Multishifts on Directed Cartesian Product of Rooted Directed Trees

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The pathways to new ideas will never close; the doors of wisdom (literature) will always be open, till the very end.\textsuperscript{1}

- Wali Deccani (1667-1707)

\textsuperscript{1}Translated by Anant Dhavale
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Abstract

We systematically develop the multivariable counterpart of the theory of weighted shifts on rooted directed trees. Capitalizing on the theory of product of directed graphs, we introduce and study the notion of multishifts on directed Cartesian product of rooted directed trees. This framework unifies the theory of weighted shifts on rooted directed trees and that of classical unilateral multishifts. Moreover, this setup brings into picture some new phenomena such as the appearance of system of linear equations in the eigenvalue problem for the adjoint of a multishift. In the first half of the paper, we focus our attention mostly on the multivariable spectral theory and function theory including finer analysis of various joint spectra and wandering subspace property for multishifts. In the second half, we separate out two special classes of multishifts, which we refer to as torally balanced and spherically balanced multishifts. The classification of these two classes is closely related to toral and spherical polar decompositions of multishifts. Furthermore, we exhibit a family of spherically balanced multishifts on $d$-fold directed Cartesian product $T$ of rooted directed trees. These multishifts turn out be multiplication $d$-tuples $M_{z,a}$ on certain reproducing kernel Hilbert spaces $H_a$ of vector-valued holomorphic functions defined on the unit ball $B^d$ in $\mathbb{C}^d$, which can be thought of as tree analogs of the multiplication $d$-tuples acting on the reproducing kernel Hilbert spaces associated with the kernels $\frac{1}{(1-\langle z, w \rangle)^a}$ ($z, w \in B^d, a \in \mathbb{N}$). Indeed, the reproducing kernels associated with $H_a$ are certain operator linear combinations of $\frac{1}{(1-\langle z, w \rangle)^a}$ and multivariable hypergeometric functions $\, _2F_1(\alpha_v + a + 1, 1, \alpha_v + 2, \cdot)$ defined on $B^d \times B^d$, where $\alpha_v$ denotes the depth of a branching vertex $v$ in $T$. We also classify joint subnormal and joint hyponormal multishifts within the class of spherically balanced multishifts.

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CHAPTER 1

Introduction

The investigations in the present work are related to the idea of shifts associated with discrete structures (e.g. directed trees), recently boosted in the theory of Hilbert space operators (\cite{65}, \cite{25}, \cite{66}, \cite{26}, \cite{67}, \cite{68}, \cite{27}, \cite{28}, \cite{29}, \cite{33}, \cite{79}). The significantly large class of weighted shift operators on directed trees contains all classical weighted shifts and has an overlap with that of composition operators. This interplay of graph theory and operator theory provides illuminating examples exhibiting subtle phenomena such as existence of non-hyponormal operators generating Stieltjes moment sequences and triviality of the domain of integral powers of densely defined subnormal operators (\cite{66}, \cite{68}, \cite{28}, \cite{29}; refer also to \cite{72}, \cite{74} for a systematic study of operator algebras associated with directed graphs). Further, this framework turns out to be a rich source of \(k\)-diagonal reproducing kernel Hilbert spaces of vector-valued holomorphic functions (\cite{1}, \cite{33}).

The motivation for the present work primarily comes from the multivariable operator theory, where the main objectives of study have been function theory and spectral theory of classical multishifts (\cite{69}, \cite{40}, \cite{41}, \cite{47}, \cite{16}, \cite{17}, \cite{42}, \cite{43}, \cite{18}, \cite{19}, \cite{80}, \cite{81}, \cite{21}, \cite{13}, \cite{8}, \cite{58}, \cite{57}, \cite{56}, \cite{5}, \cite{82}, \cite{50}, \cite{52}, \cite{23}, \cite{70}, \cite{71}, \cite{34}, \cite{61}). This work is an effort to develop the theory of weighted shifts on directed trees in several variables by implementing the methods of graph theory (\cite{84}, \cite{90}, \cite{77}, \cite{60}, \cite{54}, \cite{64}; refer also to \cite{9}, \cite{4}, \cite{6}, \cite{10}, \cite{11}, where Hardy-Besov spaces associated with certain trees have been introduced and studied). The well-established theory of product of directed graphs provides a foundation for the study of multishifts. Various notions of product of directed graphs (e.g. Cartesian product, tensor product) lead naturally to interesting counterparts of classical shifts in one and several variables. One peculiar aspect of the directed Cartesian product of directed trees is that although its not a directed tree, it admits a directed semi-tree structure. Interestingly, there is a natural shift operator on any directed semi-tree \cite{79}. Thus a single discrete structure gives rise to at least two distinct notions of shifts. This is not so easy to reveal in the classical case. One of the advantages of this setup is that any disjoint decomposition of the set \(V\) of vertices induces a natural decomposition of the associated unweighted Lebesgue space \(l^2(V)\), which in turn decomposes the multishift \(S_A\) into known objects like tuples of compact operators and classical multishifts. On the other hand, there are numerous ways of decomposing \(V\) by defining equivalence relations on \(V\) in terms of siblings, generations etc. As evident, every set up has its own set of problems. Here also in the context of subnormality of multishifts, one can ask for finite and minimal subset \(W\) of set \(V\) of vertices with the following property: A multishift is joint subnormal if and only if its moments are completely monotone at every vertex from \(W\). The notion of joint branching index plays an important role in the affirmative answer of this problem.
In the remaining part of this section, we set some standard notations and also collect various notions and facts, which are central to the present text. For a set $X$, $\text{card}(X)$ denotes the cardinality of $X$. We next recall that the symbol $\mathbb{N}$ stands for the set of nonnegative integers and that $\mathbb{N}$ forms a semigroup under addition. For a positive integer $d$, let $\mathbb{N}^d$ denote the $d$-fold Cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ of $\mathbb{N}$. Then, for $\alpha = (\alpha_1, \cdots, \alpha_d)$ and $\beta = (\beta_1, \cdots, \beta_d)$ in $\mathbb{N}^d$, we write $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for all $j = 1, \cdots, d$ and we also use $\alpha! := \prod_{j=1}^d \alpha_j!$ and $|\alpha| := \sum_{j=1}^d \alpha_j$. Throughout this paper, we follow the conventions below:

$$\sum_{i=1}^{n-1} x_i = 0 \text{ if } n = 1, \quad \prod_{j=0}^{n-1} y_j = 1 \text{ if } n = 0.$$  

More generally, the sum over empty set is understood to be 0 while the product over empty set is always 1.

Let $\mathbb{C}$ denote the field of complex numbers and let $\mathbb{C}^d$ denote the $d$-fold Cartesian product of $\mathbb{C}$. We set $t_z := (t_z_1, \cdots, t_z_d)$ for $t \in \mathbb{C}$ and $z = (z_1, \cdots, z_d) \in \mathbb{C}^d$. Whenever $\alpha \in \mathbb{N}^d$ or $z \in \mathbb{C}^d$, it is understood that $\alpha = (\alpha_1, \cdots, \alpha_d)$ and $z = (z_1, \cdots, z_d)$. The complex conjugate of $z \in \mathbb{C}^d$ is defined by $\bar{z} := (\bar{z}_1, \cdots, \bar{z}_d)$. We denote by $B^d_r$ the open ball in $\mathbb{C}^d$ centered at the origin and of radius $r > 0$:

$$B^d_r := \{ z = (z_1, \cdots, z_d) \in \mathbb{C}^d : \|z\|_2 < r \},$$

where $\|z\|_2 := \sqrt{|z_1|^2 + \cdots + |z_d|^2}$ denotes the Euclidean norm of $z = (z_1, \cdots, z_d)$ in $\mathbb{C}^d$. The sphere $\{ z \in \mathbb{C}^d : \|z\|_2 = r \}$ centered at the origin and of radius $r > 0$ is denoted by $\partial B^d_r$. For simplicity, the unit ball $B^d_1$ and the unit sphere $\partial B^d_1$ are denoted respectively by $B^d$ and $\partial B^d$. We denote by $D^d_r$ the open polydisc centered at the origin and of polyradius $r = (r_1, \cdots, r_d)$ with $r_1, \cdots, r_d > 0$:

$$D^d_r := \{ z = (z_1, \cdots, z_d) \in \mathbb{C}^d : |z_1| < r_1, \cdots, |z_d| < r_d \}.$$  

The $d$-torus $\{ z \in \mathbb{C}^d : |z_1| = r_1, \cdots, |z_d| = r_d \}$ centered at the origin and of polyradius $r$ is denoted by $T^d$. Again, for simplicity, the unit polydisc $D^d_1$ and the unit $d$-torus $T^d_1$ are denoted respectively by $D^d$ and $T^d$. For a subset $X$ of $\mathbb{C}^d$, the closure of $X$ in $\| \cdot \|_2$ is denoted by $cl(X)$.

Let $\mathcal{H}$ be a complex Hilbert space. The inner product on $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. If no confusion is likely, then we suppress the suffix, and simply write the inner product as $\langle \cdot, \cdot \rangle$. By a subspace of $\mathcal{H}$, we mean a closed linear manifold. Let $W$ be a subset of $\mathcal{H}$. Then span $W$ stands for the smallest linear manifold generated by $W$. In case $W$ is singleton $\{w\}$, we use the convenient notation $[w]$ in place of span $\{w\}$. By $\bigvee \{w : w \in W\}$, we understand the subspace generated by $W$. The orthogonal complement of a subspace $\mathcal{M}$ in $\mathcal{H}$ will be denoted either by $\mathcal{M}^\perp$ or $\mathcal{H} \ominus \mathcal{M}$. The Hilbert space dimension of the subspace $\mathcal{M}$ is denoted by $\dim \mathcal{M}$. For a subspace $\mathcal{M}$ of $\mathcal{H}$, we use $P_{\mathcal{M}}$ to denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. We use $I$ to denote the identity operator on $\mathcal{H}$. If $\mathcal{M}$ is a subspace of $\mathcal{H}$, then we use $I_{|\mathcal{M}}$ to denote the identity operator on $\mathcal{M}$.

Unless stated otherwise, all the Hilbert spaces occurring in this paper are complex infinite-dimensional separable and for any such Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators on $\mathcal{H}$ endowed with the operator norm. For $A \in B(\mathcal{H})$, the symbols $\ker A$ and $\ran A$ will stand for the kernel and the range of $A$ respectively. The Hilbert space adjoint of $A$ will be denoted by $A^\ast$. 
By a commuting \( d \)-tuple \( T \) on \( \mathcal{H} \), we mean a tuple \((T_1, \cdots, T_d)\) of commuting bounded linear operators \( T_1, \cdots, T_d \) on \( \mathcal{H} \). A commuting \( d \)-tuple \( T \) on \( \mathcal{H} \) is said to be doubly commuting if \( T_i T_j^* = T_j T_i^* \) for all \( i, j = 1, \cdots, d \) with \( i \neq j \). If \( T = (T_1, \cdots, T_d) \) is a commuting \( d \)-tuple on \( \mathcal{H} \), then we set \( T^* \) to denote \((T_1^*, \cdots, T_d^*)\) while \( T^\alpha \) represents \( T_1^{\alpha_1} \cdots T_d^{\alpha_d} \) for \( \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d \), where we use the convention that \( A^0 = I \) for \( A \in \mathcal{B}(\mathcal{H}) \). Also, we find it convenient to introduce the following operators on \( \mathcal{B}(\mathcal{H}) \). Given a commuting \( d \)-tuple \( T \) on \( \mathcal{H} \), define \( QT : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) by \( QT(X) := \sum_{j=1}^d T_j^* XT_j \) for \( X \in \mathcal{B}(\mathcal{H}) \). Further, the operator \( Q^n_T \) is inductively defined for all \( n \in \mathbb{N} \) through the relations \( Q^n_T(X) := X \) and \( Q^n_T(X) := QT(Q^{n-1}_T(X)) \) \( (n \geq 1) \) for \( X \in \mathcal{B}(\mathcal{H}) \). It is easy to see that

\[
Q^n_T(I) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} T^\alpha T^\alpha \quad (n \in \mathbb{N}).
\]

We collect below some definitions required throughout this text.

Let \( T = (T_1, \cdots, T_d) \) be a commuting \( d \)-tuple on \( \mathcal{H} \). We say that \( T \) is a

(i) \textit{toral contraction} if \( QT_j(I) \leq I \) for \( j = 1, \cdots, d \).
(ii) \textit{toral isometry} if \( QT_j(I) = I \) for \( j = 1, \cdots, d \).
(iii) \textit{toral left invertible} \( d \)-tuple if \( QT_j(I) \) is invertible for \( j = 1, \cdots, d \).
(iv) \textit{joint contraction} if \( QT(I) \leq I \).
(v) \textit{joint isometry} if \( QT(I) = I \).
(vi) \textit{joint left invertible} \( d \)-tuple if \( QT(I) \) is invertible.

A commuting \( d \)-tuple \( T = (T_1, \cdots, T_d) \) on \( \mathcal{H} \) is \textit{joint subnormal} if there exist a Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) and a commuting \( d \)-tuple \( N = (N_1, \cdots, N_d) \) of normal operators \( N_1, \cdots, N_d \) in \( \mathcal{B}(\mathcal{K}) \) such that \( N_j h = T_j h \) for every \( h \in \mathcal{H} \) and \( j = 1, \cdots, d \). It is well-known that toral isometry and joint isometry are joint subnormal [18] Propositions 1 and 2). The notion of joint subnormal tuples is closely related to a classical notion from abstract harmonic analysis, namely, the notion of completely monotone functions.

Let \( \phi \) be a real-valued map on \( \mathbb{N}^d \). For \( 1 \leq j \leq d \), define the difference operators \( \nabla_j \) by \( (\nabla_j \phi)(\alpha) := \phi(\alpha) - \phi(\alpha + \epsilon_j) \) \( (\alpha \in \mathbb{N}^d) \), where \( \epsilon_j \) is the \( d \)-tuple with 1 in the \( j \)th entry and zeros elsewhere. The operator \( \nabla^\beta \) is inductively defined for every \( \beta \in \mathbb{N}^d \) through the relations \( \nabla^0 \phi := \phi \), \( \nabla^{\beta+\epsilon_j} \phi := \nabla_j (\nabla^\beta \phi) \). A real-valued map \( \phi \) on \( \mathbb{N}^d \) is said to be \textit{completely monotone} if \( (\nabla^\beta \phi)(\alpha) \geq 0 \) for all \( \alpha, \beta \in \mathbb{N}^d \).

**Remark 1.0.1.** A toral contractive \( d \)-tuple \( T \) on \( \mathcal{H} \) is joint subnormal if and only if \( \phi(\alpha) := \|T^\alpha h\|^2 \) \( (\alpha \in \mathbb{N}^d) \) is completely monotone for every \( h \in \mathcal{H} \) ([16] Theorem 4.4)).

A commuting \( d \)-tuple \( T = (T_1, \cdots, T_d) \) is \textit{joint hyponormal} if the \( d \times d \) matrix \( ([T_i^*, T_j])_{1 \leq i,j \leq d} \) is positive definite, where \( [A, B] \) stands for the commutator \( AB - BA \) for \( A \) and \( B \) in \( \mathcal{B}(\mathcal{H}) \). A joint subnormal tuple is always joint hyponormal [17], [43].

For all notions introduced above, we skip the prefixes toral or joint in case the dimension \( d \) is 1. Although it has been a common practice to use interchangeably joint isometry with spherical isometry, we do not follow this practice.

We briefly recall from [30] the definitions of toral and spherical Cauchy dual tuples. Let \( T = (T_1, \cdots, T_d) \) be a commuting \( d \)-tuple on \( \mathcal{H} \). Assume that \( T \) is toral left invertible. We refer to the \( d \)-tuple \( T^d = (T_1^d, \cdots, T_d^d) \) as the \textit{toral Cauchy dual
of $T$, where
\begin{equation}
T_j^t := T_j(Q_T(I))^{-1} \ (j = 1, \cdots, d).
\end{equation}

Note that $(T^t)^t = T$.

Assume that $T$ is joint left-invertible. We refer to the $d$-tuple $T^* = (T_1^*, \cdots, T_d^*)$ as the spherical Cauchy dual of $T$, where
\begin{equation}
T_j^* := T_j(Q_T(I))^{-1} \ (j = 1, \cdots, d).
\end{equation}

Note that $(T^*)^* = T$.

1.1. Multivariable Spectral Theory

In what follows, the multivariable spectral theory will be central to the investigations in this paper. Thus we find it necessary to include a brief account of Taylor’s notion of invertibility and related concepts. We have freely drawn on [42] and [14] throughout this paper, particularly, in the following discussion.

The classical spectral theory deals with the problem of finding $x \in H$ such that $Tx = y$ for any given $y \in H$, where $T \in B(H)$. Note that $T$ is (boundedly) invertible if and only if the above problem is solvable for every $y \in H$ with unique $x \in H$.

Let us formulate an analog of the problem above for two operators $T_1, T_2 \in B(H)$ such that $T_1 T_2 = T_2 T_1$. One is interested in the notion of invertibility which will give “unique” solution $(x_1, x_2)$ of the problem of finding $x_1, x_2 \in H$ such that $T_1 x_1 + T_2 x_2 = y$ for every given $y \in H$. In this case we can not hope for uniqueness. Indeed, if $(x_1, x_2)$ is a solution of this problem, then for any $h \in H$, if we set $x'_1 := x_1 - T_2 h$ and $x'_2 := x_2 - T_1 h$, then it is easy to check $(x'_1, x'_2)$ is a also solution. Following [14], we will refer to $(x'_1, x'_2)$ as the tautological perturbation of the solution $(x_1, x_2)$.

Remark 1.1.1. There are no non-trivial tautological perturbations in case of single operator.

One needs to determine what happens modulo tautological perturbations. This is where homology enters into the picture.

Given a Hilbert space $H$, consider
\begin{align*}
\Lambda_0 & := H, \\
\Lambda_1 & := \{(h_1, h_2) : h_1, h_2 \in H\}, \\
\Lambda_2 & := \{(h_{ij}) : h_{ij} \in H \text{ for } 1 \leq i, j \leq 2, (h_{ij}) \text{ is skew symmetric}\}.
\end{align*}

Note that $\Lambda_2$ is isometrically isomorphic to $H$ via $\begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix} \sim h$. Consider the following short sequence
\begin{equation}
K : \{0\} \rightarrow \Lambda_2 \xrightarrow{B_2} \Lambda_1 \xrightarrow{B_1} \Lambda_0 \rightarrow \{0\},
\end{equation}
where
\begin{align*}
B_2((h_{ij})) & := (h_{ij}) \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = (T_2 h_{12}, -T_1 h_{12}), \\
B_1(h_1, h_2) & := (h_1, h_2) \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = T_1 h_1 + T_2 h_2.
\end{align*}

Note that $K$ is a complex, that is, $B_1 \circ B_2 = 0$.

Let us examine the complex $K$ as given in (1.4).
(1) \( T_1 x_1 + T_2 x_2 = y \) has a solution if and only if \( B_1(x_1, x_2) = y \) has a solution if and only if \( B_1 \) is surjective.

(2) Given a solution \( (x_1, x_2) \in A_1 \) of \( T_1 x_1 + T_2 x_2 = y \) for a given \( y \in \mathcal{H} \), \( (x'_1, x'_2) \) is also a solution if and only if \( (x_1 - x'_1, x_2 - x'_2) \in \ker B_1 \).

(3) \( \ker B_2 = \ker T_1 \cap \ker T_2 \).

(4) \([1,4]\) is exact if and only if \( T_1 \mathcal{H} + T_2 \mathcal{H} = \mathcal{H} \), \( \ker T_1 \cap \ker T_2 = \{0\} \) and solution of \( T_1 x_1 + T_2 x_2 = y \) for a given \( y \in \mathcal{H} \) is unique up to tautological perturbations.

Let us now see the Taylor invertibility in the general case. For that purpose, consider the co-ordinate linear functionals \( e_1, e_2, \ldots, e_d \) on \( \mathbb{C}^d \) with respect to the standard basis. Let \( \Lambda^d(\mathbb{C}^d) := \mathbb{C} \) and let \( \Lambda^1(\mathbb{C}^d) \) be the vector space with basis \( \{e_1, e_2, \ldots, e_d\} \). Given \( w, w' \in \Lambda^1(\mathbb{C}^d) \) let us define \( w \wedge w' \) by

\[
w \wedge w' = \sum_{\sigma \in S_n} \text{sgn}(\sigma) w(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) w'(e_{\sigma(n+1)}, \ldots, e_{\sigma(n+q)}),
\]

where \( S_n \) denotes the group of permutations on \( \{1, \ldots, n\} \). For \( i = 1, \ldots, d \), let \( \Lambda^i(\mathbb{C}^d) \) be the vector space generated by \( i \)-forms. We define \( \Lambda(\mathbb{C}^d) \) as the algebra over \( \mathbb{C} \) consisting of \( \Lambda^i(\mathbb{C}^d) \) (\( i = 1, \ldots, d \)) with identity \( e_0 \) defined by \( e_0 \wedge w = w \), where the multiplication is the wedge product. The vector space \( \Lambda(\mathbb{C}^d) \) is \( 2^d \) dimensional, which can be endowed with an inner product \( \langle \cdot, \cdot \rangle_\Lambda \) so that

\[
\{e_0\} \cup \{e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_k \leq d\}
\]

forms an orthonormal basis.

The finite dimensional Hilbert space \( \Lambda(\mathbb{C}^d) \) admits natural operators, to be referred to as, creation operators \( E_i : \Lambda \rightarrow \Lambda \) defined by \( E_i(w) := e_i \wedge w \) (\( i = 1, \cdots, d \)), and \( E_0(w) := w \). It satisfies the anti-commutation relations:

\[
E_i E_j + E_j E_i = 0 \quad (1 \leq i, j \leq d).
\]

Let \( T = (T_1, \ldots, T_d) \) be a commuting \( d \)-tuple on \( \mathcal{H} \) and let \( \Lambda(\mathcal{H}) := \mathcal{H} \otimes_\mathbb{C} \Lambda(\mathbb{C}^d) \) be a Hilbert space endowed with the inner product

\[
\langle x \otimes w, y \otimes w' \rangle_{\Lambda(\mathcal{H})} := \langle x, y \rangle_{\mathcal{H}} \langle w, w' \rangle_\Lambda.
\]

We set \( \Lambda^i(\mathcal{H}) := \mathcal{H} \otimes_\mathbb{C} \Lambda^i(\mathbb{C}^d) \) for \( i = 0, \cdots, d \). Consider the boundary operator \( \partial_T : \Lambda(\mathcal{H}) \rightarrow \Lambda(\mathcal{H}) \) given by

\[
\partial_T (h \otimes w) := \sum_{i=1}^d T_i(h) \otimes E_i(w).
\]

Note that \( \partial_T \) is a bounded linear operator on \( \Lambda(\mathcal{H}) \). Since \( T \) is commuting and \( E_1, \ldots, E_d \) are anti-commuting, \( \partial_T^2 = 0 \). This allows us to define the Koszul complex
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We define point spectrum and left spectrum. Here by inner radius \( \sigma \) we mean the largest nonnegative number \( r \) for which

\[
\sigma_1(T) \subseteq \{ w \in \mathbb{C}^d : r \leq \| w \|_2 \leq r(T) \}.
\]

For \( k = 0, \cdots, d \), let \( H^k(T) \) denote the \( k \)-th cohomology group appearing in the Koszul complex \( K(T) \). We say that \( T \) is Fredholm if \( H^k(T) \) finite dimensional for every \( k = 0, \cdots, d \). The Fredholm index \( \text{ind}(T) \) of a Fredholm \( d \)-tuple \( T \) is the Euler characteristic of \( K(T) \) given by

\[
\text{ind}(T) := \sum_{k=0}^{d} (-1)^k \dim H^k(T).
\]

The essential spectrum of \( T \) is defined as

\[
\sigma_e(T) := \{ \lambda \in \mathbb{C}^d : T - \lambda \text{ is not Fredholm} \}.
\]
Clearly, essential spectrum is a subset of the Taylor spectrum. By Atkinson-Curto Theorem \(41\), \(\sigma_e(T) = \sigma(T(\pi))\), where \(\pi\) is Calkin map and \(\pi(T) := (\pi(T_1), \ldots, \pi(T_d))\). In particular, essential spectrum is a nonempty compact set with polynomial spectral mapping property.

1.2. Classical Multishifts

For a given multisequence \(w = \{w^{(j)}_\alpha : 1 \leq j \leq d, \alpha \in \mathbb{N}^d\}\) of complex numbers and an orthonormal basis \(\{e_\alpha\}_{\alpha \in \mathbb{N}^d}\) of a Hilbert space \(H\), we define \(d\)-variable weighted shift \(S_w = (S_1, \ldots, S_d)\) as

\[S_je_\alpha := w^{(j)}_\alpha e_{\alpha+e_j} \quad (1 \leq j \leq d).\]

For convenience, we refer to \(S_w\) as the classical multishift. Notice that \(S_j\) commutes with \(S_k\) if and only if \(w^{(j)}_\alpha w^{(k)}_{\alpha+e_j} = w^{(k)}_{\alpha+e_j} w^{(j)}_\alpha\) for all \(\alpha \in \mathbb{N}^d\). Moreover, \(S_1, \ldots, S_d\) are bounded if and only if

\[
\sup \{ |w^{(j)}_\alpha| : 1 \leq j \leq d, \alpha \in \mathbb{N}^d \} < \infty.
\]

In this text, we always assume that the multisequence \(w\) consists of positive numbers and satisfies (1.8).

Let \(S_w\) be a classical multishift. Define \(\gamma_\alpha := \|S_w^\alpha e_0\| (\alpha \in \mathbb{N}^d)\), where 0 is the \(d\)-tuple in \(\mathbb{N}^d\) with all entries being zero. Consider the Hilbert space \(H^2(\gamma)\) of formal power series

\[f(z) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha\]

such that

\[\|f\|_{H^2(\gamma)}^2 := \sum_{\alpha \in \mathbb{N}^d} |a_\alpha|^2 \gamma_\alpha^2 < \infty.\]

It is worth noting that \(S_w\) is unitarily equivalent to the \(d\)-tuple \(M_z = (M_{z_1}, \ldots, M_{z_d})\) of multiplication by the co-ordinate functions \(z_1, \ldots, z_d\) on the corresponding space \(H^2(\gamma)\) ([69 Proposition 8]).

Let us discuss some basic examples of classical multishifts.

**Example 1.2.1.** For integers \(a, d > 0\), let \(H_{a,d}\) be the reproducing kernel Hilbert space of holomorphic functions on the open unit ball \(B^d\) with reproducing kernel

\[\kappa_{H_{a,d}}(z, w) = \frac{1}{(1 - \langle z, w \rangle)^a} \quad (z, w \in B^d).\]

The multiplication \(d\)-tuple \(M_z\) on \(H_{a,d}\) is unitarily equivalent to the weighted shift \(d\)-tuple \(S_{w,a}\) with weight multisequence

\[w^{(j)}_{\alpha,a} = \sqrt{\frac{\alpha_j + 1}{|\alpha| + a}} \quad (\alpha \in \mathbb{N}^d, j = 1, \ldots, d),\]

(see [56 Proof of Lemma 4.4]). The spaces \(H_{d,d}, H_{d+1,d}, H_{1,d}\) are commonly known as the Hardy space \(H^2(\partial B^d)\), the Bergman space \(A^2(B^d)\), the Drury-Arveson space \(H_d^2\) respectively. The associated classical multishifts \(S_{w,d}, S_{w,d+1}, S_{w,1}\) are referred to as the Szegö d-shift, the Bergman d-shift, the Drury-Arveson d-shift respectively.
For ready reference, we record the following proposition about various spectral parts of $S_{w,a}$ (see [57 Proposition 2.6] and [34 Theorem 3.4]).

**Proposition 1.2.2.** Let $S_{w,a}$ be as defined in Example 1.2.1. Then

$$
\sigma(S_{w,a}) = cl(\mathbb{R}^d), \quad \sigma_p(S_{w,a}) = \emptyset, \quad \sigma_p(S_{w,a}^*) = \mathbb{R}^d, \quad \sigma_e(S_{w,a}) = \partial \mathbb{R}^d = \sigma_l(S_{w,a}).
$$

We will investigate later the so-called tree analogs of $S_{w,a}$ (refer to Section 1.4).

### 1.3. Weighted Shifts on Directed Trees

In this section, we recall some basic concepts from the theory of directed graphs which will be frequently used in the subsequent chapters. The reader is referred to R. Diestel [51] for a detailed exposition on graph theory (refer also to [65] for a brief account of the theory of directed trees).

A directed graph is a pair $\mathcal{T} = (V, E)$, where $V$ is a nonempty set and $E$ is a nonempty subset of $V \times V \setminus \{(v, v): v \in V\}$. An element of $V$ (resp. $E$) is called a vertex (resp. an edge) of $\mathcal{T}$. A finite sequence $\{v_i\}_{i=1}^n$ of distinct vertices is said to be a circuit in $\mathcal{T}$ if $n \geq 2$, $(v_i, v_{i+1}) \in E$ for all $1 \leq i \leq n - 1$ and $(v_n, v_1) \in E$. We say that two distinct vertices $u$ and $v$ of $\mathcal{T}$ are connected by a path if there exists a finite sequence $\{v_i\}_{i=1}^n$ of distinct vertices of $\mathcal{T}$ $(n \geq 2)$ such that $u = v_1$, $v_n = v$ and $(v_i, v_{i+1})$ or $(v_{i+1}, v_i) \in E$ for all $1 \leq i \leq n - 1$. A directed graph $\mathcal{T}$ is said to be connected if any two distinct vertices of $\mathcal{T}$ can be connected by a path in $\mathcal{T}$.

For a subset $W$ of $V$, define

$$
\text{Chi}(W) := \bigcup_{u \in W} \{v \in V: (u, v) \in E\}.
$$

One may define inductively $\text{Chi}^{(n)}(W)$ for $n \in \mathbb{N}$ as follows:

$$
\text{Chi}^{(n)}(W) := \begin{cases} 
W & \text{if } n = 0, \\
\text{Chi}(\text{Chi}^{(n-1)}(W)) & \text{if } n \geq 1.
\end{cases}
$$

Given $v \in V$, we write $\text{Chi}(v) := \text{Chi}^{(1)}(\{v\})$, $\text{Chi}^{(n)}(v) := \text{Chi}^{(n)}(\{v\})$. A member of $\text{Chi}(v)$ is called a child of $v$. The descendents of a vertex $v \in V$ is given by

$$
\text{Des}(v) := \bigcup_{n=0}^{\infty} \text{Chi}^{(n)}(v).
$$

For a given vertex $v \in V$, consider the set $\text{Par}(v) := \{u \in V: (u, v) \in E\}$ (set of “generalized” parents). If $\text{Par}(v)$ is singleton, then the unique vertex in $\text{Par}(v)$ is called the parent of $v$, which we denote by $\text{par}(v)$. Let the subset $\text{Root}(\mathcal{T})$ of $V$ be defined as

$$
\text{Root}(\mathcal{T}) := \{v \in V: \text{Par}(v) = \emptyset\}.
$$

Then an element of $\text{Root}(\mathcal{T})$ is called a root of $\mathcal{T}$. If $\text{Root}(\mathcal{T})$ is singleton, then its unique element is denoted by $\text{root}$. We set $V^o := V \setminus \text{Root}(\mathcal{T})$. A directed graph $\mathcal{T} = (V, E)$ is called a directed tree if $\mathcal{T}$ has no circuits, $\mathcal{T}$ is connected and each vertex $v \in V^o$ has a unique parent.

**Remark 1.3.1.** It is well-known that every directed tree has at most one root [65 Proposition 2.1.1] (see Figure 1.1).

The following example is borrowed from [65 Chapter 6].
A directed graph $T_{n_0,k_0} = (V,E)$ is defined as follows:

$$
V = \{-1, \cdots, -k_0\} \cup \mathbb{N},
$$

$$
E = \{(j,j+1) : j = -k_0, \cdots, -1\} \cup \{(0,j) : j = 1, \cdots, n_0\}
$$

$$
\cup \cup^{n_0}_{j=1} \{(j + (l-1)n_0, j + ln_0) : l \geq 1\}.
$$

(see Figures 1.2 and 1.3 for the cases $(n_0,k_0) = (1,0)$ and $(n_0,k_0) = (2,0)$ respectively).

A directed graph $T$ is said to be

(i) **rooted** if it has a unique root.

(ii) **locally finite** if $\text{card}(\text{Chi}(u))$ is finite for all $u \in V$.

(iii) **leafless** if every vertex has at least one child.

Let $T = (V,E)$ be a directed tree and let $l^2(V)$ stand for the Hilbert space of square summable complex functions on $V$ equipped with the standard inner product. Note that the set $\{e_u\}_{u \in V}$ is an orthonormal basis of $l^2(V)$, where $e_u \in l^2(V)$ is the indicator function of $\{u\}$. Given a system $\lambda = \{\lambda_v\}_{v \in V^o}$ of nonzero complex numbers, we define the **weighted shift operator** $S_\lambda$ on $T$ with weights $\lambda$ by

$$
D(S_\lambda) := \{f \in l^2(V) : \Lambda_T f \in l^2(V)\},
$$

$$
S_\lambda f := \Lambda_T f, \quad f \in D(S_\lambda),
$$

where $\Lambda_T$ is the mapping defined on complex functions $f$ on $V$ by

$$
(\Lambda_T f)(v) := \begin{cases} 
\lambda_v \cdot f(\text{par}(v)) & \text{if } v \in V^o, \\
0 & \text{if } v \text{ is a root of } T.
\end{cases}
$$

**Example 1.3.2.** For a positive integer $n_0$ and $k_0 \in \mathbb{N}$, we define the directed tree $T_{n_0,k_0} = (V,E)$ as follows:

$$
V = \{-1, \cdots, -k_0\} \cup \mathbb{N},
$$

$$
E = \{(j,j+1) : j = -k_0, \cdots, -1\} \cup \{(0,j) : j = 1, \cdots, n_0\}
$$

$$
\cup \cup^{n_0}_{j=1} \{(j + (l-1)n_0, j + ln_0) : l \geq 1\}.
$$

(see Figures 1.2 and 1.3 for the cases $(n_0,k_0) = (1,0)$ and $(n_0,k_0) = (2,0)$ respectively).
Unless stated otherwise, \( \{ \lambda_v \}_{v \in V^*} \) consists of nonzero complex numbers and \( S_\lambda \) belongs to \( B(l^2(V)) \). It may be concluded from Proposition 3.1.7] that \( S_\lambda \) is an injective weighted shift on \( \mathcal{T} \) if and only if \( \mathcal{T} \) is leafless.

In what follows, we always assume that all the directed trees considered in the remaining part of this paper are leafless.

Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \). Then

\[
V = \bigcup_{n=0}^{\infty} \text{Chi}^{(n)}(\text{root}) \quad \text{(disjoint union)} \tag{1.9}
\]

(Corollary 2.1.5]). For \( u \in V \), let \( \alpha_u \) denote the unique integer in \( \mathbb{N} \) (to be referred to as the depth of \( u \) in \( \mathcal{T} \)) such that \( u \in \text{Chi}^{(\alpha_u)}(\text{root}) \). We use the convention that \( \text{Chi}^{(j)}(\text{root}) = \emptyset \) if \( j < 0 \). Similar convention holds for \( \text{par} \). The branching index \( k_\mathcal{T} \in \mathbb{N} \cup \{ \infty \} \) of a rooted directed tree \( \mathcal{T} \) is defined as

\[
k_\mathcal{T} := \begin{cases} 1 + \sup \{ \alpha_w : w \in V_\prec \} & \text{if } V_\prec \text{ is nonempty,} \\ 0 & \text{if } V_\prec \text{ is empty,} \end{cases}
\]

where \( V_\prec := \{ u \in V : \text{card(Chi}(u)) \geq 2 \} \). If \( V_\prec \) is finite, then \( k_\mathcal{T} \) is necessarily finite but converse is not true in general [33 Remark 2].

Remark 1.3.3. Let \( \mathcal{T}_{n_0,k_0} \) be as discussed in Example 1.3.2. Note that \( \mathcal{T}_{n_0,k_0} \) is a locally finite, rooted directed tree with branching index

\[
k_\mathcal{T} = \begin{cases} k_0 + 1 & \text{if } n_0 \geq 2, \\ 0 & \text{otherwise}. \end{cases}
\]

If \( S_\lambda \) is a left-invertible weighted shift on a rooted directed tree, then \( S_\lambda \) has wandering subspace property. This fact was recorded in [33 Theorem 2.7(iii)]. But it turns out that this is a general nature of a (bounded) weighted shift on a rooted directed tree and the left-invertibility is no longer required. We illustrate this fact in the following proposition.

Proposition 1.3.4. Let \( \mathcal{T} = (V, E) \) be a rooted directed tree and \( S_\lambda \in B(l^2(V)) \) be a weighted shift on \( \mathcal{T} \). Set \( E := \ker S_\lambda^* \). Then

\[
\bigvee_{k \in \mathbb{N}} S_\lambda^k(E) = l^2(V). \tag{1.10}
\]

Proof. Set \( M := \bigvee_{k \in \mathbb{N}} S_\lambda^k(E) \). We claim that \( e_v \in M \) for all \( v \in \text{Chi}^{(n)}(\text{root}) \) and for all \( n \in \mathbb{N} \). We prove this by induction on \( n \). Recall from the Proposition 3.5.1(ii)], that

\[
E = [e_{\text{root}}] \oplus \bigoplus_{v \in V} (l^2(\text{Chi}(v)) \ominus [\Gamma_v]), \tag{1.11}
\]

where \( \Gamma_v : \text{Chi}(v) \to \mathbb{C} \) is given by \( \Gamma_v = \sum_{u \in \text{Chi}(v)} \lambda_u e_u = S_\lambda^1 e_v \). Clearly, \( e_{\text{root}} \in M \).

Thus the claim holds true for \( n = 0 \). Suppose it is true for some \( n \in \mathbb{N} \). That is, \( e_u \in M \) for all \( u \in \text{Chi}^{(n+1)}(\text{root}) \). Let \( v \in \text{Chi}^{(n+1)}(\text{root}) \). Then \( v \in \text{Chi}(u) \) for some \( u \in \text{Chi}^{(n)}(\text{root}) \). By the induction hypothesis, \( e_u \in M \). Since \( M \) is \( S_\lambda \)-invariant, \( S_\lambda e_u = \Gamma_u e_u \in M \) and hence \( [\Gamma_u] \subseteq M \). Further, as \( E \subseteq M \), \( l^2(\text{Chi}(u)) \ominus [\Gamma_u] \subseteq M \). Thus \( l^2(\text{Chi}(u)) \subseteq M \), which in turn implies that \( e_v \in M \). Thus the claim stands verified. By (1.9), it follows that \( e_v \in M \) for all \( v \in V \), and hence, \( l^2(V) \subseteq M \). Thus (1.10) stands true. \( \square \)
1.4. OVERTURE

The argument above relies completely on the formula (1.11) for the kernel of $S^*_\lambda$. Clearly, this formula is associated with a system of linear equations corresponding to vertices from the branching set. In case of several variables, this correspondence becomes highly involved. This is one of the difficulties in the derivation of the wandering subspace property in several variables (see Theorem 4.0.1).

1.4. Overture

In this section, we briefly discuss some important aspects of this work. The exposition here is far from being complete, but it conveys some of the essential ideas presented in this text. Motivated by [34, Question 4.7] about the classification of so-called spherical tuples of higher multiplicity, we construct tree analogs of the multiplication $d$-tuples $M_{z,a}$ as discussed in Example 1.2.1. We outline this construction as follows.

Consider the directed Cartesian product $\mathcal{T} = (V,E)$ of locally finite, rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$ of finite joint branching index and let $S_{\lambda,\varepsilon_a}$ denote the multishift on $\mathcal{T}$ with weights given by

$$\lambda^{(j)}_w = \frac{1}{\text{card}(\text{Chi}_j(v))} \sqrt{\frac{1}{\alpha_v} + 1}$$

for $w \in \text{Chi}_j(v)$, $v \in V$ and $j = 1, \cdots, d$.

Here $a$ is a positive integer and $\alpha_u \in \mathbb{N}^d$ denotes the depth of $u \in V$ in $\mathcal{T}$. It turns out that the multishift $S_{\lambda,\varepsilon_a}$ is unitarily equivalent to multiplication $d$-tuple $M_{z,a}$ acting on reproducing kernel Hilbert space $H_{a,d}$ of $E$-valued holomorphic functions on the open unit ball in $\mathbb{C}^d$, where $E$ denotes the joint kernel of $S^*_\lambda e_a$.

The associated reproducing kernel $\kappa_{\mathcal{H}_{a,d}}: \mathbb{B}^d \times \mathbb{B}^d \to B(E)$ is given by

$$\kappa_{\mathcal{H}_{a,d}}(z,w) = \frac{1}{(1 - \langle z, w \rangle)^{d}} P_{[e_{\text{root}}]} + \sum_{F \in \mathcal{P}} \sum_{u \in \Omega_F} \kappa_{u,F}(z,w),$$

where

$$\kappa_{u,F}(z,w) = \sum_{\alpha \in \mathbb{N}^d} \left( \frac{\alpha_u!}{(\alpha_u + a)!} \right) \left( \prod_{j=0}^{\vert \alpha \vert - 1} (\vert \alpha_u \vert + a + j) \right) z^a \overline{w}^a P_{L_{u,F}}$$

with $P_{\mathcal{M}}$ being the orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{M}$ of $\mathcal{H}$. We refer the reader to Theorem 5.2.6 for a precise statement.

Remark 1.4.1. In case $\mathcal{T}_j = \mathcal{T}_{1,0}$ for $j = 1, \cdots, d$, only first series in $\kappa_{\mathcal{H}_{a,d}}(z,w)$ survives, and hence we obtain the kernel

$$\frac{I_E}{(1 - \langle z, w \rangle)^a} (z, w \in \mathbb{B}^d).$$

Let us try to understand the above formula for $\kappa_{\mathcal{H}_{a,d}}(z,w)$. The following decomposition of the joint kernel $E$ of $S^*_\lambda e_a$ is useful in this regard:

$$E = [e_{\text{root}}] + \bigoplus_{F \in \mathcal{P}} \bigoplus_{u \in \Omega_F} L_{u,F}.$$
with certain system of linear equations related to $S_{\lambda e_a}$ (the reader is referred to Chapters 4 and 5 for a detailed discussion). It is worth noting that the spaces $\mathcal{H}^*_{a,d}$ are unnoticed even in dimension $d = 1$. Indeed, in this case, the reproducing kernel

$$k_{\mathcal{H}^*_{a,1}}(z, w) = \frac{1}{(1 - z \overline{w})^a} P_{\{e_{\text{root}}\}}$$

$$+ \sum_{v \in V, w} \sum_{n=0}^{\infty} \frac{\prod_{v \in V} z^n w^n P_{\{e_{\text{root}}\} \subseteq [\Gamma_v]} (z, w \in \mathbb{D}),}$$

where $V_\prec$ denotes the set of branching vertices of $V$. One can rewrite this formula using the hypergeometric function $2F_1(a, b, c, t)$ [Pg 217]:

$$k_{\mathcal{H}^*_{a,1}}(z, w) = 2F_1(a, 1, 1, z \overline{w}) P_{\{e_{\text{root}}\}}$$

$$+ \sum_{v \in V} 2F_1(\alpha_v + a + 1, 1, \alpha_v + 2, z \overline{w}) P_{\{e_{\text{root}}\} \subseteq [\Gamma_v]} (z, w \in \mathbb{D}).$$

An alternative verification of this formula (based on Shimorin’s analytic model) will be given in Chapter 5.

**Remark 1.4.2.** We analyze below the cases in which $a = 1$ and $a = 2$ in the one-dimensional case. Note that $k_{\mathcal{H}^*_{1}}$ is the Cauchy kernel $\frac{f_z}{1 - z \overline{w}}$ while $k_{\mathcal{H}^*_{2}}$ is given by

$$k_{\mathcal{H}^*_{2}}(z, w) = \sum_{n=0}^{\infty} \frac{\prod_{v \in V} z^n w^n}{(n + 1) P_{\{e_{\text{root}}\}}}$$

$$+ \sum_{v \in V} \sum_{n=0}^{\infty} \frac{\prod_{v \in V} z^n w^n P_{\{e_{\text{root}}\} \subseteq [\Gamma_v]} (z, w \in \mathbb{D}),}$$

Note that $k_{\mathcal{H}^*_{2}} = \frac{1}{(1 - z \overline{w})^2}$ in case $\mathcal{F} = \mathcal{F}_{1,0}$.

It turns out that $S_{\lambda e_a}$ is finitely multicyclic, essentially normal $d$-tuple with Taylor spectrum being equal to the closed unit ball $cl(B^d)$. However, we would like to emphasize here that $S_{\lambda e_a}$ are, in general, not unitarily equivalent to orthogonal direct sums of any number of copies of the classical multishifts $S_{\lambda e_a}$. For instance, in case $d = 1$ and $a = 2$, the defect operator $I - 2S_{w,a}S_{w,a}^* + S_{w,a}^2 S_{w,a}^2 + S_{w,a}^2 S_{w,a}$ is always an orthogonal projection of rank 1 [Pg 618]. On the other hand, if $v \in V^\circ$ is such that $\text{card}(\text{sib}(\text{par}(v))) = 1$ and $s_v := \frac{1}{\text{card}(\text{sib}(v))} < 1$, then

$$\langle (I - 2S_{\lambda e_a}S_{\lambda e_a}^* + S_{\lambda e_a}^2 S_{\lambda e_a}^2)^j e_v, e_v \rangle = (1 - s_v)^j$$

for $j = 1, 2$, which shows that $I - 2S_{\lambda e_a}S_{\lambda e_a}^* + S_{\lambda e_a}^2 S_{\lambda e_a}^2$ is not even idempotent.

