TRANSPOSITION ANTI-INVOLUTION IN CLIFFORD ALGEBRAS AND INVARIANCE GROUPS OF SCALAR PRODUCTS ON SPINOR SPACES

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Abstract. We introduce on the abstract level in real Clifford algebras $C_{\ell}^{p,q}$ of a non-degenerate quadratic space $(V,Q)$, where $Q$ has signature $\varepsilon = (p,q)$, a transposition anti-involution $T_{\varepsilon}^{-}$. In a spinor representation, the anti-involution $T_{\varepsilon}^{-}$ gives transposition, complex Hermitian conjugation or quaternionic Hermitian conjugation when the spinor space $\tilde{S}$ is viewed as a $C_{\ell}^{p,q}$-left and $\tilde{K}$-right module with $\tilde{K}$ isomorphic to $\mathbb{R}$ or $\mathbb{C}$, or, $\mathbb{H}$ or $2\mathbb{H}$. This map and its application to SVD was first presented at ICCA 7 in Toulouse in 2005 [3].

The anti-involution $T_{\varepsilon}^{-}$ is a lifting to $C_{\ell}^{p,q}$ of an orthogonal involution $t_{\varepsilon} : V \rightarrow V$ which depends on the signature of $Q$. The involution is a symmetric correlation [18] $t_{\varepsilon} : V \rightarrow V^{*} \cong V$ and it allows one to define a reciprocal basis for the dual space $(V^{*}, Q)$. When the Clifford algebra $C_{\ell}^{p,q}$ splits into the graded tensor product $C_{\ell}^{p,0} \otimes C_{\ell}^{0,q}$, the anti-involution $T_{\varepsilon}^{-}$ acts as reversion on $C_{\ell}^{p,0}$ and as conjugation on $C_{\ell}^{0,q}$. Using the concept of a transpose of a linear mapping one can show that if $[L_{u}]$ is a matrix in the left regular representation of the operator $L_{u} : C_{\ell}^{p,q} \rightarrow C_{\ell}^{p,q}$ relative to a Grassmann basis $B$ in $C_{\ell}^{p,q}$, then matrix $[LT_{\varepsilon}^{-}(u)]$ is the matrix transpose of $[L_{u}]$, see [6].

Of particular importance is the action of $T_{\varepsilon}^{-}$ on the spinor space. The algebraic spinor space $\tilde{S}$ is realized as a left minimal ideal generated by a primitive idempotent $f$, or a sum $f + \hat{f}$ in simple or semisimple algebras as in [14]. The map $T_{\varepsilon}^{-}$ allows us to define a new spinor scalar product $\tilde{S} \times \tilde{S} \rightarrow \tilde{K}$, where $\tilde{K} = fC_{\ell}^{p,q}f$ and $\tilde{K} = K$ or $K \oplus \hat{K}$ depending whether the algebra is simple or semisimple. Our scalar product is in general different from the two scalar products discussed in literature, e.g., [14]. However, it reduces to one or the other in Euclidean and anti-Euclidean signatures. The anti-involution $T_{\varepsilon}^{-}$ acts as the identity map, complex conjugation, or quaternionic conjugation on $\tilde{K}$. Thus, the action of $T_{\varepsilon}^{-}$ on spinors results in matrix transposition, complex Hermitian conjugation, or quaternionic Hermitian conjugation. We classify automorphism group of the new product as $O(N)$, $U(N)$, $Sp(N)$, $2O(N)$, or $2Sp(N)$. 

1
1 INTRODUCTION

Let $\mathcal{C}_n$ be a universal Clifford algebra over an $n$-dimensional real quadratic space $(V, Q)$ with $Q(x) = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \cdots + \varepsilon_n x_n^2$ where $\varepsilon_i = \pm 1$ and $x = x_1 e_1 + \cdots + x_n e_n \in V$ for an orthonormal basis $B_1 = \{e_i\}_{i=1}^n$. Let $B$ be the canonical basis of $\bigwedge V$ generated by $B_1$. That is, let $[n] = \{1, 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$, by underlined Roman characters. The basis elements of $\bigwedge V$, or, of $\mathcal{C}_n$ due to the linear space isomorphism $\bigwedge V \rightarrow \mathcal{C}_n$ [14], can be indexed by these finite ordered subsets as $e_i = \lambda_{\epsilon_1} e_1$. Then, an arbitrary element of $\bigwedge V$, can be written as $u = \sum_{\lambda \subseteq [n]} u_\lambda e_\lambda$ where $u_\lambda \in \mathbb{R}$ for each $\lambda \in 2^{[n]}$. The unit element 1 of $\mathcal{C}_n$ is identified with $e_0$. Our preferred basis for $\mathcal{C}_n$ is the exterior algebra basis $B$ sorted by an admissible monomial order $\prec$ on $\bigwedge V$. We choose for $\prec$ the monomial order called InvLex, or, the inverse lexicographic order [45]. Let $B$ be the symmetric bilinear form defined by $Q$ and let $\prec, \triangleright : \bigwedge V \times \bigwedge V \rightarrow \mathbb{R}$ be an extension of $B$ to $\bigwedge V$ [14]. We will need this extension later when we define the Clifford algebra $\mathcal{C}(V^*, Q)$.

We begin by defining the following map on $(V, Q)$ dependent on the signature $\varepsilon$ of $Q$.

Definition 1. Let $t_\varepsilon : V \rightarrow V$ be the linear map defined as

$$t_\varepsilon(x) = t_\varepsilon(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i \left( \frac{e_i}{\varepsilon_i} \right) = \sum_{i=1}^n x_i (\varepsilon_i e_i)$$

(1)

for any $x \in V$ and for the orthonormal basis $B_1 = \{e_i\}_{i=1}^n$ in $V$ diagonalizing $Q$.

The $t_\varepsilon$ map can be viewed in two ways: (1) As a linear orthogonal involution of $V$; (2) As a correlation [18] mapping $t_\varepsilon : V \rightarrow V^* \cong V$. The set of vectors $B_1^* = \{t_\varepsilon(e_i)\}_{i=1}^n$ gives an orthonormal basis in the dual space $(V^*, Q)$. Furthermore, under the identification $V \cong V^*$, $t_\varepsilon$ is a symmetric non-degenerate correlation on $V$ thus making the pair $(V, t_\varepsilon)$ into a non-degenerate real correlated (linear) space [6]. Then, viewing $t_\varepsilon$ as a correlation $V \rightarrow V^*$, we can define the action of $t_\varepsilon(x) \in V^*$ on $y \in V$ for any $x \in V$ as

$$t_\varepsilon(x)(y) = \langle t_\varepsilon(x), y \rangle,$$

(2)

and we get the expected duality relation among the basis elements in $B_1$ and $B_1^*$:

$$t_\varepsilon(e_i)(e_j) = \langle \varepsilon_i e_i, e_j \rangle = \varepsilon_i \langle e_i, e_j \rangle = \varepsilon_i \delta_{i,j} = \delta_{i,j}.$$  

(3)

The extension of the duality $V \rightarrow V^*$ to the Clifford algebras $\mathcal{C}(V, Q) \rightarrow \mathcal{C}(V^*, Q)$ is of fundamental importance to defining a new transposition scalar product on spinor spaces. When we apply Porteous’ theorem [18] Thm. 15.32] to the involution $t_\varepsilon$, we get the following theorem and its corollary proven in [6].

