Learning over All Stabilizing Nonlinear Controllers for a Partially-Observed Linear System

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Abstract

We propose a parameterization of nonlinear output feedback controllers for linear dynamical systems based on a recently developed class of neural network called the recurrent equilibrium network (REN), and a nonlinear version of the Youla parameterization. Our approach guarantees the closed-loop stability of partially observable linear dynamical systems without requiring any constraints to be satisfied. This significantly simplifies model fitting as any unconstrained optimization procedure can be applied whilst still maintaining stability. We demonstrate our method on reinforcement learning tasks with both exact and approximate gradient methods. Simulation studies show that our method is significantly more scalable and significantly outperforms other approaches in the same problem setting.

Keywords: Reinforcement Learning, Nonlinear Stability, Partially-Observed Systems, Contraction, Neural Networks.

1. Introduction

Deep neural networks and reinforcement learning (RL) hold tremendous potential for continuous control applications and impressive performance has already been demonstrated on robotic locomotion, manipulation, and other benchmark control tasks (Lillicrap et al., 2016; Andrychowicz et al., 2020; Duan et al., 2016; Kalashnikov et al., 2018). Despite these successes, deep RL based controllers are not widely used in engineering practice, as they often exhibit unexpected or brittle performance and even instability. This has motivated a rapid growth in literature studying reinforcement learning with stability guarantees.

Research in stable deep RL has largely focused on learning static state feedback neural network controllers, where full state information is available (Berkenkamp et al., 2017; Chang et al., 2019; Dai et al., 2021; Khader et al., 2021). In practical systems, however, most controllers are output feedback controllers that rely on only partial state observations. In this setting, the optimal controller is a generally dynamic function of previous observations. So far there has been little research on learning neural network output feedback policies with stability guarantees. To the authors’ knowledge, the only proposed methods are Anderson (1998); Knight and Anderson (2011); Gu et al. (2021), however, these methods utilize nonconvex sets of RNN controllers that require complex projection methods at each training step.

In recent years, there has also been a significant amount of research on the learning of controllers for linear dynamical systems (Simchowitz et al., 2020; Simchowitz and Foster, 2020; Hazan
et al., 2020; Mania et al., 2018). It has been shown that the performance of a learning algorithm strongly depends on the controller parameterization (Roberts et al., 2011). In particular, the Youla parameterization (Youla et al., 1976b) has demonstrated significant benefits for stable reinforcement learning (Roberts et al., 2011), and has led to the best known bounds on regret in the online learning context (Simchowitz et al., 2020).

In this work, we construct nonlinear controller parameterizations with stability guarantees for partially-observed linear systems. Even for linear systems, nonlinear controllers are often necessary to deal with constraints, non-Gaussian disturbances, or to optimize non-quadratic cost functions. Furthermore, when applied to a nonlinear system our approach can be used to search over controllers guaranteed to be locally stabilizing at a particular operating condition. Our proposed parameterizations are unconstrained or direct, making them simple to use in a learning context by alleviating the need for projections or other constrained optimisation techniques.

2. Background

The Youla Parameterization. Developed in the late 70’s, the Youla parameterization (or the Youla-Kucera parameterization) provides a parameterization of all stabilizing controllers for a given LTI system via a so-called $Q$-parameter (Youla et al., 1976b,a; Kučera, 1975). See for instance Mahtout et al. (2020) for a review of recent developments. The key feature of the Youla parameterization is that all stabilizing controllers can be represented via a stable $Q$, and the closed-loop response is linear in $Q$. It plays a central role in robust control theory (Zhou et al., 1996), controller optimization (Boyd and Barratt, 1991), and decentralized control (Rotkowitz and Lall, 2005). Further applications have been found in, for example, controller reconfiguration for linear parameter varying control Mahtout et al. (2018), disturbance and noise rejection (Doumiati et al., 2017), closed-loop system identification (Sekunda et al., 2018), adaptive control (Wang et al., 1991), fault tolerant control (Li et al., 2019), online learning (Simchowitz et al., 2020), and reinforcement learning (Roberts et al., 2011).

