Abstract

We analyse waves that propagate along the interface between a dielectric half-space and a half-space filled with a Lorentz material. We show that the corresponding interface condition leads to a generalisation of the classical Leontovich condition on the boundary of a dielectric half-space. We study when this condition supports propagation of (dispersive) surface waves. We derive the related dispersion relation for waves propagating along the boundary of a stratified half-space and determine the relationship between the loss parameter, frequency and wavenumber for which interfacial waves exist.

Keywords: Maxwell equations, Leontovich boundary conditions, Impedance boundary conditions, Stratified media, Surface waves.

1 Introduction

The question of what boundary conditions are admissible for an electromagnetic field on the boundary of a free space has been touched upon in the mathematics literature several times since the work by Leontovich in the early 1940's, see [1], in relation to wave propagation near the earth surface. Clearly the most general conditions are obtained as the conditions on the interface between the free half space (or its curved analogue) and the electromagnetic medium on its boundary, which is a well-studied subject from the physics, analysis and numerical perspective, and does not present a challenge with the modern computational power available. However, from the point of view of understanding which interfacial waves are possible, in principle on the boundary between two electromagnetic media, the question of what form the Leontovich condition takes when neither medium is nearly perfectly conducting is worth exploring, for at least two reasons. Firstly, replacing the interface conditions by “effective” conditions on the boundary of a dielectric medium can lead to a reduction in computational demand. Secondly, the knowledge about the associated surface waves may be exploited for the design bespoke transmitting devices. These considerations motivate us to consider what effective Leontovich-type boundary conditions are possible on the boundary of a dielectric in contact with a general (dissipative) Lorentz-type medium and what form the related frequency-wavenumber diagrams (“dispersion relations”) take. Interfacial and surface waves are often amenable to a direct analysis, in view of the simplified geometry of the wave process, as the related wave solutions admit separation of variables into a product of two functions, one of which oscillates along the surface and the other exponentially decays away from it. This observation prompts us to try and obtain closed-form dispersion relations for the surface of contact between a dielectric and a general Lorentz medium, at least in the case of a flat interface. The related analysis should admit further generalisation towards the case of a curved surface, in the case when

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the typical radius of curvature is assumed small compared to the wavelength, see [2] for the case of the classical Leontovich condition (which we later explore as a limit of a more general condition).

In the present work we discuss the case of a two-component stratified dielectric in contact with a Lorentz medium where the components fill up the half space in the alternating fashion. In other words, the material properties of the dielectric half-space are periodic in the direction orthogonal to the contact surface. This allows us to consider a non-homogeneous case while retaining the ability to carry out some explicit analysis by virtue of the Floquet theory, see Section 4.

For reference we list here the conventions adopted for notation throughout this paper:

- $k$ wavenumber;
- $\omega$ (angular) frequency;
- $\varepsilon$ (with subscripts where necessary) electric permittivity ("dielectric constant");
- $\mu$ (with subscripts where necessary) magnetic permeability;
- $c$ the speed of light in vacuum, $c = 1/\sqrt{\mu_0\varepsilon_0}$, where $\varepsilon_0$ (respectively $\mu_0$) is the permittivity (respectively, permeability) of free space;
- $v_p$ phase velocity, $v_p = \omega/k$;
- $\wedge$ the standard 3-vector cross product. When taken with the $\nabla$ operator on the left, it is meant as taking the curl of a vector field: $\nabla \wedge \mathbf{A} = \text{curl} \mathbf{A}$.

We also adopt the following terminology to describe materials:

- A dielectric material refers to a material that has constant and real permittivity and permeability.
- A Lorentz material refers to a material of constant permeability, but which has $\omega$-dependent permittivity; the form of which is given in [4].
- A lossless Lorentz material refers to a Lorentz material with zero loss factor, that is $\gamma = 0$ in [4], and hence an entirely real (but potentially negative) permittivity for all $\omega \neq \omega_0$.

## 2 Problem setup

### 2.1 Maxwell system

The Maxwell equations of electromagnetism are:

\[
\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \wedge \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t},
\]

where $\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic field, respectively, and $\mathbf{D}$, $\mathbf{B}$ are the related electric displacement and magnetic flux density. We consider these equations either in a half-space $\{x \in \mathbb{R} : x_3 > 0\}$, with a boundary condition at $\{x_3 = 0\}$ and a decay condition into the half-space, or as a full-space system with an interface condition between two materials at $\{x_3 = 0\}$. We are concerned with surface (in the half-space case) or interfacial (in the full-space case) wave solutions, and without loss of generality seek waves propagating in the $x_1$ direction on the $\{x_3 = 0\}$ surface (or interface), i.e. solutions of the form

\[
\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \exp(i(kx_1 - \omega t)), \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \exp(i(kx_1 - \omega t)),
\]

where the components $E_j$, $B_j$ ($j = 1, 2, 3$) are functions of $x_3$ only. We choose to work with the components of the electromagnetic field that are continuous across the interfaces: $E_1, E_2, D_3, H_1, H_2, B_3$.  

