Extracting \( \bar{m}_c(M_c) \) and \( f_{D_s,B} \) from the pseudoscalar sum rules

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I report here on the (first) direct extraction of the running charm quark mass \( \bar{m}_c(\nu) \) from the \( D \)-meson sum rules, and on the implications of this result for the estimate of the leptonic decay constants \( f_{D_s,B} \). The outputs: \( \bar{m}_c(M_c) = (1.08 \pm 0.11) \) GeV, \( f_D \simeq (1.52 \pm 0.16) f_\pi \), \( f_{D_s} \simeq (1.75 \pm 0.18) f_\pi \) and \( f_B = (1.44 \pm 0.07) f_\pi \) are in good agreement with the existing sum rule results obtained using the pole mass. In particular, the result \( f_D \approx f_B \) supports early ’87 sum rule results \(^1\), which indicated a huge \( 1/m \) correction to the heavy quark symmetry expectation. This talk is based on the paper \texttt{hep-ph/9712386} and updates the discussions given there.

1. Introduction

One of the most important parameters of the standard model is the quark masses. However, contrary to the leptons, where the physical mass can be identified with the pole of the propagator, the quark masses are difficult to define because of confinement. Some attempts have been made in order to define the quark pole mass within perturbation theory, where it has been shown to be IR-finite \(^2\) and independent from the choice of the regularization and renormalization schemes used \(^3\). More recently, it has been noticed, in the limit of a large number of flavours, that the resummation of perturbative series can induce a non-perturbative term, which can affect the truncated perturbative result, and can, then, limit the accuracy of the pole mass determination \(^4\). However, a proper use of such a result needs a resummation, at the same level of accuracy (which is not often the case), of the Green function of a given process involving the pole mass, where some eventual cancellation between the resummed terms of the two series may occur (we plan to come back to this point in a future publication). One may bypass the previous problems, by working, at a given order of perturbative QCD, with the running quark masses, which are treated like coupling constants of the QCD Lagrangian (see e.g. \(^5\)), and where some non-perturbative-like effect is expected to be absent. A lot of efforts has been furnished in the literature \(^6\) for extracting directly from the data the running masses of the light and heavy quarks using the SVZ QCD spectral sum rules (QSSR) \(^7\). In this note, I shall consider a direct extraction of the running charm quark mass using the observed value of \( M_D = 1.865 \) MeV, and study its implication on the value of the decay constants \( f_{D_s(B)} \), which are normalized as \( f_\pi = 93.3 \) MeV where the leptonic decay width reads:

\[
\Gamma(D_s \to l\nu(\gamma)) = \frac{G_F^2 |V_{cs}|^2}{4\pi} \times f_{D_s}^2 m_l^2 M_{D_s} \left( 1 - \frac{m_l^2}{M_B^2} \right)^2
\]

In this respect, this present work is an improvement and update of the ones in \(^10\) based on the use of the perturbative pole mass, where some eventual non-perturbative effects induced by the resummation of the QCD series are not taken into account. For our discussion, we shall use the average value of the experimental widths quoted in \(^8\), from which we can deduce:

\[
f_{D_s} \simeq (1.92 \pm 0.23) f_\pi.
\]

\(^*\)Talk given at the QCD 98 Euroconference-Montpellier (2-8th July1998) – Montpellier preprint PM-98/36.
while, combining it with the most reliable sum rule result for the ratio $f_D/f_D = 1.15 \pm 0.04$.

one also obtains:

$$f_D \simeq (1.67 \pm 0.24)f_\pi.$$  \hspace{1cm} (4)

2. The QCD spectral sum rules

We shall work with the pseudoscalar two-point correlator:

$$\psi_5(q^2) \equiv i \int d^4x \; e^{iqx} \langle 0 | T J_q(x) J_q(0)|0 \rangle, \hspace{1cm} (5)$$

built from the heavy-light quark current: $J_q(x) = (m_c + m_d)\bar{c}(i\gamma_5)d$, and which has the quantum numbers of the $D$ meson. The corresponding Laplace transform sum rules are:

$$\mathcal{L}(\tau) = \int_{t_\leq}^\infty \; dt \; e^{-\tau t} \frac{1}{\pi} \text{Im} \psi_5(t), \text{ and}$$

$$\mathcal{R}(\tau) \equiv -\frac{d}{d\tau} \log \mathcal{L}(\tau), \hspace{1cm} (6)$$

