Power-Law Energy Splitting Generated By Tunneling Between Non-smooth Tori

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Abstract

We discuss the energy level splitting $\Delta \epsilon$ due to quantum tunneling between congruent tori in phase space. In analytic cases, it is well known that $\Delta \epsilon$ decays faster than power of $\hbar$ in the semi-classical limit. This is not true in non-smooth cases, specifically, when the tori are connected by line on which the Hamiltonian is not smooth. Under the assumption that the non-smoothness depends only upon the $x$- or $p$-coordinate, the leading term in the semi-classical expansion of $\Delta \epsilon$ is derived, which shows that $\Delta \epsilon$ decays as $\hbar^{k+1}$ when $\hbar \to 0$ with $k$ being the order of non-smoothness.
1 Introduction

This paper concerns the splitting of classically degenerate energy levels. The near degeneracy (ND) classically corresponds to congruent tori in phase space while the quantum tunneling between the tori causes the splitting $\Delta\varepsilon$. A well-known example occurs in the one-dimensional symmetric double-well potential, where the eigenenergies below the top of the barrier cluster into two-fold ND's with energy difference vanishes as

$$\Delta\varepsilon \sim \hbar^\alpha e^{-S/\hbar}$$

(1.1)

when $\hbar \to 0$. When turning to multi-dimensional cases, M. Wilkinson showed that $\Delta\varepsilon$ vanishes normally in the same or, in certain situation, even more singular manner than (1.1)\textsuperscript{[2]}. However, is it always true that the energy level splitting resulting from quantum tunneling is smaller than any power of $\hbar$ in the semi-classical limit? Let us see the following example.

Consider the system on one-dimensional circle defined by any of the four Hamiltonians,

$$H_1 = \frac{p^2}{2} + \cos^2 x, \quad H_2 = \frac{p^2}{2} + |\cos x|, \quad H_3 = |p| + \cos^2 x, \quad H_4 = |p| + |\cos x|.$$  

(1.2)

In classical mechanics, the above Hamiltonians determine similar phase space portraits, particularly, the motion at $H \neq 1$ contains two symmetric closed orbits: the two vibrational orbits with $H < 1$ are connected by the transition $(x, p) \rightarrow (x + \pi, p)$ and the two rotational orbits with $H > 1$ are connected by the time reversal $(x, p) \rightarrow (x, -p)$. According to the Einstein-Brillouion-Keller (EBK) semi-classical quantization rule, this classical degeneracy implies a two-fold ND structure in the spectrum of $H$. We can verify this prediction by directly diagonalizing the Hamiltonians. In Fig.1, $\Delta\varepsilon$ is plotted as the function of the mean energy of the ND pair$(\varepsilon)$. As expected, $\Delta\varepsilon$ (open dots) is much smaller than the spacing of $\varepsilon$ (approximately the dotted lines). However, contrary to the exponentially decay of $\Delta\varepsilon$ with the increase of $|\varepsilon - 1|$ in Fig.1(a), non-exponentially decay of $\Delta\varepsilon$ in some cases is obvious. From the four illustrations, we can see that the “exceptional” ND occurs when and only when the corresponding classically degenerate tori (closed orbits) in phase space are connected by line(s) where the Hamiltonian is not smooth. This fact suggests that tunneling between the degenerate tori can be greatly enhanced by the passage of non-smoothness.

In fact, M.V. Berry showed this non-smoothness-enhanced quantum transition between classically degenerate states about two decades ago\textsuperscript{[3]}. In studying the coefficient $r$ for reflection

\footnote{In this paper, the word “tunneling” refers to the quantum transition between states that classically correspond to separate tori in phase space.}
above a barrier $V(x)$ in the semi-classical limit, Berry proved that $r \sim \hbar^k$ when $V(x)$ has a discontinuous $k$-th derivatives, in contrast to the analytic case where $r$ is exponentially small. Another interesting quantum manifestation of non-smoothness, the power-law localization of eigenstates was also discussed in more recent papers (e.g. [3, 4]).

In this paper we shall investigate the energy level splitting resulting from the non-smoothness-enhanced tunneling. We first consider the case where ND is related to the time reversal symmetry. By perturbation method, a relation between $\Delta \epsilon$ and the non-smoothness of the potential is derived. Based on a geometrical interpretation, this relation is applied to a class of non-smooth systems.

2 Power-Law Energy Splitting

In this section we study systems where ND is related to the time reversal symmetry. The problem is more tractable since the projection of torus onto the coordinate space contains no singularity (caustic). By perturbation method, we obtain an explicit power-law $\hbar$-dependence of the energy splitting.

