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Analytical investigation of the dynamics of a nonlinear structure with two degree of freedom

MADELEINE PASCAL

ABSTRACT
A two degree of freedom oscillator with a colliding component is considered. The aim of the study is to investigate the dynamic behavior of the system when the stiffness obstacle changes from a finite value to an infinite value. Several cases are considered. First, in the case of rigid impact and without external excitation, a family of periodic solutions are found in analytical form. In case of soft impact, with a finite time duration of the shock, and no external excitation, the existence of periodic solutions, with an arbitrary value of the period, is proved. Periodic motions are also obtained when the system is submitted to harmonic excitation, in both cases of rigid or soft impact. The stability of these periodic motions is investigated for these four cases.

Keywords: Nonlinear vibrations, two degree of freedom oscillator, rigid and soft impact, periodic motion, stability, forced and unforced system.

1. INTRODUCTION

Vibrating systems with clearance between the moving parts are frequently encountered in technical applications. These systems with impacts are strongly nonlinear; they are usually modeled as a spring-mass system with amplitude constraint. Such systems have been the subject of several investigations, mainly in the simplest case of a one degree of freedom system [1-4] and more seldom for multi degree of freedom systems [5-9]. The system behavior during the contact between the moving parts can be described as rigid impact, usually associated with a restitution coefficient, or modeled as soft impact, with a finite time duration of the shock. Several other parameters such as damping, external excitation, influence the behavior of the system. The work is the continuation of a previous paper [10], in which a two degree of freedom oscillator is considered. The nonlinearity in this case comes from the presence of two fixed stops limiting the motion of one mass. Assuming no damping and no external excitation, the behavior of the system is investigated when the obstacles stiffness changes from a finite value to an infinite one. In both cases, a family of symmetrical periodic solutions, with two impacts per period, is obtained in analytical form.

In the present paper, a two degree of freedom system in presence of one fixed obstacle is considered. Assuming that no damping occurs, we investigate four cases: unforced system with rigid impact, unforced system with soft impact, forced system (with harmonic excitation) with rigid impact and, at last, forced system with soft impact. In all cases, periodic solutions are found and stability results of these particular motions are obtained.

2. PROBLEM FORMULATION

The system under consideration (Fig.1) is a generalization of the double oscillator investigated in the paper [10]. It consists of two masses \( m_1 \) and \( m_2 \) connected by linear springs of stiffness \( k_1 \) and \( k_2 \). The displacement \( z_i \) of the mass \( m_i \) is limited by the presence of a fixed stop. When \( z_i \) is greater than the clearance, a contact of the first mass with the stop occurs; this contact gives rise to a restoring force associated to a spring stiffness \( k_{ij} \).

![Figure 1. Double oscillator](image-url)
The mathematical model of the system is given by:

\[
M \ddot{z} + F + P \cos(\omega t + \phi) = z = (z_1, z_2, \ldots, z_n)
\]

where

\[
F = \begin{cases} \dot{f}(z_1) & \text{if } z_1 > 0 \\ 0 & \text{if } z_1 \leq 0 \end{cases}
\]

\[
M = \begin{pmatrix} m_0 & 0 & \cdots & 0 \\ 0 & m_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{pmatrix}
\]

\[
F = \begin{pmatrix} f_0(z_1) \\ \vdots \\ f_n(z_n) \end{pmatrix}
\]

\[
F = \begin{pmatrix} f(z_1) \\ \vdots \\ f(z_n) \end{pmatrix}
\]

\[
det(K - M \lambda) = 0
\]

3. UNFORCED SYSTEM

Let us consider the system without external excitation (\(P = 0\)).

