SPACES OF METRICS OF POSITIVE RICCI CURVATURE
AND BUNDLES WITH NON-MULTIPLICATIVE \( \hat{A} \)-GENUS

GEORGFRENCK AND JENSRHEINHOLD

Abstract. We construct smooth bundles with base and fiber products of
two spheres whose total spaces have non-vanishing \( \hat{A} \)-genus. We then use
these bundles to locate non-trivial rational homotopy groups of the space of
metrics of positive Ricci curvature for all manifolds of dimension 6 or at least
10 which admit such a metric and are a connected sum of some manifold
and \( S^n \times S^n \) or \( S^n \times S^{n+1} \), respectively. We also construct manifolds
\( M \) whose spaces of metrics of positive scalar curvature have homotopy groups
that contain elements of infinite order which lie in the image of the orbit
map induced by the push-forward action of the diffeomorphism group.

1. Introduction

For a closed smooth manifold \( M \), let \( R_{\text{Ric}>0}(M) \) denote the space of Rie-
mannian metrics on \( M \) whose Ricci curvature is strictly positive everywhere,
equipped with the \( C^\infty \) topology. In contrast to the case of positive scalar
curvature, very little is known about the topology of \( R_{\text{Ric}>0}(M) \) in the case
that it is non-empty; in particular, even simple non-vanishing results for its
rational homotopy or homology groups are scarce.

In this paper, we detect elements of infinite order in these groups for a large
class of manifolds. To state our results, let \( W^2_n \# g \) denote the
\( g \)-fold connected sum of \( S^n \times S^n \), and analogously \( W^{2n+1}_n \# g \).

For a closed manifold \( M \) of dimension \( d \), we define the genus of \( M \) to be the
largest number \( g \) such that there exists a manifold \( N \) with \( M \cong N \# W^g \).
The term \( \pi_j(X) \otimes \mathbb{Q} \neq 0 \) for some space \( X \) shall mean that there exists a base
point \( x \in X \) such that \( \pi_j(X,x) \otimes \mathbb{Q} \neq 0 \).

Theorem A. Let \( d \geq 10 \) with \( d \neq 13 \) and let \( M \) be a \( d \)-manifold of genus
at least 1 that admits a metric of positive Ricci curvature. Then either
\( \pi_1(R_{\text{Ric}>0}(M)) \) is infinite or \( \pi_j(R_{\text{Ric}>0}(M)) \otimes \mathbb{Q} \neq 0 \) for some \( 2 \leq j \leq 9 \).

Remark 1.1. One can refine the statement in this theorem so that one obtains a
more precise estimate on which rational higher homotopy groups are nontrivial
depending on the dimension modulo 8:

\[
\begin{align*}
\text{d} \equiv 0 \pmod{8} \quad & j \in \{2, 3, 7\} \quad \text{d} \equiv 1 \pmod{8} \quad & j \in \{2, 3, 6\} \\
\text{d} \equiv 2 \pmod{8} \quad & j \in \{2, 5\} \quad \text{d} \equiv 3 \pmod{8} \quad & j \in \{2, 4\} \\
\text{d} \equiv 4 \pmod{8} \quad & j = 3 \quad \text{d} \equiv 5 \pmod{8} \quad & j \in \{2, 3, 4, 6\} \\
\text{d} \equiv 6 \pmod{8} \quad & j \in \{2, 3, 4, 9\} \quad \text{d} \equiv 7 \pmod{8} \quad & j \in \{2, 3, 4, 8\}
\end{align*}
\]

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Remark 1.2. Botvinnik–Ebert–Wraith derive a similar result in [BEW20], but in contrast to ours it neither applies to manifolds of dimension $d \equiv 6 \pmod{8}$, nor to odd-dimensional ones. We also improved the lower bound required on the dimension and genus of the manifold.

Theorem A does not apply to dimensions smaller than 10 and its assertion is weakest in dimensions $d \equiv 6 \pmod{8}$, but in those dimensions we get the following additional result that also holds in dimension 6.

**Theorem B.** Let $d \equiv 6 \pmod{8}$ be a positive integer, and let $M$ be a $d$-dimensional manifold of genus at least 1 that admits a metric of positive Ricci curvature. Then at least one of the following is true:

(i) The map $\mathcal{R}_{\text{Ric}} > 0(W_d^1) \to \mathcal{R}_{\text{scal}} > 0(W_d^1)$ collapses infinitely many path components to one.

(ii) $H_1(\mathcal{R}_{\text{Ric}} > 0(M); \mathbb{Q}) \neq 0$.

Remark 1.3. (i) Note that statement (ii) from Theorem B is stronger than $\pi_1(\mathcal{R}_{\text{Ric}} > 0(M))$ being infinite. It implies that there is an element of infinite order in the abelianisation of this group.

(ii) For a list of manifolds to which Theorem A and Theorem B are applicable see [BEW20, Corollary 1.2] or [Bur20, Definition 1.1].

Remark 1.4. For $d = 13$ (and in fact for every $d \geq 13$ with $d \equiv 5 \pmod{8}$) we get a result that looks like a mixture of Theorem A and Theorem B. In this case at least one of the following is true.

(i) The map $\mathcal{R}_{\text{Ric}} > 0(W_d^1) \to \mathcal{R}_{\text{scal}} > 0(W_d^1)$ collapses infinitely many path components to one.

(ii) $\pi_1(\mathcal{R}_{\text{Ric}} > 0(M))$ is infinite.

(iii) $\pi_2(\mathcal{R}_{\text{Ric}} > 0(M)) \otimes \mathbb{Q} \neq 0$.

The two theorems above follow from a more general statement about the rational cohomology groups of the space $\mathcal{R}_{\text{Ric}} > 0(M)$ (see Theorem 3.5). In its proof, the main new ingredient is the construction of $W_d^1$-bundles over the product of two spheres with non-vanishing $\hat{A}$-genus. The most general version of our construction yields the following.

**Proposition 1.5.** For positive integers $p, q, i, j$ such that $2j < p < 4i$ and $2i < q < 4j$, there exists a smooth bundle $E \to S^{4i-p} \times S^{4j-q}$ with fiber $S^p \times S^q$, containing a trivialized disk bundle, whose total space admits a Spin-structure and has non-vanishing $\hat{A}$-genus.

Employing these bundles in another way, we also obtain examples of manifolds $M$ for which the image of the map on homotopy groups induced by the orbit map $\text{Diff}(M, D) \to \mathcal{R}_{\text{scal}} > 0(M)$ (associated to the push-forward action) contains elements of infinite order. Here $\text{Diff}(M, D)$ is the topological group of diffeomorphisms of $M$ fixing an embedded disk $D \subset M$ of codimension zero.

