Semiarcs with a long secant in $\text{PG}(2, q)$

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Abstract

A $t$-semiarc is a pointset $S_t$ with the property that the number of tangent lines to $S_t$ at each of its points is $t$. We show that if a small $t$-semiarc $S_t$ in $\text{PG}(2, q)$ has a large collinear subset $K$, then the tangents to $S_t$ at the points of $K$ can be blocked by $t$ points not in $K$. We also show that small $t$-semiarcs are related to certain small blocking sets. Some characterization theorems for small semiarcs with large collinear subsets in $\text{PG}(2, q)$ are given.

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1 Introduction

Ovals, $k$-arcs and semiovals of finite projective planes are not only interesting geometric structures, but they have applications to coding theory and cryptography, too [3]. For details about these objects we refer the reader to [19, 22].

SemiaRs are natural generalizations of arcs. Throughout the paper $\Pi_q$ denotes an arbitrary projective plane of order $q$. A non-empty pointset $S_t \subset \Pi_q$ is called a $t$-semiarc if for every point $P \in S_t$ there exist exactly $t$ lines $\ell_1, \ell_2, \ldots, \ell_t$ such that $S_t \cap \ell_i = \{P\}$ for $i = 1, 2, \ldots, t$. These lines are called the tangents to $S_t$ at $P$. If a line $\ell$ meets $S_t$ in $k$ points, then $\ell$ is called a $k$-secant of $S_t$. We say that a $k$-secant is long, if $q - k$ is a small number (which will be given a precise meaning later). The classical examples of $t$-semiarcs are the $k$-arcs (with $t = q + 2 - k$), subplanes (with $t = q - m$, where $m$ is the order of the subplane) and semiovals (that is semiarcs with $t = 1$).

Because of the huge diversity of semiarcs, the complete classification is hopeless. The aim of this paper is to investigate and characterize semiarcs having some additional properties. In Section 2 we consider a very special class, namely $t$-semiarcs of size $k + q - t$ having a $k$-secant. These pointsets are closely related to the widely studied structures defining few directions [1, 6, 30]. In Section 3 we prove that in $\text{PG}(2, q)$ if a small $t$-semiarc has a large collinear subset $K$, then the tangent lines at the points of $K$ belong to $t$ pencils, whose carriers are not in $K$. This result generalizes the main result in Kiss [21]. Small semiovals with large collinear subsets were studied in arbitrary projective planes as well, see Bartoli [2] and Dover [14]. The essential part of our proof is algebraic, it is based on an application of the Rédei polynomial and the Szőnyi–Weiner Lemma. In Section 4 we associate to each $t$-semiarc $S_t$ a blocking set. If $S_t$ is small and has a long secant, then the associated blocking set is small. Applying theorems about the structure of small blocking sets we prove some characterization theorems for semiarcs.

When $t \geq q - 2$, then it is easy to characterize $t$-semiarcs. If $t = q + 1$, $q$ or $q - 1$, then $S_t$ is single point, a subset of a line, or three non-collinear points, respectively; see [13] Proposition 2.1. Hence, if no other bound is specified, we usually will assume that $t \leq q - 2$. If $t = q - 2$, then it follows from [13] Proposition 2.1. If $t = q - 1$, then it follows from [13] Proposition 2.1.
Theorem 1.3 ([4, 9, 11])

If a complete quadrilateral, or a Fano subplane. Thus sometimes we may assume that $t \leq q - 3$, which we indicate individually.

Throughout the paper we use the following notation. We denote points at infinity of $\text{PG}(2, q)$, i.e. points on the line $\ell_\infty = [0 : 0 : 1]$, by $(m)$ instead of the homogeneous coordinates $(1 : m : 0)$. We simply write $Y_\infty$ and $X_\infty$ instead of $(0 : 1 : 0)$ and $(1 : 0 : 0)$, respectively. The points of $\ell_\infty$ are also called directions. For affine points, i.e. points of $\text{PG}(2, q) \setminus \ell_\infty$, we use the Cartesian coordinates $(a, b)$ instead of $(a : b : 1)$. If $P$ and $Q$ are distinct points in $\Pi_q$, then $PQ$ denotes the unique line joining them. If $A$ and $B$ are two pointsets in $\Pi_q$, then $A \triangle B$ denotes their symmetric difference, that is $(A \setminus B) \cup (B \setminus A)$.

