T Duality Between Perturbative Characters
of $E_8 \otimes E_8$ and $SO(32)$ Heterotic Strings
Compactified On A Circle

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Abstract
Characters of $E_8 \otimes E_8$ and $SO(32)$ heterotic strings involving the full internal symmetry Cartan subalgebra generators are defined after circle compactification so that they are T dual. The novel point, as compared with an earlier study of the type II case, is the appearance of Wilson lines. Using $SO(17,1)$ transformations between the weight lattices reveals the existence of an intermediate theory where T duality transformations are disentangled from the internal symmetry. This intermediate theory corresponds to a sort of twisted compactification of a novel type. Its modular invariance follows from an interesting interplay between three representations of the modular group.

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1 Introduction

In general, string theories are invariant under a certain global symmetry group $G$, of rank $R$, that commutes with the Virasoro algebra. It is thus natural to consider the generalisation of the usual partition torus functions which mathematically is the character\footnote{Of course, we are really dealing with Kac-Moody type characters. This point is not central here, and will not be considered.} of the representation of $G \otimes \text{Vir.}$, of course, we are really dealing with Kac-Moody type characters. This point is not central here, and will not be considered.
span by the free string states; and is of the type

\[ \chi(\tau \mid v_1, \ldots, v_R) = \text{Tr} \left\{ e^{2i\pi(\tau a - a)} e^{2i\pi\tau^* (L_0 - \tilde{a})} \prod_{j=1}^{R} e^{2i\pi v_j H_j} \right\}. \]

The trace is taken over all the physical string states, and \( \exp(2i\pi v_j H_j) \) form a maximal set of commuting generators of \( G \). The usual partition function \( P(\tau) = \text{Tr} \left\{ e^{2i\pi(\tau a - a)} e^{2i\pi\tau^* (L_0 - \tilde{a})} \right\} \) is clearly recovered when the \( v \)'s vanish.

The elliptic genus, which has been much studied (see, e.g. ref.[2]) corresponds to the case where \( G \) is chosen to be generated by the left- and right-moving fermionic number operators. On the other hand, \( G \) contains the group of orthogonal transverse rotations of the string components in the uncompactified directions, and the associated Lie group characters were studied some time ago[3], [4] as book keeping devices for the representation content of string theories. The recent developments give us a strong motivation to study their duality and modular properties. These aspects left over in previous discussions, were recently studied in details in ref.[1] for perturbative type II strings. The present article deals with heterotic perturbative string characters along the same line. Here it is of course natural to extend \( G \) to involve the internal symmetry group, that is, either \( SO(32) \) or \( E_8 \otimes E_8 \), as was done originally[4].

Without any compactification, the object under study is of the type

\[ \chi(\tau \mid \vec{v}, \vec{\xi}) = \text{Tr} \left\{ e^{2i\pi(\tau a - a)} e^{2i\pi\tau^* (L_0 - \tilde{a})} \prod_{j=1}^{4} e^{2i\pi v_j \mathcal{H}_j} \prod_{j=1}^{16} e^{2i\pi \xi_j A_j} \right\}, \tag{1.1} \]

where \( \{\mathcal{H}_1, \ldots, \mathcal{H}_4\} \) are four commuting transverse-space rotation generators, and \( \{A_1, \ldots, A_{16}\} \) are sixteen commuting generators of the internal symmetry group.

The novel point of the present study, as compared to ref.[1], is that duality between the two compactified heterotic theories requires[7] that one breaks their internal symmetry group, so that the ideas just summarized do not apply strictly speaking. The point of this paper is to show how to define, nevertheless, the characters involving sources for the full Cartan subalgebras of each internal symmetry group so that they are T dual. The corresponding equality generalises the well known equality between partition functions (see e.g. [5]) which is of course recovered when all the \( v \)'s and \( \xi \)'s vanish. The basic mathematical tool to derive T duality of the circle compactified heterotic strings is the \( SO(17, 1) \) transformation between the two embeddings of

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$E_8 \otimes E_8$ and spin(32)/$Z_2$ zero mode lattices into the same Lorentzian lattice $\Pi^{17,1}$. It may be factored into three transformations. This shows the existence of an intermediate theory where the internal symmetry group is broken down to $SO(16) \otimes SO(16)$ by a sort of twisted compactification of a novel type. The character is then straightforwardly defined in the intermediate theory since this unbroken gauge group has the same Cartan subalgebra as the broken ones. This intermediate theory is such that it interpolates between the two uncompactified heterotic strings which are recovered for very large and very small compactification radius, respectively. The modular invariance of the character follows from an interesting interplay between three representations of the modular group. Finally, the characters of the original $E_8 \otimes E_8$ and spin(32)/$Z_2$ heterotic theories are defined in a natural way using the $SO(17,1)$ transformations which map their weight lattices to the one of the intermediate theory.

2 The uncompactified characters of heterotic strings

This case was already discussed in ref.[4]. In the same way as for type II superstring, we introduce modified characters, by also summing over the total momentum, following the line of ref.[4]. We shall be brief and refer to the last two references for details. This section is a sort of warming up for the coming discussion of the compactified case. The characters may be factorised under the form

$$
\chi(\tau | \vec{v}, \vec{\xi}) = \chi_0(\vec{v}) \chi_L(\tau | \vec{v}) \chi_R(\tau | \vec{v}, \vec{\xi}),
$$

(2.1)

where $\chi_0$ comes from the summation over the transverse components of the total momentum, and $\chi_L, \chi_R$ are, respectively, the contributions of the left and right transverse string modes. We select the right movers to be the standard compactified bosonic modes. The $v_i$'s are the same, for left and right worldsheet fields since they transform at the same time under space rotations. The first term in the equation above was computed in ref.[4]. The result reads

$$
\chi_0(\vec{v}) = \prod_{k=1}^4 \frac{1}{\sin^2(\pi v_k)},
$$

(2.2)

4We drop overall constant factors throughout.
The second term is the same as for type II. In the notations of ref. [1], we choose the chirality sector \( \text{GS}_- \) for definiteness. The left factor of the character is thus

\[
\chi_L(\tau \mid \vec{v}) = \left( \chi_{\text{GS}_-}(\tau \mid \vec{v}) \right)^* \equiv \left( \prod_{k=1}^{4} \sin(\pi v_k) \frac{\Theta_1^{(8)}(\vec{y} \mid \tau)}{\Theta_1^{(8)}(\vec{v} \mid \tau)} \right)^*
\]

(2.3)

The notations are the same as in ref. [1] apart from minor details. The function \( \Theta^{(2N)} \) are defined in general by

\[
\Theta_i^{(2N)}(\vec{x} \mid \tau) = \prod_{\mu=1}^{N} \theta_i(x_\mu \mid \tau)
\]

(2.4)

where \( \theta_i \) are the usual theta functions with Bateman’s conventions. The vector \( \vec{y} \) is obtained from \( \vec{v} \) by the orthogonal transformation associated to one of the \( \text{SO}(8) \) triality transformations (see ref. [1]). Accordingly we have the Jacobi identity

\[
\Theta_1^{(8)}(\vec{y} \mid \tau) = \frac{1}{2} \left( \Theta_1^{(8)}(\vec{v} \mid \tau) - \Theta_2^{(8)}(\vec{v} \mid \tau) + \Theta_3^{(8)}(\vec{v} \mid \tau) - \Theta_4^{(8)}(\vec{v} \mid \tau) \right).
\]

In order to ensure modular invariance, the character is defined by taking the supertrace. The last factor of Eq. 2.1 involves the internal symmetry group, and thus should be discussed separately for the two heterotic strings.

