Graphic Method for Phase Space Calculation

Hao-Jie Jing,1,2,∗ Chao-Wei Shen,2,+ and Feng-Kun Guo1,2,†

1CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China
2School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
(Dated: May 6, 2020)

In quantum field theory, the phase space integration is an essential part in all theoretical calculations of cross sections and decay widths. It is also needed for computing the imaginary part of a physical amplitude. A key problem is to get the phase space formula expressed in terms of any chosen invariant masses in an n-body system. We propose a graphic method to quickly get the phase space formula of any given invariant masses intuitively for an arbitrary n-body system. The method also greatly simplifies the phase space calculation.

Introduction.—In quantum field theory, decay widths and cross sections are important physical observables, and the computation of each of them contains a phase space integration. Therefore, the phase space integration is essential to connect theoretical calculations with experimental observations. It also enters into the calculation of the imaginary part of a physical amplitude through unitarity. Usually, in experiments, whether there is a nontrivial structure in the invariant mass distribution is important for understanding the structure of particle spectrum. For instance, resonances are often found as peaks in the invariant mass distributions of certain final state particles. Thus, a key problem of the phase space integration is to get the formula of the phase space element expressed in terms of any given invariant masses for an n-body system. When there are only two or three particles in the final state, the corresponding phase space integration are relatively easy, as given in, e.g., the chapter of Kinematics in the Review of Particle Physics [1]. When the number of final state particles is larger than 3, the phase space integration becomes much more involved. In this Letter, we propose a novel method based on graphics, which can not only give the phase space formula of any given invariant masses intuitively, but also greatly simplifies the calculation just as what Feynman diagrams do in calculating scattering amplitudes.

Graphic Method.—The n-body phase space element is [1]

\[ \Phi_n(m; m_1, \ldots, m_n) = \delta(p - \sum_{i=1}^n p_i) \prod_{j=1}^n \frac{d^3 p_j}{(2\pi)^3 2p_j}, \tag{1} \]

where \( p \) and \( p_i \) are the four-momenta of the initial state and the i-th particle in the final state, respectively, and they satisfy the on-shell conditions \( p^2 = m^2 \) and \( p_i^2 = m_i^2 \) \( (i = 1, \cdots, n) \) with \( m \) and \( m_i \), the corresponding particle masses. One notices that the phase space element can be written in an explicitly Lorentz-invariant manner as

\[ \frac{d^3 p_j}{(2\pi)^3 2p_j} = \frac{d^4 p_j}{(2\pi)^3} \delta(p_j^0 - m_j^0) \delta(p_j^1 - m_j^1). \tag{2} \]

After integrating out the order-4 Dirac \( \delta \)-function in Eq. (1), which represents the energy-momentum conservation, \( 3n - 4 \) integral variables are left. In order to get invariant mass distributions, variable transformations from three momenta to invariant masses are needed. Thus, a natural question is how many integral variables can be replaced by invariant masses. To answer this question, we should consider the final state particles in the momentum space as shown in Fig. 1. Since the invariant mass of two particles depends only on the scalar product of their four-momenta, all invariant masses will be fixed when the magnitude of the three-momentum for each particle and the relative angle between each two momenta are fixed. Then the final state in the momentum space is just like a rigid body, and one can make a global Euler rotation of the final state, which does not change the value of any invariant mass. Thus, when considering only the scalar products of the final state momenta, among all the \( 3n - 4 \) variables, at most \( 3n - 7 \) \( (n \geq 3) \) variables can be replaced by invariant masses, and the other three variables can be replaced by Euler angles \( \alpha, \beta \) and \( \gamma \):

\[ \Phi_n \propto d(\cos \alpha)d\beta d\gamma d^{3n-7}(\text{InvariantMasses}). \tag{3} \]

Next, let us introduce the graphic method. It is based
on the recursive relation
\[
d\Phi_n(m; m_1, \ldots, m_n) = d\Phi_l(m; m_1, \ldots, m_l)(2\pi)^3 d^2m_l^n
\times d\Phi_{n-l+1}(m_l; m_{l+1}, \ldots, m_n),
\]
where \( m^2_{i,l} = (p_i + \cdots + p_n)^2 \). This relation can be represented graphically as Fig. 2. We find that the generic \( n \)-body phase space expression in terms of invariant masses can be obtained using the following drawing rules:

(1) A single particle is represented by a single line:
\[ j \]
\[ \text{a multi-particle system is represented by a double line:} \]
\[ j_1j_2 \cdots j_l \]
\[ \chi \]
\[ \text{an} \ l \text{-body phase space element is represented by a vertex:} \]
\[ i_1 \]
\[ i_2 \]
\[ i_k \]
\[ \text{where the lines can be either single or double lines and} \]
\[ m^2_{i_1, \ldots, i_k} = (p_{j_1} + \cdots + p_{j_l})^2. \]

(2) A single line can be internal or external, and a double line can only be internal.