**Theorem C.** Let $i, j$ be positive integers with $|i - j| < \min(i, j)$ and $k < 4j - 2i - 1$. Then for every $p \in \{2j + 1, \ldots, 4i - 1\}$ there exists a fiber bundle $S^p \times S^{4j-k-1} \to M \to S^{4i-p}$ with $\mathcal{R}_{\text{scal}} > 0(M) \neq \emptyset$ such that the image of the map induced by the orbit map $\pi_k(\text{Diff}(M, D), \text{id}) \to \pi_k(\mathcal{R}_{\text{scal}} > 0(M); g)$ contains an element of infinite order for every $g \in \mathcal{R}_{\text{scal}} > 0(M)$. 
Remark 1.6. Note that this gives examples for every \( k \geq 0 \) and since \( 4j - k - 1 > 2i \geq \min(p, 4i - p) \) the manifold \( M \) is \( \min(p, 4i - p) \)-connected. It follows from the proof of Theorem C pertaining to \( \text{Diff}(M, D) \) that it is also true for \( N \# M \) for any Spin-manifold \( N \) of positive scalar curvature, by extending diffeomorphisms by the identity.

Remark 1.7. Manifolds as in Theorem C were known to exist for some time by [HSS14, Corollary 2.6], however their construction “is based on abstract existence results in differential topology [and] does not yield an explicit description of the diffeomorphism type of the [...] manifold” [loc. cit. p. 3].

Recently, Kupers, Kramnich, and Randal-Williams have shown [KKR20] that the image of the map \( \pi_3(\text{Diff}(\mathbb{H}P^2), \text{id}) \to \pi_3(\mathcal{R}_{\text{scal}}(\mathbb{H}P^2), g_{st}) \) contains an element of infinite order, analogous to the manifolds from above. Here, \( g_{st} \) denotes the standard metric on \( \mathbb{H}P^2 \), which is of positive sectional curvature. Their result is especially remarkable as it goes beyond the abstract result by Hanke, Schick and Steimle insofar it also gives elements in \( \pi_3(\mathcal{R}_{\text{Ric}}(\mathbb{H}P^2), g_{st}) \) and even \( \pi_3(\mathcal{R}_{\text{sec}}(\mathbb{H}P^2), g_{st}) \) of infinite order. We do not know if the manifolds from Theorem C admit a metric of positive Ricci curvature.

2. Bundles over products of spheres with non-vanishing \( \hat{A} \)-genus

2.1. The construction of the bundle. In this section, we construct smooth bundles whose fiber and base are products of spheres and whose total spaces have non-vanishing \( \hat{A} \)-genus, thereby proving Proposition 1.5. Throughout the whole section, we assume that \( p, q, i, j \) are positive integers such that the inequalities \( 2j < p < 4i \) and \( 2i < q < 4j \) are satisfied.

Construction 2.1. We build a smooth bundle

\[
(S^p \times S^q) \setminus D^{p+q} \to E' \xrightarrow{\pi} S^{4i-p} \times S^{4j-q}
\]
as follows. Consider the plumbing of two trivial disk bundles,

\[
S^p \times D^q \cup_{D^p \times D^q} D^p \times S^q \cong S^p \times S^q \setminus D^{p+q},
\]
where \( D^d \) denotes the lower hemisphere of \( S^d \). See Figure 1 for a visualization of this manifold. The mapping spaces \( \Omega^q_{\text{sm}}SO(q) \) and \( \Omega^q_{\text{sm}}SO(p) \) of those maps

\[
(S^p, D^p) \to (SO(q), 1) \quad \text{and} \quad (S^q, D^q) \to (SO(p), 1)
\]
so that the adjoint maps \( S^p \times D^q \to D^d \) and \( D^p \times S^q \to D^d \) are smooth can be endowed with the compact-open topology and the group structure induced from point-wise composition. The adjoint maps then twist the handles of \( S^p \times S^q \setminus D^{p+q} \), which produces two commuting actions of \( \Omega^q_{\text{sm}}SO(q) \) and \( \Omega^q_{\text{sm}}SO(p) \) on \( S^p \times S^q \setminus D^{p+q} \). The inclusions \( \Omega^q_{\text{sm}}SO(p) \subset \Omega^pSO(p) \) and \( \Omega^q_{\text{sm}}SO(q) \subset \Omega^qSO(q) \) are weak equivalences. We continue by picking \( \alpha \in \pi_{4i}(BSO(q)) \) and \( \beta \in \pi_{4j}(BSO(p)) \) such that \( p_l(\alpha), p_j(\beta) \neq 0 \), which is possible by the assumptions on \( p, q, i, j \), cf. [Mil59, Lemma 5]. We then choose representatives of clutching function of the corresponding vector bundles, \( \hat{\alpha}: S^{4i-p-1} \to \text{Maps}((S^p, D^p), (SO(q), 1)) \) and \( \hat{\beta}: S^{4j-q-1} \to \text{Maps}((S^q, D^q), (SO(p), 1)) \), respectively. By the discussion above we may
assume that these maps are smooth. Finally, we define for $x \in S^{4i-p-1}$ and $y \in S^{4j-q-1}$.

\[
\alpha_x: S^p \times D^q \rightarrow S^p \times D^q \quad \alpha_x(s, t) = \hat{\alpha}(x)(s) \cdot t
\]

\[
\beta_y: D^p \times S^q \rightarrow D^p \times S^q \quad \beta_y(s, t) = \hat{\beta}(y)(t) \cdot s.
\]

Both families fix the disk $D^p \times D^q$ and hence they can be extended to $S^p \times D^q \cup D^p \times D^q \times S^q \cong S^p \times S^q \setminus \Delta^{p+q}$. Since they have disjoint support, they obviously commute, and $(\alpha_x)_{x \in S^{4i-p-1}}$, $(\beta_y)_{y \in S^{4j-q-1}}$ can be seen as clutching functions of $S^p \times S^q \setminus \Delta^{p+q}$ with disjoint supports, so they produce a smooth bundle

\[
(S^p \times S^q) \setminus \Delta^{p+q} \rightarrow E' \xrightarrow{\pi} S^{4i-p} \times S^{4j-q}
\]

which contains a trivialized $D^{p+q}$-subbundle.

\[\text{Figure 1. The manifold } S^p \times S^q \setminus \Delta^{p+q} \text{ serves as the fiber}\]

Remark 2.2. The special case of this construction for $i = j$, $p = q = 4i - 1$ already appeared in [KR20, Lem. 4.5] where it is used to determine the image of $\kappa_{\mathcal{A}_2}: \Omega^2_2(B\text{Homeo}^+(W^{4i-1})) \rightarrow \mathbb{Z}$.

