MODEL-FREE ACTIVE INPUT-OUTPUT FEEDBACK
LINEARIZATION OF A SINGLE-_LINK FLEXIBLE JOINT MANIPULATOR: AN IMPROVED ADRC APPROACH

Abstract: Traditional Input-Output Feedback Linearization (IOFL) requires full knowledge of system dynamics and assumes no disturbance at the input channel and no system's uncertainties. In this paper, a model–free Active Input-Output Feedback Linearization (AIOFL) technique based on an Improved Active Disturbance Rejection Control (IADRC) paradigm is proposed to design feedback linearization control law for a generalized nonlinear system with known relative degree. The Linearization Control Law (LCL) is composed of a scaled generalized disturbance estimated by an Improved Nonlinear Extended State Observer (INLESO) with saturation-like behavior and the nominal control law produced by an Improved Nonlinear State Error Feedback (INLSEF). The proposed AIOFL cancels in real-time fashion the generalized disturbances which represent all the unwanted dynamics, exogenous disturbances, and system uncertainties and transforms the system into a chain of integrators up to the relative degree of the system, the only information required about the nonlinear system. Stability analysis has been conducted based on Lyapunov functions and revealed the convergence of the INLESO and the asymptotic stability of the closed-loop system. Verification of the outcomes has been achieved by applying the proposed AIOFL technique on the Flexible Joint Single Link Manipulator (SLFJM). The simulations results validated the effectiveness of the proposed AIOFL tool based on IADRC as compared to the conventional ADRC based AIOFL and the traditional IOFL techniques.

Keywords: Active Input-Output feedback linearization, Extended state observer, flexible joint manipulator, generalized disturbance, nonlinear state error feedback.
I. INTRODUCTION

There are numerous classes of nonlinear models, given the following one,

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]  \hspace{1cm} (1)

where \( x = (x_1 \ x_2 \ \cdots \ x_n)^T \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R} \) is the control input and \( y \in \mathbb{R} \), is the system output. The functions \( f, g, \) and \( h \) are sufficiently smooth in a domain \( D \subset \mathbb{R}^n \). The mappings \( f: D \rightarrow \mathbb{R}^n \) and \( g: D \rightarrow \mathbb{R}^n \) are called vector fields on \( D \). Consider the Jacobian linearization of the system (1) about the equilibrium point \((x_0, y_0, u_0)\),

\[
\begin{align*}
\dot{x} &= \left[ \frac{\partial f(x_0)}{\partial x} + \frac{\partial g(x_0)}{\partial x} \right] (x - x_0) + g(x_0)(u - u_0) \\
y - y_0 &= \frac{\partial h(x_0)}{\partial x} (x - x_0)
\end{align*}
\]

It is worthy to observe that the nonlinear system is accurately represented by the Jacobian model only at the equilibrium point \((x_0, y_0, u_0)\). Consequently, any control policy built on the linearized model may produce unacceptable performance at other operating points. Input-Output Feedback Linearization (IOFL) is another class of nonlinear control methods that can yields linear models that is a precise depiction of the fundamental nonlinear model among a wide set of the equilibrium points [1]. Simply, IOFL is a technique, which eliminates all the nonlinearities with the result that the nonlinear dynamical system is represented by a chain of integrators. IOFL can be applied in three steps, the first step is transforming the nonlinear system into a linearized model, this is achieved through an appropriate nonlinear change of variables. After this stage, the equations of the system are linear but with the cost of a Linearization Control Law (LCL) \((u)\) which converts the system into a chain of integrators up to the relative degree of the system with linear control law \( (v) \). The second step is applying one of the traditional linear control methods such as state-feedback, PID control, etc., to design a linear control law \( (v) \) to control the linearized model. The third step is the stability investigation of the internal dynamics [2].

IOFL has been applied in recent years in various research and industrial fields, for example, in the control of induction motors [3], spacecraft models that include reaction wheel configuration [4], surface
permanent-magnet-synchronous generator (SPMSG) [5]. Further applications include, maximum power point tracking (MPPT) technique to achieve the desired performance under sudden irradiation drops, set-point changes, and load disturbances [6], Adaptive Input-Output Feedback Linearization to damp the low-frequency oscillations in power systems [7] and for the reduction of torque ripple of a brushless DC motors [8]. Finally, in robust nonlinear controller design for the voltage-source converters of high-voltage, direct current transmission link using IOFL and sliding-mode control approach [9].

Active methods for feedback linearization of nonlinear systems have been addressed by many researchers using adaptive control techniques, like, approximating the nonlinear function $f(x)$ using Gaussian Radial Basis Function Neural Networks (RBF-NN) [10] or a special form of a Brunovsky type Neuro-Fuzzy Dynamical System (NFDS) [11]. A simple output feedback adaptive control scheme is developed in [12] for a general class of nonlinear systems preceded by an actuator with hysteresis nonlinearity, where a new hysteresis inverse is obtained for the hysteresis and is used to efficiently cancel the hysteresis effects when developing the control scheme with the backstepping approach. An adaptive output feedback control methodology for nonaffine minimum phase nonlinear systems using nonlinearly parameterized Single-Hidden-Layer Neural Networks (SH-NNs) as approximation model is presented in [13], with the assumption that the system is globally exponentially minimum phase. Other recent scenarios for output tracking control can be found in [14, 15, 16, 17, 18] and the references therein.

Although the aforementioned approaches of adaptive control methods [10-13] have much higher power and significantly enhance the ability of a feedback system in dealing with uncertainty beyond robust control, nevertheless, this enhancement is obliged by the variation rates of the system parameters, and performance worsens quickly when the variation rates reach a certain limit [19]. This traditional method to deal with deterministic adaptive control has some intrinsic restrictions which have been very much perceived in the literature [14, 20] and the references therein. Most remarkably, if unknown parameters vary in intricate manners, it might be extremely hard to develop a “continuously
parameterized” group of competitor controllers. Moreover, adaptation continuously over a period of time may likewise be an arduous mission. These issues turn out to be particularly harsh if high performance and robustness are looked for. Therefore, the plan of adaptive control techniques includes a substantial number of particular procedures and frequently relies upon experimentation [20].

This paper proposes a new robust method for IOFL in an active manner, namely, AIOFL, in which the nonlinearities, model uncertainties, and external disturbance are excellently estimated and canceled using IADRC paradigm, such that the resulting nonlinear system is reduced into a chain of integrators up to the relative degree of the system. The key points of the proposed method are, AIOFL is a model-free and thus requires only the relative degree of the nonlinear system in contrast to conventional Input Output Feedback Linearization(IOFL), which requires complete knowledge of the nonlinear system to design the linearizing control law \( u \). The second key point is that there is no need to do any diffeomorphism transformation. Finally, the most important key point is that, in contrast to conventional IOFL, the proposed AIOFL is highly immune to system uncertainties and exogenous disturbances.

This paper is organized as follows. In section II a brief introduction to ADRC is presented. Background and problem statement are introduced in section III. In section IV Active Input-Output Feedback Linearization is discussed with detailed stability analysis and proofs. In section V, A single link flexible joint manipulator is presented as a guideway example for the proposed AIOFL method. Finally, the conclusions are drawn in section VI.

II. ADRC

ADRC is an advanced robust control strategy, which works by augmenting the mathematical model of the nonlinear dynamical system with an additional virtual state. This virtual state describes all the unwanted dynamics, uncertainties, and exogenous disturbances, named as the “generalized disturbance” or “total disturbance”. This virtual state together with the states of the dynamic system are observed in real-time fashion using the Extended State Observer (ESO) which is the core part of the ADRC. It
performs direct and active prediction and cancelation to the generalized disturbance by feeding back the estimated generalized disturbance into the input channel after simple manipulation. With ADRC, controlling a complex time-varying nonlinear system is transformed into a simple and linearized process. The superiority that makes it such a successful robust control tool is that it is an error-driven technique, rather than model-based control law. Mainly, ADRC consists of an ESO, a tracking differentiator (TD), and a nonlinear state error combination (NLSEF) as illustrated in Fig.1 [21-23], where \( r \in \mathbb{R} \) is the reference input, \( (r_1 \ r_2 \ \ldots \ r)^T \in \mathbb{R}^n \) is the transient profile, \( v \in \mathbb{R} \) is the control input for the linearized model, \( (\hat{x}_1 \ \hat{x}_2 \ \ldots \ \hat{x}_{n+1})^T \in \mathbb{R}^{n+1} \) is the augmented estimated vector which comprises the plant states \( \hat{x}_1, \ldots, \hat{x}_n \) and the estimated generalized disturbance \( \hat{x}_{n+1} \), which are produced by the ESO and \( b_0 \) is the input gain.