2
2.2 Lorentz materials

The Lorentz oscillator model was proposed by Hendrik Lorentz in the later half of the 19th century, as a model for explaining the optical properties of materials [3]. In this model electrons inside a material are considered bound to atoms, and the binding force interaction is modelled as a “mass-on-a-spring” system under the assumption that the atom is far more massive than the electron, hence does not change its position. A damping term is introduced, to account for the inherent loss of energy as the (charged) electron accelerates; under the assumption that the atom is far more massive than the electron, hence does not change its position.

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Consider a homogeneous, isotropic dielectric material of permittivity \( \varepsilon \) and permeability \( \mu \) occupying the region \( \{x \in \mathbb{R} : x_3 \in (0, d)\} \) for some constant \( d \in \mathbb{R} \). We consider the Maxwell system (1) in this region, but forgo applying boundary/interface conditions (which we shall do later on) when considering the layered half-space problem. Using the ansatz (2) yields two systems

\[
\begin{align*}
\frac{d}{dx_3} E_1 &= i\omega \mu H_2 + \frac{i k}{\varepsilon} D_3, \\
\frac{d}{dx_3} H_2 &= i\omega \varepsilon E_1, \\
-\omega D_3 &= k H_2,
\end{align*}
\]

(5)

each consisting of two differential equations and one algebraic equation. Analysis henceforth focuses on the transverse electric (TE) system\(^1\), which involves the field components \( E_1, H_2, D_3 \), which can be expressed in matrix form as

\[
\begin{pmatrix}
\frac{d}{dx_3} E_1 \\
\frac{d}{dx_3} H_2
\end{pmatrix}
= 
\begin{pmatrix}
0 & \frac{-i\alpha^2}{\omega \varepsilon} \\
\frac{i\alpha^2}{\omega \varepsilon} & 0
\end{pmatrix}
\begin{pmatrix}
E_1 \\
H_2
\end{pmatrix}, \quad \alpha^2 := k^2 - \omega^2 \mu \varepsilon.
\]

(6)

Upon obtaining solutions for (6), the component \( D_3 \) can be recovered from the algebraic equation \(-\omega D_3 = k H_2\). The \( (2 \times 2) \)-system for \( E_1 \) and \( H_2 \) is solved by diagonalising the matrix \( A \) to obtain the solution for

\(^1\)The transverse electric system, or “mode” is also referred to as the “electric wave”. Correspondingly, the transverse magnetic mode is sometimes referred to as the “magnetic wave” in the literature, see e.g. [2].
\[ U := (E_1, H_2)^\top; \]

\[
U(x_3) = \exp(Ax_3)U(0) = \begin{pmatrix} \cosh(\alpha x_3) & -i\frac{\omega \varepsilon}{\omega} \sinh(\alpha x_3) \\ i\frac{\omega \varepsilon}{\alpha} \sinh(\alpha x_3) & \cosh(\alpha x_3) \end{pmatrix} U(0). \tag{7}
\]

We shall use this form for the solution when considering the Maxwell system in a stratified material in Section 4. For completeness, we note that calculations for the transverse magnetic (TM) mode, analogous to those above for the TE case, result in a system of the form (7), with \( U \) replaced by \( V := (H_1, E_2)^\top \) and \( \varepsilon \) replaced by \(-\mu\).

### 3.2 Decaying solution in Lorentz half-space

As preparation for considering the full-space Maxwell problem, consider (1) in a lower half-space \( \{x_3 < 0\} \), occupied by a Lorentz material with permittivity described as in (4). Again, we forgo applying a boundary condition at \( \{x_3 = 0\} \), however we do impose a decay condition into the lower half-space, that is, we seek solutions that tend to zero as \( x_3 \to -\infty \). This yields

\[
E_1(x_3) = C \frac{i\alpha_L}{\omega \varepsilon_L \varepsilon_0} \exp(\alpha_L x_3), \quad H_2(x_3) = C \exp(\alpha_L x_3), \quad C \in \mathbb{C},
\]

\[
\alpha_L(k, \omega) := \sqrt{k^2 - \omega^2 \varepsilon_0 \varepsilon_L \mu_0 \mu_L}, \quad \arg(\alpha_L) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right). \tag{8}
\]

This form of solution gives the following condition at \( x_3 = 0 \):

\[
E_1(0) = \frac{i\alpha_L}{\omega \varepsilon_L \varepsilon_0} H_2(0). \tag{9}
\]

We shall refer to (9) as the Leontovich condition (LC), as it has form similar to the classical Leontovich impedance boundary condition [6]. The quantity \( \alpha_L/(\omega \varepsilon_L \varepsilon_0) \) has physical dimensions of Ohms and plays a role analogous to the impedance in the classical condition. In what follows, we refer to this quantity as the generalised impedance. We also provide the analogous result that arises from considering the TM mode in the same way, namely

\[
E_2(0) = -\frac{i\omega \mu_0 \mu_L}{\alpha_L} H_1(0), \tag{10}
\]

where \( \mu_L \) is the \( \omega \)-dependent magnetic permittivity. In Section 7 we explore the relation between the LC and the classical condition, and in Section 5.1 we show that it can be utilised in the analysis of boundary-value problems in the same way as the classical condition.

