LAGRANGIAN SUBMANIFOLDS IN STRICTLY NEARLY KÄHLER 6-MANIFOLDS

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Abstract. Lagrangian submanifolds in strictly nearly Kähler 6-manifolds are related to special Lagrangian submanifolds in Calabi-Yau 6-manifolds and coassociative cones in $G_2$-manifolds. We prove that the mean curvature of a Lagrangian submanifold $L$ in a nearly Kähler manifold $(M^{2n}, J, g)$ is symplectically dual to the Maslov 1-form on $L$. Using relative calibrations, we derive a formula for the second variation of the volume of a Lagrangian submanifold $L^3$ in a strictly nearly Kähler manifold $(M^6, J, g)$. This formula implies, in particular, that any formal infinitesimal Lagrangian deformation of $L^3$ is a Jacobi field on $L^3$. We describe a finite dimensional local model of the moduli space of compact Lagrangian submanifolds in a strictly nearly Kähler 6-manifold. If $(M^6, J, g)$ is analytic, for instance the sphere $S^6$ with the standard nearly Kähler structure, we analyze sufficient conditions for a formal infinitesimal Lagrangian deformation to be smoothly obstructed or smoothly unobstructed. As a result, we prove that the geodesic Lagrangian sphere $S^3$ and the “squashed” Lagrangian sphere $L_1^3$ in the standard nearly Kähler sphere $S^6$ are rigid up to the motion of the automorphism group $G_2$ of $S^6$.

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1. Introduction

Nearly Kähler manifolds have been appeared first time in Gray’s work [9] in the connection with Gray’s notion of weak holonomy. Nearly Kähler manifolds represent an important class in the 16 classes of almost Hermitian manifolds \((M^{2n}, J, g)\) classified by Gray and Hervella [11]. Let us recall the definition of a nearly Kähler manifold \((M^{2n}, J, g)\). Let \(\nabla^{LC}\) denote the Levi-Civita covariant derivative associated with the Riemannian metric \(g\).

**Definition 1.1.** ([9, §1, Proposition 3.5], [10]) An almost Hermitian manifold \((M^{2n}, g, J)\) is called nearly Kähler if 
\[
(\nabla^{LC}XJ)X = 0 \quad \text{for all} \quad X \in T^2M^{2n},
\]
where \(\lambda\) is a positive constant.

**Remark 1.2.** 1. It is known that any complete simply connected nearly Kähler manifold is a Riemannian product \(M_1 \times M_2\) where \(M_1\) and \(M_2\) are Kähler respectively strict nearly Kähler manifolds [17, 30]. Furthermore, a de Rham type decomposition of a strictly nearly Kähler manifold is found by Nagy [31], where the factors of the decomposition are of the following types: 3-symmetric spaces, twistor spaces over quaternionic Kähler manifolds of positive scalar curvature, and strictly nearly Kähler 6-manifolds.

2. It is easy to see that if \((M^{2n}, J, g)\) is a nearly Kähler manifold of constant type \(\lambda\), then \((M^{2n}, J, \lambda^{-1/2}g)\) and \((M^{2n}, -J, \lambda^{-1/2}g)\) are nearly Kähler manifolds of constant type 1.

On an almost Hermitian manifold \((M^{2n}, J, g)\) the fundamental 2-form \(\omega\), defined by \(\omega(X, Y) := g(JX, Y)\), measures the connection between the almost complex structure \(J\) and the Riemannian metric \(g\). A submanifold \(L^n \subset (M^{2n}, J, g)\) is called Lagrangian, if \(\omega|_{L^n} = 0\). As in symplectic geometry, the graph of a diffeomorphism of \(M^{2n}\) that preserves \(\omega\) is a Lagrangian submanifold in the almost Hermitian manifold \((M^{2n} \times M^{2n}, J \oplus (-J), g \oplus g)\). If \((M^{2n}, J, g)\) is Kähler, then \(\omega\) is symplectic. Lagrangian submanifolds in Kähler manifolds have been studied in the context of calibrated geometry [12] and of relative calibrations [19, 20], in the investigation of the Maslov
class \([29], [22]\), of the variational problem \([19], [32], [36], [38]\), and of the deformation problem/ moduli spaces \([3], [27], [14], [38], [2]\), etc. The literature on the subject is vast, and the authors omit the name of many important papers in the field.

The relation between nearly Kähler manifolds \((M^{2n}, J, g)\) and Riemannian manifolds with special holonomy is best manifested in dimension \(2n = 6\). In dimension 6, a nearly Kähler manifold is either a Kähler manifold or a strictly nearly Kähler manifold \([10, Theorem 5.2]\). It is known from Baer’s work \([1]\) that a cone without singular point over a strictly nearly Kähler manifold \((M^6, J, g)\) is a 7-manifold with \(G_2\)-holonomy. It is not hard to see that the cone over a Lagrangian submanifold \(L^3\) in a strictly nearly Kähler manifold \((M^6, J, g)\) is a coassociative cone in \(CM^6\). Thus the study of strictly nearly Kähler 6-manifolds and their Lagrangian submanifolds are essential for the study of singular points of \(G_2\)-folds as well as for the study of singular points of coassociative 4-folds. Furthermore, special Lagrangian submanifolds in Calabi-Yau 6-manifolds could be treated as as a limit case of Lagrangian submanifolds in nearly Kähler manifolds when the type constant \(\lambda\) goes to zero (Remarks 2.5, 3.15). We also note that Lagrangian submanifolds in the standard nearly Kähler manifold \(S^6\) are found to be intimately related to holomorphic curves in \(\mathbb{C}P^2\) and they present extremely rich geometry \([6], [25], [5]\).

In this paper we study Lagrangian submanifolds \(L^3\) in strictly nearly Kähler 6-manifolds \((M^6, J, g)\) in two aspects: the variation of the volume functional and Lagrangian deformations of \(L^3\). Since \(L^3\) are minimal submanifolds in \((M^6, J, g)\) (Corollary 3.6), these two aspects are related to each other. In particular, results from theory of minimal submanifolds are applicable to Lagrangian submanifolds in strictly nearly Kähler 6-manifolds, for instance see the proof of Theorem 4.15.

Our paper is organized as follows. In section 2 we collect some important results on the canonical Hermitian connection on nearly Kähler manifolds. In section 3 using a Lé’s result \([19]\), we establish a relation between the Maslov 1-form and the mean curvature of a Lagrangian submanifold in a nearly Kähler manifold \((M^{2n}, J, g)\) (Proposition 3.3) and show its consequences (Corollaries 3.4, 3.6). If \((M^{2n}, J, g)\) is a strictly nearly Kähler 6-manifold, we derive a simple formula for the second variation of a Lagrangian submanifold in \((M^6, J, g)\) using relative calibrations (Theorem 3.9). This formula implies, in particular, that any formal infinitesimal Lagrangian deformation is a Jacobi field, generalizing a result obtained by McLean for special Lagrangian submanifolds (Corollary 3.13, Remarks 3.14, 3.15, 3.16). In section 4 we study smooth deformations of compact Lagrangian submanifolds \(L^3\) in strictly nearly Kähler 6-manifolds \((M^6, J, g)\). We prove that \(C^1\)-small smooth Lagrangian deformations of \(L^3\) are solutions of an elliptic first order equation (Theorem 4.15) and hence the set of \(C^1\)-small smooth Lagrangian deformations of \(L^3\) is the kernel of a smooth map between open
subsets of finite dimensional vector spaces (Theorem 4.8). This leads us to the following

**Conjecture 1.** The group $\text{Diff}(M^6, [\omega])$ of all diffeomorphisms $g$ of a strictly nearly Kähler manifold $(M^6, J, g)$ that preserves the fundamental 2-form $\omega$ in its conformal class, $g^*(\omega) = e^f \omega$ for some $f \in C^\infty(M^6)$, is a finite dimensional Lie group.

We prove that any smooth Lagrangian deformation of a Lagrangian submanifold $L^3$ in an analytic strictly nearly Kähler 6-manifold can be written as a convergent power series (Theorem 4.11). As a result, we derive sufficient conditions for a formal infinitesimal Lagrangian deformation to be smoothly obstructed or smoothly unobstructed (Proposition 4.12, Lemma 4.13). Finally, using the developed theory we prove that the geodesic Lagrangian sphere $S^3$ and the “squashed” Lagrangian sphere $L^3_{11}$ in the standard nearly Kähler sphere $(S^6, J_0, g_0)$ are rigid up to the motion by the automorphism group $G_2$ of $(S^6, J_0, g_0)$ (Theorems 4.15, 4.18). We finish our paper with Conjecture 2, generalizing the rigidity of the Lagrangian spheres in $S^6$ to other cases of homogeneous Lagrangian submanifolds in $(S^6, J_0, g_0)$.

2. **Geometry of nearly Kähler manifolds**

In this section we collect some important results on the canonical Hermitian connection on nearly Kähler manifolds (Propositions 2.1, 2.2) and derive an important consequence (Corollary 2.3), which plays a central role in the geometry of strictly nearly Kähler 6-manifolds (Proposition 2.4, Remark 2.5).

2.1. **The canonical Hermitian connection.** Let $U(M^{2n})$ denote the principal bundle consisting of unitary frames $(e_1, Je_1, \ldots, e_n, Je_n)$ over an almost Hermitian manifold $(M^{2n}, J, g)$. Denote by $\{e^*_i, (Je_i)^*\}$ the dual frames. Then $\{\theta^i := e^*_i + \sqrt{-1}(Je_i)^*\}$ is the canonical $\mathbb{C}^n$-valued 1-form on $U(M)$.

Let $\alpha$ be a unitary connection 1-form on $U(M)$ and $T$ its torsion 2-form. The Cartan equation for $\alpha$, and $T$ [16] Chapter IX, §3 [19] §3] is expressed as follows

$$d\theta^i = -\alpha^i_j \wedge \theta^j + T^{ij}_{jk} \theta^j \wedge \theta^k + T^{ij}_{jk} \bar{\theta}^j \wedge \bar{\theta}^k + T^{ij}_{jk} \bar{\theta}^j \wedge \bar{\theta}^k,$$

$$da^i_j = -\alpha^i_k \wedge \alpha^k_j + \Omega^i_j,$$

where $\Omega$ is the curvature tensor of $\alpha$.

**Proposition 2.1.** ([21] Chapter IV, §112] Let $(M^{2n}, J, g)$ be an almost Hermitian manifold. There exists a unique unitary connection 1-form $\alpha$ on $U(M^{2n})$ such that its torsion tensor $T$ is a two-form of type $(2,0) + (0,2)$.

The canonical torsion of a nearly Kähler manifold $(M^{2n}, J, g)$ is skew-symmetric. This fact plays an important role in our study of $(M^{2n}, J, g)$. Moreover we have the following
Proposition 2.2. [10, Theorem 1] Suppose that $(M^{2n}, J, g)$ is a nearly Kähler manifold.

(1) Then $T(X, Y) = -J∇^L_C(X)Y$.

(2) The associated torsion form $T^*(X, Y, Z) := \langle T(X, Y), Z \rangle$ is skew-symmetric. In particular $T^i_{\bar{i}j} = 0$ for all $i, j$.

(3) $\nabla\text{can}T^* = 0$.

We shall derive from Proposition 2.2 the following

Corollary 2.3. On a nearly Kähler manifold $(M, J, g)$ we have $d\omega(X, Y, Z) = 3T^*(X, Y, JZ)$. Furthermore, $d\omega$ is a 3-form of type $(3, 0) + (0, 3)$.

