Non-separability without Non-separability in Nonlinear Quantum Mechanics

Waldemar Puszkarz
Department of Physics and Astronomy,
University of South Carolina,
Columbia, SC 29208
(May 15, 1999)

Abstract

We show an example of benign non-separability in an apparently separable system consisting of \( n \) free non-correlated quantum particles, solitonic solutions to the nonlinear phase modification of the Schrödinger equation proposed recently. The non-separability manifests itself in the wave function of a single particle being influenced by the very presence of other particles. In the simplest case of identical particles, it is the number of particles that affects the wave function of each particle and, in particular, the width of its Gaussian probability density. As a result, this width, a local property, is directly linked to the mass of the entire Universe in a very Machian manner. In the realistic limit of large \( n \), if the width in question is to be microscopic, the coupling constant must be very small resulting in an “almost linear” theory. This provides a model explanation of why the linearity of quantum mechanics can be accepted with such a high degree of certainty even if the more fundamental underlying theory could be nonlinear. We also demonstrate that when such non-correlated solitons are coupled to harmonic oscillators they lead to a faster-than-light nonlocal telegraph since changing the frequency of one oscillator affects instantaneously the probability density of particles associated with other oscillators. This effect can be alleviated by fine-tuning the parameters of the solution, which results in nonlocal correlations that affect only the phase. Exclusion rules of a novel kind that we term supersuperselection rules also emerge from these solutions. They are similar to the mass and the univalence superselection rules in linear quantum mechanics, but derive from different assumptions. The effects in question and the exclusion rules do not appear if a weakly separable extension to \( n \)-particles is employed.

*Electronic address: puszkarz@cosm.sc.edu
1 Introduction

Recently, we have presented the nonlinear phase modification of the Schrödinger equation called the simplest minimal phase extension (SMPE) of the Schrödinger equation \[1\] together with some one-dimensional solutions to it \[2\]. The most interesting of them is a free solitonic Gaussian solution describing a quantum particle. By assuming that the self-energy term constitutes the rest mass-energy of the soliton, we obtained a model particle whose physical size defined as the width of its Gaussian probability density is equal to its Compton wavelength. In this approach, that we call subrelativistic, the coupling constant is inevitably a particle-dependent quantity, an intrinsic property of the particle not unlike its mass. In fact, it is a function of the mass. In a more general approach in which no such use is made of the self-energy term, the size of the particle is determined both by its mass and by the coupling constant, the latter being entirely independent of the former.

The main purpose of this paper is to present a free \(n\)-particle solution to the modification in question describing uncorrelated solitons and to demonstrate some unusual property of it that one could call the non-separability without non-separability. As we will see, the \(n\)-particle equation can be separated into \(n\) one-particle equations, but some residual non-separability that manifests itself in the dependence of individual particle’s wave function on the presence and properties of other particles in the system remains. In addition, we will present a solution describing \(n\) uncorrelated particles coupled to linear harmonic oscillators and discuss the non-separability that occurs in this case which, in general, is less benign than the non-separability of free particles. It is the analysis of this case that demonstrates that a different \(n\)-particle extension from the one we concentrate on in this paper is necessary to avoid pathological situations in which the nonlocality is manifestly compromised. We suggest such an extension for factorizable wave functions.

In what follows, \(R\) and \(S\) denote the amplitude and the phase of the wave function \(\Psi = R \exp(iS/\hbar)\).

\[1\] We follow here the convention of \[1\] that treats the phase as the angle. In a more common convention \(S\) has the dimensions of action and \(\Psi = R \exp(iS/\hbar)\).

\[
\begin{split}
\hbar \frac{\partial R^2}{\partial t} + \sum_{i=1}^{n} \frac{\hbar^2}{m_i} \nabla_i \cdot \left( R^2 \nabla_i S \right) - 2C \sum_{i=1}^{n} \Delta_i \left( R^2 \sum_{i=1}^{n} \Delta_i S \right) = 0, \\
\sum_{i=1}^{n} \frac{\hbar^2}{m_i} \Delta_i R - 2R \frac{\partial S}{\partial t} - 2RV - \sum_{i=1}^{n} \frac{\hbar^2}{m_i} R \left( \nabla_i S \right)^2 - 2CR \left( \sum_{i=1}^{n} \Delta_i S \right)^2 = 0,
\end{split}
\]

and the energy functional is

\[
E = \int d^3x \left\{ \sum_{i=1}^{n} \frac{\hbar^2}{2m_i} \left[ \left( \nabla_i R \right)^2 + R^2 \left( \nabla_i S \right)^2 \right] + CR^2 \left( \sum_{i=1}^{n} \Delta_i S \right)^2 + VR^2 \right\}.
\]

