Abstract

We study linearization of lattice gauge theory. Linearized theory approximates lattice gauge theory in the same manner as the loop O(n)-model approximates the spin O(n)-model. Under mild assumptions, we show that the expectation of an observable in linearized Abelian gauge theory coincides with the expectation in the Ising model with random edge-weights. We find a similar relation between Yang-Mills theory and 4-state Potts model. For the latter, we introduce a new observable.

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Dedicated to the last real scientists, brave to face real difficulties, not sweeping them under the rug.

We prove a combinatorial identity meaning that the expectation of an observable in (pre)linearized 2-dimensional Abelian lattice gauge theory equals its expectation in the Ising model with random edge-weights, and a similar identity for Yang-Mills theory (see Proposition 1; all the notions are defined below).

This allows to translate known deep results on the Ising model and its relatives [11, 6] to linearized lattice gauge theory. E.g. we provide an observable in the latter, having conformally invariant continuum limit (see Corollary 13 and Remark 14).

Proposition 1. (See Figure 1) Let \( \Omega \) be a polygon triangulated by regular triangles, \( \beta \in \mathbb{R} \). Let \( F, E, V \) be the sets of faces, nonboundary edges, nonboundary vertices respectively; \( F, E, V \neq \emptyset \). Let \( f \) and \( g \) be real-valued integrable functions on \( U(1)^E \) and \( SU(2)^E \) respectively, invariant under the reflection in each coordinate hyperplane of \( \mathbb{R}^2 \) or \( \mathbb{R}^4 \). Then

\[
\int_{U(1)^E} dU f(U) \prod_{ABC \in F} (1 + \beta \Re (U(AB)U(BC)U(CA))) = \\
= \int_{[0,2\pi]^E} d\theta f(e^{i\theta}) \left( 1 + \beta |F| \sum_{\sigma \in \{+1,-1\}^V} \prod_{\sigma(A) = \sigma(B)} \cos^2 \theta(AB) \prod_{\sigma(A) \neq \sigma(B)} \sin^2 \theta(AB) \right);
\]

and

\[
\int_{SU(2)^E} dU g(U) \prod_{ABC \in F} (1 + \beta \Re (U(AB)U(BC)U(CA))) = \\
= \int_{SU(2)^E} dU g(U) \left( 1 + \beta |F| \sum_{H \in \{1, j, k\}^V} \prod_{AB \in E} \Re^2 (H(A)^*U(AB)H(B)) \right),
\]

where we set \( U(AB), \sigma(A), H(A) := 1 \) for each boundary edge \( AB \) or boundary vertex \( A \).

The following more abstract result expresses the relations among the models in detail; the required definitions are given in the next sections.

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1 Linearization of Abelian lattice gauge theory

Let us give a zero-knowledge introduction to gauge theory; cf. [7, 2].

We start with an informal toy model. Let 7 cities be connected by oriented roads as in Figure 1. Each city has its own type of goods in an unlimited quantity. E.g., city A has apples, city B has bananas, and city C has coconuts. On each oriented road AB, an exchange rate \( U(AB) > 0 \) is fixed. E.g., travelling from A to B, one gets 2 bananas for an apple.

A cunning citizen can travel and exchange along a triangle ABC to multiply his initial amount of goods by a factor of \( U(AB)U(BC)U(CA) \). The total speculation profit is measured by the quantity

\[
S(U) := \sum_{\text{all faces } ABC} \log^2(U(AB)U(BC)U(CA)).
\]

Here \( \log^2(x) \) is chosen as a function vanishing at \( x = 1 \) and positive for \( x \neq 1 \). A useful observation: for a city C, one can change the units of measurements, e.g., exchange dozens of coconuts instead of single ones. Such gauge transformation multiplies the rates for all the roads starting at C and divides the rates for all the roads ending at C by the same value but preserves \( S(U) \).

One can turn this economic model into a statistical-physics one by fixing the boundary rates and making the interior ones random with the probability density \( P(U) \) exponentially decreasing with the growth of the speculation profit \( S(U) \). The resulting model describes quantum electromagnetic field.

In gauge theory used in particle physics, the rates become complex numbers, quaternions etc.

Definition 3. Let \( G \) be a submanifold of one of the sets

\[
U(1) := \{z \in \mathbb{C} : |z| = 1\}, \quad SU(2) = \{z \in \mathbb{H} : |z| = 1\}, \quad \text{or} \quad S^7 = \{z \in \mathbb{O} : |z| = 1\},
\]

of norm-one complex numbers, quaternions, and octonions respectively. (To get the idea of what follows, it is suggested to start with the case when \( G = U(1) \) everywhere, known as compact Abelian gauge theory.) Let \( \Omega \) be a polygon triangulated by regular triangles. Let \( F, E, V \) be the sets of faces, nonboundary edges, nonboundary vertices in the triangulation respectively. Assume that all the edges are parallel to the 3 cubic roots of 1. Orient the edges in the directions of the roots. Denote by \( AB \) the edge oriented from A to B. Fix \( 0 \leq \beta \leq 1 \) and an element \( u(AB) \in G \) for each boundary edge \( AB \).

