MIZUNO-TYPE RESULT AND WALLIS’ FORMULA

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Abstract. Let \( \tilde{\Gamma}(z) \) be the modified gamma function introduced by the authors in a recent preprint “arXiv2106.14674”. In this note, we obtain the following Mizuno-type result:

\[
\prod_{m=0}^{\infty} \left\{ \prod_{j=1}^{n} (m + z_j) \right\}^{(-1)^m} = \frac{(\sqrt{\frac{\pi}{2}})^n}{\prod_{j=1}^{n} \tilde{\Gamma}(z_j)},
\]

which imply a Kurokawa–Wakayama type formula

\[
\prod_{m=0}^{\infty} ((m + x)^n - y^n)^{(-1)^m} = \frac{(\sqrt{\frac{\pi}{2}})^n}{\prod_{\zeta=1}^{n} \tilde{\Gamma}(x - \zeta y)}
\]

and a Lerch-type formula

\[
\prod_{m=0}^{\infty} (m + x)^{(-1)^m} = \frac{\sqrt{\frac{\pi}{2}}}{\Gamma(x)}.
\]

By setting \( x = 1 \) in the above result, we recover Wallis’ 1656 formula

\[
\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \cdots}{1 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots} = \frac{\pi}{2}.
\]

1. Introduction

1.1. Lerch’s formula. Lerch’s 1894 formula asserts that

\[
\prod_{m=0}^{\infty} (m + x) = \frac{\sqrt{2\pi}}{\Gamma(x)}
\]

in the sense of zeta regularization, where \( \Gamma(x) \) denotes the Euler gamma function.

Letting \( x = 1 \) in (1.1) we get the following interesting result

\[
\prod_{n=1}^{\infty} n = \sqrt{2\pi}.
\]

In [9, Corollary 9.13] the authors named its Riemann’s formula, because it comes from Riemann’s result in 1859:

\[
\zeta'(0) = -\frac{1}{2} \log(2\pi).
\]
Lerch himself extended (1.1) to the Gaussian quadratic field \( \mathbb{Q}(i) \) as follows:

\[
\prod_{m=0}^{\infty} \left((m+x)^2 + y^2\right) = \frac{2\pi}{\Gamma(x+iy)\Gamma(x-iy)}.
\]

Then in 2004 Kurokawa and Wakayama [8] proved the following generalization of (1.4) to any cyclotomic fields \( \mathbb{Q}(\zeta) \), where \( \zeta \) denotes the \( n \)th roots of unity:

\[
\prod_{m=0}^{\infty} \left((m+x)^n - y^n\right) = \frac{(\sqrt{2\pi})^n}{\prod_{\zeta^n=1} \Gamma(x-\zeta y)}
\]

and in 2006, by applying Stark’s summation formula [13], Mizuno [10] got a general form:

\[
\prod_{m=0}^{\infty} \left(\prod_{j=1}^{n} (m+z_j)\right) = \frac{(\sqrt{2\pi})^n}{\prod_{j=1}^{n} \Gamma(z_j)}
\]

for \( z_j \in \mathbb{C} \setminus \{0,-1,-2,\ldots\} \).

1.2. **Our results.** In a recent preprint [6], we showed a close connection between (1.3) and the following formula found by John Wallis in 1656:

\[
2 \cdot 2 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 33 \cdot 55 \cdot 57 \ldots = \frac{\pi}{2}
\]

That is, letting \( z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\} \), we showed that (1.3) can be proved from the analytic properties for the Barnes’ multiple zeta function:

\[
\zeta_N(s,z) = \sum_{m_1, \ldots, m_N=0}^{\infty} \frac{1}{(z + m_1 + \cdots + m_N)^s}, \quad \text{Re}(s) > N,
\]

while (1.7) is implied by its alternating form

\[
\zeta_{E,N}(s,z) = \sum_{m_1, \ldots, m_N=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_N}}{(z + m_1 + \cdots + m_N)^s}, \quad \text{Re}(s) > 0.
\]

Let \( \tilde{\Gamma}(z) \) denotes the modified gamma function, which is defined by the authors in [7] from the alternating Hurwitz zeta functions. In subsection 1.3 below, we will give a brief review for its definition and properties by comparing with the ordinary gamma function \( \Gamma(z) \).