We conclude this chapter with a brief description of the layout of the present work. In Chapter 2, we discuss the theory of product of directed graphs in the context of directed trees. The motivation for this chapter comes from the theory of multishifts with which we are primarily concerned. In particular, we pay attention to two important notions, namely, directed Cartesian product and tensor product of directed trees. We will see the significance of the notion of tensor product of directed trees in the context of so-called spherically balanced multishifts later in Chapter 5.
In Chapter 3, we formally introduce the notion of multishifts $S_\lambda$ on directed Cartesian product $\mathcal{T}$ of finitely many rooted directed trees. Apart from various elementary properties of multishifts, we reveal its relation with the shift operator arising from directed semi-tree structure of $\mathcal{T}$ as ensured in Chapter 2. The later half of this chapter deals with spectral properties of multishifts $S_\lambda$ on $\mathcal{T}$. A particular attention is given to circularity and analyticity of $S_\lambda$. Indeed, $S_\lambda$ turns out to be strongly circular and separately analytic. These properties are then used to show that the point spectrum of $S_\lambda$ is empty and the Taylor spectrum is Reinhardt. Further, we obtain a matrix decomposition of 2-variable multishifts and discuss some of its consequences to spectral theory. In particular, we compute essential spectra for a family of multishifts.

Chapter 4 is devoted to the description of the joint kernel of $S_\lambda^*$. This in turn relies on decompositions of vertex set of product of directed trees and that of the underlying Hilbert space. It turns out that the problem of computing the joint kernel $\ker S_\lambda^*$ of $S_\lambda^*$ is equivalent to solving a system of (possibly infinitely many) linear equations. We illustrate this with the help two instructive examples in which $\ker S_\lambda^*$ is explicitly computed. The description of $\ker S_\lambda^*$ enables to derive the wandering subspace property for $S_\lambda$ on $\mathcal{T}$. It is to be noted that the situation gets far simpler in case of either one variable weighted shifts or classical multishifts. As a consequence, we obtain a multivariable counterpart of Shimorin’s model in this context, and use it to show that these multishifts belong to the Cowen-Douglas class.

In Chapter 5, we discuss two notions of balanced multishifts, namely, spherical and toral. We use the classification of torally balanced multishifts to obtain a local analog of von Neumann’s inequality. The classification of spherically balanced multishifts is given in terms of certain integral representations. Unlike the classical case $31$, several Reinhardt measures appear in this characterization. In the classification of spherically balanced multishifts, the notion of tensor product $\mathcal{T}^\otimes$ of directed trees appears naturally. Indeed, various properties of $S_\lambda$ on $\mathcal{T}$ are reflected in the corresponding properties of the one variable shift on the component of $\mathcal{T}^\otimes$ containing root. This correspondence allows us, in particular, to compute the spectral radius of Taylor spectrum and the inner spectral radius of left spectrum for $S_\lambda$. In this chapter, we also discuss special classes of joint subnormal and joint hyponormal multishifts $S_\lambda$ on $\mathcal{T}$. In particular, we characterize these classes within the class of spherically balanced multishifts. We illustrate these results with a family of examples which can be thought of as tree analogs of the multiplication tuples on the reproducing kernel Hilbert spaces associated with the kernels $\frac{1}{(1-\langle z, w \rangle)^n} (z, w \in \mathbb{B}^d, a > 0)$.
CHAPTER 2

Product of Directed Trees

In this chapter, we discuss two well-studied notions of product of directed trees, namely, the directed Cartesian product and the tensor product ([84], [90], [77], [60], [54]). These notions can certainly be introduced in the general context of (directed) graphs. However, since the main objects of the present study are multishifts on product of directed trees, we confine ourselves to directed trees.

2.1. Directed Cartesian Product of Directed Trees

The definition of the directed Cartesian product of two directed graphs has been introduced and studied by G. Sabidussi [84] (refer also to [54]). This notion readily generalizes to the case of finitely many directed trees as given below.

Definition 2.1.1. Let \( d \) be a positive integer and let \( T_j = (V_j, E_j) \) \((j = 1, \cdots, d)\) be a collection of directed trees. The directed Cartesian product of \( T_1, \cdots, T_d \) is a directed graph \( T = (V, E) \), where

\[
V := V_1 \times \cdots \times V_d
\]

and

\[
E := \left\{ (v, w) \in V \times V : \text{there is a positive integer } k \in \{1, \cdots, d\} \text{ such that } v_j = w_j \text{ for } j \neq k \text{ and the edge } (v_k, w_k) \in E_k \right\},
\]

where we adhere to the convention that \( v \in V = V_1 \times \cdots \times V_d \) is always understood as \( v = (v_1, \cdots, v_d) \) with \( v_j \in V_j \) for \( j = 1, \cdots, d \). We sometimes use the notation \( T_1 \times \cdots \times T_d \) for the directed Cartesian product \( T \) of \( T_1, \cdots, T_d \).

Remark 2.1.2. Note that \( E \) is precisely the collection of edges \((v, w)\) such that \( w_k \) is a child of \( v_k \) for some \( k \) and \( w_j = v_j \) for all \( j \neq k \).

Remark 2.1.3. In case \( d \geq 2 \), \( T = (V, E) \) is never a directed tree. Indeed, \( \text{card}(\text{Chi}(u) \cap \text{Chi}(v)) \geq 1 \) for some \( u, v \in V \) with \( u \neq v \). This can be seen as follows. Since \( T_1, \cdots, T_d \) are leafless, for any \( w = (w_1, \cdots, w_d) \in V \), consider \( u = (u_1, w_2, \cdots, w_d) \) and \( v = (w_1, u_2, w_3, \cdots, w_d) \), where \( u_j \in \text{Chi}(w_j) \) for \( j = 1, 2 \). Note that

\[
(u_1, u_2, w_3, \cdots, w_d) \in \text{Chi}(u) \cap \text{Chi}(v).
\]

Remark 2.1.4. For \( j = 1, \cdots, d \), let \( T_j \) be a rooted directed tree with root denoted by \( \text{root}_j \). Then the directed Cartesian product \( T \) of \( T_1, \cdots, T_d \) is a rooted direct graph with root given by \( \text{root} = (\text{root}_1, \cdots, \text{root}_d) \in V \).

We discuss below three basic examples of directed Cartesian product.

Example 2.1.5. Let \( T_{1,0} \) be as discussed in Example 1.3.2 and let \( T_j = T_{1,0} \) for all \( j = 1, \cdots, d \). The directed Cartesian product \( T = T_{1,0} \) of \( T_1, \cdots, T_d \) is
The directed graph $T$ discussed above is the $d$-finite Bargmann graph in disguise. The later one was introduced in [79, Section 3].

**Example 2.1.6.** Let $T_{1,0}$, $T_{2,0}$ be as discussed in Example 1.3.2. Then the directed Cartesian product $T = T_{1,0} \times T_{2,0}$ of $T_{2,0}$ and $T_{1,0}$ is given by $T = (V, E)$, where $V = \mathbb{N} \times \mathbb{N}$ and $(m, n, (k, l)) \in E$ if and only if either $m = k$ and $l = n + 1$, or $n = l$ and

$$k = \begin{cases} m + 2 & \text{if } m \neq 0, \\ 1 \text{ or } 2 & \text{otherwise} \end{cases}$$

(see Figure 2.2).

**Example 2.1.7.** Let $T_{2,0}$ be as discussed in Example 1.3.2. Then the directed Cartesian product of $T_{2,0}$ with itself is given by $T = (V, E)$, where $V = \mathbb{N} \times \mathbb{N}$ and $(m, n, (k, l)) \in E$ if and only if either $m = k$ and

$$l = \begin{cases} n + 2 & \text{if } n \neq 0, \\ 1 \text{ or } 2 & \text{otherwise}, \end{cases}$$

or $n = l$ and

$$k = \begin{cases} m + 2 & \text{if } m \neq 0, \\ 1 \text{ or } 2 & \text{otherwise} \end{cases}$$

(see Figure 2.3).
2.1. DIRECTED CARTESIAN PRODUCT OF DIRECTED TREES

**Figure 2.2.** Directed Cartesian Product $\mathcal{I} = \mathcal{I}_{2,0} \times \mathcal{I}_{1,0}$

**Figure 2.3.** Directed Cartesian Product $\mathcal{I} = \mathcal{I}_{2,0} \times \mathcal{I}_{2,0}$

**Definition 2.1.8.** Let $\mathcal{I} = (V, E)$ be the directed Cartesian product of directed trees $\mathcal{I}_1, \cdots, \mathcal{I}_d$. For $j = 1, \cdots, d$ and $v \in V$, we set

$$\text{Chi}_j(v) := \{ w \in V : w_j \in \text{Chi}(v_j) \text{ and } w_k = v_k \text{ for } k \neq j \}.$$ 

Further, for $W \subseteq V$, we define

$$\text{Chi}_j(W) := \bigcup_{w \in W} \text{Chi}_j(w).$$
For \( k \in \mathbb{N} \), we denote \( \text{Chi}_j \cdots \text{Chi}_j(W) \) by \( \text{Chi}_j^{(k)}(W) \), where we understand that \( \text{Chi}_j^{(0)}(W) = W \). Further, for \( \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d \) and \( W \subseteq V \), we define

\[
\text{Chi}^{\langle \alpha \rangle}(W) := \text{Chi}_1^{(\alpha_1)} \cdots \text{Chi}_d^{(\alpha_d)}(W).
\]

If \( W = \{v\} \) for some \( v \in V \), then we use the simpler notation \( \text{Chi}^{\langle \alpha \rangle}(v) \) for \( \text{Chi}^{\langle \alpha \rangle}(\{v\}) \).

**Remark 2.1.9.** It may be concluded from Remark 2.1.2 that

\[
\text{Chi}(v) = \{w \in V : (v, w) \in E\} = \bigcup_{j=1}^d \text{Chi}_j(v).
\]

Further, for \( j = 1, \ldots, d \), \( \text{Chi}^{\langle \alpha \rangle}(v) = \text{Chi}_j(v) \).

The following lemma enriches the directed graph \( T = (V, E) \) with a tree-like structure.

**Lemma 2.1.10.** Let \( T = (V, E) \) be the directed Cartesian product of directed trees \( T_1, \ldots, T_d \). Then we have the following:

1. \( \text{Chi}_j \text{Chi}_i(v) = \text{Chi}_i \text{Chi}_j(v) \) for all \( v \in V \) and \( i, j = 1, \ldots, d \).
2. For each \( \alpha \in \mathbb{N}^d \) and \( v, w \in V \),

\[
\text{Chi}^{\langle \alpha \rangle}(v) \cap \text{Chi}^{\langle \alpha \rangle}(w) = \emptyset \text{ if } v \neq w.
\]
3. For each \( \alpha, \beta \in \mathbb{N}^d \) and \( v \in V \),

\[
\text{Chi}^{\langle \alpha \rangle}(v) \cap \text{Chi}^{\langle \beta \rangle}(v) = \emptyset \text{ if } \alpha \neq \beta.
\]
4. For any \( n \in \mathbb{N} \) and \( v \in V \),

\[
\text{Chi}^{(n)}(v) = \bigsqcup_{|\alpha| \leq n} \text{Chi}^{\langle \alpha \rangle}(v).
\]
5. For each \( m, n \in \mathbb{N} \) and \( v \in V \),

\[
\text{Chi}^{(m)}(v) \cap \text{Chi}^{(n)}(v) = \emptyset \text{ if } m \neq n.
\]
6. If \( T_1, \ldots, T_d \) are rooted directed trees, then

\[
\bigsqcup_{j \in \mathbb{N}} \text{Chi}_j^{(\text{root})} = V = \bigsqcup_{\alpha \in \mathbb{N}^d} \text{Chi}^{\langle \alpha \rangle}(\text{root}).
\]

**Proof.** Note that the conclusion in (i) follows from

\[
\text{Chi}_j \text{Chi}_i(v) = \begin{cases} 
\{u \in V : u_i \in \text{Chi}(v_i), u_j \in \text{Chi}(v_j) \text{ and } u_k = v_k \text{ for } k \neq i, j\} & i \neq j, \\
\{u \in V : u_i \in \text{Chi}^{(2)}(v_i) \text{ and } u_k = v_k \text{ for } k \neq i\} & i = j.
\end{cases}
\]

for \( i, j = 1, \ldots, d \). To see (ii), let \( u \in \text{Chi}^{\langle \alpha \rangle}(v) \cap \text{Chi}^{\langle \alpha \rangle}(w) \). Then \( u_j \in \text{Chi}^{\langle \alpha_j \rangle}(v_j) \cap \text{Chi}^{\langle \alpha_j \rangle}(w_j) \) for all \( j = 1, \ldots, d \). In other words, \( \text{par}^{\langle \alpha \rangle}(u_j) = v_j \) and \( \text{par}^{\langle \alpha \rangle}(u_j) = w_j \) for all \( j = 1, \ldots, d \). Since parent is unique, it follows that \( v_j = w_j \) for all \( j = 1, \ldots, d \). Thus \( v = w \). This proves (ii).

Let \( u \in \text{Chi}^{\langle \alpha \rangle}(v) \cap \text{Chi}^{\langle \beta \rangle}(v) \). Then \( u_j \in \text{Chi}^{\langle \alpha_j \rangle}(v_j) \cap \text{Chi}^{\langle \beta_j \rangle}(v_j) \) for all \( j = 1, \ldots, d \). In other words, \( \text{par}^{\langle \alpha \rangle}(u_j) = v_j \) and \( \text{par}^{\langle \beta \rangle}(u_j) = v_j \) for all \( j = 1, \ldots, d \). It may be now concluded from (1.9) that \( \alpha_j = \beta_j \) for all \( j = 1, \ldots, d \). Thus \( \alpha = \beta \). This proves (iii).
We prove (iv) by induction on \( n \). By Remark 2.1.9

\[
\text{Chi}(v) = \bigcup_{j=1}^{d} \text{Chi}_j(v) = \bigcup_{j=1}^{d} \text{Chi}^{(\langle \alpha_j \rangle)}(v) = \bigcup_{\alpha \in \mathbb{N}^d} \text{Chi}^{(\langle \alpha \rangle)}(v).
\]

Thus the result holds for \( n = 1 \). Suppose it is true for some \( n \geq 1 \). Then

\[
\text{Chi}^{(n+1)}(v) = \text{Chi}(\bigcup_{\alpha \in \mathbb{N}^d} \text{Chi}^{(\langle \alpha \rangle)}(v)) = \bigcup_{\alpha \in \mathbb{N}^d} \text{Chi}(\text{Chi}^{(\langle \alpha \rangle)}(v))
\]

\[
= \bigcup_{\alpha \in \mathbb{N}^d} \bigcup_{j=1}^{d} \text{Chi}^{(\langle \alpha_j \rangle)}(\text{Chi}^{(\langle \alpha \rangle)}(v))
\]

\[
= \bigcup_{j=1}^{d} \bigcup_{\alpha \in \mathbb{N}^d} \text{Chi}^{(\langle \alpha_j \rangle, (\langle \alpha \rangle))}(v) = \bigcup_{\alpha \in \mathbb{N}^d} \text{Chi}^{(\langle \alpha \rangle)}(v),
\]

where the last union is disjoint in view of part (iii). Part (v) is now immediate from parts (iii) and (iv).

To see the last part, let \( U := \bigcup_{\alpha \in \mathbb{N}^d} \text{Chi}^{(\langle \alpha \rangle)}(\text{root}) \), \( W := \bigcup_{j \in \mathbb{N}} \text{Chi}^{(\langle j \rangle)}(\text{root}). \) Clearly, \( U \) and \( W \) are subsets of \( V \). By (iv), \( U = W \) and the unions in \( U \) and \( W \) are disjoint. Thus it suffices to check that \( V \subseteq U \). To this end, suppose that \( v = (v_1, \ldots, v_d) \in V \). Then there exists \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) such that \( v_j \in \text{Chi}^{(\langle \alpha_j \rangle)}(\text{root}) \) for \( j = 1, \ldots, d \). This implies that \( v \in \text{Chi}^{(\langle \alpha_1 \rangle \cdots \langle \alpha_d \rangle)}(\text{root}) = \text{Chi}^{(\langle \alpha \rangle)}(\text{root}). \)

**Definition 2.1.11.** Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( v \in V \). The unique \( \alpha_v \in \mathbb{N}^d \) such that

\[
v \in \text{Chi}^{(\langle \alpha_v \rangle)}(\text{root})
\]

will be referred to as the *depth of v in \( \mathcal{T} *(see Lemma 2.1.10(vi)).* The *t*th *generation* \( \mathcal{G}_t \) of \( \mathcal{T} \) is defined as

\[
\mathcal{G}_t := \{ v \in V : |\alpha_v| = t \}.
\]

**Remark 2.1.12.** Note that \( \alpha_v = (\alpha_{v_1}, \ldots, \alpha_{v_d}) \), where \( \alpha_{v_j} \) denotes the depth of the vertex \( v_j \in V_j \) in the directed tree \( \mathcal{T}_j \) for \( j = 1, \ldots, d \). Note further that by Lemma 2.1.10(iv), \( \mathcal{G}_t = \text{Chi}^{(t)}(\text{root}). \)

Let us briefly discuss the notion of co-ordinate parent of a vertex in the directed Cartesian product of directed trees.

**Definition 2.1.13.** Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \). For \( j = 1, \ldots, d \) and \( v \in V \), we set

\[
\text{par}_j(v) := \begin{cases} 
\{ w \in V : w_j = \text{par}(v_j) \text{ and } w_k = v_k \text{ for } k \neq j \} & \text{if } v_j \neq \text{root}_j, \\
\emptyset & \text{otherwise}.
\end{cases}
\]

Further, for \( W \subseteq V \), we define

\[
\text{par}_j(W) := \bigcup_{w \in W} \text{par}_j(w).
\]
For a positive integer \( k \), we denote \( \underbrace{\text{par}_j \cdots \text{par}_j}_{k \text{ times}}(W) \) by \( \text{par}_j^{(k)}(W) \). Moreover, we understand \( \text{par}_j^{(0)}(W) = W \).

For \( \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d \) and \( W \subseteq V \), we define
\[
\text{par}_j^{<\alpha>}(W) := \text{par}_1^{(\alpha_1)} \cdots \text{par}_d^{(\alpha_d)}(W).
\]

In case \( W = \{ v \} \) for some \( v \in V \), we use \( \text{par}_j^{<\alpha>}(v) \) for \( \text{par}_j^{<\alpha>}(\{ v \}) \).

**Remark 2.1.14.** Note that
\[
\text{Par}(v) = \bigcup_{j=1}^d \text{par}_j(v).
\]

In particular, \( \text{card}(\text{Par}(v)) \) is at most \( d \).

**Remark 2.1.15.** Note that \( \text{par}_j \text{par}_i(v) = \text{par}_i \text{par}_j(v) \) for all \( v \in V \). Further, for \( i, j = 1, \cdots, d \) with \( i \neq j \),
\[
\chi_i(\text{par}_j(v)) = \text{par}_j(\chi_i(v)).
\]

Although the directed Cartesian product of directed trees need not be a directed tree, the following result shows that it has many structural similarities of directed tree. In particular, it always admits a directed semi-tree structure in the sense of [79, Definition 2.6]. This has been observed in the special context of Example 2.1.5 in [79, Lemmas 3.1, 3.2, 3.5(ii)].

**Theorem 2.1.16.** Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of directed trees \( \mathcal{T}_1, \cdots, \mathcal{T}_d \). Then

(i) \( \mathcal{T} \) has no circuits.

(ii) \( \mathcal{T} \) is connected.

(iii) \( \mathcal{T} \) can have at most one root.

(iv) \( \text{card}(\chi(u) \cap \chi(v)) \leq 1 \) for every \( u, v \in V \) with \( u \neq v \).

**Proof.** To see (i), on contrary suppose that \( \{ v^{(1)} \}_{1}^{n} \subseteq V \) is a circuit in \( \mathcal{T} \). Hence \( \{ v^{(1)}, v^{(2)} \} \in E \) implies that \( v^{(2)} \in \chi(v^{(1)}) \). Similarly, \( \{ v^{(2)}, v^{(3)} \} \in E \) implies that \( v^{(3)} \in \chi(v^{(2)}) \subseteq \chi^{(2)}(v^{(1)}) \). Consequently, \( v^{(n)} \in \chi^{(n-1)}(v^{(1)}) \). Finally, \( \{ v^{(n)}, v^{(1)} \} \in E \) implies that \( v^{(1)} \in \chi^{(n)}(v^{(1)}) \). By Lemma 2.1.10(iv), \( v^{(1)} \in \chi^{<\alpha>}(v^{(1)}) \) for some \( \alpha \in \mathbb{N}^d \) with \( |\alpha| = n \). This means that \( v^{(1)} \in \chi^{(\alpha)}(v^{(1)}) \) for all \( j = 1, \cdots, d \). Also, since \( |\alpha| = n \), at least one \( \alpha_j \) is nonzero. This is a contradiction to the fact that \( \mathcal{T}_j \) being directed tree has no circuits. Thus \( \mathcal{T} \) has no circuits.

To prove (ii), let \( v, w \in V \). Since each \( \mathcal{T}_j \) \( (1 \leq j \leq d) \) is connected, so for each \( v_j \) and \( w_j \) there is a finite sequence of vertices \( \{ u_{j,k} \}_{k=1}^{n_j} \subseteq V_j \) such that \( u_{j,1} = v_j \), \( u_{j,\alpha_j} = w_j \) and, \( (u_{j,k}, u_{j,k+1}) \) or \( (u_{j,k+1}, u_{j,k}) \) \( \in E_j \) \( (k = 1, \cdots, \alpha_j - 1) \). Let \( \alpha = (\alpha_1, \cdots, \alpha_d) \). We construct a sequence \( \{ u^{(k)} \}_{k=1}^{n} \) of vertices in \( V \) as follows. Put

\[
\begin{align*}
\mathcal{u}^{(1)} &= v, \quad \mathcal{u}^{(2)} = (u_{1,2}, v_2, \cdots, v_d), \cdots, \mathcal{u}^{(\alpha_1)} = (u_{1,\alpha_1}, v_2, \cdots, v_d), \\
\mathcal{u}^{(\alpha_1+1)} &= (u_{1,\alpha_1}, u_{2,2}, v_3, \cdots, v_d), \cdots, \mathcal{u}^{(\alpha_1+\alpha_2)} = (u_{1,\alpha_1}, u_{2,\alpha_2}, v_3, \cdots, v_d), \cdots, \\
\mathcal{u}^{(|\alpha|)} &= (u_{1,\alpha_1}, u_{2,\alpha_2}, \cdots, u_{d,\alpha_d}) = w.
\end{align*}
\]
It is evident from the construction that \((u^{(j)}, u^{(j+1)})\) or \((u^{(j+1)}, u^{(j)})\) ∈ \(\mathcal{E}\). This proves that \(\mathcal{T}\) is connected.

The part (iii) follows immediately from Lemma 2.1.10 vi. Indeed, suppose that \(\mathcal{T}\) has a root \(v \neq \text{root}\). By Lemma 2.1.10 vi, there exists some \(\alpha \in \mathbb{N}^d \setminus \{0\}\) such that \(v \in \text{Chi}^{\leq \alpha \gg}(\text{root})\). But then \(\text{Par}(v) \neq \emptyset\), which contradicts the definition of the root.

Let \(u, v \in V\) and \(u \neq v\). Without loss of generality, we may assume that \(u_1 \neq v_1\). Suppose that \(s, w \in \text{Chi}(u) \cap \text{Chi}(v)\). Then \(s \in \text{Chi}(u) \cap \text{Chi}(w)\) and \(w \in \text{Chi}(v) \cap \text{Chi}(w)\) for some \(1 \leq i, j, k, l \leq d\). Note that \(i \neq k\). For \(i = k\), then \(u_1 \neq v_1\) would imply that \(\text{Chi}(u) \cap \text{Chi}(w) = \emptyset\). Similarly, \(j \neq l\). We treat only the case in which \(i < k\) and \(j < l\). Since \(s \in \text{Chi}(u), s = (u_1, \cdots, u_{i-1}, s_i, u_{i+1}, \cdots, u_d)\), and that \(w \in \text{Chi}(v)\) implies that \(w = (u_1, \cdots, u_{j-1}, w_j, u_{j+1}, \cdots, u_d)\). Let \(\text{par}(u_k) = \hat{u}_k\) and \(\text{par}(u_l) = \hat{u}_l\). Since \(i \neq k\) and \(j \neq l\), it follows from \(\text{par}_j(s) = w = \text{par}_i(w)\) that

\[
(u_1, \cdots, s_i, \cdots, \hat{u}_k, \cdots, u_d) = v = (u_1, \cdots, w_j, \cdots, \hat{u}_l, \cdots, u_d).
\] (2.2)

In case \(i = j\), we obtain \(s_i = w_j\) in view of (2.2). But then as \(s, w \in \text{Chi}(u)\), we must have \(s = w\). Let \(i \neq j\). Then from (2.2), either \(s_i = u_i\) or \(s_i = \hat{u}_i\). As \(s \in \text{Chi}(u), s_i = \hat{u}_i\), and hence \(i = l\). Therefore, \(s = \text{par}_i(u)\). Thus \(s \in \text{Chi}(u) \cap \text{par}_i(u)\). This is not possible. Hence the case \(i \neq j\) cannot occur. This proves that \(s = w\) and hence (iv) stands verified.

**Definition 2.1.17.** Let \(\mathcal{T} = (V, \mathcal{E})\) be the directed Cartesian product of directed trees \(\mathcal{T}_1, \cdots, \mathcal{T}_d\). A vertex \(v = (v_1, \cdots, v_d) \in V\) is called a branching vertex of \(\mathcal{T}\) if \(\text{card}(\text{Chi}(v_j)) \geq 2\) for all \(j = 1, \cdots, d\). The set of all branching vertices of \(\mathcal{T}\) is denoted by \(V_{\prec}\).

**Remark 2.1.18.** If \(V_{\prec}^{(j)}\) is the set of branching vertices of \(\mathcal{T}_j\), then

\[ V_{\prec} = V_{\prec}^{(1)} \times \cdots \times V_{\prec}^{(d)}. \]

**Proposition 2.1.19.** Let \(\mathcal{T} = (V, \mathcal{E})\) be the directed Cartesian product of rooted directed trees \(\mathcal{T}_1, \cdots, \mathcal{T}_d\). If \(\mathcal{T}_j\) has finite branching index \(k_{\mathcal{T}_j}\) for \(j = 1, \cdots, d\), then for \(k_{\mathcal{T}} = (k_{\mathcal{T}_1}, \cdots, k_{\mathcal{T}_d})\), one has

\[ \text{Chi}^{\leq < k_{\mathcal{T}} \gg}(V_{\prec}) \cap V_{\prec} = \emptyset. \]

**Proof.** Assume that each \(\mathcal{T}_j\) has finite branching index \(k_{\mathcal{T}_j}\) and let \(v \in \text{Chi}^{\leq < k_{\mathcal{T}} \gg}(V_{\prec})\). Then \(v \in \text{Chi}_{1}^{(k_{\mathcal{T}_1})} \cdots \text{Chi}_{d}^{(k_{\mathcal{T}_d})}(w)\) for some \(w \in V_{\prec}\). That is, \(v = (v_1, \cdots, v_d)\) such that \(v_j \in \text{Chi}_{j}^{(k_{\mathcal{T}_j})}(w_j)\) with \(w_j \in V_{\prec}^{(j)}\). But then \(v_j \notin V_{\prec}^{(j)}\). Hence \(v \notin V_{\prec}\). This shows that \(\text{Chi}_{1}^{(k_{\mathcal{T}_1})}(V_{\prec}) \cap V_{\prec} = \emptyset\).

The multiindex \(k_{\mathcal{T}} \in \mathbb{N}^d\) appearing in Proposition 2.1.19 will be referred to as the joint branching index of \(\mathcal{T}\). Also, we say that \(\mathcal{T}\) has finite joint branching index if \(k_{\mathcal{T}}\) is finite.

We conclude this section with a brief discussion on the notion of siblings of a vertex.

Let \(\mathcal{T} = (V, \mathcal{E})\) be the directed Cartesian product of rooted directed trees \(\mathcal{T}_1, \cdots, \mathcal{T}_d\). For \(u \in V\) and \(j = 1, \cdots, d\), we set

\[ \text{sib}_j(u) := \begin{cases} \text{Chi}_j(\text{par}_j(u)) & \text{if } u_j \neq \text{root}_j, \\ \emptyset & \text{otherwise}. \end{cases} \]
For $W \subseteq V$, we define $\text{sib}_j(W) := \bigcup_{u \in W} \text{sib}_j(u)$.

**Remark 2.1.20.** Let $1 \leq i, j \leq d$ and $v \in V$. Then $\text{sib}_i \text{sib}_j(v) = \text{sib}_i \text{sib}_j(v)$. Further, $\text{sib}_i \text{sib}_j(v) = \text{sib}_j(v)$.

For future reference, we record the following simple yet useful observation.

**Proposition 2.1.21.** Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Then

$$\text{card}(\text{sib}_i(v)) \text{card}(\text{sib}_j(\text{par}_i(v))) = \text{card}(\text{sib}_j(v)) \text{card}(\text{sib}_i(\text{par}_j(v)))$$

for every $v \in \text{Chi}_i(\text{Chi}_j(w))$, $w \in V$, and $i, j = 1, \ldots, d$.

**Proof.** Clearly, the identity holds for $i = j$. In case $i \neq j$, the conclusion follows from the fact that $\text{card}(\text{sib}_j(\text{par}_i(v))) = \text{card}(\text{sib}(v_j)) = \text{card}(\text{sib}_j(v))$. \qed

## 2.2. Tensor Product of Rooted Directed Trees

In this section, we discuss another notion of the product of two directed trees to be referred to as tensor product. This notion was introduced by P. Weichsel [90] for undirected graphs, and later, it was extended to directed graphs by M. McAndrew [77] (refer also to [60]). In the literature, the tensor product is also known as categorical product, Kronecker product, cardinal product, weak direct product and even Cartesian product. We would like to emphasize that the notions of Cartesian product and tensor product, as discussed in this text, are conceptually different.

The definition of the tensor product of two directed graphs extends naturally in context of finitely many directed trees.

**Definition 2.2.1.** Let $d$ be a positive integer and let $\mathcal{T}_j = (V_j, \mathcal{E}_j)$ ($j = 1, \ldots, d$) be a collection of rooted directed trees. The tensor product of $\mathcal{T}_1, \ldots, \mathcal{T}_d$ is a directed graph $\mathcal{T}^\otimes = (V, \mathcal{E}^\otimes)$, where $V := V_1 \times \cdots \times V_d$ and

$$\mathcal{E}^\otimes := \{(v, w) \in V \times V : (v_j, w_j) \in \mathcal{E}_j \text{ for all } j = 1, \ldots, d\}.$$  

**Caution.** Since the set of vertices of the tensor product is same as that of directed Cartesian product, we use scripted letters throughout the text to distinguish the vertices (except the root) of the tensor product from those of directed Cartesian product.

**Remark 2.2.2.** Note that $(v, w) \in \mathcal{E}^\otimes$ if and only if $w \in \text{Chi}_1 \cdots \text{Chi}_d(v)$. Thus $\text{Chi}(v) = \text{Chi}_1 \cdots \text{Chi}_d(v)$ (with reference to the directed graph $\mathcal{T}^\otimes$). This should be compared with the expression for $\text{Chi}(\cdot)$ (with reference to the directed graph $\mathcal{T}$) as given in Remark 2.1.9.

Let $\mathcal{T}^\otimes = (V, \mathcal{E}^\otimes)$ be the tensor product of rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Define two vertices $u, w \in V$ to be equivalent if either $u = w$ or $u$ and $w$ can be connected by a path in $\mathcal{T}^\otimes$. Note that this defines an equivalence relation. A component of $\mathcal{T}^\otimes$ is an equivalence class corresponding to the above equivalence relation. It turns out that each component of $\mathcal{T}^\otimes$ is a rooted directed tree. We illustrate this in more detail in the following theorem.
Theorem 2.2.3. Let $\mathcal{T}^\otimes = (V,E^\otimes)$ be the tensor product of rooted directed trees $\mathcal{T}_1,\ldots,\mathcal{T}_d$ and let $\mathcal{T}^\otimes_{root} = (V^\otimes,F)$ denote the (unique) component of $\mathcal{T}^\otimes$ that contains root. Let $\mathcal{T}$ be the directed Cartesian product of $\mathcal{T}_1,\ldots,\mathcal{T}_d$. Set

$$\text{Root}^\otimes := \{v \in V : v_j = \text{root}_j \text{ for at least one } j = 1,\ldots,d\}.$$ 

Then the following statements are true.

(i) If $v \in \text{Root}^\otimes$, then there does not exist any $u \in V$ such that $(u,v) \in E^\otimes$.

(ii) For each $v \in V \setminus \text{Root}^\otimes$, there is a unique $u \in V$ such that $(u,v) \in E^\otimes$. In other words, each $v \in V \setminus \text{Root}^\otimes$ has a parent.

(iii) $\mathcal{T}^\otimes$ has no circuits.

(iv) No two distinct vertices in $\text{Root}^\otimes$ can be connected by a path in $\mathcal{T}^\otimes$.

(v) There is a bijective correspondence between the collection of components of $\mathcal{T}^\otimes$ and the elements of $\text{Root}^\otimes$. In particular, $\mathcal{T}^\otimes$ contains countably many components.

(vi) Each component is a rooted directed tree with root coming from $\text{Root}^\otimes$.

(vii) $\mathcal{T}^\otimes$ is locally finite if and only if $\mathcal{T}$ is locally finite.

(viii) $\mathcal{T}^\otimes$ is leafless.

(ix) If $\mathcal{T}$ is of finite joint branching index $k_{\mathcal{T}} = (k_{\mathcal{T}_1},\ldots,k_{\mathcal{T}_d})$, then the branching index $k_{\mathcal{T}^\otimes}^\otimes$ of $\mathcal{T}^\otimes_{root}$ is given by $k_{\mathcal{T}^\otimes}^\otimes = \max\{k_{\mathcal{T}_j} : 1 \leq j \leq d\}$.

(x) For each $v \in V$, there exists $v \in V^\otimes$ such that $|\alpha_v| = \alpha_v$, where $\alpha_v$ is the depth of $v$ in $\mathcal{T}$ and $\alpha_v$ is the depth of $v$ in the directed tree $\mathcal{T}^\otimes_{root}$.

Remark 2.2.4. The conclusion of (v) above is in contrast with the situation occurring in the case of undirected trees, where the tensor product of two undirected trees has exactly two components (see [64, Theorem 5.29]).

Proof. Let $v \in \text{Root}^\otimes$. Then for some $j = 1,\ldots,d$, $v_j = \text{root}_j$. Now, if $u \in V$ such that $(u,v) \in E^\otimes$, then $u_j = \text{par}(v_j)$, which is not possible. This proves (i).

To see (ii), let $v \in V \setminus \text{Root}^\otimes$. Then $v_j \neq \text{root}_j$ for all $j = 1,\ldots,d$. Consider $u \in V$ with $u_j = \text{par}(v_j)$ for all $j = 1,\ldots,d$. Then $(u,v) \in E^\otimes$. This proves the existence as well as the uniqueness of $u$.

To see (iii), suppose there is a finite sequence $\{w^{(i)}\}_{i=1}^n (n \geq 2)$ of distinct vertices such that $(w^{(i)},w^{(i+1)}) \in E^\otimes$ for all $1 \leq i \leq n-1$ and $(w^{(n)},w^{(1)}) \in E^\otimes$. Then $w^{(j)} \in \text{Chi}^{(n)}(w^{(1)})$ for all $1 \leq j \leq d$. This is a contradiction to the fact that $\mathcal{T}_j$ has no circuits.

Let $u$ and $v$ be two distinct vertices in $\text{Root}^\otimes$. Suppose there exists a finite sequence $\{w^{(i)}\}_{i=1}^n \subseteq V (n \geq 2)$ of distinct vertices such that $w^{(1)} = u$, $w^{(n)} = v$ and $(w^{(i)},w^{(i+1)})$ or $(w^{(i+1)},w^{(i)}) \in E^\otimes (i = 1,\ldots,n-1)$. By (i), $(w^{(2)},w^{(1)})$ can not belong to $E^\otimes$. Hence, we must have $(w^{(1)},w^{(2)}) \in E^\otimes$. Thus $w^{(2)}$ belongs to $V \setminus \text{Root}^\otimes$. Next, if $(w^{(3)},w^{(2)}) \in E^\otimes$, then by (ii), $w^{(3)} = w^{(1)}$. This contradicts that the vertices $w^{(1)},\ldots,w^{(n)}$ are distinct. Thus $(w^{(2)},w^{(3)}) \in E^\otimes$. By arguing similarly, one can see that $(w^{(i)},w^{(i+1)}) \in E^\otimes$ for all $1 \leq i \leq n-1$. Thus we get $v_j \in \text{Chi}^{(n-1)}(u_j)$ for each $j = 1,\ldots,d$. Hence $v \notin \text{Root}^\otimes$, which is a contradiction. This proves (iv).

Suppose that $v \in V$ belongs to some component $C$. There exist $k_1,\ldots,k_d \in \mathbb{N}$ such that $v_j \in \text{Chi}^{(k_j)}(\text{root}_j)$ for $j = 1,\ldots,d$. Let $k = \min\{k_j : 1 \leq j \leq d\}$. Then $u = \text{par}_1^{(k)} \cdots \text{par}_d^{(k)}(v) \in \text{Root}^\otimes$, and since $C$ is connected, $u \in C$. Thus each component contains an element from $\text{Root}^\otimes$. Further, (iv) implies that each
component contains at most one element from $\text{Root}^\otimes$. Clearly, as each element of $\text{Root}^\otimes$ belongs to some component, it follows that the correspondence between the collection of components of $\mathcal{T}^\otimes$ and the elements of $\text{Root}^\otimes$ is bijective. This completes the verification of (v).

Let $\mathcal{C}$ be any component of $\mathcal{T}^\otimes$. From (v), there exists a unique $v \in \text{Root}^\otimes$ such that $v \in \mathcal{C}$. Further, (i) implies that $v$ has no parent, in particular, in the subgraph $\mathcal{C}$. Thus $v$ is a root for $\mathcal{C}$. Clearly, $\mathcal{C}$ is connected, and by (ii), each $u \in \mathcal{C}$, with $u \neq v$, has a parent. Also, by (iii), $\mathcal{C}$ has no circuits. This proves (vi).

Let $v$ be a vertex in $V$. By Remark 2.2.2, $\text{Chi}(v) = \text{Chi}_1 \cdots \text{Chi}_d(v)$. It follows that

$$\text{card}(\text{Chi}(v)) = \prod_{j=1}^d \text{card}(\text{Chi}_j(v)) = \prod_{j=1}^d \text{card}(\text{Chi}(v_j)).$$

Thus $\mathcal{T}^\otimes$ is locally finite if and only if $\mathcal{T}$ is locally finite.

The proof of (viii) is an obvious consequence of the fact (observed in Remark 2.2.2) that $\text{Chi}(v) = \text{Chi}_1 \cdots \text{Chi}_d(v)$ for all $v \in V$.

To see (ix), first note that $V^\otimes = \bigcup_{k=0}^\infty \text{Chi}^{(k)}(\text{root}) = \bigcup_{k=0}^\infty \text{Chi}_1^{(k)} \cdots \text{Chi}_d^{(k)}(\text{root}).$

Hence, if $v$ is any vertex of $\mathcal{T}_\text{root}^\otimes$, then there exists a unique $k \in \mathbb{N}$ such that $v_j \in \text{Chi}^{(k)}(\text{root})$ for all $1 \leq j \leq d$. Thus $\alpha_\otimes = \alpha_{v_j}$ for all $1 \leq j \leq d$. Next, observe that $\text{card}(\text{Chi}(v)) \geq 2$ if and only if there exists a positive integer $j$ ($1 \leq j \leq d$) such that $\text{card}(\text{Chi}(v_j)) \geq 2$. With these two observations it is easy to see that

$$\sup \{\alpha_\otimes : \text{card}(\text{Chi}(v)) \geq 2\} = \max_{1 \leq j \leq d} \sup \{\alpha_\otimes : \text{card}(\text{Chi}(v_j)) \geq 2\}.$$

This implies that $k_{\mathcal{T}_\text{root}^\otimes} = \max \{k_{\mathcal{T}_j} : 1 \leq j \leq d\}$. We leave the verification of the last part to the reader. \qed

We will see later that certain weighted shifts acting on the rooted directed tree $\mathcal{T}_\text{root}^\otimes$ arise naturally in an integral representation of so-called spherically balanced multishifts originating from directed Cartesian product of directed trees.

**Remark 2.2.5.** Note that $\mathcal{T}_\text{root}^\otimes$ is an isomorphic invariant for $\mathcal{T}^\otimes$. In fact, let $\{\mathcal{T}_j : 1 \leq j \leq d\}$ be a collection of rooted directed trees with roots $\text{root}_j$ respectively and let $\mathcal{T}^\otimes$ be the tensor product of $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Let $\mathcal{T}_\text{root}^\otimes$ be the component of $\mathcal{T}^\otimes$ containing $\text{root}$. Suppose that $\phi_j$ is an isomorphism between $\mathcal{T}_j$ and $\mathcal{T}_j$. Then the map $(v_1, \cdots, v_d) \mapsto (\phi_1(v_1), \cdots, \phi_d(v_d))$ defines an isomorphism between $\mathcal{T}^\otimes$ and $\mathcal{T}_\text{root}^\otimes$, and hence, $\mathcal{T}_\text{root}^\otimes$ and $\mathcal{T}_\text{root}^\otimes$ become isomorphic.

For future reference, we find it necessary to describe $\mathcal{T}_\text{root}^\otimes$ in the context of Examples 2.1.5, 2.1.6, 2.1.7 (see Figures 2.4, 2.5, 2.6 respectively). In this regard, the reader is advised to recall the definition of $\mathcal{T}_{\alpha_\otimes, k_\otimes}$ as given in Example 1.3.2.

**Example 2.2.6.** Let $\mathcal{T}_{1,0}$ be the tree as discussed in Example 2.1.5. Let $d = 2$ and $\mathcal{T}_j = \mathcal{T}_{1,0}$ for $j = 1, 2$. Then

$$\text{Root}^\otimes = \{(i, 0), (0, j) : i, j \geq 0\}.$$

For $i \geq 1$, the components $\mathcal{C}_{i,(0)} = (V_{i,(0)}, \mathcal{E}_{i,(0)})$ containing $(i, 0)$ are given by

$$V_{i,(0)} = \{(i + k, k) : k \geq 0\} \text{ and } \text{Chi}((i + k, k)) = \{(i + k + 1, k + 1)\}$$
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Figure 2.4. In $T_{1,0} \otimes T_{1,0}$, $\mathcal{T}_{\text{root}}$ is represented with bold-faced edges

Figure 2.5. In $T_{2,0} \otimes T_{1,0}$, $\mathcal{T}_{\text{root}}$ is represented with bold-faced edges

for all $k \geq 0$. Similar description of $C_{(0,j)}$ is obtained for $j \geq 1$. Further, the rooted directed tree $\mathcal{T}_{\text{root}}$, with set of vertices $V^\otimes$, is given by

$$V^\otimes = \{(k, k) : k \geq 0\} \text{ and } \chi((k, k)) = \{(k + 1, k + 1)\}$$

for all $k \geq 0$. Note that $\mathcal{T}_{\text{root}}$ is isomorphic to $T_{1,0}$ via the isomorphism $(k, k) \mapsto k$.

**Example 2.2.7.** Let $T_1 = T_{2,0}$, $T_2 = T_{1,0}$ (see Example 2.1.6). Then

$$\text{Root}^\otimes = \{(i, 0), (0, j) : i, j \geq 0\}.$$

For $i \geq 1$, the components $C_{(i,0)} = (V_{(i,0)}, E_{(i,0)})$ containing $(i, 0)$ are given by

$$V_{(i,0)} = \{(i + 2k, k) : k \geq 0\} \text{ and } \chi((i + 2k, k)) = \{(i + 2k + 2, k + 1)\}$$

for all $k \geq 0$. Further, for $j \geq 1$, the components $C_{(0,j)} = (V_{(0,j)}, E_{(0,j)})$ containing $(0, j)$ are given by

$$V_{(0,j)} = \{(0, j), (1, j + 1), (2, j + 1)\} \cup \{(2k + 1, j + k + 1), (2k, j + k) : k \geq 1\}$$

and $\chi((0, j)) = \{(1, j + 1), (2, j + 1)\}$, $\chi((k, l)) = \{(k + 2, l + 1)\}$ for all $k, l \geq 1$.

Moreover, the rooted directed tree $\mathcal{T}_{\text{root}}$, with set of vertices $V^\otimes$, is given by

$$V^\otimes = \{(2k + 1, k + 1), (2k, k) : k \geq 0\}$$
Figure 2.6. In $T_{2,0} \otimes T_{2,0}$, $T_{\text{root}}$ is represented with bold-faced edges.

and $\text{Chi}((0,0)) = \{(1,1), (2,1)\}$, $\text{Chi}(k,l) = \{(k+2,l+1)\}$ for all $k,l \geq 1$. Note that $T_{\text{root}}$ is isomorphic to $T_{2,0}$ via the isomorphism $(k,l) \mapsto k$.

Example 2.2.8. Let $T_1 = T_{2,0} = T_2$. Then

$$\text{Root}^\otimes = \{(i,0), (0,j) : i,j \geq 0\}.$$ 

For $j \geq 1$, the components $C_{(0,j)} = (V_{(0,j)}, E_{(0,j)})$ containing $(0,j)$ are given by

$$V_{(0,j)} = \{(0,j), (1,j+2), (2,j+2)\} \cup \{(2k+1,j+2k+2), (2k,j+2k) : k \geq 1\}$$

and $\text{Chi}((0,j)) = \{(1,j+2), (2,j+2)\}$, $\text{Chi}(k,l) = \{(k+2,l+2)\}$ for all $k,l \geq 1$.

