Direct numerical simulation of dynamo transition for nonhelical MHD

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Abstract. Pseudospectral Direct Numerical Simulation (DNS) has been performed to simulate dynamo transition for nonhelical magnetohydrodynamics turbulence. The numerical results are compared with a recent low-dimensional model [Verma et al. [13]]. The forcing in DNS is the same as that used in the low-dimensional model. Dynamo transition is observed in DNS, but the forcing required for the transition is higher than that for the model. A qualitative similarity is observed between DNS and model results. The difference is due to the presence of large number of modes present in the DNS.

1. Introduction

Magnetic field generation or dynamo in astrophysical objects like stars, planets, and galaxy is one of the unsolved mysteries of classical physics [1, 2, 3]. There are several theoretical models have been proposed and large number of numerical studies have been done for these phenomena. Scientists have been able to generate magnetic field in a recent laboratory experiment called Von-Karman-Sodium (VKS) [4].

Fast computers have made high resolution simulations possible in recent times. There are simulations for low Prandtl number as well for large Prandtl numbers [6, 7, 8, 10, 9]. These simulations provide important insights into the dynamo mechanism.

Qualitatively, when magnetohydrodynamic (MHD) fluid is stirred (supplied energy in velocity field), the magnetic field lines are stretched thus amplifying the magnetic field. A simple dimensional analysis based on energy calculation shows that $UL/\eta$ must be greater than one. Here $U$ and $L$ are the large scale velocity and length scales of the system, and $\eta$ is the resistivity of the plasma. There are several semi-quantitative models of dynamo. They fall in two categories: (a) Kinematic dynamos, where the generated magnetic field does not react back to the velocity field; (b) Dynamic dynamos, where the back reaction of the magnetic field on the velocity field is taken into account. The dynamic dynamo models are more complex compared to kinematic dynamo models. In the recent times, the focus of most of the researchers on dynamic models.

A turbulent fluid has large number of modes. Hence one expects that dynamo would involve large number of interacting modes. However it has been conjectured by many scientists that the large-scale modes play important role in the dynamo mechanism. Verma et al. [13] constructed six model containing three velocity and three magnetic field Fourier modes. Only interactions among these modes were considered, and the higher order modes and their interactions were
ignored. The model exhibits several important features including dynamo transition. For a given Prandtl number, dynamo was observed for force field value beyond a critical value. Interestingly, the critical force value depends on the helicity of the system. The above model, also called a low-dimensional model, indicates a lower forcing for helical dynamo compared to nonhelical model. This observation of Verma et al. [13] is consistent with earlier numerical simulations of nonhelical and helical dynamo [6, 7, 8, 10, 9].

In numerical simulation, all the modes of MHD fluid are solved numerically. The present day simulations are typically performed in grid size of $256^3$ or $512^3$ that have of the tens of millions and hundreds of millions of modes. We performed numerical simulations on smaller grid size ($64^3$) and studied the time evolution of the large-scale modes and study the dynamics of dynamo transition. We perform this study as a function of Prandtl number, In this study we focus on the large-scale modes studied by Verma et al. [13]. Also, our present study is confined to nonhelical dynamo.

In typical scenario the magneto fluid is forced through the velocity field. The dynamo transition occurs for the forcing when a nonzero magnetic field is sustained in the steady-state laminar solution or in the statistically stationary turbulent solution. Typically, dynamos occur for forcing amplitudes beyond a critical value which also defines the critical Reynolds number $Re^c$ and the critical magnetic Reynolds number $Re^c_m$. Note that $Re = UL/\nu$ and $Re_m = UL/\eta$, where $U$ and $L$ are the large scale velocity and length scale respectively, and $\nu$ and $\eta$ are the kinematic viscosity and resistivity of the fluid respectively. Prandtl number $P_m$ is defined as the ratio of $\nu$ and $\eta$. One of the objectives of both the numerical simulations and the experiments [4, 5] is the determination of $Re^c_m$. It has been found that $Re^c_m$ depends on both the type of forcing and the Prandtl number (or Reynolds number), yet the range of $Re^c_m$ observed in the numerical simulations is from 10 to 500 for a wide range of $P_m$ (from $5 \times 10^{-3}$ to 2500).

In our simulation too we will force the velocity field with a nonhelical forcing, and study the dynamo transition. We will compare the numerical results with that of model calculations of Verma et al. [13]. The outline of the paper is as follows: In Sec. 2 we will briefly describe the nonhelical low-dimensional model of Verma et al. [13]. In Section 3 we describe our numerical scheme and discuss the numerical results. Section 4 contains the conclusions.