In this note, inspiring by the above considerations, we extend Wallis’ formula (1.7) as the following general form, which is an analogue of Mizuno’s formula (1.6) above.

**Theorem 1.1.** For \( z_j \in \mathbb{C} \setminus \{0,-1,-2,\ldots\} \), we have

\[
\prod_{m=0}^{\infty} \left\{ \prod_{j=1}^{n} (m+z_j) \right\}^{(-1)^m} = \frac{(\sqrt{\frac{\pi}{2}})^n}{\prod_{j=1}^{n} \Gamma(z_j)}.
\]

For \( n = 1 \), letting \( z_1 = x \) in (1.10) we get the following analogue of Lerch’s formula (1.1).
Corollary 1.2 (Lerch type formula).

\[(1.11) \quad \prod_{m=0}^{\infty} (m + x)^{(-1)^m} = \frac{\sqrt{\pi}}{\Gamma(x)}.\]

So setting \(x = 1\) in (1.11), by Lemma 1.6 below and taking squares on the both sides, we recover Wallis’ formula (1.7).

For \(n = 2\), letting \(z_1 = x + iy\) and \(z_2 = x - iy\) in (1.10), we obtain an analogue of Lerch’s formula (1.4).

Corollary 1.3 (Lerch type formula in \(\mathbb{Q}(i)\)).

\[(1.12) \quad \prod_{m=0}^{\infty} \left(\left((m + x)^2 + y^2\right)^{(-1)^m} = \prod_{m=0}^{\infty} \left((m + x + iy)(m + x - iy)\right)^{(-1)^m}\right.\]

\[= \frac{\pi}{2} \frac{1}{\Gamma(x + iy)\Gamma(x - iy)}.\]

For \(n\)th roots of unity \(\zeta\), letting \(z_j = x - \zeta^j y, (j = 0, 1, \ldots, n - 1)\) in (1.10) we get the following analogue of Kurokawa and Wakayama’s formula (1.5).

Corollary 1.4 (Kurokawa–Wakayama type formula).

\[(1.13) \quad \prod_{m=0}^{\infty} \left((m + x)^n - y^n\right)^{(-1)^m} = \prod_{m=0}^{\infty} \left\{ \prod_{j=0}^{n-1} (m + x - \zeta^j y) \right\}^{(-1)^m}\]

\[= \frac{(\sqrt{\pi})^n}{\prod_{\zeta^n = 1} \Gamma(x - \zeta y)}.\]

1.3. Gamma function and Euler’s constant. The Hurwitz zeta function

\[(1.14) \quad \zeta(s, z) = \sum_{m=0}^{\infty} \frac{1}{(m + z)^s}\]

can be viewed as a source for several special functions and mathematical constants. Setting \(z = 1\) in (1.14), it reduces to the Riemann zeta function

\[(1.15) \quad \zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.\]

The generalized Stieltjes constant \(\gamma_k(z)\) comes from the following Laurent series expansion of \(\zeta(s, z)\) around \(s = 1\)

\[(1.16) \quad \zeta(s, z) = \frac{1}{s - 1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k(z)}{k!} (s - 1)^k \]
and $\gamma_k = \gamma_k(1)$ is the original Stieltjes constant in 1885 (see Stieltjes’ original article [12] and Ferguson [3]). Letting $k = 0$, we get Euler’s constant

$$
\gamma := \gamma_0(1) = \lim_{s \to 1} \left( \zeta(s) - \frac{1}{s-1} \right)
$$

(1.17)

\[
= \lim_{\alpha \to \infty} \left( \sum_{n=1}^{\alpha} \frac{1}{n} - \log \alpha \right) = 0.5772156649 \ldots
\]

The gamma function $\Gamma(z)$ is defined by Euler from its integral representation

$$
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \text{Re}(z) > 0,
$$

(1.18)

but it can also be defined from the derivatives of $\zeta(s, z)$ as follows (e.g., [2, Definition 9.6.13(1)]),

$$
\Gamma(z) = \exp \left( \zeta'(0, z) - \zeta'(0, 1) \right) = \exp \left( \zeta'(0, z) - \zeta'(0) \right).
$$

(1.19)

The following Weierstrass–Hadamard product representation of $\Gamma(z)$ is well-known:

$$
\Gamma(z) = \frac{1}{z e^{-\gamma} \prod_{m=1}^{\infty} \left( e^{\frac{1}{m}} \left( 1 + \frac{z}{m} \right)^{-1} \right)},
$$

(1.20)

where $\gamma$ is Euler’s constant.

