The primordial non-Gaussianity of local type ($f_{\text{NL}}^{\text{local}}$) in the WMAP 5-year data: the length distribution of CMB skeleton

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ABSTRACT
We present skeleton studies of non-Gaussianity in the Cosmic Microwave Background temperature anisotropy observed in the five-year Wilkinson Microwave Anisotropy Probe (WMAP) data. The local skeleton is traced on the 2D sphere by cubic spline interpolation which leads to more accurate estimation of the intersection positions between the skeleton and the secondary pixels than conventional linear interpolation. We demonstrate that the skeleton-based estimator of non-Gaussianity of the local type ($f_{\text{NL}}^{\text{local}}$) - the departure of the length distribution from the corresponding Gaussian expectation - yields an unbiased and sufficiently converged likelihood function for $f_{\text{NL}}^{\text{local}}$.

We analyse the skeleton statistics in the WMAP 5-year combined V- and W-band data outside the Galactic base-mask determined from the KQ75 sky-coverage. The results are consistent with Gaussian simulations of the the best-fitting cosmological model, but deviate from the previous results determined using the WMAP 1-year data. We show that it is unlikely that the improved skeleton tracing method, the omission of Q-band data, the modification of the foreground-template fitting method or the absence of 6 extended regions in the new mask contribute to such a deviation. However, the application of the Kp0 base-mask in data processing does improve the consistency with the WMAP1 results.

The $f_{\text{NL}}^{\text{local}}$-likelihood functions of the data are estimated at 9 different smoothing levels. It is unexpected that the best-fit values show positive correlation with the smoothing scales. Further investigation argues against a point-source or goodness-of-fit explanation but finds that about 30% of either Gaussian or $f_{\text{NL}}^{\text{local}}$ samples having better goodness-of-fit than the WMAP 5-year data show a similar correlation. We present the estimate $f_{\text{NL}}^{\text{local}} = 47.3 \pm 34.9$ (1σ error) determined from the first four smoothing angles and $f_{\text{NL}}^{\text{local}} = 76.8 \pm 43.1$ for the combination of all nine. The former result may be overestimated at the 0.21σ-level because of point sources.

Key words: methods: data analysis – cosmic microwave background.

1 INTRODUCTION

Generic inflationary models predict that the initial conditions of the post-inflation universe can be described
by a Gaussian random-phase field with nearly scale-invariant fluctuations. These subsequently seed the perturbations that generate Cosmic Microwave Background (CMB) anisotropies and structure formation thereafter. The Gaussianity of the statistics determined from measures of the CMB anisotropy and large scale structure distribution can provide evidence that validates the inflationary scenario of the extremely-early Universe. Besides the simplest single-scalar field model that predicts a truly Gaussian initial condition (Guth 1981; Bardeen, Steinhardt & Turner 1983; Mukhanov, Feldman & Brandenberger 1992), there are a number of inflationary models predicting non-Gaussianity in two broad classifications, the equilateral type and the local type. The detection of a specific type of non-Gaussianity can shed light on the fundamental physical properties of inflation.

In this paper, we are concerned with a local type non-Gaussianity of the “simplest weak nonlinear coupling” case (Komatsu & Spergel 2001)

Φ(x) = ΦL(x) + fNL local (ΦL(x) − Φ̅(x)),

where Φ(x) denotes the primordial curvature perturbations and ΦL is its linear Gaussian part. The amplitude of the non-Gaussianity is parameterised by the dimensionless coupling constant fNL (fNL hereafter). The first observational constraint on fNL -3500 < fNL < 2000 at 95% C.L. - was discussed by Komatsu et al. (2002) using the angular bispectrum computed from the four-year COBE DMR data (Bennett et al. 1996). A reduced bispectrum technique, hereafter the KSW estimator, (Komatsu, Spergel & Wandelt 2005), was applied to the first-year and three-year WMAP data, leading to -58 < fNL < 134 (Komatsu et al. 2003) and -54 < fNL < 114 (Spergel et al. 2007), respectively. Yadav & Wandelt (2008) employed an apparently improved estimator to obtain 27 < fNL < 147 for the V+W-band data outside the Kp0 mask with ℓmax = 750 with the three-year WMAP data. The WMAP team used the same estimator to measure fNL from the five-year WMAP V+W-band outside the KQ75 mask with ℓmax = 700 and obtained -9 < fNL < 111.

The possibility of detecting CMB non-Gaussianity using a group of morphological statistics, - Minkowski functionals (MFs) (Matsubara 2003; Hikage, Komatsu & Matsubara 2006) - has also been studied. The departure of MFs from their Gaussian expectations has been tested to be an unbiased estimator for fNL and then applied to the WMAP3-year Q+V+W combined map yielding -70 < fNL < 91 at the 95% C.L. (Hikage et al. 2008). The WMAP team re-investigated the MFs estimator with the 5-year template-cleancleaned V+W map outside the KQ75 mask, yielding fNL = -57 ± 60 (68% C.L.) at resolution Nside = 128 and fNL = -68 ± 69 at Nside = 64 (Komatsu et al 2009). It is still unclear why the MFs favour a negative best-fit amplitude for fNL while the bispectrum estimator prefers a positive one, even though the MFs can be formed by the weighted sum of the bispectrum. Thus it is of great importance to use different estimators to identify and investigate the weak non-Gaussian signal in WMAP observations. In fact, the one-point probability density function (1-pdf) of the smoothed temperature field can also be implemented (Bernardeau et al. 2002) as an alternative non-Gaussianity estimator (Jeong & Smoot 2007). Indeed, as noted by Novikov, Colambi & Dore (2006), the normalised differential length of the skeleton is closely linked to this quantity, but the skeleton remains of interest due to its different sensitivity to specific aspects of the data, e.g. the noise distribution. It is likely that a complete understanding of the data can only be realised after the application of a wide range of statistical tests.

The skeleton has been considered as a probe of the filamentary structures of a 2D or 3D smooth random field. The original definition of the skeleton is non-local, making the analytical discussion difficult and the numerical evaluation costly. Novikov, Colambi & Dore (2006) first proposed a local approximation that “the local skeleton is given by the set of points where the gradient is aligned with the local curvature major axis and where the second component of the local curvature is negative”. They also presented a numerical approach to trace the local skeleton and found an approximate expression for the differential length distribution of a Gaussian field. As another morphological statistical test, the method has been applied to both large-scale structure measures (Sousbie et al. 2006, 2008) and CMB anisotropies (Eriksen et al. 2004). The latter was performed on the Q+V+W map of the first-year WMAP data outside a base-mask that is defined on the Kp0 sky-coverage. Comparing with Gaussian simulations, the length distribution of the skeleton did not show significant deviation from the Gaussian predictions. The impact of non-excluded point sources was found to be small for the statistics concerned.

In parallel to studies of non-Gaussian signal estimators, several algorithms of simulating non-Gaussian realisations have been developed. Komatsu et al. (2003) first simulated the local-type non-Gaussian component by integrating the spherical harmonics of ΦL(x) − V−1 ∫ d3xΦL(x) in spherical harmonic space. Another strategy has been developed in which a pre-computed “filter” encoding the correlation properties of Gaussian curvature perturbation multipoles boosts the computation of high-resolution temperature and polarisation Gaussian and corresponding non-Gaussian maps (Liguori, Matarrese & Moscardini 2003; Liguori et al. 2007). This method was recently improved by Elsner & Wandelt (2009). Such fNL simulation methods provide the community with powerful tools to investigate the primordial non-Gaussianity and the impact of other astrophysical and systematic effects on it.

In this paper, the skeleton length distribution is adopted as an estimator of the local-type non-Gaussianity. We adopt the cubic spline interpolation to trace the underlying local skeleton rather than the conventional linear one to make a more accurate estimation of the intersection position between the skeleton and pixel edge. Motivated by MFs studies on fNL, the statistical properties of the skeleton length distribution and the convergence of an fNL estimation methodology are investigated from the fNL simulations. We then analyse the skeleton statistics in the five-year release of the WMAP data and compare with both Gaussian and non-zero fNL samples. The results of the null Gaussian test are compared with those of Eriksen et al. (2004) for the first-year WMAP data, and then we use the skeleton estimator to compute a likelihood estimate for fNL.

