FIRST-ORDER HYPERBOLIC FORMULATION OF THE TELEPARALLEL GRAVITY THEORY

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ABSTRACT

Motivated by numerically solving the Einstein field equations, we derive a first-order reduction of the second-order \( f(T) \)-teleparallel gravity field equations in the pure-tetrad formulation (no spin connection). We then restrict our attention to the teleparallel equivalent of general relativity (TEGR) and propose a 3+1 decomposition of the governing equations that can be used in a computational code. We demonstrate that for the matter-free space-time the obtained system of first-order equations is equivalent to the tetrad reformulation of general relativity by Estabrook, Robinson, Wahlquist, and Buchman and Bardeen and therefore also admits a symmetric hyperbolic formulation. The structure of the 3+1 equations resembles a lot of similarities with the equations of relativistic electrodynamics and the recently proposed dGREM tetrad-reformulation of general relativity.

1 INTRODUCTION

The teleparallel gravity theory is one of the alternative reformulations of Einstein’s general relativity (GR) \([30, 2, 14]\). While for GR gravitational interaction is a manifestation of the curvature of a torsion-free spacetime, for teleparallel gravity theory it is realized as a curvature-free linear connection with non-zero torsion. Although different variables can be taken as the main dynamical fields in GR (tetrad fields, soldering forms, etc.), the most extended choice is the metric tensor accompanied by a Levi-Civita connection. In contrast, for teleparallel theory the metric is trivial and the main dynamical field is usually the tetrad (or frame) field.

Despite the teleparallel gravity is considered as an alternative physical theory relative to Einstein’s gravity with several promising features missing in GR, e.g. see the discussion in \([2, \text{Sec.18}] \) and \([14]\), in this paper, we are interested in the teleparallel gravity only from a pure computational viewpoint and our goal is to use its mathematical structure to develop an efficient computational framework for numerical relativity. Thus, the main goal of this paper is to derive a 3+1-split of the so-called teleparallel equivalent of general relativity (TEGR) \([2, 36]\) which is known to pass all standard tests of GR. Up to now, not many attempts have been done to obtain a 3+1 formulation of the TEGR equations, e.g. \([15, 42]\). A Hamiltonian formulation of TEGR was used in \([42]\) to obtain evolution equations for the tetrads and conjugate momenta. In \([15]\), the spatial tetrad and their first order Lie derivative along the normal vector to the foliations were chosen as the state variables. Here, we explore another line of deriving 3+1 formulation of TEGR which is rather aligned with the relativistic electrodynamics. As the result, the obtained equations are completely different from the mentioned papers.

From the numerical viewpoint, a proposed 3+1 formulation has to have the well-posed initial value problem (i.e. a solution exists, the solution is unique and changes continuously with changes in the initial data) in order to compute stable evolution of the numerical solution. In other words, the system of governing equations has to be hyperbolic\(^2\). Therefore, the second goal of the paper is to test if the proposed 3+1 formulation of TEGR is hyperbolic. As is the case with other first-order reductions of the Einstein equations \([6]\), the question of hyperbolicity of 3+1 TEGR equations considered here is not trivial and depends on the delicate use of multiple involution constraints (stationary identities). In particular, for a vacuum space-time, we have found that the proposed 3+1 TEGR equations admits an equivalent reformulation which is in turn equivalent to the symmetric hyperbolic tetrad formulation of GR by Estabrook-Robinson-Wahlquist \([22]\) and Buchman-Bardeen \([11]\).

We note that we do not consider the most general TEGR formulation as for example presented in \([2]\). The linear connection of TEGR is the sum of the two parts: the Weitzenböck connection (which is the historical connection of TEGR) and the spin connection (parametrized by Lorentz matrices) representing the inertial content of the tetrad. The spin connection is necessary to separate the inertial effect

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\(^1\)Yet, it is guaranteed that TEGR produces all the classical results of general relativity \([5, 5]\).

\(^2\)Note that in the numerical relativity the term “strong hyperbolicity” is used emphasizing that not only eigenvalues must be real but that the full basis of eigenvectors must exist.
of a chosen frame from its gravity content as well as for establishing the full covariance of the theory [2, 27, 36], i.e. with respect to both the diffeomorphisms of the spacetime and the Lorentz transformations of tangent spaces. However, being important from the theoretical viewpoint and for extensions of teleparallel gravity [27], the spin connection of TEGR introduces extra degrees of freedom which do not have evolution equations and therefore can be treated as parameters (not state variables) of the theory. Since we are interested in developing a computational framework for GR, the consideration of this paper is, therefore, restricted to the frames for which the spin connection is set to zero globally (Weitzenböck gauge), that is we deliberately work in the class of inertial frames. We thus consider TEGR in its historical, or pure tetrad, formulation [27].

Interestingly that another motivation to study the mathematical structure of the teleparallel geometry is coming from the continuum fluid and solid mechanics. In particular, the role of the torsion to describe defects in solids has been known for decades now, e.g. see [33, 31, 53, 39, 8, 37]. Moreover, the material tetrad field (called also the distortion field in our papers) is the key field for the unified hyperbolic model of fluid and solid mechanics [45, 20]. In such a theory, the concept of torsion can be connected with the inertia field in our papers) is the key field for the unified hyperbolic model of fluid and solid mechanics with torsion [46]. Furthermore, the unified theory of fluids and solids has been also extended in the general relativistic settings [49] and therefore, as being a tetrad theory by its nature, it can be straightforwardly coupled with the 3 + 1 TEGR equations discussed in this paper.

Last but not least, as we shall see in Sec. 8, the derived 3 + 1 equations of TEGR are equivalent to the recently proposed dGREM reformulation of GR [41] which is also a tetrad-based formulation but derived based on the exterior calculus formalism.

2 DEFINITIONS

2.1 Anholonomic frame field

Throughout this paper, we use the following index convention. Greek letters $\alpha, \beta, \gamma, \ldots, \lambda, \mu, \nu, \ldots = 0, 1, 2, 3$ are used to index quantities related to the spacetime manifold, the Latin letters from the beginning of the alphabet $a, b, c, \ldots = 0, 1, 2, 3$ are used to index quantities related to the spacetime manifold, and the upper case Latin letters $A, B, C, \ldots = 1, 2, 3$ index spatial components of the tensors written in a chosen frame of the tangent space.

We shall consider a spacetime manifold $M$ equipped with a general Riemannian metric $g_{\mu\nu}$ and a coordinate system $x^\alpha$. At each point of the spacetime, there is the tangent space $T_xM$ spanned by the frame (or tetrad) $\partial_a$ which is the standard coordinate basis. There is also the cotangent space $T^*_xM$ spanned by the coframe field $dx^a$. Recall that the frames $\partial_a$ and $dx^a$ are holonomic [2].

In addition to $T_xM$, we assume that at each point of $M$, there is a soldered tangent space which is a Minkowski space spanned by an orthonormal frame (tetrad) $\eta_a$ and equipped with the Minkowski metric

$$\kappa_{ab} = \text{diag}(-1, 1, 1, 1).$$

Similarly, there is the corresponding cotangent space spanned by the co-frames $h^a$. It is assumed that the frames $h^a$ and $\eta_a$ are independent of the coordinates $x^\alpha$ and, therefore, are non-holonomic in general.

The components of the non-holonomic frames $h^a$ and $\eta_a$ in the holonomic frames $dx^a$ and $\partial_a$ are denoted by $h^a_\mu$ and $\eta^a_\mu$, i.e.

$$h^a = h^a_\mu dx^\mu, \quad \eta_a = \eta^a_\mu \partial_\mu$$

with $h^a_\mu$ being the inverse of $\eta^a_\mu$, i.e.

$$h^a_\mu \eta^a_\nu = \delta^\mu_\nu, \quad \eta^a_\mu h^b_\mu = \delta^b_\nu,$$

where $\delta^a_\nu$ and $\delta^\mu_\mu$ are the Kronecker deltas.

The soldering of the spacetime and the tangent Minkowski space means that the metrics $g_{\mu\nu}$ and $\kappa_{ab}$ are related by

$$g_{\mu\nu} = \kappa_{ab} h^a_\mu h^b_\nu.$$

Note that

$$h := \det(h^a_\mu) = \sqrt{-g},$$

if $g = \det(g_{\mu\nu})$.

2.2 Observer’s 4-velocity

Observer’s 4-velocity is associated with the 0-th vector of the tetrad basis

$$u^a := \eta^a_\beta, \quad u_\mu = g_{\mu\nu} u^\nu.$$

Also, due to

$$u_\mu = g_{\mu\nu} u^\nu = \kappa_{ab} h^a_\mu h^b_\nu \eta^\nu_\kappa = \kappa_{ab} h^a_\mu = -h^a_\mu$$

the covariant components of the 4-velocity equal to entries of the 0-th vector of the co-basis with the opposite sign

$$u_\mu = -h_\mu^a.$$
2.3 Connection and torsion

Because we work in the framework of the pure-tetrad formulation of TEGR (Weitzenböck gauge), the linear connection is set to the pure Weitzenböck connection\(^3\) [2, 35]:

\[
W^\lambda{}_{\mu \nu} := \partial_\mu h^a{}_{\nu} \quad \text{or} \quad W^\lambda{}_{\mu \nu} := \eta^{\lambda}{}_{\rho} \partial_\rho h^a{}_{\nu}.
\] (9)

The torsion is then defined as

\[
T^\lambda{}_{\mu \nu} := \partial_\mu h^a{}_{\nu} - \partial_\nu h^a{}_{\mu} = W^\rho{}_{\mu \nu} - W^\rho{}_{\nu \mu}.
\] (10)

Note that while the spacetime derivatives commute

\[
\partial_\mu (\partial_\nu V^a) - \partial_\nu (\partial_\mu V^a) = 0,
\] (11)

their tangent space counterparts \(\partial_a = \eta^a{}_{\mu} \partial_\mu\) do not (for non-vanishing torsion)

\[
\partial_a (\partial_\nu V^a) - \partial_\nu (\partial_a V^a) = -T^d{}_{bc} \partial_d V^a,
\] (13)

where \(T^d{}_{bc} = T^d{}_{\mu \nu} \eta^\mu{}_{b} \eta^\nu{}_{c}.