Let \( G^E \) be the set of \( G \)-valued functions \( U \) on the set of edges which are equal to \( u \) on the boundary. Define LatticeGaugeTheory\((G, \beta, \Omega, u)\) to be the probability space \( G^E \) with the probability density

\[
P(U) = \frac{1}{Z} \exp\left( \beta \sum_{\text{faces } ABC \in E} \Re(U(AB)U(BC)U(CA)) \right).
\]

Here the sum is over all the triangles \( ABC \) (the vertices are listed in an order compatible with the orientation of the edges) and \( Z \in \mathbb{R} \) is chosen so that the total probability \( \int_{G^E} P(U) \, dU = 1 \), where \( dU \) denotes the
Table 1: Relation between lattice gauge theory and the Ising model; see Example 5

| \sin \theta_n \cos \theta_{n+1} | \cos \theta_n \sin \theta_{n+1} | \sin \theta_n \sin \theta_{n+1} | \cos \theta_n \cos \theta_{n+1} |
|-----------------|-----------------|-----------------|-----------------|
| ![Diagram 1]    | ![Diagram 2]    | ![Diagram 3]    | ![Diagram 4]    |

The expression for product of Lebesque measures (or counting measures, if \( G \) is finite, i.e., a 0-dimensional submanifold). The constants \( Z', Z'', \ldots \) below are chosen analogously.

Define \( \text{PreLinearizedGaugeTheory}(G, \beta, \Omega, u) \) to be the space \( G^E \) with the probability density

\[
P'(U) = \frac{1}{Z} \prod_{ABC \in F} (1 + \beta \Re(U(AB)U(BC)U(CA))).
\]

In the case when \( G \) is a group, given a \( G \)-valued function \( g \) on the set of nonboundary vertices, define the gauge transformation \( \phi_g : G^E \to G^E \) by the formula \( \phi_g U)(AB) = g(A)U(AB)g(B)^* \).

**Remark 4.** Definition 3 is applicable even if \( G \) is not a group, e.g., \( G = S^1 \) or \( G = S^2 = SU(2) \cap \{ \Re z = 0 \} \).

The expression for \( P(U) \) is well-defined for \( G \subset S^1 \) because \( \Re((xy)z) = \Re(x(yz)) \) for any \( x, y, z \in \mathbb{O} \) [9, Eq. (6.21)]. Thus octonion gauge theory on a triangular lattice is defined easier than the continuum one [9]. The price is that gauge transformations are not well-defined when \( G \) is not a group. Also we conjecture that Proposition 1 does not remain true literally for \( SU(2) \) replaced by \( S^7 \) or \( S^2 \); see Remark 23 and Proposition 26.

The case \( G = SU(2) \) is often called Yang-Mills theory.

**Definition 3** is applicable if \( G \) is finite, e.g., \( G = \{ +1, -1 \} \). Then the probability density is understood as the function \( P(U) \) related to the probability measure \( \mu \) via \( \mu(A) = \int_A P(U)\,dU = \sum_{U \in \mathbb{A}} P(U) \).

Definition 3 immediately generalizes to an arbitrary Lie group \( G \), e.g., \( G = SU(3) \). It is interesting to find an analogue of Proposition 1 in this setup. Generalization to higher-dimensional lattices is also easy [2, §7].

In the model, one usually studies expectations of certain random variables. Those are easy to compute (on a 2-dimensional lattice) using gauge transformations, if a random variable is invariant under the transformations, and nontrivial otherwise. Proposition 1 asserts that for a random variable having just coordinate-reflection symmetries, the computation still simplifies much: one may drop most terms in the expansion of \( P'(U) \) in \( \beta \). We illustrate this in the simplest nontrivial particular case.

**Example 5.** Let \( G = U(1) \) be identified with \([0, 2\pi]\) by the exponential mapping. Let \( \Omega \) be a regular hexagon \( A_1A_2A_3A_4A_5A_6 \) triangulated by 6 regular triangles with a common vertex \( O \); \( A_7 := A_1 \). Set

\[
u(A_1A_2) = u(A_3A_2) = -i, \quad u(A_3A_4) = u(A_1A_6) = 1, \quad u(A_5A_4) = u(A_3A_6) = i \quad \text{(see Table 1, bottom)}.
\]

In \( \text{PreLinearizedGaugeTheory}(U(1), \beta, \Omega, u) \), we have \( Z' = (2\pi)^6 + 2\pi^6 \beta^6 \) and for each integrable function \( f : [0, 2\pi]^6 \to \mathbb{R} \) invariant under the transformations \( \theta_n \mapsto \pi \pm \theta_n \) (mod 2\pi) for \( n = 1, \ldots, 6 \) we have

\[
Ef = \int_{[0,2\pi]^6} f(\theta_1, \ldots, \theta_6) P'_\beta(\theta_1, \ldots, \theta_6) \,d\theta_1 \ldots d\theta_6,
\]
where
\[ P''(\theta_1, \ldots, \theta_6) := \frac{1}{Z'} \left( 1 + \beta^6 \cos^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 \cos^2 \theta_4 \sin^2 \theta_5 \cos^2 \theta_6 \\
+ \beta^6 \sin^2 \theta_1 \cos^2 \theta_2 \sin^2 \theta_3 \sin^2 \theta_4 \cos^2 \theta_5 \sin^2 \theta_6 \right) . \]

**Remark 6.** Equation (1) is only useful if \( f \) is not gauge invariant; otherwise there is a much simpler way to compute the expectation. Likewise, there is a simpler way to compute \( Z' \) than described below; but we discuss the method which automatically gives (1) as well.

