Conductance through quantum wires with Lévy-type disorder: Universal statistics in anomalous quantum transport

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Abstract – In this letter we study the conductance $G$ through one-dimensional quantum wires with disorder configurations characterized by long-tailed distributions (Lévy-type disorder). We calculate analytically the conductance distribution which reveals a universal statistics: the distribution of conductances is fully determined by the exponent $\alpha$ of the power-law decay of the disorder distribution and the average $\langle \ln G \rangle$, i.e., all other details of the disorder configurations are irrelevant. For $0 < \alpha < 1$ we found that the fluctuations of $\ln G$ are not self-averaging and $\langle \ln G \rangle$ scales with the length of the system as $L^\alpha$, in contrast to the predictions of the standard scaling theory of localization where $\ln G$ is a self-averaging quantity and $\langle \ln G \rangle$ scales linearly with $L$.

Our theoretical results are verified by comparing with numerical simulations of one-dimensional disordered wires.

Quantum coherent electronic transport through disordered conductors has been widely studied from a fundamental and practical point of view. For instance, the pioneering ideas by Anderson on the localization of electron wave functions in disordered conductors and the one-parameter scaling approach to this phenomenon [1] have been of large impact on condensed-matter physics. The progress in the theoretical description of the electronic transport has been accompanied and stimulated by advances in the fabrication of small electronic devices, where coherent electronic transport is possible.

On the other hand, random processes characterized by probability densities (Lévy-type distributions) $\rho(x)$ with long tails, i.e., for large $x$, $\rho(x) \sim 1/x^{1+\alpha}$ with $0 < \alpha < 2$, have been found in very different phenomena in nature and human activities. For instance, it has been seen that the movement patterns of some marine predators follow a Lévy-type distribution [2]. In economy, fluctuations of the stock market indices can be described by Lévy models [3]. Mathematicians have been investigating the properties of this class of probability densities since seminal works by Lévy [4,5]. Among these properties we remark the generalized central limit theorem (GCLT). We recall that the central limit theorem states that the normalized sum of independent variables with finite mean and variance is normally distributed in the limit of a large number of variables; the GCLT gets rid of the finite variance condition and the resulting limit probability distribution is a Lévy or $\alpha$-stable distribution. In physics, Lévy-type distributions have been applied to several areas such as statistical mechanics, fluids, and dynamics of chaotic systems [5]. Therefore Lévy processes have become of a broad interdisciplinary interest.

Lévy processes have been found in different phenomena, but an experimental study of these processes in a controllable way has not been possible until very recently. New techniques in the fabrication of materials have allowed the experimental realization and study of Lévy transport [6,7]. For instance, in ref. [6] SiC nanowires have been fabricated via self-organized processes. It turns out that the diameter of the SiC nanowires shows fluctuations whose distribution follows a power-law decay. In ref. [7] Barthelemy and collaborators were able to make a disordered medium where light travels across glass microspheres whose diameter follows a Lévy-type distribution. Motivated by these kind of experiments, here we shall study the quantum coherent transport of electrons across one-dimensional (1D) disordered wires with Lévy-type disorder. Previous

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Theoretical works [8] have pointed out and studied the effects of Lévy processes in the electronic transport.

In this work we obtain the distribution of the conductance of electrons traveling through a 1D disordered wire whose configuration of impurities follows a density distribution with a power-law width and are generated accordingly to a Levy-type distribution with $\alpha = 1/2$ (eq. (2)). As we predict $\langle -\ln G \rangle \propto L^{1/2}$. The solid line is obtained from eq. (7). A good agreement is seen between numerics (squares) and theory. Inset: $\langle -\ln G \rangle$ for a disorder distribution with $\alpha = 3/2$. In agreement with our model, $\langle -\ln G \rangle \propto L$, for $\alpha \geq 1$, as in usual disorder systems.

To study the conductance we adopt the Landauer-Büttiker approach in which the electronic transport is seen as a scattering problem. In this approach the dimensionless conductance $G$ (in units of the conductance quantum $2e^2/h$) is given by the transmission coefficient. With respect to the statistical study of the conductance, we shall use some results from the standard scaling theory developed in refs. [9,10]. These works considered the weak scattering limit ($s \ll l$, being $s$ and $l$ the Fermi wavelength and the mean free path, respectively) and found that the distribution of the conductance as a function of the length of the 1D wire is governed by a Fokker-Planck equation. Thus the solution of such a diffusion equation gives the distribution of conductances $p_s(G)$ for a wire of length $L$, which is written as [11]