Similar description for $C_{(i,0)}$ is obtained for all $i \geq 1$. Further, the rooted directed tree $T_{\text{root}}^\otimes$, with set of vertices $V^\otimes$, is given by

$$V^\otimes = \{(k,k) : k \geq 0\} \cup \{(2k-1,2k), (2k,2k-1) : k \geq 1\}$$

and $\text{Chi}((0,k)) = \{(1,1), (2,1)\}$, $\text{Chi}(k,l) = \{(k+2,l+2)\}$ for all $k,l \geq 1$. It can be seen that $T_{\text{root}}^\otimes$ is isomorphic to the rooted directed tree $T_{4,0}$.

We visit the above examples once again in the context of spherically balanced multishifts in Chapter 5.
CHAPTER 3

Multishifts on Product of Rooted Directed Trees

In this chapter, we introduce and study the notion of multishifts on directed Cartesian product of finitely many rooted directed trees. In particular, we discuss some basic properties of multishifts such as boundedness, commutativity, circularity and analyticity. These are then used to describe various spectral parts of $S_\lambda$ including the Taylor spectrum.

In this paper, we are interested in the tree counterpart of classical unilateral multishifts. Hence all the directed trees discussed in the remaining part of this text will be rooted.

3.1. Definition and Elementary Properties

Let $T_j = (V_j, E_j)$ $(j = 1, \cdots, d)$ be rooted directed trees and let $T = (V, E)$ be the directed Cartesian product of $T_1, \cdots, T_d$. For a vertex $v \in V$, let $e_v : V \to \mathbb{C}$ denote the indicator function of the set $\{v\}$. Consider the complex Hilbert space $l^2(V)$ of square summable complex functions on $V$ equipped with the standard inner product. Note that $l^2(V)$ admits the orthonormal basis $\{e_v : v \in V\}$. We always assume that $\text{card}(V) = \aleph_0$. For a nonempty subset $W$ of $V$, $l^2(W)$ may be considered as a subspace of $l^2(V)$. Indeed, if one sets $\tilde{f} = f$ on $W$ and 0 otherwise, then the mapping $U : l^2(W) \to l^2(V)$ given by $Uf = \tilde{f}$ is an isometric homomorphism.

Remark 3.1.1. Consider the the category $\mathcal{T}$ of the directed Cartesian products of finitely many directed trees with morphisms being directed graph homomorphisms (or directed graph isomorphisms). Note that $l^2$ defines a covariant functor from $\mathcal{T}$ into the category $\mathcal{C}$ of Hilbert spaces with bounded linear operators (resp. unitaries) as morphisms. Indeed, any graph homomorphism (resp. isomorphism) $\phi$ induces a bounded linear operator (resp. unitary) $l^2(\phi)$ given by

$$l^2(\phi)(e_v) = e_{\phi(v)},$$

which satisfies $l^2(\phi \circ \psi) = l^2(\phi) \circ l^2(\psi)$.

Definition 3.1.2. Given a system $\lambda = \{\lambda^{(j)}_v : v \in V^o, j = 1, \cdots, d\}$ of nonzero complex numbers, we define the multishift $S_\lambda$ on $\mathcal{T}$ with weights $\lambda$ as the $d$-tuple of operators $S_1, \cdots, S_d$ on $l^2(V)$ given by

$$\mathcal{D}(S_j) := \{f \in l^2(V) : A^{(j)}_\mathcal{T} f \in l^2(V)\},$$
$$S_j f := A^{(j)}_\mathcal{T} f, \quad f \in \mathcal{D}(S_j),$$

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where $A^{(j)}_{T}$ is the mapping defined on complex functions $f$ on $V$ by

$$(A^{(j)}_{T}f)(v) := \begin{cases} \lambda^{(j)}_{v} \cdot f(\text{par}_{j}(v)) & \text{if } v \in V^{\circ}_{j}, \\ 0 & \text{if } v \text{ is a root of } T_{j}. \end{cases}$$

We note here that not all weights $\lambda^{(j)}_{v}$ in the system $\lambda$ are used in the above definition. For instance, if $v \in \text{Chi}_{1}(\text{root})$ then $\lambda^{(2)}_{v}$ will not appear in the definition of $\lambda_{V}$.

**Remark 3.1.3.** If $e_{v} \in D(S_{j})$, then

$$S_{j}e_{v} = \sum_{w \in \text{Chi}_{j}(v)} \lambda^{(j)}_{w} e_{w}. \tag{3.1}$$

**Example 3.1.4 (Classical Multishifts).** Consider the directed Cartesian product $T$ of $d$ copies of $T_{1}$, as discussed in Example 2.1.5. Assume that $S_{j}$ is bounded for $j = 1, \cdots, d$. Then

$$S_{j}e_{\alpha} = \sum_{\beta \in \text{Chi}_{j}(\alpha)} \lambda^{(j)}_{\beta} e_{\beta} = \lambda_{\alpha+e_{j}} e_{\alpha+e_{j}}.$$

If one sets $w_{\alpha}^{(j)} := \lambda^{(j)}_{\alpha}$, then $S_{\lambda}$ is nothing but the classical multishift $S_{\lambda}$ with weight multisequence $\{w_{\alpha}^{(j)} : \alpha \in \mathbb{N}^{d}, j = 1, \cdots, d\}$.

**Lemma 3.1.5.** Let $T = (V, E)$ be the directed Cartesian product of rooted directed trees $T_{1}, \cdots, T_{d}$ and let $S_{\lambda} = (S_{1}, \cdots, S_{d})$ be a multishift on $T$. Then, for $j = 1, \cdots, d$, the following statements hold:

(i) $S_{j}$ is a bounded linear operator on $l^{2}(V)$ if and only if

$$\sup_{v \in V} \left| \lambda^{(j)}_{w} \right| < \infty.$$

(ii) $S_{j}$ is injective.

**Proof.** By Lemma 2.1.10(ii), $\{e_{w} : w \in \text{Chi}_{j}(v)\}$ is orthogonal for every $v \in V$ and $j = 1, \cdots, d$. The first part now follows from (3.1). To see (ii), suppose that $S_{j}f = 0$ for some $f \in l^{2}(V)$. Then

$$\sum_{v \in V} \left| f(v) \right|^{2} \sum_{w \in \text{Chi}_{j}(v)} \left| \lambda^{(j)}_{w} \right|^{2} = \left\| S_{j}f \right\|^{2} = 0.$$

Since $\lambda$ consists of nonzero complex numbers, the above equality holds if and only if either $f(v) = 0$ or $\text{Chi}_{j}(v) = \emptyset$. However, by assumption $T_{1}, \cdots, T_{d}$ are leafless, and hence $f(v) = 0$ for all $v \in V$. \hfill $\square$

Unless stated otherwise, $S_{j}$ belongs to $B(l^{2}(V))$ for every $j = 1, \cdots, d$.

If $S_{\lambda} = (S_{1}, \cdots, S_{d})$ is the multishift on $T$, then the Hilbert space adjoint $S_{j}^{*}$ of $S_{j}$ is given by

$$S_{j}^{*}e_{v} = \lambda^{(j)}_{v} e_{\text{par}_{j}(v)} \quad \text{for all } v \in V.$$

**Remark 3.1.6.** Note that $S_{j}^{*}e_{\text{root}} = 0$ for all $j = 1, \cdots, d$. In particular, 0 belongs to the point spectrum of $S_{\lambda}^{*}$.

In the following proposition, we collect several elementary properties of the multishift $S_{\lambda}$. 
Proposition 3.1.7. Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$ and let $S_\lambda$ be a multishift on $\mathcal{T}$. For $j = 1, \cdots, d$, $w \in V$, let $\beta(j, w, 0) := 1$ and

$$\beta(j, w, n) := \lambda^{(j)}_{\par\beta)}(w) \cdots \lambda^{(j)}_{\par\beta)}(n-1)(w) \ (n \geq 1).$$

Also, let $\alpha^{(0)} = 0 \in \mathbb{N}^d$ and $\alpha^{(j)} = (\alpha_1, \cdots, \alpha_j, 0, \cdots, 0) \in \mathbb{N}^d$ for $j = 1, \cdots, d$. Then the following statements hold true:

(i) $S_\lambda$ is commuting if and only if for all $i, j = 1, \cdots, d$ and for all $v \in V$,

$$\lambda^{(j)}_{\par\beta)}(u) \lambda^{(i)}_{\par\beta)}(u) = \lambda^{(i)}_{\par\beta)}(u) \lambda^{(j)}_{\par\beta)}(u) \quad \text{for all } u \in \Chi_i\Chi_j(v). \quad (3.2)$$

(ii) $S_\lambda$ is doubly commuting if and only if [3.2] holds and for all $v \in V$ and $i, j = 1, \cdots, d$ with $i \neq j$, the following condition holds:

$$\gamma^{(j)}_{\par\beta)}(u) \lambda^{(i)}_{\par\beta)}(u) = \lambda^{(i)}_{\par\beta)}(u) \gamma^{(j)}_{\par\beta)}(u) \quad \text{for all } u \in \Chi_i(v). \quad (3.3)$$

If, in addition, $S_\lambda$ is commuting then the following statements hold true:

(iii) For all $1 \leq i, j \leq d$, for all $v \in V$ and for all $n \geq 1$,

$$\beta(j, \par\beta)(v, n) \lambda^{(i)}_{\par\beta)}(v) = \beta(j, n) \lambda^{(i)}_{\par\beta)}(v), \quad (3.4)$$

(iv) For $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ and for all $v \in V$,

$$S_\lambda^\alpha e_v = \sum_{w \in \Chi_{\alpha}} \prod_{j=1}^d \beta(j, \par\alpha^{(j-1)})(w), \alpha_j e_w. \quad (3.5)$$

(v) For $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ and for all $v \in V$,

$$S_\lambda^\alpha e_v = \prod_{j=1}^d \beta(j, \par\alpha^{(j-1)})(v), \alpha_j e_{\par\alpha^{(j)}},$$

where $\overline{\beta(\cdot)} = \beta(\cdot)$.

(vi) For $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ and for all $v \in V$,

$$S_\lambda^\alpha S_\lambda^{\alpha} e_v = \sum_{w \in \Chi_{\alpha}} \prod_{j=1}^d |\beta(j, \par\alpha^{(j-1)})(w), \alpha_j)|^2 e_v.$$

(vii) The multishift $S_\lambda$ is toral left invertible if and only if

$$\inf_{1 \leq j \leq d} \inf_{v \in V} \sum_{w \in \Chi_i(v)} |\lambda^{(j)}_{\par\beta)}|^2 > 0.$$

(viii) The multishift $S_\lambda$ is joint left invertible if and only if

$$\inf_{v \in V} \sum_{j=1}^d \sum_{w \in \Chi_j(v)} |\lambda^{(j)}_{\par\beta)}|^2 > 0.$$

(ix) If $\alpha \neq \beta$ in $\mathbb{N}^d$, then $\langle S_\lambda^\alpha e_v, S_\lambda^{\beta} e_v \rangle = 0$ for every $v \in V$.

(x) If $v \neq w$ in $V$, then $\langle S_\lambda^\alpha e_v, S_\lambda^{\alpha} e_w \rangle = 0$ for every $\alpha \in \mathbb{N}^d$. 
3. MULTISHIFTS ON PRODUCT OF ROOTED DIRECTED TREES

PROOF. Let \( i, j = 1, \ldots, d \) and \( v \in V \). Then

\[
S_j S_i e_v = S_j \sum_{w \in \text{Chi}_i(v)} \lambda_w^{(i)} e_w = \sum_{w \in \text{Chi}_i(v)} \sum_{u \in \text{Chi}_j(w)} \lambda_u^{(i)} \lambda_w^{(j)} e_u = \sum_{w \in \text{Chi}_i(v)} \sum_{u \in \text{Chi}_j(w)} \lambda_u^{(i)} \lambda^{(j)}_{\text{par}_j(u)} e_u. \tag{3.6}
\]

By symmetry,

\[
S_i S_j e_v = \sum_{w \in \text{Chi}_i(v)} \sum_{u \in \text{Chi}_i(w)} \lambda_u^{(j)} \lambda^{(i)}_{\text{par}_i(u)} e_u. \tag{3.7}
\]

By Lemma 2.1.10(i), \( \text{Chi}_j \text{Chi}_i(v) = \text{Chi}_i \text{Chi}_j(v) \). Hence, by evaluating at \( u \in \text{Chi}_j \text{Chi}_i(v) \), we may infer from equations (3.6) and (3.7) that \( S_\lambda \) is commuting if and only if (3.2) holds. To see (ii), note that (3.2) holds. Let \( \alpha \in \mathbb{N}^d \) with \( \lambda_\alpha \). Let \( \alpha \in \mathbb{N}^d \) and \( |\alpha| = n + 1 \). Then \( \alpha = \gamma + \epsilon_i \) for some \( 1 \leq i \leq d \) and some \( \gamma \in \mathbb{N}^d \) with \( |\gamma| = n \). Therefore, for \( v \in V \),

\[
S_\alpha^o e_v = S_{\lambda} S_\alpha^o e_v = S_i \sum_{w \in \text{Chi}^{\langle \gamma \rightarrow \alpha \rangle}(v)} d \beta(j, \text{par}^{\langle \gamma \rightarrow (j-1) \rangle}(w), \gamma_j) e_w = \sum_{w \in \text{Chi}^{\langle \alpha \rightarrow \gamma \rangle}(v)} d \beta(j, \text{par}^{\langle \gamma \rightarrow (j-1) \rangle}(w), \gamma_j) \sum_{u \in \text{Chi}_i(w)} \lambda_u^{(i)} e_u = \sum_{w \in \text{Chi}^{\langle \alpha \rightarrow \gamma \rangle}(v)} d \beta(j, \text{par}^{\langle \gamma \rightarrow (j-1) \rangle}(\text{par}_i(u)), \gamma_j) \lambda_u^{(i)} e_u. \tag{3.8}
\]

In view of \( \text{Chi}_i(\text{Chi}^{\langle \gamma \rightarrow \alpha \rangle}(v)) = \text{Chi}^{\langle \alpha \rightarrow \gamma \rangle}(v) \), the last equality may be justified by pointwise evaluation. It now suffices to check that

\[
\prod_{j=1}^{d} \beta(j, \text{par}^{\langle \gamma \rightarrow (j-1) \rangle}(\text{par}_i(u)), \gamma_j) \lambda_u^{(i)} = \prod_{j=1}^{d} \beta(j, \text{par}^{\langle \alpha \rightarrow (j-1) \rangle}(u), \alpha_j). \tag{3.8}
\]

This follows from repeated applications of (3.3). Indeed,

\[
\beta(1, \text{par}_i(u), \gamma_1) \lambda_u^{(i)} = \beta(1, u, \gamma_1) \lambda^{(i)}_{\text{par}_1^{\langle \gamma_1 \rightarrow \gamma \rangle}(u)}, \tag{3.9}
\]

\[
\beta(2, \text{par}_1^{\langle \gamma_1 \rangle}(\text{par}_i(u)), \gamma_2) \lambda^{(i)}_{\text{par}_1^{\langle \gamma_1 \rangle}(u)} = \beta(2, \text{par}_1^{\langle \gamma_1 \rangle}(u), \gamma_2) \lambda^{(i)}_{\text{par}_2^{\langle \gamma_2 \rangle}(\text{par}_1^{\langle \gamma_1 \rangle}(u))}. \tag{3.9}
\]

Continuing in this way and using the facts that \( \gamma + \epsilon_i = \alpha \) and \( \text{par}_i(\text{par}_j(u)) = \text{par}_j(\text{par}_i(u)) \), we obtain (3.8). This proves (iv). The proof of (v) is along the lines of (iv) and hence we skip the details. Note that (vi) is a consequence of (iv) and (v). We leave the routine verifications of (vii) and (viii) to the reader. Finally,
As per requirement, we use notations \( \ker_{i,j} \). Proposition 3.1.7(i), for all \( i, j \) and unitarily equivalent to \( S \).\( \square \)

**Corollary 3.1.8.** Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of rooted directed trees \( T_1, \ldots, T_d \) and let \( S_\lambda \) be a commuting multishift on \( \mathcal{T} \). Then \( S_\lambda \) is unitarily equivalent to \( S_{|\lambda|} \), where

\[
|\lambda| = \{|\lambda_v^{(j)}| : v \in V^o, j = 1, \ldots, d\}.
\]

**Proof.** The idea of the proof is a combination of ideas from [69 Corollary 2] and [65 Theorem 3.2.1]. For the sake of simplicity, we treat the case \( d = 2 \). By Proposition 3.1.7(i), for all \( i, j = 1, 2 \) and for all \( v \in V \),

\[
\arg_{\theta_v^{(j)}} + \arg_{\theta_{\text{par}_1}} = \arg_{\theta_v^{(i)}} + \arg_{\theta_{\text{par}_1}} \text{ for all } u \in \text{Chi}_j \text{Chi}_i(v), \tag{3.9}
\]

where \( \arg_{\theta_v^{(j)}} \) denotes the principal argument of \( \lambda_v^{(j)} \). For a subset \( \{\theta_v : v \in V\} \) of the real line \( \mathbb{R} \), define the unitary operator \( U_\theta : L^2(V) \to L^2(V) \) by

\[
U_\theta v = \exp(i \theta_v) e_v, \text{ } v \in V.
\]

Let \( (T_1, T_2) \) denote the commuting 2-tuple \( S_\lambda \). With these notations, the system \( S_j U_\theta = U_\theta T_j \) \( (j = 1, 2) \) is equivalent to the system

\[
\theta_w - \theta_{\text{par}}(u) = \arg_{\theta_w}^{(j)}, \text{ } w \in \text{Chi}_j(V) \text{ and } j = 1, 2. \tag{3.10}
\]

We will show that the above system has a solution. To see that, let \( \theta_{\text{root}} = 0 \). It is clear that \( \theta_w \) can be defined recursively using (3.10). To see that \( \theta_w \) is well-defined, it suffices to check that

\[
\arg_{\theta_w}^{(1)} + \theta_{\text{par}_1} = \arg_{\theta_w}^{(2)} + \theta_{\text{par}_2}, \text{ } w \in \text{Chi}_1(V) \cap \text{Chi}_2(V).
\]

whenever (3.10) holds for \( \text{par}_j(w) \) \( (j = 1, 2) \). Note that

\[
\arg_{\theta_w}^{(2)} + \theta_{\text{par}_2} = \arg_{\theta_w}^{(1)} + (\theta_{\text{par}_1} + \theta_{\text{par}_2}) \tag{3.10}
\]

This completes the proof. \( \square \)

*From here onwards, we assume that the weights from \( \lambda \) appearing in the definition of \( S_\lambda \) are always positive.*

Given a positive integer \( d \), we set

\[
\mathcal{H}^{\otimes d} := \bigoplus_{\text{d times}} \mathcal{H}.
\]

For a commuting \( d \)-tuple \( T = (T_1, \ldots, T_d) \) on \( \mathcal{H} \), consider the linear transformation \( D_T : \mathcal{H} \to \mathcal{H}^{\otimes d} \) given by

\[
D_T h := (T_1 h, \ldots, T_d h) \text{ for } h \in \mathcal{H}.
\]

Note that the kernel of \( D_T \) is precisely the joint kernel \( \ker T := \bigcap_{j=1}^d \ker T_j \) of \( T \). As per requirement, we use notations \( \ker D_T \) and \( \ker T \) interchangeably.
3. Multishifts on Product of Rooted Directed Trees

Corollary 3.1.9. Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S_\lambda = (S_1, \ldots, S_d) \) be a commuting multishift on \( \mathcal{T} \). For \( i \in \mathbb{N} \), let \( T^{(i)} \) denote the commuting \( d \)-tuple \( (S_1^{\ast i}, \ldots, S_d^{\ast i}) \). Then \( \bigcup_{i \in \mathbb{N}} \ker D_{T^{(i)}} \) is dense in \( l^2(V) \).

Proof. Since \( \{ \ker D_{T^{(i)}} \}_{i \in \mathbb{N}} \) is a monotonically increasing sequence of subspaces of \( l^2(V) \), it suffices to show that \( M := \bigcup_{i \in \mathbb{N}} \ker D_{T^{(i)}} \) contains \( e_v \) for every \( v \in V \). For any \( v \in V \), by Proposition 3.1.7(v), \( e_v \in \ker D_{T^{(i)}} \) for all \( i > |\alpha_v| \), where \( \alpha_v \) is the depth of \( v \) in \( \mathcal{T} \).

The following proposition is motivated by the description of kernel of the adjoint of weighted shift on directed tree as given in [65].

Proposition 3.1.10. Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S_\lambda = (S_1, \ldots, S_d) \) be a commuting multishift on \( \mathcal{T} \). Then, for \( j = 1, \ldots, d \), we have

\[
\ker S_j^* = \bigoplus_{v \in V} \left\{ l^2(\text{Chi}_j(v)) \ominus [\Gamma_v^{(j)}] \right\} \oplus \bigvee \{ e_v : v \in V \text{ and } v_j = \text{root}_j \},
\]

where \( \Gamma_v^{(j)} : \text{Chi}(v) \to \mathbb{C} \) is given by \( \Gamma_v^{(j)}(u) = \lambda_u^{(j)}(= (S_j e_v)(u)) \).

Proof. The result follows from [65] Proposition 3.5.1(ii) and [33] eq.(4).

Remark 3.1.11. Note that \( \ker S_j^* \) is infinite dimensional whenever \( d > 1 \).

It is desirable to have a description similar to (3.11) for the joint kernel of \( S_\lambda^* \). The following example shows that the situation is more intriguing than it seems.

Example 3.1.12. Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of two rooted directed trees \( \mathcal{T}_1, \mathcal{T}_2 \). Assume that there exists a vertex \( v = (v_1, v_2) \in V \) such that \( v_1 \) has two children, say \( \hat{v}_1 \) and \( \hat{v}_2 \), and \( v_2 \) has only one child \( \hat{v}_2 \). Then

\[
\text{Chi}(v) = \{(\hat{v}_1, v_2), (\hat{v}_1, v_2), (v_1, \hat{v}_2)\},
\]

and hence \( l^2(\text{Chi}(v)) \) is 3-dimensional. Let \( f \in l^2(\text{Chi}(v)) \ominus [\Gamma_v^{(1)}, \Gamma_v^{(2)}] \), and write

\[
f = \alpha e_{(\hat{v}_1, v_2)} + \beta e_{(\hat{v}_1, v_2)} + \gamma e_{(v_1, \hat{v}_2)}
\]

for some scalars \( \alpha, \beta, \gamma \in \mathbb{C} \). We claim that

\[
l^2(\text{Chi}(v)) \ominus [\Gamma_v^{(1)}, \Gamma_v^{(2)}] \not\subseteq E,
\]

where \( E \) denotes the joint kernel of \( S_\lambda^* \). Assume to the contrary that \( f \in E \). Note that \( S_2^* f = 0 \) implies

\[
\alpha \lambda^{(2)}_{(\hat{v}_1, v_2)} e_{(\hat{v}_1, \text{par}(v_2))} + \beta \lambda^{(2)}_{(v_1, \hat{v}_2)} e_{(v_1, \text{par}(v_2))} + \gamma \lambda^{(2)}_{(v_1, \hat{v}_2)} e_{(v_1, v_2)} = 0,
\]

which is true only if \( \alpha = \beta = \gamma = 0 \), that is, \( f = 0 \). On the other hand, \( [\Gamma_v^{(1)}, \Gamma_v^{(2)}] \) is at most two dimensional (as \( \Gamma_v^{(1)}, \Gamma_v^{(2)} \) could be linearly dependent), and hence

\[
\dim(l^2(\text{Chi}(v)) \ominus [\Gamma_v^{(1)}, \Gamma_v^{(2)}]) \geq 1.
\]

Thus the claim stands verified.

As evident from the preceding discussion, the exact description of the joint kernel of \( S_\lambda^* \) is not as simple as in the case \( d = 1 \), and hence we postpone it to Chapter 4. For the time being, let us see that the joint kernel of \( S_\lambda^* \) can be finite dimensional in many interesting situations (cf. Remark 3.1.11).
Proposition 3.1.13. Let $\mathcal{F} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Let $S_\Lambda = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathcal{F}$ and let $E$ denote the joint kernel of $S_\Lambda$. Then
\[
\bigoplus_{j=1}^{d} \bigoplus_{v \in D_j} \left\{ f^2(\text{Chi}_j(v)) \otimes [\Gamma_j^{(j)}] \right\} \oplus [e_{\text{root}}]
\subseteq E \subseteq \bigcup \{ e_v : v \in F_1 \times \cdots \times F_d \},
\] where $D_j := \{ v \in V : v_j \in V_j^{(j)} \text{ and } \nu_i = \text{root}_i \text{ for } i \neq j \}$, $\Gamma_j^{(j)} : \text{Chi}_j(v) \to C$ is given by $\Gamma_j^{(j)}(u) = \lambda_j^{(j)}$, and $F_j := \text{Chi}(V_j^{(j)}) \cup \{ \text{root}_j \} \ (j = 1, \ldots, d)$.

Proof. To see the first inclusion, let $f \in l^2(\text{Chi}_j(v)) \otimes [\Gamma_j^{(j)}]$ for some $v \in D_j$ and for a fixed $j = 1, \ldots, d$. Thus $f = \sum_{u \in \text{Chi}_j(v)} f(u) e_u$ satisfies $\langle f, \Gamma_j^{(j)} \rangle = 0$. Now, for any $i \neq j$,
\[
S^*_j f = \sum_{u \in \text{Chi}_j(v)} f(u) \lambda_u^{(j)} e_{\text{par}_j(u)} = 0
\]
since $v \in D_j$. Further,
\[
S^*_j f = \sum_{u \in \text{Chi}_j(v)} f(u) \lambda_u^{(j)} e_{\text{par}_j(u)} = \sum_{u \in \text{Chi}_j(v)} f(u) \lambda_u^{(j)} e_v = \langle f, \Gamma_v^{(j)} \rangle e_v = 0,
\]
where we used $\langle f, \Gamma_v^{(j)} \rangle = \sum_{u \in \text{Chi}_j(v)} f(u) \lambda_u^{(j)}$.

To see the second inclusion, let $f \in E$ be such that $f = \sum_{v \in V} f(v) e_v$. Then, for $j = 1, \ldots, d$,
\[
S^*_j f = \sum_{v \in V} f(v) \lambda_v^{(j)} e_{\text{par}_j(v)}
\]
\[
= \sum_{v \in V} f(v) \lambda_v^{(j)} e_{\text{par}_j(v)} + \sum_{v \in V} f(v) \lambda_v^{(j)} e_{\text{par}_j(v)}.
\]
Note that $e_{\text{par}_j(v)} \perp e_{\text{par}_j(w)}$ if $v \neq w$ and $\text{card}(\text{sib}_j(v)) = 1 = \text{card}(\text{sib}_j(w))$. Since $S^*_j f = 0$, we obtain $f(v) = 0$ for every $v \in V$ such that $\text{card}(\text{sib}_j(v)) = 1$. Thus $f(v) \neq 0$ implies that either $\text{card}(\text{sib}_j(v))$ is 0 or bigger than 1 for all $j = 1, \ldots, d$. However, $\text{card}(\text{sib}_j(v)) \geq 2$ if and only if $v_j \in \text{Chi}(V_j^{(j)})$. Further, $\text{card}(\text{sib}_j(v)) = 0$ if and only if $v_j = \text{root}_j$. This completes the proof. 

Corollary 3.1.14. Let $\mathcal{F}, S_\Lambda$ and $E$ be as in the preceding proposition. If $\mathcal{F}$ is locally finite with finite set of branching vertices, then $E$ is finite dimensional. Moreover,
\[
1 + \sum_{j=1}^{d} \sum_{v_j \in V_j^{(j)}} (\text{card}(\text{Chi}(v_j)) - 1) \leq \dim E \leq \prod_{j=1}^{d} (\text{card}(\text{Chi}(V_j^{(j)}))) + 1). \tag{3.13}
\]

Proof. The proof is obvious from (3.12). 

Remark 3.1.15. In Example 2.1.5, the formula (3.13) holds with equalities at all places (with $\dim E = 1$). On the other hand, in Example 2.1.6, equality holds only at left end of (3.13) (with $\dim E = 2$). Further, in Example 2.1.7, equality
may or may not hold even at left end of (3.13) (with dim $E = 3$ or $4$). The last two assertions may be concluded from Examples 4.1.8 and 4.1.9.

**Corollary 3.1.16.** Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Let $S_\lambda$ be a commuting multishift on $\mathcal{T}$ and let $E$ denote the joint kernel of $S_\lambda^\ast$. Then $E$ is finite dimensional if and only if $\mathcal{T}$ is locally finite with finite joint branching index.

**Proof.** If $E$ is infinite dimensional then by (3.12), the cardinality of $F_j = \text{Chi}(V^{(j)}_\mathcal{T}) \cup \{\text{root}_j\}$ must be infinite for some $j = 1, \cdots, d$. It follows that either $\mathcal{T}_j$ is not locally finite or $V^{(j)}_\mathcal{T}$ is infinite. To see the converse, suppose that $\mathcal{T}_j$ is either not locally finite or of infinite branching index for some $j = 1, \cdots, d$. By Proposition 3.1.13

$$\mathcal{M} := \bigoplus_{v \in D_j} \left\{ l^2(\text{Chi}_j(v)) \ominus [\Gamma^{(j)}_v] \right\} \subseteq E,$$

where $D_j = \{v \in V : v_j \in V^{(j)}_\mathcal{T} \text{ and } v_i = \text{root}_i \text{ for } i \neq j\}$. Note that $l^2(\text{Chi}_j(v)) \ominus [\Gamma^{(j)}_v]$ is nonzero for every $v \in D_j$. If $\mathcal{T}_j$ is not locally finite, then $l^2(\text{Chi}_j(v))$ is infinite dimensional for some $v \in D_j$. If $\mathcal{T}_j$ is of infinite branching index, then $D_j$ is infinite. In any case, $\mathcal{M}$ and hence $E$ is infinite dimensional. □

We have already seen in Theorem 2.1.10 that the directed Cartesian product of directed trees admits a directed semi-tree structure. Since there is a notion of shifts $\mathcal{S}_\delta$ on directed semi-trees [79], it is thus natural to reveal the relation between $\mathcal{S}_\delta$ on directed semi-tree $\mathcal{T}$ and the multishift $S_\lambda$. To see this, let us recall the notion of shift on a directed semi-tree from [79, Section 5].

Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Given a system $\delta = \{\delta_{u,v} : (u,v) \in E\}$ of positive numbers, define the weighted shift operator $\mathcal{S}_\delta$ on $\mathcal{T}$ with weights $\delta$ by

$$\mathcal{D}(\mathcal{S}_\delta) := \{f \in l^2(V) : \Delta_\mathcal{S} f \in l^2(V)\},$$
$$\mathcal{S}_\delta f := \Delta_\mathcal{S} f, \quad f \in \mathcal{D}(\mathcal{S}_\delta),$$

where $\Delta_\mathcal{S}$ is the mapping defined on complex functions $f$ on $V$ by

$$(\Delta_\mathcal{S} f)(v) := \begin{cases} \sum_{u \in \text{Par}(v)} \delta_{u,v} f(u) & \text{if } v \in V \setminus \{\text{root}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us see the precise relation between $S_\lambda$ and $\mathcal{S}_\delta$.

**Proposition 3.1.17.** Let $\mathcal{S}_\delta$ be the weighted shift operator on the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Set $\lambda^{(j)}_v := \delta_{u,v}$ for $v \in \text{Chi}_j(u)$, and let $S_\lambda = (S_1, \cdots, S_d)$ be the multishift on $\mathcal{T}$ (possibly unbounded). Then $\cap_{j=1}^d \mathcal{D}(S_j) \subseteq \mathcal{D}(\mathcal{S}_\delta)$. If, in addition, $S_1, \cdots, S_d$ are bounded linear operators on $l^2(V)$, then $\mathcal{S}_\delta$ is bounded. In this case, $\mathcal{S}_\delta = \sum_{j=1}^d S_j$.
This shows that 

\[ f \] 

is strongly circular. This may also be deduced from [\ref{78}, \ref{79}] Proposition 5.1], and hence \( \mathcal{A} \) is a bounded linear operator in this case. This completes the proof. \( \square \)

The above result shows that \( \sum_{j=1}^{d} S_j \) can be realized as a shift on \( \mathcal{T} \) endowed with the directed semi-tree structure.

In the remaining part of this chapter, we obtain some basic properties of multi-shifts \( S_\lambda \) on \( \mathcal{T} \). These include circularity and analyticity. We also obtain a matrix decomposition for \( S_\lambda \) in dimension \( d = 2 \). All these results are then used to examine various spectral parts of \( S_\lambda \) such as point spectrum, Taylor spectrum, and essential spectrum. The discussion to follow relies heavily on the multivariable spectral theory as expounded in \( \[42\] \).

### 3.2. Strong Circularity and Taylor Spectrum

Let \( T = (T_1, \cdots, T_d) \) be a commuting \( d \)-tuple on \( \mathcal{H} \). We say that \( T \) is circular if for every \( \theta = (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d \), there exists a unitary operator \( \Gamma_\theta \) on \( \mathcal{H} \) such that

\[
\Gamma_\theta^* T_j \Gamma_\theta = \exp(i \theta_j) T_j \quad \text{for all} \quad j = 1, \cdots, d.
\]

We say that \( T \) is strongly circular if in addition \( \Gamma_\theta \) can be chosen to be a strongly continuous unitary representation of \( \mathbb{R}^d \) in the following sense: For any \( h \in \mathcal{H} \), the function \( \theta \mapsto \Gamma_\theta h \) is continuous on \( \mathbb{R}^d \).

The above notion in dimension \( d = 1 \) has been introduced and studied in \( \[15\] \). These operators have been studied considerably thereafter (refer to \( \[55\], \[78\], \[86\], \[22\]). The fact that any classical multishift is polycircular is first obtained in \( \[69\] \). This may also be deduced from \( \[34\] \) Lemma 2.14.

The following generalizes \( \[69\] \) Corollary 3. Unlike the method of proof of \( \[65\] \) Theorem 3.3.1], where the unitary \( \Gamma_\theta \) comes from solution of a system of equations, our proof exhibits a formula for \( \Gamma_\theta \).

**Proposition 3.2.1.** Let \( \mathcal{T} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \cdots, \mathcal{T}_d \) and let \( S_\lambda \) be a commuting multishift on \( \mathcal{T} \). Then \( S_\lambda \) is strongly circular.

**Proof.** Let \( \theta = (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d \). For \( f = \sum_{v \in V} f(v) e_v \in l^2(V) \), define \( \Gamma_\theta : l^2(V) \rightarrow l^2(V) \) by

\[
\Gamma_\theta f := \sum_{v \in V} \exp(-i \alpha_v \cdot \theta) f(v) e_v,
\]

where \( \alpha_v \) is the depth of \( v \) in \( \mathcal{T} \) and \( \alpha \cdot \theta := \sum_{j=1}^{d} \alpha_j \theta_j \) for \( \alpha \in \mathbb{N}^d \). Clearly, \( \Gamma_\theta \) is unitary with inverse \( \Gamma_{-\theta} \). Note that \( \alpha_w = \alpha_v + \epsilon_j \) if \( w \in \text{Chi}_j(v) \) for any \( v \in V \) and
Now we show that for any integer \( n \in \mathbb{Z} \) and the preceding proposition.

\[
\left\| (T^{\vartheta(n)} - T^{\vartheta}) \right\|^2 = \sum_{v \in V} \left| \exp(-i\alpha_v \cdot \vartheta^{(n)}) - \exp(-i\alpha_v \cdot \vartheta) \right|^2 |f(v)|^2.
\]

Let \( \epsilon > 0 \). Since \( f \in L^2(V) \), there is a finite subset \( W \) of \( V \) such that \( \sum_{v \in V \setminus W} |f(v)|^2 < \epsilon \). Further, as \( \theta \mapsto \exp(-i\alpha_u \cdot \theta) \) is continuous for each \( u \in W \), there exists a positive integer \( n(u) \) such that \( \left| \exp(-i\alpha_u \cdot \theta^{(n)}) - \exp(-i\alpha_u \cdot \theta) \right|^2 < \epsilon \) for all \( n \geq n(u) \). Let \( n(0) := \max\{n(u) : u \in W\} \). Then, for all \( n \geq n(0) \),

\[
\left\| (T^{\vartheta(n)} - T^{\vartheta}) \right\|^2 = \sum_{v \in W} \left| \exp(-i\alpha_v \cdot \vartheta^{(n)}) - \exp(-i\alpha_v \cdot \vartheta) \right|^2 |f(v)|^2 + \sum_{v \in V \setminus W} \left| \exp(-i\alpha_v \cdot \vartheta^{(n)}) - \exp(-i\alpha_v \cdot \vartheta) \right|^2 |f(v)|^2 \leq \epsilon^2 \left( \sum_{v \in W} |f(v)|^2 \right) + 4\epsilon.
\]

This completes the proof. \( \square \)

The following is immediate from spectral mapping property of Taylor spectrum and the preceding proposition.

**Corollary 3.2.2.** The Taylor spectrum of a commuting multishift \( S_{\lambda} \) has poly-circular symmetry, that is, \( \zeta \cdot w \in \sigma(S_{\lambda}) \) for any \( w \in \sigma(S_{\lambda}) \) and any \( \zeta \in \mathbb{T}^d \). In particular, the Taylor spectrum of \( S_{\lambda} \) coincides with that of \( S_{\lambda}^e \).

**Remark 3.2.3.** Note that point spectrum, left spectrum and essential spectrum do also have poly-circular symmetry.

A special case of the following result, in which \( \mathcal{T} \) is the directed Cartesian product of \( \mathcal{T}_{1,0} \) with itself, has been obtained in [34].

**Proposition 3.2.4.** Let \( \mathcal{T} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S_{\lambda} \) be a commuting multishift on \( \mathcal{T} \). Then the Taylor spectrum of \( S_{\lambda} \) is connected.
3.3. Analyticity and Point Spectrum

The idea of this proof is similar to that of Lemma 3.8. By Remark 3.1.6, 0 belongs to the point spectrum \( \sigma_p(S^*_\lambda) \) of \( S^*_\lambda \). Hence 0 belongs to the Taylor spectrum of \( S^*_\lambda \). In view of Corollary 3.2.2, it suffices to check that \( \sigma(S^*_\lambda) \) is connected. Let \( K_1 \) be the connected component of \( \sigma(S^*_\lambda) \) containing 0 and let \( K_2 = \sigma(S^*_\lambda) \setminus K_1 \). By the Shilov Idempotent Theorem [42, Application 5.24], there exist invariant subspaces \( M_1, M_2 \) of \( S^*_\lambda \) such that \( I^2(V) = M_1 + M_2 \) (vector space direct sum of \( M_1 \) and \( M_2 \)) and \( \sigma(S^*_\lambda|_{M_i}) = K_i \) for \( i = 1, 2 \).

For every \( i \in \mathbb{N} \), let \( T^{(i)} \) denote the commuting \( d \)-tuple \((S_{i1}^*, \ldots, S_{id}^*)\). Let \( h \in \ker(D_{T^{(i)}}) \) for fixed \( i \in \mathbb{N} \). Then \( h = x + y \) for \( x \in M_1 \) and \( y \in M_2 \). It follows that \( S_{ij}^*x = 0 \) and \( S_{ij}^*y = 0 \) for all \( j = 1, \ldots, d \). If \( y \) is nonzero, then \( 0 \in \sigma_p(T^{(i)}|_{M_2}) \subseteq \sigma(T^{(i)}|_{M_2}) \), and hence by the spectral mapping property [42], \( 0 \in \sigma(T^{(i)}|_{M_2}) = \sigma(S_{i1}^*|_{M_2}) \). Since \( 0 \notin K_2 \), we must have \( y = 0 \). It follows that \( M_1 \) contains the linear manifold \( \bigcup_{i \in \mathbb{N}} \ker(D_{T^{(i)}}) \), which is dense in \( I^2(V) \) by Corollary 3.1.9. Hence \( M_1 = I^2(V) \). Thus the Taylor spectrum of \( S^*_\lambda \) is equal to \( K_1 \). In particular, the Taylor spectrum of \( S^*_\lambda \) is connected. □

A connected subset \( \Omega \) of \( \mathbb{C}^d \) is said to be Reinhardt if it is invariant under the action of the \( d \)-torus \( T^d \), that is, \( \zeta \cdot z := (\zeta_1 z_1, \ldots, \zeta_d z_d) \) belongs to \( \Omega \) whenever \( z \in \Omega \) and \( \zeta \in T^d \).

Combining Proposition 3.2.1 with the preceding result, we obtain the following basic fact.

Corollary 3.2.5. Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \). Then the Taylor spectrum of a commuting multishift on \( \mathcal{F} \) is Reinhardt.

Remark 3.2.6. Suppose that the Taylor spectrum \( \sigma(S^*_\lambda) \) of \( S^*_\lambda \) has spherical symmetry in the following sense: \( Uz \in \sigma(S^*_\lambda) \) whenever \( z \in \sigma(S^*_\lambda) \) for every \( d \times d \) unitary matrix \( U \). Then \( \sigma(S^*_\lambda) \) must be a closed ball centered at the origin. Indeed, \( 0 \in \sigma(S^*_\lambda) \) since \( e^{\text{root}} \) belongs to the joint kernel of \( S^*_\lambda \) in view of Remark 3.1.6. The desired conclusion now follows from the fact that every spherically symmetric, compact Reinhardt set containing 0 is a closed ball centered at 0.

3.3. Analyticity and Point Spectrum

A commuting \( d \)-tuple \( T = (T_1, \ldots, T_d) \) on a Hilbert space \( \mathcal{H} \) is called analytic if

\[
\bigcap_{\alpha \in \mathbb{N}^d} \text{ran} \ T^\alpha = \{0\}.
\]

Just like the classical case, the multishifts on \( \mathcal{F} \) are analytic. Indeed, we see that they are separately analytic in the following sense.

Proposition 3.3.1. Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S^*_\lambda = (S_1^*, \ldots, S_d^*) \) be a commuting multishift on \( \mathcal{F} \). Then for each \( j = 1, \ldots, d \), \( S_j^* \) is analytic.

Proof. Let \( j = 1, \ldots, d \) be fixed. For \( n \in \mathbb{N} \), let

\[
M_n := \bigvee \left\{ e_v : v \in \text{Chi}^{\{e^{\alpha} \rangle}(V) \right\},
\]

and note that by (3.5), \( \text{ran} \ S_j^n \subseteq M_n \). It now suffices to check that \( \bigcap_{n=0}^{\infty} M_n = \{0\} \). To see this, note that if \( f \in M_n \), then \( f(u) = 0 \) for every \( u \in V \) such that \( u_j \in \)
\[ \bigcup_{i=0}^{n} \text{Chi}^{(i)}(\text{root}_j). \] However, \[ \bigcup_{i=0}^{\infty} \text{Chi}^{(i)}(\text{root}_j) = V_j, \] and hence for any \( f \in \bigcap_{n=0}^{\infty} M_n, \) we must have \( f(u) = 0 \) for any \( u \in V. \) This completes the proof.

The next corollary generalizes [69 Theorem 15], where the method of proof relies on the description of the commutant of \( S_j \).

**Corollary 3.3.2.** Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S_{\lambda} = (S_1, \ldots, S_d) \) be a commuting multishift on \( \mathcal{F}. \) Then for each \( j = 1, \ldots, d, \) the spectrum of \( S_j \) equals \( \sigma(S_j) = 0 \) if \( \text{ran}S_j^k = \{0\} \) for some \( k \in \mathbb{N}^d \) and \( S_j \) is analytic. Then the commuting multishift \( S_{\lambda} \) on \( \mathcal{F} \) is analytic.

**Proof.** By [33 Lemma 5.2], the spectrum of \( S_j \) is connected. Since \( S_j \) is circular (Proposition 3.2.1) and \( 0 \in \sigma(S_j^*) \) (Remark 3.1.6), the spectrum of \( S_j \) must be the disc \( \sigma(D_{r(T)}) \).

**Corollary 3.3.3.** Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \). Then the commuting multishift \( S_{\lambda} \) on \( \mathcal{F} \) is analytic.

**Proof.** Note that \( \cap_{R \in \mathbb{R}^d} \text{ran} S_{\lambda}^R \subseteq \cap_{k \in \mathbb{N}^d} \text{ran} S_j^k \) for any \( j = 1, \ldots, d. \) The desired conclusion now follows from Proposition 3.3.1.

**Corollary 3.3.4.** Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S_{\lambda} = (S_1, \ldots, S_d) \) be a commuting multishift on \( \mathcal{F}. \) Then for each \( j = 1, \ldots, d, \) the point spectrum of \( S_j \) is empty. In particular, the joint kernel of \( S_{\lambda} \) is trivial.

**Proof.** By Lemma 3.1.5(ii), \( S_j \) is injective. Also, if \( S_j f = w f \) for some nonzero \( w \in \mathbb{C} \) then \( f \in \bigcap_{k \in \mathbb{N}^d} \text{ran} S_j^k = \{0\}, \) and hence \( f = 0. \) This proves that the point spectrum of \( S_j \) is empty.

**Remark 3.3.5.** Note that none of \( S_1, \ldots, S_d \) can be normal, that is, \( S_j^* S_j \neq S_j S_j^* \) for every \( j = 1, \ldots, d. \) In view of Remark 3.1.6 this may be deduced from the fact that for any normal operator \( T, \) the kernel of \( T \) and the kernel of \( T^* \) coincide.

