Properties of $Q\bar{Q}$ mesons in non-relativistic QCD formalism

Ajay Kumar Rai†, Bhavin Patel∗ and P C Vinodkumar

†Physics Section, Applied Sciences and Humanities Department, Sardar Vallabh
National Institute of Technology, Surat-395 007, Gujarat, India.

∗Department of Physics, Sardar Patel University, Vallabh Vidyanagar,
Anand - 388 120, Gujarat, India.

Email: raiajayk@rediffmail.com

Abstract

The decay rates of $Q\bar{Q}$ mesons ($Q \in c, b$) are studied in the NRQCD formalism in terms of their short distance and long distance coefficients. The long distance coefficients are obtained through phenomenological potential model description of the mesons. The model parameters that reproduce the mass spectrum of the $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$ mesons are employed to study the decay widths of these mesons. We extract the mass spectrum as well as the reproduces the respective radial wave functions from the different potential models as well as from non-relativistic phenomenological quark antiquark potential of the type $V(r) = -\frac{\alpha}{r} + Ar^\nu$, with $\nu$ varying from 0.5 to 2. The spin hyperfine and spin-orbit interactions are employed to obtain the masses of the pseudoscalar and vector mesons. The decay constants with QCD corrections are computed in this model as well as in the case of other potential models for comparison. The digamma and dileptonic decays of $c\bar{c}$, and $b\bar{b}$ mesons are investigated using some of the known potential models without and with radiative corrections up to the lowest order. These decay widths are also computed within the NRQCD formalism up to $O(\nu^4)$ by making uses of the respective spectroscopic parameters of the models. Our theoretical predictions of the decays of the $c\bar{c}$, and $b\bar{b}$ mesons and the results obtained from some of the other potential schemes are compared with the experimental values. The partial widths and life time of the $B_c$ meson are also computed using the model parameters and are found to be in good accordance with the experimental values.

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1 Introduction

Recently, there have been renewed interest in the spectroscopy of the heavy flavoured hadrons due to number of experimental facilities (CLEO, DELPHI, Belle, BaBar, LHCb etc) which have been continuously providing and expected to provide more accurate and new informations about the hadrons from low flavour to heavy flavour sector [1, 2].

The heavy flavour mesons are those in which at least one of the quark or antiquark or both the quark and antiquark belong to heavy flavour sector; particularly the charm or beauty. They are represented by $Q\bar{Q}$ mesonic systems which include the quarkonia ($c\bar{c}$ and $b\bar{b}$) and $B_c$ ($b\bar{c}$ or $c\bar{b}$) mesons. The investigation of the properties of these mesons gives very important insight into heavy quark dynamics. Heavy quarkonia have a rich spectroscopy with many narrow states lying under the threshold of open flavour production [3, 4].

The success of theoretical model predictions with experiments can provide important information about the quark-antiquark interactions. Such information is of great interest, as it is not possible to obtain the $Q\bar{Q}$ potential starting from the basic principle of the quantum chromodynamics (QCD) at the hadronic scale. In this scale it is necessary to account for non-perturbative effects connected with complicated structure of QCD vacuum. All this lead to a theoretical uncertainty in the $Q\bar{Q}$ potential at large and intermediate distances. It is just in this region of large
and intermediate distances that most of the basic hadron resonances are formed. Among many theoretical attempts or approaches to explain the hadron properties based on its quark structure very few were successful in predicting the hadronic properties starting from mass spectra to decay widths. For the mass predictions, the nonrelativistic potential models with Buchmüller and Tye [5], Martin [6, 7, 8], Log [9, 10], Cornell [11] etc., were successful at the heavy flavour sectors while the Bethe-Salpeter approach under harmonic confinement [12] was successful at low flavour sector. There exist relativistic approaches for the study of the different hadronic properties [13, 14]. The non-relativistic potential model has been successful for $\psi$ and $\Upsilon$ families, while the relativistic approaches yield better results in the lighter sector. Some potential models have also predicted the masses and various decays of the heavy-heavy mesons which are in fair agreement with the experimental results [15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. A comprehensive review of developments in heavy quarkonium physics is available in ref [25].

The new role of the heavy flavour studies as the testing ground for the non-perturbative aspects of QCD, demands extension of earlier phenomenological potential model studies on quarkonium masses to their predictions of decay widths with the non-perturbative approaches like NRQCD.

The decay rates of the heavy-quarkonium states into photons and pairs of leptons are among the earliest applications of perturbative quantum chromodynamics (QCD) [26, 27]. In these analysis, it was assumed that the decay rates of the meson factored into a short-distance part that is related to the annihilation rate of the heavy quark and antiquark, and long-distance factor containing all nonperturbative effects of the QCD. The short-distance factor calculated in terms of the running coupling constant $\alpha_s(m_Q)$ of QCD, evaluated at the scale of the heavy-quark mass $m_Q$, while the long-distance factor was expressed in terms of the meson’s nonrelativistic wave function, or its derivatives, evaluated at origin. In case of S-wave decays [28, 31, 29, 30] and in case of P-wave decays into photons [32], the factorization assumption was supported by explicit calculations at next-to-leading order in $\alpha_s$. However, no general argument advanced for its validity in higher orders of perturbation theory. These divergence cast a shadow over applications of perturbative QCD to the calculation of annihilation rates of the heavy quarkonium states.

In this context, an elegant effort was provided by the NRQCD formalism [33]. It consists of a nonrelativistic Schrodinger field theory for the heavy quark and antiquark that is coupled to the usual relativistic field theory for light quarks and gluons. NRQCD not only organize calculation of all orders in $\alpha_s$, but also elaborate systematically the relativistic corrections to the conventional formula. Furthermore, it also provides nonperturbative definitions of the long-distance factors in terms of matrix elements of NRQCD, making it possible to evaluate them in the numerical lattice calculations. Analyzing S-wave decays within this frame work, it recover, at leading order in $v^2$, standard factorization formulae, which contain a single nonperturbative parameter. At next to leading order in $v^2$, the decay rates satisfy a more general factorization formula, which contain two additional independent nonperturbative matrix elements related to their radial wave functions.

Our attempt in this paper would be then to study the heavy-heavy flavour mesons in the charm and beauty sector in a general frame work of the potential models. The model parameters used for the predictions of the masses and their radial wave functions would be used for the study of their decay properties using NRQCD formalism.

For completion, we present a detail analysis of mass spectra of $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ mesons in the potential scheme of coulomb plus power potential ($CPP_\nu$) with the power index ($\nu$), varying from 0.5 to 2. Spin hyperfine and spin orbit interactions are introduced to get the S-wave and P-wave
masses of the pseudoscalar and vector mesons. We present details of the non-relativistic treatment of the heavy quarks along with the computed results in section-2. The decay constants $f_{P,V}$ of these mesons incorporating QCD corrections up to $O(\alpha_s)$ are presented in section-3. The weak decay of the $B_c$ meson and its life time is computed in section-4, while in section-5 we present the details of the computations of the di-gamma decays of pseudoscalar states and the leptonic decay widths of the vector states of the $c\bar{c}$ and $b\bar{b}$ quarkonia in the frame work of the NRQCD formalism as well as other treatments incorporating different correction terms to the respective decay widths. Though the NRQCD formalism takes advantage of the fact that heavy quark mass is much larger than the other energy scales such as the binding energy scale, $\Lambda_{QCD}$ and $|\vec{p}|$, the energy fluctuations of the heavy quarks of the order of the light energy scale are implemented in pNRQCD \cite{34, 35, 36}. A comprehensive comparison of the results are presented in this section. Finally we draw our conclusions in section-6 of this paper.

2 Nonrelativistic Treatment for $Q\bar{Q}$ systems

Even though there are attempts based on the relativistic theory like the light front approach for the study of the heavy flavoured quarks, under non-relativistic approximations, they reproduces the results of the non-relativistic quark-potential models \cite{37}. In the center of mass frame of the heavy quark-antiquark system, the momenta of quark and antiquark are dominated by their rest mass $m_Q, m_{\bar{Q}} \gg \Lambda_{QCD} \sim |\vec{p}|$, which constitutes the basis of the non-relativistic treatment. In NRQCD, the velocity of heavy quark is chosen as the expansion parameter \cite{38}.

Hence, for the study of heavy-heavy bound state systems such as $c\bar{c}$, $c\bar{b}$, $b\bar{b}$, we consider a nonrelativistic Hamiltonian given by \cite{21, 22, 23}

$$H = M + \frac{p^2}{2m} + V(r)$$

where

$$M = m_Q + m_{\bar{Q}}, \quad \text{and} \quad m = \frac{m_Q m_{\bar{Q}}}{m_Q + m_{\bar{Q}}} \quad (2)$$

$m_Q$ and $m_{\bar{Q}}$ are the mass parameters of quark and antiquark respectively, $p$ is the relative momentum of each quark and $V(r)$ is the quark antiquark potential. Though linear plus coulomb potential is a successful well studied non-relativistic model for heavy flavour sector, their predictions for decay widths are not satisfactory owing to the improper value of the radial wave function at the origin compared to other models \cite{22}. Recently, we have considered a general power potential with colour coulomb term of the form

$$V(r) = -\frac{\alpha_c}{r} + Ar^\nu \quad (3)$$

as the static quark-antiquark interaction potential ($CPP_\nu$). Here, for the study of mesons, $\alpha_c = \frac{4}{3}\alpha_s$, $\alpha_s$ being the strong running coupling constant, $A$ is the potential parameter and $\nu$ is a general power, such that the choice, $\nu = 1$ corresponds to the coulomb plus linear potential. This potential belong to the special choices of the generality of the potentials, $V(r) = -Cr^\alpha + Dr^\beta + V_0$ \cite{39, 40, 41} with $V_0 = 0 \alpha = -1$, $\beta = \nu$. Choices of the power index in the range $0.5 \leq \nu \leq 2.0$ have been explored for the present study. The different choices of $\nu$ here, correspond to different potential forms. Thus, the potential parameter $A$ can also be different numerically and dimensionally for each choices of $\nu$. The properties of the light-heavy flavour mesons have been calculated using the Gaussian trial wave function \cite{21}. Masses and decay constant of the light-heavy systems are found to be in agreement with the experimental results for the choice of $\nu \approx 0.5$. However, in the case of heavy-heavy systems
the predictions of the masses were satisfactory but the decay constants and decay rates were not predicted satisfactorily \[22\]. Hence for the present study of heavy-heavy flavour mesons, we employ the exponential trial wave function of the hydrogenic type to generate the Schrödinger mass spectra. Within the Ritz variational scheme using the trial radial wave function we obtain the expectation values of the Hamiltonian as \((\langle H \rangle = E(\mu, \nu) )\)

\[
E(\mu, \nu) = M + \frac{\mu^2}{8m} + \frac{1}{2} \left( -\mu \alpha_c + A \frac{\Gamma(\nu + 3)}{\mu^\nu} \right) \tag{4}
\]

