Semiclassical corrections to the Einstein equation and Induced Matter Theory

P. Moyassari 1 and S. Jalalzadeh 2,3*

1Department of physics, Tafresh University, Tafresh, Iran.
2Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran.
3Research Institute for Astronomy and Astrophysics of Maragha Maragha, IRAN, P.O.Box: 55134-441.

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Abstract

The induced Einstein equation on a perturbed brane in the Induced Matter Theory is reanalyzed. We indicate that in a conformally flat background, the local quantum corrections to the Einstein equation can be obtained via the IMT. Using the FRW metric as the 4D gravitational model, we show that the classical fluctuations of the brane may be related to the quantum corrections to the classical Einstein equation. In other words, the induced Einstein equation on the perturbed brane correspond with the semiclassical Einstein equation.

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1 Introduction

One of the most fundamental problems in present-day physics concerns a quantum theory of gravitation. In this context, the concept of semiclassicality is particularly relevant in order to make it a physical theory. Applying the semiclassical theory, as the first order quantum corrections to the classical theory, one hopes to get an insight into some of the structures of the full, elusive, theory. In fact, the semiclassical approach provides the framework for some realistic scenarios which may explain some of the features of the present universe [1]. It would even allow us to say something on quantum gravity without having to build the full consistent theory. This is particularly interesting since experiments like GLAST, AUGER and others, should be able to measure effects due to a quantum gravity regime [2]. In the semiclassical theory of gravity a classical metric is coupled to the expectation value of the stress tensor

\[ G_{\mu\nu} = -8\pi G \langle T_{\mu\nu} \rangle. \]  (1)

It was found that this theory gives reliable results when the fluctuations in the stress tensor are not too large [3]. The semiclassical limit of the quantum gravity has to match the established classical theory.

*email: s-jalalzadeh@sbu.ac.ir
for consistency. The form of the semiclassical corrections to the Einstein equation is known for many important cases in the 4D spacetime [4]. In this paper we will study the relation between the quantum corrections to the Einstein equations and Induced Matter Theory (IMT) of Wesson. The physics of gravitational interactions in higher dimensional spacetime has received considerable interest in recent years. The possibility that the fundamental Planck mass within a higher dimensional setting may be as low as the electroweak scale [5, 6, 7] has stimulated extensive model building and numerous investigations aiming at signatures of extra dimensions. Central to these scenarios is that gravity lives in higher dimensions, while standard model particles are often confined to the four dimensional brane. Higher dimensional extensions to general relativity were originally started with works of Kaluza and Klein (KK) with the addition of one extra dimension and subsequently generalized to more extra dimensions by various authors. In KK theory, the components of the 5D metric tensor is independent of the extra dimension. In contrast to this model, Wesson suggested the IMT [8] which differs from KK theory by the fact that it has a noncompact fifth dimension and that the 5D bulk space is devoid of matter. In this theory the effective 4D matter is a consequence of the geometry of the 5D bulk space which is Ricci-flat while the 4D hypersurface is curved by the 4D induced matter[9]. One of the interesting futures of IMT is that it contains some quantum mechanical effects. For example it has been shown that particles and waves are merely different representations of the same underlying geometry and may be the same thing viewed in different ways [10]. Also, using 5D IMT, a form of Heisenberg’s relation that applies to real and virtual particles has been derived by Wesson [11]. We will show that the geometrical fluctuations obtained via IMT may be related to the quantum mechanical effects. In this paper we focus attention on IMT and proceed to derive the semiclassical corrections to the Einstein equation using this theory. In doing so, we will briefly review the geometrical definitions and derive the induced Einstein equation on the perturbed brane through contracting the Gauss-Codazzi equations. In section three, we will consider a FRW universe as a non-perturbed brane embedded in a 5D flat spacetime and study the Einstein equation on the perturbed brane. We will show that these equations correspond to the semiclassical Einstein equation. It means that the classical fluctuations of the perturbed brane can be interpreted as the quantum fluctuations of the matter field.

