FINITARY CODINGS FOR SPATIAL MIXING MARKOV RANDOM FIELDS

YINON SPINKA

ABSTRACT. It has been shown by van den Berg and Steif [12] that the sub-critical and critical
Ising model on $\mathbb{Z}^d$ is a finitary factor of an i.i.d. process (ffiid), whereas the super-critical model is not.
In fact, they showed that the latter is a general phenomenon in that a phase transition presents an
obstruction for being ffiid. The question remained whether this is the only such obstruction. We
make progress on this, showing that certain spatial mixing conditions (notions of weak dependence
on boundary conditions, not to be confused with other notions of mixing in ergodic theory) imply
ffiid. Our main result is that weak spatial mixing implies ffiid with power-law tails for the coding
radius, and that strong spatial mixing implies ffiid with exponential tails for the coding radius.
The weak spatial mixing condition can be relaxed to a condition which is satisfied by some critical
two-dimensional models. Using a result of the author [98], we deduce that strong spatial mixing
also implies ffiid with stretched-exponential tails from a finite-valued i.i.d. process.

We give several applications to models such as the Potts model, proper colorings, the hard-core
model, the Widom–Rowlinson model and the beach model. For instance, for the ferromagnetic
$q$-state Potts model on $\mathbb{Z}^d$ at inverse temperature $\beta$, we show that it is ffiid with exponential tails
if $\beta$ is sufficiently small, it is ffiid if $\beta < \beta_c(q,d)$, it is not ffiid if $\beta > \beta_c(q,d)$ and, when $d = 2$ and
$\beta = \beta_c(q,d)$, it is ffiid if and only if $q \leq 4$.

1. Introduction and Results

Let $(S,S)$ and $(T,T)$ be two measurable spaces, and let $X = (X_v)_{v \in \mathbb{Z}^d}$ and $Y = (Y_v)_{v \in \mathbb{Z}^d}$ be
$(S,S)$-valued and $(T,T)$-valued translation-invariant random fields (i.e., stationary $\mathbb{Z}^d$-processes)
for some $d \geq 1$. A coding from $Y$ to $X$ is a measurable function $\varphi: T^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$, which is translation-equi-variant, i.e., commutes with every translation of $\mathbb{Z}^d$, and which satisfies that $\varphi(Y)$ and $X$ are identical in distribution. Such a coding is also called a factor map or homomorphism from $Y$ to $X$, and when such a coding exists, we say that $X$ is a factor of $Y$.

The coding radius of $\varphi$ at a point $y \in T^{\mathbb{Z}^d}$, denoted by $R_\varphi(y)$, is the minimal integer $r \geq 0$ such
that $\varphi(y') = \varphi(y)$ for all $y' \in T^{\mathbb{Z}^d}$ which coincide with $y$ on the ball of radius $r$ around the origin
in the graph-distance, i.e., $y'_v = y_v$ for all $v \in \mathbb{Z}^d$ such that $\|v\|_1 \leq r$. It may happen that no such $r$
exists, in which case, $R_\varphi(y) = \infty$. Thus, associated to a coding is a random variable $R = R_\varphi(Y)$
which describes the coding radius. A coding is called finitary if $R$ is almost surely finite. When
there exists a finitary coding from $Y$ to $X$, we say that $X$ is a finitary factor of $Y$. When $X$ is a
finitary factor of an i.i.d. (independent and identically distributed) process, we say it is ffiid. We
say that a coding has exponential tails if $\mathbb{P}(R \geq r) \leq C e^{-cr}$ for some $C,c > 0$ and all $r > 0$ and
that it has power-law tails if $\mathbb{P}(R \geq r) \leq C r^{-c}$ instead.

In this paper, we always take $S$ to be a non-empty finite set (and $S$ the power set of $S$) and $X$ to
be a (finite-valued) Markov random field. With the relevant definitions given below, we state our
main result.

Theorem 1.1. Let $d \geq 2$ and let $X$ be a translation-invariant Markov random field on $\mathbb{Z}^d$.

(a) If $X$ satisfies exponential weak spatial mixing, then it is ffiid with power-law tails.

(b) If $X$ satisfies exponential strong spatial mixing, then it is ffiid with exponential tails.
Let \( X = (X_v)_{v \in \mathbb{Z}^d} \) be a random field. We say that \( X \) is a Markov random field if the conditional distribution of its values on any finite set \( V \subset \mathbb{Z}^d \), given its values on \( V^c \), depends only on its values on the boundary \( \partial V \), i.e., if \((X_v)_{v \in V}\) and \((X_v)_{v \in V^c}\) are conditionally independent given \((X_v)_{v \in \partial V}\), where \( \partial V \) is the set of vertices at distance 1 from \( V \). We say that \( X \) is translation-invariant if its distribution is not affected by translations, i.e., if \((X_{v+u})_{v \in V}\) has the same distribution as \( X \) for all \( u \in \mathbb{Z}^d \).

Let \( X \) be a Markov random field. Denote the topological support of \( X \) by

\[
\Omega := \left\{ x \in S^{\mathbb{Z}^d} : \mathbb{P}(X_V = x_V) > 0 \text{ for all finite } V \subset \mathbb{Z}^d \right\},
\]

where we write \( x_V := (x_v)_{v \in V} \) for the restriction of a configuration \( x \in S^{\mathbb{Z}^d} \) to \( V \). Thus, \( \Omega \) is the set of “feasible” configurations. Note that \( \Omega \) is a closed subset of \( S^{\mathbb{Z}^d} \) in the product topology. When \( X \) is translation-invariant, \( \Omega \) is a shift space (a closed invariant subset of \( S^{\mathbb{Z}^d} \)). For \( V \subset \mathbb{Z}^d \) and \( \tau \in \partial V \), denote

\[
\Omega_V := \{ \omega_V : \omega \in \Omega \} \quad \text{and} \quad \Omega_V^\tau := \{ \omega_V : \omega \in \Omega, \; \omega_{\partial V} = \tau \}.
\]

The specification of \( X \) is the collection \( P \) of conditional finite-dimensional distributions, namely,

\[
P := (P_V^\tau)_{V \subset \mathbb{Z}^d \text{ finite}, \; \tau \in \partial V} \quad \text{and} \quad P_V^\tau := \mathbb{P}(X_V \in \cdot | X_{\partial V} = \tau).
\]

Note that, for \( V \subset \mathbb{Z}^d \) finite and \( \tau \in \partial V \), the measure \( P_V^\tau \) is well-defined and supported on \( \Omega_V^\tau \). We write \( P_V^\tau \) for \( P_{\{v\}}^\tau \), and when \( U \subset V \), we write \( P_{V,U}^\tau \) for the marginal of \( P_V^\tau \) on \( U \).

Usually a specification \( \bar{P} \) is defined in and of itself for all boundary conditions arising from some closed subset \( \Omega \) of \( S^{\mathbb{Z}^d} \), and one then says that \( X \) is a Gibbs measure for \( \bar{P} \) if \( \Omega \subset \Omega \) and \( P_V^\tau = P_V^\tau \) for all finite \( V \subset \mathbb{Z}^d \) and \( \tau \in \partial V \) (i.e., \( \bar{P} \) agrees with \( \bar{P} \) for all feasible boundary conditions). This point of view will not be important for us and so we chose only to define the “specification of \( X \”). We also remark that while the specification of a translation-invariant Markov random field is always translation-invariant (in the obvious sense), the converse is not true — there exist non-translation-invariant Markov random fields whose specifications are translation-invariant (e.g., the Dobrushin state for the low-temperature Ising model in three dimensions).

A Markov random field and its specification are said to satisfy weak spatial mixing with rate function \( \rho \) if, for any finite sets \( U \subset V \subset \mathbb{Z}^d \),

\[
\|P_{V,U}^\tau - P_{V,U}^{\tau'}\| \leq |U| \cdot \rho(\text{dist}(U, \partial V)) \quad \text{for any } \tau, \tau' \in \partial V,
\]

where \( \| \cdot \| \) denotes total-variation distance (half the \( L^1 \) norm) and \( \text{dist}(A, B) \) is the minimum graph-distance between a vertex in \( A \) and a vertex in \( B \). We use the term rate function to refer to any monotone function \( \rho : \{1, 2, \ldots \} \to [0, \infty) \) which tends to 0. We remark that the precise definition of weak spatial mixing varies throughout the literature; our definition closely follows the one in [105] (see also [74]). It is easy to verify that if a specification satisfies weak spatial mixing (with any rate function), then there is a unique Markov random field having that specification (i.e., it has a unique Gibbs measure). We say that a specification satisfies exponential weak spatial mixing if it satisfies weak spatial mixing with an exponentially decaying rate function, i.e., if \( \rho(n) \leq Ce^{-cn} \) for some \( C, c > 0 \) and all \( n \geq 1 \).

A Markov random field and its specification are said to satisfy strong spatial mixing with rate function \( \rho \) if, for any finite sets \( U \subset V \subset \mathbb{Z}^d \),

\[
\|P_{V,U}^\tau - P_{V,U}^{\tau'}\| \leq |U| \cdot \rho(\text{dist}(U, \partial V, \Sigma_{\tau,\tau'})) \quad \text{for any } \tau, \tau' \in \partial V,
\]

where \( \Sigma_{\tau,\tau'} \) is the subset of \( \partial V \) on which \( \tau \) and \( \tau' \) disagree. Evidently, (3) implies (2). We say that a specification satisfies exponential strong spatial mixing if it satisfies strong spatial mixing with an exponentially decaying rate function. We mention that the definition of strong spatial mixing may vary in the literature, especially in the class of \((U, V)\) for which (3) is required to hold (for example, \( V \) may be restricted to be a cube). Though we state the strongest form for simplicity (requiring (3)}
to hold for arbitrary \((U,V)\), the proof of part (b) of Theorem 1.1 uses only for a restricted class of \((U,V)\). This is particularly relevant in two-dimensions as it allows us to prove ffiid with exponential tails under a weaker assumption (see Section 1.2).

The assumption of exponential weak spatial mixing in part (a) of Theorem 1.1 can be substantially relaxed. For finite sets \(U \subset V \subset \mathbb{Z}^d\), define

\[
\gamma(V,U) := \sum_{\omega \in \Omega^U} \min_{\tau \in \Omega^{\partial V}} P_{V,U}(\omega).
\]

As we will see in Section 2.2, \(\gamma(V,U)\) is the probability under an optimal coupling that configurations sampled in \(V\) with every possible boundary condition all coincide on \(U\). For \(r > 0\), denote \(\Lambda_r := [-r, r]^d \cap \mathbb{Z}^d\).

**Theorem 1.2.** Let \(X\) be a translation-invariant Markov random field. If

\[
\limsup_{n \to \infty} \gamma(\Lambda_n, \Lambda_{\delta n}) > 0 \quad \text{for some } 0 < \delta < 1,
\]

then \(X\) is ffiid. If

\[
\liminf_{n \to \infty} \gamma(\Lambda_n, \Lambda_{\delta n}) > 0 \quad \text{for some } 0 < \delta < 1,
\]

then \(X\) is ffiid with power-law tails.

We will show in Section 3 that exponential weak spatial mixing implies (6) (in fact, it implies that the limit in (6) tends to 1 for any \(0 < \delta < 1\)) so that Theorem 1.2 implies part (a) of Theorem 1.1. Although Theorem 1.2 is applicable in any dimension, our only real application for it (i.e., when (6) holds, but exponential weak spatial mixing does not hold) is for the critical two-dimensional Potts model (see Corollary 1.5 below).

The theorems presented above are concerned with sufficient conditions for being ffiid. In the other direction, van den Berg and Steif gave a necessary condition [12, Theorem 2.1] by showing that a phase transition presents an obstruction for being ffiid in systems with no hard constraints (i.e., when \(\Omega = S^{2d}\)). That is, they showed that if there exists more than one translation-invariant Markov random field with a given specification in which no hard constraints are present, then neither random field is ffiid (to be precise, the statement of their theorem is that neither field is a finitary factor of a finite-valued i.i.d. process, but the same ideas also show that neither field is ffiid). We extend this to models with hard constraints, under a mild condition on the probabilities of feasible boundary conditions (which always holds for systems with no hard constraints). A random field is **ergodic** if it is translation-invariant and it has no non-trivial translation-invariant events, i.e., every such event occurs with probability 0 or 1.

**Theorem 1.3.** Let \(X\) and \(X'\) be two distinct ergodic Markov random fields with the same specification. Suppose that, for some \(\epsilon > 0\),

\[
\inf_{T \subset \Omega_{\partial \Lambda_n}} P(X_{\partial \Lambda_n} \in T) \geq e^{-o(n^d)} \quad \text{as } n \to \infty.
\]

Then \(X\) is not ffiid.

In our applications, we shall also be concerned with models in which the phase transition is of a different nature, where there is a single translation-invariant Markov random field with the given specification, but multiple periodic ones (as occurs, for instance, in the hard-core model at high fugacity). Thus, we also mention here that **total ergodicity** (i.e., ergodicity with respect to every translation) is a necessary condition for being ffiid. This is due to the fact that an i.i.d. process is totally ergodic and that factors preserve ergodicity with respect to any translation.
Remark 1. The statements of our results mention nothing about the i.i.d. process involved in the ffiid conclusion. However, any result in this paper that shows that a translation-invariant Markov random field $X$ is ffiid, actually yields a stronger property. Specifically, any coding $\varphi$ from $Y$ to $X$ that we construct has the property that $Y_v$ is a sequence of independent finite-valued random variables $(Y_v,n)_{n \in \mathbb{N}}$, and $\varphi$ is finitary in both the $\mathbb{Z}^d$ space (as in the usual definition of finitary coding) and in the $\mathbb{N}$ space, i.e., there exists an almost surely finite $\tilde{R}$ such that $\varphi(Y)_0$ is determined by $(Y_v,n)_{|v| \leq \tilde{R}, n \leq \tilde{R}}$. Moreover, for the results whose conclusions are ffiid with exponential tails, the random variables $Y_{v,n}$ are, in addition, identically distributed, and $\tilde{R}$ is shown to have exponential tails. This allows one to apply a result of the author [98, Proposition 9] to obtain that $X$ is also a finitary factor of a finite-valued i.i.d. process (abbreviated fv-ffiid) with stretched-exponential tails, meaning that the coding radius $R$ satisfies $\mathbb{P}(R \geq r) \leq C e^{-r^c}$ for some $C,c > 0$ and all $r > 0$. In particular, every time we write “ffiid with exponential tails”, we could instead (or in addition) write “fv-ffiid with stretched-exponential tails”. We chose not to include these comments in the statements of our theorems as we preferred to keep these statements short and concise and did not want to lose sight of the main goal, which is showing that certain Markov random fields are ffiid.

Remark 2. All our results extend to $r$-Markov random fields, which are random fields $X$ satisfying that, for any finite $V \subset \mathbb{Z}^d$, $X_V$ and $X_{V^c}$ are conditionally independent given $X_{\partial^r V}$, where $\partial^r V$ is the set of vertices $u \in \mathbb{Z}^d$ having $1 \leq \text{dist}(u,V) \leq r$.

Remark 3. We chose to work with the graph $G = \mathbb{Z}^d$ and the group $\Gamma$ of its translations, though much of the proof works in greater generality. For instance, the proof shows that if one assumes (in addition to the assumptions stated in the theorems) that $X$ is invariant to all automorphisms of $\mathbb{Z}^d$ (not just translations), then the resulting coding commutes with all automorphisms. In a more general setting, $G$ is a transitive locally-finite graph, $\Gamma$ is a group acting transitively on $V(G)$ by automorphisms, $X$ is a $\Gamma$-invariant Markov random field, and a coding is a $\Gamma$-equivariant map. Our results (and proofs) readily extend to the triangular lattice, the hexagonal lattice, the line graph of $\mathbb{Z}^d$ and variants of these. However, as our proof relies on geometric considerations (namely in Section 4.3), it is unclear how wide the class of graphs for which our results hold is. In particular, we do not know whether they hold for all transitive graphs of sub-exponential growth (part [4] of Theorem 1.1 seems to easily extend to this class). We also mention that transitivity is not strictly necessary and the definitions and proofs could be adapted to work for, say, quasi-transitive actions on graphs similar to $\mathbb{Z}^d$ (e.g., when $G = \mathbb{Z}^d$ and $\Gamma$ is the group of parity-preserving translations).

1.1. Applications. Many classical discrete spin systems on $\mathbb{Z}^d$ satisfy spatial mixing properties (for appropriate model parameters). In some of the examples below, we refer to Dobrushin’s uniqueness condition [29], to van den Berg and Maes’s disagreement percolation condition [10] (in which case, we denote by $p_c(\mathbb{Z}^d)$ the critical probability for site-percolation on $\mathbb{Z}^d$) and to Häggström and Steif’s high noise condition [50], and use the fact each implies exponential strong spatial mixing. See Section 1.2 for a discussion about these conditions. We write $Z^\tau_V$ below for a normalizing constant which makes $P^\tau_V$ a probability measure.

