \neq i_3, \cdots, i_{n-1} \neq i_n$, and $\mathbb{E}(x_j) = 0$ for all $j = 1, 2, \cdots, n$.

Let $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ be an operator-valued $W^*$-probability space. The elements in $\mathcal{A}$ are called (noncommutative) random variables. We call

$$
\mathbb{H}^+(\mathcal{B}) = \{ b \in \mathcal{B} : \exists \varepsilon > 0, \Im(b) \geq \varepsilon 1 \}
$$

the Siegel upper half-plane of $\mathcal{B}$, where we use the notation $\Im(b) = \frac{1}{2i} (b - b^*)$. We set $\mathbb{H}^-(\mathcal{B}) = \{ -b : b \in \mathbb{H}^+(\mathcal{B}) \}$. The $\mathcal{B}$-valued Cauchy transform $G_X$ of any self-adjoint operator $X \in \mathcal{A}$ is defined by

$$
G_X(b) = \mathbb{E}\left[(b - X)^{-1}\right], \quad b \in \mathbb{H}^+(\mathcal{B}).
$$

Note that the $\mathcal{B}$-valued Cauchy transform $G_X$ is a map from $\mathbb{H}^+(\mathcal{B})$ to $\mathbb{H}^-(\mathcal{B})$, and it is one-to-one in $\{ b \in \mathbb{H}^+(\mathcal{B}) : ||b^{-1}|| < \varepsilon \}$ for sufficiently small $\varepsilon$. Voiculescu’s amalgamation $R$-transform is now defined for $X \in \mathcal{A}$ by

$$
R_X(b) = G_X^{-1}(b) - b^{-1}
$$

for $b$ being invertible element of $\mathcal{B}$ suitably close to zero.

Let $X, Y$ be two self-adjoint bounded operators that are free with amalgamation in $(\mathcal{A}, \mathbb{E}, \mathcal{B})$. The $R$-transform linearizes the free convolution in the sense that if $X, Y$ are free with amalgamation in $(\mathcal{A}, \mathbb{E}, \mathcal{B})$, then

$$
R_{X+Y}(b) = R_X(b) + R_Y(b)
$$

for $b$ in some suitable domain. There exist two analytic self-maps $\Omega_1, \Omega_2$ of the upper half-plane $\mathbb{H}^+(\mathcal{B})$ so that

$$
(\Omega_1(b) + \Omega_2(b) - b)^{-1} = G_X(\Omega_1(b)) = G_Y(\Omega_2(b)) = G_{X+Y}(b),
$$

for all $b \in \mathbb{H}^+(\mathcal{B})$. We refer the reader to [27, 29] for basic operator-valued free probability theory and [4, 8, 30] for operator-valued subordination functions.

2.2. **Hermitian reduction method and subordination functions.** Our approach is based on a Hermitian reduction method and (operator-valued) subordination functions. The Hermitian reduction method was used for the calculation of the Brown measure of quasi-nilpotent DT operators in Aagaard–Haagerup’s work [2]. The idea was refined in recent work [5]. The usage of such idea in Brown measure calculation also appears earlier in some physics papers (see [15, 23] for example). The idea of the Hermitian reduction method has a connection to the work of Girko [16] on circular law, though it is written in a different form.

Let $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ be an operator-valued $W^*$-probability space and let $\phi : \mathcal{A} \rightarrow \mathbb{C}$ be a tracial state on $\mathcal{A}$ such that $\phi(x) = \phi(\mathbb{E}(x))$ for any $x \in \mathcal{A}$. We equip the algebra $M_2(\mathcal{A})$, the $2 \times 2$ matrices with entries from $\mathcal{A}$, with the conditional expectation $M_2(\mathbb{E}) : M_2(\mathcal{A}) \rightarrow M_2(\mathcal{B})$ given by

$$
M_2(\mathbb{E}) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \mathbb{E}(a_{11}) & \mathbb{E}(a_{12}) \\ \mathbb{E}(a_{21}) & \mathbb{E}(a_{22}) \end{bmatrix},
$$
and \(M_2(\phi) : M_2(A) \to M_2(\mathbb{C})\) by
\[
(2.3) \quad M_2(\phi) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \phi(a_{11}) & \phi(a_{12}) \\ \phi(a_{21}) & \phi(a_{22}) \end{bmatrix}.
\]

Then the triple \((M_2(A), M_2(E), M_2(B))\) is a \(W^*\)-probability space such that \(M_2(\phi) \circ M_2(E) = M_2(\phi)\). Given \(x \in A\), let
\[
(2.4) \quad X = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \in M_2(A),
\]

which is a selfadjoint element in \(M_2(A)\). The \(M_2(B)\)-valued Cauchy transform is defined as
\[
(2.5) \quad G_X(b) = M_2(E) [(b - X)^{-1}], \quad b \in \mathbb{H}^+(M_2(B))
\]

which is an analytic function on \(\mathbb{H}^+(M_2(B))\). In particular, for \(\varepsilon > 0\) the element
\[
(2.6) \quad \Theta(\lambda, \varepsilon) = \begin{bmatrix} i\varepsilon & \lambda \\ \bar{\lambda} & i\varepsilon \end{bmatrix} \in M_2(\mathbb{C}) \subset M_2(B)
\]

belongs to the domain of \(G_X\), and
\[
(\Theta(\lambda, \varepsilon) - X)^{-1} = \begin{bmatrix} -i\varepsilon((\lambda - x)(\lambda - x)^* + \varepsilon^2)^{-1} & (\lambda - x)((\lambda - x)^*(\lambda - x) + \varepsilon^2)^{-1} \\ -i\varepsilon((\lambda - x)^*(\lambda - x)^* + \varepsilon^2)^{-1} & (\lambda - x)^*(\lambda - x) + \varepsilon^2)^{-1} \end{bmatrix}.
\]

Hence, by taking entry-wise trace on the \(M_2(B)\)-valued Cauchy transform, we have
\[
(2.7) \quad M_2(\phi)(G_X(\Theta(\lambda, \varepsilon))) = M_2(\phi)((\Theta(\lambda, \varepsilon) - X)^{-1})
\]
where
\[
\begin{align*}
g_{X,11}(\lambda, \varepsilon) &= -i\varepsilon \phi \left( ((\lambda - x)(\lambda - x)^* + \varepsilon^2)^{-1} \right), \\
g_{X,12}(\lambda, \varepsilon) &= \phi \left( (\lambda - x)((\lambda - x)^*(\lambda - x) + \varepsilon^2)^{-1} \right), \\
g_{X,21}(\lambda, \varepsilon) &= \phi \left( (\lambda - x)^*(\lambda - x)(\lambda - x)^* + \varepsilon^2)^{-1} \right), \\
g_{X,22}(\lambda, \varepsilon) &= -i\varepsilon \phi \left( (\lambda - x)^*(\lambda - x) + \varepsilon^2)^{-1} \right).
\end{align*}
\]

Recall that \(S(x, \lambda, \varepsilon)\) is defined in (1.2). We see immediately that
\[
i g_{X,11}(\lambda, \varepsilon) = \frac{1}{2} \frac{\partial S(x, \lambda, \varepsilon)}{\partial \varepsilon}, \quad g_{X,21}(\lambda, \varepsilon) = \frac{\partial S(x, \lambda, \varepsilon)}{\partial \lambda}.
\]

Hence, the entries of operator-valued Cauchy transform carry important information about the Brown measure.

Consider now two operators \(x, y \in A\) that are \(*\)-free with amalgamation over \(B\). We have to understand the \(M_2(B)\)-valued distribution of
\[
\begin{bmatrix} 0 & x + y \\ (x + y)^* & 0 \end{bmatrix} = X + Y = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix}
\]
in terms of the \(M_2(B)\)-valued distributions of \(X\) and of \(Y\). Note that \(X\) and \(Y\) are free in \((M_2(A), M_2(E), M_2(B))\) with amalgamation over \(M_2(B)\). The subordination functions
in this context are two analytic self-maps $\Omega_1, \Omega_2$ of the upper half-plane $\mathbb{H}^+(M_2(\mathcal{B}))$ so that
\begin{equation}
(\Omega_1(b) + \Omega_2(b) - b)^{-1} = G_X(\Omega_1(b)) = G_Y(\Omega_2(b)) = G_{X+Y}(b),
\end{equation}
for every $b \in \mathbb{H}^+(M_2(\mathcal{B}))$. We shall be concerned with a special type of $b$, namely $b = \Theta(\lambda, \varepsilon)$. More precisely, we want to understand the entries of the $M_2(\mathbb{C})$-valued Cauchy-transform
\[M_2(\phi)(G_{X+Y}(\Theta(\lambda, \varepsilon))) = M_2(\phi)\left(\begin{bmatrix} i\varepsilon & \lambda - x - y^* \\ (\lambda - x - y)^* & i\varepsilon \end{bmatrix}\right)^{-1}\]
The idea of calculating the Brown measure of $x + y$ is to separate the information of $X$ and $Y$ in some tractable way using subordination functions (2.9).

3. Extension of Subordination Functions to Unbounded Operators

Let $(A, \mathcal{E}, \mathcal{B})$ be an operator-valued $W^*$-probability space. Consider arbitrary $x, y \in A$ which are $*$-free over $\mathcal{B}$ with respect to $\mathcal{E}$, meaning that $W^*(\mathcal{B}, x)$ and $W^*(\mathcal{B}, y)$ are free with respect to $\mathcal{E}$ (when $x$ is unbounded, $W^*(\mathcal{B}, x)$ denotes the von Neumann algebra generated by $\mathcal{B}, u$, and the spectral projections of $\sqrt{x^*x}$ from the polar decomposition $x = u\sqrt{x^*x}$).

We denote $X = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix}$. By the similar result for bounded operators, then, $X, Y$ are free with amalgamation over $M_2(\mathcal{B})$ with respect to $M_2(\mathcal{E})\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [\mathbb{E}(a_{11}) \mathbb{E}(a_{12}) \\ \mathbb{E}(a_{21}) \mathbb{E}(a_{22})]$. The following result is an extension of Voiculescu’s subordination functions [10] in the operator-valued framework (see also [7]). The method is adapted from the approach used in [4]. It is essentially covered by [3, Corollary 6.6], and by Williams [32], but we provide here a more elementary argument. The reader is referred to [4, Section 2] for some details.

**Theorem 3.1.** If either $\mathcal{B}$ is finite dimensional or $y$ is bounded and $\exists \mathbb{E}|(b_0 - X)^{-1}| < 0$ for some $b_0 \in M_2(\mathcal{B}), \exists b_0 > 0$, then there exist analytic maps $\Omega_1, \Omega_2: \mathbb{H}^+(M_2(\mathcal{B})) \to \mathbb{H}^+(M_2(\mathcal{B}))$ such that
\begin{equation}
\Omega_1(b) + \Omega_2(b) - b = \mathbb{E}|(\Omega_1(b) - X)^{-1}|^{-1} = \mathbb{E}|(\Omega_2(b) - Y)^{-1}|^{-1} = \mathbb{E}|(b - X - Y)^{-1}|^{-1}.
\end{equation}

**Proof.** For simplicity, denote $F_{X+Y}(b) = \mathbb{E}|(b - X - Y)^{-1}|^{-1}$, and $h_{X+Y}(b) = F_{X+Y}(b) - b$.

Case one: assume that both $x$ and $y$ are unbounded and $\dim(\mathcal{B}) < \infty$. We shall obtain the $\Omega_1, \Omega_2$ as fixed points of an analytic map. We will use a slightly different trick, due to Hari Bercovici. Observe that, if one knows the two subordination functions to exist, then one has $h_X(\Omega_1(b)) + b = \Omega_2(b), h_Y(\Omega_2(b)) + b = \Omega_1(b)$. Hence, $\Omega_1$ satisfies
$$\Omega_1(b) = b + h_Y(b + h_X(\Omega_1(b)))$$
and similarly, $\Omega_2$ satisfies
$$\Omega_2(b) = b + h_X(b + h_Y(\Omega_2(b))).$$
This turns out to be equivalent to the map \( \Omega := (\Omega_1, \Omega_2) \) being the inverse to the right of another map, namely \( \Phi(w_1, w_2) = (w_1 - h_Y(w_2), w_2 - h_X(w_1)) \). More precisely, let us define the following extensions:

\[
\Omega_1(b_1, b_2) = b_1 + h_Y(b_2 + h_X(\Omega_1(b_1, b_2))),
\]
and

\[
\Omega_2(b_1, b_2) = b_2 + h_X(b_1 + h_Y(\Omega_2(b_1, b_2))).
\]

For \( N > 0 \), observe that \( \chi_{[-N,N]}(Y) \) is well-defined and belongs to \( M_2(W^*(B, y)) \). Indeed, \( -N \leq Y \leq N \iff Y^2 \leq N^2 \iff yY^*y \leq N^2 \iff yY^* \leq N^2 \).

Now, \( y = u\sqrt{y^*y}, y^* = v\sqrt{yy^*} \) for some partial isometries \( u, v \in W^*(B, y) \), so that\[
\begin{bmatrix}
y \\
y^*
\end{bmatrix} = Y = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} \sqrt{yy^*} & 0 \\ 0 & \sqrt{y^*y} \end{bmatrix}.\]
Thus, \( Y_N := \chi_{[-N,N]}(Y)Y \) is free from \( X \) over \( M_2(B) \) with respect to \( M_2(\mathbb{E}) \) as well.

The derivative of \( \Phi \) is

\[
D\Phi(w_1, w_2) = \begin{bmatrix}
\id_M(B) & -h_Y'(w_2) \\
h_X'(w_1) & \id_M(B)
\end{bmatrix},
\]
with a similar expression for \( D\Phi_N \), except that \( h_X \) is replaced by \( h_{X,N} \) and \( h_Y \) is replaced by \( h_{Y,N} \). We intend to show that \( D\Phi(\varepsilon, i\varepsilon) \) is close (in operator norm) to the identity on \( M_2(B) \times M_2(B) \) for \( \varepsilon \in (0, +\infty) \) large. Let us perform a more detailed analysis of the behavior of \( h_Y \). We recall that \( h_Y(w) = F_Y(w) - w = \mathbb{E}[(w-Y)^{-1}]^{-1} - w \). The expression of its derivative is

\[
h_Y'(w)(\alpha) = \mathbb{E}[(w-Y)^{-1}]^{-1} \mathbb{E}[(w-Y)^{-1} \alpha(w-Y)^{-1}]^{-1} \mathbb{E}[(w-Y)^{-1}]^{-1} - \alpha.
\]

For a fixed \( w = \Re w + i\Im w \in \mathbb{H}^+(M_2(B)) \), let us look at \( \mathbb{C}^+ \ni z \mapsto (\Re w + z\Im w - Y)^{-1} \). For any positive continuous linear functional \( \varphi \) on \( M_2(\mathbb{A}) \), the function \( \mathbb{C}^+ \ni z \mapsto \varphi((\Re w + z\Im w - Y)^{-1}) \) sends \( \mathbb{C}^+ \) to \( \mathbb{C}^- \) and satisfies

\[
\lim_{\varepsilon \to +\infty} i\varepsilon \varphi(((\varepsilon\Im w - (Y - \Re w))^{-1}) = \varphi((\Im w)^{-1}).
\]

Since \( \varphi \) is arbitrary, it follows that

\[
\lim_{\varepsilon \to +\infty} i\varepsilon ((\varepsilon\Im w - (Y - \Re w))^{-1} = (\Im w)^{-1}
\]
in the weak sense. However, this tells us something more: it guarantees that \( \mathbb{C}^+ \ni z \mapsto \varphi((\Re w + z\Im w - Y)^{-1}) \) is the Cauchy transform of a positive Borel measure on \( \mathbb{R} \) of total mass \( \varphi((\Im w)^{-1}) \leq \|((\Im w)^{-1}) \| \), independently of the state \( \varphi \). For given state \( \psi \) on \( M_2(B) \), recall that \( \mathbb{E} \) is weakly continuous, we then apply the above to \( \varphi = \psi \circ \mathbb{E} \) to obtain the weak convergence

\[
\lim_{\varepsilon \to +\infty} i\varepsilon \mathbb{E}(((\varepsilon\Im w - (Y - \Re w))^{-1} = (\Im w)^{-1}
\]
bounded by \( \|((\Im w)^{-1}) \| \). Since \( B \) is finite-dimensional, the weak and norm topologies coincide. Thus, one has

\[
\lim_{\varepsilon \to +\infty} i\varepsilon \mathbb{E}(((\varepsilon\Im w - (Y - \Re w))^{-1} = (\Im w)^{-1} \text{ in norm.}
\]

\(^1\)Since \( (\Im w)^{-1/2}((\Re w) - \Re w)(\Im w)^{-1/2} \) is a selfadjoint in \( M_2(\mathbb{A}) \), it has a resolution of unity \( e \). If \( \psi \) is an arbitrary positive linear functional, then \( z \mapsto \psi((z - (\Im w)^{-1/2}((\Re w) - \Re w)(\Im w)^{-1/2})^{-1} = \psi((\int_{\mathbb{S}}(z - t)^{-1} d\mu(t)) = \int_{\mathbb{S}}(z - t)^{-1} d\psi(\alpha)(t) \), and \( (\psi \circ e) \) is a positive measure of total mass \( \psi(1) = \lim_{\varepsilon \to +\infty} i\varepsilon \int_{\mathbb{S}}(\varepsilon - t)^{-1} d\psi(\alpha)(t) \). Apply these facts to \( \psi(\cdot) = \varphi((\Im w)^{-1/2} \cdot (\Im w)^{-1/2}) \).
As a consequence, \( \lim_{x \to +\infty} \frac{\varepsilon}{2\pi} \mathbb{E} \left[ (i\varepsilon 3w - (Y - \Re w))^{-1} \right] = \Im w \), again in norm.

Thus, for any \( w \in \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \), there is an \( \varepsilon_w > 0 \) such that \( \| h'_Y(\Re w + i\varepsilon 3w) \| < 1 \) for all \( \varepsilon \geq \varepsilon_w \). Using the formula of \( D\Phi(w_1, w_2) \) and elementary arithmetic operations, it follows that there exists an open set of elements \( (w_1, w_2) \in \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \times \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \) on which \( D\Phi(w_1, w_2) \) is invertible as a linear operator from \( \mathcal{M}_2(\mathcal{B}) \times \mathcal{M}_2(\mathcal{B}) \) to itself.

The implicit function theorem guarantees the existence of a right inverse in a neighborhood of each point in this set. We argue next that these inverses extend to a unique \( \Omega \) defined on all of \( \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \times \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \).

We let \( \Phi_N(w_1, w_2) = (w_1 - h_{Y_N}(w_2), w_2 - h_{X_N}(w_1)) \). Direct computation shows that \( h_{Y_N}(w_2) \rightarrow h_Y(w_2) \) pointwise in the weak topology, and hence, since \( \mathcal{B} \) is finite dimensional, uniformly on compact subsets of \( \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \) in the norm topology of \( \mathcal{M}_2(\mathcal{B}) \).

Thus, \( \Phi_N \rightarrow \Phi \) in the same topology, but this time on \( \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \times \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \). As \( X_N, Y_N \) are bounded, there exist functions

\[
\Omega_N : \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \times \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \rightarrow \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \times \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B}))
\]

such that \( \Phi_N \circ \Omega_N = \text{id}_{\mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \times \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B}))} \). We let \( N \rightarrow \infty \), so that \( \Phi_N \rightarrow \Phi \), with \( \Phi(w_1, w_2) = (w_1 - h_Y(w_2), w_2 - h_X(w_1)) \) as above. Clearly \( \{ \Omega_N \}_{N \in \mathbb{N}} \) is a normal family, so it has a convergent subsequence. By the identity principle and the existence of a local inverse for \( \Phi \), any limit point must still satisfy \( \Phi \circ \Omega = \text{id}_{\mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B})) \times \mathbb{H}^\uparrow(\mathcal{M}_2(\mathcal{B}))} \).

This completes the argument.

We have established that analytic subordination functions exist for any \( \mathcal{B} \)-free \( x, y \in \mathring{\mathcal{A}} \) if \( \dim(\mathcal{B}) < \infty \).