3.1 RIGID IMPACT

When the stiffness of the obstacle \(k_3\) tends to infinity, a rigid impact of the first mass against the stop occurs. Starting from initial positions \(z_0 = (z_1, z_2, \ldots, z_n)\) and initial velocities \(\dot{z}_0 = (\dot{z}_1, \dot{z}_2, \ldots, \dot{z}_n)\) corresponding to a contact of the first mass against the stop, assuming a perfect elastic impact, the new positions \(z\) and the new velocities \(\dot{z}\) after the shock are obtained by:

\[
\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} I & 0 \\ \vdots & \vdots \\ 0 & I \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} -I \\ \vdots \\ -I \end{pmatrix} \begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{pmatrix}
\]

\[
(\Gamma_1, \cdots, \Gamma_{n-1}) = 0
\]

\[
\begin{pmatrix} S_0 \end{pmatrix} = \begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{pmatrix}
\]

\[
\begin{pmatrix} \dot{s}_0 \end{pmatrix} = \begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{pmatrix}
\]

\[
\begin{pmatrix} \dot{s}_0 \end{pmatrix} = \begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{pmatrix}
\]

\[
\begin{pmatrix} \dot{s}_0 \end{pmatrix} = \begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{pmatrix}
\]

The following properties for the \(\Gamma_i\) matrices hold:

\[
\Gamma_i(\lambda) - \Gamma_i(\lambda_0) = 0
\]

For \(i = 1, 2, 3\)

Moreover, the coefficients \(C_i(t)\) of the 4 by 4 matrix \(C(t)\) satisfy the property:

\[
C_i(t) = C_{i-2}(t), \quad (i, j = 3, 4)
\]

Let us investigate if for a set of initial conditions \(Z_0 = (z_1, z_2, \ldots, z_n)\) related to contact of the first mass against the stop, it is possible to obtain a periodic solution of period \(T\), with one impact per period.

The free motion performed by the system after the rigid impact finishes at time \(t = T\) when \(z_1(T) = 1\) and \(z_2(T) > 0\). Let us denote by \(Z_1 = (z_1, z_2, \ldots, z_n)\) the positions and the velocities reached by the system at \(t = T\). The condition to obtain such a periodic motion is given by:

\[
Z_1 = HZ_0
\]

\[
H = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

Let us introduce the position \(Z_1\) reached by the system from the initial position \(Z_0\) after a backward motion of duration \(T\):

\[
Z_1 = C(T)Z_0
\]

\[
Z_1 = HZ_0
\]

\[
H = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

It results for the determination of the four scalar parameters \((y, u, w, T)\) the four scalar equations:

\[
(-H_1 + C(-T))Z_1 = 0, \quad Z_1 = (1, y, u, w)'\]

or equivalently:

\[
(-H_1 + C(-T))Z_1 = 0, \quad Z_1 = (1, y, u, w)'
\]

Taking into account the properties (5) of the \(\Gamma_i\) matrices, the system (8) leads to:

\[
Z_1 = C(T)Z_0
\]

\[
Z_1 = HZ_0
\]

\[
H = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

In case of rigid impact, a family of periodic solutions is obtained for which the initial conditions are defined in terms of the period and the initial velocity \(w\) of the non impacting mass is zero. For these particular motions, the conditions giving the positions and the velocities after the shock can be formulated by:

\[
z_1 = (1, y, u, w)'
\]

\[
(\Gamma_1, \cdots, \Gamma_{n-1}) = 0
\]

The last equation of (9) reduces to \(w = 0\). From the first one, \(y\) and \(u\) are obtained in terms of the period:

\[
y = (a_0z_1 - a_1z_1)\tan(\omega_1 T)/\omega_1
\]

\[
-\lambda_1 z_1 - \lambda_2 z_2 = \lambda_3 z_3 - \lambda_4 z_4
\]

In case of rigid impact, a family of periodic solutions is obtained for which the initial conditions are defined in terms of the period and the initial velocity \(w\) of the non impacting mass is zero. For these particular motions, the conditions giving the positions and the velocities after the shock can be formulated by:

\[
z_1 = z_1, \quad z_2 = -z_0
\]

These results are similar to the results obtained in [10]. The system considered in this previous paper is a symmetrical system with respect to the position \(z_1\) of the first mass and it can be expected that the obtained results are due to this
property. But it is not the real explanation because the system investigated now is not symmetrical.