Eqn(4) gives the spin average mass of the ground state. For excited states the trial wave function is multiplied by an appropriate orthogonal polynomial function such that the excited trial wave function gets orthonormalized. So, it is straightforward to assume the trial wave function for the \((n, l)\) state to be the form given by the hydrogenic radial wave function,

\[
R_{nl}(r) = \left( \frac{\mu^3(n-l-1)!}{2n(n+l)!} \right)^{1/2} (\mu r)^l e^{-\mu r/2} L_{n-l-1}^{2l+1}(\mu r) \tag{5}
\]

Here, \(\mu\) is the variational parameter and \(L_{n-l-1}^{2l+1}(\mu r)\) is Laguerre polynomial. For a chosen value of \(\nu\), the variational parameter, \(\mu\) is determined for each state using the virial theorem

\[
\left\langle \frac{P^2}{2m} \right\rangle = \frac{1}{2} \left\langle \frac{rdV}{dr} \right\rangle \tag{6}
\]

As the interaction potential assumed here does not contain the spin dependent part, Eqn (4) gives the spin average masses of the system in terms of the power index \(\nu\). The spin average mass for the ground state is computed for the values of \(\nu\) from 0.5 to 2. We have taken the quark mass parameters \(m_b = 4.66 \text{ GeV}\) and \(m_c = 1.31 \text{ GeV}\). The potential parameter \(A\) are fixed for each choices of \(\nu\) so as to get the experimental ground state masses of \(c\bar{c}, \, c\bar{b}\) and \(b\bar{b}\) mesons. The parameters and the fitted values of \(A\) for different systems are listed in Table I. The experimental spin average masses are computed from the experimental masses of the pseudoscalar and vector mesons using the relation,

\[
M_{SA} = M_P + \frac{3}{4}(M_V - M_P) \tag{7}
\]

For the \(nJ\) state, we compute the spin-average or the center of weight mass from the respective experimental values as

\[
M_{CW,nJ} = \frac{\sum J 2(2J + 1) M_{nJ}}{\sum J 2(2J + 1)} \tag{8}
\]

The fitted value of \(A\) for each case of the power index \(\nu\) along with other model parameters are tabulated in Table I for \(c\bar{c}, \, c\bar{b}\) and \(b\bar{b}\) systems. The ground state center of weight mass of 3.068 GeV, 6.320 GeV and 9.453 GeV are used to fit the \(A\) values for \(c\bar{c}, \, c\bar{b}\) and \(b\bar{b}\) systems respectively. The values of \(A(\nu)\) thus obtained for each case of mesonic systems are then used to predict the higher \(S\) and \(P\)-wave masses (See Tables 2,7). The fitted values of \(A\) for each \(\nu\) in the case of the heavy-heavy flavour mesons are plotted in Fig ???. It is interesting to note that all the three plots intersect each other at \(\nu = 1.1\) at the value of the parameter \(A\) around 0.151 GeV\(\nu+1\). It can also be seen that the parametric values of \(A\) for \(c\bar{c}\) and \(c\bar{b}\) systems are close to each other, while in the case of \(b\bar{b}\), they are distinctly different except at \(\nu = 1.1\). It reflects the fact that potential parameter \(A\) becomes independent of the distinct energy scales of these heavy mesons at around \(\nu = 1.1\).

In the case of \(c\bar{c}\) and \(c\bar{b}\) systems, the values of the parameter \(A\) are numerically very close to each other in the range of potential index 0.9 to 1.3. The predicted masses are also found to be in good agreement with the existing experimental states in range of power index 0.9 to 1.3 of the
Table 1: Parameter A for mesons in (GeV\(^{\nu+1}\))

| \(\nu\) | \(A (cc)\) | \(A (bc)\) | \(A (bb)\) |
|---|---|---|---|
| 0.5 | 0.276 | 0.250 | 0.195 |
| 0.7 | 0.225 | 0.212 | 0.179 |
| 0.9 | 0.185 | 0.179 | 0.165 |
| 1.0 | 0.167 | 0.165 | 0.158 |
| 1.1 | 0.151 | 0.152 | 0.151 |
| 1.3 | 0.124 | 0.129 | 0.138 |
| 1.5 | 0.101 | 0.109 | 0.126 |
| 1.7 | 0.082 | 0.092 | 0.114 |
| 1.9 | 0.067 | 0.077 | 0.103 |
| 2.0 | 0.060 | 0.070 | 0.098 |

\(\alpha_c(cc) = 0.40\), \(\alpha_c(bc) = 0.34\), \(\alpha_c(bb) = 0.30\),
m\(_c\) = 1.31 GeV and m\(_b\) = 4.66 GeV

potential. Fig ?? shows the behaviour of \(|R_{1S}(0)|\) with the potential index \(\nu\) for all the three (cc, cb and bb) mesons. Like other potential model predictions of the wave functions (at the origin) of bc system lie in between those of cc and bb systems. We obtained a model independent relationship similar to the one given by [42] as

\[ |\psi_{bb}|^2 \approx |\psi_{cc}|^{2(1-q)} |\psi_{bb}|^{2q} \]  

(9)

with \(q = 0.3\). This relation provides the 1S wave function at the origin within 2% variation for the choices of the potential range \(0.5 \leq \nu \leq 2\). For 2S and 3S states we find the relation hold within 5% for all values for \(\nu\) studied here. It is to be noted here that such a scaling law with smaller percentage variations exist here even though the potential contains coulomb part.

### 2.1 Spin-Hyperfine and Spin-Orbit Splitting in Heavy-Heavy Flavour Mesons

In general, the quark-antiquark bound states are represented by \(n^{2S+1}L_J\), identified with the \(J^{PC}\) values, with \(\vec{J} = \vec{L} + \vec{S}\), \(\vec{S} = \vec{S}_Q + \vec{S}_{\bar{Q}}\), parity \(P = (-1)^{L+1}\) and the charge conjugation \(C = (-1)^{L+S}\) and (n, L) is the radial quantum numbers. So the S-wave (\(L = 0\)) bound states are represented by \(J^{PC} = 0^{++}\) and \(1^{--}\) respectively. And the P-wave (\(L = 1\)) states are represented by \(J^{PC} = 1^{++}\) with \(L = 1\) and \(S = 0\) while \(J^{PC} = 0^{++}\), \(1^{++}\) and \(2^{++}\) correspond to \(L = 1\) and \(S = 1\) respectively. Accordingly, the spin-spin interaction among the constituent quarks provides the mass splitting of \(J = 0^{++}\) and \(1^{--}\) states, while the spin-orbit interaction provides the mass splitting of \(J^{PC} = 0^{++}\), \(1^{++}\) and \(2^{++}\) states. The \(J^{PC} = 1^{++}\) state with \(L = 1\) and \(S = 0\) represents the center of weight mass of the P-state as its spin-orbit contribution becomes zero, while the two \(J = 1^{++}\) singlet and the \(J = 1^{++}\) of the triplet P-states form a mixed state. We add separately the spin-dependent part of the usual one gluon exchange potential (OGEP) between the quark antiquark for computing the hyperfine and spin-orbit shifting of the low-lying S-states and P-states. Accordingly, the spin-spin and spin-orbit interactions are taken as [43]

\[ V_{S_Q} \cdot S_Q(r) = \frac{8}{3} \frac{\alpha_s}{m_Q m_{\bar{Q}}} \vec{S}_Q \cdot \vec{S}_{\bar{Q}} \frac{4\pi \delta(r)}{r^3} \]  

(10)

\[ V_L \cdot S(r) = \frac{4}{3} \frac{\alpha_s}{m_Q m_{\bar{Q}}} \frac{\vec{L} \cdot \vec{S}}{r^3} \]  

(11)
Table 2: Wave function at the origin ($|R(0)|$) and spin average masses of S-wave $c\bar{c}$ meson

| State | $\nu$ | $\bar{\mu}$ | $|R(0)|$ | $E(\bar{\mu})$ | Exp. | Theory |
|-------|-------|-------------|----------|----------------|------|--------|
|       | GeV   | GeV$^{3/2}$ | GeV      | (GeV)          | (GeV)| (GeV)  |
| 0.5   | 1.068 | 0.781       | 3.068    |                |      |        |
| 0.7   | 1.177 | 0.903       | 3.068    |                |      |        |
| 0.9   | 1.264 | 1.005       | 3.068    |                |      |        |
| 1.0   | 1.300 | 1.049       | 3.068    |                |      |        |
| 1.1   | 1.335 | 1.090       | 3.068    | 3.068          | 3.068|        |
|       |       |             |          |                |      |        |
|       | 1.3   | 1.394       | 1.164    | 3.068          |      | 3.068b |
|       | 1.5   | 1.445       | 1.228    | 3.068          |      |        |
|       | 1.7   | 1.489       | 1.285    | 3.068          |      |        |
|       | 1.9   | 1.528       | 1.336    | 3.068          |      |        |
|       | 2.0   | 1.546       | 1.360    | 3.068          |      |        |
| 2S    |       |             |          |                |      |        |
| 0.5   | 1.057 | 0.384       | 3.368    |                |      |        |
| 0.7   | 1.242 | 0.489       | 3.454    |                |      |        |
| 0.9   | 1.403 | 0.588       | 3.534    |                |      |        |
| 1.0   | 1.473 | 0.632       | 3.567    |                |      |        |
| 1.1   | 1.540 | 0.676       | 3.601    | 3.663          | 3.662|        |
|       |       |             |          |                |      |        |
|       | 1.3   | 1.660       | 0.757    | 3.661          |      | 3.674b |
|       | 1.5   | 1.765       | 0.829    | 3.713          |      |        |
|       | 1.7   | 1.856       | 0.894    | 3.756          |      |        |
|       | 1.9   | 1.939       | 0.955    | 3.796          |      |        |
|       | 2.0   | 1.976       | 0.982    | 3.814          |      |        |
| 3S    |       |             |          |                |      |        |
| 0.5   | 1.097 | 0.271       | 3.550    |                |      |        |
| 0.7   | 1.333 | 0.363       | 3.712    |                |      |        |
| 0.9   | 1.545 | 0.453       | 3.870    |                |      |        |
| 1.0   | 1.640 | 0.495       | 3.940    |                |      |        |
| 1.1   | 1.732 | 0.537       | 4.012    | 4.040          | 4.064|        |
|       |       |             |          |                |      |        |
|       | 1.3   | 1.901       | 0.618    | 4.146          |      | 4.073b |
|       | 1.5   | 2.051       | 0.692    | 4.266          |      |        |
|       | 1.7   | 2.186       | 0.762    | 4.373          |      |        |
|       | 1.9   | 2.309       | 0.827    | 4.473          |      |        |
|       | 2.0   | 2.365       | 0.857    | 4.518          |      |        |

