LAX PAIR FORMULATION OF THE W-GRAVITY THEORIES IN TWO DIMENSIONS

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Abstract

The Lax pair formulation of the two dimensional induced gravity in the light-cone gauge is extended to the more general \( w_N \) theories. After presenting the \( w_2 \) and \( w_3 \) gravities, we give a general prescription for an arbitrary \( w_N \) case. This is further illustrated with the \( w_4 \) gravity to point out some peculiarities. The constraints and the possible presence of the cosmological constants are systematically exhibited in the zero-curvature condition, which also yields the relevant Ward identities. The restrictions on the gauge parameters in presence of the constraints are also pointed out and are contrasted with those of the ordinary 2d-gravity.

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I. Introduction

Induced gravity in two dimensions, arising from the interaction of conformal matter with the gravitational field, is now being extensively studied, due mainly to its relevance to string theory\(^1\). Significant progress has been made in the weak-coupling regime (\(c \leq 1\) and \(c > 25\), \(c\) being the central charge of the matter sector), both in the direct continuum\(^2\) and the lattice model approaches\(^3\). Although the non-local effective action, given by

\[
S = \alpha \int \sqrt{-g} \left( R \Box^{-1} R - \lambda \right) d^2 x, \tag{1}
\]

becomes local both in the conformal and the light-cone gauges, the problem of quantization is much more tractable in the latter case, as was first pointed out by Polyakov\(^1\). In particular, a hidden \(sl(2, R)\) Kac-Moody symmetry becomes apparent in this approach, allowing the extraction of non-perturbative informations about the theory, e.g. the anomalous dimensions of various fields in the presence of quantum gravitational fields\(^4\). The light-cone gauge is characterised by the line element

\[
ds^2 = dx^+ dx^- + h(x^+, x^-) dx^+ dx^-, \tag{2}
\]

where the metric \(h(x^+, x^-)\) transforms under the residual coordinate transformations as

\[
\delta h = \epsilon h' + \dot{\epsilon} - h \epsilon'. \tag{3}
\]

The dot and the prime denote the derivatives with respect to \(x^+\) and \(x^-\) variables. The simplicity of the light-cone approach can be easily seen from the equation of constant curvature, which yields a constraint

\[
R = \partial^2_- h = -2\lambda, \tag{4}
\]

in contrast to the dynamical Liouville equation in the conformal gauge. The constraint equation yields the equation of motion for the induced gravity \(\partial^2_- h = 0\), allowing the expansion of \(h(x^+, x^-)\) as

\[
h(x^+, x^-) = J^-(x^+) - 2J^0(x^+)x^- + J^+(x^+)x^-^2; \tag{5}
\]

here \(J^-(x^+), J^0(x^+)\) and \(J^+(x^+)\) are the currents of the \(sl(2, R)\) Kac-Moody algebra.

A particularly clear explanation of the geometrical origin of the current algebra symmetries in the induced gravity was provided by Polyakov\(^5\), who demonstrated how to get
diffeomorphisms from the restricted gauge transformations. Considering a $sl(2, R)$ Lie algebra valued field $A_\mu dx^\mu$, a partial gauge fixing in the $A_-$ sector led to the transformation of the $A_+^+$ field as the stress-energy tensor of a conformal field theory. Suitable gauge fixing in the $A_+$ sector then led to the transformation of the $A_-^-$ component as the light-cone metric; surprisingly, the gravitational Ward identity emerged as a consistency condition of the gauge fixing or, equivalently, from the zero-curvature condition. Generalizations to the $sl(N, R)$ case are now being actively pursued in the literature$^6$.

In the works of Das et al.$^7$, the zero-curvature condition, written as a compatibility equation of a matrix Lax pair, was interpreted as a gauge anomaly equation. The anomaly equation can, in principle, be integrated to yield a suitably gauged WZWN action$^8$. Following this approach, the KdV and the Boussinesq hierarchies were related with the ordinary and the $w_3$ gravitational Ward identities respectively. Subsequently, the work of the present authors incorporated the curvature constraints in this Lax pair formulation of the 2-d gravity$^9$.

In the present work, our goal is to analyse the more general $w_N$ gravities. In particular, we will provide a general prescription to determine the $A_-$ and the $A_+$ sectors of the $sl(N, R)$ Lie algebra valued gauge connection such that the compatibility of the Lax pair, $\partial_- + A_-$ and $\partial_+ + A_+$, will yield the constraints and the dynamical equations. We will briefly explain the origin of the constraints in the $w_N$-gravity, and their relevance for the quantum regime of the theory. The elements of the $A_-$ field, $w_2$ and $w_{k,k>2}$, will transform respectively as the stress energy tensor and the spin-$k$ quasi-primary fields under the residual gauge transformations. A conjectured formula, valid to all orders, will be provided for the infinitesimal variations of these fields under diffeomorphism. The $A_+$ sector contains variables which transform as the currents of a conformal field theory. These fields will be interpreted as the metrics of the $w_N$ gravity$^{6,10}$.

In Sec. II, we give a brief introduction to the $w_N$ algebras and their connection with the integrable non-linear equations. Our approach is outlined in the well studied $sl(2, R)$ and $sl(3, R)$ cases in Sec. III. Sec. IV deals with the generalisation to an arbitrary $w_N$ gravity. As an illustration, the $w_4$ gravity is also investigated in this section, pointing out certain peculiarities in this case and showing explicitly how some non-linear evolution equations can arise as special cases of the $w_4$ Ward identities. We conclude in Sec.V with some final remarks and future directions of work. There are three appendices; the first one gives the variations of the fields in $sl(4, R)$, the second lists the variations of the $w_i$‘s in $sl(10, R)$ as a specific example of the conjectured general formula given in Sec. IV. The last appendix presents the infinitesimal variation of the metric field $h_2$ in some higher spin $w$-gravity theories.
II. $w_N$ algebras and covariant Lax operators

A generalization of the Virasoro algebra to the so-called $w_3$ algebra\(^1\) was achieved by Zamolodchikov\(^{[11]}\). This algebra contains a chiral spin-3 conserved current $w_3(z)$ in addition to the stress-energy tensor $w_2(z)$. With additional fields, consistent generalizations to higher spin algebras have been discovered in the literature\(^{[12]}\) and these $w_N$ algebras have appeared in many physical contexts. To quote a few examples, they have manifested in the gauged WZWN models\(^{[13]}\), in the cosets of affine Lie algebras\(^{[14]}\), Toda field theories\(^{[15]}\), 2 + 1 dimensional Chern-Simons theories\(^{[16]}\) and also in the context of various non-linear integrable equations\(^{[17]}\) e.g. KdV, Boussinesq, etc. It is worth emphasizing that these algebras are not Lie algebras because of quadratic defining relations as will become clear in the course of the text. The $w_N$ algebras have series of unitary representations characterised by

$$c = (n-1)\left(1 - \frac{n(n-1)}{p(p+1)}\right), \quad p = n+1, n+2, \ldots \quad (6)$$

and hence have been of interest in the construction of the “periodic table” of the conformal-invariant solutions of the two-dimensional euclidean quantum field theories. Here, we will be mainly concerned with the classical $w_N$ algebras. In particular, their connection with the non-linear integrable equations is pertinent for our work.

The much studied KdV equation

$$w_{2t} = w_{2xx} + 6w_2w_{2x}, \quad \text{(where } w_{2x} \equiv \frac{\partial w_2}{\partial x}) \quad (7)$$

and the related hierarchy of equations can be written as Hamiltonian equations

$$w_{2t} = \{w_2, \mathcal{H}_n\}(l) \quad (8)$$

in two distinct ways\(^{[18]}\). The two Hamiltonians

$$\mathcal{H}_1 = \frac{1}{2} \int dx \left(2w_2^3 - w_{2x}^2\right), \quad \text{and} \quad \mathcal{H}_2 = \int dx w_2^2 \quad (9)$$

give rise to the above equation if the respective Poisson brackets $(l = n)$ are defined as

$$\{w_2(x), w_2(y)\}_{(1)} = \partial \delta(x-y), \quad \text{and} \quad \{w_2(x), w_2(y)\}_{(2)} = \left(\frac{1}{2} \partial^3 + w_2 \partial + \partial w_2\right)\delta(x-y) \quad (9)$$

where the operator $\partial = \frac{\partial}{\partial x}$ acts on all the objects to its right. The two Poisson brackets can be recognized as the abelian current algebra and the Virasoro algebra respectively.