Blocking sets play an important role in our proofs. For the sake of completeness we collect the basic definitions and some results about these objects. A blocking set $B$ in a projective or affine plane is a set of points which intersects every line. If $B$ contains a line, then it is called trivial. A point $P$ in a blocking set $B$ is essential if $B \setminus \{P\}$ is not a blocking set, i.e. there is a tangent line to $B$ at the point $P$. A blocking set is said to be minimal when no proper subset of it is a blocking set or, equivalently, each of its points is essential. If $\ell$ is a line containing at most $q$ points of a blocking set $B$ in $\Pi_q$, then $|B| \geq q + |\ell \cap B|$. In case of equality $B$ is a blocking set of Rédei type and $\ell$ is a Rédei line of $B$. Note that we also consider a line to be a blocking set of Rédei type. A blocking set in $\Pi_q$ is said to be small if its size is less than $3(q + 1)/2$.

Theorem 1.1 ([29, Remark 3.3 and Corollary 4.8]) Let $B$ be a blocking set in $\text{PG}(2, q)$, $q = p^h$, $p$ prime. If $|B| \leq 2q$, then $B$ contains a unique minimal blocking set. If $B$ is a minimal blocking set of size less than $3(q + 1)/2$, then each line intersects $B$ in $1 \mod p$ points.

Note that a blocking set contains a unique minimal blocking set if and only if the set of its essential points is a blocking set.

Theorem 1.2 ([27, Corollary 5.1], [26], [29]) Let $B$ be a minimal blocking set in $\text{PG}(2, q)$, $q = p^h$, $p$ prime, of size $|B| < 3(q + 1)/2$. Then there exists a positive integer $e$, called the exponent of $B$, such that $e$ divides $h$, and

$$q + 1 + p^e \left\lceil \frac{q/p^e + 1}{p^e + 1} \right\rceil \leq |B| \leq \frac{1 + (p^e + 1)(q + 1) - \sqrt{D}}{2},$$

where $D = (1 + (p^e + 1)(q + 1))^2 - 4(p^e + 1)(q^2 + q + 1)$.

If $p^e \neq 4$ and $|B|$ lies in the interval belonging to $e$, then each line intersects $B$ in $1 \mod p^e$ points.

Theorem 1.3 ([4], [9], [11]) Let $B$ be a minimal blocking set in $\text{PG}(2, q)$, $q = p^h$, $p$ prime. Let $|B| = q + 1 + k$, and let $c_p = 2^{-1/3}$ for $p = 2, 3$ and $c_p = 1$ for $p > 3$. Then the following hold.

1. If $h = 1$ and $k \leq (q + 1)/2$, then $B$ is a line, or $k = (q + 1)/2$ and each point of $B$ has exactly $(q - 1)/2$ tangent lines.
2. If $h = 2d + 1$ and $k < c_p q^{2/3}$, then $B$ is a line.
3. If $k \leq \sqrt{q}$, then $B$ is a line or $k = \sqrt{q}$ and $B$ is a Baer subplane (that is a subplane of order $\sqrt{q}$).

We remark that the third point of the above theorem holds in arbitrary finite projective planes.

2 Semiarc and the direction problem

If a $t$-semiarc $S_t$ has a $k$-secant $\ell$, then its size $s$ is at least $k + q - t$, because for any point $P \in S_t \cap \ell$ there are $q + 1 - t$ non-tangent lines to $S_t$ through $P$, one of which is $\ell$, and each of the remaining $q - t$ non-tangent lines contains at least one point from $S_t \setminus \ell$. Thus we may always assume that $s = k + q - t + \varepsilon$, where $\varepsilon \geq 0$. In this section we investigate the case $\varepsilon = 0$. Notice that $t < q$ implies $k \leq q + 1 - t$. 

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Theorem 2.1 ([12, Theorem 4]) In $PG(2, q)$, a $t$-semiarc with a $(q + 1 - t)$-secant exists if and only if $t \geq (q - 1)/2$.

