Threshold Expansion and Dimensionally Regularised NRQCD

Harald W. Grießhammer

Nuclear Theory Group, Department of Physics, University of Washington, Box 351 560, Seattle, WA 98195-1560, USA

Abstract

A Lagrangean and a set of Feynman rules are presented for non-relativistic QFT’s with manifest power counting in the heavy particle velocity $v$. A régime is identified in which energies and momenta are of order $Mv$. It is neither identical to the ultrasoft régime corresponding to radiative processes with energies and momenta of order $Mv^2$, nor to the potential régime with on shell heavy particles and Coulomb binding. In this soft régime, massless particles are on shell, and heavy particle propagators become static. Examples show that it contributes to one- and two-loop corrections of scattering and production amplitudes near threshold. Hence, NRQFT agrees with the results of threshold expansion. A simple example also demonstrates the power of dimensional regularisation in NRQFT.

Suggested PACS numbers: 12.38.Bx, 12.39.Hg, 12.39.Jh.

Suggested Keywords: non-relativistic QCD, effective field theory, threshold expansion, dimensional regularisation.

1Email: hgrie@phys.washington.edu
1 Introduction

Velocity power counting in Non-Relativistic Quantum Field Theories (NRQFT) [1, 2], especially in NRQCD and NRQED, and identification of the relevant energy and momentum regimes has proven more difficult than previously believed. In a recent article, Beneke and Smirnov [3] pointed out that the velocity rescaling rules proposed by Luke and Manohar for Coulomb interactions [4], and by Grinstein and Rothstein for bremsstrahlung processes [5], as united by Luke and Savage [6], and by Labelle’s power counting scheme in time ordered perturbation theory [7], do not reproduce the correct behaviour of the two gluon exchange contribution to Coulomb scattering between non-relativistic particles near threshold. This has cast some doubt whether NRQCD, especially in its dimensionally regularised version [6], can be formulated using a self-consistent low energy Lagrangean. The aim of this article is to demonstrate that a Lagrangean establishing explicit velocity power counting exists, and to show that this Lagrangean reproduces the results in Ref. [3].

This article is confined to outlining the ideas to resolve the puzzle, postponing more formal arguments, calculations and derivations to a future, longer publication [8] which will also deal with gauge theories and exemplary calculations. It is organised as follows: In Sect. 2, the relevant regimes of NRQFT are identified. A simple example demonstrates the usefulness of dimensional regularisation in enabling explicit velocity power counting. Sect. 3 proposes the rescaling rules necessary for a Lagrangean with manifest velocity power counting. The Feynman rules are given. Simple examples in Sect. 4 establish further the necessity of the new, soft régime introduced in Sect. 2. Summary and outlook conclude the article, together with an appendix on split dimensional regularisation [9].

2 Idea of Dimensionally Regularised NRQFT

For the sake of simplicity, let us – following [3] – deal with the Lagrangean

\[ \mathcal{L} = (\partial_\mu \Phi_R)^\dagger (\partial^\mu \Phi_R) - M^2 \Phi_R^\dagger \Phi_R + \frac{1}{2} (\partial_\mu A)(\partial^\mu A) - 2 M g \Phi_R^\dagger \Phi_R A \]  

of a heavy, complex scalar field \( \Phi_R \) with mass \( M \) coupled to a massless, real scalar \( A \). The coupling constant \( g \) has been chosen dimensionless. \( \Phi_R \) will be referred to as “quark” and \( A \) as “gluon” in a slight but clarifying abuse of language. In NRQFT, excitations with four-momenta bigger than \( M \) are integrated out, giving rise to four-point interactions between quarks. The first terms of the non-relativistically reduced Lagrangean read

\[ \mathcal{L}_{\text{NRQFT}} = \Phi^\dagger \left( i \partial_0 + \frac{\vec{\partial}^2}{2M} - g c_1 A \right) \Phi + \frac{1}{2} (\partial_\mu A)(\partial^\mu A) + c_2 \left( \Phi^\dagger \Phi \right)^2 + \ldots , \]  

where the non-relativistic quark field is \( \Phi = \sqrt{2M} e^{iMt} \Phi_R \) and the coefficients \( c_i \) are to be determined by matching relativistic and non-relativistic scattering amplitudes. To lowest order, \( c_1 = 1 \) and \( c_2 = \frac{g^4}{24\pi^2M^2} \). The non-relativistic propagators are

\[ \Phi : \frac{i}{T - \frac{\vec{p}^2}{2M} + i\epsilon} , \quad A : \frac{i}{k^2 + i\epsilon} , \]
where \( T = p_0 - M = \frac{p^2}{2M} + \ldots \) is the kinetic energy of the quark.

When a Coulombic bound state of two quarks exists, the two typical energy and momentum scales in the non-relativistic system are the bound state energy \( M v^2 \) and the relative momentum of the two quarks \( M v \) (i.e. the inverse size of the bound state) \[4\]. Here, \( v = \beta \gamma \ll 1 \) is the relativistic generalisation of the relative particle velocity. Cuts and poles in scattering amplitudes close to threshold stem from bound states and on-shell propagation of particles in intermediate states. They give rise to infrared divergences, and in general dominate contributions to scattering amplitudes. With the two scales at hand, and energies and momenta being of either scale, three régimes are identified in which either \( \Phi \) or \( A \) in (2.3) is on shell:

- **soft régime:** \( A_s: k_0 \sim |\vec{k}| \sim M v \),
- **potential régime:** \( \Phi_p: T \sim M v^2, |\vec{p}| \sim M v \),
- **ultrasoft régime:** \( A_u: k_0 \sim |\vec{k}| \sim M v^2 \).