**Statement of contribution.** The contribution of this paper is proposing an AIOFL technique for an SLFJM which is a highly nonlinear uncertain system based on the IADRC with an Improved Nonlinear ESO (INLESO) of a saturation-like behavior developed in our previous work [23]. We used the INLESO not just as an estimator for the system states \( (\hat{x}_1, \ldots, \hat{x}_n) \), but also as a part of the linearization process where the generalized disturbance \( \hat{x}_{n+1} \) is rejected from the input channel in an online-manner as shown in Fig.1. The advantage of this technique is that it transforms any nonlinear uncertain system with exogenous disturbances and uncertainties into a pure chain of integrators up to the relative degree of the system. The proposed AIOFL method is effective due to its simplicity, for linearization, the only required information is the relative order of the system. Another point of contribution is the stability investigation of the AIOFL achieved via Lyapunov stability analysis for both Linear ESO (LESO) and INLESO and the study of the asymptotic behavior of the closed-loop system using Hurwitz stability Theorem. To the best of our knowledge, no previous study found in the literature that describe the linearization process for a nonlinear uncertain system within the context of ADRC with detailed stability analysis and extensive simulations on highla y nonlinear uncertain system.
III. BACKGROUND AND PROBLEM STATEMENT

To perform IOFL, conditions have to be derived and stated which allow us to do the transformation to the nonlinear system such that the input-output map is linear. Given \( \dot{y} \) as,

\[
\dot{y} = \frac{\partial h(x)}{\partial x} \dot{x} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = L_f h(x) + L_g h(x)u
\]

where \( L_f h(x) = \frac{\partial h}{\partial x} f(x) \) is called the Lie Derivative of \( h(x) \) with respect to \( f \). If \( L_g h(x) = 0 \), then \( \dot{y} = L_f h(x) \) is independent of \( u \). The second derivative of \( y \), denoted by \( \ddot{y} \) is given by,

\[
\ddot{y} = \frac{\partial L_f h}{\partial x} \dot{x} = \frac{\partial L_f h}{\partial x}[f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)u
\]

Once again, if \( L_g L_f h(x) = 0 \), then \( \ddot{y} = L_f^2 h(x) \), is also independent of \( u \). Repeating this process with \( h(x) \), one gets,

\[
\left\{ \begin{array}{l}
L_g L_f^{i-1} h(x) = 0 \quad i = 1, 2, \ldots, \rho - 1 \\
L_g L_f^{\rho-1} h(x) \neq 0
\end{array} \right. \tag{2}
\]

It can be seen that \( u \) is not included in \( y, \dot{y}, \ldots, y^{(\rho-1)} \) but \( y^{(\rho)} \) with a nonzero coefficient, \( y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)u \). The control signal \( u = \frac{1}{L_g L_f^{\rho-1} h(x)} [-L_f^\rho h(x) + v] \) reduces the input-output map to \( y^{(\rho)} = v \). The system is obviously input-output feedback linearizable, i.e. the nonlinear system (1) is represented by a chain of integrators, where \( \rho \) is denoted as the relative degree of the nonlinear system. Now let,
\[
 z = T(x) = \begin{pmatrix}
 \phi_1(x) \\
 \vdots \\
 \phi_{n-\rho}(x) \\
 -h(x) \\
 L_f^{-1}h(x)
\end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix}
 \phi(x) \\
 -\eta \\
 -\xi
\end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix}
 \eta \\
 \xi
\end{pmatrix}
\] (3)

where \( \phi_1(x) \) to \( \phi_{n-\rho}(x) \) are chosen such that \( \frac{\partial \phi_i(x)}{\partial x} g(x) = 0 \) for \( i \in \{1, 2, ..., n - \rho\} \) \( \forall x \in D \). This condition ensures that when the following equation is calculated \( \dot{\eta} = \frac{\partial \phi(x)}{\partial x} [f(x) + g(x)u] = f_0(\eta, \xi) + g_0(\eta, \xi)u \), the term \( u \) cancels out. It is now easy to verify that \( z = T(x) \) transforms the system into normal form denoted as,

\[
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1} \quad i \in \{1, 2, ..., \rho - 1\} \\
\dot{\xi}_\rho &= \alpha(x) + \beta(x)u \\
y &= \xi_1
\end{aligned}
\]

where \( \alpha(x) = L_f^\rho h(x) \) and \( \beta(x) = L_g L_f^{\rho-1}h(x) \). The internal dynamics are described by \( \dot{\eta} = f_0(\eta, \xi) \).

The zero-dynamics of the system is stated as \( \dot{\eta} = f_0(\eta, \xi) \) with \( \xi = 0 \) (i.e., \( \dot{\eta} = f_0(\eta, 0) \)). The system is called minimum phase if the zero-dynamics of the system are (globally) asymptotically stable.

**IV. PROPOSED ACTIVE INPUT-OUTPUT FEEDBACK LINEARIZATION (AIOFL)**

Consider the nonlinear SISO system given as,

\[
y^{(\rho)}(t) = f(y(t), (t), ..., y^{(\rho-1)}(t), w(t), t) + \beta(x)u(t)
\]

where \( y^{(i)}(t) \) indicates \( i \)th derivative of \( y \) (the output), and \( w \) and \( u \) represent the disturbance and the input, respectively. \( f \in C(\mathbb{R}^\rho \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) is an uncertain function. Many classes of nonlinear systems can be represented in this notation, e.g., time-varying or time-invariant systems, nonlinear or linear systems. For simpler representation and without causing any ambiguity, the time variable will be omitted from the equations. Assuming \( \xi_1 = y, \xi_2 = \dot{y}, ..., \xi_\rho = y^{(\rho-1)} \) one gets,
\[
\begin{align*}
\dot{x}_i &= x_{i+1}, i \in \{1,2, ..., \rho - 1\} \\
\dot{x}_\rho &= f(x_{1}, x_{2}, ..., x_{\rho}, w, t) + (\beta(x) - b_0)u + b_0u
\end{align*}
\]

The above representation is sometimes called Brunovsky form. Augmenting the system with additional state, \( x_{\rho+1} = f + (\beta(x) - b_0)u \Rightarrow \dot{x}_{\rho+1} = \Delta(t) = \dot{f}_T \). The coefficient \( b_0 \) is a rough approximation of \( \beta(x) \) in the plant within a \( \pm50\% \) range [14] and \( f_T = f + (\beta(x) - b_0)u \) is the generalized disturbance, which consists all of the unknown external disturbances, system uncertainties and internal dynamics. The parameter \( b_0 \) usually chosen explicitly by the user as a design parameter. The states of the system in (4) together with the generalized disturbance \( f_T \) will be estimated by an LESO, given by [24, 25],

\[
\begin{align*}
\dot{\xi}_i &= \dot{x}_{i+1} + \beta_i(y - \dot{x}_1), i \in \{1,2, ..., \rho - 1\} \\
\dot{\xi}_\rho &= \dot{x}_{\rho+1} + \beta_\rho(y - \dot{x}_1) + b_0u \\
\dot{\xi}_{\rho+1} &= \beta_{\rho+1}(y - \dot{x}_1)
\end{align*}
\]

The noteworthy feature of the basic LESO and its variants is that it needs minimum information about the dynamical system, only the relative degree \( \rho \) of the underlying system is needed to the design of the LESO. Several modifications have been developed to expand the basic features of the LESO to adapt to a broader class of dynamical systems [24]. In this section, the convergence of the LESO and the INLESO are demonstrated using Lyapunov analysis.

