Stability of the Couette flow under the 2D steady Navier–Stokes flow

Wendong Wang

School of Mathematical Sciences, Dalian University of Technology, Dalian, China

Correspondence
Wendong Wang, School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China.
Email: wendong@dlut.edu.cn

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Abstract
In this note, we investigate the stability property of shear flows under the 2D stationary Navier–Stokes equations, and obtain that the Couette flow \((y, 0)\) is stable under the space of \(\mathcal{D}^{1,q}(\mathbb{R}^2)\) for any \(1 < q < \infty\) and unstable in the space of \(\mathcal{D}^{1,\infty}(\mathbb{R}^2)\), which is sharp in this sense. A key observation is the choice of the anisotropic cut-off function. The Poiseuille flow \((y^2, 0)\) is also considered as a by-product, which is stable in the space of \(\mathcal{D}^{1,q}(\mathbb{R}^2)\) with \(\frac{4}{3} < q \leq 4\) via a lemma of Fefferman–Stein.

Keywords
Couette flow, Liouville-type theorem, Navier–Stokes equations, Poiseuille flow

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35Q30, 35B53, 76D05

1 | INTRODUCTION

Consider the incompressible steady Navier–Stokes equations in a domain \(\Omega \subset \mathbb{R}^2\):

\[
\begin{aligned}
-\mu \Delta u + u \cdot \nabla u + \nabla \pi &= 0, \\
\text{div } u &= 0,
\end{aligned}
\]

where \(\mu\) denotes the viscosity coefficient. We assume \(\mu = 1\) for simplicity.

One fundamental question is to investigate the well-posedness property of (1.1). The existence on an exterior domain attracts the attention of many mathematicians when the boundary condition is given at infinity:

\[
\lim_{|x| \to \infty} u(x) = u_\infty,
\]

where \(u_\infty\) is a constant vector, for example, see Leray [17] and Russo [27]. They constructed a solution whose Dirichlet energy is bounded:

\[
D(u) = \int_{\mathbb{R}^2} |\nabla u|^2 \, dx < \infty
\]

but it is difficult to verify that it satisfies the condition (1.2). Hence, one challenging problem is to prove the constructed solution satisfying the asymptotic behavior at \(\infty\). Gilbarg–Weinberger [2] described the asymptotic behavior of the velocity, the pressure, and the vorticity, where they showed that \(u(x) = o(\ln |x|)\) and
\[
\lim_{r \to \infty} \int_{0}^{2\pi} |u(r, \theta) - \bar{u}|^2 d\theta = 0
\]

for some constant vector \(\bar{u}\). Later, Amick [1] proved that \(u \in L^\infty\) under zero boundary condition. Recently, Korobkov–Pileckas–Russo in [24] and [23] obtained that

\[
\lim_{|x| \to \infty} u(x) = \bar{u}.
\]

For more references on the existence and asymptotic behavior of solutions in an exterior domain, we refer to [3, 10, 12, 22, 25, 28] and the references therein.

When \(\Omega\) is the whole space, an interesting question is to study the classification of solutions of (1.1). In detail, we are concerned on the solution spaces of (1.1), or Liouville properties around some special solutions such as shear flows. The shear flow is like the form of \(U(y) = (g(y), 0)\), and it follows from (1.1) that \(g(y) = c, y, \text{or} y^2\). As in [11], for a domain \(\Omega\) and \(1 \leq q \leq \infty\), we define the following linear space (without topology):

\[
D^{1,q}(\Omega) \equiv \{u \in L^1_{\text{loc}}(\Omega), \nabla u \in L^q(\Omega)\},
\]

which describes the growth of the energy. Furthermore, for \(\alpha \in [0, 1]\) and \(1 \leq p \leq \infty\), we introduce another space

\[
\chi^{\alpha,p}(\Omega) \equiv \{u \in L^1_{\text{loc}}(\Omega), |u| \left(1 + |x|\right)^\alpha \in L^p(\Omega)\},
\]

which describes the growth of \(u\). Obviously, \(\chi^{0,p}(\Omega)\) is the usual \(L^p(\Omega)\) space.

For \(g(y) = c\), let us recall some known results on this issue. Under the condition (1.2), the smooth solution \(u\) is indeed bounded and a Liouville theorem being more in the spirit of the classical one for entire analytic functions was obtained by Koch–Nadirashvili–Seregin–Sverak [9] as a by-product of their work on the nonstationary case. If \(u \in \chi^{0,p}(\mathbb{R}^2)\) for \(p > 1\), then \(u\) is trivial, see Zhang [33]. As suggested by Fuchs–Zhong in [8], the stable space may be \(\chi^{\alpha,\infty}(\mathbb{R}^2)\) with \(\alpha < 1\) as the property of harmonic functions, since the linear solutions are the counterexamples; see also Yau [31] and Petter Li–Tam [18], where they considered the space of harmonic functions on complete manifold with nonnegative Ricci curvature with linear growth. When \(\alpha \in [0, 1/7]\) and \(u \in \chi^{\alpha,\infty}(\mathbb{R}^2)\), \(u\) is a constant vector by Fuchs–Zhong [8]. The component is improved to \(\alpha < \frac{1}{3}\) with the help of the vorticity equation by Bildhauer–Fuchs–Zhang in [21].

