INFINITE FAMILIES OF SIMPLE HOLOMORPHIC ETA QUOTIENTS

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ABSTRACT. We address the problem of constructing a simple holomorphic eta quotient of a given level \( N \). Such constructions are known for all cubefree \( N \). Here, we provide such constructions for arbitrarily large prime power levels. As a consequence, we obtain an irreducibility criterion for holomorphic eta quotients in general.

1. INTRODUCTION

The Dedekind eta function is defined by the infinite product:

\[
\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{for all } z \in \mathcal{H},
\]

where \( q = e^{2\pi i z} \) for all \( r \) and \( \mathcal{H} := \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \} \). Eta is a holomorphic function on \( \mathcal{H} \) with no zeros. This function has its significance in Number Theory. For example, \( 1/\eta \) is the generating function for the ordinary partition function \( p : \mathbb{N} \rightarrow \mathbb{N} \) (see [1]) and the constant term in the Laurent expansion at 1 of the Epstein zeta function \( \zeta_Q \) attached to a positive definite quadratic form \( Q \) is related via the Kronecker limit formula to the value of \( \eta \) at the root of the associated quadratic polynomial in \( \mathcal{H} \) (see [8]). The value of \( \eta \) at such a quadratic irrationality of discriminant \(-D\) is also related via the Lerch/Chowla-Selberg formula to the values of the Gamma function with arguments in \( D^{-1}\mathbb{N} \) (see [24]). Further, eta quotients appear in denominator formula for Kac-Moody algebras, (see [14]), in "Moonshine" of finite groups (see [12]), in Probability Theory, e.g. in the distribution of the distance travelled in a uniform four-step random walk (see [5]) and in the distribution of crossing probability in two-dimensional percolation (see [16]). In fact, the eta function comes up naturally in many other areas of Mathematics (see the Introduction in [2] for a brief overview of them).

The function \( \eta \) is a modular form of weight \( 1/2 \) with a multiplier system on \( \text{SL}_2(\mathbb{Z}) \) (see [17]). An eta quotient \( f \) is a finite product of the form

\[
\prod \eta_{d}^{X_d},
\]

where \( d \in \mathbb{N} \), \( \eta_{d} \) is the rescaling of \( \eta \) by \( d \), defined by

\[
\eta_{d}(z) := \eta(dz) \quad \text{for all } z \in \mathcal{H}
\]

and \( X_d \in \mathbb{Z} \). Eta quotients naturally inherit modularity from \( \eta \). The eta quotient \( f \) in (1.2) transforms like a modular form of weight \( \frac{1}{2} \sum d X_d \) with a

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multiplier system on suitable congruence subgroups of SL₂(\(\mathbb{Z}\)): The largest among these subgroups is

\[
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\},
\]

where

\[
N := \text{lcm}\{d \in \mathbb{N} \mid X_d \neq 0\}.
\]

We call \(N\) the level of \(f\). Since \(\eta\) is non-zero on \(\mathbb{H}\), the eta quotient \(f\) is holomorphic if and only if \(f\) does not have any pole at the cusps of \(\Gamma_0(N)\).

An eta quotient on \(\Gamma_0(M)\) is an eta quotient whose level divides \(M\). Let \(f\), \(g\) and \(h\) be nonconstant holomorphic eta quotients on \(\Gamma_0(M)\) such that \(f = g \times h\). Then we say that \(f\) is factorizable on \(\Gamma_0(M)\). We call a holomorphic eta quotient \(f\) of level \(N\) quasi-irreducible (resp. irreducible), if it is not factorizable on \(\Gamma_0(N)\) (resp. on \(\Gamma_0(M)\) for all multiples \(M\) of \(N\)). Here, it is worth mentioning that the notions of irreducibility and quasi-irreducibility of holomorphic eta quotients are conjecturally equivalent (see [2]). We say that a holomorphic eta quotient is simple if it is both primitive and quasi-irreducible.

\[2. \text{The main result and conjecture}\]

For a prime \(p\) and an integer \(n > 3\), we define the eta quotient \(f_{p,n}\) by

\[
f_{p,n} := \begin{cases} 
\frac{\eta_p^n \eta_{p^{n-1}}^{(p-1)^2} \prod_{s=1}^{n/2-1} \eta_p^{2s-3p+1} \eta_p^{2p-2p+2}}{(\eta \eta_p^n)^{p-1}} & \text{if } n \text{ is even,} \\
\frac{(\eta_p \eta_{p^{n-1}})^p \prod_{s=1}^{n-1} \eta_p^{2s-3p+2}}{(\eta \eta_p^n)^{p-1}} & \text{if } n \text{ is odd and } p \neq 2,
\end{cases}
\]

