Covering 3-uniform hypergraphs by vertex-disjoint tight paths

Jie Han

School of Mathematics and Statistics, Center for Applied Math, Beijing Institute of Technology, Beijing, China

Correspondence
Jie Han, School of Mathematics and Statistics, Beijing Institute of Technology, 100081 Beijing, China.
Email: han.jie@bit.edu.cn

Funding information
Simons Foundation

Abstract
For $\alpha > 0$ and large integer $n$, let $H$ be an $n$-vertex 3-uniform hypergraph such that every pair of vertices is in at least $n/3 + \alpha(n)$ edges. We show that $H$ contains two vertex-disjoint tight paths whose union covers the vertex set of $H$. The quantity two here is best possible and the degree condition is asymptotically best possible. This result also has an interpretation as the deficiency problems, recently introduced by Nenadov, Sudakov, and Wagner: every such $H$ can be made Hamiltonian by adding at most two vertices and all triples intersecting them.

KEYWORDS
Hamilton cycle, tight path

MATHEMATICAL SUBJECT CLASSIFICATION
05C38

1 | INTRODUCTION

The study of Hamilton cycles is a central topic in graph theory with a long history. In recent years, researchers have worked on extending the classical theorem of Dirac on Hamilton cycles to hypergraphs and we refer to [3–5, 9, 13–15, 24, 25] for some recent results and to [21, 25, 29] for excellent surveys on this topic.

In this paper we confine ourselves to 3-uniform hypergraphs (3-graphs), where each (hyper) edge contains exactly three vertices. For a 3-graph $H$, the minimum codegree $\delta_2(H)$ of $H$ is the minimum of $\deg_H(S)$ over all pairs $S$ of vertices in $H$, where $\deg_H(S)$ is defined to be the number of edges containing $S$. A 3-graph $C$ is called a tight cycle if its vertices can be ordered...
cyclically such that every three consecutive vertices in this ordering define an edge of \( C \), which implies that every two consecutive edges intersect in two vertices. We say that a 3-graph contains a tight Hamilton cycle if it contains a tight cycle as a spanning subgraph. A tight path \( P \) has a sequential order of vertices \( v_1v_2\ldots v_p \) such that every consecutive triple of vertices form an edge, where the ends of \( P \) are ordered pairs \((v_2, v_1)\) and \((v_{p-1}, v_p)\).

Confirming a conjecture of Katona and Kierstead [18], Rödl, Ruciński, and Szemerédi [26, 28] determined the minimum codegree threshold for tight Hamilton cycles in 3-graphs for sufficiently large \( n \), which is \( \frac{n}{2} \). They also showed that a minimum codegree \( n/2 - 1 \) guarantees a tight Hamilton path (a spanning path). The tightness of the results can be seen from the following example given in [18]. Let \( V = X \cup Y \), where \( |X| = \lfloor n/2 \rfloor \) and \( |Y| = \lceil n/2 \rceil \). Let \( H_0 \) be a 3-graph on \( V \) obtained from the complete 3-graph on \( V \) by removing all triples that contain one vertex from \( X \) and two vertices from \( Y \). It is straightforward to check that \( \delta_2(H_0) = \min(|X| - 1, |Y| - 2) = \lfloor n/2 \rfloor - 2 \). Moreover, by construction, no tight path can connect a pair of vertices in \( X \) and a pair of vertices in \( Y \). From this it is not hard to see that \( H_0 \) has no tight Hamilton path and adding a vertex and all triples containing it results a 3-graph with no tight Hamilton cycle. We refer to [18] for details.

### 1.1 Main result

A big obstruction for obtaining the tight Hamiltonicity is the “connection”: even when the minimum codegree is close to \( n/2 \), there might be pairs of vertices that cannot be connected by a tight path. On the other hand, the example \( H_0 \) above contains two vertex-disjoint tight paths whose union covers all vertices. The aim of this paper is to show that a much weaker minimum codegree condition assures this.

**Theorem 1.1.** Given \( \alpha > 0 \), there exists \( n_0 \) such that the following holds for \( n \geq n_0 \). Let \( H \) be an \( n \)-vertex 3-graph with \( \delta_2(H) \geq n/3 + \alpha n \). Then there exist two vertex-disjoint tight paths whose union covers \( V(H) \).

The quantity “two” is the best possible as seen by \( H_0 \). The minimum codegree condition is asymptotically best possible by the following example. Let \( V = V_0 \cup V_1 \cup V_2 \), where \( |V_0| = |V_1| = |V_2| = n/3 \). Let \( H_1 \) be the 3-graph whose edges are all triples of form \( V_i V_{i+1} \), \( i = 0, 1, 2 \) where \( V_3 = V_0 \). Note that \( H_1 \) contains three “classes” of edges and no pair of edges from two of them can be arranged in a tight path. Then it is easy to see that one needs three tight paths to cover \( V(H_1) \). It is also worth mentioning that \( n/3 \) is the asymptotic threshold for covering by a bounded number of vertex-disjoint tight paths. Indeed, for any \( \alpha \in (0, 1/3) \), the 3-graph \( H_2(\alpha) \) on \( V(H_2(\alpha)) = X \cup Y \) with \( |X| = \alpha n \) and \( E(H_2(\alpha)) = \{e : e \cap X \neq \emptyset\} \) shows that one may need \( \Omega(n) \) vertex-disjoint tight paths to cover a 3-graph \( H \) with \( \delta_2(H) \geq (1/3 - o(1))n \).

We conjecture that \( n/3 \) is indeed the threshold for this property.

**Conjecture 1.2.** The minimum codegree assumption in Theorem 1.1 can be weakened to \( n/3 \).

Our problem also relates to a new type of problems, namely, “the deficiency problem,” introduced very recently by Nenadov, Sudakov, and Wagner [22]. Note that for Hamiltonicity, this problem has been studied previously in, for example, [2, 23]. Tailoring it to
our problem, it asks for a given 3-graph $H$, what is the smallest integer $t$ such that $H\ast K_t^{(3)}$ contains a tight Hamilton cycle? Here $H\ast K_t^{(3)}$ is a 3-graph obtained from adding $t$ vertices and all triples touching it to $H$. The authors of [22] solved the problem completely for Hamiltonicity in (2-)graphs with a given number of edges and asked for analogous results in 3-graphs. Theorem 1.1 says that if $H$ satisfies that $\delta_2(H) \geq n/3 + \varepsilon n$ for $\varepsilon > 0$ and $n$ large, then $t \leq 2$. Note that the minimum-degree version of the deficiency problem for Hamiltonicity is not interesting as seen by the unbalanced complete bipartite graphs—one needs as many vertices as to raise the minimum degree of the resulting graph $G\ast K_t$ to be half of its vertices. Thus, Theorem 1.1 exhibits a different behavior for $k$-graphs, $k \geq 3$.

**Problem 1.3.** Let $H$ be a 3-graph with $\delta_2(H) \geq \alpha n$. Determine the smallest $t = t(n, \alpha)$ such that $H\ast K_t^{(3)}$ contains a tight Hamilton cycle.

Theorem 1.1 shows that $t = 2$ whenever $\alpha \in (1/3, 1/2)$. This problem makes Conjecture 1.2 more appealing as an affirmative answer to the conjecture will give a bound on $t = t(n, \alpha)$ up to an additive constant. Indeed, given a 3-graph $H$ with $\delta_2(H) = \alpha n$, add a set $A$ of $\beta n$ vertices and all triples touching $A$ to $H$, where $\beta$ satisfies $\frac{\alpha + \beta}{1 + \beta} = \frac{1}{3}$. Thus, the resulting 3-graph has a path covered by two paths, and adding two more “omni” vertices results in a tight Hamilton cycle, namely, $t \leq \beta n + 2$. Similarly, Theorem 1.1 shows that $t \leq \beta n + o(n)$. On the other hand, the 3-graph $H_2(\alpha)$ shows that $t \geq \beta n$.

### 1.2 | Proof ideas

Our proof employs the absorbing method, which is shown to be effective on embedding spanning structures. For example, in [26], under the minimum codegree condition $\delta_2(H) \geq (1/2 + o(1))n$ it is shown that every vertex has many $v$-absorbers, a 4-vertex tight path that allows us to insert $v$ into the path as an interior vertex. Then towards a tight Hamilton cycle, they first build an absorbing path that contains many $v$-absorbers for every vertex $v$, which can “absorb” a small but arbitrary set of vertices and reduces the problem by finding an almost spanning tight path. Moreover, they prove a connecting lemma: every two pairs of vertices can be connected by a constant length tight path.

In contrast, with a significantly weaker codegree condition, we have to look for weaker properties.

#### 1.2.1 | Absorption

We define our absorbers (see Section 2) for triples of vertices, and although not all triples can be absorbed, we classify the triples that can be absorbed and show that we can always partition our final leftover vertices into those triples and finish the absorption. To classify the triples that have many absorbers, we use the lattice-based absorbing method recently developed by the author [10].

---

1 Or a weaker statement: minimum codegree $n/3 + C$ guarantees $C'$ vertex-disjoint tight paths that cover the vertex set, where $C$, $C'$ are absolute constants.
1.2.2 | Connection

A pseudopath in a 3-graph $H$ is a sequence $(e_1, ..., e_t)$ of not necessarily distinct edges of $H$ such that $|e_i \cap e_{i+1}| = 2$ for each $i = 1, ..., t - 1$. Then a 3-graph $H$ is connected if every two edges are connected by a pseudopath. The tight components of $H$ are the connected components of $H$, which are equivalence classes of edges.

For connecting, we show that among every set of three pairs of vertices, two of them can be connected by a constant length tight path. In fact, this is motivated by a result of Mycroft [8, Proposition 2], who proved that any $n$-vertex 3-graph with minimum codegree $n/3$ has at most two tight components. Inspired by this, we use the regularity method and prove that the reduced 3-graph $R$ has at most two tight components. Then it is straightforward to show that almost every two pairs $(v_1, v_2), (v_3, u_4)$ from the pair of clusters who lie in the same “component” in $R$ can be connected by a short path. However, this only provides a connection for certain orientations of the pairs and is not enough to prove our connecting lemma. To see it, consider a complete 3-partite 3-graph $H$ on $V_1 \cup V_2 \cup V_3$, namely, every triple that meets all three clusters is an edge. Then taking $v_1, u_1 \in V_1$ and $v_2, u_2 \in V_2$, there is no tight path $P$ in $H$ that connects the pairs as $v_1, v_2, u_2, u_1$. To overcome this, we note that when the minimum codegree of $R$ is above $|R|/3$, every edge of $R$ lies in a copy of $K_4^-$, the unique 4-vertex 3-graph with 3 edges. The copy of $K_4^-$ will help us to make the “turn.” For the regularity method, we use a recent variant—a regular slice lemma due to Allen, Böttcher, Cooley, and Mycroft [1].

Throughout the rest of the paper, by paths we mean tight paths in 3-graphs. We introduce our absorbers in Section 2 and our connecting lemma (Lemma 3.1) and path cover lemma (Lemma 3.2) in Section 3, followed by a proof of Theorem 1.1. In Section 4, we introduce the hypergraph regularity method and the regular slice lemma (Theorem 4.4) and use them to prove Lemma 3.1 in Section 5 and Lemma 3.2 in Section 6, respectively.

