Abstract. This addendum to [2] shows that the set of tautological quantum logical propositional formulas for a finite dimensional vector space $\mathbb{C}^n$ is different for every $n$, affirmatively answering a question posed therein.

The paper [2] explored the properties of Birkhoff and Von Neumann’s propositional quantum logic (see [1]) as modelled by finite dimensional Hilbert spaces. One question asked in [2] is whether the set of tautological propositional formulas uniquely determines the dimension of the underlying vector space. A partial answer was given, namely that $\mathbb{C}^n$ and $\mathbb{C}^{2n}$ give different sets of tautologies. This note gives a full answer to the question.

For our purposes, propositional formulas consist of alphabet symbols, parentheses, and the symbols meet ($\wedge$), join ($\vee$), orthocomplement ($\neg$), top ($\top$), and bottom ($\bot$). The well formed formulas are the same as those of propositional boolean logic. The symbol $\top$ is interpreted as a finite dimensional Hilbert space, $\bot$ is the trivial subspace, alphabet symbols are variables standing for vector subspaces of $\top$, $\wedge$ is intersection, $\vee$ is span of union, and $\neg$ is orthogonal complement in $\top$. With these operations, the set of subspaces of $\top$ forms a bounded modular ortholattice.

Let $\bar{v} = v_1, \ldots, v_k$ be a list of alphabet symbols and let $\bar{S} = S_1, \ldots, S_k$ be a collection of subspaces of a finite dimensional Hilbert space $U$. Given a well formed formula $\phi(\bar{v})$, the valuation $\Xi_U(\phi(\bar{v}), \bar{S})$ is the subspace resulting from instantiating each $v_i$ with the subspace $S_i$ and performing the operations described by $\phi$ with universal space $U$. As a shorthand the valuation may be implicit; for example if $S$ and $T$ are subspaces of $U$ then $\Xi_U(v \wedge w, S, T)$ is abbreviated $S \wedge T$, and $U$ is inferred from context.

**Definition 1.** Let $\phi(\bar{v})$ be a well formed formula. Define $\bar{d}_\phi : \mathbb{N} \to \mathbb{N}$ such that

$$
\bar{d}_{\phi(\bar{v})}(n) = \max_S(dim(\Xi_{\mathbb{C}^n}(\phi(\bar{v}), \bar{S}))).
$$
DEFINITION 2. A well formed formula \( \phi(\bar{v}) \) is a tautology in \( \mathbb{C}^n \) if \( d_\phi(n) = 0 \).

DEFINITION 3. \( QL(\mathbb{C}^n) \) is the set of tautologies when \( \top = \mathbb{C}^n \).

The goal is to establish the following:

THEOREM 1. \( m < n \Rightarrow QL(\mathbb{C}^n) \subsetneq QL(\mathbb{C}^m) \).

In [2] it was shown that \( QL(\mathbb{C}^{n+1}) \subset QL(\mathbb{C}^n) \), and \( QL(\mathbb{C}^n) \neq QL(\mathbb{C}^{2n}) \). A formula \( \phi_1 \) was constructed such that \( d_{\phi_1} = \lfloor \frac{n}{2} \rfloor \). From \( \phi_1 \) formulas \( \phi_k \) were constructed in stages such that \( d_{\phi_k} = \lfloor \frac{d_{\phi_{k-1}}}{2} \rfloor \).

Thus \( \phi_k \) is a tautology in \( QL(\mathbb{C}^n) \) iff \( \log_2(n) < k \). Each \( \phi_k = \phi_1|_{\phi_{k-1}} \), which is defined as follows:¹

DEFINITION 4. Suppose that \( \alpha(\bar{u}) \) is a formula in \( k \) variables \( \bar{u} = u_1, \ldots, u_k \). For convenience, use De Morgan’s laws to replace \( \alpha \) with an equivalent formula, also called \( \alpha \), in which all negations are negations of atomic variable symbols. Given a second formula \( \beta(\bar{v}) \), let \( \alpha(\bar{u})|_{\beta(\bar{v})} \) denote the modification of \( \alpha \) such that each unnegated instance of \( u_i \) is replaced with \( u_i \land \beta(\bar{v}) \) and each instance of \( \neg u_i \) is replaced with \( \neg(u_i \land \beta(\bar{v})) \land \beta(\bar{v}) \).