**Proof of Example 5.** Given \( U \in U(1)^E \), denote \( \exp(i\theta_n) := \begin{cases} 
U(OA_n), & \text{for } n = 1, 3, 5; \\
U(A_nO), & \text{for } n = 2, 4, 6.
\end{cases} \) Then
\[
Z' = \int_{U(1)^6} dU \left( 1 + \beta \Re(U(OA_1)U(A_1A_2)U(A_2O) \cdots (1 + \beta \Re(U(A_6O)U(OA_1)U(A_1A_6))
\right)
\]
\[
= \int_{[0,2\pi]^6} d\theta_1 \ldots d\theta_6 \left( 1 + \beta \sin(\theta_1 + \theta_2) \right) \left( 1 + \beta \sin(\theta_2 + \theta_3) \right) \left( 1 + \beta \cos(\theta_3 + \theta_4) \right) \left( 1 - \beta \sin(\theta_4 + \theta_5) \right) \left( 1 - \beta \sin(\theta_5 + \theta_6) \right) \left( 1 + \beta \cos(\theta_6 + \theta_1) \right)
\]
\[
= \int_{[0,2\pi]^6} d\theta_1 \ldots d\theta_6 \left( 1 + \beta \sin(\theta_1 \cos \theta_2 + \beta \cos \theta_1 \sin \theta_2) \right) \left( 1 + \beta \sin(\theta_2 \cos \theta_3 + \beta \cos \theta_2 \sin \theta_3) \right) \left( 1 - \beta \sin(\theta_3 \cos \theta_4 - \beta \sin \theta_3 \sin \theta_4) \right) \left( 1 - \beta \sin(\theta_4 \cos \theta_5 - \beta \cos \theta_4 \sin \theta_5) \right) \left( 1 - \beta \sin(\theta_5 \cos \theta_6 - \beta \sin \theta_5 \sin \theta_6) \right) \left( 1 + \beta \cos \theta_6 \cos \theta_1 - \beta \sin \theta_6 \sin \theta_1 \right)
\]
\[
= \int_{[0,2\pi]^6} d\theta_1 \ldots d\theta_6 \left( 1 + \beta^6 \cos^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 \cos^2 \theta_4 \sin^2 \theta_5 \cos^2 \theta_6 \\
+ \beta^6 \sin^2 \theta_1 \cos^2 \theta_2 \sin^2 \theta_3 \sin^2 \theta_4 \cos^2 \theta_5 \sin^2 \theta_6 \right)
\]
\[
= (2\pi)^6 + 2\pi^6 \beta^6 .
\]

Let us explain key equality (*). We expand the product in the left-hand side of (*), and to each resulting term assign a “Feynman diagram” as follows; see Table 1.

First consider a term obtained by taking a summand distinct from 1 in each factor in the left-hand side of (*). If the taken summand has form \( \sin \theta_n \cos \theta_m \), where \( m = n + 1 \), then draw a segment from the center of the triangle \( OA_mA_n \) to the midpoint of \( OA_n \) (see Table 1, top-left). If it has form \( \sin \theta_n \sin \theta_{n+1} \), draw 2 segments from the center of \( OA_mA_{n+1} \) to the midpoints of \( OA_n \) and \( OA_{n+1} \) (see Table 1, top-middle). For the summands of the form \( \cos \theta_n \cos \theta_{n+1} \) draw nothing (see Table 1, top-right). Also draw 4 segments from the centers of \( OA_nA_{n+1} \) to the midpoints of \( A_nA_{n+1} \) for \( n = 1, 2, 4, 5 \). We get an even number of segments in each triangle, thus all the segments form several broken lines with the endpoints at the edge midpoints.

Now observe that if at least one of the endpoints does not belong to the boundary of \( \Omega \) (see Table 1, bottom-right), then the term has a factor \( \sin \theta_n \cos \theta_n \) vanishing after integration over \( \theta_n \). The only two terms giving a nonzero contribution to the integral are the ones in Table 1, bottom-left and bottom-middle. They are in bijection with the 2-colorings of vertices such that \( A_n \) is black for \( n = 1, 3, 4, 6 \) and white for \( n = 2, 5 \).

Finally, a similar argument shows that taking at least one summand 1 leads to just one term contributing to the integral. We are thus left with only 3 contributing terms, which proves (*). Equation (1) is proven analogously.

This example suggests to replace \( P' \) by \( P'' \) in Definition 3. In particular, this extends the definition to \( \beta > 1 \). Beware that \( P''(\theta_1, \ldots, \theta_6) \neq P'(e^{i\theta_1}, \ldots, e^{i\theta_6}) \) in general.