Proof. We use the fact that the nearly Kähler condition is equivalent to the following condition [11, Theorem 3.1]

\[ X d\omega = 3\nabla^L_C \omega \]

for all $X \in TM^{2n}$. A straightforward calculation derives from (2.1)

\[ d\omega(X, Y, Z) = 3\langle \nabla^L_C(X)Y, Z \rangle. \]

Since $T(X, Y) = -J\nabla_X(J)Y$, we obtain immediately the first assertion of Corollary 2.3. The second assertion follows from the first one, taking into account the fact that $T$ is a 2-form of type $(2, 0) + (0, 2)$. \[ \square \]

2.2. Strictly nearly Kähler 6-manifolds. Among nearly Kähler manifolds the class of strictly nearly Kähler 6-manifolds are most well-studied. By rescaling the metric $g$ (Remark 1.2) we can assume that the metric is of constant type 1, i.e.

\[ |\nabla^L_C(J)Y|^2 = |X|^2|Y|^2 - \langle X, Y \rangle^2 - \langle X, JY \rangle^2. \]

This together with Proposition 2.2.3 and Corollary 2.3 imply that the $U(3)$-structure on strictly nearly Kähler 6-manifolds is reduced to a $SU(3)$-structure, since $d\omega$ is no-where vanishing. By Corollary 2.3 $d\omega$ is of type $(3, 0) + (0, 3)$. Comparing (2.3) with (2.2) we conclude that $d\omega$ is of comass 3. Thus we obtain immediately the following

Proposition 2.4. Assume that $(M^6, g, J)$ is a strictly nearly Kähler manifold with constant type 1 (cf. (2.3)). Then $\frac{1}{3}d\omega$ is a special Lagrangian calibration. In particular, $(M^6, g, J)$ has a $SU(3)$-structure.

In [1, §7] Baer constructed a 3-form $\varphi$ on the cone $CM^6 = M^6 \times \mathbb{R}^+ \mathbb{R}$ supplied with the warped Riemannian metric $\bar{g} = r^2 g + dr^2$ over a strictly nearly Kähler 6-manifold $(M^6, J, g)$ of constant type 1. We identify $M^6$ with $M^6 \times \{1\} \subset CM^6$. The form $\varphi$ on $CM^6$ is defined by [1, §7]

\[ \varphi(r, x) = \frac{r^3}{3} d\omega + r^2 dr \wedge \omega. \]

Since $d\omega$ is of type $(3, 0) + (0, 3)$ and of comass 3, for any $x \in M^6$ there is a local unitary basis $((e_1)^*, (Je_1)^*, \ldots, (Je_3)^*)$ at $T_x^* M^6$ such that $d\omega = 3\text{Re} (dz^1 \wedge dz^2 \wedge dz^3)$ and $2\omega = -\text{Im} (dz^1 \wedge d\bar{z}_1 + dz^2 \wedge d\bar{z}_2 + dz^3 \wedge d\bar{z}_3)$. Here
\[ dz^i := (e_i)^* + \sqrt{-1}(Je_i)^* \]. In this basis, rewriting \( dr = e^7 \) and abbreviating \( \varepsilon^{ijk} = e^i \wedge e^j \wedge e^k \), we have

\[
\varphi(r, x) = (\varepsilon^{135} - \varepsilon^{146} - \varepsilon^{236} - \varepsilon^{245}) + \varepsilon^{127} + \varepsilon^{347} + \varepsilon^{567}.
\]

Clearly, \( d\varphi = 0 \). Baer also showed that \( d^* \varphi = 0 \). Thus \( \varphi \) is a 3-form of \( G_2 \)-type and of comass 1 (cf. (4.15)). In particular, \( \varphi \) (resp. \( *\varphi \)) is an associative (resp. coassociative) calibration on \( CM^0 \). Furthermore, \( d^* \varphi = 0 \) implies the second relation in (2.6) for \( \lambda = 1 \).

**Remark 2.5.** By the above discussion, a nearly Kähler 6-manifold \( (M^6, J, g, \omega) \) of constant type \( \lambda \) satisfies the following equation (cf. \([4, \S 4]\))

\[
(2.6) \quad d\omega = 3\lambda \Re (dz^1 \wedge dz^2 \wedge dz^3) \quad \text{and} \quad d\Im (dz^1 \wedge dz^2 \wedge dz^3) = -2\lambda \omega \wedge \omega.
\]

Thus, a Calabi-Yau 6-manifold can be regarded as an almost strictly nearly Kähler manifold with \( \lambda = 0 \).

### 3. Variation of the Volume of Lagrangian Submanifolds

In this section we introduce the notion of the Maslov 1-form \( \mu(L) \) of a Lagrangian submanifold \( L \) in a Hermitian manifold \( (M^{2n}, J, g) \) and relate this notion with the classical notion of the Maslov class of a Lagrangian submanifold in \( (\mathbb{R}^{2n}, \omega_0) \) (Remark 3.2). Then we prove that \( \mu(L) \) is symplectically dual to the twice of the mean curvature \( H_L \) of a Lagrangian submanifold \( L \) in a nearly Kähler manifold \( (M^{2n}, J, g) \) (Proposition 3.3) and derive its consequences (Corollaries 3.4, 3.6). Using relative calibrations, we prove a simple formula for the second variation of the volume of a Lagrangian submanifold in a strictly nearly Kähler 6-manifolds (Theorem 3.9) and discuss its consequences (Corollary 3.13, Remarks 3.14, 3.15, 3.16). We discuss the relation between the obtained results with known results (Remark 3.8, 3.15, 3.16).

#### 3.1. Maslov 1-form and minimality of a Lagrangian submanifold in a nearly Kähler manifold

Let \( L \) be a Lagrangian submanifold in an almost Hermitian manifold \( (M^{2n}, J, g) \) and \( (\alpha^i_j) \) the canonical Hermitian connection 1-form on \( U(M^{2n}, J, g) \). The Gaussian map \( g_L \) sends \( L \) to the Lagrangian Grassmanian \( \text{Lag}(M^{2n}) \) of Lagrangian subspaces in the tangent bundle of \( M^{2n} \). Denote by \( p : U(M^{2n}) \to \text{Lag}(M^{2n}) \) the projection defined by

\[
(v_1, Jv_1, \cdots, v_n, Jv_n) \mapsto [v_1 \wedge \cdots \wedge v_n].
\]

Set

\[
\gamma := -\sqrt{-1} \sum_i \alpha^i_i.
\]

We recall the following fact

**Lemma 3.1.** (cf. [3,19, Proposition 3.1]) There exists a 1-form \( \tilde{\gamma} \) on \( \text{Lag}(M^{2n}) \) whose pull-back to the unitary frame bundle \( U(M^{2n}) \) is equal to \( \gamma \).
We call $2\bar{\gamma}$ the universal Maslov 1-form and the induced 1-form $g^*_L(2\bar{\gamma})$ on $L$ the Maslov 1-form of $L$. We also denote $g^*_L(2\bar{\gamma})$ by $\mu(L)$.

**Remark 3.2.** For $M^{2n} = \mathbb{R}^{2n}$ we have $\text{Lag}(M^{2n}) = \mathbb{R}^{2n} \times U(n)/O(n)$. In this case it is well-known that the Maslov 1-form $\mu(L)$ is a closed 1-form and represents its Maslov index of a Lagrangian submanifold $L$.

Now we relate the Maslov 1-form $\mu(L) := g^*_L(2\bar{\gamma})$ with the mean curvature of a Lagrangian submanifold $L$. We define a linear isomorphism $L_\omega : TM \to T^*M$ as follows.

\begin{equation}
L_\omega(V) := V |\omega.
\end{equation}

**Proposition 3.3.** The Maslov 1-form $\mu(L)$ is symplectic dual to the minus twice of the mean curvature $H_L$ of a Lagrangian submanifold $L$ in a nearly Kähler manifold $(M^{2n}, J, g)$. Namely we have

$$-2L_\omega(H_L) = \mu(L).$$

**Proof.** By Proposition 2.2.2 the 1-form $\sum_{ik} T_{ik}^j \bar{\theta}^k$ vanishes, where $T$ is the torsion of the connection form $\alpha$. Using [19, Lemmas 2.1, 3.1 and (3.6)], we obtain for any normal vector $X$ to $L$

\begin{equation}
\langle -H_L, X \rangle = (\mu(L)/2, JX).
\end{equation}

Since $\omega(-H_L, JX) = \langle -H_L, X \rangle$, we derive Proposition 3.3 immediately from (3.2). □

Since the curvature $d\gamma$ form of the connection form $\gamma$ is the Ricci form of a nearly Kähler manifold we obtain immediately

**Corollary 3.4.** Assume that a Lagrangian submanifold $L$ in a nearly Kähler manifold $(M, J, g)$ is minimal. Then the restriction of the Ricci form to $L$ vanishes.

In the remainder of this section we assume that $L^3$ is a Lagrangian submanifold in a strictly nearly Kähler manifold $(M^6, J, g)$. We also need to fix some notations. Set $dz := dz_1 \wedge dz_2 \wedge dz_3$ and

$$\alpha := \text{Re} dz, \quad \beta := \text{Im} dz.$$

**Lemma 3.5.** Let $\xi$ be a simple 3-vector in $\mathbb{R}^6 = \mathbb{C}^3$ and $\omega$ the standard compatible symplectic form on $\mathbb{R}^6$. Then

1. (12) Chapter III Theorem 1.7) $|dz(\xi)|^2 = \alpha(\xi)^2 + \beta(\xi)^2$.
2. (12) Chapter III (1.17)) $|dz(\xi)|^2 + \sum_{i=1}^3 |dz_i \wedge \omega(\xi)|^2 = |\xi|^2$.

In a strictly nearly Kähler manifold $(M^6, J, g, \omega)$ of constant type $\lambda$, using Remark 2.5 we also denote by $\alpha$ the 3-form $(3\lambda)^{-1} d\omega$ and by $\beta$ the 3-form $(3\lambda)^{-1} T^*$. Lemma 3.5 implies that

\begin{equation}
\alpha|_{L^3} = 0 \quad \text{and} \quad \beta|_{L^3} = \pm \text{vol}_{L^3}.
\end{equation}
In other words, depending on the orientation on $L^3$, $L^3$ is a $\pm \beta$-calibrated submanifold, see [19], [20]. For $x \in L^3$ let $\xi(x)$ denote the unit simple 3-vector associated with $T_xL^3$. By [19, Lemma 2.1], [20, Lemma 1.1] for any $V \in NL^3$ we obtain

\begin{equation}
\langle -H_{L^3}, V \rangle = (V \cdot d \pm \beta)(\xi).
\end{equation}

(In [19] Lê showed that the formula (3.4) is equivalent to the formula (3.2).) Using (2.6), we obtain immediately that $H_{L^3} = 0$.

**Corollary 3.6.** Any Lagrangian submanifold $L^3$ in a strictly nearly Kähler 6-manifold $(M^6, J, g)$ is orientable and minimal. Hence its Maslov 1-form vanishes.

**Remark 3.7.** For the remainder of our paper we shall choose the orientation on $L^3$ such that $-\beta|_{L^3} = \text{vol}_{L^3}$. This orientation agrees with the natural orientation of the Lagrangian sphere $S^3(1) = \mathbb{H} \cap S^6$ in the standard strictly nearly Kähler sphere $S^6 \subset \text{Im} \mathbb{O}$, see also subsection 4.3 below. Note that our choice of the orientation of $L^3$ agrees with that in [25, p. 2309], but differs from that in [38, p. 18].

