Abstract. In this paper, we consider infinite words that arise as fixed points of primitive substitutions on a finite alphabet and finite colorings of their factors. Any such infinite word exhibits a “hierarchal structure” that will allow us to define, under the additional condition of strong recognizability, certain remarkable finite colorings of its factors. In particular we generalize two combinatorial results by Justin and Pirillo concerning arbitrarily large monochromatic $k$-powers occurring in infinite words; in view of a recent paper by de Luca, Pribavkina and Zamboni, we will give new examples of classes of infinite words $u$ and finite colorings that do not allow infinite monochromatic factorizations $u = u_1u_2u_3\ldots$.

1. Introduction

Let $A$ be a finite alphabet and $A^*$ be the set of all finite words over $A$. In this paper we consider infinite words that arise as fixed points of primitive substitutions on $A$ and finite colorings of $A^*$. Any such infinite word exhibits a “hierarchal structure”, induced by the underlying substitution, that will allow us to define, under the additional condition of strong recognizability (in the spirit of the recognizability conditions introduced by B. Host [3] and B. Mossé [8]), certain remarkable finite colorings of its factors.

For a two letter alphabet $\{0,1\}$, J. Justin and G. Pirillo [7] constructed a finite coloring $c$ of $\{0,1\}^*$ with respect to which the Thue-Morse word $u^T$ avoids uniform monochromatic 3-powers, that is any word of the form $w_1w_2w_3$ occurring in $u^T$, where the $w_i$ are of equal length, satisfies $c(w_i) \neq c(w_j)$ for some $i$ and $j$. The Thue-Morse word is a fixed point of the substitution $\zeta_T$ defined by $\zeta_T(0) = 01$ and $\zeta_T(1) = 10$. In order to define the coloring $c$, the authors made implicit use of the recognizability properties of this substitution. In Section 3 we will give a generalization of their result: for a large class of substitutions of constant length on a finite alphabet $A$, there always exist finite colorings of $A^*$ with respect to which any fixed word avoids arbitrarily large uniform monochromatic $k$-powers.

Once this established, it is natural to consider next arbitrarily large monochromatic $k$-powers $w_1w_2\ldots w_k$ with uniformly bounded gaps, that is the lengths of the factors $w_i$ are bounded by some $p > 0$ and $p$ is independent of $k$. J. Justin and G. Pirillo [7] showed that there exists a 3-coloring of $\{0,1\}^*$ with respect to which the fixed point $u^J$ of the substitution $\zeta_J$ defined by $\zeta_J(0) = 0001$ and $\zeta_J(1) = 1110$ avoids arbitrarily large monochromatic $k$-powers with uniformly bounded gaps. The main ingredients in their argument are the following. Firstly, $u^J$ avoids abelian 5-powers [6], that is, there does not exist any finite word of the form $w = w_1\ldots w_5$ occurring in $u^J$ where any two of the factors $w_i$ are permutations of each other. Secondly, any very long factor of $u^J$ contains “about equally many 0’s and 1’s”. In Section 4 we will be able to show that, more generally, if an infinite word $u$ over a finite alphabet satisfies

1. $u$ avoids arbitrarily large abelian $k$-powers;
2. each factor of $u$ occurs in $u$ with a uniform frequency;
then \( u \) avoids arbitrarily large monochromatic \( k \)-powers with uniformly bounded gaps. Again, a large class of substitutions gives rise to infinite words satisfying the above properties. The property of uniform frequencies is close related to the property of unique ergodicity of the dynamical system associated to the action of the shift map on \( u \). It is well known that such dynamical system is uniquely ergodic if \( u \) is the fixed word of a primitive substitution [10].

In [4, 5] the authors discussed the following question: given an infinite word \( u \) over a finite alphabet, does there exist a finite coloring of its finite factors which avoids monochromatic factorizations of \( u \)? They showed that this question, which is ultimately motivated by a result of Schutzenberger [12], has a positive answer for all non-uniformly recurrent words and for various classes of uniformly recurrent words. V. Salo and I. Törmä [11] showed that any aperiodic linearly recurrent word \( u \) admits a coloring of its factors that avoids monochromatic factorizations of \( u \) into factors of increasing lengths. We will prove, in Section 5, that the question has also a positive answer for a wide class of infinite words arising as fixed points of primitive substitutions and we shall be able to construct examples that do not fit in those classes of infinite words considered in [4, 5]. Again, the “hierarchal structure” of these words and the strong recognizability condition will play a fundamental role in the construction of our colorings.

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2. Preliminaries

2.1. Monochromatic factorizations. The binary operation obtained by concatenation of two finite words endows \( \mathcal{A}^* \) with a monoid structure (the identity is the empty word \( \emptyset \)). For each \( \alpha \in \mathcal{A} \), the \( \alpha \)-length of a finite word \( w \in \mathcal{A}^* \), which we denote by \( |w|_\alpha \), is the number of occurrences of \( \alpha \) in \( w \). The length of \( w \) is the sum \( |w| = \sum_{\alpha \in \mathcal{A}} |w|_\alpha \) of all its \( \alpha \)-lengths. If a copy of \( w \) occurs in a word \( u \), we say that \( w \) is a factor of \( u \).

Given an infinite word \( u = \alpha_1\alpha_2\alpha_3 \ldots \) over \( \mathcal{A} \), \( \mathcal{L}(u) \) stands for the language of \( u \), that is the set of all nonempty finite factors of \( u \). For each \( i \) and \( j \) satisfying \( 1 \leq i \leq j \), we write

\[ u_{[i,j]} = \alpha_i \alpha_{i+1} \ldots \alpha_j \in \mathcal{L}(u) \]

Let \( c : \mathcal{L}(u) \to \{1, \ldots, r\} \) be a finite coloring of \( \mathcal{L}(u) \).

