Fifth force from fifth dimension: a comparison between two different approaches

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Abstract

We investigate the dynamics of particles moving in a spacetime augmented by one extra dimension in the context of the induced matter theory of gravity. We examine the appearance of a fifth force as an effect caused by the extra dimension and discuss two different approaches to the fifth force formalism. We then give two simple examples of application of both approaches by considering the case of a Ricci-flat warped-product manifold and a generalized Randall-Sundrum space.

I. INTRODUCTION

The idea that our ordinary spacetime might be embedded in a higher-dimensional manifold seems to pervade almost all the recent development of Cosmology and Particle Physics theory. From the old Kaluza-Klein [1,2] model to modern theories of supergravity and superstrings [3,4] the assumption that extra dimensions do exist, though not observed yet, appears to be closely connected with the belief that all forces of nature are ultimately different aspects of a single entity. Along with the search for unification there is another motivation for constructing higher-dimensional theories which goes back to Einstein and consists in regarding the physical world as a manifestation of pure geometry [5]. Of these two schemes the latter includes the so-called induced-matter theory (IMT) or non-compactified Kaluza-Klein theory of gravity, an approach which regards macroscopic matter as being geometrically “induced” by a mechanism that locally embeds our four-dimensional (4D) spacetime in a Ricci-flat five-dimensional manifold [6–10]. As we know, the original version of Kaluza-Klein theory assumes as a postulate that the fifth dimension is compact. However, in the case of the induced-matter theory (IMT) this requirement has been dropped. Moreover, it is asserted that only one extra dimension should be sufficient to explain all
the phenomenological properties of matter. More specifically, IMT proposes that the classical energy-momentum tensor, which enters the right-hand side of the Einstein equations could be, in principle, generated by pure geometrical means. In other words, geometrical curvature would induce matter in four dimensions and to observers in the physical 4D spacetime the extra dimension would appear as the matter source for gravity. One interesting point is that the matter “generated” by this process is of a very general kind, i.e. any energy-momentum tensor can be produced by choosing the appropriate embedding, a result which is mathematically supported by a powerful theorem of differential geometry due to Campbell and Magaard [11–14]. On the other hand, the question of how the dynamics of test particles is influenced by the fifth dimension has been examined recently [15–18]. It has been argued that departures from conventional 4D dynamics could be interpreted as due to the existence of a fifth force, such deviations being in principle amenable to measurement. It seems, however, that the mathematical prescription to calculate this force is not unique and, in fact, depends on basic assumptions of the five-dimensional underlying theory. It is our purpose here to examine this point in a more critical way and give alternative definitions of the concept of fifth force. As we shall see, these concepts depend strictly on whether the geometry of the four-dimensional observed spacetime is defined by an embedding mechanism or by a foliation of the five-dimensional (5D) manifold.

II. THE AXIOMS OF THE INDUCED-MATTER THEORY

Let us briefly spell out what seems to be the basic axioms or postulates implicitly assumed in the induced-matter theory or non-compactified Kaluza-Klein gravity, as it sometimes is referred to in its original formulation.
i) Our ordinary space-time $M^4$ may be represented as a four-dimensional hypersurface

$\Sigma^4$ locally and isometrically embedded in a Ricci-flat differentiable manifold $M^5$.

Now it is well known that the five-dimensional line element $ds^2 = g_{ab}dx^a dx^b$ of $M^5$ can always be put, at least locally, in the form (see, for example, [12])

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + \epsilon \phi^2 d\psi^2$$

(1)

where $x^a = (x^a, \psi)$, $g_{\alpha\beta} = g_{\alpha\beta}(x^a)$, $\phi = \phi(x^a)$ and $\epsilon = \pm 1$, depending on whether the fifth dimension is chosen timelike or spacelike. (Throughout Latin indices take value in the range $(0,1,...,4)$ and Greek indices run from $(0,1,2,3)$). We now assume that $M^4$ is identified with the hypersurface $\Sigma^4$ defined by the equation $\psi = \psi_0 = \text{constant}$. Then, we have an induced metric on $\Sigma^4$ given by

$$g^{(4)}_{\alpha\beta}(x^\mu) = g_{\alpha\beta}(x^\mu, \psi_0)$$

(2)