2 | ABSORPTION

In this section we give some preliminary results on the absorption part of our proof. The following example illustrates our idea of absorbers.

Example 1. Given a set of three vertices $S = \{v_1, v_2, v_3\}$, consider the following set of four paths $P_1, P_2, P_3, P_4$

- for $i = 1, 2, 3$, $P_i$ is a path $a_i b_i w_i c_i d_i$ and such that $a_i b_i v_i c_i d_i$ is also a path,
- $P_4$ is a path $u_1 u_2 u_3 u_4$ and such that $u_1 u_2 w_1 w_2 w_3 u_3 u_4$ is also a path.

That is, when we absorb $S$, $v_i$ will replace $w_i$ in $P_i$, $i = 1, 2, 3$, and $w_1 w_2 w_3$ will be put inside $P_4$. A known routine of the absorbing method for our problem is to show that every triple has many absorbers and then known probabilistic arguments will produce a collection of absorbers that can absorb a small but arbitrary set of triples, which gives the existence of the absorbing path (in our problem, we may obtain a set of two paths). Unfortunately, such a property may not hold: there might be triples that have no absorber at all and we have to classify the triples.

---

2In fact, we only have a weaker condition: all but $o(|R|^2)$ pairs of vertices in $R$ have codegree $|R|/3$. 
that have many absorbers. A recent scheme to deal with such classifications is the lattice-based absorbing method developed by the author.

To see that some triple may not have any absorber, consider the divisibility barrier: let \( H' \) be a 3-graph with a vertex partition \( V(H') = X \cup Y \) and the edges of \( H' \) are all triples in \( X \) and all triples that contain exactly one vertex in \( X \). When \(|X| \approx |Y|\), we have \( \delta_3(u) \approx n/2 \). Note that for any \( S = \{v_1, v_2, v_3\} \subseteq Y \), since we can exchange \( v_i \) and \( w_i \) (as in Example 1), \( w_1, w_2, w_3 \) must also be in \( Y \). However, as \( Y \) is an independent set, we cannot build the desired \( P_3 \) because any choice of \( w_1,w_2,w_3 \notin E(H') \).

Our actual absorbers are a little bit more complicated and allow more flexibility.

**Definition 2.1.** Given \( S = \{v_1, v_2, v_3\} \), a family \( \mathcal{Q} = \{P_1, ..., P_t\} \) of vertex-disjoint tight paths is an \( S \)-absorber if there exists a family of vertex-disjoint tight paths \( Q' = \{P'_1, ..., P'_t\} \) such that \( V(\mathcal{Q}) \cup S = V(Q') \) and \( P'_i \) and \( P_i \) have the same ends, for \( i = 1, ..., t \), respectively.

We give some notation for the lattice-based absorbing method. Let \( H \) be a 3-graph on a vertex set \( V \) with \( |V| = n \). Two (not necessarily distinct) vertices \( u, v \in V \) are called \((\beta, i)\)-reachable in \( H \) if there are at least \( \beta n^{3i−1}(5i − 1) \)-sets \( T \) such that

- there exist vertex-disjoint tight paths \( P_1, ..., P_t \) of length 3 such that \( V(P_1 \cup \cdots \cup P_t) = T \cup \{u\} \),
- there exist vertex-disjoint tight paths \( P'_1, ..., P'_t \) of length 3 such that \( V(P'_1 \cup \cdots \cup P'_t) = T \cup \{v\} \),
- for each \( j \in [i] \), \( P_j \) and \( P'_j \) have the same ends.

We say a vertex set \( U \) is \((\beta, i)\)-closed in \( H \) if any two vertices \( u, v \in U \) are \((\beta, i)\)-reachable in \( H \). For every \( v \in V(H) \), let \( \tilde{N}_{\beta,i}(v) \) be the set of vertices that are \((\beta, i)\)-reachable to \( v \).

We write \( \alpha \ll \beta \ll \gamma \) to mean that it is possible to choose the positive constants \( \alpha, \beta, \gamma \) from right to left. More precisely, there are increasing functions \( f \) and \( g \) such that, given \( \gamma \), whenever we choose some \( \beta \leq f(\gamma) \) and \( \alpha \leq g(\beta) \), the subsequent statement holds. Hierarchies of other lengths are defined similarly. Given a 3-graph \( H \) and a vertex \( v \in V(H) \), \( N_H(v) \) is defined as the collection of pairs of vertices \( S \subseteq \binom{V(H)}{2} \) such that \( S \cup \{v\} \in E(H) \).

**Proposition 2.2.** Suppose that \( \frac{1}{n} \ll \eta \ll \alpha \). Let \( H \) be a 3-graph with \( \delta_2(H) \geq (1/3 + \alpha)n \). Then for any \( v \in V(H) \), \( |\tilde{N}_{\eta,1}(v)| \geq (1/3 + \alpha/2)n \).

**Proof.** Take \( \eta \ll \gamma \ll \alpha \). Fix a vertex \( v \). For any other vertex \( u \neq v \), if \( |N_H(u) \cap N_H(v)| \geq \gamma n^2 \), then by the supersaturation result (see [6]), there exist \( \eta n^4 \) copies of 4-vertex (graph) paths in \( N_H(u) \cap N_H(v) \), which means that \( u \in \tilde{N}_{\eta,1}(v) \). So if \( u \notin \tilde{N}_{\eta,1}(v) \), then \( N_H(u) \cap N_H(v) \) is \( < \gamma n^2 \). By double counting, we have

\[
(1/3 + \alpha)n \cdot |N_H(v)| \leq \sum_{S \subseteq \tilde{N}_{\eta,1}(v)} \deg_S(v) < |\tilde{N}_{\eta,1}(v)| \cdot |N_H(v)| + n \cdot \gamma n^2.
\]

Moreover, we have that \( |N_H(v)| = \deg_H(v) \geq (n - 1)(1/3 + \alpha)n/2 \geq n^2 / 6 \). Thus, \( |\tilde{N}_{\eta,1}(v)| > (1/3 + \alpha)n - \gamma n^3 / |N_H(v)| \geq (1/3 + \alpha/2)n \) as \( \gamma \ll \alpha \).
We need a “partition lemma” that classifies the vertices of $V(H)$ under the reachability relation. Note that the reachability relation allows “concatenation,” under the weakening of the constants—that is, if for $u, v \in V(H)$, there exist at least $\mu n$ vertices $w$ such that $u$ is $(\beta_1, i_1)$-reachable to $w$ and $w$ is $(\beta_2, i_2)$-reachable to $v$, then $u$ is $(\mu \beta_1 \beta_2 - o(1), i_1 + i_2)$-reachable to $v$.\(^3\)

**Lemma 2.3.** Given $\delta > 0$, and $0 < \eta' \ll \delta$, there exists a constant $\beta > 0$ such that the following holds for all sufficiently large $n$. Assume $H$ is an $n$-vertex 3-graph such that $|N_{\eta',1}(v)| \geq \delta n$ for any $v \in V(H)$. Then there is a partition $\mathcal{P}$ of $V(H)$ into $V_1, \ldots, V_r$ with $r \leq 1/\delta$ such that for any $i \in [r]$, $|V_i| \geq (\delta - \eta')n$ and $V_i$ is $(\beta, 2^{1/\delta}-1)$-closed in $H$.

**Remark.** This lemma has been proved as [12, Lemma 6.3], under a different notion of reachability designed for the $F$-factor problem. We remark that the form needed in this paper follows from the proof in [12]. Indeed, the same proof works as long as the “reachability” notion satisfies the following two properties:

- **Concatenation:** Let $i_1, i_2 \in \mathbb{N}$ and $\mu, \beta_1, \beta_2 \in (0, 1)$. If for $u, v \in V(H)$, there exist at least $\mu n$ vertices $w$ such that $u$ is $(\beta_1, i_1)$-reachable to $w$ and $w$ is $(\beta_2, i_2)$-reachable to $v$, then $u$ is $(\mu \beta_1 \beta_2 - o(1), i_1 + i_2)$-reachable to $v$.
- **Inflation:** Let $i_1, i_2 \in \mathbb{N}$ and $\beta_1 \in (0, 1)$. If $u$ is $(\beta_1, i_1)$-reachable to $v$ and $i_1 \leq i_2$, then there exists $\beta_2 > 0$ such that $u$ is $(\beta_2, i_2)$-reachable to $v$.

It is easy to see that our definition of reachability in this paper satisfies both points (for the second one, just append $i_2 - i_1$ vertex-disjoint 5-vertex paths that are disjoint from the given reachable sets).

For our problem, we can take $\delta = 1/3 + \alpha/2 > 1/3$ and thus Lemma 2.3 will return either a trivial partition or a partition of two parts, so that each part has a size at least $(1/3 + \alpha/3)n$ and is $(\beta, 2)$-closed in $H$. Due to the technicality of the statement we choose not to present an absorbing lemma but rather integrate it into our proof of Theorem 1.1.

## 3 Proof of Theorem 1.1

We first present our connecting lemma and the path cover lemma whose proofs are postponed to later sections.

**Lemma 3.1 (Connecting lemma).** Given $\alpha > 0$, there exist $\zeta_0 > 0$ and integer $n_0$ such that the following holds for all $\zeta < \zeta_0$ and integers $n \geq n_0$. Let $H$ be a 3-graph with $\delta_2(H) \geq (1/3 + \alpha)n$. Suppose $P_1, \ldots, P_q$ are $q$ vertex-disjoint tight paths of $H$ such that $|V(P_1 \cup \cdots \cup P_q)| \leq \zeta n$. Moreover, for $i = 1, 2$, assume that $p_i$ is a (specified) end edge of $P_i$. Then there exist two vertex-disjoint tight paths $P'_1, P'_2$ such that $|V(P'_1 \cup P'_2)| \leq \sqrt{\zeta} n$ and they contain $P_1, \ldots, P_q$ as subpaths and contain $p_1, p_2$ as two (out of the four) ends.

Lemma 3.1 requires that the paths to be connected occupy a small proportion of the host graph which has a minimum codegree condition. To use it to connect long paths in $H$, a known

\(^3\)Note that the $o(1)$ term exists because a pair of reachable sets for $u$ and $w$, and for $w$ and $v$ may overlap.
way is to use the trick of “reservoir”: we first put aside a set $A$ of vertices chosen uniformly at random, which inherits the minimum codegree condition of $H$ even after adding a small number of other vertices in $H$ to it; then after we find the long paths, we consider $H' := H [A \cup \bigcup_i p_i]$, where $p_i$ are the ends of the paths. So we can apply Lemma 3.1 as long as $\|\bigcup_i p_i\| \leq 3\|V(H')\|$ and the connection of the ends $p_i$’s also give rise to the connection of the long paths in $H$. One may also think of the above trick as “contracting” the long paths into 4-vertex paths.

