Dirac Quantization Condition for Monopole in Noncommutative Space-Time

Masud Chaichian\textsuperscript{1,2,}* Subir Ghosh\textsuperscript{3,†} Miklos Långvik\textsuperscript{1,‡} and Anca Tureanu\textsuperscript{1,§}

\textsuperscript{1} Department of Physics, University of Helsinki,
P.O. Box 64, FIN-00014 Helsinki, Finland
\textsuperscript{2} Helsinki Institute of Physics, P.O. Box 64, FIN-00014 Helsinki, Finland
\textsuperscript{3} Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata-700108, India

Abstract

Since the structure of space-time at very short distances is believed to get modified possibly due to noncommutativity effects and as the Dirac Quantization Condition (DQC), $\mu e = \frac{N}{2} \hbar c$, probes the magnetic field point singularity, a natural question arises whether the same condition will still survive. We show that the DQC on a noncommutative space in a model of dynamical noncommutative quantum mechanics remains the same as in the commutative case to first order in the noncommutativity parameter $\theta$, leading to the conjecture that the condition will not alter in higher orders.

PACS numbers: 14.80.Hv, 02.40.Gh, 03.65.-w

\*Electronic address: masud.chaichian@helsinki.fi
\†Electronic address: subir.ghosh2@rediffmail.com
\‡Electronic address: miklos.langvik@helsinki.fi
\§Electronic address: anca.tureanu@helsinki.fi
I. INTRODUCTION

The idea of magnetic monopoles - the so-far hypothetical particles carrying magnetic charge - is one of the most influential in modern theoretical physics. The first effective theoretical proposal that magnetic charge should exist was made by Dirac [1], who argued that in quantum mechanics the unobservability of phase permits singularities which manifest themselves as sources of magnetic fields. The Dirac Quantization Condition (DQC), $\mu e = \frac{2\pi}{e} \hbar c$, is a topological property, independent of space-time points, that tells us that the mere existence of a single magnetic monopole would imply that the electric charge is quantized. Before Dirac, the surprising asymmetry in Maxwell’s equations made Poincaré and J. J. Thompson introduce the magnetic charge in the theory as an artefact for simplifying the computation, while P. Curie suggested even the actual existence of magnetic charge [2]. The idea of magnetic monopoles was later extended by the discovery of monopole solutions of classical non-Abelian theories [3, 4] and the introduction of the concept of dyons - particles carrying both electric and magnetic charge [5], and with its significant influence, eventually leading to the concept of duality [6] and on the string theory.

In recent years magnetic monopole structures have created a lot of interest in condensed matter physics. In studying the Anomalous Hall Effect, magnetic monopole structures in momentum space have been experimentally observed and theoretically explained in [7, 8].

However, till date no magnetic monopole has been found (for a comprehensive review of magnetic monopole searches, see e.g. [9]), but the theoretical interest has stayed, as nothing in the theory has ever been found to contradict the DQC.

In the recent decade there has been a growing interest in the research concerning non-commutative spaces mainly due to the results in [10] and [11]. In [11] it was shown that string theory in a constant background field leads to a noncommutative field theory as a low energy limit.

Moreover, the result of [10] has encouraged many to believe that noncommutative field theory is a step towards a more complete description of physics. In the "gedanken experiment" of [10], it was argued that in the process of measurement of space points, as the energy grows, eventually black holes are formed and consequently objects of smaller extent than the diameter of the black holes cannot be observed and one can think of space-time "points" as operators obeying a Heisenberg-like uncertainty principle from which it follows
that space-time is homogeneous and can be interpreted as being noncommutative.

Although the main interest in this field lies in the formulation of a consistent field theory on a noncommutative space-time, it is also interesting to apply the noncommutative space to pure quantum mechanics to see whether it is possible to extend ordinary quantum mechanics to the noncommutative case. Specifically, the result \[10\] is quantum mechanical in nature and some results such as the DQC \[1\], which we will be exploring in this Letter, are not obtained directly from field theory.