Remark
In more general cases, the impact is described by a restitution coefficient \( r \) (\( 0 < r < 1 \)). In this case, the new velocities after the shock are given by:

\[
Z_r = \begin{pmatrix} 0 \\ E \end{pmatrix}, \quad E = \begin{pmatrix} -r \\ 0 \end{pmatrix}
\]

(12)

The initial conditions corresponding to a periodic orbit of period \( T \) are obtained from the relation:

\[
z_r = -(T - t) \Gamma_z \Delta_z, \quad (E + t) \Delta_z = 0
\]

(13)

leading to the solution \( z_r = z_0 = 0 \). In conclusion, if the restitution coefficient \( r \) is not equal to 1, no periodic solution with one impact per period exists. This fact, of course, is due to the non conservation of the total energy in this case: it is not possible for the system to perform a periodic orbit if no external excitation occurs.

3.2. SOFT IMPACT

Let us assume that the stiffness obstacle is bounded. The mathematical model of the system is given by:

\[
M \ddot{z}_0 + k_0 \dot{z}_0 = 0
\]

(14)

For \( z_0 = 0 \),

\[
M \ddot{z}_0 + Kz_0 = k_0 \dot{z}_0 - k_0 \dot{z}_0 = 0
\]

(15)

Let us assume that the initial conditions are given by:

\[
Z_0 = (0, y_0, u_0, w_0)
\]

A periodic solution is defined in two steps:

- For \( 0 \leq t \leq \tau \), \( z_1 > 1 \), the system is defined by the motion equations (15). The time duration \( \tau \) of this constraint motion is defined by the condition:

\[
z_1(t) = 0
\]

(16)

Let us denote by \( Z_1 = Z(\tau) = (0, y_1, u_1, w_1) \) the value of the parameters at the end of shock, with the condition \( x_0 < 0 \).

- For \( \tau < t \leq T \), a free motion obtained from equations (14) and initial conditions \( Z_1 \) occurs. This motion finishes when \( \tau (r + T) = 1 \).

Let us denote by \( Z_3 = Z(\tau + T) = (0, y_3, u_3, w_3) \) the value of the parameters at the end of free motion \( (u_3 > 0) \)

The condition to obtain a periodic orbit of period \( r + T \) is given by the condition:

\[
Z_3 = Z_0
\]

(17)

The piecewise linear systems (14) and (15) give the two parts of the motion in analytic form.

- For \( 0 \leq t \leq \tau \), the constraint motion is deduced from a modal analysis of system (15):
Two possible cases of periodic solutions can be deduced from (26), namely:

\[ X_1 = X_2 = 0 \text{ or } X_1 = X_2, \det(P - P_2) = 0 \]  

(27)

3.2 EXISTENCE OF PERIODIC MOTIONS (SOFT IMPACT)

Let us discuss the first conditions (27). In this case, we deduce:

\[ z_i = z_2, \quad z_2 = -z_i \]  

(28)

The condition (16) is fulfilled and the initial conditions are obtained from the equations:

\[ (H_1 - J)z_0 - d_0 = 0 \]  

(29)

This system provides four scalar equations for the determination of the five parameters \((y, u, w, \varphi, P)\). It results that, as in the case of rigid impact, \(P\) and hence the period can be chosen arbitrarily. Moreover, the conditions (28) and (11) obtained at the end of the shock are the same for both rigid and soft impacts.

From (29), we deduce:

\[ z_0 = -H^{-1}_1 (H_1 - J)z_0 - d_0 \]  

(30)

The last equation (30), after the elimination of \(y\), provides a relation \(P, P, T\) between the time duration \(r\) of the shock and the time duration \(T\) of the free motion. The other case \(X_1 = X_2, \det(P - P_2) = 0\) leads to no solution [10].

