EXISTENCE OF SOLUTIONS TO RELATIVISTIC NON-ABELIAN
CHERN-SIMONS-HIGGS VORTEX EQUATIONS ON GRAPHS

YUANYANG HU

Abstract. Let $G = (V, E)$ be a connected finite graph. We study the relativistic non-Abelian Chern-Simons-Higgs vortex equations on the graph $G$. We establish an existence result to the relativistic non-Abelian Chern-Simons-Higgs vortex equations.

1. Introduction

Vortices play significant parts in many fields of theoretical physics including superconductivity theory, cosmology, condensed-matter physics, electroweak theory, and quantum Hall effect. A tide of research related to vortex equations has been accomplished; see, for example, [4, 17, 18, 19, 20] and the references therein. Recently, Han, Lin and Yang [12] investigated a system of relativistic non-Abelian Chern-Simons-Higgs vortex equations whose Cartan matrix $K$ is that of arbitrary simple Lie algebra, they established a general existence result for the doubly periodic solutions of the Chern-Simons-Higgs vortex equations.

In recent years, equations on graphs have attracted extensive attention; see, for example, [3, 6, 7, 10, 11, 13] and the references therein.

Recently, Huang, Lin and Yau [11] proved the existence of solutions to mean field equations

$$
\Delta u + e^u = \rho \delta_0
$$

and

$$
\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_{ij}}
$$
on graphs.

Let $G = (V, E)$ is a connected finite graph, and $V$ denotes the vetex set and $E$ denotes the edge set.

Inspired by the work of Huang-Lin-Yau [11], we study the relativistic Chern-Simons-Higgs equations

$$
(1.1) \quad \Delta u_i = \lambda \left( \sum_{j=1}^{n} \sum_{k=1}^{n} K_{kj} K_{ji} e^{u_j} e^{u_k} - \sum_{j=1}^{n} K_{ji} e^{u_j} \right) + 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}}(x), \quad i = 1, \ldots, n,
$$
on $G$, where $K = (K_{ij})$ is the Cartan matrix of a finite dimensional semisimple Lie algebra $L$, $n \geq 1$ is the rank of $L$ which is the dimension of the Cartan subalgebra of $L$, $p_{ij}, j = 1, \ldots, N_i, i = 1, \ldots, n,$ are arbitrarily chosen distinct vertices on the graph, and $\delta_{p_{ij}}$
is the Dirac mass at \( p_{ij} \). With a view to handling the system in a unified framework, we need some suitable assumption on the matrix \( K \). We suppose that
\[
K^T = PS,
\]
(1.2)

\( P \) is a diagonal matrix satisfying
\[
P := \text{diag}\{P_1, ..., P_n\}, \ P_i > 0, \ i = 1, ..., n,
\]
(1.3)

\( S \) is a positive definite matrix of the form
\[
S \equiv \begin{pmatrix}
\alpha_{11} & -\alpha_{12} & \cdots & \cdots & -\alpha_{1n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\alpha_{i1} & -\alpha_{i2} & \cdots & \alpha_{ii} & \cdots & -\alpha_{in} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-\alpha_{n1} & -\alpha_{n2} & \cdots & \cdots & -\alpha_{nn-1} & \alpha_{nn}
\end{pmatrix},
\]
(1.4)

\( \alpha_{ii} > 0, \ i = 1, \ldots, n, \ \alpha_{ij} = \alpha_{ji} \geq 0, \ i \neq j = 1, \ldots, n, \)
and
\[
\alpha_{ij} > 0, \ i = 1, \ldots, n, \ \alpha_{ij} = \alpha_{ji} \geq 0, \ i \neq j = 1, \ldots, n,
\]
(1.5)

all the entries of \( S^{-1} \) are positive.

By (1.6), we conclude that
\[
R_i := \sum_{j=1}^{n} ((K^\tau)^{-1})_{ij} > 0
\]
(1.7)

for \( i = 1, ..., n. \)

We now state our main results as follows.

**Theorem 1.1.** Assume that the matrix \( K \) satisfies (1.2)-(1.6), then we have the following conclusions:

(i) Suppose that (1.1) has a solution. Then we have
\[
\lambda > \lambda_0 \equiv \frac{16\pi \sum_{i=1}^{n} \sum_{j=1}^{n} P_i^{-1}(K^{-1})_{ji} N_j}{|V| \sum_{i=1}^{n} \sum_{j=1}^{n} P_i^{-1}(K^{-1})_{ji}}.
\]
(1.8)

(ii) There exists a constant \( \lambda_1 > \lambda_0 \) so that if \( \lambda > \lambda_1 \), then (1.1) admits a solution \( (u_1^\lambda, \ldots, u_n^\lambda) \).

The rest of the paper is arranged as below. In Section 2, We present some results that we will use frequently in the following pages. Section 3 and Section 4 are devoted to the proof of Theorem 1.1.

2. Preliminary results

For each edge \( xy \in E \), We suppose that its weight \( w_{xy} > 0 \) and that \( w_{xy} = w_{yx} \). Set \( \mu : V \to (0, +\infty) \) be a finite measure. For any function \( u : V \to \mathbb{R} \), the Laplacian of \( u \) is defined by
\[
\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{yx}(u(y) - u(x)),
\]
(2.1)
3

where \( y \sim x \) means \( xy \in E \). The gradient \( \nabla \) of function \( f \) is defined by a vector
\[
\nabla f(x) := \left( [f(y) - f(x)] \sqrt{\frac{w_{xy}}{2\mu(x)}} \right)_{y \sim x}.
\]
The gradient form of \( u \) reads
\[
(2.2) \quad \Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)).
\]

We denote the length of the gradient of \( u \) by
\[
|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left( \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2 \right)^{1/2}.
\]

Denote, for any function \( u : V \to \mathbb{R} \), an integral of \( u \) on \( V \) by
\[
\int_V u \, d\mu = \sum_{x \in V} \mu(x) u(x).
\]
Denote \( |V| = \text{Vol}(V) = \sum_{x \in V} \mu(x) \) the volume of \( V \). For \( p > 0 \), denote \( ||u||_p := ||u||_{L^p(V)} = \left( \int_V |u|^p \, d\mu \right)^{1/p} \). Define a sobolev space and a norm on it by
\[
W^{1,2}(V) = \left\{ u : V \to \mathbb{R} : \int_V \left( |\nabla u|^2 + u^2 \right) \, d\mu < +\infty \right\},
\]
and
\[
||u||_{H^1(V)} = ||u||_{W^{1,2}(V)} = \left( \int_V \left( |\nabla u|^2 + u^2 \right) \, d\mu \right)^{1/2}.
\]

To apply the variational method, we need the following Sobolev embedding, Trudinger-Moser inequality and interpolation inequality on graphs.