(2.4)

Ultrasoft gluons \( A_u \) are emitted as bremsstrahlung or from excited states in the bound system. Soft gluons \( A_s \) do not describe bremsstrahlung: Because in- and outgoing quarks \( \Phi_p \) are close to their mass shell, they have an energy of order \( M v^2 \). Therefore, overall energy conservation forbids all processes with outgoing soft gluons but without ingoing ones, and vice versa, as their energy is of order \( M v \).

The list of particles is not yet complete: In a bound system, one needs gluons which change the quark momenta but keep them close to their mass shell:

\[ A_p: k_0 \sim M v^2, |\vec{k}| \sim M v \]  

(2.5)

So far, only potential gluons and quarks, and ultrasoft gluons had been identified in the literature of power counting in NRQFT \[1, 3, 4\]. That the soft régime was overlooked cast doubts on the completeness of NRQFT after Beneke and Smirnov \[3\] demonstrated its relevance near threshold in explicit one- and two-loop calculations. In this article, the fields representing a non-relativistic quark or gluon came naturally by identifying all possible particle poles in the non-relativistic propagators, given the two scales at hand.

When a soft gluon \( A_s \) couples to a potential quark \( \Phi_p \), the outgoing quark is far off its mass shell and carries energy and momentum of order \( M v \). Therefore, consistency requires the existence of quarks in the soft régime as well,

\[ \Phi_s: T \sim |\vec{p}| \sim M v \]  

(2.6)

As the potential quark has a much smaller energy than either of the soft particles, it can – by the uncertainty relation – not resolve the precise time at which the soft quark emits or absorbs the soft gluon. So, we expect a “temporal” multipole expansion to be associated with this vertex. In general, the coupling between particles of different régimes will not be point-like but will contain multipole expansions for the particle belonging to the weaker kinematic régime. For the coupling of potential quarks to ultrasoft gluons, this has been observed in Refs. \[5, 7\].

Propagators will also be different from régime to régime: For soft quarks, \( \frac{\vec{p}^2}{2M} \) is negligible against the kinetic energy \( T \), so that the soft quark propagator may be expanded in powers
of $\frac{m^2}{2M}$, and $\Phi_s$ is expected to become static to lowest order. As the energy of potential gluons is much smaller than their momentum, the $A_p$-propagator is expected to become instantaneous.

With these five fields $\Phi_s$, $\Phi_p$, $A_s$, $A_p$, $A_u$ representing quarks and gluons in the three different non-relativistic régimes, soft, potential and ultrasoft, NRQFT becomes self-consistent. The application of these ideas to NRQCD with the inclusion of fermions and gauge particles is straightforward and will be summarised in the next publication [8]. An ultrasoft quark (which would have a static propagator) is not relevant for this paper. It is hence not considered, as is a fourth (“exceptional”) régime in which momenta are of the order $Mv^2$ and energies of the order $Mv$ or any régime in which one of the scales is set by $M$. They do not derive from poles in propagators, and hence will be relevant only under “exceptional” circumstances. A future publication [8] has to prove that the particle content outlined is not only consistent but complete.

It is worth noticing that the particles of the soft régime can neither be mimicked by potential gluon exchange, nor by contact terms generated by integrating out the ultraviolet modes: Fields in the soft régime have momenta of the same order as the momenta of the potential régime, but much higher energies. Therefore, seen from the potential scale they describe instantaneous but non-local interactions, as pointed out in [3]. Integrating out the scale $Mv$, one arrives at soft gluons and quarks as point-like multi-quark interactions in the ultrasoft régime. The physics of potential quarks and gluons will still have to be described by spatially local, but non-instantaneous interactions. The resulting theory – baptised potential NRQCD by Pineda and Soto [10] – can be derived from NRQCD as presented here by integrating out the fields $\Phi_s$, $A_s$ and $A_p$. There is no overlap between interactions and particles in different régimes.

In order to clarify this point, and before investigating the interactions of the various régimes further, the following example will demonstrate the power of dimensional regularisation in NRQFT. It also highlights some points which simplify the discussion of the following sections. The integral corresponding to a one-dimensional loop

$$I(a, b) := \int dk \frac{1}{k^2 - a^2 + i\epsilon} \frac{1}{k^2 - b^2 + i\epsilon} = \frac{i\pi}{ab(a + b)}$$  \hspace{1cm} (2.7)

is easily calculated using contour integration. Assuming $v^2 := \frac{a^2}{b^2} \ll 1$, the dominating contributions come from the regions where $|k|$ is close to $a$ (“smaller régime”) or $b$ (“larger régime”). Then, one can approximate the integral by

$$I(a, b) \approx \left[ \int_{|k| \sim a} + \int_{|k| \sim b} \right] dk \frac{1}{k^2 - a^2 + i\epsilon} \frac{1}{k^2 - b^2 + i\epsilon}.$$  \hspace{1cm} (2.8)

In the first integral, $k$ is small against $b$, so that a Taylor expansion in $\frac{k}{b} \sim v$ in that régime is applicable and yields

$$-\frac{1}{b^2} \sum_{n=0}^{\infty} \int_{|k| \sim a} \frac{dk}{k^2 - a^2 + i\epsilon} \frac{k^{2n}}{b^{2n}}.$$  \hspace{1cm} (2.9)

If $k^2$ becomes comparable to $b^2$, the expansion breaks down, so that the approximated integral cannot be solved by contour integration. In general, the (arbitrary) borders of the
integration régimes (the “cutoffs”) will enter in the result, and lead to divergences as they are taken to infinity because of contributions from regions where \(|k| \sim b \gg a\). A cutoff regularisation may hence jeopardise power counting in \(v\).

