A maintenance-replacement optimization model under a renewable hybrid warranty

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Abstract. In this paper, firstly, a renewable hybrid warranty policy is proposed from the manufacturer’s perspective. By integrating the replacement depending on number of failures and preventive maintenance at the expiration of the warranty, secondly, a maintenance-replacement policy is presented from the perspective of the customer. Thirdly, the consumer’s expected cost rate model is obtained and the related special models are offered. Finally, a numerical example is used to illustrate performance of the proposed maintenance-replacement policy and to analyse sensitively. As showed in the example, performance of the proposed maintenance-replacement policy is more superior.

1. Introduction

The warranty plays an important role in the market process of products, and benefits simultaneously the manufacturer and the customer. From the manufacturer’s perspective, the warranty can make the manufacturer gain more profit and improve reputation. From the customer’s perspective, the warranty is a reliable quality assurance.

Due to the importance of the warranty, it has been a hot research topic. Various warranty policies have been widely researched by academic researchers and industry practitioners. Blischke and Murthy [1] analyzed rather complete warranty policies up to early 1990s. Recent advances in warranty research include mainly the following two streams. The first stream is the warranty research with preventive maintenance (PM), which can be found in [2-8]. The second stream is warranty cost analysis for a multi-component system, which was offered in [9]. It is worth stressing that these literatures were developed from the manufacturer’s perspective.

Undoubtedly, the manufacturer is usually responsible for maintaining reliability of the product during the warranty period, but the consumer (user) still confronts a problem about how to keep reliability of
the product when the warranty expires. After the warranty expires, the consumer maintains usually reliability of the product by himself or outsourcing it to a third-party service (including but not limited to the manufacturer). Recently, due to increasing the post-warranty cost rate, the post-warranty maintenance policies have attracted much attention. This type of problem was researched early in [10], and was subsequently studied in [11-14, 23]. These literatures mainly focused on two types of policies. The first type is the pure replacement policy without considering imperfect maintenance action. For example, Wu and Longhurst [11], Park and Pham [12] and Park et al. [13] determined replacement age after the expired warranty. The second type is the replacement policy considering imperfect maintenance action, which can be found in [14].

With increasingly fierce competition, the manufacturer usually presents an attractive warranty policy in order to obtain more market share. Two types of more popular warranty policies are renewable free replacement warranty (RFRW) policy and renewable pro-rata replacement warranty (RPRRW) policy. Both were investigated separately in the form of single-phase (such as [10, 12-14]) and recently are frequently applied in market of smart devices. The former requires that the manufacturer replace the product at no cost to consumer when a failure occurs during the warranty period. The latter requires that the manufacturer proportionally charge for replacement cost when a failure occurs during the warranty period. Compare with the latter, the former is more popular since the consumer doesn’t shoulder any replacement cost when a replacement happens during the warranty period. From the manufacturer’s perspective, but, the manufacturer is more inclined to the latter since he/she shoulders less replacement cost when a replacement occurs during the warranty period. Under these circumstances, obviously, there is a conflict between the manufacturer and the consumer, which is a preference for both. Therefore, it is necessary to design a warranty policy which integrates characteristics of both in order to reduce this type of conflict. But there is few literatures designing this type of warranty policy.

Besides, the product through warranty essentially is a secondhand product since it has operated for a warranty period. This means that the consumer confronts a problem about how to keep reliability of the secondhand product during the post-warranty period. Reliability of the secondhand is less than one, so some PMs can be performed to improve reliability and to reduce subsequent failures. Shang et al. [14] recently considered this case and proposed a post-warranty maintenance policy. But this literature considered reduction in failure rate function as a decision variable, which is more difficulty controlled in engineering practice. Compares with reduction in failure rate function, changes of age can be estimated in engineering practice (such as [15]). Brown and Proschan [16] and Sheu et al. [17] assumed that failures could be classified type I failure (repairable failure) and type-II failure (catastrophic failure) and proposed a nice replacement policy in which replacement was dependent on number of type I failures. However, there is scarce literature integrated the replacement policy depending on number of type I failures into the post-warranty maintenance policy.