Consider a mechanic system on one-dimensional circle with Hamiltonian $H = E_k(p) + V(x)$, $V(x + 2\pi) = V(x)$. The kinetic energy $E_k(p)$ satisfies $E_k(-p) = E_k(p)$, and, for simplicity, we assume $E_k(0) = 0$, $E_k(\infty) = \infty$ and $\frac{d}{dp}E_k(p) > 0$ when $p > 0$. A familiar example of such kinetic energy is $\frac{1}{2}p^2$. Due to the time reversal symmetry, the two classical orbits at $H(x, p) = E > \max_x V(x)$, $O^+_E$ and $O^-_E$, one with $p > 0$ and the other with $p < 0$, yield identical action integral, i.e.,

$$\oint_{O^+_E} p\,dx = \oint_{O^-_E} p\,dx = S(E). \tag{2.1}$$

Consequently, EBK quantization condition $S(E) = 2n\pi\hbar$ predicts a two-fold degenerate level $E = \epsilon_n$. The two semi-classical eigenfunctions are given by

$$\Psi^\pm_n(x) = \frac{1}{\sqrt{T_n \dot{x}_n}} \exp[\pm is_n(x)/\hbar], \tag{2.2}$$

where $s_n(x) = \int_0^x p_n(x')dx'$, $p_n(x) > 0$ is determined by $E_k(p) + V(x) = \epsilon_n$, $\dot{x}_n = \frac{d}{dp}E_k(p)|_{p=p_n(x)}$ is the classical velocity and the normalization constant $T_n = \int_0^{2\pi} \frac{dx}{\dot{x}_n}$ is the period of the corresponding classical orbit [8]. (The suffix “n” of $\epsilon$, $\Psi$, $p$, $\dot{x}$, $T$ and $s$ will be hereafter dropped out for simplicity.)
Of course, in general, the two levels do not exactly coincide. The difference between $\epsilon$ and the exact eigenenergy is of order $o(h)$ in the semi-classical limit ($h \to 0$, $n \to \infty$ while $nh$ is fixed).

In the case that $V(x)$ is not smooth (infinitely differentiable), we have seen in the last section ($H_2$ and $H_4$) that the splitting of energy levels ($\Delta \epsilon$) is not exponentially small. It is therefore possible that a non-vanishing $\Delta \epsilon$ will emerge from the higher order semi-classical corrections. If we are only interested in the leading term in $\Delta \epsilon$, however, variational calculation in the space spanned by $\Psi^+$ and $\Psi^-$ will give the result. We shall consider a simple case that $V(x)$ is a $C^{k-1}$ function and

$$\int_x^k V(x) \equiv \lim_{x' \to x+0} \frac{d^k}{dx^k} V(x') - \lim_{x' \to x-0} \frac{d^k}{dx^k} V(x') \quad (2.3)$$

is well-defined, which vanishes on $[0, 2\pi]$ except at discrete points $x_j^\pm$, $j = 1, ..., N < \infty$. Then elementary calculations show that the energy splitting is given by (see Appendix)

$$\Delta \epsilon = \frac{\hbar^{k+1}}{2^{k+1}T} \left| \sum_{j=1}^N \frac{\exp(2is(x_j^\pm)/\hbar)}{p^{k+1} \frac{d}{dp} E_k |p=p(x_j^\pm)} \int_x^k V(x_j^\pm) \right| + o(\hbar^{k+1}) \equiv \Delta \epsilon^{(0)} + o(\hbar^{k+1}). \quad (2.4)$$

Define a dimensionless measurement of ND by $\eta_n = \frac{2\Delta \epsilon}{\epsilon_n + 1 - \epsilon_{n-1}}$. Noticing that the semi-classical level spacing is $2\pi \frac{dE}{ds} = 2\pi h/T$ and according to Eq. (2.4), we find

$$\eta = \frac{\hbar^k}{2^{k+1} \pi} \left| \sum_{j=1}^N \frac{\exp(2is(x_j^\pm)/\hbar)}{p^{k+1} \frac{d}{dp} E_k |p=p(x_j^\pm)} \int_x^k V(x_j^\pm) \right| + o(\hbar^k) \equiv \eta^{(0)} + o(\hbar^k). \quad (2.5)$$

**Example 2.1** $H = \frac{1}{2}p^2 + V^{(k)}(x)$, where $V^{(1)}(x) = \max\{\cos x, 0\}$ and $V^{(k)}(x) = [V^{(1)}(x)]^k$, $k = 2, 3, \ldots$.

According to Eq. (2.5), when $\epsilon > 1$,

$$\eta^{(0)} = \frac{k! \hbar^k}{2^{k+1} \pi (2\epsilon)^{\frac{k}{2} + 1}} \left| \sin \left( \frac{(2\epsilon)^{\frac{k}{2} + 1}}{\hbar} \frac{k\pi}{2} \right) \right|.$$ 

The comparison of $\eta$ and $\eta^{(0)}$ is shown in Fig. 2.