To date, these applications have relied on a linear $Q$-parameter despite several works extending the idea to a nonlinear setting (Anderson, 1998; Desoer and Liu, 1981; Paice and Moore, 1990; Lu, 1995; Fujimoto and Sugie, 2000). The main barriers to its widespread application have been (i) non-constructive parameterizations expressed in terms of coprime factors or kernel representations which are difficult to compute, and (ii) the lack of flexible parameterizations of nonlinear $Q$ parameters.

Learning Stable Models. The construction of a nonlinear $Q$ parameter requires a parameterization of nonlinear systems with stability guarantees. A common approach is to bound the maximum singular value of an RNN’s weight matrix leading to simple, yet conservative stability criteria (Jaeger, 2002; Miller and Hardt, 2019). There have been a number of attempts to reduce conservatism by using non-Euclidean metrics (Revay and Manchester, 2020), integral quadratic constraints (IQCs) (Revay et al., 2020), and local IQCs (Yin et al., 2021). A fundamental problem with these approaches is that the stability constraints take the form of (possibly nonconvex) matrix inequalities which require complex projection, barrier or penalty methods to enforce. A significant development addressing these issues was the direct parameterization developed in Revay et al. (2021) which allows for the construction of an unconstrained model parameterization.
3. Contributions

Our work builds on recent methods applying sets of stable recurrent neural networks to construct nonlinear Q-parameters (Wang and Manchester, 2021; Revay et al., 2021). This work presents three significant developments compared to prior work: Firstly, we extend the approach from Wang and Manchester (2021); Revay et al. (2021) to the learning of feedback controllers with only partial state observations. Secondly, we provide a new theoretical result that shows that a contracting, Lipschitz Q-parameter provides a “complete” parameterization of all controllers rendering the closed-loop systems contracting and Lipschitz. Finally, we investigate the method in the context of reinforcement learning with only measurement and cost information.

4. Problem Setup

We consider a discrete-time linear time-invariant system:

\[ x_{t+1} = Ax + Bu + d_x, \quad y = Cx + d_y \]

(1)

with internal state \( x \), controlled inputs \( u \), measured outputs \( y \), and disturbances/noise to the state and output \( d_x, d_y \), respectively. For brevity \( x = x_t, x_{t+1} = x_{t+1} \), and similarly for other variables. Define

\[ d := \begin{bmatrix} d_x \\ d_y \end{bmatrix}, \quad z = \begin{bmatrix} x \\ u \end{bmatrix}. \]

We assume that the system is stabilizable and detectable, and that the disturbance signal \( d \) and initial state \( x_0 \) are random variables for which a sampler is available.

The problem we consider is to find a controller, i.e. a mapping \( u = K(y) \), which may be nonlinear and/or dynamic, such that:

1. In the disturbance-free case \( d_x = d_y = 0 \), the closed-loop system is contracting, i.e. initial conditions are forgotten exponentially (Lohmiller and Slotine, 1998).

2. The closed-loop response to disturbances is Lipschitz (a.k.a. bounded incremental \( \ell^2 \) gain):

\[ \|z^a - z^b\| \leq \gamma \|d^a - d^b\| \]

for some \( \gamma > 0 \) where \( \| \cdot \| \) is the signal \( \ell^2 \) norm \( \|z\| = \sqrt{\sum_{t=0}^{\infty} |z_t|^2} \) and \( d^a, d^b \) are two realizations of the disturbance and \( z^a, z^b \) the corresponding realizations of \( z \).

3. A cost function of the following form is minimized (at least approximately and locally)

\[ J = E \left[ \sum_{t=0}^{T-1} g(x_t, u_t) + g_T(x_T) \right]. \]

(2)

where \( E[\cdot] \) is expectation over realizations of \( x_0 \) and \( d \).

Note that the first two requirements are hard requirements that must be satisfied, whereas the third is soft in the sense that we do not expect a global minimum to necessarily be found, since we do not make any assumptions about convexity of \( g, g_T \), nor the ability to precisely compute the expectation.

We will consider two scenarios that differ in terms of the information available to the learning algorithm:
• The “simulation” case, in which internal states $x$ are available to the optimizer (but not the control policy), and the functions $g, g_T$ are known along with their gradients.

• The “reinforcement learning” case, in which the optimizer has the same information as controller: only observed values of $y$ and the actual cost $J$ achieved in a given episode.

The main difference is that in the first case, gradients can be computed exactly, whereas in the second case they must be approximated.

