Correspondence between classical dynamics and energy level spacing distribution in the transition billiard systems

Soo-Young Lee and Sunghwan Rim
Basic Science Research Institute, Korea University, Seoul 136-701, Korea

Eui-Soon Yim
Department of Physics, Semyung University, Chechon 390-230, Korea

C. H. Lee
D&S Dept., R&D Centernam Semiconductor Inc., Seoul 133-120, Korea

(February 18, 2022)
Abstract

The Robnik billiard is investigated in detail both classically and quantally in the transition range from integrable to almost chaotic system. We find out that a remarkable correspondence between characteristic features of classical dynamics, especially topological structure of integrable regions in the Poincaré surface of section, and the statistics of energy level spacings appears with a system parameter $\lambda$ being varied. It is shown that the variance of the level spacing distribution changes its behavior at every particular values of $\lambda$ in such a way that classical dynamics changes its topological structure in the Poincaré surface of section, while the skewness and the excess of the level spacings seem to be closely relevant to the interface structure between integrable region and chaotic sea rather than inner structure of integrable region.

It is very important to know what characteristic properties in quantum mechanics represent classical chaos. Many authors have devoted their efforts on this subject [1,2]. It is, now, well known that for K-system its level spacing distribution has a universal form, i.e., Wigner distribution [3]. However, for a soft chaotic system, any universal property has not been reported so far. In semiclassical limit Berry and Robnik [4] suggested a level spacing distribution which contains a physical parameter $\rho_{cl}$, the phase space volume of integrable region in the soft chaotic system. This semiclassical distribution is, however, not confirmed completely by numerical or experimental analyses [5,6]. Recently, Rim et. al. [7] show numerically in the Dreitlein billiard that the level spacing distribution is very sensitive to a small change of the system parameter in the vicinity of changing points of classical phase space topology like bifurcation points of robust islands. This result is very important because it implies that a universal distribution for a soft chaotic system, if it would exist, should contain a physical parameter related to the topology of the classical phase space manifolds in addition to $\rho_{cl}$.

In this Letter, we carefully investigate the Robnik billiard classically and quantally in the soft chaotic region of $\lambda$, and confirm the sensitivity of the level spacing statistics to the topological change of phase space manifolds for the corresponding classical system. Furthermore, we try to find out a detailed correspondence between behaviors of variance, skewness, and excess of the level spacing distribution and changes of classical phase space structures. The Robnik billiard has been studied by many authors in soft chaotic range as well as hard chaotic range [8,9]. However, it has not been analysed in this viewpoint.

The Robnik billiard is given as a quadratic conformal map $w = Az + Bz^2$ of the unit circle, and the area $S = \pi(A^2 + 2B^2)$ is fixed as $\pi$. The system parameter $\lambda$ is related to $A$ and $B$ as

$$A = \cos p, \quad B = (1/\sqrt{2})\sin p, \quad p = \tan^{-1}(\lambda\sqrt{2}). \quad (1)$$

The Robnik billiard has several advantages for our purpose. First, it is continuously transformed with increasing $\lambda$ from an integrable circle system to an almost chaotic system.
through soft chaotic systems which have both integrable parts and chaotic parts. This property enables us to investigate the soft chaotic system varying continuously with $\lambda$. Second, it has an analytic boundary, while the Dreitlein billiard has non-analyticity in the boundary when the system is soft chaotic, so that the integrable parts of the Poincaré section have complex structures which change continuously with $\lambda$. This property may give a partial support for generality of the sensitivity of the level spacing statistics. Third, the method of calculating energy levels is already given so that the energy level calculation is easily performed by the diagonalization method [12] which is known to be very accurate. Final advantage is that the system with larger than $\lambda > 0.2$ has only one dominant island surrounding period two orbit in chaotic sea. This means that without effects of other islands we can analyse impacts of the structure change of the island on the statistics of level spacings.

