EXISTENCE AND NON-EXISTENCE OF MINIMAL GRAPHS

QI DING, J. JOST, AND Y.L. XIN

Abstract. We study the Dirichlet problem for minimal surface systems in arbitrary dimension and codimension via mean curvature flow, and obtain the existence of minimal graphs over arbitrary mean convex bounded $C^2$ domains for a large class of prescribed boundary data. This result can be seen as a natural generalization of the classical sharp criterion for solvability of the minimal surface equation by Jenkins-Serrin. In contrast, we also construct a class of prescribed boundary data on just mean convex domains for which the Dirichlet problem in codimension 2 is not solvable. Moreover, we study existence and the uniqueness of minimal graphs by perturbation.

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1. INTRODUCTION

The Dirichlet problem for minimal graphs is one of the classical problems in the theory of nonlinear elliptic PDEs. It has been investigated for over a century, starting with the fundamental work of Bernstein [2], Haar [10] and Rado [20], and it has inspired the development of methods for solving nonlinear elliptic PDEs and the regularity of their solutions. Many deep and important results were achieved, and for instance the papers of

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Jenkins-Serrin [13] and Lawson-Osserman [18] can be considered as classics in the field. Nevertheless, this problem still poses difficult challenges when we move from the classical case of surfaces in $\mathbb{R}^3$ to minimal graphs of arbitrary dimension and codimension. In this paper, we make a systematic new contribution to that general problem.

In codimension 1, the graphic function that describes a minimal graph over an $n$-dimensional domain $\Omega \subset \mathbb{R}^n$ satisfies the minimal surface equation

$$
(1 + |Du|^2) \Delta u - \sum_{i,j=1}^n u_i u_j u_{ij} = 0 \quad \text{in } \Omega.
$$

For $n = 2$, when the domain $\Omega$ is convex, the Dirichlet problem for (1.1) is solvable for arbitrary continuous boundary data. This was achieved by successive efforts in the papers of Bernstein [2], Haar [10] and Rado [26] already mentioned. On the other hand, the Dirichlet problem is not necessarily solvable when $\Omega$ is non-convex, as pointed out by Bernstein. In fact, Finn [7] constructed such counterexamples. For $n > 2$, Gilbarg [8] and Stampacchia [30] established the existence of solutions to (1.1) for smooth boundary data and strictly convex smoothly bounded domains. Jenkins-Serrin [13] relaxed convexity of the domain to mean convexity, and gave a sharp criterion for the solvability. When $\partial \Omega$ is not mean convex, they found smooth boundary data for which the Dirichlet problem of (1.1) is not solvable (see also [9]). When, however, some smallness condition is imposed on the boundary data, depending on the geometry of the boundary, then Jenkins-Serrin [13] could still solve the Dirichlet problem. This, in fact, also follows from earlier work of Korn [16] by interpolation, see [33,34]. Williams [34] could solve the Dirichlet problem for boundary data with small Lipschitz norm. Also, the solution of the Dirichlet problem for the minimal surface equation is unique for fixed $C^0$-boundary data. Thus, the situation for codimension 1 can be considered as well understood.

In higher codimensions, however, the situation is much more complicated, as was shown in a seminal paper by Lawson-Osserman [18]. In dimension $n = 2$, they showed the solvability of the Dirichlet problem for the minimal surface system with arbitrary continuous boundary data on a bounded convex domain in $\mathbb{R}^2$. Again, in general, these solutions are not unique. Uniqueness fails also for reasons that do not apply in codimension 1. In fact, Sauvigny [27] showed that from non-unique parametric solutions, that is, solutions that cannot be represented as graphs, in dimension 2 and codimension 1, one can obtain non-unique graphic solutions in dimension 2 and codimension $\geq 3$. Moreover, Xu-Yang-Zhang [35] obtained the existence of boundary functions on unit disks for which infinitely many analytic solutions and at least one nonsmooth Lipschitz solution exist simultaneously.

In dimension $n \geq 4$, Lawson and Osserman gave non-existence examples for Dirichlet problems on unit balls with higher codimensions (Theorem 6.1, [18]). It is therefore natural to investigate under which conditions on the boundary data the Dirichlet problem for minimal graphs with dimension $n > 2$ and codimension $m \geq 2$ can be solved.

Reflecting upon the fact that for codimension 1, there is a general existence result, while for higher codimension there are severe obstructions to the existence, it is natural to first consider situations that can be seen as some perturbation or extension of the codimension 1 result to higher codimension. That is the approach that we take in this paper.

We study the Dirichlet problem for minimal graphs with arbitrary dimensions and codimensions, and obtain existence and uniqueness of minimal graphs over arbitrary mean convex bounded $C^2$ domain for a large class of prescribed boundary data, see Theorem
Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \in C^2$. Let $d(x) = d(x, \partial \Omega)$ for all $x \in \Omega$, and $\lambda_0$ be the maximum of 0 and the largest eigenvalue of $D^2d$ on $\partial \Omega$ (see section 2 for more details). Note that $\lambda_0 = 0$ for the convex $\Omega$.

**Theorem 1.1.** For any mean convex bounded $C^2$ domain $\Omega$ with diameter $l$, $m \geq 2$ and any $\varphi \in C^2(\overline{\Omega})$, there is a constant $\epsilon_\varphi > 0$ depending on $n, m, \lambda_0 l$, $\sup \Omega |D\varphi|$ and $l \sup \Omega |D^2\varphi|$ such that if $\psi = (\psi^1, \ldots, \psi^{m-1}) \in C^2(\overline{\Omega}, \mathbb{R}^{m-1})$ satisfies

\[
\sum_{\alpha=1}^{m-1} \left( l \sup_{\Omega} |D^2\psi^\alpha| + \sup_{\Omega} |D\psi^\alpha| \right) \leq \epsilon_\varphi,
\]

then there is a solution $u = (u^1, \ldots, u^m) \in C^\infty(\Omega, \mathbb{R}^m) \cap C^1(\overline{\Omega}, \mathbb{R}^m)$ to the minimal surface system

\[
\begin{cases}
g^{ij}u_j^\alpha = 0 & \text{in } \Omega \\
u^\alpha = \psi^\alpha & \text{on } \partial \Omega
\end{cases}
\]

for $\alpha = 1, \ldots, m$, with $g_{ij} = \delta_{ij} + \sum_{\alpha} u_i^\alpha u_j^\alpha$ and $\psi^m = \varphi$, where $\gamma$ is an arbitrary constant in $(0, 1)$.

The important point here is that the smallness assumption (1.2) is only imposed on $m-1$ of the boundary components $\psi^1, \ldots, \psi^{m-1}$. The remaining component is an arbitrary $C^2$-function. Therefore, as already mentioned, our result includes those obtained earlier for the codimension 1 case. The constant $\epsilon_\varphi$ can in principle be computed explicitly, but since our value is presumably far from optimal, we do not bother to do so. In any case, by the Lawson-Osserman counterexample, some restriction on the boundary data is necessary. Moreover, the constant $\epsilon_\varphi$ should be small, but it cannot be chosen independently of $\varphi$ (see Theorem 2.1 for details). We should also mention that Wang [32] obtained some results without uniqueness when all the boundary data are small. But because of that assumption, his results do not include those known for the codimension 1 case.

Theorem 1.1 looks like a perturbation of the codimension 1 case to higher codimension, but it cannot be obtained easily from the implicit function theorem and geometric measure theory. The reason is that the constant $\epsilon_\varphi$ is independent of the upper bound for the curvature of $\partial \Omega$. In general, the elliptic system (1.3) lacks uniqueness, which makes it difficult to study with classic continuity methods. Our strategy is utilizing the (graphic) mean curvature flow (with boundary) to approach (1.3), where the flow is a parabolic version of (1.3). Although this method has been studied in some cases (see [11, 32] for instance, and see [29] for a survey on the flow of higher codimension), the difficulty here is to show there are geometric quantities uniformly bounded along the flow under our initial condition that can control the flow.

Let $f = (f^1, \ldots, f^m)$ denote a (short-time) solution of the parabolic system corresponding to (1.3). Let $v_f$ denote the slope function of graph$_{(t, 0)}$ (see (2.7)), and $\Theta_f$ denote the function related to the 2-dilation of $f$ (see (2.8) and (4.23)). For $\Theta_f > 0$, $\Theta_f$ attains its minima and $v_f$ attains its maxima both at the parabolic boundary (see Lemma 5.3 in [31] and (4.26)). Under the a priori hypothesis $\Theta_f > 0$, we derive interior gradient estimates of $f^1, \ldots, f^{m-1}$ using Hölder gradient estimates of the parabolic version and
Huiskens’s monotonicity formula \cite{12}, and boundary gradient estimates of $f_1, \cdots, f_m$ using suitable auxiliary functions. In particular, we further need $|DF f_1|, \cdots, |DF f_{m-1}|$ small to deduce an ‘effective’ boundary gradient estimate of $f_m$ (compared with $v_f$). Moreover, the interior gradient estimate of $f_m$ is derived via $v_f$. In all, the estimates of $v_f$ and $\Theta_f$ are not independent, but inseparable. After a delicate computation, we can prove that $v_f$ is uniformly bounded and $\Theta_f > 0$ along the flow under our initial condition. With \cite{5}, we are able to deduce the uniform $C^{1, \gamma}$-estimate for $f$, which implies the long-time existence of the flow. As a result, there is a subsequence $t_i \to \infty$ so that graph $f(\cdot, t_i)$ converges to a solution of (1.2).

We can also study the Dirichlet problem of the system (1.3) by perturbation of a given minimal graph of codimension one using the implicit function theorem. However, in this situation we need the bound of the curvature of $\partial \Omega$ from both above and below (see Theorem 6.2, compared with Theorem 1.1). When comparing our result with the Lawson-Osserman counterexample, we are lead to the question to what extent also non-perturbative results are possible. But this is a question for future research.

On convex domains, we can control the constants to obtain an existence result for the Dirichlet problem with a quantitative bound related to the counterexample of \cite{18} and the Bernstein results in \cite{14} (see Theorem 1.2 for the proof). Here, the role of the constant $b_0$ below lies in controlling the gradient of the graphic mean curvature flow with the boundary condition satisfying (1.4) below.

**Theorem 1.2.** For a convex bounded $C^2$ domain $\Omega$ and any constant $b_0 \in (1, 9]$, let $\psi_1, \cdots, \psi_m$ be $C^2$-functions on $\overline{\Omega}$ with

\[
\sum_{\alpha=1}^m \left( enlb_0 \sup_{\Omega} |D^2 \psi_\alpha| + \sup_{\Omega} |D\psi_\alpha| \right) < \sqrt{b_0 - 1},
\]

where $l$ is the diameter of $\Omega$. Then there is a solution $u = (u_1, \cdots, u_m) \in C^\infty(\Omega, \mathbb{R}^m) \cap C^{1, \gamma}(\overline{\Omega}, \mathbb{R}^m)$ for any $\gamma \in (0, 1)$ to the minimal surface system (1.3) with $u = \psi$ on $\partial \Omega$ and \[
\sup_{\Omega} \left( \det \left( \delta_{ij} + u_i^\alpha u_j^\alpha \right) \right)^{1/2} < \sqrt{b_0}.
\]

As indicated, the construction of Lawson-Oseerman gives many non-existence examples for higher codimensions (Theorem 6.1, \cite{18}). In section 6, we also construct many non-existence examples for the Dirichlet problem for minimal graphs in dimension $> 2$ and codimension 2 over any mean convex (but not convex) domain, see Theorem 7.1 for concrete results. In particular, this implies that the constant $\epsilon_\varphi$ in Theorem 1.1 cannot be only small but independent of $\varphi$.

In the last section we consider the uniqueness of the Dirichlet problem for minimal graphs over a mean convex domain. In \cite{18} Lawson-Osserman showed that there exists a real analytic function $\phi: \partial D \to \mathbb{R}^2$ with the property that there are at least three distinct solutions of the corresponding problem, where $D$ is the unit disk in $\mathbb{R}^2$ (Theorem 5.1, \cite{18}). Moreover, one of these solutions represents an unstable minimal surface. Lee-Wang \cite{19} also considered the uniqueness. They proved a uniqueness theorem for nonparametric minimal submanifolds whose graphic functions are both distance-decreasing and equal on the boundary. In contrast, Sauvigny \cite{27} developed a general construction to produce non-unique solutions in codimension $\geq 3$. We prove a new uniqueness result, Proposition 8.2, which states that the solution in Theorem 6.2 is unique under a certain condition on the higher regularity of the boundary.
2. Preliminaries

For an open set \( \Omega \subset \mathbb{R}^n \), we consider a \( C^2 \) isometric immersion \( X : \Omega \rightarrow \mathbb{R}^{n+m} \). Then \( X \) is a minimal immersion if and only if

\[
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{\det g_{kl}} g^{ij} \frac{\partial X}{\partial x_j} \right) = 0,
\]

where \( g_{ij} = (\partial X/\partial x_i, \partial X/\partial x_j) \), and the matrix \( g^{ij} \) is the inverse of \( (g_{ij}) \). The immersion \( X \) is called non-parametric if it has the form \( X(x) = (x, u(x)) \) for some vector-valued function \( u = (u^1, \cdots, u^m) : \Omega \rightarrow \mathbb{R}^m \). Putting \( U(x) = (U^1(x), \cdots, U^{n+m}(x)) = (x, u(x)) \), in this case the system (2.1) becomes

\[
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{\det g_{kl}} g^{ij} \frac{\partial U^a}{\partial x_j} \right) = 0 \quad \text{for all } a,
\]

where now \( g_{ij} = \delta_{ij} + \sum_{\alpha=1}^{m} \partial x_i u^\alpha \partial x_j u^\alpha \). From this, one sees [25] (or [18]) that (2.2) may also be written as

\[
\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j} = 0 \quad \text{for } \alpha = 1, \cdots, m.
\]

If \( m = 1 \), the above minimal surface system reduces to the following single equation

\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0.
\]

De Giorgi [4] showed that Lipschitz solutions to the minimal surface equation (2.4) are smooth (see also [24], [30], for instance). However, such regularity cannot extend to Lipschitz solutions of the minimal surface system (2.3) since Lawson-Osserman [18] have constructed non-parametric minimal cones, that is, nonsmooth Lipschitz solutions. Concerning interior regularity, Morrey [22],[23] showed that \( C^1 \) solutions \( u \) of the system (2.3) are smooth.

Let \( \Delta \) and \( D \) denote the Laplacian and the Levi-Civita connection of \( \mathbb{R}^n \), respectively. For any Lipschitz function \( \varphi \) on \( \Omega \), let \( |D\varphi| \) denote the Lipschitz norm of \( \varphi \) at the considered point. If \( \varphi \in C^1(\Omega) \), then \( |D\varphi| \) is the norm of the gradient of \( D\varphi \). If \( \varphi \in C^2(\Omega) \), we define \( |D^2\varphi| = \sup_{|\xi|=1} |D^2\varphi(\xi,\xi)| \). For any vector-valued function \( \phi = (\phi^1, \cdots, \phi^m) \in C^1(\Omega, \mathbb{R}^m) \), we define 2-dilation of \( \phi \) on \( \Omega \) by

\[
\sup_{\Omega} |\Lambda^2 d\phi| = \sup_{x \in \Omega} |\Lambda^2 d\phi(x)| = \sup_{x \in \Omega, 1 \leq i < j \leq n} \mu_i(x) \mu_j(x),
\]

where \( \{\mu_k(x)\}_{k=1}^{n} \) are the singular values of the Jacobi matrix \( d\phi(x) = (\partial_x \phi^\alpha(x))_{n \times m} \).

We denote the distance from \( \partial \Omega \) by

\[
d(x) = d(x, \partial \Omega) = \inf_{y \in \partial \Omega} |x - y|
\]

for each \( x \in \overline{\Omega} \) and

\[
\Omega_s = \{ y \in \Omega \mid d(y, \partial \Omega) > s \} \quad \text{for any } s > 0.
\]

We further assume \( \partial \Omega \in C^2 \). Let \( \lambda_1(D^2d), \cdots, \lambda_{n-1}(D^2d) \) denote the \( n \) eigenvalues of \( (d_{ij})_{n \times n} \) at the points in \( \overline{\Omega} \) where \( d \) is twice differentiable. (This is the case in some neighborhood of \( \partial \Omega \).) Then \( -\lambda_1(D^2d), \cdots, -\lambda_{n-1}(D^2d) \) on \( \partial \Omega \) are the principal curvatures of \( \partial \Omega \). If \( \max_{1 \leq i \leq n-1} \lambda_i(D^2d) \leq 0 \) on \( \partial \Omega \), then \( \Omega \) is convex. If \( \sum_{i=1}^{n-1} \lambda_i(D^2d) \leq 0 \) on \( \partial \Omega \),
then \( \partial \Omega \) is mean-convex, i.e., the mean curvature of \( \partial \Omega \) is nonnegative. Let \( \lambda_\Omega \) be the maximum of zero and the largest eigenvalue of \( D^2d \) on \( \partial \Omega \), i.e.,
\[
\lambda_\Omega \triangleq \max_{\partial \Omega} \left\{ 0, \max_{1 \leq i \leq n-1} \lambda_i(D^2d) \right\} \geq 0.
\]
For any \( x \in \Omega \) where \( d \) is differentiable (for instance, in some neighborhood of \( \partial \Omega \)), there exists a unique \( y_x \in \partial \Omega \) such that \( d(x) = |x - y_x| \). In particular, \( d \) is twice differentiable at \( x \). From Lemma 14.17 in [9], we have
\[
(2.6) \quad \max_{1 \leq i \leq n} \lambda_i(D^2d) \leq \lambda_\Omega \quad \text{at } x.
\]
Let \( r_\Omega \) be the infimum of the radii of exterior balls of \( \Omega \). Namely, \( r_\Omega \) is the largest constant such that for any \( p \in \partial \Omega \) there is a unique open ball \( B_{r_\Omega}(q) \subset \mathbb{R}^n \setminus \Omega \) centered at \( q \) and with the radius \( r_\Omega \) such that \( B_{r_\Omega}(q) \cap \Omega = \emptyset \). It is clear that \( r_\Omega \leq 1/\lambda_\Omega \). Thus, if \( \lambda_\Omega = 0 \), then \( 1/\lambda_\Omega = \infty \). In general, however, \( r_\Omega \neq 1/\lambda_\Omega \) when \( \lambda_\Omega > 0 \).