Proposition 2.2 ([13, Proposition 2.2]) Let $\Pi_q$ be a projective plane of order $q$, and let $t \leq q - 2$. If a $t$-semiarc $S_t$ in $\Pi_q$ is contained in the union of two lines, $\ell$ and $\ell'$, then $\ell \cap \ell' \notin S_t$ and $|\ell \cap S_t| = |\ell' \cap S_t| = q - t$.

It is easy to give a combinatorial characterization of $t$-semiarc sizes $2(q - t)$ with a $(q - t)$-secant; for semiarcs it was also proved by Bartoli [2 Corollary 9].

Proposition 2.3 Let $\Pi_q$ be a projective plane of order $q$, and let $t \leq q - 2$. If $S_t$ is a $t$-semiarc of size $2(q - t)$ with a $(q - t)$-secant $\ell$, then $S_t$ consists of the symmetric difference of two lines with $t$ further points removed from each line.

Proof. Let $R = S_t \setminus \ell$. If $\ell'$ is a line joining two points of $R$, then $\ell \cap \ell' \notin S_t$, otherwise there would be at least $t + 1$ tangents to $S_t$ at $\ell \cap \ell'$. Now suppose to the contrary that there exist three non-collinear points in $R$. They determine three lines, each of which intersects $\ell$ in $S_t$, hence at these three points of $R$ there are at most $t - 1$ tangents to $S_t$, a contradiction. Thus the points of $R$ are collinear and $\ell \cap \ell' \notin S_t$. □

The following example shows the existence of $t$-semiarc sizes $k + q - t$ with $k$ $q$-secants for odd values of $t$.

Example 2.4 Let $C$ denote the set of non-squares in the field $GF(q)$, $q$ odd. The pointset $\{(0 : 1 : s), (s : 0 : 1), (1 : s : 0) : -s \in C\}$ is a semiarc in $PG(2, q)$ of size $3(q - 1)/2$ with three $(q - 1)/2$-secants (see Blokhuis [5]). If we delete $r < (q - 1)/2 - 1$ points from each of the $(q - 1)/2$-secants, then the remaining pointset is a $t$-semiarc of size $k + q - t$ with $k$ $q$-secants, where $k = (q - 1)/2 - r$ and $t = 2r + 1$.

There also exist examples if $t$ is even. To give their construction, we need some notation. A $(k, n)$-arc is a set of $k$ points such that each line contains at most $n$ of these points. A set $T$ of $q + t$ points in $\Pi_q$ for which each line meets $T$ in $0$, $2$ or $t$ points ($t \neq 0, 2$) is either an oval (for $t = 1$), or a $(q + t, t)$-arc of type $(0, 2, t)$. Korchmáros and Mazzocca [23, Proposition 2.1] proved that $(q + t, t)$-arcs of type $(0, 2, t)$ exist in $\Pi_q$ only if $q$ is even and $t | q$. They also provided infinite families of examples in $PG(2, q)$ whenever the field $GF(q/t)$ is a subfield of $GF(q)$. It is easy to see that through each point of $T$ there passes exactly one $t$-secant. In [17] new constructions were given by Gács and Weiner, and they proved that in $PG(2, q)$ the $q/t + 1$ $t$-secants of $T$ pass through one point, called the $t$-nucleus of $T$ (for $t = 1$ and arbitrary projective plane of even order, see [19, Lemma 8.6]). Recently Vandendriessche [32] found a new infinite family with $t = q/4$.

Example 2.5 Let $T$ be a $(q + t, \tau)$-arc of type $(0, 2, \tau)$ in $PG(2, q)$. Delete $r < \tau - 1$ points from each of the $\tau$-secants of $T$. The remaining $k + q - t$ points form a $t$-semiarc with $q/\tau + 1$ $k$-secants, where $k = q - t$ and $t = q/\tau$.

Since $(q + q/2, q/2)$-arcs of type $(0, 2, q/2)$ exist, this construction gives $t$-semiarc sizes in $PG(2, q)$, $q$ even, for each $t \leq q - 4$, $t$ even. The following example is based on the combinatorial properties of subplanes.