In this paper, we first propose a hybrid warranty (RHW) policy which is a two-phase warranty policy consisted of RFRW policy and RPRRW policy. RFRW policy is carried out during the first phase. RPRRW policy is performed during the second phase. Each replacement during the warranty period (anyone of two phases) makes warranty restart from starting point of the first phase. Based on the proposed RHW policy, second, we present a maintenance-replacement (MR) policy after expiration of the warranty. The presented MR policy is that PM with reduction in age is first performed and then
corrective replacement (CR) is carried out at occurrence time of the first type II failure or at occurrence time of the \( N^{th} \) type I failure.

The remainder of this paper is organized as follows. Section 2 gives problem definition. Section 3 presents the cost rate model of the life cycle and offer some special cases. In Section 4, a numerical example is presented to demonstrate performance of the proposed MR policy. Finally, we conclude this study with a brief discussion in Section 5.

| Notations:                      |                                                                 |
|--------------------------------|-----------------------------------------------------------------|
| RFRW                           | renewable free replacement warranty                              |
| RPRRW                          | renewable pro-rata replacement warranty                          |
| RHW                            | renewable hybrid warranty                                       |
| WSP                            | warranty service period                                          |
| PM                             | preventive maintenance                                          |
| CR                             | corrective replacement                                          |
| MR                             | maintenance-replacement                                         |
| \( x, y \)                     | time                                                            |
| \( F(x) \), \( \tilde{F}(x) \), \( \gamma(x) \) | distribution/reliability/hazard(failure rate) function of \( x \) |
| \( p \)                        | probability of type I failure when a failure happens             |
| \( q \)                        | probability of type II failure when a failure happens, where \( q=1-p \) |
| \( G(y), \tilde{G}(y) \)       | distribution/reliability function of occurrence time of the first type II failure, where \( \tilde{G}(y) = \exp \left[ -\int_{y}^{\infty} q \gamma(u) \, du \right] = [\tilde{F}(y)]^q \) and \( G(y) = 1 - \tilde{G}(y) = 1 - [\tilde{F}(y)]^q \) |
| \( w \)                        | warranty period                                                 |
| \( w_1, w_2 \)                 | the first/second phase warranty period, where \( w_1 + w_2 = w \) |
| \( G_i(y), \tilde{G}_i(y) \)   | distribution/reliability function of occurrence time of the first type II failure during the warranty period \( w_2 \), where \( \tilde{G}_i(y) = 1 - G_i(y) \) and \( G_i(y) = \exp \left[ -\int_{y}^{\infty} q \gamma(w_1 + u) \, du \right] \) |
| \( W_1, W_2, W \)              | WSP of the warranty period \( w_1 / w_2 / w \)                  |
| \( EW_1, EW_2, EW \)           | the expected length of WSP \( W_1 / W_2 / W \)                 |
| \( Y_i, Y_j \)                 | occurrence time of the first type II failure for the \( i^{th} \) / \( j^{th} \) product during the warranty period \( w_1 / w_2 \) |
| \( C_{m_1}, C_{m_2}, C_w \)    | the total cost during the WSP \( W_1 / W_2 / W \)               |
| \( EC_{m_1}, EC_{m_2}, EC_w \) | the expected total cost of \( C_{m_1} / C_{m_2} / C_w \)         |
| \( C_p(m) \)                   | maintenance cost associated with maintenance level \( m \)      |
| **Symbol** | **Description** |
|------------|----------------|
| \( Y^* \)  | occurrence time of CR |
| \( Y_r \)   | occurrence time of the first type II failure during the MR period \( Y^* \) |
| \( \tilde{G}_2(y) \) | reliability function of \( Y_r \) |
| \( N \)     | number of type I failures before occurrence of the first type II failure during the MR period \( Y^* \) |
| \( S_N \)   | occurrence time of the \( N^{th} \) type I failure during the MR period \( Y^* \) |
| \( L \)     | life cycle |
| \( E[L] \)  | The expected length of the life cycle \( L \) |
| \( c_f, c_r, c_m \) | unit failure/replacement/repair cost |
| \( c_n \)    | unit cost of a new product covered by warranty |
| \( C(N) \)  | the total cost during the MR period \( Y^* \) |
| \( E[C(N)] \) | the expected total cost during the MR period \( Y^* \) |
| \( E[C(L)] \) | the expected total cost during the life cycle |
| \( C(m,N) \) | the expected cost rate |