Case two: assume that \( y \) is bounded and \( x \) is unbounded, but such that the Cauchy transform \( \mathbb{E} \left[ (b - X)^{-1} \right] \) takes values in the operator lower half-plane whenever \( b \in \mathbb{H}^\uparrow(\mathcal{B}); \) that immediately implies \( h_X \) is a well-defined analytic map, with values in \( \mathbb{H}^\uparrow(\mathcal{B}) \). Observe also that it is enough to show that \( \Im \mathbb{E} \left[ (b - X)^{-1} \right] < 0 \) for a single \( b \in \mathbb{H}^\uparrow(\mathcal{B}) \) in order to conclude that the inequality is satisfied for all \( b \in \mathbb{H}^\uparrow(\mathcal{B}) \). Pick \( b_1, b_2 \in \mathbb{H}^\uparrow(\mathcal{B}) \) (since there are no supplementary conceptual difficulties involved in proving the full analogue of the result from Case one, we choose to prove it, even though we will only apply it with \( b_1 = b_2 = b \)). Define

\[
f_{b_1, b_2}(w) = b_1 + h_Y(b_2 + h_X(w)), \quad w \in \mathbb{H}^\uparrow(\mathcal{B}).
\]

Since \( \Im \mathbb{E} \left[ (w - X)^{-1} \right] < 0 \) for all \( w \) in the operator upper half-plane, it follows that \( h_X \) is an analytic map defined on \( \mathbb{H}^\uparrow(\mathcal{B}) \) and taking values in its closure. Thus, \( b_2 + h_X(\mathbb{H}^\uparrow(\mathcal{B})) \subseteq \mathbb{H}^\uparrow(\mathcal{B}) + b_2/2 \). By \( \mathcal{H} \) Lemma 2.12 it follows that \( f_{b_1, b_2}(\mathbb{H}^\uparrow(\mathcal{B})) \) is a bounded set which is bounded away from the complement of \( \mathbb{H}^\uparrow(\mathcal{B}) \). Specifically, the lemma shows that if \( \varepsilon \in (0, +\infty) \) is fixed, then for all \( w \in \mathbb{H}^\uparrow(\mathcal{B}) + i \varepsilon 1 \), one has

\[
h_Y(\mathbb{H}^\uparrow(\mathcal{B}) + i \varepsilon 1) \subset \{ v \in \mathbb{H}^\uparrow(\mathcal{B}) : \| v \| \leq 4\| Y\| (1 + 2\varepsilon^{-1}\| Y\|) \}.
\]

Thus, if we choose \( \varepsilon > 0 \) such that \( \Im b_j > \varepsilon 1 \), \( j = 1, 2 \), then

\[
f_{b_1, b_2}(\mathbb{H}^\uparrow(\mathcal{B})) \subset b_1 + h_Y(\mathbb{H}^\uparrow(\mathcal{B}) + i \varepsilon 1)
\]

\[
\subset \{ v \in \mathbb{H}^\uparrow(\mathcal{B}) + i \varepsilon 1 : \| v \| \leq 4\| Y\| (1 + 2\varepsilon^{-1}\| Y\|) \}.
\]

Thus, one may apply the Earle-Hamilton Theorem (See \( \mathcal{H} \) Section 2.11) to the function \( f_{b_1, b_2} \) defined on the set \( \{ v \in \mathbb{H}^\uparrow(\mathcal{B}) + i \varepsilon 1 : \| v \| \leq 4\| Y\| (1 + 2\varepsilon^{-1}\| Y\|) + 2\| b_1 \| + 2\| b_2 \| \} \) in order to conclude that \( \{ f_{b_1, b_2}^n \}_{n \in \mathbb{N}} \) converges in norm to a fixed point \( \omega(1, 2) \) which depends analytically on \( b_1 \) and \( b_2 \). One simply defines \( \omega_2(b_1, b_2) = b_2 + h_X(\omega_1(b_1, b_2)) \), choosing \( b_1 = b_2 = b \) proves our claim. \( \square \)
Remark 3.2. The results in Theorem 3.1 have a fully matricial extension following [30, 31]. For any $n \in \mathbb{N}$, we define the map $G_{X,n}(b) = (\mathbb{E} \otimes I_{Q_{n+1}})[(b - X \otimes I_{n})^{-1}]$ for $b \in \mathcal{B} \otimes M_{n}$. If $\varnothing(b) > \varepsilon$ for some $\varepsilon > 0$. The map is a noncommutative (nc) function and a remarkable observation due to Voiculescu is that the family $\{G_{X,n}\}$ can retrieve the distribution of $X$. In this framework, by applying the same proof to Theorem 3.1 we deduce that there are nc functions $\Omega_j, n(j = 1, 2)$ such that $G_{X + Y,n}(b) = G_{X,n}(\Omega_1,n(b)) = G_{Y,n}(\Omega_2,n(b))$. The interested reader is referred to [4, 30, 31] for some details.

4. The sum with a triangular elliptic operator

4.1. A review on triangular elliptic operator. By enlarging the operator-valued $W^*$-probability space $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ if necessary, we assume that $\mathcal{M}, \mathcal{N}$ are unital von Neumann subalgebras of $\mathcal{A}$ such that:

- $\mathcal{M}, \mathcal{N}$ are free with amalgamation over $\mathcal{B}$ with respect to $\mathbb{E}$;
- the tracial state $\phi$ on $\mathcal{A}$ satisfies $\phi(x) = \phi(\mathbb{E}(x))$ for any $x \in \mathcal{A}$;
- $\mathbb{E}(x) = \phi(x)$ for any $x \in \mathcal{N}$.

The space we work on is the special case when $\mathcal{B} = L^\infty[0,1]$. The tracial state $\phi : \mathcal{A} \to \mathbb{C}$ satisfy $\phi(y) = \phi(\mathbb{E}(y))$ for $y \in \mathcal{A}$, and

$$\phi(f) = \int_0^1 f(s) ds$$

for $f \in \mathcal{B}$. The existence of such subalgebra $\mathcal{N}$ can be verified as follows: let $\mathcal{B} \subset \mathcal{M}$ and let $\mathcal{N}$ be a unital subalgebra that is $*$-free from $\mathcal{B}$ with respect to the tracial state $\phi$ and let $\mathcal{N}_1$ be the unital subalgebra generated by $\mathcal{N} \cup \mathcal{B}$. We then choose $\mathcal{A} = \mathcal{M} * \mathcal{B} \mathcal{N}_1$ and identify $\mathcal{N}$ as some subalgebra in $\mathcal{N}_1 \subset \mathcal{A}$. We note that $\mathcal{N}$ is $*$-free from $\mathcal{B}$ in $(\mathcal{A}, \phi)$.

Let us review some basic properties of DT-operators introduced by Dykema–Haagerup [13, 14]. Assume that $(X_i)_{i=1}^\infty \subset \mathcal{M}$ is a standard semicircular family of random variables that are $*$-free from $\mathcal{B}$ in the $W^*$-probability space $(\mathcal{A}, \phi)$. Then for $i \in \mathbb{N}$, one can construct a quasi-nilpotent DT operator $T_i$ as the norm limit of

$$T_n^{(i)} = \sum_{j=1}^n 1_{\left[\frac{j-1}{n}, \frac{j}{n}\right]} X_i 1_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}$$

as $n \to \infty$, where $1_{[a,b]}$ is viewed as an element in $\mathcal{B}$. It follows that $T_i \in W^*(\mathcal{B} \cup \{X_i\})$ for all $i \in \mathbb{N}$ and $(T_i)_{i=1}^\infty$ are free with amalgamation over $\mathcal{B}$ in $(\mathcal{A}, \mathbb{E}, \mathcal{B})$. It is shown that $(T_i, T_i^*)_{i=1}^\infty$ is a centered $\mathcal{B}$-Gaussian family in [14 Appendix A].

Recall that $\alpha, \beta > 0$, and $\gamma \in \mathbb{C}$ such that $|\gamma| \leq \sqrt{\alpha \beta}$. The triangular elliptic element $y = g_{\alpha,\beta,\gamma} \in \mathcal{M}$ is an operator, introduced in [8 Section 6], whose only nonzero free cumulants are given by

$$\kappa(y, fy)(t) = \gamma \int_0^t f(s) ds,$$

$$\kappa(y^*, fy^*)(t) = \bar{\gamma} \int_0^t f(s) ds,$$

(4.1)

Theorem 4.1. For any $t \geq 0$, there is a unique $\mathcal{N}$-valued measurable map $\Omega_1, \Omega_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\kappa(y_{\alpha,\beta,\gamma}, mo)(t) = \int_0^t \kappa(y_{\alpha,\beta,\gamma}^*, fy_{\alpha,\beta,\gamma})(s) ds$$

for all $t \geq 0$, where $mo$ denotes the subalgebra generated by $\{T_i\}_{i=1}^\infty$.

Theorem 4.2. Let $\{T_i\}_{i=1}^\infty$ be a centered $\mathcal{B}$-Gaussian family in $(\mathcal{A}, \mathbb{E}, \mathcal{B})$. For any $t \geq 0$, there is a unique $\mathcal{N}$-valued measurable map $\Omega_1, \Omega_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\kappa(y_{\alpha,\beta,\gamma}, mo)(t) = \int_0^t \kappa(y_{\alpha,\beta,\gamma}^*, fy_{\alpha,\beta,\gamma})(s) ds$$

for all $t \geq 0$, where $mo$ denotes the subalgebra generated by $\{T_i\}_{i=1}^\infty$.
for every $f \in \mathcal{B}$. We say that such triangular elliptic operator $y$ has parameter $(\alpha, \beta, \gamma)$. One can construct the triangular elliptic operator (\mathcal{B}-circular element as studied by Dykema \cite{Dykema}) as follows. Let $T$ be a quasi-nilpotent DT operator. By \cite{Dykema} Appendix A, the distribution of the pair $T, T^*$ is a $\mathcal{B}$-Gaussian distribution determined by the free cumulants given by

$$\kappa(T, fT^*)(t) = \int_0^1 f(s) ds, \quad \kappa(T^*, fT)(t) = \int_0^t f(s) ds,$$

and $\kappa(T, fT) = \kappa(T^*, fT^*) = 0$. Write $\gamma = e^{2\theta_1} |\gamma|$. Then the triangular elliptic operator $y = g_{\alpha\beta,\gamma}$ has the same $*$-moments as $e^{i\theta}|\sqrt{T} + \sqrt{T}^*|$. In particular, a quasi-nilpotent DT operator is a triangular elliptic element with parameter $\alpha = 1, \beta = \gamma = 0$. When $\alpha = \beta = t$, the operator $g_{\alpha,\beta,0}$ is just the Voiculescu’s circular operator $c_t$ of variance $t$, and the operator $g_{\alpha,\beta,\gamma}$ is the so-called twisted elliptic operator $g_{t,\gamma}$ as studied in \cite{Voiculescu} Section 2.4.

Let $x_0 \in \log^+(\mathbb{N})$ that is $*$-freely independent from $g_{\alpha,\beta,\gamma}$ with amalgamation over $\mathcal{B}$. The main object of this paper is to calculate the Brown measure of $x_0 + g_{\alpha,\beta,\gamma}$. In the $M_2(\mathcal{B})$-valued $C^*$-probability space $(M_2(A), M_2(E), M_2(B))$, we denote

$$X = \begin{bmatrix} 0 & x_0 \\ x_0^* & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & y \\ g^* & 0 \end{bmatrix}.$$ 

Then $X, Y$ are free over $M_2(\mathcal{B})$. This can be deduced from the connection between $\mathcal{B}$-valued free cumulants and $M_2(\mathcal{B})$-valued free cumulants similar to \cite{Nica} Section 9.3, Proposition 13.

The following result adapts the identification used in \cite{Belinschi} Lemma 5.6, where the case $x_0 = 0$ was considered.

**Proposition 4.1.** Given $\alpha \geq \beta = t > 0$, $|\gamma| \leq t$, and a bounded operator $x_0 \in \mathbb{N}$, let $g_{t,\gamma}$ be a twisted elliptic operator $*$-free from $\mathcal{M} \cup \mathbb{N}$ with respect to $\phi$. Then,

$$x_0 + g_{\alpha,\beta,\gamma} \stackrel{\text{dist.}}{\sim} x_0 + g_{(\alpha - \beta)0,0} + g_{t,\gamma},$$

where $g_{(\alpha - \beta)0,0}$ denotes a triangular elliptic operator in $\mathcal{M}$ with parameters $(\alpha - \beta, 0, 0)$.

In particular, when $\alpha = \beta$, the operator $g_{\alpha,\beta,\gamma}$ is $*$-free from $\mathcal{N}$ in the scalar-valued $W^*$-probability space $(A, \phi)$.

**Proof.** It is clear that $g_{\alpha,\beta,\gamma}$ has the same $*$-distribution as $g_{t,\gamma} + g_{(\alpha - \beta)0,0}$, where $g_{t,\gamma}$ and $g_{(\alpha - \beta)0,0}$ are free with amalgamation over $\mathcal{B}$. We express $g_{t,\gamma}$ as a linear combination $e^{2i\theta_1 \sqrt{t} (T_i + T_i^*)}$ for some quasi-nilpotent DT operators $T_i$. The operator-valued free cumulants of $g_{t,\gamma}$ in $(A, \mathcal{E}, \mathcal{B})$ are complex numbers. By applying \cite{Voiculescu} Theorem 3.6, it follows that $g_{t,\gamma}$ is $\mathcal{C}$-Gaussian and is $*$-free from $\mathcal{B}$ with respect to $\phi$. Since $T_j \in W^*(\mathcal{B} \cup \{X_j\})$ and $(X_j)_{j=1}^\infty \subset \mathcal{M}$ is a standard semicircular family of random variables that are $*$-free from $\mathcal{B}$ in the $W^*$-probability space $(A, \phi)$, it follows that $g_{t,\gamma} = e^{2i\theta_1 \sqrt{t} (T_i + T_i^*)}$ is $*$-free with respect to $\phi$ from other quasi-nilpotent DT operator $T_j$ in $(A, \phi)$ for $j \neq i$. Hence, $g_{t,\gamma}$ is also $*$-free from $x_0 + g_{(\alpha - \beta)0,0}$ with respect to $\phi$. \hfill $\square$

Hence, if $|\gamma| \leq \min\{\alpha, \beta\}$, the Brown measure of $x_0 + g_{\alpha,\beta,\gamma}$ is the same as the Brown measure of $y_0 + g_{t,\gamma}$ for $y_0 = x_0 + g_{(\alpha - \beta)0,0}$ where $g_{(\alpha - \beta)0,0}$ denotes a triangular elliptic operator with parameter $(\alpha - \beta, 0, 0)$ and has the same distribution as a scalar multiple of a quasi-nilpotent DT operator. By \cite{Voiculescu}, we can express the Brown measure formula in terms of certain subordination functions determined by $y_0$ if $|\gamma| \leq \min\{\alpha, \beta\}$ and $x_0$ is bounded. We will extend the method in \cite{Voiculescu} such that we are able to express the Brown measure formula for $x_0 + g_{\alpha,\beta,\gamma}$ in terms of some functions determined directly by $x_0$. 


for any $\gamma$ such that $|\gamma| \leq \sqrt{\alpha \beta}$. In addition, we show that the main results hold for any $x_0 \in \log^{-1}(N)$.

4.2. The operator-valued subordination function. For parameter $t = (\alpha, \beta, \gamma)$ where $\alpha, \beta > 0$ and $\gamma \in \mathbb{C}$ such that $|\gamma| \leq \sqrt{\alpha \beta}$, we denote $y = g_{\alpha, \beta, \gamma}$, and we take the convention that $y = 0$ if $\alpha = \beta = \gamma = 0$. For $\lambda \in \mathbb{C}$, we have

$$S(x_0 + y, \lambda, \varepsilon) = \log \Delta \left[ (x_0 + y - \lambda)^*(x_0 + y - \lambda) + \varepsilon^2 \right], \quad \varepsilon > 0.$$  \hfill (4.2)

We introduce the following notations

$$p_t^x(\varepsilon) = \phi \left\{ (\lambda - x_0 - y)^* [(\lambda - x_0 - y)(\lambda - x_0 - y)^* + \varepsilon^2]^{-1} \right\},$$  \hfill (4.3)

$$q_t^x(\lambda) = \varepsilon \phi \left\{ [(\lambda - x_0 - y)^*(\lambda - x_0 - y) + \varepsilon^2]^{-1} \right\}.$$  

and

$$P_t^x(\varepsilon) = E \left\{ (\lambda - x_0 - y)^* [(\lambda - x_0 - y)(\lambda - x_0 - y)^* + \varepsilon^2]^{-1} \right\},$$  \hfill (4.4)

$$Q_t^x(\lambda) = \varepsilon E \left\{ [(\lambda - x_0 - y)(\lambda - x_0 - y)^* + \varepsilon^2]^{-1} \right\}.$$

Note that $p_t^x(\varepsilon), p_t^x(\lambda)$ and $q_t^x(\lambda)$ are derivatives of $S(x_0 + y, \lambda, \varepsilon)$ with respect to $\lambda, \lambda$ and $\varepsilon$ (up to some constant). They are also related to entries of the Cauchy transform of the Hermitian reduction for $x_0 + y$. It is clear that, for $\lambda \in \mathbb{C}$ and $\varepsilon > 0$, we have

$$\phi(P_t^x(\varepsilon)) = p_t^x(\varepsilon), \quad \phi(P_t^x(\varepsilon)) = p_t^x(\varepsilon),$$

and, by the tracial property, we have

$$\phi(Q_t^x(\lambda)) = \phi(Q_t^x(\lambda)) = q_t^x(\lambda).$$  \hfill (4.5)

Therefore, we have

$$G_{X+Y} \left( \begin{bmatrix} i\varepsilon & \lambda \\ \lambda & i\varepsilon \end{bmatrix} \right) = M_2(E) \left( \left( X + Y - \begin{bmatrix} i\varepsilon & \lambda \\ \lambda & i\varepsilon \end{bmatrix} \right)^{-1} \right) = \begin{bmatrix} -iQ_t^x(\lambda) & P_t^x(\varepsilon) \\ P_t^x(\varepsilon) & -iQ_t^x(\lambda) \end{bmatrix}. \hfill (4.6)$$

For any $\varepsilon_1, \varepsilon_2 \in \mathbb{B}$, set

$$g_{11}(\lambda, \varepsilon_1, \varepsilon_2) = -i\varepsilon_2 E \left\{ [(\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1 \varepsilon_2]^{-1} \right\},$$

$$g_{12}(\lambda, \varepsilon_1, \varepsilon_2) = E \left\{ (\lambda - x_0)(\lambda - x_0)^* [(\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1 \varepsilon_2]^{-1} \right\},$$

$$g_{21}(\lambda, \varepsilon_1, \varepsilon_2) = E \left\{ [(\lambda - x_0)^*(\lambda - x_0) + \varepsilon_1 \varepsilon_2]^{-1} \right\},$$

$$g_{22}(\lambda, \varepsilon_1, \varepsilon_2) = -i\varepsilon_1 E \left\{ [(\lambda - x_0)^*(\lambda - x_0) + \varepsilon_1 \varepsilon_2]^{-1} \right\}. \hfill (4.7)$$

For convenience, we also denote

$$g_{ij} = g_{ij}(\lambda, \varepsilon_1, \varepsilon_2), \quad i, j \in \{1, 2\}. \hfill (4.7)$$

Then, we have

$$G_X \left( \begin{bmatrix} i\varepsilon_1 & \lambda \\ \lambda & i\varepsilon_2 \end{bmatrix} \right) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}. \hfill (4.6)$$

Proposition 4.2. The operator $Y$ is an operator-valued semicircular element in $(M_2(A), M_2(\mathbb{E}), M_2(B))$. For any $b = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(B)$, the $R$-transform of $Y$ is given by

$$R_Y(b) = M_2(\mathbb{E})(YbY) = \begin{bmatrix} \kappa(y, a_{22}y^*) & \kappa(y, a_{21}y) \\ \kappa(y^*, a_{12}y^*) & \kappa(y^*, a_{11}y) \end{bmatrix}.$$ 

Proof. It follows from a result connecting matrix-valued free cumulants with free cumulants as in [24, Section 9.3, Proposition 13], but we have to modify it appropriately by replacing scalar-valued free cumulants by operator-valued free cumulants in $(A, \mathbb{E}, B)$.