2.1 The \( \text{SO}(32) \) case.

The right character may be factorised as follows

\[
\chi_R(\tau \mid \vec{v}, \vec{\xi}) = e^{-2\pi i \tau} \prod_k \frac{\sin(\pi v_k)}{\Theta_1^{(8)}(\vec{v} \mid \tau)} \chi^O(\tau \mid \vec{\xi}),
\]

(2.5)

where the explicit factors are the contribution of the space bosonic components. The calculation of \( \chi^O \) is similar to the discussion of the NSR characters recently carried out in ref. [1]. Some standard features of the world-sheet properties of heterotic strings are summarized in appendix A. Altogether, the following is straightforwardly derived.
The P-right sector  As recalled in appendix A, the intercept, is \( a_P = -1 \) giving an overall factor \( e^{2i\pi \tau} \). One finds

\[
\chi^O(\tau | \vec{\xi}) = e^{4i\pi \tau} \frac{1}{2} \left\{ \prod_{n, \nu=1}^{16} \left( 1 + e^{2in\pi \tau} e^{2i\pi \xi_\nu} \right) \left( 1 + e^{2in\pi \tau} e^{-2i\pi \xi_\nu} \right) + \prod_{n, \nu=1}^{16} \left( 1 - e^{2in\pi \tau} e^{2i\pi \xi_\nu} \right) \left( 1 - e^{2in\pi \tau} e^{-2i\pi \xi_\nu} \right) \right\}
\]

The symbols \( \chi^{O(32)}_{0,\ldots,0,1}(\vec{\xi}) \) and \( \chi^{O(32)}_{0,\ldots,1,0}(\vec{\xi}) \) denote the irreducible characters of the two fundamental spinor representations.

A-right sector  Now the intercept \( a_A = 1 \), and one finds

\[
\chi^A(\tau | \vec{\xi}) = \frac{1}{2} \left\{ \prod_{r=1/2}^{\infty} \prod_{\mu=1}^{16} \left( 1 + e^{2ir\pi \tau} e^{2i\pi \xi_\mu} \right) \left( 1 + e^{2ir\pi \tau} e^{-2i\pi \xi_\mu} \right) + \prod_{r=1/2}^{\infty} \prod_{\mu=1}^{16} \left( 1 - e^{2ir\pi \tau} e^{2i\pi \xi_\mu} \right) \left( 1 - e^{2ir\pi \tau} e^{-2i\pi \xi_\mu} \right) \right\}
\]

Next we re-express these characters in terms of Theta functions. For this one has to simplify the expressions by getting rid of \( \chi^{O(32)}_{0,\ldots,0,1} \) and \( \chi^{O(32)}_{0,\ldots,1,0} \). In the same way as for the \( O(8) \) case discussed on ref.[1], one may verify that

\[
\chi^{O(32)}_{0,\ldots,0,1}(\vec{\xi}) = \sum_{\epsilon_1,\ldots,\epsilon_{16} = \pm 1}^{\text{odd number}} \prod_{\nu=1}^{16} e^{i\pi \xi_\nu \epsilon_\nu}
\]

\[
\chi^{O(32)}_{0,\ldots,1,0}(\vec{\xi}) = \sum_{\epsilon_1,\ldots,\epsilon_{16} = \pm 1}^{\text{even number}} \prod_{\nu=1}^{16} e^{i\pi \xi_\nu \epsilon_\nu}
\]
A straightforward calculation gives

\[
\chi_P(\tau \mid \vec{q}, \vec{\xi}) = \int_{0}^{1} \frac{1}{2} \left[ \Theta_{2}^{(32)}(\vec{\xi} \mid \tau) + \Theta_{1}^{(32)}(\vec{\xi} \mid \tau) \right] \tag{2.6}
\]

\[
\chi_A(\tau \mid \vec{q}, \vec{\xi}) = \int_{0}^{1} \frac{1}{2} \left[ \Theta_{3}^{(32)}(\vec{\xi} \mid \tau) + \Theta_{4}^{(32)}(\vec{\xi} \mid \tau) \right] \tag{2.7}
\]

where

\[
f_0 = \prod_{n \geq 1} \left( 1 - e^{2i\pi n \tau} \right). \tag{2.8}
\]

Next we introduce summations over the internal symmetry lattice, by using the series expansion of the theta functions. It is easy to derive in general the lattice expansion

\[
\frac{1}{2} \left[ \Theta_{2}^{(2N)}(\vec{\xi} \mid \tau) + \Theta_{1}^{(2N)}(\vec{\xi} \mid \tau) \right] = \sum_{m \in \Lambda_{W}^{(N)e,h}} e^{i\pi \tau} \sum_{\mu} m_{\mu}^{2} e^{2i\pi \sum_{\mu} m_{\mu} \xi_{\mu}}
\]

\[
\frac{1}{2} \left[ \Theta_{2}^{(2N)}(\vec{\xi} \mid \tau) - \Theta_{1}^{(2N)}(\vec{\xi} \mid \tau) \right] = \sum_{m \in \Lambda_{W}^{(N)o,h}} e^{i\pi \tau} \sum_{\mu} m_{\mu}^{2} e^{2i\pi \sum_{\mu} m_{\mu} \xi_{\mu}}
\]

\[
\frac{1}{2} \left[ \Theta_{3}^{(2N)}(\vec{\xi} \mid \tau) + \Theta_{4}^{(2N)}(\vec{\xi} \mid \tau) \right] = \sum_{m \in \Lambda_{W}^{(N)e,i}} e^{i\pi \tau} \sum_{\mu} m_{\mu}^{2} e^{2i\pi \sum_{\mu} m_{\mu} \xi_{\mu}}
\]

\[
\frac{1}{2} \left[ \Theta_{3}^{(2N)}(\vec{\xi} \mid \tau) - \Theta_{4}^{(2N)}(\vec{\xi} \mid \tau) \right] = \sum_{m \in \Lambda_{W}^{(N)o,i}} e^{i\pi \tau} \sum_{\mu} m_{\mu}^{2} e^{2i\pi \sum_{\mu} m_{\mu} \xi_{\mu}}, \tag{2.9}
\]

We will only consider the case where \( N \) is multiple of 4. The four above lattices are equivalence classes of the weight lattice of \( O(2N) \):

\[
\Lambda_{W}^{(N)} = \Lambda^{(N)e,h} \oplus \Lambda^{(N)o,h} \oplus \Lambda^{(N)e,i} \oplus \Lambda^{(N)o,i}
\]

The root lattice is \( \Lambda^{(N)e,i} \). The weight lattice \( \Lambda_{W}^{(N)} \) is made up with the points \( \sum_{\mu} \vec{e}_{\mu} m_{\mu} \) with \( \vec{e}_{\mu} \) basis unit vectors, such that all \( m_{\mu} \)'s are integer or half integer simultaneously. The four equivalence classes are conveniently defined by further separating the points according to whether \( \sum_{\mu=1}^{N} m_{\mu} \) is even or odd. The notation may be summarized as follows.

