LOCAL REGULARITY OF THE GREEN OPERATOR IN A CR MANIFOLD OF GENERAL "TYPE"

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Abstract. It is here proved that if a pseudoconvex CR manifold $M$ of hypersurface type has a certain “type”, that we quantify by a vanishing rate $F$ at a submanifold of CR dimension 0, then $\Box_b$ “gains $f^2$ derivatives” where $f$ is defined by inversion of $F$. Next a general tangential estimate, “twisted” by a pseudodifferential operator $\Psi$ is established. The combination of the two yields a general “$f$-estimate” twisted by $\Psi$, that is, (1.4) below. We apply the twisted estimate for $\Psi$ which is the composition of a cut-off $\eta$ with a differentiation of order $s$ such as $R_s$ of Section 3. Under the assumption that $[\partial_b, \eta]$ and $[\partial_b, [\partial_b, \eta]]$ are superlogarithmic multipliers in a sense inspired to Kohn, we get the local regularity of the Green operator $G = \Box_b^{-1}$. In particular, if $M$ has “infraexponential type” along $S \setminus \Gamma$ where $S$ is a manifold of CR dimension 0 and $\Gamma$ a curve transversal to $T^CM$, then we have local regularity of $G$. This gives an immediate proof of [1] in tangential version and of [14]. The conclusion extends to “block decomposed” domains for whose blocks the above hypotheses hold separately.

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1. Introduction

It has been proved in [9] that if the boundary of a pseudoconvex domain of $\mathbb{C}^n$ has geometric “type $F$”, then there is an “$f$-estimate” for the $\bar{\partial}$-Neumann problem for $f = F^*(t^{-1})^{-1}$ where $F^*$ is the inverse function to $F$. The converse is also true (cf. [10]), apart from a loss of accuracy in the estimate which is in most cases negligible. The successful approach in establishing the equivalence between the $F$-type and the $f$-estimate consists in triangulating through a potential theoretical condition, namely, the “$f$-property”, that is, the existence of a bounded weight whose Levi-form grows with the rate of $f^2$ at the
boundary. This generalizes former work by Kohn [12], Catlin [3], [6], McNeal [17] et alii. What we prove here is that the \( F \) type implies the \( f \)-estimate for the tangential system \( \partial_b \); this is a generalization of Kohn [15]. In greater detail, let \( M \subset \mathbb{C}^n \) be a pseudoconvex manifold of hypersurface type and \( v \) or \( u \) a form in \( M \) of a certain degree \( h \). We use the microlocal decomposition into wavelets \( \mu = \sum_{k=1}^{\infty} \Gamma_k u \) (cf. [15] proof of Theorem 6.1).

We consider a submanifold \( S \subset M \) of CR dimension \( 0 \), and a real function \( F \) satisfying \( \frac{\partial}{\partial s} \rightarrow 0 \) as the distance \( d_S \) to \( S \) decreases to \( 0 \). We also use the notation \( Id \) for the identity of the complex tangent bundle \( T^cM = TM \cap TM \). We assume that \( M \) has type \( F \) along \( S \) in a neighborhood \( U \) of point \( z_0 \in S \) in the sense that the Levi form \( (c_{ij}) \) of \( M \) satisfies \( (c_{ij}) \geq \frac{F(d_S)}{d_S} Id \). Then, there is a bounded family of weights \( \{ \phi_k \} \) by the aid of which we get the estimate of the \( f \)-norm by the Levi form \( (c_{ij}) \) of \( M \) and \( (\phi_k) \) of the \( \phi_k \)’s.

**Theorem 1.1.** Let \( M \) have type \( F \) along \( S \); then

\[
\|f(\Lambda)v\| \lesssim \int_M (c_{ij})(\Lambda^2 v, \Lambda^2 v) \, dV + \sum_{k=1}^{\infty} \int_M (\phi_k)(\Gamma_k v, \Gamma_k v) \, dV
\]

\[
+ \|v\|_0^2, \quad \text{for any } v \text{ of degree } h \in [1, \dim_{CR}(M)],
\]

\[
\|f(\Lambda)v\| \lesssim \int_M \left( \text{Trace}(c_{ij})Id - (c_{ij}) \right)(\Lambda^2 v, \Lambda^2 v) \, dV + \sum_{k=1}^{\infty} \int_M \left( \text{Trace}(\phi_k)(\Gamma_k v, \Gamma_k v) \right) \times
\]

\[
\times (\Gamma_k v, \Gamma_k v) \, dV + \|v\|_0^2, \quad \text{for any } v \text{ of degree } h \in [0, \dim_{CR}(M) - 1].
\]

(1.1)

The proof is the content of Section 2 below. We denote by \( u = u^+ + u^- + u^0 \) the microlocal decomposition of \( u \) (cf. [15] Section 2) and also use the notation \( Q^h \) for the energy \( Q^h = \|\overline{\partial}_b v\|^2 + \|\overline{\partial}_b^* v\|^2 \), and \( \mathcal{H} \) for the space of harmonic forms \( \mathcal{H} = \ker \overline{\partial}_b \cap \ker \overline{\partial}_b^* \). We apply the first of (1.1) for \( v = u^+ \), resp. the second for \( v = u^- \), and plug into a basic estimate. We also use the elliptic estimate for \( u^0 \) and conclude

**Theorem 1.2.** We have

\[
\|f(\Lambda)u\|^2 \leq Q^h(\Lambda, \overline{\partial}_b^*) + \|u\|^2_0, \quad \text{for any } u \text{ of degree } h \in [0, \dim_{CR}(M)].
\]

(1.2)

As it has been already said, (1.2) follows from (1.1) for the common range of degrees \( h \in [1, \dim_{CR}(M) - 1] \). As for the critical top and bottom degrees, we get the estimate for \( u \in \mathcal{H}^\perp \) from the estimate in nearby degree from closed range of \( \overline{\partial}_b \) and \( \overline{\partial}_b^* \) (cf. [15] proof of Theorem 7.3 p. 237).