Clearly, \(f_{p,n}\) is invariant under the Fricke involution \(W_{p^n}\). We shall show that:

**Theorem 1.** For any integer \(n > 3\), \(f_{p,n}\) is a simple holomorphic eta quotient of level \(p^n\). 

From Theorem 2 in [2], we recall that any simple holomorphic eta quotient of a prime power level is irreducible. So, in particular, the above theorem implies:

**Corollary 1.** For any integer \(n > 3\), the eta quotient \(f_{p,n}\) is irreducible.

Also, from Corollary 1 in [2], we recall that given an irreducible holomorphic eta quotient \(f\) of a prime power level, all the rescalings of \(f\) by positive integers are irreducible. Thus we obtain:

**Corollary 2** (Irreducibility criterion for holomorphic eta quotients). Given a holomorphic eta quotient \(g\), if there exist \(n, d \in \mathbb{N}\) and a prime \(p\), such that \(g\) is the rescaling of \(f_{p,n}\) by \(d\), then \(g\) is irreducible. Here \(f_{p,n}\) is as defined in (2.1).
With a good amount of numerical evidence, we conjecture that

**Conjecture 1.** For any integer \( n > 3 \) and for any odd prime \( p \), there are no simple holomorphic eta quotients of level \( p^n \) and of weight greater that of \( f_{p,n} \).

3. Notations and the basic facts

By \( \mathbb{N} \) we denote the set of positive integers. For \( N \in \mathbb{N} \), by \( \mathcal{D}_N \) we denote the set of divisors of \( N \). For \( X \in \mathbb{Z}^{\mathcal{D}_N} \), we define the eta quotient \( \eta^X \) by

\[
\eta^X := \prod_{d \in \mathcal{D}_N} \eta^X_d,
\]

where \( X_d \) is the value of \( X \) at \( d \in \mathcal{D}_N \) whereas \( \eta_d \) denotes the rescaling of \( \eta \) by \( d \). Clearly, the level of \( \eta^X \) divides \( N \). In other words, \( \eta^X \) transforms like a modular form on \( \Gamma_0(N) \). We define the summatory function \( \sigma : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z} \) by

\[
\sigma(X) := \sum_{d \in \mathcal{D}_N} X_d.
\]

Since \( \eta \) is of weight \( 1/2 \), the weight of \( \eta^X \) is \( \sigma(X)/2 \) for all \( X \in \mathbb{Z}^{\mathcal{D}_N} \).

Recall that an eta quotient \( f \) on \( \Gamma_0(N) \) is holomorphic if it does not have any poles at the cusps of \( \Gamma_0(N) \). Under the action of \( \Gamma_0(N) \) on \( \mathbb{P}^1(\mathbb{Q}) \) by Möbius transformation, for \( a, b \in \mathbb{Z} \) with \( \gcd(a, b) = 1 \), we have

\[
[a : b] \sim_{\Gamma_0(N)} [a' : \gcd(N, b)]
\]

for some \( a' \in \mathbb{Z} \) which is coprime to \( \gcd(N, b) \) (see [10]). We identify \( \mathbb{P}^1(\mathbb{Q}) \) with \( \mathbb{Q} \cup \{\infty\} \) via the canonical bijection that maps \( [\alpha : \lambda] \) to \( \alpha/\lambda \) if \( \lambda \neq 0 \) and to \( \infty \) if \( \lambda = 0 \). For \( s \in \mathbb{Q} \cup \{\infty\} \) and a weakly holomorphic modular form \( f \) on \( \Gamma_0(N) \), the order of \( f \) at the cusp \( s \) of \( \Gamma_0(N) \) is the exponent of \( q^{1/w_s} \) occurring with the first nonzero coefficient in the \( q \)-expansion of \( f \) at the cusp \( s \), where \( w_s \) is the width of the cusp \( s \) (see [10], [23]). The following is a minimal set of representatives of the cusps of \( \Gamma_0(N) \) (see [10], [20]):

\[
S_N := \left\{ \frac{a}{t} \in \mathbb{Q} \mid t \in \mathcal{D}_N, a \in \mathbb{Z}, \gcd(a, t) = 1 \right\} / \sim,
\]

where \( \frac{a}{t} \sim \frac{b}{t} \) if and only if \( a \equiv b \pmod{\gcd(t, N/t)} \). For \( d \in \mathcal{D}_N \) and for \( s = \frac{a}{t} \in S_N \) with \( \gcd(a, t) = 1 \), we have

\[
\text{ord}_s(\eta_d; \Gamma_0(N)) = \frac{N \cdot \gcd(d, t)^2}{24 \cdot d^2 \cdot \gcd(t^2, N)} \in \frac{1}{24} \mathbb{N}
\]

