Expanding Belnap: dualities for a new class of default bilattices

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Abstract. Bilattices provide an algebraic tool with which to model simultaneously knowledge and truth. They were introduced by Belnap in 1977 in a paper entitled How a computer should think. Belnap argued that instead of using a logic with two values, for ‘true’ (t) and ‘false’ (f), a computer should use a logic with two further values, for ‘contradiction’ (⊤) and ‘no information’ (⊥). The resulting structure is equipped with two lattice orders, a knowledge order and a truth order, and hence is called a bilattice.

Prioritised default bilattices include not only values for ‘true’ (t₀), ‘false’ (f₀), ‘contradiction’ and ‘no information’, but also indexed families of default values, t₁, . . . , tₙ and f₁, . . . , fₙ, for simultaneous modelling of degrees of knowledge and truth.

We focus on a new family of prioritised default bilattices: Jₙ, for n ∈ ω. The bilattice J₀ is precisely Belnap’s seminal example. We obtain a multi-sorted duality for the variety Vₙ generated by Jₙ, and separately a single-sorted duality for the quasivariety Jₙ generated by Jₙ. The main tool for both dualities is a unified approach that enables us to identify the meet-irreducible elements of the appropriate subuniverse lattices. Our results provide an interesting example where the multi-sorted duality for the variety has a simpler structure than the single-sorted duality for the quasivariety.

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1. Introduction

We describe a new class of default bilattices \( \{ J_n \mid n \in \omega \} \) for use in prioritised default logic. While the first of these bilattices \( n = 0 \) is Belnap’s original four-element bilattice \( \text{FOUR} \) \cite{belnap}, for \( n \geq 1 \) these bilattices provide new algebraic structures for dealing with inconsistent and incomplete information. In particular, the structure of the knowledge order gives a new method for interpreting contradictory responses from amongst a hierarchy of ‘default true’ and ‘default false’ responses. The first two bilattices from our new class, drawn in their knowledge order \( (\leq_k) \) and their truth order \( (\leq_t) \), are shown in Figure 1, while \( J_n \) is shown in Figure 2.

To place both our family of bilattices, and our results concerning them, in an appropriate context, we recall that bilattices were investigated in the late 1980’s by Ginsberg \cite{ginsberg1,ginsberg2} as a method for inference with incomplete and contradictory information. These investigations built on the simple example introduced by Belnap \cite{belnap} about a decade earlier.

Belnap’s idea is represented by the four-element structure \( J_0 = \text{FOUR} \) shown in Figure 1. The elements \( t_0 \) and \( f_0 \) represent ‘true’ and ‘false’, while the elements \( \top \) and \( \bot \) represent ‘contradiction’ and ‘no information’. A statement \( p \) which is assigned the truth value \( \top \) as a result of contradictory information is less true than a statement \( q \) which is assigned \( t_0 \), as there is a source saying that \( p \) is false. On the other hand, more is known about \( p \) than is known about \( q \), as there are at least two different sources providing information. (The term ‘information order’ is used by some authors to refer to what we call the knowledge order.)

Generalising this example, a bilattice has two lattice orders, \( \leq_k \) (knowledge) and \( \leq_t \) (truth)—see Definitions 2.1 and 2.2 for details. While the concept of a truth order is familiar, from multi-valued logic, for example, the knowledge order is less familiar, and we discuss it very briefly. The join \( \oplus \) in the knowledge order is called \textit{gullability}: \( a \oplus b \) represents the combined information from \( a \) and \( b \) with no concern for any inherent contradictions. The meet \( \otimes \) in the knowledge order is called \textit{consensus}: \( a \otimes b \) represents the most information

![Figure 1. The bilattices \( J_0 = \text{FOUR} \) and \( J_1 \)]
Figure 2. The knowledge order (≤_k) and truth order (≤_t) on the bilattice J_n

upon which a and b agree. (See Fitting [13] for an excellent introduction to bilattices with many motivating examples.)

In addition to values for ‘true’ (t_0), ‘false’ (f_0), ‘contradiction’ (⊤) and ‘no information’ (∇), for n > 0 our new bilattice J_n has a hierarchy of default values, t_1, ..., t_n and f_1, ..., f_n; see Figure 2.

A statement assigned a truth value t_i (i > 0) is ‘true by default’. This truth value will be lower in both the knowledge and truth order for increasing i. A truth value of f_j (j > 0) corresponds to ‘false by default’, and will be lower in the knowledge order but higher in the truth order for increasing j. Further, we propose that, for n ∈ ω, the bilattice J_n should satisfy

$$t_i ⊕ f_j = ⊤ \quad \text{and} \quad t_i ⊗ f_j = ∇,$$

for all i, j ∈ {0, ..., n}. Any contradictory response that includes some level of truth (t_i) and some level of falsity (f_j) is registered as a total contradiction (⊤) and a total lack of consensus (∇). See Definition 2.3 for the formal definition of J_n.

Ginsberg first described default values in the bilattice SENV [14, Figure 4] and then later extended this idea to include more default values (cf. [15, Figure 7]). We assert that J_1 (and, more generally, J_n) is a better model than SENV for dealing with contradictory responses. In SENV, we have $t ⊗ f = d^T = dt ⊕ df$ (see Figure 3). If an agent is told that a certain statement is both true and false, the level of agreement or consensus is modelled by the bilattice element t ⊗ f. The join dt ⊕ df represents the total knowledge that an agent has if it is told that something is both true by default and false by default. However, it is not clear that t ⊗ f should represent the same degree of knowledge as the join dt ⊕ df. The family \{ J_n | n ∈ ω \} of default bilattices is designed to overcome this criticism.

We seek representations for algebras in the variety V_n = HSP(J_n) and the quasivariety J_n = ISP(J_n), generated by J_n. For n ⩾ 1, our bilattices are not interlaced and hence we lack the much-used product representation
Figure 3. Ginsberg’s bilattice for default logic $SEVEN$

(see Davey [8]). This leads us to develop a concrete representation via the theory of natural dualities. We derive a multi-sorted duality for the variety $V_n$ and present a single-sorted duality for the quasivariety $J_n$. Our dualities are optimal in the sense that none of the structure of the dualising object can be removed without destroying the duality.

Natural duality theory was first applied to the variety of distributive bilattices by Cabrer and Priestley [3]. Initially Craig [5], and later Cabrer, Craig and Priestley [2], considered a family $\{K_n \mid n \in \omega\}$ of non-interlaced default bilattices that generalise Belnap’s and Ginsberg’s examples; indeed, $K_0$ was $FOUR$ and $K_1$ was $SEVEN$. In both [5] and [2] the authors applied natural duality theory to produce a duality for the quasivariety $ISP(K_n)$ generated by $K_n$, and in [2] they also produced a multi-sorted duality for the variety $HSP(K_n)$ generated by $K_n$.

While our family of default bilattices overcomes the criticism mentioned above of default bilattices in the style of $SEVEN$, it comes at a price. As with the dualities for the variety and the quasivariety generated by $K_n$, to obtain our dualities for $V_n$ and $J_n$ we are required to analyse the lattices of subuniverses of certain binary products of algebras from $V_n$. This turns out to be substantially more difficult in the case of $J_n$ than in the case of $K_n$ due to the sizes of the subuniverse lattices. Nevertheless, the dualities we obtain, particularly in the multi-sorted case, are quite natural—see Subsection 8.4.

The paper is structured as follows. We define the family $\{J_n \mid n \in \omega\}$ of default bilattices in Section 2. There we not only describe the algebras themselves, but also derive some properties of the variety $V_n$ generated by $J_n$. In particular, we show that, up to isomorphism, $V_n$ contains $n+1$ subdirectly irreducible members denoted by $M_0, \ldots, M_n$, each of which is a homomorphic image of $J_n$: the algebra $M_0$ has size 4 and is term equivalent to $J_0$, and the algebras $M_1, \ldots, M_n$ have size 6.

In Section 3 we state restricted versions of general theorems from the theory of natural dualities as these are all that we require. Section 4 is devoted to setting up and stating our main duality result, the multi-sorted duality for the variety $V_n$ (Theorem 4.1). There are $n+1$ sorts, one corresponding to each of the subdirectly irreducible algebras $M_k$, for $k \in \{0, \ldots, n\}$. The duality is both optimal and strong and, for $n \geq 1$, uses a total of $\frac{1}{2}(n^2 + 3n + 2)$ relations and operations. The setup and statement of a single-sorted duality for the
quasivariety $J_n$ is in Section 8 (Theorem 8.1). Again, the duality is optimal; for $n \geq 1$, it uses $\frac{1}{2}(n^2 - n + 4)$ relations.

A consequence of the underlying lattice structure of our algebras is that the main tool for proving the multi-sorted duality for the variety $V_n$ is a good description of the meet-irreducible elements of the subuniverse lattice $\text{Sub}(M_j \times M_k)$, for all $j, k \in \{0, \ldots, n\}$. Similarly, the main tool for proving the duality for the quasivariety $J_n$ is the identification of the meet-irreducible elements of the subuniverse lattice $\text{Sub}(J_n^2)$. We achieve both of these tasks simultaneously in Section 5 by studying $\text{Sub}(A \times B)$, where $A$ and $B$ are non-trivial homomorphic images of $J_n$. The proofs of the multi-sorted duality, and of its optimality, are given in Sections 6 and 7.

In a follow-up paper, the authors will study the problem of axiomatising the dual categories, the process of translating from our duals to the Priestley duals of the underlying distributive lattices, and will use the translation to examine the free algebras in $V_n$.

2. The prioritised default bilattice $J_n$

Most definitions related to bilattices are due to Ginsberg [15]. These have evolved over time and in the literature there exists some variation in notation and terminology. The recent resurgence of interest in bilattices from mathematicians was largely catalysed by the work of Rivieccio [19]. We recommend his thesis and the article by Davey [8] for additional background on both the history and logical applications of bilattices.

**Definition 2.1.** A pre-bilattice is an algebra $B = \langle B; \otimes, \oplus, \wedge, \vee \rangle$ such that $\langle B; \otimes, \oplus \rangle$ and $\langle B; \wedge, \vee \rangle$ are lattices. We denote by $\leq_k$ the order associated with $\langle B; \otimes, \oplus \rangle$ and by $\leq_t$ the order associated with $\langle B; \wedge, \vee \rangle$.

