Cosets of the $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$-algebra

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Abstract

Let $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ be the universal $\mathcal{W}$-algebra associated to $\mathfrak{sl}_4$ with its subregular nilpotent element, and let $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ be its simple quotient. There is a Heisenberg subalgebra $H$, and we denote by $C^k$ the coset $\text{Com}(H, \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}}))$, and by $C_k$ its simple quotient. We show that for $k = -4 + (m + 4)/3$ where $m$ is an integer greater than 2 and $m + 1$ is coprime to 3, $C^k$ is isomorphic to a rational, regular $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$. In particular, $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ is a simple current extension of the tensor product of $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$ with a rank one lattice vertex operator algebra, and hence is rational.

1 Introduction

Given a vertex operator algebra $V$ and a vertex operator subalgebra $W$ the subalgebra of $V$ that commutes with $W$, $C = \text{Com}(W, V)$ is called a coset vertex operator algebra of $V$. This was introduced by Frenkel and Zhu in [FZ], generalizing earlier constructions in [KP] and [GKO], where it was used to construct the unitary discrete series representations of the Virasoro algebra. It is widely believed that if both $V$ and $W$ satisfy certain nice properties, then so does $C$. For example if $V$ and $W$ are both rational or $C_2$-cofinite then one expects $C$ to be rational or $C_2$-cofinite as well. However, general results are very difficult to obtain.

For $n \geq 4$, let $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ denote the $\mathcal{W}$-algebra at level $k$ associated to $\mathfrak{sl}_n$ with its subregular nilpotent element [CM, KRW], and let $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{subreg}})$ be the simple quotient. It was recently shown by Genra [Gen] that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ coincides with the Feigin-Semikhatov algebra $\mathcal{W}_n^{(2)}$ [FS], and is strongly generated by $n + 1$ fields of conformal weights $1, 2, \ldots, n - 1, n/2, n/2$. Note that $\mathcal{W}_n^{(2)}$ is well defined for $n = 2$ and $n = 3$; $\mathcal{W}_2^{(2)}$ coincides with the affine vertex operator algebra $\mathcal{V}^k(\mathfrak{sl}_2)$, and $\mathcal{W}_3^{(2)}$ coincides with the Bershadsky-Polyakov algebra [Ber, Pol].

The weight one field of $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ generates a Heisenberg algebra $H$, and we are interested in the coset

$$C^k = \text{Com}(H, \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}}))$$

for generic values of $k$, as well as the simple quotient

$$C_k = \text{Com}(H, \mathcal{W}_k(\mathfrak{sl}_n, f_{\text{subreg}}))$$

at certain special values of $k$. It was conjectured in the physics literature over 20 years ago [B–H] that there is a sequence of levels $k$ where $C_k$ is isomorphic to a rational
\(W(\mathfrak{sl}_m, f_{\text{reg}})\)-algebra, where \(m\) depends linearly on \(k\). In the cases \(n = 2\) and \(n = 3\), where \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})\) is replaced by the simple affine algebra \(L_k(\mathfrak{sl}_2)\) and the simple Bershadsky-Polyakov algebra, respectively, this conjecture was proven recently in [ALY] and [ACL1]. We remark that subregular \(W\)-algebras of type A have recently become important due to their role in four-dimensional supersymmetric gauge theories as chiral algebras of Argyres-Douglas theories [BN, C, CS]. Interestingly, these are exactly those levels where the \(W\)-algebra has the logarithmic singlet vertex operator algebra [Ka, AM] as Heisenberg coset [CRW].

In the case \(n = 4\), the above conjecture states that for \(k = -4 + (m + 4)/3\), \(C_k\) is isomorphic to a rational \(W(\mathfrak{sl}_m, f_{\text{reg}})\)-algebra. Here \(m\) is an integer greater than 2 such that \(m + 1\) is coprime to 3. Our main result is a proof of this conjecture. In fact, we prove the stronger statement that for these values of \(k\), \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})\) is a simple current extension of \(V_L \otimes W(\mathfrak{sl}_m, f_{\text{reg}})\), where \(V_L\) is a certain rank one lattice VOA.

As a corollary, we obtain the \(C_2\)-cofiniteness and rationality of \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})\) for the above values of \(k\). Additionally, we show that \(C_k\) is of type \(W(2,3,4,5,6,7,8,9)\) for all values of \(k\) except for \(k = -2, -5/2, -8/3\). In particular, this strong generating set works for the simple quotient \(C_k\) for the above values of \(k\), so this family of rational \(W(\mathfrak{sl}_m, f_{\text{reg}})\)-algebras has the following uniform truncation property. For \(m \geq 9\) they are of type \(W(2,3,4,5,6,7,8,9)\), even though the universal \(W(\mathfrak{sl}_m, f_{\text{reg}})\)-algebra is of type \(W(2,3,\ldots,m)\). This happens because a singular vector of weight 10 in the universal algebra gives rise to decoupling relations in the simple quotient expressing the generators of weights 10, 11, \ldots, \(m\) as normally ordered polynomials in the ones up to weight 9.