This paper is organised as follows. In Section 2, we carry out numerical studies on the CMB local skeleton, includ-
2 NUMERICAL STUDIES ON CMB LOCAL SKELETON

According to the approximation made by Novikov, Colombi & Doré (2006), the local skeleton on a smooth 2D sphere \( \rho(\mathbf{r}) \), traces those points where the gradient of \( \rho \) is the eigenvector of the corresponding Hessian matrix. That is, it satisfies the characteristic equation

\[
\mathcal{H} \nabla \rho = \lambda \nabla \rho
\]

with \( \lambda (\lambda_1 > \lambda_2; \lambda_3 < 0) \) the eigenvalues, where \( \mathcal{H} \equiv \partial^2 \rho / \partial r_i \partial r_j \) is the Hessian matrix at position \( \mathbf{r} \). Identically with Eriksen et al. (2004), we do not specify the condition of eigenvalues of the local linear system. In other words, the skeleton in our analysis is considered as the set of underlying zero-contour lines of the realisation

\[
\mathcal{S} = \rho_x \rho_y (\rho_{xx} - \rho_{yy}) + \rho_{xy} (\rho_y^2 - \rho_x^2),
\]

where \( \rho_x \) and \( \rho_y \) denote the first and second derivatives of \( \rho(\mathbf{r}) \) in two orthogonal directions, \( x \) and \( y \). As for the CMB temperature field \( T(\mathbf{n}) \), the ‘skeleton map’ \( \mathcal{S} \) is re-expressed as

\[
\mathcal{S} = T_{,\theta} T_{,\phi} (T_{,\theta} - T_{,\phi}) + T_{,\phi} (T_{,\theta}^2 - T_{,\phi}^2),
\]

where the semicolons denote the covariant derivatives and the definite expression of them can be found in Schmalzing & Górski (2002).

The method for tracing the local skeleton in the HEALPix scheme has been reviewed in detail by Eriksen et al. (2004). In Appendix A we seek to optimise the method by applying the cubic spline interpolation for estimating the underlying positions of skeleton ‘knots’ on the pixelised sphere. The resulting skeleton statistics are introduced and tested for their applicability to non-Gaussian signal detection and \( f_{\text{NL}} \) estimation.

2.1 The statistics

In this work, the CMB temperature realisation intended for skeleton analysis, \( T(\mathbf{n}) \), is first normalised as,

\[
\nu(\mathbf{n}) = \frac{T(\mathbf{n})}{\sigma}.
\]

The standard deviation \( \sigma \) is computed over the valid region of each realisation after application of an adequate smoothing process (Section 2.2).

We utilise the skeleton length distribution function of the normalised temperature thresholds \( \nu \), as a probe of non-Gaussianity and to construct an estimator of \( f_{\text{NL}} \). As with any probability density function, there are two types of distributions quantifying the skeleton length, the differential pdf

\[
\mathcal{L}_d(\nu) = \frac{1}{L_{\text{tot}}} \frac{dL(\nu)}{d\nu}
\]

and the cumulative one

\[
\mathcal{L}_\nu(\nu) = \int_{\nu}^{+\infty} \mathcal{L}_d(\nu') d\nu',
\]

where the normalisation factor \( L_{\text{tot}} = \int_{-\infty}^{+\infty} dL(\nu) \) is the total length.

These two functions are equivalent and should lead to consistent results. In the first investigation of the statistical properties of the skeleton length in the WMAP data (Eriksen et al. 2004), the cumulative form was utilised and compared with the predictions of a Gaussian model. In our analysis, both the differential and cumulative functions are computed.

2.2 The idealised skeleton-\( f_{\text{NL}} \)-test

We study the signature of the local-type non-Gaussianity as a function of \( f_{\text{NL}} \) on the skeleton length distributions, \( \mathcal{L}_d(\nu) \) and \( \mathcal{L}_\nu(\nu) \). As a necessary precursor to \( f_{\text{NL}} \)-estimation, we establish that our estimators lead to an unbiased and sufficiently converged \( f_{\text{NL}} \)-likelihood by analysing noise-free full-sky realisations with a non-Gaussian signal component. The test is based on simulations of the CMB anisotropy as a function of \( f_{\text{NL}} \). We adopt the algorithm proposed by Liguori, Matarrese & Maccarrone (2003); Liguori et al. (2007) and recently improved by Elsner & Wandelt (2008) to simulate a set of Gaussian realisations \( \{a_{\ell m}^G\} \) with corresponding non-Gaussian components \( \{a_{\ell m}^N\} \). The cosmological parameters adopted for the \( f_{\text{NL}} \) simulations are those determined for the WMAP5 best-fit \( \Lambda \)CDM model (Komatsu et al. 2009). Specifically, the following parameters are adopted: \( \Omega_m = 0.142, \Omega_b h^2 = 0.02273, \Delta_k^2(k_0 = 0.002\text{Mpc}^{-1}) = 2.41 \times 10^{-9}, h = 0.719, n_s = 0.963, \) and \( \tau = 0.087 \). There are a total of 2500 simulated \( \{a_{\ell m}^G, a_{\ell m}^N\} \) pairs in this test that include power up to a maximum multipole \( \ell_{\text{max}} = 1024 \).

Pixelised skymaps with different \( f_{\text{NL}} \) values are therefore obtained following the relation

\[
T(p, f_{\text{NL}}) = \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} (a_{\ell m}^G + f_{\text{NL}} a_{\ell m}^N) b_{\ell} Y_{\ell m}(p),
\]

where \( b_{\ell} \) is a Gaussian beam transfer function with FWHM = 30' and 60' in this test. The first and second derivatives of the map can be computed by the HEALPix routine \texttt{alm2map.der}. Using the method discussed in Appendix A the skeleton length distribution \( \mathcal{L}(\nu, f_{\text{NL}}) \) can then be estimated from the skeleton map. In this process, the normalised temperature threshold is set to \( \nu \in [-4.0, 4.0] \) with 25 uniform bins.

Given the additive nature of the non-Gaussian component, it is reasonable to express \( \mathcal{L}(\nu) \) as

\[
\mathcal{L}(\nu, f_{\text{NL}}) = \mathcal{L}^G(\nu) + \mathcal{L}^N(\nu, f_{\text{NL}}).
\]
For each $L(\nu, f_{NL})$ sample, the non-Gaussian component can be estimated as

$$L^{NG}(\nu, f_{NL}) = L(\nu, f_{NL}) - \langle L^{G}(\nu) \rangle,$$  

(10)

where $\langle L^{G}(\nu) \rangle$ gives the Gaussian expectation of the skeleton length. We depict the samples of $L^{NG}(\nu, f_{NL} = 0, \pm 150)$ in Figure 1. The grey bands indicate the $1\sigma$ and $2\sigma$ confidence regions of a purely Gaussian ensemble, $f_{NL} = 0$. It is noteworthy that the behaviour of the non-Gaussian expectation values $L^{NG}(\nu, f_{NL})$ for both the differential and cumulative distributions have a characteristic variation with threshold. It is similar to MFs in that the peak-trough order and the amplitude of such features indicate the sign and the magnitude of $f_{NL}$, respectively. This suggests that the skeleton can be considered as another morphological $f_{NL}$-estimator, which may lead to deeper understanding of the underlying non-Gaussian properties of the observations. However, with respect to the $1\sigma$ error of $L^{NG}$, the fluctuation is roughly within the $2\sigma$ range of Gaussian predictions, even with $f_{NL} = 150$ which is larger than the $95\%$ confidence level upper limit for recent $f_{NL}$-estimations using WMAP data. It would still be challenging for a skeleton estimator to provide a firm Gaussian/non-Gaussian assessment using the observed data.

Considering only the diagonal elements of the covariance matrix, we use 2000 simulations to estimate $\langle L^{G}(\nu) \rangle$, the mean and the standard deviation of $L^{NG}(\nu, f_{NL})$. The 500 remaining simulations are used to compute the $\chi^2$ functions. Given a hypothetical value of $f_{NL}^{true}$ for each $f_{NL}$-skeleton sample with index $i$ ($i = 1, 2, ...500$), is computed as

$$\chi^2(f_{NL}^{true}|f_{NL}) = \sum_{\nu} \left[ \frac{L^{NG}(\nu, f_{NL}^{true}) - \langle L^{NG}(\nu, f_{NL}) \rangle}{\sigma(L^{NG}(\nu, f_{NL}^{true}))} \right]^2,$$

(11)

where the correlations between bins have not been taken into account because the full covariance matrix is not sufficiently converged for the available sample volume in our analysis. Further tests indicate that the corresponding likelihood from each sample is of bimodal or even multi-modal shape if the full covariance matrix is adopted, which causes the estimation to be unrevealing.