Dimensional regularisation overcomes this problem in a natural and elegant way: If one treats (2.3) as a \(d\)-dimensional integral with \(d \to 1\) only at the end of the calculation, the exact result will emerge as a power series in \(v = \frac{a}{b}\). First, one extends the integration régime from the neighbourhood of \(|a|\) to the whole \(d\)-dimensional space. Then, one calculates the integral order by order in the expansion, still treating \(\frac{k^2}{v^2} \sim v^2\) as formally small. Rewriting

\[
k^{2n} = \sum_{m=0}^{n} \binom{n}{m} a^{2m}(k^2 - a^2)^{n-m},
\]

only the \((m = n)\)-term contributes thanks to the fact that dimensionally regularised integrals vanish when no intrinsic scale is present,

\[
\int \frac{d^d k}{(2\pi)^d} k^\alpha = 0.
\]

The result,

\[
\frac{i\pi}{ab^2} \sum_{n=0}^{\infty} \frac{a^{2n}}{b^{2n}} \left( = \frac{i\pi}{a \frac{1}{b^2 - a^2}} \right)
\]

is exactly the contribution one obtains in the contour integration from the pole at \(|k| = a\). Albeit the integral was expanded over the whole space, dimensional regularisation missed the poles at \(\pm b\) after expansion.

The integration about \(|k| \sim b\) is treated likewise by expansion and term-by-term dimensional regularisation. Adding this contribution,

\[
\frac{-i\pi}{b^3} \sum_{n=0}^{\infty} \frac{a^{2n}}{b^{2n}} \left( = \frac{-i\pi}{b \frac{1}{b^2 - a^2}} \right),
\]

to (2.12), one obtains term by term the Taylor expansion of the exact result (2.7) in the small parameter \(v = \frac{a}{b}\). Each of the two regularised integrals sees only the pole in either of the régimes \(|k| \sim a\) or \(|k| \sim b\). Indeed, the overlap of the two régimes is zero in dimensional regularisation, even for arbitrary \(v\). But then, the expansion in the two different régimes can be terminated only at the cost of low accuracy.

One could therefore have started with the definition of two different integration variables, one formally living in the smaller régime with \(|K_a| \sim a \sim vb\), the other formally living in the larger régime, \(|K_b| \sim b\):

\[
\int dk \to \int d^d K_a + \int d^d K_b
\]

The momentum \(k\) is represented in each of the kinematic régimes by either \(K_a\) or \(K_b\). The integrands must then be expanded in a formal way as if \(|K_a| \sim a \sim vb\) and \(|K_b| \sim b\). Otherwise, the poles are double counted. If one wants to calculate to a certain order in \(v\), the expansion in the different variables \(\frac{K_a}{b}\) and \(\frac{K_b}{b}\) is just terminated at the appropriate order. No double counting of the poles can occur. Coming back to the three different régimes of NRQFT (2.4), there will therefore be no double counting between any pair of domains.
Finally, note that the limit $a \to 0$ is not smooth: For $a = 0$, dimensional regularisation of (2.9) is zero because of the absence of a scale (2.11). A pinch singularity encountered in contour integration at $k = \pm i \varepsilon$ behaves hence like a pole of second order in dimensional regularisation and is discarded, see also App. [A].

By induction, the arguments presented here can be extended to prove that for any convergent one-dimensional integral containing several scales, Laurent expansion about each saddle point and dimensional regularisation gives the same result as contour integration. A formal proof of the validity of threshold expansion does presently not exist for the case of multi-dimensional and divergent integrals, but Beneke and Smirnov [3] could reproduce the correct structures of known non-trivial two-loop integrals using threshold expansion, which is highly suggestive that such a proof can be given. This claim is supported by the observation that threshold expansion is very similar to the asymptotic expansion of dimensionally regularised integrals in the limit of loop momenta going to infinity, for which such a proof exists [11].

3 Rescaling Rules, Lagrangean and Feynman Rules

In order to establish explicit velocity power counting in the NRQFT Lagrangean, one rescales the space-time coordinates such that typical momenta in either régime become dimensionless, as first proposed in [4] for the potential régime, and in [5] for the ultrasoft régime:

- **soft**: $t = (Mv)^{-1} T_s$, $\vec{x} = (Mv)^{-1} \vec{X}_s$,
- **potential**: $t = (Mv^2)^{-1} T_u$, $\vec{x} = (Mv)^{-1} \vec{X}_s$,
- **ultrasoft**: $t = (Mv^2)^{-1} T_u$, $\vec{x} = (Mv^2)^{-1} \vec{X}_u$.

(3.1)

In order for the propagator terms in the NRQFT Lagrangean to be properly normalised, one has to set for the representatives of the gluons in the three régimes

- **soft**: $A_s(\vec{x}, t) = (Mv)^{-\frac{1}{2}} A_s(\vec{X}_s, T_s)$,
- **potential**: $A_p(\vec{x}, t) = (Mv^2)^\frac{1}{2} A_p(\vec{X}_s, T_u)$,
- **ultrasoft**: $A_u(\vec{x}, t) = (Mv^2)^\frac{1}{2} A_u(\vec{X}_u, T_u)$.

