Masses of Quarks and Leptons and Mixing Angles in Anti-GUT Theory.

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Abstract

We describe a fit to the charged fermion mass hierarchy using the chiral quantum numbers of the maximal anti-grand unification group $SMG^3 \times U(1)_f$, where $SMG \equiv SU(3) \times SU(2) \times U(1)$. This fit suggests a set of Higgs fields responsible for the breakdown, near the Planck scale, of $SMG^3 \times U(1)_f$ to the Standard Model group.

1 Introduction: The MPP and AGUT Model

Over many years we have gradually developed a model\textsuperscript{1}, in which we are able to understand or fit a large number of the parameters—coupling constants and masses—in the Standard Model (SM), based on two assumptions which we have called: Multiple Point Principle (MPP) and Anti Grand Unified Theory (AGUT) respectively. This model is not specified in full detail, but allows, for example, a lot of unspecified particles with masses of the order of the Planck scale. There is a desert, with just SM interactions, essentially all the way up to an order of magnitude or so under the Planck energy, $M_{\text{Planck}} \simeq 10^{19}$ GeV. This is a larger energy range than most physicists expect for the validity of the pure SM; in particular we assume there is no supersymmetry in the desert.

The MPP can be formulated as the requirement that there shall be many “vacua” with essentially the same energy density; in the Euclideanised version of the theory, there is a corresponding phase transition. This requirement of degenerate vacua is then used to derive the values of various coupling constants or relations between them. If this requirement is imposed\textsuperscript{2} on the pure SM with a cut-off close to $M_{\text{Planck}}$, the values of the top quark and Higgs masses ($M_t, M_H$) must lie on the so-called vacuum stability curve. In order for the vacuum degeneracy requirement to have a good chance of being physically relevant, the vacuum expectation value (VEV) of the Higgs field $\phi$ in the second vacuum (the VEV in the first or usual vacuum is of course 246 GeV) must be of the same order of magnitude as the cut-off. This strongly first order phase transition condition selects a particular point on the vacuum stability curve, giving our SM predictions\textsuperscript{2} for the top quark and Higgs boson pole masses:

$$M_t = 173 \pm 5 \text{ GeV} \quad M_H = 135 \pm 9 \text{ GeV}$$

(1)

The AGUT model is based on extending the SM gauge group, $SMG = S(U(2) \times U(3)) \approx SU(3) \times SU(2) \times U(1)$, in much the same way as grand unified $SU(5)$ but to the non-simple gauge group $SMG^3 \times U(1)_f$, where the SM gauge group is identified as the diagonal subgroup of the $SMG^3$ group. This means that, near the Planck scale, each of the three quark-lepton generations has its own set of SM-like gauge particles together with an additional abelian $U(1)_f$ gauge boson. The gauge coupling constants are not unified but their values are predicted\textsuperscript{1} using the MPP principle. The $SMG^3$ gauge quantum numbers for the three quark-lepton generations are assigned in the obvious way. We give their $U(1)_f$ charges, $Q_f$, in section 2, where we discuss the AGUT group. These new chiral gauge
quantum numbers distinguish between the three generations and we use their partial conservation to naturally generate the charged fermion mass hierarchy. The main rule of the game in our model is that any coupling constant—at the fundamental level, the Planck scale presumably—is of order unity, except for the Higgs field expectation values. Therefore every quantity—such as effective running Yukawa couplings at the Planck scale—is of order unity in the fundamental (Planck) units, except for fermion mass suppression factors; these are taken to be the product of the Higgs field VEVs, counted in Planck units, over the number of Higgs fields needed to provide the symmetry breaking (of our AGUT group $SMG^3 \times U(1)$) to make the mass matrix element in question non-zero. Here, of course, any Higgs field that is needed several times delivers its expectation value to the corresponding power. We even make the assumption that every type of, say, fermion field needed with Planck mass can be found: everything happens at the Planck scale, and with unit strength! (This is contrary to some models in which the lack of some types of, say, fermions at this scale plays an important role). Roughly it is our philosophy that everything allowed can be found at the Planck scale.

2 The Maximal AGUT Group

The $SMG^3 \times U(1)^f$ group, with its 37 generators, at first seems a rather arbitrary choice for a “unified group”. However it can be characterized uniquely as the gauge group $G$ beyond the SM containing the SM group and satisfying the following 4 postulates:

1. $G \subseteq U(45)$. Here $U(45)$ is the group of all unitary transformations of the 45 species of Weyl fields (3 generations with 15 in each) in the SM.

