Supersymmetry and the Supergravity Landscape\textsuperscript{1}

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\textbf{Abstract}

In the recent times a lot of effort has been devoted to improve our knowledge about the space of string theory vacua (“the landscape”) to find statistical grounds to justify how and why the theory selects its vacuum. Particularly interesting are those vacua that preserve some supersymmetry, which are always supersymmetric solutions of some supergravity theory. After an general introduction to how the pursuit of unification has lead to the vacuum selection problem, we are going to review some recent results on the problem of finding all the supersymmetric solutions of a supergravity theory applied to the $N = 4, d = 4$ supergravity case.

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1 Introduction: Unification and the Landscape

Unification has been one of the most fruitful guiding principles in our search for the fundamental components and forces of the Universe. It is, however, more than just a wish or a prejudice that has produced important results for a while: it is indeed a logical necessity for the human mind to understand the Universe: the history of Physics could be written as the history of the process of unification of many different concepts, entities and phenomena into an ever smaller and more fundamental number of them. However, it was only in later times that we realized what we were doing and started doing it consciously, setting explicitly the unification of all forces and particles as our major goal.

It is this (sometimes feverish) pursuit of unification that has lead us to the vacuum selection problem in Superstring Theory and similar unification schemes that include gravity. If unification is a major goal, then, the vacuum selection problem is a major problem of Superstring Theory, perhaps the most important one.

In order to get some perspective over this problem we are going to review several instances of unification in Physics. We could go back to Archimedes or Newton but we will content ourselves with the classical period of unification that starts with Faraday and Maxwell, showing also that the process of unification underlies all the main advances in Theoretical Physics and is, in particular, strongly related to the symmetry principles on which many of our theories are based.

1. Electricity $\oplus$ Magnetism $\xrightarrow{\text{Faraday, Maxwell}}$ Electromagnetism

$$\vec{E}, \vec{B} \rightarrow (F_{\mu\nu}) \equiv \begin{pmatrix} 0 & -\vec{E}^T \\ \vec{E} & \vec{B} \end{pmatrix}.$$

The unification of electricity and magnetism into a single interaction is the first paradigm of modern unification of interactions: the unification requires (or produces) a bigger group of symmetry because the equations of each field were invariant only under the Galilean group and the full set of Maxwell’s equations are invariant under the Poincaré group. This had to be so: if two interactions are different manifestations of a single interaction, there must exist transformations that do not change the equations of the theory and transform one interaction into the other.

Had the Special Theory of Relativity been proposed before Maxwell’s equations, the latter could have been discovered by imposing Poincaré invariance on the incomplete equations of electricity and magnetism. However, the importance of symmetry principles was discovered much later.

Observe that the Principle of (Special) Relativity applied to Newtonian gravity implies the existence of gravitomagnetism and the combination of both into a single relativistic field of interaction. This interaction is not yet General Relativity, but contains its seeds.

2. Space $\oplus$ Time $\xrightarrow{\text{Einstein, Minkowski}}$ Spacetime

$$t, \vec{x} \rightarrow (x^\mu) \equiv (ct, \vec{x}).$$
This is an example of unification of fundamental concepts (not interactions), although it is strongly related to our previous example because the increase in symmetry (from Galileo to Poincaré) is the same and the underlying mechanism is similar (if space and time are different aspects of spacetime, there must be transformations that take space into time and vice-versa). It is important to observe that the new symmetry is only apparent at high speeds, but it is never broken.

3. Waves $\oplus$ Particles $\xrightarrow{\text{deBroglie}}$ Quantum particles
   
   This unification of two entities always believed to be distinct is required (and led to) Quantum Mechanics. It is, to this day, mysterious, perhaps because it is different from the other instances of unification: in this case there seems to be no underlying symmetry group transforming particles into waves and vice-versa.

4. Gravity (GR) $\oplus$ Electromagnetism $\xrightarrow{\text{Kaluza, Klein, Einstein}}$ Higher – dimensional gravity
   
   $g_{\mu\nu}, A_\mu \rightarrow (\hat{g}_{\mu\nu}) \equiv \left( \begin{array}{c|c} k^2 & A_\mu \\ \hline & g_{\mu\nu} \end{array} \right)$
   
   This attempt was unsuccessful (it was, may be, too early) but introduced many new ideas that have stayed around until now. In this theory there is also an increase of symmetry, but the scheme is more complicated: the vacuum of the theory (in modern parlance) could be 5-dimensional Minkowski spacetime, invariant under the 5-dimensional Poincaré group but this symmetry is spontaneously broken (again in modern parlance) to the 4-dimensional Poincaré group times $U(1)$ due to the (completely arbitrary) choice of vacuum (4-dimensional Minkowski spacetime times a circle). General covariance implies that these symmetries are local in the resulting effective theory, a fact that can be formulated as the Kaluza-Klein Principle:
   
   Global invariances of the vacuum are local invariances of the theory.

   An, originally unwanted, feature of the theory is that a new massless field is predicted: the Kaluza-Klein scalar (or radion) $k$. Its v.e.v., related to the radius of the internal circle, can also be fixed arbitrarily because there is no potential for this scalar. Fixing (stabilizing) the v.e.v. of scalars such as $k$ that determine the size and shape of part of the vacuum spacetime (generically known as moduli) is nowadays known as the moduli problem. Explaining why the vacuum should be 4-dimensional Minkowski spacetime times a circle of all the possible classical solutions of 5-dimensional General Relativity is the simplest version of the vacuum selection problem.

5. Quantum Mechanics $\oplus$ Relativistic Field Theory $\xrightarrow{\text{Many people}}$ QFT
   
   A difficult but fruitful marriage.

6. Weak interactions $\oplus$ Electromagnetism $\xrightarrow{\text{Glashow, Salam, Weinberg}}$ EW interaction
   
   In this case, two Relativistic QFTs are unified.
• Unification is achieved by an increase of local (Yang-Mills-type) symmetry, from $U(1)$ to $SU(2) \times U(1)$.

• The symmetry is spontaneously broken by the Higgs mechanism: choice of vacuum by energetic reasons (minimization of the ad hoc Higgs potential). (This is the main difference with Kaluza-Klein and other theories including gravity in which different vacua are associated to different spacetimes and, therefore, different definitions of energy that cannot be compared.)

• The spontaneous breaking of the symmetry renders the model renormalizable.

• The symmetry is restored at high energies.

• New massive particles are predicted associated to the enhanced symmetry (gauge bosons, found) and a new massless spin-0 particle is also predicted (Higgs boson, not yet found).

This model, part of the Standard Model of Particle Physics, has had an extraordinary success and most unification schemes of relativistic QFTs have followed the same pattern. In particular

7. Electroweak interaction $\oplus$ Strong interactions Many people... $\Rightarrow$ Grand Unified Theory

This is an unsuccessful generalization of the electroweak unification scheme based on a semisimple gauge group ($SO(10), SU(5), \cdots$) spontaneously broken by a generalized Higgs mechanism to $SU(3) \times U(1)$. There are two main problems:

• New massive and massless particles predicted may mediate proton disintegration (not observed).