Here is a brief sketch of the proof of our result.

1. Since \(C_k \otimes H \cong W_k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\), studying \(C_k\) is equivalent to studying the \(U(1)\)-orbifold \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\). We start with an obvious infinite set of generators coming from classical invariant theory in weights 1, 2, 3, \ldots, and compute normally ordered relations among these generators starting in weight 10. We show that for generic values of \(k\), all generators in weights \(w \geq 10\) can be eliminated using these relations. This shows that \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\) is of type \(W(1,2,3,4,5,6,7,8,9)\), and hence that \(C_k\) is of type \(W(2,3,4,5,6,7,8,9)\) for generic \(k\).

2. Coefficients in the normally ordered relations are polynomial functions in \(k\), and zeros of coefficients correspond to non-generic values of \(k\). It turns out that this only happens for \(k = -2, -5/2, -8/3\), so these are the only non-generic values.

3. We prove the conjecture of [B–H] by showing that a certain rational \(W(\mathfrak{sl}_m, f_{\text{reg}})\)-algebra times a rank one lattice vertex operator algebra allows for a simple current extension whose OPE algebra coincides with the one of \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})\). The main ingredient is that the Jacobi identity implies that the full OPE algebra is uniquely determined by a small amount of data which is easily shown to coincide in both algebras. Therefore the two vertex operator algebras must be isomorphic.

This paper is part of a broader program of the authors to study deformable families of vertex operator algebras, i.e. vertex operator algebras that depend continuously on one or more parameters. Examples include universal affine vertex algebras and \(W\)-algebras, where the parameter is the level \(k\), as well as orbifolds and cosets of these algebras. In many situations, the question of finding a minimal strong generating set
for an orbifold or coset of such a deformable family can be decided by passing to the limit as $k$ approaches infinity, which is often an orbifold of a free field algebra [CL1]. The structure of orbifolds of free field algebras can then be determined using ideas from classical invariant theory [CL3, L1, L2, L3, L4].

Besides our pure interest in vertex operator algebra invariant theory, our findings have quite some impact on important questions we are interested in. It is widely believed that $\mathcal{W}$-algebras that are certain quantum Hamiltonian reductions of affine vertex operator algebras, can also be realized as coset algebras. The most famous example is surely the regular $\mathcal{W}$-algebra of a simply-laced Lie algebra as a coset of the affine vertex operator algebra $V^{k+1}(\mathfrak{g})$ inside $V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})$. Jointly with Tomoyuki Arakawa we are able to prove this coset realization [ACL2]. Other examples are the coset of $V^{k+1}(\mathfrak{so}_n)$ inside $V^k(\mathfrak{so}_{n+1}) \otimes \mathcal{F}(n)$ with $\mathcal{F}(n)$ the free field algebra of $n$ free fermions. This is believed to be a regular $\mathcal{W}$-algebra of type $\mathfrak{osp}(2n+1|2n)$ and Lemma 7.17 of [CL1] tells us that this conjecture is indeed consistent with minimal strong generating sets. These families of $\mathcal{W}$-algebras all carry an action of the $N=1$ super Virasoro algebra. The $N=2$ super Virasoro case corresponds to the cosets of $V^{k+1}(\mathfrak{sl}_n)$ inside $V^k(\mathfrak{sl}_{n+1}) \otimes \mathcal{F}(2n)$ and the expected $\mathcal{W}$-algebra is a regular $\mathcal{W}$-algebra of type $\mathfrak{sl}(n+1|n)$. Lemma 7.12 of [CL1] confirms this on the level of minimal strong generating sets. It is a rather ambitious future goal to indeed prove these conjectures.

Correctness of these conjectures is the starting assumption for the (super) higher spin gravity on $\text{AdS}_3$ to two-dimensional (super) conformal field theory correspondence of [GG, CHR1, CHR2].

There are many more conjectures emerging from deformable families of vertex operator algebras that come from the physics of four-dimensional supersymmetric gauge theories [CGai]. These involve cosets of affine vertex operator algebras inside certain $\mathcal{W}$-algebras. Besides their apparent importance in physics [GR], they also relate to the quantum geometric Langlands correspondence [CGai, AFO]. Here the starting point is a series of conjectural deformable families of vertex operator algebras extending the tensor product of two affine vertex operator algebras with Langlands dual Lie algebras; see the introduction of [CGai] for a list of these conjectures.

