Completely localized gravity with higher curvature terms

Ishwaree P Neupane

The Abdus Salam ICTP, Strada Costiera, 11-34014, Trieste, Italy
and
School of Physics, Seoul National University, 151-747, Seoul, Korea

Abstract

In the intersecting braneworld models, higher curvature corrections to the Einstein action are necessary to provide a non-trivial geometry (brane tension) at the brane junctions. By introducing such terms in a Gauss-Bonnet form, we give an effective description of localized gravity on the singular delta-function branes. There exists a non-vanishing brane tension at the four-dimensional brane intersection of two 4-branes. Importantly, we give explicit expressions of the graviton propagator and show that the Randall-Sundrum single-brane model with a Gauss-Bonnet term in the bulk correctly gives a massless graviton on the brane as for the RS model. We explore some crucial features of completely localized gravity in the solitonic braneworld solutions obtained with a choice ($\xi = 1$) of solutions. The no-go theorem known for Einstein’s theory may not apply to the $\xi = 1$ solution. As complementary discussions, we provide an effective description of the power-law corrections to Newtonian gravity on the branes or at the common intersection thereof.

PACS numbers: 04.50.+h, 11.10.Kk, 11.25.Mj

1 Introduction

In recent years, it has been understood that the brane-world models of non-compact extra dimensions [1] reproduce the usual Einstein gravity. This idea was quickly generalized to a discrete patch of $AdS_{4+N}$ space accommodating $N$ intersecting $(N + 2)$ branes [2]. Later, the RS singular 3-brane model with perturbations of bulk geometry was fully investigated in [3, 4, 5], and interesting discussion of the 'crystal' multi-braneworld solutions appeared in [6]. A class of solitonic brane solutions arising from a special fine tuning between the bulk cosmological term and the Gauss-Bonnet coupling was given in [7]. The behavior of gravity localized on branes or string-like defects in $N \geq 1$ dimensions was studied in [8]. These constructions have been useful in providing some realizations of trapped four-dimensional gravity to a brane embedded in an AdS space. It could be possible to localize ordinary matter and gauge fields to this $(3 + 1)$ dimensional manifold if the RS brane is a $D3$-brane-like object of string/M theory. Since a $D3$-brane in the string theory background is necessarily a solitonic object, and only the higher curvature corrections can provide a non-trivial background at the brane junction, we find it interesting to study intersecting brane-world configurations by including a Gauss-Bonnet term.
In the intersecting brane configurations of \([2, 6, 9, 5]\), a 3-brane given by the four-dimensional intersection of \((2 + N)\)-branes \((N \geq 2)\) with larger co-dimensions has been viewed as our universe. In this picture, though each extra dimension has an infinite extension, the effective size of \(N\) non-compact extra dimensions is supposed to be finite, which could be as large as \(1\mu m\) (for \(N \geq 2\)) or much larger than their natural scale \(\ell \gg \ell_{Pl}^{-1} \sim 10^{-33}\text{cm}\) \([10, 2]\). While taking a Gauss-Bonnet coupling into account, the effective four-dimensional Planck mass reads \(M_{\text{eff}}^2 = M_{*}^{2+N} V_N\), where the effective volume of \(N\) extra dimensions is

\[
V_N = \left(\frac{2}{k}\right)^N \frac{(1 + \xi)}{(N + 1)!}.
\]

Here \(\xi\) is a coupling constant arising from a GB term, and the constant \(k\) has dimension of inverse AdS length. One of the motivations here to include a GB coupling is that the low energy version of certain string theories (such as type I heterotic strings) admits the quadratic curvature corrections as a GB invariant in their effective actions \([11]\). It is remarkable that the inclusion of a GB term can consistently explain the basic features of the Randall-Sundrum type brane-world models in \(D = 5\) \([12, 13, 14]\). Some other aspects of Randall-Sundrum brane models supplemented with higher curvature terms were studied in recent papers, including \([15, 16, 17, 18, 19, 20, 21]\). Here we construct brane-world solutions for \(N \geq 1\) with a Gauss-Bonnet term. Indeed, the inclusion of a GB term in the bulk might shed more light towards the understanding of the RS models in the \(AdS_{4+N}\) spaces. One of the interesting observations is that a GB term does allow a non-trivial four-dimensional brane tension when there are two extra transverse directions. Also a Gauss-Bonnet term could be useful to circumvent some of the arguments given for the no-go theorem \([22, 23]\) applicable to the braneworld models based on the Einstein action.

The rest of the paper is organized as follows. In section 2, we give a general set-up of the effective braneworld action, and present a framework for intersecting braneworld configurations with a GB term. Section 3 explains the basic features of the linear perturbation equations in \((4 + N)\) dimensions. In section 4, we derive a relation between the effective four-dimensional Planck mass and \((4 + N)\) dimensional fundamental mass scale. In section 5, we give the expressions for graviton propagators in \(AdS_{4+N}\) space, and discuss some salient features of gravity localized on the branes. In section 6, we elucidate upon the braneworld solutions in five dimensions, which exhibit that \(\xi = 1\) is indeed a physical choice. We then give in section 7 a complementary description of Newton potential corrections for solitonic branes. In section 8, we briefly comment upon the scalar and vector modes of the metric fluctuations. Section 9 contains conclusion.

## 2 Intersecting branes setup

In the low energy effective action of the Horova-Witten type I heterotic string theory \([24]\), there arise additional interactions of the gauge fields and higher powers of the curvatures. In particular, there is
$R^2$ interaction in the Gauss-Bonnet combination [25]. In many cases a contribution of higher curvature terms is neglected as it is higher order in derivatives, but in the warped braneworld scenario in $D \geq 5$, we see the relevance of these terms if they are expressed in a Gauss-Bonnet form. We will not stick here to the physical origin of a Gauss-Bonnet term. Rather, we will try to explore the physical importance of this term in warped backgrounds. We shall start with the following braneworld effective action in $D = (3 + N) + 1$-dimensional space-time ($B$), where $\partial B$ represents the $(3 + N)$-dimensional boundary,

$$S = \int_B d^{4+N}x \sqrt{-g_{4+N}} \left( \frac{R}{\kappa} - 2\Lambda + \alpha \left( R^2 - 4RPQRPQ + R_{MNQP}R^{MNQP} \right) \right)$$

$$+ \sum_{i=1}^{N} \int_{\partial B} d^{3+N}x \sqrt{-g_{3+N}} (-\Lambda_i)$$

$$+ \sum_{i \neq j} \int_{i,j\text{'th} \text{brane}} d^{2+N}x \sqrt{-g_{2+N}} (-\Lambda_{i,j})$$

$$+ \int d^4x \sqrt{-g_{z1=0,...,z_n=0}} (-\lambda).$$

(2)

Here $P, Q, \ldots = 0, 1, \ldots, 3 + N$, $x^\mu (\mu = 0, \ldots, 3)$ parametrize brane coordinates with Lorentzian signatures and $z^i (i = 1, \ldots, N)$ parametrize extra non-compact dimensions. The action (2) characterizes an array of $N$ orthogonal $(N+2)$-spatial dimensional branes in $(4+N)$ dimensions. $\Lambda$ is the $D$-dimensional bulk cosmological term, $\Lambda_i$ are the brane tensions of the $N$ intersecting $(N+2)$-branes, $\Lambda_{i,j}$ are $(N+1)$-brane tensions, and $\lambda$ is the four-dimensional brane tension at the common intersection of higher dimensional branes. The coupling $\alpha$ has the mass dimension of $M_*^N$, and $\kappa = 16\pi G_{4+N} \equiv M_*^{2+N}$.