(3) There is one and only one route of double lines between any two vertices.

(4) Any two single lines from a vertex cannot be the same; if there are duplicate single lines for the same particle in the whole diagram, only one can be kept, and the rest are represented by dashed single lines:
\[ j \]
\[ \chi \]
\[ \text{In} \ l \text{-body phase space element is represented by a vertex:} \]
\[ i_1 \]
\[ i_2 \]
\[ i_k \]
\[ \text{where the lines can be either single or double lines and} \]
\[ m^2_{i_1, \ldots, i_k} = (p_{j_1} + \cdots + p_{j_l})^2. \]

(5) Invariant masses for all double lines in the whole diagram must be independent (for instance, \( m_{12}, m_{23} \) and \( m_{13} \) for a 3-body system are not independent as they satisfy \( m_{12}^2 + m_{23}^2 + m_{13}^2 = m^2 + \sum_{i=1}^3 m_i^2 \)).

From the above drawing rules, one can find the following topological rules:

- using Rule (2) and Rule (3), one finds that the number of vertices \( v \) and that of double lines \( d \) are related as \( v = d + 1 \), which ensures that a Dirac \( \delta \) function for the overall energy-momentum conservation of the whole diagram exists;

- the number of final state particles \( n \), the number of double lines \( d \), the number of internal single (solid and dashed) lines \( l \) and the number of outgoing lines for each vertex \( v_j \) are related as \( n + d + l = \sum_{j=1}^{d+1} v_j \);

- using Rule (4), the number of dashed lines equals to the number of internal single lines \( l \);

- using Euler’s formula: \( v - (l + d) + L = 1 \), where \( L \) is the number of loops in a diagram, one finds \( L = l \).

A diagram with \( v_j = 2 \) for all vertices is called a complete-expansion diagram; otherwise, it is called an incomplete-expansion diagram. Because an incomplete-expansion diagram can be further expanded into a complete-expansion diagram, we shall only discuss the complete-expansion diagrams in the following. For a complete-expansion diagram, one has \( d - l = n - 2 \). A diagram without an internal single line, i.e. \( l = 0 \), is called a tree diagram, and a diagram with \( l > 0 \) is called an \( l \)-loop diagram. For a tree diagram, one finds \( d = n - 2 \), which means that we can get an \( n \)-body phase space element with \( n - 2 \) invariant masses as the integral variables. From the discussions of Eq. (3), at most \( 3n - 7 \) variables can be replaced by invariant masses in the \( n \)-body \( (n \geq 3) \) phase space element. Then, if \( d \) takes its maximal value \( 3n - 7 \), one will get \( l = 2n - 5 \), which means that a \((2n - 5)\)-loop diagram corresponds to an \( n \)-body phase space element with \( 3n - 7 \) invariant masses as the integral variables.

Applications.—Using the graphic method, the \( n \)-body phase space element can be easily decomposed into the product of many 2-body phase space elements. All possible forms of the 2-body phase space element are listed in Table I, where the presence of an \( \delta_l \) means that the physical on-shell equations admit an infinity of solutions in that special case. For example, let us consider the case with \((|p_1|, \phi_1)\) being the integral variables. In the spherical coordinate system of the \( p_1 \) space, the isoline of fixed \( |p_1| \) and \( \phi_1 \) is a semicircle \((\theta_1 \in [0, \pi]) \). Meanwhile, in the c.m. frame of initial state \((\beta = 0)\), the solutions \( p_1 \) of the physical on-shell equations correspond to a sphere centered at the origin. The isoline is completely on the sphere, and thus there are infinite solutions. In such a case, the corresponding 2-body phase space element cannot be used. If we take the solid angle as the integral
variables, as given in Table I, the 2-body phase space element in any reference frame can be written as

\[
d\Phi_2(m; m_1, m_2) = \sum_{|\mathbf{p}_1|} \frac{d\Omega_1}{(2\pi)^6} \frac{|\mathbf{p}_1|^2}{4[|\mathbf{p}^0| |\mathbf{p}_1| - |\mathbf{p}^0| |\mathbf{p}| \cos \theta_{01}]} \times \theta \left( \mathbf{p}^0 - \sqrt{|\mathbf{p}_1|^2 + m_1^2} \right),
\]

