The weak decay \( B_c \) to \( Z(3930) \) and \( X(4160) \) by Bethe–Salpeter method

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Abstract Considering \( Z(3930) \) and \( X(4160) \) as \( \chi_{c2}(2P) \) and \( \chi_{c2}(3P) \) states, the semileptonic and nonleptonic of \( B_c \) decays to \( Z(3930) \) and \( X(4160) \) are studied by the improved Bethe–Salpeter (B–S) Method. The form factors of decays are calculated through the overlap integrals of meson wave functions in the whole accessible kinematical range. The influence of relativistic corrections are considered in the exclusive decays. Branching ratios of \( B_c \) weak decays to \( Z(3930) \) and \( X(4160) \) are predicted. Some of the branching ratios are: \( Br(B_c^+ \rightarrow Z(3930)e^+\nu_e) = (3.03^{+0.09}_{-0.16}) \times 10^{-4} \) and \( Br(B_c^+ \rightarrow X(4160)e^+\nu_e) = (3.55^{+0.83}_{-0.37}) \times 10^{-6} \). These results may provide useful information to discover \( Z(3930) \) and \( X(4160) \) and the necessary information for the phenomenological study of \( B_c \) physics.

1 Introduction

During the past decade years, more and more charmonium and charmonium-like states were discovered experimentally. Such as the \( X(3915) \) was reported by Belle Collaboration in \( \gamma \gamma \rightarrow \omega J/\psi \) process [1]. \( Z(3930) \) was observed in the process \( \gamma \gamma \rightarrow D \bar{D} \) by Belle Collaboration in 2006, the corresponding mass and width were \( M = 3929 \pm 5 \pm 2 \) MeV and \( \Gamma = 29 \pm 10 \pm 2 \) MeV, respectively [2]. In 2010, BABAR Collaboration also observed the \( Z(3930) \) in \( \gamma \gamma \) production of the \( D \bar{D} \) system, with the mass and width being \( M = 3926.7 \pm 2.7 \pm 1.1 \) MeV and \( \Gamma = 21.3 \pm 3.8 \pm 3.6 \) MeV, respectively [3]. Now Particle Data Group(PDG) give lists the mass and width of \( Z(3930) \) as \( M = 3927.2 \pm 2.6 \) MeV and \( \Gamma = 24 \pm 6 \) MeV [4]. And the properties of \( Z(3930) \) are consistent with the expectations for the \( \chi_{c2}(2P) \) state [5–7]. Then Belle Collaboration reported a new charmonium-like state \( X(4160) \) from the processes \( e^+e^- \rightarrow J/\psi D^+\bar{D}^* \), which has the mass and width \( M = 4156^{+25}_{-20} \pm 15 \) MeV and \( \Gamma = 139^{+111}_{-61} \pm 21 \) MeV, respectively [8].

The quark structures were still not fully understood in these charmonium-like states which are called XYZ states, thus people studied the properties of XYZ states by different methods [9–22]. In this work, we only consider two of them: \( Z(3930) \) and \( X(4160) \). The structures of \( Z(3930) \) and \( X(4160) \) were already studied by some theoretical methods. Reference [9] studied \( B_c \) semileptonic decay to \( Z(3930) \) and \( X(4160) \) which were assumed as \( \chi_{c2}(2P) \) and \( \chi_{c2}(3P) \) states. According to study the vector-vector interaction within the framework of the hidden gauge formalism, Ref. [10] found that three resonances \( Y(3940), Z(3930) \) and \( X(4160) \) which can be assigned to the states with \( J^P_C = 0^{++}, 2^{++} \) and \( 2^{++} \), respectively. Taking \( Z(3915) \) and \( Z(3930) \) as \( \chi_{c0}(2P) \) and \( \chi_{c2}(2P) \), respectively. Reference [11] investigated the \( X(3915) \) and \( Z(3930) \) decays into \( J/\psi \omega \). Reference [12] studied the strong decay of \( X(3930) \) which was considered as \( \chi_{c2}(2P) \). Reference [13] studied the mass spectra of the hidden-charm tetraquark states in the framework of QCD sum rules, and they got the \( X(4160) \) may be classified as either the scalar or tensor \( q_cq_c\bar{c}\bar{c} \) tetraquark state. Using the NRQCD factorization approach, Ref. [18] calculated the branching fractions of \( \Upsilon(nS) \rightarrow J/\psi X \) with \( X = X(3940) \) or \( X = X(4160) \). In Ref. [19], they also explored the properties and strong decays of \( X(3940) \) and \( X(4160) \) as the \( \eta_c(3S) \) and \( \eta_c(4S) \), respectively. Reference [20] calculated the strong decay of \( X(4160) \) which was assumed as \( \chi_{c0}(3P), \chi_{c1}(3P), \chi_{c2}(2D) \) or \( \eta_c(4S) \) by the \( 3P_0 \) model. Reference [23] studied the strong decays of \( X(3940) \) and \( X(4160) \) as the \( \eta_c(3S) \) and \( \eta_c(4S) \) with the \( 3P_0 \) model, and the results showed that \( \eta_c(4S) \) was not
good candidate of $X(4160)$. According to the mass spectra and the properties of $Z(3930)$ and $X(4160)$ in Refs. [9–13], $Z(3930)$ and $X(4160)$ have the possibility to be $\chi_{c2}(2P)$ and $\chi_{c2}(3P)(J^{PC} = 2^{++})$, respectively.