Consider the system (4) with the augmented state \( \dot{x}_{\rho+1} \) is given as

\[
\begin{align*}
\dot{x}_i &= x_{i+1}, i \in \{1,2, ..., \rho - 1\} \\
\dot{\xi}_\rho &= \dot{x}_{\rho+1} + b_0u \\
\dot{\xi}_{\rho+1} &= \Delta(t) = \dot{f}_T
\end{align*}
\]

**Assumption (A1):** The function \( f_T \) is continuously differentiable.

**Assumption (A2):** There exist a positive constant \( M \) such that \( |\Delta(t)| \leq M \) for \( t \geq 0 \).

**Assumption (A3) [25]:** There exist constants \( \lambda_1 \) and \( \lambda_2 \) and positive definite, continuously differentiable functions \( V, W: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^+ \) such that,
\[
\begin{align*}
\{\lambda_1 \|y\|^2 & \leq V(y) \leq \lambda_2 \|y\|^2 \\
W(y) &= \|y\|^2
\end{align*}
\] (7)

\[
\sum_{i=1}^\rho \frac{\partial v_i}{\partial y_i} (y_i - a_i y_1) - \frac{\partial v}{\partial y_{\rho+1}} a_{\rho+1} y_1 \leq -W(y)
\] (8)

**Lemma 1 (LESO Convergence):** Given the nonlinear system expressed as in (6) and the LESO given in (5). If Assumptions (A1)-(A3) are true, then, for any initial values of the system states \(\xi_i, i \in \{1, 2, \ldots, \rho\}\), the states of the LESO converges to that of the nonlinear system, i.e.,

\[
\lim_{t \to \infty} \left| \xi_i(t) - \dot{\xi}_i(t) \right| = 0, \quad i \in \{1, 2, \ldots, \rho\}
\]

\[
\lim_{t \to \infty} \left| \dot{\xi}_{\rho+1} - \left( f(\xi_1, \xi_2, \ldots, \xi_{\rho}, w, t) + (\beta(x) - b_0)u \right) \right| = 0
\]

where \(\xi_i(t)\), and \(\dot{\xi}_i(t)\) denote the solutions of (6) and (5) respectively, \(i \in \{1, 2, \ldots, \rho + 1\}\).

**Proof:** we make use of [25] to prove the convergence for the LESO. Set \(e_i(t) = \xi_i(t) - \dot{\xi}_i(t)\), for \(i \in \{1, 2, \ldots, \rho + 1\}\). Then subtracting (5) from (6), one gets

\[
\begin{align*}
\dot{\xi}_1(t) - \dot{\xi}_1(t) &= \xi_2(t) - \left( \dot{\xi}_2(t) + \beta_1 \left( y(t) - \dot{\xi}_1(t) \right) \right) \\
\dot{\xi}_2(t) - \dot{\xi}_2(t) &= \xi_3(t) - \left( \dot{\xi}_3(t) + \beta_2 \left( y(t) - \dot{\xi}_1(t) \right) \right) \\
& \quad \vdots \\
\dot{\xi}_\rho(t) - \dot{\xi}_\rho(t) &= \xi_{\rho+1}(t) + b_0 u(t) \quad - \left( \dot{\xi}_{\rho+1}(t) + b_0 u(t) + \beta_\rho \left( y(t) - \dot{\xi}_1(t) \right) \right) \\
\dot{\xi}_{\rho+1}(t) - \dot{\xi}_{\rho+1}(t) &= \Delta(t) - \beta_{\rho+1} \left( y(t) - \dot{\xi}_1(t) \right)
\end{align*}
\]

Direct computations show that the estimated error dynamics satisfy:

\[
\begin{align*}
\dot{e}_1(t) &= e_2(t) - \beta_1 e(t) \\
\dot{e}_2(t) &= e_3(t) - \beta_2 e(t) \\
& \quad \vdots \\
\dot{e}_\rho(t) &= e_{\rho+1}(t) - \beta_\rho e(t) \\
\dot{e}_{\rho+1}(t) &= \Delta(t) - \beta_{\rho+1} e(t)
\end{align*}
\] (9)

Let \(\beta_i = a_i \omega_0^i\), where \(a_i, i \in \{1, 2, \ldots, \rho + 1\}\) is a design parameters associated with each \(\omega_0^i\) and \(\omega_0\) is the bandwidth of the LESO. Expressing \(\beta_i\) as \(a_i \omega_0^i\), the system of (9) can be written as,
\[
\begin{aligned}
\dot{e}_1(t) &= e_2(t) - \omega_0 \alpha_1 e_1(t) \\
\dot{e}_2(t) &= e_3(t) - \omega_0^2 \alpha_2 e_2(t) \\
& \vdots \\
\dot{e}_\rho(t) &= e_{\rho+1}(t) - \omega_0^\rho \alpha_\rho e_1(t) \\
\dot{e}_{\rho+1}(t) &= \Delta(t) - \omega_0^{\rho+1} \alpha_{\rho+1} e_1(t)
\end{aligned}
\]  

(10)

Assuming \( \eta_i(t) = \omega_0^{\rho+1-i} e_i\left(\frac{t}{\omega_0}\right), i \in \{1, 2, \ldots, \rho + 1\} \), then

\[
e_i\left(\frac{t}{\omega_0}\right) = \frac{1}{\omega_0^{\rho+1-i}} \eta_i(t)
\]  

(11)

Then time-scaled estimation error dynamics are expressed as:

\[
\begin{aligned}
\frac{d\eta_1(t)}{dt} &= \eta_2(t) - \alpha_1 \eta_1(t) \\
\frac{d\eta_2(t)}{dt} &= \eta_3(t) - \alpha_2 \eta_1(t) \\
& \vdots \\
\frac{d\eta_\rho(t)}{dt} &= \eta_{\rho+1}(t) - \alpha_\rho \eta_1(t) \\
\frac{d\eta_{\rho+1}(t)}{dt} &= \frac{\Delta}{\omega_0} - \alpha_{\rho+1} \eta_1(t)
\end{aligned}
\]  

(12)

Assume the candidate Lyapunov functions \( V, W: \mathbb{R}^{n+1} \to \mathbb{R}^+ \) defined by \( V(\eta) = \langle P\eta, \eta \rangle \), where \( \eta \in \mathbb{R}^{\rho+1} \) and \( P \) is a symmetric positive definite matrix. Suppose that in Assumption (A3) holds true with \( \lambda_1 = \lambda_{\text{min}}(P) \) and \( \lambda_2 = \lambda_{\text{max}}(P) \), where \( \lambda_{\text{min}}(P) \) and \( \lambda_{\text{max}}(P) \) are the minimal and maximal eigenvalues of \( P \), respectively. Then, finding \( \dot{V} \) (the differentiation of \( V(\eta) \)) w.r.t \( t \) over \( \eta \) (the solution (12)) is accomplished in the following way

\[
\dot{V}(\eta)\big|_{\text{along (12)}} = \sum_{i=1}^{\rho+1} \frac{\partial V(\eta)}{\partial \eta_i} \eta_i(t)
\]

\[
= \sum_{i=1}^{\rho} \frac{\partial V(\eta)}{\eta_i} (\eta_{i+1}(t) - \alpha_i \eta_1(t)) + \frac{\partial V(\eta)}{\partial \eta_{\rho+1}} \left( \frac{\Delta}{\omega_0} - \alpha_{\rho+1} \eta_1(t) \right)
\]

Then,

\[
\dot{V}(\eta)\big|_{\text{along (12)}} = \sum_{i=1}^{\rho} \frac{\partial V(\eta)}{\eta_i} (\eta_{i+1}(t) - \alpha_i \eta_1(t)) + \frac{\partial V(\eta)}{\partial \eta_{\rho+1}} \frac{\Delta}{\omega_0} - \frac{\partial V(\eta)}{\partial \eta_{\rho+1}} \alpha_{\rho+1} \eta_1(t)
\]

If the inequality (8) in Assumption (A3) is satisfied, then
\[
\dot{V}(\eta)_{\text{along (12)}} \leq -W(\eta) + \frac{\partial V(\eta)}{\partial \eta} \frac{\Delta}{\omega_o}
\]

Since \(\left|\frac{\partial V}{\partial \eta}\right| \leq \left\|\frac{\partial V(\eta)}{\partial \eta}\right\|\) and \(V(\eta) \leq \lambda_{\text{max}}(P)\|\eta\|^2\), one can obtain \(\left|\frac{\partial V}{\partial \eta}\right| \leq 2\lambda_{\text{max}}(P)\|\eta\|\).