2. No anomalies. There should be neither gauge anomalies nor mixed anomalies. We assume that only straightforward anomaly cancellation takes place and, as in the SM itself, do not allow for a Green-Schwarz type anomaly cancellation.

3. The various irreducible representations of Weyl fields for the SM group remain irreducible under $G$. This postulate is motivated by the observation that combining SM irreducible representations into larger unified representations introduces symmetry relations between Yukawa coupling constants, whereas the particle spectrum exhibits a hierarchy between essentially all the fermion masses rather than exact degeneracies.

4. $G$ is the maximal group satisfying the other 3 postulates.

A rather complicated calculation shows that, modulo permutations of the various SM fermion irreducible representations, we are led to the result $G = SMG^3 \times U(1)^f$ with the usual SM group embedded as the diagonal subgroup of $SMG^3$. Apart from the various permutations of the particle names, the $U(1)^f$ group is unique. The $Q_f$ charges can then be chosen so that the only non-zero values are carried by the right-handed fermions of the second and third proto-generations:

$$Q_f(\tau_R) = Q_f(b_R) = Q_f(c_R) = 1 \quad Q_f(\mu_R) = Q_f(d_R) = Q_f(t_R) = -1$$ (2)

However we do have the freedom of choosing the gauge quantum numbers of the Higgs fields responsible for breaking the $SMG^3 \times U(1)^f$ group down to the SM group near the
Planck scale. So we choose their quantum numbers with a view to fitting the fermion mass and mixing angle data, extrapolated to the Planck scale using the SM renormalisation group equations. We are thereby led to introduce three Higgs fields, $W$, $T$ and $\xi$, with VEVs an order of magnitude or so below $M_{\text{Planck}}$. In addition we introduce a Higgs field $S$ with a VEV of order unity in Planck units. Furthermore we have to assign AGUT gauge quantum numbers to the Weinberg-Salam Higgs field $\phi_{WS}$. The existence of a field $S$, which does not suppress the fermion masses, means that we cannot control phenomenologically when this $S$-field is used in the mass matrices. Thus all the quantum numbers of the other Higgs fields, found by fitting data, can only have their quantum numbers predicted modulo those of the field $S$. We specify the Higgs field abelian quantum numbers as a charge vector $\vec{Q} \equiv (y_{1}/2, y_{2}/2, y_{3}/2, Q_{i})$, where $y_{i}/2$ denotes the weak hypercharge for the $i$th proto-generation. We then determine their non-abelian representations, by imposing the natural generalisation of the SM charge quantisation rule:

$$y_{i}/2 + d_{i}/2 + t_{i}/3 = 0 \pmod{1} \quad (3)$$

We also require that the non-abelian representations be the smallest possible (singlet or fundamental like the fermions) with the dualities $d_{i}$ and/or trialities $t_{i}$ determined from the quantisation rule of eq. (3).

### 3 Fermion Masses and Mixing Angles

We have chosen the Higgs field quantum numbers up to the above-mentioned ambiguity modulo those of the field $S$ and obtained\footnote{We also included a factorial factor in each matrix element keeping track of the number of permutations of the Higgs fields mediating the corresponding quantum number transition; these factorials essentially have the effect of renormalising the VEVs in the fit.} the following order of magnitude effective SM Yukawa coupling matrices:

$$Y_{U} \simeq \begin{pmatrix} W T^{2} \xi^{2} & W T^{2} \xi & W^{2} T \xi \\ W T^{2} \xi^{3} & W T^{2} & W^{2} T \\ \xi^{3} & 1 & W T \end{pmatrix} \quad Y_{D} \simeq \begin{pmatrix} W T^{2} \xi^{2} & W T^{2} \xi & T^{3} \xi \\ W T^{2} \xi & W T^{2} & T^{3} \\ W^{2} T^{4} \xi & W^{2} T^{4} & W T \end{pmatrix} \quad (4)$$

for the up and down type quarks, and for the charged leptons we have:

$$Y_{E} \simeq \begin{pmatrix} W T^{2} \xi^{2} & W T^{2} \xi^{3} & W T^{4} \xi \\ W T^{2} \xi^{5} & W T^{2} & W T^{4} \xi^{2} \\ W T^{5} \xi^{3} & W^{2} T^{4} & W T \end{pmatrix} \quad (5)$$