**Definition 7.** Assume that \( G = U(1) \). Fix \( \beta > 0 \) and a sign \( \sigma(A) = \pm 1 \) for each boundary vertex \( A \). For each boundary edge \( AB \) set \( u(AB) = 1 \), if \( \sigma(A) = \sigma(B) \), and \( u(AB) = i\sigma(B) \), if \( \sigma(A) \neq \sigma(B) \).
Let $\{+1, -1\}^V$ be the set of $\pm 1$-valued functions $\Sigma$ on vertices, equal to $\sigma$ on the boundary. Denote

$$P(U, \Sigma) := \prod_{AB: \Sigma(A) = \Sigma(B)} \cos^2 \theta(AB) \prod_{AB: \Sigma(A) \neq \Sigma(B)} \sin^2 \theta(AB),$$

where $U(AB) =: \cos \theta(AB) + i \sin \theta(AB)$. Now define $\text{LinearizedGaugeTheory}(U(1), \beta, \Omega, \sqrt{\sigma})$ and $\text{LinearizedGaugeTheory}(U(1), \infty, \Omega, \sqrt{\sigma})$ to be the space $U(1)^E$ with the probability densities

$$P'_\beta(U) = \frac{1}{Z'_\beta} \left( 1 + \beta |f| \right) \sum_{\Sigma \in \{+1, -1\}^V} P(U, \Sigma)$$

and

$$P'_{\infty}(U) = \frac{1}{Z'_{\infty}} \sum_{\Sigma \in \{+1, -1\}^V} P(U, \Sigma).$$

**Remark 8.** The origin of the notation $\sqrt{\sigma}$ and the formula for $u(AB)$ are going to become clear in the next section.

**Example 9.** Consider the hexagon $\Omega$ from Example 5. Let $\sigma(A_n)$ be $+1$ for $n = 1, 3, 4, 6$ and be $-1$ for $n = 2, 5$. Then for $\text{LinearizedGaugeTheory}(U(1), \beta, \Omega, \sqrt{\sigma})$, the function $P'_\beta$ is given by the expression from that example. The crossing probability for site percolation on $\Omega$ with the boundary condition $\sigma$ (which itself equals 1/2) equals the expectation of the random variable

$$f(\theta_1, \ldots, \theta_6) = \frac{\cos^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 \cos^2 \theta_4 \sin^2 \theta_5 \cos^2 \theta_6}{\cos^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 \cos^2 \theta_4 \sin^2 \theta_5 \cos^2 \theta_6 + \sin^2 \theta_1 \cos^2 \theta_2 \sin^2 \theta_3 \sin^2 \theta_4 \cos^2 \theta_5 \sin^2 \theta_6}$$

in $\text{LinearizedGaugeTheory}(U(1), \infty, \Omega, \sqrt{\sigma})$. The random variable $f$ is not gauge invariant.

In the random variable $f$, one recognizes the crossing probability in the Ising model with edge weights $\cot \theta_1, \ldots, \cot \theta_6$. Such cotan weights appear naturally, e.g., on isoradial graphs [1]. This observation is summarized in the following definition and Proposition 2.

**Definition 10.** Define $\text{RandomMediumIsing}(\Omega, \sigma)$ to be the disconnected manifold $U(1)^E \times \{+1, -1\}^V$ with the probability density $P(U, \Sigma)/Z'_{\infty}$. Define $\text{Percolation}(\Omega, \sigma)$ to be the space $\{+1, -1\}^V$ with the counting measure divided by $2^{|V|}$.

**Definition 11.** A semi-direct product of probability spaces $X = (X, \Sigma_X, \mu_X)$ and $Y = (Y, \Sigma_Y, \mu_Y)$ is a probability space $Z = (X \times Y, \Sigma_X \otimes \Sigma_Y, \mu_Z)$ with any measure $\mu_Z$ such that $\mu_Z(A \times Y) = \mu_X(A)$ and $\mu_Z(X \times B) = \mu_Y(B)$ for each $A \in \Sigma_X, B \in \Sigma_Y$. Notation: $Z = X \times Y$ or $Z = X \times Y$.

**Remark 12.** The notion is closely related but different from the semi-direct product of measures.

**Proof of the 1st isomorphism in Proposition 2.** This is straightforward: Let $\mu$ and $\nu$ be the probability measures in $\text{LinearizedGaugeTheory}(U(1), \infty, \Omega, \sqrt{\sigma})$ and $\text{RandomMediumIsing}(\Omega, \sigma)$ respectively, and $\lambda$ be the Lebesque measure on $U(1)^E \times \{+1, -1\}^V$. Then for each measurable $A \subset U(1)^E$

$$\nu(A) = \int_A P''_{\infty}(U) dU = \frac{1}{Z''_{\infty}} \int_A \sum_{\Sigma \in \{+1, -1\}^V} P(U, \Sigma) dU = \frac{1}{Z''_{\infty}} \int_{A \times \{+1, -1\}^V} P(U, \Sigma) d\lambda = \mu(A \times \{+1, -1\}^V).$$

Since $\int_0^{2\pi} \cos^2 \theta d\theta = \int_0^{2\pi} \sin^2 \theta d\theta$, it follows that $\int_{U(1)^E} P(U, \Sigma) dU$ is the same for each $\Sigma$, thus

$$\mu(U(1)^E \times B) = \frac{1}{Z''_{\infty}} \int_{U(1)^E \times B} P(U, \Sigma) d\lambda = \frac{|B|}{2^{|V|}}$$

for each $B \subset \{+1, -1\}^V$. By Definition 11 we get the 1st isomorphism in Proposition 2. \hfill \Box

**Corollary 13.** For each random variable $f$ on $\text{Percolation}(\Omega, \sigma)$ the random variable

$$E(f \circ p_2|p_1)(U) := \frac{\sum_{\Sigma \in \{+1, -1\}^V} f(\Sigma) P(U, \Sigma)}{\sum_{\Sigma \in \{+1, -1\}^V} P(U, \Sigma)}$$

on $\text{LinearizedGaugeTheory}(U(1), \infty, \Omega, \sqrt{\sigma})$ has the same expectation (but not variance).
Remark 14. In particular, if \( f \) is the indicator function of the crossing event in \( \Omega \), then by the Cardy’s formula proved by S. Smirnov [11, 6], \( E(f \circ p_2|p_1) \) is an observable in linearized gauge theory, which is conformally invariant in the continuum limit. (We consider the notation \( E(f \circ p_2|p_1) \) as indecomposable to avoid discussion of its ingredients.)