**Example 2.2** $H = |p| + V^{(k)}(x)$.

When $\epsilon > 1$, the semi-classical level is given by $\epsilon_n = nh + \alpha_k$ and according to Eq. (2.4)

$$\Delta \epsilon^{(0)} = \frac{k! \hbar^{k+1}}{2^{k+1} \pi \epsilon^{k+1}} \left| \sin \left( \frac{\alpha_k \pi}{\hbar} + \frac{k\pi}{2} \right) \right|$$

where

$$\alpha_k \equiv \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^k x dx = \frac{\Gamma\left(\frac{k+1}{2}\right)}{2\Gamma\left(\frac{k}{2}\right)} \frac{\Gamma\left(\frac{k}{2} + 1\right)}{\Gamma\left(\frac{1}{2}\right)}.$$ 

The comparison of $\Delta \epsilon$ and $\Delta \epsilon^{(0)}$ is shown in Fig. 3.
3 Sum Over Transition Paths

In this section we first give Eq. (2.4) a geometrical interpretation. We find the quantum transition between the semi-classical eigenstates can be classically described by the leaking of phase space points from one torus to the other via passage of non-smoothness. This picture will facilitate the generalization of Eq. (2.4).

The splitting of nearly degenerate energy levels is closely related to the transition probability between the corresponding semi-classical eigenstates. In classical picture, $\Psi^+$ describes a particle moving on the circle with $p > 0$. After one classical period, due to quantum tunneling, the particle has a non-zero probability to jump to the reflection wave $\Psi^-$ with $p < 0$. Write $\exp(HTi\hbar)|\Psi^+> = c|\Psi^+> + A|\Psi^->$. Simple calculations show that $A \approx T\hbar < \Psi^-|(H - \epsilon)|\Psi^+>$ and $\Delta \epsilon \approx 2\hbar |A|$. According to Eq. (A.17), the leading term in $A$ is the sum of contribution from each non-smooth point of $V(x)$, i.e.,

$$A \approx A^{(0)} = \sum_{j=1}^{N} r_j \exp(i\phi_j), \quad (3.1)$$

with

$$r_j = \left(\frac{i\hbar}{2p_j}\right)^k (2p_j)^{k+1} \int x^k V(x_j^*) \quad \text{and} \quad \phi_j = 2s(x_j^*)/\hbar, \quad (3.2)$$

where $p_j^* \equiv p(x_j^*)$ and $x_j^* \equiv \frac{d}{dp}E_k[p=p(x_j^*)]$. We note that $r_j$ is exactly the reflection coefficient obtained by Berry\(^2\).

As the classical representation of $\Psi^+$ and $\Psi^-$, the tori $O^+_\epsilon$ and $O^-_\epsilon$ are connected by the straight line $x = x_j^*$ where $H$ is not smooth. We shall call the vector on $x = x_j^*$ that starts from $O^+_\epsilon$ and ends at $O^-_\epsilon$ a transition path and denote it by $\gamma_j$ (Fig.4). Accordingly, we can say that $\Psi^+ \rightarrow \Psi^-$ is dominated by the tunneling along transition path(s). In fact, the reflection coefficient $r_j$ is determined by the local properties of $\gamma_j$. Besides a constant, $r_j$ consists of three ingredients. $\int x^k V(x_j^*)$ can be regarded as the intensity of non-smoothness at $\gamma_j$. $\frac{1}{(2p_j)^{k+1}}$ describes a power-law decay with the increase of path length $2p_j^* = \hbar \frac{\partial \phi_j}{\partial x_j}$. $\frac{1}{\pi_j}$, which comes from the product of amplitude of semi-classical wave functions, gives a classical weight of the transition path: the longer the particle stays in the vicinity of the non-smooth point, the more probably it jumps to the other torus. In contrast to $r_j$, the phase $\phi_j$ is not determined by the local properties of $\gamma_j$. Since only the relative phase is of physical importance, i.e., gives rise to

\(^2\)Berry’s calculation was based on $E_k = \frac{p^2}{2m}$. However, the result (Eq. (27) in [3]) is essentially identical to Eq. (3.2).
interference effect, we find
\[ \phi_j - \phi_k = \frac{2}{\hbar} (s(x_j) - s(x_k)) = \frac{1}{\hbar} \oint_{\gamma_{jk}} p \, dx, \tag{3.3} \]
where \( \gamma_{jk} \) is a closed path consists of \( \gamma_j \), \( -\gamma_k \) (\( \gamma_k \) with opposite direction) and the segments of \( O^+_\epsilon \) and \( O^-_\epsilon \) (real paths) that attached at their ends (see Fig.4). If \( \gamma_{jk} \) is contractible, \( \phi_j - \phi_k \) is simply the phase space area (in the unit of \( \hbar \)) enclosed by this closed path.