**Notational conventions:** Unless the contrary is explicitly stated, we assume that the considered minimal graphs or mean curvature flows have dimension \( n \geq 2 \). For a vector-valued function \( \phi = (\phi^1, \ldots, \phi^m) : \Omega \to \mathbb{R}^m \), \( \phi^\alpha \) denotes its \( \alpha \)-th component, \( \phi_i^\alpha \) denotes \( \partial_x \phi^\alpha \), and \( v_\phi \) denotes the slope function of \( \text{graph}_\phi \) defined by
\[
(2.7) \quad v_\phi \triangleq \sqrt{\det(\delta_{ij} + \sum_{\alpha=1}^m \phi_i^\alpha \phi_j^\alpha)}.
\]
For simplicity, we denote \( C^k(K, \mathbb{R}^m) \) by \( C^k(K) \) for any integer \( k \geq 0 \) and any open (or closed) set \( K \subset \mathbb{R}^m \). The Einstein summation convention over repeated indices will be used. Greek indices \( \alpha, \beta \) take their values in the set \( \{1, \ldots, m\} \).

3. **Boundary gradient estimates for the mean curvature flow**

Let \( \Omega \) be an open set in \( \mathbb{R}^n \), and \( T \) a positive constant. For each \( f = (f^1, \ldots, f^m) \in C^2(\Omega \times (0, T)) \), let \( F_t \) be of the form \( F_t(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f^1(x, t), \ldots, f^m(x, t)) \) with \( x = (x_1, \ldots, x_n) \in \Omega \), \( t \in (0, T) \) such that the graph \( \text{graph}_{f(\cdot, t)} = \{(x, f(x, t)) \mid x \in \Omega \} \subset \mathbb{R}^{n+m} \) moves along mean curvature flow, i.e.,
\[
\frac{d F_t}{dt} = H_t(x),
\]
where \( H_t \) denotes the mean curvature of the graph \( f_t \). Then \( f = (f^1, \ldots, f^m) \) satisfies the parabolic equations
\[
(3.1) \quad \frac{\partial f_i^\alpha}{\partial t} = \frac{df_i^\alpha}{dt} - \frac{\partial f_i^\alpha}{\partial x_j} \frac{\partial x_j}{\partial t} = \frac{1}{\sqrt{\det g_{kl}}} \partial_t \left( g^{ij} \sqrt{\det g_{kl}} f_i^\alpha \right) - \frac{f_j^\alpha}{\sqrt{\det g_{kl}}} \partial_i \left( g^{ij} \sqrt{\det g_{kl}} \right) = g^{ij} f_{ij}^\alpha
\]
on \( \Omega \times (0, T) \), where \( g_{ij} = \delta_{ij} + \sum_{\alpha} f_i^\alpha f_j^\alpha \), and \( (g^{ij}) \) is the inverse matrix of \( (g_{ij}) \).

In this section, we will study boundary gradient estimates for the mean curvature flow.

**Lemma 3.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( \partial \Omega \in C^2 \) and diameter \( l \), and \( \psi = (\psi^1, \ldots, \psi^m) \in C^2(\overline{\Omega}) \). Suppose there is a solution \( (f^1, \ldots, f^m) \in C^\infty(\Omega \times (0, T]) \cap \)
\( C^0(\Omega \times [0, T]) \) to the flow
\[
\begin{cases}
\frac{\partial f^\alpha}{\partial t} = g^{ij} f^\alpha_{ij} & \text{in } \Omega \times (0, T) \\
f^\alpha(\cdot, 0) = \psi^\alpha & \text{on } \Omega \times \{0\} \\
f^\alpha(\cdot, t) = \psi^\alpha & \text{on } \partial \Omega \times [0, T]
\end{cases}
\]
for \( \alpha = 1, \ldots, m \), where \((g^{ij})\) is the inverse matrix of \( g_{ij} = \delta_{ij} + \sum \alpha f^\alpha_{ij} \). Then the following boundary gradient estimate holds:
\[(3.3) \sup_{\partial \Omega \times [0, T]} |Df^\alpha| \leq n l (1 + \lambda_f^2) e^{1 + \frac{(n-1)l}{\rho}(1 + \lambda_f^2)} \sup_{\Omega} |D^2 \psi^\alpha| + \sup_{\Omega} |D \psi^\alpha|,
\]
where \( \lambda_f^2 \) is the supremum of the largest eigenvalue of \( Df(Df)^T \) on \( \Omega \times (0, T) \).

**Proof.** We consider a point \( p \in \partial \Omega \); without loss of generality (after a translation), we can assume \( p = 0 \) and \( B_{r_\alpha}(r_\alpha E_n) \cap \Omega = 0 \), where \( E_n = (0, 0, \ldots, 1) \in \mathbb{R}^n \) and \( r_\alpha E_n = (0, 0, \ldots, 0, r_\alpha) \). We define a function
\[ S^\alpha_\pm(x, t) = \frac{\Theta_\alpha}{\theta} \left( 1 - e^{-\theta \rho(x)} \right) \pm (f^\alpha(x, t) - \psi^\alpha(x)) \]
on \( \overline{\Omega} \times [0, T] \), where \( \rho(x) = |x + r_\alpha E_n| - r_\alpha \), and \( \theta, \Theta_\alpha \) are positive constants to be defined later. Obviously, \( \rho > 0 \) on \( \Omega \). Put \( y_i = x_i \) for \( i = 1, \ldots, n - 1 \) and \( y_n = x_n + r_\alpha E_n \). Note that every eigenvalue of \((g^{ij})\) is between \( (1 + \lambda_f^2)^{-1} \) and 1. Since the matrix \((\delta_{ij} - y_i y_j |x + r_\alpha E_n|^{-2})\) has eigenvalues 0 and 1 (of multiplicity \((n - 1)\)), there exist an orthonormal \((n \times n)\)-matrix \( P = (p_{ij}) \) and a diagonal \((n \times n)\)-matrix \( \Lambda = (\Lambda_{ij}) = \text{diag}\{0, 1, \ldots, 1\} \) so that \( \delta_{ij} - y_i y_j |x + r_\alpha E_n|^{-2} = P_{ik} \Lambda_{kl} P_{jl} \). Clearly, the matrix \((P_{ik} g^{ij} P_{jl})_{i,j=1,\ldots,n}\) is positive definite with eigenvalues \( \leq 1 \), which implies that each element of the matrix \((P_{ik} g^{ij} P_{jl})_{i,j=1,\ldots,n}\) is \( \leq 1 \). Then
\[(3.4) \ g^{ij} \left( \delta_{ij} - \frac{y_i y_j}{|x + r_\alpha E_n|^2} \right) = g^{ij} P_{ik} \Lambda_{kl} P_{jl} \leq n - 1.\]

For any \( \alpha \in \{1, \ldots, m\} \), with \( (3.4) \) we have
\[(3.5) \ \frac{\partial S^\alpha_\pm}{\partial t} - g^{ij} \partial_{ij} S^\alpha_\pm = -g^{ij} \Theta_\alpha e^{-\theta \rho} \left( \frac{\delta_{ij}}{|x + r_\alpha E_n|^3} - \frac{y_i y_j}{|x + r_\alpha E_n|^2} - \frac{y_i y_j \theta}{|x + r_\alpha E_n|^2} \right) \pm g^{ij} \partial_{ij} \psi^\alpha \geq \Theta_\alpha e^{-\theta \rho} \left( \frac{\theta}{1 + \lambda_f^2} - \frac{n - 1}{\rho + r_\alpha} \right) - n \sup_{\Omega} |D^2 \psi^\alpha| \]
on \( \Omega \times (0, T) \). Let \( l \) be the diameter of \( \Omega \) and \( \theta = \frac{1}{l} + \frac{n-1}{r_\alpha} (1 + \lambda_f^2) \). If we set
\[ \Theta_\alpha = n l (1 + \lambda_f^2) e^{1 + \frac{(n-1)l}{\rho}(1 + \lambda_f^2)} \sup_{\Omega} |D^2 \psi^\alpha|,
\]
then
\[(3.6) \ \frac{\partial S^\alpha_\pm}{\partial t} - g^{ij} \partial_{ij} S^\alpha_\pm > 0.\]

By the maximum principle, it is clear that \( S^\alpha_\pm > 0 \) on \( \Omega \times [0, T] \). Namely,
\[(3.7) \ |f^\alpha(x, t) - \psi^\alpha(x)| < \frac{\Theta_\alpha}{\theta} \left( 1 - e^{-\theta \rho(x)} \right) \]
for any \((x, t) \in \Omega \times [0, T] \).
Hence at the point $p = 0$, it follows that

\begin{equation}
|Df^\alpha(p, t) - D\psi^\alpha(p)| \leq \Theta_\alpha \quad \text{for any } t \in [0, T].
\end{equation}

As $p$ is an arbitrary point in $\partial \Omega$, the proof is complete. \hfill \Box

**Remark.** We do not use the structure of $g^{ij}$ in the above proof. Though the general boundary gradient estimates are well-known, such as in [20] for parabolic equations or in [9] for elliptic equations, we use a different auxiliary function here. In the case of a convex $\Omega$, we see that our estimate (3.3) is stronger than Theorem 3.1 in [32] if we let $r_\alpha \to \infty$ in [83].

Comparing Lemma 3.1, we have a boundary gradient estimate for the mean curvature flow depending on $\lambda_\alpha$, but independent of $r_\alpha$.

**Lemma 3.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \subset C^2$ and diameter $l$, and $\psi = (\psi^1, \ldots, \psi^m) \in C^2(\overline{\Omega})$. Let $f = (f^1, \ldots, f^m)$ be a smooth solution to (3.2) on $\Omega \times (0, T]$ with $f \in C^0(\overline{\Omega} \times (0, T])$ and boundary data $\psi$. Then

\begin{equation}
\sup_{\partial \Omega \times [0, T]} |Df^\alpha| \leq e^{1+2(n-1)\lambda_\alpha l(1+\lambda_2^\beta)} \left(l \lambda_2^\beta \sup_{\Omega} |D^2\psi^\alpha| + 3 \sup_{\Omega} |D\psi^\alpha|\right),
\end{equation}

where $\lambda_2^\beta$ is the supremum of the largest eigenvalue of $Df(Df)^T$ on $\Omega \times (0, T)$.

**Proof.** We consider a point $p \in \partial \Omega$; without loss of generality (after a translation), we can assume $p = 0$ and $E_n = (0, 0, \ldots, 1) \in \mathbb{R}^n$ is the unit normal vector (pointing into $\Omega$) to $\partial \Omega$ at $p = 0$. Let $\rho(x) = \left| x + \frac{1}{2\lambda_\alpha} E_n - \frac{1}{2\lambda_\alpha^2} \right|$ for each $x \in \mathbb{R}^n$, and $\Omega_* \subset \{x + \frac{1}{2\lambda_\alpha} E_n < \frac{1}{\lambda_\alpha^2}\}$. By the definition of $\lambda_\alpha$, $\partial \Omega_* \cap \{\rho(x) = 0\} = \{0\}$.

We define a function

$$S^\alpha_\pm(x, t) = \frac{\Theta_\alpha}{\theta} \left(1 - e^{-\theta \rho(x)}\right) \pm (f^\alpha(x, t) - \psi^\alpha(x))$$

on $\Omega' \times [0, T]$, where $\theta = \frac{1}{2} + 2(n-1)\lambda_\alpha (1+\lambda_2^\beta)$ and

$$\Theta_\alpha = e^{1+2(n-1)\lambda_\alpha l(1+\lambda_2^\beta)} \left(l \lambda_2^\beta \sup_{\Omega} |D^2\psi^\alpha| + 2 \sup_{\Omega} |D\psi^\alpha|\right).$$

Then at any point $x$ with $\rho(x) = \frac{1}{2\lambda_\alpha}$, we get

\begin{equation}
\frac{\Theta_\alpha}{\theta} \left(1 - e^{-\theta \rho(x)}\right) \geq 2 l \sup_{\Omega} |D\psi^\alpha| \left(1 - e^{-(n-1)}\right) \geq l \sup_{\Omega} |D\psi^\alpha|,
\end{equation}

which implies $S^\alpha_\pm(x, t) \geq \sup_{\Omega} |\psi^\alpha|$, and $S^\alpha_\pm(x, t) \leq -\sup_{\Omega} |\psi^\alpha|$. From the calculation in the proof of Lemma 3.1, one has

\begin{equation}
\frac{\partial S^\alpha_\pm}{\partial t} - g^{ij} \partial_i S^\alpha_\pm \geq 0 \quad \text{on } \Omega \times (0, T].
\end{equation}

By the maximum principle, we complete the proof. \hfill \Box
Now we assume that the diameter of $\Omega$ satisfies $l = 1$, and $\partial \Omega$ has nonnegative mean curvature pointing into $\Omega$ in the rest of this section. From Lemma 14.17 in [9], $\partial \Omega$ also has nonnegative mean curvature. Let $\phi$ be a $C^2$-function on $[0, \infty)$ with

$$\phi' \geq 0 \quad \text{and} \quad \phi'' \leq 0 \quad \text{on} \quad [0, \infty).$$

Let $\varphi \in C^2(\Omega)$ and $\tilde{\phi} = \phi \circ d + \varphi$, where $d(x) = d(x, \partial \Omega)$ for all $x \in \Omega$. For each $w^\alpha \in C^1(\Omega)$ with $\alpha = 1, \cdots, m - 1$, we define

$$a_{ij} \triangleq \delta_{ij} + \sum_{\alpha=1}^{m-1} w^\alpha_i w^\alpha_j + \tilde{\phi}_i \tilde{\phi}_j$$

for $1 \leq i, j \leq n$, and let $(a^{ij})$ be the inverse matrix of $(a_{ij})$. Assume that the matrix $(D\tilde{\phi}, Dw^1, \cdots, Dw^{m-1})$ has singular values $\mu_1, \cdots, \mu_n$ with

$$|\mu_i \mu_j| \leq 1 \quad \text{for any} \quad i \neq j.$$

**Lemma 3.3.** Suppose that $\phi$ is the function defined as above, and (3.12) holds. Then

$$a^{ij} \tilde{\phi}_{ij} \leq (n - 2)\phi'\lambda_\Omega \left( \frac{2}{(\phi')^2} \left( |D\varphi|^2 + \frac{n-1}{1+\mu_1^2} \right) + \frac{1}{1+\mu_1^2} \right) + \frac{\phi''}{1+\mu_1^2} + n|D^2\varphi|,$$

at all differentiable points of $d$ on $\Omega$.

**Proof.** At any fixed point $p \in \Omega$ at which $d$ is differentiable, we choose a coordinate system such that

$$a_{ij} = \delta_{ij} (1 + \mu_i^2),$$

with $\mu_1^2 \geq \mu_2^2 \geq \cdots \geq \mu_n^2$. From $a_{ii} = 1 + \sum_{\alpha=1}^{m-1} w^\alpha_i w^\alpha_i + \tilde{\phi}_i \tilde{\phi}_i = 1 + \mu_i^2$, it follows that

$$\mu_i^2 \geq |\tilde{\phi}_i|^2.$$

Combining (3.12) and (3.14), one has

$$\sum_{i=2}^n |\tilde{\phi}_i|^2 \leq \sum_{i=2}^n \mu_i^2 \leq \frac{n-1}{\mu_1^2}.$$

With the Cauchy-Schwarz inequality we get

$$\phi'^2 \sum_{i=2}^n d_i^2 \leq -\sum_{i=2}^n (2\phi' d_i \varphi_i + \varphi_i^2) + \frac{n-1}{\mu_1^2} \leq \frac{1}{2} (\phi')^2 \sum_{i=2}^n d_i^2 + \sum_{i=2}^n \varphi_i^2 + \frac{n-1}{\mu_1^2}.$$

where $d_i = \frac{\partial}{\partial x_i}$. Thus

$$1 - d_i^2 = \sum_{i=2}^n d_i^2 \leq \frac{2}{(\phi')^2} \left( |D\varphi|^2 + \frac{n-1}{\mu_1^2} \right).$$

Recall $d(x) = d(x, \partial \Omega)$. In a neighborhood of the point $p$, we choose an orthonormal basis $\{\partial_\rho\} \cup \{e_i\}_{i=1,\cdots,n-1}$, such that $\partial_\rho d = 1$, $\{e_i\}_{i=1,\cdots,n-1}$ is normal at $p$, and

$$\frac{\partial}{\partial x_1} = d_1 \partial_\rho + \sqrt{1 - d_1^2} e_1.$$

Here, ‘normal’ means $(D_{e_i} e_i)^T = 0$ at $p$, where $(\cdot)^T$ denotes the projection onto the tangent bundle of $\partial \Omega_{d(p)}$. Since the function $d$ is a constant on $\partial \Omega_{d(p)}$, then we get

$$\frac{\partial}{\partial x_1} (D_{e_i} D_{e_i} - (D_{e_i} e_i)^T) d = D_{e_i} D_{e_i} d = 0$$
for each \( i = 1, \ldots, n-1 \) at \( p \). Since \( D_{\partial \rho} d = 1 \) and \( (D_{\partial \rho} D_{\partial \rho} - D_{\partial \rho} \partial_{\rho})d = 0 \) at \( p \), combining (3.19) one has

\[
d_{11} = \text{Hess}_d \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right) = (1 - d_1^2)\text{Hess}_d (e_1, e_1) = -(1 - d_1^2)(D_{e_1} e_1) d,
\]

and

\[
\Delta d = \sum_{i=1}^{n-1} \langle D_{e_i} D_{e_i} - D_{e_i} \partial_{\rho} \rangle d = -\sum_{i=1}^{n-1} \langle D_{e_i} e_1, \partial_{\rho} \rangle d = -H_{\partial \Omega d(p)} \leq 0,
\]

where \( H_{\partial \Omega d(p)} \) denotes the mean curvature of \( \partial \Omega d(p) = \{ x \in \Omega | d(x) = d(p) \} \). Since \( \partial \Omega d(p) \) is mean convex, then \( \Delta d \leq 0 \) at \( p \). From (2.6), it follows that \( -(D_{e_i} e_1) d \leq \lambda_\Omega \) for each \( i = 1, \ldots, n-1 \) at \( p \). Then for any \( t \in [0,1] \)

\[
-t d_{11} + \Delta d = t(1 - d_1^2)(D_{e_1} e_1) d + \Delta d \leq t(1 - d_1^2)(D_{e_1} e_1) d + t(1 - d_1^2) \Delta d
\]