Example 2.6 Let $\Pi, \tau$ be a Baer subplane in the projective plane $\Pi_q$, $q \geq 9$, and let $\ell$ be an extended line of $\Pi, \sqrt{q}$. Let $P$ be a set of $t \leq q - \sqrt{q} - 2$ points in $\Pi, \sqrt{q} \setminus \ell$ such that no line intersects $P$ in exactly $\sqrt{q} - 1$ points. For example a $(t, q\sqrt{q} - 2)$-arc is a good choice for $\ell$. Let $T$ be a set of $t$ points in $\ell \setminus \Pi, \sqrt{q}$. Then the pointset $S_t := (\Pi, \sqrt{q} \Delta \ell) \setminus (T \cup \Pi)$ is a $t$-semiarc of size $k + q - t$ with a $k$-secant, where $k = q - \sqrt{q} - t$.

Proof. Recall that a Baer subplane is a blocking set. Let $P \in S_t$. If $P \in \ell$, then a line through $P$ is tangent to $S_t$ if and only if it intersects $\Pi, \sqrt{q}$ in a point of $\ell$. If $P \in \Pi, \sqrt{q}$, then a line of $\Pi, \sqrt{q}$ through $P$ intersects $S_t$ in at least $\sqrt{q} - (\sqrt{q} - 2) = 2$ points, and any other line through $P$ is tangent to $S_t$ if and only if it intersects $\ell$ in $T$. Thus there are exactly $t$ tangents to $S_t$ at $P$. □
The so-called direction problem is closely related to \( t \)-semiarcs of size \( k+q-t \) having a \( k \)-secant. We briefly collect the basic definitions and some results about this problem. Consider \( \text{PG}(2, q) = \text{AG}(2, q) \cup \ell_{\infty} \).

Let \( \mathcal{U} \) be a set of points of \( \text{AG}(2, q) \). A point \( P \) of \( \ell_{\infty} \) is called a direction determined by \( \mathcal{U} \) if there is a line through \( P \) that contains at least two points of \( \mathcal{U} \). The set of directions determined by \( \mathcal{U} \) is denoted by \( D_{\mathcal{U}} \). If \( |\mathcal{U}| = q \) and \( Y_{\infty} \notin D_{\mathcal{U}} \), then \( \mathcal{U} \) can be considered as a graph of a function, and \( \mathcal{U} \cup D_{\mathcal{U}} \) is a blocking set of Rédei type. Our next construction is based on the following result of Blokhuis et al. [6] and Ball [1].

**Theorem 2.7 (Blokhuis [6], Ball [1])** Let \( \mathcal{U} \subset \text{AG}(2, q) \), \( q = p^h \), \( p \) prime, be a pointset of size \( q \). Let \( z = p^e \) be maximal having the property that if \( P \in D_{\mathcal{U}} \) and \( \ell \) is a line through \( P \), then \( \ell \) intersects \( \mathcal{U} \) in \( 0 \) (mod \( z \)) points. Then one of the following holds:

1. \( z = 1 \) and \( (q + 3)/2 \leq |D_{\mathcal{U}}| \leq q + 1 \),
2. \( \text{GF}(z) \) is a subfield of \( \text{GF}(q) \) and \( q/z + 1 \leq |D_{\mathcal{U}}| \leq (q - 1)/(z - 1) \),
3. \( z = q \) and \( |D_{\mathcal{U}}| = 1 \).

Let \( \mathcal{B} \) be a small blocking set of Rédei type in \( \text{PG}(2, q) \), \( q = p^h \), \( p \) prime, and let \( \ell \) be one of its Rédei lines. Since \( |\mathcal{B}| < 3(q + 1)/2 \), we have \( |\ell \cap \mathcal{B}| < (q + 3)/2 \). Hence the previous theorem implies that there exists an integer \( e \) such that \( e \) divides \( h \), \( 1 < p^e \leq q \) holds and each affine line intersects \( \mathcal{B} \) in \( 1 \) (mod \( p^e \)) points. Starting from \( \mathcal{B} \), we give a generalization of Example 2.1 which is also a semiarc for similar reasons.