Lemma 4.3. Let $y = g_{\alpha,\beta,\gamma} \in \mathcal{M}$ and $x_0 \in \mathcal{N}$ be a random variable that is $\bullet$-free from $y$ with amalgamation over $B$. For any $\varepsilon > 0$ and $z \in \mathbb{C}$, set

$$\lambda = z - \kappa(y, P^t_z(\varepsilon)y),$$

where $P^t_z(\varepsilon)$ is defined in (4.4). For $\Theta(z, \varepsilon) = \begin{bmatrix} i\varepsilon & z \\ z & i\varepsilon \end{bmatrix}$ we have subordination relation

$$G_{X+Y}(\Theta(z, \varepsilon)) = G_X(\Omega_1(\Theta(z, \varepsilon))),$$

where

$$\Omega_1(\Theta(z, \varepsilon)) = \begin{bmatrix} i\varepsilon_1 & \lambda \\ \lambda & i\varepsilon_2 \end{bmatrix},$$

and

$$\varepsilon_1 = \varepsilon + \kappa(y, \tilde{Q}^t(z)y^*), \quad \varepsilon_2 = \varepsilon + \kappa(y^*, Q^t(z)y).$$

In other words, the subordination relation $G_{X+Y}(b) = G_X(\Omega_1(b))$ for $b = \Theta(z, \varepsilon)$ can be written as

$$E \left( \left[ \begin{array}{cc} i\varepsilon & z - (x_0 + y) \\ z & i\varepsilon \end{array} \right]^{-1} \right) = E \left( \left[ \begin{array}{cc} i\varepsilon_1 & \lambda - x_0 \\ \lambda - x_0 & i\varepsilon_2 \end{array} \right]^{-1} \right).$$

Moreover, the above subordination relation is equivalent to

$$g_{11} = -iQ^t(z), \quad g_{22} = -i\tilde{Q}^t(z), \quad g_{12} = P^t_z(\varepsilon), \quad g_{21} = P^t_z(\varepsilon),$$

where $\mathbf{t} = (\alpha, \beta, \gamma)$ and $g_{ij}$ is defined in (4.7).

Proof. We first verify that Theorem 3.1 applies to our context. Recall that we have $(B, \phi) = (L^\infty[0, 1], ds)$ and $E_{W^*(x_0)} = \phi$, where $W^*(x_0)$ denotes the von Neumann algebra generated by $x_0$. It follows that

$$-E[(i - X)^{-1}] = -E \left( \left[ \begin{array}{c} i \\ -x_0 \end{array} \right] \phi((1 + x_0x_0^{-1})^{-1}x_0) \right).$$

The imaginary part of the above matrix is

$$-\Im E[(i - X)^{-1}] = \begin{bmatrix} \phi((1 + x_0x_0^{-1})^{-1}) & 0 \\ 0 & \phi((1 + x_0^{-1}x_0)^{-1}) \end{bmatrix},$$

which is strictly positive by the faithfulness of the trace. Hence, the condition for Case two in Theorem 3.1 is satisfied.

For $b = \Theta(z, \varepsilon)$, by (3.1) in Theorem 3.1 the subordination functions satisfy

$$\Omega_1(b) + \Omega_2(b) = b + (G_{X+Y}(b))^{-1}.$$
The subordination relation $G_Y(Ω_2(b)) = G_{X+Y}(b)$ yields

$$Ω_2(b) = R_Y(G_{X+Y}(b)) + (G_{X+Y}(b))^{-1},$$

provided that $\|b^{-1}\|$ is small enough. It follows that

$$(4.14) \quad Ω_1(b) = b - R_Y(G_{X+Y}(b)),$$

and this is true for any $b ∈ H^+(M_2(\mathbb{B}))$ because $R_Y ∘ G_{X+Y}$ is defined for any $b ∈ H^+(M_2(\mathbb{B}))$ thanks to Proposition 4.2. Consequently, (4.14) holds for any $b = Θ(z, ε)$ with $ε > 0$.

By definitions (4.4), the Cauchy transform can be expressed as

$$G_{X+Y}(b) = M_2(\mathbb{E}) \left[(b - X - Y)^{-1}\right]$$

$$= \begin{bmatrix}
-\iota Q_b^t(z) & P_b^t(\varepsilon) \\
P_b^t(\varepsilon) & -\iota Q_b^t(z)
\end{bmatrix}.$$  

The formula for $R$-transform of $Y$ (Proposition 4.2) implies

$$R_Y(G_{X+Y}(b)) = \begin{bmatrix}
-\iota \kappa(y, \bar{Q}_b^t(z)y^*) & \kappa(y, P_b^t(\varepsilon)y) \\
\kappa(y^*, P_b^t(\varepsilon)y^*) & -\iota \kappa(y^*, \bar{Q}_b^t(z)y)
\end{bmatrix}.$$  

Hence

$$Ω_1(b) = b - R_Y(G_{X+Y}(b))$$

$$= \begin{bmatrix}
iε + \iota \kappa(y, \bar{Q}_b^t(z)y^*) & z - \kappa(y, P_b^t(\varepsilon)y) \\
\varepsilon - \kappa(y^*, P_b^t(\varepsilon)y^*) & iε + \iota \kappa(y^*, \bar{Q}_b^t(z)y)
\end{bmatrix}$$

$$= \begin{bmatrix}
iε_1 & λ \\
λ & iε_2
\end{bmatrix},$$

where $λ = z - \kappa(y, P_b^t(\varepsilon)y) \in \mathbb{C}$ (note that $\kappa(y, P_b^t(\varepsilon)y) = \kappa(y^*, P_b^t(\varepsilon)y^*)$). Therefore, we have

$$G_X(Ω_1(b)) = M_2(\mathbb{E}) \left[(Ω_1(b) - X)^{-1}\right]$$

$$= \begin{bmatrix}g_{11} & g_{12} \\
g_{21} & g_{22}\end{bmatrix} ∈ M_2(\mathbb{B}).$$

This establishes (4.9) and (4.11). By comparing (4.13) with (4.17), we have

$$g_{11} = -\iota Q_b^t(z), \quad g_{22} = -\iota \bar{Q}_b^t(z), \quad g_{12} = P_b^t(\varepsilon), \quad g_{21} = P_b^t(\varepsilon).$$

This finishes the proof. □

**Lemma 4.4.** Fix $z ∈ \mathbb{C}$ and $ε > 0$. Using notations in Lemma 4.3 let

$$D = Ε \left[((\lambda - x_0)(\lambda - x_0)^* + ε_1 ε_2)^{-1}\right],$$

and

$$\bar{D} = Ε \left[((\lambda - x_0)^*(\lambda - x_0) + ε_1 ε_2)^{-1}\right].$$

If $N$ is $*$-free from $\mathcal{B}$ in $(A, φ)$, then $ε_1 ε_2$ and $D$ are constant functions in $\mathcal{B} = L^∞[0, 1]$. Consequently,

$$(4.18) \quad D = \bar{D} = φ \left[((\lambda - x_0)^*(\lambda - x_0) + ε_1 ε_2)^{-1}\right].$$
Proof. We note that $Q^t_\xi(z), \tilde{Q}^t_\xi(z)$ are strictly positive functions in $\mathbb{B} = L^\infty(0,1)$ by the definition \eqref{4.14}. Hence, $\kappa(y, \tilde{Q}^t_\xi(z)y^*) > 0$ and $\kappa(y^*, Q^t_\xi(z)y) > 0$. Consequently, by the defining identity \eqref{4.10},

$$
\epsilon_1 = \epsilon + \kappa(y, \tilde{Q}^t_\xi(z)y^*) \geq \epsilon.
$$

Similarly, $\epsilon_2 > \epsilon$. Since for any $x \in \mathbb{N}$, we have

\begin{equation}
\mathbb{E}[p(x, x^*)] = \phi[p(x, x^*)]
\end{equation}

where $p$ is an arbitrary polynomial of two indeterminates. Given any $x \in \mathbb{N}, b \in \mathbb{B}$ and $n \in \mathbb{N}$, since $\mathbb{N}$ is $*$-free from $\mathbb{B}$ in $(A, \phi)$, we hence have

$$
\mathbb{E}((x^*xb)^n) = \mathbb{E}((xx^*b)^n),
$$

where we used the tracial property of $\phi$. If $b \in \mathbb{B}$ is invertible and $\|b\|$ is large enough, we can write

$$(xx^* + b)^{-1} = \sum_{n=0}^{\infty} (-1)^n b^{-1}(xx^*b^{-1})^n.$$}

It follows that, if $\|b\|$ is large enough, we have

\begin{equation}
\mathbb{E}[(xx^* + b)^{-1}] = \mathbb{E}[(x^*x + b)^{-1}].
\end{equation}

The function $b \mapsto \mathbb{E}[(xx^* + b)^{-1}]$ is a holomorphic function at $b$ that is strictly positive in the sense that $b \geq \delta > 0$ for some $\delta \in \mathbb{R}$ and $b \in \mathbb{H}^+(\mathbb{B})$. Hence, the identity \eqref{4.20} holds for any $b \geq \delta > 0$ by the uniqueness of holomorphic functions.

For any $x \in \mathbb{N}$, let $x = u|x|$ be the polar decomposition of $x$. Consider the truncated operator $x_N := u|x| \cdot \chi_{[0,N]}(|x|)$. It is known that $x_N \in \mathbb{N}$ and $x_N$ converges to $x$ respect to the strong operator topology. We may assume that $A$ is a subalgebra of the set of all bounded operators $B(H)$ acting on some Hilbert space $H$. By the identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$, we have

$$
\begin{align*}
\|[(xx^* + b)^{-1} - (x_Nx_N^* + b)^{-1}]h\| &\leq \|(x_Nx_N^* + b)^{-1}(x_Nx_N^* - xx^*)(xx^*b^{-1})h\| \\
&\leq \|(x_Nx_N^* - xx^*)(xx^*b^{-1})h\|
\end{align*}
$$

for any vector $h \in H$. Note that $\|(x_Nx_N^* + b)^{-1}\| \leq \|b^{-1}\|$ and $x_Nx_N^*$ converges to $xx^*$ in strong operator topology. Hence, $(x_Nx_N^* + b)^{-1}$ converges to $(xx^* + b)^{-1}$ in the strong operator topology. Similarly, we have $(x_Nx_N + b)^{-1}$ converges to $(xx^* + b)^{-1}$. In particular, by letting $x = \lambda - x_0$ and combining \eqref{4.20}, we have

\begin{equation}
\mathbb{E}[(\lambda - x_0)^*(\lambda - x_0)^* + b)^{-1}] = \mathbb{E}[(\lambda - x_0)^*(\lambda - x_0) + b)^{-1}] = \mathbb{E}[(\lambda - x_0)^*(\lambda - x_0) + b)^{-1}].
\end{equation}

Since $\epsilon_1 \epsilon_2 \in \mathbb{B}$ and $\epsilon_1 \epsilon_2 \geq \epsilon^2$, this implies that \eqref{4.21} holds for $b = \epsilon_1 \epsilon_2$. Hence, $D = \tilde{D}$.

We next show that $\epsilon_1 \epsilon_2$ is a constant. By differentiating respect to $t$, we have

$$
\epsilon'_1 = (\epsilon + k(y, \tilde{Q}^t_\xi(z)y^*')) = - (\alpha - \beta)\tilde{Q}^t_\xi(z)
$$

and

$$
\epsilon'_2 = (\epsilon + \kappa(y^*, Q^t_\xi(z)y))' = (\alpha - \beta)Q^t_\xi(z).
$$

Lemma \ref{4.3} implies that

$$
\tilde{Q}^t_\xi(z) = ig_{22} = \epsilon_1 \tilde{D} = \epsilon_1 D
$$

and similarly $Q^t_\xi(z) = ig_{11} = \epsilon_2 D$. Hence

$$
(\epsilon_1 \epsilon_2)' = (\alpha - \beta)\epsilon_1 \epsilon_2(D - D) = 0.
$$
Moreover, defined in (4.7) Lemma 4.5. Let \( \Box \)
This finishes the proof. □

\[ D = \mathbb{E} \left[ ((\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1 \varepsilon_2)^{-1} \right] = \phi \left[ ((\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1 \varepsilon_2)^{-1} \right]. \]

Lemma 4.5. Let \( g_{ij} \) be entries of the \( 2 \times 2 \) matrix-valued Cauchy transform in (4.17) as defined in (4.7) and assume \( \alpha \neq \beta \). Then \( g_{12} \) and \( g_{21} \) are constant functions given by

\[
g_{12} = \phi \left\{ (\lambda - x_0) \left[ (\lambda - x_0)^* (\lambda - x_0) + \varepsilon_1 \varepsilon_2 \right]^{-1} \right\},
\]

and

\[
g_{21} = \phi \left\{ (\lambda - x_0)^* \left[ (\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1 \varepsilon_2 \right]^{-1} \right\}.
\]

Moreover,

\[
g_{11}(t) = \frac{i \varepsilon D(\alpha - \beta)}{\beta e(\alpha - \beta) D - \alpha} e^{(\alpha - \beta) D t},
g_{22}(t) = \frac{i \varepsilon D(\beta - \alpha)}{\alpha e(\beta - \alpha) D - \beta} e^{(\beta - \alpha) D t}.
\]

Proof. By Lemma 4.4 the product \( \varepsilon_1 \varepsilon_2 \) is a constant. Recall that \( \mathbb{E}(x) = \phi(x) \) for any \( x \in \mathbb{N} \). It follows that \( g_{12}, g_{21} \) are constants given by the formulas above. We next observe that

\[
g_{11} = -i \varepsilon_2 D, \quad g_{22} = -i \varepsilon_1 D.
\]

Taking the derivative as in the proof of Lemma 4.4, we have

\[
g_{11}' = -D(\varepsilon_2)' = -D(\alpha - \beta)(Q_\varepsilon^i(z)) = (\alpha - \beta) D g_{11}
\]

where we used \( g_{11} = -i Q_\varepsilon^i(z) \) from Lemma 4.4. Hence, by solving the ODE, we see that \( g_{11}(t) = C e^{(\alpha - \beta) D t} \) for some constant \( C \). We rewrite the definition of \( \varepsilon_2 \) using the subordination relation in Lemma 4.3 as

\[
\varepsilon_2 = \varepsilon + \kappa(y^*, Q_\varepsilon^i(z)y) = \varepsilon + i \kappa(y^*, g_{11} y).
\]

Hence, \( g_{11}(t) = C e^{(\alpha - \beta) D t} \) and

\[
g_{11}(t) = -i \varepsilon_2 D = -(i \varepsilon - \kappa(y^*, g_{11} y)) D.
\]

The cumulant formula (4.1) of \( y \) reads

\[
\kappa(y^*, g_{11} y) = \alpha \int_0^t g_{11}(s) ds + \beta \int_t^1 g_{11}(s) ds,
\]

which determines the value \( C \) by solving the integral equation. A direct verification shows

\[
C = \frac{i \varepsilon D(\alpha - \beta)}{\beta e(\alpha - \beta) D - \alpha}.
\]

Similarly, one can obtain the formula for \( g_{22} \). □

Corollary 4.6. For \( \alpha \neq \beta \), the functions \( \varepsilon_1 \) and \( \varepsilon_2 \) in \( \mathcal{B} = L^\infty[0, 1] \) are given by

\[
\varepsilon_1(t) = \frac{\varepsilon(\alpha - \beta)}{(\alpha - \beta e(\alpha - \beta) D)} e^{(1-t)(\alpha - \beta) D},
\]

and

\[
\varepsilon_2(t) = \frac{\varepsilon(\alpha - \beta)}{(\alpha - \beta e(\alpha - \beta) D)} e^{(\alpha - \beta) D}
\]

Consequently,

\[
\varepsilon_1 \varepsilon_2 = \varepsilon^2 \left( \frac{\alpha - \beta}{\alpha - \beta e(\alpha - \beta) D} \right)^2 e^{(\alpha - \beta) D}
\]
Proof. It follows directly from $g_{11} = -i\varepsilon_2 D, g_{22} = -i\varepsilon_1 D$ and the formulas for $g_{11}$ and $g_{22}$ as in (4.22). □

Remark 4.7. For any $\varepsilon > 0$, we have $D < \frac{\log \alpha - \log \beta}{\alpha - \beta}$ where $D$ is defined in Lemma 4.4. Indeed, as we shown in the proof of Lemma 4.4, we have $\varepsilon_1(t) > \varepsilon > 0$, hence by the formula (4.23) for $\varepsilon_1$, we see that $\frac{\alpha - \beta}{\alpha - \beta e^{(\alpha - \beta)D}} > 0$ which yields that $D < \frac{\log \alpha - \log \beta}{\alpha - \beta}$ whenever $\alpha \neq \beta$.

It follows from Corollary 4.6 that

\[ (4.25) \]
\[ \phi(\varepsilon_1) = \phi(\varepsilon_2) = \frac{\varepsilon(e^{(\alpha - \beta)D} - 1)}{D(\alpha - \beta e^{(\alpha - \beta)D})}. \]

Then, since $D$ is constant and $g_{11} = -i\varepsilon_2 D$, we have

\[ (4.26) \]
\[ i\phi(g_{11}) = \phi(\varepsilon_2) = \phi(\varepsilon_2) \phi \left\{ (\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1 \varepsilon_2 \right\}^{-1}. \]

Since $\frac{\alpha - \beta}{\alpha - \beta e^{(\alpha - \beta)D}} > 0$, we then set

\[ (4.27) \]
\[ \varepsilon_0 = \sqrt{\varepsilon_1 \varepsilon_2} = \frac{\varepsilon(\alpha - \beta)}{\alpha - \beta e^{(\alpha - \beta)D}} e^{(\alpha - \beta)D/2}. \]

We can then rewrite $\phi(\varepsilon)$ as

\[ \phi(\varepsilon_1) = \phi(\varepsilon_2) = \varepsilon_0 \frac{e^{(\alpha - \beta)D} - 1}{D(\alpha - \beta)} e^{-(\alpha - \beta)D/2}. \]

Our approach is to fix $\varepsilon > 0$ and $\lambda \in \mathbb{C}$, and we will show that $z, \varepsilon_1, \varepsilon_2$ can be regarded as functions of $\lambda$ and $\varepsilon$.

In light of (4.24), the constant $D$ satisfies the following implicit formula

\[ (4.28) \]
\[ D = \phi \left\{ \left[ (\lambda - x_0)^*(\lambda - x_0) + \varepsilon^2 \frac{\alpha - \beta}{\alpha - \beta e^{(\alpha - \beta)D}} \right]^2 e^{(\alpha - \beta)D} \right\}^{-1}. \]

We shall show that (4.28) determines a unique solution for $D < \frac{\log \alpha - \log \beta}{\alpha - \beta}$. Put $\sigma = (\alpha - \beta)D$, and denote

\[ F(\sigma, \lambda, \varepsilon) = \frac{1}{\sigma} \cdot \phi \left\{ \left[ (\lambda - x_0)^*(\lambda - x_0) + \varepsilon^2 \frac{\alpha - \beta^2}{(\alpha - \beta e^{\alpha - \beta})^2 e^{-\sigma}} \right]^{-1} \right\}. \]

Then (4.28) is rewritten as

\[ (4.29) \]
\[ F(\sigma, \lambda, \varepsilon) = \frac{1}{\alpha - \beta}. \]

Lemma 4.8. Assume that $\alpha > \beta > 0$, let $0 < \sigma < \log(\alpha/\beta)$. For any fixed $\varepsilon > 0$ and $\lambda \in \mathbb{C}$, the function $\sigma \rightarrow F(\sigma, \lambda, \varepsilon)$ is a decreasing function on $(0, \infty)$, and

\[ \lim_{\sigma \rightarrow 0^+} F(\sigma, \lambda, \varepsilon) = \infty, \quad \lim_{\sigma \rightarrow \log(\alpha/\beta)} F(\sigma, \lambda, \varepsilon) = 0. \]

Proof. It is easy to check that $\sigma \rightarrow (\alpha - \beta e^\sigma)^2 e^{-\sigma}$ is increasing if $\sigma > \log(\alpha/\beta)$ and is decreasing if $\sigma < \log(\alpha/\beta)$. Hence, $\frac{\partial F(\sigma, \lambda, \varepsilon)}{\partial \sigma} < 0$ when $\sigma < \log(\alpha/\beta)$. □

Proposition 4.9. Fix $\lambda \in \mathbb{C}$ and $\varepsilon > 0$, the equation (4.28) determines $D$ uniquely, and the function $D = D(\lambda, \varepsilon)$ is a $C^\infty$-function of $(\lambda, \varepsilon)$ over $\mathbb{C} \times (0, \infty)$ determined by (4.28). Consequently, the function $\varepsilon_0$ defined by (4.27) is a $C^\infty$-function of $(\lambda, \varepsilon)$ over $\mathbb{C} \times (0, \infty)$, which we denoted by $\varepsilon_0 = \varepsilon_0(\lambda, \varepsilon)$. 