Note that Lemma 2.3 below shows that we can modify the construction of the bundle so that we obtain a bundle $S^p \times S^q \rightarrow E \rightarrow S^{4i-p} \times S^{4j-q}$ with closed fiber $S^p \times S^q$ by precomposing the classifying map of $E'$ with a self-map $\varphi$ of $S^{4i-p} \times S^{4j-q}$ of nonzero degree, and gluing in a disk bundle along the boundary of $\varphi^*E'$. This boundary is a smooth $(S^{p+q-1}, D^{2n+1})$-bundle over $S^{4i-p} \times S^{4j-q}$.

It follows from the inequalities on $p, q, i, j$ that $2i \leq q - 1$ and $2j \leq p - 1$. If $d = p + q - 1$ is even (and thus one of $p - 1, q - 1$ is odd), then $2(i + j) \leq p + 1 - 3$ and hence $\dim(S^{4i-p} \times S^{4j-q}) = 4(i + j) - (p + q) \leq p + q - 6 = (d - 1) - 5$ holds, which is (i) in Lemma 2.3 with $n = d - 1$. If $d = p + q - 1$ is odd then $\dim(S^{4i-p} \times S^{4j-q}) = 4(i + j) - (p + q) \leq p + q - 4 = (d - 1) - 3$ holds, which is (ii) in Lemma 2.3.

Lemma 2.3. Let $\pi: X \rightarrow B$ be a smooth $(S^n, D^n)$-bundle whose base space $B = S^{i_1} \times \cdots \times S^{i_r}$ is a product of spheres. Then the following holds.

(i) If $n$ is even and $\dim(B) \leq n - 5$, there exists a map $\varphi: B \rightarrow B$ of non-zero degree such that $\varphi^* \pi$ is trivial.
(ii) If \( n \) is odd and \( \dim(B) \leq n - 3 \), there exists a map \( \varphi: B \to B \) of non-zero degree such that \( \varphi^* \pi \) is the bundle of boundaries of a smooth \( D^{n+1} \)-bundle.

For the proof we need the following auxiliary result.

**Lemma 2.4.** Let \( B = S^{i_1} \times \cdots \times S^{i_r} \) be a product of spheres, and let \( Y \) be a connected based space such that \( \pi_j(Y, y) \) is finite for \( 1 \leq j \leq \dim(B) = i_1 + \cdots + i_r \). Then for any map \( f: B \to Y \), there exists a map \( \varphi: B \to B \) of non-zero degree such that \( f \circ \varphi \) is null.

**Proof.** We first prove the claim for the case that \( Y = K(\pi, j) \) is an Eilenberg–MacLane space, with \( \pi \) a finite group (abelian if \( j \geq 2 \)) and \( 1 \leq j \leq b \). If \( j = 1 \), homotopy classes of based maps \( f: B \to Y \) are in bijection with group homomorphisms \( \mathbb{Z}^j \to \pi \), where \( \ell \) is the number of indices \( i_k \) that are 1. Since \( \pi \) is finite, precomposing with a suitable map \( \varphi: B \to B \) of non-zero degree induces the constant homomorphism \( \mathbb{Z}^j \to \pi \), which implies that \( f \circ \varphi \) is null.

If \( 2 \leq j \leq \dim(B) \), homotopy classes of maps \( B \to Y \) are in bijection with \( H^j(B; \pi) \), and we can find a suitable map \( \varphi: B \to B \) of non-zero degree that induces multiplication by a multiple of the order of \( \pi \) on \( j \)th cohomology, which implies that \( f \circ \varphi \) is null.

Employing an inductive argument over the Moore–Postnikov tower of \( Y \) that decomposes into fiber sequences \( Y(j + 1) \to Y(j) \to K(j, \pi_j(Y)) \), we can precompose with suitable maps \( \varphi_j: B \to B \) of non-zero degree such that \( f \circ \varphi_1 \circ \cdots \circ \varphi_j : B \to Y \) can be lifted along \( Y(j + 1) \to Y(j) \). At the end, we have produced a map \( \varphi = \varphi_1 \circ \cdots \circ \varphi_{\dim(B)}: B \to B \) of non-zero degree such that \( f \circ \varphi \) can be lifted along \( X(\dim(B) + 1) \to Y \). But any map \( B \to Y(\dim(B) + 1) \) is null, which finishes the proof. \( \square \)

**Proof of Lemma 2.3.** The given bundle \( \pi \) is classified by a map \( f: B \to B\text{Diff}_0(D^n) \), well-defined up to homotopy. If \( n \) is even, then it follows from a combination of [Kup19] and [Ran17, Theorem 4.1] that the assumption on \( Y \) in Lemma 2.4 below is satisfied for \( Y = B\text{Diff}_0(D^n) \), so (i) follows from this lemma.

For (ii), we first note that the fiber sequence

\[
B\text{Diff}_0(D^{n+1}) \to B\text{Diff}(D^{n+1}, D^n_-) \to B\text{Diff}_0(D^n_+),
\]

where \( D^n_+ \) denotes the upper hemisphere of \( \partial D^{n+1} = S^n \), can be delooped with respect to the canonical \( E_d \) structure [BL74], which gives rise to an exact sequence

\[
[B, B\text{Diff}(D^{n+1}, D^n_-)] \to [B, B\text{Diff}_0(D^n)] \to [B, B^2 \text{Diff}_0(D^{n+1})]
\]

of pointed sets of homotopy classes.

Since \( n \) is odd and hence \( n + 1 \geq 6 \) is even, it follows from [Kup19] and [Ran17, Theorem 4.1] as above that the space \( B^2 \text{Diff}_0(D^{n+1}) \) has finite homotopy groups in degrees at most \( n - 3 \). Thus, Lemma 2.4 below implies that there exists a map \( \varphi: B \to B \) of non-zero degree such that \( \delta(f \circ \varphi) = \delta(f) \circ \varphi = 0 \), hence \( \varphi^* \pi \) is the bundle of boundaries of a smooth \( D^{n+1} \)-bundle, as claimed. \( \square \)
2.2. Characteristic Classes. In this section, we prove that the total space of the fiber bundle \( S^p \times S^q \to E \to S^{4i-p} \times S^{4j-q} \) constructed above has non-vanishing \( \hat{A} \)-genus.

**Lemma 2.5.** The only two potentially non-vanishing Pontryagin-numbers of \( E \) are \( p_{i+j} \) and \( p_ip_j \), and the latter is actually non-zero. Furthermore, \( \hat{A}(E) \neq 0 \).

