FINITE GROUPS OF DIFFEOMORPHISMS ARE TOPOLOGICALLY DETERMINED BY A VECTOR FIELD

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Abstract. In a previous work it is shown that every finite group $G$ of diffeomorphisms of a connected smooth manifold $M$ of dimension $\geq 2$ equals, up to quotient by the flow, the centralizer of the group of smooth automorphisms of a $G$-invariant complete vector field $X$ (shortly $X$ describes $G$). Here the foregoing result is extended to show that every finite group of diffeomorphisms of $M$ is described, within the group of all homeomorphisms of $M$, by a vector field.

As a consequence, it is proved that a finite group of homeomorphisms of a compact connected topological 4-manifold, whose action is free, is described by a continuous flow.

1. Introduction

The study of the automorphism group, or centralizer, of a complete vector field $X$ of class $C^r$, $r \geq 1$, or more precisely that of its quotient by the flow of $X$, is a classical question with a great amount of interesting results. Often these quotient groups are trivial or almost trivial if a reasonable hypothesis of transversality is imposed.

Therefore it is natural to consider the inverse point of view (the inverse Galois problem): given a group of diffeomorphisms $G$ do construct a complete vector field $X$ whose automorphism group, up to quotient by the flow, equals $G$ (shortly one will say that $X$ determines or describes $G$). Notice that this last problem can be addressed in topological manifolds and homeomorphisms by replacing the vector field by a continuous flow.

In [8] it is shown that every finite group of diffeomorphisms of a smooth connected manifold is determined, within the group of all diffeomorphisms, by a vector field. Here the foregoing result is extended to show that every finite group of diffeomorphisms can be described, within the group of all homeomorphisms, by a vector field.

Theorem 1.1. Let $M$ be a connected $C^\infty$ manifold of dimension $m \geq 2$, and $G$ be a finite subgroup of diffeomorphisms of $M$. Then there exists $X$, a complete $G$-invariant vector field...

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on $M$, such that the map

$$G \times \mathbb{R} \to \text{Aut}_0(X)$$

$$(g, t) \mapsto g \circ \Phi_t$$

is a group isomorphism, where $\Phi$ and $\text{Aut}_0(X)$ denote the flow and the group of continuous automorphisms of $X$ respectively.

An immediate consequence of Theorem 1.1 is that any smoothable finite group of homeomorphisms of a connected topological manifold is determined by a continuous flow.

**Corollary 1.2.** Let $G$ be a smoothable finite group of homeomorphisms of a connected topological manifold $E$ of dimension $\geq 2$. Then there exists a $G$-invariant continuous flow $\psi : \mathbb{R} \times E \to E$ such that the map

$$G \times \mathbb{R} \to \text{Aut}_0(\psi)$$

$$(g, t) \mapsto g \circ \psi_t$$

is a group isomorphism, where $\text{Aut}_0(\psi)$ denotes the group of continuous automorphisms of $\psi$.

**Remark 1.3.** While it is a classical result that any finite group of homeomorphisms of a compact surface is smoothable, the situation in dimension three is not straightforward: not every finite group $G$ of homeomorphisms of a 3-dimensional topological compact manifold is smoothable [2]. Indeed $G$ is smoothable if and only if it is locally linear [6, Theorem 2.1 and Remark 2.4].

Therefore the corollary above applies to connected compact surfaces and, in the case of topological connected compact 3-manifolds, if $G$ is locally linear.

For a generalization of Corollary 1.2 to some cases of non-smoothable actions see Example 6.4 and Theorem 6.5. In this last one we show that a finite group of homeomorphisms of a compact connected topological 4-manifold, whose action is free, is described by a continuous flow.

**Terminology:** We assume the reader is familiarized with our previous paper [8]. All structures and objects considered in this work are smooth, i.e. real $C^\infty$, and manifolds are without boundary, unless another thing is stated. Whenever we say a set is countable we mean the set is either a finite set or a countably infinite set. For the general questions on Differential
Geometry the reader is referred to [5] while we refer to [3] for basic facts on Differential Topology.

2. Somme preliminary notions

Given a vector field $Z$ on an $m$-manifold $M$, a continuous automorphism of $Z$ is a homeomorphism $f : M \to M$ which maps integral curves of $Z$ into integral curves of $Z$ (i.e. if $\gamma(t)$ is an integral curve of $Z$ then $f(\gamma(t))$ is so.) The set $\text{Aut}_0(Z)$ of all continuous automorphisms of $Z$ is a subgroup of the group of homeomorphisms of $M$. If $Z$ is complete and $\Phi_t$ denotes its flow, then $f \in \text{Aut}_0(Z)$ if and only if $f \circ \Phi_t = \Phi_t \circ f$ for any $t \in \mathbb{R}$.

In a more general setting, given a topological space $E$ a continuous flow is a continuous map $\psi : \mathbb{R} \times E \to E$ such that $\psi_0 = \text{Id}$ and $\psi_{t+s} = \psi_t \circ \psi_s$ for each $t, s \in \mathbb{R}$. As before, $\text{Aut}_0(\psi)$ is the group of all homeomorphisms $f : E \to E$ such that $f \circ \psi_t = \psi_t \circ f$, $t \in \mathbb{R}$. We say that a subset $S$ of $\text{Homeo}(E)$ is smoothable if there exists a structure of smooth manifold on $E$ which is compatible with the preexisting topology and makes every element of $S$ a diffeomorphism.

Returning to the smooth framework again, given a vector field $Z$ on an $m$-manifold $M$, a pseudo-circle of $Z$ is a subspace of $M$ which is homeomorphic to $S^1$ and consists of a regular trajectory of $Z$ and an isolated singularity. In this case the $\alpha$-limit and the $\omega$-limit of the regular trajectory is the singular point.

Let $B(r)$ be the open ball in $\mathbb{R}^m$ centered at the origin and radius $r > 0$. For the purpose of this work, we will say that $p \in M$ is a source of $Z$ if there exists an open neighborhood of this point which is diffeomorphic to an open ball $B(r)$, with $p \equiv 0$, such that in the coordinates given by the diffeomorphism

$$Z = \varphi \cdot \left( \sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j} \right)$$

where

(1) $\varphi$ is a non-negative function and $\varphi^{-1}(0)$ is countable, and

(2) on each ray issuing from the origin there are at most a finite number of zeros of $\varphi$.

By condition [2] for every ray issuing from the origin there exists just one regular trajectory whose $\alpha$-limit is $p \equiv 0$ and that near the origin lies along this ray.

A point $q \in M$ is called a rivet if the following hold:

(a) $q$ is an isolated singularity of $Z$, 

(1) $\varphi$ is a non-negative function and $\varphi^{-1}(0)$ is countable, and

(2) on each ray issuing from the origin there are at most a finite number of zeros of $\varphi$.
(b) around \( q \) one has \( Z = \psi \tilde{Z} \) where \( \psi \) is a function and \( \tilde{Z} \) a vector field with \( \tilde{Z}(q) \neq 0 \), and

(c) no trajectory has \( q \) as \( \alpha \)-limit and \( \omega \)-limit at the same time.

Note that by (b) and (c) any rivet is the \( \omega \)-limit of exactly one regular trajectory, the \( \alpha \)-limit of another different one and moreover, it is an isolated singularity of index zero.

A topological rivet means an isolated singularity of \( Z \) that is the \( \alpha \)-limit of a single regular trajectory, the \( \omega \)-limit of another single regular trajectory and both trajectories are different.

As one would expect, any rivet is a topological rivet.

By definition, a chain of \( Z \) is a finite and ordered sequence of three or more different regular trajectories, each of them called a link, such that:

(a) The \( \alpha \)-limit of the first link is a source or empty.
(b) The \( \omega \)-limit of the last link is a pseudo-circle.
(c) Between two consecutive links the \( \omega \)-limit of the first one equals the \( \alpha \)-limit of the second one. Moreover this set consists in a rivet.

The number of links defining a chain is called the order of the chain. The \( \omega \)-limit of a chain is that of its last link.

Given a subset \( Q \subset M \), we say that the dimension of \( Q \) does not exceed \( \ell \), or \( Q \) can be enclosed in dimension \( \ell \), if there exists a countable collection \( \{ N_\lambda \}_{\lambda \in L} \) of submanifolds of \( M \), all of them of dimension \( \leq \ell \), such that \( Q \subset \bigcup_{\lambda \in L} N_\lambda \). Note that the countable union of sets whose dimension does not exceed \( \ell \), does not exceed dimension \( \ell \) too. On the other hand, if the dimension of \( Q \) does not exceed \( \ell < m \) then \( Q \) has measure zero and therefore \( Q \) has empty interior.

Let us give the last definition of this section. A vector field \( Z \) on \( M \) is called limit (abbreviation of “with an almost controlled \( \omega \)-limit”) if the following conditions hold:

(i) The set of zeros of \( Z \) is discrete (that is with no accumulation point).
(ii) \( Z \) has exactly one pseudo-circle.
(iii) There exists a set \( Q \subset M \) whose dimension does not exceed \( m - 1 \) such that the trajectory of any point of \( M - Q \) has the pseudo-circle as \( \omega \)-limit.
(iv) \( Z \) has no chain and no periodic regular trajectory.