2 Geometrical setup

Consider the background manifold $\nabla_4$ isometrically embedded in $V_5$ by a map $Y: \nabla_4 \rightarrow V_5$ such that

\[
G_{AB}Y^A_{,\mu}Y^B_{,\nu} = \bar{g}_{\mu\nu}, \quad G_{AB}Y^A_{,\mu}N^B = 0, \quad G_{AB}N^A N^B = 1
\]  

(2)

where $G_{AB}$ ($\bar{g}_{\mu\nu}$) is the metric of the bulk (brane) space $V_5(\nabla_4)$ in an arbitrary coordinate, $\{Y^A\}$ ($\{x^\mu\}$) are the bases of the bulk (brane) and $N^A$ is a normal unit vector orthogonal to the brane. Perturbation of $\nabla_4$ in a sufficiently small neighborhood of the brane along an arbitrary transverse direction $\xi$ is given by

\[
Z^A(x^\mu, \zeta) = Y^A + (L_\zeta Y)^A,
\]

(3)

where $L$ represents the Lie derivative and $\zeta$ is a small parameter along $N^A$ parameterizing the extra noncompact dimension. By choosing $\xi$ orthogonal to the brane we ensure gauge independency [12] and have perturbations of the embedding along a single orthogonal extra direction $\bar{N}$, giving the local coordinates of the perturbed brane as

\[
Z^A_{,\mu}(x^\nu, \zeta) = Y^A_{,\mu} + \zeta \bar{N}^A_{,\mu}(x^\nu),
\]

(4)
In a similar manner, one can find that since the vectors \( \bar{N}^A \) depend only on the local coordinates \( x^\mu \), they do not propagate along extra dimension

\[
\bar{N}^A(x^\mu) = \bar{N}^A + \zeta [\bar{N}, \bar{N}]^A = \bar{N}^A.
\]  

The above assumptions lead to the embedding equations of the perturbed geometry

\[
\mathcal{G}_{\mu\nu} = \mathcal{G}_{AB} Z^A_{\mu\nu} Z^B, \quad \mathcal{G}_{\mu A} = \mathcal{G}_{AB} Z^A_{\mu N^B}, \quad \mathcal{G}_{AB} N^A N^B = \mathcal{G}_{44}.
\]  

If we set \( N^A = \delta_4^A \), the metric of the bulk space can be written in the following matrix form (Gaussian frame)

\[
\mathcal{G}_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix},
\]

where \( g_{\mu\nu} \) is the metric of the perturbed brane

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} - 2\zeta \bar{K}_{\mu\nu} + \zeta^2 \bar{g}^{\alpha\beta} \bar{K}_{\mu\alpha} \bar{K}_{\nu\beta},
\]

and

\[
\bar{K}_{\mu\nu} = -\mathcal{G}_{AB} Y^A_{\mu N^B},
\]

represents the extrinsic curvature of the original brane (second fundamental form). Any fixed \( \zeta \) signifies a new perturbed geometry, enabling us to define an extrinsic curvature similar to the original one by

\[
K_{\mu\nu} = -\mathcal{G}_{AB} Z^A_{\mu N^B} = \bar{K}_{\mu\nu} - \zeta \bar{K}_{\mu\gamma} \bar{K}^\gamma_{\nu}.
\]