Remark 15. This result extends to Smirnov’s parafermionic observable. Let us informally sketch that. In Definition 3, fix a nonboundary edge \( AB \). Let the function \( U \) be two-valued at the edge \( AB \); let the two values \( U_+(AB) \) and \( U_-(AB) \) be related by \( U_-(AB) = iU_+(AB) \). In the formulas for \( P(U) \) and \( P'(U) \), replace \( U(AB) \) by \( U_+(AB) \) in the summand corresponding to the triangle \( ABC \) bordering upon \( AB \) from the right, and by \( U_-(AB) \) — for the one from the left. In Definition 7, replace \( \{+1, -1\}^V \) by the set of loop and arc configurations with one of the endpoints at the midpoint of \( AB \). In the formula for \( P(U, \Sigma) \), take \( \cos^2 \theta(AB) \), if an arc arrives at the midpoint of \( AB \) from the right, and \( \sin^2 \theta(AB) \) otherwise; here \( \cos \theta(AB) := \Re U_+(AB) \). To Smirnov’s parafermionic percolation observable, one naturally assigns a random variable in the resulting probability space, having the same expectation.

2 Lattice Higgs field

Now we show that the random-medium Ising model itself appears naturally in a linearization of the lattice Higgs field: the former is obtained when the latter assumes only certain basis-vector values. We follow the popular-science construction of the lattice Higgs field from [7].

Definition 16. Let a subset \( S \subset G \) be either finite or of positive measure; in the former case equip it with the counting measure. Fix \( 0 \leq \beta, \lambda \leq 1 \) and a map \( h \) from the set of boundary vertices to the set \( S \). For a boundary edge \( AB \) set \( u(AB) = h(A)h(B)^* \) and assume that \( u(AB) \in G \).

Let \( S^V \) be the set of \( S \)-valued functions \( H \) on vertices which are equal to \( h \) on the boundary. Define \( \text{LatticeHiggs}(G, S, \beta, \lambda, \Omega, h) \) to be the probability space \( G^E \times S^V \) with the probability density

\[
P(U, H) = \frac{1}{Z} \exp \left( \beta \sum_{ABC} \Re (U(AB)U(BC)U(CA)) + \lambda \sum_{AB \in E} \Re (H(A)^*U(AB)H(B)) \right).
\]

In the case when \( G \) is a group acting on \( S \), given a \( G \)-valued function \( g \) on vertices, define the gauge transformation \( G^E \times S^V \to G^E \times S^V \) by the formula \( U(AB) \mapsto g(A)U(AB)g(B)^*, H(A) \mapsto g(A)H(A) \).

Define \( \text{LinearizedHiggs}(G, S, 0, \infty, \Omega, u) \) and \( \text{LinearizedHiggs}(G, S, 0, \infty, \Omega, u) \) to be the space \( G^E \times S^V \) with the probability densities respectively

\[
P'_{\beta, \lambda, S}(U, H) = \frac{1}{Z_{\beta, \lambda, S}^r} \prod_{ABC \in F} \left( 1 + \beta \Re (U(AB)U(BC)U(CA)) \right) \prod_{AB \in E} \left( 1 + \frac{1}{2} \lambda \Re (H(A)^*U(AB)H(B)) \right)^2;
\]

\[
P'_{0, \infty, S}(U, H) = \frac{1}{Z_{0, \infty, S}^r} \prod_{ABC \in F} \Re^2 (H(A)^*U(AB)H(B)).
\]

Remark 17. We square the products over edges because they give no contribution to \( Z' \) otherwise.

Proof of the 2nd isomorphism in Proposition 2. The isomorphism follows directly from Definitions 7 and 16: \( P_{0, \infty, \{1,i\}}(U, H) = P(U, \Sigma) \) identically for \( \Sigma = H^2 \) because for each \( H \in \{1,i\}^V \) and \( U \in U(1)^E \)

\[
\Re^2 (H(A)^*U(AB)H(B)) = \begin{cases} 
\cos^2 \theta(AB), & \text{if } H(A) = H(B), \\
\sin^2 \theta(AB), & \text{if } H(A) \neq H(B).
\end{cases}
\]

Example 18. Consider the hexagon \( \Omega \) and the random variable \( f \) from Example 9. Let \( h(A_n) \) be 1 for \( n = 1, 3, 4, 6 \) and be \( i \) for \( n = 2, 5 \). Then the crossing probability for site percolation on \( \Omega \) equals the expectation of the random variable

\[
f'(\exp(i\theta_1), \ldots, \exp(i\theta_6), \exp(i\eta)) := f(\theta_1 + \eta, \theta_2 - \eta, \theta_3 + \eta, \ldots, \theta_6 - \eta)
\]

in \( \text{LinearizedHiggs}(U(1), U(1), 0, \infty, \Omega, h) \). The random variable \( f' \) is now gauge invariant.
The following proposition shows that for gauge-invariant random variables the choice of $S$ is not essential. This allows to fix a particular value of $H$ instead of summation over all $H$ (just like in Example 18 where $f$ was obtained from $f'$ by fixing $\eta$ to zero).

