LONG-TERM EVOLUTION OF DOUBLE WHITE DWARF BINARIES ACCRETING THROUGH DIRECT IMPACT

KYLE KREMER1, JEREMY SEPINSKY2, AND VASSILIKI KALOGERA1

1 Center for Interdisciplinary Exploration and Research in Astrophysics (CIERA), Department of Physics and Astronomy, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208, USA; kremer@u.northwestern.edu, vicky@northwestern.edu
2 Department of Physics and Electrical Engineering, The University of Scranton Scranton, PA 18510, USA; jeremy.sepinsky@scranton.edu

Received 2015 February 16; accepted 2015 April 21; published 2015 June 9

ABSTRACT

We calculate the long-term evolution of angular momentum in double white dwarf binaries undergoing direct impact accretion over a broad range of parameter space. We allow the rotation rate of both components to vary and account for the exchange of angular momentum between the spins of the white dwarfs and the orbit, while conserving the total angular momentum. We include gravitational, tidal, and mass transfer effects in the orbital evolution, and allow the Roche radius of the donor star to vary with both the stellar mass and the rotation rate. We examine the long-term stability of these systems, focusing in particular on those systems that may be progenitors of AM CVn or SNe Ia. We find that our analysis yields an increase in the predicted number of stable systems compared to that in previous studies. Additionally, we find that by properly accounting for the effects of asynchronism between the donor and the orbit on the Roche-lobe size, we eliminate oscillations in the orbital parameters, which were found in previous studies. Removing these oscillations can reduce the peak mass transfer rate in some systems, keeping them from entering an unstable mass transfer phase.

Key words: celestial mechanics – binaries: general – accretion, accretion disks – stars: mass-loss

1. INTRODUCTION

Binary systems containing compact objects are of great interest in a variety of areas in astrophysics. Of particular importance are double white dwarf (DWD) binary systems where both components are white dwarfs, which may make up the largest fraction of close binary stars (Marsh et al. 1995).

Following common envelope evolution, DWD binaries may emerge with a sufficiently small semi-major axis, allowing gravitational radiation to drive the stars closer together on an astrophysically interesting timescale. These close DWD systems are of prime importance as low-frequency gravitational-wave sources (Hils et al. 1990; Hils & Bender 2000; Nelemans et al. 2001) as well as progenitors of Type Ia Supernovae (SNe Ia; Maoz et al. 2014).

As energy loss due to gravitational waves drives the degenerate components of a DWD binary together, it is possible for the system to enter into a stable semi-detached state, in which the less massive component will fill its Roche lobe and begin transferring matter to its companion. Such systems may result in the formation of AM CVn (Nather et al. 1981; Tutukov & Yungelson 1996; Nelemans et al. 2001), which have extremely short orbital periods. During mass transfer, a stream of matter is pulled from the donor star through the inner Lagrangian point. If the matter stream does not impact the surface of the companion star, then the mass lost from the donor is expected to settle into a disk (Frank et al. 2002). Torques exerted between the disks and the component white dwarfs allow angular momentum stored in the disk to be transferred back to the orbit (Soberman et al. 1997; Frank et al. 2002). This allows matter in the disk to fall onto the companion star while slowing the orbit contraction, increasing the potential for a stable, long-lived binary system.3

For some DWD binaries, the mass transfer stream will directly impact the surface of the companion star. In doing so, there is no longer an obvious mechanism to return the orbital angular momentum from the transferred mass to the orbit. The division between the disk accretion and direct impact has been studied before (see Marsh et al. 2004; Gokhale et al. 2007; Sepinsky & Kalogera 2014, and references therein, hereafter Paper I, Marsh et al. 2004, and Gokhale et al. 2007, respectively).

As in Marsh et al. (2004) and Gokhale et al. (2007), we showed in Paper I that mass transfer through direct impact can either increase or decrease the semi-major axis of the system, depending on the amount of angular momentum and mass exchanged between the components. If direct impact drives the system apart, both Marsh et al. (2004) and Gokhale et al. (2007) showed that a stabilizing accretion disk is likely to be created. If direct-impact mass transfer decreases the semi-major axis, the mass transfer rate may eventually become unstable, which can result in a merger. If the total mass of the system is in excess of the Chandrasekhar limit, such merger events could lead to a Type Ia supernova (Woosley & Weaver 1986).

Marsh et al. (2004) and Gokhale et al. (2007) both calculated the long-term evolution of such systems. Marsh et al. (2004) concluded that the population of DWD binaries is likely lower than previously anticipated as a result of the high percentage of systems undergoing direct-impact accretion that become unstable. Gokhale et al. (2007) improved upon the analysis of (Marsh et al. 2004) by permitting the spin of the donor to vary and by including the effects of tidal forces from the donor star Marsh et al. (2004) only examined tidal forces arising from asynchronicity between the accretor and the orbit. In their calculation, the spin of the donor was fixed to that of the orbit throughout so that no tides arose from asynchronicity between the donor and orbit.) The modifications of Gokhale et al. (2007) resulted in an increase in the number of stable systems, which can be seen in Figures 2 and 3 in Section 4. In

---

3 We note that we have not considered the possibility of the white-dwarf components having some residual, non-degenerate, hydrogen-rich outer layers. In a recent analysis, Shen (2015) showed that the inclusion of such effects and associated nova-like outbursts may predominantly lead to mergers for both direct-impact and disk accretion.
Paper I we built upon both of these studies, providing direct ballistic integrations of the mass transfer stream (as opposed to the previous method using an approximation adopted from Verbunt & Rappaport 1988). We showed that removing this approximation, which accounts for the complex three-body dynamics of the ejected mass, can have a significant impact upon the angular momentum exchange.

In this paper, we apply the results of Paper I to determine the long-term evolution of DWD binaries in a fully self-consistent manner. In Section 2, we introduce the equations that govern the long-term evolution of DWD systems and discuss the differences between this method and the method utilized by previous studies. In Section 3 we discuss some details of our long-term numerical integrations. In Section 4 we discuss the results of our solutions and analyze the results in comparison to those of previous works. We conclude in Section 5.

2. EQUATIONS OF LONG-TERM EVOLUTION

2.1. Basic Assumptions

Following Paper I, we consider a close binary system of two white dwarfs with masses $M_A$ and $M_D$, volume-equivalent radii of $R_A$ and $R_D$, and uniform rotation rates $\Omega_A$ and $\Omega_D$ with axes perpendicular to the orbital plane for the accretor and donor, respectively.\(^4\) We assume the mass of each star is distributed spherically symmetrically. As in the analyses of Marsh et al. (2004) and Gokhale et al. (2007), we assume the binary to remain in a circular Keplerian orbit throughout its evolution. The radius of each object is assigned following Eggleton’s zero-temperature mass–radius relation (Equation (15) of Verbunt & Rappaport 1988). We assume that both the donor and accretor initially rotate synchronously with the orbit (Marsh et al. 2004; Gokhale et al. 2007).\(^5\) We choose the initial semi-major axis of the orbit such that the volume-equivalent radius of the donor ($R_D$) is equal to the volume-equivalent radius of its Roche lobe as fit by Eggleton (1983).

2.2. Evolution of Angular Momentum

As described in Marsh et al. (2004) and Gokhale et al. (2007), the orbital evolution of a circular binary system can be described by its orbital angular momentum. We begin by presenting each of the components that can lead to a change in the orbital angular momentum.