**Remark 3.8.** The relation between the Maslov class and the minimality of Lagrangian submanifolds has been found for Lagrangian submanifolds in various classes of Hermitian manifolds [29], [22], [19]. Corollary 3.4 extends a previous result by Bryant [3] and partially extends a result by Lê in [19]. The minimality of Lagrangian submanifolds in a strictly nearly Kähler 6-manifolds has been proved by Schäfer and Smoczyk by studying the second fundamental form of $L^3$ in $M^6$ [38, §4], extending a previous result by Ejiri [7] for $M^6 = S^6$. The minimality of a Lagrangian submanifold $L$ in a strictly nearly Kähler manifold $M^6$ can be also obtained from the minimality of the coassociative cone $CL^3 \subset CM^6$.

### 3.2. Second variation of the volume of Lagrangian submanifolds.

The second variation of the volume of a minimal submanifold $N$ in a Riemannian manifold $M$ has been expressed by Simons [37] in terms of an elliptic second order operator $I(N, M)$ that depends on the second fundamental form of $N$ and the Riemannian curvature on $M$, see also [21], [33]. If $L^3$ is a Lagrangian submanifold in a strictly nearly Kähler manifold $M^6$, we shall derive a simple formula for $I(L^3, M^6)$ that depends entirely on the intrinsic geometry of $L^3$ supplied with the induced Riemannian metric.

**Theorem 3.9.** Assume that $(M^6, J, g)$ is a strictly nearly Kähler manifold of constant type $\lambda$. Let $V$ be a normal vector field with compact support on a Lagrangian submanifold $L^3 \subset M^6$. Then the second variation of the volume
of $L^3$ with the variation field $V$ is given by
\[
\frac{d^2}{dt^2}|_{t=0} \text{vol}(L^3_t) = \int_{L^3} (d(L_\omega(V)) - 3\lambda \ast L_\omega(V), d(L_\omega(V)) + \lambda \ast L_\omega(V)) \\
+ \int_{L^3} ||d \ast L_\omega(V)||^2.
\]
(3.5)

Proof. Let $\phi_t : L^3 \to M^6$ be a variation of $L^3$ generated by the vector field $V$.

Set $\xi_t(x) := (\phi_t)_\ast(\xi(x))$.

We observe that, to compute the second variation of the volume of $L^3$, using Lemma 3.5 and the minimality of $L^3$, it suffices to compute the second variation of the integral over $L^3$ of $\sum_{i=1}^3 |dz_i \wedge \omega(\xi)|^2$, $(\alpha(\xi))^2$ and $(\beta(\xi))^2$.

Namely, using the observation that for all $x \in L^3$
\[
|\xi_0(x)| = 1 \quad \text{and} \quad \frac{d}{dt}|_{t=0} |(\xi_t(x))| = 0
\]
we obtain
\[
\frac{d^2}{dt^2}|_{t=0} \text{vol}(\phi_t(L^3)) = \int_{L^3} \frac{d^2}{dt^2}|_{t=0} |(\xi_t(x))| \text{vol}_x \\
= \frac{1}{2} \int_{L^3} \frac{d^2}{dt^2}|_{t=0} (|\xi_t(x)|^2) \text{vol}_x.
\]
(3.7)

Lemma 3.10. For any $x \in L^3$ we have
\[
\frac{d^2}{dt^2}|_{t=0} \sum_{i=1}^3 ((dz_i \wedge \omega), \xi_t(x))^2 = 2|dL_\omega(V) - 3\lambda \ast L_\omega(V)|^2(x).
\]

Proof. Since $\omega|_{L^3} = 0$ we have for all $i$
\[
\frac{d^2}{dt^2}|_{t=0} |dz_i \wedge \omega(\xi)|^2 = 2\left[\frac{d}{dt}|_{t=0} (dz_i \wedge \omega(\xi))\right]^2.
\]
(3.8)

By Proposition 4.2, taking into account the rescaling factor $\lambda$, see also [38, Theorem 8.1], we have
\[
\frac{d}{dt}|_{t=0} \phi_t^\ast(\omega)(x) = d(L_\omega(V))(x) - 3(\lambda \ast L_\omega(V))(x).
\]
(3.9)

Since the RHS of (3.9) is a 2-form on $L^3$, there exists an orthonormal basis $f^1, f^2, f^3$ of $T_x^*L^3$ and a number $c \in \mathbb{R}$ such that
\[
d(L_\omega(V))(x) - 3(\lambda \ast L_\omega(V))(x) = c \cdot f^1 \wedge f^2.
\]

Using $\omega|_{L^3} = 0$ and the expression of the RHS of (3.9) in this basis, we obtain from (3.9)
\[
\frac{d}{dt}|_{t=0} \sum_{i=1}^3 \phi_t(dz_i \wedge \omega) = c \cdot f^1 \wedge f^2 \wedge f^3.
\]
(3.10)
Using again $\omega|_{L^3} = 0$, we obtain Lemma 3.10 immediately from (3.8) and (3.10).

**Lemma 3.11.** For all $x \in L^3$ we have
\[
\frac{d^2}{dt^2}|_{t=0}(\alpha(\xi_t(x))^2) = 2|d*L_\omega(V)|^2(x).
\]

**Proof.** By Proposition 4.2 we have (3.11)
\[
\frac{dt}{dt}|_{t=0}(\alpha(\xi_t(x))) = (d*L_\omega(V))(x).
\]
Since $\alpha(\xi(x)) = 0$, we obtain Lemma 3.11 from (3.11) immediately.

**Lemma 3.12.** We have
\[
\frac{d^2}{dt^2}|_{t=0} \int_{L^3} \beta(\xi_t(x))^2 \text{dvol}_x = 8\lambda \int_{L^3} \langle *L_\omega(V), d(L_\omega(V)) - 3\lambda * L_\omega(V) \rangle \text{dvol}_x.
\]

**Proof.** Since $(V|\beta)|_{L^3} = 0$, (see e.g. [19, Proposition 2.2.(ii)], [20, Proposition 1.2.(ii)], which is also now called the first cousin principle), using the Cartan formula we have
\[
\frac{d}{dt}|_{t=0}(\beta(x), \xi_t(x)) = (V|d\beta, \xi_t(x)),
\]
for all $x \in L^3$.

By (3.4) the RHS of (3.12) vanishes. Since $\beta(\xi(x)) = -1$ for all $x \in L^3$, we obtain
\[
\frac{d^2}{dt^2}|_{t=0}(\beta(\xi_t(x))^2) = -2\frac{d^2}{dt^2}|_{t=0}(\beta(\xi_t(x)).
\]

It follows that
\[
\frac{d^2}{dt^2}|_{t=0} \int_{L^3} \beta(\xi_t(x))^2 \text{dvol}_x = -2\frac{d^2}{dt^2}|_{t=0} \int_{L^3} (\phi^*_t(\beta), \xi) \text{dvol}_x.
\]
Using the Cartan formula, we derive from (3.14)
\[
\frac{d^2}{dt^2}|_{t=0} \int_{L^3} \beta(\xi_t(x))^2 \text{dvol}_x = -2 \int_{L^3} \mathcal{L}_V((V|d\beta) + d(V|\beta)).
\]
Since $\mathcal{L}_V(d(V|\beta)) = d(\mathcal{L}_V(V|\beta))$, we obtain from (3.15), taking into account that $d\beta = -2\lambda \omega \wedge \omega$
\[
\frac{d^2}{dt^2}|_{t=0} \int_{L^3} \beta(\xi_t(x))^2 \text{dvol}_x = 4\lambda \int_{L^3} \mathcal{L}_V(V|\omega \wedge \omega)).
\]
Taking into account $V|\omega \wedge \omega = 2(V|\omega) \wedge \omega$ and $\omega|_{L^3} = 0$ we obtain from (3.16)
\[
\frac{d^2}{dt^2}|_{t=0} \int_{L^3} \beta(\xi_t(x))^2 \text{dvol}_x = 8\lambda \int_{L^3} (V|\omega) \wedge \mathcal{L}_V(\omega).
\]
Since $(V|\omega) = L_\omega(V)$ and $\mathcal{L}_V(\omega) = dL_\omega(V) - 3\lambda * L_\omega(V)$, we obtain Lemma 3.12 immediately from (3.17).
Now let us complete the proof of Theorem 3.9. Using Lemma 3.5 we obtain from 3.7
\[ 2 \frac{d^2}{dt^2}|_{t=0} \text{vol}(\phi_t(L^3)) = \int_{L^3} \frac{d^2}{dt^2}|_{t=0} \sum_{i=1}^3 (dz_i \wedge \omega, \xi_t)^2 \, d\text{vol}_x + \int_{L^3} \frac{d^2}{dt^2}|_{t=0} (\alpha, \xi_t)^2 \, d\text{vol}_x + \int_{L^3} \frac{d^2}{dt^2}|_{t=0} (\beta, \xi_t)^2 \, d\text{vol}_x. \]
\[ (3.18) \]
Clearly Theorem 3.9 follows from (3.18) and Lemmas 3.10, 3.11, 3.12. □

Using Corollary 4.3, we obtain immediately from Theorem 3.9 the following.

**Corollary 3.13.** 1. Any formal infinitesimal Lagrangian deformation with compact support of a Lagrangian submanifold $L^3$ in a strictly nearly Kähler manifold is a Jacobi field.

2. Assume that $L^3$ is a compact Lagrangian submanifold in a strictly nearly Kähler manifold $(M^6, J, g)$ and $H^1(L^3, \mathbb{R}) \neq 0$. Let $\beta$ be a non-zero harmonic 1-form on $L^3$. Then the variation generated by $L^{-1}_{\omega} (\beta)$ decreases the volume of $L^3$.

**Remark 3.14.** There are many known examples of Lagrangian submanifolds $L^3$ in the manifold $S^6$ supplied with the standard nearly Kähler structure induced from $\mathbb{R}^7 = \text{Im} \mathbb{O}$ such that $\dim H^1(L^3)$ is arbitrary large. For instance, $L^3$ is obtained by composing the Hopf lifting of a holomorphic curve $\Sigma_g$ of genus $g$ in the projective plane $\mathbb{C}P^2$ to $S^5$ with a geodesic embedding $S^5 \rightarrow S^6$ [6, Theorem 1], see also [25, Example 6.11].

**Remark 3.15.** Letting $\lambda$ go to zero, we obtain the formula for the second variation of the volume of a special Lagrangian submanifold $L$ in a Calabi-Yau manifold $M^n$ with a variation field $V$ which is normal to $L$:
\[ (3.19) \quad \frac{d^2}{dt^2}|_{t=0} \text{vol}(L_t) = \int_L ||d(L_\omega V)||^2 + \int_L ||d^*(L_\omega V)||^2. \]

Formula (3.19) has been obtained by McLean in [27, Theorem 3.13] for special Lagrangian submanifolds in Calabi-Yau manifolds of dimension $2n$ as a consequence of his formula for the second variation of the volume of calibrated submanifolds, using moving frame method. Note that our proof of Theorem 3.9 can be easily adapted to give (3.19) for special Lagrangian submanifolds $L^n \subset M^{2n}$. Here we use the full version of Lemma 3.5 given in [12, Chapter III, Theorem 1.7, (1.17)]. The first summand in RHS of (3.19) is the second variation of the term $(|\xi|^2 - dz(\xi)^2)/2$. The second summand in the RHS of (3.19) is the second variation of the term $(\alpha(\xi))^2/2$. By [19 (4.11)], the second variation of the term $\beta(\xi)$ vanishes, if $M^{2n}$ is a Calabi-Yau manifold. This proves (3.19) for any dimension $n$. Note that (3.19) also follows from Oh’s second variation formula for Lagrangian minimal submanifolds in Kähler manifolds [32, Theorem 3.5].
Remark 3.16. Using the strategy of the proof of Theorem 3.9, we can have a (new simple proof of) formula for the second variation of the volume of \( \phi \)-calibrated submanifolds \( N^n \) in a manifold \( M^n \) provided with a relative calibration \( \phi \) such that a generalized version of Lemma 3.5 is valid, that expresses \( |\xi|^2 \) as a sum \( |\phi(\xi)|^2 + \sum_{i=1}^{k} |\alpha_k(\xi)|^2 \). Generalized versions of Lemma 3.5 have been found for \( \text{Kähler} \) 2p-vectors, coassociative 4-vectors, etc. in [12].

