Melting the Diquark Condensate in Two-Color QCD: 
A Renormalization Group Analysis

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We use a Landau theory and the $\epsilon$-expansion to study the superfluid phase transition of two-color QCD at nonzero temperature, $T$, and baryonic chemical potential, $\mu$. Based on either one-gluon exchange or instanton induced interactions, there are compelling theoretical indications that a direct attractive quark-quark channel gives rise to diquark condensation at asymptotically large $\mu$, similar in spirit to the BCS-scenario [1]. In particular, very dense quark matter is unstable to the formation of a diquark condensate, and thus is generally believed to be a color superconductor at large $\mu$. The QCD-based predictions rely on perturbation theory and are trustworthy when the coupling constant is small, $g(\mu) \ll 1$, i.e. $\mu \gg \Lambda_{QCD}$, where $\Lambda_{QCD}$ is the hadronic scale (see [2]). To analytically study the properties of QCD at more moderate densities, where the quark chemical potential is only a few times $\Lambda_{QCD}$, one has to rely on other methods. For instance, one can resort to phenomenological models that are inspired by, but not derived from, QCD. An alternative approach would be to use numerical simulations, since lattice gauge theory is a first-principle method capable of making quantitative predictions of strong-coupling QCD. At $T > 0$ and $\mu = 0$, the numerical results show that there is a phase transition between the confined and deconfined phases of pure glue matter, at a critical temperature $T_c \simeq 270$ MeV [3]. With massless quarks, there is a transition to a chirally symmetric phase at $T_c \simeq 155 - 175$ MeV [4], and recent simulations also show that deconfinement occurs simultaneously with the chiral phase transition [5]. However, at a nonzero chemical potential the results have so far been very limited, due to the severe problems lattice gauge theory encounters at nonzero $\mu$. These difficulties arise from the fact that the action in the SU(3) gauge theory becomes complex-valued when $\mu \neq 0$ [6], which in turn inhibits the importance sampling in the Monte-Carlo method. Although different computational schemes have been implemented [7–10] the problem with large-scale simulations at nonzero chemical remains unsettled.

For $N_c = 2$, the situation is much more favorable for numerical simulations. In this case, the fundamental representation is pseudoreal, and as a consequence the lattice measure remains real-valued even at $\mu \neq 0$ [11]. Since the fermion determinant is real, it is possible to perform lattice calculations for an even number of flavors using the standard Monte-Carlo algorithms. Although two-color QCD is worthwhile to study in its own right, the fact that it is a four-dimensional, confining non-Abelian theory makes it particularly interesting as a toy model for real QCD. Naturally, some caution is needed, since while there are some important commonalities between two– and three–color QCD, there are some striking differences. For example, hadron spectroscopy will be completely unrelated since for $N_c = 2$ two quarks can form a color singlet. Likewise, the diquark condensate is a color singlet for $N_c = 2$, while for $N_c = 3$ it is not. Finally, in contrast to the color superconducting state of dense quark matter for $N_c = 3$, diquark condensation for $N_c = 2$ only gives rise to superfluidity.

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Despite these differences, two-color QCD is still useful for testing general ideas and comparing analytical calculations with lattice simulations. Indeed, early lattice studies of two-color QCD yielded interesting results at both nonzero chemical potential and temperature [11]. In recent years, an extensive numerical program has been undertaken and a consistent picture of two-color QCD at nonzero $T$ and $\mu$ has emerged [12,13]. For $T = 0$, as the chemical potential is increased, a colorless diquark condensate, $\langle \bar{\psi} \psi \rangle$, forms at $\mu_c \simeq m_\pi/2$, where $m_\pi$ is the pion mass. For $\mu \geq \mu_c$, the condensate spontaneously breaks baryon number and gives rise to superfluidity. This phase transition is most likely second order, at least for one and two flavors of staggered quarks for which the simulations have been performed. Alternatively, two-color QCD can be studied analytically using chiral perturbation theory. This effective theory describes the dynamics of the Goldstone modes associated with the spontaneous breaking of chiral symmetry and is valid for energies well below the chiral symmetry breaking scale. In particular, the critical baryonic chemical potential, $\mu_c = m_\pi/2$, is well within the effective range and indeed the onset of baryon density and diquark condensation at zero temperature is fully understood in this context at both the classical [14–19] and one–loop level [20]. The analytical predictions agree well with the lattice results, thus further strengthening our current understanding as well as showing the great utility of having both analytical and numerical tools available.

