S-matrices of non-simply laced affine Toda theories by folding

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Abstract

The exact factorisable quantum S-matrices are known for simply laced as well as non-simply laced affine Toda field theories. Non-simply laced theories are obtained from the affine Toda theories based on simply laced algebras by folding the corresponding Dynkin diagrams. The same process, called classical ‘reduction’, provides solutions of a non-simply laced theory from the classical solutions with special symmetries of the parent simply laced theory. In the present note we shall elevate the idea of folding and classical reduction to the quantum level. To support our views we have made some interesting observations for S-matrices of non-simply laced theories and give prescription for obtaining them through the folding of simply laced ones.

1 Introduction

Affine Toda field theories \[1, 2\] received considerable attention in past years. At present it is one of the best understood field theories at classical and quantum levels. The renewal of interest for Toda field theories in recent years is due to the work of Zamolodchikov \[3\], where he discussed a class of deformation of conformal field theories that preserved

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integrable theory is characterised by eight masses related to the Cartan matrix of $E_8$, and by integrals of motion with spins given by the exponents of $E_8$ modulo its Coxeter number. The natural description of these resultant integrable theories with massive excitations is in terms of their $S$-matrices. Subsequently, Hollowood and Mansfield [4] considered a class of integrable field theories, namely Toda field theories and showed that for particular values of the coupling constant these theories describe the minimal models. The affine version of these theories was connected to the perturbed conformal field theory. This basically motivated people to study affine Toda field theories based on various Lie algebras [5–11]. Toda field theory is integrable at the classical level [1, 12] due to the presence of an infinite number of conserved quantities. It is firmly believed that the integrability survives quantisation. Higher-spin quantum conserved currents are discussed in Ref. [13]. Exact quantum $S$-matrices for affine Toda field theories based on simply laced algebras, $a_n^{(1)}$, $d_n^{(1)}$, $e_6^{(1)}$, $7$, $8$ were constructed successfully in Refs. [6], but after that one had to wait more than two years for the $S$-matrices of non-simply laced theories, because of their intricate nature. Delius et al. [10] came up with the beautiful idea of floating masses and constructed $S$-matrices for most of the non-simply laced theories. The remaining ones were constructed by Corrigan et al. [11] where generalised bootstrap principle is introduced and more insight to the mechanism is provided. The singularity structure of the $S$-matrices of simply laced theories, which in some cases contain poles up to 12-th order [6], is beautifully explained in terms of the singularities of the corresponding Feynman diagrams [14], so called Landau singularities.

Non-simply laced theories are obtained from the affine Toda theories based on simply laced algebras by folding the corresponding Dynkin diagrams. The same process, called classical ‘reduction’, provides solutions of a non-simply laced theory from the classical solutions with special symmetries of the parent simply laced theory. In the present paper we construct the $S$-matrices of non-simply laced theories by folding the simply laced theories. For this we have to write the $S$-matrices of the simply laced ones cleverly and just substitute the Coxeter number of non simply laced theories for obtaining the $S$-matrix. This is a new type of construction, hint of which was already present in Ref.[3], but authors left them at very early stage. We think that this kind of construction will provide better understanding of the structure of $S$-matrices and mechanism. This paper is organised as follows. In the next section we give brief introduction of Toda theories and the associated $S$-matrices. Section 3 contains our new formulae for $S$-matrices incorporating the idea of folding. In the following section we constructing the $S$-matrices of a specific example of $c_n^{(1)}$ theory. In subsequent sections we explicitly show the way of constructing $S$-matrices for
$f_4^{(1)}$, $b_n^{(1)}$ and $g_2^{(1)}$ theories through folding. In section 8 we discuss twisted theories and final section is reserved for some discussions.

## 2 Affine Toda field theory: An overview

In this section we give a quick review of Toda field theories. Affine Toda field theory is a massive scalar field theory with exponential interactions in $1+1$ dimensions described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi}. \quad (2.1)$$

The field $\phi$ is an $r$-component scalar field, $r$ is the rank of a compact semi-simple Lie algebra $g$ with $\alpha_i; i=1, \ldots, r$ being its simple roots and $\alpha_0$ is the affine root. The roots are normalised so that long roots have length 2, $\alpha_L^2 = 2$. The Kac-Coxeter labels $n_i$ are such that $\sum_{i=0}^{r} n_i \alpha_i = 0$, with the convention $n_0 = 1$. The quantity, $\sum_{i=0}^{r} n_i$, is denoted by ‘$h$’ and known as the Coxeter number. When the term containing the affine root is removed, the theory becomes conformally invariant (conformal Toda field theory). Then the theory is based on the root system of a finite Lie algebra ‘$g$’ and sometimes it is called a non-affine Toda theory in distinction with the affine one. ‘$m$’ is a real parameter setting the mass scale of the theory and $\beta$ is a real coupling constant, which is relevant only in quantum theory. The equation of motion reads,

$$\partial^2 \phi = - \frac{m^2}{\beta} \sum_{i=0}^{r} n_i \alpha_i e^{\beta \alpha_i \cdot \phi}. \quad (2.2)$$

It turns out that the data in quantum theory, such as the masses and couplings of various kinds, are also useful for the reduction of classical equation of motion. Expanding the potential part of the Lagrangian up to second order, one can extract a (mass)$^2$ matrix

$$(M^2)^{ab} = m^2 \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b. \quad (2.3)$$

The mass matrix has been studied before [1, 6, 7, 16, 17]. One important fact which underlies the present work is that the particles of the simply laced theory are associated unambiguously with the spots on the Dynkin diagram and thus to the simple roots (fundamental weights) of the associated finite Lie algebra [6, 17–19]. It is based on the observation that the set of masses computed as the $r$ eigenvalues of the mass matrix

\footnote{For an excellent review see Ref. [15].}
(2.3) actually constitute the Frobenius-Perron eigenvector of the Cartan matrix of the associated finite Lie algebra. In other words, if we set $m = (m_1, m_2, \ldots, m_r)$ then

$$Cm = \lambda_{\min}m = 4m^2 \sin^2 \frac{\pi}{2h} m,$$

(2.4)

where $C$ is the Cartan matrix $C_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_j^2$, $i, j = 1, \ldots, r$. The Coxeter numbers and (mass)$^2$ of various theories together with the Dynkin diagrams and particle labelling, can be found in Ref. [6].

Folding and reductions based on the symmetry (automorphism) of the Dynkin diagram can be understood in the following way [20, 6]: a symmetry of the Dynkin diagram, permuting the points as $\alpha \rightarrow p(\alpha)$, can be rewritten as a mapping of the field space to itself, $\phi \rightarrow p(\phi)$. This is a symmetry of the classical field equations (2.2), namely it maps a solution to another. This means that if the fields initially take values in the subspace invariant under $p$, they will remain there, at least classically. Since the subspace is of smaller dimension than the original field space, the evolution of fields within it can be described in terms of an equation with fewer variables than the original equation. The latter is obtained by projecting the variables $\alpha_i$ in eqn. (2.2) onto the invariant subspace. This process of obtaining new equations and their solutions from the old, by exploiting diagram symmetries, is known as reduction. In other words, an arbitrary solution of a reduced theory always gives solution(s) of the original theory by appropriate embedding(s).

The so-called direct reductions are those such that $\alpha \cdot p(\alpha) = 0$ for each root $\alpha$ (i.e. the symmetry does not relate points linked by a line on the Dynkin diagram). A symmetry of the unextended Dynkin diagram of a simply-laced algebra yields the diagram for one of the non simply-laced algebras on projection onto the invariant subspace, and the addition of the extra point to extend the diagram always respects such a symmetry. The resulting projected diagram is the untwisted affine diagram for the non simply-laced algebra. This is a reflection of the fact that the symmetry group of the extended diagram contains always at least that of the unextended diagram. Reductions involving any additional symmetries of the extended diagram yield affine Toda theories based on the twisted affine Dynkin diagrams. The reductions based on the symmetries of the unextended as well as extended (affine) Dynkin diagrams are discussed rather completely in earlier papers [6, 20–22]. There is an interesting distinction to be made here between the two different types of non simply-laced theories, namely twisted and untwisted theories. The foldings leading to untwisted theories turn out to remove degeneracies from the mass spectrum, the resulting non degenerate particles always being linear combinations of the degenerate particles in the parent theory. In contrast, foldings leading to twisted diagrams remove
some particles from the spectrum altogether, while leaving the others unchanged.