4. Deformations of Lagrangian submanifolds in strictly nearly Kähler 6-manifolds

In this section we assume that \( L \) is a bounded submanifold if not otherwise stated. We prove that any \( \mathcal{C}^1 \)-small smooth Lagrangian deformation of a Lagrangian submanifold \( L^3 \) in a strictly nearly Kähler 6-manifold is a solution of an elliptic first order PDE (Theorem 4.5) and hence the set of all \( \mathcal{C}^1 \)-small smooth Lagrangian deformations of \( L^3 \) is the kernel of a smooth map between open domains in finite dimensional vector spaces (Theorem 4.8). If \((M^6, J, g)\) is analytic, we show that \( L^3 \) and any smooth Lagrangian deformation of \( L^3 \) are also analytic (Proposition 4.9). We analyze sufficient and necessary conditions for an analytic deformation of \( L^3 \) to be Lagrangian (Theorem 4.11, Lemma 4.10, Proposition 4.12, Lemma 4.13). Finally we examine two homogeneous Lagrangian submanifolds \( S^3(1) \) and \( L_1^3 \) in the standard nearly Kähler sphere \((S^6, J_0, g_0)\) and show that they are rigid up to the motion of the group \( G_2 \) (Theorems 4.15, 4.18).

4.1. Smooth deformations of Lagrangian submanifolds in strictly nearly Kähler 6-manifolds. Let \( L \) be a submanifold in a Riemannian manifold \((M, g)\). It is known that a tubular neighborhood \( U(L) \subset M \) of \( L \) can be identified with a small neighborhood \( N_{\varepsilon}(L) \) of the zero section of the normal bundle \( NL \) of \( L \) via the exponential mapping \( \text{Exp}_L : NL \to M \). Then we identify a \( \mathcal{C}^1 \)-small deformation of \( L \) with a section \( s : L \to N_{\varepsilon}(L) \) via the exponential map \( \text{Exp}_L \), which we shall also denote by \( \text{Exp} \) if no confuse arises.

Now assume that \((M, J, g)\) is a Hermitian manifold and \( \omega \) is the associated fundamental 2-form. If \( L \) is a Lagrangian submanifold, then \( L_\omega \) identifies a vector in \( NL \) (resp. a section \( s_\alpha \in \Gamma(NL) \)) with a covector in \( T^*L \) (resp. a 1-form \( \alpha \in \Omega^1(L) \)). With this identification we define a non-linear map \( F : \Omega^1(L) \to \Omega^2(L) \oplus \Omega^3(L) \) as follows

\[
F(\alpha) := (\text{Exp}(s_\alpha))^*(\omega) + (\text{Exp}(s_\alpha))^*(d\omega) \in \Omega^2(L) \oplus \Omega^3(L).
\]

Note that \( F(\alpha) = 0 \) if and only if the image (the graph) of \( \text{Exp}(s_\alpha) \) is a Lagrangian submanifold in \( U(L) \). Hence we identify the set of all \( \mathcal{C}^1 \)-small Lagrangian deformations of \( L \) with the set \( F^{-1}(0) \subset \Omega^1(L) \).

There are two motivations which lead us to consider the equation \( F(\alpha) = 0 \) instead of the simpler equation \( F_0(\alpha) := (\text{Exp}(s_\alpha))^*(\omega) = 0 \) as in [38]. Firstly, the equation \( F(\alpha) = 0 \) is the prolongation of the equation \( F_0(\alpha) = 0 \).
Secondly, if \((M^6, J, g)\) is strictly nearly Kähler, the equation \(F(\alpha) = 0\) looks similar to the equation for a coassociative deformation of the cone \(CL^3\) in \(CM^6\), see [24], which will guide us in analyzing [41] later.

For the remainder of this section we assume that \((M = M^6, J, g, \omega)\) is a strictly nearly Kähler 6-manifold \(M^6\) of constant type 1 and \(L = L^3\) is a Lagrangian submanifold in \(M^6\).

Now we shall compute the linearization of \(F\) at \(\alpha = 0\). Denote by \(*\) the Hodge star operator on \(L^3\), which depends on a choice of orientation of \(L\). First we need the following

**Lemma 4.1.** With the choice of the orientation of \(L^3\) in Remark 3.7 we have
\[
(4.2) \quad 3 * \beta(x) = -(s_\beta | d\omega)|_{L^3}(x).
\]

**Proof.** As we have remarked in subsection 2.2, using the same notations, at any given point \(x \in L^3\) there exists a unitary basis \((e^i, Je^i)\) such that \(d\omega(x) = 3Re(dz_1 \wedge dz_2 \wedge dz_3)\) and \(\omega = -Im(dz_1 \wedge dz_2 + dz_2 \wedge dz_3 + dz_3 \wedge dz_1)\).

Since special Lagrangian planes in \(T_x M^6\) are transitive under \(SU(3)\)-action [12], we can assume that \(T_x L^3\) is spanned by \((Je_1, Je_2, Je_3)\). Denote by \(\{(e_1)^*, (Je_1)^*, (e_2)^*, (Je_2)^*, (e_3)^*, (Je_3)^*\}\) the dual frame to \(\{e_1, Je_1, e_2, Je_2, e_3, Je_3\}\). Since \(\omega(x), d\omega(x)\) and \(T_x L^3\) are invariant under the action \(SO(3) \subset SU(3)\), we can assume further that \(\beta = c \cdot (Je_1)^*\) for some \(c \in \mathbb{R}\). Under this assumption Lemma 4.1 is verified easily. Recall that the orientation on \(L^3\) is defined by the following equation
\[
Im (dz_1 \wedge dz_2 \wedge dz_3)|_{L^3} = -3vol|_{L^3},
\]
i.e. \((Je_1, Je_2, Je_3)\) is an oriented frame. Then we have
\[
3 * (Je_1)^* = 3(Je_2)^* \wedge (Je_3)^* = -(e_1 | d\omega)|_{L^3} = -(L^{-1}_\omega(Je_1)^*)d\omega)|_{L^3},
\]
what is required to prove. \(\square\)

**Proposition 4.2.** For all \(\beta \in \Omega^1(L^3)\) we have
\[
(4.3) \quad \partial F|_0(\beta) = (d\beta - 3 * \beta, -3d(*\beta)) \in \Omega^2(L^3) \oplus \Omega^3(L^3).
\]

**Proof.** Let \(\beta \in \Omega^1(L^3)\). Let \(D_r, r \in [0, 1]\), be a 1-parameter family of diffeomorphisms on \(U(L^3)\) with \(D_0 = Id\) that satisfy the following condition for all \(x \in L^3\) and for all \(r \in [0, 1]\)
\[
(4.4) \quad D_r(x) = Exp(s_r \beta(x)) = Exp(r s_\beta(x)).
\]

For instance, let \(\tilde{D}_r : NL^3 \to NL^3\) be defined as follows
\[
\tilde{D}_r(x, l) := (x, l + rs_\beta(x)).
\]

We set
\[
(4.5) \quad D_r(y) := Exp_L \circ \tilde{D}_r \circ Exp^{-1}_L.
\]

Then \(D_r\) defined by (4.5) satisfies (4.3).
From (4.4) and (4.1) we obtain

\[
(4.6) \quad F(r\beta) = (D_r^*(\omega), D_r^*(d\omega))|_{L^3}.
\]

Let \( \tilde{s}_\beta \) denote the vector field on a neighborhood \( U_\varepsilon(L^3) \) generated by \( D_r \), i.e.

\[
(4.7) \quad \tilde{s}_\beta(y) := \frac{d}{dr}|_{r=0} D_r(y) \quad \text{for} \quad y \in U(L^3).
\]

It follows from (4.4) and (4.7) that

\[
(4.8) \quad \tilde{s}_\beta|_{L^3} = s_\beta.
\]

Using the Cartan formula, we obtain from (4.6)

\[
\partial F|_0(\beta) = ((\mathcal{L}_{\tilde{s}_\beta}\omega)|_{L^3}, (\mathcal{L}_{\tilde{s}_\beta} d\omega)|_{L^3}) =
\]

\[
(4.9) \quad = (d(\tilde{s}_\beta|\omega)|_{L^3} + (\tilde{s}_\beta|d\omega)|_{L^3}, d(s_\beta|d\omega)|_{L}).
\]

By definition (3.1), \( s_\beta|\omega|_{L^3} = \beta \). Using this, we deduce Lemma 4.2 from (4.9) and Lemma 4.1. \( \square \)

Corollary 4.3. (cf. [38, Theorem 8.1]) A 1-form \( \beta \in \Omega^1(L^3) \) is a solution of the linearized equation \( \partial F|_0(\beta) = 0 \), iff \( d\beta - 3 * \beta = 0 \). In particular, \( d^* \beta = 0 \) and \( \triangle^d = 9\beta \).

Set \( (\partial F|_0)^* : \Omega^1(L^3) \to \Omega^2(L^3) \) as follows

\[
(\partial F|_0)^*(\beta) = d\beta + 3 * \beta.
\]

Clearly any solution \( \beta \) of \( (\partial F|_0)^*(\beta) = 0 \) is also a solution of \( \triangle^d(\beta) = 9\beta \) and \( d^* \beta = 0 \). We set

\[
\Omega^k_a(L^3) := \{\alpha^k \in \Omega^k(L^3) | \triangle^d(\alpha^k) = a \cdot \alpha\}.
\]

From now till the end of this paper we assume that \( L^3 \) is a compact submanifold.

Proposition 4.4. We have

\[
\Omega^1_0(L^3) \cap \ker(d^*) = \ker(\partial F|_0) \oplus \ker(\partial F|_0)^*.
\]

Proof. Write \( \alpha^k = d\alpha^{k-1} + d^* (\alpha^{k+1}) \).

Then

\[
\triangle^d(\alpha^k) = d d^* d\alpha^{k-1} + d^* d d^* \alpha^{k+1}.
\]

We assume that \( a > 0 \). Then

\[
\alpha^k \in \Omega^k_a(L^3) \iff d\alpha^{k-1}, d^* \alpha^{k+1} \in \Omega^k_a(L^3).
\]

Hence, for \( a > 0 \) we have

\[
(4.10) \quad \Omega^k_a(L^3) = \Omega^k_a(L^3) \cap \Omega^k_{\text{exact}} \oplus \Omega^k_a \cap (\ast \Omega^3_{\text{exact}} \ast(L^3)).
\]
Furthermore we have
\begin{equation}
(*d) \Delta^d|_{\Omega^2(L^3)} = \Delta^d(*d)|_{\Omega^2(L^3)} = (*d)^2.
\end{equation}
It follows that \((*d)\) preserves the space \(\Omega^1_0(L^3)\). Using (4.10) we obtain
\begin{equation}
(*d)(\Omega^1_0(L^3)) \subset \Omega^1_0(L^3) \cap \ker d^*.
\end{equation}
Moreover we have
\begin{equation}
\Delta^d|_{\Omega^3_0(L^3) \cap \ker d^*} = (*d)^2.
\end{equation}
To prove Proposition 4.4, taking into account Corollary 4.3, it suffices to show the following inclusion
\begin{equation}
\Omega^1_0(L^3) \cap \ker d^* \subset \ker (d + 3\ast) \oplus \ker (d - 3\ast).
\end{equation}
Let \(\beta \in \Omega^1_0 \cap \ker d^*\). Since \((*d|_{\Omega^1_0(L^3)})^\ast = *d\), we can write \(\beta = \beta_1 + \cdots + \beta_k\), where \(*d(\beta_j) = a_j \beta_j\) and \(\beta_j \in \Omega^1_0(L^3) \cap \ker d^*\) for all \(j \in [1, k]\). By (4.12) \(a_j \neq 0\). Using (4.13) we obtain
\[\Delta^d(\beta) = a_1^2 \beta_1 + \cdots + a_k^2 \beta_k.\]
Since \(\beta \in \Omega^1_0\) it follows that \(a_i^2 = 9\) for all \(i\). Thus \(a_i = \pm 3\). This proves (4.14) and completes the proof of Proposition 4.4.