It is unsurprising that in some contexts there will be some interaction between the two orders. A distributive pre-bilattice $B$ is one in which $\bullet$ distributes over $\ast$, for all $\bullet, \ast \in \{\otimes, \oplus, \wedge, \vee\}$. When each set of operations preserves the other order, i.e., $\otimes$ and $\oplus$ preserve $\leq_t$ and $\wedge$ and $\vee$ preserve $\leq_k$, then the pre-bilattice is said to be interlaced.

**Definition 2.2.** A bilattice is an algebra $B = \langle B; \otimes, \oplus, \wedge, \vee, \neg \rangle$ such that the reduct $\langle B; \otimes, \oplus, \wedge, \vee \rangle$ is a pre-bilattice and $\neg$ is a unary operation which is $\leq_k$-preserving, $\leq_t$-reversing and involutive.

We note that some authors use the terms ‘bilattice’ and ‘bilattice with negation’ to describe the objects from Definition 2.1 and 2.2, respectively. Our presentation is close to that of Jung and Rivieccio [17].

We now define formally the prioritised default bilattices $J_n$ illustrated in Figure 2. These bilattices were originally studied in the first author’s DPhil thesis [5].

**Definition 2.3.** For each $n \in \omega$, the underlying set of $J_n$ is $J_n = \{\top, f_0, \ldots, f_n, t_0, \ldots, t_n, \bot\}$. 
Figure 4. The $J_0$-reduct of $M_0$

The knowledge and truth orders, $\leq_k$ and $\leq_t$, on $J_n$ are as given in Figure 2. A unary involutive operation $\neg$ that preserves the $\leq_k$-order and reverses the $\leq_t$-order on $J_n$ is given by:

$$\neg \top = \top, \quad \neg \bot = \bot, \quad \neg f_m = t_m \text{ and } \neg t_m = f_m, \quad \text{for all } m \in \{0, \ldots, n\}.$$ 

We then add every element of $J_n$ as a constant to obtain the prioritised default bilattice

$$J_n = \langle J_n; \otimes, \oplus, \land, \lor, \neg, \top, f_0, \ldots, f_n, t_0, \ldots, t_n, \bot \rangle,$$

where $\otimes$ and $\oplus$ are greatest lower bound and least upper bound in the knowledge order $\leq_k$, and $\land$ and $\lor$ are greatest lower bound and least upper bound in the truth order $\leq_t$. To simplify the notation, we let $F_n = \{f_0, f_1, \ldots, f_n\}$ and $T_n = \{t_0, t_1, \ldots, t_n\}$.

The bilattice $J_n$ generalises Belnap’s bilattice $J_0 = \mathcal{FOUR}$ by taking the truth values $f_0$ and $t_0$ and expanding them to create a chain of truth values in each of their places. Moreover, as we now see, $J_n$ has a homomorphic image that is term equivalent to Belnap’s bilattice.

Fix $n \in \omega$. Let $M_0$ be the algebra in the signature of $J_n$ whose reduct is isomorphic to $J_0$, as shown in Figure 4, and in which

- the constants $f_0, f_1, \ldots, f_n$ take the value $f^0$, and
- the constants $t_0, t_1, \ldots, t_n$ take the value $t^0$.

Clearly, $M_0$ is term equivalent to $J_0$. The equivalence relation $\theta$ on $J_n$ with blocks $\{\top\}$, $F_n$, $T_n$ and $\{\bot\}$ is a congruence on $J_n$ with $J_n/\theta \cong M_0$. Hence $M_0$ is a homomorphic image of $J_n$.

We close this section with some remarks about the congruence lattice of $J_n$ and the structure of the variety $\mathcal{V}_n = \text{HSP}(J_n)$ generated by $J_n$.

Lemma 2.4. Let $n \in \omega$.

(1) Let $\theta$ be an equivalence relation obtained by independently collapsing any collection of the pairs $(f_0, f_1), \ldots, (f_{n-1}, f_n)$, and the corresponding pairs in $T_n$, and collapsing no other elements of $J_n$. Then $\theta$ is a congruence on $J_n$. Every non-trivial congruence on $J_n$ arises this way.

(2) $\text{Con}(J_n) \cong 2^n \oplus 1$ (i.e., $2^n$ with a new top adjoined).

Proof. We prove only (1) as (2) is an immediate consequence. It is clear that $\theta$ is a congruence on $J_n$. Now let $\alpha$ be a congruence on $J_n$. It is easily seen that if $\top/\alpha = \{\top\}$ and $\bot/\alpha = \{\bot\}$, then $\alpha$ is of the form described. It remains to prove that if $\top/\alpha \neq \{\top\}$, then $\alpha = J_n^2$ (the other case follows by duality).
Assume that \( c \in J_n \setminus \{ \top \} \) with \( c \equiv \alpha \top \). If \( c = \bot \), then we are done, so we may assume that \( c \notin \{ \top, \bot \} \). Hence \( \bot = c \otimes \neg c \equiv \alpha \top \otimes \neg \top = \top \otimes \top = \top \), and again we are done. \( \square \)

Fix \( n \in \omega \setminus \{0\} \) and let \( k \in \{1, \ldots, n\} \). Define \( M_k \) to be the algebra in the signature of \( J_n \) that has bilattice reduct isomorphic to the bilattice reduct of \( J_1 \), as shown in Figure 5, and in which

- the constants \( f_0, \ldots, f_{k-1} \) take the value \( 0^k \) and the constants \( f_{k}, \ldots, f_n \) take the value \( f^k \), while
- the constants \( t_0, \ldots, t_{k-1} \) take the value \( 1^k \) and the constants \( t_{k}, \ldots, t_n \) take the value \( t^k \).

Clearly, \( M_k \) is term equivalent to \( J_1 \). Let \( \theta_k \) be the equivalence relation on \( J_n \) with blocks

\[
\{ \top \}, \{ f_0, \ldots, f_{k-1} \}, \{ f_{k}, \ldots, f_n \}, \{ t_0, \ldots, t_{k-1} \}, \{ t_{k}, \ldots, t_n \}, \{ \bot \}.
\]

By Lemma 2.4, the relation \( \theta_k \) is a congruence on \( J_n \). Clearly, \( J_n / \theta_k \cong M_k \). Hence \( M_k \) is a homomorphic image of \( J_n \).

**Proposition 2.5.**

(1) Up to isomorphism, the only subdirectly irreducible algebra in the variety \( V_0 = HSP(J_0) \) is \( J_0 \) itself.

(2) Let \( n \in \omega \setminus \{0\} \). Up to isomorphism, the variety \( V_n = HSP(J_n) \) contains \( n + 1 \) subdirectly irreducible algebras, the four-element algebra \( M_0 \) and the six-element algebras \( M_k \), for \( k \in \{1, \ldots, n\} \).

(3) The algebras \( M_0 \) and \( M_k \), for \( k \in \{1, \ldots, n\} \), are injective in \( V_n \).

(4) Every algebra in \( V_n \) embeds into an injective algebra in \( V_n \).

(5) The variety \( V_n \) has the congruence extension property and the amalgamation property.

**Proof.** Since \( V_n \) is congruence distributive, a simple application of Jónsson’s Lemma [16, Corollary 3.4] tells us that the subdirectly irreducible algebras in \( V_n \) are the subdirectly irreducible homomorphic images of \( J_n \). We know from Lemma 2.4 that \( J_n \) has \( n + 1 \) meet-irreducible congruences: the unique coatom and its \( n \) lower covers. The corresponding subdirectly irreducible quotients of \( J_n \) are \( M_0 \) and \( M_k \), for \( k \in \{1, \ldots, n\} \).

Since each of these subdirectly irreducible algebras has no proper subalgebras, it is clear that each is injective in the class of subdirectly irreducible...
algebras in $\mathcal{V}_n$, and hence each is injective in the variety $\mathcal{V}_n$—see Davey [7, Corollary 2.3]. Consequently, every algebra in $\mathcal{V}_n$ embeds into an injective algebra, from which it follows that $\mathcal{V}_n$ satisfies both the congruence extension property and the amalgamation property—see Taylor [20, Proposition 2.1 and Theorem 2.3].

□

3. Natural dualities

It is important to note that a product representation theorem exists for both distributive bilattices [12, Proposition 8] and interlaced pre-bilattices. (See Davey [8] for a full historical account.) These representations have been used extensively in the study of bilattices and pre-bilattices. Duality and representation theorems for bilattices have largely focussed on product representations. Mobasher, Pigozzi, Slutzki and Voutsadakis [18] used the product representation of distributive bilattices to show that the category of distributive bilattices and the category of Priestley spaces are dually equivalent. Jung and Rivieccio [17] defined Priestley bispaces and showed that this new category is dually equivalent to the category of distributive bilattices. For $n \geq 1$, the bilattice $J_n$ is not interlaced: indeed, $f_0 \leq_k \top$ but $f_0 \wedge \bot = f_0 \not\leq_k f_n = \top \wedge \bot$. Hence we are not able to use a product representation to study either the variety or the quasivariety generated by $J_n$. We will turn to natural duality theory in order to study this new class of default bilattices.

For the basic concepts, results and notations of the theory of natural dualities we refer to the book by Clark and Davey [4]. Here we mention only a few key results, concepts and notations that will be important for our work.

Let $M$ be a finite algebra. We search for structures $M = \langle M; G, H, R, T \rangle$, where $T$ is the discrete topology and $G, H$ and $R$ are sets of finitary operations, partial operations and relations, respectively, such that the relations in $R$ and the graphs of the (partial) operations in $G \cup H$ are non-empty subuniverses of finite powers of $M$. If this is the case, we say that the operations, partial operations and relations are compatible with $M$ (or algebraic over $M$). We also say that the structure $M$ is compatible with $M$ (or algebraic over $M$). The structure $M$ is referred to as an alter ego of $M$.

When $M$ is a finite lattice-based algebra, we are able to apply a very powerful theorem to help us find an appropriate dualising structure $M$. The NU Duality Theorem [4, Theorem 2.3.4] is in fact much more general than the statement given below, but this special case will be sufficient for our needs. Note that, since $M$ is lattice based, it has a ternary NU term.