Quantum $S$-matrices of all affine Toda theories are also known [2, 5–9]. Based on the assumption that the infinite set of conserved quantities be preserved after quantisation, only the elastic processes are allowed and the multi-particle $S$-matrices are factorised into a product of two particle elastic $S$-matrices. A typical elastic, unitary $S$-matrix for a process $a + b \rightarrow a + b$ can be written as product of ratios of hyperbolic sines.

$$S_{ab}(\theta) = \prod_{x \in I_{ab}} \{x\}, \quad \{x\} = \frac{(x - 1)(x + 1)}{(x - 1 + B)(x + 1 - B)},$$  \tag{2.5}

for some set of integers $I_{ab}$. Block $(x)$ and the function $B(\beta)$ are given by

$$(x) = \frac{\sinh\left(\frac{\beta}{2} + \frac{i\pi}{2} x\right)}{\sinh\left(\frac{\beta}{2} - \frac{i\pi}{2} x\right)}, \quad B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2/4\pi}. \tag{2.6}$$

$\theta = \theta_a - \theta_b$ is the relative rapidity ($p_x \equiv (m_a \cosh \theta_a, m_a \sinh \theta_a)$), $h$ is the Coxeter number of the Lie algebra on which theory is based. For $x \leq h$, mod 2$h$ these $S$-matrices have physical sheet simple poles at $\theta = i\pi x/h$ and these can be interpreted as elementary particle poles from s-channel or u-channel exchange, with mass related to the value of $x$. Above $S$-matrices respect crossing symmetry and bootstrap principle [6]. $S$-matrices for various simply laced theories were constructed in Refs. [6–9]. But the ideas of simply laced theories failed for the theories based on non-simply laced algebras. The problem was resolved by Delius et al. [10] with the introduction of floating masses and the renormalised Coxeter number. Next the ‘generalised bootstrap principle’ was introduced in Ref. [11] and construction of all the $S$-matrices for various non-simply laced theories was completed. According to the ‘generalised bootstrap principle’ there is a quantum field theory corresponding to the dual pair of non-simply laced algebras together than either classical theory separately and the weak ($\beta \rightarrow 0$) and strong ($\beta \rightarrow \infty$) coupling limits of this quantum field theory would effectively lead to one or the other of the dual pair. In other words the transformation $\beta \rightarrow 4\pi/\beta$ effectively implements the inversion $\alpha_i \rightarrow 2\alpha_i/|\alpha_i|^2$, which interchanges the two extended Dynkin diagrams. For every dual pair one defines a renormalised Coxeter number $H$, which is a function of $B(\beta)$ and interpolates the Coxeter numbers of the dual algebras. In what follow we shall be following the notations of Ref. [4], which is convenient. We shall be mentioning the explicit forms of $S$-matrices for various types of theories as we proceed. For future convenience we arm ourselves with the following notations,

$$(x)_H = \frac{\sinh\left(\frac{\beta}{2} + \frac{i\pi}{2H} x\right)}{\sinh\left(\frac{\beta}{2} - \frac{i\pi}{2H} x\right)}, \quad \{x\}_{\nu} = \frac{(x - \nu B - 1)_H(x + \nu B + 1)_H}{(x + \nu B + B - 1)_H(x - \nu B - B + 1)_H}. \tag{2.7}$$
and

\[ [x]_\nu = \{x\}_\nu \{H - x\}_\nu, \]  

(2.8)

where \( H \) is the renormalized Coxeter number described in Refs. \([10, 11]\).

### 3 S-matrices

First we consider non-simply laced algebras which are not twisted, (that is Dynkin diagrams for these non-simply laced algebras can be obtained by folding the non-affine simply laced algebras using the automorphisms of Dynkin diagrams). In Ref. \([6]\) it was argued that \( S \)-matrices of the corresponding affine theories might be related by following formula.

\[
S_{\text{daughter}}^{ab} = S_{\text{parent}}^{ab} S_{\text{parent}}^{a \bar{b}}. \tag{3.1}
\]

But this formula could not explain multipole structure on the basis of daughter Lagrangian. Here we propose a slight modification to the formula (3.1), we would only replace in the right hand side of (3.1) the Coxeter number \( h \), of the parent theory by the Coxeter number \( H \) of the daughter theory. This new \( H \) is a kind of renormalized Coxeter number as suggested by Delius et al. in their paper \([10]\). So, we have,

\[
S_{\text{daughter}}^{ab} = S_{\text{parent}}^{ab} S_{\text{parent}}^{a \bar{b}} |_{h \rightarrow H}. \tag{3.2}
\]

But to show that the formula is universal first one has to write down the \( S \)-matrices of the parent theory cleverly in terms of Coxeter number \( h \). Terms written in arbitrary fashion would lead to wrong results. Although we don’t have any general rule for writing them yet, in the following we will show (case by case that) that the above formula holds good. Furthermore we would see that although the formula (3.2) is nice for checking the bootstrap, but it conceals a number of cancellations between zeros and poles (see Ref.\([11]\)). Finally we propose a set formulae, which makes more sense physically, by modifying the formula (3.2) a little further. That is

\[
S_{\text{daughter}}^{ab} = S_{\text{parent}}^{ab} S_{\text{parent}}^{a \bar{b}} |_{h \rightarrow H}, \tag{3.3}
\]

\[
S_{\text{daughter}}^{an} = S_{\text{parent}}^{an} |_{h \rightarrow H}, \tag{3.4}
\]

\[
S_{\text{daughter}}^{mn} = S_{\text{parent}}^{mn} |_{h \rightarrow H}. \tag{3.5}
\]
Above $a, b$ represent degenerate particles of the parent theory (non-degenerate in daughter) and $m, n$ are non-degenerate particles of the parent theory. For example let us consider the folding of $e_6^{(1)}$ into $f_4^{(1)}$.

\[ \begin{align*}
\text{1} & \rightarrow \text{2} \\
\text{1} & \rightarrow \text{2} \\
\text{3} & \rightarrow \text{4} \\
\text{3} & \rightarrow \text{4} \\
\text{1} & \rightarrow \text{2} \\
\text{1} & \rightarrow \text{2} \\
\text{3} & \rightarrow \text{4} \\
\text{3} & \rightarrow \text{4} \\
\text{1} & \rightarrow \text{2} \rightarrow \text{1} \\
\text{3} & \rightarrow \text{4} \rightarrow \text{3} \\
\text{1} & \rightarrow \text{2} \rightarrow \text{1} \\
\text{3} & \rightarrow \text{4} \rightarrow \text{3} \\
\end{align*} \]

In the above case $a, b = \{1, 3\}$ and $m, n = \{2, 4\}$. At this point let us compare the expression (3.2) with expressions (3.3)–(3.5). The folding, leading to untwisted theories turns out to remove degeneracies from the mass spectrum, the resulting non-degenerate particles always being linear combinations of the degenerate particles of the parent theory. Since all the particles in the daughter theory are self conjugate, expression (3.2) is manifestly crossing symmetric. This can be seen in the following way. We know

\[ S_{ab}(\theta) = S_{ab}(i\pi - \theta), \] (3.7)

at the level of the building block $\{x\}$, it is realised by,

\[ \{x\}_{i\pi-\theta} = \{h - x\}_{\theta}. \] (3.8)

As mentioned in earlier section (expression (2.5)), $S_{ab}$ is a product of these building blocks, therefore the combination $S_{ab}^{\text{parent}} S_{ab}^{\text{parent}}$, appearing in (3.2) always contains $\{x\}$ in the self conjugate combination:

\[ \{x\}\{h - x\} (\equiv x). \] (3.9)

Now coming to the formulae (3.3)–(3.5). For expression (3.3) arguments are the same as above and it is manifestly crossing symmetric because $a, b$ belong to degenerate class of particles in the parent theory and become a self conjugate and non-degenerate in the daughter theory. On the other hand $m, n$ were non-degenerate and self conjugate in the parent itself and therefore $S_{mn}^{\text{parent}}$ must have been crossing symmetric to start with. Similarly, $S_{an}^{\text{parent}}(\theta) = S_{an}^{\text{parent}}(i\pi - \theta)$ (because $n$ is self conjugate particle in parent theory), so $S_{an}^{\text{parent}}$ is also crossing symmetric. That is why (3.4) and (3.5) have only one term, whereas (3.3) has two terms on the right hand side respectively and as expected all the $S$-matrices of the daughter theory are crossing symmetric.

\[ ^3 \text{Although we talk about folding of affine algebras, the figures we present will depict the folding of non-affine Dynkin diagrams only. The folding of corresponding affine diagrams is just adding the affine vertex at proper positions \cite{6}. The purpose of giving these diagrams is to provide mass labellings.} \]
Twisted algebras are obtained by folding the affine versions of simply laced algebras. These algebras use the extra symmetry of the affine Dynkin diagrams (automorphisms). In contrast to untwisted theories, folding leading to twisted diagrams remove some particles from the spectrum altogether [3], while leaving the others unchanged. The particles surviving in the daughter form a subset of the parent theory mass spectrum. So in this case (3.3) is applicable and resulting S-matrices form a subset of the original S-matrices of the the parent theory with the Coxeter $h$ replaced by the renormalised Coxeter $H$.

4. $a_{2n-1}^{(1)} \rightarrow c_n^{(1)}$

First we take the case of $c_n^{(1)}$ theories. The Dynkin diagram of $c_n^{(1)}$ is obtained by folding $a_{2n-1}^{(1)}$. Mass labelling and folding are shown in following diagram.

$$
\begin{align*}
1 & \rightarrow 2 \\
m_1 & \rightarrow m_2 \\
n & \rightarrow m_n \\
2 & \rightarrow 1 \\
m_2 & \rightarrow m_1 \\
n & \rightarrow m_n
\end{align*}
$$

To be more specific let us take example of $a_5^{(1)} \rightarrow c_3^{(1)}$. In $a_5^{(1)}$ there are 5 particles, 1st and 2nd particles come with their mass degenerate conjugate partners, 5th (or 1) and 4th (or 2) respectively. The 3rd particle is self conjugate and the Coxeter number $h = 6$. In $c_3^{(1)}$ the mass degeneracy is removed and one has three self conjugate particles. The renormalized Coxeter $H = 6 + B$ for the dual pair $(c_3^{(1)}, d_4^{(2)})$. Now we write the S-matrices of $c_3^{(1)}$ in the following manner.

$$S_{11}^{c_3^{(1)}} = S_{11}^{a_5^{(1)}} S_{11}^{a_5^{(1)}} |_{h \rightarrow H} = \{1\} \{h - 1\} |_{h \rightarrow H} = \{1\} \{H - 1\} = [1]_0. \quad (4.2)$$

