TIME FRACTIONAL AND SPACE NONLOCAL STOCHASTIC BOUSSINESQ EQUATIONS DRIVEN BY GAUSSIAN WHITE NOISE

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Abstract. We present the time-spatial regularity of the nonlocal stochastic convolution for Caputo-type time fractional nonlocal Ornstein-Uhlenbeck equations, and establish the existence and uniqueness of mild solutions for time fractional and space nonlocal stochastic Boussinesq equations driven by Gaussian white noise.

1. Introduction. The current paper is devoted to the following time fractional nonlocal stochastic Boussinesq equations (FBEs) driven by Gaussian noise on two dimensional torus $T^2 = [0, 1] \times [0, 1]$:

\[
\begin{align*}
\mathcal{D}_t^\alpha u + (u \cdot \nabla)u + (-\Delta)^{\beta_1} u + \nabla p &= \theta e_2 + \frac{dW_1(t)}{dt}, \\
\mathcal{D}_t^\beta \theta + (u \cdot \nabla) \theta + (-\Delta)^{\beta_2} \theta &= \frac{dW_2(t)}{dt}, \\
\nabla \cdot u &= 0, \\
u(t, 0) &= u(t, 1) = 0, \quad \theta(t, 0) = \theta(t, 1) = 0, \\
u(0, x) &= u_0(x), \quad \theta(0, x) = \theta_0(x),
\end{align*}
\]

(1)

where the variable $u(t, x) = (u_1, u_2)$ represents the velocity vector field, variable $p = p(t, x)$ is the scalar pressure, variable $\theta(t, x)$ represents the scalar temperature, $\beta_1, \beta_2 \in (1, 2)$ and vector $e_2 = (0, 1)$, $W_i, (i = 1, 2)$ stands for $L^2(0, 1)$-cylindrical Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with normal filtration $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}, t \in [0, T]$. $\alpha \in (0, 1)$ is a fixed number representing the order of time fractional differential operator and $\mathcal{D}_t^\alpha$ is the Caputo fractional derivative to be introduced later. The fractional Laplacian $(-\Delta)^\gamma$ can be regarded as a pseudo differential operator with $|\xi|^{2\gamma}$ and can be realized through the Fourier transform:

\[
(-\Delta)^\gamma u(\xi) = |\xi|^{2\gamma} \hat{u}(\xi),
\]

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where $\hat{u}$ is the Fourier transform of $u$.

There are several papers on the well-posedness for time fractional partial differential equations. Lions studied the weak solution of fractional Navier-Stokes equation ([7]), Shinbrot studied the fractional derivatives of solutions of the Navier-stokes equations in 1971, see [9] for details. Recently, there are some papers on the Caputo-type time fractional Navier-Stokes equation

$$D_t^\alpha u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad (2)$$

Carvalho-Neto and Planas in [2] established the existence and uniqueness of mild solutions of time fractional Navier-Stokes equation (2). Zhou and Peng in [12] and [13] studied the weak solution for equation (2) in Besov space. Very recently, Wang, Xu & Kloeden investigated the asymptotic behavior of a stochastic lattice system with a Caputo fractional time derivative in [10]. Zeng and Yang [11] studied the mild solution of the time fractional Navier-Stokes equations driven by fractional Brownian motions. To the best of our knowledge, there is a few paper to study the regularity of time-space fractional partial differential equations.

The novelty in this paper is to establish the time regularity and space regularity for the nonlocal stochastic convolution, which can be used in other time and space fractional stochastic fluid equations. To the end, the restriction are imposed on the order of fractional derivative $\alpha$ and the order of spatial nonlocal effects $\beta_1, \beta_2$. We also prove the existence and uniqueness of mild solutions of (1) by the the Banach fixed point Theorem. Moreover, the dependence of the order of time-fractional derivative, the order of the space-fractional derivative and the regularity of the initial data are revealed.

The rest of the paper is organized as follows. Some basic concepts, the function setting and the definition of mild solution of (1) are presented in section 2. In section 3, the regularity of the nonlocal stochastic convolution are established. The existence and uniqueness of mild solutions for time fractional nonlocal stochastic Boussinesq equations are established in section 4.

2. Preliminaries. In this section, we present the definitions of fractional operators and definition of mild solutions for stochastic systems (1), which is taken from [11].