(see [20]). It is easy to check the above inclusion when \( N \) is a prime power. The general case follows by multiplicativity (see (3.13) and (3.16)). It follows that for all \( X \in \mathbb{Z}^{\mathcal{D}_N} \), we have

\[
\text{ord}_s(\eta^X; \Gamma_0(N)) = \frac{1}{24} \sum_{d \in \mathcal{D}_N} \frac{N \cdot \gcd(d, t)^2}{d^2 \cdot \gcd(t^2, N)} X_d.
\]
In particular, that implies
\[(3.7) \quad \text{ord}_{a_1/t}(\eta^X; \Gamma_0(N)) = \text{ord}_{1/t}(\eta^X; \Gamma_0(N))\]
for all \(t \in \mathcal{D}_N\) and for all the \(\varphi(\gcd(t, N/t))\) inequivalent cusps of \(\Gamma_0(N)\) represented by rational numbers of the form \(\frac{a}{t} \in \mathcal{S}_N\) with \(\gcd(a, t) = 1\).

The index of \(\Gamma_0(N)\) in \(\text{SL}_2(\mathbb{Z})\) is given by
\[(3.8) \quad \psi(N) := N \cdot \prod_{p \mid N} \left( 1 + \frac{1}{p} \right),\]
(see [10]). The valence formula for \(\Gamma_0(N)\) (see [23]) states:
\[(3.9) \quad \sum_{P \in \Gamma_0(N) \setminus \varnothing} \frac{1}{n_P} \cdot \text{ord}_P(f) + \sum_{s \in \mathcal{S}_N} \text{ord}_s(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24},\]
where \(k \in \mathbb{Z}, f : \mathfrak{h} \to \mathbb{C}\) is a meromorphic function that transforms like a modular forms of weight \(k/2\) on \(\Gamma_0(N)\) which is also meromorphic at the cusps of \(\Gamma_0(N)\) and \(n_P\) is the number of elements in the stabilizer of \(P\) in the group \(\Gamma_0(N)/\{\pm I\}\), where \(I \in \text{SL}_2(\mathbb{Z})\) denotes the identity matrix. In particular, if \(f\) is an eta quotient, then from (3.9) we obtain
\[(3.10) \quad \sum_{s \in \mathcal{S}_N} \text{ord}_s(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24},\]
because eta quotients do not have poles or zeros on \(\mathfrak{h}\).

It follows from (3.10) and from (3.7) that for an eta quotient \(f\) of weight \(k/2\) on \(\Gamma_0(N)\), the valence formula further reduces to
\[(3.11) \quad \sum_{t \mid N} \varphi(\gcd(t, N/t)) \cdot \text{ord}_{1/t}(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24}.\]

Since \(\text{ord}_{1/t}(f; \Gamma_0(N)) \in \frac{1}{27} \mathbb{Z}\) (see (3.5)), from (3.11) it follows that of any particular weight, there are only finitely many holomorphic eta quotients on \(\Gamma_0(N)\). More precisely, the number of holomorphic eta quotients of weight \(k/2\) on \(\Gamma_0(N)\) is at most the number of solutions of the following equation
\[(3.12) \quad \sum_{t \mid N} \varphi(\gcd(t, N/t)) \cdot x_t = k \cdot \psi(N)\]
in nonnegative integers \(x_t\).

We define the order map \(O_N : \mathbb{Z}^{\mathcal{D}_N} \to \frac{1}{27} \mathbb{Z}^{\mathcal{D}_N}\) of level \(N\) as the map which sends \(X \in \mathbb{Z}^{\mathcal{D}_N}\) to the ordered set of orders of the eta quotient \(\eta^X\) at the cusps \(\{1/t\}_{t \in \mathcal{D}_N}\) of \(\Gamma_0(N)\). Also, we define the order matrix \(A_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}\) of level \(N\) by
\[(3.13) \quad A_N(t, d) := 24 \cdot \text{ord}_{1/t}(\eta^d; \Gamma_0(N))\]
for all $t, d \in D_N$. For example, for a prime power $p^n$, we have

$$A_{p^n} = \begin{pmatrix} p^n & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-2} & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-4} & p^{n-3} & p^{n-2} & \cdots & p & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & p & p^2 & \cdots & p^{n-1} & p^{n-2} \\ 1 & p & p^2 & \cdots & p^{n-1} & p^n \end{pmatrix}. \tag{3.14}$$

By linearity of the order map, we have

$$O_N(X) = \frac{1}{24} \cdot A_N X. \tag{3.15}$$

For $r \in \mathbb{N}$, if $Y, Y' \in \mathbb{Z}^{D_N}$ is such that $Y - Y'$ is nonnegative at each element of $D_N$, then we write $Y \geq Y'$. In particular, for $X \in \mathbb{Z}^{D_N}$, the eta quotient $\eta^X$ is holomorphic if and only if $A_N X \geq 0$.

