Ascent Properties for Derived Functors

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Introduction

Assumption
The term “ring” is short for “commutative, noetherian ring with identity”. Fix a ring homomorphism $R \to S$.

Big Picture
- To study a ring $R$, study its modules.
- To study how close $R$ is to $S$, compare the $R$-modules and the $S$-modules.
  - Each $S$-module is an $R$-module by restriction of scalars.
  - Each $R$-module $M$ gives rise to two $S$-modules: $S \otimes_R M$ and $\text{Hom}_R(S, M)$.

Question (Ascent)
When does a given $R$-module $M$ have a compatible $S$-module structure?
Fact (B. Anderson, Frankild, SW, Wiegand; ’08×2 and in press)

Let \((R, m, k) \to (S, mS, k)\) be a flat local ring homomorphism, and let \(M\) be a finitely generated \(R\)-module. TFAE:

(i) \(M\) has a compatible \(S\)-module structure.
(ii) \(\text{Hom}_R(S, M) \xrightarrow{\cong} M\) by \(f \mapsto f(1)\).
(iii) \(M \xrightarrow{\cong} S \otimes_R M\) by \(n \mapsto 1 \otimes n\).
(iv) \(\text{Ext}^i_R(S, M) = 0\) for all \(i \geq 1\).
(v) \(\text{Ext}^i_R(S, M)\) is f.g. over \(S\) for \(i = 1, \ldots, \dim_R(M)\).
(vi) \(S \otimes_R M\) is finitely generated over \(R\).
(vii) \(R/\text{Ann}_R(M) \xrightarrow{\cong} S/\text{Ann}_R(M)S\).
(viii) For all \(p \in \text{Min}_R(M)\), we have \(R/p \xrightarrow{\cong} S/pS\).
The case $\hat{S} = \hat{R}$

Fact (B. Anderson, Frankild, SW, Wiegand; ’08×2 and in press)

Let $M$ be a finitely generated $R$-module. TFAE:

(i) $M$ is complete.
(ii) $\text{Hom}_R(\hat{R}, M) \xrightarrow{\cong} M$ by $f \mapsto f(1)$.
(iii) $\text{Ext}_R^i(\hat{R}, M) = 0$ for all $i \geq 1$.
(iv) $\text{Ext}_R^i(\hat{R}, M)$ is f.g. over $\hat{R}$ for $i = 1, \ldots, \dim_R(M)$.
(v) $\hat{M}$ is finitely generated over $R$.
(vi) $R/\text{Ann}_R(M) \xrightarrow{\cong} \hat{R}/\text{Ann}_R(M)\hat{R}$.
(vii) For all $p \in \text{Min}_R(M)$, we have $R/p \xrightarrow{\cong} \hat{R}/p\hat{R}$.

Question

Let $R = k[X, Y]_{(X, Y)}$ and $S = \hat{R} = k[[X, Y]]$. For which $i$ do we have $\text{Ext}_R^i(S, R) \neq 0$?
Questions about Ascent of Pairs of Modules

Question

Given $R$-modules $M$ and $N$, what conditions guarantee that $\text{Ext}_R^i(M, N)$ and $\text{Tor}_i^R(M, N)$ have compatible $S$-module structures?

Answer

If $M$ or $N$ has a compatible $S$-module structure.

Example

Let $k$ be a field, and set $R = k[X, Y]_{(X,Y)}$. Then $\hat{R} \cong k[X, Y]$. $R/XR$ and $R/YR$ do not have compatible $\hat{R}$-module structures. However, $\text{Ext}_R^i(R/ XR, R/ YR)$ and $\text{Tor}_i^R(R/ XR, R/ YR)$ do have compatible $\hat{R}$-module structures.
Theorem (SW ’13)

Let \((R, m, k) \rightarrow (S, mS, K)\) be a flat local ring homomorphism, and let \(M\) and \(N\) be finitely generated \(R\)-modules. TFAE:

(i) \(M \otimes_R N\) has a compatible \(S\)-module structure.

(ii) \(\text{Tor}_i^R(M, N)\) has a compatible \(S\)-module structure \(\forall i \geq 0\).

(iii) \(\text{Ext}_R^i(M, N)\) has a compatible \(S\)-module structure \(\forall i \geq 0\).

(iv) \(\text{Ext}_R^i(M, N)\) has a compatible \(S\)-module structure for \(i = 0, \ldots, \dim_R(N) - 1\).

(v) \(\text{Ext}_R^i(S \otimes_R M, N)\) is finitely generated over \(R\) for all \(i \geq 1\).

(vi) \(\text{Ext}_R^i(S \otimes_R M, N) \xrightarrow{\cong} \text{Ext}_R^i(M, N)\) for all \(i \geq 0\).

(vii) \(\forall p\) that are minimal elements of \(\text{Supp}_R(M) \cap \text{Supp}_R(N)\), we have \(R/p \xrightarrow{\cong} S/pS\).
In the previous result, if $\text{Ext}^i_R(M, N)$ has a compatible $S$-module structure for $i = 0, \ldots, \dim_R(N) - 1$, then $\text{Ext}^i_R(M, N)$ has a compatible $S$-module structure $\forall i \geq 0$, as does $M \otimes_R N$.

**Example**

Let $R = k[X_1, \ldots, X_d]_{(x_1, \ldots, x_d)}$, and choose $j \in \{0, \ldots, d - 1\}$.
Set $M = R/(X_1, \ldots, X_j)$ and $N = R/(X_{j+1}, \ldots, X_{d-1})$.

$$\text{Ext}^i_R(M, N) \cong \begin{cases} 0 & \text{for all } i \neq j = \dim_R(N) - 1 \\ N & \text{for all } i = j = \dim_R(N) - 1. \end{cases}$$

$\text{Ext}^i_R(M, N)$ is an $\hat{R}$-module for $i = 0, \ldots, \dim(N) - 2$.
$\text{Ext}^{\dim_R(N)-1}_R(M, N)$ does not have an $\hat{R}$-module structure.
Same for $M \otimes_R N \cong R/(X_1, \ldots, X_{d-1})$. 

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