\(^1\) In the present day terminology, this algebra is known as the $w_3^{(1)}$ algebra.
In Ref.[19], another interesting derivation of the KdV equation, connecting it to the covariance property of the Hill operator under reparameterisation was obtained. It has been subsequently extended to other integrable systems[20]. Briefly, the Hill operator

$$L^{(2)} = \partial^2 + w_2(x^-)$$

transforms covariantly under reparameterisation in the sense that

$$x^- \rightarrow f(x^-),$$

$$L^{(2)} \rightarrow (f')^{-\frac{2}{3}} \left( \partial^2 + \bar{w}_2(x^-) \right) (f')^{-\frac{1}{3}}$$

where

$$\bar{w}_2(x^-) = (f')^2 w_2(f) + \frac{1}{2} S_f(x^-),$$

$$S_f$$ being the Schwarzian derivative, $$S_f = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2$$. A time (represented by $$x^+$$) dependent reparameterisation then yields $$w_2(x^-) \rightarrow \bar{w}_2(x^+, x^-)$$ and

$$\dot{\bar{w}}_2 = \frac{1}{2} h''' + 2 \bar{w}_2 h' + \bar{w}_2 h,$$

where $$h$$ is defined as $$\frac{\dot{f}}{f}$$. The KdV equation in the standard form is obtained by making the special choice $$h = 2 \bar{w}_2$$. It is worth pointing out that the above non-linear evolution equation is nothing but the gravitational Ward identity for the central charge $$c = \frac{1}{2}$$ and $$h = \frac{\dot{f}}{f}$$ is the well known Beltrami equation defining the light-cone metric.

At this point, looking at the infinitesimal variation of $$w_2(x^-)$$ under $$x^- \rightarrow x^- + \epsilon^-$$. the second Poisson bracket mentioned before can be extracted, assuming $$w_2(x^-)$$ as the generator of diffeomorphism. This procedure, in principle, can be generalized to the higher covariant operators. Given an $$n$$th order differential operator of the form $$M_n = \partial^n + \sum_{i=0}^{n-2} u_i \partial^i$$, a covariant operator $$L_{(n)}$$ can be constructed[21] such that it acts on densities of weight $$\frac{1}{2} - n$$ and the functions appearing in $$L_{(n)}$$ are the generators of the $$w_N$$ algebra. Some examples of $$L_{(n)}$$ are given below.

$$L_{(0)} = 1,$$
$$L_{(1)} = \partial,$$
$$L_{(2)} = \partial^2 + w_2,$$
$$L_{(3)} = \partial^3 + 4w_2 \partial + 2w_{2x} + w_3,$$
$$L_{(4)} = \partial^4 + 10w_2 \partial^2 + 10w_{2x} \partial + 9w_2^2 + 3w_{2xx} + \partial w_3 + w_3 \partial + w_4.$$ (14)

It should be mentioned that the covariant Lax operators $$L_{(2)}$$ and $$L_{(3)}$$ have appeared in the $$w_2$$ and $$w_3$$ gravities, in the works of Zamolodchikov[22] and Matsuo[23] respectively.
For the sake of completeness, it should also be pointed out that the above mentioned Hamiltonian formulations can be related to the symplectic structures associated with the space of pseudo-differential operators\cite{24}. In particular, the second Poisson bracket structure relevant for the $w_N$ algebras can be extracted following Adler\cite{25}.

Given a Lax operator $\mathcal{L}_{(n)}$, one defines

$$\frac{\partial \mathcal{L}_{(n)}}{\partial t} = (\mathcal{L}_{(n)} F)_+ + \mathcal{L}_{(n)} - \mathcal{L}_{(n)} (\mathcal{L}_{(n)} F)_+$$

(15)

where

$$F = \partial^{-1} f_0 + \partial^{-2} f_1 + \ldots + \partial^{-n} f_{n-1},$$

and the $(\mathcal{L}_{(n)} F)_+$ is the differential part of $\mathcal{L}_{(n)} F$. Comparing this equation with

$$\frac{\partial u_i}{\partial t} = \sum_{j=0}^{n-2} D^{(2)}_{ij} f_j$$

(16)

one finds the Poisson brackets as $\{ u_i(x), u_j(y) \} = D^{(2)}_{ij} \delta(x - y)$. The Lax operators, if written in a covariant form, generate the $w_N$ algebras.

Although, we will pursue a method involving the matrix valued Lax pairs in this paper, it turns out that the specific mapping from $M_n$ to $\mathcal{L}_{(n)}$ is of relevance in this formalism. The relationship of the $u_i$’s and the $w_i$’s appearing in $M_n$ and $\mathcal{L}_{(n)}$ respectively is precisely those of the fields $W_i$’s and the primary fields $w_i$’s in this work. The scalar differential operators will be connected with the matrix Lax operator $\partial_- + A_-$ to explain this interesting relationship. It is also worth observing that the inverse mapping appears to determine the $A_+$ sector and the gauge parameters that generate the gauge preserving transformations.

**III. $w_2$ and $w_3$ gravities**

Let us first elaborate on the (matrix) Lax pair formulation of the much studied ordinary gravity. This is done in a manner which can be easily generalized to an arbitrary $w_N$ algebra. In case of the $w_2$ gravity, one considers the deformed $sl(2, R)$ algebra\cite{26}

$$[t_+, t_-] = \lambda_+ \lambda_- t_0, \quad [t_0, t_{\pm}] = \pm 2 t_{\pm}.$$  

(17)

where $t_a$’s are represented by the matrices

$$t_- = \begin{pmatrix} 0 & 0 \\ \lambda_- & 0 \end{pmatrix}, \quad t_+ = \begin{pmatrix} 0 & \lambda_+ \\ 0 & 0 \end{pmatrix}, \quad t_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

(18)
λ± being the so-called spectral parameters. In the limit of λ± going to zero, the algebra reduces to the Poincaré algebra in two dimensions. The Killing metric $g_{ab} = \frac{1}{2} f_{ac}^\ d f_{bd}^\ c$ is degenerate when either $\lambda_+$ or $\lambda_-$ is vanishing. Considering the reduced Killing metric on $sl(2, R)/U(1)$ given by $2\lambda_+\lambda_-\eta_{ab}$, we define the gauge field $A = A^a t_a$ and the line element as the coset invariant

$$ds^2 = \eta_{ab} A^a A^b.$$ (19)

The gauge fields are the zweibeins and the spin connection.

The following chiral gauge choice

$$A_- = \begin{pmatrix} 0 & 0 \\ \lambda_- & 0 \end{pmatrix}, \quad A_+ = \begin{pmatrix} -\frac{1}{2} h' & \lambda_+ \\ \lambda_- h & \frac{1}{2} h' \end{pmatrix},$$

yields the constraint equation

$$R = \partial^2 h = -2\lambda_+\lambda_-$$
as the integrability condition, $F = dA + A \wedge A = 0$. It can be easily checked that the field $h(x^+, x^-)$ transforms as the light-cone metric. Furthermore, it is worth pointing out that the same equations can be obtained as a consistency condition under the gauge transformations that maintain this gauge choice. Following the remarkable analysis of Polyakov\textsuperscript{[5]}, we can show that the gauge choice

$$A_- = \begin{pmatrix} 0 & -t \\ \lambda_- & 0 \end{pmatrix}, \quad A_+ = \begin{pmatrix} -\frac{1}{2} h' & \lambda_+ - (\frac{1}{2\lambda_-} + \kappa) h'' - th \\ \lambda_- h & \frac{1}{2} h' \end{pmatrix}$$

yields $\delta t = 2te' + e't + \frac{1}{2\lambda_-}\epsilon''$ under the residual gauge transformations that maintain $A_-$. This variation is easily recognized as the variation of the stress-energy tensor in a conformal field theory, with a central charge $c$ equal to $\frac{1}{2\lambda_-}$. Furthermore, all the parameters of the gauge variation are not independent:

$$\begin{cases} 
\epsilon^- = +\frac{1}{\lambda_-}\epsilon, \\
\epsilon^0 = -\frac{1}{2}\partial\epsilon, \\
\epsilon^+ = - \left(\frac{1}{2\lambda_-}\partial^2 + t\right)\epsilon;
\end{cases}$$