Although the different phases and associated phase transitions at $T = 0$ and $\mu \neq 0$ have been investigated in some detail, less is known about the phase structure of two–color QCD in the $(\mu, T)$–plane. Analytical results employing renormalization group arguments for the chiral symmetry restoring phase transition at nonzero temperature but vanishing chemical potential were presented in Ref. [21], and the phase diagram at nonzero $T$ and $\mu$ has been studied within a random matrix model [22]. On the lattice, simulations corresponding to $N_f = 4$ continuum flavors indicate that as $T$ increases at a fixed moderate ratio $x = 2\mu/m_\pi$, the system undergoes a second order phase transition where $\langle \bar{\psi} \psi \rangle$ vanishes [12]. As this ratio increases, this line of second order phase transitions passes through a tricritical point and the phase transition becomes first order [12,23]. Recently the one loop calculation within chiral perturbation theory was extended to nonzero temperature [20,24], and the temperature dependence of the critical chemical potential for $\mu \sim m_\pi/2$ was calculated. Furthermore, the presence of a tricritical endpoint of this line of second order phase transitions was confirmed. The tricritical point was predicted to occur at $T \sim m_\pi$ [24] in qualitative agreement with the lattice results [12].

In this paper we further study the phase diagram at nonzero temperature and baryonic chemical potential by using a Landau theory combined with one–loop renormalization group arguments [25]. The Landau theory is fundamentally different from the chiral perturbation theory analysis in that it attempts to include all modes which become massless at the critical temperature. In addition to being a different approach, we are also considering a different range of the $(\mu, T)$–plane than in Ref. [24]. Here, we study the chiral limit with a nonzero baryonic chemical potential. Since we consider the deep diquark phase a large temperature limit is of interest (in the chiral perturbation theory approach the low temperature limit was used for $\mu \sim m_\pi/2$). Our motivation is twofold. First of all, as mentioned above, the phase diagram of two-color QCD needs to be better understood, and one of our objectives is to provide a step in that direction. In particular, the emergence of the tricritical point at intermediate values of $T$ and $\mu$ requires further explanation. Although the discussion will be concerned with the case of massless quarks, $m_q = 0$, for simplicity, our results and predictions are expected to apply also for nonzero degenerate quark masses as long as $N_f$ is even and $\mu \gg m_\pi/2$. Secondly, further comparisons between different analytical approaches and numerical simulations on the lattice will hopefully increase our understanding of both methods, and lay the groundwork for future attempts in full $N_c = 3$ QCD at nonzero baryon density.

We assume throughout that the gauge group is $SU(2)$ and that there are an even number $N_f$ of massless flavors. In the next section, we discuss the pseudoreal representation of $SU(2)$ and its implications for the $N_c = 2$ QCD action and the symmetry breaking patterns. We then incorporate these properties into an effective Lagrangian. In Sec. III, we present our renormalization group analysis and discuss the order of the phase transition for various $N_f$. We first assume the absence of the chiral anomaly and later consider its implications. We end with our conclusions and an outlook in Sec. IV.

II. TWO–COLOR QCD AND EFFECTIVE THEORIES

A. The microscopic action

The matter part of the QCD action with massless quarks and at zero chemical potential is

$$S_0 = -i \sum_{f=1}^{N_f} \int d^4 x \bar{\psi}_f \left( \partial_\mu \delta_i^j - ig A^a_\mu(T^a)_i^j \right) \gamma^\mu \psi_j = -i \sum_{f=1}^{N_f} \int d^4 x \bar{\psi}_f D^\mu \psi_f ,$$

(1)
where the Minkowski metric is $g_{\mu\nu} = \text{diag}(+---)$ and the color matrices are given by $T^a = \sigma^a/2$, with $\sigma^a$ the usual Pauli matrices. Defining $\sigma_\mu = (1, \bar{\sigma})$ and $\bar{\sigma}_\mu = (1, -\bar{\sigma})$, we have in the two-component formalism,

$$\psi_{if} = \left(\frac{\xi_f}{\lambda_f^a}\right)_{if} \ , \ \gamma^\mu \equiv \left( \begin{array}{cc} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{array} \right).$$

Rewriting the action in Eq. (1) in this Weyl notation gives, after a partial integration,

$$S_0 = -i \sum_{f=1}^{N_f} \int d^4x \left[ \bar{\lambda}_{if}(T^a)_{ij}^{T} \lambda_{jf}^{T} + \bar{\xi}_{if}(T^a)_{ij} \xi_{jf} \right].$$

Using the pseudo–reality relation in SU(2), $\sigma_2 \sigma_a \sigma_2 = -\sigma_a^T$, where the superscript $T$ denotes the transpose, the interaction term for $\lambda$ can be reorganized as

$$\bar{\lambda}_{if}(T^a)_{ij}^{T} \lambda_{jf} \equiv - \bar{\lambda}_{if} T^a_{ij} \lambda_{jf} = - (\bar{\lambda}_{if} \sigma_2) T_a (\sigma_2 \lambda_{jf}).$$