| sublattice | \( \Lambda^{(N)e,i} \) | \( \Lambda^{(N)o,i} \) | \( \Lambda^{(N)e,h} \) | \( \Lambda^{(N)o,h} \) |
|------------|-----------------|-----------------|-----------------|-----------------|
| \( m_{\mu} \) | integer | integer | half int. | half int. |
| \( \sum_{\mu} m_{\mu} \) | even | odd | even | odd |
Altogether
\[
\chi_A^O(\tau | \vec{\xi}) + \chi_P^O(\tau | \vec{\xi}) = f_0^{-16} \sum_{\vec{m} \in \Gamma^{16}} e^{i\pi \tau \sum \mu (m_\mu)^2} e^{2i\pi \sum \mu m_\mu \xi_\mu}
\]
where
\[
\Gamma^{16} = \Lambda^{(16)e,h} \oplus \Lambda^{(16)e,i} \equiv \Lambda^{(16)e,h} \oplus \Lambda_R^{(16)} \quad (2.10)
\]
This lattice is of course the usual self-dual lattice (see e.g. ref.\[5\] vol 1). Let us finally put together left-, right-movers, and zero modes. One finds, using standard identities about theta functions,
\[
\chi^O(\tau | \vec{v}, \vec{\xi}) = \Theta^{(32)}_1(\vec{\xi} | \tau) + \Theta^{(32)}_2(\vec{v} | \tau) + \Theta^{(32)}_3(\vec{\xi} | \tau) + \Theta^{(32)}_4(\vec{v} | \tau)
\]
(2.11)

Our next point is modular invariance. The invariance under \(\tau \to \tau + 1\) is immediate. Concerning the other basic one, that is \(\tau \to -1/\tau\), one has
\[
\Theta^{(2N)}_i(\vec{\xi}/\tau | -1/\tau) = \sum_j S_{ij} \Theta^{(2N)}_j(\vec{\xi} | \tau) e^{i\pi \xi_j / \tau} \Theta^{(2N)}_i(\vec{\xi} | \tau) / (\theta'(0|\tau))^{N/3} \quad i = 1, 2, 3, 4,
\]
where
\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]
(2.12)
Thus
\[
\chi^O(\tau | \vec{v}, \vec{\xi}) = e^{-i\pi (\vec{\xi}^2 - \vec{v}^2)/\tau} \chi^O \left( \frac{-1}{\tau} | \vec{v}, \vec{\xi} \right)
\]
This is not quite modular invariant. The left over multiplicative factor is removed by using the simple identity
\[
\frac{1}{\Im(\tau)} - \frac{1}{\tau^2} \Im(-1/\tau) = \frac{1}{\tau}
\]
Thus we define
\[
\chi^O_{\text{modular}} = e^{i\pi (\vec{\xi}^2 - \vec{v}^2)/\Im(\tau)} \chi^O.
\]
Since the additional factor is invariant under $\tau \to \tau + 1$, we have full modular invariance:

$$\chi^O_{\text{modular}}(\tau | \vec{v}, \vec{\xi}) = \chi^O_{\text{modular}} \left( \frac{a\tau + b}{c\tau + d}, \frac{\vec{v}}{c\tau + d}, \frac{\vec{\xi}}{c\tau + d} \right)$$  \hspace{1cm} (2.13)

The internal symmetry group parameters $\vec{\xi}$ are also acted upon by the modular group.

2.2 The case of the $E_8 \otimes E_8$ heterotic string

Now we have

$$\chi_R(\tau | \vec{v}, \vec{\xi}) = e^{-2\pi \tau \prod_k \sin(\pi v_k)} \chi^E(\tau | \vec{\xi}),$$  \hspace{1cm} (2.14)

where the last term is a $E_8 \otimes E_8$ character. It is easy to see that the Cartan algebras of $SO(32)$ and $E_8 \otimes E_8$ are the same\footnote{The situation is similar to what happens between $O(2N)$ and $O(2N + 1)$}. Thus the characters of the two string theories are computed from the same trace, albeit with different GSO like projections. There are now four sectors for the right movers. Since the 16 dimensional space is split in two, we introduce the notation $\vec{\xi} \to \vec{\xi}, \vec{\xi}'$, where $\vec{\xi} = \{\xi_1, \ldots, \xi_8\}, \vec{\xi}' = \{\xi_9, \ldots, \xi_{16}\}$. The result will be expressed as products of characters of $O(16)$. A calculation similar to the one just summarised for the previous case gives the internal symmetry contribution

$$\chi^E(\tau | \vec{\xi}) =$$

$$= \frac{f_0^{-16}}{4} \left[ \Theta_2^{(16)}(\vec{\xi} | \tau) + \Theta_1^{(16)}(\vec{\xi} | \tau) + \Theta_3^{(16)}(\vec{\xi} | \tau) + \Theta_4^{(16)}(\vec{\xi} | \tau) \right] \times \left[ \Theta_2^{(16)}(\vec{\xi}' | \tau) + \Theta_1^{(16)}(\vec{\xi}' | \tau) + \Theta_3^{(16)}(\vec{\xi}' | \tau) + \Theta_4^{(16)}(\vec{\xi}' | \tau) \right]$$

According to Eq.2.9, each term is a sum over the lattice $\Lambda_W^{(16)e,h} \oplus \Lambda_W^{(16)e,i}$ which is usually denoted $\Gamma^8$. Putting together left and right movers one finds

$$\chi^E(\tau | \vec{v}, \vec{\xi}) = \left( \frac{\Theta_1^{(8)}(\vec{v} | \tau)}{\Theta_1^{(8)}(\vec{v} | \tau)} \right)^* \frac{(\theta'(0 | \tau))^{4/3}}{\Theta_1^{(8)}(\vec{v} | \tau)} \times$$
\[
\frac{1}{2} (\theta'(0|\tau))^{-8/3} \left[ \Theta_2^{(16)}(\xi|\tau) + \Theta_1^{(16)}(\xi|\tau) + \Theta_3^{(16)}(\xi|\tau) + \Theta_4^{(16)}(\xi|\tau) \right] \times
\frac{1}{2} (\theta'(0|\tau))^{-8/3} \left[ \Theta_2^{(16)}(\xi'|\tau) + \Theta_1^{(16)}(\xi'|\tau) + \Theta_3^{(16)}(\xi'|\tau) + \Theta_4^{(16)}(\xi'|\tau) \right]
\]

(2.15)

Modular invariance is discussed in the same way as before, with the same additional factor as in Eq.2.13.

3 The Narain lattices of compactification

As is well known, the two heterotic string become T dual when compactified to nine dimensions. In this situation the relevant zero mode lattices are made up by appending to either $\Gamma^{16}$ or $\Gamma^8 \otimes \Gamma^8$ the even two dimensional Lorentzian lattice $\Pi^{1,1}$. The mathematical tool behind the T duality is that this embeds them both into the same even Lorentzian self-dual lattice $\Pi^{17,1}$. The practical realisation of this fact, which we will use, was put forward in ref.[7]. Starting from the Euclidean self-dual lattices just reviewed, the two embeddings are most directly described using two different sets of basis vectors. The one associated with $\Gamma^{(16)}$ will be described by $\vec{e}_i^{(16)}$, $i = 1, \ldots, 16$, $\vec{k}^{(16)}$, $\vec{\bar{k}}^{(16)}$, where the first sixteen are unit vectors, and the last two are null vectors orthogonal to the $e$’s such that $\vec{k}^{(16)} \cdot \vec{\bar{k}}^{(16)} = 1$. The one associated with $\Gamma^8 \otimes \Gamma^8$ is similar; it will be distinguished by a superscript (8). The transformation between the two basis will mix light-like vectors and Euclidean vectors.

