An $O(D,D)$ invariant Hamiltonian action for the superstring

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Abstract

We construct $O(D,D)$ invariant actions for the bosonic string and RNS superstring, using Hamiltonian methods and ideas from double field theory. In this framework the doubled coordinates of double field theory appear as coordinates on phase space, and T-duality becomes a canonical transformation. Requiring the algebra of constraints to close leads to the section condition, which splits the phase space coordinates into spacetime coordinates and momenta.

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1. Introduction

The usual string worldsheet actions have several manifest symmetries, such as invariance under worldsheet reparametrizations and invariance under the spacetime Poincaré group or a curved space analog. In contrast, T-duality is a hidden symmetry of the action. To better understand this symmetry, one is led to search for alternative string worldsheet actions, which are written in terms of objects covariant under the $O(D,D)$ transformations of T-duality.

The development of actions with this goal in mind has a long history, with the foundations being laid in [1–5] and a recent renewal of interest including [6–16]. Two recent reviews dealing with this material and its extensions are [17, 18]. A common feature of these attempts is the introduction of new ‘dual coordinates’ in order to place momentum and winding on an equal footing, as required by T-duality. Momentum is dual to the spacetime coordinates, $X^a$, so winding should be dual to some new dual coordinates, $Y^a$. With these
extra coordinates the string is thought of as moving in a space with twice the usual number of dimensions, and the result is the \textit{doubled formalism} of the string.

In this doubled formalism the usual coordinates and their duals are packaged into an $O(D, D)$ vector. If the string moves in a background involving a metric, $G$, and Kalb–Ramond field, $B$, one can place these two fields into a ‘generalized metric’, an $O(D, D)$ tensor. The $O(D, D)$ transformations of this object match the well-known Buscher transformations [19, 20]. Duality invariant actions can then be constructed from these two $O(D, D)$ tensors. These actions must be supplemented with a constraint added by hand in order to allow us to (locally) reduce back down to $D$ physical coordinates.

The doubled formalism successfully provides a string worldsheet action with manifest T-duality invariance. The cost of doing this is of course the introduction of extra coordinates, whose precise physical significance is perhaps unclear. In addition, one would like to understand the origin of doubled formalism actions.

The geometry that is appropriate to understanding the doubled formalism is ‘generalized geometry’, introduced by Hitchin and Gualtieri [21, 22]. In generalized geometry one extends the tangent bundle to the direct sum of the tangent and cotangent bundles. This can be interpreted as the tangent bundle in the doubled formalism.

Generalized geometry is fundamental for the study of T-duality covariance from the spacetime perspective [4, 5, 23–27]. In the ‘double field theory’ approach, originally derived from string field theory, this is achieved by doubling the number of coordinates [23]. This leads to a rewriting of the low energy effective action of the string in terms of the above $O(D, D)$ vector of coordinates and their duals, and $O(D, D)$ tensors, giving a manifest $O(D, D)$ covariant form of the string’s low energy theory. In the bosonic case, at least, this action is the low-energy effective action of the \textit{doubled} string [28–30]. Double field theory has also been applied with success to the ten-dimensional supergravities [31–42] and allows important insights into string theory beyond its supergravity regime, such as non-geometric backgrounds, as for example discussed in the reviews [17, 18].

In double field theory the fields can at first depend on an $O(D, D)$ vector of coordinates. However, just as in the doubled formalism where half the coordinates must be eliminated by a constraint, to make contact with the usual low energy theories half the dependence of the fields in double field theory must be eliminated. This is achieved by the ‘section condition’. The coordinates of double field theory can be thought of as parametrizing a doubled space with some unusual properties [43–47]. The local symmetries of this space form ‘generalized diffeomorphisms’ [24, 48], and the section condition arises by requiring that their algebra closes. In this paper, we will show how the section condition also appears when studying the symmetry algebra of the \textit{worldsheet} diffeomorphisms of a doubled string.

Another profitable approach to T-duality is to treat the string from the Hamiltonian viewpoint, since the Hamiltonian of a string can naturally be expressed in an $O(D, D)$ invariant form. The spacetime momentum, $P_\mu$, and the worldsheet spatial derivatives of the spacetime coordinates, $X^\alpha$, combine naturally into an $O(D, D)$ vector. The Hamiltonian can then be written in terms of this vector and the generalized metric in a manifestly $O(D, D)$ covariant way. In this framework, T-dualities appear as canonical transformations, as has been observed before [49–53], while a phase space T-duality manifest action has also appeared in [54].

This paper has two aims. Firstly, to combine insights from both the Hamiltonian and doubled approaches to T-duality to present a new way to construct $O(D, D)$ invariant string actions. By making use of the Hamiltonian framework we hope to clarify the relationship between the doubled approach and the normal formulation of the string, particularly the status
of the extra coordinates in the former. Secondly, to construct a doubled action for the RNS superstring in a non-trivial $G$ and $B$ background.

Different but related approaches to the study of the doubled string from a Hamiltonian perspective include [55, 56], while similar structures appear in the study of two-dimensional (2D) chiral bosons [57].

We begin in section 2 with the $\sigma$-model for the bosonic string moving through a curved background with a Kalb–Ramond field. Since the Hamiltonian of the string is naturally $O(D, D)$ invariant, we rewrite the action in ‘Hamiltonian form’,

\[ S = \int \mathrm{d}r \mathrm{d}\sigma \left( \dot{X} \cdot P - \text{Ham}(X, P) \right) , \]

where $\text{Ham}$ denotes the Hamiltonian. In order to make contact with double field theory and assemble everything into $O(D, D)$ tensors we introduce the dual coordinates $Y_{\mu}$. This is done by making the replacement $P_{\mu} \rightarrow Y_{\mu}$, which has been suggested by previous work and can be justified by T-duality. The 2D doubled coordinates of double field theory are therefore interpreted as the 2D coordinates on the phase space of a $D$ dimensional system. This action naturally defines a constrained Hamiltonian system, which we analyse. We find the section condition of double field theory as a requirement for the closure of the constraint algebra, or equivalently for worldsheet reparametrization invariance.

In section 3 we analyse the RNS superstring in the same way. Starting with the corresponding $\sigma$-model, we construct the Hamiltonian form of the action and then an $O(D, D)$ invariant form as for the bosonic string. Unlike the spacetime coordinates, $X^\mu$, the fermions do not get ‘doubled’; this is consistent with the phase space interpretation of the bosonic doubled coordinates. We analyse the algebra of constraints and again find the section condition.