Next, we prove a general basic weighted estimate twisted by a pseudodifferential operator \( \Psi \), that is, (3.2) and (3.3) of Theorem 3.1 below. We have to mention that our formula is classical (cf. McNeal [18], [19]) when \( \Psi \) is a function. A recent application, in which \( \Psi \) is a family of cut-off, has been given in [2] in the problem of the local regularity of the Green operator \( G = \Box_b^{-1} \). We choose a smooth orthonormal basis of \( (1,0) \) forms
\(\omega_1, \ldots, \omega_{n-1}\), supplement by a purely imaginary form \(\gamma\) and denote the dual basis of vector fields by \(\partial_{\omega_1}, \ldots, \partial_{\omega_{n-1}}, T\). We define various constants \(c_{ij}^{b}\)'s as the coefficients of the commutator \([\partial_{\omega_i}, \partial_{\omega_j}] = c_{ij}^{b} T + \sum_{j=1}^{n-1} c_{ij}^{b} \partial_{\omega_h} - \sum_{j=1}^{n-1} c_{ji}^{b} \partial_{\omega_h};\) sometimes, we also write \(c_{ij}\) instead of \(c_{ij}^{b}\). We use the notation \(\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}\) for an operator of order smaller than \(\Psi\) whose support is contained in a conical neighborhood of that of \(\Psi\). Combination of the estimate with the basic twisted estimate yields

\[
\|f(\Lambda)\Psi v\|_2^2 \leq \int (c_{ij})(\Psi T^\frac{1}{2} v, \Psi T^\frac{1}{2} v) dV + \sum_k \int (\phi^k)(\Gamma_k \Psi v, \Gamma_k \Psi v) dV + t\|\Psi v\|_0^2 \leq Q^b_{\Psi}(v, \bar{v}) + \|[\partial_b, \Psi] \perp v\|_0^2 + \sum_h \int (c_{ij}^h)([\partial_{\omega_h}, \Psi](v), \bar{\Psi} v) dV \\
+ \left| \int [\partial_b, [\bar{\partial}_b, \Psi^2]](v, \bar{v}) dV \right| + \text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}(v, \bar{v}) + \|\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} v\|_0^2 + \|\Psi v\|_0^2.
\]

(1.3)

Here \(Q^b_{\Psi} = \|\Psi \bar{\partial}_b v\|_2^2 + \|\Psi \partial_b v\|_2^2\).

(ii) The similar equation holds for \(u^-\) in degree \([0, \dim_{CR}(M) - 1]\) if we replace \((c_{ij}), (\phi^k)\) and \([\partial_b, [\bar{\partial}_b, \Psi^2]]\) by \(-(c_{ij}), -(\phi^k)\) and \(-[\partial_b, [\bar{\partial}_b, \Psi^2]] + \text{Trace}([\partial_b, [\bar{\partial}_b, \Psi^2]])\) \(\text{Id}\) respectively.

(iii) Taking summation of the estimate for \(v = u^+, v = u^-\) together with the elliptic estimate for \(v = u^0\), and using the closed range of \(\bar{\partial}_b\) and \(\partial_b\) for the critical degrees we get for the full \(u \in \mathcal{H}^+\) in degree \(h \in [0, \dim_{CR}(M)]\)

\[
\|f(\Lambda)\Psi u\|_0^2 \lesssim Q^b_{\Psi}(u, \bar{u}) + \|[\partial_b, \Psi] \perp u\|_0^2 + \left| \int_M [\partial_b, [\bar{\partial}_b, \Psi^2]](u^+, \bar{u}^-) dV \right| \\
+ \sum_h \int (c_{ij}^h)([\partial_{\omega_h}, \Psi](u), \bar{\Psi} u) dV + \left| \int_M (\partial_b, [\bar{\partial}_b, \Psi^2])(u^-, \bar{u}^+) dV \right| + \text{Trace}([\partial_b, [\bar{\partial}_b, \Psi^2]]) \text{Id}(u^-, \bar{u}^+) dV + Q^b_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}(u, \bar{u}) + \|\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} u\|_0^2 + \|\Psi u\|_0^2.
\]

(1.4)

The proof is just the superposition of the items (i) and (ii) of Theorem 3.1 below. We have indeed, in Theorem 3.1 (i) and (ii) a more general, weighted version of this estimate. We give an application of the general twisted estimate in which \(\Psi\) includes a cut-off \(\eta\) and a differentiation of arbitrarily high order \(s\) (such as \(R^s\) of Section 4 below). To introduce it, we need the notion of superlogarithmic multipliers which are an obvious variant of the subelliptic multipliers (cf. [15] Definition 8.1). The crucial point in our discussion is
that we consider vector multipliers \( g = (g_j) \) and also require a more intense property in which energy is replaced by Levi form, that is, for any \( \epsilon \), suitable \( c_\epsilon \), and for an uniformly bounded family of weights \( \{ \phi^k \} \)

\[
|| \log(\Lambda) g \| _2^2 \leq \epsilon \left( \int_M (c_{ij}(\Lambda^j v, \Lambda^j v) dV + \sum_{k=1}^{+\infty} \int_M (\phi^k_{ij}(\Gamma_k v, \Gamma_k v) dV) \right) + c_\epsilon \| v \|_0. \tag{1.5}
\]

We also require that the same estimate holds for \( (c_{ij}) \) and \( (\phi^k_{ij}) \) replaced by \( -(c_{ij}) + \text{Trace}(c_{ij}) \text{Id} \) and \( -(\phi^k_{ij}) + \text{Trace}(\phi^k_{ij}) \text{Id} \) respectively. With this preliminary we have

**Theorem 1.4.** Assume that there is a system of cut-off \( \{ \eta \} \) at \( z_0 \) such that \( [\partial \nu, \eta] \) and \( [\partial_b, [\partial_b, \eta]] \) are vector and matrix superlogarithmic multipliers respectively, and \( (c^k_{ij}) \) are subelliptic multipliers. Then \( G \) is regular at \( z_0 \).

The proof is found in Section 4. We combine Theorem 1.4 with 1.1. This gives back the conclusion of [1] (in a tangential version) which was in turn a generalization of [14]. It also provides a larger class of hypersurfaces for which \( G \) is regular. Let \( M \) be the “block decomposed” hypersurface of \( \mathbb{C}^n \) defined by \( x_n = \sum_{j=1}^{m} h^j_I(z_I, y_n) \) where \( z = (z_1, \ldots, z_m, z_n) \) is a decomposition of coordinates.

**Theorem 1.5.** Assume that

\[
\begin{align*}
(a) \ h^j_I \text{ has infraexponential type along a totally real } S^j_I \setminus \Gamma^j_I \text{ where } S^j_I \text{ is totally real in } \mathbb{C}^j \times \mathbb{C}_{z_n} \\
(b) \ h^j_{z_j} \text{ are superlogarithmic multipliers,} \\
(c) \ c^k_{ij} \text{ are subelliptic multipliers.}
\end{align*}
\]

Then, we have local regularity of \( G \) at \( z_0 = 0 \).

In case of a single block \( x_n = h^I_i \) we regain [2] and [14]. The proof is found in Section 4 below.