**Lemma 2.1.** (\cite[Lemma 5]{3}) Let \( G = (V, E) \) be a finite graph. The sobolev space \( W^{1,2}(V) \) is precompact. Namely, if \( u_j \) is bounded in \( W^{1,2}(V) \), then there exists some \( u \in W^{1,2}(V) \) such that up to a subsequence, \( u_j \to u \) in \( W^{1,2}(V) \).

**Lemma 2.2.** (\cite[Lemma 6]{3}) Let \( G = (V, E) \) be a finite graph. For all functions \( u : V \to \mathbb{R} \) with \( \int_V u \, d\mu = 0 \), there exists some constant \( C \) depending only on \( G \) such that
\[
\int_V u^2 \, d\mu \leq C \int_V |\nabla u|^2 \, d\mu.
\]

**Lemma 2.3.** (\cite[Lemma 7]{3}) Let \( G = (V, E) \) be a finite graph. For any \( \beta \in \mathbb{R} \), there exists a constant \( C \) depending only on \( \beta \) and \( G \) such that for all functions \( v \) with \( \int_V |\nabla v|^2 \, d\mu \leq 1 \) and \( \int_V v \, d\mu = 0 \), there holds
\[
(2.3) \quad \int_V e^{\beta v^2} \, d\mu \leq C.
\]
Lemma 2.4. (Interpolation inequality for $L^r$-norms on graphs). Suppose that $\theta \in (0, 1)$, $0 < \theta r \leq s$, $0 < (1 - \theta) t \leq s$, and $\frac{1}{r} = \frac{\theta}{s} + \frac{1 - \theta}{t}$. Then we have

$$\|u\|_{L^r(V)} \leq \|u\|_{L^s(U)}^{\theta} \|u\|_{L^t(U)}^{1 - \theta}.$$  

Proof. By H"older inequality, we see that

$$\int_V |u|^r \, d\mu = \int_V |u|^{\theta r} |u|^{(1 - \theta) r} \, d\mu \leq \left( \int_V |u|^{\theta r} \, d\mu \right)^{\frac{\theta}{s}} \left( \int_V |u|^{(1 - \theta) r} \, d\mu \right)^{\frac{1 - \theta}{t}}.$$  

\[\square\]

In order to establish Theorem 1.1, we need the following result due to Huang-Lin-Yau.

Theorem 2.5. ([11, Theorem 2.2]) There is a critical value $\lambda_c$ depending on $G$ satisfying

$$\lambda_c \geq \frac{16\pi M}{|V|},$$

such that when $\lambda > \lambda_c$, the equation

$$\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^M \delta_{p_j}, \quad x \in G,$$

has a solution $u_\lambda$ on $G$, and when $\lambda < \lambda_c$, the equation (2.6) has no solution, where $M > 0$ is an integer.

In fact, we may establish the following more accurate result.

Proposition 2.6. The solution $u_\lambda$ obtained in Theorem 2.5 is maximal in the sense that if $u$ is any other solution of (1.1), then

$$u \leq u_\lambda.$$  

Proof. Suppose that $u$ is any other solution of (2.6). Then it is clear that $u$ is a subsolution and hence

$$u \leq v_n$$

as in the proof of Lemma 4.2 in [11] for every $n \in \mathbb{N}$, where $v_n$ is defined by iterative scheme (4.5) in [11]. Letting $n \to +\infty$ in (2.8), we deduce that $u \leq u_\lambda$ and hence that $u_\lambda$ is a maximal solution of (2.6).

We now complete the proof. \[\square\]

Furthermore, we have the following propositions.

Proposition 2.7. If $\lambda_1 > \lambda_2 > \lambda_c$, then $u_{\lambda_1} \geq u_{\lambda_2}$. 

Proof. By Lemma 4.4 of [11], we deduce that $u_{\lambda_2} < 0$, and hence that
\[
\Delta u_{\lambda_2} = \lambda_2 e^{u_{\lambda_2}}(e^{u_{\lambda_2}} - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j} > \lambda_1 e^{u_{\lambda_2}}(e^{u_{\lambda_2}} - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j}.
\]
Thus $u_{\lambda_2}$ is a subsolution of (2.6) with $\lambda = \lambda_1$. By the sub-supersolution argument as in the proof of Lemma 4.2 in [11], and the maximality of $u_{\lambda_1}$, this implies that $u_{\lambda_2} \leq u_{\lambda_1}$.

**Proposition 2.8.** Let $u\lambda$ be the maximal solution of (2.6) for $\lambda > \lambda_c$. We have
\[
\int e^{u\lambda}(1 - e^{u\lambda})d\mu = 0,
\]
for $\lambda \geq \lambda_0$. Let $\bar{v} := \sup_{\lambda>\lambda_0} u\lambda(x), x \in V$. We deduce that $u_{\lambda_0} \leq \bar{v} \leq 0$ in $V$ and
\[
\int e^{\bar{v}}(1 - e^{\bar{v}})d\mu = 0.
\]
It follows that $\bar{v} \equiv 0$ on $V$.