These can be derived from the Lagrangian density proposed in \[1\] and generalized to \(n\) particles as follows

\[
- L_n(R, S) = \hbar R^2 \frac{\partial S}{\partial t} + \sum_{i=1}^{n} \frac{\hbar^2}{2m_i} \left[ \left( \nabla_i R \right)^2 + R^2 \left( \nabla_i S \right)^2 \right] + CR^2 \left( \sum_{i=1}^{n} \Delta_i S \right)^2 + R^2 V.
\]
The energy functional is a conserved quantity for the potentials $V$ that do not depend explicitly on time and coincides with the quantum-mechanical energy defined as the expectation value of the Hamiltonian for this modification \[1, 3\].

2 N-Particle Solution to the SMPE

As demonstrated in \[2\], the modification under discussion possesses a free solitonic solution. Its amplitude is that of a Gaussian,

$$R(x, t) = N \exp \left[ \frac{-(x - vt)^2}{s^2} \right],$$

(5)

where $v$ is the speed of the particle and $s$ is the half-width of the Gaussian to be determined through the coupling constant $C$ and other fundamental constants of the modification. The normalization constant $N = (2/\pi s^2)^{1/4}$. The phase of the soliton has the form

$$S(x, t) = a(x - vt)^2 + bv + c(t),$$

(6)

where $a$ and $b$ are certain constants and $c(t)$ is a function of time, all of which, similarly as $v$ and $s$, are to be found from the equations of motion. These equations are satisfied provided

$$b = m/\hbar, s^2 = -8mC/\hbar^2, s^4a^2 = 1,$$

(7)

and

$$2\hbar s^4m\frac{\partial c(t)}{\partial t} + 2\hbar^2 s^2 + \hbar^2 s^4b^2v^2 + 8Ca^2s^4m = 0.$$

(8)

We see that the coupling constant $C$ has to be negative, $C = -|C|$. The energy of the particle turns out to be

$$E = E_{st} + \frac{mv^2}{2},$$

(9)

where

$$E_{st} = \frac{\hbar^2}{2ms^2} = \frac{\hbar^4}{16m^2 |C|}$$

(10)

is the stationary part of it.

Before we present the solitonic solution for $n$ uncorrelated solitons that generalizes the above one, let us first mention the existence of a two particle solution that differs from this $n$-soliton solution when the latter is specified to the two-particle case. This is so because the solution in question is not factorizable for $x$ and $y$. If we assume that the particles have the same mass $m$, their “entangled” configuration is given by the amplitude and the phase of wave function as follows

$$R(x, y, t) = N_2 \exp \left[ -\frac{(x - y - vt)^2}{s^2} \right] \exp \left[ -\frac{(x + y + vt)^2}{s^2} \right],$$

(11)

$$S(x, y, t) = a \left[ (x - y - vt)^2 + (x + y + vt)^2 \right] + v [b_-(x - y) + b_+(x + y)] + c(t),$$

(12)

where $s^2 = -32Cm/\hbar^2$, $a^2 = 1/s^4$ and $b_- = \mp b_+ = m/2\hbar$ in accord with the upper/lower signs $\mp$ in these formulas. We note that the term $-[(x - y - vt)^2 + (x + y + vt)^2]/s^2$, that appears in the
amplitude, cannot be obtained as a linear combination of $-(x - vt)^2/s^2$ and $-(y \mp vt)^2/s^2$ which we will use to build the amplitude for two particles in the general case of uncorrelated solitons. This confirms that the solution (11-12) cannot be factorized for $x$ and $y$. The energy of this configuration is $E = mv^2/2 + \hbar^2/16m^2|C|$ with the kinetic energy being, surprisingly enough, just as for one particle. We did not succeed in finding similar solutions in the presence of potentials. The most obvious choice, the potential of harmonic oscillator does not support solutions of this type. It does support though the solutions that we will discuss now.