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of function $f \in L^1([0,T]; X)$ is defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds. \quad (3)$$

**Definition 2.2.** The Caputo fractional derivative of order $\alpha \in (0,1)$ of function $f \in C([0,T]; X)$ is defined by

$$D_t^\alpha f(t) := \frac{d}{dt}[I_t^{1-\alpha} (f(t) - f(0))] = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds. \quad (4)$$

**Definition 2.3.** The Mainardi’s function is defined by

$$M_\alpha(z) = \sum_{k=1}^{\infty} \frac{(-z)^k}{k! \Gamma(-\alpha k + 1 - \alpha)}, 0 < \alpha < 1, z \in \mathbb{C}. \quad (5)$$

In addition, $M_\alpha(z) \geq 0$ for all $t \geq 0$ and satisfies the following equality

$$\int_0^\infty t^r M_\alpha(t)dt = \frac{\Gamma(r+1)}{\Gamma(\alpha r + 1)}, r > -1, 0 < \alpha < 1. \quad (6)$$
Next we give some notations of Banach spaces which the mild solutions exist in. Let $H_1 = \{ u \in (L^2_{\text{per}}(\mathbb{T}^2))^2 : \nabla \cdot u = 0 \}$, $H_2 = \{ \theta \in L^2_{\text{per}}(\mathbb{T}^2) \}$, $H = H_1 \times H_2$, with the norm

$$
\|u\|_{H_1}^2 = \int_{\mathbb{T}^2} |u|^2 dx, \quad \|	heta\|_{H_2}^2 = \int_{\mathbb{T}^2} |\theta|^2 dx, \quad \|\phi\|_{H_1}^2 = \|u\|_{H_1}^2 + \|\theta\|_{H_2}^2,
$$

where $\phi = (u, \theta) \in H$. For simplicity, we use the notations $\|u\|$, $\|	heta\|$ and $\|\phi\|$ to represent the norm for space $H_1$, $H_2$ and $H$ respectively.

Let

$$
e_{m,n}(x) = \sqrt{\frac{2}{\pi}} \sin m\pi x \sin n\pi x, \quad \lambda_{m,n} = \pi^2 (m^2 + n^2), \quad m, n \in \mathbb{N}.
$$

Then $(e_{m,n}, \lambda_{m,n})$ are the eigenvectors and eigenvalues of $(-\Delta)$ with Dirichlet boundary conditions. Denote $A = -\Delta$ and the operator $A^{\frac{\sigma}{2}}$ is well defined in the space of functions $H^\sigma = \{ f = \Sigma f_{m,n} e_{m,n} \in L^2(0, 1) : \|f\|_{H^\sigma} := (\Sigma f_{m,n} \lambda_{m,n}^\sigma)^{\frac{1}{2}} < \infty \}$.

Then we denote $\|u\|_{V_1} = \|u\|_{H^\sigma}$, $\|A^{\frac{\sigma}{2}} u\|_{H_1}$, and $\|	heta\|_{V_2} = \|	heta\|_{H^\sigma}$, $\|A^{\frac{\sigma}{2}} \theta\|_{H_2}$.

In what follows, we define the bilinear operators $B_1$ and $B_2$ by

$$
\langle B_1(u, v), w \rangle = b_1(u, v, w) = \int_D (u \cdot \nabla v) w dx = \Sigma_{i,j=1,2} \int_D u_i \partial_j v_j w_i dx,
$$

$$
\quad u, v, w \in H_1 \cap V_1 \cap (H^1)^2,
$$

and

$$
\langle B_2(u, \theta), w \rangle = b_2(u, \theta, w) = \int_D (u \cdot \nabla \theta) w dx
$$

$$
= \Sigma_{i=1,2} \int_D u_i \partial_i \theta w dx, \quad u, \theta, w \in H_2 \cap V_2 \cap H^1.
$$

Note that $b_1(u, v, w) = -b_1(w, u, v)$ and $b_2(u, \theta, w) = -b_2(u, \theta, w)$. Then Equation (1) can be rewritten in the following abstract evolution equation

$$
\begin{align*}
\left\{ \begin{array}{l}
\{ d[I_t^{1-\alpha} (u - u_0)] + (B_1(u, u) + A^{\alpha}_{\varphi} u + \nabla p - \theta e_2) dt = dW_1, \\
\{ d[I_t^{1-\alpha} (\theta - \theta_0)] + (B_2(u, \theta) + A^{\alpha}_{\varphi} \theta) dt = dW_2,
\end{array} \right.
\end{align*}
$$

(7)

where $I_t^{1-\alpha}$ is the $(1-\alpha)$-order Riemann-Liouville fractional integral operator.