Behind the simple form of Eq. (3.2) there are two non-generic facts resulting from the assumption that \( \frac{d}{dp} E_k(p) > 0 \) when \( p > 0 \): the starting and end points of \( \gamma_j \) are symmetric with respect to \( p = 0 \) and the projection of \( O^+_\epsilon \) or \( O^-_\epsilon \) onto the coordinate space contains no singularity. Now we ignore this assumption and require only \( E_k(-p) = E_k(p) \) to guarantee the time reversal symmetry. Let \( A_j = (x_j, p_j) \in O^+_\epsilon \) and \( A'_j = (x'_j, p'_j) \in O^-_\epsilon \) be the starting and end points of \( \gamma_j \). By adopting the general semi-classical eigenfunctions corresponding to the tori \( O^+_\epsilon \) and \( O^-_\epsilon \), similar calculations as that performed in Appendix show that Eq. (3.2-3) should be modified as
\[ r_j = \frac{(i\hbar)^k}{(p_j - p'_j)^{k+1}} \sqrt{|\dot{x}(A_j)\dot{x}(A'_j)|} \int_{x_j}^k V(x_j) \tag{3.4} \]
and
\[ \phi_j - \phi_k = \frac{1}{\hbar} \oint_{\gamma_{jk}} p \, dx - M_{jk} \pi/2, \tag{3.5} \]
where \( M_{jk} \) is the sum of the Maslov indices of the segments of real paths on \( \gamma_{jk} \). Having the contribution of each transition path, we need only to sum over all these paths to obtain the energy splitting \( \Delta \epsilon^{(0)} \) or \( \eta^{(0)} \).

Example 3.1 \( H = (p^2 - 1)^2 + V(x) \), where \( V(x) = 1 - (\frac{x}{\pi})^2, |x| \leq \pi \).
When \( \epsilon < 1 \), the Maslov index of \( O^+_\epsilon \) (or \( O^-_\epsilon \)), which encircles point \((\pi, 1)\) (or \((\pi, -1)\)), is 2 and EBK quantization condition reads \( S(\epsilon_n) = 2(n + \frac{1}{2})\pi \hbar \) (see inset of Fig. 5). The straight line \( x = \pi \) intersects \( O^+_\epsilon \) (or \( O^-_\epsilon \)) at points \( A_{1,2} \) (or \( A'_{1,2} \)) where \( p = (1 \pm \epsilon^{\frac{1}{2}})^{\frac{1}{2}} \) (or \(-(1 \pm \epsilon^{\frac{1}{2}})^{\frac{1}{2}}\)). There exist four transition paths, i.e., \( \gamma_1 \) \((A_1 \rightarrow A'_1)\), \( \gamma_2 \) \((A_2 \rightarrow A'_2)\), \( \gamma_3 \) \((A_1 \rightarrow A'_2)\) and \( \gamma_4 \) \((A_2 \rightarrow A'_1)\). Moreover, EBK quantization condition implies \( \phi_j - \phi_1 = 0, n\pi, n\pi \) (mod 2\( \pi \)) for \( j = 2, 3, 4 \) respectively. According to Eq. (3.4), up to a phase,
\[ A^{(0)} = \frac{i\hbar}{4\pi\epsilon^{\frac{1}{2}}(1 + \epsilon^{\frac{1}{2}})^{\frac{1}{2}}} \left[ \frac{1}{(1 + \epsilon^{\frac{1}{2}})^{\frac{1}{2}}} + \frac{1}{(1 - \epsilon^{\frac{1}{2}})^{\frac{1}{2}}} + (-1)^n \frac{4}{(1 + (1 - \epsilon^{\frac{1}{2}})(1 + \epsilon^{\frac{1}{2}})^{\frac{1}{2}})} \right]. \]
When \( \epsilon > 1 \), only \( \gamma_1 \) survives so that
\[ A^{(0)} = \frac{i\hbar}{4\pi\epsilon^{\frac{1}{2}}(1 + \epsilon^{\frac{1}{2}})^{\frac{1}{2}}}. \]
Numerical results show that \( \eta(0) = |A(0)|/\pi \) is a good approximation of \( \eta \) when \( \bar{h} \) is sufficiently small and \( \epsilon \) is not too close to 1, the energy of separatrix (Fig. 5).

The same treatment can be applied to non-smooth systems where ND is originated from spatial symmetries. By substituting \( (p, -x) \rightarrow (x, p) \), relations (3.4-5) can be directly transformed to systems where the non-smoothness that results transition path depends only upon the \( p \)-coordinate. Specifically, consider a transition path \( \gamma_j \) on the straight line \( p = p_j^* \) with starting and end points at \( A_j = (x_j, p_j^*) \) and \( A_j' = (x_j', p_j^*) \), the corresponding reflection coefficient should be

\[
\mathcal{r}_j = \frac{(ih)^k}{(x_j' - x_j)^{k+1}} \sqrt{|p(A_j)p(A_j')|} \wedge_p H(x, p_j^*). \tag{3.6}
\]

The phase difference is also given by Eq. (3.5), whereas the Maslov index should count the singularity of the projection of torus onto the momentum space \[^3\]. Despite this similarity, interesting behavior may occur when the configuration space has a non-trivial topology. We shall demonstrate it by some examples.