• Unification of coupling constants should occur at the energy at which the symmetry is restored, but this does not seems to work.

8. Bosons $\oplus$ Fermions Golfand, Likhtman, Volkov, Akulov, Soroka, Wess and Zumino $\Rightarrow$ Superfields

This is a new kind of unification based in an increase of (global spacetime) symmetry to supersymmetry, which should also be spontaneously broken by a yet unknown super-Higgs mechanism. It has many interesting properties:

• It is the most general extension of the Poincaré and Yang-Mills symmetries of the S-matrix (Haag-Lopuszanski-Sohnius theorem).

• This new symmetry can be combined with Yang-Mills-type symmetries (super-Yang-Mills theories) and with GUT models in which, in some cases, unification of coupling constants can be achieved.

• It can also be combined with g.c.t.’s, making it local (supergravity theories). We can have supergravity theories with Yang-Mills fields etc., but in most of these theories gravity is not unified with the other interactions since they belong to different supermultiplets.

• However, extended ($N > 1$) supergravities contain in the same supermultiplet of the graviton additional bosonic fields that may describe the other interactions. In this scheme all interactions would be described in a truly unified way.
These extended supergravities can in general be obtained from compactification of simpler higher-dimensional supergravities. It was also discovered that many $N = 1$ supergravities coupled to Yang-Mills fields could also be obtained in the same way, by a careful choice of compact manifold (i.e. Kaluza-Klein vacuum). This lead to a new brand of unified theories which could describe everything (Theories of Everything). The first of these is

9. Kaluza-Klein Supergravity [1, 2]

It is a combination of the Kaluza-Klein theories with supersymmetry. Now, a Kaluza-Klein vacuum is (arbitrarily) chosen that breaks spontaneously part of the (super)symmetries of the “original” vacuum (Minkowski spacetime for Poincaré supergravities and anti-De Sitter spacetime for aDS supergravities). Now, the rule of the game, the supersymmetric Kaluza-Klein Principle, is

\[ \text{Global (super)symmetries of the vacuum are local (super)symmetries of the compactified theory.} \]

In general, the theories were based on compactifications of $N = 1, d = 11$ supergravity [3], the unique supergravity that can be constructed in the highest dimension in which a consistent supergravity can be constructed. It can accommodate the bosonic part of the Standard Model with minimal supersymmetry. However, these theories are anomalous and it is impossible to obtain the chiral structure of the Standard Model by compactification on smooth manifolds [4]. The vacuum of these theories was arbitrarily chosen to recover the Standard Model. The arbitrariness in the choice of vacuum replaces that of the choice of Higgs field and potential (and gauge interactions, dimensionality...). This makes these theories, conceptually, far superior, but raises to a very prominent place the vacuum selection problem.

These problems and the advent of String Theory, in particular the Heterotic Superstring [5], which is anomaly-free and has chiral fermions, killed these theories, although they have been resurrected again by the same theory that killed them.

10. Superstring Theories

In these theories, all quantum particles are different vibration states of a single physical entity: the superstring. All known interactions could be described in this way. At low energies, one recovers an anomaly-free supergravity theory. However, there are still some problems:

- They are 10-dimensional, and require compactification. At low energies we are faced with 10-dimensional Kaluza-Klein supergravity and the vacuum selection problem.
- There are at least five superstring theories: Types IIA, Type IIB, Heterotic $SO(32)$, Heterotic $E(8) \times E(8)$ and Type I $SO(32)$. Which one should be considered?
- The theory seems to contain other extended objects besides strings: $D$-branes [4], NSNS-branes... Why should strings be fundamental [7]?
The answer to the last two questions lies on the dualities that related the different superstring theories and the different extended objects that occur in them [8]. Dualities are transformations that relate different theories: their spectra, interactions and coupling constants. Their existence allows the mapping of all scattering amplitudes of one theory into those of the other theory and vice-versa. In some cases, the mapping relates the coupling constant of one theory with the inverse coupling constant of the other theory and we talk about the non-perturbative S dualities. In other cases the non-trivial mapping affects only geometrical data of the compactification (moduli) and we talk about perturbative T dualities. These are characteristic of String Theories.

Dualities are, certainly, not symmetries of a single theory. Instead, they can be seen as symmetries in the space of theories. If two dual theories arise from two different compactifications (i.e. choices of vacua) of a given String Theory, then dualities can be seen as symmetries in the space of vacua. The extrapolation of this fact to the cases in which the theories are not known to originate from the same theory by different choices of vacuum is the basis of M Theory.

11. ⊕ Superstring Theories $\xrightarrow{\text{Witten et al.}}$ M theory

In this (super-) unification scheme, all the superstring theories are understood as different duality-related vacua of an unknown theory called M Theory, whose low energy limit is $N = 1, d = 11$ supergravity, which was discovered by Witten [2] to be related to the strong-coupling limit (S duality) of the low-energy limit of Type IIA Superstring ($N = 2A, d = 10$ Supergravity).

Now we are back, in a sense, into the old Kaluza-Klein supergravity scenario, but all the Supergravity fields have got a String Theory meaning. It is amusing to see how, in this scheme, the low-energy limit of the Heterotic Superstring, which has chiral fermions, is related to the low-energy limit of M theory: 11-dimensional supergravity, which was apparently forbidden by Witten’s no-go theorem [3]. The solution to the inconsistency is the use of non-smooth manifolds (orbifolds) [10], evading one the hypothesis of the theorem. This could have been done many years earlier, but, without Superstring Theory underlying the Supergravity theory other problems such as anomalies may never have been solved.

The unification scheme proposed by M theory is very attractive and could satisfy all our desires for unification: all particles and interactions may be explained in a unified way. Further, we no longer have different Superstring Theories to choose from. All the arbitrariness we had have disappeared, but only to be replaced by the arbitrariness in the choice of vacuum. Now there is only one theory and everything depends on that. But the theory seems to have nothing to tell us yet about how it chooses the vacuum and why our Universe is as we see it.

It has to be mentioned that, nowadays, we ask much more from a good candidate to the vacuum of our theory: it is not enough (but it is, certainly, a good starting point) that it gives the Standard Model of Particle Physics, but it should also explain the evolution of our Universe, that is, according to the most extended prejudices, it should give rise to an inflationary era and explain, in a fundamental way, dark energy.

With respect to this problem, there have been two main directions of work:

- Finding phenomenologically viable vacua (in the Particle Physics, e.g. [11] and/or cosmological, e.g. [12] sense).
• Find a vacuum-selection mechanism.

There has been no real progress in the second direction for many years.