By (iii) the union of the trajectory of any point of \( M - Q \) and the pseudo-circle is a connected set, hence the \( Z \)-saturation of \( M - Q \) together with the pseudo-circle is a connected
set too. Therefore $M = M - Q$ is connected. Moreover $\dim M \geq 2$, otherwise $M$ equals the pseudo-circle and [iii] cannot hold.

**Proposition 2.1.** Each sphere $S^k$, $k \geq 2$, supports a limit vector field.

**Proof.** On $S^k \subset \mathbb{R}^{k+1}$ consider the vector field

$$
\xi = -\sum_{j=1}^{k} x_j x_{k+1}(\partial/\partial x_j) + (1 - x_{k+1}^2)(\partial/\partial x_{k+1}),
$$

orthogonal projection of the vector field $\partial/\partial x_{k+1}$ onto the sphere, whose trajectories go from the south pole to the north one. Since $\xi$ is transverse to the equator $E = \{x \in S^k \mid x_{k+1} = 0\}$, one may identify an open neighborhood $A$ of $E$ to $(-\varepsilon, \varepsilon) \times S^{k-1}$, endowed with coordinates $(t, y) = (t, y_1, \ldots, y_k)$ where $S^{k-1} \subset \mathbb{R}^k$, in such a way that $E$ corresponds to $\{0\} \times S^{k-1}$ and $\xi = \partial/\partial t$. Thus the north band is given by $t > 0$ and the south one by $t < 0$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function such that $\varphi([(-\infty, -\varepsilon/2])] = 1$, $\varphi([\varepsilon/2, \infty)) = -1$, and $\varphi(t) = 0$ if and only if $t = 0$.

First assume $k = 2$. On $A$ consider the vector field $Z' = \varphi(t)\partial/\partial t + (1 - \varphi^2(t))(-y_2\partial/\partial y_1 + y_1\partial/\partial y_2)$ and extend it outside $A$ by $-\xi$ on the north part and by $\xi$ on the south one. Fixed a point $p_0 \in E$, consider a function $\psi : S^2 \to \mathbb{R}$ vanishing at $p_0$ and positive on $S^2 - \{p_0\}$. It is easily checked that $Z = \psi Z'$ is a limit vector field (here $Q$ is the equator plus both poles, thus the dimension of $Q$ does not exceed 1, and observe that the only sources are the poles).

Now assume $k \geq 3$; let $\tilde{Z}$ be a limit vector field on $S^{k-1}$ constructed by induction. On $A$ consider the vector field $Z = \varphi(t)\partial/\partial t + (1 - \varphi^2(t))\tilde{Z}$, where $\tilde{Z}$ is regarded as a vector field on $A$ in the obvious way, that is tangent to the second factor, and extend it outside of $A$ by $-\xi$ on the north part and by $\xi$ on the south one. We now prove that $Z$ is a limit vector field on $S^k$.

First note that the only sources of $Z$ are the poles. Moreover, $Z$ does not have any rivet, which implies that $Z$ has no chain. Indeed clearly no point of $S^k - E$ is a rivet; on the other hand if $(0, q)$ is a rivet, as $Z$ is tangent to $\{0\} \times S^{k-1}$ the only trajectory whose $\omega$-limit is this point is included in $\{0\} \times S^{k-1}$. But clearly $(-\delta, 0) \times \{q\}$ for a $\delta > 0$ sufficiently small is included in a trajectory with $\omega$-limit $(0, q)$ what leads to contradiction.

By construction, no trajectory in $S^k - E$ is regular and periodic, so $Z$ does not possess any periodic regular trajectory. On the other hand, if $\tilde{Z}$ is regarded as a vector field on $E$ and $\tilde{Q} \subset E$ satisfies condition [iii] for $\tilde{Z}$, it suffices to take as $Q$ the union of all trajectories
of $\xi$ passing through $\tilde{Q}$ plus both poles. For if $p \in (S^n - E)$ and $q \in E$ belong to the same $\xi$-trajectory, then the $\omega$-limit of the $Z$-trajectory of $p$ equals the $\omega$-limit of the $\tilde{Z}$-trajectory of $q$. \hfill \Box

3. The almost free case

Let $M$ be a connected manifold of dimension $m$. Given a diffeomorphism $\varphi : M \to M$, the isotropy of $\varphi$ is the set $I_\varphi := \{ p \in M : \varphi(p) = p \}$. A point $p \in I_\varphi$ is said to be positive or negative according to the sign of the determinant of $\varphi_*(p) : T_pM \to T_pM$. Obviously $I_\varphi = I_\varphi^+ \cup I_\varphi^-$ where the positive isotropy $I_\varphi^+$ is the set of positive points and the negative isotropy $I_\varphi^-$ that of negative ones.

If $\varphi \neq Id$ has finite order, that is to say $\varphi$ spans a finite subgroup of diffeomorphisms, then $\varphi$ is an isometry for some Riemannian metric. Thus if $p \in I_\varphi$, the use of normal coordinates with origin $p$ allows us to identify the diffeomorphism $\varphi$ with an element of $O(m)$ different from the identity. Therefore locally $I_\varphi^-$ is a regular submanifold of codimension $\geq 1$ and $I_\varphi^+$ a regular submanifold of codimension $\geq 2$.

By definition the maximal isotropy $I^\text{max}_\varphi$ is the set of those points $p \in I_\varphi^-$ such that the codimension of $I_\varphi^-$ at $p$ equals 1. It is easily seen that $I^\text{max}_\varphi$ is either empty or a closed regular submanifold of codimension 1. Notice that if $p \in I^\text{max}_\varphi$ then in normal coordinates with origin $p$ the diffeomorphism $\varphi$ is a symmetry with respect to a hyperplane (the trace of $I^\text{max}_\varphi$).

Let $G$ be a finite group of diffeomorphisms of $M$. Let $e \in G$ be the identity element and $\ell$ be the order of $G$. By the isotropy, the positive isotropy, the negative isotropy and the maximal isotropy of $G$ we mean

$$I^*_G = \bigcup_{\sigma \in G - \{e\}} I^*_\sigma$$

where * equals nothing, +, − or max respectively.

For the purpose of this work one will say that the action of $G$ is almost free if $I_G = I^\text{max}_G$.

Lemma 3.1. Assume $I_G^+ = \emptyset$. Then the following hold:

(a) If $g$ and $h$ are two different elements of $G - \{e\}$ then $I_g^- \cap I_h^- = \emptyset$.

(b) If $I_g^- \neq \emptyset$ then $g^2 = e$.

Proof. (a) If $p \in I_g^- \cap I_h^-$ then $p$ belongs to the positive isotropy of $gh^{-1}$, so $gh^{-1} = e$.

(b) If $p \in I_g^-$ then $p$ belongs to $I_{g^2}^+$, hence $g^2 = e$. \hfill \Box
In the remainder of this section the action of $G$ is assumed to be almost free. Our goal will be to prove the main theorem under this supplementary hypothesis.

The proof consists of four steps. In the first one, we construct a vector field $Z$ as the gradient of a suitable $G$-invariant Morse function $\mu$. In a second step, one modifies $Z$ for obtaining a new $G$-invariant vector field $Y$ with as many pseudo-circle as (local) minima of $\mu$.

The third part is the construction from $Y$ of a $G$-invariant vector field $X$ that possesses a countable family of chains. These chains are topological invariant of $X$ and allow us to control its continuous automorphisms. Finally, the fourth step is devoted to determine these automorphisms.

3.1. The gradient vector field. Let $\mu: M \to \mathbb{R}$ be a Morse function that is $G$-invariant, proper and non-negative, whose existence is assured by a result of Wasserman \[9\]. Let $C$ denote the set of critical points of $\mu$, which is closed, discrete and countable. As $M$ is paracompact, there exists a locally finite family of disjoint open sets \{\(A_p: p \in \mathbb{C}\}\}_{p \in \mathbb{C}} which is $G$-invariant, i.e. $A_g = g \cdot A_p$ for any $p \in \mathbb{C}$ and any $g \in G$. By shrinking each $A_p$ if necessary, one constructs a collection of charts \{(\(A_p, \rho_p\))\}_{p \in \mathbb{C}} such that:

- $\rho_p(A_p) = B(2r_p)$ for some $r_p > 0$ and $\rho_p(p) = 0$.
- $\mu = \sum_{j=1}^{k} x_j^2 - \sum_{j=k+1}^{m-1} x_j^2 + \varepsilon x_m^2 + \mu(p)$ on $A_p$ where $\varepsilon = \pm 1$. (Of course $k$ and $\varepsilon$ depend on $p$, and $x = (x_1, \ldots, x_m)$ are the coordinates associated to the chart $(A_p, \rho_p)$. Nevertheless, in order to avoid an over-elaborated notation, these facts are not indicated unless it is completely necessary.)
- $\rho_{gp} \circ g \circ \rho_p^{-1}$ equals
  - the identity map, if the $G$-orbit of $p$ has exactly $\ell$ elements, or
  - the identity or the symmetry $\Gamma(x) = (x_1, \ldots, x_{m-1}, -x_m)$, if the $G$-orbit of $p$ has less than $\ell$ elements.
- The function $\mu$ can be chosen in such a way that the $G$-orbit of every (local) minimum has $\ell$ elements.