We now complete the proof.

\[\square\]

3. The Constraints

For the sake of convenience, by applying the translation
\[
u_i \to u_i + \ln R_i, \quad i = 1, \ldots, n,
\]
in equations (1.1), we can conclude that
\[
\Delta u_i = \lambda \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{K}_{jk} \tilde{K}_{ij} e^{u_j} - \sum_{j=1}^{n} \tilde{K}_{ij} e^{u_j}\right) + 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}}(x)
\]
for $i = 1, \ldots, n$. Set $u_i^0$ be the unique solution to
\[
\Delta u_i^0 = 4\pi \sum_{s=1}^{N_i} \delta_{p_{is}} - \frac{4\pi N_i}{|V|}, \quad \int_{V} u_i^0 \, dx = 0.
\]
Denote $u_i = u_i^0 + v_i, i = 1, \ldots, n$; then $v_i(i = 1, \ldots, n)$ satisfy
\[
\Delta v_i = \lambda \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{K}_{jk} \tilde{K}_{ij} e^{u_j^0 + v_j} e^{u_k^0 + v_k} - \sum_{j=1}^{n} \tilde{K}_{ij} e^{u_j^0 + v_j}\right) + \frac{4\pi N_i}{|V|},
\]
or
\[
\Delta v = \lambda \tilde{K}U \tilde{K}(U - 1) + \frac{4\pi N}{|V|},
\]
where \( \mathbf{v} = (v_1, ..., v_n)^T, \mathbf{N} = (N_1, ..., N_n)^T, \mathbf{U} = \text{diag}\{e^{u_1^0 + v_1}, ..., e^{u_n^0 + v_n}\}, \mathbf{U} = (e^{u_1^0 + v_1}, ..., e^{u_n^0 + v_n})^T, \) (3.6)

\[ \tilde{K} := K^T R = P S R, \quad R := \text{diag}\{R_1, ..., R_n\}. \]

We next establish a necessary condition for the existence of solutions of (1.1), and then the conclusion (i) of Theorem 1.1 follows.

**Lemma 3.1.** Suppose that (1.1) admits a solution. Then

(3.7) \[ \lambda > \lambda_0 := \frac{16\pi |V|}{|V|} \sum_{i=1}^{n} \sum_{j=1}^{n} P_i^{-1} (K^{-1})_{ji} N_j. \]

**Proof.** Denote

(3.8) \[ A = P^{-1} S^{-1} P^{-1} \quad \text{and} \quad Q = R S R. \]

In view of \( S \) is positive definite, we see that \( A \) and \( Q \) are positive definite. By (1.6), we deduce that

(3.9) \[ \mathbf{b} \equiv (b_1, ..., b_n)^T := 4\pi A \mathbf{N} = 4\pi P^{-1} S^{-1} P^{-1} \mathbf{N} > 0. \]

From (3.5), we deduce that

(3.10) \[ \Delta A \mathbf{v} = \lambda U Q (\mathbf{U} - \mathbf{1}) + \frac{\mathbf{b}}{|V|}. \]

It follows that

(3.11) \[ \int_V U Q (\mathbf{U} - \mathbf{1}) d\mu + \frac{\mathbf{b}}{\lambda} = 0, \]

where \( \mathbf{1} := (1, ..., 1)^T \). Multiplying both sides of (3.11) by \( \mathbf{1}^T \), we deduce that

(3.12) \[ \int_\Omega U^T Q (\mathbf{U} - \mathbf{1}) d\mu + \frac{\mathbf{1}^T \mathbf{b}}{\lambda} = 0. \]

From (1.2) and (1.7), we conclude that

(3.13) \[ (K^T)^{-1} \mathbf{1} = S^{-1} P^{-1} \mathbf{1} = R \mathbf{1}. \]

Combining (3.12) and (3.13), we deduce that

(3.14) \[ \int_V \left( \mathbf{U} - \frac{\mathbf{1}}{2} \right)^\tau Q \left( \mathbf{U} - \frac{\mathbf{1}}{2} \right) d\mu = \frac{|V|}{4} 1^\tau P^{-1} (K^{-1})^{-1} 1 - \frac{4\pi 1^\tau P^{-1} (K^{-1})^{-1} \mathbf{N}}{\lambda}. \]

Recall that \( Q \) is positive definite, it follows from (3.14) that

(3.15) \[ \frac{|V|}{4} 1^\tau P^{-1} (K^{-1})^{-1} 1 - \frac{4\pi 1^\tau P^{-1} (K^{-1})^{-1} \mathbf{N}}{\lambda} > 0, \]

which implies that (3.7) holds. \( \square \)
4. The Proof of Theorem 1.1

In this section, we formulate a variational solution of equations (1.1) by using an equality type constraint. Define the energy functional by

\[ I(v) = \frac{1}{2} \sum_{j,k=1}^{n} \int_{V} b_{kj} \Gamma(v_{k}, v_{j}) d\mu + \frac{\lambda}{2} \int_{V} (U - 1)^{T} Q(U - 1) d\mu + \int_{V} b^{T} v |V| d\mu, \]

where \( A = (b_{ij})_{n \times n} \). Due to the fact that \( Q \) and \( A \) are symmetric, we know that if \( v \) is a critical point to \( I \), then it is a solution to (1.1).

We could work on the standard space \( H^{1}(V) := W^{1,2}(V) \). Denote

\[ H^{0} := \left\{ v \in W^{1,2}(V) \mid \int_{V} v d\mu = 0 \right\}. \]

Clearly, for any \( f \in H \), there exists a unique \( c \in \mathbb{R} \) and \( f' \in H^{0} \) such that

\[ f = c + f'. \]

In the sequel, we use \( H^{1}(V) \) to denote the spaces of both scalar and vector-valued functions.

Suppose that \( v = w + c \in H^{1}(V) \) given in (4.3) satisfies (3.11), we deduce that

\[ \text{diag} \{ e^{c_{1}}, \ldots, e^{c_{n}} \} \tilde{Q} \begin{pmatrix} e^{c_{1}} \\ \vdots \\ e^{c_{n}} \end{pmatrix} - P^{-1} R \text{diag} \{ a_{1}, \ldots, a_{n} \} \begin{pmatrix} e^{c_{1}} \\ \vdots \\ e^{c_{n}} \end{pmatrix} + \frac{b}{\lambda} = 0, \]

where

\[ a_{i} := a_{i}(w_{i}) = \int_{V} e^{u_{i}^{0} + w_{i}} d\mu, \]

\[ a_{ij} := a_{ij}(w_{i}, w_{j}) = \int_{V} e^{u_{i}^{0} + a_{j}^{0} + w_{i} + w_{j}} d\mu, \quad i, j = 1, \ldots, n, \]

\[ \tilde{Q} := \tilde{Q}(w) = R \tilde{S} R, \]

and

\[ \tilde{S} \equiv R_{i} \begin{pmatrix} a_{11} & -\alpha_{12} a_{12} & \cdots & \cdots & \cdots & -\alpha_{1n} a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_{i1} a_{i1} & -\alpha_{i2} a_{i2} & \cdots & \alpha_{ii} a_{ii} & \cdots & -\alpha_{in} a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n1} a_{n1} & -a_{n2} a_{n2} & \cdots & \cdots & \cdots & a_{nn} \alpha_{nn} \end{pmatrix}. \]

By (4.7) and (4.8), we deduce that

\[ \tilde{Q} \] is positive positive definite.