where \( p, p_1 \) and \( \theta_{01} \) are the momentum of the initial state, the momentum of particle 1 in the final state and the relative angle between the spatial components of these two momenta, respectively. The summation runs over all \(|\mathbf{p}_1|\) solutions of the on-shell equations. The effect of the Heaviside \( \theta \)-function is to limit the integration range to the physical region. One may omit it for simplicity, but needs to be careful with the integration range. In the c.m. frame of the initial state (\( \beta = 0 \)), \( p = (p^0, \mathbf{p}) = (m, \mathbf{0}) \), and one gets

\[
d\Phi_2(m; m_1, m_2) = d\Omega_1 \frac{|\mathbf{p}_1|}{(2\pi)^6 4m},
\]

where \(|\mathbf{p}_1|\) is the magnitude of the three-momentum of particle 1 in the c.m. frame of the initial state, and the integration region is given by \( \cos \theta_{11} \in [-1, 1] \) and \( \phi_1 \in [0, 2\pi] \).

Next, we give four examples depicted in Fig. 3 to show how to get the \( n \)-body phase space element by using the graphic method.

For Fig. 3(a), using the corresponding rules, one has the following building blocks:

- \( V_1 \) : \( d\Phi_2(m; m_1, m_2) \),
- \( V_2 \) : \( d\Phi_2(m_2; m_2, m_3) \), \( \ldots \), \( V_{n-1} \) : \( d\Phi_2(m_{n-1}; m_{n-1}, m_n) \),
- \( "(2)" \) : \( (2\pi)^3 dm_2^2 \), \( "(3)" \) : \( (2\pi)^3 dm_3^2 \), \( \ldots \), \( "(n-1)" \) : \( (2\pi)^3 dm_{n-1}^2 \).

Multiplying them together and using Eq. (6), an expression for the \( n \)-body phase space element can be easily obtained,

\[
d\Phi_n(m; m_1, \ldots, m_n) = \frac{|\mathbf{p}_1|}{2^n (2\pi)^3 m} \prod_{i=2}^{n-1} |\mathbf{p}_i| d\Omega_i dm_i, \]

where \(|\mathbf{p}_i|, \Omega_i\) is the three-momentum of the final-state particle \( i \) in the c.m. frame of the \((i, i+1, \ldots, n)\) particle system. The integration region for the invariant mass \( m_{(i)} \) is \( \sum_{k=i}^{n} m_k, m_{(i-1)} - m_{i-1} \) \((i = 2, 3, \ldots, n - 1)\) with \( m_{(1)} = m \). A much more lengthy derivation can be found in the appendix of Ref. [2].

For the 3-body phase space, in addition to the chain

\[
TABLE I. Integrand of the 2-body phase space element in 5 cases with different integral variables. A factor \((2\pi)^{-6}\) has been omitted in each integrand. The definitions of \( p, p_1 \) and \( \theta_{01} \) can be below Eq. (5). The last three columns give the possible numbers of solutions of the physical on-shell equations of particles 1 and 2 in the final state when the corresponding two integral variables are fixed in any reference frame, where \( \beta \) and \( \beta_1^* \) are the velocity of the initial state and that of particle 1 in the rest frame of the initial state, respectively. \( \aleph_1 \) is the second transfinite number.

| Integral measure | Integrand | \( \beta = 0 \) | \( 0 < \beta < \beta_1^* \) | \( \beta \geq \beta_1^* \) |
|-----------------|-----------|----------------|-----------------|-----------------|
| \( d\cos \theta_1 d\phi_1 \) | \( \frac{|\mathbf{p}_1|^2}{4(|\mathbf{p}^0| |\mathbf{p}_1| - |\mathbf{p}^0| |\mathbf{p}| \cos \theta_{01})} \) | 1 | 1 | 1,2 |
| \( d|\mathbf{p}_1| d\phi_1 \) | \( \frac{|\mathbf{p}_1|}{4p_1^0 |\mathbf{p}| \partial \cos \theta_{01}/\partial \cos \theta_{11}} \) | \( \aleph_1 \) | 1,2 | 1,2 |
| \( dp_1^0 d\phi_1 \) | \( \frac{1}{4|\mathbf{p}| \partial \cos \theta_{01}/\partial \cos \theta_{11}} \) | \( \aleph_1 \) | 1,2 | 1,2 |
| \( d|\mathbf{p}_1| d\cos \theta_1 \) | \( \frac{|\mathbf{p}_1|}{4p_1^0 |\mathbf{p}| \partial \cos \theta_{01}/\partial \cos \theta_{11}} \) | 1,\( \aleph_1 \) | 1,2,\( \aleph_1 \) | 1,2,\( \aleph_1 \) |
| \( dp_1^0 d\cos \theta_1 \) | \( \frac{1}{4|\mathbf{p}| \partial \cos \theta_{01}/\partial \cos \theta_{11}} \) | 1,\( \aleph_1 \) | 1,2,\( \aleph_1 \) | 1,2,\( \aleph_1 \) |