2. The bosonic string

The $\sigma$-model action for a bosonic string propagating in a $D$-dimensional background with metric $G_{\mu\nu}(X)$ and Kalb–Ramond field $B_{\mu\nu}(X)$ is\(^2\)

\[ S = -\frac{1}{2} \int \mathrm{d}r \mathrm{d}\sigma \left( \sqrt{-h} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + e^{\alpha\beta} \partial_\alpha X^\nu \partial_\beta X^\nu B_{\mu\nu}(X) \right) . \]

We will begin by putting this action into Hamiltonian form. It will be convenient to use a general parametrization of the worldsheet metric in terms of three real parameters, $\lambda$, $\tilde{\lambda}$ and $\Omega$:

\[ h_{\alpha\beta} = \Omega \begin{pmatrix} -\lambda \tilde{\lambda} & \frac{1}{2} (\tilde{\lambda} - \lambda) \\ \frac{1}{2} (\tilde{\lambda} - \lambda) & 1 \end{pmatrix} , \]

a parametrization which will be justified when we find the Hamiltonian. The conformal scale $\Omega$ drops out of the action as expected. With this parametrization, the momentum conjugate to $X^\mu$ is

\[ P_{\mu} = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{2}{\lambda + \tilde{\lambda}} G_{\mu\nu} \left( X^\nu + \frac{\lambda - \tilde{\lambda}}{2} X^\nu \right) + B_{\nu\mu} X^\nu . \]

\(^2\) $\mu$, $\nu$, ... are spacetime indices, $\alpha$, $\beta$, ... are worldsheet indices, the worldsheet metric has signature $(-, +)$ and $\epsilon^{01} = -1$ and we have set the string tension to 1.
We can Legendre transform to find the Hamiltonian

\[ \text{Ham}(X, P) = \frac{\lambda}{4}(P - (G + B)X')^2 + \frac{\tilde{\lambda}}{4}(P + (G - B)X')^2, \]  
(2.4)
suppressing indices and using \( G_{\mu\nu} \) to square. Like in any reparametrization invariant theory, the Hamiltonian is a sum of constraints with corresponding Lagrange multipliers.

To make this Hamiltonian manifestly duality invariant we need to package all the fields into representations of \( O(D, D) \). We can construct an \( O(D, D) \) vector out of \( X^\mu \) and \( P_\mu \)

\[ Z^M = \begin{pmatrix} X^\mu \\ P_\mu \end{pmatrix}, \]  
(2.5)
and we package the background fields into the generalized metric, \( H_{MN} \). We will also need to use the defining \( O(D, D) \) form \( \eta_{MN} \). These two tensors can be written

\[ H_{MN} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}, \quad \eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  
(2.6)
These will naturally combine into the projectors\(^3\)

\[ \Pi_{MN} = \frac{1}{2}(H_{MN} - \eta_{MN}), \quad \tilde{\Pi}_{MN} = \frac{1}{2}(H_{MN} + \eta_{MN}). \]  
(2.7)

Under the action of \( O(D, D) \), the transformation of the generalized metric exactly matches the Buscher transformations of the background fields. We can now write the Hamiltonian entirely in terms of \( O(D, D) \) covariant objects

\[ \text{Ham}(X, P) = \frac{\lambda}{2}Z^M H_{MN}Z^N + \frac{\tilde{\lambda}}{2}Z^M \tilde{\Pi}_{MN}Z^N. \]  
(2.8)
We will use this \( O(D, D) \) invariant Hamiltonian to construct an \( O(D, D) \) invariant action. By the definition of the Hamiltonian, the Lagrangian is equal to

\[ \mathcal{L} = X \cdot P - \text{Ham}(X, P), \]  
(2.9)
and we can integrate over the worldsheet to get the action in Hamiltonian form:

\[ S = \int d\tau d\sigma \left( X \cdot P - \frac{\lambda}{4}(P - (G + B)X')^2 - \frac{\tilde{\lambda}}{4}(P + (G - B)X')^2 \right). \]  
(2.10)
The equivalence of this action to (2.1) can be checked by eliminating \( P_\mu \) via its equation of motion.

In this formulation it is most natural to study the dynamics as of the string as a constrained Hamiltonian system. The dynamical fields are the canonical pair of \( D \)-vectors \( (X^\mu, P_\mu) \) and the constraints are

\[ \mathcal{K} = \frac{1}{4}(P - (G + B)X')^2, \quad \tilde{\mathcal{K}} = \frac{1}{4}(P + (G - B)X')^2, \]  
(2.11)
which generate reparametrizations in the lightcone directions \( \tau - \sigma \) and \( \tau + \sigma \) respectively. These constraints are imposed by Lagrange multipliers \( \lambda \) and \( \tilde{\lambda} \), the sum and difference of

\(^3\) Throughout this paper we will use the convention that \( O(D, D) \) indices are lowered and raised by the generalized metric, \( H_{MN} \) not \( \eta_{MN} \). This means in particular that \( \Pi_{MN} = -\Pi_{MN} \eta^{LN} \), whereas \( \tilde{\Pi}_{MN} = \tilde{\Pi}_{MN} \eta^{LN} \).
which are, up to factors of the string tension, the ‘lapse’ and ‘shift’ familiar from the Hamiltonian formulation of general relativity.

All $\tau$ derivatives appear in the first term of the action, and from this we can read off the fundamental Poisson bracket

$$\{X^\mu(\sigma_1), P_\nu(\sigma_2)\} = \delta_\nu^\mu \delta(\sigma_1 - \sigma_2),$$

(2.12)

while all others vanish. Calculations are made easier by working with ‘smeared’ constraints, defined for any constraint $C(\sigma)$ and test function $\alpha$ by

$$C(\alpha) \equiv \int d\sigma \, \alpha(\sigma) C(\sigma).$$

(2.13)

In this notation the algebra of the smeared constraints can be worked out to be

$$\{\mathcal{H}(\alpha), \mathcal{H}(\beta)\} = -\mathcal{H}(\alpha\beta' - \beta\alpha'),$$

$$\{\mathcal{H}(\alpha), \mathcal{H}(\beta)\} = 0,$$

$$\{\mathcal{H}(\alpha), \tilde{\mathcal{H}}(\beta)\} = \mathcal{H}(\alpha\beta' - \beta\alpha').$$

(2.14)

We recognize this as the $\text{Diff}_1 \times \text{Diff}_1$ algebra which ensures that the action is worldsheet diffeomorphism invariant. To make contact with more familiar formulations of the string, note that the two constraints can be Fourier expanded in $\sigma$, and the modes of each constraint would give an independent Virasoro algebra.