**Lemma 3.2** (Path cover lemma). Given $\alpha, \eta > 0$, there exists integer $n_0$ such that the following holds for all integers $n \geq n_0$. Let $H$ be a 3-graph with $\delta_2(H) \geq (1/3 + \alpha)n$. Then there exist two vertex-disjoint tight paths $P_1, P_2$ such that $V(P_1 \cup P_2) \supseteq n - \eta n$.

Let $P = \{V_1, \ldots, V_6\}$ be a partition of $V$. The index vector $i_P(S) \in \mathbb{Z}^6$ of a subset $S \subseteq V$ is the vector whose coordinates are the sizes of the intersections of $S$ with each part of $P$.

We recall the following Chernoff’s inequality (see, e.g., [17]). For $x > 0$ and a binomial random variable $X = \text{Bin}(n, \zeta)$, it holds that

$$
\mathbb{P}(X \geq n\zeta + x) < e^{-x^2/(2n\zeta + x/3)} \quad \text{and} \quad \mathbb{P}(X \leq n\zeta - x) < e^{-x^2/(2n\zeta)}.
$$

(1)

Now we are ready to prove Theorem 1.1. The proof follows the scheme of the absorbing–reservoir method and uses Lemmas 3.1 and 3.2 in the obvious way. The additional work comes from the fact that not all triples have many absorbers. To address this we use Lemma 2.3 to find a partition of $V(H)$ into at most two parts, and classify the triples that do have many absorbers. Then in the last step, we show that we can always partition the leftover vertices $A'$ into triples that have many absorbers in the absorbing paths.

**Proof of Theorem 1.1.** Apply Lemma 3.1 with $\alpha/4$ in place of $\alpha$ and obtain $\zeta_0 > 0$. Choose new constants $1/n \ll \eta \ll \gamma \ll \beta \ll \eta' \ll \alpha, \zeta_0$. By Proposition 2.2, Lemma 2.3 (applied with $\eta' \ll \delta := 1/3 + \alpha$) gives a partition $P$ of $V(H)$ with $|P| = 1$ or 2 such that each part of $P$ has at least $(1/3 + \alpha/4)n$ vertices and is $(\beta, 2)$-closed in $H$. We first show the following claim.

**Claim 3.3.** For any triple $S = \{v_1, v_2, v_3\}$, if $H$ has $\alpha n^3$ edges $e$ such that $i_P(S) = i_P(e)$, then $H$ contains $\beta^4 n^{34}/2$ $S$-absorbers.

**Proof:** By the supersaturation result (see [6]), $H$ contains $\beta n^7$ copies of $K_{3,3,3}^{(3)}$ using these $\alpha n^3$ edges given in the claim. Fix a copy $K$ of such $K_{3,3,3}^{(3)}$ and take any edge $e$ from it. Note that $i_P(S) = i_P(e)$. Let $e = \{w_1, w_2, w_3\}$ such that $v_i$ and $w_i$ are $(\beta, 2)$-reachable, $i = 1, 2, 3$. So we can take 9-sets $T_1, T_2, T_3$ such that for $i = 1, 2, 3$, both $H[T_i \cup \{v_i\}]$ and $H[T_i \cup \{w_i\}]$ contain two 5-vertex paths with the same ends as stated in the definition of the reachability. Thus for each $T_i$ there are $\beta n^9$ choices and overall there are $\beta^3 n^{34}$ choices for $K \cup T_1 \cup T_2 \cup T_3$. Among them, at most $3n^{33}$ of them intersect $S$ and at most $34^2 n^{33}$ of them contain repeated vertices. Thus, there are at least $\beta^4 n^{34}/2$ 34-sets such that $K, T_1, T_2, \text{and } T_3$ are disjoint.

It remains to verify that each $K \cup T_1 \cup T_2 \cup T_3$ gives an $S$-absorber. For each $i = 1, 2, 3$, take the two paths that span $T_i \cup \{v_i\}$ and the path $u_i u_2 u_3 u_4$, where
\[ \{u_1, u_2, u_3, u_4\} = V(K) \setminus \{w_1, w_2, w_3\} \] forms a copy of \( K_{1,2,1}^{(3)} \). We claim that the family of these seven paths is an \( S \)-absorber. Indeed, for \( i = 1, 2, 3 \) take the two paths that span \( T_i \cup \{v_i\} \) and then take the path \( u_1u_2w_1w_2w_3u_4u_4 \) on \( K \). This gives a family of 7 paths which span \( S \cup K \cup T_1 \cup T_2 \cup T_3 \) and have the same ends as the family of paths mentioned above. □

Our main proof proceeds as the following steps.

Build absorbing paths: Let \( S \) be the family of triples \( S \) such that \( H \) has \( \alpha n^3 \) edges \( e \) such that \( \mathbb{I}_S(e) = \mathbb{I}_P(e) \). So the above claim says that for each \( S \in S, H \) contains \( \beta^4 n^{34}/2 \) \( S \)-absorbers. We next choose a set \( \mathcal{F} \) of absorbers uniformly at random from \( H \), that is, we select a random set \( \mathcal{F} \) by including each 34-set in \( V(H) \) independently with probability \( p = \beta^5 n^{-33} \). Because of (1) (for (i) and (ii) below) and Markov's inequality (for (iii)) and the union bound, there exists a family \( \mathcal{F}' \) satisfying the following properties:

(i) for each triple \( S \in S, \mathcal{F}' \) contains at least \( (p/2)\beta^4 n^{34}/2 = \beta^9 n/4 \) \( S \)-absorbers;
(ii) \( \mathcal{F}' \) contains at least \( 2p \binom{n}{34} \leq \beta^5 n/34 \); (iii) there are at most \( 2p^2 \cdot 34 \binom{n}{34} \binom{n}{33} \leq \beta^{10} n \) pairs of overlapping members of \( \mathcal{F}' \).

By deleting one set from each overlapping pair of members of \( \mathcal{F}' \) and the members that are not \( S \)-absorbers for any triple \( S \in S \), we obtain a family \( \mathcal{F} \) of 34-sets such that (i) \( |\mathcal{F}| \leq \beta^5 n/34 \), (ii) each 34-set spans a family of seven vertex-disjoint paths that is an \( S \)-absorber for some \( S \in S \), (iii) for each \( S \in S, \mathcal{F} \) has at least \( \beta^9 n/4 - \beta^{10} n \geq \beta^{10} n \) \( S \)-absorbers.

Next we use Lemma 3.1 with \( \zeta = \beta^5 \) to connect the paths in \( \mathcal{F} \) into two vertex-disjoint paths \( P_1 \) and \( P_2 \) such that \( |V(P_1 \cup P_2)| \leq \sqrt{\beta^{15} n} \leq \beta n \). This is possible as the tight paths cover \( |V(\mathcal{F})| \leq \beta^5 n < \zeta_0 n \) vertices.

Choose a reservoir set \( A \): Now we choose a random vertex set \( A \) by including every vertex in \( V(H) \setminus V(P_1 \cup P_2) \) with probability \( \gamma \). Since \( |V(P_1 \cup P_2)| \leq \beta n \), for any \( u, v \in V(H), |N_H(uv) \setminus V(P_1 \cup P_2)| \geq (1/3 + \alpha - \beta) n \). By (1) and the union bound, there exists a choice of \( A \) such that \( (1 - 2\beta) n \gamma n \leq |A| \leq (1 + \beta) n \gamma n \) and

(a) for any \( u, v \in V(H), |N_H(uv) \cap A| \geq (1 - \beta)(1/3 + \alpha - \beta) n \gamma n \geq (1/3 + \alpha/2) |A| \),
(b) if \( P = \{X, Y\} \), namely, \( P \) has two parts, then \( |A \cap X|/|A| \in (1/3 + \alpha/5, 2/3 - \alpha/5) \).

Cover almost all vertices: Let \( V' = V(H) \setminus (V(P_1 \cup P_2) \cup A) \) and let \( H' = H[V'] \). Since \( |V(P_1 \cup P_2) \cup A| \leq \beta n + 2 \gamma n \), it holds that \( \delta_2(V(H)) \geq (1/3 + \alpha/2) n \). Then Lemma 3.2 gives two paths \( P_3 \) and \( P_4 \) that cover all but a set \( U \) of at most \( \eta n \) vertices of \( H' \). We will connect the four paths to two paths with the help of \( A \). Indeed, for \( i \in [4] \), we contract each \( P_i \) to a 4-vertex path \( \tilde{P}_i \) and consider \( H'' := H[A \cup V(\tilde{P}_1 \cup \ldots \cup \tilde{P}_4)] \). By (a), \( \delta_2(H'') \geq (1/3 + \alpha/3) |V(H'')| \). So we can apply Lemma 3.1 with \( \zeta = 16/|V(H'')| \leq 16/(\eta n/2) < \zeta_0 \) and connect the \( \tilde{P}_i \)'s into two paths, and note that this also connects the \( P_i \)'s into two paths. We take one of the paths and extend it by at most two edges so that the number of unused vertices in \( A \cup U \) is a multiple of 3. Denote the two paths by \( Q_1 \) and \( Q_2 \) and \( A' := V(H) \setminus V(Q_1 \cup Q_2) \). It remains to absorb the vertices in \( A' \). Note that \( |A'| \leq |A| + |U| \leq |A| + \eta n \leq (1 + \gamma)|A| \) and

That is, let \( \tilde{P}_i \) denote the 4-vertex path on the end vertices of \( P_i \) in order and add the possibly missing two edges to \( H \). The added edges will be removed upon the completion of the connection step.
\[ |A'| \geq |A| - \sqrt{16/|V(H'')| |V(H'')|} \geq |A| - \sqrt{16n} \geq |A| - \gamma |A| = (1 - \gamma)|A|. \tag{2} \]

That is, \( |A'| = (1 + \gamma)|A| \), and similar calculations give \( A' \cap X| = (1 + \gamma)|A \cap X| \). Together with (b) and \( \gamma \ll \alpha \) we obtain \( A' \cap X| \in (1/3 + \alpha/6, 2/3 - \alpha/6). \)

Absorb the leftover: We first assume that \( |\mathcal{P}| = 1 \), namely, every two vertices in \( V(H) \) are \((\beta, 2)\)-reachable. Since clearly \( H \) contains \( an^3 \) edges, \( S = \binom{V(H)}{3} \). In this case \( Q_1 \) and \( Q_2 \) contain \( \beta^{10n} S \)-absorbers for every triple \( S \). As \( A' \in 3\mathbb{N}, |A'| \leq |A| + |U| \leq 2\gamma n \), and \( \gamma \ll \beta \), we can partition \( A' \) arbitrarily into at most \( \gamma n \) triples and absorb these triples one by one by their absorbers in \( Q_1 \) and \( Q_2 \). Therefore, we obtain a path cover of \( H \) by two paths.