(22)

and it can be easily checked that $h(x^+, x^-)$ does transform as the light-cone metric field:

$$\delta h = -he' + \epsilon h' + \dot{\epsilon}.$$ (23)

The consistency requirement for maintaining the gauge choice or the zero curvature condition $F = 0$ provide the constraint equation and the gravitational Ward identity given respectively by

$$\partial^2 h = \frac{\lambda_+}{\kappa},$$

(24a)
and

$$\partial_+ t = \left( (c + \kappa) \partial_3^2 + \partial t + t \partial \right) h. \quad (24b)$$

The work of Das et al.\cite{7} corresponds to the case $\kappa = \lambda_+ = 0$, for which only the Ward identity exists. It is worth emphasising that for taking into account the constraint with the possible presence of a cosmological constant, the parameter $\kappa$ should be nonvanishing. It should be pointed out that it is precisely the case in the quantum Ward identity of the 2d-gravity. In this case however, the nonvanishing $\kappa$ is exactly calculable. In our gauge choice with $\kappa \neq 0$, the gauge preserving symmetries are restricted to those described by $\epsilon$ such that $\partial \epsilon = 0$ if $\lambda_+ \neq 0$ or $\partial^2 \epsilon = 0$ if $\lambda_+ = 0$. The parameter of the residual coordinate invariance is restricted as can be observed in the work of KPZ\cite{4}.

We pursue the analysis with the example of $N = 3$ for the purpose of generalization to an arbitrary $w_N$ case. We choose the following matrix representation for the $\mathfrak{sl}(3, \mathbb{R})$ algebra elements (the same as in Ref.[7]):

$$t_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t_{00} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad t_{++} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_{--} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$t_{0+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_{0-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (25)$$

and define the gauge field $A_-$:

$$A_- = \begin{pmatrix} 0 & -W_2 & -W_3 \\ c_1 & 0 & 0 \\ 0 & c_2 & 0 \end{pmatrix} \quad (26)$$

where the $c_i$’s are constants and the $W_i$’s are dynamical fields with dimension $i$ (the dimension of $\frac{\partial}{\partial x_-}$ is +1). As we have noticed in Ref.[9], one can set $c_2 = 1$ without loss of generality\footnote{This constant scales the primary field $w_3$ defined in the next page; this observation generalizes to the higher spin algebras.}.

As they stand, the dynamical fields in $A_-$ do not satisfy the Virasoro symmetry properties mentioned before. The gauge transformations that preserve Eq.(26) are found to be
described by the functions $\epsilon^-$ and $\epsilon^{--}$ or, more conveniently, by the arbitrary functions $\rho_2$ and $\rho_3$:

$$
\begin{cases}
\epsilon^- = c_1 \rho_2 - \frac{c_1}{2} \rho_3', \\
\epsilon^{--} = c_1 \rho_3,
\end{cases}
$$

(27)

and, defining the $w_i$’s by the following change of variables

$$
W_2 = w_2, \quad W_3 = w_3 - \frac{1}{2} w_2',
$$

(28)

$$
\begin{align*}
\epsilon^0 &= -\rho_2' + c_1 \left( \frac{1}{6c_1} \partial^2 - \frac{1}{3} w_2 \right) \rho_3, \\
\epsilon^+ &= -\left( \frac{1}{c_1} \partial^2 + w_2 \right) \rho_2 + \left( \frac{1}{6c_1} \partial^3 + \frac{1}{6} \partial w_2 - w_3 \right) \rho_3, \\
\epsilon^{00} &= -\rho_2' - c_1 \left( \frac{1}{6c_1} \partial^2 + \frac{2}{3} w_2 \right) \rho_3, \\
\epsilon^{0-} &= \rho_2 + \frac{1}{2} \rho_3', \\
\epsilon^{0+} &= -\rho_2'' - c_1 \left( \frac{1}{6c_1} \partial^3 - \frac{1}{6} \partial w_2 - \frac{1}{2} w_2 \partial - w_3 \right) \rho_3.
\end{align*}
$$

(29)

Indeed, the gauge preserving transformations lead to the Virasoro and the spin-3 symmetries in terms of the parameters $\rho_2$ and $\rho_3$ as defined in Eq.(27). These symmetries are generated respectively by the energy momentum tensor $w_2$ and the primary field $w_3$ defined in Eq.(28) since we find that $\delta_\epsilon(A_-)$ leads to

$$
\delta w_i = \delta(\rho_2) w_i + \delta(\rho_3) w_i
$$

with

$$
\begin{align*}
\delta(\rho_2) w_2 &= 2 w_2 \rho_2' + \rho_2 w_2' + \frac{2}{c_1} \rho_2''' , \\
\delta(\rho_2) w_3 &= 3 w_3 \rho_2' + \rho_2 w_3' , \\
\delta(\rho_3) w_2 &= 3 w_2 \rho_3' + 2 \rho_3 w_2' , \\
\delta(\rho_3) w_3 &= -\frac{1}{12} \left( 2 w_2'' \rho_3 + 9 w_2'' \rho_3' + 15 w_2' \rho_3'' + 10 w_2 \rho_3''' \right) \\
&\quad - \frac{2 c_1}{3} \left( w_2^2 \rho_3' + w_2' \rho_3 \right) - \frac{1}{6c_1} \rho_3'''' .
\end{align*}
$$

(30)

We can consistently construct a Poisson bracket structure between the $w_i$’s, starting with the Virasoro algebra satisfied by the generator of diffeomorphisms $w_2$:

$$
\{w_2(x), w_2(y)\} = \left( c \partial^3 + w_2 \partial + \partial w_2 \right) \delta(x - y)
$$

(31)
with \( c = \frac{2}{c_1} \). The primary field \( w_3 \) can be interpreted as the generator of the spin-3 transformations:

\[
\{ w_3(x), w_3(y) \} = -\frac{1}{12} (c\partial^5 + 2w_2\partial^3 + 3\partial^2w_2\partial + 2\partial^3w_2 + \frac{16}{c}w_2\partial w_2)\delta(x-y),
\]
\[
\{ w_2(x), w_3(y) \} = (w_3\partial + 2\partial w_3)\delta(x-y)
\]
\[
\{ w_3(x), w_2(y) \} = (2w_3\partial + \partial w_3)\delta(x-y).
\]

(32)

As can be seen from the Poisson bracket \( \{ w_3(x), w_3(y) \} \), there are non-linear terms which go to zero in the so-called “classical limit” \( c \to \infty \). The \( A_- \) sector being defined,

\[
A_- = \begin{pmatrix}
0 & -w_2 & -w_3 + \frac{1}{2}w_2' \\
\frac{c_1}{0} & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\]

(33)

we now study the \( A_+ \) gauge field. The components \( A_+^a \) are defined in terms of the “metric fields” \( h_i, i = 2, 3 \) and some general \( n \)th order differential operator, \textit{linear} in the \( W_i \)'s and denoted by \( D_a^{(n)} \):

\[
A_+^a = D_a^{(1+l(a))}h_2 + D_a^{(2+l(a))}h_3.
\]

(34)

Here, \( l(a) \) represents the number of nonvanishing gauge indices in “\( a \)” e.g. \( l(+) = l(0+) = 1, l(-) = l(0-) = -1, l(0) = l(00) = 0, \) and \( l(++) = -l(--) = 2 \). Finally, we introduce the possible “cosmological constants” \( \lambda_i \) through the shift

\[
A_+ \to A_+ + \lambda_2t_+ + \lambda_3t_{0+},
\]

(35)

a choice which will be justified shortly. Let us therefore write the gauge field in the form

\[
A_+ = \begin{pmatrix}
\ldots & \lambda_0 & \ldots & \ldots \\
H_2 & \ldots & \lambda_1 & \ldots \\
H_3 & \ldots & \lambda_2 & \ldots 
\end{pmatrix},
\]

(36)

where we have emphasized the components \( H_i \sim h_i + \ldots \), i.e. the components \( A_+^- \) and \( A_+^{--} \). Notice that the gauge fields are, by construction, \textit{linear} in the fields \( W_i \)'s and the prescription used to describe them is easily generalizable to higher rank \( sl(N,R) \) algebras. The significance of the constants is already known\(^9\) for the \( sl(2,R) \) and the \( sl(3,R) \) cases: the \( \lambda_i \)'s enter in the constraint equations involving the \( h_i \)'s, and, for the \( w_2 \)-gravity, \( \lambda_2 \) can be interpreted as the cosmological constant.