(3.2)

and for the quark representatives

- **soft**: $\Phi_s(\vec{x}, t) = (Mv)^{\frac{3}{2}} \phi_s(\vec{X}_s, T_s)$,
- **potential**: $\Phi_p(\vec{x}, t) = (Mv)^{\frac{3}{2}} \phi_p(\vec{X}_s, T_u)$.

(3.3)

The rescaled free Lagrangean in the three regions reads then

- **soft**: $d^3X_s\ dT_s \left[ \phi_s^\dagger \left( i \partial_0 + \frac{v}{2} \vec{\partial}^2 \right) \phi_s + \frac{1}{2} (\partial_\mu A_s)(\partial^\mu A_s) \right]$, (3.4)
- **potential**: $d^3X_s\ dT_u \left[ \phi_p^\dagger \left( i \partial_0 + \frac{1}{2} \vec{\partial}^2 \right) \phi_p + \frac{1}{2} (A_p \vec{\partial}\ A_p - v^2 A_p \partial_0^2 A_p) \right]$, (3.5)
- **ultrasoft**: $d^3X_u\ dT_u \frac{1}{2} (\partial_\mu A_u)(\partial^\mu A_u)$.

(3.6)

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1I am indebted to M. Beneke for conversation on this point.
Here, as in the following, the positions of the fields have been left out whenever they coincide with the variables of the volume element. Derivatives are to be taken with respect to the rescaled variables of the volume element. The (un-rescaled) propagators are

**soft:** \[ \Phi_s : \frac{(T,\vec{p})}{A_1} = i \frac{T}{T + i\epsilon}, \] (3.7)

**potential:** \[ \Phi_p : \frac{(T,\vec{p})}{A_2} = i \frac{T - \vec{p}^2/2M + i\epsilon}{k^2 + i\epsilon}, \] (3.8)

**ultrasoft:** \[ A_u : \frac{(T,\vec{p})}{A_5} = i \frac{k^2 + i\epsilon}{k^2 + i\epsilon}. \] (3.9)

As expected, the soft quark becomes static, resembling the quark propagator in heavy quark effective theory, and the potential gluon becomes instantaneous. In order to maintain velocity power counting, corrections of order \( v \) or higher must be treated as insertions as in the example, (2.7). Insertions are represented by the (un-rescaled) Feynman rules

\[ \frac{(T,\vec{p})}{2M} = O(v), \quad \frac{k}{k^2} = +i k_0^2 = O(v^2). \] (3.10)

Except for the physical gluons \( A_s \) and \( A_u \), there is no distinction between Feynman and retarded propagators in NRQFT: Antiparticle propagation has been eliminated by the field transformation from the relativistic to the non-relativistic Lagrangian, and both propagators have maximal support for on-shell particles, the Feynman propagator outside the light cone vanishing like \( e^{-M} \). Feynman’s perturbation theory becomes more convenient than the time-ordered formalism, as less diagrams have to be calculated.

Finally, the interaction part of the Lagrangean reads (neglecting for the moment the \( \Phi^4 \) vertex in (2.2))

**soft:** \[ d^3x \, dT_s (-g) \left[ (A_s + \sqrt{v} \, A_p(\vec{X}_s,vT_s) + v \, A_u(v \vec{X}_s,vT_s)) \phi_s^\dagger \phi_s + \right. \]

\[ \left. + (A_s \phi_s^\dagger \phi_p(\vec{X}_s,vT_s) + h.c.) \right] \] (3.11)

**potential:** \[ d^3x \, dT_u (-g) \left( \frac{1}{\sqrt{v}} A_p + A_u(v \vec{X}_s,T_u) \right) \phi_p^\dagger \phi_p. \] (3.12)

Note that the scaling régime of the volume element is set by the particle with the highest momentum and energy. Vertices like \( A_s \phi_s^\dagger \phi_p \) cannot occur as energy and momentum must be conserved within each régime to the order in \( v \) one works. Amongst the fields introduced, these are the only interactions within and between different régimes allowed. One sees that technically, the multipole expansion comes from the different scaling of \( \vec{x} \) and \( t \) in the three régimes. It is also interesting to note that there is no choice but to assign one and the same coupling strength \( g \) to each interaction. Different couplings for one vertex in different
régimes are not allowed. This is to be expected, as the example (2.14) demonstrated that the
fields in the various régimes are representatives of one and the same non-relativistic particle,
whose interactions are fixed by the non-relativistic Lagrangean (2.2).

The interaction Feynman rules are

\[ \frac{\langle T, \vec{p} \rangle}{A8} \frac{\langle T', \vec{p}' \rangle}{\uparrow k} = -ig(2\pi)^4 \delta(T + T' + k_0) \delta^{(3)}(\vec{p} + \vec{p}' + \vec{k}) = \mathcal{O}(\frac{1}{\sqrt{v}}), \quad (3.13) \]

\[ = -ig(2\pi)^4 \delta(T + T' + k_0) \left[ \exp \left( \vec{k} \cdot \frac{\partial}{\partial(\vec{p} + \vec{p}')} \right) \delta^{(3)}(\vec{p} + \vec{p}') \right] = \mathcal{O}(e^v) \quad (3.14) \]

\[ = -ig(2\pi)^4 \delta(T + T' + k_0) \delta^{(3)}(\vec{p} + \vec{p}' + \vec{k}) = \mathcal{O}(v^0), \quad (3.15) \]