Definition 1. [7] A finite word \( w \in \mathcal{L}(u) \) is a monochromatic \( k \)-power if there exists a factorization \( w = w_1 w_2 \ldots w_k \), with \( w_i \in \mathcal{L}(u) \), such that \( c(w_i) = c(w_j) \), for all \( i \) and \( j \). A monochromatic \( k \)-power is uniform if \( |w_i| = |w_j| \) for all \( i \) and \( j \). A monochromatic \( k \)-power has gaps bounded by \( p \) if \( |w_i| < p \) for all \( i \).

2.2. Substitutive words. Next we recall some fundamental facts concerning substitutive words. For further details we refer the reader to [10]. A substitution \( \zeta \) of \( \mathcal{A} \) is a map from \( \mathcal{A} \) to \( \mathcal{A}^* \) which associates the letter \( \alpha \) to some word \( \zeta(\alpha) \in \mathcal{A}^* \). This induces a morphism of the monoid \( \mathcal{A}^* \) by putting \( \zeta(\emptyset) = \emptyset \) and

\[ \zeta(\alpha_1 \alpha_2 \ldots \alpha_k) = \zeta(\alpha_1) \zeta(\alpha_2) \ldots \zeta(\alpha_k) \]

For each \( \alpha \in \mathcal{A} \), let \( l_\alpha = |\zeta(\alpha)| \) so that

\[ \zeta(\alpha) = \zeta(\alpha)_1 \zeta(\alpha)_2 \ldots \zeta(\alpha)_{l_\alpha} \].
The substitution has constant length \( l \) if \( l = l_\alpha \) for all \( \alpha \in \mathcal{A} \). The substitution \( \zeta \) is said to be primitive if there exists \( n > 0 \) such that, for every \( \alpha, \beta \in \mathcal{A} \), \( \beta \) occurs in \( \zeta^n(\alpha) \) (that is, \( n \) can be chosen independent of \( \alpha \) and \( \beta \)). Henceforth we will assume that \( \alpha = \zeta(\alpha)_1 \) for some \( \alpha \in \mathcal{A} \).

Consider the space of \( \zeta \)-substitutive infinite words \( X_\zeta = \{ u \in \mathcal{A}^\mathbb{N} : \mathcal{L}(u) \subseteq \mathcal{L}_\zeta \} \), where

\[
\mathcal{L}_\zeta = \bigcup_{n \geq 0, \alpha \in \mathcal{A}} \{ \text{factors of } \zeta^n(\alpha) \},
\]

and endow \( \mathcal{A}^\mathbb{N} \) with the metric \( d \) defined by: given \( u = \alpha_1 \alpha_2 \ldots \) and \( v = \beta_1 \beta_2 \ldots \in \mathcal{A}^\mathbb{N} \), \( d(u, v) = 0 \) if \( u = v \) and \( d(u, v) = 1/N \) if \( N \) is the smallest positive integer for which \( \alpha_N \neq \beta_N \). Denote by \( \sigma \) the shift map in \( \mathcal{A}^\mathbb{N} \),

\[
\sigma(\alpha_1 \alpha_2 \alpha_3 \ldots) = \alpha_2 \alpha_3 \ldots,
\]

which preserves \( X_\zeta \). The substitution \( \zeta \) also induces a map \( \zeta : X_\zeta \to X_\zeta \).

**Proposition 1.** \([10]\) Suppose that \( \zeta \) is a primitive substitution. We have:

1. \( \zeta \) admits a fixed point: \( \zeta(u) = u \), for some \( u \in X_\zeta \).
2. The fixed word \( u \) is uniformly recurrent, that is, for each factor \( w \) of \( u \) there exists \( R > 0 \) such that in any other factor of \( u \) of length \( R \) there is at least one occurrence of \( w \).
3. The system \( (X_\zeta, \sigma) \) is minimal, that is, \( X_\zeta \) is the closure of any orbit under \( \sigma \): \( X_\zeta = \overline{\sigma(v)} \) for all \( v \in X_\zeta \). In particular \( \mathcal{L}_\zeta = \mathcal{L}(v) \) for all \( v \in X_\zeta \).
4. Each factor of \( u \) occurs in \( u \) with a uniform frequency. More precisely, denoting, for each finite interval \( I \), the number of occurrences of the factor \( w \) in \( u_I \) by \( L_w(u_I) \), then

\[
\lim_{N \to \infty} \frac{L_w(u_{[k,k+N]})}{N}
\]

converges uniformly in \( k \) to some number \( d_w \geq 0 \).

In particular, if \( \zeta \) is primitive, for each letter \( \alpha \in \mathcal{A} \), the limit

\[
\lim_{|w| \to \infty} \frac{|w|_{\alpha}}{|w|}
\]

exists and does not depend on the factors \( w \in \mathcal{L}(u) \).

### 2.3. Recognizability.

Let \( \zeta : \mathcal{A} \to \mathcal{A}^* \) be a substitution and \( u \) a fixed point. For each \( k \geq 1 \), let

\[
E_k = \{1\} \cup \{|\zeta^k(u_{[1,p]})| + 1 : p \geq 1\}
\]

be the set of \( k \)-cutting bars of \( \zeta \). Of course, the \( k \)-cutting bars of \( \zeta \) are precisely the 1-cutting bars of \( \zeta^k \). Given an integer interval \( I = [i^-, i^+] \), we denote by \( m(I) \) the least 1-cutting bar of \( \zeta \) greater than or equal to \( i^- \) and \( M(I) \) the largest 1-cutting bar of \( \zeta \) less than or equal to \( i^+ \). Let \( p_I \) and \( q_I \) be defined by

\[
m(I) = |\zeta(u_{[1,p_I]})| + 1, \quad M(I) = |\zeta(u_{[1,q_I]})| + 1.
\]

We say that the interval \( I \) is \( k \)-fitted if \( i^- \) and \( i^++1 \) are \( k \)-cutting bars. Of course, if \( I \) is \( k \)-fitted, then: \( I \) is also \( k' \)-fitted for all \( k' < k \); \( i^- = m(I) \) and \( i^+ = M(I) - 1 \); \( u_I = \zeta(u_{[p_I+1,q_I]}) \). For a general integer interval \( I \), we also define

\[
u_{I,\zeta}^{\text{pref}} = \alpha_{i^-} \ldots \alpha_{m(I)-1}, \quad u_{I,\zeta}^{\text{suf}} = \alpha_{M(I)} \ldots \alpha_{i^+}.
\]
When it is clear in the context which substitution we are referring to, then we simply denote $u_{I,\zeta}^{\text{pref}} = u_I^{\text{pref}}$ and $u_{I,\zeta}^{\text{jiff}} = u_I^{\text{jiff}}$.