### 4 The Maxwell system for stratified media

#### 4.1 System setup

Consider the upper half-space \( \{x_3 > 0\} \) occupied by a stratified dielectric, i.e. a medium consisting of alternating layers of materials A and B, parallel to the \( x_3 \)-plane, with permittivities \( \varepsilon_A \) and \( \varepsilon_B \), and permeabilities \( \mu_A \) and \( \mu_B \) respectively. We denote the period of the structure by \( d \), so that A-layers have thickness \( dh \) and B-layers have thickness \( d(1 - h) \) for some \( h \in (0, 1) \). We wish to solve the Maxwell equations (1), in order to obtain wave solutions that decay as \( x_3 \to \infty \); subject to interface conditions (continuity of the fields) at \( x_3 = dh, d, d(1 + h), 2d, \ldots \). etc.
4.2 Floquet analysis of arbitrary whole-space solutions

In this section we review results on matrix differential equations and Floquet theory, which we shall invoke when solving the Maxwell system in the upper half-space. The results that follow are specific to our needs for the system we are considering; however more general results and further development of the theory is widely known and available, see for example [7].

In the upper (layered) half-space, the Maxwell system has the form

$$\frac{d}{dx}U(x) = A(x)U(x), \quad x > 0,$$

where we write $x$ instead of $x_3$ for brevity, $U$ is a 2-vector, and $A$ a piecewise-constant $(2 \times 2)$-matrix

$$A(x) = \begin{cases} A_1, & 0 < x < dh, \\ A_2, & dh < x < d, \end{cases}$$

extended $d$-periodically. The matrices $A_1$ and $A_2$ have the form of the matrix in [6], with the general $\varepsilon$ and $\mu$ replaced with the constants specific to materials $A$ and $B$. Given that the $(2 \times 2)$-matrix in [6] is non-degenerate, and hence it has an appropriate diagonal form and distinct eigenvalues, the matrices $A_1$ and $A_2$ also possess these properties and have diagonal forms $A_j = T_j \Lambda_j T_j^{-1}$, $j = 1, 2$, for some transformation matrices $T_j$ whose columns are eigenvectors of $A_j$. It follows that

$$U(x) = \begin{cases} T_1 \exp(A_1 x) T_1^{-1} U(0), & 0 < x \leq dh, \\ T_2 \exp(A_2(x - dh)) T_2^{-1} T_1 \exp(A_1 d h) T_1^{-1} U(0), & dh < x \leq d, \end{cases}$$

by considering the system in each layer and using the interface condition at $\{x_3 = dh\}$. In particular,

$$U(dh) = T_1 \exp(A_1 d h) T_1^{-1} U(0),$$

$$U(d) = T_2 \exp(A_2(1 - h)) T_2^{-1} T_1 \exp(A_1 d h) T_1^{-1} U(0). \quad (12)$$

We wish to show that there exists a fundamental matrix $\tilde{\Phi}$ of the $(2 \times 2)$-system [11], where $\tilde{\Phi}$ has the form

$$\tilde{\Phi}(x) = \tilde{\Psi}(x) \text{diag}\{\exp(\tilde{\lambda}_1 x), \exp(\tilde{\lambda}_2 x)\},$$

and $\tilde{\Psi}(x)$ is a $d$-periodic matrix. To show this, we begin from the general theory of linear ODEs, which implies that there exists a fundamental matrix $\Phi$ with $\Phi(0) = I$ and constant vector $C$ such that

$$U(x) = \Phi(x) C.$$

Then, by [12], one has $\Phi(d) = M \Phi(0)$ where $M$ is the monodromy (or transfer) matrix

$$M = T_2 \exp(A_2(1 - h)) T_2^{-1} T_1 \exp(A_1 d h) T_1^{-1}.$$

We diagonalise $M$ by $M = T \text{diag}(\lambda_1, \lambda_2) T^{-1}$, where $\lambda_1, \lambda_2$ are the eigenvalues of $M$, define the matrix $\Psi$ by

$$\Psi(x) = \Phi(x) T \text{diag}\left\{\exp\left(-\frac{x}{d} \ln \lambda_1\right), \exp\left(-\frac{x}{d} \ln \lambda_2\right)\right\} T^{-1}, \quad x \in (0, d),$$