Since $\lambda_f$ transforms as $\mathbf{2}$ under a gauge transformation, $\sigma_2 \lambda_f$ transforms as $\xi$, i.e. like $\mathbf{2}$. Thus, by making the following flavor decomposition within the same color representation,

$$Q_f \equiv \left\{ \begin{array}{c} \xi_f \\ \sigma_2 \lambda_f \end{array} \right\}_{f=1, \ldots, N_f} \ , \ \bar{Q}_f \equiv \left\{ \begin{array}{c} \bar{\xi}_f \\ \lambda_f \sigma_2 \end{array} \right\}_{f=1, \ldots, N_f}$$

we can write the action in Eq. (3) as,

$$S_0 = -i \sum_{f=1}^{2N_f} \int d^4x \bar{Q}_f \left( \partial_\mu - igA_\mu^a T^a \right) \bar{\sigma}_f^\mu Q_f.$$

As can be seen from Eq. (6), the naïve global symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ is in fact enlarged to $SU(2N_f)$ in two-color QCD, as a direct consequence of the pseudo-reality condition.

However, this enlarged symmetry exists only as long as $\mu = 0$. By introducing a common nonzero baryonic chemical potential $\mu$, the action in Eq. (1) is augmented by an additional term,

$$S_0 \xrightarrow{\mu \to 0} S = S_0 - \mu \sum_{f=1}^{N_f} \int d^4x \bar{\psi}_{if} \gamma^0 \psi_{if} = S_0 - \mu \int d^4x \bar{Q}BQ,$$

where $B$ is the $(2N_f) \times (2N_f)$ diagonal baryonic charge matrix,

$$B = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$

and $\mathbf{0}$ and $\mathbf{1}$ are $N_f \times N_f$ matrices. The $\mu$-dependent term is not an $SU(2N_f)$ singlet, since the block-diagonal form is only invariant under rotations that do not mix the different blocks. In other words, when $\mu \neq 0$ the global symmetry is reduced to the more familiar $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$. Furthermore, in the presence of a common quark mass, $m_q$, the symmetry is reduced even further, since the mass term explicitly breaks the axial symmetry. The global symmetry in this case is only $SU(N_f)_V \times U(1)_B$.

**B. Spontaneous breaking of symmetries**

When $\mu = m_q = 0$, we assume, as indicated by lattice simulations [12,13], that the chiral $SU(2N_f)$ symmetry is spontaneously broken at low temperatures. A Lorentz and gauge invariant chiral condensate must be of the form [26]

$$\bar{\psi}\psi \propto Q_i E^{ij} Q_j,$$

with $i$ and $j$ the flavor indices and $E$ a $(2N_f) \times (2N_f)$ antisymmetric matrix. Assuming a maximal flavor symmetry scenario [26], the standard form for $E$ is given by
\[
E \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

If the chiral condensate is nonvanishing, the global symmetry is spontaneously broken to the subgroup that leaves \(E\) invariant, namely \(\text{Sp}(2N_f)\), and the symmetry breaking pattern \(\text{SU}(2N_f) \rightarrow \text{Sp}(2N_f)\) gives rise to \((2N_f^2 - N_f - 1)\) Goldstone bosons.

On the other hand, when \(\mu > 0\) and the quark mass \(m_q \neq 0\), there is only a \(\text{SU}(N_f)_V \times \text{U}(1)_B\) symmetry. Although the chiral condensate does not break any of these symmetries, the lattice results show that the preferred ground-state configuration is still the chirally broken one in the region where \(\mu \leq m_\pi/2\) [12,13]. However, at a baryonic chemical potential \(\mu = m_\pi/2\), on-shell baryons are produced out of the vacuum, the chiral condensate begins to decrease and a diquark condensate appears:

\[
\psi \psi \propto Q^T \Sigma Q ,
\]

where \(\Sigma\) is a \((2N_f) \times (2N_f)\) matrix given by

\[
\Sigma \equiv i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \text{with} \quad I \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,
\]

and \(I\) is an \(N_f \times N_f\) matrix. A nonvanishing diquark condensate \((m_q \neq 0 \text{ and } \mu \geq m_\pi/2)\) breaks the \(\text{SU}(N_f)_V \times \text{U}(1)_B\) symmetry down to \(\text{Sp}(N_f)\) and gives rise to a Bose condensed phase.