The graviton equations of motion derived by varying the action (2) with respect to $g^{MN}$ take the following form

$$\sqrt{-g_{4+N}} (G_{MN} + \kappa H_{MN} + \kappa \Lambda g_{MN}) = -\frac{\kappa}{2} \sum_{i=1}^{N} \Lambda_i \sqrt{-g_{3+N}} \delta(z_i) \delta_M g^R_{pq} \delta_R^q g(z_i=0)$$

$$- \frac{\kappa}{2} \sum_{i \neq j} \Lambda_{i,j} \sqrt{-g_{2+N}} \delta(z_i) \delta(z_j) \delta_M^r \delta_N^s g^r_{st} (z_i,z_j=0)$$

$$- \frac{\kappa \lambda}{2} \sqrt{-g_{4}} \delta(z_1) \delta(z_2) \cdots \delta(z_N) \delta_M^{\mu} \delta_N^{\nu} g_{\mu\nu}^{z_1,z_2,\cdots,z_N=0};$$

(3)

where $H_{MN}$, an analogue of the Einstein tensor stemmed from the GB term, reads

$$H_{MN} = -\frac{\alpha}{2} g_{MN} \left( R^2 - 4RPQRPQ + R_{PQRS}R^{PQRS} \right)$$

$$+ 2\alpha \left[ RR_{MN} - 2RM_{PQ}R^{PQ} + R_{MPQR}R_N^{PQR} - 2R_M^P R_{NP} \right].$$

(4)

In equation (3), the indices $(p, q)$ take $(3 + N)$ possible values, while $(r, s)$ take only $(2 + N)$ possible values, and $\mu, \nu = 0, 1, 2, 3$. The $(2 + N)$-brane tensions $\Lambda_i$ multiply only $\delta(z_i)$ (i.e. the terms involving only one delta function), $\Lambda_{i,j}$ multiply two delta functions, and the brane tension at the common intersection ($\lambda$) multiplies a product of delta functions $\delta(z_1), \cdots, \delta(z_N)$. In $D = 7$, for example, $\lambda$
multiplies $\delta(z_1)\delta(z_2)\delta(z_3)$. One observes that when $N = 2$, $\Lambda_{1,2}$ is replaced by $\lambda$, so that $\lambda$ can have a non-zero value at the intersection of two 4-branes as we exhibit below.

We write a $(3+N)+1$-dimensional metric as

$$ds^2 = e^{-2A(z)} \left( g_{\mu\nu}(x^\lambda) dx^\mu dx^\nu + g_{ij}(z) dz^i dz^j \right).$$  \hspace{1cm} (5)

We assume that the four-dimensional spacetime is Minkowski. Then we may express the metric (5) in the following form, which represents the Poincaré half parameterization of $AdS_{4+N}$ space [2],

$$ds^2_{4+N} = \ell^2 \left( \eta_{\mu\nu} dx^\mu dx^\nu + d\bar{z}^2 + d\Omega^2_{N-1} \right).$$  \hspace{1cm} (6)

The length scale $\ell$ is fixed by the $(4+N)$-dimensional bulk cosmological term $\Lambda$. From the exact non-linear analysis in $(4+N)$-dimensions we can find the most general solution to modified Einstein field equations, admitting a normalisable 4$^d$ graviton. This solution is $A(z) = \log \left( \sum_{i=1}^{N} k_i |z_i| + 1 \right)$, where $z_i = z_1, z_2, \cdots$ count extra spaces and $k_i = k_1, k_2, \cdots$ are the inverse of $AdS$ curvature radii. The choice $k_1 = k_2 = \cdots = k$ further keeps the metric (6) manifestly symmetric under permutations of all extra $N$ dimensions, and the length scale $L \equiv (V_N)^{1/N}$ can be interpreted as the compactification size of $N$ extra dimensions [2]. The metric solution then becomes

$$ds^2_{4+N} = \frac{1}{(k \sum_{i=1}^{N} |z_i| + 1)^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu + \sum_{j=1}^{N} (dz^j)^2 \right).$$  \hspace{1cm} (7)

Here $k \equiv (\sqrt{n} \ell)^{-1}$. Since the coordinates $z^j$ parametrize the extra dimensions, the full bulk space ($z^j \neq 0$) comprises $2^N$ identical patches of the $AdS_{4+N}$ space, which are glued together along the branes ($z^j = 0$). Here we have taken AdS space as the simplest choice of the bulk spacetime. And branes are characterized by the delta-function-like singularities, which arise from the second derivative of the warp factor $A(z)$. However, there might exist other possibilities of an AdS bulk, such as the AdS Schwarzschild black-hole spacetime considered in [26]. It is also plausible that within the AdS/CFT correspondence gravity can be trapped on the brane in a manner similar to that in the original RS scenario [1], when the brane is embedded into the AdS Schwarzschild bulk [27]. It would be interesting to study the gravitational perturbations in such a black-hole background.

We have assumed that the warp factor $A(z)$ is a function of all transverse coordinates, that is, $z^i$

$$A(z) = \log \left( \sum_{i=1}^{N} k |z_i| + 1 \right).$$  \hspace{1cm} (8)

Then a straightforward calculation yields

$$A'' = e^{-2A(z)} k^2 \sum_{i=1}^{N} (\partial_{z_i} |z|)^2 = e^{-2A(z)} k^2 N, \hspace{1cm} (9)$$

$$A''' = -e^{-2A(z)} k^2 N + e^{-A(z)} 2k \sum_{i=1}^{N} \delta(z_i), \hspace{1cm} (10)$$

Here $k \equiv (\sqrt{n} \ell)^{-1}$. Since the coordinates $z^j$ parametrize the extra dimensions, the full bulk space ($z^j \neq 0$) comprises $2^N$ identical patches of the $AdS_{4+N}$ space, which are glued together along the branes ($z^j = 0$). Here we have taken AdS space as the simplest choice of the bulk spacetime.
where primes represent differentiation with respect to \( z \). The metric solution, in terms of the warp factor \( A(z) \), therefore satisfies a relation \( A'' + A^2 = e^{-A(z)} 2k \delta(z) \geq 0 \). Here, we simply note that this condition is equivalent to the weak energy condition one derives from \( G_i^i - G_z^z = \kappa (T_i^i - T_z^z) = (N + 2) (A'' + A^2) \geq 0 \).