Equations (8) and (10) are important in obtaining the Einstein equation on the perturbed brane in our approach and show the relation of original and perturbed branes. In the Induced Matter approach, the Einstein equation in the bulk is written in the form

\[
R_{\alpha\beta\gamma\delta} = 0,
\]  

where \( R_{\alpha\beta\gamma\delta} \) is the Ricci tensor of the 5D bulk space. To obtain the effective field equations in 4D, let us start by contracting the Gauss-Codazzi equations [13] \(^1\)

\[
R_{\alpha\beta\gamma\delta} = 2\epsilon K_{\gamma[\alpha} K_{\beta]\delta] + \mathcal{R}_{ABCD} Z^A_{\alpha} Z^B_{\beta} Z^C_{\gamma} Z^D_{\delta}
\]

and

\[
2K_{\mu[\nu;\rho]} = \mathcal{R}_{ABCD} Z^A_{\mu\nu} Z^C_{\rho} Z^D_{\gamma},
\]

where \( \mathcal{R}_{ABCD} \) and \( R_{\alpha\beta\gamma\delta} \) are the Riemann curvature of the bulk and perturbed brane respectively. Now by contracting Gauss equations (12), decomposing Riemann tensor of the bulk space into the Weyl and Ricci tensors and Ricci scalar and using equation (11), the Einstein equation induced on the perturbed brane becomes

\[
G_{\mu\nu} = Q_{\mu\nu} - \mathcal{E}_{\mu\nu},
\]

where \( \mathcal{E}_{\mu\nu} = \mathcal{C}_{ABCD} Z^A_{\mu\nu} N^B_{\gamma} N^C Z^D_{\rho} \) is the electric part of the Weyl tensor \( \mathcal{C}_{ABCD} \) of bulk space. One can directly show that \( Q_{\mu\nu} \) is independently a conserved quantity, that is \( Q_{\mu\nu} = 0 \). All of the above quantities in equation (14) are obtained on the perturbed brane, since in the spirit of IMT the matter field cannot exactly be confined to the original non perturbed brane. Hence from a 4D point of view, the empty 5D equations look like the Einstein equation with induced matter. The electric part of

\(^1\)Eisenhart’s convention [13] has been used in defining the Riemann tensor.
the Weyl tensor is well known from the brane point of view. It describes a traceless matter, denoted by dark radiation or Weyl matter. Since $Q_{\mu \nu}$ is a conserved quantity, according to the spirit of IMT [8], it should be related to the ordinary matter as partly having a geometrical origin

$$Q_{\mu \nu} = -8\pi G T_{\mu \nu}. \quad (15)$$

According to Wesson’s results, the IMT can contain quantum effects [9, 10]. Now, the question arises as to whether the classical fluctuations of the brane can relate to the quantum fluctuations of the matter field. In other words, can induced field equation (14) describe the semiclassical Einstein equation? In the following we focus on deriving the semiclassical corrections to the Einstein equation through the IMT.

One knows that quantum corrections to general relativity are expected to be important in regimes where the curvature is near the Planck scale ($l_{pl} = 1.6 \times 10^{-33} \text{cm}$). In a regime where the curvature approaches but always remains less than the Planck length, a semiclassical approximation to the full theory of quantum gravity should be sufficient. It may seem that if the brane perturbations are of the order of the Planck length, equation (14) may relate to the semiclassical description of quantum gravity. As we know the effective size of the extra dimension which should be smaller than 0.2 mm can be obtained from

$$L = \frac{M_p^2}{M_*^2},$$

provided $M_* > 2 \times 10^8 \text{GeV}$ [14]. Here $M_p$ and $M_*$ are the Planck mass and the fundamental scale of the energy in the bulk space respectively. On the other hand, the standard model fields are usually confined to the brane within some localization width i.e, the brane width [15, 16]. Similarly, in Induced Matter theory, if the induced matter satisfies the restricted energy condition, the particles will be stabilized around the original brane [17]. The size of the fluctuations of the induced matter corresponds with the width of the brane. Since within the ordinary scales of energy we do not see the disappearance of particles, one may assume the fluctuations of the matter field exist only around the original brane. In other words, if the brane width is $d$, it means that brane localized particles probe this length scale across the brane and therefore the observer cannot measure the distance on the brane to a better accuracy than $d$. In braneworld models with large extra dimensions, usually the width of the brane should be in order of or less than $\text{TeV}^{-1}$. On the other hand, the observational data constrains the brane width to be in order of planck length, see [18] and references therein. Hence, in this paper according to [19] we assume that the size of the fluctuations of the brane (the width of the brane) is in order of Planck length which is much smaller than the effective size of the extra dimension $L$. This assumption may help us to investigate the correct quantum phenomenology in IMT. In the following section we proceed to study the Einstein equation on the perturbed brane applying a simple gravitational model and derive the semiclassical corrections to the Einstein equation.