Next assume that \( \mathcal{P} = \{X, Y\} \). Let \( I \) be the set of indices \( v \in \{(3, 0), (2, 1), (1, 2), (0, 3)\} \) such that \( H \) contains \( an^3 \) edges \( e \) with \( i_\mathcal{P}(e) = v \). Note that we can achieve the same conclusion as in the above proof if \( I = \{(3, 0), (2, 1), (1, 2), (0, 3)\} \). We now count the edges of \( H \) with different index vectors. By \( \delta_2(H) \geq (1/3 + \alpha)n \), there are at least \( \frac{1}{3} \binom{|I'|}{2} (1/3 + \alpha)n > 2\alpha n^3 \) edges that each contain two vertices from \( X \) (recall that \(|X| \geq n/3 \)). This implies that one of \((3, 0)\) and \((2, 1)\) must be in \( I \). Similar countings derive that one of \((1, 2)\) and \((2, 1)\) must be in \( I \), and one of \((1, 2)\) and \((0, 3)\) must be in \( I \).

By symmetry (namely, exchange \( X \) and \( Y \) if necessary), it suffices to consider the following two cases.

Case 1. \((2, 1), (1, 2) \in I\).

Because \( |A' \cap X| \in (1/3 + \alpha/6, 2/3 - \alpha/6) \) and \( |A'| \in 3\mathbb{N} \), the following system:

\[
2x^* + y^* = |A' \cap X| \quad \text{and} \quad x^* + 2y^* = |A' \cap Y|
\]

has a solution \( x^*, y^* \in \mathbb{N} \). So we can partition \( A' \) into \( x^* \) triples with index vector \((2, 1)\) and \( y^* \) triples with index vector \((1, 2)\). These triples can be greedily absorbed by \( Q_1 \) and \( Q_2 \) and we are done.

Case 2. \((2, 1), (0, 3) \in I\).

We need some extra work for this case. First pick two disjoint edges \( e_1 \) and \( e_2 \) in \( A' \) such that \( e_1 \) contains at least one vertex in \( X \) and \( e_2 \) contains at least two vertices in \( Y \). The desired edges exist because \( \delta_2(H_{|A'|}) \geq (1/3 + \alpha/3)|A| \) by (2). Denote the specified vertex in \( e_1 \cap X \) by \( x \) and the two specified vertices in \( e_2 \cap Y \) by \( y_1 \) and \( y_2 \). Now connect the four paths \( e_1, e_2, Q_1, Q_2 \) into two paths \( Q'_1, Q'_2 \), so that the end with specified vertices \( x \), or \( y_1, y_2 \) are kept as the (two out of the four) ends of \( Q'_1, Q'_2 \). This can be done by contracting \( Q_1 \) and \( Q_2 \) to 4-vertex paths \( \bar{Q}_1, \bar{Q}_2 \) and applying Lemma 3.1 on \( H'' := H[A' \cup V(\bar{Q}_1 \cup \bar{Q}_2)] \), because \( \delta_2(H_{|A'|}) \geq (1/3 + \alpha/3)|A| \geq (1/3 + \alpha/4)\mathbb{A}(H''). \) Denote the set of uncovered vertices in \( A' \) by \( A'' \).

If \( |A'' \cap X| \) is odd, we remove \( x, y_1, \) and \( y_2 \) from \( Q'_1 \) and \( Q'_2 \) (this is possible as they are at the ends) and add them to \( A'' \) (and we do nothing if \( |A'' \cap X| \) is even). Thus \( A'' \cap X \) is even and clearly we still have \( |A''| \in 3\mathbb{N} \) and \( |A'' \cap X|/|A''| \in (1/3, 2/3) \). Now consider the following system:

\[
2x^* = |A'' \cap X| \quad \text{and} \quad x^* + 3y^* = |A'' \cap Y|
\]
which has a solution $x^*, y^* \in \mathbb{N}$ because $|A'' \cap X|$ is even and $|A'' \cap X|/|A''| \in (1/3, 2/3)$. So we can partition $A'$ into $x^*$ triples with index vector $(2, 1)$ and $y^*$ triples with index vector $(0, 3)$. These triples can be greedily absorbed and we obtain a path cover of $H$ by two paths.

## 4 | HYPERGRAPH REGULARITY LEMMA AND REGULAR SLICES

In this section we introduce the regularity lemma and related tools we need. The main tools needed in later proofs are the regular slice lemma (Theorem 4.4) and an extension lemma (Lemma 4.8).

### 4.1 | Regular complexes

Let $\mathcal{P}$ be a partition of $V$ into vertex classes $V_1, ..., V_s$. A subset $S \subseteq V$ is $\mathcal{P}$-partite if $S \cap V_i \leq 1$ for all $1 \leq i \leq s$. A hypergraph is $\mathcal{P}$-partite if all of its edges are $\mathcal{P}$-partite, and it is $s$-partite if it is $\mathcal{P}$-partite for some partition $\mathcal{P}$ with $|\mathcal{P}| = s$.

A hypergraph $H$ is a complex if whenever $e \in E(H)$ and $e'$ is a nonempty subset of $e$ we have that $e' \in E(H)$. All the complexes considered in this paper have the property that all vertices are contained in an edge. A complex $H$ is a 3-complex if all the edges of $H$ consist of at most 3 vertices. The edges of size $i$ are called $i$-edges of $H$. Given a 3-complex $H$, for all $i = 1, 2, 3$ we denote by $H_i$ the underlying $i$-graph of $H$: the vertices of $H_i$ are those of $H$ and the edges of $H_i$ are the $i$-edges of $H$. Given $s \geq 3$, a $(3, s)$-complex $H$ is an $s$-partite 3-complex. Given $i \leq j$, an $(i, j)$-graph is a $j$-partite $i$-graph.

Let $H$ be a $\mathcal{P}$-partite 3-complex. For $i \leq 3$ and $X \in \left(\binom{\mathcal{P}}{i}\right)$, we write $H_X$ for the subgraph of $H_i$ induced by $\bigcup X$. Note that $H_X$ is an $(i, i)$-graph. In a similar manner we write $H_{X_i}^<$ for the hypergraph on the vertex set $\bigcup X$, whose edge set is $\bigcup_{X_i \subseteq X} H_{X_i}^<$. Note that if $H$ is a 3-complex and $X$ is a 3-set, then $H_{X_i}^<$ is a $(2, 3)$-complex.

Given $i \geq 2$, consider an $(i, i)$-graph $H_i$ and an $(i - 1, i)$-graph $H_{i-1}$ on the same vertex set, which are $i$-partite with respect to the same partition $\mathcal{P}$. We write $\mathcal{K}_i(H_{i-1})$ for the family of all $\mathcal{P}$-partite $i$-sets that form a copy of the complete $(i - 1)$-graph $K_i^{i-1}$ in $H_{i-1}$. We define the density of $H_i$ with respect to $H_{i-1}$ to be

$$d(H_i|H_{i-1}) = \frac{|\mathcal{K}_i(H_{i-1}) \cap E(H_i)|}{|\mathcal{K}_i(H_{i-1})|}$$

if $|\mathcal{K}_i(H_{i-1})| > 0$,

and $d(H_i|H_{i-1}) = 0$ otherwise. More generally, if $Q = (Q_1, ..., Q_r)$ is a collection of $r$ subhypergraphs of $H_{i-1}$, we define $\mathcal{K}_i(Q) := \bigcup_{j=1}^r \mathcal{K}_i(Q_j)$ and

$$d(H_i|Q) = \frac{|\mathcal{K}_i(Q) \cap E(H_i)|}{|\mathcal{K}_i(Q)|}$$

if $|\mathcal{K}_i(Q)| > 0$,

and $d(H_i|Q) = 0$ otherwise.

We say that $H_i$ is $(d_i, \varepsilon, r)$-regular with respect to $H_{i-1}$ if for all $r$-tuples $Q$ with $|\mathcal{K}_i(Q)| > \varepsilon|\mathcal{K}_i(H_{i-1})|$ we have $d(H_i|Q) = d_i \pm \varepsilon$. Instead of $(d_i, \varepsilon, 1)$-regularity we simply refer to $(d_i, \varepsilon)$-regularity; we also say simply that $H_i$ is $(\varepsilon, r)$-regular with respect to $H_{i-1}$ if there is
some $d_i > 0$ for which $H_i$ is $(d_i, \varepsilon, r)$-regular with respect to $H_{i-1}$. Given an $i$-graph $G$ such that $V(G) \supseteq V(H_{i-1})$, we say that $G$ is $(d_i, \varepsilon, r)$-regular with respect to $H_{i-1}$ if the $i$-partite subgraph of $G$ induced by the vertex classes of $H_{i-1}$ is $(d_i, \varepsilon, r)$-regular with respect to $H_{i-1}$ (recall that $H_{i-1}$ is $i$-partite).

Finally, given $s \geq 2$ and a $(2, s)$-complex $H$ with a vertex partition $\mathcal{P}$, we say that $H$ is $(d_2, \varepsilon, r)$-regular if for every $A \in \binom{\mathcal{P}}{2}$, $H_A$ is $(d_2, \varepsilon)$-regular with respect to $(H_{A'})_1$. Given $s \geq 3$ and a $(3, s)$-complex $H$ with a vertex partition $\mathcal{P}$, we say that $H$ is $(d, d_2, \varepsilon, \varepsilon, r)$-regular if:

(i) for every $A \in \binom{\mathcal{P}}{2}$, $H_A$ is $(d_2, \varepsilon)$-regular with respect to $(H_{A'})_1$ or $d(H_A)(H_{A'}))_1 = 0$, and

(ii) for every $A \in \binom{\mathcal{P}}{3}$, $H_A$ is $(d, \varepsilon, r)$-regular with respect to $(H_{A'})_2$ or $d(H_A)(H_{A'})_2 = 0$.

Note that by the Dense Counting Lemma (see, e.g., [19, Theorem 6.5]), a $(d, d_2, \varepsilon, \varepsilon, r)$-regular $(3, 3)$-complex with $n$ vertices in each part and at least one 3-edge has at least $(dd_2^2/2)n^3$ 3-edges.

We need the following lemma which states that the restriction of regular complexes to a sufficiently large set of vertices is still regular.

**Lemma 4.1** (Restriction Lemma, Kühn et al. [20, Lemma 4.1]). Let $s, r, m$ be positive integers and $\alpha, d_2, d, \varepsilon, \varepsilon_3 > 0$ such that

\[
1/m \ll 1/r, \quad \varepsilon \leq \min\{\varepsilon, d_2\} \leq \varepsilon_3 \ll \alpha \ll d, \quad 1/s.
\]

Let $H$ be a $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular $(3, s)$-complex with vertex classes $V_1, \ldots, V_s$ of size $m$. For each $i$ let $V'_i \subseteq V_i$ be a set of size at least $\alpha m$. Then the restriction $H' = H[V'_1 \cup \cdots \cup V'_s]$ of $H$ to $V'_1 \cup \cdots \cup V'_s$ is $(d, d_2, \sqrt[3]{\varepsilon_3}, \sqrt[3]{\varepsilon}, r)$-regular.

### 4.2 Statement of the regular slice lemma

In this section we state the version of the regularity lemma (Theorem 4.4) due to Allen, Böttcher, Cooley, and Mycroft [1], which they call the *regular slice lemma*. For most of the notations in this subsection we follow those from [1] (with some simplification because we only focus on 3-complexes). A similar lemma was previously applied by Haxell, Łuczak, Peng, Rödl, Ruciński, and Skokan [16]. This lemma says that all 3-graphs $G$ admit a regular slice $\mathcal{J}$, which is a regular multipartite 2-complex whose vertex classes have an equal size such that $G$ is regular with respect to $\mathcal{J}$.

