Dilation operators in Besov spaces over local fields

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Abstract

We consider a dilation operator on Besov spaces \(B^{s}_{r,t}(K)\) over local fields and estimate an operator norm on such a field for \(s > \sigma_{r} = \max\left(\frac{1}{r} - 1, 0\right)\) which depends on the constant \(k\) unlike the case of Euclidean spaces. In \(\mathbb{R}^n\), it is independent of constant \(k\), the constant appears for limiting case \(s = 0\) and \(s = \sigma_{r}\). In local fields, the limiting case is still open.

Keywords  Local fields · Besov spaces · Dilation operators

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1 Introduction

Besov spaces \(B^{s}_{pq}(\mathbb{R}^n)\) and Triebel–Lizorkin spaces \(F^{s}_{pq}(\mathbb{R}^n)\) with \(s \in \mathbb{R}, 0 < p \leq \infty (p < \infty \text{ for the F-type spaces}), 0 < q \leq \infty\) on the Euclidean spaces \(\mathbb{R}^n\) are usually considered to be two very general scales of function spaces. These two spaces \(B^{s}_{pq}(\mathbb{R}^n)\) and \(F^{s}_{pq}(\mathbb{R}^n)\) contain well-known classical function spaces as special cases, such as Lebesgue spaces \(L_p = F^{0}_{p,q}(\mathbb{R}^n)\) (with \(1 < p < \infty\)), Sobolev spaces \(W^m_p = \leftarrow \text{ and } \rightarrow \text{ are special cases.}\) These function spaces have been extensively studied by Triebel, we refer to [21, 22] for more details. The theory of partial differential equation is one of the main applications of these two spaces. Dilation operator has been investigated on many function spaces, including the Besov spaces. We state the following results summarizing the behavior of dilation operator on Besov spaces.

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In [21, 3.4], Triebel considered the dilation operators of the form
\[(T_k f)(x) = f(2^k x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}, \quad (1.1)\]
and showed that for \(0 < p, q \leq \infty\), and \(\infty > s > \sigma_p = \max(n\left(\frac{1}{p} - 1\right), 0)\), \(T_k\) represent bounded operators from \(B^s_{pq}(\mathbb{R}^n)\) into itself and in this case we have
\[
\|T_k f\|_{B^s_{pq}(\mathbb{R}^n)} \leq c 2^{k(s-n/p)} \|f\|_{B^s_{pq}(\mathbb{R}^n)}.
\]

Boundedness of \(T_k\) for the limiting case \(s = 0\) and \(s = \sigma_p\) remained open until Vybíral in [26] and Schneider in [15], respectively, gave the final answer and showed that the additional constant \(k\) will appear in the calculation of the norm of dilation operator \(T_k\). The norm calculated by Vybíral in [26] is as follows.

For \(0 < q \leq \infty\),
\[
\|T_k \| \mathcal{L}(B^0_{pq}(\mathbb{R}^n)) \sim 2^{-k \frac{n}{p}} \begin{cases} \frac{1}{k^\frac{1}{q}} & \text{if } 1 < p < \infty, \\ \frac{1}{k^\frac{1}{p}} & \text{if } p = 1 \text{ or } p = \infty, \end{cases}
\]
and by Schneider [15] is given by,
\[
\|T_k \| \mathcal{L}(B^\sigma_{pq}(\mathbb{R}^n)) \sim 2^{k(\sigma - \frac{n}{p})} \frac{1}{k^\frac{1}{q}}, \quad \text{for } 0 < p \leq 1 \text{ and } 0 < q \leq \infty,
\]
where \(\|T_k \| \mathcal{L}(B^s_{pq}(\mathbb{R}^n))\) denotes the norm of the operator \(T_k\) from \(B^s_{pq}(\mathbb{R}^n)\) into itself.

The boundedness of the dilation operators are studied not only for theoretical aspect, but it also has major applications to several classical problems in analysis. For instance, they appear in the localization of \(B^s_{pq}\) and \(F^s_{pq}\) spaces and Hölder inequalities and sharp embeddings of \(B^s_{pq}\) and \(F^s_{pq}\) spaces (see [7, chap 2], [16, 23]).

