BORDERLINE REGULARITY FOR FULLY NONLINEAR EQUATIONS IN DINI DOMAINS

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ABSTRACT. In this paper, we prove borderline gradient continuity of viscosity solutions to Fully nonlinear elliptic equations at the boundary of a $C^{1,Dini}$-domain. These results (see Theorem 3.1) are a sharpening of the boundary gradient estimate proved in [17] following the borderline interior gradient regularity estimates established in [6]. We however mention that, differently from the approach in [6] which is based on $W^{1,q}$ estimates, our proof is slightly more geometric and is based on compactness arguments inspired by the techniques in the fundamental works of Caffarelli as in [2, 3, 4].

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1. INTRODUCTION

The aim of this paper is to obtain pointwise gradient continuity estimates up to the boundary for viscosity solutions of

\[
\begin{aligned}
F(x, D^2 u) &= f \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega,
\end{aligned}
\]

under minimal regularity assumptions on $f, g$ and $\partial \Omega$. 

2000 Mathematics Subject Classification. Primary 35J25, 35J60.
Key words and phrases. Fully Nonlinear equations, borderline gradient continuity, Dini-domains.
The first author was supported in part by the National Research Foundation of Korea grant NRF-2015R1A4A1041675.
The fundamental role of these regularity estimates in the theory of elliptic and parabolic partial differential equations is well known. In order to put our results in the correct historical perspective, we note that in 1981, E. Stein in his visionary work [19] showed the following “limiting” case of Sobolev embedding theorem.

Theorem 1.1. Let $L(n,1)$ denote the standard Lorentz space, then the following implication holds:

$$ \nabla v \in L(n,1) \implies v \text{ is continuous.} $$

The Lorentz space $L(n,1)$ appearing in Theorem 1.1 consists of those measurable functions $g$ satisfying the condition

$$ \int_0^\infty |\{x : g(x) > t\}|^{1/n} dt < \infty. $$

Theorem 1.1 can be regarded as the limiting case of Sobolev-Morrey embedding that asserts

$$ \nabla v \in L^{n+\varepsilon} \implies v \in C^{0, \frac{\varepsilon}{n+\varepsilon}}. $$

Note that indeed $L^{n+\varepsilon} \subset L(n,1) \subset L^n$ for any $\varepsilon > 0$ with all the inclusions being strict. Now Theorem 1.1 coupled with the standard Calderon-Zygmund theory has the following interesting consequence.

Theorem 1.2. $\Delta u \in L(n,1) \implies \nabla u \text{ is continuous.}$

The analogue of Theorem 1.2 for general nonlinear and possibly degenerate elliptic and parabolic equations has become accessible not so long ago through a rather sophisticated and powerful nonlinear potential theory (see for instance [7, 10, 12] and the references therein). The first breakthrough in this direction came up in the work of Kuusi and Mingione in [11] where they showed that the analogue of Theorem 1.2 holds for operators modelled after the $p$-laplacian. Such a result was subsequently generalized to $p$-laplacian type systems by the same authors in [13].

Since then, there has been several generalizations of Theorem 1.2 to operators with other kinds of nonlinearities and in context of fully nonlinear elliptic equations, the analogue of Theorem 1.2 has been established by Daskalopoulos-Kuusi-Mingione in [6]. More precisely, they showed that (see Theorem 1.1 in [6])

Theorem 1.3. Let $u$ be a viscosity solution to

$$ F(x, D^2 u) = f \quad \text{in} \quad \Omega, \quad \text{(1.2)} $$

where $F$ is uniformly elliptic fully nonlinear operator and $f \in L(n,1)$. Then there exists $\theta \in (0,1)$ depending only on $n$ and the ellipticity constants of $F$ such that if $F(.)$ has $\theta$-BMO coefficients, then $Du$ is continuous in the interior of $\Omega$.

It turns out that the key to the nonlinear theory as observed in [11] is to consider the following modified $L^q$ version of the classical Riesz potential:

$$ \tilde{I}_q^r(x, r) = \int_0^r \left( \int_{B_r(x)} |f(y)|^q dy \right)^{1/q} \rho \, d\rho, \quad \text{(1.3)} $$

and then getting gradient $L^n$ as well as moduli of continuity estimates in terms of this modified Riesz potential, which is analogous to the classical linear theory where similar estimates are known in terms of the truncated Riesz potential. In the context of fully nonlinear elliptic equations as in Theorem 1.3
above, the authors show that the following estimate holds
\[ |\nabla u(x_1) - \nabla u(x_2)| \leq c\|Du\|_{L^\infty(\Omega)} |x_1 - x_2|^{\alpha(1-\delta)} + c \tilde{I}_f(x, 4|x_1 - x_2|^\delta) \] (1.4)
where \(\alpha, \delta\) depends on \(n, q\) and the ellipticity constants of \(F\).

Estimate (1.4) from [6] is obtained by a delicate combination of \(W^{1,q}\) estimates for fully nonlinear equations established in [20] with a certain modified Morrey-Campanato type argument. Over here, the reader should note that the success of such a small perturbation type argument relies crucially on intricate scaling properties of the equation in Theorem 1.3 (which for \(f \in L(n, 1)\) is scaling “critical” as the reader will observe in our work later on) and also on the fact that at small enough scales, such an equation can be regarded as a small perturbation of
\[ F(D^2u) = 0, \]
for which apriori \(C^{1,\alpha}\) estimates are known. (see for instance [2]). It turns out that if \(f \in L(n, 1)\), then
\[ \tilde{I}_f(x, r) \to 0 \quad \text{as} \quad r \to 0, \] (1.5)
whenever \(q < n\), which combined with the estimate (1.4) gives that \(\nabla u\) is continuous.

These recent results have provided us with a natural motivation to investigate the validity of similar gradient continuity estimates up to the boundary for solutions to (1.2) in the borderline situation as in Theorem 1.3 (i.e., with \(f \in L(n, 1)\)) and with minimal regularity assumptions on the boundary and the boundary datum. Our main result (see Theorem 3.1) can be thought of as the boundary analogue of Theorem 1.3 which was established in [6]. More precisely, in the boundary situation as in (1.1), we show that if \(\Omega\) is \(C^{1,\text{Dini}}\) and \(g\) is \(C^{1,\text{Dini}}\), then \(Du\) is continuous up to \(\partial \Omega\) with a modulus of continuity similar to that in Theorem 1.3, in particular an estimate of the form (1.4) holds up to the boundary. Note that standard results on gradient continuity of solutions require \(\partial \Omega \in C^{1,\text{Dini}}\) (see for instance [14]) and \(C^1\) regularity is not true in general for \(C^1\) domains where the solution may even fail to be Lipschitz up to the boundary (see for instance [9]). Therefore in that sense, our regularity assumptions on Dirichlet boundary conditions are in a sense, optimal.

The reader should note that in order to obtain an estimate similar to (1.4) in our situation from which gradient continuity follows thanks to the convergence in (1.5), we follow an approach which is somewhat different from the approach used in [6] and therefore our work gives a slightly different viewpoint in the interior case as well. Our method is based on the adaptation of compactness arguments and is independent of the \(W^{1,q}\) estimates which is crucially used in [6] in order to establish (1.4). We note that such geometric compactness arguments have their roots in the seminal work of Caffarelli (see [2]) and is based on pointwise affine approximation of the solution at dyadic scales which is achieved in our situation by suitable rescalings that are partially inspired by those used in [6] and by appropriately comparing our boundary value problem with a relatively smooth Dirichlet problem. We would like to mention as well that although our work is inspired by some of the earlier works mentioned above, it has nonetheless required some delicate adaptations in our setting which is complicated by the presence of the Dirichlet condition. For instance, in order to ensure that our compactness lemma in Section 4 can be applied, we have to additionally ensure smallness of the boundary datum at each step of iteration. The reader can see from the analysis involved in the proof of Lemma 4.8 that this requires some subtle work in our Dirichlet situation because of additional moduli of continuities involved unlike the interior case. We also note that unlike what is conventionally done in the divergence form theory, the boundary
cannot be flattened in our situation to begin with because of the lower regularity assumption on $\Omega$ and the fact that our equation has non-divergence structure. In that sense, our techniques are also partially inspired by that in the recent paper [1], which is on boundary Schauder estimates on Carnot groups where the boundary cannot be flattened either.