Then the digamma functions can be defined from the derivatives of the log gamma functions $\log \Gamma(z)$, that is,

$$
\psi(z) := \frac{d}{dz} \log \Gamma(z)
$$

(1.21)

and more generally

$$
\psi^{(n)}(z) := \left( \frac{d}{dz} \right)^n \psi(z), \quad n = 0, 1, 2, \ldots
$$

(1.22)

(see [14, p. 33]), and [14, p. 33, Eq. (53)] shows that

$$
\psi^{(n)}(z) = (-1)^{n+1} n! \zeta(n + 1, z), \quad n = 1, 2, \ldots
$$

(1.23)

(see also [2, Proposition 9.6.41]).

Now we go to the alternating case. For details, we refer to [7]. Let

$$
\zeta_E(s, z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m + z)^s}
$$

(1.24)

be the alternating Hurwitz zeta function. Setting $z = 1$ in (1.24), it reduces to Dirichlet’s eta function

$$
\eta(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^s}.
$$

(1.25)
According to Weil’s history [15, p. 273–276] (also see a survey by Goss [5, Section 2]), Euler used (1.25) to “prove”

\[
\eta(1-s) \eta(s) = -\frac{\Gamma(s)(2^s-1)\cos(\pi s/2)}{(2^{s-1}-1)\pi^s},
\]

which leads to the functional equation of the Riemann zeta function \(\zeta(s)\).

As a result of analytic continuation, we see that \(\zeta_E(s,z)\) is non-singular at \(s = 1\). Thus we can designate a modified Stieltjes constant \(\tilde{\gamma}_k(z)\) from the Taylor expansion of \(\zeta_E(s,z)\) at \(s = 1\),

\[
\zeta_E(s,z) = \sum_{k=0}^{\infty} \frac{(-1)^k \tilde{\gamma}_k(z)}{k!} (s - 1)^k.
\]

In analogy with the classical case (1.16), \(\tilde{\gamma}_k = \tilde{\gamma}_k(1)\) is named the modified Stieltjes constant. Letting \(k = 0\), we get the modified Euler constant

\[
\tilde{\gamma}_0 := \tilde{\gamma}_0(1) = \frac{1}{2} + \frac{1}{2} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j(j+1)}.
\]

(See [7, p. 4]).

Following [16, Proposition 2], the modified digamma function \(\tilde{\psi}(z)\) is defined to be

\[
\tilde{\psi}(z) := -\tilde{\gamma}_0(z).
\]

or equivalently

\[
\tilde{\psi}(z) = -\frac{\Gamma''(z)}{\Gamma(z)} + \frac{\Gamma''(z/2)}{\Gamma(z/2)} + \log 2 = -\psi(z) + \psi(z/2) + \log 2,
\]

where \(\Gamma\) is the gamma function and \(\psi\) is the digamma function. Let

\[
\tilde{\psi}^{(n)}(z) := \left(\frac{d}{dz}\right)^n \tilde{\psi}(z), \quad n = 0, 1, 2, \ldots.
\]

As in the classical situation (1.23), we have the following representation

\[
\tilde{\psi}^{(n)}(z) = (-1)^{n+1} n! \zeta_E(n+1, z), \quad n = 0, 1, 2, \ldots
\]

(see [4, p. 957, 8.374]).