The total angular momentum, $J_{\text{tot}}$, of a binary system is given by the sum of the orbital angular momentum, $J_{\text{orb}}$, and the spin angular momenta, $J_{\text{spin},A}$ and $J_{\text{spin},D}$, of the accretor and donor, respectively:

$$J_{\text{tot}} = J_{\text{orb}} + J_{\text{spin},A} + J_{\text{spin},D}. \quad (1)$$

The orbital angular momentum is given by

$$J_{\text{orb}} = \frac{Ga}{M} M_A M_D. \quad (2)$$

where $M = M_A + M_D$, $G$ is the gravitational constant, and $a$ is the semi-major axis of the system. The spin angular momenta of the accretor and donor are $J_{\text{spin},A} = k_A M_A R_A^2 \Omega_A$ and $J_{\text{spin},D} = k_D M_D R_D^2 \Omega_D$, respectively, where $k_A$ and $k_D$ are dimensionless constants depending upon the internal structure of the accretor and donor, respectively. It follows that:

$$J_{\text{tot}} = \frac{Ga}{M} M_A M_D + k_A M_A R_A^2 \Omega_A + k_D M_D R_D^2 \Omega_D. \quad (3)$$

There are three effects that change the orbital angular momentum over time: mass transfer (MT), tides, and gravitational radiation (GR). Assuming each effect is independent of the others, we can then write the total change of the orbital angular momentum as the sum of the changes due to each of the above effects, with subscripts, as noted above, for each of the respective components:

$$\dot{J}_{\text{orb}} = \dot{J}_{\text{orb,MT}} + \dot{J}_{\text{orb,tides}} + \dot{J}_{\text{orb,GR}}. \quad (4)$$

To determine the total change in the angular momentum, then, we simply need to write the change due to each of the above effects.

The change in orbital angular momentum due to GR for a circular orbit is given by:

$$\dot{J}_{\text{orb,GR}} = \frac{32}{5} \frac{G^3 M_A M_D M}{a^5} \dot{J}_{\text{orb}} \quad (5)$$

(see for example, Landau & Lifshitz 1975).

Prior to the onset of mass transfer, the semi-major axis of the binary will have been shrinking due to the effects of GR as seen in Equation (5). During this time, tidal coupling will act to circularize the binary as well as synchronize the spins of the stars with the orbit. At the onset of mass transfer, we assume the spins and orbit to be synchronized. As mass transfer begins, angular momentum will be exchanged between the spins of the component stars and the orbit (see Paper I). Any resulting asynchronization between the spins and the orbit leads to tidal coupling, the strength of which will determine how much of the spin angular momentum of the accretor is returned to the orbit. This will ultimately affect the stability of the mass transfer process.

As in Gokhale et al. (2007), we model the change in the binary orbital angular momentum due to tides by:

$$\dot{J}_{\text{orb, tides}} = \frac{k_A M_A R_A^2}{\tau_A} \omega_A + \frac{k_D M_D R_D^2}{\tau_D} \omega_D. \quad (6)$$

The first term on the right-hand side of this equation represents the torque due to dissipative coupling upon the accretor, and the second term the torque upon the donor. These torques are parameterized in terms of the synchronization timescales of the accretor, $\tau_A$, and donor, $\tau_D$ and are linearly proportional to the difference between the component and orbit spin, $\omega_i = \Omega_i - \Omega_{\text{orb}}$, with $i \in \{A, D\}$ for the accretor and donor, respectively. Here, $\Omega_{\text{orb}}$ is the angular velocity of the circular orbit:

$$\Omega_{\text{orb}} = \sqrt{\frac{GM}{a^3}} \quad (7)$$
and \(k_i M_i R_i^2\) are the moments of inertia of each component, with \(k_i\) being the inertial constant.

The values of \(\tau_A\) and \(\tau_D\) determine the strength of tidal coupling relative to MT and GR. As in Gokhale et al. (2007), we examine both the cases of \(\tau_A = \tau_D = 10^5\) years (weak tidal coupling) and \(\tau_A = \tau_D = 10\) years (strong tidal coupling). The tidal-synchronization timescales are discussed further in Section 2.7.

To determine \(J_{\text{orb,MT}}\), we follow Paper I, which uses the ballistic mass transfer calculations of Sepinsky et al. (2010) to determine the instantaneous effect of mass transfer on DWD systems. This method uses a fully self-consistent, conservative, ballistic model of the transferred mass to determine the orbital parameters of the system after a single mass-transfer event. As seen in that paper, the mass transferred is small compared to the total mass of the binary system, the change in the orbital parameters per unit mass that results is directly proportional to the mass transfer rate:

\[
J_{\text{orb,MT}} = \frac{\Delta J_{\text{orb,b}}}{M_p} M_D,  
\]

where \(\Delta J_{\text{orb,b}}\) is the change in the orbital angular momentum for the DWD as calculated by the above ballistic model for a single mass transfer event ejecting a particle of mass \(M_p\). In this method, changes in \(J_{\text{orb,MT}}\) (and changes in all orbital parameters) are calculated at each time-step by integrating the three-body system consisting of the two stars and the discrete particle representing the mass transfer stream. The change in the \(J_{\text{orb,MT}}\) per unit mass transferred is independent of the mass of the ejected particle as long as \(M_p \ll M_D, M_A\). For the calculations here, we use \(M_p = 10^{-8}\ M_D\). The rate of change of \(J_{\text{orb,MT}}\) is then determined by multiplying by the current mass-transfer rate of our evolving DWD, \(\dot{M}_D\). The calculation of \(\dot{M}_D\) will be discussed in Section 2.5.

### 2.3. Differential Equations for Long-term Evolution

Using the above rates of change for the orbital angular momentum, we are now ready to develop the equations for long-term evolution. It follows from Equation (2) that

\[
\frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} = (1 - q) \frac{\dot{M}_D}{M_D} + \frac{1}{2a} \dot{a},
\]

for conservative mass transfer \((M = 0)\).\(^6\) We let \(q = M_D/M_A\).

#### 2.3.1. Evolution of the Semi-major Axis

As in Section 2.2 for the orbital angular momentum, we examine the changes of the semi-major axis due to mass transfer, tides, and GR. We assume that each effect is independent, and write the total change in the semi-major axis due to each of the above effects, respectively, as:

\[
\dot{a} = \dot{a}_{\text{MT}} + \dot{a}_{\text{tides}} + \dot{a}_{\text{GR}}.
\]

To calculate \(\dot{a}_{\text{MT}}\), we utilize the model for mass transfer developed in Sepinsky et al. (2010). The differences between this model and models used in other analyses will be discussed in Section 2.4. Using this method, we calculate the time rate of change of \(a\) due to mass transfer as

\[
\dot{a}_{\text{MT}} = \frac{\Delta a_b}{M_p} \dot{M}_D.  
\]

Here, \(\Delta a_b\) is the change in the semi-major axis for the DWD as calculated by the above ballistic model of Sepinsky et al. (2010) as described in Section 2.2 for the orbital angular momentum.

