C*-ALGEBRAS WITH NORM CONTROLLED DUAL LIMITS AND NILPOTENT LIE GROUPS.

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Abstract. Motivated by the description of the C*-algebras of 5 dimensional nilpotent Lie groups as algebras of operator fields defined over their spectra, we introduce the family of C*-algebras with norm controlled dual limits and we show that the C*-algebras of the 5 dimensional nilpotents Lie groups belong to this class.

1. Introduction.

1.1. In recent papers, the C*-algebra of the Heisenberg groups, of threadlike groups and “ax + b′′-like groups have been described as algebras of operator fields (see [6] and [7]). For this description a precise understanding of the topology of the spectrum of these groups was essential (see for instance [1] for the case of threadlike groups). In this paper we study the group C*-algebra of all connected nilpotent Lie groups of dimension ≤ 5 as algebra of operator fields. This family of Lie groups has been classified by several authors, a list can be found for instance in [9]. It contains the Heisenberg groups of dimensions 3 and 5 and also the threadlike groups F_4 and F_5. There are 6 simply connected nilpotent un-decomposable Lie groups of dimension 5. Thanks to Kirillov’s orbit picture of the spectrum of a connected simply connected nilpotent Lie group, we have an description of the spectrum of these groups in terms of the structure of the space of its co-adjoint orbits. But the orbit theory is only an algorithm, it does not give us any details about the result of computations. The topology of the orbit space or the behaviour of the operators π(F), F ∈ C*(G) as π varies in the spectrum is different for each of these groups and must be studied case by case.

The paper begins with section 2, where some definitions, methods and results are presented which are needed in the sequel. In section 3, a family of C*-algebras, which we call C*-algebras with norm controlled dual limits (see Definition 3.3) is introduced. This is a family of separable CCR-algebras A, for which there exists a finite increasing family S_0 ⊂ S_1 ⊂ ⋯ ⊂ S_d = Ĥ of closed subsets of the spectrum Ĥ of A, such that for i = 1, ⋯, d, the subsets Γ_0 = S_0 and Γ_i := S_i \ S_{i-1} have separated relative topologies and which have the property that for every converging sequence τ = ((γ_k, H_k))_k ⊂ S_i with limit set L(τ) ⊂ S_{i-1} there exists a sequence (σ_k)_k : CB(S_{i-1}) ↪ B(H_k) (here CB(S_{i-1}) denotes the C*-algebra of continuous bounded operator fields defined over S_{i-1}) of linear mappings, which is uniformly bounded in k, such that for every a ∈ A we have that \( \lim_k \| \gamma_k(a) - σ_k(a) \|_{op} = 0 \). These C*-algebras are then completely determined by the topology of their spectra.

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(in particular by the limit sets \( L(\gamma) \) of properly converging sequences in \( \hat{A} \)) and these mappings \( \sigma_{\gamma,k} \) (see Theorem 3.5).

We then study the 6 groups of dimension \( \leq 5 \) case by case and we show that all of them have \( C^* \)-algebras with norm controlled dual limits. For the Heisenberg and the threadlike groups this has already been shown in the paper [5]. There remains then only the 4 groups \( G_{5,2}, G_{5,3}, G_{5,4} \) and \( G_{5,6} \), which are treated separately in the sections 6,7,8 and 9. Since the structure of the dual space of these groups are different for each of them, we must determine the topology of \( \hat{G} \) group by group and construct by hand for every limit set \( \overline{\sigma} \) of a properly converging sequence in \( g^*/G \) these essential mappings \( \sigma_{\overline{\sigma},k} \).

To understand these mappings \( \sigma_{\overline{\sigma},k} \), one has to recall a theorem of Fell, (see [3]), where he shows that in the case of a properly converging net \( \overline{\sigma} = (\pi_k)_k \subset \hat{A} \) of a \( C^* \)-algebra \( A \), with limit set \( L \), one has that

\[
\lim_{k} \|\pi_k(a)\|_{op} = \sup_{\pi \in L} \|\pi(a)\|_{op}, \ a \in A.
\]

To implement that theorem we need the mappings \( \sigma_{\overline{\sigma},k} \). We shall construct for our limit sets \( L \) for every \( k \in \mathbb{N} \), an increasing sequence of countable subsets \( L_k \) of \( L \) and a sequence of positive numbers \( \varepsilon_k \) such that \( \lim_k \varepsilon_k = 0 \) and such that \( \bigcup_k L_k \) is dense in \( L \). Furthermore for every \( \pi \in L_k \) and \( k \in \mathbb{N} \), we shall find an orthogonal projection \( P_{k,\pi} \) on the Hilbert space \( \mathcal{H}_k \) of \( \pi_k \), such that \( \sum_{\pi \in L_k} P_{k,\pi} = I_{\mathcal{H}_k} \) and a linear mapping \( U_{k,\pi} : \mathcal{H}_\pi \to \mathcal{H}_{\pi_k} \) such that

\[
\sum_{\pi \in L_k} \| P_{k,\pi} \circ \pi_k(a) \circ P_{k,\pi} - U_{k,\pi} \circ \pi(a) \circ U_{k,\pi}^* \|_{op} \leq \varepsilon_k \|a\|, \ a \in C^*(G).
\]

This results for the 5 dimensional groups lead to the following question.

Do the \( C^* \)-algebras of connected nilpotent Lie groups have all this property of norm controlled dual limits?

In several forthcoming papers it will be shown that the answer to this question is yes for all groups of dimension 6.

2. Preliminaries.

2.1. Orbit picture. Kirillov’s orbit theory for a connected simply connected nilpotent Lie group \( G \) tells us that for every irreducible unitary representation \( \pi \) of \( G \) there exists an \( \ell \in g^* \) and a polarization \( p \subseteq g \) at \( \ell \) (i.e. a subalgebra \( p \) of \( g \) of dimension \( d = \frac{\dim(g) + \dim(g(\ell))}{2} \) with the property \( \langle \ell, [p,p] \rangle = \{0\} \)) such that \( \pi \) is equivalent to the induced representation \( \pi_{\ell,p} = ind_P^G \chi_\ell \) of the unitary character \( \chi_\ell = e^{-2\pi i \text{flog}|p|} \) from \( P = \text{exp}(p) \) to \( G \). Furthermore for two linear functionals \( \ell, \ell' \) on \( g \) and the Pukanszky polarizations \( p \) at \( \ell \) (resp \( p' \) at \( \ell' \)), the representations \( \pi_{\ell,p} \) and \( \pi_{\ell',p'} \) are equivalent if and only if \( \ell \) and \( \ell' \) are contained in the same \( G \)-orbit (see [2]). Let \( [\pi] \) denote the unitary equivalence class of a unitary representation \( \pi \) of \( G \). The Kirillov map

\[
K : g^*/G \to \hat{G}; Ad^*(G)\ell \to [\pi_{\ell,p}]
\]

is a homeomorphism of the orbit space \( g^*/G \) onto \( \hat{G} \) (see [5]).
2.2. **Some definitions and results.** We indicate here some definitions, methods and results, which will be needed in the sequel.

(1)

**Definition 2.1.** Let \( H = \exp(h) \) be a closed connected subgroup of a connected nilpotent Lie group \( G = \exp(g) \) and let \( \chi_\ell : H \rightarrow \mathbb{T}; \chi_\ell(h) = e^{-2\pi i \ell \cdot (\log(h))}, h \in H(\ell \in \mathfrak{g}^*) \), be a unitary character of \( H \). The quotient space \( G/H \) has a unique left invariant measure \( d\hat{\gamma} \). With this measure we can define the Hilbert space \( \mathcal{H} = \mathcal{H}_{H,\ell} = L^2(G/H, \ell) \) by

\[
L^2(G/H, \ell) := \{ \xi : G \rightarrow \mathbb{C}, \xi \text{ measurable, } \xi(g\hat{h}) = \chi_\ell(h^{-1})\xi(g), h \in H, g \in G, \quad ||\xi||_2^2 := \int_{G/H} |\xi(g)|^2 d\hat{\gamma} < \infty \}.
\]

The group \( G \) acts by left translation on this space and defines a unitary representation

\[
\sigma_{\ell,b}(g)\xi(u) := \xi(g^{-1}u), \xi \in L^2(G/H, \ell), g, u \in G
\]

called the induced representation of \( \chi_\ell \) (from \( H \) to \( G \)).

If \( \mathfrak{h} \) is polarization at \( \ell \), then the representation \( \sigma_{\ell,b} \) is irreducible and we denote it sometimes by \( \pi_{\ell,b} \). If we take a Malcev basis \( \mathcal{B} = \{X_1, \cdots, X_d\} \) of \( \mathfrak{g} \) relative to \( \mathfrak{h} \), which means that the subspaces \( \mathfrak{g}_j := \text{span}\{X_j, \cdots, X_d, \mathfrak{h}\} \) is a subalgebra of \( \mathfrak{g} \) and that \( \mathfrak{g} = \bigoplus_{j=1}^d \mathbb{R}X_j \oplus \mathfrak{h} \) is a direct sum, then the mapping \( E_{\mathcal{B}} : \mathbb{R}^d \times H \rightarrow G, E_{\mathcal{B}}(t_1, \cdots, t_d) := \exp(t_1X_1) \cdots \exp(t_dX_d)h \) is a diffeomorphism and it allows us to identify the Hilbert space \( L^2(G/H, \ell) \) with the space \( L^2(\mathbb{R}^d) \). We shall consider in the following pages always the representations \( \sigma_{\ell,b} \) as representations on the space \( L^2(\mathbb{R}^d) \).

It is well known that for \( F \in L^1(G) \) the operator \( \pi_{\ell,b} \) is a kernel operator with kernel function

\[
F_{\ell,b}(s,t) = \int_H F(sh^{-1})\chi_\ell(h)dh, s, t \in G.
\]

If \( H \) is a normal subgroup of \( G \), then the function \( F_{\ell,b} \) can be written as

\[
F_{\ell,b}(s,t) = \tilde{F}^b(st^{-1}, Ad^*(t)\ell|\mathfrak{h}), s, t \in G,
\]

where

\[
\tilde{F}^b(s, q) := \int_H F(sh)e^{-2\pi i q \cdot (\log(h))}dh, s \in G, q \in \mathfrak{h}^*.
\]

We denote for a normal subgroup \( H = \exp(h) \) by \( L^1_c = L^1_{c,b} \) the subspace of \( L^1(G) \) consisting of all \( F \)'s in \( L^1(G) \) for which \( \tilde{F}^b \in C^\infty_c(G/H \times \mathfrak{h}^*) \). This space \( L^1_c \) is dense in \( L^1(G) \) and hence it is also dense in \( C^*(G) \). The functions \( \tilde{F}^b \) satisfy the covariance condition

\[
\tilde{F}^b(sh, q) = \chi_q(h^{-1})\tilde{F}^b(s, q), s \in G, h \in H, q \in \mathfrak{h}^*.
\]

The vector space \( L^1_c \) is of course dense in \( L^1(G) \) and hence also in \( C^*(G) \). Since for every \( F \in L^1_c \) the function \( \tilde{F}^b \) is smooth with compact support on \( G/H \times \mathfrak{h}^* \), there exists a function \( \varphi \in C_c(G/H) \) such that

\[
|\tilde{F}^b(s, q)| \leq ||q|||\varphi(s)|, s \in G, q \in \mathfrak{h}^*.
\]
We shall often use in the sequel the following fact.

Let \( F: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) be a smooth function with compact support for which there exists some continuous function \( \varphi: \mathbb{R}^d \to \mathbb{R}_+ \) with compact support, such that

\[
|F(x, y)| \leq \varphi(x - y), \quad x, y \in \mathbb{R}^d.
\]

Then by Young’s inequality, for the kernel operator \( T_F \) defined on \( L^2(\mathbb{R}^d) \) by

\[
T_F(\xi)(x) := \int_{\mathbb{R}^d} F(x, y)\xi(y)dy \quad \text{for} \quad \xi \in L^2(\mathbb{R}^d), x \in \mathbb{R}^d,
\]

its operator norm \( \|T_F\|_{\text{op}} \) is bounded by the \( L^1\)-norm \( \|\varphi\|_1 \) of \( \varphi \).

Let \( G \) be a second countable locally compact group with a continuous action by homeomorphisms \( G \times S \to S, (g, s) \to g \cdot s \) on a second countable locally compact space \( S \). Denote for \( s \in S \) the stabilizer of \( s \) in \( G \) by \( G_s \).

Let \( \mathcal{O} = (\mathcal{O}_k) \) be a sequence of \( G \)-orbits in \( S \). We say that \( \mathcal{O} \) converges with multiplicity \( \ell \) to the \( G \)-orbit \( \mathcal{O} \), if there exists for every \( k \in \mathbb{N} \) an element \( s_k \in \mathcal{O}_k \) and an \( s \in \mathcal{O} \), such that \( \lim_{k} s_k = s \) and such that for any compact subset \( K \) of \( S \), for which the intersection of the interior \( \mathring{K} \) of \( K \) with \( G \cdot s \) is not empty, there exists a compact subset \( C \subset G \), such that for \( k \) large enough, for any \( g \notin CG_{s_k} \) we have that \( g \cdot s_k \notin K \).

Let \( \mathfrak{g} \) be a nilpotent Lie algebra. We fix an euclidean scalar product on \( \mathfrak{g} \). Let \( (\ell_k)_{k \in \mathbb{N}} \) be a converging sequence in \( \mathfrak{g}^* \) with limit \( \ell \). Let \( (\mathfrak{h}_k)_{k} \) be a sequence of subalgebras of \( \mathfrak{g} \), such that \( \dim(\mathfrak{h}_k) = n - d, k \in \mathbb{N} \), for some \( d \in \mathbb{N} \), and such that \( \{\ell_k, [\mathfrak{h}_k, \mathfrak{h}_k]\} = 0 \), \( k \in \mathbb{N} \). We pick for every \( k \in \mathbb{N} \) an orthonormal Malcev basis of \( \mathcal{B}_k = \{Z_1^k, \ldots, Z_n^k\} \) such that \( \mathfrak{h}_k = \text{span}\{Z_{n-d}^k, \ldots, Z_n^k\} \) and we can assume (passing if necessary to a subsequence), that the vectors \( Z_j^k \) converge for every \( j = 1, \ldots, n \) to a vector \( Z_j \). Then \( \mathcal{B} := \{Z_1, \ldots, Z_n\} \) is an orthonormal Malcev basis of \( \mathfrak{g} \) which passes through \( \mathfrak{h} := \text{span}\{Z_{n-d}, \ldots, Z_n\} \). Then \( \mathfrak{h} \) is subalgebra of \( \mathfrak{g} \) and \( \{\ell, [\mathfrak{h}, \mathfrak{h}]\} = 0 \). Let \( H_k := \exp(\mathfrak{h}_k), k \in \mathbb{N} \) and \( H = \exp(\mathfrak{h}) \). Let \( \sigma_k := \sigma_{\ell_k, \mathfrak{h}_k} \) and \( \sigma = \sigma_{\ell, \mathfrak{h}} \). This gives us the following

**Proposition 2.2.** The representations \( \sigma_k \) converge weakly to the representation \( \sigma \). This means that for every \( \xi, \eta \in L^2(\mathbb{R}^d) \), for every \( a \in C^*(G) \):

\[
\lim_{k \to \infty} \langle \sigma_k(a)\xi, \eta \rangle = \langle \sigma(a)\xi, \eta \rangle.
\]

**Proof.** Take first \( F \in C_c(G) \) and \( \xi, \eta \in C_c(\mathbb{R}^d) \). We have then that:

\[
\langle \sigma(F)\xi, \eta \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi(u)F_k(u, v)\overline{\eta(v)}dudv,
\]

where

\[
F_k(u, v) := \int_{\mathbb{R}^{n-d}} F(E_k(h)E_{k,d}(h)E_k(v)^{-1}\chi_k(E_{k,d}(h))dh, u, v \in \mathbb{R}^n
\]

and where \( E_k = E_{B_k}, E_{k,d} = E_k|_{\mathbb{R}^{n-d}}, k \in \mathbb{N} \). We get similar expression for \( k = \infty \), i.e. for \( E_B \) etc. It is easy to see that the supports of the functions \( f_k : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{n-d} \) defined by

\[
f_k(u, v) := \xi(u)F(E_k(u)E_{k,d}(h)E_k(v)^{-1})\overline{\eta(v)},
\]

are contained in \( \mathcal{O}_k \) for all \( k \in \mathbb{N} \).
are contained in a common compact set and that the functions converge point-wise to the function
\[ f(u, v, h) := F(E(u)E_d(h)E(v)^{-1})\chi(E_d(h)), \]
where \( E_d = E_{R|\mathbb{R}^n - d}. \) Therefore by Lebegue’s theorem of dominated convergence we have that
\[ \lim_{k \to \infty} \langle \sigma_k(F)\xi, \eta \rangle = \langle \sigma(F)\xi, \eta \rangle. \]
The proposition now follows from the density of \( C_c(G) \) in \( C^*(G) \) and the density of \( C_c(\mathbb{R}^d) \) in \( L^2(\mathbb{R}^d) \).

**Theorem 2.3.** Let \( (\pi_k)_k \subset \hat{G} \) be a sequence which converges with multiplicity 1 to a single limit point \( \pi \). Then there exists a subsequence (also indexed by \( k \) for simplicity of notations), a Hilbert space \( \mathcal{H} \), for every \( k \in \mathbb{N} \) a concrete realization \( (\sigma_k, \mathcal{H}) \) of \( \pi_k \) and a concrete realization \( (\sigma, \mathcal{H}) \) of \( \pi \), such that
\[ \lim_{k} \|\sigma_k(a) - \sigma(a)\|_{op} = 0, a \in C^*(G). \]

**Proof.** We write \( \pi_k \simeq \sigma_k, k \in \mathbb{N}, \) where \( \sigma_k = \text{ind}_{\mathcal{H} \chi_{\ell_k}, k \in \mathbb{N}, \ell_k} \) is an element in the Kirillov orbit of \( \pi_k \) and where \( \ell_k = \exp(p_k) \) is a polarization at \( \ell_k \). Since \( \pi_k \) converges to \( \pi \), we can assume that \( \lim_k \ell_k = \ell \) for some \( \ell \) in the orbit of \( \pi \) and that (passing to a subsequence) \( \lim_k p_k = p \) in the sense of the preceding Proposition 2.2. Then \( p \) is a polarization at \( \ell \), since \( \pi \) is the only limit of the sequence \( (\pi_k)_k \).