On the other hand, for the growth of \(\nabla u\), Gilbarg–Weinberger proved the above Liouville-type theorem by assuming (1.3) in [2], where they made use of the fact that the vorticity function satisfies a nice elliptic equation to which a maximum principle applies. The assumption on boundedness of the Dirichlet energy can be relaxed to \(\nabla u \in L^p(\mathbb{R}^2)\) with some \(p \in (\frac{6}{3}, 3]\), see Bildhauer–Fuchs–Zhang [21] for generalized Navier–Stokes equations. If \(u \in D^{1,q}(\mathbb{R}^2)\) for \(1 < p < \infty\), the constant \(u\) follows by the author in [29]. The above results also can be generalized to the shear thickening flows, for example, see [5–7, 16, 32]. For the two-dimensional (2D) steady magnetohydrodynamic (MHD) equations, the similar Liouville-type theorems were obtained by Y. Wang and the author in [30] by assuming (1.3) or \(u \in \chi^{0,p}(\mathbb{R}^2)\) with \(2 < p \leq \infty\), where the smallness conditions of the magnetic field are added. See also the recent result in [29] for \(u \in \chi^{\alpha,\infty}(\mathbb{R}^2)\) with \(\alpha < \frac{1}{3}\) by using the idea of [21] and energy estimates in an annular domain. We also refer to the recent interesting work on the classification of the \((-1)\)-homogeneous solutions of stationary Navier–Stokes equations in [19] and [20].

Next, we consider the stable space of shear flows in \(D^{1,q}\) or \(\chi^{\alpha,\infty}\). Let \((u, \pi)\) be a smooth solution of (1.1) and the vorticity \(\bar{\omega} = \delta_2 u_1 - \delta_1 u_2\), then the vorticity equations are as follows:

\[
-\Delta \bar{\omega} + u \cdot \nabla \bar{\omega} = 0. \tag{1.4}
\]

We will first study the stability of the Couette flow \(U = (y, 0)\), which is a solution of (1.1). Let \(v = u - U\) be the perturbation of the velocity satisfying

\[
\begin{cases}
-\Delta v + v \cdot \nabla v + \nabla \pi + (u_2, 0) + y \delta_2 v = 0, \\
\text{div } v = 0,
\end{cases} \tag{1.5}
\]
Let $w = \delta_x v_1 - \delta_y v_2$, then

$$-\Delta w + v \cdot \nabla w + y \delta_x w = 0. \quad (1.6)$$

Now we state our main result on the Couette flow:

**Theorem 1.1.** Let $(u, \pi)$ be a smooth solution of the 2D Navier–Stokes equations (1.1) defined over the entire plane. For \( U = (y, 0) \), assume that \( v = u - U \in D^{1, q}(\mathbb{R}^2) \) with \( 1 < q < \infty \). Then, \( v \) and \( \pi \) are constants.

**Remark 1.2.** Obviously, the above result fails in the space of \( D^{1, \infty}(\mathbb{R}^2) \), since the linear solutions are not unique (e.g., shear flow \((c y, 0)\)). The above result also shows that the stable spaces are similar as the constant solution (see [29]). It is worth mentioning that the stability threshold in Sobolev spaces for the 2D time-dependent Navier–Stokes is more complicated, for example, see Bedrossian–Germain–Masmoudi [14], Bedrossian–Wang–Vicol [15], and Chen–Li–Wei–Zhang [26], where if the initial velocity is around the Couette flow

$$\|u_0 - (y, 0)\|_{H^2} \leq c Re^{-\frac{1}{2}}$$

for a small \( c \), then the solution still stays in this space for any time.

If the velocity is largely growing around the Couette flow, we have the following stability estimate:

**Theorem 1.3.** Let \((u, \pi)\) be a smooth solution of the 2D Navier–Stokes equations (1.1) defined over the entire plane and \( v = u - U \) satisfies the growth estimates \( v \in \chi^{\alpha, \infty}(\mathbb{R}^2) \) for \( 0 < \alpha < \frac{1}{5} \), where \( U = (y, 0) \). Then, \( v \) and \( \pi \) are constants.

**Remark 1.4.** The above result generalized the Liouville-type theorems around the trivial solution in [9, 21] to the Couette flow.

The similar arguments can applied to the Poiseuille flow, which is stated as follows.

**Corollary 1.5.** Let \((u, \pi)\) be a smooth solution of the 2D Navier–Stokes equations (1.1) defined over the entire plane. For \( U = (y^2, 0) \), let \( v = u - U \). Then, \( v \) and \( \pi \) are constants, if one of the following conditions holds:

(i) \( v \in D^{1, q}(\mathbb{R}^2) \) with \( \frac{4}{3} < q \leq 4 \);

(ii) \( v \) satisfies the growth estimates \( v \in \chi^{\alpha, \infty}(\mathbb{R}^2) \) for \( 0 < \alpha < \frac{1}{8} \).

Let us recall a result of Gilbarg–Weinberger in [2] about the decay of functions with finite Dirichlet integrals.

