AUTOMATA AND AUTOMATA MAPPINGS OF SEMIGROUPS

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Abstract. The paper is devoted to two types of algebraic models of automata. The usual (first type) model leads to the developed decomposition theory (Krohn-Rhodes theory). We introduce another type of automata model and study how these automata are related to cascade connections of automata of the first type. The introduced automata play a significant role in group theory and, hopefully, in the theory of formal languages.

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1. Introduction

In this small note we consider two types of algebraic model of automata. In Section 2 we consider usual automata, i.e., triples of the form \((A, X, B)\) where \(A\) is a set of states of the automaton, \(X\) a set of inputs, and a set \(B\) is treated as the set of external states. The theory of such automata is well-known (see for example \([1, 3, 4]\)) and leads to the famous Krohn-Rhodes decomposition theory. In Section 3 we introduce the notion of automata of the second type. These are triples of the form \((A, \Gamma, \Sigma)\) where \(A\) is a set of states of the automaton, \(\Gamma\) a semigroup of inputs, and a semigroup \(Y\) is treated as outputs. The triple \((A, \Gamma, \Sigma)\) is provided by two binary operations subject to special conditions. This notion is motivated by the ideas of the paper \([2]\). One of the aims of what follows and of entire note is to show how these automata are related to usual ones, and how they appear in the process of the cascade connections of the automata of the first type. Sections 4 and 5 deal with this relation.
2. THE FIRST TYPE OF AUTOMATA

Let \((A, X, B)\) be a triple with two operations \(\circ\) and \(*\), such that \(a \circ x \in A\), \(a * x = b \in B\) for \(a \in A\), \(b \in B\). Here \(A\) is a set of states, \(B\) is a set of external states and \(X\) is a system of inputs. Such a triple is said to be a pure automaton of the first type.

We define a semigroup automaton of the first type as a triple \((A, \Gamma, B)\) with the semigroup of inputs \(\Gamma\), and operations \(\circ: A \times X \to A\) and \(*: A \times X \to B\), subject to conditions
\[
a \circ \gamma_1 \gamma_2 = (a \circ \gamma_1) \circ \gamma_2,
\]
\[
a \ast \gamma_1 \gamma_2 = (a \circ \gamma_1) \ast \gamma_2,
\]
where \(\gamma_1, \gamma_2 \in \Gamma\). There are no operations on the set \(B\). These are the usual definitions of pure and semigroup automata (see [4]).

Given sets \(A\) and \(B\), denote by \(S_A\) the semigroup of transformations of the set \(A\) and by \(\text{Fun}(A, B)\) the set of mappings from \(A\) to \(B\). Consider Cartesian product \(S_{A,B} = S_A \times \text{Fun}(A, B)\). Here \(S_{A,B}\) is a semigroup with respect to the multiplication: \((\sigma, \varphi_1)(\sigma_2, \varphi_2) = (\sigma_1 \sigma_2, \sigma_1 \varphi_2), \sigma \in S_A, \varphi \in \text{Fun}(A, B)\). Define an automaton \((A, S_{A,B}, B)\) by the rule: \(a \circ (\sigma, \varphi) = a \sigma, a \ast (\sigma, \varphi) = a \varphi\). Every automaton \((A, X, B)\) is determined by a mapping \(X \to S_{A,B}\).

The automaton \((A, S_{A,B}, B)\) is a semigroup automaton. Any semigroup automaton \((A, \Gamma, B)\) is determined by a homomorphism \(\Gamma \to S_{A,B}\). In this sense the automaton \((A, S_{A,B}, B)\) is universal.

Let again \((A, X, B)\) be a pure automaton. We have a mapping \(X \to S_{A,B}\). Let \(F(X)\) be the free semigroup over the set \(X\). The initial mapping is extended up to a homomorphism \(F(X) \to S_{A,B}\), which determines a semigroup automaton \((A, F(X), B)\). We can pass from \((A, F(X), B)\) to a faithful semigroup automaton \((A, \Gamma, B)\) where \(\Gamma\) is a result of factorization of the semigroup \(F(X)\) by the kernel of homomorphism in \(S_{A,B}\). So, any pure automaton \((A, X, B)\) gives rise to a faithful semigroup automaton \((A, \Gamma, B)\). This allows to construct a decomposition theory for pure automata (Krohn-Rhodes theory)[3], [4].