Ebert (a) $\rightarrow$ [14], Pandya (b) $\rightarrow$ [19]
Table 3: Derivative of wave function at the origin ($|R'(0)|$) and P-wave masses of $c\bar{c}$ meson

| State | $\nu$ | $\bar{\mu}$ | $|R'(0)|$ | $E(\bar{\mu})$ | Exp. | Theory | Ebert (a) → | Pandya (b) → |
|-------|-------|-------------|-------------|---------------|------|--------|-------------|-------------|
|       | GeV   | GeV$^{5/2}$ | (GeV)       | (GeV)         | (GeV)| (GeV)  | 14          | 19          |
| 0.5   | 1.024 | 0.217       | 3.313       |               |      |        |             |             |
| 0.7   | 1.180 | 0.309       | 3.373       |               |      |        |             |             |
| 0.9   | 1.331 | 0.417       | 3.426       |               |      |        |             |             |
| 1P    |       |             |             |               |      |        |             |             |
| 1.0   | 1.392 | 0.467       | 3.450       |               |      |        |             |             |
| 1.1   | 1.450 | 0.517       | 3.473       | 3.525         | 3.526$^a$ | 3.497$^b$ |             |             |
| 1.3   | 1.553 | 0.614       | 3.513       |               |      |        |             |             |
| 1.5   | 1.642 | 0.705       | 3.547       |               |      |        |             |             |
| 1.7   | 1.720 | 0.792       | 3.576       |               |      |        |             |             |
| 1.9   | 1.790 | 0.875       | 3.603       |               |      |        |             |             |
| 2.0   | 1.823 | 0.916       | 3.615       |               |      |        |             |             |
| 2P    |       |             |             |               |      |        |             |             |
| 1.0   | 1.560 | 1.013       | 3.872       |               |      |        |             |             |
| 1.1   | 1.687 | 1.232       | 3.936       |               |      |        |             |             |
| 1.3   | 1.847 | 1.545       | 4.055       |               |      |        |             |             |
| 1.5   | 1.990 | 1.862       | 4.162       |               |      |        |             |             |
| 1.7   | 2.117 | 2.174       | 4.257       |               |      |        |             |             |
| 1.9   | 2.232 | 2.481       | 4.345       |               |      |        |             |             |
| 2.0   | 2.285 | 2.631       | 4.385       |               |      |        |             |             |

Ebert (a) → [14], Pandya (b) → [19]
Table 4: Wave function at the origin ($|R(0)|$) and spin average masses of S-wave $b\bar{c}$ meson

| State | $\nu$ | $\bar{\mu}$ | $|R(0)|$ | $E(\bar{\mu})$ | Theory |
|-------|------|----------|---------|--------------|--------|
|       | GeV  | GeV$^{3/2}$ | (GeV)   | (GeV)        |        |
| 0.5   | 1.281| 1.025    | 6.320   |              |        |
| 0.7   | 1.404| 1.176    | 6.320   |              |        |
| 0.9   | 1.502| 1.301    | 6.320   |              |        |
| 1S    | 1.0  | 1.544    | 1.357   | 6.320        | 6.317$^a$ |
| 1.1   | 1.583| 1.408    | 6.320   | 6.319$^c$    |        |
| 1.3   | 1.652| 1.502    | 6.320   |              |        |
| 1.5   | 1.711| 1.583    | 6.320   |              |        |
| 1.7   | 1.763| 1.655    | 6.320   |              |        |
| 1.9   | 1.807| 1.718    | 6.320   |              |        |
| 2.0   | 1.825| 1.744    | 6.320   |              |        |
| 0.5   | 1.236| 0.486    | 6.589   |              |        |
| 0.7   | 1.453| 0.619    | 6.665   |              |        |
| 0.9   | 1.635| 0.739    | 6.730   |              |        |
| 2S    | 1.0  | 1.719    | 0.797   | 6.761        |        |
| 1.1   | 1.796| 0.851    | 6.790   |              |        |
| 1.3   | 1.938| 0.954    | 6.844   | 6.869$^a$    |        |
| 1.5   | 2.061| 1.046    | 6.890   | 6.888$^c$    |        |
| 1.7   | 2.172| 1.132    | 6.931   |              |        |
| 1.9   | 2.266| 1.206    | 6.965   |              |        |
| 2.0   | 2.307| 1.239    | 6.977   |              |        |
| 0.5   | 1.274| 0.339    | 6.746   |              |        |
| 0.7   | 1.550| 0.455    | 6.889   |              |        |
| 0.9   | 1.792| 0.565    | 7.021   |              |        |
| 3S    | 1.0  | 1.905    | 0.620   | 7.085        |        |
| 1.1   | 2.012| 0.674    | 7.147   |              |        |
| 1.3   | 2.211| 0.775    | 7.265   | 7.224$^a$    |        |
| 1.5   | 2.389| 0.870    | 7.372   | 7.271$^c$    |        |
| 1.7   | 2.550| 0.960    | 7.471   |              |        |
| 1.9   | 2.692| 1.041    | 7.555   |              |        |
| 2.0   | 2.755| 1.078    | 7.590   |              |        |

Ebert $\rightarrow [14]$; Eichten $\rightarrow [43]$
Table 5: Derivative of wave function at the origin (|R′(0)|) and P-wave masses of c\(\bar{b}\) meson

| State | ν | \(\bar{\mu}\) | \(|R′(0)|\) | \(E(\bar{\mu})\) | Theory |
|-------|---|----------------|-------------|---------------|--------|
|       | GeV | GeV⁵/² | (GeV) | (GeV) |
| 0.5   | 1.198 | 0.321 | 6.542 |
| 0.7   | 1.392 | 0.467 | 6.597 |
| 0.9   | 1.553 | 0.613 | 6.641 |
| 1P    | 1.0 | 1.625 | 0.687 | 6.662 |
|       | 1.1 | 1.692 | 0.760 | 6.682 |
|       | 1.3 | 1.814 | 0.905 | 6.718 | 6.749⁺ |
|       | 1.5 | 2.920 | 1.042 | 6.749 | 6.736⁻ |
|       | 1.7 | 2.013 | 1.173 | 6.777 |
|       | 1.9 | 2.094 | 1.296 | 6.799 |
|       | 2.0 | 2.129 | 1.349 | 6.806 |
| 0.5   | 1.260 | 0.594 | 6.720 |
| 0.7   | 1.520 | 0.950 | 6.851 |
| 0.9   | 1.751 | 1.353 | 6.969 |
| 2P    | 1.0 | 1.859 | 1.571 | 7.027 |
|       | 1.1 | 1.961 | 1.794 | 7.082 |
|       | 1.3 | 2.149 | 2.257 | 7.188 | 7.145⁺ |
|       | 1.5 | 2.317 | 2.725 | 7.283 | 7.142⁻ |
|       | 1.7 | 2.469 | 3.184 | 7.370 |
|       | 1.9 | 2.603 | 3.644 | 7.445 |
|       | 2.0 | 2.662 | 3.852 | 7.476 |

⁺Ebert → [14];⁻Eichten → [43]
Table 6: Wave function at the origin ($|R(0)|$) and spin average masses of S-wave $b\bar{b}$ meson

| State | $\nu$ | $\bar{\nu}$ | $|R(0)|$ | $E(\bar{\nu})$ | Theory |
|-------|------|---------------|--------|----------------|--------|
|       | $\text{GeV}$ | $\text{GeV}^{3/2}$ | (GeV) | (GeV) | (GeV) |
| 0.5   | 1.985 | 1.977         | 9.453  |     |     |
| 0.7   | 2.122 | 2.186         | 9.453  |     |     |
| 0.9   | 2.238 | 2.368         | 9.453  |     |     |
| 1S    | 1.0   | 2.288         | 2.447  | 9.453 |     |
|       | 1.1   | 2.336         | 2.525  | 9.453 | 9.445 |
|       | 1.3   | 2.402         | 2.662  | 9.453 | 9.453 |
|       | 1.5   | 2.491         | 2.780  | 9.453 | 9.453 |
|       | 1.7   | 2.554         | 2.885  | 9.453 |     |
|       | 1.9   | 2.611         | 2.982  | 9.453 |     |
|       | 2.0   | 2.638         | 3.030  | 9.453 |     |
| 0.5   | 1.701 | 0.784         | 9.701  |     |     |
| 0.7   | 1.979 | 0.984         | 9.758  |     |     |
| 0.9   | 2.227 | 1.175         | 9.812  |     |     |
| 2S    | 1.0   | 2.338         | 1.264  | 9.838 |     |
|       | 1.1   | 2.442         | 1.349  | 9.861 | 10.016 |
|       | 1.3   | 2.636         | 1.513  | 9.905 | 10.008 |
|       | 1.5   | 2.807         | 1.663  | 9.944 |     |
|       | 1.7   | 2.958         | 1.790  | 9.977 |     |
|       | 1.9   | 3.094         | 1.924  | 10.008 |     |
|       | 2.0   | 3.158         | 1.984  | 10.023 |     |
| 0.5   | 1.692 | 0.519         | 9.826  |     |     |
| 0.7   | 2.053 | 0.694         | 9.935  |     |     |
| 0.9   | 2.385 | 0.868         | 10.043 |     |     |
| 3S    | 1.0   | 2.538         | 0.953  | 10.095 |     |
|       | 1.1   | 2.684         | 1.036  | 10.144 | 10.348 |
|       | 1.3   | 2.957         | 1.199  | 10.241 | 10.351 |
|       | 1.5   | 3.205         | 1.353  | 10.330 |     |
|       | 1.7   | 3.427         | 1.496  | 10.410 |     |
|       | 1.9   | 3.631         | 1.631  | 10.485 |     |
|       | 2.0   | 3.727         | 1.696  | 10.522 |     |