In both cases (soft or rigid impact), a family of periodic motions is obtained, with an arbitrary value of the period. Moreover, the conditions (28) obtained at the end of the shock for soft impact are consistent with Newton rules of rigid impact (11) with a restitution coefficient equal to one, i.e., with assumption of ideal elastic impact. This rather remarkable result has been already obtained for the symmetrical system of (10).

4. FORCED SYSTEM

Let us assume that the two masses are subjected to harmonic external excitations of period \(2\pi/\omega\), constant amplitudes \(P_1, P_2\) and constant phase angle \(\varphi\). From the results obtained in the previous paragraph, where a family of periodic orbits is found with an arbitrary value of the period, it can be expected that for the forced system, periodic solutions of period \(2\pi/\omega\) exist.

4.1 RIGID IMPACT

Let us investigate the case of rigid impact, with a restitution coefficient \(r = 1\). Starting from the initial conditions \(Z_0 = (l, y, u, w, \varphi)\) \((\omega > 0)\), the conditions \(Z_\omega = (l, y, u, w, \varphi)\) after the shock are obtained from (2) and the free motion performed by the system is given by:

\[ z = \Gamma_\omega(t)z_0 - R\cos(\omega t) + \Gamma_\omega(t)z_1 + \text{R} \cos(\omega t + \varphi) \]  

\[ \dot{z} = \Gamma_\omega(t)z_0 - R\cos(\omega t) + \Gamma_\omega(t)z_1 + \text{R} \cos(\omega t + \varphi) \]  

(31)

where \(R = (R_1, R_2)\) is the amplitude of the response:

\[ R_1 = A_1 + A_2, \quad R_2 = A_1 + A_2 \]  

(32)

The free motion finishes at time \(t = T\) when \(z_1(T) = 1, \dot{z}_1(T) > 0\).

Let us denote by \(Z_j = (z_j, \dot{z}_j, \dot{z}_j)\) the positions and the velocities reached by the system at this time. The condition to obtain a periodic motion of period \(T\) is:

\[ Z_j = Z_0 \]  

(33)

or

\[ z_0 = \Gamma_\omega(t)z_0 - R\cos(\omega t) + \Gamma_\omega(t)z_1 + \text{R} \cos(\omega t + \varphi) \]  

\[ \dot{z}_0 = \Gamma_\omega(t)z_0 - R\cos(\omega t) + \Gamma_\omega(t)z_1 + \text{R} \cos(\omega t + \varphi) \]  

(34)

Taking into account the properties (5) of the \(\Gamma_\omega\) matrices, this system reduces to:

\[ \dot{z}_0 = \Gamma_\omega(t)z_0 - R\cos(\omega t) \]  

(35)

and the corresponding values of \(y\) and \(u\) are obtained:

\[ y = \left( \frac{\omega^2 - \omega^2}{\omega^2 - \omega^2} \right) A_1 - A_2 \]  

(36)

Remark:

In more general cases, the impact is described by a restitution coefficient \(r\) \((0 < r < 1)\). The initial conditions and the phase angle related to a periodic solution of period \(2\pi/\omega\) can also be obtained in analytical form. A similar solution has been studied in paper [6].
4.2 SOFT IMPACT

When the stiffness obstacle is bounded, the motion equations of the system are given by:

\[ ME + Ke = F_e + P\cos(\omega t + \phi) \quad \zeta \geq 1 \]

\[ ME + Ke = P\cos(\omega t + \phi) \quad \zeta \leq 1 \]  

From the initial condition \( z_y(0) \neq (1, y, w) \) (\( u > 0 \)), the solution is defined in two parts:

- For \( 0 \leq t < \tau \), \( \zeta \geq 1 \), the solution is given by:
  \[ z = H_x(t)(z_y - d_0 - Q\cos\phi) + H_y(t)z_0 + Q\sin\phi \]
  \[ + d_0 + Q\cos(\omega t + \phi) \]
  \[ \dot{z} = H_x(t)(z_y - d_0 - Q\cos\phi) + H_y(t)z_0 + Q\sin\phi \]
  \[ - Q\sin(\omega t + \phi) \]

The time duration \( r \) of this motion is obtained from the condition \( z_y(r) = 1 \). Let us denote \( z_y = (1, y, w) \) the value of the parameters at \( t = \tau \) (\( u > 0 \)).

