ON THE HOMOGENIZATION OF SECOND ORDER LEVEL SET PDE IN PERIODIC MEDIA

PETER S. MORFE

Abstract. This paper analyzes two classes of second order level set PDE in periodic media in the parabolic scaling. First, we study fully nonlinear geometric operators under general assumptions in dimension $d = 2$ and prove that the associated equations homogenize in this case. Next, we treat a class of quasi-linear geometric operators in arbitrary dimensions $d \geq 2$. In this setting, by adapting arguments form the study of oscillating boundary value problems, we prove that the effective coefficients are generically discontinuous in all dimensions $d \geq 3$. This necessitates a study of level set PDE driven by operators that are discontinuous at every rational direction on the sphere. We prove that, in fact, the effective operators so obtained do have a comparison principle and, thus, homogenization occurs. Finally, we investigate the connection between the effective mobility obtained in the quasi-linear case and linear response, drawing a connection between our results and those obtained in the hyperbolic scaling.

1. Introduction

In this paper, we are interested in the behavior, as $\epsilon \to 0^+$, of the solutions of the parabolically scaled, level set PDE

$$
\begin{cases}
u_t^\epsilon - F(Du^\epsilon, D^2u^\epsilon, \epsilon^{-1}x) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\qquad u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}.
\end{cases}
$$

Here $F$ is a spatially periodic, geometric operator that has the same ellipticity properties as the mean curvature operator. (See Section 2.1 for precise assumptions).

When $d = 2$ and $F$ is a general geometric operator, we prove that there is an effective nonlinearity $\overline{F} = \overline{F}(p, X)$ such that the solutions $(u^\epsilon)_{\epsilon > 0}$ of (1) converge as $\epsilon \to 0^+$ to the solution $\bar{u}$ of the effective equation

$$
\begin{cases}
\bar{u}_t - \overline{F}(D\bar{u}, D^2\bar{u}) = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
\qquad \bar{u} = u_0 & \text{on } \mathbb{R}^2 \times \{0\}.
\end{cases}
$$

In this case, due to the simpler geometry when $d = 2$, $\overline{F}$ is continuous in $(\mathbb{R}^2 \setminus \{0\}) \times S_2$.

In dimensions $d \geq 2$, we prove the homogenization of quasi-linear level set PDE with regular coefficients. Specifically, these equations take the form

$$
\begin{cases}
m(\epsilon^{-1}x, \bar{D}u^\epsilon)u_{t}^\epsilon - \text{tr} \left( A(\epsilon^{-1}x, \bar{D}u^\epsilon) D^2u^\epsilon \right) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\qquad u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}.
\end{cases}
$$


\textbf{Date:} November 4, 2021.
Here \( m : \mathbb{T}^d \times S^{d-1} \to (0, \infty) \) is a mobility coefficient and \( A \) is obtained from a uniformly elliptic matrix field \( a : \mathbb{T}^d \times S^{d-1} \to \mathcal{S}_d \) through the following relation:

\[
A(y, e) = (\text{Id} - e \otimes e)a(y, e)(\text{Id} - e \otimes e).
\]

Though the mobility \( m \) can be incorporated into \( A \), we will see below that it is illuminating to write the equation in this form.

We prove that when \( m \) and \( a \) are regular enough, there are effective coefficients \( \overline{m} \) and \( \overline{a} \) such that if \( \overline{A} \) relates to \( \overline{a} \) as in (3), then the solutions \( (u^\epsilon)_{\epsilon>0} \) converge as \( \epsilon \to 0^+ \) to the solution \( \overline{u} \) of the effective equation

\[
\overline{m}(\widehat{D}\overline{u})\overline{u}_t - \text{tr} \left( \overline{A}(\widehat{D}\overline{u})D^2\overline{u} \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty).
\]

In dimensions \( d \geq 3 \), we prove that the effective coefficients \( \overline{a} \) and \( \overline{m} \) are generically discontinuous at normal directions \( e \in \mathbb{R} \mathbb{Z}_d \), so-called rational directions. Therefore, in order to make sense of (4), we need to study the well-posedness of second order level set PDE driven by operators that are discontinuous at countably many directions on the sphere. We prove that the comparison principle does indeed extend to equations like (4) and use this to conclude that homogenization occurs.

Finally, we show that, in some cases, the effective mobility \( \overline{m} \) describes the linear response of (2), precisely as in the work of Spohn [55] and Katsoulakis and Souganidis [40], [41]. Further, when \( d = 2 \), this allows us to relate \( \overline{m} \) to the front speeds obtained in the hyperbolic scaling by Caffarelli and Monneau [14]. Again, there are pathologies in rational directions that complicate the picture.

We expect that these results can be generalized to (1) with fully nonlinear, geometric operators \( F \) in arbitrary dimensions \( d \geq 3 \). Toward that end, it will be necessary to overcome a number of difficulties that arise when the correctors used in the asymptotic analysis are no longer \( C^2 \).

1.1. Literature review. The level-set method for describing geometric flows was introduced by Ohta, Jasnow, and Kawasaki [48], Sethian [53], and Osher and Sethian [49]. The first rigorous description of the method was developed by Barles [3] for a first order model of flame propagation and, in the second order setting, by Evans and Spruck [25] and Chen, Giga, and Goto [20]. See the papers of Barles, Soner, and Souganidis [8] and Barles and Souganidis [9] for the state of the art.

Much of the literature on the homogenization of second order level set PDE treats the hyperbolic scaling, where the aim is to obtain a first order motion in the limit. For periodic media, we refer the reader to the papers by Lions and Souganidis [42], Cardaliaguet, Lions, and Souganidis [18], Caffarelli and Monneau [14], and Gao and Kim [32] and the references therein. The random setting has been treated by Armstrong and Cardaliaguet [1] and Feldman [28].

Fewer results are available in the parabolic scaling. Novaga and Valdinoci [43] proved a pinning result for mean curvature with periodic forcing in this scaling, showing that pinning is generic for forcing terms in the \( L^1(\mathbb{T}^d) \) topology and relating this phenomenon to a prescribed curvature problem. Barles, Cesaroni, and Novaga
prove homogenization of a similar problem when the forcing is laminar and the initial datum is a graph over a suitable direction.

A number of works have treated parabolically scaled geometric motions subject to small forcings in the same spirit as equation (22) below. Cesaroni, Novaga, and Valdinoci [19] proved homogenization of plane-like solutions of a forced mean curvature equation closely related to (22), obtaining homogenized speeds that are discontinuous with respect to the direction. That built on the work of Craciun and Bhattacharya [22], who had initially studied the homogenized speeds in question, and is related to a paper by Chen and Lou [21], who found planar and V-shaped solutions of equations like (2) and (22) in dimension two and with almost periodic forcings.

During the preparation of this paper, the author learned that a number of the ideas presented here were anticipated by Craciun and Bhattacharya in their study of periodically forced mean curvature flow. Compare the proofs of Theorem 8 and Proposition 16 below to the formal analysis of [22, Proposition 6].

This paper draws on a number of technical advances in the theory of viscosity solutions. First of all, our approach draws a novel connection between the homogenization of second order level set PDE and that of uniformly elliptic equations in nondivergence form. Throughout the paper, we study the asymptotics of (1) using a degenerate elliptic cell problem in the torus $T^d$. As shown in Section 4 below, this cell problem leads to the analysis of a family of uniformly elliptic equations with almost periodic coefficients, which falls under the purview of the papers by Caffarelli, Souganidis, and Wang [16] and Caffarelli and Souganidis [15]. In the study of the discontinuity properties of the homogenized coefficients, in particular, we make extensive use of the obstacle problem formulation of [16].

Secondly, where the discontinuity of the effective coefficients is concerned, our approach is inspired by recent developments in the study of oscillating boundary value problems. A non-exhaustive list of references in this area includes the papers by Barles, Da Lio, and Souganidis [5], Barles and Mironescu [7], Gérard-Varet and Masmoudi [34], Choi and Kim [CK], Feldman [27], Feldman and Kim [29], Feldman and Zhang [31], and Feldman, Kim, and Souganidis [30]. This paper adapts to the setting of geometric flows the approach of [29].

Finally, the paper builds on studies of level set PDE with discontinuous coefficients by Gurtin, Soner, and Souganidis [36], Ohnuma and Sato [47], and Ishii [37]. The major difference between our approach and theirs is the effective equations obtained in higher dimensions are generically discontinuous at a countably infinite set of directions on the sphere. Nonetheless, by exploiting the particular properties of the effective operators, we prove comparison by adapting the arguments of [37].

1.2. Organization of the Paper. In the next section, the main results of the paper are stated and the proofs are sketched. Section 4 is devoted to the analysis of the degenerate cell problem used throughout the paper. In Section 5, these correctors are used to show that the homogenized motion has the desired behavior whenever its normal vector is irrational, and we explain how to extend this to arbitrary normal
directions. Section 6 treats the continuity properties of the homogenized operator, including the proof that discontinuity in rational directions is generic. Section 7 extends the comparison principle to a class of level set PDE with discontinuous coefficients that includes the effective equations obtained in Section 5. Finally, Section 8 studies the effective mobility of the quasi-linear problem from the point of view of linear response.

A number of technical results needed in the paper are provided in the Appendix A, while Appendix B specifically treats a geometric construction used in the proof of comparison.

1.3. Notation. The Euclidean inner product in $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$; $\| \cdot \|$ is its associated norm. $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$, that is, those vectors $e$ with $\|e\| = 1$. We write $\langle e \rangle$ for the real span of $e$ and $\langle e \rangle^\perp$ for its orthogonal complement.

Given $p \in \mathbb{R}^d \setminus \{0\}$, we write $\hat{p} = \|p\|^{-1}p$.

$S_d$ is the space of real $d \times d$ symmetric matrices. Given $e \in S_1^d$, $e \otimes e$ is the orthogonal projection onto $\langle e \rangle$.

Given such an $e$ and an $X \in S_d$, we define the matrix $\tilde{X}_e$ by $\tilde{X}_e = (\text{Id} - e \otimes e)X(\text{Id} - e \otimes e)$.

$T^d$ is the $d$-dimensional torus, that is, the quotient space obtained from $\mathbb{R}^d$ by identifying points that differ by an element of the integers $\mathbb{Z}^d$. $\mathbb{R}\mathbb{Z}^d$ denotes the real span of $\mathbb{Z}^d$.

Given an $e \in S^{d-1}$, the differential operator $D^2_e$ is defined via its action on smooth functions $\varphi$ by $D^2_e \varphi = (\text{Id} - e \otimes e)D^2 \varphi(\text{Id} - e \otimes e)$.

$L^d$ and $H^{d-1}$ are the Lebesgue measure and $(d-1)$-dimensional Hausdorff measure in $T^d$, respectively, the latter normalized to coincide with surface area.

2. Main Results

2.1. Homogenization for $d = 2$. In dimension two, we consider operators $F : (\mathbb{R}^2 \setminus \{0\}) \times S_2 \times T^2 \to \mathbb{R}$ satisfying the following assumptions:

(i) (Geometric) If $\nu, \mu \in \mathbb{R}$ and $\nu > 0$, then

$$F(\nu p, \nu X + \mu p \otimes p, y) = \nu F(p, X, y) \quad \text{for} \quad (p, X, y) \in \mathbb{R}^2 \times S_2 \times T^2.$$  

(ii) (Stationary planes) For each $e \in S^1$,

$$F(e, 0, y) = 0 \quad \text{for} \quad y \in T^2.$$  

(iii) (Uniform degenerate ellipticity) There are constants $\lambda, \Lambda > 0$ such that if $(e, X, y) \in S^1 \times S_2 \times T^2$, $Y \in S_2$ satisfies $Y \geq 0$, and $\check{Y}_e$ is the matrix $\check{Y}_e = (\text{Id} - e \otimes e)Y(\text{Id} - e \otimes e)$, then

$$\lambda \|\check{Y}_e\| \leq F(e, X + Y, y) - F(e, X, y) \leq \Lambda \|\check{Y}_e\|.$$  

(iv) (Regularity) $F$ is continuous in $(\mathbb{R}^2 \setminus \{0\}) \times S_2 \times T^2$.

(v) (Comparison) $F$ satisfies a technical assumption ensuring (1) is well-posed. (This assumption is stated precisely in the appendix.)
**Theorem 1.** If $F$ satisfies assumptions (i)-(v) above, then there is a continuous function $\overline{F} : (\mathbb{R}^2 \setminus \{0\}) \times \mathcal{S}_2 \to \mathbb{R}$ satisfying the same assumptions such that if $u_0 \in UC(\mathbb{R}^2)$, $(u^\epsilon)_\epsilon>0$ are the unique viscosity solutions of (1), and $\bar{u} : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is the solution of the effective equation

\[
\begin{cases}
\bar{u}_t - \overline{F}(D\bar{u}, D^2\bar{u}) = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
\bar{u} = u_0 & \text{on } \mathbb{R}^2 \times \{0\},
\end{cases}
\]

then $u^\epsilon \to \bar{u}$ locally uniformly in $\mathbb{R}^2 \times [0, \infty)$.

Well-posedness of (1) is reviewed in Appendix A.1 below.

2.2. Homogenization of Quasi-linear Geometric Operators in $d \geq 2$. We now state the results concerning the homogenization of (2). Here and henceforth we fix $\lambda, \Lambda > 0$ and define the space $\mathcal{S}_d(\lambda, \Lambda)$ by

\[
\mathcal{S}_d(\lambda, \Lambda) = \{ a_0 \in \mathcal{S}_d \mid \lambda \text{Id} < a_0 < \Lambda \text{Id} \}.
\]

The matrix field $a : \mathbb{T}^d \times S^{d-1} \to \mathcal{S}_d$ is assumed to satisfy the following:

\[
a : \mathbb{T}^d \times S^{d-1} \to \mathcal{S}_d(\lambda, \Lambda) \text{ is continuous,}
\]

\[
\|a(\cdot, e)\|_{C^2(\mathbb{T}^d)} < \infty \quad \text{if } e \in S^{d-1},
\]

\[
\sup \left\{ \frac{|a(y, e) - a(y', e)|}{\|y - y'\|} \mid (y, y), (e, y') \in S^{d-1} \times \mathbb{T}^d, y' \neq y \right\} < \infty.
\]

Concerning the mobility $m : \mathbb{T}^d \times S^{d-1} \to (0, \infty)$, the assumptions are listed next:

\[
m(\cdot, e) \in C^2(\mathbb{T}^d) \quad \text{if } e \in S^{d-1},
\]

\[
\sup \left\{ \frac{|m(y, e) - m(y', e)|}{\|y - y'\|} \mid (y, y), (y', e) \in \mathbb{T}^d \times S^{d-1}, y' \neq y \right\} < \infty.
\]

The main result in this setting is:

**Theorem 2.** If $a$ satisfies assumptions (9), (10), and (11) and $m$ satisfies (12), (13), and (14), then there are effective coefficients $\overline{m} : S^{d-1} \setminus \mathbb{R}Z^d \to (0, \infty)$ and $\overline{a} : S^{d-1} \setminus \mathbb{R}Z^d \to \mathcal{S}_d$ such that if $\overline{F} : (\mathbb{R}^d \setminus \mathbb{R}Z^d) \times \mathcal{S}_d \to \mathbb{R}$ is given by

\[
\overline{F}(p, X) = \overline{m}(\hat{p})^{-1} \text{tr} \left( \overline{a}(\hat{p}) \hat{p} \right),
\]

$u_0 \in UC(\mathbb{R}^d)$, and $(u^\epsilon)_\epsilon>0$ are the solutions of (2), then:

(i) There is a unique viscosity solution $\bar{u} : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ of the equation

\[
\begin{cases}
\bar{u}_t - \overline{F}(D\bar{u}, D^2\bar{u}) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\bar{u} = u_0 & \text{on } \mathbb{R}^d \times \{0\}.
\end{cases}
\]

(ii) $u^\epsilon \to \bar{u}$ locally uniformly as $\epsilon \to 0^+$.

Well-posedness of (2) is reviewed in Appendix A.1. A few remarks are in order:
Remark 1. (i) Even though we have imposed considerable regularity restrictions on the matrix field \(a\), nonetheless we show below that \(F\) generically fails to admit a continuous extension to \((\mathbb{R}^d \setminus \{0\}) \times S_d\). We expect this is true more generally (i.e. fully non-linear operators) whenever \(d \geq 3\).

(ii) In certain special cases, \(F\) does extend to a continuous function. Examples of this are discussed in Section 3.1 below.

(iii) The discontinuity of \(F\) means that the usual comparison principle does not apply to (16). This is rectified in Section 7.

Remark 2. The smoothness assumptions on \(m\) and \(a\) are certainly restrictive. As we will see below, these assumptions are natural in light of the degeneracy of the cell problems used in the asymptotic analysis. At various places in the arguments, it is not clear how to proceed unless the corrector is \(C^2\), and requiring the coefficients to be \(C^2\) is a natural way to achieve this.

Nonetheless, when \(d = 2\), we are able to obtain \(C^2\) approximate correctors by regularizing the coefficients and exploiting the special structure of the cell problem in this dimension. It is for this reason that Theorem 1 is more general.

We expect that, using the same techniques, Theorem 2 could be extended to convex or concave operators that are twice continuously differentiable in the \((X, y)\) variables, but this is not so interesting since the main examples we have in mind are one-homogeneous in the Hessian and, therefore, not differentiable at the origin.

2.3. (Dis)continuity of the Homogenized Coefficients. The effective operator \(F\) in Theorem 2 is determined by the invariant probability measures associated with a certain family of diffusion processes on the torus. Toward that end, we define the sets of invariant measures \(\{I^a_e\}_{e \in S^{d-1}} \subseteq \mathcal{P}(T^d)\) so that \(\mu \in I^a_e\) if and only if, for each \(\varphi \in C^\infty(T^d)\), writing \(D^2_e \varphi = \langle Id - e \otimes e, D^2 \varphi \rangle (Id - e \otimes e)\), we have

\[
\int_{T^d} \text{tr} \left( a(y, e) D^2_e \varphi(y) \right) \mu(dy) = 0.
\]

For a given \(e \in S^{d-1}\), these are precisely the invariant probability measures of the diffusion process \(X^e\) on \(T^d\) determined by the SDE

\[
dX^e_t = (Id - e \otimes e) \sqrt{a(X^e_t, e)}(Id - e \otimes e) dB_t.
\]

The basic structure of the sets \(\{I^a_e\}_{e \in S^{d-1}}\) and their relationship to the homogenized coefficients \(\bar{a}\) and \(\bar{m}\) is summed up in the following theorem. In the rational case, the foliation of \(T^d\) into sub-tori normal to \(e\) arises naturally. That is, we will use the decomposition of \(T^d\) as

\[T^d = \bigcup_{r \in [0, r_e)} T^{d-1}_e(r),\]

where the sub-tori \(\{T^{d-1}_e(r)\}_{r \in [0, r_e)}\) and the period \(r_e > 0\) are defined by

\[
T^{d-1}_e(r) = \{ y \in T^d \mid \langle y, e \rangle = r + \langle k, e \rangle \text{ for some } k \in \mathbb{Z}^d \}, \quad r_e = \min \{ \langle k, e \rangle \mid k \in \mathbb{Z}^d \} \cap (0, \infty).
\]
When $e$ is rational, the trajectories of the process $X^e$ of (17) remain confined to $\mathbb{T}^{d-1}_e((X^0_0, e))$ and this leads to a corresponding decomposition of $\mathcal{E}_e$.

Now we are prepared to state the main result concerning the structure of the invariant measures and their relation to the effective coefficients $\tilde{\mu}$ and $\overline{m}$:

**Theorem 3.** Under the assumptions of Theorem 2 we have:

(i) If $e \notin \mathbb{R}Z^d$, then there is a unique probability measure $\mu_e$ such that $\mathcal{I}_e = \{\mu_e\}$. Furthermore, $\mu_e \ll \mathcal{L}$.

(ii) If $e \in \mathbb{R}Z^d$, then there is an $r_e$-periodic function $\mu_e : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{T}^d)$, $\mu_e : s \mapsto \mu^s_e$, such that $\mathcal{I}_e$ equals the closed convex hull of $\{\mu^s_e | s \in \mathbb{R}\}$. For each $s \in [0, r_e)$, we have $\mu^s_e \ll \mathcal{H}^{d-1} |_{T^d_e(r)}$.

(iii) For each $e \in S^{d-1} \setminus \mathbb{R}Z^d$, $\tilde{\mu}$ and $\overline{m}$ are given by

$$\overline{m}(e) = \int_{\mathbb{T}^d} m(y, e) \mu_e(dy), \quad \overline{m}(e) = \int_{\mathbb{T}^d} m(y, e) \mu_e(dy).$$

In order to prove Theorem 2 we will study the continuity properties of the set-valued maps $e \mapsto \mathcal{I}_e$. When $e \notin \mathbb{R}Z^d$, continuity at $e$ is implied directly by uniqueness and compactness. The continuity question is significantly more complicated when $e \in \mathbb{R}Z^d$. The next result addresses that issue.

First, we need a digression on analysis in $S^{d-1}$ because it turns out that the limiting behavior of $e \mapsto \mathcal{I}_e$ at a rational direction depends on the direction of approach. To make that precise, notice that if $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1}$ converges to $e$ as $n \rightarrow \infty$, then there is necessarily a a subsequence $(n_j)_{j \in \mathbb{N}}$ such that

$$-\eta = \lim_{j \rightarrow \infty} \frac{e_{n_j} - e}{\|e_{n_j} - e\|}.$$ 

Geometrically, that means $e_{n_j}$ approximately approaches $e$ along the great circle parametrized by $\theta \mapsto \cos(\theta)e + \sin(\theta)\eta$, or, put simply, $e_{n_j} \rightarrow e$ along the $\eta$ direction. We will see that it is necessary to take account of the direction $\eta$ when studying the continuity properties of $e \mapsto \mathcal{I}_e$.

Finally, in the statement, it is convenient to use the metrizability of $\mathcal{P}(\mathbb{T}^d)$. Toward that end, we fix here and henceforth a metric $D : \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d) \rightarrow [0, \infty)$ inducing the weak-* topology (e.g. Wasserstein distance).

**Theorem 4.** Under the assumptions of Theorem 2 for each $e \in \mathbb{R}Z^d$, if we define the function $\tilde{\mu} : S^{d-1} \cap \langle e \rangle^\perp \rightarrow \mathcal{P}(\mathbb{T}^d)$, $\tilde{\mu} : \eta \mapsto \tilde{\mu}_\eta$, by

$$\tilde{\mu}_\eta = \left(\int_0^{r_e} \langle a^+_e(s\eta)\eta, \eta \rangle^{-1} ds\right)^{-1} \int_0^{r_e} \langle a^+_e(s\eta)\eta, \eta \rangle^{-1} \mu^s_e ds,$$

$$a^+_e(y) = \int_{\mathbb{T}^d} a(y', e)\mu^e(y)(dy'),$$

then this function describes the continuity properties of $e' \mapsto \mathcal{I}_{e'}$ at $e$ in the sense that

$$\lim_{\delta \rightarrow 0^+} \sup_{\mu \in \mathcal{I}_{e'}} \left\{ D(\mu, \tilde{\mu}_\eta) \mid \|e' - e\| + \|e' - e\| + \eta < \delta \right\} = 0.$$
The corollary that follows shows that discontinuity is generic among coefficients $a$ satisfying the assumptions of Theorem 2 if $d \geq 3$.

**Corollary 1.** Assume $d \geq 3$. There is a residual set $\mathcal{C}_d \subseteq C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\Lambda, \Lambda))$ such that if $a$ is independent of the $e$ variable and $a \in \mathcal{C}_d$, then the following statements hold:

(a) If $\bar{\mu}$ is the function defined in Theorem 4, then

$$\forall e \in S^{d-1} \cap \mathbb{R}^d \# \{ \bar{\mu}_e^\eta \mid \eta \in S^{d-1} \cap \langle e \rangle^\perp \} = \infty.$$  

(b) The effective coefficient $\bar{a}: S^{d-1} \setminus \mathbb{R}^d \to \mathcal{S}_d(\Lambda, \Lambda)$ defined in Theorem 3 has infinitely many distinct directional limits at each $e \in \mathbb{R}^d$.

In particular, by taking $a \in \mathcal{C}_d$ and $m \equiv 1$ in Theorem 2, we find that $\bar{F}_*(e, \cdot) \neq \bar{F}_*(e, \cdot)$ for each $e \in S^{d-1} \cap \mathbb{R}^d$.

2.4. Effective Mobility and Linear Response. Finally, we discuss the relationship between the effective mobility $\bar{m}$ in (2) and linear response. Specifically, given $e \in S^{d-1}$ and defining $A$ as in (3), we are interested in the forced motion

$$\begin{cases}
m(\epsilon^1 x, \hat{D}_e u^\epsilon_{e,t}) u^\epsilon_{e,t} - \text{tr} \left( A(\epsilon^{-1} x, \hat{D}_e u^\epsilon_{e}) D^2 u^\epsilon_{e} \right) - \alpha \| Du^\epsilon_{e} \| = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\quad u^\epsilon_{e}(x, 0) = \langle x, e \rangle & \text{if } x \in \mathbb{R}^d.
\end{cases}$$

To start with, we prove homogenization of (22):

**Theorem 5.** Fix $d \geq 2$. If $a$ and $m$ satisfy the assumptions of Theorem 2 then there is an $\bar{m}_{pl}: S^{d-1} \to (0, \infty)$ such that if $e \in S^{d-1}$ and $(u^\epsilon_{e})_{\epsilon > 0}$ are the solutions of (22), then

$$\lim_{\epsilon \to 0^+} u^\epsilon_{e}(x, t) = \langle x, e \rangle + \alpha \bar{m}_{pl}(e)^{-1} t \quad \text{locally uniformly in } \mathbb{R}^d \times [0, \infty).$$

When $d = 2$, $\bar{m}_{pl}^{-1}$ is precisely the derivative of the front speeds obtained by Caffarelli and Monneau [14]. Recall that if $e \in S^1$ and $\alpha \in \mathbb{R}$ and if $u^{e,\alpha}$ is the solution of the forced problem

$$\begin{cases}
m(y, \hat{D}_e u^{e,\alpha}) u^{e,\alpha}_{e,t} - \text{tr} \left( A(y, \hat{D}_e u^{e,\alpha}) D^2 u^{e,\alpha} \right) - \alpha \| Du^{e,\alpha} \| = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
\quad u^{e,\alpha}(x) = \langle x, e \rangle & \text{if } x \in \mathbb{R}^2,
\end{cases}$$

then, by [14], there is a $\lambda_e(\alpha) \in \mathbb{R}$ such that

$$\lim_{R \to \infty} R^{-1} u^{e,\alpha}(Rx, Rt) = \langle x, e \rangle + \lambda_e(\alpha) t \quad \text{locally uniformly in } \mathbb{R}^2 \times [0, \infty).$$

The derivative of this function is $\bar{m}_{pl}^{-1}$:

**Corollary 2.** If $d = 2$ and $e \in S^1$, then $\bar{m}_{pl}(e)^{-1} = \lim_{\alpha \to 0} \alpha^{-1} \lambda_e(\alpha)$.

Notice that, up to a parabolic rescaling, the solution $u^\epsilon_{e}$ of (22) equals $u^{e,\alpha e}$. As discussed by Spohn [55] and Bellettini, Buttà, and Presutti [10], this is precisely the scaling where the forcing and the curvature are evenly matched, and the limiting
behavior of $u'$ should capture the linear response of (2) to small perturbations. In light of previous works, it is natural to expect that $\overline{m}_{pl} = \overline{m}$.

It turns out these functions only coincide in irrational directions, even when $d = 2$. To start with, we obtain an expression for $\overline{m}_{pl}$:

**Theorem 6.** Assume that $d \geq 2$ and $a$ and $m$ satisfy the assumptions of Theorem 1. The following statements hold:

(i) If $e \in S^{d-1} \setminus \mathbb{RZ}^d$, then $\overline{m}_{pl}(e) = \overline{m}(e)$.

(ii) If $e \in S^{d-1} \cap \mathbb{RZ}^d$ and $\mu_e$ is the function defined in Theorem 3, then

\[
\overline{m}_{pl}(e) = r^{-1} \int_0^{r e} \int_{\mathbb{T}^d} m(y, e) \mu_e(dy) ds.
\]

In view of the formula for $\overline{m}_{pl}$ in rational directions, a proof similar to that of Corollary 1 yields the following:

**Corollary 3.** If $d \geq 2$, then there is a residual set $\mathcal{A}_d \subseteq C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda)) \times C^{2,\alpha}(\mathbb{T}^d; (0, \infty))$ such that if $m$ and $a$ are independent of $e$ and $(a, m) \in \mathcal{A}_d$, then, for each $e \in S^{d-1} \cap \mathbb{RZ}^d$, the following hold:

(i) There is an $\eta \in S^{d-1} \cap (e)_{\perp}$ such that $\overline{m}_{pl}(e) \neq \tilde{m}_\eta$.

