Abelian groups yield many large families for the diamond problem

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Abstract

There is much recent interest in excluded subposets. Given a fixed poset $P$, how many subsets of $[n]$ can be found without a copy of $P$ realized by the subset relation? The hardest and most intensely investigated problem of this kind is when $P$ is a diamond, i.e. the power set of a 2 element set. In this paper, we show infinitely many asymptotically tight constructions using random set families defined from posets based on Abelian groups. They are provided by the convergence of Markov chains on groups. Such constructions suggest that the diamond problem is hard.

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1 Introduction

This introduction largely follows the concise and accurate description of the background and history from [20]. For posets \( P = (P, \leq) \) and \( P' = (P', \leq') \), we say \( P' \) is a weak subposet of \( P \) if there exists an injection \( f : P' \to P \) that preserves the partial ordering, meaning that whenever \( u \leq' v \) in \( P' \), we have \( f(u) \leq f(v) \) in \( P \) (see [2]). By subposet we always mean weak subposet. The height \( h(P) \) of a poset \( P \) is the length of the longest chain in \( P \). We consider a family \( \mathcal{F} \) of subsets of \( [n] \) a poset for the subset relation. If \( P \) is not a subposet of \( \mathcal{F} \), we say \( \mathcal{F} \) is \( P \)-free. We are interested in determining the largest size of a \( P \)-free family of subsets of \( [n] \), denoted \( \operatorname{La}(n, P) \). Let \( m_k \) denote the total order of \( k \) elements that we term as \( k \)-chain. The archetypal results is Sperner’s Theorem [24, 10]: \( \operatorname{La}(n, P_2) = \binom{n}{\lfloor n/2 \rfloor} \). Let \( \mathcal{B}(n, k) \) denote the middle \( k \) levels in the subset lattice of \( [n] \) and let \( \sum(n, k) := |\mathcal{B}(n, k)| \). Erdős [11, 10] proved that \( \operatorname{La}(n, P_k) = \sum(n, k) \).

For any \( \mathcal{F} \) family of subsets of \( [n] \), define its Lubell function \( h_n(\mathcal{F}) := \sum_{F \in \mathcal{F}} \frac{1}{|F|} \). The celebrated Bollobás–Lubell–Meshalkin–Yamamoto (BLYM) inequality asserts that for an \( \mathcal{F} \) \( P_k \)-free family \( h_n(\mathcal{F}) \leq k - 1 \), which was originally shown for \( k = 2 \) [4] [21, 23, 27] and extended by P. L. Erdős, Z. Füredi, G.O.H. Katona [12]. (For the ultimate generalization of the BLYM inequality, where cases of equality characterize mixed orthogonal arrays, see Aydinian, Czabarka and Székely [1].) The BLYM inequality gives the book proof to \( \operatorname{La}(n, P_k) = \sum(n, k) \). In view of this, it makes sense to study \( \lambda_n(P) = \max h_n(\mathcal{F}) \), where the maximization takes place for \( P \)-free families \( \mathcal{F} \) in \([n]\).

G.O.H Katona had a key role starting the investigation of extremal problems with excluded posets [9, 6, 7, 18, 13]. The most famous open problem about excluded posets is that of the diamond problem. The cited conjecture of Griggs and Lu would imply that in (1.2) the limit exists and is equal to 2. This is what we refer to as the diamond conjecture. Axenovich, Manske and Martin [2] reduced the upper bound in (1.2) to 2.283;
Griggs, Li, and Lu [16] further reduced it to 25/11. The current best upper bound is 2.25, achieved by Kramer, Martin, and Young [19]. They also pointed out that this is the best possible bound that can be derived from a Lubell function argument. Manske and Shen [22] have a better upper bound for 3-layered families of sets, 2.1547, improving on an earlier bound of Axenovich, Manske and Martin [2], 2.207. A similar improvement for for 3-layered families of sets was also made by Balogh et al. [3].

We want to point out that we are not aware of any construction for the diamond problem with more sets than those in the two largest levels. This number sometimes can be achieved in other ways then two consecutive levels: e.g. on 12 points take all 5-subsets, all 7-subsets and a Steiner system S(5, 6, 12).