The main aim of the present paper is to study dilation operators in the framework of Besov spaces on local fields. In particular, we extend the results obtained by Triebel in [21, 3.4]. A local field \(K\) is a locally compact, totally disconnected, non-Archimedean norm valued and non-discrete topological field, we refer [20] to basic Fourier analysis on local fields. The local fields are essentially of two types (excluding the connected local fields \(\mathbb{R}\) and \(\mathbb{C}\)) namely characteristic zero and of positive characteristic. The characteristic zero local fields include the \(p\)-adic field \(\mathbb{Q}_p\) and the examples of positive characteristic are the Cantor dyadic groups, Vilenkin \(p\)-groups and \(p\)-series fields.

Besov spaces \(B^s_{pq}(K)\) and Triebel–Lizorkin spaces \(F^s_{pq}(K)\) on local fields were introduced and studied by Onneweer and Weiyi [13]. In [17], Su, introduced, “\(p\)-type smoothness” of the functions defined on local fields. To complete the theoretical base of the Function spaces on local fields and to broaden the range of its applications, a series of studies have been carried out such as the construction theory of functions on local fields [19], the Weierstrass functions, Cantor functions and their \(p\)-adic derivatives on local fields, the Lipschitz classes on local fields etc. (see [18]). Operator theory on local fields, is quite new and lots of new topics are worth to study. Boundedness of some
fundamental operators in Harmonic analysis, like Hardy operator, Hausdorff operator, maximal operator and singular integral operator on function spaces over local fields have been studied by many researchers (see [1, 6, 8, 9, 12, 14, 24, 25]).

Wavelet theory on local fields has also been developed widely. Jiang, Li and Jin [10] have introduced the concept of multiresolution analysis and wavelet frames on local fields [11]. Later, Behera and Jahan have developed the theory of wavelets on such a field in a series of papers [2–4]. We refer [5] for more details of wavelets on local fields.

This article is organized as follows. In Sect. 2, we provide a brief introduction to local fields and test function class and distribution on such a field. We have also provided the definition of Besov spaces $B^s_{rt}(K)$ on local fields. In Sect. 3, we study the dilation operators

$$ (T_k f)(x) = f(p^{-k} x), \quad x \in K, \quad k \in \mathbb{N}, $$

in the framework of Besov spaces $B^s_{rt}(K)$ and more precisely, we shall prove the following result:

**Theorem 1.1** Let $0 < r, t \leq \infty, s > \sigma_r = \max \left( \frac{1}{t} - 1, 0 \right)$. For $k \in \mathbb{N}$, $T_k$ is defined by (1.2). Then

$$ \| T_k f \|_{B^s_{rt}(K)} \leq (c_2 + c_1 k^{1/t}) q^{k(s-\frac{1}{r})} \| f \|_{B^s_{rt}(K)}, $$

for some $c_1, c_2$ which are independent of $k$ and for all $f \in B^s_{rt}(K)$.

Note that, in the case of $B^s_{rt}(K)$ (with $s > \sigma_r$), Besov norm of operators $T_k$ (defined by (1.2)) depends on $k$, whereas in the case of $B^s_{pq}(\mathbb{R}^n)$, Besov norm of operators $T_k$ (defined by (1.1)) is independent of the constant $k$ for $s > \sigma_r$, see [21], in the case of Euclidean spaces it depends on $k$ in the limiting case $s = 0$ and $s = \sigma_p$, see [23] for more details. For local fields, the limiting case is still open.

## 2 Preliminaries

### 2.1 Local fields

Let $K$ be a field and a topological space. If the additive group $K^+$ and multiplicative group $K^*$ of $K$ both are locally compact Abelian group, then $K$ is called locally compact topological field, or local field.

If $K$ is any field and is endowed with the discrete topology, then $K$ is a local field. Further, if $K$ is connected, then $K$ is either $\mathbb{R}$ or $\mathbb{C}$. If $K$ is not connected, then it is totally disconnected. So, here local fields means a field $K$ which is locally compact, non-discrete and totally disconnected. If $K$ is of characteristic zero, then $K$ is either a $p$-adic field for some prime number $p$ or a finite algebraic extension of such a field. If $K$ is of positive characteristic, then $K$ is either a field of formal Laurent series over a finite field of characteristic $p$ or an algebraic extension of such a field.
Let $K$ be a local field. Since $K^+$ is a locally compact Abelian group, we choose a Haar measure $dx$ for $K^+$. If $\alpha \neq 0$, $\alpha \in K$, then $d(\alpha x)$ is also a Haar measure and by the uniqueness of Haar measure we have $d(\alpha x) = |\alpha|dx$. We call $|\alpha|$ the absolute value or valuation of $\alpha$ which is, in fact, a natural non-Archimedean norm on $K$. A mapping $| \cdot | : K \to \mathbb{R}$ satisfies:

(a) $|x| = 0$ if and only if $x = 0$;
(b) $|xy| = |x||y|$ for all $x, y \in K$;
(c) $|x + y| \leq \max\{|x|, |y|\}$, for all $x, y \in K$.