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Proof. Without losing generality, we may assume that $\alpha > \beta$ (the proof for the case $\alpha < \beta$ is similar by symmetric consideration). It is known from Remark 4.7 that $D < \frac{\log \alpha - \log \beta}{\alpha - \beta}$ which means $\sigma = (\alpha - \beta)D < \log(\alpha/\beta)$. The uniqueness then follows from Lemma 4.8. The function $(\sigma, \lambda, \varepsilon) \mapsto F(\sigma, \lambda, \varepsilon)$ is real analytic in $(\Re\lambda, 3\lambda)$ and complex analytic in $(\sigma, \varepsilon)$.

Hence $D(\lambda, \varepsilon)$ is a $C^\infty$ function by implicit function theorem.

For any $\varepsilon > 0$ and $\lambda \in \mathbb{C}$, the following equations

\begin{align}
\begin{cases}
D&=\phi \left\{ \left[ (\lambda - x_0)^* (\lambda - x_0) + \varepsilon_0^2 \right]^{-1} \right\} \\
\varepsilon_0^2 &= \frac{\varepsilon^2 \lambda^2}{(\alpha - \beta)^2} \cdot \varepsilon^{(\alpha - \beta)D} \\
\end{cases}
\end{align}

(4.30)

determine uniquely $\varepsilon_0 = \varepsilon_0(\lambda, \varepsilon) > 0$ and $D = D(\lambda, \varepsilon) < \frac{\log \alpha - \log \beta}{\alpha - \beta}$. We now define $\Phi_{\alpha\beta, \gamma}(\varepsilon) : \mathbb{C} \to \mathbb{C}$ as follows

\begin{align}
\Phi_{\alpha\beta, \gamma}(\varepsilon) = \lambda + \gamma \phi \left\{ (\lambda - x_0)^* \left[ (\lambda - x_0)(\lambda - x_0)^* + \varepsilon_0(\lambda, \varepsilon)^2 \right]^{-1} \right\}.
\end{align}

(4.31)

We summarize the relation between $\lambda$ and $z$:

- fix $\varepsilon > 0$, for any $z \in \mathbb{C}$, let $\lambda = z - \kappa(y, P_t^k \varepsilon) y$ as in Lemma 4.3 then by the cumulant formula, we have

\begin{align}
\lambda &= z - \int_0^1 [P_t^k (\varepsilon)] (t) dt = z - \phi(P_t^k (\varepsilon)) \\
&= z - \gamma \phi \left\{ (z - x_0 - y)^* \left[ (z - x_0 - y)(z - x_0 - y)^* + \varepsilon^2 \right]^{-1} \right\},
\end{align}

(4.32)

where $y = g_{\alpha\beta, \gamma}$.

- fix $\varepsilon > 0$, for any $\lambda \in \mathbb{C}$, we set $z = \Phi_{\alpha\beta, \gamma}(\varepsilon)(\lambda)$. We would like to see if $\lambda$ can be retrieved by the previous construction (4.32).

The following result shows that there is a natural one-to-one correspondence between $\lambda$ and $z$ via the subordination relation (4.32) and $z = \Phi_{\alpha\beta, \gamma}(\varepsilon)(\lambda)$.

**Lemma 4.10.** The map $\Phi_{\alpha\beta, \gamma}(\varepsilon)$ is a homeomorphism of the complex plane for $\varepsilon > 0$. Its inverse map is

\begin{align}
J_{\alpha\beta, \gamma}(\varepsilon)(z) &= z - \gamma \phi \left\{ (z - x_0 - y)^* \left[ (z - x_0 - y)(z - x_0 - y)^* + \varepsilon^2 \right]^{-1} \right\},
\end{align}

where $y = g_{\alpha\beta, \gamma}$.

**Proof.** Using notation in the proof of Lemma 4.3 for $b \in \mathbb{H}^+(B)$, we have (see 4.14)

\begin{align}
\Lambda_1(b) = b - R_Y(G_X \Lambda Y)(b) = b - R_Y(G_X \Lambda_1(b)).
\end{align}

Set $H_1(b) = b + R_Y(G_X(b))$, then $H_1(\Lambda_1(b)) = b$ for any $b \in \mathbb{H}^+(B)$. Hence, for any $p \in \Omega_1(\mathbb{H}^+(B))$, we have $\Omega_1(H_1(p)) = p$.

Observe that

\begin{align}
\|(z - x_0 - y)^* \left( (z - x_0 - y)(z - x_0 - y)^* + \varepsilon^2 \right)^{-1} \| \leq \frac{1}{2\varepsilon}.
\end{align}

Hence $J_{\alpha\beta, \gamma}(\varepsilon)(z)$ is a $C^\infty$ map of $\mathbb{C}$ to itself, we then deduce that $J_{\alpha\beta, \gamma}(\varepsilon)$ is surjective. By Theorem 7.3 we deduce that $\Omega_1(\{ \Theta(z, \varepsilon) : z \in \mathbb{C} \})$ contains any element of the form

\begin{align}
p(\lambda) = \begin{pmatrix}
\varepsilon_1 & \lambda \\
\lambda & \varepsilon_2
\end{pmatrix}, \quad \lambda \in \mathbb{C},
\end{align}
where \( \varepsilon_1, \varepsilon_2 \) are defined as in Lemma 4.3.

Suppose \( z = \Phi^{(e)}_{\alpha\beta\gamma}(\lambda) \) for \( \lambda \in \mathbb{C} \). We want to show that \( \lambda = J^{(e)}_{\alpha\beta\gamma}(z) \). Suppose \( p(\lambda) = \Omega_1(b) \), where \( b = \Theta(z', \varepsilon) \) for some \( z' \in \mathbb{C} \). Then \( b = H_1(\Omega_1(b)) \).

One can check that
\[
H_1(p(\lambda)) = \Theta(z, \varepsilon) = \begin{bmatrix} i\varepsilon & z \\ \bar{z} & i\varepsilon \end{bmatrix},
\]
which yields that \( b = \Theta(z', \varepsilon) = \Theta(z, \varepsilon) \). Hence, \( z = z' \) and \( p(\lambda) = \Omega_1(\Theta(z, \varepsilon)) \).

Therefore,
\[
\lambda = z - \kappa(y, P_{z}(\varepsilon)y) = J^{(e)}_{\alpha\beta\gamma}(z).
\]

This finishes the proof. \( \square \)

The relation (4.33) between \( \lambda \) and \( z \) in Lemma 4.3 plays a fundamental role in our work. We will show that the regularized Brown measure \( \mu^{(e)}_{\alpha\beta\gamma} \) is the push-forward measure of the regularized Brown measure \( \mu^{(e)}_{\alpha\beta\gamma} \) under the map \( \Phi^{(e)}_{\alpha\beta\gamma} \).

Theorem 4.11. For \( \alpha \neq \beta \), let \( y = g_{\alpha\beta\gamma} \in \mathcal{M} \) and \( x_0 \in \tilde{N} \) be a random variable that is \(*\)-free from \( y \) with amalgamation over \( \mathcal{B} \) in \((A, \mathcal{E}, \mathcal{B})\), and \( \mathcal{N} \) is \(*\)-free from \( \mathcal{B} \) in \((A, \phi)\).

For any \( \varepsilon > 0 \) and \( \lambda \in \mathbb{C} \), let \( \varepsilon_0 > 0 \) and \( D < \frac{\log \alpha - \log \beta}{\alpha - \beta} \) be the solution for the system of equations (4.30), and put
\[
z = \lambda + \gamma \cdot \phi \left( (\lambda - x_0)^* \left[ (\lambda - x_0)(\lambda - x_0)^* + \varepsilon_0 \right]^{-1} \right).
\]

We have the subordination equation
\[
M_2(\phi)[G_{X+Y}(b)] = M_2(\phi)[G_{X}(\Omega_1(b))]
\]
where
\[
\Omega_1 \left( \begin{bmatrix} i\varepsilon & z \\ \bar{z} & i\varepsilon \end{bmatrix} \right) = \left( \begin{bmatrix} i\varepsilon_1 & \lambda \\ \bar{\lambda} & i\varepsilon_2 \end{bmatrix} \right),
\]
and \( \varepsilon_1, \varepsilon_2 \) are given by Corollary 4.6. Moreover, the subordination equation (4.34) for \( b = \left[ \begin{array}{c} i\varepsilon \\ \bar{z} \\ i\varepsilon \end{array} \right] \) gives
\[
p_{X}(t_0)(\varepsilon_0) = p_{X}^{t}(\varepsilon), \quad p_{Y}(t_0)(\varepsilon_0) = p_{Y}^{t}(\varepsilon),
\]
where \( t = (\alpha, \beta, \gamma) \) and \( t_0 = (0, 0, 0) \) following notation in (4.3), and
\[
q_{e}^{t}(z) = \frac{\varepsilon_1 e^{(\alpha - \beta)D} - 1}{\alpha - \beta e^{(\alpha - \beta)D}}.
\]

Proof. By Lemma 4.10, fix \( \varepsilon > 0 \), \( \lambda \) and \( z \) are determined by each other by two maps \( \Phi^{(e)}_{\alpha\beta\gamma} \) and \( J^{(e)}_{\alpha\beta\gamma} \). We note that, by Lemma 4.5 and the subordination relation, we have
\[
z = \lambda + \kappa(y, P_{X}(\varepsilon)y) = \lambda + \kappa(y, g_{21}(\lambda, \varepsilon_1, \varepsilon_2)y) = \lambda + \gamma \cdot p_{X}^{t_0}(\varepsilon_0),
\]
where we used the fact that \( g_{21}(\lambda, \varepsilon_1, \varepsilon_2) \) is constant and \( \varepsilon_0^2 = \varepsilon_1 \varepsilon_2 \) (see Lemma 4.5 and (4.5)) to derive the second identity. The formula for subordination function \( \Omega_1 \) is the same as Lemma 4.3. The (4.36) follows by taking the trace for (4.12). Note that, by the first subordination relation in (4.12), we have
\[
i\phi(g_{11}) = \phi(Q_{e}^{t}(z)) = q_{e}^{t}(z),
\]
where \( Q^1(z) \) and \( q^z_\varepsilon(z) \) are defined in (4.4). We also have \( i\phi(g_{11}) = \phi(\varepsilon_2)D \). By (4.25), we then have
\[
q^z_\varepsilon(z) = \phi(\varepsilon_2)D = \frac{\varepsilon(\alpha - \beta)D - 1}{\alpha - \beta e^{(\alpha - \beta)D}}.
\]

Finishing the proof. \( \square \)

### 4.3. The regularized Brown measures and regularized map

The following result is a reformulation of the argument from [33, Theorem 5.2]. For completeness, we provide a proof.

**Lemma 4.12.** Let \( S_1, S_2 \) be two \( C^\infty \) subharmonic function on \( \mathbb{C} \). Given \( \gamma \in \mathbb{C} \), denote the map \( \Phi \) by
\[
\Phi(\lambda) = \lambda + \gamma \frac{\partial S_1(\lambda)}{\partial \lambda}.
\]
Assume that:
1. the map \( \Phi \) is a homeomorphism of the complex plane, and
2. for any \( \lambda \in \mathbb{C} \), we have
\[
\frac{\partial S_1(\lambda)}{\partial \lambda} = \frac{\partial S_2(z)}{\partial z}, \quad \frac{\partial S_1(\lambda)}{\partial \lambda} = \frac{\partial S_2(z)}{\partial z},
\]
where \( z = \Phi(\lambda) \).

Then, the Riesz measure \( \mu_2 \) of \( S_2 \) is the push-forward measure of the Riesz measure \( \mu_1 \) of \( S_1 \) under the map \( \Phi \). In other words, for any Borel measurable set \( E \), we have
\[
\mu_2(E) = \mu_1(\Phi^{-1}(E)).
\]

**Proof.** Let \( \gamma = \gamma_1 + i\gamma_2 \). For \( \lambda = \lambda_1 + i\lambda_2 \) and \( z = z_1 + iz_2 \), we denote the vector fields
\[
P^{(1)}(\lambda_1, \lambda_2) = \frac{1}{2} \frac{\partial S_1(\lambda)}{\partial \lambda_1}, \quad Q^{(1)}(\lambda_1, \lambda_2) = \frac{1}{2} \frac{\partial S_1(\lambda)}{\partial \lambda_2}
\]
and
\[
P^{(2)}(z_1, z_2) = \frac{1}{2} \frac{\partial S_2(\lambda)}{\partial z_1}, \quad Q^{(2)}(z_1, z_2) = \frac{1}{2} \frac{\partial S_2(\lambda)}{\partial z_2}.
\]

Then, \( z = \Phi(\lambda) \) can be expressed as
\[
z_1 = \lambda_1 + \left( \gamma_1 P^{(1)}(\lambda_1, \lambda_2) + \gamma_2 Q^{(1)}(\lambda_1, \lambda_2) \right),
\]
\[
z_2 = \lambda_2 + \left( \gamma_2 P^{(1)}(\lambda_1, \lambda_2) - \gamma_1 Q^{(1)}(\lambda_1, \lambda_2) \right).
\]

Denote the differential 1–form
\[
d_2 = -Q^{(2)}(z_1, z_2)dz_1 + P^{(1)}(z_1, z_2)dz_2.
\]

Let \( d_1 \) be the pulled-back 1–form of \( d_2 \) under the map \( \Phi \). In this case, it means that we change the variable from \( (z_1, z_2) \) to \( (\lambda_1, \lambda_2) \) because \( \Phi \) is one-to-one. We have
\[
d_1 = -Q^{(2)}(z_1, z_2)d(\lambda_1 + \gamma_1 P^{(1)} + \gamma_2 Q^{(1)})
\]
\[
\quad + P^{(2)}(z_1, z_2)d(\lambda_2 + \gamma_2 P^{(1)} - \gamma_1 Q^{(1)})
\]
\[
= -Q^{(1)}d(\lambda_1 + \gamma_1 P^{(1)} + \gamma_2 Q^{(1)})
\]
\[
\quad + P^{(1)}d(\lambda_2 + \gamma_2 P^{(1)} - \gamma_1 Q^{(1)}).
\]

We can rewrite it as
\[
d_1 = -Q^{(1)}d\lambda_1 + P^{(1)}d\lambda_2 + \left[ -\gamma_1 P^{(1)}Q^{(1)} + \frac{1}{2}\left( (P^{(1)})^2 - (Q^{(1)})^2 \right) \right].
\]
Hence, for any simply connected domain $D$ with piecewise smooth boundary, we have

$$\int_{\partial \Phi(D)} d_2 = \int_{\partial D} d_1.$$  

By Green’s formula and the definitions of $1-$form $d_1$ and $1-$form $d_2$, we have

$$\mu_2(\Phi(D)) = \frac{1}{2\pi} \int_{\Phi(D)} \nabla^2 S_2 d_2 dz_2 = \frac{1}{2\pi} \int_{\partial \Phi(D)} d_2$$

and similarly,

$$\mu_1(D) = \frac{1}{2\pi} \int_{\partial D} d_1.$$

This finishes the proof. □

**Theorem 4.13.** For $\alpha \neq \beta$, the regularized Brown measure $\mu^{(e)}_{x_0 + g_{\alpha\beta}}$ is the push-forward measure of the regularized Brown measure $\mu^{(e)}_{x_0 + g_{\alpha\beta,0}}$ under the map $\Phi^{(e)}_{\alpha\beta,\gamma}$.

**Proof.** Recall that $S(x, \lambda, \varepsilon) = \phi \left( \log \left( (x - \lambda)^* (x - \lambda) + \varepsilon^2 \right) \right)$. We set $S_1(\lambda) = S(x_0 + g_{\alpha\beta,0}, \lambda, \varepsilon)$ and $S_2(z) = S(x_0 + g_{\alpha\beta,\gamma}, z, \varepsilon)$, and $S_0(\lambda) = S(x_0, \lambda, \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_0(\lambda, \varepsilon)$ and $z = \Phi^{(e)}_{\alpha\beta,\gamma}(\lambda)$ defined as (4.31). Then the map $\Phi^{(e)}_{\alpha\beta,\gamma}$ can be rewritten as

$$\Phi^{(e)}_{\alpha\beta,\gamma}(\lambda) = \lambda + \gamma \frac{\partial S_0}{\partial \lambda}.$$

The map $\Phi^{(e)}_{\alpha\beta,\gamma}$ is a homeomorphism by Lemma 4.10. By Definition (4.3), we have

$$p^{t_0}_{\lambda}(\varepsilon_0) = \frac{\partial S_0(\lambda)}{\partial \lambda}, \quad p^t_{\lambda}(\varepsilon) = \frac{\partial S_2(z)}{\partial z},$$

where $t_0 = (0, 0, 0)$ and $t = (\alpha, \beta, \gamma)$. In addition, since $\Phi^{(e)}_{\alpha\beta,\gamma}(\lambda) = \lambda$, we also have

$$p^{t_0}_{\lambda}(\varepsilon) = \frac{\partial S_1(\lambda)}{\partial \lambda},$$

where $t(0) = (\alpha, \beta, 0)$. Hence, by choosing $\gamma = 0$ and an arbitrary eligible $\gamma$, the subordination relation $p^{t_0}_{\lambda}(\varepsilon_0) = p^t_{\lambda}(\varepsilon)$ from (4.36) shows

$$\frac{\partial S_0(\lambda)}{\partial \lambda} = \frac{\partial S_1(\lambda)}{\partial \lambda} = \frac{\partial S_2(z)}{\partial z}$$

and

$$\frac{\partial S_0(\lambda)}{\partial \lambda} = \frac{\partial S_1(\lambda)}{\partial \lambda} = \frac{\partial S_2(z)}{\partial z}.$$ 

Hence, $\Phi^{(e)}_{\alpha\beta,\gamma}(\lambda) = \lambda + \gamma \frac{\partial S_1(\lambda)}{\partial \lambda}$ and conditions in Lemma 4.12 are satisfied, which yields the desired result. □

5. **The Brown measure formulas**

In this section, we calculate the Brown measure of $x_0 + g_{\alpha\beta,0}$ where $x_0 \in \log^+ N \subset \tilde{N}$ is a random variable that is *-free from $g_{\alpha\beta,0}$ with amalgamation over $\mathcal{B}$ in the operator-valued $W^*$-probability space $(A, \mathcal{E}, \mathcal{B})$. Recall that $N$ is *-free from $\mathcal{B}$ in $(A, \phi)$. 
5.1. The limit of subordination functions. Throughout this subsection, we choose \( \alpha \neq \beta \). Recall that the subordination relation \( G_{X+Y}(h) = G_X(\Omega_1(h)) \) gives

\[
g_{11} = -iQ^t_1(z), \ g_{22} = -i\overline{Q}^t_1(z), \ g_{12} = P^t_2(\varepsilon), \ g_{21} = P^t_1(\varepsilon),
\]

where \( t = (\alpha, \beta, \gamma) \) and \( g_{ij} \) was defined in (5.1).