Let \( j(\emptyset) \) be the minimal dense ideal of \( C^*(G) \) i.e. the Pedersen ideal of \( C^*(G) \). Since \( (\pi_k)_k \) converges with multiplicity 1 to \( \pi \) it follows from \( \mathbb{N} \) that for every \( a \in j(\emptyset) \) the operators \( \sigma_k(a), k \in \mathbb{N}, \) and \( \sigma(a) \) are finite rank and that
\[ \lim_{k} \text{tr}(\pi_k(a)) = \text{tr}(\pi(a)). \]

Take now \( a = a^* \in j(\emptyset) \). Then
\[
\|\sigma_k(a) - \sigma(a)\|_{H-S}^2 = \text{tr}(\sigma_k(a)^2) + \text{tr}(\sigma(a)^2) - \text{tr}(\sigma_k(a) \circ \sigma(a)) - \text{tr}(\sigma(a) \circ \sigma_k(a))
= \text{tr}(\sigma_k(a)^2) + \text{tr}(\sigma_k(a)^2) - 2\text{tr}(\sigma_k(a) \circ \sigma(a))
\rightarrow \text{tr}(\sigma(a)^2) + \text{tr}(\sigma(a)^2) - 2\text{tr}(\sigma(a) \circ \sigma(a))
= 0,
\]
by Proposition 2.2 the continuity of the trace and the fact that \( \sigma(a) \) has finite rank. Hence
\[ \lim_{k} \|\sigma_k(a) - \sigma(a)\|_{op} = 0. \]
The theorem now follows from the density of \( j(\emptyset) \) in \( C^*(G) \).

(5) Denote for a measurable subset \( S \subset X \) of a measure space \( (X, \mu) \) the multiplication operator with the indicator of a measurable subset \( S \) on \( L^2(X, \mu) \) by \( M_S \).
Proposition 2.4. Let $(X, \mu)$ a measure space, let $(\sigma_i)_{i \in I}$ be a family of bounded linear operators on the Hilbert space $\mathcal{H} = L^2(X, \mu)$, such that $\|\sigma_i\|_{\text{op}} \leq C$, $\forall i \in I$, for some $C > 0$. Suppose furthermore that there exists families $(T_{i,j})_{i \in I} (j = 1, \cdots, N)$ and $(S_i)_{i \in I}$ of measurable subsets of $X$ such that $T_{i,j} \cap T_{i',j} = \emptyset, (j = 1, \cdots, N)$, $S_i \cap S_{i'} = \emptyset$ whenever $i \neq i'$. Then the linear operator

$$\sigma = \sum_{j=1}^N \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i}$$

is bounded by $NC$.

Proof. Let us write

$$\sigma^j := \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i}, j = 1, \cdots, N.$$ 

Then $\sigma = \sum_{j=1}^N \sigma^j$ and for $\xi \in L^2(X, \mu), j \in \{1, \cdots, N\}$ we then have:

$$\|\sigma^j(\xi)\|^2 = \int_X \left| \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i}(\xi)(x) \right|^2 d\mu(x)$$

$$\leq \sum_{i \in I} \int_{T_{i,j}} |\sigma_i(M_{S_i}(\xi))(x)|^2 d\mu(x)$$

$$\leq \sum_{i \in I} \int_X |\sigma_i(M_{S_i}(\xi))(x)|^2 d\mu(x)$$

$$\leq \sum_{i \in I} C^2 \int_X |M_{S_i}(\xi(x))|^2 d\mu(x)$$

$$= C^2 \left( \sum_{i \in I} \int_{S_i} |\xi(x)|^2 d\mu(x) \right)$$

$$\leq C^2 \int_X |\xi(x)|^2 d\mu(x)$$

$$= C^2 \|\xi\|^2.$$ 

Hence $\|\sigma\|_{\text{op}} \leq NC$. \hfill \(\square\)

Remark 2.5. Let $(X, \mu), (Y, \nu)$ be two measure spaces. Let $Y \rightarrow L^2(X, \mu); y \rightarrow \xi(y)$ be an integrable mapping. Then we have that:

$$\left\| \int_Y \xi(y) d\nu(y) \right\|_2 \leq \int_Y \|\xi(y)\|_2 d\nu(y)$$

i.e.

$$\left( \int_X \left| \int_Y \xi(y)(x) d\nu(y) \right|^2 d\mu(x) \right)^{1/2} \leq \int_Y \left( \int_X |\xi(y)(x)|^2 d\mu(x) \right)^{1/2} d\nu(y).$$ 

(7)
Definition 2.6. We say that a net \((\gamma_k)_{k \in I}\) in a topological space \(\Gamma\) goes to infinity, if the net contains no converging subnet.

In particular, a sequence of orbits \(\Omega = (\Omega_k)_{k \in \mathbb{N}} \subset \mathfrak{g}^*\) goes to infinity, if for any compact subset \(K \subset \mathfrak{g}^*\) there exists an index \(k_0\) such that \(K \cap \Omega_k = \emptyset\) whenever \(k \geq k_0\).

Proposition 2.7. (Riemann-Lebesgue Lemma) Let \(A\) be a \(C^*\)-algebra. If a net \((\pi_k)_{k \in \mathbb{N}} \subset \hat{A}\) goes to infinity, then \(\lim_k \|\pi_k(a)\|_{op} = 0\) for all \(a \in A\).

Proof. We know from [8] Proposition 3.3.7, that for every \(c > 0\) and \(a \in A\), the subset \(\{\pi \in \hat{A}; \|\pi(a)\|_{op} \geq c\}\) is quasi-compact. This shows that if \((\pi_k)_{k \in \mathbb{N}}\) goes to infinity, \(\lim_k \|\pi_k(a)\|_{op} = 0\), if the net \((\pi_k)_{k \in \mathbb{N}}\) goes to infinity.

(8) Let \(G\) be a locally compact group and \(H\) a closed normal subgroup of \(G\). Then the canonical projection \(P_{G/H}: L^1(G) \to L^1(G/H)\) defined by:

\[ P_{G/H} F(x) := \int_H F(xh)dh, F \in L^1(G), x \in G, \]

is a surjective homomorphism (see [10]). Let \(H^+ := \{\pi \in \mathcal{G}, \pi(H) = \mathbb{1}_{H^+}\}\).

Then

\[ \ker(P_{G/H}) = \{F \in L^1(G), \pi(F) = 0, \pi \in H^+\}. \]

The mapping \(P_{G/H}\) extends then to a surjective *-homomorphism (also denoted by \(P_{G/H}\)) of \(C^*(G)\) onto \(C^*(G/H)\).

3. A special class of \(C^*\)-algebras.

Definition 3.1. Let \(A\) be a \(C^*\)-algebra with spectrum \(\hat{A}\). We choose for every \(\gamma \in \hat{A}\) a representation \((\pi_\gamma, \mathcal{H}_\gamma)\) in the equivalence class \(\gamma\). Let \(l^\infty(\hat{A})\) be the algebra of all bounded operator fields defined over \(\hat{A}\) by

\[ l^\infty(\hat{A}) := \left\{ \phi = (\phi(\gamma) \in \mathcal{B}(\mathcal{H}_\gamma))_{\gamma \in \hat{A}}; \|\phi\|_\infty := \sup_\gamma \|\phi(\pi_\gamma)\|_{op} < \infty \right\}. \]

We define for \(a \in A\) its Fourier transform \(\mathcal{F}(a) = \hat{a}\) by:

\[ \mathcal{F}(a)(\gamma) = \hat{a}(\gamma) := \pi_\gamma(a), \gamma \in \hat{A}. \]

Then \(\hat{a}\) is a bounded field of operators over \(\hat{A}\), i.e. \(\hat{a} \in l^\infty(\hat{A})\). The mapping

\[ \mathcal{F} : A \to l^\infty(\hat{A}); a \mapsto \hat{a} \]

is an isometric *-homomorphism.

For a closed subset \(S \subset \hat{A}\), denote by \(CB(S)\) the *-algebra of all uniformly bounded operator fields \((\psi(\gamma) \in \mathcal{B}(\mathcal{H}_\gamma))_{\gamma \in S \cap \Gamma_i, i=1,\ldots,d}\), which are operator norm continuous on the subsets \(\Gamma_i \cap S\) for every \(i \in \{1,\ldots,d\}\) for which \(\Gamma_i \cap S \neq \emptyset\). We provide the algebra \(CB(S)\) with the infinity-norm:

\[ \|\varphi\|_S := \sup_{\gamma \in S} \|\varphi(\gamma)\|_{op}. \]
Definition 3.2. Let $A$ be a separable CCR $C^*$-algebra, i.e. for every irreducible representation $(\pi, \mathcal{H})$ of $A$, the image of $\pi$ is the algebra of compact operators $\mathcal{K} (\mathcal{H})$. We suppose that there exists a finite increasing family $S_0 \subset S_1 \subset \ldots \subset S_d = \hat{A}$ of closed subsets of the spectrum $\hat{A}$ of $A$, such that for $i = 1, \ldots, d$, the subsets $\Gamma_0 = S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$, $i = 1, \ldots, d$, are Hausdorff in their relative topologies. Furthermore we assume that for every $i \in \{0, \ldots, d\}$ there is a Hilbert space $\mathcal{H}_i$ and for every $\gamma \in \Gamma_i$ a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of $\gamma$ on the Hilbert space $\mathcal{H}_i$. The set $S_0$ is the collection $\mathcal{X}$ of all characters of $A$.

Definition 3.3. We say that our $C^*$-algebra $A$ has norm controlled dual limits (NCDL for short), if for every $a \in A$:

1. The mappings $\gamma \to F(a)(\gamma)$ are norm continuous on the different sets $\Gamma_i$.
2. For any $i = 0, \ldots, d$, and for any converging sequence contained in $\Gamma_i$ with limit set outside $\Gamma_i$, there exists a properly converging sub-sequence $\gamma = (\gamma_k)_{k \in \mathbb{N}}$, $\mathcal{A} > 0$ and for every $k \in \mathbb{N}$ a linear mapping $\tilde{\sigma}_{\gamma,k} : BC(S_{i-1}) \to B(\mathcal{H}_i)$, which is bounded by $\mathcal{A}||S_{i-1},$ such that
   \[
   \lim_{k \to \infty} \|F(a)(\gamma_k) - \tilde{\sigma}_{\gamma,k}(F(a)|_{S_{i-1}})\|_{\text{op}} = 0,
   \]
   \[
   \lim_{k \to \infty} \|\tilde{\sigma}_{\gamma,k}(F(a)|_{S_{i-1}}) - \tilde{\sigma}_{\gamma,k}(F(a)|_{S_{i-1}}^*)\|_{\text{op}} = 0.
   \]
   (By the condition 1) the restriction of $F(a)$ to the limit set $L(\gamma)$ is contained in $CB(S_{i-1})$.

Definition 3.4. Let $D^*(A)$ be the set of all operator fields $\varphi$ defined over $\hat{A}$ such that

1. $\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ for every $\gamma \in \Gamma_i, i = 1, \ldots, d$.
2. The field $\gamma$ is uniformly bounded, i.e. we have that $\|\gamma\| := \sup_{\gamma \in \hat{A}} \|\varphi(\gamma)\|_{\text{op}} < \infty$.
3. The mappings $\gamma \to \varphi(\gamma)$ are norm continuous on the different sets $\Gamma_i$.
4. We have for any sequence $\gamma = (\gamma_k)_{k \in \mathbb{N}} \subset \hat{A}$ going to infinity, that $\lim_{k \to \infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$.
5. Let $\gamma = (\gamma_k)_{k \in \mathbb{N}}$ and $\tilde{\sigma}_{\gamma,k}, k \in \mathbb{N}$, as in the point 2. of Definition 3.3.
   Then the restriction of $\varphi$ to the subset $L(\gamma)$ is contained in $CB(S_{i-1})$ by conditions 2. and 3. and we assume that:
   \[
   \lim_{k \to \infty} \|\varphi(\gamma_k) - \tilde{\sigma}_{\gamma,k}(\varphi|_{S_{i-1}})\|_{\text{op}} = 0,
   \]
   \[
   \lim_{k \to \infty} \|\tilde{\sigma}_{\gamma,k}(\varphi|_{S_{i-1}}) - \tilde{\sigma}_{\gamma,k}(\varphi|_{S_{i-1}}^*)\|_{\text{op}} = 0.
   \]

We remark immediately that for every $a \in A$, the operator field $F(a)$ is contained in the set $D^*(A)$. It turns out that $D^*(A)$ is a $C^*$-sub-algebra of $l^\infty(\hat{A})$ and that $A$ is isomorphic to $D^*(A)$.

Theorem 3.5. Let $A$ be a separable $C^*$-algebra with norm controlled dual limits. Then the subset $D^*(A)$ of the $C^*$-algebra $l^\infty(\hat{A})$ is a $C^*$-sub-algebra of $l^\infty(\hat{A})$ which is isomorphic with $A$ under the Fourier transform.

Proof. We see that the conditions 1. to 4. imply that $D^*(A)$ is a closed involution-invariant subspace of $l^\infty(\hat{A})$. For $i = 0, \ldots, d$, let $D_i^*$ be the set of all operator fields defined over $S_i$, satisfying conditions 1. to 5. on the sets $S_j$, $j = 1, \ldots, i$. Then the sets $D_i^*$ are closed sub-spaces of the $C^*$-algebra $l^\infty(S_i)$. Let $I_C$ be the closed two-sided ideal in $A$ generated by the elements of the form $ab - ba, a, b \in A$. Then
the space of characters $S_0 = \mathcal{X}$ of $A$ is the spectrum of $A/I_C$ and $D_0^*$ equals the algebra $C_0(S_0)$ of continuous functions on $S_0$ vanishing at infinity by the conditions 1., 2., 3. and 4. Since $\mathcal{F}(C^*(A)|_{S_0}) = C_0(S_0)$ it follows that $D_0^* = \mathcal{F}(A)|_{S_0}$.

Let us assume now that for some $1 \leq i < d$ we have that $D_j^* = \mathcal{F}(A)|_{S_j}$, $j = 1, \cdots, i - 1$. We shall prove then that $D_i^* = \mathcal{F}(A)|_{S_i}$. We know already that $\mathcal{F}(A)|_{S_i}$ is an algebra sitting in the closed sub-space $D_i^* \subset L^\infty(S_i)$ and it follows from its definition that the restriction of $D_i^*$ to $S_{i-1}$ is contained in $D_{i-1}^*$. Let $\varphi, \psi \in D_i^*$. By our assumption, there exists $a, b \in A$ such that $\varphi|_{S_{i-1}} = \hat{a}|_{S_{i-1}}$ and $\psi|_{S_{i-1}} = \hat{b}|_{S_{i-1}}$. The product $\varphi \circ \psi$ satisfies then also the conditions 1. to 4. for $i$. We shall show that it too satisfies condition 5. for $i$. Indeed we have for any properly converging sequence $(\gamma_k)_k \subset \Gamma_i$ with limit set outside $\Gamma_i$ that:

$$
\|\varphi(\gamma_k) \circ \psi(\gamma_k) - \sigma_{\tau,k}(\varphi \circ \psi|_{S_{i-1}})\|_{\text{op}} = \|\varphi(\gamma_k) \circ \psi(\gamma_k) - \sigma_{\tau,k}(\varphi|_{S_{i-1}} \circ \psi|_{S_{i-1}})\|_{\text{op}}
$$

$$
\leq \|\varphi(\gamma_k) \circ \psi(\gamma_k) - \sigma_{\tau,k}(\varphi|_{S_{i-1}}) \circ \sigma_{\tau,k}(\psi|_{S_{i-1}})\|_{\text{op}}
$$

$$
+ \|\sigma_{\tau,k}(\varphi|_{S_{i-1}}) \circ \sigma_{\tau,k}(\psi|_{S_{i-1}}) - \sigma_{\tau,k}(\varphi|_{S_{i-1}}) \circ \sigma_{\tau,k}(\psi|_{S_{i-1}})\|_{\text{op}}
$$

$$
+ \|\gamma_k \circ \hat{b}(\gamma_k) - \sigma_{\tau,k}(\varphi|_{S_{i-1}}) \circ \sigma_{\tau,k}(\psi|_{S_{i-1}})\|_{\text{op}}
$$

$$
+ \|\sigma_{\tau,k}(\gamma_k) - \sigma_{\tau,k}(\gamma_k)\|_{\text{op}}
$$

$$
+ \|((\hat{a} \circ \hat{b})(\gamma_k) - \sigma_{\tau,k}(\hat{a} \circ \hat{b}|_{S_{i-1}}))\|_{\text{op}}.
$$

This shows that $\lim_{k \to \infty} \|\varphi \circ \psi(\gamma_k) - \sigma_{\tau,k}(\varphi \circ \psi|_{S_{i-1}})\|_{\text{op}} = 0$ and so $\varphi \circ \psi \in D_i^*$. Hence the subspace $D_i^*$ is a $C^*$-sub-algebra of $L^\infty(S_i)$ containing the algebra $\mathcal{F}(A)|_{S_i}$. In order to prove that $D_i^* = \mathcal{F}(A)|_{S_i}$, by Stone-Weierstrass it suffices to show that the spectrum of $D_i^*$ equals that of $\mathcal{F}(A)|_{S_i}$, i.e. that every element in $D_i^*$ is an evaluation at some point in $S_i$. Let $\pi \in D_i^*$. Consider the kernel $K_{i-1}$ of $R_{i-1}$, the restriction mapping from $D_i^*$ into $L^\infty(S_{i-1})$. If $\pi(K_{i-1}) = \{0\}$, then we can consider $\pi$ as being a representation of the quotient algebra $D_i^*/K_{i-1}$.