The parameter $f_{NL}$ is uniformly sampled from $-300$ to $300$ with a step-length $\Delta f_{NL} = 5$. We estimate the likelihoods for three specific $f_{NL}^{true}$ values, $0$ and $\pm 150$. The posterior PDF for $f_{NL}^{true}$ can be obtained by Bayes’ theorem

$$P(f_{NL}^{true}|f_{NL}^{true}) \propto P(f_{NL}^{true}) \times \chi^2(f_{NL}^{true}|f_{NL}^{true}),$$

(12)

where we have conservatively set the prior $P(f_{NL}^{true})$ to be uniform and $N$ equals to 500. In fact, we have found that roughly 20 samples with FHWM= 30’ smoothing are adequate for the posterior distribution to converge sharply around $f_{NL}^{true}$ with a $1\sigma$ error $\Delta f_{NL} = 5$. However, we have only one observed CMB sample so that the convergence of the consequent posterior distribution is limited by the data resolution and the noise level. The effective likelihood functions of each sample, i.e., $\chi^2(f_{NL}^{true}|f_{NL}^{true})$, are illustrated in Figure 2 using different normalisation factors for visual convenience. The histograms depict the computed likelihoods which are perfectly fitted by Gaussian functions.
Figure 2. exp \left\{ -\frac{1}{2} \sum_{i=1}^{N} \chi^2(f_{\text{true}}^i \mid f_{\text{NL}}^i) \right\}, the effective likelihood functions computed by the differential estimator $\mathcal{L}_d$ with input parameter $f_{\text{true}}^i = 0, \pm 150$. The actual functions are renormalised by different factors for visual convenience and are shown by histograms. They are extremely well fitted by the Gaussian functions depicted by solid curves. The higher, narrower (lower, wider) histograms and curves correspond to likelihoods from simulations with FWHM = 30′ (60′).

Accordingly, the mean and the 1σ width of each likelihood are estimated as presented in Table 1.

The results demonstrate a good recovery of the input $f_{\text{NL}}$ values given the interval $\Delta f_{\text{NL}} = 5$ of our sampling. The $\chi^2(f_{\text{true}} \mid f_{\text{NL}})$ in Eq. [1] therefore constitutes an unbiased maximum likelihood position in $f_{\text{NL}}$-space and the corresponding 1σ error is determined by the likelihood function. It is noteworthy that the cumulative estimator behaves a little bit better than the differential one and therefore the former is selected for $f_{\text{NL}}$ estimation as applied to real data.

3 METHOD

Even though the literature contains theoretical predictions for the length distributions of the local skeleton on a 2D Gaussian random field, our analysis compares measures derived from simulated observations of the sky with the corresponding values for the WMAP data, since the inhomogeneous noise contribution and the complicated sky-coverage render analytical investigation difficult. Furthermore, it is also difficult to interpret the non-Gaussian component of these skeleton measures analytically. In what follows, we introduce both the instrumental properties impacting the observed data and the essential numerical processing steps required for further analysis.

3.1 The WMAP data and the simulations

The WMAP instrument measures the CMB temperature anisotropy in five frequency bands from 23 to 94 GHz (Bennett et al. 2003a). The foreground-reduced sky maps in V and W-band are used in our analysis, identical to the data selection for the WMAP five-year power spectrum estimation (Nolta et al. 2009). These maps are available in the HEALPix pixelisation scheme with $N_{\text{side}} = 512$ from the

LAMBDA website. The maps from two (four) differencing assemblies (DAs) at V- (W-) band are combined using uniform weights over the sky and equal weights for each DA. The resulting maps in V- and W-band are then combined to obtain the VW-band map using the same method. The effective beam transfer function of the VW-band map can then be easily computed from the beam functions of those DAs constituting the VW-band map. The observational data are inevitably affected by the instrumental noise, dominated by an uncorrelated component with a variance per pixel depending on the noise amplitude $\sigma_n$ and the pixel scanning strategies of each DA, $N_{\text{obs}}(p)$ (Bennett et al. 2003a).

The extended temperature analysis mask (KQ75) is adopted to minimise the contamination from the diffuse Galactic foreground and point source emission. For further investigations, the part related to the Galactic emission is separated out to form a base-mask called ‘KQ75B’ in our analysis. As a comparison of the base-masks used in the 1-, 3- and 5-year WMAP data analyses, we illustrate the KQ75B (adopted by Eriksen et al. 2004) and KQ75B mask in Figure 3. Besides the extended Galactic profile, there are six extended regions (labeled from ‘1’ to ‘6’) eliminated by the new base-mask. The impact of these regions on skeleton statistics will be considered when comparing our results with those of Eriksen et al. (2004) for WMAP1.

The 2500 pairs of Gaussian and non-Gaussian realisations \{\{a_{\ell m}^G, a_{\ell m}^{NG}\}\} introduced in Section 2.2 are used for our $f_{\text{NL}}$ studies. For each $f_{\text{NL}}$ value, we construct a map with resolution parameter $N_{\text{side}} = 512$ and WMAP instrumental properties as

$$T(p, f_{\text{NL}}) = \sum_{l=2}^{l_{\text{max}}} \sum_{m=-l}^{l} \left( a_{\ell m}^G + f_{\text{NL}} a_{\ell m}^{NG} \right) b_l p_l Y_{\ell m}(p) + \frac{\sigma_n}{N_{\text{obs}}(p)},$$

where $b_l$ is the effective beam transfer function of the

Figure 3. Comparison of the KQ75 base-mask (KQ75B) used in WMAP5 data processing with the Kq0 base-mask (KQ75B) used for WMAP1 and WMAP3 analysis. The white (light-grey) regions are excluded (included) by both KQ75B and Kq0B. The dark-grey (black) parts are excluded by the former (latter) but not excluded by the latter (former). There are 6 extended regions (labeled from ‘1’ to ‘6’) eliminated in the KQ75B mask.

\footnote{http://lambda.gsfc.nasa.gov/product/map/dr3/maps_fla_forered_5yr_5yr_get.cfm}

\footnote{http://lambda.gsfc.nasa.gov/product/map/dr3/beam_info.cfm}
WMAP VW-band data and $p_{\rm f}$ is the pixelisation window function for $N_{\rm side} = 512$. The second term on the rhs simulates the noise contribution on each pixel with Gaussian random number $\eta \sim N(0, 1)$.

In this work, we perform both a Gaussian frequentist test and $f_{\sigma_{\rm NL}}$-estimations. In the former, the Gaussian simulations are processed in the same way as Eq. [13] but free of the $f_{\sigma_{\rm NL}}$ term.

### 3.2 Data processing and the analysis

In this section, we introduce the data processing methods applied to both the observed and the simulated realisations for studies of the skeleton length distribution. The processing steps presented here follow the strategy detailed in Section 4 of Eriksen et al. (2004).

#### 3.2.1 Map processing

The base-mask is applied to the map to avoid Galactic foreground contamination. Following the methodology of Eriksen et al. (2004), we do not exclude point sources, in particular because any additional smoothing applied to the mask reduces the sky coverage available for analysis dramatically. This approach is supported by studies of the spectral parameter, $\gamma$, by Eriksen et al. (2004), which indicates that smoothing of the data renders the skeleton less sensitive to point source signal contributions for larger FWHMs. Moreover, a median-filter technique is applied to the point sources to investigate their impact on the skeleton statistics for smaller smoothing FWHMs. Specifically, for a given pixel $i$ that would be eliminated by the point-source mask, we consider all other unmasked pixels within a $1^\circ$ radius and determine the median temperature for this set of pixels. The temperature at pixel $i$ is then replaced by this median value, and the process repeated for all pixels specified in the point-source mask. The median-filtered map is then analysed in the same manner as the unfiltered data set.

Following standard procedure in CMB data analysis, the monopole and dipole components are fitted and re-estimated in the same manner as the unfiltered data set. The median-filtered map is then analysed in the same way as Eq. [13] but free of the $f_{\sigma_{\rm NL}}$ term.

While the invalid pixels are abandoned for computing the standard deviation. Using the method discussed in Appendix A the skeleton length distributions, $L_i(\nu)$ and $L_{\sigma}(\nu)$, can be estimated for each set of smoothed samples. The original distribution $L(\nu)$ in Eq. [9] is divided into 200 bins with $\nu \in [-4.0, 4.0]$ during skeleton tracing.