The components of \( A_+ \) are determined in such a way that the curvature is of the following form

\[
\begin{pmatrix}
\Phi_2 & E_1 & E_2 \\
0 & \Phi_3 - \Phi_2 & 0 \\
0 & 0 & -\Phi_3
\end{pmatrix},
\]

(37)
with
\[
\begin{align*}
\Phi_2 &= \kappa_2 \partial^2 h_2 - \lambda_2, \\
\Phi_3 &= \kappa_3 \partial^3 h_3 - \lambda_3,
\end{align*}
\]
(38)
and \( E_1 = -\partial_+ w_2 + \cdots, E_2 = -\partial_+ w_3 + \cdots \). The constants \( \kappa_i \) are free parameters. Together with the “cosmological constants” \( \lambda_i \), they represent the presence or the absence of the constraints. Let us notice that the Eqs.(38) are legitimate only if the metric fields \( h_i \) have the proper dimensions such that \( \partial^i h_i \) is dimensionless. In fact, the zero-curvature condition alone does not define a unique \( A_+ \) since we obtain
\[
\begin{align*}
H_2 &= c_1 h_2 + a h_3', \\
H_3 &= c_1 h_3,
\end{align*}
\]
(39)
where \( a \) is free. However, choosing this parameter equal to zero spoils the symmetry property of \( h_2 \) under the diffeomorphisms. These symmetries are easily calculated from Eqs.(26,28) and \( \delta A_+ = \partial_+ \epsilon + [A_+, \epsilon] \). Keeping the interpretation of \( h_2 \) as a metric field, i.e. Eq.(3), gives \( a = -\frac{b}{2} \). Then, the remaining variations are found to be
\[
\begin{align*}
\delta(\rho_2) h_3 &= -2 h_3 \rho_2' + \rho_2 h_3', \\
\delta(\rho_3) h_2 &= \frac{1}{6} h_3 \rho_3''' - \frac{1}{4} h_3' \rho_3'' + \frac{1}{4} h_3''' \rho_3' - \frac{1}{6} h_3'' \rho_3 - \frac{4}{3c} w_2 h_3 \rho_3' - \frac{4}{3c} w_2 h_3' \rho_3 \\
\delta(\rho_3) h_3 &= - h_2 \rho_3' + 2 h_2' \rho_3 + \dot{\rho}_3.
\end{align*}
\]
(40)
Finally, the dynamical equations \( E_{1,2} = 0 \), obtained from the zero-curvature condition are the following:
\[
\begin{align*}
\partial_+ w_2 &= ((c + \alpha_2) \partial^3 + w_2 \partial + \partial w_2) h_2 + (\beta_2 \partial^4 + (w_3 \partial + 2 \partial w_3) h_3, \quad (41a) \\
\partial_+ w_3 &= (\alpha_3 \partial^3 + 2 w_3 \partial + \partial w_3) h_2 \\
&- \frac{1}{12} ((c + \beta_3) \partial^5 + (2 \partial^3 w_2 + 3 \partial^2 w_2 \partial + 3 \partial w_2 \partial^2 + 2 w_2 \partial^3) + \frac{16}{c} w_2 \partial w_2) h_3. \quad (41b)
\end{align*}
\]
The \( \alpha_i \)'s and \( \beta_i \)'s are linear combinations\(^3\) of the \( \kappa_i \)'s. It is remarkable that they only occur in the shift of the central charges, as in the \( sl(2,R) \) case. Let us recall that this system leads to the Boussinesq equation by setting \( h_2 = 0 \) and \( h_3 = \frac{1}{12} \). The particular case \( \kappa_i = \lambda_i = 0 \) reproduces the analysis of Das et al\(^7\).

The previously mentioned covariant operators \( \mathcal{L}(2) \) and \( \mathcal{L}(3) \) can be easily seen to correspond to the matrix determinant of the gauge covariant operators \( D_- = \partial + A_- \) respectively for the gauge choice of Eq.(21) and Eq.(33). This observation will be very

\(^3\) Notice that the explicit expressions for the \( \alpha_i \)'s and the \( \beta_i \)'s are irrelevant in this classical analysis.
useful in the next section where we investigate the general $sl(N, R)$ case. We will adopt the approach of partial gauge fixing à la Polyakov in obtaining the $w_N$ algebras in the $A_-$ sector and the corresponding metric fields in the $A_+$ sector. The guiding principle in the $A_-$ sector is to keep the correspondence between the scalar Lax operator $L(N)$ and the matrix covariant operator $D_-$. Although this approach seems lacking in physical motivations, recent works have illustrated connection of this method with 2+1 dimensional Chern-Simons theories\textsuperscript{[16]}. Hence, assuming validity of the Chern-Simons approach for the general $sl(N, R)$ case, we will proceed with the present analysis.

IV. General Lax pair: from $sl(3, R)$ to $sl(N, R)$

This section presents a general algorithm to calculate the matrix Lax pair associated with $sl(N, R)$. This is done in three steps. First, the $A_-$ sector of the gauge field is fixed in such a way that the remaining degrees of freedom in $A_-$, the so-called $w_i$’s, are Virasoro quasi-primary fields under the remaining gauge preserving transformations. The requirement of quasi-primary nature of the $w_i$’s avoids the complicated non-linearities that would have occurred in fixing the interaction terms in $A_+$ with strictly primary $w_i$’s. In the second step, we define the $A_+$ sector by introducing a new family of fields, the $h_i$’s, which are analogs of the metric field for the $w_N$ generators. Our objective now is to obtain, as in the $sl(2, R)$ case, the relevant Ward identities and the consistent constraints involving the $h_i$’s from the zero-curvature condition. This procedure determines most of the components of $A_+$; the remaining freedom is related with the constraint equations themselves (analogs of the $\kappa$ and $\lambda$ of the $sl(2, R)$ case) and the symmetry properties of the $h_i$’s under diffeomorphisms. This leads to the last step, similar to the first one, where we require the fields $h_i, i > 2$ to transform like currents under the fundamental Virasoro symmetry:

$$\delta(\rho) h_{i,i>2} = -(i-1) h_i \rho' + h_i' \rho. \quad (42)$$

For $h_2$, we want to maintain Eq.(3), at least for the finite subgroup of diffeomorphism. This requirement and Eq.(42) imply that $\partial^i h_i$ are dimensionless quantities and, hence, the constraint equations

$$\Phi_i \equiv \kappa_i \partial^i h_i - \lambda_i \sim 0, \quad \forall i = 2, 3, ..., N \quad (43)$$

are meaningful. This gives us the opportunity to introduce $N$ “cosmological constants” $\lambda_i$ and $N$ parameters $\kappa_i$. Vanishing of the $\kappa_i$’s and the $\lambda_i$’s eliminates these constraints.

We now propose an algorithm that realizes for any $sl(N, R)$ algebra the gauge choice that satisfies the previous conditions. Keeping the form of the gauge suggested by the
sl(2, R) and sl(3, R) cases, let us write

\[
A_- = \begin{pmatrix}
0 & -W_2 & -W_3 & -W_4 & \ldots & -W_N \\
c_1 & 0 & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & & & & & \\
0 & 0 & & & 1 & 0 \\
\end{pmatrix}, \tag{44}
\]

\[
A_+ = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots \\
H_2 & \ldots & \ldots & \ldots \\
H_3 & \ldots & \ldots \\
H_4 & \ldots \\
\vdots & \\
H_N & \ldots \\
\end{pmatrix}. \tag{45}
\]

We will put \(c_1 = 1\) for convenience; this parameter scales the central charge. The other constant components of \(A_-\) could have been chosen different from unity; however, rescaling of the primary fields in \(A_-\) eliminates these parameters in both the resulting Poisson brackets and the dynamical equations.