Finally, we describe a related boundary regularity result that has been previously obtained by Ma-Wang in their interesting paper [17]. In [17], the authors establish the gradient continuity of solutions to (1.1) upto the boundary of a $C^{1,\text{Dini}}$ domain $\Omega$ under the assumption that

$$
\int_0^r \left( \int_{B_r(x)} |f(y)|^n dy \right)^{1/n} \, dp =: \tilde{I}_n(x, r) \to 0 \quad \text{as} \quad r \to 0, 
$$

(1.6)
or equivalently under the assumption that the convergence in (1.5) holds for $q = n$. Now for an arbitrary function $f \in L(n, 1)$, the convergence in (1.5) doesn’t hold when $q = n$ and hence the result from [17] doesn’t cover our regularity result. Therefore our main result is a true sharpening of the result in [17].

We note that the method in [17], which in turn is inspired by some of the the fundamental works of Wang in [21] is quite different from ours and makes clever use of barriers, Alexandroff-Bakelman-Pucci type maximum principle and Dini-continuity of the normal at the boundary using which the authors obtain appropriate estimates at each iterative step (see the proof of [17, Lemma 3.1]). Because of the use of Alexandroff-Bakelman-Pucci maximum principle, their estimate relies crucially on the $L^n$ norm of $f$ at each step and that is precisely why their gradient continuity estimate depend in an essential way on the convergence of the quantity as in (1.6) which involves the $L^n$ norm of $f$ at each scale. This heuristics shows that the approach in [17] cannot be modified to prove our borderline regularity result at the boundary.

The paper is organized as follows. In section 2, we introduce certain relevant notions and gather some known results. In section 3, we state our main result. In Section 4, we first establish a basic compactness Lemma and then consequently establish uniform affine approximation of the solution at the boundary at dyadic scales (see Lemma 4.8) and finally in Section 5, we prove main Theorem 3.1.

2. Preliminaries

In this section, we shall collect all the preliminary material that will be used in the subsequent sections.

2.1. Fully Nonlinear equations. In this subsection, let us recall some well known definitions and properties of Fully nonlinear equations. This subsection is taken from [5] (see also [22]).

**Definition 2.1.** Let $M(n)$ be the set of all symmetric $n \times n$ matrices equipped with the order $M \preceq N$ iff $M - N$ is positive semi-definite. Any function $F : \mathbb{R}^n \times M(n) \mapsto \mathbb{R}$ is said to be uniformly elliptic if there exists constants $0 < \Lambda_0 < \Lambda_1 < \infty$ such that for almost every $x \in \mathbb{R}^n$, the following holds:

$$
P^{-}(M - N) \leq F(x, N) - F(x, N) \leq P^{+}(M - N),
$$

where $P^{-}$ and $P^{+}$ are the standard Pucci’s extremal operators defined as

$$
P^{-}(M) := \Lambda_0 \sum_{\mu_j > 0} \mu_j + \Lambda_1 \sum_{\mu_j < 0} \mu_j, \quad P^{+}(M) := \Lambda_1 \sum_{\mu_j > 0} \mu_j + \Lambda_0 \sum_{\mu_j < 0} \mu_j,
$$

where $\left\{ \mu_j \right\}_{j=1}^n$ are the eigenvalues of $M$. 
Definition 2.2. Let \( F(x, M) \) be continuous in \( M \) and measurable in \( x \) and we assume \( f \in L^q_{\text{loc}} \) for some \( q > \frac{n}{2} \). A continuous function \( u \) is an \( W^{2,q} \)-viscosity subsolution (supersolution) of (1.1) if for all \( \phi \in W^{2,q}(B_r(x_0)) \) with \( B_r(x_0) \subset \Omega \) and any \( \varepsilon > 0 \) satisfying the bound
\[
F(x, D^2 \phi(x)) \leq f(x) - \varepsilon \quad \text{almost everywhere in } B_r(x_0),
\]
\[
(F(x, D^2 \phi(x)) \geq f(x) + \varepsilon) \quad \text{almost everywhere in } B_r(x_0),
\]
implies \( u - \phi \) cannot attain a local maximum (minimum) at \( x_0 \). The function \( u \) is called \( W^{2,q} \)-viscosity solution if \( u \) is both a subsolution and supersolution.

2.2. Modulus of continuity. In this subsection, we shall recall some of the properties of modulus of continuity functions.

Definition 2.3. A function \( \Psi(t) \) for \( 0 \leq t \leq S_0 \) is said to be a modulus of continuity if the following properties are satisfied:
- \( \Psi(t) \rightarrow 0 \) as \( t \searrow 0 \).
- \( \Psi(t) \) is positive and increasing as a function of \( t \).
- \( \Psi(t) \) is sub-additive, i.e., \( \Psi(t_1 + t_2) \leq \Psi(t_1) + \Psi(t_2) \).
- \( \Psi(t) \) is continuous.

We now define the notion of Dini-continuity:

Definition 2.4. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function and define the following modulus of continuity:
\[
\omega_f(t) := \sup_{|x-y| \leq t} |f(x) - f(y)|.
\]
We then say \( f \) is Dini-continuous if
\[
\int_0^1 \frac{\omega_f(t)}{t} \, dt < \infty.
\]
(2.1)

From [16, Page 44], we see that any continuous, increasing function \( \Psi(t) \) on the interval \([0, S_0]\) which satisfies \( \Psi(0) = 0 \) is a modulus of continuity if it is concave. From this, we have the following important result proved in [16, Theorem 8]:

Theorem 2.5. For each modulus of continuity \( \Psi(t) \) on \([0, S_0]\), there is a concave modulus of continuity \( \tilde{\Psi}(t) \) with the property
\[
\Psi(t) \leq \tilde{\Psi}(t) \leq 2\Psi(t) \quad \text{for all } t \in [0, S_0].
\]

We will also need the following definition which captures a certain monotonicity property of the modulus of continuity.

Definition 2.6. Given \( \eta \in (0, 1] \), we say that a modulus \( \Psi \) is \( \eta \)-decreasing if the following holds:
\[
\frac{\Psi(t_1)}{t_1^\eta} \geq \frac{\Psi(t_2)}{t_2^\eta} \quad \text{for all } t_1 \leq t_2.
\]

Remark 2.7. From [16, Page 44], we see that any continuous, increasing function \( \Psi \) on an interval \([0, S_0]\) with \( \Psi(0) = 0 \) is a modulus of continuity if it is concave. More generally, it suffices to assume that \( \frac{\Psi(x)}{x} \) is decreasing instead of concavity for \( \Psi \).
2.3. Geometric structure. Let us now make clear the geometric assumptions imposed on the boundary of the domain $\Omega$ and on the nonlinearity.

**Definition 2.8.** We say $\Omega$ is $C^{1, \text{Dini}}$ domain if after translation, rotation and scaling, we may assume that $0 \in \partial \Omega$ and $\Omega \cap B_r$ for any $r \in (0, R_0)$ is given by
$$\Omega \cap B_r := \{(x', x_n) : x_n > \Gamma(x')\},$$
where $\Gamma \in C^{1, \text{Dini}}$ function and $\Gamma(0) = 0, \nabla_x \Gamma(0) = 0$. In particular, $\nabla \Gamma$ has Dini modulus of continuity in the sense of Definition 2.4.

**Definition 2.9.** Let $F : \mathbb{R}^n \times M(k) \to \mathbb{R}^n$ be continuous in $x$ and define
$$\Theta_F(x, y) = \Theta(x, y) := \sup_{M \in S(n) \setminus \{0\}} \frac{|F(x, M) - F(y, M)|}{\|M\|}.$$  

We say $F$ is $\Theta_0$-BMO in $\Omega$ for some $\Theta_0 > 0$, if the following holds:
$$\left(\int_{B_r \cap \Omega} \Theta(x_0, x)^n \, dx\right)^{\frac{1}{n}} \leq \Theta_0 \quad \text{for all} \quad x_0 \in \Omega \text{ and } r > 0.$$

2.4. Extension Lemma. In this subsection, let us recall a standard extension Lemma proved in [15, Theorem 2.2] that will be used throughout the paper. For the sake of completeness, we include its proof.