When the quark mass \(m_q = 0\),\(m_\pi\) also vanishes. This situation requires some additional considerations for the diquark condensate. There is an interesting relationship, peculiar to two-color QCD, between the chiral and diquark invariants under the larger \(U(2N_f)\). The breaking pattern remains \(\text{SU}(2N_f) \rightarrow \text{Sp}(N_f)\) for definiteness took \(\Phi = \text{diag}(\Phi_1, \Phi_2, \ldots, \Phi_{2N_f})\) and by choosing \(m^{2f} > 0\) at \(T = 0\) the potential is minimized by a nonzero expectation value \(\Phi_0 = \rho_0 \Sigma\), with \(\rho_0^2 = -m^{2f}/(8N_f \lambda_1 + 4\lambda_2)\) for \(c = 0\). This gives the correct symmetry breaking pattern \(\text{SU}(2N_f) \rightarrow \text{Sp}(2N_f)\), where we for definiteness took \(\Phi_0\) to be proportional to \(\Sigma\) instead of \(E\). As discussed above, these are equivalent choices.

\[\text{C. The Landau theory at } \mu = 0\]

To model the \(m_q = 0\) symmetry breaking, and its restoration as the temperature is increased for \(\mu = 0\), one can start from a Landau theory [21]. We assume that the Landau field can be parameterized by an antisymmetric, complex-valued \((2N_f) \times (2N_f)\) matrix \(\Phi, \Phi \sim QQ^T\), that transforms under the \(\text{SU}(2N_f)\) symmetry as \(\Phi \rightarrow M \Phi M^T\) with \(M \in \text{SU}(2N_f)\). The effective Lagrangian\(^1\) which is renormalizable and invariant under the global \(\text{SU}(2N_f)\) flavor symmetry is

\[
L_0 = \frac{1}{2} \text{Tr} [ (\partial_\mu \Phi^\dagger)(\partial^\mu \Phi) ] - \frac{m^2}{2} \text{Tr} \left[ \Phi^\dagger \Phi \right] - \lambda_1 \left( \text{Tr} \left[ \Phi^\dagger \Phi \right] \right)^2 - \lambda_2 \text{Tr} \left[ (\Phi^\dagger \Phi)^2 \right] - c \left[ \text{Pf}(\Phi) + \text{Pf}(\Phi^\dagger) \right].
\]

The last term in Eq. (15) is the so-called Pfaffian [27],

\[
\text{Pf}(\Phi) = \frac{1}{2^{N_f} N_f!} \sum_P (-1)^P \Phi_{i_1 i_2} \cdots \Phi_{i_{(2N_f-1)} i_{(2N_f)}} ,
\]

where \(i_k = 1, \ldots, 2N_f\), and \(P\) denotes a summation over all permutations of \(\{i_1, \ldots, i_{2N_f}\}\) with \((-1)^P\) the sign of the permutation. This term ensures that the axial \(U(1)_A\) is explicitly broken, since if \(c = 0\) the Lagrangian is actually invariant under the larger \(U(2N_f)\) group. Stability of the classical potential requires that \(\lambda_2 \geq 0\) and \(\lambda_1 + \lambda_2/(2N_f) \geq 0\), and by choosing \(m^2 < 0\) at \(T = 0\) the potential is minimized by a nonzero expectation value \(\Phi_0 = \rho_0 \Sigma\), with \(\rho_0^2 = -m^2/(8N_f \lambda_1 + 4\lambda_2)\) for \(c = 0\). This gives the correct symmetry breaking pattern \(\text{SU}(2N_f) \rightarrow \text{Sp}(2N_f)\), where we for definiteness took \(\Phi_0\) to be proportional to \(\Sigma\) instead of \(E\). As discussed above, these are equivalent choices.

\[\text{\textsuperscript{1}We thank F. Sannino for access to his notes in which this effective Lagrangian was first written down.}\]
D. The Landau theory at $\mu \neq 0$

We now introduce a nonzero baryonic chemical potential into the effective theory. The global symmetry $SU(2N_f)$ is reduced to $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ and consequently there is a distinction between the chiral and the diquark condensate. Since we set $m_\pi = 0$, the ground state has $\langle \bar{\psi} \psi \rangle \neq 0$ and $\langle \bar{\psi} \psi \rangle = 0$ at any $\mu > 0$, breaking the symmetry spontaneously down to $Sp(N_f) \times Sp(N_f)$. These aspects must of course be reflected in the effective Lagrangian when a chemical potential is added to it. A general discussion of Landau theories and the $\epsilon$-expansion at nonzero chemical potential is given in Ref. [29]. The important point to note here is that due to the pseudoreality of the representation, the order parameter is the same $^{2}$ as at $\mu = 0$, namely $QQ^T$. As in mean field theory, we assume that the quartic coupling constants remain unaffected by the introduction of $\mu$, although we will later consider the possible influence of $\mu$ on the coupling $c$ in Eq. (15). The dependence of the Lagrangian on $\mu$ is fixed uniquely by the symmetries and renormalizability. The transformation properties of the order parameter field is identical to that of the Goldstone field in chiral perturbation theory. Hence, the coupling of $\mu$ to $\Phi$ enters through the covariant derivative [15]

$$
\partial_\nu \Phi \to \partial_\nu \Phi - i \mu \delta_{\nu 0} (B\Phi + \Phi B),
\partial_\nu \Phi^\dagger \to \partial_\nu \Phi^\dagger + i \mu \delta_{\nu 0} (B\Phi^\dagger + \Phi^\dagger B).
$$