But the image of $R_{i-1}$ contains $D_{i-1}^*$ and contains $\mathcal{F}(A)|_{S_{i-1}}$ and so by assumption $R_{i-1}D_i^* = D_i^*|_{S_{i-1}}$ and hence $D_i^*/K_{i-1} \cong \mathcal{F}(A)|_{S_{i-1}}$ and therefore $\pi$ is an evaluation at a point in $S_{i-1}$. If $\pi(K_{i-1}) \neq \{0\}$, then we look at the restriction of $\pi$ to this ideal. The elements in $K_{i-1}$ are operator fields defined on $S_i$ which are norm continuous, which go to 0 at infinity and by condition 5., for any properly converging sequence $\gamma \subset \Gamma_i$ with limit outside $\Gamma_i$, for every $\varphi \in D_i^*$, we have that

$$
\lim_{k \to \infty} \|\varphi(\gamma_k)\|_{\text{op}} = \lim_{k \to \infty} \|\varphi(\gamma_k) - \sigma_{\tau,k}(\varphi|_{S_{i-1}})\|_{\text{op}} = 0.
$$

This shows that $K_{i-1} \subset C_0(\Gamma_i, \mathcal{K}(H_i))$. On the other hand the elements of $\mathcal{F}(A)|_{S_i} \cap K_{i-1}$ separate the points of $\Gamma_i$ (see Proposition 2.11.2 in [3] for details). Therefore by the theorem of Stone-Weierstrass the algebras $C_0(\Gamma_i, \mathcal{K}(H_i))$ and $K_{i-1}$ coincide. This tells us that $\pi$ is an evaluation at a point in $\Gamma_i$, since the spectrum of the algebra $C_0(\Gamma_i, \mathcal{K}(H_i))$ is homeomorphic to $\Gamma_i$. We conclude that $\mathcal{F}(A)|_{S_i} = D_i^*$. 

\[\square\]
4. The list of the nilpotent Lie algebras of dimension \( \leq 5 \) according to [9].

There are 8 un-decomposable nilpotent non-abelian Lie algebras of dimension \( \leq 5 \): the Heisenberg Lie algebras of dimension 3 and 5, \( h_1 \) and \( h_2 \), the thread-like Lie algebras of dimension 4 and 5, \( f_4 \) and \( f_5 \), the step 2 Lie algebra \( g_{5,2} \), two step 3 Lie algebras, \( g_{5,3} \) and \( g_{5,4} \) and the step 4 Lie algebra \( g_{5,6} \).

(1) The Heisenberg Lie algebras \( h_n, n \geq 1 \):

Let \( h_n \) be the nilpotent Lie algebra of dimension \( 2n+1 \) spanned by the basis

\[
B = \text{span} \{ X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z \}
\]

equipped with the Lie bracket

\[
[X_i, Y_j] = \delta_{i,j} Z, i, j = 1, \cdots, n.
\]

(2) The thread-like Lie algebras \( f_n, n \geq 4 \):

Let \( f_n \) be the nilpotent Lie algebra spanned by the basis \( B = \text{span} \{ X_n, \cdots, X_1 \} \) equipped with the Lie brackets:

\[
[X_n, X_j] = X_{j-1}, j = n-1, \cdots, 2.
\]

(3) the step 2 Lie algebra \( g_{5,2} \):

Let \( g_{5,2} \) be the nilpotent Lie algebra spanned by the basis \( B = \text{span} \{ A, B, C, U, V \} \) equipped with the Lie brackets

\[
[C, A] = -U, \quad [C, B] = V.
\]

This is a semi-direct product of \( \mathbb{R} \) with \( \mathbb{R}^4 \).

(4) The step 3 Lie algebras \( g_{5,3} \) and \( g_{5,4} \):

- Let \( g_{5,3} \) be the nilpotent Lie algebra spanned by the basis \( B = \text{span} \{ A, B, C, U, V \} \) equipped with the Lie brackets

\[
[A, B] = U, \quad [A, U] = V, \quad [B, C] = V.
\]

This is a semi-direct product of \( \mathbb{R} \) with \( h_1 \times \mathbb{R} \).

- Let \( g_{5,4} \) be the nilpotent Lie algebra spanned by the basis \( B = \{ A, B, C, U, V \} \) equipped with the Lie brackets

\[
[A, B] = C, \quad [A, C] = U, \quad [B, C] = V.
\]

This is a semi-direct product of \( \mathbb{R} \) with \( f_4 \times \mathbb{R} \).

(5) The Step 4 Lie algebra \( g_{5,6} \):

Let \( g_{5,6} \) be the nilpotent Lie algebra spanned by the basis \( B = \{ A, B, C, U, V \} \) equipped with the Lie brackets

\[
[A, B] = C, \quad [A, C] = U, \quad [A, U] = V, \quad [B, C] = V.
\]

This is a semi-direct product of \( \mathbb{R} \) with \( f_4 \).

5. The \( C^* \)-algebras of \( H_n, n \geq 1 \) and \( F_n, n \geq 4 \).

The \( C^* \)-algebras of \( H_n \) and of \( F_n \) have been realized as algebras of operator fields in [6]. It is shown there that the corresponding \( C^* \)-algebras have the norm-controlled dual limit property.

We give some details on the spectrum of these groups.
5.1. The groups $H_n$. The orbit space $\mathfrak{h}_n^*/H_n$ consists of two layers:

1. the set of characters $X = \Gamma_0^n = \sum_{j=1}^n \mathbb{R}X_j^* + \mathbb{R}Y_j^*$,
2. the set $\Gamma_1^n$ of flat orbits:

$\Gamma_1^n = \left\{ \Omega_\lambda = \lambda Z^* + Z^* \parallel = \lambda Z^* + \sum_{j=1}^n \mathbb{R}X_j^* + \mathbb{R}Y_j^*, \lambda \in \mathbb{R}^* \right\}.$

It is easy to see the relative topology of $\Gamma_1^n$ is Hausdorff, that $\Omega_\lambda$ goes to infinity if $\lambda$ goes to infinity and that

$\lim_{\lambda \to 0} \Omega_\lambda = \Gamma_0^n.$

The irreducible representation $\pi_\lambda$ associated to $\lambda \in \mathbb{R}^*$ acts on $L^2(\mathbb{R}^n)$ and for $F \in L^1(H_n)$ the operator $\pi_\lambda(F)$ is a kernel operator with kernel function $\mathcal{F}_\lambda$ given by

$F_\lambda(s, t) := \widehat{F}^2,3(s - t, -\frac{\lambda}{2}(s + t), \lambda), s, t \in \mathbb{R}^n$

where

$\widehat{F}^2,3(s, t, \lambda) := \int_{\mathbb{R}^n \times \mathbb{R}} F(s, u, z)e^{-2i\pi(tu + \lambda z)}dudz.$

5.2. The groups $F_n$, $n \geq 4$. For $n \geq 4$, let $\mathfrak{f}_n$ be the $n$–dimensional real nilpotent Lie algebra with basis $\{X_1, \ldots, X_n\}$ and non-trivial Lie brackets

$[X_n, X_{n-1}] = X_{n-2}, \ldots, [X_n, X_2] = X_1.$

For all $\ell = \xi_1X_1^* + \ldots + \xi_{n-1}X_{n-1}^*$ and $t \in \mathbb{R}$ let

$t \cdot \ell = \Ad^*(\exp(tX_n))\xi$

$= \left(0, \xi_{n-1} - t\xi_{n-2} + \ldots + \frac{1}{(n-2)!}(-t)^{n-2}\xi_1, \ldots, \xi_2 - t\xi_1, \xi_1\right).$

We define the function $\widehat{\ell}$ on $\mathbb{R}$ by

$\widehat{\ell}(t) := (t \cdot \xi)\big|_{n-1} = \xi_{n-1} - t\xi_{n-2} + \ldots + \frac{1}{(n-2)!}(-t)^{n-2}\xi_1.$

The mapping $\ell \rightarrow \widehat{\ell}$ intertwines the $\Ad^*$–action and translation in the following way:

$\widehat{t \cdot \xi}(s) = (s \cdot (t \cdot \xi))\big|_{n-1} = ((s + t) \cdot \xi)\big|_{n-1} = \hat{\xi}(s + t), \text{ for } s, t \in \mathbb{R}.$

This identifies the dual space $\mathfrak{f}_n^*$ with the space $\mathcal{P}_{n-2} \oplus \mathbb{R}$, where $\mathcal{P}_{n-2}$ denotes the space $\mathbb{R}[X, \leq n-2]$ of real polynomials of degree $\leq n-2$. A co-adjoint orbit $\Omega_\ell$ then corresponds to the translates $\{\ell(t)P, t \in \mathbb{R}\}$ of some polynomial $P \in \mathbb{R}[X, \leq n-2]$.

We can parameterize the orbit space $\mathfrak{f}_n^*/F_n$ with the sets

$\bigcup_{j=1}^{n-2} \Gamma_j^n,$

where $\Gamma_j^n := \{\ell \in \mathfrak{f}_n^*, \ell(X_k) = 0, k = 1, \ldots, j-1, j+1, \ell(X_j) \neq 0\}.$
Example 5.1. (1) For \( n = 4 \): The 4-dimensional thread-like algebra let \( F_4 \) is the 4-dimensional thread-like group which the multiplication is given by

\[(a, b, c, u)(a', b', c', u') = (a + a', b + b', c + c', u + u' - a'c + \frac{a^2b}{2}).\]

Let \( \{A, B, C, U\} \) be the canonical basis of \( F_4 \). For all \( \ell = \alpha A^* + \beta B^* + \rho C^* + \mu U^* \) and \( t \in \mathbb{R} \), the co-adjoint action is given by:

\[t \cdot \ell = (\alpha, \beta - \rho t + \mu \frac{t^2}{2}, \rho - \mu t, \mu)\]

and

\[\hat{\ell}(t) := \beta - \rho t + \mu \frac{t^2}{2} \] 

We can parameterize the orbit space \( f_4^* / F_4 \) in the following way. First we have a decomposition

\[f_4^* / F_4 = \Gamma_4^2 \cup \Gamma_4^1 \cup \Gamma_4^0,\]

where

\[
\begin{align*}
\Gamma_4^2 &= \{ \ell \in f_4^*, \ell(U) \neq 0, \ell(C) = 0 \} \\
\Gamma_4^1 &= \{ \ell \in f_4^*, \ell(U) = 0, \ell(C) \neq 0, \ell(B) = 0 \} \\
\Gamma_4^0 &= \{ \ell \in f_4^*, \ell(U) = 0, \ell(C) = 0 \}.
\end{align*}
\]

(2) For \( n = 5 \): The 5-dimensional thread-like algebra let \( F_5 \) is the 5-dimensional thread-like group which the multiplication is given by

\[(a, b, c, u, v)(a', b', c', u', v') = (a + a', b + b', c + c' - a'b, u + u' - a'c + \frac{a^2b}{2}, v + v' - a'c + \frac{a^2c}{2} + \frac{a^3b}{6}).\]

Let \( \{A, B, C, U, V\} \) be the canonical basis of \( F_5 \). For all \( \ell = \alpha A^* + \beta B^* + \rho C^* + \mu U^* + \nu V^* \) and \( a \in \mathbb{R} \), the co-adjoint action is given by:

\[t \cdot \ell = (\alpha, \beta - \rho t + \mu \frac{t^2}{2} - \nu \frac{t^3}{6}, \rho - \mu t + \nu \frac{t^2}{2}, \mu - \nu t, \nu)\]

and

\[\hat{\ell}(t) := \beta - \rho t + \mu \frac{t^2}{2} - \nu \frac{t^3}{6}.\]

We can parameterize the orbit space \( f_5^* / F_5 \) in the following way. First we have a decomposition

\[f_5^* / F_5 = \Gamma_5^3 \cup \Gamma_5^2 \cup \Gamma_5^1 \cup \Gamma_5^0,\]

where

\[
\begin{align*}
\Gamma_5^3 &= \{ \ell \in f_5^*, \ell(V) \neq 0, \ell(U) = 0 \} \\
\Gamma_5^2 &= \{ \ell \in f_5^*, \ell(V) = 0, \ell(U) \neq 0, \ell(C) = 0 \} \\
\Gamma_5^1 &= \{ \ell \in f_5^*, \ell(V) = 0, \ell(U) = 0, \ell(C) \neq 0, \ell(B) = 0 \} \\
\Gamma_5^0 &= \{ \ell \in f_5^*, \ell(V) = 0, \ell(U) = 0, \ell(C) = 0 \}.
\end{align*}
\]
6. The $C^*$-algebra of the group $G_{5,2}$.

6.1. The description of the groups $G_{5,2}$ and of the three other ones of dimension 5, of their co-adjoint orbits and irreducible representations can be found in [9].

We shall describe in this and in the following sections for our 4 remaining groups the limit sets of properly converging sequences $O = (\pi_k)_k$ in their dual spaces and we are explicitly constructing the mappings $\sigma_{\pi_k}$. In this way it will turn out that the $C^*$-algebra of every connected Lie group of dimension $\leq 5$ has norm controlled dual limits.

6.2. Recall that the Lie algebra of $g_{5,2}$ is spanned by the basis $B = \text{span}\{A, B, C, U, V\}$ equipped with the Lie brackets

$$[C, A] = -U, \quad [C, B] = V.$$ 

The Lie algebra $g_{5,2}$ has a two-dimensional centre $z = \text{span}\{U, V\}$. The group $G_{5,2} = \exp(g_{5,2})$ can be realized as $\mathbb{R}^5$ with the Campbell-Baker-Hausdorff multiplication

$$((a, b, c, u, v)(a', b', c', u', v')) = (a + a', b + b', c + c', u + u' + \frac{1}{2}(ac' - a'c), v + v' + \frac{1}{2}(b'c - bc')).$$

For all $(a, b, c, u, v) \in G_{5,2}, (\alpha, \beta, \rho, \mu, \nu) \in g^*_{5,2}$ we obtain the following expression for $A^*(a, b, c, u, v)$:

$$A^*(a, b, c, u, v)(x, y, t, \mu, \nu) = (x - \mu c, y + \nu c, t + \mu a - \nu b, \mu, \nu) (6.1)$$

We give now a description of the co-adjoint orbits:

1. The generic elements $(\alpha, \beta, \rho, \mu, \nu)$ in $g^*_{5,2}$ are those for which $r^2_{\mu, \nu} = \rho^2 = 0$. It follows from equation (6.1) that a generic orbit $O$ is determined by 3 parameters $(\beta, \mu, \nu) \in \mathbb{R}^3, r_{\mu, \nu} \neq 0$:

$$O = O_{\beta, \mu, \nu} = \{x A^* + y B^* + c C^* + \mu U^* + \nu V^*, c \in \mathbb{R}, \nu x + \mu y = \beta r_{\mu, \nu}\}.$$ 

In particular, if $\mu, \nu \neq 0$, then the element

$$\frac{\beta r_{\mu, \nu}}{\nu} A^* + \mu U^* + \nu V^* (6.2)$$

is contained in $O_{\beta, \mu, \nu}$ and if $\nu = 0$ then the functional

$$\beta \text{sign}(\mu) B^* + \mu U^* (6.3)$$

belongs to $O_{\beta, \mu, 0}$.

We take a new basis $B_{\mu, \nu}$ of $g_{5,2}$. For that, letting $\tilde{\mu} := \frac{\mu}{r_{\mu, \nu}}, \tilde{\nu} := \frac{\nu}{r_{\mu, \nu}}$, we put:

$$B_{\mu, \nu} := \{A_{\mu, \nu} := \tilde{\mu} A - \tilde{\nu} B, B_{\mu, \nu} := \tilde{\nu} A + \tilde{\mu} B, C, U, V\}.$$ (6.4)

Let also

$$Z_{\mu, \nu} = \tilde{\mu} U + \tilde{\nu} V, T_{\mu, \nu} = \tilde{\nu} U - \tilde{\mu} V.$$ 

Then:

$$[A_{\mu, \nu}, C] = Z_{\mu, \nu}[B_{\mu, \nu}, C] = T_{\mu, \nu}$$

$$\langle \ell_{\beta, \mu, \nu}, [A_{\mu, \nu}, C] \rangle = r_{\mu, \nu}.$$
In this basis for any $R = xA_{\mu,\nu} + bB_{\mu,\nu} + yC + zZ_{\mu,\nu} + tT_{\mu,\nu}$, resp $R' = x'A_{\mu,\nu} + b'B_{\mu,\nu} + y'C + z'Z_{\mu,\nu} + t'T_{\mu,\nu}$ in $\mathfrak{g}$ we obtain the multiplication:

$$R \cdot R' = (x + x')A_{\mu,\nu} + (b + b')B_{\mu,\nu} + (y + y')C +$$

$$+ \left( z + z' + \frac{1}{2}(xy' - x'y) \right) Z_{\mu,\nu} + \left( t + t' + \frac{1}{2}(by' - b'y) \right) T_{\mu,\nu}. \quad (6.5)$$

We see that the vectors $A_{\mu,\nu}, B_{\mu,\nu}, Z_{\mu,\nu}$ span the three dimensional Heisenberg Lie algebra, that $B_{\mu,\nu}$ is contained in the stabilizer of the linear form $\ell_{\beta,\mu,\nu}$ and that $T_{\mu,\nu}$ is contained in the kernel of the restriction of $\ell_{\beta,\mu,\nu}$ to the centre of $\mathfrak{g}_{\mu,\nu}$. In the dual basis $\mathcal{B}_{\mu,\nu}^*$ of the basis $\mathcal{B}_{\mu,\nu}$ the orbit $\mathcal{O}_{\beta,\mu,\nu}$ of the element

$$\ell_{\beta,\mu,\nu} = \beta B^*_\mu + \mu U^* + \nu V^*,$$

is given by:

$$\mathcal{O}_{\beta,\mu,\nu} = \{ aA^*_{\mu,\nu} + \beta B^*_\mu + cC^* + \mu U^* + \nu V^*, a, c \in \mathbb{R} \}. \quad (6.6)$$

The stabilizer of $\ell_{\beta,\mu,\nu}$ is the set

$$\mathfrak{g}_{5,2}(\ell_{\beta,\mu,\nu}) = \text{span}\{ B_{\mu,\nu}, U, V \}.$$

We can take as polarization at $\ell_{\beta,\mu,\nu}$ the sub-algebra

$$p := \text{span}\{ A, B, U, V \}, \quad P := \exp(p).$$

We denote by $\Gamma_1^{5,2}$ the orbit space of this layer and we parametrize it by

$$\Gamma_1^{5,2} := \{ \ell_{\beta,\mu,\nu} \mid (\beta, \mu, \nu) \in \mathbb{R}, (\mu, \nu) \in \mathbb{R}^2, \mu^2 + \nu^2 \neq 0 \}. \quad (6.7)$$