**Lemma 2.10.** Let $k_0 \in \mathbb{N}$ be a fixed integer and let $\Omega$ be a $C^{k_0, \alpha}$ domain for some $\alpha > 0$, $f \in C^{k_0, \alpha}(\Omega \cap B_1(x_0))$ be a function for some fixed $x_0 \in \partial \Omega$. There exists a $C^{k_0, \alpha}$ function $\tilde{f}$ defined on $B_1(x_0)$ such that $\tilde{f}(x) = f(x)$ whenever $x \in \Omega \cap B_1(x_0)$ and
$$\|\tilde{f}\|_{C^{k_0, \alpha}(B_1(x_0))} \leq C\|f\|_{C^{k_0, \alpha}(\Omega \cap B_1(x_0))}.$$  

**Proof.** Let $\partial \Omega \cap B_1(x_0)$ be parametrized by
- $\phi(x_0) = 0$,
- $\Omega \cap B_1(x_0) = \{(x', x_n) \in B_1(x_0) : \phi(x') > x_n\}$.

After translating and flattening the boundary, there exists a $C^{k_0, \alpha}$ diffeomorphism such that $\Phi : \Omega \cap B_1(x_0) \mapsto B_1^+(0)$ and $\Phi : \Omega^c \cap B_1(x_0) \mapsto B_1^-(0)$.

Let us define the following extension function:
$$v(x', x_n) = f \circ \Phi^{-1}(x', x_n) \quad \text{for all} \quad (x', x_n) \in B_1^+(0).$$

Since $f$ and $\Phi$ are $C^{k_0, \alpha}$ function, we must have $v \in C^{k_0, \alpha}(B_1^+(0))$. We shall define the extension function for $(x', x_n) \in B_1(0)$ by
$$V(x', x_n) = \begin{cases} v(x', x_n) & \text{if } x_n \geq 0, \\ \sum_{i=1}^{k_0+1} c_i v\left(x', -\frac{x_n}{t}\right) & \text{if } x_n < 0, \end{cases} \quad \text{(2.2)}$$
where the constants $c_i$ are obtained by solving the linear system $\left\{ \sum_{k=1}^{k_0+1} (-1)^j k^{-j} c_k = 1 \right\}_{0 \leq j \leq k_0}$. From [15, Theorem 2.2], we see that $V \in C^{k_0, \alpha}(B_1(0))$. We now define the extension function $\tilde{f}$ by
$$\tilde{f}(x', x_n) = V \circ \Phi(x', x_n) \quad \text{for all} \quad (x', x_n) \in B_1(x_0).$$
It is easy to see that the extension function $\tilde{f} \in C^{k_0,\alpha}(\Omega \cap B_1(x_0))$ and the following bound holds:

$$\|\tilde{f}\|_{C^{k_0,\alpha}(B_1(x_0))} \leq C\|f\|_{C^{k_0}(\Omega \cap B_1(x_0))}.$$ 

This completes the proof of the Lemma.

3. MAIN THEOREM

Let us now state the main theorem that we will prove:

**Theorem 3.1.** Let $u$ be a $W^{1,q}$ viscosity solution for some $q > n - n_0$ to (1.1) in $\Omega \cap B_1$. There exists an $\Theta \in (0,1)$ depending only on $\Lambda_0, \Lambda_1, n$ such that if $F$ has $\Theta$-BMO coefficients, $f \in L(n,1)(\Omega \cap B_1)$, $g$ is $C^{1,\text{Dini}}(\partial \Omega)$ on a domain $\Omega$ with $C^{1,\text{Dini}}$ boundary, then $\nabla u$ is continuous up to the boundary.

In particular, for any two points $y, z \in \Omega \cap B_1$, there exists two universal constants $C_0$ and $C_1$ such that the following estimate holds:

$$|\nabla u(y) - \nabla u(z)| \leq C_0 K(C_1 |y - z|),$$

where $K(\cdot)$ is as defined in (4.35) and depends on Dini-modulus of $\partial \Omega$, the Dini-modulus of $g$ and the $L(n,1)$ character of $f$.

**Remark 3.2.** Note that by a standard covering argument, we conclude that $\nabla u$ is continuous in $\Omega \cap B_r$ for any $r < 1$.

4. SOME USEFUL LEMMAS

Before we begin this section, let us fix an exponent $q \in (n - n_0, n)$ where $n_0$ (denoted by $\varepsilon$ in [8]) is a small universal constant as obtained in [8] such that the Krylov-Safanov type Hölder estimate holds for $W^{2,q}$ viscosity solutions to

$$F(D^2 u, x) = f \quad \text{with} \quad f \in L^q.$$ 

See also [22] for the analogous estimate up to the boundary.

**Definition 4.1.** Let $(\Lambda_0, \Lambda_1)$ be two fixed constants. Let $\mathfrak{F}$ denote the set of all uniformly elliptic functions $\tilde{F}$ with elliptic constants $(\Lambda_0, \Lambda_1)$. Furthermore, denote $\mathfrak{U}$ to be the class of all viscosity solutions $v \in C^0(B_1^+)$ solving

$$\begin{cases}
\tilde{F}(x, D^2 v) = 0 & \text{in } B_1^+, \\
v = 0 & \text{on } B_1 \cap \{x_n = 0\}, \\
\|v\|_{L^\infty(B_1^+)} \leq 1,
\end{cases}$$

in the sense of Definition 2.2.

The following boundary regularity was proved in [18, Theorem 1.1]:

**Proposition 4.2.** There exists an $\beta = \beta(n, \Lambda_0, \Lambda_1) \in (0,1)$ such that any solution $v \in \mathfrak{U}$ has the improved regularity $v \in C^{1,\beta}(B_1^+)$. 

Let us now fix a constant $\alpha$ (see proof of Proposition 5.1) such that

$$0 < \alpha < \beta.$$ (4.1)
4.1. Compactness Lemma. We now state our first relevant compactness lemma at the boundary.

Lemma 4.3. Let $u$ be the viscosity solution of (1.1) $\Omega \cap B_1$ with $\|u\|_{L^\infty(\Omega \cap B_1)} \leq 1$. Furthermore, suppose that $\Omega$ is $C^{1,\text{Dini}}$ and furthermore assume that $\Omega$ can be parametrized as in the set up of Definition 2.8 with $R_0 = 1$ and with $f \in L^q(\Omega)$ and $g \in C^{1,\text{Dini}}(\partial \Omega)$. Consider now the local problem

\[
\begin{cases}
F(x, D^2 u) = f & \text{in } \Omega \cap B_1, \\
u = g & \text{on } \partial \Omega \cap B_1,
\end{cases}
\]

then given any $\varepsilon > 0$, there exists a $\delta = \delta(\Lambda_0, \Lambda_1, n, \varepsilon, \text{Dini}) > 0$ such that if

\[
\|f\|_{L^q(\Omega \cap B_1)} \leq \frac{1}{k}, \quad \|\Gamma\|_{C^{1,\text{Dini}}(B_1')} \leq \frac{1}{k}, \quad \|g\|_{C^0(\partial \Omega \cap B_1)} \leq \frac{1}{k}, \quad \|\Theta_F\|_{BMO} \leq \frac{1}{k},
\]

then there exists a function $h \in C^{1,\beta}(\overline{B_{1/2}})$ with $\beta$ as obtained in Proposition 4.2 such that

\[
|u(x) - h(x)| \leq \varepsilon \quad \text{in } \Omega \cap B_{1/2}.
\]

Proof. The proof is by contradiction and follows the strategy from [22, Proposition 3.2] (see also [20, Lemma 2.3]). Suppose the Lemma is false, then there exists an $\varepsilon_0$ such that for any $k \in \mathbb{N}$, there exists a function $f_k$, an operator $F_k$ with ellipticity constants $(\Lambda_0, \Lambda_1)$, boundary data $g_k$ and domains $\Omega_k$ parametrized by $\Gamma_k$ satisfying

\[
\|f_k\|_{L^q(\Omega_k \cap B_1)} \leq \frac{1}{k}, \quad \|\Gamma_k\|_{C^{1,\text{Dini}}(B_1')} \leq \frac{1}{k}, \quad \|g_k\|_{C^0(\partial \Omega_k \cap B_1)} \leq \frac{1}{k}, \quad \|\Theta_{F_k}\|_{BMO} \leq \frac{1}{k},
\]

and a corresponding local viscosity solution $u_k$ solving

\[
\begin{cases}
F_k(x, D^2 u_k) = f_k & \text{in } \Omega_k \cap B_1, \\
u_k = g_k & \text{on } \partial \Omega_k \cap B_1,
\end{cases}
\]

such that

\[
\|u_k - \phi\|_{L^\infty(\Omega_k \cap B_{1/2})} \geq \varepsilon_0 \quad \text{for any } \phi \in C^{1,\beta}(B_1).
\]

Making use of uniform bounds in (4.3), we can now use the Hölder estimate up to the boundary from [22, Theorem 1.10] to obtain

\[
\|u_k\|_{C^{0,\alpha}(\Omega_k \cap B_1)} \leq C(n, \Lambda_0, \Lambda_1, p) \left( \|u_k\|_{L^\infty(\Omega_k \cap B_1)} + \|g_k\|_{C^0(\partial \Omega_k \cap B_1)} + \|f_k\|_{L^q(\Omega_k \cap B_1)} \right)
\]

\[
\leq C(n, \Lambda_1, \Lambda_1, q).
\]

We now use an idea similar to that in the proof of Lemma 4.1 in [1]. After flattening the boundary as in the proof of Lemma 2.10, we extend $u_k$ to $B_1$ using (2.2) with $k_0 = 1$ and we still denote the extended function by $u_k$. It is easy to see that such an extension ensures that $u_k$ is uniformly bounded in $C^{0,\alpha}(B_1)$. As a consequence of the above estimates and hypothesis, we have the following convergence results:

(i) Applying Arzela-Ascoli theorem to (4.4), we see that there exists a function $u_\infty \in C^0(B_1)$ such that $u_k \to u_\infty$ uniformly in $B_1$.