2. Problem definition

Shang et al. [14] classified failure into type I failure with a probability \( p \) and type II failure with a probability \( q \). This classification is more realistic and more general. We consider also this classification in this paper. The warranty agreement activates after installment of the product sold under RHW policy. Subsequently, the consumer carries out MR policy during the MR period. PM at time \( w \) is imperfect and reduces age to a variable, and CR is implemented at occurrence time of the first type II failure or at occurrence time of the \( N^{th} \) type I failure, whichever occurs earlier. From the consumer’s perspective, obviously, the interval between two successive CRs can be considered as a cycle. Therefore, we define life cycle as an interval from installment of the product to occurrence time of CR, which is similar to [13, 14] and is depicted by Figure 1.

Our work can be summarized the followings. First, we compute the expected length \( E[L] \) of the life cycle. Second, we derive the expected total cost \( E[C(L)] \) during the life cycle. Based on the renewal theorem in [18], third, we calculate the expected cost rate as \( C(m,N) = \frac{E[C(L)]}{E[L]} \), where \( m \) and \( N \) are two decision variables. By minimizing the expected cost rate \( C(m,N) \), finally, the optimal solutions are searched numerically and performance is demonstrated in the example.

![Figure 1. Life cycle and maintenance actions.](image)
Assumptions:
- All claims are valid and accepted during the warranty period;
- Repair/replacement/maintenance time are neglectful;
- The failure rate function is continuous and known to consumers; and
- The post-warranty period is equivalent to MR period.

3. Model Formulation

The main goal of this section is to derive formulas of the expected cost rate model from the consumer’s perspective.

3.1. The expected length of the life cycle

As depicted by Figure 1, the expected length $E[L]$ of the life cycle $L$ is, from the consumer’s perspective, equal to the sum of the expected length $EW$ of the WSP $W$ and the expected length $E[Y^+]$ of occurrence time $Y^+$ of CR during the post-warranty period, i.e.,

$$E[L] = EW + E[Y^+] .$$

(1)

Obviously, we need to derive $EW$ and $E[Y^+]$, and next we derive respectively them.

3.1.1. The expected length of WSP

Let random variable $\eta - 1$ be the number of replacements until the first product through the warranty period $w_i$ is obtained, then $\eta$ follows geometric distribution, as

$$Pr(n = \eta) = [G(w_i)]^{\eta-1}[G(w_i)].$$

(2)

By the expectation property of geometric distribution, the expectation of $\eta - 1$ is given by

$$E[\eta - 1] = \frac{G(w_i)}{G(w_i)} .$$

(3)

When the first type II failure occurs during the warranty period $w_i$, the manufacturer will replace this failed product. This means that occurrence time of the first type II failure is replacement time of the product. Let random variable $Y_i$ $(i = 1, \cdots, \eta - 1)$ be occurrence time of the first type II failure of the $i^{th}$ product during the warranty period $w_i$. From the consumer’s perspective, WSP of the warranty period $w_i$ is defined as an interval from the installment point of the product to a time where the first product survives the warranty period $w_i$. And because $\eta - 1$ products are replaced until the first product through the warranty period $w_i$ is obtained, WSP $W_i$ of the warranty period $w_i$ is equal to sum of the total length $\sum_{i=1}^{\eta-1} Y_i$ resulted from $\eta - 1$ replacements and the warranty period $w_i$ of the first product through the warranty period $w_i$, i.e.,

$$W_i = \sum_{i=1}^{\eta-1} Y_i + w_i .$$

By Wald’s equation in [19], the expected length of WSP $W_i$ can be obtained as
\[ EW_i = E[\eta-1] \cdot E[Y_i] + w_i = \int_0^{y_i} \frac{G(y)dy}{G(w_i)}, \quad (4) \]

where \( E[Y_i] = \frac{\int_0^{y_i} dG(y)}{G(w_i)}. \)

Let random variable \( \kappa - 1 \) be the number of replacements until the first product through the warranty period \( w_2 \) is obtained, then \( \kappa \) likewise conforms to geometric distribution and the expectation of \( \kappa - 1 \) can be given by
\[ E[\kappa-1] = \frac{G_i(w_2)}{G_i(w_2)}. \quad (5) \]