Suppose the configuration space is a circle, i.e., \( (x, p) \) and \( (x + 2\pi, p) \) describe the same point. In this case, a path \( (x, p) \rightarrow (x', p) \) implies a family of paths \( (x, p) \rightarrow (x' + 2n\pi, p), n \in \mathbb{Z} \). If we attribute the contribution of all these paths to a representative path, say, \( (x, p) \rightarrow (x', p) \), the only change of Eq. (3.6) is that \( \frac{1}{(x_j' - x_j)^{k+1}} \) should be replaced by

\[
\sum_{q = -\infty}^{\infty} \frac{\exp(i2q\pi p_j^*/\bar{h})}{(x_j' - x_j + 2q\pi)^{k+1}} \equiv W_{k+1}(x_j', x_j, p_j^*/\bar{h}). \tag{3.7}
\]

\( W \) satisfies periodic condition \( W_k(x, y + 1) = e^{i2\pi y}W_k(x + 2\pi, y) = W_k(x, y) \). When \( y \in [0, 1] \),

\[
W_2(x, y) = \frac{1}{4\sin^2\frac{x}{2}}[1 + y(e^{ix} - 1)]e^{-ixy},
\]

\[
W_3(x, y) = \frac{1}{8\sin^3\frac{x}{2}}[\cos\frac{x}{2} + i2y\sin\frac{x}{2} - 2y^2\sin^2\frac{x}{2}e^{ix/2}]e^{-ixy}
\]

and so on. We note that \( A(0) \) is in general not invariant under the translation \( (x, p) \rightarrow (x, p + \delta_p) \) when \( \delta_p \) is not an integer multiple of \( \bar{h} \), which is however always a symmetric transformation in classical mechanics. This difference reflects the discreteness of quantum momentum space.

**Example 3.2** \( H = |p - p_c| + \cos^2 x \).

The symmetric double-well potential causes ND at \( \epsilon < 1 \). According to Eq. (3.6-7), the total

\[^3\] We use \( \omega_1 = \oint dx \) instead of \( \omega'_1 = -\oint dp \) based on two facts. Firstly, \( \int_{\gamma_{jk}} \omega_1 = \int_{\gamma_{jk}} \omega'_1 \) when \( \gamma_{jk} \) is contractible. Secondly, if the coordinate space has non-trivial topology, \( \omega_1 \) is well-defined while \( \omega'_1 \) is not. We find that this choice is justified by numerical results.
In quantum mechanics, a symplectic two form \( \omega \) is given by:

\[
\omega = \frac{i\hbar}{\varepsilon^2} [W_2(2x_c, \frac{p_c}{\hbar}) + W_2(2\pi - 2x_c, \frac{p_c}{\hbar}) + (-1)^n 2W_2(\pi, \frac{p_c}{\hbar})],
\]

where \( x_c = \cos^{-1} \frac{1}{\varepsilon} \). When \( p_c = 0 \), \( A^{(0)} = \frac{i\hbar}{2\varepsilon^2 (1 - \varepsilon)} [\frac{1}{1 - \varepsilon} + (-1)^n] \). When \( p_c = \frac{\hbar}{2} \), \( A^{(0)} = 0 \). In fact, \( \Delta \varepsilon \equiv 0 \) in this case because \( H \) is represented by the same matrix in the invariant subspaces spanned by functions \( \{e^{i2nx}\}_n \) and \( \{e^{i(2n+1)x}\}_n \) respectively.

Consider a spin system defined in classical and quantum mechanics by respectively \( \{J_j, J_k\} = \varepsilon jks, j, k = 1, 2, 3 \). When \( J^2 = J_1^2 + J_2^2 + J_3^2 \) is fixed, the classical mechanics is confined within a sphere \( S_j \). Restricting the \( su(2) \) Poisson structure to \( S_j \) yields a symplectic form \( \omega_2 = J \sin \theta \sin \phi \wedge d\phi \), where \( \phi \) is the conventional sphere coordinate. In quantum mechanics, \( J^2 = j(j+1)\hbar^2 \), \( j = \frac{1}{2}, 1, \frac{3}{2}, ... \). An eigenspace of \( J^2 \) is associated with a classical sphere \( S_j \), in which we shall assume \( J = (j + \frac{1}{2})\hbar \) so that its phase space area (integral of \( \omega_2 \) on \( S_j \)) in unit \( 2\pi \hbar \) is \( 2j + 1 \), which directly corresponds to the dimension of the eigenspace. In our treatment of non-smooth systems, a prerequisite is that the phase space is the direct product of coordinate and momentum spaces. To meet this requirement, we write \( (J \cos \theta + p_0, \phi) = (p, x) \), in which \( \omega_2 = dp \wedge dx \), and regard \( (x, p) \) as the natural coordinate of the phase space of a mechanic system on a circle. Moreover, to ensure the right spectrum of \( J_3 = p - p_0 \), we choose \( p_0 = 0 \) (or \( \frac{1}{2}\hbar \)) in the case of \( j \) is an integer (or half integer). By this transformation in classical mechanics, we can treat the non-smoothness-enhanced tunneling in some spin systems.