(17)

The Lagrangian in Eq. (15) then becomes

$$
L_0 \to L = \frac{1}{2} \text{Tr} \left[ (\partial_\mu \Phi^\dagger)(\partial^\mu \Phi) \right] + i \frac{\mu}{2} \text{Tr} \left[ (\partial_0 \Phi)(\Phi^\dagger B + B\Phi^\dagger) - h.c. \right] + \frac{\mu^2}{2} \text{Tr} \left[ (B\Phi + \Phi B)(B\Phi^\dagger + \Phi^\dagger B) \right] - \frac{m^2}{2} \text{Tr} \left[ \Phi^\dagger \Phi - \lambda_1 (\text{Tr} \left[ \Phi^\dagger \Phi \right] )^2 - \lambda_2 \text{Tr} \left[ (\Phi^\dagger \Phi)^2 \right] - c \left[ \text{Pf}(\Phi) + \text{Pf}(\Phi^\dagger) \right] \right].
$$

(18)

Given that $\Phi$ transforms as $\Phi \to M\Phi M^T$, the Lagrangian in Eq. (18) is only invariant under transformations that satisfy $BM = MB$, which reduces the global symmetry to $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ for $\mu \neq 0$. Moreover, from Eq. (18) it is straightforward to verify that the minimum of the classical potential is obtained when $\Phi_0 = \phi_0 \Sigma$, where $\phi_0 = -(m^2 - 4\mu^2)/(8N_f\lambda_1 + 4\lambda_2)$ for $c = 0$ and $m^2 < 4\mu^2$. This expectation value breaks the symmetry down to $Sp(N_f) \times Sp(N_f)$, as required. It should also be noted that the choice $\Phi_0 = \phi_0 E$ now corresponds to a local maximum, so that the effective model does distinguish between the two vacua.

III. MELTING THE DIQUARK CONDENSATE

As the temperature is increased, at a given $\mu > 0$, we expect the diquark condensate to eventually melt and the symmetry to be restored. The object is to determine the order of this phase transition by studying the fixed points of the $\beta$-functions in the $\epsilon$ expansion. The effect of the chemical potential in the Landau theory is threefold: 1) it allows for terms linear in the time derivative, 2) it explicitly breaks the $U(2N_f)$ symmetry, and 3) it induces a Bose diquark condensate.

We will assume that $T_c$ at $\mu = 0$ is large, $T_c^2 \gg |m^2|$. Then as discussed in detail in [29] we can dimensionally reduce the Landau theory to an effective theory of the massless zeroth Matsubara modes. It is in this dimensionally reduced theory we will study the stability of the $\beta$-functions. Because of the breaking of the $U(2N_f)$ symmetry not all zeroth Matsubara modes become massless at $T_c$ when $\mu > 0$. At nonzero $\mu$ the effective three dimensional theory describes the static zeroth Matsubara modes from the block diagonal modes in $\Phi$. These are the massless modes at the phase transition. Explicitly, in the three dimensional theory the field $\Phi$ is

$$
\Phi \to \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},
$$

(19)

with $A_1$ and $A_2 N_f \times N_f$ matrices given by.

$^{2}$We consider values for the chemical potential which are not asymptotically large so that the effective degrees of freedom are given by bosonic diquark bound states and not by the quarks and gluons themselves. A more restrictive upper bound on $\mu$ is given by half of the mass of the lightest vector meson. At this value of the chemical potential, the formation of a vector condensate is expected [30–32].
We then replace $\Phi$ for different matrix elements: $\beta_i \phi_i$.

The line in the phase diagram may be somewhat ambiguous. To one-loop order around the expectation value $\Phi$, this separation of mass scales to hold, although it should be emphasized that the matching to the nonzero $T$, $\mu = 0$, line in the phase diagram may be somewhat ambiguous.

Our approach is to use renormalization group arguments in $(4 - \epsilon)$ dimensions and study the existence of infra-red fixed points in the space of coupling constants for $\epsilon \to 1$. Whether or not this extrapolation is valid is of course a difficult question to answer fully. However, this approach has been successful under many different circumstances, both in the condensed matter context (see e.g. [33,34] and references therein) and in the context of relativistic field theories [35,36].

### A. Renormalization group flow in the anomaly-free case

Having defined the model, we now analyze the renormalization group flow in the space of parameters $\lambda_1$, $\lambda_2$, and $c$. As long as the quartic coupling constants are within their stability region, two situations can occur. Either the $\beta$-functions have infrared stable fixed points, in which case there is a second order phase transition, or they do not, and the phase transition is then first order, induced by fluctuations. The obvious caveat here is that the coupling constants can be adjusted to lie outside the stability region, with the result that higher dimensional terms must be added to stabilize the theory, making the phase transition first order already at the mean field level. In addition, there could also be fixed points at the nonperturbative level.