1.1.1. Proper colorings. Let $q \geq 3$ and set $S := \{1, \ldots, q\}$. A proper $q$-coloring is a configuration $x \in S^{\mathbb{Z}^d}$ satisfying that $x_u \neq x_v$ for any adjacent vertices $u$ and $v$. Let $\Omega$ be the set of proper $q$-colorings of $\mathbb{Z}^d$ and let the specification be uniform, i.e., $P^\tau_V$ is the uniform distribution on $\Omega^\tau_V$, the set of proper $q$-colorings of $V$ which are compatible with the boundary condition $\tau$. It is well-known (e.g., by Dobrushin’s uniqueness condition or by disagreement percolation) that there is a unique Gibbs measure for proper $q$-colorings when $q > 4d$, and that this measure satisfies exponential strong spatial mixing. In [44], it is shown that exponential strong spatial mixing holds in any triangle-free graph of maximum degree $\Delta \geq 3$ when $q > \alpha \Delta - \gamma$, where

$$\alpha = \text{the solution to } \alpha^x = e \quad \text{and} \quad \gamma := \frac{4\alpha^3 - 6\alpha^2 - 3\alpha + 4}{2(\alpha^2 - 1)}.$$  

(8)
(so $\alpha \approx 1.763$ and $\gamma \approx 0.47$). It is also shown there that, for $\mathbb{Z}^3$, $q \geq 10$ suffices. For $\mathbb{Z}^2$, it known that $q \geq 6$ suffices [143]. On the other hand, for any $q \geq 3$, it is known that there are multiple maximum-entropy Gibbs measures for proper $q$-colorings of $\mathbb{Z}^d$ when the dimension $d$ is sufficiently large ($d \geq C q^{10} \log^3 q$ suffices, with $C$ a universal constant) [85], and that these are not totally ergodic (they are mixtures of 2-periodic extreme Gibbs measures), so that, in particular, they are not ffiid. Thus, we obtain the following corollary.

**Corollary 1.4.** For $d \geq 2$ and $q > 2\alpha d - \gamma$, with $\alpha$ and $\gamma$ given by [8], the unique Gibbs measure for proper $q$-colorings of $\mathbb{Z}^d$ is ffiid with exponential tails. Moreover, this also holds for 6-colorings in two dimensions and 10-colorings in three dimensions. In addition, for any $q \geq 3$, when $d$ is sufficiently large, no maximum-entropy Gibbs measure for proper $q$-colorings of $\mathbb{Z}^d$ is ffiid.

**Remark 4.** The first part of the corollary extends a previous result of the author [98], where ffiid with exponential tails was shown when $q \geq 4d(d + 1)$.

### 1.1.2. The Potts model

Let $q \geq 2$ and set $S := \{1, \ldots, q\}$. Given a parameter $\beta \in \mathbb{R}$, called the inverse temperature, the specification is defined by

$$P^c_V(\omega) := \frac{1}{Z_V} \cdot e^{\beta H_V^c(\omega)},$$

where $H_V^c(\omega) := \sum_{\{u,v\}: E = E(V,\omega)} 1_{\{\omega_u = \omega_v\}}.$

When $\beta > 0$, neighboring spins tend to take the same value and the model is said to be ferromagnetic. When $\beta < 0$, neighboring spins tend to take different values and the model is said to be antiferromagnetic. In particular, the case of $\beta = -\infty$ reduces to that of proper $q$-colorings.

For the ferromagnetic Potts model, it is well-known (see, e.g., [42, Theorem 3.2]) that there exists a critical value $\beta_c = \beta_c(q,d) \in (0,\infty)$ such that there is a unique Gibbs measure at inverse temperature $0 < \beta < \beta_c$ and multiple ergodic Gibbs measures at inverse temperature $\beta > \beta_c$. Using the random-cluster representation of the Potts model and the sharp phase transition in the random-cluster model [33], it follows that at inverse temperature $\beta < \beta_c$, the unique Gibbs measure satisfies exponential weak spatial mixing. In two dimensions, more is known: there is a unique Gibbs measure at the critical $\beta = \beta_c$ if and only if $q \leq 4$ [32, 34], there is a RSW property at criticality when $q \leq 4$ [34], and exponential weak spatial mixing implies exponential strong spatial mixing for squares (see Section 1.2). Putting these results together allows us to show the following.

**Corollary 1.5.** Let $d \geq 2$ and $q \geq 2$. There exists $\beta_0(q,d) \geq 1/2d$ satisfying $\beta_0(q,d) = \beta_c(q,d)$ if either $q = 2$ or $d = 2$, such that the following holds. Let $\mu$ be a Gibbs measure for the ferromagnetic $q$-state Potts model on $\mathbb{Z}^d$ at inverse temperature $\beta > 0$.

- If $\beta < \beta_0(q,d)$ then $\mu$ is ffiid with exponential tails.
- If $\beta < \beta_c(q,d)$ then $\mu$ is ffiid with power-law tails.
- If $\beta > \beta_c(q,d)$ then $\mu$ is not ffiid.
- If $\beta = \beta_c(q,d)$ and $d = 2$, then $\mu$ is ffiid if and only if $q \leq 4$.

**Proof.** Some of the proofs here rely on the well-known connection between the Potts model and the random-cluster model via the Edwards–Sokal coupling, which we now explain (see, e.g., [45] for definitions and properties of the latter model and coupling). Let $V \subset \mathbb{Z}^d$ be finite and let $G$ be the graph with vertex-set $V \cup \partial V$ and edge-set $\{\{u,v\} \in E(\mathbb{Z}^d) \setminus \{u,v\} \cap V \neq \emptyset\}$. We denote by $\phi_{V,p,q}$ the random-cluster measure (with the parameter $p$ chosen in correspondence to $\beta$, namely, $p := 1 - e^{-\beta}$) on $G$ with wired boundary conditions (meaning that clusters intersecting $\partial V$ do not contribute a factor of $q$ to the weight of a configuration). An element $\eta$ in the configuration space of $\phi_{V,p,q}$ is a spanning subgraph of $G$. For a boundary condition $\tau \in \{1, \ldots, q\}^{\partial V}$, let $E^\tau$ be the event that the clusters of $u$ and $v$ in $\eta$ are disjoint for any $u, v \in \partial V$ such that $\tau_u \neq \tau_v$. We may obtain a sample $\omega^\tau$ from $P^c_V$ by first sampling $\eta^\tau$ from $\phi_{V,p,q}(\cdot \mid E^\tau)$ and then independently assigning each cluster of $\eta^\tau$ a uniform value in $\{1, \ldots, q\}$ if it does not intersect $\partial V$, or the deterministic
value \( \tau_v \) if it contains some \( v \in \partial V \). Moreover, since \( E^\tau \) is a decreasing event for any \( \tau \), the FKG property of \( \phi_{V,\rho} \) [45, Theorem 3.8] implies that \( \eta^\tau \) is stochastically dominated by \( \eta^1 \) for any \( \tau \). Thus, by Strassen’s theorem [100], we may couple all \( (\eta^\tau)_\tau \) so that \( \eta^\tau \subset \eta^1 \) almost surely for all \( \tau \). By carrying out the above construction simultaneously for every \( \tau \) (namely, by assigning the same value to a cluster of \( \eta^\tau \) and a cluster of \( \eta^\tau \) whenever the two clusters are identical and do not intersect \( \partial V \)), we may further couple \( (\omega^\tau)_\tau \) on the same probability space as \( (\eta^\tau)_\tau \), so that \( (\omega^\tau)_\tau \) all coincide on the event that \( U \) is not connected to \( \partial V \) in \( \eta^1 \) (i.e., the event that no cluster of \( \eta^1 \) intersects both \( U \) and \( \partial V \)).

We proceed to prove the corollary. The second part will follow from part (a) of Theorem 1.1 once we verify that \( \mu \) satisfies exponential weak spatial mixing. Indeed, by the above coupling, we have that \( \|P_{V,U}^\tau - P_{V,U}^\rho\| \) is bounded above by the probability that \( U \) is connected to \( \partial V \) in \( \eta^1 \). Since \( p < p_c(q,d) \), the probability that any given \( u \in U \) is connected to \( \partial V \) in \( \eta^1 \) decays exponentially fast in \( \text{dist}(U,\partial V) \) [33, Theorem 1.2], thus implying exponential weak spatial mixing.

The third part follows from Theorem 1.3 and the definition of \( \beta_c(q,d) \).

The last part follows using the results in [32,34]. Namely, when \( q > 4 \), there are multiple ergodic Gibbs measures [32, Theorem 1.1] so that \( \mu \) is not ffiid by Theorem 1.3. When \( q \leq 4 \), by the Russo–Seymour–Welsh estimates [34, Theorems 3 and 4] and the FKG property, the probability that \( \Lambda_{n/4} \) is connected to \( \partial \Lambda_n \) in \( \eta^1 \) is at most \( c \) for some constant \( 0 < c < 1 \). From this and the coupling above (with \( U := \Lambda_{n/4} \) and \( V := \Lambda_n \)), it is easy to see that \( \gamma(\Lambda_n,\Lambda_{n/4}) \geq 1 - c \) so that Theorem 1.2 implies that \( \mu \) is ffiid.

It remains to prove the first part of the corollary. By Dobrushin’s uniqueness condition and a direct computation, \( \mu \) is ffiid with exponential tails when \( \beta < 1/2d \). This shows that \( \beta_0(q,d) \) can be chosen to satisfy \( \beta_0(q,d) \geq 1/2d \). The fact that we may take \( \beta_0(q,d) = \beta_c(q,d) \) when \( q = 2 \) follows from the results in [12] for the Ising model. The fact that we may also take \( \beta_0(q,d) = \beta_c(q,d) \) when \( d = 2 \) is due to the fact that, in two dimensions, exponential weak spatial mixing implies exponential strong spatial mixing for squares [6,77] and this is the only assumption needed for the proof of part (b) of Theorem 1.1 (see Section 1.2).

**Remark 5.** The result in the case of the Ising model \( (q = 2) \) was already known [12]. In fact, since the Ising model is known to undergo a continuous phase transition in any dimension \( d \geq 2 \) [2,3,108], the unique Gibbs measure at the critical point \( \beta = \beta_c(2,d) \) is also ffiid. It is still unknown whether this unique Gibbs measure is f-v-ffiid, though it is known that it is not ffiid with finite expected coding volume [12].

**Remark 6.** Using the Edwards–Sokal coupling [36,102], it is a simple consequence of Corollary 1.5 that the (wired/free) random-cluster measure \( \phi_{p,q} \) on \( \mathbb{Z}^d \) with \( q \in \mathbb{N} \) is ffiid when \( p < p_c(q,d) \) (with exponential tails when \( p \) is sufficiently small, \( q = 2 \) or \( d = 2 \)), and, when \( d = 2 \) and \( p = p_c(q,d) \), is ffiid if and only if \( q \leq 4 \). Indeed, under this coupling, given the Potts configuration, the state of the edges in the random-cluster configuration are independent. In two dimensions, it follows by duality that \( \phi_{p,q} \) is ffiid when \( p > p_c(q,d) \). In particular, \( \phi_{p,q} \) is ffiid for all \( p \) when \( d = 2 \) and \( q \leq 4 \).

In a work with Matan Harel [51], we study finitary codings for the random-cluster model and show that \( \phi_{p,q} \) is ffiid with exponential tails whenever \( p < p_c(q,d) \). In particular, it follows that the unique Gibbs measure for the ferromagnetic Potts model is ffiid with exponential tails whenever \( \beta < \beta_c(q,d) \).

The picture for the anti-ferromagnetic Potts model is less complete. Dobrushin’s uniqueness condition implies that there is a unique Gibbs measure for the anti-ferromagnetic \( q \)-state Potts model on \( \mathbb{Z}^d \) at inverse temperature \( \beta < 0 \) whenever \( \beta > -c_q/d \) or \( q > 4d \) [91], where \( c_q > 0 \) is a constant depending only on \( q \), and that this Gibbs measure satisfies exponential strong spatial mixing (see also [109]). Thus, in this regime, the unique Gibbs measure is ffiid with exponential tails. In [43], the two-dimensional model is investigated and various sufficient conditions are given for exponential strong spatial mixing (thus implying ffiid with exponential tails), but we do not list...
those here. In the other direction, for every \( q \geq 3 \) and \( \beta_0 < 0 \), there exists \( d_0(q, \beta_0) \) such that there are multiple ergodic Gibbs measures at inverse temperature \( \beta \leq \beta_0 \) in dimension \( d \geq d_0(q, \beta) \) \cite{84}, and it follows from Theorem 1.3 that they cannot be ffiid.

**Remark 7.** The anti-ferromagnetic Ising model \( (q = 2, \beta < 0) \) is an interesting special case. On the one hand, it is well-known that, on \( \mathbb{Z}^d \) (or any other bipartite graph), it is equivalent to the ferromagnetic Ising model (so that, in particular, \( -\beta_c(2, d) \) is the critical inverse temperature for the existence of multiple Gibbs measures for the anti-ferromagnetic model). This is because “flipping” the spins on one of the sublattices has the effect of changing the sign of \( \beta \) in \( \mathbb{Z} \). It would therefore seem that one could transform a finitary coding for the ferromagnetic Ising model at inverse temperature \( \beta > 0 \) into a finitary coding for the anti-ferromagnetic Ising model at inverse temperature \( -\beta < 0 \). However, the operation of “flipping a sublattice” cannot be carried out in a translation-equivariant manner, so that this does not provide a recipe for constructing a finitary coding for the anti-ferromagnetic model. Nevertheless, the anti-ferromagnetic model does indeed satisfy the analogous properties as the ferromagnetic model: ffiid with exponential tails when \( 0 > \beta > -\beta_c(2, d) \), ffiid when \( \beta = -\beta_c(2, d) \) and not ffiid when \( \beta < -\beta_c(2, d) \). This is due to the fact that the anti-ferromagnetic Ising model is anti-monotone (see Section 1.2; see also Remark 8 where this is explained in more detail for the hard-core model).

1.1.3. The hard-core model. We set \( S := \{0, 1\} \) and identify configurations in \( S^{\mathbb{Z}^d} \) with subsets of \( \mathbb{Z}^d \). Let \( \Omega \) be the collection of independent sets in \( \mathbb{Z}^d \), where an independent set is a subset of vertices that contains no two adjacent vertices. Given a parameter \( \lambda > 0 \), called the fugacity or activity, the specification is defined by

\[
P_V^\tau(\omega) := \frac{1_{\{\omega \cup \tau \text{ is an independent set}\}}}{Z_V^\tau} \cdot \lambda^{|\omega|},
\]

Dobrushin’s uniqueness condition (or, alternatively, the high noise condition) implies that there is a unique Gibbs measure for the hard-core model when \( \lambda < \frac{1}{2d-1} \), and that this Gibbs measure satisfies exponential strong spatial mixing. The disagreement percolation condition broadens this regime to \( \lambda < \frac{\rho_c(\mathbb{Z}^d)}{1 - \rho_c(\mathbb{Z}^d)} \) (see \cite{11}). It has further been shown by Weitz \cite{107} that this continues to hold when \( \lambda < \lambda_c(2d - 1) \), where

\[
\lambda_c(\Delta) := \frac{\Delta}{(\Delta - 1)^{\Delta + 1}}
\]

is the critical fugacity for the hard-core model on the regular tree of degree \( \Delta + 1 \). Moreover, it is shown in \cite{94} that exponential strong spatial mixing further holds when \( \lambda < \lambda_c(\mu_d) \), where \( \mu_d \) is the connective constant of \( \mathbb{Z}^d \) (this improves Weitz’s result since \( \mu_d < 2d - 1 \) and \( \lambda_c(\Delta) \) is decreasing in \( \Delta \)). We refer the reader to \cite{94}, where the known bounds on the connective constants of several lattices are conveniently summarized in a table. For \( \mathbb{Z}^2 \), the authors of \cite{94} also show (by a different means than improving the connective constant of \( \mathbb{Z}^2 \)) that exponential strong spatial mixing holds when \( \lambda \leq 2.538 \). On the other hand, for \( d \geq 2 \) and \( \lambda > Cd^{-1/3} \log^2 d \), with \( C \) a universal constant, it is known that there are multiple Gibbs measures for the hard-core model on \( \mathbb{Z}^d \) at fugacity \( \lambda \) \cite{83}. In fact, in this regime, there are two 2-periodic extreme Gibbs measures, \( \mu^e \) and \( \mu^o \), characterized by unequal densities on the even and odd sublattices (when \( \lambda > Cd^{-1/4} \log^2 d \), it has been shown that these are the only periodic extreme Gibbs measures \cite{84}), and the translation-invariant Gibbs measure \( \frac{1}{2}(\mu^e + \mu^o) \) is not totally ergodic and hence not ffiid. Since the existence of a safe symbol ensures that (7) holds, Theorem 1.3 implies that no Gibbs measure can be ffiid in this regime. Thus, we obtain the following corollary.

**Corollary 1.6.** For \( d \geq 2 \) and \( \lambda < \lambda_c(\mu_d) \), the unique Gibbs measure for the hard-core model on \( \mathbb{Z}^d \) at fugacity \( \lambda \) is ffiid with exponential tails. Moreover, in the case \( d = 2 \), this also holds when \( \lambda \leq \)
2.538. In addition, there exists a universal constant $C$ such that for $d \geq 2$ and $\lambda > Cd^{-1/3} \log^2 d$, no Gibbs measure for the hard-core model on $\mathbb{Z}^d$ at fugacity $\lambda$ is ffiid.