2.4 Levi-Civita symbol (tensor density)

We shall also need the Levi-Civita symbol (tensor-density\(^4\) of weight +1)

\[
\varepsilon^{\lambda \mu \nu \rho} = \begin{cases} +1, & \text{if} \ \lambda \mu \nu \rho \ \text{is an even permutation of 0123}, \\ -1, & \text{if} \ \lambda \mu \nu \rho \ \text{is an odd permutation of 0123}, \\ 0, & \text{otherwise}. \end{cases}
\] (14)

Its covariant components \(\varepsilon_{\lambda \mu \nu \rho}\) define a tensor density of weight −1 with the reference value \(\varepsilon_{0123} = -1\). One could define an absolute Levi-Civita contravariant \(\varepsilon^{\lambda \mu \nu \rho}\) = \(h^{-1} \varepsilon^{\lambda \mu \nu \rho}\) and covariant \(\varepsilon_{\lambda \mu \nu \rho} = h \varepsilon_{\lambda \mu \nu \rho}\) ordinary tensors but for our further considerations (see Sec. 5), it is important that the derivatives \(\partial_\nu \varepsilon^{\lambda \mu \nu \rho} = 0\) vanish:

\[
\partial_\nu \varepsilon^{\lambda \mu \nu \rho} = 0,
\] (15)

whereas for \(\varepsilon^{\lambda \mu \nu \rho}\) one has

\[
\partial_\nu \varepsilon^{\lambda \mu \nu \rho} = \partial_\nu (h^{-1} \varepsilon^{\lambda \mu \nu \rho}) = \varepsilon^{\lambda \mu \nu \rho} \partial_\nu h^{-1} = -\varepsilon^{\lambda \mu \nu \rho} h^{-1} \eta^\rho{}_{\eta} \partial_\eta h^\rho{}_{\eta} = -\varepsilon^{\lambda \mu \nu \rho} W^\rho{}_{\eta \eta},
\] (16)

which is not zero in general.

\(^3\) Note that we use a different convention on the positioning of the lower indices of the Weitzenböck connection \(W^a{}_{\mu \nu} = \partial_\mu h^a{}_{\nu}\) than in [2]. Namely, the derivative index goes first.

\(^4\) We use the sign convention for the tensor density weight according to [52, 29], i.e., under a general coordinate change \(x^\mu \rightarrow x'^{\mu}\) the determinant \(\det(h^a{}_{\mu}) = h\) transforms as \(h' = \det\left(\frac{\partial x'^{\mu}}{\partial x^\mu}\right) W h\) with \(W = +1\). Therefore, the tetrad’s determinant \(h\) and the square root of the metric determinant \(h = \sqrt{\det h}\) have weights +1, as well as the Lagrangian density in the action integral.

3 VARIATIONAL FORMULATION

We consider a general Lagrangian (scalar-density)

\[
\Lambda(h^a{}_{\mu}, \partial_\lambda h^a{}_{\mu}, \ldots) + \Lambda^{(g)}(h^a{}_{\mu}, \partial_\lambda h^a{}_{\mu})
\] (17)

but the derivation will be performed for the total unspecified Lagrangian \(\Lambda\), so that in principle the derivation can be adopted for the extensions of the teleparallel gravity such as \(f(T)\)-teleparallel gravity, where \(f(T)\) is some function of the torsion scalar \(T\), see Section 12. We shall utilize the explicit form of the TEGR Lagrangian only in the last part of the paper.

Varying the action of the teleparallel gravity

\[
\int \Lambda(h^a{}_{\mu}, \partial_\lambda h^a{}_{\mu}) dx
\]

with respect to the tetrad, one obtains the Euler-Lagrange equations

\[
\frac{\delta \Lambda}{\delta h^a{}_{\mu}} = \partial_\lambda (\Lambda_{\lambda a} h^a{}_{\mu}) - \Lambda_{\mu} = 0,
\] (18)

where \(\Lambda_{\lambda a} = \frac{\partial \Lambda}{\partial (\partial_\lambda h^a{}_{\mu})}\) and \(\Lambda_{\mu} = \frac{\partial \Lambda}{\partial h^a{}_{\mu}}\). Equations (18) form a system of 16 second-order partial differential equations for 16 unknowns \(h^a{}_{\mu}\). Our goal is to replace this second-order system by an equivalent but larger system of only first-order partial differential equations.

4 EQUivalence TO GR

Before writing system (18) as a system of first-order equations let us first make a comment on the equivalence of the GR and TEGR formulations.

The Lagrangian density of TEGR (and its extensions) is formed from the torsion scalar \(T\), see (108). As it is known, e.g. see [2, Eq.(9.30)], the torsion scalar can be written as

\[
h T = -\sqrt{-g} R - \partial_\mu (2 h T^\nu{}_{\lambda \rho} \varepsilon^{\lambda \mu \nu \rho}),
\] (19)

where \(R\) is the Ricci scalar and \(T^\nu{}_{\lambda \rho} = \eta^\nu{}_{b} T^b{}_{\lambda \rho}\). In other words, the Lagrangians of TEGR and GR differ by the four-divergence term (surface term). The latter does not affect the Euler-Lagrange equations if there are no boundaries which is implied in this paper. Therefore, Euler-Lagrange equations of TEGR (18) in vacuum is nothing else but the Euler-Lagrange equations of GR written in terms of the tetrads and hence, their physical solutions must be equivalent because the information about the physical interaction is contained in the Euler-Lagrange equations of a theory. What is different in GR and TEGR is the way one interprets the tetrads and their first derivatives, i.e. the way one defines the linear connection of the spacetime from the gradients of tetrads, e.g., torsion-free Levi-Civita connection of GR.
and curvature-free Weitzenböck connection of TEGR. These different geometrical interpretations then define extra evolution equations (compatibility constraints/identities, e.g., see (23)) that are merely consequences of the geometrical definitions but do not define the physics of the gravitational interaction. The critical point for the numerical relativity, though, is that these extra evolution equations must be solved simultaneously with the Euler-Lagrange equations and may affect the mathematical regularity (well-posedness of the Cauchy problem) of the resulting system.

5 FIRST-ORDER EXTENSION

Our first goal is to replace second-order system (18) by a larger but first-order system. This is achieved in this section.

From now on, we shall treat the frame field \( h^a_{\mu} \) and its gradients (the Weitzenböck connection) \( \partial_{\lambda} h^a_{\mu} \) formally as independent variables and in what follows, we shall rewrite system of second-order PDEs (18) as a larger system of first-order PDEs for the extended set of unknowns \( \{ h^a_{\mu}, \partial_{\lambda} h^a_{\mu} \} \), or actually, for their equivalents.

In the setting of the teleparallel gravity, \( \Lambda \) is not a function of a general combination of the gradients \( \partial_{\lambda} h^a_{\mu} \) but of their special combination, that is torsion. Yet, we shall employ not the torsion directly but its Hodge dual, i.e. we assume that

\[
\Lambda(h^a_{\mu}, \partial_{\lambda} h^a_{\mu}) = L(h^a_{\mu}, \hat{T}^{a\mu
u}),
\]

where \( \hat{T}^{a\mu
u} \) is the Hodge dual to the torsion, i.e.

\[
\hat{T}^{a\mu
u} := \frac{1}{2} \varepsilon^{a\mu\nu\rho} T^{\rho}_{c\tau} = \varepsilon^{a\mu\nu\rho} \partial_{\rho} h^c_{\tau},
\]

\[
T^{a}_{\mu\nu} = -\frac{1}{2} \varepsilon^{a\mu\nu\rho} \hat{T}^{a\rho\tau}.
\]

It is important to emphasize that we deliberately chose to define the Hodge dual using the Levi-Civita symbol \( \varepsilon^{a\mu
u\rho} \) and not the Levi-Civita tensor \( \varepsilon^{a\mu
u\rho} = h^{-1} \varepsilon^{a\mu
u\rho} \) that will be important later for the so-called integrability condition (23). Remark that according to definition (21a), \( \hat{T}^{a\mu
u} \) is a tensor-density of weight +1.

In terms of the Lagrangian density \( L(h^a_{\mu}, \hat{T}^{a\mu
u}) \), using notations (20) and definitions (21a), we can instead rewrite Euler-Lagrange equations (18) as

\[
\partial_{\nu} (\varepsilon^{a\mu\lambda\rho} L_{\hat{T}^{a\nu\mu\rho}}) = -L_{h^a_{\nu}}.
\]

The latter has to be supplemented by the integrability condition

\[
\partial_{\lambda} \hat{T}^{a\mu\nu} = 0,
\]

which is a trivial consequence of the definition of the Hodge dual (21a), i.e. of the commutativity property of the standard spacetime derivative \( \partial_{\mu} \), and the identity (15). We note that if the Hodge dual was defined using the Levi-Civita tensor \( \varepsilon^{a\mu\nu\rho} \) instead of the Levi-Civita symbol, then one would have that \( \partial_{\mu} \hat{T}^{a\mu
u} \neq 0 \).

Another consequence of the commutative property of \( \partial_{\mu} \) and the definition of the Hodge dual (based on the Levi-Civita symbol) is that the Noether energy-momentum current density

\[
J^{a}_{\mu} := L_{h^a_{\nu}},
\]

is conserved in the ordinary sense:

\[
\partial_{\mu} J^{a}_{\mu} = 0.
\]

If equations (22), (23) and (25) are accompanied with the torsion definition

\[
\partial_{\mu} h^a_{\nu} - \partial_{\nu} h^a_{\mu} = T^a_{\mu\nu},
\]

they form the following system of first-order partial differential equations (only first-order derivatives are involved)

\[
\partial_{\nu} (\varepsilon^{a\mu\lambda\rho} L_{\hat{T}^{a\nu\mu\rho}}) = -L_{h^a_{\nu}},
\]

\[
\partial_{\nu} T^a_{\mu\nu} = 0,
\]

\[
\partial_{\mu} L_{h^a_{\nu}} = 0,
\]

\[
\partial_{\nu} h^a_{\nu} - \partial_{\mu} h^a_{\mu} = T^a_{\mu\nu},
\]

for the unknowns \( \{ h^a_{\mu}, \hat{T}^{a\mu
u} \} \).

This system forms a base on which we shall build our 3 + 1-split of TEGR in Sections 7 and 8.

6 ENERGY-MOMENTUM BALANCE LAWS

Any conservation law written as a 4-ordinary divergence is a true conservation law, meaning that it yields a time-conserved "charge" [2]. Hence, the Noether current \( J^{a}_{\mu} := L_{h^a_{\nu}} \) is a conserved charge in the ordinary sense, see (25), (27c). It expresses the conservation of the total\(^3\) energy-momentum current density.

However, its spacetime counterpart

\[
\phi^{a}_{\nu} := h^a_{\nu} L_{h^a_{\nu}},
\]

which can be called the total energy-momentum tensor density, is conserved neither ordinary nor covariantly.

Indeed, after contracting with \( h^a_{\nu} \) and adding to it \( 0 \equiv L_{h^a_{\nu}}, \partial_{\nu} h^a_{\nu} - \partial_{\mu} h^a_{\mu} = L_{h^a_{\nu}}, \partial_{\nu} h^a_{\nu} - \partial_{\mu} h^a_{\mu} = L_{h^a_{\nu}}, \partial_{\nu} h^a_{\nu} - \partial_{\mu} h^a_{\mu} = \Lambda^{a}_{\nu
\mu} \Lambda^{\lambda}_{\mu\nu} \). Noether current conservation law (25) can be rewritten in a pure spacetime form:

\[
\partial_{\mu} \phi^{a}_{\mu} = \phi^{a}_{\nu} \Lambda^{\lambda}_{\mu\nu},
\]

\(^3\)The "total" here means the gravitational + matter/electromagnetic energy-momentum current, i.e. \( J^{a}_{\mu} = J^{(g)}_{\mu} + J^{(\text{em})}_{\mu} \).
which has a production term on the right hand-side and, therefore, \( \sigma^\mu \) is not a conserved quantity in the ordinary sense. Exactly this equation and not the conservation law (27c) will be used in the 3+1-split of the TEGR equations. This is our personal preference conditioned, in particular, by the wish to put the 3+1 TEGR equations in the class of symmetric hyperbolic equations and to study their Hamiltonian structure [44, 45] in a follow-up paper.