Proposition 1. Let $A = \mathcal{C}_n$ be the universal Clifford algebra of $(V, Q)$ and let $t_\varepsilon : V \rightarrow V$ be the orthogonal involution of $V$ defined in [14]. Then there exists a unique algebra involution $T_{t_\varepsilon}$ of $A$ and a unique algebra anti-involution $T_{t_\varepsilon}$ of $A$ such that the following diagrams commute:

$$\begin{align*}
\begin{array}{ccc}
V & \xrightarrow{t_\varepsilon} & V \\
\downarrow & & \downarrow \\
A & \xrightarrow{T_{t_\varepsilon}} & A
\end{array} & \quad \text{and} \quad \begin{array}{ccc}
V & \xrightarrow{t_\varepsilon} & V \\
\downarrow & & \downarrow \\
A & \xrightarrow{T_{t_\varepsilon}} & A
\end{array}
\end{align*}$$

(4)

In particular, we can define $T_{t_\varepsilon}$ and $T_{t_\varepsilon}$ as follows:

\footnote{We view $\mathcal{C}_n$ as Porteous’ $L^\alpha$-Clifford algebra for $(V, Q)$ under the identification $L = \mathbb{R}$ and $\alpha = 1_{\mathbb{R}}$.}
Definition 2. The Clifford algebra over the dual space $V^*$ is the universal Clifford algebra $\mathcal{C}(V^*, Q)$ of the quadratic pair $(V^*, Q)$. For short, we denote this algebra by $\mathcal{C}^*_{\eta}$.\footnote{The switches are defined on the basis tensors $e_i \otimes e_j \in \mathcal{C}^*_{\eta}$ as $S(e_i \otimes e_j) = e_j \otimes e_i$ and $\hat{S}(e_i \otimes e_j) = (-1)^{|i|+|j|} e_j \otimes e_i$. Then, their action is extended by linearity to the graded product $\mathcal{C}^*_{\eta}$ as expected.}

(i) For simple $k$-vectors $e_k$ in $\mathcal{B}$, let $T_\varepsilon(e_k) = T_\varepsilon(\prod_{i \in k} e_i) = \prod_{i \in k} t_\varepsilon(e_i)$ where $k = |i|$ and $T_\varepsilon(1_A) = 1_A$. Then, extend by linearity to all of $\mathcal{A}$.

(ii) For simple $k$-vectors $e_k$ in $\mathcal{B}$, let

$$T_\varepsilon^\sim(e_k) = T_\varepsilon^\sim(\prod_{i \in k} e_i) = (\prod_{i \in k} t_\varepsilon(e_i))' = (-1)^{\frac{k(k-1)}{2}} \prod_{i \in k} t_\varepsilon(e_i) \tag{5}$$

where $k = |i|$ and $T_\varepsilon^\sim(1_A) = 1_A$. Then, extend by linearity to all of $\mathcal{A}$.

Maple code of the procedure tp which implements the anti-involution $T_\varepsilon^\sim$ in $\mathcal{C}^*_{\eta}$, was first presented at ICCA 7 in Toulouse [3]. The procedure tp requires the CLIFFORD package [9]. In the following corollary, $\alpha, \beta, \gamma$ denote, respectively, the grade involution, the reversion, and the conjugation in $\mathcal{C}^*_{\eta}$.

Corollary 1. Let $\mathcal{A} = \mathcal{C}^*_{p,q}$ and let $T_\varepsilon : \mathcal{A} \to \mathcal{A}$ and $T_\varepsilon^\sim : \mathcal{A} \to \mathcal{A}$ be the involution and the anti-involution of $\mathcal{A}$ from Proposition [7]

(i) For the Euclidean signature $(p, q) = (n, 0)$, or $p - q = n$, we have $t_\varepsilon = 1_V$. Thus, $T_\varepsilon$ is the identity map $1_{\mathcal{A}}$ on $\mathcal{A}$ and $T_\varepsilon^\sim$ is the reversion $\beta$ of $\mathcal{A}$.

(ii) For the anti-Euclidean signature $(p, q) = (0, n)$, or $p - q = -n$, we have $t_\varepsilon = -1_V$. Thus, $T_\varepsilon$ is the grade involution $\alpha$ of $\mathcal{A}$ and $T_\varepsilon^\sim$ is the conjugation $\gamma$ of $\mathcal{A}$.

(iii) For all other signatures $-n < p - q < n$, we have $t_\varepsilon = 1_{V_1} \otimes -1_{V_2}$ where $(V, Q) = (V_1, Q_1) \perp (V_2, Q_2)$. Here, $(V_1, Q_1)$ is the Euclidean subspace of $(V, Q)$ of dimension $p$ spanned by $\{e_i\}_{i=1}^p$ with $Q_1 = Q|_{V_1}$ while $(V_2, Q_2)$ is the anti-Euclidean subspace of $(V, Q)$ of dimension $q$ spanned by $\{e_i\}_{i=p+1}^q$ with $Q_2 = Q|_{V_2}$. Let $\mathcal{A}_1 = \mathcal{C}(V_1, Q_1)$ and $\mathcal{A}_2 = \mathcal{C}(V_2, Q_2)$ so $\mathcal{C}(V, Q) \cong \mathcal{C}(V_1, Q_1) \otimes \mathcal{C}(V_2, Q_2)$. Let $S$ (resp. $\hat{S}$) be the ungraded switch (resp. the graded switch) on $\mathcal{C}(V_1, Q_1) \otimes \mathcal{C}(V_2, Q_2)$. Then,

$$T_\varepsilon = 1_{\mathcal{A}_1} \otimes \alpha_{\mathcal{A}_2} \quad \text{and} \quad T_\varepsilon^\sim = (\beta_{\mathcal{A}_1} \otimes \gamma_{\mathcal{A}_2}) \circ (\hat{S} \circ S).$$

(iv) The anti-involution $T_\varepsilon^\sim$ is related to the involution $T_\varepsilon$ through the reversion $\beta$ as follows:

$$T_\varepsilon^\sim = T_\varepsilon \circ \beta = \beta \circ T_\varepsilon.$$

For an extensive discussion of the properties of the involutions $T_\varepsilon^\sim$ and $T_\varepsilon$ see [6].