Ebert (a) $\rightarrow$ [14], Gupta (d) $\rightarrow$ [44]
Table 7: Derivative of wave function at the origin (\(|R'(0)\)|) and P-wave masses of \(b\bar{b}\) meson

| State | \(\nu\) | \(\bar{\mu}\) | \(|R'(0)|\) | \(E(\bar{\mu})\) | Exp | Theory |
|-------|--------|---------------|-------------|--------------|-----|--------|
|       |        | \(\text{GeV}\) | \(\text{GeV}^{5/2}\) | \(\text{GeV}\) |     | \(\text{GeV}\) |
| 0.5   | 1.654  | 0.718         | 9.670       |              |     |        |
| 0.7   | 1.904  | 1.021         | 9.712       |              |     |        |
| 0.9   | 2.121  | 1.337         | 9.751       |              |     |        |
| 1P    | 1.0    | 2.218         | 1.446       | 9.768        |     |        |
|       |        |               |             | 9.900        | 9.901\(^a\) |        |
|       |        |               |             | 9.900\(^d\) |     |        |
| 1.1   | 2.308  | 1.653         | 9.784       | 9.900        |     |        |
|       | 1.3    | 2.474         | 1.966       | 9.815        |     | 9.900\(^d\) |
|       | 1.5    | 2.621         | 2.271       | 9.842        |     |        |
|       | 1.7    | 2.750         | 2.559       | 9.864        |     |        |
|       | 1.9    | 2.866         | 2.838       | 9.885        |     |        |
|       | 2.0    | 2.921         | 2.977       | 9.896        |     |        |
| 0.5   | 1.669  | 1.200         | 9.808       |              |     |        |
| 0.7   | 2.016  | 1.922         | 9.908       |              |     |        |
| 0.9   | 2.333  | 2.770         | 10.006      |              |     |        |
| 2P    | 1.0    | 2.478         | 3.223       | 10.053       |     |        |
|       |        |               |             | 10.097       | 10.260 |        |
|       |        |               |             | 10.261\(^a\) |     |        |
|       |        |               |             | 10.258\(^d\) |     |        |
| 1.1   | 2.617  | 3.692         | 10.097      | 10.260       |     |        |
|       | 1.3    | 2.876         | 4.677       | 10.184       |     |        |
|       | 1.5    | 3.111         | 5.691       | 10.264       |     |        |
|       | 1.7    | 3.321         | 6.699       | 10.335       |     |        |
|       | 1.9    | 3.513         | 7.708       | 10.401       |     |        |
|       | 2.0    | 3.604         | 8.218       | 10.434       |     |        |

Ebert (a) \(\rightarrow [14]\), Gupta (d) \(\rightarrow [44]\)
Table 8: S-Wave and P-Wave Masses (in GeV) of $c\bar{c}$ meson

| $\nu$ | $1^1S_0$ | $1^3S_1$ | $1^1P_1$ | $1^3P_1$ | $1^1P_3$ | $1^3P_3$ | $2^1S_0$ | $2^3S_1$ | $2^1P_1$ | $2^3P_1$ | $2^3P_3$ | $3^1S_0$ | $3^1P_1$ |
|-------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0.5   | 3.000    | 3.092    | 3.313    | 3.292    | 3.302    | 3.323    | 3.352    | 3.375    | 3.519    | 3.494    | 3.507    | 3.531    | 3.541    |
| 0.7   | 2.980    | 3.100    | 3.373    | 3.341    | 3.357    | 3.389    | 3.427    | 3.464    | 3.666    | 3.623    | 3.644    | 3.687    | 3.697    |
| 0.9   | 2.960    | 3.109    | 3.429    | 3.383    | 3.406    | 3.451    | 3.495    | 3.547    | 3.808    | 3.742    | 3.775    | 3.842    | 3.846    |
| 1.0   | 2.950    | 3.112    | 3.450    | 3.398    | 3.424    | 3.477    | 3.522    | 3.583    | 3.872    | 3.792    | 3.832    | 3.911    | 3.912    |
| 1.1   | 2.942    | 3.116    | 3.473    | 3.414    | 3.444    | 3.503    | 3.549    | 3.619    | 3.936    | 3.843    | 3.889    | 3.983    | 3.979    |
| 1.3   | 2.926    | 3.123    | 3.513    | 3.441    | 3.477    | 3.550    | 3.597    | 3.683    | 4.055    | 3.933    | 3.994    | 4.116    | 4.102    |
| 1.5   | 2.912    | 3.129    | 3.547    | 3.461    | 3.504    | 3.590    | 3.636    | 3.739    | 4.162    | 4.009    | 4.085    | 4.239    | 4.212    |
| 1.7   | 2.899    | 3.134    | 3.576    | 3.477    | 3.526    | 3.625    | 3.668    | 3.788    | 4.257    | 4.073    | 4.165    | 4.394    | 4.309    |
| 1.9   | 2.887    | 3.141    | 3.603    | 3.491    | 3.547    | 3.658    | 3.696    | 3.832    | 4.345    | 4.129    | 4.237    | 4.453    | 4.396    |
| 2.0   | 2.882    | 3.144    | 3.615    | 3.497    | 3.556    | 3.673    | 3.708    | 3.852    | 4.385    | 4.153    | 4.269    | 4.501    | 4.436    |

The value of the radial wave function $R(0)$ for $0^{-+}$ and $1^{--}$ states would be different due to their spin dependent hyperfine interaction. The spin hyperfine interaction of the heavy flavour mesons are small and this can cause a small shift in the value of the wave function at the origin. Thus, many other models do not consider this contribution to their value of $R(0)$. However, we account this correction to the value of $R(0)$ by considering

$$R_{nJ}(0) = R(0) \left[ 1 + (SF)_J \frac{\varepsilon_{SD} > nJ}{M_{SA}} \right]$$

(12)

Where $(SF)_J$ and $\varepsilon_{SD} > nJ$ is the spin factor and spin interaction energy of the meson in the $nJ$ state, while $R(0)$ and $M_{SA}$ correspond to the radial wave function at the zero separation and spin average mass respectively of the $QQ$ system. It can be seen that Eqn(12) provides the average radial wave function given by [33] as

$$R(0) = \frac{R_p + 3R_v}{4}$$

(13)

It is found that the computed mass increases with increase of $\nu$. The computed results for the pseudoscalar($P$) and Vector ($V$) mesons in the case of $c\bar{c}$, $c\bar{b}$, $b\bar{b}$ systems are tabulated in Tables 8-10. The spin-spin hyperfine and spin-orbit interactions are computed perturbatively to get the masses of $\eta_c$, $J/\psi$, $B_c$, $B_c^*$, $\eta_b$, and $\Upsilon$ states. The results are compared with known experimental values as well as with other theoretical predictions. Mass predictions with $\nu$ between 1.0 and 1.5 are found in accordance with the experimental results [2].

3 Decay constants ($f_P/V$) of the heavy flavoured mesons

The decay constants of mesons are important parameters in the study of leptonic or non-leptonic weak decay processes. The decay constants of pseudoscalar ($f_P$) and vector ($f_V$) mesons are obtained by parameterizing the matrix elements of weak current between the corresponding mesons and the vacuum as

$$\langle 0 | \bar{Q} \gamma^\mu \gamma_5 Q | P_\mu(k) \rangle = i f_P k^\mu$$

(14)
Table 9: S-Wave and P-Wave Masses (in GeV) of $\bar{c}b$ meson

| $\nu$ | $^{1}_S^0$ | $^{3}_S^1$ | $^{1}_P^1$ | $^{3}_P^1$ | $^{1}_P^3$ | $^{3}_P^3$ | $^{2}_S^0$ | $^{2}_S^1$ | $^{2}_P^0$ | $^{2}_P^1$ | $^{2}_P^3$ | $^{3}_S^0$ | $^{3}_S^1$ |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.5   | 6.291     | 6.330     | 6.542     | 6.534     | 6.538     | 6.546     | 6.582     | 6.591     | 6.720     | 6.711     | 6.715     | 6.726     | 6.743     |
| 0.7   | 6.283     | 6.334     | 6.597     | 6.584     | 6.590     | 6.603     | 6.655     | 6.685     | 6.834     | 6.840     | 6.859     | 6.884     | 6.891     |
| 0.9   | 6.273     | 6.335     | 6.641     | 6.624     | 6.633     | 6.650     | 6.715     | 6.735     | 6.944     | 6.957     | 6.982     | 7.012     | 7.024     |
| 1.0   | 6.269     | 6.337     | 6.662     | 6.642     | 6.652     | 6.672     | 6.743     | 6.767     | 7.027     | 6.997     | 7.012     | 7.042     | 7.075     |
| 1.1   | 6.265     | 6.338     | 6.682     | 6.659     | 6.671     | 6.770     | 6.797     | 7.082     | 7.047     | 7.065     | 7.099     | 7.135     | 7.151     |
| 1.3   | 6.259     | 6.341     | 6.718     | 6.691     | 6.704     | 6.732     | 6.819     | 6.852     | 7.188     | 7.142     | 7.165     | 7.211     | 7.249     |
| 1.5   | 6.252     | 6.344     | 6.749     | 7.716     | 7.733     | 7.765     | 6.860     | 6.900     | 7.283     | 7.225     | 7.254     | 7.312     | 7.351     |
| 1.7   | 6.247     | 6.347     | 6.777     | 7.739     | 7.758     | 7.796     | 6.943     | 7.730     | 7.335     | 7.405     | 7.445     | 7.479     |           |
| 1.9   | 6.241     | 6.348     | 6.799     | 7.756     | 7.777     | 7.820     | 6.978     | 7.445     | 7.363     | 7.404     | 7.486     | 7.525     | 7.565     |
| 2.0   | 6.237     | 6.348     | 6.806     | 7.762     | 7.784     | 7.829     | 6.991     | 7.476     | 7.388     | 7.519     | 7.558     | 7.601     |           |

Table 10: S-Wave and P-Wave Masses (in GeV) of $b \bar{b}$ meson

| $\nu$ | $^{1}_S^0$ | $^{3}_S^1$ | $^{1}_P^1$ | $^{3}_P^1$ | $^{1}_P^3$ | $^{3}_P^3$ | $^{2}_S^0$ | $^{2}_S^1$ | $^{2}_P^0$ | $^{2}_P^1$ | $^{2}_P^3$ | $^{3}_S^0$ | $^{3}_S^1$ |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.5   | 9.426     | 9.463     | 9.672     | 9.664     | 9.670     | 9.683     | 9.696     | 9.702     | 9.808     | 9.803     | 9.806     | 9.810     | 9.824     |
| 0.7   | 9.419     | 9.465     | 9.716     | 9.703     | 9.712     | 9.731     | 9.751     | 9.760     | 9.908     | 9.903     | 9.903     | 9.913     | 9.931     |
| 0.9   | 9.414     | 9.467     | 9.757     | 9.740     | 9.751     | 9.774     | 9.803     | 9.816     | 10.006    | 9.999     | 10.014    | 10.038    |           |
| 1.0   | 9.411     | 9.468     | 9.775     | 9.755     | 9.768     | 9.781     | 9.816     | 9.841     | 10.053    | 10.035    | 10.044    | 10.062    | 10.088    |
| 1.1   | 9.408     | 9.468     | 9.791     | 9.769     | 9.784     | 9.800     | 9.846     | 9.865     | 10.097    | 10.076    | 10.086    | 10.108    | 10.136    |
| 1.3   | 9.403     | 9.470     | 9.824     | 9.797     | 9.815     | 9.840     | 9.888     | 9.910     | 10.184    | 10.155    | 10.170    | 10.198    | 10.230    |
| 1.5   | 9.399     | 9.472     | 9.852     | 9.820     | 9.842     | 9.866     | 9.924     | 9.951     | 10.264    | 10.228    | 10.246    | 10.282    | 10.317    |
| 1.7   | 9.394     | 9.473     | 9.877     | 9.840     | 9.864     | 9.887     | 9.955     | 9.985     | 10.335    | 10.291    | 10.313    | 10.357    | 10.394    |
| 1.9   | 9.390     | 9.474     | 9.900     | 9.857     | 9.885     | 9.905     | 9.982     | 10.017    | 10.401    | 10.350    | 10.376    | 10.428    | 10.466    |
| 2.0   | 9.389     | 9.475     | 9.911     | 9.866     | 9.896     | 9.913     | 9.995     | 10.032    | 10.434    | 10.379    | 10.406    | 10.462    | 10.501    |
| [2] [1]| 9.460     | 9.860     | 9.893     | 9.913     | 10.023    | 10.232    | 10.255    | 10.268    |           |           |           |           |           |
| [14]  | 9.400     | 9.460     | 9.901     | 9.863     | 9.892     | 9.913     | 9.993     | 10.023    | 10.261    | 10.234    | 10.255    | 10.268    | 10.328    |
| [54]  | 9.414     | 9.461     | 9.900     | 9.861     | 9.891     | 9.912     | 9.999     | 10.023    | 10.262    | 10.231    | 10.255    | 10.272    | 10.345    |
\[ \langle 0 | \bar{Q} \gamma^\mu Q | V(k, \epsilon) \rangle = f_V M_V \epsilon^\mu \]  