FIG. 3. Some examples of computing the phase space element by the graphic method: (a) is an \( n \)-body phase space; (b) is a 3-body phase space; (c) and (d) correspond to the 4-body phase space in terms of two different combinations of invariant masses. Notice that these graphs should not be understood as Feynman diagrams.
tree diagram in Fig. 3(a), which reduces to the integration over a single invariant mass and angular variables, there is also a 1-loop diagram shown as Fig. 3(b). One has the following building blocks:

\[ V_1 : \text{d} \Phi_2(m; m_3, m_{12}), \quad V_2 : \text{d} \Phi_2(m; m_1, m_{23}), \]

\[ V_3 : \text{d} \Phi_2(m_{23}; m_2, m_3), \quad \text{“12” : (2\pi)^3 d}m_{12}^2, \quad \text{“23” : (2\pi)^3 d}m_{23}^2, \]

dashed single line “3” : \[ \frac{(2\pi)^3}{d^2 p_3 \delta(p_3^2 - m_3^2) \theta(p_3^0)} \]

Multiplying them together leads to

\[
\begin{align*}
\text{d} \Phi_3(m; m_1, m_2, m_3) &= \text{d} \Phi_2(m; m_3, m_{12})(2\pi)^3 d^2 m_{12}^2 \text{d} \Phi_2(m; m_1, m_{23}) d^2 m_{23}^2 \\
&\quad \times \delta [(p_23 - p_3)^2 - m_2^2].
\end{align*}
\]

(8)

Substituting Eq. (6) into Eq. (8), one gets

\[
\begin{align*}
\text{d} \Phi_3(m; m_1, m_2, m_3) &= \frac{|p_1||p_3|}{(2\pi)^6 m_1 m_2 m_3} \text{d} \Omega_1 \text{d} \Omega_2 \text{d} \Omega_3 d^2 m_{12}^2 \\
&\quad \times \delta \left[ m_2^2 + m_3^2 - m_1^2 - 2p_23 p_3^0 - 2|p_1||p_3| \cos \theta_3 \right] \\
&= \frac{1}{(2\pi)^8 m_1 m_2 m_3} \text{d} m_{12}^2 \text{d} m_{23}^2 \text{d} \Omega_1 \text{d} \Omega_2 \text{d} \Omega_3 \\
&\quad \times \theta_{[-1,1]} \left( \frac{m_2^2 + m_3^2 - m_1^2 - 2p_23 p_3^0}{2|p_1||p_3|} \right),
\end{align*}
\]

(9)

where all the integral variables are in the c.m. frame of the initial state. The effect of the boxcar function, which is defined as \( \theta_{[-1,1]}(x) = 1 \) for \( x \in [-1,1] \) and 0 otherwise, is to limit the invariant masses \( m_{12} \) and \( m_{23} \) to the physical region.

The 4-body phase space has more possibilities. In addition to the chain tree diagram, let us discuss two other diagrams shown as Fig. 3(c) and 3(d). For Fig. 3(c), one has the following building blocks:

\[ V_1 : \text{d} \Phi_2(m; m_{12}, m_{34}), \quad V_2 : \text{d} \Phi_2(m_{12}; m_1, m_2), \]

\[ V_3 : \text{d} \Phi_2(m_{34}; m_3, m_4), \quad \text{“12” : (2\pi)^3 d}m_{12}^2, \quad \text{“34” : (2\pi)^3 d}m_{34}^2, \]

and obtains using Eq. (6),

\[
\begin{align*}
\text{d} \Phi_4(m; m_1, m_2, m_3, m_4) &= \frac{|p_{12}||p_{34}||p_4|}{(2\pi)^{12} 16 m} \text{d} m_{12} \text{d} m_{34} \text{d} \Omega_{12} \text{d} \Omega_{34} \text{d} \Omega_4, \\
\end{align*}
\]

(10)

where \((|p_{12}||\Omega_{12}|)\) is the three-momentum of the final-state (1,2) particle system in the c.m. frame of the initial state, \((|p_3^4||\Omega_3^4|)\) is the three-momentum of particle 1 in the c.m. frame of particles 1 and 2, and \((|p_4^4||\Omega_4^4|)\) is the three-momentum of particle 3 in the c.m. frame of particles 3 and 4. The integration regions of \( m_{12} \) and \( m_{34} \) are \([m_1 + m_2, m - m_3 - m_4] \) and \([m_3 + m_4, m - m_{12}] \), respectively.