$$p_s(G) = \frac{s^{-\frac{\alpha}{2}} e^{-\frac{G}{s^2}}}{\sqrt{2\pi G^2}} \int_{y_0}^{\infty} \frac{y e^{-\frac{y^2}{2}}}{\sqrt{\cosh y + 1 - 2/G^\alpha}} \, dy,$$

where $y_0 = \text{arccosh}(2/G - 1)$ and $s = L/l$. We make a couple of general comments on the scaling approach of ref. [10]. On the one hand, in order to obtain the above-mentioned diffusion equation, it is assumed that for a very small sample of length $\delta L$, the resistance average is fixed and depends linearly on $\delta L$. As a consequence the average $\langle -\ln G \rangle$ of the whole sample scales linearly with $L$: $\langle -\ln G \rangle = L/l$. On the other hand, one can see from eq. (1) that the distribution of the conductance is determined by the single parameter $s$. This is a remarkable fact since it means that all the information we need to give a statistical description of the conductance is the system length measured in units of the mean free path, all other microscopic details are irrelevant. This point was referred as a central limit theorem for weakly disordered wires and eq. (1) shows explicitly the single-parameter dependence of the statistics of the transport problem. In relation to the previous comments, we shall see that for Lévy-type disorder $\langle -\ln G \rangle$ does not scale linearly with $L$ but still the conductance distribution is determined by the average $\langle -\ln G \rangle$ and the exponent of the power-law decay of the disorder-configuration distribution.

As a further motivation, let us illustrate the fact that the ensemble average $\langle -\ln G \rangle$ does not scale linearly with $L$ in the presence of long-range disorder for $\alpha < 1$. In fig. 2 we show the numerical results for $\langle -\ln G \rangle$ for two different values of $\alpha$. For the numerical simulations, we consider a succession of square-potentials barriers with fixed height, but separations and widths randomly generated from a Lévy-type distribution [12]. The statistics is collected across an ensemble of different disorder realizations where the length of the system is fixed. For $\alpha = 1/2$, we can see clearly from fig. 2 (main frame) that $\langle -\ln G \rangle$ is not a linear function of $L$, whereas for $\alpha = 3/2$ (inset of fig. 2) $\langle -\ln G \rangle \propto L$, as in the standard scaling approach. The solid line in fig. 2 is obtained from our theoretical model which we explain below. In order to describe the results shown in fig. 2 and calculate the conductance distribution, it is essential to consider the strong fluctuations in the number of scatterers in a wire of length $L$ due to the long tails of the density distribution of the disorder configurations. For instance, if the tail of the density decays slowly as

$$\rho(x) \sim e^{-x^{1+\alpha}},$$

(2)
for large $x$ and $0 < \alpha < 1$, the first and second moment diverges. Actually, we shall concentrate precisely in these values of $\alpha$ as they exhibit the most interesting results for the conductance statistics. For the case $1 < \alpha < 2$, where the first moment is finite, the statistical properties of the conductance are similar to those described by the standard scaling theory (e.g., see the inset of fig. 2). This situation can also be treated along similar lines of this work 1.

Let us start by introducing the probability density $\Pi_L(\nu)$ for the number of scattering units $\nu$ in a wire of length $L$. If the separations and widths of the scattering units in the wire follow a law like eq. (2) one can show, using the properties of the Lévy distributions that 2

$$\Pi_L(\nu) = \frac{L}{\alpha (2\nu)^{\alpha/2}} q_{\alpha,c}(L/(2\nu)^{1/\alpha}), \quad (3)$$

for $0 < \alpha < 1$, in the macroscopic limit $L \gg c^{1/\alpha}$. Here $q_{\alpha,c}$ is the probability density function (PDF) of the Lévy distribution supported in the positive semiaxis that behaves like $\rho(x)$ for large values of $x$ (see eq. (2)). The concrete form of $q_{\alpha,c}(x)$ is better expressed using the Fourier transform

$$\tilde{q}_{\alpha,c}(k) = \exp(-|k|^\alpha (B\theta(k) + B^\ast \theta(-k))), \quad (4)$$

where $\theta$ is the Heaviside step function and $B = -c(\theta-\alpha)\Gamma(\frac{\alpha}{2})$, $\Gamma$ being the Gamma function. Note that the result for $\Pi_L(\nu)$, eq. (3), depends only of the parameters $c$ and $\alpha$. All other details of the PDF $\rho$ of the disorder configurations are washed out. This fact follows from the generalization of the central limit theorem to distributions with long tails (see footnote 2).