In the usual doubled formalism of the string [6], $X^\mu$ is packaged into an $O(D, D)$ vector with a set of dual coordinates, $Y_\mu$:

$$X^M = \begin{pmatrix} X^\mu \\ Y_\mu \end{pmatrix}. $$

(2.15)

The Hamiltonian tells us that $X^\mu$ is naturally packaged with $P_\mu$, so we should investigate the formal replacement

$$P_\mu \mapsto Y_\mu',$$

(2.16)

which changes the Poisson bracket to

$$\{X^\mu(\sigma_1), Y'_\nu(\sigma_2)\} = \delta_\nu^\mu \delta(\sigma_1 - \sigma_2).$$

(2.17)

The $O(D, D)$ covariant extension of this bracket would be

$$\{X^M(\sigma_1), X^N(\sigma_2)\} = \eta^{MN} \delta(\sigma_1 - \sigma_2),$$

(2.18)

which requires that

$$\{Y_\mu(\sigma_1), X'^\nu(\sigma_2)\} = \delta_\nu^\mu \delta(\sigma_1 - \sigma_2).$$

(2.19)

However if we first differentiate (2.17) with respect to $\sigma_1$ to obtain

$$\{X'^\mu(\sigma_1), Y_\nu(\sigma_2)\} = \delta_\nu^\mu \delta(\sigma_1 - \sigma_2)$$

(2.20)

4 It is actually $TX'$ which appears in an $O(D, D)$ vector with $P$, so when the tension is restored, $P$ should be set to $TY'$. 

5
and then integrate over $\sigma_2$, after relabelling and reordering, we find that (2.19) holds up to a constant of integration

$$\{ Y_\mu(\sigma_1), X^\nu(\sigma_2) \} = \delta_\mu^\nu \left( \delta(\sigma_1 - \sigma_2) - C \right).$$

(2.21)

Now we note that the zero-mode of $Y_\mu$ is absent from the action and thus has zero bracket with everything. If we integrate both sides with respect to $\sigma_2$, we find that $C = -1/2\pi$, exactly cancelling the zero-mode of the $\delta$-function, so we do not obtain the covariant bracket (2.19). We can understand this as follows: the zero-mode of $X^\mu$ appears in the action, but the zero-mode of $Y_\mu$ does not, breaking $O(D, D)$ covariance. We will now investigate the consequences of including a term involving the zero-mode of $Y_\mu$ in the action.

It is helpful to split the variables up into zero-mode, winding and oscillator parts. To implement winding, we demand the boundary condition $X^\mu(2\pi) = X^\mu(0) = w^\mu$, where $w^\mu$ is a constant, non-dynamical vector. We can define the zero-modes

$$x^\mu = \int d\sigma \, X^\mu, \quad p_\mu = \frac{1}{2\pi} \int d\sigma \, P_\mu,$$

(2.22)

and then write

$$X^\mu = x^\mu + \frac{1}{2\pi} w^\mu(\sigma - \pi) + \bar{X}^\mu, \quad P_\mu = \frac{1}{2\pi} p_\mu + \bar{P}_\mu.$$

(2.23)

The barred quantities are periodic and integrate to zero, so can be expressed as Fourier series in $\sigma$. The quantity $\bar{Y}$ such that $Y = \bar{Y}$ and $\int d\sigma \, \bar{Y} = 0$ is therefore well-defined. We can define $Y'$ via

$$Y'_\mu = p_\mu = \frac{1}{2\pi} \bar{p}_\mu + \bar{Y}'_\mu.$$

(2.24)

This is so far just a renaming of variables, but we can integrate this with respect to $\sigma$, picking up a constant of integration,

$$Y_\mu = y_\mu + \frac{1}{2\pi} \bar{p}_\mu(\sigma - \pi) + \bar{Y}_\mu.$$

(2.25)

This introduces an extra zero-mode to the theory, but this has no effect as only $Y'$ appears in the action. The Hamiltonian is written in terms of $Z^M = X^M$, and hence in terms of $O(D, D)$ objects, but the first term requires more work. Consider

$$\int d\sigma \, \dot{x} \cdot P = \dot{x} \cdot p + \int d\sigma \, \dot{x} \cdot \bar{P}.$$

(2.26)

We notice that

$$\int d\sigma d\tau \, \dot{x} \cdot \bar{P} = \int d\tau d\sigma \left( \frac{1}{2} \dot{x} \cdot \bar{Y}' + \frac{1}{2} \bar{x}' \cdot \dot{y} + \frac{1}{2} \frac{d}{d\tau} (\dot{x} \cdot \bar{Y}') - \frac{1}{2} \frac{d}{d\sigma} (\dot{x} \cdot \bar{Y}') \right).$$

(2.27)

By periodicity in $\sigma$ we can drop the last term, and we are also free to drop the total $\tau$ derivative, so that

$$\int d\tau d\sigma \, \dot{x} \cdot P = \dot{x} \cdot p + \int d\tau d\sigma \frac{1}{2} \dot{x}^M \eta_{MN} \dot{x}^N.$$

(2.28)

Everything is now $O(D, D)$ covariant except for the zero-mode term $\dot{x} \cdot p$. To deal with this, we note from equation (2.25) that $p_\mu$ can be interpreted as a winding in the $Y_\mu$ direction. Just as the momentum $p_\mu$ is conjugate to position $x^\mu$, the winding $w^\mu$ should be conjugate to $y_\mu$. 
Furthermore, for $O(D, D)$ covariance $w^\mu$ should be put on the same footing as $p_\mu$, and so be a dynamical variable. The zero-modes naturally combine into two $O(D, D)$ vectors

$$x^M = \begin{pmatrix} x^\mu \\ y^\mu \end{pmatrix}, \quad p_M = \begin{pmatrix} p_\mu \\ w^\mu \end{pmatrix},$$

(2.29)

and we replace the term $\dot{x} \cdot p$ with the fully $O(D, D)$ invariant $\dot{x}^M p_M$.

To reiterate, we have made two modifications to the action. We have promoted $w^\mu$ to a dynamical variable, and we have added a term $\dot{y} \cdot w$ to the action. As $y^\mu$ only appears in this term, the equation of motion of $y$ sets $\dot{w} = 0$, and the equation of motion of $w$ sets the new variable $y$ to some function of the other variables. The equations of motion of the new action are identical to the equations of motion of the original action with non-dynamical $w$ plus an equation determining $y$ which does not affect the other variables. The actions are therefore classically equivalent, although there may be further implications quantum mechanically.

This issue of zero-modes has been considered by other authors in [8, 10].