Let $t_0, t_1 \in \mathbb{N}$ and $\varepsilon > 0$. Following [1], we say that a 2-complex $\mathcal{J}$ is $(t_0, t_1, \varepsilon)$-equitable if it has the following two properties:

(i) There exists a partition $\mathcal{P}$ of $V(\mathcal{J})$ into $t$ parts of equal size, for some $t_0 \leq t \leq t_1$, such that $\mathcal{J}$ is $\mathcal{P}$-partite. We refer to $\mathcal{P}$ as the ground partition of $\mathcal{J}$, and to the parts of $\mathcal{P}$ as the clusters of $\mathcal{J}$.

(ii) There exists $d_2 \geq 1/t_1$ with $1/d_2 \in \mathbb{N}$, and the 2-complex $\mathcal{J}$ is $(d_2, \varepsilon, 1)$-regular.
Let $X \in \binom{P}{3}$. We write $\hat{J}_X$ for the $(2, 3)$-graph $(J_X)^*$. A 3-graph $G$ on $V(J)$ is $(\varepsilon_3, r)$-regular with respect to $\hat{J}_X$ if there exists some $d$ such that $G$ is $(d, \varepsilon_3, r)$-regular with respect to $\hat{J}_X$. We also write $d^*_G(X)$ for the density of $G$ with respect to $\hat{J}_X$, or simply $d^*(X)$ if $J$ and $G$ are clear from the context. Now we are ready to state the definition of a regular slice from [1] (for 3-complexes).

**Definition 4.2** (Regular slice, Allen et al. [1]). Given $\varepsilon, \varepsilon_3 > 0, r, t_0, t_1 \in \mathbb{N}$, a 3-graph $G$ and a 2-complex $J$ on $V(G)$, we call $J$ a $(t_0, t_1, \varepsilon, \varepsilon_3, r)$-regular slice for $G$ if $J$ is $(t_0, t_1, \varepsilon)$-equitable and $G$ is $(\varepsilon_3, r)$-regular with respect to all but at most $\varepsilon_3 \binom{t}{3}$ of the triples of clusters of $J$, where $t$ is the number of clusters of $J$.

Given a regular slice $J$ for a 3-graph $G$, we keep track of the relative densities $d^*(X)$ for triples $X$ of clusters of $J$, which is done via a weighted 3-graph.

**Definition 4.3** (Weighted reduced 3-graph, Allen et al. [1]). Given a 3-graph $G$ and a $(t_0, t_1, \varepsilon)$-equitable 2-complex $J$ on $V(G)$, we let $R_J(G)$ be the complete weighted 3-graph whose vertices are the clusters of $J$, and where each edge $X$ is given weight $d^*(X)$. When $J$ is clear from the context we write $R(G)$ instead of $R_J(G)$.

The regular slice lemma (Theorem 4.4) guarantees the existence of a regular slice $J$ with respect to which $R(G)$ resembles $G$ in various senses. In particular, $R(G)$ inherits the codegree condition of $G$ in the following sense. Let $G$ be a 3-graph on $n$ vertices. Given a set $S \subset \binom{V(G)}{2}$, the relative degree $\overline{\deg}(S; G)$ of $S$ with respect to $G$ is defined to be

$$\overline{\deg}(S; G) = \frac{\deg_G(S)}{n-2},$$

that is, $\overline{\deg}(S; G)$ is the proportion of triples of vertices in $G$ extending $S$ which are in fact edges of $G$. To extend this definition to weighted 3-graphs $G$ with weight function $d^*$, we define

$$\overline{\deg}(S; G) = \frac{\sum_{e \in E(G): S \subseteq e} d^*(e)}{n-2}.$$

Finally, for a collection $S$ of pairs in $V(G)$, the mean relative degree $\overline{\deg}(S; G)$ of $S$ in $G$ is defined to be the mean of $\overline{\deg}(S; G)$ over all sets $S \subset S$.

We also need the “rooted counting” property in $G$ inherited by the regular slice $J$. For that we need the following definitions from [1]. Given a 3-graph $G$ and distinct “root” vertices $v_1, ..., v_\ell$ of $G$, and a 3-graph $H$ with a specified set of distinct “root” vertices $x_1, ..., x_\ell$, let $n_H(G; v_1, ..., v_\ell)$ be the number of injective maps from $V(H)$ to $V(G)$ which embed $H$ in $G$ and map $x_j$ to $v_j$ for $j \in [\ell]$. Then define

$$d_H(G; v_1, ..., v_\ell, w_\ell) := \frac{n_H(G; v_1, ..., v_\ell, w_\ell)}{\binom{v(G)-\ell}{v(H)-\ell} \cdot (v(H) - \ell)!}.$$

Next we define $H^{\text{skel}}$ to be the 2-complex on $V(H) - \ell$ vertices which are obtained from the complex generated by the down-closure of $H$ by deleting the vertices $x_1, ..., x_\ell$ and deleting all edges of size 3. Given a $(t_0, t_1, \varepsilon)$-equitable 2-complex $J$ on $V(G)$, define $n_H(G; v_1, ..., v_\ell, J)$ to be...
the number of labeled rooted copies of $H$ in $G$ such that each vertex of $H^{skel}$ lies in a distinct cluster of $\mathcal{J}$ and the image of $H^{skel}$ is in $\mathcal{J}$. We also define $n'_{H^{skel}}(\mathcal{J})$ to be the number of labeled copies of $H^{skel}$ in $\mathcal{J}$ with each vertex of $H^{skel}$ embedded in a distinct cluster of $\mathcal{J}$. Then define

$$d_H(G; v_1, ..., v_r, \mathcal{J}) := \frac{n_H(G; v_1, ..., v_r, \mathcal{J})}{n'_{H^{skel}}(\mathcal{J})}.$$  

We can now state the version of the regular slice lemma that we will use.

**Theorem 4.4** (Regular slice lemma, Allen et al. [1, Lemma 6]). For all $t_0 \in \mathbb{N}$, $\varepsilon_3 > 0$ and all functions $r : \mathbb{N} \to \mathbb{N}$ and $\varepsilon : \mathbb{N} \to (0, 1]$, there exist $t_1, n_1 \in \mathbb{N}$ such that the following holds for all $n \geq n_1$ which are divisible by $t_1!$. Let $G$ be a 3-graph on $n$ vertices. Then there exists a $(t_0, t_1, \varepsilon(t_1), \varepsilon_3, r(t_1))$-regular slice $\mathcal{J}$ for $G$ such that,

1. (Codegree) for all pairs $Y$ of clusters of $\mathcal{J}$, we have $\bar{\deg}(Y; R(G)) = \bar{\deg}(\mathcal{J}_Y; G) \pm \varepsilon_3$.
2. (Rooted counting) for each $1 \leq \ell \leq 1/\varepsilon_3$, each 3-graph $H$ is equipped with a set of distinct root vertices $x_1, ..., x_\ell$ such that $v(H) \leq 1/\varepsilon_3$, and any distinct vertices $v_1, ..., v_r \in V(G)$, we have

$$|d_H(G; v_1, ..., v_r, \mathcal{J}) - d_H(G; v_1, ..., v_r)| < \varepsilon_3.$$

### 4.3 The $d$-reduced 3-graph and the extension lemma

Once we have a regular slice $\mathcal{J}$ for a 3-graph $G$, we would like to work within triples of clusters with respect to which $G$ is both regular and dense. To keep track of those tuples, we introduce the following definition.

**Definition 4.5** (The $d$-reduced 3-graph, Allen et al. [1]). Let $G$ be a 3-graph and $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_3, r)$-regular slice for $G$. Then for $d > 0$ we define the $d$-reduced 3-graph $R_d(G)$ of $G$ to be the 3-graph whose vertices are the clusters of $\mathcal{J}$ and whose edges are all triples of clusters $X$ of $\mathcal{J}$ such that $G$ is $(\varepsilon_3, r)$-regular with respect to $X$ and $d^*(X) \geq d$. Note that $R_d(G)$ depends on the choice of $\mathcal{J}$ but this will always be clear from the context.

For $0 \leq \mu, \theta \leq 1$, we say that a $k$-graph $H$ on $n$ vertices is $(\mu, \theta)$-dense if there exists $S \subseteq \binom{V(H)}{k}$ of size at most $\theta \binom{n}{k}$ such that, for all $S \in \binom{V(H)}{k}$ \ $S$, we have $\deg_H(S) \geq \mu(n - k + 1)$. A $k$-graph $H$ on $n$ vertices is strongly $(\mu, \theta)$-dense if it is $(\mu, \theta)$-dense and, for all edges $e \in E(H)$ and all $(k - 1)$-sets $X \subseteq e$, $\deg_H(X) \geq \mu(n - k + 1)$. The next lemma was proved in [11], which states that for regular slices $\mathcal{J}$ as in Theorem 4.4, the codegree conditions are also preserved by $R_d(G)$. Note that its original version allows $G$ to be $(\mu, \theta)$-dense as well.

**Lemma 4.6** (Han et al. [11]). Let $1/n \ll 1/t_1 \leq 1/t_0 \ll 1$ and $\mu, d, \varepsilon, \varepsilon_3 > 0$. Suppose that $G$ is a 3-graph on $n$ vertices such that $\delta_2(G) \geq \mu n$. Let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_3, r)$-regular slice for $G$ such that for all pairs $Y$ of clusters of $\mathcal{J}$, we have $\bar{\deg}(Y; R(G)) = \bar{\deg}(\mathcal{J}_Y; G) \pm \varepsilon_3$. Then $R_d(G)$ is $(\mu - d - \varepsilon_3 - \sqrt{\varepsilon_3}, 3\sqrt{\varepsilon_3})$-dense.
We use the following result proved in [11, Lemma 8.8].

**Lemma 4.7.** Let \( n \geq 6 \) and \( 0 < \mu, \theta < 1 \). Any \((\mu, \theta)\)-dense 3-graph \( H \) contains a spanning subgraph \( H' \) that is strongly \((\mu - 8\theta^{1/4}, \theta + \theta^{1/4})\)-dense.

Suppose that \( G \) is a \((3, \ell')\)-complex with vertex classes \( V_1, V_2, V_3 \), and \( H \) is a \((3, \ell)\)-complex with vertex classes \( X_1, X_2, X_3 \). We say that \( G \) respects the partition of \( H \) if whenever \( H \) contains an \( i \)-edge with vertices in \( X_{j_1}, ..., X_{j_i} \), then there is an \( i \)-edge of \( G \) with vertices in \( V_{j_1}, ..., V_{j_i} \). On the other hand, a labeled copy of \( H \) in \( G \) is partition-respecting if for each \( i \in [\ell] \) the vertices corresponding to those in \( X_i \) lie within \( V_i \). We write \( |H|_G \) for the number of (labeled) partition-respecting copies of \( H \) in \( G \).