Since $(V^*, Q)$ is a non-degenerate quadratic space spanned by the orthonormal basis $B_i^*$, we can define the Clifford algebra $\mathcal{C}(V^*, Q)$ as expected.
Let $B^*$ be the canonical basis of $\bigwedge V^* \cong \mathcal{Cl}^{\ast}_n$ generated by $B_1^*$ and sorted by InvLex. That is, we define $B^* = \{ T_\varepsilon(e_\underline{\varepsilon}) | e_\underline{\varepsilon} \in B \}$ given that

$$<T_\varepsilon(e_\underline{\varepsilon}), e_\underline{\varepsilon}^\ast > = \delta_{\underline{\varepsilon} \underline{\varepsilon}}$$

(6)

for $e_\underline{\varepsilon}, e_\underline{\varepsilon}^\ast \in B$ and $T_\varepsilon(e_\underline{\varepsilon}) \in B^*$. An arbitrary linear form $\varphi$ in $\bigwedge V^* \cong \mathcal{Cl}^{\ast}_n$ can be written as

$$\varphi = \sum_{\underline{\varepsilon} \in 2^n} \varphi_{\underline{\varepsilon}} T_\varepsilon(e_\underline{\varepsilon})$$

(7)

where $\varphi_{\underline{\varepsilon}} \in \mathbb{R}$ for each $\underline{\varepsilon} \in 2^n$. Due to the linear isomorphisms $V \cong V^*$ and $\bigwedge V^* \cong \mathcal{Cl}(V^*, Q)$, we extend, by a small abuse of notation, the inner product $<\cdot, \cdot>$ defined in $\bigwedge V$ to

$$<\cdot, \cdot> : \bigwedge V^* \times \bigwedge V^* \rightarrow \mathbb{R}.$$

(8)

In this way we find, as expected, that the matrix of this inner product on $\bigwedge V^*$ is also diagonal, that is, that the basis $B^*$ is orthonormal with respect to $<\cdot, \cdot>$. We extend the action of dual vectors from $V^*$ on $V$ to all linear forms $\varphi$ in $\mathcal{Cl}^{\ast}_n$ acting on multivectors $v$ in $\mathcal{Cl}_n$ via the inner product (8) as

$$\varphi(v) = <\varphi, v> = \sum_{\underline{\varepsilon} \in 2^n} \varphi_{\underline{\varepsilon}} v_{\underline{\varepsilon}}$$

(9)

given that $\varphi = \sum_{\underline{\varepsilon} \in 2^n} \varphi_{\underline{\varepsilon}} T_\varepsilon(e_\underline{\varepsilon}) \in \mathcal{Cl}^*_n$ where $\varphi_{\underline{\varepsilon}} = \varphi(e_\underline{\varepsilon}) \in \mathbb{R}$ and $v = \sum_{\underline{\varepsilon} \in 2^n} v_{\underline{\varepsilon}} e_\underline{\varepsilon} \in \mathcal{Cl}_n$ for some coefficients $v_{\underline{\varepsilon}} \in \mathbb{R}$.

Properties of the left multiplication operator $L_u : \mathcal{Cl}_n \rightarrow \mathcal{Cl}_n$, $v \mapsto uv$, $\forall v \in \mathcal{Cl}_n$ and its dual $L_\overline{u}$ with respect to the inner product $<\cdot, \cdot> : \bigwedge V \times \bigwedge V \rightarrow \mathbb{R}$ are discussed in [6]. In particular, it is shown there that if $[L_u]$ is the matrix of the operator $L_u$ relative to the basis $B$ and $[L_Tv(u)]$ is the matrix of the operator $LTv(u)$ relative to the basis $B$, then $[L_u]^T = [L_Tv(u)] = [L_Tv(u)]^\ast$ where $[L_u]^\ast$ is the matrix transpose of $[L_u]$. However, in order to introduce a new scalar product on spinor spaces related to the involution $T_\varepsilon^\ast$, we need to discuss the action of $T_\varepsilon^\ast$ on spinor spaces.

## 2 ACTION OF THE TRANSPPOSITION INVOLUTION ON SPINOR SPACES

Stabilizer groups $G_{p,q}(f)$ of primitive idempotents $f$ are classified in [7]. The stabilizer $G_{p,q}(f)$ is a normal subgroup of Salingaros’ finite vee group $G_{p,q}$ [20, 22] which acts via conjugation on $\mathcal{Cl}_{p,q}$. The importance of the stabilizers to the spinor representation theory lies in the fact that a transversal $\mathcal{T}$ of $G_{p,q}(f)$ in $G_{p,q}$ generates spinor bases in $S = \mathcal{Cl}_{p,q}^{\ast}$ and $\mathcal{S} = \mathcal{Cl}_{p,q}^{\ast}$. In [7] it is also shown that depending on the signature $\varepsilon = (p, q)$, the real anti-involution $T_\varepsilon^\ast$ is responsible for transposition, the Hermitian complex, or the Hermitian quaternionic conjugation of a matrix $[u]$ for any $u$ in all Clifford algebras $\mathcal{Cl}_{p,q}$ with the spinor representation realized either in $S$ (simple algebras) or in $\mathcal{S} = \mathcal{S} \oplus \mathcal{S}$ (semisimple algebras). This is because $T_\varepsilon^\ast$ acts on $\mathbb{K} = f\mathcal{Cl}_{p,q}^{\ast}$ and $\mathbb{K} = \mathbb{K} \oplus \mathbb{K}$ as an anti-involution. Thus, $T_\varepsilon^\ast$ allows us to define a dual spinor space $S^\ast$ or $\mathcal{S}$, a new spinor product, and a new spinor norm. The following results are proven in [7].

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4Let $K$ be a subgroup of a group $G$. A transversal $\ell$ of $K$ in $G$ is a subset of $G$ consisting of exactly one element $\ell(bK)$ from every (left) coset $bK$, and with $\ell(K) = 1$ [19].
Proposition 2. Let $\psi, \phi \in S = \mathcal{C}(p,q,f)$. Then, $T_{\tilde{\varepsilon}}(\psi)\phi \in \mathbb{K}$. In particular, $T_{\tilde{\varepsilon}}(\psi)\psi \in \mathbb{R}f \subset \mathbb{K}$.

Thus, we can define an invariance group of the scalar product $S \times S \to \mathbb{K}$, $(\psi, \phi) \mapsto T_{\tilde{\varepsilon}}(\psi)\phi$, as follows:

Definition 3. Let $G^e_{p,q} = \{g \in \mathcal{C}(p,q) \mid T_{\tilde{\varepsilon}}(g)g = 1\}$.