LEMMA 2. Let \( \bar{u} = u_1, \ldots, u_{k_u} \) and \( \bar{v} = v_1, \ldots, v_{k_v} \) be lists of variables, \( S = S_1, \ldots, S_{k_u} \) and \( T = T_1, \ldots, T_{k_v} \) lists of subspaces of \( \mathbb{C}^n \), and \( \alpha(\bar{u}) \) and \( \beta(\bar{v}) \) formulas. Let \( \bar{P} = P_1, \ldots, P_{k_u} \) such that \( P_i = S_i \land \Xi_{\mathbb{C}^n}(\beta(\bar{v}), \bar{T}) \). Then the following holds:

\[
\Xi_{\mathbb{C}^n}(\alpha(\bar{u})|_{\beta(\bar{v})}, S, T) = \Xi_{\mathbb{C}^n}(\beta(\bar{v}), \bar{T})(\alpha(\bar{u}), \bar{P})
\]

PROOF. The construction procedure gives the result for atomic formulas and their negations. Since unions and intersections are not changed by inclusion into a larger universal space, the result follows by structural induction.

COROLLARY 3. If \( \bar{d}_{\alpha(\bar{u})} = f \), and \( \bar{d}_{\beta(\bar{v})} = g \), then \( \bar{d}_{\alpha(\bar{u})|_{\beta(\bar{v})}} = f \circ g \). In particular, \( \alpha(\bar{u})|_{\beta(\bar{v})} \) is a tautology in \( \mathbb{C}^n \) iff \( \alpha(\bar{u}) \) is a tautology in \( \mathbb{C}^g(n) \).

Because the function \( n \rightarrow \lfloor \frac{n}{2} \rfloor \) is not injective, none of the formulas constructed in [2] distinguish dimensions between \( 2^k \) and \( 2^{k+1} - 1 \). To overcome this limitation, suppose \( 2^k \leq m < n \leq 2^{k+1} - 1 \) for some \( k \), and assume there exists a formula \( \alpha \) such that \( \bar{d}_{\alpha} = \lfloor \frac{n}{2} \rfloor \).

Construct a formula \( \phi \) in stages, starting with \( \phi_0 = \top \).

¹This definition corrects a small mistake in the definition given in [2].
Suppose it is the beginning of stage $s$, and $\bar{d}_{\phi_{s-1}}(m) < \bar{d}_{\phi_{s-1}}(n)$. If $m$ is odd, or $\bar{d}_{\phi_{s-1}}(n) - \bar{d}_{\phi_{s-1}}(m) > 1$, define $\phi_s(\bar{u}, \bar{v}) = \alpha(\bar{u})|_{\phi_{s-1}(v)}$. Then by Corollary 3, $\bar{d}_{\phi_{s}} = \lfloor \frac{\bar{d}_{\phi_{s-1}}(m)}{2} \rfloor$, and $\bar{d}_{\phi_{s}}(m) < \bar{d}_{\phi_{s}}(n)$. If, on the other hand, there is some $l$ such that $\bar{d}_{\phi_{s-1}}(m) = 2l$ and $\bar{d}_{\phi_{s-1}}(n) = 2l + 1$, it will be shown that there is a formula $\beta_l$, which depends on $l$, such that $\bar{d}_{\beta_l}(2l) = l$ and $\bar{d}_{\beta_l}(2l + 1) = l + 1$. Then define $\phi_s(\bar{u}, \bar{v}) = \beta_l(\bar{u})|_{\phi_{s-1}(v)}$. Then by Corollary 3

$$d_{\phi_s}(m) = d_{\beta_l(d_{\phi_{s-1}}(m))) = l < l + 1 = d_{\beta_l(d_{\phi_{s-1}}(n))) = d_{\phi_s}(n).$$

Since at each stage $\bar{d}_{\phi_{s}}(m) = \lfloor \frac{d_{\phi_{s-1}}(m)}{2}\rfloor$ and $\bar{d}_{\phi_{s}}(m) < \bar{d}_{\phi_{s}}(n)$, the construction procedure must eventually give $\phi_s$ such that $\bar{d}_{\phi_{s}}(m) = 0$ but $\bar{d}_{\phi_{s}}(n) > 0$. It remains only to construct $\alpha$ and $\beta_l$. A suitable formula for $\alpha$ was given in [2], but here a new $\alpha$ will be constructed and then modified to give $\beta_l$.