Indeed, let $\mathcal{O} \subset \mathbb{C}$ be a $G$-orbit and fix a point $p \in \mathcal{O}$. If $|\mathcal{O}| = \ell$ one constructs such a chart around $p$ and then use the $G$-action to get a chart around any point of $\mathcal{O}$. 
If $|\mathcal{O}| < \ell$ then $p \in I_G$ and, by Lemma 3.1, $|\mathcal{O}| = \ell/2$ and there exists just one element $h \in G - \{e\}$ such that $h \cdot p = p$. Moreover $h^2 = e$. As $I_h = I_h^{\max}$ there are coordinates $(y_1, \ldots, y_m)$ around $p \equiv 0$ such that $h$ is given by the symmetry $\Gamma(y) = (y_1, \ldots, y_{m-1}, -y_m)$.

By Lemma 7.1 applied to coordinates $(y_1, \ldots, y_m)$ (observe that now $x$ and $y$ have exchanged their roles) there exist coordinates $(x_1, \ldots, x_m)$ around $p \equiv 0$ in which $h$ is still given by the symmetry $\Gamma(x) = (x_1, \ldots, x_{m-1}, -x_m)$ and $\mu = \sum_{j=1}^{k} x_j^2 - \sum_{j=k+1}^{m-1} x_j^2 + \varepsilon x_m^2 + \mu(p)$, $\varepsilon = \pm 1$. Now for another point $q \in \mathcal{O}$ choose $a \in G - \{e\}$ such that $q = g \cdot p$ and set $\rho_q = \rho_p \circ g^{-1}$.

Finally if our $p$ is a minimum of $\mu$, always with a $G$-orbit of $\ell/2$ elements, by applying Proposition 7.2 to a 0 < $r_p' < \min\{1, r_p\}$ we may modify $\mu$ inside $\rho^{-1}(B(r_p))$ to construct a new $h$-invariant Morse function, still called $\mu$, such that each one of its minima in $A_p$ is not $h$-invariant and, as before, transfer this modification to every $A_q$, $q \in \mathcal{O} - \{p\}$, by means of a $q \in G - \{e\}$ such that $g \cdot p = q$.

As every minimum in $A_p$ of the new $\mu$ is not $h$-invariant, then the $G$-orbit of any minimum in $\bigcup_{q \in \mathcal{O}} A_q$ of the new $\mu$ has $\ell$ elements. Obviously the same thing can be done with any other $G$-orbit with $\ell/2$ elements consisting of minima.

On the other hand since $|\tau(x)| \leq ||x||^2$ in Proposition 7.2 the new function $\mu$ is proper and low bounded by $-1$ (more exactly the difference between the new function $\mu$ and the old one takes its values in $[-2, 0]$). Therefore replacing $\mu$ by $\mu + 1$ shows (C.4).

On $M$ there always exists a Riemannian metric $g'$ that on each $\rho_p^{-1}(B(r_p))$ is written as $2 \sum_{j=1}^{m} dx_j \otimes dx_j$. Therefore shrinking every $A_p$ allows to assume $g' = 2 \sum_{j=1}^{m} dx_j \otimes dx_j$ on the whole $A_p$. Moreover taking into account Property (C.3) of the collection $\{(A_p, \rho_p)\}_{p \in C}$ we may assume, without losing the property above, that $g'$ is $G$-invariant by considering $(1/\ell) \sum_{h \in G} h^{*}(g')$ instead of $g'$ if necessary.

Let $Z'$ be the gradient vector field of $\mu$ with respect to $g'$ and $\varphi: M \rightarrow \mathbb{R}$ be a $G$-invariant proper function that is constant around every $p \in C$. As before, $\varphi$ can be supposed constant on each $A_p$ by shrinking these open sets if necessary. It is well known that the vector field $Z = e^{-(Z')^2} Z'$ is complete. Moreover $Z$ is the gradient of $\mu$ with respect to the $G$-invariant Riemannian metric $\tilde{g} = e^{(Z')^2} g'$. 
On the other hand $\tilde{g} = g'$ on every $A_p$, $p \in C$, since $\varphi$ is constant on these sets. Hence

$$Z = \sum_{j=1}^{k} x_j \frac{\partial}{\partial x_j} - \sum_{j=k+1}^{m-1} x_j \frac{\partial}{\partial x_j} + \varepsilon x_m \frac{\partial}{\partial x_m}$$

$\varepsilon = \pm 1$ on each $A_p$.

3.2. **Construction of pseudo-circles.** Since $\mu$ is non-negative and proper, the $\alpha$-limit of any regular trajectory of $Z$ is a (local) minimum or a saddle of $\mu$, whereas its $\omega$-limit is empty, a (local) maximum or a saddle of $\mu$. Moreover $Z$ does not possesses any pseudo-circle because no trajectory of a gradient vector field has its $\alpha$-limit equal to its $\omega$-limit. Clearly $Z$ does not have rivets nor topological rivets.

Now by modifying $Z$ we will construct a new vector field with as many pseudo-circle as minima of $\mu$.

Let $I$ be the set of minima of $\mu$ and $\hat{I}$ be that of maxima. For sake of simplicity let us identify $A_i$ with $B(2r_i)$. Denote by $E_i$, $i \in I$, the sphere in $A_i$ of radius $r_i$ and center the origin. For each $i \in I$ there exist $\varepsilon_i > 0$, $0 < r_i'' < r_i < r_i' < 2r_i$ and a diffeomorphism identifying $\tilde{A}_i = B(r_i') - B(r_i'')$ with $(-\varepsilon_i, \varepsilon_i) \times S^{m-1}$, endowed with coordinates $(t, y)$, in such a way that $E_i$ corresponds to $\{0\} \times S^{m-1}$ and $Z$ to $\partial/\partial t$.

Let $\varphi_i : \mathbb{R} \to \mathbb{R}$ be a function such that $\varphi_i((\infty, -\varepsilon_i/2]) = 1$, $\varphi_i([\varepsilon_i/2, \infty)) = -1$ and $\varphi_i(t) = 0$ if and only if $t = 0$. On every $\tilde{A}_i \equiv (-\varepsilon_i, \varepsilon_i) \times S^{m-1}$ define

$$Y = \varphi_i(t)\partial/\partial t + (1 - \varphi_i^2(t))Z_i'$$

where:

1. If $m \geq 3$ then $Z_i'$ is a limit vector field on $S^{m-1}$ regarded on $(-\varepsilon_i, \varepsilon_i) \times S^{m-1}$ in the obvious way.

2. If $m = 2$ then $Z_i' = \lambda_i(t, y)(-y_2\partial/\partial y_1 + y_1\partial/\partial y_2)$ where the function $\lambda_i : (-\varepsilon_i, \varepsilon_i) \times S^1 \to \mathbb{R}$ vanishes at some point $(0, q_i)$ of $\{0\} \times S^1$ and is positive elsewhere.

Now prolong $Y$ to each $A_i$ by $Z$ inside of $A_i - \tilde{A}_i$, that is on $B(r_i'')$, and by $-Z$ outside of $A_i - \tilde{A}_i$, that is on $A_i - B(r_i')$. In turn, extend $Y$ already defined on $\bigcup_{i \in I} A_i$ to the whole manifold $M$ by $-Z$ on $M - \bigcup_{i \in I} A_i$.

To be sure that $Y$ is $G$-invariant, in each orbit $O$ included in $I$ choose a point $i$ and construct $Y$ on $A_i$ as before. Then by means of the $G$-action construct $Y$ on every $A_j$, $j \in O$. 

As the action of $G$ on $\bigcup_{j \in O} A_j$ is free by property (C.4) of the family $\{(A_p, \rho_p)\}_{p \in C}$, this construction is coherent. Therefore from now on $Y$ will be assumed $G$-invariant.

Notice that the singularities of $Y$ in $M - \bigcup_{i \in I} E_i$ are saddles or sources. The singularities of $Y$ in $\bigcup_{i \in I} E_i$ are never sources nor rivets since each of them is the $\omega$-limit of two or more regular trajectories traced in $M - \bigcup_{i \in I} E_i$. Thus $Y$ has no topological rivet and, consequently, no chain. Besides every $E_i$ contains a single pseudo-circle of $Y$ denoted by $P_i$ henceforth; this vector field does not possess any other pseudo-circle.

It is easily checked that $Y$ is complete with no regular periodic trajectories. On the other hand the set $Y^{-1}(0)$ of singularities of $Y$ consists of $C$ plus the singularities in each $E_i$ (a finite number for every $E_i$). Since the family $\{E_i\}_{i \in C}$ is locally finite because $\{A_p\}_{p \in C}$ is, it follows that $Y^{-1}(0)$ is discrete and countable. Moreover the set of sources of $Y$ equals $I \cup \tilde{I}$.

**Lemma 3.2.** There exists a subset $Q \subset M$, which does not exceed dimension $m - 1$, such that for every point $q \in (M - Q)$, the $Y$-trajectory of $q$ is regular and included in $M - Q$, its $\alpha$-limit is a source or empty, and its $\omega$-limit a pseudo-circle.

**Proof.** As the set of zeros of $Y$ is countable and $\bigcup_{i \in I} E_i$ can be enclosed in dimension $m - 1$, it suffices to consider the points $q$ of $M - \bigcup_{i \in I} E_i$ such that $Y(q) \neq 0$. On the other hand since the outset and the inset of any saddle is enclosed in dimension $m - 1$, the set of points whose $\alpha$-limit or whose $\omega$-limit is a saddle is enclosed in dimension $m - 1$.