**Proof.** To verify the first claim, it suffices to consider the \( S^p \times S^q \setminus \tilde{D}^{p+q} \) bundle \( E_0 = S^{4i-p} \times S^{4j-q} \setminus D^{4i+p+q} \), where \( B \simeq S^{4i-p} \vee S^{4j-q} \) obtained from \( E \to S^{4i-p} \times S^{4j-q} \) previously constructed by cutting out a disk in the base, and show that the only non-zero Pontryagin classes of \( E_0 \) are \( p_i \) and \( p_j \), as all non-zero Pontryagin classes of \( E \) besides \( p_{i+j} \) remain non-zero under the map \( H^*(E, \mathbb{Q}) \to H^*(E_0, \mathbb{Q}) \). For this note that the core \( S^p \vee S^q \subset S^p \times S^q \setminus D^{p+q} \) is a deformation retract and is fixed under the handle twisting in **Construction 2.1**, hence the bundle \( E_0 \to S^{4i-p} \vee S^{4j-q} \) is trivial as a fibration.

The submanifolds \((S^p \vee S^q) \times (S^{4i-p} \vee \ast) \) and \((S^p \vee \ast) \times (\ast \vee S^{4j-q}) \) have a trivial normal bundle in \( E \), hence the stable tangent bundle of \( E \) is trivial when pulled back along the inclusion of these submanifolds. Moreover, the compositions

\[
(S^p \vee \ast) \times (S^{4i-p} \vee \ast) \to S^{4i} \xrightarrow{\alpha} BSO \quad \text{and} \quad (\ast \vee S^q) \times (\ast \vee S^{4j-q}) \to S^{4j} \xrightarrow{\beta} BSO
\]

classify the stabilized normal bundles of these spheres, where the first maps are given by collapsing the \((4i-1)\)- resp. \((4j-1)\)-skeleton. This implies that the only non-zero Pontryagin classes of \( E_0 \) are \( p_i \) and \( p_j \), and that these classes evaluate nontrivially when paired with the fundamental classes of the submanifolds \((S^p \vee S^q) \times (S^{4i-p} \vee \ast) \) and \((S^p \vee \ast) \times (\ast \vee S^{4j-q}) \). As these submanifolds intersect transversally in one point in \( E \), we deduce that \( p_ip_j(E) \) is non-zero.

The signature of any manifold that fibers over a sphere vanishes: if the dimension of the sphere satisfies \( d \geq 2 \), this follows from [CHS57]. If \( d = 1 \), note that the signature of any manifold is bounded by its middle Betti number, but also multiplies under coverings. So by pulling a bundle \( N \to S^1 \) back along finite-sheeted covering maps of \( S^1 \) implies that \( \sigma(N) = 0 \). We thus see that the signature \( \sigma(E) \) vanishes. The claim \( \hat{A}(E) \neq 0 \) then follows from the proceeding **Lemma 2.6**. \( \square \)

**Lemma 2.6.** Let \( M \) be a closed oriented manifold of dimension \( 4(i+j) \) such that all characteristic numbers of \( M \) except \( p_{i+j} \) and \( p_ip_j \) vanish, and \( p_ip_j(M) \neq 0 \). Then if the signature \( \sigma(M) \) vanishes, \( \hat{A}(M) \neq 0 \).

**Proof.** We will show that the corresponding polynomials \( L_{i+j} \) and \( \hat{A}_{i+j} \) are linearly independent in \( \mathbb{Q}[p_i,p_j,p_{i+j}]^{4(i+j)} \), which implies the assertion. Let us denote the coefficients of the \( L \)- and \( \hat{A} \)-polynomials as given in the following expressions.

\[
L = 1 + \frac{1}{3}p_1 + \cdots + s_ip_i + \cdots + s_jp_j + \cdots + s_{i+j}p_{i+j} + \cdots + s_{i,j}p_ip_j + \cdots
\]

\[
\hat{A} = 1 - \frac{1}{24}p_1 + \cdots + a_ip_i + \cdots + a_jp_j + \cdots + a_{i+j}p_{i+j} + \cdots + a_{i,j}p_ip_j + \cdots
\]
We claim that \( s_is_j + \lambda s_{i,j} \) and \( a_ia_j = a_{i+j} + \lambda a_{i,j} \) where \( \lambda = 2 \) if \( i = j \) and \( \lambda = 1 \) if \( i \neq j \). This can be derived from the the multiplicativity of the corresponding sequences [MS74], as follows. We consider the multiplicative sequence arising from \( L \), abusively denoted by the same variable, where we abbreviate the sequence \((0, \ldots, 0, 1, 0, \ldots)\) with the 1 at the \( \ell \)th position by \( e_\ell \).

Suppose first that \( i = j \). Then
\[
\begin{align*}
2s^2i,i &= ((1 + s_ie_i + s_{i,i}e_{2i} + \ldots)(1 + s_je_j + s_{j,j}e_{2j} + \ldots))_{i,j} \\
&= (L(1 + e_i)L(1 + e_i))_{i,j} \\
&= L((1 + e_i)(1 + e_i))_{i,j} \\
&= L(1 + 2e_i + e_{2i})_{i,j} \\
&= s_{2i} + 4s_{i,i},
\end{align*}
\]
and subtracting \( 2s_{i,i} \) gives the desired equation.

If \( i \neq j \), the calculation is slightly different. Without loss of generality, we may assume \( i < j \). Furthermore, we will first assume that \( i \) does not divide \( j \), in which case we have
\[
\begin{align*}
s_is_j &= ((1 + s_ie_i + s_{i,i}e_{2i} + \ldots)(1 + s_je_j + s_{j,j}e_{2j} + \ldots))_{i,j} \\
&= (L(1 + e_i)L(1 + e_j))_{i,j} \\
&= L((1 + e_i)(1 + e_j))_{i,j} \\
&= L(1 + e_i + e_j + e_{i+j})_{i,j} \\
&= s_{i+j} + s_{i,j},
\end{align*}
\]
as claimed. If \( i \) does divide \( j \), however, and \( r := j/i \) is integral, this computation has to changed in such a way that \( s_{i,i,...,i} \) (with \( i \) appearing \( r \) times) is added to both terms at the very beginning and end of the equation, which does not affect the truth of the identity. The proof of the identity for the coefficients of \( \hat{A} \) is completely analogous.