(15)

where \( k \) is the meson momentum, \( \epsilon^\mu \) and \( M_V \) are the polarization vector and mass of the vector meson. In the relativistic quark model, the decay constant can be expressed through the meson wave function \( \Phi_{P,V}(p) \) in the momentum space as [14]

\[
f_{P,V} = \sqrt{\frac{12}{M_{P,V}}} \int \frac{d^3p}{(2\pi)^3} \left( \frac{E_Q(p) + m_Q}{2E_Q(p)} \right) \left( \frac{E_Q(p) + m_Q}{2E_Q(p)} \right) \left\{ 1 + \lambda_{P,V} \frac{p^2}{|E_Q(p) + m_Q||E_Q(p) + m_Q|} \right\} \Phi_{P,V}(p) \]  

(16)

with \( \lambda_P = -1 \) and \( \lambda_V = -1/3 \). In the nonrelativistic limit \( \frac{p^2}{m^2} \to 0 \), this expression reduces to the well-known relation between \( f_{P,V} \) and the ground state wave function at the origin \( \psi_{P,V}(0) \), the Van-Royen-Weisskopf formula [46]. Though most of the models predict the mesonic mass spectrum successfully, there are disagreements in the predictions of the pseudoscalar and vector decay constants. For example, most of the cases, the ratio \( f_P/f_V \) was predicted to be \( > 1 \) as \( m_P < m_V \) and their wave function at the origin \( \psi_P(0) \sim \psi_V(0) \) [47]. The ratio computed in the relativistic models [47] predicted the ratio \( f_P/f_V < 1 \), particularly in the heavy flavour sector. The disparity of the predictions of these decay constants play a decisive role in the precision measurements of the weak decay parameters. So, we re-examine the predictions of the decay constants under different potential schemes discussed in the present work. Incorporating a first order QCD correction factor, we compute them using the relation,

\[
f_{P/V}^2 = \frac{12}{M_{P/V}} \left| \frac{\psi_{P/V}(0)}{\bar{C}(\alpha_s)} \right|^2 \]  

(17)

Where \( \bar{C}(\alpha_s) \) is the QCD correction factor given by [48, 49]

\[
\bar{C}(\alpha_s) = 1 - \frac{\alpha_s}{\pi} \left[ \delta^P - \frac{m_Q - m_{\bar{Q}}}{m_Q + m_{\bar{Q}}} \ln \frac{m_{\bar{Q}}}{m_Q} \right] \]  

(18)

Here \( \delta^P = 2 \) and \( \delta^V = 8/3 \). The computed \( f_P \) and \( f_V \) for \( c\bar{c}, \bar{c}\bar{b} \) and \( b\bar{b} \) systems using Eqn [17] & [18] and the predicted radial wave functions at the origin \( R_{nJ}(0) \) of the respective mesons are tabulated in Tables [11] - [13]. The decay constants without and with the QCD corrections are also listed as \( f_{P,V} \) and \( f_{P,V}(cor.) \) in the table. The plot of \( f_P \) vs \( (m_Q + m_{\bar{Q}}) \) shows (see fig 3) deviations from linearity as against the predictions of a linear scaling between the weak decay constant and the sum of quark antiquark masses justified within a renormalized light front QCD inspired model for quark antiquark bound states [51].

4 Weak decay of \( B_c^+ \) meson

The Decay properties of \( B_c^+ \) (\( \bar{b}c \)) meson is of interest as it decays only through weak interactions [14, 24, 52, 53]. This is due to the fact that its ground state energy lies below the (BD) threshold and has non vanishing flavour. This eliminates the uncertainties encountered due to strong decays and provides a clear decay width and lifetime for \( B_c^+ \) meson, which helps to fix more precise value of the weak decay parameters such as the CKM mixing matrix elements \( \langle V_{cb}, V_{cs} \rangle \) and the leptonic decay constant \( f_P \). Adopting the spectator model for the charm-beauty system [24], the total decay width of \( B_c^+ \) meson can be approximated as the sum of the widths of \( \bar{b} \)-quark decay keeping \( c \)-quark as spectator, the \( c \)-quark decay with \( b \)-quark as spectator, and the annihilation channel \( B_c^+ \to l^+ \nu_l (c\bar{s}, u\bar{s}) \), \( l = e, \mu, \tau \) with no interference assumed between them.
Table 11: Decay constants ($f_P$ & $f_V$) (in MeV) of $1S$ and $c\bar{c}$ mesons states.

| Models | $R_p(0)$ GeV$^{3/2}$ | $R_v(0)$ GeV$^{3/2}$ | $f_P$ MeV | $f_P$ (cor.) MeV | $f_V$ MeV | $f_V$ (cor.) MeV |
|--------|----------------------|----------------------|-----------|------------------|-----------|------------------|
| ERHM   | 0.726                | 0.752                | 410       | 317              | 418       | 323              |
| BT     | 0.874                | 0.909                | 499       | 382              | 505       | 389              |
| PL     | 0.971                | 1.009                | 550       | 399              | 560       | 407              |
| LOG    | 0.877                | 0.914                | 496       | 379              | 506       | 387              |
| Cornell| 1.171                | 1.217                | 663       | 532              | 676       | 543              |

| $\nu$ = 0.5 | 0.763 | 0.787 | 430 | 348 | 437 | 326 |
|-------------|-------|-------|-----|-----|-----|-----|
| 0.7         | 0.875 | 0.912 | 495 | 401 | 506 | 377 |
| 0.9         | 0.967 | 1.018 | 549 | 444 | 564 | 421 |
| 1.0         | 1.005 | 1.063 | 572 | 463 | 589 | 439 |
| 1.1         | 1.041 | 1.107 | 593 | 480 | 613 | 457 |
| 1.3         | 1.104 | 1.184 | 627 | 507 | 655 | 488 |
| 1.5         | 1.158 | 1.252 | 663 | 536 | 692 | 516 |
| 1.7         | 1.204 | 1.311 | 691 | 559 | 724 | 539 |
| 1.9         | 1.245 | 1.366 | 716 | 579 | 753 | 561 |
| 2.0         | 1.264 | 1.391 | 728 | 589 | 767 | 571 |

| $\nu$ | $f_P$ MeV | $f_P$ (cor.) MeV | $f_V$ MeV | $f_V$ (cor.) MeV |
|-------|-----------|------------------|-----------|------------------|
| 0.5   | 335±      | 459±             | 416±      |                  |
| 0.7   |           |                  |           |                  |
| 0.9   |           |                  |           |                  |
| 1.0   |           |                  |           |                  |
| 1.1   |           |                  |           |                  |
| 1.3   |           |                  |           |                  |
| 1.5   |           |                  |           |                  |
| 1.7   |           |                  |           |                  |
| 1.9   |           |                  |           |                  |
| 2.0   |           |                  |           |                  |

Table 12: Pseudoscalar and Vector meson decay constants ($f_P$ & $f_V$) (in MeV) of $1S$ $b\bar{c}$ meson state.

| $R_p(0)$ | $R_v(0)$ | $f_P$ MeV | $f_P$ (cor.) MeV | $f_V$ MeV | $f_V$ (cor.) MeV |
|----------|----------|-----------|------------------|-----------|------------------|
| $\nu$ = 0.5 | 1.021 | 1.027 | 398 | 356 | 399 | 336 |
| 0.7 | 1.169 | 1.178 | 456 | 408 | 457 | 385 |
| 0.9 | 1.291 | 1.304 | 504 | 451 | 506 | 426 |
| 1.0 | 1.346 | 1.361 | 525 | 470 | 528 | 445 |
| 1.1 | 1.396 | 1.413 | 545 | 488 | 548 | 461 |
| 1.3 | 1.487 | 1.507 | 581 | 520 | 585 | 492 |
| 1.5 | 1.565 | 1.589 | 612 | 548 | 616 | 519 |
| 1.7 | 1.635 | 1.662 | 639 | 572 | 645 | 542 |
| 1.9 | 1.695 | 1.725 | 663 | 594 | 669 | 563 |

| $\nu$ | $f_P$ MeV | $f_P$ (cor.) MeV | $f_V$ MeV | $f_V$ (cor.) MeV |
|-------|-----------|------------------|-----------|------------------|
| 2.0   |           |                  |           |                  |

| $433^{[14]}$ | $503^{[14]}$ | $418$ | $\pm 24^{[56]}$ |
Table 13: Decay constants \((f_P \& f_V)\) (in MeV) of \(1S \bar{b}b\) meson state.

| Models  | \(R_p(0)\) \(\text{GeV}^{3/2}\) | \(R_v(0)\) \(\text{GeV}^{3/2}\) | \(f_P\) MeV | \(f_P(\text{cor.})\) MeV | \(f_V\) MeV | \(f_V(\text{cor.})\) MeV |
|---------|----------------------------------|----------------------------------|------------|-------------------|------------|-------------------|
| ERHM    | 2.232                           | 2.235                           | 709        | 601               | 710        | 601               |
| BT      | 2.527                           | 2.551                           | 807        | 683               | 810        | 686               |
| PL      | 2.132                           | 2.146                           | 680        | 563               | 682        | 565               |
| LOG     | 2.206                           | 2.221                           | 703        | 594               | 706        | 596               |
| Cornell | 3.706                           | 3.762                           | 1185       | 1022              | 1194       | 1029              |
| \(\nu = 0.5\) |                           |                                  |            |                   |            |                   |
| 0.5     | 1.971                           | 1.979                           | 627        | 537               | 629        | 509               |
| 0.7     | 2.178                           | 2.189                           | 693        | 594               | 695        | 563               |
| 0.9     | 2.358                           | 2.371                           | 751        | 643               | 753        | 609               |
| 1.0     | 2.436                           | 2.451                           | 776        | 665               | 778        | 630               |
| 1.1     | 2.513                           | 2.529                           | 801        | 686               | 803        | 650               |
| 1.3     | 2.648                           | 2.667                           | 844        | 723               | 847        | 685               |
| 1.5     | 2.764                           | 2.785                           | 881        | 755               | 884        | 715               |
| 1.7     | 2.867                           | 2.891                           | 914        | 783               | 918        | 743               |
| 1.9     | 2.962                           | 2.989                           | 945        | 809               | 949        | 768               |
| 2.0     | 3.009                           | 3.037                           | 960        | 822               | 964        | 780               |