To find the already mentioned $n$-soliton solution, we will proceed in the same manner as for the one particle solution. To begin with, we assume that the solitons are one-dimensional, but we will remove this condition later on. The amplitude for the system of $n$ such solitons is chosen by analogy to (5) as

$$R_n(x_1, x_2, ..., x_n, t) = N_n \prod_{i=1}^{n} \exp \left[ -\frac{(x_i - v_i t)^2}{s_i^2} \right],$$

(13)

and its phase is taken in the form

$$S_n(x_1, x_2, ..., x_n, t) = \sum_{i=1}^{n} \left[ a_i(x_i - v_i t)^2 + b_i v_i x_i \right] + c(t).$$

(14)

The energy equation (2) is separable if and only if

$$s_i^4 = \frac{1}{a_i^2},$$

(15)

and

$$b_i = \frac{m_i}{\hbar},$$

(16)

whereas the continuity equation (1) requires that

$$s_i^2 = \frac{-8m_i C (\sum a_j)}{a_i \hbar^2},$$

(17)

and

$$s_i^2 a_i m_j = s_j^2 a_j m_i.$$

(18)

Even if the last formula is valid for any $i$ and $j$, it is trivial unless $i \neq j$. This comment applies also to other relationships that like (18) contain $i$ and $j$ that we will deal with throughout the rest of this paper. Now, (15) and (18) lead to a consistency condition implying that $m_1 = m_2 = ... = m_n = m$. This is somewhat reminiscent of superselection rules. However, whereas the superselection rules stem from some kinematical or dynamical symmetries, the rule in question is the consequence of the assumption that particles are uncorrelated and separable and forbids free of different masses to belong to the same quantum-mechanical Universe. Thus, this rule is much stronger than the superselection rules of linear quantum mechanics that merely exclude certain types of superpositions, such as between particles of different mass or between different species of particles as, for instance, bosons and fermions. The latter exclusion is called the univalence superselection rule. We suggest that this new type of rules

$^2$On the other hand, the amplitudes containing only $-(x - y \pm vt)^2/s^2$ or $-(x + y \pm vt)^2/s^2$ would give rise to wave functions that are not normalizable.
be called supersuperselection rules.\footnote{Or simply $S^2S$ rules.} The mass, the univalence, and the particle number along with other superselection rules are believed to be firmly established in linear quantum mechanics although objections against some of them have been raised in the literature \cite{5}.

Moreover, (17) entails that all $a_i$’s have to be of the same sign or at least one of $s_i^2$’s will be negative, and that $C = -|C|$. Combining (17) and (15) produces another consistency condition,

$$\sum_{i=1}^{n} a_i = \pm \frac{\hbar^2}{8m|C|}, \quad (19)$$

with the sign of the RHS of (19) depending on the sign of $a_i$’s, but otherwise $a_i$’s being unspecified. It is also a straightforward implication of (17) that

$$\sum_{i=1}^{n} \frac{1}{s_i^2} = \frac{\hbar^2}{8m|C|} \quad (20)$$

which has important consequences for the shape of the probability density of a three-dimensional particle, as we will see later on.

The energy equation (2) together with (15-18) gives

$$\frac{\hbar}{\hbar} \frac{\partial c(t)}{\partial t} = -\frac{1}{2} \sum_{i=1}^{n} m v_i^2 - \frac{\hbar^2}{2ms_i^2a_i} = -\frac{1}{2} m \sum_{i=1}^{n} v_i^2 - \frac{1}{n} \sum_{i=1}^{n} \frac{\hbar^2 (\sum a_j)}{2ms_i^2a_i}. \quad (21)$$

Now, as seen from (2-3), the energy $E = -\hbar \langle \partial S/\partial t \rangle$, where $<>$ denotes the expectation value of the quantity embraced. Therefore, we find that for $R_n$ normalized to unity,

$$E = \frac{1}{2} \sum_{i=1}^{n} m v_i^2 + \frac{1}{n} \sum_{i=1}^{n} \frac{\hbar^2 (\sum a_j)}{2ms_i^2a_i} = E_{\text{kin}} + \frac{\hbar^4}{16m^2|C|}. \quad (22)$$

What we have arrived at is quite remarkable: the system consisting of $n$ objects, in principle different, even if of the same mass, seemingly weakly non-separable due to the $(\Delta S)^2$-coupling between them turns out to be separable and the energy of the system is just the sum of the kinetic energy of its constituents and some constant self-energy term. In the subrelativistic approach, this constant term is identified with the rest mass-energy of the system of $n$ particles equal $nm c^2$. In a more general nonrelativistic scheme, the term in question represents the internal energy of this system not necessarily equal to its rest mass-energy. In either approach though, this term is, surprisingly enough, independent of the number of particles; it is some constant, the same for one particle and for a gazillion of them.