The interpretations of $Z(3930)$ and $X(4160)$ are not the major work in this paper, we only consider $Z(3930)$ and $X(4160)$ as charmonium states with the possible quantum numbers, then study their production in $B_c$ decays. We will consider $Z(3930)$ and $X(4160)$ as $P$-wave charmonium states $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$, respectively. Then we focus on the productions of $Z(3930)$ and $X(4160)$ in exclusive weak decays of $B_c$ meson by the improved the Bethe–Salpeter (B–S) Method. On the one hand, the $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ have larger relativistic correction than that of $\chi_{c2}(1P)$, so a relativistic model is needed in a careful study; on the other hand, this study can improve the knowledge of $B_c$ meson, which is an ideal particle to study the weak decays, since it decays weakly only. The properties of $B_c$ meson have been studied by different relativistic constituent quark models [24–32], such as the covariant light-front quark model [33,34], the perturbative QCD factorization approach [35] and so on. We also have discussed the properties of $B_c$ meson by the improved B–S method, include $B_c$ decays to $P$-wave mesons, the rare weak decays and rare radiative decays of $B_c$, the nonleptonic charmless decays of $B_c$, and so on [36–42]. In previous work, we only studied $B_c$ decays to $\chi_{c2}(2P)$ state [36], because when the final states are $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, the corresponding branching ratios are very small, and there were only limited data of $B_c$ available. Now the large hadron collider (LHC) will produce as many as $5 \times 10^{10}$ $B_c$ events per year [43,44]. The huge amount of $B_c$ events will provide us a chance to study $B_c$ decay to $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, and some channels also provide an opportunities to discover new particles in $B_c$ decays.

The paper is organized as follows. In Sect. 2, we give the formulations of the exclusive semileptonic and nonleptonic decays. We show the hadronic weak-current matrix elements in Sect. 3. The wave functions of initial and final mesons are given in Sect. 4. The corresponding results and conclusions are presented in Sect. 5. Finally in the Appendix, we present the instantaneous Bethe–Salpeter equation.

### 2 The formulations of semileptonic decays and nonleptonic decays of $B_c$

In this section we present the formulations of semileptonic decays and nonleptonic decays of $B_c$ meson to $Z(3930)$ and $X(4160)$ which are considered as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, respectively.
differential width i.e. $\Gamma_1$ and have the structure as follows:

For the nonleptonic decay $B_c \to X M_2$, the decay can be reduced to:

$$\Gamma = \frac{1}{2 M^2 (2\pi)^9} \int \frac{d^3 \vec{p}_f d^3 \vec{p}_l}{E_f E_l} \times \frac{d^3 \vec{p}_i}{2 E_v} (2\pi)^4 \delta^4 (P - P_f - p_l - p_v) \Sigma_{s_v, s_f, s_X} |T|^2,$$

(5)

where $E_f$, $E_l$ and $E_v$ are the energies of the charmonium, the charged lepton and the neutrino respectively. If we define $x \equiv E_f / M$, $y \equiv (P - P_f)^2 / M^2$, the differential width of the decay can be reduced to:

$$\frac{d^2 \Gamma}{dx dy} = |V_{bc}|^2 \frac{G_F M^5}{64\pi^3} \left\{ \frac{2\alpha}{M^2} (y - m_f^2) \right\}$$

$$+ \beta_{++} \left[ 4 \left( 2x (1 - \frac{M_f^2}{M^2} + y) - 4x^2 - y \right) \
+ \frac{m_f^2}{M^2} \left( 8x + 4 \frac{M_f^2}{M^2} - 3y - \frac{m_f^2}{M^2} \right) \right]$$

$$+ (\beta_{+-} + \beta_{-+}) \frac{m_f^2}{M^2} \left( 2 - 4x + y - 2 \frac{M_f^2}{M^2} + \frac{m_f^2}{M^2} \right)$$

$$+ \beta_{--} \frac{m_f^2}{M^2} \left( y - \frac{m_f^2}{M^2} \right)$$

$$- 2\gamma \left[ 1 - \frac{M_f^2}{M^2} - 4x + y + \frac{M_f^2}{M^2} \right]$$

$$+ 2\gamma \frac{M_f^2}{M^2} \left( 1 - \frac{M_f^2}{M^2} \right) \right\}. \tag{6}$$

The total width of the decay is just an integration of the differential width i.e. $\Gamma = \int dx \int dy \frac{d^2 \Gamma}{dx dy}$.