Thus it suffices to study those points $q$ in $M - \bigcup_{i \in I} E_i$ whose trajectory is regular and intersects some $A_i$.

By construction $Y$ is tangent to $E_i \equiv \{0\} \times S^{m-1}$ and a limit vector field on this submanifold. Therefore there exists $\{0\} \times \tilde{Q}_i \subset E_i$, which can be enclosed in dimension $m - 2$, such that for any $q \in (E_i - \{0\} \times \tilde{Q}_i)$ its $Y$-trajectory has the pseudo-circle $P_i$ as $\omega$-limit. Consequently, the pseudo-circle $P_i$ is the $\omega$-limit of the $Y$-trajectory of each point of $(-\varepsilon_i, \varepsilon_i) \times (S^{m-1} - \tilde{Q}_i)$.

Let $\Phi_t$ be the flow of $Y$. The set $(-\varepsilon_i, \varepsilon_i) \times \tilde{Q}_i$ does not exceed dimension $m - 1$ and, since $Q$ is countable, neither does $\bigcup_{i \in Q} \Phi_t((-\varepsilon_i, \varepsilon_i) \times \tilde{Q}_i)$. In other words, taking into account that $I$ is countable follows that the set of points $q \in (M - \bigcup_{j \in I} E_j)$, with $Y(q) \neq 0$, whose $Y$-trajectory intersects some $A_i$ but whose $\omega$-limit is not a pseudo-circle may be enclosed in dimension $m - 1$. 
Finally if $M - Q$ is not $Y$-saturated, since all points of the trajectory of $q \in M - Q$ have the same properties, we may replace $Q$ by $Q' = M - M'$ where $M'$ is the $Y$-saturation of $M - Q$. □

3.3. **Construction of chains.** First notice that $Y$ is tangent to $I_G = I_G^{\max}$ because it is $G$-invariant. Consider a set $Q$ as in Lemma 3.2 which is $Y$-saturated since $M - Q$ is so. On the other hand as $G$ is finite the set $G \cdot Q$ still has the properties of Lemma 3.2. In short, we may suppose that $Q$ is $G$ and $Y$-saturated. Since $Q = Q^0 = \emptyset$ and $I_G$ is closed, there exists a countable set $N \subset M - (Q \cup I_G)$ that is dense in $M$ and $G$-saturated. Observe that any orbit of the action of $G$ on $N$ has $\ell$ elements because this action is free. Let $F$ be the family of all trajectories of $Y$ that intersect $N$. Observe that $F$ is infinite since no trajectory of $Y$ is locally dense.

**Lemma 3.3.** Let $U$ be a $G$-invariant vector field on $M$, $\varphi_t$ be its flow and $q$ be a point whose $U$-trajectory is non-periodic. Given $(g, t), (h, s) \in G \times \mathbb{R}$, if $(g \circ \varphi_t)(q) = (h \circ \varphi_s)(q)$ then $t = s$ and either $g = h$ or $q \in I_G$.

**Proof.** From hypotheses, it immediately follows that $((h^{-1}g) \circ \varphi(t-s))(q) = q$. Therefore

$$q = ((h^{-1}g) \circ \varphi(t-s))^\ell(q) = ((h^{-1}g)^\ell \circ \varphi(t-s))(q) = \varphi(t-s)(q)$$

since $\ell$ is the order of $G$. Hence $\varphi(t-s)(q) = q$, which implies $\ell(t-s) = 0$ and $t = s$.

Thus $(h^{-1}g) \cdot q = q$. If $h^{-1}g \neq e$ then $q \in I_{(h^{-1}g)} \subset I_G$. □

**Corollary 3.4.** The natural action of $G$ on $F$ is free.

**Proof.** Assume $g \cdot T = T$ for some $g \in G$ and $T \in F$. Then given $q \in T$ there exists $t \in \mathbb{R}$ such that $\Phi_t(q) = g \cdot q$ and, applying Lemma 3.3 to $Y$ and $q$, it follows that $t = 0$ and $g \cdot q = q$. As $q \not\in I_G$, then $g = e$ must hold. □

The set $F$ is a disjoint union of $G$-orbits, say $F_n$, $n \geq 3$ (by technical reasons we start at natural three). Let $N' = N - \{0, 1, 2\}$. Since by Corollary 3.4 each $F_n$ consists of $\ell$ different trajectories one set $F = \{ T_{nk} : n \in N', k = 1, \ldots, \ell \}$, where $F_n = \{ T_{nk} : k = 1, \ldots, \ell \}$, in such a way that $T_{nk} \neq T_{n'k'}$ if $(n, k) \neq (n', k')$. (That is to say first one numbers the $G$-orbits in $F$ and then, with a second subindex, the elements of every orbit.)
Consider a sequence of $G$-invariant compact sets $\{K_n\}_{n \in \mathbb{N}}$ such that $K_n \subset \overset{\circ}{K}_{n+1}$ and $\bigcup_{n \in \mathbb{N}} K_n = M$. For every trajectory $T_{nk}$ let $W_{nk}$ be a set of $n - 1$ different points of $T_{nk}$ in such a way that:

(a) If $g \cdot T_{nk} = T_{nk'}$ then $g \cdot W_{nk} = W_{nk'}$.

(b) $W_{nk} \subset M - K_n$ if the $\alpha$-limit of $T_{nk}$ is empty.

(c) $W_{nk} \subset \rho_i^{-1}(B(r_i/n))$ if the $\alpha$-limit of $T_{nk}$ is $i$.

Let $W = \bigcup_{n \in \mathbb{N}', k = 1, \ldots, \ell} W_{nk}$; then $C \cup W$ is a countable set whose accumulation points are the minima and the maxima of $\mu$, i.e. the elements of $I \cup \overset{\circ}{I}$. Therefore $C \cup W$ is closed and there exists a function $\tau : M \to [0, 1] \subset \mathbb{R}$ such that $\tau^{-1}(0) = C \cup W$. Set $X = \tau Y$. It easily seen that:

(1) $X^{-1}(0) = Y^{-1}(0) \cup W$ is countable and closed. The set of its accumulation points equals $I \cup \overset{\circ}{I}$.

(2) $X$ is complete and has no periodic regular trajectory.

(3) $\{P_i\}_{i \in I}$ is the family of all pseudo-circles of $X$.

(4) Let $C_{nk}$ be the family of $X$-trajectories of $T_{nk} - W_{nk}$ endowed with the order induced by that of $T_{nk}$ as $Y$-trajectory. Then $C_{nk}$ is a chain of $X$ of order $n$ whose rivets are the points of $W_{nk}$. Besides $C_{n1}, \ldots, C_{n\ell}$ are the only chains of $X$ of order $n$ and hence $\{C_{nk}\}, ~k = 1, \ldots, \ell, ~n \in \mathbb{N}'$, is the set of all the chains of $X$.

Denote by $H_{nk}$ the last link of $C_{nk}$ and by $P_{\lambda(n,k)}$ the $\omega$-limit of $H_{nk}$ (therefore $\lambda$ is a map from $\mathbb{N}' \times \{1, \ldots, \ell\}$ to $I$).

(5) $\bigcup_{n \in \mathbb{N}', k = 1, \ldots, \ell} H_{nk}$ is dense in $M$.

**Remark 3.5.** Notice that the chains $C_{nk}$ given by (4) can be described in topological terms as finite sequences of $X$-trajectories such that:

(i) Between two consecutive links, the $\omega$-limit of the first one equals the $\alpha$-limit of the second one. Moreover this set consists in a topological rivet.

(ii) The $\alpha$-limit of the first link is an accumulation point of $X^{-1}(0)$ or empty.

(iii) The $\omega$-limit of the last link is a pseudo-circle.

Observe that $\{C_{nk}\}, ~k = 1, \ldots, \ell, ~n \in \mathbb{N}'$, is the set of all the objects satisfying (i), (ii) and (iii) above because $W$ is the set of topological rivets of $X$ ($Y$ has no rivet). Therefore any continuous automorphisms of $X$ maps chains to chains.
By definition the roll $R_i$, $i \in I$, is the union of all $H_{nk}$ whose $\omega$-limit equals $P_i$.

3.4. **$X$ is a suitable vector field.** In this subsection $\Phi_t$ will be the flow of $X$. Consider a homeomorphism $f : M \rightarrow M$ such that $f \circ \Phi_t = \Phi_t \circ f$ for any $t \in \mathbb{R}$.

**Proposition 3.6.** For each roll $R_i$ there exist $t_i \in \mathbb{R}$ and $g_i \in G$ such that $f = g_i \circ \Phi_{t_i}$ on $R_i$.

**Proof.** Fixed a $R_i$ consider a chain $C_{nk}$ whose last link $H_{nk}$ has $P_i$ as $\omega$-limit. By Remark 3.5 $f(C_{nk})$ is a chain of of order $n'$, so $f(C_{nk}) = C_{nk'}$ and $f(H_{nk}) = H_{nk'}$ for some $k' \in \{1, \ldots, \ell\}$. Moreover $f(P_i)$ is the $\omega$-limit of $H_{nk'}$.

As $\mathcal{F}_n = \{T_{n1}, \ldots, T_{n\ell}\}$ is an orbit of the action of $G$ on $\mathcal{F}$, there exists $h \in G$ such that $h \cdot T_{nk} = T_{nk'}$, hence $h \cdot H_{nk} = H_{nk'}$. Now by composing $f$ on the left with $h^{-1}$ we may assume $f(C_{nk}) = C_{nk}$, $f(H_{nk}) = H_{nk}$ and $f(P_i) = P_i$.