**Definition 2.** The substitution $\zeta$ is strongly recognizable if there exists an integer $K$ such that: if $I$ is a 1-fitted interval, $|I| > K$ and $u_I = u_J$, then the integer interval $J$ is also 1-fitted and

$$u_{[p_I+1,q_I]} = u_{[p_J+1,q_J]}.$$ 

The smallest integer $K$ enjoying this property is called the (strong) recognizability index of $\zeta$. Given a strongly recognizable substitution $\zeta$, if $w \in L(u)$, $|w| > K$ and $w = u_I$ for some 1-fitted interval $I$, then we say that $w$ is 1-fitted and we denote $\zeta^{-1}(w) = u_{[p_I+1,q_I]}$ (it is clear that this definition does not depend on the interval $I$).

**Example 1.** The Thue-Morse substitution $\zeta_T$ on $A = \{0,1\}$ defined by $\zeta_T(0) = 01$ and $\zeta_T(1) = 10$ is strongly recognizable. As a matter of fact, if $|I| = 4$, $I$ is 1-fitted if, and only if, $u_I$ coincides with one of the following words: $1001$, $0110$, $1010$, $0101$. On the other hand, $\zeta_T$ has constant length and is one-to-one on letters; consequently, if $I$ is 1 fitted, $|I| \geq 4$ and $u_I = u_J$, then $J$ is also 1-fitted and $w = u_I = u_J$ can be unequivocally “desubstituted”, that is, $u_{[p_I+1,q_I]} = u_{[p_J+1,q_J]}$.

**Lemma 1.** Let $\zeta : A \to A^*$ be a strongly recognizable substitution with recognizability index $K$. Let $L = \max\{|\zeta(\alpha)| : \alpha \in A\}$. Suppose that $u_I = u_J$, with $|I| = |J| > 2L + K$. Then

1. $u_I^{\text{pref}} = u_J^{\text{pref}}$
2. $u_I^{\text{jiff}} = u_J^{\text{jiff}}$
3. $u_{[m(I),M(I)-1]} = u_{[m(J),M(J)-1]}$

**Proof.** For $I = [i^-,i^+]$ and $J = [j^-,j^+]$, take $n = j^- + m(I) - i^-$. Since $|I| > 2L + K$, it is clear that $[m(I), M(I) - 1]$ has length $r_I = M(I) - m(I)$ larger than $K$. Observe that $[n, n + r_I - 1] \subset J$ and write

$$u_I = u_{[i^-,m(I)-1]}u_{[m(I),M(I)-1]}u_{[M(I),i^+]}, \quad u_J = u_{[j^-,n-1]}u_{[n,n+r_I-1]}u_{[n+r_I,j^+]}.$$ 

From the definition of $n$, we have $|u_{[i^-,m(I)-1]}| = |u_{[j^-,n-1]}|$. Hence, since $u_I = u_J$, we get

$$u_{[i^-,m(I)-1]} = u_{[j^-,n-1]}, \quad u_{[m(I),M(I)-1]} = u_{[n,n+r_I-1]}, \quad u_{[M(I),i^+]} = u_{[n+r_I,j^+]}. $$

Consequently, by recognizability, the interval $[n, n + r_I - 1]$ is 1-fitted. Moreover, we must have $n = m(J)$. Otherwise, by definition of $m(J)$, we would have $n > m(J)$; but in this case, again by recognizability and reversing the rules of $I$ and $J$, $n' = i^- + m(J) - j^-$ would be a 1-cutting bar satisfying $i^- \leq n' < m(I)$, which is in contradiction with the definition of $m(I)$. Hence $n = m(J)$, $u_I^{\text{pref}} = u_J^{\text{pref}}$, and we conclude that $u_I^{\text{pref}} = u_J^{\text{pref}}$. The remaining assertions of the lemma follow as a consequence of this.

Given a strongly recognizable substitution $\zeta$ with recognizability index $K$ and a 2-fitted interval $I$ (equivalently, $I$ is 1-fitted with respect to $\zeta^2$), assume that $|I| > LK$, where $L = \max\{|\zeta(\alpha)| : \alpha \in A\}$. Suppose that $u_I = u_J$. By recognizability, the interval $J$ is 1-fitted and $u_{[p_I+1,q_I]} = u_{[p_J+1,q_J]}$. Set $I' = [p_I+1,q_I]$ and $J' = [p_J+1,q_J]$. Since $I$ is 2-fitted, $I'$ is 1-fitted. On the other hand, $|I'| > |I|/L > K$. Hence, again by recognizability, $J'$ is 1-fitted (consequently, $J$ is 2-fitted) and $u_{[p_I+1,q_I]} = u_{[p_J+1,q_J]}$. This proves that $\zeta^2$ is strongly recognizable. More generally, this argument can be extended in order to prove by induction the following.
Proposition 2. Suppose that \( \zeta \) is strongly recognizable. Then \( \zeta^k \) is strongly recognizable for each integer \( k > 1 \).

We say that a substitution is \textit{admissible} if it is primitive and strongly recognizable.