\(711^{19}_{19}\) \(\pm 8^{37}_{37}\)

Table 14: Decay widths (in \(10^{-4} \text{ eV}\)) and lifetime \(\tau\) (in ps) of \(B_c^+\) meson

| Model  | \(\Gamma(\text{Anni})\) a b | \(\Gamma(B_c \rightarrow X)\) a b | \(\tau\) (in PS) a b |
|--------|-------------------------------|-------------------------------|-------------------|
| \(\nu = 0.5\) | 0.370 0.370 12.47 13.18 0.530 0.499 |                                   |                   |
| 0.7    | 0.486 0.484 12.59 13.30 0.523 0.495 |                                   |                   |
| 0.9    | 0.596 0.593 12.70 13.41 0.518 0.491 |                                   |                   |
| 1.0    | 0.644 0.642 12.75 13.46 0.516 0.489 |                                   |                   |
| 1.1    | 0.693 0.690 12.79 13.51 0.515 0.487 |                                   |                   |
| 1.3    | 0.786 0.783 12.89 13.60 0.511 0.484 |                                   |                   |
| 1.5    | 0.871 0.867 12.89 13.68 0.507 0.481 |                                   |                   |
| 1.7    | 0.951 0.950 12.97 13.76 0.504 0.478 |                                   |                   |
| 1.9    | 1.023 1.020 13.05 13.83 0.502 0.476 |                                   |                   |
| 2.0    | 1.053 1.050 13.15 13.87 0.500 0.475 |                                   |                   |
| \[\text{1}^{16}_{10}\] | 0.46^{+0.18}_{-0.16} |                                   |                   |
| \[\text{24}^{16}_{10}\] | 1.40 14.00 0.47 |                                   |                   |
| \[\text{52}^{16}_{10}\] | 0.67 8.8 0.75 |                                   |                   |
Accordingly, the total width is written as

\[ \Gamma(B_c \rightarrow X) = \Gamma(b \rightarrow X) + \Gamma(c \rightarrow X) + \Gamma(Anni) \]  

(19)

Neglecting the quark binding effects, we obtain for the \( b \) and \( c \) inclusive widths in the spectator approximation as

\[ \Gamma(b \rightarrow X) = \frac{9 \, G_F^2 \, |V_{cb}|^2 \, m_b^5}{192 \pi^3} = 7.97 \times 10^{-4} eV(a) \]

\[ = 8.66 \times 10^{-4} eV(b) \]  

(20)

\[ \Gamma(c \rightarrow X) = \frac{5 \, G_F^2 \, |V_{cs}|^2 \, m_c^5}{192 \pi^3} = 4.13 \times 10^{-4} eV(a) \]

\[ = 4.15 \times 10^{-4} eV(b) \]  

(21)

Here we have used the model quark masses and the two values (a) and (b) correspond to the two set of values for the CKM matrix elements (a)→|\( V_{cs} \)| = 0.97296, |\( V_{cb} \)| = 0.04221 as used in reference [1] and (b)→ |\( V_{cs} \)| = 0.975, |\( V_{cb} \)| = 0.044 as the upper bound provided by particle data group. The values of \( \Gamma(B \rightarrow X) \) and \( \Gamma(c \rightarrow X) \) in Bethe-Salpeter model [24] and relativized quark model [52] are 7.5 & 5.1 and 4.8 & 3.3 (widths are in \( 10^{-4} \) eV) respectively.

Employing the computed mass of the \( 1^1S_0 \) state (\( M_{B_c} \)) and \( f_{B_c} \) values obtained from the present study, the width of the annihilation channel is computed using the expression given by [24],

\[ \Gamma(Anni) = \frac{G_F^2 \, |V_{bc}|^2 \, f_{B_c}^2 \, M_{B_c} \, m_i^2}{8 \pi} \left( 1 - \frac{m_q^2}{M_{B_c}^2} \right)^2 \, C_q, \]  

(22)

Where \( C_q = 3 \, |V_{cs}|^2 \) for \( c\bar{s} \), and \( m_q \) is the mass of the heaviest fermions.

The computed widths and lifetime in our CPP\( \nu \) model are listed in Table [14]. Our predictions for the life time with the potential index \( 0.5 \leq \nu \leq 2 \) lie well within the experimental error bar.

### 5 Decay rates of quarkonia

Along with the mass spectrum, successful predictions of various decay widths of heavy flavoured systems have remained as testing ground for the success of phenomenological models. Experimentally, the excited states and the leptonic, di-gamma and other hadronic decay width, of the heavy flavour mesons have been reported. However, experimentally, the pseudoscalar \( bb \) bound state \( \eta_b \) is still elusive though experimental search for this state at the di-gamma decay channel has been initiated recently [47].

As an attempt to improve the theoretical predictions involving the phenomenological description of the meson, using the redial wave functions and other model parameters of the different potential models we study the decay of \( 1^1S_0 \) quarkonium into di-gamma and the decay of \( 3^3S_1 \) into lepton pairs using the NRQCD formalism [33]. It is expected that the NRQCD formalism has all the corrective contributions for the right predictions of the decay rates. NRQCD factorization expressions for the decay rates of quarkonium and decay are given by [31].
\[ \Gamma(1S_0 \rightarrow \gamma \gamma) = \frac{F_{\gamma\gamma}(1S_0)}{m_Q^2} |\langle 0|\chi^\dagger \psi |1S_0 \rangle|^2 + \frac{G_{\gamma\gamma}(1S_0)}{m_Q} \text{Re} \left[ \langle 1S_0 |\psi^\dagger \chi |0 \rangle < 0 |\chi^\dagger (-\frac{i}{2} \hat{D})^2 \psi |1S_0 \rangle \right] \\
+ \frac{H_{\gamma\gamma}(1S_0)}{m_Q^2} < 1S_0 |\psi^\dagger (-\frac{i}{2} \hat{D})^2 \chi |0 \rangle < 0 |\chi^\dagger (-\frac{i}{2} \hat{D})^2 \psi |1S_0 \rangle \\
+ \frac{H_{\gamma\gamma}(1S_0)}{m_Q^2} \text{Re} \left[ \langle 1S_0 |\psi^\dagger \chi |0 \rangle < 0 |\chi^\dagger (-\frac{i}{2} \hat{D})^2 \psi |1S_0 \rangle \right] \] (23)

\[ \Gamma(3S_1 \rightarrow e^+ e^-) = \frac{F_{ee}(3S_1)}{m_Q^2} |\langle 0|\chi^\dagger \sigma \psi |3S_1 \rangle|^2 + \frac{G_{ee}(3S_1)}{m_Q} \text{Re} \left[ \langle 3S_1 |\psi^\dagger \sigma \chi |0 \rangle < 0 |\chi^\dagger \sigma (-\frac{i}{2} \hat{D})^2 \psi |3S_1 \rangle \right] \\
+ \frac{H_{ee}(3S_1)}{m_Q^2} < 3S_1 |\psi^\dagger \sigma (-\frac{i}{2} \hat{D})^2 \chi |0 \rangle < 0 |\chi^\dagger \sigma (-\frac{i}{2} \hat{D})^2 \psi |3S_1 \rangle \\
+ \frac{H_{ee}(3S_1)}{m_Q^2} \text{Re} \left[ \langle 3S_1 |\psi^\dagger \sigma \chi |0 \rangle < 0 |\chi^\dagger \sigma (-\frac{i}{2} \hat{D})^4 \psi |3S_1 \rangle \right] \] (24)

The short distance coefficients F’s and G’s of the order of \( \alpha_s^2 \) and \( \alpha_s^3 \) are given by [31]

\[ F_{\gamma\gamma}(1S_0) = 2\pi Q^4 \alpha_s^2 \left[ 1 + \left( \frac{\pi^2}{4} - 5 \right) C_F \frac{\alpha_s}{\pi} \right] \] (25)

\[ G_{\gamma\gamma}(1S_0) = -\frac{8\pi Q^4}{3} \alpha_s^2 \] (26)

\[ H_{\gamma\gamma}(1S_0) + H_{ee}(1S_0) = \frac{136\pi}{45} Q^4 \alpha_s^2 \] (27)

\[ F_{ee}(3S_1) = \frac{2\pi Q^2 \alpha_s^2}{3} \left\{ 1 - 4C_F \frac{\alpha_s(m)}{\pi} \right\} \]

\[ \left[ -117.46 + 0.82n_f + \frac{140\pi^2}{27} \ln \left( \frac{2m}{\mu_A} \right) \right] \] (28)

\[ G_{ee}(3S_1) = -\frac{8\pi Q^2}{9} \alpha_s^2 \] (29)

\[ H_{ee}(3S_1) + H_{ee}(3S_1) = \frac{58\pi}{54} Q^2 \alpha_s^2 \] (30)

The matrix elements that contributes to the decay rates of the S wave states into \( \eta_Q \rightarrow \gamma \gamma \) and \( \psi \rightarrow e^+ e^- \) through next-to-leading order in \( v^2 \), the vacuum-saturation approximation gives [33]

\[ \langle 1S_0 |\mathcal{O}(1S_0) |1S_0 \rangle = |\langle 0|\chi^\dagger \psi |1S_0 \rangle|^2 \left[ 1 + O(v^4) \right] \] (31)

\[ \langle 3S_1 |\mathcal{O}(3S_1) |3S_1 \rangle = |\langle 0|\chi^\dagger \sigma \psi |3S_1 \rangle|^2 \left[ 1 + O(v^4) \right] \] (32)

\[ \langle 1S_0 |\mathcal{P}_1(1S_0) |1S_0 \rangle = \text{Re} \langle 1S_0 |\psi^\dagger \chi |0 \rangle \]
\[ < 0 | \chi^\dagger (\frac{-i}{2} \overrightarrow{D})^2 \psi |^1 S_0 > \] + \mathcal{O}(v^4 \Gamma) \quad (33) \\

\[ <^3 S_1 | \mathcal{P}_1 (^3 S_1) | ^3 S_1 > = Re [<^3 S_1 | \psi^\dagger \sigma \chi | 0 > < 0 | \chi^\dagger \sigma (\frac{-i}{2} \overrightarrow{D})^2 \psi | ^3 S_1 > ] + \mathcal{O}(v^4 \Gamma) \quad (34) \]

\[ <^1 S_0 | \mathcal{Q}_1^1 (^1 S_0) | ^1 S_0 > = < 0 | \chi^\dagger (\frac{-i}{2} \overrightarrow{D})^2 \psi | ^1 S_0 > \quad (35) \]

\[ <^3 S_1 | \mathcal{Q}_1^1 (^3 S_1) | ^3 S_1 > = < 0 | \chi^\dagger \sigma (\frac{-i}{2} \overrightarrow{D})^2 \psi | ^3 S_1 > \quad (36) \]

\[ <^1 S_0 | \mathcal{Q}_1^2 (^1 S_0) | ^1 S_0 > = < 0 | \chi^\dagger (\frac{-i}{2} \overrightarrow{D})^4 \psi | ^1 S_0 > \quad (37) \]

\[ <^3 S_1 | \mathcal{Q}_1^2 (^3 S_1) | ^3 S_1 > = < 0 | \chi^\dagger \sigma (\frac{-i}{2} \overrightarrow{D})^4 \psi | ^3 S_1 > \quad (38) \]