### 2.2 Nonleptonic decays of $B_c$

For the nonleptonic decay $B_c \to X + M_2$ in Fig. 2, the relevant effective Hamiltonian $H_{eff}$ is [45,46]:

$$H_{eff} = \frac{G_F}{\sqrt{2}} \left\{ V_{bc} [c_1(\mu) O^{bc}_1 + c_2(\mu) O^{bc}_2] + h.c. \right\}, \tag{7}$$

where $c_i(\mu)$ are the scale-dependent Wilson coefficients. $O_i$ are the operators responsible for the decays constructed by four quark fields and have the structure as follows:

![Fig. 2 Feynman diagram of the nonleptonic decay $B_c \to XM_2$.](image)

### 3 The hadronic weak-current matrix elements

The calculation of the hadronic weak-current matrix element are different for different models. In this paper, we...
combine the B–S method which is based on relativistic B–S equation with Mandelstam formalism [50] and relativistic wave functions to calculate the hadronic matrix element. The numerical values of wave functions have been obtained by solving the full Salpeter equation which we will introduce in Appendix. As an example, we consider the semileptonic decay $B_c \to X \ell^+ \nu_\ell$ in Fig. 1. In this way, at the leading order the hadronic matrix element can be written as an overlap integral over the wave functions of initial and final mesons [51],

$$
\langle X(P_f, \ell) | J^{\mu} | B_c(P) \rangle = \int \frac{d\vec{q}}{(2\pi)^3} \text{Tr} \left[ \bar{\psi}_{r_f}^{++}(\tilde{\vec{q}}_f) \frac{P}{M} \psi_r^{+++}(\vec{q}) \gamma^\mu (1 - \gamma_5) \right],
$$

where $\vec{q}$ ($\vec{q}_f$) is the relative three-momentum between the quark and anti-quark in the initial (final) meson and $\vec{q}_f = \vec{q} - \frac{m_{f1}^2 + m_{f2}^2}{m_{f1}^2 + m_{f2}^2} \vec{P}_f$. $\vec{P}_f$ is the three dimensional momentum of $X$, $\vec{q}_f^{++}$ is the positive Salpeter wave function of $B_c$ meson and $\vec{q}_f^{+++}$ is the positive Salpeter wave function of $X$ meson, $\vec{q}_f^{++} = \gamma_0 (\vec{q}_f^{++})^\dagger \gamma_0$. The detailed calculation of the hadronic matrix element Eq. (10) which is a function of final meson momentum $P_f$ were discussed by Ref. [51], so the Eq. (10) is suitable for the whole kinetic region. We have calculated $B_c$ weak decays to $S$-wave and $P$-wave mesons [36,37,42] with this hadronic matrix element in previous work, and the results were consistent with the results of some other different models. So the $B_c$ weak decays to $Z(3930)$ and $X(4160)$ are calculated by the same method in this work. The corresponding Salpeter wave functions for the different mesons are shown in the next section.

4 The relativistic wave functions of meson

4.1 For $B_c$ meson with quantum number $J^P = 0^-$

The general form for the relativistic wave function of pseudoscalar meson $B_c$ can be written as [52]:

$$
\psi^{++}_0(\vec{q}) = \left[ f_1(\vec{q}) P + f_2(\vec{q}) M + f_3(\vec{q}) \not\! q + f_4(\vec{q}) \frac{P \not\! q}{M} \right] \gamma_5,
$$

where $M$ is the mass of the pseudoscalar meson, and $f_i(\vec{q})$ are functions of $|\vec{q}|^2$. Due to the last two equations of Eq. (A7): $\psi^{++}_0 = \psi^{--}_0 = 0$, we have:

$$
f_3(\vec{q}) = \frac{f_2(\vec{q}) M (-\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2},
$$

$$
f_4(\vec{q}) = -\frac{f_1(\vec{q}) M (\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2},
$$

where $m_1, m_2$ and $\omega_1 = \sqrt{m_1^2 + \vec{q}^2}$, $\omega_2 = \sqrt{m_2^2 + \vec{q}^2}$ are the masses and the energies of quark and anti-quark in $B_c$ mesons, $q_1 = q - (q \cdot P/M^2) P$, and $q_1^2 = -|\vec{q}|^2$.