Define the Mittag-Leffler families operators based on the analytic semigroup $S(t)$ generated by the space fractional operator $A^{\frac{\sigma}{2}}$:

$$
T_{\alpha, \beta}(t) = \int_0^\infty M_\alpha(s) S(st^\alpha) ds,
$$

and

$$
S_{\alpha, \beta}(t) = \int_0^\infty \alpha s M_\alpha(s) S(st^\alpha) ds.
$$

Then we can define the mild solution of (7) through the Mittag-Leffler families operators.
Definition 2.4. An $\mathbb{F}_t$-adapted random field $\{(u(t, x), \theta(t, x)), t \geq 0, x \in \mathbb{T}^2\}$ is said to be a mild solution of (8) with initial value $(u_0, \theta_0)$ if the following integral equation is fulfilled:

$$
\begin{aligned}
\begin{cases}
 u(t, x) = T_{\alpha, \beta} u_0 - \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_1} (t-\tau) B_1 (u, u) d\tau \\
 - \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_1} (t-\tau) \theta_2 d\tau \\
 + \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_1} (t-\tau)dW_1 (\tau),
\end{cases}
\end{aligned}
$$

$$
\theta(t, x) = T_{\alpha, \beta_2} \theta_0 - \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_2} (t-\tau) B_2 (u, \theta) d\tau \\
+ \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_2} (t-\tau) dW_2 (\tau).
$$

3. Regularity of the nonlocal stochastic convolution. Here, we consider the nonlocal stochastic convolution

$$
z_{\beta}(t) = \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta} (t-\tau) dW (\tau),
$$

where

$$
W (t) = \sum_{k=1}^{\infty} \beta_k e_k(t), \quad t \in [0, T],
$$

where $e_k, k \in \mathbb{N}$ is an orthonormal basis of $L^2(0, 1)$ and $\beta_k, k \in \mathbb{N}$ is a family of independent real-valued Brownian motions.

Lemma 3.1. For $\alpha \in (\frac{1}{4}, 1)$ and $2k_1 \alpha < \beta \leq 2$ with $1 < k_1 < \frac{4}{\alpha}$. Then $z_{\beta}$ has a version $z_{\beta}(t, \xi), t \geq 0, \xi \in (0, 1), \text{Hölder continuous with respect to } t \geq 0, \xi \in [0, 1]$.

Proof. For $t_1, t_2 \in [0, T], t_1 > t_2$, we have

$$
\mathbb{E}|z_{\beta}(t_1, x) - z_{\beta}(t_2, x)|^2
\leq \sum_{m,n=1}^{\infty} \int_0^{t_1} \int_0^{t_2} |(t_1 - \tau)^{\alpha-1} \int_0^{\infty} \alpha M_{\alpha}(s) e^{-s(t_1-\tau)^{\alpha} A^\beta} e_{m,n}(x) ds|^2 d\tau
\leq \sum_{m,n=1}^{\infty} \int_0^{t_1} \int_0^{t_2} |(t_1 - \tau)^{\alpha-1} \int_0^{\infty} \alpha M_{\alpha}(s) e^{-s(t_1-\tau)^{\alpha} A^\beta} ds
\leq -(t_2 - \tau)^{\alpha-1} \int_0^{\infty} \alpha M_{\alpha}(s) e^{-s(t_2-\tau)^{\alpha} A^\beta} ds |e_{m,n}(x)|^2 d\tau := I_1 + I_2.
$$

Minkowski’s inequality yields that

$$
I_1 \leq \sum_{m,n=1}^{\infty} \left[ \int_0^{\infty} \alpha M_{\alpha}(s) \left( \int_0^{t_1} (t_1 - \tau)^{2(\alpha-1)} e^{-2s(t_1-\tau)^{\alpha} \lambda_{m,n}^\beta} d\tau \right)^{\frac{1}{2}} ds \right]^2
\leq \sum_{m,n=1}^{\infty} \left[ \int_0^{t_1} (t_1 - \tau)^{2(\alpha-1)-\alpha \theta_1 s^{-\theta_1} \lambda_{m,n}^{-\frac{\theta_1}{2}}} d\tau \right]^2
\leq C \sum_{m,n=1}^{\infty} \left[ \int_0^{t_1} (t_1 - \tau)^{4(\alpha-1)-(2-k_1) \alpha \theta_1 s^{-\theta_1} \lambda_{m,n}^{-\frac{\theta_1}{2}}} d\tau \right]^2
+ \int_0^{t_1} (t_1 - \tau)^{-k_1 \alpha \theta_1 s^{-\theta_1} \lambda_{m,n}^{-\frac{\theta_1}{2}}} d\tau \right]^2 ds\right]^2
$$

$$
+ \int_0^{t_1} (t_1 - \tau)^{-k_1 \alpha \theta_1 s^{-\theta_1} \lambda_{m,n}^{-\frac{\theta_1}{2}}} d\tau \right]^2 ds\right]^2
$$