On the other hand, if
\[
D^\lambda V^\mu = \partial_\lambda V^\mu + V^\rho W^\mu_{\lambda \rho,} \\
D^\lambda V^\mu = \partial_\lambda V^\mu - V_\rho W^\mu_{\lambda \rho}
\]
(30a)
(30b)
is the covariant derivative of the Weitzenböck connection, and keeping in mind that \( h^\lambda, L^\mu_\rho \) is a tensor density of weight +1, balance law (29) can be rewritten as a covariant divergence with a production term
\[
D_\mu \sigma^\mu_\nu = -\sigma^\mu_\nu T^\rho_{\mu \rho,}
\]
and hence, \( \sigma^\mu_\nu \) does not considered also in the covariant sense.

For later needs, the following expression of the energy momentum \( \sigma^\mu_\nu = h^\lambda, L^\mu_\rho \) is required
\[
\sigma^\mu_\nu = 2\hat{T}^{\nu \lambda \mu} L_{\lambda \mu \rho} - \left(\hat{T}^{\nu \lambda \mu} L_{\lambda \mu \rho} - \hat{T}\delta^\mu_\nu\right).
\]
which is valid for the TEGR Lagrangian discussed in Sec. 12. This formula will be used later in the 3+1-split and is analogous to [2, Eq.(10.13)].

7 PRELIMINARIES FOR 3+1 SPLIT

7.1 Transformation of the torsion equations

Before performing a 3+1-split [1] of system (27), we need to do some preliminary transformations of every equation in (27).

Similar to electromagnetism, we define the following quantities
\[
E^\rho_\mu := T^\rho_{\mu \nu} u^\nu, \quad B^\rho_\mu := \hat{T}^{\rho \nu \lambda \mu} u^\nu
\]
(32)
Note that \( E^\rho_\mu \) is a rank-1 spacetime tensor, while \( B^\rho_\mu \) is a tensor-density.

It is known that for any skew-symmetric tensor, its Hodge dual, and a time-like vector \( u^\mu \) the following decompositions hold
\[
\hat{T}^{\rho \nu \lambda \mu} = u^\rho B^{\nu \mu} - u^\nu B^{\rho \mu} + \epsilon^{\rho \nu \lambda \mu} u^\lambda E^\mu_\nu, \\
T^{\rho \nu \lambda \mu}_{\mu} = u^\rho E^\mu_\nu - u^\nu E^\mu_\rho - \epsilon^{\rho \nu \lambda \mu} u^\lambda B^{\mu \rho}.
\]
(33a)
(33b)
Furthermore, we assume that the Lagrangian density can be equivalently expressed in different sets of variables, i.e.
\[
L(h^\nu_\mu \hat{T}^{\rho \nu \lambda \mu} = L(h^\nu_\mu, T^{\rho \nu \lambda \mu} = L(h^\nu_\mu, B^{\rho \mu}, E^\mu_\nu). 
\]
(34)

It then can be shown that the derivatives of these different representations of the Lagrangian are related as
\[
L_{T^{\rho \nu \lambda \mu}_\mu} u^\nu = -\frac{1}{2} \left( \epsilon_{\nu \lambda \rho} u^\mu L_{\nu \lambda} u^\nu \right), \\
L_{T^{\rho \nu \lambda \mu}_\mu} u^\nu = -\frac{1}{2} \left( \epsilon_{\nu \lambda \rho} u^\mu L_{\nu \lambda} u^\nu \right),
\]
and hence, (27a) and (27b) can be written as (see Appendix (C))
\[
\partial_\nu (u^\rho L_{\nu \lambda} u^\nu - u^\nu L_{\nu \lambda} u^\nu + \epsilon^{\rho \nu \lambda \mu} u^\lambda L_{\nu \mu} = f^\nu_\mu, \\
\partial_\nu (u^\rho B^{\nu \mu} - u^\nu B^{\rho \mu} + \epsilon^{\rho \nu \lambda \mu} u^\lambda E^\mu_\nu) = 0,
\]
where the source \( f^\nu_\mu \) has yet to be developed.

Let us now introduce a new potential \( U(h^\nu_\mu, B^{\rho \mu}, D^\rho_\mu) \) as a partial Legendre transform of the Lagrangian \( L \)
\[
U(h^\nu_\mu, B^{\rho \mu}, D^\rho_\mu) := E^\rho_\mu L_{E^\rho_\mu} - L.
\]
(37)
By abusing a little bit notations for \( B^{\rho \mu} \) (we shall use the same letter for \( B^{\rho \mu} \) and \( B^{\rho \mu}_b \), this is an intermediate change of variables and will not appear in the final formulation), we introduce the new state variables
\[
D^\rho_\mu := L_{E^\rho_\mu} = -B^{\rho \mu}, \quad B^{\rho \mu}_b := h^\mu, \quad \partial_\rho \mu := h^\rho_\mu.
\]
(38)
Note that both \( D^\rho_\mu \) and \( B^{\rho \mu}_b \) are tensor-densities. For derivatives of the new potential, we have the following relations
\[
U_{D^\rho_\mu} = E^\rho_\mu, \quad U_{B^{\rho \mu}_b} = L_{B^{\rho \mu}_b}, \quad U_{B^{\rho \mu}_b} = -L_{B^{\rho \mu}_b}. \quad (39)
\]
This allows us to rewrite equations (36) in the form similar to the non-linear electrodynamics of moving media [40, 21, 32]
\[
\partial_\nu (u^\rho D^\nu_\rho - u^\nu D^\nu_\rho + \epsilon^{\rho \nu \lambda \mu} u^\lambda U_{B^{\rho \mu}}) = f^\nu_\mu, \\
\partial_\nu (u^\rho B^{\nu \mu} - u^\nu B^{\rho \mu} + \epsilon^{\rho \nu \lambda \mu} u^\lambda E^\mu_\nu) = 0,
\]
with \( B^{\rho \mu}_b \) and \( D^\rho_\mu \) being the analog of the magnetic and electric fields, accordingly.

Finally, we need to express also the Noether current \( f^\mu_\rho \) in terms of the new potential \( U \) and the fields \( D^\rho_\mu \) and \( B^{\rho \mu}_b \). One has (see details in Appendix A)
\[
f^\mu_\rho = -U_{D^\rho_\mu} + u^\rho B^{\nu \mu} u_{\nu \lambda} B^{\lambda \mu} - u^\rho B^{\nu \mu} u_{\nu \lambda} U_{B^{\lambda \mu}} + \epsilon^{\nu \rho \lambda \mu} u^\lambda U_{B^{\rho \mu}}, \\
f^\mu_\rho = -U_{D^\rho_\mu} + u^\rho B^{\nu \mu} u_{\nu \lambda} B^{\lambda \mu} - u^\rho U_{D^\rho_\mu}, \\
\]
(41a)
(41b)
7.2 Transformation of the tetrad equations

Contracting (27d) with the 4-velocity $u^\mu$, and then after change of variables (38) and (39), the resulting equation reads as
\[
(\partial_\nu h^\mu_{\nu} - \partial_\nu h^\nu_{\mu}) u^\nu = U_{\nu} D^\rho_{\nu}.
\]
Furthermore, using the identity $\eta^\mu_{\nu} \partial_\nu h^\rho_{\sigma} = -h^\mu_{\nu} \partial_\nu \eta^\rho_{\sigma}$ and the definition $u^\mu = \eta^\mu_{\nu} u^\nu$, the latter equation can be rewritten as
\[
u^\mu \partial_\nu h^\rho_{\mu} + h^\rho_{\mu} \partial_\nu u^\nu = - U_{\rho} D^\rho_{\nu},
\]
that later will be used in the $3 + 1$-split.

7.3 Transformation of the energy-momentum

Finally, we express the gravitational part of the energy-momentum tensor $\rho^\nu_{\mu}$ (31) in terms of new variables (38) and the potential $U(h^\rho_{\nu}, B^\rho_{\nu}, B^\mu_{\nu})$, while we keep energy-momentum equation (28) unchanged. It reads
\[
u^\mu = - B^\rho_{\nu} U_{\rho \nu} - D^\mu_{\nu} U_{\nu} - \eta^\mu_{\nu} \partial_\nu \eta^\rho_{\sigma} u^\rho u^\sigma + \epsilon_{\nu \rho \sigma \tau} u^\rho B^\sigma D^\tau_{\nu} + \epsilon_{\nu \rho \sigma \tau} u^\rho u^\sigma U_{\rho \tau} U_{\nu} + (B^\rho_{\nu} U_{\rho \nu} + D^\mu_{\nu} U_{\nu} - \eta^\rho_{\sigma} u^\rho u^\sigma U_{\rho \nu} U_{\mu \nu} - U \eta^\rho_{\nu} u^\rho).
\]
We shall need this expression for $\rho^\nu_{\mu}$ in the last part of the derivation of the $3 + 1$ equations.

Therefore, the TEGR system in its intermediate form for the unknowns $\{h^\rho_{\nu}, D^\mu_{\nu}, B^\mu_{\nu}\}$ reads
\[
\partial_\nu (u^\mu D^\nu_{\mu} - u^\tau D^\rho_{\mu} + \epsilon_{\nu \rho \sigma \tau} u^\rho u^\sigma U_{\rho \tau}) = j^\rho_{\mu}, \quad (45a)
\]

\[
\partial_\nu (u^\mu B^\nu_{\mu} - u^\tau B^\rho_{\mu} - \epsilon_{\nu \rho \sigma \tau} u^\rho u^\sigma U_{\rho \tau}) = 0, \quad (45b)
\]

\[
\partial_\nu j^\rho_{\mu} = 0, \quad (45c)
\]

\[
u^\mu \partial_\nu h^\rho_{\mu} + h^\rho_{\mu} \partial_\nu u^\nu = - U_{\rho} D^\rho_{\nu}, \quad (45d)
\]
with $j^\rho_{\mu}$ given by (41). After we introduce a particular choice of the observer’s 4-velocity $u^\mu$ at the beginning of the next section, we shall finalize transformation of system (45) to present the final $3 + 1$ equations of TEGR.

8 $3 + 1$ SPLIT OF THE TEGR EQUATIONS

In this section, we derive a $3 + 1$ version of system (45) that can be used in a computational code for numerical relativity.