**Corollary 3.3.6.** Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S_{\lambda} = (S_1, \ldots, S_d) \) be a commuting multishift on \( \mathcal{F}. \) Then for each \( j = 1, \ldots, d, \)

\[ \bigcup_{k \in \mathbb{N}} \ker S_j^k = l^2(V) = \bigcup_{\alpha \in \mathbb{N}^d} \ker S_{\lambda}^\alpha. \]

**Proof.** After taking orthogonal complement, first equality may be deduced from the analyticity of \( S_j \) while the second one follows from the analyticity of \( S_{\lambda}. \)

### 3.4. A Matrix Decomposition and Essential Spectrum

In this section, we discuss a matrix decomposition of multishifts \( S_{\lambda} \) on \( \mathcal{F} \) (cf. [33 Lemma 5.3]). The building blocks in this decomposition include classical multishifts and tuples with entries as weighted shifts on directed trees. We will use this decomposition to relate the spectral parts of \( S_{\lambda} \) with the spectral parts of the building blocks appearing in the matrix decomposition of \( S_{\lambda}. \) For simplicity, we treat the case \( d = 2. \) Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_j = (V_j, \mathcal{E}_j), j = 1, 2. \) Assume that \( \mathcal{F} \) is locally finite with finite joint branching index \( k_{\mathcal{F}} = (k_{\mathcal{T}_1}, k_{\mathcal{T}_2}). \) Let us observe the following:
3.4. A Matrix Decomposition and Essential Spectrum

(A) Fix \( v_1 \in V_1 \). If \( W = \{ v_1 \} \times V_2 \), then \( l^2(W) \) is invariant under \( S_2 \). Moreover,
\[
P_{l^2(W)} S_1 |_{l^2(W)} = 0,
\]
and \( S_2 |_{l^2(W)} \) is unitarily equivalent to a weighted shift on the directed tree \( \mathcal{T}_2 \). A similar observation holds for \( V_1 \times \{ v_2 \} \) for any \( v_2 \in V_2 \).

(B) Fix \( j \in \{ 1, 2 \} \). Define \( G_j := \{ \text{root}_j \} \) if \( V^{(j)}_2 \) is \( \emptyset \). Otherwise, let
\[
G_j := \{ v_j \in \text{Chi}(V^{(j)}_2) : \text{card}(\text{Chi}^{(n)}(v_j)) = 1 \text{ for all } n \geq 1 \}.
\]
Let \( v \in G_1 \times G_2 \). Then \( L_v := \bigcup_{n \in \mathbb{N}} \text{Chi}^{(n)}(v) \) is a directed graph isomorphic to \( \mathcal{T}_{1,0} \times \mathcal{T}_{1,0} \). Note first that for any two distinct vertices \( u_j, v_j \in G_j \), there is no positive integer \( n_j \) such that \( \text{Chi}^{(n_j)}(u_j) = \{ v_j \} \).

Indeed, if \( \text{Chi}^{(n_j)}(u_j) = \{ v_j \} \) then \( \text{Chi}^{(n_j-1)}(u_j) \cap V_2^{(j)} = \{ \text{par}(u_j) \} \), and hence we obtain \( \text{card}(\text{Chi}^{(n_j)}(u_j)) \geq 2 \), which is a contradiction. We next check that for \( v, w \in G_1 \times G_2 \) such that \( v \neq w \), \( L_v \cap L_w = \emptyset \). Without loss of generality, assume that \( v_1 \neq w_1 \). Suppose that \( u \in L_v \cap L_w \). Then \( \text{Chi}^{(\alpha)}(v_1) \cap \text{Chi}^{(\beta)}(w_1) = \{ u_1 \} \) for some \( \alpha_1, \beta_1 \in \mathbb{N} \), and hence \( \text{Chi}^{(\alpha)}(v_1) = \text{Chi}^{(\beta)}(w_1) \). It follows that either \( \text{Chi}^{(\alpha)}(v_1) = \{ v_1 \} \) or \( \text{Chi}^{(\beta)}(w_1) = \{ w_1 \} \) for some \( \alpha_1, \beta_1 \in \mathbb{N} \). This contradicts the above observation.

(C) Let \( W_j := \bigcup_{n=1}^{\infty} \text{par}^{(n)}(G_j) \) for \( j = 1, 2 \). Note that \( W_1, W_2 \) are finite sets. Consider the disjoint sets
\[
F_1 := \bigsqcup_{w_1 \in W_1} \{ w_1 \} \times V_2, \quad F_2 := \bigsqcup_{w_2 \in W_2} (V_1 \setminus W_1) \times \{ w_2 \}.
\]

(D) Note that \( V = F_1 \sqcup F_2 \sqcup F_3 \), where
\[
F_3 := \bigsqcup_{v \in G_1 \times G_2} L_v.
\]
This gives the decomposition \( l^2(V) = l^2(F_1) \oplus l^2(F_2) \oplus l^2(F_3) \), where
\[
l^2(F_1) = \bigoplus_{w_1 \in W_1} N_{w_1} \quad \text{and} \quad N_{w_1} := l^2(\{ w_1 \} \times V_2),
\]
\[
l^2(F_2) = \bigoplus_{w_2 \in W_2} M_{w_2} \quad \text{and} \quad M_{w_2} := l^2((V_1 \setminus W_1) \times \{ w_2 \}),
\]
\[
l^2(F_3) = \bigoplus_{v \in G_1 \times G_2} l^2(L_v).
\]
We now decompose \( (S_1, S_2) \) with respect to the decomposition \( l^2(V) = l^2(F_1) \oplus l^2(F_2) \oplus l^2(F_3) \). Indeed, \( S_1 = (A_{ij})_{1 \leq i, j \leq 3} \) and \( S_2 = (B_{ij})_{1 \leq i, j \leq 3} \), where
(a) \( A_{1i} = 0 \) \( (i = 2, 3) \), \( A_{22} = 0 = A_{32} \), \( B_{1i} = 0 \) \( (i = 2, 3) \), \( B_{2j} = 0 \) \( (j = 1, 3) \), \( B_{31} = 0 \),
(b) \( A_{11} = 0 \) if \( \text{Chi}(W_1) \cap W_1 = \emptyset \) (if and only if \( k_{\mathcal{T}_1} \leq 1 \)) and otherwise of infinite rank, \( B_{22} = 0 \) if \( \text{Chi}(W_2) \cap W_2 = \emptyset \) (if and only if \( k_{\mathcal{T}_2} \leq 1 \)), and otherwise of infinite rank,
(c) \( A_{21} \) is the matrix with generic entry \( P_{M_{w_2}} S_1 |_{N_{w_1}} \) (finite rank operator),
Assume further that \( T \) finite rooted directed trees

\[
\text{(d)}\ A_{22} \text{ is the diagonal matrix with generic entry } S_1|_{M_{w_2}}, \text{ } B_{11} \text{ is the diagonal matrix with generic entry } S_2|_{N_{w_1}} \text{ (one variable shifts on directed trees),}
\]

\[
\text{(e)}\ A_{33} \text{ is the diagonal matrix with generic entry } S_1|_{\omega(L_v)}, \text{ } B_{33} \text{ is the diagonal matrix with generic entry } S_2|_{\omega(L_v)} \text{ (entries of the classical multishift } S_w),
\]

\[
\text{(f)}\ A_{31} \text{ is the matrix with generic entry } P_{\omega(L_v)}S_1|_{N_{w_1}}, \text{ } B_{32} \text{ is the matrix with generic entry } P_{\omega(L_v)}S_2|_{M_{w_2}} \text{ (infinite rank non-shifts)}.
\]

Since \( S_1 \) and \( S_2 \) are commuting, a plain calculation shows that

\[
A_{21}B_{11}=0, \quad A_{31}B_{11}=B_{32}A_{21}+B_{33}A_{31}, \quad A_{33}B_{32}=B_{32}A_{22}, \quad A_{33}B_{33}=B_{33}A_{33}.
\]

Thus the building blocks for \( S_\lambda \) consist of 2-tuples of the form \((A_{11},B_{11})\) or \((A_{22},B_{22})\) for single variable weighted shifts \( B_{11}, A_{22} \) on directed trees, commuting classical multishifts \((A_{33},B_{33})\), finite rank 2-tuple \((A_{21},0)\), and infinite rank non-shifts \((A_{31},0)\), \((0,B_{32})\).

It is worth noting that the situation in case \( d=1 \) is entirely different in the sense that all non-diagonal entries in the matrix decomposition of \( S_\lambda \) are of finite rank (see [33] Lemma 5.3).

Before we see applications of the above decomposition, we would like to discuss convergence of nets associated with directed Cartesian product of directed trees. Let \( \mathcal{T}_j = (V_j, E_j) \) \((j = 1, \cdots, d)\) be rooted directed trees and let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of \( \mathcal{T}_1, \cdots, \mathcal{T}_d \). Define the relation \( \leq \) on \( V \) as follows:

\[
v \leq w \text{ if } \alpha_v \leq \alpha_w,
\]

where \( \alpha_v \) denotes the depth of \( v \) in \( \mathcal{T} \). Note that \( V \) is a partially ordered set with partial order relation \( \leq \) (that is, \( \leq \) is reflexive and transitive). Note that given two vertices \( v, w \in V \), there exists \( u \in V \) such that \( v \leq u \) and \( w \leq u \). In this text, we will be interested in the nets \( \{\alpha_v\}_{v \in V} \) of complex numbers induced by the above partial order (the reader is referred to [73] for the definition and elementary facts pertaining to nets).

\textbf{Remark 3.4.1.} One can also endow \( V \) with the following partial order relations:

\( \text{(1)} \) \( v \leq w \) if \( \alpha_v \) is less than or equal to \( \alpha_w \) with respect to the dictionary ordering,

\( \text{(2)} \) \( v \leq w \) if \( |\alpha_v| \leq |\alpha_w| \).

Note that convergence of net in (1) is weaker than and that in (2) is stronger than the convergence defined prior to the remark. All these notions agree in case \( d = 1 \).

We now see an application of the matrix decomposition of multishifts as discussed above.

\textbf{Proposition 3.4.2.} Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of locally finite rooted directed trees \( \mathcal{T}_1, \mathcal{T}_2 \) of finite joint branching index \( k_{\mathcal{T}} = (k_{\mathcal{T}_1}, k_{\mathcal{T}_2}) \). Let \( S_\lambda \) be the commuting multishift on \( \mathcal{T} \). Then we have the following statements.

\( \text{(i)} \) \( \sigma(S_\lambda) \subseteq \sigma((A_{11},B_{11})) \cup \sigma((A_{22},B_{22})) \cup \sigma((A_{33},B_{33})). \)

\( \text{(ii)} \) \( \sigma((A_{33},B_{33})) \subseteq \sigma(S_\lambda). \)

Assume further that \( \max\{k_{\mathcal{T}_1}, k_{\mathcal{T}_2}\} \leq 1 \). Then

\( \text{(iii)} \) \( \sigma(S_\lambda) \subseteq \{(0) \times \sigma(B_{11})\} \cup (\sigma(A_{22}) \times \{0\}) \cup \sigma((A_{33},B_{33})). \)
(iv) If, in addition,
\[
\lim_{u_2 \in \text{Des}(v_2)} \lambda^{(1)}_{(v_1, u_2)} = 0 = \lim_{u_1 \in \text{Des}(v_1)} \lambda^{(2)}_{(u_1, v_2)} \text{ for all } v \in G_1 \times G_2,
\] (3.14)
then \(\sigma_e(S_\lambda)\) is union of essential spectra of finitely many 2-tuples of the form \((0, U_\lambda)\) or \((U_\lambda, 0)\) for a weighted shift \(U_\lambda\) on a directed tree, and the essential spectra of finitely many commuting classical 2-variable shifts.

**Proof.** Note that (i) and (iii) are particular consequences of part (b) of (D) of the previous decomposition and Lemmas 4.4 and 4.5. To see (ii), note that if any commuting \(d\)-tuple \(T\) on \(\mathcal{H}\) is bounded below then so is its restriction to any joint invariant subspace \(\mathcal{M}\) of \(\mathcal{H}\). Applying this fact to \(T := S_\lambda - \omega (\omega \in \mathbb{C}^2)\) and \(\mathcal{M} := L^2(F_3)\) yields the conclusion in (ii).

To see the remaining part, assume further that (3.14) holds. We first note that \(S_\lambda\) is a commuting compact perturbation of orthogonal direct sum of finitely many 2-tuples of the form \((0, U_\lambda)\) or \((U_\lambda, 0)\) for a single variable weighted shift \(U_\lambda\) on a directed tree, and finitely many commuting classical 2-variable shifts. This may be drawn once we observe that \(A_{31} = P_{L(v)}S_1|_{B_{v_1}}\) and \(B_{32} = P_{L(v)}S_2|_{B_{v_2}}\) are compact for every \(v \in G_1 \times G_2\). But this follows from (3.14). To complete the proof, in view of Atkinson-Curto Theorem 4.1 Theorem 2, we need the fact that the essential spectrum \(\sigma_e(A \oplus B)\) of orthogonal direct sum of \(A\) and \(B\) is union of \(\sigma_e(A)\) and \(\sigma_e(B)\), where \(A\) and \(B\) denote commuting \(d\)-tuples of bounded linear operators on \(\mathcal{H}\) and \(\mathcal{K}\) respectively. We include elementary verification of this fact. Note that the boundary operators \(\partial_{A \oplus B}\) appearing in the Koszul complex of \(A \oplus B\) are orthogonal direct sum of boundary operators \(\partial_A\) and \(\partial_B\) appearing in the Koszul complexes of \(A\) and \(B\) respectively (refer to Section 1.1). That is, \(\partial_{A \oplus B} + \partial_{A \oplus B}^* = (\partial_A + \partial_A^*) \oplus (\partial_B + \partial_B^*)\). On the other hand, by 4.2 Theorem 6.2, a \(d\)-tuple \(T\) is Fredholm if and only if \(\partial_T + \partial_T^*\) is Fredholm. The desired conclusion is now immediate. \(\square\)

We illustrate the previous result with the help of an example.

**Example 3.4.3.** Let \(\mathcal{T} = \mathcal{T}_{2,0} \times \mathcal{T}_{1,0}\) be as discussed in Example 2.1.6. Note that \(G_1 = \{1, 2\}, G_2 = \{0\}, W_1 = \{0\}, W_2 = \emptyset, F_1 = \{0\} \times V_2, F_2 = \emptyset, F_3 = L(1,0) \cup L(2,0)\).

Let \(S_\lambda\) be a multishift on \(\mathcal{T}\) with weights \(\lambda\) such that
\[
\lim_{k \to \infty} \lambda^{(1)}_{(1,k)} = \lim_{k \to \infty} \lambda^{(1)}_{(2,k)} = 0.
\]
By the above result, the essential spectrum \(\sigma_e(S_\lambda)\) of \(S_\lambda\) is equal to the union of essential spectra of \((0, U_\lambda), S_{w_1}, S_{w_2}\). In particular, this is applicable to the commuting multishift \(S_\lambda = (S_1, S_2)\) with weights given by
\[
\lambda^{(1)}_{m,n} = \frac{1}{\text{card(sib}_1(m,n))} \sqrt{\frac{m}{2} + n}, \quad \lambda^{(2)}_{m,n} = \sqrt{\frac{n}{2} + n},
\]
where \(m, n \in \mathbb{N}\), and
\[
\left[\frac{m}{2}\right] = \begin{cases} \frac{m}{2} & \text{if } m \text{ is an even integer} \\ \frac{m+1}{2} & \text{otherwise}. \end{cases}
\]
Note that none of \(S_1, S_2\) is compact. Further, \(U_\lambda\) is the unilateral unweighted shift with essential spectrum the unit circle \([\mathbb{T}]\) (refer to [86]). By spectral mapping
property, the essential spectrum of \((0, U_\lambda)\) is \(\{0\} \times \mathbb{T}\). Further, \(S_{w^{(1)}}\) is the 2-variable classical multishift with weights

\[
w^{(1)}_{(2k+1,l)} = \sqrt{\frac{k+1}{k+l+1}}, \quad w^{(2)}_{(2k-1,l+1)} = \sqrt{\frac{l+1}{k+l+1}} \quad (k \geq 1, l \geq 0).
\]

It is easy to see that \(S_{w^{(1)}}\) is unitarily equivalent to the Drury-Arveson 2-shift, and hence the essential spectrum of \(S_{w^{(1)}}\) equals the unit sphere \(\partial \mathbb{B}^2\) in \(\mathbb{C}^2\) (Proposition 1.2.2). Similarly, the essential spectrum of \(S_{w^{(2)}}\) is equal to \(\partial \mathbb{B}^2\). It follows that \(\sigma_e(S_\lambda) = \partial \mathbb{B}^2\). Also, since the left-spectrum of Drury-Arveson shift is the unit sphere (Proposition 1.2.2), it may be concluded from Proposition 3.4.2(ii) that \(\sigma_l(S_\lambda)\) contains \(\partial \mathbb{B}^2\).

**Remark 3.4.4.** We will see later in Chapter 5 that \(\sigma_l(S_\lambda)\) is contained in the unit sphere in \(\mathbb{C}^2\) (see (5.31)).

We conclude this section with a recipe to construct non-compact multishifts \(S_\lambda\) on \(\mathcal{T}\) having Taylor spectra with empty interior. One such family of classical multishifts has been exhibited in [49, Example 2]. We now capitalize on it to construct examples of multishifts \(S_\lambda\) on \(\mathcal{T}_{2,0} \times \mathcal{T}_{2,0}\) with Taylor spectra of empty interior.

**Example 3.4.5.** Let \(\mathcal{T} = \mathcal{T}_{2,0} \times \mathcal{T}_{2,0}\) be as discussed in Example 2.1.7. Note that \(G_1 = \{1, 2\} = G_2, W_1 = \{0\} = W_2, F_1 = \{0\} \times V_2, F_2 = V_1^* \times \{0\}, F_3 = L_{(1,1)} \cup L_{(1,2)} \cup L_{(2,1)} \cup L_{(2,2)}.\) By Proposition 3.4.2 we have

\[
\sigma(S_\lambda) \subseteq (\{0\} \times \sigma(B_{11})) \cup (\sigma(A_{22}) \times \{0\}) \cup \sigma((A_{33}, B_{33})).
\]

We now choose weights of \(S_\lambda\), so that all four classical multishift appearing in \((A_{33}, B_{33})\) are unitarily equivalent to some classical multishift with Taylor spectrum of empty interior. Since \((\{0\} \times \sigma(B_{11})) \cup (\sigma(A_{22}) \times \{0\})\), being contained in \(\{0\} \times \mathbb{D}_r \cup (\mathbb{D}_s \times \{0\})\) for some \(r, s > 0\), has always empty interior, we conclude that \(\sigma(S_\lambda)\) has empty interior.
CHAPTER 4

Wandering Subspace Property

Let $T = (T_1, \ldots, T_d)$ be a commuting $d$-tuple on a Hilbert space $\mathcal{H}$. A subspace $W$ of $\mathcal{H}$ is said to be a wandering subspace for $T$ if $T^\alpha W$ is orthogonal to $W$ for every $\alpha \in \mathbb{N}^d \setminus \{0\}$. Note that the joint kernel $E = \cap_{j=1}^d \ker T_j^*$ of $T^*$ is always a wandering subspace for $T$. Following [87, Definition 2.4], we say that $T$ possesses wandering subspace property if $\mathcal{H} = [E]_T$, where

$$[E]_T := \bigvee_{\alpha \in \mathbb{N}^d} T^\alpha E.$$

The main result of this chapter ensures the wandering subspace property for multishift $S_\lambda$ on $\mathcal{T}$ under some modest assumptions. Unlike the cases either of classical multishifts or of one variable weighted shifts on rooted directed trees, this fact lies deeper.

**Theorem 4.0.1.** Let $\mathcal{T} = (V,E)$ be the directed Cartesian product of locally finite, rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$ and let $S_\lambda$ be a commuting multishift on $\mathcal{T}$. Then $S_\lambda$ possesses wandering subspace property.

The proof of Theorem 4.0.1, as presented below, relies heavily on the analysis of the joint kernel of $S_\lambda^*$ carried out in the next section. Before we start preparing for the proof of this result, we would like to discuss Shimorin's approach to the wandering subspace property for left invertible analytic operators. Note that the wandering subspace property for a left invertible analytic operator $T$ on $\mathcal{H}$ is a simple consequence of the duality relation

$$(\bigcap_{k \in \mathbb{N}} T^k \mathcal{H})^\perp = [\ker T^*]_T,$$

(4.1)

which, in turn, relies on the identity

$$I - T^n T^* T^n = \sum_{k=0}^{n-1} T^{k} (I - T T^*) T^{*k},$$

(4.2)

where $T' = T(T^* T)^{-1}$ denotes the Cauchy dual of $T$ and $I - TT^*$ is the orthogonal projection $P_{\ker T^*}$ onto $\ker T^*$. In order not to distract the reader from the main line of development, we have relegated the discussion on some of the difficulties arising in finding a multivariable counterpart of Shimorin's approach to the Appendix.

**4.1. The Joint Kernel and a System of Linear Equations**

In this section, we show that finding the joint kernel of $S_\lambda^*$ is equivalent to solving certain system of linear equations. This information is then used to derive wandering subspace property for $S_\lambda$. 

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We now introduce a framework suitable for decomposing the joint kernel of $S^*_\lambda$ into smaller subspaces of $l^2(V)$. These subspaces are induced by a system of linear equations arising from the action of $S^*_\lambda$.

For a set $A$, let $\mathcal{P}(A)$ denote the collection of all subsets of $A$. In case $A = \{1, \ldots, d\}$, we sometimes use the simpler notation $\mathcal{P}$ for $\mathcal{P}(\{1, \ldots, d\})$.

Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Consider the set-valued function $\Phi : \mathcal{P} \to \mathcal{P}(V)$ given by $\Phi(F) = \Phi_F$, where

$$\Phi_F := \{v \in V : v_j \in V^s_j \text{ if } j \in F, \text{ and } v_j = \text{root}_j \text{ if } j \notin F\}, \quad F \in \mathcal{P}. \tag{4.3}$$

Note that $\Phi_F \cap \Phi_G = \emptyset$ if $F \neq G$. Further, if $v \in V$ then $v \in \Phi_F$ for

$$F = \{j \in \{1, \ldots, d\} : v_j \neq \text{root}_j\}.$$  

This shows that

$$V = \bigcup_{F \in \mathcal{P}} \Phi_F. \tag{4.4}$$

Let $F := \{i_1, \cdots, i_k\} \subseteq \{1, \cdots, d\}$ be fixed. For $u \in \Phi_F$, define

$$\text{sib}_F(u) := \text{sib}_{i_1} \text{sib}_{i_2} \cdots \text{sib}_{i_k}(u). \tag{4.5}$$

As a convention, we set $\text{sib}_\emptyset(u) = \{u\}$ for all $u \in V$.

Define a relation $\sim$ on $\Phi_F$ by $u \sim v$ if $u \in \text{sib}_F(v)$, and note that $\sim$ is an equivalence relation on $\Phi_F$. Moreover, for any $u \in \Phi_F$, the equivalence class containing $u$ is precisely $\text{sib}_F(u)$. An application of axiom of choice \cite{59} allows us to form a set $\Omega_F$ (to be referred to as an indexing set corresponding to $F$) by picking up exactly one element from each of the equivalence classes $\text{sib}_F(u)$. Thus we have the disjoint union

$$\Phi_F = \bigsqcup_{u \in \Omega_F} \text{sib}_F(u). \tag{4.6}$$

This combined with \eqref{4.4} yields

$$V = \bigsqcup_{F \in \mathcal{P}} \bigsqcup_{u \in \Omega_F} \text{sib}_F(u).$$

As a consequence, we obtain the following decomposition of $l^2(V)$.

**Proposition 4.1.1.** Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Then

$$l^2(V) = \bigoplus_{F \in \mathcal{P}} \bigoplus_{u \in \Omega_F} l^2(\text{sib}_F(u)),$$

where $\Omega_F$ is the indexing set corresponding to $F$ and $\text{sib}_F(u)$ is given by \eqref{4.5}.

In order to describe the joint kernel $E$ of $S^*_\lambda$, one needs to understand the subspace $l^2(\text{sib}_F(u))$. Before that, let us see some definitions.

**Definition 4.1.2.** For $F \in \mathcal{P}$ and $v = (v_1, \cdots, v_d) \in V$, let $v_F$ denote the $d$-tuple with $j$th coordinate given by

$$(v_F)_j := \begin{cases} v_j & \text{if } j \in F, \\ \text{root}_j & \text{if } j \notin F. \end{cases}$$
Further, for fixed $1 \leq i \leq d$ such that $i \notin F$, and $u_i \in V_i$, we define $v_F|u_i$ to be $(w_1, \cdots, w_d)$, where

$$w_j = \begin{cases} u_i & \text{if } j = i, \\ (v_F)_j & \text{otherwise.} \end{cases}$$

**Remark 4.1.3.** Note that $v_F$ is obtained from $v$ by replacing $j^{th}$ coordinate by root$_j$ whenever $j \notin F$. On the other hand, $v_F|u_i$ is obtained from $v_F$ by replacing its $i^{th}$ coordinate by $u_i$.

For subsets $F, G$ of $\{1, \cdots, d\}$ such that $G \subseteq F$, and $u \in \Phi_F$, we define

$$\text{sib}_{F,G}(u) := \{v_G : v \in \text{sib}_F(u)\}.$$  \hfill (4.7)

**Remark 4.1.4.** Note that different vertices $v$ in $\text{sib}_F(u)$ may correspond to single $v_G \in \text{sib}_{F,G}(u)$.

**Lemma 4.1.5.** Let $F \in \mathcal{P}$ and let $i \in F$. Let $\Omega_F$ be the indexing set corresponding to $F$ and let $\text{sib}_F(u)$ be given by (4.5). For $u \in \Omega_F$ and $G := F \setminus \{i\}$, we have the following:

1. $\text{sib}_F(u) = \bigsqcup_{v_G \in \text{sib}_{F,G}(u)} \text{sib}_i(v_G|u_i)$.
2. For all $v_G \in \text{sib}_{F,G}(u)$, $\text{card}(\text{sib}_i(v_G|u_i))$ is constant.
3. $\text{card}(\text{sib}_{F,G}(u)) = \prod_{j \in F, j \neq i} \text{card}(\text{sib}_j(u))$.
4. $\text{card}(\text{sib}_F(u)) = \prod_{j \in F} \text{card}(\text{sib}_j(u))$.

**Proof.** Let $v_G, w_G \in \text{sib}_{F,G}(u)$ such that $v_G \neq w_G$. Then there exists $j \in G$ such that $v_j \neq w_j$. Suppose that $\eta \in \text{sib}_i(v_G|u_i) \cap \text{sib}_i(w_G|u_i)$. Then $v_j = \eta_j = w_j$, which is a contradiction. Hence $\text{sib}_i(v_G|u_i) \cap \text{sib}_i(w_G|u_i) = \emptyset$ if $v_G \neq w_G$. Next, observe that $\text{sib}_i(v_G|u_i) \subseteq \text{sib}_F(u)$ for all $v_G \in \text{sib}_{F,G}(u)$. To see the other inclusion in (i), note that if $w \in \text{sib}_F(u)$ then $w \in \text{sib}_i(v_G|u_i)$. This completes the proof of the first part. The second part follows from the fact that $\text{card}(\text{sib}_i(v_G|u_i)) = \text{card}(\text{sib}(u))$ while the third part follows from (4.7). The last part is immediate from (4.5). \qed

The following lemma describes the action of $S_X^\ast$ on $l^2(\text{sib}_F(u))$.

**Lemma 4.1.6.** Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$ and let $S_X = (S_1, \cdots, S_d)$ be a commuting multishift on $\mathcal{T}$. Let $F \in \mathcal{P}$, $i \in F$ and $G := F \setminus \{i\}$. If $u \in \Phi_F$, then for any $f \in l^2(\text{sib}_F(u))$,

$$S_X^\ast(f) = \sum_{v_G \in \text{sib}_{F,G}(u)} \left( \sum_{w \in \text{sib}_i(v_G|u_i)} f(w) \lambda^{(i)}_w \right) e_{\text{par}_F(v_G|u_i)}.$$  \hfill (4.8)

**Proof.** Let $u \in \Phi_F$. By (i) of the preceding lemma, we obtain the orthogonal decomposition

$$l^2(\text{sib}_F(u)) = \bigoplus_{v_G \in \text{sib}_{F,G}(u)} l^2(\text{sib}_i(v_G|u_i)).$$
Let $f \in l^2(\text{sib}_F(u))$. Then $f = \sum_{v_G \in \text{sib}_{F,G}(u)} \sum_{w \in \text{sib}_i(v_G)} f(w)e_w \in l^2(V)$. It follows that

$$S_i^*(f) = \sum_{v_G \in \text{sib}_{F,G}(u)} \sum_{w \in \text{sib}_i(v_G)} f(w)\lambda_w^{(i)} e_{\text{par}_i(w)}$$

$$= \sum_{v_G \in \text{sib}_{F,G}(u)} \left( \sum_{w \in \text{sib}_i(v_G)} f(w)\lambda_w^{(i)} \right) e_{\text{par}_i(v_G)},$$

where we used the fact that $\text{par}_i(\text{sib}_i(v)) = \text{par}_i(v)$ for any $v \in V$. \hfill \Box

Let $i \in F$ be fixed and let $G := F \setminus \{i\}$. In view of (4.8), finding solution of $S_i^*(f) = 0$, $f \in l^2(\text{sib}_F(u))$ amounts to solve the following system of $N_{i,u,F}$ equations in $M_{i,u,F}$ unknowns:

$$\sum_{w \in \text{sib}_i(v_G)} f(w)\lambda_w^{(i)} = 0, \quad v_G \in \text{sib}_{F,G}(u),$$

(4.9)

where, in view of Lemma 4.1.5, $N_{i,u,F}, M_{i,u,F} \in \mathbb{N} \cup \{\infty\}$ are given by

$$N_{i,u,F} = \text{card}(\text{sib}_{F,G}(u)) = \prod_{j \in F, j \neq i} \text{card}(\text{sib}_j(u)),$$

$$M_{i,u,F} = \text{card}(\text{sib}_i(v_G)) N_{i,u,F} = \text{card}(\text{sib}_F(u)) = \prod_{j \in F} \text{card}(\text{sib}_j(u)).$$

Note that $M_{u,F} := M_{i,u,F}$ is independent of $i$. Further, by Lemma 4.1.5(i), the set of unknowns in the system (4.9) is equal to $\{f(w) : w \in \text{sib}_F(u)\}$ for each $i \in F$. Thus varying $i$ over $F$, we get the following system of $N(u,F) := \sum_{i \in F} N_{i,u,F}$ equations in $M_{u,F}$ number of unknowns:

$$\sum_{w \in \text{sib}_i(v_G)} f(w)\lambda_w^{(i)} = 0, \quad i \in F \text{ and } v_G \in \text{sib}_{F,G}(u).$$

(4.10)

Let $L_{u,F}$ denote the linear manifold of $l^2(\text{sib}_F(u))$ given by

$$L_{u,F} := \{f \in l^2(\text{sib}_F(u)) : f \text{ is a solution of (4.10)}\}.$$  

(4.11)

If $\mathcal{F}_1, \ldots, \mathcal{F}_d$ are locally finite then $L_{u,F}$ is a subspace. In this case, by Proposition 4.1.4, the joint kernel $E$ of $S_\lambda^*$ is given by

$$E = [e_{\text{root}}] \oplus \bigoplus_{F \in \mathcal{P}} \bigoplus_{u \in \Omega_F} L_{u,F}.$$  

(4.12)

**Remark 4.1.7.** Let us discuss the system (4.10) in following special cases:

1. In case $S_\lambda$ is the classical multishift $S_w$, the system (4.10) has only trivial solution, and hence $E = [e_\lambda]$.

2. In case $d = 1$, $\mathcal{P} = \{\emptyset, \{1\}\}$, and hence the system (4.10) takes the form

$$\sum_{w \in \text{sib}_u} f(w)\lambda_w = 0 \quad (u \in V^\circ).$$


However, linear equations associated with vertices outside $\text{Chi}(V_\prec)$ have trivial solutions, and hence

$$E = [e_{\text{root}}] \oplus \bigoplus_{u \in \text{Chi}(V_\prec)} \mathcal{L}_{u, \{1\}}.$$

This expression should be compared with \[\text{[4.11]}\].

To understand the above description of the joint kernel of $S^*_\Lambda$, we include a couple of instructive examples.

**Example 4.1.8.** Let $\mathcal{F}$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1 = \mathcal{T}_{2,0}, \mathcal{T}_2 = \mathcal{T}_{1,0}$ as described in Example 2.1.6. Note that

$$\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

It follows from \[\text{[4.3]}\] that

$$\Phi_\emptyset = \{(0, 0)\}, \Phi_{\{1\}} = \{(i, 0) : i \geq 1\},$$

$$\Phi_{\{2\}} = \{(0, j) : j \geq 1\}, \Phi_{\{1, 2\}} = \{(i, j) : i, j \geq 1\}.$$

Let us now understand $\text{sib}_F(u)$ for $F \in \mathcal{P}$ and $u \in \Phi_F$. By convention, $\text{sib}_\emptyset((0, 0)) = \{(0, 0)\}$. Note that

$$\text{sib}_{\{1\}}((1, 0)) = \{(1, 0), (2, 0)\} = \text{sib}_{\{1\}}((2, 0)), \text{ sib}_{\{1\}}((i, 0)) = \{(i, 0) : i \geq 3\},$$

$$\text{sib}_{\{2\}}((0, j)) = \{(0, j)\} \text{ for all } j \geq 1, \text{ and}$$

$$\text{sib}_{\{1, 2\}}((i, j)) = \begin{cases} \{(i, j), (2, j)\} & \text{if } i \in \{1, 2\} \text{ and } j \geq 1, \\ \{(i, j)\} & \text{if } i \geq 3, j \geq 1. \end{cases}$$

One may form $\Omega_F$ by picking up one element from each of the equivalence classes $\text{sib}_F(u)$ as follows:

$$\Omega_\emptyset = \{(0, 0)\}, \Omega_{\{1\}} = \{(1, 0)\} \cup \{(i, 0) : i \geq 3\}, \Omega_{\{2\}} = \{(0, j) : j \geq 1\},$$

$$\Omega_{\{1, 2\}} = \{(i, j), (i, j) : i \geq 3, j \geq 1\}.$$ Let us calculate $\text{sib}_{F,G}(u)$ for possible choices of $F,G$, and $u \in \Omega_F$. If $F = \{1\}$, then $G = \emptyset$. In this case,

$$\text{sib}_{\{1\}, \emptyset}(1, 0) = \{(0, 0)\} = \text{sib}_{\{1\}, \emptyset}((i, 0) : i \geq 3).$$

This together with \[\text{[4.9]}\] yields the following equations:

$$f(1, 0)\lambda^{(1)}_{(1,0)} + f(2, 0)\lambda^{(1)}_{(2,0)} = 0,$$

$$f(i, 0)\lambda^{(1)}_{(i,0)} = 0 \text{ for } i \geq 3.$$

In case $F = \{2\}$, $G = \emptyset$ and $\text{sib}_{\{2\}, \emptyset}(0, j) = \{(0, 0)\}$ for $j \geq 1$, and hence we obtain the equations

$$f(0, j)\lambda^{(2)}_{(0,j)} = 0 \text{ for } j \geq 1.$$

In case $F = \{1, 2\}$, $G = \{1\}$ or $\{2\}$. Then for all $i \geq 3$ and $j \geq 1$,

$$\text{sib}_{\{1, 2\}, \{1\}}((1, j)) = \{(0, j)\}, \text{ sib}_{\{1, 2\}, \{1\}}((1, j)) = \{(1, 0), (2, 0)\},$$

$$\text{sib}_{\{1, 2\}, \{2\}}((i, j)) = \{(0, j)\}, \text{ sib}_{\{1, 2\}, \{1\}}((i, j)) = \{(i, 0)\}. $$
This gives following equations for $i \geq 3$ and $j \geq 1$,
\[
\begin{aligned}
f(1,j)\lambda_{(1,j)}^{(1)} + f(2,j)\lambda_{(2,j)}^{(1)} &= 0, \\
f(1,j)\lambda_{(1,j)}^{(2)} &= 0, \\
f(i,j)\lambda_{(i,j)}^{(1)} &= 0, \\
f(i,j)\lambda_{(i,j)}^{(2)} &= 0.
\end{aligned}
\]
Solving above, we get that 
\[
f(1,0) = \alpha\lambda_{(2,0)}^{(1)}, \\
f(2,0) = -\alpha\lambda_{(1,0)}^{(1)}, \\
\text{for } \alpha \in \mathbb{C}, \\
f(i,j) = 0 \text{ for all } i, j \geq 1, \\
f(0,j) = 0 \text{ for all } j \geq 1 \text{ and } f(i,0) = 0 \text{ for all } i \geq 3.
\]
Thus,
\[
E = [e_{\text{root}}] \oplus [\lambda_{(2,0)}^{(1)} e_{(1,0)} + \lambda_{(1,0)}^{(1)} e_{(2,0)}].
\]
The situation in the preceding example resembles more like the situation occurring in dimension $d = 1$ (cf. (1.11)). We see below an example which gives an idea of the complications which can occur in dimension more than 1.

**Example 4.1.9.** Consider the directed Cartesian product $\mathcal{T} = (V,E)$ of the directed tree $\mathcal{T}_0$ with itself (see Example 2.1.7). Note that 
\[
\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.
\]
It follows that
\[
\begin{aligned}
\Phi_{\emptyset} &= \{(0,0)\}, \\
\Phi_{\{1\}} &= \{(i,0) : i \geq 1\}, \\
\Phi_{\{2\}} &= \{(0,i) : i \geq 1\}, \\
\Phi_{\{1,2\}} &= \{(i,j) : i,j \geq 1\}.
\end{aligned}
\]
Let us now understand $\text{sib}_F(u)$ for $F \in \mathcal{P}$ and $u \in \Phi_F$. By convention, $\text{sib}_{\emptyset}((0,0)) = \emptyset$. Note that
\[
\text{sib}_{\{1\}}((1,0)) = \{(1,0),(2,0)\}, \\
\text{sib}_{\{1\}}((i,0)) = \{(i,0) : i \geq 3\}, \\
\text{sib}_{\{2\}}((0,1)) = \{(0,1),(0,2)\}, \\
\text{sib}_{\{2\}}((0,i)) = \{(0,i) : i \geq 3\}.
\]
Similarly,
\[
\begin{aligned}
\text{sib}_{\{1,2\}}((i,j)) &= \left\{\begin{array}{ll}
\{(1,1),(1,2),(2,1),(2,2)\} & \text{if } i,j \in \{1,2\}, \\
\{(1,j),(2,j)\} & \text{if } i \in \{1,2\} \text{ and } j \geq 3, \\
\{(i,1),(i,2)\} & \text{if } i \geq 3 \text{ and } j \in \{1,2\}, \\
\{(i,j)\} & \text{if } i,j \geq 3.
\end{array}\right.
\end{aligned}
\]
One may form $\Omega_F$ by picking up one element from each of the equivalence classes $\text{sib}_F(u)$ as follows:
\[
\begin{aligned}
\Omega_\emptyset &= \{(0,0)\}, \\
\Omega_{\{1\}} &= \{(1,0)\} \cup \{(i,0) : i \geq 3\}, \\
\Omega_{\{2\}} &= \{(0,1)\} \cup \{(0,j) : j \geq 3\}, \\
\Omega_{\{1,2\}} &= \{(1,1)\} \cup \{(1,i),(1,j),(i,j) : i,j \geq 3\}.
\end{aligned}
\]
Let us calculate $\text{sib}_{F,G}(u)$ for possible choices of $F,G$, and $u \in \Omega_F$. If $F = \{1\}$, then $G = \emptyset$. In this case,
\[
\text{sib}_{\{1\},\emptyset}((1,0)) = \{(0,0)\}, \\
\text{sib}_{\{1\},\emptyset}((i,0)) = \{(0,0)\} : (i \geq 3).
\]
This yields the following equation:
\[
\begin{aligned}
\sum_{w \in \text{sib}_1((v_0)_{\{1\}})} f(w)\lambda_w^{(1)} &= 0, \\
\sum_{w \in \text{sib}_2((v_0)_{\{1\}})} f(w)\lambda_w^{(1)} &= 0.
\end{aligned}
\]
which is same as
\[
\begin{align*}
  f(1, 0)\lambda_{(1,0)}^{(1)} + f(2, 0)\lambda_{(2,0)}^{(1)} &= 0, \\
  f(i, 0)\lambda_{(i,0)}^{(1)} &= 0 \quad (i \geq 3).
\end{align*}
\]

Similarly, in case \( F = \{2\} \), we obtain the equations
\[
\begin{align*}
  f(0, 1)\lambda_{(0,1)}^{(2)} + f(0, 2)\lambda_{(0,2)}^{(2)} &= 0, \\
  f(0, j)\lambda_{(0,j)}^{(2)} &= 0 \quad (j \geq 3).
\end{align*}
\]

In case \( F = \{1, 2\} \) then either \( G = \{1\} \) or \( \{2\} \), and hence for \( i, j \geq 3 \),
\[
\begin{align*}
  \text{\text{sib}_{\{1,2\},\{2\}}(1,1) &= \{(0,1), (0,2), \},} \\
  \text{\text{sib}_{\{1,2\},\{1\}}(1,1) &= \{(1,0), (2,0)\)} \\
  \text{\text{sib}_{\{1,2\},\{2\}}(i,1) &= \{(0,1), (0,2), \},} \\
  \text{\text{sib}_{\{1,2\},\{1\}}(i,1) &= \{(i,0), \}} \\
  \text{\text{sib}_{\{1,2\},\{2\}}(1,j) &= \{(0,j), \},} \\
  \text{\text{sib}_{\{1,2\},\{1\}}(1,j) &= \{(1,0), (2,0)\},} \\
  \text{\text{sib}_{\{1,2\},\{2\}}(i,j) &= \{(0,j), \},} \\
  \text{\text{sib}_{\{1,2\},\{1\}}(i,j) &= \{(i,0), \}.
\end{align*}
\]

Thus we obtain the equations for \( i, j \geq 3 \),
\[
\begin{align*}
  \sum_{w \in \text{\text{sib}}_{\{1\},\{v_{(2)}\}}(1)} f(w)\lambda_{w}^{(1)} &= 0, \quad v_{(2)} \in \{(0,1), (0,2)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{2\},\{v_{(1)}\}}(1)} f(w)\lambda_{w}^{(2)} &= 0, \quad v_{(1)} \in \{(1,0), (2,0)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{1\},\{v_{(2)}\}}(i)} f(w)\lambda_{w}^{(1)} &= 0, \quad v_{(2)} \in \{(0,1), (0,2)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{2\},\{v_{(1)}\}}(1)} f(w)\lambda_{w}^{(2)} &= 0, \quad v_{(1)} \in \{(i,0)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{1\},\{v_{(2)}\}}(1)} f(w)\lambda_{w}^{(1)} &= 0, \quad v_{(2)} \in \{(0,j)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{2\},\{v_{(1)}\}}(1)} f(w)\lambda_{w}^{(2)} &= 0, \quad v_{(1)} \in \{(1,0), (2,0)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{2\},\{v_{(2)}\}}(i)} f(w)\lambda_{w}^{(1)} &= 0, \quad v_{(2)} \in \{(0,j)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{1\},\{v_{(2)}\}}(1)} f(w)\lambda_{w}^{(2)} &= 0, \quad v_{(1)} \in \{(i,0)\}, \\
  \sum_{w \in \text{\text{sib}}_{\{2\},\{v_{(1)}\}}(1)} f(w)\lambda_{w}^{(2)} &= 0, \quad v_{(1)} \in \{(i,0)\},
\end{align*}
\]
which is same as
\[ f(1, 1)\lambda_{(1,1)}^{(1)} + f(2, 1)\lambda_{(2,1)}^{(1)} = 0, \quad f(1, 2)\lambda_{(1,2)}^{(1)} + f(2, 2)\lambda_{(2,2)}^{(1)} = 0. \]
\[ f(1, 1)\lambda_{(1,1)}^{(2)} + f(1, 2)\lambda_{(1,2)}^{(2)} = 0, \quad f(2, 1)\lambda_{(2,1)}^{(2)} + f(2, 2)\lambda_{(2,2)}^{(2)} = 0. \]
\[ f(i, 1)\lambda_{(i,1)}^{(1)} = 0, \quad f(i, 2)\lambda_{(i,2)}^{(1)} = 0 \]
\[ f(i, 1)\lambda_{(i,1)}^{(2)} + f(i, 2)\lambda_{(i,2)}^{(2)} = 0 \]
\[ f(1, j)\lambda_{(1,j)}^{(1)} + f(2, j)\lambda_{(2,j)}^{(1)} = 0 \]
\[ f(1, j)\lambda_{(1,j)}^{(2)} + f(2, j)\lambda_{(2,j)}^{(2)} = 0 \]
\[ f(i, j)\lambda_{(i,j)}^{(1)} = 0 \]
\[ f(i, j)\lambda_{(i,j)}^{(2)} = 0. \]

Let \( W := \{(i, j) \in V : i \geq 3 \text{ or } j \geq 3\} \cup \{(0, 0)\}. \) Then \( f \in E \oplus \{e_{\text{root}}\} \) if and only if \( f \in L^2(V \setminus W) \) satisfies the following systems of equations:
\[ L_{(1,0),\{1\}}[f(1, 0), f(2, 0)]^T = 0, \]
\[ L_{(0,1),\{2\}}[f(0, 1), f(0, 2)]^T = 0, \]
\[ L_{(1,1),\{1,2\}}[f(1, 1), f(2, 1), f(1, 2), f(2, 2)]^T = 0, \]
where \( X^T \) denotes the transpose of a column vector \( X \). Here \( L_{(1,0),\{1\}} = [\lambda_{(1,0),1}^{(1)} \quad \lambda_{(1,0),2}^{(1)} \quad 0 \quad 0] \), \( L_{(0,1),\{2\}} = [\lambda_{(0,1),1}^{(2)} \quad \lambda_{(0,1),2}^{(2)} \quad 0 \quad 0] \), and

\[
L_{(1,1),\{1,2\}} = \begin{bmatrix}
\lambda_{(1,1)}^{(1)} & \lambda_{(2,1)}^{(1)} & 0 & 0 \\
0 & 0 & \lambda_{(1,2)}^{(1)} & \lambda_{(2,2)}^{(1)} \\
\lambda_{(1,1)}^{(2)} & 0 & \lambda_{(1,2)}^{(2)} & 0 \\
0 & \lambda_{(2,1)}^{(2)} & 0 & \lambda_{(2,2)}^{(2)}
\end{bmatrix}.
\]

Note that the rank of \( L_{(1,1),\{1,2\}} \) is at least 3. By Schur’s formula \[91\] Theorem 1.1, the determinant of \( L_{(1,1),\{1,2\}} \) is zero if and only if
\[ \lambda_{(1,2)}^{(1)}\lambda_{(2,1)}^{(1)}\lambda_{(1,1)}^{(2)} = \lambda_{(1,1)}^{(1)}\lambda_{(2,2)}^{(1)}\lambda_{(1,2)}^{(2)}. \]

Thus any \( f \in E \) takes the form
\[
f = f(0, 0)e_{(0,0)} + \sum_{v \in V \setminus W} f(v)e_v
\]
\[
= f(0, 0)e_{(0,0)} + f(1, 0)(e_{(1,0)} + a_1e_{(2,0)})
+ f(0, 1)(e_{(0,1)} + a_2e_{(0,2)}) + g_{(1,1)},
\]
where \( g_{(1,1)} \) is given by
\[
g_{(1,1)} = \begin{cases}
0 & \text{if rank } L_{(1,1),\{1,2\}} = 4, \\
f(1, 1)(e_{(1,1)} + b_1e_{(2,1)} + b_2e_{(1,2)} + b_3e_{(2,2)}) & \text{otherwise}.
\end{cases}
\]
Further, the scalars $a_1, a_2, b_1, b_2, b_3$ are given by

\[
\begin{align*}
a_1 &= -\frac{\lambda^{(1)}_{(1,0)}}{\lambda^{(1)}_{(1,2)}}, & a_2 &= -\frac{\lambda^{(2)}_{(0,1)}}{\lambda^{(2)}_{(0,2)}}, \\
b_1 &= -\frac{\lambda^{(1)}_{(1,1)}}{\lambda^{(1)}_{(1,2)}}, & b_2 &= -\frac{\lambda^{(2)}_{(1,1)}}{\lambda^{(2)}_{(1,2)}}, & b_3 &= \frac{\lambda^{(2)}_{(2,1)} \lambda^{(1)}_{(1,1)}}{\lambda^{(2)}_{(2,2)} \lambda^{(1)}_{(2,1)}}.
\end{align*}
\]

Thus the dimension of $E$ is either 3 or 4.