Next we calculate \(\dot{a}_{\text{tides}}\) and \(\dot{a}_{\text{GR}}\). We cannot use the standard tidal prescription for changes in the semi-major axis following Hut (1981) because, following Marsh et al. (2004) and Gokhale et al. (2007), we used a different metric for changing the spin angular momentum due to tides (Equation (6)). Instead, we must develop \(\dot{a}_{\text{tides}}\) from that angular momentum change. Holding the donor mass constant \((M_D = 0)\) for the final two terms of Equation (10) we can combine Equation (4) with Equation (9) to obtain:

\[
\frac{\dot{J}_{\text{orb,tides}}}{J_{\text{orb}}} + \frac{\dot{J}_{\text{orb,GR}}}{J_{\text{orb}}} = \frac{1}{2a} \dot{a}_{\text{tides}} + \frac{1}{2a} \dot{a}_{\text{GR}}.
\]

Using Equations (2), (5), and (6), it follows that:

\[
\dot{a}_{\text{tides}} = 2a \frac{1}{M_A M_D} \frac{M}{Ga} \left( k_A M_A R_A^2 \frac{\Omega_A}{\tau_A} + k_D M_D R_D^2 \frac{\Omega_D}{\tau_D} \right),  
\]

and

\[
\dot{a}_{\text{GR}} = -\frac{64}{5} \frac{G^3}{c^5} \frac{M_A M_D M}{a^3}.  
\]

For our purposes, we rewrite the two tidal components in terms of the rotation rates of the accretor and the donor relative to the orbital angular velocity, which we define as \(f_A\) and \(f_D\), respectively. The rotation rates can be written in terms of the rotational angular velocities of the stars, \(\Omega_i\) and the angular velocity of the circular orbit, \(\Omega_{\text{orb}}\):

\[
f_i - 1 = \frac{\Omega_i - \Omega_{\text{orb}}}{\Omega_{\text{orb}}} = \frac{a_i}{\Omega_{\text{orb}}}. 
\]

Using Equations (7) and (16), we can rewrite Equation (14) as

\[
\dot{a}_{\text{tides}} = \frac{2M}{a M_A M_D} (a_A + a_D),
\]

where

\[
a_A = \frac{k_A M_A R_A^2}{\tau_A} (f_A - 1)
\]

\(^6\) Recall that we assume the orbit remains circular throughout.
and
\[ a_D = \frac{k_D M_D R_D^2}{\tau_D} (f_D - 1). \]  

(19)

Finally, we can insert Equations (11), (15), and (17) into Equation (10) to obtain the equation for the evolution of the semi-major axis with time:
\[ \dot{a} = \Delta a_D \frac{M_D}{M_P} + \frac{2M}{a M_A M_D} (\alpha_A + \alpha_D) \]
\[ - \frac{64}{5} \frac{G^3 M_A M_D M}{a^5}. \]  
\[ (20) \]

2.3.2. Evolution of the Component Rotation Rates

Next, we find the equations for the evolution of the component spins, \( \dot{f_A} \) and \( \dot{f_D} \). Like the changes to the semi-major axis (Equation (10)), the changes in \( \dot{f} \) and \( \dot{f}_D \) can be separated into three components:
\[ \dot{f}_i = \dot{f}_{MT} + \dot{f}_{\text{tides}, i} + \dot{f}_{\text{GR}} \]  
\[ (21) \]

with \( i \in \{ A, D \} \).

Analogous to \( \dot{a}_{MT} \) in Equation (11), the change in \( \dot{f}_i \) due to mass transfer, \( \dot{f}_{\text{MT}, i} \), can be written as:
\[ \dot{f}_{\text{MT}, i} = \frac{\Delta f_{a,i}}{M_P M_D}. \]  
\[ (22) \]

Here \( \Delta f_{a,i} \) is the change in the rotation rates of star \( i \) resulting from a single mass transfer event in the formulation of Sepinsky et al. (2010) as described in Section 2.2.

As in Equation (3), the spin angular momentum of each component can be written as
\[ J_{\text{spin}, i} = k_i M_i R_i^2 f_i \Omega_{\text{orb}}, \]  
\[ (23) \]

where we have substituted \( f_i \) from Equation (16).

Since we have already determined the change in \( \dot{f}_i \) due to mass transfer Equation (22), we can determine \( \dot{f}_{\text{tides}, i} \) and \( \dot{f}_{\text{GR}} \) by differentiating Equation (23) with the mass held constant:
\[ J_{\text{spin}, i} \bigg|_{M_i} = k_i M_i R_i^2 \Omega_{\text{orb}} (\dot{f}_{\text{tides}, i} + \dot{f}_{\text{GR}}) \]
\[ - \frac{3}{2} k_i M_i R_i^2 \Omega_{\text{orb}} \dot{f}_i \left( \frac{1}{a} (\dot{a}_{\text{tides}} + \dot{a}_{\text{GR}}) \right). \]  
\[ (24) \]

where the second term arises due to the dependence of \( \Omega_{\text{orb}} \) on the semi-major axis (Equation (7)). We note that, because we are holding the mass constant, the second term depends only upon changes due to tides and GR, and not changes due to mass transfer. Changes to \( f_i \) due to the effect of mass transfer on the semi-major axis are fully accounted for by Equation (22).

Since we do not include any GR effects on the spin angular momentum of the components, conservation of angular momentum dictates that any changes in the spin angular momentum of a component must be equal and opposite to the changes in the orbital angular momentum of the system due to tides acting on that component. Combining Equations (6), (7), and (16), we have:
\[ \dot{J}_{\text{spin}} \bigg|_{M_i} = \frac{k_i M_i R_i^2}{\tau_i} \Omega_{\text{orb}} (\dot{f}_i - 1). \]  
\[ (25) \]

By combining Equations (24) and (25) and rearranging, we can write \( \dot{f}_{\text{GR}, i} + \dot{f}_{\text{tides}, i} \) as:
\[ \dot{f}_{\text{GR}, i} + \dot{f}_{\text{tides}, i} = \frac{f_i - 1}{\tau_i} \]
\[ + \frac{3}{2} \frac{f_i}{a} \left[ -\beta + \frac{2}{a} \frac{M}{M_A M_D} (\alpha_A + \alpha_D) \right]. \]  
\[ (26) \]

where we have used Equations (15), (17)–(19), and let:
\[ \beta = \frac{64}{5} \frac{G^3 M_A M_D M}{a^5}. \]  
\[ (27) \]

Following the form of Equation (21), we can combine Equations (22) and (26) to write the equations for the evolution of \( \dot{f}_A \) and \( \dot{f}_D \):
\[ \dot{f}_A = \frac{\Delta f_{a,A}}{M_P} M_D - \frac{f_A - 1}{\tau_A} \]
\[ + \frac{3}{2} \frac{f_A}{a} \left[ -\beta + \frac{2}{a} \frac{M}{M_A M_D} (\alpha_A + \alpha_D) \right] \]  
\[ (28) \]
\[ \dot{f}_D = \frac{\Delta f_{a,D}}{M_P} M_D - \frac{f_D - 1}{\tau_D} \]
\[ + \frac{3}{2} \frac{f_D}{a} \left[ -\beta + \frac{2}{a} \frac{M}{M_A M_D} (\alpha_A + \alpha_D) \right]. \]  
\[ (29) \]

2.3.3. Evolution of the Eccentricity

We assume that the eccentricity of these systems is zero prior to the onset of direct-impact accretion. During the three-body integration of a single mass-transfer event, it is possible for the binary to develop a small eccentricity. However, for simplicity and in accordance with Marsh et al. (2004) and Gokhale et al. (2007), we force the eccentricity to remain zero throughout. Because the system begins in a circular orbit, and due to the action of tidal forces and gravitational wave emission, which both act to circularize the orbit, it is unlikely for any significant eccentricity to develop. In order to keep this orbital angular momentum in the orbit, we manually set our new semi-major axis to:
\[ a = a_I (1 - e^2), \]  
\[ (30) \]

where \( e \) is the eccentricity gained by the system during mass transfer and \( a_I \) is the semi-major axis of the eccentric orbit. Using this modification, we force:
\[ \dot{e} = 0, \]  
\[ (31) \]

and still conserve orbital angular momentum. A more thorough analysis in which we allow eccentricity to vary throughout the entire calculation will be presented in a forthcoming paper.
2.3.4. Mass Transfer Rates

Finally, the changes in the masses of the accretor and donor are given, respectively, by:

\[
\frac{dM_A}{dt} = -M_D \\
\frac{dM_D}{dt} = M_D
\]

(32)

(33)

The mass loss rate of the donor, \( \dot{M}_D \) is obtained as described in Section 2.5.