We first recall that Latin indices from the middle of the alphabet $i, j, k, \ldots = 1, 2, 3$ are used to denote the spatial components of the spacetime tensors, and Latin indices $\alpha, \beta, \gamma, \ldots = 0, 1, 2, 3$ to denote the spatial directions in the tangent Minkowski space. Additionally, we use the hat on top of a number, e.g. $\hat{0}$, for the indices marking the time and space direction in the tangent space in order to distinguish them from the indices of the spacetime tensors. Also recall that observer’s 4-velocity $u^\mu$ is associated with the 0-th column of the inverse tetrad $\eta^\mu_{\nu}$, while the covariant components $u^\mu$ of the 4-velocity with the 0-th row of the frame field. For $u^\mu$ and $u_\mu$ we standardly assume [1, 48]:
\[
u^\mu = \eta^\mu_{\nu} = \alpha^{-1}(1, -\beta^i), \quad (46a)
\]

\[
u_\mu = -h^0_{\mu} = (-\alpha, 0, 0, 0), \quad (46b)
\]
with $\alpha$ being the lapse function, and $\beta^i$ being the shift vector.

One can write down $h^\mu_{\nu}$ and $\eta^\mu_{\nu}$ explicitly:
\[
h = \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
\beta^i & h^i_1 & \cdots & \cdots \\
\beta^i & h^i_2 & \cdots & \cdots \\
\beta^i & h^i_3 & \cdots & \cdots \\
\end{pmatrix},
\]

\[
h^{-1} = \begin{pmatrix}
1/\alpha & 0 & 0 & 0 \\
-\beta^i/\alpha & \gamma_{ij} \\
-\beta^i/\alpha & \gamma_{ij} \\
-\beta^i/\alpha & \gamma_{ij} \\
\end{pmatrix},
\]
where $\beta^i = h^i_1 \beta^1$. The metric tensor and its inverse are (e.g. see [28])
\[
\g_{\mu \nu} = \begin{pmatrix}
-\alpha^2 + \beta^i \beta^i & \beta^i \\
\beta^i & \gamma_{ij} \\
\beta^i & \gamma_{ij} \\
\beta^i & \gamma_{ij} \\
\end{pmatrix},
\]

\[
\g^{\mu \nu} = \begin{pmatrix}
-1/\alpha^2 & \beta^i / \alpha^2 \\
\beta^i / \alpha^2 & \gamma_{ij} - \beta^i \beta^j / \alpha^2 \\
\beta^i / \alpha^2 & \gamma_{ij} - \beta^i \beta^j / \alpha^2 \\
\beta^i / \alpha^2 & \gamma_{ij} - \beta^i \beta^j / \alpha^2 \\
\end{pmatrix},
\]
where $\gamma_{ij} = \kappa h \eta^\mu_{\nu} h^\nu_{\rho} \gamma^{\rho}_{ij} = (\gamma_{ij})^{-1}$, and $\beta^i = \gamma_{ij} \beta^j$.

In the rest of the paper, $h_3$ stands for $\det(h^\mu_{\nu})$ so that
\[
h := \det(h^\mu_{\nu}) = \alpha h_3,
\]
and we use the following convention for the three-dimensional Levi-Civita symbol
\[
\epsilon^{ijk} = -\epsilon^{ijk}, \quad \epsilon^{0ijk} = -\epsilon^{ijk}. \quad (50)
\]

8.1 $3 + 1$ split of the torsion PDEs

**CASE** $\mu = i, j, k$. For our choice of observer’s velocity (46), equations (45a) and (45b) read
\[
\partial_\gamma (\alpha^{-1} D^i_{\gamma}) + \partial_\delta (\beta^i \alpha^{-1} D^k_{\gamma} - \beta^k \alpha^{-1} D^i_{\gamma} - \epsilon^{ikj} \alpha U_{\beta \gamma}) = - j^i_{\alpha}, \quad (51a)
\]

\[
- \partial_\gamma (\alpha^{-1} B^i_{\gamma}) + \partial_\delta (\beta^i \alpha^{-1} B^k_{\gamma} - \beta^k \alpha^{-1} B^i_{\gamma} + \epsilon^{ikj} \alpha U_{\beta \gamma}) = 0. \quad (51b)
\]
Because of the factor $a^{-1}$ everywhere in these equations, it is convenient to rescale the variables:

$$D^a := a^{-1} D^a_a, \quad B^{\mu
u} := -a^{-1} B^{\mu
u}$$

so that the derivatives of the potential $U(h^a, D^a_a, B^{\mu
u}) := U(h^a, D^a_a, B^{\mu
u})$ transform as

$$U_{D^a} = a U_{D^a}, \quad U_{B^{\mu
u}} = -a U_{B^{\mu
u}}.$$  

Hence, (51) can be rewritten as

$$\partial_a D^a + \partial_\mu (\beta^a D^a - \beta^i D^a - \varepsilon^{ijk} B_{ijk}) = -f^a, \quad \partial_a B^{\mu
u} + \partial_\mu (\beta^i B^{\mu
u} - \beta^k B^{\mu
u} + \varepsilon^{ijk} U_{D^a}) = 0.$$  

Finally, extending formally the shift vector as $\beta = (-1, \beta^i)$ and introducing the change of variables

$$D^a := D^a_a + \tilde{D}^a, \quad B^{\mu
u} := \tilde{B}^{\mu
u}$$

which are essentially 3-by-3 matrices ($\tilde{D}^a = 0, B^{\mu
u} = \tilde{B}^{\mu
u} = 0$, and $D^a$ are given by (87a)), we arrive at the final form of the 3 + 1 equations for the fields $D^a$ and $B^{\mu
u}$

$$\partial_a D^a + \partial_\mu (\beta^a D^a - \beta^i D^a - \varepsilon^{ijk} B_{ijk}) = -f^a, \quad \partial_a B^{\mu
u} + \partial_\mu (\beta^i B^{\mu
u} - \beta^k B^{\mu
u} + \varepsilon^{ijk} U_{D^a}) = 0.$$  

where

$$E^j = U_{D^a}, \quad H_{\alpha\beta} = -a U_{B^{\alpha\beta}}.$$  

In terms of $\{B^{\mu\nu}, D^a\}$ the potential $U(h^a, D^a_a, B^{\mu
u})$ will be denoted as $U(h^a, B^{\mu\nu}, D^a_a)$ and it reads (see (112))

$$U(h^a, B^{\mu\nu}, D^a_a) = -\frac{a}{2\lambda} \left( x^a \left( D^a_a D^a_a - \frac{1}{2} (D^a_a)^2 \right) + \frac{1}{x^a} \left( B^{\mu\nu} B^{\mu\nu} - \frac{1}{2} (B^{\mu\nu})^2 \right) \right).$$  

Alternatively, it can be computed as

$$U = \frac{1}{2} (D^a_a E^a + B^{\mu\nu} H_{\mu\nu}).$$  

CASE $\mu = 0$. The 0-th equations (45b) and (45d) are actually not time-evolution equations but pure spatial (stationary) constraints

$$\partial_0 B^{\mu\nu} = 0, \quad \partial_0 B^{\mu\nu} = 0.$$  

Their possible violation during the numerical integration of the time-evolution equations (56) is a well-known problem in the numerical analysis of hyperbolic equations with involution constraints. Different strategies to preserve such stationary constraints are known, e.g. constraint-cleaning approaches [38, 16, 17] or constraint-preserving integrators [41, 10].

### 8.2 Tetrad PDE

Because of our choice for the tetrad (47), we are only interested in the evolution equations for the components $h^a_k$. Thus, using the definition of the 4-velocity (46), equation (45d) can be written as

$$\partial_t h^a_k - \beta^i \partial_t h^i_k - h^\sigma \partial_\sigma h^a_k = -E^a_k.$$  

This equation can also be written in a slightly different form. After adding $0 = -\beta^i \partial_t h^i_k + \beta^k \partial_k h^a_k$ to the left hand side of (61), one has

$$\partial_t h^a_k - \beta^k \partial_k h^a_k = -E^a_k.$$  

Finally, using the definition of $B^{\mu\nu}$ and $U_{\alpha\beta}$, we have that

$$B^{\mu\nu} = \tilde{T}^{\mu\nu} + u^\nu \varepsilon^{\nu\rho\sigma \delta} \partial_\sigma h^\rho_\delta = -u^\nu \varepsilon^{\nu\rho\sigma \delta} \partial_\sigma h^\rho_\delta = \alpha \varepsilon^{\rho\sigma \delta \beta} \partial_\delta h^\rho_\beta$$

and hence (use that $\varepsilon^{\rho\sigma \delta \beta} = \varepsilon^{\rho\delta \beta \sigma}$)

$$B^{\mu\nu} = \varepsilon^{\mu\nu} h^a_k.$$  

Therefore, (62) can be also rewritten as

$$\partial_t h^a_k - \beta^\mu (\partial_\mu h^a_k - \partial_k h^a) = \varepsilon_{\mu\nu} \beta^\nu h^a.$$  

It is not clear yet which one of these equivalent forms (61), (62), or (66) at the continuous level is more suited for the numerical discretization. However, for a structure-preserving integrator, e.g. [10], which is able to preserve (64) up to the machine precision, all these forms are equivalent.

### 8.3 Energy-momentum PDE

We now turn to the energy-momentum evolution equation

$$\partial_u \sigma^\mu_{\nu} = \sigma^\mu_{\nu} W^1_{\mu\nu}$$  

and provide its $3 + 1$ version. Because the Lagrangian density in (20) represents the sum of the gravitational and matter Lagrangian, the energy-momentum tensor is also assumed to be the sum of the gravity and matter parts: $\sigma^\mu_{\nu} = (m)\sigma^\mu_{\nu} + (g)\sigma^\mu_{\nu}$. However, in what follows, we shall omit the matter part keeping in mind that the energy and momentum equations discussed below are equations for the total (matter+gravity) energy-momentum.

We first, explicitly split (67) into the three momentum and one energy equation:

$$\partial_\mu \sigma^0_{\nu} + \partial_\nu \sigma^0_{\mu} = \sigma^0_{\lambda} W^1_{\lambda\mu\nu}$$  

$$\partial_\mu \sigma^0_0 + \partial_\nu \sigma^0_{\nu} = \sigma^0_{\lambda} W^1_{\lambda0\nu}.$$  

Applying the change of variables (52), (53), and (55), expression (44) for the energy-momentum reduces to the following expressions.
• for the total momentum density, \( \sigma^0_i = \varrho_i \) (the matter part is left unspecified and omitted in this paper):

\[
\varrho_i := - \varepsilon_{ij} \mathcal{B}^{jl} \mathcal{D}^j_a \tag{69}
\]

• for the momentum flux, \( \sigma^k_i \):

\[
\sigma^k_i = - \mathcal{B}^{ak} \mathcal{H}_a - \mathcal{D}^k_a \mathcal{E}^j + \varepsilon_{ij} \mathcal{B}^{bl} \mathcal{D}^l_a \\
+ \mathcal{B}^{j} \mathcal{H}_a + \mathcal{D}^j_a \mathcal{E}^i - \mathcal{U} \delta_{ij} \tag{70}
\]

• for the energy density:

\[
\rho_0 := \sigma^0_0 = - \varepsilon_{ij} \beta^j \mathcal{D}^j_a - \mathcal{U} = \beta^i \varrho_i - \mathcal{U}, \tag{71}
\]