**Example** Let

\[
(i) \quad x_n = \sum_{j=1}^{n} e^{-\frac{1}{|x_j|^p}} e^{-\frac{1}{|x_j|^q}} \quad \text{for any } a \geq 0 \text{ and for } b < 1.
\]

Then, (1.6) (a) is obtained starting from \( h^j_{z_j, z_j} \geq e^{-\frac{1}{|x_j|^p}} \), that is, the condition of type \( F_j := e^{-\frac{1}{|x_j|^p}} \) along \( S_j = \mathbb{R}_{y_j} \times \{0\} \). This yields the estimate of the \( f \) norm for \( f(t) = \log^\frac{1}{2}(t); \) since \( \frac{1}{p} > 1 \), this is superlogarithmic. (1.6) (b) follows from \( |h^j_{z_j}|^2 \leq h^j_{z_j, z_j} \) which says that the \( h^j_{z_j} \)’s are not only superlogarithmic, but indeed \( \frac{1}{2} \)-subelliptic, multipliers.
Finally, (c) follows from $c^h_{jj} \sim c_{jj}$ (a consequence of the “rigidity” of $M$) which shows that these constant are $\frac{1}{2}$ subelliptic multipliers.

(1.6) is the ultimate step of a long sequence of criteria of regularity of $G$, not reducible in one another, described by the hypersurface models below, in which $a > 0$ and $0 < b < 1$,

(ii) $x_n = \sum_{j=1}^{n-1} e^{-\frac{|x_j|}{2}}$ Kohn [15],

(iii) $x_n = e^{-\sum_{j=1}^{n-1} \frac{|x_j|}{2}}$ Kohn [14],

(iv) $x_n = e^{-\sum_{j=1}^{n-1} \frac{|x_j|}{2}} \left( \sum_{j=1}^{n-1} e^{-|x_j|^a} \right)$ Baracco-Khanh-Zampieri [1],

(v) $x_n = \sum_{j=1}^{n-1} e^{-\frac{|x_j|}{2}}$ Baracco-Pinton-Zampieri [3],

(vi) $x_n = \sum_{j=1}^{n-1} e^{-\frac{|x_j|}{2}} x_j^a$ Baracco-Pinton-Zampieri [2].

Thus, the degeneracy in our model (i) comes as the combination of those of (ii) with (v) (or (vi)).

2. Estimate of the $f$-norm by the Levi form

Let $M$ be a $C^\infty$ CR-manifold of $\mathbb{C}^n$ of hypersurface-type, $z_o$ a point of $M$, $U$ an open neighborhood of $z_o$. Our setting being local, we can find a local CR-diffeomorphism which reduces $M$ to a hypersurface of $TM + iTM$; therefore, it is not restrictive to assume that $M$ is a hypersurface of $\mathbb{C}^n$ from the beginning. We choose a smooth orthonormal basis of $(1, 0)$ forms $\omega_1, ..., \omega_{n-1}$, supplement by a purely imaginary form $\gamma$ and denote the dual basis of vector fields by $\partial_{\omega_1}, ..., \partial_{\omega_{n-1}}, T$. We also use the notation $\partial_b$ for the tangential CR-system. For a smooth real function $\phi$, we denote by $(\phi_{ij})$ the matrix of the Levi form $\partial_b \partial_b \phi$. Note that $\phi_{ij}$ differs from $\partial_{\omega_i} \partial_{\omega_j} (\phi)$ because of the presence of the derivatives of the coefficients of the forms $\partial_{\omega_j}$. Let $(c_{ij})_{i,j=1,...,n-1}$ be the Levi-form $d\gamma|_{T^cM}$ where $T^cM = TM \cap iTM$.

Let $S \subset M$ be a submanifold of CR-dimension 0, $d_S$ the Euclidean distance to $S$, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ a smooth monotonic increasing function such that $f < t^\frac{1}{2}$. We use the notation $a_k$ for the constant $a_k := f^{-1}(2^k)$ and $S_{a_k}$ for the strip $S_{a_k} := \{ z \in M : d_S(z) \leq a_k \}$.

Lemma 2.1. There is an uniformly bounded family of smooth weights $\{ \phi^k \}$ with supp $\phi^k \subset S_{2a_k}$ whose Levi-form satisfies

$$
\partial_b \partial_b \phi^k \sim \begin{cases} 
  f^2(2^k) & \text{on } S_{a_k} \\
  -f^2(2^k) & \text{on } S_{2a_k} \setminus S_{a_k}, \\
  0 & \text{on } M \setminus S_{2a_k}.
\end{cases}
$$

(2.1)
This also readily implies the same inequalities as (2.1) with \(\partial_b \bar{\partial}_b \phi^k\) replaced by \(\text{Trace}(\partial_b \bar{\partial}_b \phi^k) \text{Id} – \partial_b \bar{\partial}_b \phi^k\).

Note that there is no assumption about the behavior of \(M\) at \(S\) in this Lemma.

**Proof.** Set
\[
\phi^k = c \chi \left( \frac{d_{S}(z)}{a_k} \right) \log \left( \frac{d_{S}^2(z)}{a_k^2} + 1 \right),
\]
(2.2)
where \(c\) is a constant that will be specified later and \(\chi \in C^\infty(0, 2)\) is a decreasing cut-off function which satisfies
\[
\begin{cases}
\chi \equiv 1 \quad &\text{on } [0, 1], \\
0 \leq \chi \leq 1 \quad &\text{on } [1, \frac{3}{2}], \\
\chi \equiv 0 \quad &\text{on } [\frac{3}{2}, 2].
\end{cases}
\]

Remark that
\[
\partial_b \bar{\partial}_b d_{S}^2 = 2 \partial_b d_{S} \otimes \bar{\partial}_b d_{S} + 2 d_{S} \partial_b \bar{\partial}_b d_{S} \\
\geq 2 \partial_b d_{S} \otimes \bar{\partial}_b d_{S} \\
\sim \text{Id},
\]
where the last inequality follows from \(\dim_{CR}(M) = 0\) (with the agreement that \(\text{Id}\) denotes the identity of \(T^{\mathbb{C}}M\)).