Let random variable \( Y_j \) \( (j = 1, \ldots, \kappa - 1) \) be occurrence time of the first type II failure of the \( j \)th product during the warranty period \( w_2 \). From the consumer’s perspective, Similarly, WSP of the warranty period \( w_2 \) can be calculated as \( W_2 = \sum_{j=1}^{\kappa-1} Y_j + w_2 \). Further, the expected length of WSP \( W_2 \) can be further obtained as
\[ EW_2 = E[\kappa-1] \cdot E[Y_j] + w_2 = \int_0^{y_i} \frac{G_i(y)dy}{G_i(w_2)}, \quad (6) \]

where \( E[Y_j] = \frac{\int_0^{y_i} dG_i(y)}{G_i(w_2)}. \)

As mentioned above, each replacement during the warranty period \( w \) makes warranty restart from starting point of the first phase. So each replacement during the warranty period \( w_1 \) needs one WSP \( \sum_{j=1}^{\kappa-1} Y_j + w_i \) that is corresponding to the warranty period \( w_1 \). Because \( \kappa - 1 \) replacements occur during the warranty period \( w_1 \) until the first product through the warranty period \( w_1 \) is obtained, the corresponding WSP is \( \sum_{j=1}^{\kappa-1} \left( \sum_{i=1}^{\kappa-1} (Y_j + w_i) \right) \). Besides, the WSP generated by \( \kappa - 1 \) replacements during the warranty period \( w_2 \) is \( \sum_{j=1}^{\kappa-1} Y_j \) and WSP of the first product through the warranty period \( w \) is \( w \). Therefore, WSP of the warranty period \( w \) can be obtained as
\[ W = \sum_{j=1}^{\kappa-1} \left( \sum_{i=1}^{\kappa-1} (Y_j + w_i) \right) + \sum_{j=1}^{\kappa-1} Y_j + w_1 + w_2. \quad (7) \]

Obviously, WSP \( W \) of the warranty period \( w \) is greater than the warranty period \( w \), i.e., \( W > w \). Finally, the expected length of WSP \( W \) can be calculated as
\[ EW = E \left[ \sum_{j=1}^{N-1} \sum_{i=1}^{j-1} (Y_j + w_i) + w_i + E \sum_{j=1}^{N-1} Y_j + w_j \right] + \frac{G(0)}{G(w)} - \frac{G(w) \gamma(\delta(m)w + u)}{G(w)} \]

\[ = \sum_{i=1}^{N-1} \sum_{j=1}^{i-1} (Y_j + w_i) + w_i + E \sum_{j=1}^{N-1} Y_j + w_j \]

3.1.2. The expected length of the MR period. As mentioned earlier, PM is an imperfect PM and its effect is to reduce past (actual) age to a virtual age. Here we adopt the modeling framework proposed in [20] to describe the PM effect. PM at time \( w \) reduces past age \( w \) to a virtual age \( \delta(m)w \), where \( \delta(m) \) (\( 0 \leq \delta(m) \leq 1, m = 0, 1, 2, \ldots, M \)) is a decreasing function with respect to \( m \) with \( \delta(0) = 1 \) and \( \delta(M) = 0 \). Obviously, PM is perfect if \( m = M \); PM is minimal repair if \( m = 0 \); and PM is imperfect if \( 0 < m < M \). So failure rate function after PM is \( \gamma(\delta(m)w + u) (u > 0) \). Let random variable \( Y_g \) be occurrence time of the first type II failure during the MR period \( Y^* \), then reliability function of \( Y_g \) can be expressed as

\[ \tilde{G}_2(y) = \exp \left\{ -\int_0^y q(\delta(m)w + u)du \right\} \]

As discussed above, the product is correctly replaced by the consumer at occurrence time \( Y_g \) of the first type II failure or occurrence time \( S_n \) of the \( N^{th} \) type I failure, whichever takes place first. Let random variable \( Y^* \) be occurrence time of CR during the post-warranty period, then

\[ Y^* = \min(Y_g, S_n) \]

and reliability function of \( Y^* \) can be given by

\[ \tilde{R}(y) = P[Y^* > y] = \sum_{k=0}^{N-1} P[Y_k > y, N(y) = k] = \sum_{k=0}^{N-1} G_k(y)P_k(y), \]

where \( P_k(y) = P[N(y) = k] = \frac{\exp\left\{ -\int_0^y q(\delta(m)w + u)du \right\} \left( \int_0^y q(\delta(m)w + u)du \right)^k}{k!} \).