For Fig. 3(d), the building blocks are

\[ V_1 : \text{d} \Phi_2(m; m_{12}, m_{34}), \quad V_2 : \text{d} \Phi_2(m_{12}; m_1, m_2), \]

\[ V_3 : \text{d} \Phi_2(m_{234}; m_4, m_{23}), \quad V_4 : \text{d} \Phi_2(m_{23}; m_2, m_3), \]

\[ “12” : (2\pi)^3 d^2 m_{12}^2, \quad “23” : (2\pi)^3 d^2 m_{23}^2, \quad “34” : (2\pi)^3 d^2 m_{34}^2, \]

dashed single line “2” : \( (2\pi)^3 \left[ d^2 p_2 \delta(p_2^2 - m_2^2) \theta(p_2^0) \right]^{-1} \).

Multiplying them together leads to

\[
\begin{align*}
\text{d} \Phi_4(m; m_1, m_2, m_3, m_4) &= \frac{|p_{12}||p_{34}||p_4|}{(2\pi)^{12} 16 m m_{12} m_{34}} \text{d} m_{12} \text{d} m_{34} \text{d} \Omega_{12} \text{d} \Omega_{34} \text{d} \Omega_4 \\
&\quad \times \delta \left[ m_{23}^2 + m_3^2 - m_1^2 - 2p_{23} p_3^0 - 2|p_1||p_3| \cos \theta_3 \right] \\
&= \frac{1}{(2\pi)^{12} 16 m m_{12} m_{34}} \text{d} m_{12} \text{d} m_{34} \text{d} \Omega_{12} \text{d} \Omega_{34} \text{d} \Omega_4 \\
&\quad \times \theta_{[-1,1]} \left( \frac{m_{23}^2 + m_3^2 - m_1^2 - 2p_{23} p_3^0}{2|p_1||p_3|} \right),
\end{align*}
\]

(11)

where \((|p_{12}||\Omega_{12}|)\) and \((|p_{34}||\Omega_{34}|)\) are as those in Eq. (10), and the quantities labelled by a “2” are defined in the c.m. frame of \((2,3,4)\) particle system in the final state. The invariant mass of particles 2, 3 and 4, \( m_{34} \), is apparently a function of \( m_{12} \) and \( m_{34} \). The integration regions of \( m_{12} \) and \( m_{34} \) are \([m_1 + m_2, m - m_3 - m_4] \) and \([m_3 + m_4, m - m_{12}] \), respectively, and the integration region of \( m_{23} \) is limited by the boxcar function.

Summary.—In this Letter, we propose a graphic method which can greatly simplify the phase space calculation. The method is generic for evaluating an arbitrary n-body phase space. By combining the result of the 2-body phase space element with the graphic method, one can obtain the n-body phase space element in terms of any given invariant masses intuitively and efficiently: one simply follow the rules to draw the diagram containing double lines for these invariant masses. The phase space element is an essential part in computing cross sections and decay widths, and it also enters into the calculation of the imaginary part of a physical amplitude through unitarity. A broad use of this method is foreseen.

This work is supported in part by the National Natural Science Foundation of China (NSFC) under Grants No. 11835015, No. 11947302, No. 11961141012 and No. 11621131001 (the Sino-German Collaborative Research Center CRC110 “Symmetries and the Emergence
of Structure in QCD”), by the Chinese Academy of Sciences (CAS) under Grants No. XDB34030303 and No. QYZDB-SSW-SYS013, and by the CAS Center for Excellence in Particle Physics (CCEPP).

* jinghaojie@itp.ac.cn
† shencw@ucas.ac.cn
‡ fkguo@itp.ac.cn

[1] M. Tanabashi et al. [Particle Data Group], Phys. Rev. D 98, 030001 (2018).
[2] H.-J. Jing, S. Sakai, F.-K. Guo and B.-S. Zou, Phys. Rev. D 100, 114010 (2019) [arXiv:1907.12719 [hep-ph]].