As a key step in the proof of Theorem 4.0.1, we need to understand the orthogonal complement of the subspace $\mathcal{L}_{u,F}$ of $l^2(\text{sib}_F(u))$.

**Lemma 4.1.10.** Let $\mathcal{T} = (V,E)$ be the directed Cartesian product of locally finite, rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$ and let $S_\lambda = (S_1, \cdots, S_d)$ be a commuting multishift on $\mathcal{T}$. Let $\mathcal{L}_{u,F}$ be the subspace as given in (4.11). Then

\[
l^2(\text{sib}_F(u)) \ominus \mathcal{L}_{u,F} = \bigvee \{S_\lambda e_{\text{par}(u_i)} : v_G \in \text{sib}_{F,G}(u) \text{ with } G = F \setminus \{i\}, i \in F\}.
\]

**Proof.** For $v_G \in \text{sib}_{F,G}(u)$, note that

\[
S_\lambda e_{\text{par}(u_i)} = \sum_{w \in \text{Chi}(v_G|\text{par}(u_i))} \lambda^{(i)}_w e_w = \sum_{w \in \text{sib}_i(v_G|u_i)} \lambda^{(i)}_w e_w.
\]

Thus, for $f \in l^2(\text{sib}_F(u))$, by Lemma 4.1.5(ii),

\[
\langle f, S_\lambda e_{\text{par}(u_i)} \rangle = \left( \sum_{w \in \text{sib}_i(v_G|u_i)} \sum_{w \in \text{sib}_i(v_G|u_i)} f(w) \lambda^{(i)}_w, \sum_{w \in \text{sib}_i(v_G|u_i)} \lambda^{(i)}_w e_w \right) = \sum_{w \in \text{sib}_i(v_G|u_i)} f(w) \lambda^{(i)}_w.
\]

In particular, $f \in l^2(\text{sib}_F(u))$ is orthogonal to $S_\lambda e_{\text{par}(u_i)}$ for every $v_G \in \text{sib}_{F,G}(u)$ and $i \in F$ if and only if $f$ satisfies the system (4.10). The latter one holds if and only if $f \in \mathcal{L}_{u,F}$. This yields the desired formula.

We are now ready to complete the derivation of the wandering subspace property for $S_\lambda$.

**Proof of Theorem 4.0.1.** Let $E$ denote the joint kernel of $S_\lambda^*$. Since $e_{\text{root}} \in E$, it is enough to show that for every nonempty $F \subseteq \{1, \cdots, d\}$,

\[
l^2(V_F) \subseteq [E]_{S_\lambda} = \bigcup_{\alpha \in \mathbb{N}^d} \{S_\lambda^* f : f \in E\},
\]

where $V_F = \bigcup_{G \in \mathcal{P}(F)} \Phi_G$. Fix a nonempty subset $F$ of $\{1, \cdots, d\}$. For $l = 0, \cdots, \text{card}(F)$, set

\[
\mathcal{F}_l := \{e_v \in l^2(V) : v \in \Phi_G \text{ with } G \in \mathcal{P}(F) \text{ and } \text{card}(G) \leq l\}.
\]

Since $\bigvee_{\mathcal{F}_{\text{card}(F)}} = l^2(V_F)$, it suffices to check that

\[
\mathcal{F}_{l-1} \subseteq [E]_{S_\lambda} \Rightarrow \mathcal{F}_l \subseteq [E]_{S_\lambda}, \quad l = 1, \cdots, \text{card}(F).
\]

To this end, fix $1 \leq l \leq \text{card}(F)$, and assume that $\mathcal{F}_{l-1} \subseteq [E]_{S_\lambda}$. Let $G \in \mathcal{P}(F)$ with $\text{card}(G) = l$. In particular,

\[
e_{v_{G \setminus \{i\}}} \in [E]_{S_\lambda} \text{ for every } i \in G \text{ and } v \in \Phi_G.
\]

(4.13)
We must check that \( e_v \in [E]_{S_\lambda} \) for every \( v \in \Phi_G \).

We prove by induction on \( k \in \mathbb{N} \) the following statement: For every \( i \in G \) and \( v \in \Phi_G \), \( e_{v_{G\setminus\{i\}^{(w_i)}}} \in [E]_{S_\lambda} \) for all \( w_i \in \text{Chi}^{(k)}(\text{root}_i) \). In view of Lemma 4.1.10, this statement holds trivially for \( k = 0 \) since \( v_{G\setminus\{i\}^{(\text{root}_i)}} \). Let us assume the inductive statement for an integer \( k \geq 0 \) and let \( w_i \in \text{Chi}^{(k+1)}(\text{root}_i) \). By induction hypothesis, \( e_{v_{G\setminus\{i\}^{(\text{par}(w_i)^})}} \in [E]_{S_\lambda} \) for every \( i \in G \) and \( v \in \Phi_G \). It follows from Lemma 4.1.10 with \( u := v_{G\setminus\{i\}^{(w_i)}} \) that

\[
I^2(\text{sib}_G(u)) \ominus L_{u,G} = \bigvee \left\{ S_i e_{v_{G\setminus\{i\}^{(\text{par}(w_i)^})}} : v_{G\setminus\{i\}^{(w_i)}} \in \text{sib}_G, G\setminus\{i\}(u), i \in G \right\}
\subseteq [E]_{S_\lambda}.
\]

But we already know that \( L_{u,G} \subseteq E \) and hence

\[
e_{v_{G\setminus\{i\}^{(w_i)}}} \in I^2(\text{sib}_G(u)) = L_{u,G} \oplus (L_{u,G})^\perp \subseteq [E]_{S_\lambda}.
\]

This completes the proof of induction on \( k \in \mathbb{N} \).

To complete the proof, let \( v \in \Phi_G \). Thus \( v = (v_1, \cdots, v_d) \) with \( v_j \in V_j^\circ \) for \( j \in G \) and \( v_j = \text{root}_j \) for \( j \notin G \). Since \( v_i \in \text{Chi}^{(\alpha_i)}(\text{root}_i) \), we obtain \( e_v = e_{v_{G\setminus\{i\}^{(w_i)}}} \in [E]_{S_\lambda} \). This completes the proof of the theorem. \( \square \)

In the remaining part of this section, we discuss some immediate consequences of Theorem 4.1.1.

Let \( T = (T_1, \cdots, T_d) \) be a commuting \( d \)-tuple on a Hilbert space \( \mathcal{H} \). A subspace \( \mathcal{M} \) of \( \mathcal{H} \) is said to be cyclic for \( T \) if

\[
\mathcal{H} = \bigvee \{T^\alpha h : h \in \mathcal{M}, \alpha \in \mathbb{N}^d \}.
\]

We say that \( T \) is finitely multicyclic if there exists a finite dimensional cyclic subspace for \( T \).

**Corollary 4.1.11.** Let \( \mathcal{T} = (V, \mathcal{E}) \) be the directed Cartesian product of locally finite, rooted directed trees \( \mathcal{T}_1, \cdots, \mathcal{T}_d \) and let \( S_\lambda \) be a commuting multishift on \( \mathcal{T} \). If \( \mathcal{T} \) has finite joint branching index, then \( S_\lambda \) is finitely multicyclic with cyclic subspace the joint kernel \( E \) of \( S_\lambda^r \). In this case, \( \dim \ker(S_\lambda^r - \omega) \leq \dim E \) for every \( \omega \in \mathbb{C}_d \).

**Proof.** Assume that \( \mathcal{T} \) has finite joint branching index. The first part follows from Theorem 4.0.1 and Corollary 3.1.14. The idea of proof of second part seems to be known (see, for instance, [63] for the case \( d = 1 \)). Assume that \( \mathcal{T} \) has finite joint branching index and let \( \omega \in \mathbb{C}_d \). By Theorem 4.0.1

\[
\bigvee_{\alpha \in \mathbb{N}^d} S_\lambda^r(E) = I^2(V).
\]

Since \( S_\lambda^r \) is a finite linear combination of terms of the form \( (S_\lambda - \omega)^\beta \) for \( \beta \in \mathbb{N}^d \), we must have

\[
\bigvee_{\alpha \in \mathbb{N}^d} (S_\lambda - \omega)^\alpha(E) = I^2(V).
\]
If $P_\omega$ denotes the orthogonal projection of $l^2(V)$ onto $\ker(S_\lambda^* - \omega)$, then $P_\omega(S_j - \overline{\omega}_j) = 0$ for any $j = 1, \cdots, d$. It follows that

$$\ker(S_\lambda^* - \omega) = P_\omega l^2(V) = P_\omega \bigvee_{\alpha \in \mathbb{N}^d} (S_\lambda - \overline{\omega})^\alpha(E)$$

$$= P_\omega E + P_\omega \left( \bigvee_{\alpha \in \mathbb{N}^d} (S_\lambda - \overline{\omega})^\alpha(E) \right)$$

$$= P_\omega E + \bigvee_{j=1}^d \bigvee_{\alpha \in \mathbb{N}^d} P_\omega (S_j - \overline{\omega}_j)^{\alpha_j} (S_\lambda - \overline{\omega})^{\alpha - \alpha_j}(E) = P_\omega E,$$

since $P_\omega(S_j - \overline{\omega}_j)^k = 0$ for $k \neq 0$ and $j = 1, \cdots, d$. Hence the dimension of $\ker(S_\lambda^* - \omega)$ is at most $\dim E$. \hfill \Box

Recall that $l^2_\mathcal{M}(\mathbb{N}^d)$ is defined as the Hilbert space of square-summable multisequence $\{h_\alpha\}_{\alpha \in \mathbb{N}^d}$ in $\mathcal{M}$, where $\mathcal{M}$ is a nonzero complex Hilbert space. If $\{W_\alpha^{(j)}\}_{\alpha \in \mathbb{N}^d} \subseteq B(\mathcal{M})$ for $j = 1, \cdots, d$, then the linear operator $W_j$ in $l^2_\mathcal{M}(\mathbb{N}^d)$ is defined by $W_j(h_\alpha)_{\alpha \in \mathbb{N}^d} = (k_\alpha)_{\alpha \in \mathbb{N}^d}$ for $(h_\alpha)_{\alpha \in \mathbb{N}^d} \in \mathcal{D}$, where

$$k_\alpha = \begin{cases} W_\alpha^{(j)} h_{\alpha - \epsilon_j} & \text{if } \alpha_j \geq 1, \\ 0 & \text{if } \alpha_j = 0 \end{cases}$$

and $\mathcal{D} := \{(h_\alpha)_{\alpha \in \mathbb{N}^d} \in l^2_\mathcal{M}(\mathbb{N}^d) : (k_\alpha)_{\alpha \in \mathbb{N}^d} \in l^2_\mathcal{M}(\mathbb{N}^d)\}$. If we use the convention that $W^{(j)}_\alpha = 0 = h_\alpha$ whenever $\alpha_j < 0$, then the definition of $W_j$ can be rewritten as $W_j(h_\alpha)_{\alpha \in \mathbb{N}^d} = (W_\alpha^{(j)} h_{\alpha - \epsilon_j})_{\alpha \in \mathbb{N}^d}$. We refer to the $d$-tuple $W = (W_1, \cdots, W_d)$ as an operator valued multishift with operator weights $\{W_\alpha^{(j)} : \alpha \in \mathbb{N}^d, j = 1, \cdots, d\}$ (for one variable counter-part of operator valued multishift, the reader is referred to [76]). Note that $W$ is the classical multishift $S_\omega$ in case $\mathcal{M}$ is the one dimensional complex Hilbert space.

**Corollary 4.1.12.** Let $\mathcal{I} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Let $S_\lambda$ be a total left invertible multishift on $\mathcal{I}$ and let $E$ denote the joint kernel of $S_\lambda$. If the multisequence $\{S_\lambda^* E\}_{\alpha \in \mathbb{N}^d}$ of linear manifolds of $l^2(V)$ is mutually orthogonal, then $S_\lambda$ is unitarily equivalent to a commuting operator valued multishift $W$ on $l^2_E(\mathbb{N}^d)$ with invertible weights.

**Proof.** The proof relies on the technique employed in [7] Theorem 3.3]. Assume that the multisequence $\{S_\lambda^* E\}_{\alpha \in \mathbb{N}^d}$ is mutually orthogonal. Since the operator $S_\lambda$ is total left invertible, $S_\lambda^*$ is left invertible for every $\alpha \in \mathbb{N}^d$. Hence we deduce that $\mathcal{M}_\alpha := S_\lambda^*(E) (\alpha \in \mathbb{N}^d)$ is a subspace of $l^2(V)$ and $\dim \mathcal{M}_\alpha = \dim E$ for every $\alpha \in \mathbb{N}^d$. For $\alpha \in \mathbb{N}^d$, let $U_\alpha : \mathcal{M}_\alpha \rightarrow E$ be any isometric isomorphism. By Theorem 4.0.1 $l^2(V) = \bigoplus_{\alpha \in \mathbb{N}^d} S_\lambda^*(E) = \bigoplus_{\alpha \in \mathbb{N}^d} \mathcal{M}_\alpha$. We can now define the isometric isomorphism $U : l^2(V) \rightarrow l^2_E(\mathbb{N}^d)$ by

$$U(\oplus_{\alpha \in \mathbb{N}^d} h_\alpha) := (U_\alpha h_\alpha)_{\alpha \in \mathbb{N}^d}, \oplus_{\alpha \in \mathbb{N}^d} h_\alpha \in l^2(V).$$

Consider the operator valued multishift $W$ on $l^2_\mathcal{M}(\mathbb{N}^d)$ with weights $\{W_\alpha^{(j)} := U_{\alpha + \epsilon_j} S_j | \mathcal{M}_\alpha U_{\alpha}^{-1}\}_{\alpha \in \mathbb{N}^d} \subseteq B(E)$ for $j = 1, \cdots, d$. Then for $\oplus_{\alpha \in \mathbb{N}^d} h_\alpha \in l^2(V)$ with
at most finitely many nonzero terms \( h_\alpha \),
\[
US_j(\oplus_{\alpha \in \mathbb{N}^d} h_\alpha) = U(\oplus_{\alpha \in \mathbb{N}^d} S_j | M \alpha h_\alpha) = (U \alpha S_j | M \alpha^{-1} h_\alpha^{-1})_{\alpha \in \mathbb{N}^d} = (W^{(j)}_{\alpha^{-1}}, U \alpha^{-1} h_\alpha^{-1})_{\alpha \in \mathbb{N}^d} = W_j U(\oplus_{\alpha \in \mathbb{N}^d} h_\alpha).
\]
This shows that \( US_j U^* \) agrees with \( W_j \) on a dense linear manifold of \( L_E^2(\mathbb{N}^d) \), and hence \( W_j \) must be a bounded linear operator on \( L_E^2(\mathbb{N}^d) \). Since \( S_\alpha \) is commuting, so is \( W \). Finally, since \( S_\alpha \) is toral left-invertible, each \( W^{(j)}_{\alpha} \) is invertible in \( B(E) \). \( \square \)

Remark 4.1.13. The converse of the above result is trivially true. We note further that in case \( E \) is finite dimensional, the conclusion of the corollary holds without the assumption of toral left invertibility of \( S_\alpha \). A slight modification of the argument above together with the injectivity of \( S_1, \ldots, S_d \), as ensured by Corollary 3.3.4, yields the desired conclusion.

4.2. Multishifts admitting Shimorin’s Analytic Model

In this section, we discuss a subclass of multishifts \( S_\alpha \) on \( \mathcal{T} \) satisfying a kernel condition (cf. \[30]\], Equation (5.3)). In particular, we obtain an analytic model of Shimorin’s analytic model \[87\] does not naturally extend to all toral left invertible multishifts. Indeed, the toral left invertible multishifts admitting Shimorin’s model must belong to the aforementioned class (see Remark 4.2.5). Before we introduce this class, we need a lemma pertaining to toral Cauchy dual of multishifts.

Lemma 4.2.1. Let \( \mathcal{T} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \). Let \( S_\alpha = (S_1, \ldots, S_d) \) be a toral left invertible multishift on \( \mathcal{T} \). Then the toral Cauchy dual \( S_\alpha^* = (S_1^*, \ldots, S_d^*) \) of \( S_\alpha \) is given by
\[
S_j^* e_v = \frac{1}{\|S_j e_v\|^2} \sum_{w \in \text{Chi}_j(v)} \lambda^{(j)}_w e_w \text{ for all } v \in V \text{ and } j = 1, \ldots, d.
\]
In particular, \( S_\alpha^* \) is a multishift with weights
\[
\left\{ \frac{\lambda^{(j)}_w}{\|S_j e_v\|^2} : w \in \text{Chi}_j(v), \ v \in V, \ j = 1, \ldots, d \right\}.
\]

Proof. This follows from the definition of \( S_j^* \) (see (1.2)). \( \square \)

Definition 4.2.2. Let \( T = (T_1, \ldots, T_d) \) be a toral left invertible commuting \( d \)-tuple on \( \mathcal{H} \) and let \( E \) denote the joint kernel of \( T^* \). Let \( T^I \) be the toral Cauchy dual of \( T \). We say that \( T \) satisfies kernel condition (\( \mathfrak{K} \)) if
\[
E \subseteq \ker T_j^* T_{\alpha}^* \text{ for all } j = 1, \ldots, d \text{ and for all } \alpha \in \mathbb{N}^d,
\]
where \( T_{\alpha}^* := \prod_{\alpha \neq j} T_j^* \) for \( \alpha \in \mathbb{N}^d \) and \( j = 1, \ldots, d \).

Remark 4.2.3. Note that the kernel condition (\( \mathfrak{K} \)) is satisfied in any one of the following cases:

(i) the dimension \( d = 1 \).
(ii) \( T \) is doubly commuting.
(iii) \( T \) is a commuting operator valued multishift \( W \).
In case dimension 2, \( S_\lambda \) satisfies kernel condition \((\mathfrak{R})\) if and only if
\[
E \subseteq (\ker S_1^1 S_2^{\alpha_2}) \cap (\ker S_2^2 S_1^{\alpha_1}) \quad \text{for all } (\alpha_1, \alpha_2) \in \mathbb{N}^2.
\] (4.14)

In general, \( S_\lambda \) does not satisfy the kernel condition \((\mathfrak{R})\). Indeed, a rather tedious calculation shows that for \( S_\lambda \) on \( \mathcal{T}_{2,0} \times \mathcal{T}_{2,0} \), as discussed in Example 2.1.7, \( f := \lambda^{(1)}(2,0) \epsilon^{(1)}(2,0) - \lambda^{(1)}(1,0) \epsilon^{(2,0)} \in E \) does not belong to \( \ker S_1^1 S_2^2 \) for suitable choice of weights \( \lambda \).

The following provides a multivariable counterpart of [33] Theorem 2.2.

**Theorem 4.2.4.** Let \( \mathcal{T} = (V, \mathcal{E}) \) be the directed Cartesian product of locally finite, rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) and let \( S_\lambda = (S_1, \ldots, S_d) \) be a toral left invertible multishift on \( \mathcal{T} \). Let \( E \) be the joint kernel of \( S_\lambda^* \). Assume that the toral Cauchy dual \( S_\lambda^* = (S_1^1, \ldots, S_d^1) \) of \( S_\lambda \) is commuting and let
\[
r := (r(S_1^1)^{-1}, \ldots, r(S_d^1)^{-1}),
\]
where \( r(T) \) denotes the spectral radius of a bounded linear operator \( T \). If \( S_\lambda \) satisfies the kernel condition \((\mathfrak{R})\), then there exist a reproducing kernel Hilbert space \( \mathcal{H} \) of \( E \)-valued holomorphic functions defined on the polydisc \( \mathcal{D}^d \) and a unitary \( U : l^2(V) \rightarrow \mathcal{H} \) such that \( US_j = \mathcal{M}_{z_j}U \) for \( j = 1, \ldots, d \). If, in addition, \( \mathcal{T} \) has finite joint branching index \( k_{\mathcal{T}} \), then the reproducing kernel \( \kappa \) of \( \mathcal{H} \) is given by
\[
\kappa_{\mathcal{H}}(z, w) = \sum_{\alpha, \beta \in \mathbb{N}^d \mid \alpha_j = 0, 1 \leq j \leq d} P_E S_\lambda^{(\alpha)} S_\lambda^{(\beta)} |_{E} z^{\alpha} w^{\beta} \quad (z, w \in \mathbb{D}_r^d). \] (4.15)

**Proof.** The proof relies on Shimorin’s technique as presented in [37] and the wandering subspace property of \( S_\lambda \) as obtained in Theorem 4.0.1. Assume that \( S_\lambda \) satisfies the kernel condition \((\mathfrak{R})\). For \( f \in l^2(V) \), define
\[
U_f(z) := \sum_{\alpha \in \mathbb{N}^d} (P_E S_\lambda^{(\alpha)} f) z^{\alpha}, \quad z \in \mathbb{C}^d.
\]
Then the power series \( U_f \) converges absolutely on the polydisc \( \mathbb{D}_r^d \) of polyradius \( r \). Let \( \mathcal{H} \) denote the complex vector space of \( E \)-valued holomorphic functions of the form \( U_f \). Thus \( U : l^2(V) \rightarrow \mathcal{H} \) defines a map from \( l^2(V) \) onto \( \mathcal{H} \) given by \( U(f) = U_f \). Now we show that \( U \) is injective.

To this end, let \( U_f = 0 \) for some \( f \in l^2(V) \). Then \( \sum_{\alpha \in \mathbb{N}^d} (P_E S_\lambda^{(\alpha)} f) z^{\alpha} = 0 \) which implies that \( P_E S_\lambda^{(\alpha)} f = 0 \) for all \( \alpha \in \mathbb{N}^d \). Note that \( S_1^1 \) is also a multishift and the joint-kernel of \( S_\lambda^* \) is equal to \( E \). Hence by Theorem 4.0.1 we get that \( V_{\alpha \in \mathbb{N}^d} S_\lambda^{(\alpha)} (E) = l^2(V) \). By taking orthogonal complement on both sides, we get \( \bigcap_{\alpha \in \mathbb{N}^d} (S_\lambda^{(\alpha)} (E))^\perp = \{0\} \). It is easy to see that \( (S_\lambda^{(\alpha)} (E))^\perp = \ker P_E S_\lambda^{(\alpha)} \) for any \( \alpha \in \mathbb{N}^d \). Hence \( \bigcap_{\alpha \in \mathbb{N}^d} \ker P_E S_\lambda^{(\alpha)} = \{0\} \). Since \( P_E S_\lambda^{(\alpha)} f = 0 \) for all \( \alpha \in \mathbb{N}^d \), we must have \( f \in \bigcap_{\alpha \in \mathbb{N}^d} \ker P_E S_\lambda^{(\alpha)} \). This shows that \( f = 0 \), and hence \( U \) is injective.
We now define the inner product on \( \mathcal{H} \) as \( \langle U_f, U_g \rangle = \langle f, g \rangle_{L^2(V)} \) for all \( f, g \in L^2(V) \). Then \( \mathcal{H} \) becomes a Hilbert space and \( U \) a unitary. Also, for \( f \in L^2(V) \),

\[
(U S_j f)(z) = \sum_{\alpha \in \mathbb{N}^d} (P_E S_{\lambda}^{\alpha_\lambda} S_j f) z^\alpha = \sum_{\alpha \in \mathbb{N}^d} (P_E S_{\lambda}^{\alpha_\lambda} S_j f) z^\alpha + \sum_{\alpha \in \mathbb{N}^d \at \alpha_j > 1} (P_E S_{\lambda}^{\alpha_\lambda} S_j f) z^\alpha = \sum_{\alpha \in \mathbb{N}^d} (P_E S_{\lambda}^{\alpha_\lambda} S_j f) z^\alpha + \sum_{\alpha \in \mathbb{N}^d} (P_E S_{\lambda}^{\alpha_\lambda+\epsilon_j} S_j f) z^\alpha+\epsilon_j = \sum_{\alpha \in \mathbb{N}^d} (P_E S_{\lambda}^{\alpha_\lambda+\epsilon_j} S_j f) z^\alpha+\epsilon_j, \tag{4.16}
\]

where we used the kernel condition \((\mathfrak{R})\) to get the last equality. Since the toral Cauchy dual \( S_{\lambda}^{\alpha_\lambda} \) is commuting and \( S_{\lambda}^{\alpha_\lambda} S_j = I \), the sum on the right hand side of \((4.16)\) is equal to \( z_j \sum_{\alpha \in \mathbb{N}^d} (P_E S_{\lambda}^{\alpha_\lambda} f) z^\alpha = z_j U_f(z) \). Thus we get \( U S_j = \mathcal{M}_z U \).

We skip the verification of

\[
\kappa_{\mathcal{F}}(z, w) = \sum_{\alpha, \beta \in \mathbb{N}^d} P_E S_{\lambda}^{\alpha_\lambda} S_{\beta}^{\beta_\beta} |E z^\alpha w^\beta = P_E \prod_{i=1}^d (I - z_i S_i^{\approx})^{-1} \prod_{j=1}^d (I - w_j S_j^{\approx})^{-1} |E,
\]

for \( z, w \in \mathbb{D}^d \), since it is along the lines of [87 Proposition 2.13]. It is now easy to see that

\[
\langle U_f, \kappa_{\mathcal{F}}(\cdot, w) g \rangle_{\mathcal{F}} = \langle U_f(w), g \rangle (f, g \in E, \ w \in \mathbb{D}^d).
\]

Thus \( \mathcal{H} \) is a reproducing kernel Hilbert space with kernel \( \kappa \).

Assume further that \( \mathcal{F} \) has finite joint branching index \( k_{\mathcal{F}} \). To check that \( \kappa \) has the form given in \((4.15)\), let \( \alpha, \beta \in \mathbb{N}^d \) be such that \( |\alpha_j - \beta_j| > k_{\mathcal{F}} \) for some \( j = 1, \cdots, d \). In view of Proposition \( 3.1.13 \), it suffices to check that \( P_E S_{\lambda}^{\alpha_\lambda} S_{\beta}^{\beta_\beta} e_u = 0 \) for all \( v \in F_1 \times \cdots \times F_d \), where \( F_1 := \text{Chi}(V_{\lambda}^{(j)}) \cup \{\text{root}_j\} \) \((j = 1, \cdots, d)\). To see this, let \( v \in F_1 \times \cdots \times F_d \). Since the depth \( \alpha_v \) of \( v \) equals \((\alpha_v_1, \cdots, \alpha_v_d)\) with \( \alpha_v_j \) being the depth of \( v_j \) in \( \mathcal{F}_j \), we obtain \( 0 \leq \alpha_v_j \leq k_{\mathcal{F}_j} \) for every \( j = 1, \cdots, d \). An application of Proposition \( 3.1.7 \) \( C_i \) shows that

\[
S_{\lambda}^{\alpha_\lambda} S_{\beta}^{\beta_\beta} e_u = \sum_{u \in \text{par}^{<\alpha_\lambda>}(\text{Chi}^{<\beta_\beta>(v))} \gamma_u e_u
\]

for some scalars \( \gamma_u \in \mathbb{C} \). It follows that \( \alpha_u = \alpha_v + \beta - \alpha \) for \( u \in \text{par}^{<\alpha_\lambda>}(\text{Chi}^{<\beta_\beta>(v))} \). Note that \( \beta_j - \alpha_j = \alpha_u_j - \alpha_v_j \geq \alpha_u_j - k_{\mathcal{F}_j} \) and hence

\[
\beta_j - \alpha_j + |\beta_j - \alpha_j| > \beta_j - \alpha_j + k_{\mathcal{F}_j} \geq \alpha_u_j \geq 0.
\]

This shows that \( \beta_j - \alpha_j > 0 \), and consequently,

\[
\alpha_u_j = \alpha_v_j + \beta_j - \alpha_j = \alpha_v_j + |\alpha_j - \beta_j| > k_{\mathcal{F}_j}.
\]

Thus \( u \notin F_1 \times \cdots \times F_d \), and hence by Proposition \( 3.1.13 \) \( P_E S_{\lambda}^{\alpha_\lambda} S_{\beta}^{\beta_\beta} e_u = 0 \). \( \square \)
Remark 4.2.5. The kernel condition \((\mathcal{R})\) is used only in obtaining the intertwining relation \(US_{\lambda}^1 = \mathcal{M}_zU\). Conversely, if one assumes the above intertwining relation then the calculations in (4.16) shows that the kernel condition \((\mathcal{R})\) is also necessary.

It is interesting to know the maximum value of \(r\) for which \(\kappa_{\mathcal{M}}(z, w)\) converges on \(D_r^d \times D_r^d\). Unfortunately, we do not know this even in the one dimensional case (see Equation (2.3)). Before we proceed to the next result, it is convenient to introduce some terminology.

Let \(S_{\lambda}\) be a toral left invertible multishift on \(\mathcal{T}\). We refer to the pair \((\mathcal{M}_z, \kappa_{\mathcal{M}})\) as Shimorin’s analytic model.

If one relaxes the toral left invertibility of \(S_{\lambda}\) then it may happen that the interior of the Taylor spectrum of \(S_{\lambda}\) is empty (Example 3.4.5). This is not possible otherwise.

Corollary 4.2.6. If \(S_{\lambda}\) has Shimorin’s analytic model, then the polydisc \(D_r^d\) is contained in the point spectrum of \(S_{\lambda}\), where \(r := (r(S_{\lambda}^1)^{-1}, \cdots, r(S_{\lambda}^d)^{-1})\).

Remark 4.2.7. Since \(\text{cl}(\sigma_p(S_{\lambda}^1)) \subseteq \sigma(S_{\lambda}^1) = \sigma(S_{\lambda})\), we must have
\[
(r(S_{\lambda}^1)^{-2} + \cdots + r(S_{\lambda}^d)^{-2})^{\frac{1}{2}} \leq r(S_{\lambda}),
\]
where \(r(S_{\lambda})\) is the spectral radius of \(S_{\lambda}\).

Let us analyze Theorem 4.2.4 in case \(S_{\lambda}\) is a doubly commuting toral isometry.

Corollary 4.2.8. Let \(\mathcal{T} = (V, E)\) be the directed Cartesian product of locally finite, rooted directed trees \(\mathcal{T}_1, \cdots, \mathcal{T}_d\) and let \(S_{\lambda}\) be a toral isometry multishift on \(\mathcal{T}\) and let \(E\) denote the joint kernel of \(S_{\lambda}^1\). Then \(S_{\lambda}\) is doubly commuting if and only if \(S_{\lambda}\) is unitarily equivalent to the multiplication \(d\)-tuple \(\mathcal{M}_z\) on the \(E\)-valued Hardy space of the unit polydisc \(D_r^d\). In particular,
\[
\kappa_{\mathcal{M}}(z, w) = \prod_{j=1}^{d} \frac{I_E}{1 - z_j \overline{w}_j} (z, w \in D_r^d),
\]
where \(I_E\) denotes the identity operator.

Proof. Assume that \(S_{\lambda}\) is doubly commuting. Since \(S_{\lambda}\) is a toral isometry, \(S_{\lambda}^1 = S_{\lambda}\). Thus
\[
P_E S_{\lambda} \mathcal{M}_z S_{\lambda}^1 |_E = \delta_{\alpha\beta} I_E \ (\alpha, \beta \in \mathbb{N}^d),
\]
where \(\delta_{\alpha\beta}\) denotes the Kronecker delta. It now follows from (4.15) that \(\kappa\) has the desired form. The conclusion can now be drawn from Theorem 4.2.4 and the fact that the reproducing kernel uniquely determines the reproducing kernel Hilbert space [12]. We leave the converse to the interested reader.

Now we discuss a large class of multishifts \(S_{\lambda}\) (not covered by Remark 4.2.3), which always satisfy the kernel condition \((\mathcal{R})\).

Corollary 4.2.9. Let \(\mathcal{T}_1 = (V_1, E_1)\) be a locally finite, rooted directed tree and let \(\mathcal{T}_2 = \mathcal{T}_{1,0}\) be the rooted directed trees as described in Example 2.1.3. Consider the directed Cartesian product \(\mathcal{T}\) of \(\mathcal{T}_1\) and \(\mathcal{T}_2\). Let \(S_{\lambda} = (S_{\lambda}^1, S_{\lambda}^2)\) be a toral left invertible multishift on \(\mathcal{T}\) and let \(E\) be the joint kernel of \(S_{\lambda}^1\). Assume that the toral Cauchy dual \(S_{\lambda}^r = (S_{\lambda}^1, S_{\lambda}^2)\) of \(S_{\lambda}\) is commuting and let \(r := (r(S_{\lambda}^1)^{-1}, r(S_{\lambda}^2)^{-1})\),
where \( r(T) \) denotes the spectral radius of a bounded linear operator \( T \). Then \( S_\chi \) has Shimorin's analytic model \((\mathcal{M}_z, \kappa, \mathcal{F})\). If, in addition, \( \mathcal{F}_1 \) has finite branching index \( k_{\mathcal{F}_1} \), then the reproducing kernel \( \kappa \) of \( \mathcal{F} \) is given by

\[
\kappa_{\mathcal{F}}(z, w) = \sum_{\alpha \in \mathbb{N}^2} P_\alpha S_\chi^\alpha S_\chi^\alpha |_E \ z^{\alpha} \overline{w}^\alpha
+ \sum_{\substack{\alpha, \beta \in \mathbb{N}^2 \\ \alpha \geq \beta}} P_\alpha S_\chi^\alpha S_\chi^\beta |_E \ z^{\alpha} \overline{w}^\beta \ (z, w \in D_r^2).
\]

**Proof.** We first compute the joint kernel \( E \) of \( S_\chi \). The argument is similar to that of Example 4.1.8. Note that \( \mathcal{P} = \emptyset \). Also,

\[
\begin{align*}
\Phi_\emptyset &= \{(\text{root}_1, 0)\}, \\
\Phi_{\{1\}} &= \{(v, 0) : v \in V_1^\circ\}, \\
\Phi_{\{2\}} &= \{(\text{root}_1, j) : j \geq 1\}, \Phi_{\{1,2\}} = \{(v, j) : v \in V_1^\circ, j \geq 1\}.
\end{align*}
\]

Note further that

\[
\begin{align*}
\text{sib}_1(\text{root}_1, 0) &= \{(\text{root}_1, 0)\}, \\
\text{sib}_{\{1\}}(v, 0) &= \{(w, 0) : w \in \text{sib}(v) \} \ (v \in V_1^\circ), \\
\text{sib}_{\{2\}}(\text{root}_1, j) &= \{(\text{root}_1, j)\} \text{ for all } j \geq 1, \text{ and} \\
\text{sib}_{\{1,2\}}(v, j) &= \{(w, j) : w \in \text{sib}(v) \} \ (v \in V_1^\circ, j \geq 1).
\end{align*}
\]

Let us form \( \Omega_F \) by picking up one element from each of the equivalence classes \( \text{sib}_F(u) \) for every \( F \in \mathcal{P} \). We next calculate \( \text{sib}_{F,G}(u) \) for possible choices of \( F, G \), and \( u \in \Omega_F \). If \( F = \{1\} \), then \( G = \emptyset \). In this case,

\[
\text{sib}_{\{1\}, \emptyset}(v, 0) = \{(\text{root}_1, 0)\} \ (v \in V_1^\circ).
\]

This together with (4.1.9) yields the following equations:

\[
\begin{align*}
\sum_{w \in \text{Chi}(v)} f(w, 0) \lambda_{(w, 0)}^{(1)} &= 0 \ (v \in V_1^{(1)}), \\
f(w, 0) \lambda_{(w, 0)}^{(1)} &= 0 \ (w \in \text{Chi}(V_1 \setminus V_1^{(1)})).
\end{align*}
\]

In case \( F = \{2\} \), \( G = \emptyset \) and \( \text{sib}_{\{2\}, \emptyset}(\text{root}_1, j) = \{(\text{root}_1, 0)\} \) for \( j \geq 1 \), and hence we obtain the equations

\[
\lambda_{(\text{root}_1, j)}^{(2)} = 0 \ (j \geq 1).
\]

In case \( F = \{1,2\} \), \( G = \{1\} \) or \( \{2\} \). Then for all \( w \in V_1^\circ \) and \( j \geq 1 \),

\[
\text{sib}_{\{1,2\}}(w, j) = \{(\text{root}_1, j)\} \text{ and } \text{sib}_{\{1,2\}}(u, 0) = \{(u, 0) : u \in \text{sib}(w)\}.
\]

This gives following equations for \( j \geq 1 \),

\[
\begin{align*}
\sum_{u \in \text{Chi}(w)} f(u, j) \lambda_{(u, j)}^{(1)} &= 0 \ (w \in V_1^{(1)}), \\
f(w, j) \lambda_{(w, j)}^{(1)} &= 0 \ (w \in \text{Chi}(V_1 \setminus V_1^{(1)})), \\
f(w, j) \lambda_{(w, j)}^{(2)} &= 0 \ (w \in \text{Chi}(V_1^{(1)})), \\
f(w, j) \lambda_{(w, j)}^{(1)} &= 0 \ (w \in \text{Chi}(V_1 \setminus V_1^{(1)})).
\end{align*}
\]

Solving all the above equations, we get that

\[
E = \left[ e_{(\text{root}_1, 0)} \right] \bigoplus_{w \in V_1^{(1)}} \left( l^2(\text{Chi}(w) \times \{0\}) \oplus \{0 \} \right),
\]
where $\Gamma^{(1)}_{(w,0)} : \text{Chi}_1((w,0)) \to \mathbb{C}$ defined as $\Gamma^{(1)}_{(w,0)}(u,0) = \lambda^{(1)}_{(u,0)}$.

We next check that $S_{\lambda}$ satisfies the kernel condition (R). Since $E \subseteq \ker S_2^{\alpha_2}$ for all $\alpha_1 \in \mathbb{N}$, in view of (4.14), it suffices to check that

$$E \subseteq \ker S_1^*S_2^{\alpha_2} \text{ for all } \alpha_2 \in \mathbb{N}.$$  

Clearly, $e_{(\text{root},0)} \in \ker S_1^*S_2^{\alpha_2}$. For $w \in V_\lambda^{(1)}$, let $f = \sum_{u \in \text{Chi}(w)} f((u,0))e_{(u,0)} \in l^2(\text{Chi}(w) \times \{0\})$ be such that

$$\sum_{u \in \text{Chi}(w)} f((u,0))\lambda^{(1)}_{(u,0)} = 0 \ (w \in V_\lambda^{(1)}). \tag{4.17}$$

Note that

$$S_1^*S_2^{\alpha_2}f = S_1^* \left( \sum_{u \in \text{Chi}(w)} f((u,0)) \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \right) e_{(u,\alpha_2)} \lambda^{(1)}_{(u,\alpha_2)}.$$

Thus $S_1^*S_2^{\alpha_2}f = 0$ if and only if

$$\sum_{u \in \text{Chi}(w)} f((u,0)) \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \lambda^{(1)}_{(u,\alpha_2)} = 0.$$

Note that any general solution of (4.17) is of the form

$$f = \sum_{u \in \text{Chi}(w), u \neq v} f((u,0)) \left( e_{(u,0)} - e_{(v,0)} \frac{\lambda^{(1)}_{(u,0)}}{\lambda^{(1)}_{(v,0)}} \right)$$

for some fixed $v \in \text{Chi}(w)$. It follows that

$$\sum_{u \in \text{Chi}(w)} f((u,0)) \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \lambda^{(1)}_{(u,\alpha_2)} = \sum_{u \in \text{Chi}(w)} f((u,0)) \left( \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \right) \left( \frac{\lambda^{(1)}_{(u,\alpha_2)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} - \frac{\lambda^{(1)}_{(v,0)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(v,k)}} \right),$$

and hence it suffices to see that for every $u \in \text{Chi}(w) \setminus \{v\}$,

$$\frac{\lambda^{(1)}_{(u,\alpha_2)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} - \frac{\lambda^{(1)}_{(v,0)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(v,k)}} = 0.$$

However, by repeated applications of (3.2), we obtain

$$\lambda^{(1)}_{(u,\alpha_2)} \left( \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \right) = \lambda^{(1)}_{(v,\alpha_2)} \left( \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \lambda^{(1)}_{(v,\alpha_2)} \right),$$

$$\lambda^{(1)}_{(v,\alpha_2)} \left( \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \right) = \lambda^{(1)}_{(v,\alpha_2)} \left( \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)} \lambda^{(1)}_{(u,\alpha_2)} \right),$$

which shows that $S_{\lambda}$ satisfies the kernel condition (R). The desired conclusion now follows from Theorem 4.2.4 once we observe that $k_{\beta_2} = 0$. \hfill \Box

**Remark 4.2.10.** We discuss two special cases of the preceding corollary.
WANDERING SUBSPACE PROPERTY

(i) In case \( T_1 = T_{1,0} \), \( S_{\lambda} \) is nothing but the classical multishift and the associated kernel \( \kappa(z, w) \) is diagonal.

(ii) In case \( T_1 \) is \( T_{2,0} \), \( k_{T_1} = 1 \), and hence the kernel \( \kappa(z, w) \) is given by

\[
\kappa(z, w) = \sum_{\alpha \in \mathbb{N}^2} P_E S^\alpha \kappa_S |_E z^\alpha w^{\alpha+1} + \sum_{\alpha \in \mathbb{N}^2} P_E S^\alpha \kappa_S |_E z^\alpha w^{\alpha+1} \quad (z, w \in \mathbb{D}_r^2).
\]

We conclude this chapter with one application to the Cowen-Douglas theory.

Let \( \Omega \) be an open connected subset of \( \mathbb{C}^d \). For a positive integer \( n \), let \( B_n(\Omega) \) denote the set of all commuting \( d \)-tuples \( T \) on \( H \) satisfying the following conditions:

1. For every point \( \omega = (\omega_1, \cdots, \omega_d) \in \Omega \), we have
   a) the map \( D_{T-\omega}(x) = ((T_j - \omega_j)x) \) from \( H \) into \( H^d \) has closed range.
   b) \( \dim \ker(T - \omega) = n \).

2. The subspace \( \bigvee_{\omega \in \Omega} \ker(T - \omega) \) of \( H \) equals \( H \).

We will call the set \( B_n(\Omega) \) the Cowen-Douglas class of degree \( n \) with respect to \( \Omega \) (refer to [39, 47] for the basic theory of Cowen-Douglas class in one and several variables).

**Corollary 4.2.11.** Let \( \mathcal{I} = (V, E) \) be the directed Cartesian product of locally finite, rooted directed trees \( \mathcal{T}_1, \cdots, \mathcal{T}_d \) with finite joint branching index. If \( S_{\lambda} \) is a toral left invertible multishift which satisfies the kernel condition \( (K) \), then \( S_{\lambda}^* \) belongs to Cowen-Douglas class \( B_{\dim E}(\mathbb{D}_r^d) \), where \( E \) denotes the joint kernel of \( S_{\lambda} \) and \( r := (\|S_{\lambda 1}\|^{-1}, \cdots, \|S_{\lambda d}\|^{-1}) \).