Obviously, the expected length of occurrence time \( Y^* \) of CR can be given by

\[ E[Y^*] = \int_0^\infty \tilde{R}(y)dy = \sum_{k=0}^{N-1} \int_0^\infty G_k(y)P_k(y)dy. \]

3.1.3. The expected length of the life cycle. The expected length of WSP \( W \) and the expected length \( E[Y^*] \) have been obtained respectively in (8) and (10). By (1), therefore, the expected length of the life cycle can be expressed as

\[ E[L] = EW + \sum_{k=0}^{N-1} \int_0^\infty \tilde{G}_2(y)P_k(y)dy. \]

3.2. The expected total cost during the life cycle
The expected total cost \( E[C(L)] \) during the life cycle is equal to the sum of the expected total cost \( EC \) during the WSP \( W \), maintenance cost \( C_P(m) \) at time \( w \) and the expected total cost \( E[C(N)] \) during the MR period \( Y^* \), i.e.,

\[ E[C(L)] = EC_r + C_P(m) + E[C(N)]. \]
Obviously, we need to derive \( EC_w \) , \( C_r(m) \) and \( E[C(N)] \), and we next derive respectively.

### 3.2.1. The expected total cost during the WSP \( W \)

During the WSP \( W_i \), the total cost resulted from the \( i^{\text{th}} \) product is \( c_f \int_0^r p_f(x)dx \) and the total cost resulted from the first product through the warranty period \( w_i \) is \( c_f \int_0^{w_i} p_f(x)dx \). And because \( \eta-1 \) products are replaced until the first product through the warranty period \( w_i \) is obtained, the total cost \( C_{w_i} \) during the WSP \( W_i \) is equal to the sum of the total cost \( \sum_{i=1}^{w_i-1} \left( c_f \int_0^r p_f(x)dx + c_s \frac{Y_f}{w_2} \right) \) resulted from \( \eta-1 \) replacements and the total cost \( c_f \int_0^{w_i} p_f(x)dx \) resulted from the first product through the warranty period \( w_i \), i.e.,

\[
C_{w_i} = \sum_{i=1}^{w_i-1} \left( c_f \int_0^r p_f(w_i+x)dx + c_s \frac{Y_f}{w_2} \right) + c_f \int_0^{w_i} p_f(w_i+x)dx. \tag{13}
\]

During the WSP \( W_2 \), the total cost resulted from the \( j^{\text{th}} \) product is \( c_f \int_0^r p_f(w_j+x)dx + c_s \frac{Y_f}{w_2} \) represents replacement cost of the \( j^{\text{th}} \) product) and the total cost resulted from the first product through the warranty period \( w_2 \) is \( c_f \int_0^{w_2} p_f(w_1+x)dx \). Similarly, the total cost \( C_{w_2} \) is given by

\[
C_{w_2} = \sum_{j=1}^{w_1} \left( c_f \int_0^r p_f(w_j+x)dx + c_s \frac{Y_f}{w_2} \right) + c_f \int_0^{w_2} p_f(w_1+x)dx. \tag{14}
\]

Further, the expected total cost that of \( C_{w_2} \) can be calculated as

\[
EC_{w_2} = E \left[ \sum_{i=1}^{w_i-1} \left( c_f \int_0^r p_f(w_i+x)dx + c_s \frac{Y_f}{w_2} \right) + c_f \int_0^{w_i} p_f(w_i+x)dx \right] + c_f \int_0^{w_2} p_f(w_1+x)dx = \frac{G(w_i) pc_f}{G(w_i) q} + \frac{\int_0^{w_i} G(w_i) dy}{w_2 G_i(w_2)}. \tag{15}
\]

Similar to analysis method of (7), further, the total cost during the WSP \( W \) can be obtained as

\[
E \left[ \sum_{i=1}^{w_1} \left( c_f \int_0^r p_f(x)dx + c_s \frac{Y_f}{w_2} \right) + \int_0^{w_1} p_f(x)dx \right] + \sum_{j=1}^{w_1} \left( c_f \int_0^r p_f(w_j+x)dx + c_s \frac{Y_f}{w_2} \right) + c_f \int_0^{w_2} p_f(w_1+x)dx. \tag{15}
\]