(ii) From (4.3), we see that $f_k \to 0$ as $k \to \infty$.

(iii) From (4.3), we see that $\Omega_k \cap B_1 \to B_1^+$ as $k \to \infty$.

(iv) From (4.3), we see that $g_k \to 0$ as $k \to \infty$ which implies $u_\infty = 0$ on $B_1 \cap \{x_n = 0\}$.

(v) Since $F_k$ is uniformly elliptic and $\|\Theta_{F_k}\|_{BMO} \to 0$, we see that $F_k(0, \cdot) \to F_\infty(\cdot)$ uniformly over compact subsets of $S(n)$. 
From the above convergence results along with an argument similar to [20, Lemma 2.3], we get that $u_\infty$ is a $C^0$ viscosity solution of
\[
\begin{aligned}
F_\infty(D^2 u_\infty) &= 0 \quad \text{in } B_1^+,
\quad u_\infty = 0 \quad \text{on } x_n = 0.
\end{aligned}
\]

(4.5)

We can now make use of the estimate from Proposition 4.2 to get $u_\infty \in C^{1,\beta}(\overline{B_{1/2}^+})$ for some $\beta \in (0, 1)$. Now from the following expression of $u_\infty$ in $\{x_n < 0\}$
\[
u(x', x_n) = \sum_{i=1}^2 c_i u_\infty(x', -\frac{x_n}{i}),
\]
which follows from the uniform convergence of extended $u_k$ to $u_\infty$, we conclude that
\[
\|u_\infty\|_{C^{1,\beta}(B_{1/2})} \leq C\|u_\infty\|_{C^{1,\beta}(B_{1/2})}.
\]

(4.6)

Thus, from (i) and (4.6), we get $u_k \to u_\infty$ uniformly in $B_1$. In particular, this implies
\[
\|u_k - u_\infty\|_{L^\infty(B_1)} \to 0 \quad \text{as } k \to \infty,
\]
which is a contradiction for large enough $k$. This completes the proof of the Lemma.

We now have an important Corollary which proves a boundary affine approximation to viscosity solutions of (1.1).

**Corollary 4.4.** Let $\alpha$ be as in (4.1), then for any $u$ with $\|u\|_{L^\infty(\Omega \cap B_1)} \leq 1$ solving (1.1) in the viscosity sense, there exist universal constants $\delta_0 = \delta_0(u, \Lambda_0, \Lambda_1, q, \alpha) \in (0, 1)$ and $\lambda = \lambda(u, \Lambda_0, \Lambda_1, q, \delta_0, \alpha) \in (0, 1/2)$ such that if (4.2) holds for some $\delta \in (0, \delta_0)$, then there exists an affine function $L = bx_n$ on $B_{1/2}$ such that
\[
\|u - L\|_{L^\infty(\Omega \cap B_1)} \leq \lambda^{1+\alpha}.
\]

**Proof.** From Lemma 4.3, we see that for any $\varepsilon > 0$, there exists constant $\delta$ (depending on $\varepsilon$) and a function $u_\infty \in C^{1,\beta}(\Omega \cap B_{1/2})$ such that if (4.2) holds, then
\[
\|u - u_\infty\|_{L^\infty(\Omega \cap B_{1/2})} \leq \varepsilon.
\]

(4.7)

Since $u_\infty$ solves (4.5), we see from the observation $u_\infty = 0$ on $B_1 \cap \{x_n = 0\}$, there exists an affine function $L = bx_n$ such that for all $r \in (0, 1/2)$, there holds
\[
\|u_\infty - L\|_{L^\infty(\Omega \cap B_r)} \leq C_{\text{Hd}} r^{1+\beta}.
\]

(4.8)

Combining (4.7) and (4.8), we get
\[
\|u - L\|_{L^\infty(\Omega \cap B_r)} \leq \|u - u_\infty\|_{L^\infty(\Omega \cap B_r)} + \|u_\infty - L\|_{L^\infty(\Omega \cap B_r)}
\leq \|u - u_\infty\|_{L^\infty(\Omega \cap B_{1/2})} + \|u_\infty - L\|_{L^\infty(\Omega \cap B_r)}
\leq \varepsilon + C_{\text{Hd}} r^{1+\beta}.
\]

(4.9)

We now make the following choice of exponents:

- From the choice $\alpha < \beta$ (see (4.1)) and the observation $r^{1+\beta} \to 0$ as $r \to 0$, there exist $\lambda \in (0, 1/2)$ such that
  \[
  C_{\text{Hd}} \lambda^{1+\beta} = \frac{1}{2} \lambda^{1+\alpha}.
  \]

- Now choose $\varepsilon = \frac{1}{2} \lambda^{1+\alpha}$ which in turn fixes $\delta$.  

Using these constants in (4.9), we get
\[ \| u - L \|_{L^\infty(\Omega \cap B_1)} \leq \lambda^{1+\alpha}, \]
which completes the proof of the Corollary.

\[ \square \]

4.2. Reductions. By rotation, translation and scaling, we shall henceforth always assume everything is centred at 0 ∈ ∂Ω and Ω satisfies the set up in Definition 2.8 with \( R_0 = 1 \). From the observation that \( u(x) - u(0) - \langle \nabla_x g(0), x \rangle \) is also a solution of (1.1), without loss of generality, we can assume that \( u(0) = 0 \) and \( \nabla_x g(0) = 0 \).

For any fixed \( r_0 \), let us define the following rescaled functions:
\[
\begin{align*}
    u_{r_0}(x) &:= u(r_0 x), \\
    g_{r_0}(x) &:= g(r_0 x), \\
    f_{r_0}(x) &:= r_0^2 f(r_0 x), \\
    \Gamma_{r_0}(x') &:= \frac{1}{r_0} \Gamma(r_0 x').
\end{align*}
\]  

(4.10)

We make the following observations about the rescaled functions defined in (4.10).

**Observation 1:** Using (1.1), the rescaled functions from (4.10) solves the following equation:
\[
\begin{align*}
    F_{r_0}(x, D^2 u_{r_0}) &= r_0^2 F \left( r_0 x, \frac{1}{r_0^2} D^2 u_{r_0} \right) = r_0^2 f(r_0 x) \quad \text{in } \Omega_{r_0} \cap B_1, \\
    u_{r_0}(x) &= g_{r_0}(x) \quad \text{on } \partial \Omega_{r_0} \cap B_1.
\end{align*}
\]

Here we have set \( \Omega_{r_0} := \{ x \in \mathbb{R}^n : r_0 x \in \Omega \} \).

**Observation 2:** Computing the \( C^1 \) norm of \( \Gamma_{r_0} \), we get
\[
|\nabla \Gamma_{r_0}(x') - \nabla \Gamma_{r_0}(y')| = \left| \frac{1}{r_0} \nabla \Gamma(r_0 x') - \frac{1}{r_0} \nabla \Gamma(r_0 y') \right| \leq \omega(\delta(r_0 x' - r_0 y')) \to 0 \quad \text{as } r_0 \to 0.
\]  

(4.11)

**Observation 3:** Analogous to the calculation leading to (4.11), we can compute the \( C^1 \) norm of \( g_{r_0} \) to get
\[
\omega \nabla g_{r_0}(\cdot) \to 0 \quad \text{as } r_0 \to 0.
\]

**Observation 4:** For any \( r < 1 \), the following estimate holds:
\[
\frac{1}{r_0} \left( \int_{B_r} |f_{r_0}|^q \, dx \right)^{\frac{1}{q}} \leq r \left( \int_{B_r} |f_{r_0}|^n \, dx \right)^{\frac{1}{n}} = r_0 \left( \int_{B_{r_0 r}} |f(x)|^n \, dx \right)^{\frac{1}{n}}.
\]

Therefore, if \( r_0 \) is small enough, then \( r \left( \int_{B_r} |f_{r_0}|^q \, dx \right)^{\frac{1}{q}} \) can be made uniformly small for all \( r < 1 \).