- For \( \tau \leq t \leq \tau + \bar{T} \), the motion of the system is defined by:
  \[ z = \Gamma_1(t - \tau)(z_y - R\cos\phi) + \Gamma_2(t - \tau)x_1 + \Gamma_3(t - \tau)x_2 - R\cos\phi \]
  \[ + \Gamma_4(t - \tau)x_3 - R\cos\phi \]
  \[ \dot{z} = \Gamma_1(t - \tau)(z_y - R\cos\phi) + \Gamma_2(t - \tau)x_1 + \Gamma_3(t - \tau)x_2 - R\cos\phi \]
  \[ - \Gamma_4(t - \tau)x_3 - R\cos\phi \]  

The time duration \( r \) of this motion is obtained from the condition \( z_y(r) = 1 \). Let us denote \( z_y = (1, y, w) \) the value of the parameters at \( t = \tau + \bar{T} \) (\( u > 0 \)).

The conditions of periodicity are reformulated as:

\[ \bar{X}_i = \bar{X}_i, \quad \bar{Y}_i = \bar{Y}_i \]  

\[ \bar{X}_i = z_y - z_0 = (H_x - \Gamma_1 x_1 - d_0 - Q\cos\phi) + H_y z_0 + Q\sin\phi \]

\[ \bar{Y}_i = z_0 + d_0 = H_x(z_y - d_0 - Q\cos\phi) + (H_y + \Gamma_1 x_1 - Q\sin\phi) \]

As in the case of unforced system taking into account the properties of the \( H \) and \( \Gamma \) matrices, the solution of system (43) is given by:

\[ \bar{X}_i = 0, \quad \bar{Y}_i = 0 \]  

5. STABILITY OF PERIODIC MOTIONS (RIGID IMPACT)

5.1 UNFORCED SYSTEM

Let us consider a periodic motion of period \( T \) related to initial conditions \( z_{00} = (1, y_0, w_0) \) where \( y_0 \neq 0 \).

Assuming small perturbations \( \Delta y, \Delta w \) of the initial conditions,
The stability of the periodic impact solution is determined by the eigenvalues of the matrix $\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$. By elimination of $dT$, (48) gives:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

From the relations

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

By elimination of $dT$, (48) gives:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

From (49) and the second equation related to $dx$, we deduce:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

or

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

The stability of the periodic impact solution is determined by the eigenvalues of the matrix $A^T B$. All the eigenvalues are inside the unit circle, the periodic solution is stable. Critical cases occur if some eigenvalues lie on the unit circle, the other ones being strictly inside this circle.

Let us consider a perturbed motion related to initial conditions (47) and phase angle $\phi = \phi_0 + z_0$. The corresponding free motion performed by the system for $t > 0$, is obtained from (29), with $z = E_2$. This motion ends at $t = \frac{2\pi}{\omega} + 2\pi \omega T$, when $z(t) = E_2$. Let us denote by $z(t)$, the positions and the velocities of the system at this time.

Assuming small perturbations of the initial conditions and of the phase angle:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

From (52), we deduce:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

From the property $det(A) = det(B)$, we deduce

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

In this case, it is impossible that all the eigenvalues of the matrix $N_1$ lie strictly inside the unit circle. The periodic solution is unstable except if all these eigenvalues lie on the unit circle.