We find that $G_{p,q}(f) \trianglelefteq G_{p,q} \leq G^e_{p,q} \triangleleft \mathcal{C}(p,q)$ (the group of units in $\mathcal{C}(p,q)$). Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a set of $N = 2^k$, $k = q - r_{q-p}$ mutually annihilating primitive idempotents adding up to 1 in a simple Clifford algebra $\mathcal{C}(p,q)$.

Proposition 3. Let $\mathcal{C}(p,q)$ be a simple Clifford algebra, $p - q \neq 1 \bmod 4$ and $p + q \leq 9$. Let $\psi_i \in S_i = \mathcal{C}(p,q)f_i$, $f_i \in \mathcal{F}$, and let $[\psi_i]$ (resp. $[T_{\tilde{\varepsilon}}(\psi_i)]$) be the matrix of $\psi_i$ (resp. $T_{\tilde{\varepsilon}}(\psi_i)$) in the spinor representation with respect to the ordered basis $S_1 = [m_1 f_1, \ldots, m_N f_1]$ with $\alpha_i = m_i^2$.

Then,

$$[T_{\tilde{\varepsilon}}(\psi_i)] = \begin{cases} [\psi_i]^T & \text{if } p - q = 0, 1, 2 \bmod 8; \\ [\psi_i] & \text{if } p - q = 3, 7 \bmod 8; \\ [\psi_i]^\dagger & \text{if } p - q = 4, 5, 6 \bmod 8; \end{cases}$$

(10)

where $T$ denotes transposition, $\dagger$ denotes Hermitian complex conjugation, and $\hat{\dagger}$ denotes Hermitian quaternionic conjugation.

This action of $T_{\tilde{\varepsilon}}$ on $S = S_i$ extends to a similar action on $\hat{S}$, hence to $\hat{\hat{S}} = S \oplus \hat{S}$ as it is shown in [7],[8]. In particular, the product $(\psi, \phi) \mapsto T_{\tilde{\varepsilon}}(\psi)\phi$ is invariant under two of the subgroups of $G^e_{p,q}$. The Salingaros’ vee group $G_{p,q} < G^e_{p,q}$ and the stabilizer group $G_{p,q}(f)$ of a primitive idempotent $f$. Since the stabilizer group $G_{p,q}(f)$ and its subgroups play an important role in constructing and understanding spinor representation of Clifford algebras, we provide here a brief summary of related definitions and findings. See [8] for a complete discussion.

Primitive idempotents $f \in \mathcal{F} \subset \mathcal{C}(p,q)$ formed out of commuting basis monomials $e_{2i}, \ldots, e_{2k}$ in $B$ with square 1 have the form $f = \frac{1}{2}(1 \pm e_{2i})\frac{1}{2}(1 \pm e_{2j}) \cdots \frac{1}{2}(1 \pm e_{2k})$ where $k = q - r_{q-p}$. With any primitive idempotent $f$, we associate the following groups:

(i) The stabilizer $G_{p,q}(f)$ of $f$ defined as

$$G_{p,q}(f) = \{m \in G_{p,q} \mid fm = m^{-1} f\} < G_{p,q}.$$  

(11)

The stabilizer $G_{p,q}(f)$ is a normal subgroup of $G_{p,q}$. In particular,

$$|G_{p,q}(f)| = \begin{cases} 2^{1+p+r_{q-p}}, & p - q \neq 1 \bmod 4; \\ 2^{2+p+r_{q-p}}, & p - q = 1 \bmod 4. \end{cases}$$

(12)

(ii) An abelian idempotent group $T_{p,q}(f)$ of $f$, a subgroup of $G_{p,q}(f)$ defined as

$$T_{p,q}(f) = \langle \pm 1, e_{2i}, \ldots, e_{2k} \rangle < G_{p,q}(f),$$

(13)

where $k = q - r_{q-p}$.

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3Here, $r_i$ is Radon-Hurwitz number defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0, r_1 = 1, r_2 = r_3 = 2, r_4 = r_5 = r_6 = r_7 = 3$ [13],[14].

4For the sake of consistency with a proof of this proposition given in [7] we remark that $\alpha_i$ is just the square of the monomial $m_i^2 \in \{\pm 1\}$. 

5
(iii) A field group $K_{p,q}(f)$ of $f$, a subgroup of $G_{p,q}(f)$, related to the (skew double) field $\mathbb{K} \cong f\mathcal{Cl}_{p,q}f$, and defined as

$$K_{p,q}(f) = \langle \pm 1, m \mid m \in \mathbb{K} \rangle < G_{p,q}(f)$$

(14)

where $\mathbb{K}$ is a set of Grassmann monomials in $\mathcal{B}$ which provide a basis for $\mathbb{K} = f\mathcal{Cl}_{p,q}f$ as a real subalgebra of $\mathcal{Cl}_{p,q}$.

The following theorem proven in [8] relates the above groups to $G_{p,q}$ and its commutator subgroup $G'_{p,q}$.

**Theorem 1.** Let $f$ be a primitive idempotent in a simple or semisimple Clifford algebra $\mathcal{Cl}_{p,q}$ and let $G_{p,q}$, $G_{p,q}(f)$, $T_{p,q}(f)$, $K_{p,q}(f)$, and $G'_{p,q}$ be the groups defined above. Furthermore, let $S = \mathcal{Cl}_{p,q}f$ and $\mathbb{K} = f\mathcal{Cl}_{p,q}f$.

(i) Elements of $T_{p,q}(f)$ and $K_{p,q}(f)$ commute.

(ii) $T_{p,q}(f) \cap K_{p,q}(f) = G'_{p,q} = \{\pm 1\}$.

(iii) $G_{p,q}(f) = T_{p,q}(f)K_{p,q}(f) = K_{p,q}(f)T_{p,q}(f)$.

(iv) $|G_{p,q}(f)| = |T_{p,q}(f)K_{p,q}(f)| = \frac{1}{2}|T_{p,q}(f)||K_{p,q}(f)|$.

(v) $G_{p,q}(f) < G_{p,q}$, $T_{p,q}(f) < G_{p,q}$, and $K_{p,q}(f) < G_{p,q}$ in particular, $T_{p,q}(f)$ and $K_{p,q}(f)$ are normal subgroups of $G_{p,q}(f)$.

(vi) $G_{p,q}(f)/K_{p,q}(f) \cong T_{p,q}(f)/G'_{p,q}$ and $G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/G'_{p,q}$.

(vii) $(G_{p,q}(f)/G'_{p,q})/(T_{p,q}(f)/G'_{p,q}) \cong G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/\{\pm 1\}$ and the transversal of $T_{p,q}(f)$ in $G_{p,q}(f)$ spans $\mathbb{K}$ over $\mathbb{R}$ modulo $f$.