and extend $\Psi$ by periodicity to $(0, \infty)$. We now claim that

$$\Phi(x) = \Psi(x) T \text{diag}\left\{\exp\left(\frac{x}{d} \ln \lambda_1\right), \exp\left(\frac{x}{d} \ln \lambda_2\right)\right\} T^{-1}, \quad x \in (0, \infty). \quad (13)$$
Note that this form for \( \Phi \) satisfies the equation
\[
\frac{d}{dx} \Phi = A\Phi
\]
at all points \( x \) except \( x = d, 2d, 3d, \ldots \), so it suffices to check continuity at these points. Since \( \Psi \) was extended by \( d \)-periodicity, it is sufficient to check that \( \Psi(d) = \Psi(0) \), which follows from Indeed,
\[
\Psi(d) = \Phi(d)T \text{diag}\{\exp(-\ln \lambda_1), \exp(-\ln \lambda_2)\}^{-1} = \Phi(d)M^{-1} = M\Phi(0)M^{-1} = M1M^{-1} = I,
\]
and \( \Psi(0) = \Phi(0) = I \). Using (13) and multiplying through on the right by \( T \) we find that
\[
\Phi(x) := \Phi(x)T = \Psi(x)T \text{diag}\{\exp\left(-\frac{x}{d} \ln \lambda_1\right), \exp\left(-\frac{x}{d} \ln \lambda_2\right)\}
\]
where \( \tilde{\Psi}(x) := \Psi(x)T \) is \( d \)-periodic by construction of \( \Psi \), so \( \Phi(x) \) is the sought fundamental system. An immediate consequence of the above is that any solution \( U \) to (11) has the form
\[
U(x) = \tilde{\Phi}(x)C = \bar{c}_i \exp\left(-\frac{x}{d} \ln \lambda_1\right) \bar{\phi}_1 + \bar{c}_2 \exp\left(-\frac{x}{d} \ln \lambda_2\right) \bar{\phi}_2,
\]
for some constant column vector \( C = (\bar{c}_1, \bar{c}_2)^\top \) and \( \bar{\phi}_j \), \( j = 1, 2 \), denoting the \( j \)th-column of the matrix \( \tilde{\Phi} \).

### 4.3 Decaying solution in the stratified half-space

Following the reasoning of the previous section and explicitly performing the required calculations, we find that
\[
U(x_3) = \Phi(x_3)U(0), \quad x_3 \in (0, d),
\]
where
\[
\Phi(x_3) = \begin{cases}
\exp(A_1 x_3), & 0 \leq x \leq dh, \\
\exp(A_2(x_3 - dh)) \exp(A_1 dh), & dh \leq x \leq d,
\end{cases}
\]
with \( \alpha_A, \alpha_B \) having the form as in (6) but taking the appropriate material constants in place of the general \( \epsilon, \mu \). Also note that the exponents in (15) have form analogous to (7). In addition \( \Phi(0) = I \) and the periodic extension of \( \Phi \) across the entire half-space gives \( \Phi(x_3 + d) = \Phi(d)\Phi(x_3) \), for \( x_3 \in (0, \infty) \). Hence we invoke the result of the previous section to conclude that a solution of the form (14) exists.

We now define \( \rho_j = \ln(\lambda_j), \ j = 1, 2 \), where \( \lambda_j, \ j = 1, 2 \) are the eigenvalues of \( \Phi(d) \), where we use the principal value of the logarithm As can be seen from (6) the ODE consists of a traceless matrix in both layers, implying that \( \lambda_1 \lambda_2 = 1 \) (equivalently, \( \rho_1 + \rho_2 = 0 \)) and giving two possible cases: either \( \lambda_1, \lambda_2 = \lambda_1^{-1} \in \mathbb{R} \) or \( \lambda_1, \lambda_2 = \lambda_1 \in i\mathbb{R} \), where the bar denotes complex conjugation. In the second case solutions correspond to (non-decaying) oscillatory waves, and so for the purposes of this investigation we ignore these solutions. Therefore, we only pursue the first case, when find that \( \rho_1 = -\rho_2 \), i.e. the general solution \( U \) is a linear combination of two exponentials that decay at \( -\infty \) or \( \infty \). Hence, the application of the decay condition as \( x_3 \to \infty \) implies that the physical solutions \( U \) are those for which
\[
\Phi(d)U(0) = \exp(-\rho I)U(0), \quad \rho > 0.
\]
Assuming non-trivial boundary data at \( x_3 = 0 \), it follows that \( \det(\Phi(d) - \exp(-\rho I)) = 0 \), i.e.
\[
\exp(-2\rho) - \left(S_A S_B \left[\frac{\alpha_A \varepsilon_B}{\alpha_B \varepsilon_A} - \frac{\alpha_B \varepsilon_A}{\alpha_A \varepsilon_B}\right] + 2C_A C_B\right) \exp(-\rho) + 1 = 0,
\]
(17)
where

\[ S_A := \sinh(\alpha_A d h), \quad S_B := \sinh(\alpha_B d (1 - h)), \quad C_A := \cosh(\alpha_A d h), \quad C_B := \cosh(\alpha_B d (1 - h)), \]

and \( \alpha_A, \alpha_B \) are the values of the parameter \( \alpha \) (see (6)) for the materials A and B, respectively.