Above is the correct answer for $S_{11}^{c_3^{(1)}}$ (see Ref. [11]). Similarly,

$$
\begin{align*}
S_{12}^{c_3^{(1)}} &= S_{12}^{a_5^{(1)}} S_{12}^{a_5^{(1)}} |_{h \rightarrow H} = \{2\} \{h - 2\} |_{h \rightarrow H} = \{2\} \{H - 2\} = [2]_0, \\
S_{22}^{c_3^{(1)}} &= S_{22}^{a_5^{(1)}} S_{22}^{a_5^{(1)}} |_{h \rightarrow H} = \{1\} \{3\} \{h - 1\} \{h - 3\} |_{h \rightarrow H} = \{1\} \{H - 1\} \{H - 3\} = [1]_0 [3]_0, \\
S_{31}^{c_3^{(1)}} &= S_{31}^{a_5^{(1)}} S_{31}^{a_5^{(1)}} |_{h \rightarrow H} = \{3\} \{h - 3\} |_{h \rightarrow H} = \{3\} \{H - 3\} = [3]_0, \\
S_{32}^{c_3^{(1)}} &= S_{32}^{a_5^{(1)}} S_{32}^{a_5^{(1)}} |_{h \rightarrow H} = \{2\} \{4\} \{h - 2\} \{h - 4\} |_{h \rightarrow H} = \{2\} \{H - 2\} \{H - 4\} = [2]_0 [4]_0, \\
S_{33}^{c_3^{(1)}} &= S_{33}^{a_5^{(1)}} S_{33}^{a_5^{(1)}} |_{h \rightarrow H} = \{1\} \{3\} \{5\} \{h - 1\} \{h - 3\} \{h - 5\} |_{h \rightarrow H} = \{1\} \{3\} \{5\} \{H - 1\} \{H - 3\} \{H - 5\} = [1]_0 [3]_0 [5]_0. \quad (4.3)
\end{align*}
$$
Now we show that the above expressions can also be expressed as (3.3)–(3.5). The first three $S$-matrices remain the same because now $a, b = 1, 2$ and $m, n = 3$. In the last three cancellations take place and they can be written in the following way. For this one has to go to the basic building blocks and write them cleverly to obtain the answer.

$$S_{31}^{(1)} = S_{31}^{(1)} \big|_{h \to H} = \{3\} \big|_{h \to H} = \frac{(2)(4)}{(2 + B)(4 - B)} \big|_{h \to H} = \frac{(2)(h - 2)}{(h - 4 + B)(4 - B)} \big|_{h \to H}$$

$$= \frac{(2)(H - 2)_{H}}{(H - 4 + B)_{H}(4 - B)_{H}} = \frac{(2)(4 + B)_{H}}{(2 + 2B)_{H}(4 - B)_{H}}$$

$$= \frac{(2 + B)_{H}(4 - B)_{H}}{(2 + B)_{H}(4 + B)_{H}} \frac{(2)(4)_{H}}{(4 + B)_{H}}$$

$$= \frac{(2)(H - 2)_{H}}{(H - 4 + B)_{H}(4 - B)_{H}} = \{3\} \{H - 3\} = [3]_{0},$$

$$S_{32}^{(1)} = S_{32}^{(1)} \big|_{h \to H} = \{2\} \big|_{h \to H} = \frac{(1)(3)}{(1 + B)(3 - B)(3 + B)(5 - B)} \big|_{h \to H}$$

$$= \frac{(1)(h - 3)}{(h - 5 + B)(3 - B)(h - 3 + B)(5 - B)} \big|_{h \to H}$$

$$= \frac{(H - 5 + B)_{H}(3 - B)_{H}}{(H - 3 + B)_{H}(5 - B)_{H}} \frac{(3)(h - 1)}{(1)(H - 3)_{H}(3)(H - 1)_{H}}$$

$$= \frac{(1 + 2B)_{H}(3 - B)_{H}}{(3 + B)_{H}(5 - B)_{H}} \frac{(3)(5)}{(1 + 2B)_{H}(3)_{H}}$$

$$= \frac{(3 + 2B)_{H}(5)}{(1 + 2B)_{H}(3)} \frac{(3)(5)}{(1 + 2B)_{H}(3)}$$

$$= \{2\} \{4\} \{2 + B\} = \{2\} \{H - 2\} \{4\} \{H - 4\} = [2]_{0} [4]_{0},$$

$$S_{33}^{(1)} = S_{33}^{(1)} \big|_{h \to H} = \{1\} \big|_{h \to H}$$

$$= \frac{(0)(2)(4)(6)}{(2 - B)(2 + B)(4 - B)(6 - B)} \big|_{h \to H}$$

$$= \frac{(0)(h - 4)(2)(4)(6)}{(h - 6 + B)(2 - B)(6 - B)} \big|_{h \to H}$$

$$= \frac{(0)(H - 4)_{H}(2)(4)(6)}{(2)(H - 2)_{H}(6 - B)_{H}} \big|_{h \to H}$$

$$= \frac{(H - 6 + B)_{H}(2 - B)_{H}(6 - B)_{H}}{(2)(H - 2)_{H}H_{H}(6 - B)_{H}} \big|_{h \to H}$$

$$= \frac{(2)(H - 2)_{H}H_{H}(4 - B)_{H}(6 - B)_{H}}{(2)(H - 2)_{H}H_{H}(6 - B)_{H}} \big|_{h \to H}$$

$$= \frac{(2)(H - 2)_{H}H_{H}(4 - B)_{H}(6 - B)_{H}}{(2)(H - 2)_{H}H_{H}(6 - B)_{H}} \big|_{h \to H}$$
\[
\frac{(2 + B)H(4 + B)H}{(2 + 2B)H(4)H} \times \frac{(4)H(6)H}{(4 + B)H(6 - B)H} \times \frac{(B)H(2 + B)H}{(2B)H(2)H}
\]
\[=
\{1\}_0 \{5 + B\}_0 \{3\}_0 \{3 + B\}_0 \{5\}_0 \{1 + B\}_0
\]
\[=
\{1\}_0 \{H - 1\}_0 \{3\}_0 \{H - 3\}_0 \{5\}_0 \{H - 5\}_0 = [1]_0 [3]_0 [5]_0.
\]

Note that to show above we made replacements at the very basic building block level. We have added some extra terms in the numerators and denominators which eventually cancel. For making replacements, although we do not have some universal rules but certain patterns can be observed within a theory. In the above example, the second block of each unit is replaced by a linear function of \(h\) with integer coefficients, for example (2) by \((h - 4)\) and (3) by \((h - 3)\) and so on.

One can verify that expression (3.2) holds for any \(a \rightarrow c\), so that one can write
\[
S_{ab}^{(1)} = S_{ab}^{(1)} S_{ab}^{(1)} \Big|_{h \rightarrow H}
\]
\[=
\prod_{p = |a - b| + 1}^{a + b - 1} \{p\} \prod_{p = |a - b| + 1}^{a + b - 1} \{h - p\} \Big|_{h \rightarrow H}
\]
\[=
\prod_{p = |a - b| + 1}^{a + b - 1} \{p\}_0 \prod_{p = |a - b| + 1}^{a + b - 1} \{H - p\}_0
\]
\[=
\prod_{p = |a - b| + 1}^{a + b - 1} [p]_0,
\]
and which is the same as the one given in expression (5.3) of Ref. [11].

Alternatively one can also write S-matrices according to (3.3)–(3.5), after cancelling zeros and poles, in the following fashion.
\[
S_{ab}^{(1)} = S_{ab}^{(1)} S_{ab}^{(1)} \Big|_{h \rightarrow H}, \quad a, b = 1, 2, \ldots, n - 1,
\]
\[
S_{ab}^{(1)} = S_{ab}^{(1)} \Big|_{h \rightarrow H},
\]
\[
S_{ab}^{(1)} = S_{ab}^{(1)} \Big|_{h \rightarrow H}.
\]

In this case one would arrive at the expressions given in (5.4) of Ref. [11].

\[
5 \quad e_6^{(1)} \rightarrow f_4^{(1)}
\]

The folding of \(e_6^{(1)}\) Dynkin diagram with labelling is already given in (3.6). In this case the Coxeter number of \(e_6^{(1)}\) is 12, whereas the renormalised Coxeter number for the dual
pair \((f_4^{(1)}, e_6^{(2)})\) according to the notations of Ref. [1] is \(H = 12 + 3B\). \(S\)-matrices of \(e_6^{(1)}\) can be found in Table 1 of Ref. [3]. We have,

\[
S_{ab}^{f_4^{(1)}} = S_{ab}^{e_6^{(1)}} S_{ab}^{e_6^{(1)}} |_{h \to H},
\]

(5.1)

and \(S_{11}^{f_4^{(1)}}\) and \(S_{21}^{f_4^{(1)}}\) are given as

\[
S_{11}^{f_4^{(1)}} = S_{11}^{e_6^{(1)}} S_{11}^{e_6^{(1)}} |_{h \to H} = \{1\} \{7\} \{5\} \{11\} |_{h \to H} = \{1\} \{2h/3 - 1\} \{h/3 + 1\} \{h - 1\} |_{h \to H} = \{1\}_0 \{2H/3 - 1\}_0 \{H/3 + 1\}_0 \{H - 1\}_0 = [1]_0 \{H/3 + 1\}_0,
\]

\[
S_{21}^{f_4^{(1)}} = S_{21}^{e_6^{(1)}} S_{21}^{e_6^{(1)}} |_{h \to H} = \{4\} \{8\} \{4\} \{8\} |_{h \to H} = \{h/6 + 2\} \{5h/6 - 2\} \{h/2 - 2\} \{h/2 + 2\} |_{h \to H} = \{H/6 + 2\}_0 \{5H/6 - 2\}_0 \{H/2 - 2\}_0 \{H/2 + 2\}_0 = [H/6 + 2]_0 \{H/2 + 2\}_0. \tag{5.2}
\]