(ii) If $d \geq 3$, then $m : S^{d-1} \setminus \mathbb{RZ}^d \to (0, \infty)$ does not have a limit at $e$.

In particular, if $(a, m) \in \mathcal{A}_d$, then $\overline{m}_{pl}$ is not a continuous function.

2.5. **Strategy of Proof.** Let us briefly review the main elements of the proof of Theorems 1, 2, and 5. Inspired by the approach of Barles and Souganidis [9] and Barles, Cesaroni, and Novaga [4], we employ the ansatz

\[
u'(x, t) = \bar{u}(x, t) + \epsilon^2 V(\epsilon^{-1} x) + \ldots
\]

which leads to the following cell problems for correctors $V$ depending on the normal vector $e \in S^{d-1}$ and second fundamental form $X \in \mathcal{S}_d$:

\[
- F(e, X + D^2 V, y) = -\overline{F}(e, X) \quad \text{if } \mathbb{T}^d.
\]

In view of (17), this is a degenerate elliptic PDE. The analysis below shows that it is ill-posed when $e \in \mathbb{RZ}^d$, which presents a significant obstacle in what follows.

Nonetheless, in *irrational directions*, that is, if $e \notin \mathbb{RZ}^d$, it turns out that if we let $(V^\delta)_{\delta > 0}$ be the solutions of the penalized cell problem

\[
\delta V^\delta - F(e, X + D^2 V^\delta, y) = 0 \quad \text{in } \mathbb{T}^d,
\]

then there is an $\overline{F}(e, X) \in \mathbb{R}$ such that

\[
\overline{F}(e, X) = \lim_{\delta \to 0^+} \delta V^\delta \quad \text{uniformly in } \mathbb{T}^d.
\]

As we show below, this can be seen by rewriting (26) as a one-parameter family of uniformly elliptic $(d-1)$-dimensional problems and applying homogenization results for almost periodic, uniformly elliptic operators in nondivergence form.

Once (27) is proved, we construct approximate correctors of (26) following a well-worn approach in homogenization theory. The approximate correctors are then used
to show that the solutions \((u^\epsilon)_{\epsilon>0}\) behave appropriately wherever the limiting level set has an irrational normal.

That leaves the question of whether or not a function that solves a level set PDE at points where its normal vector is irrational is actually a genuine solution. In this case, as long as we are studying the unforced problems (1) or (2), the answer is yes and follows from the results of the companion paper [46], as shown below.

Two problems remain. First, it is not at all clear that \(F\) extends to a continuous function. Toward that end, we adapt ideas from the study of oscillating boundary value problems to characterize all possible limits of \(\overline{F}\) at rational directions. When \(d = 2\), it is continuous and, thus, the usual comparison principle for level set PDE applies and implies homogenization. In the setting of Theorem 2, however, the operator is generically discontinuous in higher dimensions, necessitating an extension of the comparison principle to more exotic, discontinuous operators.

The remaining issue is the homogenization of the forced motion (22). Here additional ideas are needed to treat rational directions. If \(e \in \mathbb{R}\mathbb{Z}^d\) and we let \((V^\delta)_{\delta>0}\) be the solutions of

\[
(28) \quad m(y, e) + \delta V^\delta - \text{tr} \left( A(y, e) D^2 V^\delta \right) = 0 \quad \text{in} \quad \mathbb{T}^d,
\]

then there is an \(m^\perp_e \in C(\mathbb{T}^d)\) varying only in the \(e\) direction such that

\[
m^\perp_e = \lim_{\delta \to 0^+} \left( -\delta V^\delta \right) \quad \text{uniformly in} \quad \mathbb{T}^d.
\]

In general, \(m^\perp_e\) is not a constant function. The reason is that, at the level of (22), sending \(\delta \to 0^+\) in (28) only averages the fluctuations of the front relative to a moving reference plane. As we show below, the height of this plane oscillates around its mean and needs to be corrected through another, one-dimensional cell problem.

3. Examples

In this section, we discuss a few examples of interest that fit into the assumptions of either Theorem 1 or Theorem 2.

3.1. Anisotropic Curvature Flows with Periodic Mobilities. Let \(\varphi : \mathbb{R}^d \to [0, \infty)\) be a \(C^2\) Finsler norm, that is, a convex, positively one-homogeneous function that is positive and \(C^2\) in \(\mathbb{R}^d \setminus \{0\}\). Given a mobility coefficient \(m\) as above, the equation

\[
(29) \quad m(y, \hat{D}u) u_t - \text{tr} \left( D^2 \varphi(\hat{D}u) D^2 u \right) = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, \infty)
\]

is referred to as an anisotropic curvature flow. Formally, this is the gradient flow of the anisotropic perimeter determined by \(\varphi\) with respect to the \(L^2\)-Riemannian metric determined by \(m\) (cf. [56]). The geometric flows associated with these equations are of interest in materials science.

In this setting, our results show that, at large scales, the mobility gets averaged:
Proposition 1. Suppose that \( \varphi \) is a \( C^2 \) Finsler norm that is uniformly convex in the following sense:

\[
\lambda(\text{Id} - e \otimes e) \leq D^2 \varphi(e) \leq \Lambda(\text{Id} - e \otimes e) \quad \text{if } e \in S^{d-1}.
\]

If \( m \) satisfies the assumptions of Theorem 2 and \( A(y,e) = D^2 \varphi(e) \) for all \( e \in S^{d-1} \), then the solutions \( (u^e)_{e>0} \) of (2) converge locally uniformly to the solution \( \bar{u} \) of the averaged equation

\[
\left\{ \begin{array}{l}
\overline{m}(Du)\bar{u}_t - tr\left(D^2 \varphi(Du)D^2 \bar{u}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \\
\bar{u} = u_0 \quad \text{on } \mathbb{R}^d \times \{0\}.
\end{array} \right.
\]

Here \( \overline{m}(e) = \int_{\mathbb{T}^d} m(y,e) \, dy \). Furthermore, in this case, \( \overline{m}_{pl} = \overline{m} \) in \( S^{d-1} \).

A few remarks are in order:

Remark 3. It is interesting to note that the effective mobility \( \overline{m}_{pl} \) is continuous in Proposition 7. By contrast, given a positive function \( c \in C^{0,1}(\mathbb{T}^d) \), the structurally similar forced problem

\[
\left\{ \begin{array}{l}
u_t^e - tr\left(D^2 \varphi(Du^e)D^2 u^e\right) - c(\epsilon^{-1}x)\|Du^e\| = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \\
u^e(x,0) = \langle x,e \rangle \quad \text{on } \mathbb{R}^d \times (0,\infty),
\end{array} \right.
\]

also homogenizes with \( u^e(x,t) \to \langle x,e \rangle + \bar{c}(e)t \) in the limit. (This was proved when \( d = 2 \) and \( \varphi = \| \cdot \| \) in [19] and follows by arguing as in Section 8 below in general.) However, as in [19], one finds that \( \bar{c} \) is generically discontinuous. Put slightly differently, pathologies arise if we replace constant forcing by periodic forcing in (22).

Remark 4. Let us observe that by exploiting the regularity theory of the Laplacian and using approximation arguments as in [10], it is not hard to show that Proposition 7 still holds if (13) is dropped.

Remark 5. Of course, it would also be interesting to analyze the case when \( \varphi \) depends on \( y \) as well as \( e \). In that case, to get an anisotropic curvature flow, (29) should include another term involving \( D\varphi \) and no longer fits into the framework of (2). The homogenization of such equations, which are the natural divergence form analogue of (2), is largely open. However, see [4] for results that apply to the case where \( \varphi(y,e) = 1 + \langle \psi(y),e \rangle \) for some vector field \( \psi \) with \( \| \psi \|_{L^{\infty}(\mathbb{T}^d)} < 1 \).

Remark 6. In general, the operator \( \overline{F} \) obtained in Theorem 2 extends continuously to \( (\mathbb{R}^d \setminus \{0\}) \times S_d \) whenever \( a \) is independent of the spatial variable, as in the last proposition. Another (actually equivalent) case is when \( a(y,e) = \bar{a}(y,e)\text{Id} \) for some positive function \( \bar{a} \in C^{0,1}(\mathbb{T}^d) \). In this case, \( \bar{a}(e) \) equals the harmonic mean of \( \bar{a}(\cdot,e) \), consistent with [21] Theorem C).

3.2. Fully Nonlinear Operators in Dimension Two. Notice that if \( F \) satisfies assumptions (i)-(iv) of Theorem 11 and also is Lipschitz continuous in the following sense

\[
|F(p,X,y) - F(p,X,y')| \leq C\|\bar{X}_e\|\|y - y'\| \quad \text{if } (p,X) \in (\mathbb{R}^d \setminus \{0\}) \times S_d, \ y, y' \in \mathbb{T}^d,
\]
then we can argue as in [38] to see that F also satisfies (v).

A natural class of fully nonlinear operators satisfying these assumptions take the following form:

\[ F(p, X, y) = \sup_{\alpha \in A} \inf_{\beta \in B} \text{tr} \left( (\text{Id} - \hat{p} \otimes \hat{p}) a_{\alpha, \beta}(y, \hat{p})(\text{Id} - e \otimes e) X \right). \]

Here we assume that the matrix fields \( \{a_{\alpha, \beta} \mid \alpha \in A, \beta \in B\} \) map \( \mathbb{T}^d \times S^{d-1} \) continuously into \( S_d(\lambda, \Lambda) \) and that there is an \( L > 0 \) such that, for each \((\alpha, \beta)\in A \times B\), we have

\[ \sup \left\{ \left\| a_{\alpha, \beta}(y, e) - a_{\alpha, \beta}(y', e) \right\| \mid (y, e), (y', e) \in \mathbb{T}^d \times S^{d-1}, y \neq y' \right\} \leq L. \]

When \( A \) is a singleton so that \( F \) is concave, such operators describe stochastic target problems, which are of interest in financial mathematics (see [54], [12]).

4. Approximate Correctors

In this section, we construct approximate correctors that will be used in the analysis of the asymptotics of (1) and (2). The use of approximate correctors in homogenization is by now standard (cf. [42] and [16]). The main difficulty here is the operators of interest are degenerate elliptic and, therefore, as we will see below, the cell problem is not well behaved in rational directions.

The result we need concerning approximate correctors is stated next. The majority of this section is devoted to its proof.

**Theorem 7.** If \( F \) satisfies the assumptions of Theorem 1 or \( F \) takes the form

\[ F(p, X, y) = m(y, \hat{p})^{-1} \text{tr} \left( a(y, \hat{p}) X \right) \]

with \( m \) and \( a \) satisfying the assumptions of Theorem 3, then, for each \((e, X) \in S^{d-1} \times S_d\), there is a unique continuous function \( F_e(X, \cdot) \) varying only in the \( e \) direction such that, for each \( \nu > 0 \), there is a (non-unique) \( V_\nu \in C^2(\mathbb{T}^d) \) satisfying

\[ -\nu \leq F_e(X, (y, e)e) - F(e, \hat{X}_e + D^2V_\nu, y) \leq \nu \quad \text{in } \mathbb{T}^d. \]

If \( e \in S^{d-1} \setminus \mathbb{R} \mathbb{Z}^d \), then there is a constant \( \bar{F}(e, X) \) such that \( F_e(X, \cdot) \equiv \bar{F}(e, X) \) in \( \mathbb{T}^d \). Furthermore, \( \bar{F}(e, \cdot) \) is continuous, it satisfies [12], \( |\bar{F}(e, X)| \leq \Lambda \|X\| \), and

\[ \bar{F}(e, X) = \bar{F}(e, \hat{X}_e) \quad \text{if } X \in S_d. \]

Following the proof of Theorem 7 we describe the relation between the functions \( \{F_e\}_{e \in S^{d-1}} \) and \( \bar{F} \) and the invariant measures of Theorem 3. Finally, at the end of the section, we prove results specific to the rational case that are used in the proofs of Theorem 5 and Corollary 1.

In Appendix A.4 we discuss sufficient conditions for the existence of a corrector (i.e. a solution of (31) with \( \nu = 0 \)) in the case when \( a \) is independent of the spatial variable.

In this section and elsewhere in the paper, we will extensively use the differential operator \( D^2_e \) defined by its action on smooth functions \( \varphi \) in \( \mathbb{R}^d \) by

\[ D^2_e \varphi = (\text{Id} - e \otimes e) D^2 \varphi (\text{Id} - e \otimes e). \]
4.1. Cell Problem and Ergodic Constant. To start with, we establish the existence of penalized correctors and discuss their convergence as $\delta \to 0^+$.

**Theorem 8.** Given $e \in S^{d-1}$, $X \in S_d$, and $\delta > 0$, there is a unique $V^\delta \in C(\mathbb{T}^d)$ solving the penalized cell problem

\begin{equation}
\delta V^\delta - F(e, X + D^2 V^\delta, y) = 0 \quad \text{in } \mathbb{T}^d.
\end{equation}

Furthermore, there is a function $F_e^\perp(X, \cdot) : \mathbb{T}^d \to \mathbb{R}$ varying only in the $e$ direction such that

\begin{equation}
\lim_{\delta \to 0^+} \sup \{|\delta V^\delta(y) + F_e^\perp(X, y)| \mid y \in \mathbb{T}^d\} = 0.
\end{equation}

In particular, if $e \notin \mathbb{R}Z^d$, then there is a constant $\overline{F}(e, X) \in \mathbb{R}$ such that $F_e^\perp(X, \cdot) \equiv \overline{F}(e, X)$ in $\mathbb{T}^d$.

**Proof.** The existence and uniqueness of $V^\delta \in C(\mathbb{T}^d)$ is proved in Appendix A.

Given $y \in \mathbb{T}^d$, define $\tilde{V}_y^\delta : \langle e \rangle^\perp \to \mathbb{R}$ by

\begin{equation}
\tilde{V}_y^\delta(x') = V^\delta(y + x').
\end{equation}

A perturbation argument (cf. [45, Appendix B]) shows that, no matter the choice of $y$, the function $\tilde{V}_y^\delta$ is solution of

\begin{equation}
\delta \tilde{V}_y^\delta - F(e, \bar{X}_e + D^2 \tilde{V}_y^\delta, y + x') = 0 \quad \text{in } \langle e \rangle^\perp.
\end{equation}

(Here $D_e^2$ is given by (32).) This is the penalized cell problem associated with a uniformly elliptic operator with quasi-periodic coefficients. Therefore, by [15, Lemma 9.1], there is a constant $F_e^\perp(X, y) \in \mathbb{R}$ such that

\begin{equation}
\lim_{\delta \to 0^+} \sup \{|\tilde{V}_y^\delta(x') + F_e^\perp(X, y)| \mid x' \in \langle e \rangle^\perp\} = 0.
\end{equation}

Notice that $y \mapsto F_e^\perp(X, y)$ is well-defined and only varies in the $e$ direction. Indeed, if $k \in \mathbb{Z}^d$, then $\tilde{V}_{y+k}^\delta = \tilde{V}_y^\delta$ so $F_e^\perp(X, y + k) = F_e^\perp(X, y)$. To see that $F_e^\perp(X, y) = F_e^\perp(X, \langle y, e \rangle)$, observe that

\begin{align*}
-F_e^\perp(X, y) &= \lim_{\delta \to 0^+} \delta \tilde{V}_y^\delta(0) = \lim_{\delta \to 0^+} \delta \tilde{V}_{\langle y, e \rangle}^\delta(y - \langle y, e \rangle) = -F_e^\perp(X, \langle y, e \rangle).
\end{align*}

This proves $F_e^\perp(X, \cdot)$ varies only in the $e$ direction.

It remains to prove the uniform convergence of $(\delta V^\delta)_{\delta > 0}$. If $e \in \mathbb{R}Z^d$, then (36) is a periodic homogenization problem in $\langle e \rangle^\perp$. (Here the coefficients are invariant under the group $M_e = \mathbb{Z}^d \cap \langle e \rangle^\perp$, which is a rank $(d-1)$-subgroup of $\langle e \rangle^\perp$.) Thus, arguing as in [15, Section 8], we see that the rate of convergence of $(\delta \tilde{V}_{y}^\delta)_{\delta > 0}$ is uniform with respect to $y$. From this, it is immediate that $(\delta V^\delta)_{\delta > 0}$ converges uniformly in $\mathbb{T}^d$.

On the other hand, if $e \notin \mathbb{R}Z^d$, then the density of each leaf of the foliation $\mathbb{T}^d \{r \in \mathbb{R} Z_e\}$ implies that, for each $y' \in \mathbb{T}^d$,

\begin{align*}
\sup \{|\delta V^\delta(y) + F_e^\perp(X, y')| \mid y \in \mathbb{T}^d\} &= \sup \{|\delta \tilde{V}_{y'}^\delta(x') + F_e^\perp(X, y')| \mid x' \in \langle e \rangle^\perp\}.
\end{align*}

Thus, $\lim_{\delta \to 0^+} \delta V^\delta(y) = -F_e^\perp(X, y')$ uniformly in $\mathbb{T}^d$, no matter the choice of $y'$. Note that this shows $F_e^\perp(X, \cdot)$ is a constant in this case. \qed
Now we prove the $\mathbf{F}(e, \cdot)$ claimed in Theorem 7.

**Corollary 4.** $\mathbf{F}(e, \cdot) : S_d \to \mathbb{R}$ is continuous, it satisfies (7), $|\mathbf{F}(e, X)| \leq \Lambda \|X\|$, and $\mathbf{F}(e, X) = \mathbf{F}(e, \tilde{X}_e)$ if $X \in S_d$.

**Proof.** By the proof of Theorem 8, $\tilde{X}_e$ is the homogenized operator associated with the (e.g. Dirichlet) homogenization problem

$$
\begin{align*}
-F(e, D^2 U^e, \epsilon^{-1} x') &= 0 \quad \text{in } \langle e \rangle^\perp \\
U^e &= G \quad \text{on } \langle e \rangle^\perp
\end{align*}
$$

Therefore, the claims follow directly from (16). □

Finally, we define the homogenized operator $\mathbf{F} : (\mathbb{R}^d / \mathbb{Z}^d) \times S_d \to \mathbb{R}$ by

$$
\mathbf{F}(p, X) = \|p\| \mathbf{F}(\hat{p}, \|p\|^{-1} X).
$$

### 4.2. Regularity Estimates.

We now turn to the regularity of the penalized correctors $(V^\delta)_{\delta > 0}$ when the operator $F$ is sufficiently regular. To start with, we consider the two dimensional case since it benefits from special structure. We then turn to the operators of Theorem 2, which admit more-or-less smooth penalized correctors due to the strong assumptions imposed on the matrix field $a$.

Here is the result when $d = 2$:

**Proposition 2.** Let $d = 2$ and assume that $F(e, \cdot) : S_d \times \mathbb{T}^d \to \mathbb{R}$ is twice continuously differentiable. Given $\delta > 0$, $e \in S^1$, and $X \in S_2$, the penalized corrector $V^\delta$ solving (33) is in $C^2(\mathbb{T}^2)$.

**Proof.** Let $\{\tilde{V}^\delta_y\}_{y \in \mathbb{T}^2}$ be the functions defined in (33). Since $|F(e, X, y)| \leq \Lambda \|X\|$, it follows that $\|V^\delta\|_{L^\infty(\mathbb{T}^2)} \leq \Lambda \delta^{-1} \|X\|$. From this, a straightforward computation involving (7) (and the fact that $\langle e \rangle^\perp$ is one-dimensional) shows that

$$
-\frac{2 \Lambda \|X\|}{\Lambda} \leq -D^2 e \tilde{V}^\delta y \leq \frac{2 \Lambda \|X\|}{\Lambda}
$$

in the viscosity sense in $\langle e \rangle^\perp$.

Hence $\{\tilde{V}^\delta_y\}_{y \in \mathbb{T}^2}$ are bounded in $C^{1,1}$. The equation uniquely and continuously determines the functions $\{D^2 e \tilde{V}^\delta y\}_{y \in \mathbb{T}^2}$ by (7) so actually we are working with $C^2$ functions.

Manipulating difference quotients and invoking assumptions (7) and the $C^2$ assumption on $F$, we apply the Krylov-Safonov Theorem to find that $\{\tilde{V}^\delta_y\}_{y \in \mathbb{T}^2} \subseteq C^{2,\alpha}(\langle e \rangle^\perp)$.

Next, we show that $y \mapsto \tilde{V}^\delta_y$ is $C^{2,\alpha}$. Fix $y' \in \langle e \rangle^\perp$. Using difference quotients again, we see that the function $\frac{\partial \tilde{V}^\delta y}{\partial y_e}$ given by

$$
\frac{\partial \tilde{V}^\delta y}{\partial y_e}(x') = \lim_{h \to 0} \frac{\tilde{V}^\delta y_{y+h e}(x') - \tilde{V}^\delta y_y(x')}{h} \quad (x' \in \langle e \rangle^\perp)
$$

is solution of the uniformly elliptic linear PDE

$$
\delta \frac{\partial \tilde{V}^\delta y}{\partial y_e} - D_A F(e, \tilde{X}_e + D^2 e \tilde{V}^\delta y, y + x') : D^2 e \left( \frac{\partial \tilde{V}^\delta y}{\partial y_e} \right) - \langle D_y F(e, \tilde{X}_e + D^2 e \tilde{V}^\delta y, y + \tilde{x}), e \rangle = 0 \quad \text{in } \langle e \rangle^\perp.
$$
Therefore, the previous considerations show that $V$ is $C^2$ in $(e)^{1}$ and the uniform ellipticity gives a constant $C > 0$ such that

$$\sup \left\{ \delta \left\| \frac{\partial V_{\delta}}{\partial y_{e}} \right\|_{L^{\infty}((e)_{+}^{1})} + \left\| D^{2} \left( \frac{\partial V_{\delta}}{\partial y_{e}} \right) \right\|_{L^{\infty}((e)_{+}^{1})} \mid y \in \mathbb{T}^{2} \right\} \leq C.$$  

Differentiating again, we find that the second derivative $\frac{\partial^{2} V_{\delta}}{\partial y_{e}^{2}}$ (defined analogously) is $C^{2}$ and satisfies

$$\sup \left\{ \delta \left\| \frac{\partial^{2} V_{\delta}}{\partial y_{e}^{2}} \right\|_{C((e)_{+}^{1})} + \left\| D^{2} \left( \frac{\partial^{2} V_{\delta}}{\partial y_{e}^{2}} \right) \right\|_{L^{\infty}((e)_{+}^{1})} \mid y \in \mathbb{T}^{2} \right\} \leq C^{2}.$$  

Finally, observe that if $y \in \mathbb{T}^{d}$ and $e' \in S^{d-1} \cap (e)_{+}^{1}$, then

$$D_{e}^{2} V_{\delta}(y) = D_{e}^{2} V_{\delta}(0), \quad \langle D^{2} V_{\delta}(y)e, e' \rangle = \frac{\partial V_{\delta}}{\partial y_{e}^{2}}(0),$$

$$\langle D^{2} V_{\delta}(y)e, e' \rangle = \left\langle D \left( \frac{\partial V_{\delta}}{\partial y_{e}} \right) (0), e' \right\rangle.$$  

Therefore, the previous considerations show that $V_{\delta} \in C^{2}(\mathbb{T}^{d})$. □

Finally, in the quasi-linear set-up, we have

**Proposition 3.** If $a$ and $m$ satisfy the assumptions of Theorem 2 and $F(p, X, y) = m(y, \hat{p})^{-1} \text{tr}(a(y, \hat{p})X_{\hat{p}})$, then, for each $\delta > 0$, the penalized corrector $V_{\delta}$ of (33) satisfies $V_{\delta} \in C^{2, \alpha}(\mathbb{T}^{d})$.

**Proof.** The proof proceeds as in the last proposition. To obtain $C^{2, \alpha}$ estimates, we use Schauder estimates for linear elliptic equations (cf. [33, Chapter 6] or [13, Chapter 8]). □

### 4.3. Approximate Correctors

This section is devoted to the proof of Theorem 2. In general, in the setting of Theorem 1, if $F$ is not twice differentiable, we approximate it by a family of regularized operators $(F_{\mu})_{\mu > 0}$ that are. This is made precise in the next result:

**Proposition 4.** Given $e \in S^{1}$ and $F$ satisfying the assumptions of Theorem 2, there is a family $(F_{\mu})_{\mu > 0}$ of operators, twice continuously differentiable in $S_{2} \times \mathbb{T}^{2}$, such that, for each $\mu > 0$,

\[
\lambda \| \tilde{N}_{e} \| - \Lambda \| \tilde{N}_{e} \| \leq F_{\mu}(M + N, x) - F_{\mu}(M, x) \leq \lambda \| \tilde{N}_{e} \| - \lambda \| \tilde{N}_{e} \| \quad \text{if } M, N \in S_{2},
\]

\[
|F_{\mu}(M, y') - F_{\mu}(M, y)| \leq C(1 + \mu + \| M \|) \| x' - x \| \quad \text{if } M \in S_{2}, \ y, y' \in \mathbb{T}^{2},
\]

\[
|F_{\mu}(M, y) - F(e, M, y)| \leq \Lambda \mu + \omega_{R}(\mu) \quad \text{if } (M, y) \in S_{2} \times \mathbb{T}^{2}.
\]

Here, for each $R > 0$, $\omega_{R}$ is the modulus of $F$ in $\{e\} \times B(0, R) \times \mathbb{T}^{d}$, that is, $\omega_{R}(\delta) = \sup \left\{ |F(e, X, y) - F(e, X', y')| \mid (X, y), (X', y') \in B(0, R) \times \mathbb{T}^{d}, \| y - y' \| \leq \delta \right\}.$
Proposition 4 can be proved by mollifying the operator \((X, y) \mapsto F(e, X, y)\). The details are left to the interested reader.

**Proof of Theorem 7.** We will assume that \(F\) satisfies the assumptions of Theorem 1. If instead \(F\) satisfies the assumptions of Theorem 2, then Proposition 3 shows that the penalized correctors of Theorem 8 are \(C^2\) so there is no need to regularize.

Fix \(\nu > 0\). Let \(\delta, \mu > 0\) be free variables and let \(V^{\delta, \mu} \in C^2(\mathbb{T}^2)\) be the solution of (33) with \(F\) replaced by \(F^{\mu}\). Recall that

\[
\left\|D^2eV^\delta\right\|_{L^\infty(\mathbb{T}^2)} \leq \frac{2\Lambda\|X\|}{\lambda}.
\]

Thus, if \(y \in \mathbb{T}^2\), then

\[
|\mathcal{F}(e, X) - F(e, \tilde{X}_e + D^2eV^\delta, y)| \leq (I) + (II) + (III),
\]

where

\[
(I) = \|\mathcal{F}^{\mu}(e, X) + \delta V^\delta\|_{L^\infty(\mathbb{T}^2)},
\]

\[
(II) = |F(e, \tilde{X}_e + D^2eV^\delta, y) - F^{\mu}(\tilde{X}_e + D^2eV^\delta, y)|
\leq \Lambda \mu + \omega_{1+(1+2\lambda-1\Lambda)}\|X\|\(\mu\),
\]

\[
(III) = |\mathcal{F}(e, X) - \mathcal{F}^{\mu}(e, X)|.
\]

Using Lemma 7 below and (41), it is not hard to show that

\[
|\mathcal{F}(e, X) - \mathcal{F}^{\mu}(e, X)| \leq \|\mathcal{F}^{\mu}(e, X) + \delta V^\delta\|_{L^\infty(\mathbb{T}^2)} + \Lambda \mu + \omega_{1+(1+2\lambda-1\Lambda)}\|X\|\(\mu\).
\]

Indeed, the approximate corrector \(V^{\delta, \mu}\) satisfies

\[
\delta V^{\delta, \mu} - F^{\mu}(e, \tilde{X}_e + D^2eV^\delta) = 0 \quad \text{in} \quad \mathbb{T}^2.
\]

Thus, (III) and the choice of \(F^{\mu}\) yield

\[
|\mathcal{F}(e, \tilde{X}_e + D^2eV^\delta) + \mathcal{F}^{\mu}(e, X)| \leq \|\mathcal{F}^{\mu}(e, X) + \delta V^\delta\|_{L^\infty(\mathbb{T}^2)} + \Lambda \mu + \omega_{1+(1+2\lambda-1\Lambda)}\|X\|\(\mu\) \quad \text{in} \quad \mathbb{T}^2.
\]

Hence, by Lemma 7 we have

\[
(III) = |\mathcal{F}(e, X) - \mathcal{F}^{\mu}(e, X)| \leq \|\mathcal{F}^{\mu}(e, X) + \delta V^\delta\|_{L^\infty(\mathbb{T}^2)} + \Lambda \mu + \omega_{1+(1+2\lambda-1\Lambda)}\|X\|\(\mu\).
\]

We conclude that if we first choose \(\mu > 0\) small enough that

\[
\Lambda \mu + \omega_{1+(1+2\lambda-1\Lambda)}\|X\|\(\mu\) \leq \frac{\nu}{3}
\]

and then choose \(\delta > 0\) so small that \(\|\mathcal{F}^{\mu}(e, X) + \delta V^\delta\|_{L^\infty(\mathbb{T}^2)} \leq \nu/3\), then we obtain

\[
-\nu \leq \mathcal{F}(e, X) - F(e, \tilde{X}_e + D^2eV, y) \leq \nu \quad \text{in} \quad \mathbb{T}^2.
\]

\(\square\)
4.4. Invariant Measures. This section makes the connection between the homogenized operators $\overline{F}$ and oscillating functions $F^\perp_e$ obtained in Theorem 7 and the invariant measures of Theorem 3.