The failure to solve the vacuum selection problem through some dynamical mechanism has favored recently a purely statistical approach in which one first has to explore and chart ("classify") the space of vacua a.k.a. Landscape. In this approach, our Universe is the way it is because the probability of this kind of Universe is overwhelming. Of course, this way of thinking can be combined with different forms of the Anthropic Principle.

Charting the superstring landscape is a very difficult problem and some simplifications have been suggested: for instance, one could consider all supersymmetric String Theory vacua, which correspond to different kinds of supergravities [13] or only the vacua with 4-dimensional Poincaré symmetry and a Calabi-Yau internal space, which correspond to $N = 1, d = 4$ supergravities and give Standard-Model-like theories [14]. One could also consider, as proposed by Van Proeyen [15], all possible supergravities, even if the stringy origin of many of them is unknown (the supergravity landscape).

In this talk, which is based on Refs. [16, 17, 18], we are going to review some recent general results on the classification of supersymmetric String Theory vacua and new techniques that can be used to find them, presenting some particular results on the classification of the supersymmetric vacua of the toroidally compactified Heterotic String Theory ($N = 4, d = 4$ SUGRA). First, we are going to define what is a supersymmetric configuration and its symmetry superalgebra, describing some useful special identities that they satisfy (Killing spinor identities). Then we will move on to define the problem of finding all the supersymmetric configurations of a given supergravity theory Tod’s problem and we will explain the strategy to solve it in most (4-dimensional) cases. Finally, we will consider the case of $N = 4, d = 4$ supergravity.

2 Supersymmetric configurations and solutions

Supersymmetric configurations\(^2\) (a.k.a. configurations with residual or unbroken or preserved supersymmetry) are classical bosonic configurations of supergravity (SUGRA) theories which are invariant under some supersymmetry transformations. Let us see what this definition implies.

Generically, the supersymmetry transformations take, schematically, the form

\[ \delta_s \phi^b \sim \bar{\epsilon} \phi^f, \quad \delta_s \phi^f \sim \partial \bar{\epsilon} + \phi^b \epsilon, \]

(2.1)

where $\phi^b$ stands for bosonic fields (or products of an even number of fermionic fields) and $\phi^f$ for the fermionic fields (or products of an odd number of fermionic fields) and $\epsilon$ are the infinitesimal, local, parameters of the supersymmetry transformations, which are fermionic.

Then, a bosonic configuration (i.e. a configuration with vanishing fermionic fields $\phi^f = 0$) will be invariant under the infinitesimal supersymmetry transformation generated by the parameter $\epsilon^\alpha(x)$ if it satisfies the Killing spinor equations (one equation for each $\phi^f$), which have the generic form

\[^2\text{It will be very important for our discussion to distinguish between general field configurations and (classical) solutions of a given theory. General field configurations may or may not satisfy the classical equations of motion and, therefore, may or may not be classical solutions. As we are going to see, supersymmetry does not ensure that the equations of motion are satisfied.}\]
\[ \delta \epsilon \phi^f \sim \partial \epsilon + \phi^b \epsilon = 0. \] (2.2)

The concept of unbroken supersymmetry is a generalization of the concept of isometry, an infinitesimal general coordinate transformation generated by \( \xi^\mu(x) \) that leaves the metric \( g_{\mu\nu} \) invariant because it satisfies the Killing (vector) equation

\[ \delta \xi g_{\mu\nu} = 2\nabla_{(\mu} \eta_{\nu)} = 0. \] (2.3)

As it is well known, in this case, to each bosonic symmetry we associate a generator

\[ \xi^\mu_{(I)}(x) \rightarrow P_I, \] (2.4)

of a symmetry algebra

\[ [P_I, P_J] = f_{IJ}^K P_K, \quad \Leftrightarrow [\xi_{(I)}, \xi_{(J)}] = f_{IJ}^K \xi_{(K)}, \] (2.5)

where the brackets in the right are Lie brackets of vector fields.

In our case, the unbroken supersymmetries are associated to the odd generators

\[ \epsilon^\alpha_{(n)}(x) \rightarrow Q_n, \] (2.6)

of a superalgebra

\[ [Q_n, P_I] = f_{nI}^m Q_m, \quad \{ Q_n, Q_m \} = f_{nm}^I P_I. \] (2.7)

The calculation of these commutators and anticommutators is explained in detail in Refs. [19, 20] and the consistency of the scheme was proven in [21]. According to the Kaluza-Klein principle we enunciated at the beginning, conveniently generalized to the supersymmetric case, this global supersymmetry algebra becomes the algebra of the local symmetries of the field theories constructed on this field configuration.

Of course, we do not want to construct field theories on just any field configuration but only on vacua of the theory. In general, for a field configuration to be considered a vacuum, we require that it is a classical solution of the equations of motion of the theory. Apart from this requirement, it is not clear what \textit{a priori} characteristics a good vacuum must have except for classical and quantum stability, which are difficult to test in general, but which are, under certain conditions, guaranteed by the presence of unbroken supersymmetry. This is one of the reasons that makes supersymmetric vacua interesting. We also prefer highly symmetric vacua (such as Minkowski or anti-De Sitter space) since, on them, we can define a large number of conserved quantities, but it is uncertain why Nature should have the same prejudices.

Sometimes, when a vacuum solution has a clear (possibly warped) product structure, we can distinguish internal and spacetime (super-) symmetries and, if we choose this vacuum, our choice implies spontaneous compactification.

### 3 Tod’s problem

This is the problem of finding \textit{all} the supersymmetric bosonic field configurations, i.e. all the bosonic field configurations \( \phi^b \) for which a SUGRA’s Killing spinor equations

\[ \delta \epsilon \phi^f \bigg|_{\phi^f=0} \sim \partial \epsilon + \phi^b \epsilon = 0, \] (3.1)
have a solution $\epsilon$, which includes all the possible supersymmetric vacua and compactifications.

Observe that, as we announced, not all supersymmetric bosonic field configurations satisfy the classical bosonic equations of motion for which we use the notations $\frac{\delta S}{\delta \phi^b} \big|_{\phi^f=0} \equiv S_{,b}^{\phi^f=0} \equiv E(\phi^b)$. Actually, the bosonic equations of motion of supersymmetric bosonic field configurations satisfy the so-called Killing spinor identities (KSIs) [16, 17] that relate different equations of motion of a supersymmetric theory. These identities can be derived as follows: The supersymmetry invariance of the action implies, for arbitrary local supersymmetry parameters $\epsilon$

$$\delta_{\epsilon} S = \int d^d x \left( S_{,b} \delta_{\epsilon} \phi^b + S_{,f} \delta_{\epsilon} \phi^f \right) = 0. \quad (3.2)$$

Taking the functional derivative w.r.t. the fermions and setting them to zero

$$\int d^d x \left[ S_{,bf_1} \delta_{\epsilon} \phi^b + S_{,b} \left( \delta_{\epsilon} \phi^b \right)_{f_1} + S_{,ff_1} \delta_{\epsilon} \phi^f + S_{,f} \left( \delta_{\epsilon} \phi^f \right)_{f_1} \right] \bigg|_{\phi^f=0} = 0, \quad (3.3)$$