The equation \(F(\alpha) = 0\) is an overdetermined equation: it is the restriction of an elliptic first order equation to a subspace as we shall see now. We shall add to \(F\) another parameter, so the new equation becomes elliptic. Set
\[F_1 : \Omega^1(L^3) \oplus C^\infty(L^3) \rightarrow \Omega^2(L^3) \oplus \Omega^3(L^3), (\alpha, f) \mapsto (F(\alpha) + *df).\]
Note that
\[F^{-1}(0) \cong \{(\alpha, f) \mid F_1(\alpha, f) = 0 \text{ and } f = 0\}.
\]

**Theorem 4.5.** If \((\alpha, f)\) is a solution of the equation \(F_1(\alpha, f) = 0\) then \(\alpha\) is a solution of the equation \(F(\alpha) = 0\) and \(f = \text{const.}\)

**Proof.** Recall that \((\alpha, f)\) is a solution of \(F_1(\alpha, f) = 0\) iff
\begin{equation}
(\exp(s_\alpha))^\ast(d\omega) = 0 \text{ and } (\exp(s_\alpha))^\ast(\omega) = *df.
\end{equation}
Since \((\exp(s_\alpha))^\ast(d\omega) = d((\exp(s_\alpha))^\ast(\omega))\), it follows from (4.15) that \(d*df = 0\). Hence \(df\) is a harmonic form. Since \(df\) is an exact 1-form and \(L^3\) is compact, we have \(df = 0\). Hence \(f\) is a constant and therefore \(\alpha\) is a solution of the equation \(F(\alpha) = 0\). This proves Theorem 4.5.

Let us denote by \(\Omega^i_{\text{exact}}(L^3)\) the space of exact forms on \(L^3\). We note that
\[F_1(\Omega^1(L^3) \oplus C^\infty(L^3)) \subset \Omega^2(L^3) \oplus \Omega^3_{\text{exact}}(L^3),\]
since \((\exp(s_\alpha))^* (d\omega) = d((\exp(s_\alpha))^*(\omega))\). Thus, we also denote by \(F_1\) the same map whose target is \(\Omega^2(L^3) \oplus \Omega^3_{\text{exact}}(L^3)\):
\[F_1 : \Omega^1(L^3) \oplus C^\infty(L^3) \rightarrow \Omega^2(L^3) \oplus \Omega^3_{\text{exact}}(L^3)\]
\begin{equation}
(\alpha, f) \mapsto (\exp(s_\alpha))^*(\omega) + d((\exp(s_\alpha))^*(\omega)) + *df.
\end{equation}
It follows from (4.3) and (4.16) that the differential
\[ \partial F_1|_{(0,0)} : \Omega^1(L^3) \oplus C^\infty(L^3) \to \Omega^2(L^3) \oplus \Omega^3_{\text{exact}}(L^3) \]
has the following form
\[ (4.17) \quad \partial F_1|_{(0,0)}(\beta, g) = (d\beta - 3 * \beta + *dg, -3d(\star \beta)). \]

**Proposition 4.6.** We have \( \ker \partial F_1|_{(0,0)} = \ker \partial F_0 \times \mathbb{R} \).

**Proof.** Using the Hodge decomposition we write
\[ \beta = dh + \beta_{\text{harm}} + *d\alpha, \]
where \( h \in C^\infty(L^3), \beta_{\text{harm}} \in \Omega^1_{\text{harm}}(L^3) \) and \( \alpha \in \Omega^1(L^3) \). Then
\[ (4.18) \quad \partial F_1|_{(0,0)}(\beta, g) = (d*d(\alpha) - 3 * \beta_{\text{harm}} - 3d(\alpha) + *dg, -3d*dh). \]

From (4.18), using the Hodge decomposition, it follows that \( \partial F_1|_{(0,0)}(\beta, g) = 0 \) iff
\[ (4.19) \quad dh = 0, \quad \beta_{\text{harm}} = 0, \quad dg = 0 \iff g = \text{const} \]
\[ d(*d\alpha) - 3 * (*d\alpha) = 0. \]

Note that (4.19) is equivalent to \( *d\alpha \in \ker \partial F_1 \). This completes the proof of Proposition 4.6. \( \square \)

**Proposition 4.7.** The differential \( \partial F_1|_{(0,0)} \) is an elliptic first order differential operator. Furthermore, \( \dim \text{coker} \partial F_1|_{(0,0)} = \dim \ker \partial F|_0 \) and
\[ \langle \text{coker} \partial F_1|_{(0,0)} *, \text{ker} \partial F_1|_{(0,0)} \rangle = 0. \]

**Proof.** The first assertion of Proposition 4.7 follows from the observation that the symbol of \( \partial F_1|_{(0,0)} \) is equivalent to the symbol of the Dirac operator \( d + *d* \).

Now let us examine the image of \( \partial F_1|_{(0,0)} \). Recall that \( \Omega^0_0(L^3) \) denotes the space of harmonic forms on \( L^3 \). We have the following diagram, using the Hodge decomposition.
Set:
\[ V := \{ (df, -3f) | f \in C^\infty(L^3) \} \subset \Omega^1(L^3) \oplus C^\infty(L^3). \]
Since \( \partial F_1|_{(0,0)}(df, -3f) = (0, 3d(*df)) \), using the Hodge decomposition for \( \Omega^3(L^3) \) and \( \Omega^2(L^3) \), we obtain
\[(4.20)\]
\[ \Omega^3_{\text{exact}}(L^3) = \partial F_1|_{(0,0)}(V). \]
A direct computation shows that
\[(4.21)\]
\[ *d(C^\infty(L^3)) = \partial F_1|_{(0,0)}(0, C^\infty(L^3)), \]
\[(4.22)\]
\[ \Omega^2_{\text{harm}}(L^3) = \partial F_1|_{(0,0)}(\Omega^1_{\text{harm}}(L^3), 0). \]
Next, for \( \gamma \in \Omega^2(L^3) \) we have
\[(4.23)\]
\[ \partial F_1|_{(0,0)}(*d*\gamma, 0) = (d*d*\gamma - 3d*\gamma, 0). \]
Note that \( d*d*\gamma - 3d*\gamma = 0 \) iff \( *d*\gamma \in \ker \partial F_0|_0 \). Taking into account \( (4.20), (4.21), (4.22), (4.23) \), this proves the second assertion of Proposition 4.7.

The last assertion of Proposition 4.7 follows from the observation that \( *dF_1|_{(0,0)} \) is a self-adjoint linear elliptic operator and therefore its cokernel is the \( L^2 \)-complement to its kernel. This completes the proof of Proposition 4.7.

Set
\[ Ob(L^3) := \Omega^2_0(L^3) \cap \ker(d * -3 \cdot \text{Id}). \]
The discussion above yields
\[(4.24)\]
\[ \Omega^2(L^3) \oplus \Omega^3_{\text{exact}}(L^3) = \text{Im} \partial F_1|_{(0,0)} \oplus (Ob(L^3), 0). \]
Denote by \( \pi_1 \) (resp. \( \pi_2 \)) the projection of the RHS of \( (4.24) \) to the first factor (resp. the second factor) in RHS of \( (4.24) \). We define a smooth map \( O : \ker \partial F_1|_{(0,0)} \to Ob(L^3) \) as follows
\[ O(\alpha) = \pi_1 \circ \pi_2 \circ F_1, \]
where \( \pi \) is the projection to the first factor.

**Theorem 4.8.** There exists a neighborhood \( B(L^3) \) of \( L^3 \) in the set \( \text{Def}(L^3) \) of all smooth Lagrangian deformations of \( L^3 \) in \( M^6 \) such that \( B(L^3) \) is homeomorphic to \( O^{-1}(0) \cap U_\varepsilon(0) \), where \( U_\varepsilon(0) \) is a small neighborhood of the zero section in \( \Omega^1(L^3) \).

**Proof.** To prove Theorem 4.8 we use the implicit mapping theorem for Banach spaces. The Banach spaces under consideration are the completion \( L^k_2(\Lambda^* T^* L^3) \) of \( \Omega^* L^3 \) in the \( L^2 \)-norm, for \( k \geq 4 \), so that \( L^k_2(\Lambda^* T^* L^3) \) are \( C^2 \)-differential forms. The map \( F_1 \) is extended to a smooth map \( F^k_1 \) between the completion of the spaces under consideration. Since \( F_1 \) is elliptic, for all \( k \geq 4 \) we have
\[(4.25)\]
\[ \ker \partial F^k_1 = \ker \partial F_1 \text{ and } \langle Ob(L^3), \text{Im} \partial F^k_1 \rangle_{L^2} = 0. \]
We write $F_1 = (\pi_1 \circ F_1, \pi_2 \circ F_1)$. Note that $\pi_1 \circ F_1$ is a smooth map between Banach spaces whose derivative $\partial(\pi_1 \circ F_1)|_{(0,0)}$ is onto. Note that $\ker \partial(\pi_1 \circ F_1)|_{(0,0)} = \ker \partial F_1|_{(0,0)}$. Denote by $W_k$ the $L_2$-complement to $\ker \partial F_1|_{(0,0)}$ in $L^2_k(T^*L^3)$. The implicit mapping theorem says that, there exists an open neighborhood $U_1(0) \subset \ker \partial(\pi_1 \circ F_1)|_{(0,0)}$ and a smooth map $\chi_k : \ker \partial F_1|_{(0,0)} \to W_k$ such that
\[
\pi_1 F^k_1(x, \chi_k(x)) = 0
\]
for all $x \in U_1(0)$, see for instance [13] Theorem 5.9. It follows that
\[
(F^k_1)^{-1}(0) \cap U_\varepsilon(0) = \{x \in U_1(0) | \pi_2 \circ F^k_1(x) = 0\}.
\]
Since $F_1$ is elliptic, $x \in U_1(0)$ implies that $x$ is smooth, and hence $F^k_1(x) = F_1(x)$. Furthermore, $(F^k_1)^{-1}(0) = F^{-1}_1(0)$ for $k \geq 4$. Hence, setting $B(L^3) := F^{-1}_1(0) \cap U_\varepsilon(0)$, we derive from (4.26) Theorem 4.8 immediately. 

4.2. Deformation of compact Lagrangian submanifolds in analytic strictly nearly Kähler 6-manifolds.

**Proposition 4.9.** Let $L^3$ be a smooth compact Lagrangian submanifold in an analytic strictly nearly Kähler 6-manifold $M^6$.