The equation (17) is solved for those values of \( k, \omega \) that yield real-valued solutions \( \exp(\rho), \exp(-\rho) \) to (17). In general, imposing a specific boundary condition results in a set of curves (“one-dimensional manifolds”) in the \((k, \omega)\) plane for which there exists a surface wave satisfying the boundary conditions.

### 5 LC at the boundary of a half-space

#### 5.1 Two half-spaces in contact

We consider the situation, see Figure 1, where the half-space \( \{x_3 < 0\} \) occupied by a Lorentz material with \( \omega \)-dependent permittivity \( \varepsilon_L \) as in (4) and constant permeability \( \mu_L \), while the complementary half-space \( \{x_3 \geq 0\} \) is occupied by a stratified dielectric as described in Section 4. We study the Maxwell problem in

![Figure 1: Diagram of the full-space Maxwell system.](image)

the entire space where \( \varepsilon, \mu \) in (1) take the values corresponding to the material occupying each region of the space and seek interfacial wave solutions of the form (2). At each interface between materials we impose the standard conditions that the quantities \( E \cdot n, H \cdot n, D \wedge n, B \wedge n \) be continuous, where \( n \) denotes the normal the interface. Seeking solutions of the surface-wave type, we additionally impose the condition of (exponential) decay away from the plane \( \{x_3 = 0\} \). We focus on the TE-mode, using the notation \( U = (E_1, H_2)\top \) (as the component \( D_3 \) can be recovered from an algebraic equation, see Section 3.1). Our approach is to solve the lower and upper half-space problems, and couple the two systems via the shared boundary at \( \{x_3 = 0\} \).

#### 5.2 Equivalent problem in single half-space

Having obtained the solution in the lower half-space (see Section 3.2), we find that the Lorentz material supports a decaying wave when the fields are related at \( \{x_3 = 0\} \) via the LC (9) and when \( \Re(\alpha_L) > 0 \). These conditions capture the effect of the Lorentz material on the full-space system; and so we may solve the equivalent half-space problem with a stratified dielectric, using the LC as a boundary condition (in place of the interface conditions in the full-space system) and seeking surface wave solutions in the upper half-space propagating along \( \{x_3 = 0\} \).

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2As discussed later, it is customary to impose a Leontovich boundary condition \( E \wedge H = \eta \), see [2], [6]. Alternatives include the “metallic” condition \( E_3 = 0 \) is part of the case when the impedance is set to 0.
In Section 4 the solution in the upper half-space was obtained, hence all that remains is to apply [9] as a boundary condition. Substituting into (16) and equating components yields

\[
\exp(-\rho) = C_A C_B + \frac{\varepsilon_A \alpha B}{\varepsilon_B \alpha A} S_A S_B - \frac{\varepsilon_L \varepsilon_0}{\alpha_L} \left( \frac{\alpha_A}{\varepsilon_A} S_A C_B + \frac{\alpha_B}{\varepsilon_B} S_B C_A \right),
\]

(18)

\[
\exp(-\rho) = C_A C_B + \frac{\varepsilon_B \alpha A}{\varepsilon_A \alpha B} S_A S_B - \frac{\alpha_L}{\varepsilon_L \varepsilon_0} \left( \frac{\alpha_A}{\varepsilon_A} S_A C_B + \frac{\alpha_B}{\varepsilon_B} S_B C_A \right).
\]

(19)

We eliminate the variable \(\rho\) from (18–19) and express the dispersion relation in terms of the dimensionless wavenumber \(k = dk\) and phase velocity \(v_p = \omega / k\):

\[
\begin{bmatrix}
\chi_A \varepsilon_B - \chi_B \varepsilon_A \\
\chi_B \varepsilon_A - \chi_A \varepsilon_B
\end{bmatrix}
\hat{S}_A \hat{S}_B + \begin{bmatrix}
\varepsilon_L \varepsilon_0 \chi_A - \chi_L \varepsilon_0 \chi_A \\
\chi_L \varepsilon_0 - \varepsilon_L \varepsilon_0 \chi_B
\end{bmatrix}
\hat{S}_A \hat{C}_B + \begin{bmatrix}
\varepsilon_L \varepsilon_0 \chi_B - \chi_L \varepsilon_0 \chi_B \\
\chi_L \varepsilon_0 - \varepsilon_L \varepsilon_0 \chi_A
\end{bmatrix}
\hat{C}_A \hat{C}_B = 0,
\]

(20)

employing the notation

\[
\chi_A := \sqrt{1 - v_{p}^2 \mu A \varepsilon_A}, \quad \chi_B := \sqrt{1 - v_{p}^2 \mu B \varepsilon_B}, \quad \chi_L := \sqrt{1 - v_{p}^2 \mu_0 \mu_L \varepsilon_L \varepsilon_0},
\]

\[
\hat{S}_A := \sinh(\chi_A \hat{k} h), \quad \hat{S}_B := \sinh(\chi_B \hat{k}(1 - h)), \quad \hat{C}_A := \cosh(\chi_A \hat{k} h), \quad \hat{C}_B := \cosh(\chi_B \hat{k}(1 - h)).
\]

Equation (20) serves as an implicit dispersion relation for solutions of the interfacial-wave type. Those pairs of values \((k, v_p)\) corresponding to surface wave solutions must further satisfy the conditions \(\Re(\alpha_L) > 0\) and \(\rho > 0\), see (8) and (16).