Moreover, since \(\lambda_{\text{min}}(P)\|\eta\|^2 \leq V(\eta)\), this leads to \(\|\eta\| \leq \sqrt{\frac{V(\eta)}{\lambda_{\text{min}}(P)}}\). This results in

\[
\left|\frac{\partial V}{\partial \eta}\right| \leq 2\lambda_{\text{max}}(P)\sqrt{\frac{V(\eta)}{\lambda_{\text{min}}(P)}}
\]

Given that \(V(\eta) \leq \lambda_{\text{max}}(P)\|\eta\|^2 = \lambda_{\text{max}}(P)W(\eta)\), then,

\[
-W(\eta) \leq \frac{-V(\eta)}{\lambda_{\text{max}}(P)}
\]

Given that the rate of change \(\Delta(t)\) of the generalized disturbance \(f_T\) is bounded, namely, Assumption (A2) is fulfilled, and substituting (14) and (15) in (13), we get,

\[
\dot{V}(\eta) \leq -\frac{V(\eta)}{\lambda_{\text{max}}(P)} + \frac{M}{\omega_o} 2\lambda_{\text{max}}(P) \sqrt{\frac{V(\eta)}{\lambda_{\text{min}}(P)}}
\]

Knowing that \(\frac{d}{dt} \sqrt{V(\eta)} = \frac{1}{2} \frac{1}{\sqrt{V(\eta)}} V(\eta)\), then (16) is an ordinary 1st order differential equation (16) and its solution can be found as,

\[
\sqrt{V(\eta)} \leq \frac{2M\lambda_{\text{max}}^2(P)}{\omega_o \sqrt{\lambda_{\text{min}}(P)}} \left(1 - e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}\right) + \sqrt{V(\eta(0))} e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}
\]

From assumption (A3), \(\lambda_{\text{min}}(P)\|\eta\|^2 \leq V(\eta)\), this leads to \(\|\eta\| \leq \sqrt{\frac{V(\eta)}{\lambda_{\text{min}}(P)}}\), then

\[
\|\eta(t)\| \leq \frac{1}{\sqrt{\lambda_{\text{min}}(P)}} \left(\frac{2M\lambda_{\text{max}}^2(P)}{\omega_o \sqrt{\lambda_{\text{min}}(P)}} \left(1 - e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}\right) + \sqrt{V(\eta(0))} e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}\right)
\]

\[
\|\eta(t)\| \leq \frac{2M\lambda_{\text{max}}^2(P)}{\omega_o \sqrt{\lambda_{\text{min}}(P)}} \left(1 - e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}\right) + \frac{V(\eta(0))}{\sqrt{\lambda_{\text{min}}(P)}} e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}
\]

It follows from (11) that \(|\dot{\xi}_i(t) - \dot{\xi}_i(t)| = \frac{1}{\omega_o^{\rho+1-i}} |\eta_i(\omega_o t)| \Rightarrow |\dot{\xi}_i(t) - \dot{\xi}_i(t)| \leq \frac{1}{\omega_o^{\rho+1-i}} \|\eta(t)\|\).

Then, (19) becomes,

\[
|\dot{\xi}_i(t) - \dot{\xi}_i(t)| \leq \frac{1}{\omega_o^{\rho+1-i}} \left(\frac{2M\lambda_{\text{max}}^2(P)}{\omega_o \sqrt{\lambda_{\text{min}}(P)}} \left(1 - e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}\right) + \sqrt{V(\eta(0))} e^{-\frac{\tau}{2\lambda_{\text{max}}(P)}}\right)
\]
Finally,

\[
\lim_{t \to \infty} \left| \xi_i(t) - \dot{\xi}_i(t) \right| = 0,
\]

\[
\lim_{\omega_0 \to \infty} \left| \ddot{\xi}_{i+1} - \left( f(\xi_1, \xi_2, ..., \xi_{\rho}, w, t) + (\beta(x) - b_0)u \right) \right| = 0
\]

\[\blacksquare\]

The states of the system in (4) together with the generalized disturbance \( f_T \) will be estimated by the INLESO given by

\[
\begin{aligned}
\dot{\xi}_i &= \xi_{i+1} + \beta_i \varrho (y - \dot{\xi}_1), \quad i \in \{1, 2, ..., \rho - 1\} \\
\dot{\xi}_\rho &= \xi_{\rho+1} + \beta_\rho \varrho (y - \dot{\xi}_1) + b_0 u \\
\dot{\xi}_{\rho+1} &= \beta_{\rho+1} \varrho (y - \dot{\xi}_1)
\end{aligned}
\]  

(21)

The nonlinear function \( \varrho : \mathbb{R} \to \mathbb{R} \) is designed as,

\[
\varrho(e) = K_\alpha |e|^{\alpha_f} \text{sign}(e) + K_\beta |e|^{\beta} e
\]  

(22)

**Remark 1:** It is worthy to remember that in our work we have assumed that the dynamics of the plant \( \Delta(t) = f_T \) in (6) is mostly unknown. Consequently, the steady state observer estimation errors \( e_i(t) = \xi_i(t) - \dot{\xi}_i(t) \), for \( i \in \{1, 2, ..., \rho + 1\} \) are bounded and their upper bound is monotonically shrinking with the bandwidth of the observer as evident from (20). Then, as \( e_i(t) \to 0 \) in (20), the closed-loop system would satisfy the Lipchitz condition, the stability analysis with Lyapunov function is still valid, and \( \varrho(e) \) in (22) does not exhibit any numerical problems whatever the values of \( \alpha_f \) and \( \beta \) are.

Since \( \text{sign} \ (e) = e / |e| \), for \( |e| \neq 0 \), then

\[
\varrho(e) = \begin{cases} 
0 & e = 0 \\
|k(e)| e & e \neq 0
\end{cases}
\]  

(23)

The function \( k : \mathbb{R}/\{0\} \to \mathbb{R}^+ \) is an even nonlinear gain function with

\[
k(e) = K_\alpha |e|^{\alpha_f - 1} + K_\beta |e|^{\beta}
\]

where \( K_\alpha, K_\beta, \alpha_f \) and \( \beta \) are the positive design parameters. The convergence of the INLESO will be investigated in the next Lemma, before that, the following assumption is needed.
**Assumption 4 (A4):** There exist constants \( \lambda_1 \) and \( \lambda_2 \) and positive definite, continuously differentiable functions \( V, W : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^+ \) such that:

\[
\lambda_1 \|y\|^2 \leq V(y) \leq \lambda_2 \|y\|^2,
\]

\[
W(y) = \|y\|^2,
\]

\[
\sum_{i=1}^{\rho} \frac{\partial V_i}{\partial y_i} (y_i - a_i k \left( \frac{y_i}{\omega_0^\rho} \right) y_1) - \frac{\partial V}{\partial y_{\rho+1}} a_{\rho+1} k \left( \frac{y_i}{\omega_0^\rho} \right) y_1 \leq -W(y)
\]

(24)

(25)

(26)

**Lemma 2 (Convergence of the INLESO):** Given the system set out in (4), and the INLESO (21), it follows that under Assumptions A1, A2, and A4, for any initial values of the system states \( \xi_i, i \in \{1,2,\ldots,\rho\} \),

\[
\lim_{t \to \infty} \lim_{\omega_0 \to \infty} |\xi_i(t) - \hat{\xi}_i(t)| = 0, \ i \in \{1,2,\ldots,\rho\}
\]

\[
\lim_{t \to \infty} |\dot{\xi}_{\rho+1} - (f(\xi_1, \xi_2, \ldots, \xi_{\rho}, w, t) + (\beta(x) - b_0)u)| = 0
\]

where \( \xi_i \) and \( \hat{\xi}_i \) denote the solutions of (4) and (21), respectively, \( i \in \{1,2,\ldots,\rho + 1\} \).