• and for the energy flux, \( \sigma^k_0 \):

\[
\sigma^k_0 = - \beta^i (\mathcal{B}^{ik} \mathcal{H}_a + \mathcal{D}^k_a \mathcal{E}^j) \\
+ \beta^j (\mathcal{B}^{jl} \mathcal{H}_a + \mathcal{D}^l_a \mathcal{E}^i) \\
+ \varepsilon_{ij} \beta^j \mathcal{D}^j_a - \mathcal{U} \delta_{ij}, \tag{72}
\]

It is convenient to split the momentum and energy fluxes \( \sigma^k_i \) and \( \sigma^0_0 \) in (68) into advective and constitutive parts, so that the final form of the 3+1 equations for the total energy-momentum (68) reads

\[
\partial_t \varrho_i + \partial_k (\beta^k \varrho_i + s^k_i) = f_i, \tag{73a}
\]

\[
\partial_t \rho_0 + \partial_k (\beta^k \rho_0 + \beta^i s^k_i + \varepsilon_{ij} \mathcal{H}_a \mathcal{E}^j) = f_0, \tag{73b}
\]

where the non-matter (gravity + inertia) part of \( s^k_i \) is given by

\[
s^k_i = - \mathcal{B}^{ak} \mathcal{H}_a - \mathcal{D}^k_a \mathcal{E}^j + (\mathcal{B}^{jl} \mathcal{H}_a + \mathcal{D}^l_a \mathcal{E}^i - \mathcal{U}) \delta_{ij}. \tag{73c}
\]

The source terms in (73a) and (73b) are given by

\[
f_i = - \rho \eta^j \mathcal{E}^a_i + \varrho_i \partial^j \beta^l + s^j_i W^j_i, \tag{73d}
\]

\[
f_0 = - (\partial_t \mathcal{H}_a (\kappa - \beta^k \partial_k \mathcal{H}_a)) \mathcal{U} \\
+ (\partial_t \beta^l - \beta^k \partial_k \beta^l) \rho \eta^j \mathcal{D}^j_i \\
+ \varepsilon_{ij} \mathcal{H}_a \mathcal{E}^j \partial_k \mathcal{H}_a + s^j_i \eta^j \mathcal{D}^j_i \partial^l \beta^l. \tag{73e}
\]

### 8.3.2 Alternative form of the energy-momentum equations

In (67), we deliberately use the energy-momentum tensor with both spacetime indices because we would like to use the SHTC and Hamiltonian structure [44] of these equations for designing numerical schemes in the future, e.g. [13, 34]. However, the resulting PDEs (73) do not have a fully conservative form preferable for example when dealing with the shock waves in the matter fields. Therefore, one may want to use the true conservation law (25) for the total (matter+gravity) Noether current \( j^0_a \) instead of (73). Thus, in the new notations introduced above, four conservation laws \( \partial_{\mu} j^0_\mu = 0 \) now read

\[
\partial_t \rho_a + \partial_k (\beta^k \varrho_a + s^k_a) = 0, \tag{76a}
\]

\[
\partial_t \rho_0 + \partial_k (\beta^k \rho_0 + \beta^i s^k_i + \varepsilon_{ij} \mathcal{H}_a \mathcal{E}^j) = 0, \tag{76b}
\]

with \( \rho_0 = j^0_0 = -\alpha^{-1} \mathcal{U} \).

### 8.3.3 Evolution of the space volume

As in the computational Newtonian mechanics [20, 10], the evolution of the tetrad field at the discrete level has to be performed consistently with the volume/mass conservation law. In TTEGR, the equivalent to the volume conservation in the Newtonian mechanics is

\[
\partial_t (h \mathcal{N}^\mu) = - h E^\mu \tag{77}
\]

which can be obtained after contracting (45d) with \( \partial h / \partial \mathcal{N}^\mu = h \eta^\mu_a \), and where \( E^\mu = \eta^\mu_a \mathcal{E}^a \).

After using (46), (47), and (80b), this balance law becomes

\[
\partial_t h_3 - \partial_k (h_3 \beta^k) = - \alpha \gamma^j \mathcal{D}^j_i \tag{78}
\]

where \( h_3 = \det(h^\kappa_\kappa) = \sqrt{\det (\gamma^\mu_\mu)} \).

### 9 SUMMARY OF THE 3+1 TTEGR EQUATIONS

Here, we summarize the 3+1 TTEGR equations and give explicit expressions for the constitutive fluxes \( E^a_k \) and \( \mathcal{B}_k \), which then close the specification of the entire system.
9.1 Evolution equations

The system of 3 + 1 TEGR governing equations reads

\[ \partial_t \rho^\alpha_{\beta} + \partial_i (\beta^i \rho^\beta_{\alpha} - \beta^0 \rho^\beta_{\alpha} - \epsilon^{\beta ij} H_{ij}) = \beta^0 \rho_{\alpha} - s^\alpha, \]  
\[ \partial_t B^{\alpha i} + \partial_k (\beta^k B^{\alpha i} - \beta^0 B^{\alpha i} + \epsilon^{\alpha k} E^k) = 0, \]  
\[ \partial_t \rho_{\alpha} + \partial_k (-\beta^k \rho_{\alpha} + s^\alpha) = 0, \]  
\[ \partial_t \rho_{\beta} + \partial_k (-\beta^k \rho_{\beta} + s^\beta) = 0, \]  
\[ \partial_t h^k_{\alpha} - \beta^i \partial_i h^k_{\alpha} - h^k_{\alpha} \partial_i \beta^i = -\epsilon_{\alpha k}, \]  
\[ \partial_t h_{\alpha} - \beta^i \partial_i h_{\alpha} - h_{\alpha} \partial_i \beta^i = -\epsilon_{\alpha}, \]  

with the total (matter+gravity) momentum \( \rho_{\alpha} = \eta_{\alpha} \rho_{\alpha} \) and the total energy density \( \epsilon_{\alpha} \) computed from (75), and the gravitational part of the momentum flux \( s^\alpha = \eta^\alpha \partial^a \) computed from (73).

In particular, the structure of this system resembles very much the structure of the nonlinear electrodynamics of moving media already solved numerically in [21] for example, as well as the structure of the continuum mechanics equations with torsion [46]. Moreover, despite deep conceptual differences, it is identical to the new dGREM formulation of the Einstein equations recently pushed forward in [41]. However, it is important to mention that the Lagrangian approach adopted here in principle permits to generalize the formulation to other \( f(T) \) theories. A detailed comparison of these two formulations will be a subject of a future paper.

It is also important to note that the structure of system (79) remains the same independently of the Lagrangian in use as it can be seen from system (45) where all the constitutive parts are defined through the derivatives of the potential (57). If the Lagrangian changes, then only the constitutive functions \( E^\mu_j \) and \( H_{\beta \alpha} \) have to be recomputed.

9.2 Constitutive relations

For the TEGR Lagrangian (108), \( E^\alpha_j \) and \( H_{\beta \alpha} \) read

\[ H_{\beta \alpha} := -\frac{\alpha}{\sqrt{\lambda}} \kappa_{\beta \alpha} \frac{1}{\kappa_{ij}} \left( h^\mu_i h_{\mu j} - \frac{1}{2} h^\mu_i h^\mu j \right) B^\beta, \]  
\[ \epsilon_{\alpha} := -\frac{\alpha}{\sqrt{\lambda}} \kappa_{\alpha} \left( h^\mu_i h_{\alpha} - \frac{1}{2} h^\mu_i h^\mu \right) D^\beta, \]  
\[ \epsilon_{\alpha} := -\frac{\alpha}{\sqrt{\lambda}} \kappa_{\alpha} \left( h^\mu_i h_{\alpha} - \frac{1}{2} h^\mu_i h^\mu \right) D^\beta, \]  

where \( \kappa_{\beta \alpha} = E^\beta_j \) and \( \Omega_{\beta \alpha} = H_{\beta \alpha} \). Note that these relations can also be written as

\[ H_{\beta \alpha} = \frac{\partial U}{\partial B^\beta_{\alpha}}, \]  
\[ \epsilon_{\alpha} = \frac{\partial U}{\partial D^\beta}, \]  

with the potential \( U \) given by (58).

9.3 Stationary differential constraints

System (79) is supplemented by several differential constraints. Thus, as already was mentioned, the 0-th equations (\( \mu = 0 \)) of (45a) are not actually time-evolution equations but reduce to the so-called Hamiltonian and momentum stationary divergence-type constraints that must hold on the solution to (79) at every time instant:

\[ \partial_t \rho^\beta_{\alpha} = \rho_{\alpha}, \]  
\[ \partial_t B^{\alpha i} = \rho_{\alpha}, \]  
\[ \partial_t h^k_{\alpha} = h^k_{\alpha}, \]  
\[ \partial_t h_{\alpha} = h_{\alpha}, \]  

with \( \rho_{\alpha} = -\alpha^{-1} U = \rho_{\alpha} \).

Accordingly, the 0-th equation of (45b) gives the divergence constraint on the \( B^\alpha j \) field

\[ \partial_t B^\beta = 0, \]  

Finally, from the definition of \( B^\alpha j \), we also have a curl-type constraint on the spatial components of the tetrad field:

\[ \epsilon_{\beta i} \partial_t h^i_{\beta} = B^\beta, \]  

9.4 Algebraic constraints

As a consequence of our choice of observer’s reference frame (46), and the fact that \( E^\mu_j u^\mu = 0 \) and \( B^{\mu \nu} u_\mu = 0 \), we have the following algebraic constraints

\[ B^\beta = 0, \]  
\[ E^\alpha_j = \frac{\partial E^\alpha_j}{\partial \Omega_{\beta \alpha}}, \]  
\[ D^\beta = -\frac{1}{\sqrt{\lambda}} \epsilon_{\beta i} B^i, \]  
\[ D^\beta = D^\beta, \]  
\[ H_{\beta \alpha} = \beta^2 H_{\beta \alpha}, \]  

where \( D^\beta = \gamma_{\beta \alpha} D^\alpha j \). As already noted in [2], the gravitational part of the energy-momentum \( u_{\nu}^\nu \) is trace-free:

\[ u_{\nu}^\nu = 0. \]  

10 HYPERBOLICITY OF THE VACUUM EQUATIONS

Despite the structure of system (79) shares a lot of similarities with equations of nonlinear electrodynamics of moving media which belong to the class of symmetric hyperbolic thermodynamically compatible (SHTC) equations [44, 26, 50, 51, 25], we cannot immediately apply the machinery of SHTC systems to symmetrize the TEGR equations.
We remark that due to our choice of the 3 + 1 split (47), i.e. that observer’s time vector, \( \eta \), is aligned with the normal vector to the spatial hypersurfaces, matrices \( K_{AB} \) and \( D_{AB} \) are symmetric.