**Theorem 3.1.** (Special NU Duality Theorem). Let $M$ be a finite lattice-based algebra and let $R_M$ denote the set of all binary relations compatible with $M$. Then $M = \langle M; R_M, T \rangle$ yields a duality on $A = \text{ISP}(M)$.

The first application of natural duality to bilattices was by Cabrer and Priestley [3], who looked at both bounded and unbounded distributive bilattices. They showed that the knowledge order alone yields a duality on the class $\text{ISP}(J_0)$ of bounded distributive bilattices. (Except for $J_0$, in our class...
of bilattices the truth operations do not preserve $\leq_k$, and hence $\leq_k$ is not a compatible relation and cannot be used in the alter ego.)

**Theorem 3.2** [3, Theorem 4.2]. Consider the four-element bilattice

$$J_0 = \langle \{ \top, f_0, t_0, \bot \}; \otimes, \oplus, \land, \neg, \top, f_0, t_0, \bot \rangle \cong \text{FOUR}.$$  

The alter ego

$$J_0 = \langle \{ \top, f_0, t_0, \bot \}; \leq_k, \mathcal{T} \rangle$$

yields a strong, and therefore full, duality on $V_0 = \text{ISP}(J_0)$.

It is easy to see that for $n \in \omega \setminus \{0\}$ there is no finite algebra $M$ such that $V_n = \text{ISP}(M)$. Hence we cannot apply Theorem 3.1 to obtain a single-sorted duality for $V_n$, and we are led naturally to develop a multi-sorted duality for $V_n$. We refer to Davey and Priestley [10, Section 2], where multi-sorted dualities were first introduced, and to Clark and Davey [4, Chapter 7] for a detailed presentation of multi-sorted dualities in general. See Davey and Talukder [11, Section 4] for a discussion of strong dualities in the multi-sorted context along with the proofs missing from [4].

Let $\{M_0, \ldots, M_n\}$ be a set of finite, pairwise non-isomorphic lattice-based algebras of the same signature and let $A = \text{ISP}(\{M_0, \ldots, M_n\})$ be the quasivariety generated by them. We shall refer to a non-empty subuniverse of $M_j \times M_k$ as a compatible relation from $M_j$ to $M_k$, for all $j, k \in \{0, \ldots, n\}$. As an alter ego for the set $\{M_0, \ldots, M_n\}$, we will use a multi-sorted structure of the following kind:

$$M = \langle M_0 \cup \cdots \cup M_n; G, R, \mathcal{T} \rangle,$$

where, for each $g \in G$, there exist $j, k \in \{0, \ldots, n\}$, such that $g: M_j \rightarrow M_k$ is a homomorphism, each relation $R \in R$ is a compatible relation from $M_j \rightarrow M_k$, for some $j, k \in \{0, \ldots, n\}$, and $\mathcal{T}$ is the disjoint-union topology obtained from the discrete topology on the sorts.

We now present a version of the Multi-sorted NU Strong Duality Theorem [4, Theorem 7.1.2] that is tailored to the variety $V_n$. (In general, multi-sorted partial operations are needed to produce a strong duality. The fact that each algebra $M_k$ is subdirectly irreducible and has no proper subalgebras guarantees that we can restrict to total homomorphisms between the sorts.)

**Theorem 3.3.** (Special Multi-sorted NU Strong Duality Theorem). Assume that $M_0, \ldots, M_n$ are finite, pairwise non-isomorphic lattice-based algebras of the same signature, and let $A = \text{ISP}(\{M_0, \ldots, M_n\})$. Assume also that, for all $k \in \{0, \ldots, n\}$, the algebra $M_k$ is subdirectly irreducible and every element of $M_k$ is a constant. Define

$$M = \langle M_0 \cup \cdots \cup M_n; G, R, \mathcal{T} \rangle,$$

where $G = \bigcup \{ A(M_j, M_k) \mid j, k \in \{0, \ldots, n\} \}$ is the set of all homomorphisms between the sorts and $R = \bigcup \{ \text{Sub}(M_j \times M_k) \mid j, k \in \{0, \ldots, n\} \}$ is the set of all compatible relations between the sorts. Then $M$ yields a multi-sorted strong, and therefore full, duality on $A$. 
In general, the set $\text{Sub}(M^2)$ of subuniverses of $M^2$ as well as the set $\text{Sub}(M_j \times M_k)$ of all compatible relations between the sorts can be extremely large, even when $M$, $M_j$ and $M_k$ are small algebras. For example, computer calculations reveal that $|\text{Sub}(J^3_2)| = 200$. Although we are guaranteed a duality via the entire set $\mathcal{R}_M$ of compatible binary relations, we want to reduce the size of the set of relations, ideally to some minimal set.

The concept of entailment [4, Section 2.4] is crucial to understanding how and why it is possible to reduce the number of compatible relations required to yield a duality. We will briefly discuss the single-sorted variant.

If $M$ is a finite algebra and $\mathcal{R} \cup \{S\}$ is a set of compatible finitary relations on $M$, then we say that $\mathcal{R}$ entails $S$ and write $\mathcal{R} \vdash S$ if, for all algebras $A \in \text{ISP}(M)$, all continuous maps from the dual of $A$ to $M$ that preserve the relations in $\mathcal{R}$ also preserve $S$. (For details see [4, Section 2.4].)

The significance of entailment is that if $M = \langle M; R, T \rangle$ yields a duality on $A$ and $\mathcal{R} \setminus \{S\} \vdash S$, then $M' = \langle M; \mathcal{R} \setminus \{S\}, T \rangle$ also yields a duality on $A$. Admissible constructs for entailment were investigated by Davey, Haviar and Priestley [9]. An extensive list of constructs is given in [4, 2.4.5]. For example, every compatible binary relation, $R$, on $M$ entails its converse, $R^\sim$, and every pair $R, S$ of compatible binary relations on $M$ entail their intersection: $R \vdash R^\sim$ and $\{R, S\} \vdash R \cap S$.

Thus, we can remove all meet-reducible members of the subuniverse lattice $\text{Sub}(M^2)$ without destroying the duality.

There is an obvious extension of the concept of entailment to the multi-sorted setting that can be used to simplify a multi-sorted duality; it will be applied in Section 6.

Our task is to describe the meet-irreducible members of the subuniverse lattices $\text{Sub}(M^2)$ and $\text{Sub}(M_j \times M_k)$. This is done in Sections 5 and 6.

**Remark 3.4.** Assume that $M$ yields a single-sorted duality on the quasivariety $\mathcal{A} = \text{ISP}(M)$ and that $M'$ yields an $(n+1)$-sorted duality on the variety $\mathcal{V} = \text{HSP}(M) = \text{ISP}(\{M_0, \ldots, M_n\})$. The $S$-generated free algebra $F_{\mathcal{V}}(S)$ in the variety $\mathcal{V}$ is isomorphic to the subalgebra of $M^{MS}$ generated by the projections and so lies in the quasivariety $\mathcal{A}$. We can therefore use either duality to find the free algebras in $\mathcal{V}$. Indeed, $F_{\mathcal{V}}(S)$ is isomorphic to the dual of $M^S$ in the single-sorted case and the dual of $M'^S$ in the multi-sorted case. The difference is that the dual of $M^S$ is a subalgebra of $M^{MS}$ while the dual of $M'^S$ is a subalgebra of $M_0^{M_0^S} \times \cdots \times M_n^{M_n^S}$—see Section 8.4.

4. A natural duality for the variety $\mathcal{V}_n$

Fix $n \in \omega \setminus \{0\}$. In this section, we describe a strong, multi-sorted natural duality for the variety $\mathcal{V}_n$ generated by $J_n$. The proof that the alter ego yields a strong duality is contained in Section 6 and its optimality is proved in Section 7.
The multi-sorted alter ego

Example 4.2. The multi-sorted alter ego

\[ M_1 = \langle M_0 \cup M_1; g_1, \leq^0, \leq^1, \mathcal{T} \rangle \]
yields a strong and optimal duality on the variety $V_1 = \text{HSP}(J_1)$, and the multi-sorted alter ego

$$M_2 = \langle M_0 \cup M_1 \cup M_2; g_1, g_2, \leq^0, \leq^1, \leq^2, \leq^{12}, \top \rangle$$

yields a strong and optimal duality on the variety $V_2 = \text{HSP}(J_2)$.

## 5. Subuniverses of products of homomorphic images of $J_n$

The Special NU Duality Theorem 3.1 and the Special Multi-sorted NU Strong Duality Theorem 3.3 tell us that the set of all subuniverses of $J_n^2$ yields a duality on the quasivariety $J_n$ and that the set of all subuniverses of $M_j \times M_k$, for $j, k \in \{0, \ldots, n\}$, yields a multi-sorted duality on the variety $V_n$. It is always possible to restrict to subuniverses that are meet-irreducible in Sub($J_n^2$) and in Sub($M_j \times M_k$)—see the discussion of entailment after Theorem 3.3. Since $M_k$ is a non-trivial homomorphic image of $J_n$, we will treat both cases simultaneously and describe the meet-irreducible members of the lattice Sub($A \times B$), where $A$ and $B$ are non-trivial homomorphic images of $J_n$.

We shall use the following observations, usually without comment.

(a) Let $u : J_n \rightarrow A$ and $v : J_n \rightarrow B$ be homomorphisms. Since each element of $J_n$ is a constant, every subuniverse of $A \times B$ contains the set $K = \{ (u(c), v(c)) \mid c \in J_n \}$ of constants of $A \times B$.

(b) Let $u : J_n \rightarrow A$ be a homomorphism with $A$ non-trivial. For all $c \in J_n$, if $c \not\in \{ \top, \bot \}$, then $u(c) \not\in \{ \top, \bot \}$. (For example, if $u(c) = \top$, then in $A$ we would have $\bot = u(\bot) = u(c \otimes \neg c) = u(c) \otimes \neg u(c) = \top \otimes \bot = \bot$.)

(c) Let $u : J_n \rightarrow A$ be a surjective homomorphism with $A$ non-trivial.

(i) The bilattice reduct of $A$ is isomorphic to the bilattice reduct of $J_k$, for some $k \in \{0, \ldots, n\}$,

(ii) $u$ is the unique homomorphism from $J_n$ to $A$ (since $u$ preserves the constants).