**Example 3.3**

\[
H(J_1, J_2, J_3) = \begin{cases} 
J_1^2 - J_2^2 + J_3^2 & \text{if } J_3 \geq 0, \\
J_1^2 - J_2^2 & \text{if } J_3 < 0.
\end{cases}
\]

The corresponding classical system on a circle is

\[
H(x, p) = \begin{cases} 
[J^2 - (p - p_0)^2] \cos 2x + (p - p_0)^2 & p \geq p_0, \\
[J^2 - (p - p_0)^2] \cos 2x & p < p_0.
\end{cases}
\]

From phase space portrait we know that energy levels in \( (-J^2, 0) \) consist of 2-fold ND and according to Eq. (3.6-7),

\[
A^{(0)} = \frac{-\hbar^2}{J^2 \sin 2x_c} [W_3(2\pi - 2x_c, \frac{p_0}{\hbar}) + W_3(2x_c, \frac{p_0}{\hbar}) e^{2i(\phi - \frac{\pi}{2})} + 2W_3(\pi, \frac{p_0}{\hbar}) e^{i(\phi - \frac{\pi}{2})}],
\]

where \( x_c = \frac{1}{3} \cos^{-1} \xi \) with \( \xi \equiv \epsilon/J^2 \) and \( \phi = \pi J(1 - \sin x_c)/\hbar = \pi (j + \frac{1}{2})[1 - (1 - \frac{1}{2}\xi)^2] \). When \( j \) is an integer,

\[
|A^{(0)}| = \frac{\cos \phi}{2(j + \frac{1}{2})^2(1 - \xi)^2}
\]
and when \( j \) is a half integer,

\[
|A(0)| = \frac{1}{4(j + \frac{1}{2})^2(1 - \xi^2)^{\frac{3}{2}}} \left| \frac{3 + \xi}{\sqrt{2}(1 - \xi)^{\frac{3}{2}}} \sin \phi + \frac{1}{2} \right|
\]

(In this case, \( \frac{1}{j + \frac{1}{2}} \) can be regarded as an effective \( \hbar \).) These relations give a good description of the energy splitting when \( j \gg 1 \) (Fig. 6).

4 Discussion

When the non-smooth system is controlled by a parameter \( \lambda \), e.g., \( V(x) \to \lambda V(x) \), it is easy to obtain a zero of \( A(0) \) when \( \lambda \) is continuously varied. One can naturally ask that whether the zero of \( A(0) \) predicts an exact degeneracy of energy level or it merely corresponds to a minimum of \( \Delta \epsilon \). The answer turns out to be dependent upon the symmetry of the system. If the eigenstates involved in ND can be distinguished by different symmetries irrespective of the parameter, the energy difference between the two eigenstates should be a smooth function of \( \lambda \), which is approximately given by \( \frac{2\hbar A(0)}{T} \) or similar expression. In this case, the zero of \( A(0) \) indicates a nearby exact degeneracy. Of course, because of the symmetry of \( H \), this conclusion cannot be regarded as a violation of the well-known theorem of von Neumann and Wigner, which states that generically we must vary two parameters to create a degeneracy [8]. On the other hand, if the eigenstates cannot be restricted within different parameter-independent invariant subspaces, e.g., when \( H = p^2/2 + \cos x + \lambda|\sin x| \), the zero of \( A(0) \) generally corresponds to a minimum of \( \Delta \epsilon \) where we must take the higher order corrections into account.

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Figure Captions

Fig.1 Splitting of nearly degenerate energy levels. (a)-(d) for $H = H_1$, $H_2$, $H_3$ and $H_4$ respectively. The numerical result of $\Delta \epsilon$ (open circles), the spacing of semi-classical levels (dotted lines) and the semi-classical approximation of $\Delta \epsilon$ (solid lines) are shown at $\hbar = 0.02$. The insets show the degenerate tori (solid line) in phase-space where $H$ is not smooth on the dotted lines.

Fig.2 Scaled energy splitting $\eta$ (open circles) and $\eta^{(0)}$ (connected solid dots) in example 2.1 at $k = 1$ to 4 and $\hbar = 0.05$.

Fig.3 Energy splitting $\Delta \epsilon$ (open circles) and $\Delta \epsilon^{(0)}$ (solid lines) in example 2.2 at $k = 1$ to 4 and $\hbar = 0.04$.

Fig.4 Schematic figure show transition paths $\gamma_1$ ($A \rightarrow A'$), $\gamma_2$ ($B \rightarrow B'$) and closed path $\gamma_{21}$ ($B \rightarrow B' \rightarrow A' \rightarrow A' \rightarrow B'$). EBK quantization rule guarantees that $\phi_2 - \phi_1$ (mod $2\pi$) is independent on the choice of real paths $B' \rightarrow A'$ on $O_{\epsilon^-}$ and $A \rightarrow B$ on $O_{\epsilon^+}$.

Fig.5 $\eta$ (open circles) and $\eta^{(0)}$ (solid lines) in example 3.1 at $\hbar = 0.02$. The inset shows three types of tori in phase space. The tori encircling point (0, 0) produce a semi-classically non-degenerate component of energy spectrum at $1 < \epsilon \leq 2$, which has been excluded according to semi-classical criterion that the expectation value of $p^2$ at the corresponding eigenstates is less than unity.