**Proposition 19.** Let $G = U(1)$ or $SU(2)$, and $S \subset G$. Then for each gauge-invariant integrable function $f : G^E \times G^V \to \mathbb{R}$, we have

$$\int_{G^E \times G^V} f(U, H) P'_{\beta, \lambda, S}(U, H) \, dU dH = \int_{G^E \times G^V} f(U, H) P'_{\beta, \lambda, G}(U, H) \, dU dH.$$  

**Proof.** Fix any $H' \in S^V$. By the gauge invariance, the change of variables $U(AB) = g(A)U'(AB)g(B)^*$, $H(A) = g(A)H'(A)$ takes the right-hand side to

$$\int_{G^E \times G^V} f(U', H') P'_{\beta, \lambda, G}(U', H') \, dU' dg = \mu(G)^{|V|} \int_{G^E} f(U', H') P'_{\beta, \lambda, G}(U', H') \, dU' = \mu(S)^{|V|} \int_{G^E} f(U', H') P'_{\beta, \lambda, S}(U', H') \, dU'.$$

Averaging over $H' \in S^V$ gives the left-hand side. \qed

## 3 Linearization of non-Abelian lattice gauge theory

Now generalize Example 5 and Definition 7 to non-Abelian and nonassociative gauge theories.

**Definition 20.** Let $G = U(1)$, $SU(2)$ or $S^7$. Let $S$ be the set of $\{i, j\}$, $\{i, j, k\}$ or $\{e_1, \ldots, e_8\}$ of the complex, quaternionic or octonionic units respectively. Fix $0 \leq \beta, \lambda \leq 1$ and a map $h$ from the set of boundary vertices to the set $S$. Recall that $G^E$ is the set of $G$-valued functions $U$ on the set of all edges such that $U(AB) = h(A)h(B)^*$ for each boundary edge $AB$. Define $\text{LinearizedGaugeTheory}(G, \beta, \Omega, h)$ and $\text{LinearizedGaugeTheory}(G, \infty, \Omega, h)$ to be the space $G^E$ with the probability densities respectively

$$P''_{\beta, S}(U) = \frac{1}{Z''_{\beta, S}} \left(1 + \beta \sum_{H \in S^V} P'_{0, \infty, S}(U, H)\right)$$

and

$$P''_{\infty, S}(U) = \sum_{H \in S^V} P'_{0, \infty, S}(U, H).$$

A coordinate reflection $G^E \to G^E$ is a reflection in a coordinate hyperplane of space $\mathbb{R}^{2|E|} = \mathbb{C}^{|E|}$, $\mathbb{R}^{4|E|} = \mathbb{H}^{|E|}$, or $\mathbb{R}^{8|E|} = \mathbb{O}^{|E|}$ containing $G^E$.

Let us restate Proposition 1 in a slightly more general form.

**Proposition 21.** Let $G = U(1)$ or $SU(2)$, $S = \{i, j\}$ or $\{i, j, k\}$ respectively. If an integrable function $f : G^E \to \mathbb{R}$ is invariant under each coordinate reflection $G^E \to G^E$, then

$$\int_{G^E} f(U) P'(U) \, dU = \int_{G^E} f(U) P''_{\beta, S}(U) \, dU.$$  

We present a proof for $G = SU(2)$; the one for $U(1)$ is simpler and is easily obtained from that. The argument is similar to Example 5 but uses regularly edge-3-colored graphs instead of broken lines.

**Lemma 22.** (Cf. [10, p. 235]) Let $h$ be a map from the set of boundary vertices of $\Omega$ to $SU(2)$. Take $U \in SU(2)^E$ such that $U(AB) = h(A)h(B)^*$ for each boundary edge $AB$ and $U(AB)U(BC)U(CA) = \pm 1$ for each face $ABC$. Then $\prod_{ABC} U(AB)U(BC)U(CA) = 1$.

**Proof of Lemma 22.** Use induction over the number of faces. If $\Omega$ has a single face $ABC$, then

$$U(AB)U(BC)U(CA) = h(A)h(B)^*h(B)h(C)^*h(C)h(A)^* = 1.$$
Otherwise let \(ABC\) be a face such that the edge \(AB\) is on the boundary and \(BC\) is not on the boundary. If \(U(AB)U(BC)U(CA) = -1\) then change the sign of \(U(BC)\); this does not affect the product over all faces in question. We get \(U(AB)U(BC)U(CA) = +1\). Set \(h(C) = U(CA)h(A)\). Then \(U(CA) = h(C)h(A)^*\) and
\[
U(BC) = U(AB)^* \cdot U(AB)U(BC)U(CA) \cdot U(CA)^* = h(B)h(A)^* \cdot 1 \cdot h(A)h(C)^* = h(B)h(C)^*.
\]
Remove the triangle \(ABC\) from \(\Omega\). Applying the inductive hypothesis to the remaining polygon(s), we arrive at the required assertion.