We begin with a lemma that gives simple sufficient conditions for a subuniverse of $A \times B$ to contain large rectangular blocks, that is, large subsets of the form $A' \times B'$, for some $A' \subseteq A$ and $B' \subseteq B$.

Given $A \in V_n$, let $F_A$ and $T_A$ denote, respectively, the sets of ‘false’ constants and ‘true’ constants in $A$. Note that $F_n = F_{J_n}$ and $T_n = T_{J_n}$.

**Assumption 5.1.** In Lemmas 5.2 to 5.7, we fix $n \in \omega$ and surjective homomorphisms $u : J_n \rightarrow A$ and $v : J_n \rightarrow B$ with $A$ and $B$ non-trivial.

**Lemma 5.2.** Let $S$ be a subuniverse of $A \times B$.

(a) The following are equivalent:

(i) $(a, \top) \in S$ for some $a \in A \setminus \{ \top \}$;

(ii) $(\bot, b) \in S$ for some $b \in B \setminus \{ \bot \}$;

(iii) $A \times \{ \top \} \subseteq S$;

(iv) $\{ \bot \} \times B \subseteq S$.

(b) The following are equivalent:

(i) $(\top, b) \in S$ for some $b \in B \setminus \{ \top \}$;

(ii) $(a, \bot) \in S$ for some $a \in A \setminus \{ \bot \}$.
(iii) \( \{\top\} \times B \subseteq S \);
(iv) \( A \times \{\bot\} \subseteq S \).

(c) If \( S \) satisfies any of the eight conditions listed in (a) and (b), then \( F_A \times F_B \subseteq S \) and \( T_A \times T_B \subseteq S \).

Proof. By symmetry it suffices to prove (a). Since the bilattice reduct of a non-trivial homomorphic image of \( J_n \) is isomorphic to the bilattice reduct of \( J_k \), for some \( k \in \{0, \ldots, n\} \), both \( A \) and \( B \) satisfy

\[
x \neq \top \implies x \otimes \neg x = \bot \quad \text{and} \quad x \neq \bot \implies x \oplus \neg x = \top.
\]

The implications (a)(iii) \( \Rightarrow \) (a)(i) and (a)(iv) \( \Rightarrow \) (a)(ii) are of course trivial. We now use \( (\dagger) \) to prove the implications (a)(i) \( \Rightarrow \) (a)(iii) and (a)(ii) \( \Rightarrow \) (a)(iv).

(a)(i) \( \Rightarrow \) (a)(iii): Assume that (a)(i) holds, i.e., there exists \( a \in A \setminus \{\top\} \) with \( (a, \top) \in S \). Let \( a' \in A \) and choose \( c \in J_n \) with \( u(c) = a' \). By \( (\dagger) \) we have \( a \otimes \neg a = \bot \) and thus

\[
(a', \top) = (\bot, \top \uplus (a', v(c)) = ((a, \top) \otimes \neg (a, \top)) \oplus (u(c), v(c)) \in S,
\]

as \( (a, \top), (u(c), v(c)) \in S \). So (a)(iii) holds. The implication (a)(ii) \( \Rightarrow \) (a)(iv) is similar.

(a)(i) \( \Rightarrow \) (a)(ii): Assume again that (a)(i) holds and let \( a \in A \setminus \{\top\} \) with \( (a, \top) \in S \). If \( a = \bot \), then we can conclude (a)(ii) immediately as \( B \) is non-trivial. Assume now that \( a \neq \bot \) and let \( c \in J_n \) with \( u(c) = a \). Note that \( c \neq \bot \).

As \( a \neq \top \), by \( (\dagger) \) we have \( a \otimes \neg a = \bot \) and thus

\[
(\bot, \neg v(c)) = (a, \top) \otimes \neg (a, v(c)) = (a, \top) \otimes \neg (u(c), v(c)) \in S,
\]

as \( (a, \top), (u(c), v(c)) \in S \). Note that \( c \neq \bot \) implies \( v(c) \neq \bot \), since \( B \) is non-trivial, and consequently \( \neg v(c) \neq \bot \). Hence (a)(ii) holds. The implication (a)(ii) \( \Rightarrow \) (a)(i) holds by symmetry and duality.

(c) By symmetry, it is enough to assume that the equivalent conditions in (a) hold. Let \( (a, b) \in F_A \times F_B \). Then \( (a, b) = (a, \top) \land (\bot, b) \in S \) as \( (a, \top), (\bot, b) \in S \), whence \( F_A \times F_B \subseteq S \). Similarly, \( T_A \times T_B \subseteq S \). \( \square \)

The following lemma provides a test for whether a subuniverse of \( A \times B \) is proper.

Lemma 5.3. The following are equivalent for every subuniverse \( S \) of \( A \times B \).

(i) \( S = A \times B \);
(ii) \( (\top, \bot), (\bot, \top) \in S \);
(iii) \( (F_A \times T_B) \cap S \neq \emptyset \);
(iv) \( (T_A \times F_B) \cap S \neq \emptyset \).

Proof. (i) implies (iv) is trivial and (iv) is equivalent to (iii) by applying \( \neg \).

Now assume (iii), and let \( (a, b) \in (F_A \times T_B) \cap S \). Let \( c \in J_n \) with \( u(c) = a \).

We have \( c \in F_n \) since \( a \in F_A \), thus \( v(c) \in F_B \), whence \( v(c) \otimes b = \bot \). Hence

\[
(a, \bot) = (a, v(c)) \otimes (a, b) = (u(c), v(c)) \otimes (a, b) \in S,
\]

\[
(a, \bot) = (a, v(c)) \otimes (a, b) = (u(c), v(c)) \otimes (a, b) \in S.
\]
since \((u(c), v(c)) \in S\), and so
\[
(\top, \bot) = (a, \bot) \oplus (\neg a, \bot) \in S.
\]
Similarly, \((\bot, \top) \in S\). Hence (ii) holds. Finally, assume (ii). Let \((a, b) \in A \times B\) and assume that \(c, d \in J_n\) with \(u(c) = a\) and \(v(d) = b\). Then
\[
(a, b) = (u(c), v(d)) = ((u(c), v(c)) \ominus (\top, \bot)) \oplus ((u(d), v(d)) \ominus (\bot, \top)) \in S,
\]
since \((u(c), v(c)), (u(d), v(d)) \in S\). Hence (i) holds.
\[
\Box
\]

5.1. The relations \(S_\leq\) and \(S_\geq\)

Define subsets \(S_\leq\) and \(S_\geq\) of \(A \times B\) by
\[
S_\leq = (A \times \{\top\}) \cup (\{\bot\} \times B) \cup (\mathbb{F}_A \times \mathbb{F}_B) \cup (T_A \times T_B),
\]
\[
S_\geq = (A \times \{\bot\}) \cup (\{\top\} \times B) \cup (\mathbb{F}_A \times \mathbb{F}_B) \cup (T_A \times T_B).
\]

Since, up to isomorphism, \(\mathbb{A}\) has the same bilattice reduct as \(J_k\), for some \(k \in \{0, \ldots, n\}\), there is a unique homomorphism \(u_0: \mathbb{A} \rightarrow \mathbb{M}_0\). Similarly there is a unique homomorphism \(v_0: \mathbb{B} \rightarrow \mathbb{M}_0\). We shall use these homomorphisms to show that \(S_\leq\) and \(S_\geq\) are subuniverses of \(A \times B\). We will now add superscripts and denote the knowledge order on \(J_n\) by \(\leq_k^n\) and the knowledge order on \(\mathbb{M}_0\) by \(\leq_0^n\).

**Lemma 5.4.** \(S_\leq\) and \(S_\geq\) are subuniverses of \(A \times B\). Indeed,
\[
\text{sg}_{A \times B} ((u, v)(\leq_k^n)) = S_\leq = (u_0, v_0)^{-1}(\leq_0^n),
\]
\[
\text{sg}_{A \times B} ((u, v)(\geq_k^n)) = S_\geq = (u_0, v_0)^{-1}(\geq_0^n).
\]

**Proof.** We prove the result for \(S_\leq\). As \(u\) satisfies \(u(F_n) \subseteq F_A\) and \(u(T_n) \subseteq T_A\), and similarly for \(v\), we have \((u, v)(\leq_k^n) \subseteq S_\leq\). It follows from Lemma 5.2 that \(S_\leq \subseteq \text{sg}_{A \times B} ((u, v)(\leq_k^n))\). Hence
\[
(u, v)(\leq_k^n) \subseteq S_\leq \subseteq \text{sg}_{A \times B} ((u, v)(\leq_k^n)).
\]
Since \(\leq_0^n\) is a subuniverse of \(\mathbb{M}_0^n\), it follows that \((u_0, v_0)^{-1}(\leq_0^n)\) is a subuniverse of \(A \times B\). As the knowledge order on the bilattice \(\mathbb{M}_0\) is given by \(\leq_k = (M_0 \times \{\top\}) \cup (\{\bot\} \times M_0) \cup \{(f_k^0, f_k^0), (t_k^0, t_k^0)\}\), it is clear that
\[
(u_0, v_0)^{-1}(\leq_k^n) = (A \times \{\top\}) \cup (\{\bot\} \times B) \cup (F_A \times F_B) \cup (T_A \times T_B) = S_\leq.
\]
Hence \(S_\leq\) is a subuniverse of \(A \times B\), and \(\text{sg}_{A \times B} ((u, v)(\leq_k^n)) = S_\leq\) follows immediately.
\[
\Box
\]

**Lemma 5.5.** \(S_\leq\) and \(S_\geq\) are the only maximal proper subuniverses of \(A \times B\).

**Proof.** Let \(S\) be a proper subuniverse of \(A \times B\). We have \((F_A \times T_B) \cap S = \emptyset\) and \((T_A \times F_B) \cap S = \emptyset\), by Lemma 5.3. Hence
\[
S \subseteq (A \times \{\top, \bot\}) \cup (\{\top, \bot\} \times B) \cup (F_A \times F_B) \cup (T_A \times T_B). \quad (*)
\]
Suppose that $S \nsubseteq S_\leq$ and $S \nsubseteq S_\geq$. Thus, there exist $(a, b) \in S \setminus S_\leq$ and $(c, d) \in S \setminus S_\geq$. By (*) we have

$$(a, b) \in (A \times \{\bot\}) \setminus \{(\bot, \bot)\} \text{ or } (a, b) \in (\{\top\} \times B) \setminus \{(\top, \top)\}.$$

By Lemma 5.5, both cases yield $(\top, \bot) \in S$. Similarly, $(c, d) \in S \setminus S_\geq$ yields $(\bot, \top) \in S$. By Lemma 5.3, this gives $S = A \times B$, a contradiction. □

### 5.2. The relations $S_{ab}$

By Lemma 5.5, both $S_\leq$ and $S_\geq$ are meet-irreducible in $\text{Sub}(A \times B)$. Our next step is to describe the non-maximal meet-irreducibles. To do this we first require a simple lemma. Recall that we define

$$K = \{ (u(c), v(c)) \mid c \in J_n \} \subseteq A \times B.$$

Given $C \in V_n$, let $F_C = \langle F_C; \leq_k \rangle$ and $T_C = \langle T_C; \leq_k \rangle$ be the chains consisting of the ‘false’ constants and the ‘true’ constants of $C$, respectively, in their knowledge order. We shall abbreviate $F_{J_n}$ and $T_{J_n}$ to $F_n$ and $T_n$, respectively.