(3.22)

\[
= -t(1 - d_1^2) \sum_{i=2}^{n-1} (D_{e_i} e_i) d \leq (n-2) t(1 - d_1^2) \lambda_\Omega.
\]

We shall now compute \( a^{ij} \phi_{ij} \) at \( p \). From \( \mu_1^2 \geq \mu_2^2 \geq \cdots \geq \mu_n^2 \), (2.6), (3.21), (3.22), one has

\[
a^{ij} d_{ij} = \sum_{i=1}^{n} \frac{1}{1 + \mu_i^2} d_{ii} = \frac{d_{11}}{1 + \mu_1^2} - \frac{d_{11}}{1 + \mu_2^2} + \frac{\Delta d}{1 + \mu_2^2} + \sum_{i=3}^{n} \left( \frac{1}{1 + \mu_i^2} - \frac{1}{1 + \mu_2^2} \right) d_{ii} \leq \frac{1}{1 + \mu_2^2} \left( -\frac{\mu_1^2 - \mu_2^2}{1 + \mu_1^2} d_{11} + \Delta d \right) + \sum_{i=3}^{n} \left( 1 - \frac{1}{1 + \mu_2^2} \right) \lambda_\Omega \leq \frac{(n-2) \mu_1^2}{1 + \mu_1^2} (1 - d_1^2) \lambda_\Omega + \frac{n - 2}{1 + \mu_1^2} \lambda_\Omega.
\]

Noting \( \phi'' \leq 0 \) and the definition of \( |D^2 \varphi| \), we have

(3.24)

\[
a^{ij} (\phi'' d_{ij} + \varphi_{ij}) \leq \phi'' \sum_{i=1}^{n} \frac{d_{i}^2}{1 + \mu_i^2} + n|D^2 \varphi| \leq \phi'' \sum_{i=1}^{n} \frac{d_{i}^2}{1 + \mu_i^2} + n|D^2 \varphi| = \frac{\phi''}{1 + \mu_1^2} + n|D^2 \varphi|.
\]

Combining (3.17) (3.23) and (3.24), we obtain

\[
a^{ij} \phi_{ij} = a^{ij} (\phi' d_{ij} + \phi'' d_{ij} + \varphi_{ij}) \leq (n-2) \phi' \lambda_\Omega \left( \frac{2}{(\phi')^2} \left( |D^2 \varphi|^2 + \frac{n - 1}{1 + \mu_1^2} \right) + \frac{1}{1 + \mu_1^2} \right) + \phi'' + n|D^2 \varphi|.
\]

This completes the proof. \( \square \)

Denote \( |D \varphi|_\Omega \triangleq \sup_{x \in \Omega} |D \varphi| \) and \( |D^2 \varphi|_\Omega \triangleq \sup_{x \in \Omega} |D^2 \varphi| \). Let us deduce another boundary gradient estimate using the structure of \( g^{ij} \).

**Theorem 3.4.** Let \( \Omega \) be a mean convex bounded domain in \( \mathbb{R}^n \) with diameter \( l = 1 \) and \( \partial \Omega \subset C^2 \). Let \( \psi = (\psi_1, \ldots, \psi_m) \subset C^2(\overline{\Omega}) \) and \( f = (f^1, \ldots, f^m) \) be a smooth solution to (3.2) on \( \Omega \times (0,T] \) with \( f \in C^0(\overline{\Omega} \times [0,T]) \) and boundary data \( \psi \). Denote \( \varphi = \psi^m \) and \( \nu = 16n(|D^2 \varphi|_\Omega + 1) \). Let \( \kappa \) be the constant defined by

\[
(3.26) \quad \kappa = \max \left\{ 64(n-2) \lambda_\Omega (1 + |D \varphi|_\Omega^2) e^{3|D \varphi|_\Omega \nu}, 2
\nu \left( \sqrt{n} + |D \varphi|_\Omega \right) e^{|D \varphi|_\Omega \nu} \right\}.
\]
If \( \sup_{\Omega \times (0,T)} |\nabla^2 f| \leq 1 \) and \( |Df^\alpha| \leq \frac{1}{m-1} \) on \( \Omega_{1/\kappa} \times [0,T] \) for \( \alpha = 1, \ldots, m-1 \), and
\[
\sup_{\Omega \times (0,T)} \det g_{ij} \leq \frac{2\kappa^2}{\nu},
\]
then we have
\[
(3.27) \quad \sup_{(x,t) \in \partial \Omega \times [0,T]} |Df^m(x,t)| \leq \frac{\kappa}{\nu} + |D\varphi|_\Omega.
\]

**Proof.** By the maximum principle for parabolic equations,
\[
(3.28) \quad \inf_{y \in \Omega} \varphi(y) \leq f^m(x,t) \leq \sup_{y \in \Omega} \varphi(y) \quad \text{for all } x \in \Omega \times [0,T].
\]
Set
\[
\phi(d) = \frac{1}{\nu} \log (1 + \kappa d) \quad \text{on } \Omega,
\]
then
\[
(3.29) \quad \phi' = \frac{\kappa}{\nu(1 + \kappa d)} > 0 \quad \text{and} \quad \phi'' = -\frac{\kappa^2}{\nu(1 + \kappa d)^2} < 0.
\]
Set
\[
\tilde{\varphi} = \phi \circ d + \varphi \quad \text{on } \Omega.
\]
Denote \( g_{ij} = \delta_{ij} + \sum_{\alpha=1}^m u^\alpha_i u^\alpha_j \). We claim
\[
(3.30) \quad g^{ij} \tilde{\varphi}_{ij} = g^{ij} \left( \phi' d_{ij} + \phi'' d_i d_j + \varphi_{ij} \right) < 0
\]
at each considered point \( (q,t) \in \Omega \times (0,T) \) with \( D\tilde{\varphi}(q) = Df^m(q,t) \) and \( f^m(q,t) > \tilde{\varphi}(q) \).
By the maximum principle for \( (3.30) \) and \( \frac{\partial f^m}{\partial t} - g^{ij} f^m_{ij} = 0 \), we obtain
\[
\phi(d(x)) + \varphi(x) \geq f^m(x,t) \quad \text{for any } x \in \Omega \times [0,T].
\]
Analogously to the above argument, one has
\[
\phi(d(x)) - \varphi(x) \geq -f^m(x,t) \quad \text{for any } x \in \Omega \times [0,T].
\]
Therefore, for any \( (x,t) \in \partial \Omega \times [0,T] \) it follows that
\[
(3.31) \quad |Df^m(x,t)| \leq \sup_{\partial \Omega} |D(\phi \circ d)| + |D\varphi|_\Omega \leq \frac{\kappa}{\nu} + |D\varphi|_\Omega.
\]

For completing the proof of this theorem, we only need to show the claim \((3.30)\) at the point \( q \) with \( D\tilde{\varphi}(q) = Df^m(q,t) \) and \( f^m(q,t) > \tilde{\varphi}(q) \). Denote
\[
d_0 = \frac{1}{\kappa} e^{|D\varphi|_\Omega \nu}.
\]
From \( \phi(d_0) > |D\varphi|_\Omega \geq \sup_{x \in \Omega} \varphi(x) - \inf_{x \in \Omega} \varphi(x) \), there holds \( q \in \Omega \setminus \overline{\Omega_{d_0}} \). Let \( \lambda_1, \ldots, \lambda_n \) be the singular values of \( f^m_{ij} \) at \( (q,t) \) such that \( \lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2 \). Since \( 0 < d(q) < d_0 \), then by the definition of \( \kappa \) we have
\[
\phi' = \frac{\kappa}{\nu(1 + \kappa d)} \geq \frac{1}{2d_0 \nu} \geq \sqrt{n} + |D\varphi|_\Omega.
\]
Combining the definition of $\kappa, d_0$ and $1 + \lambda_1^2 \leq \det g_{ij} \leq \frac{2k^2}{\nu^2}$, first we have

$$
(n - 2)\lambda_\Omega \left( \frac{2}{(\phi')^2} \left( |D\phi|^2 + \frac{n - 1}{1 + \lambda_1^2} \right) + \frac{1}{1 + \lambda_1^2} \right) - \frac{\phi''}{2\phi'} \frac{1}{1 + \lambda_1^2}
$$

$$
\leq (n - 2)\lambda_\Omega \left( \frac{2|D\phi|^2}{(\phi')^2} + \frac{2(n - 1)}{n(1 + \lambda_1^2)} + \frac{1}{1 + \lambda_1^2} \right) - \frac{\kappa}{2(1 + \kappa d)} \frac{1}{1 + \lambda_1^2}
$$

(3.32)

$$
\leq 8(n - 2)\lambda_\Omega |D\phi|^2 \frac{\nu^2}{\kappa^2} e^{2|D\phi|\alpha\nu} - \left( \frac{1}{4d_0} - 3(n - 2)\lambda_\Omega \right) \frac{\nu^2}{2\kappa^2}
$$

$$
= \frac{\nu^2}{8\kappa^2} \left( - \frac{1}{d_0} + 64(n - 2)\lambda_\Omega |D\phi|^2 e^{2|D\phi|\alpha\nu} + 12(n - 2)\lambda_\Omega \right) \leq 0.
$$

Next, let us estimate the remaining terms in (3.33).

- Case 1: $q \in \Omega \setminus \Omega_{1/\kappa}$. Since $\kappa d(q) \leq 1$ and $1 + \lambda_1^2 \leq \frac{2k^2}{\nu^2}$, it follows that

$$
\frac{\phi''}{2(1 + \lambda_1^2)} + n|D^2\phi|\Omega \leq \frac{\kappa^2}{2\nu^2} \frac{\nu^2}{2\kappa^2} + |D^2\phi|\Omega \leq -\frac{\nu}{16} + n|D^2\phi|\Omega \leq 0;
$$

- Case 2: $q \in \Omega_{1/\kappa}$. From the assumption in this theorem, one has

$$
1 + \lambda_1^2 \leq 1 + \left( \sum_{a=1}^m |Df^a| \right)^2 \leq 1 + \left( |Df^m| + (m - 1) \frac{1}{m - 1} \right)^2
$$

(3.34)

$$
\leq 1 + \left( |D\phi| + 1 \right)^2 \leq 1 + (\phi' + |D\phi| + 1)^2 \leq 5(\phi')^2.
$$

Then

$$
\frac{\phi''}{2(1 + \lambda_1^2)} + n|D^2\phi|\Omega \leq -\frac{\phi''}{10(\phi')^2} + n|D^2\phi|\Omega = -\frac{\nu}{10} + n|D^2\phi|\Omega \leq 0.
$$

Combining Lemma 3.13 and (3.32) (3.33) (3.35), the claim (3.30) is true. We complete the proof. \hfill \Box

4. The Dirichlet problem on mean convex domains

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the singular values of a matrix $\{u_j^i\}_{1 \leq i \leq n, 1 \leq \alpha \leq m}$ ($\lambda_j = 0$ if $\min\{m, n\} < j \leq n$). Let us first prove an algebraic lemma that will be needed in the sequel.

Lemma 4.1. If $|D u|^2 \sum_{\alpha=2}^m |D u^\alpha|^2 \leq K^2$ for some constant $K > 0$, and $\sum_{\alpha=2}^m |D u^\alpha|^2 \leq K$, then $|\lambda_i\lambda_j| \leq 2K$ for all $i \neq j$.

Proof. By scaling, we only need prove the lemma for $K = 1$. For any considered point $x$, we choose an orthonormal coordinate system in its neighborhood such that $D_1 u^1 = |D u^1|$ at $x$. Now we assume that $D_1 u^1 = t$ for some constant $t \geq 1$. Then by the assumption of the lemma, it follows that

$$
\sum_{\alpha=2}^m |D u^\alpha|^2 \leq \frac{1}{t^2}.
$$
For any $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{S}^n$, we have

\begin{equation}
\sum_{\alpha,\beta,j} u_\alpha^{i} u_j^{\alpha} \xi_i \xi_j = u_1^{i} u_1^{i} \xi_1 + \sum_{\alpha \geq 2, i,j} u_\alpha^{i} u_j^{\alpha} \xi_i \xi_j \leq t^2 \xi_1^2 + \frac{1}{t^2}.
\end{equation}

By a rearrangement, we can assume that $\lambda_2^2$ is the maximal eigenvalue of the matrix $(\sum_{\alpha} u_\alpha^{i} u_j^{\alpha})_{n \times n}$ with the corresponding eigenfunction $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{S}^n$. Then from (4.1) it follows that

\begin{equation}
t^2 \xi_1^2 \leq t^2 \xi_1^2 + \frac{1}{t^2},
\end{equation}

which implies

\begin{equation}
\xi_1^2 \geq 1 - \frac{1}{t^4}.
\end{equation}

For any $\eta = (\eta_1, \cdots, \eta_n) \in \mathbb{S}^n$ with $\xi \perp \eta = 0$, one has

\begin{equation}
\xi_1^2 \eta_1^2 = \left(\sum_{i \geq 2} \xi_i \eta_i\right)^2 \leq \sum_{i \geq 2} \xi_i^2 \eta_i^2 = (1 - \xi_1^2)(1 - \eta_1^2),
\end{equation}

which implies

\begin{equation}
\eta_1^2 \leq 1 - \xi_1^2 \leq \frac{1}{t^4}.
\end{equation}

If $\eta = (\eta_1, \cdots, \eta_n) \in \mathbb{S}^n$ is the eigenfunction of $(\sum_{\alpha} u_\alpha^{i} u_j^{\alpha})_{n \times n}$ with respect to the second eigenvalue $\lambda_2^2$, then combining $D_1 u_1 = |Du_1| = t$ and $\sum_{\alpha=2}^{m} |Du_\alpha|^2 \leq \frac{1}{t^2}$ one has

\begin{equation}
\lambda_2^2 = \sum_{\alpha,\beta,j} u_\alpha^{i} u_j^{\alpha} \eta_i \eta_j = t^2 \eta_1^2 + \sum_{\alpha \geq 2, i,j} u_\alpha^{i} u_j^{\alpha} \eta_i \eta_j \leq t^2 \eta_1^2 + \frac{1}{t^2} \leq \frac{2}{t^2}.
\end{equation}

So we obtain

\begin{equation}
\lambda_1^2 \lambda_2^2 \leq \left(t^2 + \frac{1}{t^2}\right) \frac{2}{t^2} = 2 + \frac{2}{t^4} \leq 4.
\end{equation}

If $D_1 u_1 \leq 1$, then $\lambda_2^2 \leq 2$ clearly. In all, we always have

$$\lambda_1 \lambda_2 \leq 2.$$

Hence, this completes the proof of Lemma 4.1. \hfill \Box

Now let us deduce an interior gradient estimate for the mean curvature flow.

**Lemma 4.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \in C^2$ and diameter $l = 1$. Let $\psi = (\psi^1, \cdots, \psi^m) \in C^2(\overline{\Omega})$ and $f = (f^1, \cdots, f^m)$ be a smooth solution to (4.2) on $\Omega \times (0, T]$ with $f \in C^0(\overline{\Omega} \times [0, T])$ and boundary data $\psi$ such that $|A^2 df| \leq 1 - \epsilon$ on $\Omega \times [0, T]$ for some constant $\epsilon \in (0, 1)$. Let $\kappa$ be the constant in Theorem 3.4. If $\sup_{\Omega \times [0, T]} |Df| \leq \Lambda$ for some constant $\Lambda > 0$, then for any $0 < s \leq 1$ there exists a constant $C_{s, \epsilon, \Lambda, \psi}$ depending only on $n, m, s, \epsilon, \Lambda, |D\psi|_{|\Omega}$ and $|D^2 \psi|_{|\Omega}$ (but independent of $T$) such that

\begin{equation}
|D f^{\alpha}|_{\Omega \times [0, T]} \leq \frac{1}{m} + C_{s, \epsilon, \Lambda, \psi} |D \psi^\alpha|_{\Omega} \quad \text{for every } \alpha = 1, \cdots, m.
\end{equation}

**Proof.** For a point $x = (x, t) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, we set $|x| = \max\{|x|, |t|^{1/2}\}$ and the cylinder

$$Q_R(x) = \{y = (y, \tau) \in \mathbb{R}^{n+1} | |x - y| < R, \tau < t\}.$$
From Lemma 3.1 and Lemma 3.2 in [5], there is a general constant $C_{e,L}$ depending only on $n, m, \epsilon, \Lambda$ such that for any $Q_{\tau}(x_0) \subset \Omega \times (0, T)$ with $x_0 = (x_0, t_0)$

$$
(4.7) \quad \sup_{Q_{\tau}(x_0)} (r|D^2 f| + r^2|\partial_t D f|) \leq C_{e,L},
$$

and

$$
(4.8) \quad \sup_{Q_{\tau/2}(x_0)} |D f - \xi| \leq C_{e,L} \left( \frac{1}{2} + \sup_{x \in \Omega_{\tau}} \frac{1}{2} \sup_{Q_{\tau/2}^r} \right) \sup_{Q_{\tau/2}^r} \n.
$$

for any $\xi \in \mathbb{R}^n \times \mathbb{R}^m$ and $\tau \in \mathbb{R}^m$. Let $s$ be a constant in $(0, 1/\kappa]$ with $\kappa$ defined in [3.26]. By translation, we assume $0 \in \Omega_s$. For any $R > 0$, denote

$$
Q_R^* = \{ y = (y, \tau) \in \mathbb{R}^{n+1} \mid |y| < R, \ 0 \leq \tau < R^2 \}.
$$

For the fixed $\alpha \in \{1, \cdots, m\}$, we define $w(x) = f^\alpha(x) - D\psi^\alpha \cdot x - \psi^\alpha(0)$ for all $x = (x, t) \in \Omega_s \times (0, T)$. From Lemma 12.6 in [20], for all $0 < r < R$ and $Q_R^* \subset \Omega \times (0, T)$

$$
(4.9) \quad \sup_{Q_r^*} \left( |D^2 \psi| |D f| + r \sup_{Q_r^*} \right) \sup_{Q_r^*} \n.
$$

Denote $y_0 = (0, t) \in \Omega_s \times \mathbb{R}$. Combining (4.8), we have

$$
(4.10) \quad \sup_{Q_{\tau/2}(y_0)} |D f - D\psi(0)| \leq C_{e,L} \left( \frac{1}{2} + \sup_{Q_{\tau/2}^r} \right) \sup_{Q_{\tau/2}^r} \n.
$$

Hence, there is a general constant $C_{s,\epsilon,\Lambda,\psi} \geq 1$ depending only on $n, m, s, \epsilon, \Lambda, |D\psi|_{\Omega}$ and $|D^2 \psi|_{\Omega}$ such that for any $x \in \Omega_{s/2}$ and $y = (y, t) \in \Omega_{s/2} \times (0, T)$ we have

$$
(4.11) \quad |D f(y, \tau) - D f(x, 0)| \leq C_{s,\epsilon,\Lambda,\psi} \max\{|x - y|, \sqrt{\tau}\}.
$$

Let $\delta$ be a positive constant satisfying $C_{s,\epsilon,\Lambda,\psi} \sqrt{\delta} = \frac{1}{m}$. If $T \leq \delta$, then (4.11) implies (4.10).