2. Details of a low-dimensional dynamo model

Here we describe low-dimensional model proposed by Verma et al. [13]. This model exhibits dynamo transition. A brief review of the model is given below.

The incompressible MHD equations are

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla p_{\text{tot}} - (\mathbf{u} \cdot \nabla) \mathbf{u} + (\mathbf{b} \cdot \nabla) \mathbf{b} + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{1}$$

$$\frac{\partial \mathbf{b}}{\partial t} = - (\mathbf{u} \cdot \nabla) \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b} \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{b} = 0 \tag{4}$$

where $\mathbf{u}$ and $\mathbf{b}$ are the velocity and magnetic field respectively, $p_{\text{tot}}$ is the sum of hydrodynamic and magnetic pressure, and $\mathbf{f}$ is the forcing function. The parameters $\nu$ and $\eta$ are kinematic viscosity and resistivity respectively.

We rewrite the above equations in truncated Fourier basis. We consider only three incompressible basis vectors given below.
\[ e_1 = \sqrt{2} \begin{bmatrix} -\sin k_0 x \cos k_0 z \\ 0 \\ \cos k_0 x \sin k_0 z \end{bmatrix} \]  
\[ e_2 = \sqrt{2} \begin{bmatrix} 0 \\ -\sin k_0 y \cos k_0 z \\ \cos k_0 y \sin k_0 z \end{bmatrix} \]  
\[ e_3 = 2 \sqrt{\frac{2}{3}} \begin{bmatrix} -2 \sin k_0 x \cos k_0 y \cos 2k_0 z \\ -\cos k_0 x \sin k_0 y \cos 2k_0 z \\ \cos k_0 x \cos k_0 y \sin 2k_0 z \end{bmatrix} \]

where \( k_0 = 2\pi/l \) with \( l \) being the box size. The projections of variable \( u, b \) and \( f \) on these basis vectors are

\[ u^\prec = (u_1 e_1 + u_2 e_2 + u_3 e_3)\nu k_0 \]  
\[ b^\prec = (b_1 e_1 + b_2 e_2 + b_3 e_3)\nu k_0 \]  
\[ f^\prec = (f_1 e_1 + f_2 e_2 + f_3 e_3)\nu^2 k_0^3 \]

where \( u_i, b_i \) and \( f_i \) are dimension less real numbers. Now the new space is 6-dimensional, formed by 3-velocity field vectors of \( u^\prec \) and 3-magnetic field vectors of \( b^\prec \). The kinetic helicity \( u \cdot (\nabla \times u) \) and the magnetic helicity \( b \cdot a \) (\( a \) is the vector potential) of the model is zero. Therefore, this low-dimensional model is called nonhelical model. The dynamo models for which the kinetic and magnetic helicities are nonzero are called helical models. In the present paper we focus on nonhelical model only.

The projections of velocity equation (1) and magnetic field equation (2) on \( e_1, e_2 \) and \( e_3 \) produce dynamical equations for \( u_i \)’s and \( b_i \)’s. So the incompressible MHD equations take the following form in this new space.

\[ \dot{u}_1 = \frac{1}{\sqrt{6}}(u_2 u_3 - b_2 b_3) - 2u_1 + f \]  
\[ \dot{u}_2 = \frac{1}{\sqrt{6}}(u_1 u_3 - b_1 b_3) - 2u_1 + f \]  
\[ \dot{u}_3 = -\sqrt{\frac{2}{3}}(u_1 u_2 - b_1 b_2) - 6u_1 \]  
\[ \dot{b}_1 = \frac{1}{\sqrt{6}}(u_2 b_3 - b_2 u_3) - 2P_m^{-1}b_1 \]  
\[ \dot{b}_2 = -\frac{1}{\sqrt{6}}(u_3 b_1 - b_3 u_1) - 2P_m^{-1}b_2 \]  
\[ \dot{b}_3 = 0 \]

The above model has an exact solution. The equations have three fixed points, fluid A, fluid B, MHD. The three fixed points are stable in the \((P_m, f)\) regions drawn in Fig 1 Dynamo does not exist for \( P_m < 1 \), and it is possible for \( f > f_c \) for \( P_m \geq 1 \). The line \( f = f_c \)

\[ f_c = 12\sqrt{2}\frac{(P_m+1)}{P_m^{3/2}} \]
is the critical force line at which dynamo transition takes place. We observe dynamo for all $f > f_c$, and the bifurcation is pitchfork.