For identical particles the above formulas become somewhat simpler. For instance, (17) turns into

$$s^2 = \frac{8nm|C|}{\hbar^2}, \quad (23)$$

and since $M = nm$ is the total mass of the Universe filled with the identical solitonic particles, we see that the size of the particle, a local quantity, is determined by the mass of the entire Universe! One can view it as a truly Machian effect \cite{6}. To recall, Mach ca. 1883 conjectured that the inertia of an object, a local property, is due to the rest of the Universe, the foremost global object. This came to
be known as Mach’s principle, and the situations where a local physical quantity is related to some global one can be thought of as manifestations of the generalized Mach principle. As we see, in the case under discussion, the physical size of one particle defined as the width of its Gaussian probability density,

\[ L_{ph} = \sqrt{2s} = \frac{4\sqrt{nm|C|}}{\hbar} = \frac{4\sqrt{M|C|}}{\hbar}, \]  

is affected by the presence of other particles even though they remain uncorrelated, and its square is directly proportional to the mass of the Universe. In reality, a free single particle is never alone, it is part of a larger entity, the Universe. Assuming that the Universe consists of \( n \) such non-interacting particles, the formalism of linear quantum mechanics allows us to analyze the equation for the single particle independently of others since the Schrödinger equation for the entire Universe can be separated into \( n \) equations for each of its constituents. There are no side effects to this formal procedure. A similar procedure, as just demonstrated, can be applied to the case of \( n \) particles in the modification under study. Yet, in this case the presence of other particles affects the physical size of each separate particle. In the limit of large \( n \), their size would become macroscopic, even gigantic. The experiment however suggests that quantum particles do not come in sizes greater than some \( l_{max} \) that is assumed to be microscopic. To meet this experimental constraint, \( C \) should not be greater than \( l_{max}^2 \hbar^2/M \), where \( M \) stands for the total mass of identical particles in a model Universe consisting of only one kind of particles. As already observed, free particles of different masses constitute separate Universes in our scheme. Consequently, if one assumes that there is a soliton associated with every free quantum system in this Universe and that their number is large, but their size is small as it is in the actual Universe we live in where microscopic quantum particles abound, then this constant becomes unimaginably small and the theory becomes “almost linear.” Nevertheless, the solitons persist. On the other hand, if we could somehow determine (the upper bound on) the coupling constant \( C \), then knowing the size of one particle (say \( s \)) we would be able to tell (the lower bound on) the mass of the model Universe of our modification!

Let us now discuss the case of \( n \) one-dimensional particles coupled to harmonic oscillators. The wave function for this system of particles is taken in the form of (13-14) and the potential is assumed to be

\[ V(x_1, x_2, ..., x_n, t) = \sum_{i=1}^{n} k_i (x_i - v_it)^2 = \frac{1}{2} \sum_{i=1}^{n} m_i \omega_i^2 (x_i - v_it)^2. \]  

The energy equation (2) is separable if and only if

\[ b_i = \frac{m_i}{\hbar} \]  

and

\[ s_i^4 = \frac{2\hbar^2}{2a_i^2 \hbar^2 + \frac{1}{2}m_i^2 \omega_i^2}. \]  

The continuity equation (1) is soluble for each \( x_i \) separately provided

\[ s_i^2 = \frac{-8m_iC (\sum a_j)}{a_i \hbar^2} \]  

and

\[ s_i^2 a_i m_j = s_j^2 a_j m_i \]
for any $i$ and $j$. Equation (28) implies that either all $a_j$ are positive or negative whereas equations (27) and (29) lead to the first consistency condition,

\[
\left( \frac{s_i}{s_j} \right)^4 = \frac{2\hbar^2 a_j^2 + \frac{1}{2}m_j^2 \omega_j^2}{2\hbar^2 a_i^2 + \frac{1}{2}m_i^2 \omega_i^2} = \left( \frac{m_j a_j}{m_i a_i} \right)^2 ,
\]

valid for any $i$ and $j$ which in the free particle case produces the condition that the masses of particles must be equal. Now, however, this condition does not have to be met, which, in other words, means that the potential can remove the degeneration in the mass. This last formula can be transformed into

\[
\left[ \left( \frac{m_i}{m_j} \right)^2 - 1 \right] = \frac{m_j^2 \omega_j^2}{4\hbar^2 a_j^2} \left[ 1 - \left( \frac{m_i}{m_j} \right)^4 \left( \frac{a_j}{a_i} \right)^2 \left( \frac{\omega_i}{\omega_j} \right)^2 \right].
\]