Note that the following lemma says nothing when at least one of $A$ and $B$ is isomorphic to $M_0$. For example, if $A \cong M_0$, then $|F_A| = 1$ and so $F_A \times F_B \subseteq K$; whence (i), (ii) and (iii) of Part (1) of the lemma are false.

**Lemma 5.6.** Let $(a, b) \in F_A \times F_B$.

1. The following are equivalent:
   (i) $(a, b) \in (F_A \times F_B) \setminus K$;
   (ii) $u^{-1}(a) \cap v^{-1}(b) = \emptyset$;
   (iii) $(du) \max_k(u^{-1}(a)) <_k \min_k(v^{-1}(b))$ or $(ud) \min_k(u^{-1}(a)) >_k \max_k(v^{-1}(b))$.

2. Condition (du) holds if and only if $(\downarrow_{F_A} a \times \downarrow_{F_B} b) \cap K = \emptyset$.

3. Condition (ud) holds if and only if $(\uparrow_{F_A} a \times \uparrow_{F_B} b) \cap K = \emptyset$.

Conditions (2) and (3) explain the notation: (du) and (ud) are abbreviations for down-up and up-down, respectively.

**Proof.** We have

$$(a, b) \in (F_A \times F_B) \setminus K \iff (\forall c \in F_n) \ (u(c), v(c)) \neq (a, b) \iff u^{-1}(a) \cap v^{-1}(b) = \emptyset.$$

As $u^{-1}(a)$ and $v^{-1}(b)$ are intervals in $F_n$, we have $u^{-1}(a) \cap v^{-1}(b) = \emptyset$ if and only if

$$\max_k(u^{-1}(a)) <_k \min_k(v^{-1}(b)) \text{ or } \min_k(u^{-1}(a)) >_k \max_k(v^{-1}(b)).$$

This proves (1). Since $u^{-1}(a')$ and $v^{-1}(b')$ are intervals in $F_n$ for all $a' \in F_A$ and all $b' \in F_B$, we have

$$\max_k(u^{-1}(a')) <_k \min_k(v^{-1}(b')) \iff (\forall a' \in \downarrow_{F_A} a)(\forall b' \in \uparrow_{F_B} b) \max_k(u^{-1}(a')) <_k \min_k(v^{-1}(b')) \iff (\downarrow_{F_A} a \times \uparrow_{F_B} b) \cap K = \emptyset.$$

Hence (2) holds, and therefore (3) holds by symmetry. □
Figure 7. The subuniverse $S_{ab}$ of $A \times B$

Given $(a, b) \in (F_A \times F_B) \setminus K$, precisely one of the conditions (du) and (ud) in Lemma 5.6(1)(iii) holds. If $(a, b) \models (du)$, then we define

$$S_{ab} := \{(\top, \top), (\bot, \bot)\} \cup ((F_A \times F_B) \setminus (\downarrow F_A a \times \uparrow F_B b))$$

$$\cup ((T_A \times T_B) \setminus (\downarrow T_A \neg a \times \uparrow T_B \neg b)),$$

and if $(a, b) \models (ud)$, then we define

$$S_{ab} := \{(\top, \top), (\bot, \bot)\} \cup ((F_A \times F_B) \setminus (\uparrow F_A a \times \downarrow F_B b))$$

$$\cup ((T_A \times T_B) \setminus (\uparrow T_A \neg a \times \downarrow T_B \neg b)).$$

Assume $(a, b) \models (du)$. As $F_A$ and $F_B$ are chains, $F_{ab} := (F_A \times F_B) \setminus (\downarrow F_A a \times \uparrow F_B b)$ and $T_{ab} := (T_A \times T_B) \setminus (\downarrow T_A \neg a \times \uparrow T_B \neg b)$ form sublattices of $F_A \times F_B$ and $T_A \times T_B$, respectively. The knowledge and truth orders on the subset $S_{ab} = \{(\top, \top), (\bot, \bot)\} \cup F_{ab} \cup T_{ab}$ of $A \times B$ are shown in Figure 7. Note that in Figure 7, and in later figures, we abbreviate $(a, b)$ to $ab$ for readability. With this diagram in hand, the following lemma is almost immediate.

**Lemma 5.7.** $S_{ab}$ is a subuniverse of $A \times B$, for all $(a, b) \in (F_A \times F_B) \setminus K$.

Let $\mathcal{F}$ be a topped intersection structure on a non-empty set $X$, that is, $\mathcal{F}$ contains $X$ and is closed under intersections of non-empty families, and let $x \in X$. An element $Y$ of $\mathcal{F}$ is a value at $x$ if $Y$ is maximal in $\mathcal{F}$ with respect to not containing $x$. The following simple lemma will help us to identify the meet-irreducible elements of the lattice $\text{Sub}(A \times B)$.

**Lemma 5.8.** Let $\mathcal{F}$ be a topped intersection structure on a non-empty set $X$. An element $Y$ of $\mathcal{F}$ is completely meet-irreducible in the lattice $\mathcal{F}$ if and only if $Y$ is a value at $x$ for some $x \in X$.

Given a topped intersection structure $\mathcal{F}$ on $X$ and $x \in X$, let $\text{Val}(x)$ denote the set of values of $\mathcal{F}$ at $x$. Note that Case (d) in the following theorem arises only when neither $A$ nor $B$ is isomorphic to $M_0$. 


Theorem 5.9. Let \( n \in \omega \) and let \( u: J_n \to A \) and \( v: J_n \to B \) be surjective homomorphisms with \( A \) and \( B \) non-trivial. The meet-irreducible elements in the lattice \( \text{Sub}(A \times B) \) are the sets \( S_\leq \) and \( S_\geq \), and \( S_{ab} \), for all pairs \( (a,b) \in (F_A \times F_B) \setminus K \). Indeed,

(a) \( \text{Val}(a,b) = \{S_\leq, S_\geq\} \), for all \( (a,b) \in (A \times B) \setminus (S_\leq \cup S_\geq) \),

(b) \( \text{Val}(a,b) = \{S_\leq\} \), for all \( (a,b) \in S_\geq \setminus S_\leq \),

(c) \( \text{Val}(a,b) = \{S_\geq\} \), for all \( (a,b) \in S_\leq \setminus S_\geq \),

(d) \( \text{Val}(a,b) = \{S_{ab}\} \), for all \( (a,b) \in (S_\leq \cap S_\geq) \setminus K \),

(e) \( \text{Val}(a,b) = \emptyset \), for all \( (a,b) \in K \).

Proof. (a) By Lemma 5.5, \( S_\leq \) and \( S_\geq \) are the only maximal subuniverses of \( A \times B \). It is therefore trivial that, for all \( (a,b) \in (A \times B) \setminus (S_\leq \cup S_\geq) \), we have \( \text{Val}(a,b) = \{S_\leq, S_\geq\} \).

(b) Let \( (a,b) \in S_\geq \setminus S_\leq = \{(\top) \times B\} \cup (A \times \{\bot\}) \setminus ((\top, \top), (\bot, \bot)) \) and assume that \( S \) is a value at \( (a,b) \). Since \( S \) is a proper subuniverse, by Lemma 5.5 we have either (i) \( S \subseteq S_\leq \) or (ii) \( S \subseteq S_\geq \). Assume that (ii) holds. By Lemma 5.2, \( S \) is disjoint from \( ((\top) \times B) \cup (A \times \{\bot\}) \setminus ((\top, \top), (\bot, \bot)) \) and so

\[
S \subseteq ((\top, \top), (\bot, \bot)) \cup (F_A \times F_B) \cup (T_A \times T_B) \subseteq S_\leq.
\]

Hence, in both cases (i) and (ii) we have \( S \subseteq S_\leq \). As \( (a,b) \not\in S_\leq \), the maximality of \( S \) yields \( S = S_\leq \).

(c) If \( S \) is a value at \( (a,b) \in S_\leq \setminus S_\geq \), then similarly we derive \( S = S_\geq \).

