Swarm Modeling With Dynamic Mode Decomposition

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ABSTRACT Modelling biological or engineering swarms is challenging due to the inherently high dimension of the system, despite the often low-dimensional emergent dynamics. Most existing swarm modelling approaches are based on first principles and often result in swarm-specific parameterisations that do not generalise to a broad range of applications. In this work, we apply a purely data-driven method to (1) learn local interactions of homogeneous swarms through observation data and to (2) generate similar swarming behaviour using the learned model. In particular, a modified version of dynamic mode decomposition with control, called swarmDMD, is developed and tested on the canonical Vicsek swarm model. The goal is to use swarmDMD to learn inter-agent interactions that give rise to the observed swarm behaviour. We show that swarmDMD can faithfully reconstruct the swarm dynamics, and the model learned by swarmDMD provides a short prediction window for data extrapolation with a trade-off between prediction accuracy and prediction horizon. We believe the proposed swarmDMD approach will be useful for studying multi-agent systems found in biology, physics, and engineering.

INDEX TERMS Swarms, multi-agent systems, reduced-order models, dynamic mode decomposition, control, optimisation.

I. INTRODUCTION

Emergent behaviours, such as swarming and flocking, are ubiquitous in natural and engineered systems. Fascinating phenomena including birds flocking, fish schooling, and ants colonising have intrigued generations of biologists to discover and understand the rules of life [1]–[3]. Fueled by the interplay between the intra-swarm and swarm-environment interactions, swarms often exhibit complex multi-scale dynamics, making them an interesting research subject for physicists and mathematicians as well. Inspired largely by the biology community, engineers and roboticians study swarms in order to understand the emergence of macroscopic organised behaviours for dynamics modeling and prediction [4]–[7], to build robotic platforms to directly interact with biological swarms [8]–[10], and ultimately to design scalable control and planning algorithms for artificial swarms [11]–[17].

One long-standing challenge in the study of swarms and emergent behaviours is the discovery of optimal interaction laws that lead to various macroscopic phenomena resembling our observations in nature. Although several mathematically simple swarm interaction laws have been proposed to generate compelling swarm or flocking behaviour [18]–[23], there remains a need for a principled approach that allows the extraction of the fundamental interaction law from observations alone. In other fields, data-driven methods have proven effective in identifying unknown dynamics from observation [24]–[26]. However, data-driven discovery of swarm interactions is still nascent [27], [28].

Collective behaviour in biological swarms has been studied in depth across organisms ranging from cellular systems [29] and midges [7], to jackdaw flocks [30] and humans playing baseball [31]. Studies on the decision making processes of biological swarms have shown
geometry often plays a role, resulting in a binary process used across biological scales [32]; whereas other studies have shown that further structure is often needed to make general statements across scales [29]. Much research has been conducted on the initiation of large collective movements [30], showing that environmental cues play an important role in their emergence [33]; and on disruptions to collective behaviour showing that genetic mutations in zebra fish can cause reduced cohesion in formations [34]. Such studies inform the development and evaluation of methods for determining governing laws of swarm interactions [35].

Many methods to learn swarm laws take a model-based approach [35]. Some of the simplest models, aimed only at recreating the basic structure of emergent patterns, include the alignment-based Vicsek model and its modifications [36], [37], a stochastic model by Aoki [38], and attraction-repulsion models [18], [39]. Although biologically inspired, these models were developed with the aim of recreating similar dynamics to those seen in biological swarms, and not at determining the actual underlying governing equations of an observed swarm. Model-based methods that do aim to discover governing equations often make simplifying assumptions, such as imposing a structure on the governing equations [40], [41]. In recent years, machine learning [42] and game theoretic [43] approaches are becoming more widely used.

Fully capturing the governing laws of a swarm system with a model-based approach would likely require in a very complicated model. This promotes the use of model-free, or data-driven methods, where the goal is to accurately capture dynamics without the need for restrictive assumptions. Many model-free approaches exploit known properties of swarms, such as the interchangeability of agents [27], the inherent network dynamical interactions [31], and how pairwise distances between agents play an important role in interactions [44].

In our proposed method, we take a fully data-driven approach inspired by dynamic mode decomposition (DMD) [26], [45]–[49], a method originating in fluid dynamics and capable of capturing system dynamics across timescales. Using DMD with control as a starting point [50], we exploit the general idea of how swarm agents update their states — using information about their own state and their neighbours — to develop dynamic mode decomposition for swarms (swarmDMD). The resulting modelling framework produces an optimal linear approximation, distinguishing our approach from others which may seek nonlinear estimates. As swarmDMD is a purely data-driven approach, with no assumptions made for specific swarm systems, it is generalisable to many kinds of multi-agent systems and can use different types of state data that are relevant to the dynamics. For data generated by the Vicsek model, it was shown in a preliminary comparison of data types that swarmDMD performs the best with relative position information (i.e., inter-agent distance), resulting in accurate recreations of swarm dynamics during the training period and a window of accurate prediction post-training.

