On the Theoretical Period-Radius Relation of Classical Cepheids

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ABSTRACT

We present the results of a comprehensive theoretical investigation on the Period-Radius (PR) relation of classical Cepheids based on new sequences of full amplitude, nonlinear convective models constructed by adopting a wide range of both stellar masses and chemical compositions. In the period range $0.9 \leq \log P \leq 1.8$ a very good agreement is found between theoretical predictions and current available data whereas outside this range, both at shorter and at longer periods, nonlinear radii attain intermediate values between empirical relations based on different Baade-Wesselink (BW) methods and photometric bandpasses.

Subject headings: Cepheids — galaxies: stellar content — stars: fundamental parameters — stars: oscillations

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1. Introduction

The BW method (Baade 1926; Wesselink 1946) has been receiving growing attention from the astronomical community since it allows direct measurement of both radii and absolute magnitudes. Even though some physical assumptions of this method were questioned (Karp 1975; Gautschy 1987; Bono, Caputo & Stellingwerf 1994; Butler et al. 1996), in the last years a paramount effort has been undertaken for improving its accuracy and consistency (Barnes & Evans 1976; Sollazzo et al. 1981; Laney & Stobie 1995, hereinafter LS; Ripepi et al. 1997, hereinafter RBMR). At the same time, Krockenberger, Sasselov & Noyes (1997, hereinafter KSN) have recently developed a new BW method, based on Fourier coefficients, for evaluating the uncertainty on mean stellar radii due to individual measurement errors.

Substantial improvements in the measurements of both Cepheid radii and distances were thoroughly discussed in several outstanding papers by LS and more recently by Laney (1997), Di Benedetto (1997) and Gieren, Fouquè & Gómez (1997, hereinafter GFG). Despite this ongoing observational effort, theoretical investigations devoted to the Cepheid PR relation based on up-to-date evolutionary and pulsational models are lagging. In fact LS, by comparing the PR relation derived for a sample of 49 Galactic Cepheids with Fernie’s (1984) weighted mean theoretical PR relation, found that the slope of the empirical relation is steeper than the theoretical one and that BW radii are 12% smaller than the theoretical ones for a period equal to 10 d. On the other hand RBMR, by adopting a new version of the CORS method (Sollazzo et al. 1981, and references therein) found, as expected (LS), that the slope of their PR relation is slightly shallower if compared either with empirical BW relations based on IR photometry, or with theoretical relations.

The reason why so far only few investigations have been devoted to the evaluation of the mean theoretical PR relation is that its slope depends on the intrinsic width of the instability strip. The cool edge of the instability strip can be evaluated only by coupling the local conservation equations with a nonlocal and time-dependent equation for turbulent-convective motions (Stellingwerf 1982; Gehmeyr 1992). Theoretical PR relations available in the literature (Karp 1975; Cogan 1978) are based on radiative models and therefore cannot be considered “pure” theoretical relations. In fact, radiative models can only fix the location of the blue edge, whereas the temperature width of the instability strip is inferred from observational data. As a consequence, both the zero-point and the slope of these “semi-theoretical” PR relations depend on the completeness of the adopted sample and on the relations used for transforming the mean colors into mean effective temperatures. Moreover, Karp’s and Cogan’s relations have been derived by assuming that the width of the instability strip is constant when moving from short to long-period Cepheids. However,
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This assumption is not supported by observational estimates, and indeed Pel (1980) in a seminal investigation showed that the Cepheid instability region is not a rectangular-shaped but a wedge-shaped strip, i.e. the color range narrows toward short-period Cepheids. The main aim of this investigation is to establish the Cepheid PR relation on a genuine theoretical basis by adopting the mean radii and the periods predicted by full amplitude, nonlinear convective models and then to compare theoretical with empirical PR relations.