Our final action is therefore

$$S = \int d\tau d\sigma \left( \frac{1}{2\pi} \dot{x}^M p_M + \frac{1}{2} x^M \eta_{MN} \dot{x}^N - \frac{1}{2} x^M \Pi_{MN} x^N - \frac{1}{2} x^M \tilde{\Pi}_{MN} x^N \right).$$

(2.30)

The form of this action is a little strange, given the separate treatment of zero-modes and oscillators, but the extra term is exactly that needed for the natural bracket (2.18), and to treat momentum and winding equally. We stress again that the normal string action, with boundary conditions involving non-dynamical winding, is equivalent to the doubled action (2.30) where winding is allowed to be dynamical. We will later see that this action not only describes configurations with constant winding but, after T-duality transformations, situations with non-constant winding along isometry directions.

Up to this point the background fields, expressed either as the generalized metric, $H$, or as $G$ and $B$, have been functions of $X^\mu$ only. This is not $O(D, D)$ covariant. However, allowing the fields to depend on the full $X^M$ would be the same as allowing the fields of the normal string to depend on both the spacetime coordinates and momenta. If one were to then compute the algebra of constraints, extra terms proportional to the momentum derivatives of the background fields would appear, preventing the algebra from closing. The momentum independence of the background fields is therefore required for consistency.

We will temporarily allow the generalized metric to depend on the full $X^M$, and find that similar extra terms appear in the algebra, preventing its closure. Requiring that these terms vanish will give us an analogous consistency condition. Crucially, this condition will be $O(D, D)$ invariant.

With this extra step, the usual string action (2.10) and the manifestly $O(D, D)$ invariant action (2.30) differ only by the classically irrelevant zero-mode term and the covariant way in which momentum dependence of the background fields is prevented.

We now analyse (2.30) as a constrained Hamiltonian system. Denoting by $\tilde{P}_M$ the momentum conjugate to $\tilde{X}^M$, we find a set of second-class constraints,

$$C_M = \tilde{P}_M - \frac{1}{2} \eta_{MN} \tilde{X}^N.$$

(2.31)

We pass to (a priori) partial Dirac brackets in these second-class constraints, finding as promised the integrated version of (2.18)

$$\left\{ X^M(\sigma_1), X^N(\sigma_2) \right\}^* = -\eta^{MN} \epsilon(\sigma_1 - \sigma_2),$$

(2.32)
where $\epsilon$ is such that $\epsilon = \delta$, the Dirac delta. The remaining constraints are
\[ \mathcal{H} = \frac{1}{2} X^M \mathcal{H}_{MN} X^N, \quad \tilde{\mathcal{H}} = \frac{1}{2} X^M \tilde{\mathcal{H}}_{MN} X^N. \] (2.33)
We can now use the brackets (2.32) to compute the algebra of the remaining constraints, finding
\[
\begin{align*}
\{ \mathcal{H}(\alpha), \mathcal{H}(\beta) \}^\ast &= -\mathcal{H}(\alpha \beta' - \beta \alpha') + \Delta, \\
\{ \tilde{\mathcal{H}}(\alpha), \tilde{\mathcal{H}}(\beta) \}^\ast &= \Delta, \\
\{ \tilde{\mathcal{H}}(\alpha), \tilde{\mathcal{H}}(\beta) \}^\ast &= \tilde{\mathcal{H}}(\alpha \beta' - \beta \alpha') + \Delta.
\end{align*}
\] (2.34)
This is the $\text{Diff} \times \text{Diff}_1$ algebra up to the extra term
\[
\Delta = \frac{1}{16} \int d\sigma_1 d\sigma_2 \, \alpha(\sigma_1) \beta(\sigma_2) \, X^M(\sigma_1) X^{1N}(\sigma_1) X^P(\sigma_2) X^{Q}(\sigma_2) \\
\times \left\{ H_{MN}(\sigma_1), H_{PQ}(\sigma_2) \right\}^\ast.
\] (2.35)
As discussed earlier, the appearance of this term is a direct consequence of allowing the generalized metric to depend on the full $X^M$. It therefore tells us how to restrict the dependence in an $O(D, D)$ covariant way. Note that
\[
\left\{ H_{MN}(\sigma_1), H_{PQ}(\sigma_2) \right\}^\ast = -\eta^{RS} \partial_R H_{MN}(\sigma_1) \partial_S H_{PQ}(\sigma_2) \epsilon(\sigma_1 - \sigma_2),
\] (2.36)
and so a sufficient condition for $\Delta$ to vanish, and for all brackets with $\Delta$ in to vanish is
\[
\eta^{RS} \partial_R F_1(\sigma_1) \partial_S F_2(\sigma_2) = 0 \quad \forall \, \sigma_1, \sigma_2,
\] (2.37)
where $F_1$ and $F_2$ are any combinations of $H_{MN}$ and its spacetime derivatives. We recognize this as the section condition of double field theory in a form studied previously in [30]; it essentially requires the generalized metric to only depend on at most half of the coordinates. The consistency of this condition will have to be checked upon quantization. It is possible that a slightly weaker condition on the generalized metric could cause $\Delta$ to vanish [58–63], which should be studied in the future; in this paper we will impose the section condition in its full strength. This condition should be applied before performing the Hamiltonian analysis, although in this case it does not make any difference to the result.

Note that we are not treating $\Delta$ as a new constraint in the Hamiltonian sense. We are viewing it as a condition on the admissible background fields, not on the dynamical variables. This is analogous to requiring, for the normal string action,
\[
\frac{\partial G_{\mu\nu}}{\partial P_\alpha} = 0,
\] (2.38)
which one would certainly not treat as a constraint.

There are other reasons why we do not treat $\Delta$ as a constraint on the dynamical variables. Firstly, the algebra of constraints would no longer be the $\text{Diff}_1 \times \text{Diff}_1$ algebra, breaking the reparametrization invariance of the string. Secondly, adding additional constraints would change the number of physical degrees of freedom, which we do not want to do.

After imposing the section condition, the remaining constraints, $\mathcal{H}$ and $\tilde{\mathcal{H}}$, are first-class with respect to the Dirac brackets (2.32). These brackets are thus actually full Dirac brackets and we have completed our analysis of constraints.
The section condition allows us to reduce our action (2.30) back to the ‘undoubled’ action (2.10). It implies that we can go to a frame
\[
X^M = \begin{pmatrix} X^\mu \\ Y_\mu \end{pmatrix},
\]  
(2.39)
where the generalized metric is independent of \(Y_\mu\). The second term in (2.30) can be integrated by parts so that it becomes \(X^{\mu}Y_{\mu}\), and the \(\gamma \cdot w\) can be ignored classically if \(w^\mu\) is set constant. As \(Y_\mu\) then appears in the action only as \(Y_{\mu}\) we may rename \(Y_{\mu} \mapsto P_{\mu}\) and recover the undoubled action.