The numerical values of radial wave functions $f_1$, $f_2$ and eigenvalue $M$ can be obtained by solving the first two Salpeter equations in Eq. (A7). According to the Eq. (A6) the relativistic positive wave function of pseudoscalar meson $B_c$ in C.M.S can be written as [52]:

$$
\psi^{++}_0(\vec{q}) = b_1 \left[ b_2 + \frac{P}{M} + b_3 \not\! q + b_4 \frac{P \not\! q}{M} \right] \gamma_5,
$$

where the $b_i$s ($i = 1, 2, 3, 4$) are related to the original radial wave functions $f_1$, $f_2$, quark masses $m_1, m_2$, quark energy $w_1, w_2$, and meson mass $M$:

$$
b_1 = \frac{M}{2} \left( f_1(\vec{q}) + f_2(\vec{q}) \frac{m_1 + m_2}{\omega_1 + \omega_2} \right),
$$

$$
b_2 = \frac{\omega_1 + \omega_2}{m_1 + m_2},
$$

$$
b_3 = -\frac{(m_1 - m_2)}{m_1 \omega_2 + m_2 \omega_1},
$$

$$
b_4 = \frac{(\omega_1 + \omega_2)}{(m_1 \omega_2 + m_2 \omega_1)}.
$$

4.2 For $Z(3930)$ and $X(4160)$ mesons with quantum number $J^P = 2^+-$

Considering $Z(3930)$ and $X(4160)$ as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$, the general expression of the relativistic wave function can be written as [53]

$$
\psi^{++}_0(\vec{q}_f) = \epsilon_{\mu \nu} q_{f \perp} \left[ f_1'(\vec{q}_f) + \frac{P_f}{M_f} f_2'(\vec{q}_f) + \frac{q_{f \perp} q_{f \perp}}{M_f^2} f_3'(\vec{q}_f) \right] \gamma_\nu
$$

$$
+ \frac{P_f}{M_f} f_4'(\vec{q}_f) + \frac{q_{f \perp} q_{f \perp}}{M_f^2} \left[ f_5'(\vec{q}_f) + \frac{P_f}{M_f} f_6'(\vec{q}_f) + \frac{q_{f \perp} q_{f \perp}}{M_f^2} \right] \gamma_\mu
$$

$$
+ \frac{i}{M_f^2} f_7'(\vec{q}_f) e^{\mu \alpha \beta \delta} P_f q_{f \perp} g_{\beta \gamma} \gamma_5 \gamma_\nu
$$

with the constraint on the components of the wave function:

$$
f_1'(\vec{q}_f) = \left[ q_{f \perp} f_2'(\vec{q}_f) + M_f^2 f_3'(\vec{q}_f) \right]/M_f m_{f1}^2,
$$

$$
f_2'(\vec{q}_f) = 0, \quad f_3'(\vec{q}_f) = 0, \quad f_4'(\vec{q}_f) = \frac{f_6'(\vec{q}_f) M_f}{m_{f1}^2}.
$$

Then we have the reduced wave function $\psi^{++}_0(\vec{q}_f)$ as:

$$
\psi^{++}_0(\vec{q}_f) = \epsilon_{\mu \nu} q_{f \perp} \left[ q_{f \perp} \left[ a_1 + a_2 \frac{P_f}{M_f} + a_3 \frac{q_{f \perp} q_{f \perp}}{M_f^2} \right] + a_4 \frac{q_{f \perp} P_f}{M_f^2} \right] + q_{f \perp} \left[ a_5 + a_6 \frac{P_f}{M_f} + a_7 \frac{q_{f \perp} q_{f \perp}}{M_f^2} + a_8 \frac{P_f q_{f \perp} q_{f \perp}}{M_f^2} \right],
$$

with

$$
a_1 = \frac{q_{f \perp} q_{f \perp}}{2m_{f1} m_{f1}} + \frac{(f_2'(\vec{q}_f) w_1 - f_6'(\vec{q}_f) m_{f1}) m_{f2}}{2m_{f1} m_{f2}},
$$

$$
a_2 = \frac{q_{f \perp} q_{f \perp}}{2m_{f2} m_{f1}} + \frac{(f_2'(\vec{q}_f) w_2 - f_6'(\vec{q}_f) m_{f1}) m_{f2}}{2m_{f1} m_{f2}}.
$$
Fig. 3 The wave functions of $Z(3930)$ and $X(4160)$