We now write (4.4) as the component form:

\[ e^{2\epsilon_{i}} R_{i}^{2} \alpha_{ii} a_{ii} - e^{\epsilon_{i}} \left( \frac{R_{i} a_{i}}{P_{i}} + \sum_{j \neq i} e^{\epsilon_{j}} R_{i} R_{j} \alpha_{ij} a_{ij} \right) + \frac{b_{i}}{\lambda} = 0, \quad i = 1, \ldots, n. \]
Of course, (3.4) is a quadratic equation in $t = e^c$ which admits a solution if and only if
\begin{equation}
\left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} e^{c_j} R_i R_j \alpha_{ij} a_{ij} \right)^2 \geq \frac{4 R_i^2 b_i \alpha_{ii} a_{ii}}{\lambda}, \quad i = 1, \ldots, n.
\end{equation}

It is clear that (4.10) follows from the following inequalities
\begin{equation}
\frac{a_i^2}{a_{ii}} \geq \frac{4 \alpha_{ii} P_i^2 b_i}{\lambda}, \quad i = 1, \ldots, n.
\end{equation}

Denote
\begin{equation}
\mathcal{A} \equiv \{ w \mid w \in H^0(V) \text{ such that (4.12) holds} \}.
\end{equation}

In this case, we may select $c = c(w) := (c_1, \ldots, c_n)$ in (4.10) to satisfy
\begin{equation}
\text{such that (4.10) holds}.
\end{equation}

To proof Lemma 4.6 and Lemma 4.7, we give a priori estimates.

**Lemma 4.1.** For any $w \in \mathcal{A}$ and $\epsilon \in [0, 1]$, if $t$ satisfies the following equations
\begin{equation}
F(\epsilon, t) \equiv t - f(\epsilon, t) = 0, \quad t \in \mathbb{R}^n_+, \quad \epsilon \in [0, 1],
\end{equation}

where
\begin{equation}
f(\epsilon, t) \equiv (f_1(\epsilon, t), \ldots, f_n(\epsilon, t))^T,
\end{equation}

and
\begin{equation}
f_i(\epsilon, t) \equiv \frac{1}{2 R_i^2 \alpha_{ii} a_{ii}} \left\{ \left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} e^{c_j} R_i R_j \alpha_{ij} a_{ij} \right)^2 - \frac{4 b_i R_i^2 \alpha_{ii} a_{ii}}{\lambda} \right\},
\end{equation}

then
\begin{equation}
0 < a_i t_i \leq |V|, \quad i = 1, \ldots, n
\end{equation}

and
\begin{equation}
0 < t_i \leq 1, \quad i = 1, \ldots, n.
\end{equation}

**Proof.** From (4.15), (4.16) and (4.17), we deduce that the left hand side of (4.18) and (4.19) holds.
By (4.15), we see that
\begin{equation}
(4.20) \quad t_i = f_i(\epsilon, t) \leq \frac{R_{a_i}}{P_i} + \sum_{j \neq i} R_j \alpha_{ij} a_{ij} \frac{t_j}{R_i^2 \alpha_{ii} a_{ii}}, \quad i = 1, \ldots, n,
\end{equation}
which implies that
\begin{equation}
(4.21) \quad \tilde{Q} t \leq P^{-1} R a,
\end{equation}
where
\begin{equation}
(4.22) \quad a := (a_1, \ldots, a_n)^T.
\end{equation}
From (4.5), (4.6) and Hölder inequality, we conclude that
\begin{equation}
(4.23) \quad a_{ij}^2 \leq a_{ii} a_{jj}, \quad a_i \leq |V|^\frac{1}{2} a_i^\frac{1}{2}, \quad i, j = 1, \ldots, n.
\end{equation}
By (4.23) and the fact that \( t_i > 0 \), for \( i = 1, \ldots, n \), we conclude that
\begin{equation}
(4.24) \quad \text{diag}\left\{ \frac{a_{11}^\frac{1}{2}}{\ldots}, \ldots, \frac{a_{nn}^\frac{1}{2}}{\ldots}\right\} Q \text{diag}\left\{ \frac{a_{11}^\frac{1}{2}}{\ldots}, \ldots, \frac{a_{nn}^\frac{1}{2}}{\ldots}\right\} t \leq \tilde{Q} t.
\end{equation}
From (1.6) and (3.8), we conclude that
\begin{equation}
(4.25) \quad (Q^{-1})_{ij} > 0, \quad i, j = 1, \ldots, n.
\end{equation}
Thus, combining (4.23), (4.24) and (4.25), we conclude that
\begin{equation}
(4.26) \quad \text{diag}\left\{ a_{11}^{-\frac{1}{2}}, \ldots, a_{nn}^{-\frac{1}{2}}\right\} Q^{-1} \text{diag}\left\{ a_{11}^{-\frac{1}{2}}, \ldots, a_{nn}^{-\frac{1}{2}}\right\} P^{-1} R a
\end{equation}
Therefore, by (3.13), (4.23), (4.25) and (4.26), we deduce that
\begin{equation}
(4.27) \quad \text{diag}\left\{ a_1, \ldots, a_n\right\} t
\end{equation}
\begin{equation}
\leq \text{diag}\left\{ a_1 a_{11}^{-\frac{1}{2}}, \ldots, a_n a_{nn}^{-\frac{1}{2}}\right\} Q^{-1} \text{diag}\left\{ a_1 a_{11}^{-\frac{1}{2}}, \ldots, a_n a_{nn}^{-\frac{1}{2}}\right\} P^{-1} R 1
\end{equation}
\begin{equation}
= |V| Q^{-1} P^{-1} R 1
\end{equation}
Thus, we get the right hand side of (4.18) holds. It follows from Jensen’s inequality that
\begin{equation}
\frac{a_i}{|V|} \geq e^{\int \frac{u^\theta v + w d\mu}{|V|}}, \quad i = 1, \ldots, n.
\end{equation}
Then the right hand sides of (4.19) follows from this and (4.27).