Now we assume $T > \delta$. Let $x = (x, t), y = (y, \tau) \in \Omega_{s/2} \times (0, T)$, and we denote $y_x = (y, t) \in \Omega_{s/2} \times (0, T)$. Note $l = 1$, and the definition of $\kappa$ in [5.20]. For $|x - y| \leq \min\{t, \tau\}$, from (4.7) we have

$$
(4.12) \quad |D f(x) - D f(y)| \leq |D f(x) - D f(y_x)| + |D f(y) - D f(y_x)|
$$

$$
\leq \frac{C_{s,\epsilon,\Lambda,\psi}}{\min\{\sqrt{t}, \sqrt{\tau}\}} |x - y| + \frac{C_{s,\epsilon,\Lambda,\psi}}{\min\{t, \tau\}} |t - \tau| \leq C_{s,\epsilon,\Lambda,\psi} |x - y|^{1/2}.
$$

For $|x - y| > \min\{t, \tau\}$, from (4.11) we have

$$
(4.13) \quad |D f(x) - D f(y)| \leq |D f(x) - D f(x, 0)| + |D f(y) - D f(x, 0)|
$$

$$
\leq C_{s,\epsilon,\Lambda,\psi} \left( \sqrt{t} + \min\{|x - y|, \sqrt{\tau}\} \right) \leq C_{s,\epsilon,\Lambda,\psi} |x - y|^{\frac{1}{2}}.
$$

Let $\eta$ be a Lipschitz function with support in $\Omega_{s/4}$ such that $\eta = 1$ on $\Omega_{s/2}$, $|D \eta| \leq \frac{c}{s}$ and $|D^2 \eta| \leq \frac{c}{s^2}$ for some absolute constant $c \geq 1$. Let $M_t$ be the graph of $f(\cdot, t)$ in $\mathbb{R}^{n+m}$. We will see $\eta$ and $f^\alpha(\cdot, t)$ as the functions on $M_t$ by identifying $\eta(x, f(x, t)) = \eta(x)$ and $f^\alpha(x, f(x, t)) = f^\alpha(x, t)$. Since $t \in (0, t) \mapsto M_t$ is a mean curvature flow, then

$$
(4.14) \quad \frac{df^\alpha}{dt} - \Delta_{M_t} f^\alpha = 0,
$$
where $\Delta_{M_t}$ is the Laplacian of $M_t$. For simplicity, let $\nabla$, $\bar{\nabla}$ denote Levi-Civita connections of $M_t$ and $\mathbb{R}^{n+m}$, respectively. Let $e_1, \ldots, e_n$ be a local orthonormal tangent frame of $M_t$ at any considered point. Then from the definition of $\eta$ and $\{1.14\}$,

$$
\left(\frac{d}{dt} - \Delta_{M_t}\right) \eta^2 = 2\eta \bar{\nabla} \eta \cdot H_{M_t} - 2|\nabla \eta|^2 - 2\eta \Delta_{M_t} \eta
$$

(4.15)

$$
= -2|\nabla \eta|^2 - 2\eta \sum_i \bar{\nabla}^2 \eta(e_i, e_i) \leq \frac{2c^2(n+1)}{s^2}.
$$

So we have

$$
\left(\frac{d}{dt} - \Delta_{M_t}\right) (\eta^2) \leq \frac{2c^2(n+1)}{s^2} (f^\alpha)^2 - 2\eta^2|\nabla f^\alpha|^2 + 16(f^\alpha)^2|\nabla \eta|^2 \leq \frac{2c^2(n+9)}{s^2} (f^\alpha)^2 - \eta^2|\nabla f^\alpha|^2.
$$

(4.16)

Let $\Phi(X, t) = (-4\pi t)^{-\frac{n}{2}} e^{|X|^2/t}$ for all $X \in \mathbb{R}^{n+m}$, $t < 0$, and $\Phi_{X_0,t_0}(X, t) = \Phi(X - X_0, t - t_0)$ for all $t_0 > 0$, $X_0 \in \mathbb{R}^{n+m}$. Combining Huisken’s monotonicity formula [12] (see also (7) in [5], or (1.2) in [3]) and (4.16), we have

$$
\frac{d}{dt} \int_{M_t} (f^\alpha)^2 \eta^2 \Phi_{X_0,t_0} \leq \int_{M_t} \left(\frac{d}{dt} - \Delta_{M_t}\right) (\eta^2) \Phi_{X_0,t_0}
$$

(4.17)

$$
\leq \int_{M_t} \left(\frac{2c^2(n+9)}{s^2} (f^\alpha)^2 - \eta^2|\nabla f^\alpha|^2\right) \Phi_{X_0,t_0}.
$$

Integrating the above inequality on $[t_1, t_2] \subset [0, \min\{t_0, T\}]$ implies

$$
\int_{t_1}^{t_2} \int_{M_t} \eta^2 |\nabla f^\alpha|^2 \Phi_{X_0,t_0} \leq \int_{M_t} (f^\alpha)^2 \eta^2 \Phi_{X_0,t_0} \bigg|_{t_1}^{t_2} + \frac{2c^2(n+9)}{s^2} \int_{t_1}^{t_2} \int_{M_t} (f^\alpha)^2 \Phi_{X_0,t_0}.
$$

(4.18)

Note that $\sup_{\Omega \times [0, T]} |f^\alpha| \leq \sup_{\Omega} |\psi^\alpha|$, and $\sup_{\Omega \times [0, T]} |Df| \leq \Lambda$. Then by choosing suitable $X_0, t_0$, for any $0 < s \leq 1/\kappa$ there holds

$$
\int_{t_1}^{t_2} \int_{\Omega_{s/2}} |Df^\alpha|^2 \leq C_{s, \epsilon, \Lambda} (t_2 - t_1 + 1)|\psi^\alpha|_{\Omega}^2
$$

(4.19)

for any $\alpha = 1, \ldots, m$,

where $C_{s, \epsilon, \Lambda} > 0$ is a general constant depending only on $n, m, s, \epsilon, \Lambda$. Let $\omega_n$ be the volume of the unit ball in $\mathbb{R}^n$. For any $(x, t) \in \Omega_s \times [\delta, T]$ and $0 < s_0 < \min\{s/2, \sqrt{\delta}\}$, from (4.12), (4.13), (4.19) we get

$$
\omega_n s_0^{n+2} |f^\alpha_1(x, t)| \leq \int_{t-s_0^2}^{t} \int_{B_{s_0}(x)} |f^\alpha_1(y, \tau) - f^\alpha_1(x, t)| dy d\tau + \int_{t-s_0^2}^{t} \int_{B_{s_0}(x)} |f^\alpha_1(y, \tau)| dy
$$

(4.20)

$$
\leq \omega_n s_0^{n+2} C_{s, \epsilon, \Lambda, \psi} s_0^{1/2} + \left(\omega_n s_0^{n+2} \int_{t-s_0^2}^{t} \int_{B_{s_0}(x)} |f^\alpha_1|^2\right)^{1/2}
$$

$$
\leq \omega_n s_0^{n+2} C_{s, \epsilon, \Lambda, \psi} + \sqrt{C_{s, \epsilon, \Lambda} \omega_n s_0^{n+2} (s_0^2 + 1)|\psi^\alpha|_{\Omega}}.
$$
Let $\sqrt{s_s} = \frac{1}{mC_{s,e,\lambda}}$, we have
\begin{equation}
|f_\alpha^\alpha(x, t)| \leq \frac{1}{m} + C_{s,e,\lambda}s_s^{-\frac{2+\epsilon}{2}} |\psi^\alpha|_{\Omega}.
\end{equation}
For any $(x, t) \in \Omega_s \times [0, \delta)$, using (4.11) we can also get (4.6). This completes the proof. □

With the mean curvature flow, we can now prove Theorem 1.1.

\textbf{Proof.} By scaling, we may assume that the diameter $l = 1$. According to Theorem 8.2 in [20] (see also Lemma 5.1 in [5] for instance), there is a constant $T > 0$, $\gamma \in (0, 1)$, a solution $(f_1, \ldots, f_m) \in C^\infty(\Omega \times (0, T)) \cap C^{1,\gamma}([\Omega \times [0, T])$ to the mean curvature flow
\begin{equation}
\begin{cases}
\frac{\partial f_\alpha}{\partial t} = g^{ij} f_\alpha_{ij} \\
f_\alpha(\cdot, 0) = \psi_\alpha \\
f_\alpha(\cdot, t) = \psi_\alpha
\end{cases}
\end{equation}
for $\alpha = 1, \ldots, m$, such that $|f_\alpha|_{1+\gamma; [0, T]} \leq C_{\Omega, \psi}$ for each $\alpha = 1, \ldots, m$, where $C_{\Omega, \psi}$ is a constant depending only on $m, \Lambda_{\Omega}, \sup_{[0, T]} |D \psi|_{\Omega}, |D^2 \psi|_{\Omega}$ and the curvature of $\partial \Omega$. Here, $| \cdot |_{1+\gamma}$ denotes the (higher order) H"older norm in the parabolic case (see chapter IV in [20] or [5]). Let $\lambda_1, \ldots, \lambda_n$ be the singular values of the matrix $(f_i^\alpha)$ at each point in $\overline{\Omega} \times [0, T]$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Denote
\begin{equation}
\Theta_f = 1 - \frac{\lambda_1^2}{1 + \lambda_1^2} + 1 - \frac{\lambda_2^2}{1 + \lambda_2^2} = \frac{2(1 - \lambda_1^2\lambda_2^2)}{(1 + \lambda_1^2)(1 + \lambda_2^2)}\text{ on } \overline{\Omega} \times [0, T].
\end{equation}
Let $\kappa, \nu$ be the constants in Theorem 3.1 and $\Psi \triangleq \frac{2\nu}{\kappa}$. There is a constant $\epsilon_\varphi > 0$ depending on $n, m, \lambda_1, \sup_{\Omega} |D \varphi|$ and $\sup_{\Omega} |D^2 \varphi|$ such that if (1.2) holds, then $\sup_{\Omega} \Theta_f(\cdot, 0) > \Psi^{-1}$ from Lemma 3.1 and $v_f = \delta_{ij} + \sum_\alpha f_i^\alpha f_j^\alpha < \Psi$ on $\Omega$ from Lemma 10.1 in Appendix II. Let $t_1$ be the maximal time $\leq T$ such that $\Theta_f \geq \Psi^{-1}$ and $v_f \leq \Psi$ on $\overline{\Omega} \times [0, t_1]$. For a suitable constant $\epsilon_\varphi > 0$, we can assume that
\begin{equation}
\sup_{0 \leq t \leq t_1} \left( \sup_{\partial \Omega} \sum_{\alpha=1}^{m-1} |D f_\alpha|^2 \right) \leq \frac{\nu^2}{25(m-1)\kappa^2}
\end{equation}
from Lemma 3.2 and
\begin{equation}
|D f_\alpha| \leq \frac{1}{m-1} \text{ on } \Omega_{1/\kappa} \times [0, t_1]
\end{equation}
for $\alpha = 1, \ldots, m-1$ from Lemma 4.2.

Now we assume that there is a time $t_0 \in (0, t_1]$ such that
\begin{itemize}
  \item $\sup_{x \in \Omega} \Theta_f(x, t_0) = \Psi^{-1}$ or $\sup_{x \in \Omega} v_f^2(x, t_0) = \Psi$;
  \item $\Theta_f(x, t) > \Psi^{-1}$ and $v_f^2(x, t) < \Psi$ for all $(x, t) \in \overline{\Omega} \times [0, t_0)$.
\end{itemize}

Let $E_1, \ldots, E_{n+m}$ be a standard basis of $\mathbb{R}^n \times \mathbb{R}^m$. We can see $v_f(x, t)$ as a function on $\text{graph}_{f(\cdot, t)}$ for each $t \in [0, T]$ defined by
\begin{equation}
v_f^{-1} = \left\langle n_f^1 \bigwedge \cdots \bigwedge n_f^m, E_{n+1} \bigwedge \cdots \bigwedge E_{n+m} \right\rangle,
\end{equation}
where $n_f^1, \ldots, n_f^m$ are the local orthonormal normal vector fields of $\text{graph}_{f(\cdot, t)}$ in $\mathbb{R}^{n+m}$. Let $\Delta_f$ and $A_f$ be the Laplacian and the second fundamental form of $\text{graph}_{f(\cdot, t)}$ for each $t \in [0, T]$, respectively. Note that $\text{graph}_{f(\cdot, t)} = \{(x, f(x, t)) \in \mathbb{R}^n \times \mathbb{R}^m | x \in \Omega\}$ moves
by mean curvature. Then the Gauss map of graph \( f(t) \) satisfies the harmonic heat flow equation, which is a parabolic version of the Ruh-Vilms theorem. (2.8) in \([14]\) then yields
\[
(4.26) \quad \left( \frac{\partial}{\partial t} - \Delta_f \right) v_f = -v_f \left( |A_f|^2 + \sum_{i,j} \lambda_i \lambda_j \left( h^i_{ik} h^j_{jk} + h^i_{ik} h^j_{jk} \right) \right),
\]
where \( h^\alpha_{ij} \) are the components of the second fundamental form defined by \( h^\alpha_{ij} = \langle \nabla e_i, n^\alpha_f \rangle \), \( \{e_i\} \) is a tangent basis at the considered point in graph \( f(t) \), and \( \nabla \) is the Levi-Civita connection of \( \mathbb{R}^{n+m} \). From \( \Theta_f \geq \Psi^{-1} \) for each \( t \in [0, t_0] \), one have \( \lambda_1 \lambda_2 \leq 1 \), and
\[
\frac{1}{\lambda_1 \lambda_2} \leq \frac{2(1-\lambda_1^2 \lambda_2^2)}{4 \lambda_1^2 \lambda_2^2},
\]
which implies \( \lambda_1 \lambda_2 \leq \sqrt{\frac{\Psi}{\Psi+2}} \). Combining \((4.26)\) and \( \lambda_i \lambda_j < \frac{\Psi}{\Psi+2} \) for all \( i \neq j \), we obtain
\[
(4.27) \quad \left( \frac{\partial}{\partial t} - \Delta_f \right) v_f \leq -\frac{2v_f}{\Psi+2} |A_f|^2 \quad \text{for each } t \in [0, t_0].
\]
From Theorem \(3.4\) and \((4.25)\) one has
\[
(4.28) \quad \sup_{\partial \Omega \times [0, t_0]} \sup_{\partial \Omega \times [0, t_0]} |Df|^m \leq \frac{\kappa}{\nu} + |D\varphi| \Omega.
\]
Since
\[
(4.29) \quad \frac{\kappa}{\nu} \geq 2 \left(1 + |D\varphi| \Omega \right) e^{16n|D\varphi| \Omega} \geq 2 \left(1 + |D\varphi| \Omega \right) (1 + 16n|D\varphi| \Omega) \geq 66 |D\varphi| \Omega + 2,
\]
then \( \frac{\kappa}{\nu} + |D\varphi| \Omega \leq \frac{67 \kappa}{66 \nu} \). Combining Lemma \(4.3\) and \((4.24)\), we obtain \( \lambda_1 \lambda_2 \leq \frac{1}{2} \) on \( \partial \Omega \times [0, t_0] \) and then
\[
\sup_{\partial \Omega \times [0, t_0]} \Theta_f \geq \sup_{\partial \Omega \times [0, t_0]} 2 \frac{1 - \lambda_1^2 \lambda_2^2}{v_f^2} > \Psi^{-1}.
\]
From Lemma \(5.3\) in \([31]\), \( \Theta_f \) satisfies the maximum principle as \( \lambda_i \lambda_j < 1 \) on \( \Omega \times [0, t_0] \) for all \( i \neq j \), which implies \( \sup_{\Omega} \Theta_f (\cdot, t_0) > \Psi^{-1} \). Combining \((4.28)\), \((4.29)\) and Lemma \(10.1\) one has
\[
(4.30) \quad v_f^2 \leq 1 + \left( \sum_{\alpha=1}^m |Df^\alpha| \right)^2 \leq 1 + \left( |Df|^m + \sqrt{\frac{m-1}{m}} \sum_{\alpha=1}^m |Df^\alpha|^2 \right)^2
\]
\[
\leq 1 + \left( \frac{\kappa}{\nu} + |D\varphi| \Omega + \frac{\nu}{5 \kappa} \right)^2 \leq 1 + \left( \frac{\kappa}{\nu} + \frac{\kappa}{20 \nu} \right)^2 < \Psi
\]
on \( \partial \Omega \times [0, t_0] \). With the maximum principle for \((4.27)\), we have \( \sup_{\Omega \times [0, t_0]} v_f^2 < \Psi \). Therefore, such \( t_0 \) does not exist provided \((1.2)\) holds for such constant \( \epsilon \varphi > 0 \).