\[ = -ig(2\pi)^4 \left[ \exp \left( \frac{T'}{\partial(T + k_0)} \right) \delta(T + k_0) \right] \delta^{(3)}(\vec{p} + \vec{p}' + \vec{k}) = \mathcal{O}(e^v), \quad (3.16) \]

\[ = -ig(2\pi)^4 \left[ \exp \left( \frac{k}{\partial(T + T')} \right) \delta(T + T') \right] \delta^{(3)}(\vec{p} + \vec{p}' + \vec{k}) = \mathcal{O}(\sqrt{v} e^v), \quad (3.17) \]

\[ = -ig(2\pi)^4 \left[ \exp \left( \frac{k}{\partial(T + T')} \right) \delta(T + T') \right] \times \]

\[ \times \left[ \exp \left( \vec{k} \cdot \frac{\partial}{\partial(\vec{p} + \vec{p}')} \right) \delta^{(3)}(\vec{p} + \vec{p}') \right] = \mathcal{O}(v e^v). \quad (3.18) \]

The exponents representing the multipole expansion have to be expanded to the desired order
in \( v \). Double counting is prevented by the fact that in addition to most of the propagators,
all vertices are distinct because of different multipole expansions.

Using the equations of motion, the temporal multipole expansion may be re-written such
that energy becomes conserved at the vertex. Now, both soft and potential or ultrasoft
energies are present in the propagators, making it necessary to expand it in ultrasoft and
potential energies. An example would be to restate the vertex (3.17) as

\[ \frac{\langle T, \vec{p} \rangle}{\uparrow k} = -ig(2\pi)^4 \delta(T + T' + k_{p,0}) \delta^{(3)}(\vec{p} + \vec{p}' + \vec{k}) = \mathcal{O}((\sqrt{v} e^v)), \quad (3.19) \]

and the soft propagator as containing insertions \( \mathcal{O}(v) \) for potential energies \( k_{p} \)

\[ \frac{\langle T + k_{p,0}, \vec{p} \rangle}{\uparrow k} = \frac{i}{T} \frac{1}{\Gamma} \sum_{n=0}^{\infty} \left( \frac{-k_{p,0}}{T} \right)^n. \quad (3.20) \]

The same holds of course for the momentum-non-conserving vertices.

In the renormalisation group approach, there is therefore only one relevant coupling (i.e.
only one which dominates at zero velocity): As expected, it is the \( \Phi_p \Phi_p A_p \) coupling providing
the binding. The $\Phi_s\Phi_s A_p$-, $\Phi_s\Phi_s A_s$-couplings and both insertions (3.10) are irrelevant. The marginal couplings $\Phi_p\Phi_p A_u$, $\Phi_s\Phi_s A_s$ and $\Phi_s\Phi_p A_s$ are irrelevant in gauge theories in carefully chosen gauges like the Coulomb gauge. This point will be elaborated upon in the future [8].

The velocity power counting is not yet complete. As one sees from the volume element used in (3.11), the vertex rules for the soft régime count powers of $v$ with respect to the soft régime. One hence retrieves the velocity power counting of Heavy Quark Effective Theory [12, 13] (HQET), in which the interactions between one heavy (and hence static) and one or several light quarks are described. Usually, HQET counts inverse powers of mass in the Lagrangean, but because in the soft régime $Mv \sim \text{const.}$, the two approaches are actually equivalent. HQET becomes a sub-set of NRQCD, complemented by interactions between soft (HQET) and potential or ultrasoft particles.

In NRQCD with two potential quarks as initial and final states, the soft régime can occur only inside loops, as noted above. Therefore, the power counting in the soft sub-graph has to be transferred to the potential régime. Because soft loop momenta scale like $[d\Psi_s] \sim v^4$, while potential ones like $[d\Psi_p] \sim v^5$, each largest sub-graph which contains only soft quarks and no potential ones (a “soft blob”) is enhanced by an additional factor $1/v$. Couplings between soft quarks and any gluons inside a blob take place in the soft régime and hence are counted according to the rules of that régime. Each soft blob contributes at least four orders of $g$, but only one inverse power of $v \sim g^2$. Power counting is preserved. These velocity power counting rules in loops are verified in explicit calculations of the exemplary graphs (see also below), but a rigorous derivation is left for a future publication [8].

With rescaling, multipole expansion and loop counting, the velocity power counting rules are established, and one can now proceed to check the validity of the proposed Lagrangean, matching NRQFT to the relativistic theory in the examples given by Beneke and Smirnov [3].

4 Model Calculations

The first example is the lowest order correction to the two quark production graph. Without proof, it will be used that in dimensional regularisation, one can match NRQFT and the relativistic theory graph by graph, so that not the whole scattering amplitude has to be considered [3]. The collection of graphs to be matched to the relativistic diagram is depicted in Fig. 1. Here and in the following, hard (ultraviolet) contributions will not be shown explicitly. They are taken care of by the four-quark interaction of the non-relativistic

Figure 1: Matching of the $\mathcal{O}(g^2)$ correction to two quark production off an external current to lowest order in $v$ in each of the three régimes.
Lagrangean (2.2) and renormalisation of the external currents [6].

The energy and momentum routing has been chosen to be the one of the non-relativistic center of mass system, with $2T$ the total kinetic energy, and $y = -(\vec{p})^2 \propto -v^2$ the relative four-momentum squared of the outgoing quarks as indicator for the thresholdness of the process considered. Thanks again to dimensional regularisation, any other assignment can be chosen and reproduces the result.