We now complete the proof. \(\square\)

The following result implies that we can solve constraints (4.14), so constraints (4.10) could be solved.

**Lemma 4.2.** For any \( w \in \mathcal{W} \), the equations
\begin{equation}
(4.28) \quad F(t) \equiv t - f(t) = 0, \quad t \equiv (t_1, \ldots, t_n)^T \in \mathbb{R}_n^+,
\end{equation}
admits a solution \( t \in (0, \infty)^n \), where \( \mathbb{R}_n^+ \equiv (\mathbb{R}_+)^n \), \( f(t) \equiv (f_1(t), \ldots, f_n(t))^T \).
Proof. For the sake of convenience, we write
\begin{equation}
(\alpha_1, \ldots, \alpha_n)^T < (\leq) (\beta_1, \ldots, \beta_n)^T \text{ if } \alpha_i < (\leq) \beta_i, i = 1, \ldots, n,
\end{equation}
and we use the same notation for matrices. We next find a solution to \((4.28)\) with \(\epsilon = 1\).

By \((4.19)\), we conclude that \(F(\epsilon, t)\) has no zero on the boundary of \(\Omega\) for all \(w \in \mathcal{A}\) and \(\epsilon \in [0, 1]\), where \(\Omega := (0, r_0)^n\) and \(r_0 > 1\) is a constant. Thus, we can define the Brouwer degree \(\text{deg}(F(\epsilon, t), \Omega, 0)\). Clearly,
\begin{equation}
F(0, t) = 0
\end{equation}
is equivalent to
\begin{equation}
t_i - \frac{R_i \alpha_i + \sum_{j \neq i} t_j R_j \alpha_{ij} a_{ij}}{R_i^2 \alpha_i a_{ii}} = 0, \quad i = 1, \ldots, n.
\end{equation}
We write \((4.31)\) in its vector form
\begin{equation}
\tilde{Q} t = P^{-1} R a.
\end{equation}
Since \(\tilde{Q}\) is invertible, we know that \((4.30)\) admits a unique solution
\begin{equation}
t = \tilde{Q}^{-1} P^{-1} R a,
\end{equation}
By \((4.19)\), we see that it belongs to the interior of \(\Omega\). By the fact that \(\tilde{Q}\) is positive definite, we deduce that the Jacobian of \(F(0, t)\) is positive everywhere, and hence that \(\text{deg}(F(0, t), \Omega, 0) = 1\). It is easy to check that \(F(\epsilon, t)\) is a smooth function for any \(\epsilon \in [0, 1]\).

Thus by homotopy invariance, we deduce that
\begin{equation}
\text{deg}(F(1, t), \Omega, 0) = \text{deg}(F(0, t), \Omega, 0).
\end{equation}
Now we complete the proof. \(\square\)

The following Lemma follows from Lemma \((4.2)\) immediately.

Lemma 4.3. For any \(w \in \mathcal{A}\), \((4.10)\) admits a solution \(c(w) = (c_1(w), \ldots, c_n(w))^T\) which satisfies \((4.14)\), so that \(v = w + c(w) = (w_1 + c_1(w), \ldots, w_n + c_n(w))^T\) satisfies \((3.11)\).

Define the constrained functional
\begin{equation}
J(w) := I(w + c(w)), \quad w \in \mathcal{A}.
\end{equation}
For all \(w \in \mathcal{A}\), since \(v = w + c(w)\) satisfies \((3.11)\), we conclude that
\begin{align}
\int_V (U - 1)^T Q(U - 1) d\mu &= \int_V 1^T Q(1 - U) d\mu - \frac{1^T b}{\lambda} \\
&= \int_V 1^T P^{-1} R(1 - U) d\mu - \frac{1^T b}{\lambda}.
\end{align}

By (4.1), we deduce that

\[
J(w) = \frac{1}{2} \sum_{j,k=1}^{n} b_{kj} \Gamma(v_k, v_j) d\mu + \frac{\lambda}{2} R \int_{V} \frac{1}{2} T \Gamma^{-1} \left(1 - U \right) d\mu + b^T c - \frac{1}{2} T b
\]

(4.37)

\[
= \frac{1}{2} \sum_{j,k=1}^{n} b_{kj} \Gamma(v_k, v_j) d\mu + \frac{\lambda}{2} \sum_{i=1}^{n} R_i \int_{V} \left(1 - e^{u_i^0 + w_i} \right) d\mu 
+ \sum_{i=1}^{n} b_i c_i - \frac{1}{2} \sum_{i=1}^{n} b_i.
\]

To prove Lemma 4.5, we need the following result.

**Lemma 4.4.** Suppose that \( w \in A \) and \( \tau \in (0, 1) \). Then

\[
\int_{V} e^{u_i^0 + w_i} d\mu \leq \left( \frac{\lambda}{4 P_i^2 b_i \alpha_{ii}} \right)^{\frac{1-\tau}{\tau}} \left( \int_{V} e^{u_i^0 + \tau w_i} d\mu \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n
\]

(4.38)

**Proof.** Let \( a = \frac{1}{2\tau} \). By (4.12) and Lemma 2.4 we conclude that

\[
\int_{V} e^{u_i^0 + w_i} d\mu \leq \left[ \int_{V} \left( e^{u_i^0 + w_i} \right)^{\tau} d\mu \right]^a \left[ \int_{V} \left( e^{2(u_i^0 + w_i)} \right) d\mu \right]^{1-a}
\]

\[
\leq \left[ \int_{V} \left( e^{u_i^0 + w_i} \right)^{\tau} d\mu \right]^a \left[ \frac{\lambda}{4 P_i^2 b_i \alpha_{ii}} \left( \int_{V} e^{u_i^0 + w_i} d\mu \right) \right]^{\frac{1-a}{2a-1}}, \quad i = 1, \ldots, n,
\]

and hence that

\[
\int_{V} e^{u_i^0 + w_i} d\mu \leq \left( \frac{\lambda}{4 P_i^2 b_i \alpha_{ii}} \right)^{\frac{1-a}{2a-1}} \left( \int_{V} e^{\tau (u_i^0 + w_i)} d\mu \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n.
\]

(4.40)

**Lemma 4.5.** Suppose that \( \gamma \) is the smallest eigenvalues of \( A \). Then

\[
J(w) \geq \frac{\gamma}{4} \sum_{i=1}^{n} \int_{V} \Gamma(w_i, w_i) d\mu - C(\ln \lambda + 1),
\]

(4.41)

for all \( w \in A \), where \( C > 0 \) is a constant independent of \( \lambda \).