In order to exhibit localized behavior of gravity to the brane intersections, one may look at the linear perturbations about the background given by (7). For this purpose, we perform a conformal transformation \( g_{MN} = e^{-2A(z)} \tilde{g}_{MN} \), where \( A(z) = \log(k \sum_i |z_i| + 1) \). Then, we adopt a background subtraction technique as in [9]. Being specific, we use the linear equation \( \delta \tilde{G}_{MN} + \kappa \delta \tilde{H}_{MN} = 0 \), where \( \delta \tilde{G}_{MN} = \delta G_{MN} - \tilde{G}_{MP} h_N^P \), \( \delta \tilde{H}_{MN} = \delta H_{MN} - \tilde{H}_{MP} h_N^P \). Note that \( \tilde{G}_{MN} = \kappa \tilde{T}_{MN} \) gives the background Einstein equations. What we are equating here with \( \tilde{G}_{MN} \) are only the vacuum energy contributions to the energy-momentum tensor. In our approach, the Randall-Sundrum-type fine tunings will only be implicit. If we wish, we can see them by considering the difference between \( (\delta G_{\mu \nu} + \kappa \delta H_{\mu \nu}) \) and \( (\delta \tilde{G}_{\mu \nu} + \kappa \delta \tilde{H}_{\mu \nu}) \). This gives after some simplifications,

\[
-\frac{1}{2} \left[ \tilde{R} h_{\mu \nu} - 2 \tilde{R}_\mu^\lambda h_{\nu \lambda} - \frac{\alpha \kappa}{2} \left( \tilde{R}^2 - 4 \tilde{R}_{MN}^2 + \tilde{R}_{MN}^2 \right) h_{\mu \nu} \right. \\
\left. - 4 \left( \tilde{R} \tilde{R}_\mu^\lambda - 2 \tilde{R}_\mu^P \tilde{R}_P^Q + 2 \tilde{R}_\mu S \tilde{R}^{\lambda S} + \tilde{R}_\mu PQR \right) h_{\nu \lambda} \right]. 
\]  

(11)

One could fine-tune the contributions coming from these terms to the vacuum energy contributions on the branes, that is, the source terms on the right-hand side of equation (3). For this purpose, we parameterize \( h_{\mu \nu} = e^{(N+2)A(z)/2} \tilde{h}_{\mu \nu} \) and obtain

\[
\left[ N(N + 2)(N + 3) k^2 e^{-2A} (1 - \xi/2) - 8(N + 2) k e^{-A} (1 - \xi/3) \sum_{i=1}^N \delta(z_i) \\
- 16(N + 1)(N + 2) \alpha \kappa k^2 \sum_{i \neq j} \delta(z_i) \delta(z_j) \right] \tilde{h}_{\mu \nu}(x, z) = \\
- 2 \kappa \left[ e^{-2A(z)} \Lambda + e^{-A(z)} \sum_{i=1}^N \Lambda_i \delta(z_i) + \sum_{i \neq j} \Lambda_{i,j} \delta(z_i) \delta(z_j) \right] \tilde{h}_{\mu \nu}(x, z),
\]  

(12)

where \( \xi = 2N^2(N + 1) \alpha' \kappa M_*^{-2} k^2 \) and we have replaced \( \alpha \) by \( M_*^N \alpha' \), so that \( \alpha' \) is a dimensionless Gauss-Bonnet coupling. Here, it might be relevant to note that in the full non-linear analysis the fine-tuned relations emerge as boundary conditions to be satisfied on the brane(s), while the bulk part amounts to express \( \Lambda \) in terms of \( k^2 \); see [17] for the \( N = 2 \) case. For arbitrary \( N \), we obtain

\[
\Lambda = - \frac{N(N + 2)(N + 3) k^2 (1 - \xi/2)}{2 \kappa}, 
\]  

(13)

\[
\Lambda_i = \frac{4(N + 2) k (1 - \xi/3)}{\kappa}, 
\]  

(14)

\[
\Lambda_{i,j} = 8(N + 1)(N + 2) \alpha' M_*^2 k^2. 
\]  

(15)

For \( N = 2 \), one may replace the brane tension \( \Lambda_{1,2} \) by \( \lambda \), because there is only one four-dimensional
brane intersection for two 4-branes that mutually intersect. A clear message is that the brane tension \( \Lambda_{1,2} \) is non-vanishing for \( \alpha' > 0 \), but it vanishes for the Einstein gravity (\( \alpha = 0 \)).

We may perturb the background metric as

\[
ds_{4+N}^2 = \frac{1}{(k \sum_{i=1}^{N} |z_i| + 1)^2} \left( (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu \, dx^\nu + \sum_{j=1}^{N} (dz_j)^2 \right).
\]

In this gauge, however, there may be some additional gravitational degrees of freedom coming from off-diagonal components (\( h_{\mu m} \)) and diagonal components (\( h_{mm} \)) of the perturbed equations for \( h_{MN} \), where \( m \) could run from \( z_1 \) to \( z_N \). We will comment upon these modes in the section 8. One may also use the approximation where the tensor modes either decouple from the perturbed equations for \( h_{\mu m} \) and \( h_{mm} \), or only the non-vanishing components of the fluctuations are \( h_{\mu\nu} \) [2, 5]. In fact, in the ordinary two-derivative gravity supplemented by a GB term there are no extra degrees of freedom than that of Einstein’s theory.

We find reasonable to study first the linear tensor fluctuations, \( h_{\mu\nu} \) in the gauge \( h_{\mu\nu} = 0 \), \( \partial^\lambda h_{\lambda\mu} = 0 \). The linearized equations for \( h_{\mu\nu} \) then take the following form

\[
\frac{1}{\kappa} \left[ -\Box_4 - \Box_z + (N + 2) k \, e^{-A(z)} \sum_{i=1}^{N} sgn(z_i) \partial_{z_i} \right] h_{\mu\nu} \\
+ 2\alpha N \left[ N(N + 1) k^2 \left( \Box_4 + \Box_z \right) - 4k \, e^{A(z)} \sum_{i=1}^{N} \delta(z_i) \Box_4 - 4k \, e^{A(z)} \sum_{i \neq j}^{N} \delta(z_i) \partial_{z_j}^2 \right] \\
+ (N + 1) k^2 \left( 4 \sum_{i=1}^{N} \delta(z_i) - N(N + 2) k \, e^{-A(z)} \right) \sum_{j=1}^{N} sgn(z_j) \partial_{z_j} \right] h_{\mu\nu} = 0
\]

where \( \Box_z = \partial_{z_1}^2 + \partial_{z_2}^2 + \cdots + \partial_{z_N}^2 \). When \( \alpha = 0 \), our results will reproduce the results appeared in [2]. But here we find more interesting new features which are unavailable if the gravity action does not contain higher curvature corrections.

### 3 Localized gravity in \((4 + N)\)-dimensions

One can remove from equation (17) the single (linear in) derivative term from first square bracket by re-scaling the metric: \( h_{\mu\nu} = e^{(N+2)A(z_i)/2} \tilde{h}_{\mu\nu} \). We should note, however, that single derivatives of \( \tilde{h}_{\mu\nu} \), and terms like \( \delta(z) \, sgn(z) \partial_z \) still survive from the second square bracket in (17). Fortunately, suitable combinations of these terms satisfy the necessary jump condition(s) on the brane(s). In fact, such terms do not appear if one has started only with the Einstein action, i.e. \( \alpha = 0 \). We can make a change of variables \( \tilde{h}_{\mu\nu}(x, z) \equiv \tilde{h}_{\mu\nu}(x) \psi(z) = \epsilon_{\mu\nu} e^{ipx} \psi(z) \), where \( \epsilon_{\mu\nu} \) is the constant polarization tensor of the graviton wave function, and \( p^2 = -q^2 \equiv m^2 \). Then, to the leading order,

\[
S \sim \int d^N z \, |\psi(z)|^2 \int d^4 x \, \partial^\lambda \tilde{h}_{\mu\nu}(x) \partial_\lambda \tilde{h}_{\mu\nu}(x) + \cdots.
\]
This actually implies that (i) $\psi(z)$ is the conventional quantum-mechanical wavefunction, and (ii) if we had not chosen the coordinates in a conformally flat form, the kinetic terms in the $x$ and $z$ directions would have had different conformal factors, but the redefinition of the metric $h_{\mu\nu} = e^{(N+2)A(z)/2} \tilde{h}_{\mu\nu}$ puts the action in the canonical form by absorbing the conformal factors. This is similar to what happens in a model without the Gauss-Bonnet interaction term [5].