Now, when \(\partial_b \bar{\partial}_b\) hits log, we have
\[
\partial_b \bar{\partial}_b \log \left( \frac{d_{S}^2(z)}{a_k^2} + 1 \right) \sim \frac{\partial_b d_{S} \otimes \bar{\partial}_b d_{S} + d_{S} \partial_b \bar{\partial}_b d_{S}}{a_k^2} \\
\geq \frac{\text{Id}}{a_k^2} = f^2(2^k) \text{Id}.
\]
(2.3)

On the other hand, on \(S_{a_k}\), the function \(\chi\) is constant and therefore \(\partial_b \bar{\partial}_b \phi^k = \partial_b \bar{\partial}_b \log\). Thus (2.3) yields the first of (2.1). When, instead, \(\partial_b \bar{\partial}_b\) hits \(\chi\), we have
\[
\left| \partial_b \bar{\partial}_b \chi \left( \frac{d_{S}(z)}{a_k} \right) \right| \leq |\chi| \frac{\partial_b d_{S} \otimes \bar{\partial}_b d_{S}}{a_k^2} + |\chi| \frac{\partial_b \bar{\partial}_b d_{S}}{a_k} \\
\sim \frac{\text{Id}}{a_k^2},
\]
(2.4)
since \(\dim_{CR}(S) = 0\).
On the other hand, log stays bounded on $S_{2a_k}$ and therefore $\partial_b \bar{\partial}_b (\chi) \log \gtrsim -a_k^{-2} = -f^2(2^k)$. Finally, when $\partial_b$ and $\bar{\partial}_b$ hit $\chi$ and log separately, we get

$$
|2\Re \partial_b \chi \left( \frac{d_S}{a_k} \right) \bar{\partial}_b \log \left( \frac{d_S^2}{a_k^2} + 1 \right)| \lesssim |2\Re \chi \partial_b d_S \left( \frac{a_k^2 \partial_b \bar{\partial}_b d_S}{2d_S^2 a_k^2} \right)|
\lesssim \frac{\partial_b d_S \otimes \bar{\partial}_b d_S}{a_k^2} = f^2(2^k) \Id.
$$

(2.5)

Thus, again, $2\Re \partial_b \chi \bar{\partial}_b \log \gtrsim -f^2(2^k) \Id$. 

As we have seen in the proof of Lemma 2.1, when $\chi$ and $\bar{\chi} \neq 0$, the Levi form of $\phi^k$ can get negative. However, this annoyance can be well behaved by the aid of the Levi coordinates $x$. Let $F$ be a smooth real function such that $F(d) \downarrow 0$ as $d \downarrow 0$, denote by $F^*$ the inverse to $F$ and define $f(t) := (F^*(\delta))^{-1}$, for $\delta = t^{-1}$. Let $f(\Lambda)$ be the tangential pseudodifferential operator with symbol $f$. This is defined by introducing a local straightening $M \simeq \mathbb{R}^{n-1} \times \{0\}$ for a defining function $r = 0$ of $M$, taking local coordinates $x \in M$, dual coordinates $\xi$ of $x$ and setting

$$
f(\Lambda)(u) = \int \left( e^{i\xi} f(\sqrt{1 + \xi^2}) \int e^{-i\eta \eta} u(y) d\eta \right) d\xi.
$$

In particular $\Lambda$ is the standard elliptic pseudodifferential operator with symbol $\sqrt{1 + \xi^2}$. 

**Definition 2.2.** We say that $M$ has type $F$ along $S$ in a neighborhood $U$ of $z_0$, if

$$
(c_{ij}) \gtrsim \frac{F(d_S)}{d_S^2} \Id \text{ on } U.
$$

(2.6)

Note that (2.6) implies

$$
\left( \text{Trace}(c_{ij}) \Id - (c_{ij}) \right) \gtrsim \frac{F(d_S)}{d_S^2} \Id \text{ on } U.
$$

(2.7)

**Proposition 2.3.** Let $M$ have type $F$ along $S$ of CR dimension 0. Then

$$
\begin{aligned}
\|f(\Lambda) \Gamma_k v\|_0^2 &\lesssim \int_M (c_{ij})(\Gamma_k \Lambda^\perp v, \Gamma_k \Lambda^\perp v) dV + \int_M (\phi_{ij}^k)(\Gamma_k v, \Gamma_k v) dV + \|\Gamma_k v\|_0^2, \ h \in [1, n - 1], \\
\|f(\Lambda) \Gamma_k v\|_0^2 &\lesssim \int_M \left( \text{Trace}(c_{ij}) \Id - (c_{ij}) \right)(\Gamma_k \Lambda^\perp v, \Gamma_k \Lambda^\perp v) dV \\
&\quad + \int_M \left( \text{Trace}(\phi_{ij}^k) \Id - (\phi_{ij}^k) \right)(\Gamma_k v, \Gamma_k v) dV + \|\Gamma_k v\|_0^2, \ h \in [0, n - 2].
\end{aligned}
$$

(2.8)

**Proof.** We set $a_k = f^{-1}(2^k) = F^*(2^{-k})$, $S_{a_k} = \{z : d_S(z) < a_k\}$ and denote by $\lambda(z)$ the minimum of the $n - 1$ eigenvalues of $(c_{ij})$ at $z$. We start from the first of (2.8). We...
We also use the notation $\bar{M}$ is in fact a hypersurface. For a neighborhood $U$ of a point $z_0 \in M$, we identify $U \cap M$ to $\mathbb{R}^{2n-1}$ with coordinates $x$ and dual coordinates $\xi$, and consider a pseudodifferential operator $\Psi$ with symbol $\mathcal{S}(\Psi)(x, \xi)$. For notational convenience we assume that the symbol is real. We also use the notation $L^2_\phi$ for the $L^2$ space weighted by $e^{-\phi}$, $Q^b = \|\bar{\partial}_b u\|_2 +$
∥v∥² for the energy, and \( Q^b_{\phi} = \|\Psi \bar{\partial}_b v\|_{\phi}^2 + \|\Psi \bar{\partial}_b v\|_{\phi}^2 \) for the energy weighted by \( \phi \) and twisted by \( \Psi \). We consider the pseudodifferential decomposition of the identity by Kohn \( \text{Id} = \Phi^\dagger + \Phi - \Phi^0 \) modulo \( \text{Op}^{-\infty} \). We consider a basis of \((1,0)\) forms \( \omega_1, \ldots, \omega_{n-1} \) the conjugate basis \( \bar{\omega}_1, \ldots, \bar{\omega}_{n-1} \) and complete by a purely imaginary form \( \gamma \). We denote by \( \partial_{\omega_1}, \ldots, \partial_{\omega_{n-1}}, \bar{\partial}_{\omega_1}, \ldots, \bar{\partial}_{\omega_{n-1}}, T \) the dual basis of vector fields. \( M \) being a hypersurface defined, say, by \( r = 0 \), we can supplement the \( \omega_j \)'s to a full basis of \((1,0)\) forms in \( \mathbb{C}^n \) by adding \( \omega_n = \partial r \). Then \( \gamma = \omega_n - \bar{\omega}_n \) and \( T = \partial_{\omega_n} - \bar{\partial}_{\omega_n} \). We describe the commutators by

\[
[\partial_{\omega_i}, \partial_{\omega_j}] = \sum_{j=1}^{n} c_{ij}^h \partial_{\omega_h} - \sum_{j=1}^{n} \bar{c}_{ij}^h \bar{\partial}_{\omega_h} = c_{ij}^h T + \sum_{j=1}^{n-1} c_{ij}^h \partial_{\omega_h} - \sum_{j=1}^{n-1} \bar{c}_{ij}^h \bar{\partial}_{\omega_h};
\]

We also write \( c_{ij} \) instead of \( c_{ij}^h \).