To start with, we give the basic existence result for invariant measures:

**Proposition 5.** Assume that $e \in S^{d-1}$. For each $e \in S^{d-1}$, $\mathcal{I}^a_e$ is non-empty. Furthermore, for $e \in \mathbb{R}Z^d$ there is an $r_e$-periodic function $\mu_e : \mathbb{R} \rightarrow \mathcal{I}^a_e$, $\mu_e : s \mapsto \mu^s_e$, such that, for each $r \in [0, \rho_0)$, the support of $\mu_e$ equals $\mathbb{T}_{e^{-1}}(r)$ and $\mu^s_e \ll \mathcal{H}^{d-1} |_{\mathbb{T}_{e^{-1}}(r)}$.

**Proof.** First, assume that $e \in \mathbb{R}Z^d$. Given $r \in [0, \rho_0)$, the trajectories of the SDE (17) with $\langle X_0, e \rangle = r$ satisfy $\langle X_t, e \rangle = r$ for all $t > 0$. Hence, as long as we restrict to initial distributions concentrated in $\mathbb{T}_{e^{-1}}(r)$, we can consider (17) as a process in $\mathbb{T}_{e^{-1}}(r)$. Restricted to $\mathbb{T}_{e^{-1}}(r)$, (17) is uniformly non-degenerate. Therefore, it has a unique invariant measure $\mu^e_r$ that has full support in $\mathbb{T}_{e^{-1}}(r)$ and $\mu^r_e \ll \mathcal{H}^{d-1} |_{\mathbb{T}_{e^{-1}}(r)}$ (cf. 11 or 51).

If $e \notin \mathbb{R}Z^d$, then there is a sequence $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \cap \mathbb{R}Z^d$ such that $e = \lim_{n \to \infty} e_n$. By the Banach-Alaoglu Theorem, $(\mu^{e_n}_e)_{n \in \mathbb{N}}$ has a weak-* accumulation point $\mu$. A quick argument shows that $\mu \in \mathcal{I}^a_e$ necessarily holds. Further, Lemma 8 in the appendix shows that $\mu \ll \mathcal{L}^d$.

We still need to understand the structure of the sets $\{\mathcal{I}^a_e\}_{e \in S^{d-1}}$. Here and in Section 8 it will be convenient to use the following variant of Theorem 7:

**Proposition 6.** If $e$ satisfies the assumptions of Theorem 2 and $f \in C(\mathbb{T}^d)$, then, for each $e \in S^{d-1}$ and $\delta > 0$, there is a unique $V^\delta \in C(\mathbb{T}^d)$ solving the degenerate elliptic PDE:

$$
\delta V^\delta - \text{tr}(a(y, e)D^2_y V^\delta) = f(y) \quad \text{in } \mathbb{T}^d.
$$

Furthermore, there is an $f^\perp_e \in C(\mathbb{T}^d)$ varying only in the $e$ direction such that

$$
\lim_{\delta \to 0^+} \|\delta V^\delta - f^\perp_e\|_{L^\infty(\mathbb{T}^d)} = 0.
$$

If $e \notin \mathbb{R}Z^d$, then there is a constant $\bar{f}(e) \in \mathbb{R}$ such that $f^\perp_e \equiv \bar{f}(e)$.

Finally, if $f \in C^{2,\alpha}(\mathbb{T}^d)$, then $V^\delta \in C^{2,\alpha}(\mathbb{T}^d)$.

This readily leads to the proof of Theorem 3. In addition, we will show that the operator $F^\perp_e$ of Theorem 7 is determined in this case by the formula

$$
F^\perp_e(X, y) = m^\perp_y(Xe)^{-1} \text{tr} \left( a^\perp_y(X, Ye) \right),
$$

where $a^\perp_y$ is given by (21) and $m^\perp_y$ is defined analogously.

**Proof of Theorem 3.** We start with (i). Assume that $e \notin \mathbb{R}Z^d$. By the Riesz Representation Theorem, we can define a Borel probability measure $\bar{\mu}_e$ in $\mathbb{T}^d$ by

$$
\int_{\mathbb{T}^d} f(y) \bar{\mu}_e(dy) = \bar{f}(e),
$$

where $\bar{f}(e)$ is the constant in Theorem 6. We claim that $\mathcal{I}^a_e = \{\bar{\mu}_e\}$.
Indeed, if $\mu \in \mathcal{I}^a_e$, $f \in C^{2,\alpha}(\mathbb{T}^d)$, and $V^\delta$ is the associated solution of (42), then
\[
\int_{\mathbb{T}^d} \delta V^\delta(y) \mu(dy) = \int_{\mathbb{T}^d} f(y) \mu(dy).
\]
Sending $\delta \to 0^+$, we find $\int_{\mathbb{T}^d} f(y) \mu(dy) = \bar{f}(e)$. This proves $\mu = \bar{\mu}_e$ by definition.

Since $\mathcal{I}^a_e$ is non-empty by Proposition 4, it also shows that $\bar{\mu}_e \in \mathcal{I}^a_e$.

Next, we turn to (ii). It remains to show that $\mathcal{I}^a_e$ is the closed convex hull of $\{ \mu^s_e \mid s \in \mathbb{R} \}$. By the Krein-Milman Theorem, we only have to prove that the latter equals the set of extreme points.

First, notice that if $f \in C(\mathbb{T}^d)$, $(V^\delta)_{\delta > 0}$ are the associated solutions of (42), and $s \in [0, r_e)$, then Proposition 2 and the definition of $\mathcal{I}^a_e$ give
\[
\int_{\mathbb{T}^d} f(y) \mu^s_e(dy) = \lim_{\delta \to 0^+} \int_{\mathbb{T}^d} \left( \delta V^\delta(y) - \text{tr} \left( A(y, e) D^2 V^\delta(y) \right) \right) \mu^s_e(dy)
= \lim_{\delta \to 0^+} \int_{\mathbb{T}^d} \delta V^\delta(y) \mu^s_e(dy) = f^\perp_e(se).
\]

Now assume $\mu$ is an extreme point of $\mathcal{I}^a_e$. Repeating the previous computation with $\mu^s_e$ replaced by $\mu$, we find, for each $f \in C(\mathbb{T}^d)$,
\[
\int_{\mathbb{T}^d} f(y) \mu(dy) = \int_{\mathbb{T}^d} f^\perp_e(\langle y, e \rangle) \mu(dy) = \int_{\mathbb{T}^d} \left( \int_{\mathbb{T}^d} f(y') \mu^s_e(dy') \right) \mu(dy).
\]
In particular, letting $\nu \in \mathcal{P}(r_e \mathbb{T})$ be the push-forward of $\mu$ onto $r_e \mathbb{T}$ given by
\[
\nu(A) = \mu(\{ y \in \mathbb{T}^d \mid \langle y, e \rangle \in A \}),
\]
we obtain $\mu = \int_{r_e \mathbb{T}} \mu^s_e \nu(ds)$. Notice that if $A \subseteq r_e \mathbb{T}$ and $\nu(A) > 0$, then
\[
\mu = \nu(A) \int_A \mu^s_e \nu(ds) + (1 - \nu(A)) \int_{A^c} \mu^s_e \nu(ds).
\]
Hence, since $\mu$ is an extreme point of $\mathcal{I}^a_e$, either $\nu(A) = 1$ or $\nu(A) = 0$. From this, an elementary argument shows that $\nu = \delta_{s'}$ for some $s' \in [0, r_e)$. In particular, $\mu = \mu^s_e$. This proves $\{ \mu^s_e \mid s \in \mathbb{R} \}$ equals the set of extreme points of $\mathcal{I}^a_e$.

Finally, we prove (iii). If $m \equiv 1$ in Theorem 7, $\delta > 0$, and $X \in S_d$, then the solution $V^\delta$ of (33) solves (42) with $f(y) = \text{tr}(a(y, e)X)$. Hence
\[
\overline{F}(e, X) = \int_{\mathbb{T}^d} \text{tr}(a(y, e)X) \bar{\mu}_e(dy) \quad \text{if } e \notin \mathbb{R} \mathbb{Z}^d,
\]
\[
F^\perp_e(X, y) = \int_{\mathbb{T}^d} \text{tr}(a(y', e)X) \mu^s_e(dy') \quad \text{if } e \in \mathbb{R} \mathbb{Z}^d.
\]
This shows that the operator $\overline{F}$ is given by (15) with $\bar{a}$ and $\bar{a}_e^\perp$ as in (19) and (21).

When $m \not\equiv 1$, the solution $V^\delta$ of (33) is also solution of
\[
\delta m(y, e)V^\delta - \text{tr} \left( a(y, e) D_e^2 V^\delta \right) = \text{tr} \left( a(y, e)X \right) \quad \text{in } \mathbb{T}^d.
\]
Given $\epsilon > 0$, for small enough $\delta$, this yields
\[
| F^\perp_e(X, y)m(y, e) - \text{tr} \left( a(y, e) D_e^2 V^\delta \right) - \text{tr} \left( a(y, e)X \right) | \leq \epsilon \quad \text{in } \mathbb{T}^d.
\]
When \( e \in \mathbb{RZ}^d \), we integrate with respect to \( \mu^s_e \) for a given \( s \in [0, r_e) \) to find
\[
\left| F_e^\perp(X, se) \int_{\mathbb{T}^d} m(y', \epsilon) \mu^s_e(dy') - \text{tr} \left( a^\perp_e(se) X \right) \right| \leq \epsilon.
\]
The arbitrariness of \( \epsilon \) implies (43). A similar computation applies in the case that \( e \notin \mathbb{RZ}^d \), giving (15).

4.5. Correctors in Rational Directions. Finally, we build correctors of the cell problem when \( e \in \mathbb{RZ}^d \) and study the regularity of the oscillating function \( f^\perp_e \). These will be convenient in the analysis of the forced problem (22) as well as the proof of Corollary 1 concerning generic discontinuities.

This section is devoted to the study of the following cell problem:
\[
(44) \quad -\text{tr} \left( a(y, e) D^2_e \tilde{V}_e \right) = f(y) - f^\perp_e((y, e)e) \quad \text{in } \mathbb{T}^d.
\]
Notice that if a solution exists, then \( f^\perp_e \) is necessarily as in Proposition 6.

**Proposition 7.** If a satisfies the assumptions of Theorem 2 and \( f \in C^{2,\alpha}(\mathbb{T}^d) \), then there is a solution \( V_e \in C^{2,\alpha}(\mathbb{T}^d) \) of (44) and \( f^\perp_e \in C^{2,\alpha}(\mathbb{T}^d) \). Further, if \( \nabla V_e \in C^{2,\alpha}(\mathbb{T}^d) \) is any other solution of (44), then \( V_e - \nabla V_e \) varies only in the \( e \) direction.

Note that this implies \( a^\perp_e \) and \( m^\perp_e \) of (43) are both \( C^{2,\alpha} \).

**Proof of Proposition 7.** To start with, let \( \varphi \in C^{2,\alpha}([0, r_e]) \) be a function with \( \varphi(0) = 0 \). For each \( s \in [0, r_e) \), let \( \tilde{V}_{se} : \mathbb{T}^{d-1}_e(0) \to \mathbb{R} \) be the solution of the cell problem
\[
\begin{cases}
-\text{tr} \left( a(se + x') D^2_e \tilde{V}_{se} \right) = f(se + x') - f^\perp_e(se) & \text{in } \mathbb{T}^{d-1}_e(0) \\
\tilde{V}_{se}(0) = \varphi(s)
\end{cases}
\]
Notice that \( \tilde{V}_{se} \) is unique.

**Step 1: Preliminary regularity of \( f^\perp_e \)**

We claim that there is a constant \( C > 0 \) such that
\[
|f^\perp_e((s + h)e) - f^\perp_e(se)| \leq C|h| \quad \text{if } s, h \in \mathbb{R}.
\]
Indeed, first, recall (cf. [24] Lemma 3.1 or [17] Lemma 2.1) that there is a \( C_\ast > 0 \) depending only on \( \lambda, \Lambda \), and \( \|f\|_{L^\infty(\mathbb{T}^d)} \) such that
\[
\sup \left\{ \|\tilde{V}_{se} - \tilde{V}_{se}(0)\|_{L^\infty(\mathbb{T}^{d-1}_e(0))} \mid s \in [0, r_e] \right\} \leq C_\ast.
\]
Therefore, by Schauder estimates,
\[
C_0 := \sup \left\{ \|D^2_e \tilde{V}_{se}\|_{L^\infty(\mathbb{T}^{d-1}_e(0))} \mid s \in [0, r_e] \right\} < \infty.
\]
Therefore, by [10], given \( s \in [0, r_e) \), the function \( \tilde{V}_{(s+h)e} \) satisfies
\[
-f(se + x') - \text{tr} \left( a(se + x') D^2_e \tilde{V}_{(s+h)e} \right) \leq -f^\perp_e((s + h)e) + (C_0\|a(\cdot, e)\|_{C^{1,\alpha}(\mathbb{T}^d)} + \|f\|_{C^{1,\alpha}(\mathbb{T}^d)})|h|
\]
\[
-f(se + x') - \text{tr} \left( a(se + x') D^2_e \tilde{V}_{(s+h)e} \right) \geq -f^\perp_e((s + h)e) - (C_0\|a(\cdot, e)\|_{C^{1,\alpha}(\mathbb{T}^d)} + \|f\|_{C^{1,\alpha}(\mathbb{T}^d)})|h|
\]
From this and Lemma 7, we find
\[
|f^+_{e}(s + h)e) - f^+_{e}(se)| \leq \left( C_0 \|a(\cdot, e)\|_{C^1, \alpha(\mathbb{T}^d)} + \|f\|_{C^1, \alpha(\mathbb{T}^d)} \right) |h|
\]

**Step 2: Convenient extension of \(\{\tilde{V}_{se}\}_{se\in[0,r_e]}\)**

At this stage, it is more-or-less inevitable to extend the function \(s \mapsto \tilde{V}_{se}\) from \([0, r_e]\) to \(\mathbb{R}\). Given \(r \in \mathbb{R}\), define \(\tilde{V}_{re} \in C(\mathbb{T}^{d-1}_e(0))\) by
\[
\tilde{V}_{re}(x') = \tilde{V}_{(r-Nr_e)e}(x' + Nr_e e) \quad \text{if} \ 0 \leq r - Nr_e < r_e \quad \text{and} \quad N \in \mathbb{Z}.
\]

We refine the choice of \(\varphi\) to make \(r \mapsto \tilde{V}_{re}\) continuous. Since it coincides with \(\varphi\), \(r \mapsto \tilde{V}_{re}(0)\) is differentiable at each \(r \in (0, r_e)\). To get a derivative at 0 or \(r_e\), we constrain \(\varphi\) so that the following identity holds:
\[
\varphi(r_e) = \tilde{V}_0(r_e e).
\]

Note this isn’t circular since \(\tilde{V}_0\) is determined by \(\varphi(0)\) alone. We leave it to the reader to check that now \(r \mapsto \tilde{V}_{re}\) takes \(\mathbb{R}\) continuously into \(C(\mathbb{T}^{d-1}_e(0))\).

**Step 3: Differentiability in \(e\) direction**

To get a derivative of \(r \mapsto \tilde{V}_{re}\), we further refine the choice of \(\varphi\). First of all, notice that, with the Lipschitz estimate on \(f^+_{e}\) in hand, we deduce that there is a \(C_1 > 0\) such that, for each \(s \in [0, r_e]\) and \(h > 0\) small enough, the function \(\tilde{W}^h_{se} = \tilde{V}_{(s+h)e} - \tilde{V}_{se}\) satisfies
\[
\begin{cases}
-C_1 |h| \leq -\text{tr} \left( a(se + x') D^2 \tilde{W}^h_{se} \right) \leq C_1 |h| & \text{in} \ \mathbb{T}^{d-1}_e(0) \\
\tilde{W}^h_{se}(0) = \varphi(s + h) - \varphi(s)
\end{cases}
\]

From this, the compactness of \(\mathbb{T}^{d-1}_e(0)\), and the Krylov-Safonov Theorem, it follows that there is a \(B > 0\) depending only on \(C_1\) such that
\[
\|\tilde{W}^h_{se}\|_{L^\infty(\mathbb{T}^{d-1}_e(0))} \leq B|h|.
\]

We claim that \(\tilde{W}_{se} = \lim_{h \to 0^+} h^{-1}\tilde{W}^h_{se}\) exists in \(C(\mathbb{T}^d)\) for each \(s \in [0, r_e]\). Indeed, suppose \((h_n)_{n \in \mathbb{N}} \subseteq (0, \infty)\) is a sequence such that \(\lim_{n \to \infty} h_n = 0\) and \(h_n^{-1}\tilde{W}^h_{se} \to \tilde{W}\) uniformly in \(\mathbb{T}^{d-1}_e(0)\) for some \(\tilde{W} \in C(\mathbb{T}^{d-1}_e(0))\). Passing to a subsequence, if necessary, we can assume that there is a \(\tilde{c} \in \mathbb{R}\) such that
\[
\tilde{c} = \lim_{n \to \infty} \frac{f^+_{e}(se + h_n)e) - f^+_{e}(se)}{h_n}.
\]

It follows that \(\tilde{W}\) is a viscosity solution of
\[
-\langle Df(se + x'), e \rangle - \text{tr} \left( a(se + x', e) D^2 \tilde{W} \right) = -\tilde{c} + \text{tr} \left( \langle D_y a(se + x', e), e \rangle D^2 \tilde{V}_{se} \right) \quad \text{in} \ \mathbb{T}^{d-1}_e(0).
\]

By Lemma 7, \(\tilde{c}\) does not depend on the sequence \((h_n)_{n \in \mathbb{N}}\), and it follows from the normalization \(\tilde{W}(0) = \varphi'(s)\) that \(\tilde{W}\) is also unique. In particular, there is a function \(\tilde{W}_{se}\) such that, for each \(s \in [0, r_e]\),
\[
h^{-1}(\tilde{V}_{(s+h)e} - \tilde{V}_{se}) \to \tilde{W}_{se} \quad \text{uniformly in} \ \mathbb{T}^{d-1}_e(0) \ \text{as} \ h \to 0^+.
\]
Note that \( \tilde{W}_{se} \) is the unique viscosity solution of
\[ -\langle Df(se+x'), e \rangle - \text{tr} \left( a(se+x') D^2 \tilde{W}_{se} \right) = -\langle Df^\perp_e(se), e \rangle + \text{tr} \left( D_g a(se+x'), e \right) D^2 \tilde{V}_{se} \] in \( \mathbb{T}^{d-1}_e(0) \).

Note that above we considered one-sided derivatives. It is straightforward to see that, for \( s \in (0, r_e) \), the restriction \( h > 0 \) was unnecessary. To get a genuine (two-sided) derivative at 0 or \( r_e \), however, we need to add a compatibility condition to \( \phi \).

We require the following one:
\[ \varphi'(r_e) = \tilde{W}_0(r_e e). \] (45)

We leave it to the reader to verify that (45) implies that \( s \mapsto \tilde{V}_{se} \) is differentiable at each \( s \in [0, r_e] \). In fact, it implies that \( s \mapsto \tilde{V}_{se} \) is differentiable in \( \mathbb{R} \) with
\[ \lim_{h \to 0} \frac{\tilde{V}_{(s+h)e} - \tilde{V}_{se}}{h} = \tilde{W}_{(s-Nr_e)e}(\cdot + Nr_e e) \] uniformly in \( \mathbb{T}^d \) if \( 0 \leq s - Nr_e < r_e \).

Furthermore, \( \| \tilde{W}_{se} \|_{L^\infty(\mathbb{T}^{d-1}_e(0))} \leq B \) independent of \( s \in \mathbb{R} \).

Finally, notice that if we define \( V_e : \mathbb{T}^d \to \mathbb{R} \) by
\[ V_e(x) = \tilde{V}_{(x,e)e}(x - \langle x, e \rangle e) \]
then \( V_e \) solves (44) and
\[ \langle DV_e(x), e \rangle = \tilde{W}_{(x,e)e}(x - \langle x, e \rangle e) \] if \( x \in \mathbb{T}^d \).

In particular, \( V_e \in C^1(\mathbb{T}^d) \).

**Step 4: Second derivative in \( e \) direction** Since \( \tilde{W}_{se}(0) = \varphi'(s) \) for all \( s \in [0, r_e] \) by construction, by imposing compatibility conditions on \( \varphi'' \), we can repeat the previous arguments to show that \( V_e \in C^{2,\alpha}(\mathbb{T}^d) \). (The Hölder continuity of \( D^2 f^\perp_e \) follows by arguing as in the proof that it is Lipschitz; from this, the linearity of the equation can be used to show \( \langle D^2 V_e e, e \rangle \) is Hölder continuous with respect to the \( e \) variable.) \( \square \)

5. **Homogenization in Irrational Directions**

In this section, we undertake the main step in the proofs of Theorems 1 and 2. We prove that the solutions \( (u^\epsilon)_{\epsilon>0} \) are described in the limit \( \epsilon \to 0^+ \) by the homogenized equation at any contact point where the level set of the normal vector is irrational. It turns out that once this is proved, the remaining viscosity inequalities follow directly.

More precisely, in this section, we show that if \( \bar{u}^* = \lim \sup \ u^\epsilon \) and \( \underline{u}^*_e = \lim \inf u^\epsilon \), then \( \bar{u}^* \) and \( \underline{u}_e \) are respectively sub- and supersolutions of (8) or (16) depending on the context.

### 5.1. Solutions in Irrational Directions

Due to the difficulty analyzing (11) at contact points with rational normals, we are led to consider an a priori weaker notion of viscosity solution. In what follows, we are interested in functions \( \tilde{u} \) that satisfy one or more of the viscosity inequalities
\[ \tilde{u}_t - F^s(D\tilde{u}, D^2 \tilde{u}) \leq 0, \quad \tilde{u}_t - F^s(D\tilde{u}, D^2 \tilde{u}) \geq 0 \] in \( \mathbb{R}^d \times (0, \infty) \).

(46)
Here, as usual in the theory of viscosity solutions, the operators $\overline{F}^*, \overline{F} : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ are obtained from the effective operator $\overline{F}$ of (37) by the formulas

$$\overline{F}^*(p, X) = \lim_{\delta \to 0^+} \sup \{ \overline{F}(p', X') \mid (p', X') \in (\mathbb{R}^d \setminus \mathbb{R}^d) \times \mathcal{S}_d, \| p' - p \| + \| X' - X \| < \delta \},$$

$$\overline{F}^*(p, X) = \lim_{\delta \to 0^+} \inf \{ \overline{F}(p', X') \mid (p', X') \in (\mathbb{R}^d \setminus \mathbb{R}^d) \times \mathcal{S}_d, \| p' - p \| + \| X' - X \| < \delta \}.$$

The difficulties encountered in rational directions motivate the following definition:

**Definition 1.** Given an open set $U \subseteq \mathbb{R}^d \times (0, \infty)$, we say that a locally bounded, upper semi-continuous function $\tilde{u} : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ is a subsolution of (16) in irrational directions in $U$ if there is a $K > 0$ such that, given a smooth function $\varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ such that $\tilde{u} - \varphi$ has a strict local maximum at $(x_0, t_0) \in U$, the following conditions are met:

(a) If $D\varphi(x_0, t_0) \in \mathbb{R}^d \setminus \mathbb{R}^d$, then

$$\varphi_t(x_0, t_0) - \overline{F}^*(D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \leq 0.$$

(b) If $D\varphi(x_0, t_0) \in \mathbb{R}^d \setminus \{0\}$ and

$$\left\| \left( \text{Id} - \nabla D\varphi(x_0, t_0) \otimes \nabla D\varphi(x_0, t_0) \right) D^2\varphi(x_0, t_0) \right\| \leq \delta \| D\varphi(x_0, t_0) \|,$$

then

$$\varphi_t(x_0, t_0) \leq K\delta \| D\varphi(x_0, t_0) \|.$$

(c) If $\| D\varphi(x_0, t_0) \| = \| D^2\varphi(x_0, t_0) \| = 0$, then

$$\varphi_t(x_0, t_0) \leq 0.$$

Similarly, a locally bounded, lower semi-continuous function $\tilde{v} : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ is a supersolution of (16) in irrational directions in $U$ if $-\tilde{v}$ satisfies the definition of subsolution in irrational directions with $\overline{F}^*$, replacing $\overline{F}^*$ in (a).

Combining this definition with the definition of viscosity solution of Barles and Georgelin [6], we can prove that a sub- or supersolution of (16) in irrational directions is actually a sub- or supersolution in the usual sense. Let us first recall the definition of [6]:

**Definition 2.** Given an open set $U \subseteq \mathbb{R}^d \times (0, \infty)$, we say that a locally bounded, upper semi-continuous function $\tilde{u} : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ is a viscosity subsolution of (16) in $U$ if, for each smooth function $\varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ and each point $(x_0, t_0) \in U$ such that $\tilde{u} - \varphi$ has a strict local maximum at $(x_0, t_0)$, the following inequalities hold:

(i) If $D\varphi(x_0, t_0) \neq 0$, then

$$\varphi_t(x_0, t_0) - \overline{F}^*(D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \leq 0.$$

(ii) If $\| D\varphi(x_0, t_0) \| = \| D^2\varphi(x_0, t_0) \| = 0$, then

$$\varphi_t(x_0, t_0) \leq 0.$$
Analogously, we say that a locally bounded, lower semi-continuous function \( \tilde{v} : \mathbb{R}^d \times (0, \infty) \to \mathbb{R} \) is a viscosity supersolution of (16) in \( U \) if \( -\tilde{v} \) is a viscosity subsolution in \( U \) with \( F_* \) replacing \( F^* \). A locally bounded, continuous function in \( \mathbb{R}^d \times (0, \infty) \) is a viscosity solution of (16) if it is both a sub- and supersolution.

The fact that this definition is equivalent to the usual one is proved arguing exactly as in [6, Proposition 2.2].

Using Definition 2 and the comparison machinery for viscosity solutions of second order parabolic equations, we can prove the following:

**Theorem 9.** Given \( U \subseteq \mathbb{R}^d \times (0, \infty) \), if \( \tilde{u} : \mathbb{R}^d \times (0, \infty) \to \mathbb{R} \) is a subsolution (resp. supersolution) of (16) in irrational directions in \( U \), then it is a viscosity subsolution (resp. supersolution) in \( U \).

**Proof.** Modulo cosmetic changes, this is precisely [46, Theorem 2]. \( \square \)

### 5.2. Homogenization

In view of the previous theorem, our first step in the direction of Theorems 1 and 2 is to show that the half-relaxed limits \( \bar{u}^* \) and \( \bar{u}_* \) are sub- and supersolution, respectively, of (46) in irrational directions. In particular, this eliminates any non-trivial analysis of the behavior of those functions at contact points where the normal is rational.

At contact points where the normal vector is irrational, we will use the approximate correctors of the previous section and Evans's perturbed test function method [24].

Where rational normals are concerned, we only have to check that conditions (b) and (c) are satisfied. As we will see, this follows directly from the assumptions on the operator \( F \).