The velocity and magnetic fields can be analytically calculated as a function of $f$ and $P_m$. In the MHD regime

$$b_1 = b_2 = \pm \frac{P_m}{4(P_m+1)} \sqrt{f^2 - f_c^2}$$  

(18)

$$b_3 = 0$$  

(19)

We define the critical magnetic Reynolds number based on the velocity scale

$$U_L = \sqrt{u_1^2 + u_2^2 \nu k_0}$$  

(20)

and the box size. Verma et al. [13] showed that

$$Re_m^{Mo} = P_m Re = 6\sqrt{2P_m}$$  

(21)

Note that the magnetic field modes $b_1$ and $b_2$ are constant in this dynamo model for all $f > f_c$. Dynamo models with large number of modes yield chaotic timeseries. In the following sections we will report the results of MHD simulation and compare the simulation results with those of low-dimensional model.

### 3. Simulation Procedure and Results

We numerically solve the MHD equations using pseudospectral method in a periodic box of size $(2\pi)^3$. The grid size used are $32^3$ and $64^3$. The equations are solved using fourth-order Runge-Kutta method. We dealiase our code using 2/3 rule. The time-step is computed using CFL criterion. We use FFTW for Fourier transformations in the code.

In order to make comparison with the low-dimensional model, we force the DNS Fourier modes corresponding to the basis functions $e_1$ and $e_2$ only (modes $[\pm 1, 0, \pm 1]$ and $[0, \pm 1, \pm 1]$). As discussed in the previous section we keep the amplitude of forcing for these basis functions to be the same. Clearly the helicity of the forcing function is zero. We fix the kinematic viscosity $\nu$ to 1.0, and vary resistivity $\eta$ from 5.0 to 0.01. Hence, the Prandtl number varies from 1/10 to 100.

We find that the numerical results for $64^3$ table 1 and $32^3$ [table 1 - column 5th] are not the same, but the difference is around 25 percent. In this paper we report the results for $64^3$ since it is of higher resolution. We observe that dynamo exists for all $P_m$, but the value of critical force increases drastically as we decrease $P_m$ below 1. For example, the critical $f^{Sim}$ for $P_m = 1/5$ is approximately 15000. There does not appear to be a critical Prandtl number in our simulation. We are investigating this aspect theoretically. Note that these results differ from the low-dimensional dynamo model of Verma et al. [13], where the critical Prandtl number is 1. However the increase of critical $f$ with lowering $P_m$ is consistent with the low-dimensional results and also with the nonhelical dynamo simulation of Schekochihin et al. [11]

We study the evolution of the kinetic and magnetic modes. The time series of these modes vary drastically in the initial phase, but it fluctuates randomly after the system attains steady state. We observe random magnetic field for $f$ beyond $f_c$. As a sample, we illustrate the time evolution magnetic mode $b_{101}$ in Fig 2. The model line $b_1 = b_2$ is somewhat larger than the numerical values (see Eq. (18)).

In figures 3 and 4 we illustrate the time series of kinetic energy and magnetic energy respectively for two sets of parameters. The result shows certain amount of randomness around a mean value.
In Table 1, we list $f^{m\text{Sim}}$, $f^{m\text{Mo}}$, $b^{m\text{Sim}}$, $b^{m\text{Mo}}$, $u_{rms\text{Sim}}$, and $b_{rms\text{Sim}}$. Here the superscripts Sim and Mo represent simulation and model results respectively. For a given parameter $f^{m\text{Sim}}$ is the largest forcing parameter for which we find fluid (nonmagnetic) solution in DNS, $f^{m\text{Sim}}$ is the smallest forcing parameter for which we observe dynamo in DNS, and $f^{m\text{Mo}}$ is the critical forcing function for the low-dimensional model. As described above the evolution of magnetic modes is random. Here we report the rms value of as a function for the low-dimensional model. As described above the evolution of magnetic modes is random. Here we report the rms value of $b_{rms\text{Sim}} = \sqrt{2E_b}$, where $E_b$ is the total magnetic energy.

In Fig. 5 we plot the entries of the Table 1 for all $P_m \geq 1$. Clearly the critical forcing of numerical simulation is one order of magnitude larger than the corresponding value for the model. The difference between the low-dimensional model and the DNS is expected since DNS involves large number of modes that have been ignored in low-dimensional model. We need larger forcing in DNS to excite large number of these modes in addition to the large-scale modes of the low-dimensional model. However we lack quantitative model to estimate the forcing as a function of $P_m$ for simulations. We are exploring this aspect using energy flux, the details of which will be reported in future.

We study the rms value of the magnetic field as a function of Prandtl number. The numerical values of $b_{rms\text{Sim}}$ for various $P_m$ have been plotted in Fig. 6. We also plot the corresponding value for the low-dimensional model. These results do not match, yet they grow as a function of $P_m$ in qualitatively similar manner.