Let $P_0$ denote the linear operator that projects onto the subspace given by the variable $c$. The following formula evaluates to the image of $P_0 \circ P_a$:

$$P(a, b) = (a \lor \neg b) \land b.$$ 

In a distributive lattice $P(a, b) = a \land b$, but in a modular lattice one only has $P(a, b) \geq a \land b$. The following are true when the lattice is subspaces of $\mathbb{C}^n$:

**Lemma 4.** for all subspaces $S$ and $T$, the following hold:
1. $\dim(P(S, T)) = \min(\dim(P(T, S)), \dim(T))$,
2. $S \land P(S, T) = T \land P(T, S) = P(S, T) \land P(T, S) = S \land T$.

The proof is easy and omitted.

Define the formula

$$\alpha(a, b) = P(b, a) \land \neg(a \land b).$$

When the distributive law holds $\alpha$ is a tautology, but $\alpha$ is not a tautology in $\mathbb{C}^n$ for $n \geq 2$.

**Lemma 5.** The formula $\alpha(a, b)$ has $\bar{d}_\alpha = \lfloor \frac{n}{2} \rfloor$ and for spaces $S$ and $T$, $\dim(\Xi_{\mathbb{C}^n}(\alpha(a, b), S, T)) = \frac{n}{2}$ iff the following hold:
1. $n$ is even,
2. $\dim(S) = \frac{n}{2}$,
3. $S \land T = \bot$,
4. $S \land \neg T = \bot$. 

Proof. It is easy to verify that the above conditions on $S$ and $T$ give dimension $\frac{n}{2}$. For the other direction, if $\min(\dim(S), \dim(T)) < \frac{n}{2}$, $\dim(\alpha(S, T)) \leq \dim(P(S, T)) < \frac{n}{2}$. Also, since $S \land T \subset P(T, S)$, one gets the following:

1. $\dim(\alpha(S, T)) = \dim(P(T, S)) - \dim(S \land T)$
2. $\leq \min(\dim(T), \dim(S)) - \dim(S \land T)$
3. $\leq \min(\dim(T), \dim(S)) - \dim(S) - \dim(T) + \dim(\top)$
4. $= \dim(\top) - \max(\dim(S), \dim(T))$.

Therefore, if $\dim(\alpha(S, T)) = \frac{n}{2}$, $\dim(S) = \dim(T) = \frac{n}{2}$. Thus $\dim(P(T, S)) \leq \frac{n}{2}$, so line 1 implies that $\dim(P(T, S)) = \frac{n}{2}$ and $\dim(S \land T) = 0$. Since $\dim(P(T, S)) = \frac{n}{2}$, $\dim(S \land \neg T) = 0$. Hence, $\dim(\alpha(S, T)) = \frac{n}{2}$ implies that $\alpha(S, T) = P(T, S)$. Thus $\dim(\alpha(S, T)) = \frac{n}{2} \Rightarrow \alpha(S, T) = P(T, S)$.

Corollary 6. $\dim(\alpha(S, T)) = \frac{n}{2} \Rightarrow \alpha(S, T) = P(T, S)$.

To define $\beta_l$, restrict $\alpha$ to itself $\lfloor \log_2(l) \rfloor - 1$ times to obtain a formula $\gamma$ such that $\bar{d}_\gamma(2l) = \bar{d}_\gamma(2l + 1) = 1$. Define $\bar{\beta}_l(a, b, \bar{c}) = \neg(P(b, a) \lor P(a, b)) \land \bar{\gamma}(\bar{c})$ and $\beta_l(a, b, \bar{c}) = \bar{\beta}_l(a, b, \bar{c}) \lor \alpha(a, b)$.

Lemma 7. The formula $\beta_l$ satisfies $\bar{d}_{\beta_l}(2l) = l$ and $\bar{d}_{\beta_l}(2l + 1) = l + 1$.

Proof. Clearly, $\bar{d}_{\beta_l}(2l) = \bar{d}_{\beta_l}(2l + 1) = 1$. The conditions of Lemma 5 imply that $\bar{d}_{\beta_l}(2l) = \bar{d}_{\alpha}(2l) = l$, while $\bar{d}_{\beta_l}(2l + 1) = \bar{d}_{\alpha}(2l + 1) + 1 = l + 1$.

References

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[2] J. Michael Dunn, Tobias J. Hagge, Lawrence S. Moss, and Zhenghan Wang, Quantum logic as motivated by quantum computing, J. Symbolic Logic, vol. 70 (2005), no. 2.

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