On the other hand, in terms of the 5D Christoffel symbols the 5D Ricci tensor is given by

$$R_{ab} = (\Gamma^c_{ab})_.,c - (\Gamma^c_{ac})_.,b + \Gamma^c_{ab} \Gamma^d_{cd} - \Gamma^c_{ad} \Gamma^d_{bc}$$

(3)

Putting $a \rightarrow \alpha, b \rightarrow \beta$ in the above equation will give us the 4D components of the 5D Ricci tensor. It is not difficult to verify that we can rewrite (3) as

$$R_{\alpha\beta} = (4) R_{\alpha\beta} + \Gamma^4_{\alpha\beta,4} - \Gamma^4_{\alpha4,\beta} + \Gamma^\lambda_{\alpha\beta} \Gamma_4^\lambda + \Gamma^4_{\alpha\beta} \Gamma_4^D - \Gamma_4^\lambda \Gamma_4^\beta - \Gamma_4^D \Gamma_4^\lambda$$

(4)

where $(4) R_{\alpha\beta}$ may be looked upon as the 4D Ricci tensor calculated with the metric $(4) g_{\alpha\beta}(x^\mu)$ provided that in the above equation we set $\psi = \psi_0$. Finally, we can show that the 5D vacuum Einstein equations $G_{ab} = 0$ can be written, separately, in the following way:

$$(4) G_{\alpha\beta} = \kappa (4) T_{\alpha\beta}$$

(5)

where $(4) T_{\alpha\beta}$ is now interpreted as the energy-momentum tensor of ordinary 4D matter, and is given explicitly by

$$(4) T_{\alpha\beta} = \frac{\phi_{\alpha\beta}}{\phi} - \frac{\epsilon}{2\phi^2} \left[ \phi_{,\alpha\beta,4} + g^{\lambda\mu} g_{\alpha\lambda,4} g_{\beta\mu,4} - \frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2} + \frac{g_{\alpha\beta}}{4} \left( g_{,4}^\mu g_{\mu,4} + (g_{,4}^\mu g_{\mu,4})^2 \right) \right]$$

(6)
\[ \varepsilon \phi \Box \phi = -\frac{g_{\lambda \beta} g_{\Lambda \beta, 4}}{4} - \frac{g^{\lambda \beta} g_{\Lambda \beta, 44}}{2} + \frac{\phi_{,4} g^{\lambda \beta} g_{\Lambda \beta, 4}}{2\phi} \] 

which may be viewed as an equation for a scalar field \( \phi \); and

\[ P^\beta_{\alpha;\beta} = 0 \]  

an equation that has the appearance of a conservation law, where \( P_{a\beta} \) is defined by

\[ P_{a\beta} = \frac{1}{2\sqrt{g_{44}}} (g_{\alpha\beta, 4} - g_{\alpha\beta} g_{\mu\nu} g_{\mu\nu, 4}) \]  

We now state a second postulate:

ii) The energy-momentum tensor which describes the matter content of the four-dimensional Universe will be given by the equation (6).

For completion of the theory we need a third postulate concerning the motion of free-falling test particles and light rays. This point will be dealt with in the next section.

**III. THE FIFTH FORCE IN THE INDUCED-MATTER THEORY**

The induced-matter theory has often been regarded in the literature as an embedding theory, i.e. a theory which assumes as a first principle that our ordinary spacetime corresponds to some hypersurface embedded in some higher-dimensional manifold (in this case, a 5D Ricci-flat space). In the same sense, the recently proposed brane-world theory [19] may also be called an embedding theory since the brane which models our observable Universe is viewed as a four-dimensional hypersurface embedded in a five-dimensional anti-de Sitter manifold (the so-called bulk). (The relationship between the induced-matter and brane-world theories is discussed in ref. [20].)

It turns out, however, that when it comes to the dynamics of test particles in IMT it has been implicitly assumed that the paths of these particles correspond to curves in the five-dimensional manifold \( M^5 \), not necessarily confined to the hypersurface \( \Sigma^4 \). In this respect
let us recall that in the brane-world model of the Universe matter and radiation are confined
to the brane, although in high energy regime particles, as well as gravitons, can in principle
leave the brane. In this case the effects of the motion in the fifth dimension are expected to
appear as a force affecting the motion of particles in four dimensions.