The property (c) is called the ultrametric inequality. It follows that

$$|x + y| = \max\{|x|, |y|\} \text{ if } |x| \neq |y|.$$ 

The set $\mathcal{O} = \{x \in K : |x| \leq 1\}$ is called the ring of integers in $K$. It is the unique maximal compact subring of $K$. The set $\mathfrak{P} = \{x \in K : |x| < 1\}$ is called the prime ideal in $K$. It is the unique maximal ideal in $\mathcal{O}$. It can be proved that $\mathcal{O}$ is compact and open and hence $\mathfrak{P}$ is also compact and open. Therefore, the residue space $\mathcal{O}/\mathfrak{P}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime number $p$ and $c \in \mathbb{N}$.

Let $p$ be a fixed element of maximum absolute value in $\mathfrak{P}$ and it is called a prime element of $K$. For a measurable subset $E$ of $K$, let $|E| = \int_K 1_E(x)dx$, where $1$ is the indicator function of $E$ and $dx$ is the Haar measure of $K$ normalized so that $|\mathcal{O}| = 1$. Then it is easy to prove that $|\mathfrak{P}| = q^{-1} = |p| = p^{-1}$ (see [20]).

Let $\mathcal{U} = \{a_i\}_{i=0}^{r-1}$ be any fixed full set of coset representative of $\mathfrak{P}$ in $\mathcal{O}$ then each $x \in K$ can be written uniquely as $x = \sum_{i=k}^{\infty} c_i p^i$, where $c_i \in \mathcal{U}$. If $x(\neq 0) \in K$ then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathcal{O}^* = \mathcal{O} \setminus \mathfrak{P} = \{x \in \mathcal{O} : |x| = 1\}; \mathcal{O}^*$ is the group of units in $K^*$. If $x \neq 0$, we can write as $x = p^k x', x' \in \mathcal{O}^*$. Let $\mathfrak{P}^k = p^k \mathcal{P} = \{x \in K : |x| \leq q^{-k}\}, k \in \mathbb{Z}$. These are called fractional ideals.

The set $\{\mathfrak{P}^k \subset K : k \in \mathbb{Z}\}$ satisfies the following:

(i) $\{\mathfrak{P}^k \subset K : k \in \mathbb{Z}\}$ is a base for neighborhood system of identity in $K$, and $\mathfrak{P}^{k+1} \subset \mathfrak{P}^k, k \in \mathbb{Z}$;
(ii) $\mathfrak{P}^k, k \in \mathbb{Z},$ is open, closed and compact in $K$;
(iii) $K = \bigcup_{k=-\infty}^{+\infty} \mathfrak{P}^k$ and $\{0\} = \bigcap_{k=+\infty}^{-\infty} \mathfrak{P}^k$.

Let $\Gamma$ be the character group of the additive group $K^+$. There is a nontrivial character $\chi \in \Gamma$ which is trivial on $\mathcal{O}$ but is non-trivial on $\mathfrak{P}^{-1}$. Let $\chi$ be a non-trivial character on $K^+$, then the corresponding relationship $\lambda \in K \longleftrightarrow \chi_\lambda \in \Gamma$ is determined by $\chi_\lambda(x) = \chi(\lambda x)$, and the topological isomorphism is established for $K$ and $\Gamma$, moreover, we have $\Gamma = \{\chi_\lambda : \lambda \in K\}$.

For $k \in \mathbb{Z}$, let $\Gamma^k$ be the annihilator of $\mathfrak{P}^k$, that is,

$$\Gamma^k = \{\chi \in \Gamma : \forall x \in \mathfrak{P}^k \implies \chi(x) = 1\},$$

subset in the character group $\Gamma$.  