Inspiring by the classical formula (1.21), we define the modified gamma function \(\tilde{\Gamma}(z)\) from the differential equation

\[
\tilde{\psi}(z) = \frac{d}{dz} \log \tilde{\Gamma}(z), \quad \text{Re}(z) > 0
\]

and the following analogue of the Weierstrass–Hadamard product (1.20) has been shown in [7, Theorem 1.12]:

\[
\tilde{\Gamma}(z) = \frac{1}{z} e^{\tilde{\gamma}_0 z} \prod_{m=1}^{\infty} \left( e^{-\frac{z}{m}} \left(1 + \frac{z}{m}\right) \right)^{(-1)^{m+1}},
\]

where \(\tilde{\gamma}_0\) is the modified Euler constant (see (1.28)).
The following two lemmas on the properties of the Dirichlet’s eta function \( \eta(s) \) and the modified gamma function \( \tilde{\Gamma}(z) \) shall be used in the proof of the main result.

**Lemma 1.5.**

(1) \( \eta(1) = \gamma_0. \)

(2) \( \eta'(0) = \log \sqrt{\frac{\pi}{2}}. \)

*Proof.* (1) Setting \( s = 1 \) and \( z = 1 \) in (1.27), by (1.25) and (1.28) we have
\[
\eta(1) = \zeta_E(1, 1) = \tilde{\gamma}_0(1) = \tilde{\gamma}_0.
\]

(2) Since \( \eta(s) = (1 - 2^{1-s}) \zeta(s), \) by taking the derivatives on both sides of the above equality and noticing that
\[
\zeta(0) = -\frac{1}{2},
\]
([11, Theorem 12.16]) and
\[
\zeta'(0) = -\frac{1}{2} \log(2\pi),
\]
we get
\[
\eta'(0) = \log \sqrt{\frac{\pi}{2}},
\]
which is what we want. \( \Box \)

**Lemma 1.6.**

\( \tilde{\Gamma}(1) = \frac{\pi}{2}. \)

*Proof.* By [7, p. 19, Eq. (2.31)], we have
\[
\log \tilde{\Gamma}(z) = -\log z + \tilde{\gamma}_0 z + \sum_{k=1}^{\infty} (-1)^k \left( \frac{z}{k} - \log \left( 1 + \frac{z}{k} \right) \right).
\]

Letting \( z = 1 \) in (1.37), we see
\[
\log \tilde{\Gamma}(1) = \tilde{\gamma}_0 + \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right).
\]

By Lemma 1.5 (1) we have
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\eta(1) = -\tilde{\gamma}_0,
\]
so (1.38) implies
\[
\log \tilde{\Gamma}(1) = \log \prod_{k=1}^{\infty} \left( \frac{k}{k+1} \right)^{(-1)^k}.
\]
From Wallis’ formula (1.7) we have

\[ \prod_{k=1}^{\infty} \left( \frac{k}{k+1} \right)^{(-1)^k} = \frac{\pi}{2}. \]

Here it may be necessary to mention that [6, Section 3] points out that Wallis formula can be derived from the alternating multiple Hurwitz zeta functions (1.9) directly. Substituting (1.40) into (1.39), we get

\[ \log \tilde{\Gamma}(1) = \log \frac{\pi}{2} \]

and

\[ \tilde{\Gamma}(1) = \frac{\pi}{2} \]

which is what we want. \(\square\)

2. Proof of the main result

In this section, we prove Theorem 1.1 by modifying the method of Mizuno [10]. The Weiestrass-Hadamard product of the modified gamma function \(\tilde{\Gamma}(z)\) (1.34) will play a key role in our approach.

Let \(c \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\). Define

\[ \Lambda_c^*(s) = \sum_{m=c+1}^{\infty} (-1)^{m+1} \prod_{j=1}^{n} (m + z_j)^{-s}, \quad \text{Re}(s) > 0. \]  

(2.1)

It is easy to see that

\[ \Lambda_c^*(s) - \sum_{m=c+1}^{\infty} (-1)^{m+1} m^{-ns} + s \left( \sum_{j=1}^{n} z_j \right) \sum_{m=c+1}^{\infty} (-1)^{m+1} m^{-(ns+1)} \]

\[ = \sum_{m=c+1}^{\infty} (-1)^{m+1} m^{-ns} \left\{ \prod_{j=1}^{n} \left( 1 + \frac{z_j}{m} \right)^{-s} - 1 + s \left( \sum_{j=1}^{n} z_j \right) \frac{1}{m} \right\}. \]

(2.2)

For \(c \in \mathbb{N}\) big enough, we have \(|\frac{z_j}{m}| < 1\) for any \(m \geq c+1\) and \(j = 1, 2, \ldots, n\).