**Example 2.8** Let \( \mathcal{B} \) be a small blocking set of Rédei type in \( \text{PG}(2, q) \) and let \( \ell \) be one of its Rédei lines. Denote by \( z = p^e \) the maximal number such that each line intersects \( \mathcal{B} \) in \( 1 \) (mod \( z \)) points and suppose \( z \geq 3 \). Let \( \mathcal{P} \) be a set of \( t \leq q - |\mathcal{B} \cap \ell| - 1 \) points in \( \mathcal{B} \setminus \ell \) such that for each line \( \ell' \) intersecting \( \mathcal{B} \) in more than one point we have \( |\ell' \cap \mathcal{P}| \neq |\ell' \cap \mathcal{B}| - 2 \). For example a \((t, z-2)\)-arc is a good choice for \( \mathcal{P} \). Also let \( \mathcal{T} \) be a set of \( t \) points in \( P \setminus \mathcal{B} \). Then the pointset \( \mathcal{S}_k := (\mathcal{B} \Delta \ell) \setminus (\mathcal{T} \cup \mathcal{P}) \) is a \( t \)-semiarc of size \( k + q - t \) with a \( k \)-secant, where \( k = 2q + 1 - |\mathcal{B}| - t \).

Note that if \( \mathcal{B} \) is a line, then Example 2.8 gives the example seen in Proposition 2.3. To characterize the examples above, we need results about the number of directions determined by a set of \( q \) affine points, and results about the extendability of a set of almost \( q \) affine points to a set of \( q \) points such that the two pointsets determine the same directions. The first theorem about the extendability was proved by Blokhuis [5]; see also Szőnyi [30].

**Theorem 2.9 ([5, Proposition 2], [30, Remark 7])** Let \( \mathcal{U} \subset \text{AG}(2, q) \), \( q \geq 3 \), be a pointset of size \( q - 1 \). Then there exists a unique point \( P \) such that the pointset \( \mathcal{U} \cup \{P\} \) determines the same directions as \( \mathcal{U} \).

Slightly extending a result of Szőnyi [30, Theorem 4], Sziklai proved the following.

**Theorem 2.10 ([28, Theorem 3.1])** Let \( \mathcal{U} \subset \text{AG}(2, q) \) be a pointset of size \( q - n \), where \( n \leq \alpha \sqrt{q} \) for some \( 1/2 \leq \alpha < 1 \). If \( |D_{\mathcal{U}}| < (q + 1)(1 - \alpha) \), then \( \mathcal{U} \) can be extended to a set \( \mathcal{U}' \) of size \( q \) such that \( \mathcal{U}' \) determines the same directions as \( \mathcal{U} \).

The three cases of the next theorem were proved by Lovász and Schrijver [25], by Gács [15], and by Gács, Lovász and Szőnyi [16], respectively.

**Theorem 2.11 ([25, 15, 16])** Let \( \mathcal{U} \) be the set of \( q \) affine points in \( \text{AG}(2, q) \), \( q = p^h \), \( p \) prime.

1. If \( h = 1 \) and \( |D_{\mathcal{U}}| = \frac{(p + 3)}{2} \), then \( \mathcal{U} \) is affinely equivalent to the graph of the function \( x \mapsto x^{\frac{p+1}{2}} \).
2. If \( h = 1 \) and \( |D_{\mathcal{U}}| > \frac{(p + 3)}{2} \), then \( |D_{\mathcal{U}}| \geq \left\lceil \frac{2(p - 1)}{3} \right\rceil + 1 \).
3. If \( h = 2 \) and \( |D_4| \geq (p^2 + 3)/2 \), then either \( |D_4| = (p^2 + 3)/2 \) and \( U \) is affinely equivalent to the graph of the function \( x \mapsto x^{p^2+1} \), or \( |D_4| \geq (p^2 + p)/2 + 1 \).

For the characterization of semiarcs in Example 2.3, we also need the following lemma.

**Lemma 2.12** Let \( z \) and \( t \) be two positive integers such that \( z \geq 3 \) and \( t \leq \sqrt{q(z-1)/z} \). Also let \( U \subset AG(2,q) \) be a set of \( q-t \) affine points and let \( E \subset F \) be two sets of directions satisfying the following properties:

1. There are at least \( t \) tangents to \( U \) with direction in \( F \) through each point of \( U \);
2. there exists a suitable set of \( t \) affine points, \( P \), such that \( U \cap P = \emptyset \) and each tangent to \( U \) with direction not in \( E \) intersects \( U \cup P \) in \( 0 \) (mod \( z \)) points.

Then \( |E| \geq t \).