This constitutes the equation from which to determine $a_j^2$ as a function of the other parameters. In what follows, we will consider only the simplest case that corresponds to $m_i = m_j = m$ for any $i$ and $j$. Then

\[
a_j = \pm \frac{\omega_j}{\omega_i}
\]

for arbitrary $i$ and $j$. Consequently,

\[
s_i^2 = -\frac{8Cm \sum \omega_j}{\hbar^2 \omega_i},
\]

where $C = -|C|$. The last formula and (27) lead to another consistency condition

\[
a_i^2 = \omega_i^2 \left[ -\frac{\hbar^4}{64C^2 m^2 (\sum \omega_j)^2} - \left( \frac{m}{2\hbar} \right)^2 \right]
\]

which essentially is an equation for $a_1$ in terms of the parameters of the system. Knowing the value of $a_1$ one can determine other $a_i$'s from (32). Moreover, since $a_i^2 > 0$ (the case of $a_i^2 = 0$ corresponds to linear quantum mechanics), we obtain that

\[
\sum_{i=1}^{n} \omega_i < \frac{\hbar^3}{4|C|m^2}.
\]

This formula replaces (19) valid for free particles, but (20) still holds in this case. Now, the energy equation (2) yields

\[
\hbar \frac{\partial c(t)}{\partial t} = -\frac{1}{2} \sum_{i=1}^{n} mv_i^2 - \frac{\hbar^2 (\sum a_j)}{2m s_i^2 a_i} - \frac{1}{2} \sum_{i=1}^{n} m \omega_i^2 s_i^2
\]

so that the total energy of the system is

\[
E = -\hbar \left\langle \frac{\partial S}{\partial t} \right\rangle = E_{\text{kin}} + \frac{\hbar^4}{16m^2 |C|} + \frac{4|C|m^2}{\hbar^2} \left( \sum_{i=1}^{n} \omega_i \right)^2 < E_{\text{kin}} + \frac{5\hbar^4}{16m^2 |C|}.
\]

We observe that the wave function of one particle is affected not only by the frequency of the oscillator it is coupled to but also by the frequencies of other oscillators. One can thus wonder if this violates causality. One can imagine that, for simplicity, in a system of two particles coupled to oscillators of different frequencies changing the frequency of one of them will affect the probability
density of the particle coupled to the other one. The change in question means a global change in the potential of one of the particles as the potential is affected in its entire domain. In this sense, this differs from a telegraph that employs spins on which operations are local. Nevertheless, in a system like that one can transmit information instantaneously from one observer to another one separated by an arbitrary distance and thus the causality in this system can indeed be violated. The energy changes in this process, but one cannot tell whether it is transmitted or not for it is not localized.

This is not necessarily unexpected for, as already pointed out in [1], the equations of motion are not separable in the weak sense [7] that assumes a factorizable wave function for a compound system. On the other hand, the discussed model constitutes an elementary example of a causality violation mechanism in nonlinear quantum mechanics which does not involve the spin [10, 11, 12, 13] or any complex reasoning. If only because of that, it may deserve some attention. It should also be noted that situations like that are rather generic in nonlinear systems. Nevertheless, the consensus is that situations of this kind are unphysical and because of that they are dismissed.

The problem in question can, in fact, be alleviated within the theory itself by selecting the parameters of the solution in such a way that the demonstrated causality violation does not occur. One does that by assuming that all \(a_i\)'s are equal. As a result, (30) implies now that

\[
\omega_j^2 = \left(\frac{m_1}{m_j}\right)^4 \omega_1^2 + 2h^2 a^2 \left[\left(\frac{m_1}{m_j}\right)^2 - 1\right],
\]

(38)

whereas (27) and (28) yield that

\[
\omega_i^2 = \frac{4h^6}{64m_i^4 C^2 n^2} - \frac{4a^2 h^2}{m_i^2}.
\]

(39)

Applying the last formula for \(i = 1\) and using it in (38) gives

\[
\omega_j^2 = \frac{4h^6}{64m_j^4 C^2 n^2} - \frac{4m_j^2 a^2 h^2}{m_j^4} + 2h^2 a^2 \left[\left(\frac{m_1}{m_j}\right)^2 - 1\right],
\]

(40)

which is equal to (39) specialized for \(i = j\) only if \(m_j = m_1\). In view of the generality of this argument one is lead to the conclusion that all the masses \(m_i\) must be the same and so the particles-oscillators are identical. The parameter \(a\) is determined by the frequency \(\omega\) and other constants,

\[
a^2 = \frac{h^4}{64m^2 C^2 n^2} - \frac{m^2 \omega^2}{4h^2}.
\]