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For $k \in \mathbb{Z}$, let $\chi \in \Gamma^k \subset \Gamma$ which implies that $\chi \in \Gamma$ and therefore $\chi = \chi_\lambda$ for some $\lambda \in K$. For all $x \in \mathbb{P}^k$, we have $\chi_\lambda(x) = 1$, i.e., $\chi(\lambda x) = 1$, since a character $\chi$ is trivial on $\mathcal{O}$, this implies $\lambda x \in \mathcal{O}$ and hence $|\lambda| \leq q^k$, i.e., $\lambda \in \mathbb{P}^{-k}$. Therefore, every $\chi \in \Gamma^k$ is of the form $\chi_\lambda$ with $|\lambda| \leq q^k$. Conversely, if we take any $\chi_\lambda \in \Gamma$ with $|\lambda| \leq q^k$ then $\chi_\lambda(x) = \chi(\lambda x) = 1$, $\forall x \in \mathbb{P}^k$ which implies that $\chi_\lambda \in \Gamma^k$. Hence, we can write

$$\Gamma^k = \{\chi_\lambda \in \Gamma : |\lambda| \leq q^k\}, \quad k \in \mathbb{Z}.$$  

For the Haar measure $dx$ of $K^+$, let $d\xi$ be the Haar measure on $\Gamma$, chosen such that

$$|\Gamma^0| = 1, \quad \text{and} \quad |\Gamma^k| = q^k.$$  

We refer to [18, 20] for details of local fields and proof of statements discussed in this section.

**Test function class $S(K)$:** It is the linear space in which functions have the form

$$\phi(x) = \sum_{j=1}^{n} c_j \Phi_{\mathbb{P}^j}(x - h_j), \quad c_j \in \mathbb{C}, h_j \in K, j \in \mathbb{Z}, n \in \mathbb{N},$$

where $\Phi_{\mathbb{P}^j}$ is the characteristic function of $\mathbb{P}^j$. The space $S(K)$ is an algebra of continuous functions with compact support that separates points. Consequently, $S(K)$ is dense in $C_0(K)$ as well as $L^r(K)$, $1 \leq r < \infty$. Similarly, the test function class $S(\Gamma)$ on $\Gamma$ can be defined. However, since $K$ is isomorphic to $\Gamma$, so $S(K)$ and $S(\Gamma)$ can be regarded as equivalent with respect to absolute value or valuation.

The space $S(K)$ is equipped with a topology as a topological vector space as follows: Define a null sequence in $S(K)$ as a sequence $\{\phi_n\}$ of functions on $S(K)$ in such a way that each $\phi_n$ is constant on cosets of $\mathbb{P}^l$ and is supported on $\mathbb{P}^k$ for a fixed pair of integers $k$ and $l$ and the sequence converges to zero uniformly. The space $S(K)$ is complete and separable and is called the space of testing function.

Since $S(K)$ is dense in $L^1(K)$, thus the Fourier transformation of $\phi(x) \in S(K)$ is defined by

$$\hat{\phi}(\xi) \equiv (\mathcal{F}\phi)(\xi) = \int_{K} \phi(x) \overline{\chi_\xi(x)} dx, \quad \xi \in \Gamma,$$

and the inverse Fourier transformation of $\phi \in S(K)$ is defined by the formula

$$\tilde{\phi}(x) \equiv (\mathcal{F}^{-1}\phi)(x) = \int_{\Gamma} \phi(\xi) \chi_\xi(\xi) d\xi, \quad x \in K.$$

$S'(K)$, the space of distributions, is a collection of continuous linear functional on $S(K)$. $S'(K)$ is also a complete topological linear space. The action of $f$ in $S'(K)$ on an element $\phi$ in $S(K)$ is denoted by $\langle f, \phi \rangle$. The distribution space $S'(K)$ is given the
weak* topology. Convergence in $S'(K)$ is defined in the following way: $f_k$ converges to $f$ in $S'(K)$ if $(f_k, \phi)$ converges to $(f, \phi)$ for any $\phi \in S(K)$.

The Fourier transformation $\hat{f}$ of a distribution $f \in S'(K)$ is defined by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle \quad \text{for all } \phi \in S(K),$$

the inverse Fourier transformation $\check{g}$ is defined by

$$\langle \check{g}, \psi \rangle = \langle g, \check{\psi} \rangle \quad \text{for all } \psi \in S(K).$$

### 2.2 Function spaces on local fields

**Definition 2.1** Let $\{\phi_j\}_{j=0}^\infty = \{\Phi_{\Gamma^0}, \Phi_{\Gamma \backslash \Gamma_{j-1}}\}_{j=1}^\infty$, we define the operators $\Delta_j$ as follows

$$\Delta_j f = F^{-1}(\phi_j F f), \quad j \in \mathbb{N}_0, \quad f \in S'(K).$$

Then we obtain the Littlewood–Paley decomposition,

$$f = \sum_{j=0}^\infty \Delta_j f,$$

of all $f \in S'(K)$.