**Remark 8.** At any fugacity, the hard-core model is anti-monotone (see Section 1.2 for the definition; see also [11, 48]). Using that $\mathbb{Z}^d$ is bipartite and that the hard-core model has only nearest-neighbor interactions, allows to take advantage of the anti-monotonicity in a strong manner. Define a partial order $\preceq$ on $\{0, 1\}^{\mathbb{Z}^d}$ in which $\omega \preceq \omega'$ if and only if $\omega_v \leq \omega'_v$ for all even $v$ and $\omega_v \geq \omega'_v$ for all odd $v$ (where $v \in \mathbb{Z}^d$ is said to be *even* or *odd* according to the sum of its coordinates). Let $\omega^e$ and $\omega^o$ be unique minimal and maximal elements of $\{0, 1\}^{\mathbb{Z}^d}$ with respect to this partial order and note that $\omega^e, \omega^o \in \Omega$ (these elements correspond to the independent sets induced by the even and odd sublattices). It follows (see, e.g., [11, Lemma 3.2]) that $P^e_{\{V\}' \{\omega\}}$ and $P^o_{\{V\}' \{\omega\}}$ converge weakly as $V$ increases to $\mathbb{Z}^d$, giving rise to two Gibbs measures, $\mu^e$ and $\mu^o$, which are the smallest and largest Gibbs measures in the sense of stochastic domination (with respect to the partial order $\preceq$), and that there is a unique Gibbs measure if and only if $\mu^e = \mu^o$. Thus, when there is more than one Gibbs measure, Theorem 1.3 implies that no Gibbs measure is ffiid. On the other hand, when there is a unique Gibbs measure, the arguments in [12] (adapted to anti-monotone systems instead of monotone systems) yield that the unique Gibbs measure is ffiid (see Section 1.2). We remark that, since the existence of multiple Gibbs measures in the hard-core model is not known to be monotonic with respect to the fugacity ([17, 47]), we do not know that there exists a critical fugacity for the property of being ffiid.

1.1.4. The Widom–Rowlinson model with $q \geq 2$ types of particles. Set $S := \{0, 1, \ldots, q\}$ and let $\Omega$ be the set of configurations $x \in S^{\mathbb{Z}^d}$ satisfying $x_u x_v = 0$ or $x_u = x_v$ for any adjacent $u$ and $v$. Given a parameter $\lambda > 0$, called the *activity*, the specification is defined by

$$P^\nu_V(\omega) := \frac{1_{\Omega'_V(\omega)}}{Z_V^\nu} \cdot \lambda^{\sum_{v \in V} 1_{(\omega_v \neq 0)}}.$$  

The high noise condition implies that there is a unique Gibbs measure when $q \lambda < \frac{1}{2d-1}$, and that this measure satisfies exponential strong spatial mixing. Dobrushin’s unique condition implies the same whenever $q \lambda < \min\{\frac{1}{2d-1}, \frac{1}{2d-1 - 2d/q}\}$, and the disagreement percolation condition implies this whenever $q \lambda < \frac{p_c(\mathbb{Z}^d)}{1 - p_c(\mathbb{Z}^d)}$. Thus, in these regimes, Theorem 1.1 implies that the unique Gibbs measure is ffiid with exponential tails. On the other hand, when $\lambda$ is sufficiently large as a function of the dimension ($\lambda > Cd^{-1/8} \log d$ suffices [84]), it has been shown that there are multiple ergodic Gibbs measures [20, 67, 74, 90], and since the existence of a safe symbol implies that (7) holds for any two such measures (see Section 1.2), Theorem 1.3 implies that these Gibbs measures are not ffiid. We remark that, in the special case of $q = 2$, the model is monotone, so that whenever there is a unique Gibbs measure it is ffiid (see Section 1.2).

1.1.5. The beach model. Set $S := \{-2, -1, 1, 2\}$ and let $\Omega$ be the set of configurations $x \in S^{\mathbb{Z}^d}$ satisfying that $x_u x_v \geq -1$ for any adjacent $u$ and $v$. Given a parameter $\lambda > 0$, called the *activity*, the specification is defined by

$$P^\nu_V(\omega) := \frac{1_{\Omega'_V(\omega)}}{Z_V^\nu} \cdot \lambda^{\sum_{v \in V} |\omega_v|}.$$

Häggström [46] showed that there exists a critical activity $\lambda_c(d) \in (0, \infty)$ such that there is a unique Gibbs measure for the beach model on $\mathbb{Z}^d$ at activity $\lambda < \lambda_c(d)$ and multiple ergodic Gibbs measures at activity $\lambda > \lambda_c(d)$. Moreover, there exists a universal constant $C$ such that

$$\frac{2}{2d^2 + d - 1} \leq \lambda_c(d) \leq 1 + Cd^{-1/4} \log^2 d.$$  

The lower bound is shown in [46] and the upper bound follows from the results in [84] (the upper bound $\lambda_c(d) \leq 4e \cdot 28^d$ was first shown by Burton and Steif [19]). As we show below, no Gibbs
measure is i.i.d. when $\lambda > \lambda_c(d)$. On the other hand, showing that the unique Gibbs measure is i.i.d. when $\lambda < \lambda_c(d)$ is not straightforward, even for small $\lambda$. This is because neither Dobrushin’s uniqueness condition nor disagreement percolation can be directly applied to this model, as their assumptions do not hold for any $\lambda > 0$. Nevertheless, we can show the following.

**Corollary 1.7.** Let $d \geq 2$ and let $\lambda_c(d)$ be the critical activity for the beach model on $\mathbb{Z}^d$. Let $\lambda > 0$ and let $X$ be a Gibbs measure for the beach model on $\mathbb{Z}^d$ at activity $\lambda$. If $\lambda < 2/(2d^2 + d - 1)$ then $X$ is i.i.d. with exponential tails, if $\lambda < \lambda_c(d)$ then $X$ is i.i.d., and if $\lambda > \lambda_c(d)$ then $X$ is not i.i.d.

**Proof.** The proof of the first part of the corollary is based on a correspondence between the beach model and the so-called site-centered ferromagnetic, which is an Ising-type model (with finite-range ferromagnetic interactions on more than just pairs of sites). We briefly describe some properties of this correspondence; see [46, Section 4] for details and proofs (though the model is defined in [46] only for rational $\lambda$, the proofs there extend to any positive $\lambda$). Define a $\pm 1$-valued random field $Z$ by $Z_v := \text{sign}(X_v)$ so that $X = (Z_v, X_{u|v})_{v \in \mathbb{Z}^d}$. Then $Z$ is a Gibbs measure for the site-centered ferromagnet (one may take the induced specification of $Z$ as the definition of the site-centered ferromagnet). Moreover, conditioned on $Z$, the random field $(|X_v|)_{v \in \mathbb{Z}^d}$ is an i.i.d. process, where, for every $v$, we have that $|X_v| = 1$ almost surely if $Z_u \neq Z_v$ for some $u \in \partial v$, and $|X_v| - 1$ is a Bernoulli random variable with parameter $\lambda/(1 + \lambda)$ otherwise.

When $\lambda < 2/(2d^2 + d - 1)$, it is shown in [46] (see the proof of Lemma 4.9 there) that $Z$ satisfies Dobrushin’s uniqueness condition. Thus, $Z$ satisfies exponential strong spatial mixing so that, by Theorem 1.1, it is i.i.d. with exponential tails (actually, $Z$ is a 2-Markov random field, but Theorem 1.1 extends to this case; see Remark 2). Let $W = (W_v)_{v \in \mathbb{Z}^d}$ be an i.i.d. process, independent of $Z$, with $W_v$ distributed Ber($\lambda/(1 + \lambda)$). Since $X$ is a finitary factor of $(Z, W)$ with (deterministic) coding radius 1, it easily follows that $X$ is i.i.d. with exponential tails.

When $\lambda < \lambda_c(d)$, there is a unique Gibbs measure. Since the beach model is monotone, it follows that $X$ is i.i.d. (see Section 1.2).

When $\lambda > \lambda_c(d)$, there are two distinct ergodic Gibbs measures (characterized by an equal density of positive and negative values) [46,84]. Thus, Theorem 1.3 will imply that $X$ is not i.i.d. once we show that (7) holds for any two ergodic Gibbs measures. In fact, we shall show that the stronger (14) holds. Specifically, we show that for any finite $U \subset \mathbb{Z}^d$ and any $\tau \in \Omega_U$, 

$$\mathbb{P}(X_U = \tau) \geq c^{(2d+1)^2|U|}$$

where $c := \min \min_{\tau \in \Omega_{\partial U}} P^\tau V = \frac{\min\{1, \lambda\}}{2 + \lambda}$.

In fact, this follows from [16, Proposition 4.9] (perhaps with a different constant $c$), since $\Omega$ satisfies a property called topological strong spatial mixing. Nevertheless, we provide a short self-contained proof. Denote $W := U \cup \partial U$ and $V := W \cup \partial W$. It suffices to show that, for any $\xi \in \Omega_{\partial W}$,

$$\mathbb{P}(X_U = \tau \mid X_{\partial V} = \xi) \geq c^{1/2} \geq c^{(2d+1)^2|U|}.$$ 

This will follow if we show that there exists $\omega \in \Omega^S_V$ such that $\omega_U = \tau$, and that, for any $\omega \in \Omega^S_V$, $P^\omega V(\omega) \geq c^{1/V}$. The latter easily follows from (10), since $Z' v \leq (\lambda(2 + \lambda))^{1/V}|V|$. To see the former, first observe that, for any $\eta \in \Omega_{\partial W}$, we have $1 \in \Omega^\eta_V$ (we slightly abuse notation here by identifying $\Omega^\eta_V$ with a subset of $S$) if and only if $\eta_u \neq -2$ for all $u \in \partial v$. A similar property holds for $-1$. Thus, there exists $\tau' \in \Omega_{\partial V}$ such that $\tau'_v \in \{1, -1\}$ for all $v \in \partial U$, and similarly, there exists $\xi' \in \Omega_{\partial W}$ such that $\xi' \in \{1, -1\}$ for all $v \in \partial v$. It follows that the concatenation of $\tau'$ and $\xi'$ is a feasible configuration on $V \cup \partial V$. That is, the element $\omega \in S^V$ defined by $\omega_w := \tau'$ and $\omega_{\partial W} = \xi' \in \Omega^\omega_{\partial W}$ belongs to $\Omega^\omega_V$. \]

**Remark 9.** In light of the correspondence established in [46] between the beach model and the site-centered ferromagnet (which is monotone and enjoys an additional monotonicity property with respect to $\lambda$), it seems plausible that one may be able to obtain a sharp transition for the existence
of a finitary coding (in a similar manner as for the Ising model). Namely, that for $\lambda < \lambda_c(d)$, the unique Gibbs measure is ffiid with exponential tails. We do not pursue this here.

1.1.6. **The monomer-dimer model.** This model consists of a random matching in $\mathbb{Z}^d$, i.e., a collection of disjoint edges (also called dimers), in which a matching $\omega$ in finite-volume is chosen with probability proportional to $\omega$ to the power of the number of dimers. In particular, configurations in this model are elements of $\{0, 1\}^{E(\mathbb{Z}^d)}$ (in fact, this model is precisely the hard-core model on the line graph of $\mathbb{Z}^d$). Strictly speaking, such models do not fall into the setting of our theorems, but the results extend to this case (see Remark 3) (alternatively, we could instead regard configurations as elements of, say, $\{0, 1, \ldots, 2d\}^{\mathbb{Z}^d}$ by encoding the direction of the dimer incident to a vertex, where 0 means no dimer is present). Van den Berg [9] showed that this model satisfies exponential strong spatial mixing for any finite value of $\lambda > 0$. It follows that the monomer-dimer model on $\mathbb{Z}^d$ is ffiid with exponential tails for any $\lambda$.

1.1.7. **Graph homomorphisms.** In [15, 18], it is shown that a large class of nearest-neighbor spin systems satisfy exponential weak spatial mixing. Theorem 1.2 shows that the unique Gibbs measure in such a system is ffiid. We do not describe the general class of these systems (they are certain weighted homomorphisms to dismantlable graphs), but instead give one simple example in which weak spatial mixing is satisfied, but strong spatial mixing is not. Let $\Omega$ be the set of all configurations $x \in \{0, 1, 2, 3\}^{\mathbb{Z}^d}$ such that $\{x_u, x_v\} \notin \{\{0, 2\}, \{1, 3\}, \{2, 2\}\}$ for any adjacent $u$ and $v$, and let $P^x_V$ be the probability measure on $\Omega^V$ in which $P^x_V(\omega)$ is proportional to $\lambda^{\sum_{\omega \in \Omega} 1_{\{\omega_0 = 0\}}}$. It is shown in [15, Proposition 9.2] that this specification satisfies exponential weak spatial mixing when $\lambda$ is sufficiently large (as a function of $d$), but does not satisfy strong spatial mixing (with any rate function) for any $\lambda$. Thus, for any $d \geq 2$, when $\lambda$ is sufficiently large, the unique Gibbs measure is ffiid; but we do not know whether it is ffiid with exponential tails. On the other hand, using the results in [84], one may check that, for any fixed $\lambda \neq 1$, when $d$ is sufficiently large, no Gibbs measure is ffiid; for $\lambda < 1$, the reason being that there is a unique translation-invariant Gibbs measure and this measure is not totally ergodic (it is a mixture of two 2-periodic extreme Gibbs measures, characterized by having an unequal density of $\{0, 2\}$ and $\{1, 3\}$ on the even and odd sublattices), and for $\lambda > 1$, the reason being that there are precisely two ergodic Gibbs measures (characterized by having a very high density of either $\{0, 1\}$ or $\{0, 3\}$) so that Theorem 1.3 applies (although $\Omega$ does not have a safe symbol and does not satisfy topological strong spatial mixing, one may check in a similar manner as in the proof of Corollary 1.7 that 14 holds, so that assumption (7) of the theorem is satisfied).

1.2. Discussion. This work deals with a relation between finitary codings and spatial mixing properties. We give a short survey on each of these.

Finitary codings and finitary isomorphisms (invertible finitary codings, whose inverses are also finitary) have a long history (see, e.g., [93] for a survey), beginning with Meshalkin’s example of finitarily isomorphic i.i.d. processes [79], and culminating in the work of Keane and Smorodinsky [62] who showed that i.i.d. processes of the same entropy are finitarily isomorphic, strengthening the celebrated Ornstein isomorphism theorem [82]. The theory of finitary codings and finitary isomorphisms for one-dimensional processes is rather well established [4, 25, 37, 52, 53, 61, 63, 66, 88, 89, 92, 92, 96]. Finitary codings of random fields in one or more dimensions have also been studied both in particular models such as the Ising model [12, 98], proper colorings [7, 54–56, 104], random walk in random scenery [26, 64, 99] and perfect matchings [72, 97, 103], and in general settings [12, 24, 39, 50, 52, 92]. While any finite-state mixing Markov chain in one dimension is isomorphic to an i.i.d. process (and is a finitary factor of any i.i.d. process with larger entropy [52]), this is not the case for Markov random fields in two or more dimensions [95]. Moreover, there exists a Markov random field which is isomorphic to an i.i.d. process, but which is not finitarily isomorphic to such a process. On the one hand, del Junco showed that there always exists a finitary coding from any ergodic Markov...
random field to any finite-valued i.i.d. process with lower entropy \cite{24}. On the other hand, however, there does exist a Markov random field which is a factor of an i.i.d. process (and is thus isomorphic to an i.i.d. process \cite{60}), but which is not ffiid. The plus-state of the Ising model above the critical inverse temperature is such an example \cite{12}.

Spatial mixing properties have been of interest, among other reasons, because of their implications on uniqueness of Gibbs measures and on efficient approximation of counting problems. They have been studied both in particular models such as the Potts model \cite{43,70,80,109}, proper colorings \cite{13,15,18,35,68,71,73,75,81}, the hard-core (and monomer-dimer) model \cite{9,94,107}, and in general settings \cite{5,6,13,15,18,35,68,71,73,75,81}. In two-dimensional models with no hard constraints, exponential weak spatial mixing implies exponential strong spatial mixing for squares \cite{77} (we elaborate on this below). Exponential strong (and in some cases also weak) spatial mixing is intimately related to optimal temporal mixing of natural local dynamics (such as Glauber dynamics and various block dynamics) \cite{35,74–76,101,105}, and has even been used to establish cutoff, a sharp transition in the convergence to equilibrium \cite{70,71}.

Rapid temporal mixing of local dynamics is closely related to perfect sampling (also called exact simulation), a subject which has received much attention (see, e.g., \cite{28,86}). A simple and useful method, called coupling-from-the-past, was devised by Propp and Wilson \cite{87} for perfectly sampling from the stationary distribution of a one-dimensional Markov chain on a finite state-space (e.g., a stochastic spin system on a finite graph). This method works particularly well for monotone spin systems and even for anti-monotone ones \cite{48}. In the absence of monotonicity, auxiliary processes (these have been given several names, including bounding chains, dominating processes, super-PCA and envelope-PCA) have been used together with coupling-from-the-past to provide efficient perfect sampling algorithms \cite{21,49,50,57,58,65}. An extension of coupling-from-the-past to Markov random fields (nearest-neighbor spin systems on infinite graphs) was a basic ingredient underlying the constructions of finitary codings in \cite{12,50,98}, and is such in the present paper as well. In particular, these constructions also provide an algorithm for perfectly sampling the marginal \( X_v \) of a Markov random field \( X \) (for which the results apply) on any finite set \( V \subset \mathbb{Z}^d \). This is somewhat surprising since, in such random fields, the value \( X_v \) at any given site \( v \) is usually correlated with the value at infinitely many other sites. We mention that perfect sampling of non-Markov random fields (having infinite-range interactions) has also been studied \cite{22,23,39,40}.