Note that \( |f^\alpha|_{1+\gamma' \Omega \times [0, T]} \leq C_{\Omega, \psi} \) for each \( \alpha = 1, \ldots, m \). By Theorem 8.2 in \([20]\) (or Lemma 5.1 in \([1]\), for each \( t \in (0, T) \) the flow \((1.2)\) from the time \( t \) with boundary data \( f(\cdot, t) \) has a short-time existence on \( [t, t+t'] \) for some \( t' > 0 \) depending only on \( m, n, \Psi, |D\psi| \Omega, |D^2 \psi| \Omega \) and the curvature of \( \partial \Omega \). Hence, \( |f^\alpha|_{1+\gamma' \Omega \times [0, t}] < \infty \) for each \( \alpha = 1, \ldots, m \). From Theorem 3.3 in \([31]\), \( |f^\alpha|_{1+\gamma' \Omega \times [0, T]} \leq C_{\Omega, \psi} \) for each \( \alpha = 1, \ldots, m \), where \( \gamma' \in (0, \gamma) \) and \( C_{\Omega, \psi} \) are constants depending only on \( m, n, \Psi, |D\psi| \Omega, |D^2 \psi| \Omega \) and the curvature of \( \partial \Omega \). The constant \( \Psi \) depends only on \( m, n, |D\psi| \Omega, |D^2 \psi| \Omega \) and \( \lambda_0 \), but is independent of the time. Hence, the flow has long-time existence and does not have any finite time singularity. Let \( H_M \) denote the mean curvature of \( M_t \equiv \text{graph} f(t) \). From
\[ \frac{\partial}{\partial t} v_f = -|H_{M_t}|^2 v_f \quad \text{and} \quad \partial M_t = \partial \Omega \times \{0\}, \]

one has

\[ (4.31) \quad \int_0^\infty \left( \int_{M_t} |H_{M_t}|^2 \right) dt < \infty. \]

From Lemma 3.1 in [5], we obtain the interior curvature estimates for \( M_t \) with any \( t > 0 \). Then one can get the estimates of the higher order derivatives of \( f \), which consequently are uniformly bounded in each compact set of \( \Omega \). Combining this with the above uniform boundary estimates of \( M_t \), there is a sequence \( t_i \to \infty \) such that \( M_{t_i} \) converges to a smooth minimal graph over \( \Omega \) with the graphic function \( u = (u^1, \cdots, u^m) \in C^{1, \gamma}(\bar{\Omega}) \) and \( u = \psi \) on \( \partial \Omega \). It is then standard to obtain \( W^{2,p} \)-estimates (see [32] for instance). From these estimates and the Sobolev imbedding theorem, \( u \in C^{1, \gamma}(\bar{\Omega}) \) for any \( \gamma_s \in (0, 1) \). This completes the proof.

5. THE DIRICHLET PROBLEM ON CONVEX DOMAINS

We now turn our attention to the smaller class of domains that are convex, and not only mean convex. This will allow us to obtain better bounds. In fact, as explained in the Introduction, the example of [18] and the Bernstein result of [14] suggest an explicit quantitative result in this direction stated in Theorem 1.2.

Proof. According to Theorem 8.2 in [20] (see also Lemma 5.1 in [5] for a version in the present context), there are a constant \( T > 0, \gamma \in (0,1) \) and a solution \((f^1, \cdots, f^m) \in C^\infty(\Omega \times (0, T)) \cap C^{1, \gamma}(\bar{\Omega} \times [0, T])\) to the mean curvature flow

\[ (5.1) \quad \begin{cases} 
\frac{\partial f^\alpha}{\partial t} = g_{ij} f_{ij}^\alpha & \text{in } \Omega \times (0, T) \\
 f^\alpha(\cdot, 0) = \psi^\alpha & \text{on } \Omega \times \{0\} \\
 f^\alpha(\cdot, t) = \psi^\alpha & \text{on } \partial \Omega \times [0, T]
\end{cases} \]

such that \(|f^\alpha|_{1+\gamma, \bar{\Omega} \times [0, T]} \leq C_{\Omega, \psi} \) for each \( \alpha = 1, \cdots, m \), where \( C_{\Omega, \psi} \) is a constant depending only on \( m, n, |D\psi|_\Omega, |D^2\psi|_\Omega, \operatorname{diam}(\Omega) \) and the curvature of \( \partial \Omega \).

Recall \( v_f^2(x,t) = \det \left( \delta_{ij} + \sum_\alpha f_i^\alpha f_j^\alpha \right) \) for each \( (x,t) \in \Omega \times [0, T] \). From Lemma 10.2 in Appendix II, \( \sup_{x \in \Omega} v_f^2(x,0) < b_0 \). We assume that there is a time \( t_0 \in (0, T) \) such that \( \sup_{x \in \Omega} v_f^2(x,t) < b_0 \) for each \( t \in [0, t_0) \) and \( \sup_{x \in \Omega} v_f^2(x,t_0) = b_0 \). Note that the right hand side of (4.25) is the same as the right hand side of (3.7) in [14], and so, the estimates (3.13), (3.16), Lemma 3.1 and Lemma 3.2 in [14] apply (those estimates are derived algebraically without using the minimal surface system). Therefore, for \( t \in [0, T) \) one has

\[ (5.2) \quad \left( \frac{\partial}{\partial t} - \Delta_f \right) v_f \leq 0, \]

where \( \Delta_f \) is the Laplacian of graph \( f \). Since \( \Omega \) is convex, we can allow \( r_\alpha \to \infty \) in Lemma 3.1. From Lemma 3.1 with \( |D\psi|_\Omega, |D^2\psi|_\Omega, \operatorname{diam}(\Omega) \) and \( v_f^2 \leq b_0 \) on \( \Omega \times [0, t_0] \), we obtain

\[ (5.3) \quad \sum_{\alpha=1}^m \sup_{\partial \Omega \times [0, t_0]} |Df^\alpha| < \sqrt{b_0} - 1. \]

By Lemma 10.2 in Appendix II, one has \( \sup_{\partial \Omega \times [0, t_0]} v_f^2 < b_0 \). From the parabolic maximum principle, it follows that \( \sup_{\Omega \times [0, t_0]} v_f^2 < b_0 \), which is a contradiction to \( \sup_{x \in \Omega} v_f^2(x,t_0) = b_0 \).
EXISTENCE AND NON-EXISTENCE OF MINIMAL GRAPHS

\[ b_0. \] Hence \( t_0 \) does not exist. The remaining argument is similar to the last part of the proof of Theorem 1.1. Hence there is a sequence \( t_i \to \infty \) such that \( f(x, t_i) \) converges to a smooth minimal graph with the graphic function \( u = (u^1, \cdots, u^m) \) and boundary data \( \psi \) and \( u \in C^{1, \gamma}(\Omega) \) for any \( \gamma \in (0, 1) \). It is clear that \( \sup_{\Omega} \det \left( \delta_{ij} + u_i^a u_j^a \right) \leq b_0 \). Combining (1.4) and Lemma 3.2, it follows that

\[
\sum_{a=1}^{m} \sup_{\partial \Omega} |Du^a| < \sqrt{b_0 - 1}. \tag{5.4}
\]

Let \( \Delta_u \) denote the Laplacian of \( \text{graph}_u \). Using Lemma 10.2 and the maximum principle for \( \Delta_u \), we obtain \( \sup_{\Omega} \det \left( \delta_{ij} + u_i^a u_j^a \right) < b_0 \) and complete the proof. \( \square \)

For any \( C^2 \) map \( \phi : S^{n+k} \to S^n \subset \mathbb{R}^{n+1}, n \geq 4, k > 0 \) which is not homotopic to zero as a map to \( S^n \), Lawson-Osserman \[15\] constructed boundary data on the unit ball in \( \mathbb{R}^n \), for which the corresponding Dirichlet problem has no solution. Their example arises from the map

\[ u : \mathbb{R}^4 \to \mathbb{R}^3 \]

\[ (z_1, z_2) \in C^2 \mapsto (|z_1|^2 - |z_2|^2, 2z_1 \bar{z}_2) \in \mathbb{R}^3 \]

which is an extension of the classical Hopf map \( S^3 \to S^2 \), and they used it as a counterexample to the Bernstein problem in higher codimension. Since local regularity results, underlying for instance solutions to Dirichlet problems, and global Bernstein theorems are related via scaling arguments, they could then also use this example to infer non-solvability of certain Dirichlet problems. For this particular \( u \), its slope satisfies \( \sup_{\mathbb{R}^4} \det \left( \delta_{ij} + u_i^a u_j^a \right)^{1/2} = 9 \), and it is a fundamental open question whether the constant 9 occurring here is sharp for the Bernstein problem. So far, the best value for which a general Bernstein theorem in higher codimensions could be derived \[14\] is 3 instead of 9. It is currently unclear whether one can go substantially beyond 3, because the convex geometry of the Gauss regions in Grassmannians on which all such results depend breaks down for values \( > 3 \). (Actually, as we can see by a contradiction argument, there is a constant \( t > 0 \) (depending on the dimension) so that the Bernstein theorem still holds for the value \( 3 + t \). In fact, if not, there is a sequence \( t_i \) converging to 0, and a sequence of nontrivial \( n \)-minimal graphs \( M_i \) with slope \( v_i < 3 + t_i \). We consider a tangent cone \( C_i \) of \( M_i \) at infinity. The slope of \( C_i \) then is also less than \( 3 + t_i \). Choosing a subsequence, we may assume that \( C_i \) converges to a stationary varifold \( C \) in the varifold sense. It is not hard to see that \( C \) has multiplicity one. Moreover, \( C \) is a cone, and its slope is \( \leq 3 \). By \[14\], \( C \) is flat. Since, however, the \( C_i \) are not flat with a singular point 0 at least, we get a contradiction by Allard’s regularity theorem.) – Conversely, one may also investigate the rigidity of the Lawson-Osserman cone, see \[15\]. Since the Bernstein problem is structurally more restricted than the Dirichlet or the local regularity problem, it makes sense to look for the optimal constant there. Because of the relation between local regularity and global Bernstein alluded to above, we should expect that the optimal value of the constant \( \sqrt{b_0} \) in our Theorem 1.2 and the optimal value for the Bernstein problem coincide. Thus, our theorem reaches the best known value and is quantitatively explicit.
6. The Dirichlet Problem by Perturbation

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \in C^2$. Let $M$ be a smooth minimal graph over $\Omega$ in $\mathbb{R}^{n+1}$ with the graph function $u = (u^1, \ldots, u^m) \in C^\infty(\Omega)$. For any function $\phi = (\phi^1, \ldots, \phi^m) \in C^2(\Omega)$, and a constant $\delta \in (0,1]$, let $M_\delta$ denote a graph over $\Omega$ with the graphic function $(u^1_s, \ldots, u^m_s) = (u^1 + s\phi^1, \ldots, u^m + s\phi^m)$ for each $|s| \leq \delta$. Let $g_s^{ij}dx_idx_j$ be the metric of $M_\delta$ defined by

$$g_s^{ij} = \delta_{ij} + \sum_\alpha \partial_x^\alpha u_s^\alpha \partial_{x_j} u_s^\alpha.$$  

Let $(g_s^{ij})$ be the inverse matrix of $(g_s^{ij})$. Let $v_{us} = \sqrt{\det g_s^{ij}}$, and $\Delta_{M_\delta}$ be the Laplacian of $M_\delta = \text{graph}_{u_s}$. We omit the index $s$ for $s = 0$, and write $\partial_i$ instead of $\partial_{x_i}$ for convenience. Let $L$ be the linear differential operator of the second order defined by $L\phi = (L\phi^1, \ldots, L\phi^m)$ with

$$L\phi^\alpha = \frac{\partial}{\partial s} \bigg|_{s=0} \Delta_{M_\delta} u_s^\alpha.$$  

Note that $\Delta_{M_\delta} u^\alpha = 0$. A direct computation implies

$$L\phi^\alpha = \frac{1}{v_{us}} \frac{\partial}{\partial s} \bigg|_{s=0} \left( g_s^{ij} v_{us} \partial_j u_s^\alpha \right) = \frac{1}{v_{us}} \left( g_s^{ij} \phi_s^\alpha \partial_j u_s^\alpha + u_s^j \left( -2g_s^{ik} g_s^{lj} u_s^\beta \phi_s^\beta + g_s^{ij} g_s^{kj} u_s^\beta \phi_s^\beta \right) \phi_s^\alpha \right).$$  

**Remark 6.1.** If we further assume $u^\alpha, \phi^\alpha \in C^1(\bar{\Omega})$, $\phi^\alpha = 0$ on $\partial \Omega$ for all $\alpha = 1, \ldots, m$, then $-\int_{\Omega} \langle \phi, L\phi \rangle v_u$ has the following geometric meaning.

$$-\int_{\Omega} \langle \phi, L\phi \rangle v_u = \int_{\Omega} \frac{\partial}{\partial s} \bigg|_{s=0} \left( g_s^{ij} \phi_s^\alpha \partial_j u_s^\alpha \right) v_{us} = \frac{\partial}{\partial s} \bigg|_{s=0} \int_{\Omega} v_{us}.$$  

In particular, for the codimension $m = 1$,

$$-\int_{M} \langle \phi, L\phi \rangle = \int_{M} \left( |\nabla_M \phi|^2 - |\langle \nabla_M u, \nabla_M \phi \rangle|^2 \right),$$  

where $\nabla_M$ is the Levi-Civita connection of $M = \text{graph}_{u_s}$.

It’s not hard to see that there is a constant $c_n > 0$ depending only on $n$ such that

$$\left| \frac{\partial^2}{\partial s^2} \Delta_{M_\delta} u_s^\alpha \right| \leq c_n (1 + |Du|^2 + |D\phi|^2)^3 \left( |D^2 u| + |D^2 \phi| \right).$$  

Moreover, suppose $\hat{\phi} = (\hat{\phi}^1, \ldots, \hat{\phi}^m)$, $\hat{u}_s = (\hat{u}_s^1, \ldots, \hat{u}_s^m) = (u^1 + s\hat{\phi}^1, \ldots, u^m + s\hat{\phi}^m)$ for each $0 \leq s \leq \delta$, and $\hat{u}_0 = (u^1, \ldots, u^m)$. Denote $\hat{M}_s = \text{graph}_{\hat{u}_s}$. Then with a suitable constant $c_n > 0$

$$\left| \frac{\partial^2}{\partial s^2} \left( \Delta_{M_\delta} u_s^\alpha - \Delta_{\hat{M}_s} \hat{u}_s^\alpha \right) \right| \leq c_n (1 + |Du|^2 + |D\phi|^2 + |D\hat{\phi}|^3 |D^2 (\phi - \hat{\phi})| + c_n |D(\phi - \hat{\phi})| (1 + |Du|^2 + |D\phi|^2 + |D\hat{\phi}|^2)^{5/2} \left( |D^2 u| + |D^2 \phi| + |D^2 \hat{\phi}| \right).$$  


Let \( Q_{\delta,u,\phi} = (Q_{\delta,u,\phi}^1, \ldots, Q_{\delta,u,\phi}^m) \) with

\[
Q_{\delta,u,\phi}^\alpha = -\frac{1}{\delta} \int_0^\delta \left( \int_0^\tau \frac{\partial^2}{\partial s^2} \Delta_M u_s^\alpha \, ds \right) \, d\tau.
\]

Then for each \( \alpha = 1, \ldots, m \)

\[
|Q_{\delta,u,\phi}^\alpha| \leq c_n \delta (1 + |Du|^2 + |D\phi|^2)^3 (|D^2 u| + |D^2 \phi|).
\]

From (6.7), we have

\[
|Q_{\delta,u,\phi}^\alpha - Q_{\delta,u,\phi}^\alpha| \leq c_n \delta (1 + |Du|^2 + |D\phi|^2)^3 |D^2 (\phi - \phi)| + c_n \delta |D(\phi - \phi) (1 + |Du|^2 + |D\phi|^2 + |D\phi|^2)|^{5/2} (|D^2 u| + |D^2 \phi| + |D^2 \phi|).
\]

Suppose that \( M_\delta \) is a minimal graph, then

\[
0 = \Delta_M u_\delta - \Delta_M u_0 = \int_0^\delta \frac{\partial}{\partial s} \Delta_M u_s^\alpha = \int_0^\delta \left( \int_0^\tau \frac{\partial^2}{\partial s^2} \Delta_M u_s^\alpha \, ds + L\phi^\alpha \right) \, d\tau
\]

which implies

\[
L\phi^\alpha = Q_{\delta,u,\phi}^\alpha.
\]

Now we assume that \( \Omega \) is mean convex. Let \( \kappa_\Omega \) denote the largest absolute principle curvature of \( \partial \Omega \), i.e.,

\[
\kappa_\Omega = \sup_{\partial \Omega} \sup_{r=1,\ldots,n-1} |\lambda_i(D^2 d)|,
\]

where \( d = d(\cdot, \partial \Omega) \) on \( \overline{\Omega} \). For any \( \varphi \in C^2(\overline{\Omega}) \), from Jenkins-Serrin [13] (see also Theorem 16.8 and Theorem 14.9 in [9]), there is a smooth solution \( w \) to the minimal surface equation (1.1) with \( w = \varphi \) on \( \partial \Omega \) such that \( \sup_\Omega |Dw| \leq C_{\Omega,\varphi} \) for a constant \( C_{\Omega,\varphi} > 0 \) depending only on \( n, \text{diam} \Omega, \sup_\Omega (|D\varphi| + |D^2 \varphi|) \). From Theorem 13.7 in [9] and \( W^{2,p} \)-estimates (see [32] for instance), for any \( p > n \) there is a constant \( C_{p,\Omega,\varphi} \) depending only on \( n, p, \text{diam} \Omega, \kappa_\Omega, \sup_\Omega (|D\varphi| + |D^2 \varphi|) \) such that

\[
||w||_{W^{2,p}(\Omega)} \triangleq \int_\Omega \left( |w|^p + |Dw|^p + |D^2 w|^p \right) \leq C_{p,\Omega,\varphi}.
\]

By the Sobolev embedding theorem, we can assume the Hölder norm \( |Dw|_{\gamma,\Omega} \leq C_{p,\Omega,\varphi} \) with \( \gamma = 1 - n/p \).