We now set

\[
\sigma = (\alpha - \beta)D > 0.
\]

Similarly, if \( \alpha \leq \beta \), then \( \sigma = (\alpha - \beta)D > 0 \). We have

\[
\frac{1}{\alpha - \beta} = F(\sigma, \lambda, \varepsilon) = \frac{1}{\sigma} \cdot \phi \left\{ \left( (\lambda - x_0)^*(\lambda - x_0) + \varepsilon^2 \frac{(\alpha - \beta)^2}{(\alpha - \beta e^\sigma)^2 e^{-\sigma}} \right)^{-1} \right\} \leq \frac{1}{\sigma H},
\]

which implies that

\[
\sigma \leq \frac{\alpha - \beta}{H}.
\]

Similarly, if \( 0 \leq \alpha < \beta \), then

\[
\sigma \geq \frac{\alpha - \beta}{H}.
\]

We now set

\[
\Xi_{\alpha, \beta} = \left\{ \lambda : H < \frac{\alpha - \beta}{\log \alpha - \log \beta} \right\},
\]

and denote \( \sigma_0(\lambda) = \min\{\log \left( \frac{\alpha - \beta}{H} \right) \}, \frac{\alpha - \beta}{H} \} \) if \( \alpha > \beta > 0 \).

For any \( \varepsilon \geq 0 \), denote the function \( f_\varepsilon \) of \( \lambda \) by

\[
f_\varepsilon(\lambda) = \phi \left( \left( |x_0 - \lambda|^2 + \varepsilon \right)^{-1} \right).
\]

For \( \varepsilon > 0 \), the function \( f_\varepsilon \) is a continuous function of \( \lambda \). As the increasing limit of the \( (f_\varepsilon)_{\varepsilon > 0} \), we deduce that \( f_0 \) is lower semi-continuous. Hence, the set \( \Xi_{\alpha, \beta} \) is open.

**Lemma 5.1.** If \( \lambda \in \Xi_{\alpha, \beta} \), then \( \lim_{\varepsilon \to 0^+} \varepsilon_0(\lambda, \varepsilon) \) exists and the limit is a finite positive number \( s \) determined by

\[
\frac{\log \alpha - \log \beta}{\alpha - \beta} = \phi \left\{ \left( (\lambda - x_0)^*(\lambda - x_0) + s^2 \right)^{-1} \right\}.
\]

If \( \lambda \notin \Xi_{\alpha, \beta} \), then \( \lim_{\varepsilon \to 0^+} \varepsilon_0(\lambda, \varepsilon) = 0 \). Denote the limit by \( s = s(\lambda) \) for \( \lambda \in \mathbb{C} \), then \( s(\lambda) \) is a \( C^\infty \) function of \( \lambda \in \Xi_{\alpha, \beta} \).

**Proof.** For \( \varepsilon > 0 \), recall that \( \sigma = (\alpha - \beta)D \) is the unique solution of \( F(\sigma) = 1/(\alpha - \beta) \) such that \( \sigma < \log(\alpha/\beta) \). By symmetric consideration, we may assume that \( \alpha > \beta > 0 \).

Then, \( \lambda \in \Xi_{\alpha, \beta} \) if and only if

\[
\log \left( \frac{\alpha}{\beta} \right) \leq \frac{\alpha - \beta}{H},
\]

which implies \( \sigma_0(\lambda) = \log(\alpha/\beta) \). By Proposition 5.1, \( \sigma = \sigma(\lambda, \varepsilon) \) is a \( C^\infty \)-function of \( (\lambda, \varepsilon) \) over \( \mathbb{C} \times (0, \infty) \). We first claim that \( \lim_{\varepsilon \to 0^+} \sigma(\lambda, \varepsilon) = \log(\alpha/\beta) \). Assume that there exists some \( x_k \to 0^+ \) such that \( \sigma = \sigma(\lambda, x_k) = (\alpha - \beta)D(\lambda, x_k) \) converges to some \( \theta < \log(\alpha/\beta) \). Then, we have

\[
\lim_{x_k \to 0^+} \varepsilon_0(\lambda, \varepsilon) = \lim_{x_k \to 0^+} \frac{\varepsilon(\alpha - \beta)}{\alpha - \beta e^{(\alpha - \beta)D/2} e^{(\alpha - \beta)D/2}} = 0.
\]
 Consequently,

\[
\lim_{x_k \to 0^+} \sigma F(x_k) = H^{-1}.
\]

On the other hand,

\[
\lim_{x \to 0^+} \sigma F(x) = \frac{\theta}{\alpha - \beta} < \frac{\log \alpha - \log \beta}{\alpha - \beta}.
\]

This contradicts to \( \lambda \in \Xi_{\alpha,\beta} \). Hence, \( \lim_{x \to 0^+} \sigma = \log(\alpha/\beta) = \sigma_0(\lambda) \) if \( \lambda \in \Xi_{\alpha,\beta} \). By using the identity \( D = \sigma F(\sigma) = \frac{\sigma}{\alpha - \beta} \), we have

\[
\frac{\log \alpha - \log \beta}{\alpha - \beta} = \lim_{x \to 0^+} \sigma F(x) = \lim_{x \to 0^+} \phi \{ [\lambda - x_0^2 + \varepsilon_0^2]^{-1} \}
\]

which implies that \( \lim_{x \to 0^+} \varepsilon_0 \) exists and is determined by (5.2).

For any \( s > 0 \) and \( \lambda \in \mathbb{C} \), the map \( s \mapsto \phi \{ [\lambda - x_0^2 + s^2]^{-1} \} \) is \( C^\infty \) on \([x, \infty)\), and for any \( s > x \), the map \( \lambda \mapsto \phi \{ [\lambda - x_0^2 + s^2]^{-1} \} \) is \( C^\infty \) on \( \mathbb{C} \). Then, the implicit function theorem implies that as unique solution of (5.2), the function \( s(\lambda) \) is \( C^\infty \) on \( \Xi_{\alpha,\beta} \).

Similarly, if \( \lambda \notin \Xi_{\alpha,\beta} \), by assuming \( \lim_{x \to 0^+} \varepsilon_0(\lambda, \varepsilon) \neq 0 \), we must have \( \lim_{x \to 0^+} \sigma = \log(\alpha/\beta) \). This yields that \( H < \frac{\alpha - \beta}{\log \alpha - \log \beta} \). This contradiction implies that if \( \lambda \notin \Xi_{\alpha,\beta} \), then \( \lim_{x \to 0^+} \varepsilon_0(\lambda, \varepsilon) = 0 \).

\[\square\]

**Remark 5.2.** The function \( s(\lambda) \) can be viewed as the boundary value of scalar-valued subordination function parameterized by \( \lambda \in \mathbb{C} \). See Definition 7.2 and Proposition 3.5 for details.

**Lemma 5.3.** The convergence \( \lim_{x \to 0^+} \varepsilon_0(\lambda, \varepsilon) = s(\lambda) \) is uniform over any compact subset in \( \lambda \in \mathbb{C} \). If \( x_0 \) is bounded, the convergence is uniform over \( \mathbb{C} \).

**Proof.** We first show that \( s(\lambda) \) is a continuous function in \( \mathbb{C} \). Since \( s(\lambda) \) is \( C^\infty \) in the open set \( \Xi_{\alpha,\beta} \), to show \( s(\lambda) \) is continuous, it remains to show that for any \( \lambda_0 \in \mathbb{C} \setminus \Xi_{\alpha,\beta} \) and a sequence \( \{\lambda_n\} \subset \Xi_{\alpha,\beta} \) converging to \( \lambda_0 \), we have

\[
\lim_{n \to \infty} s(\lambda_n) = s(\lambda_0) = 0.
\]

Assume this is not true, by dropping to a subsequence if necessary, we may assume that there exists \( \delta > 0 \) such for all \( n \), \( s(\lambda_n) > \delta \). In this case, note that

\[
\phi \{ [(\lambda_n - x_0)\ast(\lambda_n - x_0) + s(\lambda_n)^2]^{-1} \} < \phi \{ [(\lambda_n - x_0)\ast(\lambda_n - x_0) + \delta^2]^{-1} \}.
\]

By using (5.2) and passing to the limit, we have

\[
\frac{\log \alpha - \log \beta}{\alpha - \beta} \leq \phi \{ [(\lambda_0 - x_0)\ast(\lambda_0 - x_0) + \delta^2]^{-1} \}
\]

which yields that \( s(\lambda_0) \geq \delta > 0 \). It contradicts to our choice \( \lambda_0 \in \mathbb{C} \setminus \Xi_{\alpha,\beta} \). Hence \( \lim_{\lambda \to \infty} s(\lambda_n) = s(\lambda_0) = 0 \) and the function \( s(\lambda) \) is continuous in \( \mathbb{C} \).

Without losing generality, we now assume that \( \alpha > \beta \). For any \( \lambda \in \mathbb{C} \), for \( 0 < \varepsilon_1 < \varepsilon_2 \), note that \( F(\sigma, \lambda, \varepsilon) \) decreases as \( \varepsilon \) increases, hence we have

\[
\frac{1}{\alpha - \beta} = F(\sigma_1, \lambda, \varepsilon_1) = F(\sigma_2, \lambda, \varepsilon_2) < F(\sigma_2, \lambda, \varepsilon_1)
\]

where \( \sigma_i = \sigma(\lambda, \varepsilon_i) \) (\( i = 1, 2 \)). By Lemma 4.8, we deduce that \( \sigma_1 > \sigma_2 \). Recall that \( \sigma(\lambda, \varepsilon_i) = (\alpha - \beta)D(\lambda, \varepsilon_i) \) and

\[
D(\lambda, \varepsilon_i) = \phi \{ [(\lambda - x_0)\ast(\lambda - x_0) + \varepsilon(\lambda, \varepsilon_i)^2]^{-1} \}.
\]
Hence, $\varepsilon_0(\lambda, \varepsilon_1) < \varepsilon_0(\lambda, \varepsilon_2)$. We then conclude that $\varepsilon_0(\lambda, \varepsilon)$ converges to $s(\lambda)$ uniformly in any compact subset of $\mathbb{C}$ as $\varepsilon$ tends to zero by Dini’s theorem.

Assume now that $x_0$ is a bounded operator. Since $\Xi_{\alpha, \beta}$ is a bounded set, to prove the convergence is uniform over $\mathbb{C}$, it suffices to show that $\varepsilon_0(\lambda, \varepsilon) < 2\varepsilon$ if $|\lambda|$ is sufficiently large. Recall that

$$
\varepsilon_0(\lambda, \varepsilon) = \frac{\varepsilon(\alpha - \beta)}{\alpha - \beta e^\sigma} e^{\sigma/2},
$$

where $\sigma = \sigma(\lambda, \varepsilon)$. Hence, $\varepsilon_0(\lambda, \varepsilon) \geq 2\varepsilon$ if and only if

$$
\alpha - \beta > 2 \left( \alpha e^{-\sigma/2} - \beta e^{\sigma/2} \right).
$$

But $\sigma = (\alpha - \beta)D \leq (\alpha - \beta)/(M^2 + 4\varepsilon^2)$ if $|\lambda| > M + ||x_0||$ and $\varepsilon_0(\lambda, \varepsilon) \geq 2\varepsilon$ by the formula for $D$ in (4.30). It follows that $\sigma$ could be arbitrarily small by choosing $M$ large enough. Observe that (5.3) can not hold when $\sigma$ is sufficiently small. Hence, we may choose $M$ large enough, so that $\varepsilon_0(\lambda, \varepsilon) < 2\varepsilon$ for any $|\lambda| > M + ||x_0||$.

5.2. The Brown measure formula. We adapt the proof of [33, Theorem 4.6] to calculate the density formula $\mu_{x_0 + g_{\alpha, \beta, 0}}$ in $\Xi_{\alpha, \beta}$.

**Theorem 5.4.** The support of the Brown measure $\mu_{x_0 + g_{\alpha, \beta, 0}}$ of $x_0 + g_{\alpha, \beta, 0}$ is the closure of the set $\Xi_{\alpha, \beta}$ defined by (5.1), which can be rewritten as

$$
\Xi_{\alpha, \beta} = \left\{ \lambda : \phi(1 - 1/|\lambda|^{-2}) > \frac{\log \alpha - \log \beta}{\alpha - \beta} \right\},
$$

Let $s(\lambda)$ be the function determined by

$$
\frac{\log \alpha - \log \beta}{\alpha - \beta} = \phi \left\{ \left( \lambda - x_0 \right)^*(\lambda - x_0) + s(\lambda)^2 \right\}^{-1},
$$

as in Lemma 5.1 and set

$$
h(\lambda, s) = (\lambda - x_0)^*(\lambda - x_0) + s^2
$$

and

$$
k(\lambda, s) = \lambda - x_0 + s^2.
$$

The probability measure $\mu_{x_0 + g_{\alpha, \beta, 0}}$ is absolutely continuous in the open set $\Xi_{\alpha, \beta}$ with density given by

$$
\frac{1}{\pi} s^2(\lambda) \phi(h^{-1}k^{-1}) + \frac{1}{\pi} \frac{\phi((\lambda - x_0)(h^{-1})^2)}{\phi(h^{-1})^2}.
$$

We need the following result. It should be well-known, but we include a proof for the sake of completeness.

**Lemma 5.5.** (1) Let $x \in \mathbb{N}$, for any $a \in (0, \infty)$ we have

$$
x(x^* x + a)^{-1} = (xx^* + a)^{-1} x.
$$

(2) For $h(\lambda, s)$ and $k(\lambda, s)$ given in Theorem 5.4 we have

$$
\frac{\partial}{\partial \lambda} \phi(h^{-1}) = -\phi \left[ h^{-1} \left( (\lambda - x_0) + 2s \frac{\partial s}{\partial \lambda} \right) h^{-1} \right],
$$

and

$$
\frac{\partial}{\partial \lambda} \phi(h^{-1}(\lambda - x_0)^*) = \phi \left[ h^{-1} - h^{-1} \left( (\lambda - x_0) + 2s \frac{\partial s}{\partial \lambda} \right) h^{-1} \right].$$
Proof. For (1), it is equivalent to show that \((xx^*+a)x(x^*x+a)^{-1} = x\). Since the affiliated algebra \(\mathcal{N}\) is an algebra, this follows from the following calculation
\[
x x^* x (x^* x + a)^{-1} = x (x^* x + a - a) (x^* x + a)^{-1} = x - ax (x^* x + a)^{-1}.
\]

For (2), if \(x_0\) is bounded, we can obtain the result by applying [19] Lemma 3.2. We denote
\[
h = h(\lambda, \overline{\lambda}, s) = (\lambda - x_0)^*(\lambda - x_0) + s^2,
\]
\[
k = k(\lambda, \overline{\lambda}, s) = (\lambda - x_0)(\lambda - x_0)^* + s^2,
\]
where \(s = s(\lambda, \overline{\lambda})\) determined by (5.4). By the identity \(a^{-1} - b^{-1} = b^{-1}(b - a)a^{-1}\) we have
\[
h(\lambda, \overline{\lambda}, s)^{-1} - h(\lambda, \overline{\lambda}, s)^{-1} = -h(\lambda, \overline{\lambda}, s)^{-1}((\overline{\lambda} - \lambda)(\lambda - x_0) + s(\lambda, \overline{\lambda})^2 - s(\lambda, \overline{\lambda})^2)h(\lambda, \overline{\lambda}, s)^{-1}
\]
Since \(s\) is a \(C^\infty\) function of \(\lambda\) and \(s > 0\), it follows that
\[
\lim_{\overline{\lambda} \to \lambda} h(\lambda, \overline{\lambda}, s(\lambda, \overline{\lambda}))^{-1} = h(\lambda, \overline{\lambda}, s(\lambda, \overline{\lambda}))^{-1}.
\]
Therefore, by using the notation of complex derivative, the partially derivation with respect to \(\overline{\lambda}\) can be calculated as
\[
\frac{\partial}{\partial \overline{\lambda}} \phi (h^{-1}) = \lim_{\overline{\lambda} \to \lambda} \frac{\phi(h(\lambda, \overline{\lambda}, s)^{-1}) - \phi(h(\lambda, \overline{\lambda}, s)^{-1}))}{\overline{\lambda} - \lambda}
\]
\[
= -\lim_{\overline{\lambda} \to \lambda} \phi[h(\lambda, \overline{\lambda}, s)^{-1}(\lambda - x_0)h(\lambda, \overline{\lambda}, s)^{-1}]
\]
\[
- \lim_{\overline{\lambda} \to \lambda} \left[ \frac{s(\lambda, \overline{\lambda})^2 - s(\lambda, \overline{\lambda})^2}{\overline{\lambda} - \lambda} \cdot \phi[h(\lambda, \overline{\lambda}, s)^{-1}] \right]
\]
\[
= -\phi \left[ h^{-1} \left( (\lambda - x_0) + 2s \frac{\partial s}{\partial \lambda} \right) h^{-1} \right].
\]
The proof for (5.8) is similar. \(\square\)

Proof of Theorem 5.4. Set \(t = (\alpha, \beta, 0), t_0 = (0, 0, 0)\) and \(b = \begin{bmatrix} i\varepsilon & \lambda \\ \overline{\lambda} & i\varepsilon \end{bmatrix}\), we have
\[
M_2(\phi)[G_{X + Y}(b)] = \begin{bmatrix} -iq_t^3(\lambda) & p_t^5(\varepsilon) \\ p_t^3(\varepsilon) & -iq_t^5(\lambda) \end{bmatrix}
\]
Applying (4.36) for \(\gamma = 0\), we have
\[
p^t_\lambda(\varepsilon) = p^t_\lambda(\varepsilon_0) = \phi \left\{ (\lambda - x_0)^* [(\lambda - x_0)(\lambda - x_0)^* + \varepsilon_0^2]^{-1} \right\}.
\]
By Lemma 5.1 we have
\[
\lim_{\varepsilon_0 \to 0^+} \phi \left\{ (\lambda - x_0)^* [(\lambda - x_0)(\lambda - x_0)^* + \varepsilon_0^2]^{-1} \right\}
\]
\[
= \phi \left\{ (\lambda - x_0)^* [(\lambda - x_0)(\lambda - x_0)^* + s(\lambda)^2]^{-1} \right\}
\]
\[
= \phi \left\{ [(\lambda - x_0)(\lambda - x_0) + s(\lambda)^2]^{-1} (\lambda - x_0)^* \right\}
\]
\[
= \phi \left( h^{-1}(\lambda - x_0)^* \right).
\]
By \cite{10} Lemma 4.19 (up to a sign change), the Brown measure of \( x_0 + g_{\alpha,\beta,0} \) is equal to
\begin{equation}
\mu_{x_0+g_{\alpha,\beta,0}} = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \frac{\partial}{\partial \lambda} \pi^\varepsilon_\lambda (\varepsilon) = \frac{1}{\pi} \frac{\partial}{\partial \lambda} \phi \left( h^{-1}(\lambda - x_0)^* \right)
\end{equation}
in the distribution sense. Applying Lemma \cite{55} we have
\begin{equation}
\frac{\partial}{\partial \lambda} \phi \left( h^{-1}(\lambda - x_0)^* \right) = \phi \left[ h^{-1} \left( 1 - (\lambda - x_0)h^{-1}(\lambda - x_0)^* \right) \right] - 2s(\lambda) \frac{\partial s(\lambda)}{\partial \lambda} \phi \left[ (\lambda - x_0)^*(h^{-1})^2 \right].
\end{equation}
Applying the identity \( x(x^*x + \varepsilon)^{-1}x^* = (x^*x + \varepsilon)^{-1}xx^* \), we obtain
\begin{equation}
\phi \left[ h^{-1} \left( 1 - (\lambda - x_0)h^{-1}(\lambda - x_0)^* \right) \right] = s(\lambda)^2 \phi(h^{-1}k^{-1}).
\end{equation}
The function \( s(\lambda) \) is the implicit function determined by \( 5.4 \) when \( \lambda \in \Xi_{\alpha,\beta} \), which can be rewritten as (by tracial property)
\[ \phi(h^{-1}) = \frac{\log \alpha - \log \beta}{\alpha - \beta}. \]
Applying implicit differentiation \( \frac{\partial}{\partial \lambda} \), we have
\[ \phi \left[ h^{-1} \left( (\lambda - x_0) + 2s(\lambda) \frac{\partial s(\lambda)}{\partial \lambda} \right) h^{-1} \right] = 0 \]
which yields
\begin{equation}
-2s(\lambda) \frac{\partial s(\lambda)}{\partial \lambda} = \frac{\phi((\lambda - x_0)(h^{-1})^2)}{\phi((h^{-1})^2)}.
\end{equation}
Since \( \phi((\lambda - x_0)(h^{-1})^2) = \phi((\lambda - x_0)^*(h^{-1})^2) \), we hence get the density formula \cite{55} by plugging \( 5.15 \) and \( 5.13 \) to \( 5.11 \).