Additionally, we have the relations for \( a_A, \omega^A \), and \( A_k = E_k = \partial_t a \) and \( \Omega_k = B_k \)

\[
a_A = a^{-1} A_A, \quad \omega^A = \kappa a^{-1} \kappa^A \Omega B.
\]

Finally, expression of the constitutive fluxes \( E^A_k \) and \( B^A_k \) in terms of \( \{ k_{AB}, K_{AB}, a_A, \omega^A \} \) read

\[
E_{AB} = -\kappa K_{AB} + a a_{AB} a^C, \quad (95a)
\]

\[
H_{AB} = a^C a_{AB} - \kappa \epsilon_{ABC} a^B. \quad (95b)
\]

Unfortunately, system (79) is not immediately hyperbolic even for the Minkowski spacetime (it is actually only weakly hyperbolic because some eigenvectors are missing) and therefore it can not be equivalent to the symmetric hyperbolic ERWBB formulation of GR right away. In this section, we discuss what modifications of equations (79) are necessary in order to make it equivalent to the ERWBB formulation. These modifications rely on the use of Hamiltonian and momentum constraints (85) and symmetry of \( D_{AB} \).

Because we are considering a vacuum space-time, the momentum and energy equations (79c), (79d) can be omitted. Moreover, modifications concern only torsion equations (79a), (79b), and therefore for the sake of simplicity, we also omit the tetrad equations (79e).

If one considers 18 equations (79a), (79b) written in a quasilinear form

\[
\partial_t Q + M^A(Q) \partial_A Q = S
\]

for 18 unknowns \( Q = \{ \rho, \rho^a, \omega^A \} \) then the matrix \( M = \xi Q \xi^T \) has all real eigenvalues but only 10 linearly independent eigenvectors already for the flat Minkowski spacetime, that is the system is only weakly hyperbolic. Here, \( M^A = \partial Ê / \partial Q \) is the Jacobian of the k-th flux vector.

Evolution equations (79a) and (79b) for the torsion fields have a conservative form (i.e. their differential terms are the 4-divergence) which is in particular convenient for developing of high-order methods for hyperbolic equations. However, this flux-conservative form of equations has a flaw in the eigenvalues already for the flat spacetime with non-zero shift vector \( \beta^k \). Namely, some characteristic velocities are always 0 despite the non-zero shift vector. The same issue is well known for the magnetohydrodynamics equations [47]. From our experience with the non-relativistic equations of nonlinear electrodynamics of moving media [21] and continuum mechanics with torsion [46] that have the same structure as (79). Instead, the following non-conservative form does not have such a pathology

\[
\partial_t \beta^F_A + \partial_k (\beta^F_A - \beta^B A_k - 2i E^B + H_i) - \beta^B \partial_k \beta^F_A = -s^F_A,
\]

\[
\partial_t \beta^F + \partial_k (\beta^F k - \beta^F \omega^F + E^B + H_i) - \beta^B \partial_k \beta^F = 0,
\]

\[\text{Due to the rotational invariance, it is actually sufficient to consider the system in any preferred coordinate direction, say in } x^i, \text{ without lose of generality.}\]
and which is obtained from (79a) and (79b) by turning the momentum \( p_a \) on the right hand-side of (79a) into a differential term according to the momentum constraint \( \rho_a = \partial_a \beta^k \), and adding formally a zero, \( 0 = \partial_0 B^k \), to (79b). These modifications are legitimate for smooth solutions thanks to involution constraints (85) and (75). After this, the characteristic velocities have the desired values equal to \( \sim \beta^k \) or \( \sim \beta^k \pm c \), if \( c \) is the light speed. However, this only allows to fix the issue with the eigenvalues, while some eigenvectors are still missing.

In order to recover all missing eigenvectors, we shall use the Hamiltonian constraint (83)\( _1 \) and the symmetry of \( D_{ij} \). Thus, we shall turn the potential \( U = -\alpha p_0 \) in the expression of the stress tensor \( \tilde{s}_a \) into the differential term \( U = -\alpha D_0 B^k \) according to the Hamiltonian constraint (83)\( _1 \). Additionally, we use expression (87)a\( _2 \) for \( D^k_0 \), i.e. \( D_0^k = -\epsilon^{kl} h_0^{-1} \epsilon^lj_0 \). Similarly, we are adding formally a zero \( 0 = \partial_k (h_0^{-1} \epsilon^lj_0) \) to the equation for the \( B^k \) field. This expression is zero due to the fact that the contraction of antisymmetric Levi-Civita symbol with the symmetric tensor \( D_{ij} \) is zero.

The resulting equations read

\[
\partial_t D^k_0 + \partial_k (\beta^2 D^k_0 - \beta^k D^k_0 - \epsilon^lj_0 \ H_{ij}) = -\beta^2 \partial_t D^k_0 + \alpha \eta^j \partial_j \beta^k = -\tilde{s}_a - \partial_0 \eta^j, \quad (98a)
\]

\[
\partial_0 B^{kl} + \partial_k (\beta^2 B^{kl} - \beta^k B^{kl} + \epsilon^lj_0 \ E^l_j) = -\beta^2 \partial_0 B^{kl} - \alpha \eta^j \partial_k \epsilon^lj_0 = 0, \quad (98b)
\]

where we have defined new vectors

\[
E^k := -\epsilon^{-1} \epsilon^lj_0 \ D_{ij}, \quad H^k := -\epsilon^{-1} h_0^{-1} \epsilon^lj_0 \ B_{ij}. \quad (98c)
\]

Now, using relations (93)–(95) and after a lengthy but rather straightforward sequence of transformations, equations (98) can be written in the ERWBB form

\[
\partial_t K_{ab} - \kappa_{ab} c E^m_0 \partial_m K_{ab} - \partial_a \alpha_b = \text{L.o.t.} \quad (99a)
\]

\[
\partial_0 K_{ab} + \kappa_{bc} c E^m_0 \partial_m K_{ab} + \partial_a \omega_b = \text{L.o.t.} \quad (99b)
\]

where ‘L.o.t.’ stands for ‘low-order terms’ (i.e. algebraic terms that do not contain space and time derivatives), \( \partial_0 = \eta^j \partial_j = \eta^j \partial_0 \partial_j \), \( \partial_a = -\partial_a \partial_0 \partial_j + \alpha^{-1} \partial_a \partial_0 = \eta^k \partial_a \), and \( \partial_0 = \eta^k \partial_0 \partial_k \).

Similar to \( A_k \) and \( \Omega_k \) in TEGR, the vectors \( \alpha_b \) and \( \omega_b \) are not provided with particular evolution equations and therefore, one could try to choose their evolution equations in such a way as to guarantee the well-posedness of the enlarged system for the unknowns \( \{ K_{ab}, \eta_{ab}, \alpha_b, \omega_b \} \). Thus, as shown in [22, 11], if coupled with equations whose differential part is

\[
\partial_t \alpha_a - \partial_a \beta^k = 0, \quad (100)
\]

\[
\partial_0 \omega_a + \partial_a B^k = 0, \quad (101)
\]

system (99) is symmetric hyperbolic, and thus (98) can also be put into a symmetric hyperbolic form. However, we cannot immediately conclude the same for the original 3 + 1 TEGR equations (79) because the differential operators of (79) and (98) are different even though the overall equations are equivalent. We recall that the hyperbolicity (well-posedness of the initial value problem) is defined by the leading-order terms of the differential operator (all differential terms in the case of first-order systems). Nevertheless, this is an important result demonstrating that inside of the class of equations equivalent to TEGR equations (79) there is a possibility, at least one given by (98), to rewrite the TEGR vacuum equations as a symmetric hyperbolic system.

11 RELATION BETWEEN THE TORSION AND EXTRINSIC CURVATURE

It is useful to relate the state variables of TEGR to the conventional quantities used in numerical general relativity [4, 6, 28]. In what follows, we relate the spatial extrinsic curvature of GR to the \( B^k \) field. We remark that the two fields are conceptually different due to the different geometry interpretations in GR and TEGR. Therefore, the obtained relation is possible only due to the equivalence of TEGR and GR.

In GR, the evolution equation of the spatial metric \( \gamma_{ij} \) is (see [48, Eq.(7.64)])

\[
\partial_t \gamma_{ij} - \beta^l \partial_l \gamma_{ij} - \gamma_{il} \partial_i \beta^j - \gamma_{lj} \partial_l \beta^i = -2 \kappa_{ij}, \quad (102)
\]

where \( K_{ij} \) is the spatial extrinsic curvature. On the other hand, in TEGR, contracting (61) with \( \kappa_{ab} h^a_j \), one obtains the following evolution equation for the spatial metric:

\[
\partial_t \gamma_{ij} - \beta^l \partial_l \gamma_{ij} - \gamma_{il} \partial_i \beta^j - \gamma_{lj} \partial_l \beta^i = - \kappa_{ab} (h^a_i E^j_0 + h^a_j E^i_0). \quad (103)
\]

Therefore, one can deduce an expression for the extrinsic curvature in terms of \( E^a_i \):

\[
K_{ij} = \frac{1}{2 \kappa} \kappa_{ab} (h^a_i E^j_0 + h^a_j E^i_0). \quad (104)
\]

To obtain another expression for \( K_{ij} \) in terms of the primary state variable \( D^i_0 \), one needs to use the the constitutive relationship (80b) to deduce

\[
K_{ij} = \frac{\kappa}{h_3} \left( \partial_0 \frac{1}{2} B^i_k \gamma_{ij} \right), \quad (105)
\]

which, if contracted, gives the relationship for the traces

\[
K_i^i = \gamma^j K_{ji} \quad \text{and} \quad D_i^i = B_i^k h^a_i \quad (106)
\]

Remark that if written in the tetrad frame \( K_{ab} = \eta^j \eta^k K_{ij} \) then the extrinsic curvature is exactly \( K_{ab} \) introduced in (93) apart from the opposite sign

\[
K_{ab} = -K_{ab}. \quad (107)
\]
12 TORSION SCALAR

In TEGR and its \( f(T) \)-extensions, the Lagrangian density is a function of the torsion scalar \( T \), e.g. in TEGR, the Lagrangian density is

\[
\mathcal{L}(\sigma^\mu, T_{\mu\nu}) = \frac{\hbar}{2} T, \tag{108a}
\]

\[
T(\sigma^\mu, T_{\mu\nu}) := \frac{1}{4} \delta^\alpha_\beta \delta^\gamma_\delta g_{\alpha\delta} T^\lambda_\gamma T^\mu_\nu \tag{108b}
\]

\[
+ \frac{1}{2} \delta^\alpha_\beta T^\lambda_\mu T^\gamma_\nu \tag{108c}
\]

\[
- \delta^\alpha_\beta T^\gamma_\nu T^\lambda_\mu \tag{108d}
\]

where \( \kappa = 8\pi Ge^{-4} \) is the Einstein gravitational constant, \( G \) is the gravitational constant, and \( c \) is the speed of light in vacuum.