For a cut-off \( \eta \in C_c^\infty(U \cap M) \) we write \( u^+ := \eta \Phi^+ u \), \( u^- := \eta \Phi^- u \), \( u^0 = \eta \Phi^0 u \), \( T^\dagger = \eta T \Phi^\dagger \). We note that \( S(T) > 0 \) on \( \text{supp} S(\Phi^+) \) (resp. \( S(T^-) > 0 \) on \( \text{supp} S(\Phi^-) \)) and therefore \( T^\dagger \) (resp. \( (T^-)^\dagger \)) makes sense when acting on \( u^+ \) (resp. \( u^- \)). We make the relevant remark that

\[
\begin{align*}
&\{S(T) \sim \Lambda \text{ on } \text{supp } S(\Phi^+), \quad S(T^-) \sim \Lambda \text{ on } \text{supp } S(\Phi^-), \quad \{S(\partial_{\omega_j})_{j=1,\ldots,n-1} \sim \Lambda \text{ and } S(\bar{\partial}_{\omega_j})_{j=1,\ldots,n-1} \sim \Lambda \text{ on } \text{supp } S(\Phi^0). \n\end{align*}
\]

We denote by \( \text{Op}^{\text{ord}(\Psi) - \frac{1}{2}} \), resp. \( \text{Op}^0 \), an operator of order \( 2\text{ord}(\Psi) - \frac{1}{2} \), resp. 0, whose support is contained in \( \text{supp } \Psi \); we also assume that \( \text{Op}^0 \) only depends on the \( C^2 \)-norm of \( M \) and, in particular, is independent of \( \phi \) and \( \Psi \).

**Theorem 3.1.** (i) We have for every smooth form \( v = u^+ \) of degree \( h \in [1, n-1] \)

\[
\int_M e^{-\phi} (c_{ij})(T^\dagger \Psi v, \overline{T^\dagger \Psi v})dV + \int_M e^{-\phi} ((\phi_{ij}) - \frac{1}{2}(c_{ij})T(\phi))(\Psi v, \overline{\Psi v})dV + ||\Psi \nabla v||_{\phi}^2
\]

\[
\leq Q^b_{\psi}(v, \overline{v}) + ||[\partial_{\bar{b}}, \Psi] L v||_{\phi}^2 + ||\partial_{\bar{b}}, \phi \nabla v||_{\phi}^2 + \sum_{h=1}^{n-1} \int (c_{ij}^h)([\partial_{\omega_h}, \Psi](v, \overline{\Psi v}))dV
\]

\[
+ \int_M e^{-\phi} ([\partial_{\bar{b}}, \phi]^2)(v, \overline{v})dV + Q^{b_{\phi}}_{\text{Op}^{\text{ord}(\Psi) - \frac{1}{2}}(v, \overline{v}) + ||\text{Op}^{\text{ord}(\Psi) - \frac{1}{2}}(v, \overline{v})||_{\phi}^2 + ||\Psi v||_{\phi}^2.
\]

Here we are using the notation \( Q^b_{\psi} = ||\Psi \bar{\partial}_b v||_{\phi}^2 + ||\Psi \bar{\partial}_b v||_{\phi}^2. \)
(ii) We also have, for $v = u^-$ smooth of degree $h \in [0, n - 2]$

$$\int_M e^{-\phi} \left( - (c_{ij} + \sum_j c_{jj} Id) ((T^-)^2 \Psi v, (T^-)^2 \Psi v) + ||\Psi \nabla v||^2_\phi \right)$$

$$+ \int_M e^{-\phi} \left( \left( - (\phi_{ij} + \sum_j \phi_{jj} Id) + \frac{1}{2} \left( (c_{ij}) T(\phi) - (\sum_j c_{jj}) T(\phi) \right) \right) (\Psi v, \overline{\Psi v}) dV$$

$$\leq Q^b_\Psi(v, \overline{\Psi v}) + ||[\partial_b, \Psi] \Psi v||^2_\phi + ||[\partial_b, \phi] \Psi v||^2_\phi + \sum_{h=1}^{n-1} \int \left( - (c_{ij} + \sum_j c_{jj} Id) ([\partial_{\omega_h}, \Psi] v, x \right. \times \left. \overline{\Psi v}) dV \right) + \left| \int_M e^{-\phi} \left( - [\partial_{\omega_i}, \Psi v], \Psi v + Trace([\partial_{\omega_i}, [\partial_{\omega_j}, \Psi^2]]) Id \right) (v, \overline{\Psi v}) dV \right|$$

$$+ Q_{Op^{ord(\Psi)}}^{b, \overline{b}}(v, \overline{v}) + ||Op^{ord(\Psi)}^{b, \overline{b}} v||^2_\phi + ||\Psi v||^2_\phi.$$  (3.3)

Clearly $u^0$ is subject to elliptic estimates. These, combined with (3.2), (3.3) yield an estimate for the full $u$ in degrees $[1, n - 2]$ and then also for $u \in H^+$ in degree $k \in [0, n - 1]$ by closed range.

**Remark 3.2.** The formula also holds for $\Psi$ complex: in this case one replaces $\Psi^2$ by $|\Psi|^2$ and add the additional error term $[\partial_b, \Psi] \Psi$ to the already existing $[\partial_b, \Psi] \Psi$.