Since $f$ commutes with any $\Phi_t$ and the regular trajectory of $P_i$ is not periodic, there exists a single $t_i \in \mathbb{R}$ such that $f = \Phi_{t_i}$ on $P_i$.

Now consider any $H_{ab}$, $(a, b) \in \mathbb{N} \times \{1, \ldots, \ell\}$, with $\omega$-limit $P_i$. Then $f(H_{ab})$, which is the last link of $f(C_{ab})$, has $P_i$ as $\omega$-limit too.

Recall that the $G$-orbit of $i$ possesses $\ell$ elements, that is to say if $i \not\in I_G$. Therefore there is a single $H_{ab'}$ with $\omega$-limit $P_i$; $H_{ab}$ itself. Indeed, there are only $\ell$ chains of order $a$ and the $\omega$-limits of their last links are included in the disjoint union $\bigcup_{g \in G}(g \cdot A_i)$. In other words $f(H_{ab}) = H_{ab}$.

But $H_{ab}$ is a non-periodic regular trajectory and $f$ commutes with the flow $\Phi_t$, so there exists $t' \in \mathbb{R}$ such that $f = \Phi_{t'}$ on $H_{ab}$. As $P_i$ is the $\omega$-limit of $H_{ab}$ one has $f = \Phi_{t'}$ on $P_i$ too, hence $t' = t_i$. \hfill $\square$

From Proposition 3.6 it immediately follows:

**Corollary 3.7.** For every roll $R_i$ there exist $t_i \in \mathbb{R}$ and $g_i \in G$ such that $f = g_i \circ \Phi_{t_i}$ on $R_i$.

**Lemma 3.8.** The family $\{\overline{R}_i\}_{i \in I}$ is locally finite and $\bigcup_{i \in I} \overline{R}_i = M$.

**Proof.** For the first part it suffices to show that $\{R_i\}_{i \in I}$ is locally finite. From the fact that $P_i$ is included in $A_i$ follows that $\mu(R_i)$ is low bounded by $\mu(i)$. But $I$ is a discrete set and $\mu$ a non-negative proper Morse function, so in every compact set $\mu^{-1}((-\infty, a])$ there are only
a finite number of elements of \( I \). Therefore \( \mu^{-1}((-\infty,a]) \) and of course \( \mu^{-1}(-\infty,a) \) only intersect a finite number of rolls \( R_i \). Finally, observe that \( M = \bigcup_{a \in \mathbb{R}} \mu^{-1}(-\infty,a) \).

By construction of \( X \) (see Property \([5]\)) \( \bigcup_{n \in \mathbb{N}',k=1,\ldots,\ell} H_{nk} \) is dense in \( M \). On the other hand, \( \bigcup_{i \in I} \overline{R}_i \) is closed because \( \{\overline{R}_i\}_{i \in I} \) is locally finite, so \( \bigcup_{i \in I} \overline{R}_i \) is a closed set that includes \( \bigcup_{n \in \mathbb{N}',k=1,\ldots,\ell} H_{nk} \).

\[ \square \]

**Lemma 3.9.** All scalars \( t_i \) given by Corollary 3.7 are equal.

**Proof.** Assume that the family \( \{t_i\}_{i \in I} \) possesses two or more elements. Fixed one of them, say \( t \), set \( D_1 \) the union of all \( \overline{R}_i \) such that \( t_i = t \) and \( D_2 \) the union of all \( \overline{R}_i \) such that \( t_i \neq t \). By Lemma 3.8, \( D_1 \) and \( D_2 \) are closed and \( M = D_1 \cup D_2 \).

On the other hand if \( p \in D_1 \cap D_2 \) then there exist \( \overline{R}_i \subset D_1 \) and \( \overline{R}_j \subset D_2 \) such that \( p \in \overline{R}_i \cap \overline{R}_j \); so \( f(p) = (g_i \circ \Phi_{t_i})(p) = (g_j \circ \Phi_{t_j})(p) \). As \( t_i \neq t_j \) from Lemma 3.3 applied to \( X \) and \( p \) follows that the \( X \)-orbit of \( p \) is periodic. Hence \( X(p) = 0 \) since \( X \) has no periodic regular trajectories, which implies that \( D_1 \cap D_2 \) is countable. Consequently \( M = D_1 \cap D_2 \) is connected. But \( M - D_1 \cap D_2 = (D_1 - D_1 \cap D_2) \cup (D_2 - D_1 \cap D_2) \) where the terms of this union are non-empty, disjoint and closed in \( M - D_1 \cap D_2 \), contradiction. \[ \square \]

Now composing \( f \) with \( \Phi_{-t} \) where \( t \) is the scalar given by Corollary 3.7 and Lemma 3.9 we may assume, without lost of generality, that \( f(x) = g_x \cdot x \) for any \( x \in M \) where \( g_x \in G \). For finishing the proof of the existence of \( (g,t) \in G \times \mathbb{R} \) such that \( f = g \circ \Phi_t \) it suffices to apply the following result:

**Lemma 3.10.** Consider a continuous and injective map \( \tau : M \to M \). Assume for every \( x \in M \) there exists \( g_x \in G \) such that \( \tau(x) = g_x \cdot x \). Then \( \tau = g \) for some \( g \in G \).

**Proof.** Given \( g \in G \) set \( D_g := \{ p \in M - I_G : \tau(p) = g \cdot p \} \). As the space is Hausdorff \( G_g \) is closed in \( M - I_G \). Moreover \( D_g \cap D_h = \emptyset \) when \( g \neq h \) since the action of \( G \) on \( M - I_G \) is free. As \( G \) is finite and \( M - I_G = \bigcup_{h \in G} D_h \), every \( D_g \) is open too so union of connected components of \( M - I_G \).

Some of the sets \( D_g \), \( g \in G \), has to be non-empty and by composing \( \tau \) on the left with a suitable element of \( G \) one may assume \( D_e \neq \emptyset \). Let us see that \( \overline{D}_e \) is open in \( M \). If so the proof is finished since \( M \) is connected and necessarily \( \overline{D}_e = M \).

Consider any \( p \in \overline{D}_e \). If \( p \in M - I_G \) then \( p \in D_e \) and it is an interior point. Now assume \( p \in I_h = I_h^{\max} \) for some \( h \in G - \{e\} \). Then around \( p \) there exist coordinates \( (x_1, \ldots, x_m) \)
whose domain \( D \) is diffeomorphic through these coordinates to an open ball \( B(r) \) such that \( p \equiv 0 \) and \( h \) is given by the symmetry \( \Gamma(x) = (x_1, \ldots, x_{m-1}, -x_m) \).

Let \( S^+ \) be the open half domain defined by \( x_m > 0 \) and \( S^- \) that given by \( x_m < 0 \). If \( r \) is sufficiently small then \( S^+ \cup S^- \subset M - I_G \), and \( I_G \cap D \) and \( I_h \cap D \) are equal and defined by \( x_m = 0 \) (see \( (a) \) of Lemma 3.1). As \( p \in D \) and \( D \) is an union of connected components of \( M - I_G \) necessarily one at least of the foregoing open half domains is included in \( D \).

Assume \( S^+ \subset D \), the other case is similar. If \( S^- \not\subset D \) then \( S^- \cap D = \emptyset \) and there is some \( \bar{g} \in G - \{e\} \) such that \( S^- \subset D \bar{g} \) because \( S^- \) is connected.

By continuity \( \bar{g} \cdot p = \tau(p) \) and \( e \cdot p = \tau(p) \), hence \( \bar{g} \cdot p = p \) and by \( (a) \) of Lemma 3.1 one has \( \bar{g} = h \). Therefore \( \tau \) on \( S^- \) equals \( h \). Thus \( \tau(0, \ldots, 0, -\varepsilon) = (0, \ldots, 0, \varepsilon) = e \cdot (0, \ldots, 0, \varepsilon) = \tau(0, \ldots, 0, \varepsilon), \varepsilon > 0 \), and \( \tau \) is not injective, contradiction. In short \( S^- \subset D \) and necessarily \( p \in D \subset \overline{D} \). \( \square \)

Finally, if \( g \circ \Phi_t = Id_M \) then \( Id_M = (g \circ \Phi_t)^\ell = g^\ell \circ \Phi_{\ell t} = \Phi_{\ell t} \). As \( X \) has regular non-periodic trajectories \( t = 0 \), so \( g = e \). This fact implies the injectivity of the morphism from \( G \times \mathbb{R} \) to \( \text{Aut}_0(\mathbb{X}) \). Therefore the main theorem is proved under the supplementary hypothesis \( I_G = I_G^{\max} \).

**Remark 3.11.** Consider a function \( \varphi : M \to \mathbb{R} \) that is \( G \)-invariant, positive and bounded. Then \( \varphi X \) is a complete vector field. Besides \( X \) and \( \varphi X \) have the same trajectories (with different speeds but the same orientation by the time), \( \alpha \) and \( \omega \)-limits, pseudo-circles, rolls, rivets and chains. Therefore reasoning as before but this time with \( \varphi X \) shows that \( (g, t) \in G \times \mathbb{R} \to g \circ \Phi_t \in \text{Aut}_0(\varphi X) \) is a group isomorphism where \( \Phi_t \) is the flow of \( \varphi X \).