IV. TWO APPROACHES TO THE FIFTH FORCE

Let us now consider two distinct approaches to the fifth force which will be referred to
as the foliating and embedding approaches. They are defined as follows:

i) The foliating approach makes use of a congruence of a given vector field \( V \) defined in
\( M^5 \) and implicitly assumes that the equations governing the 4D observed physical laws are
in a way ”projections” of 5D equations onto a foliation of hypersurfaces \( \{\Sigma\} \) orthogonal to
\( V \). In this approach the fifth force is determined by inducing the metric of \( M^5 \) on the leaves.

ii) In the embedding approach it is also assumed that the fundamental 5D manifold \( M^5 \)
can be foliated by a set of hypersurfaces \( \{\Sigma\} \) orthogonal to a vector field \( V \). However, here
the geometry of the 4D space-time is not supposed to be determined by the entire foliation,
but by a particular leaf \( \Sigma^4 \) selected from the set \( \{\Sigma\} \), on which a metric tensor is induced
by the embedding manifold \( M^5 \). In this approach the fifth force is determined in terms of
geometrical quantities which are defined exclusively in \( \Sigma^4 \).

We then add a third postulate concerning the motion of particles and light:

The paths corresponding to the motion of free-falling test particles and light rays are
godesic lines in the 5D fundamental Ricci-flat space \( M^5 \).

V. THE 5D DYNAMICS IN THE FOLIATING APPROACH

The distinction made in the previous section between two possible approaches to the
induced-matter theory is crucial when one attempts at defining which has been called a
”fifth force” acting on test particles. Traditionally this has been done by examining how the
geodesic equations in 5D splits up when they are "projected" orthogonally onto the leaves of the foliation defined by the vector field $\frac{\partial}{\partial \psi}$ [15]. As we shall see, the equation of the projected geodesics contains a term which can be viewed as a "force". It turns out that this force depends in general on the 4D components of the 5D Christoffel symbols $\Gamma^a_{bc}(x^a, \psi)$, which are not to be identified with the Christoffel symbols $(4)^{\Gamma^a_{bc}}(x^a, \psi_0)$ of the 4D hypersurface $\Sigma^4$ (these latter are calculated with the induced metric $(4)^{g_{\alpha\beta}}(x^\mu) = (4)^{g_{\alpha\beta}}(x^\mu, \psi_0)$ defined on $\Sigma^4$). Thus, in the traditional approach (in accordance with the third postulate), the motion of particles and light rays is assumed to be governed by the five-dimensional geodesic equation:

$$\frac{d^2 x^a}{dS^2} + \Gamma^a_{bc} \frac{dx^b}{dS} \frac{dx^c}{dS} = 0$$

(10)

where $S$ is an affine parameter.

Let us now show that when the equation of motion of a particle is augmented by one dimension, extra terms may appear which can have an interesting interpretation as an anomalous acceleration or a force (per unit mass). (One would plausibly argue here that to detect deviations of 4D general relativity dynamics could provide an indirect way of testing the space-time dimensionality). In order to simplify the dynamics we set $\phi = 1$ in the equation (1), which amounts to disregard the effects of a possible scalar field in the usual interpretation of Kaluza-Klein theory [10]. With respect to the congruence defined by $V$ let us define the projector

$$h_{ab}(x, \psi) = g_{ab}(x, \psi) - V_aV_b$$

(11)

This projector allow us to split the five-dimensional metric $g_{ab}(x, \psi)$ into a part parallel to $\frac{\partial}{\partial \psi}$ and a part orthogonal to the leaves of the foliation defined by $\psi = \text{const.}$ We thus define the 4D metric on each leaf by

$$(4)^{\tilde{g}_{\mu\nu}}(x, \psi) \equiv h^c_{\mu}h^d_{\nu}g_{cd}(x, \psi) = g_{\mu\nu}(x, \psi)$$