2 Vertex algebras

In this section, we define vertex algebras, which have been discussed from various points of view in the literature (see for example [Bor, FLM, K]). We will follow the formalism developed in [LZ] and partly in [LiI]. Let $V = V_0 \oplus V_1$ be a super vector space over $\mathbb{C}$, and let $z, w$ be formal variables. By $\text{QO}(V)$, we mean the space of linear maps

$$V \rightarrow V((z)) = \{ \sum_{n \in \mathbb{Z}} v(n)z^{-n-1} | v(n) \in V, \ v(n) = 0 \text{ for } n \gg 0 \}.$$

Each element $a \in \text{QO}(V)$ can be represented as a power series

$$a = a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \in \text{End}(V)[[z, z^{-1}]].$$

We assume that $a = a_0 + a_1$ where $a_i : V_j \rightarrow V_{i+j}((z))$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$, and we write $|a_i| = i$. 

3
For each $n \in \mathbb{Z}$, we have a nonassociative bilinear operation on $\text{QO}(V)$, defined on homogeneous elements $a$ and $b$ by

$$a(w)_{(n)}b(w) = \text{Res}_z a(z)b(w) \, \langle z \rangle^n - (-1)^{|a||b|} \text{Res}_z b(w)a(z) \, \langle z \rangle^n.$$  

Here $\langle z \rangle = \sum_{n \geq 0} z^n$. We often omit the formal variable $z$.

As a matter of notation, we say that a vertex algebra $A$ is spanned by words in the letters $\{i\}$ if it has a minimal strong generating set consisting of one field in each weight.

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Often, (2.1) is written as

$$a(z)b(w) \sim \sum_{n \geq 0} a(w)_{(n)}b(w) \, (z - w)^{-n-1},$$

where $\sim$ means equal modulo the term $a(z)b(w)$, which is regular at $z = w$.

Note that $a(w)b(w)$ is a well-defined element of $\text{QO}(V)$. It is called the Wick product or normally ordered product of $a$ and $b$, and it coincides with $a_{(-1)}b$. For $n \geq 1$ we have

$$n! \, a(z)_{(-n-1)}b(z) = : (\partial^n a(z))b(z) :,$$

where $\partial = \frac{d}{dz}$.

For $a_1(z), \ldots, a_k(z) \in \text{QO}(V)$, the $k$-fold iterated Wick product is defined inductively by

$$: a_1(z)a_2(z) \cdots a_k(z) : = : a_1(z)b(z) :, \quad b(z) = : a_2(z) \cdots a_k(z) :. \quad (2.2)$$

We often omit the formal variable $z$ when no confusion can arise.

A subspace $\mathcal{A} \subset \text{QO}(V)$ containing 1 which is closed under all the above products will be called a quantum operator algebra (QOA). We say that $a, b \in \text{QO}(V)$ are local if

$$(z - w)^N[a(z), b(w)] = 0$$

for some $N \geq 0$. Here $[,]$ denotes the super bracket. This condition implies that $a_{(n)}b = 0$ for $n \geq N$, so (2.1) becomes a finite sum. Finally, a vertex algebra will be a QOA whose elements are pairwise local. This notion is well known to be equivalent to the notion of a vertex algebra in the sense of [FLM].

A vertex algebra $\mathcal{A}$ is said to be generated by a subset $S = \{a_i \mid i \in I\}$ if $\mathcal{A}$ is spanned by words in the letters $a_i$, and all products, for $i \in I$ and $n \in \mathbb{Z}$. We say that $S$ strongly generates $\mathcal{A}$ if $\mathcal{A}$ is spanned by words in the letters $a_i$, and all products for $n < 0$. Equivalently, $\mathcal{A}$ is spanned by

$$\{ : \partial^{k_1}a_{i_1} \cdots \partial^{k_m}a_{i_m} : \mid i_1, \ldots, i_m \in I, \ k_1, \ldots, k_m \geq 0 \}.$$ 

As a matter of notation, we say that a vertex algebra $\mathcal{A}$ is of type

$$W(d_1, \ldots, d_r)$$

if it has a minimal strong generating set consisting of one field in each weight $d_1, \ldots, d_r$. 
Given fields $a, b, c$ in any vertex algebra $\mathcal{V}$, and integers $m, n \geq 0$, the following identities are known as Jacobi relations of type $(a, b, c)$.

$$a_{(r)}(b_{(s)}c) = (-1)^{|a||b|}b_{(s)}(a_{(r)}c) + \sum_{i=0}^{r} \binom{r}{i}(a_{(i)}b)_{(r+s-i)}c. \quad (2.3)$$

For a fixed choice of fields $a, b, c$, these identities are nontrivial for only finitely many integers $m, n$.