$$
+ \int_0^{t_1} (t_1 - \tau)^{-k_1 \alpha \theta_1 s^{-\theta_1} \lambda_{m,n}^{-\frac{\theta_1}{2}}} d\tau \right]^2 ds\right]^2
$$
\[
\begin{align*}
&\leq C \sum_{m,n=1}^{\infty} \left[ \int_0^\infty \alpha s M_\alpha(s) \left( \lambda_{m,n}^{-\frac{\theta_1}{2}} s^{-\theta_1} |t_1 - t_2|^{4\alpha - 3 - (2-k_1)\alpha \theta_1} \right. \\
&\quad \left. + \int_{t_2}^{t_1} \lambda_{m,n}^{-\frac{\theta_1}{2}} (t_1 - \tau)^{-k_1\alpha \theta_1} s^{-\theta_1} d\tau \right) \frac{1}{2} ds \right]^2 \\
&\leq C \sum_{m,n=1}^{\infty} \int_0^\infty \alpha s^{-\frac{\theta_1}{2}} M_\alpha(s) ds \left( |t_1 - t_2|^{4\alpha - 3 - (2-k_1)\alpha \theta_1} \right.\\
&\quad \left. + |t_1 - t_2|^{1 - k_1\alpha \theta_1} \right) \right]^2 \\
&\leq C \sum_{m,n=1}^{\infty} (m^2 + n^2)^{-\frac{\theta_1}{2}} \left( |t_1 - t_2|^{4\alpha - 3 - (2-k_1)\alpha \theta_1} + |t_1 - t_2|^{1 - k_1\alpha \theta_1} \right),
\end{align*}
\]

where \( 1 < k_1 < \frac{3}{4} \).

Since for \( s > 1 \), it holds that

\[
\sum_{m,n=1}^{\infty} \frac{1}{(m^2 + n^2)^s} < \infty.
\]

Choosing \( \frac{\epsilon}{2} < \theta_1 < \frac{1}{k_1\alpha} \) and \( \alpha > \frac{2}{3} \) ensures that \( I_1 < \infty \).

Similarly, let \( \theta_2 > 2 + \frac{\epsilon}{3} \), then

\[
\begin{align*}
I_2 &\leq \sum_{m,n=1}^{\infty} \left[ \int_0^\infty \alpha s M_\alpha(s) \left( \int_0^{t_2} ||(t_1 - \tau)^{\alpha - 1} e^{-s(t_2-\tau)^\alpha \lambda_{m,n}^{\frac{\theta_1}{2}}} \right. \\
&\quad \left. - (t_2 - \tau)^{\alpha - 1} e^{-s(t_2-\tau)^\alpha \lambda_{m,n}^{\frac{\theta_1}{2}}} |^2 d\tau \right) \frac{1}{2} ds \right]^2 \\
&\leq \sum_{m,n=1}^{\infty} \left[ \int_0^\infty \alpha s M_\alpha(s) \left( \int_0^{t_2} ||(t_1 - \tau)^{\alpha - 1} e^{-s(t_2-\tau)^\alpha \lambda_{m,n}^{\frac{\theta_1}{2}}} \\
&\quad - (t_1 - \tau)^{\alpha - 1} e^{-s(t_2-\tau)^\alpha \lambda_{m,n}^{\frac{\theta_1}{2}}} |^2 d\tau \right) \frac{1}{2} ds \right]^2 \\
&\quad + \int_0^{t_2} e^{-2s(t_2-\tau)^\alpha \lambda_{m,n}^{\frac{\theta_1}{2}}} \left( (t_1 - \tau)^{\alpha - 1} - (t_2 - \tau)^{\alpha - 1} \right)^2 d\tau \right) \frac{1}{2} ds \right]^2 \\
&\leq C \sum_{m,n=1}^{\infty} \left[ \int_0^\infty \alpha s M_\alpha(s) \left( \int_0^{t_2} \lambda_{m,n}^{\beta} s^{-\theta_1} |t_1 - t_2|^2 (t_1 - \tau)^{2(\alpha - 1)} (t \\
&\quad - \tau)^{2(\alpha - 1)} e^{-2s(t_2-\tau)^\alpha \lambda_{m,n}^{\frac{\theta_1}{2}}} d\tau \\
&\quad + \int_0^{t_2} s^{-\theta_1} (t_2 - \tau)^{-\alpha \theta_1 \lambda_{m,n}^{-\frac{\theta_1}{2}}} |t_1 - t_2|^2 (t - \tau)^{2(\alpha - 2)} d\tau \right) \frac{1}{2} ds \right]^2 \\
&\leq C \sum_{m,n=1}^{\infty} \left[ \int_0^\infty \alpha s^{2-\frac{\theta_1}{2}} M_\alpha(s) ds \left( \int_0^{t_2} |t_1 - t_2|^2 (t_1 - \tau)^{2\alpha - 2} (t \\
&\quad - \tau)^{2(\alpha - 2) - \alpha \theta_2} d\tau \right) \right] \frac{1}{2}
\end{align*}
\]
Theorem 4 implies that Lemma 3.1 holds. The proof of Lemma 3.1 is complete.