In other words \( \varphi X \) is a suitable vector field too.

### 4. THE GENERAL CASE

In this section the main result will be proved in the general case by reducing it to the almost free one.

Given \( g \in G - \{e\} \) let \( J_g \) be the set of those points \( p \in I_g \) such that the dimension of \( I_g \) at \( p \) is \( \leq m - 2 \). Set \( J_G = \bigcup_{g \in G - \{e\}} J_g \). It is easily seen that \( J_G \) is a \( G \)-invariant closed set and \( M - J_G \) a \( G \)-invariant dense open set.
One will say that the dimension of a point $p \in J_g$ is zero if the dimension at $p$ of every $I_g$ such that $p \in J_g$, is zero. Let $S_0$ be the set of all points of $J_G$ of dimension zero. Clearly $S_0$ is $G$-invariant and $S_0 \subset J_G$.

By making use of normal coordinates with respect to a $G$-invariant Riemannian metric, centered at points of $J_G$, it is easily checked that:

1. $S_0$ has no accumulation point so it is countable and closed.
2. Every $q \in S_0$ possesses a neighborhood whose intersection with $J_G$ equals $\{q\}$.
3. For any $q \in J_G - S_0$ and any neighborhood $A$ of $q$ the set $A \cap J_G$ is uncountable.

Since $M$ is paracompact (even more $\sigma$-compact) from [1] and [2] follows the existence of a locally finite family of disjoint open sets $\tilde{A} = \{\tilde{A}_q\}_{q \in S_0}$ such that every $\tilde{A}_q \cap S_0 = \{q\}$. As $S_0$ is $G$-invariant and $G$ is finite shrinking the elements of $\tilde{A}$ allows us to assume that this family is $G$-invariant. Even more one can suppose that each $\tilde{A}_q$ is a domain of normal coordinates centered at $q$ of some $G$-invariant Riemannian metric.

For every $q \in S_0$ consider a set $q \in C_q \subset \tilde{A}_q$ that in the normal coordinates mentioned before is a closed non-trivial segment sufficiently small. Set $D_q : = \{(g, p) \in G \times S_0 : g \cdot p = q\}$ and $E_q : = \bigcup_{(g, p) \in D_q} g \cdot C_p$. Then $E_q \subset \tilde{A}_q$ so the family $\{E_q\}_{q \in S_0}$ is locally finite, hence $S_1 : = \bigcup_{q \in S_0} E_q$ is closed. Moreover $S_1$ is $G$-invariant and any neighborhood of any point of $S_1$ includes uncountably many points of $S_1$. On the other hand we may assumed that $M - S_1$ is connected without loss of generality.

Thus the set $\tilde{M} : = M - (J_G \cup S_1)$ is $G$-invariant, connected, dense and open, since $S_1$ can be enclosed in dimension one and $J_g$ in dimension $m - 2$ and $M - S_1$ was connected. Besides each neighborhood of every point of $J_G \cup S_1$ contains uncountably many elements of $J_G \cup S_1$.

On the other hand the action of $G$ on $\tilde{M}$ is almost free. Indeed, if $q \in I_g \cap \tilde{M}$ for some $g \in G - \{e\}$ then $q \notin J_g$ so the dimension of $I_g$ at $q$ equals $m - 1$ and $q \in I_g^{\text{max}}$.

By proposition 5.5 of [4] there exists a bounded function $\varphi : M \to \mathbb{R}$, which is positive on $\tilde{M}$ and vanishes on $M - \tilde{M}$, such that the vector field $\hat{X}$ on $M$ defined by $\hat{X} = \varphi X$ on $\tilde{M}$ and $\hat{X} = 0$ on $M - \tilde{M}$ is differentiable.

Notice that $g^\prime\inv(\hat{X})$ equals $(\varphi \circ g)X$ and zero on $M - \tilde{M}$. Therefore by taking $\ell^{-1} \sum_{g \in G}(\varphi \circ g)$ instead of $\varphi$ we may assume that $\varphi$ is $G$-invariant, which implies that $\hat{X}$ is $G$-invariant too.

Given a singularity $p$ of $\hat{X}$ one has two possibilities:
(a) Any neighborhood of $p$ contains uncountably many zeros of $\hat{X}$; that is to say $p \in M - \bar{M}$.

(b) There is a neighborhood of $p$ that only includes countably many zeros of $\hat{X}$; that is to say $p \in \bar{M}$ and $X(p) = 0$.

Consider $f \in \text{Aut}_0(\hat{X})$. From (a) and (b) follows that $f(M - \bar{M}) = M - \bar{M}$ and $f(\bar{M}) = \bar{M}$.

Thus the restriction of $f$ to $\bar{M}$ is a continuous automorphism of $\hat{X}|_M = \varphi_X$ and, by Section 3, there exist $g \in \mathbb{G}$ and $t \in \mathbb{R}$ such that $f = g \circ \hat{\Phi}_t$ on $\bar{M}$ where $\hat{\Phi}_t$ is the flow of $\hat{X}$. By continuity $f = g \circ \hat{\Phi}_t$ everywhere. The uniqueness of $g$ and $t$ is obvious. In short the main result is proved in the general case.

5. ACTIONS ON MANIFOLDS WITH BOUNDARY

Let $P$ be a $m$-manifold with nonempty boundary $\partial P$. Then each homeomorphism $f: P \to P$ induces a homeomorphism $f: \partial P \to \partial P$. Therefore the same reasoning as in Section 4 of [8] shows that the main result of the present paper also holds for a connected manifold $P$, of dimension $m \geq 2$, with nonempty boundary and a finite subgroup $G$ of $\text{Diff}(M)$.

6. EXAMPLES

Example 6.1. On $\mathbb{R}^2$ consider the group of two elements $G = \{e, g\}$ where $g(x) = (-x_1, x_2)$. Then the action of $G$ on $\mathbb{R}^2$ is almost free and $I^\text{max}_G = \{0\} \times \mathbb{R}$. For constructing a suitable vector field $X$ as in Section 3 one can start with the Morse function $\mu = (x_1^2 - 1)^2 + x_2^2$ that has two minima at $(1, 0)$ and $(-1, 0)$ respectively and a saddle at the origin.

Therefore at the end of the process $X$ has two pseudo-circles around $(1, 0)$ and $(-1, 0)$ respectively. Moreover the set of singularities of $X$ is countable and accumulates towards $(1, 0), (-1, 0)$ and the infinity. Observe that $I^\text{max}_G$ consists of a singular point and two regular trajectories with the singular point as $\alpha$-limit and empty $\omega$-limit.

In a similar way, on $S^2$ one may consider the group $G = \{e, g\}$ where now $g(x) = (x_1, x_2, -x_3)$ and the Morse function $\mu = 2x_1^2 + x_2^2$ that has two minima at $(0, 0, \pm 1)$, two maxima at $(\pm 1, 0, 0)$ and two saddles at $(0, \pm 1, 0)$. The action of $G$ is almost free and there is no minimum on $I^\text{max}_G = S^2 \cap (\mathbb{R}^2 \times \{0\})$.

Example 6.2. Let $M$ be a connected compact manifold of dimension $m \geq 2$. Given $G$, a finite group of diffeomorphisms of $M$, and a $G$-invariant Morse function $\mu$, let $X$ be the
gradient vector field of $\mu$ with respect to a $G$-invariant Riemannian metric. Then, although the group of smooth automorphisms of $X$, namely $\text{Aut}(X)$, may equal $G \times \mathbb{R}$ (e.g. in Example 5.2), the group $\text{Aut}_0(X)$ is strictly greater than $G \times \mathbb{R}$.

Indeed, first note that there always exist a minimum and a maximum of $\mu$, that we denote by $p$ and $q$, and a trajectory $\gamma$ of $X$ whose $\alpha$-limit and $\omega$-limit are $p$ and $q$ respectively. Consider a closed sufficiently small $(m - 1)$-disk $D$ transverse to $X$ and intersecting $\gamma$ just once. We may suppose, without lost of generality, that every trajectory of $X$ intersects $D$ at most once and if so its $\alpha$-limit equals $p$ and its $\omega$-limit $q$.

Let $E$ be the set of those points of $M$ whose trajectory meets $D$. Then $E$ is diffeomorphic to $D \times \mathbb{R}$ in such a way that $X$ becomes $\partial/\partial s$ where $D \times \mathbb{R}$ is endowed with coordinates $(x, s) = (x_1, \ldots, x_{m-1}, s)$.

For each continuous function $\lambda: D \to \mathbb{R}$ such that $\lambda(\partial D) = 0$ one defines $f \in \text{Aut}_0(X)$ to be $f = \text{Id}$ on $M - E$ and $f(x, s) = \Phi_{t\lambda(x)}(x, s) = (x, s + \lambda(x))$ on $E$, where $\Phi_t$ is the flow of $X$. As $\overline{E} - E = ((\partial D) \times \mathbb{R}) \cup \{p, q\}$ our $f$ is continuous. Its inverse is given by $-\lambda$ and obviously $f$ is an automorphism of $X$, which in general does not belong to $G \times \mathbb{R}$.