1. Then $L^3$ is also an analytic submanifold of $M^6$.

2. Assume that $\text{Exp } s_{\alpha(t)}$ is a smooth Lagrangian deformation of $L^3$ for some $\alpha(t) \in \Omega^1(L^3) \times (-\varepsilon, \varepsilon)$. Then $\alpha(t)$ can be written as a convergent power series $\alpha(t) = \sum_{k=1}^{\infty} t^k \alpha_k$ for $t$ in some open interval $(-\varepsilon', \varepsilon') \subset (-\varepsilon, \varepsilon)$, and $\alpha_k \in \Omega^1(L^3)$. 

**Proof.** First we note that any Lagrangian submanifold $L^3 \subset (M^6, J, g)$ is a minimal submanifold in $(M^6, g)$. This implies the first assertion of Proposition 4.9 from the Morrey regularity theorem for vector solutions of class $C^1$ of a regular variational problem [28] (cf. [12], IV.2.B).

Now assume that $s_{\alpha(t)}$ is a smooth Lagrangian deformation of $L_0$. Let us consider the following map
\[
L^3 \times (-\varepsilon, \varepsilon) \xrightarrow{\hat{s}} N_\varepsilon(L^3) \times (-\varepsilon, \varepsilon) \xrightarrow{\text{Exp}} M^6 \times (-\varepsilon, \varepsilon).
\]
Since the image $\text{Exp} \circ \hat{s}(L^3 \times (-\varepsilon, \varepsilon))$ is a minimal submanifold in the analytic Riemannian manifold $(M^6 \times (-\varepsilon, \varepsilon), g \oplus dt^2)$ the composition $\text{Exp} \circ \hat{s}$ is an analytic map.

Since the map $\text{Exp}^{-1}$ is analytic at each point in $L^3$, and $L^3$ is compact, we can assume that $t \in (-\varepsilon', \varepsilon')$ is so small such that the map $\hat{s}$ is also analytic. It follows that $\alpha(t)$ can be written as a convergent power series at the zero section. Since $\hat{s}$ is a smooth mapping, we obtain the second assertion of Proposition 4.9 immediately, noting that
\[
s_{\alpha_k} = \frac{1}{k!} \frac{d^k s_{\alpha(t)}}{dt^k}
\]
is smooth for each $k$ and hence $\alpha_k = L_\omega(s_{\alpha_k})$ is also smooth. 

\[
\square
\]
Now we shall say that a convergent power series

\[ \alpha(x,t) = \sum_{k=1}^{\infty} t^k \alpha_k(x), \]

where \( \alpha_k(x) \in \Omega^1(L^3) \) for \( k \geq 1 \), is \textit{Lagrangian}, if the image of \( \text{Exp}(s_{\alpha(x,t)}) : L^3 \to M^6 \) is a Lagrangian submanifold for all \( t \in (-\varepsilon, \varepsilon) \).

Next, we shall find a necessary and sufficient condition under which a convergent power series in (4.27) is Lagrangian. Denote by \( \pi : L^3 \times (-\varepsilon, \varepsilon) \to L^3 \) the canonical projection. We shall compute the value \( \tilde{\alpha} \mapsto (\text{Exp}(s_{\tilde{\alpha}})^* (\pi^* \omega)) \).

Write \( \alpha_t(x) := \tilde{\alpha}(x,t) \). Since \( \text{Exp} \) is analytic at the zero section, we can write

\[ \tilde{\alpha}(x) = \sum_{k=0}^{\infty} t^k \omega_k(x), \]

where \( \sum_{k=0}^{\infty} t^k \omega_k(x) \) is a convergent power series in \( t \). Note that

\[ \omega_0 = \omega|_{L^3} = 0, \]

\[ \omega_1 = \frac{d}{dt}|_{t=0} F_0(\alpha_t) = \partial F_0|_0(\alpha_1). \]

\[ \omega_k = \frac{d^k}{(dt)^k}|_{t=0} F_0(\alpha_t). \]

To find \( \omega_k \) for \( k \geq 2 \) it is necessary to compute the value

\[ \partial F_0|_{(\alpha_0)(\beta)} \in \Omega^2(L^3) \]

for any \( \alpha, \beta \in \Omega^1(L^3) \). Let \( \alpha, \beta \in \Omega^1(L^3) \) be given. Let \( g_t^{(\alpha,\beta)} : U(L^3) \to U(L^3) \) be a 1-parameter family of diffeomorphisms on \( U(L^3) \) defined by the following equation

\[ g_t^{(\alpha,\beta)}(x,l) = \text{Exp}_{L^3}(x,l + s_{(\alpha + t\beta)}(x)). \]

Since

\[ g_t^{(\alpha,\beta)}(x) = \text{Exp}_{L^3}(s_{(\alpha(x) + t\beta(x))}), \]

for all \( x \in L^3 \) we obtain

\[ F_0(\alpha + t\beta) = (g_t^{(\alpha,\beta)})^* (\omega)|_{L^3} \]

for all \( t \). Hence

\[ \partial F_0|_{(\alpha_0)(\beta)} = \frac{d}{dt}|_{t=0} (g_t^{(\alpha,\beta)})^* (\omega)|_{L^3}. \]
Let \( V(\alpha, \beta) \) be the vector field associated with \( g_t(\alpha, \beta) \), i.e.
\[
V(\alpha, \beta)(x) = \frac{d}{dt} |_{t=0} (g_t(\alpha, \beta)(x)) = \frac{d}{dt} |_{t=0} (g_t(0, \beta)(x)).
\]
Hence \( V(\alpha, \beta)(x) \) does not depend on \( \alpha \). So we abbreviate \( V(\alpha, \beta) \) as \( V_\beta \). By \( (4.32) \) for \( x \in L^3 \) we have
\[
(4.34) \quad V_\beta(\text{Exp}(s\alpha(x))) = d\text{Exp}|_{s\alpha(x)}(s\beta(x)).
\]
(To compute the RHS of \( (4.34) \) we can use the formula given in \([13, \text{Theorem 6.5}]\).)

Note that
\[
(4.35) \quad \frac{d}{dt} |_{t=0} ((g_t(\alpha, \beta))^* \omega) = (g_0(\alpha, 0))^* (\mathcal{L}_{V_\beta} \omega).
\]
Combining \( (4.32) \), \( (4.33) \) and \( (4.35) \) we obtain
\[
(4.36) \quad \partial F_0(\beta) = (\text{Exp} \alpha)^* (\mathcal{L}_{V_\beta} \omega).
\]

Next, observe that
\[
(4.37) \quad \frac{d}{dt} |_{t=0} F_0(\alpha_t) = \partial F_0|_{\alpha_0}(\dot{\alpha}_0).
\]

**Lemma 4.10.** The form \( \omega_k(\alpha) \) depends only on the first \( k \)-terms \( \alpha_1, \cdots, \alpha_k \).
More precisely, we have
\[
\omega_k(\alpha) = \frac{1}{k!} \sum_{p_1 + \cdots + p_k = k} (\mathcal{L}_{V_{\alpha_1}} \cdots \mathcal{L}_{V_{\alpha_k}} \omega)|_{L^3}.
\]

**Proof.** We derive Lemma 4.10 from \( (4.30) \) by induction on \( k \), using the following formula
\[
\frac{d}{dt} (g_t^* \omega_t^{k-1}) = g_t^* (\mathcal{L}_{g_t^*} \omega_t^{k-1} + \frac{d}{dt} \omega_t^{k-1})
\]
where
\[
\omega_t^{k-1} = (g_t^{-1})^* (\frac{d}{dt})^{k-1} (g_t^* \omega),
\]
and taking into account \( (4.31) \) which implies that the vector fields \( \mathcal{L}_{V_{\alpha_i}} \) commute with each other. \( \square \)

We summarize our discussion in the following

**Theorem 4.11.** Let \( L^3 \) be a compact Lagrangian submanifold in an analytic strictly nearly Kähler 6-manifold \( M^6 \). Assume that \( \text{Exp} s_{\alpha_t}, \alpha_t \in \Omega^1(L^3) \) is a Lagrangian smooth deformation of \( L^3 \). Then there exists \( \varepsilon > 0 \) such that on \( (-\varepsilon, \varepsilon) \) \( \alpha_t \) can be written as a convergent sequence
\[
\alpha_t(x) = \sum_{k=1}^{\infty} \alpha_k(x) t^k,
\]
where \( \omega_k(\alpha_1, \cdots, \alpha_k) = 0 \) for all \( k \). Conversely, if \( \alpha_t \) is a convergent series such that \( \omega_k(\alpha_1, \cdots, \alpha_k) = 0 \) for all \( k \) then \( \text{Exp} s_{\alpha_t} \) is a Lagrangian deformation.
An element $\alpha \in \ker \partial F_0|_0$ is called \textit{smoothly obstructed}, if there is no smooth deformation $s_t : L^3 \to M^6$ such that $L_\omega(s_0) = s_\alpha$. An element $\alpha \in \ker \partial F_0|_0$ is called \textit{smoothly unobstructed}, if $s_\alpha$ is an infinitesimal Lagrangian deformation, i.e. $\alpha$ is tangent to a smooth curve of Lagrangian deformations of $L^3$.

To find a sufficient condition for an element $\alpha \in \ker \partial F_0|_0$ to be smoothly unobstructed or smoothly obstructed we define the following Kuranishi map

$$K : \ker \partial F_0|_0 \times \ker \partial F_0|_0 \to \Omega^2(L^3), \; (\alpha_1, \alpha_2) \mapsto (L_{\alpha_1}L_{\alpha_2}\omega)|_{L^3}.$$  \hspace{1cm} (4.38)

Clearly, $K$ is a symmetric, $\mathbb{R}$-bilinear map. The construction of $K$ is in a sense analogous to the construction of the Kuranishi map in $[23]$, see also $[23]$.

\begin{proposition} \label{proposition 4.12}
Assume that $(M^6, J, g)$ is an analytic strictly nearly Kähler manifold. An element $\alpha \in \ker \partial F_0|_0$ is smoothly obstructed, if there exists $\beta \in \ker \partial F_0|_0$ such that

$$\langle K(\alpha, \alpha), \beta \rangle_{L^3} \neq 0.$$ \hspace{1cm} (4.39)

If $K(\alpha, \alpha) = 0$ then $\alpha$ is smoothly unobstructed.
\end{proposition}

\begin{proof}
Assume that $\alpha \in \ker \partial F_0|_0$ and $L^3_t(x)$ is a smooth deformation of $L$ such that $(d/dt)|_{t=0}L^3_t = \alpha$. By Theorem \ref{lemma 4.11} for $t$ small, $L^3_t$ can be written as the image $\text{Exp}(\alpha_t(x))$ for some convergent power series $\alpha_t$ such that $\alpha_0 = 0$ and $\alpha_1 = \alpha$. By Theorem \ref{lemma 4.11} and Lemma \ref{lemma 4.10}

$$\omega_2(\alpha_t) = K(\alpha, \alpha) + (L_{\alpha_2}\omega)|_{L^3} = 0.$$ \hspace{1cm} (4.40)

This implies that $K(\alpha, \alpha)$ lies on the image of the map $\partial F_0$. Hence the condition in (4.39) does not hold for $\alpha$. This proves the first assertion of Proposition \ref{proposition 4.12}

Now assume that $K(\alpha, \alpha) = 0$. Then $\alpha_t(x) := t\alpha$ is a convergent power series and $\omega_k(\alpha_t) = 0$ for all $k$. By Theorem \ref{lemma 4.11} the deformation $\text{Exp}(\alpha_t)$ is Lagrangian for small $t$.