### 3 FRW model with quantum corrections

Consider a FRW universe embedded (as a non perturbed brane) in an 5D flat bulk space so that the extra dimension is spacelike. The FRW line element is written as

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right], \quad (16)$$

where $k$ takes the values $\pm 1$ or 0 and $a(t)$ is the scale factor. Now, we proceed to analyze the Einstein equation (14) on the perturbed brane. To do this, we first compute the extrinsic curvature through
solving the Codazzi equations (13) \cite{20}

\begin{align*}
\bar{K}_{00} &= -\frac{1}{\dot{a}} \frac{d}{dt} \left( \frac{b}{a} \right), \\
\bar{K}_{\alpha\beta} &= \frac{b}{a^2} g_{\alpha\beta}, \quad \alpha, \beta \neq 0. 
\end{align*}

(17)

Here, \( b \) is an arbitrary function of \( t \). Consequently, the components of \( \bar{Q}_{\mu\nu} \) become

\begin{align*}
\bar{Q}_{00} &= -\frac{3}{a^4} b^2, \\
\bar{Q}_{\alpha\beta} &= \frac{1}{a^4} \left( \frac{2bb}{H} - b^2 \right) g_{\alpha\beta} \quad \alpha, \beta \neq 0, 
\end{align*}

(18)

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. Note that the electric part of the Weyl tensor vanishes in the case of the flat bulk. To proceed with geometrical interpretation of the energy momentum tensor, let us consider an analogy between \( \bar{Q}_{\mu\nu} \) and a simple example of matter consisting of free radiation field and cosmological constant \( \Lambda \), that is

\( \bar{Q}_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \),

(19)

where

\( T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}, \quad p = \frac{1}{3} \rho. \)

(20)

Using equations (18-20) the energy density and cosmological constant take the following forms

\begin{align*}
\rho &= \frac{3}{16\pi G a^4} \left( -\frac{bb}{H} + 2b^2 \right), \\
\Lambda &= \frac{3}{2} \frac{bb}{a^4 H}.
\end{align*}

(21)

For a radiative universe, which we have \( \rho a^4 = \rho_0 a_0^4 \), equations (21) lead to

\( b = \sqrt{\frac{\rho^*}{2} + \frac{\Lambda a^4}{3}}, \)

(22)

where \( \rho_0 \) and \( a_0 \) are the radiation density and the scale factor at the present epoch and

\( \rho^* = \frac{16\pi G}{3} \rho_0 a_0^4. \)

(23)

Using equations (19-22), the 4D induced components of the Einstein equation (14) on the non perturbed brane become

\begin{align*}
3 \frac{\dot{a}^2 + k}{a^2} &= \frac{2a^4 \Lambda + 3\rho^*}{2a^4}, \\
\frac{2a\ddot{a} + k + \dot{a}^2}{1 - kr^2} &= \frac{2a^4 \Lambda - \rho^*}{2a^2(1 - kr^2)}.
\end{align*}

(24)

(25)
Now, one can obtain the components of the tensors $G_{\mu\nu}$ and $Q_{\mu\nu}$ on the perturbed brane using the components of the metric and the extrinsic curvature of the perturbed brane through (8) and (10). After some manipulations we obtain

\[ Q_{00} = -\frac{2a^4\Lambda + 3\rho^* - \rho^*\sqrt{12a^4\Lambda + 18\rho^*}}{2a^4} \zeta - \frac{243\rho^{*4} + 378a^4\Lambda\rho^* + 360a^8\Lambda^2\rho^* + 72a^2\Lambda^3\rho^* + 16a^4\Lambda^4}{3a^8(2a^4\Lambda + 3\rho^*)^2} \zeta^2, \]  