**Proof:** Following the same steps of the proof of Lemma 1 with the assumptions A1 and A2, we get the following

\[
\dot{V}(\eta) = \sum_{i=1}^{\rho} \frac{\partial V(\eta)}{\eta_i} \left( \eta_{i+1}(t) - a_i k \left( \frac{\eta_i(t)}{\omega_0^\rho} \right) \eta_1(t) \right) - \frac{\partial V(\eta)}{\partial \eta_{\rho+1}} a_{\rho+1} k \left( \frac{\eta_1(t)}{\omega_0^\rho} \right) \eta_{1}(t) + \frac{\partial V(\eta)}{\partial \eta_{\rho+1}} \frac{\Delta h}{\omega_0^2}
\]

If Assumption A 4 is satisfied, then,

\[
\dot{V}(\eta) \leq -W(\eta) + \frac{\partial v(\eta)}{\partial \eta_{\rho+1}} \frac{\Delta h}{\omega_0^2}.
\]

(27)

Substituting (14) and (15) in (27) we reach to (16), solving (16) with simple manipulations on its solutions (17) as done in the proof of Lemma 1 we get,

\[
\lim_{t \to \infty} \lim_{\omega_0 \to \infty} |\xi_i(t) - \hat{\xi}_i(t)| = 0, \ i \in \{1,2,\ldots,\rho\}
\]

\[
\lim_{t \to \infty} |\dot{\xi}_{\rho+1} - (f(\xi_1, \xi_2, \ldots, \xi_{\rho}, w, t) + (\beta(x) - b_0)u)| = 0
\]

\[\blacksquare\]
Theorem 1(AIOFL): Given the nonlinear system of (6) and the LESO or INLESO in (5) or (21) respectively. Then the nonlinear system is of (4) or (6) is reduced to a chain of integrators described as,
\[ \dot{\xi}_i = \xi_{i+1}, \quad i \in \{1,2,\ldots,\rho - 1\}, \]
\[ \dot{\xi}_\rho = b_0 v. \]

Proof: Based on the result of Lemma 1 and Lemma 2 and if the LCL \( u \) is selected as,
\[ u = v - \frac{\xi_{\rho+1}}{b_0} \]
Then,
\[ \dot{\xi}_\rho = \xi_{\rho+1} + b_0 \left( v - \frac{\dot{\xi}_{\rho+1}}{b_0} \right) \]
\[ \dot{\xi}_\rho = \xi_{\rho+1} + b_0 v - \dot{\xi}_{\rho+1} = e_{\rho+1} + b_0 v \]
\[ \dot{\xi}_\rho \approx b_0 v \quad \text{(For large } \omega_0) \]

Remark 2: The main differences between IOFL and AIOFL are, for the AIOFL, there is no need to obtain the transformation of (3). The only required information is the relative degree of the system (\( \rho \)) for the nonlinear system to be linearized. While, for the IOFL, transformation (3) is the key step to linearize the system, it is based on exact mathematical cancelation of the nonlinear terms \( \alpha(x) \) and \( \beta(x) \), which requires knowledge of \( \alpha, \beta, \) and \( T \). Furthermore, AIOFL in addition to linearizing the nonlinear system, it lumps the external disturbances, uncertainties, and unmodelled dynamics, into a single term for online and active estimation and cancelation later on.

The stability of the closed-loop system with the ADRC is considered in the following theorem. Before that, the following assumptions are needed.

Assumption 5 (A5): The states \( x_i \) (\( i = 1,2,\ldots,n \)) and the generalized disturbance \( f \) of a \( n \)-dimensional uncertain nonlinear SISO system (4) are estimated by a convergent LESO which produces the estimated
states $\hat{x}_i$ ($i = 1,2, ..., \rho$) of the plant and the estimated generalized disturbance $\hat{x}_{\rho+1}$ as $t \to \infty$ respectively, i.e.,

$$\lim_{t \to \infty} |x_i - \hat{x}_i| = 0, i \in \{1,2, ..., \rho\},$$

(28)

and

$$\lim_{t \to \infty} |f - \hat{x}_{\rho+1}| = 0.$$

(29)

**Assumption 6 (A6):** A Tracking Differentiator (TD) produces a trajectory $r_i$, $i \in \{1,2, ..., \rho\}$ with minimum set point change. The trajectory converges to a reference trajectory $r^{(l-1)}_i$ for $i \in \{1,2, ..., \rho\}$ as $t \to \infty$, i.e.,

$$\lim_{t \to \infty} |r^{(l-1)}_i - r_i| = 0, i \in \{1,2, ..., \rho\}.$$

(30)

**Theorem 2 (Closed–Loop Stability):** Consider an $n$-dimensional uncertain nonlinear SISO system given in (4). The system (4) is controlled by the Linearization Control Law (LCL) $u$ given by:

$$u = v - \hat{x}_{\rho+1}.$$ 

(31)

where $v$ is given as,

$$v = \mathcal{K}_1(\tilde{e}_1)\tilde{e}_1 + \mathcal{K}_2(\tilde{e}_2)\tilde{e}_2 + ... + \mathcal{K}_n(\tilde{e}_n)\tilde{e}_n.$$ 

(32)

where $\mathcal{K}_i: \mathbb{R} \to \mathbb{R}^+$ is an even nonlinear gain function, $\tilde{e}_i = r_i - \hat{x}_i$, $i \in \{1,2, ..., \rho\}$ is the tracking error. Assuming that Assumptions A5 and A6 hold true, then, the closed-loop system is asymptotically stable, i.e., $\lim_{t \to \infty} |\tilde{e}_i| = 0, i \in \{1,2, ..., \rho\}$.

**Proof:** The tracking error between the reference trajectory and the corresponding plant estimated states is given as:

$$\tilde{e}_i = r_i - \hat{x}_i, i \in \{1,2, ..., \rho\}.$$ 

(33)

With LESO and TD as in assumptions A4 and A5 respectively, the tracking error can be described as,

$$\tilde{e}_i = r^{(l-1)} - x_i, i \in \{1,2, ..., \rho\}.$$

(34)
For the system given in (4), the states $x_i$ are expressed in term of the plant output,

$$x_i = y^{(i-1)} , i \in \{1,2,...,\rho\}.$$  \hfill (35)

Substitute (35) in (34), and the tracking error is given by

$$\tilde{e}_i = r^{(i-1)} - y^{(i-1)} , i \in \{1,2,...,\rho\}$$ \hfill (36)

Differentiating (36) w.r.t $t$, gives

$$\dot{\tilde{e}}_i = r^{(i)} - y^{(i)} = \tilde{e}_{i+1} , i \in \{1,2,...,\rho\}.$$

It follows that the tracking error dynamics $\tilde{e}_i , i \in \{1,2,...,\rho\}$ are given below

$$\begin{align*}
\hat{e}_1 &= \hat{e}_2, \\
\hat{e}_2 &= \hat{e}_3, \\
\vdots & \\
\hat{e}_\rho &= r^{(\rho)} - y^{(\rho)} = r^{(\rho)} - \tilde{\xi}_\rho. 
\end{align*}$$ \hfill (37)

This together with (4) gives:

$$\begin{align*}
\hat{e}_1 &= \hat{e}_2, \\
\hat{e}_2 &= \hat{e}_3, \\
\vdots & \\
\hat{e}_\rho &= r^{(\rho)} - (f + u)
\end{align*}$$ \hfill (38)

From (35), we get

$$\begin{align*}
\hat{e}_1 &= \hat{e}_2, \\
\hat{e}_2 &= \hat{e}_3, \\
\vdots & \\
\hat{e}_n &= r^{(n)} - v + \hat{z}_{n+1} - f
\end{align*}$$ \hfill (39)