The terms $\delta_{\epsilon} \phi^b \big|_{\phi^f=0}$, $S_{,f} \big|_{\phi^f=0}$, $(\delta_{\epsilon} \phi^f)_{f_1} \big|_{\phi^f=0}$ vanish automatically because they are odd in fermion fields $\phi^f$ and so we are left with

$$\left\{ S_{,b} \left( \delta_{\epsilon} \phi^b \right)_{f_1} + S_{,ff_1} \delta_{\epsilon} \phi^f \right\} \bigg|_{\phi^f=0} = 0. \quad (3.4)$$

This is valid for any fields $\phi^b$ and any supersymmetry parameter $\epsilon$. For a supersymmetric field configuration $\epsilon$ is a Killing spinor $\delta_{\epsilon} \phi^f \big|_{\phi^f=0}$ and we obtain the KSIs

$$E(\phi^b) \left( \delta_{\epsilon} \phi^b \right)_{f_1} \bigg|_{\phi^f=0} = 0. \quad (3.5)$$

These non-trivial identities are linear relations between the bosonic equations of motion and can be used to solve Tod’s problem, obtain BPS bounds etc. Let’s see some examples.

### 3.1 Example: $N = 1, d = 4$ Supergravity.

This is the simplest supergravity theory. Its field content is $\{e^a_{\mu}, \psi_{\mu}\}$. The bosonic action (Einstein-Hilbert’s) and the equations of motion (Einstein’s) are

$$S |_{\psi_{\mu}=0} = \int d^4 x \sqrt{|g|} R, \quad \Rightarrow E^\mu_{\epsilon}(\epsilon) \sim G^\mu_{a}. \quad (3.6)$$

The supersymmetry transformations of the graviton and gravitino are

$$\delta_{\epsilon} e^a_{\mu} = -ie\gamma^a \psi_{\mu}, \quad \delta_{\epsilon} \psi_{\mu} = \nabla_{\mu} \epsilon = \partial_{\mu} \epsilon - \frac{i}{4} \omega_{\mu}^{ab} \gamma_{ab} \epsilon. \quad (3.7)$$

The KSIs can be readily computed from the general formula Eq. (3.5) and simplified

$$-ie\gamma^a G^\mu_{a} = 0, \quad \Rightarrow R = 0, \quad -ie\gamma^a R^\mu_{a} = 0. \quad (3.8)$$

On the other hand, in trying to solve the Killing spinor equations (KSEs) which, here, take the form $\delta_{\epsilon} \psi_{\mu} = \nabla_{\mu} \epsilon = 0$, we can consider first their integrability conditions:

$$[\nabla_{\mu}, \nabla_{\nu}] \epsilon = -\frac{1}{4} R_{\mu\nu}^{ab} \gamma_{ab} \epsilon = 0, \quad \Rightarrow R^\mu_{a} \gamma^{a} \epsilon = 0. \quad (3.9)$$
Thus, at least in the case, the KSIs are contained in the integrability conditions. We will see later how to obtain more information from these identities.

### 3.2 Example: $N = 2, d = 4$ Supergravity.

This is the next simplest supergravity theory, if we do not consider adding matter supermultiplets to the $N = 1$ theory. Its field content is $\{e^a_{\mu}, A_\mu, \psi_\mu\}$ (but now $\psi_\mu$ is a Dirac spinor, instead of a Majorana spinor as in the $N = 1$ case). The bosonic action (Einstein-Maxwell’s) and the equations of motion (Einstein’s and Maxwell’s) are

$$S|_{\psi_\mu = 0} = \int d^4x \sqrt{|g|} \left[ R - \frac{1}{4} F^2 \right], \quad \Rightarrow \left\{ \begin{array}{l} E_a^{\mu}(e) = -2\{G_a^\mu - \frac{1}{2} T_a^\mu\}, \\ E^{\mu}(A) = \nabla_\alpha F^{\alpha\mu}. \end{array} \right. \quad (3.10)$$

The supersymmetry transformations are

$$\delta_\epsilon e^a_\mu = -i\bar{\epsilon}\gamma^a \psi_\mu + \text{c.c.}, \quad \delta_\epsilon A_\mu = -2i\bar{\epsilon} \psi_\mu + \text{c.c.}, \quad \delta_\epsilon \psi_\mu = \nabla_\mu \epsilon - \frac{1}{8} F^{ab}\gamma_{ab}\epsilon \equiv \tilde{D}_\mu \epsilon. \quad (3.11)$$

Using the bosonic fields supersymmetry transformations, we find that the KSIs take the form

$$\bar{\epsilon}\{E_a^{\mu}(e)\gamma^a + 2E^{\mu}(A)\} = 0. \quad (3.12)$$

On the other hand, the integrability conditions of the KSEs $\delta_\epsilon \psi_\mu = \tilde{D}_\mu \epsilon = 0$ are

$$[\tilde{D}_\mu, \tilde{D}_\nu] \epsilon = -\frac{1}{4} \left\{ [R_{\mu\nu}^{\ ab} - e^a_{\ [\mu} T_{\nu]}^\ b] \gamma^{ab} + \nabla^a (F_{\mu\nu} + *F_{\mu\nu} \gamma_5) \gamma_a \right\} \epsilon = 0, \quad (3.13)$$

$$\Rightarrow \left\{ E_a^{\mu}(e)\gamma^a + 2[\mathcal{E}^{\mu}(A) + \mathcal{B}^{\mu}(A)\gamma_5]\} \epsilon = 0. \right.$$  

In this case we get a more general formula from the integrability conditions, valid for the case in which the Bianchi identities are not satisfied. When they are satisfied we recover the KSIs, which is consistent since we have explicitly used the supersymmetry variations of the vector field in order to derive them, assuming, then, implicitly, that the Bianchi identities are satisfied.

The last formula (which we are also going to call KSI) has one important advantage over the original KSI: it is covariant under the $U(1)$ group of electric-magnetic duality rotations of the Maxwell and Bianchi identities that act as chiral rotations of the spinors.

### 4 Solving Tod’s problem

In 1983 showed in Ref. [22] that in $N = 2, d = 4$ SUGRA the problem could be completely solved using just integrability and consistency conditions. However, he used the Newman-Penrose formalism, unfamiliar to most particle physicists and suited only for $d = 4$. Thus, there were no further results until 1995, when Tod, using again the same methods, solved partially the problem in $N = 4, d = 4$ SUGRA [23]. Then, in 2002, Gauntlett, Gutowski, Hull, Pakis and Reall proposed to translate the Killing spinor equation to tensor language
and they solved the problem in minimal $N = 1, d = 5$ SUGRA [24]. This opened the gates to new results: in 2002 the problem was solved in gauged minimal $N = 1, d = 5$ SUGRA [25], in 2003 in minimal $N = (1, 0), d = 6$ SUGRA [26, 27] and gauged $N = 2, d = 4$ SUGRA [28], and in 2004 and 2005 in gauged minimal $N = 1, d = 5$ SUGRA coupled to Abelian vector multiplets [29, 30] and in $N = 4, d = 4$ SUGRA [18], completing the work started by Tod on this theory.