6. STABILITY OF PERIODIC MOTIONS (SOFT IMPACT).

6.1 UNFORCED SYSTEM

When the stiffness of the obstacle is bounded and when there is no external excitation, the mathematical model of the system is given by (14) for the free motion and (15) for the constraint motion. Let us consider a periodic motion of period $T_s = 2\pi / \omega$, where $\omega$ is an arbitrary positive value in this case. This periodic motion is related to the initial conditions $z_0 = \begin{pmatrix} z_0 \\ y_0 \end{pmatrix}$, where $(y_0, u_0, v_0, w_0)$ are defined in terms of $\omega$ by

Let us consider a perturbed motion related to initial conditions (47) and phase angle $\phi = \phi_0 + z_0$. The corresponding free motion performed by the system for $t > 0$, is obtained from (29), with $z = E_2$. This motion ends at $t = \frac{2\pi}{\omega} + 2\pi \omega T$, when $z(t) = E_2$. Let us denote by $z(t)$, the positions and the velocities of the system at this time.

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Let us consider a perturbed motion related to initial conditions (47) and phase angle $\phi = \phi_0 + z_0$. The corresponding free motion performed by the system for $t > 0$, is obtained from (29), with $z = E_2$. This motion ends at $t = \frac{2\pi}{\omega} + 2\pi \omega T$, when $z(t) = E_2$. Let us denote by $z(t)$, the positions and the velocities of the system at this time.

Assuming small perturbations of the initial conditions and of the phase angle:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

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Let us consider a perturbed motion related to initial conditions (47) and phase angle $\phi = \phi_0 + z_0$. The corresponding free motion performed by the system for $t > 0$, is obtained from (29), with $z = E_2$. This motion ends at $t = \frac{2\pi}{\omega} + 2\pi \omega T$, when $z(t) = E_2$. Let us denote by $z(t)$, the positions and the velocities of the system at this time.

Assuming small perturbations of the initial conditions and of the phase angle:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

From (52), we deduce:

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

From the property $det(A) = det(B)$, we deduce

$$\Gamma_1 \Gamma_1 \Gamma_1 \Gamma_1$$

In this case, it is impossible that all the eigenvalues of the matrix $N_1$ lie strictly inside the unit circle. The periodic solution is unstable except if all these eigenvalues lie on the unit circle.
\[
\begin{align*}
\dot{z}_m &= -H_i^2(H_i - I)(z_m - d_i) \\
(\Gamma_i - I + \Gamma_i H_i^2(H_i - I))z_m &= \Gamma_i H_i^2(H_i - I)d_i \\
H_i &= H_i(z_2) \\
\Gamma_i &= \Gamma_i(T_i), (i = 1, 2)
\end{align*}
\]
(54)

and the condition \( T_i = 2\pi / \omega \).

Let us consider the perturbed motion defined by a set of new initial conditions (47). This motion is defined in two steps:

-For \( 0 \leq t \leq r = t_i + d\tau \):
\[
z = H_i(t)(z_2 - d_i) + \dot{H}_i(t)\dot{z}_2 + d_a \\
\dot{z} = \dot{H}_i(t)(z_1 - d_1) + \dot{H}_i(t)\dot{z}_2
\]

This motion ends when \( z_1(t) = 1 \) and \( \dot{z}_1(t) < 0 \). Let us denote by \( z(t) = z_m + d\tau \), \( \dot{z}(t) = \dot{z}_m + d\dot{\tau} \) the positions and the velocities reached by the system at this final time.