(12)
In this approach all the geometric quantities which are relevant in 4D are to be constructed from \((4)\hat{g}_{\mu\nu}(x,\psi)\), and not from the induced metric \(g_{\alpha\beta}(x^\mu,\psi_0)\). Clearly, the mathematical formalism developed in Section 2 for the case of an embedding theory, which leads to the equations (4-9), is exactly the same for the case of a foliating theory. Our aim now is to look at the five-dimensional geodesic equation (10) as being constituted of an equation of motion in four dimensions plus an equation for the fifth coordinate \(\psi\). We shall first consider timelike curves in 5D and also assume that \(\epsilon = -1\), although these restrictions are by no means essential, i.e. null curves may also be considered. Thus, given a 5D timelike curve \(x^a = x^a(\lambda)\) we define the 5D proper time function \(S = S(\lambda)\) by

\[
S(\lambda) = \int_{\lambda_0}^{\lambda} \left[ g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \right]^{1/2} d\tau = \int_{\lambda_0}^{\lambda} \left[ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \left( \frac{d\psi}{d\lambda} \right)^2 \right]^{1/2} d\tau \tag{13}
\]

where \(\lambda\) is a real parameter and \(\lambda_0\) an arbitrary constant. Since we are assuming a positive integrand in the above equation \(S\) has inverse, \(\lambda = \lambda(S)\), and we have the equation

\[
\frac{d\lambda}{dS} = \left[ g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \left( \frac{d\psi}{d\lambda} \right)^2 \right]^{-1/2} \tag{14}
\]

In the same way, for the same curve \(x^a = x^a(\lambda)\) we define the 4D proper time function by

\[
s(\lambda) = \int_{\lambda_0}^{\lambda} \left[ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2} d\tau \tag{15}
\]

from which we have

\[
\frac{d\lambda}{ds} = \left[ g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right]^{-1/2} \tag{16}
\]

Here we make two important assumptions: i) although the world-line of a particle is a curve in 5D what we directly observe is its “4D part” \(x^\mu = x^\mu(\lambda)\); ii) accordingly, \(s(\lambda)\) gives the proper time elapsed along that curve as measured by a 4D clock. Note that \(s(\lambda)\) is calculated with the 4D components of 5D metric \(g_{ab}(x,\psi)\), i.e. \(g_{\mu\nu}(x,\psi)\), not being restricted to any particular hypersurface of the foliation.
By taking \( \lambda = s \) we deduce from (14) and (16) the following equation

\[
\frac{ds}{dS} = \left[ 1 - \left( \frac{d\psi}{ds} \right)^2 \right]^{-1/2}
\] (17)

Using (17) to reparametrise (10) we easily obtain the equations below

\[
\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = -\left( \frac{d^2s}{dS^2} \right) \left( \frac{ds}{dS} \right)^{-2} \frac{dx^\mu}{ds} - 2\Gamma^\mu_{\nu} \frac{dx^\nu}{ds} \frac{d\psi}{ds}
\] (18)

\[
\frac{d^2\psi}{ds^2} + \Gamma^4_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = -\left( \frac{d^2s}{dS^2} \right) \left( \frac{ds}{dS} \right)^{-2} \frac{d\psi}{ds}
\] (19)

In the foliating approach the right-hand side of equation (18), which depends on the fifth coordinate \( \psi \) is interpreted as a fifth force (per unit mass). In terms of the 5D metric components this quantity may be expressed as

\[
f^\mu = -\frac{dx^\nu}{ds} \frac{d\psi}{ds} \left( \left[ 1 - \left( \frac{d\psi}{ds} \right)^2 \right]^{-1} \delta^\nu_\mu \frac{d^2\psi}{ds^2} + g^{\mu\alpha} \frac{\partial g_{\alpha\nu}}{\partial \psi} \right)
\] (20)

It follows immediately that if the 5D curve lies entirely in a hypersurface \( \psi = \text{const} \), then no fifth force arises in this formulation.

**VI. THE 5D DYNAMICS IN THE EMBEDDING APPROACH**

In the embedding approach our 4D spacetime is regarded as the four-dimensional hypersurface \( \Sigma^4 \) defined by \( \psi = \psi_0 = \text{const} \). Therefore all physical quantities which in principle can be measured in 4D should be expressed in terms of the coordinates \( x^\mu \) (which cover the hypersurface \( \Sigma^4 \)). According to this view a curve \( x^a = x^a(\lambda) \) lying on \( M^5 \) is accessible to four-dimensional observers through its projection onto \( M^4 \), \( x^\mu = x^\mu(\lambda) \). On the other hand, as we have pointed out in Section II, the geometry of \( M^4 \) is to be identified to the geometry of the hypersurface \( \Sigma^4 \), defined by \( \psi = \psi_0 \), with a metric tensor given by

\[
(4) \quad g_{\alpha\beta}(x^\mu) = g_{\alpha\beta}(x^\mu, \psi_0)
\] (21)