(d) Let \( (a,b) \in (S_\leq \cap S_\geq) \setminus K = ((F_A \times F_B) \cup (T_A \times T_B)) \setminus K \). By symmetry, we may assume that \( (a,b) \in (F_A \times F_B) \setminus K \). By Lemma 5.6 we may assume without loss of generality that \( (a,b) \models (d_u) \), in which case

\[
S_{ab} = ((\top, \top), (\bot, \bot)) \cup F_{ab} \cup T_{ab}.
\]

Assume \( S \) is a value at \( (a,b) \). By Lemma 5.2, we know that \( S \) is disjoint from

\[
((A \times \{\bot, \top\}) \cup (\{\bot, \top\} \times B)) \setminus ((\top, \top), (\bot, \bot)).
\]

As \( S \) is a proper subuniverse of \( A \times B \), by Lemma 5.3 it is also disjoint from \( (F_A \times T_B) \cup (T_A \times F_B) \). Hence

\[
S \subseteq ((\top, \top), (\bot, \bot)) \cup ((F_A \times F_B)) \cup ((T_A \times T_B)).
\]

We prove that \( (\downarrow_{F_A} a \times \uparrow_{F_B} b) \cap S = \emptyset \). Suppose \( (c,d) \in (\downarrow_{F_A} a \times \uparrow_{F_B} b) \cap S \), whence \( c \leq_k a \) and \( d \geq_k b \). Choose \( a' \in u^{-1}(a) \), \( b' \in v^{-1}(b) \). As \( (a,b) \models (d_u) \), by Lemma 5.6(1) we have \( a' <_k b' \), whence \( u(b') \geq_k u(a') = a \geq_k c \) and \( v(a') \leq_k v(b') = b \leq_k d \). Thus,

\[
(a,b) = (c \oplus a, b \oplus v(a')) = ((c \oplus u(b')) \oplus u(a'), (d \ominus v(b')) \oplus v(a'))
\]

\[
= ((c,d) \ominus (u(b'), v(b'))) \oplus (u(a'), v(a')).
\]

It follows that \( (a,b) \in S \) since \( (c,d) \in S \) by assumption, and since we have \( (u(b'), v(b')) \), \( (u(a'), v(a')) \in K \), and \( K \subseteq S \). This contradiction shows that \( (\downarrow_{F_A} a \times \uparrow_{F_B} b) \cap S = \emptyset \). By applying \( \neg \), we get \( (\downarrow_{T_A} a \times \uparrow_{T_B} b) \cap S = \emptyset \). We conclude that

\[
S \subseteq ((\top, \top), (\bot, \bot)) \cup F_{ab} \cup T_{ab} = S_{ab}.
\]
Since \((a,b) \notin S_{ab}\) and \(S\) is maximal with respect to not containing \((a,b)\), we have \(S = S_{ab}\).

(e) It is trivial that \(\text{Val}(a,b) = \emptyset\), for all \((a,b) \in K\), as \(K\) is the set of constants of \(A \times B\).

\[\square\]

6. The proof that \(M_n\) yields a multi-sorted duality on \(V_n\)

By the Special Multi-sorted NU Strong Duality Theorem 3.3, the structure

\[M' = \langle M_0 \cup M_1 \cup \cdots \cup M_n; \mathcal{G}, \mathcal{R}, \mathcal{J} \rangle,\]

where

- \(\mathcal{G} = \bigcup \{ \mathcal{A}(M_j, M_k) \mid j, k \in \{0,1,\ldots,n\} \} \) and
- \(\mathcal{R} = \bigcup \{ \text{Sub}(M_j \times M_k) \mid j, k \in \{0,1,\ldots,n\} \},\)

yields a multi-sorted strong duality on the variety \(V_n\). Our first step in refining this into a proof of Theorem 4.1 is to describe the meet-irreducibles in the lattice \(\text{Sub}(M_j \times M_k)\), for \(j, k \in \{0,\ldots,n\}\) with \(j \leq k\).

We have already introduced the relations \(\leq^k\) (and their converses \(\geq^k\)), for \(k \in \{0,\ldots,n\}\), and the relations \(\leq^j_k\), for \(j, k \in \{1,\ldots,n\}\) with \(j < k\). In addition to these, we also require the compatible relations \(S_{\leq}^k\) and \(S_{\geq}^k\) (from Subsection 5.1) with \(A = M_j\) and \(B = M_k\), for \(j = 0\) and \(k \in \{1,\ldots,n\}\), and for \(j, k \in \{1,\ldots,n\}\) with \(j \leq k\). We shall denote these multi-sorted relations from \(M_j\) to \(M_k\) by \(S_{\leq}^{jk}\) and \(S_{\geq}^{jk}\):

\[
S_{\leq}^{0k} = (M_0 \times \{\top^k\}) \cup \{\bot^0 \times M_k\} \cup \{(f^0, f^k), (f^0, 0^k)\} \\
\cup \{(t^0, t^k), (t^0, 1^k)\},
\]

\[
S_{\geq}^{0k} = (M_0 \times \{\bot^k\}) \cup \{\top^0 \times M_k\} \cup \{(f^0, f^k), (f^0, 0^k)\} \\
\cup \{(t^0, t^k), (t^0, 1^k)\},
\]

\[
S_{\leq}^{jk} = (M_j \times \{\top^k\}) \cup \{\bot^j \times M_k\} \cup \{(f^j, 0^k) \times \{f^j, 0^k\}\} \\
\cup \{t^j, 1^k \times \{t^j, 1^k\}\},
\]

\[
S_{\geq}^{jk} = (M_j \times \{\bot^k\}) \cup \{\top^j \times M_k\} \cup \{(f^j, 0^k) \times \{f^j, 0^k\}\} \\
\cup \{t^j, 1^k \times \{t^j, 1^k\}\}.
\]

**Theorem 6.1.** Let \(n \in \omega \setminus \{0\}\).

1. The meet-irreducible elements of \(\text{Sub}(M_0 \times M_0)\) are \(\leq^0\) and \(\geq^0\).
2. For all \(k \in \{1,\ldots,n\}\), the meet-irreducible elements of \(\text{Sub}(M_k \times M_k)\) are \(\leq^k\), \(\geq^k\), \(S_{\leq}^{0k}\) and \(S_{\geq}^{0k}\).
3. For all \(k \in \{1,\ldots,n\}\), the meet-irreducible elements of \(\text{Sub}(M_0 \times M_k)\) are \(S_{\leq}^{0k}\) and \(S_{\geq}^{0k}\).
4. For all \(j, k \in \{1,\ldots,n\}\) with \(j < k\), the meet-irreducible elements of \(\text{Sub}(M_j \times M_k)\) are \(\leq^{jk}\), \(S_{\leq}^{jk}\) and \(S_{\geq}^{jk}\).

**Proof.** Let \(j, k \in \{0,\ldots,n\}\) with \(j \leq k\), and let \(u: J_n \to M_j\) and \(v: J_n \to M_k\) be the unique homomorphisms. Hence, \(u\) maps \(f_0, \ldots, f_{j-1}\) to \(0^j\) and maps
Given $f_j, \ldots, f_n$ to $f^j$, and $v$ maps $f_0, \ldots, f_{k-1}$ to $0^k$ and maps $f_k, \ldots, f_n$ to $f^k$ (and similarly for the ‘true’ constants).

By Theorem 5.9, the meet-irreducibles in $\text{Sub}(M_j \times M_k)$ are the appropriate versions of $S_{\leq}$, $S_{\geq}$ and $S_{ab}$, for $(a, b) \in (F_{M_j} \times F_{M_k}) \setminus K$. Inspection shows that $S_{\leq}$ and $S_{\geq}$ yield the relations $\leq^0$ and $\geq^0$, when $j = k = 0$, and yield $S^j_{ik}$ and $S^i_{jk}$ otherwise. It remains to calculate the relations $S_{ab}$, for $(a, b) \in (F_{M_j} \times F_{M_k}) \setminus K$. Since $(F_{M_0} \times F_{M_k}) \setminus K = \emptyset$, for all $k \in \{0, \ldots, n\}$, we must calculate $S_{ab}$, for $(a, b) \in (F_{M_j} \times F_{M_k}) \setminus K$, with $0 < j < k$. We need to distinguish two cases: $j < k$ and $j = k$.

First consider the case where $j < k$. We then have
\[
K = \{ (u(c), v(c)) \mid c \in J_n \} = \{(\top^j, \top^k), (0^j, 0^k), (f^j, f^k),
(1^j, 1^k), (t^j, 1^k), (t^j, t^k), (\bot^j, \bot^k)\}.
\]
Thus, $(F_{M_j} \times F_{M_k}) \setminus K = \{ (0^j, f^k) \}$, whence $S_{0j} f^k = \leq^k$ is the only meet-irreducible of the form $S_{ab}$ that occurs in this case.

Now consider the case where $j = k$. We then have
\[
K = \{ (u(c), v(c)) \mid c \in J_n \} = \{(\top^k, \top^k), (0^k, 0^k), (f^k, f^k),
(1^k, 1^k), (t^k, t^k), (\bot^k, \bot^k)\}.
\]
Thus, $(F_{M_k} \times F_{M_k}) \setminus K = \{ (0^k, f^k), (f^k, 0^k) \}$, whence $S_{0k} f^k = \leq^k$ and $S_{f^k 0^k} = \geq^k$ (and no others) occur as meet-irreducibles of the form $S_{ab}$ in this case.

We are now ready to establish the duality statement in Theorem 4.1. We will need a multi-sorted generalisation of an entailment construct known as action by an endomorphism—see [4, 2.4.5(15)]. If $A$, $B$, $C$ and $D$ are sorts, $g: A \to C$, $h: B \to D$ and $S \subseteq C \times D$, then define
\[
(g, h)^{-1}(S) := \{ (a, b) \in A \times B \mid (g(a), h(b)) \in S \}.
\]
A simple calculation shows that $\{g, h, S\} \vdash (g, h)^{-1}(S)$.

**Proof of Theorem 4.1:** duality. As already observed, the Special Multi-sorted NU Strong Duality Theorem 3.3 implies that the alter ego $M'$ yields a multi-sorted duality on the variety $V_n$. Since each relation $R$ from $M_j$ to $M_k$ entails its converse $R^\prime$ from $M_k$ to $M_j$, it suffices to restrict to relations in $\text{Sub}(M_j \times M_k)$, for $j \leq k$. As each set of relations in $\text{Sub}(M_j \times M_k)$ entails its intersection, we can further restrict to the meet-irreducibles in $\text{Sub}(M_j \times M_k)$. By comparing the relations and maps in $S_{(n)} \cup G_{(n)}$ with the meet-irreducibles listed in Theorem 6.1, we see that it remains to show that $S_{(n)} \cup G_{(n)}$ entails the following relations
\[
\leq^0 \text{ and } \geq^k, S^k_{\leq}, S^k_{\geq}, S^0_{\leq}, S^0_{\geq}, S^k_{\leq}, S^k_{\geq}, \text{ for } j, k \in \{0, \ldots, n\} \text{ with } j < k.
\]
Since $S^k_{\leq} \in S_{(n)}$ and $\geq^k$ is the converse of $\leq^k$, it is clear that $S_{(n)} \cup G_{(n)}$ entails $\geq^k$, for all $k \in \{0, \ldots, n\}$. We now turn to the relations of the form $S^k_{\leq}$ or $S^k_{\geq}$. By Lemma 5.4, each of these relations is of the form $(g_j, g_k)^{-1}(\leq^0)$ or $(g_j, g_k)^{-1}(\geq^0)$, for some $j, k \in \{0, \ldots, n\}$, and hence is entailed by $S_{(n)} \cup G_{(n)}$. This completes the proof that $M_n$ yields a duality on the variety $V_n$. 
Finally, to show that the duality is strong we need to compare
\[ \mathcal{G} = \bigcup \{ \mathcal{A}(M_j, M_k) \mid j, k \in \{0, 1, \ldots, n\} \} \]
with the set \( \mathcal{G}_{(n)} \). The values of the constants in each of the algebras \( M_k \), for \( k \in \{0, \ldots, n\} \), guarantee that, for all \( j, k \in \{0, \ldots, n\} \), the only homomorphisms \( u : M_k \rightarrow M_j \) are the identity maps \( \text{id}_{M_k} \) along with the maps \( g_k : M_k \rightarrow M_0 \). Since the identity maps can be removed from any alter ego without destroying a strong duality, we are done.