The \(N-1\) fields \(W_i\) and \(H_i\) have been considered in the Refs.[6,7]. It is well-known that the \(W_i\)'s are not quasi-primary fields and the \(H_i\)'s do not transform as definite spin currents. Let us consider the following change of variables:

\[
W_i = \sum_{j=2}^{i} (-1)^{i-j} A_{ij}(N) \partial^{i-j} w_j, \quad \text{with} \quad A_{ij}(N) = \frac{(i-1)(N-j)}{(i+j-1)!},
\]

\[
H_i = \sum_{j=i}^{N} B_{ij}(N) \partial^{i-j} h_j, \quad \text{with} \quad B_{ij}(N) = (-1)^{i-j} \frac{(i-1)(N-j)}{(2i-2)!},
\]

that we will justify later. The gauge field \(A_-\) being thus completely defined, the zero-curvature condition yields the following recurrent relations on the matrix elements \(a_{ij}\) of \(A_+\)

\[
\begin{aligned}
a_{i,j+1} - a_{i-1,j} &= a'_{i,j} - H_i W_j, \\
a_{k,k} &= \sum_{i=2}^{k} (a'_{i,i-1} - H_i W_{i-1}) - \sum_{i=2}^{k} \frac{N+1-i}{N} (a'_{i,i-1} - H_i W_{i-1}), \\
a_{N-i,N} &= -(a'_{N-i+1,N} - H_{N-i+1} W_N).
\end{aligned} \tag{47}
\]

These equations determine all the elements of \(A_+\) in terms of the \(N-1\) fields \(H_i\)'s. Let us ignore the constraints for the moment and concentrate on the symmetries. It is rather obvious that the issue of the gauge preserving symmetries \(\delta A_- \equiv \partial_- \epsilon + [A_-, \epsilon]\) is similar to
the zero-curvature condition since, on the constant elements of the gauge field in Eq.(44), the gauge preserving condition amounts to solving $\partial_- \epsilon - \partial_+ A_- + [A_-, \epsilon] = 0$. Therefore, the matrix $\epsilon$ is determined by its first column

$$
\epsilon = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\epsilon_2 & \epsilon_3 & \epsilon_4 & \vdots & \vdots \\
\epsilon_3 & \epsilon_4 & \vdots & \vdots & \vdots \\
\epsilon_4 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\epsilon_N & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix},
$$

(48)

and the recurrent equations (47) can be used to compute all the other components. The knowledge of the $\epsilon$ leads to the proper variations of the $w_i$’s and of the $h_i$’s. It is remarkable that the generalization of Eq.(27), in order to fulfill the symmetry requirements previously described, is the following redefinition of the gauge parameter:

$$
\epsilon_i = \sum_{j=1}^N B_{ji}(N) \partial^{i-j} \rho_j,,
$$

(49)

with the matrix $B(N)$ defined in Eq.(46). More precisely, looking only at the variations induced by $\rho_2$, the gauge transformations lead to the following expressions

$$
\begin{aligned}
\delta w_2 &= 2w_2'\rho_2 + w_2''\rho_2 + C(N)\rho_2''', \\
\delta w_{2k+1} &= (2k + 1)w_{2k+1}'\rho_2 + w_{2k+1}''\rho_2 + \sum_{l=1}^{k-1} C_{2k+1}^l(N) \Omega_{l}^{2k+1} \\
\delta w_{2k} &= 2k w_{2k}'\rho_2 + w_{2k}''\rho_2 + \sum_{l=1}^{k-1} C_{2k}^l(N) \Omega_{l}^{2k}.
\end{aligned}
$$

(50)

We have defined the central charge $C(N) = \frac{(N-1)N(N+1)}{12}$ and

$$
\Omega_l^q = \sum_{i=0}^{2(l-1)} \frac{(-1)^i}{{i-4l+1\choose i}} w_q^{(i)} \rho_2^{(2l-i+1)},
$$

(51)

where $\sigma_i(l)$ are given by:

$$
\sigma_0(l) = 1, \quad \sigma_1(l) = 2(l-1)(2l+1), \quad \sigma_2(l) = l(l-1)(2l+1)(2l-3),
$$

and,

$$
\sigma_{i\geq 3}(l) = \frac{(2l+1) {2l-1\choose i-3} {2l-1\choose 3} {2l+1-i\choose 3}}{i\choose 3}.
$$

(52)
Let us note that the derivatives appearing on the parameter $\rho_2$ are always greater or equal to three which implies that we have defined the gauge field $A_-$ in terms of quasi-primary fields. Furthermore, the dependance on $N$, i.e. on the central charge, is completely contained in the coefficients $C^l_k(N)$. A lengthy calculation leads to the following conjecture\footnote{28}

$$C^l_k(N) = \frac{(l-1)!}{2^l(2l+1)} \left( \frac{N-k+2l}{2l} \right) \left( \frac{k-l-2}{l-1} \right) \frac{(2(k-l)-1)l N + (k-l)(k-l-1) + l^2}{\prod_{i=1}^{l} (2(k-i)-1)}$$

(53)

Finally, let us mention that we obtain the correct symmetry variations of the $h_i$’s for $i > 2$:

$$\delta(\rho_2) h_{i>2} = -(i-1) h_i \rho'_2 + h'_i \rho_2. $$

However, the variation of $h_2$ involves higher derivative terms which vanish only in the finite subgroup of the diffeomorphisms. This point will be illustrated below with some particular cases $N \geq 4$.

The “cosmological constants” can now be added into the $A_+$ sector. We consider the following shift of the previous gauge field $A_+$:

$$A_+ \rightarrow A_+ + \left( \begin{array}{cccccc}
0 & \lambda_2 - \kappa_2 \partial^2 h_2 & 0 & 0 & \ldots & 0 \\
0 & 0 & \lambda_3 - \kappa_3 \partial^3 h_3 & 0 & \ldots & 0 \\
0 & 0 & 0 & \lambda_4 - \kappa_4 \partial^4 h_4 & \ldots & 0 \\
\vdots & & & & & \vdots \\
0 & \ldots & & & 0 & \lambda_N - \kappa_N \partial^N h_N \\
0 & \ldots & & & & 0
\end{array} \right).$$

(54)

The resulting zero curvature equations exhibit the expected dynamical equations on the fields $w_k$ with, as already noticed in the $N = 2$ and $N = 3$ cases, a shift in the coefficients of the higher derivative of the $h_i$’s. The diagonal entries of $F_{+-}$ now exhibit the expected $N - 1$ equations $\Phi_i \sim 0$.

It is easy to show that the presence of constraints restricts the gauge preserving symmetry to that of the finite subgroup of the diffeomorphisms. Of course, such restriction is also seen in the quantum regime where the relevance of the constraints is well understood in terms of the ghost system. Let us only mention that, similar to Eq.(5), the operators $h_i$’s are expanded in terms of $sl(N, R)$ current operators

$$h_i = \sum_{n=2}^{2i-2} x^{-n} J_i^{(n)}(x^+).$$

(55)

Such expansions are legitimate by the equations $\partial^{2i-1} h_i = 0$ that generalize the constant curvature equation and are easily obtainable from the dynamical equations with the $w_i$’s
equal to zero. Then, the equations $\Phi_i \sim 0$ lead to the following system of constraints

$$J_i^{(i)} \sim \text{const}, \quad \text{and} \quad \begin{cases} J_i^{(i+1)} \sim 0, \\ J_i^{(i+2)} \sim 0, \\ \vdots \\ J_i^{(j)} \sim 0, \\ J_{i,i>2}^{(i)} \sim 0, \\ \vdots \\ J_i^{(2i-2)} \sim 0. \end{cases}, i = 2, 3, \ldots, N \quad (56)$$

Dimensional analysis of this set of constraints completely describes the weight spectrum of the necessary ghost systems. The resulting central charge contribution is\textsuperscript{10,28} $c_{\text{ghost}} = -n^2(n + 1)^2 + n(n + 1) + 2$.