#### 3.2.2 Non-Gaussian detector and estimator

From the processed Gaussian simulations, we compute the Gaussian expectation of the skeleton statistics for each smoothing scale, $\langle L^G(\nu, \theta_{\rm FWHM}) \rangle$. The departure from these expectation values is then obtained for both the observed data and each Gaussian sample as

$$\Delta L(\nu, \theta_{\rm FWHM}) = L(\nu, \theta_{\rm FWHM}) - \langle L^G(\nu, \theta_{\rm FWHM}) \rangle,$$

where we omit the $\langle \Delta L \rangle$ term since it is definitely zero. In the $f_{\sigma_{\rm NL}}$ analysis, the non-Gaussian departure, $L^{NG}(\nu, f_{\sigma_{\rm NL}}, \theta_{\rm FWHM})$, and the $\chi^2$ statistics are estimated by Eqs [10] and [11].

The best-fit value and error of $f_{\sigma_{\rm NL}}$ can then be obtained by analysing the likelihood function as discussed in Section 4.

Before we provide final estimates of $f_{\sigma_{\rm NL}}$ from the different smoothing scales, we combined the estimators, $\Delta L(\nu)$ of the data and $L^{NG}(\nu, f_{\sigma_{\rm NL}})$ of each set of $f_{\sigma_{\rm NL}}$ sample, to

$$\Delta L_C(\nu, f_{\sigma_{\rm NL}}) = \sum_{i=1}^{N_{\rm fwhm}} w_i(\nu, f_{\sigma_{\rm NL}}) \Delta L^i(\nu)$$

and

$$L^{NG}_C(\nu, f_{\sigma_{\rm NL}}) = \frac{\sum_{i=1}^{N_{\rm fwhm}} w_i(\nu, f_{\sigma_{\rm NL}}) L^{NG,i}(\nu, f_{\sigma_{\rm NL}})}{N_{\rm fwhm}}$$

respectively with the inverse-variance weighting

$$w_i(\nu, f_{\sigma_{\rm NL}}) = \frac{1/\sigma_i^2 L^{NG,i}(\nu, f_{\sigma_{\rm NL}})}{\sum_{i=1}^{N_{\rm fwhm}} 1/\sigma_i^2 L^{NG,i}(\nu, f_{\sigma_{\rm NL}})}$$

where $i$ corresponds to one smoothing scale and $N_{\rm fwhm}$ represents the number of scales used in the combination. The combined $\chi^2$ is then computed

$$\chi^2(f_{\sigma_{\rm NL}}) = \sum_{\nu} \left\{ \frac{\Delta L_C(\nu, f_{\sigma_{\rm NL}}) - \langle L^{NG}_C(\nu, f_{\sigma_{\rm NL}}) \rangle}{\sigma[L^{NG}_C(\nu, f_{\sigma_{\rm NL}})]} \right\}^2.$$

This combination makes an integrated estimation of $f_{\sigma_{\rm NL}}$ which includes the non-Gaussian signal at several different scales with a mild weighting.

### 4 RESULTS AND DISCUSSIONS

#### 4.1 Gaussian frequentist results

We first compare the observed results with our Gaussian model predictions. In this case, we perform 10240 Gaussian simulations of the WMAP VW-band properties. Different base-masks, as well as the median-filter, are applied independently to both the real and the simulated skies to study the foreground effect on the skeleton results. The corresponding $\chi^2$ values are then computed to enable the frequentist test.
4.1.1 Results of KQ75B processing

For each smoothing scale, the skeleton length departure from the Gaussian expectation, \( \Delta L(\nu, \theta_{\text{FWHM}}) = L(\nu, \theta_{\text{FWHM}}) - \langle L^0(\nu, \theta_{\text{FWHM}}) \rangle \), is computed from samples obtained with the KQ75B masked maps. The results are shown in the left two columns (for both the differential and cumulative distributions) of Figure 5 for \( \theta_{\text{FWHM}} = 0^\circ.64, 0^\circ.85, 1^\circ.28, 1^\circ.70, 2^\circ.98 \) and \( 3^\circ.40 \). The grey bands demonstrate the 1\( \sigma \) and 2\( \sigma \) confidence regions of the Gaussian prediction. The observed ones are rebinned to 25 bins and depicted by filled circles with the 1\( \sigma \)-error bar of each bin. The rebinning is necessary since the differential skeleton distribution is relatively noisy.

In the case of the cumulative distributions, \( \Delta L_a(\nu) \) for WMAP5, some features consistent with a positive \( f_{\text{NL}} \) value are observed, albeit within the 1\( \sigma \) Gaussian confidence level. The behaviour of the differential distribution, \( \Delta L_a(\nu) \), supports this inference despite the existence of a higher level of fluctuations. However, there are differences between the new results and the corresponding WMAP1 ones (Eriksen et al. 2004). For each smoothing scale, the latter show a 1\( \sigma \)-level peak around \( \nu = 0^\circ \) while the neighbouring troughs show less fluctuations especially in the \( \nu > 1 \) region. In contrast, as shown in Figure 5 (the left two columns), the former’s peak is less apparent but the troughs are much more distinct particularly for \( \theta_{\text{FWHM}} = 1^\circ.28 \) and \( 1^\circ.70 \). The comparison between WMAP1 and our new results is shown in Figure 7 for \( \theta_{\text{FWHM}} = 0^\circ.64, 0^\circ.85 \) and \( 1^\circ.28 \).

There are several possibilities associated with such a discrepancy.

(1) Change of the skeleton-tracing method. Utilising cubic spline interpolation in the skeleton tracing algorithm yields a more accurate estimation of the quantities than the previously adopted linear algorithm (see Appendix B). We computed \( \Delta L_a(\nu) \) for the template-cleaned WMAP1 data using the same band-selection, mask and processing steps as in Eriksen et al. (2004), and tracing the underlying skeleton by both linear and cubic spline interpolation strategies. The Gaussian expectation is also estimated in both cases using the same band-selection, mask and processing steps as in Eriksen et al. (2004), and tracing the underlying skeleton by both cubic spline interpolation strategies. The Gaussian expectation is also estimated in both cases using the same band-selection, mask and processing steps as in Eriksen et al. (2004), and tracing the underlying skeleton by both cubic spline interpolation strategies.

(2) Band-selection. In the analysis of Eriksen et al. (2004), the Q-, V- and W-band maps are combined with a spatially-invariant inverse-noise-variance weighting. The resulting map is dominated by the Q-band since it has the lowest noise of the three. However, since it is the band for which Galactic foreground residuals remain significant, it is plausible that these have an impact on the skeleton results. We repeated our analysis using the appropriately weighted WMAP5 Q-, V- and W-band data, but retaining the KQ75B base-mask. Corresponding Gaussian simulations are also performed. The results are shown as the black connected-filled-circles in the right column of Figure 7. The profile shows modest deviation from our VW-results (black filled-squares), however, it does not result in the discrepancy level required. On the contrary, the difference becomes less significant for large \( \theta_{\text{FWHM}} \).

(3) Difference of the foreground subtraction method between WMAP1 and WMAP5. The foreground templates used for the former (Bennett et al. 2003b) are the FDS 94 GHz dust prediction, the H\( \alpha \) map for free-free emission and the 408 MHz Haslam map for synchrotron emission. The three-year WMAP foreground analysis (Hinshaw et al. 2007) and beyond replace the 408 MHz data with a template based on the the K-Ka difference map. The difference between the two foreground models at V-band utilising the coefficients for the first-year fits of Bennett et al. (2003b) and the five-year analysis of Gold et al. (2009) is shown in Figure 6. The profile demonstrates a dipole-like structure in the large-scale temperature distribution outside both the Kp0B or KQ75B masks, which may affect the skeleton statistics and the corresponding inferences of \( f_{\text{NL}} \). We subtract the five-year foreground model from the one-year raw maps at Q-, V- and W-bands, which are then combined and processed identically with Eriksen et al. (2004) using the Kp0B mask. The corresponding skeleton statistic, \( \Delta L_a(\nu) \), is depicted by the connected open-circles in the left column of Figure 7. They demonstrate consistency with the original WMAP1 results. Similarly, another independent test has been carried out on the five-year raw maps from which the one-year foreground model is subtracted before the data are combined and processed using the KQ75B mask. The results are depicted as the dashed grey line in the right column of Figure 6 and demonstrate consistency with our five-year templated-cleaned VW-KQ75B results (black filled-squares). We conclude that it is difficult to attribute the observed discrepancy to the change of foreground subtraction method.