**Observation 5:**
\[
\Theta_{F_{r_0}}(x, y) = \Theta_F(r_0 x, r_0 y).
\]

4.3. Boundary Approximation by Affine function. Let \( \delta_0 \) be as obtained in Corollary 4.4 which in-turn fixes \( \lambda \) which depends on \( \delta_0 \) and is independent of \( \delta \in (0, \delta_0) \). Now we choose another exponent \( \tilde{\delta} \) (satisfying (4.26)) such that
\[
\tilde{\delta} < \delta.
\]  

(4.12)

Furthermore, we assume (4.2) holds with \( \delta = \tilde{\delta} \) which in view of the above discussion can be ensured by choosing \( r_0 \) small enough. In view of **Observation 5**, we note that the rescaling preserves the
\textit{BMO}-norm of the nonlinearity. Therefore by letting $F_{r_0}$ as our new $F$, $\Omega_{r_0}$ as our new $\Omega$, $u_{r_0}$ as our new $u$ and so on and finally by letting $\Theta \leq \tilde{\delta}$ where $\Theta$ is the bound on BMO norm of $F$ as in Theorem 3.1, we can assume that (4.2) is satisfied for $r_0$ small enough.

Let us define a few more functions:

\textbf{ModI:} With $\alpha$ from (4.1), let us define
\[
\tilde{\omega}_1(r) := \max\{\omega_T(r), \omega_g(r), r^{\alpha}\}.
\]

After normalizing and using Theorem 2.5, we can assume $\tilde{\omega}_1(\cdot)$ is concave and $\tilde{\omega}_1(1) = 1$. We now define
\[
\omega_1(r) = \tilde{\omega}_1(r^{\alpha}),
\]
from which we see that $\omega_1$ is $\alpha$-decreasing in the sense of Definition 2.6. This modulus $\omega_1$ is still Dini-continuous which can be ascertained by a simple of change o of variables in (2.1).

\textbf{ModII:} We define
\[
\omega_2(r) := |\Omega \cap B_1|^{\frac{1}{n}} r \left( \int_{\Omega \cap B_r} |f|^q \, dx \right)^{\frac{1}{q}} = C_{\Pi} r \left( \int_{\Omega \cap B_r} |f|^q \, dx \right)^{\frac{1}{q}}.
\]  \hspace{1cm} (4.13)

\textbf{ModIII:} With $\lambda$ obtained from Corollary 4.4 and $\tilde{\delta}$ from (4.12), we define
\[
\omega_3(\lambda^k) := \frac{1}{\tilde{\delta}} \sum_{i=0}^{k} \omega_1(\lambda^{k-i}) \omega_2(\lambda^i).
\]  \hspace{1cm} (4.14)

\textbf{ModIV:} Finally, we define
\[
\omega_4(\lambda^k) := \max\{\omega_3(\lambda^k), \lambda^{ak}\}.
\]  \hspace{1cm} (4.15)

Let us first prove a preliminary Lemma that follows from [6].

\textbf{Lemma 4.5.} The following bound holds:
\[
\sum_{i=0}^{\infty} \omega_4(\lambda^i) \leq C_{bnd}.
\]  \hspace{1cm} (4.16)

\textit{Proof.} From (4.15), we have the trivial bound
\[
\sum_{i=0}^{\infty} \omega_4(\lambda^i) \leq \sum_{i=0}^{\infty} \omega_3(\lambda^i) + \sum_{i=0}^{\infty} \lambda^{ia} \leq \sum_{i=0}^{\infty} \omega_3(\lambda^i) + C_{gm}.
\]

In the above estimate, the constant $C_{gm}$ is the sum of geometric progression $\sum_{i=0}^{\infty} \lambda^{ia}$. Hence, in order to prove the Lemma, it suffices to bound the first term, to do this, using (4.14) along with the fact that $\omega_1(\cdot)$ is increasing, we get
\[
\sum_{i=0}^{\infty} \omega_3(\lambda^i) = \frac{1}{\delta} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \omega_1(\lambda^{i-j}) \omega_2(\lambda^{j}) = \frac{1}{\delta} \left( \sum_{i=0}^{\infty} \omega_1(\lambda^i) \right) \left( \sum_{i=0}^{\infty} \omega_2(\lambda^i) \right).
\]  \hspace{1cm} (4.17)

\textbf{Estimate for $\sum_{i=0}^{\infty} \omega_1(\lambda^i)$:} Using the fact that $\omega_1(\cdot)$ is Dini-continuous, we get
\[
\sum_{i=0}^{\infty} \omega_1(\lambda^i) \leq \frac{1}{-\log \lambda} \sum_{i=1}^{\infty} \int_{\lambda^{i-1}}^{\lambda^{i}} \frac{\omega_1(s)}{s} \, ds = \int_{0}^{1} \frac{\omega_1(s)}{s} \, ds < \infty.
\]  \hspace{1cm} (4.18)
Estimate for $\sum_{i=0}^{\infty} \omega_2(\lambda^i)$: From [6, Equation (3.4)], there exists a constant $\tilde{c}$ such that
\[
\sum_{i=0}^{\infty} \omega_2(\lambda^i) \leq \tilde{c} \int_0^2 \left( \int_{\Omega \cap B_{\rho}} |f(x)|^q \, dx \right)^{\frac{1}{q}} \, d\rho =: \tilde{c} \mathcal{I}_q^f(0, 2). \tag{4.19}
\]
Further making use of [6, Equation (3.13)], we get
\[
\sup_x \tilde{I}_q^f(x, r) \leq \frac{1}{|B_1|^{1/2}} \int_{|B_2|} \left[ f^{**}(\rho)^{1/2} \right] \, d\rho. \tag{4.20}
\]
Thus using (4.20), (4.19) and (4.18) into (4.17), we get the bound in (4.16). This completes the proof of the lemma.

\[\square\]

Remark 4.6. From the estimates (4.19) and (4.20), we see that
\[
\sum_{i=0}^{\infty} \omega_2(\lambda^i) \leq \tilde{c} \frac{1}{|B_1|^{1/2}} \int_{|B_2|} \left[ f^{**}(\rho)^{1/2} \right] \, d\rho. \tag{4.21}
\]

We need to prove another crucial bound given in the following lemma:

Lemma 4.7. For a fixed $k$, the following bound holds:
\[
\lambda^a \omega_4(\lambda^k) \leq \omega_4(\lambda^{k+1}).
\]

Proof. From (4.15), if $\omega_4(\lambda^k) = \lambda^{a_k}$, then trivially, we get
\[
\lambda^a \omega_4(\lambda^k) = \lambda^{a(k+1)} \leq \omega_4(\lambda^{k+1}).
\]
Hence, we only have the case $\omega_4(\lambda^k) = \omega_3(\lambda^k)$. In this case, we proceed as follows:
\[
\lambda^a \omega_3(\lambda^k) = \frac{1}{\delta} \sum_{i=0}^{k} \lambda^a \omega_1(\lambda^{k-i}) \omega_2(\lambda^i) \leq \frac{1}{\delta} \sum_{i=0}^{k} \omega_1(\lambda^{k+1-i}) \omega_2(\lambda^i) \leq \omega_3(\lambda^{k+1}).
\]
To obtain (a), we made use of the fact that $\omega_1(\cdot)$ is $\alpha$-decreasing in the sense of Definition 2.6 and to obtain (b), we used the definition of the expression in (4.14).

We now prove the following important lemma which gives a linear approximation to the solution $u$ of (1.1) at the boundary.