Other way for constructing such a $f$ is to consider a homeomorphism $\tau: D \to D$ with $\tau|_{\partial D} = \text{Id}|_{\partial D}$ and set $f(x, s) = (\tau(x), s)$ on $E$, $f = \text{Id}$ elsewhere.

Observe that an analogous construction can be done if the gradient field is slightly modified, namely if one adds a finite number of new singularities of index zero. Thus, in general, the group of continuous automorphisms of vector fields constructed in [3] is strictly greater than $G \times \mathbb{R}$ (for the non-compact case the reasoning above can be easily adapted if there is at least a maximum). In other words, these vector fields determine $G$ in the smooth category but not in the continuous one.

The next two examples are extensions of Corollary 1.2 to situations where differentiability at every point is not assured. One has:

**Proposition 6.3.** Consider a finite group $G$ of homeomorphisms of a connected $C^\infty$ manifold $M$ of dimension $\geq 2$. Let $A$ be a $G$-invariant open set of $M$. Assume that:

(a) $A$ is connected and dense, and every neighborhood of each point of $M - A$ contains uncountably many points of $M - A$.

(b) Under restriction each element of $G$ is a diffeomorphism of $A$. 

Then there exists a $G$-invariant smooth, and therefore continuous, flow $\psi: \mathbb{R} \times M \to M$ such that the map

$$G \times \mathbb{R} \to \text{Aut}_0(\psi)$$

$$(g,t) \mapsto g \circ \psi_t$$

is a group isomorphism.

Proof. Consider $G$ under restriction as a group of diffeomorphisms of $A$ and define $J_G$ and $S_1$ as in Section 4 (for $A$ of course). Set $\tilde{M} = M - (J_G \cup S_1 \cup (M - A)) = A - J_G \cup S_1$. Then the action of $G$ on $\tilde{M}$ is almost free, which gives rise to a suitable vector field $X$ on $\tilde{M}$.

Since $J_G \cup S_1 \cup (M - A)$ is a closed set of $M$ with empty interior, a vector field $\hat{X}$ on $M$ that

1. vanishes in $J_G \cup S_1 \cup (M - A)$ and
2. on $\tilde{M}$ equals $\varphi X$ for a suitable function $\varphi: M \to \mathbb{R}$,

can be constructed as in Section 4.

The flow of $\hat{X}$ has the required properties. Indeed, any neighborhood of any point of $J_G \cup S_1 \cup (M - A)$ includes uncountably many points of this set, and one can reason as in the second part of Section 4. □

Example 6.4. Following the notation in [4, pp. 23–24], in $\mathbb{C}^4$ endowed with coordinates $z = (z_0, z_1, z_2, z_3)$ the equations

$$z_0^3 + z_1^2 + z_2^2 + z_3^2 = 0$$

$$\sum_{k=0}^{3} z_k \bar{z}_k = 1$$

define a smooth real submanifold that is diffeomorphic to the standard sphere $S^5$ such that $\beta(z) = (e^{2\pi i/3} z_0, z_1, z_2, z_3)$ defines a diffeomorphism of $S^5$ of order three, whose set of fixed points is (diffeomorphic to) $\mathbb{R}P^3$ [4, REMARKS p. 24].

The (topological) suspension of $S^5$ and that of $\beta$ give rise to a homeomorphism $f: S^6 \to S^6$ of order three, whose set of fixed points is (homeomorphic to) the suspension of $\mathbb{R}P^3$ in such a way that the vertices are the poles. Therefore $f$ cannot be smoothed otherwise the suspension of $\mathbb{R}P^3$ has to be a differentiable manifold, which is not the case.

Let $G$ be the group of homeomorphisms of $S^6$ spanned by $f$, whose order equals three. Clearly $G$ cannot be smoothed. However, away of the poles $G$ is a group of diffeomorphisms.
Consider a meridian (that is the intersection of $S^6 \subset \mathbb{R}^7$ with a plane passing through the origin and the poles) and saturate it under the action of $G$ for constructing a $G$-invariant compact set $C$. Finally set $A = S^6 - C$ and apply Proposition 6.3 for concluding that, even if $G$ cannot be smoothed, there exists a differentiable flow on $S^6$ that determines $G$.

**Theorem 6.5.** Let $G$ be a finite group of homeomorphisms of a connected compact topological 4-manifold $M$ with no boundary. Assume that the action of $G$ is free. Then there exist a continuous flow $\tilde{\Phi}$ that determines $G$.

We devote the rest of this section to the proof of Theorem 6.5.

Let $P = M/G$ be the topological quotient manifold and $\pi: M \to P$ the canonical projection, which is a covering.

Fix a point $a$ of $P$. Then $P' = P - \{a\}$ has a structure of smooth manifold (Quinn [7]). The pull-back of this structure defines a smooth structure in $M' = M - \pi^{-1}(a) = \pi^{-1}(P')$ in such a way that the natural action of $G$ on $M'$ is smooth and $\pi: M' \to P'$ is a smooth covering.

Consider a set $a \in C \subset P$ that with respect to some topological coordinates centered at $a$ is a closed non-trivial segment sufficiently small; assume that $a$ is one of its vertices. Then $\pi^{-1}(C)$ is a disjoint union of compact sets $C_1 \cup \ldots \cup C_\ell$ where $\ell$ is the order of $G$ and each $\pi: C_j \to C$ a homeomorphism. Notice that $P - C$ is connected and dense in $P$, and $M - \pi^{-1}(C)$ is connected, dense in $M$ and $G$-invariant.

Now construct a suitable vector field $X$ on $M - \pi^{-1}(C)$. Since $\pi^{-1}(C - \{a\})$ is closed in $M'$, a vector field $\hat{X}$ that vanishes on $\pi^{-1}(C - \{a\})$ and equals $\varphi X$ on $M - \pi^{-1}(C)$ for a suitable function $\varphi: M' \to \mathbb{R}$ can be constructed as in Section 4.

On the other hand the flow $\hat{\Phi}$ of $\hat{X}$ can be extended into a continuous flow $\tilde{\Phi}: \mathbb{R} \times M \to M$ by setting $\tilde{\Phi}(\mathbb{R} \times \{b\}) = b$ for every $b \in \pi^{-1}(a)$. Accept this fact by the moment; we will prove it later on. If $f: M \to M$ is a continuous automorphism of $\tilde{\Phi}$, then $f(\pi^{-1}(C)) = \pi^{-1}(C)$ by the same reason as in Section 4 (replace singularities of $\hat{X}$ by stationary points of the flow $\hat{\Phi}$ and take into account that any neighborhood of any point of $\pi^{-1}(C)$ contains uncountably many points of $\pi^{-1}(C)$). Therefore $f: M - \pi^{-1}(C) \to M - \pi^{-1}(C)$ is a continuous automorphism of $\hat{X}$ and hence $f = g \circ \tilde{\Phi}_t = g \circ \tilde{\Phi}_t$ on $M - \pi^{-1}(C)$ for some $g \in G$ and $t \in \mathbb{R}$. Since $M - \pi^{-1}(C)$ is dense in $M$, by continuity $f = g \circ \tilde{\Phi}_t$ everywhere.

Clearly if $g \circ \tilde{\Phi}_t = Id$ then $g = e$ and $t = 0$. 


In short, even if $M$ has no smooth structure, the continuous flow $\tilde{\Phi}$ determines $G$.

Let us prove that $\tilde{\Phi}$ is a continuous flow. The only difficult point is the continuity. Note that as $\tilde{\Phi}$ is $G$-invariant there is a smooth flow $\psi: \mathbb{R} \times P' \to P'$ such that $\pi \circ \tilde{\Phi} = \psi \circ (Id \times \pi)$ (it is the flow associated to the projection of $\tilde{X}$ onto $P'$). Denote by $\tilde{\psi}: \mathbb{R} \times P \to P$ the extension of $\psi$ defined by setting $\tilde{\psi}(\mathbb{R} \times \{a\}) = a$. Observe that $\pi \circ \tilde{\Phi} = \tilde{\psi} \circ (Id \times \pi)$.

Lemma 6.6. $\tilde{\psi}$ is continuous.

Proof. As before the difficult point is the continuity. For checking it one will show that $\tilde{\psi}: [a,b] \times P \to P$ is continuous for any $a < b$ belonging to $\mathbb{R}$.

Consider a map $s: E_1 \to E_2$ between locally compact but not compact topological spaces. Denote by $A(E_k)$, $k = 1, 2$, the Alexandroff compactification of $E_k$ and by $A(s): A(E_1) \to A(E_2)$ the extension of $s$ that maps the infinity point of $A(E_1)$ to that of $A(E_2)$. Recall that $A(s)$ is continuous if and only if $s$ is proper.

The map $h: [a, b] \times P' \to [a, b] \times P'$ given by $h(t, x) = (t, \psi(t, x))$ is a homeomorphism and hence $A(h): A([a, b] \times P') \to A([a, b] \times P')$ is continuous.

In turn the second projection $\pi_2: [a, b] \times P' \to P'$ is proper, so $A(\pi_2): A([a, b] \times P') \to A(P')$ is continuous.

Since the Alexandroff compactification is the smallest one among the Hausdorff compactifications, the map $g: [a, b] \times A(P') \to A([a, b] \times P')$ that equals the identity on $[a, b] \times P'$ and maps $[a, b] \times \{\infty\}$ to $\infty$ is continuous.

