Solitons in a chiral quark model with non-local interactions

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Abstract

Hedgehog solitons are found in a chiral quark model with non-local interactions. The solitons are stable without the chiral-circle constraint for the meson fields, as was assumed in previous Nambu-Jona–Lasinio model with local interactions.

Key words: Effective chiral models, chiral solitons, models of baryons

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In recent years a lot of efforts have been undertaken to describe baryons as solitons of effective chiral models. Of particular interest are models which include the Dirac sea, since they allow for a unified description of mesons and baryons. So far solitons have only been obtained in the Nambu–Jona-Lasinio model with the proper-time or Pauli-Villars regularization of the Dirac sea \cite{1–3}.\footnote{Recently solitons have also been reported in the Nambu–Jona-Lasinio model with the sharp 3-momentum cut-off, but only in the Thomas-Fermi approximation for the Dirac sea \cite{4}. These solitons cease to exist when the mean-field calculation of the Dirac sea is carried out exactly.} One problem encountered with the proper-time regularization, by far the most commonly used, is that solitons turn out to be unstable unless the sigma and pion fields are constrained to the chiral circle \cite{5,6}. Such a constraint is external to the model defined at the quark level. Furthermore, somewhat artificially, the regularization in these models is applied only to the real (non-anomalous) part of the Euclidean quark loop term. The (finite) imaginary part is left unregularized in order to properly describe anomalous processes. The above-mentioned problems disappear in models with non-local quark interactions, which result, for example, from QCD derivations.

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of low-energy effective theories [7–9]. In such models, the quark propagators have 4-momentum-dependent masses which regularize the theory in the Euclidean domain. This regularization is in fact far more satisfying than in local models. Indeed, it is no longer required to regularize differently the real and imaginary parts. Furthermore, anomalous processes turn out not to depend on the cut-off, and are properly described [10–12]. Finally, the momentum-dependent regulator makes the theory finite to all orders in the $1/N_c$ expansion. This is in contrast to local models, where inclusion of higher-order-loop effects requires extra regulators [13], and the predictive power is thereby reduced. The price to pay is a greater difficulty in calculating hadron properties. In view of the aforementioned advantages, it is a challenge to discover whether solitons exist in these theories. The purpose of this Letter is to report that indeed stable solitons do exist in a chiral quark model with non-local interactions.

Our model is defined by the Euclidean action:

$$I = -\text{Tr} \log (-i\partial_\mu \gamma_\mu + m + r\Phi) + \frac{1}{2G^2} \int d^4x \left( S^2 + P_a^2 \right)$$

(1)

where $\Phi = S + i\gamma_5 P_a \tau_a$ is the SU(2) chiral field. The chiral field is the dynamical variable of the system and it is local in coordinate space. The trace is in color, isospin, Dirac indices and space-time. The average $u$ and $d$ current quark mass is denoted by $m$. The coupling constant $G$ has the dimension of inverse energy. The operator $r$ is the regulator, chosen to be a scalar function diagonal in the 4-momentum space: $\langle k|r|k'\rangle = \delta(k - k')r_k$. It introduces a non-locality in the interaction between quarks and the chiral field. The particular way of introducing the non-locality is the one which results from instanton-induced interactions [7]. It has also been used in Ref. [14].

We choose the Gaussian form $r_k = e^{-k^2/\Lambda^2}$, where $\Lambda$ is a cut-off parameter. The form of $r$ is not critical for results presented here.

Our treatment of model (1) is the same as in other soliton calculations in effective chiral models: we treat the chiral $S$ and $P_a$ fields classically, which amounts to keeping the leading-order contribution in the number of colors, $N_c$. For sufficiently large values of $G$ the model leads to dynamical chiral symmetry breaking. The vacuum value of the scalar field, $S_0$, is determined by the stationary-point condition $\delta I/\delta S|_{S=S_0, P_a=0} = 0$, which explicitly

\[ \{S(x), P_a(x)\} = -G^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{-i(p_1+p_2)\cdot x} \bar{\psi}(p_1) r_{p_1} \{1, i\gamma_5 \tau_a\} r_{p_2} \psi(p_2). \]

\[ \{S(x), P_a(x)\} = -G^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{-i(p_1+p_2)\cdot x} \bar{\psi}(p_1) r_{p_1} \{1, i\gamma_5 \tau_a\} r_{p_2} \psi(p_2). \]
gives:

\[
\frac{1}{G^2} = 4N_cN_f \int \frac{d_4k}{(2\pi)^4} \frac{r_k^2 \left( r_k^2 + \frac{m}{S_0} \right)}{k^2 + (m + r_k^2S_0)^2}. \tag{2}
\]