Let us illustrate the previous results with the case of $N = 4$. The definitions in Eq.(46) give

$$W_2 = w_2,$$
$$W_3 = w_3 - w'_2,$$
$$W_4 = w_4 - \frac{1}{2} w'_3 + \frac{3}{10} w''; \quad (57)$$

and

$$H_2 = h_2 - h'_3 + \frac{1}{5} h''_4,$$
$$W_3 = h_3 - h'_4,$$
$$W_4 = h_4. \quad (58)$$

The gauge is then uniquely determined through the relations (47) and, finally, the shift (54) introduces the cosmological constants. Inspecting the gauge transformations that maintain the gauge so defined, we obtain the $c = 5$ Virasoro algebra for $w_2$ together with the following infinitesimal variations:

$$\delta(\rho_2) w_3 = 3 w_3 \rho'_2 + \rho_2 w'_3,$$
$$\delta(\rho_2) w_4 = 4 w_4 \rho'_2 + \rho_2 w'_4 + \frac{9}{10} w_2 \rho''_2,$$
$$\delta(\rho_2) h_2 = -h_2 \rho'_2 + \rho_2 h'_2 - \frac{9}{10} h_4 \rho''_2 + \dot{\rho}_2,$$
$$\delta(\rho_2) h_3 = -2 h_3 \rho'_2 + \rho_2 h'_3,$$
$$\delta(\rho_2) h_4 = -3 h_4 \rho'_2 + \rho_2 h'_4, \quad (59)$$

and the higher spin variations listed in the Appendix A. The variation $\delta(\rho_2) w_4$ is a consequence of our gauge choice linear in the $w_i$’s. However, we can redefine the parameterisation.
of the gauge and introduce a non-linear term in \( W_4 \):

\[
W_4 = w_4 - \frac{1}{2} w'_3 + \frac{3}{10} w''_2 + \frac{9}{100} w_2^2
\]  
(60)

such that \( w_4 \) is now primary, like \( w_3 \). This remark leads us to the justification of the parameterisation of \( W_i \) in Eq.(46). It mainly relies on a theorem due to Itzykson et al.\cite{21} in the context of the scalar Lax operator \( \mathcal{L}_{(N)} \); the matrix determinant being the operation that connects our approach to theirs. Similar to Ref\cite{21}, our gauge is thus parameterised with quasi-primary fields. The previous redefinition (60) illustrates the missing part of the theorem, i.e. the non-linear contribution in \( W_i \) that makes all the \( w_i > 2 \) primary. Using the variations in Eq.(50), the expression of \( W_4 \) given by Eq.(60) can be generalised to higher \( N \geq 4 \):

\[
W_4 = w_4 + \sum_{j=2}^{3} (-1)^{4-j} A_{4j}(N) \partial^{4-j} w_j + \frac{C_4^1(N)}{C(N)} w_2^2,
\]

Further generalizations to higher dimensional \( W_i \)'s are certainly possible. For example,

\[
W_5 = w_5 + \sum_{j=2}^{4} (-1)^{5-j} A_{5j}(N) \partial^{5-j} w_j + \frac{C_5^1(N)}{C(N)} w_2 w_3,
\]

\[
W_6 = w_6 + \sum_{j=2}^{5} (-1)^{6-j} A_{6j}(N) \partial^{6-j} w_j + \frac{C_6^1(N)}{C(N)} w_2 w_4 + \frac{C_6^2(N)}{4 C(N)} (4 w_2 w''_2 - 5 w_2^2)
\]

\[
+ \frac{C_6^1(N) C_4^1(N) - 9 C_6^2(N)}{6 C(N)^2} w_2^3,
\]

\[
W_7 = w_7 + \sum_{j=2}^{6} (-1)^{7-j} A_{7j}(N) \partial^{7-j} w_j + \frac{C_7^1(N)}{C(N)} w_2 w_5 + \frac{C_7^2(N)}{21 C(N)} (21 w_3 w''_2 - 35 w'_3 w''_2
\]

\[
+ 10 w''_3 w_2) + \frac{7 C_7^1(N) C_5^1(N) - 24 C_7^2(N)}{14 C(N)^2} w_2^2 w_3,
\]

etc...

(62)

A complete formula for this parameterisation of the \( W_k \)'s in terms of \( w_2 \) and primary \( w_i \)'s is under investigation.

The parameterisation of \( H_i \) in Eq.(46) is more intriguing since, as shown in Ref\cite{21}, the matrices \( A_{ij}(N) \) and \( B_{ij}(N) \) satisfy \( A.B = 1 \). Whereas the fields \( h_i \)'s satisfy the symmetry conditions described previously, the Virasoro variation of the metric field \( h_2 \) exhibits a contribution that only vanishes on the finite subgroup of the diffeomorphisms. This observation is also true for higher \( w_N \) gravities (see Appendix C). However, it is possible to introduce coupling terms in the field \( H_2 \)

\[
H_2 = h_2 - h'_3 + \frac{1}{5} (\partial^2 + \frac{9}{2} w_2) h_4
\]

(63)
such that the field $h_2$ now exactly transforms as in Eq.(3).

Let us come back to our $N = 4$ example and consider the zero curvature equations. They exhibit the constraints on the $h_i$'s and the following dynamical equations

$$\partial_+ w_2 = \left( (5 + \alpha_2) \partial^3 + w_2 \partial + \partial w_2 \right) h_2 + \left( \beta_2 \partial^4 + (w_3 \partial + 2 \partial w_3) \right) h_3 + \left( \gamma_2 \partial^5 + \frac{9}{10} \partial^3 w_3 + (w_4 \partial + 3 \partial w_4) \right) h_4$$

$$\partial_+ w_3 = (\alpha_3 \partial^4 + (2 w_3 \partial + \partial w_3)) h_2 - \frac{1}{5} \left( (5 + \beta_3) \partial^5 + (2 \partial^3 w_2 + 3 \partial^2 w_2 \partial + 3 \partial w_2 \partial^2 \\
+ 2 w_2 \partial^3) - 10 (w_4 \partial + \partial w_4) + 5 w_2 \partial w_2 \right) h_3 - \frac{1}{10} \left( \gamma_3 \partial^6 + 5 \partial^3 w_3 + 5 \partial^2 w_3 \partial \\
+ 3 w_3 \partial^3) + 5 w_2 \partial w_3 \right) h_4$$

and

$$\partial_+ w_4 = \left( \alpha_4 \partial^5 + \frac{9}{10} w_2 \partial^3 + 3 w_3 \partial + \partial w_4 \right) h_2 - \frac{1}{10} \left( \beta_4 \partial^6 + \partial^3 w_3 + 3 \partial^2 w_3 \partial + 5 \partial w_3 \partial^2 \\
+ 5 w_3 \partial^3 + 5 w_3 \partial w_2 \right) h_3 + \frac{1}{100} \left( (5 + \gamma_4) \partial^7 + 3 \partial^5 w_2 + 5 \partial^4 w_2 \partial + 6 \partial^3 w_2 \partial^2 \\
+ 4 \partial^2 w_2 \partial^3 + 5 \partial w_2 \partial^4 + 3 w_2 \partial^5 + 10 (\partial^3 w_4 + 2 \partial^2 w_4 \partial + 2 \partial w_4 \partial^2 + w_4 \partial^3) \\
+ 5 \left( \partial^3 w_4^2 + w_2^2 \partial^3 + 4 \partial w_2 \partial w_2 \partial + 4 w_2 \partial^3 w_2 \right) \right) h_4$$

We notice that equations (64a,b) with $h_2 = h_3 = 0$, $h_4 = \frac{1}{2}$ and $w_4 = w_2^2$ lead to a coupled KdV system:

$$\begin{cases}
\partial_+ w_2 - \frac{1}{2} w_2'' - 3 w'_2 w_2 = 0, \\
\partial_+ w_3 - \frac{1}{4} w_3''' - \frac{1}{4} w'_3 w_2 = 0,
\end{cases}$$

whereas Eq.(64c) becomes the following constraint equation:

$$\frac{1}{100} w_2'''' + \frac{1}{20} w_2''''' - \frac{3}{5} w_2 w_2'' + \frac{1}{3} w_2' = \frac{1}{8} w_3'.$$
V. Conclusion