Finally, the expected total cost that corresponds to \( C_{w} \) can be calculated as

\[
EC_w = E(\eta - 1)EC_n + c_f \int_0^{w_1} p_f(x)dx + EC_{w_2} = \frac{pc_f G(w_1)}{Q} \left( \frac{G(w_i) pc_f}{G(w_i) q} + \frac{\int_0^{w_i} G(w_i) dy}{w_2 G_i(w_2)} + c_f \int_0^{w_2} p_f(x)dx \right). \tag{16}
\]
3.2.2. Maintenance cost. Ja et al. [21] considered the minimal repair cost as an increasing nonlinear function of age. Based on this literature, Shafiee and Chukova [22] considered maintenance cost as the sum of the cost of the minimal repair plus the term modelled by the increasing power function of the age reducing factor. As presented in Subsection 3.1.2, virtual age after PM is $w_{\delta(m)}$, so reduction in past age is $(1-\delta(m))w$. Similar to [22], we further model maintenance cost as a power function with respect to reduction $(1-\delta(m))w$ and maintenance age $w$, i.e.,

$$C_p(m) = c_m + c(1-\delta(m))^\alpha w^\beta = c_m + c(1-\delta(m))^\alpha w^{\alpha+\beta},$$

(17)

where $c > 0$, $\alpha > 0$ and $\beta > 0$.

3.2.3. The expected total cost during the MR period. During the MR period $Y^*$, CR is implemented at occurrence time of the $N^{th}$ type I failure or at occurrence time of the first type II failure. So the expected number of type I failures until occurrence of CR can be calculated as

$$E[N] = (N-1)p^N + \sum_{k=1}^{N}(k-1)p^{k-1}q = \frac{(p-p^N)}{q}.$$  
(18)

Further, the expected total cost $E[C(N)]$ during the MR period $Y^*$ can be obtained as

$$E[C(N)] = E[N] \cdot c_m + c_r = \frac{c_m(p-p^N)}{q} + c_r.$$  
(19)

3.2.4. The expected total cost during the life cycle. The expected total cost $EC$ during the WSP $W$, maintenance cost $C_p(m)$ at time $w$ and the expected total cost $E[C(N)]$ during the MR period $Y^*$ have derived in (16), (17) and (19). By (12), finally, the expected total cost $E[C(L)]$ during the life cycle can be expressed as

$$E[C(L)] = EC + c_m + c(1-\delta(m))w^{\alpha+\beta} + \frac{c_m(p-p^N)}{q} + c_r.$$  
(20)

3.3. The expected cost rate

The expected total cost $E[C(L)]$ during the life cycle and the expected length $E[L]$ of the life cycle have been derived respectively in (11) and (20). By the renewal rewarded theorem in [18], finally, the expected cost rate can be calculated as

$$C(m, N) = \frac{E[C(L)]}{E[L]} = \frac{EC + c_m + c(1-\delta(m))w^{\alpha+\beta} + c_r(p-p^N)}{q + c_r} + \int_{y=0}^{\infty} G_2(y)P_2(y)dy.$$  
(21)

The consumer can determine the optimal $m^*$ and $N^*$ by minimizing the expected cost rate presented in (21). But the joint optimization is mathematically intractable. So the optimal values will be numerically searched in the numerical example.

3.4. Special cases
When some parameters of (21) take special values, (21) can be transformed into some special models, as follows.

1) When \( w_i = 0 \), (21) can be rewritten as
\[
C(m, N) = EC_m + c_m + c(1 - \delta(m)) w^\alpha + c_m(p - p^N)/q + c_e, \\
EW_2 + \sum_{k=0}^{N-1} \int_0^\infty G_s(y) \cdot p_k(y) dy
\]
where \( \hat{G}_s(y) = \exp \left[ -\int_0^y q\gamma(\delta(m)w_i + u) du \right] \) and \( p_k(y) = \frac{\exp \left[ -\int_0^y p\gamma(\delta(m)w_i + u) du \right] \left( \int_0^y p\gamma(w_i + u) du \right)^k}{k!} \).

This model is the expected cost rate model in which the RPRRW policy is only considered while the RPRRW policy is not taken into account.