5. The Youla-REN Controller Architecture

We will construct a parameterization of all possible nonlinear controllers such that the closed-loop is contracting and the mapping $d \mapsto z$ is Lipschitz (bounded incremental gain) in terms of a “parameter” which is an arbitrary contracting and Lipschitz nonlinear system.

Since the system is stabilizable and detectable, one can easily construct (see e.g. Zhou et al. (1996)) gain matrices $K, L$ such that $(A − BK)$ and $(A − LC)$ are stable. We assume some such gains are known and define the “base” output-feedback linear controller

$$\hat{x}_+ = A\hat{x} + Bu_K + L\tilde{y}, \quad u_K = -K\hat{x},$$

where $\tilde{y} = y - C\hat{x}$. The proposed control architecture is an extension of the classical Youla parameterization for linear systems, and is based on augmenting the base controller:

$$\hat{x}_+ = A\hat{x} + Bu + L\tilde{y}, \quad u = -K\hat{x} + Q(\tilde{y}).$$

where $Q$ is an arbitrary contracting and Lipschitz nonlinear system. Note that since $\tilde{y}$ represents the difference between expected and observed outputs, the Youla parameterization could be interpreted as prescribing a stable response to surprises.

In this paper, $Q$ will be constructed as a recurrent equilibrium network (REN), introduced in Revay et al. (2021). A REN is a nonlinear system consisting of an interconnection of a linear system and nonlinear “activation functions”

$$\tilde{u} = Q(\tilde{y}) : \left\{ \begin{bmatrix} \chi_{t+1} \\ v_t \\ \tilde{u}_t \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \chi_t \\ w_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} b_x \\ b_v \\ b_y \end{bmatrix}, \right. \quad w_t = \sigma(v_t) := \begin{bmatrix} \sigma(v_{t1}^1) & \sigma(v_{t2}^2) & \cdots & \sigma(v_{tq}^q) \end{bmatrix}^T,$$

here $\chi_t$ is the internal state of the REN, and $\tilde{y}_t$ is its input and $\tilde{u}_t$ its output. $v_t, w_t \in \mathcal{R}^{n_v}$ are the input and output of the neuron layer. We assume that the activation function is $\sigma : \mathcal{R} \to \mathcal{R}$ with slope restricted in $[0, 1]$. When $D_{22} \neq 0$ the relationship between $w$ and $v$ defines an equilibrium network a.k.a. implicit network. In this work, however, we will restrict to $D_{22}$ being strictly lower-triangular, so the mapping from $v \mapsto w$ can be written in explicit form.

The learnable parameters in (5) are $W, b$, but the key feature of RENs for the purposes of the present paper is they admit a direct parameterization of contracting systems, i.e. there is a smooth
mapping \( \theta \mapsto (W, b) \) from an unconstrained parameter \( \theta \in \mathbb{R}^q \), such that for all \( \theta \) the resulting REN is contracting. A further benefit is that they are very flexible, including many previously-used models as special cases. See Revay et al. (2021) for details. In fact, combined with the classical results of Boyd and Chua (1985), it can easily be shown that with bounded inputs, RENs densely cover the set of all contracting nonlinear systems.

5.1. Theoretical Results

We will show that our proposed parameterization is in a sense universal. Consider an arbitrary feedback controller \( u = K(y) \) admitting a state-space realisation

\[
\begin{align*}
\zeta_+ &= f(\zeta, y), \quad u = g(\zeta, y) \\
\end{align*}
\]

leading to the closed-loop dynamics:

\[
\begin{align*}
x_+ &= Ax + dx + Bg(\zeta, Cx + dy), \quad \zeta_+ &= f(\zeta, Cx + dy).
\end{align*}
\]

We give the following version of the well-known Youla parameterization result:

**Proposition 1** Consider the following control architecture (4), parameterized by \( Q \).

1. For any contracting and Lipschitz \( Q \), the closed-loop system with the controller (4) is contracting and Lipschitz.

2. Any controller of the form (6) that achieves contracting and Lipschitz closed-loop can be written in the form (4) with contracting and Lipschitz \( Q \).

Moreover, the closed-loop response with this controller structure is

\[
z = T_0d + T_1Q(T_2d)
\]

where \( T_0, T_1, T_2 \) are stable linear systems.