In this note, we studied the Lax pair formulation of the $w_N$ gravity theories; the analysis being strictly at the classical level. One of the matrix Lax operator $\partial + A_-$ yielded the basic differential operator associated with the $n$th reduction of the KP hierarchy, after taking the matrix determinant or, equivalently, solving the differential equation $(\partial + A_-)\Psi = 0$ where $\Psi$ takes its value in a jet bundle[6]. Hence, interestingly, the partial gauge fixing of the $A_-$ sector, designed for obtaining fields transforming as higher spin objects under the residual gauge transformations, turned out to be connected with the process of covariantization of this $n$th order differential operator. It will be of interest to find out the geometry behind the other members of the $w^{(l)}_N$ algebras. Although these fields were chosen to be quasi-primary for the sake of convenience, the generalisation to primary fields can be achieved by using the above mentioned connection with the scalar covariant operators. We gave a conjectured formula, valid to all orders, for the infinitesimal variations of the $w_i$’s under diffeomorphism. By studying these variations, it is possible, in principle, to construct the primary fields out of the quasi primary ones. It is of deep interest to obtain an all order formulation for the gauge defined by Eq.(44) in terms of primary fields; this problem is currently under investigation. Furthermore, the work of Zamolodchikov has shown the connection of these covariant operators (in the absence of higher spin fields) with the induced Liouville action, which indicates a large-N analysis of this problem can be carried out using the matrix valued connections thereby clarifying the relationship between the continuum and the lattice model approaches to the two dimensional gravity. The $A_+$ sector, related with the $sl(N,R)$ current algebra, has been determined modulo scalings of the metric fields $h_i$’s by constant parameters, by requiring $h_{i>2}$ to transform as currents under the Virasoro symmetry and $h_2$ to transform as the light-cone metric. The scaling freedom is tied with the level of the Kac-Moody algebra. The relationship of the original $H_i$’s and the $h_i$’s is of great interest: it happens to be the inverse of the transformation relating the noncovariant fields in $M_\langle N \rangle$ to the covariant ones in $L_\langle N \rangle$. The same connection also holds for the gauge parameters; namely the infinitesimal parameters associated with the Virasoro and the higher spin symmetries and the $\epsilon^{-1}$’s appearing in the gauge transformations are connected to each other by an identical relation. We suspect that this intriguing connection originates from the coadjoint orbits of the Lax operator. This point needs further study for clarification.

The other entries in $A_+$ were determined by demanding that the zero curvature condition should yield the relevant constraints and the Ward identities. It is worth mentioning that the same equations follow from the requirement of the consistency of the gauge fixing. This is natural because the Lie derivative of a gauge field can be described as a gauge transformation if the curvature tensor vanishes. The $F_{+-} = 0$ condition immediately
suggests a connection with the Chern-Simons theories, where similar condition appears as the Gauss law constraint. In fact, this connection has been already illustrated for the \( w_2 \) and \( w_3^{(2)} \) cases\[^{16}\]; the proof for the \( w_N \) case is of obvious interest. In light of the beautiful geometry associated with the compact non-abelian Chern-Simons theories, this point needs deeper understanding. Another open question is the construction of the covariant action like that of the KPZ analysis of the \( \mathfrak{sl}(2,R) \) case and understanding of the precise relationship between the non-linear hierarchies appearing in the continuum approach and those of the matrix models.

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Appendix A:
The infinitesimal higher spin variations of the \( w_i \)'s and the \( h_i \)'s in the \( \mathfrak{sl}(4,R) \) case are the following:

\[
\delta(\rho_3) w_2 = 3w_3\rho_3' + 2\rho_3 w_3',
\]
\[
\delta(\rho_3) w_3 = 4w_4\rho_3' + 2w_4\rho_3 + 2w_2\rho_3'' + 3w_2\rho_3'' + \frac{9}{5}w_2\rho_3' + \frac{2}{5}w_2'\rho_3
\]
\[
- (w_2^2\rho_3' + w_2 w_2\rho_3) - \rho_3''',
\]
\[
\delta(\rho_3) w_4 = -\frac{7}{5}w_3\rho_3'' - \frac{7}{5}w_3'\rho_3' - \frac{3}{5}w_3\rho_3'' - \frac{1}{10}w_3\rho_3
\]
\[
- \frac{1}{2}(w_2 w_3\rho_3' + w_3 w_2\rho_3),
\]
\[
\delta(\rho_3) h_2 = \frac{2}{5}h_3\rho_3''' - \frac{3}{5}h_3'\rho_3'' + \frac{3}{5}h_3''\rho_3' - \frac{2}{5}h_3''' \rho_3
\]
\[
- \frac{1}{2}(h_4 w_3\rho_3 + h_4 w_3'\rho_3) + h_3 w_2\rho_3' - h_3' w_2\rho_3,
\]
\[
\delta(\rho_3) h_3 = 2h_3\rho_3' - h_2\rho_3' + \dot{\rho}_3
\]
\[
+ \frac{1}{2}h_4\rho_3'' - \frac{1}{2}h_4'\rho_3'' + \frac{3}{10}h_4''\rho_3' - \frac{1}{10}h_4''' \rho_3
\]
\[
+ \frac{1}{2}(h_4 w_2\rho_3 + h_4 w_2'\rho_3),
\]
\[
\delta(\rho_3) h_4 = 2h_3\rho_3' - 2h_3\rho_3',
\]
\[ \delta(\rho_4) w_2 = 4w_4\rho'_4 + 3\rho_4 w'_4 + \frac{9}{10}(w_2\rho'_4 + 3w'_2\rho_4' + 3w'_2\rho_4' + 3w''\rho_4), \]

\[ \delta(\rho_4) w_3 = -\frac{1}{10}(14w_3\rho'_4 + 28w'_3\rho_4 + 20w'_3\rho_4 + 5w''\rho_4) - \frac{1}{2}(w_2w_3\rho_4 + w_2w_3\rho_4), \]

\[ \delta(\rho_4) w_4 = \frac{1}{10}(w_4\rho'_4 + 9w'_4\rho_4 + 5w'_4\rho_4 + 6w''\rho_4) \]
\[ \quad + \frac{7}{25}w_2\rho''_4 + \frac{7}{25}w'_2\rho''_4 + \frac{21}{25}w'_2\rho''_4 + \frac{14}{25}w''_2\rho'_4 + \frac{1}{5}w''_2\rho_4 + \frac{3}{100}w'''''_4 \rho_4 + 3 \]
\[ - \frac{3}{4}(w_3^2\rho'_4 + w_3w_3\rho_4) + 2w_4w_2\rho_4 + w'_4w_2\rho_4 + w_4w'_4\rho_4 + \frac{1}{2}w_2\rho''_4 \]
\[ + \frac{3}{2}w_2w'_2\rho''_4 + \frac{1}{2}w_2\rho'_4 + \frac{11}{10}w_2w'_2\rho_4 \]
\[ + \frac{3}{10}w'w_2\rho_4 + \frac{3}{10}w_2w''\rho_4 + \frac{1}{20}\rho''''_4 \]

\[ \delta(\rho_4) h_2 = \frac{9}{10}h''_2\rho_4 - \frac{3}{10}h_4\rho''_4 + \frac{1}{20}h'_4\rho''_4 - \frac{3}{50}h'_4\rho''_4 + \frac{3}{50}h''''_4\rho_4 - \frac{1}{20}h''''_4 \rho_4 \]
\[ + \frac{3}{100}h'''''_4 \rho_4 - \frac{3}{10}w_2h_4\rho''_4 - \frac{3}{5}w_2h_4\rho'_4 + \frac{1}{5}w_2h'_4\rho''_4 - \frac{1}{5}w_2h''_4\rho''_4 - \frac{3}{5}w'_2h_4\rho''_4 \]
\[ + \frac{3}{5}w'_2h'_4\rho_4 + \frac{3}{10}w_2h''''_4 \rho_4 + w_4h_4\rho'_4 + w'_4h_4\rho_4 + \frac{1}{2}h_3w_3\rho_4 + \frac{1}{2}h_3w_3\rho'_4, \]

\[ \delta(\rho_4) h_3 = \frac{1}{10}(h_3\rho''''_4 - 3h'_3\rho''''_4 + 5h''''_4 - 5h''''_4 \rho_4) \]
\[ - \frac{1}{2}w_2h'_3\rho_4 - \frac{1}{2}w_2h_3\rho_4 + \frac{3}{4}w_3h_4\rho_4 + \frac{3}{4}w_3h_4\rho'_4, \]

\[ \delta(\rho_4) h_4 = 3h'_2\rho_4 - h_2\rho_4 + \rho_4 \]
\[ - \frac{1}{10}(h_4\rho''''_4 - 2h'_4\rho''''_4 - 2h''''_4 \rho_4 + h''''_4 \rho_4 + h'_4w_4\rho_4 - h_4w_4\rho_4). \]

**Appendix B:**
Looking at the gauge symmetries preserving the gauge as defined in Eq.(44) for \(sl(10, R)\), we get the following Virasoros transformations of the \(w_i\)'s, \(2 \leq i \leq 10\):

\[ \delta(\rho_2) w_2 = 2w_2\rho'_2 + w'_2\rho_2 + \frac{165}{2}\rho''_2, \]

\[ \delta(\rho_2) w_3 = 3w_3\rho'_2 + w'_3\rho_2, \]

\[ \delta(\rho_2) w_4 = 4w_4\rho'_2 + w'_4\rho_2 + \frac{266}{5}w_2\rho''_2, \]