The vanishing of the ultrasoft gluon exchange diagram and the value of the potential gluon exchange diagram have already been calculated in [6]. The soft exchange diagram vanishes, so that no new contribution is obtained. It is not even necessary to specify how soft quarks couple to external sources: If energy is conserved at the production vertex, the integral to be calculated is

$$\int \frac{d^4k}{(2\pi)^d} \frac{1}{T + k_0 + i\epsilon} \frac{1}{T - k_0 + i\epsilon} \frac{1}{k_0^2 - \vec{k}^2}.$$ (4.1)

As the gluon is soft, $T \ll k_0$ and the quark propagators must be expanded in $T/k_0 \sim v$, giving zero to any order as no scale is present in the dimensionally regularised integral. If energy is not conserved at the production vertex, the soft quark propagator is $\frac{1}{\pm k_0}$, and the contribution vanishes again. Therefore, there is no coupling of soft subgraphs to external sources to any order in $v$. Soft quarks in external lines are far off their mass shell and hence violate the assumptions underlying threshold expansion and NRQFT. In general, we conclude that soft quarks are present only in internal lines, and that the first non-vanishing contribution from the soft régime for the production vertex occurs not earlier than at $O(g^4)$.

The first soft non-zero contribution comes actually from the two gluon direct exchange diagram of Fig. 2 calculated by Beneke and Smirnov [3] using threshold expansion. The Mandelstam variable $t = -(\vec{p} - \vec{p}')^2$ describes the momentum transfer in the center of mass system. The ultraviolet behaviour of this graph is mimicked in NRQFT by a four-fermion

\[\text{exchange given by the vertex } i\bar{c}_2 = \frac{-ig^4}{24\pi^4M^2} = O(t^0, y^0) \text{ of the Lagrangean (2.2), which using the rescaling rules is seen to be } O(v).\]
The Feynman rules \([3,14]\) give that the \(A_u A_u\)-diagram is of order \(e^0\) with a leading loop integral contribution (similar to \([3, \text{fl. (32)}]\))

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{k_0^2 - \bar{k}^2} \frac{1}{k_0^2 - \bar{k}^2} \frac{1}{T + k_0 - \frac{\bar{p}^2}{2M}} \frac{1}{T - k_0 - \frac{\bar{p}^2}{2M}} .
\] (4.2)

The diagram is expected to be zero to all orders since the ultrasoft gluons do not change the quark momenta and therefore the scattering takes place only in the forward direction, \(\bar{p} = \bar{p}'\). Upon employing the on-shell condition for potential quarks, \(T = \frac{\bar{p}^2}{2M}\) to leading order, it indeed vanishes as no scale is present. Since \(T - \frac{\bar{p}^2}{2M} \sim M \nu^4 \ll |\bar{k}| \sim M \nu\) (and \(k_0 \sim M \nu^2\)) in the potential régime, this is a legitimate expansion. The \(A_u A_p\) and \(A_p A_u\) contributions (\(O(\frac{1}{e^0})\)) are zero for the same reason. The lowest order contribution to the \(A_p A_p\) graph (\(O(\frac{1}{e^1})\)) is

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{k_0^2 - i\epsilon} \frac{1}{(\bar{p} - \bar{p}' + \bar{k})^2 - i\epsilon} \frac{1}{T + k_0 - \frac{(k + \bar{p})^2}{2M} + i\epsilon} \frac{1}{T - k_0 - \frac{(k + \bar{p})^2}{2M} + i\epsilon} .
\] (4.3)

In the light of the discussion at the end of Sect. \([3]\) it is most consistent to perform the \(k_0\) integration by dimensional regularisation, using \(\int \frac{d^d k}{(2\pi)^d} = \int \frac{d^d k_0}{(2\pi)^d} \frac{d^{d-\epsilon} k}{(2\pi)^d}\), \(\sigma \to 1\) [14]. Chap. 4.1. Split dimensional regularisation was introduced by Leibbrandt and Williams [1] to cure the problems arising from pinch singularities in non-covariant gauges. Appendix A shows that in the case at hand, it has the same effect as closing the \(k_0\)-contour and picking the quark propagator poles prior to using dimensional regularisation in \(d - 1\) Euclidean dimensions. To achieve \(O(v^1)\) accuracy, one must also consider one insertion (4.10) at the potential gluon lines, giving rise to a contribution

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{k_0^2 - i\epsilon} \frac{1}{(\bar{p} - \bar{p}' + \bar{k})^2 - i\epsilon} k_0^2 \left( \frac{1}{k_0^2 - i\epsilon} + \frac{1}{(\bar{p} - \bar{p}' + \bar{k})^2 - i\epsilon} \right) \times
\]

\[
\times \frac{1}{T + k_0 - \frac{(k + \bar{p})^2}{2M} + i\epsilon} \frac{1}{T - k_0 - \frac{(k + \bar{p})^2}{2M} + i\epsilon} .
\] (4.4)

The \(k_0\) integration is naïvely linearly divergent, and hence closing the contour is not straightforward. As App. A demonstrates, split dimensional regularisation circumvents this problem. The sum of both contributions \((4.3) + (4.4)\),

\[
\frac{i}{8\pi t} \frac{M + T}{\sqrt{g}} \left( \frac{2}{4 - d} - \gamma_E - \ln \frac{-t}{4\pi \mu^2} \right) ,
\] (4.5)

agrees with \([3, \text{fl. (31)}]\) when one keeps in mind that in that reference, heavy particle external lines were normalised relativistically, while a non-relativistic normalisation was chosen here. Also, this article uses the \(\overline{MS}\) rather than the \(\overline{MS}\) scheme. Near threshold, the scale is set by the total threshold energy \(4\pi \mu^2 = 4(M + T)^2\).