It remains to show that the matrix
\[
\begin{pmatrix}
s_{i,j} & s_{i+j} \\
a_{i,j} & a_{i+j}
\end{pmatrix}
\]
or equivalently the matrix
\[
\begin{pmatrix}
s_is_j & s_{i+j} \\
a_is_j & a_{i+j}
\end{pmatrix}
\]
is non-singular. Since for all \( k \geq 0 \),
\[
s_k = \frac{2^{2k}(2^{2k-1} - 1)B_{2k}}{(2k)!} \quad \text{and} \quad a_k = -\frac{B_{2k}}{2 \cdot (2k)!}
\]
(see [Hir66, Ch. 1, 3]), it thus suffices to check that the matrix
\[
\begin{pmatrix}
\frac{1}{4} & 2^{2(i+j)-1} - 1 \\
2^{2i-1} - 1 & -\frac{1}{2}
\end{pmatrix}
\]
is non-singular, which is clear since its determinant \( -\frac{1}{4}(2^{2i} - 1)(2^{2j} - 1) \) is always negative. \( \square \)

\(^1\)For \( L \), this identity is mentioned in the appendix of a preliminary version of [Wei16].
Remark 2.7. For a far more general formula of the coefficients of $L$- and $\hat{A}$-polynomials, see [FS16; BB18].

In order to make use of the bundle $E \to S^{4i-p} \times S^{4j-q}$ from above with applications to positive curvature in mind, we need the existence of a Spin-Structure on its vertical tangent bundle, or equivalently the total space of $E$. This follows from the next lemma since the base space $S^{4i-p} \times S^{4j-q}$ is Spin and the fiber $S^p \times S^q$ is 2-connected as $p, q > 2$ by assumption.

Lemma 2.8. For a 2-connected smooth $d$-manifold $F$ and a Spin-manifold $B$, the total space of any oriented smooth bundle $E \to B$ with fiber $F$ that contains a trivialized disk bundle admits a Spin-structure.

Proof. Let $\iota: B \times D^d \hookrightarrow E$ be the bundle embedding of trivial disk bundle. Then $B = B \times \{0\} \to B \times D^d \to E$ is a section of $E$. The map $E \to B$ is 3-connected and hence $\iota$ induces an isomorphism $H^2(B \times D^d) \to H^2(E)$. The fiber $F$ has a canonical Spin structure since it is 2-connected, and we get the following extension problem:

$$
\begin{array}{ccc}
F & \to & B\text{Spin}(d) \\
\downarrow & & \downarrow \\
E & \to & B\text{SO}(d)
\end{array}
$$

The obstruction to extending the map $F \to B\text{Spin}(d)$ lies in the group $H^2(E, F; \mathbb{Z}/2)$, since the fiber of $B\text{Spin}(d) \to B\text{SO}(d)$ is a $K(\mathbb{Z}/2, 1)$. The map $\iota$ also induces an isomorphism $H^2(E, F; \mathbb{Z}/2) \to H^2(B, B \times D^d; \mathbb{Z}/2)$ by the long exact sequence and the five-lemma. By naturality, the extension problem described above is thus equivalent to the one that arises if we try to extend a Spin-structure from $D^d$ to $B \times D^d$, which is possible since $B$ was assumed to be Spin. □

Remark 2.9. In Lemma 2.8, the assumption that $E$ contains a trivial disk bundle is essential: Let $\pi: V \to S^2$ be a vector bundle of rank $n \geq 3$ with nontrivial second Stiefel–Whitney class and let $S^{n-1} \to E \xrightarrow{\pi} S^2$ be the embedded sphere bundle. The stabilized tangent bundle of $E$ is given by

$$
TE \oplus \mathbb{R}^2 \cong (\pi^* TS^2 \oplus \mathbb{R}) \oplus (T\pi E \oplus \mathbb{R}) \cong \mathbb{R}^3 \oplus \pi^* V
$$

and hence $w_2(E) \neq 0$ and $E$ does not admit a Spin-structure.

The following proposition combines what we proved so far and is the essential result needed for the geometric applications in Section 3.

Proposition 2.10. The total space $E$ from Lemma 2.5 admits no metric of positive scalar curvature.

Proof. By the above discussion (Lemma 2.8), there is a Spin-structure on $E$. Using that we know $\hat{A}(E) \neq 0$ by Lemma 2.5, the statement about metrics of positive scalar curvature is now a well-known consequence of the Lichnerowicz formula [Lic63] and the Atiyah–Singer index theorem [AS63]. □
Recall the definition of the generalized Miller-Morita-Mumford-classes $\kappa_c \in H^*(B \text{Diff}^+(M))$, also known as tautological or simply $\kappa$-classes: for a smooth $M$-bundle $\pi: E \to B$, let $T_\pi E$ denote the vertical tangent bundle. Then for any class $c \in H^*(BSO(\dim(M)))$, $\kappa_c := \pi_! c(T_\pi E) \in H^{* - \dim(M)}(B)$, where $\pi_!$ stands for the Gysin map\(^2\). Note that for the bundle $\pi: E \to S^{4i-p} \times S^{4j-q}$, we have

\[
\langle \kappa_{\hat{A}_k}(\pi), [S^{4i-p} \times S^{4j-q}] \rangle = \langle \pi_! \hat{A}_k(T_\pi E), [S^{4i-p} \times S^{4j-q}] \rangle \\
= \langle \hat{A}(T(S^{4i-p} \times S^{4j-q})) \cup \pi_! \hat{A}_k(T_\pi E), [S^{4i-p} \times S^{4j-q}] \rangle \\
= \langle \hat{A}(\pi^* T(S^{4i-p} \times S^{4j-q})) \cup \hat{A}_k(T_\pi E), [E] \rangle \\
= \langle \hat{A}(TE), [E] \rangle = \hat{A}(E) \neq 0.
\]

where $\hat{A}_k$ is the degree $k$ homogeneous part of the $\hat{A}$-class. We conclude that $0 \neq \kappa_{\hat{A}_k}(E) \in H^{4k-d}(S^{4i-p} \times S^{4j-q}; \mathbb{Q})$. Since $E$ contains a trivial $D^d$-subbundle we deduce the following by glueing trivial bundles onto $\pi$.

**Corollary 2.11.** For any Spin-manifold $M$ of dimension $d = p + q$, there exists a smooth bundle $\pi_M: E_M \to S^{4i-p} \times S^{4j-q}$ with fiber $(S^p \times S^q)\# M$ whose vertical tangent bundle has Spin-structure and which satisfies $\kappa_{\hat{A}_k}(\pi_M) \neq 0$.

3. Applications to spaces of metrics

3.1. Metrics of positive Ricci-Curvature on $S^p \times S^q$. For the implications on the space $\mathcal{R}_{\text{Ric}>0}(S^p \times S^q)$, we use a slight upgrade of the detection principle from [BEW20]. To make it easier to follow the line of thought, it seems most natural to include the proof from loc. cit. and adjust it slightly.