Note that $\Delta_j f \in S'(K)$ for any $f \in S'(K)$.

We are providing the proof of convergence of Littlewood–Paley decomposition on local fields.

**Proposition 2.1** Let $u$ be in $S'(K)$. Then, we have $u = \sum_{j=0}^\infty \Delta_j u$, in the sense of the convergence in the space $S'(K)$.

**Proof** Let $u \in S'(K)$ and

$$S_n u = \sum_{j=0}^n \Delta_j u,$$

we have

$$\langle (I - S_n)u, f \rangle = \langle u, (I - S_n) f \rangle, \quad \text{for all } f \in S(K).$$

Thus, it is enough to prove

$$f = \lim_{n \to \infty} S_n f, \quad \text{for all } f \in S(K).$$
For all $\xi \in K$, we have

$$
\mathcal{F}(S_n f - f)(\xi) = \mathcal{F}\left( \sum_{j=0}^{n} \Delta_j f - f \right)(\xi) = \sum_{j=0}^{n} \phi_j \mathcal{F}(f)(\xi) - \mathcal{F}(f)(\xi).
$$

Since $\sum_{j=0}^{\infty} \phi_j = 1$, hence

$$
\lim_{n \to \infty} \mathcal{F}(S_n f - f)(\xi) = 0,
$$

and therefore,

$$
f = \lim_{n \to \infty} S_n f.
$$

\hfill \Box

**Definition 2.2** (*B-type space*) For $0 < r \leq +\infty$, $0 < t \leq +\infty$, $s \in \mathbb{R}$, we define B-type spaces or Besov spaces on local fields $K$ as

$$
B_{rt}^s(K) = \{ f \in S'(K) : \| f \|_{B_{rt}^s(K)} < \infty \},
$$

with norm

$$
\| f \|_{B_{rt}^s(K)} = \| q^{sjs} \Delta_j f \|_{\ell_t(L_r(K))} = \left\{ \sum_{j=0}^{\infty} q^{sjs} \left\{ \int_K |\Delta_j f|^r \, dx \right\}^{\frac{t}{r}} \right\}^{\frac{1}{t}}.
$$

**Remark 2.1** The spaces $B_{rt}^s(K)$ are quasi-Banach spaces (Banach spaces for $r, t \geq 1$), and $S(K) \subset B_{rt}^s(K) \subset S'(K)$, where the first embedding is dense if $r < \infty$ and $t < \infty$. The theory of the spaces $B_{rt}^s(K)$ has been developed in [18].

Note that members of $B_{rt}^s(K)$ are tempered distributions and which can only be interpreted as regular distributions for sufficiently high smoothness. More precisely, we have

$$
B_{rt}^s(K) \subset L_{1}^{hc}(K) \quad \text{if and only if} \quad s > \sigma_r \text{ for } 0 < r \leq \infty, \ 0 < t \leq \infty,
$$

see [18, Theorem 4.1.3]. Since, for $0 < s < \sigma_r$, the $\delta$-distribution belongs to $B_{rt}^s(K)$, which is a singular distribution, so, in general one cannot interpret $f \in B_{rt}^s(K)$ as a regular distribution.
Definition 2.3 Let $\Omega$ be a compact subset of $\Gamma$ and $0 < r \leq \infty$, then we define

$$S^\Omega(K) = \{ \phi : \phi \in S(K), \supp \mathcal{F}\phi \subset \Omega \}$$

$$L_r^\Omega(K) = \{ \psi : \psi \in S'(K), \supp \mathcal{F}\psi \subset \Omega, \| \psi \|_{L_r(K)} < \infty \}.$$

If $\sigma$ is a real number, then

$$H_2^\sigma = \{ f : f \in S'(K), \| f \|_{H_2^\sigma} = \| \langle \cdot \rangle^\sigma \mathcal{F}f \|_{L_2(K)} < \infty \}.$$

By the well-known fact that $\mathcal{F}$ is a unitary operator on $L_2$, we have

$$\| f \|_{H_2^\sigma} = \| \mathcal{F}^{-1} \langle \cdot \rangle^\sigma \mathcal{F}f \|_{L_2(K)}.$$

We will be using the following theorem in further sections. We refer [27, Theorem 1.1.5] to the proof of the theorem.