**Lemma 1.6** (Lemma 2.1, 2.2, [2]). Let a \( C^1 \) vector-valued function \( f(x) = (f_1, f_2)(x) = f(r, \theta) \) with \( r = |x| \) and \( x_1 = r \cos \theta \). There holds finite Dirichlet integral in the range \( r > r_0 \), that is,

$$\int_{r > r_0} |\nabla f|^2 \, dx \, dy < \infty.$$

Then, we have

$$\lim_{r \to \infty} \frac{1}{\ln r} \int_0^{2\pi} |f(r, \theta)|^2 d\theta = 0.$$

If, furthermore, we assume \( \nabla f \in L^p(\mathbb{R}^2) \) for some \( 2 < p < \infty \), then the above decay property can be improved to be point-wise uniformly. More precisely, we have

**Lemma 1.7** (Theorem II.9.1 [11]). Let \( \Omega \subset \mathbb{R}^2 \) be an exterior domain.
(i) Let
\[ \nabla f \in L^2 \cap L^p(\Omega), \]
for some \(2 < p < \infty\). Then,
\[ \lim_{|x| \to \infty} \frac{|f(x)|}{\sqrt{\ln(|x|)}} = 0, \]
uniformly.

(ii) Let
\[ \nabla f \in L^p(\Omega), \]
for some \(2 < p < \infty\). Then,
\[ \lim_{|x| \to \infty} \frac{|f(x)|}{|x|^{\frac{p}{2}} - \frac{p}{p}} = 0, \]
uniformly.

Throughout this article, \(C(\alpha_1, \ldots, \alpha_n)\) denotes a constant depending on \(\alpha_1, \ldots, \alpha_n\), which may be different from line to line.

2 | PROOF OF THEOREM 1.1

In this section, we are aimed to prove Theorem 1.1. First, let us prove a similar result as Gilbarg–Weinberger in [2] about the decay of functions with finite \(D^{1,q}\) gradient integrals.

**Lemma 2.1.** Let \(C^1\) be vector-valued function \(f(x) = (f_1, f_2)(x) = f(r, \theta)\) with \(r = |x|\) and \(x_1 = r \cos \theta\). There holds
\[ \int_{r > r_0} |\nabla f|^q \, dx < \infty, \quad 1 < q < 2. \]

Then, we have
\[ \limsup_{r \to \infty} \int_0^{2\pi} |f(r, \theta)|^q \, d\theta < \infty. \]

**Proof of Lemma 2.1.** By Hölder inequality, we have
\[ \frac{d}{dr} \left( \int_0^{2\pi} |f(r, \theta)|^q \, d\theta \right)^{\frac{1}{q}} \leq \left( \int_0^{2\pi} |f|^q \, d\theta \right)^{\frac{1}{q} - 1} \int_0^{2\pi} |f|^{q-1} |f_r| \, d\theta \]
\[ \leq \left( \int_0^{2\pi} |f_r|^q \, d\theta \right)^{\frac{1}{q}}. \]
Integrating from $r_1$ with $r_1 \geq r_0$, we get

\[
\left( \int_0^{2\pi} |f(r, \theta)|^q d\theta \right)^{\frac{1}{q}} - \left( \int_0^{2\pi} |f(r_1, \theta)|^q d\theta \right)^{\frac{1}{q}} \leq \int_{r_1}^{r} \left( \int_0^{2\pi} |f_r|^q d\theta \right)^{\frac{1}{q}} dr
\]

\[
\leq \left( \int_{r>r_0} |\nabla f|^q dx \right)^{\frac{1}{q}} \left( \int_{r_1}^{r} r^{-\frac{1}{q-1}} dr \right)^{1-\frac{1}{q}}
\]

\[
\leq \left( \int_{r>r_0} |\nabla f|^q dx \right)^{\frac{1}{q}} \left( \frac{q-1}{2-q} \right)^{1-\frac{1}{q}} r_{\frac{q-2}{q}},
\]

which yields the required result. \hfill \Box

Proof of Theorem 1.1. Step I. Case of $2 < q < \infty$. Let $\eta(x, y) \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function with $0 \leq \eta \leq 1$ satisfying $\eta(x, y) = \eta_1(x) \eta_2(y)$, where

\[
\eta_1(x) = \begin{cases} 
1, & |x| \leq R^\beta, \\
0, & |x| > 2R^\beta,
\end{cases}
\]

where $1 < \beta < (1 - \frac{2}{q} )^{-1}$, and

\[
\eta_2(y) = \begin{cases} 
1, & |y| \leq R, \\
0, & |y| > 2R.
\end{cases}
\]

Multiply $q|w|^{q-2}w$ on both sides of (1.6), and we have

\[
I = \frac{4(q-1)}{q} \int_{\mathbb{R}^2} |\nabla (|w|^\frac{q}{2})|^2 \eta dx dy
\]

\[
\leq \int_{\mathbb{R}^2} |w|^q \Delta \eta dx dy + \int_{\mathbb{R}^2} |w|^q y \partial_x \eta dx dy
\]

\[
+ \int_{\mathbb{R}^2} |w|^q v \cdot \nabla \eta dx dy \equiv I_1 + I_2 + I_3. \tag{2.1}
\]

Since $v \in D^{1,q}$ and $\beta > 1$, obviously $I_1 \to 0$ and

\[
I_2 \leq CR^{1-\beta} \to 0,
\]

as $R \to \infty$. About the term $I_3$, due to Lemma 1.7, for large $R > 0$, we have

\[
|v(x, y)| \leq |(x, y)|^{1-\frac{2}{q}}.
\]

Thus, we have

\[
I_3 \leq CR^{\beta(1-\frac{2}{q})-1} \to 0,
\]
as $R \to \infty$, since
\[ \beta(1 - \frac{2}{q}) < 1. \]

Consequently, we get $\nabla(|w|^q) \equiv 0$, which implies that $w \equiv C$. Due to $\text{div } v = 0$, it follows that
\[ \nabla \equiv 0 \]

and the known condition $v \in D^{1,q}$ yield that
\[ \nabla v \equiv 0. \]

Hence, $v$ and $\pi$ are constant.