Together, the set of Equations (20), (28), (29), (31)–(33) can be integrated in time to calculate the evolution of the system.

2.4. Differences from Previous Analyses

There are several differences between our treatment of mass transfer and those of Marsh et al. (2004) and Gokhale et al. (2007). In order to calculate the angular momentum exchange during mass transfer, the studies of Marsh et al. (2004) and Gokhale et al. (2007) both utilize a numerical prescription based on Verbunt & Rappaport (1988). In that formulation, it is assumed that the angular momentum transferred from the orbit to the spin of the accretor is exactly equal to the angular momentum of the ballistic particle in a circular orbit around the donor at its average radius during its motion from donor to accretor. Where the analysis of Marsh et al. (2004) keeps the spin of the donor fixed, Gokhale et al. (2007) does allow the spin of the donor to vary, and notes that in doing so, the number of stable systems increases. However, the allowance of variation in donor spin is not done self-consistently. By introducing the spin angular momentum of the donor, there are now three separate sources/sinks of angular momentum: the spin of the donor, the spin of the accretor, and the orbit. Paper I showed that angular momentum is transferred between each during mass transfer, and that the fraction of angular momentum transferred between each is strongly dependent on the ballistic trajectory of the transferred mass, and hence the system properties. In this paper, we apply the ballistic calculations of Paper I to lift the dependence on the Verbunt & Rappaport (1988) approximations to more accurately determine the flow of angular momentum between the component spins and the orbit.

Since orbital angular momentum changes are directly linked to changes in the mass ratio and semi-major axis, much can be learned from analyses such as that of Gokhale et al. (2007) and Marsh et al. (2004). However, a more complete analysis demands a thorough consideration of the spin angular momenta of the stars as well due to their indirect effect upon the system properties through tidal coupling. This is accomplished via the ballistic calculations described above, which evaluate the three-body problem throughout the evolution of the system to determine the precise effect mass transfer has on the evolution of the system. These calculations take into account not only the immediate feedback on the orbit and spin of the accretor during ejection, but also (1) the gravitational effect on the orbit during the mass transfer process, (2) a calculation of the precise moment when the particle impacts the surface of the accretor, (3) the instantaneous properties of both the accretor and particle at impact, including the angle of impact, and (4) correctly divides the momentum of the impacting particle between linear and angular momentum based upon the angle of impact and the rotation rate of the accretor. We can then examine the angular momentum exchange between all components, the spins of both stars and the orbit, and do so in a way that allows the spins of both stars to vary self-consistently.

2.5. Calculation of the Mass-transfer Rate

As in Marsh et al. (2004), we define the overfill of the Roche lobe as:

\[
\Delta = R_D - R_L. 
\]

(34)

where \( R_D \) is the radius of the donor and \( R_L \) is the radius of the donor’s Roche lobe. The way in which the mass-transfer rate varies with \( \Delta \) has been investigated in many analyses (see, for example, Paczynski & Sienkiewicz 1972, Webbink 1977, Savonije 1978). In accordance with Marsh et al. (2004), we approximate the mass transfer as adiabatic (Webbink 1984). In the adiabatic regime the mass transfer rate is given by:

\[
\dot{M}_D = \frac{8\pi^3}{9} \left( \frac{5Gm_e}{h^2} \right)^{3/2} (\mu_e m_n)^{5/2} \\
\times \frac{1}{P_{\text{orb}}} \left( \frac{3\mu M_D}{5\mu M_D} \right)^{3/2} \frac{1}{\sqrt{a_2(a_2 - 1)}} \Delta^3
\]

(35)

for \( \Delta > 0 \) and zero for \( \Delta < 0 \) (using results from Chandrasekhar 1967; Webbink 1977, 1984). Here, \( m_n \) is the mass of an electron, \( m_e \) is the mass of a nucleon, \( \mu_n \) is the mean number of nucleons per free electron in the outer layers of the donor (assumed here to be two), \( P_{\text{orb}} \) is the orbital period, \( a_L = R_L/a_2 \), and \( \mu_n \) and \( a_2 \) are given by:

\[
\mu = \frac{M_D}{M_A + M_D} \\
a_2 = \frac{\mu}{x_{L1}} + \frac{1 - \mu}{(1 - x_{L1})^3}
\]

(36)

(37)

where \( x_{L1} \) is the distance from the center of the donor to the inner Lagrangian point of the donor, in units of the semi-major axis (Webbink 1977).

2.6. Calculation of Roche Lobe

In the case of Marsh et al. (2004), where the rotational velocity of the donor is fixed to the orbital velocity and where the orbit is circular throughout, the shape and volume of the Roche lobe depends only upon the mass ratio of the system. In that case, the approximation from Eggleton (1983),

\[
R_{L,\text{Egg}} = a \frac{0.49 q^{2/3}}{0.6 q^{2/3} + \ln(1 + q^{1/3})}
\]

(38)

is sufficient. However, for eccentric and/or non-synchronous binaries, the shape and volume of the roche lobe also depend upon the eccentricity, the true anomaly, \( \nu \), and the rotation rate, which can be expressed by the parameter:

\[
A_e(e, f, \nu) = \frac{f^2 (1 + e)^4}{(1 + e \cos \nu)^3}
\]

(39)
are the initial synchronization timescales. As in Gokhale et al. (2007), we retain the scaling with mass ratio and orbital separation, but, because of the uncertainty of the overall magnitude of the timescale, we allow the magnitude to vary. We define the overall magnitude by the timescale at the moment of first contact and assume, as in Marsh et al. (2004) and Gokhale et al. (2007), that

\[ \tau_c = 1.3 \times 10^7 \left( \frac{M_A}{M_D} \right)^2 \left( \frac{a}{R_A} \right)^6 \frac{M_A/M_\odot}{L/A/L_\odot} \right)^{5/7} \quad \text{yr} \quad (40) \]

Following Marsh et al. (2004), we retain the scaling with mass ratio and orbital separation, but, because of the uncertainty of the overall magnitude of the timescale, we allow the magnitude to vary. We define the overall magnitude by the timescale at the moment of first contact and assume, as in Marsh et al. (2004) and Gokhale et al. (2007), that

\[ \tau_A(t) = C_A \left( \frac{M_A}{M_D} \right)^2 \left( \frac{a}{R_A} \right)^6 \quad (41) \]

and

\[ \tau_D(t) = C_D \left( \frac{M_D}{M_A} \right)^2 \left( \frac{a}{R_B} \right)^6 \quad (42) \]

where \( C_A \) and \( C_D \) are constants defined such that \( \tau_A(0) = \tau_{A,I} \) and \( \tau_D(0) = \tau_{D,I} \). Here, \( \tau_{A,I} \) and \( \tau_{D,I} \) are the initial synchronization timescales. As in Gokhale et al. (2007), we explore two different values for the synchronization timescale at contact: \( 10^{15} \) years (very weak tidal coupling) and 10 years (very strong tidal coupling). Much work on tidal synchronization in DWDs has been done (see, for example, Valsecchi et al. 2013, Fuller & Lai 2014, Burkart et al. 2014). These analyses have shown that shorter synchronization timescales may be a better approximation for systems with short orbital periods, as we consider in this analysis, which may lend credence to our 10 years synchronization timescale. For simplicity, we use the same initial timescale for the donor and accretor \( (\tau_{A,I} = \tau_{D,I}) \) and refer to the initial synchronization timescale for both stars as simply \( \tau_c \).