Proposition 3. If \( \zeta \) is admissible, any infinite word \( v \) in \( X_\zeta \) is not shift periodic, that is there does not exist a finite word \( w \) with \( v = w^\infty = ww \ldots \).

\textit{Proof.} The argument is standard. Let \( u = \alpha_1 \alpha_2 \alpha_3 \ldots \) be a fixed point of \( \zeta \) and suppose that \( v \in X_\zeta = \overline{\mathcal{O}(u)} \) is periodic. Since \( v \) is periodic, there exists \( n_0 \) so that \( \sigma^{n_0}(v) = v \). By primitivity, we can choose \( n \) satisfying \( |\zeta^n(\alpha)| > 2n_0 \) for all \( \alpha \in A \). By minimality, there exists an arbitrarily large interval \( I = [i^-, i^+] \) so that \( v = u_I v \), where \( v \) is an infinite factor of \( v \). Assume that \( |I| \) is much larger than the recognizability index of \( \zeta^n \). Write
\[
u_I = u_{I,\zeta^n}^{\text{pref}} \zeta^n(\alpha_{p_I+1}) \zeta^n(\alpha_{p_I+2}) \ldots \zeta_n(\alpha_{q_I}) u_{I,\zeta^n}^{\text{suf}},
\]
where \( p_I \) and \( q_I \) are defined with respect to the substitution \( \zeta^n \). Write now two cases. Firstly, if \( n_0 \leq |u_{I,\zeta^n}^{\text{pref}}| \), then, since \( \sigma^{n_0}(v) = v \), we have
\[
u_{I,\zeta^n}^{\text{pref}} \zeta^n(\alpha_{p_I+1}) \zeta^n(\alpha_{p_I+2}) \ldots = w \zeta^n(\alpha_{p_I+1}) \zeta^n(\alpha_{p_I+2}) \ldots,
\]
where \( w \) is the prefix of \( u_{I,\zeta^n}^{\text{pref}} \) with \( |w| = |u_{I,\zeta^n}^{\text{pref}}| - n_0 < |u_{I,\zeta^n}^{\text{pref}}| \), which contradicts the recognizability property of Lemma 1. Secondly, if \( n_0 > |u_{I,\zeta^n}^{\text{pref}}| \), then we can write, since \( \sigma^{n_0}(v) = v \),
\[
u_{I,\zeta^n}^{\text{pref}} \zeta^n(\alpha_{p_I+1}) \zeta^n(\alpha_{p_I+2}) \ldots = w \zeta^n(\alpha_{p_I+2}) \zeta^n(\alpha_{p_I+3}) \ldots,
\]
where \( w \) is the suffix of \( \zeta^n(\alpha_{p_I+1}) \) with
\[|w| = |\zeta^n(\alpha_{p_I+1})| - (n_0 - |u_{I,\zeta^n}^{\text{pref}}|).
\]
Since \( |\zeta^n(\alpha)| > 2n_0 \) for all \( \alpha \in A \), we see from this that \( |w| > n_0 > |u_{I,\zeta^n}^{\text{pref}}| \), which contradicts the recognizability property of Lemma 1. We conclude that \( v \) cannot be periodic. \( \square \)

3. AVOIDING ARBITRARILY LARGE UNIFORM MONOCHROMATIC \( k \)-POWERS

J. Justin and G. Pirillo in [7] constructed a finite coloring of the factors of the Thue-Morse word which avoids uniform monochromatic 3-powers. The Thue-Morse word is a fixed point of the admissible substitution \( \zeta_T \) on \( A = \{0, 1\} \) defined by \( \zeta_T(0) = 01 \) and \( \zeta_T(1) = 10 \). Next we generalize this construction to a large class of substitutions \( \zeta \) on a finite alphabet.

Theorem 1. Let \( \zeta \) be an admissible substitution of constant length \( L \) on a two-letter alphabet \( A \) with recognizability index \( K \). Let \( u = \alpha_1 \alpha_2 \ldots \) be a fixed point of \( \zeta \). Then there exists a finite coloring of \( \mathcal{L}(u) \) avoiding uniform monochromatic \( k \)-powers for all \( k \) sufficiently large.

\textit{Proof.} We start by observing that an admissible substitution \( \zeta \) on a two-letter alphabet \( A = \{0, 1\} \) is always one-to-one on letters (otherwise the fixed word would be periodic, in contradiction with Proposition 3). Hence, for some \( r > 0 \), we have \( \zeta(0)_r \neq \zeta(1)_r \), which means that the substitutions \( \zeta_r^{\text{pref}} \) and \( \zeta_r^{\text{suf}} \) on \( A \) defined by
\[
\zeta_r^{\text{pref}}(\alpha) = \zeta(\alpha_1) \zeta(\alpha_2) \ldots \zeta(\alpha)_r, \quad \zeta_r^{\text{suf}}(\alpha) = \zeta(\alpha)_r \zeta(\alpha)_{r+1} \ldots \zeta(\alpha)_L,
\]
with \( \alpha \in \{0, 1\} \), are both one-to-one on letters.

Define a finite coloring \( c \) of \( \mathcal{L}(u) \) as follows. For \( w \in \mathcal{L}(u) \):
(1) if \(|w| \leq 2L + K\), then \(c(w) = w\).
(2) if \(|w| > 2L + K\) and \(w = u_I\) for some integer interval \(I\), we have three cases.
   (a) if \(0 < |u_I^{\text{suf}}| + |u_I^{\text{pref}}| \neq L\), then \(c(w) = (u_I^{\text{suf}}, u_I^{\text{pref}})\);
   (b) if \(I\) is 1-fitted, that is \(|u_I^{\text{suf}}| = |u_I^{\text{pref}}| = 0\), then \(c(w) = c(\zeta^{-1}(u_I))\);
   (c) if \(|u_I^{\text{suf}}| + |u_I^{\text{pref}}| = L\), then \(c(w) = c(u_{\tau(I)})\), where

\[
\tau(I) = \begin{cases} 
    [m(I), \overline{M}(I) - 1] & \text{if } |u_I^{\text{suf}}| \geq r \\
    [\underline{m}(I), M(I) - 1] & \text{if } |u_I^{\text{suf}}| < r 
\end{cases}
\]

Here \(\overline{M}(I)\) is the successor of \(M(I)\) and \(\underline{m}(I)\) is the predecessor of \(m(I)\) in the sequence of 1-cutting bars of \(\zeta\).