Given a vertex algebra $\mathcal{V}$ and a vertex subalgebra $\mathcal{A} \subset \mathcal{V}$, the coset (or commutant) of $\mathcal{A}$ in $\mathcal{V}$, denoted by $\text{Com}(\mathcal{A}, \mathcal{V})$, is the subalgebra of elements $v \in \mathcal{V}$ such that $[a(z), v(w)] = 0$ for all $a \in \mathcal{A}$. Equivalently, $v \in \text{Com}(\mathcal{A}, \mathcal{V})$ if and only if $a_{(n)}v = 0$ for all $a \in \mathcal{A}$ and $n \geq 0$.

3 The algebra $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$

Let $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ denote that $\mathcal{W}$-algebra at level $k$ associated to $\mathfrak{sl}_4$ with its subregular nilpotent element $f_{\text{subreg}}$. By a theorem of Genra [Gen], it coincides with the algebra $\mathcal{W}^{(2)}_4$ [FS] in the family $\mathcal{W}^{(2)}_n$ constructed by Feigin and Semikhatov [FS]. It is freely generated by fields $J, T, W, G^\pm$ of weights $1, 2, 3, 2, 2$, respectively, satisfying the following OPEs.

\[
\begin{align*}
T(z)T(w) &\sim -\frac{(8 + 3k)(17 + 8k)}{2(4 + k)}(z - w)^{-4} + 2T(w)(z - w)^{-2} + \partial T(w)(z - w)^{-1}, \\
T(z)J(w) &\sim J(w)(z - w)^{-2} + \partial J(w)(z - w)^{-1}, \\
T(z)W(w) &\sim 3W(w)(z - w)^{-2} + \partial W(w)(z - w)^{-1}, \\
T(z)G^\pm(w) &\sim 2G^\pm(w)(z - w)^{-2} + \partial G^\pm(w)(z - w)^{-1}, \\
J(z)J(w) &\sim (2 + \frac{3k}{4})(z - w)^{-2}, \\
J(z)G^\pm(w) &\sim \pm G^\pm(w)(z - w)^{-1}, \\
W(z)G^\pm(w) &\sim \pm \frac{2(4 + k)(7 + 3k)(16 + 5k)}{(8 + 3k)^2}G^\pm(w)(z - w)^{-3} \\
&\quad + \left( \pm \frac{3(4 + k)(16 + 5k)}{2(8 + 3k)}\partial G^\pm - \frac{6(4 + k)(16 + 5k)}{(8 + 3k)^2} : JG^\pm : \right)(w)(z - w)^{-2} \\
&\quad + \left( - \frac{8(3 + k)(4 + k)}{(2 + k)(8 + 3k)} : J\partial G^\pm - \frac{4(4 + k)(16 + 15k + 3k^2)}{(2 + k)(8 + 3k)^2} : (\partial J)G^\pm : \right) \\
&\quad \pm \frac{(3 + k)(4 + k)}{2 + k} \partial^2 G^\pm \pm \frac{2(4 + k)^2}{(2 + k)(8 + 3k)} : TG^\pm : \\
&\quad \pm \frac{4(4 + k)(16 + 5k)}{(2 + k)(8 + 3k)^2} : J\partial JG^\pm : \right)(w)(z - w)^{-1}, \quad (3.2)
\end{align*}
\]


\[G^+(z)G^-(w) \sim (2+k)(5+2k)(8+3k)(z-w)^{-4} + 4(2+k)(5+2k)J(w)(z-w)^{-3} + \left( -(2+k)(4+k)T + 6(2+k) : JJ : +2(2+k)(5+2k)\partial J \right)(w)(z-w)^{-2} + \left( (k+2)W + \frac{8(2+k)(32+11k)}{3(8+3k)^2} : JJJ : -\frac{4(2+k)(4+k)}{8+3k} : TJ : +6(2+k) : (\partial J)J : -\frac{1}{2}(2+k)(4+k)\partial T + \frac{4(2+k)(26+17k+3k^2)}{3(8+3k)}\partial^2 J \right)(w)(z-w)^{-1}. \]

(3.3)

\[W(z)W(w) \sim \frac{2(k+4)(2k+5)(3k+7)(5k+16)}{3k+8}(z-w)^{-6} + \cdots , \]

(3.4)

The remaining terms in the OPE of \(W(z)W(w)\) have been omitted but can be found in the paper [FS]. As in [ACL1], a strong generating set for \(\text{gr}(\mathcal{A})\) is a \(\mathbb{Z}_{\geq 0}\)-filtration

\[\mathcal{A}(0) \subset \mathcal{A}(1) \subset \cdots , \quad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}(d) \]  

(3.5)

such that for \(a \in \mathcal{A}(r), b \in \mathcal{A}(s)\), we have

\[a \circ_n b \in \mathcal{A}(r+s), \quad n \in \mathbb{Z}. \]