We have now determined the set of differential equations that govern the evolution of the DWDs undergoing direct-impact accretion, including the methods for calculating the Roche lobe size, the mass transfer rate, and the synchronization timescales, which determine the strength of tidal coupling. We now have all the steps in place to solve for the long-term evolution of the systems.

3. NUMERICAL SOLUTIONS

We integrate Equations (20), (28), (29), (31)–(33) using an 8th order Runge–Kutta ordinary differential equation solver (Galassi et al. 2003). Excluding the losses due to gravitational radiation, the total energy and momentum of the system are conserved throughout the integration over the entire parameter space to one part in \( 10^{-2} \) or better. At the end of the evolution, we calculate \( J_{\text{conserv}} = J_{\text{total}} - J_{\text{total,GR}} \) where \( J_{\text{total}} \) is the initial total angular momentum and \( J_{\text{total,GR}} \) is the total angular momentum lost to gravitational radiation. If angular momentum is perfectly conserved, we expect \( J_{\text{conserv}} \) to be equal to zero.

Figure 1 shows the final value of \( J_{\text{conserv}} \) for both synchronization timescales at contact over the entire parameter space of interest for solutions calculated using both the Eggleton approximation (left), \( R_{L,Egg} \) and \( R_{L,asynch} \) (right) for the calculation of the Roche lobe. The colors correspond to different values of \( J_{\text{conserv}} \) as described in the caption. In general, systems with a smaller total mass conserve total
angular momentum better than systems with a higher mass. For many of the systems in the lower right, this is due to the fact that they reach a stable configuration (disk accretion) early in the integration. Therefore, the integration needs to run for only a few orbits. Systems where \( J_\text{conserv} \) is larger tend to be systems where the numerical integration runs for many orbits, allowing systematic errors in the numerical integration to accumulate. Even so, all systems presented in this paper conserve total angular momentum to better than 1% throughout the entire evolution.

We integrate over a period of 1 Gyr. As in Marsh et al. (2004), if \( M_D \) exceeds 0.01 \( M_\odot \text{ yr}^{-1} \) at any point during the integration, the integration stops and the system is discarded as unstable.

As a system evolves, it is possible to pass back and forth through phases of mass transfer and phases of no mass transfer as the semi-major axis and the two masses change. If the semi-major axis increases enough for mass transfer to stop altogether, the integration proceeds (with \( M_D = 0 \)) until the action of gravitational radiation shrinks the orbit sufficiently for mass transfer to resume.

As in Paper I, in this work we are only interested in direct-impact mass transfer where the particle impacts the surface of the accretor within one orbital period. In this case, the evolution of the orbital parameters is determined by the differential equations presented in Section 2. If the particle does not accrete within one orbital period, it is likely that the accretion stream will eventually intersect with itself, ultimately leading to the formation of an accretion disk (Sepinsky et al. 2010).

Compared to direct-impact accretion, disk accretion is known to provide a much more efficient mechanism for redistributing spin angular momentum in the accretor back into the orbit through tidal coupling (Frank et al. 2002). As a result, it is likely that once a system enters a phase of disk accretion it will remain in this phase or perhaps even become detached as the orbital separation continues to grow and the mass ratio decreases due to continued mass transfer. If at any point during our numerical integrations a disk is formed under these circumstances, the integration stops and the system is assumed to be stable throughout its lifetime. A more thorough analysis, which continues to track the evolution through potential disk phases, will be performed in a future paper. This issue is discussed further in Section 4.

3.1. Maximum Spin-rate of the Accretor

For weak tidal coupling, it is possible for the accretor to be spun up to its breakup rate \( (\Omega_b = \sqrt{GM_\odot/R_\odot^2}) \). We track \( \Omega_b \) throughout our calculations and note that only one system ever reaches this maximum spin for the accretor. This system, which has an initial donor mass of 0.1 \( M_\odot \) and an initial accretor mass of 0.275 \( M_\odot \), is marked by a yellow dot in Figure 2. The accretor spin-rates in all other systems stay below the breakup rate throughout their evolution.

3.2. Super-Eddington Accretion

As noted by Marsh et al. (2004), there are likely ranges of parameter space where systems, despite being stable, experience super-Eddington accretion at some point during their evolution. It is expected that a possible ultimate consequence of sustained super-Eddington accretion is a merger (Han & Webbink 1999, Nelemans et al. 2001, Marsh et al. 2004), the same result as a dynamically unstable system, as they eventually can reach very high mass-transfer rates.

As in Marsh et al. (2004), we calculate the Eddington accretion rate using a modified form of the calculation used by Han & Webbink (1999):

\[
M_{\text{Edd}} = \frac{8\pi G m_p c M_A}{\sigma_T \left( \phi_{l1} - \phi_\alpha - \frac{1}{2} v_i^2 + \frac{1}{2} (v_i - v_o)^2 \right)},
\]

where \( \sigma_T \) is the Thomson cross-section of the electron, \( m_p \) is the mass of a proton, \( v_i \) is the impact velocity of the accreted particle, and \( v_o \) is the spin–velocity of the accretor’s surface at the point of impact, both measured in the co-rotating frame of reference.

In the Marsh et al. (2004) analysis, a set of impact velocities and locations were pre-computed and were interpolated during the calculations to calculate \( v_i \) and \( v_o \). Instead we are able to calculate \( v_i \) and \( v_o \) explicitly as part of the three-body integration of the mass transfer event. We track the mass-transfer and flag systems that eventually exceed the maximum mass-transfer as unstable.

4. RESULTS

Using the techniques described above, we computed evolution over a grid in \( M_A, M_D \) parameter space to determine the long-term stability of various systems. The grid was computed for two different tidal synchronization timescales at contact: \( 10^{15} \) years and 10 years.

4.1. Evolution of Systems Using Eggleton Roche Lobe

We first present the results of our numerical integrations using the standard Eggleton approximation for the calculation of the Roche lobe (Equation (38)), which does not take into
account asynchronicity between the donor and orbit. The Eggleton approximation was used in the analyses of both Marsh et al. (2004) and Gokhale et al. (2007).

Figure 2 shows the end result of systems with an initial synchronization timescale of \( \tau = 10^{15} \) years using \( R_{\text{Egg}} \). As mentioned in Section 3, we assume that once a disk is reached, the system will remain stable for the remainder of its evolution, therefore we stop the integration once a disk is formed. With the exception of the unstable black systems (whose evolution stops when the mass transfer rate exceeds the maximum limit of 0.01 \( M_\odot \text{yr}^{-1} \)), the evolution of all systems in this plot is stopped when the system reaches a phase of disk accretion. The time it takes to reach this phase of disk accretion varies from system to system.

The solid black line in Figure 2 shows the boundary between disk and direct-impact accretion for initially synchronous and circular binaries. All systems to the right of and below this line begin evolution in a disk phase. Therefore, these systems are categorized as stable systems increased compared to the analysis of Marsh et al. (2004) as observed by comparing the dotted versus dashed lines in Figures 2 and 3.