Group theoretically, the light-like vectors correspond to extensions of the root systems either of $D_{16}$, or $E_8 \otimes E_8$. Let recall the principle of these extensions for completeness, following ref.[8]. Consider in general an ordinary simply laced Lie algebra, with roots $\vec{\alpha}_i$, $i = 1, \ldots, N$. The extended root system is defined by introducing the additional root $\vec{\alpha}_0 = -\sum_i n_i \vec{\alpha}_i + \vec{k}$, where $\vec{k}^2 = 0$, $\vec{\alpha}_i \cdot \vec{k} = 0$, and $\sum_i n_i \vec{\alpha}_i$ is the biggest root. The over extended root system involves yet another root $\vec{\alpha}_{-1} = -\left( \vec{k} + \vec{\bar{k}} \right)$, with another light-like vector $\vec{k}$, such $\vec{k} \cdot \vec{\bar{k}} = 1$. 
3.1 Embedding of $\Gamma^{16}$ into $\Pi^{17}$:

Given the orthonormal set of $e_i^{(16)}$ we may construct the simple roots of $D_{16}$ as

$$\pi_i^{(16)} = e_i^{(16)} - e_{i+1}^{(16)}, \quad i = 1, \ldots, 15, \quad \pi_{16}^{(16)} = e_{15}^{(16)} + e_{16}^{(16)}$$

Applying, the procedure just recalled gives the extensions

$$\pi_0^{(16)} = -(e_1^{(16)} + e_2^{(16)}) + \bar{k}^{(16)}, \quad \pi_{-1}^{(16)} = -(\bar{k}^{(16)} + \bar{\pi}^{(16)})$$

In order to arrive at the $\Pi^{17}$ Dynkin diagram, we introduce another vector

$$\pi_{17}^{(16)} = -\frac{1}{2} \sum_{i=1}^{16} e_i^{(16)} + \bar{k}^{(16)} - \bar{\pi}^{(16)}$$

Denote the Cartan matrix of $\Pi^{17}$ by $K_{\mu\nu} = \pi_\mu^{(16)} \cdot \pi_\nu^{(16)}$, with $\mu = -1, 0, 1, \ldots, 17$. Then, apart from $K_{\mu\mu} = 2$, the non zero entries above the diagonal are

$$K_{ii+1} = -1, \quad i = 1 \ldots 14, \quad K_{14,16} = -1$$

which, of course, give the diagram of $D_{16}$: and

$$K_{0,2} = -1, \quad K_{-1,0} = -1, \quad K_{17,16} = -1.$$ 

The Dynkin diagram is depicted on the following figure, where the original $D_{16}$ points are in white.

The vectors $\pi_i^{(16)}$, $i = -1, 0, 1, \ldots, 16$, are 18 linearly independent vectors, so that all others may be re-expressed in terms of them. Thus we may invert the transformation from the basis vectors to these root vectors. One finds

$$e_{16}^{(16)} = \frac{1}{2} \left( \pi_{16}^{(16)} - \pi_{15}^{(16)} \right), \quad e_{15}^{(16)} = \frac{1}{2} \left( \pi_{16}^{(16)} + \pi_{15}^{(16)} \right)$$

\(^6\)If you have a couloured picture, the extended root is in blue, and the overextended ones in red.
\[ e_i^{(16)} = \sum_{j=1}^{14} \vec{\pi}_j^{(16)} + \frac{1}{2} \left( \vec{\pi}_{16}^{(16)} + \vec{\pi}_{15}^{(15)} \right) , \quad i = 1, \ldots , 14. \]

\[ \overrightarrow{k}^{(16)} = \vec{\pi}_0^{(16)} + \sum_{i=1}^{16} n_i^{(16)} \vec{\pi}_i^{(16)} , \quad \vec{\pi}^{(16)} = -\vec{\pi}_{-1}^{(16)} - \vec{\pi}_0^{(16)} - \sum_{i=1}^{16} n_i^{(16)} \vec{\pi}_i^{(16)} \quad (3.1) \]

Moreover \( \vec{\pi}_{17}^{(16)} \) is not an independent vector. An explicit computation indeed shows that

\[ \vec{\pi}_{-1}^{(16)} + 2 \vec{\pi}_0^{(16)} + \frac{3}{2} \vec{\pi}_1^{(16)} + \sum_{j=2}^{14} \left( 4 - \frac{j}{2} \right) \vec{\pi}_j^{(16)} - \frac{3}{2} \vec{\pi}_{15}^{(15)} - 2 \vec{\pi}_{16}^{(16)} - \vec{\pi}_{17}^{(16)} = 0 \quad (3.2) \]

### 3.2 Embedding of \( \Gamma^8 \otimes \Gamma^8 \) into \( \Pi^{17,1} \):

We denote by \( \vec{\pi}_{i}^{(1,8)} \), \( \vec{\pi}_{j}^{(2,8)} \), \( i, j = -1, 0, 1 \ldots 8 \) the two sets of hyperextended simple roots of \( E_8 \). According to ref. \[7\] the same \( \Pi^{17,1} \) Dynkin diagrams comes out from two hyperextended \( E_8 \) Dynkin diagram glued together if we use the correspondence

\[ \vec{\pi}_{i}^{(1,8)} \leftrightarrow \vec{\pi}_{i}^{(16)} , \quad i = 1, \ldots , 6, \quad \vec{\pi}_7^{(1,8)} \leftrightarrow \vec{\pi}_0^{(16)} , \quad \vec{\pi}_8^{(1,8)} \leftrightarrow \vec{\pi}_{-1}^{(16)} \]

\[ \vec{\pi}_{i}^{(2,8)} \leftrightarrow \vec{\pi}_{16-i}^{(16)} , \quad i = 1, \ldots , 6, \quad \vec{\pi}_7^{(2,8)} \leftrightarrow \vec{\pi}_{16}^{(16)} , \quad \vec{\pi}_8^{(2,8)} \leftrightarrow \vec{\pi}_{17}^{(16)} \]

\[ \vec{\pi}_0^{(1,8)} \leftrightarrow \vec{\pi}_7^{(16)} , \quad \vec{\pi}_0^{(2,8)} \leftrightarrow \vec{\pi}_9^{(16)} , \quad \vec{\pi}_{-1}^{(1,8)} \leftrightarrow \vec{\pi}_8^{(16)} \quad (3.3) \]

The two diagrams have the \(-1\) point in common. Thus we write \( \vec{\pi}_{-1}^{(8)} \) instead of \( \vec{\pi}_{-1}^{(1,8)} \) or \( \vec{\pi}_{-1}^{(2,8)} \). This realisation is depicted in the following picture with the same (colour) conventions as the preceding one.