2. Pulsational Models

Several sequences of envelope models were constructed by adopting four different stellar masses \((M/M_\odot=5.0, 7.0, 9.0, 11.0)\) and two luminosity levels for each mass value. The luminosity levels were fixed according to the Mass-Luminosity (ML) relations predicted both by canonical (no overshooting) and noncanonical (mild overshooting, \(\lambda_{\text{over}}=0.5\)) evolutionary models. The former relation was chosen from the calculations of Castellani, Chieffi & Straniero (1992), whereas the latter was fixed by increasing the canonical luminosity level by 0.25, i.e. \(\log L/L_\odot (\text{NC}) = \log L/L_\odot (C) + 0.25\) (see e.g. Chiosi et al. 1992 and Chiosi, Wood & Capitanio 1993). Our investigation is also focussed on the dependence of the PR relation on He and metal contents, and therefore calculations were performed by adopting three different chemical compositions which are representative of Cepheids in the Small \((Y=0.25, Z=0.004)\), the Large \((Y=0.25, Z=0.008)\) Magellanic Cloud (MC), and in the Galaxy \((Y=0.28, Z=0.02)\). The models were arranged in sequences characterized by constant mass, luminosity and chemical composition but by different values of the effective temperature \((4000 \leq T_e \leq 7000 \text{ K})\). Physical and numerical assumptions adopted for performing the linear, nonadiabatic analysis as well as the nonlinear, full amplitude analysis have already been the subjects of previous papers (Bono & Stellingwerf 1994; Bono & Marconi 1997, and references therein), and therefore they are not further discussed here. The theoretical framework we developed proved to be successful in reproducing observational properties (amplitudes and modal stability) of Cepheids characterized by periods shorter than 40 d. This notwithstanding, we found that high-mass models \(-M/M_\odot=11.0-\) present peculiarities in the nonlinear limit cycle stability. In fact, light and velocity curves display irregularities such as sharp bumps and sudden dips during both contraction and expansion phases. Moreover, pulsational properties undergo substantial changes over consecutive periods. A similar behavior was found by Christy (1975), who only pointed out that both pulsation irregularities and very large amplitudes take place in models with high radius/mass, period/radius, and period/luminosity ratios.

For investigating the intimate nature of this phenomenon a detailed analysis of the
dependence of the limiting amplitude behavior on physical and/or numerical assumptions was undertaken. We found that pulsation irregularities are caused by the coarse spatial resolution in the H and first He (HeI) ionization regions. In fact these layers, due to their large back and forth motion, over a full cycle undergo a large excursion both in temperature and density. The coarse spatial resolution causes a sudden increase in the temperature and density gradients and consequently the formation and propagation of strong spurious shocks. This is a typical limit of the Lagrangian models when compared with the adaptive grid models. For solving this problem we constructed a new sequence of linear models. The main differences between these equilibrium models and the standard ones are the following:

standard models are constructed, following Stellingwerf (1975), by anchoring the opacity peak of the H ionization regions (HIR) and by locating a proper number of zones (20 ÷ 30) between this peak and the surface layer. Instead of improving accuracy by simply increasing the number of zones located above this peak we developed a new method which, by means of a multiple iteration on the mass of the surface zone, ensures via a secant method a uniform sampling (∆T = 500 ÷ 650 K) of the layers located between the surface and the base of the H and HeI ionization regions (T ≃ 2.1 × 10⁴ K). The left and the right panels of Figure 1 show the opacity and the adiabatic exponent Γ₁ of two models located close to the blue and to the red edge of the instability strip. The fine models show two substantial differences when compared to the coarse ones: a) their adiabatic index attains smaller values and resolve the HeI ionization zone. The HeI region is dynamically unstable, and indeed close to logT ≃ 4.2 the Γ₁ is smaller than 4/3. b) Their opacity peak attains larger values, the differences are a factor of four in the blue models and of the order of 10% in the red models. The same differences were found by Gehmeyr (1992) in his comparison of two static RR Lyrae models constructed by adopting a Lagrangian and an adaptive grid code respectively. It is hardly necessary to point out the role played by these changes in the instability and pulsation amplitudes of long-period Cepheids.