The soft gluon part is to lowest order \((O(v^{-1})\) because of one soft blob) given by

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{k_0^2 - \bar{k}^2 + i\epsilon} \frac{1}{k_0^2 - \bar{k}^2} \frac{1}{k_0 + i\epsilon} \frac{1}{-k_0 + i\epsilon} ,
\] (4.6)
which corresponds to \[ \text{fl. (33)} \]. Now, split dimensional regularisation must be used if no ad-hoc prescription for the pinch singularity at \( k_0 = 0 \) is to be invoked. That the pinch is accounted for by potential gluon exchange and hence must be discarded, agrees with the intuitive argument that zero four-momentum scattering in QED is mediated by a potential only, and no retardation or radiation effects occur. On the other hand, the model Lagrangean contains three marginal couplings as seen at the end of Sect. \[ \text{fl. (33)} \] which may give finite contributions as energies and momenta of the scattered particles go to zero. Although the prescription and the result from split dimensional regularisation coincide in the present case as demonstrated at the end of Sect. \[ \text{fl. (33)} \] and in App. \[ \text{fl. (33)} \] this may not hold in general. The result to \( \mathcal{O}(v^1) \) exhibits another collinear divergence,

\[
\frac{-i}{4\pi^2} \left( \frac{2}{4-d} - \gamma_E - \ln \frac{-t}{4\pi\mu^2} \right) + \frac{i}{24\pi^2M^2} \left[ 1 + \frac{2y}{t} \left( \frac{2}{4-d} - \gamma_E - \ln \frac{-t}{4\pi\mu^2} \right) \right],
\]

and agrees with the one given by Beneke and Smirnov \[ \text{fl. (36)} \]. The second term comes from insertions and multipole expansions to achieve \( \mathcal{O}(v) \) accuracy.

It is easy to see that the power counting proposed works. As expected, the potential diagram is \( \sqrt{y} \propto v \) stronger than the leading soft contribution, and \( t\sqrt{y} \propto v^3 \) stronger than the four-fermion interaction.

In conclusion, the proposed NRQFT Lagrangean reproduces the result for the planar graph of the relativistic theory only if the soft gluon and the soft quark are accounted for: The four-fermion contact interaction produces just a \( \frac{1}{\sqrt{y}} \)-term, graphs containing ultrasoft gluons were absent, and the potential gluon \( (4.5) \) gave no \( \mathcal{O}(y^0) \) contribution. This shows the necessity of soft quarks and gluons. The coupling strength of the \( \Phi_s A_s \Phi_p \) vertex is also seen to be identical to the other vertex coupling strengths, \( g \).

The planar fourth order correction to two quark production (Fig. \[ \text{fl. (33)} \]) was also compared to the result of \[ \text{fl. (33)} \], and is correctly accounted for when the Feynman rules proposed above are used to \( \mathcal{O}(v^1) \).

\[
\begin{align*}
\text{Figure 3: The non-vanishing contributions to the planar fourth order correction to two quark production. Diagrams with insertions or four-point interactions not displayed.}
\end{align*}
\]

\section{5 Conclusions and Outlook}

The objective of this article was a simple presentation of the ideas behind explicit velocity power counting in dimensionally regularised NRQFT. It started with the identification of three different régimes of scale for on-shell particles in NRQFT from the poles in the non-relativistic propagators. This leads in a natural way to the existence of a new quark field
and a new gluon field in the soft scaling régime $E \sim |\vec{p}| \sim Mv$. In it, quarks are static and gluons on shell, and HQET becomes a sub-set of NRQCD. Neither of the five fields in the three régimes should be thought of as “physical particles”. Rather, they represent the “true” quark and gluon in the respective régimes as the infrared-relevant degrees of freedom. None of the régimes overlap. An NRQFT Lagrangean has been proposed which leads to the correct behaviour of scattering and production amplitudes. It establishes explicit velocity power counting which is preserved to all orders in perturbation theory. The reason for the existence of such a Lagrangean, once dimensional regularisation is chosen to complete the theory, was elaborated upon in a simple example: the non-commutativity of the expansion in small parameters with dimensionally regularised integrals.

Due to the similarity between the calculation of the examples in the work presented here and in [3], one may get the impression that the Lagrangean presented is only a simple reformulation of the threshold expansion. Partially, this is true, and a future publication [8] will indeed show the equivalence of the two approaches to all orders in the threshold and coupling expansion. A list of other topics to be addressed there contains: the straightforward generalisation to NRQCD; a proof whether the particle content outlined above is not only consistent but complete, i.e. that no new fields (e.g. an ultrasoft quark) or “exceptional” régimes arise; an investigation of the influence of soft quarks and gluons on bound state calculations in NRQED and NRQCD; a full list of the various couplings between the different régimes and an exploitation of their relevance for physical processes. The formal reason why double counting between different régimes and especially between soft and ultrasoft gluons does not occur, a derivation of the way soft quarks couple to external sources, and the rôle of soft gluons in Compton scattering deserve further attention, too.