**Step II. Case of $1 < q \leq 2$.** We take a cut-off function $\phi$ as follows.

(i) Let $r = \sqrt{x^2 + y^2}$. $\phi$ be radially decreasing and satisfying
\[ \phi(x, y) = \phi(r) = \begin{cases} 1, & r \leq \rho, \\ 0, & r \geq \tau, \end{cases} \]

where $0 < \frac{R}{2} \leq \frac{2}{3} \tau < \frac{3}{4} R \leq \rho < \tau \leq R$;

(ii) $|\nabla \phi|(x, y) \leq \frac{C}{r - \rho}$ for all $(x, y) \in \mathbb{R}^2$.

Multiplying both sides of (1.6) by $\phi w$, respectively, and then applying integration by parts, we arrive at
\[
\int_{\mathbb{R}^2} \phi |\nabla w|^2 \, dx \, dy = -\int_{\mathbb{R}^2} \nabla w \cdot \nabla \phi w \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} v \cdot \nabla \phi w^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} y \partial_x \phi w^2 \, dx \, dy \equiv I'_1 + I'_2 + I'_3. \tag{2.2}
\]

In what follows, we shall estimate $I'_j$ for $j = 1, 2, 3$ one by one.

For the term $I'_1$, by Hölder's inequality we have
\[ I'_1 \leq \frac{C}{\tau - \rho} \|\nabla w\|_{L^q(B_\tau)} \|w\|_{L^2(B_\tau)}. \]

Using the following multiplicative Gagliardo–Nirenberg inequality (see, e.g., Lemma II.3.3 in [11])
\[ \|w\|_{L^q(B_\tau)} \leq C \|\nabla w\|_{L^2(B_\tau)}^{1 - \frac{q}{2}} \|w\|_{L^q(B_\tau)}^{\frac{q}{2}} + C \tau^{1 - \frac{2}{q}} \|w\|_{L^q(B_\tau)}, \tag{2.3} \]

which yields that
\[ I'_1 \leq \frac{1}{8} \int_{B_\tau} |\nabla w|^2 \, dx + \frac{C}{(\tau - \rho)^{\frac{q}{2}}} + \frac{C \tau^{-\frac{2}{q}}}{(\tau - \rho)^2}, \tag{2.4} \]

by noting that $\|w\|_{L^q(B_\tau)} \leq \|\nabla v\|_{L^q(\mathbb{R}^2)} < \infty$. 
For the terms $I'_2$, let

$$
\tilde{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta,
$$

then by Wirtinger's inequality (e.g., for $p = 2$ see chapter II.5 [11]), we have

$$
\int_0^{2\pi} |f - \tilde{f}|^p d\theta \leq C(p) \int_0^{2\pi} |\partial_\theta f|^p d\theta,
$$

(2.5)

for $1 \leq p < \infty$.

Then, by using (2.5), Lemma 1.6, and Lemma 2.1 we have

$$
I'_2 \leq \left| \int_{\mathbb{R}^2} w^2 (v - \bar{v}) \cdot \nabla \phi \, dx \, dy \right| + \left| \int_{\mathbb{R}^2} w^2 \bar{v} \cdot \nabla \phi \, dx \, dy \right|
$$

\begin{align*}
&\leq C \left( \int_{B_\tau} w^{2q} \right)^{\frac{1}{q'}} \left( \int_{\frac{\tau}{2} < r < \tau} |v(r, \theta) - \bar{v}|^q d\theta \, dr \right)^{\frac{1}{q'}} \\
&+ C \left( \int_{B_\tau} w^{2q} \right)^{\frac{1}{q'}} \left( \int_{\frac{\tau}{2} < r < \tau} \frac{1}{r^q} \int_0^{2\pi} |\partial_\theta v|^q d\theta \, dr \right)^{\frac{1}{q'}} \\
&\leq \frac{CR}{\tau - \rho} \left( \int_{B_\tau} w^{2q} \right)^{\frac{1}{q'}} \left( \int_{\frac{\tau}{2} < r < \tau} \frac{1}{r^q} \int_0^{2\pi} |\partial_\theta v|^q d\theta \, dr \right)^{\frac{1}{q'}} \\
&+ \frac{C(\ln R)^{\frac{3}{4}}}{\tau - \rho} \int_{B_\tau} w^2 \, dx \, dy.
\end{align*}

Using Gagliardo–Nirenberg inequality again,

$$
\|w\|_{L^{2q}(B_{\tau})} \leq C\|\nabla w\|_{L^{1-q}(B_{\tau})} ^{1-q} \|w\|_{L^{q}(B_{\tau})} ^q + C\tau^{1-\frac{3}{q}}\|w\|_{L^{q}(B_{\tau})},
$$

(2.6)

which together with (2.3) imply that

\begin{align*}
I'_2 &\leq \frac{1}{8} \left( \int_{B_\tau} |\nabla w|^2 \right) + C \left( \frac{R}{\tau - \rho} \|\nabla w\|_{L^{4}(B_{\tau})} \right)^{\frac{2q}{q'}} + \frac{CR^{2-\frac{6}{q}}}{\tau - \rho} \\
&+ C \left( \frac{\ln R}{\tau - \rho} \right)^{\frac{2}{q}} + C \left( \frac{\ln R}{\tau - \rho} \right)^{2-\frac{3}{q}},
\end{align*}

(2.7)

where we used the boundedness of $D^{1,q}$ integral.