In order to continue we introduce some notation: let $(\ln G)_\nu$ and $(\ln G)_L$ be the expectation values for samples

1The results for $1 < \alpha < 2$ will be published elsewhere.

2Let us approximate the number of scatterers units $n$ by a continuous variable $\nu$ such that the probability of having exactly $n$ scatterers in a system of length $L$ is given by $\int_0^{\nu+1/2} \Pi_L(\nu) d\nu$. Therefore we have that

$$\int_0^{\nu+1/2} \Pi_L(\nu) d\nu = \int_0^\infty \rho_{2n+1}(x) dx,$$

where $\rho_{2n}(x)$ is the nth auto-convolution of $\rho(x)$ and represents the PDF of the total length after $m$ drawings.

Introducing the new variables $\varphi = L/(2\nu)^{1/\alpha}$, $\varphi = x/(2n+1)^{1/\alpha}$, and $\Pi_{\varphi}(\varphi) = (a/2)\varphi^\alpha \varphi^{-1/2} \Pi_L(\varphi^{1/2}/2)$. The previous relation reads

$$\int_A^{\infty} \Pi_{\varphi}(\varphi) d\varphi = \int_A^{\infty} (2n+1)^{-1/\alpha} \rho_{2n+1}((2n+1)^{1/\alpha} \varphi) d\varphi,$$

where $A = L(2n+1)^{-1/\alpha}$. If we take now the limit of large $L$ keeping $A$ constant and use the generalization of the central limit theorem which states that

$$\lim_{m \to \infty} m^{1/\alpha} \rho_{m}(m^{1/\alpha} \varphi) = q_{\alpha,c}(\varphi);$$

we obtain the relation

$$\lim_{L \to \infty} \frac{1}{A} \int_A^{\infty} \Pi_{\varphi}(\varphi) d\varphi = \int_A^{\infty} q_{\alpha,c}(\varphi) d\varphi,$$

from which our result, eq. (3), follows.

with a fixed number of scatterers $\nu$ and fixed length $L$, respectively. From the standard scaling theory of localization $(-\ln G)_\nu$ is proportional to $\nu$: $(-\ln G)_\nu = a\nu$, $a$ being a constant. Thus we have that

$$(-\ln G)_L = \int_0^\infty (-\ln G)_\nu \Pi_L(\nu) d\nu \quad (5)$$

$$= \int_0^\infty a\nu \frac{L}{(2\nu)^{1/\alpha}} q_{\alpha,c}(L/(2\nu)^{1/\alpha}) d\nu, \quad (6)$$

where we have used eq. (3). Using the scaling property of the Lévy distributions, namely $c^{1/\alpha} q_{\alpha,c}(c^{1/\alpha} x) = q_{\alpha,1}(x)$ and making the change of variable $z = L/(2\nu)^{1/2}$, we finally obtain

$$(-\ln G)_L = a \frac{L}{c} \frac{1}{2} \int_0^\infty z^{-\alpha} q_{\alpha,1}(z) dz = a \frac{L}{c} I_\alpha, \quad (7)$$

where $I_\alpha = (1/2) \int_0^\infty z^{-\alpha} q_{\alpha,1}(z) dz$. The last equality in (7) stands to emphasize the rôle played by the different elements ($c, \alpha$) of the original distribution and the physical characteristics ($\alpha, L$) of the system. We have shown, therefore, that $(-\ln G)_L \propto L^\alpha$, in contrast to the linear behavior with $L$ expected from the usual scaling theory. This is true for any value of $0 < \alpha < 1$, while the standard linear behavior is recovered for $\alpha \geq 1$. This was already illustrated in the inset of fig. 2. Moreover, $\ln G$ for $\alpha < 1$ is not self-averaging. That is, in the limit of large samples we found that the ratio $R = \lim_{L \to \infty} \text{var}(\ln G)/(\ln G)^2$ is zero for $\alpha \leq 1$, whereas $R = 0$ for $\alpha > 1$, as in the conventional scaling theory.

We now calculate the conductance distribution. As we mentioned before, in the standard scaling theory the complete conductance statistics is fully determined $(\ln G)_L$. We shall show that the same is true when the configuration of scattering units follows a PDF with long tails, with the only additional information of the parameter $\alpha$ that characterizes the distribution tail.