This completes the proof of Proposition \ref{proposition 4.12} \hfill $\square$

We can construct other analogous Kuranishi obstruction maps, using any diffeomorphism $D : N_\varepsilon L^3 \to U_\varepsilon(L^3) \subset M^6$ that is analytic at the zero section and satisfies $D(x, 0) = x$, $\partial D(x, 0)(v) = v$ for all $v \in N_\varepsilon L^3$, or modify the construction slightly as follows. We lift the vector field $s_\alpha$ to a vector field $\tilde{s}_\alpha$ on the cone $CL^3 \subset CM^6$ as follows (cf. (2.31))

$$\tilde{s}_\alpha(r, x) := (0, s_\alpha(x)) \in N_{(r,x)}CL^3.$$ \hspace{1cm} (4.40)

We assume that the value $r$ in (4.40) belongs to an open interval $(1 - \varepsilon, 1 + \varepsilon)$. Denote by $\tilde{\text{Exp}}$ the Riemannian exponential mapping on the cone $CM^6$. Note that $N_{(r,x)}TL^3 = N_\varepsilon L^3$. Let $\tilde{V}_\alpha$ be the vector field defined on an open neighborhood $U(CL^3) \subset CM^6$ by (4.40) as $V_\alpha$ but with respect to the Riemannian exponential map $\tilde{\text{Exp}}$ (we shall see in the next subsection.
that $\tilde{\text{Exp}}$ is much simpler than the map $\text{Exp}$. Recall that $\varphi$ is defined in (2.4). Now we define a map

$$K : \ker \partial F_0|_0 \times \ker \partial F_0|_0 \to \Omega^2(L^3),$$

$$(\alpha_1, \alpha_2) \mapsto (\partial r|_{L_{\tilde{\alpha}_1}} L_{\tilde{\alpha}_2} \varphi)|_{L^3}.$$ 

**Lemma 4.13.** Assume that $(M^6, J, g)$ is an analytic strictly nearly Kähler manifold. An element $\alpha \in \ker \partial F_0|_0$ is smoothly obstructed, if there exists $\beta \in \ker \partial F_0|_0$ such that

$$\langle \tilde{K}(\alpha, \alpha), \ast \beta \rangle_{L^2} \neq 0.

If $K(\alpha, \alpha) = 0$ then $\alpha$ is smoothly unobstructed.

**Proof.** Assume that $\alpha \in \ker \partial F_0|_0$ and $L^3(x)$ is a smooth deformation of $L^3$ such that $(d/dt)|_{t=0}L^3 = \tilde{s}_\alpha$. Then $C\varepsilon(L^3)$ is a smooth coassociative deformation of $C\varepsilon L^3$ and hence this deformation is analytic in $t$. Now we write $C\varepsilon(L^3) = \tilde{\text{Exp}}(\tilde{s}_\alpha(t,r,x))$, where

$$\tilde{s}_\alpha(t,r,x) = \sum_{k=1}^{\infty} t^k \tilde{s}_{\alpha_k}(r,x).$$

Since $d\tilde{\text{Exp}}_{(r,x)}(v) = v$ for any $v \in T_{(r,x)} CL^3 = T_x L^3$ it follows that

$$\tilde{s}_{\alpha_1}(r,x) = \tilde{s}_\alpha(r,x).$$

Using the argument in the proof of Theorem 4.11 and Lemma 4.10 we have

$$(\mathcal{L}_{\tilde{\varphi}} \tilde{\varphi}^2 + \mathcal{L}_{\tilde{\varphi}} \varphi)|_{C\varepsilon L^3} = 0.$$

By (4.40) we have

$$\mathcal{L}_{\tilde{\varphi}} (\partial r) = 0.$$ 

It follows from (2.4), (4.42) and (4.43)

$$K(\alpha, \alpha) + L^2(\alpha_2(1,x)) \varphi = 0.$$ 

Comparing (4.39) with (4.44) we obtain immediately the first assertion of Lemma 4.13.

Now assume that $K(\alpha, \alpha) = 0$. Set

$$\tilde{\alpha}(t,r,x) := (0, t\alpha(x)) \in N_{(r,x)} CL^3.$$

Then the smooth deformation $\tilde{\text{Exp}}(\tilde{\alpha}(t))$ is coassociative for each $t$. Since the metric on $CM^6$ is warped, the image $\tilde{\text{Exp}}(\tilde{\alpha}(t))$ is a cone for each $t$. Hence the set $\{(1, M^6)\} \cap \tilde{\text{Exp}}(\tilde{\alpha}(t))$ is a Lagrangian submanifold in $M^6$ for each $t$. This completes the proof of Lemma 4.13. $\square$
4.3. **Examples.** Denote by $\times$ the octonionic multiplication. For coordinate fixing, we provide the multiplication law for octonions in the table below taken from [26].

| 1 | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_1$ | $e_1$ | $-1$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $e_7$ | $-e_6$ |
| $e_2$ | $e_2$ | $-e_3$ | $-1$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_3$ | $e_2$ | $-e_1$ | $-1$ | $e_7$ | $e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $1$ | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | $-1$ | $-e_3$ | $e_2$ |
| $e_6$ | $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $-e_2$ | $e_3$ | $-1$ | $-e_1$ |
| $e_7$ | $e_7$ | $-e_6$ | $e_5$ | $e_4$ | $-e_3$ | $-e_2$ | $e_1$ | $-1$ |

Recall that the $G_2$-form $\varphi$ on $\mathbb{R}^7 = \text{Im} \mathbb{O}$ is defined by $\varphi(X,Y,Z) := \langle X \times Y, Z \rangle$. In the basis $(e_1, \cdots, e_7)$, using the multiplication table above, $\varphi$ has the following expression

$$
(4.45) \quad \varphi = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}.
$$

(The expression in (4.45) after the transformation $(e_1 \mapsto -e_1, e_2 \mapsto -e_2, e_3 \mapsto e_7, e_4 \mapsto -e_3, e_6 \mapsto -e_6, e_7 \mapsto e_4)$ coincides with the expression given in [25].)

The almost complex structure $J_0$ on the unit sphere $S^6 \subset \text{Im} \mathbb{O}$ is defined by

$$
J_0|_p(u) := p \times u
$$
for $p \in S^6$ and $u \in T_p S^6 \subset \text{Im} \mathbb{O}$. Let $g_0$ denote the standard metric on $S^6$ as the unit sphere in $\text{Im} \mathbb{O}$. It is well known that $(S^6, J_0, g_0)$ is a nearly Kähler manifold. The group $G_2$ acts transitively on $(S^6, J_0, g_0)$ with isotropy group $SU(3)$.

The simplest example of Lagrangian submanifolds in $(S^6, J_0, g_0)$ is the geodesic sphere $S^3(1)$ in $S^6 \subset \text{Im} \mathbb{O}$, that is defined by the intersection of $S^6$ with the coassociative 4-plane $\mathbb{H} \subset \text{Im} \mathbb{O}$. The geodesic sphere $S^3(1)$ is an orbit of the action of the maximal subgroup $SO(4)_{3,4}$ of the group $G_2$ and $S^3(1)$ is also an orbit of the actions of two non-conjugate subgroups of $SO(4)_{3,4}$. Let us recall a description of the maximal subgroup $SO(4)_{3,4}$ of the group $G_2$, whose representation on $\text{Im} \mathbb{H} \oplus \mathbb{H}$ is given as follows [12, Chapter IV (1.9)]

$$
(4.46) \quad \chi(q_1, q_2)(a, b) = (q_1 a\bar{q}_1, q_2 b\bar{q}_1)
$$
where $(q_1, q_2) \in Sp(1) \times Sp(1)$. Let us denote by $SU(2)_{3,4}$ the Lie subgroup in $SO(4)_{3,4} \subset G_2$ that is equal to $\{ \chi(q_1, 1) | q_1 \in Sp(1) \}$ and by $SU(2)_{0,4}$ the Lie subgroup in $SO(4)_{3,4}$ that is equal to $\{ \chi(1, q_2) | q_2 \in Sp(1) \}$. Then $S^3(1)$ is also an orbit of $SU(2)_{3,4}$ and an orbit of $SU(2)_{0,4}$.

Denote by $g_0$ the Riemannian metric on $S^3(1)$. In what follows we shall describe the eigenvalues of $*_{g_0} d$ and the corresponding 1-forms on $S^3(1)$ and
therefore we shall write \( \ast \) instead of \( \ast_{g_0} \). Our description is based on the work by Folland \[8\], which generalized previous results by Korany-Vagi and Reimann. In Folland’s paper the sphere \( S^3(1) \) is regarded as the quotient \( SO(4)/SO(3) \). The action of \( SO(4) \) on \( S^3 \) extends naturally to an action on the space \( \Omega^1(S^3) \) and its complexification \( \Omega^1(S^3) \otimes \mathbb{C} \).

Denote by \( \mathcal{H}_m \) the space of complex homogeneous harmonic polynomials of degree \( m \) on \( \mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H} \). We write \( z_1 = x_4 + ix_5 \) and \( z_2 = x_6 + ix_7 \). It is known that \( \mathcal{H}_m \) is \( SO(4) \)-invariant and irreducible under the action of \( SO(4) \). Its highest weight is \( (m, 0) \) and its highest vector is \( z_m \).

Denote by \( \Lambda^2_+ \) the space of constant self-dual 2-forms on \( \mathbb{R}^4 = \mathbb{H} \). The orientation on \( \mathbb{H} = \mathbb{C}^2 \) is given by the oriented frame \((e_4, e_5, e_6, e_7)\). Set

\[
\partial r := \sum_{i=1}^{4} x_i \partial x_i
\]

\[
\Theta^+ := \{ \partial r | \alpha | \alpha \in \Lambda^2_+ \}.
\]

Then \( \Theta^+ \) is a representation space of \( SO(4) \) which is equivalent to the representation space \( \text{Im} \mathcal{H} \).

Let \( \vartheta \) be the highest vector of \( \Lambda^2_+ \) under the representation of \( SO(4) \). Denote by \( \mathcal{G}_m \) the smallest invariant \( SO(4) \)-subspace of 1-forms on \( \mathbb{R}^4 = \mathbb{H} \) containing \( z_1^m \otimes (\partial r) \vartheta \subset \mathcal{H}_m \otimes \Theta^+ \).

Let \( \Phi_m \) be the restriction of the space \( \mathcal{G}_m \) to \( S^3(1) \).

**Proposition 4.14.** ([8, Theorems B, C]) The eigenvalues of the operator \( (\ast d) \) on \( (S^3(1), g_0) \) are non-negative integers. Furthermore, for each positive integer \( m \), the set of all 1-forms \( \alpha \in \Omega^1(S^3) \) that satisfies the equation \( \ast d\alpha = m \cdot \alpha \) coincides with the space \( \Phi_{m-1} \).

Now we are ready to prove the following.

**Theorem 4.15.** Let \( \alpha \) be a formal infinitesimal Lagrangian deformation of \( S^3(1) \). Then \( \tilde{K}(\alpha, \alpha) = 0 \). Hence \( \alpha \) is smoothly unobstructed. The Lagrangian sphere \( S^3(1) \) is rigid up to the motion of the automorphism group \( G_2 \) of \( (S^6, J_0, g_0) \).