The condition (20) in general provides two constraints on \(\hat{k}\) and \(v_p\), one each from the real and imaginary components being zero. The first term is always real and is not zero identically (with the exception of a homogeneous dielectric occupying \(\{x_3 > 0\}\)), so in general we do not obtain dispersion branches relating \(\hat{k}\) and \(v_p\), but rather a sequence of points. Should the imaginary part of (20) be identically zero, one obtains dispersion branches, as in this case one has a single equation in \((k, \omega)\). One such situation is if the Lorentz material is lossless \((\gamma = 0)\), in which case we obtain a dispersion relation and multiple dispersion branches (see Section 6).

In general, if the imaginary component of the dispersion relation has the form \(F(\gamma, \omega, k)\) for some function \(F\), and one can express (at least locally in \(\omega, k\)) the loss \(\gamma\) in terms of the frequency and wavenumber and substitute it into the real part of (20), to obtain a dispersion relation in \(k\) and \(\omega\) only. We do not pursue this approach analytically, however a selection of numerical solutions \((\gamma, \omega, k)\) to (20) are provided in Figure 2 to illustrate that the plots of such dispersion relations can be obtained.

5.3 Further generalisations to the electromagnetic system

The analysis so far has assumed the magnetic permeability \(\mu_L\) of the Lorentz material to be a fixed material constant, however it can be more generally modeled as \(\omega\)-dependent. In this more general case, \(\mu_L\) takes a form analogous to \(\varepsilon_L\) in (3–4) and is treated as a function of the frequency \(\omega\). This further restricts the triples \((\gamma, \omega, k)\) that support decaying waves, mentioned in the discussion at the end of Section 5.2.

6 Interfacial waves between a lossless Lorentz material and a stratified dielectric

In this section we consider the system where the Lorentz material is assumed lossless, to illustrate that dispersion branches can be obtained provided there is a suitable expression for the loss factor. In the lossless

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3. The TM mode gives a dispersion relation which has similar form to the above, see a discussion at the end of Section 5.3.

4. We can interchange between the pairs \((k, \omega)\) and \((k, v_p)\) freely, so a dispersion relation between one pair gives an equivalent one between the other.
(a) $(\gamma, \omega, k)$ triples for $\gamma \leq 10^{12}$.

(b) $(\gamma, \omega, k)$ triples for $\gamma \geq 10^{21}$.

Figure 2: Solutions $(\gamma, \omega, k)$ to (20). The plots show the dependence of $(\omega, k)$ on the values of $\gamma$, which are provided in the legend. The region $10^{12} \leq \gamma \leq 10^{21}$ is omitted because there is no discernible change in $(\omega, k)$ within this region. Similarly for $\gamma \geq 10^{24}$ and $\gamma \leq 10^{9}$. Parameter values used: $d = 1 \times 10^{-8}$, $h = 0.5$, $\mu_A = \mu_B = \mu_0$, $\varepsilon_A = 5 \varepsilon_0$, $\varepsilon_B = 10 \varepsilon_0$, $\omega_p = 1.3 \times 10^{16}$, and $\omega_0 = 6.077 \times 10^{15}$.

regime, regions in $(\gamma, \omega, k)$ where (20) has non-zero imaginary part are exactly the regions where we do not have a wave decaying into the Lorentz material.\(^5\) The equation (20) takes the form (cf. (3)–(4))

\[
\left[ \frac{\chi A\varepsilon B - \chi B\varepsilon A}{\chi B\varepsilon A} \right] \vec{S}_A \vec{S}_B + \left[ \frac{\varepsilon_r \varepsilon_0 \chi A}{\chi L\varepsilon A} - \frac{\chi L\varepsilon A}{\varepsilon_r \varepsilon_0 \chi A} \right] \vec{S}_A \vec{C}_B + \left[ \frac{\varepsilon_r \varepsilon_0 \chi B}{\chi L\varepsilon B} - \frac{\chi L\varepsilon B}{\varepsilon_r \varepsilon_0 \chi B} \right] \vec{S}_B \vec{C}_A = 0. \tag{21}
\]

The pairs $(\hat{k}, \nu_p)$ that yield surface-wave solutions must further satisfy the conditions $\Re(\alpha_L) > 0$ and $\rho > 0$.

\(^5\)This region is precisely where $\chi_L$ is imaginary due to the term $\nu_p \mu_0 \varepsilon_0 \chi_L$ becoming larger than 1.
Analysing the limit $\omega \to 0$ in (21) reduces the above dispersion relation to a (wholly real) transcendental equation in $k$, showing the potential for standing wave modes to exist at the ends of dispersion branches (that is, surface waves with finite $\hat{k}$ and zero $v_p^6$). Existence of such branches is supported by numerical solutions, one of which is displayed in Figure 3b.