**Proof.** We start from

$$\partial_{\overline{\partial}} \partial_{\partial} \phi = \partial_b \left( \sum_j \bar{\partial}_{\omega_j}(\phi) \bar{\omega}_j \right)$$

$$+ \sum_{i,j} \left( \partial_{\omega_i} \bar{\partial}_{\omega_j}(\phi) + \sum_h c_{ij}^h \bar{\partial}_{\omega_h}(\phi) \right) \omega_i \wedge \bar{\omega}_j.$$  (3.4)

Similarly,

$$\bar{\partial}_{\bar{\partial}} \partial_{\partial} \phi = \bar{\partial}_b \left( \sum_j \partial_{\omega_j}(\phi) \omega_i \right)$$

$$= \sum_{i,j} \left( - \bar{\partial}_{\omega_i} \partial_{\omega_j}(\phi) - \sum_h c_{ij}^h \bar{\partial}_{\omega_h}(\phi) \right) \omega_i \wedge \bar{\omega}_j.$$  (3.5)
Differently from the ambient $\bar{\partial}$-system on $\mathbb{C}^n$, we do not have $\bar{\partial}_b \bar{\partial}_b = \bar{\partial}_b \partial_b$ and in fact, combining (3.4) with (3.5), we can describe $(\phi^b_{ij})$, the matrix of $\frac{1}{2}(\bar{\partial}_b \partial_b - \bar{\partial}_b \partial_b)(\phi)$, by

$$
\phi^b_{ij} = \frac{1}{2} (\bar{\partial}_b \partial_b - \bar{\partial}_b \partial_b)(\phi), \partial_{\omega_i} \wedge \bar{\partial}_{\omega_j}
$$

by (3.4), (3.5)

$$
= \bar{\partial}_{\omega_i} \partial_{\omega_i}(\phi) + \frac{1}{2} \left( \left[ \partial_{\omega_i}, \bar{\partial}_{\omega_j} \right](\phi) + \sum_h c^h_{ij} \partial_{\omega_h}(\phi) + \sum_h c^h_{ji} \bar{\partial}_{\omega_h}(\phi) \right)
$$

(3.6)

We consider now

$$
e^{\phi} \Psi^{-2}[\bar{\partial}_{\omega_i}, e^{-\phi} \Psi^2] = -\phi_{\omega_i} + 2 \left[ \partial_{\omega_i}, \Psi \right] \Psi + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2},
$$

(3.7)

whose sense is fully clear when both sides are multiplied by $\Psi^2$. In other terms, we have

$$
\bar{\partial}^*_{e^{-\phi} \Psi^2} = \bar{\partial}^* + \partial \Phi - 2 \left[ \partial, \Psi \right] \Psi - \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0.
$$

(3.8)

This leads us to define the transposed operator $\delta_{\omega_i}$ to $\bar{\partial}_{\omega_i}$ by

$$
\delta_{\omega_i} := \bar{\partial}_{\omega_i} - \phi_{\omega_i} + 2 \left[ \partial_{\omega_i}, \Psi \right] \Psi + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0.
$$

(3.9)

With these preliminaries we have

$$
\left[ \delta_{\omega_i}, \bar{\partial}_{\omega_j} \right] = c_{ij} T + \sum_{h=1}^{n-1} c^h_{ij} \partial_{\omega_h} - \sum_{h=1}^{n-1} c^h_{ji} \bar{\partial}_{\omega_h} + \left( \phi^b_{ij} - \frac{1}{2} c^b_{ij} T(\phi) \right)
$$

$$
- 2 \sum_h c^h_{ij} \left[ \partial_{\omega_h}, \Psi \right] \Psi + \left[ \partial_{\omega_i}, [\bar{\partial}_{\omega_j}, \Psi] \right] \Psi + \left[ \partial_{\omega_i}, \Psi \right] \left[ \bar{\partial}_{\omega_j}, \Psi \right] \Psi^2 + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0.
$$

(3.10)

We remember now that there are two equally reasonable definition of the pseudodifferential action

$$
\Psi(w) = \begin{cases}
(i) & \int e^{i\xi \cdot x} S(\Psi)(x, \xi) \tilde{w}(\xi) d\xi \\
(ii) & \int e^{i\xi \cdot x} (S(\Psi)(\cdot, \xi) \ast \tilde{w}) d\xi,
\end{cases}
$$

(3.11)
where \( \tilde{w} \) denotes the Fourier transform. Up to error terms of type \( \text{Op}^{\omega(\Psi)-\frac{1}{2}} \), we have

\[
||\Psi(w)||^2 \sim (\Psi w, \Psi w)
\]

Plancherel and \( \text{(3.11) (ii)} \)

\[
\sim \int \widehat{\Psi}(w)(\xi) \overline{\mathcal{S}(\Psi)(\xi, \eta)} \tilde{w}(\eta) d\eta d\xi
\]

\[
= \int \left( \int \widehat{\Psi}(w)(\xi) \overline{\mathcal{S}(\Psi)(\eta, \xi)} d\xi \right) \tilde{w}(\eta) d\eta
\]

\[
\sim \int (3.11) (i) \int \widehat{\Psi}(w)(\eta) \tilde{w}(\eta) d\eta
\]

Plancherel \( (|\Psi|^2 w, w) \).

For the same reason \( (\Psi^2 w, w) \sim \int |\Psi|^2 |w|^2 dV \) and therefore

\[
||\Psi(w)||^2 \sim \int |\Psi|^2 |w|^2 dV.
\]

Adding the weight \( \phi \) and recalling that in our discussion \( \Psi \) is real,

\[
||\Psi \partial^* \partial_{b}\rangle v||^2_{\phi} = \int e^{-\phi} \Psi^2 |\partial^* \partial_{b}\rangle v|^2 dV + ||\text{Op}^{\omega(\Psi)-\frac{1}{2}} (\partial^* \partial_{b} v)||^2_{\phi}, \tag{3.12}
\]

where \( \partial^* \partial_{b} \) denotes either \( \partial_b \) or \( \partial_b^* \). We are ready for the proof of (3.2); we prove it only for \( v = u^+ \), the proof of (3.3) for \( v = u^- \) being similar. We have

\[
\int \Omega e^{-\phi} (c_{ij})(T v, \overline{v}) + \int [\partial_b, [\partial_b, e^{-\phi} \Psi^2]](v, \overline{v}) dV
\]

\[
- ||[\partial_b, \phi] \rangle v||^2_{\phi} - ||[\partial_b, \Psi] \rangle v||^2_{\phi} + ||\Psi \nabla v||^2_{\phi}
\]

\[
< ||\Psi \partial_b v||^2_{\phi} + ||\Psi (\overline{\partial_b})^* e^{-\phi} \Psi^2 v||^2_{\phi} + sc||\Psi \nabla v||^2_{\phi} + \sum_{h} \int e^{-\phi} (c_{ij}^h)([\partial_{\omega_h}, \Psi] v, \overline{\Psi v}) dV
\]

\[
+ Q^h_{\text{Op}^{\omega(\Psi)-\frac{1}{2}}(v, v)} + ||\text{Op}^{\omega(\Psi)-\frac{1}{2}} v||^2_{\phi} + ||\Psi v||^2_{\phi}, \tag{3.13}
\]
or, according to (3.10) and after absorbing the term which comes with $s$,
\[
\int_M e^{-\phi}c_{ij}(T\psi v, \psi v) dV + \int_M e^{-\phi}\phi_{ij}(\psi v, \psi v) dV - ||[\partial_b, \phi ] \psi v||^2_\phi + \int_M e^{-\phi}[(\partial_i, [\bar{\partial}_j, \psi]) v, \psi] dV - ||[\partial_b, \psi ] \psi v||^2_\phi + ||\psi \bar{\psi} v||^2_\phi \\
\leq ||\psi \bar{\partial}_v||^2_\phi + ||\psi (\bar{\partial}_b^* \psi v)||^2_\phi + \sum_h \int_M e^{-\phi}(c_{ij}^h)((\partial_{\psi^h}, \psi) v, \psi v) dV \\
+ Q_{Op^{ord(\psi)}-\frac{1}{2}}(v, v) + ||Op^{ord(\psi)}-\frac{1}{2} v||^2_\phi + ||\psi v||^2_\phi. 
\]
(3.14)