$$a_2 = \frac{f'_i(q_f)w'_2 - f'_j(q_f)m'_1^2)}{2m'_1w'_2},$$

$$a_3 = \frac{1}{2}n_1 + \frac{f'_i(q_f)M_f^2}{2m'_1w'_2},$$

$$a_4 = \frac{1}{2} \left( - \frac{w'_1}{m'_1} \right) n_1 + \frac{f'_i(q_f)M_f^2}{2m'_1w'_2},$$

$$a_5 = \frac{M_f}{2} n_2, \quad a_6 = \frac{M_f m'_1}{2w'_1} n_2, \quad a_7 = 0, \quad a_8 = \frac{M_f^2}{2w'_1} n_2,$$

$$n_1 = \frac{1}{2} \left( f'_i(q_f) + f'_j(q_f) \frac{m'_1}{w'_1} \right),$$

$$n_2 = \frac{1}{2} \left( f'_i(q_f) - f'_j(q_f) \frac{w'_1}{m'_1} \right),$$

where $M_f, P_f, f'_i(q_f)$ are the mass, momentum and the radial wave functions of $Z(3930)$ and $X(4160)$, respectively. $m'_1, m'_2$ and $\omega'_1 = \sqrt{m'_1^2 + q'_f^2}, \omega'_2 = \sqrt{m'_2^2 + q'_f^2}$ are the masses and the energies of quark and anti-quark in $Z(3930)$ and $X(4160)$. To show the numerical results of wave functions explicitly, we plot the wave functions of $Z(3930)$ and $X(4160)$ states in Fig. 3.

5 Number results and discussions

In order to fix Cornell potential in Eq. (A11) and masses of quarks, we take these parameters: $a = e = 2.7183, \lambda = 0.21$ GeV$^2, \Lambda_{QCD} = 0.27$ GeV, $\alpha = 0.06$ GeV, $m_0 = 4.96$ GeV, $m_e = 1.62$ GeV, etc. [53], which are best to fit the mass spectra of $B_c$ and other heavy meson states. Taking these parameters to B–S equation, and solving the B–S equation numerically, we get the masses of $Z(3930), X(4160)$ and $B_c$ as: $M_{Z(3930)} = (3.926 \pm 0.167)$ GeV, $M_{X(4160)} = (4.156 \pm 0.170)$ GeV, $M_{B_c} = (6.276 \pm 0.303)$ GeV, varying all the input parameters ($\lambda, \Lambda_{QCD}, \alpha, etc$) simultaneously within $\pm 5\%$ of the central values, we also obtain the uncertainties of masses, and the corresponding wave functions were obtained.
In order to calculate the semileptonic decays of $B_c$ to $Z(3930)$ and $X(4160)$.

5.1 The semileptonic decays

In order to calculate the semileptonic decays of $B_c$ to $Z(3930)$ and $X(4160)$, we use the central values of the CKM matrix elements: $V_{cb} = 0.0406$, and other constants: $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$, which are taken from PDG [4]. Taking the masses and the corresponding wave functions to Eq. (10), we represent the hadronic transition weak-current matrix elements as proper integrations of the components of the $B$–$S$ wave functions. And the hadronic weak-current matrix element can be written as the form factors $k, c_1, c_2, h$. The form factors are related to four-momentum transfer squared $t = (P - P_f)^2 = M_c^2 + M_f^2 - 2ME_f$, which provides the kinematic range for the semileptonic decay of $B_c$. It varies from $t = 0$ to $t = 5.52$ GeV$^2$ for the decays to $Z(3930)$ and from $t = 0$ to $t = 4.48$ GeV$^2$ for the decays to $X(4160)$. In Fig. 4 we give the relations of $(t_m - t)(t_m = (M - M_f)^2$ is the maximum of $t$ and the form factors. Taking the form factor to the Eq. (6), then we will get the leptonic energy spectra $d\Gamma/d|P|$ for semileptonic $B_c$ decay to $Z(3930)$ and $X(4160)$, the leptonic energy spectra are plotted in Fig. 5 which are related to the momentum of the final mesons.