The goal of this paper is to provide infinitely many exotic examples that show the asymptotical tightness of the diamond conjecture. These constructions are based on Abelian groups and are very different from the usual extremal set systems. The proofs use the theory of Markov chains on groups, allowing citations of theorems instead of making analytic proofs from scratch.

2 Strongly diamond-free Cayley posets

Let us be given a finite group $\Gamma$ and a set of generators $H \subseteq \Gamma$. Recall that the Cayley graph $\bar{G}(\Gamma, H)$ has vertex set $\Gamma$ and edge set \{$(g \rightarrow gh) : h \in H, g \in \Gamma$\}. We do not assume $H = H^{-1}$, an assumption often made for Cayley graphs, but we do assume $e \notin H$. We define the infinite Cayley poset $P(\Gamma, H)$ as follows:

the vertices of the poset are ordered pairs $(\gamma, i)$, for $\gamma \in \Gamma$ and $i \in \mathbb{Z}$, and $(\gamma, i) \preceq (\delta, j)$, if $j \geq i$ and $\gamma = \delta \eta_1 \eta_2 \cdots \eta_{j-i}$ for some $\eta_1, \eta_2, \ldots, \eta_{j-i} \in H$. It is easy to see that $P(\Gamma, H)$ is a partial order indeed. Furthermore, mapping the vertices of the infinite Cayley poset $P(\Gamma, H)$ to the vertices of the Cayley graph $\bar{G}(\Gamma, H)$ by projection to the first coordinate, upward oriented edges of the Hasse diagram map to the edges of the Cayley graph. We term finite subposets of the infinite Cayley poset as Cayley posets. We say that $H$ is aperiodic, if for $L = \{\ell : \exists \eta_1, \ldots, \eta_\ell \in H$ such that $\eta_1 \eta_2 \cdots \eta_\ell = e\}$, the greatest common divisor of elements of $L$ is 1. We say that a (finite) Cayley poset is aperiodic, if the generating set is aperiodic.

Assume now that $\Gamma$ is abelian of order $m$ and $|H| = h$ with $H = \{\eta_1, \eta_2, \ldots, \eta_h\}$. For convenience, as we focus on abelian groups, in the rest of the paper we use additive notation. Let us be given an $n$-element set $N$ partitioned into classes $N_1, N_2, \ldots, N_h$, such that $|N_i| = n_i$. Assign for $x \in N$ a weight $w(x) \in H$, such that for all $x \in N_i$, $w(x) = \eta_i$. We will refer to $N$ as a weighted set. For $A \subseteq N$, define $w(A) = \sum_{x \in A} w(x)$. For every $i \geq 0$ and $\gamma \in \Gamma$, define

$s_i(i) := \{A \subseteq N : |A| = \lfloor n/2 \rfloor + i$ and $w(A) = \gamma\}.$

Let $(\gamma_1, i_1) \prec (\gamma_2, i_2) \prec (\gamma_3, i_3)$ be three distinct elements of a Cayley poset $\Pi$. If some $\eta \in H$ can be used in both an $i_2 - i_1$ term sum of elements of $H$ representing $\gamma_2 - \gamma_1$ and an $i_3 - i_2$ term sum of elements of $H$ representing $\gamma_3 - \gamma_2$, we say that the three elements form a strong chain in $\Pi$. We call a Cayley poset $\Pi$ strongly diamond-free, if (1) $\Pi$ is diamond-free, and (2) it has no strong chains. We need the following easy lemma:

Lemma 1 If a Cayley poset $\Pi$ with elements $\{(\gamma_i, i_i) : i = 1, 2, \ldots, \ell\}$ is strongly diamond-free, then for a weighted $n$-element set $N$, the family of sets

$\mathcal{F}(N, w, \Pi) := \bigcup_{(\gamma_i, i_i) \in \Pi} s_{i_i}(j_i) = \{A \subseteq N : |A| = \lfloor n/2 \rfloor + j_i$ and $w(A) = \gamma_i$, for $i = 1, 2, \ldots, \ell\}$

is diamond-free.