where $t_\leq$ is the hadronic threshold. The latter sum rule, or its slight modification, is also useful, as it is equal to the resonance mass squared, in the simple duality ansatz parametrization of the spectral function:

$$\frac{1}{\pi} \text{Im} \psi_5(t) \simeq 2f_D^2 M_D^4 \delta(t - M_D^2) + \text{"QCD continuum"} \Theta(t - t_c), \hspace{1cm} (7)$$

where the "QCD continuum" comes from the discontinuity of the QCD diagrams, which is expected to give a good smearing of the different radial excitations. The decay constant $f_D$ is analogous to $f_\pi = 93.3$ MeV; $t_c$ is the QCD continuum threshold, which is, like the sum rule variable $\tau$, an a priori arbitrary parameter. In this paper, we shall impose the $t_c$ and $\tau$ stability criteria for extracting our optimal results. The QCD expression of the correlator is known to two-loop accuracy (see e.g. [3]) and the explicit expressions

$$\psi_5(q^2) = \int d^4x \; e^{iqx} \langle 0 | T J_q(x) J_q(0)|0 \rangle,$$  \hspace{1cm} (5)

given in [10], in terms of the perturbative pole mass $M_c$, and including the non-perturbative condensates of dimensions less than or equal to six.

The sum rule reads:

$$\mathcal{L}(\tau) = M_D^2 \left\{ \int_{1/4M_D^2}^\infty dt \; e^{-\tau t} \frac{3}{8\pi^2} t(1 - x)^2 \left[ 1 + \frac{4}{3} \left( \frac{\alpha_s}{\pi} \right) f(x) \right] \right. \left. + \left[ C_4(\Delta_4) + C_6(\Delta_6) \right] e^{-M_D^2 \tau} \right\}, \hspace{1cm} (8)$$

with:

$$x \equiv M_D^2/t,$$

$$f(x) = \frac{9}{4} + 2\log(1 - x) + \log x \log(1 - x) + x \log(1 - x) - \frac{3}{2} \log(1 - x) + x \log(1 - x) - (x/(1 - x)) \log x,$$

$$C_4(\Delta_4) = -M_c(d\bar{d}) - \frac{\alpha_s G_4^2}{12\pi},$$

$$C_6(\Delta_6) = \frac{M_D^2}{2} \bar{g}(d\bar{d} \nu \lambda_\nu \lambda_\nu G_2^4 \bar{d} d) \hspace{1cm} (9)$$

It can be expressed in terms of the running mass $\bar{m}_c(\nu)$ through the perturbative two-loop relation [3]:

$$M_c(\nu) = \bar{m}_c(\nu) \left\{ 1 + \left( \frac{\alpha_s}{\pi} \right) \left( \frac{4}{3} + 2 \log \frac{\nu}{M_c} \right) \right\} \hspace{1cm}, (10)$$

with: [3,4,5,6,7]:

$$\bar{m}_c(\nu) = \bar{m}_c \left\{ -\beta_1 \left( \frac{\alpha_s}{\pi} \right)(\nu)^{-\gamma_1/\beta_1} + \beta_2 \left( \frac{\alpha_s}{\pi} \right)(\nu)^{\gamma_2/\beta_2} \right\} \hspace{1cm}, (11)$$

\[4\text{We shall use the corrected coefficient of the quark-gluon mixed condensate given in [11]. This change affects only slightly the result. We shall also skip the negligible contribution from the dimension six four-quark and three-gluon condensates. Notice that there is some discrepancy on the value of the four-quark coefficient in the literature.}

\[5\text{It is clear that, for the non-perturbative terms which are known to leading order of perturbation theory, one can use either the running or the pole mass. However, we shall see that the non-perturbative effects are not important in the analysis, such that this distinction does not affect the result.}]}
where $\gamma_m$ and $\beta$ are respectively the QCD $\beta$-function and mass anomalous dimension normalized as:

$$\beta(\alpha_s) = \sum_{i=1}^{n} \beta_i \left( \frac{\alpha_s}{\pi} \right)^i, \quad \gamma_m = \sum_{i=1}^{n} \gamma_i \left( \frac{\alpha_s}{\pi} \right)^i , \quad (12)$$

which, read for $n_f$ flavours [8],

$$\beta_1 = -\frac{1}{2} \left( 11 - \frac{2}{3} n_f \right),$$
$$\beta_2 = -\frac{1}{4} \left( 51 - \frac{19}{3} n_f \right),$$
$$\gamma_1 = 2, \quad \gamma_2 = \frac{1}{6} \left( \frac{101}{2} - \frac{5}{3} n_f \right) . \quad (13)$$