Using the leptonic energy spectra, we calculate the decay widths of the semileptonic $B_c \rightarrow X \ell^+ \nu_\ell$ ($X = Z(3930)$ or $X(4160)$, $\ell = e, \mu, \tau$) and give the results in Table 1. Since $m_\tau$ is very large and $m_\ell \approx m_\mu$ is quite a good approximation for the $B_c$ meson decays, thus only the cases where the lepton is an electron or $\tau$ are given in Table 1. Because of the larger kinematic ranges and the different wave functions in Fig. 3, the corresponding decay widths of $B_c^+ \rightarrow Z(3930)$ are larger than those of $B_c^+ \rightarrow X(4160)$.

| Mode                              | Ours     |
|----------------------------------|----------|
| $B_c^+ \rightarrow Z(3930)$      | $e^+ \bar{\nu}_e$ | $(4.39^{+0.11}_{-0.22}) \times 10^{-1}$ |
| $B_c^+ \rightarrow Z(3930)$      | $\tau^+ \bar{\nu}_\tau$ | $(0.78^{+0.15}_{-0.42}) \times 10^{-3}$ |
| $B_c^+ \rightarrow X(4160)$      | $e^+ \bar{\nu}_e$ | $(5.14^{+1.06}_{-0.46}) \times 10^{-3}$ |
| $B_c^+ \rightarrow X(4160)$      | $\tau^+ \bar{\nu}_\tau$ | $(3.80^{+0.30}_{-0.30}) \times 10^{-6}$ |

5.2 The nonleptonic decays

We only consider two-body nonleptonic decays of $B_c^+$ to $Z(3930)$ and $X(4160)$, and another meson is light meson. Thus, the hadronic transition matrix elements of weak currents have a fixed momentum transfer. To calculate the decay widths basis on Eq. 9, we only need to calculate the annihilation matrix element $\langle M_2 | J_\mu | 0 \rangle$ which is related to the decay constant of $M_2$. The masses and decay constants are: $M_\pi = 0.140$ GeV, $f_\pi = 0.130$ GeV, $M_\rho_0 = 0.775$ GeV, $M_\rho = 0.775$ GeV.
Table 3: The branching ratio (in %) of exclusive semileptonic decay $B_c$ to $Z(3930)$, $X(4160)$ with the lifetime of $B_c: \tau_B = 0.453$ ps

| Mode | Results | Mode | Results |
|------|---------|------|---------|
| $B_c^+ \to Z(3930) e^+ \bar{\nu}_e$ | $(3.03^{+0.69}_{-0.16}) \times 10^{-2}$ | $B_c^+ \to X(4160) e^+ \bar{\nu}_e$ | $(3.55^{+0.83}_{-0.30}) \times 10^{-4}$ |
| $B_c^0 \to Z(3930) \tau^+ \bar{\nu}_\tau$ | $(0.55^{+0.23}_{-0.30}) \times 10^{-3}$ | $B_c^+ \to X(4160) \tau^+ \bar{\nu}_\tau$ | $(2.62^{+0.32}_{-0.20}) \times 10^{-7}$ |
| $B_c^+ \to Z(3930) + \pi$ | $(1.68^{+0.44}_{-0.58}) \times 10^{-4}$ | $B_c^+ \to X(4160) + \pi$ | $(6.17^{+1.35}_{-1.02}) \times 10^{-6}$ |
| $B_c^+ \to Z(3930) + K$ | $(1.24^{+0.33}_{-0.46}) \times 10^{-5}$ | $B_c^+ \to X(4160) + K$ | $(4.21^{+0.75}_{-0.60}) \times 10^{-7}$ |
| $B_c^+ \to Z(3930) + \rho$ | $(5.61^{+1.28}_{-1.33}) \times 10^{-4}$ | $B_c^+ \to X(4160) + \rho$ | $(2.12^{+0.42}_{-0.32}) \times 10^{-4}$ |
| $B_c^+ \to Z(3930) + K^*$ | $(3.43^{+0.54}_{-0.78}) \times 10^{-5}$ | $B_c^+ \to X(4160) + K^*$ | $(1.47^{+0.41}_{-0.25}) \times 10^{-5}$ |

In order to compare the numerical values with experimental measurements in the future, taking the values $\alpha_1 = 1.14$ for nonleptonic decays [45,46], combining the life time of $B_c$ meson, we calculate the branching ratios of the decays and list them in Table 3. Because of $B_c \to Z(3930)$, $X(4160)$ have small kinematic ranges and the wave functions have some minus parts in $Z(3930)$, and $X(4160)$, comparing our results with $B_c$ decays to $\chi_{c2}(1P)$ in Ref. [36], the results are smaller than the results of $B_c$ decay to $\chi_{c2}(1P)$. The uncertainties of decay widths and branching ratios shown in Tables 1, 2 and 3, are very large. The large uncertainties not only come from the phase spaces, but also from the variation of the node of the $2P$ and $3P$ wave functions, which means that a small change of node location will result in large uncertainties.