(3.6)

Then the associated graded algebra \(\text{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}(d)/\mathcal{A}(d-1)\) is a vertex algebra, and a strong generating set for \(\text{gr}(\mathcal{A})\) lifts to a strong generating set for \(\mathcal{A}\); see Lemma 4.1 of [ACL1]. We define a filtration

\[\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_0 \subset \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_1 \subset \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_2 \subset \cdots \]  

(3.7)

on \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})\) as follows: \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{-1} = \{0\}\), and \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{r}\) is spanned by iterated Wick products of the generators \(J, T, W, G^\pm\) and their derivatives, such that at most \(r\) of the fields \(W, G^\pm\) and their derivatives appear. It is clear from the defining OPE relations that this is a weak increasing filtration.

### 4 The \(U(1)\)-orbifold of \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\)

The action of the zero mode \(J_0\) integrates to a \(U(1)\)-action on \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})\), and the orbifold \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\) is just the kernel of \(J_0\). Since \(J, T, W\) lie in \((\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\) and \(J_0(G^\pm) = \pm G^\pm\), it is immediate that \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\) is spanned by all normally ordered monomials of the form

\[(\partial^{a_1} T) \cdots (\partial^{a_r} T)(\partial^{b_1} J) \cdots (\partial^{b_s} J)(\partial^{c_1} W) \cdots (\partial^{c_t} W)(\partial^{d_1} G^+) \cdots (\partial^{d_r} G^+)(\partial^{e_1} G^-) \cdots (\partial^{e_s} G^-), \]

(4.1)

where \(i, j, r, s \geq 0\) and \(a_1 \geq \cdots \geq a_r \geq 0, b_1 \geq \cdots \geq b_s \geq 0, c_1 \geq \cdots \geq c_t \geq 0, d_1 \geq \cdots \geq d_r \geq 0,\) and \(e_1 \geq \cdots \geq e_s \geq 0.\)

The filtration (3.7) on \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})\) restricts to a weak increasing filtration on \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\) where

\[\left(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}\right)_r = \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)} \cap \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_r. \]
Define

\[ U_{i,j} = : \partial^i G^+ \partial^j G^- : , \]

which lies in \((\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{(U(1)}(2)\right)\) and has weight \(i + j + 4\). By the same argument as the proof of Lemma 5.1 of [ACL1], we have

**Lemma 4.1.** \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{(U(1)}\) is strongly generated as a vertex algebra by

\[ \{ J, T, W, U_{0,m} \mid m \geq 0 \} . \quad (4.2) \]

However, \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{(U(1)}\) is not freely generated by (4.2). There is a relation of weight 10 of the form

\[
\frac{(2 + k)(5 + 2k)(8 + 3k)}{360} U_{0,6} = : U_{0,0} U_{1,1} : - : U_{0,1} U_{1,0} : + \cdots ,
\]

where the remaining terms are normally ordered monomials in \(T, J, W, U_{0,i}\) and their derivatives, for \(i \leq 5\). We can regard \( : U_{0,0} U_{1,1} : - : U_{0,1} U_{1,0} : \) as the analogue of a classical relation which does not vanish due to the nonassociativity of the Wick product, and the remaining terms provide the necessary corrections to make it a genuine relation.

This relation is unique up to scalar multiples, and the coefficient of \(U_{0,6}\) is canonical in the sense that it does not depend on any choices of normal ordering in the expression on the right side. In particular, we see that \(U_{0,6}\) decouples for all \(k \neq -2, -5/2, -8/3\).

Similarly, for all \(n > 1\) we have relations

\[
\frac{n(9 + n)(2 + k)(5 + 2k)(8 + 3k)}{120(4 + n)(5 + n)} U_{0,n+5} = : U_{0,0} U_{1,n} : - : U_{0,n} U_{1,0} : + \cdots ,
\]

where the remaining terms are normally ordered monomials in \(T, J, W, U_{0,i}\) and their derivatives, for \(i \leq 5\). The proof is similar to the proof of Theorem 5.4 of [ACL1]. Again, the coefficient of \(U_{0,n+5}\) is canonical, and this shows that \(U_{0,n+5}\) can be decoupled for all \(n > 1\) whenever \(k \neq -2, -5/2, -8/3\). We obtain

**Theorem 4.2.** For all \(k \neq -2, -5/2, -8/3\), \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{(U(1)}\) has a minimal strong generating set

\[ \{ J, T, W, U_{0,i} \mid i \leq 5 \} , \]

and in particular is of type \(\mathcal{W}(1, 2, 3, 4, 5, 6, 7, 8, 9)\).