**Proof.** Suppose that \( S_{\lambda} \) is a toral left invertible multishift satisfying the kernel condition \( (K) \). By Corollary 3.1.16, \( \dim E \) is finite. Also, by Theorem 4.0.1, the toral Cauchy dual \( S_{\lambda}^* \) possesses wandering subspace property. It may now be concluded from [30] Theorem 5.4 [2] that for any \( s \leq r \),

\[
I^2(V) = \bigvee_{\omega \in \mathbb{D}_r^d} \ker(S_{\lambda}^* - \omega) \quad \text{and} \quad \dim \ker(S_{\lambda}^* - \omega) \geq \dim E.
\]

However, by Corollary 4.1.11 we get \( \dim \ker(S_{\lambda}^* - \omega) = \dim E \). Thus it remains only to check (1)(a) of the definition of the Cowen-Douglas class. Also, \( \sigma_l(S_j) \cap \mathbb{D}_r^d \) = \( \emptyset \) for \( j = 1, \cdots, d \). Thus for any \( \omega \in \mathbb{D}_r^d \), \( S_j - \omega_j \) has closed range, and hence range of \( S_j^* - \omega_j \) is closed for \( j = 1, \cdots, d \). This shows that the range of \( D_{S_{\lambda}^* - \omega} \) is also closed for every \( \omega \in \mathbb{D}_r^d \). \( \square \)
CHAPTER 5

Special Classes of Multishifts

In this chapter, we discuss two classes of so-called balanced multishifts, namely torally balanced multishifts and spherically balanced multishifts (cf. [31, 75]). These generalize largely the classes of toral and spherical isometries ([18], [50], [53], [3]). In particular, we introduce tree analogs of the classical multishifts $S_w,a$ as discussed in Example 1.2.1. We show that these multishifts are unitarily equivalent to multiplication tuples acting on reproducing kernel Hilbert spaces of vector valued holomorphic functions defined on the unit ball. We also provide a compact formula for the associated reproducing kernels involving finitely many hypergeometric functions. We further investigate some known classes of multishifts which include mainly the well-studied class of joint subnormal tuples ([48] [16], [18], [88], [37], [19], [53], [8], [58]), and comparatively less understood class of joint hyponormal tuples [17], [45], [46], [43], [44]. We emphasize, in particular, on characterizations of these classes within spherically balanced multishifts.

5.1. Torally Balanced Multishifts

Before we introduce the class of torally balanced multishifts, let us understand somewhat related class of multishifts with commuting toral Cauchy dual. The later class admits a polar decomposition in the following sense.

**Proposition 5.1.1.** Let $T = (V, E)$ be the directed Cartesian product of rooted directed trees $T_1, \cdots, T_d$. Let $S_\lambda = (S_1, \cdots, S_d)$ be a toral left invertible multishift on $T$ with toral Cauchy dual $S_\lambda^t$. Then $S_\lambda^t$ is commuting if and only if there exist a toral isometry multishift $U_\theta = (U_1, \cdots, U_d)$ on $T$ and a commuting $d$-tuple $D = (D_1, \cdots, D_d)$ of diagonal, positive, invertible bounded linear operators on $l_2(V)$ such that

$$S_j = U_j D_j, \ j = 1, \cdots, d. \quad (5.1)$$

Further, this decomposition is unique.

**Proof.** Let us first see the uniqueness of the above decomposition. Indeed, if (5.1) holds then for $j = 1, \cdots, d,$

$$S_j^* S_j = D_j U_j^* U_j D_j = D_j^2,$$

and hence $D_j$ must be the positive square root of $S_j^* S_j$. Also, since $D_j$ is invertible, $U_j = S_j D_j^{-1}$ for $j = 1, \cdots, d.$

By Proposition 3.1.7 and Lemma 4.2.1, $S_\lambda^t$ is commuting if and only if

$$\|S_j e_{par_j(v)}\| \|S_i e_{par_i, par_j(v)}\| = \|S_i e_{par_i(v)}\| \|S_j e_{par_i, par_j(v)}\| \quad (5.2)$$
for all \( v \in V \) and \( i, j = 1, \cdots, d \). Consider now the multishift \( U_\theta = (U_1, \cdots, U_d) \) with weights given by
\[
\theta^{(j)}_w := \frac{\lambda^{(j)}_w}{\|S_j e_v\|} \quad \text{for} \ w \in \text{Chi}_j(v), \ v \in V \ \text{and} \ j = 1, \cdots, d.
\]
(5.3)

Since \( S_i S_j = S_j S_i \) \((i, j = 1, \cdots, d)\), by an application of Proposition 3.1.7(i), \( S_\lambda^t \) is commuting if and only if \( U_i U_j = U_j U_i \) \((i, j = 1, \cdots, d)\).

To see the sufficiency part, assume that \( S_\lambda^t \) is commuting. Thus \( U_\theta \) is commuting. Also, since \( U_j^* U_j = I \) for \( j = 1, \cdots, d \), \( U_\theta \) is a toral isometry. Thus \( S_\lambda \) admits the decomposition (5.1), where \( D_j \) \((j = 1, \cdots, d)\) is given by
\[
D_j e_v := \|S_j e_v\| e_v, \ v \in V.
\]

Further, by Proposition 3.1.7(vii), \( D_1, \cdots, D_d \) are diagonal, positive, invertible bounded linear operators.

Finally, since \( S_\lambda^t \) is commuting if and only if \( U_\theta \) is commuting, the necessary part follows from the uniqueness of (5.1).

For the sake of convenience, we refer to \( U_\theta \) and \( D = (D_1, \cdots, D_d) \) as toral isometry part and diagonal part of the multishift \( S_\lambda \) respectively.

**Remark 5.1.2.** Let \( S_w \) be a toral left invertible classical multishift. Then the operator tuple \( S_w^t \) toral Cauchy dual of \( S_w \) is given by
\[
S_w^t e_\alpha = \frac{1}{w^{(j)}_\alpha} e_{\alpha + j}, \ (1 \leq j \leq d).
\]

Note that \( S_w^t \) is also a commuting \( d \)-variable weighted shift with weight multisequence
\[
\left\{ \frac{1}{w^{(j)}_\alpha} : 1 \leq j \leq d, \ \alpha \in \mathbb{N}^d \right\}.
\]

Thus the toral Cauchy dual of a classical multishift \( S_w \) always commutes. Indeed, (5.2) is equivalent to the commutativity of \( S_w \) in this case. Moreover, the toral isometry part can be identified with the multiplication tuple \( \mathcal{M}_z \) of the Hardy space of the unit polydisc (commonly known as Cauchy d-shift). Indeed, \( \theta^{(j)}_\alpha = 1 \) for all \( v \in V^0 \) and \( j = 1, \cdots, d \).

Since toral isometry part and diagonal part in the above decomposition need not commute, prima facie the relation between the moments \( \{\|S_\lambda^t e_v\|^2\}_{\alpha \in \mathbb{N}^d} \) of \( S_\lambda \) and that of toral isometry part \( U_\theta \) is not visible. Nevertheless, there is a subclass of multishifts in which we are facilitated to get a nice formula for \( \{\|S_\lambda^t e_v\|^2\}_{\alpha \in \mathbb{N}^d} \) (see (5.9)).

**Definition 5.1.3.** Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \cdots, \mathcal{T}_d \) and let \( S_\lambda = (S_1, \cdots, S_d) \) be a commuting multishift on \( \mathcal{T} \). For \( j = 1, \cdots, d \), define \( \mathcal{C}_j : V \to (0, \infty) \) by
\[
\mathcal{C}_j(v) := \|S_j e_v\| \quad (v \in V, \ i, j = 1, \cdots, d).
\]

We say that \( S_\lambda \) is *torally balanced* if for each \( j = 1, \cdots, d \), \( \mathcal{C}_j \) is constant on every generation \( G_t, t \in \mathbb{N} \). We denote the constant value of \( \mathcal{C}_j(v) \) by \( c_j^{(j)}(\alpha_v) \), where \( \alpha_v \) is the depth of \( v \) in \( \mathcal{T} \) (see Definition 2.1.11).
Remark 5.1.4. Note that $S_\lambda$ is a toral isometry if and only if
\[
\sum_{w \in \text{Chi}_j(v)} (\lambda_w^{(j)})^2 = 1 \text{ for all } v \in V \text{ and } j = 1, \cdots, d.
\]
Clearly, any toral isometry multishift is torally balanced with $\mathcal{C}_j$ being constant function with value 1 for every $j = 1, \cdots, d$.

The following proposition leads to an interesting family of torally balanced multishifts. In particular, any directed Cartesian product of locally finite rooted directed trees supports a toral isometry (cf. [65 Proposition 8.1.3]).

**Proposition 5.1.5.** Let $\mathcal{F} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. For $i = 1, \cdots, d$, let $\{c(t, i)\}_{t \in \mathbb{N}}$ be a bounded sequence of positive real numbers such that
\[
c(t, i)c(t - 1, j) = c(t, j)c(t - 1, i) \text{ for all integers } t \geq 1 \text{ and } i, j = 1, \cdots, d. \tag{5.4}
\]
Consider the multishift $S_{\lambda_e} = (S_1, \cdots, S_d)$ with weights
\[
\lambda_{c}^{(j)} = \sqrt{\frac{c(\alpha_v^i, j)}{\text{card}({\text{Chi}_j(v)})}} \text{ for all } v \in \text{Chi}_j(v), \ v \in V \text{ and } j = 1, \cdots, d.
\]
Then $S_{\lambda_e}$ defines a torally balanced multishift. In case $c(t, j) = 1$ for all $j = 1, \cdots, d$ and $t \in \mathbb{N}$, $S_{\lambda_e}$ is a toral isometry.

**Proof.** Since $\{c(t, j)\}_{t \in \mathbb{N}}$ is a bounded sequence, by Lemma 3.1.5(i), $S_j$ defines a bounded linear operator on $l^2(V)$ for every $j = 1, \cdots, d$. Let $w \in V$ and $i, j = 1, \cdots, d$. By Proposition 2.1.21 for every $v \in \text{Chi}(\text{Chi}_j(w))$, we get
\[
\text{card}({\text{sib}_i(v)})\text{card}({\text{sib}_j(\text{par}_i(v))}) = \text{card}({\text{sib}_j(v)})\text{card}({\text{sib}_i(\text{par}_j(v))}). \tag{5.5}
\]
We now check the commutativity of $S_{\lambda_e}$. Note that for $w \in \text{Chi}_j(v)$, $\lambda_{c}^{(j)}$ can be rewritten as $\lambda_{c}^{(j)} = \sqrt{\frac{c(\alpha_v^i, j)}{\text{card}({\text{Chi}_j(v)})}}$. Now for $u \in \text{Chi}_j(v)$, we have
\[
\lambda_{c}^{(i)}\lambda_{c}^{(j)} = \sqrt{\frac{c(\alpha_u^i - 1, i)}{\text{card}({\text{sib}_i(u)})}} \sqrt{\frac{c(\alpha_{\text{par}_i(u)} - 1, j)}{\text{card}({\text{sib}_j(\text{par}_i(u))})}} = \sqrt{\frac{\text{card}({\text{sib}_i(u)})\text{card}({\text{sib}_j(\text{par}_i(u))})}{\text{card}({\text{sib}_i(u)})\text{card}({\text{sib}_j(\text{par}_i(u))})}},
\]
\[
\lambda_{c}^{(j)}\lambda_{c}^{(i)} = \sqrt{\frac{c(\alpha_u^i - 1, j)}{\text{card}({\text{sib}_j(u)})}} \sqrt{\frac{c(\alpha_{\text{par}_j(u)} - 1, i)}{\text{card}({\text{sib}_i(\text{par}_j(u))})}} = \sqrt{\frac{\text{card}({\text{sib}_j(u)})\text{card}({\text{sib}_i(\text{par}_j(u))})}{\text{card}({\text{sib}_j(u)})\text{card}({\text{sib}_i(\text{par}_j(u))})}}.
\]
This together with Proposition 3.1.7(i), (5.4) and (5.5) shows that $S_{\lambda_e}$ is commuting. Further,
\[
\|S_{\lambda_e}v\|^2 = \sum_{w \in \text{Chi}_j(v)} (\lambda_w^{(j)})^2 = \sum_{w \in \text{Chi}_j(v)} \frac{c(|\alpha_v|, j)}{\text{card}(\text{Chi}_j(v))} = c(|\alpha_v|, j),
\]
which is a function of $|\alpha_v|$. This shows that $S_{\lambda_e}$ is torally balanced. The last assertion is immediate from the above equality.
Remark 5.1.6. Assume that $S_{\lambda_c}$ is toral left invertible. Then $S_{\lambda_c}^4$ is commuting. Indeed, the weights of $S_{\lambda_c}^4$ are
\[
\left\{ \frac{1}{\sqrt{c(\alpha_v,j)}} \frac{1}{\sqrt{\text{card}(\text{Chi}_v)}} : w \in \text{Chi}_j(v), \; v \in V \right. \left. \; \text{and} \; j = 1, \cdots, d \right\}.
\]

By arguing as above, it can be seen that $S_{\lambda_c}^4$ is commuting.

Example 5.1.7. Let $\mathcal{F} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. For positive numbers $a, b \in \mathbb{R}$, let
\[
c_{a,b}(t, j) = \frac{t + b}{t + a} \quad (t \in \mathbb{N}, \; j = 1, \cdots, d).
\]

Then $S_{\lambda_c}$ is a torally balanced multishift on $\mathcal{F}$. In case $\mathcal{F}$ is the $d$-fold directed Cartesian product of $\mathcal{T}_{a,0}$ with itself, the choice $a = 1 = b$ yields the Cauchy $d$-shift. In case $b = 1$, we denote $c_{a,b}$ by a simpler notation $c_a$.

In dimension $d = 1$, the multishifts $S_{\lambda_c}$ with $b = 1$ can be realized as multiplication operators on reproducing kernel Hilbert spaces. Indeed, these shifts can be looked upon as tree counter-part of Agler-type shifts [2]. This is made precise in the following proposition.

Proposition 5.1.8. Let $\mathcal{F} = (V, \mathcal{E})$ be a locally finite rooted directed tree of finite branching index. For a positive integer $a$, let $S_{\lambda_{ca}}$ denote the weighted shift on $\mathcal{F}$ with weights given by
\[
\lambda_u = \frac{\alpha_u + 1}{\alpha_v + a} \frac{1}{\sqrt{\text{card}(\text{Chi}_v)}} \quad \text{for} \; u \in \text{Chi}(v), \; v \in V,
\]

where $\alpha_v$ denotes the depth of $v$ in $\mathcal{F}$. Then $S_{\lambda_{ca}}$ is unitarily equivalent to the multiplication operator $M_{z,a}$ on a reproducing kernel Hilbert space $\mathcal{H}_a$ of $E$-valued holomorphic functions on unit disc $\mathbb{D}$, where $E := \ker S_{\lambda_{ca}}^*$ (see (1.11)). Moreover, the reproducing kernel $\kappa_{\mathcal{H}_a}(z, w)$ associated with $\mathcal{H}_a$ is given by
\[
\kappa_{\mathcal{H}_a}(z, w) = \sum_{n=0}^{\infty} \binom{n + a - 1}{n} \frac{1}{z^n w^n} P_{[\text{root}]} + \sum_{v \in V} \sum_{n=0}^{\infty} \binom{\alpha_v + n + a}{\alpha_v + a} \binom{\alpha_v + n + 1}{\alpha_v + a} \frac{1}{z^n w^n} P_{\mathcal{E}(\text{Chi}(v) \cap [v])} (z, w \in \mathbb{D}),
\]

where $P_M$ denotes the orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{M}$ of $\mathcal{H}$.

Proof. Note that
\[
\inf_{v \in V} \sum_{u \in \text{Chi}(v)} \lambda_u^2 = \inf_{v \in V} \frac{\alpha_v + 1}{\alpha_v + a} = \frac{1}{a}.
\]

It now follows from Proposition 3.1.7(vii) that $S_{\lambda_{ca}}$ is left-invertible. Hence the first part follows from Remark 4.2.3 and Theorem 4.2.4. To see the remaining part, we need the following identity:
\[
\sum_{u \in \text{Chi}^{(k)}(v)} \frac{1}{\text{card}(\text{sib}(\text{par}^{(0)}(u)))} = 1 \quad \text{for} \; v \in V \; \text{and} \; k \geq 1. \quad (5.6)
\]
We prove this by induction on integers $k \geq 1$. For $k = 1$, the identity (5.6) follows from the fact that $\text{card}(\text{Chi}(v)) = \text{card}(\text{sib}(u))$, where $u \in \text{Chi}(v)$. Suppose that (5.6) holds for some integer $k \geq 1$. Then

$$
\sum_{u \in \text{Chi}^{(k+1)}(v)} \prod_{l=0}^{k} \frac{1}{\text{card}(\text{sib}(\text{par}^{(l)}(u)))}
= \sum_{\eta \in \text{Chi}^{(k)}(v)} \sum_{u \in \text{Chi}(\eta)} \prod_{l=0}^{k} \frac{1}{\text{card}(\text{sib}(\text{par}^{(l)}(u)))}
= \sum_{\eta \in \text{Chi}^{(k)}(v)} \left( \sum_{u \in \text{Chi}(\eta)} \frac{1}{\text{card}(\text{sib}(u))} \prod_{l=0}^{k} \frac{1}{\text{card}(\text{sib}(\text{par}^{(l)}(\eta)))} \right)
= \sum_{\eta \in \text{Chi}^{(k)}(v)} \prod_{l=0}^{k-1} \frac{1}{\text{card}(\text{sib}(\text{par}^{(l)}(\eta)))} = 1,
$$

where the last equality follows from the induction hypothesis.

Note that the weights of $S_{\lambda_{ca}}$ can be rewritten as

$$
\lambda_v = \sqrt{\frac{\alpha_v}{\alpha_v + a - 1}} \frac{1}{\sqrt{\text{card}(\text{sib}(v))}} \text{ for } v \in V^\circ.
$$

Let $S_{\lambda'_v}$ denote the Cauchy dual of $S_{\lambda_{ca}}$. By Lemma 4.2.1, the weights $\lambda'$ of $S_{\lambda'_v}$ are given by

$$
\lambda'_v = \sqrt{\frac{\alpha_v + a - 1}{\alpha_v}} \frac{1}{\sqrt{\text{card}(\text{sib}(v))}} \text{ for all } v \in V^\circ.
$$

Now for $v \in V$ and $j,k \geq 1$, an application of parts (iv) and (v) of Proposition 3.1.7 yields

$$
S_{\lambda'_v}^{k_j} e_v = \sqrt{\frac{(\alpha_v + k + a - 1)!\alpha_v!}{(\alpha_v + a - 1)!\alpha_v!}} \sum_{u \in \text{Chi}^{(k)}(v)} \prod_{l=0}^{k-1} \frac{1}{\sqrt{\text{card}(\text{sib}(\text{par}^{(l)}(u)))}} e_u,
$$

$$
S_{\lambda'_v}^{(j)} e_v = \sqrt{\frac{(\alpha_v + a - 1)!\alpha_v!}{(\alpha_v + a - j - 1)!\alpha_v!}} \prod_{l=0}^{j-1} \frac{1}{\sqrt{\text{card}(\text{sib}(\text{par}^{(l)}(v)))}} e_{\text{par}^{(j)}(v)}.
$$

For a positive integer $i$ and $v \in V$, set $s_{i,v} := \text{card}(\text{sib}(\text{par}^{(i)}(v)))$. Also, for positive integers $j, k$ and $v \in V$ such that $\text{par}^{(j-k)}(v)$ is nonempty, set

$$
\beta_{j,k}(v, a) := \prod_{i=0}^{j-k-1} \frac{1}{\sqrt{s_{i,v}}} \sqrt{\frac{(\alpha_v + k + a - 1)!\alpha_v!}{(\alpha_v + a - 1)!\alpha_v!}} \times \sqrt{\frac{(\alpha_v + k + a - 1)!\alpha_v + k - j)!}{(\alpha_v + k + a - j - 1)!\alpha_v + k)!}.
$$
Let \( v \in V \) and \( j \geq k \). It is easily seen that if \( \text{par}^{(j-k)}(v) \) is empty, then \( S_{\lambda_v}^{\ast j} S_{\lambda_v}^k e_v = 0 \). Otherwise

\[
S_{\lambda_v}^{\ast j} S_{\lambda_v}^k e_v = \sqrt{\frac{(\alpha_v + k + a - 1)!\alpha_v!}{(\alpha_v + a - 1)!((\alpha_v + k)!}} \sum_{u \in \text{Chi}(v)} \prod_{l=0}^{k-1} \frac{1}{\sqrt{S_{\lambda_v}^l}} S_{\lambda_v}^{\ast j} e_u
\]

\[
= \sqrt{\frac{(\alpha_v + k + a - 1)!\alpha_v!}{(\alpha_v + a - 1)!((\alpha_v + k)!}} \sum_{u \in \text{Chi}(v)} \prod_{l=0}^{k-1} \frac{1}{\sqrt{S_{\lambda_v}^l}} e_{\text{par}^{(j-k)}(u)}
\]

\[
= \beta_j,k(v,a) \sum_{u \in \text{Chi}(v)} \prod_{l=0}^{k-1} \frac{1}{\sqrt{S_{\lambda_v}^l}} e_{\text{par}^{(j-k)}(u)}
\]

where the last equality follows from (5.6).

Let \( E \) be the kernel of \( S_{\lambda_v} \) and let \( f \in E \). Note that \( f \) takes the form \( f = f_{\text{root}} + \sum_{v \in V_{\lambda_v}} f_v \), where \( f_{\text{root}} = \gamma e_{\text{root}} \) for some \( \gamma \in \mathbb{C} \) and \( f_v = \sum_{u \in \text{Chi}(v)} f(u) e_u \) such that \( \sum_{u \in \text{Chi}(v)} f(u) \lambda_u = 0 \) for \( v \in V_{\lambda_v} \). Since \( \lambda_u \) is constant on \( \text{Chi}(v) \), we obtain that

\[
\sum_{u \in \text{Chi}(v)} f(u) = 0 \quad \text{for all } v \in V_{\lambda_v}.
\]

(5.7)

It follows that for \( v \in V_{\lambda_v} \) and \( j > k \),

\[
S_{\lambda_v}^{\ast j} S_{\lambda_v}^k \sum_{u \in \text{Chi}(v)} f(u) e_u = \sum_{u \in \text{Chi}(v)} f(u) S_{\lambda_v}^{\ast j} S_{\lambda_v}^k e_u
\]

\[
= \sum_{u \in \text{Chi}(v)} f(u) \beta_j,k(u,a) e_{\text{par}^{(j-k)}(u)}
\]

\[
= 0,
\]

where we used (5.7) and the fact that \( \beta_j,k(u,a) \) is constant on \( \text{Chi}(v) \). Almost the same calculations show that

\[
S_{\lambda_v}^{\ast j} S_{\lambda_v}^k f_v = \frac{(\alpha_v + k + a)!((\alpha_v + 1)!}{(\alpha_v + a)!((\alpha_v + k + 1)!} f_v \quad \text{for every } k \in \mathbb{N}.
\]

It may now be concluded from (4.15) that the reproducing kernel \( \kappa_{\lambda_v} \) takes the required form. Finally, since \( E \) is finite dimensional (Corollary 3.1.16), we note that for every \( w \in \mathbb{D} \), \( \kappa_{\lambda_v}(\cdot,w) \) is a sum of finitely many power series in \( z \) converging on the unit disc.

We now classify all torally balanced multishifts. For this, we must understand their moments.

**Lemma 5.1.9.** Let \( \mathcal{T} = (V,E) \) be the directed Cartesian product of rooted directed trees \( T_1, \cdots, T_d \). Let \( S_{\lambda} \) be a toral left invertible multishift on \( \mathcal{T} \) with commuting toral Cauchy dual \( S_{\lambda}^* \) and let \( U_{\theta} \) be the toral isometry part of \( S_{\lambda} \) governed
by \([5.1]\). If \(S_{\lambda}\) is torally balanced, then, for any \(v \in V\) and \(\alpha \in \mathbb{N}^d\),

\[
S_{\lambda}^n e_v = \prod_{j=1}^{d} \left( \prod_{k=0}^{\alpha_j-1} c^{(j)}_{|\alpha_v|+\sum_{i=1}^{j-1} \alpha_i+k} \right) U^n_j e_v, \tag{5.8}
\]

where \(c^{(j)}_{|\alpha_v|}\) denotes the constant value of \(c_j(v)\) for \(v \in V\) and \(j = 1, \cdots, d\). In this case,

\[
\|S_{\lambda}^n e_v\|^2 = \prod_{j=1}^{d} \left( \prod_{k=0}^{\alpha_j-1} c^{(j)}_{\alpha_v|+\sum_{i=1}^{j-1} \alpha_i+k} \right)^2 (v \in V, \alpha \in \mathbb{N}^d). \tag{5.9}
\]

**Proof.** Assume that \(S_{\lambda}\) is torally balanced. Let us first verify

\[
S^n_j e_v = \left( \prod_{k=0}^{\alpha_j-1} c^{(j)}_{\alpha_v|+k} \right) U^n_j e_v (n \geq 1, j = 1, \cdots, d)
\]

by induction on integers \(n \geq 1\). Note that

\[
S_j e_v = \sum_{w \in \text{Ch}_j(v)} \lambda^{(j)}_w e_w = \sum_{w \in \text{Ch}_j(v)} \frac{\lambda^{(j)}_w}{c_j(\text{par}_j(w))} c_j(\text{par}_j(w)) e_w \tag{5.3},
\]

\[
\text{Ch}_j(v) \sum_{w \in \text{Ch}_j(v)} \theta^{(j)}_w e_w = c^{(j)}_{|\alpha_v|} U_j e_v. \tag{5.10}
\]

This verifies the induction statement for \(n = 1\). Assume the induction hypothesis for \(n \geq 1\), and consider

\[
S^{n+1}_j e_v = \prod_{k=0}^{\alpha_j-1} c^{(j)}_{\alpha_v|+k} S_j U^n_j e_v = \prod_{k=0}^{\alpha_j-1} c^{(j)}_{\alpha_v|+k} S_j \sum_{w \in \text{Ch}_j^{(n)}(v)} \theta^{(j)}_w \cdots \theta^{(j)}_{\text{par}_j^{(n-1)}(w)} e_w \tag{5.10}
\]

\[
= \prod_{k=0}^{\alpha_j-1} c^{(j)}_{\alpha_v|+k} \sum_{w \in \text{Ch}_j^{(n)}(v)} \theta^{(j)}_w \cdots \theta^{(j)}_{\text{par}_j^{(n-1)}(w)} U_j e_w \quad \text{(since } \alpha_w = \alpha_v + n \epsilon_j) \]

\[
= \left( \prod_{k=0}^{n} c^{(j)}_{\alpha_v|+k} \right) U^{n+1}_j e_v.
\]

This completes the inductive argument. A similar inductive argument on \(m \geq 1\) together with the commutativity of \(S_{\lambda}^i\) yields

\[
S^m_i S^n_j e_v = \prod_{k=0}^{m-1} c^{(i)}_{\alpha_v|+k} \prod_{k=0}^{n-1} c^{(j)}_{\alpha_v|+m+k} U^m_i U^n_j e_v
\]

for \(m, n \in \mathbb{N}\) and \(1 \leq i, j \leq d\). The desired conclusion may now be deduced from the above identity by a finite inductive argument. Finally, we note that \((5.9)\) follows from \((5.8)\) and the fact that \(U_{\theta}\) is a toral isometry. \(\square\)
\section*{5. Special Classes of Multishifts}

**Definition 5.1.10.** Let $c := \{c(t, j) : t \in \mathbb{N}, \ j = 1, \cdots, d\}$ be a bounded multisequence of positive real numbers such that (5.4) holds. For $s \in \mathbb{N}$, set

\[ \gamma_{\alpha,s} := \prod_{j=1}^{d} \prod_{k=0}^{\alpha_j-1} c(s + \alpha_1 + \cdots + \alpha_{j-1} + k, j) \ (\alpha \in \mathbb{N}^d). \]

We refer to the Hilbert space of formal series $H^2(\gamma_s)$ as the Hilbert space associated with $c$.

**Remark 5.1.11.** Note that the multiplication $d$-tuple on $H^2(\gamma_s)$ is a commuting $d$-tuple of bounded linear operators $M_{z_1}, \cdots, M_{z_d}$.

We are now in a position to present the main result of this section.

**Theorem 5.1.12.** Let $\mathcal{F} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. For $v \in V$, let

\[ f = \sum_{\beta \in \mathbb{N}^d} a_{\beta} S_{\lambda}^{\beta} e_v \in l^2(V), \quad \tilde{f}(w) = \sum_{\beta \in \mathbb{N}^d} a_{\beta} w^\beta. \]

Let $S_{\lambda}$ be a toral left invertible multishift on $\mathcal{F}$ with commuting toral Cauchy dual $S_{\lambda}'$. Then $S_{\lambda}$ is a torally balanced multishift on $\mathcal{F}$ if and only if for every $v \in V$, there exists a Hilbert space $H^2(\gamma_{|\alpha_v|})$ of formal power series in the variables $w_1, \cdots, w_d$ associated with a bounded multisequence $c$ such that

\[ \|f\|_{l^2(V)} = \|\tilde{f}\|_{H^2(\gamma_{|\alpha_v|})}. \]

**Proof.** Suppose that $S_{\lambda}$ is a torally balanced multishift. Set $c(t, j) := c^{(j)}_t$ for $t \in \mathbb{N}$ and $j = 1, \cdots, d$, where $c^{(j)}_t$ denotes the constant value of $\mathcal{E}_j(v)$ on the generation $\mathcal{G}_t$. Consider the Hilbert space $H^2(\gamma_{|\alpha_v|})$ of formal power series in $w_1, \cdots, w_d$. By taking norms on both sides of (5.8), we obtain for every $\alpha \in \mathbb{N}^d$,

\[ \|S_{\lambda}^\alpha e_v\|_{l^2(V)} = \prod_{j=1}^{d} \left( \prod_{k=0}^{\alpha_j-1} c^{(j)}_t \right) \|U_{\theta}^\alpha e_v\|_{l^2(V)} = \gamma_{\alpha,v} \|w^\alpha\|_{H^2(\gamma_{|\alpha_v|})}, \tag{5.11} \]

where we used that $U_{\theta}$ is a toral isometry. By orthogonality of $\{S_{\lambda}^\alpha e_v\}_{\alpha \in \mathbb{N}^d}$ (Proposition 3.1.7(ix)), the above formula holds for all pairs $f$ and $\tilde{f}$. To see the converse, let $f = S_j e_v$ and $\tilde{f} = w_j$ in $\|f\|_{l^2(V)} = \|\tilde{f}\|_{H^2(\gamma_{|\alpha_v|})}$ to obtain $\|S_j e_v\| = c(|\alpha_v|, j)$, which is clearly constant on $\mathcal{G}_{|\alpha_v|}$. \hfill $\square$

Here we present a local analog of von Neumann’s inequality for torally balanced multishifts.

**Corollary 5.1.13.** Let $\mathcal{F} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$ and let $S_{\lambda}$ be a toral left invertible, torally balanced multishift with commuting toral Cauchy $S_{\lambda}'$. If $S_{\lambda}$ is a toral contraction, then for any positive integer $k$ and scalars $a_{\beta}$, $|\beta| \leq k$,

\[ \sup_{v \in V} \left\| \sum_{|\beta| \leq k} a_{\beta} S_{\lambda}^\beta e_v \right\| \leq \sup_{z \in \mathbb{D}^d} \left| \sum_{|\beta| \leq k} a_{\beta} z^{\beta} \right|. \]
5.2. Spherically Balanced Multishifts

Proof. Assume that $S_\lambda$ is a toral contraction. Fix $v \in V$. By the preceding theorem, there exists a Hilbert space $H^2(\gamma_{\alpha_v})$ of formal power series in the variables $w_1, \ldots, w_d$ such that

$$
\| \sum_{|\beta| \leq k} a_\beta S_\lambda^\beta e_v \|_{L^2(V)} = \| \sum_{|\beta| \leq k} a_\beta w^\beta \|_{H^2(\gamma_{\alpha_v})} = \| \sum_{|\beta| \leq k} a_\beta M_w^\beta \|_{H^2(\gamma_{\alpha_v})},
$$

where $M_w$ denotes the $d$-tuple of operators of multiplication by the coordinate functions $w_1, \ldots, w_d$. However, since $S_\lambda$ is a toral contraction, by (5.11), so is $M_w$. It follows from $\|1\|_{H^2(\gamma_{\alpha_v})} = 1$ that

$$
\| \sum_{|\beta| \leq k} a_\beta S_\lambda^\beta e_v \|_{L^2(V)} \leq \| \sum_{|\beta| \leq k} a_\beta M_w^\beta \|_{L^2(V)} = \| \sum_{|\beta| \leq k} a_\beta M_w^\beta \|_{L^2(V)}.
$$

By a result of M. Hartz [61, Theorem 1.1], von Neumann’s inequality holds for any torally contractive classical multishift. Hence

$$
\| \sum_{|\beta| \leq k} a_\beta S_\lambda^\beta e_v \|_{L^2(V)} \leq \sup_{z \in \mathbb{D}^d} \sum_{|\beta| \leq k} a_\beta |z|^\beta.
$$

Taking supremum over $v \in V$ on the left hand side, we get the desired inequality. □

5.2. Spherically Balanced Multishifts

Definition 5.2.1. Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$ and let $S_\lambda = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathcal{T}$. Define the function $\mathcal{C} : V \to (0, \infty)$ by

$$
\mathcal{C}(v) := \sum_{j=1}^d \| S_j e_v \|^2 \text{ for } v \in V. \quad (5.12)
$$

We say that $S_\lambda$ is spherically balanced if $\mathcal{C}$ is constant on every generation $G_t$, $t \in \mathbb{N}$. We denote the constant value of $\mathcal{C}(v)$ by $\mathcal{C}_{|\alpha_v|}$, where $\alpha_v$ is the depth of $v$ in $\mathcal{T}$.

Remark 5.2.2. Note that $S_\lambda$ is a joint isometry if and only if

$$
\sum_{j=1}^d \| S_j e_v \|^2 = \sum_{j=1}^d \sum_{w \in \text{Chi}_i(v)} (\lambda(w)^{\beta})^2 = 1 \text{ for all } v \in V.
$$

It is now clear that every joint isometry multishift is spherically balanced with $\mathcal{C}$ being the constant function 1.

The following proposition yields examples of spherically balanced multishifts apart from joint isometries.

Proposition 5.2.3. Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$ and let $\{c_t\}_{t \in \mathbb{N}}$ be a bounded sequence of positive real numbers. Consider the multishift $S_{\lambda_x} = (S_1, \ldots, S_d)$ with weights

$$
\lambda_w^{(i)} = \sqrt{\frac{c_{|\alpha_v|}}{\text{card}(\text{Chi}_i(v))}} \sqrt{\frac{\alpha_{v_i} + 1}{|\alpha_v| + d}} \text{ for } w \in \text{Chi}_i(v), \ v \in V \text{ and } i = 1, \ldots, d.
$$

Then $S_{\lambda_x}$ defines a spherically balanced multishift. In case $c_t = 1$ for all $t \in \mathbb{N}$, $S_{\lambda_x}$ is a joint isometry.
PROOF. One may conclude from Lemma 3.1.5(i) that $S_I, \cdots, S_d$ are bounded linear operators on $l^2(V)$ whenever the sequence $\{c_t\}_{t \in \mathbb{N}}$ is bounded. Let $v \in V$ and $i, j = 1, \cdots, d$. First we check the commutativity of $S_{\chi_v}$. Note that for $w \in \chi_i(v)$, $\lambda_w^{(i)}$ can be rewritten as

$$\lambda_w^{(i)} = \frac{\sum_{\alpha} c_{\alpha_w} \alpha_i}{\text{card}(\text{par}_i(w))} \|\alpha_w\| + d - 1.$$ 

Now for $u \in \chi_i \chi_j(v)$, we have

$$\lambda_u^{(i)} \lambda_{\text{par}_i(u)}^{(j)} = \frac{\sum_{\alpha} c_{|\alpha_u|} \alpha_i}{\text{card}(\text{par}_i(u))} \|\alpha_u\| + d - 1 \times \frac{\sum_{\alpha} c_{|\alpha_u|} \alpha_j}{\text{card}(\text{par}_j(u))} \|\alpha_{\text{par}_j(u)}\| + d - 1,$$

$$\lambda_u^{(j)} \lambda_{\text{par}_j(u)}^{(i)} = \frac{\sum_{\alpha} c_{|\alpha_v|} \alpha_j}{\text{card}(\text{par}_j(u))} \|\alpha_v\| + d - 1 \times \frac{\sum_{\alpha} c_{|\alpha_v|} \alpha_i}{\text{card}(\text{par}_i(u))} \|\alpha_{\text{par}_i(u)}\| + d - 1.$$ 

Since $\alpha_u = \alpha_{\text{par}_j(u)}$, for $i \neq j$, by (5.5), $S_{\chi_v}$ is commuting. Note further that

$$\sum_{j=1}^{d} \|S_j v\|^2 = \sum_{j=1}^{d} \sum_{w \in \chi_i(v)} (\lambda_u^{(j)})^2 = \sum_{j=1}^{d} \sum_{w \in \chi_i(v)} \frac{c_{|\alpha_u|} \alpha_{\text{par}_j(w)} + 1}{\text{card}(\chi_j(v)) \|\alpha_u\| + d},$$

which is a function of $|\alpha_v|$. Thus $S_{\chi_v}$ is spherically balanced. The above calculation also shows that $S_{\chi_v}$ is joint isometry if and only if $c_t = 1$ for all $t \in \mathbb{N}$. \hfill $\square$

REMARK 5.2.4. The choice $c_t = \frac{t+d}{t+a} \frac{t+2}{t+1}$ ($t \in \mathbb{N}$) with $\mathcal{T}$ being the $d$-fold directed Cartesian product of $\mathcal{T}_{1,0}$ with itself yields the Dirichlet $d$-shift on the unit ball [58, Example 1].

We discuss below a family of examples of spherically balanced multishifts, which is a tree analog of the multiplication $d$-tuples on the reproducing kernel Hilbert spaces associated with the reproducing kernels $\frac{1}{1-\langle z, w \rangle^a}$ defined on the unit ball in $\mathbb{C}^d$, where $a$ is a positive number.

EXAMPLE 5.2.5. Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. For a positive real number $a$, consider the sequence $\{c_{a,t}\}_{t \in \mathbb{N}}$ given by

$$c_{a,t} = \frac{t+d}{t+a} \frac{t+2}{t+1} \frac{t+3}{t+2} \cdots$$

Then the multishift $S_{\chi_v}$ on $\mathcal{T}$ is spherically balanced. In case $\mathcal{T}$ is the $d$-fold directed Cartesian product of $\mathcal{T}_{1,0}$ with itself then the choices $a = d, a = d+1, a = 1$ yield Szegö $d$-shift, Bergman $d$-shift, Drury-Arveson $d$-shift respectively on the unit ball (see Example 1.2.1).
We refer to the multishifts $S_{\lambda_{e_1}}$ on $\mathcal{T}$ as the tree analog of Szeg"{o} $d$-shift, Bergman $d$-shift, Drury-Arveson $d$-shift respectively in case $a = d, a = d+1, a = 1$. It is worth noting that $S_{\lambda_{e_1}}$ and $S_{\lambda_{e_{d+1}}}$ are joint contractions while $S_{\lambda_{e_1}}$ is a row contraction (that is, $S_{\lambda_{e_1}}^* S_{\lambda_{e_1}}$ is a joint contraction).

Although there is no known satisfactory counter-part of Shimorin’s analytic model for joint left invertible analytic tuples, we are able to show, as in the classical case, that the multishifts $S_{\lambda_{e_1}}$ on $\mathcal{T}$ can be realized as multiplication tuples $\mathcal{M}_z$ on reproducing kernel Hilbert spaces at least in case the joint kernel $E$ of $S_{\lambda_{e_1}}$ is finite dimensional. Before we make this precise, we recall from (4.12) that $E$ is given by

$$E = [e_{\text{root}}] \oplus \bigoplus_{\alpha \in \Omega_F} \bigoplus_{u \in \Omega_F} \mathcal{L}_{u,F}$$

(refer to Section 4.1 for the definitions of $\Omega_F$ and $L_{u,F}$). We record that $\Omega_F$ is finite whenever $V_{\prec}$ is finite. For instance, this happens whenever $E$ is finite dimensional.

**Theorem 5.2.6.** Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$ of finite joint branching index. Let $S_{\lambda_{e_1}}$ be as introduced in Example 5.2.5 and let $E$ denote the joint kernel of $S_{\lambda_{e_1}}$. Then $S_{\lambda_{e_1}}$ is unitarily equivalent to the multiplication $d$-tuple $\mathcal{M}_{z,a} = (\mathcal{M}_{z_1}, \cdots, \mathcal{M}_{z_d})$ on a reproducing kernel Hilbert space $\mathcal{H}_{a,d}$ of $E$-valued holomorphic functions defined on the open unit ball $B^d$ in $C^d$. Further, the reproducing kernel $\kappa_{\mathcal{H}_{a,d}} : B^d \times B^d \to B(E)$ associated with $\mathcal{H}_{a,d}$ is given by

$$\kappa_{\mathcal{H}_{a,d}}(z, w) = \sum_{\alpha \in \mathbb{N}^d} \frac{a(a+1) \cdots (a+|\alpha|-1)}{\alpha!} z^\alpha \overline{w}^\alpha P_{|\alpha_{\text{root}}|} + \sum_{\alpha \in \mathbb{N}^d} \sum_{u \in \Omega_F} \kappa_{u,F}(z, w),$$

where $\kappa_{u,F}(z, w)$ is given by

$$\kappa_{u,F}(z, w) = \sum_{\alpha \in \mathbb{N}^d} \frac{\alpha_u!}{\alpha_u + \alpha} |\alpha|^{-1} \prod_{j=0}^{\alpha_u} (|\alpha_u| + a + j) z^\alpha \overline{w}^\alpha P_{\mathcal{L}_{u,F}},$$

with $P_{\mathcal{M}}$ being the orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{M}$ of $\mathcal{H}$.

The proof of the above theorem, as presented below, turns out to be more involved as compared to that of Proposition 5.1.8. Perhaps one reason may be that no counter-part of Shimorin’s model is known for joint left invertible analytic tuples. This proof relies on the description of joint kernel $E$ as provided in Chapter 4. A key observation required is the following lemma.

**Lemma 5.2.7.** Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$ of finite joint branching index. Let $S_{\lambda_{e_1}}$ be as introduced in Example 5.2.5 and let $E$ denote the joint kernel of $S_{\lambda_{e_1}}$. Then the following are true:

(i) $E$ is invariant under $S_{\lambda_{e_1}}^\alpha S_{\lambda_{e_1}}^\beta$ and $S_{\lambda_{e_1}}^\alpha S_{\lambda_{e_1}}^\beta |_E$ is boundedly invertible for every $\alpha, \beta \in \mathbb{N}^d$.

(ii) The multisequence $\{S_{\lambda_{e_1}}^\alpha E\}_{\alpha \in \mathbb{N}^d}$ of subspaces of $L^2(V)$ is mutually orthogonal.
This yields the formula
\[ \beta(j, w, n) = \lambda_w^{(j)} \lambda_{\text{par}_j(w)}^{(j)} \cdots \lambda_{\text{par}_j(n-1)(w)}^{(j)} (n \geq 1). \]

It is easy to see using \( \lambda^{(i)}_w = \frac{1}{\sqrt{\text{card}(\text{sib}_w(w))}} \sqrt{\alpha_{w_j}} \) that
\[
\beta(j, w, n)^2 = \left( \prod_{k=0}^{n-1} \frac{1}{\text{card}(\text{sib}_j \text{par}_j^{(k)}(w))} \right) \frac{\alpha_{w_j}!}{\alpha_{w_j} - n - a - 1)!} \frac{(\alpha_{w_j} - n)!}{\alpha_{w_j} + a - 1)!} \]  
\[
\times \prod_{j=1}^{k} \frac{\alpha_{v_{ij}}}{\alpha_{v_{ij}} + \alpha_{i_j}!} \frac{(\alpha_{v_{ij}} + \sum_{m=j+1}^{k} \alpha_{i_m} + a - 1)!}{\alpha_{v_{ij}} + \sum_{m=j}^{k} \alpha_{i_m} + a - 1)!} \]  
\[
\times \prod_{w \in \text{Chi}^\alpha \supset (v)} \frac{1}{\text{card}(\text{sib}_j \text{par}_j^{(l)} \text{par}_j^{(a_{j-1})} \ldots \text{par}_1^{(a_1)}(w))} e_v, \]
where we used that \( |\alpha_{\text{par}_j^{(a_{j-1})} \supset (v)} = |\alpha_w| - \sum_{l=1}^{j-1} \alpha_{i_l} \) and \( \alpha_w = \alpha_v + \alpha \) for any \( w \in \text{Chi}^\alpha \supset (v) \). As in the proof of Proposition 5.1.8 one can verify by induction on \( |\alpha| \geq 1 \) that
\[
\sum_{w \in \text{Chi}^\alpha \supset (v)} \prod_{j=1}^{d} \prod_{l=0}^{a_{j-1} - 1} \frac{1}{\text{card}(\text{sib}_j \text{par}_j^{(l)} \text{par}_j^{(a_{j-1})} \ldots \text{par}_1^{(a_1)}(w))} = 1. \]
This yields the formula
\[ S_{\alpha}^{\alpha} S_{\lambda \alpha}^{\alpha} e_v = \prod_{j=1}^{k} \frac{(\alpha_{v_{ij}} + \alpha_{i_j}!)}{\alpha_{v_{ij}}!} \frac{(\alpha_i + \sum_{m=j+1}^{k} \alpha_{i_m} + a - 1)!}{(\alpha_i + \sum_{m=j}^{k} \alpha_{i_m} + a - 1)!} e_v (v \in V). \]

Since depth is constant on \( \text{sib}_F(u) \) and \( E \) is finite dimensional (Corollary 3.1.16), the desired conclusion in (i) is immediate from this formula. For future reference,
we also note the following expression for moments of $S_{\lambda_{v}}$ deduced from \((5.16)\):

\[
\|S_{\lambda_{v}}^a e_v\|^2 = \prod_{j=1}^{d} \frac{(\alpha_{v_j} + \alpha_j)!}{\alpha_{v_j}!} \frac{(|\alpha_v| + \sum_{i=j+1}^{d} \alpha_i + a - 1)!}{(|\alpha_v| + \sum_{i=j}^{d} \alpha_i + a - 1)!} \\
= \left( \prod_{j=1}^{d} \frac{(\alpha_{v_j} + \alpha_j)!}{\alpha_{v_j}!} \right) \frac{1}{(|\alpha_v| + a)(|\alpha_v| + a + 1) \cdots (|\alpha_v| + a + |\alpha| - 1)}
\]

(5.17)

for all $v \in V$.