In summary, considering $Z(3930)$ and $X(4160)$ as $\chi_{c2}(2P)$ and $\chi_{c2}(3P)$ states, respectively, we study the semileptonic and nonleptonic $B_c$ decays to $Z(3930)$ and $X(4160)$ by the improved B–S method which consider the relativistic correction. According to the Mandelstam formalism and the relativistic wave functions of heavy mesons, we get the corresponding decay form factors, and obtain the corresponding decay widths and branching ratios. Because of the minus value in the wave functions of $Z(3930)$ and $X(4160)$ and the small CKM $V_{bc}$, the decay widths and branching ratios are very small. But now the large hadron collider (LHC) will produce as many as $5 \times 10^{10}$ $B_c$ events per year [43,44]. If sufficient events can be observed, some channels will provide us a sizable ratios, such as the branching ratios of the order of $(10^{-6})$ could be measured precisely at the LHC, and maybe they will detect the productions of $Z(3930)$ and $X(4160)$ in $B_c$ exclusive weak semileptonic and nonleptonic decay. Then our results will provide a new way to observe the $Z(3930)$ and $X(4160)$ and the necessary information for the study of $B_c$ meson.

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Data Availability Statement This manuscript has associated data in a data repository. [Authors’ comment: The experimental data used in the present study was published by Belle Collaboration [1,2], BABAR Collaboration [3], Particle Data Group [4].]

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Appendix A: Instantaneous Bethe–Salpeter equation

In this section, we briefly review the Bethe–Salpeter (B–S) equation and its instantaneous one, the Salpeter equation.

The B–S equation is read as [55]:

$$\left(\hat{p}_1 - m_1\right)\chi(q)\left(\hat{p}_2 + m_2\right) = i \int\frac{d^4k}{(2\pi)^4} V(P, k, q)\chi(k),$$

(A1)

where $\chi(q)$ is the B–S wave function, $V(P, k, q)$ is the interaction kernel between the quark and antiquark, and $p_1, p_2$ are the momentum of the quark 1 and anti-quark 2.
We divide the relative momentum $q$ into two parts, $q_{\parallel}$ and $q_{\perp}$,

$$
q^\mu = q^\mu_{\parallel} + q^\mu_{\perp},
q^\mu_{\parallel} \equiv (P \cdot q / M_{\perp}) P^\mu, \quad q^\mu_{\perp} \equiv q^\mu - q^\mu_{\parallel}.
$$

B–S equation Eq. (A1) is a four dimension covariant equation, in order to solve the Eq. (A1), we will take the instantaneous approximation in the interaction kernel $V(P, k, q)$, then the B–S equation will lose the covariance. The effect of instantaneous approximation in $V(P, k, q)$ could be corrected by the retardation effects in $V(P, k, q)$. But the retardation effects in $V(P, k, q)$ are very small for the heavy mesons [56–58], this means that the influence of the instantaneous approximation on the covariance of B–S equation are very small for the heavy mesons. The instantaneous approximation in $V(P, k, q)$ almost don’t influence the wave functions, and the decay matrix elements which involve the heavy mesons mostly unchanged [56]. Our model mostly keeps the covariance in the calculation, and the weak decay results also satisfy the Lorentz-covariance.

In instantaneous approach, the kernel $V(P, k, q)$ takes the simple form [59]:

$$
V(P, k, q) \Rightarrow V(|\vec{k} - \vec{q}|).
$$

Let us introduce the notations $\varphi_p(q^\mu_{\perp})$ and $\eta(q^\mu_{\parallel})$ for three dimensional wave function as follows:

$$
\varphi_p(q^\mu_{\perp}) \equiv i \int \frac{dq^\perp}{2\pi} \chi(q^\mu_{\parallel}, q^\mu_{\perp}),
\eta(q^\mu_{\parallel}) \equiv \int \frac{dk^\perp}{2\pi} V(k^\perp, q^\perp) \varphi_p(k^\mu_{\perp}). \quad (A2)
$$

Then the BS equation can be rewritten as:

$$
\chi(q^\mu_{\parallel}, q^\perp) = S_1(p_1) \eta(q^\mu_{\parallel}) S_2(p_2). \quad (A3)
$$

The propagators of the two constituents can be decomposed as:

$$
S_i(p_i) = \frac{\Lambda^+_i(q^\mu_{\perp})}{J(i) q^\perp + \alpha_i M - \omega_i + i \epsilon} + \frac{\Lambda^-_i(q^\mu_{\perp})}{J(i) q^\perp + \alpha_i M + \omega_i - i \epsilon}, \quad (A4)
$$

with

$$
\omega_i = \sqrt{m_i^2 + q^\perp}, \quad \Lambda^+_i(q^\mu_{\perp})
$$

$$
= \frac{1}{2\omega_p} \left[ \frac{p_i}{M} \omega_i \pm J(i)(m_i + q^\perp) \right], \quad (A5)
$$

where $i = 1, 2$ for quark and anti-quark, respectively, and $J(i) = (-1)^{i+1}$.