Here $W$, $T$ and $\xi$ denote the VEVs of the Higgs field in Planck units. By including the order of one $S$ field VEV in the fit\footnote{We also included a factorial factor in each matrix element keeping track of the number of permutations of the Higgs fields mediating the corresponding quantum number transition; these factorials essentially have the effect of renormalising the VEVs in the fit.} we get a set of results dependent on the ambiguity in the quantum number choice. In table \footnote{We also included a factorial factor in each matrix element keeping track of the number of permutations of the Higgs fields mediating the corresponding quantum number transition; these factorials essentially have the effect of renormalising the VEVs in the fit.}, we present the results for the following set of quantum numbers, chosen on the principle of using small representations:

$$\vec{Q}_{\phi_{WS}} = (1/6, 1/2, -1/6, 0) \quad \vec{Q}_{W} = (-1/6, -1/3, 1/2, -1/3) \quad \vec{Q}_{T} = (-1/6, 0, 1/6, 1/3)$$

$$\vec{Q}_{\xi} = (0, 0, 0, 1) \quad \vec{Q}_{S} = (1/6, -1/6, 0, -1) \quad (6)$$

The quantum numbers $\vec{Q}_{\phi_{WS}}$ have been chosen to ensure that the order of one top quark Yukawa coupling corresponds to an off-diagonal element of $Y_{U}$.
Table 1: Best fit to experimental data. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

|            | \(m_u\)         | \(m_d\)         | \(m_e\)         | \(m_c\)         | \(m_s\)         | \(m_\mu\)        |
|------------|-----------------|-----------------|-----------------|-----------------|-----------------|------------------|
| Fitted     | 2.9 MeV         | 9.9 MeV         | 0.71 MeV        | 0.98 GeV        | 426 MeV         | 88 MeV           |
| Experimental | 4 MeV           | 9 MeV           | 0.5 MeV         | 1.4 GeV         | 200 MeV         | 105 MeV          |

|            | \(M_t\)         | \(m_b\)         | \(m_\tau\)      | \(V_{us}\)      | \(V_{cb}\)      | \(V_{ub}\)       |
|------------|-----------------|-----------------|-----------------|-----------------|-----------------|------------------|
| Fitted     | 153 GeV         | 7.4 GeV         | 1.35 GeV        | 0.16            | 0.030           | 0.0033           |
| Experimental | 180 GeV         | 6.3 GeV         | 1.78 GeV        | 0.22            | 0.041           | 0.0035           |

The most characteristic feature of the AGUT Yukawa matrices \(Y_U\), \(Y_D\) and \(Y_E\) is that their diagonals are equal order of magnitudewise. This feature follows from the quark-lepton quantum numbers, which all follow from the general structure of the model, and is independent of the choice of Higgs fields. Apart from the top and charm quarks, the fermion mass eigenvalues are given in order of magnitude by the diagonal elements and hence the AGUT model simulates the GUT SU(5) mass predictions, namely the degeneracy of the \(dsb\)-quarks with the charged leptons in the corresponding generations. Note, however, that we only get the prediction of these degeneracies at the Planck scale as far as order of magnitude is concerned, and not exactly! This gives much better agreement with experiment than exact SU(5) predictions, which are rather bad unless more Weinberg-Salam Higgs fields are included a la Georgi-Jarlskog’s factor 3 mechanism.

Also note that we in addition predict that the up-quark is degenerate with the down-quark and the electron. This does not follow just from GUT SU(5), although the up-quark is equally, not to say better, degenerate with the electron than the down quark!

It is possible to obtain rather simple relations from our model by eliminating the suppression factors. First one gets the already mentioned degeneracy of the masses in the same generation, except for the top and the charm quarks (all after transport by the renormalisation group to the Planck scale). In addition we have the following order of magnitude Planck scale relations:

\[
m^3_b \simeq m_t m_c m_s \\
V_{ub} \simeq V_{td} \simeq V_{us} V_{cb} \\
V_{us} \simeq V_{cd} \simeq \sqrt{\frac{m_d}{m_s}} \\
V_{cb} \simeq V_{ts} \simeq \frac{m^2_s}{m_c m_b}
\]  

We also predict the CP-violating area of the “unitarity triangle” to be given order of magnitudewise by \(J \simeq V_{us} V_{cb} V_{ub}\).

References

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