2) When \( m = 0 \), (21) can be rewritten as
\[
C(0, N) = EC_m + c_m(p - p^N)/q + c_e, \\
EW + \sum_{k=0}^{m-1} \int_0^\infty G_s(y) \cdot p_k(y) dy
\]
where \( \hat{G}_s(y) = \exp \left[ -\int_0^y q\gamma(w + u) du \right] \) and \( p_k(y) = \frac{\exp \left[ -\int_0^y p\gamma(w + u) du \right] \left( \int_0^y p\gamma(w + u) du \right)^k}{k!} \).

This model is the expected cost rate model, where PM is not considered.

3) When \( w_i = 0 \) and \( m = 0 \), (21) can be rewritten as
\[
C(0, N) = EC_m + c_m(p - p^N)/q + c_e, \\
EW_2 + \int_0^\infty G_s(y) \cdot p_k(y) dy
\]
where \( \hat{G}_s(y) = \exp \left[ -\int_0^y q\gamma(w_i + u) du \right] \) and \( p_k(y) = \frac{\exp \left[ -\int_0^y p\gamma(w_i + u) du \right] \left( \int_0^y p\gamma(w_i + u) du \right)^k}{k!} \).

This model is the expected cost rate model, where the RPRRW policy is only considered while PM is not considered.

4) When \( EW = 0 \) and \( EC_m = 0 \), (21) can be rewritten as
\[
C(m, N) = c_m + c(1 - \delta(m)) w^\alpha + c_m(p - p^N)/q + c_e, \\
\sum_{k=0}^{N-1} \int_0^\infty G_s(y) \cdot p_k(y) dy
\]

This model is the expected cost rate model that corresponds to a secondhand product with a known past age \( w \). And meanwhile, the proposed MR policy is considered and any warranty is not considered.

At last, it is worth pointing out that three special cases are only presented by \( w_i \) or/and \( m \) taking special values and other special cases can be obtained by \( p(q) \), \( w_i \) or/and \( m \) taking special values. They can be derived if necessary.

4. Numerical example

To illustrate our approaches obtained in the previous section, this section presents a simple numerical example.
We assume that failure rate function is two-parameter Weibull failure rate function, as
\[ \gamma(x) = \theta \theta x^{\theta - 1} \]
with shape parameter \( \theta > 1 \) (meaning an increasing failure rate function) and scale parameter \( \theta > 0 \).

The following parameters \( c_m = 0.1, c_f = 0.5, c_i = 1, c_u = 5, c_r = 10, \theta = 1 \) and \( \theta = 2 \) are used in the example.

4.1. Performance analysis
In order to compare performance, we consider the replacement policy without PM at time \( w \) as a benchmark policy.

From Table 1, we can conclude the following remarks:

- The optimal expected cost rate \( C(m', N') \) of MR policy is lower than that of benchmark policy (Exp. 23), while the optimal number \( N' \) of type I failures under MR policy is larger than that of benchmark policy. These monotonic laws can be directly depicted in Figure 2, where \( c_r = 10, w_1 = 1 \) and \( w_2 = 1 \). This implies that performance of MR policy outperforms that of benchmark policy.

| \( p \) | MR policy | Benchmark policy |
|-------|-----------|------------------|
| \( m' \) | \( N' \) | \( C(m', N') \) | \( N' \) | \( C(0, N') \) |
| 0.8   | 1         | 5                | 3       | 5.4293 |
| 0.9   | 1         | 7                | 6       | 6.3155 |
| 0.99  | 2         | 11               | 8       | 6.9488 |

**Figure 2.** The expected cost rate versus number of type I failures under different policies.

4.2. Sensitivity analysis
From Table 2 and Table 3, we can derive the following remarks:

- For a fixed $p$, both the optimal number $N'$ of type I failures and the optimal expected cost rate $C(m',N')$ are decreased with respect to $w_1$ ($w_2$). This implies that when the first (second) phase warranty $w_1$ ($w_2$) is much larger, replacement action is more frequent while the cost rate becomes lower.

- For a fixed $w_1$ ($w_2$), both the optimal number $N'$ of type I failures and the optimal expected cost rate $C(m',N')$ are increased (decreased) with respect to $p$ ($q$). This situation means that when the probability $p$ ($q$) of type I (II) failure is much larger (smaller), replacement frequency is much lower while the cost rate gets bigger.