Let $M$ be a $d$-dimensional closed Spin-manifold that admits a metric of positive Ricci-curvature and let $\pi: E \to B$ be an $M$-bundle over a connected space $B$ with a Spin-structure on the vertical tangent bundle. We get an associated $\text{Diff}(M)$-principal bundle $Q \to B$ such that $Q \times_{\text{Diff}(M)} M \cong E$ and we define

$$\mathcal{R}_{\text{Ric}>0}(\pi) := Q \times_{\text{Diff}(M)} \mathcal{R}_{\text{Ric}>0}(M) \xrightarrow{\Pi} B$$

**Theorem 3.1.** [BEW20, Theorem 2.1] In the situation described above, let us assume that

(i) The action of $\pi_1(B)$ on $H^0(\mathcal{R}_{\text{Ric}>0}(M))$ induced by the fiber transport factors through a finite group.

(ii) For some $k > \frac{d}{4}$ the class $\kappa_{\hat{A}_k}(E) \in H^{4k-d}(B; \mathbb{Q})$ is nontrivial.

Then there exists an $r \in \{2, \ldots, 4k - d\}$ such that $H^{4k-d-r}(B; \mathbb{Q}) \neq 0$ and $H^{r-1}(\mathcal{R}_{\text{Ric}>0}(M); \mathbb{Q}) \neq 0$.

**Proof.** If the action $\pi_1(B) \curvearrowright H^0(\mathcal{R}_{\text{Ric}>0}(M))$ factors through a finite group there is a finite cover $p: \tilde{B} \to B$ such that the action $\pi_1(B) \curvearrowright H^0(\mathcal{R}_{\text{Ric}>0}(M))$ is trivial. The induced map $p^*: H^k(B; \mathbb{Q}) \to H^k(\tilde{B}; \mathbb{Q})$ is injective and so the bundle $p^* E \to \tilde{B}$ satisfies assumption (ii), too. Therefore we may assume that the action $\pi_1(B) \curvearrowright H^0(\mathcal{R}_{\text{Ric}>0}(M))$ is trivial.

Consider the following diagram of fibrations.

\(^2\)This is often called fiber integration.
The bundle $\Pi^* E$ admits a canonical fiberwise metric of positive Ricci and hence positive scalar curvature. Therefore $\Pi^* \text{ind}(E) \in KO^{-d}(\mathcal{R}_{\text{Ric}>0}(\pi))$ vanishes by the Schrödinger–Lichnerowicz formula. By the cohomological version of the Atiyah–Singer family index theorem we have

$$\Pi^* \kappa_{\hat{A}_k} = (\text{ch}_{2k} \circ c)(\Pi^* \text{ind}(E)) = 0 \in H^{4k-d}(\mathcal{R}_{\text{Ric}>0}(\pi); \mathbb{Q})$$

where $\text{ch}_{2k}$ denotes the $2k$-th component of the Chern character and $c$ is the complexification map. So the nontrivial class $\kappa_{\hat{A}_k}$ lies in the kernel of

$$\Pi^*: H^{4k-d}(B; \mathbb{Q}) \rightarrow H^{4k-d}(\mathcal{R}_{\text{Ric}>0}(\pi); \mathbb{Q}).$$

Consider the Serre spectral sequence for the fibration $\mathcal{R}^{\text{Ric}>0}(\pi) \xrightarrow{\Pi} B$. The homomorphism $\Pi^*$ agrees with the composition

$$H^{4k-d}(B; \mathbb{Q}) \rightarrow H^{4k-d}(B; H^0(\mathcal{R}_{\text{Ric}>0}(M); \mathbb{Q})) = E_2^{4k-d,0} \rightarrow E_r^{4k-d-r,r-1} \rightarrow H^{4k-d}(\mathcal{R}_{\text{Ric}>0}(\pi); \mathbb{Q})$$

Since the action of $\pi_1(B)$ on $H^0(\mathcal{R}_{\text{Ric}>0}(M); \mathbb{Q})$ is trivial the coefficient system is simple and the first map is injective. So there is an element in the kernel of $E_2^{4k-d,0} \rightarrow E_r^{4k-d-r,r-1}$ and hence there exists an $r \in \{2, \ldots, 4d-k\}$ such that the differential $d^r: E_r^{4k-d-r,r-1} \rightarrow E_r^{4k-d,0}$ is nontrivial. It follows that $0 \neq E_r^{4k-d-r,r-1} = H^{4k-d-r}(B; H^{r-1}(\mathcal{R}_{\text{Ric}>0}(M); \mathbb{Q}))$ which enforces $H^{4k-d-r}(B; \mathbb{Q}) \neq 0$ and $H^{r-1}(\mathcal{R}_{\text{Ric}>0}(M); \mathbb{Q}) \neq 0$. \hfill \Box

**Proposition 3.2.** Theorem 3.1 holds moreover for every nonempty, $\text{Diff}(M)$-invariant subset $\mathcal{R}_C(M) \subset \mathcal{R}(M)$ which admits a $\text{Diff}(M)$-equivariant, continuous map $\mathcal{R}_C(M) \rightarrow \mathcal{R}_{\text{scal}>0}(M)$.

**Proof.** Let $f: \mathcal{R}_C(M) \rightarrow \mathcal{R}_{\text{scal}>0}(M)$ be continuous and $\text{Diff}(M)$-equivariant. Then any $M$-bundle $S \rightarrow X$ that admits a fiberwise metric $(g_x)_{x \in X}$ with $g_x \in \mathcal{R}_C(\pi^{-1}(x))$ also admits a fiberwise metric of positive scalar curvature: If $\alpha_x: M \cong \pi^{-1}(x)$ are some identifications of the fibers with $M$, then $((\alpha_x)^* f(g_x))_{x \in X}$ is a fiberwise metric of positive scalar curvature and since $f$ is $\text{Diff}(M)$-equivariant the above family does not depend on the choice of $\alpha_x$ and is hence well-defined.

