Stability Analysis of a Pedestrian’s Walking Motion Based on the Inverted Pendulum Model

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ABSTRACT

In this paper we use an inverted pendulum with harmonic vibrations of the pivot point to model the walking motion of a pedestrian. The stability analysis of the inverted pendulum is carried out to understand dynamics of a pedestrian. Our theoretical research shows that not only the ratio between the lateral and vertical amplitudes of the pivot point, but also the ratio between the lateral and vertical frequencies of the pivot point has significant effects on the stability of the inverted pendulum. The theoretical results are consistent with everyday experiences and observations as well as numerical simulations. Our research indicates that it is proper to applying the inverted pendulum model to analyze to dynamics of a pedestrian’s walking motion.\(^1\)

INTRODUCTION

The problem of pedestrian-footbridge interaction has received great attention in this decade [1]. The pedestrians can be regarded as external forces applying on a footbridge, which cause vibrations of the bridge. The pedestrians are able to change the stiffness and damping of the pedestrian-footbridge system. It is known now that the zigzag movements of pedestrians can result in excessive lateral vibrations of a footbridge [2]. The pedestrian-footbridge interaction problem is very different from the coupled vibration of vehicle-bridge problem. The determination of the force exerted by pedestrians on a footbridge becomes one of the keys to understand the mechanism of the coupled vibrations of pedestrian-footbridge problem.

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When a pedestrian walks on a footbridge, vertical and horizontal dynamic time-varying forces are both acted on the surface of the bridge as a consequence of oscillations of the gravity center of the body. A pedestrian standing on a rigid surface need a quiet standing postural control, in which the body attempts to center its center of mass and reduce sway. The real biomechanical model for a pedestrian’s walking motion is accurate but complicated. In order to reduce the modeling difficulty to discuss the dynamic interaction between pedestrians and a footbridge in theory, it is more feasible to use a simply inverted pendulum to model a pedestrian [3]. According to the experimentally observed phenomena, Erlicher et al. [4] suggested that a pedestrian’s walking motion is self-sustained. Therefore, it is reasonable to consider the pivot point of the inverted pendulum vibrates harmonically up, down, left and right to model the pedestrian’s walking motion.

In this paper we will analyze the stability of the inverted pendulum model based on the theory of nonlinear dynamics. The stability conditions are derived by calculating the transition curves in the parametric plane by using the perturbation method. Our research reveals that the ratio between lateral and vertical frequencies of a pedestrian’s walking motion has significant effect on the body stability. Our research could contribute to the understanding of the mechanism of the dynamic interaction between pedestrians and a footbridge.

THE INVERTED PENDULUM MODEL

A pedestrian walking on the rigid surface is modeled by an inverted pendulum as shown in Figure 1. The lumped mass represents the center of mass of the pedestrian, which is supported by a rigid massless rod representing a leg. We consider that the pivot point of the inverted pendulum moves harmonically up, down, left and right to model the pedestrian walking. The mass of the ball, the length of the rod, the vertical frequency of the pivot point’s motion are denoted by \( m \), \( L \) and \( \omega \), respectively. \( A, B \) are the amplitudes of vertical and lateral vibrations of the pivot point, respectively, and \( \phi \) is the phase difference between two directions. \( n \) is the ratio between lateral and vertical vibration frequencies of the pivot point, and \( g \) is the acceleration due to the gravity force.

![Figure 1. The diagram of the inverted pendulum model.](image)
Let $\theta$ be the angular coordinate of $m$ measured counterclockwise from the down position. According to Lagrange’s method, the governing equation of the inverted pendulum in Figure 1 can be approximately given by

$$\ddot{\theta} + (\delta + \varepsilon \sin t)(\theta - \frac{1}{6} \theta^3) + \nu \varepsilon^2 \sin(nt + \phi)(1 - \frac{1}{2} \theta^2) = 0,$$

(1)

where $\delta = -\frac{g}{L \omega^2}$, $\varepsilon = \frac{A}{L}$, $\nu = \frac{n^2 BL}{A^2}$, and $0 < B < A < L$, $0 < n < 1$. Therefore, $\varepsilon$ is a small quantity.

**TRANSITION CURVES**

We write the parameter $\delta$ in the form of an expansion series

$$\delta = \delta_0 + \delta_1 \varepsilon + \delta_2 \varepsilon^2 + \cdots,$$

(2)

in which $\delta_i$ are constants to be determined to prevent the secular terms. We focus on the stability conditions near $\delta = 0$, $\delta = n^2 (n \neq 1/2)$ and $\delta = 1/4$. The transition curves can be given by

$$\delta = -\frac{1}{2} \varepsilon^2,$$

(3)

$$\delta = n^2 - \frac{\nu}{a} \varepsilon + \frac{4a^2 n^4 - a^2 n^2 + 4}{8(4n^2 - 1)} \varepsilon^2,$$

(4)

![Figure 2](image)

Figure 2. The transition curves emanating from points $\delta = 0$, $\delta = n^2$ and $\delta = 1/4$ on the $\delta$–axis when $\nu = 0.01$, $\phi = 0.15$, $a = 0.1$, $b = -0.2$. (a) $n = 0.25$ (b) $n = 0.45$ (c) $n = 0.55$ (d) $n = 0.75$. 

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\delta = \frac{1}{4} \pm \frac{1}{2} \varepsilon + \left(\frac{b^2}{16} - \frac{1}{8}\right) \varepsilon^2, \quad (5)
\]

Where \(a\), \(b\) are arbitrary constants dependent on the initial conditions. By considering the values of 0.01, 0.15, 0.1, \(-0.2\) for the parameters \(\nu\), \(\phi\), \(a\) and \(b\), respectively, the transition curves are shown in Figure.2 for different values of the parameter \(n\). From Figure.2, the area of the stable regions increases with the increase of \(n\) in the range of \(n < 1/2\), however, the area has no change once the value of \(n\) exceeds 1/2. In addition, the stable regions are discontinuous when \(n < 1/2\).

CONCLUSIONS

To better understand the dynamics of a pedestrian’s walking motion, we investigate the stability of an inverted pendulum model with the pivot point vibrating harmonically. According to stability charts of the inverted pendulum, a walking pedestrian with a fixed vertical step frequency should increase the lateral step frequency such that the ratio between the lateral and vertical step frequencies exceeds 1/2. Otherwise, the area and continuity of the stable regions are very easily affected by variations of lateral sway amplitude of the body. On the other hand, the stability of the body is not enhanced with the increase of the ratio. From the minimal energy principle the pedestrian should keep the ratio between the lateral and vertical step frequencies near 1/2. Our research can contribute to the study of mechanism of pedestrian-footbridge interaction.

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