The remainder of this paper is organised as follows. Section II provides an overview of DMD for data-driven approximation of nonlinear dynamics. Section III introduces swarmDMD and how it is developed from DMD. Section IV covers the numerical simulation set-up, including the Vicsek model and modification used for the ground truth swarm data sets, and evaluation metrics for the swarmDMD results. Section V discusses the experimental results and major findings.

II. PRELIMINARIES

In this section, we introduce important background material and concepts used in the development of swarmDMD. Beginning with DMD, we then describe the DMD with control variant, on which swarmDMD is based.

We focus on swarm behaviours that are induced by local interactions between swarm agents. These emergent behaviours often give rise to coherent swarm structures that can persist with inherently low-dimensional dynamics. For such systems, a method like DMD, which captures low-order dynamics in high-order systems, is promising in modelling the dominant modes of the systems and providing a short window for prediction.

DMD approximates the dynamics of a high dimensional, and potentially nonlinear, system with a linear model that advances the system state forward in time [45], [46]. Although DMD was originally developed to study fluid systems [45], [47], it has since been applied to a wide range of physics systems, such as neuroscience [51], [52], and it has also been rigorously connected to nonlinear systems through Koopman operator theory [46], [49], [53].

Let us consider \( \mathbf{x}(t) \in \mathbb{R}^n \) as the state vector of an \( n \)-dimensional dynamical system at time \( t \). In a discrete-time representation with uniform interval \( \Delta t \), we can denote the system state at each time step with subscript \( k \in \mathbb{Z} \) such as \( \mathbf{x}_k = \mathbf{x}(k \Delta t) \). Now consider \( T \) time samples of the state, i.e., \( k \in \{1, \ldots, T\} \), and define two state snapshot matrices \( \mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_{T-1}] \) and \( \mathbf{X}' = [\mathbf{x}_2 \ldots \mathbf{x}_T] \). The goal of DMD is to compute a state transition matrix, \( \mathbf{A} \in \mathbb{R}^{n \times n} \), that propagates the data forward by one time step, i.e., \( \mathbf{A} \) such that \( \mathbf{X}' = \mathbf{AX} \). To do so, DMD uses a singular value decomposition (SVD) to compute the pseudo-inverse of the data matrix \( \mathbf{X} \), which has the added benefit of allowing a reduced-order approximation to \( \mathbf{X} \) based on how many modes, or singular values, are kept. Thus, \( \mathbf{A} \) provides a linear approximation of the system dynamics.

The process of calculating \( \mathbf{A} \) can be broken into the four steps outlined below [26], [54]:

1) Perform SVD on \( \mathbf{X} \):

\[
\mathbf{X} = \mathbf{U} \Sigma \mathbf{V}^*.
\]

where \( \mathbf{V}^* \) denotes the complex conjugate transpose of \( \mathbf{V} \).
2) Consider instead, the projection of A onto the first r most dominant modes:
\[ \tilde{A} = U_r^T A U_r = U_r^T X V_r \Sigma_r^{-1}, \]
where \( U_r \in \mathbb{R}^{n \times r}, V_r \in \mathbb{R}^{(r-1) \times r}, \) and \( \Sigma_r \in \mathbb{R}^{r \times r} \) are the r-truncated singular vector and value matrices.

3) Compute the eigenvalues and eigenvectors of \( \tilde{A} \):
\[ \tilde{A} \tilde{W} = \tilde{W} \Lambda. \]

The eigenvalues of \( \tilde{A} \) are also eigenvalues of A.

4) Compute the eigenvectors of A:
\[ W = X V_r \Sigma_r^{-1} \tilde{W}. \]

With the eigenvalues and eigenvectors of A, the future state can be predicted. Knowing \( A = W \Lambda W^* \), and that A provides a one time step forward approximation, we can write \( x_{k+1} = Ax_k \) and, we can write \( x_k \) as:
\[ x_{k+1} \approx W e^{\Lambda \Delta t} W^* x_1. \]

By projecting onto the dominant modes in Step 2, DMD provides a computationally efficient way to compute the transition matrix A, thus making it a useful method when dealing with high-dimensional dynamics [55], [56], such as for multi-agent systems. The dominant modes are those singular vector groups with the largest magnitude singular values, indicating that those corresponding singular vectors have the most influence on the structure of the matrix. By choosing to keep only the most dominant modes, one still retains the key properties of the matrix, without the computational burden of performing an eigen-decomposition on a full-rank matrix.