**Proof.** If \( \ell \) is a tangent to \( U \) that intersects \( F \setminus E \), then \( |P \cap \ell| \equiv -1 \) (mod \( z \)). The maximum number of such tangent lines is \( \frac{t(z-1)}{(z-1)(z-2)} \). Hence at least \( (q-t)t - \frac{t(t-1)}{(z-1)(z-2)} \) tangents to \( U \) have direction in \( E \). This implies

\[
|E|q \geq (q-t)t - \frac{t(t-1)}{(z-1)(z-2)}, \quad \text{thus} \quad (|E|-t)q \geq -t^2 - \frac{t(t-1)}{(z-1)(z-2)}.
\]

If \( |E|-t \) is a negative integer, then this inequality gives \( q < t^2 \frac{(z-1)(z-2)+1}{(z-1)(z-2)} \leq t^2 z/(z-1) \), contradicting the assumption \( t \leq \sqrt{q(z-1)/z} \).

**Theorem 2.13** Let \( S_t \) be a \( t \)-semiarc in \( PG(2,q) \), \( q = p^k \), \( p \) prime, of size \( k + q - t \) and let \( \ell \) be a \( k \)-secant of \( S_t \). Then the conditions

- \( t = 1, \ q > 4 \) and \( k > (q-1)/2 \), or
- \( 2 \leq t \leq \alpha \sqrt{q} \) and \( k > \alpha (q+1) \) for some \( 1/2 \leq \alpha \leq \sqrt{(p-1)/p} \) if \( p \) is an odd prime, and \( 1/2 \leq \alpha \leq 3/2 \) if \( p = 2 \)

imply that \( S_t \) is a semiarc described in Example 2.3.

**Proof.** Take \( \ell \) as the line at infinity and let \( U = S_t \setminus \ell \subset AG(2,q) \). The directions in \( S_t \setminus \ell \) are not determined by \( U \), hence \( |D_\ell| < (q+1)(1-\alpha) \) holds for \( t \geq 2 \). We can apply Theorem 2.9 when \( t = 1 \); if \( t \geq 2 \), then the conditions of Theorem 2.10 hold since \( |U| = q-t \) and \( t \leq \alpha \sqrt{q} \). Let \( \mathcal{P} = \{ P_1, P_2, \ldots, P_t \} \) be a set of \( t \) points such that \( \mathcal{U} \cup \mathcal{P} \) determines the same directions as \( U \).

First consider the case \( t \geq 2 \). We have \( |D_\ell| < (q+1)/2 \), thus applying Theorem 2.4, we get that there exists an integer \( z = p^k \geq 3 \) such that each affine line with direction in \( D_\ell \) intersects \( \mathcal{U} \cup \mathcal{P} \) in \( 0 \) (mod \( z \)) points. We can apply Lemma 2.12 with \( F = \ell \setminus S_t \) and \( E = \ell \setminus (S_t \cup D_\ell) \) to obtain \( |E| \geq t \). On the other hand, the lines joining any point of \( E \) with any point of \( U \) are tangent to \( S_t \), thus \( |E| \leq t \). The same observation implies that each of the tangents to \( S_t \) at the points of \( U \) meets \( E \). Let \( \mathcal{B} = \mathcal{U} \cup \mathcal{P} \cup D_\ell \), which is a small blocking set of Rédei type. Let \( \ell' \neq \ell \) be a line intersecting \( \mathcal{B} \) in more than one point and let \( M = \ell' \cap \ell \). Then \( M \subset D_\ell \subset \mathcal{B} \) and \( M \notin E \). If \( |\ell' \cap \mathcal{P}| = |\ell' \cap \mathcal{B}| - 2 \), then \( \ell' \) would be a tangent to \( S_t \) at the unique point of \( \ell' \cap U \), but this is a contradiction since \( M \notin E \). We obtained Example 2.8.

If \( t = 1 \), then in the same way we get that there exists an integer \( z = p^k \geq 2 \) such that each affine line with direction in \( D_\ell \) intersects \( \mathcal{U} \cup \{ P_1 \} \) in \( 0 \) (mod \( z \)) points. If \( z \geq 3 \), then we can finish the proof as above, otherwise Theorem 2.7 implies \( |D_\ell| \geq q/2 + 1 \). Compared to \( |D_\ell| < (q+3)/2 \), we get \( |D_\ell| = q/2 + 1 \) and hence \( k = q/2 \). This means that each of the \( q-1 \) tangent lines at the points of \( U \) passes through \( P_1 \). If \( q > 4 \), then \( q-1 > q/2 + 1 \), thus at least one of these tangents would intersect \( \ell \) in \( S_t \), that is a contradiction.