From the binomial theorem,

\[ \left( 1 + \frac{z_j}{m} \right)^{-s} = 1 - s \frac{z_j}{m} + O \left( \frac{1}{m^2} \right). \]

(2.3)

so the right hand side of (2.2) converges absolutely and uniformly on compact subset of \(\{s \in \mathbb{C} : \text{Re}(s) > -\frac{1}{n}\}\).

Denote by

\[ \Lambda^*(s) = \sum_{m=0}^{\infty} (-1)^{m+1} \prod_{j=1}^{n} (m + z_j)^{-s}, \quad \text{Re}(s) > 0. \]

(2.4)
We have
\[(2.5)\]
\[\Lambda^*(s) = \sum_{m=0}^{c} (-1)^{m+1} \prod_{j=1}^{n} (m + z_j)^{-s} + \Lambda_c^*(s) \]
\[= \sum_{m=0}^{c} (-1)^{m+1} \prod_{j=1}^{n} (m + z_j)^{-s} \]
\[+ \sum_{m=c+1}^{\infty} (-1)^{m+1} m^{-ns} \cdot s \left( \sum_{j=1}^{n} z_j \right) \sum_{m=c+1}^{\infty} (-1)^{m+1} m^{-(ns+1)} \]
\[+ \sum_{m=c+1}^{\infty} (-1)^{m+1} m^{-ns} \left\{ \prod_{j=1}^{n} \left( 1 + \frac{z_j}{m} \right)^{-s} - 1 + s \left( \sum_{j=1}^{n} z_j \right) \frac{1}{m} \right\} \]
\[= \sum_{m=0}^{c} (-1)^{m+1} \prod_{j=1}^{n} (m + z_j)^{-s} + \left( \eta(ns) - \sum_{m=1}^{c} (-1)^{m+1} m^{-ns} \right) \]
\[\times \left( \sum_{j=1}^{n} z_j \right) - s \left( \sum_{j=1}^{n} z_j \right) \left( \eta(ns + 1) - \sum_{m=1}^{c} (-1)^{m+1} m^{-(ns+1)} \right) \]
\[+ \sum_{m=c+1}^{\infty} (-1)^{m+1} \left\{ \prod_{j=1}^{n} (m + z_j)^{-s} - m^{-ns} + s \left( \sum_{j=1}^{n} z_j \right) m^{-(ns+1)} \right\} . \]

Thus taking the derivatives on both sides of the above equality, we have
\[(2.6)\]
\[\left. \frac{\partial}{\partial s} \Lambda^*(s) \right|_{s=0} = - \sum_{m=0}^{c} (-1)^{m+1} \sum_{j=1}^{n} \log(m + z_j) \]
\[+ n\eta'(0) - \sum_{m=1}^{c} (-1)^{m+1} (-n) \log m \]
\[\times \left( \sum_{j=1}^{n} z_j \right) + \left( \sum_{j=1}^{n} z_j \right) \sum_{m=1}^{c} (-1)^{m+1} \frac{1}{m} \]
\[+ \sum_{m=c+1}^{\infty} (-1)^{m+1} \left\{ - \sum_{j=1}^{n} \log(m + z_j) + n \log m + \left( \sum_{j=1}^{n} z_j \right) \frac{1}{m} \right\} \]
\[= - \sum_{m=0}^{c} (-1)^{m+1} \sum_{j=1}^{n} \log(m + z_j) + \sum_{j=1}^{n} \sum_{m=1}^{c} (-1)^{m+1} \left( \log m + \frac{z_j}{m} \right) \]
\[+ \sum_{j=1}^{n} \sum_{m=c+1}^{\infty} (-1)^{m+1} \left\{ \log \left( 1 + \frac{z_j}{m} \right) - \frac{z_j}{m} \right\} \]
\[+ n\eta'(0) - \left( \sum_{j=1}^{n} z_j \right) \eta(1) . \]
By Lemma 1.5 \( \eta'(0) = \log \sqrt{\pi} \) and \( \eta(1) = \tilde{\gamma}_0 \), we have