Standard DMD does not explicitly model the influence of control inputs. However, it is often convenient to represent the dynamics of an individual swarm agent using a state-space model with a control input that is dependent on the states of its neighboring agents. DMD with control (DMDc) [50] captures this structure and is used as the base for the development of swarmDMD. DMDc is used to determine the state transition matrix, A, and the control matrix, B, of a linear system approximation of the form:
\[ x_{k+1} = Ax_k + Bu_k, \]
where \( x_k \) is the system state and \( u_k \) is the control input at time step k. Depending on whether B is known, DMDc can be used to learn either just the A matrix or both A and B. As we will show in Section III, (1) motivates the development of swarmDMD.

III. DMD FOR SWARMS

The swarmDMD algorithm is designed to use data we can observe/measure from the swarm, such as agent position, velocity, and inter-agent distance, to create an augmented “state” vector, which is used as the control input to an agent’s position dynamics. We assume that the coupled multi-agent dynamics can be considered control-affine. The goal is to determine the feedback control matrix, which determines the inter-agent interactions.

Consider a swarm of \( N \) agents. We define the 2D location of agent-i at time t as \( p_i^t = [x_i^t, y_i^t] \) for \( i \in \{1, 2, \ldots, N\} \), the velocity as \( v_i^t = [v_i^x(t), v_i^y(t)] \), and the heading as \( \theta_i^t = \theta_i(t) \). The state of agent-i at time t is denoted by \( x_i \), which could be comprised of any of the aforementioned variables. By stacking the states of every agent we obtain the state vector \( x_t = [(x_1^T)^T, \ldots, (x_N^T)^T]^T \in \mathbb{R}^{Nw} \), where \( w \) is the number of state variables chosen for each agent. For any variable written without a superscript, it is used to keep only the most dominant modes, one still retains the key influence on the structure of the matrix. By choosing to keep only the most dominant modes, one still retains the key properties of the matrix, without the computational burden of performing an eigen-decomposition on a full-rank matrix.

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\[ x_{k+1} = Ax_k + Bu_k, \]
where \( B \) is an augmented state vector used to determine the control input, and \( F \) is the feedback matrix for agent-i. The augmented state vector \( y_k^i \in \mathbb{R}^{m_i} \) can include measurement data of individual agents, such as agent position, velocity, and heading as defined before, as well as derived quantities such as relative position \( d_k^i = \|p_k^i - p_k^j\| \), relative distance \( d_k^i = \|p_k^i - p_k^j\| \), relative heading \( \Delta \theta_k^i = \theta_k^i - \theta_k^j \), and relative speed \( \Delta v_k^i = \|v_k^i - v_k^j\| \), relative speed \( \Delta v_k^i = \|v_k^i - v_k^j\| \), and relative speed \( \Delta v_k^i = \|v_k^i - v_k^j\| \). By stacking the control vectors we obtain \( y_k = [y_1^T, \ldots, y_N^T]^T \in \mathbb{R}^{Nm} \). Thus, (1) may be rewritten as:
\[ x_{k+1} = x_k + Ky_k, \]
where \( K \in \mathbb{R}^{Nm \times Nm} \), and \( K = BF \). Note that we have fixed the state dynamics to be the identity, \( A = I \), so that all interactions influencing an agent are considered as external inputs and don’t depend on its own internal state. Rearranging (2) yields:
\[ x_{k+1} - x_k = Ky_k, \]
resulting in a system that resembles the original DMD regression. Thus, DMD can be used to determine \( K \). As in standard DMD, T snapshots are arranged into matrices \( X = [x_1 \ldots x_{T-1}] \), \( X' = [x_2 \ldots x_T] \), and \( Y = [y_1 \ldots y_{T-1}] \). Equation (2) may be written in matrix form as:
\[ X' - X = KY. \]

By defining \( S = X' - X \), we arrive at a linear approximation of the swarm dynamics:
\[ S = KY. \]
Taking the rank-r SVD of Y to obtain \( U \in \mathbb{R}^{Nm \times r} \), \( \Sigma \in \mathbb{R}^{r \times r} \), and \( V \in \mathbb{R}^{(T-1) \times r} \), it is possible to approximate \( K \) as:
\[ K \approx SV \Sigma^{-1} U^*, \]
which can be used to predict future dynamics as:
\[ x_{k+1} = x_k + Ky_k. \]
The swarmDMD process, from data collection to the computation of $K$, is outlined in Fig. 1.