**Proof.** From (3.8) and (4.35), we deduce that

\[
J(w) \geq \frac{\gamma}{2} \sum_{i=1}^{n} \| \nabla w_i \|_{2}^{2} + \sum_{i=1}^{n} b_i c_i.
\]

(4.42)
From (4.12) and (4.14), we deduce that
\[ e^{c_i} \geq \frac{a_i}{2R_i P_i \alpha_i a_i} \geq \frac{2b_i P_i}{\lambda R_i \alpha} = \frac{2P_i b_i}{\lambda R_i \int e^{u_i^0 + w_i} \, dx}, \quad i = 1, \ldots, n. \]

It follows from that
\[ c_i \geq \ln \frac{2P_i b_i}{R_i} - \ln \lambda \int e^{u_i^0 + w_i} \, dx, \quad i = 1, \ldots, n. \]

By Cauchy inequality with \( \varepsilon (\varepsilon_i) > 0 \) and (2.3), we deduce that
\[ \int e^w \, d\mu \leq \int e^{||\nabla w||^2_2} \, d\mu \leq \int e^{4\varepsilon||\nabla w||^2_2} \, d\mu e^{||\nabla w||^2_2} =: C(\varepsilon, G) e^{||\nabla w||^2_2}. \]

It is easy to check that
\[ ||\nabla u_i^0||^2_2 = -\int \nabla u_i^0 \Delta u_i^0 \, d\mu \]
\[ = -\int \nabla u_i^0 4\pi \sum_{s=1}^{N_i} \delta_{p_s} \, d\mu \]
\[ = -4\pi \sum_{s=1}^{N_i} u_i^0 (p_s) \]
\[ \leq 4\pi N_i \max_V |u_i^0|, \quad i = 1, \ldots, n. \]

By Lemma (4.4), (4.6) and (4.5), we conclude that
\[ \ln \int e^{u_i^0 + w_i} \, d\mu \leq \frac{1 - \tau}{\tau} \left\{ \ln \lambda - \ln \left( 4P_i^2 b_i \alpha_{ii} \right) \right\} + \frac{1}{\tau} \ln \int e^{\tau u_i^0 + sw_i} \, d\mu \]
\[ \leq 2\varepsilon \tau \|\nabla w_i\|^2_2 + 2\varepsilon \tau 4\pi N_i \max_V |u_i^0| + \frac{1 - \tau}{\tau} \left\{ \ln \lambda - \ln \left( 4P_i^2 b_i \alpha_{ii} \right) \right\} \]
\[ + \frac{\ln C}{\tau}, \quad i = 1, \ldots, n. \]

It follows from (4.47), (4.44) and (4.42) that
\[ J(w) \geq \left( \frac{\gamma}{2} - 2\varepsilon \tau \max_{1 \leq i \leq n} \{ b_i \} \right) \sum_{i=1}^n \|\nabla w_i\|^2_2 - \frac{1}{s} \sum_{i=1}^n b_i \left\{ \ln \lambda - \ln \left( 4P_i^2 b_i \alpha_{ii} \right) + \ln C \right\} \]

\[ - \sum_{i=1}^n b_i \left\{ \ln (2R_i P_i b_i \alpha_{ii}) + 2\varepsilon \tau 4\pi N_i \max_V |u_i^0| \right\} \]

Taking \( \tau \) sufficiently small in (4.48), we get the desired conclusion (4.41). \( \square \)
By Lemma 4.5, we can select a minimizing sequence \( \{w^k\} := \{(w_1^{(k)}, ..., w_n^{(k)})\} \) of the constrained minimization problem

\[
\eta = \inf \{ J(w) | w \in A \}.
\]

By Lemma 4.5 and Lemma 2.1, we see that, there exists \( w^0 := \{(w_1^{(0)}, ..., w_n^{(0)})\} \in H^0(V) \) such that, by passing to a subsequence, denoted still by \( \{w^k\} \),

\[
w_i^{(k)} \to w_i^{(0)}
\]

uniformly for \( x \in V \) as \( k \to +\infty \) for \( i = 1, ..., n \). By the fact that

\[
\lim_{k \to +\infty} J(w^k) = J(w^0),
\]

we deduce that \( J(w^0) = \eta \). Thus \( w^0 \) is a minimizer of \( J \). Next, we prove that the minimizer belongs to the interior of \( \mathcal{A} \).

**Lemma 4.6.** There holds the following inequalities

\[
\inf_{w \in \partial \mathcal{A}} J(w) \geq \frac{|V|\lambda}{2} \min_{1 \leq i \leq n} \left\{ \frac{R_i}{P_i} \right\} - C(1 + \ln \lambda + \sqrt{\lambda}),
\]

where \( C \) is constant independent of \( \lambda \).