3. THE SECOND TYPE OF AUTOMATA

We define a pure automaton of the second type as a triple \((A, X, Y)\), where \(A\) is a set of states, \(X\) a system of inputs and \(Y\) a system of outputs. This triple is equipped with operations \(\circ: A \times X \to A\) and \(*: A \times X \to Y\).

The axiom for the operation \(*\) for the second type automaton is different from that of the first type. The difference comes up from the fact that in the second case the set of outputs \(Y\) is intended to be used as input signals in the serial connection of two automata of the first type, and thus should satisfy the conditions below.

Define a semigroup automaton of the second type as a triple \((A, \Gamma, \Sigma)\) with the set of states \(A\), semigroup of inputs \(\Gamma\), semigroup of outputs \(\Sigma\) and
operations \( \circ : A \times \Gamma \to A \) and \( * : A \times \Gamma \to \Sigma \), subject to conditions
\[
a \circ \gamma_1 \gamma_2 = (a \circ \gamma_1) \circ \gamma_2,
\]
\[
a * \gamma_1 \gamma_2 = (a * \gamma_1)((a \circ \gamma_1) * \gamma_2).
\]
Let us study how an arbitrary pure automaton of the second type \((A, X, Y)\) gives rise to a semigroup automaton of the second type \((A, \Gamma, \Sigma)\).

Consider first the situation when \(\Gamma = F(X)\) and \(\Sigma = F(Y)\), the free semigroups. The transition \(a \to a \circ x\) determines the mapping \(X \to S_A\) and then the homomorphism \(F(X) \to S_A\). We have \(a \circ u \in A\) and thus \(a \circ u_1 u_2 = (a \circ u_1) \circ u_2\).

Proceed now from the mapping \(\alpha : A \times F(X) \to F(Y)\) with the condition \(\alpha(a, u_1 u_2) = \alpha(a, u_1) \alpha(a \circ u_1, u_2)\). This condition arises from the definition of the cascade connection of automata of the first type.

Denote \(\alpha(a, u)\) by \(a * u\). Then \(a * u_1 u_2 = (a * u_1)((a \circ u_1) * u_2)\). We got a semigroup automaton of the second type \((A, F(X), F(Y))\). As a rule, this kind of automata with free semigroups \(F(X), F(Y)\) is used in applications (as in [2]). We are interested here in a more general case of arbitrary semigroups \(\Gamma, \Sigma\).

Suppose the automaton \((A, F(X), F(Y))\) is given and we need to define \((A, \Gamma, \Sigma)\). Proceed from surjections (homomorphisms) \(\mu : F(X) \to \Gamma\) and \(\nu : F(Y) \to \Sigma\) and point out conditions which lead to the automaton \((A, \Gamma, \Sigma)\). Define \(a \circ u = a \circ u^\mu, a \in A, u \in F(X)\). Then the semigroup \(\Gamma\) acts in \(A\). Define the relation between \(\mu\) and \(\nu\) as
\[
a * u^\mu = (a * u)^\nu.
\]
Then we calculate
\[
a * (u_1 u_2)^\mu = a * u_1^\mu u_2^\nu = (a * u_1 u_2)^\nu = ((a * u_1)((a \circ u_1) * u_2))^\nu = (a * u_1)^\nu((a \circ u_1) * u_2)^\nu = (a * u_1)((a \circ u_1) * u_2).
\]
So, \(a * u_1^\mu u_2^\nu = (a * u_1)((a \circ u_1) * u_2)\), as required. Thus, the defined above relation between \(\mu\) and \(\nu\) allows us to construct the semigroup automaton \((A, \Gamma, \Sigma)\), grounding on a pure automaton of the second type \((A, X, Y)\). Various other automata \((A, \Gamma, \Sigma)\) can be constructed using cascade connections of the automata of the first type (see Section 5).

4. Cascade connections of automata of the first type

We start the topic of constructions in automata theory. Cascade connections discussed here generalize parallel and serial connections of automata.