In the noncommutative case, the space-time is particularly sensitive to the short-distance effects. Since the DQC in its essence probes the singularity structure of the magnetic field, one would think that this condition could no longer remain valid in the noncommutative case. This is the main motivation for the present work.

We shall start by briefly reviewing one known method for deriving the DQC in the commutative case. Then, starting from a classical Lagrangian corresponding to a dynamical model of noncommutative quantum mechanics, we shall derive the DQC to first order in the noncommutativity parameter $\theta$, and finally we shall discuss the result and its possible generalizations.

II. ONE WAY OF DERIVING THE DQC

In the commutative case, there is an ingenious way to derive the DQC, first introduced by Jackiw \[12\], which uses a gauge-invariant algebra, dependent only on the magnetic field. The derivation is an example of the three-cocycles which appear when a representation of a transformation group is nonassociative; in particular, when the translations group is represented by gauge-invariant operators in the presence of a magnetic monopole, the Jacobi identity among the translation generators fails. The restoration of the associativity of finite translations leads to the DQC. A sketch of the derivation \[12\] will be quite illuminating: for a nonrelativistic particle with the electric charge $e$, moving in a magnetic field $\mathbf{B}(\mathbf{x})$, one starts by finding the non-canonical quantum brackets

\[
[x_i, x_j] = 0, \quad [x_i, \pi_j] = i\hbar \delta_{ij}, \quad (1)
\]

\[
[\pi_i, \pi_j] = i\hbar \frac{e}{c} \epsilon_{ijk} B_k(\mathbf{x}), \quad (2)
\]
where one defines the operators \( \pi_i \) in the \( x \)-representation as
\[
\pi_i = -i\hbar \partial_i - \frac{e}{c} A_i(x),
\]
with \( A_i(x) \) being the vector potential \( \mathbf{A} \). These commutation relations, along with the Hamiltonian
\[
H = \frac{\pi^2}{2m}, \quad \pi = m\dot{x},
\]
yield the well-known Lorentz-Heisenberg equations of motion
\[
\dot{x} = \frac{i}{\hbar} [H, x] = \frac{\pi}{m},
\]
\[
\dot{\pi} = \frac{i}{\hbar} [H, \pi] = \frac{e}{2mc} [\pi \times \mathbf{B} - \mathbf{B} \times \pi],
\]
where \( \pi \) is the gauge-invariant mechanical momentum. So far there is no restriction on \( \mathbf{B} \) but the following Jacobi identity violation,
\[
\frac{1}{2} \varepsilon_{ijk} \{[\pi_i, \pi_j], \pi_k\} = \frac{e\hbar^2}{c} \nabla \cdot \mathbf{B},
\]
indicates that the magnetic field has to be source-free. Otherwise, \( \nabla \cdot \mathbf{B} \neq 0 \) will lead to a loss of associativity of the translation operators \( T(a) \equiv \exp \left( -\frac{ie}{\hbar} \mathbf{a} \cdot \pi \right) \),
\[
\left(T(a_1)T(a_2)\right)T(a_3) = \exp \left( -\frac{ie}{\hbar c} \omega(x; a_1, a_2, a_3) \right) \times T(a_1) \left(T(a_2)T(a_3)\right).
\]
Here \( a_i \) are constant vectors and the non-trivial phase factor turns out to be the magnetic flux coming out of the tetrahedron formed by \( a_i \):
\[
\exp \left( -\frac{ie}{\hbar c} \omega(x; a_1, a_2, a_3) \right),
\]
which is nonzero if a magnetic monopole is enclosed by the tetrahedron. The phase factor \( \omega \) becomes 1 and thus the associativity of finite translations in the presence of the magnetic monopoles can be re-established for
\[
\int d^3x \, \nabla \cdot \mathbf{B} = 2\pi \frac{\hbar c}{e} N,
\]
where \( N \) is an integer. This condition, together with the Gauss equation for a monopole of magnetic charge \( \mu \), \( \nabla \cdot \mathbf{B} = 4\pi \mu \delta^3(x) \), yields the celebrated DQC
\[
\mu e = \frac{1}{2} N\hbar c.
\]
Note that the Jacobi identity is still violated at the location of each monopole and these points are conventionally excluded from the manifold.
III. THE NONCOMMUTATIVE DQC