Introducing the notations $\varphi^\pm_p(q^\perp)$ as:

$$
\varphi^\pm_p(q^\perp) \equiv \Lambda^+_p(q^\mu_{\perp}) \frac{p_i}{M} \varphi_p(q^\perp) \frac{p_i}{M} \Lambda^\pm_p(q^\mu_{\perp}). \quad (A6)
$$

With contour integration over $q^\perp$ on both sides of Eq. (A3), we obtain:

$$
\varphi_p(q^\perp) = \frac{\Lambda^+_p(q^\mu_{\perp}) \eta_p(q^\perp) \Lambda^+_p(q^\mu_{\perp})}{(M - \omega_1 - \omega_2)} - \frac{\Lambda^-_p(q^\mu_{\perp}) \eta_p(q^\perp) \Lambda^-_p(q^\mu_{\perp})}{(M + \omega_1 + \omega_2)},
$$

and the full Salpeter equation:

$$
(M - \omega_1 - \omega_2) \varphi^+_p(q^\perp) = \Lambda^+_p(q^\mu_{\perp}) \eta_p(q^\perp) \Lambda^+_p(q^\mu_{\perp}),
$$

$$
(M + \omega_1 + \omega_2) \varphi^-_p(q^\perp) = -\Lambda^-_p(q^\mu_{\perp}) \eta_p(q^\perp) \Lambda^-_p(q^\mu_{\perp}),
$$

$$
\varphi^+_p(q^\perp) = \varphi^-_p(q^\perp) = 0. \quad (A7)
$$

For the different $J^{PC}$ (or $J^P$) states, we give the general form of wave functions. Reducing the wave functions by the last equation of Eq. (A7), then solving the first and second equations in Eq. (A7) to get the wave functions and mass spectrum. We have discussed the solution of the Salpeter equation in detail in Refs. [52, 53].

The normalization condition for BS wave function is:

$$
\int \frac{q^2 dq}{2\pi^2} Tr \left[ \frac{1}{\sqrt{M}} \frac{P}{M} \varphi^+ \frac{P}{M} - \varphi^- \frac{P}{M} \frac{P}{M} \right] = 2P_0. \quad (A8)
$$

In our model, the instantaneous interaction kernel $V$ is Cornell potential, which is the sum of a linear scalar interaction and a vector interaction:

$$
V(r) = V_s(r) + \gamma_0 \otimes \gamma^0 V_v(r) = \frac{\lambda}{r} + \frac{V_0 - \gamma_0 \otimes \gamma^0 \frac{4}{3} \frac{\alpha_s}{r}}{r}. \quad (A9)
$$

where $\lambda$ is the string constant and $\alpha_s(\bar{q})$ is the running coupling constant. In order to fit the data of heavy quarkonia, a constant $V_0$ is often added to confine potential. We introduce a factor $e^{-ar}$ to avoid the infrared divergence in the momentum space:

$$
V_s(r) = \frac{\lambda}{\alpha_s(1 - e^{-ar})}, \quad V_v(r) = -4 \frac{\alpha_s}{3} r e^{-ar}. \quad (A10)
$$

It is easy to know that when $\alpha_s r \ll 1$, the potential becomes to Eq. (A9). In the momentum space and the C.M.S of the bound state, the potential reads:

$$
V(\bar{q}) = V_s(\bar{q}) + \gamma_0 \otimes \gamma^0 V_v(\bar{q}),
$$

$$
V_s(\bar{q}) = -\left( \frac{\lambda}{\alpha_s} + V_0 \right) \delta^3(\bar{q}) + \frac{\lambda}{\pi^2} \frac{1}{(\bar{q}^2 + \alpha^2)^2},
$$

$$
V_v(\bar{q}) = -\frac{2}{3\pi^2} \frac{\alpha_s(\bar{q})}{(\bar{q}^2 + \alpha^2)}. \quad (A11)
$$
where the running coupling constant $\alpha_s(\bar{q})$ is:

$$\alpha_s(\bar{q}) = \frac{12\pi}{33 - 2N_f} \frac{1}{\log \left( \frac{a + \bar{q}^2}{\Lambda_{QCD}^2} \right)}.$$

We introduce a small parameter $a$ to avoid the divergence in the denominator. The constants $\lambda$, $\alpha$, $V_0$ and $\Lambda_{QCD}$ are the parameters that characterize the potential. $N_f = 3$ for $\bar{b}q$ (and $\bar{c}q$) system.

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