We note that (4) is different from the conventional DMD formulation in that $S$ may have a different shape than $Y$, leading to a rectangular $K$. There are several reasons for this formulation, as opposed to simply stacking the desired data (velocity, heading, inter-agent distance, etc.) in the state vector and using standard DMD. First, the swarmDMD formulation yields a particular interpretation, as discussed below. Further, it can be shown that for any combination of data in swarmDMD that includes at least one data type that is not position, the computational complexity of swarmDMD is less than that of a traditional stacked DMD. Moreover, stacked DMD would seek a model for the evolution of all of the variables, which is known to have closure issues [49].

Note that additional regularisers $R(\cdot)$ may be included in the swarmDMD optimisation problem:

$$
\arg\min_K \|S - KY\| + \lambda R(K).
$$

For example, sparsity may be useful to promote an influence matrix $K$ where each agent uses minimal information in its control law. Similarly, group sparsity may be used to promote a group of agents having similar influence laws. Both are promising avenues of ongoing research.

### A. INTERPRETING $K$

To interpret the swarm influence matrix $K$, it is important to note how the variables are grouped in the vector $y$. Each variable type, such as the $x$ or $y$ component of the agent position, are grouped together across all agents, and then these variable groups are concatenated as in Fig. 1. The state corresponds to the agents’ positions, then $w = 2$. From the definition of $K$ in (5), recall $S = X' - X$ a matrix of component-wise speeds of each agent at each time step. The rows of $S$ contain, for each agent $i$, the $x$ and $y$-components of the agent’s speed, $s^T_{i,x}$ and $s^T_{i,y}$, respectively, for $i \in \{1, \ldots, N\}$. Define the concatenated vector as $s^T_i$ and let

$$
S = \begin{bmatrix} s^T_1 \\ \vdots \\ s^T_N \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix},
$$

$$
\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r), \quad U = \begin{bmatrix} u^T_1 \\ \vdots \\ u^T_{r,n} \end{bmatrix}.
$$

The individual entries, $k_{i,j}$, of $K$ can be written as:

$$
k_{i,j} = \sum_{\ell=1}^r \sigma_{\ell}^{-1} u^T_{\ell,j} s^T_{\ell} v_{\ell} = u^T_{\ell,j} \Sigma^{-1} v^T_{\ell} s^T_{\ell},
$$

where $\bar{u}^T_{\ell,j} \in \mathbb{R}^r$ is the $j$-th row of $U$. Each entry $k_{i,j}$ of $K$ is a projection of the speed onto the $j$-th component of each column of $U$. Since the columns of $U$ are left singular vectors correspond to modes, and these modes are ordered from most influential to least influential, then each subsequent entry in $\bar{u}^T_{\ell,j}$ carries less influence than the last. Thus, each row of $K$, 

![Diagram of swarmDMD process](image-url)
which determines the combination of data that each agent receives for its x- and y-positions, is given by:

\[ k_i = [\hat{u}_i^T \Sigma^{-1} V^T s_1 \cdots \hat{u}_i^T \Sigma^{-1} V^T s_n]. \]  

(9)

Recall \( s_1^T \) is the vector of speed snapshots at each time step and \( V \) encodes the temporal correlation within the snapshot data. Starting with \( V^T s_i \), the columns \( v_k \) of \( V \) indicate which instants are most influential to the change of state-\( i \). This influence is scaled by \( \Sigma^{-1} \). Finally, multiplying with \( \hat{u}_i \) determines which components \( j \) will have the greatest influence on component \( i \), looking over the entire row \( i \) in (9). In a sense, \( K \) selects the states, neighbours, and time instants that are most influential to an agent’s position change, acting as a control matrix that encodes the multi-agent interaction laws inherent to the observation data. And from this perspective, \( K \) is quite similar to a Laplacian matrix.

B. ALTERNATIVE DYNAMICS FORMULATIONS

In addition to the dynamics given in (2), which we will call the standard dynamics, we also considered implementations which assumed other basic structure on the dynamics, namely first-order Cartesian and polar dynamics configurations. The reason for considering alternate dynamics in (2) is twofold: certain applications may benefit from a specific dynamics type depending agent manoeuvrability and sensing capabilities; and to show that swarmDMD is capable of working with a variety of system types.

The first-order Cartesian dynamics formulation is set up as

\[ \mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{v}_k \Delta t + \mathbf{K} \mathbf{y}_k, \]  

(10)

where

\[ \mathbf{x}_k = \mathbf{p}_k, \mathbf{y}_k = \begin{bmatrix} \Delta \mathbf{p}_k \\ \Delta v_k \end{bmatrix}, \]

and \( \mathbf{v}_k \) is defined as before. Similarly, the first-order polar dynamics formulation is set up as

\[ \mathbf{x}_{k+1} = \mathbf{x}_k + \begin{bmatrix} v_k \cos \theta_k \\ v_k \sin \theta_k \end{bmatrix} \Delta t + \mathbf{K} \mathbf{y}_k, \]  

(11)

where

\[ \mathbf{x}_k = \mathbf{p}_k \text{ and } \mathbf{y}_k = \begin{bmatrix} \Delta d_k \\ \Delta v_k \end{bmatrix}. \]

These alternative formulations provide flexibility in capturing various types of swarm behaviour and dynamics.