**Theorem 2.4** Let $\Omega$ be a compact subset of $K$ and $0 < r \leq \infty$. If $\sigma > \left( \frac{1}{\min(r, 1)} - \frac{1}{2} \right)$ then there exists a constant $c$ such that

$$\| \mathcal{F}^{-1} M \mathcal{F}h \|_{L_r(K)} \leq c \| M \|_{H_2^\sigma} \| h \|_{L_r(K)},$$

holds for all $h \in L_r^\Omega(K)$ and all $M \in H_2^\sigma$.

### 3 Dilation operator

Our main result is the following.

**Theorem 3.1** Let $0 < r, t \leq \infty$, $s > \sigma_r = \max \left( \frac{1}{r} - 1, 0 \right)$. For $k \in \mathbb{N}$, $T_k$ is defined by (1.2). Then

$$\| T_k f \|_{B_r^s(K)} \leq (c_2 + c_1 k^{1/t}) q^{k(s-\frac{1}{t})} \| f \|_{B_r^s(K)},$$

for some $c_1, c_2$ which are independent of $k$ and for all $f \in B_r^s(K)$.

**Proof** Recall from Definition 2.1,

$$\{ \phi_j \}_{j=0}^{+\infty} = \{ \Phi_1^0, \Phi_{\Gamma_j \setminus \Gamma_{j-1}} \}_{j=1}^{+\infty},$$

the non-homogeneous unit decomposition on $\Gamma$. We may assume that

$$\phi_j(\xi) = \phi_1(p^{j-1} \xi), \quad j \in \mathbb{N}.$$
We have

\[
(\mathcal{F} f(p^{-k} \cdot)) (\xi) = \int_K f(p^{-k} x) \bar{\chi}_\xi(x) \, dx \\
= \int_K f(t) \bar{\chi}_\xi(p^{k} t) q^{-k} \, dt \\
= q^{-k}(\mathcal{F} f)(p^{k} \xi).
\]

Also

\[
\mathcal{F}^{-1}\{\phi_j(\xi)(\mathcal{F} f(p^{-k} \cdot))(\xi)\}(x) = q^{-k} \mathcal{F}^{-1}\{\phi_j(\xi)(\mathcal{F} f)(p^{k} \xi)\}(x) \\
= q^{-k} \int_\Gamma \phi_j(\xi)(\mathcal{F} f)(p^{k} \xi) \chi_x(\xi) \, d\xi \\
= q^{-k} \int_\Gamma \phi_j(p^{-k} \xi)(\mathcal{F} f)(\xi) \chi_x(p^{-k} \xi) q^{k} \, d\xi \\
= q^{-k} q^{k} \int_\Gamma \phi_j(p^{-k} \xi)(\mathcal{F} f)(\xi) \chi_{p^{-k} x}(\xi) \, d\xi \\
= \mathcal{F}^{-1}\{\phi_j(p^{-k} \xi)(\mathcal{F} f)(\xi)\}(p^{-k} x). \tag{3.2}
\]

From the definition of Besov spaces with \( f(p^{-k} x) \) in place of \( f(x) \) and using (3.2) along with \( \|g(p^{-k} \cdot) \mid L_r(K)\| = q^{-k/r} \|g \mid L_r(K)\| \), we obtain

\[
\|f(p^{-k} x) \mid B^s_{r,t}(K)\| = \left( \sum_{j=0}^{\infty} q^{sjt} \|\mathcal{F}^{-1}\{\phi_j(\xi)(\mathcal{F} f(p^{-k} \cdot))(\xi)\}(x) \mid L_r(K)\|^t \right)^{1/t} \\
= \left( \sum_{j=0}^{\infty} q^{sjt} \|\mathcal{F}^{-1}\{\phi_j(p^{-k} \xi)(\mathcal{F} f)(\xi)\}(p^{-k} x) \mid L_r(K)\|^t \right)^{1/t} \\
= q^{-k/r} \left( \sum_{j=0}^{\infty} q^{sjt} \|\mathcal{F}^{-1}\phi_j(p^{-k} \cdot)\mathcal{F} f \mid L_r(K)\|^t \right)^{1/t} \tag{3.3}
\]