For (discrete nearest-neighbor) monotone spin systems, van den Berg and Steif \cite{12} established Theorem \ref{thm:main1} under weaker assumptions (they did not formulate the results in the precise manner that we now describe, but these follow from their results). Let us first explain what we mean by a monotone system. A specification \( P \) is said to be monotone if there exists a linear ordering \( \leq \) on \( S \) such that, for any vertex \( v \in \mathbb{Z}^d \) and any boundary conditions \( \tau, \tau' \in \Omega_{\partial v} \) such that \( \tau \leq \tau' \) pointwise (i.e., \( \tau_u \leq \tau'_u \) for all \( u \in \partial v \)), we have that \( P_v^\tau \) is stochastically dominated by \( P_v^{\tau'} \), i.e.,

\[
P_v^{\tau} (\omega_v \geq s) \leq P_v^{\tau'} (\omega_v \geq s) \quad \text{for all } s \in S.
\]

Many well-known spin systems are monotone, including the Ising model (see Section \ref{sec:ising}), the Widom-Rowlinson model with two types of particles (see Section \ref{sec:wdm}) and the beach model (see Section \ref{sec:beach}). Let us also mention a similar anti-monotonic property. A specification \( P \) is said to be anti-monotone if there exists a linear ordering \( \leq \) on \( S \) such that \( P_v^{\tau} \) is stochastically dominated by \( P_v^{\tau'} \) whenever \( \tau \geq \tau' \) pointwise. The hard-core model (see Section \ref{sec:hardcore}) is an example of an anti-monotone spin system. Van den Berg and Steif showed that when there is a unique Markov random field with a given monotone specification, this random field is ffiid. Furthermore, using a result of Martinelli and Olivieri \cite{75}, they showed that, for monotone systems with no hard constraints, exponential weak spatial mixing implies ffiid with exponential tails (they also showed that this implies fv-ffiid, i.e., finitary factor of a finite-valued i.i.d. process, which was improved to fv-ffiid with stretch-exponential tails in \cite{98}). An inspection of the proofs in \cite{12,75} reveals that these results carry over to anti-monotone systems (for this it is important that \( \mathbb{Z}^d \) is bipartite and that
the spin system has only nearest-neighbor interactions, rather than any finite-range interaction; see \cite{48}). Thus, for monotone and anti-monotone systems with no hard constraints, Theorem 1.1 yields weaker results than those in \cite{12}. On the other hand, when exponential strong spatial mixing is satisfied, though Theorem 1.1 does not yield a new result, we provide a self-contained proof (whereas the result in \cite{12} appeals to \cite{73}). Also, when \cite{6} holds (e.g., for the critical two-dimensional Ising model), the result in \cite{12} gives no information on the coding radius in general (though specifically for the Ising model, one could get such information by appealing to results about the mixing time \cite{69}), whereas our proof yields a more explicit power-law bound on its tails.

Häggström and Steif \cite{50} showed how one could drop the monotonicity assumption in exchange for a high noise condition, which is, with our definitions,

$$\gamma(\{v\},\{v\}) > 1 - \frac{1}{2d}, \tag{11}$$

The quantity \(\gamma(\{v\},\{v\})\) is called the multigamma admissibility, and, as mentioned, is the probability under an optimal coupling of \((P^\tau_v)_{\tau \in \Omega_{\partial V}}\) that the value at \(v\) is the same for all boundary conditions (see Section 2.2). The authors of \cite{50} proved that any translation-invariant Markov random field satisfying (11) is füid with exponential tails (as in the case above, they also show fü-ffiid and this was improved to fü-ffiid with stretched-exponential tails in \cite{98}).

Let us further compare the high noise condition (11) to two well-known weaker conditions existing in the literature, which are sufficient for exponential strong spatial mixing. These are Dobrushin’s uniqueness condition \cite{29} and van den Berg and Maes’s disagreement percolation condition \cite{10} (we formulate these for translation-invariant Markov random fields, though both are more general). As will immediately be evident, the main advantage of these conditions over the above high noise condition is that they involve comparisons between pairs of boundary conditions, whereas (11) involves a simultaneous comparison between all boundary conditions. In particular, it is the case that each of the two conditions is implied by (11). Recall that \(\Sigma_{\tau,\tau'}\) is the set on which \(\tau\) and \(\tau'\) disagree. The uniqueness condition of Dobrushin is

$$\sum_{u \in \partial V} \max_{\tau,\tau' \in \Omega_{\partial V}} \|P^\tau_v - P^{\tau'}_v\| < 1. \tag{12}$$

Since \(\|P^\tau_v - P^{\tau'}_v\| \leq 1 - \gamma(\{v\},\{v\})\) for any \(\tau,\tau' \in \Omega_{\partial V}\), it is immediate that (11) implies (12). It is also known that (12) implies exponential strong spatial mixing (see \cite{106} Theorem 3.2 and \cite{105} Theorem 3.3, Corollary 3.12, Theorem 4.6). One may interpret (12) as a bound on the total influence (of the neighbors a site) on a site. Similar conditions have been studied \cite{30,31,101,105,106}, allowing for other metrics, larger finite-size criteria (rather than a single-site criterion), and total influence of a site (dual to the notion of total influence on a site). The disagreement percolation condition of van den Berg and Maes is, in its simplest form,

$$\|P^\tau_v - P^{\tau'}_v\| < p_c(Z^d) \quad \text{for all } \tau,\tau' \in \Omega_{\partial V}, \tag{13}$$

where \(p_c(Z^d)\) is the critical probability for site-percolation on \(Z^d\). Using that \(p_c(Z^d) > 1/(2d-1)\) for \(d \geq 2\), one sees that (11) implies (13). Moreover, the proof in \cite{10} shows that (13) implies exponential strong spatial mixing. Neither (12) nor (13) follow from one another, and there are situations in which either one provides better bounds than the other (see \cite{10} for a discussion and comparison between the two conditions). As both (12) and (13) imply exponential strong spatial mixing, by Theorem 1.2, they are both sufficient conditions for a translation-invariant Markov random field to be füid with exponential tails.

Strong spatial mixing, by definition, allows to couple any two measures \(P^\tau_v\) and \(P^{\tau'}_v\) so that, with high probability, they agree on the region of \(V\) which is far away from the set \(\Sigma_{\tau,\tau'}\) of disagreement between \(\tau\) and \(\tau'\). Such a coupling is “optimal” for the pair of boundary conditions \(\{\tau,\tau'\}\). The main challenge in the proof of part (b) of Theorem 1.1 is to construct a coupling of the measures
(\(P^\tau_V\))\_\tau, for all possible boundary conditions \(\tau \in \Omega_{\partial V}\), which is simultaneously nearly optimal for any pair of boundary conditions (in fact, we need this for more than just pairs of boundary conditions). It is not immediately clear that this is at all possible (see Remark 10 in Section 2.2; this is somewhat analogous to the difference between stochastic monotonicity and realizable monotonicity [38]). Although versions of the proof of Theorem 4.1 have appeared before [50,98] (the proof is based on a contracting property for an auxiliary “bounding” random field [27,49,50,58,98]), the key ingredient, Theorem 4.2 which establishes the existence of such a coupling, seems to be new.

As mentioned above, in two-dimensional models with no hard constraints (i.e., when the topological support \(\Omega\) is \(S^Z\)) exponential weak spatial mixing implies exponential strong spatial mixing for squares [77] (see also [6]). In fact, for this conclusion, it even suffices to assume exponential weak spatial mixing only for subsets of a sufficiently large square (see the remark just after Theorem 1.1 in [77]). More precisely, for any exponentially decaying rate function \(\rho\), there exists \(L_0\) and an exponentially decaying rate function \(\rho'\) for which, if (2) holds for all \((U,V)\) such that \(V \subset \Lambda_{L_0}\), then (3) holds (with rate function \(\rho'\)) for all \((U,V)\) such that \(V = \Lambda_L\) for some \(L\). An inspection of the proof of part (b) of Theorem 1.1 for the two-dimensional case, shows that we only require exponential strong spatial mixing for squares (see Remark 11). Thus, we obtain that a translation-invariant Markov random field on \(Z^2\) that has no hard constraints and satisfies exponential weak spatial mixing (even just for subsets of a sufficiently large square) is ffiid with exponential tails.

As we have also mentioned, van den Berg and Steif [12] showed that a phase transition presents an obstruction for being ffiid in models with no hard constraints (their proof is based on the so-called blowing-up property, which was established for finitary factors of i.i.d. processes by Marton and Shields [78]). One explanation behind this phenomenon is that the ergodic theorem holds at an exponential rate for i.i.d. processes, and ffiid preserves this property, while a phase transition prohibits an exponential rate in the ergodic theorem, as the configuration may “jump” from one phase to another at a sub-exponential “cost” (such a transition comes with a boundary effect, but no cost in the bulk) (see [12] for a more detailed discussion). Theorem 1.3 generalizes this phenomenon to models with hard constraints (the idea of the proof is based on the explanation above and on some remarks in [12]). Indeed, it is easy check that (7) holds in the absence of hard constraints, since, in this case, \(P^\tau_V(s) > 0\) for any \(\tau \in \Omega_{\partial V}\) and \(s \in S\), implying the stronger property that, for some \(c > 0\),

\[
P(\tau_{\partial \Lambda_n} = \tau) \geq e^{-cn^d-1} \quad \text{for any } n \geq 1 \text{ and any } \tau \in \Omega_{\partial \Lambda_n}.
\] (14)

One may similarly check that (14) holds whenever there is a so-called “safe symbol”, that is, a value \(s_0 \in S\) for which \(P^\tau_V(s_0) > 0\) for any \(\tau \in \Omega_{\partial V}\). In fact, (14) holds in much greater generality, namely, whenever the topological support \(\Omega\) satisfies topological strong spatial mixing [16, Proposition 4.9] (see Section 1.1.5 for a hands-on proof in the case of the beach model). Moreover, (14) also holds in other natural situations in which there is neither a safe symbol nor topological strong spatial mixing (e.g., the example given in Section 1.1.7). Thus, for such models, Theorem 1.3 says that a phase transition (i.e., the existence of more than one ergodic Markov random field with the given specification) is an obstruction for being ffiid.

We conclude with some open problems:

1. Does exponential weak spatial mixing imply ffiid with finite expected coding radius? (our proof gives a coding radius having heavy power-law tails)
2. Does exponential weak spatial mixing imply fv-ffiid? (we showed that it implies ffiid)
3. Does weak spatial mixing with some rate function imply ffiid?
4. Does exponential strong spatial mixing imply fv-ffiid with exponential tails? (we showed that it implies ffiid with exponential tails and also fv-ffiid with stretched-exponential tails)
5. Does exponential strong spatial mixing imply the existence of a finitary coding from every finite-valued i.i.d. process with larger entropy?
6. Does ffiid with exponential tails imply (exponential) weak spatial mixing?
(7) Can assumption (7) be removed from Theorem 1.3 (instead assuming only that $X$ and $X'$ have the same topological support)?

1.3. Notation. We consider $\mathbb{Z}^d$ as a graph in which two vertices $u$ and $v$ are adjacent if $|u - v| = 1$, where $|v| = |v|_1 := |v_1| + \cdots + |v_d|$ denotes the $\ell_1$-norm. For sets $U, V \subset \mathbb{Z}^d$, we write $\text{dist}(U, V) := \min_{u \in U, v \in V} |u - v|$ and $\text{dist}(u, V) := \text{dist}(\{u\}, V)$. We denote $\partial V := \{u \in \mathbb{Z}^d : \text{dist}(u, V) = 1\}$ and $\partial v := \partial\{v\}$. We use $0$ to denote the origin $(0, \ldots, 0) \in \mathbb{Z}^d$.

1.4. Organization. In Section 2, we provide the basic technique of proof used to construct the finitary codings. We then apply this in Section 3 to prove Theorem 1.2 and part (a) of Theorem 1.1. In Section 4, we again apply this technique in order to prove part (b) of Theorem 1.1. Finally, in Section 5, we prove Theorem 1.3.

1.5. Acknowledgments. I would like to thank Raimundo Briceño, Nishant Chandgotia, Hugo Duminil-Copin and Jeff Steif for useful discussions. I am especially grateful to Matan Harel for many discussions throughout the preparation of this paper.

2. Proof technique and overview

There is a common thread lying at the base of the proofs of the two parts of Theorem 1.1 (and also Theorem 1.2), which is the use of block dynamics in conjunction with the method of coupling-from-the-past. We begin by explaining the dynamics in Section 2.1. We then discuss couplings in Section 2.2, an important ingredient for the implementation of coupling-from-the-past. Next, in Section 2.3, we give a formal definition of the dynamics, coupled from all starting configurations. Finally, we show in Section 2.4 how these coupled dynamics can be used together with coupling-from-the-past to provide a recipe for constructing finitary codings, and we indicate in Section 2.5 how this method is later applied in Section 3 and Section 4 to obtain Theorem 1.1 and Theorem 1.2.

2.1. Dynamics. Our proofs rely on a dynamics called the heat-bath block dynamics. On a finite graph, this is the following procedure:

- Choose a block $\Delta$ (a set of vertices) at random according to some distribution.
- Resample the configuration on $\Delta$ according to the conditional distribution given by the current configuration on $\partial\Delta$.

On an infinite graph, we may follow a similar procedure, as long as we restrict ourselves to blocks $\Delta$ which have no infinite connected components. As we are considering Markov random fields, this restriction ensures that the resampling step is well-defined by the specification (the configuration is resampled independently in each component).

As we aim to construct a coding, our block distributions cannot be arbitrary and should satisfy certain properties such as translation invariance. We thus restrict ourselves specialized block distributions which arise from the following procedure:

- Let $\Delta_0 \subset \mathbb{Z}^d$ be finite and contain $0$ – this is the “base” block.
- Declare each vertex of $\mathbb{Z}^d$, independently, to be “active” with probability $p_0$.
- Declare each active vertex to be “chosen” if there are no other active vertices within distance $\text{diam}(\Delta_0) + 1$ from it.
- Let $\Delta$ be the union of the translates of $\Delta_0$ by the chosen vertices.

Note that the definition of “chosen” ensures that every connected component of $\Delta$ is almost surely a translate of $\Delta_0$. It is instructive to note that the block distribution obtained is translation-invariant and is $\text{f\-ffiid}$ (in fact, $\text{fv\-ffiid}$ with bounded coding radius).
2.2. Couplings. Let μ and ν be two probability measures on a common space A, which we assume to be finite for simplicity. A coupling of μ and ν is a probability measure π on A×A which has marginals μ and ν, i.e., π(A×A) = μ(A) and π(A×A) = ν(A) for any A ⊆ A. The total-variation distance between μ and ν is

\[ \|μ − ν\| := \max_{A ⊆ A} |μ(A) − ν(A)| = \frac{1}{2} \sum_{a ∈ A} |μ(a) − ν(a)|. \]

It is well-known that \( \|μ − ν\| \) equals the probability that a sample from μ and ν are different under an optimal coupling. That is, if (Z, Z') is sampled from a coupling of μ and ν, then

\[ \mathbb{P}(Z = Z') \leq 1 − \|μ − ν\|, \]

and there exists a coupling for which equality holds. Noting that

\[ 1 − \|μ − ν\| = \sum_{a ∈ A} \min\{μ(a), ν(a)\}, \]

the above immediately generalizes to any number of distributions μ₁, ..., μₖ on A. Namely, if (Z₁, ..., Zₖ) is sampled from a coupling of μ₁, ..., μₖ, then

\[ \mathbb{P}(Z₁ = Z₂ = \cdots = Zₖ) \leq \sum_{a ∈ A} \min\{μ₁(a), ..., μₖ(a)\}, \tag{15} \]

and there exists a coupling for which equality holds. A coupling for which there is equality in (15) is called an optimal coupling of μ₁, ..., μₖ. When k ≥ 3, we often refer to a coupling of μ₁, ..., μₖ as a grand coupling in order to stress the fact that it is a coupling of multiple measures, and not just of two measures.

We shall be interested in grand couplings of (Pₚₕ)τ∈Ωₜv or (Pₚₕ,U)τ∈Ωₜv for various finite sets U ⊆ V ⊆ ℤᵈ. Note that such a grand coupling simultaneously couples samples of the Markov random field (on V or U) under all boundary conditions, and not just under two boundary conditions. Recall the definition of γ(V, U) from (4) and observe that a grand coupling π of (Pₚₕ,U)τ∈Ωₜv is an optimal coupling if and only if

\[ π(ωτ = ωτ' \text{ for all } τ, τ' ∈ Ωₜv) = γ(V, U). \]

Since any such coupling can be extended to a coupling of (Pₚₕ)Vτ∈Ωₜv, it follows that there is also a grand coupling π of (Pₚₕ)Vτ∈Ωₜv for which

\[ π(ωₚV = ωₚUr \text{ for all } τ, τ' ∈ Ωₜv) = γ(V, U). \tag{16} \]

We say that such a coupling is optimal for the marginals on U. In particular, such couplings exist.