The metric fluctuations in terms of $\tilde{h}_{\mu\nu}$ take the following form

$$\frac{1}{\kappa} \left[ -\Box - (N + 2) k e^{-A(z)} \sum \delta(z_i) + \frac{N(N+2)(N+4)k^2}{4} e^{-2A(z)} \right] \tilde{h}_{\mu\nu}$$

$$+ N \alpha e^{2A(z)} \left[ 2N(N-1)k^2 e^{-2A(z)} (\Box_4 + \Box) - 8ke^{-A(z)} \sum \delta(z_i) \Box_4 \right]$$

$$+ 8k^2 e^{-2A(z)} \left( (N+1) \sum \text{sgn}(z_i) \delta(z_i) \partial_{z_i} - \sum \delta(z_j) \text{sgn}(z_i) \partial_{z_i} \right)$$

$$- 8ke^{-A(z)} \sum_{i \neq j} \delta(z_i) \delta(z_j) - 4(N+2)k^2 e^{-2A(z)} \left( \frac{N^2(N+1)(N+4)}{8} k^2 e^{-2A(z)} \right)$$

$$+ 2 \sum_{i \neq j} \delta(z_i) \delta(z_j) - (N^2 + 2) e^{A(z)} \sum_i \delta(z_i) \right] \tilde{h}_{\mu\nu} = 0. \quad (19)$$

Here we have defined $\Box_4 \tilde{h}_{\mu\nu} = m^2 \tilde{h}_{\mu\nu}$. In terms of $\psi(z)$, we may express the above equation in the form

$$(1 - \xi) \left[ -m^2 - \Box - (N + 2) k e^{-A(z)} \sum_{i=1}^N \delta(z_i) + \frac{N(N+2)(N+4)k^2}{4(k \sum_i |z_i| + 1)^2} \right] \psi(z)$$

$$+ 8N \alpha k \kappa e^{A(z)} \left[ - \sum_{i=1}^n \delta(z_i) m^2 - \sum_{i \neq j} \delta(z_i) \partial_{z_j}^2 - e^{A(z)} k \left( \sum_{i \neq j} \delta(z_j) \text{sgn}(z_i) \partial_{z_i} \right) \right. $$

$$- (N+1) \sum_{i=1}^N \delta(z_i) \text{sgn}(z_i) \partial_{z_i} \left. + e^{2A(z)} \frac{(N+2)(N^2 - N + 4)k^2}{4} \sum_{i=1}^n \delta(z_i) \right] \psi(z)$$

$$- 8N(N+2) \alpha k^2 \sum_{i \neq j}^N \delta(z_i) \delta(z_j) \psi(z) = 0, \quad (20)$$

where $\xi = 2N^2(N+1)\alpha k^2$. The modified Einstein equations in the bulk amount to giving a solution for $k^2$:

$$1 - \xi = \mp \sqrt{1 + \frac{8N(N+1)\alpha k^2 \Lambda}{(N+2)(N+3)}}. \quad (21)$$

An interesting special case arises for $k = M_* \sqrt{2N^2(N+1)\alpha}$ (and hence $\xi = 1$), where the two branches of the bulk solution (21) coincide. Of course, one require $\alpha > 0$ to ensure $k > 0$. Moreover, a fine tuning condition between the bulk cosmological term and the GB coupling $\alpha$,

$$\Lambda = -\frac{(N+2)(N+3)}{8N(N+1)\alpha k^2}. \quad (22)$$
is useful to study the localization of gravity to a solitonic 3-brane \[7\]. For \(N = 2\), the choice \(\xi = 1\) allows one to study a solitonic 3-brane solution with the supersymmetric interpretation \[7\]. The readers will note that \((1 - \xi) = 0\) implies trivial corrections to the graviton propagator in \((4 + N)\) dimensions, since all sub-leading-order corrections involve this factor. But one still recovers the RS-type braneworld solutions in one lower dimensions (i.e., in \(3 + N\) dimensions). In fact, the choice \(\xi = 1\) helps us to study gravity confined to a solitonic brane. It is interesting that the solutions with \(\xi = 1\) have structures which are similar to the original Randall-Sundrum braneworld solutions.

4 Finiteness of Newton’s constant

A finite four-dimensional gravitational coupling (or the Planck mass) ensures that the ordinary four-dimensional graviton is present as a massless and normalizable zero-mode solution of the theory. In the RS three-brane model of co-dimension one, the four dimensional Newton constant is determined to be

\[
\frac{1}{G_{N}^{(4)}} \sim M_{Pl}^{2} \simeq M_{*}^{3} \int_{0}^{\infty} dz e^{-2A(z)} \left( A^{2} - A'' \right)
\]

(23)

where \(\xi = 4\alpha\kappa k^{2}A^{2}\). The last term \(e^{-A(z)}A'_{0}^{\infty}\) has to be finite for \(G^{(4)}_{N}\) to be finite. This is finite for a solution of the type \(A(z) = \log(|z|/L + 1)\), since \(A'(0_{+}) = 1/L\) and the background is invariant under \(z \rightarrow -z\) symmetry.

Next consider the general case where \(N\) is arbitrary. In this case, one can find the four-dimensional Planck mass by reading off the coefficients of \(-m^{2}\) in the linearized expression, equation (19) or equation (20), and then integrating over the extra dimensions. The result is

\[
M_{Pl}^{2} = M_{*}^{2+N} \int_{-\infty}^{+\infty} d^{N}z e^{-(N+2)A(z)} \left( 1 - \xi + 8N\alpha\kappa k e^{A(z)} \sum_{i=1}^{N} \delta(z_{i}) \right)
\]

(24)

We will shortly justify this result by integrating the 4\(d\) part of the metric, as in the original RS model. For this purpose, one may replace \(\eta_{\mu\nu}\) in equation (7) by \(g_{(4)}^{(4)}(x)\), insert the metric into the action (2), and finally integrate over the \(z_{i}\) coordinates. The net result is that

\[
S_{eff} = \tilde{M}_{Pl}^{2} \int d^{4}x \sqrt{-\tilde{g}^{(4)}} \left[ \tilde{R} + \tilde{\alpha}(\tilde{R}^{2} - 4\tilde{R}_{\mu\nu}^{2} + \tilde{R}_{\mu\nu\lambda\rho}^{2}) \right] + \text{fine-tuned terms},
\]

(25)

where the effective 4\(d\) Planck mass \(\tilde{M}_{Pl}\) and \(\tilde{\alpha}\), the 4\(d\) analog of the \(D\)-dimensional GB coupling \(\alpha\), are defined by

\[
\tilde{M}_{Pl}^{2} = M_{*}^{2+N} \int_{-\infty}^{+\infty} d^{N}z e^{-(N+2)A(z)} \left[ 1 - 2N(N + 1)(N + 2)\alpha\kappa k^{2} \right]
\]

\[+8(N + 1)\alpha\kappa k e^{A(z)} \sum_{i=1}^{N} \delta(z_{i}) \]
\[ M_{s}^{2+N} \frac{2^{N}}{(N+1)!} (1 + \xi) k^{-N}, \] \hspace{1cm} (26)

and,
\[ \hat{\alpha} = \alpha \frac{M_{s}^{2+N}}{M_{Pl}^{2}} \int_{-\infty}^{+\infty} d^{N}z \ e^{-N A(z)}. \] \hspace{1cm} (27)

Observe that to ensure a positivity of the four-dimensional graviton coupling (i.e. \( G_{4} > 0 \)) and to preserve unitarity (i.e. a definite positive-norm state) condition a with GB term, one must take \((1 + \xi) > 0\) and \(\alpha > 0\).