There is by now a well-defined recipe to attack this problem (at least in low dimensions) starting with only one assumption: the existence of one Killing spinor $\epsilon$. The recipe consists in the following steps:

I Translate the Killing spinor equations and KSIs into tensorial equations.

With the Killing spinor $\epsilon$ one can construct scalar, vector, and $p$-form bilinears $M \sim \bar{\epsilon}\epsilon, \quad V_\mu \sim \bar{\epsilon}\gamma_\mu\epsilon, \cdots$ that are related by Fierz identities. These bilinears satisfy certain equations because they are made out of Killing spinors, for instance, if the KSE is of the general form

$$\delta_\epsilon \psi_\mu = \bar{\partial}_\mu \epsilon = [\nabla_\mu + \Omega_\mu] \epsilon = 0, \quad \Rightarrow \nabla_\mu M + 2\Omega_\mu M = 0,$$

(4.1)

The set of all such equations for the bilinears should be equivalent to the original spinorial equation or at least it should contain most of the information contained in it (but, certainly, not all of it).

II One of the vector bilinears (say $V_\mu$) is always a Killing vector which can be timelike or null. These two cases are treated separately.

III One can get an expression of all the gauge field strengths of the theory using the Killing equation for those scalar bilinears: $\Omega_\mu$ is usually of the form $F_{\mu\nu}V^\nu$ and, then Eq. (4.1) tells us that $F_{\mu\nu}V^\nu \sim \nabla_\mu \log M$. When $V$ is timelike this determines completely $F$ and, when it is null, it determines the general form of $F$. Of course, Eq. (4.1) is an oversimplified KSE and in real-life situations there are additional scalar factors, $SU(N)$ indices etc.

IV The Maxwell and Einstein equations and Bianchi identities are imposed on those field strengths $F$, getting second order equations for the scalar bilinears $M$.

V The KSIs guarantee that these three different sets of equations (plus the equations of the scalar fields, if any) are complicated combinations a a reduced number of simple equations involving a reduced number of scalar unknowns. Solving these equations for the scalar unknowns gives full solutions of the theory. The tricky part is, usually, identifying the right variables that satisfy simple equations and finding these equations as combinations of the Maxwell, Einstein etc. equations.

VI Finally, with the results obtained, the KSEs have to be solved, which may lead to additional conditions on the fields.

Let us see how this recipe works in the examples considered before.
4.1 Example: $N = 1, d = 4$ Supergravity.

With one (Majorana) Killing spinor $\epsilon$ the only bilinear that one can construct is a real vector bilinear $V_\mu$ which is always null. $V_\mu$ is also covariantly constant (i.e. it is a Killing vector and $V_\mu d\sigma^\mu$ is an exact 1-form, which allows us to write $V_\mu d\sigma^\mu = d\sigma$):

$$\delta_\epsilon \psi_\mu = \nabla_\mu \epsilon = 0, \Rightarrow \nabla_\mu V_\nu = 0, \quad R^\mu_{\nu\rho\sigma} \epsilon^\rho = 0. \quad (4.2)$$

All the metrics with covariantly constant null vectors are Brinkmann pp-waves and have the form

$$ds^2 = 2d\sigma (d\tau + Kd\sigma + A_\tau dx^i) + \tilde{g}_{ij} dx^i dx^j, \quad (4.3)$$

where all the components are independent of $v$, where $v$ is defined by $V^\mu \partial_\mu \equiv \partial / \partial v$.

It can be checked that for all these metrics the KSE has solutions. These, then, are all the supersymmetric field configurations of $N = 1, d = 4$ SUGRA, but only those with $R_{\mu\nu} = 0$ are supersymmetric solutions.

4.2 Example: $N = 2, d = 4$ Supergravity.

With two Weyl spinors $\epsilon^I$ one can construct the following independent bilinears

- A complex scalar $\bar{\epsilon}^I \epsilon^J \equiv M \epsilon^I \epsilon^J$
- A Hermitean matrix of null vectors $V_{IJ}^I \equiv i \bar{\epsilon}^I \gamma_\mu \epsilon^J$

The KSEs imply the following equations for the bilinears:

$$\nabla_\mu M \sim F^+_{\mu\nu} V^I_{J\nu}, \quad (4.4)$$

$$\nabla_\mu V^I_{J\nu} \sim \delta^I_J [MF^+_{\mu\nu} + M^* F^-_{\mu\nu}] - \Phi_{KJ} (\rho) \epsilon^K F^+_{\nu\rho} - \Phi_{IK} (\rho) \epsilon^K F^+_{\nu\rho}, \quad (4.5)$$

$$\nabla_\mu V_{IJ\nu} \sim \Phi_{IK} (\rho) \epsilon^K F^+_{\nu\rho}, \quad (4.6)$$

so the vector $V^\mu \equiv V^I I^\mu$ is Killing and the other three are exact forms. The Fierz identities tell us that $V^\mu V_\mu \sim |M|^2 \geq 0$ can be timelike or null. When it is timelike, $V^\mu \partial_\mu \equiv \sqrt{2} \partial / \partial t$ and the metric can be put in the conformastationary form

$$ds^2 = |M|^2 (dt + \omega)^2 - |M|^{-2} d\tilde{x}^2, \quad (4.7)$$

where, for consistency, the 1-form $\omega$ has to be related to $M$ by

$$d\omega = i |M|^{-2} [M dM^* - \text{c.c.}], \quad (4.8)$$

On the other hand, Eq. (4.4) gives

$$F^+ \sim |M|^{-2} \{ V \wedge dM + i^* [V \wedge dM] \}. \quad (4.9)$$

In this theory one can use pairs of Majorana or Weyl spinors or single Dirac spinors. We now use, for convenience, pairs of Weyl spinors.
The KSIs are satisfied if Eq. (4.8) is satisfied. It can be seen that, then, any metric and 2-form field strength of the above form admit Killing spinors. On the other hand, all the equations of motion are combinations of the simple equation in 3-dimensional Euclidean space

$$\nabla^2 M^{-1} = 0.$$  

Thus, solving this equation for some $M$ gives us a supersymmetric solution of all the equations of motion (all the fields are determined by $M$). These solutions of the Einstein-Maxwell theory are the Israel-Wilson-Perjes family [31, 32].

The case in which $V$ is null is very similar to the $N = 1$ case and we will not study it here in detail for lack of space.