From (3.13) and (3.5), we note that $A_N(t, d)$ is multiplicative in $N, t$ and $d$. Hence, it follows that

$$A_N = \bigotimes_{p^n \mid N} A_{p^n}, \tag{3.16}$$

where by $\otimes$, we denote the Kronecker product of matrices.*

It is easy to verify that for a prime power $p^n$, the matrix $A_{p^n}$ is invertible with the tridiagonal inverse:

$$A_{p^n}^{-1} = \frac{1}{p^n \cdot (p - \frac{1}{p})} \begin{pmatrix} p & -p & & & & \\ -1 & p^2 + 1 & -p^2 & & & \\ & -p & p \cdot (p^2 + 1) & -p^3 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & \ddots \\ 0 & & & & -p^2 & p^2 + 1 & -1 \\ & & & & -p & p \end{pmatrix}, \tag{3.17}$$

where for each positive integer $j < n$, the nonzero entries of the column $A_{p^n}^{-1}(\_ , p^j)$ are the same as those of the column $A_{p^j}^{-1}(\_ , p)$ shifted down by

*Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing $N$ induces a lexicographic ordering on $D_N$ with which the entries of $A_N$ are indexed, Equation (3.16) makes sense for all possible orderings of the primes dividing $N$. 
$j - 1$ entries and multiplied with $p^{\min\{j-1,n-j-1\}}$. More precisely,

$$p^n \cdot (p - \frac{1}{p}) \cdot A_{p^n}^{-1}(p^j, p^j) =
\begin{cases}
p & \text{if } i = j = 0 \text{ or } i = j = n \\
-p^{\min\{j,n-j\}} & \text{if } |i - j| = 1 \\
p^{\min\{j-1,n-j-1\}} \cdot (p^2 + 1) & \text{if } 0 < i = j < n \\
0 & \text{otherwise.}
\end{cases}
$$

(3.18)

For general $N$, the invertibility of the matrix $A_N$ now follows by (3.16). Hence, any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$. In particular, for distinct $X, X' \in \mathbb{Z}^{\mathcal{D}_N}$, we have $\eta^X \neq \eta^{X'}$. The last statement is also implied by the uniqueness of $q$-series expansion: Let $\eta^\hat{X}$ and $\eta^\hat{X'}$ be the *eta products* (i.e., $\hat{X}, \hat{X'} \geq 0$) obtained by multiplying $\eta^X$ and $\eta^{X'}$ with a common denominator. The claim follows by induction on the weight of $\eta^\hat{X}$ (or equivalently, the weight of $\eta^\hat{X'}$) when we compare the corresponding first two exponents of $q$ occurring in the $q$-series expansions of $\eta^\hat{X}$ and $\eta^\hat{X'}$.

### 4. Proof of Theorem 1

We shall only prove the theorem for the case where $n$ is even. The proof for the case where $n$ is odd is quite similar. Let $A_N$ be the order matrix of level $N$ (see (3.13)). Since all the entries of $A_N^{-1}$ are rational (see (3.16) and (3.17)), for each $t \in \mathcal{D}_N$, there exists a smallest positive integer $m_{t,N}$ such that $m_{t,N} \cdot A_N^{-1}(-, t)$ has integer entries, where $A_N^{-1}(-, t)$ denotes the column of $A_N$ indexed by $t \in \mathcal{D}_N$. We define $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ by

$$(4.1) \quad B_N(-, t) := m_{t,N} \cdot A_N^{-1}(-, t) \text{ for all } t \in \mathcal{D}_N.$$

From the multiplicativity of $A_N^{-1}(d, t)$ in $N$, $d$ and $t$ (see (3.16)), it follows that $B_N(d, t)$ (see (4.1)) is also multiplicative in $N$, $d$ and $t$. That implies:

$$(4.2) \quad B_N = \bigotimes_{p \in \mathcal{P}_N \atop p^n \parallel N} B_{p^n},$$

where $\mathcal{P}_N$ denotes the set of prime divisors of $N$. For a prime $p$, from (4.1) and (3.17), we have