The sub-Chandrasekhar, sub-Eddington systems shown in red in Figures 2 and 3 are expected to remain stable throughout their lifetimes and are therefore characterized as likely AM CVn progenitors. As observed from the dotted and dashed lines in Figures 2 and 3, the region of parameter space occupied by such systems in Gokhale et al. (2007) (dashed line) increased the number of stable systems compared to Marsh et al. (2004) (dotted line); our analysis reveals a further increase in the extent of this region of parameter space. This observation holds true for both the 10\(^{15}\) years timescale (Figure 2) and the 10 year timescale (Figure 3).

Gokhale et al. (2007) noted that by allowing the spin of the donor to vary and by including the resulting effects of tidal coupling between the donor’s spin and the orbit, the number of stable systems increased compared to the analysis of Marsh et al. (2004) (as observed by comparing the dotted versus dashed lines in both Figures 2 and 3). As described in Section 2.4, the main difference between our analysis and that of Gokhale et al. (2007) is in the treatment of the way we handle angular momentum evolution during the direct impact accretion process. In observing the increase in the number of stable systems compared to the analysis of Gokhale et al. (2007), we conclude that by utilizing a mass transfer treatment that allows the rotation rates of both components to vary and self-consistently accounts for the exchange of angular momentum between the spins of the components and the orbit, we are able to increase the number of stable systems.

Figures 2 and 3, produced using \( R_{\text{L,asynch}} \) for the Roche lobe calculation, isolate the effect that our different treatment of the mass transfer process has upon the stability of systems in comparison with Marsh et al. (2004) and Gokhale et al. (2007). We will discuss the differences resulting from using \( R_{\text{L,asynch}} \) in Section 4.4.

Forming a boundary between the stable (red and blue) and unstable (black) systems are the green and magenta systems, which experience super-Eddington accretion at some point during the integration. Unlike the black systems, whose mass transfer exceeds the allowed limit of 0.01 \( M_\odot \text{yr}^{-1} \) and are therefore categorized as unstable, these green/magenta super-Eddington systems are categorized as stable. However, it is likely that systemic mass loss is needed for a binary to survive phases of super-Eddington accretion. Han & Webbink (1999) argues that such mass loss may lead to a common envelope surrounding the binary, which will ultimately lead to a merger. In this case, systems that experience super-Eddington accretion would be considered unstable systems. The analyses of Marsh et al. (2004) and Gokhale et al. (2007) both note that super-Eddington systems are expected to be unstable. We make the same assumption but note that a more thorough treatment of the physics governing systems as they pass through any phases of super-Eddington accretion is necessary to state with certainty whether or not such systems are ultimately stable or unstable.

The green super-Eddington systems have a total mass, which is sub-Chandrasekhar, and the magenta super-Eddington systems are super-Chandrasekhar. If we assume super-Eddington systems are ultimately unstable, the magenta systems can be categorized as possible SNe Ia progenitors.

As the tidal synchronization timescale is reduced from \( \tau = 10^{15} \) years in Figure 2 to \( \tau = 10 \) years in Figure 3, we see the parameter space occupied by stable systems grows (red). This is to be expected and in agreement with the analyses of Marsh et al. (2004) and Gokhale et al. (2007). Stronger tidal coupling will allow more spin angular momentum from the accretor to be transferred back into the orbit, which increases the semi-major axis, causing the mass-transfer rate to decrease. This results in an increase the stability of the systems in general.

### 4.2. Disk Accretion

In both Figures 2 and 3 we see a portion of stable, super-Chandrasekhar, sub-Eddington systems (blue) in the top right corners of our plots. In the analyses of both Marsh et al. (2004) and Gokhale et al. (2007), this portion of parameter space is occupied by super-Eddington systems, which would be magenta circles in our plots. Notice that the dotted and dashed lines of Marsh et al. (2004) and Gokhale et al. (2007) decrease as we move right across the plot, whereas our stability boundary is more extended. This is due to the fact that we do not follow the evolution of the systems past the development of disk accretion, while Marsh et al. (2004) and Gokhale et al. (2007) both continue to track the evolution of systems through phases of disk accretion.
Here we acknowledge the main limitation of our decision to assume that systems are stable once they reach a phase of disk accretion. The blue systems in the region of parameter space of interest here begin the integration in a disk phase, and are therefore labeled as stable. However, it is feasible that if these high donor-mass systems were allowed to evolve through the initial disk phase, they would eventually become super-Eddington and potentially unstable. In this case, this region of parameter space would more closely conform with the shape of Marsh et al. (2004) and Gokhale et al. (2007) analyses, both of which allow the systems to continue evolving through any phases of disk accretion. As we noted earlier, a more thorough analysis of this region of parameter space, which includes treatment of disk accretion, will be studied in a forthcoming paper.

4.3. Comparison of System Evolution with Different Tidal Timescales

Figure 4 shows the evolution in time of a system with initial masses of $M_A = 0.8 M_\odot$ and $M_B = 0.38 M_\odot$ for the two different synchronization timescales at contact as well as a third much stronger tidal timescale, $\tau = 0.1$ years, for further illustration. These solutions are obtained using $R_{L,Eg}$ to calculate the Roche lobe. For $\tau = 10^{15}$ years, the system is unstable, for $\tau = 10$ years the system is stable, but passes through a phase of super-Eddington accretion, and for $\tau = 0.1$ years the system is stable and sub-Eddington.

For weak tidal coupling ($\tau = 10^{15}$ years), shown in green, mass transfer causes the rotation rates of the donor and accretor to rapidly decrease and increase, respectively. For the case of stronger tidal coupling ($\tau = 10$ years and $\tau = 0.1$ years; shown in red and green, respectively), tidal forces exist to redistribute angular momentum, working to keep the spins of the donor and accretor synchronous with the orbit ($f_i = 1$).

The rapid increase in the semi-major axis for $\tau = 10^{15}$ years is a direct result of the conservation of angular momentum: the angular momentum lost during the spin-down and mass loss of the donor is greater than the angular momentum gained by the accretor. The net decrease in spin angular momentum corresponds to an increase in the orbital angular momentum, increasing the semi-major axis. The mass transfer rate increases rapidly as the mass loss from the donor causes the radius to increase, even as the Roche lobe grows due to the increasing semi-major axis (see Equations (34) and (35)).

For the case of strong tidal coupling ($\tau = 10, 0.1$ year) we see a significant decrease in the mass transfer rate, with a correspondingly slower increase in the semi-major axis. The rotation rates of the component white dwarfs remain closer to synchronous, and we do not see the rapid runaway that is evident in the case of weak tidal coupling.

Finally, we note that oscillations in the orbital parameters are seen in the $\tau = 10$ years case, which have been observed before by Gokhale et al. (2007). These oscillations are sensitive to small changes in the initial orbital parameters. We discuss this phenomenon further in Section 4.5.

4.4. Evolution of Systems Including the Effect of Asynchronism on the Roche-lobe Size

Here, we show the results of our numerical integrations using $R_{L,asyn}$ to calculate the Roche lobe size. As discussed in Section 2.6, the Eggleton approximation is dependent upon only the mass ratio of the binary, whereas $R_{L,asyn}$ accounts for deviations from synchronism.

Figure 5 shows the end results of the systems with initial synchronization timescales of $\tau = 10^{15}$ years and evolved using $R_{L,asyn}$. The color code remains the same as in Figures 2 and 3 and the dotted and dashed lines from Marsh et al. (2004) and Gokhale et al. (2007), respectively, for the $10^{15}$ year timescale are included for reference. As can be seen, this plot is nearly identical to Figure 2, which was calculated using the Eggleton approximation.