**Proof:** We first construct \( T_0, T_1, T_2 \), which parallels exactly the linear case (Zhou et al., 1996). Defining the observer-error as \( \tilde{x} = x - \hat{x} \), the observer and plant dynamics can be rewritten as

\[
\begin{align*}
\tilde{x}_+ &= (A - LC)\tilde{x} + dx - Ldy, \quad x_+ = (A - BK)x + dx + BK\hat{\xi}, \quad \hat{y} = C\tilde{x}.
\end{align*}
\]

The first and second equations define a stable linear system \( d \mapsto z \) which we denote \( T_0 \), i.e. the closed-loop response with the base controller, while the first and third define a stable linear system from disturbance to innovations \( d \mapsto \tilde{y} \), which we denote \( T_2 \).

To construct \( T_1 \), we make a change of variables to a virtual control input: \( \tilde{u} := u - u_K = u + K\tilde{x} = u + K(x - \hat{x}) \) Now, regardless of whether the base controller is in the loop or not, the plant dynamics can be rewritten in terms of \( \tilde{u} \) as (9) (first equation) coupled with

\[
x_+ = (A - BK)x + dx + BK\tilde{x} + B\tilde{u}
\]

So by superposition we have: \( z = T_0w + T_1\tilde{u} \) where \( T_1 \) is the stable system

\[
\xi_+ = (A - BK)\xi + B\tilde{u}, \quad \eta = \begin{bmatrix} \xi \\ \tilde{u} - K\xi \end{bmatrix}.
\]
Hence if an operator \( Q \) maps \( \tilde{y} \) to \( \tilde{u} \), we have the closed loop (8).

**Proof of Claim 1.** This follows from the stability of \( T_0, T_1, T_2 \) and the contraction and Lipschitz condition on \( Q \), and the composition properties of contracting systems and Lipschitz mappings.

**Proof of Claim 2.** Assuming a controller \( u = K(y, r) \) exists in the form (6) we can equivalently augment it with the state estimator and rewrite it in terms of \( \tilde{y} \) and \( \tilde{u} \) as follows:

\[
\begin{align*}
\dot{x} &= A\tilde{x} + L\tilde{y} + B\phi(C\tilde{x} + \tilde{y}) \\
\phi &= f(\phi, C\tilde{x} + \tilde{y}) \\
\tilde{u} &= K\tilde{x} + u = K\tilde{x} + g(\phi, C\tilde{x} + \tilde{y})
\end{align*}
\]  

defining the closed-loop mapping \( \tilde{y} \mapsto \tilde{u} \), which we denote \( Q_K \). I.e. the closed-loop with arbitrary controllers (not necessarily incorporating the base controller) can be written as \( z = T_0w + T_1Q_K(T_2w, r) \).

Now, note that taking (7), relabelling \( x \) as \( \tilde{x} \), and taking \( d_x = L\tilde{y} \) and \( d_y = \tilde{y} \) as particular inputs, gives (11), (12). Hence if the closed-loop with \( K \) is contracting then so is \( Q_K \). Moreover, the fact that the mapping from \( d \to z \) is Lipschitz with (7) implies that the mapping \( \tilde{y} \to v \) is too, since by (13) \( v = [K, I]z \) under this relabelling so \( |\tilde{u}| \leq \alpha|z| \) for some \( \alpha \). Hence the closed-loop system is contracting and Lipschitz if and only if \( Q_K \) is.

**6. Numerical Experiments**

**6.1. Problem Setup**

We will compare the proposed policy parameterization to a recently proposed stable policy learning method via LMI projection (Gu et al., 2021) in various control tasks. First we consider a linearized pendulum with mass \( m = 0.15 \), length \( l = 0.5 \), and friction coefficient \( \mu = 0.5 \), whose discretized dynamics with sampling time \( \delta = 0.02 \) can be written as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -mpg\delta & 0 \\
0 & 0 & 1 & \delta \\
0 & 0 & (m_c + m_p)g\delta & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -m_p g & 0 \\
0 & 0 & 1 & \delta \\
0 & 0 & (m_c + m_p)g\delta & 1 \\
\end{bmatrix}
\begin{bmatrix}
u \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -m_p & 0 \\
0 & 0 & 1 & \delta \\
0 & 0 & (m_c + m_p)g\delta & 1 \\
\end{bmatrix}
\begin{bmatrix}
u \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
+ d_x, \quad y = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -m_p & 0 \\
0 & 0 & 1 & \delta \\
0 & 0 & (m_c + m_p)g\delta & 1 \\
\end{bmatrix}
\begin{bmatrix}
u \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
+ d_y,
\]

where \( m_p = 0.2, m_c = 1.0, l = 0.5, g = 9.81 \) and \( \delta = 0.02 \).