Let (pure) automata \((A_1, X_1, B_1)\) and \((A_2, X_2, B_2)\) be given. Their cascade connection has the form \((A_1 \times A_2, X, B_1 \times B_2)\). In order to make this triple an automaton it is assumed that the mappings
\[
\alpha : A_2 \times X \to X_1, \quad \beta : X \to X_2.
\]
are defined. We set:
\[
(a_1, a_2) \circ x = (a_1 \circ \alpha(a_2, x), a_2 \circ \beta(x)),
\]
Hence, the cascade connection \((A_1 \times A_2, X, B_1 \times B_2)\) is determined by a triple \((X, \alpha, \beta)\). Define further a category of such triples. Let \(\mu : (X, \alpha, \beta) \to (X', \alpha', \beta')\) be a morphism. We have commutative diagrams of mappings

\[
\begin{array}{ccc}
X \times A_2 & \xrightarrow{\alpha} & X_1 \\
\downarrow{\mu} & & \downarrow{\alpha'} \\
X' \times A_2 & \xrightarrow{\beta} & X_2
\end{array}
\]

Here, \(\mu(x, a_2) = (\mu(x), a_2)\). The category of triples determines the category of cascade connections of the given automata.

Proceed to semigroup automata. Given \((A_1, \Gamma_1, B_1)\) and \((A_2, \Gamma_2, B_2)\), pass to \((A_1 \times A_2, \Gamma, B_1 \times B_2)\). We need here a triple \((\Gamma, \alpha, \beta)\), where the mapping \(\beta : \Gamma \to \Gamma_2\) is a homomorphism of semigroups and the mapping \(\alpha : A_2 \times \Gamma \to \Gamma_1\) satisfies condition similar to those for a homomorphism, namely, \(\alpha(a_2, \gamma_1\gamma_2) = \alpha(a_2, \gamma_1)\alpha(a_2 \circ \beta(\gamma_1), \gamma_2)\).

We define actions \(\circ\) and \(*\) in a cascade connection of the automata \((A_1, \Gamma_1, B_1)\) and \((A_2, \Gamma_2, B_2)\) as above, and obtain a category of triples \((\Gamma, \alpha, \beta)\). It is checked that an automaton \((A_1 \times A_2, \Gamma, B_1 \times B_2)\) satisfies the axioms of a semigroup automata, and we have a category of such automata. It is proved that such a category has the universal terminal object, called wreath product of the given automata and denoted by

\[
(A_1, \Gamma_1, B_1) \ wr (A_2, \Gamma_2, B_2).
\]

By the definition of a terminal object, every cascade connection of the given automata is embedded into wreath product.

Recall that the terminal object is realized as follows [4]. We consider a triple \((\Gamma, \alpha, \beta)\), where \(\Gamma\) is a wreath product of semigroups

\[
\Gamma = \Gamma_1 \ wr^{A_2} \Gamma_2.
\]

It is defined as follows: take the semigroup \(\Gamma_1^{A_2}\) whose elements are mappings \(\tilde{\gamma}_1 : A_2 \to \Gamma_1, \tilde{\gamma}_1(a_2) = \gamma_1 \in \Gamma_1\). The semigroup \(\Gamma_2\) acts in \(\Gamma_1^{A_2}\) by the rule \((\tilde{\gamma}_1 \circ \gamma_2)(a_2) = \tilde{\gamma}_1(a_2 \circ \gamma_2)\). Let \(\Gamma\) be the Cartesian product \(\Gamma = \Gamma_1^{A_2} \times \Gamma_2\) with the multiplication defined by the rule \((\tilde{\gamma}_1, \gamma_2)(\tilde{\gamma}_1', \gamma_2') = (\tilde{\gamma}_1 \cdot (\tilde{\gamma}_1' \circ \gamma_2), \gamma_2 \gamma_2')\). Then \(\Gamma \ wr^{A_2} \Gamma_2\). Setting \(\alpha(a_2, (\tilde{\gamma}_1, \gamma_2)) = \tilde{\gamma}_1(a_2)\) we define \(\alpha : A_2 \times \Gamma \to \Gamma_1\). Setting \(\beta(\tilde{\gamma}_1, \gamma_2) = \gamma_2\) we get \(\beta : \Gamma \to \Gamma_2\). The necessary conditions are checked and we come to the automaton

\[
(A_1 \times A_2, \Gamma_1 \ wr^{A_2} \Gamma_2, B_1 \times B_2) = (A_1, \Gamma_1, B_1) \ wr (A_2, \Gamma_2, B_2).
\]

The wreath product construction works in the Krohn-Rhodes theory which leads to the decomposition of pure automata and to the definition of complexity of this decomposition.
5. Automata of the second type and cascade connections

Now we want to relate the automata of the second type with the cascade connection operation defined in the previous section. Let us show that any automaton of the second type can be built through the serial connection of automata.