Note that if \( \lambda \notin \Xi_{\alpha,\beta} \), then \( s(\lambda) = 0 \) in some neighborhood of \( \lambda \), and
\begin{equation}
\phi(|\lambda - x_0|^{-2}) \leq \frac{\log \alpha - \log \beta}{\alpha - \beta},
\end{equation}
which implies that the above limit \( 5.9 \) is finite by an estimation using Cauchy-Schwartz inequality. It follows that we can take the limit
\begin{equation}
\lim_{\varepsilon \to 0^+} \pi^\varepsilon_\lambda (\varepsilon) = \phi \left( \lambda - x_0 \right)^* \left[ (\lambda - x_0)(\lambda - x_0)^* \right]^{-1} \right) \right) \right) = \lim_{\varepsilon \to 0^+} \pi^\varepsilon_\lambda (\varepsilon).
\end{equation}
Hence, \( \mu_{x_0 + g_{\alpha,\beta,0}} \) and \( \mu_{x_0} \) coincide in the open set \( \mathbb{C}\backslash \Xi_{\alpha,\beta} \). By the inequality \( 5.14 \) and \cite{33} Theorem 4.5], for any \( \lambda \in \mathbb{C}\backslash \Xi_{\alpha,\beta} \), then \( \lambda \) is not in the support of \( \mu_{x_0} \). Consequently the density of \( \mu_{x_0 + g_{\alpha,\beta,0}} \) is zero on \( \mathbb{C}\backslash \Xi_{\alpha,\beta} \) by \( 5.10 \) and \( 5.15 \).

Remark 5.6. Let \( t = \frac{\alpha - \beta}{\log \alpha - \log \beta} \). The proof for Theorem \cite{54} shows that the partial derivatives of \( \log S(x_0 + g_{\alpha,\beta,0}, \lambda, 0) \) are the same as the partial derivatives of \( \log S(x_0 + c_0, \lambda, 0) \) by comparing the proof of Theorem \cite{7,10} (see also \cite{33} Theorem 4.2) for bounded case. Hence, the Brown measure of \( x_0 + g_{\alpha,\beta,0} \) is the same as the Brown measure of \( x_0 + c_0 \). In particular, by \cite{33} Theorem 4.6], the measure \( \mu_{x_0 + g_{\alpha,\beta,0}} \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{C} \). We refer to Corollary \cite{7,8} and Theorem \cite{7,10} for \( x_0 \) to be unbounded operator.
5.3. **The push-forward map.** In this section, we show that the Brown measure \( \mu_{x_0 + \alpha \beta, \gamma} \) is the push-forward measure of \( \mu_{x_0 + \alpha \beta, 0} \) under the map \( \Phi_{\alpha \beta, \gamma} \) defined by

\[
\Phi_{\alpha \beta, \gamma}(\lambda) = \lambda + \gamma \phi \left\{ (\lambda - x_0)^* \left[ (\lambda - x_0)(\lambda - x_0)^* + s(\lambda)^2 \right]^{-1} \right\},
\]

where \( s(\lambda) = \lim_{\varepsilon \to 0} \varepsilon \phi(\lambda, \varepsilon) \) as in Lemma 5.1. However, this map might be singular. We explain how the strategy in [33, Section 5] applies to our case.

**Lemma 5.7.** The function \( \Phi_{\alpha \beta, \gamma}^{(e)} \) converges uniformly to \( \Phi_{\alpha \beta, \gamma} \) in any compact subset of \( \mathbb{C} \) as \( \varepsilon \) tends to zero. The convergence is uniform over \( \mathbb{C} \) if \( x_0 \) is bounded.

**Proof.** For \( \varepsilon_2 > \varepsilon_1 > 0 \), apply the same estimation as in [33, Lemma 5.3], we have

\[
|p_{\lambda}^{t_0}(\varepsilon_0(\lambda, \varepsilon_2)) - p_{\lambda}^{t_0}(\varepsilon_0(\lambda, \varepsilon_1))| \leq \frac{\log \alpha - \log \beta}{\alpha - \beta} |\varepsilon_0(\lambda, \varepsilon_2) - \varepsilon_0(\lambda, \varepsilon_1)|.
\]

Since \( \Phi_{\alpha \beta, \gamma}^{(e)} \) can be written as

\[
\Phi_{\alpha \beta, \gamma}^{(e)}(\lambda) = \lambda + \gamma p_{\lambda}^{t_0}(\varepsilon_0(\lambda, \varepsilon))
\]

where \( t_0 = (0, 0, 0) \). The result then follows from convergence result of \( \varepsilon_0(\lambda, \varepsilon) \) to \( s(\lambda) \) showed in Lemma 5.3.

**Theorem 5.8.** The Brown measure \( \mu_{x_0 + \alpha \beta, \gamma} \) is the push-forward measure of \( \mu_{x_0 + \alpha \beta, 0} \) under the map \( \Phi_{\alpha \beta, \gamma} \). Hence, we have the following commutative diagram.

\[
\begin{array}{ccc}
\mu_{x_0 + \alpha \beta, 0} & \xrightarrow{\Phi_{\alpha \beta, \gamma}^{(e)}} & \mu_{x_0 + \alpha \beta, \gamma} \\
\varepsilon \to 0 & & \varepsilon \to 0 \\
\mu_{x_0 + \alpha \beta, 0} & \xrightarrow{\Phi_{\alpha \beta, \gamma}} & \mu_{x_0 + \alpha \beta, \gamma}
\end{array}
\]

**Proof.** The regularized Brown measure \( \mu_{x_0 + \alpha \beta, \gamma}^{(e)} \) is the push-forward measure of the regularized Brown measure \( \mu_{x_0 + \alpha \beta, 0}^{(e)} \) under the map \( \Phi_{\alpha \beta, \gamma}^{(e)} \). For any \( x \in \log^+(A) \), the regularized Brown measure \( \mu_{x}^{(e)} \) converges to the Brown measure \( \mu_{x} \) weakly as \( \varepsilon \) tends to zero. Since \( \Phi_{\alpha \beta, \gamma}^{(e)} \) converges to \( \Phi_{\alpha \beta, \gamma} \) uniformly in any compact subset of \( \mathbb{C} \), we then conclude the desired result.

**Example 5.9** (The Brown measure of triangular elliptic operator). Given \( \alpha, \beta > 0 \), set \( t = \frac{\alpha - \beta}{\log \alpha - \log \beta} \). When \( x_0 = 0 \), the set \( \Xi_{\alpha, \beta} \) is the circle centered at the origin with radius \( \sqrt{t} \),

\[
\Xi_{\alpha, \beta} = \{ \lambda \in \mathbb{C} : |\lambda| < t \}
\]

and \( s(\lambda)^2 + |\lambda|^2 = t \) for \( \lambda \in \Xi_{\alpha, \beta} \). Using notation in Theorem 5.4,

\[
h(\lambda, s) = k(\lambda, s) = |\lambda|^2 + s(\lambda)^2 = t.
\]

Hence, the density of the Brown measure of \( g_{\alpha \beta, 0} \) is

\[
d\mu_{g_{\alpha \beta, 0}} = \frac{1}{\pi t^2} + \frac{1}{\pi t^2} |\lambda|^2 d\lambda dy = \frac{1}{\pi t} d\lambda dy.
\]

In other words, it has the same Brown measure as \( c_t \), the circular operator with variance \( t \). If \( T \) is a quasi-nilpotent DT operator, and \( c_t \) is a circular operator with variance \( t \), let
\(\alpha = 1 + \varepsilon, \beta = \varepsilon,\) then \(T + c_\varepsilon\) has the same \(*\)-moments as \(g_{\alpha,\beta}\). Hence, the Brown measure \(T + c_\varepsilon\) is the uniform measure on the circle \(\{\lambda : |\lambda| \leq \frac{1}{\sqrt{\log(1 + \varepsilon - 1)}}\}\). This recovers a result of Aagaard-Haagerup [2, Theorem 4.3].

For \(\gamma \in \mathbb{C}\) such that \(|\gamma| \leq \sqrt{\alpha\beta}\), we now have

\[\Phi_{\alpha,\beta,\gamma}(\lambda) = \lambda + \gamma \cdot \frac{\lambda}{|\lambda|^2 + s^2},\]

for \(|\lambda| < t\). Hence, the Brown measure of \(g_{\alpha,\beta,\gamma}\) is supported in the ellipse with parametrization

\[\sqrt{t}e^{i\theta} + \frac{|\gamma|}{\sqrt{t}}e^{i(\psi - \theta)}\]

where \(\psi = \arg(\gamma)\). For \(0 \leq r < \sqrt{t}\), we have

\[\Phi_{\alpha,\beta,\gamma}(re^{i\theta}) = re^{i\theta} + \frac{|\gamma|}{t}e^{i(\psi - \theta)} .\]

Hence, for \(\gamma = \gamma_1 + i\gamma_2\), the Jacobian matrix of \(\Phi_t\) at \(\lambda = \lambda_1 + i\lambda_2\) is given by

\[\text{Jacobian}(\Phi_{\alpha,\beta,\gamma}) = \begin{bmatrix} 1 + \frac{2t}{\pi} & \frac{2t}{\pi} \\ \frac{2t}{\pi} & 1 - \frac{2t}{\pi} \end{bmatrix}\]

whose determinant is equal to \(1 - \frac{|\gamma|^2}{t}\). Hence, the Brown measure of \(g_{\alpha,\beta,\gamma}\) is the uniform measure in the ellipse by Theorem 5.8. This recovers the result obtained in [5, Section 6].

The interested reader is referred to [20, 33] for more examples of the push-forward maps and explicit Brown measure formulas.

6. SOME REGULARITY RESULTS ON THE PUSHFORWARD MAP

The push-forward map \(\Phi_{\alpha,\beta,\gamma}\) is the limit of the family of homeomorphisms \(\Phi^{(c)}_{\alpha,\beta,\gamma}\), which could be singular in general. In this section, we study its properties under some regularity assumptions. Recall that \(N\) is \(*\)-free from \(M\) with amalgamation over \(B\) in the operator-valued \(W^*\)-probability space \((A, \mathcal{E}, B)\) constructed in Section 4.1. The unital subalgebra \(N\) is also \(*\)-free from \(B\) in \((A, \phi)\).

**Lemma 6.1.** Let \(y = g_{\alpha,\beta,0} \in M\) and \(x_0 \in \log^+(N)\). The function \(\lambda \mapsto S(x_0 + y, \lambda, 0) = \log \Delta(|x_0 + y - \lambda|^2)\) is a \(C^\infty\)-function of \(\lambda\) in the open set \(\Xi_{\alpha,\beta}\).

*Proof.* By the definition (4.2) of the function \(S\), we observe that

\[\frac{1}{2} \frac{d}{d\varepsilon} S(x_0 + y, \lambda, \varepsilon) = q_\varepsilon^t(\lambda),\]

where \(t = (\alpha, \beta, 0)\), and by (4.3)

\[q_\varepsilon^t(\lambda) = \varepsilon \phi \left\{[(\lambda - x_0 - y)^*(\lambda - x_0) + \varepsilon^2]^{-1} \right\} .\]

We now apply Lemma 4.3 to the case \(\gamma = 0\). In this case, the free cumulant \(\kappa(y, f y) = 0\) for any \(f \in B\), and hence \(z = \Phi^{(c)}_{0,\beta,0}(\lambda) = \lambda\). The subordination relation (4.12) yields

\[g_{11}(\lambda, \varepsilon_1, \varepsilon_2) = -iQ_2^t(\lambda),\]

and by (4.4) and (4.7)

\[g_{11}(\lambda, \varepsilon_1, \varepsilon_2) = -i\varepsilon_2 E \left\{[(\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1 \varepsilon_2]^{-1} \right\} .\]
and 

\[ Q^\epsilon_\lambda(x) = \epsilon \mathbb{E}\left\{ \left[ (\lambda - x_0 - y)(\lambda - x_0 - y)^* + \epsilon^2 \right]^{-1} \right\}. \]

By taking trace, we showed in Section 5.1 (see (4.25) and (4.26)) that

\[ i\phi(g_{11}) = \phi(\epsilon_2)D = \phi(\epsilon_2)\phi\left\{ \left[ (\lambda - x_0)(\lambda - x_0)^* + \epsilon_1 \epsilon_2 \right]^{-1} \right\}, \]

and \( \epsilon_0^2 = \epsilon_1 \epsilon_2, \) and

\[ \phi(\epsilon_2) = \epsilon_0 \frac{(e\sigma - 1)}{\sigma} e^{-\sigma/2}, \]

where \( \sigma = \sigma(\lambda, \epsilon) = (\alpha - \beta)D. \) We now put

\[ f(\sigma) = \frac{e\sigma/2 + e^{-\sigma/2}}{\sigma}. \]

Then we can rewrite \( i\phi(g_{11}) \) as

\[ i\phi(g_{11}) = \left( \epsilon_0 \phi \left\{ \left[ (\lambda - x_0)(\lambda - x_0)^* + \epsilon_0^2 \right]^{-1} \right\} \right) \cdot f(\sigma(\lambda, \epsilon)). \]

Recall that \( \phi(Q^\epsilon_\lambda(\lambda)) = q^\epsilon_\lambda(\lambda). \) Denote \( t_0 = (0, 0, 0). \) Following definition (4.3), the identity \( i\phi(g_{11}) = \phi(Q^\epsilon_\lambda(\lambda)) \) can be rewritten as

\[ q^{t_0}_{\epsilon_0(\lambda, \epsilon)}(\lambda) \cdot f(\sigma(\lambda, \epsilon)) = q^\epsilon_\lambda(\lambda). \]

Therefore, by taking the integration of the above identity over some interval \([\epsilon, \delta],\) we obtain

\[(6.1) \quad S(x_0 + y, \lambda, \delta) - S(x_0 + y, \lambda, \epsilon) = 2 \int_\epsilon^\delta f(\sigma(\lambda, u))q^{t_0}_{\epsilon_0(\lambda, \epsilon)}(\lambda) \, du. \]

By Proposition 4.9 we know that \( \epsilon_0(\lambda, \epsilon) \) and \( D(\lambda, \epsilon) \) are \( C^\infty \)-functions of \((\lambda, \epsilon).\) Hence, \((\lambda, \epsilon) \to f(\sigma(\lambda, \epsilon))\) is a \( C^\infty \)-function of \((\lambda, \epsilon)\) over \( \Xi_{\alpha, \beta} \times [0, \infty). \) Moreover, by Lemma 5.1 we recall that, for \( \lambda \in \Xi_{\alpha, \beta}, \) we have \( \lim_{\epsilon \to 0^+} \epsilon_0 = s(\lambda) > 0 \) and \( s(\lambda) \) is a \( C^\infty \)-function of \( \lambda. \) Therefore,

\[ S(x_0 + y, \lambda, 0) = S(x_0 + y, \lambda, \delta) - \lim_{\epsilon \to 0^+} 2 \int_\epsilon^\delta f(\sigma(\lambda, u))q^{t_0}_{\epsilon_0(\lambda, u)}(\lambda) \, du = S(x_0 + y, \lambda, \delta) - 2 \int_0^\delta f(\sigma(\lambda, u))q^{t_0}_{\epsilon_0(\lambda, u)}(\lambda) \, du. \]

By the above discussion, the right hand side of the above equation is a \( C^\infty \)-function of \( \lambda \in \Xi_{\alpha, \beta}. \) We remark that it is clear that \( S(x_0 + y, \lambda, \delta) \) is a \( C^\infty \)-function of \( \lambda \) due to the regularity of \( S(x_0 + y, \lambda, \epsilon) \) for \( \epsilon > 0. \) Hence, \( S(x_0 + y, \lambda, 0) \) is a \( C^\infty \)-function of \( \lambda \) in \( \Xi_{\alpha, \beta}. \)

**Proposition 6.2.** Given \( t(\gamma) = (\alpha, \beta, \gamma) \) and set \( t = t(0) = (\alpha, \beta, 0) \) and \( t_0 = (0, 0, 0), \) let \( x_0 \in \log^+ (N) \) be an operator that is \( \ast \)-free from \( \{ g_{\alpha, \beta}, g_{\alpha, \alpha} \} \) with amalgamation over \( \mathbb{F}. \) If the map \( \Phi_{\alpha, \beta, \gamma} \) is non-singular at some \( \lambda \in \Xi_{\alpha, \beta}, \) then the map \((z, \epsilon) \mapsto S(x_0 + y, z, \epsilon) \) has a \( C^\infty \)-extension in some neighborhood of \((\Phi_{\alpha, \beta, \gamma}(\lambda), 0).\) Hence,

\[(6.2) \quad \lim_{\epsilon \to 0^+} \rho^{t(\gamma)}_{\Phi_{\alpha, \beta, \gamma}(\lambda)}(\epsilon) = \lim_{\epsilon \to 0^+} \rho^t_{\lambda}(\epsilon) = \lim_{\epsilon \to 0^+} p_{\lambda}(\epsilon)_{t_0(\lambda, \epsilon)}. \]

Moreover, if the map \( \Phi_{\alpha, \beta, \gamma} \) is non-singular at any \( \lambda \in \Xi_{\alpha, \beta}, \) then it is also one-to-one.
Proof. Again, by the definition (4.2) of the function \( S(x_0 + y, z, \varepsilon) \), we observe that
\[
(6.3) \quad \frac{1}{2} \frac{d}{d\varepsilon} S(x_0 + g_{\alpha, \beta, \gamma}, z, \varepsilon) = q_{z}^{t(\gamma)}(z).
\]
Recall that the subordination relation in (4.12) reads
\[
g_{11}(\lambda, \varepsilon_1, \varepsilon_2) = -iQ_{z}^{t(\gamma)}(z)
\]
where
\[
z = \Phi_{\alpha, \beta, \gamma}(\lambda) = \lambda + \gamma p_{\lambda}^{t_0}(\varepsilon_0).
\]
If \( \gamma = 0 \), then \( \Phi_{\alpha, \beta, \gamma}(\lambda) = \lambda \). By choosing \( \gamma = 0 \) and an arbitrary eligible \( \gamma \) respectively, we have
\[
Q_{z}^{t(\gamma)}(z) = Q_{z}^{t}(\lambda) = ig_{11}(\lambda, \varepsilon_1, \varepsilon_2),
\]
By taking the trace, we have \( q_{z}^{t(\gamma)}(z) = q_{z}^{t}(\lambda) \). Hence, for \( z = \Phi_{\alpha, \beta, \gamma}(\lambda) \), by integrating (6.3), we obtain
\[
(6.4) \quad S(x_0 + g_{\alpha, \beta, \gamma}, z, \varepsilon) - S(x_0 + g_{\alpha, \beta, \gamma}, z, 1) = S(x_0 + g_{\alpha, \beta, \gamma}, \lambda, \varepsilon) - S(x_0 + g_{\alpha, \beta, \gamma}, \lambda, 1).
\]
Lemma 6.1 shows that the right hand side of the above equation is a \( C^\infty \)-function of \( \lambda \) when \( \varepsilon \) goes to zero. Recall that, for \( \lambda \in \Xi_{\alpha, \beta} \), \( \lim_{\varepsilon \to 0^+} \varepsilon \Phi_{\alpha, \beta, \gamma}(\lambda, \varepsilon) = s(\lambda) \). We then have
\[
\lim_{\varepsilon \to 0^+} \Phi_{\alpha, \beta, \gamma}(\lambda) = \lim_{\varepsilon \to 0^+} \lambda + \gamma \cdot p_{\lambda}^{t_0}(\varepsilon_0) = \lambda + \gamma \cdot p_{\lambda}^{t_0}(s(\lambda)) = \Phi_{\alpha, \beta, \gamma}(\lambda).
\]
Thus, the assumption that \( \Phi_{\alpha, \beta, \gamma} \) is non-singular at \( \lambda \) implies that the map \( (\lambda, \varepsilon) \mapsto (z, \varepsilon) \) is non-singular at \( (\lambda, 0) \). This yields that the analyticity of right hand side of (6.4) implies that the left hand side of (6.4) has a \( C^\infty \)-extension in some neighborhood of \( (\Phi_{\alpha, \beta, \gamma}(\lambda), 0) \).