(4) Change of the base-mask in processing. It is very suspicious that the residual foreground components around the dark-grey regions in Figure 3 bias the skeleton results of WMAP1, although mild smoothing and mask thresholding are applied before skeleton tracing. We discuss this issue in Section 4.1.2 by investigating the Galactic plane region and the extragalactic sources (labeled from 1–6 in Figure 3) separately.
Table 2. The processing elements for the styles of lines and symbols in Figure 2

| Band  | Maska | Fore-redb | Interpc | Style                  |
|-------|-------|-----------|---------|------------------------|
| QVW   | Kp0B  | 1yr       | Cubic   | Connected open-circles |
| QVW   | KQ75B | 5yr       | Cubic   | Connected filled-circles |
| QVW   | KQ75B | 1yr       | Linear  | Solid grey line        |
| VW    | KQ75B | 1yr       | Cubic   | Dashed black line      |

a The base-mask applied in map-processing and analysis.

b The templates and the corresponding coefficients applied for foreground-reducing before our map-processing.

c The interpolation method used for tracing the underlying local skeleton.

d The V+W combined data with uniform weighting, while spatial invariant inverse-noise-variance weighting for QVW.

Table 3. The χ²-based frequentist results for the WMAP5 skeleton analysis derived using different processing masks and methods on 10 smoothing scales. We list the fraction of the simulations with a χ² values less extreme than the observed one. The letters ‘M’ correspond to ‘median-filter’. The values are determined from 10240 Gaussian samples. The corresponding results for WMAP1 (Eriksen et al. 2004) are also listed for easy comparison.

| FWHM  | WMAP1 | KQ75B | KQ75M | Kp0B | KQhybrid |
|-------|-------|-------|-------|------|----------|
| 0°53  | 0.234 | 0.1220| 0.1115| 0.0812| 0.1310   |
| 0°64  | 0.286 | 0.1503| 0.1345| 0.1539| 0.1604   |
| 0°85  | 0.354 | 0.2608| 0.2148| 0.2147| 0.2720   |
| 1°28  | 0.293 | 0.3490| 0.3167| 0.2481| 0.3590   |
| 1°70  | 0.284 | 0.4258| 0.3761| 0.1360| 0.4504   |
| 2°13  | 0.248 | 0.3691| 0.3745| 0.1669| 0.3822   |
| 2°55  | 0.208 | 0.3205| 0.3352| 0.1379| 0.3361   |
| 2°98  | 0.166 | 0.2728| 0.2684| 0.1343| 0.2892   |
| 3°40  | 0.113 | 0.2119| 0.2389| 0.1023| 0.2237   |
| 3°83  | 0.081 | 0.1866| 0.2410| 0.0923| 0.1963   |

expected for a positive-fNL. In particular, both ∆L_a and ∆L_d, rebinned for FWHM = 1°28 and 1°70, demonstrate consistent features with the solid lines shown in Figure 4 for fNL = -150. However, the troughs in the ν > 1 region (hot region) seem relatively less depressed. It is likely that the point sources and foreground components contribute to this asymmetry between the two troughs. The results from the median-filtered map yield insights implications into this issue.

We computed the χ² values of ∆L_a for both the observed and the simulated samples. We list the fraction of the simulations with a χ² values less extreme than the observed one in Table 3. The corresponding WMAP1 results are also listed (Table 3 in Eriksen et al. 2004). Generally speaking, there is no qualitative difference between the five-year and one-year results. But our results show a unimodal dependence on the smoothing scales. The fNL-signal seems more significant around the angular scales FWHM = 1°28, 1°70 and 2°13.

4.1.2 Results of Kp0B and KQhybrid processing

We applied the one-year Kp0B mask used in Eriksen et al. (2004) in our analysis with all other operations remaining unchanged. We also create a new base-mask called ‘KQhybrid’ which excludes the same Galactic plane with KQ75B but handles the six extended sources (Figure 3) identically to Kp0B. The KQhybrid mask is then included in the data processing too as an independent test. Some of the results are shown in Figure 6 and the right column of Figure 7.

In general, the ∆L_a profiles of the Kp0B processing are generally consistent with the previous WMAP1 results, although the peak-trough structure is not identical in detail. The KQhybrid mask yields a consistent set of results with those of KQ75B as shown in Figure 3. Moreover, we have applied the KQ75B mask to the WMAP1 data and found that the results (dashed grey line in the left column of Figure 7) show a similar discrepancy from the Kp0B processed ones and consistency with results from our 5-year VW data processing. This indicates that modifications of the mask do significantly affect the skeleton estimation in the WMAP1 analysis. Although the reason can be easily found by examining the area ratio of the dark-grey regions in Figure 3 it is important to make a separate investigation on the impact of residual Galactic foreground and extragalactic sources since ∆L_a analysis exhibits different responses to different types of foreground contamination (Cabella et al. 2010). This separate analysis motivates future skeleton studies on the effects of different Galactic foreground templates.

It is noteworthy that the skeleton discrepancies caused by base-mask selection indicate that residual Galactic foregrounds bias the non-Gaussian analyses for WMAP1 and even WMAP3 since the Kp0 mask was the standard temperature analysis window then and the KP2 mask excluded even less area around Galactic plane. This issue may have implications on the bispectrum analysis because the additional smoothing operation, which smears the local structures of foreground templates, is not necessary for bispectrum estimation.

The foreground issue is also assessed as a complement to the mask-changing analysis. We subtracted the five-year (one-year) foreground templates from the raw maps of WMAP1 (WMAP5) data. The subtracted maps are then combined and processed using the KQ75B (Kp0B) mask and the skeleton results are depicted as the connected filled-circles (dashed black line) in the left (right) column of Figure 7. They are consistent with the results from the standard foreground subtraction processing with the same corresponding base-mask. It is therefore confirmed that the foreground model is not responsible for the discrepancy of the skeleton statistics as seen.

The corresponding results are listed in Table 3. It is straightforward to infer that the KQhybrid processing results are more consistent with the corresponding KQ75B ones. The differences of a few percent come from the 6 extended regions. The χ² results from the Kp0B-processing are somewhat different to the WMAP1 inference although the profiles of ∆L_a are quite similar. Besides the band selection, it is most probably due to the modified template-fitting of the Galactic foreground in five-year data processing, as well as the better S/N level in 5-year data.
have applied a median filter to those pixels located at positions in the point source mask before smoothing, then processed the filtered map to obtain the skeleton statistics. Some results are plotted in the right two columns of Figure 5, and listed in Table 3. In general, the median filtered results show good consistency with the KQ75B results even for the first few smoothing scales, implying that the base-mask processing is safe for skeleton analysis on the scales considered in this work.

Nevertheless, small visual differences suggest further investigation into how point sources modify the skeleton statistics and the $f_{NL}$ estimations. We make a comparison of the WMAP5 $\Delta L_a$ between the KQ75B and median-filter processing. The differences between them are plotted in Figure 5 for FWHM = 0°53, 0°64, 0°85 and 1°28 by solid, dotted, dashed and dotted-lines, respectively. It is suggested that the point sources do have asymmetric impacts on the skeleton for positive and negative temperature thresholds - negative biasing is seen for the range $-2.0 < \nu < 0.0$ and positive biasing is apparent for $0.0 < \nu < 2.0$. In particular for the dotted-line, a 30% lower depression is observed over $0.0 < \nu < 0.5$. This could bias the best-fit $f_{NL}$ value though the bins around this range are assigned lower weights according in the combined $\chi^2$ computation. Although the plot suggests that the magnitude of potential biasing seems to increase with smoothing scale, the larger smoothing still reduces sensitivity to point sources. Moreover, the profiles seen in Figure 8 become increasingly noise-like within the range $\nu \in [-2.5, 2.5]$ at larger smoothing scales.

4.2 $f_{NL}$ estimation

4.2.1 General results

Using the method introduced in Section 2.2, the likelihood function for $f_{NL}$ is estimated for each smoothing scale based on the 2500 sets of $f_{NL}$ samples, $L^NC(\nu, \{f_{NL}\})$. We sample the parameter within the range $f_{NL} \in [-200, 400]$ with step-length $\Delta f_{NL} = 2.5$. The KQ75B-processed data are utilised from FWHM = 0°53 to 3°40 with the median-filter-processed data from FWHM = 0°53 to 1°28 compared for reference. We use the cumulative estimator $\Delta L_a$ because it leads to 10% more converged estimations than the differential one according to a mock test (Section 2.2). Before $\chi^2$
computation, the estimator resulting from both the observed data and simulations are rebinned to 25 bins.

The results are shown in the top panel of Figure 7 with each curve depicting the likelihood (without normalisation) for each smoothing scale. The likelihood functions are fitted by Gaussian functions so that the best-fitting $f_{NL}$ and the corresponding 1σ error are obtained and then marked in the same plot. The likelihood at the highest resolution indicates that the Gaussian hypothesis ($f_{NL} = 0$) is rejected only at 0.8σ-level, while it increases to 2.7σ for FWHM = 2°13. It is apparent that the best-fitting $f_{NL}$ values show a positive correlation with the smoothing scale, which is unexpected since $f_{NL}$ is scale-independent according to the local-type non-Gaussian model and our simulations.