Assume that we have an ensemble of disordered wires with $(-\ln G)_L = \xi$ whose scattering units are distributed according to a long-tailed probability, as described before. The conductance distribution $P_G(G)$ is given by an integral that involves $p_s(G)$, given in eq. (1) with the parameter $s$ related to the number of scatterers, $s = a\nu$, and $\Pi_L(\nu)$ in eq. (3), i.e.,

$$P_G(G) = \int_0^\infty p_s(G) \Pi_L(\nu) d\nu. \quad (8)$$

Now, using eqs. (3) and (7) as well as the scaling properties of the Lévy distribution, we finally write the distribution of conductances as

$$P_G(G) = \int_0^\infty p_s(\alpha, \xi, z) q_{\alpha,1}(z) dz, \quad (9)$$

where we have made again the change of variable $z = L/(2\nu)^{1/2}$. We remark

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that our result for $P_\xi(G)$ (eq. (9)) depends only on two parameters $(-\ln G)_L = \xi$ and $\alpha$, as we have previously announced. Thus, other details of the disorder configurations are irrelevant.

Next we show some specific examples of the distribution of conductances as given by eq. (9). The theoretical results are compared with numerical simulations using the model of potential barriers introduced previously. First we illustrate the fact that $P_\xi(G)$ is only determined by the values of $\xi$ and $\alpha$. With this purpose we have calculated numerically the conductivity distributions from two differently generated ensembles of wires: the value of $\xi$ is the same in both cases, as well as the value of $\alpha$, but the disorder configurations of each ensemble follow different density distribution\(^3\). We thus expect the same conductance distribution for both cases. This is shown in fig. 3 where the distribution $P_\xi(G)$ for each case (histograms in black and red) are statistically the same. In addition, in fig. 3, we plot in solid line our theoretical result, eq. (9). A good agreement with the numerical simulations is seen. On the other hand, we point out that the conductance distributions exhibit two peaks, at $G \sim 0$ and $G \sim 1$. This is an unconventional behavior in 1D disordered systems and reveals the coexistence of insulating and metallic (ballistic) regimes. Interestingly, a similar behavior of the conductance distribution has been found in the so-called random-mass Dirac model, where anomalously localized electronic states might occur due to rare trapping disorder configurations. In the inset of fig. 3 we show a further example of the conductance distribution for wires with Lévy-type disorder with $\alpha = 3/4$. Again, the agreement between theory and numerics is very good.

Finally, we show an example of $P_\xi(G)$ in the insulating regime, i.e., for very long wires. From the standard scaling theory, it is known that $p_s(G)$ follows a log-normal distribution whose variance is twice the mean value $(-\ln G)$. An example of this standard case is shown in the inset of fig. 4. For long-range disorder configurations the situation is different. In the insulating regime, the integral in eq. (9) can be performed analytically using saddle point approximation. We plot the resulting distribution in fig. 4 (main frame) for $\alpha = 1/2$ and the corresponding result from the numerical simulation (histogram). We can see clearly that $P_\xi(\ln G)$ is not Gaussian, as we have remarked previously. Finally, we point out that in all previous comparisons between theory and simulations there are no free-fitting parameters: given the value of the exponent $\alpha$, all we need is $(-\ln G)_L = \xi$, which we extract from the numerical experiments.

Conclusions. – Lévy processes have been found in very different phenomena in nature, but recent transport experiments indicate the possibility of studying these processes in a controllable manner. Motivated by these experimental achievements, we have studied the statistical properties of the conductance across one-dimensional quantum wires whose disorder configurations are characterized by probability densities with a power-law tail, i.e., Lévy-type disorder. We have shown that the conductance distribution is completely determined by only two parameters: the average $\langle \ln G \rangle_L$ and the scaling exponent $\alpha$ of the power-law tail. Therefore the distribution of conductances is universal in the sense that all wires with the same $\langle \ln G \rangle_L$, but different disorder distributions obeying

\(^3\)Density distributions used for the numerical simulations: $p_1(x) = q_{1/2,c}(x) = (c/\sqrt{2\pi})x^{-3/2} \exp(-c^2/2x)$, $p_2(x) = 2^{-3}(1 + x)^{-3/2}$.
the same scaling exponent $\alpha$ have the same conductance statistics. Since we provide the complete conductance distribution other quantities of interest such as the moments of the conductance and the shot-noise power can be calculated from our results. We have shown that for $\alpha \leq 1$ the strong fluctuations of the conductance allow for the coexistence of localized and delocalized regimes and the fluctuations of $\ln G$ do not exhibit self-averaging. For $\alpha > 1$ the logarithm of the conductance is a self-averaged quantity, as in the standard scaling theory. We think that the experimental observation of the different phenomena described here such as the universality of the conductance statistics, in optical materials or electronic transport experiments, should be of considerable interest. Finally, we leave for future investigations the extension of our results to higher-dimensional wires.

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