Remark 10. A word of caution regarding the optimality of grand couplings: there typically does not exist an optimal coupling of μ₁, ..., μₖ which induces an optimal coupling of every subcollection μ₁ᵢ, ..., μᵢₖ. As an example, consider the case where μᵢ, for 1 ≤ i ≤ k, is the uniform distribution on \( \{1, 2, ..., k\} \setminus \{i\} \). Observe that any coupling of μ₁, ..., μₖ is optimal since the right-hand side of (15) is zero for this choice of μ₁, ..., μₖ. Moreover, one may check that if (Z₁, ..., Zₖ) is sampled from any coupling of μ₁, ..., μₖ, then \( \mathbb{P}(Zᵢ ≠ Zⱼ) ≥ 2/k \) for some i ≠ j. In particular, since \( \|μᵢ − μⱼ\| = 1/(k−1) \) for i ≠ j, there does not exist a grand coupling for which \( \mathbb{P}(Zᵢ ≠ Zⱼ) = \|Zᵢ − Zⱼ\| \) for all i ≠ j. In fact, the same is true for any 2 ≤ m ≤ k − 1. Namely, for any grand coupling, \( \mathbb{P}(Zᵢ₁ = \cdots = Zᵢₘ) ≤ 1 − m/k \) for some 1 ≤ i₁ < ... < iₘ ≤ k, which is strictly less than \( \sum_{j=1}^{ₘ} \min\{\mathbb{P}(Zᵢᵢ = j), ..., \mathbb{P}(Zᵢᵢₘ = j)\} \) = \( 1 − (m − 1)/(k − 1) \). Thus, no grand coupling induces an optimal coupling of μᵢ₁, ..., μᵢₘ for all 1 ≤ i₁ < ... < iₘ ≤ k.

This example shows that it is not obvious that one can find a grand coupling of (Pₚₕ)Vτ∈Ωₜv which reflects “good” properties the subcollections (Pₚₕ)Vτ∈T, T ⊂ Ωₜv. This will be a primary difficulty in the proof of part (b) of Theorem 1.1 in Section 4, where we will use strong spatial mixing, a
property stated in terms of pairs of boundary conditions, to construct a grand coupling which has a similar property for all pairs (and other \( T \)) simultaneously.

Lastly, we mention that it has recently been shown \cite{8} that there always exists a grand coupling which induces nearly optimal pairwise couplings, but that this is not sufficient for our purposes, as we also need the induced coupling to be nearly optimal for the marginals on arbitrary \( U \subset V \).

2.3. The coupled dynamics. We shall require a simultaneous coupling of the dynamics started from all possible starting states. Let us define these coupled dynamics formally.

The \( n \)-th step of the dynamics depends on three parameters: a number \( p_n \in (0, 1) \) indicating the probability to be active, a base block \( \Delta_n \subset \mathbb{Z}^d \) and a grand coupling \( \pi_n \) of \( (P^\tau_{\Delta_n})_{\tau \in \partial \Delta_n} \). Given these parameters, we let \( (A_n, B_n)_{n \geq 1} \) be a sequence of independent random variables, where, for each \( n \geq 1 \), \( A_n \) and \( B_n \) are independent, \( A_n \) is a Bernoulli random variable with parameter \( p_n \) and \( B_n = (B^\tau_n)_{\tau \in \partial \Delta_n} \) is a sample from \( \pi_n \). We place an independent copy of this sequence at each vertex of \( \mathbb{Z}^d \) so as to obtain an i.i.d. process

\[
Y = (Y_v)_{v \in \mathbb{Z}^d}, \quad \text{where } Y_v := ((A_{v,n}, B_{v,n}))_{n \geq 1}.
\]

This will ultimately be the process from which we will obtain a finitary coding to our target Markov random field \( X \).

We regard \( A_{v,n} \) as indicating whether \( v \) is active at time \( n \). We then let \( A'_{v,n} \) be the indicator of the event that \( v \) is the only active vertex in a ball of radius

\[
r_n := \text{diam}(\Delta_n) + 1
\]

around \( v \) at time \( n \), so that \( A'_{v,n} \) indicates whether \( v \) is chosen at time \( n \). Define

\[
U_{v,n} := \{ u \in v - \Delta_n : A'_{u,n} = 1 \},
\]

and observe that \( |U_{v,n}| \leq 1 \) almost surely. If \( U_{v,n} = \emptyset \) then \( v \) will not be updated at time \( n \), while, if \( U_{v,n} = \{ u \} \), then \( v \) will be updated as part of the update to the block centered at \( u \).

This and the description in Section 2.1 then give rise to the random function

\[
f_n : \Omega \to \Omega
\]

defined by

\[
f_n(\xi)_v := \begin{cases} 
\xi_v & \text{if } U_{v,n} = \emptyset \\
B^\tau_{u,n}(v-u) & \text{if } U_{v,n} = \{ u \} \text{ and } \tau = (\xi_{w+u})_{w \in \partial \Delta_n}.
\end{cases}
\]

It is immediate that \( f_n \) is almost surely well-defined. The function \( f_n \) describes a coupling of one step of the dynamics (at time \( n \)) started from all possible starting states (see \cite{27} for more on this point of view). We remark that these dynamics may be seen as a (non-time-homogeneous) probabilistic cellular automaton (PCA).

Lemma 2.1. For any \( n \geq 1 \), \( f_n \) preserves the distribution of \( X \). More precisely, if \( X \) and \( Y \) are independent, then \( f_n(X) \) has the same distribution as \( X \).

Proof. The proof is standard (and straightforward) and thus omitted. \( \square \)

2.4. Constructing finitary codings using coupling-from-the-past. Let \( \xi \in \Omega \) be the starting configuration. The usual dynamics is loosely described by the evolution:

\[
\xi \to f_1(\xi) \to (f_2 \circ f_1)(\xi) \to \cdots \to (f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)(\xi) \to \cdots.
\]

In other words, one starts with the configuration \( \xi \) at time zero and then, by iteratively composing functions on the left, follows its trajectory forward in time until some positive time \( n \). The method of coupling-from-the-past suggests to reverse this process by starting with the configuration \( \xi \) at some negative time and then following its trajectory (still forward in time) until time zero. Rather
than using negative times, one may instead realize this idea by composing functions on the right. This leads us to consider the evolution:

$$\cdots \leftarrow (f_1 \circ f_2 \circ \cdots \circ f_{n-1} \circ f_n)(\xi) \leftarrow \cdots \leftarrow (f_1 \circ f_2)(\xi) \leftarrow f_1(\xi) \leftarrow \xi.$$  

The difference between the two evolutions is subtle, but significant. In the first process, when \((f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)(\cdot)_v\) becomes a constant (but still random) function, even though the evolution ceases to depend on the starting configuration, it still continues to evolve in time. The main advantage of the second process is that once \((f_1 \circ f_2 \circ \cdots \circ f_{n-1} \circ f_n)(\cdot)_v\) becomes constant, the evolution not only becomes independent of the starting configuration, but also stabilizes.

The following proposition summarizes the usefulness of coupling-from-the-past for our purposes. Denote

$$\omega^n := f_1 \circ f_2 \circ \cdots \circ f_{n-1} \circ f_n$$  

and

$$T_v := \min \{ n \geq 0 : (\omega^n(\cdot))_v \text{ is a constant function} \}. \quad (20)$$

**Proposition 2.2.** Suppose that \(T_v\) is almost surely finite for all \(v\). Then the random field

$$\omega^* = (\omega^{T_v}(\cdot))_{v \in \mathbb{Z}^d}$$

has the same distribution as \(X\).

**Proof.** Let \(\xi\) be sampled from the distribution of \(X\), independently of \(Y\). Fix a finite \(V \subset \mathbb{Z}^d\). Since \(\omega^t(\cdot)_v = \omega^*_v\) for all \(t \geq T_v\), we have \(\omega^t(\xi)_V = \omega^*_V\) for all \(t \geq \max_{v \in V} T_v\). Since, by Lemma 2.1, \(\omega^t(\xi)_V\) has the same distribution as \(X_V\) for all \(t \geq 0\), and since \(\omega^t(\xi)_V = \omega^*_V\) for sufficiently large \(t\), we conclude that \(\omega^*_V\) has the same distribution as \(X_V\). As \(V\) was arbitrary, the proposition follows. \(\square\)

**Proposition 2.3.** Suppose that \(T_v\) is almost surely finite for all \(v\). Then \(\omega^*\) defines a finitary coding from \(Y\) to \(X\) whose coding radius \(R\) satisfies

$$R \leq 2(r_1 + r_2 + \cdots + r_{T_0}).$$

**Proof.** The fact that \(\omega^*\) defines a coding from \(Y\) to \(X\) is a consequence of Proposition 2.2 and the observation that \(f_n\) is translation-equivariant by (19). We proceed to bound the coding radius. By (18) and (19), for any \(n \geq 1\) and \(v \in \mathbb{Z}^d\), \(f_n(\xi)_v\) depends only on \(\xi_{v+\Lambda_{2r_n}}\) and \((Y_{u+u,n})_{u \in \Lambda_{2r_n}}\). Thus, for any \(k \geq 1\), \(f_n(\xi)_{\Lambda_k}\) depends only on \(\xi_{\Lambda_{k+2r_n}}\) and \((Y_{u,n})_{u \in \Lambda_{k+2r_n}}\). Proceeding by induction, we see that

$$\omega^n(\xi)_0 \text{ depends only on } \xi_{\Lambda_{2r_1+\cdots+2r_n}} \text{ and } (Y_{u,i})_{u \in \Lambda_{2r_1+\cdots+2r_i}, 1 \leq i \leq n}. \quad (21)$$

The bound on \(R\) follows. \(\square\)

### 2.5. Outline of proof of theorems.

Proposition 2.3 provides a recipe for constructing a finitary coding from \(Y\) to \(X\). Thus, the method of proof of both parts of Theorem 1.1 (and of Theorem 1.2) consists of showing that a suitable choice of \((p_n, \Delta_n, \pi_n)_{n \geq 1}\) yields a coding whose coding radius has the desired properties (either power-law tails for part (a) of the theorem or exponential tails for part (b)).

For part (a) of the theorem, we shall choose a rapidly growing sequence of blocks \(\Delta_n\). The probabilities \(p_n\) will be chosen to be of order \(|\Delta_n|^{-1}\) so that a vertex has a constant probability of being updated at any step. Moreover, the grand coupling \(\pi_n\) will be chosen so that there is also a constant probability to reach the stopping time \(T_0\) at each step – the existence of which is made possible by the assumption (6). This will give exponential tails for \(T_0\), but since \(r_n\) will grow exponentially fast, it will yield heavy power-law tails for the coding radius.

For part (b) of the theorem, we shall choose \(\Delta_n\) to be a large, but fixed (i.e., independent of \(n\)), box \(\Lambda_{n_0}\). The probabilities \(p_n\) will also be fixed, say to 1/2. The choice of \(n_0\), and of course the assumption of exponential strong spatial mixing, will guarantee the existence of a grand coupling with “contracting” properties, which will in turn ensure that the stopping time \(T_0\) is reached exponentially fast, thus leading to exponential tails for the coding radius.
3. Weak spatial mixing implies fFID with power-law tails

In this section, we prove Theorem 1.2 and part (a) of Theorem 1.1. Let us first explain how Theorem 1.2 implies part (a) of Theorem 1.1. For this, we require an additional notion of spatial mixing. A Markov random field and its specification are said to satisfy ratio weak spatial mixing with rate function $\rho$ if, for any finite sets $U \subset V \subset \mathbb{Z}^d$,

$$\max_{\omega \in \Omega_U} \left( 1 - \frac{P_{\tau,U}^\tau(\omega)}{P_{\tau,V,U}^\tau(\omega)} \right) \leq |U| \cdot \rho(\text{dist}(U, \partial V)) \quad \text{for any } \tau, \tau' \in \Omega_{\partial V}, \tag{22}$$

where the maximum is only over those $\omega$ having $P_{\tau,U}^\tau(\omega) > 0$. It may strike the reader as strange that there are parenthesis in the left-hand side of (22) and not absolute value. However, the two are essentially equivalent, since when the right-hand side of (22) is, say, less than $1/2$, the parenthesis can be replaced with absolute value at the price of introducing a multiplicative factor of 2 in the right-hand side. The difference between the two is then only apparent when the right-hand side is not small. In particular, the definition given in (22) does not convey any information when the right-hand side is greater than 1. For this reason, we find this definition more convenient.

Ratio weak spatial mixing is a priori stronger than weak spatial mixing, as (22) clearly implies (2). Surprisingly, exponential weak spatial mixing is equivalent to exponential ratio weak spatial mixing (see [5, Theorem 3.3] or Section 1.2 below for a short self-contained proof of an analogous statement for strong spatial mixing). Observe that exponential ratio weak spatial mixing easily implies that, for any fixed $0 < \delta < 1$, for sufficiently large $n$, we have

$$\min_{\tau,\tau' \in \Omega_{\Lambda_n}} \min_{\omega \in \Omega_{\Lambda_n}} \frac{P_{\Lambda_n,\Lambda_n}^\tau(\omega)}{P_{\Lambda_n,\Lambda_n}^{\tau'}(\omega)} \geq \delta, \tag{23}$$

where the second minimum is over $\omega$ having $P_{\Lambda_n,\Lambda_n}^{\tau'}(\omega) > 0$ (in fact, the left-hand side in (23) tends to 1 exponentially fast as $n \to \infty$). It is immediate that (23) implies (6).

We now turn to the proof of Theorem 1.2. Let $X$ be a translation-invariant Markov random field satisfying (5). It follows from (5) that there exist $0 < \delta, \epsilon < 1/3$ and a sequence $(\ell_n)_{n \geq 1}$ of positive integers such that, for all $n \geq 1$,

$$\delta \ell_{n+1} > 4d(\ell_1 + \cdots + \ell_n) \tag{24}$$

and

$$\gamma(\Lambda_{\ell_n}, \Lambda_{3d\ell_n}) > \epsilon. \tag{25}$$

In addition, if (6) holds, then $(\ell_n)$ can be chosen to also satisfy

$$\ell_n \leq c\epsilon^n \quad \text{for some } c > 0 \text{ and all } n \geq 1. \tag{26}$$

Set $\Delta_n := \Lambda_{\ell_n}$ and $p_n := \ell_n^{-d}$. Recall (17) and note that $r_n = 2d\ell_n$. Let $\pi_n$ be any grand coupling of $(P_{\Delta_n}^\tau)_{\tau \in \Omega_{\partial \Delta_n}}$ which is optimal for the marginals on $\Lambda_{3d\ell_n}$, so that, by (25) and (16),

$$\pi_n(\mathcal{E}_n) \geq \epsilon, \quad \text{where } \mathcal{E}_n := \left\{ \omega_{\Lambda_{3d\ell_n}}^\tau = \omega_{\Lambda_{3d\ell_n}}^{\tau'} \text{ for all } \tau, \tau' \in \Omega_{\partial \Delta_n} \right\}. \tag{27}$$

In light of Proposition 2.3, Theorem 1.2 boils down to establishing bounds on the tails of $T_v$. These are provided by the following two lemmas. Define

$$T'_v := \min \left\{ n \geq 0 : U_{v,n} = \{u\} \text{ and } B_{u,n} \in \mathcal{E}_n \text{ for some } u \in v + \Lambda_{\delta \ell_n} \right\}.$$

In words, $T'_v$ is the first time at which $v$ is updated by the dynamics in such a way that (i) it falls deep within the inner region of the updated block and (ii) the updated block successfully couples its inner region (thus shielding its inner region from the boundary conditions). See Figure 1.

**Lemma 3.1.** $T'_v$ has exponential tails. In particular, $T'_v$ is almost surely finite.
By translation invariance, it suffices to prove that 

\[ \omega_r \mid T \]

Lemma 3.2.

In light of Proposition 2.3, the first part of Theorem 1.2 follows immediately from the fact that \( T_v \) is almost surely finite, as is shown by the above two lemmas. Similarly, for the second part of the theorem, it suffices to show that \( \omega_T(\xi)_0 \) does not depend on the starting state \( \xi \).

Proof. Fix \( n \geq 1 \). Let \( F_n \) be the event that \( U_{0,n} = \{u\} \) and \( B_{u,n} \in \mathcal{E}_n \) for some \( u \in \Lambda_\delta n \). Then, using that \( \{(A_{u,n},B_{u,n})\}_{u \in \mathbb{Z}^d} \) are i.i.d., that \( A_{u,n} \) and \( B_{u,n} \) are independent, and (18),

\[
\mathbb{P}(F_n) \geq \mathbb{P}(B_{0,n} \in \mathcal{E}_n) \cdot \mathbb{P}\left( \{ u \in \Lambda_\delta n : A_{u,n} = 1 \} = 1 \right) \cdot \mathbb{P}\left( \{ u \in \Lambda_\delta n + r_n : A_{u,n} = 1 \} = \emptyset \right)
\]

\[
= \pi_n(\mathcal{E}_n) \cdot \mathbb{P}\left( \text{Bin}(\lfloor \Lambda_\delta n \rfloor, \ell_n^{-d}) = 1 \right) \cdot \mathbb{P}\left( \text{Bin}(\lfloor \Lambda_\delta n + r_n \rfloor, \ell_n^{-d}) = 0 \right) \geq c > 0,
\]

where we used (27) and the fact that \( \lfloor \Lambda_\delta n \rfloor = \Theta(\ell_n^d) \) and \( \lfloor \Lambda_\delta n + r_n \rfloor = \Theta(\ell_n^d) \). Thus, by translation invariance, \( \mathbb{P}(T_v > n) = \mathbb{P}(T_0 > n) = \mathbb{P}(F_1) \cdots \mathbb{P}(F_v) \leq (1-c)^n \). \( \square \)

Lemma 3.2. \( T_v \leq T'_v \) almost surely.

Proof. By translation invariance, it suffices to prove that \( T_0 \leq T'_0 \) almost surely. By the definition of \( T_0 \), it suffices to show that \( \omega^{T'_0}(\xi)_0 \) does not depend on the starting state \( \xi \).