Thus the Christoffel symbols of \( M^4 \) are to be calculated with the metric above.
Therefore one has to consider the 4D projection of the 5D geodesic equation (10), which will be given by

\[
\frac{d^2 x^\mu}{dS^2} + \Gamma^\mu_{bc} \frac{dx^b}{dS} \frac{dx^c}{dS} = 0
\]  

(22)

where now the 4D proper time

\[
\sigma(\lambda) = \int_{\lambda_0}^{\lambda} \left[ (4) g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2} d\tau
\]  

(23)

must be calculated with the metric \((4) g_{\mu\nu}(x^\alpha) = g_{\mu\nu}(x^\alpha, \psi_0)\). Note that in this approach both the metric tensor \((4) g_{\alpha\beta}\) and the proper time \(\sigma\) are quantities which are perfectly measurable in \(\Sigma^4\) and do not depend on the fifth coordinate \(\psi\). If we also define \(\Delta \Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta}(x^\nu, \psi) - (4) \Gamma^\mu_{\alpha\beta}(x^\nu, \psi_0)\), where \((4) \Gamma^\mu_{\alpha\beta}(x^\nu, \psi_0)\) denotes the Christoffel symbols of \(M^4\) calculated with \((4) g_{\mu\nu}(x^\alpha)\), then the equation (22) can be put in the form

\[
\frac{d^2 x^\mu}{d\sigma^2} + (4) \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = - \left( \frac{d^2 \sigma}{dS^2} \right) \left( \frac{d\sigma}{dS} \right)^{-2} \frac{dx^\mu}{d\sigma} - 2 \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\sigma} \frac{d\psi}{d\sigma} - \Delta \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}
\]  

(24)

Clearly, the left-hand side of this equation corresponds to the absolute derivative \(DV^\mu/Ds\) of the vector \(V^\mu\) tangent to the 4D projected curve \(x^\mu = x^\mu(\sigma)\). Hence it would be natural to consider, in the embedding approach, the fifth force (per unit mass) as being given by

\[
f^\mu = - \left( \frac{d^2 \sigma}{dS^2} \right) \left( \frac{d\sigma}{dS} \right)^{-2} \frac{dx^\mu}{d\sigma} - 2 \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\sigma} \frac{d\psi}{d\sigma} - \Delta \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}
\]  

(25)

or, in terms of the metric components,

\[
f^\mu = - \frac{dx^\nu}{d\sigma} \frac{d\psi}{d\sigma} \left( 1 - \left( \frac{d\psi}{d\sigma} \right)^2 \right)^{-1} \delta^\mu_\nu \frac{d^2 \psi}{d\sigma^2} + g^{\mu\alpha}(x, \psi_0) \left( \frac{\partial g_{\alpha\nu}}{\partial \psi} \right)_{\psi = \psi_0} - \Delta \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}
\]  

(26)

Likewise, the equation governing the motion in the extra dimension will be given by

\[
\frac{d^2 \psi}{d\sigma^2} + (4) \Gamma^4_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = - \left( \frac{d^2 \sigma}{dS^2} \right) \left( \frac{d\sigma}{dS} \right)^{-2} \frac{d\psi}{d\sigma} - \Delta \Gamma^4_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}
\]  

(27)
VII. APPLICATIONS OF THE TWO FORMALISMS

In this section we wish to work out the equations obtained in the preceding sections through two examples: the case of a Ricci-flat warped-product manifold and a generalized Randall-Sundrum space. Our aim is to compare the results obtained for the fifth force as prescribed by the foliating and embedding approaches.

Let us firstly consider the five-dimensional metric

\[ dS^2 = \Lambda \frac{\psi^2}{3} \left( dt^2 - e^{2\sqrt{\frac{\Lambda}{3}}t} \left[ dx^2 + dy^2 + dz^2 \right] \right) - d\psi^2 \]  (28)

This is a 5D Ricci-flat manifold which may be viewed as an embedding space for the 4D de Sitter space-time with the metric

\[ ds^2 = dt^2 - e^{2\sqrt{\frac{\Lambda}{3}}t} \left[ dx^2 + dy^2 + dz^2 \right] \]  (29)

the embedding taking place at the branes \( \psi = \pm \psi_0 = \pm \sqrt{\frac{3}{\Lambda}} \) [9] [13]. The nonvanishing Christoffel symbols calculated from (28) are \( \Gamma^0_{04} = 1/\psi, \ \Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = \sqrt{\Lambda/3} e^{2\sqrt{\frac{\Lambda}{3}}t}, \ \Gamma^i_{0j} = \delta^i_j \sqrt{\Lambda/3}, \ \Gamma^i_{j4} = \delta^i_j/\psi, \ \Gamma^0_{j0} = \Lambda \psi/3, \ \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\Lambda/3 \psi e^{2\sqrt{\frac{\Lambda}{3}}t} \) (\( i = 1, 2, 3 \)).