In fact our action can also describe non-constant winding by picking a different frame. We may pick as one of the \(X^\mu\) a direction upon which the generalized metric does not depend, in other words an isometry direction. The winding in this direction need not be constant, but the momentum, the winding of the dual direction, must be.

We can also relate our action (2.30) to a form of doubled action used in the literature. By going to conformal gauge, \(\lambda = \bar{\lambda} = 1\), we obtain essentially the original doubled action introduced by Tseytlin in [2]. The only difference lies in our treatment of the zero modes which is essential to obtain the expected Dirac bracket.

The first-class constraints generate gauge transformations associated to the diffeomorphism symmetry which the action enjoys. Under a gauge transformation \(X^M\) transforms as
\[
\{ \mathcal{H} (\alpha), X^M (\sigma) \}^\# = a \Pi^M_N X^N + \frac{1}{4} \int d\sigma' \left( aX^P X^Q \eta^{MN} \partial_N H_{PQ} \right)(\sigma') \epsilon (\sigma' - \sigma),
\]
\[
\{ \bar{\mathcal{H}} (\alpha), X^M (\sigma) \}^\# = -a \Pi^M_N X^N + \frac{1}{4} \int d\sigma' \left( aX^P X^Q \eta^{MN} \partial_N H_{PQ} \right)(\sigma') \epsilon (\sigma' - \sigma). \tag{2.40}
\]
The gauge transformations induce non-local contributions, but this problem is also solved by the section condition. If the generalized metric, \(H_{MN}\), depends on a coordinate \(X_\tau\), then its \(\eta_{MN}\)-dual coordinate \(Y_\tau\) will transform non-locally. If we want the generalized metric to depend only on coordinates which transform locally on the worldsheet, then it can depend on at most \(D\) coordinates and must be independent of their duals. These coordinates then transform locally, and their duals only appear in the action through their \(\sigma\)-derivatives, which are also local.

3. The RNS superstring

The \(\sigma\)-model action for an RNS superstring propagating in a \(D\)-dimensional background with metric \(G_{\mu \nu} (X)\) and Kalb–Ramond field \(B_{\mu \nu} (X)\) is [64]\(^5\)

\(^5\) The flat \(\gamma\)-matrices are \(\gamma^0 = i\sigma_2, \gamma^1 = -\sigma_1\) and \(\gamma_5 = -\sigma_3\), and \(z = x^5 \gamma^0, T_{\mu \nu} = 3 \partial_{[\mu} B_{\nu]}\) is the field strength of the Kalb–Ramond field, \(\Gamma^\mu_{\nu \rho}\) is the Levi–Civita connection of the metric and the associated Riemann tensor in our conventions is \(R^\mu_{\nu \rho} = 2 \partial_{[\nu} \Gamma^\mu_{\rho]} + 2 \Gamma^\mu_{[\nu | \rho]} \Gamma^{|\nu | \rho} \).
This extends the bosonic action (2.1) by the addition of $\Psi^\mu$, a worldsheet two-component Majorana spinor and a spacetime vector, and a worldsheet gravitino, $\chi^a$, a worldsheet vector-spinor and spacetime scalar. To derive the Hamiltonian form of this action we use the same parametrization of the worldsheet metric as before. We also choose a convenient parametrization of the worldsheet vielbein, so that

$$h_{\alpha\beta} = \begin{pmatrix} -\lambda\bar{\lambda} & \frac{1}{2}(\lambda - \bar{\lambda}) \\ \frac{1}{2}(\lambda - \bar{\lambda}) & 1 \end{pmatrix}, \quad L^{-1} = \Omega^{-\frac{1}{2}} \begin{pmatrix} 2 & 0 \\ -\frac{\bar{\lambda}}{\lambda - \bar{\lambda}} & 1 \end{pmatrix}. \quad (3.2)$$

If we rescale the fermions by

$$\Psi \mapsto \Omega^{-1/4}\Psi, \quad \chi \mapsto \Omega^{1/4}\chi, \quad (3.3)$$

then the conformal factor again drops out. It is convenient to split the worldsheet spinor $\Psi^\mu$ into its components as

$$\Psi^\mu = \begin{pmatrix} \psi^\mu \\ \tilde{\psi}^\mu \end{pmatrix}. \quad (3.4)$$

We also introduce a vielbein $e^a_{\mu}$ in spacetime, so that $G_{\mu\nu} = e^a_{\mu}e^b_{\nu}\eta_{ab}$ where $\eta_{ab}$ is the flat Minkowski metric. We use this vielbein to flatten the indices on the fermions so that $\psi^a = e^a_{\mu}\psi^\mu, \tilde{\psi}^a = e^a_{\mu}\tilde{\psi}^\mu$.

We now follow the same procedure as in the bosonic case. First, we find the momentum conjugate to $X^\mu$. We then Legendre transform to find the Hamiltonian and write the action in Hamiltonian form. We do not Legendre transform in the fermionic variables. Again two components of the metric remain and enforce two bosonic constraints; they are now joined by two surviving components of the worldsheet gravitino, which we call $\xi$ and $\bar{\xi}$, that enforce two new fermionic constraints. The resulting Hamiltonian form action is

$$S = \int d\tau d\sigma \left( \left. \mathcal{H} \right|_{\Psi^a} + \mathcal{H} \right), \quad (3.5)$$

Superconformal gauge corresponds to setting $\lambda = \bar{\lambda} = 1, \xi = \bar{\xi} = 0$. 


where

\[
4\mathcal{H} = \left( P_\mu - (G + B)_\mu X^{\nu} + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b \right)^2
\]

\[
+ 2i \left( \eta_{ab} \psi^a \psi^b + \omega_{ab} \psi^a X^{\nu} \right) + 2R_{\xi \chi \rho \sigma} \tilde{\psi}^a \psi^b \tilde{\psi}^c \psi^d,
\]

\[
4\mathcal{H} = \left( P_\mu + (G - B)_\mu X^{\nu} + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b \right)^2
\]

\[
- 2i \left( \eta_{ab} \psi^a \psi^b + \alpha_{ab} \psi^a X^{\nu} \right) + 2R_{\xi \chi \rho \sigma} \tilde{\psi}^a \psi^b \tilde{\psi}^c \psi^d,
\]

\[
2\mathcal{J} = \left( P_\mu - (G + B)_\mu X^{\nu} + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b \right) \psi^\mu,
\]

\[
2\mathcal{J} = \left( P_\mu + (G - B)_\mu X^{\nu} + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b + \frac{i}{2} \alpha^{ab} \psi_1 a b \psi_2 b \right) \tilde{\psi}^\mu.
\]

(3.6)

We have here the following torsionful connections and associated spin connections

\[
\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} \pm \frac{1}{2} \nabla^\mu \rho,
\]

\[
\omega_{\mu b} = \partial_{\mu} e^a_b + \Gamma^a_{\mu b}.
\]

(7.7)

We denote by \(R_{\xi \chi \rho \sigma}^\mu\) the Riemann tensors associated to \(\Gamma^\mu_{\nu\rho}\). These are in fact equal as a result of the Bianchi identity \(\partial_\mu T_{\nu\rho\sigma} = 0\). The action (3.5) can of course be converted back into (3.1) by eliminating \(P_\mu\).