If \( j \geq k + 1 \), then \( \phi_j(p^{-k} \xi) = \phi_1(p^{j-k-1} \xi) = \phi_{j-k}(\xi) \). This gives

\[
q^{-k/r} \left( \sum_{j=k+1}^{\infty} q^{sjt} \|\mathcal{F}^{-1}\phi_j(p^{-k} \cdot)\mathcal{F} f \mid L_r(K)\|^t \right)^{1/t} \\
= q^{-k/r} \left( \sum_{j=k+1}^{\infty} q^{(j-k)t} q^{skt} \|\mathcal{F}^{-1}\phi_{j-k}\mathcal{F} f \mid L_r(K)\|^t \right)^{1/t} \\
= q^{-k/r} q^{sk} \left( \sum_{l=1}^{\infty} q^{slt} \|\mathcal{F}^{-1}\phi_l\mathcal{F} f \mid L_r(K)\|^t \right)^{1/t}
\]
Using this fact we have
\[ \phi \]
where
\[ \sigma \]
which gives that
\[ M = j \]
For the remaining terms, \( j = 0, 1, \ldots, k \) we will use Theorem 2.4. Since \( \phi_j (p^{-k} \xi) = 1 \) when \( p^{-k} \xi \in \Gamma^j \setminus \Gamma^{j-1} \), otherwise it is zero, so
\[ p^{-k} \xi \in \Gamma^j \setminus \Gamma^{j-1} \implies |p^{-k} \xi| = q^j \]
\[ |\xi| = q^{j-k} \leq 1 \quad (\because j \leq k) \]
which gives that \( \xi \in \Gamma^0 \) and hence \( \phi_0 (\xi) = 1 \), therefore \( \phi_j (p^{-k} \xi) \phi_0 (\xi) = \phi_j (p^{-k} \xi) \).
Using this fact we have
\[ \mathcal{F}^{-1} \phi_j (p^{-k} \cdot) \mathcal{F} f = \mathcal{F}^{-1} \phi_j (p^{-k} \cdot) \phi_0 \mathcal{F} f = \mathcal{F}^{-1} \phi_j (p^{-k} \cdot) (\mathcal{F}^{-1} \phi_0 \mathcal{F} f). \]  
(3.5)

We put \( h = \mathcal{F}^{-1} \phi_0 \mathcal{F} f \), where \( \text{supp} \mathcal{F} h \subset \text{supp} \phi_0 = \Gamma^0 \). If \( j = 0 \), we take \( M = \phi_0 (p^{-k} \cdot) \) and calculate
\[ q^{-k/r} \| \mathcal{F}^{-1} \phi_0 (p^{-k} \cdot) \mathcal{F} f \| L_r (K) \| \leq c q^{-k/r} \| \phi_0 (p^{-k} \cdot) \| H^s_2 \| \| \mathcal{F}^{-1} \phi_0 \mathcal{F} f \| L_r (K) \|, \]
(3.6)

where \( \sigma \) is an arbitrary number with \( \sigma > \left( \frac{1}{\min(r, 1)} - \frac{1}{2} \right) \). It is easy to see that
\[ \| \phi_0 (p^{-k} \cdot) \| H^s_2 \| = \| (\xi)^{\sigma} (\mathcal{F} \phi_0 (p^{-k} \cdot)) (\xi) \| L_2 (K) \| \]
\[ = q^{-k} \| (\xi)^{\sigma} (\mathcal{F} \phi_0) (p^k \xi) \| L_2 (K) \| \]
\[ = q^{-k} q^{k/2} \| (p^{-k} \xi)^{\sigma} (\mathcal{F} \phi_0) (\xi) \| L_2 (K) \| \]
\[ = q^{-k/2} q^{k \sigma} \| (\mathcal{F} \phi_0) (\xi) \| L_2 (K) \| \]
\[ = q^{-k/2} q^{k \sigma} \| \phi_0 \| L_2 (K) \| \]
\[ = q^{k (\sigma - 1/2)}. \]

We may assume that \( s > \sigma - 1/2 \). This gives
\[ q^{-k/r} \| \mathcal{F}^{-1} \phi_0 (p^{-k} \cdot) \mathcal{F} f \| L_r (K) \| \leq c q^{-k/r} q^{k s} \| \mathcal{F}^{-1} \phi_0 \mathcal{F} f \| L_r (K) \| \]
\[ \leq c' q^{k (s - 1/r)} \| f \| B^s_{r,t} (K) \|. \]  
(3.7)
Finally, it remains to consider $1 \leq j \leq k$. This is the crucial step leading to $k^{1/t}$. In this case, $\phi_j(p^{-k}\xi) = \phi_1(p^{j-k-1}\xi)$ and

\[
\mathcal{F}^{-1} \phi_j(p^{-k}\cdot) \mathcal{F} f = \mathcal{F}^{-1} \phi_1(p^{j-k-1}\cdot) \phi_0 \mathcal{F} f = \mathcal{F}^{-1} \phi_1(p^{j-k-1}\cdot) \mathcal{F}(\mathcal{F}^{-1} \phi_0 \mathcal{F} f). \tag{3.8}
\]