Lemma 3.2. Let 

\[ \| A^{\frac{\beta}{2}} Z_{\beta}(t, x) \|_{H}^2 < \infty. \]  

Next, we study the space regularity of equation (9).

Lemma 3.2. Let \( \frac{4}{3} < k_3 < 2 \) and \( 0 < \sigma < \frac{\beta}{k_3} - 1 \), we have

\[ \| A^{\frac{\beta}{2}} Z_{\beta}(t, x) \|_{H}^2 < \infty. \]  

(10)
Proof. It follows from the fact \( \|e_{m,n}\|_{L^\infty} < 1 \) that
\[
\mathbb{E}[A^\frac{\sigma}{2} z_\beta(t, x)]^2_H
= \mathbb{E} \sum_{m,n=1}^\infty \langle A^\frac{\sigma}{2} \int_0^t (t-\tau)^{\alpha-1} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t-\tau)^\alpha} A^\frac{\sigma}{2} ds d\mathcal{W}(\tau), e_{m,n} \rangle^2
\leq \sum_{m,n=1}^\infty \int_0^t [\int_D A^\frac{\sigma}{2} (t-\tau)^{\alpha-1} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t-\tau)^\alpha} A^\frac{\sigma}{2} e_{m,n} ds dx]^2 d\tau
\leq C \sum_{m,n=1}^\infty \int_0^t \int_0^\infty \lambda_{m,n} - \beta \theta_4 \sum_{1+k_3 = \alpha \theta_4} (t-\tau)^{\alpha-1-\alpha \theta_4} \alpha^{1-\alpha \theta_4} M_\alpha(s) ds^2 d\tau
\leq C \sum_{m,n=1}^\infty (m^2 + n^2)^{\sigma-\beta \theta_4} \int_0^t [(t-\tau)^{4\alpha-4-k_3 \alpha \theta_4} + (t-\tau)^{-(4-k_3) \alpha \theta_4}] d\tau
\leq C \sum_{m,n=1}^\infty \sum_{1+k_3 = \alpha \theta_4} (m^2 + n^2)^{\sigma-\beta \theta_4} \big(T^{4\alpha-k_3 \alpha \theta_4-3} + T^{1-(4-k_3) \alpha \theta_4}\big),
\]
where \( \frac{\sigma+1}{\beta} < \theta_4 < \min\{\frac{4\alpha-3}{k_3 \alpha}, \frac{1}{4-k_3} \alpha\}. \) \( \square \)

4. Local well-posedness of mild solution for equations (1). Let \( R(\omega) \) be a large positive random variable such that
\[
R(\omega) = \sup_{t \in [0, T]} \{ \|z_{\beta_1}\|_{V_1}, \|z_{\beta_2}\|_{V_2}\}. \quad (11)
\]
Denote \( v(t) = u(t) - z_{\beta_1}(t), \eta(t) = \theta(t) - z_{\beta_2}(t) \). Then we have
\[
\left\{\begin{array}{l}
\{d[I_t^{-\alpha}(v - u_0)] + (B_1(v + z_{\beta_1}, v + z_{\beta_1}) + A^\frac{\sigma}{2} v - (\eta + z_{\beta_2}) e_2) dt = 0,
\{d[I_t^{-\alpha}(\eta - \theta_0)] + (B_2(v + z_{\beta_1}, \eta + z_{\beta_2}) + A^\frac{\sigma}{2} \eta) dt = 0.
\end{array}\right. \quad (12)
\]
Then (12) has a mild solution in the following form
\[
\left\{\begin{array}{l}
v = T_{\alpha, \beta_1} u_0 - \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_1}(t-\tau) B_1(v + z_{\beta_1}, v + z_{\beta_1}) d\tau
- \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_1}(t-\tau) (\eta + z_{\beta_2}) e_2 d\tau,
\eta = T_{\alpha, \beta_2} \theta_0 - \int_0^t (t-\tau)^{\alpha-1} S_{\alpha, \beta_2}(t-\tau) B_2(v + z_{\beta_1}, \eta + z_{\beta_2}) d\tau.
\end{array}\right.
\]
Denote the space \( B^T_R \)
\[
B^T_R = \{ (v, \eta) : (v, \eta) \in C([0, T]; V), \|v\|_{V_1} + \|\eta\|_{V_2} \leq R, \forall t \in [0, T]\},
\]
where \( V_1 = (H^{\sigma_1})^2 \) and \( V_2 = H^{\sigma_2} \).