(2) The second layer, denoted by

$$\Gamma_0^{5,2} = (\mathfrak{g}_{5,2}^*/G_{5,2})_{\text{char}} \simeq \mathbb{R}^3 \quad (6.8)$$

is the collection of all characters $\ell_{\alpha,\beta,\rho} = \alpha A^* + \beta B^* + \rho C^*$, $\alpha, \beta, \rho \in \mathbb{R}$. Their orbits are the one point sets $\{ \ell_{\alpha,\beta,\rho} \}$

**Theorem 6.1.**

1. On the set $\Gamma_1^{5,2}$ the dual topology is Hausdorff.
2. Let $\overline{O} = (\mathcal{O}_{\beta_k,\mu_k,\nu_k})_k$ be a sequence such that $\lim_k r_k = 0$. Then this sequence has a converging sub-sequence if and only if $\lim \inf_k |\beta_k|$ is finite. If $\overline{O}$ is properly converging, then, passing to a sub-sequence (also denoted by the same symbol for simplicity of notations) we can assume that $|\lim_k \beta_k| = \beta$ exists, that the sequences of vectors $(A^*_k = A^*_{\mu_k,\nu_k})_k$, resp. $(B^*_k = B^*_{\mu_k,\nu_k})_k$ converges to a $A^*_\infty$, resp. $B^*_\infty$ in $\mathfrak{g}_{5,2}^*$ and then

$$L(O) = RA^*_\infty + \beta B^*_\infty + RC^* \subset \Gamma_0^{5,2}.$$ 

*Proof.*

1. The point 1) is evident.
2. If $\lim \inf_k |\beta_k|$ exists in $\mathbb{R}$, then we take a sub-sequence (indexed also by $(\beta_k)_k$ for simplicity of notation) such that $\lim_k \beta_k = \beta$ exists in $\mathbb{R}$ and such that the sequences of vectors $(A^*_k = A^*_{\mu_k,\nu_k})_k$, resp. $(B^*_k = B^*_{\mu_k,\nu_k})_k$ converges to a $A^*_\infty$, resp. $B^*_\infty$ in $\mathfrak{g}_{5,2}^*$. It follows then from the description (6.6) of the coadjoint orbits that the limit set $L(O)$ of the sub-sequence is the set described in the theorem. If $\lim \inf_k |\beta_k| = +\infty$, then $\lim_k \{ |O_k, B_k| \} = \{ |\beta_k| \} = +\infty$. Hence $\overline{O}$ goes to infinity.
6.3. The unitary dual of $G_{5,2}$. The spectrum of the group $G_{5,2}$ can be identified by Kirillov’s orbit theory with the orbit space $g_{5,2}^* / G_{5,2} = \hat{G}_{5,2} := \Gamma_1^{5,2} \cup \Gamma_0^{5,2}$. For every $(\beta, \mu, \nu) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{R})^*$, we take the irreducible representation $\pi_{\beta, \mu, \nu} = \text{ind}^G_{G_{5,2}} \chi_{\beta, \mu, \nu}$ which is associated to $G_{\beta, \mu, \nu}$. This representation acts on the Hilbert space $L^2(\mathbb{R})$ and is given by the formula

$$\pi_{\beta, \mu, \nu}(x, y, z, t) \xi(s) = e^{-2\pi i (\beta b + r_{\mu, \nu} z + \nu s - \frac{r_{\mu, \nu} x y}{x})} \xi(s - y), \quad s \in \mathbb{R}, \quad \xi \in L^2(\mathbb{R}), \quad (x, y, z, t) \in G_{5,2},$$

using the coordinates coming from the basis $G_{\beta, \mu, \nu}$.

For $F \in L^1(G_{5,2})$, let

$$\hat{F}(\alpha, \beta) := \chi_{\alpha, \beta, \rho}(F) = \int_{G_{5,2}} F(a, b, c, u, v) e^{-2\pi i (\alpha a + \beta b + \rho c)} dadbdc, \quad \alpha, \beta, \rho \in \mathbb{R},$$

and

$$\|F\|_{\infty, 0} := \sup_{\alpha, \beta, \rho \in \mathbb{R}} |\chi_{\alpha, \beta, \rho}(F)| = \|\hat{F}\|_{\infty}.$$

**Definition 6.2.** Define for $a \in C^*(G_{5,2})$ its Fourier transform $F(a) \in L^\infty(\hat{G}_{5,2})$ by

$$\mathcal{F}(a)(\beta, \mu, \nu) := \pi_{\beta, \mu, \nu}(a) \in \mathcal{B}(L^2(\mathbb{R})), \quad (\beta, \mu, \nu) \in \Gamma_1^{5,2}$$

and

$$\mathcal{F}(a)(\alpha, \beta, \rho) := \chi_{\alpha, \beta, \rho}(a), \quad \alpha, \beta, \rho \in \mathbb{R}.$$

**Definition 6.3.** For all $F \in L^1(G_{5,2})$, and $(\beta, \mu, \nu) \in \Gamma_1^{5,2}$ the operator $\pi_{\beta, \mu, \nu}(F)$ is a kernel operator with kernel function $F_{\beta, \mu, \nu} \in \mathcal{B}(L^1(\mathbb{R}))$ given by:

$$F_{\beta, \mu, \nu}(s, x) = \hat{F}_{\beta, \mu, \nu} \left( \frac{\mu \nu}{2} (s + x) \right) A_{\mu, \nu}^* + \beta B_{\mu, \nu}^* + (s - y) C^* + \rho_{\mu, \nu} Z_{\mu, \nu}^*$$

in the coordinates coming from the basis $B_{\mu, \nu}$. Here the symbol $\hat{F}_{\beta, \mu, \nu}$ denotes the function

$$\hat{F}_{\beta, \mu, \nu}(s, q) := \int_p F(\exp(sC)p) e^{-2\pi i (q, \log p)} dp, \quad s \in \mathbb{R}, \quad q \in \mathfrak{p}^*.$$

Indeed, for $s \in L^2(\mathbb{R})$ and $s \in \mathbb{R}$ we have

$$\pi_{\beta, \mu, \nu}(F)(\xi)(s) = \int_{G_{5,2}} F(x, b, y, z, t) \pi_{\beta, \mu, \nu}(x, b, y, z, t) \xi(s) dxdbdydzdt$$

$$= \int_{G_{5,2}} F(x, b, y, z, t) e^{-2\pi i (\beta b + r_{\mu, \nu} z + \nu s - \frac{r_{\mu, \nu} x y}{x})} \xi(s - y) dxdbdydzdt$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^4} F(x, b, y, z, t) e^{-2\pi i (\beta b + r_{\mu, \nu} z + \nu s + \frac{r_{\mu, \nu} x y}{x})} \xi(y) dxdbdydzdt \right) dy$$

$$= \int_{\mathbb{R}} \hat{F}_{\beta, \mu, \nu} \left( \frac{\mu \nu}{2} (s + x) \right) A_{\mu, \nu}^* + \beta B_{\mu, \nu}^* + (s - y) C^* + \rho_{\mu, \nu} Z_{\mu, \nu}^* \xi(y) dy.$$
The following proposition is a consequence of Formula (6.9).

**Proposition 6.8.** For any \( a \in C^*(G_{5,2}) \) and \((\beta, \mu, \nu) \in \Gamma_{1,2}^5\), the operator \( \pi_{\beta, \mu, \nu}(a) \) is compact, the mapping \( \Gamma_{1,2}^5 \to B(L^2(\mathbb{R})) : (\beta, \mu, \nu) \mapsto \pi_{\beta, \mu, \nu}(a) \) is norm continuous in \((\beta, \mu, \nu)\) and tending to 0 for \( r_{\mu, \nu} \) going to infinity.

**Definition 6.5.** Let as before for \( \mu^2 + \nu^2 \neq 0 \), \( A_{\mu, \nu} := \bar{\mu}A - \bar{\nu}B, B_{\mu, \nu} := \bar{\nu}A + \bar{\mu}B \). Choose a Schwartz-function \( \eta \) in \( S(\mathbb{R}) \) with \( L^2 \)-norm equal to 1. For \((\alpha, \rho) \in \mathbb{R}^2 \) we define the function \( \eta_{\mu, \nu}(\alpha, \rho) \) by

\[
\eta_{\mu, \nu}(\alpha, \rho)(s) := \frac{r_{\mu, \nu}}{r_{\mu, \nu}} e^{2\pi is \rho} \eta \left( \frac{r_{\mu, \nu}}{r_{\mu, \nu}}(s + \frac{\alpha}{r_{\mu, \nu}}) \right), \quad s \in \mathbb{R}.
\]

**6.4. A \( C^* \)-condition.** The \( C^* \)-conditions for the group \( G_{5,2} \) can be copied from the corresponding conditions for the Heisenberg groups (see [6]).

**Lemma 6.6.** Let \( \xi \in S(\mathbb{R}) \). Then,

\[
\xi = \frac{1}{r_{\mu, \nu}} \int_{\mathbb{R}^2} (\xi, \eta_{\mu, \nu}(\alpha, \rho)) \eta_{\mu, \nu}(\alpha, \rho) d\alpha d\rho.
\]

**Proof.** The proof is the same as the proof of Lemma 2.8 in [Lud,Tur]. \( \square \)

**Definition 6.7.** 1. For all \((\alpha, \rho) \in \mathbb{R}^2 \) and \( k \in \mathbb{N} \), let \( P_{\mu, \nu}(\alpha, \rho) \) be the orthogonal projection onto the one dimensional subspace \( C\eta_{\mu, \nu}(\alpha, \rho) \).

2. Define for \( h \in C_0(\mathbb{R}^2) \) the linear operator

\[
\sigma_{\beta, \mu, \nu}(h) := \frac{1}{r_{\mu, \nu}} \int_{\mathbb{R}^2} h(\alpha A_{\mu, \nu}^* + \beta B_{\mu, \nu}^* + \rho C^*) P_{\mu, \nu}(\alpha, \rho) \, d\alpha d\rho.
\]

**Proposition 6.8.** (see Proposition 2.11 in [6])

1. For every \((\beta, \mu, \nu) \in \Gamma_{1,2}^5 \) and \( h \in S(\mathbb{R}^2) \) the integral (6.11) converges in operator norm.

2. \( \sigma_{\beta, \mu, \nu}(h) \) is compact and \( \|\sigma_{\beta, \mu, \nu}(h)\|_{op} \leq \|h\|_{\infty} \).

3. The mapping \( \sigma_{\beta, \mu, \nu} : C_0(\mathbb{R}^2) \to \mathcal{F}_1 \) is involutive, i.e. \( \sigma_{\beta, \mu, \nu}(h^*) = \sigma_{\beta, \mu, \nu}(h)^*, h \in C^*(\mathbb{R}^2) \), where by \( \sigma_{\beta, \mu, \nu} \) we denote also the extension of \( \sigma_{\beta, \mu, \nu} \) to \( C_0(\mathbb{R}^2) \).

**Theorem 6.9.** Let \( a \in C^*(G_{5,2}) \) and let \( \varphi \) be the operator field \( \varphi = F(a) \). Then the function \( \varphi(\beta) : (\alpha, \rho) \to F(a)(\alpha, \rho) \) is contained \( C_0(\mathbb{R}^2) \). Let \( \mathcal{T} = (\gamma_k = (\beta_k, \mu_k, \nu_k))_k \) be a properly converging sequence in \( \Gamma_{1,2}^5 \) having its limit set \( L(\mathcal{T}) = \mathbb{R}A_{\infty}^* + \beta B_{\infty}^* + \mathbb{R}C^* \) in \( \Gamma_{0,2}^5 \). Then

\[
\lim_{k \to \infty} \|\varphi(\gamma_k) - \sigma_{\gamma_k}(\varphi(\beta))\|_{op} = 0.
\]

**Proof.** The proof is the same as that of Theorem 2.12 in [Lud,Tur]. \( \square \)

7. **The \( C^* \)-algebra of the group \( G_{5,3} \).**

Recall that the Lie algebra of \( g_{5,3} \) is spanned by the basis \( \mathcal{B} = \text{span}\{A, B, C, U, V \} \) equipped with the Lie brackets

\[
[\mathcal{A}, \mathcal{B}] = U, \quad [\mathcal{A}, \mathcal{U}] = V, \quad [\mathcal{B}, \mathcal{C}] = V.
\]

This Lie algebra has a one-dimensional centre \( \mathcal{Z} = \mathbb{R}V \). The group \( G_{5,3} \) can be realized as \( \mathbb{R}^3 \) with the Campbell-Baker-Hausdorff multiplication

\[
(a, b, c, u, v)(a', b', c', u', v') = (a + a', b + b', c + c', u + u' - a'b + v + v' - a'u + \frac{a'^2b}{2} - \frac{b'c}{2} + \frac{bc'}{2}).
\]
For all \((a, b, c, u, v) \in G_{5,3}\), \((\alpha, \beta, \rho, \mu, \nu) \in g_{5,3}\) we obtain the following expression for \(\text{Ad}^*(a, b, c, u, v)\):

\[
\text{Ad}^*(a, b, c, u, v)(\alpha, \beta, \rho, \mu, \nu) = (\alpha - \mu b - \nu u - \nu \frac{ab}{2}, \beta + \mu a - \nu c + \nu \frac{a^2}{2}, \rho + \nu b, \mu + \alpha v, \nu).
\]

We give now a description of the co-adjoint orbits:

1. The generic orbits: They have a non-zero value \(\nu\) on the central element \(V\). It follows from (7.1) that we can characterize such an orbit \(O\) by \(\nu \in \mathbb{R}\).
2. The second layer is given by the set of linear functionals, which are 0 on all characters \(\ell\), parameterized by the set \(\Gamma^{5,3}_2 := \{\ell \equiv \nu; \nu \in \mathbb{R}\}\).
3. The last layer, denoted by \(\Gamma^{5,3}_0 = (g_{5,3}/G_{5,3})_{\text{char}} \simeq \mathbb{R}^3\) is the collection of all characters \(\ell_{\alpha, \beta, \rho} = \alpha A^* + \beta B^* + \rho C^*\), \(\alpha, \beta, \rho \in \mathbb{R}\). Their orbits are the point set \(\{\ell_{\alpha, \beta, \rho}\}\).

**Theorem 7.1.** (1) On the set \(\Gamma^{5,3}_2\) (resp on the set \(\Gamma^{5,3}_1\)) the dual topology is Hausdorff.

(2) Let \(\mathcal{O} = (O_{\nu_k})_k \subset \Gamma^{5,3}_2\) be a sequence, such that \(\lim_{k \to \infty} \nu_k = 0\). Then \(\mathcal{O}\) is properly converging and \(L(\mathcal{O}) = \Gamma^{5,3}_1 \cup \Gamma^{5,3}_0\).

(3) Let \(\mathcal{O} = (O_{\rho_k, \mu_k})_k\) be a sequence such that \(\lim_{k \to \infty} \mu_k = 0\). If \(\mathcal{O}\) has a limit then \(\rho := \lim_{k \to \infty} \rho_k\) exists in \(\mathbb{R}\). Conversely, if \(\lim_{k \to \infty} \rho_k = \rho\) exists, then the sequence \(\mathcal{O}\) converges and \(L(\mathcal{O}) = \mathbb{R}A^* + \mathbb{R}B^* + \rho C^*\).

**Proof.** The proof is straight forward. \(\square\)

**7.1. The Fourier transform for \(C^*(G_{5,3})\).** The spectrum of the group \(G_{5,3}\) can be identified by Kirillov’s orbit theory with the orbit space \(g_{5,3}/G_{5,3} = \overline{G_{5,3}} = \Gamma^{5,3}_2 \cup \Gamma^{5,3}_1 \cup \Gamma^{5,3}_0\).

1. Let \(\ell \in \Gamma^{5,3}_2\), its orbit \(O_{\ell}\) is of dimension 4. A polarization at \(\ell \equiv \nu V^*\) is given by \(p = p_{\nu} := \text{span}\{C, U, V\}\). We realize then \(\pi_{\nu} := \text{ind}_{P_{\nu}}^{G_{5,3}} \chi_{\nu}\).

The Hilbert space \(L^2(G_{5,3}/P_{\nu}, \chi_{\nu})\) is in fact isometric to \(L^2(\mathbb{R}^2)\), let \(E : \mathbb{R}^2 \to G_{5,3}\), \(E(a, b) := \exp(aA)\exp(bB)\) and \(S = \exp(\mathbb{R}A)\exp(\mathbb{R}B) = E(\mathbb{R}^2)\). Then \(G_{5,3} = S.P_{\nu}\) as topological product and the mapping \(\lambda : L^2(G_{5,3}/P_{\nu}, \chi_{\nu}) \to L^2(\mathbb{R}^2)\) defined by \(\lambda(t) := \xi(E(t)), \ t \in \mathbb{R}^2\), is unitary. Let us compute the operator \(\pi_{\ell}(F)\) for \(F \in C^*(G_{5,3})\) explicitly. For \(\xi \in L^2(\mathbb{R}^2)\), \(s \in S\), \(p \in P_{\nu}\) we have

\[
\pi_{\ell}(F)\xi(t) = \int_{G_{5,3}/P_{\nu}} \xi(s) \left( \int_{P_{\nu}} F(ts^{-1}p)e^{-2i\pi(s,t,p)}dp \right)ds.
\]
For \( t = E(a, b) \) and \( s = E(a', b') \) we get
\[
\pi_t(F)\xi(a', b') = \int_{\mathbb{R}^2} \hat{F}^{\rho, \mu}(a' - a, b' - b, E(a, b), \xi) e^{-2\pi i \ell a' b' / 2} \xi(a, b) dadb.
\]

(2) Let \( \ell = \ell_{\rho, \mu} \in \Gamma_1^{5,3} \). A polarization at \( \ell \) is given by \( p_{\rho, \mu} = \text{span}\{B, C, U, V\} \).