**Lemma 2.** The finite coloring \(c\) is well-defined.

**Proof.** We have to check that, for \(|w| > 2L + K\), the definition of \(c(w)\) does not depend on \(I\). This is a direct consequence of our recognizability condition (cf. Lemma 1) when \(0 < |u_I^{\text{suf}}| + |u_I^{\text{pref}}| \neq L\).

If \(|u_I^{\text{suf}}| = |u_I^{\text{pref}}| = 0\), that is \(I = [m(I), M(I) - 1]\), we have

\[u_I = \zeta(\alpha_{p+1}) \ldots \zeta(\alpha_q)\]

and \(\zeta^{-1}(u_I) = \alpha_{p+1} \ldots \alpha_q\). Take any interval \(J\) such that \(u_I = u_J\). By recognizability, \(J\) is also 1-fitted and and we have \(\zeta^{-1}(u_J) = \zeta^{-1}(u_I)\).

For the case (c), we can write

\[u_{[m(I), \overline{M}(I) - 1]} = u_{[m(I), M(I) - 1]} u_{[M(I), \overline{M}(I) - 1]}\]

with \(u_{[M(I), \overline{M}(I) - 1]} = \zeta(\alpha)\) for some letter \(\alpha \in A\). If \(|u_I^{\text{suf}}| \geq r\), such letter is uniquely determined by \(u_I^{\text{suf}}\), since \(\zeta^{\text{pref}}\) is one-to-one on letters. Hence, if \(u_I = u_J\), from Lemma 1 we conclude that

\[u_{[m(I), \overline{M}(I) - 1]} = u_{[m(I), M(I) - 1]} \zeta(\alpha) = u_{[m(I), M(I) - 1]} \zeta(\alpha) = u_{[m(I), \overline{M}(I) - 1]}\]

Similarly, if \(|u_I^{\text{suf}}| < r\) and \(u_I = u_J\) we have \(u_{[\underline{m}(I), M(I) - 1]} = u_{[\underline{m}(I), M(I) - 1]}\). Hence the definition of \(c(w)\) does not depend on \(I\). \(\square\)

Next we prove that the finite coloring \(c\) avoids uniform monochromatic \(k\)-powers for all \(k\) sufficiently large. Suppose that, contrary to our claim, there exists a uniform monochromatic \(k\)-power \(w_1 w_2 \ldots w_k\), with \(|w_i| \leq 2L + K\), for each \(k\). By definition of \(c\), we have in this case \(c(w_i) = c(w_j)\) if and only if \(w_i = w_j\). Hence \(w_1 w_2 \ldots w_k = w_1^k\). Since the subset of factors \(w_1 \in L(u)\) with \(w_1 \leq 2L + K\) is finite, there exists \(w \in L(u)\) such that \(w^k\) occurs in \(u\) for all \(k > 0\). But this means that the periodic word \(w^\infty = w w \ldots\) is in the closure of the \(\sigma\)-orbit of \(w\), which contradicts Proposition 3.

Assume now that \(w_1 w_2 \ldots w_k\) is a uniform \(k\)-power of \(u\) with \(|w_i| > 2L + K\). Set \(w_i = u_{I_i}\), with \(I_1, I_2, \ldots, I_k\) consecutive intervals. If \(|u_{I_i}|\) is not a multiple of \(L\), then, since the substitution \(\zeta\) is of constant length \(L\), we must have \(|u_{I_i}^{\text{suf}}| + |u_{I_i}^{\text{pref}}| \neq L\) for all \(i\). In this case, it is clear from the definition of \(c\) that \(c(w_i) \neq c(w_{i+1})\): as a matter of fact, since \(|u_{I_i}^{\text{suf}}| + |u_{I_i}^{\text{pref}}| = L\) and \(|u_{I_i}^{\text{suf}}| + |u_{I_i}^{\text{pref}}| \neq L\), we have \(|u_{I_i}^{\text{pref}}| \neq |u_{I_{i+1}}^{\text{pref}}|\). Note that, in this particular case, it is sufficient to take \(k = 2\).

If \(|u_{I_i}|\) is multiple of \(L\), then we have either \(|u_{I_i}^{\text{suf}}| + |u_{I_i}^{\text{pref}}| = L\) for all \(i\) or \(|u_{I_i}^{\text{suf}}| + |u_{I_i}^{\text{pref}}| = 0\) for all \(i\). In both cases, after a finite number of “translations” \(\tau\) (observe that, for a such uniform
Remark 1. Theorem 1 holds for admissible substitutions of constant length $L$ on an arbitrary finite alphabet $A$ if one considers the following additional condition: for some $r \in \{1, \ldots, L\}$, the substitutions $\zeta_{\text{pref}}^r$ and $\zeta_{\text{suf}}^r$ on $A$ defined by (1) are both one-to-one on letters. The proof is exactly the same.

Remark 2. If $\zeta$ is admissible, then $L_\zeta = L(u)$, where $u$ is the fixed point of $\zeta$, and $X_\zeta = \overline{O(u)}$. Consequently, with respect to the finite coloring $c$ of $L(u)$ defined above, any $v \in \overline{O(u)}$ avoids arbitrarily large uniform monochromatic $k$-powers.