From the above discussion, the $c = 0$ effective Lagrangian around criticality is

$$L = \frac{1}{2} \text{Tr} \left[ (\partial_\mu \Phi^\dagger)(\partial^\mu \Phi) \right] - \lambda_1 \left( \text{Tr} [\Phi^\dagger \Phi] \right)^2 - \lambda_2 \text{Tr} \left[ (\Phi^\dagger \Phi)^2 \right],$$

(21)

where $\Phi$ is defined as in Eq. (19). To study the existence of fixed points, we expand the Lagrangian in Eq. (21) to one-loop order around the expectation value $\Phi_0 = \phi_0 \Sigma$ and calculate the $\beta$-functions using the background field method. To more easily separate the one-loop corrections to $\lambda_1$ and $\lambda_2$, we will also label the constant $\phi_0$ differently for different matrix elements:

$$\Phi_0 \to i \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_0 I \end{pmatrix}.$$

(22)

We then replace $\Phi \to \Phi_0 + \Phi$ in Eq. (21) and split the fields into their real and imaginary parts, $\phi_{mn} = (\phi_{mn}^r + i \phi_{mn}^i)/\sqrt{2}$. Up to terms quadratic in the fluctuations, we find that

$$L = -V_0 + \frac{1}{2} \sum_{k,l=1}^{N_f} \left[ (\partial_\nu \phi_{kl}^{(r)})^2 + (\partial_\nu \phi_{kl}^{(i)})^2 + (\partial_\nu \phi_{(k+N_f)(l+N_f)}^{(r)})^2 + (\partial_\nu \phi_{(k+N_f)(l+N_f)}^{(i)})^2 \right]_{k<l}$$

$$- \lambda_1 \left[ 8 \phi_0^2 \left( \sum_{k=1}^{N_f/2} \phi_{k(N_f/2+k)}^{(i)} \right)^2 + 8 \phi_0^2 \left( \sum_{k=N_f+1}^{3N_f/2} \phi_{k(N_f/2+k)}^{(i)} \right)^2 + 16 \phi_0 \phi_0 \left( \sum_{k=1}^{N_f/2} \phi_{k(N_f/2+k)}^{(i)} \right) \left( \sum_{k=N_f+1}^{3N_f/2} \phi_{k(N_f/2+k)}^{(i)} \right) + 2N_f (\phi_{0}^{(r)} + \phi_{0}^{(i)}) \right]_{k<l}$$

$$- \lambda_2 \left[ 4 \phi_0^2 \left( \sum_{k,l=1}^{N_f} \left( \phi_{kl}^{(r)} \right)^2 + \phi_{kl}^{(i)} \right) \right]_{k<l} + 4 \phi_0^2 \left( \sum_{k,l=1}^{2N_f} \left( \phi_{kl}^{(r)} \right)^2 + \phi_{kl}^{(i)} \right) \right]_{k<l} - 2 \phi_0^2 \left( \sum_{k=1}^{N_f/2} \left( \phi_{k(N_f/2+k)}^{(r)} \right)^2 - \phi_{k(N_f/2+k)}^{(i)} \right)^2$$

$$+ \sum_{k,l=1}^{N_f/2} \left( \phi_{k(N_f/2+k)}^{(r)} \phi_{l(N_f/2+k)}^{(r)} - \phi_{k(N_f/2+k)}^{(i)} \phi_{l(N_f/2+k)}^{(i)} \right)_{k<l} + \phi_{kl}^{(r)} \phi_{kl}^{(r)} - \phi_{kl}^{(i)} \phi_{kl}^{(i)} \right]_{k<l}$$

$$= \frac{1}{2} \text{Tr} \left[ (\partial_\mu \phi^\dagger)(\partial^\mu \phi) \right] - \lambda_1 \left( \text{Tr} [\phi^\dagger \phi] \right)^2 - \lambda_2 \text{Tr} \left[ (\phi^\dagger \phi)^2 \right].$$

(20)
Since we are examining the long-range fluctuations, the fixed points mass scale, we find the eigenvalues of the stability matrix, \( \delta \lambda \) where we have only kept the ultra-violet divergent terms and \( V \) where the classical potential Tr\( N \) has a fixed point that is infra-red stable for an orthonormal transformation, dimensional regularization in \( -\)\( + \)\( -\)\( + \). Consider now the different fixed points that can be inferred from Eq. (26)–(27), i.e. the solutions to \( \lambda \). The renormalization group flow in the \( (\phi, \lambda) \) order calculation is necessary to make a definite prediction. However, at \( \mu = 0 \), the temperature induced phase transition is first order for \( N_f = 2 \) and \( c = 0 \) \([21]\), so it is likely that this first order transition extends to small but nonzero values of \( \mu \). If the phase transition becomes second order as \( \mu \) increases further, it should correspond to the O(4) critical point (i.e. case iii above).