We take \( \pi_{\rho, \mu} := \text{ind}_{p_{\rho, \mu}}^\Gamma \xi_{\rho, \mu} \). This representation acts on the Hilbert space \( L^2(G_{5,3}/p_{\rho, \mu}) \simeq L^2(\mathbb{R}) \) and for \( F \in L^1(G_{5,3}) \), \( \xi \in L^2(\mathbb{R}) \), we have:
\[
\pi_{\rho, \mu}(F)\xi(a') = \int_{\mathbb{R}} \hat{F}^{p_{\rho, \mu}}(a' - a, a \cdot p_{\rho, \mu}) da, \quad \text{where } p_{\rho, \mu} = \ell_{\rho, \mu}|p_{\rho, \mu}.
\]

(3) Any one-dimensional representation is a unitary character \( \chi_{\alpha, \beta, \rho} \), \( (\alpha, \beta, \rho) \in \mathbb{R}^3 \), of \( G_{5,3} \) which is given by
\[
\chi_{\alpha, \beta, \rho}(a, b, c, u, v) = e^{-2\pi i (\alpha a + \beta b + \rho c)}, \quad (a, b, c, u, v) \in G_{5,3}.
\]

For \( F \in L^1(G_{5,3}) \), let
\[
\hat{F}(\alpha, \beta, \rho) := \chi_{\alpha, \beta, \rho}(F) = \int_{G_{5,3}} F(a, b, c, 0, 0) e^{-2\pi i (\alpha a + \beta b + \rho c)} dadb, \quad \alpha, \beta, \rho \in \mathbb{R}.
\]

**Definition 7.2.** Define for \( a \in \mathcal{C}^*(G_{5,3}) \) its Fourier transform \( F(a) \in \mathcal{C}^*(G_{5,3}) \) by
\[
\hat{a}(\nu) = F(a)(\nu) := \pi_{\nu}(a) \in \mathcal{K}(L^2(\mathbb{R}^3)), \quad \nu \in \Gamma_2^{5,3};
\]
\[
\hat{a}(\rho, \mu) = F(a)(\rho, \mu) := \pi_{\rho, \mu}(a) \in \mathcal{K}(L^2(\mathbb{R})), \quad (\rho, \mu) \in \Gamma_1^{5,3};
\]
\[
\hat{a}(\alpha, \beta, \rho) = F(a)(\alpha, \beta, \rho) := \chi_{\alpha, \beta, \rho}(a) \in \mathcal{C}^*(\mathbb{R}^3).
\]

**Proposition 7.3.** For every \( a \in \mathcal{C}^*(G_{5,3}) \) and \( \nu \in \Gamma_2^{5,3} \) (resp \( \rho, \mu \in \Gamma_1^{5,3} \)), the operator \( \pi_{\nu}(a) \) (resp the operator \( \pi_{\rho, \mu} \)) is compact, the mapping \( \Gamma_2^{5,3} \rightarrow \mathcal{K}(L^2(\mathbb{R}^3)) : \nu \mapsto \pi_{\nu}(a) \) (resp the mapping \( \Gamma_1^{5,3} \rightarrow \mathcal{K}(L^2(\mathbb{R})) : (\rho, \mu) \mapsto \pi_{\rho, \mu}(a) \)) is norm continuous in \( \nu \) (resp in \( \rho, \mu \)) and tending to 0 for \( \nu \) going to infinity (resp for \( \rho \) or \( \mu \) going to infinity).

7.2. The changing of layers condition.

- **Passing from** \( \Gamma_2^{5,3} \) to \( \Gamma_1^{5,3} \cup \Gamma_1^{5,3} \). Let \( \mathcal{O} = \{O_{t_k}\}_k \subset \Gamma_2^{5,3} \) be a properly converging sequence where \( t_k = (0, 0, 0, 0, v_k) \), \( k \in \mathbb{N} \) such that \( \lim_k v_k = 0 \). Let \( p_k := (\ell_k)|p \).

By Theorem 4.1, the restriction of the limit set \( L(O) \) to \( p \) is the closed set \( L = L(O)_p = \{\rho, \mu, 0\}, \rho \in \mathbb{R}, \mu \in \mathbb{R} \).

**Definition 7.4.** For \( k \in \mathbb{N} \) let:
\[
\epsilon_k := |v_k|^\frac{1}{2},
\]
\[
I_{k, j} := \left\{(c, u, v_k) \in \mathbb{R}^3 ; i\epsilon_k^\frac{1}{2} \leq c < i\epsilon_k^\frac{1}{2} + \epsilon_k\right\},
\]
\[
U_{k, i, j} := \{(x, y) \in \mathbb{R}^2 ; (xA + yB) \cdot p_k \in I_{k, j}\}, \quad j \in \mathbb{Z}^*.
\]

Finally:
\[
U_k := \bigcup_{i, j \in \mathbb{Z}^*} U_{k, i, j}.
\]

Let also for \( k \in \mathbb{N} \), \( i, j \in \mathbb{Z} \):
\[
x_{i, j}^k := \frac{j \epsilon_k}{v_k}, \quad y_{i, j}^k := \frac{i \epsilon_k^\frac{1}{2}}{v_k}, \quad g_{i, j}^k = x_{i, j}^k A + y_{i, j}^k B.
\]
Let for $i, j \in \mathbb{Z}^*$, $k \in \mathbb{N}^*$:

$$p^k_{i,j} := (i\varepsilon_k^+, j\varepsilon_k, 0).$$

An easy computation gives:

$$y^k_{i,j}p_k = (i\varepsilon_k^+, j\varepsilon_k, \nu_k) = p^k_{i,j} + (0, 0, \nu_k).$$

**Proposition 7.5.** Let $K$ be a compact subset, for $k$ large enough we have that

$$K(U^k_{i,j} \subset \bigcup_{i', j' = -1}^{1} U^k_{i'+i,j'+j} =: V^k_{i,j}).$$

**Proof.** We can suppose that $KP$ is contained in $[-M, M]^2P$ for some $M > 0$. For $r = (u, v) \in K \subset G_{5,3}/P$ and $s = (x, y) \in U^k$ we have that

$$(rs) \cdot p_k = (\nu_kv + \nu_ky, \nu_kx + \nu_ku, \nu_k)$$

and

$$(x, y) \in U^k_{i,j} \iff (x, y) \cdot p_k \in I^k_{i,j} \Rightarrow$$

$$\begin{cases}
    j\varepsilon_k \leq \nu_kx < j\varepsilon_k + \varepsilon_k, \\
    i\varepsilon_k^+ \leq \nu_ky < i\varepsilon_k^+ + \varepsilon_k^+,
\end{cases} \Rightarrow$$

$$\begin{cases}
    (j-1)\varepsilon_k \leq \nu_kx + \nu_ku < (j+1)\varepsilon_k + \varepsilon_k, \\
    (i-1)\varepsilon_k^+ \leq \nu_ky + \nu_kv < (i+1)\varepsilon_k^+ + \varepsilon_k^+,
\end{cases}$$

It follows that $KU^k_{i,j} \subset \bigcup_{i', j' = -1}^{1} U^k_{i'+i,j'+j}.$ \qed

**Definition 7.6.** For $k \in \mathbb{N}^*$ Let

$$R^k = \left[ -\varepsilon_k \frac{\varepsilon_k}{|\nu_k|}, \varepsilon_k \frac{\varepsilon_k}{|\nu_k|} \right] \times \left[ -\varepsilon_k^+ \frac{\varepsilon_k^+}{|\nu_k|}, \varepsilon_k^+ \frac{\varepsilon_k^+}{|\nu_k|} \right].$$

**Lemma 7.7.** For $k \in \mathbb{N}^*$ large enough, for any $i, j \in \mathbb{Z}^*$ we have the set $U^k_{i,j}$ is contained in $R^k + g^k_{i,j}$.

**Proof.** Let $s = (x, y) \in U^k_{i,j}$. Then:

$$(x, y) \cdot p_k \in I^k_{i,j} \iff$$

$$\begin{cases}
    j\varepsilon_k \leq \nu_kx < j\varepsilon_k + \varepsilon_k \Rightarrow |x - x^k_j| \leq \frac{\varepsilon_k}{|\nu_k|} \Rightarrow x \in \left[ -\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right] + x^k_j, \\
    i\varepsilon_k^+ \leq \nu_ky < i\varepsilon_k^+ + \varepsilon_k^+ \Rightarrow |y - y^k_i| \leq \frac{\varepsilon_k^+}{|\nu_k|} \Rightarrow y \in \left[ -\frac{\varepsilon_k^+}{|\nu_k|}, \frac{\varepsilon_k^+}{|\nu_k|} \right] + y^k_i.
\end{cases}$$

$$\Rightarrow \quad s \in R^k + g^k_{i,j}.$$

\qed

**Lemma 7.8.** For $k \in \mathbb{N}^*$ large enough, for $i, j \in \mathbb{Z}^*$ and any $(x, y) \in U^k_{i,j}$ we have that

$$\|(xA + yB) \cdot p_k - ((xA + yB) \cdot (g^k_{i,j})^{-1}) \cdot p^k_{i,j}\| \leq 3\varepsilon_k^+. $$
Proof. For \((x, y) \in U_{k,j}^k\) we have that \((x, y) = (x' + x_j^k, y' + y_j^k)\) where \(|\nu_k x'| \leq \varepsilon_k\) and \(|\nu_k y'| \leq \varepsilon_k^2\). Therefore

\[
\left\|((x' + x_j^k)A + (y' + y_j^k)B)p_k - (x' A + y'B) \cdot p_k^k\right\|
\]

\begin{align*}
&= \left\|((\nu_k y' + i\varepsilon_k^k, j\varepsilon_k + \nu_k x', \nu_k) - (i\varepsilon_k^2, j\varepsilon_k, 0))\right\|
\end{align*}

\begin{align*}
&= \left\|((\nu_k y' + i\varepsilon_k^k, j\varepsilon_k + \nu_k x', \nu_k) - (i\varepsilon_k^2, j\varepsilon_k, 0))\right\|
\end{align*}

\begin{align*}
&= |\nu_k y'| + |\nu_k x'| + |\nu_k|
\end{align*}

\[
\leq \varepsilon_k^2 + \varepsilon_k + |\nu_k| \leq 3\varepsilon_k^2.
\]

\[\blacksquare\]

**Definition 7.9.** Let for \(\beta, \rho, \mu \in \mathbb{R}\), let

\[
\ell_{\beta,\rho,\mu} = \beta B^* + \rho C^* + \mu U^*, \quad \ell_{\rho,\mu} = \beta B^* + \mu U^*, \quad \text{and} \quad \ell_{\rho,\mu} = \rho C^* + \mu U^* \in \mathfrak{g}_5^3.
\]

The sub-algebra \(\mathfrak{p} := \mathfrak{p}_{\beta,\mu} = \text{span}\{B, \mathfrak{p}\}\) is a polarization at \(\ell_{\beta,\mu}\) and at \(\ell_{\beta,\rho,\mu}\), which gives us the equivalent representations \(\pi_{\beta,\mu} = \text{ind}_{\mathfrak{p}} G_{5,3} \chi_{\ell_{\beta,\mu}} \in \tilde{G}_{5,3}\) and \(\pi_{\beta,\rho,\mu} = \text{ind}_{\mathfrak{p}} G_{5,3} \chi_{\ell_{\beta,\rho,\mu}}\). Let \(\sigma_{\rho,\mu}\) be the unitary operator which gives the equivalence between both representations. We take the direct integral representation

\[
(7.2) \quad \tau_{\rho,\mu} := \left(\int_{\mathbb{R}} \pi_{\ell_{\beta,\rho,\mu}} d\beta, \int_{\mathbb{R}} L^2(G_{5,3}/P, \chi_{\ell_{\beta,\rho,\mu}}) d\beta\right).
\]

This representation \(\tau_{\rho,\mu}\) is in fact equivalent to the representation \(\sigma_{\rho,\mu} := \text{ind}_{\mathfrak{p}} G_{5,3} \chi_{\ell_{\beta,\rho,\mu}}\) and a unitary intertwining \(U_{\rho,\mu}\) operator is given by:

\[
U_{\rho,\mu} : L^2(G_{5,3}/P, \chi_{\rho,\mu}) \to \int_{\mathbb{R}} L^2(G_{5,3}/P, \chi_{\ell_{\rho,\mu}}) d\beta,
\]

\[
U_{\rho,\mu}(\xi)(\beta)(g) = \int_{\mathbb{R}} \xi(\exp(sB)) e^{-2i\pi \beta s} ds, g \in G_{5,3}, \beta \in \mathbb{R}.
\]

Hence for every \(a \in C^*(G_{5,3})\) we have that

\[
(7.3) \quad \|\sigma_{\rho,\mu}(a)\|_{\text{op}} = \sup_{\beta \in \mathbb{R}} \|\pi_{\ell_{\beta,\rho,\mu}}(a)\|_{\text{op}}.
\]

**Definition 7.10.**

- Let \(C_{\mathcal{O}} = CB(L(\mathcal{O}), B(L^2(\mathbb{R}^2)))\) be the \(C^*\)-algebra of all continuous, uniformly bounded mappings \(\phi : L(\mathcal{O}) \to B(L^2(\mathbb{R}^2))\) from the locally compact space \(L(\mathcal{O})\) into the algebra of bounded linear operators \(B(L^2(\mathbb{R}^2))\) on the Hilbert space \(L^2(\mathbb{R}^2)\). By the Theorem \(\square\) we observe that for any \(a \in C^*(G_{5,3})\), the operator field \(a_{L(\mathcal{O})}\) is contained in \(C_{\mathcal{O}}\). Furthermore, for \(\ell = \rho C^* + \mu U^* \in \mathfrak{g}_5^3\), we obtain a representation \(\tilde{\sigma}_{\rho,\mu} = \tilde{\sigma}_\ell\) on the Hilbert space \(L^2(\mathbb{R}^2)\) of the algebra \(C_{\mathcal{O}}\) defined by:

\[
(7.4) \quad \tilde{\sigma}_\ell(\phi) \xi := U_{\ell}^{-1} \left(\int_{\mathbb{R}} u_{\sigma,\beta,\mu}^* \circ \phi(\beta, \mu) \circ u_{\beta,\rho,\mu}(U_{\ell}(\xi)(\beta)) d\beta\right), \phi \in C_{\mathcal{O}}.
\]

- Define for \(k \in \mathbb{N}\) and \(\phi \in C_{\mathcal{O}}\) the linear operator \(\tilde{\sigma}_{k,\mathcal{O}}(\phi)\) by

\[
(7.5) \quad \tilde{\sigma}_{k,\mathcal{O}}(\phi) := \sum_{i \in \mathbb{Z}} M_{V_{k,i}} \circ \tilde{\sigma}_{(g_{k,i})^{-1}}(\phi) \circ M_{V_{k,i}}^{-1},
\]
where $\hat{\sigma}_k$ is an in Equation \([7.4]\). For $a \in C^*(G_{5,3})$ we have that $\sigma_{k,\overline{\sigma}}(a) = \hat{\sigma}_k(\overline{a}_{1L(\overline{\sigma})})$.

**Proposition 7.11.** Let $a \in C^*(G_{5,3})$. Then:

$$
\lim_{k \to \infty} \|\pi_{tk}(a) - \sigma_{k,\overline{\sigma}}(a)\|_{op} = 0.
$$

**Proof.** Let $\varepsilon > 0$. Take first $F \in L^1_c(G_{5,3})$. Let us choose a compact subset $K \subset p^*$ and an $M > 0$ such that the function $\mathbb{R}^2 \times p^* \ni ((x, y), p) \to \hat{F}^P(E(x, y), p)$ is supported in $[-M, M]^2 \times K$. By Proposition \([6.3]\) we have for $k$ large enough:

$$
\pi_{tk}(F) \circ M_{V_{k,i}} = M_{V_{k,i}} \circ \pi_{tk}(F) \circ M_{V_{k,i}}, \quad i, j \in \mathbb{Z}.
$$

The kernel function $F_k$ of the operator $\pi_{tk}(F) \circ M_{V_{k,i}} - M_{V_{k,i}} \circ \pi_{tk}(F)$. Since the function $(s, p) \to |\hat{F}^P(s, p)|^2$ is in $C^\infty_c(G_{5,3}/P, p^*)$ there exists a non-negative continuous function with compact support $\varphi : G_{5,3}/P \to \mathbb{R}_+$ such that for any $q, p \in p^*$, $s \in G_{5,3}/P$:

$$
|\hat{F}^P(s, q) - \hat{F}^P(s, p)| \leq \varphi(s)\|q - p\|.
$$

It follows then from Lemma \([7.7]\) and Lemma \([7.8]\) that for $k \in \mathbb{N}$ large enough, $i, j \in \mathbb{Z}$, $s \in G_{5,3}/P$ :

$$
|F_k(s, t)| \leq \left|\hat{F}^P(st^{-1}, t \cdot p_k) - \hat{F}^P(st^{-1}, t(\xi_{k,i,j})^{-1} \cdot p_k)\right| \\
\leq 3\varepsilon \|\varphi(st^{-1})\|
$$

Using now Young’s estimate, we see that for $k$ large enough and $i, j \in \mathbb{Z}$ :

$$
\|\pi_{tk}(F) - \hat{\sigma}_k(\overline{\sigma}(F))\|_{op} \leq 3\varepsilon_1 \|\varphi\|_1.
$$

\[\square\]

**Passing from $\mathbb{R}^{5,3}_1$ to $\Gamma^{5,3}_0$.**

**Definition 7.12.** Choose a Schwartz-function $\eta$ in $\mathcal{S}(\mathbb{R})$ with $L^2$-norm equal to 1. For $(\alpha, \beta) \in \mathbb{R}^2$ we define the function $\eta_{\mu}(\alpha, \beta)$ by

$$
(\sigma_4) \quad \eta_{\mu}(\alpha, \beta)(s) := |\mu|^{-\frac{1}{2}} e^{2\pi i s \alpha} \eta(|\mu|^{\frac{1}{2}}(s + \frac{\beta}{\mu})), \quad s \in \mathbb{R}.
$$

**Lemma 7.13.** Let $\xi \in \mathcal{S}(\mathbb{R})$. Then,

$$
\xi = \frac{1}{|\mu|} \int_{\mathbb{R}^2} \langle \xi, \eta_{\mu}(\alpha, \beta) \rangle \eta_{\mu}(\alpha, \beta) d\alpha d\beta.
$$

**Definition 7.14.** 1. For all $(\alpha, \beta) \in \mathbb{R}^2$ and $k \in \mathbb{N}$, let $P_{\mu(\alpha, \beta)}$ be the orthogonal projection onto the one dimensional subspace $\mathbb{C} \eta_{\mu}(\alpha, \beta)$.

2. Define for $h \in C_0(\mathbb{R}^2)$ the linear operator

$$
(\sigma_{\mu,\nu}) \quad \sigma_{\mu,\nu}(h) := \frac{1}{|\mu|} \int_{\mathbb{R}^2} h(\alpha A^* + \beta B^* + \rho C^*) P_{\mu(\alpha, \beta)} d\alpha d\beta.
$$

**Proposition 7.15.** (see Proposition 2.11 in \([\mathbb{R}]\))
(1) For every \((\rho, \mu) \in \Gamma^{5,3}_1\) and \(h \in \mathcal{S}(\mathbb{R}^2)\) the integral \(\int \sigma_{\rho, \mu}(h)\) converges in operator norm.