Roughly speaking, the Extension Lemma says that if \( G' \) is an induced subcomplex of \( G \), and \( H \) is suitably regular, then almost all copies of \( H \) in \( G' \) are respect respect copies of \( H \) in \( G \). We use the following version from \([1]\) which allows each triple of clusters to have different densities.

**Lemma 4.8** (Extension Lemma, Allen et al. [1, Lemma 25]). Let \( \ell, r, t, t', n_0 \) be positive integers, where \( t < t' \), and let \( \beta, d_2, d, \varepsilon, \varepsilon_3 \) be positive constants such that \( 1/d_2, 1/d \in \mathbb{N} \) and

\[
1/n_0 \ll 1/r, \varepsilon \ll c \ll \min(\varepsilon_3, d_2) \leq \varepsilon_3 \ll \beta, d, 1/\ell, 1/t'.
\]

Then the following holds for all integers \( n \geq n_0 \). Suppose that \( H' \) is a \((3, \ell)\)-complex on \( t' \) vertices with vertex classes \( Y_1, ..., Y_\ell \) and let \( H \) be an induced subcomplex of \( H' \) on \( t \) vertices. Suppose also that \( G \) is a \((3, \ell')\)-complex with vertex classes \( V_1, ..., V_\ell \), all of size \( n \), which respects the partition of \( H' \), such that the 2-complex formed by 2-edges and 1-edges in \( G \) is \((t_0, t_1, \varepsilon)\)-equitable with density \( d_2 \). Suppose further that for each 3-edge \( e \) of \( H' \) with index \( A \in \left( \frac{|\ell|}{3} \right) \), the \((3, 3)\)-graph \( G_A \) is \((d', \varepsilon_3, r)\)-regular with respect to \((G_{A < e})_2 \) for some \( d' \geq d \). Then all but at most \( \beta |H|_G \) labeled partition-respecting copies of \( H \) in \( G \) can extend to \( cn^{t-t'} \) labeled partition-respecting copies of \( H' \) in \( G \).

## 5 CONNECTION

In this section we prove Lemma 3.1. We first use the extension lemma (Lemma 4.8) to prove the following result.

**Lemma 5.1** (Connector). Let \( \ell, r, n_0 \) be positive integers, and let \( \beta, d_2, d, \varepsilon, \varepsilon_3 \) be positive constants such that \( 1/d_2, 1/d \in \mathbb{N}, \beta \leq d_2/18 \) and

\[
1/n_0 \ll 1/r, \varepsilon \ll c \ll \min(\varepsilon_3, d_2) \leq \varepsilon_3 \ll \beta, d.
\]

Then the following holds for all integers \( n \geq n_0 \). Suppose that \( G \) is a \((3, \ell)\)-complex with a vertex partition \( P = \{V_1, ..., V_\ell\} \), each of size \( n \), such that the 2-complex formed by 2-edges and 1-edges in \( G \) is \((t_0, t_1, \varepsilon)\)-equitable with density \( d_2 \). Let \( R \) be a 3-graph on \( P \) such that for each triple \( T \in E(R) \), \( G_T \) is \((d', \varepsilon_3, r)\)-regular with respect to \((G_{T < e})_2 \) for some \( d' \geq d \). Let
As mentioned in Section 1, the assumption that $X_0$ and $X_1$ can be connected by a pseudopath guarantees that we can connect most of the 2-edges $v_1v_2$ and $v_3v_4$ but only under certain orderings. That is where we need the existence of a copy of $K_4^-$ to overcome the issue.

Proof: Without loss of generality, suppose $S_1 = \{V_a, V_{a'}\}$ and $S_0 = \{V_b, V_{b'}\}$, where $a, a', b, b' \in [\ell]$ are such that $a \neq a'$ and $b \neq b'$. There are four cases for the pair of labeled edges, namely, for example, for $(v_1, v_2) \in V_a \times V_{a'}$ or $V_a \times V_a$ and $(v_3, v_4) \in V_b \times V_{b'}$ or $V_{b'} \times V_{b'}$, we will only show that for all but at most $\beta n^4/4$ pairs of labeled 2-edges $(v_1, v_2) \in V_a \times V_{a'}$ and $(v_3, v_4) \in V_{b'} \times V_{b'}$, there exists a tight path $P$ of length at most $15 + \ell^3$ with $(v_2, v_1)$ and $(v_3, v_4)$ as ends, because the same proof also treats the other three cases.

Let $X_1X_2\cdots X_mX_0$ be the pseudopath $P$ in $R$ of minimum length that connects $X_1$ and $X_0$, where $X_i \in E(R)$. Note that $|V(P)| \leq 3 + m$. First we define a sequence $S = Y_1Y_2\cdots Y_p$ of not necessarily distinct clusters of $P$ (So each $Y_i = V_j$ for some $j \in [\ell]$) such that $(V_{a_i}, V_{a_i}) = (Y_1, Y_2)$, $S_0 = \{Y_p, Y_p\}$, $X_1 = \{Y_1, Y_2, Y_3\}$, $X_0 = \{Y_p\}$ and every consecutive three clusters in the sequence form one of the edges $X_0, X_1, \ldots, X_m$ as follows. We start the sequence with $Y_1Y_2Y_3$. After we have arranged the clusters of $X_i$ in the sequence, say $Y_0Y_1Y_2Y_3\cdots Y_q$, if $X_i \cap X_{i+1} \neq \{Y_{q+1}, Y_{q+2}\}$, then we “wind around” $X_i$ which is, let $Y_{q+3} = Y_{q+1}$ and so on, until the last two vertices of the sequence are exactly the vertices in $X_i \cap X_{i+1}$, and then put down the last vertex of $X_{i+1}$. After having arranged $X_0$ in the sequence, wind around at most two more times if necessary so that the last two clusters in $S$ are elements of $S_0$. Note that each time before we insert an edge $X_i$, we may need to add at most two clusters (to wind around), which implies that $|S| \leq 3|V(P)| \leq 3(3 + m) \leq 9 + \ell^3$.

So the problem is that $(V_b, V_{b'})$ may equal $(Y_p, Y_{p-1})$ or $(Y_{p-1}, Y_p)$. In this case we use the copy of $K_4^-$ to make the “turn.” Indeed, assume that the clusters for the $K_4^-$ are $Y_{p-2}, Y_{p-1}, Y_p$ and $Y_0$. It is straightforward to check that we can extend $S$ from the end $Y_{p-1}Y_p$ as

- $Y_{p-1}Y_pY_0Y_{p-1}Y_{p-2}Y_pY_{p-1}$ if the missing edge of $K_4^-$ is $Y_{p-2}Y_pY_0$;
- $Y_{p-1}Y_pY_{p-1}Y_0Y_pY_{p-1}$ if the missing edge of $K_4^-$ is $Y_{p-2}Y_{p-1}Y_0$;
- $Y_{p-1}Y_pY_{p-2}Y_0Y_{p-1}Y_{p-2}Y_pY_{p-1}$ if the missing edge of $K_4^-$ is $Y_{p-1}Y_pY_0$.

Denote the resulting sequence by $S' = (Y_1, Y_2, \ldots, Y_q)$, and thus $|S'| = q \leq 15 + \ell^3$.

Let $H'$ be the $(3, \ell')$-complex on distinct vertices $(x_1, x_2, \ldots, x_q)$ where $\ell' = |V(P)|$ and $q = |S'|$, such that

- $E(H')$ is generated by the down-closure from a tight path on $(x_1, x_2, \ldots, x_q)$,
- for $i, j \in [q]$ vertices $x_i, x_j$ are in the same cluster if and only if $Y_i = Y_j$.

Since $P$ is a pseudopath in $R$, by the definition of $R$, $G(\cup V(P))$ is a $(3, \ell')$-complex which respects the partition of $H'$ and satisfies the regularity assumptions in the lemma. So if we let $H$ be the subcomplex of $H'$ induced on $(x_1, x_2, x_{q-1}, x_q)$, it looks like that we
may apply Lemma 4.8 to embed $H'$. However, this does not work because $\ell'$ might be too large, namely, we may not have, say, $\varepsilon_2 \ll 1/\ell'$.

What we actually do is to chop $H'$ into segments and apply Lemma 4.8 on them separately. More precisely, let $\ell_0 = \lfloor q/9 \rfloor$. We define $H^{(1)}, ..., H^{(\ell_0)}$ such that for $i \in \{\ell_0 - 1\}$, $H^{(i)}$ is the subcomplex of $H'$ induced on the vertices $(x_{9(i-1)+1}, ..., x_{9i+2})$, and $H^{(\ell_0)}$ is the subcomplex of $H'$ induced on $(x_{9(\ell_0-1)+1}, ..., x_q)$. So each of these complexes has 11 vertices except $H^{(\ell_0)}$ which has at most $17 + 2 = 19$ vertices. By Lemma 4.8, all but at most $\beta n^4$ choices of pairs of labeled 2-edges $e_i \in Y_{9(i-1)+1} \times Y_{9(i-1)+2}, e_{i+1} \in Y_{9i-1} \times Y_{9i}$, where $s_i = 9i + 2$ for $i \in [\ell_0 - 1]$ and $s_{\ell_0} = q$ can be connected by at least $cn^6$ tight paths in $G$, for each $i \in [\ell_0 - 1]$, where $t_i = \cdots = t_{\ell_0-1} = 7$ and $t_{\ell_0} = q - 9(\ell_0 - 1) - 4$.

We claim that for pairs of labeled 2-edges $e \in Y_1 \times Y_2$ and $e' \in Y_{p-1} \times Y_p$, if $e$ can be connected to all but at most $8\beta n^2$ edges in $Y_{10} \times Y_{11}$ by $cn^6$ paths and $e'$ can be connected to all but at most $8\beta n^2$ edges in $Y_{9(\ell_0-1)+1} \times Y_{9(\ell_0-1)+2}$ by $cn^6$ paths, then $e$ and $e'$ can be connected by a desired path as stated in the lemma. This clearly finishes the proof as the number of pairs of edges violating the properties is at most $(\beta n^2/8) \cdot n^2 + (\beta n^2/8) \cdot n^2 = \beta n^4/4$. Now we prove the claim. Indeed, for each $j \in \{2, ..., \ell_0 - 1\}$, we will only consider the 2-edges $e_j \in Y_{9(j-1)+1} \times Y_{9(j-1)+2}$ that can be connected to all but at most $8\beta n^2$ edges in $Y_{9j+1} \times Y_{9j+2}$. Thus, we can pick labeled 2-edges $e_2, e_3, ..., e_{\ell_0-1}, e_{\ell_0}$ greedily so that each consecutive pair of 2-edges can be connected by at least $cn^6$ paths: indeed, when choosing $e_2, e_3, ..., e_{\ell_0-1}$ we have at least $(d_2 - \varepsilon)n^2 - 8\beta n^2 - \beta n^2/8 > \beta n^2$ choices, and we have at least $(d_2 - \varepsilon)n^2 - 2 \cdot 8\beta n^2 > \beta n^2$ choices for $e_{\ell_0}$. Together with the choices for the internal vertices that connect these $e_i$’s, we have at least $\beta n^6-1 \cdot \ell \cdot n^{p-4}$ such candidates, of which at most $p^2 n^{p-5} + 4n^{p-5} < \beta n^6-1 \cdot \ell \cdot n^{p-4}$ can include repeated vertices or intersect $e$ or $e'$. So we conclude the existence of the desired path.