The Poisson brackets for the bosonic variables are as before. The canonical momenta of the fermions are proportional to the fermions themselves; this leads to a set of second-class constraints which must be dealt with by passing to Dirac brackets. This is a standard situation so we just give the results

\[
\{ X^\mu (\sigma_1), P_\nu (\sigma_2) \}^* = \delta^\mu_\nu \delta (\sigma_1 - \sigma_2),
\]

\[
\{ \psi^a (\sigma_1), \psi^b (\sigma_2) \}^* = \eta^{ab} \delta (\sigma_1 - \sigma_2),
\]

\[
\{ \tilde{\psi}^a (\sigma_1), \tilde{\psi}^b (\sigma_2) \}^* = \eta^{ab} \delta (\sigma_1 - \sigma_2).
\]

(3.8)

All brackets between bosonic variables and fermionic variables with flat indices vanish. We now work out the algebra of constraints [65], again using smeared constraints. We use Grassmann odd test functions for the Grassmann odd constraints

\[
\{ 3 (a), 3 (b) \}^* = i \mathcal{H} (b a),
\]

\[
\{ 3 (a), \mathcal{H} (b) \}^* = -3 (a^\beta - \frac{1}{2} a b^\beta),
\]

\[
\{ \mathcal{H} (a), \mathcal{H} (c) \}^* = -3 (a^\beta - \frac{1}{2} a b^\beta),
\]

(3.9)

all brackets between tilded and untilded constraints vanish.

To package everything into representations of \(O(D, D)\), we again make the substitution

\[
P \mapsto Y',
\]

(3.10)

and add the zero-mode term. This raises the same issues as in the bosonic case but there are no new complications; we do not double the fermions as they only transform under the action of \(O(D, D)\) when contracted with a vielbein. We in fact introduce two double-vielbeine, \(L_M^\mu\) and \(R_M^\mu\) [4, 5, 32–38] such that
These vielbeine have an $O(D, D)$ index $M, N, \ldots$ which like all $O(D, D)$ indices we lower and raise by the generalized metric, $H_{MN}$, as well as a flat $O(d - 1, 1)$ index $a, b, \ldots$ which is lowered/raised by the Minkowski metric $\eta_{ab}$. Furthermore, these vielbeine are eigenstates of the projectors onto left-/right-moving subspaces, $\Pi_{MN}, \tilde{\Pi}_{MN}$

\[
\Pi_{MN} L^a_{\ b} = L^a_{\ b}, \quad \Pi_{MN} R^a_{\ b} = 0,
\]

\[
\tilde{\Pi}_{MN} R^a_{\ b} = R^a_{\ b}, \quad \tilde{\Pi}_{MN} L^a_{\ b} = 0,
\]

and can thus be parameterized by spacetime fields as \(^8\)

\[
L^a_{\ M} = \frac{1}{\sqrt{2}} \left( e^a_{\ \mu} - B_{\mu a} e^\mu \right), \quad R^a_{\ M} = \frac{1}{\sqrt{2}} \left( \tilde{e}^a_{\ \mu} + B_{\mu a} \tilde{e}^\mu \right).
\]

We use these vielbeine to define

\[
\omega_{Mab} = L_{Na} \partial_M L^N_{\ b} + L^N_{\ (a} L^p_{\ b)} \partial_F H_{MN},
\]

\[
\tilde{\omega}_{Mab} = R_{Na} \partial_M R^N_{\ b} + R^N_{\ (a} R^p_{\ b)} \partial_F H_{MN},
\]

which are closely related to the normal spin-connections but are not themselves spin-connections as we will discuss a little further on.

With these objects we can rewrite the Hamiltonian action $O(D, D)$ covariantly

\[
S = \int \mathrm{d}\tau \mathrm{d}\sigma \left( \frac{1}{2\pi} \chi^M P_M + \frac{1}{2} \chi^M \eta_{MN} \dot{X}^N - \frac{i}{2} \left( \psi^a \dot{\psi}^b + \dot{\psi}^a \psi^b \right) \eta_{ab} \right.
\]

\[
- \lambda \mathcal{K} - \lambda \mathcal{K} - i \xi \right) \bigg),
\]

where

\[
2\mathcal{K} = \chi^M \Pi_{MN} X^N + i \eta_{ab} \psi^a \psi^b + i \chi^M \left( \Pi^N_{\ (a} \partial_{\ b)} \psi^ab - \Pi^N_{\ (a} \partial_{\ b)} \psi^{a\ b} \right)
\]

\[
- \frac{1}{4} \Pi^M_{\ b} \partial_{Mab} \partial_{p} \psi^{a} \psi^{b} \psi^{d} - \frac{1}{4} \Pi^M_{\ b} \partial_{Mab} \partial_{p} \psi^{a} \psi^{b} \psi^{d}
\]

\[
- \frac{1}{2} F_{abcd} \psi^{a} \psi^{b} \psi^{c} \psi^{d},
\]

\[
2\mathcal{K} = \chi^M \Pi_{MN} X^N - i \eta_{ab} \psi^a \psi^b + i \chi^M \left( \Pi^N_{\ (a} \partial_{\ b)} \psi^{ab} - \Pi^N_{\ (a} \partial_{\ b)} \psi^{a\ b} \right)
\]

\[
- \frac{1}{4} \Pi^M_{\ b} \partial_{Mab} \partial_{p} \psi^{a} \psi^{b} \psi^{d} - \frac{1}{4} \Pi^M_{\ b} \partial_{Mab} \partial_{p} \psi^{a} \psi^{b} \psi^{d}
\]

\[
- \frac{1}{2} \tilde{F}_{abcd} \psi^{a} \psi^{b} \psi^{c} \psi^{d},
\]

\[
\sqrt{2} \mathcal{K} = \chi^M \eta_{MN} L_{\ a} \psi^a + \frac{i}{2} L_{\ c} \partial_{Mab} \psi^a \psi^b \psi^c + \frac{i}{2} L_{\ c} \partial_{Mab} \psi^a \psi^b \psi^c,
\]

\[
\sqrt{2} \mathcal{K} = \chi^M \eta_{MN} R^a_{\ b} \psi^a + \frac{i}{2} R_{\ c} \partial_{Mab} \psi^a \psi^b \psi^c + \frac{i}{2} R_{\ c} \partial_{Mab} \psi^a \psi^b \psi^c,
\]

\(^7\) The flat indices actually transform under different copies of $O(d - 1, 1)$ \(^32-38\) and thus could be labelled by different indices. We do not use this convention to avoid the inevitable index clutter.