-For \( r \leq t \leq 2\pi / \omega + d\theta \), the motion is defined by:
\[
z = \Gamma_i(t - r)z_2 + \Gamma_i(t - r)\dot{z}_2 \\
\dot{z} = \dot{\Gamma}_i(t - r)\dot{z}_2 + \dot{\Gamma}_i(t - r)\dot{\dot{z}}_2
\]

This motion ends when \( z(t) = \Gamma(t(2\pi/\omega + d\theta)) = 1 \). Let us denote by \( \tau_i = z_m + d\tau_i \), \( \dot{\tau}_i = \dot{z}_m + d\dot{\tau}_i \) the positions and the velocities reached by the system at this time. Assuming small perturbations \( d\dot{z}_m, d\dot{z}_2 \) of the initial conditions:
\[
\begin{align*}
\dot{p}_i &= H_1d_1 + H_2d_2 + p_1d\theta \\
\dot{d}_i &= H_1d_1 + H_2d_2 + p_1d\theta \\
p_i &= H_1(z_2 - d_1) + H_2\dot{z}_2 \\
p_1 &= H_1(z_2 - d_1) + H_2\dot{z}_2
\end{align*}
\]
(55)

From the properties of the \( H_i \) matrices and the results obtained in paragraph 3.2, we deduce:
\[
\begin{align*}
p_i &= -z_m, p_2 &= -H_i^{-1}(H_i + I)d_m \\
\dot{p}_1 &= \Gamma_i d_2 + \Gamma_i d_2 + p_1 d\theta \\
\dot{d}_1 &= \Gamma_i d_2 + \Gamma_i d_2 + p_1 d\theta \\
p_i &= \Gamma_i(z_2 - d_1) + \dot{H}_i \\
p_2 &= \Gamma_i(z_2 - d_1) + \dot{H}_i
\end{align*}
\]
(56)

In a same way:

\[
\begin{align*}
\dot{p}_1 &= \Gamma_i d_2 + \Gamma_i d_2 + p_1 d\theta \\
\dot{d}_1 &= \Gamma_i d_2 + \Gamma_i d_2 + p_1 d\theta \\
p_i &= \Gamma_i(z_2 - d_1) + \dot{H}_i \\
p_2 &= \Gamma_i(z_2 - d_1) + \dot{H}_i
\end{align*}
\]
(57)

From the system (55), after the elimination of \( d\tau \), we deduce:
\[
\begin{bmatrix}
y_f \\
w_f \\
y_i \\
w_i \\
x_f \\
x_i
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y \\
w \\
x_f \\
x_i \\
x_f \\
x_i
\end{bmatrix} + \begin{bmatrix}
A_1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega_f \\
\omega_i \\
\omega_f \\
\omega_i \\
\omega_f \\
\omega_i
\end{bmatrix}
\]
(58)

From the system (57), after the elimination of \( d\theta \), we obtain:
\[
\begin{bmatrix}
y_f \\
w_f \\
y_i \\
w_i \\
x_f \\
x_i
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y \\
w \\
x_f \\
x_i \\
x_f \\
x_i
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\omega_f \\
\omega_i \\
\omega_f \\
\omega_i \\
\omega_f \\
\omega_i
\end{bmatrix}
\]
(59)

The stability of the periodic solution is determined by the eigenvalues of the matrix \( A_1 \) giving the linear correspondence between the initial perturbations and the final ones:
\[
\begin{bmatrix}
y_f \\
w_f \\
y_i \\
w_i \\
x_f \\
x_i
\end{bmatrix} = A_1 y \\
A_2 u \\
A_3 i
\]
(60)

The periodic motion is stable if all the eigenvalues of \( A_1 \) lie inside the unit circle.

6.2 FORCED SYSTEM

Let us consider a periodic solution of system (37), of period \( T_0 + T_0 = 2\pi / \omega \), with \( \varphi = -\omega T / 2 \), related to the initial conditions:
\[
z_m = \begin{bmatrix}
y_0 \\
w_0 \\
x_f \\
x_i \\
x_f \\
x_i
\end{bmatrix}
\]
where \( \varphi = -\omega T / 2 \).