(41)

As seen from the last formula or (35), the frequency, which is the same for all oscillators, has to satisfy the condition

\[
\omega < \frac{h^3}{4|C| nm^2}
\]

(42)

corresponding to \(a^2 > 0\). The energy of the field of oscillators is now

\[
E = E_{\text{kin}} + \frac{h^4}{16m^2 |C|} \frac{4|C| m^2 n^2}{h^2} \omega^2 < E_{\text{kin}} + \frac{5h^4}{16m^2 |C|}.
\]

(43)

The picture that emerges from this is that of a field of \(n\) identical particles oscillating in unison. Changing the frequency of a single oscillator is possible only if the same change takes place in the
entire field. This can happen instantaneously but does not lead to any quantum-mechanical violation of causality because the probability density of the particles is not changed in the process, as opposed to their phase. The situation in question is somewhat similar to the EPR phenomenon \[8, 9\] in that the observer measuring or changing the frequency of one oscillator knows what the frequency of other oscillators is going to be at the very same moment. However, unlike in the standard EPR experiment, the nonlocal impact of measurement is not confined to just one particle but has a global sweep. Moreover, the particles are not entangled as is the case for the EPR type of correlations. Yet, by changing the frequency of her oscillator, one observer can change the frequency of all the other oscillators in the Universe. If there are any observers associated with them, they will notice this change. It is obvious that one can instantaneously transmit information in this way, although it is not clear if the same is true about the energy for the latter is not localized. It does change though in this process.

It should be stressed that the discussed solitonic solution does not invalidate nor replaces the multi-oscillator solutions of linear quantum mechanics. Such solutions still exist in the modification under study. However, if the wave function of the Universe is to be factorizable, the oscillators of linear theory cannot share this Universe with their solitonic brethren. A wave function that would accommodate both kinds of oscillators is not supported by the equations of motion as seen from (29). This relation cannot be satisfied if one of \(a_i\)'s is zero which would correspond to a “linear” quantum oscillator. Either all \(a_i\)'s must be zero or none. As a matter of fact, even the free solitons and the solitons coupled to harmonic oscillators cannot belong to the same Universe if they are to be uncorrelated. Let us now demonstrate this fact. Without any loss of generality, we can assume that the Universe contains only one free particle, all others being oscillators. Putting \(\omega_1 = \omega_i = 0\) in (30) and solving it for \(a_j\) gives

\[
a_j = \pm \frac{m_j \omega_j}{2\hbar \sqrt{1 - \eta^2}},
\]

where we need to assume that \(\eta = m_1/m_j < 1\). \(\eta\) equal 1 implies that \(\omega_j = 0\) and so is not interesting, but this observation will prove useful later on. Inserting this into (27) leads to

\[
s_j^4 = \frac{4\hbar^2 \left[1 - \eta^2\right]}{m_j^2 \omega_j^2 \left[2 - \eta^2\right]}.
\]

Now, from (27) and (28) applied to the free particle we obtain that

\[
1 = \left(s_1^2 a_1\right)^2 = \left(\frac{8m_1 |C|}{\hbar^2}\right)^2 \left(\sum a_j\right)^2.
\]

Applying (28) to the \(j\)-th oscillator and using (46) and (44) yields

\[
s_j^4 = \left(\frac{8m_j |C|}{a_j \hbar^2}\right)^2 \left(\sum a_j\right)^2 = \frac{m_j^2}{m_1^2 a_j^2} = \frac{4\hbar^2 \left[1 - \eta^2\right]}{m_j^2 \omega_j^2}.
\]

If (45) and (47) are to be equal, \(\eta\) must be 1, which means that the \(j\)-th particle is also free. Therefore, we conclude that the factorizable wave function of the Universe can describe either only the free or only the coupled solitons, but not both those species together.

Finally, following the scheme for the \(n\) one-dimensional particles, it is straightforward to find a \(d\)-dimensional one-particle solitonic solution. The wave function of such a particle is given by (13-14)

9
with \( n \) replaced by \( d \). The energy of a single \( d \)-dimensional quanton is

\[
E = \frac{1}{2} \sum_{i=1}^{d} mv_i^2 + \frac{\hbar^2}{2s^2} d = E_{\text{kin}} + \frac{\hbar^4}{16m^2|C|},
\]

(48)

where

\[
s^2 = \frac{8d|C|m}{\hbar^2}.
\]

(49)

The last formula assumes that the particle is spherical. One can now treat even a more general case of \( n \) particles in \( d \) dimensions, which would remove the assumption of one-dimensional particles. Instead of (49) we can also use (20) to obtain a greater variety of shapes for one three-dimensional particle. For instance, if, say, \( s_1 \) and \( s_2 \) are much greater than \( s_3 \) we obtain a sheet-like configuration extended in the \( x_1-x_2 \) plane, while for \( s_1 \) and \( s_2 \) being small but of the same order and \( s_3 \) being much greater we obtain a string-like configuration extended along the direction of \( x_3 \). Even if these configurations have different shapes they are topologically equivalent to a three-dimensional spherical particle. Moreover, they all have the same energy.