Recall that a serial connection of automata \((A_1, \Gamma_1, B_1)\) and \((A_2, \Gamma_2, B_2)\) is a particular case of the cascade connection and defined by the triple \((\Gamma, \alpha, \beta)\) where \(\Gamma = \Gamma_2\), the mapping \(\beta : \Gamma \to \Gamma_2\) is defined as \(\beta(x) = x\) and the mapping \(\alpha : A_2 \times \Gamma_2 \to \Gamma_1\) satisfies \(\alpha(a_2, \gamma_1\gamma_2) = \alpha(a_2, \gamma_1)\alpha(a_2 \circ \gamma_1, \gamma_2)\).

It is enough to consider a serial connection of semiautomata of the form \((A, \Gamma)\) and \((B, \Sigma)\). Given a map \(\alpha : A \times \Gamma \to \Sigma\) with the condition
\[
\alpha(a, \gamma_1\gamma_2) = \alpha(a, \gamma_1)\alpha((a \circ \gamma_1), \gamma_2),
\]
define \(\alpha(a, \gamma) = a \ast \gamma\). Then
\[
a \ast \gamma_1\gamma_2 = (a \ast \gamma_1)((a \circ \gamma_1) \ast \gamma_2).
\]
Here the map \(\bar{a} : \Gamma \to \Sigma\) is defined by \(\bar{a}(\gamma) = a \ast \gamma\) for each \(a \in A\). This map is not correlated with the multiplications in \(\Gamma\) and \(\Sigma\). We call such a mapping an automaton one. It is not a homomorphism of semigroups but something similar. Automata mappings of semigroups are determined by automata of the second type.

Action of the semigroup \(\Gamma\) in \(A \times B\) is defined by the rule
\[
(a, b) \circ \gamma = ((a \circ \gamma), (b \circ (a \ast \gamma))).
\]
So we have a triple \((A, \Gamma, \Sigma)\) with the action of \(\Gamma\) in \(A\) defined by \(a \circ \gamma\) and with \(a \ast \gamma \in \Sigma\). This is the automaton of the second type corresponding to the serial connection \((A, \Gamma)\) and \((B, \Sigma)\).

Since we proceed from the automaton \((A, X, Y)\), the condition \(\bar{a}(x) = y \in Y\) holds for every \(x \in X\). Note that if \(\bar{a} : X \to Y\) is a bijection here, then \(\bar{a} : F(X) \to F(Y)\) is a bijection as well. We use the fact that if \(u = v\) is an equality in the free semigroup \(F(Y)\), then it is an identity there, and \(u\) and \(v\) coincide graphically. Induction by the length of the word finishes the proof. In particular, for the words \(wx\) and \(w' x'\) we have \(a \ast wx = (a \ast w)((a \circ w) \ast x) = (a \ast w)y\) and \(a \ast w' x' = (a \ast w')y'\). If \(a \ast wx = a \ast w' x'\), then \(y = y'\) and \(a \ast w = a \ast w'\), which leads to the uniqueness.

6. Applications

Let, for example, \(\Gamma\) and \(\Sigma\) coincide with the free semigroup \(F(X)\) and the mapping \(\bar{a} : F(X) \to F(X)\) is bijective. Take a group generated by all such mappings \(\bar{a}\). For the finite sets \(A\) and \(X\) this group is called automaton group. Numerous applications of automata groups in group theory are described in the seminal paper by Grigorchuk-Nekrashevich-Sushchanskii ([2]). This group is used also in the theory of formal languages, since there are some transformations of words which are elements of \(F(X)\) (see [5]).
Remark 6.1. In the paper [2] also the case of linear automata with acting free semigroup is treated. For this case one can introduce the notion of a linear automaton of the second type. Replacing serial connection of pure automata by the triangular product of linear automata (see [4]) the result similar to that of Section 5 can be obtained. Of course, the axioms for the linear automaton of the second type will be different from the axioms for the pure case.

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