However, to close system (79), we need not the Lagrangian \( \mathcal{L}(\sigma^\mu, T_{\mu\nu}) \) directly but we need to perform a sequence of variable and potential changes: \( \mathcal{L}(\sigma^\mu, T_{\mu\nu}) = \mathcal{L}(\sigma^\mu, E^\mu_\nu, B^\mu) \rightarrow E^\mu_\nu D^\nu_\lambda \mu - \mathcal{L} = U(\sigma^\mu, E^\mu_\nu, B^\mu) = U(\sigma^\mu, D^\nu_\lambda, B^\mu). \) Thus, we have

\[
\mathcal{L}(\sigma^\mu, E^\mu_\nu, B^\mu) = \frac{\hbar}{2 \kappa} \left( - \frac{1}{2} (E^\lambda_\mu E^\mu_\lambda - 2E^\lambda_\mu E^\mu_\lambda + E^\mu_\beta E^\beta_\lambda) + \xi \right) \left( E^\gamma_\beta E^\lambda_\gamma + 2E^\lambda_\beta E^\beta_\gamma \right) \]

\[
- \frac{1}{2\kappa^2} \left( B^\lambda_\alpha B^\alpha_\lambda + B^\lambda_\beta B^\beta_\lambda - 2B^\lambda_\beta B^\beta_\lambda \right), \tag{109}
\]

In turn, if we perform the Legendre transform \( U(\sigma^\mu, D^\nu_\lambda, B^\mu) = E^\mu_\nu D^\nu_\lambda - \mathcal{L} \) then the new potential \( U \) reads

\[
U(h^\nu_\mu, D^\mu_\nu, B^\mu) = \frac{1}{4\hbar} \left( \right) \left( D^\mu_\nu D^\nu_\lambda - 2D^\mu_\nu D^\nu_\lambda \right) + \frac{1}{\kappa} \left( B^\mu_\nu B^\nu_\lambda - 2B^\mu_\nu B^\nu_\lambda \right) + \frac{1}{\kappa} \left( B^\mu_\nu B^\nu_\lambda - 4B^\mu_\nu B^\nu_\lambda \right), \tag{110}
\]

where \( D^\mu_\nu = D^\mu_\nu h^\nu_\mu \) and \( B^\mu_\nu = B^\mu_\nu h^\nu_\mu \), and in the last two terms, one should pay attention that the new field \( D^\mu_\nu \) introduced in (55) appears there.

Apparentely, apart from the last terms in (110) depending on \( D^\mu_\nu \) the potential \( U \) is more symmetric in the variables \( D^\mu_\nu \) and \( B^\mu \) rather than the Lagrangian \( \mathcal{L} \) in \( E^\mu_\nu \) and \( B^\mu \). This in particular, justifies the introduction of the new and final variables (55)

\[
D^\mu_\nu = D^\mu_\nu + B^\mu D^\nu_\lambda, \quad B^\mu = B^\mu, \tag{111}
\]

so that the potential \( U(h^\nu_\mu, D^\mu_\nu, B^\mu) = U(h^\nu_\mu, D^\mu_\nu, B^\mu) \) becomes just

\[
U(h^\nu_\mu, D^\mu_\nu, B^\mu) = -\frac{\kappa}{2\hbar} \left( \frac{1}{\kappa} \left( D^\mu_\nu D^\nu_\lambda - 2D^\mu_\nu D^\nu_\lambda \right) + \frac{1}{\kappa} \left( B^\mu_\nu B^\nu_\lambda - 1 \left( B^\mu_\nu B^\nu_\lambda \right)^2 \right) \right). \tag{112}
\]

13 CONCLUSION AND DISCUSSION

We have presented a 3 + 1-split of the TEGR equations in their historical pure tetrad version, i.e. with the spin connection set to zero. To the best of our knowledge, there were not many attempts to obtain a 3 + 1-split of the TEGR equations, e.g. [15, 42], and we are not aware of any attempts to solve numerically the full TEGR system of equations for general spacetimes. This paper, therefore, may help to cover this gap.

The derivation was done for a general Lagrangian and therefore extensions of TEGR such as \( f(T) \)-teleparallel theories also can be covered in the future.

The 3 + 1 governing partial differential equations have appeared to have the same structure as equations of nonlinear electrodynamics [21] and equations for continuum fluid and solid mechanics with torsion [46]. Moreover, it has appeared that the derived equations are equivalent to the recently proposed dGREL tetrad formulation of GR [41].

The derived 3 + 1 TEGR equations are not immediately hyperbolic as usually the case for many first-order reductions of the Einstein equations. We demonstrated that the differential operator of the equations can be transformed into a different but equivalent form which, in turn and surprisingly is equivalent to the symmetric hyperbolic tetrad reformulation of GR by Estabrook-Robinson-Wahlquist [22] and Buchanan-Bardeen [11]. The main idea of these transformations relies on the use of multiple stationary identities of the theory, e.g. Hamiltonian and momentum constraints. It, therefore, should be very critical to fulfill these constraints on the discrete level.

Despite it is argued that TEG it is fully equivalent to Einstein’s general relativity, the proposed 3 + 1 TEGR equations have yet to be carefully tested in a computational code and have yet to be proved to pass all the standard benchmark tests of GR. Therefore, further research will concern implementation of the TEGR equations in a high-order discontinuous Galerkin code [18, 12], with a possibility of constraint cleaning [17] and well-balancing [24]. This in particular would allow a direct comparison of the TEGR with other 3 + 1 equations of GR, such as Z4 formulations [5] forwarded by Bona et al in [5], and FO-CCZ4 by Dumbser et al [19], a strongly hyperbolic formulations of GR, within the same computational code. Another numerical strategy would be to use staggered grids and to develop a structure-preserving
discretization \[10, 41, 25\] that should allow to keep errors of div and curl-type involutional constraints of TGR at the machine precision.

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A TRANSFORMATION OF NOETHER’S CURRENT \[J^\mu_a\]

Here, we express Noether’s current \[J^\mu_a \equiv L_{b\nu} \text{ for the gravitational part of the Lagrangian (i.e. the matter part is ignored in this section) in terms of the potential } \mathcal{U} \text{ and new variables } D^\mu_a \text{ and } B^{\mu
u}.

Thus, for the parametrization \[L(h_{\mu\nu}, \hat{\tau}^{\mu\nu}) = \mathcal{L}(h^{\mu\nu}, B^{\mu\nu}, E_{\mu\nu}^a),\] one has

\[
L_{b\nu} = L_{h\nu} + \mathcal{L}_{B^{\mu\nu}} \frac{\partial B^{\mu\lambda}}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\mu\nu}^a} \frac{\partial E_{\mu\nu}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\lambda}^a} \frac{\partial E_{\lambda}^a}{\partial h^{\nu\lambda}}. \tag{113}
\]

Then, using the definitions of the frame 4-velocity \[u^\nu = \eta^\nu_{\lambda} u_\lambda\] and \[u_\nu = -\hat{h}_\nu \] and the torsion fields \[b_{\mu\lambda} \equiv \hat{T}_{\mu\lambda\nu} u_\nu = -\hat{T}_{\mu\lambda\nu h} u_\nu\] and \[E_{\lambda}^a = \lambda_{\lambda}^a u^\nu = -\frac{1}{2} \epsilon_{\lambda\nu\sigma} \hat{T}^\mu_{\nu\sigma} \] and the fact that \[\partial \eta^\nu_{\lambda}/\partial h^{\nu\lambda} = -\eta^\nu_{\lambda} \] we can rewrite (113) as

\[
L_{b\nu} = L_{h\nu} + \mathcal{L}_{B^{\mu\nu}} \frac{\partial B^{\mu\lambda}}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\mu\nu}^a} \frac{\partial E_{\mu\nu}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\lambda}^a} \frac{\partial E_{\lambda}^a}{\partial h^{\nu\lambda}} = \mathcal{L}_{E_{\mu\nu}^a} \frac{\partial E_{\mu\nu}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\lambda}^a} \frac{\partial E_{\lambda}^a}{\partial h^{\nu\lambda}}
\]

\[
- \mathcal{L}_{E_{\lambda}^a} (u_\lambda E^{\nu}_{\nu} - u_\nu E^{\lambda}_{\lambda} + \epsilon_{\lambda\nu\sigma} u_\sigma b^{\mu\nu}) - u_\nu \eta^\nu_{\lambda} \frac{\partial b^{\mu\nu}}{\partial h^{\nu\lambda}}. \tag{114}
\]

Using the definitions (38) and (39), the latter can be rewritten as

\[
L_{b\nu} = -\mathcal{L}_{E_{\mu\nu}^a} \frac{\partial E_{\mu\nu}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\lambda}^a} \frac{\partial E_{\lambda}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\nu}^a} \frac{\partial E_{\nu}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\lambda}^a} \frac{\partial E_{\lambda}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\nu}^a} \frac{\partial E_{\nu}^a}{\partial h^{\nu\lambda}}
\]

\[
- \mathcal{L}_{E_{\lambda}^a} (u_\lambda E^{\nu}_{\nu} + u_\nu E^{\lambda}_{\lambda} - \epsilon_{\lambda\nu\sigma} u_\sigma b^{\mu\nu}) - u_\nu \eta^\nu_{\lambda} \frac{\partial b^{\mu\nu}}{\partial h^{\nu\lambda}}. \tag{115}
\]

B EXPRESSION FOR THE ENERGY–MOMENTUM

In this section, we derive expression (44) for the gravitational part of the energy-momentum

\[
\sigma^\mu_v = 2\hat{T}^{\mu\lambda\nu} L_{\tau\lambda\nu} - (\hat{T}^{\mu\nu} L_{\tau\nu\lambda} - L) \sigma^\mu_v \tag{118}
\]

Because \[\hat{T}^{\mu\nu}\] is antisymmetric tensor, to compute the derivative \[L_{\tau\lambda\nu}\] one needs to use its parametrization via the Weitzenböck connection, which is not symmetric, i.e. \[\hat{T}^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} W_{\rho\sigma}.\] Thus, for Lagrangians \[L(h_{\mu\nu}, W_{\mu\nu}) = L(h_{\mu\nu}, \hat{T}^{\mu\nu})\] one can write

\[
\Lambda_{W_{\mu\nu}} = \epsilon_{\mu\nu\rho\sigma} W_{\rho\sigma} \tag{119}
\]

or using the identity \[\epsilon_{\mu\nu\rho\sigma} W_{\rho\sigma} = -2(\delta^\sigma_{\mu} \delta^\rho_{\nu} - \delta^\rho_{\mu} \delta^\sigma_{\nu})\]

\[
\epsilon_{\mu\nu\rho\sigma} W_{\nu\rho} = -4L_{\tau\nu\rho}. \tag{120}
\]