Proof. Referring to a diamond in this proof, we assume that $a_1$ is its lowest element, $a_4$ is its largest element, and $a_2, a_3$ are the middle (uncomparable) elements.

We will show that if $\mathcal{F}(N, w, \Pi)$ is not diamond free, then $\Pi$ is not strongly diamond-free.

If there are four different sets $A_1, A_2, A_3, A_4$ in $\mathcal{F}(N, w, \Pi)$ that correspond to a diamond $a_1, a_2, a_3, a_4$ resp., then, $j_1 < j_2 < j_3 < j_4$. Now we have that either $(j_2 \neq j_3)$ or $(j_2 = j_3$ and $\gamma_2 \neq \gamma_3$) or $(j_2 = j_3$ and $\gamma_2 = \gamma_3$).

If $(j_2 \neq j_3)$ or $(j_2 = j_3$ and $\gamma_2 \neq \gamma_3$), then the four elements $(\gamma_i, i_i) i \in [4]$ form a diamond in $\Pi$ so $\Pi$ is not strongly diamond-free.
When \( j := j_2 = j_3 \) and \( \gamma := \gamma_2 = \gamma_3 \), then we have that \( w(A_1) = \gamma_1, w(A_4) = \gamma_4, |A_1| = j_1, |A_4| = j_4 \)

\( w(A_2) = w(A_3) = \gamma \) and \( |A_2| = |A_3| = j \).

Now clearly for \( i \in \{2,3\} \) we have that \( A_1 \subseteq A_2 \cap A_3 \subseteq A_i \subseteq A_2 \cup A_3 \subseteq A_4 \).

Using \((A_2 \cap A_3) \setminus A = \{x_1, \ldots, x_3\} \) (possibly empty) and \( A_4 \setminus (A_2 \cup A_3) = \{y_1, \ldots, y_k\} \) (possibly empty) and \( A_1 \setminus (A_2 \cap A_3) = \{z_1, z_1', \ldots, z_3\} \) (nonempty!)

for \( i \in \{2,3\} \) it follows that for \( j \in \{(2,3) \setminus \{i\}\} \) we have that \((A_2 \cup A_3) \setminus A = \{z_1, z_1', \ldots, z_3\} = A_4 \setminus (A_2 \cap A_3) \).

It follows that \( j = j_1 + s + r \) and \( j_4 = j + r + k = j_1 + 2r + s + k \) and that

\[
\gamma = \gamma_1 + \left( \sum_{\ell=1}^{s} w(x_\ell) \right) + \left( \sum_{\ell=1}^{r} w(z_\ell^2) \right)
\]

and

\[
\gamma_4 = \gamma + \left( \sum_{\ell=1}^{k} w(y_\ell) \right) + \left( \sum_{\ell=1}^{r} w(z_\ell^2) \right),
\]

where \( j - j_1 = s + r \) and \( j_4 - j = k + r \).

Now since \( r \geq 1 \), we can chose \( h := w(x_3^2) \)

In particular, in this case we have found \((\gamma_1, j_1) < (\gamma, j) < (\gamma_4, j_4)\) in \( \Pi \) such that \( \gamma - \gamma_1 \) can be written as a \((j - j_1)\)-term sum of elements of \( H \) containing the term \( h \) and and \( \gamma_4 - \gamma \) is a \((j_4 - j)\)-term sum of elements of \( H \) containing the term \( h \). This, in this case \( \Pi \) is not strongly diamond-free either.

\[\square\]

## 3 Markov chains on \( \Gamma \)

Let us be given the set \( N = [1, n] \). Assign for every \( x \in N \) i.i.d. \( H \)-valued random variables \( \omega(x) \). Assume \( \Pr[\omega(x) = \eta] > 0 \) for all \( \eta \in H \) and extend the probability distribution to \( \gamma \in \Gamma \setminus H \) by \( \Pr[\omega(x) = \eta] = 0 \). For an arbitrary \( A \subseteq N \), assume \( A = \{a_1 < a_2 < \cdots < a_{|A|}\} \). Now we associate a finite Markov chain \( X_j^A \) on \( \Gamma \) for \( j = 0, 1, \ldots, |A| \) with \( A \): define it with \( X_0^A = 0 \) for sure, and

\[
X_i^A = \gamma \text{ iff } \exists \delta \in \Gamma \text{ such that } X_{i-1}^A = \delta \text{ and } \omega(a_i) = \gamma - \delta.
\]