For the term $I'_3$, we have

$$
I'_3 \leq \left| \int_{\mathbb{R}^2} w^2 y \partial_x \phi \, dx \, dy \right| \leq \frac{CR}{\tau - \rho} \left( \int_{B_\tau \setminus B_{\frac{\tau}{2}}} w^2 \, dx \, dy \right).
$$
and using the Gagliardo–Nirenberg inequality in an annular domain, a slightly different version of (2.3) is

\[ \|w\|_{L^q(B, B_{2\tau})} \leq C \|\nabla w\|_{L^q(B, B_{2\tau})}^{\frac{q}{1-\frac{q}{2}}} \|w\|_{L^q(B, B_{2\tau})}^{\frac{q}{q}} + C \tau^{\frac{1-\frac{q}{2}}{q}} \|w\|_{L^q(B, B_{2\tau})}, \]

which implies that

\[ I_3' \leq \frac{1}{8} \left( \int_{B_\tau} |\nabla w|^2 \right)^{\frac{1}{2}} + C \left( \frac{R}{\tau - \rho} \right)^{\frac{q}{2}} \|w\|_{L^q(B, B_{2\tau})} \]

\[ + C \left( \frac{R}{\tau - \rho} \right)^{\frac{q}{2}} \|w\|_{L^q(B, B_{2\tau})} \]  

(2.8)

Collecting the estimates of \(I_1, \ldots, I_3\), by (2.4), (2.7), and (2.8), we have

\[ \int_{B_\rho} |\nabla w|^2 \, dx \, dy \]

\[ \leq \frac{1}{2} \int_{B_\tau} |\nabla w|^2 + \frac{C}{(\tau - \rho)^{\frac{q}{2}}} \int_{B_\rho} |\nabla w|^2 + \frac{C}{(\tau - \rho)^{2}} \int_{B_\rho} |\nabla w|^2 \]

\[ + C \left( \frac{\sqrt{\ln R}}{\tau - \rho} \right)^{\frac{q}{2}} \|w\|_{L^q(B, B_{R/2})} + C \left( \frac{\sqrt{\ln R}}{\tau - \rho} \right)^{\frac{q}{2}} \|w\|_{L^q(B, B_{R/2})} \]

Then, an application of Giaquinta's iteration lemma [13, Lemma 3.1] yields

\[ \int_{B_{R/2}} |\nabla w|^2 \, dx \, dy \leq CR^{\frac{1}{\frac{q}{2}}} + C \left( \frac{\sqrt{\ln R}}{R} \right)^{\frac{q}{2}} \]

\[ + C \left( \|\nabla w\|_{L^q(B, B_{R/2})} \right)^{\frac{q}{2}} + C \left( R^{\frac{1}{\frac{q}{2}}} + 1 \right) \|w\|_{L^q(B, B_{R/2})} \]

Letting \(R \to \infty\), we have

\( \nabla w \equiv 0 \)

and \(w \equiv C\). With similar arguments as in Step I, we complete the proof.

\[ \square \]

3 | PROOF OF THEOREM 1.3

In this section, we will prove Theorem 1.3.

Proof. Let \(\eta(x, y) \in C_0^\infty(\mathbb{R}^2)\) be a cut-off function on a cylinder domain with \(0 \leq \eta \leq 1\) satisfying \(\eta(x, y) = \eta_1(x)\eta_2(y)\), where

\[ \eta_1(x) = \begin{cases} 1, & |x| \leq R^\beta, \\ 0, & |x| > 2R^\beta, \end{cases} \]

\[ \eta_2(y) = \begin{cases} 1, & |y| \leq R^\beta, \\ 0, & |y| > 2R^\beta. \end{cases} \]
where $\beta > 1$, to be decided, and

$$
\eta_2(y) = \begin{cases} 
1, & |y| \leq R, \\
0, & |y| > 2R.
\end{cases}
$$

Write $w^{2q} = (w^2)^q$. As in [21](see also [29]), for $q \geq 2$, $\ell \geq q$, we have

$$
\int_{\mathbb{R}^2} w^{2q}\eta^{2\ell} \, dx \, dy = \int_{\mathbb{R}^2} (\partial_2 v_1 - \partial_1 v_2)w^{2q-2}w\eta^{2\ell} \, dx \, dy
$$

$$
= \int_{\mathbb{R}^2} (v_2, -v_1) \cdot \nabla [w^{2q-2}w\eta^{2\ell}] \, dx \, dy
$$

$$
\leq (2q - 1) \int_{\mathbb{R}^2} |v||\nabla w|w^{2q-2}\eta^{2\ell} \, dx \, dy + 2\ell \int_{\mathbb{R}^2} |v||\nabla \eta||w|^{2q-1}\eta^{2\ell-1} \, dx \, dy
$$

$$
\leq \frac{1}{2} \int_{\mathbb{R}^2} w^{2q}\eta^{2\ell} \, dx \, dy + C(q) \int_{\mathbb{R}^2} |v|^2|\nabla w|^2w^{2q-4}\eta^{2\ell} \, dx \, dy
$$

$$
+ 2\ell \int_{\mathbb{R}^2} |v||\nabla \eta||w|^{2q-1}\eta^{2\ell-1} \, dx \, dy.
$$