(viii) A transversal of $G_{p,q}(f)$ in $G_{p,q}$ spans $S$ over $\mathbb{K}$ modulo $f$.

(ix) $(G_{p,q}(f)/T_{p,q}(f)) < (G_{p,q}/T_{p,q}(f))$ and $(G_{p,q}/T_{p,q}(f))/(G_{p,q}(f)/T_{p,q}(f)) \cong G_{p,q}/G_{p,q}(f)$ and a transversal of $T_{p,q}(f)$ in $G_{p,q}$ spans $S$ over $\mathbb{R}$ modulo $f$.

(x) The stabilizer $G_{p,q}(f) = \bigcap_{x \in T_{p,q}(f)} C_{G_{p,q}}(x) = C_{G_{p,q}}(T_{p,q}(f))$ where $C_{G_{p,q}}(x)$ is the centralizer of $x$ in $G_{p,q}$ and $C_{G_{p,q}}(T_{p,q}(f))$ is the centralizer of $T_{p,q}(f)$ in $G_{p,q}$.

Recall that in CLIFFORD [9] information about each Clifford algebra $\mathcal{Cl}_{p,q}$ for $p + q \leq 9$ is stored in a built-in data file. This information can be retrieved in the form of a seven-element list with the command clidata([[p, q]]). For example, for $\mathcal{Cl}_{3,0}$ we find:

$$\text{data} = [\text{complex}, 2, \text{simple}, \frac{1}{2}\text{Id} + \frac{1}{2}e_1, [\text{Id}, e_2, e_3, e_{23}], [\text{Id}, e_{23}], [\text{Id}, e_2]]$$

(15)

where Id denotes the identity element of the algebra. In particular, from the above we find that: (i) $\mathcal{Cl}_{3,0}$ is a simple algebra isomorphic to $\text{Mat}(2, \mathbb{C})$; (data[1], data[2], data[3]) (ii) The expression $\frac{1}{2}\text{Id} + \frac{1}{2}e_1$ (data[4]) is a primitive idempotent $f$ which may be used to generate a spinor ideal $S = \mathcal{Cl}_{3,0};$ (iii) The fifth entry data[5] provides, modulo $f$, a real basis for $S$, that is, $S = \text{span}_\mathbb{R}\{f, e_2, e_3, e_{23}, e_{23}f\};$ (iv) The sixth entry data[6] provides, modulo $f$, a real basis for $\mathbb{K} = f\mathcal{Cl}_{3,0}f \cong \mathbb{C}$, that is, $\mathbb{K} = \text{span}_\mathbb{R}\{f, e_{23}, e_{23}f\};$ and, (v) The seventh entry data[7] provides, modulo $f$, a basis for $S$ over $\mathbb{K}$, that is, $S = \text{span}_\mathbb{K}\{f, e_{23}f\}$.

The above theorem yields the following corollary:

7We have $G'_{p,q} = \{1, -1\}$ since any two monomials in $G_{p,q}$ either commute or anticommute.
8See [12] how to use CLIFFORD.
Corollary 2. Let data be the list of data returned by the procedure clidata in CLIFFORD. Then, data[5] is a transversal of \( T_{p,q}(f) \) in \( G_{p,q} \); data[6] is a transversal of \( T_{p,q}(f) \) in \( G_{p,q} \); and data[7] is a transversal of \( G_{p,q} \) in \( G_{p,q} \). Therefore, \[ |\text{data}[5]| = |\text{data}[6]| |\text{data}[7]| \]. This is equivalent to \[ |G_{p,q}(f)| = |G_{p,q}(f)| \].

The theorem and the corollary are illustrated with examples in [8]. Maple worksheets verifying this and other results from [6–8] can be accessed from [10].

3 TRANSPOSITION SCALAR PRODUCT ON SPINOR SPACES

In [14, Ch. 18], Lounesto discusses scalar products on \( S = \mathcal{C}l_{p,q} f \) for simple Clifford algebras and on \( \tilde{S} = S \oplus \hat{S} = \mathcal{C}l_{p,q} e, e = f + \hat{f} \), for semisimple Clifford algebras where \( \hat{f} \) denotes the grade involution of \( f \). It is well known that in each case the spinor representation is faithful. Following Lounesto, we let \( K \) be either \( K \) or \( \mathbb{K} \) and \( \tilde{S} \) be either \( S \) or \( S \oplus \hat{S} \) when \( \mathcal{C}l_{p,q} \) is simple or semisimple, respectively. Then, in the simple algebras, the two \( \beta \)-scalar products are

\[
S \times S \to K, \quad (\psi, \phi) \mapsto \begin{cases} 
\beta_+ (\psi, \phi) = s_1 \tilde{\psi} \phi \\
\beta_- (\psi, \phi) = s_2 \psi \phi 
\end{cases}
\]

(16)

whereas in the semisimple algebras they are

\[
\tilde{S} \times \tilde{S} \to \mathbb{K}, \quad (\tilde{\psi}, \tilde{\phi}) \mapsto \begin{cases} 
(\beta_+ (\tilde{\psi}, \tilde{\phi}), \beta_+ (\psi_g, \phi_g)) = (s_1 \tilde{\psi} \phi, s_1 \tilde{\psi} \phi) \\
(\beta_- (\tilde{\psi}, \tilde{\phi}), \beta_- (\psi_g, \phi_g)) = (s_2 \psi \phi, s_2 \psi \phi) 
\end{cases}
\]

(17)

for \( \tilde{\psi} = \psi + \psi_g \) and \( \tilde{\phi} = \phi + \phi_g \), \( \psi, \phi \in S, \psi_g, \phi_g \in \tilde{S} \), and where \( \tilde{\psi}, \tilde{\psi}_g \) (resp. \( \tilde{\phi}, \tilde{\phi}_g \)) denotes reversion (resp. Clifford conjugation) of \( \psi, \psi_g \). Here \( s_1, s_2 \) are special monomials in the Clifford algebra basis \( \mathcal{B} \) which guarantee that the products \( s_1 \tilde{\psi} \phi, s_2 \psi \phi \), hence also \( s_1 \tilde{\psi}_g \phi_g, s_2 \psi_g \phi_g \), belong to \( K \approx \mathbb{K}^9 \). In fact, the monomials \( s_1, s_2 \) belong to the chosen transversal of the stabilizer \( G_{p,q} (f) \) in \( G_{p,q} \) [8]. The automorphism groups of \( \beta_+ \) and \( \beta_- \) are defined in the simple case as, respectively, \( G_+ = \{ s \in \mathcal{C}l_{p,q} \mid ss = 1 \} \) and \( G_- = \{ s \in \mathcal{C}l_{p,q} \mid ss = 1 \} \), and as \( 2G_- \) and \( 2G_+ \) in the semisimple case. They are shown in [14, Tables 1 and 2, p. 236].