The preceding discussion leads to the following result:

**Proposition 8.** If \( (u^\epsilon)_{\epsilon > 0} \) are the functions in Theorem 1 or 2 and \( \bar{u}^* = \limsup \* u^\epsilon \) and \( \bar{u}_* = \liminf \* u^\epsilon \) are the half-relaxed limits defined by

\[
\bar{u}^*(x,t) = \lim_{\delta \to 0^+} \sup \{ u^\epsilon(y,s) \mid \epsilon + \| y - x \| + |s-t| \leq \delta \},
\]

\[
\bar{u}_*(x,t) = \lim_{\delta \to 0^+} \inf \{ u^\epsilon(y,s) \mid \epsilon + \| y - x \| + |s-t| \leq \delta \},
\]

then \( \bar{u}^* \) is a subsolution of (46) in irrational directions in \( \mathbb{R}^d \times (0, \infty) \) and \( \bar{u}_* \) is a supersolution of (46) in irrational directions in \( \mathbb{R}^d \times (0, \infty) \).

**Proof.** We break the proof down into three steps, one corresponding to each condition in Definition 1. Since the argument for \( \bar{u}^* \) is analogous, we restrict attention to \( \bar{u}_* \).

Assume that \( \varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R} \) is a smooth function and \( (x_0, t_0) \in \mathbb{R}^d \times (0, \infty) \) is a point where \( \bar{u}_* - \varphi \) attains a strict local minimum.

**Step 1: Irrational Normal**

Assume that \( D\varphi(x_0, t_0) \notin \mathbb{R} \mathbb{Z}^d \). Let us argue by contradiction. Assume that there is a \( \nu > 0 \) such that

\[
\varphi_t(x_0, t_0) - \mathcal{F}(D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \leq -2\nu \| D\varphi(x_0, t_0) \|.
\]
Define $e = D\varphi(x_0, t_0)$ and $X = \|D\varphi(x_0, t_0)\|^{-1}D^2\varphi(x_0, t_0)$. By Theorem 7 we can let $\nu \in C^2(\mathbb{T}^d)$ be an approximate corrector satisfying
\[-\nu \leq \mathcal{F}(e, X) - F(e, \tilde{X}_e + D^2\nu \epsilon^1 x) \leq \nu \quad \text{in } \mathbb{T}^d.
\]

For each $\epsilon > 0$, define $\varphi_\epsilon$ by
\[\varphi_\epsilon(x, t) = \varphi(x, t) + \epsilon^2\|D\varphi(x_0, t_0)\|V^\nu(\epsilon^{-1}x).
\]
Let $(x_\epsilon, t_\epsilon)$ be a local minimum of $u^\epsilon - \varphi_\epsilon$ close to $(x_0, t_0)$. Since $(x_0, t_0)$ is a strict minimum, we can fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ such that
\[\lim_{n \to \infty} \epsilon_n = 0, \quad \lim_{n \to \infty} (x_\epsilon, t_\epsilon) = (x_0, t_0), \quad \inf \{\|D\varphi(x_\epsilon, t_\epsilon)\| \mid n \in \mathbb{N}\} > 0.
\]
Invoking the equation satisfied by $u^\epsilon$ and using the continuity of $F$, we find, for sufficiently large $n$,
\[\varphi_t(x_\epsilon, t_\epsilon) - F(D\varphi(x_\epsilon, t_\epsilon), D^2\varphi(x_\epsilon, t_\epsilon)) + \|D\varphi(x_0, t_0)\|D^2V^\nu(\epsilon^{-1}x_\epsilon, \epsilon^{-1}x_\epsilon) = o(1).
\]
The continuity and geometric properties of $F$ then yield
\[0 \leq \varphi_t(x_\epsilon, t_\epsilon) - F(D\varphi(x_0, t_0), D^2\varphi(x_\epsilon, t_\epsilon)) + \|D\varphi(x_0, t_0)\|D^2V^\nu(\epsilon^{-1}x_\epsilon, \epsilon^{-1}x_\epsilon) = o(1)
\]
\[\leq \varphi_t(x_\epsilon, t_\epsilon) - \|D\varphi(x_0, t_0)\|F(e, \tilde{X}_e + D^2\nu(\epsilon^{-1}x_\epsilon), \epsilon^{-1}x_\epsilon) + o(1)
\]
\[\leq \varphi_t(x_0, t_0) - \|D\varphi(x_0, t_0)\|F(e, \tilde{X}_e) + \nu\|D\varphi(x_0, t_0)\| + o(1).
\]
Sending $\epsilon \to 0^+$ and recalling the definition of $\tilde{F}$ in (37), we contradict (47). Since $\varphi$ was arbitrary, we conclude that $\tilde{u}_*$ satisfies condition (a) of Definition 1.

**Step 2: Rational Normal, Flat Level Set**

Next, we assume that $D\varphi(x_0, t_0) \neq 0$ and define $\delta$ by
\[\delta = \|D\varphi(x_0, t_0)\|^{-1}\|\text{Id} - \tilde{D}\varphi(x_0, t_0) \otimes \tilde{D}\varphi(x_0, t_0)\|D^2\varphi(x_0, t_0)\|.
\]
Using the equation directly and invoking assumptions (9) and (11) or (3) and (9), we find $\epsilon_n \to 0^+$ and $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ so that
\[0 \leq \varphi_t(x_\epsilon, t_\epsilon) - F(D\varphi(x_\epsilon, t_\epsilon), D^2\varphi(x_\epsilon, t_\epsilon), \epsilon^{-1}x_\epsilon),
\]
\[\leq \varphi_t(x_\epsilon, t_\epsilon) + \Lambda\|D^2\varphi(x_\epsilon, t_\epsilon)\|,
\]
where $e_n = \tilde{D}\varphi(x_\epsilon, t_\epsilon)$. Sending $n \to \infty$, we recover
\[\varphi_t(x_0, t_0) \geq -\Lambda\|D^2\varphi(x_0, t_0)\| \geq -\Lambda\delta\|D\varphi(x_0, t_0)\|.
\]
We conclude that $\tilde{u}_*$ satisfies condition (b) in Definition 1 with $K = \Lambda$.

**Step 3: Vanishing Normal**

Finally, if $\|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0$, then we can find $\epsilon_n \to 0^+$ and $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ such that
\[0 \leq \varphi_t(x_\epsilon, t_\epsilon) - F(x_\epsilon, D\varphi(x_\epsilon, t_\epsilon), D^2\varphi(x_\epsilon, t_\epsilon), \epsilon^{-1}x_\epsilon).
\]
In the limit $n \to \infty$, we use (7) to recover $\varphi_t(x_0, t_0) \geq 0$. This proves $\tilde{u}_*$ satisfies condition (c) in Definition 1. \qed
6. Continuity and Discontinuity of Homogenized Coefficients

In this section, we study the continuity properties of the homogenized operator $\bar{F}$ of Theorem 1 and the effective coefficients $\bar{m}$ and $\bar{a}$ of Theorem 2. The main results are Theorem 4 and Corollary 1 concerning the (disc)continuity properties of $\bar{m}$ and $\bar{a}$ in the quasi-linear setting.

In dimension two, it turns out that the homogenized operator obtained in Theorem 1 is always continuous, a result that is stated next.

**Proposition 9.** Under the assumptions of Theorem 1, the homogenized operator $\bar{F} : (\mathbb{R}^2 \setminus \mathbb{R}Z^2) \times S_2 \to \mathbb{R}$ (see (37)) extends continuously to $(\mathbb{R}^2 \setminus \{0\}) \times S_2$.

For the sake of completeness, we show how Proposition 9 completes the proof of Theorem 1.

**Proof of Theorem 1.** Let $\bar{u}^*$ and $\bar{u}_*$ be the half-relaxed limits defined in Proposition 8. That proposition shows that $\bar{u}^*$ and $\bar{u}_*$ are respectively sub- and supersolution of (16) in $\mathbb{R}^d \times (0, \infty)$. Furthermore, using (7) and the assumption $u_0 \in UC(\mathbb{R}^d)$, we can build sub- and supersolutions that show $\bar{u}^* = \bar{u}_* = u_0$ on $\mathbb{R}^d \times \{0\}$. Since $\bar{f}^* = \bar{f}_*$ in $(\mathbb{R}^2 \setminus \{0\}) \times S_2$, the comparison principle implies $\bar{u}^* \leq \bar{u}_*$ (cf. Appendix A). Therefore, $\bar{u}^* = \bar{u}_* = \bar{u}$ and we conclude that $u^e \to \bar{u}$ locally uniformly in $\mathbb{R}^d \times [0, \infty)$.

By contrast, in higher dimensions, the effective coefficients $\bar{a}$ and $\bar{m}$ are generically discontinuous at each rational direction on $S^{d-1}$. Thus, the final step in the proof of Theorem 2 will be deferred until Section 7 where we extend comparison to (16).

6.1. Strategy of Proof. Before entering into the details, let us briefly give a heuristic explanation of the strategy of the proof and the core technical issues that arise. The main idea of the proof and the discussion that follows are inspired by [29].

Suppose $e \in S^{d-1} \cap \mathbb{R}Z^d$. In the proof of Theorem 4, we proceed by studying the behavior of $(\bar{\mu}_{e_n})_{n \in \mathbb{N}}$ along a sequence of irrational directions $(e_n)_{n \in \mathbb{N}}$ with $e_n \to e$ and $\frac{\bar{\mu}_{e_n} - \bar{\mu}_e}{||e_n - e||} \to -\eta$.

$\bar{\mu}_{e_n}$ captures the behavior of the diffusion $X^{e_n}$ of (17) after long times. Therefore, it is natural to pass to the diffusions $X^{\bar{e}_n}$ and consider their behavior. Further, in light of the structure of $\mathcal{I}_e$, the only question is the $e$ marginal of any limit point of $(\bar{\mu}_{e_n})_{n \in \mathbb{N}}$, that is, we only need to study $f((X^{e_n}_t, e))$, where $f$ is an $r_e$-periodic function of one variable and $t > 0$ is large.

We may as well assume that $\langle X^{e_n}_0, e_n \rangle = 0$, which implies that $\langle X^{e_n}_t, e_n \rangle = 0$ for all $t \geq 0$. Therefore, if we write $e_n = \cos(\theta_n)e - \sin(\theta_n)\eta_n$, for some $\theta_n \in (-\pi, \pi]$ and $\eta_n \in S^{d-1} \cap \{e\}^\perp$, we have

$$\langle X^{e_n}_t, e \rangle = \langle X^{e_n}_t, e - e_n \rangle = (1 - \cos(\theta_n)\langle X^{e_n}_t, e \rangle + O(\theta_n^2).$$

Hence to recover anything meaningful from $f((X^{e_n}_t, e))$, we need to wait until $||X^{e_n}_t|| \approx \theta_n^{-1}$. Since this takes a time proportional to $\theta_n^{-2}$, we should study $f((X^{e_n}_{\theta_n^2 T}, e))$ in the simultaneous limit $n, T \to \infty$. 
Put another way, at the level of the PDE, we would like to understand the behavior as \( n \to \infty \) and \( \delta \to 0^+ \) of the penalized correctors \((V_n^\delta)_{\delta,n} \in (0,\infty) \times \mathbb{N} \) solving

\[
\delta V_n^\delta - \text{tr} \left( a(\theta_n^{-1} y, e_n) D^2 V_n^\delta \right) = f(\theta_n^{-1} \langle x, e \rangle) \quad \text{in} \quad \langle e_n \rangle^\perp.
\]

A simple homogenization argument shows that, for a fixed \( \delta > 0 \), \( V_n^\delta \to \bar{V}^\delta \) as \( n \to \infty \), where \( \bar{V}^\delta \) is the solution of the problem

\[
\delta \bar{V}^\delta - \text{tr} \left( a(\langle \eta, x \rangle e) D^2 \bar{V}^\delta \right) = f(\langle \eta, x \rangle e) \quad \text{in} \quad \langle e \rangle^\perp.
\]

It turns out that extracting the limit of \( \delta \bar{V}^\delta \) as \( \delta \to 0^+ \) leads to the correct limit of \( \int_{\mathbb{T}^d} f(\langle y, e \rangle) \mu_{e_n}(dy) \). The question is simply how to show that the limits \( n \to \infty \) and \( \delta \to 0^+ \) commute.

The argument below shows how to do this working at the level of the obstacle problems introduced in [16] rather than the penalized correctors. Working with penalized correctors is difficult since it requires understanding the rate at which \( \delta V_n^\delta \) converges as \( \delta \to 0^+ \), independently of \( n \). We circumvent this by passing to the obstacle problem approach of [16] and leveraging an upper semi-continuity property proved there. In this way, it is possible to first send \( n \to \infty \) and then \( \delta \to 0^+ \) without explicitly quantifying the rates of convergence of the associated almost periodic homogenization problems.

6.2. Preliminaries. We start by introducing some notation that will be used later.

Henceforth let \( F \) be the operator in (1) satisfying either the assumptions of Theorem 1 or Theorem 2 with \( F(p, X, y) = m(y, \bar{p})^{-1} \text{tr}(a(y, \bar{p})X_{\bar{p}}) \). It will be convenient to define the family of shifted operators \( \{F^x\}_{x \in \mathbb{T}^d} \) by

\[
F^x(p, X, y) = F(p, X, y + x).
\]

The reader familiar with stochastic homogenization can think of the shift \( x \) like an element \( \omega \) of a probability space \((\Omega, \mathbb{P})\). In our case, \( \Omega = \mathbb{T}^d \) and \( \mathbb{P} = \mathcal{L}^d \), although we will also have occasion to work with the surface area measures on the sub-tori \( \{\mathbb{T}^{d-1}_e \times \mathbb{R}^d\}_{e \in \mathbb{S}^{d-1} \cap \mathbb{R}^d} \).

To prove Proposition 11 and Theorem 4 we start with the following preliminary result:

**Proposition 10.** Fix \( e \in \mathbb{S}^{d-1} \cap \mathbb{R}^d \) and \( \eta \in \mathbb{S}^{d-1} \cap \langle e \rangle^\perp \). There is a function \( \gamma_{e,\eta}^e : \mathcal{S}_d \to \mathbb{R} \) such that if \( (e_n)_{n \in \mathbb{N}} \subseteq \mathbb{S}^{d-1} \) satisfies

\[
\lim_{n \to \infty} \left( \|e_n - e\| + \left\| \frac{e_n - e}{\|e_n - e\|} + \eta \right\| \right) = 0.
\]

then

\[
\lim_{n \to \infty} \sup \left\{ |F^x(e_n)(X, s) + \gamma_{e,\eta}^e(X)| \mid s \in \mathbb{R} \right\} = 0.
\]

The main thrust of the section is the proof of this result. For the rest of the section, fix such an \( e, \eta, \) and \( (e_n)_{n \in \mathbb{N}} \).

By assumption, we can fix \((\eta_n)_{n \in \mathbb{N}} \subseteq \mathbb{S}^{d-1}\) and \((\theta_n)_{n \in \mathbb{N}} \subseteq (-\pi, \pi]\) such that

\[
e_n = \cos(\theta_n)e - \sin(\theta_n)\eta_n.
\]
We will assume without loss of generality that \( (\theta_n)_{n \in \mathbb{N}} \subseteq (-\pi, \pi) \setminus \{0\} \).

For each \( n \in \mathbb{N} \), let \( O_n : \mathbb{R}^d \to \mathbb{R}^d \) be the orthogonal transformation such that
\[
O_n(e_n) = e, \quad O_n(\sin(\theta_n)e + \cos(\theta_n)\eta_n) = \eta_n, \quad O_n|_{\langle e_n, e_n \rangle} = \text{Id} |_{\langle e_n, e_n \rangle}
\]
Notice that \( \lim_{n \to \infty} O_n = \text{Id} \) and \( O_n(\langle e_n \rangle) = \langle e \rangle \).

Finally, we let \( \{v_1, \ldots, v_{d-1}\} \) be an orthonormal basis of \( \langle e \rangle \) and define, for each \( R > 0 \), the cube \( Q_R^e \subseteq \langle e \rangle \) by
\[
Q_R^e = \{y \in \langle e \rangle \mid |\langle y, v_i \rangle| \vee \cdots \vee |\langle y, v_{d-1} \rangle| < R/2\}.
\]
Given \( n \in \mathbb{N} \), we define the analogous cubes \( \{Q_R^{(n)} \mid R > 0\} \) in \( \langle e \rangle \) by \( Q_R^{(n)} = O_n^{-1}(Q_R^e) \).

6.3. Obstacle Problems. It is technically very convenient to replace the penalized correctors \( (V_n^{(\delta)})_{(\delta,n)} \) of the previous discussion by solutions of a related family of obstacle problems. In this section, we describe the set-up.

Given \( n \in \mathbb{N}, \gamma \in \mathbb{R}, R > 0, X \in \mathcal{S}_d, \) and \( x \in \mathbb{T}^d \), we set \( \nu = (n, \gamma, R, X) \) and define the obstacle sub- and supersolutions \( u_{\nu,x}^\mu \) and \( u_{\nu,x} \) as the solutions of the equations
\[
\max \left\{ -F^\mu(e_n, \bar{X}_n + D_{e_n}^2 u_{\nu,x}^\mu, \theta_n^{-1} y) - \gamma, u_{\nu,x}^\mu \right\} = 0 \text{ in } Q_R^{(n)}, \quad u_{\nu,x} = 0 \text{ on } \partial Q_R^{(n)},
\]
\[
\min \left\{ -F^\gamma(e_n, \bar{X}_n + D_{e_n}^2 u_{\nu,x}, \theta_n^{-1} y) - \gamma, u_{\nu,x} \right\} = 0 \text{ in } Q_R^{(n)}, \quad u_{\nu,x} = 0 \text{ on } \partial Q_R^{(n)}.
\]

Existence and uniqueness for these problems can be proved using a comparison principle or through a penalization procedure as in [16].

When \( \nu = (\ast, \gamma, R, X) \), \( u_{\nu,x}^\mu \) and \( u_{\nu,x} \) are the solutions of the homogenized problems
\[
\max \left\{ -F_e^\mu(\bar{X}_e + D_{e}^2 u_{\nu,x}^\mu, \langle y, \eta \rangle e + \langle x, e \rangle e) - \gamma, u_{\nu,x}^\mu \right\} = 0 \text{ in } Q_R^e, \quad u_{\nu,x}^\mu = 0 \text{ on } \partial Q_R^e,
\]
\[
\min \left\{ -F_e^\gamma(\bar{X}_e + D_{e}^2 u_{\nu,x}, \langle y, \eta \rangle e + \langle x, e \rangle e) - \gamma, u_{\nu,x} \right\} = 0 \text{ in } Q_R^e, \quad u_{\nu,x} = 0 \text{ on } \partial Q_R^e.
\]

For these equations, well-posedness is clear in the context of Theorem [1] since \( \langle e \rangle \) is one-dimensional in this case. Where Theorem [2] is concerned, the regularity of \( a_{\nu}^\mu \) and \( m_{\nu}^\mu \) proved in Proposition [2] is more than enough to guarantee well-posedness.

Next, for \( n \in \mathbb{N} \cup \{\ast\} \), we define the contact sets \( K_{\nu,x}^\mu \) and \( K_{\nu,x} \) by
\[
K_{\nu,x}^\mu = \{y \in Q_R^{(n)} \mid u_{\nu,x}^\mu(y) = 0\}, \quad K_{\nu,x} = \{y \in Q_R^{(n)} \mid u_{\nu,x}(y) = 0\}.
\]

By the results of [16], for each \( n \in \mathbb{N} \cup \{\ast\} \), there are functions \( \ell_n^\mu, \ell_n^\gamma : \mathbb{R} \times \mathcal{S}_d \times \mathbb{T}^d \to [0, 1] \) such that, for each \( (\gamma, X) \in \mathbb{R} \times \mathcal{S}_d, \nu(R) = (n, \gamma, R, X) \), and \( x \in \mathbb{T}^d \), we have
\[
\ell_n^\mu(\gamma, X, x) = \lim_{R \to \infty} R^{1-d} \mathcal{H}^{d-1}(K_{\nu}^\mu(R, x)) \quad \text{and} \quad \ell_n^\gamma(\gamma, X, x) = \lim_{R \to \infty} R^{1-d} \mathcal{H}^{d-1}(K_{\nu}^\gamma(R, x)).
\]
Proposition 11. Given $\nu$ and $x$ in $N$ and (49), homogenize to the problems (50) and (51) as

$$
\lim_{n \to \infty} \bar{v}_n^\perp(\gamma, X, \cdot) = \inf \left\{ R^{1-d} \int_{T^d} \mathcal{H}^{d-1}(K^\nu_n(R), y) \, dy \mid R > 0 \right\},
$$

$$
\lim_{n \to \infty} \bar{v}_n^\perp(\gamma, X, \cdot) = \inf \left\{ R^{1-d} \int_{T^d} \mathcal{H}^{d-1}(K^\nu_n(R), y) \, dy \mid R > 0 \right\},
$$

while the case $e_n \in \mathbb{R}^d$ (cf. [14 Proposition 53]), for each $s \in [0, r_{e_n}]$,

$$
\bar{v}_n(\gamma, X, se_n) = \inf \left\{ R^{1-d} \int_{T^d} \mathcal{H}^{d-1}(K^\nu_n(R), \xi) \mathcal{H}^{d-1}(d\xi) \mid R > 0 \right\},
$$

$$
\bar{v}_n(\gamma, X, se_n) = \inf \left\{ R^{1-d} \int_{T^d} \mathcal{H}^{d-1}(K^\nu_n(R), \xi) \mathcal{H}^{d-1}(d\xi) \mid R > 0 \right\}.
$$

All of this follows from the sub-additive ergodic theorem (cf. [16, Appendix A]).

Finally, notice that, when $n = \ast$, we can argue as in the rational case. Since the coefficients of the associated obstacle problems vary only in the $\eta$ direction, it follows that the integrals over $T^{d-1}(s)$ so obtained do not depend on $s$. Therefore, as in the irrational case, we find, for each $x \in T^d$,

$$
\bar{v}_n^\perp(\gamma, X, x) = \inf \left\{ R^{1-d} \int_{T^d} \mathcal{H}^{d-1}(K^\nu_n(R), y) \, dy \mid R > 0 \right\},
$$

$$
\bar{v}_n^\perp(\gamma, X, x) = \inf \left\{ R^{1-d} \int_{T^d} \mathcal{H}^{d-1}(K^\nu_n(R), y) \, dy \mid R > 0 \right\}.
$$

6.4. Homogenization of Obstacle Problems. By a direct analogy to the discussion of the penalized correctors above, we now show that the obstacle problems (48) and (49) homogenize to the problems (50) and (51) as $n \to \infty$.

**Proposition 11.** Given $(\gamma, X, R) \in \mathbb{R} \times S_d \times (0, \infty)$ and $x \in T^d$, if $\nu_n = (n, \gamma, X, R)$ and $\nu = (\ast, \gamma, X, R)$, and if $(x_n)_{n \in \mathbb{N}} \subseteq T^d$ is such that $x = \lim_{n \to \infty} x_n$, then

$$
0 = \lim_{\delta \to 0^+} \sup \left\{ \left| u^{\nu_n, x_n}(y_1) - u^{\nu, x}(y_2) \right| \mid (y_1, y_2) \in Q_R^{(n)} \times Q_R^\ast, \ n^{-1} + \|y_1 - y_2\| < \delta \right\},
$$

$$
0 = \lim_{\delta \to 0^+} \sup \left\{ \left| u^{\nu_n, x_n}(y_1) - u^{\nu, x}(y_2) \right| \mid (y_1, y_2) \in Q_R^{(n)} \times Q_R^\ast, \ n^{-1} + \|y_1 - y_2\| < \delta \right\}.
$$

**Proof.** To prove this, we will work with half-relaxed limits. The proof for supersolutions follows by analogous arguments so we will restrict attention to subsolutions.
Define $\bar{u}^*$ and $\bar{u}_*$ in $Q^*_R$ by
\[
\bar{u}^*(y) = \lim_{\delta \to 0^+} \sup \left\{ u^{v_n,x_n}(y') \mid y' \in Q^{(n)}_R, \ n^{-1} + ||y' - y|| < \delta \right\},
\]
\[
\bar{u}_*(y) = \lim_{\delta \to 0^+} \inf \left\{ u^{v_n,x_n}(y') \mid y' \in Q^{(n)}_R, \ n^{-1} + ||y' - y|| < \delta \right\}.
\]

It suffices to show that $\bar{u}^* = \bar{u}_* = u^{v_n,x_n}$. To do this, we only need to prove that $\bar{u}^*$ and $\bar{u}_*$ are sub- and supersolution of (50) and apply the comparison principle. Since the proof for $\bar{u}^*$ is almost identical, we will restrict attention to $\bar{u}_*$.

Assume that $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a smooth function and $y_0 \in Q^*_R$ is a point where $\bar{u}_* - \varphi$ has a strict local minimum. We claim that
\[
\max \left\{ -F_e^\perp(\bar{X}_e + D^2_e\varphi(y_0)), -\langle y_0, \eta \rangle e + \langle x, e \rangle e - \gamma, \bar{u}_*(y_0) \right\} \geq 0.
\]

Clearly, we can assume that $\bar{u}_*(y_0) < 0$.

Let us argue by contradiction. If (55) fails to hold, then there is a $\zeta > 0$ such that
\[
-F_e^\perp(\bar{X}_e + D^2_e\varphi(y_0)), -\langle y_0, \eta \rangle e + \langle x, e \rangle e - \gamma < -\zeta.
\]

Let $V \in C^2(\mathbb{T}^d)$ be an approximate corrector satisfying the equation
\[
-\zeta \leq F_e^\perp(\bar{X}_e + D^2_e\varphi(y_0)), -\langle y_0, \eta \rangle e + \langle x, e \rangle e - F^x(e, \bar{X}_e + D^2_e\varphi(y_0) + D^2_eV, y) \leq \zeta \quad \text{in } \mathbb{T}^d.
\]

For each $n \in \mathbb{N}$, let $y_n \in Q^{(n)}_R$ be a point where the function $y \mapsto u^{v_n,x_n}(y) - \varphi(y) - \theta_n^2V(\theta_n^{-1}y)$ attains its minimum in $Q^{(n)}_R$. By classical arguments, we can pass to a subsequence $(y_{n_j})_{j \in \mathbb{N}}$ such that
\[
\lim_{j \to \infty} y_{n_j} = y_0, \quad \lim_{j \to \infty} u^{v_{n_j},x_{n_j}}(y_{n_j}) = \bar{u}_*(y_0) < 0.
\]

There is no loss of generality in assuming that both $y_{n_j} \in Q^{(n_j)}_R$ and $u^{v_{n_j},x_{n_j}}(y_{n_j}) < 0$ for all $j \in \mathbb{N}$. Thus, the equation satisfied by $u^{v_{n_j},x_{n_j}}$ gives
\[
0 \leq -F^{\perp}(e_{n_j}, \bar{X}_{e_{n_j}} + D^2_{e_{n_j}}\varphi(y_{n_j}) + D^2_{e_{n_j}}V(\theta_n^{-1}y_{n_j}), \theta_n^{-1}y_{n_j}) - \gamma
\]
\[
= -F^x(e, \bar{X}_e + D^2_e\varphi(y_0) + D^2_eV(\theta_n^{-1}y_{n_j}), \theta_n^{-1}y_{n_j}) - \gamma + o(1)
\]
\[
\leq -F_e^\perp(\bar{X}_e + D^2_e\varphi(y_0), \theta_n^{-1}y_{n_j}, e) + \langle x, e \rangle e - \gamma + \zeta + o(1)
\]

Since $y_{n_j} \in (e_{n_j})^\perp$, we have
\[
\theta_n^{-1}
\]
\[
\langle y_{n_j}, e \rangle = \theta_n^{-1}
\]
\[
\langle y_{n_j}, e - e_{n_j} \rangle = \left( \frac{1 - \cos(\theta_n)}{\theta_n} \right) \langle e, y_{n_j} \rangle + \left( \frac{\sin(\theta_n)}{\theta_n} \right) \langle \eta_{n_j}, y_{n_j} \rangle.
\]

Thus, $\theta_n^{-1}
\]
\[
\langle y_{n_j}, e \rangle \to \langle y_0, \eta \rangle$ and we find, in the limit $n \to \infty$,
\[
-\zeta \leq -F_e^\perp(\bar{X}_e + D^2_e\varphi(y_0), \langle y_0, \eta \rangle e + \langle x, e \rangle e) - \gamma.
\]

This directly contradicts (55).