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\[
\delta(\rho_2)w_5 = 5w_5'\rho_2 + w_5'\rho_2 + \frac{83}{2} w_3'\rho_2''',
\]
\[
\delta(\rho_2)w_6 = 6w_6'\rho_2 + w_6'\rho_2 + \frac{185}{6} w_4'\rho_2'''
+ \frac{52}{3} w_2'\rho_2''' - \frac{130}{3} w_2'\rho_2'' + \frac{130}{3} w_2''\rho_2'',
\]
\[
\delta(\rho_2)w_7 = 7w_7'\rho_2 + w_7'\rho_2 + \frac{235}{11} w_5'\rho_2'''
+ \frac{119}{11} w_3'\rho_2''' - \frac{595}{33} w_3'\rho_2'' + \frac{170}{11} w_3''\rho_2'',
\]
\[
\delta(\rho_2)w_8 = 8w_8'\rho_2 + w_8'\rho_2 + \frac{173}{13} w_6'\rho_2'''
+ \frac{762}{143} w_4'\rho_2''' - \frac{1905}{286} w_4'\rho_2'' + \frac{635}{429} w_4''\rho_2''
+ \frac{23}{33} w_2'\rho_2''' - \frac{161}{33} w_2'\rho_2'' + \frac{483}{55} w_2''\rho_2'' - \frac{161}{33} w_2''\rho_2''' + \frac{23}{33} w_2'''\rho_2'',
\]
\[
\delta(\rho_2)w_9 = 9w_9'\rho_2 + w_9'\rho_2 + \frac{173}{13} w_7'\rho_2'''
+ \frac{762}{143} w_5'\rho_2''' - \frac{1905}{286} w_5'\rho_2'' + \frac{635}{429} w_5''\rho_2''
+ \frac{23}{33} w_3'\rho_2''' - \frac{161}{33} w_3'\rho_2'' + \frac{483}{55} w_3''\rho_2'' - \frac{161}{33} w_3''\rho_2''' + \frac{23}{33} w_3'''\rho_2'',
\]
\[
\delta(\rho_2)w_{10} = 10w_{10}'\rho_2 + w_{10}'\rho_2 + \frac{173}{13} w_8'\rho_2'''
+ \frac{762}{143} w_6'\rho_2''' - \frac{1905}{286} w_6'\rho_2'' + \frac{635}{429} w_6''\rho_2''
+ \frac{23}{33} w_4'\rho_2''' - \frac{161}{33} w_4'\rho_2'' + \frac{483}{55} w_4''\rho_2'' - \frac{161}{33} w_4''\rho_2''' + \frac{23}{33} w_4'''\rho_2'''
+ \frac{23}{33} w_2'\rho_2''' - \frac{161}{33} w_2'\rho_2'' + \frac{483}{55} w_2''\rho_2'' - \frac{161}{33} w_2''\rho_2''' + \frac{23}{33} w_2'''\rho_2'''
- \frac{161}{33} w_2'''\rho_2'' + \frac{23}{33} w_2'''\rho_2''.
\]

Appendix C:

As mentioned in the text, the Virasoro variation of \(h_i\) with \(i > 2\) is that of a current with dimension \(i\),
\[
\delta(\rho_2)h_{i,i>2} = -(i-1)h_i' \rho_2' + h_i' \rho_2.
\]

However, the variation of \(h_2\) involves higher derivative terms for \(N \geq 4\). Denoting
\[
\mathcal{L}_{\rho_2} h_2 = -h_2' \rho_2' + \rho_2 h_2' + \rho_2,
\]

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we obtain for $N \geq 4$
\[
\delta(\rho_2) h_2 = \mathcal{L}_{\rho_2} h_2 + \Gamma_N,
\]
where $\Gamma_N$ is vanishing in the finite subgroup of diffeomorphism, in which case $\rho_2''' = 0$. In addition to $\delta(\rho_2) h_2$ given in Eq.(59), some explicit expressions are given below

\[
\begin{align*}
\Gamma_5 &= -\frac{16}{5} h_4 \rho_2''', \\
\Gamma_6 &= -\frac{37}{5} h_4 \rho_2''' - \frac{5}{7} h_6 \rho_2'''' - \frac{5}{7} h_6' \rho_2''' - \frac{10}{63} h_6'' \rho_2'', \\
\Gamma_7 &= -14 h_4 \rho_2''' - \frac{57}{14} h_6 \rho_2'''' - \frac{57}{14} h_6' \rho_2''' - \frac{19}{21} h_6'' \rho_2'', \\
\Gamma_8 &= -\frac{47}{2} h_4 \rho_2''' - \frac{96}{7} h_6 \rho_2'''' - \frac{96}{7} h_6' \rho_2''' - \frac{64}{21} h_6'' \rho_2'', \\
&\quad - \frac{7}{12} h_8 \rho_2'''' - \frac{7}{6} h_8' \rho_2''' - \frac{21}{26} h_8'' \rho_2'' - \frac{35}{156} h_8''' \rho_2' - \frac{35}{1716} h_8'''' \rho_2'', \\
\Gamma_9 &= -\frac{182}{5} h_4 \rho_2''' - \frac{71}{2} h_6 \rho_2'''' - \frac{71}{2} h_6' \rho_2''' - \frac{71}{9} h_6'' \rho_2'', \\
&\quad - \frac{68}{15} h_8 \rho_2'''' - \frac{136}{15} h_8' \rho_2''' - \frac{408}{65} h_8'' \rho_2'' - \frac{68}{39} h_8''' \rho_2' - \frac{68}{429} h_8'''' \rho_2'', \\
\Gamma_{10} &= -\frac{256}{5} h_4 \rho_2''' - \frac{78}{2} h_6 \rho_2'''' - \frac{78}{2} h_6' \rho_2''' - \frac{52}{3} h_6'' \rho_2'', \\
&\quad - \frac{299}{15} h_8 \rho_2'''' - \frac{598}{15} h_8' \rho_2''' - \frac{138}{5} h_8'' \rho_2'' - \frac{23}{3} h_8''' \rho_2' - \frac{23}{33} h_8'''' \rho_2'', \\
&\quad - \frac{27}{55} h_{10} \rho_2'''' - \frac{81}{35} h_{10}' \rho_2''' - \frac{324}{187} h_{10}'' \rho_2'' - \frac{189}{187} h_{10}''' \rho_2' - \frac{189}{187} h_{10}'''' \rho_2'', \\
&\quad - \frac{567}{1870} h_{10}'''' \rho_2'' - \frac{81}{1870} h_{10}''' \rho_2' - \frac{27}{12155} h_{10}'''' \rho_2''.
\end{align*}
\]

We notice that $\Gamma_N$ involves only even dimensional $h_i$'s, e.g. $h_4, h_6, \ldots$. As already mentioned in the $sl(4,R)$ case, these higher derivative terms can be removed by a redefinition of $H_2$ that introduces coupling terms. For example,

\[
N = 5 : H_2 = h_2 - \frac{3}{2} h_3' + \frac{3}{5} (\partial^2 - \frac{8}{15} w_2) h_4 - \frac{1}{14} h_5'',
\]
\[
N = 6 : H_2 = h_2 - 2 h_3' + \frac{6}{5} (\partial^2 - \frac{37}{105} w_2) h_4 - \frac{2}{7} h_5'',
\]
\[
+ \frac{1}{42} (\partial^4 - \frac{12}{7} (\frac{2}{9} w_2 \partial^2 + w_2 \partial + w_2'' - 4 w_2^2)) h_6,
\]
\[
N = 7 : H_2 = h_2 - \frac{5}{2} h_3' + 2 (\partial^2 - \frac{1}{4} w_2) h_4 - \frac{5}{7} h_5''
\]
\[
+ \frac{5}{42} (\partial^4 + \frac{171}{140} (\frac{2}{9} w_2 \partial^2 + w_2 \partial + w_2'' - 4 w_2^2)) h_6 - \frac{1}{132} h_7''',
\]
will lead to $\delta(\rho_2) h_2 = \mathcal{L}_{\rho_2} h_2$. A general all order expression for $H_2$ is under investigation.
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