The pion mass, \( m_\pi = 139 \text{ MeV} \), is identified with the location of the pole of the pion propagator, and the pion decay constant, \( F_\pi = 93 \text{ MeV} \), is easily obtained via the Gell-Mann-Oakes-Renner relation \([15,16]\). We keep the leading terms in the chiral expansion:

\[
F_\pi^2 = 2N_cN_f f + \mathcal{O} \left( m^2 \right), \quad m_\pi^2 = \frac{2mg}{S_0f} + \mathcal{O} \left( m^2 \right), \tag{3}
\]

where

\[
g = \int \frac{d_4k}{(2\pi)^4} \frac{r_k^2}{k^2 + (r_k^2S_0)^2} \quad \text{and} \quad f = \int \frac{d_4k}{(2\pi)^4} \frac{r_k^4 - k^2r_k^2\frac{dr_k^2}{dk^2} + k^4 \left( \frac{dr_k^2}{dk^2} \right)^2}{\left( k^2 + (r_k^2S_0)^2 \right)^2}. \tag{4}
\]

The model (1) has three parameters: \( G \), \( \Lambda \), and \( m \). The stationary-point condition (2) and the equations (3) allow us to express them in terms of one parameter, \( S_0 \). Our numerical results are presented for various choices of \( S_0 \).\textsuperscript{2}

The static meson fields are determined self-consistently by solving the Euler-Lagrange equations deduced from the action (1). In order to evaluate the trace in functional space it is convenient to introduce the energy-dependent Dirac Hamiltonian, \( h (\nu^2) \), given by

\[
h \left( \nu^2 \right) = -i\bar{\alpha} \cdot \nabla + \beta r \left( \nu^2 - \bar{V}^2 \right) \Phi r \left( \nu^2 - \bar{V}^2 \right) + \beta m, \tag{5}
\]

with eigenstates \( |\lambda_\nu\rangle \) and eigenvalues \( e_\lambda (\nu^2) \) satisfying \( h (\nu^2) |\lambda_\nu\rangle = e_\lambda (\nu^2) |\lambda_\nu\rangle \). We obtain

\[
\{ S (\vec{x}) , P_a (\vec{x}) \} = G^2 \int \frac{d\nu}{2\pi} \sum_\lambda \frac{\langle \lambda_\nu | r | \vec{x} \rangle \langle \beta, i\beta \gamma_5 \tau_a | \vec{x} \rangle \langle \vec{x} | r | \lambda_\nu \rangle}{i\nu + e_\lambda (\nu^2)}. \tag{6}
\]

The integration contour over \( \nu \) in Eq. (6) has to be chosen in such a way as to include the valence quark orbit. This point requires a careful explanation. It is well known that in hedgehog solitons a \( 0^+ \) (grand-spin zero, positive-parity) orbit becomes bound and well

\textsuperscript{3} It should be noted that unlike the NJL model with local interactions, \( S_0 \) does not have the interpretation of the constituent quark mass. It does not correspond to the location of the pole in the quark propagator, which is obtained by the condition \( k^2 + (r_k^2S_0 + m)^2 = 0 \). In fact, in the vacuum sector, for sufficiently large \( S_0 \) the quark propagator has no poles on the physical axis, \( \text{i.e.} \) for real negative \( k^2 \). See the discussion following of Fig. 2.
isolated from other orbits [17]. This feature persists in the present model. We search for a pole of the quark propagator in the complex $\nu$ plane by solving the equation

$$i\nu + e_0(\nu^2) = 0,$$

where $e_0(\nu^2)$ is the eigenvalue of the valence $0^+$ orbit calculated with the Dirac Hamiltonian $h(\nu^2)$. The value of $\nu$ which solves Eq. (7) is denoted by $i\nu_{val}$, and the corresponding orbit by $|\nu_{val}\rangle$. We find that it is always possible to find a solution of Eq. (7) in the background of soliton fields, even for choices of model parameters for which the quarks do not materialize on-shell in the vacuum [3,14,18]. The solution of Eq. (7) corresponds to a pole of the quark propagator which lies on the imaginary (i.e., physical) axis close to $\nu = 0$. It is well separated from other (unphysical) poles in the complex plane. Such poles are associated with the presence of the regulator [19]. In all cases described in this Letter $\nu_{val} > 0$. The inclusion of the valence orbit in Eq. (6) yields

$$\{ S(\vec{x}), P_a(\vec{x}) \} = G^2(q_{\text{val}}) \{ \beta, i\beta\gamma_5\tau_a \} \langle \vec{x}|r|\nu_{\text{val}}\rangle + G^2 \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \sum_{\lambda} \langle \lambda_{\nu}|r|\vec{x}\rangle \{ \beta, i\beta\gamma_5\tau_a \} \langle \vec{x}|r|\lambda_{\nu}\rangle.$$  

(8)

The second term is the Dirac sea contribution, where the integral over $\nu$ is performed numerically along the real $\nu$-axis. Note that in the absence of the regulator ($r = 1$) the eigenvalues are energy-independent, the integral over $\nu$ can be done via the Cauchy theorem, and the Dirac-sea part of Eq. (6) reduces to the usual formula expressed by the sum over the negative-energy spectrum.