It follows from (29) and (35) that

$$\begin{align*}
\hat{e}_1 &= \hat{e}_2, \\
\hat{e}_2 &= \hat{e}_3, \\
\vdots & \\
\hat{e}_\rho &= r^{(\rho)} - v
\end{align*}$$ \hfill (40)

The tracking error dynamics given in (40) associated with the control law $v$ designed in (32) produces the following closed-loop error dynamics
\[
\begin{bmatrix}
\dot{\varepsilon}_1 = \varepsilon_2, \\
\dot{\varepsilon}_2 = \varepsilon_3, \\
\vdots \\
\dot{\varepsilon}_\rho = -\kappa_1(\varepsilon_1)\varepsilon_1 - \kappa_2(\varepsilon_2)\varepsilon_2 - \cdots - \kappa_\rho(\varepsilon_\rho)\varepsilon_\rho.
\end{bmatrix} \tag{41}
\]

The dynamics given in (41) can be represented as
\[
\dot{\varepsilon} = A\varepsilon.
\]

where
\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-\kappa_1(\varepsilon_1) & -\kappa_2(\varepsilon_2) & -\kappa_3(\varepsilon_3) & \cdots & -\kappa_{\rho-1}(\varepsilon_{\rho-1}) & -\kappa_\rho(\varepsilon_\rho)
\end{pmatrix}
\]

and \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\rho)^T \). The characteristic polynomial of \( A \) is given by
\[
|\lambda I - A| = \lambda^\rho + \kappa_\rho(\varepsilon_\rho)\lambda^{\rho-1} + \kappa_{\rho-1}(\varepsilon_{\rho-1})\lambda^{\rho-2} + \cdots + \kappa_1(\varepsilon_1) \tag{42}
\]

The design parameters of \( \kappa_i \) are selected to ensure that the roots of the characteristic polynomial (42) have strictly negative real parts i.e. Hurwitz (stable) polynomial.

\[\blacksquare\]

**V. Guideway Example**

In this example, a SISO single-link flexible joint manipulator (SLFJM) offered by [26] is studied and shown in Fig. 2. The state-space representation of the SLFJM system in the form of the nonlinear system given in (1) is described as,
\[
\begin{aligned}
\dot{x} &= f(x) + bu + b_d\tau_d, \\
y &= Cx,
\end{aligned}
\]

where \( x = (x_1 \ x_2 \ x_3 \ x_4)^T = (\theta \ \dot{\alpha} \ \dot{\theta} \ \ddot{\alpha})^T \in \mathbb{R}^4 \) is the plant state, \( u \in \mathbb{R} \) the plant input, \( \tau_d \in \mathbb{R} \) the exogenous disturbance, \( y = (\theta + \alpha) \in \mathbb{R} \) the plant output, \( f: \mathbb{R}^4 \to \mathbb{R}, \theta \) is the motor angular displacement and \( \alpha \) is the joint twist or link deflection. The components of \( f, b, b_d, C \) are denoted, respectively, by
\[
\begin{aligned}
\begin{bmatrix}
f(x) = \begin{bmatrix} x_3 \\
x_4 \\
-K_s \frac{J_h}{h} x_2 - K_s \frac{K_m J_h}{R_m J_l} x_3 \\
+ K_s \frac{K_m J_l}{R_m J_h} x_3 + \frac{m g h}{J_l} \sin(x_1 + x_2)
\end{bmatrix} \\
\end{aligned}
\]

where \( K_s \) is the link stiffness, \( J_h \) is the inertia of hub, \( m \) is the link mass, \( h \) is the height of hub, \( K_m \) is the motor constant, \( K_g \) is the gear ratio, \( J_l \) is the load inertia, and \( R_m \) is the motor resistance. The values of the coefficients for SLFJM are [26]: \( K_s = 1.61 \), \( J_h = 0.0021 \), \( m = 0.403 \), \( g = -9.81 \), \( h = 0.06 \), \( K_m = 0.00767 \), \( K_g = 70 \), \( J_l = 0.0059 \), and \( R_m = 2.6 \). Applying the Lie derivative on equation (5), we get the following set of equations [27],

\[
L_g h(x) = 0, L_g L_f h(x) = 0,
\]

\[
L_g L_f^2 h(x) = 0, L_g L_f^3 h(x) \neq 0
\]

It can be noticed that the SLFJM system in (43) satisfies (2); consequently, the relative order of SLFJM is 4, i.e., \( \rho = 4 \) [27]. Three control configurations have been considered for linearization in this work and applied on SLFJM, these are:

**A. The traditional IOFL transformation**

The IOFL is given as,

\[
v = \alpha(x) + \beta(x) u
\]

such that \( u = \beta^{-1}(x)(v - \alpha(x)) \). In this method, we assume that the system states are available for feedback with no need for state observer. Thus,

\[
\alpha(x) = -\frac{1}{R_m J_h J_l} (\alpha_{t4} x_4(t) + \alpha_{t3} x_3(t) + \alpha_{t2} x_2(t) + \alpha_{t1})
\]

\[
\alpha_{t1} = -K_s \frac{J_l^2}{R_m J_h} m g h \sin(x_1(t) + x_3(t))
\]
\[ \alpha_{l2} = K_m^6 K_g^6 J_l - K_s R_m^2 J_h K_m^2 K_g^2 J_l \]

\[ \alpha_{l3} = -K_m^4 K_g^4 J_l K_s R_m + K_s^2 R_m^3 J_h J_l + K_s^2 R_m^2 J_h^2 \]

\[ \alpha_{l4} = K_s R_m^2 J_h K_m^2 K_g J_l \]

and \( \beta(x) = \frac{k_m k_g}{R_m J_h^3} - \frac{k_m k_g K_s}{R_m J_h^3} \)

The linear control law \( v \) is taken as the standard linear PID controller given as,

\[ v = k_p e + k_d \frac{de}{dt} + k_i \int e \, dt \quad (44) \]

**B. The Conventional ADRC (CADRC) based AIOFL**

The CADRC is the combination of the Linear ESO(LESO) given by (45), the NLSEF was given by (46), and TD given by (47). According to [24], [25], an LESO observer can be designed as,

\[
\begin{align*}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \beta_1 (y - \tilde{\xi}_1), \\
\dot{\tilde{\xi}}_2 &= \tilde{\xi}_3 + \beta_2 (y - \tilde{\xi}_1), \\
\dot{\tilde{\xi}}_3 &= \tilde{\xi}_4 + \beta_3 (y - \tilde{\xi}_1), \\
\dot{\tilde{\xi}}_4 &= \tilde{\xi}_5 + \beta_4 (y - \tilde{\xi}_1) + b_0 u \\
\dot{\tilde{\xi}}_5 &= \beta_5 (y - \tilde{\xi}_1)
\end{align*}
\]

where \( \tilde{\xi} = (\tilde{\xi}_1 \quad \tilde{\xi}_2 \quad \tilde{\xi}_3 \quad \tilde{\xi}_4 \quad \tilde{\xi}_5)^T \) is the observer's state vector, and \( \beta = (\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5)^T = (\alpha_1 \omega_0 \quad \alpha_2 \omega_0^3 \quad \alpha_3 \omega_0^3 \quad \alpha_4 \omega_0^4 \quad \alpha_5 \omega_0^5)^T \) is the observer gain vector. The control law for the AIOFL is given as,

\[ u = v - \frac{\tilde{\xi}_5}{b_0} \quad (46) \]

where \( v = \text{fal}(e_1, \alpha_1, \delta_1) + \text{fal}(e_2, \alpha_2, \delta_2) \), with \( \text{fal}(.) \text{ defined as [21],} \)

\[
\text{fal}(e, \alpha_v, \delta) = \begin{cases} 
\frac{e}{\delta^{1-\alpha_v}} & |e| \leq \delta \\
|e|^{\alpha_v} \text{sgn}(e) & |e| > \delta 
\end{cases}
\]

where \( e = (e_1 \quad e_2)^T \) is the tracking error vector which can be defined as \( e_i = r_i - \tilde{\xi}_i \), \( i = 1, 2 \) and \( \alpha_v \) and \( \delta \) are design parameters. The conventional second order differentiator is given as [21],

\[
\begin{align*}
\dot{r}_1 &= r_2 \\
\dot{r}_2 &= -R \text{ sign}(r_1 - r(t) + \frac{r_2 |r_2|}{2r})
\end{align*}
\]

(47)
where $r_1$ is the tracking signal of the input $r$, and $r_2$ tracking signal of the derivative of the input $r$. To speed up or slow down the system during the transient, the coefficient $R$ is adapted according to this, it is an application dependent.