The extension of the approach in [12] to the noncommutative case can be achieved once one finds the algebra of coordinate and gauge-invariant momentum operators for a charged quantum mechanical particle in motion in a magnetic field in the noncommutative spacetime, i.e. the analogue of the non-canonical algebra (2). It is expected that the noncommutativity of space-time coordinates would change the dynamics of the charged particle in the magnetic field (i.e. the Lorentz force), and this in turn will require a change in the commutation relations (2). However, we can find the new noncommutative algebra by starting from a classical Lagrangian, for example the one for the model [13], and deriving the corresponding Dirac brackets and then quantizing them. We therefore consider a Lagrangian of the form

\[ L = \left( P_i + \frac{e}{c} A_i \right) \dot{X}_i - \frac{1}{2} \epsilon_{ijk} P_i \dot{P}_j \theta_k - \frac{1}{2m} P^2 + eA_0, \]

where \( P_i \) is the momentum, \( \theta_k \) - the noncommutativity parameter, of dimension (length)^2/action, and \( A_i, A_0 \) - the magnetic and electric potential, respectively. The Lagrangian (12) is a straightforward generalization to three-dimensions of the one considered in [14], which is a Lagrangian for the model [13].

The Lagrangian (12) is a Lagrangian of dynamical noncommutativity of the space coordinates. This claim is better understood once we derive the Dirac brackets from this Lagrangian. For this, we need the canonical momenta which are given by

\[ \pi_i = \frac{\partial L}{\partial \dot{X}_i} = P_i + \frac{e}{c} A_i, \quad \pi_i^P = \frac{\partial L}{\partial \dot{P}_i} = \frac{1}{2} \epsilon_{ijk} P_j \theta_k. \]

These lead to the constraints

\[ \eta_i \equiv \pi_i - P_i - \frac{e}{c} A_i, \quad \psi_i \equiv \pi_i^P - \frac{1}{2} \epsilon_{ijk} P_j \theta_k. \]

In the classical framework, with \( \{ X_i, \pi_j \} = \delta_{ij}, \{ P_i, \pi_j^P \} = \delta_{ij} \), we calculate the constraint algebra,

\[
\begin{align*}
\{ \eta_i, \eta_j \} &= \frac{e}{c} (\partial_i A_j - \partial_j A_i) = \frac{e}{c} F_{ij} = \frac{e}{c} \epsilon_{ijk} B_k, \\
\{ \psi_i, \psi_j \} &= -\epsilon_{ijk} \theta_k, \quad \{ \eta_i, \psi_j \} = -\delta_{ij}.
\end{align*}
\]

From this algebra we find that the constraints are second class and, performing the Dirac
constraint analysis [15, 16], we obtain the classical Dirac brackets as

\[
\{X_i, X_j\} = \frac{\epsilon_{ijk}\theta_k}{1 - \frac{\epsilon}{c} \theta \cdot B},
\]
\[
\{X_i, P_j\} = \frac{\delta_{ij} - \frac{\epsilon}{c} B_i \theta_j}{1 - \frac{\epsilon}{c} \theta \cdot B},
\]
\[
\{P_i, P_j\} = \frac{\epsilon_{ijk} \frac{\epsilon}{c} B_k}{1 - \frac{\epsilon}{c} \theta \cdot B}.
\] (14)

This is exactly how interactions have been introduced in the model of [14]. In this model the noncommutativity of coordinate operators is dynamical in the sense that it is generated within the system. Thus the gauge field cannot be affected by the noncommutativity which emerges upon quantization. Therefore the field \( A_i \) is the Abelian \( U(1) \) gauge field in this model of noncommutativity.