IV. EXPERIMENT SET-UP

In this section, we introduce the swarm simulation and the metrics that were used to assess the performance of swarmDMD. Swarm data is generated from the flocking behaviour following the Vicsek model. We provide information about the swarmDMD set-up, the ground truth models used for training, and the metrics used in evaluating the results.

A. VICSEK MODEL

We use the Vicsek swarm model to generate data for training and analysis. The Vicsek model is a simple and well-studied model that produces flocking behaviour similar to that exhibited in biological swarm systems [36]. In the original Vicsek model, agents possess a constant forward speed and interact with the swarm by aligning their heading direction with the average heading of their neighbours in a certain Euclidean radius.

The agent dynamics of the Vicsek model follow:

\[ \mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{v}_k \Delta t, \]  

(12)

where \( \mathbf{v}_k \) is the velocity of agent-\( i \), which acts like a control input to the linear system, and \( \Delta t \) is the length of the time step. The velocity at time step \( k \) is determined from the heading angle \( \theta_k^i \) and the constant speed \( v \), where \( \theta_k^i = \langle \theta_{k-1}^j \rangle_r + \hat{\theta}^i \) and \( \langle \theta_{k-1}^j \rangle_r \) is the average direction of the agents, including agent-\( i \), in a radius \( r \) about agent-\( i \), formally defined as:

\[ \langle \theta_{k-1}^j \rangle_r = \arctan((\sin \theta_{k-1}^j)_r/(\cos \theta_{k-1}^j)_r). \]

Here, \( \hat{\theta}^i \) is a random heading perturbation chosen with uniform probability from the interval \([-\eta/2, \eta/2]\). The values of the interaction radius, \( r \), and randomness, \( \eta \), characterise the emergent behaviour of a Vicsek swarm.

In addition to conventional flocking behaviours, we are also interested in how swarmDMD performs on systems with more complicated dynamics, such as milling. As a result, we also consider the modification to the original Vicsek model presented by Costanzo and Hemelrijk [37]. In this modification, the field of view of each agent is restricted and a bound is placed on the agents’ angular velocity, such that

\[ \theta_k^i = \begin{cases} \langle \theta_{k-1}^j \rangle_r + \hat{\theta}^i & |\Delta \theta^i| < \omega \Delta t \\ \theta_{k-1}^j + \omega \Delta t + \hat{\theta}^i & \Delta \theta^i \geq \omega \Delta t \\ \theta_{k-1}^j - \omega \Delta t + \hat{\theta}^i & \Delta \theta^i \leq -\omega \Delta t. \end{cases} \]  

(13)

Here, \( \langle \theta_{k-1}^j \rangle_r \) is the average heading agent, as before, except now only agents in the field of view \( \phi \in [0, 2\pi] \) of agent-\( i \) are considered, \( \omega \in [0, \pi/\Delta t] \) is the maximal angular velocity, and \( \Delta \theta^i \in [-\pi, \pi] \) is the difference between the current orientation and the average orientation. Please see [37] for precise definitions of the field of view and difference in orientation.

The parameter values used in the ground truth simulations are given in Table 1. A time step of 0.1s is used in the standard Vicsek model simulation and the swarmDMD recreation, and a time step of 1s is used in the modified Vicsek model for milling. The training period is 5s and the prediction period is 5s post-training. Before being used in swarmDMD, the milling swarm data was interpolated to have a time step of 0.1s, and the number of agents was reduced to 200 via random sampling. We use the first eight most dominant DMD modes in our swarmDMD recreation and prediction.
FIGURE 2. A visual representation of key parameters used in simulations: domain size, interaction radius, noise in heading; and four of the metrics used to analyse the results: heading error, position error, and polarisation. In the first frame at \( t = 0 \), swarmDMD is initialised with the ground truth agent positions. In a future frame (\( t = k \)) when positions predicted by swarmDMD have diverged from the ground truth, the ground truth agent position is shown in orange, and the swarmDMD prediction in blue. Here, \( l \) is the width of the square simulation domain, and \( \rho \) the width of the square domain within which agents are initialised following a uniform distribution. The large grey arrows in the background of the Polarisation frames indicate the average direction of the swarm.