### C. The Improved ADRC (IADRC) based AIOFL

The IADRC has been designed in our previous works [23], [28], [29] and tested on the differential drive mobile robot model [30] and on Permeant Magnet DC Motor (PMDC) [31] and [32]. It is structured from the Improved Nonlinear ESO (INLESO) given by (48) [23], the Improved NLSEF (INLSEF) given by (50) [28], and the Improved TD (ITD) given by (51) [29]. The INLESO is the second type of observers used in these numerical simulations. The INLESO is described as,

\[
\begin{cases}
\dot{\xi}_1 = \dot{\xi}_2 + \beta_1 g(y - \xi_1), \\
\dot{\xi}_2 = \dot{\xi}_3 + \beta_2 g(y - \xi_1), \\
\dot{\xi}_3 = \dot{\xi}_4 + \beta_3 g(y - \xi_1), \\
\dot{\xi}_4 = \dot{\xi}_5 + \beta_4 g(y - \xi_1) + b_0 u
\end{cases}
\]  

where

\[
g(e) = k_\alpha |e|^{\alpha_f} \text{sign}(e) + k_\beta |e|^{\beta} e,
\]

(49)

The two vectors $\xi$ and $\beta$ are defined previously as in LESO case. The control law for the AIOFL is defined as,

\[
u = v - \frac{\dot{\xi}_5}{b_0}
\]

(50)

where $v$ is the INLSEF given in our previous work as [28],

\[
v = \delta_l \tanh \left( \frac{v_1 + v_2}{\delta_i} \right)
\]

\[
v_1 = \left( k_{11} + \frac{k_{12}}{1 + \exp(\mu_1 e_1^2)} \right) |e_1|^{\alpha_3} \text{sign}(e_1)
\]

\[
v_2 = \left( k_{21} + \frac{k_{22}}{1 + \exp(\mu_2 e_2^2)} \right) |e_2|^{\alpha_4} \text{sign}(e_2)
\]

where $k_{11}, k_{12}, \mu_1, \alpha_3, k_{21}, k_{22}, \mu_2, \alpha_4$, and $\delta_l$ are the design parameters of the INLSEF controller.

The third part of the IADRC is the ITD and is described as [29],
\[
\begin{align*}
\dot{r}_1 &= r_2 \\
\dot{r}_2 &= -\rho t^2 \tanh \left( \frac{br_1 - (1-a)r}{c} \right) - \rho t r_2
\end{align*}
\]

(51)

where the coefficients \(a, b, c, \) and \(\rho\) are suitable design factors, where \(0 < a < 1, \ b > 0, \ c > 0,\) and \(\rho t > 0\). The AIOFL based on the classical LESO and INLESO is applied on the SLFJM given in (43). An Objective Performance Index (OPI) is proposed to evaluate the performance of the LESO and the INESO observers, which is represented as,

\[
\text{OPI} = w_1 \frac{\text{ITAE}}{N_1} + w_2 \frac{\text{ISU}}{N_2} + w_3 \frac{\text{IAU}}{N_3}
\]

(52)

where \(\text{ITAE} = \int_0^{t_f} t|y - r| dt\) is the integration of the time absolute error for the output signal, \(\text{ISU} = \int_0^{t_f} u_0^2 dt\) is the integration of square of the control signal, and \(\text{IAU} = \int_0^{t_f} |u_o| dt\) is the integration of the absolute of the control signal. The weights must satisfy \(w_1 + w_2 + w_3 = 1\), are defined as the relative emphasis of one objective as compared to the other. The values of \(w_1, w_2,\) and \(w_3\) are chosen to increase the pressure on the selected objective functions. The \(N_1, N_2,\) and \(N_3\) are included in the performance index to insure that the individual objectives have comparable values, and are treated equally likely by the tuning algorithm. Because, if a certain objective is of very high value, while the second one has very low value, then the tuning algorithm will pay much consideration to the highest one and leave the other with little reflection on the system.

The tuning process of both observers is achieved using Genetic Algorithm (GA) under MATLAB environment with \(w_1 = 0.6, w_2 = 0.2, w_3 = 0.6, N_1 = 10, N_2 = 2, N_3 = 2.7,\) and \(t_f = 6 \text{ sec}.\) Based on this, the parameters of the control law \(u\) for the IOFL control scheme are \(k_p = 18369.94, k_d = 3.45, k_l = 1824382.96, a_{l1} = 6.2163 \times 10^{-8} \sin(x_1(t) + x_3(t)), a_{l2} = 0.0001, a_{l3} = -0.0014,\) and \(a_{l4} = 4.2758 \times 10^{-5}.\) While the tuned parameters values for the CADRC are. **LESO:** \(\omega_0 = 513.8283,\ a_1 = 8.772, \ a_2 = 0.1946, \ a_3 = 0.7384, \ a_4 = 9.6881 \times 10^{-3}, \ a_5 = 2.2651 \times 10^{-6}, \text{and} \ b_0 = 22.771.\) **TD:** \(R = 2408.6918.\) **NLSEF:** \(\delta_1 = 16.6108, \delta_2 = 14.6238, \alpha_{v1} = 0.3804,\) and \(\alpha_{v2} = 0.4583.\) The values of tuned parameters for IADRC are. **INLESO:** \(\omega_0 = 104.6131, \ a_1 = 0.1364, \ a_2 = 0.6691,\)
\[ \alpha_3 = 0.6893, \quad \alpha_4 = 0.0155, \quad \alpha_5 = 14.3801 \times 10^{-6}, \quad b_0 = 8.74500, \quad \alpha_f = 0.6906, \quad \beta = 0.1880, \quad k_\alpha = 0.3682, \text{and} \quad k_\beta = 0.1290. \]

**ITD:** \[ a = 0.9153, \quad b = 8.7141, \quad c = 0.0813, \text{and} \quad \rho_t = 22.89333. \]

**INLSEF:**
\[ k_{11} = 1.7741, \quad k_{12} = 1.2147, \quad k_{21} = 0.00115, \quad k_{22} = 0.3312, \quad \delta_l = 3.3900, \quad \mu_1 = 3.8297, \quad \mu_2 = 10.9415, \quad \alpha_{v_3} = 0.8244, \text{and} \quad \alpha_{v_4} = 1.8079. \]

It is worthy to note that in the AIOFL, the nonlinear system is linearized by either LESO or INLESO and represented by a chain of integrators. In this case, the LESO or INLESO will estimate the states of the chain of integrators up to the relative degree of the nonlinear system. With this arrangement, the higher order estimated states represent signals with higher derivative degrees, they contain high-frequency components which in turn increase the control signal activity and leading to the chattering phenomena. Based on the above reasoning, only the first two estimated states (\( \hat{\xi}_1 \) and \( \hat{\xi}_2 \)) are feedback to either NLSEF or INLSEF in the numerical simulations. The entire estimated states of the system (except the augmented state) can be provided for feedback to the NLSEF. In our case, with the first two estimated states, it was sufficient to produce the individual control laws (\( v_1 \) and \( v_2 \)) which in turn produced the required control law (\( v \)). With this scenario, eliminating the states from the feedback that do not affect on the performance of the system will reduce the number of the parameters of both the NLSEF controller and the TD. We expect that the total energy required for the controller to produce the control law (\( v \)) will be reduced. Runge-Kutta ODE45 solver in MATLAB environment has been used for the numerical simulations of the continuous models. Three different scenarios are conducted in this work, these are:

**1. Reference Tracking Scenario**

In this scenario, a sinusoidal signal with frequency 2 rad/sec and amplitude of 45 has been chosen as a reference input. The simulation time is selected to be 20 sec. The results of the numerical simulation are shown in Figs. 3-5. The results are collected based on evaluating two indices listed in table I, where

\[ \text{ITAE} = \int_0^{20} t |y - r| dt \quad \text{is the integration of the time absolute error for the output signal, and} \]

\[ \text{ISU} = \int_0^{20} u^2 dt \quad \text{is the integration of the square of the control signal.} \]
which represents the energy delivered to the SJFLM motor, has been decreased by 23.82% and a noticeable improvement in the transient response (ITAE is reduced by 23.7%).