The following lemma strengthens a result of Mycroft slightly, and actually follows from the same proof. We include its (short) proof for completeness.

**Lemma 5.2.** Let $\theta \in (0, 1)$. Let $H$ be an $n$-vertex 3-graph which is strongly $(1/3, \theta)$-dense. Then $H$ has at most two tight component.

**Proof.** We regard the tight components of $H$ as an edge coloring of $H$, namely, all edges in a tight component share the same color. Consider an edge-coloring of $K_n$, where an edge $uv$ gets the color from any 3-edge in $H$ that contains $uv$. This coloring is well-defined as all 3-edges containing $uv$ are in the same tight component and thus have the same color (note that an edge $uv$ may receive no color, if $\deg_H(uv) = 0$).

Given a vertex $v$ and a color $c$, let $N_c(v)$ be the set of vertices that are connected to $v$ by an edge of color $c$. Note that if $uv$ is colored with color $c$ then it has degree $n/3$ in $H$, which implies that it is adjacent to another $n/3$ edges from each of $u$ and $v$ all of color $c$. This implies that if $uv$ and $vw$ have different colors $c_1, c_2$, then we have $|N_{c_1}(v)| = |N_{c_2}(v)| \geq n/3$, and $N_{c_1}(v) \cap N_{c_2}(v) = \emptyset$. Therefore, there is no vertex $v$ that sees three colors, and a similar argument shows that there are no three-colored triangles.

---

$^5$Recall that the $(d_2, \varepsilon)$-regularity implies the existence of $(d_2 - \varepsilon)n^2$ edges. Among them, at most $8\beta n^2$ are not well connected to the previous 2-edge we chose, and at most $\beta n^2/8$ of them are not well connected to the next 2-edge to be chosen.
We may assume that \( H \) has three tight components, with colors \( r, b, \) and \( g \). Since each color class of \( E(K_n) \) contains a star of size \( n/3 \), there is a vertex, say \( v \), that sees two colors, say, \( r \) and \( b \). Note that \( |N_r(v)|, |N_b(v)| \geq n/3 \). Since edges of \( K_n \) of color \( g \) contain a star of size \( n/3 \) and \( N_r(v) \cap N_b(v) = \emptyset \), there is an edge \( uv \) of color \( g \) such that \( u \in N_r(v) \cup N_b(v) \). Note that as there is no three-colored triangle, there is no edge of color \( g \) between \( N_r(v) \) and \( N_b(v) \). So without loss of generality, assume that \( u \in N_r(v) \) and \( w \in R := V(H) \setminus (N_b(v) \cup \{v\}) \). Since \( N_b(v) \geq n/3 \), we have \( R \leq |V(H)| - |N_b(v)| - 1 < 2n/3 \). Note that there exist \( n/3 \) vertices \( x \) which form a 3-edge of color \( r \) with \( uv \). Moreover, as for such \( x, xv \) also has color \( r \), we infer \( x \in N_r(v) \subseteq R \). These together imply that \( |N_r(u) \cap R| \geq n/3 \). Moreover, because \( N_g(u) \subseteq R, |N_g(u)| \geq n/3 \) and \( N_r(u) \cap N_g(u) = \emptyset \), we derive that \( |R| \geq 2n/3 \), a contradiction. 

Now we can prove our connecting lemma. The idea is to host the ends of all paths in the regular partition and then connect the pairs that lie in the same tight component and have the “correct” (labeled) ends required by Lemma 5.1. We remark that the technical restriction on the end edges is needed for the absorption in the proof of Theorem 1.1.

**Proof of Lemma 3.1.** Choose constants

\[
1/n_0 \ll \zeta_0 \ll 1/t_1 \ll 1/r, \varepsilon \ll c \ll \min \{\varepsilon_3, d_2\} \ll \varepsilon_3 \ll d \ll \theta \ll \alpha
\]

and suppose \( n \geq n_0 + t_1 \) and \( \zeta \leq \zeta_0 \). Let \( \beta = d_2^2/400 \). Let \( H' \) be an induced subgraph of \( H \) on \( n' \) vertices such that \( n' \geq n - t_1, t_1 \ll n' \) and \( V(P_1 \cup \cdots \cup P_0) \subseteq V(H') \). Then we will focus on \( H' \) and note that \( \delta_2(H') \geq (1/3 + \alpha)n - t_1 \geq (1/3 + \alpha - \theta)n \) as \( 1/n \leq 1/n_0 \ll 1/t_1 \ll \theta \). Then Theorem 4.4 applied with \( \varepsilon(t_1) = \varepsilon \) and \( r(t_1) = \varepsilon \) gives a \((t_0, t_1, \varepsilon, \varepsilon, r)\)-regular slice \( J \) for \( H' \). Let \( R_d(H') \) be the \( d \)-reduced graph which is \((1/3 + \alpha - 2\varepsilon, \varepsilon)\)-dense by Lemma 4.6. Then let \( R \) be the strongly \((1/3 + \alpha/2, 2\varepsilon^{1/4})\)-dense spanning subgraph of \( R_d(H') \) given by Lemma 4.7. By Lemma 5.2, \( R \) has at most two tight components. Let \( \mathcal{P} \) be the ground partition of \( J \) with \( |\mathcal{P}| = t \) and let \( n_* := n'/t \).

Let \( F = x_1x_2x_3x_4 \) be the 4-vertex tight path with \( x_1, x_2 \) as root vertices. Since \( \delta_2(J_X) \geq (1/3 + \alpha - \theta)n \), for any \( \nu, \nu' \in V(H') \), by Theorem 4.4(2), we have that

\[
d_{F}(H; \nu, \nu', J) \geq d_{F}(H; \nu, \nu') - \varepsilon_3 \geq \frac{(1/3 + \alpha/2)^2 n_3^2}{(n - 2)(n - 3)} - \varepsilon_3 \geq \frac{1}{9}.
\]

By the regularity, for any \( X \in \binom{\mathcal{P}}{2} \), it holds that \( K_2(J_X) = (1 \pm \varepsilon)d_2 n_*^2 \). Note that \( F^{\text{skel}} \) is a 2-edge (together with two singletons), and thus

\[
n_{\text{skel}}(J) = 2 \sum_{\mathcal{P}} K_2(J_X) = t(t - 1) \cdot (1 \pm \varepsilon)d_2 n_*^2,
\]

where the factor of 2 is because \( n_{\text{skel}}(J) \) counts labeled copies. These imply that

\[
n_{F}(H; \nu, \nu', \nu'') = d_{F}(H; \nu, \nu', J) \cdot n_{\text{skel}}(J) \geq \frac{1}{9}(1 - \varepsilon)t(t - 1)d_2 n_*^2.
\]

Since \( R \) is strongly \((1/3 + \alpha/2, 2\varepsilon^{1/4})\)-dense, the number of labeled copies of \( F \) that are

- rooted at \( \nu, \nu' \) and
- with \( x_3, x_4 \) mapped to a pair \( S \) of distinct clusters of \( J \) satisfying \( \deg_R(S) > 0 \)

...
is at least \( n_F(H; v_1, v_2, \mathcal{J}) - 2\delta^{1/4} t (t - 1)(1 + \varepsilon)d_2n_*^2 \geq t(t - 1)d_2n_*^2/10 \). Therefore, there exists a pair \( S := S(v_1, v_2) \) of clusters of \( \mathcal{J} \) such that \( \text{deg}_R(S) > 0 \) and \( \mathcal{J}_S \) supports at least \( d_2n_*^2/10 \) labeled copies of \( F \) rooted at \( v_1, v_2 \).

Let \( H'' \) be the subgraph of \( H' \) which consists of the edges supported on triples of clusters in \( E(R) \) only. Since \( R \) has at most two tight components, we will show that as long as there are at least three paths (which are not too long) we can connect two of them by Lemma 5.1. Note that as we iteratively connect the paths, to guarantee the property on the end edges as stated in the lemma, it suffices to consider connecting the paths \( P_1, P_2 \) “at last” and when considering them with a third path we will only connect them from the end other than \( P_1, P_2 \).

Note also that we will use at most \( q(15 + t^3) \) vertices for connection and thus the collection of paths will cover at most \( q(15 + t^3) + (16 + t^3)|\mathcal{J}| \leq \sqrt{\xi} n \) vertices, as \( \xi \leq \xi_0 \ll 1/t_1 \leq 1/t \). So throughout the process, by Lemma 4.1, for each \( X \in E(R) \), the restriction of \( H'' \cup \mathcal{J} \) on the set of unused vertices in \( X \) is \((d', d_3, \sqrt{\xi}, \sqrt{\xi}, r)\)-regular for some \( d' \geq d \), so that we can apply Lemma 5.1 on the subcomplex of \( H'' \cup \mathcal{J} \) induced on the unused vertices.

Without loss of generality suppose we have paths \( P_1, P_2, P_3 \) and consider one end pair from each of them (but not any of \( P_1, P_2 \)), denoted by \((v_i^1, v_i^2)\), \( i = 1, 2, 3 \). Let \( S_i = S(v_i^1, v_i^2) \) be the pair of clusters defined above. Since \( \text{deg}_R(S_i) > 0 \), \( i = 1, 2, 3 \), take \( X_i \), \( i = 1, 2, 3 \), such that \( S_i \subseteq X_i \in E(R) \). As \( R \) has at most two tight components, there exists \( \{i, j\} \in \binom{[3]}{2} \) such that \( X_i \) and \( X_j \) are in the same tight component. Write \( X_j := \{w_1, w_2, w_3\} \). Since \( R \) is strongly \((1/3 + \alpha/2, 2\delta^{1/4})\)-dense and \( 3(1/3 + \alpha/2)n > n \), we derive that two of \( N_R(w_1, w_2), N_R(w_2, w_3) \), and \( N_R(w_1, w_3) \) have nonempty intersection, implying the existence of a copy of \( K_4^- \) containing \( X_j \). Recall that each \( S_i \) hosts at least \( d_2n_*^2/10 \) labeled copies of \( F \) rooted at \( v_i^1, v_i^2 \), and among them, there are at least \( d_2n_*^2/10 - \sqrt{\xi} n \cdot n_* > d_2n_*^2/20 = \sqrt{\beta} n_*^2 \) such copies that do not intersect the existing paths. As \((\sqrt{\beta} n_*^2)^2 = \beta n_*^4 \), there are at least \( \beta n_*^4 + 1 \) pairs of labeled copies of \( F \), one rooted at \( v_i^1, v_i^2 \) and the other rooted at \( v_i^1, v_i^2 \). If we regard the nonroot vertices as labeled 2-edges, then Lemma 5.1 says at least one of the pairs can be connected by a tight path of length \( 15 + t^3 \), which gives the desired path connecting \((v_i^1, v_i^2)\) and \((v_i^1, v_i^2)\). \( \square \)

6 PATH COVER

We use the following result [7, Lemma 4.3] in a slightly relaxed form.\(^6\)

**Lemma 6.1** (Almost perfect matching). For any \( \alpha, \beta > 0 \), there exist \( \varepsilon_0 > 0 \) and \( n_0 \) such that the following holds for \( \varepsilon \leq \varepsilon_0 \) and \( n \geq n_0 \). Let \( H = (V, E) \) be an \( n \)-vertex 3-graph which is \((1/3 + \alpha, \varepsilon)\)-dense. Then \( H \) contains a matching that covers all but at most \( \beta n \) vertices of \( V \).