Suppose \((u^1, \ldots, u^m) = (0, \ldots, 0, w)\). In this case \( g_{ij} = g^{0}_{ij} = \delta_{ij} + w_i w_j \) from (6.1). Moreover, \( g^{ij} = \delta_{ij} - w_i w_j v^-_w \), and \( g^{ij} w_j = w_i v^-_w \). From (6.3), one has

\[
L\phi^\alpha = \frac{1}{v^-_w} \partial_i \left( g^{ij} \delta^\alpha_j v^-_w + w^\alpha_j \left( -2 g^{ij} w_i \phi^m_i + g^{ij} w_i \phi^m_i \right) v^-_w \right).
\]

Let \( L_* \) be a linear differential operator of the second order defined by

\[
L_* \xi = \frac{1}{v^-_w} \partial_i \left( v^-_w \left( \delta_{ij} - w_i w_j (v^-_w + v^-_w) \right) \xi \right)
\]

for any \( \xi \in C^2(\Omega) \). Then from (6.14) we have

\[
L\phi = (L\phi^1, \ldots, L\phi^m) = (\Delta_M \phi^1, \ldots, \Delta_M \phi^m, L_* \phi^m).
\]

Now we shall use the Schauder fixed point theorem to show the existence of minimal graphs by perturbation of a given minimal graph of codimension one.
Theorem 6.2. For any mean convex bounded $C^2$ domain $\Omega \subset \mathbb{R}^n$ with diameter $l = 1$, $m \geq 2$, $p > n$ and any $\varphi \in C^2(\Omega)$, there is a constant $\delta_{m,p,\Omega,\varphi} > 0$ depending on $n, m, p, \kappa_\Omega, \sup_{\Omega} |D\varphi|$ and $\sup_{\Omega} |D^2\varphi|$ such that if the functions $\psi^1, \ldots, \psi^{m-1} \in C^{1,\gamma}(\Omega)$ with $\gamma = 1 - n/p$ satisfy $\sum_{\alpha=1}^{m-1} ||\psi^\alpha||_{W^{2,p}(\Omega)} \leq \delta_{m,p,\Omega,\varphi}$, then there is a solution $u = (u^1, \ldots, u^m) \in C^\infty(\Omega) \cap C^{1,\gamma}(\Omega)$ to the minimal surface system (6.13) with $u = (\psi^1, \ldots, \psi^{m-1}, \varphi)$ on $\partial\Omega$.

Proof. From the above argument, there is a smooth solution $w$ to the minimal surface equation (6.13) with boundary $\varphi$ such that $\sup_{\Omega} |Du| + |Du|_{\gamma,\Omega} + ||u||_{W^{2,p}(\Omega)} \leq C_{p,\Omega,\varphi}$, where $C_{p,\Omega,\varphi}$ is a constant depending only on $n, p, \text{diam} \Omega, \kappa_\Omega, \sup_{\Omega} (|D\varphi| + |D^2\varphi|)$, $\gamma = 1 - n/p$. Let $M$ be a graph over $\Omega$ in $\mathbb{R}^{n+m}$ with the graphic function $(0, \ldots, 0, w)$. Let $\delta$ be a constant in $(0, 1)$ to be defined later, and $\psi^1, \ldots, \psi^{m-1} \in C^{1,\gamma}(\Omega)$ with $\gamma = 1 - n/p$ such that $\sum_{\alpha=1}^{m-1} ||\psi^\alpha||_{W^{2,p}(\Omega)} = \delta^2$. Put $\hat{\psi} = (\psi^1, \ldots, \psi^m) = (\psi^{1/\delta}, \ldots, \psi^{m-1/\delta}, 0)$. For any $t > 0$, let

$$\mathcal{S}_t = \left\{ \phi = (\phi^1, \ldots, \phi^m) \in C^{1,\gamma}(\Omega, \mathbb{R}^m) \middle| \sum_{\alpha} ||\phi^\alpha||_{W^{2,p}(\Omega)} \leq t \right\},$$

which is a compact set in the Banach space $W^{2,p}(\Omega, \mathbb{R}^m)$ for any $t > 0$. Moreover, $\mathcal{S}_t$ is convex by the Minkowski inequality.

For any $\phi = (\phi^1, \ldots, \phi^m) \in \mathcal{S}_1$, let $w_s = (w^1_s, \ldots, w^m_s) = (s\phi^1, \ldots, s\phi^{m-1}, w + s\phi^m)$, and $M_s = \text{graph} w_s$ for $s \in [0, \delta]$. Let $Q_{\delta,w,\psi}^\alpha$ be defined as in (6.15), where $u$ is replaced by $(0, \ldots, 0, w)$, $u^s$ is replaced by $w^s$. For any $\alpha = 1, \ldots, m - 1$, there is a unique solution $\xi^\alpha \in C^\infty(\Omega) \cap C^{1,\gamma}(\Omega)$ to

$$(6.17) \begin{cases} \Delta M\xi^\alpha = Q_{\delta,w,\psi}^\alpha & \text{in } \Omega \\ \xi^\alpha = \hat{\psi} & \text{on } \partial\Omega, \end{cases}$$

such that

$$(6.18) ||\xi^\alpha||_{W^{2,p}(\Omega)} \leq C_{p,\Omega,\varphi} \left( ||Q_{\delta,w,\psi}^\alpha||_{L^p(\Omega)} + ||\hat{\psi}||_{W^{2,p}(\Omega)} \right).$$

where $C_{p,\Omega,\varphi}$ is a general constant depending only on $n, p, \text{diam} \Omega, \kappa_\Omega, \sup_{\Omega} (|D\varphi| + |D^2\varphi|)$. From (6.18) and the definition of $\hat{\psi}$, for a suitable constant $C_{p,\Omega,\varphi}$ we get

$$(6.19) ||\xi^\alpha||_{W^{2,p}(\Omega)} \leq C_{p,\Omega,\varphi}.\delta.$$ 

Let $L_s$ be the operator defined in (6.15), then $L_s$ is uniformly elliptic since $1 - |Dw|^2(v_w^{-2} + v_w^{-4}) = v_w^{-4}$. Then for a suitable constant $C_{p,\Omega,\varphi}$, there is a unique solution $\xi^m \in C^\infty(\Omega) \cap C^{1,\gamma}(\Omega)$ to

$$(6.20) \begin{cases} L_s\xi^m = Q_{\delta,w,\psi}^m & \text{in } \Omega \\ \xi^m = 0 & \text{on } \partial\Omega, \end{cases}$$

such that

$$(6.21) ||\xi^m||_{W^{2,p}(\Omega)} \leq C_{p,\Omega,\varphi} ||Q_{\delta,w,\psi}^m||_{L^p(\Omega)} \leq C_{p,\Omega,\varphi}.\delta.$$ 

Let $T_\psi$ be the operator defined by letting $\xi = (\xi^1, \ldots, \xi^m) = T_\psi \phi$ be the unique solution in $W^{2,p}(\Omega)$ of the following linear Dirichlet problem,

$$(6.22) \begin{cases} L\xi^\alpha = Q_{\delta,w,\psi}^\alpha & \text{in } \Omega \\ \xi^\alpha = \hat{\psi}^\alpha & \text{on } \partial\Omega, \end{cases}$$
for each $\alpha = 1, \cdots, m$. Combining (6.19) and (6.21), we have

$$(6.23) \quad \sum_{\alpha=1}^{m} ||\xi^{\alpha}||_{W^{2,p}(\Omega)} \leq mC_{p,\Omega,\varphi}\delta.$$ 

Put $\delta = \frac{1}{mC_{p,\Omega,\varphi}}$, then $\xi \in \mathcal{S}_1$. Combining this with (6.11), $T_{\psi}$ is a continuous mapping from $\mathcal{S}_1$ into $\mathcal{S}_1$. From (6.19) (6.18) (6.21), $T_{\psi}$ is a compact operator, i.e., $T_{\psi}(K)$ is pre-compact for any compact $K$ in $W^{2,p}(\Omega, \mathbb{R}^m)$. Hence by the Schauder fixed point theorem (see also [9]), there is a fixed point $\hat{\phi}_* = (\hat{\phi}_1, \cdots, \hat{\phi}_m) \in \mathcal{S}_1$ for the operator $T_{\psi}$. Namely,

$$\phi_* = T_{\psi}\phi_*.$$ 

Let $u^\alpha = \delta\phi^\alpha_*$ for $\alpha = 1, \cdots, m - 1$ and $u^m = \delta\phi^m_* + w$. From (6.11) (6.12), $(u^1, \cdots, u^m)$ is a smooth solution to the minimal surface system (1.3) with $u^m = \delta\phi^m_* + w = \varphi$ and $u^\alpha = \delta\phi^\alpha_* = \delta\hat{\psi} = \psi^\alpha$ on $\partial\Omega$ for $\alpha = 1, \cdots, m - 1$. \hfill $\Box$

7. Non-existence results for solutions of Dirichlet problems

**Theorem 7.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n (n \geq 3)$ with a smooth mean convex boundary, and suppose that there is a point $q \in \partial\Omega$ such that $\partial\Omega$ is not convex, but has zero mean curvature at $q$. Then for any constant $0 < \epsilon \leq 1$, there exists a vector-valued function $\psi = (\psi^1, \psi^2) \in C^2(\overline{\Omega}, \mathbb{R}^2)$ with $|D\psi^j| \leq \epsilon$ such that the minimal surface system (1.3) has no classical solution with boundary data $(\psi^1, \psi^2)$.

**Proof.** Without loss of generality, we may assume that $q$ is the origin, the unit normal vector $\nu$ at $0$ to $\partial\Omega$ is parallel to the axis $x_n$, and $\langle D_{e_1}e_1, \nu \rangle < 0$ at $0$. Here, $e_1$ is a unit tangent vector field of $\partial\Omega$ in a neighborhood of the origin, such that $e_1$ is parallel to the axis $x_1$ at $0$.

Let $\psi$ be a linear function on $\overline{\Omega}$ such that $D\psi = \epsilon E_1$ for some fixed constant $0 < \epsilon \leq 1$. Here, $\{E_i\}^{n-1}_{i=1}$ is a standard basis of $\mathbb{R}^n$ such that $E_n$ is parallel to the axis $x_n$. Let $\varphi$ be a smooth function on $\overline{\Omega}$ to be defined later. Assume that there is a smooth solution $(u, v)$ of the minimal surface system

$$(7.1) \quad \begin{cases} g^{ij}u_{ij} = g^{ij}v_{ij} = 0 & \text{in } \Omega \\ u = \varphi, \ v = \psi & \text{on } \partial\Omega \end{cases}$$

with $g_{ij} = \delta_{ij} + u_iu_j + v_iv_j$.

Let $p_{ij} = \delta_{ij} + u_iu_j + \psi_i\psi_j$, and $(p^{ij})$ be the inverse matrix of $(p_{ij})$. Note that $D^2\psi = 0$, which means that

$$\sum_{i,j=1}^{n} p^{ij}\psi_{ij} = 0.$$ 

Then by the maximum principle, $v = \psi$ on $\overline{\Omega}$.

Put $P_{ij} = \delta_{ij} + \psi_i\psi_j + w_iw_j$ for some function $w \in C^2(\overline{\Omega})$. Then

$$P_{11} = 1 + \epsilon^2 + w_1^2,$$

and

$$P_{ij} = \delta_{ij} + w_iw_j, \quad \text{for } 1 \leq i, j \leq n, \ i + j \neq 2.$$
Set
\[|w|_{\epsilon,1}^2 = \frac{w_1^2}{1 + \epsilon^2} + \sum_{i=2}^{n} w_i^2.\]
The components of the inverse matrix of \((P_{ij})\) are
\[(7.2)\]
\[P^{11} = \frac{1}{1 + \epsilon^2} - \frac{w_1^2}{(1 + |w|_{\epsilon,1}^2)(1 + \epsilon^2)^2}, \quad P^{i1} = -\frac{w_i w_1}{(1 + |w|_{\epsilon,1}^2)(1 + \epsilon^2)} \quad \text{for } i \geq 2,
\]
and
\[(7.3)\]
\[P^{ij} = \delta_{ij} - \frac{w_i w_j}{1 + |w|_{\epsilon,1}^2}, \quad \text{for } 2 \leq i, j \leq n.
\]
One can check this easily as follows. We see \(P_{ij}\) and \(P^{ij}\) as smooth functions of \(\epsilon^2\). Let \(P_{11}(t) = 1 + t + w_1^2\) and \(P_{ij}(t) = \delta_{ij} + w_i w_j\) for all \(t \geq 0\), then \((P_{11})' = \frac{d}{dt}P_{11} = 1\) and \((P_{ij})' = 0\) for all \(i, j \geq 2\). Let \((P^{ij}) = (P^{ij}(t))\) be the inverse matrix of \((P_{ij}(t))\), then
\[(7.4)\]
\[(P^{ij})' = -P^{ij}P^{ij}.
\]
We solve ODE \((7.4)\) and get \(P^{11}(t) = P^{11}(0)/(1 + P^{11}(0)t)\), \(P^{i1}(t) = P^{i1}(0)/(1 + P^{11}(0)t)\) for all \(i \geq 2\), \(P^{ij}(t) = P^{ij}(0) - P^{i1}(0)P^{j1}(0)/1 + P^{11}(0)t\) for all \(i, j \geq 2\). Taking \(t = \epsilon^2\), we can show \((7.2)\) and \((7.3)\).

Let \(d(x) = d(x, \partial \Omega)\) for \(x \in \overline{\Omega}\) as before. Set \(d_{ij} = \partial x_i \partial x_j\) for \(1 \leq i, j \leq n\). Let \(\{e_i\}_{i=1}^{n-1}\) be a local orthonormal basis in a neighborhood of the origin, such that \(e_1\) is parallel to the axis \(x_1\) at 0. Set \(\Omega_t = \{x \in \Omega \mid d(x, \Omega) > t\}\) for \(t \geq 0\) as before. Let \(e_n\) be the unit normal vector field to \(\partial \Omega\) so that \(e_n\) points into \(\Omega_t\). Since \(d\) is a constant on \(\partial \Omega\), then at the point \(0 \in \partial \Omega\) we get \((D_{e_1}D_{e_1} - (D_{e_1}e_1)^T) d = 0\), and
\[
\partial x_i \partial x_j\ d = d_{11} = (D_{e_1}e_1)^T d = (D_{e_1}e_1)d - (D_{e_1}e_1, e_n)D_{e_n}d = (D_{e_1}e_1, e_n) < 0.
\]
Since \(\Delta d = H_{\partial \Omega}\) on \(\partial \Omega\), from the assumption there is a positive constant \(a > 0\) such that
\[(7.5)\]
\[d_{11} < -a \quad \text{and} \quad \Delta d \geq -\frac{a \epsilon^2}{2(1 + \epsilon^2)}\]
on \(B_a(0) \cap \Omega\).

Let \(\chi\) be a \(C^2\)-function on \((0, a)\) such that \(\chi(2a) = 0\), \(\chi' \leq 0\), \(\chi'(0) = -\infty\). Let \(\chi_{\delta}(t) = \chi(t - \delta)\) for any \(t \in (\delta, a)\). Set \(\Omega_{\delta, a} = \{x \in B_a(0) \cap \Omega \mid \delta < d(x) < a\}\), and
\[w(x) = \chi_{\delta}(d(x)) + \sup_{|y| = a} u(y) \quad \text{for } x \in \Omega_{\delta, a}.
\]
Put \(|d|_{\epsilon,1}^2 = \frac{d_i^2}{1 + \epsilon^2} + \sum_{i=2}^{n} d_i^2\), then \(|w|_{\epsilon,1}^2 = (\chi_{\delta}')^2|d|_{\epsilon,1}^2\). For such \(w\), from \((7.2)\) \((7.3)\) we have
\[(7.6)\]
\[P^{11} = \frac{1}{1 + \epsilon^2} - \frac{(\chi_{\delta}')^2|d|_{\epsilon,1}^2}{(1 + (\chi_{\delta}')^2|d|_{\epsilon,1}^2)(1 + \epsilon^2)^2},
\]
and for \(2 \leq i, j \leq n\)
\[(7.7)\]
\[P^{ij} = \delta_{ij} - \frac{(\chi_{\delta}')^2 d_i d_j}{1 + (\chi_{\delta}')^2|d|_{\epsilon,1}^2}.\]
So one has

\begin{equation}
\sum_{i,j=1}^{n} P^{ij} d_{ij} d_{j} = \frac{d_{1}^{2}}{1 + \epsilon^{2}} - \frac{(\chi_{\delta}')^{2} d_{1}^{2}}{1 + (\chi_{\delta}')^{2} |d_{1}|^{2}_{\epsilon,1}} (1 + \epsilon^{2})^{2} - 2 \sum_{i=2}^{n} \frac{(\chi_{\delta}')^{2} d_{i}^{2} d_{i}^{2}}{1 + (\chi_{\delta}')^{2} |d_{i}|^{2}_{\epsilon,1}} (1 + \epsilon^{2})
+ \sum_{i,j=2}^{n} \left( \delta_{ij} d_{ij} - \frac{(\chi_{\delta}')^{2} d_{i}^{2} d_{j}^{2}}{1 + (\chi_{\delta}')^{2} |d_{i}|^{2}_{\epsilon,1}} \right)
= |d_{1}|^{2}_{\epsilon,1} - \frac{(\chi_{\delta}')^{2} |d_{1}|^{4}_{\epsilon,1}}{1 + (\chi_{\delta}')^{2} |d_{1}|^{2}_{\epsilon,1}} = \frac{|d_{1}|^{2}_{\epsilon,1}}{1 + (\chi_{\delta}')^{2} |d_{1}|^{2}_{\epsilon,1}} \leq \frac{1}{(\chi_{\delta}')^{2}}.
\end{equation}