By (21), we have that \( \omega^{n-1}(\xi)_0 \) depends on \( \xi \) only through \( \xi_{\Lambda_2 \ell_n + \cdots + 2r_n - 1} \). By (24) and since \( r_n = 2d \ell_n \), we have that \( \omega^{n-1}(\xi)_0 \) depends on \( \xi \) only through \( \xi_{\Lambda_\delta n} \). Since \( \omega^{n}(\xi)_0 = \omega^{n-1}(f_n(\xi))_0 \), we conclude that \( \omega^n(\xi)_0 \) depends on \( \xi \) only through \( f_n(\xi)_{\Lambda_\delta n} \).

Suppose now that \( U_{0,n} = \{u\} \) and \( B_{u,n} \in \mathcal{E}_n \) for some \( n \geq 1 \) and \( u \in \Lambda_\delta n \). Observe that \( U_{v,n} = \{u\} \) for all \( v \in u + \Lambda_\delta n \). Thus, by (19), \( f_n(\xi)_{w+u} w \in \Lambda_\delta n = B^\ast_{u,n} \) for some \( \tau \in \Omega_{\Lambda_\delta n} \) which depends on \( \xi \). But since \( B_{u,n} \in \mathcal{E}_n \), \( f_n(\xi)_{w+u} w \in \Lambda_\delta n \) implies that \( (B^\ast_{u,n})_{\Lambda_\delta n} \) does not depend on \( \tau \). Hence, \( (f_n(\xi)_{w+u} w \in \Lambda_\delta n) \) or equivalently, \( f_n(\xi)_{u+\Lambda_\delta n} \), does not depend on \( \xi \). Since \( u \in \Lambda_\delta n \), we have \( \Lambda_\delta n \subset u + \Lambda_\delta n \), and thus, \( f_n(\xi)_{\Lambda_\delta n} \) does not depend on \( \xi \). We conclude that \( \omega^{T'_0}(\xi)_0 \) does not depend on \( \xi \). \( \square \)

Proof of Theorem 1.2. In light of Proposition 2.3, the first part of Theorem 1.2 follows immediately from the fact that \( T_v \) is almost surely finite, as is shown by the above two lemmas. Similarly, for the second part of the theorem, it suffices to show that \( \omega_T(\xi)_0 \) does not depend on the starting state \( \xi \).
tails. Indeed, since the above two lemmas imply that $\mathbb{P}(T_v > t) \leq C e^{-ct}$ for some $C,c > 0$ and all $t > 0$, and since (24) and (26) imply that $r_1 + r_2 + \cdots + r_n \leq e^{c'n}$ for some $c' > 0$ and all $n \geq 1$, we see that
\[
\mathbb{P}(r_1 + r_2 + \cdots + r_{T_v} > t) \leq \mathbb{P}(e^{c'T_v} > t) = \mathbb{P}(T_v > \frac{1}{c'} \log t) \leq C e^{-\frac{c'}{c'} \log t} = C t^{-c/c'}.
\]
\[
\square
\]

4. Strong spatial mixing implies ffiid with exponential tails

In this section, we prove part (b) of Theorem 1.1. Let us begin with a simple observation. For a finite set $V \subset \mathbb{Z}^d$ and a grand coupling $\pi$ of $(P^\tau_U)_{\tau \in \partial V}$, define
\[
\kappa_\pi := \frac{1}{|V|} \sum_{v \in V} \sum_{s \in S} \pi(\omega_v^\tau = s \text{ for all } \tau \in \partial V).
\]

One may regard $\kappa_\pi$ as the average ratio of vertices in $V$ that are successfully coupled (i.e., whose value does not depend on the boundary condition $\tau$) by $\pi$. As such, recalling (16), it is easy to see that every optimal grand coupling $\pi$ satisfies
\[
\kappa_\pi \geq \gamma(V,U) \cdot \frac{|U|}{|V|} \text{ for any } U \subset V.
\]

We aim to prove a statement of the form “if there exists a grand coupling $\pi$ for which the average ratio of vertices in $V$ that are successfully coupled by $\pi$ is sufficiently close to 1, then $X$ is ffiid with exponential tails”. In particular, we will show that if there exists a grand coupling $\pi$ for which
\[
\kappa_\pi > 1 - \frac{1}{|\partial V|},
\]
then $X$ is ffiid with exponential tails (showing that the high noise condition (11) can be somewhat relaxed). Unfortunately, this result is not sufficient in order to deduce part (b) of Theorem 1.1 as, under the assumption of exponential strong spatial mixing, we cannot guarantee that (28) holds for some grand coupling. We thus require a more refined version of this, which we now formulate.

For a boundary condition $\tau \in \partial V$ and a subset $A \subset \partial V$, define
\[
\lambda_\pi(\tau,A) := \frac{1}{|V|} \sum_{v \in V} \pi(\omega_v^\tau \neq \omega_v^\tau' \text{ for some } \tau' \in \partial V \text{ having } \tau_{\partial V \setminus A} = \tau_{\partial V \setminus A}').
\]

Note that $\lambda_\pi(\tau,A) \leq \lambda_\pi(\tau,A')$ for any $\tau \in \partial V$ and $A \subset A' \subset \partial V$, and that
\[
\lambda_\pi(\tau,\partial V) = 1 - \kappa_\pi \text{ for any } \tau \in \partial V.
\]

One may regard the role of $A$ in $\lambda_\pi(\tau,A)$ as indicating which vertices on the boundary $\partial V$ are still uncoupled (i.e., whose values depend on the initial configuration in the coupled dynamics and are thus unknown), so that $\lambda_\pi(\tau,A)$ measures the average ratio of vertices in $V$ that are not successfully coupled by $\pi$ (and hence, whose values are unknown after applying one step of the coupled dynamics), given that we know that the values of the vertices in $\partial V \setminus A$ are given by $\tau$. Thus, $\lambda_\pi(\tau,A)$ allows us to relax (28), or equivalently, $\lambda_\pi(\tau,A) < 1/|\partial V|$, to a condition which takes into account the known/available information. When little or no information is available (i.e., when $A$ is large or even equal to $\partial V$), we shall require a much more modest bound on $\lambda_\pi(\tau,A)$. As more information becomes available (i.e., as $A$ becomes smaller), we shall require a better bound, which will match the original one precisely when $|A| = 1$.

**Theorem 4.1.** Let $X$ be a translation-invariant Markov random field with specification $P$. If for some finite set $V \subset \mathbb{Z}^d$ and some grand coupling $\pi$ of $(P^\tau_U)_{\tau \in \partial V}$,
\[
\lambda_\pi(\tau,A) < \frac{|A|}{|\partial V|} \text{ for all } \tau \in \partial V \text{ and non-empty } A \subset \partial V,
\]
then $X$ is ffiid with exponential tails.
One may regard a coupling satisfying \( \eta_\pi \) as a contraction, since it reduces the density of vertices whose values are unknown. Indeed, \( \eta_\pi \) says that if we know the values of the vertices in \( \partial V \setminus A \), then, on average, after applying one step of the coupled dynamics, the ratio of vertices in \( V \) whose values are unknown is strictly less than it was on the boundary.

The following theorem implies that, under the assumption of exponential strong spatial mixing, one may always find a grand coupling satisfying \( \eta_\pi \), at least when \( V \) is a sufficiently large box.

**Theorem 4.2.** Let \( P \) be the specification of a Markov random field satisfying exponential strong spatial mixing. Then there exists a grand coupling \( \pi_n \) of \( (P_{\Lambda_n}^\tau)_{\tau \in \Omega_{\partial \Lambda_n}} \) such that

\[
\lambda_{\pi_n}(\tau, A) \leq \frac{C \log d}{n^d} \cdot |A| \quad \text{for all } \tau \in \Omega_{\partial \Lambda_n} \text{ and } A \subset \partial \Lambda_n,
\]

where \( C > 0 \) depends on \( P \), but does not depend on \( n \).

Part \( \text{[b]} \) of Theorem 1.1 is an immediate consequence of the two theorems above. The proof of Theorem 4.1 is given in Section 4.1. In Section 4.2, we prove that exponential strong spatial mixing implies exponential ratio strong spatial mixing. We then use this in Section 4.3 to prove Theorem 4.2.

### 4.1. Contraction implies fiid with exponential tails

It is instructive to consider a more general setting than that of Theorem 4.1. Let \( \psi : 2^S \setminus \{\emptyset\} \to [0, \infty) \) be any function satisfying

\[
\psi(A) \leq \psi(B) \quad \text{for any non-empty } A \subset B \subset S
\]

and

\[
\psi(A) = 0 \iff |A| = 1 \quad \text{for any non-empty } A \subset S.
\]

We think of \( \psi \) as a measure of how much uncertainty there is about the value at a vertex. The simplest such \( \psi \) (and the one which is relevant for Theorem 4.1 and Theorem 4.2) is \( \psi_1 \) defined by

\[
\psi_1(A) := 1_{\{|A| > 1\}}.
\]

For finite \( V \subset \mathbb{Z}^d \) and \( \eta \in (2^S)^{\partial V} \), denote

\[
\Omega_{\partial V, \eta} := \{ \tau \in \Omega_{\partial V} : \tau_v \in \eta_v \text{ for all } v \in \partial V \}.
\]

For a grand coupling \( \pi \) of \( (P_V^\tau)_{\tau \in \Omega_{\partial V}} \) and \( \eta \in (2^S)^{\partial V} \) such that \( \Omega_{\partial V, \eta} \neq \emptyset \), define

\[
\lambda_{\pi, \psi}(\eta) := \frac{1}{|V|} \sum_{v \in V} \lambda_{\pi}(\tau, \{ \tau_v \})
\]

(Note that \( \lambda_{\pi, \psi}(\eta) = \lambda_{\pi}(\tau, A) \), where \( A := \{ v \in \partial V : |\eta_v| > 1 \} \) and \( \tau \in \Omega_{\partial V} \) is any element satisfying \( \eta_v = \{ \tau_v \} \) for all \( v \in \partial V \setminus A \). Thus, the following is a generalization of Theorem 4.1.)

**Theorem 4.3.** Let \( X \) be a translation-invariant Markov random field with specification \( P \). If for some finite set \( V \subset \mathbb{Z}^d \), some grand coupling \( \pi \) of \( (P_V^\tau)_{\tau \in \Omega_{\partial V}} \) and some \( \epsilon > 0 \),

\[
\lambda_{\pi, \psi}(\eta) \leq \frac{1 - \epsilon}{|\partial V|} \sum_{v \in \partial V} \psi(\eta_v) \quad \text{for all } \eta \in (2^S)^{\partial V} \text{ such that } \Omega_{\partial V, \eta} \neq \emptyset,
\]

then \( X \) is fiid with exponential tails.

**Proof.** Let \( V, \pi \) and \( \epsilon \) be as in the statement of the theorem. Set \( \Delta_n := V, p_n = 1/2 \) and \( \pi_n := \pi \) for all \( n \geq 1 \). In light of Proposition 2.3, the theorem will follow once we show that \( T_v \) has exponential tails. Observe that \( \omega^n = f_1 \circ \cdots \circ f_n \) has the same distribution as \( f_n \circ \cdots \circ f_1 \). Thus,

\[
\mathbb{P}(T_v > n) = \mathbb{P}(\omega^n(\cdot)_v \text{ is not constant}) = \mathbb{P}((f_n \circ \cdots \circ f_1)(\cdot)_v \text{ is not constant}).
\]

Denote

\[
Z_n(v) := \{ (f_n \circ \cdots \circ f_1)(\xi)_v : \xi \in \Omega \}\quad \text{and} \quad \Psi_n := \psi(Z_n(0)).
\]
By (31), we have
\[ \mathbb{P}(T_0 > n) = \mathbb{P}(|Z_n(0)| \neq 1) = \mathbb{P}(|\Psi_n| > 0). \]

Since \( \psi \) takes finitely many values, it remains to show that \( \mathbb{E}(|\Psi_n|) \) decays exponentially in \( n \).

Let \( \alpha := \mathbb{P}(A_{v,i}^c) \) denote the probability that a vertex is chosen at any given time and let \( \beta := \mathbb{P}(U_{v,n} \neq 0) = \alpha |V| \) denote the probability that it is updated at any given time. Let \( F_n \) denote the \( \sigma \)-algebra generated by \( \{A_{v,i}, B_{v,i}\}_{v \in \mathbb{Z}^d, 1 \leq i \leq n} \). Then
\[
\mathbb{E}(|\Psi_{n+1}| | F_n) = \mathbb{E}\left[ \mathbb{E}[|\Psi_{n+1}| | U_{0,n+1}, F_n] | F_n \right] = (1 - \beta) \mathbb{E}[|\Psi_{n+1}| | U_{0,n+1} = \emptyset, F_n] + \alpha \sum_{u \in V} \mathbb{E}[|\Psi_{n+1}| | U_{0,n+1} = \{u\}, F_n] = (1 - \beta) \mathbb{E}[|\Psi_n|] + \alpha \sum_{u \in V} \mathbb{E}[\psi(\{\omega_u^\tau : \tau \in \{f_n \circ \cdots \circ f_1(\xi)_{w-u} \}_{w \in \partial V} \text{ for some } \xi \in \Omega})].
\]

Thus, by the definition of \( Z_n \) and by (30),
\[
\mathbb{E}[\Psi_{n+1} | F_n] \leq (1 - \beta) \mathbb{E}[\Psi_n] + \alpha \sum_{u \in V} \mathbb{E}[\psi(\{\omega_u^\tau \in \partial V, Z_n(w-u)_{w \in \partial V} \text{ for some } \xi \in \Omega})].
\]

Using that the distribution of \( Z_n \) is translation-invariant, (32) and (33), we obtain
\[
\mathbb{E}[\Psi_{n+1}] = \mathbb{E}[\mathbb{E}[\Psi_{n+1} | F_n]] \leq (1 - \beta) \mathbb{E}[\Psi_n] + \beta \mathbb{E}[\lambda_{\pi,\psi}(Z_n|\partial V)] \leq (1 - \beta) \mathbb{E}[\Psi_n] + \beta \frac{(1 - \epsilon)}{|\partial V|} \sum_{v \in \partial V} \mathbb{E}[\psi(Z_n(v))] = (1 - \epsilon \beta) \mathbb{E}[\Psi_n].
\]

Thus, \( \mathbb{E}[\Psi_n] \) decays exponentially in \( n \).

\[ \square \]

### 4.2. Ratio strong spatial mixing

Here we prove that strong spatial mixing implies ratio strong spatial mixing. The proof is essentially contained in [5] Theorem 3.3, where it is shown (in a slightly more general setting than Markov random fields) that exponential weak spatial mixing implies exponential ratio weak spatial mixing (see also [6] Theorem 3.10 and [74]). We present a short proof with some simplifications.

We say that a Markov random field and its specification satisfy ratio strong spatial mixing with rate function \( \rho \) if, for any finite sets \( U \subset V \subset \mathbb{Z}^d \),
\[
\max_{\omega \in \Omega_U} \left( 1 - \frac{P_{V,U}^\tau(\omega)}{P_{V,U}^{\tau'}(\omega)} \right) \leq |U| \cdot \rho(\text{dist}(U, \Sigma_{\tau,\tau'})) \quad \text{for any } \tau, \tau' \in \Omega_{\partial V},
\]
where the maximum is only over those \( \omega \) having \( P_{V,U}^\tau(\omega) > 0 \), and \( \Sigma_{\tau,\tau'} \) was defined just after (3).

Let \( s_r \) be the number of vertices \( v \in \mathbb{Z}^d \) such that \( |v| = r \), and note that \( s_r = \Theta(r^{d-1}) \) as \( r \to \infty \). Given a rate function \( \rho \), define the rate function \( \rho_* \) by
\[
\rho_*(r) := \begin{cases} \infty & \text{if } r = 1 \\ 3s_{\lfloor r/2 \rfloor} \cdot \sqrt{\rho(\lfloor r/2 \rfloor)} & \text{if } r \geq 2 \end{cases}
\]

Formally, \( \rho_*(1) \) is required to be finite, but this is clearly not important (it could instead be taken to equal, say, \( \max\{1, \rho^*(2)\} \)), and we write \( \infty \) just to emphasize that (34) has no content when \( \text{dist}(U, \Sigma_{\tau,\tau'}) = 1 \). This is not a mere artifact of the proof, and cannot be avoided in general. Indeed, there are models with hard constraints (e.g., proper \( q \)-colorings with large \( q \)), which satisfy strong spatial mixing with a rate function \( \rho \) with arbitrarily small \( \rho(1) \), and for which the left-hand side of (34) equals 1 when \( \text{dist}(U, \Sigma_{\tau,\tau'}) = 1 \), due to the hard constraints.

**Proposition 4.4.** If a Markov random field satisfies strong spatial mixing with rate function \( \rho \), then it satisfies ratio strong spatial mixing with rate function \( \rho_* \).
Figure 2. The set $\Sigma_{\tau,\tau'}$ of vertices in $\partial V$ where the two boundary conditions $\tau$ and $\tau'$ disagree is depicted in dashed red, while the vertices of agreement are depicted in green. Strong spatial mixing allows one to couple samples from $P^\tau_V$ and $P^{\tau'}_V$ so that they agree on $B$ with high probability. Since $B$ is far from both $U$ and $\Sigma_{\tau,\tau'}$, using strong spatial mixing again, one may deduce that the densities of $P^\tau_{V,U}$ and $P^{\tau'}_{V,U}$ are similar as in (34).