Fig.6 $\eta$ (open circles) and $\eta^{(0)}$ (connected dots) in example 3.3 at (a) $j = 100$ and (b) $j = 99 \frac{1}{2}$. 
Appendix: Semi-classical Calculation of Energy Splitting

We first consider the conventional Hamiltonian \( H = \frac{1}{2}p^2 + V(x) \). Direct calculation show that

\[
\left(-\frac{\hbar^2}{2} \frac{d^2}{dx^2} + V(x)\right)\Psi^\pm(x) = (\epsilon + Q(x))\Psi^\pm(x),
\]

(A.1)

with \( Q = -\frac{\hbar^2}{2}p^{1/2}(p^{-1/2})'' \), where the prime denotes derivation with respect to \( x \) at fixed \( \epsilon \).

Because \(<\Psi_+|\Psi_+> = <\Psi_-|\Psi_-> = 1 \) and \(<\Psi_-|\Psi_+> \sim 0 \), the energy splitting calculated in the space spanned by \( \Psi^+ \) and \( \Psi^- \) is given by

\[
\Delta \epsilon = 2|<\Psi^-|Q|\Psi^+>| = \frac{\hbar^2}{2T} \left| \int_0^{2\pi} \frac{V''}{p^3} + \frac{5(V')^2}{2p^5} \exp(i2s(x)/\hbar)dx \right|.
\]

(A.2)

Before evaluating \( \Delta \epsilon \) according to Eq. (A.2), it is helpful to recall an useful mathematical result on asymptotic behavior of the Fourier coefficients of a non-smooth function. Let \( f(x) \) be a sufficiently regular \( 2\pi \)-periodic function on \( R \). How its Fourier coefficients, defined by

\[
\hat{f}(n) = \int_0^{2\pi} f(x) \exp(inx)dx, \quad n \in Z,
\]

(A.3)

decay when \( n \to \pm \infty \) is basically determined by the analytic property of \( f(x) \). If it is smooth, then \( \hat{f}(n) \) for large \( n \) will approach zero faster than any power of \( |n|^{-1} \), i.e., \( \lim_{|n| \to \infty} \hat{f}(n)|n|^\alpha = 0 \) for arbitrary \( \alpha > 0 \). On the other hand, if \( f(x) \) is not smooth, the decay of \( \hat{f}(n) \) may follow a power law. In the simple case when \( f(x) \) is the union of \( N \) smooth segments on intervals \([x_1^\pm, x_2^\pm], x_1 < x_2 \ldots < x_{N+1} = x_1 + 2\pi\), \( \hat{f}(n) \) can be expressed by asymptotic series

\[
\hat{f}(n) = \sum_{l=0}^{\infty} \frac{n^{l+1}}{l+1} \sum_{j=1}^{N} \exp(inx_j^\pm) \bigwedge_x f(x_j^\pm).
\]

(A.4)

Let \( s(x) = nh\theta(x) \), we rewrite Eq. (A.2) as

\[
\Delta \epsilon = \frac{\hbar^2}{2T} \left| \int_0^{2\pi} \frac{V''}{p^3} + \frac{5(V')^2}{2p^5} \frac{n\hbar}{p} \exp(i2n\theta)d\theta \right|.
\]

(A.5)

Noticing the integrand apart from \( \exp(i2n\theta) \) is unchanged in the semi-classical limit, according to Eq. (A.4), we have

\[
\Delta \epsilon = \frac{\hbar^{k+1}}{2kT} \sum_{j=1}^{N} \frac{\exp|2is(x_j^\pm)/\hbar|}{p^{k+2}V(x_j^\pm)} \bigwedge_x V(x_j^\pm) + o(\hbar^{k+1}).
\]

(A.6)

Then we consider Hamiltonian \( H = E_k(p) + V(x) \). In order to evaluate energy splitting according to \( \Delta \epsilon = 2|<\Psi^-|H - \epsilon|\Psi^+>| \), it is instructive to go into some details about the
momentum representation of $\Psi^\pm$. Write

$$\Psi^\pm = \sum_{r=-\infty}^{\infty} \phi_r^\pm |r>, \quad <x|r> = \frac{1}{\sqrt{2\pi}} \exp(irx), \quad (A.7)$$

with

$$\phi_r^\pm = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Psi^\pm(x) \exp(-irx)dx. \quad (A.8)$$

Since $\phi_{-r} = \phi_r^{*}$, we shall focus on $\Psi^+$. The semi-classical limit of Eq. (A.8) should be calculated in two separate cases. In the classically permissible region (CPR), where $p(x)-r\hbar = 0$ is satisfied by some $x \in [0, 2\pi)$, stationary phase approximation can be adopted, which results

$$\phi_r^+ \approx \sqrt{\frac{\hbar}{T}} \sum_{m} \frac{1}{|V'(x_m)|} \exp[i(s(x_m)/\hbar - rx_m - \sigma_m \pi/2)], \quad (A.9)$$

where $\{x_m\}$ are solutions of $p(x) - r\hbar = 0$ and $\sigma_m = \text{sign}(V''(x_m))$. When $r\hbar$ is beyond CPR, by using expansion (A.4), we find

$$\phi_r^+ \approx \frac{(i\hbar)^{k+1}}{\sqrt{2\pi T}} \sum_{j=1}^{N} \frac{\exp[i(s(x_j^+)/\hbar - rx_j^+)]}{(p-r\hbar)^{k+1}E_k} \frac{d}{dp} \left[ \frac{-1}{(p-r\hbar)\sqrt{\frac{d}{dp}E_k}} \right]_{p=p(x_j^+)} \phi_{r+m}^+. \quad (A.10)$$