There is a non-trivial problem associated with theories with non-local interactions: Noether currents, in particular the baryon current, acquire extra contributions induced by the presence of the regulator $r$ [10,16,15]. By performing the gauge transformation, $\psi(x) \rightarrow e^{i\eta(x)}\psi(x)$, the quark part of the action is transformed into:

$$- \text{Tr} \log e^{-i\eta}(-i\partial_{\mu}\gamma_{\mu} + r\Phi_{\mu})e^{i\eta} = -\text{Tr} \log \left(-i\partial_{\mu}\gamma_{\mu} + \frac{\partial\eta}{\partial x_{\mu}}\gamma_{\mu} + e^{-i\eta}r\Phi_{\mu}e^{i\eta} \right) + \frac{1}{2\pi i N_c} \int d\nu \sum_{\lambda} \frac{i + e_{\lambda}(\nu^2)}{i\nu + e_{\lambda}(\nu^2)}.$$  

(9)

The extra contributions are due to the term $e^{-i\eta}r\Phi_{\mu}e^{i\eta}$. When these are taken into account, the expression for the baryon number becomes:

$$B = \frac{1}{i} \frac{\delta I(\eta)}{\delta \partial\eta/\partial x_{\mu}} = -\frac{1}{2\pi i N_c} \int d\nu \sum_{\lambda} \frac{i + e_{\lambda}(\nu^2)}{i\nu + e_{\lambda}(\nu^2)}.$$  

(10)

The non-local regulator contributes the extra term $\frac{e_{\lambda}(\nu^2)}{\partial\nu}$, which causes the residues of poles of the quark propagator to be normalized to 1. As a result, the baryon number is properly quantized, independently of the shape of the regulator.\footnote{Note that the baryon number in not quantized (although it is enforced to be equal to one)
The energy of the time independent soliton can be deduced from the euclidean action (1):

\[ E_{\text{sol}} = N_c e_{\text{val}} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \nu d\nu \sum_{\lambda} \frac{i + \frac{e_{\lambda}}{\nu}}{i\nu + e_{\lambda} (\nu^2)} + \frac{1}{2G^2} \int d^3x \left( S^2 + P_{\text{a}}^2 \right) - \text{vac}, \]  

(11)

where \( \text{vac} \) denotes the vacuum subtraction. The same expression can be derived from the Noether construction of the energy-momentum tensor.

![Graph](image)

**Fig. 1.** Self consistent fields for \( S_0 = 400 \text{ MeV} \)

Our numerical calculations were performed with the hedgehog-shaped chiral field, in which \( S(\vec{x}) = S(|\vec{x}|) \) and \( P_{\alpha}(\vec{x}) = \hat{x}_{\alpha}P(|\vec{x}|) \). The Euler-Lagrange equations (8) are solved by iteration. The Kahana-Ripka spherical plane-wave basis [17] is well suited to calculate the matrix elements of the Dirac Hamiltonian \( h(\nu^2) \). The quark orbits are calculated by diagonalizing \( h(\nu^2) \) in this basis, successively for each value of the energy \( \nu \). We find convergence without forcing the fields to remain on the chiral circle, as was the case of solitons in Refs. [1–3]. Figure 1 shows the self-consistent fields obtained for \( S_0 = 400 \text{ MeV} \). The departure from the chiral circle is substantial. Increasing \( S_0 \) brings the fields closer to the chiral circle. It is important to stress that our non-local regularization cuts off effects due to high gradients in the chiral fields. Such effects were responsible for the instability in the case of the proper-time regularization [5,6].

Figure 2 shows that our soliton is energetically stable above \( S_0 \simeq 280 \text{MeV} \). We plot the soliton mass (dot-dashed line) and 3 times the vacuum quark mass, \( 3M_q \) (solid line). In the approach of Ref. [20], where the proper-time regularization is applied to the imaginary (anomalous) part of the effective action.
The quantity $M_q$ is defined as the pole of the quark propagator in the vacuum, i.e. corresponds to the solution of the equation $k^2 + (r^2 S_0 + m)^2|_{k^2 = -M_q^2} = 0$. Real solutions to this equation exist only below the critical point denoted by a circle at $S_0 = 309\text{MeV}$. Thus in the region $280\text{MeV} < S_0 < 309\text{MeV}$ the mass of the soliton is lower than the mass of three quarks, hence the soliton is bound. Above $S_0 = 309\text{MeV}$ the soliton is also stable, since there are no asymptotic quark states (no real quark poles) into which it could decay to. Below $S_0 = 280\text{MeV}$ the soliton is not energetically stable.