**Theorem 4.1.** Assume \( (u_0, \theta_0) \in V, \|u_0\|_{V_1} + \|\theta_0\|_{V_2} \leq \frac{R}{T} \), and the conditions in Lemma 3.1 and Lemma 3.2 hold. Let \( \frac{3}{2} < k_3 < 1 \). For \( \alpha \in (\frac{3}{2}, 1) \) and \( 0 \leq \sigma_1 < \frac{\beta_1}{k_1} - 2 \) and \( 0 \leq \sigma_2 < \frac{\beta_2}{k_2} - 2 \), there exists a random variable \( T > 0 \) such that (12) has a unique mild solution in \( B^T_R \).
Proof. Taking any \((v, \eta)\) in \(B^2_{H_1}\), and define
\[
L v = T_{\alpha, \beta} u_0 - \int_0^t (t - \tau)^{-1/2} S_{\alpha, \beta_1}(t - \tau)B_1(v + z_{\beta_1}, v + z_{\beta_1})d\tau
- \int_0^t (t - \tau)^{-1} S_{\alpha, \beta_1}(t - \tau)(\eta + z_{\beta_2})e_2d\tau,
\]
\[
L \eta = T_{\alpha, \beta_2} \theta_0 - \int_0^t (t - \tau)^{-1} S_{\alpha, \beta_2}(t - \tau)B_2(v + z_{\beta_1}, \eta + z_{\beta_2})d\tau.
\]
Then
\[
\|L v\|_{V_1} \leq \|T_{\alpha, \beta} u_0\|_{V_1} + \| \int_0^t (t - \tau)^{-1/2} S_{\alpha, \beta_1}(t - \tau)B_1(v + z_{\beta_1}, v + z_{\beta_1})d\tau \|_{V_1}
+ \| \int_0^t (t - \tau)^{-1} S_{\alpha, \beta_1}(t - \tau)(\eta + z_{\beta_2})e_2d\tau \|_{V_1} := I_1 + I_2 + I_3,
\]
and
\[
\|L \eta\|_{V_2} \leq \|T_{\alpha, \beta_2} \theta_0\|_{V_2} + \| \int_0^t (t - \tau)^{-1} S_{\alpha, \beta_2}(t - \tau)B_2(v + z_{\beta_1}, \eta + z_{\beta_2})d\tau \|_{V_2}
= J_1 + J_2.
\]
We deduce that
\[
I_1 = \|A^{\alpha s} \int_0^\infty M_\alpha(s) e^{-st\alpha} A^\beta_{\alpha, \beta_1} u_0 ds\|_{H_1}
\leq \|A^{\alpha s} u_0\|_{H_1} \leq \|A^{\alpha s} u_0\|_{H_1} = \|u_0\|_{V_1}.
\]
It is clear that
\[
I_2 \leq \int_0^t (t - \tau)^{-1/2} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t - \tau)^\alpha} A^\beta_{\alpha, \beta_1} B_1(v + z_{\beta_1}, v + z_{\beta_1})ds d\tau.
\]
It follows from Hölder inequality and \(\|\nabla e_{m,n}\|_{L^\infty} \leq C\lambda_m^2\) that
\[
\| \int_0^\infty \alpha s M_\alpha(s) e^{-s(t - \tau)^\alpha} A^\beta_{\alpha, \beta_1} B_1(v + z_{\beta_1}, v + z_{\beta_1})ds \|_{V_1}^2
= \|A^{\alpha s} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t - \tau)^\alpha} A^\beta_{\alpha, \beta_1} B_1(v + z_{\beta_1}, v + z_{\beta_1})ds \|_{H_1}^2
\leq \sum_{m,n=1}^\infty \|\alpha s M_\alpha(s) B_1(v + z_{\beta_1}, v + z_{\beta_1})ds \|_{L^\infty}^2
\leq C \sum_{m,n=1}^\infty (m^2 + n^2)^{\beta_1 \theta_5} \|B_1(v + z_{\beta_1}, v + z_{\beta_1}, e_{m,n})\|_{L^\infty}^2
\leq C \sum_{m,n=1}^\infty (m^2 + n^2)^{\beta_1 \theta_5} \|B_1(v + z_{\beta_1}, v + z_{\beta_1})\|_{H_1}^2 \|\nabla e_{m,n}\|_{L^\infty}^2
\leq C \sum_{m,n=1}^\infty (m^2 + n^2)^{\beta_1 \theta_5} \|B_1(v + z_{\beta_1}, v + z_{\beta_1})\|_{H_1}^2 \|\nabla e_{m,n}\|_{V_1}^2.
\]
Then
\[
I_2 \leq C \left( \sum_{m,n=1}^{\infty} (m^2 + n^2)^{1+\sigma_1 - \beta_1 \theta_5} \right) \frac{1}{2} \int_0^t (t-\tau)^{\alpha - 1 - \alpha \theta_5} d\tau \|v + z_{\beta_1} \|^2_{V_1} + (t- \tau)^{(2-k_4)\alpha \theta_5} d\tau \|v + z_{\beta_1} \|^2_{V_1} \\
\leq C \left( \sum_{m,n=1}^{\infty} (m^2 + n^2)^{1+\sigma_1 - \beta_1 \theta_5} \right) \frac{1}{2} \int_0^t (t-\tau)^{2\alpha - 2 - k_4 \alpha \theta_5} d\tau \|v + z_{\beta_1} \|^2_{V_1} + (t- \tau)^{(2-k_4)\alpha \theta_5} d\tau \|v + z_{\beta_1} \|^2_{V_1} \\
\leq C \left( \sum_{m,n=1}^{\infty} (m^2 + n^2)^{1+\sigma_1 - \beta_1 \theta_5} \right) \frac{1}{2} \left[ T^{2\alpha - 1 - k_4 \alpha \theta_5} + T^{1 - (2-k_4)\alpha \theta_5} \right] \|v + z_{\beta_1} \|^2_{V_1},
\]
where \( \frac{2+\sigma_1}{\beta_1} < \theta_5 < \min\left( \frac{2\alpha - 1}{\alpha \theta_5}, \frac{1}{(2-k_4)\alpha} \right) \).