As discussed in Section 4.1.3, although the estimation is inevitably biased by the point sources or other types of foreground, large angle smoothing renders the estimation insensitive to those effects. We repeat the estimation using median-filtered samples from the first four smoothing scales. As shown in the middle panel of Figure 7 the results are consistent in general, and the positive correlation between $f_{NL}^{\text{est}}$ and the smoothing scales is identical to the unfiltered analysis. It is therefore suggested that the point sources contribute little to such correlation. The 1σ errors are robust according to the median-filter reference but the best-fit values of $f_{NL}$ from the KQ75B processing seem to be over-estimated by levels of 0.04σ, 0.26σ, 0.39σ and 0.22σ for FWHM = 0°53, 0°64, 0°85 and 1°28, respectively.

In principle, different heights of the the likelihoods represent variations in the goodness-of-fit if the corresponding $\chi^2$ values have the same number of degrees-of-freedom. A higher likelihood implies the $f_{NL}$ expectation fits the data better and it does appear that the likelihoods from larger-angle smoothing (FWHM = 2°55, 2°98 and 3°40) show better results than for smaller FWHMs. However, in our analysis, we pick up only the diagonal elements of the covariance matrix to compute the $\chi^2$. It is inappropriate to make a theoretical interpretation of the goodness-of-fit. Consequently, the correlation found above would be a false appearance because there might be some bad fittings. For each FWHM, the $\chi^2$ value at the maximum likeli-

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3 It has been tested that 25 is the best number for rebinning in our analysis. More bins will make the estimator more noisy so that the resulting likelihood is bimodal or even multimodal, whereas less bins will make the likelihood less converged.

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Figure 7. The skeleton statistics, $\Delta L_a(\nu)$, computed from different processing methods. **Left Column**: the results obtained from the one-year WMAP data. **Right Column**: the results obtained from the five-year WMAP data. The lines and symbols denoting different processing elements are noted in Table 2.
Figure 8. The distribution difference between KQ75B and KQ75 median-filter processed estimator, $\Delta C_\nu$ of WMAP5 data. The cases of different smoothing scales are distinguished by different line-styles.

Table 4. The goodness of fit, i.e., the probabilities that the simulated samples with $\chi^2_{\min}(f_{NL}|f_{ML}^{NL}) < \chi^2_{\min}$, $f_{NL}$ and $\chi^2_{\min}$ are the maximum likelihood $f_{NL}$ of the data and its corresponding $\chi^2$ value of each case, respectively. The results in the case of combined data (KQ75B Comb.) are also listed.

| FWHM   | KQ75B $\chi^2_{\min}$ $P(\%)$ | KQ75M $\chi^2_{\min}$ $P(\%)$ | KQ75B Comb. $\chi^2_{\min}$ $P(\%)$ |
|--------|-------------------------------|-------------------------------|-----------------------------------|
| 0°53   | 6.50 27.1 6.28 20.9           |                               |                                   |
| 0°64   | 6.34 25.8 6.57 22.3           |                               |                                   |
| 0°85   | 7.89 35.6 8.18 30.5           | 6.80 30.4                     |                                   |
| 1°28   | 7.85 33.5 7.88 27.3           |                               |                                   |
| 1°70   | 9.42 40.5 N/A N/A            | 6.24 28.0                     |                                   |
| 2°13   | 6.15 19.5 N/A N/A            |                               |                                   |
| 2°55   | 5.20 12.1 N/A N/A            |                               |                                   |
| 2°98   | 5.26 11.7 N/A N/A            |                               |                                   |
| 3°40   | 4.37 6.0 N/A N/A             |                               |                                   |

This number may be underestimated because the underlying $\chi^2$ minima of some samples lay outside our $f_{NL}$ sampling range, i.e., their corresponding $f_{ML}^{NL} > 400$. Similar cases are also found for FWHM = 2°98 and 3°40.

Figure 9. The likelihood functions of $f_{NL}$ from the skeleton statistic for WMAP5 data. Top: The $f_{NL}$ likelihood functions computed by KQ75B processed $\Delta C_\nu(\nu)$ and $C_{NG}(\nu, f_{NL})$ on 9 different smoothing scales. The estimator is rebinned to 25 bins before analysis. The best-fittings and 1$\sigma$ errors are obtained by fitting the likelihoods using Gaussian functions. Middle: Similar cases for statistic derived by KQ75 median-filter processing on 4 smoothing scales. Bottom: The $f_{NL}$ likelihood functions estimated from the combined estimator $\Delta C_{NG}(\nu, f_{NG})$ and $C_{NG}(\nu, f_{NG})$. The solid (dot-dashed) curve shows the likelihood computed from KQ75B processed estimator with 4 (9) FWHMs combined. The dashed curve corresponds to the KQ75 median-filter processed likelihood.
4.2.2 Estimation from the combined $\Delta L_a$

As presented in Section 3.2.2, the combinations on different smoothing scales are applied separately to the rebinned $\Delta L_a(\nu)$ of the data and $L^{NL}_{GW}(\nu, f_{NL})$ of the $f_{NL}$ samples. It is verified that such a combination still leads to an unbiased estimation of $f_{NL}$ (Appendix E).

In our analysis, the first 4 and all 9 scales are combined, yielding estimates of $f_{NL,C} = 47.3 \pm 34.9$ and $f_{NL,C} = 76.8 \pm 43.1$ respectively, by fitting the likelihood using a Gaussian function. The likelihoods are shown in the bottom panel of Figure 9 and the goodness-of-fit is also listed in Table 4.2.2. The estimates are consistent with the results discussed in Section 4.2.1 and the moderate probabilities (30.4% and 28.0%) validate the best-fit results.

The median-filtered results are also combined over the first 4 FWHMs and the corresponding likelihood is depicted by the dashed curve, resulting in the estimate $f_{NL,C}^{best} = 39.8 \pm 34.9$. The point sources lead to an over-estimate of $f_{NL,C}^{best}$ at the 0.21σ-level according to this comparison. The combined estimators, $\Delta L_a, C(\nu)$ for the QK75B processed data and $L_{NG}^{NL}(\nu, f_{NL}) = 0.475, 77.5$ for the corresponding $f_{NL}$ simulations, are illustrated in Figure 10 for comparison.

4.2.3 Cosmic variance and $f_{NL}^{best}$

It is interesting that $f_{NL}^{best}$ shows a monotonic correlation with smoothing scale. The discussions above argue against the explanation based on point sources or goodness-of-fit. We search for this kind of correlation in our mock samples to investigate whether cosmic variance is a possible source of such a correlation. In order to make a comprehensive and reliable interpretation, we pick up those Gaussian and $f_{NL}$ samples which show $f_{NL}$ features at least to the same extent as the WMAP5 data. The selection method is introduced below.

(1) Gaussian samples. Similar to the WMAP5 data, each of the 10240 Gaussian samples of $\Delta L_a(\nu)$ is input into $f_{NL}$-estimations on all 9 FWHMs as introduced in Section 4.2.2. The chi-square for each FWHM $\chi^2_{Gauss}(f_{NL}, \nu_{FWHM})$, is obtained as a function of smoothing scale and $f_{NL}$ before we combine the estimators of all 9 FWHMs to $\Delta L_a(\nu, f_{NL})$. The combined chi-square, $\chi^2_{C,Gauss}(f_{NL})$, and likelihood are then computed by the combined estimator. We find 3111 samples whose minimum $\chi^2_{C,Gauss}(f_{NL})$ are less than $\chi^2_{C,min}$ from the WMAP5 data. It is believed that these samples demonstrate better $f_{NL}$-like features than the WMAP5 data for all 9 smoothing scales even though there is no non-Gaussian component encoded in the simulations.

(2) $f_{NL}$ samples. For the 9-FWHM combination discussed in Section 4.2.2, samples with $\chi^2_{C,min}(f_{NL}|f_{NL}^{best} = f_{NL,C}^{best})$ less than the WMAP5 $\chi^2_{C,min}$ are selected from 2500 groups of $f_{NL}$ samples. The 701 selected samples form the $f_{NL}$-reference for investigating the correlation between $f_{NL}^{best}$ and smoothing scale.

In 3111 selected Gaussian samples, we find 844 that feature a monotonic correlation with smoothing scale (~27.1%). Similarly, there are 222 $f_{NL}$ samples from 701 showing the same behaviour (~31.7%). According to our tests, they show similar properties to that illustrated in the top panel of Figure 9 where the maximum likelihood for large-scale smoothing is ‘pulled’ significantly to the non-Gaussian region. There is a considerable probability (around 30%) of such a correlation so that cosmic variance is a highly probable explanation.