The above \(S\)-matrices appear after the expression (4.5) in Ref. [1]. The rest of the \(S\)-matrices are calculated in appendix A (A.1-A.8). Again one can show that (3.3)–(3.5) holds for \(S\)-matrices. For obtaining those one has to write down \(S\)-matrices (after cancelling zeros and poles present in the above ones, see expressions (4.6) of Ref. [1]) in terms of elementary building blocks, \((x)\), and replace \(x\) in terms of linear function of \(h\) appropriately there itself. To demonstrate this we take a sample \(S\)-matrix element \((S_{12})\) here, rest of the calculations can be found in the appendix A (A.9-A.14).

\[
S_{12}^{f_4^{(1)}} = S_{12}^{e_6^{(1)}} |_{h \to H} = 4 |_{h \to H} = \{4\} \{8\} |_{h \to H} = \frac{(3)(5)}{(3 + B)(5 - B)(7 + B)(9 - B)} |_{h \to H} = \frac{(7)(9)}{(h/6 + 1)(h/2 - 1)} \frac{(h/2 + 1)(5h/6 - 1)}{(h/2 - 3 + B)(h/6 + 3 - B)(5h/6 - 3 + B)(h/2 + 3 - B)} |_{h \to H} = \{h/6 + 1\}' \{h/2 + 1\}' |_{h \to H} = \{H/6 + 1\}'_0 \{H/2 + 1\}'_0 = \frac{(3 + B/2)_H(5 + 3B/2)_H(7 + 3B/2)_H(9 + 5B/2)_H}{(3 + 5B/2)_H(5 - B)_H(7 + 7B/2)_H(9 + B/2)_H} = \{4 + B\}_1/2 \{8 + 2B\}_1/2 = \{H/3\}_1/2 \{2H/3\}_1/2 = [H/3]_1/2. \tag{5.3}
\]

Where we have used modified blocks \(\{x\}'\) and \(\{x\}'_0\), given by

\[
\{x\}' = \frac{(x)(h/3 - 2 + x)}{(h/3 - 4 + x + B)(x + 2 - B)}, \quad \{x\}'_0 = \frac{(x)_H(H/3 - 2 + x)_H}{(H/3 - 4 + x + B)_H(x + 2 - B)_H}. \tag{5.4}
\]

\[
6 \quad d_{n+1}^{(1)} \to b_{n}^{(1)}
\]

Folding of Dynkin diagram and mass labellings are given in the following figure.
Again to make life simple we take a specific example of this family and show the results. It can be seen that generalisation will follow immediately. Here we discuss the case $d^{(1)}_5 \rightarrow b^{(1)}_4$. Apart from the particles 1, 2 and 3 in $d^{(1)}_5$ one has mass degenerate conjugate pair of particles $s$ and $\bar{s}$. After the folding degeneracy is lost and 4th particle of $b^{(1)}_4$ is a linear combination of $s$ and $\bar{s}$. In this case first we show the formulae (3.3)–(3.5) and postpone the discussion of (3.2) till the end of this section. The Coxeter number for $d^{(1)}_5$ is 8 and the renormalised Coxeter for the pair $(b^{(1)}_4, a^{(2)}_7)$ is $8 - B/2$. According to the expression (3.3), $a, b = 1, 2, 3$ and $m, n = 4$. So,

\[
\begin{aligned}
S_{41}^{d^{(1)}_5} &= S_{ss}^{d^{(1)}_5} S_{s\bar{s}}^{d^{(1)}_5} |_{h \rightarrow H} = \{1\}\{5\}\{3\}\{7\} \\
&= (0)(2)(4)(6)(2)(4)(6)(8) |_{h \rightarrow H} \\
&= (h - 8 + B)(2 - B)(h - 4 + B)(6 - B) \\
&\times (h - 6 + B)(4 - B)(h - 2 + B)(8 - B) |_{h \rightarrow H} \\
&= (H - 8 + B)H(2 - B)H(H - 4 + B)H(6 - B)H \\
&\times (H - 6 + B)H(4 - B)H(H - 2 + B)H(8 - B)H \\
&= (0)(2)(4 - B/2)H(4 - B/2)H(6 - B)H \\
&\times (2 + B/2)H(4 - B)H(H - 2 + B)H(8 - B)H \\
\end{aligned}
\]

\footnote{Notice that in $d^{(1)}_{n+1}$, $n = odd$, one has mass degenerate pair $s$ and $s'$ which are self conjugate. But the folding $d^{(1)}_{n+1} \rightarrow b^{(1)}_n$, identifies the particles $s$ and $s'$ to produce $h^{(1)}_n$ particle of $b^{(1)}_n$. In this case we write $S_{nn}^{d^{(1)}_n}$ as,

\[
S_{nn}^{d^{(1)}_n} = S_{ss}^{d^{(1)}_n} S_{ss'}^{d^{(1)}_n},
\]

which is analogous to (3.3). In the next section a similar situation arises again when three self conjugate particles of $d^{(1)}_4$ are identified to produce $g^{(1)}_2$ theory. There too one should understand the formula (3.3) accordingly.
\[
\begin{align*}
&= \{1 - \frac{B}{4}\}^{-1/4}\{5 - \frac{B}{4}\}^{-1/4}\{3 - \frac{B}{4}\}^{-1/4}\{7 - \frac{B}{4}\}^{-1/4} \\
&= \{\frac{H}{2} - 3\}^{-1/4}\{\frac{H}{2} + 1\}^{-1/4}\{\frac{H}{2} - 1\}^{-1/4}\{\frac{H}{2} + 3\}^{-1/4} \\
&= \prod_{p = -3}^{3} \{\frac{H}{2} - p\}^{1/4}.
\end{align*}
\]

(6.3)

For \(S_{a_{b}^{1}}\) and \(S_{a_{4}^{1}}^{b}\), \(a, b = 1, 2, 3\) matrix elements we begin with the matrix elements of \(d_{5}^{1}\) as given in the expressions (4.13) and (4.16) of Ref. 3, respectively.

\[
S_{a_{b}^{1}}^{d_{5}^{1}} = S_{a_{b}^{1}}^{d_{5}^{(1)}}|_{h \rightarrow H}
\]

\[
= \prod_{p = |a - b| + 1}^{a + b - 1} \{p\}\{h - p\}|_{h \rightarrow H}
\]

(6.4)

\[S_{a_{4}^{1}}^{d_{5}^{1}} = S_{a_{4}^{1}}^{d_{5}^{(1)}}|_{h \rightarrow H}
\]

\[= \prod_{p = |a - b| + 1}^{2a - 2} \{5 - a + p\}|_{h \rightarrow H}
\]

\[= \prod_{p = 0}^{2a - 2} \{\frac{h}{2} + 1 - a + p\}|_{h \rightarrow H} = \prod_{p = 1}^{2a - 1} \{\frac{h}{2} - a + p\}|_{h \rightarrow H}
\]

\[= \prod_{p = 1}^{2a - 1} \{\frac{H}{2} - a + p\}_0.
\]

(6.5)

To show the formula (3.2), one has to go to the elementary building block level and make substitutions their itself. Of course \(S_{a_{4}^{1}}\) is the same as expression (6.3), for the rest one proceeds in the following manner,

\[
S_{a_{4}^{1}}^{d_{5}^{1}} = S_{a_{4}^{1} S_{a_{4}^{1}}}^{d_{5}^{(1)}}|_{h \rightarrow H}
\]

\[
= \{4\}\{4\}|_{h \rightarrow H} = \frac{(3)(5)}{(3 + B)(5 - B)}\frac{(3)(5)}{(3 + B)(5 - B)}|_{h \rightarrow H}
\]

\[= \frac{(h/2 - 1)(3h/2 - 7)}{(3h/2 - 9 + B)(h/2 + 1 - B)}\frac{(7 - h/2)(h/2 + 1)}{(h/2 - 1 + B)(9 - h/2 - B)}|_{h \rightarrow H}
\]
To show (3.3)–(3.5) we proceed followingly. Points were identified. For the dual pair \((g^1, d^1)\) the renormalised Coxeter \(H = 6 + 3B\) and the Coxeter for \(d^1\) is 6. In this we show the \((3.3)–(3.5)\) for the \(S\)-matrix elements.

\[ S_{11}^{d_2} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = \{1\} \{5\} \{3\} |_{h\to H} = \frac{(H/2 - 1)_H (3H/2 - 7)_H}{(3H/2 - 9 + B)_H (H/2 + 1 - B)_H (H/2 - 1 + B)_H (9 - H/2 - B)_H} \frac{(7 - H/2)_H (H/2 + 1)_H}{(H/2 - 1)_H (5 - 3B/4)_H (3 + B/4)_H (H/2 + 1 + B)_H} \frac{(H/2 - 1)_H (H/2 + 1 - B)_H}{(H/2 - 1 + B)_H (H/2 + 1 - B)_H} = \{H/2\}_0. \] (6.6)

Remaining ones are evaluated in a similar fashion. Calculations are straightforward but tedious so we do not produce them here.