Example 2. The one-to-one substitution $\zeta$ on the alphabet $\{0, 1\}$ defined by
$$\zeta(0) = 0100, \quad \zeta(1) = 1101$$
is an admissible substitution of constant length $L = 4$ with fixed word
$$u = 0100110100011001 \ldots$$
Since the factors 0100 and 1101 can only occur in 1-fitted intervals, $\zeta$ is strongly recognizable, with recognizability index $K = 0$. Taking $r = 1$ and the coloring $c$ of $L(u)$ as defined in the proof of the previous theorem, we see that $u$ avoids arbitrarily large uniform monochromatic $4$-powers. More precisely, we can take $k = 4$. In order to check that $u$ avoids uniform monochromatic 4-powers, one only has to investigate factors of $u$ of the form $w_1w_2w_3w_4$ with $l = |w_i| \leq 8$. In this case, $w_1w_2w_3w_4$ is a uniform monochromatic 4-power if, and only if, $w_1w_2w_3w_4 = w_4^1$. For example, factors of the form $1^4$ and $0^4$ do not occur in $u$, and a similar analysis can be done for the remaining values of $l \leq 8$.

4. AVOIDING ARBITRARILY LARGE MONOCHROMATIC $k$-POWERS WITH UNIFORMLY BOUNDED GAPS.

J. Justin and G. Pirillo [7] considered the infinite word
$$u_J = 00001000010000100001111 \ldots,$$
which is a fixed point of the substitution $\zeta_J$ defined by $\zeta_J(0) = 00001$ and $\zeta_J(1) = 11110$, and constructed a 3-coloring of its factors which avoids arbitrarily large monochromatic $k$-powers with uniformly bounded gaps. In this section we generalize this construction to a certain class of infinite words over a finite alphabet $A$. Recall that $w = w_1w_2 \ldots w_k \in A^*$ is an abelian $k$-power if any two of the factors $w_i$ are permutations of each other.

Theorem 2. Let $u$ be an infinite word over $A$. Assume that each factor of $u$ occurs in $u$ with a uniform frequency and that $u$ avoids abelian $k_0$-powers for some $k_0 > 0$. Then there exists a finite coloring $c$ of $L(u)$ which avoids arbitrarily large monochromatic $k$-powers with uniformly bounded gaps.

Proof. By hypothesis, any letter $\alpha \in A$ occurs in $u$ with uniform frequency, say $d_\alpha > 0$. We have $\sum_{\alpha \in A} d_\alpha = 1$. Define the following finite coloring of $L(u)$:
$$c(w) = \{\alpha \in A : \frac{|w_\alpha|}{|w|} > d_\alpha\}.$$
In particular, if \( c(w) = \emptyset \) then \( \frac{|w|}{|w|} = d_\alpha \) for all \( \alpha \in \mathcal{A} \). Toward a contradiction, suppose that \( c \) admits arbitrarily large monochromatic \( k \)-powers with uniformly bounded gaps. Then there exists \( p > 0 \) such that \( u \) admits a monochromatic \( k \)-power \( w = w_1 w_2 \ldots w_k \) with gaps bounded by \( p \), for arbitrarily large \( k \).

We claim that, in the monochromatic \( k \)-power \( w = w_1 w_2 \ldots w_k \), we must have \( c(w_i) = \emptyset \) for all \( i \). As a matter of fact, otherwise we would have, for some letter \( \alpha \), \( |w_i|_\alpha > d_\alpha |w_i| \) for all \( i \) and arbitrarily large \( k \). Since we are supposing that \( |w_i| < p \), then there would exist some \( \epsilon_0 > 0 \) such that \( |w_i|_\alpha > d_\alpha |w_i| + \epsilon_0 \) for all \( i \) and arbitrarily large \( k \). Consequently,

\[
\frac{|w_i|_\alpha}{|w_i|} > d_\alpha + \frac{\epsilon_0}{p},
\]

which contradicts the hypothesis on the uniformity of letter frequencies.

Therefore we must have \( c(w_i) = \emptyset \) for all \( i \), and we can mimic the argument used by J. Justin and G. Pirillo (Theorem 2, [7]) in order to conclude our proof. For completeness, next we present the details. Consider the alphabet \( \mathcal{A}_p = \{ v \in \mathcal{A}^* : |v| < p \} \) and the morphism \( \varphi : \mathcal{A}_p^* \to \mathbb{N} \) given by \( \varphi(v) = |v| \), where the length of \( v \) is taken with respect to the alphabet \( \mathcal{A} \). By Theorem 4.2.1 of [9], \( \varphi \) is repetitive. Recall that, given an alphabet \( \mathcal{A} \), a mapping \( \tilde{\varphi} : \mathcal{A}^* \to S \) to a set \( S \) is repetitive if for each \( k_0 \), there exists an integer \( k \), such that each word \( v \in \mathcal{A}^* \) of length \( k \) contains a factor of the form \( v_1 \ldots v_{k_0} \), with \( v_i \in \mathcal{A}^* \) and \( \tilde{\varphi}(v_1) = \cdots = \tilde{\varphi}(v_{k_0}) \). In our setting, this means that for \( k \) sufficiently large, there exist integers \( i_1, i_2, \ldots, i_{k_0}, i_{k_0+1} \) such that the consecutive factors \( v_1, v_2 \) and \( v_3 \) of \( w = w_1 w_2 \ldots w_k \) defined by

\[
v_1 = w_{i_1} w_{i_1+1} \ldots w_{i_2-1}, \quad v_2 = w_{i_2} w_{i_2+1} \ldots w_{i_3-1}, \ldots, \quad v_{k_0} = w_{i_{k_0}} w_{i_{k_0}+1} \ldots w_{i_{k_0+1}-1}
\]

have the same length \( L = |v_1| = \ldots = |v_{k_0}| \). On the other hand, since \( c(w_i) = \emptyset \), we have \( |w_i|_\alpha = d'_\alpha |w_i| \) for all \( \alpha \in \mathcal{A} \). This fact implies that \( |v_i|_\alpha = d_\alpha |v_i| = d_\alpha L \), for all \( i \). Hence any two of the factors \( v_i \) are permutations of each other, that is \( v_1 v_2 \ldots v_{k_0} \) is an abelian \( k_0 \)-power, in contradiction with our hypothesis.