5 Tod’s problem in $N = 4, d = 4$ supergravity

This theory can be obtained by toroidal compactification on $T^6$ of $N = 1, d = 10$ SUGRA [33] (the effective field theory of the Heterotic String) and subsequent (consistent) truncation of the matter vector fields. The 10- and 4-dimensional fields are related as indicated in Fig. 1.

$$d = 10, N = 1 \{e^a_{\mu}, B_{\mu
u}, \phi, \psi_{\mu}, \chi\} \{V_{R \mu}, \psi^{R}\}$$

$$d = 4, N = 4 \{e^a_{\mu}, A_{I \mu}, \tau, \psi_{I \mu}, \chi_I\} \{V_{R \mu}, \phi^{R I J}, \psi^{R I}\}$$

Figure 1: Relation between the fields of $N = 1, d = 10$ SUGRA $N = 4, d = 4$ SUGRA. The fields in curly brackets belong to the same supermultiplet. Both in $d = 10$ and $d = 4$ there a supergravity multiplet containing the graviton and vector supermultiplets, but the 4-dimensional vector supermultiplets originate from both the $d = 10$ supergravity and vector supermultiplets. The $I, J = 1, \cdots, 4$ indices are $SU(4)$ indices related to the six internal dimensions using the isomorphism between $SO(6)$ and $SU(4)$. The $R, S = 1, \cdots, 22$ indices count the vector supermultiplets: 6 of them coming from the supergravity multiplet and 16 from 10-dimensional vector supermultiplets.

A special role is played by the axidilaton field $\tau = a + i e^{-\phi}$, where $a$ is dual to the 4-dimensional Kalb-Ramond 2-form and plays the role of local $\theta$ parameter and $\phi$ is the 4-dimensional dilaton, which plays its usual role of local coupling constant.

It is convenient to start by studying the pure supergravity theory (without the vector supermultiplets) [34], for simplicity. The theory has global $SU(4)$ symmetry (duality) and, furthermore, only at the level of the equations of motion, an $SL(2, \mathbb{R})$ invariance (S duality) that rotates Maxwell equations into Bianchi identities and acts on the axidilaton according to

$$\tau' = \frac{\alpha \tau + \beta}{\gamma \tau + \delta}, \quad \alpha \delta - \gamma \beta = 1.$$

Observe that the $N = 2$ and $N = 1$ are included as truncations.

The bosonic action of the theory is
\[ S = \int d^4 x \sqrt{|g|} \left[ R + \frac{1}{2} \frac{\partial_\mu \partial_\nu \tau^*}{(3 m \tau)^2} - \frac{1}{16} \Im \tau F^{IJ \mu \nu} F_{IJ \mu \nu} - \frac{1}{16} \Re \tau F^{IJ \mu \nu} F_{IJ \mu \nu} \right]. \quad (5.2) \]

It is convenient to denote the equations of motion by

\[ \mathcal{E}_a^\mu \equiv - \frac{1}{2} \sqrt{|g|} \frac{\delta S}{\delta e_a^\mu}, \quad \mathcal{E} \equiv \frac{2 \Im \tau}{\sqrt{|g|}} \frac{\delta S}{\delta \tau}, \quad \mathcal{E}^{IJ \mu} \equiv \frac{8 \sqrt{|g|}}{\delta A_{IJ \mu}}. \quad (5.3) \]

The Maxwell equation \( \mathcal{E}^{IJ \mu} \) transforms as an \( SL(2, \mathbb{R}) \) doublet together with the Bianchi identity

\[ \mathcal{B}^{IJ \mu} \equiv \nabla_\nu^* F^{IJ \nu \mu}. \quad (5.4) \]

For vanishing fermions, the supersymmetry transformation rules of the gravitini and dilatini, generated by 4 spinors \( \epsilon_I \) of negative chirality, are

\[ \delta \psi_I^\mu = D_\mu \epsilon_I - \frac{i}{2 \sqrt{2}} (3 m \tau)^{1/2} F_{IJ}^{+ \mu \nu} \gamma^\nu \epsilon^J, \quad (5.5) \]

\[ \delta \chi_I = \frac{1}{2 \sqrt{2}} \frac{\partial \tau}{3 m \tau} \epsilon_I - \frac{1}{8} (3 m \tau)^{1/2} F_{IJ}^- \epsilon^J, \quad (5.6) \]

where \( D \) is the Lorentz plus \( U(1) \) covariant derivative and where the \( U(1) \) connection is given by

\[ Q_\mu \equiv \frac{i}{4} \frac{\partial_\mu \Re \tau}{3 m \tau}. \quad (5.7) \]

The supersymmetry transformation rules of the bosonic fields take the form

\[ \delta \epsilon_\mu = - \frac{i}{4} (\bar{\epsilon}^I \gamma^a \psi_I^\mu + \bar{\epsilon}_I \gamma^a \psi^I_\mu), \quad (5.8) \]

\[ \delta \bar{\epsilon}_\tau = - \frac{i}{\sqrt{2}} \frac{3 m \tau}{\bar{\epsilon}_I \chi_I}, \quad (5.9) \]

\[ \delta \epsilon_{A_{IJ} \mu} = \frac{\sqrt{2}}{(3 m \tau)^{1/2}} \left[ \bar{\epsilon}_I [\psi^\mu_\tau] + \frac{1}{\sqrt{2}} \bar{\epsilon}_I \gamma_\mu \chi_I \right] + \frac{i}{4} \bar{\epsilon}_{IJKL} \left( \bar{\epsilon}_K \psi^L_\mu + \frac{1}{\sqrt{2}} \bar{\epsilon}_K \gamma_\mu \chi^L \right). \quad (5.10) \]

Given \( N \) chiral commuting spinors \( \epsilon_I \) and their complex conjugates \( \epsilon^I \) we can constructed the following independent bilinears:

1. A complex, antisymmetric, matrix of scalars

\[ M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J, \quad M^{IJ} \equiv \bar{\epsilon}^I \epsilon^J = (M_{IJ})^*, \quad (5.11) \]

2. A complex matrix of vectors

\[ V_{IJ a} \equiv i \bar{\epsilon}^I \gamma_a \epsilon_J, \quad V^I_{Ja} \equiv i \bar{\epsilon}_I \gamma_a \epsilon^J = (V_{IJ a})^*, \quad (5.12) \]
which is Hermitean:

$$(V^I J a)^* = V^I J a = V^J I a = (V^I J a)^T.$$  \hspace{1cm} (5.13)

Using the supersymmetry transformation rules of the bosonic fields, one can find the KSIs of this theory, associated to the gravitini and dilatini, respectively. However, just as in the $N = 2, d = 4$ example, since the Bianchi identities do not appear in these equations, they break S-duality covariance. This covariance can be restored by hand or re-deriving the KSIs from the KSEs integrability conditions. The result is

$$E^\mu a \gamma^a \epsilon_I - \frac{i}{\sqrt{2}(3m \tau)^{1/2}} (\mathcal{E}_{I J}^\mu - \tau^* B_{I J}) \epsilon^J = 0,$$  \hspace{1cm} (5.14)

$$E^* \epsilon_I - \frac{1}{\sqrt{2}(3m \tau)^{1/2}} (\mathcal{E}_{I J}^* - \tau B_{I J}) \epsilon^J = 0.$$  \hspace{1cm} (5.15)