Lemma 4.8. Let $\alpha$ be as in (4.1), then for any $u$ with \(\|u\|_{L^\infty(\Omega \cap B_1)} \leq 1\) solving (1.1) in the viscosity sense, there exist universal constants $\tilde{\delta} = \tilde{\delta}(n, \Lambda_0, \Lambda_1, q, \alpha) \in (0, \delta_0)$ where $\delta_0$ is from Corollary 4.4 such that if (4.3) holds with $\delta = \tilde{\delta}$, then there exists a sequence of affine functions $L_k := b_k x_n$ for $k = 0, 1, 2 \ldots$ such that the following holds:

A1: $|u - L_k| \leq \lambda^k \omega_4(\lambda^k)$,
A2: $|b_k - b_{k+1}| \leq C \omega_4(\lambda^k)$ in $\Omega \cap B_{\lambda^k}$.

Here $\lambda$ is from Corollary 4.4.

Proof. The proof is by induction. In the case $k = 0$, we have $L_\infty = 0$ and trivially get
\[
\|u\|_{L^\infty(\Omega \cap B_1)} \leq 1 \leq \omega_4(1).
\]
Let $k \in \mathbb{N}$ be fixed and assume (A1) and (A2) holds for all $i = 0, 1, \ldots, k$. In order to prove the lemma, it suffices to show (A1) and (A2) holds true for $k + 1$. In order to do this, we rescale and make use of Corollary 4.4.
Let us define the following rescaled functions defined for $x \in \tilde{\Omega} \cap B_1$ where $\tilde{\Omega} = \Omega_{\lambda^k}$ (Note that in view of the discussion in Observation 2, $\tilde{\Omega}$ is more "flat" than $\Omega$ since $\lambda < 1$).

\[
v(x) := \frac{u(\lambda^k x) - L_k(\lambda^k x)}{\lambda^k \omega_4(\lambda^k)}, \quad \tilde{F}(x, M) := \frac{\lambda^k}{\omega_4(\lambda^k)} F \left( \lambda^k x, \frac{\omega_4(\lambda^k) M}{\lambda^k} \right),
\]

\[
\tilde{f}(x) := \frac{\lambda^k}{\omega_4(\lambda^k)} f(\lambda^k x), \quad \tilde{g}(x) := \frac{g(\lambda^k x)}{\lambda^k \omega_4(\lambda^k)}|_{\partial \tilde{\Omega} \cap B_1}, \quad \tilde{L}_k(x) := \frac{L_k(\lambda^k x)}{\lambda^k \omega_4(\lambda^k)}|_{\partial \tilde{\Omega} \cap B_1}.
\]

From the induction hypothesis on (A1), we see that $\|v\|_{L^\infty(\tilde{\Omega} \cap B_1)} \leq 1$. Furthermore, $v$ solves the equation

\[
\left\{ \begin{array}{ll}
\frac{\lambda^k}{\omega_4(\lambda^k)} F \left( \lambda^k x, \frac{\omega_4(\lambda^k) M}{\lambda^k} \right) = \frac{\lambda^k}{\omega_4(\lambda^k)} f(\lambda^k x) & \text{in } \tilde{\Omega} \cap B_1 \\
v(x) = \frac{g(\lambda^k x)}{\lambda^k \omega_4(\lambda^k)} & \text{on } \partial \tilde{\Omega} \cap B_1.
\end{array} \right.
\]

We shall now show that (4.3) is satisfied for some $\tilde{\delta}$ (this is where we make a choice for $\tilde{\delta}$) which enables us to apply Corollary 4.4 and complete the induction argument. Let us now check each of the terms in (4.3) are satisfied:

**Bound for $\Theta_{\tilde{F}}$:** We have the following bound:

\[
\Theta_{\tilde{F}}(x, y) = \frac{\lambda^k}{\omega_4(\lambda^k)} \sup_{\lambda \in \mathcal{F}} \left| F \left( \lambda^k x, \frac{\omega_4(\lambda^k) M}{\lambda^k} \right) - F \left( \lambda^k y, \frac{\omega_4(\lambda^k) M}{\lambda^k} \right) \right| = \Theta_{F}(\lambda^k x, \lambda^k y).
\]

In particular, the following estimate holds

\[
\Theta_{\tilde{F}}(x, y) = \Theta_{F}(\lambda^k x, \lambda^k y).
\]

**Bound for $\tilde{f}$:** In this case, we get the sequence of estimates

\[
\left( \int_{\tilde{\Omega} \cap B_1} |\tilde{f}(x)|^q \, dx \right)^{\frac{1}{q}} \leq \frac{\lambda^k}{\omega_4(\lambda^k)} \left( \int_{\tilde{\Omega} \cap B_1} |f(x)|^q \, dx \right)^{\frac{1}{q}} \overset{(4.13)}{=} \frac{1}{C_{\mathcal{II}}} \frac{\omega_2(\lambda^k)}{\omega_4(\lambda^k)} \overset{(\text{Mod}, 11)}{\leq} \frac{C_{\mathcal{II}}}{\delta v_{1, 2}(\lambda^k)} \overset{(4.14)}{\leq} \frac{1}{\delta v_{1, 2}(\lambda^k)} \overset{(4.15)}{\leq} \frac{\omega_2(\lambda^k)}{\omega_4(\lambda^k)} \overset{(4.16)}{\leq} \delta |y - z|.
\]

**Bound for $\tilde{g}$:** By using the fact that $\nabla g$ has modulus of continuity given by $\tilde{\delta} \omega_1(\cdot)$ and also that $\nabla g(0) = 0$, we get from the mean value theorem that the following holds for any $y, z \in \partial \tilde{\Omega} \cap B_1$:

\[
|\tilde{g}(y) - \tilde{g}(z)| \leq \frac{|g(\lambda^k y) - g(\lambda^k z)|}{\lambda^k \omega_4(\lambda^k)} \leq \frac{\tilde{\delta} \omega_1(\lambda^k) |\lambda^k y - \lambda^k z|}{\lambda^k \omega_4(\lambda^k)} \overset{(4.14, 4.15)}{\leq} \tilde{\delta} |y - z|.
\]

**Bound for $\tilde{L}_k$:** Again since $\nabla \Gamma$ has modulus of continuity given by $\tilde{\delta} \omega_1(\cdot)$ and $\nabla x \Gamma(0) = 0$, we obtain by an analogous computation and from the expression of $\tilde{L}_k$ as in (4.22) that the following holds for $y, z \in \partial \tilde{\Omega} \cap B_1$:

\[
\tilde{L}_k(y) - \tilde{L}_k(z) = \frac{b_k(\lambda^k y_n - \lambda^k z_n)}{\lambda^k \omega_1(\lambda^k)} = \frac{b_k(\Gamma(\lambda^k y') - \Gamma(\lambda^k z'))}{\lambda^k \omega_1(\lambda^k)} \leq b_k \tilde{\delta} |y - z|.
\]

Hence

\[
\|\tilde{L}_k\|_{C^{0,1}(\partial \tilde{\Omega} \cap B_1)} \leq b_k \tilde{\delta}.
\]
Bound for $b_k$: From the induction hypothesis applied to (A2), we get
\[ |b_k| \leq \sum_{i=1}^{k} |b_i - b_{i-1}| \leq C_b \sum_{i=0}^{k-1} \omega_4(\lambda^i) \overset{\text{Lemma } 4.5}{\leq} C_b \delta. \]  

(4.25)

Choice of $\tilde{\delta}$: Combining the estimate from (4.23), (4.24) and (4.25), we get
\[ \frac{g(\lambda^k x) - L_k(\lambda^k x)}{\lambda^k \omega(\lambda^k)} \implies C^{0,1}(\partial \Omega \cap B_1) \leq (C_b \delta + 1) \tilde{\delta}. \]

We will choose $\tilde{\delta}$ smaller than $\delta$ from (4.12) satisfying
\[ (C_b \delta + 1 + \frac{1}{C_Y}) \tilde{\delta} \leq \delta. \]  

(4.26)

Thus all the hypothesis of Corollary 4.4 are satisfied and thus, we can find an affine function $\tilde{L} = bx_n$ in $\tilde{\Omega} \cap B_\lambda$ such that
\[ |u(x) - \tilde{L}(x)| \leq \lambda^{1+\alpha} \quad \text{for all} \quad x \in \tilde{\Omega} \cap B_\lambda. \]  

(4.27)

There also holds the following bound
\[ |b| \leq C. \]  

(4.28)

In particular, if we define
\[ L_{k+1}(y) := L_k(y) + \lambda^k \omega_4(\lambda^k) \tilde{L} \left( \frac{y}{\lambda^k} \right) \quad \text{for} \quad y \in \Omega \cap B_{\lambda^{k+1}}, \]
then clearly after scaling back, we get
\[ |u(y) - L_{k+1}(y)| \leq \lambda^{k+1} \omega_4(\lambda^k) \overset{\text{Lemma } 4.7}{\leq} \lambda^{k+1} \omega_4(\lambda^{k+1}). \]

Moreover, it follows from (4.28) and the expression of $L_{k+1}$ as above that
\[ |b_k - b_{k+1}| \leq C \omega_4(\lambda^k). \]

This completes the proof of the lemma. \hfill \square

With $L_k$ as in Lemma 4.8, letting $k \to \infty$, we see that $L_k \to L_\infty$ for some linear function $L_\infty$. In the following lemma, we show that $L_\infty$ is an affine approximation to the viscosity solution $u$ of (1.1) at the boundary.