## 5 The Heisenberg coset of \(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})\)

Let \(\mathcal{H} \subset \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})\) denote the copy of the Heisenberg vertex algebra generated by \(J\), and let \(\mathcal{C}^k\) denote the commutant \(\text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}}))\). Note that

\[ \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{(U(1)} = \mathcal{H} \otimes \mathcal{C}^k \]

and \(\mathcal{C}^k\) has a Virasoro element

\[ T^C = T - \frac{2}{8 + 3k} : JJ : \]

of central charge

\[ c = -\frac{4(5 + 2k)(7 + 3k)}{4 + k} . \]

Also, it is clear from the OPE algebra that \(W \in \mathcal{C}^k\). By a straightforward computer calculation, we obtain
Theorem 5.1. For $0 \leq i \leq 5$, and $k \neq -2, -5/2, -8/3$, there exist correction terms $\omega_i \in \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$ such that $U^C_i = U_0, i + \omega_i$ lies in $\mathcal{C}^k$. Therefore $\mathcal{C}^k$ has a minimal strong generating set $\{T^C, W, U^C_i \mid 0 \leq i \leq 5\}$, and is therefore of type $\mathcal{W}(2, 3, 4, 5, 6, 7, 8, 9)$.

Next, let $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ denote the simple quotient of $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ by its maximal proper ideal graded by conformal weight, and let $\mathcal{C}_k = \text{Com}(\mathcal{H}, \mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}}))$. Evidently we have a surjective map

$$\mathcal{C}^k \to \mathcal{C}_k,$$ 

so for $k \neq -2, -5/2, -8/3$, $\mathcal{C}_k$ is strongly generated by the fields above.

6 Simple current extensions and $\mathcal{W}_\ell(\mathfrak{sl}_n, f_{\text{reg}})$

Vertex operator algebra extensions of a given vertex algebra $V$ can be efficiently studied using commutative, associative algebras with injective unit in the representation category of $V$. This has been developed in [KO, HKL, CKM] and especially structure about parafermionic cosets, i.e. cosets by a Heisenberg or lattice vertex algebra, has been derived in [CKL, CKLR, CKM]. Here we use these ideas to construct simple current extensions of rational, regular $\mathcal{W}$-algebras of type $A$ tensored with certain lattice vertex operator algebras. Recall that a simple current is an invertible object in the tensor category of the vertex operator algebra.

Let $n, r$ be in $\mathbb{Z}_{>1}$ such that $n + 1$ and $n + r$ are coprime (so that especially $nr$ is even) and define

$$\mathcal{W}(n, r) := \mathcal{W}_\ell(\mathfrak{sl}_n, f_{\text{reg}}), \quad \ell + n = \frac{n + r}{n + 1}.$$ 

By [Ar1], $\mathcal{W}(n, r)$ is rational and $C_2$-cofinite. Let $L = \sqrt{nr}\mathbb{Z}$ and $V_L$ the lattice vertex operator algebra of $L$. Modules and their fusion rules for $\mathcal{W}(n, r)$ are essentially known due to [FKW, AvE]. Modules are parameterized by modules of $L_r(\mathfrak{sl}_n)$, i.e. by integrable positive weights of $\mathfrak{sl}_n$ at level $r$. Fusion rules (Theorem 4.3, Proposition 4.3 of [FKW] together with Corollary 8.4 of [AvE]) imply that the group of simple currents is $\mathbb{Z}/n\mathbb{Z}$ and these simple currents correspond to the modules $L_{\nu, \omega_i}$ with $\omega_i$ the fundamental weights of $\mathfrak{sl}_n$. The question of extending a given regular vertex algebra by a group of simple currents to a larger one is entirely decided by conformal dimension and quantum dimension of the involved simple currents. One gets a vertex operator superalgebra if and only if conformal dimensions of a set of generators of the group of simple currents are in $\frac{1}{2}\mathbb{Z}$. Moreover the quantum dimension of generators of the group of simple currents decide whether this is even a vertex operator algebra. See [CKL] for details.

By the quantum dimension of a module $M$ we mean the categorical dimension of $M$. By Verlinde’s formula [H1, H2] one has

$$\text{qdim}(M) = \frac{S_{M, V}}{S_{V, V}}$$ 

with $S$-matrix of the modular transformation of torus one-point functions $ch[M](v, \tau) := \text{tr}_M(o(v)q^{L_0-c/24})$ ($v$ in $V$ of conformal weight $k$ and $o(v)$ the zero-mode of $v$)

$$ch[M](v, -1/\tau) = \tau^k \sum_{N} S_{M, N} ch[N](v, \tau).$$
The sum here is over all inequivalent modules of $V$. See [CG] for a review on modular and categorical aspects of vertex algebras.