![Diagram](image)

We make use of the explicit realisation of ref.\[7\]:

\[ \vec{\pi}_i^{(1,8)} = e_i^{(8)} - e_{i+1}^{(8)} , \quad i = 1, \ldots , 6, \quad \vec{\pi}_7^{(1,8)} = -e_1^{(8)} - e_2^{(8)} , \quad \vec{\pi}_8^{(1,8)} = \frac{1}{2} \sum_{i=1}^{8} e_i^{(8)} \]
\[ \vec{\pi}^{(2,8)}_i = \vec{e}^{(8)}_{16-i} - \vec{e}^{(8)}_{17-i}, \ i = 1, \ldots, 6, \ \vec{\pi}^{(2,8)}_7 = \vec{e}^{(8)}_{15} + \vec{e}^{(8)}_{16}, \ \vec{\pi}^{(2,8)}_8 = -\frac{1}{2} \sum_{i=9}^{16} \vec{e}^{(8)}_i \]

\[ \vec{\pi}^{(1,8)}_0 = \vec{e}^{(8)}_7 - \vec{e}^{(8)}_8 + \vec{k}^{(8)}, \ \vec{\pi}^{(2,8)}_0 = \vec{e}^{(8)}_9 - \vec{e}^{(8)}_{10} + \vec{k}^{(8)}, \ \vec{\pi}^{(8)}_0 = -\vec{k}^{(8)} - \vec{\pi}^{(8)}_1 \]

Then one may verify that this set of vectors does generate the same \( \Pi^{17,1} \) Cartan matrix. Concerning the linear dependence, one may check that the (8) root vectors satisfy, for \( \ell = 1, 2, \)

\[ \vec{\pi}^{(\ell,8)}_8 + 2\vec{\pi}^{(\ell,8)}_7 + \frac{3}{2} \vec{\pi}^{(\ell,8)}_1 + \sum_{j=2}^{6} (4 - \frac{j}{2})\vec{\pi}^{(\ell,8)}_j + \frac{1}{2} \vec{\pi}^{(\ell,8)}_0 = \frac{1}{2} \vec{k}^{(8)}, \] (3.4)

so that

\[ \vec{\pi}^{(1,8)}_8 + 2\vec{\pi}^{(1,8)}_7 + \frac{3}{2} \vec{\pi}^{(1,8)}_1 + \sum_{j=2}^{6} (4 - \frac{j}{2})\vec{\pi}^{(1,8)}_j + \frac{1}{2} \vec{\pi}^{(1,8)}_0 = \vec{k}^{(8)} \]

\[ -\vec{\pi}^{(2,8)}_8 - 2\vec{\pi}^{(2,8)}_7 - \frac{3}{2} \vec{\pi}^{(2,8)}_1 - \sum_{j=2}^{6} (4 - \frac{j}{2})\vec{\pi}^{(2,8)}_j - \frac{1}{2} \vec{\pi}^{(2,8)}_0 = 0 \] (3.5)

This coincides with the transformed of Eq.3.2 through the correspondence 3.3.

### 3.3 The transformation between basis vectors:

Since the linear dependence of the extended root vectors is the same, we may look for a transformation between the basis vectors, such that the root vectors coincide following the correspondence displayed on Eq.3.3. Making use of Eq.3.1, one finds

\[ \vec{e}^{(16)}_i = \vec{e}^{(8)}_i, \ i = 10, \ldots, 16 \] (3.6)

\[ \vec{e}^{(16)}_9 = \vec{e}^{(8)}_9 + \vec{k}^{(8)}, \ \vec{e}^{(16)}_8 - \frac{1}{2} \vec{\pi}^{(16)}_1 = \vec{e}^{(8)}_8 - \vec{k}^{(8)} \] (3.7)

\[ \vec{e}^{(16)}_i - \frac{1}{2} \vec{\pi}^{(16)}_1 = \vec{e}^{(8)}_i, \ i = 1, \ldots, 7 \] (3.8)

\[ \vec{k}^{(16)} = 2 \left( \vec{e}^{(8)}_9 - \vec{e}^{(8)}_8 + \vec{k}^{(8)} - \vec{\pi}^{(8)}_1 \right) \] (3.9)

\[ \vec{\pi}^{(16)}_1 = -\frac{1}{2} \sum_{i=1}^{8} \vec{e}^{(8)}_i - 2 \left( \vec{e}^{(8)}_9 - \vec{e}^{(8)}_8 + \vec{k}^{(8)} - \vec{\pi}^{(8)}_1 \right) \] (3.10)
For future discussion, let us consider the transformation of a generic lattice point. Imposing that

\[ \sum_{\mu=1}^{16} m_{\mu}^{(16)} \vec{e}_{\mu}^{(16)} + m_0^{(16)} \vec{k}^{(16)} = \sum_{\mu=1}^{16} m_{\mu}^{(8)} \vec{e}_{\mu}^{(8)} + m_0^{(8)} \vec{k}^{(8)} + \bar{m}_0^{(8)} \vec{\bar{k}}^{(8)} \]

gives

\[ m_{\mu}^{(8)} = m_{\mu}^{(16)} - \frac{1}{2} \bar{m}_0^{(16)}, \quad \mu = 1, \ldots, 7, \quad (3.11) \]

\[ m_8^{(8)} = -\sum_{\mu=1}^{7} m_{\mu}^{(16)} - 2m_0^{(16)} + \frac{3}{2} \bar{m}_0^{(16)} \quad (3.12) \]

\[ m_9^{(8)} = \sum_{\mu=1}^{7} m_{\mu}^{(16)} + m_8^{(16)} + m_9^{(16)} + 2m_0^{(16)} - 2\bar{m}_0^{(16)} \quad (3.13) \]

\[ m_{\mu}^{(8)} = m_{\mu}^{(16)}, \quad \mu = 10, \ldots, 16, \quad (3.14) \]

\[ m_0^{(8)} = \sum_{\mu=1}^{7} m_{\mu}^{(16)} + m_9^{(16)} + 2m_0^{(16)} - 2\bar{m}_0^{(16)} \quad (3.15) \]

\[ \bar{m}_0^{(8)} = -\sum_{\mu=1}^{7} m_{\mu}^{(16)} - m_8^{(16)} - 2m_0^{(16)} + 2\bar{m}_0^{(16)} \quad (3.16) \]

Note, in particular that

\[ \sum_{\mu=1}^{8} m_{\mu}^{(8)} = -2 \left( m_0^{(16)} + \bar{m}_0^{(16)} \right) \quad (3.17) \]

\[ \sum_{\mu=9}^{16} m_{\mu}^{(8)} = \sum_{\mu=1}^{16} m_{\mu}^{(16)} + 2m_0^{(16)} - 2\bar{m}_0^{(16)} \quad (3.18) \]

One thus sees that, if \( m_0^{(16)} \), and \( \bar{m}_0^{(16)} \) are integer, and \( \sum_{\mu=1}^{16} m_{\mu}^{(16)} \) is even, \( \sum_{\mu=1}^{8} m_{\mu}^{(8)} \), and \( \sum_{\mu=9}^{16} m_{\mu}^{(8)} \) are also both even. Moreover, it follows from the above that \( m_0^{(8)} \), and \( \bar{m}_0^{(8)} \) are both integer. Clearly, by construction, we have the orthogonality relations

\[ \sum_{\mu=1}^{16} \left( m_{\mu}^{(16)} \right)^2 + 2m_0^{(16)} \bar{m}_0^{(16)} = \sum_{\mu=1}^{16} \left( m_{\mu}^{(8)} \right)^2 + 2m_0^{(8)} \bar{m}_0^{(8)}. \]

Let us denote by \( \Gamma^{(1,1)} \) the set of points \( pk + q\bar{k}, p, q \in \mathbb{Z} \). Then Eqs. 3.11–3.16 define a \( SO(17, 1) \) transformation such that \( \Gamma^8 \otimes \Gamma^8 \otimes \Gamma^{(1,1)} \leftrightarrow \Gamma^{16} \otimes \Gamma^{(1,1)} \).
### 3.4 Intermediate basis vectors:

Quite generally, given a set of basis vectors $\vec{e}_i, i = 1, \ldots, 16, \vec{k}$ and $\vec{\bar{k}}$, we get another set satisfying the same orthogonality relations by letting

$$\vec{e}'_i = \vec{e}_i - \beta_i \vec{k}, \quad \vec{\bar{k}}' = \sum_j \beta_j \vec{e}_j + \vec{k} - \frac{1}{2} \sum_j \beta_j^2 \vec{k}, \quad \vec{k}' = \vec{\bar{k}}.$$

where $\beta_i$ are arbitrary. Looking at the transformation formulae 3.6 – 3.10, one sees that such transformations do appear on each side with

$$\beta^{(16)}_i = \frac{1}{2}, \text{ for } i = 1, \ldots, 8, \quad \beta^{(16)}_i = 0, \text{ for } i = 9, \ldots, 16 \quad (3.19)$$

$$\beta^{(8)}_8 = -\beta^{(8)}_9 = 1, \quad \beta^{(16)}_i = 0, \text{ for } i \neq 9, 8 \quad (3.20)$$

Let us introduce, the corresponding basis vectors with a prime. Eqs.3.6 – 3.10 leads to

$$\vec{e}'^{(16)}_i = \vec{e}'^{(8)}_i, \quad i = 1, \ldots, 16, \quad \vec{k}'^{(16)} = -2\vec{\bar{k}}'^{(8)}, \quad \vec{k}'^{(16)} = -\frac{1}{2}\vec{k}'^{(8)}. \quad (3.21)$$

With these intermediate bases, the transformation reduces to simple exchange and rescaling of the light-like vectors.

Next, it is easy to determine the range of the transformed coordinates also distinguished by a prime. We will denote by $\Gamma^{(17,1)'}$ and $\Gamma^{(17,1)''}$ the corresponding lattices. They are best visualised using the $SO(16) \otimes SO(16)$ weight lattices obtained by separating the $m'_\mu$ variables with $\mu \neq 0$ into two sets made up with the eight first and with the eight last variables, respectively. One finds:

| $m_0^{(16)}$ | $m_0^{(16)}$ | $D_8 \otimes D_8$ weight space |
|--------------|--------------|---------------------------------|
| $\mathcal{Z}$ even | $\Lambda^{(8)ei} \otimes \Lambda^{(8)ei}$ | $\Lambda^{(8)eh} \otimes \Lambda^{(8)eh}$ |
| $\mathcal{Z}$ odd | $\Lambda^{(8)ei} \otimes \Lambda^{(8)eh}$ | $\Lambda^{(8)eh} \otimes \Lambda^{(8)ei}$ |
| $\mathcal{Z} + \frac{1}{2}$ even | $\Lambda^{(8)oi} \otimes \Lambda^{(8)oi}$ | $\Lambda^{(8)oh} \otimes \Lambda^{(8)oh}$ |
| $\mathcal{Z} + \frac{1}{2}$ odd | $\Lambda^{(8)oi} \otimes \Lambda^{(8)oh}$ | $\Lambda^{(8)oh} \otimes \Lambda^{(8)oi}$ |

$\mathcal{Z}$ denotes the set of non negative integers.
Let us turn to the transformation between the $m$’s. One verifies that

$$m_\mu'(16) = m_\mu'(8), \quad m_0'(16) = -\frac{1}{2}m_0'(8), \quad \bar{m}_0'(16) = -2\bar{m}_0'(8) \quad (3.22)$$

This is indeed a mapping $\Gamma^{(17,1)'}_{(16)} \leftrightarrow \Gamma^{(17,1)'}_{(8)}$. It corresponds to

| $m^{(8)'}_0$ | $\bar{m}^{(8)'}_0$ | $D_8 \otimes D_8$ weight space |
|-------------|-------------|--------------------------------|
| $Z$ even    | $\Lambda^{(8)ei} \otimes \Lambda^{(8)ei}$ | $\Lambda^{(8)eh} \otimes \Lambda^{(8)eh}$ |
| $Z$ odd     | $\Lambda^{(8)oi} \otimes \Lambda^{(8)oh}$ | $\Lambda^{(8)oh} \otimes \Lambda^{(8)oh}$ |
| $Z + \frac{1}{2}$ even | $\Lambda^{(8)ei} \otimes \Lambda^{(8)ei}$ | $\Lambda^{(8)eh} \otimes \Lambda^{(8)eh}$ |
| $Z + \frac{1}{2}$ odd | $\Lambda^{(8)oi} \otimes \Lambda^{(8)oh}$ | $\Lambda^{(8)oh} \otimes \Lambda^{(8)oh}$ |

The two lines in the middle are interchanged.

## 4 Compactified heterotic characters

### 4.1 Definition

The characters of course involve summations over the discrete momentum and winding number in a way similar to the purely bosonic and type II cases, described in ref. [1]. The left movers, and the space-time part of the right movers contributions are straightforwardly written down, following the line of this last reference. Thus Eq.2.11 are replaced by equations of the form

$$\chi^O(\tau, R | \vec{v}^{(3)}, \vec{\xi}) = \left(\theta'(0 | \tau)\right)^{2/3} \frac{\Theta^{(8)}(\vec{v}^{(3)} | \tau)}{\Theta^{(8)}(\vec{v}^{(3)} | \tau)} \chi_{(16)}(\tau R | \vec{\xi}),$$

with the same form for Eq.2.13 (for $\chi^E$) involving $\chi^{(8)}(\tau R | \vec{\xi})$ instead of $\chi_{(16)}(\tau R | \vec{\xi})$. For the following it is convenient to write both formulae at once with an index $\ell = 16$ for $SO(32)$, or 8 for $E_8 \otimes E_8$, respectively. We
have pulled out explicit modular invariant factors, so that when we come to it, we will only have to check the modular invariance of $\chi(\ell)$. In the above formulae, we have gone from 8 to 7 transverse dimensions. Choosing to compactify the eighth direction, the corresponding breaking of rotational invariance forces us to let $v_4 = 0$. We write $\vec{v}^{(3)} = \{v_1, v_2, v_3, 0\}$, and note $\vec{y}^{(3)}$, its transformed by triality, that is

$$
y^{(3)}_1 = \frac{1}{2}(v_1 - v_2 + v_3), \quad y^{(3)}_2 = \frac{1}{2}(-v_1 + v_2 + v_3),$$

$$
y^{(3)}_3 = \frac{1}{2}(v_1 + v_2 + v_3), \quad y^{(3)}_4 = \frac{1}{2}(-v_1 - v_2 + v_3),$$

Using the same convention as in ref.[1], we have let $\Theta^{(8)}_1(0|\tau) \equiv \theta_1(0|\tau) \prod^{3}_{\mu=1} \theta_1(v_\mu|\tau)$.

Let us now turn to the characters $\chi(\ell)$ that involve the summation over internal symmetry and compactified model lattices. Their definition is most natural using the lattices $\Gamma^{(17,1)'}(\ell)$. Indeed, the corresponding theories are explicitly invariant under $SO(16) \otimes SO(16)$, so that we may use the standard mathematical definition of characters which is based on the corresponding 16 dimensional Cartan algebra. Moreover, with this lattice, we have seen that the T duality transformations does not involve the internal symmetry modes. Duality will thus be very simply ensured. With these motivations, we introduce the definitions

$$
\chi(\ell)(\tau | R, \vec{\xi}) = \frac{1}{\sqrt{\tau - \tau^*}} \left( (\theta'_1(0|\tau))^* \right)^{-2/3} (\theta'_1(0|\tau))^{-18/3} \times \sum_{\Gamma^{(17,1)'}(\ell)} e^{i\pi \tau (m^{(\ell)}_0/2R + \bar{m}^{(\ell)}_0 R)^2 - \tau^* (m^{(\ell)}_0/2R - \bar{m}^{(\ell)}_0 R)^2} \frac{i\pi \tau}{2} \sum^{16}_{\mu=1} \left( m^{(\ell)}_\mu \right)^2 e^{2i\pi \sum^{16}_{\mu=1} m^{(\ell)}_\mu \xi_\mu}
$$