\(^8\) Here $e^a_{\ \mu}$ and $e^\mu_{\ a}$ are both vielbeine for the spacetime metric \(^4, 5, 32-38\). We define $\tilde{\psi}^a = \tilde{e}^a_{\ \mu} \psi^\mu$, while $\tilde{\psi}^\mu = e^\mu_{\ a} \tilde{\psi}^a$. 

\[12\]
and we have defined
\[
F_{abcd} = 2L^M_{[a} \partial_M \left( L^N_{b]} \omega_{Ncd} \right) + 2L^M_{[a} L^N_{b]} \omega_{Mce} \partial^{e}_{cd} + 3L^M_{[e} \partial_M \left( L^N_{e} \omega_{Ncd} \right),
\]
\[
\bar{F}_{abcd} = 2R^M_{[a} \partial_M \left( R^N_{b]} \omega_{Ncd} \right) + 2R^M_{[a} R^N_{b]} \omega_{Mce} \omega^{e}_{cd} + 3R^M_{[e} \partial_M \left( R^N_{e} \omega_{Ncd} \right). \tag{3.17}
\]

For the undoubled string, the Hamiltonian constraints involved the Riemann curvature tensor and the torsionful spin connections \( \omega \) (3.7). A natural question to ask is whether the doubled action contains objects which have a geometric interpretation in double field theory \cite{33, 43, 59, 62, 63}.

The objects \( \omega_{Mab} \) and \( \tilde{\omega}_{Mab} \) are connections allowing us to covariantize the local \( O(D) \times O(D) \) symmetry. They consist of a Weitzenböck term \cite{59, 62, 63} plus a term which is an \( O(D) \times O(D) \) tensor. This extra term allows one to construct generalized diffeomorphism scalars; \( L^M_{[e} \partial_M \omega^{e}_{ab} \), \( R^M_{[e} \partial_M \omega^{e}_{ab} \), \( L^N_{[e} \partial^{e}_{ab} \), and \( R^N_{[e} \partial^{e}_{ab} \) are generalized scalars, even though \( \omega_{Mab} \) and \( \tilde{\omega}_{Mab} \) themselves are not generalized tensors.

The above means that every term appearing in \( Q, \bar{Q} \), \( F_{abcd} \) and \( \bar{F}_{abcd} \) is a scalar under generalized diffeomorphisms. In addition, the latter two consist of an \( O(D) \times O(D) \) tensor plus a non-tensorial part:
\[
F_{abcd} = 2L^M_{[a} \partial_M \left( L^N_{b]} \omega_{Ncd} \right) + 2L^M_{[a} L^N_{b]} \omega_{Mce} \partial^{e}_{cd} + 3L^M_{[e} \partial_M \left( L^N_{e} \omega_{Ncd} \right),
\]
\[
\bar{F}_{abcd} = 2R^M_{[a} \partial_M \left( R^N_{b]} \omega_{Ncd} \right) + 2R^M_{[a} R^N_{b]} \omega_{Mce} \omega^{e}_{cd} + 3R^M_{[e} \partial_M \left( R^N_{e} \omega_{Ncd} \right), \tag{3.18}
\]

where in the tensorial part the covariant derivative \( \nabla_M \) acts on the \( O(D) \) indices
\[
\nabla_M L^N_a = \partial_M L^N_a - \omega^b_{Ma} L^N_b, \quad \nabla_M R^N_a = \partial_M R^N_a - \omega^b_{Ma} R^N_b. \tag{3.19}
\]

One can also relate \( \omega_{Mab} \) and \( \tilde{\omega}_{Mab} \) to the ‘semi-covariant’ derivative \cite{33, 43} of double field theory, by appropriately projecting the latter.

We now return to the analysis of the Hamiltonian system (3.15). The Dirac brackets are
\[
\{ X^M (\sigma_1), X^N (\sigma_2) \}^a = -\eta^{MN}_a (\sigma_1 - \sigma_2),
\]
\[
\{ \psi^a (\sigma_1), \psi^b (\sigma_2) \}^a = i \delta^{ab}_a (\sigma_1 - \sigma_2),
\]
\[
\{ \psi^a (\sigma_1), \psi^b (\sigma_2) \}^a = i \delta^{ab}_a (\sigma_1 - \sigma_2). \tag{20.20}
\]

Now consider the algebra of constraints. The Dirac bracket of the \( X^M \) implies that every bracket of constraints will generate many \textit{a priori} non-zero terms of the form
\[
\{ F_1 (\sigma_1), F_2 (\sigma_2) \}^a , \tag{3.21}
\]
where \( F_1 \) and \( F_2 \) are combinations of background fields and their derivatives. These will prevent the algebra from closing, so we will demand
\[
\eta^{RS} \partial_R F_1 (\sigma_1) \partial_S F_2 (\sigma_2) = 0 \quad \forall \sigma_1, \sigma_2, \tag{3.22}
\]
clearly analogous to the section condition in the bosonic case. This condition again requires the background fields to be independent of half the coordinates and allows us to reduce to (3.1) just as in the bosonic case. It is again possible that this condition could be relaxed, an idea for further study \cite{58–63}.
With this condition we can now study the algebra of constraints; a series of long and
tedious calculations and judicious use of the Jacobi identity verifies that the algebra still
holds:

\[
\begin{align*}
\{ \partial(\alpha), \partial(\beta) \} &= \mathcal{H}(\beta \alpha), \\
\{ \partial(\alpha), \mathcal{H}(\beta) \} &= \mathcal{H}(\alpha' - \frac{1}{2} \alpha \beta'), \\
\{ \mathcal{H}(\alpha), \mathcal{H}(\beta) \} &= \mathcal{H}(\alpha' - \alpha \beta'), \\
\end{align*}
\]

(3.23)

with vanishing brackets between tilded and untilded constraints.