**Remark 23.** Lemma 22 does not hold for \(SU(2)\) replaced by \(S^7\). E.g., if \(\Omega\) has a single face \(ABC\), \(h(A) = ab, h(B) = b, h(C) = c\), where \(a, b, c\) are octonion units satisfying \((ab)c = -a(bc)\), then \((U(AB)U(BC))U(CA) = ((ab)b^*)(bc^*)i(c(ab)^*) = (-c(ab))^2 = -1\).

**Proof of Propositions 1 and 21 for \(G = SU(2)\).** The propositions follow from the set of equalities
\[
Z' \int_{G^E} f(U)P'(U) \, dU = \int_{G^E} f(U) \, dU \prod_{ABC} (1 + \beta \Re(U(AB)U(BC)U(CA)))
\]
\[
= \int_{G^E} f(U) \, dU \prod_{ABC} \left(1 + \beta \sum_{a,b,c \in S; abc = \pm 1} abcU_a(AB)U_b(BC)U_c(CA)\right)
\]
\[
= \int_{G^E} f(U) \, dU \left(1 + \beta^{[F]} \sum_{u \in S^F; u(AB)u(BC)u(CA) = \pm 1} \prod_{ABC} u(AB)u(BC)u(CA) \prod_{AB \in E} U_u(AB)^2\right)
\]
\[
= \int_{G^E} f(U) \, dU \left(1 + \beta^{[F]} \sum_{u \in S^F; u(AB)u(BC)u(CA) = \pm 1} \prod_{AB \in E} U_u(AB)^2\right)
\]
\[
= \int_{G^E} f(U) \, dU \left(1 + \beta^{[F]} \sum_{H \in S^V} \prod_{AB \in E} \Re^2 (H(A)^*U(AB)H(B))\right)
\]
\[
= Z''_{\beta,S} \int_{G^E} f(U)P''_{\beta,S}(U) \, dU.
\]

Setting \(f(U) = 1\), we get \(Z' = Z''_{\beta,S}\), and we are done. It remains to explain the equalities.

Here the first and the last equalities follow from Definitions 3 and 20.

In the 2nd equality we use basis decomposition \(U(AB) =: U_1(AB) + iU_1(AB) + jU_j(AB) + kU_k(AB)\).

In the 3rd equality the summation is over all functions \(u\) on edges such that

- \(u(AB) = U(AB) = h(A)h(B)^* \in Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}\) for each boundary edge \(AB\),
- \(u(AB) \in S = \{1, i, j, k\}\) for each nonboundary edge \(AB\), and
- \(u(AB)u(BC)u(CA) = \pm 1\) for each face \(ABC\).

The sum is obtained by expanding the product in the previous line. By the coordinate reflection invariance, the terms containing the first power of \(U_a(AB)\) for some nonboundary edge \(AB\) do not contribute to the integral. Hence only the product of square factors \(U_a(AB)^2\) over all nonboundary edges \(AB\) survives (and the free term). For a boundary edge \(AB\), the factor \(aU_a(AB)\) is nonzero, only if \(a = \pm U(AB)\) and hence \(aU_a(AB) = u(AB)\).

The 4th equality follows from Lemma 22.

In the 5th equality, to every \(u \in S^F\) such that \(u(AB)u(BC)u(CA) = \pm 1\) for each face \(ABC\), we assign the unique \(H \in S^V\) such that \(u(AB) = \pm H(A)H(B)^*\) for each edge \(AB\) (and \(H(A) = h(A)\) for each boundary vertex \(A\)). This is possible by a standard “homological” argument (the existence of \(H\) is equivalent to the assertion \(H^1(\Omega; \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) = 0\) because \(Q_8 / \{1\} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\)). Then \(U_u(AB)(AB) = \pm \Re (H(A)^*U(AB)H(B))\), which completes the proof. \(\square\)
**Definition 24.** Define Potts\(|\mathcal{S}|, \infty, \Omega, h\) to be the space \(S^V\) with counting measure divided by \(|\mathcal{S}|^V\).

**Proof of the 3rd isomorphism in Proposition 2.** This is straightforward: Let \(\nu\) and \(\mu\) be the probability measures in \(\text{LinearizedGaugeTheory}(SU(2), \infty, \Omega, h)\) and \(\text{LinearizedHiggs}(SU(2), S, \infty, \Omega, h)\) respectively, and \(\lambda\) be the Lebesgue measure on \(SU(2)^E \times S^V\), where \(S = \{1, i, j, k\}\). Then for each measurable \(A \subset SU(2)^E\)

\[
\nu(A) = \int_{A} P'_{0,0,S}(U) \, dU = \int_{A} \sum_{H \in S^V} P'_{0,0,S}(U,H) \, dU = \int_{A \times S^V} P'_{0,0,S}(U,H) \, d\lambda = \mu(A \times S^V)
\]

Since \(\int_{SU(2)^E} P'_{0,0,\{1,i,j,k\}}(U,H) \, dU\) is the same for each \(H\), we also have \(\mu(SU(2)^E \times B) = |B|/4^V\). \(\square\)

### 4 Variations

In a sense, Proposition 21 holds for \(G = S^2 = \{z \in \mathbb{H} : |z| = 1, \text{Re} \, z = 0\}\). Surprisingly, although \(S^2\) lives in 3-dimensional space, we still have the Potts model with 4 colors, but now antiferromagnetic.