**Proof.** Let \( s_\alpha \) be a formal Lagrangian deformation of \( S^3(1) \). By Corollary 4.3 and Proposition 4.14 \( \alpha \) belongs to \( \Phi_2 \). The Weyl formula implies that the dimension of \( \Phi_2 \) is equal to 8. By Corollary 3.14 \( s_\alpha \) is a Jacobi field. Denote by \( V_{3,4} \) the orthogonal complement to the Lie algebra \( \mathfrak{so}(3) \oplus \mathfrak{so}(4) \) in \( \mathfrak{so}(7) \) w.r.t. the Killing metric, where \( \mathfrak{so}(3) \) is the Lie algebra of \( SO(\text{Im} \mathbb{H}) \) and \( \mathfrak{so}(4) \) is the Lie algebra of \( SO(\mathbb{H}) \). Elements in \( V_{3,4} \) generate Killing vector fields on \( S^6 \) whose value at \( S^3(1) \) are non-trivial Jacobi fields on \( S^3(1) \). Note that we also have the orthogonal decomposition \( \mathfrak{g}_2 = \mathfrak{so}(4)_{3,4} + \mathbb{R}^3_1 \). (From now on we always omit “Killing” when we say about a metric on \( \mathfrak{so}(7) \) and its compact Lie subalgebras.)
Lemma 4.16. The subspace $\mathbb{R}^8_1$ belongs to $V_{3,4}$. Any formal Lagrangian deformation $s_\alpha$ is generated by a Killing vector field associated with an element in $\mathbb{R}^8_1$.

Proof. We consider $so(7)$ as a $so(4)_{3,4}$-module. Using the table 5 in [34] it is not hard to decompose $so(7)$ into irreducible components as follows

\begin{equation}
so(7) = su(2)_{3,4} \oplus su(2)_{0,4} \oplus \mathbb{R}^3 \oplus \mathbb{R}^8 \oplus \mathbb{R}^4,
\end{equation}

where

- $su(2)_{3,4}, su(2)_{0,4}$ are the Lie algebra of $SU(2)_{3,4}$ and $SU(2)_{0,4}$ respectively,
- $\mathbb{R}^3_1$ is the orthogonal complement to $so(4)_{3,4}$ in $so(3) \oplus so(4),$
- $\mathbb{R}^8 \oplus \mathbb{R}^4$ is the decomposition of $V_{3,4}$ into irreducible components.

Since $\mathbb{R}^8_1$ is a $so(4)_{3,4}$-module, which is orthogonal to $su(2)_{3,4} \oplus su(2)_{0,4}$, we conclude that $\mathbb{R}^8 = \mathbb{R}^8_1$. This proves the first assertion of Lemma 4.16.

The second assertion follows from the fact that Killing fields generated by $\alpha \in \mathbb{R}^9_1$ are infinitesimal Lagrangian deformations of $S^3(1)$. □

Now assume that $s_\alpha$ is a formal infinitesimal deformation of $S^3(1)$ associated with $\bar{\alpha} \in \mathbb{R}^8_1 = V_{3,4} \cap g_2$. Denote by $\Pi_H$ the element in $gl(Im \mathcal{O})$ that defines the orthogonal projection $Im \mathcal{O} \to \mathbb{H}$. Then for $x \in Im \mathcal{O}$ we have

\[ \tilde{V}_\alpha(x) = \bar{\alpha} \circ \Pi_H \cdot x \]

where $\cdot$ denotes the natural action of $gl(7)$ on $Im \mathcal{O}$. It follows that

\[ L_{\tilde{V}_\alpha}(\varphi)(x) = 0 \forall x \in Im \mathcal{O}. \]

This proves the second assertion of Theorem 4.15.

The last assertion of Theorem 4.15 is a consequence of the fact that $s_\alpha$ are generated by elements in $g_2$. □

Remark 4.17. The rigidity of the Lagrangian sphere $S^3(1)$ also follows from the Simons rigidity theorem which states that each geodesic sphere in $S^n$ is rigid as minimal submanifold up to the motion of the isometry group $SO(n + 1)$ [37, Theorem 5.2.3].

We now examine another homogeneous Lagrangian submanifold, the “squashed” sphere $L^3_1$ in $(S^6, J_0, g_0)$. This example has been first considered by Harvey-Lawson in [12, Chapter IV Theorem 3.2] and then later by many others [20, 5, 25]. It is the orbit of $SU(2)_{3,4}$ through the point $(\sqrt{5}/3)e_1 + (2/3)e_5$. The group $SU(2)_{3,4}$ acts on $L^3_1$ without fixed point, therefore we shall identify $L^3_1$ with the Lie group $SU(2)$ provided with a left-invariant Riemannian metric. The sphere $L^3_1$ can be seen as the graph of the Hopf map $S^3 \to S^2$.

Theorem 4.18. The space of formal infinitesimal Lagrangian deformations $s_\alpha$ of $L^3_1$ is identified with the space \{ $\alpha \in \Phi_1$ \}. Any formal Lagrangian deformation $s_\alpha$ of $L^3_1$ is generated by a Killing vector field associated with some element $\bar{\alpha}$ in $g_2$. 
Proof. It follows immediately from the expression
\begin{equation}
(\ast_g d) : \Omega^1(SU(2)) \to \Omega^1(SU(2)), \alpha \mapsto \ast_g d(\alpha),
\end{equation}
that the eigenvalues of \((\ast_g d)\) with respect to a left-invariant Riemannian metric \(g\) coincide with the eigenvalues of \(\ast g_0 d\) after multiplication the latter ones with the ratio \(\text{vol}(SU(2), g)/\text{vol}(SU(2), g_0)\). Here \(g_0\) is the standard round metric of curvature 1 on \(S^3(1) = SU(2)\) that has been induced from the Riemannian metric \(g_0\) on \(S^6 \subset \text{Im} \mathcal{O}\). Straightforward calculations yield that \(\text{vol}(L^3_1)/\text{vol}(S^3(1)) = 2/3\). Taking into account Corollary 4.3 and Proposition 4.14 this implies the first assertion of Theorem 4.18. To prove the second assertion it suffices to observe that the each normal vector to \(T_x L^3_1\), for each \(x \in L^3_1\), is the value of some Killing vector field generated by elements in \(g_2\). This completes the proof of Theorem 4.18 \(\square\)

Theorems 4.15, 4.18 lead us to the following.

Conjecture 2. Each homogeneous Lagrangian submanifold in \((S^6, J_0, g_0)\) is rigid up to the motion of \(G_2\).

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References

[1] C. Baer, Real Killing Spinors and Holonomy, Comm. Math. Phys. 154 (1993), 509-521.
[2] A. Butscher, Deformations of minimal Lagrangian submanifolds with boundary, Proc. Amer. Math. Soc. 131 (2003), 1953-1964.
[3] R. L. Bryant, Minimal Lagrangian submanifolds of Kähler-Einstein manifolds, Lect. Notes in Math., 1255 (1987), 1-12.
[4] S. Chiossi and S. Salamon, The intrinsic torsion of \(SU(3)\) and \(G_2\)-structures, Differential geometry, Valencia, 2001, 115-133, World Sci. Publ., River Edge, NJ, 2002.
[5] F. Dillen, L. Verstraelen and L. Vrancken, Classification of totally real 3-dimensional submanifolds of \(S^6(1)\) with \(K \geq 1/16\), J. Math. Soc. Japan, 42(1990), 565-584.
[6] F. Dillen, L. Verstraelen, Totally real submanifolds of \(S^6\) satisfying Chen’s Equality, Trans. AMS, 348(1996), 1633-1646.
[7] N. Ejiri, Totally real submanifolds in a 6-sphere, Proc. Amer. Math. Soc., 83/4,(1981), 759-763.
[8] G.B. Folland, Harmonic analysis of the de Rham complex on the sphere, J. Reine Angew. Math. 398 (1989), 130-143.
[9] A. Gray, Nearly Kähler manifolds, JDG 4(1970), 283-309.
[10] A. Gray, Structure of nearly Kähler manifolds, Math. Ann., 223 (1976), 233-248.
[11] A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifold and their linear invariants, Ann. Math. Pura App. 123(1980), 35-58.
[12] R. Harvey and H. B. Lawson, Calibrated geometry, Acta Math. 148(1982), 47-157.
[13] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, (1978).
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[14] N. Hitchin, The moduli space of special Lagrangian submanifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 3-4, 503-515 (1998).
[15] D. Joyce, Riemannian holonomy groups and calibrated geometry, Oxford, 2007.
[16] S. Kobayashi and K. Nomizu, Foundations of differential geometry, vol. II, Interscience Publishers, New York-London-Sydney 1969.
[17] V. F. Kirichenko, K-spaces of maximal rank, Mat. Zametki 22 (1977), 465-476.
[18] S. Lang, Fundamental of Differential Geometry, Springer, 1999.
[19] H. V. Lê, Minimal Φ-Lagrangian surfaces in almost Hermitian manifolds, Math USSR Sbornik, 67 (1990), 379-391.
[20] H. V. Lê, Relative calibration and the problem of stability of minimal surfaces, Lect. Notes in Math., Springer-Verlag, 1990, v1453, 245-262.
[21] H.V. Lê, Jacobi equations on minimal homogeneous submanifolds in homogeneous Riemannian spaces, Funct. Anal. Appl. 24 (1990), no. 2, 125-135.
[22] H.V. Lê, A. T. Fomenko, A criterion for the minimality of Lagrangian submanifolds in Kählerian manifolds. Math. Notes, 1987, v.42, 810-816.
[23] H. V. Lê and Y. G. Oh, Deformations of coisotropic submanifolds in locally conformal symplectic manifolds, [arXiv:1208.3590].
[24] A. Lichnerowicz, Theorie globale des connexions et des groupes d’holonomie, Roma, Edizioni Cremonese 1955.
[25] J. Lotay, Ruled Lagrangian submanifolds of 6-sphere, T.A.M.S. 363(2011), 2305-2339.
[26] K. Mashimo, Homogeneous totally real submanifolds of $S^6$, Tsukuba J. Math., 9 (1985), 185-202.
[27] R. McLean, Deformations of Calibrated submanifolds, Comm. in Analysis and Geom. 6 (1998), 705-747.
[28] C. B. Morrey, Second order elliptic system of partial differential equations, 101-160, in Contribution to the theory of Partial differential equations, Ann. of Math. Study, 33, Princeton Univ. Press, Princeton, 1954.
[29] J. M. Morvan, Classes de Maslov d’une immersion lagrangienne et minimalite, C. R. Acad. Sci. Paris, (292)1981, 633-636.
[30] P. A. Nagy, On nearly Kähler manifolds, Ann. Glob. An. Geom. ; 22 (2002), 167-178.
[31] P. A. Nagy, Nearly Kähler geometry and Riemannian foliations, Asian J. Math., 6(2002), 481-504.
[32] Y. G. Oh, Second variation and stability of minimal lagrangian submanifolds in Kähler manifolds, Invent. Math. 101(1990), 501-519.
[33] Y. Ohnita, On stability of minimal submanifolds in compact symmetric spaces, Compositio Mathematica, 64(1987), 157-189.
[34] A.L. Onishchik, E.B. Vinberg, Lie Groups and Algebraic Groups, Springer, New York, 1990.
[35] Y. G. Oh and J. S. Park, Deformations of coisotropic submanifolds and strong homotopy Lie algebroids, Invent. Math. 161(2005), 287-360.
[36] R. Schoen, J. Wolfson, The volume functional for Lagrangian submanifolds. Lectures on partial differential equations, 181-191, New Stud. Adv. Math., 2, Int. Press, Somerville, MA, 2003.
[37] J. Simons, Minimal varieties in Riemannian manifolds, Annals of Math., 88(1968), 62-105.
[38] L. Schäffer and K. Smoczyk, Decomposition and minimality of Lagrangian submanifolds in nearly Kähler manifolds, Annals of Global Analysis and Geometry 37 (2010), 221-240.

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