On the other hand, using the definitions \[E^\mu_{\mu} = (W_{\mu\nu} - W_{\nu\mu}) u^\nu, B^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} W_{\rho\sigma},\] and the parametrization \[L(h_{\mu\nu}, W_{\mu\nu}) = \mathcal{L}(h^\mu_{\nu}, B^{\mu\nu}, E^\mu_{\mu}, L_{\tau\nu\rho})\] one can write

\[
\Lambda_{W_{\mu\nu}} = \mathcal{L}_{E_{\nu}^a} \frac{\partial E_{\nu}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{B^{\mu\nu}} \frac{\partial B^{\mu\nu}}{\partial h^{\nu\lambda}} = u_\nu \mathcal{L}_{E_{\nu}^a} - \mathcal{L}_{E_{\nu}^a} \frac{\partial E_{\nu}^a}{\partial h^{\nu\lambda}} + \mathcal{L}_{E_{\nu}^a} \frac{\partial E_{\nu}^a}{\partial h^{\nu\lambda}}. \tag{121}
\]
and hence, from (120) and (121), one can deduce
\[
L_{\tilde{T}_{\lambda\rho}} = \frac{1}{2} \left( u Lam \mathcal{B}_{\nu} - u \mathcal{B}_{\lambda \rho} - \varepsilon_{\lambda \rho \nu \sigma} \mathcal{U}_{\sigma} \right). \tag{122}
\]
Then, after contracting the later equation with $\tilde{T}^{\lambda \rho}$, one obtains
\[
2 \hat{T}_{\lambda \rho} L_{\tilde{T}_{\lambda\rho}} = \mathcal{B}_{\lambda \rho} \mathcal{B}_{\nu} - \mathcal{E}_{\lambda \rho} \mathcal{E}_{\nu}. \tag{123}
\]
This can be used to demonstrate that the full contraction $\tilde{T}_{\lambda \rho} L_{\tilde{T}_{\lambda\rho}}$ results in
\[
\tilde{T}_{\lambda \rho} L_{\tilde{T}_{\lambda\rho}} = \mathcal{B}_{\lambda \rho} \mathcal{B}_{\nu} + \mathcal{E}_{\lambda \rho} \mathcal{E}_{\nu}. \tag{124}
\]
Collecting together (123) and (124) and using the change of variables and potential (37)–(39), we arrive at
\[
2 \hat{T}_{\lambda \rho} L_{\tilde{T}_{\lambda\rho}} - \left( \hat{T}_{\lambda \rho} L_{\tilde{T}_{\lambda\rho}} - L \delta_{\lambda \rho} \right) = - \mathcal{B}_{\lambda \rho} \mathcal{B}_{\nu} - \mathcal{E}_{\lambda \rho} \mathcal{E}_{\nu}.
\]
are Hodge duals of each other:
\[
\tilde{T}_{\lambda \rho} := \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \mathcal{T}_{\mu \nu}, \tag{129a}
\]
\[
\mathcal{T}_{\mu \nu} := \frac{1}{2} \varepsilon_{\rho \sigma \mu \nu} \tilde{T}_{\rho \sigma}. \tag{129b}
\]
Therefore, one can write the following identities
\[
\tilde{T}_{\lambda \rho} = \varepsilon_{\mu \nu \rho \sigma} \mathcal{H}_{\mu \nu} - \varepsilon_{\mu \nu \rho \sigma} \mathcal{T}_{\mu \nu} \mathcal{H}_{\rho \sigma}, \tag{130a}
\]
\[
\mathcal{T}_{\mu \nu} = \varepsilon_{\mu \nu \rho \sigma} \mathcal{H}_{\rho \sigma} - \varepsilon_{\mu \nu \rho \sigma} \mathcal{T}_{\rho \sigma} \mathcal{H}_{\mu \nu}, \tag{130b}
\]
where
\[
\mathcal{H}_{\mu \nu} := \tilde{T}_{\mu \nu} \varepsilon, \quad \mathcal{D}_{\mu} := \mathcal{T}_{\mu \nu} \varepsilon_{\nu}, \tag{131}
\]
Hence, the Euler-Lagrange equation (126) now reads
\[
\partial_{\nu} \mathcal{T}_{\mu \nu} = - \frac{1}{2} L \delta_{\mu \nu}, \tag{132}
\]
or, according to (130b), it can be written as
\[
\partial_{\nu} (\varepsilon_{\mu \nu \rho \sigma} \mathcal{H}_{\rho \sigma} - \varepsilon_{\mu \nu \rho \sigma} \mathcal{T}_{\rho \sigma} \mathcal{H}_{\mu \nu}) = - \frac{1}{2} L \delta_{\mu \nu}. \tag{133}
\]
It remains to express $\mathcal{D}_{\mu}$ and $\mathcal{H}_{\mu \nu}$ in terms of $\mathcal{L}(\mathcal{B}_{\mu \nu}, \mathcal{E}_{\mu \nu})$. Thus, using the fact that $\Lambda_{\nu \mu \rho} = 2 \Lambda_{\mu \nu \rho} \mathcal{E}_{\mu \nu}$ and $\Lambda_{\nu \rho \nu} = \varepsilon_{\mu \nu \rho \sigma} \mathcal{H}_{\mu \nu}$, and the expression (121), one can derive that
\[
\mathcal{H}_{\mu \nu} = - \frac{1}{2} \left( \mathcal{L}_{\mu \nu} + \mathcal{L}_{\nu \mu} \delta_{\mu \nu} \right), \tag{134a}
\]
\[
\mathcal{D}_{\mu} = - \frac{1}{2} \left( \mathcal{L}_{\nu \mu} + \mathcal{L}_{\mu \nu} \delta_{\mu \nu} \right). \tag{134b}
\]
Finally, plugging these expressions in (133), one obtains the desired result
\[
\partial_{\nu} (\varepsilon_{\mu \nu \rho \sigma} \mathcal{H}_{\rho \sigma} - \varepsilon_{\mu \nu \rho \sigma} \mathcal{T}_{\rho \sigma} \mathcal{H}_{\mu \nu}) = L \delta_{\mu \nu}. \tag{135}
\]

\section{D SOME EXPRESSIONS OF THE TORSION SCALAR}

Denoting the scalars in the right-hand side of (108b) as $(T = T_{1} + T_{2} + T_{3})$
\[
T_{1} = \frac{1}{2} k_{a} k_{b} k_{c} \frac{\partial \mathcal{L}_{\mu \nu \rho \sigma}}{\partial k_{d}} \frac{\partial k_{e}}{\partial k_{f}}, \tag{136a}
\]
\[
T_{2} = \frac{1}{2} k_{a} k_{b} k_{c} \frac{\partial \mathcal{L}_{\mu \nu \rho \sigma}}{\partial k_{d}} \frac{\partial k_{e}}{\partial k_{f}}, \tag{136b}
\]
we can also write
\[
T_{3} = - \frac{1}{2} \varepsilon_{\nu \rho \sigma} \frac{\partial \mathcal{L}_{\mu \nu \rho \sigma}}{\partial k_{e}} \frac{\partial k_{e}}{\partial k_{f}} \tag{136c}
\]
\[
T_{1} = Q_{1} + \frac{1}{2} C_{4}, \tag{137a}
\]
\[
T_{2} = - Q_{1} + Q_{4} + L_{1} - C_{1} + C_{2} + \frac{1}{2} C_{4}, \tag{137b}
\]
\[
T_{3} = 2 Q_{1} + Q_{2} + L_{2} + 2 C_{1} + C_{3} - C_{4}. \tag{137c}
\]
where the scalars \( Q, L, \) and \( C \) are scalars made of \( E^i_\mu \) and \( B^a_\mu \) as follows.

**Quadratic in \( E^a_\mu \):**

\[
Q_1 = -\frac{1}{2} E^a_\mu S^a_\rho E^\rho_\mu \quad (138a)
\]

\[
Q_2 = a^a_\alpha E^a_\rho \eta^\rho_\beta E^\beta_\beta = a^a_\alpha E^\beta_\beta \quad (138b)
\]

\[
Q_3 = -\frac{1}{2} a_{ab} E^a_\alpha E^b_\beta E^\mu_\mu \quad (138c)
\]

\[
Q_4 = -\frac{1}{2} h^\lambda_\rho E^a_\rho \eta^\beta_\beta h^b_\lambda E^\lambda_\beta = E^a_\beta E^\beta_\lambda \quad (138d)
\]

**Mixed scalars (linear in \( E^a_\mu \)):**

\[
L_1 = \varepsilon_{\lambda \tau \rho \eta} \eta^a_\alpha E^\lambda_\beta E^\rho_\rho \quad (138e)
\]

\[
L_2 = 2 \varepsilon_{\lambda \tau \rho \eta} \eta^a_\alpha E^\lambda_\beta E^\rho_\rho \quad (138f)
\]

**Quadratic in \( B^a_\mu \) (constant in \( E^a_\mu \)):**

\[
C_1 = -\frac{1}{2} h^{-2} g_{\lambda \mu} B^a_\lambda B^a_\mu \quad (138g)
\]

\[
C_2 = -\frac{1}{2} h^{-2} g_{\lambda \mu} \eta^a_\alpha B^a_\lambda \eta^a_\beta B^a_\mu \quad (138h)
\]

\[
C_3 = h^{-2} \eta^a_\alpha B^a_\nu g_{\nu \lambda} h^a_\beta B^a_\lambda \quad (138i)
\]

\[
C_4 = h^{-2} \eta^a_\alpha B^a_\nu g_{\nu \lambda} h^a_\beta B^a_\lambda \quad (138j)
\]

In terms of the Hodge dual \( \hat{\nu}^{\rho \mu} \), the torsion scalar \( T \) can be rewritten as

\[
T = H_1 + H_2 + H_3 + H_4 \quad (139a)
\]

where

\[
H_1 = -\frac{1}{2h^2} \varepsilon_{\alpha \beta \gamma} h^a_\alpha \rho \lambda \epsilon d \varepsilon_{\tau \rho} \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (139b)
\]

\[
H_2 = -\frac{1}{2h^2} \varepsilon_{\alpha \beta \gamma} h^a_\beta \epsilon d \varepsilon_{\tau \rho} \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (139c)
\]

\[
H_3 = -\frac{1}{4h^2} \varepsilon_{\alpha \beta \gamma} h^a_\rho \epsilon d \varepsilon_{\tau \rho} \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (139d)
\]

\[
H_4 = -\frac{1}{4h^2} \varepsilon_{\alpha \beta \gamma} h^a_\rho \epsilon d \varepsilon_{\tau \rho} \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (139e)
\]

Finally, in terms of the Weitzenböck connection, the torsion scalar reads:

\[
T = \sum_{n=1}^{8} \mathcal{W}_n \quad (140a)
\]

\[
\mathcal{W}_1 = \frac{1}{2} \varepsilon_{\alpha a} h^a_\alpha \varepsilon d \varepsilon_{\gamma \beta \lambda} \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140b)
\]

\[
\mathcal{W}_2 = -\frac{1}{2} \varepsilon_{\alpha a} h^a_\alpha \varepsilon d \varepsilon_{\gamma \beta \lambda} \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140c)
\]

\[
\mathcal{W}_3 = \frac{1}{2} S^{a \gamma \eta \rho} \eta^a_\alpha \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140d)
\]

\[
\mathcal{W}_4 = \frac{1}{2} S^{a \gamma \lambda} \eta^a_\alpha \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140e)
\]

\[
\mathcal{W}_5 = -S^{a \gamma \lambda} \eta^a_\alpha \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140f)
\]

\[
\mathcal{W}_6 = -S^{a \gamma \lambda} \eta^a_\alpha \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140g)
\]

\[
\mathcal{W}_7 = -S^{a \gamma \lambda} \eta^a_\alpha \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140h)
\]

\[
\mathcal{W}_8 = 2S^{a \gamma \lambda} \eta^a_\alpha \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \quad (140i)
\]

where \( \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} = \eta^a_\alpha \hat{\nu}^{\rho \tau \gamma} \hat{\nu}^{\beta \lambda \tau} \).