(4.1)

### 4.2 Duality

Let us use the mapping between the two lattices spelled out on Eqs.3.11 – 3.16. One gets after the corresponding change of variable,

$$
\chi(16)(\tau | R, \vec{\xi}) = \frac{1}{\sqrt{\tau - \tau^*}} \left( (\theta'_1(0|\tau))^* \right)^{-2/3} (\theta'_1(0|\tau))^{-18/3} \times
$$

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\[
\sum_{\Gamma^{(17),1'}} e^{i\pi [\tau(2m_0^{(8)} R + \bar{m}_0^{(8)}/4R)^2 - \tau^*(2m_0^{(8)} R - \bar{m}_0^{(8)}/4R)^2]} e^{i\pi \sum_{\mu=1}^{8} \left( m^{(8)}_\mu \right)^2} e^{2i\pi \sum_{\mu=1}^{16} m^{(8)}_\mu \xi_\mu},
\]
which gives
\[
\chi(16)(\tau | R, \xi) = \chi(8)(\tau | 1/4R, \xi).
\]

### 4.3 Modular invariance

Applying Eqs.2.9, it is straightforward to re-express \( \chi(\ell) \) in terms of \( \Theta_i^{(16)} \). The result is conveniently written under the form
\[
\chi(\ell)(\tau | R, \xi) = (\theta'(0|\tau))^{-16/3} \sum_{i,j,k} N_{i,j,k}^{(\ell)} F_i(R, \tau) \Theta_j^{(16)}(\xi | \tau) \Theta_k^{(16)}(\xi' | \tau)
\]
where \( \ell = 16, 8, \) and
\[
F_i(R, \tau) = \frac{1}{\sqrt{\tau - \tau^*}} \left| \frac{1}{\theta'(0|\tau)} \right|^{2/3} \sum_{m,n \in \Lambda_i^{(2)}} e^{i\pi(\tau - \tau^*)[m^2/4R^2+n^2R^2]} e^{-i\pi(\tau + \tau^*)mn}.
\]

The lattices \( \Lambda_i^{(2)}, i = 1, \ldots, 4 \) are two dimensional lattices of \( R^2 \), with points of coordinates \((m, n)\) that are (integer, even), (integer, odd), (half integer, even), (half integer, odd) respectively. The numerical coefficients \( N_{i,j,k}^{(\ell)} \), considered as matrices in the last two indices are defined as follows.

\[
N_1^{(16)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad N_2^{(16)} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}
\]
\[
N_3^{(16)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad N_4^{(16)} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}
\]
\[
N_1^{(8)} = N_1^{(16)}, \quad N_2^{(8)} = N_3^{(16)}, \quad N_3^{(8)} = N_2^{(16)}, \quad N_4^{(8)} = N_4^{(16)}
\]
4.3.1 The transformation $\tau \rightarrow -1/\tau$

The behaviour of the Theta functions is summarised by the matrix $S$ of Eq.2.12. After some computation, one verifies that, for both $\ell = 16, 8$,

$$\chi^{(\ell)}(\tau) = \sum_{\ell} S_{kj} SN^{(\ell)} S,$$

and

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad (4.4)$$

Concerning the functions $F_i$, They all may be deduced from the twisted characters.

$$F(R, \tau, \vec{\epsilon}, \vec{\eta}) = \frac{1}{\sqrt{\tau - \tau^*}} \left| \frac{1}{(\theta'_1(0|\tau))^{2/3}} \right|^2 \times \sum_{m,n\in\mathbb{Z}} e^{i\pi(\tau-\tau^*)(m+\epsilon_1/2)^2/4R^2+(n+\epsilon_2/2)^2/4R^2} e^{-i\pi(\tau+\tau^*)(m+\epsilon/2)(n+\epsilon_2/2)} e^{i\pi(m+n)}$$

where $\vec{\epsilon}, \vec{\eta}$ have components one or zero. Applying Poisson resummation technics, to this twisted character, one verifies that the functions $F_i$ satisfy

$$\mathcal{F}(R, \tau) = S\mathcal{F}(R, -1/\tau), \quad \text{where } \mathcal{F} \equiv \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \quad (4.5)$$

Finally, making use of Eqs.4, 12, the last equation, and the fact that $S^2 = 1$ one verifies that $\chi^{(\ell)}(\tau|R, \xi)$ is invariant under $\tau \rightarrow -1/\tau$.

4.3.2 The transformation $\tau \rightarrow \tau + 1$

The discussion is similar but simpler. One has

$$\Theta^{(2N)}_i(\vec{\xi} | \tau + 1) = T_{ij} \Theta^{(2N)}_j(\vec{\xi} | \tau), \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$TN^{(\ell)}_i T = T_{ij} N^{(\ell)}_j, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathcal{F}(\tau + 1) = T\mathcal{F}(\tau).$$

Modular invariance follows from $T^2 = 1$. 

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4.4 The decompactification limits

In Eq. (4.3), for \( R \to \infty \) only \( n = 0 \) may give a non vanishing contribution. Thus \( F_2 \), and \( F_4 \) vanish for \( R \to \infty \), and \( F_1 \sim F_3 \). Thus one gets, taking \( \ell = 16 \) for definiteness,

\[
\sum_{i,j,k} N^{(16)}_{i,jk} F_i(R, \tau) \Theta^{(16)}_j(\xi|\tau) \Theta^{(16)}_k(\xi'|\tau)
\]

\[
\to_{R \to \infty} F_1(\infty, \tau) \sum_{jk} \left( N^{(16)}_{1,jk} + N^{(16)}_{3,jk} \right) \Theta^{(16)}_j(\xi|\tau) \Theta^{(16)}_k(\xi'|\tau)
\]

\[
= F_1(\infty, \tau) \sum_i \Theta^{(16)}_i(\xi|\tau) \Theta^{(16)}_i(\xi'|\tau) = F_1(\infty, \tau) \sum_i \Theta^{(32)}_i(\xi|\tau).
\]

Indeed this gives back the lattice \( \Gamma^{(16)} \). On the other hand, for \( R \to 0 \), only \( m = 0 \) may give a non vanishing contribution. Thus \( F_3 \), and \( F_4 \) disappear, and \( F_1 \sim F_2 \). Thus one gets,

\[
\sum_{i,j,k} N^{(16)}_{i,jk} F_i(0, \tau) \Theta^{(16)}_j(\xi|\tau) \Theta^{(16)}_k(\xi'|\tau)
\]

\[
\to_{R \to 0} F_1(0, \tau) \sum_{jk} \left( N^{(16)}_{1,jk} + N^{(16)}_{2,jk} \right) \Theta^{(16)}_j(\xi|\tau) \Theta^{(16)}_k(\xi'|\tau)
\]

\[
= F_1(\infty, \tau) \sum_{i,j} \Theta^{(16)}_i(\xi|\tau) \Theta^{(16)}_j(\xi'|\tau).
\]

This gives back the character associated with the lattice \( \Gamma^{(8)} \otimes \Gamma^{(8)} \).

One sees that the characters we have introduced define a one parameter set of theories which interpolate between the uncompactified \( SO(32) \) heterotic string (obtained for \( R \to \infty \)) and the \( E_8 \otimes E_8 \) one (obtained for \( R \to 0 \)). These theories should be considered as obtained by a sort of twisted compactification, where a non trivial “coupling” is introduced between the various sectors of the compactified modes (the functions \( F_i \) and the \( SO(16) \) modular blocks \( \Theta_i \).