$$(4.3) \quad B_{p^n} = \begin{pmatrix}
p & -p & \cdots & 0 \\
-1 & p^2 + 1 & -p & \cdots \\
p & p^2 + 1 & -p & \cdots \\
0 & -p & p^2 + 1 & -p \\
& & & 0
\end{pmatrix}.$$
From [3], we recall that if \( \eta^X \) is an irreducible holomorphic eta quotient on \( \Gamma_0(N) \), then \( X \in \mathbb{Z}_N \cap \mathbb{Z}^{D_N} \), where

\[
Z_N = \left\{ \sum_{d \mid N} C_d v_d \left| C_d \in [0, 1] \text{ for all } d \mid N \right. \right\}
\]

and for all \( d \in \mathcal{D}_N \), \( v_d = m_d u_d \), where \( u_d \) is the column of \( B_N \) indexed by \( d \) and \( m_d \) is the smallest positive integer such that \( v_d \in \mathbb{Z}^{D_N} \). Let \( v := \sum_{d \mid N} v_d \) and let \( F_N = \eta^v \). Again, from [3], we recall that given a holomorphic eta quotient \( g \) on \( \Gamma_0(N) \), the eta quotient \( F_N/g \) on \( \Gamma_0(N) \) is holomorphic if and only if \( g \) corresponds to some point in \( \mathbb{Z}_N \cap \mathbb{Z}^{D_N} \).

In particular, for \( N = p^{2m} \), the eta quotient \( F_{p^{2m}} = \eta^v \) is given by

\[
F_{p^{2m}}(z) = \begin{cases} 
\eta(pz)^{p^2-1} & \text{if } m = 1, \\
(\eta(pz)\eta(p^{2m-1}z))^{p(p-1)} \prod_{r=2}^{2m-2} \eta(p^rz)(p-1)^2 & \text{if } m > 1.
\end{cases}
\]

Let \( f_{p,2m} \) be the eta quotient defined in (2.1). Then we have

\[
\frac{F_{p^{2m}}(z)}{f_{p,2m}(z)} = \begin{cases} 
\eta(z)^{p-1} \eta(pz)^{p-2} \eta(p^2z)^{p-1} & \text{if } m = 1, \\
\eta(z)^{p-1} \eta(pz)^{p-2} \eta(p^{2m}z)^{p-1} \prod_{r=1}^{m-1} \eta(p^{2r-1}z) \eta(p^{2r+1}z)^{p-1} & \text{if } m > 1.
\end{cases}
\]

Since \( \frac{\eta(z)\eta(p^2z)^{p-1}}{\eta(pz)} \) is a holomorphic eta quotient of level \( p^2 \), it follows that \( \frac{F_{p^{2m}}(z)}{f_{p,2m}(z)} \) is a holomorphic eta quotient of level \( p^{2m} \) for all \( m \in \mathbb{N} \). Let \( X \in \mathbb{Z}^{D_N} \) be such that \( f_{p,2m} = \eta^X \). From (4.6), we conclude that \( X \in \mathbb{Z}_N \). In other words, \( Y := A_N X \) has all its entries in the interval \([0, 1]\). From (2.1), it easily follows that \( \text{ord}_\infty(f_{2m,p}) = 1/24 \). Since \( f_{2m,p} \) is invariant under the Fricke involution on \( \Gamma_0(p^{2m}) \), we also have \( \text{ord}_0(f_{2m,p}) = 1/24 \), since the Fricke involution interchanges the cusps 0 and \( \infty \) of \( \Gamma_0(p^{2m}) \). Since 0 and 1 (resp. \( \infty \) and \( 1/p^{2m} \)) represent the same cusp of \( \Gamma_0(p^{2m}) \), from (3.17) we get that both the first and the last entries of \( Y \) are equal to \( \frac{1}{p^{2m-1}(p^2-1)} \).

There exists \( U_N, V_N \in GL_{\sigma_0(N)}(\mathbb{Z}) \) and a diagonal matrix \( D_N \) such that \( D_N = U_N \times B_N \times V_N \). We shall see in the next section that if \( N = p^n \) for some prime \( p \) and some integer \( n > 2 \), then

\[
D_N = \text{diag}(1, 1, \ldots, 1, p^{n-1}, p^{n-1}(p^2-1))
\]

and the last two columns of \( V_N \) are respectively

\[
C_{n,1} := \begin{cases} 
(1, 0)^t & \text{if } n = 1, \\
(-1, 0, 1)^t & \text{if } n = 2, \\
(1, 1, p, p^2, \ldots, p^{n-3}, p^{n-2}, 0)^t & \text{if } n > 2.
\end{cases}
\]
and

$$C_{n,2} := \begin{cases} (p, 1)^t & \text{if } n = 1, \\ (p^2, 1)^t & \text{if } n = 2, \\ (p^n, p^{n-2}, p^{n-3}, \ldots, p, 1)^t & \text{if } n > 2. \end{cases}$$