It should also be noted that the decaying-wave dispersion branches can only take values of $(\omega, k)$ that satisfy $k^2 \geq \omega^2 \varepsilon \varepsilon_0 \mu_0 \mu_L$. This is a consequence of $\Re(\alpha_L) > 0$ in the lossless Lorentz case and the effect of this condition can be seen in Figure 3a where the dispersion branches can only lie in the $(\omega, k)$ region as defined. Therefore each wavenumber $k$ does not necessarily support a corresponding decaying wave - taking the limit $k \to 0$ in (21) whilst maintaining finite, non-zero $\omega$ clearly violates this condition; and the point $(\omega, k) = (0, 0)$ does not correspond to a wave solution.

(a) Dispersion branches that support decaying surface waves, when a lossless Lorentz material occupies the lower half-space.

(b) The lowest dispersion branch, supporting a standing wave at $\hat{k} \approx 1.7$.

Figure 3: Dispersion branches obtained numerically for the lossless Lorentz system. Parameter values used: $d = 1 \times 10^{-8}$, $h = 0.5$, $\mu_A = \mu_B = \mu_0$, $\varepsilon_A = 5\varepsilon_0$, $\varepsilon_B = 10\varepsilon_0$, $\omega_p = 1.3 \times 10^{16}$, and $\omega_0 = 6.077 \times 10^{15}$. The values for $\omega_p, \omega_0, \gamma$ are the same as those used for the examples given in [3].

7 Half-space impedance condition

7.1 Classical Leontovich condition as limit of LC

It is common to make approximations to the exact interface conditions in systems similar to those in section 5 by invoking a boundary condition at $\{x_3 = 0\}$ for one of the two half-spaces. One such approximation is the classical Leontovich (or impedance) boundary condition, which requires that at the interface $\{x_3 = 0\}$, the tangential components of the $E$ field ($E_t$) and the $H$ field are related via $E_t = Z(n \times H)$. Here, $n$ denotes the normal vector to the interface (pointing out from the material that is to be neglected, into the remaining material). The quantity $Z$ represents an impedance, and more generally has the form $Z = \sqrt{\mu/\varepsilon}$ for some permittivity and permeability. In this section we show that the LC can be used to recover the classical Leontovich condition.

Recall that the LC (and in particular the generalised impedance) has the form in [9] for the TE mode.
and (10) for the TM mode. One can express the impedance as follows:

\[-\frac{i\alpha_L}{\omega\varepsilon_0} = -\frac{i}{\omega\varepsilon_0} \sqrt{k^2 - \omega^2 \mu_L \mu_0 \varepsilon_L \varepsilon_0} = -i \sqrt{\frac{\mu_0 \mu_L}{\varepsilon_0 \varepsilon_0}} \sqrt{\frac{k^2}{\omega^2 \mu_L \mu_0 \varepsilon_L \varepsilon_0} - 1},\]

in the case of the TE mode, and

\[-\frac{i\mu_0 \mu_L}{\alpha_L} = -i \left( \frac{\mu_0 \mu_L}{\varepsilon_0 \varepsilon_0} \left( \frac{k^2}{\omega^2 \mu_0 \mu_L \varepsilon_0} - 1 \right)^{-\frac{1}{2}} \right)\]

for the TM mode. For the case of constant \(\mu_L\) and provided

\[\left| \frac{k^2}{\omega^2 \varepsilon_L} \right| \ll 1,\]

we obtain the classical Leontovich condition as an approximation, up to the order \(O(\left| \varepsilon_L^2 - k^2 \right|)\), to the condition presented in (9) and (10).\(^7\) To conclude, we note that in [6] the Leontovich condition is purported to be derived under the condition that \(|\varepsilon_L| \ll 1\), which coincides with (22) under the assumption that \(\omega/k\) is bounded above and below.

### 7.2 Homogeneous half-space with classical Leontovich boundary condition

Under the assumption that the stratified dielectric half-space is actually homogeneous, (20) is reduced to the equation

\[\sinh(\chi A k) \left( \frac{\varepsilon_0 \varepsilon_A \chi A}{\varepsilon_A \chi L} - \frac{\varepsilon_A \chi L}{\varepsilon_L \varepsilon_0 \chi A} \right) = 0,\]

by setting \(\varepsilon_A = \varepsilon_B\) and \(\mu_A = \mu_B\) in (20).\(^8\) This equation has solutions for when either factor is zero, in the case of the sinh factor this happens only when \(\chi_A = 0\) which does not correspond to a decaying wave into the dielectric (the solution (7) has constant amplitude). In the case of the latter factor we may rearrange and obtain the dispersion relation in \(\omega\) and \(k\) as

\[\omega^2 = \frac{k^2}{\mu_A \varepsilon_0 \varepsilon_L - \mu_0 \mu_L \varepsilon_A} \left( \frac{\varepsilon_0 \varepsilon_L}{\varepsilon_A} - \frac{\varepsilon_A}{\varepsilon_0 \varepsilon_L} \right).\]

Note that for a general Lorentz material (\(\tau\)) still has non-zero real and imaginary part, due to the presence of \(\varepsilon_L\). Therefore, the discussion in section 5.2 is still applicable here, although it is now possible to rearrange and obtain \(\gamma\) as a function of \(\omega\) and \(k\).