We compute the critical magnetic Reynolds number $Re_m^c = U_{rms\text{Sim}} L/\eta$ using the numerical values from the simulation. In Fig. 7 we plot $Re_m^c$ vs. $P_m$. The trend indicates that $Re_m^c$ increases monotonically with increasing $P_m$. On the contrary, $Re_m^c$ of the low-dimensional model increases rather slowly ($Re_m^c \propto \sqrt{P_m}$ according to Eq. (21)). This difference need further investigation.

4. Conclusions
In this paper we numerically simulated MHD equations under specific forcing akin to that used in low-dimensional model proposed by Verma et al. [13]. We studied the dynamo transition as a function of the Prandtl number for nonhelical forcing. We observe that the critical forcing of the simulation increases as we decrease $P_m$. The increase is dramatic below $P_m = 1$. For $P_m > 1$, the DNS forcing is one order of magnitude larger than that of the low-dimensional model. This is due to the fact that the large number modes in DNS need larger forcing to get excited. The DNS results agree with the low-dimensional models only qualitatively. We are investigating the difference between the model and simulations.

We observe that the time series of the magnetic field in dynamo region is random unlike low-dimensional model’s magnetic field that remains constant. We also calculated the critical magnetic Reynolds number for the simulation, and observe it to be much higher compared to the model. Qualitatively these differences are due to the large number of modes present in DNS.

Our present study indicate qualitative similarities between the DNS and low-dimensional results. These observations compels us to the importance of large-scale modes. However the present study also illustrates the inadequacy of low-dimensional model in many aspects, mainly due to the neglect of large number of modes. A study of these differences will shed important light to the dynamics of dynamo. Some of these work are in progress at present.

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| $\eta$ | $P_m$ | $f_c^{32\text{sim}}$ | $f_{b\text{sim}}^3$ | $f_{b\text{Mo}}^3$ | $U_{\text{sim}}^3$ | $U_{\text{Mo}}^3$ | $Re_{\text{sim}}^m$ | $Re_{\text{Mo}}^m$ |
|-------|-------|-------------------|-------------------|----------------|----------------|----------------|----------------|----------------|
| 0.10  | 10    | -                 | -                 | -              | 4.0x10^4*     | -              | 590            | 70             |
| 0.05  | 20    | -                 | -                 | -              | 1.3x10^4*     | -              | 208            | 9              |
| 0.033 | 30    | -                 | -                 | -              | 1.2x10^4      | 2.3            | 64             | 238            |
| 0.025 | 40    | -                 | -                 | -              | 0.6x10^4      | 3.2            | 64             | 238            |
| 0.020 | 50    | -                 | -                 | -              | 0.3x10^4      | 1.2            | 64             | 238            |
| 0.015 | 65    | 3                 | -                 | -              | 0.1          | 1.3            | 64             | 238            |
| 0.013 | 75    | 1                 | -                 | -              | 2.1          | 3.6            | 64             | 238            |
| 0.011 | 85    | 1                 | -                 | -              | 2.1          | 3.6            | 64             | 238            |
| 0.01  | 100   | 1                 | -                 | -              | 2.1          | 3.6            | 64             | 238            |

Table 1. Various entries of $64^3$ data for $\nu=1$, column 5th for $32^3$. The critical forcing for $P_m = 1/5$ and 1/10 were calculated for $32^3$ simulations (rows 1st and 2nd).

**Figure 1.** Plot of critical force $f_c$ as a function of $P_m$ for the nonhelical model. Dynamo is excited for $f > f_c$, only for $P_m > 1$. 

Fluid A

MHD

Fluid B
Figure 2. Time series of $b_{101}$ mode in steady state. The Model result and the simulation results are labelled in the right and left of $y$-axis respectively.

Figure 3. Time series of kinetic energy in steady state

Figure 4. Time series of magnetic energy in steady state
Dynamo Transition In DNS Simulation

Figure 5. Dynamo transition for our numerical simulation for various $P_m$. The grid size is $64^3$. Here 'empty circles' represent the highest $f$ that yields nonmagnetic solution, while 'stars' represent the lowest $f$ that yield dynamo solution. We also plot model critical force $f_c$ vs. $P_m$ curve for the low-dimension model as a comparison.

Figure 6. Variation of $b_{rms}$ as a function of $P_m$. 
Figure 7. Variation of $Re_m^c$ as a function of $P_m$. Large change in magnitudes of $Re_m^c$ - model and $Re_m^c$ - simulation but there is a qualitative agreement.