The four-dimensional massless graviton corresponding to a bound state with the wavefunction, up to a normalization factor, is
\[ \psi_{\text{bound}} \sim e^{-(N+2) A(z)/2}. \] \hspace{1cm} (28)

For the massless graviton mode \(m^{2} = 0\), the second square bracket in equation (20) then gives a non-trivial term
\[ 8N(N+2) \alpha \kappa k^{2} \sum_{i \neq j} \delta(z_{i}) \delta(z_{j}) \psi(z). \] \hspace{1cm} (29)

This exactly cancels with the last term in equation (20). Thus, the massless mode of the graviton fluctuation is unaffected by a Gauss-Bonnet action. And for \(|z| >> 1/k\) one always recovers the \(N\)-dimensional Randall-Sundrum-type volcano potential. This, in turn, implies that the gravity in the bulk is not delocalized by a Gauss-Bonnet term even if \(N > 1\).

### 5 Graviton propagator in \((4 + N)\) dimensions

In order to study the graviton propagator, let us set \(N = 1\) in (19) and momentarily change the bulk coordinate to \(z = e^{k|y|/k}\). Then, in terms of \(A(y)(= k|y|)\), the Fourier modes \(e^{ip \cdot x} \psi(x, y)\) of the tensor fluctuation \(h_{\mu\nu}\) in five dimensions satisfy
\[ \left[ (1 - 4 \alpha \kappa_{5} \ddot{A}) \left( e^{2k|y|} m^{2} + \partial_{y}^{2} - 4 \ddot{A}^{2} + 2 \dot{A} \dot{A} \right) + 4 \alpha \kappa_{5} \ddot{A} e^{2k|y|} m^{2} - 8 \alpha \kappa_{5} \ddot{A} (\partial_{y} + 2 \dot{A}) \right] \psi(y) = 0. \] \hspace{1cm} (30)

Here dot represents differentiation with respect to \(y\). To study graviton propagator for the matter-localized brane, one may place a test mass \(2\pi m_{*}\) on the brane \(z = 1/k\) (or on the intersection of \((N + 2)\)-branes) and ask for the corresponding Newtonian potential at a distant point on the brane. This may be done by inserting a source term \(G_{4+N} (2\pi m_{*}) \delta^{3}(x) \delta(z)\) on the rhs of equation (30). The five-dimensional graviton propagator \(G_{5}(x, z; x', z')\) is obtained by integrating over the Fourier modes
\[ G_{5}(x, z; x', z') = \int \frac{d^{4}p}{(2\pi)^{4}} e^{ip \cdot (x - x')} \psi(z, z'). \] \hspace{1cm} (31)

We also need appropriate Neumann-type boundary conditions on the brane(s) to proceed further. For this purpose, we need to regularize the \(\delta\)-function\(^{1}\). Also note that \(\partial_{y} \psi(0+) = -\partial_{y} \psi(0-),\) while

\(^{1}\)This issue has also been explained in Ref. [20], where a similar analysis is carried out by considering arbitrary order of curvature correction terms.
\[ \partial_y \psi(0) = 0. \] Then the last term of equation (30) gives

\[- \frac{16 \alpha \kappa_5 k^2}{3} \delta(y) (\partial_y + 2k) \psi(0_+).\]

Similarly, two other non-trivial terms arising from the first two brackets in equation (30), namely

\[- 4 \alpha \kappa_5 \hat{A} \partial_y^2 \psi(y) \text{ and } -8 \alpha \kappa_5 \hat{A} \psi(y),\]

give

\[- \frac{8}{3} \alpha \kappa_5 k^2 \delta(y) (\partial_y + 2k) \psi(0_+).\]

The total non-trivial contribution at \( y = 0 \), from equation (30), is therefore

\[ 2 (\partial_y + 2k) \psi(0_+) - 8 \alpha \kappa_5 k^2 (\partial_y + 2k) \psi(0_+) + 8 \alpha \kappa_5 k m^2 \psi(0_+). \]

(32)

This gives the correct Neumann-type boundary condition on the brane after integrating from just below to the just above the brane (i.e., in the neighborhood of \( y = 0 \)). Hence a Neumann type boundary condition that one must impose on the brane at \( z = 1/k \) is

\[ \left( z \partial_z + 2 + \frac{\xi}{1 - \xi} q^2 z^2 \right) \psi(z, z') \big|_{z = 1/k} = 0, \]

(33)

where \( \xi = 4 \alpha' M_*^{-2} k^2 \), and \( m^2 = q^2 = -p^2 \). By the same token, one can derive \( (N + 4) \) dimensional Neumann boundary condition at \( z = 1/k \). As the calculation is much involved, we avoid here the technical details and simply give the final expression

\[ \left( z \partial_z + \frac{N + 3}{2} + \frac{2}{(N + 1)} \frac{\xi q^2 z^2}{(1 - \xi)} \right) \psi \big|_{z = 1/k} = 0, \]

(34)

This boundary condition was first introduced in Ref. [4], but for \( \xi = 0 \) case. One should also satisfy the following matching conditions at \( z = z' \):

\[ \psi_\downarrow \big|_{z = z'} = \psi_\uparrow \big|_{z = z'}, \]

\[ \partial_z \left( \psi_\uparrow - \hat{\psi}_\downarrow \right) \big|_{z = z'} = (1 - \xi)^{-1} \frac{1}{k z'}. \]

(35)

By satisfying (35) and (34) we can find the Neumann propagator in \((4 + N)\) dimensions. To this end, we follow the derivations given in the Ref. [4, 14]. The formula for the Green function in five dimensions \((N = 1)\), for both the arguments of the propagator on the brane, was given, for example, in [14], so here we give the expression valid for arbitrary \( N \). The general expression for the Green function in \((4 + N)\) dimensions for both the arguments on the brane is

\[ G_{4+N}(x, \frac{1}{k}; x', \frac{1}{k}) \simeq (1 - \xi)^{-1} \int \frac{d^{3+N} p}{(2\pi)^{3+N}} e^{ip(x-x')} \frac{1}{q} \left[ \frac{1 - \xi}{1 + \xi} \frac{N + 1}{w} \right. \]

\[ + \left. \frac{(N + (N - 2) \xi) (1 - \xi)}{2(N - 1)(1 + \xi)^2} w + O(w^3) + \ldots + \frac{(1 - \xi)^2 w^N}{(1 + \xi)^2 c_1} \ln \left( \frac{w}{w} \right) \right], \]

(36)
where \( q/k \equiv w \), \( c_1 \) is a dimensional constant (\( c_1 = 1, 4, 64, \cdots \), for \( N = 1, 3, 5, \cdots \)), but the last expression involving a logarithmic term will be absent for even numbers of extra dimensions (\( N = 2, 4, 6, \cdots \)). We should mention that the second expression above will appear only for \( N > 1 \), though we have written it for arbitrary \( N \). Note that only the first two terms do not vanish for \( \xi = 1 \).