To ensure \( I_2 < \infty \), it is enough to choose
\[
\frac{2}{3} < k_4 < 1, \quad 0 < \sigma_1 < \frac{\beta_1}{k_4} - 2.
\]
Similarly,
\[
\|S_{\alpha,\beta_1}(t-\tau)(\eta + z_{\beta_2}) d\tau\|^2_{V_1} \leq C \left( \sum_{m,n=1}^{\infty} (m^2 + n^2)^{\sigma_1 - \beta_1 \theta_5} \right) \frac{1}{2} \left[ T^{2\alpha - 1 - k_4 \alpha \theta_5} + T^{1 - (2-k_4)\alpha \theta_5} \right] \|\eta + z_{\beta_2} \|^2_{V_2},
\]
and
\[
I_3 \leq C \left( \sum_{m,n=1}^{\infty} (m^2 + n^2)^{\sigma_1 - \beta_1 \theta_5} \right) \frac{1}{2} \left[ T^{2\alpha - 1 - k_4 \alpha \theta_5} + T^{1 - (2-k_4)\alpha \theta_5} \right] \|\eta + z_{\beta_2} \|^2_{V_2}.
\]
Similarly to \( I_1 \) and \( I_2 \)
\[
J_1 \leq \|\theta_0\|_{V_2},
\]
and
\[
J_2 \leq C \left( \sum_{m,n=1}^{\infty} (m^2 + n^2)^{1+\sigma_1 - \beta_2 \theta_5} \right) \frac{1}{2} \left[ T^{2\alpha - 1 - k_4 \alpha \theta_5} + T^{1 - (2-k_4)\alpha \theta_5} \right] \|v + z_{\beta_1} \|_{V_1} \|\eta + z_{\beta_2} \|_{V_2}.
\]
By the estimates of \( I_1-I_3 \) and \( J_1-J_2 \), we infer that
\[
\|\mathcal{L}v\|_{V_1} + \|\mathcal{L}\eta\|_{V_2} \leq \frac{R}{2} + C \left[ T^{2\alpha - 1 - k_4 \alpha \theta_5} + T^{1 - (2-k_4)\alpha \theta_5} \right] R^2. \quad (14)
\]
Now let \((v_1, \eta_1), (v_2, \eta_2) \in \mathcal{B}_{R}^T\). Then
\[
\mathcal{L}v_1 - \mathcal{L}v_2 = \int_0^t (t-\tau)^{\alpha - 1} S_{\alpha,\beta_1}(t-\tau)(B_1(v_2 + z_{\beta_1}, v_2 + z_{\beta_1}) - B_1(v_1 + z_{\beta_1}, v_1 + z_{\beta_1})) d\tau \\
+ \int_0^t (t-\tau)^{\alpha - 1} S_{\alpha,\beta_1}(t-\tau)((\eta_2 + z_{\beta_2}) - (\eta_1 + z_{\beta_2})) e_2 d\tau, \quad (15)
\]
and
\[
\mathcal{L}\eta_1 - \mathcal{L}\eta_2 = \int_0^t (t-\tau)^{\alpha - 1} S_{\alpha,\beta_2}(t-\tau)(B_2(v_2 + z_{\beta_1}, \eta_2 + z_{\beta_2}) - B_2(v_1 + z_{\beta_1}, \eta_1 + z_{\beta_2})) d\tau. \quad (16)
\]
We can deduce that
\[
\|L_{1} - L_{2}\|v_{1} \\
\leq \int_{0}^{t} (t - \tau)^{\alpha - 1} \|S_{\alpha, \beta_{1}}(t - \tau)(B_{1}(v_{2} + z_{\beta_{1}}, v_{2} + z_{\beta_{1}}) - B_{1}(v_{1} + z_{\beta_{1}}, v_{1} + z_{\beta_{1}}))\|v_{1} d\tau \\
+ \int_{0}^{t} (t - \tau)^{\alpha - 1} \|S_{\alpha, \beta_{1}}(t - \tau)((\eta_{2} + z_{\beta_{2}}) - (\eta_{1} + z_{\beta_{2}}))\|v_{2} d\tau := K_{1} + K_{2},
\]
(17)
\[
\|L_{m} - L\eta_{2}\|v_{2} \\
= \int_{0}^{t} (t - \tau)^{\alpha - 1} \|S_{\alpha, \beta_{2}}(t - \tau)(B_{2}(v_{2} + z_{\beta_{1}}, \eta_{2} + z_{\beta_{2}}) - B_{2}(v_{1} + z_{\beta_{1}}, \eta_{1} + z_{\beta_{2}}))d\tau\|v_{2} \\
= H_{1}.
\]
(18)