\[\square\]

**Example 3.** Consider the substitution \( \zeta \) on three letters defined by

\[
\zeta(a) = aabc, \quad \zeta(b) = bbc, \quad \zeta(c) = acc.
\]

This is a primitive substitution whose fixed word

\[
u = aabcaabcbbccaccaabcaabcbacccbcbcbbc\ldots
\]

avoids abelian 3-powers [2]. By Proposition 11 each factor of \( u \) occurs in \( u \) with a uniform frequency. Hence, there exists a finite coloring of \( \mathcal{L}(u) \) which avoids arbitrarily large monochromatic \( k \)-powers with uniformly bounded gaps.

5. **Factorizations of infinite words**

Let \( u \) be an infinite word over a finite alphabet \( \mathcal{A} \), and \( c \) be any finite coloring of \( \mathcal{L}(u) \). A known consequence of Ramsey's Theorem (for infinite complete graphs) is the following.

**Theorem 3.** [12] There exists a factorization \( u = vu_1 u_2 u_3 \ldots \) with \( c(u_1) = c(u_2) = c(u_3) = \ldots \).
Inspired by this result, T. Brown [1] and L. Zamboni [13] posed independently the following question: Given an infinite word over the alphabet $A$, does there exist a finite coloring of $\mathcal{L}(u)$ which avoids monochromatic factorizations of $u$? Let $P$ denote the set of all infinite words over a finite alphabet for which this question has a positive answer. That is, for each $u \in P$ there exists a finite coloring $c$ of $\mathcal{L}(u)$ satisfying the property that for each factorization $u = u_1 u_2 u_3 \ldots$ there exist $i, j$ for which $c(u_i) \neq c(u_j)$. In [5] the authors proved that all non-uniformly recurrent words and various classes of non-periodic uniformly recurrent words are in $P$: non-periodic balanced words, and all words $u \in A^\mathbb{N}$ satisfying $\lambda_u(n+1) - \lambda_u(n) = 1$ for all sufficiently large $n$, where $\lambda_u(n)$ denotes the number of distinct factors of $u$ of length $n$. Recall that an infinite word $u$ is balanced if for any two finite factors $w_1$ and $w_2$ of $u$ with the same length, we have $|w_1|_\alpha - |w_2|_\alpha \leq 1$ for any letter $\alpha \in A$. Next we prove that all substitutive words associated to admissible substitutions on a finite alphabet avoiding arbitrarily large $k$-powers also belong to $P$.

**Theorem 4.** Let $\zeta$ be an admissible substitution on a finite alphabet $A$, with recognizability index $K$. Let $u$ be a fixed point of $\zeta$. Then $u \in P$.

**Proof.** Let $L_k = \max\{|\zeta^k(\alpha)| : \alpha \in A\}$. For $k = 1$ we also denote $L = L_1$. In view of Proposition 3 we can fix $k_0$ such that no word of the form $w^{k_0}$ with $|w| \leq 2L + K$ occurs in $u$. Fix an integer $q$ satisfying $|\zeta^q(\alpha)| > k_0(K + 2L)$ for all $\alpha \in A$, and fix another integer $P > q$ satisfying $|\zeta^P(\alpha)| > 2Lq + Kq$ for all $\alpha \in A$, where $K_q$ is the recognizability index of $\zeta^q$ (cf. Proposition 2).

Define a finite coloring $c$ of $\mathcal{L}(u)$ as follows. Observe first that if $v$ is a prefix of $u$, then $v$ is also a prefix of $\zeta^k(v)$, since $u$ is a fixed point of $\zeta$. For $w \in \mathcal{L}(u)$:

1. if $w$ is not a prefix of $u$, then $c(w) = 0$.
2. if $w$ is a prefix of $u$, we have three cases.
   a. if $|w| \leq 2L + K$, then $c(w) = w$;
   b. if $w = \zeta^k(v)$ for some $k > 0$ and some prefix $v$ of $u$ satisfying $|v| \leq 2L + K < |\zeta(v)|$, then $c(w) = (v, k \mod P)$;
   c. otherwise $c(w) = 1$.

We claim that $u$ avoids monochromatic factorizations with respect to $c$. Suppose, on the contrary, that $u = u_1 u_2 u_3 \ldots$ is a monochromatic factorization. Since $u_1$ is a prefix of $u$, we must have $c(u_1) \neq 0$ and consequently $c(u_i) \neq 0$ for all $i$. On the other hand, if $|u_i| < 2L + K$ for some $i$, then, by definition of $c$, we would have $w = u_i$ for all $i$ and consequently $u = w^\infty$, in contradiction with Proposition 3.

So we can suppose that each $u_i$ is a prefix of $u$ and $|u_i| > 2L + K$. Write

$$u_1 = u_{[1,|u_1|]}, \quad u_2 = u_{[|u_1|+1,|u_1|+|u_2|]} = u_{[1,|u_2|]}, \quad u_3 = u_{[|u_1|+|u_2|+1,|u_1|+|u_2|+|u_3|]} = u_{[1,|u_3|]}, \ldots$$

By recognizability (cf. Lemma 1), this means that $u_i^{\text{pref}} = \emptyset$ for each $i$, which implies that each factor $u_i$ is 1-fitted.