We put $h = \mathcal{F}^{-1} \phi_0 \mathcal{F} f$, where supp $\mathcal{F} h \subset$ supp $\phi_0 = \Gamma^0$, and we take $M = \phi_1(p^{j-k-1}\cdot)$. This gives

\[
\| \mathcal{F}^{-1} \phi_j(p^{-k}\cdot) \mathcal{F} f \mid L_r(K) \| = \| \mathcal{F}^{-1} \phi_1(p^{j-k-1}\cdot) \mathcal{F}(\mathcal{F}^{-1} \phi_0 \mathcal{F} f) \mid L_r(K) \| \\
\leq c \| \phi_1(p^{j-k-1}\cdot) \| \cdot \| \mathcal{F}^{-1} \phi_0 \mathcal{F} f \mid L_r(K) \|, \tag{3.9}
\]

where $\sigma$ is an arbitrary number with $\sigma > \left( \frac{1}{\min(r, 1)} - \frac{1}{2} \right)$ and

\[
\| \phi_1(p^{j-k-1}\cdot) \| \cdot H_2^\sigma = \| (\mathcal{F} \phi_1(p^{j-k-1}\cdot))(\xi) \mid L_2(K) \| \\
= q^{-(k-j+1)} \| (\mathcal{F} \phi_1)(p^{(k-j+1)}\xi) \mid L_2(K) \| \\
= q^{-(k-j+1)} q^{(k-j+1)/2} \| (\mathcal{F} \phi_1)(\xi) \mid L_2(K) \| \\
= q^{-(k-j+1)/2} q^{(k-j+1)\sigma} \| \phi_1 \mid L_2(K) \| \\
= (q - 1)^{1/2} q^{(k-j+1)(\sigma - 1/2)}. \tag{3.10}
\]

Using (3.9) and (3.10), we obtain

\[
q^{-k/r} \left( \sum_{j=1}^{k} q^{s_1} \| \mathcal{F}^{-1} \phi_j(p^{-k}\cdot) \mathcal{F} f \mid L_r(K) \| \right)^{1/t} \\
\leq c q^{-k/r} \left( \sum_{j=1}^{k} q^{s_1} q^{(k-j)(\sigma - 1/2)t} (q - 1)^{1/2} q^{(\sigma - 1/2)t} \| \mathcal{F}^{-1} \phi_0 \mathcal{F} f \mid L_r(K) \| \right)^{1/t} \\
\leq c q^{-k/r} \left( \sum_{j=1}^{k} q^{s_1} q^{(k-j)^s} (q - 1)^{1/2} q^{s} \| \mathcal{F}^{-1} \phi_0 \mathcal{F} f \mid L_r(K) \| \right)^{1/t} (s > \sigma - 1/2) \\
= c q^{-k/r} \left( \sum_{j=1}^{k} q^{s_1} (q - 1)^{1/2} q^{s} \| \mathcal{F}^{-1} \phi_0 \mathcal{F} f \mid L_r(K) \| \right)^{1/t} \\
= c q^{-k/r} q^{s_1} (q - 1)^{1/2} q^{s} \| \mathcal{F}^{-1} \phi_0 \mathcal{F} f \mid L_r(K) \| \left( \sum_{j=1}^{k} \right)^{1/t} \\
\leq c_1 q^{(s-1/2)} k^{1/t} \| f \mid B_{r,t}^s(K) \|. \tag{3.11}
\]
Finally, (3.11) together with (3.7), (3.4) and (3.3) gives the estimate i.e,

\[ \| f(\beta^{-k}x) \|_{B_{rt}^s(K)} \leq (c+c'+c_1k^{1/t})q^{k(s-1/r)} \| f \|_{B_{rt}^s(K)}, \]

\[ = (c_2+c_1k^{1/t})q^{k(s-1/r)} \| f \|_{B_{rt}^s(K)}, \]

which is the desired result. \( \square \)

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Data availability The authors confirm that the data supporting the findings of this study are available within the article.

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