Table 1 lists several soliton properties with various values of the vacuum scalar field. Parameters $\Lambda$ and $m$ are determined from (3), and $G$ from the stationarity point condition (2). Different contributions to the soliton energy are given: the energy of the valence orbit obtained from (7), the Dirac sea contribution (the second term in (11)). The contribution of the third term in (11) is not explicitly given since it can be deduced from the total energy and other components given in Table 1. The soliton mean square baryon radius $\langle r^2 \rangle$ is calculated from the baryon density. Its main contribution is due to the
valence quarks, and the Dirac sea contributes only about 2 %. The axial-vector charge of
the nucleon, \( g_A \), is deduced from the asymptotic behavior of the pion field, given by
\[ P(r) \sim -D \left( \frac{1}{r^2} + \frac{m_\pi}{r} \right) \exp \left( -\frac{m_\pi}{r} \right), \]
and equals to \([21,22]\) \( g_A = \frac{8\pi DF_\pi}{3} \). For \( S_0 \) above
300 MeV the value of \( g_A \) is somewhat smaller than the experimental value of 1.26, but
50 % larger than the leading-\( N_c \) value obtained using proper time regularization and
constraining the chiral field to remain on the chiral circle \([1]\). Our value of \( g_A \) is intermediate
between too low Skyrme values (on the chiral circle) and too high values obtained in
T. D. Lee solitons (where the pion field vanishes) which are more akin to the MIT bag
model.

Our soliton is a mean-field hedgehog solution which, of course, should not be identi-
fied with the nucleon. Quantizing the collective coordinates, or projecting the soliton
wavefunction on the subspace with nucleon quantum numbers, will reduce its energy by
eliminating the spurious rotational and translational motion. It is therefore not undesir-
able that the energies in Table 1 be higher than the experimental nucleon mass. We may
similarly expect that the large radius of the soliton will be reduced when the center-of-
mass corrections are calculated. The aim in this Letter was to investigate the mechanism
of soliton formation and its stability in chiral models with non-local regulators, rather
than to make detailed predictions for various nucleon observables. Further properties of
the soliton and a more detailed account of the calculation will be published shortly.

References

[1] C. Christov, A. Blotz, H. Kim, P. Pobylitsa, T. Watabe, Th. Meissner, E. Ruiz Arriola, and
K. Goeke, Prog. Part. Nucl. Phys. 37 (1996) 1
[2] R. Alkofer, H. Reinhardt, and H. Weigel, Phys. Rep. 265 (1996) 139
[3] G. Ripka, Quarks Bound by Chiral Fields (Oxford University Press, Oxford, 1997)
[4] M. Fiolhais, J. da Providência, M. Rosina, and C. A. de Sousa, Phys. Rev. C56 (1997) 3311
[5] P. Sieber, T. Meissner, F. Grümmer, and K. Goeke, Nucl. Phys. A 547 (1992) 459
[6] Th. Meissner, G. Ripka, R. Wünsch, P. Sieber, F. Grümmer, and K. Goeke, Phys. Lett. B
299 (1993) 183
[7] D. I. Diakonov and V. Y. Petrov, Nucl. Phys. B 272 (1986) 457
[8] R. D. Ball, Int. Journ. Mod. Phys. A 5 (1990) 4391
[9] C. D. Roberts, in QCD Vacuum Structure (World Scientific, Singapore, 1992), p. 114
[10] R. D. Ball and G. Ripka, in Many Body Physics (Coimbra 1993), edited by C. Fiolhais, M.
Fiolhais, C. Sousa, and J. N. Urbano (World Scientific, Singapore, 1993)
[11] J. Praschifka, C. D. Roberts, and R. T. Cahill, Phys. Rev. D 36 (1987) 209
[12] B. Holdom, J. Terning, and K. Verbeek, Phys. Lett. B 232 (1989) 351
[13] E. N. Nikolov, W. Broniowski, C. Christov, G. Ripka, and K. Goeke, Nucl. Phys. A 608 (1996) 411

[14] M. Buballa and S. Krewald, Phys. Lett. B 294 (1992) 19

[15] R. S. Plant and M. C. Birse, Nucl. Phys. A 628 (1998) 607

[16] R. D. Bowler and M. C. Birse, Nucl. Phys. A 582 (1995) 655

[17] S. Kahana and G. Ripka, Nucl. Phys. A 429 (1984) 462

[18] C. Burden, C. Roberts, and A. Williams, Phys. Lett. B 285 (1992) 347

[19] W. Broniowski, G. Ripka, E. N. Nikolov, and K. Goeke, Zeit. Phys. A 354 (1996) 421

[20] J. Schlienz, H. Weigel, H. Reinhardt, and R. Alkofer, Phys. Lett. B 315 (1993) 6

[21] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. B 228 (1983) 552

[22] T. D. Cohen and W. Broniowski, Phys. Rev. D 34 (1986) 3472