We deduce that $\bar{u}_*$ is a supersolution of (50) in the interior of $Q^*_R$. Using barriers, it is not hard to show that $\bar{u}_* \geq 0$ in $\partial Q^*_R$. Thus, $\bar{u}_*$ is a supersolution.
6.5. Densities of Contact Sets. Once we state some properties of the densities $\overline{\ell}_n$ and $\overline{\ell}_n^*$, we will have all the tools necessary to prove Proposition 9 and Theorem 4.

The following result follows by arguing as in [16] (also see [2]). In the statement, we write $e_*=e$ and assume we are in case (i).

**Proposition 12.** For each $(X,x) \in S_d \times \mathbb{T}^d$, there is a sequence $(\gamma_n(X,x))_{n \in \mathbb{N} \cup \{\ast\}} \subseteq \mathbb{R}$ such that, for each $n \in \mathbb{N} \cup \{\ast\}$, the following statements hold:

1. $\overline{\ell}_n^*(\gamma, X, x) = 0$ if $\gamma < \gamma_n(X,x)$ and $\overline{\ell}_n^*(\gamma, X, x) \geq c|\gamma - \gamma_n|^{d-1}$ if $\gamma > \gamma_n(X,x)$.
2. $\overline{\ell}_n^*(\gamma, X, x) = 0$ if $\gamma > \gamma_n(X,x)$ and $\overline{\ell}_n^*(\gamma, X, x) \geq c|\gamma - \gamma_n|^{d-1}$ if $\gamma < \gamma_n(X,x)$.
3. If $n \in \mathbb{N} \cup \{\ast\}$ and, for each $\nu = (n, \mu, X, R)$, $v^{\nu,x}$ is the solution of the Dirichlet problem

$$
\begin{cases}
-F^x(e_n, \tilde{X}_{e_n} + D_{e_n}^2 v^{\nu,x}, y) = \gamma_n(X,x) & \text{in } Q^{(n)}_R, \\
v^{\nu,x} = 0 & \text{on } \partial Q^{(n)}_R,
\end{cases}
$$

then

$$
\lim_{R \to \infty} \sup \left\{ R^{-2} |v^{\nu(R),x}(y)| \mid y \in Q^{(n)}_R \right\} = 0.
$$

(The constant $c > 0$ depends on $\lambda$, $\Lambda$, and $d$, but not on $n$.)

In addition, we will need the following fact, adapted from [16], that follows from the homogenization result above:

**Proposition 13.** For each $(\gamma, X, x) \in \mathbb{R} \times S_d \times \mathbb{T}^d$, we have

$$
\overline{\ell}_n^*(\gamma, X, x) \geq \limsup_{n \to \infty} \left\{ \overline{\ell}_n^*(\gamma, X, x') \mid x' \in \mathbb{T}^d \right\},
$$

$$
\ell_n^*(\gamma, X, x) \geq \limsup_{n \to \infty} \left\{ \ell_n^*(\mu, X, x') \mid x' \in \mathbb{T}^d \right\}.
$$

The proof uses the upper semi-continuity properties of the contact sets $\{K^{\nu,x}\}$. When $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \setminus \mathbb{R}Z^d$, this can be combined with Fatou’s Lemma by adapting the idea of [16] directly to our setting using Proposition 11.

When $(e_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}Z^d$, the situation is more delicate since the probability measures in the variational principle (53) depend on $n$. As pointed out by W.M. Feldman, the same argument still applies if we replace Fatou’s Lemma by the generalization due to Feinberg, Kasyanov, and Zadoianchuk [26].

**Proof.** We give the details for $(\overline{\ell}_n^*)_{n \in \mathbb{N}}$; the same basic idea also applies to $(\ell_n^*)_{n \in \mathbb{N}}$.

**Step 1: Property of Contact Sets**

Given $(\mu, X, R) \in \mathbb{R} \times S_d \times (0, \infty)$ and $x \in \mathbb{T}^d$, let $\nu_n = (n, \mu, X, R)$ and $\nu_* = (\ast, \mu, X, R)$, and assume that $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{T}^d$ satisfies $\lim_{n \to \infty} x_n = x$. We claim that

$$
\mathcal{H}^{d-1}(K^{\nu_*}) \geq \limsup_{n \to \infty} \mathcal{H}^{d-1}(K^{\nu_n,x_n}).
$$

To see this, first, define $\tilde{K} \subseteq Q_R^*$ by

$$
\tilde{K} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{ y \in Q_R^* \mid u^{\nu_m,x_m}(O_m^{-1}(y)) = 0 \}.
$$
Notice that Proposition 11 implies that \( \tilde{K} \subseteq K^{\nu,x} \). Therefore, using the measure preserving property of the orthogonal transformations \( \{O_m\}_{m \in \mathbb{N}} \), we find
\[
\mathcal{H}^{d-1}(K^{\nu,x}) \geq \lim_{n \to \infty} \mathcal{H}^{d-1}(\tilde{K}) \\
= \lim_{n \to \infty} \mathcal{H}^{d-1}\left( \bigcup_{m=0}^{\infty} \left\{ y \in Q^*_R \mid u^m(O^{-1}m(y)) = 0 \right\} \right) \\
\geq \limsup_{n \to \infty} \mathcal{H}^{d-1}(K^{\nu_n,x_n}).
\]

**Step 2: Convenient Reformulation**

The result of the previous step can be reformulated slightly. Define Borel functions \( \{f_n\}_{n \in \mathbb{N}} \) and \( f_* \) in \( \mathbb{T}^d \) by
\[
f_n(x) = \mathcal{H}^{d-1}(K^{\nu_n,x}), \quad f_*(x) = \mathcal{H}^{d-1}(K^{\nu_*x}).
\]
By Step 1, for each \( x \in \mathbb{T}^d \), we have
\[
\lim_{\delta \to 0^+} \sup \left\{ f_n(y) \mid n^{-1} + |x - y| < \delta \right\} \leq f_*(x).
\]

**Step 2: Conclusion**

We conclude the proof by considering two cases: (i) \( (e_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \setminus \mathbb{R}^d \) and (ii) \( (e_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^d \). Note that there is no loss of generality in assuming that either (i) or (ii) holds since we can always pass to subsequences if necessary.

In either case, we start by fixing an \( \epsilon > 0 \). By invoking (54), we can fix an \( R > 0 \) such that
\[
\bar{\ell}^*_n(\gamma, X, x) \geq R^{1-d} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_*x}) \, dx - \epsilon.
\]
Now we consider cases (i) and (ii) separately.

In case (i), Fatou’s Lemma, the conclusion of Step 1, and (52) combine to give
\[
\int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_n-y}) \, dy \geq \limsup_{n \to \infty} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_n-y}) \, dy \geq \limsup_{n \to \infty} R_{\mathbb{R}}^{d-1} \sup \left\{ \bar{\ell}_n(\gamma, X, x') \mid x' \in \mathbb{T}^d \right\}.
\]
Thus,
\[
\bar{\ell}_*(\gamma, X) \geq \limsup_{n \to \infty} \sup \left\{ \bar{\ell}_n(\gamma, X, x') \mid x' \in \mathbb{T}^d \right\} - \epsilon.
\]

In case (ii), we can combine (53), (58), Lemma 8 and the generalization of Fatou’s Lemma in [26, Theorem 1.1] to find, for each \( (s_n)_{n \in \mathbb{N}} \subseteq [0, \infty) \),
\[
\int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_n,y}) \, dy \geq \limsup_{n \to \infty} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_n,y}) \mathcal{H}^{d-1}(d\xi) \geq \limsup_{n \to \infty} R_{\mathbb{R}}^{d-1} \bar{\ell}_n(\gamma, X, s_n e_n).
\]
Since \( (s_n)_{n \in \mathbb{N}} \) was arbitrary, we conclude
\[
\bar{\ell}_*(\gamma, X) \geq \limsup_{n \to \infty} \sup \left\{ \bar{\ell}_n(\gamma, X, x') \mid x' \in \mathbb{T}^d \right\} - \epsilon.
\]
In any case, the arbitrariness of \( \epsilon > 0 \) gives the desired result. \( \square \)

Finally, combining Propositions 12 and 13, we obtain
Proposition 14. For each \((X, x) \in S_d \times \mathbb{T}^d\), we have
\[
\lim_{n \to \infty} \sup \{|\gamma_n(X, x') - \gamma_\ast(X, x)| \mid x' \in \mathbb{T}^d\} = 0.
\]

Proof. Note that it is enough to prove the following two inequalities:
\[
\gamma_\ast(X, x) \geq \limsup_{n \to \infty} \sup \{\mu_n(X, x') \mid x' \in \mathbb{T}^d\},
\]
\[
\gamma_\ast(X, x) \leq \liminf_{n \to \infty} \inf \{\mu_n(X, x') \mid x' \in \mathbb{T}^d\}.
\]

We will only prove the first inequality since the second one follows by a similar argument in which \((\bar{\ell}_n)_{n \in \mathbb{N}}\) replaces \((\ell_n)_{n \in \mathbb{N}}\).

Choose a sequence \((x_n)_{n \in \mathbb{N}} \subseteq \mathbb{T}^d\) such that
\[
\limsup_{n \to \infty} \sup \{\gamma_n(X, x') \mid x' \in \mathbb{T}^d\} = \lim_{n \to \infty} \gamma_n(X, x_n).
\]
To obtain the desired result, we will show that if \(\gamma < \lim_{n \to \infty} \gamma_n(X, x_n)\), then \(\gamma < \gamma_\ast(X, x)\).

To see this, suppose that \(\gamma < \limsup_{n \to \infty} \gamma_n(X, x_n)\). By Proposition 12
\[
\ell_n(\gamma, X, x) \geq \limsup_{n \to \infty} \ell_n(\gamma, X, x_n) \geq c \limsup_{n \to \infty} (\gamma_n - \gamma)^{d-1} > 0.
\]
From this, the same proposition with \(n = *\) yields \(\gamma < \gamma_\ast(X, x)\).

\[\Box\]

6.6. Proof of Proposition 10. The results of the previous section directly imply Proposition 10 as we now show.

To start with, we note that the functions \((\gamma_n)_{n \in \mathbb{N}}\) of Proposition 12 are precisely the oscillating functions \((F_{\epsilon_n})_{n \in \mathbb{N}}\).

Lemma 1. For each \(n \in \mathbb{N}\) and \((X, x) \in S_d \times \mathbb{T}^d\), the following identity holds:
\[
\gamma_n(X, x) = -F_{\epsilon_n}(X, x).
\]

Proof. Fix \(n \in \mathbb{N}\) and \((X, x) \in S_d \times \mathbb{T}^d\). By Proposition 12 if we let \((v_\epsilon)_{\epsilon > 0}\) be the solutions of the Dirichlet problem
\[
\begin{cases}
-F(x_n, X_{\epsilon_n} + D_{\epsilon_n}^2 v_\epsilon, \epsilon^{-1} x') = \mu_n(X) & \text{in } Q_1^{(n)}, \\
v_\epsilon = 0 & \text{on } \partial Q_1^{(n)},
\end{cases}
\]
then \(v_\epsilon \to 0\) uniformly in \(Q_1^{(n)}\) as \(\epsilon \to 0^+\).

At the same time, this is an almost periodic elliptic homogenization problem in \((\epsilon_n)_\perp\) and we know (e.g. using the approximate correctors of Theorem 7) that \(v_\epsilon \to \bar{v}\), where \(\bar{v}\) is the solution of the constant coefficient equation
\[
\begin{cases}
-F_{\epsilon_n}(X_{\epsilon_n} + D_{\epsilon_n}^2 \bar{v}, x) = \gamma_n(X, x) & \text{in } Q_1^{(n)}, \\
\bar{v} = 0 & \text{on } \partial Q_1^{(n)}
\end{cases}
\]
The previous paragraph says that \(\bar{v} \equiv 0\) in \(Q_1^{(n)}\). Therefore,
\[
\gamma_n(X, x) = -F_{\epsilon_n}(X_{\epsilon_n}, x).
\]
\[\Box\]
Now Proposition 10 follows by combining Lemma 1 with Proposition 14.

Before moving on to the proofs of Proposition 9 and Theorem 1, let us identify the function $\gamma_*$. We begin with the setting of Theorem 1.

**Lemma 2.** If $d = 2$, then, for each $x \in \mathbb{T}^d$, the function $-\gamma_*(\cdot, x) : \mathcal{S}_d \to \mathbb{R}$ is the homogenized coefficient associated with the (e.g. Dirichlet) elliptic homogenization problem:

$$
\begin{cases}
U - F_e^\perp(D^2 U, e^{-1}(x', e^\perp)e) = 0 & \text{in } Q_1^*

U = G & \text{on } \partial Q_1^*
\end{cases}
$$

(59)

Here $e^\perp$ is any unit vector with $\langle e^\perp, e \rangle = 0$. In particular, $\gamma_*$ is independent of $\eta$.

**Proof.** Fix $e^\perp \in S^1 \cap \langle e \rangle^\perp$ and note that $S^1 \cap \langle e \rangle^\perp = \{e^\perp, -e^\perp\}$.

Arguing as in Lemma 1 and replacing $e^\perp$ by $-e^\perp$ if necessary in (59), we see that $-\mu_*(X)$ is the homogenized coefficient associated with (59). At the same time, a direct manipulation of (59) shows this coefficient is unchanged if $e^\perp$ is replaced by $-e^\perp$. Therefore, $-\mu_*(X)$ does not depend on $\eta$. □

The quasi-linear problem can be treated in the same way. However, as soon as $d \geq 3$, the limiting coefficient $\mu_*$ has a non-trivial dependence on the direction $\eta$.

**Lemma 3.** If $m$ and $a$ satisfy the assumptions of Theorem 2 and $F(p, X, y) = m(y, \hat{p})^{-1}\text{tr}(a(y, \hat{p})\hat{X}_p)$, then, for each $x \in \mathbb{T}^d$,

$$
-\gamma_*(X, x) = (\tilde{m}_e^*)^{-1}\text{tr}(\tilde{a}_e^* \hat{X}_p),
$$

where $\tilde{a}_e^*$ and $\tilde{m}_e^*$ are given by

$$
\tilde{a}_e^* = \int_{\mathbb{T}^d} a(y, e) \hat{\mu}_e^*(dy), \quad \tilde{m}_e^* = \int_{\mathbb{T}^d} m(y, e) \hat{\mu}_e^*(dy).
$$

(Here $\hat{\mu}_e^*$ is the measure defined in Theorem 4.)

**Proof.** As in Lemma 2, $-\gamma_*(X, x)$ is the homogenized coefficient associated with the homogenization problem

$$
\begin{cases}
m_e^\perp(e^{-1}(x', \eta)e) - \text{tr}(a_e^\perp(e^{-1}(x', \eta)e)D^2 U) = 0 & \text{in } Q_1^*,

U = G & \text{on } \partial Q_1^*
\end{cases}
$$

Since the coefficients only vary in the $\eta$ direction, the homogenized coefficients are the same as if we consider the one-dimensional problem. In particular, by a well-known computation, $-\gamma_*(X, x)$ is given by (60), where $\tilde{a}_e^*$ and $\tilde{m}_e^*$ are defined by

$$
\tilde{a}_e^* = \left(\int_0^{r_e} \langle a^\perp_e(se)\eta, \eta \rangle^{-1} ds\right)^{-1} \int_0^{r_e} a^\perp_e(se)\langle a^\perp_e(se)\eta, \eta \rangle^{-1} ds,
$$

$$
\tilde{m}_e^* = \left(\int_0^{r_e} \langle a^\perp_e(se)\eta, \eta \rangle^{-1} ds\right)^{-1} \int_0^{r_e} m^\perp_e(se)\langle a^\perp_e(se)\eta, \eta \rangle^{-1} ds.
$$

We conclude by recalling the definitions of $a^\perp_e$ and $m^\perp_e$ (see (21) and (13)). □
6.7. **Proofs of Propositions 9 and Theorem 4** By applying Proposition 10, we readily deduce the continuity of $\overline{F}$ in $d = 2$ and the limiting behavior of the set-valued maps $e \in \mathcal{J}_a^n$ in the quasi-linear setting.

**Proof of Proposition 10.** Since Proposition 10 applies to an arbitrary sequence and $\gamma_{s,\eta}$ is independent of $\eta$ by Lemma 2, we deduce that, for each $e \in S^1 \cap \mathbb{R}\mathbb{Z}^2$,

$$\lim_{\delta \to 0^+} \sup \left\{ |\overline{F}(e', X) + \gamma_{s,e}^\perp(X)| \mid e' \in S^1 \setminus \mathbb{R}\mathbb{Z}^2, \|e' - e\| < \delta \right\} = 0.$$ 

Hence $\overline{F}$ extends continuously to $(\mathbb{R}^2 \setminus \{0\}) \times S_2$. □

In light of the form of $\{F_e^\perp\}$ in the quasi-linear case, a similar argument implies Theorem 4 as we now show:

**Proof of Theorem 4.** Notice that to obtain the conclusion of Theorem 4, it suffices to show that if $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1}$ is such that $e_n \to e$ and $\frac{e_n - e}{\|e_n - e\|} \to -\eta$ as $n \to \infty$, then, for each positive $m \in C^{\infty}(\mathbb{T}^d)$ and each $(s_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$, we have

$$\lim_{n \to \infty} \int_{\mathbb{T}^d} m(y)\mu_{e_n}^m(dy) = \int_{\mathbb{T}^d} m(y)\mu_{e}^m(dy).$$

(Here define $\mu_{e'}^m = \mu_{e'}$ if $s \in \mathbb{R}$ and $e' \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^{d,*}$. This follows directly from Proposition 10 Lemmas 1 and 3 and (43) by varying $m$ while $a$ remains fixed. □

6.8. **Generic Discontinuities.** In light of the formulas obtained for the limiting measures in the previous section, it is natural to expect that $\bar{a}$ and $\bar{m}$ are generically discontinuous at some rational directions. In this section, we prove that, in fact, $\bar{a}$ and $\bar{m}$ are generically discontinuous at every rational direction when $d \geq 3$.

**Proof of Corollary 1.** To start with, since $d \geq 3$, we can fix $e \in S^{d-1}$ and let $(\eta_n)_{n \in \mathbb{N}}$ be a sequence of points in $S^{d-1} \cap (e)^\perp$ with $\eta_n \notin \{\eta_m, -\eta_m\}$ for all $n \neq m$. The goal is to prove that, for each $n, m \in \mathbb{N}$ with $n \neq m$, the following sets are open and dense in $C^{2,\alpha}(\mathbb{T}^d; S_d(\lambda, \Lambda))$ in the $C^{2,\alpha}$ norm topology:

$$\mathcal{U}_e(n, m) = \left\{ a \in C^{2,\alpha}(\mathbb{T}^d; S_d(\lambda, \Lambda)) \mid \mu_{e}^m \neq \tilde{\mu}_e^m \right\},$$

$$\mathcal{V}_e(n, m) = \left\{ a \in C^{2,\alpha}(\mathbb{T}^d; S_d(\lambda, \Lambda)) \mid \tilde{a}_{e}^m \neq \tilde{\bar{a}}_e^m \right\}.$$

That these sets are open is immediate. It only remains to show they are dense.

We start with $\mathcal{U}_e(n, m)$. It turns out that we only need to understand the derivative of the map $a \mapsto a_e^\perp$. Toward that end, for each $n \in \mathbb{N}$, define $u_n \in C^{2,\alpha}(\mathbb{T}^d)$ to be the solution of the cell problem (44) with $f(y) = \langle a(y)\eta_n, \eta_n \rangle$. Notice that the oscillating function $f_e^\perp$ equals $\langle a_e^\perp \eta_n, \eta_n \rangle$ in this case.

**Step 1: Perturb so that $D^2_e u_n \neq D^2_e u_m$**

To start with, we claim that there is no loss of generality in assuming that $D^2_e u_n \neq D^2_e u_m$. Indeed, if $D^2_e u_n = D^2_e u_m$, then (44) implies $\langle [a - a_e^\perp] \eta_n, \eta_n \rangle = \langle [a - a_e^\perp] \eta_m, \eta_m \rangle$. That is,

$$\langle a(y)\eta_n, \eta_n \rangle - \langle a(y)\eta_m, \eta_m \rangle = \langle a_e^\perp(\langle y, e \rangle) \eta_n, \eta_n \rangle - \langle a_e^\perp(\langle y, e \rangle) \eta_m, \eta_m \rangle$$

if $y \in \mathbb{T}^d$. 

P.S. MORFE
In particular, the left-hand side varies only in the $e$ direction. This symmetry is easily broken, for instance, by replacing $a$ by $y \mapsto a(y) + \nu \cos(2\pi(k,y))\eta_h \otimes \eta_h$ for some $k \in \mathbb{Z}^d \setminus \langle e \rangle$ and $\nu \in \mathbb{R}$ sufficiently small.

**Step 2: Restrict attention to** $a_e^\perp$

First, observe that $\tilde{\mu}_{e}^{h} = \tilde{\mu}_{e}^{m}$ if and only if there is a $C > 0$ such that $\langle a_e^\perp \eta_h, \eta_n \rangle = C\langle a_e^\perp \eta_m, \eta_m \rangle$ in $\mathbb{T}^d$. This is a direct consequence of the formula (20) proved in the previous section. We claim that if $\langle a_e^\perp \eta_h, \eta_n \rangle = C\langle a_e^\perp \eta_m, \eta_m \rangle$ for some $C > 0$, then this symmetry is broken by some arbitrarily small perturbation of $a$.

To see this, we will differentiate the function $a \mapsto a_e^\perp$. Given $a_s \in C^2,\alpha(\mathbb{T}^d;\mathcal{S}_d)$ and $h \in \mathbb{R}$ small enough, define $a_h = a + ha_s$ and let $(a_h)_e^\perp$ be the associated averaged tensor.

Employing arguments similar to those in the proof of Proposition 7, we see that, for each $r \in [0,r_e)$, $h \in \mathbb{R}$, and $j \in \{n,m\}$, if $\tilde{U}_{j, \alpha}^h$ is the solution of the equation

$$-\text{tr} \left( a_h(re + x') D^2\tilde{U}_{j, \alpha}^h \right) = \langle a_h(re + x') \eta_j, \eta_j \rangle - \langle (a_h)_e^\perp(re) \eta_j, \eta_j \rangle \quad \text{in } \mathbb{T}^d_1(0)$$

then there are functions $\tilde{U}_{n, \alpha}^h, \tilde{U}_{m, \alpha}^h \in C(\mathbb{T}^d_1(0))$ and a function $a_e^\perp \in C(\mathbb{T}^d)$ varying only in the $e$ direction such that, for $j \in \{n,m\}$ and $r \in [0,r_e)$,

$$\tilde{U}_{j, \alpha}^h = \lim_{h \to 0} \frac{\tilde{U}_{j, \alpha}^h - \tilde{U}_{j, \alpha}^0}{h} \quad \text{uniformly in } \mathbb{T}^d_1(0), \quad a_e^\perp = \lim_{h \to 0} \frac{(a_h)_e^\perp - a_e^\perp}{h} \quad \text{uniformly in } \mathbb{T}^d.$$

Furthermore, $\tilde{U}_{j, \alpha}^h$ is a solution of the equation

$$-\text{tr} \left( (a(re + x')) D^2\tilde{U}_{j, \alpha}^h \right) = \text{tr} \left( a_s(re + x') D^2 u_j(re + x') \right) - \langle a_e^\perp(re) \eta_j, \eta_j \rangle \quad \text{in } \mathbb{T}^d_1(0).$$

Notice that $a_e^\perp(y) = \int_{\mathbb{T}^d} \text{tr} (a_s(y') D^2 u_j(y')) \mu_e^{(y,c)}(dy')$.

By the previous step, there is no loss in generality assuming that $D^2 u_n \neq D^2 u_m$. In fact, since these functions are $C^\alpha$, we can fix $y_1, y_2 \in \mathbb{T}^d$ so that $\langle y_1, e \rangle \neq \langle y_2, e \rangle$ and $D^2 u_n(y_i) \neq D^2 u_m(y_i)$ for $i \in \{1,2\}$. Since $\mu_e^{(y,c)}$ is supported on $\mathbb{T}^d_1(s)$ for each $s \in [0,r_e)$, we can we fix $a_s \in C^\alpha(\mathbb{T}^d;\mathcal{S}_d)$ so that

$$\int_{\mathbb{T}^d} \text{tr} (a_s(y') D^2 u_n(y')) \mu_e^{(y_1,c)}(dy') > 0 > \int_{\mathbb{T}^d} \text{tr} (a_s(y') D^2 u_m(y')) \mu_e^{(y_2,c)}(dy'),$$

$$\int_{\mathbb{T}^d} \text{tr} (a_s(y') D^2 u_n(y')) \mu_e^{(y_2,c)}(dy') < 0 < \int_{\mathbb{T}^d} \text{tr} (a_s(y') D^2 u_m(y')) \mu_e^{(y_2,c)}(dy').$$

From this, we find that, for all $h > 0$ sufficiently small,

$$\frac{\langle (a_h)_e^\perp(y_1) \eta_m, \eta_m \rangle}{\langle (a_h)_e^\perp(y_2) \eta_m, \eta_m \rangle} < \frac{\langle (a_h)_e^\perp(y_1) \eta_m, \eta_m \rangle}{\langle (a_h)_e^\perp(y_2) \eta_m, \eta_m \rangle} \quad \text{for } a - a_h \in C^\alpha(\mathbb{T}^d) \leq Ch.$$

Therefore, $\langle (a_h)_e^\perp \eta_n, \eta_n \rangle$ is not a constant multiple of $\langle (a_h)_e^\perp \eta_m, \eta_m \rangle$ even while $\|a - a_h\|_{C^\alpha(\mathbb{T}^d)} \leq Ch$.

We conclude from the preceding that $\mathcal{U}_e(n, m)$ is dense in $C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda))$. It remains to prove the same thing for $V_e(n, m)$.

**Step 3: Ensure** $\tilde{\alpha}_{e}^{mn} \neq \tilde{\alpha}_{e}^{mn}$
We want to show that $\mathcal{V}_e(n, m)$ is dense. Given that $\tilde{a}_e^\eta = \int_{\mathbb{T}_d} a(y) \tilde{\mu}_e^\eta(dy)$, this is intuitively clear given what we just proved.

The previous arguments show that we can assume that $a \in \mathcal{U}_e(n, m) \setminus \mathcal{V}_e(n, m)$ to start with. We will show that there is an $m \in C^{2,\alpha}(\mathbb{T}_d)$ varying only in the $e$ direction such that the function $a_h = (1 + hm)^{-\alpha}$ is in $\mathcal{V}_e(n, m)$ for all $h \in \mathbb{R}$ sufficiently small.

To start with, notice that, by arguing as in Section 4.4, we see that, for each $C$ in $\mathbb{R}$, the set of $\mathbb{R}$ is residual, being a countable intersection of open, dense sets. Further, since $u(\mathbb{R})$, this is $\mathcal{U}_e(n, m)$, we know that $\langle a_e^\perp(\iota e, \iota m) \rangle$ for all $\mathbb{R}$ small enough. This proves $a$ is a limit point of $\mathcal{V}_e(n, m)$.

**Conclusion**

We showed that $\mathcal{U}_e(n, m)$ and $\mathcal{V}_e(n, m)$ are both open and dense in $C^{2,\alpha}(\mathbb{T}_d; \mathcal{S}_d(\lambda, \Lambda))$ in the $C^{2,\alpha}$ norm topology. Define $\mathcal{C}_d$ by

$$
\mathcal{C}_d = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N} \setminus \{n\}} \mathcal{U}_e(n, m) \cap \mathcal{V}_e(n, m).
$$

This set is residual, being a countable intersection of open, dense sets. Further, since $C^{2,\alpha}(\mathbb{T}_d; \mathcal{S}_d(\lambda, \Lambda))$ is an open subset of the Banach space $C^{2,\alpha}(\mathbb{T}_d; \mathcal{S}_d)$, $\mathcal{C}_d$ is itself dense. \hfill \square

7. Comparison Principle

In this section, we prove the well-posedness of (16), completing the proof of Theorem 2.