3.1 Simple Clifford algebras

In Example 3 [7] it was shown that the transposition scalar product in \( S = \mathcal{C}l_{2,2} f \) is different from each of the two Lounesto’s products whereas Example 4 showed that the transposition product in \( S = \mathcal{C}l_{2,0} f \) coincided with \( \beta_+ \). Furthermore, it was remarked that \( T_{\varepsilon}^e (\psi) \phi \) always equaled \( \beta_+ \) for Euclidean signatures \((p,0)\) and \( \beta_- \) for anti-Euclidean signatures \((0,q)\). We formalize this in the following proposition. For all proofs see [8].

Proposition 4. Let \( \psi, \phi \in S = \mathcal{C}l_{p,q} f \) and \((\psi, \phi) \mapsto T_{\varepsilon}^e (\psi) \phi = \lambda f, \lambda \in K \), be the transposition scalar product. Let \( \beta_+ \) and \( \beta_- \) be the scalar products on \( S \) shown in (16). Then, there exist monomials \( s_1, s_2 \) in the transversal \( \ell \) of \( G_{p,q} (f) \) in \( G_{p,q} \) such that

\[
T_{\varepsilon}^e (\psi) \phi = \begin{cases} 
\beta_+ (\psi, \phi) = s_1 \tilde{\psi} \phi, & \forall \psi, \phi \in \mathcal{C}l_{p,0} f, \\
\beta_- (\psi, \phi) = s_2 \psi \phi, & \forall \psi, \phi \in \mathcal{C}l_{0,q} f.
\end{cases}
\]

(18)

In simple Clifford algebras, the monomials \( s_1 \) and \( s_2 \) also satisfy: (i) \( \tilde{f} = s_1 f s_1^{-1} \) and (ii) \( f = s_2 f s_2^{-1} \). The identity (i) (resp. (ii)) is also valid in the semisimple algebras provided \( \beta_+ \neq 0 \) (resp. \( \beta_- \neq 0 \).
Table 1 (Part 1): Automorphism group $G^e_{p,q}$ of $T^e_\psi(\phi)$
in simple Clifford algebras $\mathcal{C}l_{p,q} \cong \text{Mat}(2^k, \mathbb{R})$

$k = q - r_{q-p}, p - q \neq 1 \mod 4, p - q = 0, 1, 2 \mod 8$

| $(p, q)$ | $G^e_{p,q}$ |
|---------|-------------|
| $(0, 0)$ | $O(1)$      |
| $(1, 1)$ | $O(2)$      |
| $(2, 0)$ | $O(2)$      |
| $(2, 2)$ | $O(4)$      |
| $(3, 1)$ | $O(8)$      |
| $(3, 3)$ | $O(8)$      |
| $(0, 6)$ |             |

Table 1 (Part 2): Automorphism group $G^e_{p,q}$ of $T^e_\psi(\phi)$
in simple Clifford algebras $\mathcal{C}l_{p,q} \cong \text{Mat}(2^k, \mathbb{R})$

$k = q - r_{q-p}, p - q \neq 1 \mod 4, p - q = 0, 1, 2 \mod 8$

| $(p, q)$ | $G^e_{p,q}$ |
|---------|-------------|
| $(4, 2)$ | $O(8)$      |
| $(5, 3)$ | $O(16)$     |
| $(1, 7)$ | $O(16)$     |
| $(0, 8)$ | $O(16)$     |
| $(4, 4)$ | $O(16)$     |
| $(8, 0)$ | $O(16)$     |

Let $u \in \mathcal{C}l_{p,q}$ and let $[u]$ be a matrix of $u$ in the spinor representation $\pi_S$ of $\mathcal{C}l_{p,q}$ realized in the spinor $(\mathcal{C}l_{p,q}, \mathbb{K})$-bimodule $\mathcal{C}l_S \mathbb{K} \cong \mathcal{C}l_{p,q} \mathbb{K}$. Then, by [7] Prop. 5,

\[
[T^e_\psi(u)] = \begin{cases} 
[u]^T & \text{if } p - q = 0, 1, 2 \mod 8; \\
[u]^\dagger & \text{if } p - q = 3, 7 \mod 8; \\
[u]^\ddagger & \text{if } p - q = 4, 5, 6 \mod 8;
\end{cases}
\]  

where $T$, $\dagger$, and $\ddagger$ denote, respectively, transposition, complex Hermitian conjugation, and quaternionic Hermitian conjugation. Thus, we immediately have:

**Proposition 5.** Let $G^e_{p,q} \subset \mathcal{C}l_{p,q}$ where $\mathcal{C}l_{p,q}$ is a simple Clifford algebra. Then, $G^e_{p,q}$ is: The orthogonal group $O(N)$ when $\mathbb{K} \cong \mathbb{R}$; the complex unitary group $U(N)$ when $\mathbb{K} \cong \mathbb{C}$; or, the compact symplectic group $Sp(N)$ when $\mathbb{K} \cong \mathbb{H}$\footnote{See Fulton and Harris [12] for a definition of the quaternionic unitary group $U_{\mathbb{H}}(N)$. In our notation we follow loc. cit. page 100, ‘Remark on Notations’}. That is,

\[
G^e_{p,q} = \begin{cases} 
O(N) & \text{if } p - q = 0, 1, 2 \mod 8; \\
U(N) & \text{if } p - q = 3, 7 \mod 8; \\
Sp(N) & \text{if } p - q = 4, 5, 6 \mod 8;
\end{cases}
\]  

where $N = 2^k$ and $k = q - r_{q-p}$.

The scalar product $T^e_\psi(\psi)\phi$ was computed with CLIFFORD [9] for all signatures $(p, q), p+q \leq 9$ [10]. Observe that as expected, in Euclidean (resp. anti-Euclidean) signatures $(p, 0)$ (resp. $(0, q)$) the group $G^e_{p,0}$ (resp. $G^e_{0,q}$) coincides with the corresponding automorphism group of the scalar product $\beta_+$ (resp. $\beta_-$) listed in [14, Table 1, p. 236] (resp. [14, Table 2, p. 236]). This is indicated by a single (resp. double) box around the group symbol in Tables 1–5. For example, in Table 1, for the Euclidean signature $(2, 0)$, we show $G^e_{2,0}$ as $O(2)$ like for $\beta_+$ whereas for the anti-Euclidean signature $(0, 6)$, we show $G^e_{0,6}$ as $O(8)$ like for $\beta_+$. \footnote{See Fulton and Harris [12] for a definition of the quaternionic unitary group $U_{\mathbb{H}}(N)$. In our notation we follow loc. cit. page 100, ‘Remark on Notations’}. 