(2) \(\sigma_{\rho, \mu}(h)\) is compact and \(\|\sigma_{\rho, \mu}(h)\|_\infty \leq \|h\|_\infty\).

(3) The mapping \(\sigma_{\rho, \mu} : C_0(\mathbb{R}^2) \to \mathcal{F}_1\) is involutive, i.e., \(\sigma_{\rho, \mu}(h^*) = \sigma_{\rho, \mu}(h)^*, h \in C^*(\mathbb{R}^2)\), where by \(\sigma_{\rho, \mu}\) we denote also the extension of \(\sigma_{\rho, \mu}\) to \(C_0(\mathbb{R}^2)\).

**Theorem 7.16.** Let \(a \in C^*(G_{5,3})\) and let \(\varphi\) be the operator field \(\varphi = \mathcal{F}(a)\). Then the function \(\varphi(0) : (\alpha, \beta) \to \mathcal{F}(a)(\alpha, \beta)\) is contained \(C_0(\mathbb{R}^2)\). Let \(\gamma = (\gamma_k = (\rho_k, \mu_k))_k\) be a properly converging sequence in \(\Gamma^{5,3}_1\) having its limit set \(L(\gamma) = \mathbb{R}A^* + \mathbb{R}B^* + \rho C^*\) in \(\Gamma^{5,3}_0\). Then

\[
\lim_{k \to \infty} \|\varphi(\gamma_k) - \sigma_{\gamma_k}(\varphi(\rho))\|_\infty = 0.
\]

**Proof.** The proof is the same as that of Theorem 2.12 in [Lud-Tur]. \(\square\)

#### 8. The \(C^*\)-Algebra of the Group \(G_{5,4}\)

Recall that the Lie algebra of \(g_{5,4}\) is spanned by the basis \(B = \{A, B, C, U, V\}\) equipped with the Lie brackets

\[
[A, B] = C, [A, C] = U, [B, C] = V.
\]

This Lie algebra has a two-dimensional centre \(\mathfrak{z} = \text{span}\{U, V\}\). The group \(G_{5,4} = \text{exp}(g_{5,4})\) can be realized as \(\mathbb{R}^5\) with the Campbell-Baker-Hausdorff multiplication

\[
(a, b, c, u, v) \cdot (a', b', c', u', v') = (a + a', b + b', c + c' - a'b, u + u' - a'c + \frac{a^2b}{2}, v + v' + \frac{bc'}{2} - \frac{b'c}{2} + \frac{a'b'c}{2}).
\]

For all \((a, b, c, u, v) \in G_{5,4}\), \((\alpha, \beta, \rho, \mu, \nu) \in g_{5,4}^*\) we obtain the following expression for \(\text{Ad}^* (a, b, c, u, v)\):

\[
\text{Ad}^* (a, b, c, u, v) ((\alpha, \beta, \rho, \mu, \nu) = (a - bp - c\mu - \mu \frac{ab}{2} - \frac{b^2}{2}, \beta + \rho a - \nu c + \nu \frac{ab}{2} + \frac{a^2}{2}, \rho + \mu a + \nu b, \mu, \nu)).
\]

We give now a parameterization of the co-adjoint orbits: The generic elements \(\ell = (\alpha, \beta, \rho, \mu, \nu)\) in \(g_{5,4}^*\), are those for which a non-zero value \(r_{\mu, \nu} = \sqrt{\mu^2 + \nu^2}\). As in \([6,3]\) we take a new basis \(B_{\mu, \nu}\) of \(g_{5,4}\). For that, letting \(\tilde{\mu} := \frac{\mu}{r_{\mu, \nu}}, \tilde{\nu} := \frac{\nu}{r_{\mu, \nu}}\) we put:

\[
B_{\mu, \nu} := \{A_{\mu, \nu} := A_{\ell} := \tilde{\mu} A + \tilde{\nu} B, B_{\mu, \nu} := B_{\ell} := -\tilde{\nu} A + \tilde{\mu} B, C, U, V\}.
\]

In the dual basis \(B_{\mu, \nu}^*\) of the basis \(B_{\ell}\) the orbit \(O_{\ell}\) of the element \(\ell = \ell_{\mu, \nu} = \beta B_{\mu, \nu}^* + \mu U^* + \nu V^*\) is given by:

\[
O_{\ell} = \left\{aA_{\ell}^* + (\beta + \frac{e^2}{2\ell_{\mu, \nu}})B_{\ell}^* + cC^* + \mu U^* + \nu V^*, a, c \in \mathbb{R}\right\}.
\]

It follows from this description that the function

\[
Q : \ell = aA_{\ell}^* + bB_{\ell}^* + cC^* + \mu U^* + \nu V^* \to 2b\ell - c^2
\]

is \(G_{5,4}\)-invariant on this set. The stabilizer \(g_{5,4}(\ell)\) of \(\ell\) is the sub-algebra

\[
g_{5,4}(\ell) = \text{span}\{B_{\ell}, U, V\}.
\]
We denote by $\Gamma_{5,4}^2$ the orbit space of this family of generic co-adjoint orbits parameterized by the set

$$\Gamma_{5,4}^2 := \{ (\ell, \mu, \nu) \in (\beta, \mu, \nu), \beta \in \mathbb{R}, (\mu, \nu) \in \mathbb{R}^2 \setminus \{(0, 0)\} \}$$

Since $G_{5,4}/\exp(\mathfrak{g}) = H_1$ we can decompose the orbit space $\mathfrak{g}_{5,4}/G_{5,4}$, and hence also the dual space $\hat{G}_{5,4}$, into the disjoint union

$$\mathfrak{g}_{5,4}/G_{5,4} = \Gamma_{5,4}^2 \cup \Gamma_1^4 \cup \Gamma_0^1,$$

where $\Gamma_0^1$ and $\Gamma_1^1$ are as in (5.1).

### 8.1. Limit sets of properly converging sequences in $\Gamma_{5,4}^2$.

**Theorem 8.1.** The sequence $(O_{\beta_k, \mu_k, \nu_k})_k$ goes to infinity if and only if the real sequence $(\sqrt{\mu_k^2 + \nu_k^2} + \varepsilon_k \beta_k + (\varepsilon_k - 1)\beta_k \nu_k)_k$ goes to infinity, where for $k \in \mathbb{N}$:

$$\varepsilon_k := \begin{cases} 
1 & \text{if } \beta_k > 0 \\
0 & \text{if } \beta_k \leq 0.
\end{cases}$$

**Proof.** Suppose that the sequence of orbits does not tend to infinity. Then there is a convergent sub-sequence $(r_k)$ and convergent sequence $(c_j)$ such that $(2\beta_k \nu_k + \varepsilon_k \beta_k + (\varepsilon_k - 1)\beta_k \nu_k)_k$ goes to infinity. Multiplying by $2r_k$, we see that $(2\beta_k \nu_k + \varepsilon_k \beta_k + (\varepsilon_k - 1)\beta_k \nu_k)_k$ is convergent and hence $(\beta_k \nu_k)_k$ is also convergent. Hence $(\sqrt{\mu_k^2 + \nu_k^2} + \varepsilon_k \beta_k + (\varepsilon_k - 1)\beta_k \nu_k)_k$ does not tend to infinity. Conversely, suppose that $(\sqrt{\mu_k^2 + \nu_k^2} + \varepsilon_k \beta_k + (\varepsilon_k - 1)\beta_k \nu_k)_k$ does not tend to infinity. Then there is a convergent sub-sequence $(r_k)$ such that $(\beta_k \nu_k)_k$ is also convergent. We may choose convergent sequence $(c_j)$ such that $2\beta_k \nu_k + \varepsilon_j \beta_k + (\varepsilon_j - 1)\beta_k \nu_k = 0$ for all $j$. Then $(0, 0, \varepsilon_j, \mu_k, \nu_k) \in O_k$, and the sequence of functionals converges as $j \to \infty$. Hence the sequence of orbits $(O_k)$ does not tend to infinity. □

**Theorem 8.2.**

1. On the set $\Gamma_{5,4}^2$ the dual topology is Hausdorff.
2. Let $\overline{O} = (O_{\ell_k} = O_{\beta_k B^*_\nu + \mu_k v^* + \nu_k v^*})_k$ (where $B_k := B_{\ell_k}, k \in \mathbb{N}$) be a sequence in $\Gamma_{5,4}^2$ with $\lim_k r_k = \lim_k \sqrt{\mu_k^2 + \nu_k^2} = 0$. We can assume (passing if necessary to a sub-sequence) that the real sequence $(\tilde{\mu}_k)_k$ (resp. $(\tilde{\nu}_k)_k$) converges to $\tilde{\mu}$ (resp. to $\tilde{\nu}$). Then the sequence of vectors $(A_k = A_{\ell_k})_k$, resp. $(B_k = B_{\ell_k})_k$ converges to the vector $A_\infty = \tilde{\mu} A + \tilde{\nu} B$ (resp. to $B_\infty = -\tilde{\nu} A + \tilde{\mu} B$).

- If $\overline{O}$ has a limit, then $d := \lim_k \rightarrow -\infty (2\beta_k r_k) =: d_k$ exists and $d \geq 0$.
- Suppose now that $(\overline{O} = (O_{\ell_k} = O_{\beta_k B^*_\nu + \mu_k v^* + \nu_k v^*})_k)$ is a properly converging sequence in $\Gamma_{5,4}^2$. If $\overline{O}$ admits a limit in $\Gamma_1^1 \cup \Gamma_0^1$, then its limit set $L(\overline{O})$ is given by:

$$L(\overline{O}) = \begin{cases} 
\mathbb{R} A^*_\infty + [\beta_\infty, +\infty] [B^*_\infty] & \text{if } \beta_\infty \in \mathbb{R} \\
\mathbb{R} A^*_\infty + [\beta_\infty, +\infty] [B^*_\infty] - \infty, +\infty [B^*_\infty] = \Gamma_0^1 & \text{if } \beta_\infty = -\infty.
\end{cases}$$

**Proof.**

1. The point 1) is evident.
(2) • Let $\mathcal{O}$ be such a sequence in $\Gamma_{2}^{5,4}$ having a limit. Let $\ell$ be a point in a limit orbit of the sequence $\mathcal{O}$ and let $m_k = a_kA_k^* + (\beta_k + \frac{c_k^2}{2r_k})B_k^* + c_kC^* + \mu_kU + \nu_kV^* \in \mathcal{O}_k, k \in \mathbb{N}$, such that $\lim_k m_k = \ell$. Then:

$$\ell(B_\infty) = \lim_k m_k(B_k) = \lim_k \frac{2\beta_k r_k + m_k(C)^2}{2r_k}.$$ 

This shows that $\lim_k (2\beta_k r_k + m_k(C)^2) = 0$ since $\lim_k r_k = \lim_k \sqrt{\mu_k^2 + \nu_k^2}$, i.e. $\lim_k (-2\beta_k r_k) = \ell(C)^2 =: d$.

• (a) Let now $\mathcal{O}$ be properly convergent and suppose that $d > 0$. For $k$ large enough, we have that $\beta_k > 0$ and then the element $m_{k,\pm} := \pm \sqrt{-2\beta_k r_k C^* + \mu_k U^* + \nu_k V^*}$ of $\mathcal{O}_k$ converges to $\pm \sqrt{d} C^*$. Hence the orbits $\mathcal{O}_{k,\sqrt{d}}$ are contained in $L(\mathcal{O})$. On the other hand, every other $\ell$ in the limit set $L(\mathcal{O})$ satisfies the relation $d = \ell(C)^2$, which means that $\ell(C) = \pm \sqrt{d}$. Hence $L(\mathcal{O}) = \{O_{\sqrt{d}}, O_{-\sqrt{d}}\}$.

(b) If now $d = 0$, we see in a similar manner that the limit of the sequence $\mathcal{O}$ must be characters, i.e. vanish on $C$. Choose a sub-sequence (also denoted by $\mathcal{O}$ for simplicity), such that $\lim_k \beta_k = \beta_\infty$. Take any sequence

$$m_k = a_kA_k^* + (\frac{2\beta_k r_k + c_k^2}{2r_k})B_k^* + c_k C^* + \mu_k U^* + \nu_k V^* \in \mathcal{O}_k, k \in \mathbb{N}$$

which converges to $\ell = aA^* + bB_\infty^* \in L(\mathcal{O})$ for some $a, b \in \mathbb{R}$. Since

$$b = \lim_{k \to \infty} \left( \beta_k + \frac{c_k^2}{2r_k} \right) = \lim_k \beta_k + \lim_k \frac{c_k^2}{2r_k} = \lim_k \beta_k = \beta_\infty,$$

it follows that $\beta_\infty \leq b < +\infty$. On the other hand, for any $a \in \mathbb{R}$, $b > \beta_\infty$, we have that $b \geq \beta_k$ for $k$ large enough and the sequence

$$m_k = a_kA_k^* + bB_k^* + \sqrt{(b - \beta_k)2r_k C^* + \mu_k U^* + \nu_k V^*} \in \mathcal{O}_k,$$ 

converges to $aA_\infty^* + bB_\infty^*$, since $\lim_k \beta_k r_k = 0$. Hence, since $L(\mathcal{O})$ is closed,

$$L(\mathcal{O}) = \mathbb{R}A_\infty^* + [\beta_\infty, +\infty[B_\infty^*].$$

8.2. The Fourier transform. Let $\ell \in \Gamma_{2}^{5,4}$, then $\dim(\mathcal{O}_\ell) = 2$. A polarization at $\ell = \ell_{\beta,\mu,\nu} = \beta B_\mu^* + \mu U^* + \nu V^*$ is given by $P_\ell = P_{\mu,\nu} = \text{span}\{B_\mu, C, U, V\}$. Then $G_{5,4} = \exp(\mathbb{R}A_\ell)P_\ell$ and $P_\mu = \exp(\mathbb{R}B_{\mu,\nu})P$ as topological products. We take $\pi_\ell := \text{id}_{G_{5,4}} \chi_\ell$. Its Hilbert space is isomorphic to $L^2(\mathbb{R})$ and for $g = \exp(s A_\ell)p, s, u \in \mathbb{R}, p \in P_\ell$ and $\xi \in L^2(\mathbb{R})$ we have

$$\pi_\ell(g)\xi(u) = e^{-2\pi i(\exp(u-s)A_\ell)\cdot \ell, \log(p_\ell))\xi(u-s)},$$
and so for $F \in L^1(G_{5,4})$:

\[
\pi_\ell(F)\xi(t) = \int_{G_{5,4}} F(g)\pi_\ell(g)\xi(t)dg = \int_\mathbb{R} \widehat{F_\pi}(s-t, s, \ell_\pi)\xi(s)ds.
\]

(8.3)

Here

\[
\widehat{F_\pi}(s, q) = \int_{\pi} F(\exp(sA_\ell)p)\chi_q(p)dp, q \in \mathfrak{p}^*_t, s \in \mathbb{R}.
\]

and $s \cdot q = \exp(sA_{\mu,\nu}) \cdot q$, $q \in \mathfrak{p}^*_t$. For $F \in L^1_0(G_{5,4})$, the function $\widehat{F_\pi}$ is of compact support in $s \in \mathbb{R}$ and Schwartz in the variable $q \in \mathfrak{p}^*_t$.

**Definition 8.3.** The Fourier transform $\hat{a} = F(a)$ of an element $a \in C^*(G_{5,4})$ is defined as the field of bounded linear operators over the dual space of $G_{5,4}$, but where we put all the unitary characters of $G_{5,4}$ together to form the representation $\pi_0$ (here $\pi_0$ is the left regular representation of the group $G_{5,4}$). This gives us the set $\Gamma_{2,4}^5 \cup \Gamma_1^1 \cup \Gamma_0^1$ and we define for $a \in C^*(G_{5,4})$ the operator field:

\[
\hat{a}(\beta, \mu, \nu) = F(a)(\beta, \mu, \nu) := \pi_{\beta,\mu,\nu}(a) \in \mathcal{K}(L^2(\mathbb{R})), (\beta, \mu, \nu) \in \Gamma_{2,4}^5;
\]

\[
\hat{a}(\rho) = F(a)(\rho) := \pi_\rho(a) \in \mathcal{K}(L^2(\mathbb{R})), \rho \in \Gamma_1^1;
\]

\[
\hat{a}(0) = F(a)(0) := \pi_0(a) \in C^*(\mathbb{R}^2) \subset \mathcal{B}(L^2(\mathbb{R}^2)).
\]

**8.3. The continuity condition.** We have seen in Theorem 8.2 that the topology of the sub-set $\Gamma_{2,4}^5$ is Hausdorff. This means for the $F$’s in $C^*(G_{5,4})$, that the functions $\pi \rightarrow \|\pi(F)\|_{\text{op}}$ are continuous on this set.

**Theorem 8.4.** The mapping $\Gamma_{2,4}^5 \rightarrow \mathcal{B}(L^2(\mathbb{R})): \ell \mapsto \pi_\ell(F)$ is norm-continuous for all $F \in C^*(G_{5,4})$.