To clarify the exposition, we will consider the following general framework. Given operators $G^*, G_* : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ and a countable set $\{e_n\}_{n \in \mathbb{N}} \subseteq \mathcal{S}^{d-1}$ of “bad” directions, we consider sub- and supersolutions of the viscosity inequalities

$$
 u_t - G^*(D^2 u) \leq 0, \quad u_t - G_* (D^2 u) \geq 0 \quad \text{in } \mathbb{R}^d \times (0, T).
$$

Here we assume that the operators $G^*, G_* : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ satisfy the following assumptions:

(i) (Geometric) If $G \in \{G^*, G_*\}$, $(p, X) \in \mathbb{R}^d \times \mathcal{S}_d$, $\mu \in \mathbb{R}$, and $\kappa > 0$, then

$$
 G(\kappa p, \kappa X + \mu p \otimes p) = \kappa G(p, X).
$$

(ii) (Strongly degenerate elliptic) There are constants $\lambda, \Lambda > 0$ such that if $G \in \{G^*, G_*\}$, $p \in \mathbb{R}^d \setminus \{0\}$, $X, Y \in \mathcal{S}_d$, and $Y \geq 0$, then

$$
 \lambda \|\tilde{Y}_p\| \leq G(p, X + Y) - G(p, X) \leq \Lambda \|\tilde{Y}_p\|.
$$
(iii) (Stationary planes) \( G^*(e,0) = G_*(e,0) = 0 \) for each \( e \in S^{d-1} \).

(iv) (Semi-continuity) \( G^* \) is upper semi-continuous, \( G_* \) is lower semi-continuous, \( (G^*)_r = G_* \), and \( (G_*)^* = G^* \).

(v) (Continuity at “good” directions) If \( (p,X) \in (\mathbb{R}^d \setminus \{0\}) \times S_d \) and \( \hat{p} \notin \{e_n\}_{n \in \mathbb{N}} \), then

\[
G^*(p,X) = G_*(p,X).
\]

(vi) (Controlled oscillation) The discontinuities of \( G^* \) and \( G_* \) can be controlled in the following manner:

\[
\lim_{N \to \infty} \sup_{X \in S_d} \left\{ \frac{G^*(e_n,X) - G_*(e_n,X)}{1 + \|X\|} \right\} = 0.
\]

In the next subsection, we will show that the assumptions above are satisfied when \( G^* = F^* \) and \( G_* = F_* \). Thus, a comparison result for operators satisfying (i)-(vi) implies homogenization in Theorem 2 just as the usual one did in Theorem 1.

The remainder of the section is devoted to the proof of just such a comparison principle, stated next:

**Theorem 10.** Assume that \( u : \mathbb{R}^d \times (0,T) \to \mathbb{R} \) is a locally bounded, upper semi-continuous subsolution of (62) and \( v : \mathbb{R}^d \times (0,T) \to \mathbb{R} \) is a locally bounded, lower semi-continuous supersolution. If (i)-(vi) all hold and \( u \) and \( v \) satisfy the following condition

\[
\lim_{\delta \to 0^+} \sup_{\|x - y\| < \delta} \left\{ u^*(x,0) - v_*(y,0) \right\} \leq 0,
\]

then \( u \leq v \) in \( \mathbb{R}^d \times (0,T) \).

The idea of the proof is this: assumption (v) implies that, for each \( \beta > 0 \), \( F^* \) and \( F_* \) almost coincide (up to a \( \beta \) error) except at finitely many rational directions. The papers of Gurtin, Soner, and Souganidis [36], Ohnuma and Sato [47], and Ishii [37] show how to prove a comparison principle in the case when \( G^* \) and \( G_* \) coincide at all but finitely many directions. Therefore, if we can manage the \( \beta \) error, a comparison principle should hold in our setting as well.

### 7.1. Application to \( \overline{F} \).

Let us verify that the semi-continuous envelopes \( \overline{F}^* \) and \( \overline{F}_* \) of the effective operator \( \overline{F} \) of (37) satisfy assumptions (i)-(vi) above with \( \{e_n\}_{n \in \mathbb{N}} \) being any enumeration of \( S^{d-1} \cap \mathbb{R}Z^d \). For the sake of completeness, we will then recall how Theorem 10 can be combined with what we previously proved to give Theorem 2. Finally, in a concluding remark, we note that the limiting behavior of the forced motion (22) cannot be described by a level set PDE with a comparison principle.

Let us start with assumptions (i)-(v).

**Lemma 4.** If \( \{e_n\}_{n \in \mathbb{N}} \) is any enumeration of \( S^{d-1} \cap \mathbb{R}Z^d \), then the pair \( (\overline{F}^*, \overline{F}_*) \) satisfies assumptions (i)-(v) with “bad directions” \( \{e_n\}_{n \in \mathbb{N}} \).

**Proof.** (i) follows by the definition (37) and (ii) and (iii) were proved in Corollary 4. (iv) is a direct consequence of the definition of \( \overline{F}^* \) and \( \overline{F}_* \) (see Section 3).

It only remains to prove (v). If \( e \in S^{d-1} \setminus \mathbb{R}Z^d \) and \( (e_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \), then, for any \( (s_n)_{n \in \mathbb{N}} \subseteq (0,\infty) \), each accumulation point of the measures \( (\mu_{e_n}^{s_n})_{n \in \mathbb{N}} \) is necessarily in
Lemma 5. The pair $(\omega^a_\varepsilon, \mu^a_\varepsilon)$, we have just prior to Theorem 4. Notice that, in view of Theorem 4 and the definition (37), we expect that there is a modulus $\omega : [0, \infty) \to [0, \infty)$ with $\lim_{\delta \to 0^+} \omega(\delta) = 0$ such that, for each $e \in S^{d-1} \cap \mathbb{R}Z^d$, the following estimate holds:

\[
\sup \left\{ \frac{F^\perp(e, X) - F^\perp(e, X)}{1 + \|X\|} \mid X \in S_d \right\} \leq \omega(r_e).
\]

When $d = 2$, it is not hard to show that if $F$ satisfies (30), then there is a constant $A > 0$ such that

\[
\sup \left\{ \frac{F^\perp(e, x) - F^\perp(e, y)}{1 + \|X\|} \mid X \in S_d, x, y \in \mathbb{T}^d \right\} \leq Ar_e.
\]

If such an estimate were to hold in higher dimensions (possibly with $Ar_e$ replaced by $\omega(r_e)$), then it would imply (63). However, this remains to be seen.

Instead, we employ a soft argument pointed out by I.C. Kim:

Lemma 5. The pair $(\mathcal{F}^a, \mathcal{F}^\ast_a)$ satisfies (vi) with $\{e_n\}_{n \in \mathbb{N}}$ any enumeration of $S^{d-1} \cap \mathbb{R}Z^d$.

Proof. We claim that

\[
\lim_{N \to \infty} \sup \left\{ \text{diam}(\mathcal{J}_e^a) \mid e \geq N \right\} = 0.
\]

Here $\text{diam}(\mathcal{J}_e^a)$ is the diameter of $\mathcal{J}_e^a$ with respect to $D$, the metric on $\mathcal{P}(\mathbb{T}^d)$ chosen just prior to Theorem 4. Notice that, in view of Theorem 4 and the definition (37) of $\mathcal{F}^a$, the claim implies (vi) holds.

To prove it, we argue by contradiction, exploiting the compactness of $S^{d-1}$. If (vi) fails, then we can find $\zeta > 0$, $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1}$, and sequences $(s_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that

\[
\inf \left\{ D(\mu^{s_n}_{e_n}, \mu^{t_n}_{e_n}) \mid n \in \mathbb{N} \right\} \geq \zeta.
\]

To see this is impossible, note that, up to extraction, we can assume that there is an $e \in S^{d-1}$ such that $\lim_{n \to \infty} e_n = e$.

If $e \in S^{d-1} \setminus \mathbb{R}Z^d$, then any accumulation point of $(\mu^{s_n}_{e_n})_{n \in \mathbb{N}}$ or $(\mu^{t_n}_{e_n})_{n \in \mathbb{N}}$ is in $\mathcal{J}_e^a$, hence must equal $\mu_e$. In particular, $\mu^{s_n}_{e_n} \overset{a}{\to} \mu_e$ and $\mu^{t_n}_{e_n} \overset{a}{\to} \mu_e$, contradicting (64).

On the other hand, if $e \in S^{d-1} \setminus \mathbb{R}Z^d$, then, passing to another subsequence if necessary, we can assume that there is an $\eta \in S^{d-1} \cap \langle e \rangle^\perp$ such that $\frac{e_n - e}{\|e_n - e\|} \to -\eta$ as $n \to \infty$. By Theorem 4, this implies $\mu^{s_n}_{e_n} \overset{a}{\to} \mu_e^\eta$ and $\mu^{t_n}_{e_n} \overset{a}{\to} \mu_e^\eta$, another contradiction. \qed
Now, for the sake of completeness, observe that we can combine Theorem 10 with the results of Section 5 to prove Theorem 2.

**Proof of Theorem 2.** By Proposition 8 and Theorem 9, the half-relaxed limits \( \bar{u}^* \) and \( \bar{u}_* \) of \( (u^\epsilon)_\epsilon > 0 \) are, respectively, sub- and supersolution of (16) in the usual viscosity sense, and we can show that \( \bar{u}^* \leq u_0 \leq \bar{u}_* \) by constructing appropriate \( \epsilon \)-independent sub- and super-solutions of (2). Hence Theorem 10 implies \( \bar{u}^* \leq \bar{u}_* \), from which \( \bar{u}_* \) necessarily follows. Denoting this function by \( \bar{u}_* \), we see that it is a continuous viscosity solution of (16), necessarily unique, and \( u^\epsilon \to \bar{u}_* \).

A last remark is in order:

**Remark 7.** Since \( m_{\text{pl}} \) can be discontinuous, Theorem 5 shows that, in general, the limiting behavior of (22) with \( \alpha \neq 0 \) and arbitrary initial data \( u_0 \in UC(\mathbb{R}^d) \) cannot be described by an effective equation with a comparison principle, unlike the \( \alpha = 0 \) case. The reason is that comparison forces the solution map to be continuous with respect to the topology of local uniform convergence, whereas Theorem 5 shows that any solution map, if well-defined, has to act discontinuously on linear functions when \( m_{\text{pl}} \) is discontinuous. It remains to be seen what can be said about these problems.

### 7.2. Proof of Theorem 10

As in [37], the proof of Theorem 10 proceeds by replacing the Euclidean norm by some other Finsler norm in a variable doubling argument. (Recall that \( \psi : \mathbb{R}^d \to [0, \infty) \) is a Finsler norm if it is convex, positively one-homogeneous, and positive away from zero.) To improve the result of [37] from finitely many discontinuity points to our setting, we use the following fact:

**Proposition 15.** There is a universal constant \( c_0 > 0 \) such that if \( \{e_n\}_{n \in \mathbb{N}} \subseteq S^{d-1} \), then, for each \( N \in \mathbb{N} \), there is a Finsler norm \( \psi_N \in C^2(\mathbb{R}^d \setminus \{0\}) \) such that

\[
1 \leq \psi_N(e) \leq \frac{5}{4}, \quad D^2\psi_N(e) \leq c_0(\text{Id} - e \otimes e) \quad \text{if } e \in S^{d-1}
\]

and the following property holds: given \( p \in \mathbb{R}^d \setminus \{0\} \), if \( \hat{D}\psi_N(p) = e_i \) for some \( i \in \{1, 2, \ldots, N\} \), then \( D^2\psi_N(p) = 0 \).

Before proving Proposition 15, which follows from a relatively simple but tedious geometric construction, let us see how it implies Theorem 10.

**Proof of Theorem 10.** By (i), if \( \varphi : \mathbb{R} \to \mathbb{R} \) is a smooth, non-decreasing function, then \( \varphi(u) \) remains a subsolution and \( \varphi(v) \), a supersolution. Therefore, we can assume that \( u \) and \( v \) are bounded.

We argue by contradiction, assuming that the following inequality holds:

\[
\sup \{ u(x, t) - v(x, t) \mid (x, t) \in \mathbb{R}^d \times (0, T) \} > 0.
\]

It follows that we can fix \( \sigma > 0 \) small enough that

\[
\sup \{ u(x, t) - v(x, t) - \sigma t \mid (x, t) \in \mathbb{R}^d \times (0, T) \} > 0.
\]

Let \( \zeta, \beta, \gamma > 0 \) be free variables. By (v), we can fix an \( N = N(\gamma) \in \mathbb{N} \) such that

\[
\sup \{ G^*(e_i, X) - G_*(e_i, X) \mid i \in \mathbb{N} \setminus \{1, 2, \ldots, N\} \} \leq \gamma(1 + \|X\|) \quad \text{if } X \in S_d.
\]
Letting $\psi_N$ be the Finsler norm of Proposition 15 with $\{e_n\}_{n \in \mathbb{N}}$ the set of “bad” directions associated with the pair $(G^*, G_*)$, define $\Phi = \Phi_{\xi, \beta} : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$ by

$$\Phi(x, y, t) = u(x, t) - v(y, t) - \frac{\psi_N(x - y)^4}{4\zeta} - \frac{\beta||y||^2 - \sigma t}{2}.$$ 

Since $u$ and $v$ are bounded, $\Phi$ is bounded above and attains its maximum in $\mathbb{R}^d \times \mathbb{R}^d \times [0, T]$. Let $(\bar{x}, \bar{y}, \bar{t}) = (\bar{x}_{\xi, \beta}, \bar{y}_{\xi, \beta}, \bar{t}_{\xi, \beta})$ be such a global maximum, that is,

$$\Phi(x, y, t) = \max \{ \Phi(x, y, t) \mid (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \}.$$

By the boundedness of $u$ and $v$ and the lower bound $\psi_N \geq \| \cdot \|$, there is a $\gamma$-independent constant $C > 0$ such that

$$(66) \quad \sup \left\{ \frac{\beta\|\bar{y}_{\xi, \beta}\|^2}{2} + \frac{||\bar{x}_{\xi, \beta} - \bar{y}_{\xi, \beta}||^4}{\zeta} \mid (\zeta, \beta) \in (0, \infty) \times (0, 1) \right\} \leq C.$$

In view of (65) and the assumptions on $u$ and $v$, there are constants $\zeta_0, \beta_0 > 0$ such that $\tilde{t}_{\xi, \beta} > 0$ for all $(\zeta, \beta) \in (0, \zeta_0) \times (0, \beta_0)$ and all $\gamma \in (0, 1)$. Henceforth let $(\zeta, \beta) \in (0, \zeta_0) \times (0, \beta_0)$ and assume $\gamma < 1$.

Since $t > 0$, we can invoke the maximum principle for semi-continuous functions, [37] Lemma 1], and the equations satisfied by $u$ and $v$. This gives matrices $X, Y \in S_d$ and numbers $a, b \in \mathbb{R}$ so that if $\bar{A} = \bar{A}(\bar{x} - \bar{y}) \in S_d$ and $\bar{p} = \bar{p}(\bar{x} - \bar{y}) \in \mathbb{R}^d$ are defined by

$$\bar{p}(\bar{x} - \bar{y}) = \zeta^{-1}\psi_N(\bar{x} - \bar{y})^3D\psi_N(\bar{x} - \bar{y}),$$

$$\bar{A}(\bar{x} - \bar{y}) = \begin{cases} 
\zeta^{-1}\psi_N(\bar{x} - \bar{y})^3D^2\psi_N(\bar{x} - \bar{y}) + 3\zeta^{-1}\psi_N(\bar{x} - \bar{y})^2D\psi_N(\bar{x} - \bar{y})^2, & \text{if } \bar{x} \neq \bar{y}, \\
0, & \text{otherwise},
\end{cases}$$

then

$$\sigma = a - b, \quad -3 \begin{pmatrix} \bar{A} & 0 \\
0 & \bar{A} \end{pmatrix} \leq \begin{pmatrix} X & 0 \\
0 & -(Y + \beta \text{Id}) \end{pmatrix} \leq 3 \begin{pmatrix} \bar{A}^{-1} & -\bar{A}^{-1} \\
-\bar{A}^{-1} & \bar{A}^{-1} \end{pmatrix},$$

$$a - G^*(\bar{p}, X) \leq 0, \quad b - G^*(\bar{p} - \beta \bar{Y}, Y) \geq 0.$$

Note, in addition, that Proposition 15 and (66) yield the following $\beta$-independent estimates on $\|\bar{p}\|$ and $\|\bar{A}\|:

$$(67) \quad \|\bar{p}\| \leq \frac{5}{4}C^4\zeta^{-\frac{5}{4}}, \quad \|\bar{A}\| \leq \sqrt{C}\zeta^{-\frac{1}{4}}.$$ 

Hence, we can send $\beta \to 0^+$ and invoke (66) to obtain $\xi \in \mathbb{R}^d$, $\bar{p} = \bar{p}(\xi) \in \mathbb{R}^d$, $\bar{A} = \bar{A}(\xi) \in S_d$, and $\bar{X}, \bar{Y} \in S_d$ such that

$$\sigma + G^*(\bar{p}, \bar{Y}) - G^*(\bar{p}, \bar{X}) \leq 0, \quad -3 \begin{pmatrix} \bar{A} & 0 \\
0 & \bar{A} \end{pmatrix} \leq \begin{pmatrix} \bar{X} & 0 \\
0 & -\bar{Y} \end{pmatrix} \leq 3 \begin{pmatrix} \bar{A}^{-1} & -\bar{A}^{-1} \\
-\bar{A}^{-1} & \bar{A}^{-1} \end{pmatrix}.$$

There are four cases left to consider: (i) $\xi = 0$, (ii) $\bar{D}\psi_N(\xi) \in \{e_1, \ldots, e_N\}$, (iii) $\bar{D}\psi_N(\xi) \in \{e_{N+1}, e_{N+2}, \ldots\}$, and (iv) $\bar{D}\psi_N(\xi) \in S^{d-1} \setminus \mathbb{R}Z_d$. 

Case (i): $\xi = 0$
In this case, we have $\tilde{p} = 0$ and $\tilde{A} = 0$, hence $\tilde{X} = 0$ and $\tilde{Y} = 0$. This yields the estimate
\begin{equation}
(68) \quad \sigma \leq \sigma + G_*(0, 0) - G^*(0, 0) \leq 0.
\end{equation}

**Case (ii):** $\widehat{D\psi}_N(\xi) \in \{e_1, e_2, \ldots, e_N\}$

In this case, Proposition 15 implies that $D^2\psi_N(\xi) = 0$. Thus, $\tilde{A} = ce_i \otimes e_i$ for some $c > 0$ and $\|\tilde{p}\|^{-1} \tilde{p} = e_i$ so (i) and (vi) give
\begin{align*}
G^*(\tilde{p}, X) &\leq G^*(\tilde{p}, 3ce_i \otimes e_i) = 0, \quad G_*(\tilde{p}, \tilde{Y}) \geq G_*(\tilde{p}, -3ce_i \otimes e_i) = 0.
\end{align*}
Combining these estimates, we obtain
\begin{equation}
(69) \quad \sigma \leq \sigma + G_*(\tilde{p}, \tilde{Y}) - G^*(\tilde{p}, \tilde{X}) \leq 0.
\end{equation}

**Case (iii):** $\widehat{D\psi}_N(\xi) \in \{e_{N+1}, e_{N+2}, \ldots\}$

By the choice of $N$,
\begin{align*}
G_*(\tilde{p}, \tilde{Y}) \geq G^*(\tilde{p}, \tilde{Y}) - \gamma \|\tilde{Y}\| - \gamma \|\tilde{p}\| &\geq G^*(\tilde{p}, \tilde{Y}) - \gamma \|\tilde{p}\| - 3\gamma \|\tilde{A}\|.
\end{align*}
From this and the inequality $\tilde{X} \leq \tilde{Y}$, we find
\begin{equation}
(70) \quad \sigma - \gamma \|\tilde{p}\| - 3\gamma \|\tilde{A}\| \leq \sigma + G_*(\tilde{p}, \tilde{Y}) - G^*(\tilde{p}, \tilde{X}) \leq 0.
\end{equation}

**Case (iv):** $\widehat{D\psi}_N(\xi) \in S^{d-1} \setminus \mathbb{R}Z^d$

Here $G^*(\tilde{p}, \tilde{X}) = G_*(\tilde{p}, \tilde{X})$ and, thus,
\begin{align*}
G^*(\tilde{p}, \tilde{Y}) - G^*(\tilde{p}, \tilde{X}) = G_*(\tilde{p}, \tilde{Y}) - G_*(\tilde{p}, \tilde{X}) \geq 0.
\end{align*}
This gives our last estimate:
\begin{equation}
(71) \quad \sigma \leq \sigma + G_*(\tilde{p}, \tilde{Y}) - G^*(\tilde{p}, \tilde{X}) \leq 0.
\end{equation}
Combining (68), (69), (70), and (71), we conclude that
\begin{equation}
(72) \quad \sigma \leq (\|\tilde{p}\| + 3\|\tilde{A}\|)\gamma.
\end{equation}
However, in view of (67), this is a contradiction as soon as $\gamma$ is small enough compared to $\zeta$ and $\sigma$. \hfill $\square$

8. Effective Mobility as Linear Response

In this section, we prove Theorems 5 and 6 and their corollaries. As already indicated in the introduction, the proofs are relatively routine when $e \notin \mathbb{R}Z^d$. We will start by motivating the modifications that are necessary in the case that $e \in \mathbb{R}Z^d$. 

8.1. **Strategy of proof.** As already mentioned in Section 2, the analysis of the forced problem (22) requires an additional correction. Fix $e \in \mathbb{R}Z^d$, $\alpha \in \mathbb{R} \setminus \{0\}$, and let $(u_e^t)_{t>0}$ be the solutions of (22). Proceeding as in the proof of Theorems 1 and 2, we seek to identify the limit of $(u_e^t)_{t>0}$ by showing that the functions $\bar{u}_e^*$ and $\bar{u}_{e,e}$ are sub- and supersolutions of an effective equation.

Let us see what can be deduced naively about $\bar{u}_e^*$ using the perturbed test function method. As in Section 4.5, we let $\tilde{V}_e$ be the solution of the following cell problem:

\[(73) \quad m(y, e) - m_e^+(y) - \text{tr} \left( A(y, e) D^2 \tilde{V}_e \right) = 0 \quad \text{in} \, \mathbb{T}^d.\]

Suppose that $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$, $\varphi$ is smooth, and $\bar{u}_e^* - \varphi$ has a strict local maximum at $(x_0, t_0)$. Since we expect that $\bar{u}_e^*(x, t) = \langle x, e \rangle + \alpha \bar{m}_{pl}(e) t$ for some $\bar{m}_{pl}(e) > 0$, we may as well assume that $D\bar{u}(x_0, t_0) = e$.

With $\tilde{V}_e$ a solution of (73), define the perturbed test function $\varphi^\epsilon$ by

\[\varphi^\epsilon(x, t) = \varphi(x, t) + \epsilon^2 \varphi_t(x, t_0) \tilde{V}_e(\epsilon^{-1} x).\]

Letting $(x_\epsilon, t_\epsilon)$ be a local maximum of $u_e^* - \varphi^\epsilon$ close enough to $(x_0, t_0)$, we know that $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ (along a subsequence) and, thus, we can invoke the equation satisfied by $u_e^*$, which gives, after some simplification,

\[
\limsup_{\epsilon \to 0^+} \left[ m_e^+ (\epsilon^{-1} \langle x_\epsilon, e \rangle) \varphi_t (x_\epsilon, t_\epsilon) - \alpha \| D \varphi^\epsilon(x_\epsilon, t_\epsilon) \| \right] \leq 0.
\]

This shows that to homogenize $(u_e^t)_{t>0}$, it is not enough to treat the transversal fluctuations of the front using $\tilde{V}_e$; the position of the front along the $e$ axis continues to oscillate. Nonetheless, we have lost $(d - 1)$-degrees of freedom by averaging, and the equation obtained so far suggests that we should be able to proceed by exploiting the asymptotic behavior of the averaged, one-dimension equation:

\[(74) \quad \begin{cases} m_e^+(se)\mathcal{U}_t - \alpha |\mathcal{U}_s| = 0 \quad \text{in} \, \mathbb{R} \times (0, \infty), \\ \mathcal{U}(s, 0) = s \quad \text{if} \, \ s \in \mathbb{R}. \end{cases} \]

This is precisely the approach taken in what follows.

8.2. **Intermediate results.** Let us state precisely some of the results used in the proof of Theorem 5.

First, we construct traveling wave sub- and supersolutions of (22) in rational directions:

**Proposition 16.** Under the same assumptions as in Theorems 1 and 2, if $e \in \mathbb{R}Z^d$ and $\alpha \in \mathbb{R} \setminus \{0\}$, then:

(i) The cell problem (73) has a solution $\tilde{V}_e \in C^{2,\alpha}(\mathbb{T}^d)$ and $m_e^+ \in C^{2,\alpha}(\mathbb{T}^d)$.

(ii) There is a $C^2$, $r_c$-periodic function $P_e : \mathbb{R} \to \mathbb{R}$ and a constant $\bar{m}_{pl}(e)$ such that

\[\bar{m}_{pl}(e)^{-1} m_e^+(se) \left| 1 + P_e(s) \right| = 0 \quad \text{in} \, \mathbb{R}.\]

In particular, for each $\alpha \in \mathbb{R} \setminus \{0\}$, the function $(s, t) \mapsto s + P_e(s) + \alpha \bar{m}_{pl}(e)^{-1} t$ is a pulsating wave solution of (74). Moreover, $\bar{m}_{pl}(e)$ is given explicitly by (21).
(iii) For each $\alpha \in \mathbb{R} \setminus \{0\}$, there is an $\epsilon_0^+ > 0$ and a family $(\epsilon_0^+)_{\epsilon \in (0, \epsilon_0^+)}$ such that the functions $(u^{+, \epsilon})_{\epsilon \in (0, \epsilon_0^+)}$ defined by

$$u^{+, \epsilon}(x, t) = \langle x, e \rangle + \epsilon P_e(e^{-1}(x, e)) + \alpha_+ m^{-1}(e) \left( \epsilon^2 \tilde{V}_e(e^{-1}x) + t \right)$$

are super-solutions of (22). Furthermore, there is a constant $C_+ > 0$ depending only on $\tilde{V}_e$ and $P_e$ such that, for each $(x, t) \in \mathbb{R}^d \times \mathbb{R}$,

$$|\alpha_+^+ - \alpha| \leq C_+ \epsilon$$

$$|u^{+, \epsilon}(x, t) - \langle x, e \rangle - \alpha m^{-1}t| \leq C_+ \epsilon (1 + |t|)$$

(iv) For each $\alpha \in \mathbb{R} \setminus \{0\}$, there is an $\epsilon_0^- > 0$ and a family $(\epsilon_0^-)_{\epsilon \in (0, \epsilon_0^-)}$ such that the functions $(u^{-, \epsilon})_{\epsilon \in (0, \epsilon_0^-)}$ defined by

$$u^{-, \epsilon}(x, t) = \langle x, e \rangle + \epsilon P_e(e^{-1}(x, e)) + \alpha_- m^{-1}(e) \left( \epsilon^2 \tilde{V}_e(e^{-1}x) + t \right)$$

are sub-solutions of (22) in $\mathbb{R}^d \times \mathbb{R}$. Furthermore, there is a constant $C_- > 0$ depending only on $\tilde{V}_e$ and $P_e$ such that, for each $(x, t) \in \mathbb{R}^d \times \mathbb{R}$,

$$|\alpha_-^- - \alpha| \leq C_- \epsilon$$

$$|u^{-, \epsilon}(x, t) - \langle x, e \rangle - \alpha m^{-1}t| \leq C_- \epsilon (1 + |t|)$$

A similar construction works when $e \notin \mathbb{R}Z^d$ and (73) has a $C^2$ solution. Here is the result in that case:

**Proposition 17.** If there is a solution $V_e \in C^2(\mathbb{T}^d)$ of (73) and $e \in S^{d-1} \setminus \mathbb{R}Z^d$, then there is an $\epsilon_0 > 0$ and families $(\alpha_0^+ e \epsilon \in (0, \epsilon_0^+))$ and $(\alpha_0^- e \epsilon \in (0, \epsilon_0^-))$ such that the functions $(u^{+, \epsilon})_{\epsilon \in (0, \epsilon_0^+)}$ and $(u^{-, \epsilon})_{\epsilon \in (0, \epsilon_0^-)}$ defined by

$$u^{+, \epsilon}(x, t) = \langle x, e \rangle + \alpha_0^+ m^{-1}(e) \left( \epsilon^2 V_0 e^{-1}x \right)$$

are, respectively, super- and sub-solutions of (22) in $\mathbb{R}^d \times \mathbb{R}$. Furthermore, there is a $C > 0$ such that, for each $(x, t) \in \mathbb{R}^d \times \mathbb{R}$,

$$|\alpha_0^+ - \alpha| + |\alpha_0^- - \alpha| \leq C \epsilon$$

$$|u^{+, \epsilon}(x, t) - \langle x, e \rangle - \alpha m^{-1}t| + |u^{-, \epsilon}(x, t) - \langle x, e \rangle - \alpha m^{-1}t| \leq C \epsilon (1 + |t|)$$

Since the proof of Proposition 17 is effectively a simpler version of that of Proposition 16, we omit it. In general, it is far from clear whether or not (73) has a solution when $e \notin \mathbb{R}Z^d$ due to the loss of compactness; cases where this is possible are discussed in Appendix A.