Due to the growth estimates $v \in \chi^{\alpha, \infty}$, we have

$$
\int_{\mathbb{R}^2} w^{2q}\eta^{2\ell} \, dx \, dy \leq C(q)R^{2\alpha\beta} \int_{\mathbb{R}^2} |\nabla w|^2w^{2q-4}\eta^{2\ell} \, dx \, dy
$$

$$
+ C(\ell)R^{2\beta-1} \int_{\mathbb{R}^2} |w|^{2q-1}\eta^{2\ell-1} \, dx \, dy. \quad (3.1)
$$

On the other hand, multiply $\eta^{2\ell}w^{2q-4}w$ on both sides of (1.6), and we have

$$
II \doteq (2q - 3) \int_{\mathbb{R}^2} |\nabla w|^2w^{2q-4}\eta^{2\ell} \, dx \, dy
$$

$$
\leq \frac{1}{2q - 2} \int_{\mathbb{R}^2} w^{2q-2} \Delta (\eta^{2\ell}) \, dx \, dy + \frac{1}{2q - 2} \int_{\mathbb{R}^2} w^{2q-2}v \cdot \nabla (\eta^{2\ell}) \, dx \, dy
$$

$$
+ \frac{1}{2q - 2} \int_{\mathbb{R}^2} w^{2q-2}y\partial_x(\eta^{2\ell}) \, dx \, dy. \quad (3.2)
$$

Then, it follows from (3.1), (3.2) that

$$
\int_{\mathbb{R}^2} w^{2q}\eta^{2\ell} \, dx \, dy
$$

$$
\leq C(q, \ell)R^{2\alpha\beta-2} \int_{\mathbb{R}^2} w^{2q-2}\eta^{2\ell-2} \, dx \, dy + C(q, \ell)R^{3\alpha\beta-1} \int_{\mathbb{R}^2} w^{3q-2}\eta^{2\ell-1} \, dx \, dy
$$

$$
+ C(q, \ell)R^{2\beta-1} \int_{\mathbb{R}^2} |w|^{2q-1}\eta^{2\ell-1} \, dx \, dy + C(q, \ell)R^{2\alpha\beta+1-\beta} \int_{\mathbb{R}^2} w^{2q-2}\eta^{2\ell-2} \, dx \, dy
$$

$$
\doteq II_1 + \cdots + II_4.
$$

Noting $\ell \geq q$, by Young inequality we have
\[ II_1 \leq \delta \int_{\mathbb{R}^2} w^{2q} \left( \frac{2\ell-2}{q-1} \right) dxdy + C(\delta, \ell, q)R^{1+\beta+q(2\alpha\beta-2)}, \]
\[ II_2 \leq \delta \int_{\mathbb{R}^2} w^{2q} \left( \frac{2\ell-1}{q-1} \right) dxdy + C(\delta, \ell, q)R^{1+\beta+q(3\alpha\beta-1)}, \]
\[ II_3 \leq \delta \int_{\mathbb{R}^2} w^{2q} \left( \frac{2\ell-1}{q-1} \right) \frac{2\ell}{2q-1} dxdy + C(\delta, \ell, q)R^{1+\beta+2q(\alpha\beta-1)}, \]
and
\[ II_4 \leq \delta \int_{\mathbb{R}^2} w^{2q} \left( \frac{2\ell}{q} \right) dxdy + C(\delta, \ell, q)R^{1+\beta+q(2\alpha\beta+1-\beta)}. \]

Hence, first take \( \delta < \frac{1}{32} \); second, for fixed \( \alpha < \frac{1}{5} \) and \( \beta = \frac{5}{3} \), we take \( q_0 = \frac{1+\beta}{\beta-2\alpha\beta-1} \). Then, for any \( q > q_0 \), we have
\[ 1 + \beta + q(2\alpha\beta - 2) < 0, \quad 1 + \beta + q(3\alpha\beta - 1) < 0, \]
\[ 1 + \beta + 2q(\alpha\beta - 1) < 0, \quad 1 + \beta + q(2\alpha\beta + 1 - \beta) < 0. \]

Consequently, we get
\[ \int_{\mathbb{R}^2} w^{2q} dxdy = 0, \]
as \( R \to \infty \). Thus, we have \( \Delta v = 0 \), which implies \( v \equiv C \), since \( v \in \chi^{\alpha,\infty} \). The proof is complete. \( \square \)

4 | STABILITY OF POISEUILLE FLOW

In this section, we will consider the stable space of the Poiseuille flow \( U = (y^2, 0) \) under the Navier–Stokes flow and prove Corollary 1.5. For the Poiseuille flow \( U = (y^2, 0) \), which is a solution of (1.1), let \( v = u - U \) be the perturbation of the velocity, which satisfies
\[ \begin{cases} -\Delta v - 2 + v \cdot \nabla v + \nabla \pi + (2yv_2, 0) + y^2 \partial_x v = 0, \\ \text{div } v = 0. \end{cases} \]
\[ \quad (4.1) \]

Let \( w = \partial_y v_1 - \partial_1 v_2 \), then
\[ -\Delta w + v \cdot \nabla w + y^2 \partial_x w = 0. \]
\[ \quad (4.2) \]

To overcome the singularity of the term with \( y^2 \partial_x \), we have to estimate the growth of the functions in \( D^{1,q}(\mathbb{R}^2) \).