We may split equation (36) into zero mode and KK mode

\[
G_{4+N}(x, \frac{1}{k}, x', \frac{1}{k}) = \frac{(N+1)k}{1+\xi} \int \frac{d^{3+N}p}{(2\pi)^{3+N}} \frac{e^{ip(x-x')}}{q^2} + G_{KK}(x,x'),
\]

(37)

where

\[
G_{KK}(x,x') = -(1+\xi)^{-1} \int \frac{d^{3+N}p}{(2\pi)^{3+N}} e^{ip(x-x')} \left[ \frac{N+(N-2)\xi}{2(N-1)} + \frac{1}{1+\xi} \frac{1}{k} + \frac{1-\xi}{q c_1} - \xi \frac{q/k}{N c_1} \ln \left( \frac{q}{2k} \right) + \cdots \right].
\]

(38)

Equation (37) reflects a useful relation

\[
G_{N+3} = \frac{(N+1)k}{1+\xi} G_{N+4}
\]

(39)

One observes from equation (37) that in the low-energy scale \( q/k (\equiv w) \ll 1 \) and for \( N > 1 \) the contribution to the Newton potential from the continuum KK modes is almost negligible compared to the contribution of zero mode. One finds \( N = 1 \) as the simplest example. The limiting behavior of the Newtonian potential due to a point source of mass \( 2\pi m_* \) on the brane is therefore [14]

\[
V(r) = \frac{2\pi m_*}{M_5^3} \int dt G_5 \left( r, \frac{1}{k}, 0, \frac{1}{k} \right)
\]

\[
\approx -\frac{G_4 m_*}{r} \frac{1}{(1+\xi)} \frac{1}{2k^2 r^2} + \cdots
\]

(40)

For \( \xi = 0 \), this expression agrees with the result found in [3]. A difference of factor 1/2 to the leading-order potential correction from that of the original RS work [1] (which is the case of \( \xi = 0 \)) arises from the convention in a relation between four- and five-dimensional Newton constants. If one defines [1]

\[
G_4 = \frac{k}{(1+\xi)} G_5
\]

\[
M^2_{(4)} = \frac{1+\xi}{k} M^2_{(5)}
\]

(41)

instead of relation (39) the factor of 1/2 would be absent. To reconcile the result with an exact graviton propagator analysis, however, one may have to relate \( G_{N+3} \) and \( G_{N+4} \) according to relation (39). For \( \xi = 1 \), there is no correction from the continuum KK modes. Moreover, to avoid anti-gravity effect (\( G_4 < 0 \)), one requires \( \xi > -1 \), and another limit \( \xi \leq 1 \) arises from the propagator analysis; this is consistent with the previous observation. Thus, for \( \xi = 1 \) solution, there are no bulk propagation of graviton in \( AdS_{4+N} \) space. By taking \( \xi = 0 \), from (41), one recovers the relation \( M^2_{pl} = M^2_{(5)} k^{-1} \) and also recovers the Newtonian potential in a form identical to that obtained by Randall and Sundrum [1].
For $r >> 1/k$, the continuum KK modes with $m >> k$, which are normally suppressed on the brane, would only sub-dominantly contribute to the potential generated by the $4d$ graviton bound state. For distances $r << 1/k$ or at the origin, the continuum modes with $m >> k$ have unsuppressed wavefunction, thus as anticipated we get the $(4 + N)$-dimensional potential even if $\xi > 0$.

A remark is in order. For $\xi = 1$ and $N > 1$, only the first term in (38) can survive, but this is negligible as compared to the zero-mode contribution. One, therefore, always recovers a localized gravity on the brane which precisely follows the Newton’s law, and there may almost be trivial correction to the gravitational potential. We will show in next section that $\xi = 1$ is indeed a viable solution of the RS-type braneworld models.

6 Effective solutions with $\xi = 1$

It is not difficult to show that $\xi = 1$ explains the RS warped braneworld compactification. It looks reasonable first to solve the modified Einstein equations in five dimensions. To this end, we shall introduce a brane action corresponding to a RS singular 3-brane with a positive brane tension $\lambda > 0$. Thus the low-energy effective action in five dimensions can be taken to be

$$S = \int d^5 x \sqrt{-g_5} \left( \frac{R}{\kappa_5} - \Lambda + \alpha \left( R^2 - 4 R_{pq} R^{pq} + R_{pqrs} R^{pqrs} \right) \right) + \int d^4 x \sqrt{|g_4|} (-\lambda).$$  \hspace{1cm} (42)

The RS single brane action is recovered for $\alpha = 0$. The modified Einstein field equations following from (42) would simplify to give

$$\left( A'^2 + A'' \right) (1 - \xi) = \frac{\lambda}{6} \kappa_5 \delta(z)$$ \hspace{1cm} (43)

$$A'^2 - \frac{\xi}{2} A'^2 = \frac{\kappa_5}{12} \Lambda e^{-2A(z)}$$ \hspace{1cm} (44)

where $\xi \equiv 2 \epsilon e^{2A(z)} A'^2$, and $\epsilon \equiv 2 \alpha \kappa_5$. One might have already noted that appealing to $\xi = 0$ in the bulk essentially implies $A'' + A'^2 = 0$. In fact, $\xi = 0$ kills another possible solution, i.e. $\xi = 1$, which indeed gives a physical solution. As is known, any braneworld compactification with $\xi = 0$ suffers from the so-called c-theorem [22, 23] based on Einstein theory. We should mention that the c-theorem may not be available to $\xi = 1$ solution. We will explain this below only briefly, and detail discussions on this issue will appear elsewhere.

One clearly sees from (43) that $\xi \equiv 2 \epsilon A'^2 e^{2A(z)} = 1$ is a bulk solution of the field equations. From this we find

$$e^{A(z)} = \int_0^z \frac{1}{\sqrt{2\epsilon}} dz$$

$$e^{-A(z)} = \frac{\sqrt{2\epsilon}}{|z| + \sqrt{2\epsilon}} = \frac{l}{|z| + l},$$  \hspace{1cm} (45)

where we have normalized the solution so that $A(0) = 0$. Since we have a non-trivial brane tension $\lambda > 0$ at the brane $z = 0$, equation (43) implies that we must satisfy the following relation on the
brane \( z = 0 \):

\[
\frac{12\delta(z)}{l + |z|} \left( 1 - \frac{2\epsilon}{L^2} \text{sgn}(z)^2 \right) = \lambda \kappa_5 \delta(z).
\]  

(46)

This, after regularizing the \( \delta \)-function, uniquely determines the 3-brane tension,

\[
\lambda = \frac{1}{2\pi G_5 l}.
\]  

(47)

Next, equation (44) can be used to fix the bulk cosmological constant

\[
\Lambda = -\frac{3}{8\pi G_5 l^2}.
\]  

(48)

The readers can easily check that \( \xi = 1 \) consistently gives a physical solution even if the number of extra dimensions \( N > 1 \). It is interesting that \( e^{-A(z)} \) will converge as \( z \to \infty \), this is what we need for the solution to be physical. If one interprets the singularity at \( A'' + A'^2 \) as the infrared cut-off (or the RS singular brane), then we may interpret the \( z \to \infty \), \( \xi = 1 \) as the ultraviolet cutoff or anti-de Sitter boundary where gravity is strong. Changing to \( y \)-coordinate \( dy = e^{-A(z)} dz \), the weak energy condition reads \( A''(y) \geq 0 \), and the warp factor reads \( A(y) = \pm |y|/l \). Since \( A'(y) \) takes a positive (negative) value for \( y \to +\infty (-\infty) \), the no-go theorem \[22\] may not apply to the \( \alpha > 0 \) case.