Classical analysis on the Robnik billiard has been performed and several bifurcation points are reported [13, 14]. We also investigate the Poincaré surface of section and confirm those points. Additionally we find a new pattern of the structure change of island within soft chaotic range of $\lambda$. We summarize results of the analyses for the classical dynamics below. At about $\lambda = 0.175$, three kinds of islands dominate the Poincaré section, which are corresponding to period two, period three, and period four orbits. The period four orbit is bifurcated at $\lambda = 0.176$ and the period five orbit at $\lambda = 0.185$. After then, the only island surrounding period two orbit remains alone in chaotic sea, and the period two orbit is bifurcated at $\lambda = 0.207$. As usual, this bifurcation does not mean that the island is divided into two isolated islands by chaotic sea. Just after the bifurcation the bifurcated orbits are still wrapped by invariant tori as described in Fig. 1 (a). These invariant tori break out at about $\lambda = 0.219$ so that two isolated islands centered at the bifurcated orbits appear in chaotic sea (Fig. 1 (b)). These bifurcated orbits are again bifurcated at $\lambda = 0.266$ which is reported by Hayli et. al. [14].

In order to see the structural change of the island surrounding period two orbit, we plot carefully the island in the range of $0.19 < \lambda < 0.25$ where the structures of integrable parts are effectively given by those of the period two island. From this plotting we find that the structure of the period two island evolves after a certain pattern with increasing $\lambda$. The pattern is shown in Fig. 2. The first step of the pattern is that resonances appear in mid of the island, and secondly, the size of resonances grow gradually and the position goes to outer part of the island. As the final step, all invariant tori wrapping the resonances break out so that the resonances become independent islands embedded in chaotic sea and, then, these new islands disappear rapidly. This pattern is repeated several times in the range of $0.19 < \lambda < 0.25$. We note that the final step of the pattern is a topological changing point of the interface structure between the island and chaotic sea. This point as well as bifurcation points would play an important role in understanding the correspondence between classical chaos and quantum chaos. In practice, it is not simple to determine numerically the precise value of $\lambda$ at which such break of invariant tori appears. We, therefore, assume that not all invariant tori wrapping the resonances would be broken if a trajectory starting from the thin chaotic region between the resonances and the island does not reach chaotic sea within 10000 boundary collisions. Using this way, we can determine such breaking points as $\lambda = 0.203, 0.227, 0.231, 0.238$. In addition to the pattern, we find another structure change of the island at $\lambda = 0.245$. Before and after this point the shape of island is inverted like mirror images, and right at the point the size of island becomes very small as shown in Fig. 3.
3. As explained above the Robnik billiard has various structural behaviors compared with the Dreitlein billiard where the period two island does not show such complex pattern due to the simplicity and non-analyticity of the boundary. This relatively complicated behavior of robust island structure enables us to investigate a detailed correspondence between classical phase space structure and quantal level spacing statistics.

Using the diagonalization method [12], we obtain energy eigenvalues for the Robnik billiard. We calculate 1100 energy levels at discrete values of $\lambda$ with an interval $\Delta \lambda = 0.001$. Among 1100 energy levels the lowest 500 levels, which are assured to be reliable, are taken for calculating the level spacing statistics. Of course, calculation of further energy levels is desirable and it may give more precise statistics. We, however, believe that the lowest 500 levels are enough to see characteristic behavior of level spacing statistics along $\lambda$.

We obtain the Brody distribution exponent $\nu$ making a comparison with the normalised variance $\sigma^2$ of the level spacings as

$$\sigma^2 = 2(\nu + 1)\Gamma(2/(\nu + 1))/[\Gamma(1/(\nu + 1))]^2 - 1.$$

This method was used by Robnik[ref]. The results are shown in Fig. 4. The skewness and the excess are also obtained and shown in Fig.5. In Figs. 4 and 5, the vertical dotted lines indicate the $\lambda$ values at which the wrapping invariant tori break out, and the dashed line does the structural changing point of island at $\lambda = 0.245$. We omitted the line of $\lambda = 0.231$ because this point seems to give a minor effect on the statistics. The vertical arrows point out the bifurcation points of period four, three, and two orbits. As shown in Fig.5, the exponent $\nu$ increases globally up to about $\nu = 1$ indicating the transition from the Poisson to the Wigner distribution. However, it can be seen clearly that the detailed glimpse of the increasing behavior discloses a piece of veil; it is not gradual and smooth, but is rather staircaselike. This behavior implies that there must be some competing process independent of the global parameter $\rho_{cl}$ which alone would smoothly increases the variance of level spacings with increasing $\lambda$. It is surprising that the starting points showing steep increase of $\nu$ are located on the points indicated by the dotted lines or arrows. Since the dotted lines and arrows denote the positions at which structural changes of islands appear in classical phase space, these coincidence strongly suggest that the level spacing statistics are affected by the topology of the phase space manifolds as well as the global parameter $\rho_{cl}$ for the corresponding classical system. This staircaselike behavior is very similar to the case of first order phase transition; temperature($\nu$) does not change even under external injection of energy($\lambda$) while the phase of matter(topological structure of classical phase space) is changing. More plausible physical situation for this seems that changing procedure of the phase space structure, i.e., occurrence and growing of resonances, may give the effect of lowering the exponent $\nu$, and whose effect would end at topological changing point of the phase space manifolds.