\(^6\)The original statement of [7, Lemma 4.3] requires that the 3-graph \( H \) has no independent set of size \((2/3 - o(1))n\), whose existence would imply that all pairs in the independent set has a degree at most \((1/3 + o(1))n\). This is indeed ruled out by our stronger degree assumption.
Proof of Lemma 3.2. Apply Lemma 3.1 with $\alpha/3$ in place of $\alpha$ and obtain $\zeta_0 > 0$. Choose constants

$$1/n_0 < 1/t_1 < 1/r, \varepsilon < \min\{\varepsilon_3, d_2\} \leq \varepsilon_3 < d < \gamma < \theta < \alpha, \eta, \zeta_0$$

and suppose $n \geq 2n_0$. We first choose a random set $A$ of vertices by including every vertex with probability $\gamma$. By (1) and the union bound, there exists a choice of $A$ such that $(1 - \beta)n \leq |A| \leq (1 + \beta)n$ and

(a) for any $u, v \in V(H), |N_{H} (uv) \cap A| \geq (1 - \beta)(1/3 + \alpha)yn \geq (1/3 + \alpha/2)|A|$.

Let $H'$ be an induced subgraph of $H - A$ on $n'$ vertices such that $n' \geq n - t_1! - |A|$ and $t_1! |n'|$. Note that $\delta_{2(H')} \geq (1/3 + \alpha)n - |A| - t_1! \geq (1/3 + \alpha - \delta)n$ as $1/n \leq 1/n_0 < 1/t_1 \leq \theta$. Then Theorem 4.4 applied with $\varepsilon(t_1) = \varepsilon$ and $r(t_1) = r$ gives a $(t_0, t_1, \varepsilon, \varepsilon_3, r)$-regular slice $\mathcal{J}$ for $H'$. Let $R_{d(H')} \equiv$ the $d$-reduced graph which is $(1/3 + \alpha/2, 3, \varepsilon)$-dense by Lemma 4.6 and the choice of the constants. Let $\mathcal{P}$ be the ground partition of $\mathcal{J}$ with $|\mathcal{P}| = t \leq t_1$ and let $n_* := n'/t$. By Lemma 6.1 applied with $\alpha/2$ in place of $\alpha$ and $3, \varepsilon_3$ in place of $\varepsilon$, $R_{d(H')} \equiv$ has a matching $M$ that covers all but $\theta t$ vertices of $R_{d(H')}$, and clearly $|\mathcal{P}| \leq t/3$.

Note that each edge in $M$ corresponds to a $(d', d_2, \varepsilon, \varepsilon, r)$-regular complex $G$ for some $d' \geq d$. We now find a collection of vertex-disjoint paths in $G$. Indeed, note that by Lemma 4.1, for any subcomplex $G'$ of $G$ with $n_0 \geq \theta n_*$ vertices from each cluster, $G'$ is $(d', d_2, \sqrt{\varepsilon_3}, \sqrt{\varepsilon}, r)$-regular. Thus, $G'$ has at least $(dd_2^2/2)n_0^3$ 3-edges, and by [27, Claim 4.1] it contains a tight path on at least $(dd_2^2/2)n_0 \geq (dd_2^3/2)n_*$ vertices. Therefore, we can greedily construct a family of at most $6/(dd_2^3 \theta)$ vertex-disjoint tight paths that together covers all but at most $3 \theta n_*$ vertices. We do the same for all edges in $M$, which altogether gives a family of at most $(t/3) \cdot 3 \theta n_* + \theta t \cdot n_* \leq 2 \theta n$ vertices of $H'$.

Next we connect these paths $Q_1, ..., Q_q, q \leq 2t/(dd_2^3 \theta)$ into two tight paths by the vertices of $A$. To see it, for $i \in [q]$, let $p_i^1$ and $p_i^2$ be the ends of $Q_i$ and consider $H_a \equiv H [A \cup \bigcup_{i \in [q]} (p_i^1 \cup p_i^2)]$. By (a), we have that $\delta_2(H_a) \geq (1/3 + \alpha/2)|A| \geq (1/3 + \alpha/3)|V(H_a)|$ as $|V(H_a)| \leq |A| + 4q \leq |A| + 8t/(dd_2^3 \theta) \leq (1 + \alpha/3)|A|$. We regard each $p_i^1, p_i^2, i \in [q]$ as a 4-vertex path $Q_i'$. Because $|V(Q_i' \cup \cdots \cup Q_q')| = 4q \leq \zeta_0 |V(H_a)|$, we can use Lemma 3.1 to connect them to two paths. This gives rise to a connection of $Q_1, ..., Q_q$ into two tight paths $P_1, P_2$. Note that $|V(H) \setminus V(P_1 \cup P_2)| \leq 2 \theta n + |A| + t_1! \leq \eta n$ and we are done.

Orcid

Jie Han http://orcid.org/0000-0002-2013-2962

REFERENCES

1. P. Allen, J. Böttcher, O. Cooley, and R. Mycroft, Tight cycles and regular slices in dense hypergraphs, J. Combin. Theory Ser. A. 149 (2017), 30–100.

2. K. Bari and M. O'Sullivan, The Hamiltonian problem and t-path traceable graphs, Involve. 10 (2017), 801–812.
3. J. O. Bastos, G. O. Mota, M. Schacht, J. Schnitzer, and F. Schulenburg, *Loose Hamiltonian cycles forced by large (k – 2)-degree—approximation version*, SIAM J. Discrete Math. 31 (2017), 2328–2347.

4. J. O. Bastos, G. O. Mota, M. Schacht, J. Schnitzer, and F. Schulenburg, *Loose Hamiltonian cycles forced by large (k – 2)-degree—sharp version*, Contrib. Discrete Math. 13 (2019), no. 2, 88–100.

5. E. Buß, H. Hán, and M. Schacht, *Minimum vertex degree conditions for loose Hamilton cycles in 3-uniform hypergraphs*, J. Combin. Theory Ser. B. 103 (2013), no. 6, 658–678.

6. P. Erdős, *On extremal problems of graphs and generalized graphs*, Israel J. Math. 2 (1964), no. 3, 183–190.

7. W. Gao and J. Han, *Minimum codegree threshold for $C_6^3$-factors in 3-uniform hypergraphs*, Combin. Probab. Comput. 26 (2017), no. 4, 536–559.

8. A. Georgakopoulos, J. Haslegrave, and R. Montgomery, *Forcing large tight components in 3-graphs*, European J. Combin. 77 (2019), 57–67.

9. R. Glebov, Y. Person, and W. Weps, *On extremal hypergraphs for Hamiltonian cycles*, European J. Combin. 33 (2012), no. 4, 544–555.

10. J. Han, *Decision problem for perfect matchings in dense $k$-uniform hypergraphs*, Trans. Amer. Math. Soc. 369 (2017), no. 7, 5197–5218.

11. J. Han, A. Lo, and N. Sanhueza-Matamala, *Covering and tiling hypergraphs with tight cycles*, Combin. Probab. Comput. 30 (2021), 288–329.

12. J. Han and A. Treglown, *The complexity of perfect matchings and packings in dense hypergraphs*, J. Combin. Theory Ser. B. 141 (2020), 72–104.

13. J. Han and Y. Zhao, *Minimum codegree threshold for Hamilton $\ell$-cycles in $k$-uniform hypergraphs*, J. Combin. Theory Ser. A. 132 (2015), no. 0, 194–223.

14. J. Han and Y. Zhao, *Minimum degree thresholds for loose Hamilton cycle in 3-graphs*, J. Combin. Theory Ser. B. 114 (2015), 70–96.

15. J. Han and Y. Zhao, *Forbidding Hamilton cycles in uniform hypergraphs*, J. Combin. Theory Ser. A. 143 (2016), 107–115.

16. P. E. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, and J. Skokan, *The Ramsey number for 3-uniform tight hypergraph cycles*, Combin. Probab. Comput. 18 (2009), no. 1–2, 165–203.

17. S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

18. G. Katona and H. Kierstead, *Hamiltonian chains in hypergraphs*, J. Graph Theory. 30 (1999), no. 2, 205–212.

19. Y. Kohayakawa, V. Rödl, and J. Skokan, *Hypergraphs, quasi-randomness, and conditions for regularity*, J. Combin. Theory Ser. A. 97 (2002), no. 2, 307–352.

20. D. Kühn, R. Mycroft, and D. Osthus, *Hamilton $\ell$-cycles in uniform hypergraphs*, J. Combin. Theory Ser. A. 117 (2010), no. 7, 910–927.

21. D. Kühn and D. Osthus, *Hamilton cycles in graphs and hypergraphs: an extremal perspective*, Proceedings of the International Congress of Mathematicians 2014, Seoul, Korea, vol. 4, 2014, pp. 381–406.

22. R. Nenadov, B. Sudakov, and A. Z. Wagner, *Completion and deficiency problems*, J. Combin. Theory Ser. B. 145 (2020), 214–240.

23. S. Noorvash, *Covering the vertices of a graph by vertex-disjoint paths*, Pacific J. Math. 58 (1975), 159–168.

24. C. Reiher, V. Rödl, A. Ruciński, M. Schacht, and E. Szemerédi, *Minimum vertex degree condition for tight Hamiltonian cycles in 3-uniform hypergraphs*, Proc. Lond. Math. Soc. 119 (2019), 409–439.

25. V. Rödl and A. Ruciński, *Dirac-type questions for hypergraphs—a survey (or more problems for Endre to solve)*, An Irregular Mind, Bolyai Soc. Math. Studies, vol. 21, 2010, pp. 561–590.

26. V. Rödl, A. Ruciński, and E. Szemerédi, *A Dirac-type theorem for $3$-uniform hypergraphs*, Combin. Probab. Comput. 15 (2006), no. 1–2, 229–251.

27. V. Rödl, A. Ruciński, and E. Szemerédi, *An approximate Dirac-type theorem for $k$-uniform hypergraphs*, Combinatorica. 28 (2008), no. 2, 229–260.

28. V. Rödl, A. Ruciński, and E. Szemerédi, *Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs*, Adv. Math. 227 (2011), no. 3, 1225–1299.
29. Y. Zhao, *Recent advances on Dirac-type problems for hypergraphs*, Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications (A. Beveridge, J. R. Griggs, L. Hogben, G. Musiker, and P. Tetali, eds.), vol. 159, Springer, New York, 2016.

**How to cite this article:** J. Han, *Covering 3-uniform hypergraphs by vertex-disjoint tight paths*, J. Graph Theory. 2022;101:782–802. [https://doi.org/10.1002/jgt.22853](https://doi.org/10.1002/jgt.22853)