(ii) It suffices to check that $S_{j}^{\beta_j+1}S_{\lambda_{v}} |_{E} = 0$ for $j = 1, \ldots, d$. We will verify this only for $j = d$. The verification for the rest of the coordinates is invariably the same. Let $f \in E$. Clearly, for $f = e_{\text{root}}$, we have $S_{d}^{\beta_d+1}S_{\lambda_{v}} f = 0$. Let $F \in \mathcal{B}$ such that $F \neq \emptyset$ and let $f \in \mathcal{L}_{u,F}$ for $u \in \Omega_F \subseteq \Phi_F$ (see \((4.3)\) and \((4.6)\)). If $d \notin F$ then once again $S_{d}^{\beta_d+1}S_{\lambda_{v}} f = 0$ since $f \in I^2(\text{sib}_F(u))$ is supported on a subset of $V_1 \times \cdots \times V_{d-1} \times \{\text{root}\}$. Hence we may assume that $d \in F$. Since $f \in \mathcal{L}_{u,F}$, by \((4.10)\), $f = \sum_{v \in \text{sib}_F(u)} f(v) e_v$ satisfies

\[
\sum_{w \in \text{sib}_i(u_G|u_i)} f(w)\lambda_{w}^{(i)} = 0, \quad i \in F \text{ and } v_G \in \text{sib}_{F,G}(u).
\]

However, since $\lambda_{w}^{(i)} = \frac{1}{\sqrt{\text{card}(\text{sib}_i(u))}} \sqrt{\frac{a_w}{|a_w|+a-1}}$ is constant on $w \in \text{sib}_i(u_G|u_i)$, we obtain

\[
\sum_{w \in \text{sib}_i(u_G|u_i)} f(w) = 0, \quad i \in F \text{ and } v_G \in \text{sib}_{F,G}(u).
\]

(5.18)

Let $\alpha = \beta - \beta_d \epsilon_d$. One may now argue as in (i) to see that

\[
S_{d}^{\beta_d+1}S_{\lambda_{v}} f = S_{d}^{\alpha} \sum_{v \in \text{sib}_F(u)} f(v) S_{d}^{\beta_d} S_{\lambda_{v}}^a e_v \\
= \sum_{v \in \text{sib}_F(u)} \prod_{j=1}^{d} \frac{(\alpha_{v_j} + \alpha_j)!}{\alpha_{v_j}!} \frac{f(v)}{\sqrt{(|\alpha_v| + a)(|\alpha_v| + a + 1) \cdots (|\alpha_v| + a + |\alpha| - 1)}} \\
\times \sum_{w \in \text{Chi}^{\alpha \gg \beta}(v)} \prod_{j=1}^{d} \frac{\alpha_{v_j}!}{\alpha v_i!} \frac{S_{d}^{\beta_d} S_{d}^{\beta_d} e_w}{\sqrt{\text{card}(\text{sib}_j(\text{par}_{j}(\text{par}_{j-1}(\cdots \text{par}_{1}(w))))}} \\
\times \sum_{w \in \text{sib}_F(u)} f(v) \|S_{\lambda_{v}}^a e_v\| \sum_{w \in \text{Chi}^{\alpha \gg \beta}(v)} \gamma(w, \alpha) S_{d}^{\beta_d} S_{d}^{\beta_d} e_w, \quad (5.19)
\]

where $\gamma(w, \alpha)$ is given by

\[
\gamma(w, \alpha) := \prod_{j=1}^{d} \prod_{i=0}^{\alpha_{v_j} - 1} \frac{1}{\sqrt{\text{card}(\text{sib}_j(\text{par}_{j}(\text{par}_{j-1}(\cdots \text{par}_{1}(w))))}}
\]

(5.20)

However, by \((5.16)\) and $\alpha_{wd} = \alpha_{vd}$,

\[
S_{d}^{\beta_d} S_{d}^{\beta_d} e_w = \frac{(a_{wd} + \beta_d)!}{\alpha_{wd}!} \frac{(|\alpha_w| + a - 1)!}{(|\alpha_w| + \beta_d + a - 1)!} e_w \\
= \frac{(a_{vd} + \beta_d)!}{\alpha_{vd}!} \frac{(|\alpha_v| + |\alpha| + a - 1)!}{(|\alpha_v| + |\alpha| + \beta_d + a - 1)!} e_w
\]
This combined with (5.19) yields
\[ S^* d + 1 \big( \sum_{v \in \text{sib}_d(u)} f(v) \big) \begin{pmatrix} S^\alpha_{\lambda'e} e_v \end{pmatrix} \frac{(\alpha_{vd} + \beta_d)!}{\alpha_{vd}!} \frac{(|\alpha_{u}| + |\alpha| + a - 1)!(|\alpha| + |\beta_d| + a - 1)!}{(|\alpha| + |\beta_d| + a - 1)} \times \sum_{w \in \text{Chi} \langle \alpha \rangle \rangle (v)} \gamma(w, \alpha) S^* d e_w. \]

Since \( w_d = v_d \) and \( \alpha_w = \alpha_v + \alpha \), we have
\[ S^* d e_w = \frac{1}{\sqrt{\text{card}(\text{sib}_d(w))}} \sqrt{\frac{|\alpha_{w_d}| + a - 1}{|\alpha_v| + |\alpha| + a - 1}} e_{\text{par}_d(w)}. \]

This gives
\[ S^* d + 1 \big( \sum_{v \in \text{sib}_d(u)} f(v) \big) \begin{pmatrix} S^\alpha_{\lambda'e} e_v \end{pmatrix} \frac{(\alpha_{vd} + \beta_d)!}{\alpha_{vd}!} \frac{(|\alpha_{u}| + |\alpha| + a - 1)!(|\alpha| + |\beta_d| + a - 1)!}{(|\alpha| + |\beta_d| + a - 1)} \times \sum_{w \in \text{Chi} \langle \alpha \rangle \rangle (v)} \gamma(w, \alpha) e_{\text{par}_d(w)}. \]

It is clear from the definition of depth and siblings that the expression \( \Gamma(v, \alpha) \) below is independent of \( v \in \text{sib}_F(u) \):
\[ \Gamma(v, \alpha) := \begin{pmatrix} S^\alpha_{\lambda'e} e_v \end{pmatrix} \frac{(\alpha_{vd} + \beta_d)!}{\alpha_{vd}!} \frac{(|\alpha_{u}| + |\alpha| + a - 1)!(|\alpha| + |\beta_d| + a - 1)!}{(|\alpha| + |\beta_d| + a - 1)} \times \sum_{w \in \text{Chi} \langle \alpha \rangle \rangle (v)} \gamma(w, \alpha) e_{\text{par}_d(w)}. \]

Further, since \( \alpha_d = 0 \), we conclude from (5.20) that \( \gamma(w, \alpha) \) is independent of \( w_d = v_d \). Also, since \( v = v_G[v_d] \) for \( G = F \setminus \langle d \rangle \) and \( v \in \text{sib}_F(u) \), it follows that
\[ S^* d + 1 \big( \sum_{v \in \text{sib}_F(u)} f(v) \big) \begin{pmatrix} S^\alpha_{\lambda'e} e_v \end{pmatrix} \frac{(\alpha_{vd} + \beta_d)!}{\alpha_{vd}!} \frac{(|\alpha_{u}| + |\alpha| + a - 1)!(|\alpha| + |\beta_d| + a - 1)!}{(|\alpha| + |\beta_d| + a - 1)} \times \sum_{w \in \text{Chi} \langle \alpha \rangle \rangle (v)} \gamma(w, \alpha) e_{\text{par}_d(w)} \]
\[ = \sum_{v_G \in \text{sib}_F, \langle G \rangle (u)} \left( \sum_{v_d \in \text{sib}(u_d)} \left( \sum_{w \in \text{Chi} \langle \alpha \rangle \rangle (v)} \gamma(w, \alpha) e_{\text{par}_d(w)} \right) \right) \]
\[ = \sum_{v_G \in \text{sib}_F, \langle G \rangle (u)} \left( \sum_{v_d \in \text{sib}(u_d)} \left( \sum_{v_e \in \text{sib}(u_e)} \gamma(w, \alpha) e_{\text{par}_d(w)} \right) \right) \]
\[ = \sum_{v_G \in \text{sib}_F, \langle G \rangle (u)} \left( \sum_{w \in \text{Chi} \langle \alpha \rangle \rangle (v)} \left( \sum_{v_d \in \text{sib}(u_d)} \gamma(w, \alpha) e_{\text{par}_d(w)} \right) \right) \]
where the sum in the inner bracket is 0 in view of (5.18). \( \square \)
We are now in a position to complete the proof of Theorem 5.2.6.

**Proof of Theorem 5.2.6** We divide the proof into several steps.

**Step I.** In this step, we prove that $S_{\lambda_{\epsilon_a}}$ can be modeled as a multiplication tuple on a Hilbert space of $E$-valued formal power series. Note that by Theorem 4.0.1 and Lemma 5.2.7(ii), we have

$$l^2(V) = \bigoplus_{\alpha \in \mathbb{N}^d} S_{\lambda_{\epsilon_a}}^\alpha E.$$ 

Thus for any $f \in l^2(V)$, there exists a multisequence $\{f_\alpha\}_{\alpha \in \mathbb{N}^d}$ in $E$ such that

$$f = \sum_{\alpha \in \mathbb{N}^d} S_{\lambda_{\epsilon_a}}^\alpha f_\alpha.$$ 

Also, since $S_1, \ldots, S_d$ are injective (Corollary 3.3.4), the multisequence $\{f_\alpha\}_{\alpha \in \mathbb{N}^d}$ with the above property is unique. This unique representation allows us to form the inner product space $\mathcal{H}_{a,d}$ of $E$-valued formal power series by

$$\mathcal{H}_{a,d} := \left\{ F(z) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha : f_\alpha \in E \ (\alpha \in \mathbb{N}^d), \ \sum_{\alpha \in \mathbb{N}^d} \|S_{\lambda_{\epsilon_a}}^\alpha f_\alpha\|^2 < \infty \right\}$$

endowed with the inner product

$$\langle F(z), G(z) \rangle := \sum_{\alpha \in \mathbb{N}^d} \langle S_{\lambda_{\epsilon_a}}^\alpha f_\alpha, S_{\lambda_{\epsilon_a}}^\alpha g_\alpha \rangle,$$

where $G(z) = \sum_{\alpha \in \mathbb{N}^d} g_\alpha z^\alpha$. Since $S_{\lambda_{\epsilon_a}}^\alpha S_{\lambda_{\epsilon_a}}^\alpha$ is bounded below on $E$ (Lemma 5.2.7(i)), $\mathcal{H}_{a,d}$ is a Hilbert space. We now define unitary $U : l^2(V) \to \mathcal{H}_{a,d}$ by $U(f) = F$. Further, if $\mathbb{M}_{z_j}$ denotes the $d$-tuple of (densely defined) multiplication operators $\mathbb{M}_{z_1}, \ldots, \mathbb{M}_{z_d}$ in $\mathcal{H}_{a,d}$ then

$$(S_{\lambda_{\epsilon_a}}^\alpha f_\alpha)_j = U S_{\lambda_{\epsilon_a}}^{\alpha + \epsilon_j} f_\alpha = f_\alpha z^{\alpha + \epsilon_j} = \mathbb{M}_{z_j} f_\alpha z^\alpha = \mathbb{M}_{z_j} U(S_{\lambda_{\epsilon_a}}^\alpha f_\alpha) \ (j = 1, \ldots, d).$$

Note that $U S_j = \mathbb{M}_{z_j} U$ holds on a dense set. Since $S_j$ is bounded, it follows that $\mathbb{M}_{z_j}$ is bounded for every $j = 1, \ldots, d$.

**Step II.** In this step, we check that $\mathcal{H}_{a,d}$ is a reproducing kernel Hilbert space associated with the reproducing kernel

$$\kappa_{\mathcal{H}_{a,d}}(z, w) = \sum_{\alpha \in \mathbb{N}^d} D_\alpha z^\alpha \overline{w}^\alpha \ (z, w \in \Omega),$$

where $D_\alpha$ is the inverse of $S_{\lambda_{\epsilon_a}}^\alpha S_{\lambda_{\epsilon_a}}^\alpha |_E$ on $E$ as ensured by Lemma 5.2.7(i), and $\Omega$ denotes the domain of convergence of $\kappa_{\mathcal{H}_{a,d}}$ (possibly $\{0\}$). Note that for $F(z) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathcal{H}_{a,d}$, $g \in E$ and $w \in \Omega$,

$$\langle F, \kappa_{\mathcal{H}_{a,d}}(\cdot, w)g \rangle = \sum_{\alpha \in \mathbb{N}^d} \langle S_{\lambda_{\epsilon_a}}^\alpha f_\alpha, S_{\lambda_{\epsilon_a}}^\alpha (D_\alpha g) \overline{w}^\alpha \rangle$$

$$= \sum_{\alpha \in \mathbb{N}^d} \langle f_\alpha w^\alpha, S_{\lambda_{\epsilon_a}}^\alpha S_{\lambda_{\epsilon_a}}^\alpha D_\alpha g \rangle$$

$$= \langle F(w), g \rangle_E,$$

where we used that $S_{\lambda_{\epsilon_a}}^\alpha S_{\lambda_{\epsilon_a}}^\alpha D_\alpha = I_E$ for every $\alpha \in \mathbb{N}^d$. This completes the verification of Step II.
Step III. We note that $D_\alpha$ is a block diagonal operator on $E$ with diagonal entries \( \frac{a(a+1)\cdots(a+|\alpha|-1)}{a!} \) (corresponding to \( e_{\text{root}} \)) and
\[
\left( \prod_{j=1}^{d} \frac{\alpha_{u_j}}{(\alpha_{u_j} + \alpha_j)!} \right) (|\alpha_u| + a)(|\alpha_u| + a + 1) \cdots (|\alpha_u| + a + |\alpha| - 1)
\]
(corresponding to the component of $E$ from $l^2(\mathbb{S}_F(u))$). This is immediate from \[5.17\] and the definition of $D_\alpha$.

Step IV. We verify that the domain of convergence of $\kappa_{\mathcal{H}_a,d}$ equals the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. Indeed, by the preceding two steps, $\kappa_{\mathcal{H}_a,d}$ takes the form
\[
\kappa_{\mathcal{H}_a,d}(z, w) = \sum_{\alpha \in \mathbb{N}^d} \frac{a(a+1)\cdots(a+|\alpha|-1)}{a!} z^\alpha \overline{w}^\alpha \ P_{\text{root}} + \sum_{F \in \mathcal{P}} \sum_{u \in \Omega_F} \kappa_{u,F}(z, w),
\]
where $\kappa_{u,F}(z, w)$ is given by \[5.14\]. Since the first series in the expression for $\kappa_{\mathcal{H}_a,d}(z, w)$ is precisely $P_{\text{root}} \prod_{u \in \Omega_F} \kappa_{u,F}(z, w)$, it suffices to check that the domain of convergence of $\kappa_{u,F}(\cdot, w)$ equals $\mathbb{B}^d$ for every $w \in \mathbb{B}^d$. However, the coefficients in this series are obtained (modulo some scalars) by adding the constant $d$-tuple $\alpha_u$ to the coefficients of $\frac{1}{(1-(z,w))^a}$, and hence the domain of convergence is $\mathbb{B}^d$. \hfill \square

Remark 5.2.8. In case $F_j = \Omega_{1,0}$ for $j = 1, \ldots, d$, then the reproducing kernel spaces $\mathcal{H}_a,d$ are precisely the spaces appearing in [23] (1.11) (see Table 1 below).

**Table 1.** Tree analogs of reproducing kernels $\kappa_{\mathcal{H}_a,d}(z, w)$

| Kernel $\kappa_{\mathcal{H}_a,d}(z, w)$ | $F_1 \times F_2$ |
|-------------------------------------|------------------|
| $\frac{1}{(1-\langle z, w \rangle)^a} P_{\text{root}(0,0)}$ | $\Omega_{1,0} \times \Omega_{1,0}$ |
| $\frac{1}{(1-\langle z, w \rangle)^a} P_{\text{root}(0,0)} + \sum_{\alpha \in \mathbb{N}^d} \frac{1}{|\alpha|-1} \prod_{j=0}^{\infty} \frac{1}{\alpha_j!} (a+|\alpha|+1) z^\alpha \overline{w}^\alpha \ P_{\text{root}(0,1),(1)}$ | $F_2 \times \Omega_{1,0}$ |
| $\frac{1}{(1-\langle z, w \rangle)^a} P_{\text{root}(0,0)} + \sum_{\alpha \in \mathbb{N}^d} \frac{1}{|\alpha|-1} \prod_{j=0}^{\infty} \frac{1}{\alpha_j!} (a+|\alpha|+1) z^\alpha \overline{w}^\alpha \ P_{\text{root}(0,1),(1)}$ | $F_2 \times \Omega_{1,0}$ |
| $+ \sum_{\alpha \in \mathbb{N}^d} \frac{1}{|\alpha|-1} \prod_{j=0}^{\infty} (a+|\alpha|+1) z^\alpha \overline{w}^\alpha \ P_{\text{root}(0,1),(2)}$ | $F_2 \times \Omega_{1,0}$ |
| $+ \sum_{\alpha \in \mathbb{N}^d} \frac{1}{|\alpha|-1} \prod_{j=0}^{\infty} (a+|\alpha|+1+2) z^\alpha \overline{w}^\alpha \ P_{\text{root}(1,1),(1,2)}$ | $F_2 \times \Omega_{1,0}$ |

Corollary 5.2.9. Let $\mathcal{F} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathcal{F}_1, \ldots, \mathcal{F}_d$ of finite joint branching index and let $S_{\lambda, \epsilon}$ be as introduced in Example 5.2.5. Then the point spectrum of $S_{\lambda, \epsilon}$ contains the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$.

**Proof.** By Theorem 5.2.6, $S_{\lambda}$ is unitarily equivalent to multiplication $d$-tuple $\mathcal{M}_\epsilon = (\mathcal{M}_{\epsilon,1}, \ldots, \mathcal{M}_{\epsilon,d})$ acting on the reproducing kernel Hilbert space $\mathcal{H}_{a,d}$ of $E$-valued holomorphic functions defined on unit ball $\mathbb{B}^d$. The desired conclusion now follows from the fact that $\mathcal{M}_{\epsilon}(\kappa_{\mathcal{H}_a,d}(\cdot, w)f) = \overline{w} \kappa_{\mathcal{H}_a,d}(\cdot, w)f$ for any $f \in E$, 

\[\]
We now turn our attention to the classification of spherically balanced multi-
shifts. The notion of spherically balanced multishifts is closely related to that of
spherical Cauchy dual tuple. Before we make this precise, note that by (1.3), \( S^*_i = S_i \left( \sum_{i=1}^{d} S_i^* S_i \right)^{-1} \). It follows that the spherical Cauchy dual \( S^*_\lambda = (S^*_1, \ldots, S^*_d) \) of a joint left invertible multishift \( S_\lambda \) is given by

\[
S^*_i e_v = \left( \sum_{i=1}^{d} \|S_i e_v\|^2 \right)^{-1} \sum_{w \in \text{Chi}_i(v)} \lambda^{(i)}_w e_w \quad \text{for all } v \in V, \ i = 1, \ldots, d.
\]

Note that \( S^*_\lambda \) is also a multishift on \( \mathcal{F} \) with weights

\[
\lambda^{(i)}_w \left( \sum_{i=1}^{d} \|S_i e_{\text{par}_i(w)}\|^2 \right)^{-1}, \ w \in V^\circ, \ i = 1, \ldots, d.
\]

In the next proposition, we show that every joint left invertible spherically
balanced multishift admits a polar decomposition in the following sense.

**Proposition 5.2.10.** Let \( \mathcal{F} = (V, \mathcal{E}) \) be the directed Cartesian product of rooted
directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \). Let \( S_\lambda = (S_1, \ldots, S_d) \) be a joint left invertible multishift
on \( \mathcal{F} \) and let \( S^*_\lambda \) denote the spherical Cauchy dual of \( S_\lambda \). Then the following
statements are equivalent:

(i) \( S^*_\lambda \) is commuting.

(ii) For every \( v \in V^\circ \), \( \mathcal{E} \) is constant on \( \text{Par}(v) \), where \( \mathcal{E} \) is as defined in (5.12).

(iii) There exists a joint isometry multishift \( T_\lambda = (T_1, \ldots, T_d) \) and a block
diagonal, positive, invertible, bounded linear operator \( D_\lambda \) on \( l^2(V) \) such that

\[
S^*_j T^*_j D^*_j, \ j = 1, \ldots, d.
\]

In this case, the above decomposition is unique.

**Proof.** We do not include the verification of the uniqueness part as it is similar
to that of Proposition 5.1.1.

(i) \( \iff \) (ii): Fix \( v \in V \). By the discussion prior to the proposition, we have

\[
S^*_i e_v = \mathcal{E}(v)^{-1} \sum_{w \in \text{Chi}_i(v)} \lambda^{(i)}_w e_w.
\]

Therefore,

\[
S^*_i S^*_j e_v = \mathcal{E}(v)^{-1} \sum_{w \in \text{Chi}_i(v)} \lambda^{(i)}_w S^*_j e_w
\]

\[
= \mathcal{E}(v)^{-1} \sum_{w \in \text{Chi}_i(v)} \lambda^{(i)}_w \mathcal{E}(w)^{-1} \sum_{u \in \text{Chi}_j(w)} \lambda^{(j)}_u e_u
\]

\[
= \mathcal{E}(v)^{-1} \sum_{u \in \text{Chi}_j \text{Chi}_i(v)} \lambda^{(j)}_{\text{par}_i(u)} \lambda^{(i)}_u \mathcal{E}(\text{par}_j(u))^{-1} e_u.
\]

Similarly,

\[
S^*_i S^*_j e_v = \mathcal{E}(v)^{-1} \sum_{u \in \text{Chi}_i \text{Chi}_j(v)} \lambda^{(j)}_{\text{par}_i(u)} \lambda^{(i)}_u \mathcal{E}(\text{par}_j(u))^{-1} e_u.
\]
Since $S_{\lambda}$ is commuting, by Proposition 3.1.7(i), $S_{\lambda}^i S_{\lambda}^j e_v = S_{\lambda}^j S_{\lambda}^i e_v$ for all $i,j = 1,\cdots,d$ if and only if
\[ \mathcal{C}(\text{par}_i(u)) = \mathcal{C}(\text{par}_j(u)) \text{ for all } u \in \text{Chi}_i \text{Chi}_j(v) \text{ and } i,j = 1,\cdots,d. \]
This yields the desired equivalence of (i) and (ii).

(ii) $\iff$ (iii): We introduce a multishift $T_\lambda = (T_1,\cdots,T_d)$ on $\mathcal{F}$ with weights
\[ \frac{\lambda^{(i)}_w}{\sqrt{\mathcal{C}(v)}}, \quad w \in \text{Chi}_i(v), \quad v \in V, \quad \text{and } i = 1,\cdots,d. \]
Note that for $v \in V$,
\[ \sum_{j=1}^{d} \|T_j e_v\|^2 = \mathcal{C}(v)^{-1} \sum_{j=1}^{d} \|S_j e_v\|^2 = 1. \]
One may argue as in the previous paragraph to see that $T_\lambda$ is commuting if and only (ii) holds. We now define a block diagonal operator $D_c$ on $l^2(V)$ as follows:
\[ D_c e_v := \sqrt{\mathcal{C}(v)} e_v \text{ for any } v \in V. \] (5.21)
Since $S_{\lambda}$ is joint left invertible, by Proposition 3.1.7(viii), $D_c$ is a block diagonal, positive, invertible bounded linear operator. If (ii) holds then by the above argument $S_{\lambda}$ has the decomposition given in (iii). Conversely, if (iii) holds then by the uniqueness of the decomposition, $T_\lambda$ and $D_c$ must be of the form as defined above. The desired conclusion in (ii) now follows from the commutativity of $T_\lambda$. \hfill $\square$

**Remark 5.2.11.** The spherical Cauchy dual $S_{w}^a$ of a joint left invertible classical multishift $S_w$ is the $a$-variable weighted shift given by
\[ S_{w}^a e_{\alpha} = \frac{w_{(j)}^{(j)}}{\delta_{\alpha,w}} e_{\alpha+j} \quad (1 \leq j \leq d), \]
where
\[ \delta_{\alpha,w} := \sum_{j=1}^{d} (w_{(j)}^{(j)})^2 \quad (\alpha \in \mathbb{N}^d). \]
It is easily seen that that $S_{w}^a$ is commuting if and only if $\delta_{\alpha+\epsilon_j,w} = \delta_{\alpha+\epsilon_k,w}$ for all $1 \leq j,k \leq d$ [30 Section 6]. In this case, the function $\mathcal{C}$ turns out to be a function of $|\alpha|, \alpha \in \mathbb{N}^d$ [51 Lemma 3.1]. This may also be deduced from the result above once we note that for any integer $t \geq 1$, one can order $G_t$ as $\{u_1,\cdots,u_{\text{card}(G_t)}\}$ such that $\text{Par}(u_i) \cap \text{Par}(u_{i+1}) \neq \emptyset$ for every $i = 1,\cdots,\text{card}(G_t) - 1$.

We refer to $T_\lambda$ and $D_c$ as joint isometry part and diagonal part of the multishift $S_{\lambda}$ respectively.

**Corollary 5.2.12.** Let $\mathcal{F} = (V,E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1,\cdots,\mathcal{T}_d$ and let $S_{\lambda}$ be a joint left invertible multishift on $\mathcal{F}$. If $S_{\lambda}$ is spherically balanced, then for any $\beta \in \mathbb{N}^d$ and any $v \in V$, we have
\begin{align*}
(i) \quad & T_{\lambda}^\beta e_v = \left( \prod_{p=0}^{[\beta]-1} \frac{1}{\mathcal{C}(v_{(p)})} \right) S_{\lambda}^\beta e_v, \\
(ii) \quad & \|T_{\lambda}^\beta e_v\|^2 = \left( \prod_{p=0}^{[\beta]-1} \frac{1}{\mathcal{C}(v_{(p)})} \right) \|S_{\lambda}^\beta e_v\|^2,
\end{align*}
where $\mathcal{C}_t$ denotes the constant value of $\mathcal{C}$ on the generation $G_t$ and $T_{\lambda}$ is the joint isometry part of $S_{\lambda}$.
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Proof. Assume that \( S \) is spherically balanced. Note that the weights of \( T \) takes the form

\[
\frac{\lambda^{(i)}(w)}{\sqrt{\mathcal{C}_{|v_i|}}} \quad w \in \text{Chi}(v_i), \ v \in V, \text{ and } i = 1, \cdots, d.
\]

To see (i), let \( v \in V \). Using induction on \( k \in \mathbb{N} \), one can verify that for \( i = 1, \cdots, d \),

\[
T_i^k e_v = \prod_{p=0}^{k-1} \frac{1}{\sqrt{\mathcal{C}_{|v_i|}+p}} S_i^k e_v.
\]

Further, in a similar fashion one can get

\[
T_i^k S_i^k e_v = \prod_{p=0}^{k-1} \frac{1}{\sqrt{\mathcal{C}_{|v_i|}+p}} S_i^k e_v.
\]

Continuing this, we obtain (i). Finally, (ii) is immediate from (i).

Note that part (ii) above gives precise relation between the moments of \( S \) and that of \( T \). The multiplicative factor \( \left( \prod_{p=0}^{[\beta]} \frac{1}{\sqrt{\mathcal{C}_{|\alpha_p|}+p}} \right) \) appearing in (ii) suggests one to introduce a classical unilateral weighted shift \( S \) on some Hilbert space \( H^2(\gamma) \) of formal power series, so that \( \| T_\lambda^k e_v^v \| = \| S_\alpha^k f_\alpha \| \| S_\lambda^d \| \) for some orthonormal basis \( \{ f_k \} \) of \( H^2(\gamma) \). Unfortunately, in this formulation, the tree like structure of \( T \) does not reflect in the construction of \( S \) on \( H^2(\gamma) \). However, there is an alternate way to construct a shift on a directed tree arising naturally from \( T \) which at the same time gives the above relation between moments of \( S \) and that of \( T \).

Definition 5.2.13. Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of locally finite, rooted directed trees \( \mathcal{T}_1, \cdots, \mathcal{T}_d \). Consider the component \( \mathcal{T}^\otimes = (V^\otimes, F) \) of the tensor product \( \mathcal{T}^\otimes \) of \( \mathcal{T}_1, \cdots, \mathcal{T}_d \), which contains root (see Theorem 2.2.3). For a bounded sequence \( \{ c_t \} \) of positive real numbers, consider the (one variable) weighted shift \( S_\theta \) on the rooted directed tree \( \mathcal{T}^\otimes_{\text{root}} \) with weights given by

\[
\theta_w := \frac{\sqrt{c_w \mathcal{C}_{\text{sib}(w)}-1}}{\sqrt{\text{card}(\text{sib}(w))}} \quad (w \in V^\otimes \setminus \text{root}),
\]

where \( \alpha_w \) is the depth of \( w \) in \( \mathcal{T}^\otimes_{\text{root}} \) and \( \text{sib}(w) \) is the set of siblings of \( w \) in the directed tree \( \mathcal{T}^\otimes_{\text{root}} \). We refer to the weighted shift \( S_\theta \) on \( \mathcal{T}^\otimes_{\text{root}} \) as the balanced weighted shift associated with \( \{ c_t \} \).

Example 5.2.14. If \( \mathcal{T}_1 = \mathcal{T}_{1,0} = \mathcal{T}_2 \), then as seen in Example 2.2.6, \( \mathcal{T}^\otimes \) is (isomorphic to) \( \mathcal{T}_{1,0} \), and hence \( S_\theta \) is (unitarily equivalent to) classical unilateral shift on \( l^2(\mathbb{N}) \).

If \( \mathcal{T}_1 = \mathcal{T}_{2,0}, \mathcal{T}_2 = \mathcal{T}_{1,0} \), then as seen in Example 2.2.7, \( \mathcal{T}^\otimes \) is \( \mathcal{T}_{2,0} \), and hence \( S_\theta \) is a weighted shift on the rooted directed tree \( \mathcal{T}_{2,0} \) (see Figure 5.1).

Finally, if \( \mathcal{T}_1 = \mathcal{T}_{2,0} = \mathcal{T}_2 \) then as seen in Example 2.2.8, \( \mathcal{T}^\otimes \) is \( \mathcal{T}_{4,0} \), and hence \( S_\theta \) is a weighted shift on the rooted directed tree \( \mathcal{T}_{4,0} \) (see Figure 5.2).

Before we present the main result of this section, recall that a finite positive Borel measure \( \mu \) supported in the unit sphere \( \partial B^d \) in \( \mathbb{C}^d \) is said to be the \( T^d \)-invariant if for any Borel measurable subset \( \Delta \) of \( \partial B^d \),

\[
\mu(\zeta \cdot \Delta) = \mu(\Delta) \quad \text{for all } \zeta \in T^d,
\]
where $\zeta \cdot \Delta = \{ \zeta \cdot z : z \in \Delta \}$ with $\zeta \cdot z$ denoting the dot product of $\zeta$ and $z$. A Reinhardt measure is a $T^d$-invariant probability measure on the unit sphere.

The following generalizes [31, Theorem 1.10], [75, Theorem 1.3] (the case in which $T = T_{1,0}^d$, as described in Example 2.1.5, with $v = \text{root}$).

**Theorem 5.2.15.** Let $T = (V, E)$ be the directed Cartesian product of locally finite, rooted directed trees $T_1, \ldots, T_d$ and let $T_{\text{root}} = (V^\circ, F)$ be the component of $T^\circ$ containing $\text{root}$. For $v \in V$, choose $v \in V^\circ$ so that $|\alpha_v| = \alpha_v$ (see the last part of Theorem 2.2.3). If $S_\lambda$ is a joint left invertible multishift on $T$, then the following statements are equivalent:

(i) The multishift $S_\lambda$ is spherically balanced.

(ii) For every $v \in V$, there exists a Reinhardt measure $\mu_v$ supported in the unit sphere $\partial B_d$ and a balanced weighted shift $S_\theta$ on $T_{\text{root}}^\circ$ associated with a bounded sequence $\{c_t\}_{t \in \mathbb{N}}$ such that

$$\|f_k\|^2_{L^2(V)} = \int_{\partial B_d} \|f_{\theta,k}(z)\|^2_{L^2(V^\circ)} d\mu_v(z), \quad (5.23)$$
where
\[ f_k = \sum_{\beta \in \mathbb{N}^d, |\beta| \leq k} a_{\beta} S_{\lambda}^\beta e_v \in \ell^2(V), \quad f_{\theta,k}(z) = \sum_{\beta \in \mathbb{N}^d, |\beta| \leq k} a_{\beta} z^\beta S_{\theta}^\beta e_v \in \ell^2(V^\circ) \] (k \in \mathbb{N}, z \in C^d).

If any of the above equivalent statements holds, then we have the following:
(a) If \( \{f_k\}_{k \in \mathbb{N}} \) converges to \( f \) in \( \ell^2(V) \), then \( \{f_{\theta,k}(z)\}_{k \in \mathbb{N}} \) converges to some \( f_{\theta}(z) \) in \( \ell^2(V^\circ) \) for \( \mu_v \)-a.e. \( z \in \partial B^d \), and
\[
\|f\|_2^2(V) = \int_{\partial B^d} \|f_{\theta}(z)\|_2^2(V^\circ) \, d\mu_v(z).
\] (5.24)

(b) If \( Q_{\lambda}^n(I) \) is as defined in (1.1), then for \( n \in \mathbb{N} \),
\[
\langle Q_{\lambda}^n(I) e_v, e_v \rangle = \|S_{\theta}^n e_v\|^2, \quad \|Q_{\lambda}^n(I)\| = \|S_{\theta}^n\|^2.
\] (5.25)

PROOF. Suppose that \( S_{\lambda} \) is a joint left invertible multishift on \( \mathcal{T} \). To see the implication (i) \( \implies \) (ii), assume that \( S_{\lambda} \) is spherically balanced and let \( c_1 := \mathcal{C}_1 \), the constant value of \( \mathcal{C} \) on the generation \( G_t \) for \( t \in \mathbb{N} \). Consider the balanced weighted shift \( S_{\theta} \) on \( \mathcal{T}_{\text{root}} \) associated with \( \{c_t\}_{t \in \mathbb{N}} \). We first prove the formula
\[
\|S_{\theta}^k e_v\|^2 = \prod_{p=0}^{k-1} c_{\alpha_u + p} \quad (k \geq 1, \ v \in V^\circ)
\] (5.27)
by induction on integers \( k \geq 1 \). If \( k = 1 \), then
\[
\|S_{\theta} e_v\|^2 = \sum_{u \in \text{Chi}(v)} \theta_u^2 \sum_{u \in \text{Chi}(v)} \frac{c_{\alpha_u - 1}}{\text{card}(\text{sib}(u))} = c_{\alpha_v}.
\]
Assume the formula for some integer \( k \geq 1 \). By the induction hypothesis, we obtain
\[
\|S_{\theta}^{k+1} e_v\|^2 = \left\| \sum_{u \in \text{Chi}(v)} \theta_u S_{\theta}^k e_u \right\|^2 = \sum_{u \in \text{Chi}(v)} \theta_u^2 \|S_{\theta}^k e_u\|^2
\]
\[
= \sum_{u \in \text{Chi}(v)} \frac{c_{\alpha_u - 1}}{\text{card}(\text{sib}(u))} \left( \prod_{p=0}^{k-1} c_{\alpha_u + p} \right)
\]
\[
= c_{\alpha_v} \left( \prod_{p=0}^{k-1} c_{\alpha_v + p} \right) \sum_{u \in \text{Chi}(v)} \frac{1}{\text{card}(\text{sib}(u))}
\]
\[
= \prod_{p=0}^{k} c_{\alpha_v + p}.
\]
This completes the induction. We now verify that there exists a Reinhardt measure \( \mu_v \) supported in the unit sphere \( \partial B^d \) such that
\[
\|S_{\lambda}^\beta e_v\|^2_{2(V)} = \|S_{\theta}^\beta e_v\|^2_{2(V^\circ)} \int_{\partial B^d} |z|^\beta^2 \, d\mu_v, \ \beta \in \mathbb{N}^d.
\] (5.28)
In view of Corollary 5.2.12(ii), it suffices to find a Reinhardt measure \( \mu_v \) supported in the unit sphere \( \partial B^d \) such that \( \|T_{\lambda}^\beta e_v\|^2 = \int_{\partial B^d} |z|^\beta^2 \, d\mu_v \), where \( T_{\lambda} \) is the joint
isometry part of $S_{\lambda}$. Consider the $T_{\lambda}$-invariant subspace $M := \bigvee \{ T_{\lambda}^{\beta} e_{v} : \beta \in \mathbb{N}^d \}$ of $L^2(V)$. By Proposition 3.1.7(ix), $(T_{\lambda}^{\beta} e_{v}, T_{\lambda}^{\gamma} e_{v}) = 0$ if $\beta \neq \gamma$. It follows that $T_{\lambda}|_{M}$ is a joint isometry classical multishift (up to unitary equivalence). The existence of the desired measure is now a consequence of the well-known fact that any joint isometry is joint subnormal [18] Proposition 2 (see the proof of [31] Theorem 1.10 for more details).

We now check the integral representation appearing in (5.23). Note that $\langle S_{\lambda}^{\beta} e_{v}, S_{\lambda}^{\gamma} e_{v} \rangle = 0$ if $\beta \neq \gamma$ and $\langle S_{\lambda}^{\beta} e_{v}, S_{\lambda}^{\beta} e_{v} \rangle = 0$ if $k \neq l$ (see Proposition 3.1.7(ix)). It follows that

$$
\| f_k \|^2_{L^2(V)} = \sum_{\beta \in \mathbb{N}^d, \| \beta \| \leq k} |a_{\beta}|^2 \| S_{\lambda}^{\beta} e_{v} \|^2_{L^2(V)}
$$

(5.28)

$$
\sum_{\beta \in \mathbb{N}^d, \| \beta \| \leq k} |a_{\beta}|^2 \| S_{\lambda}^{\beta} e_{v} \|^2_{L^2(V \otimes \cdot)} \int_{\partial \mathbb{B}^d} |z|^2 d\mu_v.
$$

(5.29)

Since $\mu_v$ is a Reinhardt measure, by [31] Lemma 2.3, the monomials $\{z^\alpha\}_{\alpha \in \mathbb{N}^d}$ are orthogonal in $L^2(\partial \mathbb{B}^d, \mu_v)$. Thus

$$
\int_{\partial \mathbb{B}^d} \| f_{\theta,k}(z) \|^2_{L^2(V \otimes \cdot)} d\mu_v(z)
$$

$$
= \int_{\partial \mathbb{B}^d} \sum_{\beta, \gamma \in \mathbb{N}^d, \| \beta \|, \| \gamma \| \leq k} a_{\beta \gamma} \overline{S_{\lambda}^{\beta} e_{v}, S_{\lambda}^{\gamma} e_{v}} z^{\beta} \overline{z}^{\gamma} d\mu_v(z)
$$

$$
= \sum_{\beta \in \mathbb{N}^d, \| \beta \| \leq k} |a_{\beta}|^2 \| S_{\lambda}^{\beta} e_{v} \|^2_{L^2(V \otimes \cdot)} \int_{\partial \mathbb{B}^d} |z|^2 d\mu_v(z)
$$

(5.29) $\| f_k \|^2_{L^2(V)}$.

This proves that (i) implies (ii). To see that (ii) implies (i), let $f_k = S_j e_{v}$ and $f_{\theta,k} = z_j S_{\theta} e_{v}$ in (5.23) and sum over $j = 1, \ldots, d$ to see that

$$
\sum_{j=1}^{d} \| S_j e_{v} \|^2_{L^2(V)} = \| S_{\theta} e_{v} \|^2_{L^2(V \otimes \cdot)} \sum_{j=1}^{d} \int_{\partial \mathbb{B}^d} |z_j|^2 d\mu_v
$$

$$
= \| S_{\theta} e_{v} \|^2_{L^2(V \otimes \cdot)} = c_{\alpha_v} = c_{|\alpha_v|},
$$

which is constant on $G_{|\alpha_v|}$.

To see the remaining part of the proof, assume that $S_{\lambda}$ is spherically balanced.

(a) Note that $\| f_k \|^2_{L^2(V)} \uparrow \| f \|^2_{L^2(V)}$ and $\| f_{\theta,k}(z) \|^2_{L^2(V \otimes \cdot)} \uparrow g(z)$ (possibly in the extended real line) as $k \to \infty$, where

$$
g(z) := \left( \sum_{\beta \in \mathbb{N}^d} |a_{\beta}|^2 |z|^2 \| S_{\lambda}^{\beta} e_{v} \|^2 \right)^{\frac{1}{2}} (z \in \partial \mathbb{B}^d).
$$

Applying monotone convergence theorem to (5.23), we obtain

$$
\| f \|^2_{L^2(V)} = \int_{\partial \mathbb{B}^d} g(z)^2 d\mu_v(z).
$$
Since the left hand side is a finite positive number, $0 \leq g(z) < \infty$ except for $z$ in a set of $\mu_v$ measure 0. It follows that $f_\theta(z) := \sum_{\beta \in \mathbb{N}^d} a_\beta z^\beta S_\theta^\beta e_v \in l^2(V \otimes)$ except for $z$ in a set of $\mu_v$ measure 0 and $g(z) = \|f_\theta(z)\|_{l^2(V \otimes)}$. This also yields (5.24).

(b) Note that

\begin{align*}
\sum_{\alpha \in \mathbb{N}^d} \frac{n!}{\alpha!} |a_{\alpha}|^2 &\leq \sum_{\alpha \in \mathbb{N}^d} \frac{n!}{\alpha!} |S_\theta^\alpha e_v|^2 \int_{\mathbb{R}^d} |z^\alpha|^2 d\mu_v \\
&= \|S_\theta^n e_v\|^2_{l^2(V \otimes)} \int_{\mathbb{R}^d} \sum_{\alpha \in \mathbb{N}^d} \frac{n!}{\alpha!} |z^\alpha|^2 d\mu_v,
\end{align*}

which is same as $\|S_\theta^n e_v\|^2_{l^2(V \otimes)}$ since

\begin{align*}
\sum_{\alpha \in \mathbb{N}^d} \frac{n!}{\alpha!} |z^\alpha|^2 &= \|z\|_{l^2}^{2n} (z \in \mathbb{C}^d)
\end{align*}

and $\mu$ is a probability measure supported in the unit sphere. It follows that $\langle Q_{S_\lambda}^n (I)e_v, e_v \rangle = \|S_\theta^n e_v\|^2$. Note further that

\begin{align*}
\|Q_{S_\lambda}^n (I)\| &= \sup_{v \in V} \langle Q_{S_\lambda}^n (I)e_v, e_v \rangle = \sup_{v \in V \otimes} \|S_\theta^n e_v\|^2 = \|S_\theta^n\|^2.
\end{align*}

This completes the verification of (b). \hfill \square

For convenience, we refer to the balanced weighted shift $S_\theta$ on $\mathcal{T}_\text{root}^\otimes$ as the shift associated with the multishift $S_\lambda$ on $\mathcal{T}$ (cf. \[24, Definition 2.3\]).

Here we discuss some consequences of the preceding theorem. The first of which is a local spherical analog of von Neumann’s inequality (cf. \[75, Proposition 2.5\] and \[71, Theorem 7.6\]).

**Corollary 5.2.16.** Let $\mathcal{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Let $S_\lambda$ be a joint left invertible, spherically balanced multishift on $\mathcal{T}$. If $S_\lambda$ is a joint contraction, then for any positive integer $k$ and finite sequence $\{a_\beta : \beta \in \mathbb{N}^d, |\beta| \leq k\}$, we have

\[
\sup_{v \in V} \left\| \sum_{\beta \in \mathbb{N}^d, |\beta| \leq k} a_\beta S_\lambda^\beta e_v \right\| \leq \sup_{z \in \mathbb{R}^d} \left\| \sum_{\beta \in \mathbb{N}^d, |\beta| \leq k} a_\beta z^\beta \right\|.
\]

**Proof.** Suppose that $S_\lambda$ is a joint contraction. Let $S_\theta$ be the balanced weighted shift associated with $S_\lambda$ and fix $v \in V$. Note that by (5.26), $S_\theta$ is a contraction. On the other hand, by the preceding theorem, there exists a Reinhardt
measure $\mu_v$ supported in $\partial B^d$ such that

$$
\left\| \sum_{\beta \in \mathbb{N}^d \mid \beta \leq k} a_\beta S^\beta_\lambda e_v \right\|_{L^2(V)}^2 \leq \int_{\partial B^d} \left\| \sum_{\beta \in \mathbb{N}^d \mid \beta \leq k} a_\beta z^\beta S^\beta_\theta e_v \right\|_{L^2(V^\oplus)}^2 \, d\mu_v(z)
$$

where we used that monomials are orthogonal in $L^2(\partial B^d, \mu_v)$. After taking supremum over $v \in V$, the desired conclusion now follows from the maximum modulus principle in several complex variables [83, pg 5].