4.5 Return to the self-dual lattices.

Finally, we consider the theories compactified in the usual way, using the lattices \( \Gamma^{16} \) and \( \Gamma^8 \oplus \Gamma^8 \). The way to ensure T duality is to define the characters as equal to the ones just introduced, and to retransform back the
lattice summation variables to the variables $m^{(\ell)}_\mu$ using the formulae displayed in section 3. One finds

$$
\chi^{(\ell)}(\tau | R, \xi) = \frac{1}{\sqrt{\tau - \tau^*}} \left( (\theta'_1(0|\tau))^* \right)^{-2/3} (\theta'_1(0|\tau))^{-18/3} \times
\sum_{m_0^{(\ell)}, \bar{m}_0^{(\ell)} \in \mathbb{Z}} e^{i\pi[\tau(p_R^{(\ell)} - p_L^{(\ell)})^2 - \tau^* (\bar{p}_R^{(\ell)} - \bar{p}_L^{(\ell)})^2]} e^{2\pi i \left[ (\bar{m}^{(\ell)} + \bar{A}^{(\ell)} \omega^{(16)}) \xi \right]} \tag{4.6}
$$

\begin{align*}
(p_R^{(\ell)})^2 &= \left\{ \frac{1}{2} p^{(\ell)} - \frac{1}{4} \left( \bar{A}^{(\ell)} \right)^2 \omega^{(\ell)} - \frac{1}{2} \bar{A}^{(\ell)} \bar{m}^{(\ell)} + \omega^{(\ell)} \right\}^2 + \left( \bar{m}^{(\ell)} + \bar{A}^{(\ell)} \omega^{(16)} \right)^2, \\
(p_L^{(\ell)})^2 &= \left\{ \frac{1}{2} p^{(\ell)} - \frac{1}{4} \left( \bar{A}^{(\ell)} \right)^2 \omega^{(\ell)} - \frac{1}{2} \bar{A}^{(\ell)} \bar{m}^{(\ell)} - \omega^{(\ell)} \right\}^2, \\
A^{(16)}_\mu &= -1/2R, \quad \mu = 1, \ldots, 8, \quad A^{(16)}_\mu = 0, \quad \mu = 9, \ldots, 16, \\
A^{(8)}_\mu &= -1/R, \quad A^{(8)}_9 = 1/R, \quad A^{(8)}_\mu = 0, \quad \mu \neq 8, 9.
\end{align*}

Of course Wilson lines appear, and these formulae show how to define characters when non zero background gauge fields are turned on. Note the presence of a coupling between the $\xi$ variables and winding number: $\exp \left( 2\pi i \left( \bar{A}^{(\ell)} \bar{\omega}^{(\ell)} \right) \right)$.

At this point one should keep in mind that, although the characters are the same as in the intermediate theories, the string theories are of course different, since the mass operators is not invariant under $SO(17,1)$.

## 5 Outlook

In the same way as for type II superstrings the present study of Lie group characters gives an interesting insight at the perturbative level. The basic advantage, as compared to partition functions, is that the characters we have studied encode the full structure of the perturbative states. Moreover the $SO(17,1)$ that relate the weight lattices act naturally on them, giving a detailed relationship between perturbative spectra. This revealed the existence of the intermediate theories where T duality is simply the inversion of the compactification radius. It is possible that this theory will play a useful role in connecting the various string/M theories, since in that game the
two heterotic strings theories compactified on a circle are usually not distinguished, owing to their exchange by T duality although their structures are very different as we have seen.

We have studied the simplest compactification. Of course it would be interesting to study compactifications on higher dimensional manifolds. One may expect that a similar discussion could be carried out, but the needed mathematical properties are much more involved and much less developed.

So far we did not select out the BPS states. This may be done trivially, by simply retaining only the zero mode contribution in $\Theta^{(8)} (\vec{y} | \tau)$ in Eq.2.3. Then modular invariance is destroyed. Nevertheless, one might try to extend the character formulae (as well as the ones written in ref.[4]) to non perturbative states, so that S duality could be verified by change of variables in the summations that define them. This very strong requirement would open the way towards understanding the characters of M theory.

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A The Heterotic String

For completeness, I summarise here some elementary facts about heterotic strings in the light cone formulation. Notations are the same as in [5], except that it is notationally simpler to exchange left and right movers. Thus the left movers are described by the same operators $\tilde{\alpha}_n^\mu (\mu = 1 \ldots 8)$ and $S_n^a (n \in \mathbb{Z}, a O(8) spinor index)$ as the type II superstrings. The mass operator is

$$\tilde{N} = \sum_{n \geq 1} \left( \tilde{\alpha}_n^\mu \tilde{\alpha}_n^\mu + n \tilde{S}_n^a \tilde{S}_n^a \right).$$

The $SO(32)$ heterotic string revisited

For the right movers, we have the right-moving part of $X^\mu$ and $\lambda^A$, with $A = 1, \ldots 32$ where the latter are world sheet fermions. There are two sectors:
The (periodic) P sector: There
\[ \lambda^A(\sigma) = \sum_{\text{integer}} e^{-i n \sigma} \lambda^A_n, \quad [\lambda^A_n, \lambda^B_m]_+ = \delta_{A,B} \delta_{n,-m}. \]
The right-right matching condition is
\[ \tilde{N} = N - 1, \quad N = \sum_{n \geq 1} \left( \tilde{\alpha}^i_{-n} \tilde{\alpha}^i_n + n \lambda^A_{-n} \lambda^A_n \right). \]
One defines
\[ (-1)^F = \lambda_0 (\tilde{\alpha}^i_{-n} \tilde{\alpha}^i_n + n \lambda^A_{-n} \lambda^A_n) \]
where
\[ \lambda_0 = \lambda^1_0 \cdots \lambda^{32}_0 \]
and only keeps the eigenvalue +1.

The (antiperiodic) A sector: There
\[ \lambda^A(\sigma) = \sum_{r \text{ half integer}} e^{-i r \sigma} \lambda^A_r, \quad [\lambda^A_r, \lambda^A_s]_+ = \delta_{A,B} \delta_{r,-s}. \]
The matching condition is
\[ N = \tilde{N} - 1, \quad \tilde{N} = \sum_{n \geq 1} \left( \tilde{\alpha}^i_{-n} \tilde{\alpha}^i_n + r \lambda^A_{-r} \lambda^A_r \right). \]
One defines
\[ (-1)^F = (-1)^{\frac{1}{2}} \lambda^A_r \lambda^A_r \]
and only keeps the eigenvalue +1.

The \( E_8 \otimes E_8 \) heterotic string
The right sector is described by the same oscillators with different boundary conditions for the \( \lambda^A \) operators. Separate \( A = 1, \ldots, 16 \), denoted \( \lambda^A \) and \( A = 17, \ldots, 32 \) denoted \( \lambda^{tA} \). One assigns independently \( A \) and \( P \) boundary conditions to each set, and GSO projectors \( (-1)^F_i, i = 1, 2 \) for each set. The physical states are taken to have eigenvalue +1 for both parity charges. The level matching conditions are
\[
\begin{cases}
\tilde{N} = N - 1 & \text{in the } AA \text{ sector} \\
\tilde{N} = N & \text{in the } AP \text{ and } PA \text{ sectors} \\
\tilde{N} = N + 1 & \text{in the } PP \text{ sector}
\end{cases}
\]
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