Our next step is to quantize the brackets. We do this by promoting the classical variables \( X_i \) and \( P_i \) in the Dirac brackets (14) to the status of operators \( \hat{X}_i, \hat{P}_i \) and multiplying the right-hand side of the Dirac brackets by \( i\hbar \). This is the standard procedure [15, 16]. We consider the Dirac brackets (14) expanded to first order in \( \theta \) and hereafter we perform all our calculations to this order only. We resort to this approximation because we will need to find representations for our operators in order to have a well-defined quantum theory [16], and this is a task that is difficult to do exactly for the algebra (14). The quantization of (14) gives

\[
[\hat{X}_i, \hat{X}_j] = i\hbar\epsilon_{ijk}\theta_k + \mathcal{O}(\theta^2),
\]
\[
[\hat{X}_i, \hat{P}_j] = i\hbar\left[\delta_{ij} - \frac{\epsilon}{c} B_i(\hat{X})\theta_j + \frac{\epsilon}{c} \delta_{ij} \theta \cdot B(\hat{X})\right] + \mathcal{O}(\theta^2),
\]
\[
[\hat{P}_i, \hat{P}_j] = i\hbar \frac{\epsilon}{c} \epsilon_{ijk} B_k(\hat{X})\left[1 + \frac{\epsilon}{c} \theta \cdot B(\hat{X})\right] + \mathcal{O}(\theta^2).
\] (15)

The algebra (15) poses a twofold problem. Firstly, the operator \( \hat{P}_j \) in (15) does not represent the translation generator, since there are extra terms on the right-hand side of \([\hat{X}_i, \hat{P}_j]\), other than \( i\hbar \delta_{ij} \). Secondly, we face the problem of how to represent the operators \( \hat{X}_i \), since they do not commute to first order in \( \theta \). This problem becomes much simpler if we are able to define some new operators \( x_i \), in terms of the old ones \( \hat{X}_i \) and \( \hat{P}_i \), such that they commute to first order in \( \theta \). An appropriate definition for our purpose is

\[
x_i = \hat{X}_i + \frac{1}{2} \epsilon_{ijk} \hat{P}_j \theta_k.
\] (16)
Then the functions of the operator $\hat{X}_i$ can be expanded in terms of the new coordinate operator $x_i$ as, e.g.,

$$B_i(\hat{X}) = B_i(x) - \frac{1}{2} \epsilon_{njk} \hat{P}_j \partial_n B_i(x) + O(\theta^2). \quad (17)$$

We use the operator $x_i$ (16) and the expansion (17) to obtain an intermediate algebra of $x_i$ and $\hat{P}_j$, and further define the generator of translations corresponding to $x_i$:

$$p_j = \hat{P}_j - \frac{1}{2c} \left( \hat{P}_j (B \cdot \theta) - \hat{P} \cdot B \theta_j \right). \quad (18)$$

The newly-defined operators $p_i$ and $x_i$ obey the algebra

$$[x_i, x_j] = 0 + O(\theta^2)$$
$$[x_i, p_j] = i\hbar \delta_{ij} + O(\theta^2), \quad (19)$$
$$[p_i, p_j] = i\hbar \frac{e}{c} \epsilon_{ijk} B_k - \frac{e}{2c} \left[ i\hbar \delta_{ij} (B \cdot \theta) + p_i \theta_j \nabla \cdot B + p \cdot \theta_i \partial_j B \right]$$
$$+ p_j [p_i, B] \cdot \theta + p \cdot [B, p_i] \theta_j ] + O(\theta^2),$$

where the indices in brackets are anti-symmetrized.

To have properly quantized the algebra (19), we need a representation of its operators. From the similarity of the algebras (19) and (2), we infer that in the $x$-representation, we can realize the translation generators as (3) plus an extra term involving the first order noncommutativity contribution. Explicitly,

$$p_i = -i\hbar \partial_i - \frac{e}{c} A_i(x) + T_i(\theta, x) + O(\theta^2). \quad (20)$$

Inserting (20) into the commutator $[p_i, p_j]$ of the algebra (19), it simplifies to

$$[p_i, p_j] = i\hbar \frac{e}{c} \epsilon_{ijk} B_k + \frac{1}{2c} \left[ (i\hbar \partial_i + \frac{e}{c} A_i(\theta)) \nabla \cdot B \right] + O(\theta^2). \quad (21)$$