It is useful to derive tensorial equations from these KSIs. Combining them we arrive to the following, which are chosen among the many possible tensorial KSIs by their interest. For timelike $V^a \equiv V^I I$ we get

$$\mathcal{E}^{ab} - \frac{1}{2} 3m \epsilon^a V^b - \frac{1}{\sqrt{2}} (3m \tau)^{1/2} 3m (M^{I J} B_{I J}^a) V^b = 0,$$  \hspace{1cm} (5.16)

$$\mathcal{E}^* V^a - \frac{i}{\sqrt{2} (3m \tau)^{1/2}} M^{I J} (\mathcal{E}_{I J}^a - \tau B_{I J}^a) = 0,$$  \hspace{1cm} (5.17)

$$3m [M^{I J} (\mathcal{E}_{I J}^a - \tau^* B_{I J}^a)] = 0.$$  \hspace{1cm} (5.18)

Observe that the first equation implies the off-shell vanishing of all the Einstein equations with one or two spacelike components. Further, the Einstein equation is automatically satisfied when the Maxwell, Bianchi and complex scalar equations are satisfied and the scalar equation is automatically satisfied when the Maxwell and Bianchi are.

When $V^a$ is null (we denote it by $I^a$), all the spinors $\epsilon_I$ are proportional and we can parametrize all of them by $\epsilon_I = \phi_I \epsilon$, where $\phi^I \phi_I = 1$. In order to construct tensor bilinears we define an auxiliary spinor $\eta$ normalized by $\bar{\epsilon} \eta = \frac{1}{2}$. With these two spinors we can construct a standard complex null tetrad

$$l_\mu = i \bar{\epsilon}^*  \gamma_\mu \epsilon, \quad n_\mu = i \bar{\eta}^*  \gamma_\mu \eta, \quad m_\mu = i \bar{\epsilon}^*  \gamma_\mu \eta = i \bar{\eta} \gamma_\mu \epsilon^*, \quad m_\mu^* = i \bar{\epsilon} \gamma_\mu \eta^* = i \bar{\eta}^*  \gamma_\mu \epsilon. \hspace{1cm} (5.19)$$

Then, in the null case, the KSIs take the form

$$\mathcal{E}^\mu_a - \frac{1}{2} \mathcal{E}^a \mu \mathcal{E}^\rho_\rho l^a = (\mathcal{E}^\mu_a - \frac{1}{2} \mathcal{E}^a \mu \mathcal{E}^\rho_\rho) m^a = 0,$$  \hspace{1cm} (5.20)

$$\mathcal{E} = 0,$$  \hspace{1cm} (5.21)

$$\mathcal{E}_{I J}^\mu - \tau^* B_{I J}^\mu \phi^J = 0.$$  \hspace{1cm} (5.22)
In this case supersymmetry implies that the scalar equations of motion are automatically satisfied. We are not going to work out here the null case, since it was treated completely in Ref. [23].

We are now ready to follow the recipe to find all the supersymmetric configurations of this theory. The first step consists in finding (Killing) equations for the spinor bilinears. From the vanishing of the gravitini supersymmetry transformation rule we find

\[
D_\mu M_{IJ} = \frac{1}{\sqrt{2}} (\Im m \, \tau)^{3/2} F_K[I]^- K^+_{[J]} \,,
\]

(5.23)

and from that of the dilatini, we find

\[
D_\mu V^I_{J\nu} = -\frac{1}{2\sqrt{2}} (\Im m \, \tau)^{1/2} \left[ M_{KJ} F^{K[I} - \mu \nu + M^{IK} F_{JK} + \mu \nu \right.
\]

\[-\Phi_{KJ}(\rho \rho F^{K[I} - \nu \rho - \Phi_{IK}(\mu \rho F^{K[I} + \nu \rho) ) \,.
\]

(5.24)

It is immediate to see that \( V^I \equiv V^I_{I} \) is a Killing vector and that \( V^\mu \partial_\mu \tau = 0 \).

(5.27)

Further, using Eq. (5.23) and the antisymmetric part of Eq. (5.26) we find

\[
F_{SR}^-_{\mu \nu} V^\nu = -\frac{\sqrt{2} i}{(\Im m \, \tau)^{3/2}} M_{SR} \partial_\mu \tau - \frac{\sqrt{2}}{(\Im m \, \tau)^{1/2}} \varepsilon_{SRJI} D_\mu M_{IJ} \,,
\]

(5.28)

which determines completely the vector field strengths in terms of the scalar bilinears, \( \tau \) and the Killing vector \( V^a \) when this is timelike. In the null case, this equation gives us important constraints on the form of the field strengths, but does not completely determine them. From now on we will focus on the timelike case since it illustrates our procedure best. In this case we can write the metric in the conformastationary form Eq. (4.7), but, while in the \( N = 2, d = 4 \) case one could show that three of the vector bilinears where exact 1-forms and then the metric on the constant-time slices could be chosen to be Euclidean, in the \( N = 4, d - 4 \) case this is not possible and we have to live with a non-trivial 3-dimensional metric \( \gamma_{ij} \). Thus

\[
ds^2 = |M|^2 (dt + \omega)^2 - |M|^{-2} \gamma_{ij} dx^i dx^j \,, \quad i, j = 1, 2, 3 \,,
\]

(5.29)

where \( \omega \) has to satisfy the equation

\[
d \omega = \frac{1}{\sqrt{2}} \Omega = \frac{1}{2\sqrt{2}} |M|^{-4} \left[ (M^{1J} D M_{1J} - M_{1J} D M^{1J}) \wedge \hat{V} \right] \,.
\]

(5.30)

Having the field strengths expressed in terms of the scalars \( M^{IJ}, \tau \), we move on to the next step and impose the Maxwell equations and Bianchi identities on them, to obtain equations that only involve those scalars. We also substitute the field strengths into the \( \tau \) equation,
obtaining another equation that only involves $M^{IJ}$ and $\tau$. Now comes the magic of supersymmetry: these three sets of equations are combinations of just two sets of much simpler equations in the 3-dimensional metric $\gamma_{ij}$:

$$n_{(3)}^{IJ} \equiv (\nabla_i + 4i\xi_i) \left( \partial^i N^{IJ} \right) / |N|^2,$$

$$e_{(3)}^* \equiv (\nabla_i + 4i\xi_i) \left( \partial^i \tau \right) / |N|^2,$$

where $N^{IJ} \equiv (\Im \tau)^{1/2} M^{IJ}$ and $\xi$ is defined by

$$\xi \equiv \frac{i}{4} |M|^{-2} (M_{I,J} dM^{IJ} - M^{IJ} dM_{IJ}),$$

and acts as a $U(1)$ connection.