The first type of control task is the classic LQG problem where \( d_x, d_y \) are Gaussian noise with zero mean and covariance matrices of \( \Sigma_x \) and \( \Sigma_y \), respectively. The cost function is

\[
J_q = E \left[ \|x_T\|^2_Q + \sum_{t=0}^{T-1} (\|x_t\|^2_Q + \|u_t\|^2_R) \right]
\]

where \( x_t, u_t \) are the state and input at time step \( t \), respectively, \( Q, R, Q_f \succ 0 \) are the weighting matrices, and \( T \) is the trajectory length. For the experimental study, we consider the weight matrices

[1, 1, 1, 1]
$Q_f = Q = \text{diag}(100, 10), R = 1$ for the pendulum system and $Q_f = Q = \text{diag}(100, 10, 10, 10), R = 1$ for the cartpole system. The covariance matrices of the two systems are chosen as $\Sigma_x = 0.0005I$ and $\Sigma_y = 0.0001I$. We use trajectory length $T$ of 200 and 400 for the pendulum and cartpole, respectively.

The second type of control problem is disturbance rejection subject to soft input bound. To be specific, we will consider the following convex but non-quadratic cost function

$$J_c = E \left[ J_q + \rho \sum_{t=0}^{T-1} \max(|u_t| - \overline{\pi}, 0) \right] \quad (17)$$

where $\overline{\pi}$ is the input bound and $\rho > 0$ is the penalty coefficient. The optimal policy for this problem is nonlinear due to the input constraint. The soft constraint can be interpreted as allowing excessive control actions for short periods of time, since it corresponds to the $\ell^1$ norm of the portion exceeding the bound. This is relevant e.g. for electric motors which allow larger peak currents for short periods before reverting to their continuous current capacity, to deal with large state transitions or input disturbance changes. Here we choose penalty coefficient $\rho = 500$ and bounds of $\overline{\pi} = 0.5$ and $\overline{\pi} = 2$ for the pendulum and cartpole, respectively. The disturbances are mixed signals of Gaussian noise and piecewise constant input perturbation $w_d$ with random duration and magnitude. The trajectory length is chosen as $T = 1000$.

6.2. Training details

Exact Gradients. To learn the optimal closed-loop behaviour using our method, the base controller (3) is constructed by choosing $L$ as the steady-state Kalman filter and the state-feedback gain $K$ as the solution of the infinite-horizon LQR problem with identity weighting matrices. For the LMI projection method, we follow the algorithm in Gu et al. (2021) excepting that the initial guess of $P_0$ is chosen based on the closed-loop stability for the base linear controller. Both approaches are implemented in Julia (Bezanson et al., 2017) and models are learned via exact gradients obtained from Flux.jl (Innes, 2018). The costs $J_q$ and $J_c$ are approximated via a Monte-Carlo method, i.e., the averaged cost over a batch of initial states and disturbance realizations. The default controller configuration for both methods is 16 states and 20 neurons with $\tanh$ activation for all experiments. We use the ADAM optimiser (Kingma and Ba, 2015) and gradient clipping to a maximum magnitude of 10 for each parameter. The learning rate is initialised at $10^{-3}$ and then reduced to $10^{-4}$ for the last 100 epochs. We use batch size of 40 and 100 for training and testing in the LQG task. For the disturbance rejection experiment, we choose a batch size of 10 and 20 for training and testing.

Approximate Gradients. The LQG and disturbance-rejection experiments are also examined using reinforcement learning to demonstrate the flexibility of our approach when only observed outputs and costs are available. We focus on the linear pendulum model (Eqn. 14), and approximate the gradients of $J_q$ and $J_c$ with respect to the model parameters using Augmented Random Search (ARS) (v1) (Mania et al., 2018; Recht, 2019), which is a modified version of the classic stochastic approximation algorithm of Robbins and Monro (1951). ARS approximates gradients with a finite-difference approach, perturbing the policy parameter vector by $\pm \nu$ for some small value $\nu$ in random directions. As one of the simplest RL algorithms, ARS makes no assumptions on the controller structure or state information, and approximates gradients purely from policy evaluations.
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It is therefore well-suited to training dynamic controllers on partially observed systems, whereas other popular continuous RL algorithms like policy-gradient methods require special treatment and substantial modification (Sutton and Barto, 2018).