For simple Clifford algebras, the automorphism groups $G^e_{p,q}$ are displayed in Tables 1, 2, and 3. In each case the form is positive definite and non-degenerate. Also, unlike in the case of the forms $\beta_+$ and $\beta_-$, there is no need for the extra monomial factor like $s_1, s_2$ in (16) (and (17)) to guarantee that the product $T^e_\epsilon(\psi)\phi$ belongs to $\mathbb{K}$ since this is always the case [6][7]. Recall that the only role of the monomials $s_1$ and $s_2$ is to permute entries of the spinors $\psi\phi$ and $\tilde{\psi}\phi$ to assure that $\beta_+(\psi, \phi)$ and $\beta_-(\psi, \phi)$ belong to the (skew) field $\mathbb{K}$. That is, more precisely, that $\beta_+(\psi, \phi)$ and $\beta_-(\psi, \phi)$ have the form $\lambda f = f \lambda$ for some $\lambda \in \mathbb{K}$. The idempotent $f$ in the spinor basis in $S$ corresponds uniquely to the identity coset $G_{p,q}(f)$ in the quotient group $G_{p,q}/G_{p,q}(f)$. Based on [7] Prop. 2) we know that since the vee group $G_{p,q}$ permutes entries of any spinor $\psi$, the monomials $s_1$ and $s_2$ belong to the transversal of the stabilizer $G_{p,q}(f) < G_{p,q}$ [7] Cor. 2][4].

One more difference between the scalar products $\beta_+$ and $\beta_-$, and the transposition product $T^e_\epsilon(\psi)\phi$ is that in some signatures one of the former products may be identically zero whereas the transposition product is never identically zero. The signatures $(p, q)$ in which one of the products $\beta_+$ or $\beta_-$ is identically zero can be easily found in [14], Tables 1 and 2, p. 236] as the automorphism group of the product is then a general linear group.

### 3.2 Semisimple Clifford algebras

Faithful spinor representation of a semisimple Clifford algebra $C\ell_{p,q}$ ($p - q = 1 \mod 4$) is realized in a left ideal $\tilde{S} = S \oplus \tilde{S} = C\ell_{p,q} e$ where $e = f + \tilde{f}$ for any primitive idempotent $f$. Recall that $\tilde{\cdot}$ denotes the grade involution of $u \in C\ell_{p,q}$. We refer to [14] pp. 232–236] for some of the concepts. In particular, $S = C\ell_{p,q} f$ and $\tilde{S} = C\ell_{p,q} \tilde{f}$. Thus, every spinor $\tilde{\psi} \in \tilde{S}$ has unique components $\psi \in S$ and $\tilde{\psi}_g \in \tilde{S}$. We refer to the elements $\tilde{\psi} \in \tilde{S}$ as “spinors” whereas to its components $\psi \in S$ and $\tilde{\psi}_g \in \tilde{S}$ we refer as “$1\overline{2}$-spinors”.

For the semisimple Clifford algebras $C\ell_{p,q}$, we will view spinors $\tilde{\psi} \in \tilde{S} = S \oplus \tilde{S}$ as ordered pairs $(\psi, \tilde{\psi}_g) \in S \times \tilde{S}$ when $\tilde{\psi} = \psi + \tilde{\psi}_g$. Likewise, we will view elements $\tilde{\lambda}$ in the double

---

**Table 2 (Part 1): Automorphism group $G^e_{p,q}$ of $T^e_\epsilon(\psi)\phi$ in simple Clifford algebras $C\ell_{p,q} \cong \text{Mat}(2^k, \mathbb{C})$**

| $(p, q)$ | $(0, 1)$ | $(1, 2)$ | $(3, 0)$ | $(2, 3)$ | $(0, 5)$ | $(4, 1)$ | $(1, 6)$ | $(7, 0)$ |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $G^e_{p,q}$ | $U(1)$ | $U(2)$ | $U(2)$ | $U(4)$ | $U(4)$ | $U(4)$ | $U(8)$ | $U(8)$ |

**Table 2 (Part 2): Automorphism group $G^e_{p,q}$ of $T^e_\epsilon(\psi)\phi$ in simple Clifford algebras $C\ell_{p,q} \cong \text{Mat}(2^k, \mathbb{C})$**

| $(p, q)$ | $(5, 2)$ | $(3, 4)$ | $(4, 5)$ | $(6, 3)$ | $(2, 7)$ | $(0, 9)$ | $(8, 1)$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| $G^e_{p,q}$ | $U(8)$ | $U(8)$ | $U(16)$ | $U(16)$ | $U(16)$ | $U(16)$ | $U(16)$ |

---

11In [14] Page 233], Lounesto states correctly that “the element $s$ can be chosen from the standard basis of $C\ell_{p,q}$.” In fact, one can restrict the search for $s$ to the transversal of the stabilizer $G_{p,q}(f)$ in $G_{p,q}$ which has a much smaller size $2^{q-r_{s-p}}$ compared to the size $2^{p+q}$ of the Clifford basis.
The two known spinor scalar products

4 CONCLUSIONS

All results in these tables, like in Tables 1, 2, and 3, have been verified with CLIFFORD [9] and the corresponding Maple worksheets are posted at [10].

Table 3 (Part 1): Automorphism group $G^e_{p,q}$ of $T^\sim_\psi(\phi)$ in simple Clifford algebras $\mathcal{C}l_{p,q} \cong \text{Mat}(2^k, \mathbb{H})$

| (p, q)   | (0, 2) | (0, 4) | (4, 0) | (1, 3) | (2, 4) | (6, 0) |
|----------|--------|--------|--------|--------|--------|--------|
| $G^e_{p,q}$ | $Sp(1)$ | $Sp(2)$ | $Sp(2)$ | $Sp(2)$ | $Sp(4)$ | $Sp(4)$ |

Table 3 (Part 2): Automorphism group $G^e_{p,q}$ of $T^\sim_\psi(\phi)$ in simple Clifford algebras $\mathcal{C}l_{p,q} \cong \text{Mat}(2^k, \mathbb{H})$

| (p, q)   | (1, 5) | (5, 1) | (6, 2) | (7, 1) | (2, 6) | (3, 5) |
|----------|--------|--------|--------|--------|--------|--------|
| $G^e_{p,q}$ | $Sp(4)$ | $Sp(4)$ | $Sp(8)$ | $Sp(8)$ | $Sp(8)$ | $Sp(8)$ |

fields $\bar{\mathbb{K}} = \mathbb{K} \oplus \mathbb{K}$ as ordered pairs $(\lambda, \lambda_q) \in \mathbb{K} \times \mathbb{K}$ when $\lambda = \lambda + \lambda_q$. As before, $\mathbb{K} = f\mathcal{C}l_{p,q}f$ while $\bar{\mathbb{K}} = \bar{f}\mathcal{C}l_{p,q}\bar{f}$. Recall that $\bar{\mathbb{K}} \cong 2\mathbb{R} \overset{\text{def}}{=} \mathbb{R} \oplus \mathbb{R}$ or $\bar{\mathbb{K}} \cong 2\mathbb{H} \overset{\text{def}}{=} \mathbb{H} \oplus \mathbb{H}$ when, respectively, $p - q = 1$ mod 8, or $p - q = 5$ mod 8.