7 Completely localised gravity with \( \xi = 1 \)

As we already emphasized by considering \( \xi = 1 \), we can obtain equivalent \( N \) copies of the RS-type non-relativistic Schrödinger equations, but in one lower spacetime dimensions. We shall express the metric fluctuations again in a canonical form by changing the variables in equation (17) as

\[
h_{\mu\nu} = e^{(N+1)A(z)/2} \psi(z_i) e^{ip \cdot x} \epsilon_{\mu\nu}.
\]  

Then the non-trivial part of linearized equations takes the following form

\[
8N\alpha \kappa k \sum_{i=1}^{N} \delta(z_i) \left[ -m^2 - \sum_{i \neq j} (N+1)ke^{-A(z)} \delta(z_j) \right. \\
\left. + \frac{(N-1)(N+1)(N+3)k^2}{4(k \sum_{i \neq j} |z_j| + 1)^2} \right] \psi(z) + (N + 1) \text{sgn}(z_i) \left( \partial_{z_i} + \frac{(N+1)}{2} A'(z) \right) \psi(z) = 0.
\]  

(49)

Equation (49) clearly implies that the so called 'solitonic' solutions defined for \( \xi = 1 \) have the structure which is formally the same as the “normal” or RS-type brane solutions, other than the last term. Some of the clear differences here are that (i) there is no bulk propagation of the graviton in \( (4 + N) \)-dimension, (ii) the solitonic brane is \( \delta \)-function like, which is also seen from the overall coefficient in (49). The last term in equation (49) implies an appropriate jump condition(s) to be satisfied across the branes at \( z_i = 0 \), but does not contribute for \( z > 0 \). Thus, for \( \xi = 1 \), the \( (N - 1) \)-dimensional Schrödinger equation can also be written in the form

\[
\frac{4\xi \delta(z_i)}{N(N+1)k} \left[ -\partial_{z_j}^2 + \frac{(N + 1)^2}{4} A'^2 - \frac{N + 1}{2} A'' \right] \psi = m^2 \psi,
\]  

(50)
where \( A = \log(k \sum_{p} |z_p| + 1) \) and \( i, j = 1, 2, \cdots, N, \ i \neq j \). Thus, the bulk equations along \( z_j \)-directions are

\[
\frac{4 \xi \delta(z_i)}{N(N+1)k} \left[ -\partial_{z_j}^2 + \frac{(N-1)(N+1)(N+3)}{4(|z_j| + 1/k)^2} \right] \hat{\psi} = m^2 \hat{\psi}.
\] (51)

The simplest choice is \( N = 2 \). Then (51) gives two five dimensional bulk equations of graviton, for each \( i, j = 1, 2 \) but \( i \neq j \). For the continuum of KK modes propagating along the solitonic 4-branes located at the \( z_1 \) and \( z_2 \) axes, the continuum mode solutions are given by a linear combination

\[
\hat{\psi}^{(j)}(z_j) \sim N^{(j)}_m \sqrt{(|z_j| + 1/k)} \left[ J_2(m(|z_j| + 1/k)) + B^{(j)}_m Y_2(m(|z_j| + 1/k)) \right],
\] (52)

where \( j = 1, 2 \). The coefficients \( B^{(j)}_m \) and \( N^{(j)}_m \) are determined from the boundary conditions and \( \delta \)-function normalization condition. These are readily evaluated to be

\[
B^{(j)}_m = -\frac{Y_1(m/k)}{J_1(m/k)}, \quad N^{(j)}_m = \sqrt{\frac{m}{4(1 + B^{(j)}_m)^2}}.
\] (53)

We consider the contribution of KK graviton exchange to the non-relativistic gravitational potential between two point masses \( m_1, m_2 \) placed on the intersection of two orthogonal solitonic 4-branes with a separation of \( r \). A total contribution of the Newtonian potential is therefore

\[
\begin{aligned}
-\frac{V(r)}{m_1 m_2} &= \left[ \frac{G_4}{r} + \frac{3k}{2} M_4^{-4} \int_{m_0}^{\infty} dm \left| \hat{\psi}_m(0) \right|^2 \frac{e^{-mr}}{r} \right] \\
V(r) &= -G_4 \frac{m_1 m_2}{r} \left[ 1 + \frac{1}{k^2 r^2} \right].
\end{aligned}
\] (54)

where \( G_4 = (3/4) \times k^2 M_4^{-4} \) is used. In arriving at the second line it was important to use \( |\hat{\psi}_m(0)|^2 \sim m/(4k) \), which one can easily find from plane wave normalization conditions of the Bessel functions as in [5].

For \( N = 3 \), equation (51) would rise to give the following three six-dimensional bulk equations of the graviton, each bulk mode propagating along the solitonic 5-branes located at \( u(= z_p - z_q) \) and \( v(= z_p + z_q) \) axes. Hence

\[
\frac{2 \delta(z_i)}{3k} \left[ -\partial_u^2 - \partial_v^2 + \frac{6}{(|v| + 1/k)^2} \right] \hat{\psi}(u, v) = \frac{1}{2} m^2 \hat{\psi}(u, v),
\] (55)

where \( i = 1, 2, 3 \), and \( i \neq p \neq q \). In order to satisfy the appropriate boundary condition(s) implied by the \( \delta \)-function potential at \( z_i = 0 \), we must choose the linear combinations

\[
\begin{aligned}
\phi_m(u) &\sim N_1(m) \sqrt{|u| \sin(m_u u) + A_m \cos(m_u u)} \\
\phi_m(v) &\sim N_2(m) \sqrt{(|v| + 1/k)} \left[ J_{5/2}(m_v(|v| + 1/k)) + B_m Y_{5/2}(m_v(|v| + 1/k)) \right].
\end{aligned}
\] (56, 57)

These solutions must satisfy the Neumann type boundary conditions at the brane junction \( u = 0, v = 0 \):

\[
\begin{aligned}
\left( u \frac{\partial_v - \partial_u}{2} + \frac{5}{2} \right) \hat{\psi} = 0, \quad &\left( v \frac{\partial_u + \partial_v}{2} + \frac{5}{2} \right) \hat{\psi} = 0.
\end{aligned}
\] (58)
For \( m_u^2, m_v^2 > 0 \), satisfying the boundary conditions, the coefficients \( A_m \) and \( B_m \) are determined to be
\[
A_u = \cot(m_u u), \quad B_v = \frac{Y_{3/2}(m_v/k)}{J_{3/2}(m_u/k)}.
\]
(59)
The continuum wavefunction therefore reads
\[
\hat{\psi}_m(j) = \varphi_m(u_j) \times \varphi_m(v_j).
\]
(60)
By considering the contribution of KK graviton exchange between two unit point masses \( m_1, m_2 \), placed on the intersection of three intersecting 5-branes with a separation of \( r \), one may find the total contribution of the Newtonian potential. This is given by
\[
-\frac{V(r)}{m_1 m_2} \simeq \frac{G_4}{r} + \frac{3k^2}{2} \tilde{m}^{-5} \int_0^\infty dm_u dm_v \frac{e^{-\tilde{m}r}}{r} |\hat{\psi}_m(0)|^2
\]
\[
V(r) \simeq -G_4 \frac{m_1 m_2}{r} \left[ 1 + \frac{c_1}{(kr)^3} \right].
\]
(61)
where \( \sqrt{m_u^2 + m_v^2} \equiv \tilde{m}, G_4 = (3/2) \times k^3 \tilde{m}^{-5} \) and \( c_1 \) is some constant of order 1. Again in arriving at this result, we have used the value \( |\hat{\psi}_m(0)|^2 \sim m_v^2/(6k^2) \) determined from the plane wave normalization conditions of the Bessel functions, and specialized to the case where \( m_u \) would extend down to \( m_u = 0 \), and \( \psi_{m_u}(0) = 1 \).