When applicable, Propositions 16 and 17 allow us to quantify the convergence in Theorem 5. More precisely, we have

**Proposition 18.** If $e \in S^{d-1} \cap \mathbb{R}Z^d$ or there is a $V_e \in C^2(\mathbb{T}^d)$ solving (25), then there is an $\epsilon_0 \in (0, 1)$ and a $C_\epsilon > 0$ such that if $\epsilon \in (0, \epsilon_0)$ and $u^\epsilon_e$ solves (22) with $u^\epsilon_e(x, 0) = \langle x, e \rangle$, then

$$|u^\epsilon_e(x, t) - \langle x, e \rangle - \alpha m^{-1}t| \leq C_\epsilon \epsilon (1 + t) \quad \text{if} \ (x, t) \in \mathbb{R}^d \times (0, \infty)$$
Since the proof of Proposition 18 follows directly from Propositions 16 and 17 and the comparison principle, we omit it.

It only remains to consider the case when \( e \not\in \mathbb{R}^d \) yet the hypotheses of Proposition 17 fail. Using approximate correctors, it is still possible to construct traveling wave sub- and supersolutions and use these to give a qualitative proof of Theorem 5. This is briefly treated in Section 8.4 below.

8.3. Proof of Proposition 16. Most of the work in proving the proposition lies in (iii) and (iv) since (i) is Proposition 7 and (ii) is an exercise in ODE theory. More precisely, (ii) is covered by the following lemma, the proof of which is left to the interested reader:

**Lemma 6.** Fix \( r > 0 \). If \( \tilde{m} : r \mathbb{T} \to (0, \infty) \) is \( C^1 \), then the solution \( X^{s,\tilde{m}} \) of the one-dimensional ODE

\[
\begin{cases}
\tilde{m}(X^{s,\tilde{m}}_t)\dot{X}^{s,\tilde{m}}_t = 1, \\
X^{s,\tilde{m}}_0 = s,
\end{cases}
\]

satisfies

\[
\lim_{t \to \infty} \frac{X^{s,\tilde{m}}_t - s}{t} = \frac{1}{\int_0^r \tilde{m}(u) \, du}.
\]

Furthermore, the function \( \hat{P} : r \mathbb{T} \to \mathbb{R} \) given by

\[
\hat{P}(s) = \left( \int_0^s \tilde{m}(u) \, du \right)^{-1} \int_0^s \tilde{m}(s) \, ds - 1
\]

solves the pulsating wave equation

\[
\frac{\tilde{m}(s)}{\int_0^r \tilde{m}(u) \, du} - |1 + P'(s)| = 0 \quad \text{in} \quad r \mathbb{T}.
\]

Now we proceed with the

**Proof of Proposition 16.** (i) follows from Proposition 7 above, and (ii) is an application of Lemma 6. We will only prove (iii) since (iv) follows similarly.

Let \( \epsilon > 0 \) and \( \alpha_\epsilon^+ \in \mathbb{R} \) be free variables for the moment and define \( u^{+,\epsilon} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) by

\[
(75) \quad u^{+,\epsilon}(x, t) = \langle x, e \rangle + \epsilon P_\epsilon(\epsilon^{-1} \langle x, e \rangle) + \alpha_\epsilon \tilde{m}_{pl}(e)^{-1} \left( \epsilon^2 \tilde{V}_e(\epsilon^{-1} x) + t \right).
\]

We will show that if \( \epsilon > 0 \) is small enough and \( \alpha_\epsilon^+ = \alpha + C_0 \epsilon \) for some large enough constant \( C_0 > 0 \), then \( u^{+,\epsilon} \) is a super-solution of (22).

Let us study the equation for \( u^{+,\epsilon} \) term by term. First, the term with the time derivative. To declutter the notation, we define \( p_\epsilon : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) by

\[
p_\epsilon(x, t) = Du^{+,\epsilon}(x, t) = (1 + P_\epsilon'(\epsilon^{-1} \langle x, e \rangle)) e + \epsilon \alpha_\epsilon \tilde{m}_{pl}(e)^{-1} D\tilde{V}_e(\epsilon^{-1} x).
\]

Observe that since \( 1 + P_\epsilon' > 0 \), it follows that \( \hat{p}_\epsilon = e + O(\epsilon) \). Thus,

\[
m(\epsilon^{-1} x, Du^{+,\epsilon}) u^{+,\epsilon}_t = \alpha_\epsilon \tilde{m}_{pl}(e)^{-1} m(\epsilon^{-1} x, e) + O(\epsilon).
\]
Next, the curvature term. First, notice that if we write \( \mathcal{N}(p) = \hat{p} \otimes \hat{p} \), then, by Taylor expansion,
\[
\text{tr} \left( A(\epsilon^{-1} x, \tilde{D} u^{+, \epsilon}) D^2 u^{+, \epsilon} \right) = \text{tr} \left( a(\epsilon^{-1} x, p_\epsilon)(\text{Id} - \hat{p}_\epsilon \otimes \hat{p}_\epsilon) D^2 u^{+, \epsilon}(\text{Id} - \hat{p}_\epsilon \otimes \hat{p}_\epsilon) \right) \\
= \text{tr} \left( a(\epsilon^{-1} x, e)(\text{Id} - e \otimes e) D^2 u^{+, \epsilon}(\text{Id} - e \otimes e) \right) \\
+ \text{tr} \left( D_p a(\epsilon^{-1} x, e)[p_\epsilon - e](\text{Id} - e \otimes e) D^2 u^{+, \epsilon}(\text{Id} - e \otimes e) \right) \\
+ \text{tr} \left( a(\epsilon^{-1} x, e)(\text{Id} - e \otimes e) D^2 u^{+, \epsilon} D_p N(e)[p_\epsilon - e] \right) \\
+ \| D^2 u^{+, \epsilon} \|_{L^\infty(T^d)} O(\| \hat{p}_\epsilon - e \|)
\]
\[
\text{tr} \left( A(\epsilon^{-1} x, \tilde{D} u^{+, \epsilon}) D^2 u^{+, \epsilon} \right) = \alpha \overline{m}_{pl}(e)^{-1} \left( \text{tr} \left( A(\epsilon^{-1} x, e) D^2 \tilde{V}_e \right) + \| D^2 \tilde{V}_e \|_{L^\infty(T^d)} O(\| \hat{p}_\epsilon - e \|) \right) \\
+ \| D^2 u^{+, \epsilon} \|_{L^\infty(T^d)} O(\| \hat{p}_\epsilon - e \|)
\]
\[
\text{tr} \left( A(\epsilon^{-1} x, \tilde{D} u^{+, \epsilon}) D^2 u^{+, \epsilon} \right) = \alpha \overline{m}_{pl}(e)^{-1} \left( \text{tr} \left( A(\epsilon^{-1} x, e) D^2 \tilde{V}_e \right) + O(\epsilon) \right).
\]
Finally, we treat the first-order term. Since \( \| e \| = 1 \), we have
\[
\alpha \| D u^{+, \epsilon} \| = \alpha |1 + P^e_\epsilon| + O(\epsilon).
\]
Putting it all together and invoking the equations satisfied by \( \tilde{V}_e \) and \( P_\epsilon \), we find, for some \( \overline{C} > 0 \),
\[
m(\epsilon^{-1} x, \tilde{D} u^{+, \epsilon}) u^{+, \epsilon} - \text{tr} \left( A(\epsilon^{-1} x, \tilde{D} u^{+, \epsilon}) D^2 u^{+, \epsilon} \right) - \alpha \| D u^{+, \epsilon} \| \geq \alpha \overline{m}_{pl}(e)^{-1} \left( m(\epsilon^{-1} x, e) - \text{tr} \left( A(\epsilon^{-1} x, e) D^2 \tilde{V}_e \right) \right) - \alpha |1 + P^e_\epsilon| - (\overline{C} + o(1)) e = \alpha \overline{m}_{pl}(e)^{-1} m^e_\epsilon(\epsilon^{-1} x, e) - \alpha \overline{m}_{pl}(e)^{-1} m^e_\epsilon(\epsilon^{-1} x, e) - (\overline{C} + o(1)) e = (C_0 - C + o(1)) e \overline{m}_{pl}(e)^{-1} m^e_\epsilon(\epsilon^{-1} x, e).
\]
Setting \( C_0 = C + 1 \), we deduce that there is an \( \epsilon_0 > 0 \) such that \( u^{+, \epsilon} \) is a super-solution of \( (22) \) in \( \mathbb{R}^d \times \mathbb{R} \).

8.4. **Proofs of Theorems 5 and 6.** Finally, we complete the proof of Theorem 5 and address Theorem 6.

In view of Propositions 16 and 17, Theorem 5 is proved as soon as we analyze the case that \( e \notin \mathbb{RZ}^d \) and yet (73) has no smooth solution. Not surprisingly, we argue using approximate correctors; the argument is included only for the sake of completeness.

Theorem 6 follows directly from Proposition 16 when \( e \in \mathbb{RZ}^d \) and the proof of Theorem 5 otherwise.
Proof of Theorem 5. Assume \( e \in \mathbb{S}^{d-1} \setminus \mathbb{R} \mathbb{Z}^d \); otherwise, Proposition 10 applies. Given \( \delta > 0 \), let \( V^\delta \) be the solution of (42) with \( f(y) = -m(y, e) \). By Schauder estimates, there is a constant \( C > 0 \) such that

(76) \[ \delta \| V^\delta \|_{L^\infty(\mathbb{T}^d)} + \delta^2 \| D V^\delta \|_{L^\infty(\mathbb{T}^d)} + \delta^3 \| D^2 V^\delta \|_{L^\infty(\mathbb{T}^d)} \leq C. \]

To start with, we claim that

(77) \[ \limsup^* u^\epsilon(x, t) \leq \langle x, e \rangle + \alpha m(e)^{-1} t. \]

To see this, fix \( \beta \in (\alpha, \infty) \), let \( \epsilon \in (0, 1) \), set \( \delta(\epsilon) = \epsilon^{1/4} \), and define \( v^\epsilon : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) by

\[ v^\epsilon(x, t) = \langle x, e \rangle + \beta m(e)^{-1} \left( t + \epsilon^2 V_{e}^{\delta(\epsilon)}(\epsilon^{-1} x) \right). \]

We claim there is an \( \epsilon_0 > 0 \) depending only on \( m \) and \( \beta \) such that \( v^\epsilon \) is a super-solution of (22) if \( \epsilon \in (0, \epsilon_0) \).

Indeed, invoking (76), we find

\[
m(\epsilon^{-1} x, \overline{Dv^\epsilon}) v_t^\epsilon - \text{tr} \left( A(\epsilon^{-1} x, \overline{Dv^\epsilon}) D^2 v^\epsilon \right) - \alpha \| D v^\epsilon \| = (\beta - \alpha) + \beta m(e)^{-1} \left( -\delta V_{e}^{\delta(\epsilon)}(\epsilon^{-1} x) - m(e) \right) + O(\epsilon^{7/4})
\]

\[ = \beta - \alpha + o(1). \]

Thus, there is an \( \epsilon_0 \in (0, 1) \) such that if \( \epsilon \in (0, \epsilon_0) \), then

\[ m(\epsilon^{-1} x, \overline{Dv^\epsilon}) v_t^\epsilon - \text{tr} \left( A(\epsilon^{-1} x, \overline{Dv^\epsilon}) D^2 v^\epsilon \right) - \alpha \| D v^\epsilon \| \geq \frac{\beta - \alpha}{2} \text{ in } \mathbb{R}^d \times \mathbb{R}. \]

Now we prove (77). First, notice that, by the choice of \( \delta(\epsilon) \),

\[ \langle x, e \rangle \leq v^\epsilon(x, 0) + \beta m(e)^{-1} \| \delta V_{e}^{\delta(\epsilon)} \|_{L^\infty(\mathbb{T}^d)} \epsilon^{7/4}. \]

Thus, the comparison principle implies

\[ u^\epsilon(x, t) \leq v^\epsilon(x, t) + \beta m(e)^{-1} \| m \|_{L^\infty(\mathbb{T}^d)} \epsilon^{7/4}. \]

Sending \( \epsilon \to 0^+ \), we deduce that

\[ \limsup^* u^\epsilon(x, t) \leq \langle x, e \rangle + \beta m(e)^{-1} t. \]

At the same time, \( \beta = \text{ an arbitrary number in } (\alpha, \infty) \). Therefore, sending \( \beta \to \alpha^+ \), we recover (77).

Replacing \( \beta \in (\alpha, \infty) \) by \( \beta \in (-\infty, \alpha) \), we similarly prove that

\[ \liminf^* u^\epsilon(x, t) \geq \langle x, e \rangle + \alpha m(e)^{-1} t. \]

\( \square \)
8.5. Derivatives of Front Speeds. This section treats the proof of Corollary 2. In the proof, we use the fact that there is a pulsating wave solution of (23). This is constructed using a viscosity solution $P_{e,\alpha} \in C(\mathbb{T}^d)$ of the equation

$$\lambda_e(\alpha) m(y, (e + DP_{e,\alpha})) - \text{tr} \left( A(y, (e + DP_{e,\alpha})) D^2 P_{e,\alpha} \right) - \alpha \| e + DP_{e,\alpha} \| = 0 \quad \text{in } \mathbb{T}^d.$$ 

Since $d = 2$, the existence of such a function follows from [14].

Proof of Corollary 2 As in the previous section, the proof is neater depending on whether or not (73) has a smooth solution. We only give the arguments in the case when $e \in S^{d-1} \setminus \mathbb{R}Z^d$, but (73) does not have a smooth solution.

Fix $\beta \in (1, \infty)$. Let $v^e(x,t) = \langle x, e \rangle + \beta \overline{m}(\epsilon^{-1}(\epsilon^2 V^\delta(e)(\epsilon^{-1}x) + t))$ for $\delta(e) = \epsilon^\frac{1}{4}$.

Arguing as in the proof of Theorem 5 we see that there is an $\epsilon_0 \in (0, 1)$ such that $v^e$ is a super-solution of (22) with $\alpha = 1$ if $\epsilon \in (0, \epsilon_0)$.

Fix $P_{e,\epsilon}$ as above and set $u^e(y,t) = \langle y, e \rangle + P_{e,\epsilon}(y) + \lambda_e(\epsilon) t$. Notice that this is a viscosity solution of (23). Hence $(x,t) \mapsto \epsilon u^e(\epsilon^{-1}x, \epsilon^{-2}t)$ is a viscosity solution of the first equation in (22) with $\alpha = 1$. Since $P_{e,\epsilon}$ is bounded,

$$\epsilon u^e(\epsilon^{-1}x, 0) \leq v^e(x, 0) + \epsilon \| P_{e,\epsilon} \|_{L^\infty(\mathbb{T}^d)}$$

and, thus, the comparison principle implies that, for each $(x,t) \in \mathbb{R}^d \times (0, \infty)$,

$$\langle x, e \rangle + \epsilon P_{e,\epsilon}(\epsilon^{-1}x) + (e^{-1} \lambda_e(\epsilon)) t = \epsilon u^e(\epsilon^{-1}x, \epsilon^{-2}t) \leq v^e(x, t) + \epsilon \| P_{e,\epsilon} \|_{L^\infty(\mathbb{T}^d)}$$

$$= \langle x, e \rangle + \beta \overline{m}(\epsilon^{-1}e^2 V^\delta(e)(\epsilon^{-1}x) + \epsilon \| P_{e,\epsilon} \|_{L^\infty(\mathbb{T}^d)} + \beta \overline{m}(\epsilon)^{-1} t.$$ 

Since $P_{e,\epsilon}$ is bounded and $t > 0$ is arbitrary, we deduce that $\epsilon^{-1} \lambda_e(\epsilon) \leq \beta \overline{m}(\epsilon)^{-1}$. Sending first $\epsilon \rightarrow 0^+$ and then $\beta \rightarrow 1^+$, we conclude

$$\limsup_{\epsilon \rightarrow 0^+} \epsilon^{-1} \lambda_e(\epsilon) \leq \overline{m}(\epsilon)^{-1}.$$ 

Arguing using subsolutions instead of supersolutions, we find

$$\liminf_{\epsilon \rightarrow 0^+} \epsilon^{-1} \lambda_e(\epsilon) \geq \overline{m}(\epsilon)^{-1}.$$ 

It remains to show that $\tilde{\alpha}^{-1} \lambda_e(\tilde{\alpha}) \rightarrow \overline{m}(\epsilon)^{-1}$ as $\tilde{\alpha} \rightarrow 0^-$. Here we repeat the previous proof, replacing $\alpha = 1$ by $\alpha = -1$. □

Remark 8. When (73) has a smooth solution, it is possible to prove the following rate: $|\epsilon^{-1} \lambda_e(\epsilon) - \overline{m}(\epsilon)^{-1}| \leq C_\epsilon \epsilon$ for some $C_\epsilon > 0$ depending on bounds on the derivatives of the corrector.

Appendix A. Technical Lemmata

This appendix covers some of the technical results that were needed above.
A.1. **Comparison and Well-posedness.** In this section, we briefly review some of the comparison principles and well-posedness results that were used above. To start with, following [14], we make assumption (v) of Theorem 1 precise:

(v) (comparison) There is a $K \geq 9$ and a modulus $\omega_K : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\lim_{\delta \rightarrow 0^+} \omega_K(\delta) = 0$$

such that if $\alpha \geq 0$, $X, Y \in S_d$, and $x, y \in \mathbb{R}^d$ satisfy

$$-K\alpha \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq K\alpha \begin{pmatrix} \text{Id} & -\text{Id} \\ -\text{Id} & \text{Id} \end{pmatrix}$$

then

$$F^*(\alpha(x - y), X, x) - F^*(\alpha(x - y), Y, y) \leq \omega_K(\|x - y\|(1 + \alpha\|x - y\|))$$

with the understanding that $\alpha = 0$ if $x = y$.

**Proposition 19.** If $F$ satisfies the assumptions of Theorem 7 or $F$ is given by

$$F(p, X, y) = m(y, \hat{p})^{-1}\text{tr}(a(y, \hat{p})\hat{X}_\hat{p})$$

with $a$ and $m$ satisfying the assumptions of Theorem 2, then, for each $u_0 \in UC(\mathbb{R}^d)$ and $\epsilon > 0$, there is a unique $u^\epsilon$ solving (1) or (2), respectively.

**Proof.** In the case of (1) and Theorem 1, the assumptions of the theorem imply that (1) satisfies a comparison principle. See [14, Theorem 3.3]. This guarantees uniqueness of the solution, and then existence follows through Perron’s Method.

The same idea pertains to (2) and Theorem 2. The key point is the matrix $\sqrt{\alpha}$ is uniformly Lipschitz continuous in the spatial variable and, thus, one can obtain the inequality in (v) by arguing as in [23, Example 3.6]. \(\square\)

Next, we turn to well-posedness of the penalized cell problem studied in Section 4.

**Proposition 20.** If $F$ satisfies the assumptions of Theorem 7 or

$$F(p, X, y) = m(y, \hat{p})^{-1}\text{tr}(a(y, \hat{p})\hat{X}_\hat{p})$$

with $a$ and $m$ satisfying the assumptions of Theorem 2, then, for each $\delta > 0$, $e \in S^{d-1}$, and $X \in S_d$, the penalized cell problem (33) has a unique solution $V_\delta \in C(\mathbb{T}^d)$.

**Proof.** If $V_1$ and $V_2$ are respectively bounded sub- and super-solutions of $\delta V_1 - F(e, X + D_e^2 V_1, y) \leq 0$ and $\delta V_2 - F(e, X + D_e^2 V_2, y) \geq 0$ in $\mathbb{R}^d$, then the function $\{\tilde{V}_{1,y}\}_{y \in \mathbb{R}^d}$ and $\{\tilde{V}_{2,y}\}_{y \in \mathbb{T}^d}$ defined by analogy with (33) lead to sub- and supersolutions of the equations

$$\delta \tilde{V}_{1,y} - F(e, \tilde{X}_e + D_e^2 \tilde{V}_{1,y}, y + x') \leq 0, \quad \delta \tilde{V}_{2,y} - F(e, \tilde{X}_e + D_e^2 \tilde{V}_{2,y}, y + x') \geq 0 \quad \text{in } \langle e \rangle^\perp.$$ 

Thus, the functions $\{\tilde{V}_{0,y}\}_{y \in \mathbb{T}^d}$ given by $\tilde{V}_{0,y} = \tilde{V}_{1,y} - \tilde{V}_{2,y}$ satisfy

$$\delta \tilde{V}_{0,y} - \mathcal{P}^e_{\frac{\lambda}{d-1}, \Lambda}(D_e^2 \tilde{V}_{0,y}) \leq 0 \quad \text{in } \langle e \rangle^\perp.$$ 

Here the Pucci maximal operator $\mathcal{P}^e_{\frac{\lambda}{d-1}, \Lambda}$ is given by

$$\mathcal{P}^e_{\frac{\lambda}{d-1}, \Lambda}(X) = \sup \left\{ \text{tr}(A\tilde{X}_e) \mid A \in S_d, \ (d-1)^{-1}\lambda \text{Id} \leq A \leq \Lambda \text{Id} \right\}.$$
Since we are working in $\langle e \rangle^\perp$, this is a uniformly elliptic, translationally invariant operator. Therefore, a standard comparison argument shows that $\bar{V}_0,y \leq 0$ for all $y \in \mathbb{R}^d$. In particular, $V_1 \leq V_2$ in $\mathbb{R}^d$.

Applying Perron’s Method, we find a continuous $V$ satisfying $\delta V - F(e,X + D_e^2V,y) = 0$ in $\mathbb{R}^d$. By uniqueness, $V$ descends to a function in $C(\mathbb{T}^d)$. □

A.2. Comparison Principle for Ergodic Constants. Here we state and prove the comparison principles for ergodic constants that were used in Sections 4 and 6. The idea goes back at least as far as [24].

Lemma 7. Assume that $F$ satisfies the assumptions of Theorem 1 or $F(p,X,y) = m(y,\hat{p})^{-1}\text{tr}(a(y,\hat{p})\vec{X}_p)$ with $a$ and $m$ satisfying the assumptions of Theorem 2. Fix $e \in S_{d-1}$, $g : \langle e \rangle^\perp \to \mathbb{R}$ is a bounded, uniformly continuous function, $X \in S_d$, and $C_1, C_2 \in \mathbb{R}$. If $V_1 : \langle e \rangle^\perp \to \mathbb{R}$ is a bounded, upper semi-continuous subsolution of the differential inequality
\[ C_1 + g(x') - F(e,\vec{X}_e + D_e^2V_1, x') \leq 0 \quad \text{in } \langle e \rangle^\perp \]
and $V_2 : \langle e \rangle^\perp \to \mathbb{R}$ is a bounded, lower semi-continuous supersolution of the differential inequality
\[ C_2 + g(x') - F(e,\vec{X}_e + D_e^2V_2, x') \geq 0 \quad \text{in } \langle e \rangle^\perp , \]
then $C_1 \leq C_2$.

Proof. The assumptions of Theorem 1 and 2 imply that the difference $V_0 = V_1 - V_2$ is a bounded, upper semi-continuous subsolution of the differential inequality
\[ (C_1 - C_2) - \mathcal{P}_{e,\frac{\lambda}{C_1}A}^+(D_e^2V_0) \leq 0 \quad \text{in } \langle e \rangle^\perp , \]
where the Pucci maximal operator $\mathcal{P}_{e,\frac{\lambda}{C_1}A}^+$ is as in the previous proof. Therefore, another standard comparison argument shows that $C_1 \leq C_2$. □

A.3. Equi-distribution of codimension one sub-tori. We will be interested in certain probability measures supported on the sub-tori $(T_{e_n}^{d-1}(s))_{s \in [0, r_{e_n})}$ defined by (18). Toward that end, the result that follows is fundamental.

In this section, if $\mu$ is a finite measure, we write $\int_A \mu(dy) = \frac{1}{\mu(A)} \int_A \mu(dy)$.

Lemma 8. If $(e_n)_{n \in \mathbb{N}} \subseteq S_{d-1} \cap \mathbb{R}^d$ is any infinite sequence and $(s_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$ satisfies $s_n \in [0, r_{e_n})$ for each $n \in \mathbb{N}$, then
\[ \mathcal{H}^{d-1}(T_{e_n}^{d-1}(s_n))^{-1} \mathcal{H}^{d-1} \mid_{T_{e_n}^{d-1}(s_n)} \overset{\ast}{\Rightarrow} \mathcal{L}^d \]
Furthermore, if $(f_n)_{n \in \mathbb{N}}$ are $\mathcal{H}^{d-1}$-measurable functions in $\mathbb{T}^d$, $p \in [1, \infty]$, and $C > 0$ is such that
\[ \int_{T_{e_n}^{d-1}(s_n)} |f_n(\xi)|^p \mathcal{H}^{d-1}(d\xi) \leq C^p \]
and if we define measures $(\mu_n)_{n \in \mathbb{N}}$ on $\mathbb{T}^d$ by
\[ \int_{\mathbb{T}^d} g(y) \mu_n(dy) = \int_{T_{e_n}^{d-1}(s_n)} g(\xi) f_n(\xi) \mathcal{H}^{d-1}(d\xi), \]
then there is a subsequence \((n_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}\) and a measure \(\tilde{\mu}\) such that \(\tilde{\mu} = \lim_{j \to \infty} \mu_{n_j}\) weakly-\(*\), \(\tilde{\mu} \ll \mathcal{L}^d\), and

\[
(79) \quad \left\| \frac{d\tilde{\mu}}{d\mathcal{L}^d} \right\|_{L^p(\mathbb{T}^d)} \leq C.
\]

**Proof.** First, we prove that the normalized surface measures converge to \(\mathcal{L}^d\). Assume that \(g \in C(\mathbb{T}^d)\) satisfies \(\sum_{k \in \mathbb{Z}^d} |\hat{g}(k)| < \infty\). An exercise in Fourier analysis shows that if \(k \in \mathbb{Z}^d\), then

\[
\int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} e^{i2\pi(k,\xi)} \mathcal{H}^{d-1}(d\xi) = \begin{cases} e^{i2\pi(k,\mathbf{e}_n)\mathbf{e}_n}, & \text{if } k \in \{\mathbf{e}_n\}, \\ 0, & \text{otherwise}. \end{cases}
\]

Therefore,

\[
\left| \int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} g(\xi) \mathcal{H}^{d-1}(d\xi) - \int_{\mathbb{T}^d} g(y) dy \right| \leq \sum_{k \in \mathbb{Z}^d \setminus \{\mathbf{e}_n\}} |\hat{g}(k)| \int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} e^{i2\pi(k,\xi)} \mathcal{H}^{d-1}(d\xi) \leq \sum_{k \in \mathbb{Z}^d \setminus \{\mathbf{e}_n\}} |\hat{g}(k)|.
\]

Since \((\mathbf{e}_n)_{n \in \mathbb{N}}\) is infinite, it follows that for each \(R > 0\), there is an \(N \in \mathbb{N}\) such that if \(n \geq N\), then

\[
\mathbb{Z}^d \cap \{\mathbf{e}_n\} \setminus \{0\} \subseteq \mathbb{R}^d \setminus B(0, R).
\]

Thus,

\[
\limsup_{n \to \infty} \left| \int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} g(\xi) \mathcal{H}^{d-1}(d\xi) - \int_{\mathbb{T}^d} g(y) dy \right| \leq \lim_{R \to \infty} \sum_{k \in \mathbb{Z}^d \setminus B(0, R)} |\hat{g}(k)| = 0.
\]

Recalling that functions with summable Fourier coefficients are dense in \(C(\mathbb{T}^d)\), we conclude that (7.8) holds as claimed.