The nonlinear radii and periods discussed in this investigation were evaluated by adopting fine zoning models. Figure 2 shows the comparison between the theoretical PR relations at solar metallicity provided by Cogan (1978) and Karp (1975) and our models constructed by adopting canonical (triangles) and noncanonical (squares) ML relations. The nonlinear radii of canonical Cepheids are larger than the noncanonical ones, with a difference ranging from 4% at logP ≃ 0.6 to 7% at logP ≃ 0.6. Interesting enough, our nonlinear radii are quite similar to the radii predicted by Cogan’s relation in the range 0.6 ≤ logP ≤ 1.2, whereas toward longer periods they first attain values similar to those given by Karp’s relation and then become systematically smaller than the radii predicted by the quoted relations. This difference is mainly due to the proper location of red boundaries without invoking ad hoc assumptions and, more marginally, to new opacities. Table 1
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summarizes the zero points and the slopes of the linear regression obtained by adopting different compositions and ML relations. An interesting result is that the average PR relations show a mild but non negligible dependence on metal content. In fact, for canonical radii an increase in the metal content from $Z=0.004$ to $Z=0.008$ leads to a decrease which ranges from 2% at $\log P \approx 0.6$ to almost 4% at $\log P \approx 2.0$. An increase in both He and metal contents ($Y=0.28, Z=0.02$ against $Y=0.25, Z=0.004$) implies a decrease which ranges from 4% at $\log P \approx 0.6$ to 9% at $\log P \approx 2.0$. A similar outcome results for noncanonical radii.

3. Comparison Between Theory and Observations

Figure 3 shows the empirical PR relations for Galactic Cepheids obtained by GFG (dashed line), Laney (1997, long-dashed lines), and CORS (dotted line). Theory and observations were also compared by plotting canonical periods and radii of models with $Z=0.008$ and $Z=0.02$. We adopted two different compositions for accounting for the spread in metal content recently found by Fry & Carney (1997) among calibrating Galactic Cepheids. The comparison brings out two major results: 
a) theoretical predictions are, within the observational errors, in good agreement with average empirical PR relations obtained by adopting different methods and different photometric bands. In the period range $0.9 \leq \log P \leq 1.8$ observed and theoretical radii are almost identical. b) Theoretical radii are systematically larger than the observed ones in the period range $0.4 \leq \log P \leq 0.6$, whereas they are smaller toward longer periods, i.e. $\log P > 1.8$. Both the paucity of long-period Cepheids detected in the Galaxy and the lack of a detailed analysis of the systematic errors involved in empirical PR relations based on different methods and/or photometric bands prevent a quantitative explanation of this discrepancy.

However, KSN have recently provided a thorough analysis of the uncertainty in the radius estimates introduced by individual measurement errors. Their results on the slope of the PR relation for Galactic Cepheids fairly agree with those of RBMR who adopted a CORS method which accounts for the loop performed by the variable in a color-color plane, the main advantage of this method being its independence of reddening corrections. Moreover, Di Benedetto (1997) obtained a very precise general PR relation by adopting both Galactic and MCs Cepheids for which high precision photometric and spectroscopic data were available. In this method the use of both magnitude and colors in evaluating stellar angular sizes ensures a marginal dependence of radii on both reddening and metallicity.

Figure 4 shows the last two empirical PR relations, the results obtained by KSN as well as the theoretical predictions for the three chemical compositions. The comparison discloses once again a remarkable agreement between theory and observations. The major
4 CONCLUSIONS

Discrepancy is in the short-period range, in which theoretical radii are smaller than the radii obtained by KSN and RBMR, and larger than the radii provided by Di Benedetto’s relation. However, firm constraints on this observational discrepancy cannot be drawn since, as KSN clearly stated, the uncertainty in the mean radii are dominated by the error in the phase difference between color index and magnitude. Moving toward short-period Cepheids this difference becomes smaller, and in turn the uncertainty becomes larger. This trend is reversed in the long-period range, and indeed for periods longer than 30 d the radii obtained by KSN and RBMR are systematically smaller than the estimates of other authors, theoretical radii being located once again between these two different estimates. This finding confirms the results obtained by LS concerning the systematic error which affects radius estimates, i.e. by neglecting the variation of the effective gravity over the pulsation cycle, the radii based on optical bands systematically underestimate (overestimate) the radii of long (short) period Cepheids. Since Cepheid radii are proportional to the $p$-factor, i.e. the factor adopted for converting observed radial velocities into pulsational velocities, we suspect that this parameter is not only phase-dependent and that its value depends on both the BW method and the data sets adopted for estimating the radii (see e.g. KSN), but also that it should attain smaller values in long-period Cepheids observed in the IR bands. In fact, data in Figure 4 suggest that the dependence of $p$ on period is stronger than predicted by Gieren, Barnes & Moffett (1989) relation.