7 \[ d_4^{(1)} \rightarrow g_2^{(1)} \]

The case of \(g_2^{(1)}\) is a little different from the rest. There are three mass degenerate self conjugate particles, viz. \(m_1, m_s, m_s'\) in \(d_4^{(1)}\). In this case three points of the parent theory are identified in the process of folding unlike previous cases where at most two points were identified. For the dual pair \((g_2^{(1)}, d_4^{(1)})\) the renormalised Coxeter \(H = 6 + 3B\) and the Coxeter for \(d_4^{(1)}\) is 6. In this we show the \((3.3)–(3.5)\) for the \(S\)-matrix elements. The expression \((3.2)\) can be shown without much difficulty and will not be produced here. To show \((3.3)–(3.5)\) we proceed followingly.

\[ S_{11}^{d_2} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = \{1\} \{5\} \{3\} |_{h\to H} = \frac{(H/2 - 1)_H (3H/2 - 7)_H}{(3H/2 - 9 + B)_H (H/2 + 1 - B)_H (H/2 - 1 + B)_H (9 - H/2 - B)_H} \frac{(7 - H/2)_H (H/2 + 1)_H}{(H/2 - 1)_H (5 - 3B/4)_H (3 + B/4)_H (H/2 + 1 + B)_H} \frac{(H/2 - 1)_H (H/2 + 1 - B)_H}{(H/2 - 1 + B)_H (H/2 + 1 - B)_H} = \{H/2\}_0. \] (6.6)

\[ S_{11}^{d_2} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = S_{11}^{d_4} S_{11}^{d_4} |_{h\to H} = \{1\} \{5\} \{3\} |_{h\to H} = \frac{(H/2 - 1)_H (3H/2 - 7)_H}{(3H/2 - 9 + B)_H (H/2 + 1 - B)_H (H/2 - 1 + B)_H (9 - H/2 - B)_H} \frac{(7 - H/2)_H (H/2 + 1)_H}{(H/2 - 1)_H (5 - 3B/4)_H (3 + B/4)_H (H/2 + 1 + B)_H} \frac{(H/2 - 1)_H (H/2 + 1 - B)_H}{(H/2 - 1 + B)_H (H/2 + 1 - B)_H} = \{H/2\}_0. \] (6.6)
For the dual theories, \( S_{22}^{d_3} \) and \( S_{15}^{d_1} \), we notice that the daughter theory is just a subset of that of parent theory. In the same way (as mentioned in Ref. [6]) the mass spectrum of the twisted theories obtained by exploiting the symmetries of extended Dynkin diagrams of simply laced theories. In these cases as described in the Ref. [6] the mass spectrum of the corresponding parent theories with the Coxeter number \( h \) replaced by the renormalised \( H \).

\[
S_{22}^{d_3} = S_{15}^{d_1} = \{1\} \{5\} \{3\} \{3\} |_{h \rightarrow H} \\
= \frac{(H/3 + 1)_H (H - 1)_H}{(H - 3)_H (H/3 - 5)_H} \\
= \{H/3\}_1 \{2H/3\}_1.
\]

\[
S_{22}^{d_2} = S_{15}^{d_1} = 1 \{3\} \{3\} |_{h \rightarrow H} = \{1\} \{5\} \{3\} \{3\} |_{h \rightarrow H} \\
= \frac{(B)(2 - B) (4 + B)(6 - B)}{(2)(4)(4)} \\
\times \frac{(2h/3 - 2) (4 - B)(2 + B)(4 - B) |_{h \rightarrow H}}{(0)(2h/3 - 2) (h/3 + 2)(h)} \\
= \frac{(2h/3 - 2 + B)(4 - B)(2 + B)(h/3 + 2 - B) |_{h \rightarrow H}}{(2h/3 - 2) (2)(h/3)(h - 2)(h)} \\
= \frac{(2h/3 - 2 + B)(4 - B)(2 + B)(h/3 + 2 - B) |_{h \rightarrow H}}{(2h/3 - 2) (2)(h/3)(h - 2)(h)} \\
= \{H/3 - 1\}_1 \{2H/3 + 1\}_1 \{H/3 + 1\}_1 \{2H/3 - 1\}_1.
\]

\[7.2\]

**8 Twisted theories**

For the dual theories, \( d_{n+1}^{(2)} \), \( c_n^{(2)} \), \( d_4^{(3)} \) and \( d_{2n-1}^{(2)} \) the story is a little different. These are twisted theories obtained by exploiting the symmetries of extended Dynkin diagrams of simply laced theories. In these cases as described in the Ref. [6] the mass spectrum of the daughter theory is just a subset of that of parent theory. In the same way (as mentioned at the end of the section 3) we notice that the \( S \)-matrices of these theories are just subsets of the corresponding parent theories with the Coxeter \( h \) replaced by the renormalised \( H \).

**8.1 \( d_{n+2}^{(1)} \rightarrow d_{n+1}^{(2)} \)**

Here two degenerate particles \( s \) and \( s' \) (\( s \) and \( s' \) for \( d_{odd}^{(1)} \) are lost. All other particles survive folding and labelled by \( 1, 2, \ldots, n \). The Coxeter number \( h \) for \( d_{n+2}^{(1)} \) theory is \( 2n + 2 \), whereas the renormalized Coxeter for the pair \( (d_{n+1}^{(2)}, c_n^{(1)}) \), \( H = 2n + B \). \( S \)-matrices are calculated...
as (for \( d_n^{(1)} \) S-matrices, see expression (4.13) of Ref. [6]),

\[
S_{n+1}^{d^{(2)}}_{ab} = S_{n+2}^{d^{(1)}}_{ab}, \quad a, b = 1, 2, \ldots, n. \\
= \prod_{p=[a-b]+1}^{a+b-1} \{p\} \{h - p\} |_{h \to H} \\
= \prod_{p=[a-b]+1}^{a+b-1} \{p\}_0 \{H - p\}_0 \\
= \prod_{p=[a-b]+1}^{a+b-1} \{[p]_0. \\
\]

These are the same S-matrices obtained in the expression (4.5) of section 4 for the dual theory \( e_n^{(1)} \).

**8.2 \( e_7^{(1)} \rightarrow e_6^{(2)} \)**

In this case masses \( m_2, m_4, m_5, \) and \( m_7 \) of \( e_7^{(1)} \) survive and become particle 1,2,3 and 4 respectively in \( e_6^{(2)} \) after relabelling. Here \( h = 18 \) for \( e_7^{(1)} \) and \( H = 12 + 3B \) for the dual pair \( (e_6^{(2)}, f_4^{(1)}) \). S-matrices for \( e_6^{(2)} \) are given in Ref. [11] (following equation (4.5), before cancelling zeros and poles) can be obtained very easily from the \( e_7^{(1)} \) S-matrices shown in Table 2 of the Ref. [6] as follows. Here we give just the \( S_{11}^{e_6^{(2)}} \) rest of the calculations are presented in appendix B.

\[
S_{11}^{e_6^{(2)}} = S_{22}^{e_7^{(1)}} |_{h \to H} = 17 |_{h \to H} = \{1\} \{17\} \{7\} \{11\} |_{h \to H} \\
= \{1\} \{h - 1\} \{h/3 + 1\} \{2h/3 - 1\} |_{h \to H} \\
= \{1\}_0 \{H - 1\}_0 \{H/3 + 1\}_0 \{2H/3 - 1\}_0 \\
= [1]_0 [H/3 + 1]_0. \\
\]

Again notice that these S-matrices match with the S-matrices of the dual theory, \( f_4^{(1)} \), obtained in section 5 (expressions (5.2) and (A.1–A.8)).

**8.3 \( e_6^{(1)} \rightarrow d_4^{(3)} \)**

In this case the particles 2 and 4 of \( e_6^{(1)} \) survive and relabelled as particle 1 and 2 in \( d_4^{(3)} \). The Coxeter number \( h = 12 \) for \( e_6^{(1)} \) and the renormalized Coxeter, as mentioned earlier, for the pair \( (d_4^{(3)}, g_2^{(1)}) \) is \( 6 + 3B \). The corresponding S- matrix elements,
Similarly,

\[ S_{d_1}^{(3)} \mid _{h \to H} = 1 \ 5 \mid _{h \to H} = \{1\} \{1\} \{5\} \{1\} \{5\} \{7\} \mid _{h \to H} \]

\[ = \{1\} \{h - 1\} \{h/3 + 1\} \{2h/3 - 1\} \mid _{h \to H} \]

\[ = \{1\} \{H - 1\} \{H/3 + 1\} \{2H/3 - 1\} \mid _{h \to H} \]

\[ = \{1\} \{5 + 3B\} \{3 + B\} \{3 + 2B\} \mid _{h \to H} \]

\[ = \frac{(0)_{H}(2)_{H}}{(B)_{H}(2 - B)_{H}} \frac{(4 + 3B)_{H}(6 + 3B)_{H}}{(4 + 4B)_{H}(6 + 2B)_{H}} \]

\[ \times \frac{(2 + B)_{H}(4 + B)_{H}(2 + 2B)_{H}(4 + 2B)_{H}}{(2 + 2B)_{H}(4)_{H}} \frac{(2 + 3B)_{H}(4 + B)_{H}}{(6 + 3B)_{H}(2 + 2B)_{H}} \]

\[ = \frac{(0)_{H}(2)_{H}}{(B)_{H}(2 - B)_{H}} \frac{(4 + 3B)_{H}(6 + 3B)_{H}(2 + 2B)_{H}(4 + 2B)_{H}}{(4 + 4B)_{H}(6 + 2B)_{H}(2 + 3B)_{H}(4)_{H}} \]

\[ = \frac{(H/3 - 2)_{H}(4 - H/3)_{H}(2 + 2H/3)_{H}(4H/3 - 4)_{H}(H/3)_{H}(2H/3)_{H}}{(H - 4)_{H}(4)_{H}} \]

\[ = \{1\} \{H - 1\} \{1\} \{H/2\} \{1/2\}. \]  

(8.3)