(4.9)

Next we briefly recall an useful tool from Linear Algebra:

By elementary row and column operations [13], one can reduce any matrix $B \in \text{GL}_n(\mathbb{Z})$ to a diagonal matrix $D$. In other words, there exists $U, V \in \text{GL}_n(\mathbb{Z})$ and $D = \text{diag}(d_1, d_2, \ldots, d_n) \in \text{GL}_n(\mathbb{Z})$ such that $D = U \cdot B \cdot V$. Since $U, V \in \text{GL}_n(\mathbb{Z})$, we have $U^{-1} \cdot Z^n = Z^n$ and $V \cdot Z^n = Z^n$. Therefore,

$$Z^n/(B \cdot Z^n) = U^{-1} \cdot Z^n/(B \cdot V \cdot Z^n) \simeq Z^n/(U \cdot B \cdot V \cdot Z^n) = Z^n/(D \cdot Z^n) = \bigoplus_{i=1}^n \mathbb{Z}/d_i \mathbb{Z}^n.$$

The above isomorphism maps the element $\ell := (\ell_1 \ldots \ell_n)^t$ of $\bigoplus_{i=1}^n \mathbb{Z}/d_i \mathbb{Z}^n$ to the element

$$U^{-1} \cdot \ell \pmod{B \cdot Z^n}$$

of $Z^n/(B \cdot Z^n)$. Since $B$ is invertible, there is a bijection between $Z^n/(B \cdot Z^n)$ and $[0, 1)^n \cap B^{-1} \cdot Z^n$, given by

$$X \pmod{B \cdot Z^n} \mapsto B^{-1} \cdot X \pmod{Z^n}.$$

Composing this bijection with the isomorphism above, we get a bijection between $\bigoplus_{i=1}^n \mathbb{Z}/d_i \mathbb{Z}^n$ and $[0, 1)^n \cap B^{-1} \cdot Z^n$, given by

$$\ell \mapsto B^{-1} \cdot U^{-1} \cdot \ell \pmod{Z^n} = V \cdot D^{-1} \cdot \ell \pmod{Z^n}.$$

Now multiplication by $B$ maps $[0, 1)^n \cap B^{-1} \cdot Z^n$ bijectively to $B \cdot [0, 1)^n \cap Z^n$.

Let $N = p^{2m}$ and suppose, $\eta^X = f_{2m,p}$ is reducible. Let $Y = A_N X$. Since $\eta^X$ is reducible there exists $Y', Y'' \in \mathbb{Z}P_N \setminus \{0\}$ with $Y' \geq 0$ and $Y'' \geq 0$ such that $Y = Y' + Y''$ and both $B_N Y'$ and $B_N Y''$ have integer entries. Since $B_N$ is an integer matrix with determinant $d_N := p^{2m-1}(p^2 - 1)$, we see that $\frac{1}{d_N}$ is the least possible entry for $Y'$ and $Y''$. Since $Y' + Y'' = Y$ has $\frac{1}{d_N}$ as its first entry, either the first entry of $Y'$ or that of $Y''$ is zero. Similarly, either the last entry of $Y'$ or that of $Y''$ is zero. But it is easy to show that if both the first and the last entries of $Y'$ (resp. $Y''$) is zero, then $Y'$ (resp. $Y''$) is entirely zero. So, without loss of generality, we may assume that the first entry of $Y'$ is $\frac{1}{d_N}$ and the last entry of $Y'$ is 0. From the previous section and from the entries of the diagonal matrix $D_N$, we know that there exists $\ell_1 \in \{0, 1, \ldots, p^{2m-1} - 1\}$ and $\ell_2 \in \{0, 1, \ldots, p^{2m-1}(p^2 - 1) - 1\}$ such that

$$\frac{\ell_1}{p^{2m-1}} \cdot C_{2m,1} + \frac{\ell_2}{p^{2m-1}(p^2 - 1)} \cdot C_{2m,2} \equiv Y' \pmod{Z^n}.$$ 

(4.10)

Case 1. $(m = 1)$

*From the congruence relation (4.10) (resp. replacing $Y'$ with $Y''$ in (4.10)).
Equating only the first and the last entries from both sides of (4.10), we obtain
\[ \frac{\ell_1}{p} + \frac{p \ell_2}{p^2 - 1} \equiv \frac{1}{dN} \pmod{Z} \quad \text{and} \quad \frac{\ell_1}{p} + \frac{\ell_2}{p(p^2 - 1)} \equiv 0 \pmod{Z}, \]
which together implies that
\[ \frac{\ell_1}{p} \equiv \frac{1}{dN} \pmod{Z}. \]
But this modular equation has no solution in \( \ell_1 \in \{0, 1, \ldots, p - 1\} \). Thus we get a contradiction!