Recall that \( g_{21} = g_{21}(\lambda, \varepsilon_1, \varepsilon_2) \) is a constant by Lemma 4.5. Again by choosing \( \gamma = 0 \) and an arbitrary \( \gamma \) respectively in the subordination relation \( g_{21} = P_{z}^{t(\gamma)}(\varepsilon) = p_{z}^{t(\gamma)}(\varepsilon) \), we have
\[
p_{z}^{t(\gamma)}(\varepsilon) = \lambda, \varepsilon_0 = \varepsilon_0(\lambda, \varepsilon) \text{ is given by (4.30).}
\]
Recall that, by definitions (4.3) and (4.2), for \( y = g_{\alpha, \beta, \gamma} \), we have
\[
p_{z}^{t(\gamma)}(\varepsilon) = \phi\left\{ (z - x_0 - y)^* \left[ (z - x_0 - y)(z - x_0 - y)^* + \varepsilon^2 \right]^{-1} \right\}
\]
\[
= \frac{\partial S(x_0 + y, z, \varepsilon)}{\partial z}.
\]
Hence, the regularity of \( S(x_0 + g_{\alpha, \beta, \gamma}, z, \varepsilon) \) in the neighborhood of \( (\Phi_{\alpha, \beta, \gamma}(\lambda), 0) \) implies that
\[
(6.5) \quad \lim_{\varepsilon \to 0^+} p_{\Phi_{\alpha, \beta, \gamma}(\lambda)}^{t(\gamma)}(\varepsilon) = \lim_{\varepsilon \to 0^+} p_{\lambda}^{t}(\varepsilon) = \lim_{\varepsilon \to 0^+} p_{\lambda}^{t_0}(\varepsilon_0(\lambda, \varepsilon)).
\]
Assume now that the map \( \Phi_{\alpha, \beta, \gamma} \) is non-singular at any \( \lambda \in \Xi_{\alpha, \beta} \), then (6.2) holds at any \( \lambda \in \Xi_{\alpha, \beta} \). Hence, if \( \lambda_1, \lambda_2 \in \Xi_{\alpha, \beta} \) and \( \Phi_{\alpha, \beta, \gamma}(\lambda_1) = \Phi_{\alpha, \beta, \gamma}(\lambda_2) = z \), we have
\[
z = \lambda_1 + \gamma p_{\lambda_1}^{t_0}(s(\lambda_1)) = \lambda_2 + \gamma p_{\lambda_2}^{t_0}(s(\lambda_2)).
\]
Then (6.5) shows
\[
\lim_{\varepsilon \to 0^+} p_{\lambda_1}^{t_0}(\varepsilon_0(\lambda_1, \varepsilon)) = \lim_{\varepsilon \to 0^+} p_{\lambda_2}^{t_0}(\varepsilon_0(\lambda_2, \varepsilon)) = p_{z}^{t(\gamma)}(0).
\]
That is \( p_{\lambda_1}^{\alpha}(s(\lambda_1)) = p_{\lambda_2}^{\beta}(s(\lambda_2)) \). Hence, we deduce that \( \lambda_1 = \lambda_2 \) and the map \( \Phi_{\alpha,\beta,\gamma} \) is one-to-one in \( \Xi_{\alpha,\beta} \).

7. The sum of an elliptic operator and an unbounded operator

We proceed to study the case when \( \alpha = \beta \), in which case \( g_{\alpha,\beta,\gamma} \) is a twisted elliptic operator. Consider now an operator \( x_0 \in \text{log}^+(A) \) and a twisted elliptic operator \( g_{t,\gamma} \) that is *-free from \( x_0 \). By using operator-valued subordination functions for unbounded operators from Lemma 4.3, we are able to study the Brown measure of \( x_0 + g_{t,\gamma} \). Since \( x_0 \) could be unbounded, we extend main results in [33]. We denote

\[
(7.1) \quad X = \begin{bmatrix} 0 & x_0 \\ x_0^* & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & g_{t,\gamma} \\ g_{t,\gamma}^* & 0 \end{bmatrix}.
\]

Then \( X \) is affiliated with \( M_2(A) \), and \( \{X, Y\} \) are free with amalgamation over \( M_2(\mathbb{C}) \) in the noncommutative probability space \( (M_2(A), M_2(\phi), M_2(\mathbb{C})) \). By Theorem 3.1 there exist two analytic self-maps \( \Omega_1, \Omega_2 \) of upper half-plane \( \mathbb{H}^+(M_2(\mathbb{C})) \) of \( M_2(\mathbb{C}) \) such that

\[
(7.2) \quad (\Omega_1(b) + \Omega_2(b) - b)^{-1} = G_X(\Omega_1(b)) = G_Y(\Omega_2(b)) = G_{X+Y}(b),
\]

for all \( b \in M_2(\mathbb{C}) \) with \( \Re b > 0 \). In other words, subordination relation (2.1) for \( X + Y \) is still available, where \( X \) could be unbounded.

We introduce a new approach via regularized Brown measure to study the distinguished measure \( \mu_{x_0+g_{t,\gamma}} \). This allows us to prove several regularity results for this Brown measure. In particular, we show that \( \mu_{x_0+g_{t,\gamma}} \) is absolutely continuous with respect to the Lebesgue measure.

7.1. The subordination functions

We first calculate a formula for \( \Omega_1(\Theta(\lambda, \varepsilon)) \) and study its entries following the strategy in [33] Section 3. The next result is [33] Proposition 3.5 and we provide a direct proof for convenience.

**Lemma 7.1.** Let \( \mu_1 \) be a symmetric probability measure, not necessarily compactly supported, and let \( \mu_2 \) be the semicircular distribution with variance \( t \). Denote \( \mu = \mu_1 \boxplus \mu_2 \). Let \( \omega_1, \omega_2 \) be subordination function such that

\[
G_{\mu_1}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z)).
\]

Set \( w(\varepsilon) = \Im \omega_1(i\varepsilon) \). Then, \( w = w(\varepsilon) \) is the unique solution in \( (0, \infty) \) of the following equation

\[
(7.3) \quad w = \varepsilon + tw \int_{\mathbb{R}} \frac{1}{u^2 + w^2} d\mu_1(u).
\]

**Proof.** Note that the \( R \)-transform of \( \mu_2 \) is \( R_{\mu_2}(z) = tz \). Since \( R_{\mu} = R_{\mu_1} + R_{\mu_2} \), we have

\[
G_{\mu_1}^{-1}(z) = G_{\mu_1}^{-1}(z) + R_{\mu_2}(z) = G_{\mu_2}^{-1}(z) + tz.
\]

By replacing \( z \) with \( G_{\mu}(z) \), we obtain \( \omega_1(z) = z - tG_{\mu}(z) \) which holds for all \( z \in \mathbb{C}^+ \) by analytic continuation. In particular, for \( z = i\varepsilon \), we have

\[
(7.4) \quad \omega_1(i\varepsilon) = \varepsilon - tG_{\mu}(i\varepsilon) = \varepsilon - tG_{\mu_1}(\omega_1(i\varepsilon)).
\]

It is clear that \( \omega_1(i\varepsilon) \) is a pure imaginary number and hence \( \omega_1(i\varepsilon) = iw(\varepsilon) \). Note that,

\[
G_{\mu_1}(iw) = \int_{\mathbb{R}} \frac{1}{iu - u} d\mu_1(u) = -iw \int_{\mathbb{R}} \frac{1}{u^2 + w^2} d\mu_1(u)
\]

by the symmetry of \( \mu_1 \). Taking the imaginary part of (7.4), we see that \( w(\varepsilon) \) satisfy (7.3).
We next show that (7.3) has a unique solution in \((0, \infty)\). Denote
\[
(7.5) \quad k(w, \varepsilon) = \frac{w}{w - \varepsilon} \left( \int \frac{1}{u^2 + w^2} d\mu_1(u) \right).
\]
Then (7.5) is equivalent to \(k(w, \varepsilon) = 1/t\). Observe that \(w \mapsto k(w, \varepsilon)\) is a strictly decreasing function in \((0, \infty)\). Hence, (7.3) has a unique solution in \((0, \infty)\). \(\square\)

**Definition 7.2.** For any probability measure \(\mu\) on \(\mathbb{R}\), we denote by \(\tilde{\mu}\) the symmetrization of \(\mu\) defined by
\[
\tilde{\mu}(B) = \frac{1}{2}(\mu(B) + \mu(-B)),
\]
for any Borel measurable set \(B\) on \(\mathbb{R}\). Given \(\lambda \in \mathbb{C}\), let \(\mu_1 = \tilde{\mu}_{|\lambda - x_0|}\) and let \(\mu_2\) be the semicircular distribution with variance \(t\). Let \(\omega_1\) be the subordination function (depending on \(\lambda\)) such that
\[
G_{\mu_1 \ast \mu_2}(z) = G_{\mu_1}(\omega_1(z)).
\]
We denote \(w(\varepsilon; \lambda; t) = \Im \omega_1(i\varepsilon)\).

We choose
\[
b = \Theta(z, \varepsilon) = \begin{bmatrix} i\varepsilon & z \\ \lambda & i\varepsilon \end{bmatrix}
\]
where \(\varepsilon > 0\) and \(z \in \mathbb{C}\). The following result extends [33, Theorem 3.8] to unbounded operator \(x_0\).

**Theorem 7.3.** Let \(x_0 \in \log^+(A)\) be an operator that is \(*\)-free from the elliptic operator \(g_{t, \gamma}\). For any \(\varepsilon > 0\) and \(z \in \mathbb{C}\), we set
\[
\lambda = z - \gamma \cdot \phi \left( (z - x_0 - g_{t, \gamma})^* ((z - x_0 - g_{t, \gamma})(z - x_0 - g_{t, \gamma})^* + \varepsilon^2)^{-1} \right).
\]
Then \(\Omega_1(\Theta(z, \varepsilon)) = \Theta(\lambda, w(\varepsilon; \lambda; t))\). That is,
\[
(7.7) \quad \Omega_1 \left( \begin{bmatrix} i\varepsilon & z \\ \lambda & i\varepsilon \end{bmatrix} \right) = \begin{bmatrix} iw(\varepsilon; \lambda, t) & \lambda \\ \lambda^{-1} & iw(\varepsilon; \lambda, t) \end{bmatrix}.
\]
The subordination relation \(G_{X + Y}(\Theta(z, \varepsilon)) = G_X(\Omega_1(\Theta(z, \varepsilon)))\) is expressed as
\[
(7.8) \quad \mathbb{E} \left( \left[ z - (x_0 + g_{t, \gamma})^* \right]^{-1} \right) = \mathbb{E} \left( \left[ \lambda - x_0 \overline{\lambda - x_0} - 1 \right]^{-1} \right),
\]
which is also equivalent to
\[
\varepsilon \phi \left( ((z - x_0 - g_{t, \gamma})(z - x_0 - g_{t, \gamma})^* + \varepsilon^2)^{-1} \right) = w(\varepsilon; \lambda, t) \phi \left( ((\lambda - x_0)(\lambda - x_0)^* + w(\varepsilon; \lambda, t)^2)^{-1} \right),
\]
\[
\phi \left( (z - x_0 - g_{t, \gamma})^* ((z - x_0 - g_{t, \gamma})(z - x_0 - g_{t, \gamma})^* + \varepsilon^2)^{-1} \right) = \phi \left( (\lambda - x_0)^* ((\lambda - x_0)(\lambda - x_0)^* + w(\varepsilon; \lambda, t)^2)^{-1} \right).
\]
Proof. Apply Lemma 4.3 to the case $\alpha = \beta = t$. In this case, 

$$Q_t^\ast(z) = \tilde{Q}_t^\ast(z)$$

and hence $\varepsilon_1 = \varepsilon_2 \in (\varepsilon, \infty)$. Moreover, by (4.12) and the definition (4.7), then $\varepsilon_1$ can be expressed as

$$\varepsilon_1 = \varepsilon + \varepsilon_1 \phi \left( \left( (\lambda - x_0)(\lambda - x_0)^* + \varepsilon_1^2 \right)^{-1} \right).$$

This shows that $\varepsilon_1 = \varepsilon_2 = w(\varepsilon; \lambda, t)$ by (7.3) and Definition 7.2. \qed

Recall that the open set $\Xi_t$ is defined in (1.3) as

$$\Xi_t = \left\{ \lambda \in \mathbb{C} : \phi \left[ \left( (x_0 - \lambda)^*(x_0 - \lambda) \right)^{-1} \right] > \frac{1}{t} \right\}.$$ 

**Definition 7.4.** For $\lambda \in \Xi_t$, let $w(0; \lambda, t)$ be the unique solution $w \in (0, \infty)$ to the following equation

(7.10) \[ \phi \left[ \left( (x_0 - \lambda)^*(x_0 - \lambda) + w(0; \lambda, t)^2 \right)^{-1} \right] = \frac{1}{t}. \]

For $\lambda \in \mathbb{C} \setminus \Xi_t$, set $w(0; \lambda, t) = 0$.

We define the function $\Phi^{(\varepsilon)}_{t, \gamma}$ on $\mathbb{C}$ by

(7.11) \[ \Phi^{(\varepsilon)}_{t, \gamma}(\lambda) = \lambda + \gamma \cdot p^{(0)}_\lambda (w(\varepsilon; \lambda, t)), \quad \lambda \in \mathbb{C} \]

where

$$p^{(0)}_\lambda (w(\varepsilon; \lambda, t)) = \phi \left( \lambda - x_0 \right)^* \phi \left( (\lambda - x_0)(\lambda - x_0)^* + w(\varepsilon; \lambda, t)^2 \right)^{-1}. \]

This can also be rewritten as

$$\Phi^{(\varepsilon)}_{t, \gamma}(\lambda) = \lambda + \gamma \cdot \frac{\partial S}{\partial \lambda}(x_0, \lambda, w(\varepsilon; \lambda, t)).$$

The map $\Phi_{t, \gamma}$ is defined as

$$\Phi_{t, \gamma}(\lambda) = \lambda + \gamma \cdot p^{(0)}_\lambda (w(0; \lambda, t)), \quad \lambda \in \mathbb{C}.$$

The following result is a special case of Lemma 4.10 by letting $\alpha = \beta = t$.

**Corollary 7.5.** The map $\Phi^{(\varepsilon)}_{t, \gamma}$ is a homeomorphism of the complex plane for any $\varepsilon > 0$. Its inverse map is

(7.12) \[ J^{(\varepsilon)}_{t, \gamma}(z) = z - \gamma \cdot p^{(t, \gamma)}_z (\varepsilon), \]

where

$$p^{(t, \gamma)}_z (\varepsilon) = \phi \left( (z - x_0 - g_{t, \gamma})^* \left( (z - x_0 - g_{t, \gamma})(z - x_0 - g_{t, \gamma})^* + \varepsilon^2 \right)^{-1} \right).$$

**Lemma 7.6.** For any $\varepsilon > 0$, the function $\lambda \mapsto w(\varepsilon; \lambda, t)$ is a $C^\infty$ function. The function $\lambda \mapsto w(0; \lambda, t)$ is a continuous function of $\lambda$ on $\mathbb{C}$ and is a $C^\infty$ function in the open set $\Xi_t$.

Moreover, the function $w(\varepsilon; \lambda, t)$ converges uniformly to $w(0; \lambda, t)$ in any compact subset of $\mathbb{C}$ as $\varepsilon$ tends to zero.
Proof. By Definition 7.2, for any \( \varepsilon > 0 \), the function \( w(\varepsilon; \lambda, t) \) is the imaginary part of the subordination function parameterized by \( \lambda \). In particular, \( w(\varepsilon; \lambda, t) = \omega_1(\varepsilon) > \varepsilon \), where \( \omega_1 \) is the subordination as in Definition 7.2. By Lemma 7.1, the function \( w(\varepsilon; \lambda, t) \) is the unique \( w \) such that

\[
(7.13) \quad w = \varepsilon + tw\int_{\mathbb{R}} \frac{1}{u^2 + w^2} d\tilde{\mu}|_{\lambda - x_0}(u) = \varepsilon + tw\phi[[((\lambda - x_0)^*(\lambda - x_0) + w^2)^{-1}].
\]

Following (7.5), we rewrite it as

\[
(7.14) \quad k(w, \varepsilon) = \frac{w}{w - \varepsilon}\left(\int_{\mathbb{R}} \frac{1}{u^2 + w^2} d\tilde{\mu}|_{\lambda - x_0}(u)\right) = \frac{w}{w - \varepsilon}\phi[[((\lambda - x_0)^*(\lambda - x_0) + w^2)^{-1}] = \frac{1}{t}.
\]

Note that \( \frac{\partial}{\partial \varepsilon}k(w, \varepsilon) < 0 \) for \( w > \varepsilon \) and \( \lambda \mapsto k(w, \varepsilon) \) is a smooth function. By implicit function theorem, \( \lambda \mapsto w(\varepsilon; \lambda, t) \) is a \( C^\infty \) function. Similarly, for \( \lambda \in \Xi_t \), the function \( w(0; \lambda, t) \) is determined by (7.10), which yields that \( \lambda \mapsto w(0; \lambda, t) \) is a \( C^\infty \) function in the open set \( \Xi_t \) by applying the implicit function theorem.

Recall that \( w(\varepsilon; \lambda, t) \) is the unique solution of \( k(w, \varepsilon) = 1/t \) as in (7.14). We observe that

\[
k(w(\varepsilon_1; \lambda, t), \varepsilon_1) < k(w(\varepsilon_2; \lambda, t), \varepsilon_2)
\]

if \( 0 < \varepsilon_1 < \varepsilon_2 < w(\varepsilon_1; \lambda, t) \). Hence, \( w(\varepsilon_1; \lambda, t) < w(\varepsilon_2; \lambda, t) \) since \( \frac{\partial}{\partial \varepsilon}k(w, \varepsilon) < 0 \).

By Dini’s theorem, the convergence of \( w(\varepsilon; \lambda, t) \) to \( w(0; \lambda, t) \) is uniform in any compact subset of \( \mathbb{C} \). □

7.2. The Fuglede-Kadison determinant formula. By using Theorem 7.3 we can apply the exact same methods as in [33, Lemma 3.11, Theorem 3.12] to obtain a formula for \( \Delta((x_0 + g_{t, \gamma})^*(x_0 + g_{t, \gamma}) + \varepsilon^2) \). We only state a result for circular operator below (See [6, Section 5.2] for an alternative proof).

Theorem 7.7. [33, Theorem 3.12] For \( \lambda \in \mathbb{C} \), we have the following Fuglede-Kadison determinant formulas.

1. If \( \lambda \in \Xi_t \), then

\[
(7.15) \quad \Delta(x_0 + ct - \lambda)^2 = \Delta((x_0 - \lambda)^*(x_0 - \lambda) + w(0; \lambda, t)^2) \times \exp\left(-\frac{(w(0; \lambda, t))^2}{t}\right).
\]

2. If \( \lambda \notin \Xi_t \), then

\[
(7.16) \quad \Delta(x_0 + ct - \lambda) = \Delta(x_0 - \lambda).
\]

The following proof is adapted from [33, Theorem 4.6]. We include a proof for the reader’s convenience.

Corollary 7.8. For any \( \lambda \in \mathbb{C} \), we have

\[
\log \Delta(x_0 + ct - \lambda) > -\infty.
\]

The Brown measure \( \mu_{x_0 + ct} \) has no atom and the support of \( \mu_{x_0 + ct} \) is contained in the closure \( \overline{\Xi_t} \).
Proof. If \( \lambda \in \Xi_t \), since \( w(0; \lambda, t) > 0 \), it follows from (7.15) that \( \log \Delta(x_0 + c_t - \lambda) > -\infty \). If \( \lambda \notin \Xi_t \), then \( \phi(|x_0 - \lambda|^2) \leq 1/t \), and hence
\[
2 \int_0^1 \log t d\mu_{|x_0 - \lambda|}(t) > -\int_0^1 \frac{1}{t^2} d\mu_{|x_0 - \lambda|}(t) > -\phi(|x_0 - \lambda|^2) \geq -1/t.
\]
Therefore, \( \lambda \notin \Xi_t \),
\[
\log \Delta(x_0 + c_t - \lambda) = \log \Delta(x_0 - \lambda) > -\infty.
\]
By [17] Proposition 2.14, 2.16, we deduce that \( \mu_{x_0+c_t}(\{\lambda\}) = \mu_{x_0+c_t-\lambda}(\{0\}) = 0 \) for any \( \lambda \in \mathbb{C} \).