Similarly,

\[ S_{d_2}^{(3)} \mid _{h \to H} = 2 \ 4 \ 6 \mid _{h \to H} = \{2\} \{10\} \{4\} \{8\} \{6\} \mid _{h \to H} \]

\[ = \{2\} \{h - 2\} \{h/3\} \{2h/3\} \{h/3 + 2\} \{2h/3 - 2\} \mid _{h \to H} \]

\[ = \{2\} \{H - 2\} \{H/3\} \{2H/3\} \{H/3 + 2\} \{2H/3 - 2\} \mid _{h \to H} \]

\[ = \{2\} \{H/3\} \{H/3 + 2\} \mid _{h \to H} = \{H/3\} \{2H/3\}. \]

\[ S_{d_2}^{(3)} \mid _{h \to H} = 1 \ 3^2 \ 5^2 \mid _{h \to H} = \{1\} \{h - 1\} \{3\} \{h - 3\} \{h/3 - 1\} \{2h/3 + 1\} \]

\[ \times \{2h/3 - 3\} \{h/3 + 3\} \{h/3 + 1\} \{2h/3 - 1\} \mid _{h \to H} \]

\[ = \{1\} \{H - 1\} \{3\} \{H - 3\} \{H/3 - 1\} \{2H/3 + 1\} \]

\[ \times \{2H/3 - 3\} \{H/3 + 3\} \{H/3 + 1\} \{2H/3 - 1\} \]

\[ = \{1\} \{3\} \{H/3 - 1\} \{H/3 + 3\} \{H/3 + 1\} \]

\[ = \{H/3 - 1\} \{2H/3 + 1\} \{H/3 + 1\} \{2H/3 - 1\} \]. \]  

(8.4)

where in the last line of each element we have omitted few steps and given simplified result. Results are again same as the ones obtained in the section 7 for the untwisted dual pair \( g_2^{(1)} \).
8.4 $d_{2n}^{(1)} \rightarrow a_{2n-1}^{(2)}$

Here we consider the example of $d_6^{(1)} \rightarrow a_5^{(2)}$, and show the result for just one element viz. $S_{33}^{a_5^{(2)}}$. For the rest we don’t have the results yet, but we have strong hope that they can be manipulated in the similar fashion. For $d_{6}^{(1)}$ the Coxeter $h=10$ and the renormalised Coxeter for the pair $(a_5^{(2)}, b_3^{(1)})$ is $6-B/2$.

$$
S_{33}^{a_5^{(2)}}|_{h\rightarrow H} = \{1\}{9\{5\}|_{h\rightarrow H}
= \frac{(B)(2-B)(8+B)(10-B)(4+B)(6-B)}{(0)(h-8)(8)(h)(4)(h-4)}|_{h\rightarrow H}
= \frac{(h-10+B)(2-B)(h-2+B)(10-B)(h-6+B)(6-B)}{(0)(H-8)(8)(H)(4)(H-4)}|_{h\rightarrow H}
= \frac{(H-10+B)(2-B)(H-2+B)(10-B)(H-6+B)(6-B)}{(0)(-2-B/2)(-10+B)(6-B/2)(4)(2-B/2)}
\frac{(-4+B/2)(2-B)}{(2)(6-B/2)(4)(2-B/2)}
= \frac{(2+B/2)(2-B)(4+B/2)(4-B)(B/2)(6-B)}{(0)(2-B/2)(2)(4-B/2)(4)(6-B/2)}
= \{1-B/4\}_{-1/4}\{3-B/4\}_{-1/4}\{5-B/4\}_{-1/4}
= \{H/2-2\}_{-1/4}\{H/2\}_{-1/4}\{H/2+2\}_{-1/4}
= \prod_{p=2}^{\infty} \{H/2-p\}_{-1/4}.
$$

(8.5)

9 Summary and discussion

In the present paper we made an attempt to extend the idea of folding and classical reduction of Toda field theories to the quantum case. We have shown that with the help of simple formulae viz. (3.2)–(3.5) one can construct the exact quantum $S$-matrices of the non-simply laced Toda field theories from the $S$-matrices of simply laced Toda theories by just replacing the Coxeter number appropriately. Here we concentrated on the $S$-matrices, but fact is that this idea of folding also works for the three point couplings.\(^{5}\) Couplings of non-simply laced theories can be evaluated from the couplings of simply laced theories by replacing the Coxeter number accordingly. For example, let us consider the coupling,\(^{5}\) For mass ratios folding will work in a straight forward manner and one will have floating mass ratios like the ones mentioned in Refs. 11, 11.
\( U_{12}^{1} = 5H/6 - 1 \) (in units of \( \pi/H \)) for the dual pair \( f_{4}^{(1)} \) and \( e_{6}^{(2)} \) (see expression (4.4) in the Ref. [11]). We show in the following that this can be obtained in two ways from the parent theories, \( e_{6}^{(1)} \) and \( e_{7}^{(1)} \).

\[
U_{12}^{1} (f_{4}^{(1)}, e_{6}^{(2)}) = C_{12}^{1} e_{6}^{(1)} |_{h \to H} = 9|_{h \to H} = 5h/6 - 1|_{h \to H} = 5H/6 - 1. \tag{9.1}
\]

Alternatively,

\[
U_{12}^{1} (f_{4}^{(1)}, e_{6}^{(2)}) = C_{24}^{2} e_{7}^{(1)} |_{h \to H} = 14|_{h \to H} = 5h/6 - 1|_{h \to H} = 5H/6 - 1. \tag{9.2}
\]

This may be one of reasons why folding works for the quantum \( S \)-matrices. One point lacking in our formalism is that there is no universal way of writing the block \( \{x\} \) in terms of \( h \). \( x \) should be written as a linear function of \( h \). Although certain patterns within a theory may be visible but for clarity one would seek some universal rule. At present we are investigating along these points. To conclude we think this is a new approach and in future will be able to shed some light on structure of \( S \)-matrices of Toda field theories.

**Acknowledgements**

I would like to thank Prof. Ryu Sasaki for reading the manuscript carefully and for making valuable comments and suggestions at various stages of the work.

**Appendix A:** \( e_{6}^{(1)} \rightarrow f_{4}^{(1)} \)

\[
S_{31}^{(4)} = S_{31}^{(6)} e_{6}^{(1)} |_{h \to H} = \{4\} \{6\} \{10\} \{2\} \{6\} \{8\} |_{h \to H} \\
= \{h/3\} \{h/3 + 2\} \{h - 2\} \{2\} \{2h/3 - 2\} \{2h/3\} |_{h \to H} \\
= \{H/3\} \{H/3 + 2\} \{H - 2\} \{2\} \{2H/3 - 2\} \{2H/3\} |_{h \to H} \\
= \{2\} |_{0}[H/3] |_{0}[H/3 + 2] \tag{A.1}
\]

\[
S_{41}^{(4)} = S_{41}^{(6)} e_{6}^{(1)} |_{h \to H} = 3 5 3 5 |_{h \to H} = \{3\} \{9\} \{5\} \{3\} \{9\} \{7\} |_{h \to H} \\
= \{3\} \{h - 3\} \{h/3 + 1\} \{2h/3 - 1\} \{h/3 - 1\} \times \{2h/3 + 1\} \{2h/3 - 3\} \{h/3 + 3\} |_{h \to H} \\
= \{3\} \{H - 3\} \{H/3 + 1\} \{2H/3 - 1\} |_{0} \times \{H/3 - 1\} |_{0} \{2H/3 + 1\} |_{0} \{2H/3 - 3\} |_{0} \{H/3 + 3\} |_{0} \\
= [3] |_{0}[H/3 + 1] |_{0}[H/3 - 1] |_{0}[H/3 + 3] \tag{A.2}
\]
\[ S^{i(1)}_{22} = S^{e(1)}_{22} S^{e(1)}_{22} |_{h \to H} = 15 1\overline{5}\ 1\overline{5} |_{h \to H} = \{1\} \{11\} \{5\} \{7\} \{1\} \{11\} \{5\} \{7\} |_{h \to H} \]

\[ \times \{2h/3 + 3\} \{2h/3 - 3\} \{h/3 + 3\} \]

\[ = \{1\} \{H - 1\} \{H/3 + 1\} \{2H/3 - 1\} \{H/3 - 3\} \]

\[ \times \{2H/3 + 3\} \{2H/3 - 3\} \{H/3 + 3\} \]

\[ = [1] \{H/3 + 1\} \{H/3 - 3\} \] (A.3)

\[ S^{i(1)}_{23} = S^{e(1)}_{23} S^{e(1)}_{23} |_{h \to H} = 3\ 5\ 3\ 5 |_{h \to H} = \{3\} \{9\} \{5\} \{7\} \{3\} \{9\} \{5\} \{7\} \]

\[ \times \{h/6 + 1\} \{5h/6 - 1\} \{h/6 + 3\} \{5h/6 - 3\} \]

\[ = [H/6 + 1\overline{0}] \{H/6 + 3\overline{0}\} \{H/6 + 3\overline{0}\} \] (A.4)

\[ S^{i(1)}_{24} = S^{e(1)}_{24} S^{e(1)}_{24} |_{h \to H} = 2\ 4\ 6\ 2\ 4\ 6 |_{h \to H} \]

\[ = \{2\} \{10\} \{4\} \{8\} \{6\} \{2\} \{10\} \{4\} \{8\} \{6\} \]

\[ = [H/6\overline{0}] \{H/6 + 2\overline{0}\} \{H/6 + 4\overline{0}\} \] (A.5)

\[ S^{i(1)}_{33} = S^{e(1)}_{33} S^{e(1)}_{33} |_{h \to H} = \{1\} \{3\} \{5\} \{7\} \{9\} \{3\} \{5\} \{7\} \{9\} \{11\} \]

\[ \times \{h/3 + 1\} \{2h/3 - 3\} \{2h/3 - 1\} \{2h/3 + 1\} \{h/3 + 1\} \]

\[ = [1] \{3\} \{2H/3 - 3\} \{2H/3 - 1\} \{2H/3 + 1\} \{H/3 - 1\} \] (A.6)

\[ S^{i(1)}_{43} = S^{e(1)}_{43} S^{e(1)}_{43} |_{h \to H} = 2\ 4\ 2\ 4\ 2\ 6 |_{h \to H} \]

\[ = \{2\} \{10\} \{4\} \{8\} \{6\} \{2\} \{10\} \{4\} \{8\} \{6\} \]

\[ = [2h - 2\overline{0}] \{h/3 + 1\} \{2h/3 - 2\} \{2h/3 - 2\} \{2h/3 - 2\} \]
For showing the (3.3)–(3.5) the calculations are done in following fashion.