As discussed earlier, non-perturbative terms induced by the resummation of the perturbative series can affect this relation [3]. However, within the approximation at which the spectral function is given, the use of this perturbative relation should provide the correct expression of the spectral function in terms of the running quark mass. In this way, unlike the analysis in [10, 1], the one done in this paper is not affected by the eventual existence of such non-perturbative terms related to the implicit use of the pole mass in the previous sum rule analysis. One should also notice that, to the order we are working, the expression of the spectral function in terms of the running and pole masses differ in the $\alpha_s$ correction, induced by the overall leading $M^2$ term appearing in Eq. (6). Throughout this paper we shall use the values of the parameters [8][12] given in Table 1. We have used for the mixed condensate the parametrization:

$$g(\bar{d}\sigma_{\mu\nu} \frac{\lambda_a}{2} G^{\mu\nu}_{a} d) = M^2_0 \langle \bar{d} d \rangle , \quad (14)$$

where $M^2_0 = (0.8 \pm 1)$ GeV$^2$ [3]. We shall also use, for four active flavours $[\bar{3}]$:

$$\Lambda = (325 \pm 100) \text{ MeV}. \quad (15)$$

One can also inspect that the dominant non-perturbative contribution is due to the dimension-four $m_c(\bar{d}d)$ light quark condensate, while the other non-perturbative effects remain a small correction at the optimization scale, which corresponds to $\tau \simeq 0.6 \sim 1$ GeV$^{-2}$ and $t_c \simeq 6 \sim 8$ GeV$^2$.

### 3. Discussions and results

![Figure 1. Behaviour of $f_D$ versus $\bar{m}_c(\nu)$. The horizontal band is the experimental domain of $f_D$. The theoretical band is limited by the two curves: (a) $\Lambda=0.425$ GeV, $\nu=1$ GeV $\simeq \tau^{-1/2}$, $t_c \geq 7$ GeV$^2$, $\langle \bar{d}d \rangle^{1/3}(1 \text{ GeV})=238$ MeV, $M^2_0 = 0.9$ GeV$^2$, $\langle \alpha_sG^2 \rangle = 0.06$ GeV$^2$ and (b) $\Lambda=0.225$ GeV, $\nu \approx \bar{m}_c = 1.42$ GeV, $t_c = 6$ GeV$^2$, $\langle \bar{d}d \rangle^{1/3}(1 \text{ GeV})=220$ MeV, $M^2_0 = 0.6$ GeV$^2$, $\langle \alpha_sG^2 \rangle = 0.08$ GeV$^2$.](image)

Given the experimental value on $M_D = 1.865$ GeV, we present our results on $f_D$ from the first sum rule for different values of the charm quark running mass evaluated at $p^2 = \nu^2$ in Fig. 1. The second sum rule gives the prediction on $M_D$ for each value of the charm mass (Fig. 2). The theoretical band is limited by the two extremal values of the QCD parameters used in Table 1. Notice that the effect of the errors of the different input parameters is much smaller in the ratio of sum rule $\mathcal{R}(\tau)$. The previous analysis leads to the prediction from a two-loop calculation in the $\overline{\text{MS}}$ scheme:

$$\bar{m}_c(M_c) = (1.08 \pm 0.11) \text{ GeV}, \quad (16)$$

where one should notice that, within the present accuracy of the data on $f_D$, the result comes...
| Sources                  | $|\Delta(f_D/f_\pi)|$ |
|-------------------------|-------------------|
| $\Lambda = (325 \pm 100)\text{ MeV}$ | 0.12 |
| $\nu = (1.20 \pm 0.22)\text{ GeV}$ | 0.08 |
| $\bar{m}_c(M_c) = (1.20 \pm 0.05)\text{ GeV}$ | 0.05 |
| $t_c = (6.5 \pm 0.5)\text{ GeV}^2$ | 0.04 |
| $\langle \bar{d}d \rangle^{1/3}(1\text{ GeV})=-(229 \pm 9)\text{ MeV}$ | 0.02 |
| $M_J^2 = (0.8 \pm 0.1)\text{ GeV}^2$ | 0.02 |
| $\langle \alpha_s G^2 \rangle = (0.07 \pm 0.01)\text{ GeV}^2$ | 0.01 |
| **Total**               | **0.16**         |