TABLE 1. Parameters used in the Vicsek models for ground truth simulations. Parameters are compiled from [36], [37].

| Parameter | Standard | Milling |
|-----------|----------|---------|
| \( N \)   | 50       | 1000    |
| \( \Delta t \) | 0.1  | 1       |
| \( \rho \) | 16       | 2.5     |
| \( L \)   |          |         |
| \( r \)   | 0.05, 0.25, 0.5 | 1       |
| \( \phi \) | N/A      | \( \pi/2 \) |
| \( \omega \) | N/A   | \( \pi/18 \) |
| \( \eta \) | 0, \( \pi/12 \) | 0.5\( \omega/\Delta t \) |
| \( \nu \) | 0.03     | 1.03\( r \) |

B. EVALUATION METRICS

We now discuss the metrics used to evaluate reconstruction and prediction. The metrics we consider are: position error, heading error, polarisation error, error in angular momentum, and agent distribution about a focal agent. The first three metrics and the key simulation parameters are shown in Fig. 2.

The position and heading errors are direct comparisons between the absolute position measurements from the ground truth and swarmDMD reconstruction. An agent’s position error is the distance between the reconstructed and actual position, scaled by the domain length \( L \). Similarly, an agent’s heading error is defined as the absolute heading difference between the swarmDMD reconstruction and the ground truth. These two errors are averaged over all agents.

The calculations of polarisation and angular momentum are borrowed from [42]. We are interested in these two metrics as polarisation and angular momentum describe the collective motion of the swarm, and the degree to which the swarm is acting as one cohesive unit. Polarisation is calculated as:

\[
P_k = \frac{\left\| \frac{\sum_{i=1}^{N} \mathbf{v}_i^k}{\sum_{i=1}^{N} ||\mathbf{v}_i^k||_2} \right\|_2}{\in [0, 1]},
\]

and it provides a characterisation of the orientation of the swarm and takes values between 0 and 1. A value of \( P_k \) close to 1 means the majority of agents in the swarm are travelling in a similar direction, i.e., they have similar orientation. Angular momentum is calculated as:

\[
M_k = \frac{\left\| \frac{\sum_{i=1}^{N} \mathbf{p}_i^k \times \mathbf{v}_i^k}{\sum_{i=1}^{N} ||\mathbf{p}_i^k||_2 ||\mathbf{v}_i^k||_2} \right\|_2}{\in [0, 1]},
\]

and it takes values between 0 and 1, describing the normalised momentum of the swarm. Angular momentum gives a notion of the rotational motion within a swarm. The polarisation and angular momentum errors are calculated as the absolute difference between the ground truth and swarmDMD results.

The final metric considered is the distribution of neighbours, which helps to characterise the general structure of the swarm. Our distribution calculations are based on those in [57]. To calculate the neighbour distribution for a single focal agent, a neighbourhood about the agent is divided into bins, and the number of neighbours in each bin is counted and divided by the total number of agents in the neighbourhood. Then, these bin counts are averaged over all agents in the swarm (one by one, treating each agent as a focal agent), and over the desired time frame. An illustration of the process for each time step \( k \) is given in Fig. 3.

V. RESULTS

We now discuss the performance of swarmDMD on several example scenarios and in two different implementation styles.
FIGURE 3. An illustration of the process for computing the density distribution of neighbours for each time step. The domain about each agent is discretised into bins, the number of neighbours in each bin is counted, converted to a percentage of the total number of agents in the neighbourhood, and then averaged over all agents in the swarm. The bins are distance $d$ apart and have width $l$.

The ground truth swarms in the first three scenarios are created from the standard Vicsek model, one scenario per interaction radius in Table 1 and fixing $\eta = 0$. The fourth scenario uses the Vicsek milling model. The two implementation styles presented here both use the standard swarmDMD dynamics, but are initialised in different ways. The first implementation initialises the swarmDMD agents at the start time of the training period (termed Basic), whereas the second re-initialises the agents with the positions and headings of the ground truth agents every $g$ seconds, and propagates this for $h$ seconds (termed Re-initialisation); we take $g = 0.5s$ and $h = 10s$. The re-initialisation implementation was only applied to the standard Vicsek model scenarios. A short discussion of the performance of swarmDMD with the first-order Cartesian and polar dynamics is given at the end of the section, with the corresponding figures in the Appendix.

FIGURE 4. The resulting position error of the standard dynamics is compared across ground truth interaction radii, $r$, 0.05 (blue), 0.25 (orange), and 0.5 (green), and milling results (purple). Ground truth has $\eta = 0$ for basic and re-initialisation. Three things are of note: (1) the low error during training across all interaction radii for both the basic and re-initialisation implementations, (2) the error is so small for $r = 0.5$ that it is off the plot, and (3) initialising the algorithm outside the training period does not result in an increase in error.