For $K_{1}$, we have
\[
\|S_{\alpha, \beta_{1}}(t - \tau)(B_{1}(v_{2} + z_{\beta_{1}}, v_{2} + z_{\beta_{1}}) - B_{1}(v_{1} + z_{\beta_{1}}, v_{1} + z_{\beta_{1}}))\|^{2}
\leq C \sum_{m,n=1}^{\infty} (m^{2} + n^{2})^{\alpha - \beta_{1}\theta_{5}}(t - \tau)^{-2\alpha\theta_{5}}(\|B_{1}(v_{1} + z_{\beta_{1}}, v_{1} - v_{2})ds, e_{m,n})^{2}
\]
\[+(B_{1}(v_{1} - v_{2}, v_{2} + z_{\beta_{1}})ds, e_{m,n})^{2}
\]
\[\leq C \sum_{m,n=1}^{\infty} (m^{2} + n^{2})^{\alpha - \beta_{1}\theta_{5}}(t - \tau)^{-2\alpha\theta_{5}}(\|v_{1} + z_{\beta_{1}}\|^{2} + \|v_{2} + z_{\beta_{1}}\|^{2})\|v_{1} - v_{2}\|^{2}.
\]
Therefore
\[K_{1} \leq CR[T^{2\alpha - 1 - k_{4}\alpha\theta_{5}} + T^{1 - (2-k_{4})\alpha\theta_{5}}]\|v_{1} - v_{2}\|v_{1}.
\]

Similarly,
\[
\|S_{\alpha, \beta_{1}}(t - \tau)((\eta_{2} + z_{\beta_{2}}) - (\eta_{1} + z_{\beta_{2}}))\|^{2}
\leq C \sum_{m,n=1}^{\infty} (m^{2} + n^{2})^{\alpha - \beta_{1}\theta_{5}}(t - \tau)^{-2\alpha\theta_{5}}\|\eta_{1} - \eta_{2}\|^{2},
\]
which implies that
\[K_{2} \leq CR[T^{2\alpha - 1 - k_{4}\alpha\theta_{5}} + T^{1 - (2-k_{4})\alpha\theta_{5}}]\|\eta_{1} - \eta_{2}\|v_{2}.
\]

Similarly to $K_{1}$, we obtain
\[H_{1} \leq CR[T^{2\alpha - 1 - k_{4}\alpha\theta_{5}} + T^{1 - (2-k_{4})\alpha\theta_{5}}]\|v_{1} - v_{2}\|v_{1} + \|\eta_{1} - \eta_{2}\|v_{2}.
\]
Combining $K_{1}, K_{2}$ and $H_{1}$ gives
\[
\|L_{v_{1}} - L_{v_{2}}\|v_{1} + \|L\eta_{1} - L\eta_{2}\|v_{2} \\
\leq CR[T^{2\alpha - 1 - k_{4}\alpha\theta_{5}} + T^{1 - (2-k_{4})\alpha\theta_{5}}]\|v_{1} - v_{2}\|v_{1} + \|\eta_{1} - \eta_{2}\|v_{2}.
\]
(19)

Let $T$ be small enough such that $CR[T^{2\alpha - 1 - k_{4}\alpha\theta_{5}} + T^{1 - (2-k_{4})\alpha\theta_{5}}] < 1$. Then it follows from (14) and (19) that $L$ maps $B_{R}^{\theta}$ into itself and it is a contraction in $B_{R}^{\theta}$. So $L$ has a unique point in $C([0, T]; V)$ which is the unique solution of (12) on $[0, T]$.

\[\square\]

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