To carry out our proof we need to replace $(\bar{\partial}_b^*)e^{-\psi_{\phi 2} v}$ by $\bar{\partial}_b^*$. We have from (3.10)
\[
||\psi (\bar{\partial}_b^*)e^{-\psi_{\phi 2} v}||^2_\phi \leq ||\psi \bar{\partial}_v||^2_\phi + ||\psi \partial_\phi \psi^2 v||^2_\phi + ||[\partial_b, \psi ] \psi v||^2_\phi + ||Op^{ord(\psi)}-\frac{1}{2} v||^2_\phi \\
+ 2|Re(\psi \bar{\partial}_b^*, \psi \bar{\partial}_b \psi v)\partial_\phi| + 2|Re(\psi \bar{\partial}_b^*, \psi \bar{\partial}_b \psi v)\partial_\psi| + 2|Re(\psi \bar{\partial}_b^*, \psi \bar{\partial}_b \psi v)\partial_\psi|.
\]
(3.15)

We next estimate by Cauchy-Schwarz inequality
\[
\# \leq ||\psi \bar{\partial}_v||^2_\phi + ||\psi \partial_\phi \psi v||^2_\phi + ||[\partial_b, \psi ] \psi v||^2_\phi.
\]
We move the third, forth and fifth terms from the left to the right of (3.14), and get (3.2) with $(T\psi v, \psi v)$ instead of $(T^{\frac{1}{2}}\psi v, T^{\frac{1}{2}}\psi v)$. But they only differ for
\[
|\int_M e^{-\phi } \left( (c_{ij}, T^{\frac{1}{2}}(T^{\frac{1}{2}}\psi v, \psi v) \right) dV | \leq ||\psi v||^2_0,
\]
which is negligible.

We go back to the family of weights of Theorem 1.1 and Proposition 2.3. We apply (3.2) (resp. (3.3)) for $\phi = \phi^k + t|z'|^2$ (resp. $\phi = \phi^k - t|z'|^2$). First, we note that they are absolutely uniformly bounded with respect to $k$; they can be made bounded in $t$ by taking $U = \{ z : |z'| < \frac{1}{k} \}$. (In particular, by boundedness, they can be removed from the norms.) Possibly by raising to exponential, boundedness implies “selfboundedness of the gradient” when $\phi$ is plurisubharmonic. In our case, in which to be positive is not $(\phi^k_{ij})$ itself but $2^k(c_{ij}) + (\phi^k_{ij})$, we have, for $|z'|$ small
\[
|\partial_b \phi|^2 = |\partial_b (\phi^k + t|z'|^2)|^2 \\
\leq |\partial_b \phi^k|^2 + t^2 |z|^2 \\
\leq 2^k(c_{ij}) + (\phi^k_{ij}) + t.
\]
(3.16)
So \( \| \partial_b \phi \| \Psi u^\pm \|^2 \) can be removed from the right side of both (3.2) and (3.3). Also, the term \( -\frac{1}{2}(c_{ij})T(\phi)(v, \bar{v}) \) is controlled by \((c_{ij})(T^\frac{1}{2}v, T^\frac{1}{2}v)\) by Sobolev interpolation. We then combine Proposition \( 2.3 \) with Theorem \( 3.1 \) formula (3.2) for the weight \( \phi \) to \( \| \| \| v \| \|_\phi \) (resp. formula (3.10) for the weight \( \phi^k + t|z'|^2 \) (resp. formula (3.10) for the weight \( \phi^k - t|z'|^2 \)) and notice that \( T^\frac{1}{2} \sim \Lambda^\frac{1}{2} \) on \text{supp} \( \Psi^+ \) (resp. \( (T^-)^\frac{1}{2} \sim \Lambda^\frac{1}{2} \) on \text{supp} \( \Psi^- \)). Also, on the right of (3.2) and (3.3), one reduces \( \| \text{Op}^\text{ord}(\Psi)^{-\frac{1}{2}}v\|_\phi^2 \) to \( \| v \|_\phi^2 \) by induction and estimates all terms \( Q^{\phi}_{\text{Op}^\text{ord}(\Psi)^{-\frac{1}{2}}} \) and \( Q^{\phi}_{\text{Op}^\text{ord}(\Psi)^{-j}} j \geq 1 \) by a common \( Q^{\phi}_{\Psi} \).

\textbf{Proof of Theorem 1.3.} We have to use (3.2) with the above choice of the weight \( \phi \) and take summation over \( k \); this yields (1.3) for \( v = u^+ \). The twin estimate for \( v = u^- \) follows from (3.3) by similar procedure. Finally, (1.4) comes as the combination of (1.3) for \( v = u^+ \), the twins for \( v = u^- \) and the elliptic estimate for \( v = u^0 \).

\( \Box \)

4. A CRITERION OF HYPOELLIPTICITY OF THE KOHN LAPLACIAN

Let \( M \) be a pseudoconvex, hypersurface type manifold of \( \mathbb{C}^n \), \( \square_b = \partial_b \partial_b^* + \partial_b^* \partial_b \) the Kohn Laplacian of \( M \), and \( G := \square_b^{-1} \) the Green operator.