Lemma 4.9. The linear function $L_\infty := \lim_{k \to \infty} L_k$ is the affine approximation of $u$ on $\partial \Omega$. In particular, given any $x_0 \in \partial \Omega \cap B_{1/2}$, then there exists a modulus of continuity $K(\cdot)$ such that
\[ |u(x) - L_{x_0}(x)| \leq C_{\alpha}[x - x_0] K(|x - x_0|). \]

Moreover, $K(\cdot)$ can be chosen to $\alpha$-decreasing in the sense of Definition 2.6 with $\alpha$ as in (4.1).

Proof. Without loss of generality, we can assume that $x_0 = 0$ and also that we are in the setup of Lemma 4.3. Now let $x \in \Omega \cap B_1$ such that $|x| \approx \lambda^k$ with $\lambda$ coming from Corollary 4.4. In particular, we pick a point satisfying $\lambda^k \leq |x| \leq 2\lambda^k$ for some $k \in \mathbb{N}$. We have the following sequence of estimates:
\[ |u(x) - L_\infty(x)| \overset{(a)}{=} |u(x) - L_k(x)| + |L_k(x) - L_\infty(x)| \]
\[ \leq \lambda^k \omega_4(\lambda^k) + \sum_{i=0}^{\infty} \lambda^k |b_{k+i} - b_{k+i+1}| \]
\[ \overset{(b)}{=} \lambda^k \omega_4(\lambda^k) + C_b \lambda^k \sum_{i=k}^{\infty} \omega_4(\lambda^i), \]  

(4.29)
where to obtain (a), we made use of (A1) from Lemma 4.8 and to obtain (b), we made use of (A2) from Lemma 4.8.

From (4.15) and (4.14), for a fixed $i \in \mathbb{N}$, we get
\[
\omega_4(\lambda^i) \leq \frac{1}{\delta} \sum_{j=0}^{i/2} \omega_1(\lambda^{i-j}) \omega_2(\lambda^j) + \frac{1}{\delta} \sum_{j=i/2}^{i} \omega_1(\lambda^{i-j}) \omega_2(\lambda^j) + \lambda^{i\alpha}.
\]
Recall that $\omega_1(\cdot)$ is monotone from (Mod1) and making use of (4.21), we get
\[
\sum_{i=k}^{\infty} \omega_4(\lambda^i) \leq \frac{C}{\delta} \sum_{i=k}^{\infty} \omega_1(\lambda^{i/2}) + \sum_{i=k}^{\infty} \lambda^{i\alpha} + \frac{1}{\delta} \sum_{i=k}^{\infty} \sum_{j=i/2}^{i} \omega_1(\lambda^{i-j}) \omega_2(\lambda^j)
\]
\[
= I + II + III.
\]

Before we estimate each of the terms of terms of (4.30), we note that $\delta$ is fixed. Let us also define
\[
K_1(\varepsilon) := \sup_{a \geq 0} \int_a^{a+\varepsilon/2} \frac{\omega_1(t)}{t} \, dt, \quad K_2(\varepsilon) := e^\varepsilon, \quad K_3(\varepsilon) := \sup_{a \geq 0} \int_a^{a+\varepsilon} \left[ f^{**}(\rho) \rho^{\frac{\alpha}{2}} \right]^{\frac{1}{\alpha}} \frac{d\rho}{\rho}.
\]

**Estimate for $I$**: We estimate as follows:
\[
I \leq C \int_0^{\lambda^{k\alpha}} \frac{\omega_1(t)}{t} \, dt \overset{4.31}{\leq} CK_1(\lambda^k).
\]
From (4.32) and the choice $|x| \approx \lambda^k$, it is easy to see that $I \to 0$ as $|x| \to 0$.

**Estimate for $II$**: We use the standard formula for Geometric progressions to get
\[
II \leq C \lambda^{k\alpha} \overset{4.31}{=} CK_2(\lambda^k).
\]
From (4.33) and the choice $|x| \approx \lambda^k$, it is easy to see that $II \to 0$ as $|x| \to 0$.

**Estimate for $III$**: In this case, we get
\[
III \leq C \left( \sum_{i=k/2}^{\infty} \omega_2(\lambda^i) \right) \left( \sum_{i=1}^{\infty} \omega_1(\lambda^i) \right) \overset{4.18}{\leq} C \left( \sum_{i=k/2}^{\infty} \omega_2(\lambda^i) \right) \overset{4.34}{=} C \int_0^{\lambda^{k\alpha/2}} \left[ f^{**}(\rho) \rho^{\frac{\alpha}{2}} \right]^{\frac{1}{\alpha}} \frac{d\rho}{\rho} \overset{4.31}{\leq} CK_3(\lambda^k).
\]
To obtain (a), we made use of the estimates [6, Equations (3.4) and (3.13)] and the fact that since $\lambda < 1$, we have
\[
\lambda^{nk/2} \leq \lambda^k \text{ since } n \geq 2
\]
and hence
\[
\int_0^{\lambda^{kn/2}} \left[ f^{**}(\rho) \rho^{\frac{\alpha}{2}} \right]^{\frac{1}{\alpha}} \frac{d\rho}{\rho} \leq \int_0^{\lambda^k} \left[ f^{**}(\rho) \rho^{\frac{\alpha}{2}} \right]^{\frac{1}{\alpha}} \frac{d\rho}{\rho}.
\]
From (4.34) and the choice $|x| \approx \lambda^k$, it is easy to see that $III \to 0$ as $|x| \to 0$.

**Claim**: Without loss of generality, we can assume that $K_1(\cdot)$’s are $\alpha$-decreasing in the sense of Definition 2.6 with $\alpha$ as in (4.1).

To prove the claim, we proceed as follows:

**$\alpha$-decreasing property of $K_1(\cdot)$**: From the fact that $\omega_1(\cdot)$ is a modulus of continuity and concave, we have that $K_1(\cdot)$ satisfies all the properties of Definition 2.3 and hence is also a modulus of continuity. Using Theorem 2.5, without loss of generality, we can assume $K_1(\cdot)$ is also concave. Now replacing
\( K_1(s) \) with \( K_1(s^\alpha) \) if necessary, we can also assume \( K_1(\cdot) \) is \( \alpha \)-decreasing in the sense of Definition 2.6.

**\( \alpha \)-decreasing property of \( K_2(\cdot) \):** This follows trivially from the definition of \( K_2(\cdot) \) in (4.31).

**\( \alpha \)-decreasing property of \( K_3(\cdot) \):** From Definition 2.3, it is easy to see that \( K_3(\cdot) \) is a modulus of continuity. Using Theorem 2.5, without loss of generality, we can assume \( K_3(\cdot) \) is also concave. Now replacing \( K_3(s) \) with \( K_3(s^\alpha) \), we can also assume \( K_3(\cdot) \) is \( \alpha \)-decreasing in the sense of Definition 2.6.

With the new \( K_i(\cdot) \)'s which are now \( \alpha \)-decreasing, we define
\[
K(r) := K_1(r) + K_2(r) + K_3(r),
\]
which is again \( \alpha \)-decreasing. Combining (4.32), (4.33), (4.34) and (4.30) along with (4.29) and making use of the choice \( |x| \approx \lambda^k \), we get
\[
|u(x) - L_\infty(x)| \leq C\lambda^k K(\lambda^k) = C|x|K(|x|).
\]
This completes the proof of the lemma. \( \square \)

**Remark 4.10.** The \( \alpha \)-decreasing property of \( K(\cdot) \) although not important in the proof of the above lemma, but nevertheless it is crucially used in the proof of the main result when the interior and the boundary estimates are combined.

5. **Proof of the Main Theorem**

In order to combine the interior regularity estimates proved in [6, Theorem 1.1] with our boundary estimates, we need the following rescaled version of the interior estimates.