The plots for the skewness and the excess of the level spacing distribution also give a similar correspondence with classical dynamics. Similar behaviors with $\lambda$ are shown in the skewness plot and the excess plot(Fig. 5). These plots have several local maxima and show tendency to globally decrease up to about $\lambda = 0.25$. It is worthy to note the dotted lines, which indicate the topological change of interfaces of island with chaotic sea, are located at local minima while the bifurcation point of the period two orbit is not coincident with any maximum or minimum positions. It gives an evidence for the fact that the skewness and the
excess are sensitive to the outer structural change of island rather than inner change such as bifurcation point. Another interesting feature of the level statistics is that, except the plateau just before the bifurcation point, all plateaus of $\nu$ (or variance) are located on the rapid decreasing range after maxima in the skewness and the excess. From this observation we may speculate that during the procedure of topological change of island the $\nu$ (or variance) has the almost same value and the skew and the excess are decreased rapidly.

In the conclusion, we show the correspondence between classical dynamics, particularly, the topology of the phase space manifolds, and energy level spacing statistics for the soft chaotic range in the Robnik billiard. The topological change of phase space manifolds delivers a sensitive impact upon the level spacing distribution. The variance of the distribution seems to be affected by every topological change, while the skewness and the excess have higher sensitivity to the outer structural changes of integrable parts than inner changes.

It is natural and important to raise the question as ‘Does this correspondence appear in the statistics for very high energy levels?’ A clear answer for this question would be given by future work. Our conjecture tells that it would be the case even for semiclassical limit.

The authors would like to express many thanks to other GTP members, C.S. Park, D.H. Yoon, S.K. Yoo, and D.K. Park, for very sincere and informative discussions.
REFERENCES

[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer Verlag, New York, 1990).
[2] N. E. Hurt, *Quantum Chaos and Mesoscopic Systems* (Kluwer Academic Publishers, Dordrecht, 1997).
[3] G. M. Zaslavsky, Phys. Rep. **80**, 157 (1981).
[4] M. V. Berry and M. Robnik, J. Phys. A: Math. Gen. **17**, 2413 (1984).
[5] A. Hönig and D. Wintgen, Phys. Rev. A **39**, 5642 (1989).
[6] T. Prosen and M. Robnik, J. Phys. A: Math. Gen. **26**, 2371 (1993).
[7] S. Rim, S. Y. Lee, E. S. Yim, and C. H. Lee, chao-dyn/9808019 (1998).
[8] B. Li and M. Robnik, J. Phys. A: Math. Gen. **29**, 4387 (1996).
[9] B. Li, M. Robnik, and B. Hu, Phys. Rev. E **57**, 4095 (1998).
[10] B. Li and M. Robnik, J. Phys. A: Math. Gen. **27**, 5509 (1994).
[11] A. D. Stone and H. Bruus, Physica **189B**, 43 (1993).
[12] M. Robnik, J. Phys. A: Math. Gen. **17**, 1049 (1984).
[13] M. Robnik, J. Phys. A: Math. Gen. **16**, 3971 (1983).
[14] A. Hayli, T. Dumont, J. Moulin-Ollagnier, and J. M. Strelony, J. Phys. A: Math. Gen. **20**, 3237 (1987).
FIGURES

FIG. 1. (a) λ = 0.210. (b) λ = 0.220.

FIG. 2. (a) λ = 0.210. (b) λ = 0.202. (c) λ = 0.205.

FIG. 3. (a) λ = 0.242. (b) λ = 0.245. (c) λ = 0.248.

FIG. 4. The Brody exponent ν versus λ.

FIG. 5. The skewness(circles) and the excess(triangles) with λ.
Fig. 1 (b)
Fig. 2 (c)
Fig. 3 (a)
Fig. 3 (b)
Fig. 3 (c)
Fig. 4
Fig. 5