Finally if one identifies $P$ to $A(P')$ by regarding $a$ like the infinity point, then $\tilde{\psi} = A(\pi_2) \circ A(h) \circ g$. □

Corollary 6.7. $\tilde{\Phi}$ is continuous.

Proof. Consider the map $l: \mathbb{R} \times M \to P$ given by $l(t, x) = \tilde{\psi}(t, \pi(x))$ and a point $v \in M'$. As $\mathbb{R}$ is contractile, then $l$ is homotopic to the map

$$\begin{align*}
\mathbb{R} \times M & \to P \\
(t, x) & \mapsto \pi(x)
\end{align*}$$

Therefore $l_2(\pi_1(\mathbb{R} \times M, (0, v))) = \pi_2(\pi_1(M, v))$ and hence, with respect to the covering $\pi: M \to P$, there exists a lift $L: \mathbb{R} \times M \to M$ of $l$, with initial condition $L(0, v) = v$. Moreover $L = \tilde{\Phi}$ on $\mathbb{R} \times M'$ since one knows that $\tilde{\Phi}$ is continuous on $\mathbb{R} \times M'$ and $\pi \circ \tilde{\Phi} = \tilde{\psi} \circ (Id \times \pi)$. 
Finally as $\hat{X}$ vanishes on $\pi^{-1}(C - \{a\})$, it follows that $L = \tilde{\Phi} = Id$ on $\mathbb{R} \times \pi^{-1}(C - \{a\})$. By continuity $L(\mathbb{R} \times \{b\}) = b$ for each $b \in \pi^{-1}(a)$. Thus $L = \tilde{\Phi}$ everywhere.

7. Appendix

In this section $B(r), r > 0$, will be the ball in $\mathbb{R}^m$, endowed with coordinates $(x_1, \ldots, x_m)$, of center the origin and radius $r$, and $\Gamma: \mathbb{R}^m \to \mathbb{R}^m$ the symmetry given by $\Gamma(x_1, \ldots, x_m) = (x_1, \ldots, x_{m-1}, -x_m)$.

Lemma 7.1. Consider a function $\mu$ defined around the origin of $\mathbb{R}^m$ and $\Gamma$-invariant. Assume that the origin is a non-degenerated singularity. Then about the origin there exist coordinates $(y_1, \ldots, y_m)$ such the coordinates of the origin are still $(0, \ldots, 0)$,

$$
\mu = \sum_{j=1}^{k} y_j^2 - \sum_{j=k+1}^{m-1} y_j^2 + \varepsilon y_m^2 + \mu(0)
$$

where $\varepsilon = \pm 1$, and $\Gamma(y_1, \ldots, y_m) = (y_1, \ldots, y_{m-1}, -y_m)$.

Proof. As $\mu$ is $\Gamma$-invariant its restriction to the hyperplane $H$ defined by $x_m = 0$ has a non-degenerated singularity at the origin. Therefore coordinates $(x_1, \ldots, x_m)$ can be replaced by coordinates $(y_1, \ldots, y_{m-1}, x_m)$ in such a way that $\Gamma(y_1, \ldots, y_{m-1}, x_m) = (y_1, \ldots, y_{m-1}, -x_m)$ and

$$
\mu(y_1, \ldots, y_{m-1}, 0) = \sum_{j=1}^{k} y_j^2 - \sum_{j=k+1}^{m-1} y_j^2 + \mu(0).
$$

On the other hand from the Taylor expansion in variable $x_m$ transversely to $H$ it follows

$$
\mu(y_1, \ldots, y_{m-1}, x_m) = \mu(y_1, \ldots, y_{m-1}, 0) + x_m \frac{\partial \mu}{\partial x_m}(y_1, \ldots, y_{m-1}, 0) + x_m^2 f(y_1, \ldots, y_{m-1}, x_m).
$$

By the $\Gamma$-invariance $\frac{\partial \mu}{\partial x_m}(y_1, \ldots, y_{m-1}, 0) = 0$ and $f(y_1, \ldots, y_{m-1}, -x_m) = f(y_1, \ldots, y_{m-1}, x_m)$. Moreover $2f(0) = \frac{\partial^2 \mu}{\partial x_m^2}(0) \neq 0$ since the origin is a non-degenerated singularity. Therefore close to the origin $(y_1, \ldots, y_{m-1}, y_m)$, where $y_m = x_m |f(y_1, \ldots, y_{m-1}, x_m)|^{1/2}$, is a system of coordinates as required. □

Proposition 7.2. For every $r > 0$ there exists a Morse function $\tau: \mathbb{R}^m \to \mathbb{R}$ such that:

(a) $\tau$ is $\Gamma$-invariant.
(b) If \( p \) is a minimum of \( \tau \) then \( \Gamma(p) \neq p \), that is to say \( p \) does not belong to the hyperplane \( x_m = 0 \).

(c) \( |\tau(x)| \leq ||x||^2 \) on \( \mathbb{R}^m \) and \( \tau(x) = ||x||^2 \) on \( \mathbb{R}^m - B(r) \).

For proving the foregoing proposition we need:

**Lemma 7.3.** There exists a smooth function \( \rho: \mathbb{R} \to \mathbb{R} \) such that

(a) \( \rho(t) = 1 \) if \( t \geq 1 \), \( \rho(t) = -1 \) if \( t \leq 0 \), \( 0 < \rho < 1 \) on \( (1/2, 1) \) and \(-1 < \rho < 0 \) on \( (0, 1/2) \). Moreover \( \rho(1-t) = -\rho(t) \), \( t \in \mathbb{R} \), that \( t \) is to say \( \rho \) is anti-symmetrical with respect to \( t = 1/2 \).

(b) \( \rho' \geq 0 \) on \( \mathbb{R} \) and \( \rho' > 0 \) on \( (0, 1) \). Moreover \( \rho'(1-t) = \rho'(t) \), \( t \in \mathbb{R} \).

(c) \( \rho'' > 0 \) on \( (0, 1/2) \), \( \rho'' < 0 \) on \( (1/2, 1) \) and \( \rho'' = 0 \) on \( (\mathbb{R} - (0, 1)) \cup \{1/2\} \). Moreover \( \rho''(1-t) = -\rho''(t) \), \( t \in \mathbb{R} \).

**Proof.** Let \( \varphi \) be a smooth function meeting the requirements of (c). Denote by \( \varphi_1(0) = 0 \) and by \( \varphi_2 \) the primitive of \( \varphi_1 \) such that \( \varphi_2(1/2) = 0 \).

Then \( \varphi_2 \) is constant and positive on \( [1, \infty) \) while it is constant and negative on \( (-\infty, 0] \). Moreover \( \varphi_2(0) = -\varphi_2(1) \). The function \( \rho = (\varphi_2(1))^{-1}\varphi_2 \) meets the requirements of the lemma (draw the graphics of \( \varphi \), \( \varphi_1 \) and \( \varphi_2 \)). \( \square \)

**Corollary 7.4.** Consider the function \( \lambda \) defined by \( \lambda(t) = t\rho(t) \). If \( \lambda'(c) = 0 \) then \( c \in (0, 1/2) \) and \( \lambda''(c) > 0 \).

**Proof of Proposition 7.2.** First observe that if \( \tau \) is like in Proposition 7.2 for \( r = 1 \), then for any other \( r > 0 \) it suffices to take \( \tau_r(x) = r^2 \tau(r^{-1}x) \).

Consider a function \( \rho \) as in Lemma 7.3 and set \( \tau(x) = x_1^2 + \cdots + x_{m-1}^2 + x_m \rho(||x||^2) \). Then \( |\tau(x)| \leq ||x||^2 \) everywhere and \( \tau(x) = ||x||^2 \) if \( ||x|| \geq 1 \). On the other hand an elementary computation making use of Corollary 7.4 shows that the singularities of \( \tau \) are always non-degenerate and belong to the last axis, while the origin is a saddle. \( \square \)

**Remark 7.5.** As in one variable between two consecutive minima there always exists a maximum, the function \( \lambda \) of Corollary 7.4 has a single singularity, which is a minimum. A more careful computation shows that function \( \tau \) of the proof of Proposition 7.2 has just three singular points: a saddle and two minima.
[1] Bredon, G.E., “Introduction to compact transformation groups”, Academic Press, Pure and Applied Mathematics, Vol. 46, 1972.

[2] Bing, R., *A homeomorphism between the 3-sphere and sum of the solid horned spheres*, Ann. of Math. (2). **56** (1952), 354–362.

[3] Hirsch, M.W., “Differential topology”, Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg, 1976.

[4] Hirzebruch, F., *Singularities and exotic spheres*, Séminaire Bourbaki **314** (1966/67), 13-32.

[5] Kobayashi, S. and Nomizu, K., “Foundations of differential geometry Vol. I”, Publishers, a division of John Wiley & Sons, New York-London 1963.

[6] Kwasik, S. and Lee, K.B., *Locally linear actions on 3-manifolds*, Math. Proc. Cambridge Phil. Soc. **104** (1998) 253–260.

[7] Quinn, F., *Ends of maps. III: dimension 4 and 5*, J. Differential Geometry **17** (1982) 503–521.

[8] Turiel, F.-J. and Viruel, A., *Finite $C^\infty$-actions are described by a single vector field*, Rev. Mat. Iberoam. **30** (2014), 317–330.

[9] Wasserman, A., *Equivariant differential topology*, Topology **8** (1969), 127–150.

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