**Proposition 5.1.** Let \( u \) be a local viscosity solution of
\[
F(x, D^2u) = f(x) \quad \text{in} \quad B_r \quad \text{for some} \quad r \in (0, 1).
\]
Then with modulus function \( K(\cdot) \) as given in (4.35), there exists a universal constant \( \Theta_0 \in (0, 1) \) and \( C > 0 \) such that if \( F \) has \( \Theta_0 \)-BMO coefficients, then the following estimate holds:
\[
|\nabla u(0)| \leq \frac{C}{r} \left( ||u||_{L^\infty(B_r)} + rK(r) \right).
\]
Analogously, for any \( y \in B_{r/2} \), there holds
\[
|\nabla u(y) - \nabla u(0)| \leq C \left( K(|y|) + \frac{||u||_{L^\infty(B_r)}}{r^{1+\alpha}}|y|^\alpha + |y|^\alpha \right).
\]
**Proof.** We will first recall a scale invariant version of the interior estimates. Define
\[
A(r) := \max\{r^{1+\alpha}, ||u||_{L^\infty(B_r)}\} \quad \text{and} \quad v(x) := \frac{u(rx)}{A(r)},
\]
where \( \alpha \) is the minimum of the exponent from (4.1) and [6, Theorem 1.2]. It is easy to see that \( ||v||_{L^\infty(B_1)} \leq 1 \) and \( v \) solves
\[
\tilde{F}(x, M) := \frac{r^2}{A(r)} F \left( rx, \frac{A(r)}{r^2} D^2v \right) = \frac{r^2}{A(r)} f(rx) \quad \text{in} \quad B_1.
\]
Since \( \alpha < 1 \), we have \( r^2 \leq r^{1+\alpha} \leq A(r) \). From Observation 5, we see that the rescaled problem has the same BMO-coefficients. Hence using either the estimates from before or from [6, Theorem 1.2], we get
\[
|\nabla v(0)| \overset{(4.35)}{\leq} C \left( 1 + \frac{r}{A(r)} K(r) \right).
\]
Analogously, from [6, Theorem 1.3] or from our estimates specialized to the interior case, we also get

\[ |\nabla v(x) - \nabla v(0)| \leq C \left( \frac{r}{A(r)} K(r|x|) + |x|^\alpha \right) \quad \text{for any } x \in B_{1/2}. \]

Rescaling back to \( u \), we get

\[ |\nabla u(0)| \leq \frac{C}{r} \left( \|u\|_{L^\infty(B_r)} + rK(r) \right), \]

where we used \( r^\alpha \leq K(r) \).

Analogously, for \( y \in B_{r/2} \) (using \( y = rx \)), there holds

\[ |\nabla u(y) - \nabla u(0)| \leq C \left( K(|y|) + \frac{\|u\|_{L^\infty(B_{r/2})}}{r^{1+\alpha}} |y|^\alpha + |y|^\alpha \right), \]

which completes the proof of the proposition. \( \square \)

The next lemma establishes that "\( \nabla u \)" is continuous at the boundary.

**Lemma 5.2.** Given any two points \( y, z \in \partial \Omega \cap B_{1/2} \), there exists a universal constant \( C \) such that the following estimate holds:

\[ |\nabla L_y - \nabla L_z| \leq CK(|y - z|). \]

Here \( L_y \) and \( L_z \) denotes the linear function constructed in Lemma 4.9 at the boundary points \( y \) and \( z \) respectively and \( K(\cdot) \) is the modulus defined in (4.35).

**Proof.** Let \( |y - z| = r \) and choose a "non tangential" point \( x \in \Omega \) such that \( |x - y| \approx r \) and \( |x - z| \approx r \). Furthermore, let \( B_{\beta r}(x) \subset \Omega \) for a universal \( \beta \) which can be chosen independent of \( r \) and depending only on the Lipschitz character of \( \Omega \).

We see that \( v := u - L_y \) solves the same equation from (1.1), thus from Proposition 5.1, we get

\[ |\nabla v(x)| = |\nabla u(x) - \nabla L_y| \leq \frac{C}{r} \left( \|u - L_y\|_{L^\infty(B_{\beta r}(x))} + rK(r) \right). \]  (5.1)

From the boundary regularity estimate in Lemma 4.9, we have

\[ \|u - L_y\|_{L^\infty(B_{\beta r}(x))} \leq CrK(r). \]  (5.2)

Thus combining (5.1) and (5.2), we get

\[ |\nabla u(x) - \nabla L_y| \leq \frac{C}{r} (rK(r) + rK(r)) \leq CK(r). \]  (5.3)

Likewise, since \( |x - z| \approx r \), we also get

\[ |\nabla u(x) - \nabla L_z| \leq \frac{C}{r} (rK(r) + rK(r)) \leq CK(r). \]  (5.4)

Combining (5.3) and (5.4) with an application of triangle inequality, we get

\[ |\nabla L_y - \nabla L_z| \leq CK(r) = CK(|y - z|), \]

which completes the proof of the lemma. \( \square \)

### 5.1. Proof of Theorem 3.1.

Let \( y, z \in \Omega \cap B_{1/4} \) be given. We shall denote the points \( y_0 \in \partial \Omega \) and \( z_0 \in \partial \Omega \) to be the points such that the following holds:

\[ d(y, y_0) = \min d(y, \partial \Omega) \quad \text{and} \quad d(z, z_0) = \min d(z, \partial \Omega). \]  (5.5)

Without loss of generality, let us assume that

\[ \delta := d(y, \partial \Omega) = \max \{d(y, \partial \Omega), d(z, \partial \Omega)\}, \]  (5.6)

and split the proof into two cases.
Case \(d(y, z) \leq \frac{\delta}{2}\): From (5.6), we see that \(z \in B_\delta(y) \subset \Omega\). Using the notation from (5.5), let us consider the function \(v := u - L_{y_0}\) which still solves the same problem from (1.1). Then from the rescaled estimates in Proposition 5.1, we get

\[
|\nabla v(y) - \nabla v(z)| = |\nabla u(y) - \nabla u(z)| \\
\leq C \left( K(|y - z|) + \frac{\|u - L_{y_0}\|_{L^\infty(B_\delta)}}{\delta} |y - z|^\alpha + |y - z|^\alpha \right). \tag{5.7}
\]

Using Lemma 4.9, we know that \(\|u - L_{y_0}\|_{L^\infty(B_\delta)} \leq \delta K(\delta)\) and from (4.35), we have the bound \(|y - z|^\alpha \leq K(|y - z|)\) which combined with (5.7) gives

\[
|\nabla u(y) - \nabla u(z)| \leq C \left( K(|y - z|) + \frac{K(\delta)}{\delta^\alpha} |y - z|^\alpha \right). \tag{5.8}
\]

Since \(K(\cdot)\) is \(\alpha\)-decreasing, therefore this implies

\[
\frac{K(\delta)}{\delta^\alpha} |y - z|^\alpha \leq K(|y - z|). \tag{5.9}
\]

Thus, combining (5.8) and (5.9), we get

\[
|\nabla u(y) - \nabla u(z)| \leq CK(|y - z|)
\]

Case \(d(y, z) > \frac{\delta}{2}\): Using the notation from (5.5) and making use of triangle inequality along with (5.6), we have

\[
d(y_0, z) \leq d(y, y_0) + d(y, z) = \delta + d(y, z) < 3d(y, z). \tag{5.10}
\]

From the choice of \(z_0\) in (5.5), we also get

\[
d(z_0, z) \leq d(y_0, z) \quad \text{(5.10)} \leq 3d(y, z) \tag{5.11}
\]

Combining (5.10) and (5.11), we get

\[
d(z_0, y_0) \leq d(z_0, z) + d(z, y) + d(y, y_0) \leq 7d(y, z). \tag{5.12}
\]

We now have the following sequence of estimates:

\[
|\nabla u(y) - \nabla u(z)| \quad \text{\(\leq\)} \quad |\nabla u(y) - \nabla L_{y_0}| + |\nabla L_{y_0} - \nabla L_{z_0}| + |\nabla L_{z_0} - \nabla u(z)| \\
\quad \text{\(\leq\)} \quad C K(|y - y_0|) + |\nabla L_{y_0} - \nabla L_{z_0}| + CK(|z - z_0|) \\
\quad \text{\(\leq\)} \quad C K(|y - z|) + CK(|y_0 - z_0|) \\
\quad \text{\(\leq\)} \quad CK(C|y - z|).
\]

This completes the proof of the Theorem.

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