Note that \( \sum_{i=1}^{n} d_{ij} d_{i} = \frac{1}{2} D_{j} |D d|^{2} = 0 \). Then

\begin{equation}
\sum_{i,j=1}^{n} P^{ij} d_{ij} = \frac{d_{11}}{1 + \epsilon^{2}} - \frac{(\chi_{\delta}')^{2} d_{11} d_{11}^{2}}{1 + (\chi_{\delta}')^{2} |d_{11}|^{2}_{\epsilon,1}} (1 + \epsilon^{2})^{2} - 2 \sum_{i=2}^{n} \frac{(\chi_{\delta}')^{2} d_{i} d_{i}^{2} d_{i}}{1 + (\chi_{\delta}')^{2} |d_{i}|^{2}_{\epsilon,1}} (1 + \epsilon^{2})
+ \sum_{i,j=2}^{n} \left( \delta_{ij} d_{ij} - \frac{(\chi_{\delta}')^{2} d_{i} d_{j} d_{j}}{1 + (\chi_{\delta}')^{2} |d_{i}|^{2}_{\epsilon,1}} \right)
= \frac{d_{11}}{1 + \epsilon^{2}} + \sum_{i=2}^{n} d_{ii} - \frac{(\chi_{\delta}')^{2} |d_{11}|^{2}_{\epsilon,1}}{1 + (\chi_{\delta}')^{2} |d_{11}|^{2}_{\epsilon,1}} (1 + \epsilon^{2})^{2} + 2 \sum_{i=2}^{n} \frac{d_{i} d_{i} d_{i}}{1 + \epsilon^{2}} - \sum_{i=2}^{n} d_{i} d_{i} d_{i}
= \frac{d_{11}}{1 + \epsilon^{2}} + \sum_{i=2}^{n} d_{ii} - \frac{(\chi_{\delta}')^{2} |d_{11}|^{2}_{\epsilon,1}}{1 + (\chi_{\delta}')^{2} |d_{11}|^{2}_{\epsilon,1}} (1 + \epsilon^{2})^{2}
\end{equation}

Combining this with (7.5) we have

\begin{equation}
\sum_{i,j=1}^{n} P^{ij} d_{ij} \geq \frac{d_{11}}{1 + \epsilon^{2}} + \sum_{i=2}^{n} d_{ii} = - \frac{\epsilon^{2}}{1 + \epsilon^{2}} d_{11} + \Delta d > \frac{\epsilon^{2} a}{1 + \epsilon^{2}} - \frac{a \epsilon^{2}}{2(1 + \epsilon^{2})} = \frac{a \epsilon^{2}}{2(1 + \epsilon^{2})}
\end{equation}
on \( B_{\alpha}(0) \cap \Omega \). Set

\[ \chi(d) = \frac{2\sqrt{1 + \epsilon^{2}}}{\sqrt{a \epsilon}} \left( \sqrt{2a - d} \right) \]

then \( \chi' = -\frac{\sqrt{1 + \epsilon^{2}}}{\sqrt{a \epsilon} \sqrt{d}} \) and \( \chi'' = \frac{\sqrt{1 + \epsilon^{2}}}{2a \epsilon \sqrt{d^{3}}} \geq 0 \) on \((0, a)\). Hence on \( \Omega_{\delta, a} \)

\begin{equation}
\sum_{i,j=1}^{n} P^{ij} \partial_{x_{i}} \partial_{x_{j}} \chi_{\delta} = \chi_{\delta} \sum_{i,j=1}^{n} P^{ij} d_{ij} + \chi_{\delta}' \sum_{i,j=1}^{n} P^{ij} d_{ij} < \chi_{\delta}' \frac{a \epsilon^{2}}{2(1 + \epsilon^{2})} + \frac{\chi_{\delta}''}{(\chi_{\delta}')^{2}}
= - \frac{\sqrt{1 + \epsilon^{2}}}{\sqrt{a \epsilon} \sqrt{d - \delta}} \frac{a \epsilon^{2}}{2(1 + \epsilon^{2})} + \frac{\sqrt{a \epsilon}}{2\sqrt{1 + \epsilon^{2}} \sqrt{d - \delta}} = 0.
\end{equation}

Since \( \chi_{\delta} = -\infty \) on \( \partial \Omega_{\delta, a} \setminus \partial B_{\alpha}(0) \), then by Theorem 13.10 in [9], we have

\[ u \leq \chi_{\delta} + \sup_{|y| = a} u(y) \quad \text{on} \quad \Omega_{\delta, a}. \]

Letting \( \delta \to 0 \), then

\begin{equation}
\frac{u \leq 2\sqrt{2(1 + \epsilon^{2})}}{\epsilon} + \sup_{|y| = a} u(y) \quad \text{on} \quad B_{\alpha}(0) \cap \Omega.
\end{equation}
Let \( \rho(x) = |x| \), and \( l = \text{diam}(\Omega) \). Then for every \( i \in \{1, \cdots, n\} \),
\[
0 \leq \rho_{ii} = \partial_x x_i \partial_x x_i \rho = \frac{1}{|x|} \frac{x_i^2}{|x|^3} \leq \frac{1}{|x|},
\]
and \( \Delta \rho = \frac{n-1}{|x|} \). Choose \( \phi \in C^2((a, l)) \) such that \( \phi(l) = 0 \), \( \phi' \leq 0 \) and \( \phi'(a) = -\infty \). Set
\[
w(x) = \phi(\rho(x)) + \sup_{\partial \Omega \setminus B_a(0)} u \quad \text{for any } x \in \Omega \setminus B_a(0).
\]
Note that \( |D \rho| = 1 \), then on \( \Omega \setminus B_a(0) \), analogously to the proof of (7.8)-(7.10), one has
\[
\sum_{i,j=1}^n P^{ij} \rho_i \rho_j \leq \frac{1}{1 + (\phi')^2},
\]
and
\[
\sum_{i,j=1}^n P^{ij} \rho_i \rho_j = \frac{\rho_{11}}{1 + \epsilon^2} + \sum_{i=2}^n \rho_{ii} - \frac{(\phi')^2 \rho_{11} \rho_1^2}{1 + (\phi')^2 \rho_1^2 + (1 + \epsilon^2) \sum_{2 \leq i \leq n} \rho_i^2} \frac{\epsilon^4}{1 + \epsilon^2}
\]
\[
\geq -\frac{\epsilon^2}{1 + \epsilon^2} \rho_{11} + \Delta \rho - \frac{\epsilon^4}{1 + \epsilon^2} \rho_{11} \geq -\frac{\epsilon^2}{|x|} + \frac{n - 1}{|x|}.
\]
By assumption \( \epsilon \leq 1 \) and \( n \geq 3 \), we have
\[
\sum_{i,j=1}^n P^{ij} \rho_i \rho_j \geq \frac{1}{\rho}.
\]
Let
\[
\phi(\rho) = \int_{\rho}^l \left( \log \left( \frac{t}{a} \right) \right)^{-\frac{1}{2}} \frac{\rho}{a} \ dt.
\]
Then \( \phi' = -\left( \log \left( \frac{\rho}{a} \right) \right)^{-\frac{1}{2}} \frac{1}{\rho} \) and \( \phi'' = \frac{1}{2} \left( \log \left( \frac{\rho}{a} \right) \right)^{-\frac{3}{2}} \frac{1}{\rho^2} \). On \( \Omega \setminus B_a(0) \), we have
\[
\sum_{i,j=1}^n P^{ij} \phi_{ij} = \phi' \sum_{i,j=1}^n P^{ij} \rho_{ij} + \phi'' \sum_{i,j=1}^n P^{ij} \rho_i \rho_j \leq \frac{\phi'}{\rho} + \frac{\phi''}{(\phi')^2}
\]
\[
= -\left( \log \left( \frac{\rho}{a} \right) \right)^{-\frac{1}{2}} \frac{1}{\rho} + \frac{1}{2\rho \sqrt{\log \left( \frac{\rho}{a} \right)}} < 0.
\]
Since \( \phi'(a) = -\infty \), then by Theorem 13.10 in [9], we have
\[
u \leq \phi + \sup_{\partial \Omega \setminus B_a(0)} u(y) \quad \text{on } \Omega \setminus B_a(0).
\]
Combining (7.12), we obtain
\[
u \leq \frac{2\sqrt{2(1 + \epsilon^2)}}{\epsilon} + \phi(a) + \sup_{\partial \Omega \setminus B_a(0)} u(y) \quad \text{on } \Omega \cap B_a(0).
\]
Hence on \( \partial \Omega \cap B_a(0) \), \( u \) cannot be arbitrary. For instance, if there is a point \( x \in \partial \Omega \cap B_a(0) \) such that
\[
\varphi(x) > \frac{2\sqrt{2(1 + \epsilon^2)}}{\epsilon} + \phi(a) + \sup_{\partial \Omega \setminus B_a(0)} \varphi(y),
\]
then the minimal surface system (7.1) has no classical solution. \( \square \)
### 8. Uniqueness and non-existence of Dirichlet graphs

In this section, we will study the uniqueness of strictly stable minimal submanifolds with fixed boundary. For simplicity, we assume that $M$ is an $n$-dimensional smooth minimal submanifold in $\mathbb{R}^{n+m}$ with smooth boundary. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field of $M$ at the considered point. Let $A_{ij} = \nabla_{e_i}e_j - \nabla_{e_j}e_i$ be the components of the second fundamental form of $M$. Let $\Phi$ be a smooth vector field in the normal space $NM$. Assume that $\varepsilon_0$ is sufficiently small, such that the hypersurface $M_s = M + s\Phi$, given by

$$F(s) : M \to \mathbb{R}^{n+m} \quad \text{with} \quad F(p, s) = p + s\Phi(p)$$

is smooth for every $|s| < \varepsilon_0$. Let $H_s$ be the mean curvature vector of $M_s$, and $\Delta_M^\perp$ be the normal Laplacian on $M$ for the normal bundle $NM$. By a standard computation,

$$\frac{\partial}{\partial s} H_s \bigg|_{s=0} = \Delta_M^\perp \Phi + \sum_{i,j} \langle \Phi, A_{ij} \rangle A_{ij}. \quad (8.1)$$

Furthermore, from the appendix I,

$$\frac{1}{s} H_s = \Delta_M^\perp \Phi + \sum_{i,j} \langle \Phi, A_{ij} \rangle A_{ij} + s Q_s \left( \Phi, \nabla^\perp \Phi, (\nabla^\perp)^2 \Phi \right), \quad (8.2)$$

where $Q_s, \Phi \triangleq Q_s (\Phi, \nabla^\perp \Phi, (\nabla^\perp)^2 \Phi)$ is a vector-valued function defined by the following combination

$$Q_s, \Phi = Y_0 * \Phi * \Phi + Y_1 * \Phi * \nabla^\perp \Phi + Y_2 * \nabla^\perp \Phi * \nabla^\perp \Phi + Y_3 * \Phi * (\nabla^\perp)^2 \Phi + Y_4 * \nabla^\perp \Phi * (\nabla^\perp)^2 \Phi. \quad (8.3)$$

Here, $Y_i$ are smooth vector fields depending on $s, \Phi, \nabla^\perp \Phi, (\nabla^\perp)^2 \Phi$, and $\nabla^\perp$ is the normal connection. Hence, if $\sum_{i=0}^{3}|(\nabla^\perp)^i \Phi|$ is bounded and $s$ is sufficiently small, then $\sum_{i=0}^{3}|Y_i|$ is bounded, and

$$|Q_s, \Phi| \leq C_M \left( |\Phi|^2 + |\nabla^\perp \Phi|^2 + |\Phi| + |\nabla^\perp \Phi| \right) |(\nabla^\perp)^2 \Phi|, \quad (8.4)$$

where $C_M$ depends only on the bounds of $R_M, \nabla R_M$ and $\nabla^2 R_M$. Here, $R_M$ is the curvature tensor of $M$. Let $L_M$ be the second order operator defined by

$$L_M \xi = \Delta_M^\perp \xi + \sum_{i,j} \langle \xi, A_{ij} \rangle A_{ij} \quad (8.5)$$

for each vector field $\xi \in C^2(M \setminus \partial M, NM)$. From (8.1), $M$ is strictly stable if and only if the first eigenvalue of $L_M$ is positive. Namely, there exists a constant $\epsilon_M > 0$ such that

$$\epsilon_M \int_M |\xi|^2 \leq - \int_M \langle \xi, L_M \xi \rangle \quad (8.6)$$

for any $\xi \in C_0^\infty(M \setminus \partial M, NM)$.

**Lemma 8.1.** Let $M$ be an $n$-dimensional smooth strictly stable compact minimal submanifold with smooth boundary $\partial M$ in $\mathbb{R}^{n+m}$. There exists a constant $\delta_M > 0$ such that for any smooth minimal submanifold $S$ with boundary $\partial M$, if $S = \{p + \Psi(p) | p \in M\}$ for some vector field $\Psi \in C_0^\infty(M, NM)$ with $|\nabla^\perp \Psi|_\Omega + |(\nabla^\perp)^2 \Psi|_\Omega \leq \delta_M$, then $S = M$.

**Proof.** Let $\delta = |\nabla^\perp \Psi|_\Omega + |(\nabla^\perp)^2 \Psi|_\Omega$ with $\delta \in (0, 1]$, and $\Phi = \frac{1}{\delta} \Psi$. Then from (8.2) (8.3), one has

$$L_M \Phi = \Delta_M^\perp \Phi + \sum_{i,j} \langle \Phi, A_{ij} \rangle A_{ij} = -\delta Q_\delta, \Phi \quad (8.7)$$
with \(|Q_{\delta,\phi}| \leq C_M' \left( |\Phi| + |\nabla_M^1 \Phi| \right)|, where \(C_M'\) is a constant depending only on \(C_M\), and the diameter of \(M\). From (8.6), we have

\[
\epsilon_M \int_{M} |\Phi|^2 \leq - \int_{M} \langle \Phi, L_M \Phi \rangle = \delta \int_{M} \langle \Phi, Q_{\delta,\phi} \rangle \leq C_M' \delta \int_{M} \left( |\Phi|^2 + |\Phi| + |\nabla_M^1 \Phi| \right) \leq 2C_M' \delta \int_{M} \left( |\Phi|^2 + |\nabla_M^1 \Phi|^2 \right).
\]

Let \(\{n^a\}_{a=1}^m\) be a fixed orthonormal frame for the normal space \(NM\) such that \(\nabla^1_n n^a = 0\). Then we write \(\Phi = \sum_{\alpha} \phi^a n^a\) for some vector-valued function \((\phi^1, \cdots, \phi^m)\) on \(M\) with \(\phi^a = 0\) on \(\partial M\). Denote \(A_{ij} = h_{ij}^a n^a\). From (8.7) (see also (9.18)), one has

\[
\Delta_M \phi^a + \sum_{\alpha, i, j} h_{ij}^\alpha h_{ij}^\beta \phi^\beta + \delta Q_{\delta,\phi}^a = 0
\]

with \(|Q_{\delta,\phi}| \leq C_M' \left( |\Phi| + |\nabla_M^1 \Phi| \right)|). From \(W^2,p\)-estimates,

\[
\sum_{\alpha} \|\phi\|_{W^{1,2}(M)} \leq C_M' \sum_{\alpha} \|\phi^\alpha\|_{L^2(M)},
\]

where \(C_M'\) is a positive constant depending on the geometry of \(M\). Combining (8.8) and (8.10), we deduce \(\Phi = 0\) if we choose \(\delta\) sufficiently small depending on \(M\). This completes the proof.

Let \(\Omega\) be a smooth mean convex bounded domain in \(\mathbb{R}^n\). Let \(\varphi\) be a smooth function on \(\partial \Omega\). Let \(\psi^1, \cdots, \psi^{m-1}\) be smooth functions on \(\partial \Omega\) with \(\sum_{\alpha=1}^{m-1} \sum_{i=0}^{3} |D^i \psi^\alpha|_{\partial \Omega} = 1\). Let

\[
\Gamma_t = \{(x, t\psi^1(x), \cdots, t\psi^{m-1}(x), \varphi(x)) \in \mathbb{R}^n \times \mathbb{R}^m : x \in \partial \Omega\}
\]

for all \(t \in [0, 1]\).

**Proposition 8.2.** There exists a constant \(\delta_{\Omega, \varphi} > 0\) depending only on \(\Omega\) and \(\varphi\) such that for each \(|t| \leq \delta_{\Omega, \varphi}\) there is a unique minimal submanifold with boundary \(\Gamma_t\), and it coincides with the smooth solution obtained in Theorem 6.2.