First of all, for \( 1 < q < 2 \), we have the following lemma (e.g., see Theorem II.6.1 in [11]).

**Lemma 4.1.** Let \( \Omega \subseteq \mathbb{R}^2 \) be an exterior domain of locally Lipschitz and let
\[ f \in D^{1,q}(\Omega), \quad 1 \leq q < 2. \]

Then, there exists a unique \( f_0 \in \mathbb{R} \) such that
\[ f - f_0 \in L^s(\Omega), \quad s = \frac{2q}{2-q} \]
and for some \( \gamma_1 \) independent of \( f \)
\[ \|f - f_0\|_s \leq \gamma_1 \|\nabla (f - f_0)\|_q. \]
For the critical case \( q = 2 \), it is obvious that \( f \in BMO(\mathbb{R}^2) \) if \( f \in D^{1,2}(\mathbb{R}^2) \) and \( \| f \|_* \leq C \| f \|_{D^{1,2}} \), where

\[
\| f \|_* \doteq \sup_{x_0 \in \mathbb{R}^2, r > 0} \left( \frac{1}{|Q_r(x_0)|} \int_{Q_r(x_0)} |f - f_{Q_r(x_0)}| \, dx \right)^{\frac{1}{s}} < \infty,
\]

where \( Q_r(x_0) \) is the cube whose sides have length \( r \) centered at \( x_0 \). It is well known that for the bounded mean oscillation (BMO) space, we have

\[
\Gamma(s) = \sup_{x_0 \in \mathbb{R}^2, r > 0} \left( \frac{1}{|Q_r(x_0)|} \int_{Q_r(x_0)} |f - f_{Q_r(x_0)}|^s \, dx \right)^{\frac{1}{s}} < \infty,
\]

for any \( s \in [1, \infty) \).

The integrable property of \( f \) is stated as follows, which is a slightly different version from (1.2) in [4] by Fefferman–Stein.

**Lemma 4.2.** Let \( f \in D^{1,2}(\mathbb{R}^2) \). For \( p \geq 1 \) and \( \varepsilon \in (0, 1) \), we have

\[
\int_{\mathbb{R}^2} \frac{|f - f_{Q_0}|^p}{1 + |x|^{2+\varepsilon}} \, dx \leq C(\varepsilon, p) \| f \|_*^p \leq C(\varepsilon, p) \| \nabla f \|_{L^2(\mathbb{R}^2)}^p,
\]

where \( Q_0 \) is the cube whose sides have length 1, and is centered at the origin.

**Proof of Lemma 4.2.** It is similar as in [4]. Let \( S_k = Q_k \setminus Q_{k-1} \), where \( Q_k \) is the cube centered at the origin whose sides have length \( 2^k \). Since \( \| f \|_* \leq C \| f \|_{D^{1,2}} \), it suffices to prove that

\[
I_k = \int_{S_k} \frac{|f(x) - f_{Q_0}|^p}{1 + |x|^{2+\varepsilon}} \, dx \leq C_k \| f \|_*^p,
\]

and \( \sum_{k \geq 1} C_k \) is summable.

In fact,

\[
I_k \leq \frac{4^{2+\varepsilon}}{2^{k(2+\varepsilon)}} \int_{Q_k} |f(x) - f_{Q_k} + f_{Q_k} - f_{Q_0}|^p \, dx
\]

\[
\leq \frac{4^{2+\varepsilon}}{2^{k(2+\varepsilon)}} \left( 2^{2k} + k^p 2^{2k+2p} \right) \| f \|_*^p \leq C_k \frac{k^p}{2^{k\varepsilon}} \| f \|_*^p,
\]

where we used \( |f_{Q_k} - f_{Q_{k-1}}|^p \leq 4^p \| f \|_*^p \). The proof is complete. \( \square \)

**Proof of Corollary 1.5.** It is similar as the Couette flow.

(i) It is similar as Step I in the proof of Theorem 1.1.

For \( 1 < q \leq 4 \), let \( \eta(x, y) \in C_0^\infty(\mathbb{R}^2) \) be a cut-off function with \( 0 \leq \eta \leq 1 \) satisfying \( \eta(x, y) = \eta_1(x)\eta_2(y) \), where

\[
\eta_1(x) = \begin{cases} 1, & |x| \leq R^\delta, \\ 0, & |x| > 2R^\delta, \end{cases}
\]

where \( \delta \geq 2 \), is to be decided, and

\[
\eta_2(y) = \begin{cases} 1, & |y| \leq R, \\ 0, & |y| > 2R. \end{cases}
\]