5 CONCLUSIONS

In this paper, we have studied the local-approximation to the skeleton on a 2D sphere pixelised in the HEALPix scheme, refined the method of tracing the quantity. The statistical properties of the skeleton estimator have subsequently been investigated using mock CMB temperature anisotropy maps.

The cubic spline interpolation method locates the skeleton knots more accurately than the simple linear method, which makes the local linear system more robust at the
knots. This is of great importance for finer analysis of the local system. For example, the studies on skeleton classification (Pokosyan et al. 2009), which is performed by analysing the eigenvalues of the linear characteristic equation, request highly accurate estimation of such eigenvalues in particular around the demarcation point between two types of skeleton. Our modification provides a more reliable basis for this kind of study. The departure of the skeleton length distribution from its Gaussian expectation shows connections with both the sign and the magnitude of $N_{NL}$, so that it would yield a $N_{NL}$-likelihood function. Based on simulated sets of CMB temperature anisotropy with a local type of non-Gaussian component, it has been tested that both the differential and cumulative skeleton estimators provide unbiased and sufficiently converged likelihood function for $N_{NL}$, but the latter yields a likelihood 10% more converged than the former.

The estimator was applied to the five-year WMAP data release and the results compared with not only the Gaussian predictions, but also the results from the first-year WMAP data processing by Eriksen et al. (2004). An $N_{NL}$-likelihood function has been estimated by computing the $x^2$ on the basis of 2500 sets of $N_{NL}$ samples. We have also investigated the goodness of fit, the impact of the point sources and the cosmic variance effect on the best-fit amplitudes of $N_{NL}$. The analysis is carried out on the $V+W$ combined map for various sky coverages.

The processing steps in our analysis follow closely those of Eriksen et al. (2004) but utilise the new five-year KQ75 mask and combined V- and W-band data. The smoothing scales adopted in our data processing are also identical to those selected in Eriksen et al. (2004). Our skeleton results show an apparent deviation from the first-year ones. According to an extensive series of tests, it is the difference between the two Galactic plane regions defined by the KQ75 and KP0 masks that contributes mostly to the shifts. Generally, the KQ75 mask excludes a more extended region close to the Galactic plane than the Kp0 mask, and this should be more conservative for temperature analysis. This kind of deviation (Pogosyan et al. 2009), which is performed by analysing the eigenvalues of the linear characteristic equation, request highly accurate estimation of such eigenvalues in particular around the demarcation point between two types of skeleton. Our modification provides a more reliable basis for this kind of study. The departure of the skeleton length distribution from its Gaussian expectation shows connections with both the sign and the magnitude of $N_{NL}$, so that it would yield a $N_{NL}$-likelihood function. Based on simulated sets of CMB temperature anisotropy with a local type of non-Gaussian component, it has been tested that both the differential and cumulative skeleton estimators provide unbiased and sufficiently converged likelihood function for $N_{NL}$, but the latter yields a likelihood 10% more converged than the former. The combination of samples for the first 4 and all 9 smoothing scales lead to the best-fit amplitudes with $1\sigma$ errors, $N_{NL} = 47.3\pm34.9$ and $N_{NL} = 76.8\pm43.1$, respectively. The median-filter studies suggest that the best-fit over 4 scales may be over-estimated at the 0.21$\sigma$-level because of point sources. An investigation has been carried out on the unexpected correlation between $N_{NL}$'s and smoothing scales using both Gaussian and $N_{NL}$ samples with a goodness-of-fit better than that for WMAP5. About 30% of them show the behaviour seen in our analysis, so that cosmic variance may be an appropriate explanation for this issue.

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APPENDIX A: THE LOCAL SKELETON IN HEALPIX FRAME

We construct a local coordinate system on the 2D HEALPix sky map shown in Figure A1 where the direction to the Galactic north-pole is depicted as ‘N’. Following the HEALPix coordinate convention
d two orthogonal axes, x and y in Eq. 3 are set to be aligned with the polar-angle θ and azimuth φ axes, respectively.

Our starting point is identical with that of Eriksen et al. (2004) in that we determine a pair of vertices on the edge of the pre-constructed secondary pixels on S, with one vertex value lower but the other higher than zero (canceling vertices, hereafter). It is suggested that the underlying skeleton crosses over those edges connecting pairs of canceling vertices. Figure A1 illustrates an exaggerated version of this process. Interpolation is then necessary to determine the positions of the intersections on edges (skeleton knots, hereafter). Linear interpolation has been adopted previously by Novikov, Feldman & Shandarin (1996). Shandarin et al. (2002) Eriksen et al. (2004), since it has been widely employed in morphological studies on both the CMB and large scale structures, eg, the length and genus quantities of MFs, which are related to the contour lines of the random fluctuation field and its derivatives. However, the accuracy of an interpolation method is limited by the topological properties of the random field and the pixel size of the corresponding realisation. Linear interpolation is accurate enough for an analysis of the MFs at current observational resolutions (Nside = 512, 1024), however, it is inadequate to provide precise positions of ‘knots’ on the skeleton map in Eq. 3 as a higher-order (cubic) random field. This may introduce not only bias in to the statistics of the skeleton length for a specific realisation, but also could result in a false determination of the eigenvalue of the local linear system, in particular around the demarcation point between the two types of skeletons considered for studies of skeleton classifications (Pogosyan et al. 2009). Given the cubic nature of the skeleton field, we therefore apply a cubic spline interpolation in this analysis, as introduced in the following text in detail. A comparison between the linear and cubic spline strategy is presented in Appendix B.

Once a pair of canceling vertices has been found, e.g., x2 and x3 in Figure A1 the 6 pixels are then picked up with canceling vertices in the middle, as x0, x1, ..., x5. The connection lines of the 6-pixel centres must cross over the pairs of opposite sides of the quadrangular pixels and be parallel with the connection line of canceling vertices (e.g., x2x3). The values of these 6 pixels (vertices of secondary pixels), y_i = S(x_i)(i = 0, 1, 2, 3, 4, 5), are utilised to find the spline functions along the connection lines,

\begin{equation}
S(x) = \begin{cases}
S_0(x) & x \in [x_0, x_1] \\
S_1(x) & x \in [x_1, x_2] \\
\vdots & \vdots \\
S_5(x) & x \in [x_4, x_5]
\end{cases}
\end{equation} (A1)

where each S_i is the piecewise cubic polynomial between the pixel-centres

\begin{align}
S_i(x) &= \frac{z_{i+1}(x - x_i)^3 + z_i(x_{i+1} - x)^3}{6h_i} \\
&+ \left( \frac{y_{i+1} - y_i}{h_i} - \frac{h_i}{6} z_{i+1} \right) (x - x_i) \\
&+ \left( \frac{y_i}{h_i} - \frac{h_i}{6} z_i \right) (x_{i+1} - x).
\end{align} (A2)

h_i is equal to |x_{i+1} - x_i| corresponding to the radial distance of the two pixel-centres. The coefficients \{z_i\} can be obtained by solving the linear system

\begin{align}
h_{i-1}z_{i-1} + 2(h_{i-1} + h_i)z_i + h_iz_{i+1} &= 6 \left( \frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}} \right), i = 1, 2, 3, 4
\end{align} (A3)

z_0 = z_5 = 0.

where y_i corresponds to the pixel value at x_i, i.e., the skeleton value, S(x_i), in this work.

Note that this 6-point system on the sphere has been approximated by a 1D straight line since the pixel-size in our analysis is so small (Nside = 1024, θpix ~ 3.44°). In fact, we only need S_2(x) to determine the locations of the knots, e.g., p_3 in Figure A1 by solving the cubic equation

\begin{equation}
S_2(x) = 0.
\end{equation} (A4)

There is one and only one real root, x_k, satisfying the condition x_2 < x_k < x_3. Then the vector of the underlying knot

\begin{equation}
\text{SEE ‘The HEALPix Primer’ in software package, version 2.10}
\end{equation}
can be obtained as
\[ x_k = \frac{x_3 - x_k}{x_3 - x_2} x_2 + \frac{x_k - x_2}{x_3 - x_2} x_3, \]  
and the corresponding temperature value at \( x_k \) is
\[ T_k = \frac{x_3 - x_k}{x_3 - x_2} T_2 + \frac{x_k - x_2}{x_3 - x_2} T_3. \]

According to Figure A1 after determining the vector of \( p_2 \) and \( p_3 \) (i.e., \( x_{k2} \) and \( x_{k3} \)), the skeleton length within the secondary pixel, \( x_2 x_3 x_4 x_5 \), can be estimated by the dot-product of these two vectors,
\[ \delta L(T_s) = \arccos \left( \frac{x_{k2}}{|x_{k2}|} , \frac{x_{k3}}{|x_{k3}|} \right). \]

The corresponding temperature value of this piece of skeleton length, \( T_s \), is approximately the simple average of \( T_{k2} \) and \( T_{k3} \).