One can also obtain results concerning homogeneous dielectric systems with classical Leontovich boundary conditions from the more general system presented in section 5, using the fact that the dispersion relation collapses to (\(\tau\)). In the regime of bounded \(k/\omega\) and with \(|\varepsilon_L| \ll 1\), the dispersion relation (\(\tau\)) reduces, to leading order, to the relation

\[\omega = \frac{k c}{\sqrt{\frac{\varepsilon_A \mu_A}{\varepsilon_0 \mu_0} \left( 1 - \frac{\mu_0 \mu_L}{\varepsilon_0 \varepsilon_L} \frac{\varepsilon_A}{\mu_A} \right)}},\]

where \(c\) denotes the light speed in vacuum. This regime is the same as that explored in [2]; and indeed we can perform transformations on the material constants to reconcile the differences in notation:

\[\mu_A \rightarrow \mu, \quad \varepsilon_A \rightarrow \varepsilon, \quad \eta \rightarrow \sqrt{\frac{\mu_0 \mu_L}{\varepsilon_0 \varepsilon_L}}.\]

\(^7\)Noting that the form for \(Z\) in our case is \(Z = \sqrt{\mu_0 \mu_L / \varepsilon_0 \varepsilon_L}\).

\(^8\)This isn’t the only way to obtain a homogeneous half space as a limit of the stratified system; we could take \(h \rightarrow 0\) (or 1).

\(^9\)By substituting for \(\chi_L\) in and then \(v_p\).
Together with the condition of entirely imaginary impedance assumed in [2], namely \( \eta = i |\eta| \), and recalling that \( c = 1/\sqrt{\mu_0 \varepsilon_0} \), we obtain [2, Eq. 9]. One can consider the TM mode analogously and obtain [2, Eq. 7].

### 8 Conclusion

In our analysis of the full-space problem, we have shown that it is possible to obtain a dispersion relation as in [20] that has in general real and imaginary part, providing only finitely many points \((\omega, k)\) that support surface waves. We postulate that the imaginary component of the dispersion relation has the form \( F(\gamma, \omega, k) = 0 \), and that if is permits manipulation for \( \gamma \) in terms of \( \omega \) and \( k \), one can obtain dispersion branches via substitution into the real part of the dispersion relation. In general however, we note that the form of \( F(\gamma, \omega, k) = 0 \) is unlikely to admit a closed form for \( \gamma \) as a function of the other two variables. Under certain restrictions, such as the case of a lossless Lorentz material (see section 6) one can obtain dispersion branches \((\omega, k)\) that support (decaying) surface waves. Examples are given in Figure 3, and the possibility of obtaining standing waves is noted.

The more general interface conditions [9], [10] for the Maxwell system [1] have been derived under no more general assumptions on the lower (Lorentz) half-space than it having \( \omega \)-dependent permittivity as in [3] and constant permeability. This boundary condition serves an analogous role to the classical Leontovich condition [7,1], in that it allows one to reduce a full-space problem with an interface to a half-space problem, with a boundary condition derived from one of the constituent media. In the more general case the data for the generalised impedance comes from the exterior Lorentz material; and other material parameters from the half-space to which the problem is reduced. In this sense, the approach can be viewed as a combination of the perspectives of [6] and [2]: in the former, the impedance boundary condition is derived for the Maxwell equations in what would be the analogue of our Lorentz half-space and in the latter these conditions were postulated in the complementary, dielectric, half-space. Here we have derived the boundary conditions on the interface between the two, by considering the corresponding full-space problem.

As a final remark, we are hopeful that this analysis admits further generalisation to a system with a curved interface; and motivates further study into the surface waves supported by the general dispersion relation [20] and its TM-mode analogue.

### Appendix: Lorentz medium replaced by homogeneous dielectric

If the stronger assumption is made that the Lorentz material is simply a dielectric, we obtain dispersion relations as plotted in Figure 4.

### 9 Acknowledgements

KC is supported by Engineering and Physical Sciences Research Council: Grant EP/L018802/2 “Mathematical foundations of metamaterials: homogenisation, dissipation and operator theory”.

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Figure 4: Dispersion branches for the Dielectric/Stratified Dielectric system. There is both a cut-off frequency and wavenumber for all branches in this system, where the branch no longer corresponds to interfacial waves. Parameter values used: \(d = 1 \times 10^{-9}\), \(h = 0.5\), \(\mu_A = \mu_B = 1\), \(\varepsilon_A = 10\varepsilon_0\), \(\varepsilon_B = 4.9\varepsilon_0\), and the (now dielectric) Lorentz material taken as vacuum.

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