- For \( N_f = 2 \), the order of the phase transition is not determined by the renormalization group flow, and a higher order calculation is necessary to make a definite prediction. However, at \( \mu = 0 \), the temperature induced phase transition is first order for \( N_f = 2 \) and \( c = 0 \) \([21]\), so it is likely that this first order transition extends to small but nonzero values of \( \mu \). If the phase transition becomes second order as \( \mu \) increases further, it should correspond to the O(4) critical point (i.e. case iii above).

- For \( N_f \geq 4 \), there is a stable fixed point. Thus, if this theory evolves from a point within the stability region \( \lambda_2 \geq 0 \) and \( \lambda_1 + \lambda_2/(2N_f) \geq 0 \), the phase transition in two–color QCD with more than four flavors, and without any influence from the axial anomaly, is likely to be second order.

The renormalization group flow in the \((\lambda_1, \lambda_2)\)-plane for \( N_f = 4 \) is illustrated in figure 1.
FIG. 1. The flow in the \((\lambda_1, \lambda_2)\)-plane for \(N_f = 4\) and without the anomaly term. The classical potential is stable for \(\lambda_2 \geq 0\) and \(\lambda_1 + \lambda_2/(2N_f) \geq 0\). The vectors shown have a length given by the magnitude of the eigenvalues of the stability matrix at the fixed point. The direction is away from the fixed point for negative eigenvalues and towards the fixed point for positive eigenvalues. The orientation of the vectors is given by the corresponding eigenvectors of the stability matrix. There is a stable fixed point and the phase transition is predicted to be of second order. For \(N_f > 4\) the flow diagram has a similar topology. For \(N_f = 2\) the upper right hand stable fixed point merges with the unstable fixed point on the \(\lambda_1\) axis and becomes marginally stable.

B. Adding the chiral anomaly

The above conclusions must be reconsidered if the chiral anomaly is present around \(T_c\). Lattice results for \(N_c = 3\) QCD indicate that although the effects of the chiral anomaly decrease, they do not vanish in the vicinity of the chiral symmetry restoring phase transition \[37\]. Since there is no reason to believe that the situation is drastically different in \(N_c = 2\) QCD, the anomaly can be expected to be present near \(T_c\) and \(\mu = 0\). At a nonzero chemical potential, there is an additional suppression of the anomaly \[38\] apart from the nonzero \(T\) effects. However, since our model is limited to the region \(\mu \lesssim \Lambda_{QCD}\), it is still likely that the effects of the chiral anomaly are not negligible at the phase transition.

For two flavors, the anomaly term is a mass-like operator and has the effect that only half of the number of the original real-valued fields in \(A_1\) and \(A_2\) become critical. At \(T = 0\), there is an \(SU(2) \times SU(2) \times U(1)\) symmetry that is spontaneously broken to \(Sp(2) \times Sp(2)\), but since \(Sp(2) \sim SU(2)\), only the \(U(1) \sim O(2)\) symmetry is actually broken. Keeping only those fields in Eq. (18) which become critical at \(T_c\), the theory becomes \(O(2)\)-symmetric and the diquark condensate breaks this symmetry completely. Thus, if the phase transition indeed is second order, it is characterized by \(O(2)\) critical exponents. Since the phase transition at \(\mu = 0\) can be second order as well \[21\], but with \(O(6)\) critical exponents, there could thus be a change of the critical exponents as \(\mu\) increases from zero.

For \(N_f = 4\), the anomaly term represents another quartic coupling. Repeating the calculation from the previous section with \(c \neq 0\), we find the \(\beta\)-functions,

\[
\beta_1 = \kappa \frac{\partial \lambda_1}{\partial \kappa} = -\epsilon \lambda_1 + \frac{1}{\pi^2} \left[16\lambda_1^2 + 6\lambda_1\lambda_2 + \frac{c^2}{64}\right],
\]

\[
\beta_2 = \kappa \frac{\partial \lambda_2}{\partial \kappa} = -\epsilon \lambda_2 + \frac{5}{\pi^2} \left[15\lambda_1^2 + 6\lambda_1\lambda_2 + \frac{c^2}{32}\right],
\]

\[
\beta_3 = \kappa \frac{\partial c}{\partial \kappa} = -\epsilon c + \frac{2c^2}{\pi^2} \left[3\lambda_1 - \lambda_2\right].
\]

When the fixed point \(c^*\) vanishes, the solutions of course reduce to the results of the previous section. Surprisingly, there are no real-valued solutions to the three equations that satisfy \(c^* \neq 0\). Since there is then no fixed point, we conclude that the phase transition is first order, induced by fluctuations \[39\]. The presence of a first order phase transition for \(N_f = 4\) is also in agreement with the existing lattice results \[12\]. Note, however, (see Ref. \[26,40\]) that
the roles of the fundamental and the adjoint representations are reversed for the staggered fermions as used in Ref. [12]. Hence, this comparison is not fully justified.