**Corollary 5.2.17.** Let $\mathcal{F} = (V, E)$ be the directed Cartesian product of locally finite, rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Let $S_\lambda = (S_1, \ldots, S_d)$ be a joint left invertible, spherically balanced multishift on $\mathcal{F}$ and let $S_\theta$ be the weighted shift on the rooted directed tree $\mathcal{F}_\theta^\oplus = (V^\oplus, \mathcal{F})$ associated with $S_\lambda$. Let $c_\ell$ be the constant value of $\sum_{j=1}^d \| S_j e_v \|^2$ on the generation $G_t$ of $\mathcal{F}$. Then we have the following:

(i) $r(S_\lambda) = \lim_{n \to \infty} \sup_{\ell \in \mathbb{N}} \left( \prod_{p=0}^{n-1} c_{k+p} \right)^{1/n} = r(S_\theta)$, where $r(T)$ denotes the spectral radius of any commuting $d$-tuple $T$. In particular, the Taylor spectrum of $S_\lambda$ is a Reinhardt set containing 0 and contained in the closed ball centered at the origin and of radius $r(S_\theta)$.

(ii) $m_\infty(S_\lambda) = \sup_{n \geq 1} \inf_{k \in \mathbb{N}} \left( \prod_{p=0}^{n-1} c_{k+p} \right)^{1/n} = m_\infty(S_\theta)$. In particular, the left spectrum $\sigma_l(S_\lambda)$ of $S_\lambda$ is contained in the closed ball centered at the origin with inner radius $m_\infty(S_\theta)$ and outer radius $r(S_\theta)$.

**Proof.** The first part follows from (5.26), (5.27), and the spectral radius formula (1.5) for the Taylor spectrum:

$$
r(S_\lambda) = \lim_{n \to \infty} \| Q^n_{S_\lambda}(I) \|^{1/2n} = \lim_{n \to \infty} \| S^n_\theta \|^{1/n} = r(S_\theta).
$$

To see (ii), note that by (1.6),

$$
m_\infty(S_\lambda) \leq \sup_{n \geq 1} \inf_{v \in V} (Q^n_{S_\lambda}(I) e_v, e_v)^{1/n}.
$$
Let $M_n := \inf_{v \in V} \langle Q^n_{S_{\lambda}}(I)e_v, e_v \rangle^{1/n}$ for $n \geq 1$. Then for any $f = \sum_{v \in V} f(v)e_v \in l^2(V)$ of unit norm, by parts (vi) and (x) of Proposition 3.1.7,

$$\langle Q^n_{S_{\lambda}}(I)f, f \rangle^{1/n} = \left( \sum_{v \in V} |f(v)|^2 \langle Q^n_{S_{\lambda}}(I)e_v, e_v \rangle \right)^{1/n} \geq M_n,$$

and hence $m_\infty(S_{\lambda}) = \sup_{n \geq 1} \inf_{v \in V} \langle Q^n_{S_{\lambda}}(I)e_v, e_v \rangle^{1/n}$. Similar observation holds for $S_{\theta}$. The desired conclusion in (ii) may now be drawn from (5.25) and (5.27). \( \square \)

**Example 5.2.18.** Consider the multishift $S_{\lambda_{\epsilon_a}}$ as discussed in Example 5.2.5. Recall that $\{c_{a,t}\}_{t \in \mathbb{N}}$ is given by

$$c_{a,t} = \frac{t + d}{t + a} \quad (t \in \mathbb{N}).$$

Thus weights of $S_{\lambda_{\epsilon_a}} = (S_1, \cdots, S_d)$ are given by

$$\lambda^{(j)}_w = \frac{1}{\sqrt{\text{card}(\text{Chi}_j(v))}} \left( \frac{\sqrt{\alpha_v + 1}}{|\alpha_v| + a} \right) \quad \text{for } w \in \text{Chi}_j(v) \text{ and } j = 1, \cdots, d. \quad (5.30)$$

By the preceding corollary,

$$r(S_{\lambda_{\epsilon_a}}) = \lim_{n \to \infty} \sup_{k \in \mathbb{N}} \left( \prod_{p=0}^{n-1} c_{a,k+p} \right)^{1/n} = \lim_{n \to \infty} \sup_{k \in \mathbb{N}} \left( \prod_{p=0}^{n-1} \left( \frac{k + p + d}{k + p + a} \right)^{1/n} \right).
$$

We follow the argument in [34] Lemma 3.9 to see that $r(S_{\lambda_{\epsilon_a}}) = 1$. Let $F(n, k) = \prod_{p=0}^{n-1} \left( \frac{k + p + d}{k + p + a} \right)^{1/n}$ and note that $F(n, k)$ is increasing (resp. decreasing) in $k$ if and only if $a > d$ (resp. $a < d$). Thus the following possibilities occur:

$$\prod_{p=0}^{n-1} \left( \frac{p + d}{p + a} \right)^{1/n} = F(n, 0)^{1/n} \leq F(n, k)^{1/n} \leq \prod_{p=0}^{n-1} \left( \frac{p + d}{p + a} \right)^{1/n}.$$

In either case, $\lim_{n \to \infty} \sup_{k \in \mathbb{N}} \left( \prod_{p=0}^{n-1} \left( \frac{k + p + d}{k + p + a} \right)^{1/n} \right) = 1$. This shows that $r(S_{\lambda_{\epsilon_a}}) = 1$.

One can argue similarly to see that $m_\infty(S_{\lambda_{\epsilon_a}}) = 1$. In particular,

$$\sigma(S_{\lambda_{\epsilon_a}}) \subseteq \text{cl}(\mathbb{B}^d), \quad \sigma_l(S_{\lambda_{\epsilon_a}}) \subseteq \partial \mathbb{B}^d. \quad (5.31)$$

Finally, by Corollary 5.2.9 we must have $\sigma(S_{\lambda_{\epsilon_a}}) = \text{cl}(\mathbb{B}^d)$.

We conclude this section with a brief discussion on the essential spectrum of the multishift $S_{\lambda_{\epsilon_a}}$. Recall first that a commuting $d$-tuple $T$ is *essentially normal* if $[T^*_i, T_j] = T^*_iT_j - T_jT^*_i$ is compact for every $i, j = 1, \cdots, d$.

**Proposition 5.2.19.** Let $S_{\lambda_{\epsilon_a}}$ be a multishift as discussed in Example 5.2.3. If $\mathcal{T}$ is of finite joint branching index $k_{\mathcal{T}}$, then $S_{\lambda_{\epsilon_a}} = (S_1, \cdots, S_d)$ is essentially normal.

**Proof.** Assume that $\mathcal{T}$ is of finite joint branching index $k_{\mathcal{T}}$. By the Putnam-Fuglede Theorem [36], it suffices to check that $[S^*_j, S_j]$ is compact for every $j = 1, \cdots, d$. For fixed $j = 1, \cdots, d$, define

$$W_j := \{ v \in V : \text{card(Chi}_j(v)) = 1 \text{ and card(sib}_j(v)) = 1 \}.$$
Note that \([S^*_j, S_j]\) decomposes into \(A_j \oplus B_j\) on \(l^2(V) = l^2(W_j) \oplus l^2(V \setminus W_j)\), where \(A_j, B_j\) are block diagonal operators given by
\[
A_j e_v = (\lambda_v^{(j)})^2 - (\lambda_v^{(j)})^2 e_v \ (v \in W_j, \text{Chi}_j(v) = \{w\}),
\]
\[
B_j e_v = \sum_{w \in \text{Chi}_j(v)} (\lambda_w^{(j)})^2 e_v - \sum_{u \in \text{sib}_j(v)} \lambda_u^{(j)} \lambda_w^{(j)} e_u \ (v \in V \setminus W_j).
\]

By (5.30), \(A_j\) is a diagonal operator with diagonal entries
\[
(\lambda_w^{(j)})^2 - (\lambda_v^{(j)})^2 = \frac{|\alpha_v| - |\alpha_v| + a - 1}{(|\alpha_v| + a)(|\alpha_v| + a - 1)},
\]
which tends to 0 as \(|\alpha_v| \to \infty\). This shows that \(A_j\) is a compact operator. To see that \(B_j\) is a compact operator, note first that
\[
\begin{align*}
B_j e_v \quad &\overset{(5.30)}{=} \sum_{u \in \text{Chi}_j(v)} \frac{1}{\text{card} \left( \text{Chi}_j(v) \right)} \frac{\alpha_{v_j} + 1}{|\alpha_v| + a} e_v \\
&\quad - \sum_{u \in \text{sib}_j(v)} \frac{1}{\text{card} \left( \text{sib}_j(v) \right)} \frac{\alpha_{v_j}}{|\alpha_v| + a - 1} e_w \\
&= \frac{\alpha_{v_j} + 1}{|\alpha_v| + a} e_v - \sum_{u \in \text{sib}_j(v)} \frac{1}{\text{card} \left( \text{sib}_j(v) \right)} \frac{\alpha_{v_j}}{|\alpha_v| + a - 1} e_w. \quad (5.32)
\end{align*}
\]

We next decompose \(V \setminus W_j\) as \(\bigcup_{v \in \Omega} \text{sib}_j(v)\), where \(\Omega\) is formed by picking up only one element (as ensured by axiom of choice) from every \(\text{sib}_j(v)\) for \(j = 1, \ldots, d\). Note that \(l^2(\text{sib}_j(v))\) is reducing for \(B_j\) for every \(v \in \Omega\). This immediately yields the decomposition
\[
B_j = \bigoplus_{v \in \Omega} B_{jv} \quad \text{on} \quad l^2(V \setminus W_j) = \bigoplus_{v \in \Omega} l^2(\text{sib}_j(v)),
\]
where \(B_{jv}\) is a finite rank operator (since \(\tilde{T}_j\) is locally finite) for \(j = 1, \ldots, d\). It now suffices to check that \(\|B_{jv}\| \to 0\) as \(|\alpha_v| \to \infty\) (see Remark 3.4.1). Before proceeding to this end, observe that
\[
\sup_{v \in V \setminus W_j} \text{card} \left( \text{sib}_j(v) \right) \leq M_j := \text{card} \left( \text{Chi}^{(l_{\tilde{T}_j})} \left( \text{root}_j \right) \right) < \infty, \quad (5.33)
\]
\[
\sup_{v \in V \setminus W_j} \alpha_{v_j} \leq k_{\tilde{T}_j}. \quad (5.34)
\]

Let \(f = \sum_{u \in \text{sib}_j(v)} f(u) e_u \in l^2(\text{sib}_j(v))\) and let \(\Upsilon_v := \sum_{u \in \text{sib}_j(v)} f(u)\). Note that
\[
\begin{align*}
\text{B}_j f \overset{(5.32)}{=} \sum_{u \in \text{sib}_j(v)} f(u) \left( \frac{\alpha_{u_j} + 1}{|\alpha_u| + a} e_u - \frac{1}{\text{card} \left( \text{sib}_j(u) \right)} \frac{\alpha_{u_j}}{|\alpha_u| + a - 1} \sum_{w \in \text{sib}_j(u)} e_w \right) \\
&= \frac{\alpha_{v_j} + 1}{|\alpha_v| + a} \sum_{u \in \text{sib}_j(v)} f(u) e_u - \frac{\Upsilon_v}{\text{card} \left( \text{sib}_j(v) \right)} \frac{\alpha_{v_j}}{|\alpha_v| + a - 1} \left( \sum_{w \in \text{sib}_j(v)} e_w \right) \\
&= \sum_{u \in \text{sib}_j(v)} \beta_u e_u,
\end{align*}
\]
where \(\beta_u\) is given by
\[
\beta_u = \frac{\alpha_{v_j} + 1}{|\alpha_v| + a} f(u) - \frac{\Upsilon_v}{\text{card} \left( \text{sib}_j(v) \right)} \frac{\alpha_{v_j}}{|\alpha_v| + a - 1}.
\]
Since |Υv| ≤ ∥f∥Mj, by (5.34) and Cauchy-Schwarz inequality,

\[ |β_u| \leq \frac{(k_β + 1)|f(u)|}{|α_v| + a} + \frac{|Υv|}{\text{card}(\text{si}b_j(v))} \frac{k_β}{|α_v| + a - 1} \]

\[ \leq \frac{k_β + 1}{|α_v| + a - 1} \left(1 + \frac{M_j}{\text{card}(\text{si}b_j(v))}\right)\|f\| . \]

It follows from (5.33) and that

\[ \|B_j f\|^2 = \sum_{u \in \text{si}b_j(v)} |β_u|^2 \leq \frac{(k_β + 1)^2(1 + M_j)^3}{(|α_v| + a - 1)^2} \|f\|^2 . \]

This shows that \( \|B_j f\| \rightarrow 0 \) as \( |α_v| \rightarrow \infty \). □

The conclusion of the preceding proposition no more holds true if we relax the assumption of finite joint branching index.

**Example 5.2.20.** Consider the n-ary tree \( T^{(n)} \) given by \( V^{(n)} = \{ v_{k,l} : k \in \mathbb{N}, l = 1, \cdots, 2^k \} \), \( \text{Ch}(v_{k,l}) = \{ v_{k+1,j} : n(l - 1) + 1 \leq j \leq nl \} \).

Let \( T = (V, \mathcal{E}) \) denote the directed product of \( T^{(n)} \) with itself. Note that for \( v_{k,l}, v_{p,q} \in V^{(n)} \),

\[ \text{Chi}_1((v_{k,l}, v_{p,q})) = \{ (v_{k+1,j}, v_{p,q}) \in V : n(l - 1) + 1 \leq j \leq nl \} , \]

\[ \text{Chi}_2((v_{k,l}, v_{p,q})) = \{ (v_{k,l}, v_{p+q,j}) \in V : n(q - 1) + 1 \leq j \leq nq \} , \]

so that \( \text{card}((\text{Chi}_1((v_{k,l}, v_{p,q})))) = n \) for \( j = 1, 2 \). Let \( S_{\lambda^{(n)}} \) be as discussed in Example 5.2.5 with \( d = 2 \). Note that the system \( \lambda \) is given by

\[ \lambda^{(1)}_w = \frac{1}{\sqrt{n}} \sqrt{k + 1 \over k + p + a} , w \in \text{Chi}_1((v_{k,l}, v_{p,q})). \]

\[ \lambda^{(2)}_w = \frac{1}{\sqrt{n}} \sqrt{p + 1 \over k + p + a} , w \in \text{Chi}_2((v_{k,l}, v_{p,q})). \]

We claim that \( S_{\lambda^{(n)}} \) on \( T^{(n)} \) is essentially normal if and only if \( n = 1 \). In case \( n = 1 \), \( S_{\lambda^{(n)}} \) are classical multishifts. The essential normality in this case is well-known (see, for example, [34]). To see the converse, assume that \( n \geq 2 \). Let \( B_j := [S_j^*, S_j] \) for \( j = 1, 2 \). It suffices to check that \( \|B_1 e_{(v_{k,l}, v_{p,q})}\| \rightarrow 0 \) as \( k = p \rightarrow \infty \). For \( v = (v_{k,l}, v_{p,q}) \), note that

\[ \langle B_1 e_v, e_v \rangle = \sum_{u \in \text{Chi}_1(v)} \frac{1}{n} \frac{k + 1}{k + p + a} - \sum_{w \in \text{si}b_j(v)} \frac{k}{n k + p + a - 1} \langle e_w, e_v \rangle \]

\[ = \frac{k + 1}{k + p + a} - \frac{1}{n} \frac{k}{k + p + a - 1} , \]

which converges to \( {1 \over 2} (1 - {1 \over n}) \) as \( k = p \rightarrow \infty \).

**Corollary 5.2.21.** Let \( S_{\lambda^{(n)}} \) be multishift as discussed in Example 5.2.5. Assume that \( T \) is of finite joint branching index \( k_β \). Then

\[ \sigma_r(S_{\lambda^{(n)}}) \subseteq \partial \mathbb{B}^d . \]
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PROOF. It may be concluded from the proof of [34] Lemma 3.5 that for any essentially normal $d$-tuple $T$, the essential spectrum $\sigma_e(T)$ is contained in
$$\{w \in \mathbb{C}^d : \|w\|^2_2 \in \sigma_e(Q_T(I))\},$$
where $Q_T(\cdot)$ is as defined in [1]. In view of the last result, it now suffices to check that $\sigma_e(Q_{S_{\lambda\varepsilon_{\alpha}}} (I))$ is equal to $\{1\}$. However, $Q_{S_{\lambda\varepsilon_{\alpha}}} (I)$ is diagonal operator with diagonal entries $\frac{|\alpha|+d}{|\alpha|+d} (\text{repeated card}(\text{Chi}_{\langle \alpha \rangle, \varepsilon}(\text{root}))$ times) for $v \in V$. Since the only limit point of these eigenvalues of $Q_{S_{\lambda\varepsilon_{\alpha}}} (I)$ is $1$, the essential spectrum of $Q_{S_{\lambda\varepsilon_{\alpha}}} (I)$ must be $\{1\}$ [36].

REMARK 5.2.22. Since the point spectrum of $S_{\lambda}$ is empty (Corollary 3.3.4, in dimension $d = 2$, the dimension of the cohomology group at the middle stage in the Koszul complex of $S_{\lambda\varepsilon_{\alpha}} - \omega$ is same for every $\omega \in \mathbb{B}^d$ (see [1.7]).

5.3. Joint Subnormal Multishifts

We begin this section with a simple characterization of joint subnormal multishift in terms of complete monotonicity of its moments.

PROPOSITION 5.3.1. Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Let $S_{\lambda}$ be a toral contractive multishift on $\mathcal{T}$. Then $S_{\lambda}$ is joint subnormal if and only if for every $v \in V$, the multisequence $\{\|S_{\lambda}^\alpha e_v\|^2\}_{\alpha \in \mathbb{N}^d}$ is completely monotone.

PROOF. As recorded earlier, by [16] Theorem 4.4, a toral contractive $d$-tuple $T$ on a complex Hilbert space $\mathcal{H}$ is joint subnormal if and only if for every $h \in \mathcal{H}$, the multisequence $\{|T^\alpha h|^2\}_{\alpha \in \mathbb{N}^d}$ is completely monotone. Since $\{S_{\lambda}^\alpha e_v\}_{v \in V}$ is mutually orthogonal (Proposition 3.1.7(x)), for $f = \sum_{v \in V} f(v)e_v$,
$$\|S_{\lambda}^\alpha f\|^2 = \sum_{v \in V} |f(v)|^2 \|S_{\lambda}^\alpha e_v\|^2.$$
By the general theory [24] Chapter 4, we conclude that $S_{\lambda}$ is joint subnormal if and only if for every $v \in V$, $\{\|S_{\lambda}^\alpha e_v\|^2\}_{\alpha \in \mathbb{N}^d}$ is completely monotone.

Although the preceding result characterizes all joint subnormal contractive multishifts on $\mathcal{T}$, the necessary and sufficient conditions include information at all vertices. On the other hand, information at single vertex (namely, $\{\|S_{\lambda}^\alpha e_{\text{root}}\|^2\}_{\alpha \in \mathbb{N}^d}$ is completely monotone) is sufficient to ensure joint subnormality in the context of classical multishifts. Thus a natural question arises whether joint subnormality of $S_{\lambda}$ can be recovered from complete monotonicity at finitely many vertices. This question has an affirmative answer in case each $\mathcal{T}_j$ is locally finite with finite branching index.

THEOREM 5.3.2. Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \cdots, \mathcal{T}_d$. Let $S_{\lambda}$ be a toral contractive multishift on $\mathcal{T}$. Let
$$W := \bigcup_{\alpha \in \mathbb{N}^d, \alpha \leq_k \varepsilon} \text{Chi}_{\langle \alpha \rangle, \varepsilon}(\text{root})$$
and let $\bar{W}$ be the set $\bar{W} := W_1 \times \cdots \times W_d$, where
$$W_j := \text{Chi}(V_j^{(j)}) \cup \{\text{root}_j\}, \ j = 1, \cdots, d.$$
Then the following statements are equivalent:

(i) \(S_\lambda\) is joint subnormal.

(ii) For every \(v \in W\), \(\|S_\lambda^a e_v\|_\alpha\in\mathbb{N}^d\) is completely monotone.

(iii) For every \(v \in \tilde{W}\), \(\|S_\lambda^a e_v\|_\alpha\in\mathbb{N}^d\) is completely monotone.

**Proof.** The implication that (i) implies (ii) is clear from the previous result while that (ii) implies (iii) is obvious in view of the inclusion \(W \subseteq \tilde{W}\). Let us check that (ii) implies (i). Assume that for every \(v \in W\), the multisequence \(\{\|S_\lambda^a e_v\|_\alpha\in\mathbb{N}^d\}\) is completely monotone. Fix \(v \in V \setminus \tilde{W}\). We contend that there exist \(w \in W\), \(\tilde{\alpha} \in \mathbb{N}^d\) and \(\gamma \in \mathbb{C}\) such that

\[
e_v = \gamma S_\lambda^{\tilde{\alpha}} e_w.
\]

Note that there exists a subset \(\{i_1, \ldots, i_k\}\) of \(\{1, \ldots, d\}\) such that \(\alpha_{v_{i_j}} > k_{S_{i_j}}\) for every \(j = 1, \ldots, k\). Let \(l_j = \alpha_{v_{i_j}} - k_{S_{i_j}}\) (\(j = 1, \ldots, k\)) and set

\[
w := \text{par}_{i_1}^{(l_1)} \cdots \text{par}_{i_k}^{(l_k)}(v).
\]

Then \(\alpha_{w_{i_j}} \leq k_{S_{i_j}}\) for every \(j = 1, \ldots, d\), and hence \(w \in W\). Now put

\[
\tilde{\alpha} := l_1 \epsilon_i + \cdots + l_k \epsilon_i.
\]

Since \(\alpha_{w_{i_j}} = k_{S_{i_j}}\) for \(j = 1, \ldots, d\), we must have \(\gamma S_\lambda^{\tilde{\alpha}} e_w = e_v\) for some scalar \(\gamma \in \mathbb{C}\). This completes the verification of (5.35). It follows that for any \(\beta \in \mathbb{N}^d\),

\[
\sum_{|\alpha| \leq n} (-1)^{|\alpha|} \binom{n}{\alpha} \|S_\lambda^{\alpha+\beta} e_v\|^2 = \sum_{|\alpha| \leq n} (-1)^{|\alpha|} \binom{n}{\alpha} (\gamma)^2 \|S_\lambda^{\alpha+\beta+\tilde{\alpha}} e_w\|^2 \geq 0,
\]

since \(\{\|S_\lambda^{\alpha} e_w\|_\alpha\in\mathbb{N}^d\}\) is given to be completely monotone. Now apply the preceding proposition to complete the verification of (ii) implies (i).

Finally, we check the implication that (iii) implies (ii). Let \(v \in W \setminus \tilde{W}\). Define

\[\mathcal{F}_v := \{w \in \tilde{W} : \text{Chi}^{<\alpha(w)}(w) \text{ contains } v \text{ for some } \alpha(w) \in \mathbb{N}^d\}\].

Then \(\mathcal{F}_v\) is nonempty as root in \(\mathcal{F}_v\). Now consider the set

\[\mathcal{G}_v := \{w \in \mathcal{F}_v : |\alpha_w| \geq |\alpha_u| \text{ for all } u \in \mathcal{F}_v\}\].

We claim that for all \(w \in \mathcal{G}_v\), there exists \(\alpha(w) = (\alpha_1^{(w)}, \ldots, \alpha_d^{(w)}) \in \mathbb{N}^d\) such that \(\text{Chi}^{<\alpha(w)}(w) = \{v\}\). If possible, suppose that there are distinct vertices \(v, v' \in \text{Chi}^{<\alpha(w)}(w)\) for some \(w \in \mathcal{G}_v\). Without loss of generality, assume that \(v_1 \neq v'_1\).

As \(v_1, v'_1 \in \text{Chi}^{(\alpha_1^{(w)})}(w_1)\), there exists a positive integer \(k, 1 \leq k \leq \alpha_1^{(w)}\), such that \(u_1 := \text{par}^{(k)}(v_1) \in V_1^{(k)}\). Let \(u_1 \in \text{Chi}(u_1)\) be such that \(v_1 \in \text{Chi}^{(k-1)}(u_1)\). Note that \(u_1 \in W_1\). Consider \(w' := (u_1, w_2, \ldots, w_d)\). Then \(w' \in \tilde{W}\) and \(v \in \text{Chi}^{<\beta}(w')\), where \(\beta = (k-1, \alpha_2^{(w)}, \ldots, \alpha_d^{(w)})\). Thus \(w' \in \mathcal{F}_v\), and hence \(|\alpha_w| \geq |\alpha_w'|\). On the other hand, \(|\alpha_w'| = |\alpha_{w_1}| + \alpha_1^{(w_1)} - k + 1 > |\alpha_{w_1}|\), which is a contradiction. This proves the claim that \(\text{Chi}^{<\alpha(w)}(w) = \{v\}\) for all \(w \in \mathcal{G}_v\). Now one may argue as in the preceding paragraph to see that for every \(v \in W \setminus \tilde{W}\), there exist \(w \in W\), \(\tilde{\alpha} \in \mathbb{N}^d\) and \(\gamma \in \mathbb{C}\) such that \(e_v = \gamma S_\lambda^{\tilde{\alpha}} e_w\). This gives immediately the complete monotonicity of \(\{\|S_\lambda^a e_v\|_\alpha\}_{\alpha \in \mathbb{N}^d}\). \(\square\)

**Remark 5.3.3.** If \(\mathcal{F}\) is locally finite with finite joint branching index, then \(W\) (and hence \(\tilde{W}\)) is finite.
It is well-known that there is a class of tuples antithetical to joint subnormal tuples commonly known as (toral or joint) completely hyperexpansive tuples (refer to \[20\] and \[30\] for definitions and basic properties). A characterization similar to one given above can be obtained for toral completely hyperexpansive multishifts as well, where, as expected, the moments being completely monotone is replaced by moments being completely alternating (refer to \[24\] for the definition of completely alternating functions). Similar characterization can be obtained for the class of joint \( q \)-isometries as introduced and studied in \[56\].

The class of joint subnormal multishifts within the class of spherically balanced multishifts admits a handy characterization (cf. \[34\] Theorem 5.3(1)).

**Proposition 5.3.4.** Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of rooted directed trees \( \mathcal{T}_1, \ldots, \mathcal{T}_d \). Let \( S_\mathcal{A} = (S_1, \ldots, S_d) \) be a joint left invertible, spherically balanced multishift on \( \mathcal{T} \) and let \( S_0 \) be the weighted shift on the rooted directed tree \( \mathcal{T}_\text{root} = (V^\circ, E) \) associated with \( S_\mathcal{A} \). If \( S_\mathcal{A} \) is a joint contraction, then the following statements are equivalent:

(i) \( S_\mathcal{A} \) is joint subnormal.

(ii) \( \left\{ \prod_{p=0}^d c_p \right\}_{n \in \mathbb{N}} \) is completely monotone, where \( c_i \) is the constant value of \( \sum_{j=1}^d \| S_j v \|^2 \) on the generation \( \mathcal{G}_t \) of \( \mathcal{T} \).

(iii) \( S_0 \) is subnormal.

**Proof.** Assume that \( S_\mathcal{A} \) is a joint contraction. We proved the following formula in \[5.25\]:

\[
\langle Q^n_{S_\mathcal{A}} (I) e_v, e_v \rangle = \| S_0^n e_v \|^2 \quad (n \in \mathbb{N}),
\]

where \( Q^n_{S_\mathcal{A}} (-) \) is as given by \[5.1\]. By \[19\] Theorem 5.2, \( S_\mathcal{A} \) is joint subnormal if and only if \( \{ \langle Q^n_{S_\mathcal{A}} (I) e_v, e_v \rangle \}_{n \in \mathbb{N}} \) is completely monotone for every \( v \in V \), and hence by the formula above, this is equivalent to the complete monotonicity of \( \{ \| S_0^n e_v \|^2 \}_{n \in \mathbb{N}} \) for every \( v \in V^\circ \). This yields the equivalence of (i) and (iii). The equivalence of (ii) and (iii) is immediate from \[5.27\]. This completes the proof. \( \square \)

Let us illustrate the previous result with the help of the family of multishifts discussed in Example 5.2.5.

**Example 5.3.5.** Let \( S_{\mathcal{A} \epsilon_a} \) be as introduced in Example 5.2.5. Note that \( S_{\mathcal{A} \epsilon_a} \) is a joint contraction if and only if \( d \leq a \). Assume that \( a \) is an integer such that \( d \leq a \).

By the preceding proposition, \( S_{\mathcal{A} \epsilon_a} \) is joint subnormal if and only if \( \{ \prod_{p=0}^d c_a p \}_{n \in \mathbb{N}} \) is completely monotone, where

\[
c_{a,t} = \frac{t + d}{t + a} \quad (t \in \mathbb{N}).
\]

Let us verify the last statement. Recall the fact that the product of completely monotone sequences \( \{ \frac{i}{i+n} \}_{n \in \mathbb{N}}, \ i = d, \ldots, d+k \) is completely monotone. Since for \( k = a - d \in \mathbb{N}, \)

\[
\prod_{p=0}^n c_{a,p} = \left\{\begin{array}{ll}
1 & \text{if } k = 0, \\
\frac{d(d+1)-(d+k-1)}{(d+n+1)(d+n+2)-d+k+n} & \text{otherwise,}
\end{array}\right.
\]

\( \{ \prod_{p=0}^n c_{a,p} \}_{n \in \mathbb{N}} \) is completely monotone.
**5.4. Joint Hyponormal Multishifts**

In this short section, we discuss the class of joint hyponormal tuples.

**Proposition 5.4.1.** Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Let $S_\lambda = (S_1, \ldots, S_d)$ be a multishift on $\mathcal{T}$. Then $S_\lambda$ is joint hyponormal if and only if for every $t \in \mathbb{N}$ and for every $f_1, \ldots, f_d \in l^2(V)$ supported on generation $G_t$, 

$$
\sum_{i,j=1}^{d} \langle [S^*_i, S_j] f_j, f_i \rangle \geq 0.
$$

**Proof.** Note that $\langle [S^*_i, S_j] e_v, e_w \rangle = 0$ (for any $v, w \in V$) for any $i, j = 1, \ldots, d$ for any $v, w \in V$ such that $|\alpha_v| \neq |\alpha_w|$ (see Lemma 2.1.10(vii)). It follows that for $f_t, g_s \in l^2(V)$ with supports on $G_t$ and $G_s$ respectively with $s \neq t$, 

$$
\langle [S^*_j, S_i] f_t, g_s \rangle = 0 \quad \text{for every } i, j = 1, \ldots, d. \tag{5.36}
$$

For $j = 1, \ldots, d$, let $f_j \in l^2(V)$ and write $f_j = \sum_{t \in \mathbb{N}} f_{j,t}$, where $f_{j,t}$ is supported on $G_t$. Then 

$$
\sum_{i,j=1}^{d} \langle [S^*_i, S_j] f_j, f_i \rangle = \sum_{t,s \in \mathbb{N}} \sum_{i,j=1}^{d} \langle [S^*_i, S_j] f_{j,t}, f_{i,s} \rangle = \sum_{t \in \mathbb{N}} \sum_{i,j=1}^{d} \langle [S^*_j, S_i] f_{j,t}, f_{i,t} \rangle. \tag{5.36}
$$

The desired equivalence is now immediate. \hfill \square

In case of spherically balanced multishifts, the preceding characterization can be made more explicit (cf. Theorem 5.3(5)).

**Theorem 5.4.2.** Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Let $S_\lambda = (S_1, \ldots, S_d)$ be a joint left invertible, spherically balanced multishift on $\mathcal{T}$ and let $S_0$ be the weighted shift on the rooted directed tree $\mathcal{T}^\circ \mathcal{T}_\text{root} = (V^\circ, \mathcal{F})$ associated with $S_\lambda$. Then the following statements are equivalent:

(i) $S_\lambda$ is joint hyponormal.

(ii) $\{\mathcal{E}_t\}_{t \in \mathbb{N}}$ is monotonically increasing, where $\mathcal{E}_t$ is the constant value of $
\sum_{j=1}^{d} \|S_j \mathcal{E}_v\|^2$

on the generation $G_t$ of $\mathcal{T}$.

(iii) $S_0$ is hyponormal.

**Proof.** We first verify the implication that (i) implies (ii). Recall from Lemma 4.10 that any joint hyponormal $d$-tuple $T$ satisfies the inequality 

$$
Q_T^2(I) \geq Q_T(I)^2,
$$

where $Q_T^2(\cdot)$ is as given in (1.1). In particular, for any $v \in V$, 

$$
\langle Q_{S_\lambda}^2(I) e_v, e_v \rangle \geq \|Q_{S_\lambda}(I) e_v\|^2.
$$

However, by the polar decomposition obtained in Proposition 5.2.10

$$
S_j = T_j D_v, \ j = 1, \ldots, d, \tag{5.37}
$$

In case of spherically balanced multishifts, the preceding characterization can be made more explicit (cf. Theorem 5.3(5)).

**Theorem 5.4.2.** Let $\mathcal{T} = (V, E)$ be the directed Cartesian product of rooted directed trees $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Let $S_\lambda = (S_1, \ldots, S_d)$ be a joint left invertible, spherically balanced multishift on $\mathcal{T}$ and let $S_0$ be the weighted shift on the rooted directed tree $\mathcal{T}^\circ \mathcal{T}_\text{root} = (V^\circ, \mathcal{F})$ associated with $S_\lambda$. Then the following statements are equivalent:

(i) $S_\lambda$ is joint hyponormal.

(ii) $\{\mathcal{E}_t\}_{t \in \mathbb{N}}$ is monotonically increasing, where $\mathcal{E}_t$ is the constant value of 

$$
\sum_{j=1}^{d} \|S_j \mathcal{E}_v\|^2
$$

on the generation $G_t$ of $\mathcal{T}$.

(iii) $S_0$ is hyponormal.

**Proof.** We first verify the implication that (i) implies (ii). Recall from Lemma 4.10 that any joint hyponormal $d$-tuple $T$ satisfies the inequality 

$$
Q_T^2(I) \geq Q_T(I)^2,
$$

where $Q_T^2(\cdot)$ is as given in (1.1). In particular, for any $v \in V$, 

$$
\langle Q_{S_\lambda}^2(I) e_v, e_v \rangle \geq \|Q_{S_\lambda}(I) e_v\|^2.
$$

However, by the polar decomposition obtained in Proposition 5.2.10

$$
S_j = T_j D_v, \ j = 1, \ldots, d, \tag{5.37}
$$

In case of spherically balanced multishifts, the preceding characterization can be made more explicit (cf. Theorem 5.3(5)).
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where $T_\lambda = (T_1, \cdots, T_d)$ and $D_c$ are joint isometry and diagonal part of $S_\lambda$ respectively. Since $T_\lambda$ is a joint isometry, $Q_{S_\lambda}(I) = D_c^2$. It is now easy to see that

$$e_\alpha v e_\alpha + 1 = \langle Q_{S_\lambda}(I) e_\alpha, e_\alpha \rangle = \langle D_c^2 e_\alpha, e_\alpha \rangle = C_{|\alpha|}^2,$$

which yields that $\{C_t\}_{t \in \mathbb{N}}$ is monotonically increasing.

We next verify the implication that (ii) implies (i). In view of Proposition 5.4.1, it suffices to check that for every $t \in \mathbb{N}$ and for every $f_1, \cdots, f_d \in l^2(V)$ supported on $G_t$,

$$\sum_{i,j=1}^d \langle [S_j^*, S_i] f_j, f_i \rangle \geq 0.$$

To see this, let $f_1, \cdots, f_d \in l^2(V)$ be supported on $G_t$ for some $t \in \mathbb{N}$. A routine verification using (5.37) shows that

$$[S_j^*, S_i] e_\alpha = e_\alpha v e_\alpha + (C_t - C_{t-1}) \sum_{j=1}^d \langle T_j^* f_j, f_i \rangle,$$

Hence

$$\sum_{i,j=1}^d \langle [S_j^*, S_i] f_j, f_i \rangle = C_t \sum_{i,j=1}^d \langle [T_j^*, T_i] f_j, f_i \rangle + (C_t - C_{t-1}) \sum_{i,j=1}^d \langle T_i T_j^* f_j, f_i \rangle,$$

(5.38)

where we used the convention that $C_{-1} = 0$. However,

$$\sum_{i,j=1}^d \langle [T_j^*, T_i] f_j, f_i \rangle \geq 0$$

since $T_\lambda$, being joint subnormal, is joint hyponormal, and

$$\sum_{i,j=1}^d \langle T_i T_j^* f_j, f_i \rangle = \left\| \sum_{j=1}^d T_j^* f_j \right\|^2.$$

Since $\{C_t\}_{t \in \mathbb{N}}$ is monotonically increasing, we conclude from (5.38) that

$$\sum_{i,j=1}^d \langle [S_j^*, S_i] f_j, f_i \rangle \geq 0.$$

We finally check the equivalence of (ii) and (iii). In view of [65, Theorem 5.1.2], it suffices to check that

$$\sum_{m \in \text{Chi}(\nu)} \frac{\theta_m^2}{\|S_{\theta_m} e_\nu\|^2} \leq 1 \text{ for every } \nu \in V^\perp$$
if and only if \( \{ c_t \}_{t \in \mathbb{N}} \) is monotonically increasing. Since \( \| S_{\theta} e_m \|^2 = C_{\alpha_m} \) and 
\[
\theta_m = \frac{\sqrt{C_{\alpha_m-1}}}{\text{card(sib}(m))} \quad (m \in V^\odot \setminus \text{root})
\]
(see (5.22) and (5.27)), we obtain
\[
\sum_{w \in \text{Chi}(v)} \theta^2_w \frac{1}{\| S_{\theta} e_w \|^2} = \sum_{w \in \text{Chi}(v)} \frac{\text{card(sib}(w))}{C_{\alpha_w}} = \frac{C_{\alpha_v}}{C_{\alpha_v+1}} \sum_{w \in \text{Chi}(v)} \frac{1}{\text{card(sib}(w))} = \frac{C_{\alpha_v}}{C_{\alpha_v+1}}.
\]
The equivalence of (ii) and (iii) is now clear. This completes the proof. □

**Remark 5.4.3.** Assume that \( S_\lambda \) is a joint hyponormal multishift. It is well-known that the spectral radius and norm of a hyponormal operator coincide \[38\]. One may now conclude from Corollary 5.2.17(i) and (5.26) that the spectral radius of \( S_\lambda \) equals \( \| S_1^tS_1 + \cdots + S_d^tS_d \|^{1/2} \).

**Example 5.4.4.** Let \( S_{C_\lambda a} \) be as introduced in Example 5.2.5. Note that \( S_{C_\lambda a} \) is a joint hyponormal if and only if \( \{ c_{a,t} \}_{t \in \mathbb{N}} \) is monotonically increasing, where
\[
c_{a,t} = \frac{t + d}{t + a} \quad (t \in \mathbb{N}).
\]
This holds if and only if \( d \leq a \). In view of Example 5.3.5 we have the following equivalent statements (cf. \[8\] Lemma 3.3):

1. \( S_{C_\lambda a} \) is joint subnormal.
2. \( S_{C_\lambda a} \) is joint hyponormal.
3. \( S_{C_\lambda a} \) is joint contraction.

**Afterword**

Needless to say, the work presented in this paper provides a framework to unify the theories of classical multishifts and weighted shifts on rooted directed trees. This framework also enables one to pose and peruse a diverse range of problems. More importantly, this framework allows the rich interplay of graph theory, complex function theory, and operator theory. One of the important outcomes of these investigations is perhaps the tree analogs \( S_{C_\lambda a} \) of extensively studied classical multishifts like Szegö, Bergman, and Drury-Arveson \( d \)-shifts. On one hand, the tree analogs of these \( d \)-shifts share many important properties of their classical counterparts like \( S_{C_\lambda d} \) (Szegö \( d \)-shift) is a joint isometry, \( S_{C_\lambda d+1} \) (Bergman \( d \)-shift) is joint subnormal, and \( S_{C_\lambda 1} \) (Drury-Arveson \( d \)-shift) is a row contraction. On the other hand, due to abundance of directed tree structures, various complicacies may arise. For instance, unlike their classical counterparts, for suitable choice of \( T \), the defect operator \( \sum_{k=0}^{\infty} (-1)^k \binom{a}{k} Q_k S_{C_\lambda a} I \) fails to be an orthogonal projection. Further, in the matrix decomposition of \( S_{C_\lambda a}^* \), non-diagonal tuples of infinite rank operators appear naturally. We believe that the class of multishifts \( S_{C_\lambda a} \) warrants further attention as they may play the role of building blocks in the classification of \( G \)-homogeneous tuples associated with the action of various linear groups \( G \) such as diagonal unitaries and unitaries.
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Appendix

We are thankful to V. M. Sholapurkar for kindly providing a multivariable analog of the identity given in [4.2] (along with proof).

**Lemma 1.** Let \( n \) be a positive integer and let \( X = (x_1, \cdots, x_d), Y = (y_1, \cdots, y_d) \) be \( d \)-tuples such that the variables \( x_i, y_i \) \( (i = 1, \cdots, d) \) belong to a unital complex algebra. Then

\[
1 - \sum_{\alpha \in \mathbb{N}^d \atop |\alpha| = n} \binom{|\alpha|}{\alpha} X^\alpha Y^\alpha = \sum_{\beta \in \mathbb{N}^d \atop |\beta| \leq n-1} \binom{|\beta|}{\beta} X^\beta (1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d) Y^\beta,
\]

where \( \binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1! \cdots \alpha_m!} \) \( (\alpha \in \mathbb{N}^d) \).

**Proof.** We prove the result by induction on \( n \). For \( n = 1 \), both the sides of the identity reduce to \( (1 - x_1 y_1 - x_2 y_2) \) and hence the result holds. Suppose the result holds for some \( n \geq 1 \). We now prove the identity for \( n + 1 \). Starting with the right hand side, we split the sum

\[
A := \sum_{\beta \in \mathbb{N}^d \atop |\beta| \leq n-1} \binom{|\beta|}{\beta} X^\beta (1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d) Y^\beta
\]
as \( A_1 + A_2 \), where

\[
A_1 := \sum_{\beta \in \mathbb{N}^d \atop |\beta| \leq n-1} \binom{|\beta|}{\beta} X^\beta (1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d) Y^\beta,
\]

\[
A_2 := \sum_{\beta \in \mathbb{N}^d \atop |\beta| = n} \binom{n}{\beta} X^\beta Y^\beta.
\]

Now by induction hypothesis, \( A_1 = 1 - \sum_{|\beta|=n} \binom{n}{\beta} X^\beta Y^\beta \). Observe also that

\[
A_2 = \sum_{\beta \in \mathbb{N}^d \atop |\beta| = n} \binom{n}{\beta} X^\beta Y^\beta - \sum_{\beta \leq \epsilon \atop \beta \in \mathbb{N}^d \atop |\beta| = n} \binom{n}{\beta} \left[ \sum_{i=1}^{d} X^{\beta + \epsilon_i} Y^{\beta + \epsilon_i} \right].
\]
Thus

\[ A_1 + A_2 = 1 - \sum_{\beta \in \mathbb{N}^d \mid |\beta| = n} \binom{n}{\beta} \left[ \sum_{i=1}^{d} X^{\beta + \epsilon_i} Y^{\beta + \epsilon_i} \right] \]

\[ = 1 - \sum_{\beta \in \mathbb{N}^d \mid |\beta| = n + 1} \left[ \sum_{i=1}^{d} \binom{n}{\beta - \epsilon_i} \right] X^{\beta} Y^{\beta} \]

\[ = 1 - \sum_{\beta \in \mathbb{N}^d \mid |\beta| = n + 1} \left( \binom{n+1}{\beta} \right) X^{\beta} Y^{\beta}. \]

This is the left hand side of the identity with \( n \) replaced by \( n + 1 \). Thus the result holds for all integers \( n \geq 1 \). \( \square \)

Let us discuss some of the difficulties in obtaining appropriate analog of Shi-morin’s formula (4.1) in several variables. The very first difficulty which arises is the appropriate notion of Cauchy dual \( S' \) in several variables, where \( S = (S_1, \ldots, S_d) \) is the given \( d \)-tuple of bounded linear operators on \( \mathcal{H} \). To see what a correct choice would be, note that (4.2) may be rewritten as

\[ I - T^n T'^* = \sum_{k=0}^{n-1} T^k P_E T'^* T^k, \]

where \( T \) is a left invertible operator on \( \mathcal{H} \) and \( P_E = I - TT'^* \) is the orthogonal projection onto the kernel of \( T'^* \). Hence, in view of the preceding lemma, the choice of Cauchy dual in several variables should necessarily ensure that \( I - \sum_{i=1}^{d} S_i S'^* \) an orthogonal projection. Examples show that none of the notions of Cauchy dual (toral and spherical) apply with success in this context. Keeping this aside and assuming this condition for a moment, what we obtain is the following formula:

\[ \bigcap_{|\alpha| = n} \ker S'^* \alpha \subseteq \bigvee \{ S^{\beta} f : f \in \ker S^*, |\beta| \leq n - 1 \}, \]

where it is not clear whether equality holds. In case of single operator, we obtain equality, which can be then used to derive the duality formula (4.1) crucial in obtaining the wandering subspace property for \( T \).
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