4. Conclusions

We developed a new theoretical scenario of the actual properties of classical Cepheids in the Galaxy and in the MCs. By adopting both radii and periods predicted by full amplitude, nonlinear, convective models we found that the use of two different ML relations based on canonical and noncanonical (mild overshooting) evolutionary models has a marginal effect on the PR relation, and indeed in the mean PR relation the difference is of the order of 3%. At the same time, we also found that an increase in the metal content implies a decrease in the mean radius. This effect is not constant but increases when moving from short to long-period Cepheids. In particular, a change in the chemical composition from $Y=0.25$, $Z=0.004$ to $Y=0.28$, $Z=0.02$ implies at $\log P \approx 2$ a decrease in the mean radius of the order of 9%. This result prompts that, within the current accuracy of both photometric and spectroscopic data, the dependence of the PR relation on metallicity could be detected and measured if a proper number of long-period variables ($P > 40$ d) are included in the sample.

Theoretical and empirical radii are found in very good agreement in the period range $0.9 \leq \log P \leq 1.8$, but present some discrepancies toward short and long-period Cepheids.
No firm conclusion was reached on the intimate nature of this discrepancy since current mean stellar radii estimated by adopting different BW methods, photometric bands, and data sets present a large scatter both at $\log P < 0.7$ and $\log P > 1.8$. Comparison between theory and observations suggests that the value of the $p$-factor could change when moving from short to long-period Cepheids. At the same time, the results of this investigation disclose a new approach for testing the internal accuracy and the consistency of the assumptions adopted by the different BW methods. In fact, observables predicted by nonlinear, convective models can be fed to the progeny of the BW method for assessing the intervening effects of systematic errors and/or of possible biases in the radius measurements.

It is a pleasure to thank D. Laney as a referee for his clarifying comments and valuable suggestions on current observational data.
REFERENCES

Alexander, D. R. & Ferguson, J. W. 1994, ApJ, 437, 879
Baade, W. 1926, Astron. Nachr., 228, 359
Barnes, T. G., & Evans, D. S. 1976, MNRAS, 174, 489
Bono, G., Caputo, F., & Stellingwerf, R. F. 1994, ApJ, 432, L51
Bono, G., & Marconi, M. 1997, MNRAS, 290, 353
Bono, G., & Stellingwerf, R. F. 1994, ApJS, 93, 233
Butler, R. P., Bell, R. A., & Hindsley, R. B. 1996, ApJ, 461, 362
Castellani, V., Chieffi, S., & Straniero, O. 1992, ApJS, 78, 517
Cogan, B. C. 1978, ApJ, 221, 635
Chiosi, C., Wood, G., Bertelli, G., & Bressan, A. 1992, ApJ, 387, 320
Chiosi, C., Wood, P. R., & Capitanio, N. 1993, ApJS, 86, 541
Christy, R. F. 1975, in Cepheid Modeling, eds. D. Fischel, & W.M. Sparks, (Washington: NASA SP-383), 85
Di Benedetto, G. P. 1997, ApJ, 486, 60
Gehmeyr, M. 1992, ApJ, 399, 272
Fernie, J.D. 1984, ApJ, 282, 641
Fry, A. M. & Carney, B. W. 1997, AJ, 113, 1073
Gautschy, A. 1987, Vistas Astron., 30, 197
Gieren, W. P., Barnes, T. G. & Moffett, T. J. 1989, ApJ, 342, 467
Gieren, W. P., Fouquè, P., & Gòmez, M. 1997, ApJ, 488, 74 (GFG)
Iglesias, C. A., & Rogers, F. J. 1996, ApJ, 464, 943
Karp, A. H. 1975, ApJ, 199, 448
Krockenberger, M., Sasslov, D. D., & Noyes, R. W. 1997, ApJ, 479, 875 (KSN)
Laney, C. D. 1997, in: A Half Century of Stellar Pulsation Interpretations, eds. P. A. Bradley & J.A. Guzik, (San Francisco: ASP), in press