**Case 2.** \((m > 1)\)

Since the last entries of \( Y' \) and \( C_{2m,1} \) are 0, whereas the last entry of \( C_{2m,2} \) is 1, it follows that \( \ell_2 = 0 \). Since the first entry of \( C_{2m,1} \) is 1, we get
\[ \frac{\ell_1}{p^{2m-1}} \equiv \frac{1}{dN} \pmod{Z} \]
as in the previous case. Since as before, this has no solution in \( \ell_1 \in \{0, 1, \ldots, p^{2m-1} - 1\} \), we get a contradiction.

Hence, \( f_{2m,p} = \eta^X \) is irreducible. \( \square \)

**5. The matrix identities**

We continue to prove that the matrix identities \( B = UDV \), \( UU' = 1 \) and \( VV' = 1 \) with \( B = B_{p^n} \) as defined in (4.3) and \( D = D_{p^n} \) as defined in (4.7) holds if we define \( U = U_{p^n} \), \( V = V_{p^n} \), \( U' = U'_{p^n} \) and \( V' = V'_{p^n} \) as follows for \( n = 1, 2, 3 \) or \( n \geq 4 \):

For \( n = 1 \), we define
\[
U := \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix}, \quad V := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad U' := \begin{pmatrix} p & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad V' := \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}.
\]

For \( n = 2 \), we define
\[
U := \begin{pmatrix} 0 & 1 & 0 \\ 0 & p & 1 \\ 1 & p & 1 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & -1 & p^2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad U' := \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -p & 1 & 0 \end{pmatrix} \quad \text{and} \quad V' := \begin{pmatrix} -1 & p^2 + 1 & -1 \\ -1 & p^2 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

For \( n = 3 \), we define
$$U := \begin{pmatrix} 0 & -1 & -p & -p^2 \\ 0 & 0 & -1 & -p \\ 0 & 0 & -p & -(p^2 + 1) \\ 1 & p & p^2 & p^3 \end{pmatrix}, \quad V := \begin{pmatrix} 1 & 0 & 1 & p^3 \\ 0 & 0 & 1 & p \\ 0 & -1 & p & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U' := \begin{pmatrix} p & 0 & 0 & 1 \\ -1 & p & 0 & 0 \\ 0 & -(p^2 + 1) & p & 0 \\ 0 & p & -1 & 0 \end{pmatrix} \quad \text{and} \quad V' := \begin{pmatrix} 1 & -1 & 0 & -p(p^2 - 1) \\ 0 & p & -1 & -(p^2 - 1) \\ 0 & 1 & 0 & -p \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $n > 3$:

We define $U = (U_{i,j})_{0 \leq i,j \leq n}$ by

\[
j = 0 \quad \begin{array}{|c|c|}
\hline
i < n - 1 & 0 \\
\hline
i = n - 1 & 0 \\
\hline
i = n & 1 \\
\hline
\end{array}
\begin{cases}
-j^{j-1} & \text{if } j > i, \\
0 & \text{otherwise.}
\end{cases}
\]

We define $V = (V_{i,j})_{0 \leq i,j \leq n}$ by

\[
\begin{array}{|c|c|c|c|}
\hline
i = 0 & j = 0 & 0 < j < n - 1 & j = n - 1 \\
\hline
0 < i < n & 1 & 0 & 1 \\
\hline
i = n & 0 & 0 & 1 \\
\hline
\end{array}
\begin{cases}
-p^{i-j-1} & \text{if } i > j, \\
0 & \text{otherwise.}
\end{cases}
\]

We define $U' = (U'_{i,j})_{0 \leq i,j \leq n}$ by

\[
\begin{array}{|c|c|c|c|c|}
\hline
i = 0 & j = 0 & 0 < j < n - 2 & j = n - 2 & j = n - 1 & j = n \\
\hline
p & 0 & 0 & 0 & 1 \\
\hline
0 < i < n - 1 & -1 & 0 \\
\hline
i = n - 1 & 0 & -p^{n-j} & -(p^2 + 1) & p & 0 \\
\hline
i = n & 0 & p^{n-j-1} & p & -1 & 0 \\
\hline
\end{array}
\begin{cases}
p & \text{if } i = j, \\
-1 & \text{if } i = j + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