I would like to stress that the diagrammatic threshold expansion derived here allows for a more automatic and intuitive approach and makes it easier to determine the order in $\sqrt{-y} \propto v$ to which a certain graph contributes. On the other hand, the NRQFT Lagrangean can easily be applied to bound state problems. As the threshold expansion of Beneke and Smirnov starts in a relativistic setting, it may formally be harder to treat bound states there. Indeed, I believe that even if one may not be able to prove the conjectures of the one starting from the other, both approaches will profit from each other in the wedlock of NRQFT and threshold expansion.

Acknowledgments

It is my pleasure to thank J.-W. Chen, D. B. Kaplan and M. J. Savage for stimulating discussions. The work was supported in part by a Department of Energy grant DE-FG03-97ER41014.

A Some Details on Split Dimensional Regularisation

This appendix presents the part of the calculations in the examples of Sect. 4 which makes use of split dimensional regularisation as introduced by Leibbrandt and Williams [9]. In its results, split dimensional regularisation agrees with other methods to compute loop integrals in non-covariant gauges, such as the non-principal value prescription [15], but two features
make it especially attractive: It treats the temporal and spatial components of the loop integrations on the equal footing, and no recipes are necessary. Rather, it uses the fact that, like in ordinary integration, the axioms of dimensional regularisation [14, Chap. 4.1] allow to split the integration into two separate integrals:

$$\int \frac{d^q k}{(2\pi)^d} = \int \frac{d^\sigma k_0}{(2\pi)^\sigma} \frac{d^{d-\sigma} \vec{k}}{(2\pi)^{d-\sigma}}$$ \hfill (A.1)

Both integrations can be performed consecutively, and the limit $\sigma \to 1$ can – if finite – be taken immediately, because the integration over the spatial components of the loop momentum in (A.1) is still regularised in $d-1$ dimensions. Finally, the limit $d \to 4$ is taken at the end of the calculation.

Equation (4.3) contains the simplest $k_0$ sub-integral:

$$\int \frac{d^\sigma k_0}{(2\pi)^\sigma} \frac{1}{k_0 + T - \frac{(\vec{k} + \vec{p})^2}{2M} + i\epsilon} \frac{1}{-k_0 + T - \frac{(\vec{k} + \vec{p})^2}{2M} + i\epsilon} \hfill (A.2)$$

Using standard formulae for dimensional regularisation in Euclidean space [16, App. B], the result is finite as $\sigma \to 1$:

$$\int \frac{d^\sigma k_0}{(2\pi)^\sigma} \frac{1}{k_0 + T - \frac{(\vec{k} + \vec{p})^2}{2M} + i\epsilon} \frac{1}{-k_0 + T - \frac{(\vec{k} + \vec{p})^2}{2M} + i\epsilon} \to -\frac{i}{2} \left( T - \frac{(\vec{k} + \vec{p})^2}{2M} + i\epsilon \right)^{-1} \hfill \text{as } \sigma \to 1 \hfill (A.3)$$

It is no surprise that closing the contour produces the same result, because for any finite integral, the answer of all regularisation methods have to coincide. The integral over the spatial components of the loop momentum is now straightforward.

The potential gluon diagram with one insertion at a gluon leg (4.4) yields a split dimensional integral which diverges linearly in $k_0$, so that naive contour integration is not legitimate.

$$\int \frac{d^\sigma k_0}{(2\pi)^\sigma} \frac{k_0^2}{|T - \frac{(\vec{k} + \vec{p})^2}{2M} + i\epsilon|^2 - k_0^2} \to -\frac{i}{2} \left( T - \frac{(\vec{k} + \vec{p})^2}{2M} \right) \hfill \text{as } \sigma \to 1 \hfill (A.4)$$

To arrive at this result, the numerator was re-written as $(k_0^2 - (T - \frac{(\vec{k} + \vec{p})^2}{2M})^2) + (T - \frac{(\vec{k} + \vec{p})^2}{2M})^2$. Its first term cancels the denominator, yielding an integral without scale which therefore vanishes in dimensional regularisation. The second term has been calculated in (A.3). The integral over the spatial components of the loop momentum provides again no complications, leading to (4.5).

Finally, it was already shown at the end of Sect. 2 that dimensional regularisation discards pinch singularities encountered in contour integrations. This is validated again by looking at the split dimensional integral for $k_0$ in the soft gluon contribution (4.6),

$$\int \frac{d^\sigma k_0}{(2\pi)^\sigma} \frac{1}{k_0^2 - a^2} \frac{1}{k_0^2 - b^2} \frac{1}{-k_0^2} \hfill (A.5)$$
where \( a^2 := \vec{k}^2 - i\epsilon \) and \( b^2 := (\vec{p} - \vec{p}' + \vec{k})^2 - i\epsilon \). After combining denominators, the resulting integral is simple:

\[
-2 \frac{\Gamma[3 - \frac{\alpha}{2}]}{(4\pi)^{\frac{\alpha}{2}} \Gamma[3]} \int_0^1 dx \, dy \, x \left( -a^2(1-x) - b^2xy \right)^{\frac{\alpha}{2} - 3} \rightarrow \frac{i}{2} \frac{1}{a^2 - b^2} \left( \frac{1}{a^3} - \frac{1}{b^3} \right) \quad \text{as} \, \sigma \rightarrow 1 \tag{A.6}
\]

This agrees with the result of Beneke and Smirnov \([3, \text{fl. (34)}]\) who use contour integration and drop the contribution from the pinch singularity. The integral over the spatial components of the loop momentum provides again no unfamiliar complications, leading to (4.7).

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