For $N_f \geq 6$, the critical behavior in the presence of the chiral anomaly is identical to the $c = 0$ case. In renormalization group terms, the operator $c$ is irrelevant, since its mass dimension in four Minkowski space-time dimensions is $[c] = 4 - N_f$, and the critical behavior is determined by $\lambda_1$ and $\lambda_2$ only.

Finally, we stress again that in this approach we can not exclude first order phase transitions induced dynamically by higher order operators. Hence, even if there is a stable fixed point within the $\epsilon$-expansion, the transition may still be first order.

IV. CONCLUSIONS

In this work, we have studied the diquark phase transition in two-color QCD at zero quark mass and nonzero $T$ and $\mu$ using an effective Landau theory. To improve upon mean-field theory, we studied the renormalization group flow in $4 - \epsilon$ dimensions, and then extrapolated our results to $\epsilon = 1$. The conclusions to be drawn are the following.

In the absence of the anomaly term and for $N_f \geq 4$, the phase transition could be second order, whereas the case of $N_f = 2$ is inconclusive at the one-loop level studied here. The effect of the anomaly on the phase transition has been shown to be substantial. In the presence of the anomaly, the $N_f = 2$ case exhibits a second order transition characterized by $O(2)$ critical exponents, which may change to $O(6)$ as $\mu \to 0$. For $N_f = 4$, there is no infrared stable fixed points and hence the phase transition is first order. For $N_f \geq 6$, the phase transition may be of second order, since this situation is identical to the $c = 0$ case. Of course, the statement for $N_f \geq 6$ is only true as long as $N_c = 2$ QCD is asymptotically free. From the one-loop running coupling constant, this limits the number of flavors to $N_f \leq N_f^{(\text{max})} = 11$.

Our analysis has focused solely on the case of massless quarks, and it is interesting to speculate on the consequences of adding nonzero masses. Defining $x = 2\mu/m_\pi \propto \mu/\sqrt{m_q}$, the phase transition discussed in this manuscript corresponds to the limit $x \to \infty$. For a common, nonzero quark mass $m_q$, the diquark condensate competes with the chiral condensate. Such a term alters the symmetry breaking pattern from $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \to \text{Sp}(N_f) \times \text{Sp}(N_f)$ in the massless case to $SU(N_f)_V \times U(1)_B \to \text{Sp}(N_f)$ when $m_q \neq 0$. For $\mu \gg m_\pi$, the first order phase transition for $N_f = 4$ and $c \neq 0$ should only weaken, whereas the second order phase transitions for $N_f \geq 6$ presumably stay second order. When $N_f = 2$ the number of broken symmetries is independent of the introduction of a quark mass, since only the baryon number symmetry is broken. It is therefore possible that the phase transition is second order for all $x$. For $N_f \geq 4$, however, it is unlikely that the massless results should be representative at such a small ratio of $\mu$ and $m_\pi$. Instead, the mass term should be incorporated explicitly into the effective Lagrangian, and the effects could very well change the order of the phase transition as $x$ approaches 1.

The presence of a tricritical point in the phase diagram of two color QCD has so far only been observed on the lattice for $N_f = 4$ flavors. Chiral perturbation theory [24] predicts that the tricritical point is present for any even number of flavors ($N_f \leq N_f^{(\text{max})} = 11$). In contrast, the possibility of a fluctuation induced first order phase transition depends very sensitively on the number of flavors as well as the strength of the anomaly. By combining the results of future lattice simulations for different numbers of flavors with these complementary but quite different analytical approaches, the nature of the superfluid phase transition will certainly become more clear.

In recent lattice investigations [41,42], it was found that the phase diagram of $N_c = 3$ QCD at nonzero temperature and isospin chemical potential is very similar to that of $N_c = 2$ at nonzero temperature and baryonic chemical potential. In particular, the existence of a tricritical point was also demonstrated. Chiral perturbation theory also predicts the existence of a tricritical point for $N_c = 3$ QCD at nonzero isospin chemical potential [24]. The approach followed in this work has been applied to this physically relevant case [29] and for $N_f = 2$ one marginal fixed point was found for the anomaly free case.

By carefully combining the analytical and numerical results for a variety of parameters (including not the least of which is the case where $N_f$ is odd), we should ultimately be able to grasp the complex phase diagram of $N_c = 2$ QCD. Hopefully, this experience will also further enhance our knowledge of dense $N_c = 3$ QCD.

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