We make three minor changes to the original ARS algorithm to improve numerical convergence. Firstly, we use the ADAM optimiser and gradient clipping instead of free stochastic gradient descent. We also consider the average cost over $b$ batches of initial conditions as an approximation of the expected cost. Finally, we use the same batch of initial conditions when evaluating the cost for $\pm$ perturbations of the model parameters for a fair comparison between policy rollouts. The estimated cost gradient with respect to the policy parameters $\theta$ is then

$$\nabla_{\theta} J_{\theta} = \frac{1}{m} \sum_{i=1}^{m} \frac{J_{\theta + \nu \delta_i}(x_{0_i}) - J_{\theta - \nu \delta_i}(x_{0_i})}{2\nu \sigma_R}$$

with $J_{\theta}(x_{0}) = \frac{1}{b} \sum_{j=1}^{b} J_{\theta}(x_{0_j})$, (18)

where $x_{0_j}$ are initial state vectors, $\nu$ is a small perturbation, $\delta_i$ is a normally-distributed random vector with zero mean, and $\sigma_R$ is the standard deviation of the $2m$ average costs collected for each gradient approximation, as per Mania et al. (2018). For the LQG/disturbance-rejection problems (respectively), we use: a learning rate of $10^{-4}/10^{-3}$, decimated after 400/50 iterations; $m = 128/10$ to approximate the gradient; and $\nu = 0.02$, $b = 20$, and ReLU activation functions in both cases. Gradients are clipped to an $\ell^2$ norm of 10.0 over all parameters.

6.3. Results

Exact Gradients. The experimental results for the LQG and disturbance rejection problems using the LMI projection method (Gu et al., 2021) and our approach are reported in Fig. 1. In all experiments, our approach outperforms the projection method in terms of final performance and convergence speed. For the pendulum system, the LMI projection based controller seems to fall into local minima after the first projection and its cost is higher than the base linear controller. For the cartpole system, the test cost of both projection-based and Youla-REN control policies decrease, where the Youla-REN controller reaches lower cost and the LMI projection method exhibits seemingly unstable behavior in the LQG case.

In Figure 2 we compare the per-epoch computational cost of each method as a function of the number of neurons in the control policy. It is clear that the cost of the LMI projections grows rapidly, whereas for our proposed Youla-REN method no projections are required and computation time remains small even for larger networks.

In Figure 3 we show some trajectory samples in the scenarios with soft input constraints. It can be seen the controller found via LMI projection of Gu et al. (2021) regularly exceeds the bounds, whereas the proposed Youla-REN controller is almost always within the bounds.

Approximate Gradients. Figs. 4a and 4b demonstrate that using approximate gradients from as simple an algorithm as random search is sufficient to learn stabilising nonlinear feedback policies with the Youla-REN on partially-observed linear systems. In both the LQG and disturbance-rejection problems, we achieve similar final costs on the linear pendulum environment to the models trained with exact gradients, yet only require policy evaluations to do so, rather than detailed knowledge of the system.

The ease of which we are able to switch to an approximate learning method is a direct consequence of our unconstrained parameterization. Each perturbation of the Youla-REN policy used by
Figure 1: Test cost vs epochs while learning the proposed Youla-REN and LMI projection (Gu et al., 2021) controllers via exact gradients.

Figure 2: Computation time per epoch vs number of neurons for the LMI projection method of Gu et al. (2021) (left) and the proposed Youla-REN method (right). Note the different vertical scales.
ARS is naturally guaranteed to be stabilising, allowing us to take arbitrary perturbations and remain within the set of stabilising controllers. This is a key feature of our approach, and provides a significant advantage over projection-based methods such as those in Gu et al. (2021), where even small perturbations in the policy parameters could produce unstable controllers. It is worth noting that our natural stability guarantees make the Youla-REN architecture far more suitable to online learning in safety-critical systems, where trialling unstable policies at any point in the training process is not appropriate and must be avoided at all costs.
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