**Definition 25.** Define AntiferromagneticPotts\(|\mathcal{S}|, 0, \Omega, h\) to be the subspace of Potts\(|\mathcal{S}|, \infty, \Omega, h\) formed by \(H \in S^V\) such that \(H(A) \neq H(B)\) for each edge \(AB\).

**Proposition 26.** For each polygon \(\Omega\) triangulated by regular triangles, each map \(h\) from the set of boundary vertices to \(\{1, i, j, k\}\) such that \(h(A) \neq h(B)\) for each edge \(AB\), and each function \(f : (S^2)^E \rightarrow \mathbb{R}\) invariant under each coordinate reflection of \(\mathbb{R}^{3|E|} \supset (S^2)^E\), we have

\[
\int_{(S^2)^E} dU \, f(U) \prod_{ABC \in F} (1 + \beta \Re (U(AB)U(BC)U(CA))) = \int_{(S^2)^E} dU \, f(U) \left(1 + \beta^{|F|} \sum_{H \in \text{AntiferromagneticPotts}(4,0,\Omega,h)} \prod_{AB \in E} \Re^2 (H(A)^* U(AB) H(B)) \right)
\]

**Proof.** The proof is the same as that of Proposition 21, but \(S\) should be replaced by \(\{i, j, k\}\) in the 2nd and the 3rd equality, whereas \(S^V\) should be replaced by \(\text{AntiferromagneticPotts}(4,0,\Omega,h)\) in the 5th equality, because \(\pm H(A)H(B)^* = u(AB) \in \{i, j, k\}\) implies that \(H(A) \neq H(B)\). \(\square\)

![Figure 2: Simultaneous crossing; see Conjecture 27](image)

The above results suggest that the 4-state Potts model might have a special interest. In that model, the simplest event not “reducing” to 2-state ones is the following “simultaneous crossing”.

**Conjecture 27.** (see Figure 2) Take a polygon \(\Omega\) with the boundary composed of the edges of a hexagonal lattice of mesh \(h\). Let \(A_1, A_2, A_3, A_4, A_5, A_6\) be some boundary vertices. To each hexagon of the lattice, assign one of the 4 colors independently with probability 1/4. Consider the 3 events

\[
E_1 = (A_1A_2 \text{ and } A_4A_5 \text{ are joined by a path formed by hexagons of colors 1 or 4}),
E_2 = (A_2A_3 \text{ and } A_5A_6 \text{ are joined by a path formed by hexagons of colors 2 or 4}),
E_3 = (A_3A_4 \text{ and } A_6A_1 \text{ are joined by a path formed by hexagons of colors 3 or 4}).
\]

Then these 3 events become mutually independent in the limit when \(h \rightarrow 0\) and \((\Omega, A_1, A_2, A_3, A_4, A_5, A_6)\) approaches a planar domain (having rectifiable Jordan boundary) with 6 distinct boundary points, in the Carathéodory sense. That is, \(P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3) \rightarrow 0\) under this limit.
Remark 28. Informally, there are 3 percolating fluids, each hexagon is open either for just one fluid or for three altogether, and we study simultaneous percolation of the fluids.

Notice that the 3 events are obviously pairwise independent but not mutually independent in general (take \( \Omega \) to be just one hexagon). There is some numerical evidence for the conjecture.

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References

[1] D. Chelkak, S. Smirnov, Discrete complex analysis on isoradial graphs, Adv. Math. 228 (2011), 1590-1630.

[2] M. Creutz, Quarks, Gluons and Lattices, Cambridge Univ. Press, 1983 - Science - 169 pp.

[3] H. Duminil-Copin, R. Peled, W. Samotij, Y. Spinka, Exponential decay of loop lengths in the loop O(n) model with large n, arXiv:1412.8326v3.

[4] M. Fedorov, Some aspects of probability distribution for percolation of several fluids on the hexagonal lattice, preprint, 2019, arXiv:1908.11783.

[5] A.H. Guth, Existence proof of a nonconfining phase in four-dimensional U(1) lattice gauge theory, Phys.Rev.D 21 (1980), 2291–2307.

[6] M. Khristoforov, S. Smirnov, Percolation and O(1) loop model, preprint, 2020.

[7] J. Maldacena, The symmetry and simplicity of the laws of physics and the Higgs boson, Europ.J.Phys.37:1(2016), arxiv:1410.6753.

[8] I. Novikov, Percolation of three fluids on a honeycomb lattice, preprint, 2019, arXiv:1912.01757.

[9] T. Ootsuka, E. Tanaka, E. Loginov, Non-associative Gauge Theory, 2005, arXiv:hep-th/0512349v2

[10] R. Penrose, Applications of negative dimensional tensors, in D. J. A. Welsh (ed.), Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), Academic Press, 1971, 221–244.

[11] S. Smirnov, Critical percolation in the plane: Conformal invariance, Cardy’s formula, scaling limits. C. R. Math. Acad. Sci. Paris, 333(3):239-244, 2001.

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