(26)

\[ Q_{\alpha\beta} = \left[ \frac{2a^4\Lambda - \rho^*}{2a^4} + \frac{2(3\rho^* - 2a^4\Lambda)}{a^6 \sqrt{12a^4\Lambda + 18\rho^*}} \right] \zeta + \frac{54\rho^{*4} + 126a^4\Lambda\rho^* + 228a^8\Lambda^2\rho^* + 40a^12\Lambda^3\rho^* + 16a^16\Lambda^4}{3a^8(2a^4\Lambda + 3\rho^*)^2} \zeta^2 \right] g_{\alpha\beta}, \quad \alpha, \beta \neq 0, \]  

(27)

and

\[ G_{00} = -3\frac{\dot{a}^2 + k}{a^2} - 36\frac{(\dot{a}^2 + k)\rho^*}{a^4 \sqrt{12a^4\Lambda + 18\rho^*}} \zeta - 12\rho^* \frac{3\rho^* \dot{a}^2 + 3\rho^* \dot{a}^2 + 4a^4\Lambda \rho^* + k\Lambda a^4}{(2a^4\Lambda + 3\rho^*)a^6} \zeta^2, \]  

(28)

\[ G_{\alpha\beta} = \left[ \frac{2a\ddot{a} + k + \ddot{a}}{a^2} - \frac{12\rho^* \ddot{a}}{a^3 \sqrt{12a^4\Lambda + 18\rho^*}} \zeta + \frac{2\rho^* \ddot{a}(3\rho^* - 2a^4\Lambda)}{(2a^4\Lambda + 3\rho^*)a^5} \zeta^2 \right] g_{\alpha\beta}, \quad \alpha, \beta \neq 0. \]  

(29)

Using the above equations we can derive the $(0,0)$ component of the Einstein equation on the perturbed brane up to third order in $\zeta$

\[ -3\frac{\dot{a}^2 + k}{a^2} = -\Lambda - \frac{3\rho^*}{2a^4} - \left( \frac{2a^4\Lambda + 3\rho^*)}{3a^8} \right) \zeta^2 + \frac{4a^4\Lambda \rho^*}{a^8} \zeta^2 - \frac{144\Lambda^2 \rho^*}{3(2a^4\Lambda + 3\rho^*)^2} \zeta^2 + O(\zeta^3). \]  

(30)

It is obvious that this equation includes some correction terms of order $\zeta^2$ in comparison with the Einstein equation (24) on the original brane. Since the solution of (30) is a correction to solution of (24) in power of curvature we expect that at late times, when the solution on the original brane is nearly flat, these corrections will be flat. At early times, when the curvature is below the Planck scale, we expect the corrections to be significant and at very early times, when the original brane curvature is near or above the planck scale, we expect that this approach will break down because neglected higher order corrections would dominate.

According to Wesson [19], we assume that the region of the brane fluctuations is in order of Planck length ($c = 1, \Gamma = 1$)

\[ \zeta^2 \sim l_{pl}^2 \sim h. \]  

(31)

Now we can redefine (30) using (23) and (31)

\[ 3\frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} - \Lambda - \frac{\kappa \rho_0 a_0^4}{a^4} + \frac{\hbar 8\kappa \Lambda \rho_0 a_0^4}{3a^4} + \frac{\hbar 2\kappa^2 \rho_0^2 a_0^8}{3a^8} = 0, \]  