Assume that $c(u_i) = 1$ for all $i$. Take $r_i > 0$ such that the factor $\zeta^{-r_i}(u_i)$ is not 1-fitted. Clearly $|\zeta^{-r_i}(u_i)| > 2L + K$ (otherwise $c(u_i) \neq 1$) and $c(\zeta^{-r_i}(u_i)) = 1$. Take $r = \min \{r_i\}$. The factorization $u = \zeta^{-r}(u_1) \zeta^{-r}(u_2) \zeta^{-r}(u_3) \ldots$ is also monochromatic (all factors with color 1), and some factor $\zeta^{-r}(u_i)$ is not 1-fitted, which, as we have seen, is impossible.
So we must have \( c(u_i) = (w, k) \) for some prefix \( w \), with \( |w| \leq 2L + K < |\zeta(w)| \), and \( 0 \leq k < P \). In this case the factorization is of the form
\[
(2) \quad u = \prod_{i \geq 1} (\zeta^{k+n_i}P(w))^{s_i},
\]
with \( n_i \geq 0 \).

We claim that the set \( \{i \colon n_i+1 > n_i\} \) is nonempty. Towards a contradiction, suppose that \( n_i+1 \leq n_i \) for all \( i \). In this case, there exists \( i_0 \) such that \( n_i = n_{i_0} \) for all \( i \geq i_0 \), and, in view of (2), we can write \( u = v(\zeta^{k+n_{i_0}}P(w))^\infty \), for some prefix \( v \) of \( u \), which contradicts Proposition 3.

Let \( i_0 = \min\{i : n_{i+1} > n_i\} \).

For simplicity of exposition we assume that \( i_0 = 1 \), but the argument below holds for any other possible value of \( i_0 \). Applying \( \zeta^{-(k+n_{i_0})} \) to (2), we obtain
\[
u = w^{s_1}(\zeta^{(n_2-n_1)}P(w))^{s_2} \ldots .
\]

Now, since \( \zeta^{(n_2-n_1)}P(w) \) is a prefix of the fixed point \( u \) and \( P \leq (n_2 - n_1)P \), we have \( \zeta^{(n_2-n_1)}P(w) = u_{\lfloor (\zeta^{(n_2-n_1)}P(w)) \rfloor} \) and \( |\zeta^{(n_2-n_1)}P(w)| \geq |\zeta^P(w)| > 2Lq + Kq \).

Hence, by recognizability (see Lemma 2), we see that \( |w^{s_1}| + 1 \) is a \( q \)-cutting bar. But
\[
1 \leq |w^{s_1}| = s_1|w| \leq k_0(K + 2L) < |\zeta^q(\alpha)|,
\]
for all \( \alpha \in A \), which means that \( |w^{s_1}| + 1 \notin E_q \). This is a contradiction, and consequently we conclude that \( u \) avoids monochromatic factorizations with respect to \( c \). \( \square \)

**Example 4.** Consider the primitive substitution \( \zeta \) on the alphabet \( \{0, 1\} \) defined by \( \zeta(0) = 0100 \) and \( \zeta(1) = 1101 \), as in Example 2. Let \( u \) be a fixed point of \( \zeta \). This is a uniformly recurrent, non-periodic and non-balanced infinite word. As a matter of fact: uniform recurrence and non-periodicity is a consequence of \( \zeta \) being admissible; one can easily check that \( \lambda(1) = 2^n \), whereas \( \lambda(0) = 4^n \), which means that \( u \) is non-balanced. On the other hand, we have
\[
\lambda_u(2) - \lambda_u(1) = 2, \quad \lambda_u(3) - \lambda_u(2) = 4, \quad \lambda_u(4) - \lambda_u(3) = 5, \ldots .
\]
But, by Theorem 3 of [8] (which holds only for substitutions of constant length), we know that the sequence \( \lambda_u(n+1) - \lambda_u(n) \) takes values in a finite set and each of these values occurs for infinitely many \( n \). Hence the infinite word \( u \) certainly does not satisfy \( \lambda_u(n+1) - \lambda_u(n) = 1 \) for all sufficiently large \( n \). We conclude, by Theorem 4, that \( u \) provides an example of a non-periodic infinite word in \( \mathcal{P} \) which is not considered in the paper of de Luca, Pribavkina, and Zamboni [3].

**Example 5.** Consider the substitution \( \zeta \) on the alphabet \( \{a, b, c\} \) defined by \( \zeta(a) = aabc \), \( \zeta(b) = bbc \), and \( \zeta(c) = acc \), as in the Example 3. This is an admissible substitution, and consequently \( u \) is non-periodic. Since \( aabc, bbc \) and \( acc \) only occur in 1-fitted intervals, \( \zeta \) is strongly recognizable. Hence \( u \in \mathcal{P} \). It is known that \( u \) avoids abelian 3-powers [2].

**Remark 3.** In [4], the authors considered the class \( \mathcal{P}_1 \) of all infinite words over finite alphabets admitting a prefixal factorization. They associated to each \( u \in \mathcal{P}_1 \) a “derived” infinite word \( \delta(u) \), which may or may not belong to \( \mathcal{P}_1 \), and defined the class \( \mathcal{P}_\infty \) of all words \( u \) in \( \mathcal{P}_1 \) such that \( \lambda_\delta(u) \in \mathcal{P}_1 \) for all \( n \geq 1 \). de Luca and Zamboni [4] showed that any word \( u \) which does not belong to \( \mathcal{P}_\infty \) admits a finite coloring of its factors which avoids monochromatic factorizations of \( u \), that is \( u \in \mathcal{P}_\infty \).
Recall that the Fibonacci word and the Tribonacci word, respectively, are fixed points of the admissible substitutions $\zeta_f(0) = 01, \zeta_f(1) = 0$ and $\zeta_t(1) = 12, \zeta_t(2) = 13, \zeta_t(3) = 1$, respectively. Consequently, taking Theorem 4 into account, we see that $u \in \mathcal{P}$. On the other hand, the Fibonacci word and the Tribonacci word belong to $\mathcal{P}_\infty$ [4]. Hence, Theorem 4 does not follow from de Luca and Zamboni results.

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