\[
\frac{\partial}{\partial s} \Lambda^*(s) \bigg|_{s=0} = - \sum_{m=0}^{c} (-1)^{m+1} \sum_{j=1}^{n} \log(m + z_j) \\
+ \sum_{j=1}^{n} \sum_{m=1}^{c} (-1)^{m+1} \left( \log m + \frac{z_j}{m} \right) \\
- \sum_{j=1}^{n} \sum_{m=c+1}^{\infty} (-1)^{m+1} \left\{ \log \left( 1 + \frac{z_j}{m} \right) - \frac{z_j}{m} \right\} \\
+ n \log \sqrt{\pi} - \left( \sum_{j=1}^{n} z_j \right) \tilde{\gamma}_0.
\]

(2.7)

On the other hand, since

\[
\Lambda^*(s) = \sum_{m=0}^{\infty} (-1)^{m+1} \prod_{j=1}^{n} (m + z_j)^{-s}, \quad \text{Re}(s) > 0,
\]

(2.8) by taking the derivatives on the both sides directly, we have

\[
\frac{\partial}{\partial s} \Lambda^*(s) \bigg|_{s=0} = - \sum_{m=0}^{\infty} (-1)^{m+1} \sum_{j=1}^{n} \log(m + z_j) \\
= \sum_{m=0}^{\infty} \log \left( \prod_{j=1}^{n} (m + z_j)^{(-1)^{m}} \right).
\]

(2.9)
Then comparing (2.7) and (2.9) we have

\begin{equation}
\prod_{m=0}^{\infty} \left\{ \prod_{j=1}^{n} (m + z_j) \right\}^{(-1)^m} = \exp \left( \frac{\partial}{\partial s} \Lambda^*(s) \right)_{s=0}
\end{equation}

\begin{align*}
&= \prod_{j=1}^{n} \left\{ \prod_{m=0}^{c} (m + z_j) \right\}^{(-1)^m} \prod_{j=1}^{n} \prod_{m=1}^{c} m^{-m+1} \prod_{j=1}^{n} \prod_{m=1}^{c} e^{(-1)^m z_j \frac{m}{m} - m + 1} \\
&\times \prod_{j=1}^{n} \left\{ \prod_{m=0}^{\infty} \left( 1 + \frac{z_j}{m} \right) \right\}^{(-1)^m} \prod_{j=1}^{n} \prod_{m=1}^{\infty} e^{(-1)^m z_j \frac{m}{m} - \frac{1}{2}} \\
&\times \left( \sqrt{\frac{\pi}{2}} \right)^n \times e^{\left( \sum_{j=1}^{n} z_j \right) (-\gamma_0)}
\end{align*}

\begin{align*}
&= \prod_{j=1}^{n} z_j \times \prod_{j=1}^{n} \left\{ \prod_{m=1}^{c} (m + z_j) \right\}^{(-1)^m} \\
&\times \prod_{j=1}^{n} \prod_{m=1}^{c} m^{-m+1} \prod_{j=1}^{n} \prod_{m=1}^{c} e^{(-1)^m z_j \frac{m}{m} - m + 1} \\
&\times \left( \sqrt{\frac{\pi}{2}} \right)^n \times e^{\left( \sum_{j=1}^{n} z_j \right) (-\gamma_0)}
\end{align*}

\begin{align*}
&= \left( \sqrt{\frac{\pi}{2}} \right)^n \prod_{j=1}^{n} \left\{ z_j e^{-\gamma_0 z_j} \prod_{m=1}^{\infty} \left( e^{-\frac{z_j}{m}} \left( 1 + \frac{z_j}{m} \right) \right) \right\}^{(-1)^m}.
\end{align*}

Finally from the Weierstrass–Hadamard product representation of \( \tilde{\Gamma}(z) \) (1.34) we get

\begin{equation}
\prod_{m=0}^{\infty} \left\{ \prod_{j=1}^{n} (m + z_j) \right\}^{(-1)^m} = \frac{\left( \sqrt{\frac{\pi}{2}} \right)^n}{\prod_{j=1}^{n} \tilde{\Gamma}(z_j)},
\end{equation}

which is what we want.

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