We define $V' = (V'_{i,j})_{0 \leq i,j \leq n}$ by

\[
\begin{array}{|c|c|c|c|}
\hline
i = 0 & j = 0 & 0 < j < n & j = n \\
\hline
1 & -1 & 0 & \ldots & 0 & -p^{n-2}(p^2 - 1) \\
\hline
0 < i < n - 1 \quad & 0 & \{ \begin{array}{l}
p & \text{if } i = j, \\
-1 & \text{if } i = j - 1, \\
0 & \text{otherwise.}
\end{array}
\}
\end{array}
\begin{cases}
p^{n-i-2}(p^2 - 1) & \text{if } i = j, \\
-p^{n-i-2} & \text{if } i = j - 1, \\
1 & \text{otherwise.}
\end{cases}
\]

\[
\begin{array}{|c|c|}
\hline
i = n - 1 & 0 & \ldots & 0 & -p^{n-2} \\
\hline
i = n & 0 & 0 & 1 \\
\hline
\end{array}
\begin{cases}
1 & \text{if } i = j, \\
0 & \text{if } i = j - 1, \\
0 & \text{otherwise.}
\end{cases}
\]

\[
\begin{array}{|c|c|}
\hline
j = 0 & 0 < j < n \\
\hline
1 & -1 & 0 & \ldots & 0 \\
\hline
-p^{n-2}(p^2 - 1) & \text{if } i = j, \\
-p^{n-i-2}(p^2 - 1) & \text{if } i = j - 1, \\
1 & \text{otherwise.}
\end{array}
\]
**Proposition 1.** Given \( n \in \mathbb{N} \), let \( U, V, U' \) and \( V' \) be the matrices as defined above. For \( N = p^n \), we set the matrices \( B = B_N \) and \( D = D_N \) as in equations equations (4.3) and (4.7). Then we have

\[
UU' = I, \quad VV' = I, \quad \text{and} \quad D = UBV.
\]

**Proof.** If \( n \leq 3 \), these identities hold trivially. If \( n > 3 \), the proofs of the equalities of the corresponding matrix entries in each of these matrix relations involve (at most) summation of some geometric series. For example, consider the identity \( D = UBV \). It is equivalent to \( DV' = UB \), assuming \( VV' = I \). Now from (4.7) and (5.4), we see that for \( i, j \in \{0, \ldots, n\} \), the \((i, j)\)-th entry of \( DV' \) is given by

\[
\begin{align*}
&\begin{cases}
1 & \text{if } j = 0, \\
-1 & \text{if } j > 0,
\end{cases} \\
&\begin{cases}
0 & \text{if } j < n, \\
p & \text{if } j = n,
\end{cases} \\
&\begin{cases}
-1 & \text{if } j = n - 1,
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

If we consider the case \( 0 < i < n - 1 \) and \( 0 < j < n \), then from (5.1) and (4.3) it follows that the product of the \( i \)-th row of \( U \) and the \( j \)-th column of \( B \) is

\[
-\sum_{k=1}^{n} p^{i-k-1} B_k, j = \sum_{k=i+1}^{n} p^{i-k-1}(p^{2(k-1)} - (p^2 + 1) \delta_{k,j})
\]

\[
= \sum_{k=\max\{i+1, j-1\}}^{j+1} p^{i-k-1}(p^{2(k-1)} - (p^2 + 1) \delta_{k,j})
\]

\[
= \begin{cases}
0 & \text{if } j < i, \\
p & \text{if } j = i, \\
-1 & \text{if } j = i + 1, \\
0 & \text{if } j > i + 1,
\end{cases}
\]

where \( \delta \) is the usual Kronecker delta function. So, the claim holds in this case.

Again, if we consider the case \( i = n - 1 \) and \( 0 < j < n \), then the product of the \( i \)-th row of \( U \) and the \( j \)-th column of \( B \) is

\[
-\sum_{k=1}^{n} p^{n-k} \left(p^{2(k-1)} - 1\right) B_k, j = \sum_{k=2}^{n} p^{n-k} \left(p^{2(k-1)} - 1\right) \left(p^{2(k-1)} - (p^2 + 1) \delta_{k,j}\right)
\]

\[
= \sum_{k=\max\{2, j-1\}}^{j+1} p^{n-k} \left(p^{2(k-1)} - 1\right) \left(p^{2(k-1)} - (p^2 + 1) \delta_{k,j}\right)
\]

\[
= \begin{cases}
p^{n-1} & \text{if } j = 1, \\
0 & \text{otherwise},
\end{cases}
\]

where the first equality holds since \( p^{2(k-1)} = 1 \) for \( k = 1 \). Thus, the claim also holds in this case. The rest of the proof is quite similar and only requires some more straightforward checks as above. \( \square \)
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