\[ S_{44}^{(3)} = S_{44}^{(1)} S_{44}^{(1)} |_{h \to H} = 1^2 3^4 5^6 |_{h \to H} = \{1\}^2 \{9\}^4 \{5\}^6 \{7\}^6 |_{h \to H} \]

\[ = \{1\} \{h - 1\} \{h/3 - 3\} \{2h/3 + 3\} \{3\} \{h - 3\} \]

\[ \times \{h - 3 - 1\}^2 \{2h/3 + 1\}^2 \{2h/3 - 5\} \{h/3 + 5\} \{5\} \]

\[ \times \{h - 5\} \{h/3 + 1\}^3 \{2h/3 - 1\}^3 \{2h/3 - 3\}^2 \{h/3 + 3\}^2 |_{h \to H} \]

\[ = \{1\} \{H - 1\} \{H/3 - 3\} \{2H/3 + 3\} \{H - 3\} \{H/3 - 1\} \]

\[ \times \{2H/3 + 1\}^2 \{2H/3 - 5\} \{H/3 + 5\} \{5\} \{H - 5\} \]

\[ \times \{H/3 + 1\}^3 \{2H/3 - 1\}^3 \{2H/3 - 3\}^2 \{H/3 + 3\}^2 |_{h \to H} \]

\[ = [1]_0 [3]_0 [5]_0 [H/3 - 3]_0 [H/3 - 1]_0 [H/3 + 3]_0 [H/3 + 5]_0 \]  

(A.8)

For showing the (3.3)–(3.5) the calculations are done in following fashion.

\[ S_{14}^{(3)} = S_{14}^{(1)} |_{h \to H} = 3^5 |_{h \to H} = \{3\} \{9\} \{5\} \{7\} |_{h \to H} \]

\[ = (2) \{4\} \{8\} \{10\} \{4\} \{6\} \{6\} \{8\} \]

\[ = (2 + B) \{4 - B\} \{8 + B\} \{10 - B\} \{4 + B\} \{6 + B\} \{6 - B\} \{6 + B\} \{8 + B\} \]

\[ \times (2h/3) \{h/3 - 3\} \{2h/3 + 3\} \{h/3 - 1\} \{2h/3 + 1\} \{2h/3 - 5\} \{h/3 + 5\} \{5\} \}

\[ \times (h/3 - 3 + 2B) \{h/3 + 2 - B\} \{h/3 + 2 + B\} \{h/3 + 4 - B\} \]

\[ = \{2\} \{2h/3\} \{h/3\} \{h/3 + 2\} |_{h \to H} = \{2\} \{2H/3\} \{H/3\} \{H/3 + 2\} \]

\[ = \{H/6 + 1\} \{5H/6 - 1\} \{H/2 - 1\} \{H/2 + 1\} \]

\[ = \{H/6 + 1\} \{5H/6 - 1\} \{H/2 - 1\} \{H/2 + 1\} \]

(A.9)

\[ S_{22}^{(3)} = S_{22}^{(1)} |_{h \to H} = 1^5 |_{h \to H} = \{1\} \{11\} \{5\} \{7\} |_{h \to H} \]

\[ = \{0\} \{2\} \{10\} \{12\} \{4\} \{6\} \{6\} \{8\} \]

\[ = (B) \{2 - B\} \{10 + B\} \{12 - B\} \{4 + B\} \{6 + B\} \{6 + B\} \{8 - B\} \]

\[ \times (h/3 - 2) \{2h/3 + 2\} \{h\} \]

\[ \times (h/3 - 4 + B) \{2 - B\} \{h - 2 + B\} \{2h/3 + 4 - B\} \]

\[ \times (h/3) \{2h/3 - 2\} \{h/3 + 2\} \{2h/3\} \]

\[ = \{0\} \{2h/3 + 2\} \{h/3\} \{h/3 + 2\} |_{h \to H} \]

\[ = \{0\} \{2H/3 + 2\} \{H/3\} \{H/3 + 2\} \]
\[
S_{23}^{(i)} = S_{23}^{(i)}|_{h \rightarrow H} = 34_{h \rightarrow H} = \{3\} \{9\} \{5\} \{7\}|_{h \rightarrow H}
\]
\[
= \left(\frac{(2 + B)(4 - B)(8 + B)(10 - B)(4 + B)(6 - B)(6 + B)(8 - B)}{(2)(4)(8)(10)(4)(6)(6)(8)}\right)^{h \rightarrow H}
\]
\[
= \{h/6\}' \{h/2 + 2\}' \{h/6 + 2\}' \{h/2\}'|_{h \rightarrow H}
\]
\[
= \{H/6\}'_0 \{H/2 + 2\}'_0 \{H/6 + 2\}'_0 \{H/2\}'_0
\]
\[
= \{H/3 - 1\}'_{1/2} \{2H/3 + 1\}'_{1/2} \{H/3 + 1\}'_{1/2} \{2H/3 - 1\}'_{1/2}
\]
\[
= [H/3 - 1]_{1/2} [H/3 + 1]_{1/2}
\]  
\[
(\text{A.10})
\]

\[
S_{24}^{(i)} = S_{24}^{(i)}|_{h \rightarrow H} = 24_{h \rightarrow H} = \{2\} \{10\} \{4\} \{8\} \{6\} \{6\}|_{h \rightarrow H}
\]
\[
= \{h/6 - 1\}' \{h/2 + 3\}' \{h/6 + 1\}' \{h/2 + 1\}' \{h/6 + 3\}' \{h/2 - 1\}'|_{h \rightarrow H}
\]
\[
= \{H/6 - 1\}'_0 \{H/2 + 3\}'_0 \{H/6 + 1\}'_0 \{H/2 + 1\}'_0 \{H/6 + 3\}'_0 \{H/2 - 1\}'_0
\]
\[
= \{H/3 - 2\}'_{1/2} \{2H/3 + 2\}'_{1/2} \{H/3\}'_{1/2} \{2H/3\}'_{1/2} \{H/3 + 2\}'_{1/2} \{2H/3 - 2\}'_{1/2}
\]
\[
= [H/3 - 2]_{1/2} [H/3 + 2]_{1/2}
\]  
\[
(\text{A.11})
\]

\[
S_{34}^{(i)} = S_{34}^{(i)}|_{h \rightarrow H} = 24_{h \rightarrow H} = \{2\} \{10\} \{4\} \{8\} \{6\} \{6\}|_{h \rightarrow H}
\]
\[
= \{1\}' \{2H/3 + 1\}' \{2h/3 + 1\}' \{2h/3 - 1\}' \{h/3 + 1\}' \{h/3 + 1\}'|_{h \rightarrow H}
\]
\[
= \{1\}'_0 \{2H/3 + 1\}'_0 \{2h/3 + 1\}'_0 \{2h/3 - 1\}'_0 \{h/3 + 1\}'_0 \{h/3 + 1\}'_0
\]
\[
= \{H/6\}'_{1/2} \{5H/6\}'_{1/2} \{H/6 + 2\}'_{1/2} \{5H/6 - 2\}'_{1/2}
\]
\[
\times \{H/2 - 2\}'_{1/2} \{H/2 + 2\}'_{1/2} \{H/2\}'_{1/2}
\]
\[
= [H/6]'_{1/2} [H/6 + 2]'_{1/2} [H/2 + 2]'_{1/2} [H/2]'_{1/2}
\]  
\[
(\text{A.12})
\]

\[
S_{44}^{(i)} = S_{44}^{(i)}|_{h \rightarrow H} = 13_{h \rightarrow H} = \{1\} \{11\} \{3\} \{2\} \{9\} \{2\} \{5\} \{7\} \{3\}|_{h \rightarrow H}
\]
\[
= \{0\}' \{2h/3 + 2\}' \{2h/3 - 2\}' \{h/3 + 2\}' \{h/3 + 2\}'|_{h \rightarrow H}
\]
\[
= \{0\}'_0 \{2H/3 + 2\}'_0 \{2H/3 - 2\}'_0 \{H/3 + 2\}'_0 \{H/3 + 4\}'_0
\]
\[
\times \{4\}'_0 \{H/3\}'_0 \{2H/3 - 2\}'_0 \{H/3 + 2\}'_0 \{H/3 + 4\}'_0
\]
\[
= \{H/6 - 1\}'_{1/2} \{5H/6 + 1\}'_{1/2} \{H/6 + 1\}'_{1/2} \{H/2 - 3\}'_{1/2} \{5H/6 - 1\}'_{1/2}
\]
\[
\times \{H/2 + 3\}'_{1/2} \{H/6 + 3\}'_{1/2} \{H/2 - 1\}'_{1/2} \{5H/6 - 3\}'_{1/2} \{H/2 + 1\}'_{1/2}
\]
\[
= [H/6 - 1]_{1/2} [H/6 + 1]_{1/2} [H/2 - 3]_{1/2} [H/6 + 3]_{1/2} [H/2 + 1]_{1/2}
\]
\[
= [H/6 - 1]_{1/2} [H/6 + 1]_{1/2} [H/3]'_0 [H/2 + 1]_{1/2}
\]  
\[
(\text{A.13})
\]