\( \square \)

7. Proving that the duality on \( V_n \) given by \( M_n \) is optimal

In this section we shall prove that, for all \( n \in \omega \setminus \{0\} \), the multi-sorted duality for the variety \( V_n \) given in Theorem 4.1 is optimal, that is, none of the operations and relations in

\[ \mathcal{G}_{(n)} = \{ g_k \mid k \in \{1, \ldots, n\} \} \quad \text{and} \quad \mathcal{S}_{(n)} = \{ \leq^k \mid k \in \{0, \ldots, n\} \} \cup \{ \leq^j_k \mid j, k \in \{1, \ldots, n\} \text{ with } j < k \} \]

can be removed from the alter ego \( M_n = (M_0 \cup M_1 \cup \cdots \cup M_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T}) \) without destroying the duality.

Recall that, for every algebra \( B \) in \( V_n \), the underlying set of the multi-sorted dual of \( B \) is given by (cf. [4, Chapter 7])

\[ D(B) = V_n(B, M_0) \cup V_n(B, M_1) \cup \cdots \cup V_n(B, M_n). \]

We shall see that if \( B \) is finite, then the sorts \( V_n(B, M_k) \) of \( D(B) \) have a very simple structure. We need the following consequence of Jónsson’s Lemma [16, Lemma 3.1].

We note again that, for \( j, k \in \{1, \ldots, n\} \), the only endomorphism of \( M_k \) is \( \text{id}_M \), there are no homomorphisms from \( M_j \) to \( M_k \) when \( j \neq k \), and the only homomorphism from \( M_k \) to \( M_0 \) is \( g_k \).

Lemma 7.1. Let \( B \) be a subalgebra of \( \prod_{i \in I} A_i \) with the set \( I \) finite and \( A_i \in \{M_0, \ldots, M_n\} \), for all \( i \in I \). Then, for all \( k \in \{0, \ldots, n\} \), every homomorphism from \( B \) to \( M_k \) is the restriction of a projection or, when \( k = 0 \), is the restriction of a projection followed by one of the homomorphisms in \( \mathcal{G}_{(n)} \).

Proof. Let \( k \in \{0, \ldots, n\} \) and let \( u : B \rightarrow M_k \) be a homomorphism. Since every element of \( M_k \) is the value of a constant, the map \( u \) is surjective. Similarly, the restricted projection \( \pi_i|_B : B \rightarrow A_i \) is surjective. Since \( M_k \) is subdirectly irreducible, it follows from Jónsson’s Lemma that \( u = g \circ \pi_i|_B \), for some \( i \in I \) and some homomorphism \( g : A_i \rightarrow M_k \). If \( k \neq 0 \), then we must have \( A_i = M_k \) and \( g = \text{id}_{M_k} \), whence \( u = \pi_i|_B \). If \( k = 0 \), then either \( A_i = M_0 \), in which case \( g = \text{id}_{M_k} \) and hence \( u = \pi_i|_B \), or \( A_i = M_j \), for some \( j \neq 0 \), in which case \( g = g_j \) and so \( u = g_j \circ \pi_i|_B \), as claimed. \( \square \)

Let \( M \) be a finite set of finite algebras. A compatible multi-sorted binary relation \( R \) on \( M \) is said to be absolutely unavoidable within the set \( R_M \) of all compatible multi-sorted binary relations on \( M \) if, for every subset \( \mathcal{R} \) of
\( \mathcal{R}_\mathcal{M} \) such that \( \mathcal{M} = (\bigcup \{ M \mid M \in \mathcal{M} \}; \mathcal{R}, \mathcal{T}) \) yields a multi-sorted duality on \( \text{ISP}(\mathcal{M}) \), we have \( \mathcal{R} \cap \{ R, R^\ast \} \neq \emptyset \).

**Proposition 7.2.** Let \( \mathcal{M} = \{ M_0, \ldots, M_n \} \). The relation \( \preceq^0 \) is absolutely unavoidable within \( \mathcal{R}_\mathcal{M} \).

**Proof.** Let \( \preceq^0 \) denote the algebra with underlying set \( \preceq^0 \) and let \( \mathcal{R} \) be any set of compatible multi-sorted binary relations on \( \mathcal{M} \) that yields a duality on \( \mathcal{V}_n \), and therefore yields a duality on the algebra \( \preceq^0 \). By Lemma 7.1, every sort of \( \text{D}(\preceq^0) \), other than the \( M_0 \)-sort is empty. It follows that if \( R \) is a compatible multi-sorted binary relation from \( M_j \) to \( M_k \), with at least one of \( j \) and \( k \) not equal to 0, then \( R^\diamond(\preceq^0) = \emptyset \). Hence \( \mathcal{R} \) must include a binary relation on \( M_0 \). Since the only subuniverses of \( M_0 \) are \( \preceq^0 \), \( \succeq^0 \) and the trivial relations \( \Delta \) and \( M_0^2 \), it follows that \( \mathcal{R} \) must include \( \preceq^0 \) or \( \succeq^0 \), that is, \( \preceq^0 \) is absolutely unavoidable within \( \mathcal{R}_\mathcal{M} \). \( \square \)

**Proof of Theorem 4.1: optimality.** Given Proposition 7.2, it remains to show that none of the relations in \( \mathcal{S}_{(n)} \backslash \{ \preceq^0 \} \) and operations in \( \mathcal{G}_{(n)} \) can be removed from the alter ego \( \mathcal{M}_n = \langle M_0 \cup M_1 \cup \ldots \cup M_n; \mathcal{G}_{(n)}; \mathcal{S}_{(n)}; \mathcal{T} \rangle \) without destroying the duality.

Let \( j, k \in \{ 1, \ldots, n \} \) with \( j < k \) and consider the relation \( \preceq^{jk} \). Let \( \preceq^{jk} \) be the algebra with underlying set \( \preceq^{jk} \) and \( \rho_1: \preceq^{jk} \rightarrow M_j \), \( \rho_2: \preceq^{jk} \rightarrow M_k \) be the restrictions of the projections. By Lemma 7.1, the non-empty sorts of \( \text{D}(\preceq^{jk}) \) are \( \mathcal{V}_n(\preceq^{jk}, M_0) = \{ g_j \circ \rho_1, g_k \circ \rho_2 \} \), \( \mathcal{V}_n(\preceq^{jk}, M_j) = \{ \rho_1 \} \), and \( \mathcal{V}_n(\preceq^{jk}, M_k) = \{ \rho_2 \} \). Define \( \gamma: \text{D}(\preceq^{jk}) \rightarrow M_0 \cup \ldots \cup M_n \) by

\[
\gamma(g_j \circ \rho_1) = \gamma(g_k \circ \rho_2) = f^0, \quad \gamma(\rho_1) = 0^j, \quad \text{and} \quad \gamma(\rho_2) = f^k.
\]

We show that \( \gamma \) preserves each relation and operation in \( (\mathcal{S}_{(n)} \backslash \{ \preceq^{jk} \}) \cup \mathcal{G}_{(n)} \) and does not preserve \( \preceq^{jk} \), whence \( \preceq^{jk} \) cannot be removed without destroying the duality.

Since \( (\rho_1, \rho_2) \in \preceq^{jk} \) in \( \text{D}(\preceq^{jk}) \), but \( (\gamma(\rho_1), \gamma(\rho_2)) = (0^j, f^k) \notin \preceq^{jk} \), the map \( \gamma \) does not preserve \( \preceq^{jk} \). As \( g_j(f^j) = g_k(f^k) = f^0 \), the map \( \gamma \) preserves the action of the map \( g_k \), for all \( k \in \{ 1, \ldots, n \} \). It remains to prove that \( \gamma \) preserves \( \preceq^j \) and \( \preceq^k \), as all other relations in \( \mathcal{S}_{(n)} \) are empty on \( \text{D}(\preceq^{jk}) \); but this is trivial as \( (0^j, 0^j) \in \preceq^j \) and \( (f^k, f^k) \in \preceq^k \).