By directly computing the commutator of the operators $p_i$ in the representation (20), we have to reproduce the result (21), which holds true if we set

$$T_i(\theta, x) = -\frac{1}{2c} \theta_i \nabla \cdot B + G_i,$$  \quad (22)

where

$$\partial_j G_i = \frac{1}{2\hbar} \left( \frac{e}{c} \right)^2 A_j \theta_i \nabla \cdot B.$$  \quad (23)

Thus, the quantized algebra (19) is given by

$$[x_i, x_j] = 0 + O(\theta^2),$$
$$[x_i, p_j] = i\hbar \delta_{ij} + O(\theta^2), \quad (24)$$
$$[p_i, p_j] = i\hbar \frac{e}{c} \epsilon_{ijk} B_k + \frac{e}{2c} \left[ (i\hbar \partial_i + \frac{e}{c} A_i(\theta)) \nabla \cdot B \right] + O(\theta^2)$$
in the $x$-representation.

We can now calculate the Jacobi identities of the algebra (24), and find that the only non-vanishing one is:

$$
\frac{1}{2} \varepsilon_{ijk} [[p_i, p_j], p_k] = -\hbar^2 \frac{e}{c} \nabla \cdot B + \frac{i\hbar}{2} \left( \frac{e}{c} \right)^2 \varepsilon_{ijk} \partial_k (A_i \theta_j \nabla \cdot B) + O(\theta^2). \tag{25}
$$

Since the nonvanishing terms in the right-hand side of (25) are proportional to $\nabla \cdot B$, for a divergenceless magnetic field there are no Jacobi identity violations. However, if the magnetic field is produced by monopoles, $\nabla \cdot B = 4\pi \mu \delta^3(x)$, the Jacobi identity (25) is violated, meaning nonassociativity of the translation generators $p_i$.

We would like to remark at this point that although the Lagrangian (12) contains no magnetic sources, the algebra (24) is valid whether the magnetic field is source-free or not. The reason is simply that the Lorentz-force describes the movement of electrically charged particles in a magnetic field, but does not set any requirement on how the magnetic field is produced.

Thus in the noncommutative space we end up with a Jacobi identity violation consisting of the original commutative space term plus a $\theta$-dependent total-derivative term. Let us recall that DQC appears in the commutative case [12] through a volume integration (see (9)) over the tetrahedron formed by the three translation vectors $a_1, a_2, a_3$. Now, the $\theta$-term in (25), being a total derivative, should contribute at the boundary of the tetrahedron. However, this contribution will be necessarily zero, because the integrand contains the $\delta$-function coming from $\nabla \cdot B = 4\pi \mu \delta^3(x)$, which has support only at the origin, i.e. on the monopole. Hence these two features conspire to cancel the effect of the $\theta$-term. DQC remains unchanged in the presence of spatial noncommutativity, since the argument for restoring the associativity of the noncommutative translation operators goes through in the same manner as in the commutative case [12], but now with the translation operators

$$
T_{NC}(a) = \exp \left( -\frac{i}{\hbar} a \cdot p \right), \tag{26}
$$
generated by $p$ as the element of the algebra (24), valid to first order in $\theta$ and with the $x$-representation (20).
IV. SUMMARY AND DISCUSSION

We have explicitly shown that the DQC (11) remains unaltered in noncommutative space to first order in $\theta$. Based on the structure of the classical algebra (14) and the representation of the quantum algebra (20), and also considering the fact that the form of any topological correction is strongly constrained, we conjecture that the DQC will hold true in all orders in the noncommutative space-time. We intend to elucidate this issue in the future.

It would be interesting to obtain the same kind of indication of a DQC to first order in $\theta$ using a noncommutative non-Abelian vector potential [11, 17], especially since a gauge-covariant noncommutative Aharonov-Bohm effect has been formulated in [18]. However, this formulation gives the required phase factor with the help of non-Abelian noncommutative Wilson lines which are notoriously tedious to work with even to first order in $\theta$, due to the path ordering appearing in the Wilson line. Therefore, obtaining a possible DQC in this approach stands as a challenge for the future.