Table 2. The optimal $m^*$, $N'$ and $C(m',N')$ at $w_2 = 1$ and $c_r = 10$ versus $p$ and $w_1$.

| $p$  | $w_1 = 0.8$ | $w_1 = 1$ | $w_1 = 1.2$ |
|------|-------------|-----------|-------------|
| $m^*$ | $N'$ | $C(m',N')$ | $m^*$ | $N'$ | $C(m',N')$ | $m^*$ | $N'$ | $C(m',N')$ |
| 0.7  | 5  | 5.2576  | 3  | 4.5374  | 1  | 3.8444  |
| 0.8  | 6  | 5.9051  | 5  | 5.3822  | 3  | 4.8623  |
| 0.9  | 10 | 6.3032  | 7  | 6.1491  | 6  | 5.8642  |

Table 3. The optimal $m^*$, $N'$ and $C(m',N')$ at $w_1 = 1$ and $c_r = 10$ versus $p$ and $w_2$.

| $p$  | $w_2 = 0.8$ | $w_2 = 1$ | $w_2 = 1.2$ |
|------|-------------|-----------|-------------|
| $m^*$ | $N'$ | $C(m',N')$ | $m^*$ | $N'$ | $C(m',N')$ | $m^*$ | $N'$ | $C(m',N')$ |
| 0.7  | 6  | 5.2508  | 3  | 4.5374  | 1  | 3.8919  |
| 0.8  | 8  | 5.9204  | 5  | 5.3822  | 3  | 4.8510  |
| 0.9  | 10 | 6.3176  | 7  | 6.1491  | 6  | 5.8406  |

From Table 4, we can derive the following remarks:

- For a fixed $p$, both the optimal number $N'$ of type I failures and the optimal expected cost rate $C(m',N')$ are increased with respect to $c_r$. This implies that when replacement cost $c_r$ is much larger, replacement action is less frequent while the optimal cost rate becomes greater.
For a fixed \( c_r \), both the optimal number \( N^* \) of type I failures and the optimal expected cost rate \( C(m^*,N^*) \) are increased (decreased) with respect to \( p(q) \). This situation means that when the probability \( p(q) \) of type I (II) failure is much larger (smaller), replacement action is less frequent while the optimal cost rate gets greater.

### Table 4.
The optimal \( m^* \), \( N^* \) and \( C(m^*,N^*) \) at \( w_1 = 1 \) and \( w_2 = 0.8 \) versus \( p \) and \( c_r \).

| \( p \) | \( c_r = 8 \) | \( c_r = 10 \) | \( c_r = 12 \) |
|-------|--------------|--------------|--------------|
| \( m^* \) | \( N^* \) | \( C(m^*,N^*) \) | \( m^* \) | \( N^* \) | \( C(m^*,N^*) \) | \( m^* \) | \( N^* \) | \( C(m^*,N^*) \) |
| 0.7   | 1            | 4            | 4.7595       | 1            | 6            | 5.2508       | 1            | 7            | 5.7363       |
| 0.8   | 1            | 6            | 5.3728       | 2            | 8            | 5.9204       | 2            | 10           | 6.4192       |
| 0.9   | 2            | 8            | 5.8125       | 3            | 10           | 6.3176       | 10           | 12           | 6.7628       |

5. **Conclusion**

In this paper, we considered a renewable hybrid warranty (RHW) policy, which combined renewable free replacement warranty (RFRW) policy and renewable pro-rata replacement warranty (RPRRW) policy. Compared with RFRW policy or RPRRW policy, advantage of RHW policy is that the preferences between the manufacturer and the consumer are taken into consideration at the same time and the renewable characteristic of the warranty period is existed. Under RHW policy, we presented the consumer’s maintenance-replacement (MR) policy during the post-warranty period. The presented MR policy is that preventive maintenance was first performed when warranty expired and then the product was correctly replaced at occurrence time of the \( N^\text{th} \) type I failure or at occurrence time of type II failure, whichever taken place earlier.

Based on RHW policy and the proposed MR policy, a general expected cost rate model was derived. Some special cases of the expected cost rate model were offered. As showed in the example, performance of the proposed MR policy outperformed that of corrective replacement that PM was not considered. This superiority could be indicated by comparing the optimal number of type I failures and the optimal expected cost rate per unit time.

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