FIGURE 5. Agent distribution analysis of swarmDMD with standard dynamics during the training period where the ground truth has $\eta = 0$. On the left is the ground truth distribution, and on the right is the swarmDMD recreation. SwarmDMD captures the most prominent features in the case $r = 0.05$, and is almost identical in the case $r = 0.5$.

FIGURE 6. Agent distribution analysis of swarmDMD with standard dynamics during the training period where the ground truth is the milling case. On the left is the ground truth distribution, and on the right is the swarmDMD recreation. SwarmDMD captures the ring-like pattern and three of the higher-likelihood areas.

FIGURE 7. Plots the trajectories of the agents during the training period under different inter-agent interaction radii settings, and the difference between the ground truth and the swarmDMD recreation is imperceptible during the training period. It is important to note that the level of acceptable error in the error plots (Figures 4, 8, 11, 12, and others) is context dependent on the swarm being modelled, and the required accuracy in recreation and prediction. It should also be noted the error is rarely above $10^{-4}$, demonstrating the ability of swarmDMD to accurately reconstruct the swarm dynamics. This is further reinforced by the agent density distribution shown in Fig. 5 for standard flock behaviour and Fig. 6 for milling, which indicate that the swarmDMD recreation captures the general structure of agents within the swarm. In addition, Fig. 7 plots the trajectories of the agents during the training period under different inter-agent interaction radii settings, and the difference between the ground truth and the swarmDMD recreation is imperceptible during the training period.
that in Fig. 4, the error associated with the \( r = 0.5 \) case is so small that it is not included in the plot.

Considering the post-training period of 5-10s, there is a window where the position error is still sufficiently small, indicating that swarmDMD can accurately predict agents’ dynamics post-training for a short horizon. This is most evident in the position and heading error in Figs. 4 and 8, but can also be seen to some extent across all error figures. In addition, the density distributions during prediction, given in Figs. 9 and 10, show that the general structure is captured by swarmDMD, though not as accurately as during training. This is a promising result that indicates the potential for swarmDMD models to be used for the control of swarms. It should be noted that of the “basic” scenario plots are only included for \( r = 0.05 \) and \( r = 0.5 \), to avoid cluttering. It is encouraging that in all scenarios presented, swarmDMD captures the dominant distributions amongst agents during both training and prediction.

The fact that the position error is very small for the milling scenario (Fig. 4) is significant; not only can swarmDMD recreate basic swarm flocking behaviours, it can reproduce milling motion and combined motions of milling and flocking in a single simulation. Figs. 11 and 12 also indicate that swarmDMD performs well with these more complicated dynamics, as the milling error is comparable to the “basic” scenario errors with \( r = 0.25 \). However, swarmDMD does have limitations, as seen in Fig. 8, where the algorithm struggles with keeping the heading error low in the milling scenario. Importantly, by examining the re-initialisation results
in Figs. 4, 8, and 12, we observe that initialising the agents outside of the training period does not noticeably increase error, providing evidence of robustness in the model.

Tables 2 and 3 summarise the results shown before based on two metrics: the average error during the training period, and the amount of time the error stays below 1e-01 during the post-training prediction period. In these tables we also include the data from the scenarios with $\eta = \pi/12$, for which the results are shown in Fig. 13 in the Appendix. We can see that swarmDMD does perform better in the scenarios with less randomness in the motion of the agents. During training this is most noticeable in the position and heading, as the average polarisation and angular momentum errors during training for the scenarios with $\eta = \pi/12$ are of the same magnitude as those in the scenarios with $\eta = 0$. This is expected, as the randomness in the agents’ positions, if small enough, should only affect the motion of individual agents and not the general behaviour of the swarm.

Tables 4 and 5 give the average error during the training period, and the amount of time post-training for which the error stays below 1e-01, respectively, for the first-order Cartesian and polar dynamics, given by (10) and (11), respectively. It is clear that the first-order Cartesian and polar dynamics do not perform as well as the standard implementation. Further discussion of these results can be found in the Appendix.
From left to right, the resulting position, heading, polarisation, and angular momentum errors of the standard dynamics are compared across ground truth interaction radii, $r$, 0.05 (blue), 0.25 (orange), and 0.5 (green). Ground truth has $\eta = \pi/12$ for basic and re-initialisation.

TABLE 5. Length of time (in seconds) post-training that error remains below $10^{-01}$ for the Cartesian and polar algorithms with $\eta = 0$.

| $r$    | $t_x$  | $t_y$  | $t_p$  | $t_{M}^{x,y}$ |
|--------|--------|--------|--------|---------------|
| Cartesian |
| 0.05   | 8.1    | 1      | 3.8    | 3.4           |
| 0.25   | 0      | 0      | 0      | 0             |
| 0.5    | 0      | 0      | 0      | 0             |
| 0.05   | 14     | 1.8    | 7.8    | 6.7           |
| Polar  |
| 0.25   | 1.9    | 0      | 0      | 0             |
| 0.5    | 4.8    | 1.1    | 2.2    | 1.8           |

Agent distribution analysis of swarmDMD with standard dynamics over the training period, ground truth has $\eta = \pi/12$. On the left is the ground truth distribution, and on the right is the swarmDMD recreation.