In this section we classify automorphism groups of the transposition scalar product

$$\hat{S} \times \hat{S} \rightarrow \bar{\mathbb{K}}, \quad (\tilde{\psi}, \tilde{\phi}) \mapsto T^\sim_\psi(\tilde{\psi}, \tilde{\phi}) \overset{\text{def}}{=} (T^\sim_\psi(\psi, \phi), T^\sim_\psi(\psi_g, \phi_g) \in \bar{\mathbb{K}}$$

when $\tilde{\psi} = \psi + \psi_g$ and $\tilde{\phi} = \phi + \phi_g$.

Proposition 6. Let $G^e_{p,q} \subset \mathcal{C}l_{p,q}$ where $\mathcal{C}l_{p,q}$ is a semisimple Clifford algebra. Then, $G^e_{p,q}$ is:

The double orthogonal group $2O(N) \overset{\text{def}}{=} O(N) \times O(N)$ when $\bar{\mathbb{K}} \cong 2\mathbb{R}$ or the double compact symplectic group $2Sp(N) \overset{\text{def}}{=} Sp(N) \times Sp(N)$ when $\bar{\mathbb{K}} \cong 2\mathbb{H}$.

That is,

$$G^e_{p,q} = \begin{cases} 2O(N) = O(N) \times O(N) & \text{when } p - q = 1 \text{ mod 8}; \\ 2Sp(N) = Sp(N) \times Sp(N) & \text{when } p - q = 5 \text{ mod 8}; \end{cases}$$

where $N = 2^{k-1}$ and $k = q - r_{q-p}$.

The automorphism groups $G^e_{p,q}$ for semisimple Clifford algebras $\mathcal{C}l_{p,q}$ for $p + q \leq 9$ are shown in Tables 4 and 5. All results in these tables, like in Tables 1, 2, and 3, have been verified with CLIFFORD [9] and the corresponding Maple worksheets are posted at [10].

4 CONCLUSIONS

The transposition map $T^\sim_\psi$ allowed us to define a new transposition scalar product on spinor spaces. Only in the Euclidean and anti-Euclidean signatures, this scalar product is identical to the two known spinor scalar products $\beta_+$ and $\beta_-$ which use, respectively, the reversion and the

\[12\] Recall that $Sp(N) = U_{\mathbb{H}}(N)$ where $U_{\mathbb{H}}(N)$ is the quaternionic unitary group [12].
Table 4: Automorphism group $G_{p,q}^\varepsilon$ of $T_\varepsilon(\psi)\phi$

in semisimple Clifford algebras $\mathcal{Cl}_{p,q} \cong 2^{\text{Mat}(2^{k-1}, \mathbb{R})}$

\[
k = q - r_{q-p}, \quad p - q = 1 \mod 4, \quad p - q = 1 \mod 8
\]

| $(p, q)$ | $(1, 0)$ | $(2, 1)$ | $(3, 2)$ | $(4, 7)$ | $(4, 3)$ | $(1, 8)$ | $(5, 4)$ | $(9, 0)$ |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $G_{p,q}^\varepsilon$ | $^2O(1)$ | $^2O(2)$ | $^2O(4)$ | $^2O(8)$ | $^2O(8)$ | $^2O(16)$ | $^2O(16)$ | $^2O(16)$ |

Table 5: Automorphism group $G_{p,q}^\varepsilon$ of $T_\varepsilon(\psi)\phi$

in semisimple Clifford algebras $\mathcal{Cl}_{p,q} \cong 2^{\text{Mat}(2^{k-1}, \mathbb{H})}$

\[
k = q - r_{q-p}, \quad p - q = 5 \mod 4, \quad p - q = \mod 8
\]

| $(p, q)$ | $(0, 3)$ | $(1, 4)$ | $(5, 0)$ | $(2, 5)$ | $(6, 1)$ | $(3, 6)$ | $(7, 2)$ |
|----------|----------|----------|----------|----------|----------|----------|----------|
| $G_{p,q}^\varepsilon$ | $^2Sp(1)$ | $^2Sp(2)$ | $^2Sp(2)$ | $^2Sp(4)$ | $^2Sp(4)$ | $^2Sp(8)$ | $^2Sp(8)$ |

conjugation and it is different in all other signatures. This new product is never identically zero and it does not require extra monomial factor to assure it is $\mathbb{K}$- or $\mathbb{K}$-valued. This is because the $T_\varepsilon^-$ maps any spinor space to its dual. Then, we have identified the automorphism groups $G_{p,q}^\varepsilon$ of this new product in Tables 1–5 for $p + q = n \leq 9$. The classification is complete and sufficient due to the mod 8 periodicity.

We have observed the important role played by the idempotent group $T_{p,q}(f)$ and the field group $K_{p,q}(f)$ as normal subgroups in the stabilizer group $G_{p,q}(f)$ of the primitive idempotent $f$ and their coset spaces $G_{p,q}/T_{p,q}(f)$, $G_{p,q}(f)/T_{p,q}(f)$, and $G_{p,q}/G_{p,q}(f)$ in relation to the spinor representation of $\mathcal{Cl}_{p,q}$. These subgroups allow to construct very effectively non-canonical transversals and hence basis elements of the spinor spaces and the (skew double) field underlying the spinor space. This approach to the spinor representation of $\mathcal{Cl}_{p,q}$ based on the stabilizer $G_{p,q}(f)$ of $f$ leads to a realization that the Clifford algebras can be viewed as a twisted group ring $\mathbb{R}^\ell(\mathbb{Z}_2^n)$. In particular, we have observed that our transposition $T_\varepsilon^-$ is then a ‘star map’ of $\mathbb{R}^\ell(\mathbb{Z}_2^n)$ which on a general twisted group ring $\ast : \mathbb{K}^\ell[G] \rightarrow \mathbb{K}^\ell[G]$ is defined as

\[
(\sum a_x \bar{x})^\ast = \sum a_x \bar{x}^{-1}.
\]

This is because we recall properties of the transposition anti-involution $T_\varepsilon^-$, and, in particular, its action $T_\varepsilon^-(m) = m^{-1}$ on a monomial $m$ in the Grassmann basis $\mathcal{B}$ which is, as we see now, identical to the action $\ast(m) = m^{-1}$ on every $m \in \mathcal{B}$. For a Hopf algebraic discussion of Clifford algebras as twisted group algebras, see [11,15] and references therein.

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