For \( \frac{4}{3} < q_0 \leq q \), multiply \( q_0 \eta^2 |w|^{q_0-2} w \) on both sides of (4.2), and we have
\begin{equation}
I'' \doteq \frac{4(q_0 - 1)}{q_0} \int_{\mathbb{R}^2} |\nabla(|w|^{q_0/2})|^2 \eta^2 \, dx \, dy \leq \int_{\mathbb{R}^2} |w|^{q_0} \nabla \triangle \eta^2 \, dx \, dy + \int_{\mathbb{R}^2} |w|^{q_0} y^2 \partial_x \eta^2 \, dx \, dy + \int_{\mathbb{R}^2} |w|^{q_0} u \cdot \nabla \eta^2 \, dx \, dy = I''_1 + I''_2 + I''_3. \tag{4.4}
\end{equation}

Case of $2 < q \leq 4$. Take $q_0 = q$ and $2 \leq \beta \leq (1 - \frac{2}{q})^{-1}$. Since $v \in D^{1,q}$, obviously $I''_1 \to 0$ as $R \to \infty$. When $\beta \geq 2$,

$$I''_2 \leq C(q) R^{2 - \beta} \int_{R^2 \leq |x| \leq 2R^\beta} |w|^q \, dx \, dy \to 0.$$ 

About the term $I_3$, due to Lemma 1.7, we have

$$|v(x, y)| \leq |(x, y)|^{1 - \frac{2}{q}}.$$

Then, we have

$$I''_3 \leq C(q) R^{\beta(1 - \frac{2}{q}) - 1} \int_{R^2 \leq |x| \leq 2R^\beta} |w|^q \, dx \, dy \to 0,$$

as $R \to \infty$, since

$$\beta(1 - \frac{2}{q}) \leq 1.$$ 

Consequently, we get $V(|w|^{\frac{q}{2}}) \equiv 0$, which implies that $w \equiv C$. The same arguments hold.

Case of $q = 2$. Take $q_0 = \frac{9}{5}$ and $\beta = 3$. Obviously $I''_1 + I''_2 \to 0$ as $R \to \infty$ by Hölder inequality. Next, we estimate the term $I''_3$. With the help of Lemma 4.2 as $\epsilon = 1$, there holds

$$\int_{\mathbb{R}^2} \frac{|v - v_0|^10}{1 + |(x, y)|^3} \, dx \, dy < C ||v||_{10}^1.$$ 

Thus,

$$I''_3 \leq C(q) R^{-1 + \frac{1}{2} - \frac{1}{q}} \to 0,$$

as $R \to \infty$.

Case of $\frac{4}{3} < q < 2$. At this time, take $q_0 = \frac{3q - 2}{2} \in (1, q)$ and $\beta = 1 + \frac{2}{q_0} \in (2, 3)$. Then,

$$2 - \beta + (1 + \beta)(1 - \frac{q_0}{q}) = 0$$

and hence

$$I''_2 \leq C(q) \left(\int_{R^2 \leq |x| \leq 2R^\beta} |w|^q \, dx \, dy\right)^{\frac{q_0}{q}} R^{2 - \beta + (1 + \beta)(1 - \frac{q_0}{q})} \to 0$$

as $R \to \infty$.

For the term $I''_3$, by Lemma 4.1, there exists a constant vector $v_0$ such that

$$||v - v_0||_{L^{\frac{2q}{2q-3}}} \leq C ||\nabla v||_{L^q}.$$
Thus,
\[
I_3'' \leq C(q) \left( \int_{R^2 : |x| \leq 2R^\beta} |w|^q \, dx \, dy \right)^{\frac{q_0}{q}} \| \nabla v \|_{L^q} \, R^{-1+(1+\beta)(1-\frac{q_0}{2q})} \to 0
\]
as \( R \to \infty \), since
\[
1 - \frac{q_0}{q} - \frac{2 - q}{2q} = 0.
\]

(ii) Then, it follows from (3.3) that
\[
\int_{\mathbb{R}^2} w^{3q} \eta^{2\ell} \, dx
\leq C(q, \ell) R^{2\alpha \beta - 2} \int_{\mathbb{R}^2} w^{2q - 2} (\eta^{2\ell - 2}) \, dx \, dy + C(q, \ell) R^{3\alpha \beta - 1} \int_{\mathbb{R}^2} w^{2q - 2} (\eta^{2\ell - 1}) \, dx \, dy
\]
\[
+ C(q, \ell) R^{\alpha \beta - 1} \int_{\mathbb{R}^2} |w|^{2q - 1} (\eta^{2\ell - 1}) \, dx \, dy + C(q, \ell) R^{2\alpha \beta + 2 - \beta} \int_{\mathbb{R}^2} w^{2q - 2} (\eta^{2\ell - 2}) \, dx \, dy
\]
\[
= II_1'' + \cdots + II_4''.
\]

At this time, since \( 0 < \alpha < \frac{1}{8} \), we can choose
\[
\beta = \frac{8}{3},
\]
thus
\[
1 + \beta + q(3\alpha \beta - 1) < 0, 1 + \beta + q(2\alpha \beta + 2 - \beta) < 0,
\]
for \( q \) sufficiently large. Since the similar arguments as Theorem 1.3 hold, we omitted it. The proof is complete. \( \square \)

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CONFLICT OF INTEREST
The author declares no potential conflict of interests.

ORCID
Wendong Wang \( \text{https://orcid.org/0000-0001-5861-1208} \)

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