In this scenario, we can notice that the improvement of the accuracy of the reference tracking in the case of CADRC and IADRC with respect to the conventional IOFL, see Figs. 3-5. This is due to the effectiveness of the ADRC as a robust tracking tool. Moreover, the control signal $v$ produced by the IADRC (Fig. 5-(b)) is less chattering than that of the CADRC (Fig. 4-(b)) where the control signal $v$ suffers from the peaking phenomenon due to the large values of the gain parameters of the LESO. The reason of the IADRC superiority is that the proposed nonlinear state error feedback of the IADRC produces the most economical control signal $v$ satisfying the rule “small error, large gain and large error, small gain”.

2. Inertia Uncertainty and Exogenous Disturbance Scenario

A second simulation scenario is conducted in this work which included the presence of an exogenous disturbance $\tau_d$ of type step at $t = 10$ sec with an amplitude of 0.5 $N.m$ and an increase of 40% in the load inertia. The results of the numerical simulation are shown in Figs 6-8. The numerical results of the two performance indices of the second scenario are listed in table II. As shown in the table, the ITAE significantly reduced for the IADRC case. This improvement in the transient response, which is reflected by the value of ITAE occur with an insignificant increase in the delivered energy to the actuation as compared to CADRC, where both IADRC and CADRC witnessed a big reduction in the control energy with respect to IOFL technique.

Adding inertia uncertainty and exogenous disturbance highly affect the performance of the IOFL (see Fig. 6) against the other two control schemes as described previously in this paper. The presence of LESO and NLESO in both CADRC and IADRC, respectively, is the reason for the improvement of the reference tracking in these techniques, see Fig. 7-(a) and fig. 8-(a), where both the uncertainty in the inertia and the exogenous disturbance are lumped all together and estimated by the LESO and NLESO and canceled from the input channel of the SLFJM. This process does not exist in the
traditional IOFL. Moreover, the superiority of the IADRC over the CADRC in terms of reference tracking is the existence of the saturation-like behavior that the error function of the NLESO has, where higher estimated accuracy is obtained with the NLESO than the LESO, this is reflected in the ITAE values of table II and in the reference tracking of Fig 7-(a) and Fig. 8-(a) beyond 10 sec.

3. Measurement Noise Scenario

The final scenario that has been demonstrated in this work is testing the immunity of the system against measurement noise. A Gaussian measurement noise at the output is considered, the variance and mean of the Gaussian noise are 0.0001 and 0, respectively. To actively counteract the effect of the noise, both ESOs are re-tuned again using GA under the existence of noise based on the OPI defined in (52). The newly tuned parameters of the LESO are \( \omega_0 = 851.0106, \alpha_1 = 5.40326, \alpha_2 = 0.2871, \alpha_3 = 0.7644, \alpha_4 = 0.01, \alpha_5 = 1.22\times10^{-6}, \) and \( b_0 = 33.7432. \) While the new tuned parameters of the INLESO are \( \omega_0 = 121.020, \alpha_1 = 0.205, \alpha_2 = 0.6, \alpha_3 = 0.42, \alpha_4 = 0.0232, \alpha_5 = 7.19\times10^{-6} \) and \( b_0 = 9.7. \) The other parameters of both control schemes (CADRC and IADRC) are not changed. The results of the numerical simulation are shown in Figs. 9-11. The numerical results of the two performance indices of the second scenario are listed in table III. As shown in Table III, both of the ITAE and ISU are reduced significantly using IADRC scheme. This improvement in the transient response and the reduced control energy is noticeable in Fig. 11.

The presence of measurement noise has approximately no effect on the response of the IOFL due to the nonexistence of any type of observer in this technique as can be seen from Fig. 9. In the case of CADRC, the noise is simply bypassed by the LESO which acts a high gain observer with the noise appears on the output channel of LESO where its components are amplified by the gain values of the LESO and consequently deteriorated the control signal \( v, \) see Fig. 10-(b). More specifically, the INLESO of the IADRC attenuates the noise due to its saturation-like behavior with little effect of chattering appears on the control signal \( v \) as depicted in Fig. 11-(b).
VI. CONCLUSIONS

This paper addressed the problem of AIOFL for SLFJM which is a highly nonlinear uncertain system subjected to external disturbances and measurement noise. It differs from the traditional IOFL which assumes a nominal nonlinear system to work on it. The AIOFL has been implemented by both CADRC and IADRC paradigms, which transforms the nonlinear uncertain system into a chain of integrators. The key point of the proposed method is that it requires only the relative degree of the nonlinear uncertain system. It can be concluded that the proposed IADRC based AIOFL method transformed the SLFJM uncertain system into a linear one and excellently estimated and canceled the generalized disturbance in a real-time manner. The steady-state observer estimation error is inversely proportional to the bandwidth of the ESO and the closed-loop system with the proposed AIOFL schemes is globally asymptotically stable based on Lyapunov Stability analysis. While both ADRC based AIOFL versions presented good tracking, the IADRC based AIOFL exhibited better performance than CADRC based AIOFL and traditional IOFL and provided the actuator with a more stable control signal, it has less fluctuations with small amplitude. Finally, the IADRC based AIOFL had more immunity to noise than other schemes.

Conflict of Interest Statement

The authors declare that they have no conflicts of interest.

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### Table I

The results of the numerical simulation

| Structure | ITAE     | ISU       |
|-----------|----------|-----------|
| IOFL      | 430.901166 | 19.442615 |
| CADRC     | 126.120273 | 7.982831  |
| IADRC     | 96.225965  | 6.080829  |

### Table II

The results of the numerical simulation

| Structure | ITAE     | ISU       |
|-----------|----------|-----------|
| IOFL      | 1573.526748 | 95.973932 |
| CADRC     | 1309.213956 | 58.214189 |
| IADRC     | 298.143303  | 69.471044 |

### Table III

The results of the numerical simulation

| AIOFL structure | ITAE     | ISU       |
|-----------------|----------|-----------|
| IOFL            | 431.096732 | 19.449649 |
| ADRC            | 332.443873 | 799.520367 |
| IADRC           | 102.578228 | 19.959797  |
Fig. 1 Structure of conventional ADRC.

Fig. 2. Definition of generalized coordinates for the SLFJM.
Fig. 3 The curves of the numerical simulations for the first scenario using IOFL, (a) the output response of the SLFJM, $y$ (b) the control signal, $v$.

Fig. 4 The curves of the numerical simulations for the first scenario using CADRC, (a) the output response of the SLFJM, $y$ (b) the control signal, $v$.

Fig. 5 The curves of the numerical simulations for the first scenario using IADRC, (a) the output response of the SLFJM, $y$ (b) the control signal, $v$. 
Fig. 6 The curves of the numerical simulations for the second scenario using IOFL, (a) the output response of the SLFJM, \( y \) (b) the control signal, \( v \).

Fig. 7 The curves of the numerical simulations for the second scenario using CADRC, (a) the output response of the SLFJM, \( y \) (b) the control signal, \( v \).

Fig. 8 The curves of the numerical simulations for the second scenario using IADRC, (a) the output response of the SLFJM, \( y \) (b) the control signal, \( v \).
Fig. 9 The curves of the numerical simulations for the third scenario using IOFL, (a) the output response of the SLFJM, $y$ (b) the control signal, $v$.

Fig. 10 The curves of the numerical simulations for the third scenario using CADRC, (a) the output response of the SLFJM, $y$ (b) the control signal, $v$.

Fig. 11 The curves of the numerical simulations for the third scenario using IADRC, (a) the output response of the SLFJM, $y$ (b) the control signal, $v$. 