From Eq. (A.9-10) we conclude that $\Psi^+$ consists of the main part distributed within CPR and two power-law-like long tails beyond CPR. (As the non-smoothness of eigenfunction is resulted via the eigen equation from $V(x)$, this picture is also true for exact eigenfunction.) Furthermore, if the semi-classical momentum representation of $V\Psi^+$ is calculated in the similar procedure, one can find that the main part of $\Psi^+$ within CPR but its long tails approximately satisfies eigen equation $(E_k(p) + V(x))\Psi = \epsilon\Psi$, i.e.,

$$\sum_{m=-\infty}^{\infty} (E_k(r\hbar)\delta_{m,0} + V_m)\phi_{r+m}^+ \approx \epsilon\phi_r^+, \quad (A.11)$$

when $r\hbar \in$ CPR, where

$$V_m = <0|V|m> = \frac{V^{k+1}}{2m^{k+1}\pi} \sum_{j=1}^{N} \exp(imx_j^+) \bigwedge_x V(x_j^+) \quad (|m| \to \infty). \quad (A.12)$$

Based on the above discussion, we know that

$$<\Psi^-|E_k(p)|\Psi^+> = \sum_{r=-\infty}^{\infty} \phi_{-r}^+ E_k(r\hbar) \phi_{r}^+ \approx \sum_{|r\hbar|\in\text{CPR}} \phi_{r}^+ E_k(r\hbar) \phi_{r}^+$$

$$\approx (\sum_{\nu\hbar\in\text{CPR}} \sum_{\mu=-\infty}^{\infty} + \sum_{-\mu\hbar\in\text{CPR}} \sum_{\nu=-\infty}^{\infty} ) \phi_{-\nu}^+ (\epsilon \delta_{\nu,\mu} - V_{\mu-\nu}) \phi_{\nu}^+ \quad (A.13)$$
Compare the last expression with
\[
< \Psi^- | \epsilon - V | \Psi^+ > = \sum_{\mu, \nu = -\infty}^{\infty} \phi^+_\nu (\epsilon \delta_{\nu, \mu} - V_{\mu - \nu}) \phi^+_\mu . \tag{A.14}
\]

The main contribution of Eq. (A.14) consists of three parts come from regions, (1) $\mu \hbar, \nu \hbar \in \text{CPR}$, (2) $-\mu \hbar, -\nu \hbar \in \text{CPR}$ and (3) $\mu \hbar, -\nu \hbar \in \text{CPR}$ respectively. Eq. (A.13) contains only the former two parts while we can screen the last contribution by making a high frequency cut off of $V(x)$, i.e., replacing it by
\[
V^{(0)}(x) = \sum_{|m| \leq k_c} V_m \exp(-imx), \tag{A.15}
\]
where $k_c$ is a large but fixed integer so that $V^{(1)}(x) = V(x) - V^{(0)}(x)$ is negligibly small. Therefore,
\[
< \Psi^- | E_k(p) | \Psi^+ > \approx < \Psi^- | \epsilon - V^{(0)} | \Psi^+ > ,
\]
and consequently
\[
< \Psi^- | H - \epsilon | \Psi^+ > \approx < \Psi^- | V^{(1)} | \Psi^+ > = \frac{1}{T} \int_0^{2\pi} \frac{V^{(1)}}{dE_k|_{p=p(x)} \exp(2is(x)/\hbar)} dx . \tag{A.16}
\]
Observing that $V^{(1)}(x) \approx 0$ and $\land_j \land_j V^{(1)}(x) = \land_j V(x)$ for arbitrary $x \in [0, 2\pi]$ and $j \geq 0$, by partial integrating Eq. (A.16) for successive $k + 1$ times we obtain
\[
< \Psi^- | H - \epsilon | \Psi^+ > = \frac{(ih)^{k+1}}{2^{k+1}T} \sum_{j=1}^N \exp(2is(x_j^*)/\hbar) \land_j^k V(x_j^*) + o(\hbar^{k+1}), \tag{A.17}
\]
which immediately leads to Eq. (2.4).

Finally, it is worth pointing out that although the exact eigenstates have power-law tails beyond CPR, the leading term of $\Delta \epsilon$ actually does not relies on this detail. In fact, Eq. (A.16) essentially equals to
\[
\sum_{\mu \hbar, -\nu \hbar \in \text{CPR}} \phi^+_\nu V_{\mu - \nu} \phi^+_\mu ,
\]
which is in nature controlled by the power-law decay of $\{V_m\}$ but $\{\phi^+_r\}$. Therefore, Eq. (A.16) can be reproduced from the highly localized semi-classical eigenfunctions corresponding to smoothed Hamiltonian $H^{(0)} = E_k(p) + V^{(0)}(x)$. 