Thus, if \( x^a = x^a(\lambda) \) is the curve corresponding to the worldline of a test particle, then the 5D proper time function \( S = S(\lambda) \) will be given by

\[ S(\lambda) = \int_{\lambda_0}^{\lambda} \left[ \Lambda \frac{\psi^2}{3} \left( (dt/d\tau)^2 - e^{2\sqrt{\frac{\Lambda}{3}}t} [(dx/d\tau)^2 + (dy/d\tau)^2 + (dz/d\tau)^2] \right) - (d\psi/d\tau)^2 \right]^{1/2} d\tau \]  (30)

In the foliating and embedding approaches the 4D proper time functions will be given, respectively, by

\[ s(\lambda) = \int_{\lambda_0}^{\lambda} \left[ \Lambda \frac{\psi^2}{3} \left( (dt/d\tau)^2 - e^{2\sqrt{\frac{\Lambda}{3}}t} [(dx/d\tau)^2 + (dy/d\tau)^2 + (dz/d\tau)^2] \right) \right]^{1/2} d\tau \]  (31)

\[ \sigma(\lambda) = \int_{\lambda_0}^{\lambda} \left[ \left( (dt/d\tau)^2 - e^{2\sqrt{\frac{\Lambda}{3}}t} [(dx/d\tau)^2 + (dy/d\tau)^2 + (dz/d\tau)^2] \right) \right]^{1/2} d\tau \]  (32)
Our next step is to obtain the geodesic equations for the 5D warped geometry (28). These will be given by

\[
\frac{d^2 t}{dS^2} + \frac{2}{\psi} \frac{dt}{dS} \frac{d\psi}{dS} + \left(\Lambda/3\right)^{1/2} e^{2\left(\Lambda/3\right)^{1/2} t} \left[(dx/dS)^2 + (dy/dS)^2 + (dz/dS)^2\right] = 0 \tag{33}
\]

\[
\frac{d^2 x^i}{dS^2} + \frac{2}{\psi} \frac{dx^i}{dS} \frac{d\psi}{dS} + 2 \left(\Lambda/3\right)^{1/2} \frac{dt}{dS} \frac{dx^i}{dS} = 0, \quad (i = 1, 2, 3) \tag{34}
\]

\[
\frac{d^2 \psi}{dS^2} + \left(\Lambda \psi/3\right) \left(\frac{dt}{dS}\right)^2 - \psi \left(\Lambda/3\right) e^{2\left(\Lambda/3\right)^{1/2} t} \left[(dx/dS)^2 + (dy/dS)^2 + (dz/dS)^2\right] = 0 \tag{35}
\]

A particular solution of the above system is the curve whose parametric equations are

\[
t = (1/2) \sqrt{3/\Lambda} \log(aS + b) \tag{36}
\]

\[
x^i = c^i = \text{const} \tag{37}
\]

\[
\psi = \sqrt{aS + b} \tag{38}
\]

where \(a, b\) and \(c^i\) are integration constants. It is easy to verify that the curve above is a null geodesic in 5D. As it has been pointed out by some authors [21], null geodesics in 5D may appear as timelike curves in 4D, which would suggest interpreting the rest mass of particles in terms of a fifth dimension. Surely the fact that in the case of null curves the affine parameter \(S\) can no longer be interpreted as proper time in 5D does not affect the formulation of fifth force theory in both approaches, except that now the equation (14) is not well defined. In this case one must work with (15) and (23), which are perfectly well defined.