Figure 6 shows the end results of the systems that begin with synchronization timescales of $\tau = 10$ years. By comparing with Figure 3, we observe that, unlike the $10^{15}$ years timescale, $R_{L,asyn}$ has a significant effect upon the end-states of the systems for the 10 years timescale. In particular, the number of unstable black systems is reduced substantially.

![Figure 4](image-url)
The cause of this difference is explained in details below (Section 4.5).

In Figure 6, we also observe a slight increase in the number of stable systems across the entire range of accretor masses compared to Figure 3. Additionally, there is a small group of stable red systems beyond the widely used \( q > \frac{2}{3} \) boundary for instability (for \( q > \frac{2}{3} \), the donor radius will expand faster than the Roche lobe, which suggests, simplistically, that such systems will become unstable). However, it is possible for systems with an initial mass ratio higher than \( \frac{2}{3} \) to survive, provided that the mass ratio is close to the boundary (see, for example, D’Souza et al. 2006). Our results, as shown in Figure 6, confirm this observation.

### 4.4.1. Comparison of the Evolution of Systems Using \( R_{L,Egg} \) and \( R_{L,asynch} \) Solutions for \( \tau = 10^{15} \) years

Figure 7 compares the evolution of a system with an initial donor mass of \( 0.3 \, M_\odot \) and an initial accretor mass of \( 0.8 \, M_\odot \) for \( \tau = 10^{15} \) years. On left we show the evolution using \( R_{L,asynch} \) instead of \( R_{L,Egg} \). As seen on the plots, the evolution of the orbital parameters happens on a much shorter timescale for the \( R_{L,Egg} \) solution than for the \( R_{L,asynch} \) solution. This is due to the fact that, as mass is transferred from the donor to the accretor, the rotation rate of the donor decreases relative to the orbit and the rotation rate of the accretor increases. As the donor spin \( (f_D) \) decreases, \( A \) (which is proportional to \( f_D^2 \); Equation \( 39) \) also decreases. When using \( R_{L,asynch} \), the size of the Roche lobe is inversely proportional to \( A \). So, as \( A \) decreases, the Roche lobe will increase, which ultimately reduces the mass transfer rate. However, in the \( R_{L,Egg} \) calculation, the Roche lobe is insensitive to changes in \( f_D \), so the mass transfer rate is higher relative to that calculated in the solution using \( R_{L,asynch} \). Since the mass transfer rate is higher in the Eggleton case, the orbital parameters change more quickly than in the \( R_{L,asynch} \) case, as seen in Figure 7.

### 4.4.2. Comparison of the Evolution of Systems Using \( R_{L,Egg} \) and \( R_{L,asynch} \) Solutions for \( \tau = 10 \) years

Figure 8 compares the evolution of a system with an initial donor mass of \( 0.38 \, M_\odot \) and an initial accretor mass of \( 0.8 \, M_\odot \) with \( \tau = 10 \) years for the first 1000 years of evolution. Figure 9 shows the full evolution of this system, which evolved out to approximately \( 6 \times 10^4 \) years before reaching a phase of disk accretion. As in Figure 7, the left panel shows the evolution using \( R_{L,Egg} \) and the right panel shows the evolution using \( R_{L,asynch} \). As can be seen in Figures 3 and 5, this system is stable under the \( R_{L,asynch} \) calculation and super-Eddington under the \( R_{L,Egg} \) calculation. Unlike the \( 10^{15} \) years timescale, for the 10 years timescale, the orbital parameters follow approximately the same path in both the \( R_{L,Egg} \) and \( R_{L,asynch} \) cases, with the exception being the oscillations seen when using \( R_{L,Egg} \). We discuss these oscillations in more detail below.
4.5. Oscillations of Orbital Parameters

As seen in Figure 8, using $R_{\text{L, asynch}}$ to calculate the Roche lobe damps the oscillations observed when using the $R_{\text{L,Egg}}$. In both cases, the mass transfer rate increases initially, which causes the accretor to spin up and the donor to spin down. In the $R_{\text{L, asynch}}$ case, this causes the mass transfer rate to increase at a slower rate relative to the $R_{\text{L,Egg}}$ case.\(^7\) As discussed above, when using $R_{\text{L,Egg}}$, the mass transfer rate will continue to increase until tides have transferred sufficient angular momentum from the spin of the components to the orbit to widen the orbit and reduce the mass transfer rate. Once this occurs, the mass transfer rate is reduced and tides continue to redistribute angular momentum between the component spins and the orbit (in this case the donor is spun up while the accretor is spun down). But again, because changes in the donor spin are not accounted for when using $R_{\text{L,Egg}}$, the mass transfer rate will again “over shoot” and the oscillations continue. The dependence of the Roche lobe radius upon the donor spin when using $R_{\text{L, asynch}}$ slows the changes in the mass transfer rate and damps the oscillations.

Although Figure 8 shows that the short-term evolution of this particular system differs substantially between the $R_{\text{L,Egg}}$ and $R_{\text{L, asynch}}$ Roche lobe calculations, Figure 9 shows that the long-term evolution of this system is very similar. Although the oscillations do not impact the long-term evolution of many of the systems, the oscillations do impact whether a system is categorized as sub-Eddington or super-Eddington or whether it reaches unstable levels of mass transfer. In the case of the system illustrated in Figures 8 and 9, the oscillations cause the system briefly to become super-Eddington, while the system would remain sub-Eddington in the absence of oscillations.

In general, as the mass transfer rates spike initially under the $R_{\text{L,Egg}}$ calculation before ultimately settling to the $R_{\text{L, asynch}}$ value, it may reach super-Eddington or unstable levels, while, in the $R_{\text{L, asynch}}$ case, the mass transfer rate may remain stable instead of super-Eddington, or super-Eddington instead of unstable. This explains the lack of unstable black systems in Figure 5 compared to Figure 3. For the systems shown in Figure 3, the oscillations in the orbital parameters arising from the use of $R_{\text{L,Egg}}$ may cause the mass transfer rate to reach artificially high values before the oscillations relax, leading to the large population of black systems at the top of the figure. Meanwhile, since using $R_{\text{L, asynch}}$ damps the oscillations, the mass transfer rate is able to stay below the limit for stability of $0.01 M_\odot$ yr\(^{-1}\).

Both Marsh et al. (2004) and Gokhale et al. (2007) noted the presence of oscillations in the orbital parameters near the boundary between stable and unstable systems. These analyses observed that, for such systems, small changes in the synchronization timescales and in the mass ratio will cause changes in the frequency and amplitude of the oscillations.

---

\(^7\) As described previously, as $f_D$ decreases, $A$ decreases, which causes the Roche lobe to increase and reduces the mass transfer rate.
Figures 10 and 11 show the evolution of the orbital period for various synchronization timescales and mass ratios, respectively, using both $R_{\text{L, Egg}}$ and $R_{\text{L, asynch}}$ to calculate the size of the Roche lobe. These figures confirm the observations of Marsh et al. (2004) and Gokhale et al. (2007) that the amplitude and frequency of the oscillations is dependent upon both the mass ratio and the synchronization timescale.

The left-hand panel of Figure 10 illustrates that the amplitude of the oscillations increases with increasing initial synchronization timescale, while the frequency of the oscillations decreases. These oscillations can be explained as follows: Once mass transfer begins, if the system is unstable or nearly unstable, the timescale for changes in the mass transfer rate becomes greater than the timescale at which the semi-major axis increases due to tides. As a result, $M_p$ increases rapidly and the accretor spin increases significantly. The result is a large asynchronicity between the accretor, the orbit, and the donor. As the orbital separation increases due to the effect of tides, $M_p$ decreases, until the timescale for changes in the mass transfer rate becomes less than the timescale at which the semi-major axis increases due to tides. At this point, tides are able to redistribute angular momentum from the spin of the accretor back into the orbit. If enough angular momentum is transferred back into the orbit, the donor becomes detached. Once tides effectively re-synchronize the spins of the stars with the orbit, gravitational wave losses act to bring the binary back into contact, and the entire cycle repeats.