The first-class constraints \( \partial, \tilde{\partial}, \mathcal{H}, \tilde{\mathcal{H}} \) generate symmetries of the action (3.15) up to the
section condition. From their algebra we can identify \( \partial \) and \( \tilde{\partial} \) as supersymmetry generators
under which the fields transform as

\[
\begin{align*}
\{ \partial(\alpha), X^M \} &= -\frac{1}{\sqrt{2}} \alpha L^{M}_{a} \psi^a + N^M(Q(\alpha), X), \\
\{ \partial(\alpha), \psi^a \} &= \frac{1}{\sqrt{2}} \alpha \omega^a_{Mb} \psi^b L^M_{c} \psi^c - \frac{1}{2 \sqrt{2}} \alpha L^{M}_{a} \omega_{Mb} \psi^b \psi^c \\
&\quad - \frac{1}{2 \sqrt{2}} \alpha L^{M}_{a} \omega_{Mb} \psi^b \psi^c - i \frac{1}{\sqrt{2}} \alpha X^M L^a, \\
\{ \partial(\alpha), \tilde{\psi}^a \} &= \frac{1}{\sqrt{2}} \alpha \tilde{\omega}^a_{Mb} \tilde{\psi}^b L^M_{c} \psi^c, \\
\{ \tilde{\partial}(\alpha), X^M \} &= -\frac{1}{\sqrt{2}} \alpha R^M_{a} \psi^a + N^M(\tilde{\partial}(\alpha), X), \\
\{ \tilde{\partial}(\alpha), \tilde{\psi}^a \} &= \frac{1}{\sqrt{2}} \alpha \tilde{\omega}^a_{Mb} \tilde{\psi}^b R^M_{c} \tilde{\psi}^c, \\
\{ \tilde{\partial}(\alpha), \psi^a \} &= \frac{1}{\sqrt{2}} \alpha \tilde{\omega}^a_{Mb} \tilde{\psi}^b \tilde{R}^M_{c} \psi^c - \frac{1}{2 \sqrt{2}} \alpha R^{M}_{a} \omega_{Mb} \psi^b \psi^c \\
&\quad - \frac{1}{2 \sqrt{2}} \alpha R^{M}_{a} \omega_{Mb} \tilde{\psi}^b \tilde{\psi}^c + \frac{i}{\sqrt{2}} \alpha X^M R^a, \\
\end{align*}
\]

(3.24)

where the non-local contributions are

\[
\begin{align*}
N^M(Q(\alpha), X)(\sigma_1) &= -\frac{1}{\sqrt{2}} \int d\sigma_2 \eta^{MN} \alpha(\sigma_2) \psi^a(\sigma_2) \left( \frac{i}{2} \partial_N \left( L^p_{a} \omega_{Pb} \right) \psi^b \psi^c \\
&\quad + \frac{i}{2} \partial_N \left( L^p_{a} \omega_{Pb} \right) \psi^b \psi^c - X^P \partial_N L^a \right)(\sigma_2)e(\sigma_1 - \sigma_2), \\
N^M(\tilde{\partial}(\alpha), X)(\sigma_1) &= -\frac{1}{\sqrt{2}} \int d\sigma_2 \eta^{MN} \alpha(\sigma_2) \tilde{\psi}^a(\sigma_2) \left( \frac{i}{2} \partial_N \left( R^p_{a} \omega_{Pb} \right) \tilde{\psi}^b \psi^c \\
&\quad + \frac{i}{2} \partial_N \left( R^p_{a} \omega_{Pb} \right) \tilde{\psi}^b \psi^c + X^P \partial_N R^a \right)(\sigma_2)e(\sigma_1 - \sigma_2). \\
\end{align*}
\]

(3.26)

These supersymmetry transformations reduce to the usual ones upon undoubling [64]. Again,
the section condition solves the problem of the non-local contributions. By requiring that
the generalized metric only depends on \( D \) coordinates and not on their \( \eta_{MN} \)-duals, we are
guaranteed that the coordinates it does depend on always transform locally. The constraints
and $\tilde{\mathcal{H}}$ generate worldsheet reparametrizations of the fields. As for the bosonic case, the section condition is again needed to have a section where the coordinates transform locally. One can pick superconformal gauge, $\lambda = \tilde{\lambda} = 1$, $\xi = \tilde{\xi} = 0$ to see our action as a supersymmetrization of the action in [2], up to the previously noted differences with bosonic zero-modes.

4. Conclusions

In this paper we have shown how the natural $O(D, D)$ invariance of the bosonic string Hamiltonian can be used to construct an $O(D, D)$ invariant string action, which up to a subtlety with zero-modes is that found in the literature [2]. In doing so we interpreted the $2D$ doubled coordinates of double field theory as the $2D$ coordinates of the string’s phase space. There are some potential inequivalencies between our action and the normal string action. Firstly, there is an extra zero-mode term, which may have an effect quantum mechanically and should be studied in future work. Secondly, if the background fields are allowed to depend on all $2D$ coordinates, then the worldsheet diffeomorphism algebra fails to close. This provided a natural occurrence of the section condition as a way to restore worldsheet reparametrization invariance, which we subsequently imposed.

This restricts the background fields to depend only on $D$ coordinates, but does so in an $O(D, D)$ covariant manner. In such a background we can identify $D$ coordinates as spacetime coordinates, and the other $D$ appear only through their $\sigma$-derivatives and can be interpreted as momenta; this reduces our action to the standard ones.

We then applied the same method to the RNS superstring, leading to the supersymmetrization of the action found in [2]. Only the bosonic degrees of freedom needed to be doubled as expected from a phase space point of view. We imposed the section condition on the background fields to ensure worldsheet parametrization invariance.

The natural next step is to quantize this theory. We will have to investigate its equivalence with the standard formulation of string theory; the total derivative term may play an important role [8, 10]. String backgrounds should appear out of massless modes of the string, and there should be some way to generate only backgrounds which obey the section condition. It will also be very interesting to see how T-duality switches between Type IIA and IIB string theory [37, 40, 41].

Another important avenue of future research is to investigate possible relaxations of the section condition, perhaps to Scherk–Schwarz reductions [58–63]. The extension to non-trivial boundary conditions involving $O(D, D)$ twists is also of interest as it would allow one to study the string in non-geometric backgrounds, where one expects to find modified Dirac brackets and non-commutativity of physical coordinates [66]. Finally, the method presented in this paper might also lead to duality invariant actions for the Green–Schwarz string and perhaps even the membrane.

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