Let us now calculate the fifth force prescribed by the foliating approach, that is, from the equation (20), (or, equivalently, directly from the right-hand side of (18)) and (36), (37), (38). A straightforward calculation leads to the following expression for \(f^\mu\):
\[ f^0 = -\sqrt{3/\Lambda(aS + b)^{-1}}, \quad f^i = 0, \quad i = 1, 2, 3 \] (39)

On the other hand, the fifth force according to the embedding approach may be evaluated from the equations (24) plus (36), (37), (38). It is easy to verify that in this case

\[ f^\mu = 0, \quad \mu = 0, 1, 2, 3 \] (40)

Therefore from the equations above we see that the fifth force clearly depends on which approach is chosen. Let us also note that one can express the components of the fifth force in terms of \( s \) or \( \sigma \) (the 4D proper time in the foliation or in the embedding approach, respectively). This is simply done with the help of the equations (15) and (23).

As a second example let us consider a space with a generalized Randall-Sundum metric given by

\[ ds^2 = e^{F(\psi)}(dt^2 - dx^2 - dy^2 - dz^2) - d\psi^2 \] (41)

where \( F(\psi) \) is an arbitrary warp factor. Clearly the above manifold can be foliated by a set \( \{\Sigma\} \) of hypersurfaces \( \psi = \text{const} \), each leaf corresponding to a 4D Minkowski spacetime. If we extend the framework of the induced-matter theory to include the case when the embedding 5D space is an Einstein space [22], i.e. a manifold in which \( R_{ab} = \Lambda g_{ab} \) (Ricci-flat is a particular case of an Einstein space when \( \Lambda = 0 \)), then clearly the formalism that defines the fifth force can be kept unaltered in both approaches. The same is true if we consider a more general embedding space such as the one given by (41). Let us now show that for the latter (which includes the Randall-Sundrum as a particular case) the foliating approach leads to a nonzero fifth force, while if we employ the embedding approach particles lying on a brane fell no fifth force at all.

Let us consider 5D timelike geodesics in the geometry given by (41). The timelike geodesic equations can be readily put in the form

\[ \frac{dx^\alpha}{dS} = e^{-F(\psi)}a^\alpha \] (42)
where $a^\alpha$ we denote four integration constants and $\eta_{\alpha\beta}$ stands for the Minkowski metric tensor.

It is easily seen that the equations relating the 5D proper time with the 4D proper time in both the foliating and embedding approaches can be written, respectively, as

$$\frac{dS}{ds} = e^{F(\psi)/2}/\sqrt{a^\alpha a^\alpha}$$

and

$$\frac{dS}{d\sigma} = e^{F(\psi)}/\sqrt{a^\alpha a^\alpha}$$

where we are using the convention $a_\alpha = \eta_{\alpha\beta}a^\beta$ and, as the integration constants are arbitrary, they can be chosen such that $a^2 \equiv a^\alpha a_\alpha > 0$.

Let us now calculate the fifth force in the foliating approach by considering the left-hand side of the equation (18). Since $\Gamma^\mu_{\alpha\beta} = 0$ we have

$$f^\mu = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \frac{d^2x^\mu}{ds^2} = -\frac{a^\mu F'}{2a^2} \left[a^2 e^{F(\psi)} - 1\right]^{1/2}$$

where $F' = \frac{dF}{d\psi}$.

In a similar manner we obtain, taking the embedding approach,

$$f^\mu = \frac{d^2x^\mu}{d\sigma^2} + (4) \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = \frac{d^2x^\mu}{d\sigma^2} = 0$$

since in this case we have $\frac{dx^\mu}{d\sigma} = \frac{a^\mu}{a} = \text{const}$ and $(4) \Gamma^\mu_{\alpha\beta} = 0$.

Therefore we conclude that in the case of the Randall-Sundrum model a fifth force may, in principle, arise if one follows the foliating approach. On the other hand, if one assumes the embedding approach, no fifth force can be detected on the brane.

**VIII. FINAL COMMENTS**

As far as the methodology employed in this work is concerned two comments are in order. Firstly, it would be interesting to work out a more general formulation of the fifth
force which takes into account the usual terms corresponding to scalar and electromagnetic fields as in Kaluza-Klein theories. In this respect, some significant results have been achieved basically in the framework of the foliating approach [23]. Secondly, it should be added that the character of the extra dimension (timelike or spacelike) does not seem to be an essential part of the formalisms discussed in this paper. Thus, both approaches may be applied, for instance, to investigate the fifth force in the context of five-dimensional relativity models with two times, a recent subject of research [24].

IX. ACKNOWLEDGEMENTS

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