Table 1

Different sources of errors in the estimate of $f_D$

mainly from the one used to reproduce the experimental value of $M_D$. The radiative correction increases the value of $\bar{m}_c(M_c)$ by 10%, whilst the non-perturbative term tends to decrease its value by a small amount of about 3%, which indicates that the running mass is mainly of the perturbative origin. We have varied the subtraction scale $\nu$ in the range from $\tau^{-1/2} \approx 1\text{ GeV}$ to $M_c \approx 1.42\text{ GeV}$, which gives a significant effect (see Table 1) due to the log $\nu$ term appearing in Eq. (10). Such effect can be eliminated by choosing the subtraction point at $\nu = M_c$, at which, we shall extract the results in this paper. One should also notice, in Table 1, that the effect of $t_c$ on the result is relatively small from the value $t_c \approx 6\text{ GeV}^2$, where one starts to have a $\tau$ stability until $t_c \geq 7\text{ GeV}^2$, where one has $t_c$ stability. The main sources of errors are due to $\Lambda$ and $\nu$. Within the errors, the present result is in good agreement (though less accurate) with the value [13]:

$$\bar{m}_c(M_c) = (1.23^{+0.04}_{-0.05})\text{ GeV}$$  \hspace{1cm} (17)

obtained, within the same two-loop approximation, from $M_{J/\psi}$. Inversely, we can use the combined value:

$$\bar{m}_c(M_c) = (1.20 \pm 0.05)\text{ GeV}$$  \hspace{1cm} (18)

from $M_D$ and from $M_{J/\psi}$ systems on the curve $f_D$ as function of $\bar{m}_c$ given in Fig. 1. Then, one can deduce:

$$f_D \approx (1.52 \pm 0.16)f_\pi,$$  \hspace{1cm} (19)

where, as can be seen in Table 1, the errors in this determination come mainly from the perturbative parameters: $\Lambda$, $m_c$ and the subtraction scale $\nu$, and, to a lesser extent, from $t_c$ and the non-perturbative terms. One should notice that the radiative $\alpha_s$ correction and the non-perturbative terms have respectively increased the value of $f_D$ by about 25% and 10%. We can consider that this result is an improvement of the previous value:

$$f_D \approx (1.35 \pm 0.07)f_\pi,$$  \hspace{1cm} (20)

obtained by using the perturbative pole mass of the charm quark propagator [10,1]. However, the good agreement (within the errors) between the two results in Eqs (19) and (20) may be an indirect indication that the pole mass defined at a given order of perturbation theory, can provide a good description of the physical process in this channel, and that the eventual non-perturbative

Figure 2. Behaviour of $M_D$ versus $\bar{m}_c(\nu)$. The horizontal band is the experimental value of $M_D$. The theoretical band is limited by the two curves: (a) the same as (a) of Fig. 1 but $\nu=1\text{ GeV}$ and (b) the same as (b) of Fig. 1 but $\nu=1.42\text{ GeV}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Behaviour of $M_D$ versus $\bar{m}_c(\nu)$. The horizontal band is the experimental value of $M_D$. The theoretical band is limited by the two curves: (a) the same as (a) of Fig. 1 but $\nu=1\text{ GeV}$ and (b) the same as (b) of Fig. 1 but $\nu=1.42\text{ GeV}$.}
\end{figure}
power corrections induced by the resummation of the perturbative series remain small corrections. This observation can also be supported by the agreement of the value of the perturbative pole mass obtained here and of the one from the \( J/\psi \) systems, where both values have been obtained at two-loop accuracy. A further support of this argument can also be provided by the agreement of the pole mass deduced from Eq. (10) using the value of the running mass, with the one extracted directly in [13]. Finally, using the previous value of the running mass, with the one extracted of the pole mass deduced from Eq. (10) using the argument can also be provided by the agreement at two-loop accuracy. A further support of this systems, where both values have been obtained mass obtained here and of the one from the agreement of the value of the perturbative pole mass. This observation can also be supported by the perturbative series remain small corrections.

5. Extension of the analysis to \( f_D \)

We have also extended the previous analysis to the case of the vector current having the quantum number of the \( D^* \). In this case, and working with the correlator having the same dimension as \( \psi_5(q^2) \)\(^6\), we do not obtain a \( \tau \) stability as among other things, the coefficients of the chiral condensates are opposite of the pseudoscalar ones. Then, we conclude that, from this quantity, we cannot have a good determination of \( f_D \), within our approximation.

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Notice that, in this channel, the correlator which has less power of \( q^2 \) than \( \psi_5(q^2) \), and which might present a \( \tau \) stability, can have singularities and then should be used with a great care.