Our conclusion is that the DQC remains unchanged in the noncommutative case, to the first order in $\theta$ and expectably to all orders. This is of significance, since a vast amount of work has been devoted to studying various effects of noncommutative space only to the lowest order in $\theta$. Finally, we would like to mention that our work reinforces similar topological results in the noncommutative case for other nonperturbative monopole-, soliton- and dyon-solutions [19].

Acknowledgments

We are indebted to Claus Montonen and Shin Sasaki for illuminating discussions. A. T. acknowledges Projects Nos. 121720 and 127626 of the Academy of Finland.

[1] P. A. M. Dirac, Proc. Roy. Soc. Lond. A 133, 60 (1931).
[2] P. Curie, Séances Soc. Phys. (Paris), 76-77 (1894).
[3] T. T. Wu and C. N. Yang, Phys. Rev. D 12, 3845 (1975).
[4] G. ’t Hooft, Nucl. Phys. B79, 276 (1974);
    A. M. Polyakov, JETP Lett. 20, 194 (1974);
Y. Nambu, Phys. Rev. D 10, 4262 (1974).

[5] J. Schwinger, Science 165, 757 (1969);  
B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975);  
M. K. Prasad and C. M. Sommerfeld, Phys. Rev. Lett. 35, 760 (1975);  
E. B. Bogomol’nyi, Sov. J. Nucl. Phys. 24, 449 (1976).

[6] C. Montonen and D. Olive, Phys. Lett. B 72, 117 (1977).

[7] M. Onoda and N. Nagaosa, J. Phys. Soc. Jpn. 71, 19 (2002).

[8] Z. Fang et al., Science 302, 92 (2003).

[9] G. Giacomelli and L. Patrizii, Magnetic Monopole Searches, Bologna preprint DFUB-2003-1 (2003). In Trieste 2002, Astroparticle physics and cosmology (ICTP, Trieste 2003) p. 121;  
K. A. Milton, Rept. Prog. Phys. 69, 1637 (2006).

[10] S. Doplicher, K. Fredenhagen, and J. E. Roberts, Phys. Lett. B 331, 39 (1994);  
S. Doplicher, K. Fredenhagen, and J. E. Roberts, Commun. Math. Phys. 172, 187 (1995),  
hep-th/0303037.

[11] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032, hep-th/9908142.

[12] R. Jackiw, Phys. Rev. Lett. 54, 159 (1985);  
R. Jackiw, Int. J. Mod. Phys. A 19S1, 137 (2004), hep-th/0212058.

[13] C. Duval and P. A. Horvathy, Phys. Lett. B 479, 284 (2000), hep-th/0002233.

[14] J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, Annals Phys. 306, 78 (2003),  
hep-th/0207149.

[15] P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University Press, New York, 1964).

[16] D. M. Gitman and I. V. Tyutin, Quantization of fields with constraints (Springer, 1990);  
M. Henneaux and C. Teitelboim, Quantization of gauge systems (Princeton University Press, 1991).

[17] M. Hayakawa, Phys. Lett. B 478, 394 (2000), hep-th/9912094.

[18] M. Chaichian, M. Långvik, S. Sasaki, and A. Tureanu, Phys. Lett. B 666, 199 (2008),  
arXiv:0804.3565 [hep-th].

[19] A. Hashimoto and K. Hashimoto, J. High Energy Phys. 11 (1999) 005, hep-th/9909202;  
D. J. Gross and N. A. Nekrasov, J. High Energy Phys. 03 (2001) 044, hep-th/0010090;  
L. Cieri and F. A. Schaposnik, Res. Lett. Phys. 2008 (2008) 890916, arXiv:0706.0449 [hep-th].

[20] When deriving the DQC using gauge-variant vector potentials, one still defines the magnetic
field as $B = \nabla \times A$ even though this cannot now remain true in the whole space without the potential having singularities [3]. Instead one may consider the vector potential as being defined in many overlapping regions of space connected by singularity-free gauge transformations [3].