**Proof.** For any \( w \in \partial \mathcal{A} \), from (4.13), we know that at least one of the following equalities

\[
a_i e_i \leq \frac{R_i}{P_i} (Q^{-1})_{11} a_i^2 a_{11}^{-1} + \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j} a_1 a_j a_{11}^{-1} a_{jj}^{-1} \\
\leq \frac{R_i}{P_i} (Q^{-1})_{11} a_i^2 a_{11}^{-1} + \sqrt{|V|} \sqrt{\lambda} \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j} a_1 a_j a_{11}^{-1}.
\]

By (4.52) and (4.51), we deduce that

\[
a_i e_i \leq (Q^{-1})_{11} \frac{4P_i R_i b_1 \alpha_{11}}{\lambda} + \frac{2P_i \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j} b_1 |V| \alpha_{11}}{\sqrt{\lambda}}.
\]

If other cases occur, we can establish the similar estimate. From (4.18) and (4.53), we deduce that

\[
\frac{\lambda}{2} \sum_{i=1}^{n} \frac{R_i}{P_i} \int_{V} \left( 1 - e^{e_i w_i^{(0)} + w_i} \right) \, d\mu
\]

\[
\geq \frac{|V| \lambda R_i}{2P_i} - 2 (Q^{-1})_{11} P_i R_i b_1 \alpha_{11} - P_i \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j} \sqrt{b_1 \lambda |V| \alpha_{11}}.
\]

From (4.47), (4.44) and (4.54), we obtain the desired conclusion (4.50). \( \square \)
Lemma 4.7. There exists $w_{r_e} \in \text{int } \mathcal{A}$ such that
\begin{equation}
J(w_{r_e}) - \inf_{w \in \partial \mathcal{A}} J(w) < -1.
\end{equation}

Proof. By Proposition 2.8, we see that, for all sufficiently large $r > 0$, the problem
\begin{equation}
\Delta w = r e^{u_0 + v} \left( e^{u_0 + v} - 1 \right) + \frac{4\pi N_i}{|V|}, \quad i = 1, \ldots, n,
\end{equation}
has solutions $v_{i,r}(i = 1, \ldots, n)$ so that $v_{i,r} \to -u^0_0$ as $r \to +\infty$ uniformly for $x \in V$. Let $c_{i,r} := \frac{1}{|V|} \int v_{i,r} d\mu$. It follows that $w_{i,r} := v_{i,r} - c_{i,r} \to -u^0_0$ as $r \to +\infty$, $i = 1, \ldots, n$. Thus we have
\begin{equation}
\lim_{\mu \to \infty} a_i(w_{i,r}) = |V|, \quad \lim_{\mu \to \infty} a_{ij}(w_{i,r}, w_{j,r}) = |V|, \quad i, j = 1, \ldots, n.
\end{equation}
By (4.7), we obtain
\begin{equation}
\lim_{r \to \infty} Q(w_r) = |V| Q.
\end{equation}
By (4.58) and (4.57), we can find $\sigma > 0$ such that for any $\epsilon \in (0, 1)$, there exists $r_\epsilon > 0$ so that
\begin{equation}
w_{r_\epsilon} = (w_{1, r_\epsilon}, \ldots, w_{n, r_\epsilon})^T \in \text{ int } \mathcal{A}
\end{equation}
for all $\lambda > \sigma$, and
\begin{equation}
a_{ij}(w_{i, r_\epsilon}, w_{j, r_\epsilon}) < (1 + \epsilon)|V| < 2|V|, \quad i, j = 1, \ldots, n
\end{equation}
\begin{equation}
\frac{1-\epsilon}{|V|} Q^{-1} < \tilde{Q}(w_{r_\epsilon}) < \frac{1+\epsilon}{|V|} Q^{-1} < \frac{2}{|V|} Q^{-1}.
\end{equation}
Since $w_{r_\epsilon} \in \text{ int } \mathcal{A}$, by (4.11) and the fact that
\[ \sqrt{1 - 2x} \geq 1 - 2x \text{ for } x \in [0, \frac{1}{2}], \]
we deduce that
\begin{equation}
e^{c_i(w_{r_\epsilon})} = \frac{R_{\alpha_i}}{P_i} + \sum_{j \neq i} e^{c_j(w_{r_\epsilon})} R_j R_{ij} \alpha_{ij} a_{ij}
\end{equation}
\begin{equation}
\times \left( 1 + \sqrt{1 - \frac{4b_i R^2_{\alpha_i} \alpha_{ii} a_{ii}}{\lambda \left( \frac{R_{\alpha_i}}{P_i} + \sum_{j \neq i} e^{c_j(w_{r_\epsilon})} R_j R_{ij} \alpha_{ij} a_{ij} \right)^2}} \right)
\end{equation}
\begin{equation}
\geq \frac{R_{\alpha_i}}{P_i} + \sum_{j \neq i} e^{c_j(w_{r_\epsilon})} R_j R_{ij} \alpha_{ij} a_{ij}
\left( \frac{2b_i}{R^2_{\alpha_i} \alpha_{ii} a_{ii}} \right)
\end{equation}
\begin{equation}
\frac{2b_i}{\lambda \left( \frac{R_{\alpha_i}}{P_i} + \sum_{j \neq i} e^{c_j(w_{r_\epsilon})} R_j R_{ij} \alpha_{ij} a_{ij} \right)}. \end{equation}
By (4.21) and Jensen inequility, we conclude that
\begin{equation}
e^{c_i(w_{r_\epsilon})} \geq \frac{R_{\alpha_i}|V|}{P_i} + \sum_{j \neq i} e^{c_j(w_{r_\epsilon})} R_j R_{ij} \alpha_{ij} a_{ij}
\left( \frac{2P_i b_i}{\lambda |V| R_i} \right), \quad i = 1, \ldots, n.
\end{equation}
From now on, we understand
\[(4.63)\quad a_i = a_i (w_{i, r_e}), \quad a_{ij} = a_{ij} (w_{i, r_e}, w_{j, r_e}), \quad i, j = 1, \ldots, n.\]

Then by \((4.62)\) and \((4.60)\), we deduce that
\[
(4.64)\quad R_i \alpha_{ii} a_{ii} e^{c_i (w_{r_e})} - \sum_{j \neq i} e^{c_j (w_{r_e})} R_i R_j \alpha_{ij} a_{ij} \geq \frac{R_i |V|}{P_i} - \frac{2 P_i R_j b_j}{\lambda |V|} \alpha_{ii} a_{ii} \\
\geq \frac{R_i |V|}{P_i} - \frac{4 \alpha_{ii} P_i R_i b_i}{\lambda}, \quad i = 1, \ldots, n.
\]