The quantum dimension and conformal dimension of $L_{r\omega_1}$ are now easily computed using the recent results of van Ekeren and Arakawa [AvE]:

$$q\text{dim} (L_{r\omega_1}) = \frac{S_{r\omega_1,0}}{S_{0,0}} = e^{2\pi i r (\omega_1, \rho)} q\text{dim} (L_{r\omega_1}) = e^{2\pi i r (\omega_1, \rho)}$$

$$= e^{2\pi i \frac{nr}{2}} = (-1)^r (n-1) = (-1)^r$$

Here $q\text{dim} (L_{r\omega_1})$ is the quantum dimension of the $L_r(\mathfrak{sl}_n)$ module $L_{r\omega_1}$. We firstly used that the modular $S$-matrices of $\mathcal{W}(n, r)$ and of $L_r(\mathfrak{sl}_n)$ only differ by the factor $e^{2\pi i r (\omega_1, \rho)}$ with $\rho$ the Weyl vector of $\mathfrak{sl}_n$. Secondly we used that $L_r(\mathfrak{sl}_n)$ is unitary and hence all quantum dimensions are positive and so every simple current of $L_r(\mathfrak{sl}_n)$ must have quantum dimension one. Finally, in the last equality we used that $nr$ is even. The conformal dimension is

$$\Delta (L_{r\omega_1}) = \frac{(n+1)}{2(n+r)} (r\omega_1, r\omega_1 + 2\rho) - (\omega_1, \rho) = \frac{(n-1)r}{2n}$$

since $\omega_1^2 = \frac{n-1}{n}$ and $(\omega_1, \rho) = \frac{n-1}{2}$. We denote by $V_{L+\gamma}$ the $V_L$-module corresponding to the coset $L + \gamma$ of $L$ in the dual lattice $L' = \frac{1}{\sqrt{nr}} \mathbb{Z}$. Then $V_{L+\frac{r}{\sqrt{nr}}}$ has conformal dimension $\frac{r}{\sqrt{nr}}$ and quantum dimension one since $V_L$ is unitary. It follows from [CKL] (see the Theorems listed in the introduction of that work) that

$$A(n, r) \cong \bigoplus_{s=0}^{n-1} V_{L+\frac{s}{\sqrt{nr}}} \otimes L_{r\omega_1} \boxtimes L_{W(n, r)} \cdots \boxtimes L_{W(n, r)} L_{r\omega_1}$$

is a vertex operator algebra extending $V_L \otimes \mathcal{W}(n, r)$. If $r$ is even, this is a $\mathbb{Z}$-graded vertex operator algebra, while for odd $r$ it is only $\frac{1}{2}\mathbb{Z}$-graded. The subspace of lowest conformal weight in each of the $L_{r\omega_1} \boxtimes \mathcal{W}(n, r) \cdots \boxtimes \mathcal{W}(n, r) L_{r\omega_1}$ is one-dimensional, and we denote the corresponding vertex operators by $X_s$. By Proposition 4.1 of [CKL] the OPE of $X_s$ and $X_{n-s}$ has a non-zero multiple of the identity as leading term. Without loss of generality, we may rescale $X_1$ and $X_{n-1}$ so that

$$X_1(z) X_{n-1}(w) \sim \prod_{i=1}^{n-1} (i(k+n-1)-1)(z-w)^{-r} + \ldots$$

Let $J$ be the Heisenberg field of $V_L$ and we normalize it such that

$$J(z) J(w) \sim \left( \frac{(n-1)k}{n} + n-2 \right) (z-w)^{-2}.$$ 

Then we have

$$J(z) X_1(w) \sim X_1(w)(z-w)^{-1}, \quad J(z) X_{n-1}(w) \sim -X_{n-1}(w)(z-w)^{-1}.$$ 

Let $\tilde{A}(n, r)$ be the vertex algebra generated by $X_1$ and $X_{n-1}$ under operator products.

We now rephrase a physics conjecture [B–H],
Conjecture 6.1. Let \( n, r \) as above and \( k \) defined by \( k + r = \frac{n}{r} \). Then
\[
A(n, r) \equiv \tilde{A}(n, r) \cong \mathcal{W}_k(\mathfrak{sl}_r, f_{\text{subreg}}).
\]
In particular, \( \mathcal{W}_k(\mathfrak{sl}_r, f_{\text{subreg}}) \) is rational and \( C_2 \)-cofinite.