Next, we prove the claim concerning \((\mu_n)_{n \in \mathbb{N}}\). First, observe that Hölder’s inequality implies

\[
\|\mu_n\|_{(\mathbb{T}^d)} = \int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} |f_n(\xi)| \mathcal{H}^{d-1}(d\xi) \leq C \quad \text{if } n \in \mathbb{N}.
\]

Thus, \((\mu_n)_{n \in \mathbb{N}}\) is pre-compact in \(C(\mathbb{T}^d)^*\), which gives the desired sub-sequence \((n_j)_{j \in \mathbb{N}}\) and limit point \(\tilde{\mu}\).

Note that if \(g \in C(\mathbb{T}^d)\) and \(q \in [1, \infty)\) is the conjugate exponent of \(p\) (i.e. the \(q\) so that \(p^{-1} + q^{-1} = 1\)), then

\[
\left| \int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} g(\xi) f_n(\xi) \mathcal{H}^{d-1}(d\xi) \right| \leq C \left( \int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} |g(\xi)|^q \mathcal{H}^{d-1}(d\xi) \right)^{\frac{1}{q}}.
\]

Assume first that \(q < \infty\). Since \(|g|^q \in C(\mathbb{T}^d)\), we have

\[
\left| \int_{\mathbb{T}^d} g(y) \tilde{\mu}(dy) \right| \leq C \lim_{n \to \infty} \left( \int_{\mathbb{T}^d \setminus \{\mathbf{e}_n\}} |g(\xi)|^q \mathcal{H}^{d-1}(d\xi) \right)^{\frac{1}{q}} = C \|g\|_{L^q(\mathbb{T}^d)}.
\]
This proves \( \bar{\mu} \ll L^d \) and \((\mathbb{I})\) holds when \( q < \infty \). When \( q = \infty \), a similar argument applies. \( \square \)

A.4. **Smooth Correctors in a Particular Case.** In this section, we prove that \((\mathbb{I})\) does have solutions in the setting of Theorem 2 when \( \alpha \) is constant, \( m \) is sufficiently regular, and \( e \) is well-chosen.

**Proposition 21.** Fix \( e \in S^{d-1} \). If \( m(\cdot, e) \in H^s(\mathbb{T}^d) \) for some \( s > \frac{d}{2} + \frac{1}{d-1} + 2 \) and there is a \( C \in (0, 1) \) and \( \frac{1}{d-1} < \tau < s - \frac{d}{2} - 2 \) such that

\[
\| k - \langle k, e \rangle e \| \geq C\| k \|^{-\tau} \quad (\forall k \in \mathbb{Z}^d \setminus \{0\}),
\]

then there is a solution \( V \in C^2(\mathbb{T}^d) \) of the equation

\[
m(\cdot, e) - \text{tr}(\text{Id} - e \otimes e)D^2 V = \bar{m}(e).
\]

Furthermore, \( V \) is the unique such solution among all functions \( U \in L^2(\mathbb{T}^d) \) with \( \int_{\mathbb{T}^d} U(y) \text{dy} = \int_{\mathbb{T}^d} V(y) \text{dy} \).

Concerning the generality of the assumption \((80)\), see [34], where it is shown that \( \mathcal{H}_{d-1} \)-a.e. \( e \in S^{d-1} \) satisfies such an estimate for any given \( \tau > \frac{1}{d-1} \). More precisely, if \( A(C, \tau) \) is the set of all such \( e \), then there is a constant \( B(d, \tau) > 0 \) such that

\[
\mathcal{H}_{d-1}(S^{d-1} \setminus A(C, \tau)) \leq B(d, \tau)C_{d-1}.
\]

**Proof.** Define \( \hat{V}_e : \mathbb{Z}^d \rightarrow \mathbb{C} \) by \( \hat{V}_e(0) = 0 \) and

\[
\hat{V}_e(k) = -\frac{\hat{m}(k)}{4\pi^2\| k - \langle k, e \rangle e \|^2}.
\]

Since \( m(\cdot, e) \in H^s(\mathbb{T}^d) \) and \( s > \tau + \frac{d}{2} + 2 \), for each \( i \in \{0, 1, 2\} \), we have

\[
\sum_{k \in \mathbb{Z}^d} \| k \|^i |\hat{V}_e(k)| \leq C_{i}^{-2} \left( \sum_{k \in \mathbb{Z}^d} \| k \|^{2(\tau+i-s)} \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^d} \| k \|^{2s} |\hat{m}(k, e)|^2 \right)^{\frac{1}{2}} < \infty.
\]

Thus, we can define \( V \in C^2(\mathbb{T}^d) \) by

\[
V_e(y) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \hat{V}_e(k)e^{i2\pi(k,y)}
\]

and then

\[
(\text{Id} - e \otimes e)D^2 V_e(y) = -\sum_{k \in \mathbb{Z}^d} 4\pi^2(k \otimes k - \langle k, e \rangle e \otimes k)\hat{V}_e(k)e^{i2\pi(k,y)}.
\]

In particular, by construction, \( V \) is a solution of \((81)\). Since \( k - \langle k, e \rangle e \neq 0 \) for each \( k \in \mathbb{Z}^d \), a straightforward argument shows that \( V \) is unique up to the addition of a constant. \( \square \)
Appendix B. Construction of Some Finsler Norms

To construct the Finsler norms of Proposition 15, we proceed by constructing appropriate convex perturbations of the unit ball $B(0, 1)$.

Recall that if $\mathcal{O}$ is a convex, open subset of $\mathbb{R}^d$ and $0 \in \mathcal{O}$, then there is a unique Finsler norm $\psi_\mathcal{O}$ such that $\{ \psi_\mathcal{O} < 1 \} = \mathcal{O}$. We will prove Proposition 15 by building a sequence $\{ \mathcal{O}_N \}$ of such sets, each of which is obtained from $B(0, 1)$ by flattening its boundary at the points $\{ e_1, \ldots, e_N \}$. Once this is done, it will be easy to see that the corresponding Finsler norms $\{ \psi_N \}_{N \in \mathbb{N}}$ have the desired properties.

In what follows, we define $\varphi : [-1, 1] \to [0, 1]$ by $\varphi(x) = \frac{1 - x^2}{1 - x^2}$. We will take advantage of the fact that if $e \in S^{d-1}$, then the map $B(0, 1) \cap \langle e \rangle \ni x' \mapsto x' + \varphi(x')e$ parametrizes a neighborhood of $e$ in $S^{d-1}$. In particular, we build $\mathcal{O}_N$ by perturbing these maps.

B.1. Graphs. To start with, we modify $\varphi$ so that it is flat at the points $\{ e_1, \ldots, e_N \}$. Recall that $\varphi'$ and $\varphi''$ are given by

$$\varphi'(x) = -\frac{x}{\sqrt{1 - x^2}}, \quad \varphi''(x) = -\frac{1}{\sqrt{1 - x^2}} + \frac{x^2}{(1 - x^2)^2}.$$

Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth function satisfying the following:

$\eta(x) = 1$ if $|x| \leq 1$, \quad $\eta(x) = 0$ if $|x| \geq 2$, \quad $\eta(-x) = \eta(x)$, \quad $\eta' > 0$ in $(-2, -1)$.

Given $\epsilon > 0$, define the modified parametrization $\varphi_\epsilon : [-1, 1] \to \mathbb{R}$ by

$$\varphi_\epsilon(x) = \int_{-1}^{x} (1 - \eta(\epsilon^{-1}y)) \varphi'(y) \, dy.$$

**Proposition 22.** $\varphi_\epsilon$ is $C^2$ in $[-1, 1]$ and it satisfies the following conditions:

(i) $0 \leq \varphi_\epsilon \leq 1$ in $[-1, 1]$,
(ii) $\varphi''_\epsilon$ is given by $\varphi''_\epsilon(x) = (1 - \eta(\epsilon^{-1}x))\varphi''(x) - \epsilon^{-1} \eta'(\epsilon^{-1}x)\varphi'(x)$ and, for $\epsilon < 1/4$, we have

$$\| \varphi''_\epsilon \|_{L^\infty([-1, 1])} \leq \| \varphi'' \|_{L^\infty([-1, 1])} + \frac{4}{\sqrt{3}} \| \eta' \|_{L^\infty(\mathbb{R})}.$$

(iii) For each $x \in [-2\epsilon, 2\epsilon]$, $\varphi_\epsilon(x) \geq \sqrt{1 - 4\epsilon^2}$.

**Proof.** First, notice that $\varphi_\epsilon(-1) = 0$. Furthermore, since $\varphi'_\epsilon(-x) = -\varphi'_\epsilon(x)$, it follows that $\varphi_\epsilon(x) = \varphi_\epsilon(-x)$, which gives $\varphi_\epsilon(1) = 0$.

Next, observe that $\varphi'_\epsilon(x) = 0$ if and only if $|x| \leq \epsilon$. Thus, the maximum of $\varphi_\epsilon$ is attained in $[-\epsilon, \epsilon]$. Furthermore,

$$\| \varphi_\epsilon \|_{L^\infty([-1, 1])} = \varphi_\epsilon(0) \leq \int_{-1}^{0} \varphi'(y) \, dy = 1.$$

Concerning (ii), the equation for $\varphi''_\epsilon$ is a direct consequence of the chain rule, and the second term can be estimated as follows:

$$\epsilon^{-1} |\eta'(\epsilon^{-1}x)\varphi'(x)| \leq \| \eta' \|_{L^\infty(\mathbb{R})} \frac{2}{\sqrt{1 - x^2}} \chi_{[-2\epsilon, 2\epsilon]}(x) \leq \frac{4}{\sqrt{3}} \| \eta' \|_{L^\infty(\mathbb{R})}.$$
Finally, if \( x \in [-2\epsilon, 0] \), then
\[
\varphi_{\epsilon}(x) \geq \int_{-1}^{-2\epsilon} \varphi_{\epsilon}'(y) \, dy = \varphi(-2\epsilon) = \sqrt{1 - 4\epsilon^2}.
\]
Since \( \varphi_{\epsilon}(-x) = \varphi_{\epsilon}(x) \), this proves (iii). \( \square \)

**B.2. Construction of \( O_N \).** Let \( \epsilon_N \in (0, 1/4) \) be a free variable.

We define the convex set \( O_N \subseteq B(0, 1) \) as follows: given \( j \in \{1, 2, \ldots, N\} \), define \( \varphi_{\epsilon}^j : B(0, 1) \cap \langle e_j \rangle = \mathbb{R} \) by
\[
\varphi_{\epsilon}^j(x') = \varphi(\|x'\|).
\]
Notice that \( \varphi_{\epsilon}^j(x')e_j + x' = \sqrt{1 - \|x'\|^2}e_j + x' \) if \( \|x'\| \geq 2\epsilon_N \). Thus, it follows that there is an \( \epsilon_N \in (0, 1/4) \) such that if \( \epsilon_N \in (0, \epsilon_N) \) and we define \( G_N, B_N, \) and \( S_N \) by
\[
G_N = \bigcup_{j=1}^{N} \left\{ \varphi_{\epsilon}^j(x')e_j + x' \mid x' \in \overline{B(0, 2\epsilon_N)} \right\},
\]
\[
B_N = \bigcup_{j=1}^{N} \left\{ \varphi(\|x'\|)e_j + x' \mid x' \in B(0, 2\epsilon_N) \right\},
\]
\[
S_N = G_N \cup (S^{d-1} \setminus B_N),
\]
then \( S_N \) is a compact, connected \( C^2 \) hypersurface in \( \mathbb{R}^d \). Furthermore, for each \( j \in \{1, 2, \ldots, N\} \), the function \( x' \mapsto x' + \varphi_{\epsilon}^j(x')e_j \) is a parametrization of \( S_N \).

Being connected, \( S_N = \partial O_N \) for some bounded, connected open set \( O_N \) (see [35] Section 2.5), [50], or argue directly using the fact that \( S_N \) is obtained by perturbing parametrizations of \( S^{d-1} \). Since \( \varphi_{\epsilon}^j \) is concave for each \( j \), it follows that, for each \( x \in \overline{O_N} \), there is an \( r > 0 \) such that \( \overline{O_N} \cap B(x, r) \) is convex. Therefore, by the Tietze-Nakajima Theorem (cf. [39] Section 2), \( O_N \) is a convex subset of \( \mathbb{R}^d \).

Next, we note that, by making \( \epsilon_N > 0 \) smaller if necessary, we can ensure that the outward normal \( \nu \) to \( S_N \) points in the \( e_j \) direction for some \( j \in \{1, 2, \ldots, N\} \) if and only if the corresponding point is in the flat part of \( \varphi_{\epsilon}^i \). This is the content of the next result:

**Proposition 23.** Let \( \nu \) denote the outward normal vector to \( \partial O_N \). Making \( \epsilon_N \) smaller if necessary, given \( p \in S_N \), we have \( \nu(p) = e_j \) for some \( j \in \{1, 2, \ldots, N\} \) if and only if \( p = x' + \varphi_{\epsilon}^j(x')e_j \) for some \( x' \in \overline{B(0, \epsilon_N)} \cap \langle e_j \rangle \).

**Proof.** Fix \( k \in \{1, 2, \ldots, N\} \). If \( x' \in B(0, 1) \cap \langle e_k \rangle \), \( \|x'\| \geq 2\epsilon_N \), and \( \tilde{p} := \varphi_{\epsilon}^k(x')e_k + x' \in S_N \), then \( \nu(\tilde{p}) = \tilde{p} \). Thus, by construction, \( \nu(p) = e_k \) only if \( p = \varphi_{\epsilon}^k(x')e_k + x' \) for some \( x' \in B(0, 2\epsilon_N) \cap \langle e_k \rangle \).

It follows that \( \nu(p) \) is given by
\[
\nu(p) = \frac{e_k - D\varphi_{\epsilon}^k(x')}{\sqrt{1 + \|D\varphi_{\epsilon}^k(x')\|^2}}.
\]
However, if $\epsilon_N$ is small enough, then the relation $e_\ell \neq e_j$ for $\ell \neq j$ forces $k = j$. Indeed,

$$\|\nu(p) - e_k\| \leq (\sqrt{1 + \|D\varphi_{\epsilon_N}(x')\|^2} - 1) + \|D\varphi_{\epsilon_N}(x')\|.$$ 

Since there is a $C > 0$ independent of $\epsilon_N$ and $k$ such that $\|D\varphi_{\epsilon_N}(x')\| \leq C\epsilon_N$ for $x' \in B(0, 2\epsilon_N) \cap \langle e_k \rangle$, it follows that there is an $\epsilon''_N \in (0, 1/2)$ such that if $\epsilon_N < \epsilon''_N$, then $\nu(p) = e_j$ only if $k = j$.

Finally, we claim that $\|x'\| \leq \epsilon_N$. Indeed, if $\epsilon_N < \|x'\| \leq 2\epsilon_N$, then $\nu(p)$ is given by

$$\nu(p) = \frac{e_j + (1 - \eta(\epsilon_N^{-1}x'))(1 - \|x'\|^2)^{-\frac{1}{2}}}{{\sqrt{1 + \|D\varphi_{\epsilon_N}(x')\|^2}}},$$

and this gives $(\text{Id} - e_j \otimes e_j)\nu(p) \neq 0$. Therefore, $\nu(p) = e_j$ only if $\|x'\| \leq \epsilon_N$. □

B.3. Proof of Proposition 15. Let $\psi_N : \mathbb{R}^d \rightarrow [0, \infty)$ be the Minkowski gauge associated with $O_N$, that is,

$$\psi_N(p) = \inf \{\alpha > 0 \mid \alpha^{-1}p \in O_N\}.$$ 

$\psi_N$ is the unique Finsler norm with $\{\psi_N < 1\} = O_N$. It has the following properties:

**Proposition 24.** (i) For each $e \in S^{d-1}$, $1 \leq \psi_N(e) \leq (1 - 4\epsilon_N^2)^{-\frac{1}{2}}$.

(ii) Given $p \in \mathbb{R}^d \setminus \{0\}$, if $D\psi_N(p) = e_i$ for some $i \in \{1, 2, \ldots, N\}$, then

$$D^2\psi_N(p) = 0.$$

(iii) There is a $c_0 > 0$ such that $D^2\psi_N(e) \leq c_0(\text{Id} - e \otimes e)$ for each $e \in S^{d-1}$.

Notice that this implies Proposition 15.

**Proof.** First, we prove (i). Assume that $e \in S^{d-1}$. Choose $\kappa > 0$ such that $\kappa e \in \partial O_N$. If $\kappa e = \varphi_{\epsilon_N}(x')e_i + x'$ for some $i \in \{1, 2, \ldots, N\}$ and $x' \in B(0, 1) \cap \langle e_i \rangle$, then

**Proposition 22** (iii) implies

$$1 \geq \|\kappa e\|^2 = \|x'\|^2 + \varphi_{\epsilon_N}(x')^2 \geq 1 - 4\epsilon^2.$$ 

From this, we find

$$1 \leq \kappa^{-1} = \psi_N(e) \leq \frac{1}{\sqrt{1 - 4\epsilon^2}}.$$ 

Otherwise, if $\kappa e \in S^{d-1} \cap \partial O_N$, then $\kappa = 1$ and $\psi_N(e) = \psi_N(\kappa e) = 1$.

Next, we tackle (ii). Suppose that $p \in \mathbb{R}^d$ and $D\psi_N(p) = e_i$ for some $i \in \{1, 2, \ldots, N\}$. By homogeneity, there is a $\gamma > 0$ such that $D\psi_N(p) = \gamma\psi(\psi_N(p)^{-1}p)$. Hence $\nu(\psi_N(p)^{-1}p) = e_i$ and we can invoke **Proposition 23** to find that $p = x' + \varphi_{\epsilon_N}(x')e_i$ for some $x' \in B(0, \epsilon_N) \cap \langle e_i \rangle$. From this, the flatness of $\varphi_{\epsilon_N}$ in $B(0, \epsilon_N) \cap \langle e_i \rangle$ implies that $D^2\psi_N(\psi_N(p)^{-1}p) = 0$, and then homogeneity implies $D^2\psi_N(p) = 0$.

Finally, concerning (iii), we note that the $\epsilon$-independent bounds on $\varphi''_{\epsilon_N}$ in **Proposition 22** give corresponding $N$-independent bounds on $D^2\varphi''_{\epsilon_N}$, and then this readily shows that $D^2\psi_N$ is bounded on $\{\psi_N = 1\}$ independently of $N$. Using homogeneity and (i), this gives an $N$-independent bound on $D^2\psi_N$ in $S^{d-1}$. □
Acknowledgements

It is a pleasure to acknowledge P.E. Souganidis for helpful discussions and considerable patience. The author would also like to thank W.M. Feldman and I.C. Kim, whose correspondence and suggestions led to a number of significant improvements to the paper, ultimately culminating in the proof of Theorem 10.

References

[1] S.N. Armstrong and P. Cardaliaguet, “Stochastic homogenization of quasilinear Hamilton-Jacobi equations and geometric motions,” J. Eur. Math. (JEMS) 20-4 (2018): 797-864.
[2] S.N. Armstrong and C.K. Smart, “Quantitative stochastic homogenization of elliptic equations in nondivergence form,” Arch. Rational Mech. Anal. 214-3 (2014): 867-911.
[3] G. Barles, “Remarks on a flame propagation model,” Rapport INRIA #464 (1985).
[4] G. Barles, A. Cesaroni, and M. Novaga, “Homogenization of fronts in highly heterogeneous media,” SIAM J. Math. Anal. 43-1 (2011): 212-227.
[5] G. Barles, F. Da Lio, P.L. Lions, and P.E. Souganidis, “Ergodic Problems and Periodic Homogenization for Fully Nonlinear Equations in Half-Space Type Domains with Neumann Boundary Conditions,” Indiana Univ. Math J. 57-5 (2008): 2355-2375.
[6] G. Barles and C. Georgelin, “A simple proof of convergence for an approximation scheme for computing motions by mean curvature,” SIAM Journal on Numerical Analysis 32.2 (1995): 484-500.
[7] G. Barles and E. Mironescu, “On Homogenization Problems for Fully Nonlinear Equations with Oscillating Dirichlet Boundary Conditions,” Asymptot. Anal. 82-3-4 (2013): 187-200.
[8] G. Barles, H.M. Soner, and P.E. Souganidis, “Front Propagation and Phase Field Theory,” SIAM J. Control Optim. 31-2 (1993): 439-469.
[9] G. Barles and P.E. Souganidis, “A new approach to front propagation: theory and applications,” Arch. Rational Mech. Anal. 141.3 (1998): 237-296.
[10] G. Bellettini, P. Buttà, and E. Presutti, “Sharp interface limits for non-local anisotropic interactions,” Arch. Rational Mech. Anal. 159-2 (2001): 109-135.
[11] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, American Mathematical Society, Providence (1978).
[12] R. Buckdahn, P. Cardaliaguet, and M. Quincampoix, “A Representation Formula for the Mean Curvature Motion,” SIAM J. Math. Anal. 33-4 (2001): 827-846.
[13] L.A. Caffarelli and X. Cabré, Fully Nonlinear Elliptic Equations, American Mathematical Society, Providence (2011).
[14] L.A. Caffarelli and R. Monneau, “Counter-example in three dimension and homogenization of geometric motions in two dimension,” Arch. Rational Mech. Anal. 212-2 (2014): 503-574.
[15] L.A. Caffarelli and P.E. Souganidis, “Rates of convergence for the homogenization of fully nonlinear uniformly elliptic partial differential equations in random media,” Inventiones mathematicae 180-2 (2010): 301-360.
[16] L.A. Caffarelli, P.E. Souganidis, and L. Wang, “Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media,” Communications on Pure and Applied Mathematics 58-3 (2005): 319-361.
[17] F. Camilli and C. Marchi, “Rates of convergence in periodic homogenization of fully nonlinear uniformly elliptic PDEs,” Nonlinearity 22 (2009): 1481-1498.
[18] P. Cardaliaguet, P.-L. Lions, and P.E. Souganidis, “A discussion about the homogenization of moving interfaces,” J. Math. Pures Appl. 91 (2009): 339-363.
[19] A. Cesaroni, M. Novaga, and E. Valdinoci, “Curve shortening flow in heterogeneous media,” Interfaces and Free Boundaries 13.4 (2011): 485-505.
[20] Y.G. Chen, Y. Giga, and S. Goto, “Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,” *J. Differential Geometry* **33**, 3 (1991): 749-786.

[21] X. Chen and B. Lou, “Traveling waves of a curvature flow in almost periodic media,” *J. Diff. Eq.* **247**-8 (2009): 2189-2208.

[CK] S. Choi and I. Kim, “Homogenization for nonlinear PDEs in general domains with oscillatory Neumann boundary data,” *J. Math. Pures Appl.* **102** (2014): 419-448.

[22] B. Craciun and K. Bhattacharya, “Effective motion of a curvature-sensitive interface through a heterogeneous medium,” *Interfaces Free Bound.* **6**-2 (2004): 151-173.

[23] M.G. Crandall, H. Ishii, P.-L. Lions, “User’s Guide to Viscosity Solutions of Second Order Partial Differential Equations,” *Bulletin of the American Mathematical Society* **27**-1 (1992): 1-67.

[24] L.C. Evans, “The perturbed test function method for viscosity solutions of nonlinear PDE,” *Proceedings of the Royal Society of Edinburgh A* **111**, 3/4 (1989): 359-375.

[25] L.C. Evans and J. Spruck, “Motion of level sets by mean curvature I,” *J. Differential Geometry* **33**.3 (1991): 635-681.

[26] E.A. Feinberg, P.O. Kasyanov, and N.V. Zadoianchuk, “Fatou’s Lemma for Weakly Converging Probabilities,” *Theory Probab. Appl.* **58**-4 (2014): 683-689.

[27] W.M. Feldman, “Homogenization of the oscillating Dirichlet boundary condition in general domains,” *J. Math. Pure Appl.* **101**.5 (2014): 599-622.

[28] W.M. Feldman, “Mean curvature flow with positive random forcing in 2-d,” arXiv preprint: arxiv:1911.00488 (2019).

[29] W.M. Feldman and I.C. Kim, “Continuity and Discontinuity of the Boundary Layer Tail,” *Ann. Sci. Éc. Norm. Supér* **50**-4 (2017): 599-622.

[30] W.M. Feldman, I.C. Kim, and P.E. Souganidis, “Quantitative homogenization of elliptic partial differential equations with random oscillatory boundary data,” *J. Math. Pure Appl.* **103**.4 (2015): 958-1002.

[31] W.M. Feldman and Y.P. Zhang, “Continuity properties for divergence form boundary data homogenization problems,” *Analysis & PDE* **12**-8 (2019): 287-354.

[32] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer (2015).

[33] D. Gerard-Varet, N. Masmoudi, “Homogenization and boundary layers,” *Acta Math* **209**-1 (2012): 133-178.

[34] V. Guillemin and A. Pollack, *Differential Topology*, American Mathematical Society, Providence (2010).

[35] M.E. Gurtin, H.M. Soner, and P.E. Souganidis, “Anisotropic Motion of an Interface Relaxed by the Formation of Infinitesimal Wrinkles,” *J. Diff. Eq.* **119** (1995): 54-108.

[36] H. Ishii, “Degenerate Parabolic PDEs with Discontinuities and Generalized Evolutions of Surfaces,” *Adv. Differential Equations* **1**-1 (1996): 51-72.

[37] H. Ishii and P.-L. Lions, “Viscosity solutions of fully nonlinear second-order elliptic partial differential equations,” *J. Diff. Eq.* **83**.1 (1990): 26-78.

[38] Y. Karshon and C. Bjorndahl, “Revisiting Tietze-Nakajima - Local and Global Convexity for Maps,” *Canad. J. Math.* **62**-5 (2010): 975-993.

[39] M.A. Katsoulakis and P.E. Souganidis, “Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interactions and Glauber dynamics,” *Comm. Math. Phys* **169** (1995): 61-97.

[41] M.A. Katsoulakis and P.E. Souganidis, “Stochastic Ising models and anisotropic front propagation,” *J. Stat. Phys.* **87**, 1-2 (1997): 63-89.

[42] P.-L. Lions and P.E. Souganidis, “Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications,” *Ann. I. H. Poincaré* **22** (2005): 667-677.
[43] M. Novaga and E. Valdinoci, “Closed curves of prescribed curvature and a pinning effect,” Networks and Heterogeneous Media 6.1 (2011): 77-88.

[44] P.S. Morfe, “A Variational Principle for Pulsating Standing Waves and an Einstein Relation in the Sharp Interface Limit,” arXiv preprint (2020: arXiv:2003.07298).

[45] P.S. Morfe, “Convergence and Rates for Hamilton-Jacobi Equations with Kirchoff Junction Conditions,” NoDEA Nonlinear Differential Equations Appl. 27-10 (2020): 1-69.

[46] P.S. Morfe, “Homogenization of the Allen-Cahn equation with periodic mobility,” arXiv preprint (2020): arXiv:2011.15012.

[47] M. Ohnuma and M.H. Sato, “Singular Degenerate Parabolic Equations with Applications to Geometric Evolutions,” Diff. Int. Eq. 6-5 (1993): 1265-1280.

[48] T. Ohta, D. Jasnow, and K. Kawasaki, “Universal Scaling in the Motion of Random Interfaces,” Phys. Rev. Lett. 49 (1982): 1223-1226.

[49] S. Osher and J. Sethian, “Fronts Propagating with Curvature-Dependent Speed: Algorithms Based on Hamilton-Jacobi Formulations,” J. Comp. Physics 79 (1988): 12-49.

[50] E. Outerelo and J.M. Ruiz, Mapping Degree Theory, American Mathematical Society, Providence (2009).

[51] G.C. Papanicolaou and S.R.S. Varadhan, “Diffusions with random coefficients,” Statistics and probability: essays in honor of C.R. Rao, eds. P.R. Krishnaiah, G. Kallianpur, J.K. Ghosh, and C.R. Rao, North Holland (1982).

[52] W.M. Senn, “Differentiability properties of the minimal average action,” Calc. Var. Partial Differential Equations 3-3 (1995): 343-384.

[53] J. Sethian, “An analysis of flame propagation,” University of California, Berkeley, PhD dissertation (1985).

[54] M. Soner and N. Touzi, “Dynamic programming for stochastic target problems and geometric flows,” J. Eur. Math. (JEMS) 4 (2006): 201-236.

[55] H. Spohn, “Interface motion in models with stochastic dynamics,” J. Stat. Phys. 71.5-6 (1993): 1081-1132.

[56] J.E. Taylor, J.W. Cahn, “Linking anisotropic sharp and diffuse surface motion laws via gradient flows,” J. Stat. Phys. 77.1-2 (1994): 183-197.