Proof. Let $U \subset V \subset \mathbb{Z}^d$ be finite and let $\tau,\tau' \in \Omega_{\partial V}$. Denote $r := \text{dist}(U, \Sigma_{\tau,\tau'})$ and $\epsilon := |U| \cdot \rho_\ast(r)$. Let $\omega$ and $\sigma$ denote samples from $P^\tau_V$ and $P^{\tau'}_V$, respectively. We must show that

$$P(\sigma_U = \eta) \geq (1 - \epsilon) \cdot P(\omega_U = \eta) \quad \text{for all } \eta \in \Omega_U.$$  

To this end, it suffices to show that one may couple $\omega$ and $\sigma$ so that

$$P(\omega_B = \sigma_B | \omega_U) \geq 1 - \epsilon \quad \text{almost surely.} \quad (36)$$

Let us explain how to construct such a coupling. Since (36) holds trivially when $\epsilon \geq 1$, we may assume that $\epsilon < 1$. In particular, we have that $r_\ast := \lfloor r/2 \rfloor \geq 1$. Denote

$$B := \{v \in V : \text{dist}(v,U) = r_\ast\} \quad \text{and} \quad W := \{v \in V : \text{dist}(v,U) < r_\ast\}.$$  

Observe that

$$U \subset W, \quad \partial W \subset B \cup (\partial V \setminus \Sigma_{\tau,\tau'}), \quad \text{dist}(B, U \cup \Sigma_{\tau,\tau'}) \geq r_\ast. \quad (37)$$

We first sample $(\omega_B, \sigma_B)$ from an optimal coupling of $P^\tau_{V,B}$ and $P^{\tau'}_{V,B}$. Next, we sample $(\omega_W, \sigma_W)$ from an optimal coupling of $P^\tau_V(\omega_W \in \cdot | \omega_B)$ and $P^{\tau'}_V(\sigma_W \in \cdot | \omega_B')$. Finally, we independently sample $\omega_{V \setminus (B \cup W)}$ from $P^\tau_V(\omega_{V \setminus (B \cup W)} \in \cdot | \omega_B)$ and $\sigma_{V \setminus (B \cup W)}$ from $P^{\tau'}_V(\sigma_{V \setminus (B \cup W)} \in \cdot | \sigma_B)$. It is straightforward to verify that this indeed yields a sample $(\omega, \sigma)$ whose marginals are $P^\tau_V$ and $P^{\tau'}_V$.

It remains to check that this coupling satisfies (36). Note that, by construction and (37),

$$\{\omega_B = \sigma_B\} \subset \{\omega_W = \sigma_W\} \subset \{\omega_U = \sigma_U\}.$$  

This is the only use we make of the optimality of the coupling chosen in the second step of the construction. Thus, it suffices to show that

$$P(\omega_B = \sigma_B | \omega_U) \geq 1 - \epsilon \quad \text{almost surely.} \quad (38)$$

Denote

$$G := P(\omega_B \neq \sigma_B | \omega_B).$$
Denote \( s := \sqrt{\rho(r_s)} \) and note that \( |s| \leq \epsilon/3 \), since \(|B| \leq s \leq |U|\). By Markov’s inequality, the first step in the construction of the coupling, and strong spatial mixing,

\[
\mathbb{P}(G \geq s) \leq \frac{\mathbb{E}[G]}{s} = \frac{\mathbb{P}(\omega_B \neq \sigma_B)}{s} = \frac{\|P_{V,B} - P'_{V,B}\|}{s} \leq \frac{|B| \cdot \rho(r_s)}{s} = s|B| \leq \epsilon/3. \tag{39}
\]

Using again strong spatial mixing, and since \( s \leq 1 \), we similarly obtain, almost surely,

\[
|\mathbb{P}(G \geq s | \omega_U) - \mathbb{P}(G \geq s)| \leq |\mathbb{P}(\omega_B \in \cdot | \omega_U) - \mathbb{P}(\omega_B \in \cdot)| \leq s^2|B| \leq \epsilon/3. \tag{40}
\]

Note that, by construction, \( \omega_U \) and \( \sigma_B \) are conditionally independent given \( \omega_B \). Indeed, this follows from the fact that, given \( (\omega_B, \sigma_B) \), the distribution of \( \omega_W \) is almost surely \( P_{V,B}^{\tau} (\omega_W \in \cdot | \omega_B) \). Thus, almost surely,

\[
\mathbb{P}(G \leq s, \omega_B \neq \sigma_B | \omega_U) \leq \mathbb{P}(\omega_B \neq \sigma_B | G \leq s, \omega_U) = \mathbb{E}[G | G \leq s, \omega_U] \leq s \leq \epsilon/3.
\]

Putting this together with (39) and (40) yields (38). \( \square \)

### 4.3. Existence of a contracting grand coupling.

Theorem 4.2 follows by taking \( r \) to be a large multiple of \( \log n \) in the next proposition. Recall the definition of \( \rho_* \) from (35).

**Proposition 4.5.** Let \( P \) be a specification satisfying strong spatial mixing with rate function \( \rho \). Then, for every \( 1 \leq r \leq n/4 \), there exists a grand coupling \( \pi_n \) of \( (P_{\Lambda_n}^\tau)_{\tau \in \Omega_{\Lambda_n}} \) such that

\[
\lambda_{\pi_n}(\tau, A) \leq \rho_*(r) + C_d r^d \cdot \frac{|A|}{|\Lambda_n|} \quad \text{for all } \tau \in \Omega_{\partial \Lambda_n} \text{ and } A \subset \partial \Lambda_n, \tag{41}
\]

where \( C_d \) is a positive constant depending only on the dimension.

**Proof.** Fix \( 1 \leq r < s \leq 4r \leq n \) and denote \( V := \Lambda_s \) and \( U := \Lambda_{s-r} \). In order to ease notation, we make the convention that whenever \( \tau \) is used as an index with an unspecified domain, it runs over all elements in \( \Omega_{\partial V} \).

Let \( L_1, L_2, \ldots, L_m \subset V \setminus U \) be disjoint (to be suitably chosen later). The main idea in the construction of \( \pi_n \) is to couple the configurations, in order, first on \( U \), then on \( L_1 \), then on \( L_2 \), and so on, and finally on each connected component of \( V \setminus (U \cup L_1 \cup \cdots \cup L_m) \). See Figure 3.

Set \( U_0 := U \) and \( V_0 := V \), and for \( 1 \leq i \leq m \), denote

\[
U_i := U \cup L_1 \cup \cdots \cup L_i \quad \text{and} \quad V_i := V \setminus U_i.
\]

For \( 1 \leq i \leq m \), denote

\[
T_i := \{v \in \partial V_{i-1} : \text{dist}(v, L_i) < s\}
\]

and, for each \( \eta \in \Omega_{T_i} \), denote

\[
\Omega_{\partial V_{i-1}, \eta} := \{ \tau \in \Omega_{\partial V_{i-1}} : \tau T_i = \eta\}.
\]

Note that, for every \( 1 \leq i \leq m \),

\[
\{\Omega_{\partial V_{i-1}, \eta} \}_{\eta \in \Omega_{T_i}} \text{ is a partition of } \Omega_{\partial V_{i-1}}.
\]

Let \( Q \) be an optimal coupling of \( (P^\tau_{V,U})_\tau \). For each \( 1 \leq i \leq m \) and \( \eta \in \Omega_{T_i} \), let \( Q^\eta_i \) be an optimal coupling of \( (P^\tau_{V,L_i})_\tau \in \Omega_{\partial V_{i-1}, \eta} \) with marginals \( (Q^\eta_i)_\tau \in \Omega_{\partial V_{i-1}, \eta} \), e.g., the product coupling. Note that, by ratio strong (or weak) spatial mixing (which follows from the assumption and Proposition 4.4),

\[
Q\left(\omega_{U}^\tau = \omega_{U}^{\tau'} \text{ for all } \tau, \tau' \right) \geq 1 - |U| \cdot \rho_*(r), \tag{42}
\]

and that, by ratio strong spatial mixing, for any \( 1 \leq i \leq m \) and \( \eta \in \Omega_{T_i} \),

\[
Q_i\left(\omega_{L_i}^\tau = \omega_{L_i}^{\tau'} \text{ for all } \tau, \tau' \in \Omega_{\partial V_{i-1}, \eta}\right) \geq 1 - |L_i| \cdot \rho_*(s). \tag{43}
\]

The grand coupling \( \pi_n \) is constructed as follows. First, we sample \( (\omega_{U}^\tau)_{\tau} \) from \( Q \). Next, we sample \( (\omega_{L_i}^\tau)_{\tau} \), for each \( i = 1, \ldots, m \), one after another, as follows. Let \( i \in \{1, \ldots, m\} \) and suppose
we have already sampled \((\omega^\tau_{L_i})_\tau\). We then sample \((\omega^\tau_{L_i})_\tau\) by setting \(\omega^\tau_{L_i} := \sigma^{\omega^\tau_{L_i}}\), for all \(\tau\), where \((\sigma^\tau)_\tau\) is sampled from \(Q_\tau\), independently of anything else previously sampled. Thus, at the end of this process, we have sampled \((\omega^\tau_{L_i})_\tau\). To complete the construction, for each connected component \(C\) of \(V_m\), we independently sample \((\omega^\tau_C)_\tau\) from an optimal coupling of \((P^\tau_{V,C}(. \mid \omega^\tau_{V_m}))_\tau\). Define \(\pi_n\) to be the distribution of the sample \((\omega^\tau)_\tau\) thus obtained. It is straightforward to check that \(\pi_n\) is a grand coupling of \((P^\tau_{V})_\tau\).

Let us observe some simple properties that follow from this construction. First, by \([42]\),

\[
\pi_n(E) \geq 1 - |U| \cdot \rho_s(r), \quad \text{where } E := \left\{ \omega^\tau_U = \omega^\tau_{U'} \text{ for all } \tau, \tau' \right\}.
\] (44)
For $\mathcal{T} \subset \partial \Omega_V$, let $\Sigma_\mathcal{T}$ be the subset of $\partial \Omega$ on which some boundary conditions in $\mathcal{T}$ do not agree, and let $E^T_i$ be the event that all boundary conditions in $\mathcal{T}$ are coupled on $L_i$, i.e.,

$$\Sigma_\mathcal{T} := \{ v \in \partial \Omega : \tau_v \neq \tau'_v \text{ for some } \tau, \tau' \in \mathcal{T} \}, \quad E^T_i := \{ \omega^*_{L_i} = \omega^*_{L_i} \text{ for all } \tau, \tau' \in \mathcal{T} \}.$$ 

By [43], for any $1 \leq i \leq m$ and $\mathcal{T} \subset \partial \Omega_V$ such that $\text{dist}(L_i, \Sigma_\mathcal{T}) \geq s$,

$$\pi_n\left(E^T_i \mid (\omega^*_{U_i})_{\tau} \right) \geq 1 - |L_i| \cdot \rho_n(s) \quad \text{almost surely on the event } E \cap \bigcap_{1 \leq i' < i} E^T_{i'}.$$ 

(45)

In addition, for any connected component $C$ of $V_m$, 

$$\omega^*_{\tau} = \omega^*_{\tau'} \quad \text{almost surely for any } \tau, \tau' \text{ such that } \tau_{\partial C} = \tau'_{\partial C}.$$ 

Equations (44), (45) and (46) summarize the properties we need from the construction of $\pi_n$. We claim that, for a suitable choice of $L_1, \ldots, L_m$, $\pi_n$ satisfies (41). In fact, there are many possible such choices, and any choice with, say, property (47) below will suffice.

For $v \in \partial \Omega$, denote 

$$I^v := \{ 1 \leq i \leq m : \text{dist}(v, L_i) < s \}.$$ 

Say that $i$ is a child of $i'$ if $i > i'$ and $\text{dist}(L_i, L_{i'}) < s$. Write $i' \rightarrow i$ if $i$ is a descendant (child, grandchild, etc.) of $i'$ or if $i = i'$. Denote 

$$\mathcal{T}^v := \{ 1 \leq i \leq m : i' \rightarrow i \text{ for some } i' \in \mathcal{T}^v \}.$$ 

We think of $I^v$ as indicating those $L_i$ which are directly affected by $v$, and $\mathcal{T}^v \setminus I^v$ as those which are indirectly affected (as the effect of $v$ may propagate). Denote 

$$\mathcal{L}^v := \bigcup_{i \in \mathcal{T}^v} L_i.$$ 

Let $C^v$ be the union of the connected components of $V_m$ which are adjacent to $\mathcal{L}^v \cup \{v\}$. Then we shall show that the following suffices to ensure that $\pi_n$ satisfies (41):

$$|C^v \cup \mathcal{L}^v| \leq C d^{-d} \quad \text{for all } v \in \partial \Omega.$$ 

(47)

Intuitively, this property implies that the total negative effect caused by a disagreement at a single vertex $v \in \partial \Omega$ cannot be too large, since (47) provides a bound on the number of vertices affected (via the propagation of failed coupling events) by any such disagreement.

A choice of $L_1, \ldots, L_m$ which satisfies this property can be obtained as follows (we give an informal description; see also Figure 3). In each of the $2d$ directions, there is a “thick $(d - 1)$-dimensional slab” (of side-length $2(n - r)$ and of thickness $r$) between the faces of $U$ and $V$ in that direction. We divide each such slab into thick $(d - 1)$-dimensional cubes of side-length between $s$ and $10s$ (e.g., $[0, r] \times [0, s]^{d-1}$). We then continue to slabs of one less dimension: there are $\binom{2d}{2}$ “thick $(d - 2)$-dimensional slabs” (of side-length $2(n - r)$ in $d - 2$ of the dimensions and thickness $r$ in the two other dimensions), each of which is divided into thick $(d - 2)$-dimensional cubes of side-length between $s$ and $10s$ (e.g., $[0, r]^2 \times [0, s]^{d-2}$). We continue this process to lower-dimensional slabs in a similar manner. At the end of the process, we obtain a partition of $V \setminus \partial \Omega$ into cubes of the form $[0, r]^k \times [0, s]^{d-k}$ with $1 \leq k \leq d$ and $s \leq s' \leq 10s$ (up to translations and coordinate permutations).

The $(d - 1)$-dimensional faces of these cubes (where, for example, $[0, r]^k \times [0, s]^{d-k-1} \times \{0\}$ is such a face of $[0, r]^k \times [0, s]^{d-k}$) will be the $\{L_i\}_i$ (except for those faces which are contained in $U$ or $\partial \Omega$, that is, we only consider faces which “connect” between $U$ and $\partial \Omega$), chosen in a suitable order. Such an order is, for instance, obtained as follows. Associate to each face the index $z \in \{1, \ldots, 2d\}$ of the zero-length coordinate and the set $J \subset \{1, \ldots, 2d\}$ of coordinates in which the face has side-length at least $s$ (e.g., $[0, s]^k \times [0, r]^{d-k-1} \times \{0\}$ has $z = 2d$ and $J = \{1, \ldots, k\}$). Note that the adjacency relation induced on the set of faces with a given $J$ and $z$ is that of (disjoint copies of) a $|J|$-dimensional cube. We let $p = (p_1, \ldots, p_d)$ denote the face’s position in this $|J|$-dimensional
grid (e.g., if \([0, s]^2 \times [0, r] \times \{0\}\) has \(p = (0, 0)\) then \([s, 2s] \times [5s, 6s] \times [0, r] \times \{0\}\) has \(p = (1, 5)\)). The faces are then ordered first in decreasing order according to \([J]\) (i.e., in increasing order according to the number of sides of length \(r\)), then in (say, increasing) order according to \(z\) and then in lexicographical order according to \((p_1 \mod 2, \ldots, p_{|J|} \mod 2)\). We complete this partial order to a linear order in an arbitrary manner. This choice of ordering ensures that \(i' \to i\) implies that the “grid-distance” between the faces \(L_i\) and \(L_{i'}\) (i.e., the \(l_i\)-distance between the \(p_i\)'s corresponding to the two faces) is at most \(C_d'\). In particular, since it is clear that \(|I'\| \leq C_d''\), we have \(|T'| \leq C_d'''\). Since the faces have size at most \(r(10s)^{d-2}\) and the connected components of \(V_m\) have size at most \(r(10s)^{d-1}\), we have \(|L^v \cup C^v| \leq C_d d^d\), as required.

The above construction is particularly simple (even somewhat degenerate) in two dimensions (see Remark 11 below). However, writing it down precisely for \(d \geq 3\) is a bit cumbersome and we omit the details.

We proceed to show that (47) implies that \(\tau_n\) satisfies (41). Towards showing this, let \((\omega^r)_\tau\) be sampled from \(\pi_n\), let \(\tau \in \Omega_{\partial V}\) and \(A \subset \partial V\), and define

\[
\mathcal{T} := \{\tau' \in \Omega_{\partial V} : \tau_{\partial V \setminus A} = \tau'_{\partial V \setminus A}\} \quad \text{and} \quad \mathcal{W} := \left\{v \in V : \omega^r_v = \omega^r_{v'} \text{ for all } \tau' \in \mathcal{T}\right\}.
\]

We must show that \(\mathbb{E}[\mathcal{W}] \geq |V| - |A| \cdot C_d d^d - |V| \cdot \rho_s(r)\).