Now let \( k \in \{ 1, \ldots, n \} \), let \( \preceq^k \) be the algebra with underlying set \( \preceq^k \), and let \( \rho_1, \rho_2: \preceq^k \rightarrow M_k \) be the restrictions of the projections. Again by Lemma 7.1, the non-empty sorts of \( \text{D}(\preceq^k) \) are

\[
\mathcal{V}_n(\preceq^k, M_0) = \{ g_k \circ \rho_1, g_k \circ \rho_2 \} \quad \text{and} \quad \mathcal{V}_n(\preceq^k, M_k) = \{ \rho_1, \rho_2 \}.
\]

Define \( \gamma: \text{D}(\preceq^k) \rightarrow M_0 \cup \ldots \cup M_n \) by

\[
\gamma(g_k \circ \rho_1) = \gamma(g_k \circ \rho_2) = f^0, \quad \gamma(\rho_1) = 0^k, \quad \text{and} \quad \gamma(\rho_2) = f^k.
\]

Again it is easy to see that \( \gamma \) does not preserve \( \preceq^k \) (as \( (0^k, f^k) \notin \preceq^k \)), that \( \gamma \) preserves (by construction) the action of the map \( g_k \), for all \( k \in \{ 1, \ldots, n \} \), that \( \gamma \) preserves \( \preceq^0 \) (as \( (f^0, f^0) \in \preceq^0 \)), and that \( \gamma \) preserves all other relations in \( \mathcal{S}_{(n)} \) (as they are empty on \( \text{D}(\preceq^k) \)). Consequently, \( \preceq^k \) cannot be removed without destroying the duality.
Finally, fix \( k \in \{1, \ldots, n\} \). We shall show that \( g_k \) cannot be removed from \( G_n \) without destroying the duality. A third application of Lemma 7.1 shows that the non-empty sorts of \( D(M_k) \) are
\[
\mathcal{V}_n(M_k, M_0) = \{g_k\} \quad \text{and} \quad \mathcal{V}_n(M_k, M_k) = \{\text{id}_{M_k}\}.
\]
Define \( \gamma : D(M_k) \to M_0 \cup \ldots \cup M_n \) by \( \gamma(g_k) = f^0 \) and \( \gamma(\text{id}_{M_k}) = t^k \). Clearly, \( \gamma \) does not preserve the action of \( g_k \) since
\[
\gamma(g_k \circ \text{id}_{M_k}) = g_k(\gamma(\text{id}_{M_k})) = g_k(f^0) \neq t^0 = g_k(t^k) = g_k(\gamma(\text{id}_{M_k})).
\]
The map \( \gamma \) preserves \( \leq_k \) since \( (f^0, f^0) \in \leq_0 \), preserves \( \leq_k \) since \( (t^k, t^k) \in \leq_k \), and preserves all other relations in \( S_{n,n} \) as they are empty on \( D(M_k) \). Hence \( g_k \) cannot be deleted from \( G_n \) without destroying the duality. \( \square \)

8. A natural duality for the quasivariety \( J_n \)

In this section, we describe an alter ego of \( J_n \) that yields an optimal duality on the quasivariety \( J_n = \text{ISP}(J_n) \) generated by \( J_n \). Here we sketch only the main ideas of the proof. (A complete proof that the alter ego yields a duality is in Section 7 of the extended arXiv version of this paper [6] and its optimality is completely proved in Section 8 of [6].)

8.1. The relation \( S_{n,n} \)

Let \( n \in \omega \). Define the subset \( S_{n,n} \) of \( J_n^2 \) by
\[
S_{n,n} := (J_n \times \{\top\}) \cup (\{\bot\} \times J_n) \cup F_n^2 \cup T_n^2.
\]
The relation \( S_{n,n} \) is a quasi-order on \( J_n \)—see Figure 8. (When depicting a quasi-order \( R \), we draw \( x \) and \( y \) in the same block if \( x R y \) and \( y R x \).)

Note that the relation \( S_{0,0} \) is just the knowledge order \( \leq_k \) on \( J_0 \). For \( n > 0 \), the quasi-order \( S_{n,n} \) is not an order.

8.2. The relation \( S_{n,i} \)

For \( n \in \omega \setminus \{0\} \) and \( i \in \{0, \ldots, n - 1\} \), define the subset \( S_{n,i} \) of \( J_n^2 \) by:
\[
S_{n,i} := \{(\top, \top), (\bot, \bot)\} \cup \left(F_n^2 \setminus \{f_0, \ldots, f_i\} \times \{f_{i+1}, \ldots, f_n\}\right) \cup \left(T_n^2 \setminus \{t_0, \ldots, t_i\} \times \{t_{i+1}, \ldots, t_n\}\right).
\]
The relation $S_{n,i}$ is also a quasi-order on $J_n$—see Figure 9. Note that, unlike the quasi-order $S_{n,n}$, both $\top$ and $\bot$ are isolated in the quasi-order $S_{n,i}$. The quasi-order $S_{n,i}$ is an order if and only if $n = 1$ and $i = 0$.

8.3. The relation $R_{n,i,j}$

For $n \in \omega \setminus \{0, 1\}$ and $i, j \in \{0, \ldots, n - 1\}$, we also need the union

$$R_{n,i,j} := S_{n,i} \cup S_{n,j}.$$ 

It is easily seen that, if $i < j$, then

$$R_{n,i,j} = \{(\top, \top), (\bot, \bot)\} \cup \left( F_n^2 \setminus (\{f_0, \ldots, f_i\} \times \{f_{j+1}, \ldots, f_n\}) \right) \cup \left( T_n^2 \setminus (\{t_0, \ldots, t_i\} \times \{t_{j+1}, \ldots, t_n\}) \right).$$

Each of the relations $S_{n,n}$, $S_{n,i}$ and $R_{n,i,j}$ defined above is a compatible relation on $J_n$ and hence may be used as part of the structure on an alter ego of $J_n$—see Lemma 5.4 for $S_{n,n}$ and Lemma 5.7 for $S_{n,i}$ and $R_{n,i,j}$.

We can now state our single-sorted duality theorem. The dualities for $J_1$ and $J_2$ were obtained via computer calculations in the first author’s DPhil thesis [5].

**Theorem 8.1.** Let $n \in \omega$. Define the alter ego $\mathcal{J}_n = \langle J_n; R_{(n)}, \mathcal{T} \rangle$ of $J_n$, where $R_{(n)}$ is the set of compatible binary relations on $J_n$ given by

$$R_{(0)} = \{S_{0,0}\}, \quad R_{(1)} = \{S_{1,0}, S_{1,1}\}, \quad R_{(2)} = \{S_{2,0}, S_{2,1}, S_{2,2}\}, \quad R_{(3)} = \{S_{3,0}, S_{3,1}, S_{3,2}, S_{3,3}, R_{3,0,2}\},$$

and, in general, for $n \geq 3$,

$$R_{(n)} = \{S_{n,i} \mid 0 \leq i \leq n\} \cup \{R_{n,i,j} \mid i, j \in \{0, \ldots, n - 1\} \text{ with } i < j - 1\}.$$ 

1. The alter ego $\mathcal{J}_n$ yields an optimal duality on $\mathcal{J}_n = \text{ISP}(J_n)$.
2. $\mathcal{J}_0$ and $\mathcal{J}_1$ yield strong, and therefore full, dualities on $\mathcal{J}_0$ and $\mathcal{J}_1$, respectively.
3. For all $n \geq 2$, the duality on $\mathcal{J}_n$ can be upgraded to a strong, and therefore full, duality by adding all compatible $n$-ary partial operations on $J_n$ to the structure of the alter ego $\mathcal{J}_n$.
4. $|R_{(0)}| = 1$ and $|R_{(n)}| = \frac{1}{2}(n^2 - n + 4)$, for all $n \in \omega \setminus \{0\}$.

Note that when $n = 0$, the duality is the strong duality given by Cabrera and Priestley [3] as stated in our Theorem 3.2.
The proof of the above theorem rests on the identification of the meet-irreducible elements of the lattice \( \text{Sub}(J^2_n) \). Using Theorem 5.9 for the case \( A = B = J_n \) we identify the meet-irreducibles of the lattice \( \text{Sub}(J^2_n) \) to be \( S_{n,i} \) (for \( 0 \leq i \leq n \)) and \( R_{n,i,j} \) (for \( 0 \leq i < j \leq n - 1 \) when \( n \geq 2 \)) and their converses. Using entailment we are able to remove not only converses but also some of the \( R_{n,i,j} \). We have shown that if \( n \in \omega \setminus \{0, 1\} \) and \( 0 \leq i < n - 1 \), then \( S_{n,i} \cdot S_{n,i+1} \) is a homomorphic relational product and equals \( R_{n,i,i+1} \). Consequently, \( \{S_{n,i}, S_{n,i+1}\} \vdash R_{n,i,i+1} \).

The optimality of the duality is proved using the Test Algebra method (see [4, 8.1.3]) and also using the following new result.

**Proposition 8.2.** Let \( M \) be a finite algebra and let \( S \) be a compatible binary relation on \( M \).

1. If \( S \) is hom-minimal, is a value at \( (a, b) \) and satisfies \( a \in \rho_1(S) \) and \( b \in \rho_2(S) \), then \( S \) is absolutely unavoidable within \( \text{Sub}(M^2) \).
2. If \( S \) is hom-minimal, diagonal and meet-irreducible in \( \text{Sub}(M^2) \), then \( S \) is absolutely unavoidable within \( \text{Sub}(M^2) \).

### 8.4. Briefly comparing the dualities

Perhaps because we are more accustomed to working with orders rather than quasi-orders, the multi-sorted duality for the variety appears to be simpler than the single-sorted duality for the quasivariety. For example, the duality for the quasivariety \( J_1 \) has an alter ego consisting of the order \( S_{1,0} \) and the quasi-order \( S_{1,1} \) on the six-element base set \( J_1 \). The multi-sorted duality for the variety \( V_1 \) has an alter ego with two sorts: the four-element set \( M_0 \), equipped with the knowledge order \( \leq^0 \), and the six-element set \( M_1 \), equipped with the order \( \leq^1 \) along with a connecting map \( g_1 : M_1 \rightarrow M_0 \). Since the free algebras in the variety \( V_n \) lie in the quasivariety \( J_n \), we can use either duality to find the free algebras in \( V_n \). The authors used both dualities to verify that the size of the free algebra \( F_{V_1}(1) \) is 266. That is, we found all maps from \( J_1 \) to \( J_1 \) that preserve \( S_{1,0} \) and \( S_{1,1} \), thus representing \( F_{V_1}(1) \) as a subalgebra of \( J^J_1 \), and we found all multi-sorted maps from \( M_0 \cup M_1 \) to \( M_0 \cup M_1 \) that preserve \( \leq^0 \), \( \leq^1 \) and \( g_1 \), thus representing \( F_{V_1}(1) \) as a subalgebra of \( M^{M_0}_0 \times M^{M_1}_1 \). We found the latter calculation much easier as we were first able to find the 36 maps from \( M_0 \) to \( M_0 \) that preserve \( \leq^0 \) and then to link each of these via \( g_1 \) to a number of \( \leq^1 \)-preserving maps from \( M_1 \) to \( M_1 \).

In our follow-up paper, more will be said about the advantages of the multi-sorted duality for the variety over the single-sorted duality for the quasivariety.

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