\textbf{Proof of Theorem 1.4} Our program is to prove that for any cut-off \( \eta_o \in C^\infty_c(U) \) with \( \eta_o \equiv 1 \) in a neighborhood of \( z_o \), for suitable \( \eta \prec \eta_o \), that is \( \eta|_{\text{supp} \eta_o} \equiv 1 \), for any \( s \) and suitable \( U \), we have

\[
\| \eta_o u \|_s \prec \| \eta \partial_b u \|_s + \| \eta \partial_b^* u \|_s + \| u \|_0 \quad \text{for any } u \in \mathcal{H}^s \cap C^\infty(M \cap U)
\]

\text{in any degree } k \in [0, n - 1]. \quad (4.1)

If we are able to prove (4.1), we have immediately the exact local \( H^s \)-regularity of \( \partial_b^*G \) and \( \partial G \) over \( \ker \partial \) and \( \ker \partial^* \) respectively. From this, we get the (non-exact) regularity of the Szegö \( S = \text{Id} - \partial_b^*G \partial_b \) and anti-Szegö \( S^* = \text{Id} - \partial_bG \partial_b^* \) projection respectively. (At this stage we need to apply the method of the elliptic regularization to pass from \( C^\infty \)- to \( H^s \)-forms.) From this the (non-exact) regularity of \( G \) itself follows (cf. e.g. the proof of Theorem 2.1 of [1]). Along with \( \eta_o \prec \eta \), we consider an additional cut-off \( \sigma \) with \( \eta_o \prec \sigma \prec \eta \) and denote by \( R^s \) the pseudodifferential operator with symbol \( (1 + |\xi|^2)^{-\sigma(a)} \). According to Proposition 2.1 of [1], there is no restriction on the degree of \( u \); thus \( u \) can be either a form or a function. By Section 3 above, we can prove (4.1) separately on each term of the microlocal decomposition of \( u = u^+ + u^- + u^0 \); since \( u^0 \) has elliptic estimate and \( u^- \) can be reduced to \( u^+ \) by star-Hodge correspondence, we prove the result only for...
\[ v = u^+. \] We start from

\[
\| \Lambda^s \eta_0 v \| \lesssim \| R^s \eta_0 v \| + \| v \|
\]

\[
= \| R^s \eta_0 \eta^2 v \| + \| v \|
\]

\[
\leq \| R^s \eta^2 v \| + \| [R^s, \eta] \eta^2 v \| + \| v \|
\]

\[
\lesssim \| R^s \eta^2 v \| + \| v \|
\]

\[
\lesssim \| \eta R^s v \| + \| [R^s, \eta] v \| + \| v \|
\]

\[
\lesssim \| \eta R^s v \| + \| v \|, \tag{4.2}
\]

(cf. [15] Section 7). Next, we apply Theorem 3.1 for \( \Psi = \eta R^s \eta \). What we have to describe are the error terms in the right of (3.2), (3.3), that is, \([\partial_b, \eta R^s \eta]\) and \([\partial_b, [\bar{\partial}_b, \eta R^s \eta]]\). Since the argument is similar for the two, we only treat the first. We have by Jacobi identity

\[
[\partial_b, \eta R^s \eta] = [\partial_b, \eta] R^s \eta + \eta [\partial_b, R^s] \eta + \eta R^s [\partial_b, \eta]
\]

\[
= [\partial_b, R^s] + \text{Op}^{-\infty}. \tag{4.3}
\]

In fact, since \( \text{supp} \partial_b \eta \cap \text{supp} \sigma = \emptyset \), then the first and last terms in the right of the first line of (4.3) are operators of order \(-\infty\) and can therefore be disregarded. As for the central term, we have

\[
[\partial_b, R^s] = \partial_b(\sigma) \log(\Lambda) R^s. \tag{4.4}
\]

Now, our hypothesis is that

\[
\| \log(\Lambda) \partial_b \sigma \eta R^s \eta v \|^2 \leq \epsilon \left( \int_M (c_{ij})(\Lambda^\frac{1}{2} R^s \eta_0 v, \Lambda^\frac{1}{2} R^s \eta_0 v) dV \right.
\]

\[
+ \sum_{k=1}^{+\infty} \int_M \left( (\phi_{ij})^k(\eta R^s \eta \Gamma_k v, \eta R^s \eta \Gamma_k v) \right) dV \bigg) + c_\epsilon \| \eta R^s \eta v \|^2. \tag{4.5}
\]
Altogether, we get

\[ t||\Lambda^s \eta v||_0^2 < t||\eta R^s \eta v||_0^2 + ||v||_0^2 \]

by the second of (4.3)

\[ \sim \quad Q^b_{\eta R^s \eta}(v, \bar{v}) + ||[\partial_b, \eta R^s \eta] \chi v||_0^2 + Q^b_{\operatorname{Op}^{\operatorname{ord}(\Psi)} - \frac{1}{2}}(v, \bar{v}) + ||\eta' v||_{s-\epsilon}^2 \]

absorption in the second line

\[ \sim \quad Q^b_{\eta R^s \eta}(v, \bar{v}) + Q^b_{\operatorname{Op}^{\operatorname{ord}(\Psi)} - \frac{1}{2}}(v, \bar{v}) + ||\eta R^s \eta v||_0^2 + ||\eta' v||_{s-\epsilon}^2 \]

absorption by means of \( t \)

(4.6)

Now, the \( s - \epsilon \) norm is reduced to 0 norm by induction over \( j \) with \( j \epsilon > s \), and \( Q_{\eta R^s \eta} \) and the various \( Q_{\operatorname{Op}^{\operatorname{ord}(\Psi)} - \frac{1}{2}} \) are estimated by a common \( Q_{\eta' \Lambda^s} \). In conclusion, we have got (1.1) with the notational difference of \( \eta' \) instead of \( \eta \).

\[ \square \]

**Proof of Theorem 1.5.** We choose our cut-off starting from a cut-off \( \chi \) in one real variable and setting \( \eta = \Pi_j \chi(||z_j||)\chi(||y_n||) \). We have

(a) \( \operatorname{supp} \partial z_j \chi(||z_j||) \) is contained in \( z_j \neq 0 \) in particular, outside the “critical” curve \( \Gamma \) where superlogarithmic estimates hold by Theorem 1.1 and Theorem 1.2 thus \( \partial_b(\Pi_j \chi(||z_j||)) \) are superlogarithmic multipliers.

(b) \( \partial_b \chi(||y_n||) \sim (h_j^{||y||})_j \) and hence it is by hypothesis a superlogarithmic multiplier.

Altogether, \( \partial_b \eta \chi \) are superlogarithmic multipliers. Remember that we are assuming that (\( c_{ij} \)) are subelliptic multipliers. Finally, \( \operatorname{supp} [\partial_b, [\partial_b, \chi(z_j)]] \) is contained in \( z_j \neq 0 \) and
\[
[\partial_b, [\bar{\partial}_b, \chi(y_n)]] \sim h^{P}_{-P, -P}
\]
are subelliptic multipliers; in conclusion, \([\partial_b, [\bar{\partial}_b, \eta]]\) are super-logarithmic multipliers. We can then apply Theorem 1.4 and this completes the proof of Theorem 1.5.

\[\square\]

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