**8.4. Passing from $\Gamma_{2,4}^5$ to $\Gamma_1^1 \cup \Gamma_0^1$.**

**Definition 8.5.** Let $\mathcal{O} = (\mathcal{O}_k=\beta_kB_k^*+\mu_kU^*+\nu_kV^*)$ be a properly converging sequence in $\Gamma_{2,4}^5$ such that $\lim_k (r_k = \sqrt{\mu_k^2 + \nu_k^2}) = 0$. We can assume (passing if necessary to a subsequence) that $\lim_k \tilde{u}_k = \tilde{u}$ and $\lim_k \tilde{v}_k = \tilde{v}$ exist too. Let

\[
d_k := -2\beta_kr_k > 0 \text{ and } t_k = \sqrt{-\frac{2\beta_k}{r_k}} \frac{d_k}{r_k}, k \in \mathbb{N}.
\]

By Theorem 8.2 and its notations, $d = \lim_k d_k$ exists. If $d = 0$, then $\beta_\infty := \lim_k \beta_k$ exists in $[\infty, +\infty]$ and the limit set of $\mathcal{O}$ is the set $L(\mathcal{O}) = \{\mathbb{R}A_{\mu,\nu}^* + [\beta_\infty, +\infty]B_{\mu,\nu}^* \}$ where $A_{\mu,\nu}^* = \tilde{u}A^* + \tilde{v}B^*$ and $B_{\mu,\nu}^* = -\tilde{v}A^* + \tilde{u}B^*$. Otherwise, i.e. if $d \neq 0$, the limit set $L(\mathcal{O})$ is given by $L(\mathcal{O}) = \{\sqrt{d}C^*, -\sqrt{d}C^* \}$. Recall that:

\[
\exp((t_k + s)A_k) \cdot p_k = \left(\beta_k + \frac{r_k(s + t_k)^2}{2}\right)B_k^* + ((s + t_k)r_k)C^* + \mu_kU^* + \nu_kV^*
\]

\[
= s \left(\frac{r_k}{2} + r_k t_k\right)B_k^* + ((s + t_k)r_k)C^* + \mu_kU^* + \nu_kV^*, \quad k \in \mathbb{N}.
\]
If \(d \neq 0\). We consider first the case where \(d := \lim d_k \neq 0\). Let
\[ q_k = \sqrt{d_k}C^*, \; k \in \mathbb{N}^*. \]

Let us compute:
\[
\exp((s \pm t_k)A_k) \cdot p_k - \exp(sA_k) \cdot (\pm q_k)
= \left( \beta_k + \frac{r_k(s \pm t_k)^2}{2} - \pm \sqrt{d_k}s \right) B_k^* + \left( r_k(s \pm t_k + s) - \pm \sqrt{d_k} \right) C^* + \mu_k U^* + \nu_k V^*
= \left( \frac{r_k s^2}{2} \right) B_k^* + (r_k s)C^* + \mu_k U^* + \nu_k V^*. \]

Let \((R_k)_k\) be a sequence in \(\mathbb{R}_+\) such that \(\lim R_k^2r_k = 0, \lim R_k = +\infty\). Let for \(k \in \mathbb{N}\):
\[
J_k = [-R_k, R_k], \quad I_{k, \pm} := (\pm t_k + J_k), \quad I_{k, 2, \pm} := (\pm t_k + 2J_k).
\]

Our condition on \(R_k\) tells us that \(\lim \frac{R_k}{t_k} = 0\) and so \(I_{k, +} \cap I_{k, -} = \emptyset\) for \(k\) large enough.

**Lemma 8.6.** Let \(K\) be a compact subset of \(p^*\). Then \(\exp((\mathbb{R} \setminus (I_{k, +} \cup I_{k, -}))A_k) \cdot \ell_{k|p} \cap K = \emptyset\) for \(k\) large enough.

**Proof.** Take \(R > 0\) such that \(K \subset [-R, R]\). For \(s > 0, s \notin I_{k, +}\) we have either \(s > t_k + R_k\) or \(0 \leq s \leq t_k - R_k\) (for \(k\) large enough): In the first case:
\[
|\exp(sA_k) \cdot \ell_k(B_k)| = \left| \left( (s - t_k) \cdot (t_k \cdot \ell_k) \right)(B_k) \right|
= \left| (s - t_k) \left( r_k \frac{(s - t_k)}{2} + r_k t_k \right) \right|
\geq R_k \sqrt{d_k} \geq \frac{R_k \sqrt{d}}{2} \quad (\text{for } k \text{ large enough}).
\]

In the second case:
\[
|\exp(sA_k) \cdot \ell_k(B_k)| = \left| \left( (s - t_k) \cdot (t_k \cdot \ell_k) \right)(B_k) \right|
= \left| (s - t_k) \left( r_k \frac{(s - t_k)}{2} + r_k t_k \right) \right|
\geq R_k \frac{t_k r_k - (t_k - s_k) r_k}{2} \geq \frac{R_k \sqrt{d}}{4} \quad (\text{for } k \text{ large enough}).
\]

Similarly for \(s < 0, s \notin I_{k, -}\). This means that for \(k\) large enough, \(\exp(tA_k) \cdot \ell_k \notin K\) for \(t \notin I_{k, \pm}\). \(\square\)

**Definition 8.7.** Let \(C_{\square} = CB(L(\square), \mathcal{B}(L^2(\mathbb{R}))\)) be the \(C^*\)-algebra of all continuous, uniformly bounded mappings \(\phi : L(\square) \mapsto \mathcal{B}(L^2(\mathbb{R}))\) from the locally compact space \(L(\square)\) into the algebra of bounded linear operators \(\mathcal{B}(L^2(\mathbb{R}))\) on the Hilbert space \(L^2(\mathbb{R})\).

Let for \(k \in \mathbb{N}^*,\)
\[
\sigma_{k, \pm} = \text{ind}_{\mu_k} G_{\mu_k} \chi_{(\pm t_kA_k)(\sqrt{d_k}C^*)}, \quad \pi_{\sqrt{d_k}C^*} = \text{ind}_{\mu_k} G_{\mu_k} \chi_{\sqrt{d_k}C^*},
\]
and let \(u_{k, \pm}\) be the unitary intertwining operator between \(\pi_{\sqrt{d_k}C^*}\) and \(\sigma_{k, \pm}.\)
(1) Let for $\phi \in C_0^\infty$, $k \in \mathbb{N}$,
\[
\tilde{\sigma}_k(\phi) := M_{k,2,+} \circ u_{k,+} \circ \phi(\sqrt{d_k}) \circ u_{k,+} \circ M_{k,+} + M_{k,2,-} \circ u_{k,-} \circ \phi(-\sqrt{d_k}) \circ u_{k,-} \circ M_{k,-} \in \mathcal{B}(L^2(\mathbb{R})).
\] (8.4)

(2) For $a \in C^*(G_{5,4})$ let
\[
\sigma_{k,G}(a) := \tilde{\sigma}_{k,G}([G_1^g]).
\]

**Theorem 8.8.** Let $\mathcal{O} = (\mathcal{O}_{k=\beta_kB_k^+ \mu_kU^* + v_kV^*})$ be a properly converging sequence in $\Gamma_{5.4}^5$ with limit set $L = \{\mathcal{O}_x, \mathcal{O}_-\sqrt{\mathcal{O}}\}$ where $\mathcal{O}_x = -2 \lim \beta_k \sqrt{\mu_k^2 + v_k^2} = \lim_k (-2 \beta_k v_k), k \in \mathbb{N}$. Then for every $a \in C^*(G_{5,4})$, we have that
\[
\lim_{k \to \infty} \|\pi_{tk}(a) - \sigma_{k,G}(a)\|_{op} = 0.
\]

**Proof.** Let $F \in L^1_c(G_{5,4})$. Let for $k \in \mathbb{N}^*$
\[p_k := p_k, \quad P_k = P_k = \exp(\mathbb{R}B_k) \cdot P = \exp(p_k).\]
The normal subgroup $P_k = \exp(p_k)$ is a polarization at $\ell_k$ for every $k$. Let for $k \in \mathbb{N}^*$:
\[F_k(u,t) := \hat{F}_{P_k}(\exp(uA_k), \exp(tA_k) \cdot \ell_k | p_k), u, t \in \mathbb{R}.
\]
Then the kernel function $K_k$ of the linear operator $\pi_{tk}(F)$ is given by:
\[K_k(s,t) = F_k(s-t,t), s, t \in \mathbb{R}.
\]
Since $F \in L^1_c(G_{5,4})$, there is an $M > 0$ such that $F_k(s-t, t) = 0$, if $|s-t| > M$ and together with Lemma 8.6 we have therefore for $k$ large enough that
\[\pi_{tk}(F) = M_{k,2,+} \circ \pi_{tk}(F) \circ M_{k,+} + M_{k,2,-} \circ \pi_{tk}(F) \circ M_{k,-}.
\]
The kernel function $F_k$ of the operator $\pi_{tk}(F) - \tilde{\sigma}_{k,G}(F)$ is given by
\[F_k(s,t) = 1_{tk,2,+}(s,1_{tk,2,-}(t)\left(\hat{F}_{P_k}(\exp((s-t)A_k), \exp((t-t)A_k) \cdot \ell_k | p_k) - \hat{F}_{P_k}(\exp((s-t)A_k), \exp(tA_k) \cdot q_k)\right)
\]
\[+ 1_{tk,2,-}(s,1_{tk,2,-}(t)\left(\hat{F}_{P_k}(\exp((s-t)A_k), \exp((t-t)A_k) \cdot \ell_k | p_k) - \hat{F}_{P_k}(\exp((s-t)A_k), \exp(tA_k) \cdot (-q_k))\right)
\]
Since $F \in L^1_c(G_{5,4})$, there exists a continuous function $\varphi \geq 0$ on $G_{5,4}/P_k$ with compact support, such that
\[|\hat{F}_{P_k}(\exp(sA_k), \ell_k | p_k) - \hat{F}_{P_k}(\exp(sA_k), \ell_k' | p_k)| \leq \varphi(s)\|\ell_k | p_k - \ell_k' | p_k\|, s \in G_{5,4}, \ell, \ell' \in g_{5,4}^*.
\]
It follows for $t = v + t_k \in I_{k,+}, s = u + t_k \in I_{k,2,+}$ that
\[|\hat{F}_{P_k}(\exp(u-v)A_k), (v+t_k) \cdot \ell_k | p_k) - \hat{F}_{P_k}(\exp(u-v)A_k), v \cdot q_k)\|
\leq \varphi(u-v)\|\exp((v+t_k)A_k) \cdot p_k - \exp(vA_k) \cdot q_k\|
\leq \left(\frac{r_k v^2}{2} + |r_k v| + |\mu_k| + |\nu_k|\right) \varphi(u-v)
\]
for $k$ large enough. Similarly for $t \in I_{k,-}, s \in I_{k,2,-}$. Since $\lim_k r_k R_k^2 = 0$, Young’s inequality implies that

$$\lim_{k \to \infty} \| \pi_{t_k}(F) - \tilde{\sigma}_k \overline{\psi}(F) \|_{\text{op}} = 0.$$  

$L_c^1(G_{5,4})$ being dense in $C^*(G_{5,4})$ the theorem follows. \hfill \Box

If $d = 0$. We suppose now that $\lim_k (d_k = -2\beta kr_k) = 0$.

**Definition 8.9.** Let $\overline{\psi} = (O_{t_k - \beta_k B_k^* + \mu_k U^* + \nu_k V^*})$ be a properly converging sequence in $\Gamma_{2,4}^5$. We can suppose that $\lim_k \beta_k = \beta_\infty$ exists in $[-\infty, +\infty]$ and that $\lim_k \hat{\mu}_k = \hat{\mu}$, $\lim_k \hat{\nu}_k = \hat{\nu}$. Let as before $A_k := \mu_k A + \hat{\nu}_k B$, $B_k := -\hat{\nu}_k A + \hat{\mu}_k B$ and $A_t = \mu A + \hat{\nu} B$, $B_t = -\hat{\nu} A + \mu B$. The limit set is given by $L(\overline{\psi}) = \mathbb{R} A_k^* + [\beta_\infty, \infty] B_k^*$ if $\beta_\infty \in \mathbb{R}$ otherwise $L(\overline{\psi}) = \Gamma_0$.

1. Let $(\varepsilon_k)_k \subset \mathbb{R}_+$ be a decreasing sequence converging to 0 such that $\lim_k \frac{\varepsilon_k}{r_k} = +\infty$.
2. Let

$$t^k_j := j \sqrt{\frac{2 \varepsilon_k}{r_k}}, k \in \mathbb{N}, j \in \mathbb{Z}.$$

Then:

$$\exp((t^k_j + s)A_k) \cdot p_k = \left( \beta_k + r_k \frac{(t^k_j + s)^2}{2} \right) B_k^* + (r_k (t^k_j + s)) C^* + \mu_k U^* + \nu_k V^*$$

$$= \left( \beta_k + j^2 \varepsilon_k + r_k j \sqrt{\frac{2 \varepsilon_k}{r_k} s + r_k \frac{s^2}{2}} \right) B_k^* + \left( r_k j \sqrt{\frac{2 \varepsilon_k}{r_k}} - r_k s \right) C^* + \mu_k U^* + \nu_k V^*$$

$$= \left( \beta_k + j^2 \varepsilon_k + j \sqrt{2 \varepsilon_k r_k s + r_k \frac{s^2}{2}} \right) B_k^* + \left( j \sqrt{2 \varepsilon_k r_k} + r_k s \right) C^* + \mu_k U^* + \nu_k V^*.$$

Let

$$p^{(j)}_k := \exp(t^k_j A_k) \cdot p_k = (\beta_k + j^2 \varepsilon_k) B_k^* + (j \sqrt{2 \varepsilon_k r_k}) C^* + \mu_k U^* + \nu_k V^*,$$

(8.5)

$$q^{(j)}_k := (\beta_k + j^2 \varepsilon_k) B_k^* + j \sqrt{2 \varepsilon_k r_k} C^*.$$

Let $s \in [t^k_j, t^k_{j+1}], j \in \mathbb{Z}$. Then for $k$ large enough:

$$\| \exp((t^k_j + s)A_k) \cdot p_k - \exp(sA_k) \cdot q^{(j)}_k \| = |r_k \frac{s^2}{2} + |r_k s| + |\mu_k| + |\nu_k|$$

$$\leq |r_k \frac{(t^k_{j+1} - t^k_j)^2}{2} + |r_k (t^k_{j+1} - t^k_j)| + |\mu_k| + |\nu_k|$$

$$\leq \varepsilon_k + \sqrt{2 \varepsilon_k r_k} + |\mu_k| + |\nu_k|$$

$$< \varepsilon^2_k.$$

(8.6)

**Definition 8.10.** Let for $k \in \mathbb{N}$ and $j \in \mathbb{Z}$:

$$I_{k,j} := [t^k_j, t^k_{j+1}],$$

$$I_k := \bigcup_{j \in J_k} I_{k,j}.$$
Lemma 8.11. For all $R > 0$, $j \in \mathbb{Z}$, we have that $R + I_{k,j} \subset I_{k,j} \cup I_{k,j+1}$, for $k$ large enough.

Proof. This follows from the fact that $\lim_{k}(t_{j+1}^k - t_j^k) = \sqrt{\frac{2\pi}{r_y}} = \infty$. \hfill \square

Definition 8.12.
(1) Let for $k \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$
\sigma_{k,j} := \text{ind}_{P,y}^{\mathcal{G}_{5,4}} \chi_{\exp(\pm t_y A_k)} q_j^k
$$

and let $u_{k,j}$ be the unitary intertwining operator between $\pi_{-j} \mathcal{G}_{5,4} \subset \mathcal{G}_{5,4}$ and $\sigma_{k,j}$.

(2) Let as before $C_{\mathcal{G}}$ be the $C^*$-algebra of all continuous bounded mappings from $L(\mathcal{G})$ into $B(L^2(\mathbb{R}))$. Let for $\phi \in C_{\mathcal{G}}$.

$$
\tilde{\sigma}_{k,\mathcal{G}}(\phi) = \sum_{j \in \mathbb{Z}} (M_{I_{k,j+1}} + M_{I_{k,j}}) \circ u_{k,j} \circ \sigma_{k,j}(\phi) \circ u_{k,j}^* \circ M_{I_{k,j}}.
$$

(3) For $a \in C^*(G_{5,4})$ let

$$
\sigma_{k,\mathcal{G}}(a) = \tilde{\sigma}_{k,\mathcal{G}}(\hat{a}_{L(\mathcal{G})}).
$$

The proof of the next proposition is similar to that of Proposition 2.4.

Proposition 8.13. The linear mappings $\sigma_{k,\mathcal{G}}, k \in \mathbb{N}$, are bounded by 2.

Theorem 8.14. Let $\mathcal{G} = (O_{\ell_k = A_{5,4}^* + \mu_k U^* + \nu_k V^*})_{k}$ be a properly converging sequence in $\Gamma_{2,5}^4$ with the properties of Definition 8.9. Then for every $a \in C^*(G_{5,4})$, we have that

$$
\lim_{k \to \infty} \|\pi_{\ell_k}(a) - \sigma_{k,\mathcal{G}}(a)\|_{op} = 0.
$$

Proof. Let $F \in L^2(G_{5,4})$. Then, for $k$ large enough, we have by Lemma 8.11 that

$$
\pi_{\ell_k}(F) = \sum_{j \in \mathbb{Z}} (M_{I_{k,j}} + M_{I_{k,j+1}}) \circ \pi_k(F) \circ M_{I_{k,j}}.
$$

Therefore the kernel function $F_{k,j}$ of the operator $(M_{I_{k,j}} + M_{I_{k,j+1}}) \circ (\pi_k(F) - \tilde{\sigma}_{k,j}(F)) \circ M_{I_{k,j}}$ is given by:

$$
F_{k,j}(s,t) = 1_{I_{k,j} \cup I_{k,j+1}}(u + t_j^k) (\hat{F}_{\ell_k}(u - v, (v + t_j^k) \cdot p_k) - \hat{F}_{\ell_k}(s - t, v \cdot q_j^k)),
$$

with $s = u + t_j^k$, $t = v + t_j^k$.