Finally, let us note that if the multi-particle equations are derived from the Lagrangian

\[
-L_n(R, S) = \hbar R^2 \frac{\partial S}{\partial t} + \sum_{i=1}^{n} \frac{\hbar^2}{2m_i} \left[ (\vec{\nabla}_i R)^2 + R^2 (\vec{\nabla}_i S)^2 \right] + CR^2 \sum_{i=1}^{n} (\Delta_i S)^2 + R^2 V,
\]

(50)

they are weakly separable and do not lead to any strange nonlocal effects in the systems we examined above. In particular, one can assume here that \( C = C_i \), i.e., the coupling constant is particle-dependent, similarly as the mass. For \( n = 1 \), the Lagrangian in question reduces to the same form as the Lagrangian of (4), but in the multi-particle case these Lagrangians describe different theories. The nonlocal effects may however appear when nonfactorizable wave functions are considered.

3 Conclusions

We have presented the \( n \)-particle free solitonic solution and the solution describing \( n \) particles coupled to harmonic oscillators in the nonlinear modification of the Schrödinger equation put forward recently [1]. In addition, a particular solution for two free solitons of the same mass that does not stem from this general scheme has also been presented. This solution represents two particles whose compound wave function does not factorize and whose kinetic energy is, surprisingly enough, the same as the kinetic energy of one particle.

The most remarkable property of the free \( n \)-particle solution is the remnant inseparability that manifests itself in the wave function of individual particles depending on the properties of other particles of the system. In the simplest case of identical particles, it is the total number of particles that appears in the wave function of an individual particle. As a result, the width of each particle’s Gaussian probability density, a local property, is directly linked to the mass of the entire Universe in a very Machian manner. Nevertheless, they all evolve independently. In the case of solitons coupled to harmonic oscillators the probability density of one particle is affected not only by the frequency of its own oscillator, but by the frequencies of other oscillators as well. This leads to a causality violation as any change in the frequency of one of the oscillators is instantaneously detected by the observers related to other oscillators. By measuring the probability density of the particles coupled to their
oscillators, they realize that someone is “messing” with these particles. We have shown, however, that it is possible to avoid this particular problem by an appropriate choice of the parameters of the solution. Namely, if the ratio of $a_i/a_j$ is one, the nonseparability becomes somewhat similar to the EPR kind, but without the entanglement typical for the EPR correlations. The probability density of a single oscillator is then not related to what is happening to the other oscillators, but they all have to oscillate with the same frequency and so in this sense they remain correlated. It is possible to transmit information in this way, just by changing the frequency of one oscillator, similarly as in the more generic case. The simplest way to avoid the discussed nonlocality problems is to adopt a weakly separable multi-particle extension of the basic one-particle nonlinear equation, although this does not guarantee that similar problems will not occur for nonfactorizable wave functions.

It would be interesting to see how one can circumvent this problem in the strong separability framework advocated by Czachor [14] that originated in the work of Polchinski [1] and Jordan [13]. This approach chooses as its starting point the nonlinear von Neumann equation for the density matrices [16] and proceeds from there to the $n$-particle extension. In a sense, one can call this approach “effective” as opposed to the “fundamentalist” one. The latter approach does not resort to reformulating in terms of density matrices various proposed in the literature nonlinear modifications of the Schrödinger equation that, similarly as the linear equation itself, are postulated to be obeyed by pure states. The problem of nonlocality is confounded since these approaches, even though they belong to the same strong separability framework, yield different results. For instance, the Białynicki-Birula and Mycielski modification that is both weakly and strongly separable in the effective approach turns out to be essentially nonlocal in the fundamentalist approach for a general nonfactorizable two-particle wave function, as recently demonstrated by Lücke [17]. The same applies to the Doebner-Goldin modification that being weakly separable and strongly in the effective approach fails to maintain the separability for a more general nonfactorizable wave function describing a system of two particles if the fundamentalist approach is used. Only in some special case, that corresponds to the linearizable Doebner-Goldin equation, are the particles separable in the latter approach [18].