The Vacuum saturation allows the matrix elements of some four fermion operators to be expressed in terms of the regularized wave-function parameters given by \[33\]

\[ <^1 S_0 | \mathcal{O} (^1 S_0) | ^1 S_0 > = \frac{3}{2 \pi} | R_P(0) |^2 \quad (39) \]

\[ <^3 S_1 | \mathcal{O} (^3 S_1) | ^3 S_1 > = \frac{3}{2 \pi} | R_V(0) |^2 \quad (40) \]

\[ <^1 S_0 | \mathcal{P}_1 (^1 S_0) | ^1 S_0 > = -\frac{3}{2 \pi} | R_P^2 \nabla^2 R_P | \quad (41) \]

\[ <^3 S_1 | \mathcal{P}_1 (^3 S_1) | ^3 S_1 > = -\frac{3}{2 \pi} | R_V^2 \nabla^2 R_V | \quad (42) \]

\[ <^1 S_0 | \mathcal{Q}_1^1 (^1 S_0) | ^1 S_0 > = -\sqrt{\frac{3}{2 \pi} \nabla^2 R_P} \quad (43) \]

\[ <^3 S_1 | \mathcal{Q}_1^1 (^3 S_1) | ^3 S_1 > = -\sqrt{\frac{3}{2 \pi} \nabla^2 R_V} \quad (44) \]

\[ <^1 S_0 | \mathcal{Q}_1^2 (^1 S_0) | ^1 S_0 > = \frac{3}{2 \pi} \nabla^2 (\nabla^2 R_P) \quad (45) \]

\[ <^3 S_1 | \mathcal{Q}_1^2 (^3 S_1) | ^3 S_1 > = \frac{3}{2 \pi} \nabla^2 (\nabla^2 R_V) \quad (46) \]
Table 15: Decay rates (in keV) of $0^{-+} \rightarrow \gamma \gamma$ and the relevant correction terms of $\eta_c$ and $\eta_b$ mesons.

| Systems Models | $\Gamma_0$ | $\Gamma_R$ | $\Gamma$ | $\Gamma_{NRQCD}$ up to $O(v^2)$ | $\Gamma_{NRQCD_{frs}}$ up to $O(v^4)$ | $\Gamma_{Others}$ |
|----------------|-----------|-----------|---------|-------------------------------|--------------------------------------|------------------|
| ERHM           | 7.460     | -2.855    | 4.605   | 4.005                         | 4.225                                | –                |
| BT             | 10.870    | -4.206    | 6.664   | 6.555                         | 6.561                                | –                |
| PL(Martin)     | 13.406    | -6.196    | 7.210   | 8.434                         | 10.691                               | –                |
| Log            | 10.937    | -4.349    | 6.588   | 6.691                         | 6.697                                | –                |
| Cornell        | 19.512    | -6.581    | 12.931  | 13.779                        | 17.447                               | –                |
| $\eta_c$       |           |           |         |                               |                                      | 7.2±0.7±2.0     |
| CPP $\nu$=0.5  | 8.173     | -2.635    | 5.538   | 2.511                         | 6.078                                | 2.992            |
| 0.7            | 10.918    | -3.521    | 7.397   | 3.706                         | 7.810                                | 4.087            |
| 0.9            | 13.465    | -4.342    | 9.123   | 4.925                         | 9.391                                | 5.102            |
| 1.0            | 14.649    | -4.724    | 9.925   | 5.552                         | 10.077                               | 5.549            |
| 1.1            | 15.812    | -5.099    | 10.713  | 6.171                         | 10.761                               | 6.012            |
| 1.3            | 17.987    | -5.800    | 12.187  | 7.387                         | 12.016                               | 7.095            |
| 1.5            | 19.971    | -6.440    | 13.531  | 8.556                         | 13.142                               | 7.536            |
| 1.7            | 22.788    | -7.026    | 15.762  | 9.700                         | 14.166                               | 8.170            |
| 1.9            | 23.502    | -7.578    | 16.924  | 10.789                        | 15.139                               | 8.736            |
| 2.0            | 24.297    | -7.835    | 16.462  | 11.295                        | 15.596                               | 9.006            |
| $\eta_b$       |           |           |         |                               |                                      | 0.364  |
| CPP $\nu$=0.5  | 0.406     | -0.118    | 0.288   | 0.312                         | 0.340                                | 0.490  |
| 0.7            | 0.422     | -0.106    | 0.316   | 0.310                         | 0.311                                | 0.353             |
| 0.9            | 0.495     | -0.124    | 0.371   | 0.365                         | 0.366                                | 0.321             |
| 1.0            | 0.529     | -0.132    | 0.397   | 0.390                         | 0.391                                | 0.353             |
| 1.1            | 0.563     | -0.141    | 0.422   | 0.416                         | 0.416                                | 0.386             |
| 1.3            | 0.626     | -0.156    | 0.470   | 0.462                         | 0.463                                | 0.435             |
| 1.5            | 0.683     | -0.171    | 0.512   | 0.505                         | 0.505                                | 0.510             |
| 1.7            | 0.735     | -0.184    | 0.551   | 0.544                         | 0.545                                | 0.570             |
| 1.9            | 0.786     | -0.196    | 0.590   | 0.582                         | 0.582                                | 0.628             |
| 2.0            | 0.811     | -0.203    | 0.608   | 0.600                         | 0.601                                | 0.657             |

ERHM [19, 20], BT [5], PL (Martin) [6], Log [9], Cornell [11]
Table 16: Decay rates (in keV) of $1^{-+} \rightarrow l^+ l^-$ and the relevant correction terms of $J/\psi$ and $\Upsilon$ mesons.

| Systems | Models          | $\Gamma_{\text{VW}}$ | $\Gamma_{\text{rad}}$ | $\Gamma$     | $\Gamma_{\text{NRQCD \text{upto } O(v^2)}}$ | $\Gamma_{\text{NRQCD \text{upto } O(v^4)}}$ | $\Gamma_{\text{EXP}}$[	extsuperscript{[1]}] |
|---------|-----------------|----------------------|----------------------|-------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
|         | ERHM            | 5.595                | -3.381               | 2.214       | 2.543                                       | 3.246                                       | –                                           |
| $J/\psi$| BT              | 8.152                | -4.982               | 3.170       | 2.539                                       | 2.809                                       | –                                           |
|         | PL (Martin)     | 10.055               | -7.341               | 2.714       | 3.311                                       | 4.698                                       | –                                           |
|         | Log             | 8.203                | -0.171               | 3.057       | 1.967                                       | 2.094                                       | –                                           |
|         | Cornell         | 14.634               | -7.701               | 6.933       | 7.920                                       | 10.294                                      | –                                           |
|         | $CPP_{P_L}=0.5$ | 6.130                | -0.624               | 5.506       | 4.212                                       | 4.973                                       | 0.973                                       |
|         | 0.7             | 8.189                | -0.845               | 7.344       | 6.199                                       | 7.701                                       | 1.676                                       |
|         | 0.9             | 10.153               | -1.065               | 9.088       | 8.320                                       | 10.815                                      | 2.447                                       |
|         | 1.0             | 11.053               | -1.165               | 9.888       | 9.353                                       | 12.398                                      | 2.822                                       |
|         | 1.1             | 11.946               | -1.268               | 10.678      | 10.430                                      | 14.089                                      | 3.227                                       |
|         | 1.3             | 13.621               | -1.463               | 12.158      | 12.558                                      | 17.550                                      | 3.996                                       |
|         | 1.5             | 15.165               | -1.645               | 13.520      | 12.605                                      | 19.037                                      | 4.749                                       |
|         | 1.7             | 16.582               | -1.813               | 14.769      | 16.643                                      | 24.587                                      | 5.467                                       |
|         | 1.9             | 17.920               | -1.982               | 15.938      | 18.659                                      | 28.232                                      | 6.185                                       |
|         | 2.0             | 18.549               | -2.061               | 16.488      | 19.634                                      | 30.032                                      | 6.534                                       |
| $\Upsilon$| ERHM            | 1.320                | -0.540               | 1.303       | 1.221                                       | 1.228                                       | –                                           |
|         | B.T.            | 1.720                | -0.076               | 1.644       | 1.249                                       | 1.267                                       | –                                           |
|         | PL (Martin)     | 1.218                | -0.761               | 0.457       | 0.693                                       | 0.774                                       | –                                           |
|         | Log             | 1.305                | -0.032               | 1.273       | 0.924                                       | 0.943                                       | –                                           |
|         | Cornell         | 3.733                | -0.232               | 3.501       | 2.025                                       | 2.270                                       | –                                           |
|         | $CPP_{P_L}=0.5$ | 1.035                | -0.010               | 1.025       | 0.935                                       | 0.938                                       | 0.710                                       |
|         | 0.7             | 1.266                | -0.013               | 1.253       | 1.144                                       | 1.148                                       | 0.933                                       |
|         | 0.9             | 1.485                | -0.015               | 1.470       | 1.344                                       | 1.349                                       | 1.165                                       |
|         | 1.0             | 1.587                | -0.017               | 1.570       | 1.436                                       | 1.442                                       | 1.279                                       |
|         | 1.1             | 1.690                | -0.018               | 1.678       | 1.529                                       | 1.535                                       | 1.397                                       |
|         | 1.3             | 1.878                | -0.020               | 1.858       | 1.702                                       | 1.709                                       | 1.575                                       |
|         | 1.5             | 2.047                | -0.022               | 2.025       | 1.857                                       | 1.865                                       | 1.844                                       |
|         | 1.7             | 2.206                | -0.024               | 2.182       | 2.002                                       | 2.010                                       | 2.057                                       |
|         | 1.9             | 2.357                | -0.026               | 2.331       | 2.141                                       | 2.149                                       | 2.266                                       |
|         | 2.0             | 2.433                | -0.026               | 2.407       | 2.210                                       | 2.219                                       | 2.371                                       |

ERHM \textsuperscript{[19, 20]}, BT \textsuperscript{[5]}, PL (Martin) \textsuperscript{[6]}, Log \textsuperscript{[9]}, Cornell \textsuperscript{[11]}
We have computed the $\nabla^2 R_{p/v}$ term as per ref [57]. Accordingly,

$$\nabla^2 R = \epsilon_B \frac{M}{2}, \quad \text{as} \ r \to 0$$

(47)

where $\epsilon_B$ is the binding energy and $M$ is the mass of the respective mesonic state. The binding energy is computed as $\epsilon_B = M - (2m_Q)$. The RHS of the Eqn(15) and (46) are computed by assuming that $< p^2 >^2 \approx < p^4 >$. For comparison, we also compute the decay widths with the conventional V-W formula with and without the radiative corrections.