By \((4.60)\) and \((4.64)\), we conclude that
\[
\begin{align*}
\left( e^{c_1 (w_{r_e})}, \ldots, e^{c_n (w_{r_e})} \right)^T \\
\geq |\Omega| Q^{-1} (w_{r_e}) P^{-1} R 1 - \frac{4 \tilde{Q}^{-1} (w_{r_e}) PR}{\lambda} \text{diag} \{ \alpha_{11}, \ldots, \alpha_{nn} \} b \\
\geq (1 - \varepsilon) Q^{-1} P^{-1} R 1 - \frac{8 Q^{-1} PR}{\lambda |\Omega|} \text{diag} \{ \alpha_{11}, \ldots, \alpha_{nn} \} b \\
= (1 - \varepsilon) 1 - \frac{8 Q^{-1} PR}{\lambda |\Omega|} \text{diag} \{ \alpha_{11}, \ldots, \alpha_{nn} \} b.
\end{align*}
\]

It follows that
\[
(4.65)\quad \int_V \left( 1 - e^{c_i (w_{r_e})} e^{w_{0} + w_{r_e}} \right) d \mu \leq |V| \varepsilon + \frac{8}{\lambda} \sum_{i=1}^{n} (Q^{-1})_{ij} P_j R_j b_j \alpha_{jj}, \quad i = 1, \ldots, n.
\]

By \((4.19)\), there exists a constant \(C_{\varepsilon}\) such that
\[
(4.66)\quad J (w_{r_e}) \leq \frac{|V| |\lambda| \varepsilon}{2} \sum_{j=1}^{n} R_i P_i + C_{\varepsilon}.
\]

By Lemma 4.6, this implies that there exists \(C\) independent of \(\lambda\) so that
\[
(4.67)\quad J (w_{r_e}) - \inf_{w \in \partial \mathcal{A}} J (w) \leq \frac{|V| |\lambda| \varepsilon}{2} \left( \sum_{j=1}^{n} \frac{R_i}{P_i} \varepsilon - \min_{1 \leq i \leq n} \left\{ \frac{R_i}{P_i} \right\} \right) + C(\sqrt{\lambda} + \ln \lambda + 1).
\]

We can get \((4.7)\) by taking \(\varepsilon\) suitably small and \(\lambda\) sufficiently large in \((4.68)\).

We now prove the lemma. \(\square\)

From Lemmas 4.5 and 4.7, there exists \(\lambda_2 := \max\{ \sigma, \lambda_0 \}\) such that for all \(\lambda > \lambda_2\), we can find \(w_0 \in \text{int} \mathcal{A}\) such that \(w_0\) is a minimizer of \(J\). It is easy to check that \(v_0 := w_0 + c(w_0)\) is a critical point of \(J\), which implies that \(v_0\) is a solution of equations \((3.5)\). Thus, we could establish the conclusion (ii) of Theorem 1.1.

REFERENCES

[1] A. A. Abrikosov, On the magnetic properties of superconductors of the second group, Sov. Phys. JETP 5, (1957): 1174–1182.
[2] E. B. Bogomol’nyi, The stability of classical solutions, Soviet J. Nuclear Phys., 24 (1976), pp. 449-454.
[3] Grigor’yan, Alexander, Yong Lin, Yun Yan Yang, Kazdan–Warner equation on graph, Calculus of Variations and Partial Differential Equations, 55 (2016): 1-13.
[4] D. Chae, O. Y. Imanuvilov, Non-topological solutions in the generalized self-dual Chern-Simons-Higgs theory, Calc. Var. Partial Differential Equations, 16 (2003), no. 1, 47-61.
[5] L. Caffarelli, Y. Yang, Vortex condensation in the Chern-Simons-Higgs model: an existence theorem, Comm. Math. Phys., 168 (1995) 321–336.
[6] Y. Hu, Existence of solutions to a generalized self-dual Chern-Simons equation on finite graphs, arXiv preprint arXiv:2202.09546 (2022).
[7] Y. Hu, Existence and uniqueness of solutions to the Bogomol'nyi equation on graphs, arXiv preprint arXiv:2202.05039 (2022).
[8] Y. Hu, Existence and uniqueness of solutions to Bogomol'nyi-Prasad-Sommerfeld equations on graphs, arXiv preprint arXiv:2202.09546 (2022).
[9] Y. Hu, Existence and uniqueness of solutions to non-Abelian multiple vortex equations on graphs, arXiv preprint arXiv:2203.01498 (2022).
[10] H. Huang, J. Wang, W. Yang, Mean field equation and relativistic Abelian Chern-Simons model on finite graphs, Journal of Functional Analysis, (2021): 109218.
[11] A. Huang, Y. Lin, S. Yau, Existence of Solutions to Mean Field Equations on Graphs, Communications in Mathematical Physics, 377 (2019): 613-621.
[12] X. Han, C. S. Lin, Y. Yang, Resolution of Chern–Simons–Higgs Vortex Equations, Communications in Mathematical Physics, 2016, 343(2): 701-724.
[13] Y. Lü, P. Zhong, Existence of solutions to a generalized self-dual Chern-Simons equation on graphs, arXiv preprint arXiv:2107.12555 (2021).
[14] Y. Lin Y. Wu, Blow-up problems for nonlinear parabolic equations on locally finite graphs, Acta Mathematica Scientia, 38(3) (2018): 843-856.
[15] C. Lin, Y. Yang, Non-Abelian multiple vortices in supersymmetric field theory. Communications in mathematical physics, 304.2 (2011): 433-457.
[16] H. B. Nielsen, P. Olesen, Vortex line models for dual strings, Nuclear Phys. B, 61 (1973): 45–61.
[17] G. Tarantello, Multiple condensate solutions for the Chern-Simons–Higgs theory, J. Math. Phys, 37 (1996): 3769–3796.
[18] D. H. Tchrakian, Y. Yang, The existence of generalised self-dual Chern-Simons vortices, Lett. Math. Phys., 36 (1996), no. 4, 403-413.
[19] S. Wang, Y. Yang, Abrikosov’s vortices in the critical coupling, Siam Journal on Mathematical Analysis, 23 (1992): 1125-1140.
[20] Y. Yang, Chern-Simons solitons and a nonlinear elliptic equation, Helv. Phys. Acta, 71 (1998), no. 5, 573-585.