One of the consequences of the discussed residual inseparability of free particles is the smallness of the coupling constant $C$. In the simplest case of identical particles, the width of the probability density of a single particle is proportional to the total number of particles and the constant in question. In the limit of large $n$ which is the most natural to consider if one assumes that every quantum system is associated with a particle, this width can be macroscopic if not just gigantic unless $C$ is really very small. Since the experiments do not support the view that quantum particles are of macroscopic size, the nonlinear coupling constant must indeed be very small. One needs to realize though that this is only a model which rests on the assumptions that there exist a particle associated with every quantum system, that quantum systems are microscopic, and that all of them have the same mass. None of these assumptions have to be true, but if there existed a Universe where these assumptions were correct then the coupling constant of the discussed nonlinear extension of the Schrödinger equation would be incredibly small.

A novel property that also emerged from the free multi-particle solution is an exclusion rule that forbids particles of different masses to be part of the same Universe if they are to be uncorrelated and separated. The same holds true for the solitonic oscillators of common $a$ which are supposed to share the frequency. This is somewhat characteristic of superselection rules that stem from the symmetry considerations, but is much stronger in its implications for the standard superselection rule applicable in this case only forbids superpositions of quantum mechanical systems of different masses. We propose that the rules of this type be called supersuperselection rules. Similar to it is
the univalence supersuperselction rule that under the same conditions prevents different species of particles, such as the free solitons and the solitonic oscillators in our case, from belonging to the same Universe. The same rule applies also to the oscillators of linear quantum mechanics and the “nonlinear” oscillators of the modified Schrödinger equation. Similarly as the nonlocal effects, the rules in question are native to only one of the discussed n-particle extensions; they do not emerge if the multi-particle equations are weakly separable.

Acknowledgments

I would like to thank Professor P. O. Mazur for bringing my attention to the work of Professor Staruszkiewicz that started my interest in nonlinear modifications of the Schrödinger equation. A correspondence with Dr. M. Czachor and his comments on the preliminary versions of this paper are gratefully acknowledged as is a correspondence with Professor Wolfgang Lücke concerning his recent work. This work was partially supported by the NSF grant No. 13020 F167 and the ONR grant R&T No. 3124141.

References

[1] W. Puszkarz, Nonlinear Phase Modification of the Schrödinger Equation, preprint quant-ph/9710010.

[2] W. Puszkarz, On Solutions to the Nonlinear Phase Modification of the Schrödinger Equation, preprint quant-ph/9903010.

[3] W. Puszkarz, Energy Ambiguity in Nonlinear Quantum Mechanics, preprint quant-ph/9802001.

[4] C. Cisneros, R. P. Martínez-y-Romero, H. N. Núñez-Yépez, and A. L. Salas-Brito, Eur. J. Phys. 19, 237 (1998); preprint quant-ph/9809059.

[5] R. Mirman, Found. Phys. 9, 283 (1979).

[6] Mach’s Principle: From Newton’s Bucket to Quantum Gravity, edited by J. B. Barbour and H. Pfister, (Birkhäuser, Boston, 1995).

[7] I. Białynicki-Birula and J. Mycielski, Ann. Phys. (N.Y.) 100, 62 (1976).

[8] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).

[9] D. Bohm, Quantum Theory, (Dover, New York, 1957).

[10] N. Gisin, Phys. Lett. A 143, 1 (1990).

[11] J. Polchinski, Phys. Rev. Lett. 66, 397 (1991).

[12] M. Czachor, Found. Phys. Lett. 4, 351 (1991).

[13] M. Czachor, in Bell’s Theorem and the Foundations of Modern Physics, edited by A. van der Merwe, F. Selleri, and G. Tarozzi (World Scientific, Singapore, 1993), p. 135.
[14] M. Czachor, Phys. Rev. A. 57, 4122 (1998).

[15] T. F. Jordan, Ann. Phys. (N. Y.) 225, 83 (1993).

[16] M. Czachor, M. Kuna, S. B. Leble, and J. Naundts, Nonlinear von Neumann-Type Equations, preprint quant-ph/9904110.

[17] W. Lücke, Nonlocality in Nonlinear Quantum Mechanics, preprint quant-ph/9904016.

[18] W. Lücke, Gisin Nonlocality of the Doebner-Goldin 2-Particle Equation, preprint quant-ph/9710033.