It is always the case that the four edges of one secondary pixel are connecting canceling vertices. Most of these cases indicate a stationary point (maxima or minima or saddle point) within this secondary pixel, implying two skeletons cross inside. There are still a few exceptions but they will become very rare due to the small pixel-size and the smoothing applied afterwards. We therefore make the same assumption as in Eriksen et al. (2004) that all of the cases indicate a pair of skeletons crossing over each other. The possible deviation from the length distribution is totally negligible according to various tests.

**APPENDIX B: COMPARISON BETWEEN LINEAR AND CUBIC SPLINE INTERPOLATION FOR SKELETON ANALYSIS**

On a pixelised 2D random field, the key step in tracing the local skeleton is to locate the skeleton knot which is always within the line connecting the centres of the two canceling neighbouring pixels (one edge of the secondary pixel), and whose position is conventionally estimated by linear interpolation, since the skeleton realisation \( S \) can be considered as a linear function along the line connecting just a few pixels at a very high resolution-level. This is an approximation that makes things easier to handle, especially for the HEALPix pixelization scheme. However, the skeleton is actually a cubic function, so that it is necessary to test whether linear interpolation is sufficient for its computation. In this appendix, we investigate the linear properties at the skeleton knots derived by linear and cubic spline interpolation methods.

The characteristic equation (Eq. 2) for the 2D random field must be satisfied at the skeleton knots. It can be reexpressed for a CMB temperature field as
\[ \begin{pmatrix} T_{\theta\theta} & T_{\theta\phi} \\ T_{\phi\theta} & T_{\phi\phi} \end{pmatrix} \begin{pmatrix} T_{\theta} \\ T_{\phi} \end{pmatrix} = \lambda \begin{pmatrix} T_{\theta} \\ T_{\phi} \end{pmatrix}. \]  
We define
\[ r_1 = \frac{T_{\theta\theta}T_{\theta} + T_{\theta\phi}T_{\phi}}{T_{\theta}}, \quad r_2 = \frac{T_{\phi\phi}T_{\theta} + T_{\phi\phi}T_{\phi}}{T_{\phi}}, \]
and \( \lambda \) should satisfy the following
\[ \begin{vmatrix} T_{\theta\theta} - \lambda & T_{\theta\phi} \\ T_{\phi\theta} & T_{\phi\phi} - \lambda \end{vmatrix} = 0 \]
with two real roots \( \lambda_1 \) and \( \lambda_2 \) (\( \lambda_1 \geq \lambda_2 \)). In principle, \( r_1 \) should be equal to \( r_2 \) and also equal to \( \lambda_1 \) or \( \lambda_2 \) along the

\[ f_{NL}^{\text{estimation by CMB skeleton}} \]
Hou et al. produced by linear and cubic spline interpolation from line-styles.

The positions of the skeleton knots estimated by cubic splines (linear lines) are located by the filled (open) circles. The small $x_1$, ...,$x_5$, where the values of $S$ are marked by filled triangles. The numerical robustness of the equivalence between $r$ and the eigenvalue indicates accurate and unbiased classification, in particular around the underlying demarcation point between two types of skeleton where the two eigenvalues are quite close to each other. The cases for FWHM = 60' are listed below

$P_f : r_1 = -0.8887, r_2 = -0.7466, r = -0.7174$

$P : r_1 = -0.7099, r_2 = -0.7116, r = -0.7108$

$P_f$ point ($P$ point) is the estimated skeleton knot determined by a linear (cubic spline) interpolation method. The linear properties at the two points are quantified as

$P_f : r_1 = 0.0428, r_2 = -0.4198, r = -0.2313$

$P : r_1 = -0.1916, r_2 = -0.2425, r = -0.2170$

$P_f$ point ($P$ point) is the estimated skeleton knot determined by a linear (cubic spline) interpolation method. The linear properties at the two points are quantified as

$P_f : r_1 = -0.9264, r_2 = -0.7597, r = -0.8430$

$P : r_1 = -0.7981, r_2 = -0.8004, r = -0.7993$

$P_f$ point ($P$ point) is the estimated skeleton knot determined by a linear (cubic spline) interpolation method. The linear properties at the two points are quantified as

$\Delta L_a$ of WMAP5 data between the cubic spline (cub) and linear (lin) interpolation processing. The cases of different smoothing scales are distinguished by different line-styles.
Figure C1. $\text{exp}\left[-\frac{1}{2N}\sum_{j=1}^{N} \chi^2\left(f_{NL}^{j}\right)\right]$, the effective likelihood functions computed by the combined accumulative estimator $L_C^{NG}$ from KQ75B processed noisy simulations with input parameter $f_{NL}^{\text{true}} = 0, 200$. The functions are renormalised by different factors for visual convenience and are shown by histograms. The Gaussian-fitting functions are depicted by solid curves. The higher, narrower (lower, wider) histograms and curves correspond to the combinations with $N_{\text{FWHM}} = 4 (9)$. It is noteworthy that the magnitude of such a difference contributes less than 10% to the discrepancy between the WMAP5 and WMAP1 skeleton length distribution profile. However, the structure shown in Figure 12 suggests that the linear method would lead to an over-enhanced peak and under-depressed trough, which for the positive-$f_{NL}$ structure of $\Delta L_a$ suggested by the data may bias the best-fitting value of $f_{NL}$.

APPENDIX C: TEST OF THE LIKELIHOODS FROM THE COMBINED ESTIMATOR

In this section we test for the presence of bias in our combined estimator. Given simulated noisy realisations from the KQ75B processing and the predetermined expectation $\langle L_C^{NG}(\nu, f_{NL}) \rangle$, we randomly pick up $N = 250$ sets of $f_{NL}$-samples, $L_C^{NG}(\nu, f_{NL}^{j})$ ($j = 1, 2, ..., 250$) with $N_{\text{FWHM}} = 4$ and 9, to form the conditional $\chi^2$ functions

$$
\chi^2(f_{NL}^{j}, f_{NL}^{true}) = \sum_{\nu} \left\{ \frac{L_C^{NG}(\nu, f_{NL}^{j}) - \langle L_C^{NG}(\nu, f_{NL}^{true}) \rangle}{\sigma[L_C^{NG}(\nu, f_{NL}^{true})]} \right\}^2,
$$

and the effective likelihood function for each sample,

$$
L_C(f_{NL}^{true}, f_{NL}^{true}) \propto \text{exp}\left[-\frac{1}{2N}\sum_{j=1}^{N} \chi^2(f_{NL}^{j}, f_{NL}^{true})\right].
$$

We plot $L_C(f_{NL}^{true}, f_{NL}^{true})$ as histograms for two given $f_{NL}^{true}$ values (0 and 200) in Figure C1 for $N_{\text{FWHM}} = 4$ and 9, noticing that the sampling width $\Delta f_{NL}$ is 2.5. Again, the likelihoods are perfectly fitted by Gaussian functions with the parameters listed in Table C1. Despite the noise contribution and sky-cut, it is demonstrated that the inverse-variance-combination still leads to an unbiased skeleton estimator for $f_{NL}$.

| $N_{\text{FWHM}}$ | $f_{NL}^{true}$ | $f_{NL}^{ML}$ | $f_{NL}^{best}$ | $\sigma f_{NL}$ |
|-------------------|----------------|--------------|----------------|---------------|
| 4                 | 0.0            | -2.5         | 200.0          | 35.5          |
| 9                 | 0.0            | -2.5         | 199.4          | 42.3          |
| 4                 | 200.0          | 197.5        | 197.8          | 43.4          |
| 9                 | 200.0          | 197.5        | 197.8          | 43.4          |

Table C1. The Maximum-Likelihood ($f_{NL}^{ML}$), best-fitting ($f_{NL}^{best}$) values and 1$\sigma$ error from the likelihood $L_C(f_{NL}^{true}, f_{NL}^{true})$ (Figure C1) computed from the combined estimator derived from $N = 250$ KQ75B processed noisy simulations with given parameter $f_{NL}^{true} = 0, 200$. 

$\odot$ 2010 RAS, MNRAS 000, 1–17