(32)

\[ -4\hbar \left[ \frac{\Lambda^2}{3} + \frac{2\kappa \Lambda \rho_0 a_0^4}{3a^4} + \frac{\kappa^2 \rho_0^2 a_0^8}{3a^8} \right] - \frac{16}{3} \hbar \left[ \frac{\kappa \Lambda \rho_0 a_0^4}{(a^4\Lambda + \kappa \rho_0 a_0^4)^2} \right]^2 = 0, \]

where $\kappa = 8\pi$. One may compare equation (32) with the same result which is obtained through the semiclassical approach to Einstein equations in [21]. There, the semiclassical Einstein equation for the conformally flat spacetimes take the form

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha_1 \hbar (1) H_{\mu\nu} + \alpha_3 \hbar (3) H_{\mu\nu} = -\kappa T_{\mu\nu}. \]  

(33)
In this theory the parameters $\alpha_1$ and $\alpha_3$ depend on the particular form of matter and regularization scheme and

$$
(1)H_{\mu\nu} = \frac{1}{2} R^{2}g_{\mu\nu} - 2R R_{\mu\nu} - 2\Box R g_{\mu\nu} + 2\nabla_{\mu} \nabla_{\nu} R ,
$$

(34)

$$
(3)H_{\mu\nu} = -\frac{1}{2} R^{2} g_{\mu\nu} + R^\alpha\sigma R_{\rho\mu\sigma\nu} .
$$

(35)

In this approach, although the quantum corrections contain up to fourth order derivatives of metric, the physically relevant equations are obtained using the self-consistent method [21]. In a FRW background metric, the (0, 0) component of the Einstein equations with first-order semiclassical quantum corrections is obtained in the form [21]

$$
3\dot{a}^2 + \frac{3k}{a^2} - \Lambda - \frac{\kappa \rho_0 a_0^4}{a^4} = \alpha_1 h \frac{8 \kappa \Lambda \rho_0 a_0^4}{a^4} + \alpha_3 h \left[ \frac{\Lambda^2}{3} + \frac{2 \kappa \Lambda \rho_0 a_0^4}{3a^3} + \frac{\kappa \rho_0 a_0^8}{3a^5} \right] = 0 .
$$

(36)

Comparing (32) and (36) indicates that if we choose the parameter $\alpha_1 = -\frac{1}{3}$ and $\alpha_3 = -4$ for the radiation matter, the result of the both methods are the same up to the last term in (32). This term cannot be obtained using the self-consistent method. This comparison may imply that one can interpret (32) as the Einstein equation with the first order quantum corrections.

In 4D, for conformally flat classical background, when the quantum state is constructed from the conformal vacuum (conformally trivial case), the semiclassical corrections to Einstein equation are completely determined by local geometry and there is no additional non-local contribution to the stress tensor [4]. Computing the semiclassical Einstein equation with quantum corrections in a general case is difficult because it includes the state dependent part of the expectation value of quantum matter fields. As was indicated we can derive the local corrections to the Einstein equations via IMT with a 5D flat bulk space ($E_{\mu\nu} = 0$). Studying a general case in Induced Matter approach may be possible via computing the Weyl tensor $E_{\mu\nu}$, that carriers non-local effects onto the brane [22]. Hence, one can say the non-local quantum contributions to the stress tensor from the point of view of IMT may be related to the non-local effects of bulk space. One simple case is when we attempt to obtain the quantum effects in black holes. In this case the quantum effects diverge near the singularity. This allows for the possibility that black holes without singularities might occur in nature [22]. On the other hand, it is possible to obtain non-singular black hole solutions in the brane world model by solving the effective field equations for the induced metric on the brane [23]. Hence obtaining the quantum corrections via IMT in the case of non vanishing $E_{\mu\nu}$ is an interesting subject worth studying.

## 4 Conclusions

We have considered a FRW universe embedded in a 5D flat bulk space with a space like extra dimension. We have shown that the corrections to Einstein equation through the fluctuations of brane can correspond to the semiclassical quantum corrections to the Einstein equation for the conformally trivial case. Although this correspondence was obtained in a conformally flat background, it may be generalized to a general case by considering the non-local effects of $E_{\mu\nu}$.

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