8 Comments on scalar and vector modes

So far we have analysed the tensor mode \( h_{\mu\nu} \) of the metric perturbations, leading to graviton propagators and corrections to Newton potential. This analysis was important because a high-dimensional graviton may represent a continuum of four-dimensional states and the gravitational potential on the brane is mediated by an effective four-dimensional graviton. It may be relevant to know whether the scalar/vector modes of the metric perturbations have any significant roles in the higher dimensional brane background. One of the motivation to look after these modes is that in the conventional Kaluza-Klein theory of compact extra spaces, because of the isometries of a factorizable geometry, scalar (vector) modes correspond to the physical gravi-scalar (gravi-photon) modes [28]. But this does not seem to be the case for a non-factorizable geometry.

The scalar and vector modes of the metric fluctuations, without any gauge choice, satisfy
\[
(1 - \xi) \left[ \partial^\mu \partial'^\nu (h_{\mu\nu} - \eta_{\mu\nu} h) + (N + 2) \partial_z A \left( \partial_z h - 2\partial^\lambda h_{\lambda z} + (N + 3) \partial_z A h_{zz} \right) \right] = 0. \quad (62)
\]
\[
(1 - \xi) \left[ \partial_z \left( \partial^\lambda h_{\mu\lambda} - \partial_\mu h \right) - (N + 2) \partial_z A \partial_\mu h_{zz} + \partial^\lambda \partial_\mu h_{\lambda z} \right] = 0. \quad (63)
\]
Here \( h \equiv h_{\lambda z}, \partial_z A = A' \partial_z |z| = k (2\Theta(z) - 1) = k \sum_{i=1}^N \text{sgn}(z_i) \), and \( \partial_z = \partial_{z_1}, \ldots, \partial_{z_N} \). (These equations were reported in [12, 14] but for \( N = 1 \) and in the axial gauge, i.e., \( h_{zz} = 0 = h_{\mu z} \)). Equations (62, 63) are trivially satisfied for \( \xi = 1 \) solution. There are no terms having second
derivatives of the warp factor $A(z)$, and hence there are no delta-function sources for scalar/vector modes. This is not the case for the tensor mode $h_{\mu\nu}$ (see, for example, equation (20)), even after the setting $\xi = 1$. Thus the scalar and vector modes of the metric fluctuations alone do not appear as dynamical fields of the theory [1, 7, 12], and hence may not be localized on the brane. This result is consistent with the observations made in [7, 21].

Since a massless graviton on the brane is assumed to be transverse and tracefree, it is reasonable to impose the gauge $h^\mu_\mu = 0 = \nabla^\nu h_{\mu\nu}$ to simplify the expressions (62), (63). For $N = 1$, since $Z_2$ symmetry on the brane already fixes $h_{\lambda z} = 0$, one finds $h_{zz} = 0$, thus the graviscalar and graviphoton components are pure gauge degrees of freedom. If the brane gravity is coupled to brane matter, one may still gauge them away using the brane-bending formalism discussed in [3, 4, 14].

One can, of course, couple the scalar modes of the metric fluctuations to the scalar fluctuation of a non-trivial bulk scalar field. But the effective theory will be of scalar-tensor nature rather than conventional tensor gravity. Unlike the tensor modes, the scalar/vector modes of the metric perturbations are not automatically invariant in a gauge invariant manner. Indeed, for $N > 1$, one may not set all the scalar/vector modes to zero or gauge them away, and in principle one can analyse them using some gauge-invariant formalism. The analysis is relatively simpler in a RS-type warped brane background with one extra dimension (i.e. $N = 1$) [21, 31], but the diagonalisation of linearized fluctuations of all metric fields for $N > 1$ is too hard, if not impossible. Some progress can be made by taking the simplest choice, $N = 2$, and using the gauge invariant formalism developed in [30]. We take the viewpoint that consideration of scalar and vector components would not change the results, at least the momentum structures of the graviton propagators, of this paper.

9 Conclusion

We have studied gravity localization to the RS branes for a more general class of $N$ intersecting $(2+N)$ brane configurations by including higher order curvature terms in a Gauss-Bonnet combination. For $N = 2$, a GB term can support a feasible singularity at the brane junction, thereby allowing a non-trivial brane tension at the four-dimensional brane intersection. But this would not have been possible if the braneworld action had a contribution only from the Einstein term and the cosmological constant term [2]. The effective four-dimensional Planck scale $M_{Pl}$ has been derived, and also the explicit expressions for the graviton propagator in $(4+N)$ spacetime dimensions are obtained. We have explained in some detail interesting features of localized gravity with a Gauss-Bonnet term for 'normal' or RS-type branes supported by $\delta$-function-like sources, and also so-called solitonic branes defined for the class of solution $\xi = 1$.

2See, for example, [29] for related comments about the cancellation of the scalar mode by the 'brane-bending' mode in the framework of a five dimensional 3-brane model. These comments may not be directly applicable to the 3-brane defined as the common four dimensional intersection of higher dimensional branes.
The present analysis might reveal some important features of the localized RS type braneworld solutions or solitonic braneworld gravity created by non-dynamical sources (i.e. in the absence of fields other than gravity). We have shown that the RS single brane model with a Gauss-Bonnet term in the bulk correctly gives a massless graviton on the brane as for the RS model. We provided a complementary description of the Newton potential corrections to the long-range interaction due to continuum modes living in the AdS space. Basically, our work extends the previous work in the literature [2, 4, 14] to the general (4+N) dimensional anti-de Sitter spacetimes, but the results reported here also include the contribution of the Gauss-Bonnet interaction term. The latter is the only viable combination of higher curvature corrections that one can introduce in the RS braneworld scenario of warped extra dimensions.

For the class of solution $\xi = 1$, the scalar and vector fields do not appear as the propagating degrees of freedom in the transverse $z$ direction, though the zero mode of the tensor fluctuation is localised on the brane. This reveals that the physical brane could be a solitonic 3-brane living in six-dimensional spacetimes, rather than in five-dimensional AdS space. The number of non-compact extra dimensions, $N = 2$, is itself more interesting in the scenario of [2, 10].

In nutshell, the physical relevance of our analysis is two-fold – it is a meaningful extension of the RS model with an extra non-compact dimension to an arbitrary $N$, and – a natural generalization of the ADDK [2] scenario of infinite large new dimensions including the contribution of a Gauss-Bonnet term into the effective braneworld (gravitational) action. The present analysis also provides hints for a possible creation of solitonic 3-brane in a six dimensional warped bulk background, which would not be possible without the higher curvature terms. It is plausible that the solitonic braneworld solution defined with $\xi = 1$ in $N \geq 1$ has a structure which is formally the same as for ‘normal’ or RS-type solutions.

Acknowledgments

I would like to thank M. Blau, Gregory Gabadadze, N. Kaloper and Carlos Nunez for fruitful discussions and correspondences. This work was supported in part by Seoam Foundation, Korea, and by the BK21 Project of the Ministry of Education. I would like to acknowledge a kind hospitality of Abdus Salam ICTP, where the manuscript was revised.

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