This will follow if we show that, for some \(W \subset V\),

\[
|\mathcal{W}| \geq |V| - |A| \cdot C_d d^d \quad \text{and} \quad \mathbb{P}(W \subset \mathcal{W}) \geq 1 - |V| \cdot \rho_s(r).
\]

Denote \(I^A := \bigcup_{v \in A} I^v\), \(\mathcal{I}^A := \bigcup_{v \in A} \mathcal{I}^v\), \(\mathcal{L}^A := \bigcup_{v \in A} \mathcal{L}^v\) and \(\mathcal{C}^A := \bigcup_{v \in A} \mathcal{C}^v\). Let \(C_1, \ldots, C_k\) be the connected components of \(V_m\) which are not adjacent to \(\mathcal{L}^A \cup A\). Note that

\[
\bigcup_{j=1}^k C_j = V_m \setminus \mathcal{C}^A.
\]

Set

\[W := V \setminus (\mathcal{L}^A \cup \mathcal{C}^A)\]

and note that (47) implies the first part of (48). By (46), we have

\[E \cap \bigcap_{i \in I} E^r_i \subset \{W \subset \mathcal{W}\}, \quad \text{where } I := \{1, \ldots, m\} \setminus \mathcal{I}^A.\]

Since \(\mathbb{P}(E^c) \leq |U| \cdot \rho_s(r)\) by (44), the second part of (48) will follow if we show that, for any \(i \in I\),

\[
\mathbb{P}\left((E^r_i)^c \mid E \text{ occurs and } E^r_i \text{ occurs for all } \tau' \in I \text{ with } \tau' < i\right) \leq |L_i| \cdot \rho_s(s).
\]

Indeed, this follows from (45), since \(\text{dist}(L_i, A) \geq s\) and \(\Sigma_{\mathcal{I}^A} \subset A\), and since, if \(1 \leq \tau' < i\) satisfies \(\text{dist}(L_i, L_{\tau'}) < s\), then \(\tau' \in I\) (otherwise \(\tau'' \to i' \to i\) for some \(\tau'' \in I^A\), implying that \(i \in \mathcal{I}^A\)).

Remark 11. In the two-dimensional case, the proof of Proposition 4.5 is slightly simplified due to the fact that the constructed collection \(\{L_i\}_i\) satisfies \(\text{dist}(L_i, L_{\tau'}) \geq s\) whenever \(i \neq i'\), except when \(L_i\) and \(L_{\tau'}\) are both at one of the same four corners (see Figure 3a). In fact, one may just as well remove the eight \(L_i\) which are at the corners so that this holds for all \(i \neq i'\). Thus, regardless of the order in which \((L_i)_i\) is chosen, we have \(i' \to i\) if and only if \(i' = i\), and the event on the right-hand side of (45) is just \(E\). On the other hand, although these are convenient simplifications, they are not of significant importance. Instead, what is important in two dimensions is that in order to deduce (13), we only need exponential strong spatial mixing for a restricted class of sets \((U, V)\) and boundary conditions \((\tau, \tau')\), that is, we only need to assume that (3) holds for certain \((U, V)\) and \((\tau, \tau')\). Indeed, the proof of Proposition 4.4 reveals that if one only assumes exponential strong spatial mixing for \((U, V) = \{(0) \times S_r, S_r \times S_r\}\) and \((\tau, \tau')\) for which \(\Sigma_{\tau, \tau'} \subset \{-r - 1, r + 1\} \times \{s_r\}\), where \(S_r := \{-r, \ldots, r\}\), then one obtains as a consequence exponential ratio strong spatial mixing.
exponential rate for \( W \).

Note that for \((\tau, \tau')\) for which \( \Sigma_{\tau, \tau'} \subset \{-2r-2, 2r+2\} \times S_r \) (see Figure 2b). Finally, this is all that is needed to obtain (43), so that the conclusion of Proposition 4.5 remains true under this weaker hypothesis.

5. AN OBSTRUCTION FOR FFID

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. Let \( X \) and \( X' \) be as in the theorem. Denote the distributions of \( X \) and \( X' \) by \( \mu \) and \( \nu \). We write \( \omega \) for a generic random element of \( \Omega \). Let \( A \) be a cylinder event satisfying \( \mu(A) < \nu(A) \), which is determined by the values of \( \omega \) on a box \( \Lambda_k \). For \( n > k \), denote

\[
Z_n := \frac{1}{|\Lambda_{n-k}|} \sum_{\omega \in \Lambda_{n-k}} 1_A((\omega_{u+v})_{u \in \mathbb{Z}^d}).
\]

Note that \( Z_n \) is measurable with respect to \( \omega_{\Lambda_n} \). By the pointwise ergodic theorem,

\[
\mu(Z_n \to \mu(A) \text{ as } n \to \infty) = 1 \quad \text{and} \quad \nu(Z_n \to \nu(A) \text{ as } n \to \infty) = 1.
\]  

(49)

Assume towards a contradiction that \( X \) is ffid. Then the random field \( W = (W_v)_{v \in \mathbb{Z}^d} \) defined by \( W_v := 1_A((X_{u+v})_{u \in \mathbb{Z}^d}) \) is also ffid, so that the convergence in the ergodic theorem occurs at an exponential rate for \( W \) [14] (see also [12, 78]). Hence, denoting \( a := \frac{1}{2}(\mu(A) + \nu(A)) \),

\[
\mu(Z_n \geq a) \leq Ce^{-2cn^d} \quad \text{for some } C, c > 0 \text{ and for all } n > k.
\]

By Markov’s inequality,

\[
\mathbb{P}(X_{\partial \Lambda_n} \in T_n) \leq Ce^{-cn^d}, \quad \text{where } T_n := \left\{ \tau \in \Omega_{\partial \Lambda_n} : P_{\Lambda_n}^\tau(Z_n \geq a) \geq e^{-cn^d} \right\}.
\]

Thus, since \( \mathbb{P}(X_{\partial \Lambda_n} \in T_n) = 1 - o(1) \) by (49), we have reached a contradiction to (7). \( \square \)

REFERENCES

[1] Dimitris Achlioptas, Mike Molloy, Cristopher Moore, and Frank Van Bussel, Rapid mixing for lattice colourings with fewer colours, Journal of Statistical Mechanics: Theory and Experiment 2005 (2005), no. 10, P10012.

[2] M Aizenman and R Fernández, On the critical behavior of the magnetization in high-dimensional Ising models, Journal of Statistical Physics 44 (1986), no. 3-4, 393–454.

[3] Michael Aizenman, Hugo Duminil-Copin, and Vladas Sidoravicius, Random currents and continuity of Ising models spontaneous magnetization, Communications in Mathematical Physics 334 (2015), no. 2, 719–742.

[4] M A Akcoglu, Andrés del Junco, and M Rahe, Finitary codes between Markov processes, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 47 (1979), no. 3, 305–314.

[5] Kenneth S Alexander, On weak mixing in lattice models, Probability theory and related fields 110 (1998), no. 4, 441–471.

[6] Kenneth S Alexander, Mixing properties and exponential decay for lattice systems in finite volumes, The Annals of Probability 32 (2004), no. 1A, 441–487.

[7] Omer Angel, Itai Benjamini, Ori Gurel-Gurevich, Tom Meyerovitch, and Ron Peled, Stationary map coloring, Annales de l’institut Henri Poincaré, Probabilités et Statistiques, 2012, pp. 327–342.

[8] Omer Angel and Yinon Spinka, Pairwise optimal coupling of multiple random variables, arXiv preprint arXiv:1903.00632 (2019).

[9] Jacob van den Berg, On the absence of phase transition in the monomer-dimer model, Perplexing problems in probability, 1999, pp. 185–195.

[10] Jacob van den Berg and Christian Maes, Disagreement percolation in the study of Markov fields, The Annals of Probability 22 (1994), no. 2, 749–763.

[11] Jacob van den Berg and Jeffrey E Steif, Percolation and the hard-core lattice gas model, Stochastic Processes and their Applications 49 (1994), no. 2, 179–197.

[12] Jacob van den Berg and Jeffrey E Steif, On the existence and nonexistence of finitary codings for a class of random fields, Annals of probability (1999), 1501–1522.

[13] Antonio Blanca, Pietro Caputo, Alistair Sinclair, and Eric Vigoda, Spatial mixing and non-local Markov chains, Proceedings of the twenty-ninth annual ACM-SIAM symposium on discrete algorithms, 2018, pp. 1965–1980.
Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion, Sharp phase transition for the random-cluster and Potts models, Journal of Statistical Physics 147 (2012), no. 2, 231–251.

Emilio De Santis and Mauro Piccioni, Exact simulation for discrete time spin systems and unilateral fields, Methodology and Computing in Applied Probability 10 (2008), no. 1, 105–120.

Andrés del Junco, Finitary coding of Markov random fields, Probability Theory and Related Fields 52 (1980), no. 2, 193–202.

Andrés del Junco, Bernoulli shifts of the same entropy are finitarily and unilaterally isomorphic, Ergodic Theory and Dynamical Systems 10 (1990), no. 4, 687–715.

Frank den Hollander and Jeffrey E. Steif, Random walk in random scenery: a survey of some recent results, Lecture Notes-Monograph Series (2006), 53–65.

Persi Diaconis and David Freedman, Iterated random functions, SIAM review 41 (1999), no. 1, 45–76.

Xeni K. Dimakos, A guide to exact simulation, International statistical review 69 (2001), no. 1, 27–48.

Roland L Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, Theor. Probab. Appl. 13 (1968), 197–224.

Roland L Dobrushin and Senya B Shlosman, Completely analytical Gibbs fields, Statistical physics and dynamical systems, 1985, pp. 371–403.

Roland L Dobrushin and Senya B Shlosman, Constructive criterion for the uniqueness of Gibbs field, Statistical physics and dynamical systems, 1985, pp. 347–370.

Hugo Duminil-Copin, Maxime Gagnebin, Matan Harel, Ioan Manolescu, and Vincent Tassion, Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$, arXiv preprint arXiv:1611.09877 (2016).

Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion, Sharp phase transition for the random-cluster and Potts models via decision trees, arXiv preprint arXiv:1705.03104 (2017).

Hugo Duminil-Copin, Vladis Sidoravicius, and Vincent Tassion, Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$, Communications in Mathematical Physics 349 (2017), no. 1, 47–107.

Martin Dyer, Alistair Sinclair, Eric Vigoda, and Drr Weitz, Mixing in time and space for lattice spin systems: a combinatorial view, Random Structures & Algorithms 24 (2004), no. 4, 461–479.

Robert G Edwards and Alan D Sokal, Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm, Physical review D 38 (1988), no. 6, 2009.

U R Fiebig, A return time invariant for finitary isomorphisms, Ergodic Theory and Dynamical Systems 4 (1984), no. 2, 225–231.

James A Fill and Motoya Machida, Stochastic monotonicity and realizable monotonicity, Annals of probability (2001), 938–978.

A Galves, NL Garcia, and E Löcherbach, Perfect simulation and finitary coding for multicolor systems with interactions of infinite range, arXiv preprint arXiv:0809.3494 (2008).

Antonio Galves, Eva Löcherbach, and Enza Orlandi, Perfect simulation of infinite range Gibbs measures and coupling with their finite range approximations, Journal of Statistical Physics 138 (2010), no. 1-3, 476–495.

David Gamarnik, Dmitriy Katz, and Sidhant Misra, Strong spatial mixing of list coloring of graphs, Random Structures & Algorithms 46 (2015), no. 4, 599–613.

Hans-Otto Georgii, Olle Häggström, and Christian Maes, The random geometry of equilibrium phases, Phase transitions and critical phenomena 18 (2001), 1–142.
[43] Leslie A Goldberg, Markus Jalsenius, Russell Martin, and Mike Paterson, *Improved mixing bounds for the anti-ferromagnetic Potts model on \( \mathbb{Z}^2 \)*, LMS Journal of Computation and Mathematics **9** (2006), 1–20.

[44] Leslie A Goldberg, Russell Martin, and Mike Paterson, *Strong spatial mixing with fewer colors for lattice graphs*, SIAM Journal on Computing **35** (2005), no. 2, 486–517.

[45] Geoffrey R Grimmett, *The random-cluster model*, Vol. 333, Springer Science & Business Media, 2006.

[46] Olle Håggström, *On phase transitions for subshifts of finite type*, Israel Journal of Mathematics **94** (1996), no. 1, 319–352.

[47] Olle Håggström, *A monotonicity result for hard-core and Widom-Rowlinson models on certain d-dimensional lattices*, Electronic Communications in Probability **7** (2002), 67–78.

[48] Olle Håggström and Karin Nelander, *Exact sampling from anti-monotone systems*, Statistica Neerlandica **52** (1998), no. 3, 360–380.

[49] Olle Håggström and Karin Nelander, *On exact simulation of Markov random fields using coupling from the past*, Scandinavian Journal of Statistics **26** (1999), no. 3, 395–411.

[50] Olle Håggström and Jeffrey E Steif, *Propp–Wilson algorithms and finitary codings for high noise Markov random fields*, Combinatorics, Probability and Computing **9** (2000), no. 5, 425–439.

[51] Matan Harel and Yinon Spinka, *Finitary codings for the random-cluster model and other infinite-range monotone models*, arXiv preprint arXiv:1808.02333 (2018).

[52] Nate Harvey, Alexander E Holroyd, Yuval Peres, and Dan Romik, *Universal finitary codes with exponential tails*, Combinatorics, Probability and Computing **45** (2017), no. 5, 2867–2898.

[53] Nate Harvey and Yuval Peres, *An invariant of finitary codes with finite expected square root coding length*, Ergodic Theory and Dynamical Systems **31** (2011), no. 1, 77–90.

[54] Alexander E Holroyd, *One-dependent coloring by finitary factors*, Annales de l’institut henri poincaré, probabilités et statistiques, 2017, pp. 753–765.

[55] Alexander E Holroyd, Tom Hutchcroft, and Avi Levy, *Mallows permutations and finite dependence*, arXiv preprint arXiv:1706.09526 (2017).

[56] Alexander E Holroyd, Oded Schramm, and David B Wilson, *Finitary coloring*, The Annals of Probability **45** (2017), no. 5, 2867–2898.

[57] Mark Huber, *Exact sampling and approximate counting techniques*, Proceedings of the thirtieth annual acm symposium on theory of computing, 1998, pp. 31–40.

[58] Mark Huber, *Perfect sampling using bounding chains*, The Annals of Applied Probability **14** (2004), no. 2, 734–753.

[59] Markus Jalsenius, *Strong spatial mixing and rapid mixing with five colours for the Kagome lattice*, LMS Journal of Computation and Mathematics **12** (2009), 195–227.

[60] Yitzhak Katznelson and Benjamin Weiss, *Commuting measure-preserving transformations*, Israel Journal of Mathematics **12** (1972), no. 2, 161–173.

[61] Michael Keane and Meir Smorodinsky, *A class of finitary codes*, Israel Journal of Mathematics **26** (1977), no. 3-4, 352–371.

[62] Michael Keane and Meir Smorodinsky, *Bernoulli schemes of the same entropy are finitarily isomorphic*, Annals of Mathematics **109** (1979), no. 2, 397–406.

[63] Michael Keane and Meir Smorodinsky, *Finitary isomorphisms of irreducible Markov shifts*, Israel Journal of Mathematics **34** (1979), no. 4, 281–286.

[64] Michael Keane and Jeffrey E Steif, *Finitary coding for the one-dimensional \( T, T^{-1} \) process with drift*, Annals of probability (2003), 1979–1985.

[65] Wilfrid S Kendall and Jesper Møller, *Perfect simulation using dominating processes on ordered spaces*, with application to locally stable point processes, Advances in Applied Probability **32** (2000), no. 3, 844–865.

[66] Wolfgang Krieger, *On the finitary isomorphisms of Markov shifts that have finite expected coding time*, Probability Theory and Related Fields **65** (1983), no. 2, 323–328.

[67] JL Lebowitz and G Gallavotti, *Phase transitions in binary lattice gases*, Journal of Mathematical Physics **12** (1971), no. 7, 1129–1133.

[68] Liang Li, Pinyan Lu, and Yitong Yin, *Correlation decay up to uniqueness in spin systems*, Proceedings of the twenty-fourth annual ACM-SIAM symposium on discrete algorithms, 2013, pp. 67–84.

[69] Eyal Lubetzky and Allan Sly, *Critical Ising on the square lattice mizes in polynomial time*, Communications in Mathematical Physics **313** (2012), no. 3, 815–836.

[70] Eyal Lubetzky and Allan Sly, *Cutoff for the Ising model on the lattice*, Inventiones mathematicae **191** (2013), no. 3, 719–755.

[71] Eyal Lubetzky and Allan Sly, *Cutoff for general spin systems with arbitrary boundary conditions*, Communications on Pure and Applied Mathematics **67** (2014), no. 6, 982–1027.

[72] Russell Lyons and Fedor Nazarov, *Perfect matchings as IID factors on non-amenable groups*, European Journal of Combinatorics **32** (2011), no. 7, 1115–1125.
[104] Adam Timar, *Invariant colorings of random planar maps*, Ergodic Theory and Dynamical Systems **31** (2011), no. 2, 549–562.

[105] Dror Weitz, *Mixing in time and space for discrete spin systems*, University of California, Berkeley, 2004.

[106] Dror Weitz, *Combinatorial criteria for uniqueness of Gibbs measures*, Random Structures & Algorithms **27** (2005), no. 4, 445–475.

[107] Dror Weitz, *Counting independent sets up to the tree threshold*, Proceedings of the thirty-eighth annual ACM symposium on theory of computing, 2006, pp. 140–149.

[108] Chen Ning Yang, *The spontaneous magnetization of a two-dimensional Ising model*, Physical Review **85** (1952), no. 5, 808.

[109] Yitong Yin and Chihao Zhang, *Spatial mixing and approximate counting for Potts model on graphs with bounded average degree*, arXiv preprint arXiv:1507.07225 (2015).

Tel Aviv University, School of Mathematical Sciences, Tel Aviv, 69978, Israel. 
E-mail address: yinonspi@post.tau.ac.il