Consequently

\[
\Pr[X_j^A = \gamma] = \sum_{\delta \in \Gamma} \Pr[X_{j-1}^A = \delta] \cdot \Pr[\omega(a_j) = \gamma - \delta].
\]

If we defined analogously the infinite Markov chain \( X_j^A \) on \( \Gamma \) for \( j = 0, 1, \ldots \), for an infinite \( A \subseteq \mathbb{N} \), the Markov chain would be irreducible if and only if \( H \) is a generating set, and in this case the Markov chain would be aperiodic if and only if \( H \) is not contained by a coset of a proper normal subgroup of \( \Gamma \) (see Proposition 2.3 in [8]). Hence assuming that \( H \) is an aperiodic generating set, \( X_j^A \) converges to the unique stationary distribution on \( \Gamma \), which is the uniform distribution (see p. 271 in [8]). The same results hold as well for \( X_j^A \) for a finite set \( A \), if \(|A|\) is sufficiently large for a fixed \( \Gamma \).

The Markov chains \( X_j^A \) with different \( A \)'s do correlate, but we only will use the linearity of expectation. Define \( \omega(A) = \sum_{x \in A} \omega(x) \). For a fixed \( i \) and a large \( n \), set

\[
S_\gamma(i) = \{ A \subseteq N : |A| = n/2 + i \text{ and } \omega(A) = \gamma \},
\]

a random family of sets. Note that \( \omega(A) = X_{|A|}^A \) and \( S_\gamma(i) \) are random variables, unlike \( w(A) \) and \( s_\gamma(i) \) in the previous section. By the convergence to uniform distribution recalled above, we have that for all \( \epsilon > 0 \), for all sufficiently large \( n \)

\[
\forall \gamma \in \Gamma \forall A \quad \frac{1}{|\Gamma|} - \epsilon < \Pr[\omega(A) = \gamma] < \frac{1}{|\Gamma|} + \epsilon.
\]

Hence

\[
\forall \gamma \in \Gamma \quad \frac{1}{|\Gamma|} - \epsilon < \frac{\mathbb{E}[|S_\gamma(i)|]}{\binom{n}{n/2 + i}} < \frac{1}{|\Gamma|} + \epsilon.
\]

We reformulate (3.3) above as a theorem:
Theorem 2 [Equidistribution theorem]
Assume that $H$ is an aperiodic generating set of a finite abelian group of order $m$. Under the model above, for $i$ fixed as $n \to \infty$, we have
\[
\lim_{n \to \infty} \frac{E[|S_i(i)|]}{\binom{n}{\lfloor n/2 \rfloor + i}} = \frac{1}{m}.
\]
Observe that $|\mathcal{F}(N, w, \Pi)| \leq La(n, D_2)$. Combining Lemma 1, Theorem 2, and the fact that for $i$ is fixed, the asymptotic formula
\[
\left( \frac{n}{\lfloor n/2 \rfloor + i} \right) \sim \left( \frac{n}{\lfloor n/2 \rfloor} \right)
\]
holds as $n \to \infty$, we immediately obtain the following theorem:

Theorem 3 Assume that $H$ is an aperiodic generating set of a finite abelian group of order $m$. If a fixed Cayley poset $\Pi$ with elements $(\gamma_i, j_i) : i = 1, 2, ..., \ell$ is strongly diamond-free, then
\[
\frac{\ell}{m} = \lim_{n \to \infty} \frac{E[|\mathcal{F}(N, w, \Pi)|]}{\binom{n}{\lfloor n/2 \rfloor}} \leq \liminf_{n \to \infty} \frac{La(n, D_2)}{\binom{n}{\lfloor n/2 \rfloor}}.
\]
The conclusion of this theorem is that if one constructs an aperiodic generating set of a finite abelian group of order $m$ and strongly diamond-free Cayley poset of $\ell$ elements with this generating set, then for a large $n$, a $\mathcal{F}(N, w, \Pi)$ with some weighting $w$ has size at least $\left( \frac{\ell}{m} - \epsilon \right) \binom{n}{\lfloor n/2 \rfloor}$. A construction with $\ell > 2m$ would even refute the diamond conjecture.