Accordingly, the two photon decay width of the pseudoscalar meson is given by [22]

$$\Gamma(0^{-+}\rightarrow 2\gamma) = \Gamma_0 + \Gamma_R$$

(48)

Here $\Gamma_0$ is the conventional Van Royen-Weisskopf term for the $0^{-+} \rightarrow \gamma\gamma$ decays [46], where $\Gamma_R$ is due to the radiative corrections for this decay which is given by

$$\Gamma_0 = \frac{12\alpha_e^2e^4_Q}{M_P^2} R_P^2(0)$$

(49)

and

$$\Gamma_R = \frac{\alpha_s}{\pi} \left( \frac{\pi^2 - 20}{3} \right) \Gamma_0$$

(50)

Similarly, the leptonic decay width of the vector meson is computed as

$$\Gamma(1^{--}\rightarrow ll^-) = \Gamma_{VW} + \Gamma_{rad}$$

(51)

where

$$\Gamma_{VW} = \frac{4\alpha_e^2e^2_Q}{M_V^2} R_V^2(0)$$

(52)

$\Gamma_{rad}$, the radiative correction is given by

$$\Gamma_{rad} = -\frac{16}{3\pi} \alpha_s \Gamma_{VW}$$

(53)

It is obvious to note that the computations of the decay rates and the radiative correction term described here require the right description of the meson state through its radial wave function at the origin, $R(0)$ and its mass $M$ along with other model parameters like $\alpha_s$ and the model quark masses. Generally, due to lack of exact solutions for colour dynamics, $R_{P/V}(0)$ and $M$ are considered as free parameters of the theory [57]. However, it is appropriate to employ the phenomenological model spectroscopic parameters such as of the predicted mesonic mass and the corresponding wave function for the computations of the decay widths. In many cases of potential model predictions, the radial wave function at the origin are over estimated as for the decay rates are concerned. In such cases, it is argued that the decay of $Q\bar{Q}$ occurs not at zero separation, but at some finite $Q - \bar{Q}$ radial separation. Then arbitrary scaling of the radial wave function at zero separation are done to estimate the decay rates correctly [11]. In the present computation of the decay rates using the NRQCD formalism we present our results obtained by using the radial wave function and their derivatives at zero separation ($\Gamma_{NRQCD}$) as well as at a finite radial separation of $r_o$, ($\Gamma_{NRQCD}_{r_o}$).

We defined $r_o$ by

$$r_o = \frac{N_c|e_Q|}{M_{P/V}}$$

(54)
of the mesonic state. It is similar to the compton radius and we call it as color compton radius of the \( Q\bar{Q} \) systems. Here, \( N_c = 3 \) and \( e_Q \) is the charge of the quark in terms of the electron charge.

The computed decay widths for \( 0^{-+} \rightarrow \gamma \gamma \), are presented in Table 15 and for \( 1^{--} \rightarrow l^+ l^- \) are listed in Table 16. In the case of \( \Gamma_{NRQCD} \) terms up to \( O(v^2) \) and terms up to \( O(v^4) \) are separately tabulated to highlight their contributions in the respective decays.

6 Conclusion and discussion

In this paper, we have made a comprehensive study of the heavy-heavy flavour mesonic systems in the general frame work of potential models. The potential model parameters and the masses of the charmed and beauty quark obtained from the respective quarkonia mass predictions have been employed to study their decay properties in the frame work of NRQCD formalism as well as using the conventional Van-Royen-Weisskopf nonrelativistic formula. We have also made parameter free prediction of the weak decay properties of \( B_c \) meson. The weak decay constants of the pseudoscalar \( (f_P) \) and the vector meson \( (f_V) \) computed here are is found to be in accordance with the recent predictions based on relativistic Bethe-Salpeter method [56]. The departure from the predicted linear dependence of \( f_P \) with the mesonic masses within the effective light-front model in the heavy flavour sector suggest the requirement of more refined mechanism related to their wavefunctions incorporating the confinement and hyperfine splitting.

Masses of the pseudoscalar and vector mesons and the values of the radial wave function at the origin for \( c\bar{c} \), \( c\bar{b} \) and \( b\bar{b} \) systems are computed in different potential schemes. The respective decay constants \( (f_P, f_V) \) are computed with and without QCD corrections. Using the predicted masses and radial wave functions at the origin, the digamma, leptonic, light hadronic decays of quarkonia and the weak decay properties of \( B_c^+ \) mesons are studied. For the mass predictions and for the decay rates the present results based on \( (CPP) \) are found to be in accordance with other potential model predictions as well as with the experimental values.

The theoretical \( (CPP) \) predictions of the decay widths for \( J/\psi \rightarrow l^+ l^- \) and \( \Upsilon \rightarrow l^+ l^- \) as presented in Table 16 are found to be in accordance with other potential model predictions with the radiative correction as well as with the widths computed using NRQCD formalism. Though the radiative corrections are found to be important in most of the phenomenological models, the NRQCD predictions with their matrix elements computed at finite radial separation defined through the 'color compton radius' are found to be in better agreement with the experimental values for most of the cases.

It is interesting to note that the ERHM[20] predictions of the di-gamma decay widths of \( \eta_c \) and leptonic decay widths of \( J/\psi \) and \( \Upsilon \) are in good agreement with the respective experimental results with out any correction to the Van-Royen-Weisskopf formula.

The NRQCD width for \( \eta_c \rightarrow \gamma \gamma \) predicted in the present study based on the potential model parameters of BT [5], Log [9], CPP,=0.7,0.9 are close to the experimental value of 7.2\( \pm 0.7 \pm 2.0 \) keV reported by PDG2006[11]. However for the \( \eta_b \rightarrow \gamma \gamma \) case, most of the model predictions based on NRQCD formalism are very close to similar theoretical predictions of [33]. The predictions based on V-W formula with radiative corrections are also found to be in close agreement with the prediction of[33] and [59] respectively.
The predictions of $\eta_b$ mass spectra, its hyperfine mass split ($\Upsilon - \eta_b$) of 60 MeV, its decay constant $f_P$ and the digamma width etc are important for the experimental hunting of $\eta_b$ state.

In the case of the dileptonic width of $c\bar{c}$ state, our predictions based on the NRQCD formalism with the finite range correction for the inter quark potential index $1.5 \leq \nu \leq 1.7$ are in fair agreement with the experimental value of $5.55 \pm 0.14$ keV; while that $b\bar{b}$ system the NRQCD$_{frs}$ prediction is in good agreement with value of $1.340 \pm 0.018$ keV for the potential index $\nu = 1.1$. The CPP$_{\nu=0.5}$ predictions based on V-W with radiative correction is also found to be in good agreement with the expected values while in all other choices of $\nu$ over estimates the decay width. It indicates the importance of the computation from of the decay width at finite range of quark-antiquark separation.

In the case of the leptonic decay width of $\Upsilon(1S)$ state, most of the models do provide the decay widths in close agreement with the expected value either using NRQCD formalism or using V-W with radiative corrections. Here, again the ERHM prediction for both $J/\psi$ and $\Upsilon$ are found to be very close to the respective experimental values with the conventional V-W formula only. It is suggests the adequacy of the model parameters that provide the spectroscopy as well as the decay properties.

To summarize, we find that the spectroscopy of $c\bar{c}$ system (1S to 3S) studied here are in good agreement with the respective experimental values in the potential range of $1.1 \leq \nu \leq 1.3$. However, the spectroscopic predictions with potential index $\nu = 1.5$ for $b\bar{b}$ system are found to be in agreement with the respective experimental value. The spectroscopic predictions of the $b\bar{c}$ system in the potential range $1.1 \leq \nu \leq 1.3$ are found to be in accordance with other model predictions.

In the case of the di-gamma decay widths of $c\bar{c}$ system, better agreement occurs for the potential index $\nu = 0.7$ under the NRQCD and conventional V-W formula with radiative correction. However, the NRQCD$_{frs}$ provide the experimental value of the decay width in the potential index range of $1.3 \leq \nu \leq 1.5$ only. For the $b\bar{b}$ system, better consistency in the predictions of both the leptonic and di-gamma widths are observed around the potential index $0.7 \leq \nu \leq 1.1$.

The present study of the decay rates of quarkonia clearly indicates the relative importance of QCD related corrections on the phenomenological potential models. The success of potential models in the determination of the $S$ and $P$ wave masses and decay rates of $c\bar{c}$ and $b\bar{c}$ and $b\bar{b}$ systems provide future scopes to study various transition rate and excited states of these mesonic systems. With the masses and wave functions of the heavy flavour mesons at hand, it would be rather simple to compute various transition rates such as $E1$ and $M1$ in these mesons. Such computations largely form the future applications of the present study. The decay rates and branching ratios of heavy flavour mesonic bound states are important ingredients in our present understanding of QCD.

The semileptonic decays offer an extremely favorable testing ground for both perturbative QCD, radiative corrections and nonperturbative QCD effects such as decay constants, form factors, and the best possible estimations of the CKM matrix elements. With the mass parameters of the beauty and the charm quark fixed from the study of its spectra, we have successfully computed the semi-leptonic decay width of $B_c$-meson.

The partial widths obtained here within the spectator model are compared with those obtained though the Bethe-Salpeter approach [24] as well as that from a relativistic quark model [52] in Table 14. We obtained a higher branching ratio in the $b$-decay channel compared to other approaches as seen from in Table 14. We get about 64% as the branching fractions of $b$-quark decay, about 33% as that of $c$-quark decay and about 3% in the annihilation channel. However, the CKM mixing matrix elements $V_{cb}$ and $V_{cs}$ used as free parameters in all the three models are different but lie within the range given in particle data group [2]. The lifetime of $B_c^+$ predicted by the present
calculation is found to be in good agreement with the experimental values as well as that by the Bethe Salpeter method \cite{24}. The predicted values from relativistic model \cite{52} is found to be far from the experimental values as well as other theoretical models.

Another aspect of the present study is that the decay of $QQ$ system occurs at a finite range of its separation provided by the color Compton radius. This enable us to understand at least qualitatively the importance of various processes that occur at different radial separation.

In conclusion, we have studied the importance of the spectroscopic parameters of different potential models in the predictions of the low-lying states of $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ systems as well as their decay properties in the framework of NRQCD formalism.

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