FIGURE 15. Agent distribution analysis of swarmDMD with standard dynamics over a period of 5s prediction, ground truth has $\eta = \pi/12$. On the left is the ground truth distribution, and on the right is the swarmDMD prediction.

VI. CONCLUSION

In this paper, we introduced for the first time the method of swarmDMD to learn local interaction laws that give rise to swarm motions from pure observation data. We demonstrated the performance of swarmDMD on both flocking and milling swarm motions generated with the Vicsek model and its variants. It was concluded that swarmDMD can only reconstruct the swarm behaviour faithfully but also predict the swarm motion sufficiently accurately within a short window after the training period. These results suggest that swarmDMD can be a powerful tool in analysing biological swarms and engineering intelligent multi-agent systems.

There are a number of future directions that are suggested by this work. First, it may be possible to learn nonlinear interaction laws using nonlinear generalisations of DMD, such as the sparse identification of nonlinear dynamics (SINDy) [25] or tensor DMD formulations [58], [59]. Moreover, sparsity promoting algorithms may be useful for inferring minimal network connectivity and causality [60]; sparsity-promoting DMD has already been introduced to identify a minimal set of modes [61], but may be adapted for sparse dynamics matrices. For higher order systems, or systems with partially observed dynamics, incorporating time delays will also be important [49], [53]. Similarly, if data is collected in partial and...
overlapping domains, it may be possible to merge these data by aligning the phase between different DMD models [62]. It may also be important to cluster the swarm into distinct regions, where different local rules are applied, for example using cluster reduced order modeling [63], [64]. Finite-time Lyapunov exponents and Lagrangian coherent structures may also provide further insights into swarm dynamics and behavioural regimes [65]–[69]. It will also be interesting to explore the effectiveness of swarmDMD models for active feedback control of the swarm, for example by manipulating the behaviour of a small subset of agents. Finally, it will be important to apply these methods to other canonical swarm models and also to real-world data to explore the strengths and weaknesses of these various algorithms.

CODE AVAILABILITY
The code for this work has been made available on GitHub at https://github.com/e-vic/swarmDMD.

APPENDIX. ALTERNATIVE COORDINATES
Here we will briefly discuss the performance of standard swarmDMD when trained on the ground truth cases where $\eta = \pi/12$, and of the first-order (FO) Cartesian and polar implementations.

A. STANDARD DYNAMICS
Position, heading, polarisation, and angular momentum — seen in Fig. 13 — all show increases in error values from the $\eta = 0$ cases, with heading error increasing multiple orders of magnitude. Polarisation and angular momentum errors show a significant increase in noisiness of the plots. These are all expected results as the uncertainty in the headings of the ground truth agents is now non-zero. It is also interesting to note that with the increase in uncertainty in the ground truth swarm, changes in interaction radius seem to have less effect on the results.

Figs. 14 and 15 show the agent density distribution during the training and prediction periods, respectively. Even with increased uncertainty in the ground truth swarms, swarmDMD still does a reasonable job of capturing the structure of the agents within the swarm, with the general patterns showing up in both training and predictions for $r = 0.05$, and the almost identical pattern during training for $r = 0.5$.

B. FIRST-ORDER DYNAMICS
Figs. 16 shows the position, heading, polarisation, and angular momentum plots for the FO Cartesian and polar dynamics. In general, FO polar performs better than FO
and polar dynamics over a period of 5s prediction, ground truth has \( r = 0.050 \) and \( r = 0.500 \) with \( r \) = 0. On the left are the distributions for \( r = 0.05 \), and on the right the distributions for \( r = 0.5 \). The ground truth distributions are shown in the first row.

Cartesian. This is reasonable since the Vicsek model uses heading to align agents, FO Polar would be using information more similar to that in the underlying model. It is interesting to note that across error metrics the difference in error magnitude between training and prediction periods is not as significant as when the standard dynamics are used; the error often begins to increase during the training period.

Figs. 17 and 18 give the agent density distribution for the FO Cartesian and polar dynamics. The FO polar distributions are almost identical for both the training and prediction periods. The FO polar distributions are nearly fuel-optimal flock guidance in strong background flows,'’ *IEEE Trans. Robot.*, vol. 29, no. 2, pp. 383–399, Apr. 2013.

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