Therefore the bundle $\Pi^*_C E \rightarrow \mathcal{R}_C(\pi)$ analogous to the bundle $\Pi^* E \rightarrow \mathcal{R}_{\text{Ric}>0}(\pi)$ admits a fiberwise metric of positive scalar curvature and the rest of the proof goes through without change. \hfill \Box

**Example 3.3.** The above is true for any nonnegative strict lower bound on sectional, Ricci, and scalar curvature, since the inclusion is continuous and $\text{Diff}(M)$-equivariant.
Example 3.4. By [BW07] a metric of nonnegative sectional curvature on a manifold with finite fundamental group evolves immediately into a metric of positive Ricci and hence positive scalar curvature under the Ricci flow. Furthermore, by [BGI20] the solution to the Ricci-flow depends continuously on the initial metric with respect to the $C^\infty$-topology on $\mathcal{R}(M)$ and we get a continuous map $f: \mathcal{R}_{\text{Sec} \geq 0}(M) \to (0, \infty)$ with the following property: If $g_t$ is the solution of the Ricci-flow with initial metric $g \in \mathcal{R}_{\text{Sec} \geq 0}(M)$ then $g_t \in \mathcal{R}_{\text{Ric} > 0}(M)$ for all $t < f(g)$. Since the Ricci flow preserves isometries, $f$ can be chosen to be $\text{Diff}(M)$-invariant and we get the following $\text{Diff}(M)$-equivariant continuous map

$$
\mathcal{R}_{\text{Sec} \geq 0}(M) \to \mathcal{R}_{\text{Ric} > 0}(M), \quad g \mapsto g_{f(g)/2}.
$$

This shows that Theorem 3.1 and as a consequence Theorem 3.5, Theorem A and Theorem B also hold for nonnegative sectional instead of positive Ricci curvature, provided the manifold $M$ has finite fundamental group.

Using the bundles constructed in Section 2 we derive our main result:

Theorem 3.5. Let $p, q, i, j \in \mathbb{N}$ be such that $2j < p < 4i$ and $2i < q < 4j$. Moreover, let $M$ be a Spin-manifold of dimension $(p + q)$ such that $M \# S^p \times S^q$ admits a metric of positive Ricci curvature. If $4i - p$ and $4j - q$ are both at least 2, then $H^m(\mathcal{R}_{\text{Ric} > 0}(M \# S^p \times S^q); \mathbb{Q}) \neq 0$ for some $m \in \{4i - p - 1, 4j - q - 1, 4(i + j) - (p + q) - 1\}$. If $4i - p = 1$ and the action of $\pi_0(\text{Diff}(S^p \times S^q, D^{p+q}))$ on $\pi_0(\mathcal{R}_{\text{Ric} > 0}(S^p \times S^q))$ factors through a finite group, then $H^m(\mathcal{R}_{\text{Ric} > 0}(M \# S^p \times S^q); \mathbb{Q}) \neq 0$ holds for some $m \in \{4j - q - 1, 4j - q\}$. If $4j - q = 1$ as well, then $m = 1$.

Proof. By the assumptions on $p, q, i, j$ there exists a $M \# (S^p \times S^q)$-bundle $E \to S^{4i-p} \times S^{4j-q}$ with a Spin-structure on the vertical tangent bundle and non-vanishing $\kappa_{A_k}$ by Corollary 2.11. The claim then follows immediately from Theorem 3.1. \qed

Remark 3.6. With the precise statement it is possible to illustrate two more improvements over the result from [BEW20]: Firstly we can give more precise estimates which rational cohomology groups are possibly nontrivial and secondly it is possible to show that multiple cohomology groups are nontrivial. The space $\mathcal{R}_{\text{Ric} > 0}(W^4_{12})$ for example has at least 5 non-vanishing rational cohomology groups, namely one out of each of $\{2, 5\}$, $\{6, 13\}$, $\{10, 21\}$, $\{14, 29\}$ and $\{18, 37\}$.

In order to deduce Theorem A and Theorem B, we will recall the rational Hurewicz theorem.

Theorem 3.7. Let $X$ be a simply connected space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $2 \leq i \leq r$. Then the Hurewicz map induces an isomorphism

$$
H: \pi_i(X) \otimes \mathbb{Q} \to H_i(X; \mathbb{Q})
$$

for $1 \leq i < 2r + 1$ and a surjection for $i = 2r + 1$.

Proof of Theorem A. Let us abbreviate $\mathcal{R} := \mathcal{R}_{\text{Ric} > 0}(M)$ for this proof. We start by considering the case $d = 7 + 8k$ for some $k \geq 1$. Let us first assume that $\mathcal{R}$ is simply connected. For $p = 4k + 3$, $q = 4k + 4$, $i = k + 2 = j$, Theorem 3.5 implies that $H^k(\mathcal{R}; \mathbb{Q}) \neq 0$ for some $k \in \{3, 4, 8\}$ and by the universal coefficient
If \( \pi_i(\mathcal{R}) \otimes \mathbb{Q} = 0 \) for \( i = 2, 3, 4 \), the rational Hurewicz theorem enforces \( \pi_8(\mathcal{R}) \otimes \mathbb{Q} = 0 \). If \( \pi_1(\mathcal{R}) \) is finite, consider the universal covering \( \tilde{\mathcal{R}} \to \mathcal{R} \). This yields a transfer map \( p_1: H^k(\tilde{\mathcal{R}}; \mathbb{Q}) \to H^k(\mathcal{R}; \mathbb{Q}) \) with \( p_1 \circ p^* = \text{id} \) and hence \( p_1 \) is surjective. If \( \pi_i(\mathcal{R}) \otimes \mathbb{Q} = 0 \) for \( i = 2, 3, 4 \), the same holds for \( \tilde{\mathcal{R}} \). Therefore by the same argument as above we get \( \pi_8(\mathcal{R}) \otimes \mathbb{Q} = \pi_8(\tilde{\mathcal{R}}) \otimes \mathbb{Q} \neq 0 \). The other cases are completely analogous and the required choices for \( p, q, i, j \) are given in the table below for \( k \geq 1 \) (with the exception for \( d = 13 \)).

| \( d \) | \( p \) | \( q \) | \( i \) | \( j \) | \( 4i - p - 1 \) | \( 4j - q - 1 \) | \( k \in \) |
|---|---|---|---|---|---|---|---|
| 8k + 2 | 4k + 1 | 4k + 1 | \( k + 1 \) | \( k + 1 \) | 2 | 2 | \{2, 5\} |
| 8k + 3 | 4k + 1 | 4k + 2 | \( k + 1 \) | \( k + 1 \) | 2 | 1 | \{2, 4\} |
| 8k + 4 | 4k + 2 | 4k + 2 | \( k + 1 \) | \( k + 1 \) | 1 | 1 | \{3\} |
| 8k + 5 | 4k + 2 | 4k + 3 | \( k + 1 \) | \( k + 2 \) | 1 | 4 | \{4, 6\} |
| 8k + 6 | 4k + 3 | 4k + 3 | \( k + 2 \) | \( k + 2 \) | 4 | 4 | \{4, 9\} |
| 8k + 7 | 4k + 3 | 4k + 4 | \( k + 2 \) | \( k + 2 \) | 4 | 3 | \{3, 4, 8\} |
| 8k + 8 | 4k + 4 | 4k + 4 | \( k + 2 \) | \( k + 2 \) | 3 | 3 | \{3, 7\} |
| 8k + 9 | 4k + 4 | 4k + 5 | \( k + 2 \) | \( k + 2 \) | 3 | 2 | \{2, 3, 6\} |

Note that the case \( d = 13 \) has to be excluded since it forces \( j = 2 \) and hence \( 4j - q - 1 = 0 \). Therefore the assumption that the action of \( \pi_0(\text{Diff}(S^6 \times S^7, D^{13})) \) on \( \pi_0(\mathcal{R}_{\text{Ric} > 0}(S^6 \times S^7)) \) factors through a finite group is required in this case (cf. Remark 1.4).