\[
S_{44}^{(i)} = S_{44}^{(i)}\|_{h \rightarrow H} = 13_{h \rightarrow H} = \{1\} \{11\} \{3\} \{2\} \{9\} \{2\} \{5\} \{7\} \{3\}|_{h \rightarrow H}
\]
\[
= \{0\}' \{2h/3 + 2\}' \{2h/3 - 2\}' \{h/3 + 4\}' \{4\}'
\]
\[
\times \{h/3\}'^2 \{2h/3 - 2\}'^2 \|_{h \rightarrow H}
\]
\[
= \{0\}'_0 \{2H/3 + 2\}'_0 \{2H/3 - 2\}'_0 \{H/3 + 4\}'_0
\]
\[
\times \{4\}'_0 \{H/3\}'^2 \{2H/3 - 2\}'^2 \|_{h \rightarrow H}
\]
\[
= \{H/6 - 1\}'_{1/2} \{5H/6 + 1\}'_{1/2} \{H/6 + 1\}'_{1/2} \{H/2 - 3\}'_{1/2} \{5H/6 - 1\}'_{1/2}
\]
\[
\times \{H/2 + 3\}'_{1/2} \{H/6 + 3\}'_{1/2} \{H/2 - 1\}'_{1/2} \{5H/6 - 3\}'_{1/2} \{H/2 + 1\}'_{1/2}
\]
\[
= [H/6 - 1]_{1/2} [H/6 + 1]_{1/2} [H/2 - 3]_{1/2} [H/6 + 3]_{1/2} [H/2 + 1]_{1/2}
\]
\[
= [H/6 - 1]_{1/2} [H/6 + 1]_{1/2} [H/3]'_0 [H/2 + 1]_{1/2}
\]  
\[
(\text{A.14})
\]
Appendix B: $e_7(1) \rightarrow e_6(2)$

\[ S_{12}^{(2)} = S_{24}^{(1)} \big|_{h \rightarrow H} = 5, 7|_{h \rightarrow H} = \{5\}\{13\}\{7\}\{11\}|_{h \rightarrow H} \]
\[ = \{h/6 + 2\}\{5h/6 - 2\}\{h/2 - 2\}\{h/2 + 2\}|_{h \rightarrow H} \]
\[ = \{H/6 + 2\}_0\{5H/6 - 2\}_0\{H/2 - 2\}_0\{H/2 + 2\}_0 \]
\[ = [H/6 + 2]_0[\frac{H}{2} + 2]_0 \]  
(B.1)

\[ S_{13}^{(2)} = S_{25}^{(1)} \big|_{h \rightarrow H} = 2, 6, 8|_{h \rightarrow H} = \{2\}\{16\}\{6\}\{12\}\{8\}\{10\}|_{h \rightarrow H} \]
\[ = \{2\}\{h - 2\}\{h/3\}\{2h/3\}\{h/3 + 2\}\{2h/3 - 2\}|_{h \rightarrow H} \]
\[ = \{2\}_0\{H - 2\}_0\{H/3\}_0\{2H/3\}_0\{H/3 + 2\}_0\{2H/3 - 2\}_0 \]
\[ = [2]_0[\frac{H}{3} + 2]_0 \]  
(B.2)

\[ S_{14}^{(2)} = S_{27}^{(1)} \big|_{h \rightarrow H} = 3, 5, 7, 9|_{h \rightarrow H} = \{3\}\{15\}\{5\}\{13\}\{7\}\{11\}\{9\}\{9\}|_{h \rightarrow H} \]
\[ = \{3\}\{h - 3\}\{h/3 - 1\}\{2h/3 + 1\}\{h/3 + 1\} \]
\[ \times \{2h/3 - 1\}\{h/3 + 3\}\{2h/3 - 3\}|_{h \rightarrow H} \]
\[ = \{3\}_0\{H - 3\}_0\{H/3 - 1\}_0\{2H/3 + 1\}_0\{H/3 + 1\}_0 \]
\[ \times \{2H/3 - 1\}_0\{H/3 + 3\}_0\{2H/3 - 3\}_0 \]
\[ = [3]_0[\frac{H}{3} - 1]_0[\frac{H}{3} + 1]_0[\frac{H}{3} + 3]_0 \]  
(B.3)

\[ S_{22}^{(2)} = S_{44}^{(1)} \big|_{h \rightarrow H} = 1, 3, 7, 9|_{h \rightarrow H} = \{1\}\{17\}\{3\}\{15\}\{7\}\{11\}\{9\}\{9\}|_{h \rightarrow H} \]
\[ = \{1\}\{h - 1\}\{h/3 - 3\}\{2h/3 + 3\}\{h/3 + 1\} \]
\[ \times \{2h/3 - 1\}\{h/3 + 3\}\{2h/3 - 3\}|_{h \rightarrow H} \]
\[ = \{1\}_0\{H - 1\}_0\{H/3 - 3\}_0\{2H/3 + 3\}_0\{H/3 + 1\}_0 \]
\[ \times \{2H/3 - 1\}_0\{H/3 + 3\}_0\{2H/3 - 3\}_0 \]
\[ = [1]_0[\frac{H}{3} - 3]_0[\frac{H}{3} + 1]_0[\frac{H}{3} + 3]_0 \]  
(B.4)

\[ S_{23}^{(2)} = S_{45}^{(1)} \big|_{h \rightarrow H} = 4, 6, 2, 8|_{h \rightarrow H} = \{4\}\{14\}\{6\}\{12\}\{6\}\{12\}\{8\}\{10\}|_{h \rightarrow H} \]
\[ = \{h/6 + 1\}\{5h/6 - 1\}\{h/6 + 3\}\{5h/6 - 3\} \]
\[ \times \{h/2 - 3\}\{h/2 + 3\}\{h/2 - 1\}\{h/2 + 1\}|_{h \rightarrow H} \]
\[ = \{H/6 + 1\}_0\{5H/6 - 1\}_0\{H/6 + 3\}_0\{5H/6 - 3\}_0 \]
\[ \times \{H/2 - 3\}_0\{H/2 + 3\}_0\{H/2 - 1\}_0\{H/2 + 1\}_0 \]
\[ = [H/6 + 1]_0[\frac{H}{6} + 3]_0[\frac{H}{2} + 1]_0[\frac{H}{2} + 3]_0 \]  
(B.5)

\[ S_{24}^{(2)} = S_{47}^{(1)} \big|_{h \rightarrow H} = 3, 5, 2, 7^2, 9|_{h \rightarrow H} = \{3\}\{15\}\{5\}^2\{13\}^2\{7\}^2\{11\}^2\{9\}\{9\}|_{h \rightarrow H} \]
\[ S^{(2)}_{33} = |h_{-H} = 1 3 5 7^2 9|_{h_{-H}} = \{1\}17\{3\}15\{5\}13\{7\}2\{11\}2\{9\}|_{h_{-H}} \]
\[ = \{1\}1\{3\}3\{h/3 - 1\}\{2h/3 + 1\} \times \{h/3 + 1\}2\{2h/3 - 1\}2\{h/3 + 3\}\{2h/3 - 3\}|_{h_{-H}} \]
\[ = \{1\}0\{H - 1\}0\{3\}0\{H - 3\}0\{H/3 - 1\}0\{2H/3 + 1\}0 \times \{H/3 + 1\}3\{2H/3 - 1\}3\{H/3 + 3\}0\{2H/3 - 3\}0 \]
\[ = [1]0[3]0[H/3 - 1]0[H/3 + 1]3[H/3 + 3]0 \] (B.7)

\[ S^{(2)}_{34} = S^{(1)}_{37} \]
\[ = \{2\}1\{4\}2\{14\}2\{6\}2\{12\}2\{8\}2\{10\}2|_{h_{-H}} \]
\[ = \{2\}0\{H - 2\}0\{4\}0\{H - 4\}0\{H/3 - 2\}0\{2H/3 + 2\}0 \times \{H/3 + 1\}3\{2H/3 - 1\}3\{2H/3 - 2\}3\{H/3 + 4\}\{H/3 + 4\}0 \]
\[ = [H/6]0[H/6 + 2]0[H/2 + 4]0[H/6 + 4]0[H/2 + 2]0[H/2]0 \] (B.8)

\[ S^{(2)}_{44} = S^{(1)}_{47} \]
\[ = |h_{-H} = 1 3^2 5^3 7^4 9^2|_{h_{-H}} \]
\[ = \{1\}1\{3\}2\{15\}2\{5\}3\{13\}3\{7\}4\{11\}4\{9\}2\{9\}|_{h_{-H}} \]
\[ = \{1\}1\{3\}3\{h/3 - 3\}\{2h/3 + 3\}\{5\} \times \{h/3 - 5\}3\{h/3 + 1\}3\{2h/3 + 1\}3\{h/3 - 1\}3 \times \{2h/3 + 5\}3\{h/3 + 3\}3\{2h/3 - 3\}3 |_{h_{-H}} \]
\[ = \{1\}0\{H - 1\}0\{3\}0\{H - 3\}0\{H/3 - 3\}0\{2H/3 + 3\}0\{5\}0 \times \{H - 5\}0\{H/3 - 1\}3\{2H/3 + 1\}3\{H/3 + 1\}3\{2H/3 - 1\}3 \times \{2H/3 - 5\}0\{H/3 + 3\}3\{2H/3 - 3\}3 \]
\[ = [1]0[3]0[H/3 - 3]0[5]0[H/3 - 1]3[H/3 + 1]3[H/3 + 5]0[H/3 + 3]3 \] (B.9)

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