The right-hand panel of Figure 10 demonstrates that, for this particular mass ratio, the oscillations are not present when using $R_{\text{L, asynch}}$ to calculate the size of the Roche lobe. Furthermore, the right-hand panel illustrates that, in the absence of oscillations, the orbital period, and therefore, the semi-major axis increases more rapidly for lower $\tau$ values of the synchronization timescale. For lower $\tau$ values, the tidal coupling is stronger, which means spin angular momentum is able to be re-distributed more efficiently, which allows the semi-major axis to remain larger compared to the semi-major axis of systems evolved using higher values of $\tau$.

Figure 11 demonstrates the effect of mass ratio upon the amplitude of the oscillations. As Gokhale et al. (2007) observes, we see that for a fixed $\tau$ value, a lower mass ratio (top two graphs of Figure 11) yields oscillations of lower amplitude compared to a higher mass ratio (lower two graphs of Figure 11). In Figure 3, we see that switching from a mass ratio of 0.475 to 0.425 moves us away from the boundary which separates super-Eddington systems from stable systems. This demonstrates that, in general, we move away from the stability boundary, the oscillations are reduced.

The two right-hand panels of Figure 11 show the evolution using $R_{\text{L, asynch}}$ for the Roche lobe. This again demonstrates that oscillations are not present when the Roche lobe depends on the asynchronicity between the donor and orbit.

5. CONCLUSIONS

We have studied the long-term evolution of DWD binary systems undergoing direct-impact mass transfer, including the effects due to mass transfer, gravitational radiation, and tidal forces arising from asynchronicity between the donor, accretor, and the orbit. We implemented the ballistic mass-transfer treatment developed in (Sepinsky et al. 2010) to calculate the changes to orbital parameters during direct-impact mass transfer. By implementing this method, we found that the number of stable DWD systems increased for both weak and strong tidal coupling compared with the results of Marsh et al. (2004) and Gokhale et al. (2007), as shown in Figures 2 and 3.

For the first time, we also account for the modification of the Roche-lobe size due to the asynchronicity of the donor. As a result, we find that the number of stable systems increases, particularly for the case of strong tidal coupling, as shown in Figures 5 and 6. When not accounting for the asynchronicity effects on the Roche-lobe size, we reproduce the oscillations in the orbital parameters first noted by Gokhale et al. (2007). We found that, when the size of the Roche lobe was permitted to vary with donor asynchronicity, these oscillations were dampened, as shown in Figure 8. We conclude that the oscillations created when using the Eggleton approximation for the Roche lobe calculation create artificially high mass transfer rates, which lead to an artificially high number of super-Eddington and unstable systems. By eliminating the oscillations, our treatment yields a higher number of sub-Eddington and stable systems.
We expect systems that are stable throughout our calculations (red systems in Figures 2, 3, 5, and 6) to be AM CVn progenitors. As a result of the increase in stable systems shown here compared to previous analyses, we conclude that DWD evolution may be a more likely avenue for the creation of AM CVn than previously expected.

In future analyses, we intend to investigate the case of initially eccentric binaries with asynchronous component stars, as we expect there may be a significant population of DWDs that have not had time to circularize and synchronize by the time mass transfer occurs (Willems et al. 2007). Additionally, we intend to implement a treatment of disk accretion so that we can track the evolution of the DWD systems through all types of accretion processes.

K. K. and V. K. acknowledge support from Northwestern University that made this project possible. V. K. is also grateful for the hospitality of the Aspen Center for Physics. This work used computing resources at CIERA funded by NSF PHY-1126812.

REFERENCES

Bours, M. C. P., Marsh, T. R., Parsons, S. G., et al. 2014, MNRAS, 438, 3399
Burkart, J., Quataert, E., Arras, P., et al. 2014, MNRAS, 443, 2957
Campbell, C. G. 1984, MNRAS, 1984, 433
Chandrasekhar, S. 1967, An Introduction to the Study of Stellar Structure (New York: Dover)
Eggleton, P. P. 1983, ApJ, 268, 368
D’Souza, M. C. R., Motl, P. M., Tohline, J. E., & Frank, J. 2006, ApJ, 643, 381
Frank, J., King, A., & Raine, D. 2002, Accretion Power in Astrophysics (Cambridge: Cambridge Univ. Press)
Fuller, J., & Lai, D. 2014, MNRAS, 444, 3488
Galassi, M. J. D., Theiler, J., Gough, B., et al. 2003, GNU Scientific Library Reference Manual (2nd ed.; Bristol: Network Theory Ltd.)
Gokhale, V., Peng, X. M., & Frank, J. 2007, ApJ, 655, 1010
Han, Z., & Webbink, R. F. 1999, A&A, 349, L17
Hils, D., Bender, P. L., & Webbink, R. F. 1990, ApJ, 360, 75
Hils, D., & Bender, P. L. 2000, ApJ, 537, 334
Hut, P. 1981, A&A, 99, 126
Kruszewski, A. 1963, AcA, 13, 106
Landau, L. D., & Lifshitz, E. M. 1975, The Classical Theory of Fields (Oxford: Pergamon)
Maoz, D., Mannucci, F., & Nelemans, G. 2014, AARA, 52, 107
Marsh, T. R., Dhillon, V. S., Duck, S. R., et al. 1995, MNRAS, 275, 828
Marsh, T. R., Nelemans, G., Steeghs, D., et al. 2004, MNRAS, 350, 113
Motl, P. M., Frank, J., Tohline, J. E., & D’Souza, M. C. R. 2007, ApJ, 670, 1314
Nather, R. E., Robinson, E. L., & Stover, R. J. 1981, ApJ, 244, 269
Nelemans, G., Yungelson, L. R., Portegies Zwart, S. F., et al. 2001, A&A, 375, 890
Paczynski, B., & Sienkiewicz, R. 1972, AcA, 22, 73
Plavec, M. 1958, MSRSL, 20, 411
Savonije, G. J. 1978, A&A, 62, 517
Sepinsky, J., & Kalogera, V. 2014, ApJ, 785, 157 (Paper I)
Sepinsky, J., Willems, B., Kalogera, V., et al. 2007, ApJ, 660, 1624
Sepinsky, J., Willems, B., Kalogera, V., & Rasio, A. F. 2010, ApJ, 724, 546
Shen, K. 2015, ApJL, submitted (arXiv:1502.05052)
Soberman, G. E., Phinney, G. S., van den Huevel, E. P. J., et al. 1997, A&A, 327, 620
Tutukov, A., & Yungelson, L. 1996, MNRAS, 280, 1035
Valsecchi, F., Farr, W. M., Willems, B., Rasio, F. A., & Kalogera, V. 2013, ApJ, 773, 39
Verbunt, F., & Rappaport, S. 1988, ApJ, 332, 193
Webbink, R. F. 1977, ApJ, 211, 486
Webbink, R. F. 1984, ApJ, 277, 355
Willems, Kalogera, V., Vecchio, A., et al. 2007, ApJ, 665, L59
Woosley, S. E., & Weaver, T. A. 1986, ARA&A, 24, 205