**Proof of Theorem B.** If the pullback action \( \pi_0(\text{Diff}(W^d_1, D)) \curvearrowright \pi_0(\mathcal{R}_{\text{Ric} > 0}(W^d_1)) \) factors through a finite group, then Theorem 3.5 implies \( H^1(\mathcal{R}_{\text{Ric} > 0}(M); \mathbb{Q}) \neq 0 \) and hence \( \pi_1(\mathcal{R}_{\text{Ric} > 0}(M))^{ab} \otimes \mathbb{Q} \neq 0 \). If the pullback action does not factor through a finite group, then the orbit of this action gives an infinite family of components of \( \mathcal{R}_{\text{Ric} > 0}(W^d_1) \). By [Fre19, Theorem B] the action is trivial on \( \pi_0(\mathcal{R}_{\text{scal} > 0}(W^d_1)) \) or factors through \( \mathbb{Z}/2 \) depending on the dimension. Hence this infinite family of components of \( \mathcal{R}_{\text{Ric} > 0}(W^d_1) \) gets mapped to the the same component. \( \square \)

### 3.2. The action of Diff(M) on \( \mathcal{R}_{\text{scal} > 0}(M) \)

Let \( F \) be a compact orientable manifold and let us consider an oriented fiber bundle of the form \( F \to E \to S^k \) for \( k \geq 0 \). This is classified by the homotopy class of a map \( \alpha: S^{k-1} \to \text{Diff}^+(F) \) based at the identity by clutching together two trivial \( F \)-bundles over the disk \( D^k \). This means \( P \) is given by

\[
E = (D^k \times F \amalg D^k \times F) / (s, f) \sim (s, \alpha(s)(f))
\]

for \( s \in S^{k-1} \). Assuming that \( F \) carries a metric \( g_F \) of positive scalar curvature, we get an orbit map \( \rho: \text{Diff}^+(F) \to \mathcal{R}_{\text{scal} > 0}(F) \) to the space of psc metrics on \( F \) via pullback \( \rho(f) = f^*g_F \). We have the following well known observation (see e.g. [HSS14, Remark 1.5]). The proof consists of constructing a metric on \( E \) that has fiberwise positive scalar curvature, shrinking the fibers and using the O’Neill formulas or by an explicit computation.

**Proposition 3.8.** If \( \rho_\ast[\alpha] = 0 \in \pi_{k-1}(\mathcal{R}_{\text{scal} > 0}(F), g_F) \), then \( E \) admits a psc-metric.
Now let $S^p \times S^q \to E \xrightarrow{\pi} S^{4i-p} \times S^{4j-q}$ be the bundle from Section 2. Let $M := \pi^{-1}(S^{4i-p} \times \{1\})$. Then $E$ is an $M$-bundle over $S^{4j-q}$ with a $\text{Spin}$-structure on the vertical tangent bundle, non-vanishing $\hat{A}$-genus and hence no metric of positive scalar curvature. In order to apply Proposition 3.8 we need the existence of a positive scalar curvature metric on $M$ which is guaranteed by the following Lemma.

**Lemma 3.9.** Let $F' \to F \to S^n$ be an oriented smooth fiber bundle with simply connected, stably parallelizable fiber $F'$ that admits a metric of positive scalar curvature. If $F$ is non-spinnable or if $F$ is $\text{Spin}$ and $d \not\equiv 1, 2 \pmod{8}$, then $F$ admits a metric of positive scalar curvature as well. If $F$ is $\text{Spin}$ and $d \equiv 1, 2 \pmod{8}$, then a sufficient condition for $F$ to admit a positive scalar curvature metric is that the element $\alpha \in \pi_n(\text{BDiff}^+(F'))$ classifying the bundle is divisible by 2.

**Proof.** For $n = 1$ the Lemma follows in both cases from [Fre19] as the mapping class group acts trivially on $\pi_0(\mathcal{R}_{\text{scal}} > 0(F'))$ or factors through $\mathbb{Z}/2$ depending on the dimension. Hence the mapping torus admits a psc-metric under the named assumptions. For $n \geq 2$ we note that $F$ is simply connected and thus admits a psc-metric if it is either non-spinnable, or $\text{Spin}$ and the $\alpha$-invariant of $F$ vanishes (cf. [GL80] and [Sto92]). If $d \not\equiv 1, 2 \pmod{8}$, the $\alpha$-invariant agrees with the $\hat{A}$-genus which vanishes because stably parallelizable manifolds are $\hat{A}$-multiplicative by [HSS14, Proposition 1.9]. If $d \equiv 1, 2 \pmod{8}$ and $F$ is $\text{Spin}$, $\alpha(F) \in KO^{-d}(*) \cong \mathbb{Z}/2\mathbb{Z}$ vanishes because of the given condition since the canonical map $\pi_n(\text{BDiff}^+(F')) \to \Omega^SO_\ast(\text{BDiff}^+(F')) \to \Omega^SO_d$ is a homomorphism. □

Theorem C now follows immediately from Proposition 2.10, Proposition 3.8 and Lemma 3.9.

**Remark 3.10.** Theorem C holds for every Riemannian condition that is satisfied by $M$ and satisfies the hypothesis of Proposition 3.2. Therefore, having a more explicit construction compared to the one from [HSS14] could yield non-triviality results for the induced map $\pi_k(\text{Diff}(M)) \otimes \mathbb{Q} \to \pi_k(\mathcal{R}_{C}(M)) \otimes \mathbb{Q}$.

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Email address: georg.frenck@kit.edu
Email address: math@frenck.net

INSTITUT FÜR ALGEBRA UND GEOMETRIE, ENGELSTR. 2, 76131 KARLSRUHE, GERMANY
Email address: jens.reinhold@uni-muenster.de

MATHEMATISCHES INSTITUT, EINSTEINSTR. 62, 48149 MÜNSTER, GERMANY