Since $F \in L^1(G_{5,4})$ there exists a continuous function $\varphi \geq 0$ on $\mathbb{R}$ with compact support, such that

$$
|\hat{F}_{\ell_k}(s - t, p) - \hat{F}_{\ell_k}(s - t, q)| \leq \varphi(s - t)\|p - q\|
$$

for every $k \in \mathbb{N}$ and every $p, q \in p_k$. Hence, by (8.6)

$$
|F_{k,j}(s,t)| \leq 1_{I_{k,j} \cup I_{k,j+1}}(u + t_j^k) 1_{I_{k,j}}(v + t_j^k) \\
\times \left|\hat{F}_{\ell_k}(u - v, (v + t_j^k) \cdot p_k) - \hat{F}_{\ell_k}(s - t, v \cdot q_j^k)\right| \\
\leq \varphi(u - v)\|(v + t_j^k) \cdot p_k - v \cdot q_j^k\| \\
\leq \varepsilon^k \varphi(u - v).
$$
It follows now from Young’s inequality and from the properties of the sequence 
\((I_{k,j})\) that

\[
\|\pi_{\ell_k}(F) - \hat{\sigma}_{k,\infty}(F)\|_{op} \leq 2\varepsilon_k^2 \|\varphi\|_1.
\] (8.8)

Since \(L^1_\infty(G_{5,4})\) is dense in \(C^*(G_{5,4})\) and since the mappings \(\sigma_{k,\infty}\) are all bounded in \(k\) by a fixed constant, it follows that relation (8.8) also holds for \(a \in C^*(G_{5,4})\). \(\square\)

9. The \(C^*\)-Algebra of the Group \(G_{5,6}\).

Recall that the Lie algebra \(g_{5,6}\) is spanned by the basis \(B = \{A, B, C, U, V\}\) equipped with the Lie brackets

\[
\{A, B\} = C, \{A, C\} = U, \{A, U\} = V, \{B, C\} = V.
\]

It has a one-dimensional centre \(\mathfrak{z} = \mathbb{R}V\). The group \(G_{5,6} = \exp(g_{5,6})\) can be realized as \(\mathbb{R}^3\) with the multiplication

\[
(a, b, c, u, v) \cdot (a', b', c', u', v') = (a + a', b + b', c + c' - a'b, u + u' - a'c + \frac{a'^2b}{2}, v + v' - a'u + \frac{bc'}{2} - \frac{b'c}{2} + \frac{a'bb'}{2} + \frac{a'^2c}{2} - \frac{a'^3b}{6}).
\]

We use the euclidean scalar product on \(g_{5,6}\) to identify \(g_{5,6}^*\) with \(g_{5,6} = \mathbb{R}^3\) and we obtain the following expression for \(Ad^*(a, b, c, u, v)\):

\[
Ad^*((a, b, c, u, v))((\alpha, \beta, \rho, \mu, \nu) = (\alpha - \rho b - \mu c - \frac{ab}{2} - \nu u - \frac{b^2}{2} - \nu \frac{a^2b}{2}, \beta + \rho a + \mu \frac{a^2}{2} - \nu c + \frac{ab}{2} + \frac{a^3}{6}, \\
\rho + \mu a + \nu c + \frac{a^2}{2}, \mu + \nu a, \nu).
\]

We give now a description of the co-adjoint orbits:

The generic orbits: if \(\nu \neq 0\). The orbit \(O_{\nu}\) of the element \(\ell_{\nu} = (0, 0, 0, 0, \nu)\) is given by:

\[
O_{\nu} = \{ (a, b, c, u, \nu), \ a, b, c, u \in \mathbb{R} \}.
\]

The stabilizer of \(\ell_{\nu}\) is the set \(g_{5,6}(\ell_{\nu}) = \text{span}\{V\}\), we denote by \(\Gamma_{3}^{5,6}\) the orbit space of this layer and we parametrize it by

\[
\Gamma_3^{5,6} := \{ \ell_{\nu} = \nu, \nu \in \mathbb{R}^* \}.
\]

Since \(G_{5,6}/V = F_4\) we can decompose the orbit \(g_{5,6}^*/G_{5,6}\) and hence also the dual space \(\tilde{G}_{5,6}\), into the disjoint union

\[
g_{5,6}^*/G_{5,6} = \Gamma_3^{5,6} \cup \Gamma_4^4 \cup \Gamma_1^4 \cup \Gamma_0^4.
\]

**Theorem 9.1.** Let \(\overline{O} = (O_{\nu_k})_k \subset \Gamma_3^{5,6}\) be a sequence, such that \(\lim_{k \to \infty} \nu_k = 0\). Then \(\overline{O}\) is properly converging and \(L(\overline{O}) = \Gamma_2^4 \cup \Gamma_1^4 \cup \Gamma_0^4\).
9.1. The Fourier transform.

**Definition 9.2.** Let \( F \in L^1(G_{5,6}) \), the operator \( \pi_\ell \) is a kernel operator with kernel function
\[
\hat{\pi}_\ell(t,\ell|P_t) = \int_{P_t} F(spt^{-1})\chi_\ell(p)dp, \quad s, t \in G_{5,6}/P_t,
\]
where \( P_t = \exp(p_t) \) and \( p_t \) is a polarization at \( \ell \).

For \( \ell = (0,0,0,0,\nu) \in \Gamma^{5,6}_3 \), therefore the abelian sub-algebra \( p = \text{span}\{C, U, V\} \) is a polarization at \( \ell \). We realize then \( \pi_\ell.P = \pi_\ell \) as \( \pi_\ell := \text{ind}_{P}^{G_{5,6}} \chi_\ell \). The Hilbert space \( L^2(G_{5,6}/P, \ell) \) is in fact isomorphic to \( L^2(\mathbb{R}^2) \): let \( E : \mathbb{R}^2 \to G_{5,6}, E(a, b) := \exp(aA)\exp(bB) \) and \( S = \exp(RA)\exp(RB) = E(\mathbb{R} \times \mathbb{R}) \). Then \( G = SP \) as topological product and the mapping \( U : L^2(G_{5,6}/P, \ell) \to L^2(\mathbb{R}^2) \) defined by
\[ U\xi(t) := \xi(E(t)), t \in \mathbb{R}^2, \]
is unitary. We identify now \( \pi_\ell \) with the corresponding representation on \( L^2(\mathbb{R}^2) \). Let us compute the operator \( \pi_\ell(F) \) for \( F \in C^*(G_{5,6}) \) explicitly. For \( \xi \in L^2(\mathbb{R}^2), t = (a',b') \in S, p \in P \) we have:
\[
\pi_\ell(F)(\xi(t)) = \int_{G_{5,6}/P} \xi(s) \left( \int_{P} F(tp^{-1})e^{-2i\pi(t,s.p)}dp \right) ds \quad \text{(where } s, \ell = Ad^*(s,\ell))
\[
= \int_{G_{5,6}/P} \hat{F}(s,\ell|p)\xi(s)ds
\]
\[
= \int_{G_{5,6}/P} \hat{F}(a' - a, b' - b, (a,b),p)e^{-2i\pi(\nu ab(b' - b) + \frac{a'(b' - b)}{2} + \frac{\nu a^2}{6})} \xi(a,b)dadb.
\]

(9.2)

**Definition 9.3.** Let as before \( P = \exp(p) \) and \( \pi_0 = \text{ind}_{P}^{G_{5,6}} \chi_0 \) be the left regular representation of \( G_{5,6} \) on the Hilbert space \( L^2(G_{5,6}/P) \simeq L^2(\mathbb{R}^2) \). Then the image \( \pi_0(C^*(G_{5,6})) \) is just the \( C^* \)-algebra of \( \mathbb{R}^2 \) considered as an algebra of convolution operators on \( L^2(\mathbb{R}^2) \) and \( \pi_0(C^*(G_{5,6})) \) is isomorphic to the algebra \( C(\mathbb{R}^2) \) of continuous functions vanishing at infinity on \( \mathbb{R}^2 \) via the abelian Fourier transform
\[
\hat{F}(a,b) := \int_{G_{5,6}} F(g)e^{-2\pi i a, u \log g} dg, \quad a, b \in \mathbb{R}, F \in L^1(G_{5,6}).
\]

**Definition 9.4.** The Fourier transform \( \hat{a} = \mathcal{F}(a) \) of an element \( a \in C^*(G_{5,6}) \) is defined as to the field of bounded linear operators over the dual space of \( G_{5,6} \), but where we put all the unitary characters of \( G_{5,6} \) together to form the representation \( \pi_0 \). This gives us the set \( \Gamma^0 \cup \Gamma^1 \cup \Gamma^2 \cup \Gamma^{5,6}_3 \) and we define for \( a \in C^*(G_{5,6}) \) the operator field:
\[
\hat{a}(\nu) = \mathcal{F}(a)(\nu) := \pi_\nu(a) \in K(L^2(\mathbb{R}^2)), \quad \nu \in \Gamma^{5,6}_3;
\]
\[
\hat{a}(\beta, \mu) = \mathcal{F}(a)(\beta, \mu) := \pi_{\beta, \mu}(a) \in K(L^2(\mathbb{R}^2)), \quad (\beta, \mu) \in \Gamma^2_3;
\]
\[
\hat{a}(\rho) = \mathcal{F}(a)(\rho) := \pi_{\rho}(a) \in K(L^2(\mathbb{R}^2)), \quad \rho \in \Gamma^1_3;
\]
\[
\hat{a}(0) = \mathcal{F}(a)(0) := \pi_0(a) \in C^*(\mathbb{R}^2) \subset B(L^2(\mathbb{R}^2)).
\]

**Theorem 9.5.** The mapping \( \Gamma^{5,6}_3 \to B(L^2(\mathbb{R}^2)) : \ell \to \pi_\ell(F) \) is norm-continuous for all \( F \in C^*(G_{5,6}) \).
Theorem 9.6. For every sequence \((\mathcal{O}_{\ell_k})_k \subset \mathfrak{g}_{5,6}^p/G_{5,6}\) going to infinity we have that
\[
\lim_k \|\pi_{\ell_k}(F)\|_{op} = 0.
\]

9.2. Passing from \(\Gamma_{3,6}^5\) to \(\Gamma_{3,4}^5 \cup \Gamma_{4,4}^4\). Let \(\mathfrak{T} = (\mathcal{O}_{\ell_k})_k \subset \Gamma_{3,6}^5\) be a properly converging sequence where \(\ell_k = (0,0,0,0,\nu_k)\), \(k \in \mathbb{N}\) such that \(\lim_k \nu_k = 0\). Let \(p_k := (\ell_k)_{|p}\).

By Theorem 9.1 the restriction of the limit set \(L(\mathfrak{T})\) to \(p\) is the closed set \(L = L(\mathfrak{T})_{|p} = \{(\rho, \mu, 0), \rho \in \mathbb{R}, \mu \in \mathbb{R}\}\).

Definition 9.7. For \(k \in \mathbb{N}\) let:
\[
\varepsilon_k := |\nu_k|^{\frac{1}{4}},
\]
\[
I_{i,j}^k := \left\{(c, u, \nu_k) \in p^* \mid \varepsilon_k \leq j^2 \varepsilon_k^2 - \frac{j^2 \varepsilon_k^2}{2\nu_k} - c - \frac{\nu^2}{2\nu_k} < \varepsilon_k^2 + \frac{1}{2} \varepsilon_k^2 \right\},
\]
\[
U_{i,j}^k := \left\{(x, y) \in \mathbb{R}^2 \mid (xA + yB) \cdot p_k \in I_{i,j}^k\right\}, \quad j \in \mathbb{Z}.
\]

Finally:
\[
U^k := \bigcup_{i,j \in \mathbb{Z}} U_{i,j}^k.
\]

Choose now the sequence \(R_k\), such that \(\lim_k R_k = +\infty\), \(\lim_k R_k \delta_k = 0\). Let also for \(k \in \mathbb{N}, i, j \in \mathbb{Z}\):
\[
x_j^k := \frac{j \varepsilon_k}{\nu_k}, \quad g_{i,j}^k := \frac{(x_j^k)^2}{2} + \frac{i \varepsilon_k}{\nu_k}, \quad g_{i,j}^k = x_j^k A + y_j^k B.
\]

Let for \(i, j \in \mathbb{Z}\), \(k \in \mathbb{N}^*\):
\[
p_{i,j}^k := (i \varepsilon_k^4, j \varepsilon_k, 0).
\]

An easy computation gives:
\[
g_{i,j}^k \cdot p_k = (i \varepsilon_k^4, j \varepsilon_k, \nu_k) = p_{i,j}^k + (0, 0, \nu_k).
\]

Proposition 9.8. Let \(K\) be a compact subset, for \(k\) large enough we have that
\[
KU_{i,j}^k \subset \bigcup_{i', j' = -1} U_{i'+i, j'+j}^k =: V_{i,j}^k.
\]

Proof. We can suppose that \(KP\) is contained in \([-M, M]^2P\) for some \(M > 0\). For \(r = (u, v) \in KP \subset G_{5,6}/P\) and \(s = (x, y) \in U^k\) we have that
\[
(rs).p_k = (\nu_k v + \nu_k y + \nu_k xu + \frac{x^2}{2} + \nu_k \frac{y^2}{2}, \nu_k x + \nu_k u, \nu_k)
\]

we have
\[
(x, y) \in U_{i,j}^k \quad \iff \quad (xA + yB) \cdot p_k \in I_{i,j}^k
\]
\[
\Rightarrow \left\{\begin{array}{l}
\frac{j \varepsilon_k}{\nu_k} \leq \nu_k x < \frac{j \varepsilon_k + \varepsilon_k}{\nu_k}, \\
\frac{j \varepsilon_k + \varepsilon_k}{\nu_k} - \frac{\nu^2}{2\nu_k} \leq \nu_k y < \frac{j \varepsilon_k}{\nu_k} - \frac{\nu^2}{2\nu_k} + \varepsilon_k,
\end{array}\right.
\]
\[
\Rightarrow \left\{\begin{array}{l}
(j - 1) \varepsilon_k \leq \nu_k x + \nu_k u < (j + 1) \varepsilon_k + \varepsilon_k, \\
(i - 1) \varepsilon_k - \frac{\nu^2}{2\nu_k} \leq \nu_k y + \nu_k v < (i + 1) \varepsilon_k - \frac{\nu^2}{2\nu_k} + \varepsilon_k.
\end{array}\right.
\]
Lemma 9.11. That

\[ \mathcal{K} \mathcal{U}_{i,j}^k \subset \bigcup_{i',j'=1}^{1} \mathcal{U}_{i'+i,j'+j}^k. \]

\[ \square \]

Definition 9.9. For \( k \in \mathbb{N}_0 \) Let

\[ R^k = \left[ -\frac{\varepsilon_k}{|\nu_k|}, -\frac{\varepsilon_k}{|\nu_k|} \right] \times \left[ -\frac{\varepsilon_k}{|\nu_k|}, -\frac{\varepsilon_k}{|\nu_k|} \right]. \]

Lemma 9.10. For \( k \in \mathbb{N}_0 \) large enough, for any \( i, j \in \mathbb{Z} \), the set \( \mathcal{U}_{i,j}^k \) is contained in \( R^k + g_{i,j}^k \).

Proof. Let \( s = (x, y) \in \mathcal{U}_{i,j}^k \). Then:

\[ (xA + yB) \cdot p_k \in I_{i,j}^k \]

\[ \iff \begin{cases} j\varepsilon_k \leq \nu_k x < j\varepsilon_k + \varepsilon_k \Rightarrow |x - x_j^k| \leq \frac{\varepsilon_k}{|\nu_k|} \Rightarrow x \in \left[ \frac{-\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right] + x_j^k, \\ j\varepsilon_k + \frac{\varepsilon_k^2}{2\nu_k} \leq \nu_k y < j\varepsilon_k + \frac{\varepsilon_k^2}{2\nu_k} + \frac{\varepsilon_k}{2} \Rightarrow |y - y_j^k| \leq \frac{\varepsilon_k}{|\nu_k|} \Rightarrow y \in \left[ \frac{-\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right] + y_j^k, \end{cases} \]

\[ \Rightarrow s \in R^k + g_{i,j}^k. \]

\[ \square \]

Lemma 9.11. For \( k \in \mathbb{N}_0 \) large enough, for \( i, j \in \mathbb{Z} \) and any \( (x, y) \in \mathcal{U}_{i,j}^k \) we have that

\[ \| (xA + yB) \cdot p_k - ((xA + yB) \cdot (g_{i,j}^k)^{-1}) \cdot p_{i,j}^k \| \leq 4\varepsilon_k^2. \]

Proof. For \( (x, y) \in \mathcal{U}_{i,j}^k \) we have that \( (x, y) = (x' + x_j^k, y' + y_j^k) \) where \( |\nu_k x'| \leq \varepsilon_k \) and \( |\nu_k y'| \leq \varepsilon_k \). Therefore

\[ \| (x' + x_j^k)A + (y' + y_j^k)B) \cdot p_k - (x' A + y' B) \cdot p_{i,j}^k \| \]

\[ = \| (\nu_k y' + \nu_k y_j^k + \nu_k x'x_j^k + \nu_k \frac{x_j^k}{2} + \nu_k x_j^k + \nu_k x'x_j^k + \nu_k) - (i\varepsilon_k^j + j\varepsilon_k x', j\varepsilon_k, 0) \|
\]

\[ = \| (\nu_k y' + i\varepsilon_k^j + j\varepsilon_k x' + j\varepsilon_k x_j^k + \nu_k \frac{x_j^k}{2} + \nu_k x_j^k) - (i\varepsilon_k^j + j\varepsilon_k x', j\varepsilon_k, 0) \|
\]

\[ = \| (\nu_k y' + \nu_k x_j^k + \nu_k) \|
\]

\[ = |\nu_k y'| + |\nu_k x_j^k| + |\nu_k|
\]

\[ \leq \varepsilon_k^j + \frac{|\nu_k|^j}{2} + \varepsilon_k + |\nu_k|
\]

\[ \leq 4\varepsilon_k^2. \]

\[ \square \]

Definition 9.12. Define for \( k \in \mathbb{N} \) and \( \phi \in C_{\mathcal{K}} \) the linear operator \( \tilde{\sigma}_{k,\mathcal{K}}(\phi) \) by

\[ \tilde{\sigma}_{k,\mathcal{K}}(\phi) := \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} M_{\mathcal{U}^k_{i,j}} \circ \tilde{\sigma}_{(g_{i,j}^k)^{-1}, \sigma_{i,j}^k} \circ M_{\mathcal{U}^k_{i,j}}. \]

where \( \tilde{\sigma}_{\ell} \) for \( \ell = (g_{i,j}^k)^{-1} \cdot p_{i,j}^k \), is an in Equation (9.4). For \( a \in C^*(G_{5,6}) \) we have that \( \sigma_{k,\mathcal{K}}(a) = \tilde{\sigma}_{k,\mathcal{K}}(\tilde{\sigma}_{L,\mathcal{K}}(a)). \)

The proof of the next proposition is similar to that of Proposition 7.11.
Proposition 9.13. Let \( a \in C^*(G_{5,6}) \). Then:
\[
\lim_{k \to \infty} \| \pi_{k\cdot a} - \sigma_k, O(a) \|_{\text{op}} = 0.
\]

We have treated now all simply connected, connected undecomposable Lie groups of dimension \( \leq 5 \). The other simply connected connected groups of dimension \( \leq 5 \) are of the form \( G_1 \times \mathbb{R}^d \), with \( G_1 \) undecomposable and \( \dim(G_1) + d \leq 5 \). It is easy to extend our methods to these groups to. We have thus established the following theorem:

Theorem 9.14. The \( C^* \)-algebra of every connected nilpotent Lie group of dimension \( \leq 5 \) has norm controlled dual limits.

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