We remark that Conjecture 6.1 is true for \( r = 2, 3 \) by [ALY] and [ACL1] and we will now prove it for \( r = 4 \) under some extra condition on \( n \). For this, we now assume that \( n-1 \) is co-prime to at least one of \( n+1 \) and \( n+r \) so that especially \( n \) even would work. Under this condition the formula for fusion rules is more explicit, and we know from the fusion rules of \( \mathcal{W}(n, r) \) [AvE] that

\[
A(n, r) \cong \bigoplus_{s=0}^{n-1} V_{L+\frac{s r}{\sqrt{n}}} \otimes \mathbb{I}_{r \omega_1} \otimes \mathbb{I}_{r \omega_2} \otimes \cdots \otimes \mathbb{I}_{r \omega_n} \tag{6.6}
\]

The lowest conformal weight of the \( s \)-th summand is \( \min\left\{ \frac{sr}{r}, \frac{(n-s)r}{2} \right\} \) and so in this instance \( \tilde{A}(n, r) \) is strongly generated by \( X_1, X_{n-1} \) together with the Heisenberg field \( J \) and some fields of \( \mathcal{W}(n, r) \).

Theorem 6.2. Conjecture 6.1 holds for \( r = 4 \) and all \( n \) such that \( n-1 \) is co-prime to at least one of \( n+1 \) and \( n+4 \).

Proof. Let \( L \) be the Virasoro field of \( \mathcal{W}(n, r) \) and let \( T = L + \frac{2}{8+3k} : JJ : \) be the Virasoro field of \( V_L \otimes \mathcal{W}(n, r) \). Also, let \( W \) be the weight 3 field of \( \mathcal{W}(n, r) \) which is known to generate \( \mathcal{W}(n, r) \). Since the OPE of \( X_1(z)X_{n-1}(w) \) can be expressed in terms of \( J, T, W \), the most general form is

\[
X_1(z)X_{n-1}(w) \sim (2 + k)(5 + 2k)(8 + 3k)(z - w)^{-4} + a_1J(w)(z - w)^{-3} \tag{6.7}
\]

where the \( a_i \) are constants. By imposing all Jacobi relations of the form \( (J, X_1, X_{n-1}) \) and \( (T, X_1, X_{n-1}) \) we obtain all the above coefficients uniquely except for \( a_5 \), that is,

\[
X_1(z)X_{n-1}(w) \sim (2 + k)(5 + 2k)(8 + 3k)(z - w)^{-4} + 4(2 + k)(5 + 2k)J(w)(z - w)^{-3} \tag{6.8}
\]

\[
+ \left( - (2 + k)(4k + 1)T + 6(2 + k) : JJ : +2(2 + k)(5 + 2k)\partial J \right)(w)(z - w)^{-2}
\]

\[
+ \left( a_5 W + \frac{8(2 + k)(32 + 11k)}{3(8 + 3k)^2} : JJ J : +a_7 : (\partial J)J : +a_8 : (\partial^2 J) : +a_9 : (\partial T)J : +a_{10} : (\partial^2 T)J \right)(w)(z - w)^{-1}.
\]
Using the OPE relations (6.5), and the Jacobi relations of type \((X_1, X_1, X_{n-1})\), we see that \(a_5 \neq 0\). Since we are free to rescale the field \(W\), we may assume without loss of generality that

\[ a_5 = (k + 2). \]

This completely determines \(X_1(z)X_{n-1}(w)\). Also, since \(W\) appears in \(\widetilde{A}(n,4)\) and generates \(\mathcal{W}(n,4)\) (see Proposition A.3 of [ALY]), we must have \(\widetilde{A}(n,4) = A(n,4)\).

Next, imposing all Jacobi relations of type \((T, W, X_1), (J, W, X_1), (T, W, X_{n-1})\) and \((J, W, X_{n-1})\) uniquely determines the OPEs

\[ W(z)X_1(w), \quad W(z)X_{n-1}(w). \quad (6.9) \]

Finally, using (6.3)-(6.5) and (6.8)-(6.9) and imposing all Jacobi relations of type \((W, X_1, X_{n-1})\) uniquely determines the OPE of \(W(z)W(w)\). In particular, these OPE relations are precisely the OPE relations in \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})\) with \(X_1, X_{n-1}\) replaced by \(G^+, G^-\). Since \(A(n,4)\) and \(W_k(\mathfrak{sl}_4, f_{\text{subreg}})\) are simple vertex algebras with the same strong generating set and OPE algebra, they must be isomorphic. \(\Box\)

**Corollary 6.3.** Let \(k\) be defined by \(k + 4 = \frac{n+4}{3}\), and assume that \(n-1\) is co-prime to at least one of \(n+1\) and \(n+4\). Then \(\mathcal{W}(n,4)\) is strongly generated by the fields in weights \(2, 3, 4, 5, 6, 7, 8, 9\) even though the universal regular \(\mathcal{W}\)-algebra of \(\mathfrak{sl}_n\) is of type \(W(2, 3, \ldots, n)\).

**Proof.** This is immediate from Theorems 5.1 and 6.2 and the fact that the map \(C_k \rightarrow C_k\) is surjective. \(\Box\)

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