The stability of this periodic solution is investigated by considering the motion related to the new initial conditions \( z_m = z_m + d\tau, z_i = z_i + d\dot{\tau} \) and new phase angle
\[
\varphi = -\omega T / 2 + d\varphi
\]
(57)

This motion is defined in two steps:

-For \( 0 \leq t \leq r = r_0 + d\tau \), \( z \) are defined by (38). This motion ends when \( z_1(t) = 1 \) and \( \dot{z}_1(t) < 0 \). Let us denote by \( z_i = z_m + d\tau, z_i = z_m + d\dot{\tau} \) the positions and \( z_i = z_m + d\dot{\tau} \) the velocities reached by the system at this final time.

-For \( r \leq t \leq 2\pi / \omega + d\theta \), \( d\theta = dT \), the motion is defined by
This motion ends when \( z_j = \frac{2\pi}{\omega} + d\theta = 1 \). Let us denote by \( z_j = z_m + dz_j \), the positions and the velocities reached by the system at this time.

Assuming small perturbations \( dz_m, dz_j \) of the initial conditions, \( dz_m = H_1 dz_m + H_2 dz_j + p_1 dt + q_1 d\theta \) \( dz_j = H_1 dz_m + H_3 dz_j + p_2 dt + q_2 d\theta \)

\[
\begin{align*}
\dot{p}_1 &= -z_a, \quad \dot{p}_2 = -H_1^2 (H_1 + 1) z_m - Q \sin \phi_1 \\
\dot{q}_1 &= Q \omega^2 \cos \phi_1 \\
\dot{q}_2 &= (H_1 + 1) Q \sin \phi_1 + H_2 Q \omega \cos \phi_1 \\
q_1 &= (H_1 + 1) \phi_1 \\
q_2 &= H_2 \phi_1
\end{align*}
\]

From systems (61) and (63), after the elimination of \( dt \) and \( d\theta \), we deduce the correspondence between the initial perturbations \( \delta q_0, \delta p_0 \) and the final ones, \( \delta q, \delta p \):

\[
\frac{\delta q}{\delta q_0} = \frac{\delta p}{\delta p_0} = \frac{1}{Q \omega^2} \left( H_1 + 1 \right) q_0 + 1
\]

The stability of the periodic motion is determined by the eigenvalues of the matrix \( \tilde{A} \).

Some numerical investigations are performed for the following values of the parameters [6]:

\[ k_1 = k_2 = 1, \quad k_3 = 5, \quad m_1 = 1, \quad m_2 = 2, \quad P_1 = 2/3, \quad P_2 = 0 \]

The corresponding eigenvalues of the free system are:

\[ \alpha_0 = 1.7958, \alpha_1 = 0.8835 \]

In Figures 2 and 3, the behavior of the periodic solutions is compared for unforced (ZiF) and forced system (ZiNF), in the rigid impact case. Figures 4 and 5 are related to soft impact: the bold part of the curves shows the free motion, the other part (ZiCF or ZiNF) the constraint motion. Other results about the stability conditions when \( \alpha \) varies, will be presented at the Conference.

8. CONCLUSIONS

The main objective of this paper is to compare the behavior of the system when the stiffness of the obstacle changes from a finite value (soft impact) to an infinite value (rigid impact). The results obtained for unforced systems about the existence of periodic solutions show that there is a smooth transition between the two cases. For both cases, the period \( T \) can be chosen arbitrarily, the initial conditions related to the periodic motion are obtained in terms of \( T \). Moreover, in case of rigid impact, the initial velocity of the non impacting mass being zero, we can formulate the conditions giving the positions and the velocities after the shock by formula (11). In case of soft impact, we showed analytically that the conditions obtained at the end of the constraint motion (when the contact of the first mass with the stop finishes), that the positions and the velocities at this time are obtained by the same rule. The case of forced systems is a more standard one. The investigations performed in this paper show how the results obtained in the case of unforced systems can lead very simply to obtain the periodic solutions in case of forced systems. The stability of periodic motions is based on mapping. In all cases, (rigid or soft impact, unforced or forced systems), the matrix of the linear correspondence between the initial perturbations and the final ones, is obtained in close form. In several cases, some interesting properties of the corresponding eigenvalues are also obtained.

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