Laney, C. D. & Stobie, R.S. 1995, MNRAS, 274, 337 (LS)

Pel, J. W. 1980, in Current Problems in Stellar Pulsation Instabilities, ed. D. Fischel, J.R. Lesh & W.M. Sparks, (Washington: NASA SP-625), 1

Ripepi, V., Barone, F., Milano, L., & Russo, G. 1997, A&A, 318, 797 (RBMR)

Sollazzo, C., Russo, G., Onnembo, A., & Caccin, B. 1981, A&A, 99, 66

Stellingwerf, R. F. 1975, ApJ, 195, 441

Stellingwerf, R. F. 1982, ApJ, 262, 339

Wesselink, A. J. 1946, BAN, 368, 91
Fig. 1.— Opacity (top) and adiabatic exponent (bottom) as a function of the logarithmic temperature for two models located close to the blue (left) and to the red (right) edge of the instability strip. Solid and dashed lines refer to linear models with fine and coarse spatial resolutions in the H and HeI ionization regions, respectively. The dotted lines plotted in the bottom panels display the edge between dynamically stable ($\Gamma_1 > 4/3$) and unstable ($\Gamma_1 \leq 4/3$) regions. The arrows mark the main features of the opacity and of the adiabatic exponent.

Fig. 2.— Comparison between different theoretical PR relations at solar metallicity. Triangles and squares show the nonlinear radii obtained by adopting a canonical and a noncanonical ML relation, respectively. The dashed line refers to the PR relation obtained by Cogan (1978), while the long-dashed line refers to the PR relation provided by Karp (1975). Fernie’s relation has not been plotted here since it is almost identical to Cogan’s.

Fig. 3.— Comparison between current empirical PR relations for Galactic Cepheids and theoretical nonlinear radii obtained by adopting two different chemical compositions.

Fig. 4.— Comparison between different empirical PR relations and theoretical results. Solid and dashed line show Di Benedetto’s and RBMR’s relations, respectively. The former is based on both Galactic and MCs Cepheids, whereas the latter on Galactic Cepheids only. Open circles refer to the mean radii for Galactic Cepheids obtained by KSN on the basis of visual magnitudes and B-V colors.
Table 1. Theoretical PR Relations \((\log R = \alpha + \beta \log P)\).

| $Z^a$ | $\alpha^b$        | $\beta^c$     | $r^d$    |
|-------|------------------|--------------|---------|
|       |                  |              |         |
| **Canonical** |                  |              |         |
| 0.02  | 1.188\(\pm0.008\) | 0.655\(\pm0.006\) | 0.999   |
| 0.008 | 1.192\(\pm0.009\) | 0.666\(\pm0.007\) | 0.998   |
| 0.004 | 1.199\(\pm0.010\) | 0.670\(\pm0.008\) | 0.998   |
|       |                  |              |         |
| **Noncanonical** |                  |              |         |
| 0.02  | 1.174\(\pm0.009\) | 0.647\(\pm0.006\) | 0.999   |
| 0.008 | 1.183\(\pm0.009\) | 0.653\(\pm0.006\) | 0.999   |
| 0.004 | 1.183\(\pm0.009\) | 0.661\(\pm0.006\) | 0.998   |

$^a$ Metallicity. $^b$ Zero Points of the PR relations. $^c$ Slopes of the PR relations. $^d$ Correlation coefficients of the linear regression. $^e$ The errors refer to the intrinsic dispersion.