In fact, we can write all the components of the equations of motion defined above in terms of these two

$$\mathcal{E}_{00} = |M|^2 \left[ |M|^2 3m e_{(3)}^* - 2\text{Re} \left( N_{KL} n_{(3)}^{KL} \right) + \frac{1}{2} e_k^k \right],$$

$$\mathcal{E}_{0i} = 0,$$

$$\mathcal{E}_{ij} = |M|^2 (e_{ij} - \frac{1}{2} \delta_{ij} e_k^k),$$

$$B^{IJ a} = -\sqrt{2} |M|^2 V^a \left\{ \frac{N^{IJ} + \tilde{N}^{IJ}}{\text{Im} \tau} \text{Re} e_{(3)} - i(n_{(3)}^{IJ} - \tilde{n}_{(3)}^{IJ}) \right\},$$

$$\mathcal{E}^{IJ a} = -\sqrt{2} |M|^2 V^a \left\{ \frac{N^{IJ} + \tilde{N}^{IJ}}{\text{Im} \tau} \text{Re} (\tau e_{(3)}) - i(\tau^* n_{(3)}^{IJ} - \tau \tilde{n}_{(3)}^{IJ}) \right\},$$

and a set of equations $e_{ij}$ defined by

$$e_{ij} \equiv R_{ij}(\gamma) - 2\partial(i) \left( \frac{N^{IJ}}{|N|} \right) \partial(j) \left( \frac{N_{KL}}{|N|} \right) (\delta^{KL}_{IJ} - J^K_{IJ} J^L_{IJ}),$$

and which have to vanish in order to satisfy the KSI and have supersymmetry$.^4$ These equations are conditions for the 3-dimensional metric $\gamma_{ij}$, but are not easy to solve directly. We have to substitute our results into the original KSEs or into their integrability conditions. The solution one finds is that, in order to solve the $e_{ij} = 0$ equations have supersymmetry, the 3-dimensional metric has to take the form

$$\gamma_{ij} dx^i dx^j = dx^2 + 2e^{2U(z,z^*)} dz dz^*,$$

$^4$The integrability condition of the equation for $\omega$ has to be satisfied as well in order to have supersymmetry. WE are going to discuss it later.
and the connection $\xi$ has to take the form

$$
\xi = \pm \frac{i}{2}(\partial_\underline{z}Udz - \partial_\underline{z}^*Udz^*) + \frac{1}{2}d\lambda(x, z, z^*). \quad (5.42)
$$

Since $\xi$ is defined in terms of the $M^{I\bar{J}}$ scalars, this is a condition that these scalars have to fulfill, on top of Eqs. (5.31, 5.32).

Further, to have supersymmetry, the integrability condition for the equation defining $\omega$ has to be satisfied as well. It takes the form

$$
\nabla_\underline{z} \left( \frac{Q_i - \xi_i}{\mid M \mid^2} \right) = 0. \quad (5.43)
$$

The timelike case now has been completely solved. Let us put together the results: any supersymmetric configuration of $N = 4, d = 4$ supergravity in this class is given by a set of 7 complex functions $M^{I\bar{J}}, \tau$ which have to satisfy the following conditions:

1. $M^{I\bar{J}}M^{K\bar{L}} = 0$. This is a condition that the scalar bilinears satisfy due to the Fierz identities.

2. $\mid M \mid^2 \neq 0$. We have assumed this, as definition of the timelike case ($V^2 \sim \mid M \mid^2 > 0$).

3. Eq. (5.43) has to be satisfied.

4. $\xi$ has to take the form Eq. (5.42).

Given 7 complex functions satisfying these conditions, then, a supersymmetric field configuration of $N = 4, d = 4$ is given by the metric Eqs. (5.29, 5.41) and the field strengths Eq. (5.28). These field configurations will be supersymmetric solutions if the expressions Eqs. (5.31, 5.32) vanish.

This is the main result in the timelike case.

Now comes the problem of finding sets of 7 complex functions satisfying the above conditions, which is not an easy. We have been able to find two families of supersymmetric solutions based on the Ansatz for the $M^{I\bar{J}}$s

$$
M_{I\bar{J}} = e^{i\lambda(x, z, z^*)}M(x, z, z^*)k_{I\bar{J}}(z), \quad M = M^*, \quad \lambda = \lambda^*, \quad k_{I\bar{J}}k_{K\bar{L}} = 0. \quad (5.44)
$$

which give a connection $\xi$ of the form Eq. (5.42) with

$$
U = +\ln \mid k \mid, \quad \mid k \mid^2 \equiv k^{I\bar{J}}(z*)k_{I\bar{J}}(z). \quad (5.45)
$$

This Ansatz satisfies all the conditions except for Eq. (5.43). In the following two cases, at least, this last condition is also satisfied:

1. If the $k_{I\bar{J}}$ are constants, then, normalizing $\mid k \mid^2 = 1$ for simplicity, $\xi = \frac{1}{2}d\lambda$ and $U = 0$. This case was considered by Tod in Ref. [23] and studied in detail in Ref. [35]. Defining $H_1 \equiv [(3m\tau)^{1/2}e^{-i\lambda}M]^{-1}$, and $\tau = H_1/H_2$ we get solutions if $\partial_\underline{z}\partial_\underline{z}H_1 = \partial_\underline{z}\partial_\underline{z}H_2 = 0$. 


2. With $e^{i\lambda} = M = 1$ and constant $\tau$ we solve all constraints and all equations using the holomorphy of the $k_{I,JS}$s. The metric takes the form

$$ds^2 = |k|^2(dt + \omega)^2 - |k|^{-2}dx^2 - 2dzdz^*.$$  \hspace{1cm} (5.46)

The metric and the supersymmetry projectors correspond to stationary strings lying along the coordinate $x$, in spite of the trivial axion field $\Re \tau$. These solutions clearly deserve more study. Observe that this family is precisely the one that cannot be embedded in $N = 2, d = 4$ supergravity plus matter fields \cite{36} and it is genuinely $N = 4$.

6 Conclusions

The landscape approach offers an interesting, even if controversial, point of view over the vacuum selection problem. It also gives additional reasons to work on the problem of classification of supersymmetric solutions, whose 4-dimensional structure we have reviewed in this talk, emphasizing the difference between general supersymmetric configurations and solutions and showing how the KSIs can be used in this problem. We have applied the recipes to an interesting case: pure $N = 4, d = 4$ supergravity, but is should be clear that the same procedure could be used in more general contexts ($N = 4, d = 4$ coupled to matter, gauged etc. and other 4-dimensional theories \cite{49, 50}). We also expect some of the techniques could also be of use in solving the much more complicated 11- and 10-dimensional problems \cite{37, 48}.

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