The Brown measure of \( \mu_{(x_0-\lambda)^{-1}} \) is the pushforward measure of \( \mu_{x_0} \) under the map \( z \mapsto (z - \lambda)^{-1} \). By Weil’s inequality for operators in \( L^p(A) \) (see [17] Theorem 2.19), we have
\[
\int_{\mathbb{C}} \frac{1}{|z - \lambda|^2} d\mu_{x_0}(z) = \int_{\mathbb{C}} |z|^2 d\mu_{(x_0 - \lambda)^{-1}}(z) \leq \| (x_0 - \lambda)^{-1} \|^2 = \phi(|x_0 - \lambda|^2).
\]
Consequently, if \( \lambda \notin \Xi_t \), then
\[
\int_{\mathbb{C}} \frac{1}{|z - \lambda|^2} d\mu_{x_0}(z) \leq \frac{1}{t}
\]
in some neighborhood of \( \lambda \). This implies that \( \lambda \) is not in the support of \( \mu_{x_0} \). On the other hand, by (7.16), \( \mu_{x_0} \) and \( \mu_{x_0+c_t} \) coincide when restricting to \( \mathbb{C} \setminus \Xi_t \). Hence, \( \text{supp}(\mu_{x_0+c_t}) \subset \Xi_t \). □

The following regularity result extends [26] Proposition 5 to unbounded operators. This is an analogue of [18] Corollary 4.8 where we use a circular operator in place of a circular Cauchy operator.

**Proposition 7.9.** The function \( t \mapsto \Delta(x_0 + c_t - \lambda) \) is strictly increasing for \( t \geq 0 \), and

\[
(7.17) \quad \lim_{t \to 0^+} \Delta(x_0 + c_t - \lambda) = \Delta(x_0 - \lambda)
\]

for any \( \lambda \in \mathbb{C} \).

**Proof.** By the definition of the function \( w(0; \lambda, t) \) in (7.10), we see that
\[
\lim_{t \to 0^+} w(0; \lambda, t) = 0.
\]
If \( \lambda \in \Xi_{t_0} \), for some \( t_0 > 0 \), then by Part (1) of Theorem 7.7, we have
\[
\lim_{t \to 0^+} \Delta(x_0 + c_t - \lambda)^2 = \lim_{t \to 0^+} \left[ \Delta((x_0 - \lambda)^* + w(0; \lambda, t)^2) \exp \left( -\frac{(w(0; \lambda, t)^2)}{t} \right) \right] = \Delta(x_0 - \lambda)^2.
\]
This establishes (7.17).

We note that \( \Xi_{t_1} \subset \Xi_{t_2} \) for any \( t_2 > t_1 > 0 \). We then use (7.10) to write
\[
\frac{w(0; \lambda, t)^2}{t} = w(0; \lambda, t)^2 \phi \left[ ((x_0 - \lambda)^* + w(0; \lambda, t)^2)^{-1} \right].
\]
Hence, if \( \lambda \in \Xi_t \) for some \( t > 0 \), write \( w = w(0; \lambda, t) \), then
\[
\log \Delta(x_0 + c_t - \lambda)^2 = \log \Delta((x_0 - \lambda)^*(x_0 - \lambda) + w^2)
- w^2 \phi \left( (x_0 - \lambda)^*(x_0 - \lambda) + w^2 \right)^{-1}
= \int_0^\infty \left( \log(u^2 + w^2) - \frac{w^2}{u^2 + w^2} \right) d\mu_{x_0 - \lambda}(u).
\]
Note that the integrand on the right hand side of the above equation is a strictly increasing function of \( w \in \mathbb{R}^+ \). From the defining equation (7.10), we see that \( w(0; \lambda, t_2) > w(0; \lambda, t_1) \) for any \( t_2 > t_1 \) and \( \lambda \in \Xi_t \). It follows that \( \Delta(x_0 + c_{t_2} - \lambda) > \Delta(x_0 + c_{t_1} - \lambda) \) for any \( t_2 > t_1 \). If \( \lambda \notin \Xi_{t_1} \), then
\[
\lim_{t \to (t_0)^+} \log \Delta(x_0 + c_t - \lambda) = \log \Delta(x_0 + c_{t_0} - \lambda),
\]
and again we have \( \Delta(x_0 + c_{t_2} - \lambda) > \Delta(x_0 + c_{t_1} - \lambda) \). This finishes the proof.

7.3. The Brown measure formulas. In this section, we study the Brown measures \( \mu_{x_0 + c_t} \) and \( \mu_{x_0 + g_n} \). Our first result strengthens [33, Theorem 4.2] to unbounded operator \( x_0 \) and addresses the issue of nonexistence of singular continuous part which was left open in [33]. The upper bound result generalizes [22, Theorem 3.14] for an arbitrary operator, not necessarily selfadjoint.

**Theorem 7.10.** The Brown measure of \( x_0 + c_t \) is absolutely continuous with respect to the Lebesgue measure. It is supported in the closure set \( \overline{\Xi_t} \). The density function \( \rho_{t,x_0} \) of the Brown measure at \( \lambda \in \Xi_t \) is given by
\[
\rho_{t,x_0}(\lambda) = \frac{1}{\pi} \left( \frac{\left| \phi((h^{-1})^2(\lambda - x_0))^2 + w^2 \phi(h^{-1}k^{-1}) \right|}{\phi((h^{-1})^2)} \right),
\]
where \( w = w(0; \lambda, t) \) is determined by
\[
\phi((x_0 - \lambda)^*(x_0 - \lambda) + w^2)^{-1} = \frac{1}{t},
\]
and \( h = h(\lambda, w(0; \lambda, t)) \) and \( k = k(\lambda, w(0; \lambda, t)) \) for
\[ h(\lambda, w) = (\lambda - x_0)^*(\lambda - x_0) + w^2 \]
and
\[ k(\lambda, w) = (\lambda - x_0)(\lambda - x_0)^* + w^2. \]
The density of the Brown measure of \( x_0 + c_t \) is strictly positive in the set \( \Xi_t \). Moreover,
\[
0 < \rho_{t,x_0}(\lambda) \leq \frac{1}{\pi t}
\]
for any \( \lambda \in \Xi_t \).

**Lemma 7.11.** The regularized Brown measure \( \mu_{x_0 + c_t}^{(\varepsilon)} \) of \( x_0 + c_t \) is absolutely continuous with respect to the Lebesgue measure. The density function \( \rho_{t,x_0}^{(\varepsilon)} \) of the Brown measure at \( \lambda \in \mathbb{C} \) is given by
\[
\rho_{t,x_0}^{(\varepsilon)}(\lambda) = \frac{1}{\pi} \left( \frac{\left| \phi((h^{-1})^2(\lambda - x_0))^2 + w^2 \phi(h^{-1}k^{-1}) \right|}{\phi((h^{-1})^2) + \varepsilon/(2tw^2)} \right),
\]
where \( w = w(\varepsilon; \lambda, t) \) is determined by
\[ w = \varepsilon + tw\phi((\lambda - x_0)^*(\lambda - x_0) + w^2)^{-1}, \]
and \( h = h(\lambda, w(\varepsilon; \lambda, t)) \) and \( k = k(\lambda, w(\varepsilon; \lambda, t)) \) for
\[
h(\lambda, w) = (\lambda - x_0)^*(\lambda - x_0) + w^2
\]
and
\[
k(\lambda, w) = (\lambda - x_0)(\lambda - x_0)^* + w^2.
\]
Moreover,
\[
(7.20)\quad 0 < \rho^{(e)}_{t, x_0}(\lambda) < \frac{1}{\pi t}
\]
for any \( \lambda \in \mathbb{C} \).

Proof. Recall that \( \lambda \mapsto w(\varepsilon; \lambda, t) \) is a \( C^\infty \) function by Lemma 7.6. Put
\[
g(\lambda) = \log \Delta((\lambda - x_0 - c_t)^*(\lambda - x_0 - c_t) + \varepsilon^2).
\]
Then, the density of the regularized Brown measure can be calculated as
\[
\rho^{(e)}_{t, x_0}(\lambda) = \frac{1}{\pi} \frac{\partial^2}{\partial \lambda \partial \lambda^*} g(\lambda).
\]
By choosing \( \gamma = 0 \), the subordination relation (7.9) can be rewritten as
\[
\frac{\partial}{\partial \lambda} g(\lambda) = \phi \left( (\lambda - x_0 - c_t)^*(\lambda - x_0 - c_t)(\lambda - x_0 - c_t)^* + \varepsilon^2 \right)^{-1}
\]
\[
= \phi \left( (\lambda - x_0)^*(\lambda - x_0)(\lambda - x_0)^* + w(\varepsilon; \lambda, t)^2 \right)^{-1}
\]
\[
= \phi \left( (\lambda - x_0)^*(\lambda - x_0) + w(\varepsilon; \lambda, t)^2 \right)^{-1}(\lambda - x_0)^*\big).
\]
Note that \( g(\lambda) \) is a \( C^\infty \) function of \( \lambda \). We apply Lemma 5.5 to get
\[
\frac{\partial^2}{\partial \lambda \partial \lambda^*} g(\lambda) = \frac{\partial}{\partial \lambda} \phi \left( h^{-1}(\lambda - x_0)^* \right)
\]
\[
= \phi \left( -h^{-1} \left( (\lambda - x_0) + 2w \frac{\partial w}{\partial \lambda} \right) h^{-1}(\lambda - x_0)^* + h^{-1} \right)
\]
\[
= \phi \left( -h^{-1}(\lambda - x_0)h^{-1}(\lambda - x_0)^* + h^{-1} \right) - 2w \frac{\partial w}{\partial \lambda} \phi \left( (\lambda - x_0)^*(h^{-1})^2 \right).
\]
We now apply the identity \( x(x^*x + a)^{-1}x^* = (xx^* + a)^{-1}xx^* \) to \( x = \lambda - x_0 \) and \( a = w^2 \), we find that
\[
1 - (\lambda - x_0)h^{-1}(\lambda - x_0)^* = 1 - x(x^*x + w^2)^{-1}x^*
\]
\[
= 1 - (xx^* + w^2)^{-1}xx^*
\]
\[
= w^2(\lambda - x_0)^*(h^{-1})^2
\]
\[
= w^2 k^{-1}.
\]
Therefore, the first term in the right hand side of (7.21) can be rewritten as
\[
(7.22)\quad \phi \left( -h^{-1}(\lambda - x_0)h^{-1}(\lambda - x_0)^* + h^{-1} \right) = w^2 \phi(h^{-1}k^{-1}).
\]
We next calculate \( \frac{\partial w}{\partial \lambda} \) by taking implicit differentiation. Recall that \( w = w(\varepsilon; \lambda, t) \) satisfies (7.13), which can be rewritten as
\[
(7.23)\quad w - \varepsilon = tw\phi(h^{-1}).
\]
By taking the derivative for both sides, we obtain
\[ \frac{\partial w}{\partial \lambda} = -tw \left[ \phi \left( h^{-1} \left( (\lambda - x_0) + 2w \frac{\partial w}{\partial \lambda} \right) h^{-1} \right) \right] + t\phi(h^{-1}) \frac{\partial w}{\partial \lambda}, \]
which yields
\[ \phi((h^{-1})^2(\lambda - x_0)) = -2w \frac{\partial w}{\partial \lambda} \left( \phi((h^{-1})^2) - \frac{1}{2w^2}\phi(h^{-1}) + \frac{1}{2w^2} \right) \]
\[ = -2w \frac{\partial w}{\partial \lambda} \left( \phi((h^{-1})^2) + \frac{\varepsilon}{2tw^3} \right), \]
where we used \( \phi(h^{-1}) = \frac{\varepsilon}{tw} \) to derive the last identity above. Therefore, for the second term in the right hand side of (7.21) we have
\[ (7.24) \quad -2w \frac{\partial w}{\partial \lambda} \phi((h^{-1})^2(\lambda - x_0)^*) = \frac{\phi((h^{-1})^2(\lambda - x_0))^2}{\phi((h^{-1})^2) + \varepsilon/(2tw^3)}. \]

By plugging (7.22) and (7.24) to (7.21), we obtain the density formula as desired.

We now turn to prove the upper bound of the density function. We write \( x_\lambda = \lambda - x_0 \) and recall that \( h = x_\lambda^* x_\lambda + w^2 \). Again by using Lemma 5.5 we have \( x(x_\lambda^* x + a)^{-1} = (x x^* + a)^{-1} x \) for any \( x \) and \( a \in (0, \infty) \), it follows that
\[ \phi((h^{-1})^2(\lambda - x_0)) = \phi((h^{-1})^2 x_\lambda) \]
\[ = \phi(x_\lambda h^{-1} x_\lambda) \]
\[ = \phi(k^{-1} x_\lambda h^{-1}). \]

By the Cauchy-Schwarz inequality, we have
\[ |\phi(k^{-1} x_\lambda h^{-1})|^2 = |\phi((h^{-1/2} k^{-1/2})(x_\lambda^* x_\lambda)^{-1/2})|^2 \]
\[ \leq \phi(h^{-1} k^{-1}) \cdot \phi(k^{-1/2} x_\lambda^* x_\lambda^{-1/2}) \]
\[ = \phi(h^{-1} k^{-1}) \cdot \phi(x_\lambda h^{-1} x_\lambda^{-1/2}) \]
\[ = \phi(h^{-1} k^{-1}) \cdot \phi(x_\lambda h^{-1} x_\lambda^{-1}) \]
\[ = \phi(h^{-1} k^{-1}) \cdot \phi(k^{-1} x_\lambda h^{-1}). \]

Therefore,
\[ \frac{\phi((h^{-1})^2(\lambda - x_0))^2}{\phi((h^{-1})^2) + \varepsilon/(2tw^3)} + w^2 \phi(h^{-1} k^{-1}) \]
\[ < \frac{\phi(h^{-1} k^{-1}) \cdot \phi(h^{-1} h^{-1} x_\lambda^* x_\lambda)}{\phi((h^{-1})^2)} \]
\[ + \frac{\phi(h^{-1} k^{-1}) \cdot \phi(h^{-1} k^{-1} x_\lambda^* x_\lambda)}{\phi((h^{-1})^2)} + w^2 \phi(h^{-1} k^{-1}) \]
\[ = \frac{\phi(h^{-1} k^{-1}) \cdot \phi(h^{-1})}{\phi((h^{-1})^2)} \frac{w - \varepsilon}{wt} \leq \frac{1}{t} \phi((h^{-1})^2). \]

Finally, since \( \phi((h^{-1})^2) = \phi((k^{-1})^2) \), by the Cauchy-Schwarz inequality, we have
\[ \phi(h^{-1} k^{-1}) \leq \phi((h^{-1})^2). \]

This establishes that \( 0 < \rho_{1,x_0}^{(z)}(\lambda) < 1/(\pi t) \) for any \( \lambda \in \mathbb{C} \). \( \square \)

We are now ready to prove Theorem 7.4.10.
Proof of Theorem 7.10. Let $E$ be any Borel measurable set in $\mathbb{C}$ with zero Lebesgue measure. Let $U$ be an open set such that $E \subset U$. Since $\mu_{x_0 + \varepsilon}$ converges to $\mu_{x_0 + \varepsilon}$ weakly as $\varepsilon$ tends to zero, it follows that

$$\mu_{x_0 + \varepsilon}(U) \leq \liminf_{\varepsilon \to 0} \mu_{x_0 + \varepsilon}(U) \leq \frac{m(U)}{\pi t},$$

by (7.20), where $m(U)$ denotes the Lebesgue measure of $U$. Hence, $\mu_{x_0 + \varepsilon}(E) = 0$ and the Brown measure $\mu_{x_0 + \varepsilon}$ is absolutely continuous.

By Lemma 7.6 $w(\varepsilon; \lambda, t) \to w(0; \lambda, t)$ uniformly in any compact set of $\mathbb{C}$. Since $w(0; \lambda, t) > 0$ for $\lambda \in \Xi_t$, Hence, the formula for the density function $\rho_{t,x_0}$ follows from (7.19) by letting $\varepsilon$ go to zero. In addition, by the upper bound for the regularized Brown measure, we have $\rho_{t,x_0}(\lambda) \leq 1/(\pi t)$.

**Theorem 7.12.** For any $\varepsilon > 0$, the regularized Brown measure $\mu_{x_0 + \varepsilon}$ is the push-forward measure of the regularized Brown measure of $\mu_{x_0 + \varepsilon}$ under that map $\Phi_{t, \gamma}$. The map $\Phi_{t, \gamma}$ converges uniformly to $\Phi_{t, \gamma}$ in any compact subset of $\mathbb{C}$ as $\varepsilon$ tends to zero. Consequently, the Brown measure of $\mu_{x_0 + \varepsilon}$ is the push-forward measure of the Brown measure $\mu_{x_0 + \varepsilon}$ under the map $\Phi_{t, \gamma}$.

Moreover, if the map $\Phi_{t, \gamma}$ is non-singular at any $\lambda \in \Xi_t$, then it is also one-to-one in $\Xi_t$.

**Proof.** Apply the proof for [33] Theorem 5.2 (see also Theorem 4.13) to get the first part. We set $S_1(\lambda) = S(x_0 + c_1, \lambda, \varepsilon)$ and $S_2(z) = S(x_0 + \varepsilon z, \lambda, \varepsilon)$, and $S_0(\lambda) = S(x_0, \lambda, \varepsilon_0)$, where $\varepsilon_0 = w(\varepsilon; \lambda, t)$ and $z = \Phi_{t, \gamma}(\lambda)$. Similar to the proof for Theorem 4.13 the map $\Phi_{t, \gamma}$ can be rewritten as

$$\Phi_{t, \gamma}(\lambda) = \lambda + \gamma \frac{\partial S_0(\lambda)}{\partial \lambda} = \lambda + \gamma \frac{\partial S_1(\lambda)}{\partial \lambda},$$

by choosing $\gamma = 0$ in the subordination relation (7.9). The map $\Phi_{t, \gamma}$ is a homeomorphism by Corollary 7.5. In addition, the subordination relation (7.9) shows

$$\frac{\partial S_0(\lambda)}{\partial \lambda} = \frac{\partial S_1(\lambda)}{\partial \lambda} = \frac{\partial S_2(z)}{\partial z}$$

and

$$\frac{\partial S_0(\lambda)}{\partial \lambda} = \frac{\partial S_1(\lambda)}{\partial \lambda} = \frac{\partial S_2(z)}{\partial z}.$$

Hence, conditions in Lemma 4.12 are satisfied. The pushforward result for regularized Brown measures then follows.

The proof of [33] Lemma 5.3 shows

$$|p_{\lambda}^{(0)}(w(\varepsilon_2; \lambda, t)) - p_{\lambda}^{(0)}(w(\varepsilon_1; \lambda, t))| \leq \frac{2}{t}(w(\varepsilon_2; \lambda, t) - w(\varepsilon_1; \lambda, t)).$$

Hence, local uniform convergence of $w(\varepsilon; \lambda, t)$ in Lemma 7.6 implies that $\Phi_{t, \gamma}$ converges uniformly to $\Phi_{t, \gamma}$ in any compact subset of $\mathbb{C}$ as $\varepsilon$ tends to zero. We then conclude that push-forward connection holds after passing to the limit.

Finally, if the map $\Phi_{t, \gamma}$ is non-singular, then the last claim follows from a similar argument for the proof of Proposition 6.2.

**Remark 7.13.** By Theorem 7.10 and Theorem 7.12 there is no problem to extend examples from [33] to unbounded case when $x_0$ is selfadjoint or $R$-diagonal. Hence, all results from
Section 6 and 7] still hold for unbounded $x_0$. In particular, the pushforward result obtained in \[21\] Theorem 6.4, Corollary 6.9] are very special examples of Theorem 7.12.

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