Unfortunately, the $\ell/m$ lower bound of Theorem 3 never exceeds two. The proof is the following. Take any $0 \neq h \in H$ and partition the infinite Cayley poset $P(\Gamma, H)$ into $|\Gamma|$ chains as
\[
\left\{ \{g + ih : i \in \mathbb{Z} \} : g \in \Gamma \right\}.
\]
If a finite Cayley poset $\Pi$ is free of strong chains, which is part of the requirement to be strongly diamond-free, than $\Pi$ cannot have more than two elements from any of the $\{g + ih : i \in \mathbb{Z} \}$, for any $g$.

Therefore in the next section we focus on constructing finite Cayley posets with $\ell = 2m$ or with just slightly fewer elements.

4 Constructions

Example 1 [The classic example.] For any $\Gamma$ and $H$, take two levels from the infinite Cayley poset. This is a strongly diamond-free Cayley poset with $2m$ vertices.

We leave the verification of the correctness of the following constructions to the readers, where $\Gamma_m$ denotes the additive group of modulo $m$ residue classes.

Example 2 Take $\Gamma = \mathbb{Z}_m$ with $H = \{a, b\}$ such that $\gcd(a, b) = 1$. The following is a strongly diamond-free Cayley poset with $2m$ vertices:
\[
(g, 3) : g \not\equiv a + b \mod m,
(a, 2), (b, 2),
(g, 1) : g \not\equiv 0 \mod m.
\]
This poset if often aperiodic, for example, $a = 1, b = 2$ are such for $m = 3$, or $a = 2, b = 3$ are such for $m = 4$. The exact condition for aperiodicity is that $H$ is not contained by a coset of a proper subgroup of $\Gamma$. (See Proposition 2.3 in \[\S\].)
Example 3 \(\) Take \(\Gamma = \mathbb{Z}_7\) with \(H = \{2, 3, 5\}\). The following is a strongly diamond-free, aperiodic Cayley poset with 13 vertices:

- \((g, 3) : g \not\equiv 0, 1, 5 \mod 7,\)
- \((2, 2), (3, 2), (5, 2)\)
- \((g, 1) : g \not\equiv 0 \mod 7.\)

Note that 1, 2 mod 3 and 2, 3, 5 mod 7 are difference sets. However, bigger difference sets do not seem to offer good constructions. On four levels, we still can construct ”close” constructions.

Example 4 \(\) For \(m = 4k - 1\), take \(\Gamma = \mathbb{Z}_m\) with \(H = \{2k - 1, 2k\}, k \geq 2.\) The following is a strongly diamond-free, aperiodic Cayley poset with \(2m - 2\) vertices:

- \((i, 4) : i = k + 2, \ldots, 3k - 3,\)
- \((i, j) : i = k, k + 1, \ldots, 3k - 1; \text{ for } j = 1, 2, 3.\)

Example 5 \(\) For \(m = 4k + 1\), take \(\Gamma = \mathbb{Z}_m\) with \(H = \{2k, 2k + 1\}, k \geq 2.\) The following is a strongly diamond-free, aperiodic Cayley poset with \(2m - 2\) vertices:

- \((i, 4) : i = k + 2, \ldots, 3k - 2,\)
- \((i, j) : i = k, k + 1, \ldots, 3k; \text{ for } j = 1, 2, 3.\)

The last two constructions still allow close approximations of the conjectured maximum \((2 + o(1))(\binom{n}{\lfloor n/2 \rfloor}).\) For any fixed \(\epsilon,\) set \(k > 1 + \frac{1}{\epsilon}\) to have \(\frac{2m-2}{m} > 2 - \frac{2}{\epsilon}.\) Fixing this \(k,\) for sufficiently large \(n,\) a set system is obtained from Example 4 or 5 with at least \((2 - \epsilon)(\binom{n}{\lfloor n/2 \rfloor})\) elements.

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