**Proof.** From Jenkins-Serrin [13], there is a smooth solution \(w\) to the minimal surface equation (1.1) with \(w = \varphi\) on \(\partial \Omega\). Let \(M\) denote the graph over \(\Omega\) of the graphic function \((0, \cdots, 0, \varphi)\) in \(\mathbb{R}^{n+m}\). From (6.14)-(6.16), \(M\) is a strictly stable minimal submanifold. Let us prove this proposition by contradiction. Suppose that there are two sequences of smooth minimal submanifolds \(\Sigma_k\) and \(\Sigma'_k\) with \(\Sigma_k \neq \Sigma'_k\) and \(\partial \Sigma_k = \partial \Sigma'_k = \Gamma_{t_k}\) for some sequence \(t_k \to 0\). By the Sobolev inequality on minimal submanifolds (see [28] for instance), the volumes of \(\Sigma_k, \Sigma'_k\) are uniformly bounded. By compactness of varifolds (see [21][28]), after choosing subsequences we may assume that \(\Sigma_k\) converges to \(\Sigma\) and \(\Sigma'_k\) converges to \(\Sigma'\) in the varifold sense, respectively. Since \(\partial \Sigma = \partial \Sigma' = \Gamma_0\), by the maximum principle, \(\Sigma\) and \(\Sigma'\) live in an \((n+1)\)-dimensional Euclidean space with boundary \(\Sigma(x, \varphi(x)) \in \mathbb{R}^{n+1}\). Hence, \(\Sigma = \Sigma' = M\). From Allard’s regularity theorem [1] and the reflection principle (after flattening the boundary), \(\Sigma_k\) and \(\Sigma'_k\) both converge to \(M\) smoothly since \(\Sigma_k, \Sigma'_k\) are graphs over \(\Omega\). In other words, there are smooth solutions \(u_k, u'_k\) to the minimal surface system with \(u_k = w_k = (t_k \psi^1, \cdots, t_k \psi^{m-1}, \varphi)\) on \(\partial \Omega\) such that \(\Sigma_k = \text{graph} \, u_k, \Sigma'_k = \text{graph} \, u'_k\), and \(u_k\) and \(u'_k\) both converge smoothly to \((0, \cdots, 0, w)\) on \(\overline{\Omega}\). So, the strict stability of \(M\) implies that \(\Sigma_k\) and \(\Sigma'_k\) both are strictly stable for sufficiently large \(k > 0\). Moreover, for sufficiently large \(k > 0\), \(S_k'\) can be seen as a graph over \(S_k\) with \(S'_k = \{p + \Psi_k(p) | p \in S_k\}\) for some vector field \(\Psi_k \in C^\infty_0(S_k, NS_k)\) such that \(\Psi_k\) converges to zero smoothly as \(k \to \infty\). From Lemma 8.1, we deduce \(\Psi_k = 0\) for sufficiently large \(k > 0\). This is a contradiction. We complete the proof. \(\square\)
9. Appendix I. Calculations for graphs over a submanifold

In this appendix, we will calculate the mean curvature vectors for a one-parameter family $M_s$ over an $n$-dimensional smooth embedded submanifold $M$ in $\mathbb{R}^{n+m}$ (see the case of hypersurfaces by Colding-Minicozzi in [3]). Let $\nabla_\bot$ denote the normal connection in $NM$ defined by

$$\nabla_X^\bot \nu = (\nabla_X \nu)^\bot$$

for any $X \in \Gamma(TM)$ and $\nu \in \Gamma(NM)$. Let $\{n^\alpha\}_{a=1}^m$ be a fixed orthonormal frame for the normal space $NM$ such that $\nabla_\bot n^\alpha = 0$, and $(\phi^1, \cdots, \phi^m)$ be a vector-valued function on $M$. Let $M_s$ be defined by

$$\tilde{F}(.,s) : M \to \mathbb{R}^{n+m} \text{ with } F(p,s) = p + s\phi^\alpha n^\alpha.$$ 

Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame for the tangent space $TM$. Let $A_{ij} = \nabla_{e_i}e_j - \nabla_{e_j}e_i$ be the components of the second fundamental form of $M$, and

$$h_{ij}^\alpha = \langle A_{ij}, n^\alpha \rangle = \langle \nabla_{e_i}e_j, n^\alpha \rangle.$$ 

We extend both the functions $\phi^1, \cdots, \phi^m$ and the frame $\{e_i\}_{i=1}^n$ to a small neighborhood of $M$ by parallel translation along the normal frame, so that $\langle n^\alpha, \nabla \phi^\beta \rangle = 0$ and $\nabla_\bot n^\alpha e_i = 0$.

The tangent space of $M_s$ is spanned by $\{F_i\}_{i=1}^n$, where

$$F_i(p,s) = dF_{(p,s)}(e_i(p)) = e_i(p) + s\phi_1n^\alpha(p) - s\phi^\alpha h_{ik}^\beta e_k(p).$$

Note that $\{F_i\}$ is not orthonormal in general. The metric $g_{ij}(p,s)$ of $M_s$ at $F(p,s)$ related to the frame $F_i$ is

$$g_{ij}(p,s) = \langle F_i, F_j \rangle = \delta_{ij} - 2s\phi^\alpha h_{ij}^\alpha + s^2\phi_i^\alpha \phi_j^\alpha + s^2\phi^\alpha \phi^\beta h_{ik}^\alpha h_{jk}^\beta.$$ 

Let $a$ be the matrix with elements $a_{ij} = \delta_{ij} - s\phi^\alpha h_{ij}^\alpha$, then $a$ is positive definite if $s\phi$ is sufficiently small. Let $a^{-1}$ be the inverse matrix of $a$. Put

$$\tilde{n}_s^\alpha(p) = n^\alpha(p) - s\nabla \phi^\alpha - s^2(a^{-1})_{ij} \phi_j^\beta h_{ik}^\alpha \phi_k^\alpha e_j(p),$$

then

$$\langle F_i(p,s), \tilde{n}_s^\alpha(p) \rangle = s\phi_i^\alpha - s\phi_j^\alpha a_{ij} - s^2 \phi^\beta h_{ik}^\alpha \phi_k^\alpha = 0$$

for any $i = 1, \cdots, n$ and $\alpha = 1, \cdots, m$.

Let $P_{s,\phi}$ and $Q_{s,\phi}$ stand for general functions of the form

$$P_{s,\phi} = f_0 \ast \phi^\alpha \ast \phi^\beta + f_1 \ast \phi^\alpha \ast \nabla \phi^\beta + f_2 \ast \nabla \phi^\alpha \ast \nabla \phi^\beta$$

and

$$Q_{s,\phi} = \tilde{f}_0 \ast \phi^\alpha \ast \phi^\beta + \tilde{f}_1 \ast \phi^\alpha \ast \nabla \phi^\beta + \tilde{f}_2 \ast \nabla \phi^\alpha \ast \nabla \phi^\beta$$

$$+ \tilde{f}_3 \ast \phi^\alpha \ast \nabla^2 \phi^\beta + \tilde{f}_4 \ast \nabla \phi^\alpha \ast \nabla^2 \phi^\beta,$$

where $f_i$ are smooth vector fields depending on $s$, $\phi^\alpha$, $\nabla \phi^\alpha$, and $\tilde{f}_i$ are smooth vector fields depending on $s$, $\phi^\alpha$, $\nabla \phi^\alpha$, $\nabla^2 \phi^\alpha$ such that if $\sum_{0 \leq k \leq 2, \alpha} |\nabla^k \phi^\alpha|$ is bounded, then $\sum_{i=0}^2 (|f_i| + |\nabla f_i|) + \sum_{i=0}^4 |\tilde{f}_i|$ is bounded for the sufficiently small $s > 0$. Note that the precise form of $P_{s,\phi}$ and $Q_{s,\phi}$ may be different even in the same line.

From now on, assume that $s$ is sufficiently small. Then

$$|\tilde{n}_s^\alpha(p)| = \sqrt{1 + s^2 P_{s,\phi} = 1 + s^2 P_{s,\phi}}.$$
Let
\[ n_s^\alpha(p) = \frac{n_s^\alpha(p)}{|n_s^\alpha(p)|}, \]
then \{n_s^\alpha(p)\} forms a basis (not necessarily orthonormal) for the normal space \(NM_s\) at the point \(F(p, s)\) and \(n_s^0(p) = n^\alpha(p)\). Then one has
\[
(9.7) \quad n_s^\alpha(p) = n^\alpha(p) - s\nabla^\alpha + s^2P_{s,\phi}.
\]
From \(\nabla^\alpha e_i = 0\), a direct computation implies
\[
(9.8) \quad \frac{\partial}{\partial s} \langle \nabla_{e_i} n^\alpha, e_j \rangle = \phi^\beta \nabla_{\nabla_{e_i} n^\alpha, e_j} = \phi^\beta \langle \nabla_{\nabla_{e_i} n^\alpha, e_j} \rangle
\]
at the point \(F(p, s)\). Since
\[
(9.9) \quad \langle \nabla_{e_i} n^\alpha, e_j \rangle = - \langle n^\alpha, \nabla_{e_i} e_j \rangle = 0,
\]
then with \(\nabla^\perp n^\alpha = 0\) one has
\[
(9.10) \quad \langle \nabla_{e_i} \nabla_{e_i} n^\alpha, e_j \rangle = - \langle \nabla_{\nabla_{e_i} n^\alpha, e_i} e_j \rangle = 0,
\]
and then
\[
(9.11) \quad \frac{\partial}{\partial s} \langle \nabla_{e_i} n^\alpha, e_j \rangle = \phi^\beta \langle \nabla_{\nabla_{e_i} n^\alpha, e_j} \rangle = - \phi^\beta \langle \nabla_{\nabla_{e_i} n^\alpha, e_j} \rangle
\]
hence
\[
(9.12) \quad \frac{\partial}{\partial s} \mid_{(p, 0)} \nabla_{e_i} n^\alpha = - \phi^\beta h_{ik}^\alpha h_{jk}^\alpha e_j.
\]
Taking the derivative \(\frac{\partial}{\partial s}\) again on both sides of (9.11) implies
\[
(9.13) \quad \nabla_{e_i} n^\alpha = - h_{ij}^\alpha e_j(p) - s^2h_{ik}^\alpha h_{jk}^\alpha e_j(p) + s^2P_{s,\phi}.
\]
Denote \(F' = F_i(p, 0) = \phi^\alpha n^\alpha(p) - \phi^\alpha h_{ik}^\alpha e_k(p)\), then
\[
(9.14) \quad s\nabla_{F_i} n^\alpha = - s^2 h_{ik}^\alpha \nabla_{e_k} n^\alpha = s^2 h_{ik}^\alpha h_{jk}^\alpha e_j(p) + s^2P_{s,\phi}.
\]
Hence at the point \(F(p, s)\) we get
\[
(9.15) \quad - \nabla_{F_i} n^\alpha(p) = - \nabla_{e_i} n^\alpha + s \nabla_{e_i} \nabla^\alpha - s \nabla_{F_i} n^\alpha + s^2Q_{s,\phi}
\]
\[= h_{ij}^\alpha e_j(p) + s\operatorname{Hess}^\alpha (e_i(p), e_j(p)) e_j(p) + s^2Q_{s,\phi}.\]
Therefore, at the point \(F(p, s)\) one has
\[
(9.16) \quad \langle \nabla_{F_i} F_j, n_s^\alpha(p) \rangle = - \langle \nabla_{F_i} n_s^\alpha(p), F_j \rangle = h_{ij}^\alpha + s\operatorname{Hess}^\alpha (e_i(p), e_j(p)) - s^2 h_{kj}^\alpha h_{ik}^\alpha + s^2Q_{s,\phi}.
\]
Let \(H_s(p) = (H_s^1(p), \cdots, H_s^n(p))\) denote the mean curvature vector of \(M_s\) at \(F(p, s)\). Since
\[g^{ij} = \delta_{ij} + 2\phi^\alpha h_{ij}^\alpha + s^2P_{s,\phi},\]
then
\[
(9.17) \quad H_s^\alpha(p) = \frac{\delta g^{ij}}{2} \langle \nabla_{F_i} F_j, n_s^\alpha(p) \rangle
\]
\[= H^\alpha(p) + s \left( \Delta_M^\phi + \phi h_{ij}^\alpha h_{ij}^\alpha \right) + s^2Q_{s,\phi}.
\]
Let
\[\tau_{\alpha\beta}(p, s) = \langle n_s^\alpha(p), n_s^\beta(p) \rangle = \delta_{\alpha\beta} + s^2P_{s,\phi}\]
at \(F(p, s)\), and \((\tau^\alpha \beta(p, s))\) be the inverse matrix of \((\tau_{\alpha \beta}(p, s))\). Then

\[
(9.18) \quad H_s(p) = \tau^\alpha \beta H_s^\alpha(p) \mathbf{n}_s^2 = H^\alpha(p) \mathbf{n}_s^\alpha(p) + s \left( \Delta_M \phi^\alpha + \phi^\beta h_i^\beta h_j^\alpha \right) \mathbf{n}_s^\alpha(p) + s^2 Q_s, \phi.
\]

Put \(\Delta_M^\perp \xi = (\nabla^\perp)^2 \xi(e_i, e_j)\) for \(\xi \in \Gamma(NM)\), and \(\Phi = \phi^\alpha \mathbf{n}_s^\alpha\). In particular, if \(M\) is a minimal submanifold, then

\[
(9.19) \quad H_s(p) = s \left( \Delta_M^\perp \Phi + \langle \Phi, A_{ij} \rangle A_{ij} \right) + s^2 Q_s, \phi.
\]

10. Appendix II: Algebraic Inequalities

Here we state an algebraic result, which is sharp for \(m = 1\).

**Lemma 10.1.** Let \(S_{\alpha\alpha}\) be an \((n \times m)\)-real matrix. Put \(S_\alpha = \left( \sum_{i=1}^n S^2_{\alpha i} \right)^{1/2}\) for each \(\alpha = 1, \ldots, m\). If \(\sum_{\alpha=1}^m S_\alpha \leq \min\{2\sqrt{2}, 2/S_m\}\), then

\[
(10.1) \quad \det \left( \delta_{ij} + \sum_{\alpha=1}^m S_{\alpha i} S_{\alpha j} \right) \leq 1 + \left( \sum_{\alpha=1}^m S_\alpha \right)^2.
\]

**Proof.** Let us prove this lemma by induction. Clearly, the inequality \((10.1)\) holds for \(m = 1\). Assume that \((10.1)\) holds for \(m - 1\) with \(m \geq 2\). Let \(A\) be a diagonal \((n \times n)\)-matrix with eigenvalues \(1, \ldots, 1, \sqrt{1 + S_m^2}\). Then there is an orthonormal matrix \(P = (p_{ij})_{n \times n}\) such that the matrix \((\delta_{ij} + S_{im} S_{jm}) = P A^2 P^T\). Let \(S'_{\alpha i} = \sum_{k=1}^n p_{ki} S_{\alpha k}\) for \(i = 1, \ldots, n - 1\), and \(S'_{\alpha} = 1 + S_m^2/\sum_{k=1}^n p_{kn} S_{\alpha k}\). Let \(Q = (q_{ij})\) be a matrix defined by \(q_{ij} = \sum_{\alpha=1}^{m-1} S_{\alpha i} S_{\alpha j}\), and \(Q' = (q_{ij}')\) be a matrix defined by \(q_{ij}' = \sum_{\alpha=1}^{m-1} S'_{\alpha i} S'_{\alpha j}\). Then \(Q' = A^{-1} P^T Q P A^{-1}\), and

\[
(10.2) \quad \det \left( \delta_{ij} + \sum_{\alpha=1}^m S'_{\alpha i} S'_{\alpha j} \right) = |\det(A P^T)|^2 \det(A^{-1} P^T (P A^2 P^T + Q) P A^{-1})
= (1 + S_m^2) \det (I + Q')
\]

Moreover, let \(S''_{\alpha} = \left( \sum_{i=1}^n (S'_{\alpha i})^2 \right)^{1/2}\) for each \(\alpha = 1, \ldots, m - 1\), then

\[
(10.3) \quad (S''_{\alpha})^2 = \sum_{i=1}^n (S'_{\alpha i})^2 \leq \sum_{i=1}^n \left( \sum_{k=1}^n p_{ki} S_{\alpha k} \right)^2 = \sum_{i=1}^n \sum_{k,l=1}^n p_{ki} p_{li} S_{\alpha k} S_{\alpha l} = \sum_{k=1}^n S_{\alpha k}^2 = S_{\alpha}^2.
\]

Since \(\sum_{\alpha=1}^{m-1} S_\alpha \leq \min\{2\sqrt{2}, 2/S_m\}\), then for \(m \geq 3\) we have \(\sum_{\alpha=1}^{m-2} S''_{\alpha} \leq \sum_{\alpha=1}^{m-2} S_\alpha \leq 2\sqrt{2}\) and

\[
(10.4) \quad \sum_{\alpha=1}^{m-2} S'_{\alpha} S''_{\alpha} \leq \frac{1}{4} \left( \sum_{\alpha=1}^{m-2} S'_{\alpha} + S''_{\alpha} \right)^2 \leq \frac{1}{4} \left( \sum_{\alpha=1}^{m-1} S_\alpha \right)^2 \leq 2.
\]

By assumption, \((10.1)\) holds for \(m - 1\) with \(m \geq 2\), which implies

\[
(10.5) \quad \det (I + Q') \leq 1 + \left( \sum_{\alpha=1}^{m-1} S_\alpha \right)^2.
\]
for $m \geq 3$. It is clear that (10.5) holds for $m = 2$. Substituting (10.5) into (10.2) implies
\[
(10.6) \quad \det \left( \delta_{ij} + \sum_{\alpha=1}^{m} S_{i\alpha} S_{j\alpha} \right) \leq 1 + \left( \frac{m-1}{4} \right)^2 \left( \sum_{\alpha=1}^{m-1} S_{\alpha} + S_m \right) \leq 1 + \left( \frac{m-1}{4} \right)^2 \left( \sum_{\alpha=1}^{m} S_{\alpha} \right) + \left( \sum_{\alpha=1}^{m} S_{\alpha} \right)^2,
\]
where we have used $\sum_{\alpha=1}^{m-1} S_{\alpha} S_m \leq 2$ in the above inequality. This completes the proof. □

If we assume $\sum_{\alpha=1}^{m} S_{\alpha} \leq 2\sqrt{2}$, then from the Cauchy-Schwarz inequality,
\[
\sum_{\alpha=1}^{m-1} S_{\alpha} S_m \leq \frac{1}{4} \left( \sum_{\alpha=1}^{m} S_{\alpha} + S_m \right)^2 \leq \frac{1}{4} \times 8 = 2.
\]
From Lemma 10.1, we immediately have the following result.

**Lemma 10.2.** Let $S_{i\alpha}$ be an $(n \times m)$-real matrix. Put $S_{\alpha} = \left( \sum_{i=1}^{n} S_{i\alpha}^2 \right)^{1/2}$ for $\alpha = 1, \cdots, m$. If $\sum_{\alpha=1}^{m} S_{\alpha} \leq 2\sqrt{2}$, then
\[
(10.7) \quad \det \left( \delta_{ij} + \sum_{\alpha=1}^{m} S_{i\alpha} S_{j\alpha} \right) \leq 1 + \left( \sum_{\alpha=1}^{m} S_{\alpha} \right)^2.
\]

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SHANGHAI CENTER FOR MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI 200438, CHINA

Email address: dingqi@fudan.edu.cn

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22, 04103 LEIPZIG, GERMANY

Email address: jost@mis.mpg.de

INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

Email address: ylxin@fudan.edu.cn