Next, as a corollary of Theorem 2.11, we get the characterization of the semiival \( (t = 1) \) cases of Examples 2.9 and 2.10 in planes of prime or prime square order.
Corollary 2.14 Let $S_1$ be a semi-oval of size $k + q - 1$ in $\text{PG}(2, q)$, $3 \leq q = p^h$, $p$ prime, $h \leq 2$, and let $\ell$ be a $k$-secant of $S_1$. Then we have the following.

1. If $h = 1$ and $k > (p + 4)/3$, then there are two possibilities:
   - $k = q - 1$ and $S_1$ is the semi-oval characterized in Proposition 2.3.
   - $S_1$ is the semi-oval described in Example 2.4.

2. If $h = 2$ and $k > (p^2 - p)/2$, then there are four possibilities:
   - $k = q - 1$ and $S_1$ is the semi-oval characterized in Proposition 2.3.
   - $S_1$ is the semi-oval described in Example 2.4.
   - $S_1$ is the semi-oval described in Example 2.6.
   - $p = 2$ and $S_1$ is an oval in $\text{PG}(2, 4)$.

Proof. Consider $\ell$ as the line at infinity and let $U = S_1 \setminus \ell$. The points of $\ell \cap S_1$ are not determined directions, hence we have $k + |DU| \leq q + 1$. As the pointset $U$ has size $q - 1$, it follows from Theorem 2.9 that there exists a point $P$ such that $U \cup \{P\}$ determines the same directions as $U$.

First consider the case $h = 1$. If $k > (p + 4)/3$, then $|DU| < [2(p - 1)/3] + 1$ and thus Theorems 2.7 and 2.11 imply that $|DU| = 1$ and $U$ is contained in a line, or $|DU| = (p + 3)/2$ and $U \cup \{P\}$ is affinely equivalent to the graph of the function $x \mapsto \frac{x + 1}{x + 2}$. In the first case it is easy to see that $S_1$ is the semi-oval characterized in Proposition 2.3. In the latter case the graph of $x \mapsto \frac{x + 1}{x + 2}$ is contained in two lines, namely $[1 : 1 : 0]$ and $[1 : -1 : 0]$, and these lines are $(q - 1)/2$-secants. Hence $S_1$ is contained in the union of three lines and it has two $(q - 1)/2$-secants. These semi-ovals were characterized by Kiss and Ruff [23, Theorem 3.3]; they proved that the only possibility is the semi-oval described in Example 2.4.

Now suppose that $h = 2$. If $k > (p^2 - p)/2$, then $|DU| < (p^2 + p)/2 + 1$, thus $|DU| \in \{1, (p^2 + 3)/2\}$ or $1 < |DU| < (p^2 + 3)/2$. If $|DU| = 1$ or $|DU| = (p^2 + 3)/2$, then we can argue as before. In the remaining case it follows from Theorems 2.7 and 1.3 (or already from [29, Theorem 5.7]), that $|DU| = p + 1$ and $U \cup \{P\} \cup DU$ is a Baer subplane. If $p > 2$, then $S_1$ has exactly $p^2 - p - k$ tangents at the points of $U$, hence $k = p^2 - p - 1$ and $S_1$ is the semi-oval described in Example 2.6. Finally, if $p = 2$, then $k \geq 2$ and $|DU| = p + 1 = 3$, thus $k = 2$ and $S$ is an oval in $\text{PG}(2, 4)$.

3 Proof of the main lemma

First we collect the most important properties of the Rédei polynomial. Consider a subset $U = \{(a_i, b_i) : i = 1, 2, \ldots, |U|\}$ of the affine plane $\text{AG}(2, q)$. The Rédei polynomial of $U$ is

$$H(X, Y) = \prod_{i=1}^{|U|}(X + a_iY - b_i) = \sum_{j=0}^{|U|} h_j(Y)X^{|U| - j} \in \text{GF}(q)[X, Y],$$

where $h_j(Y)$ is a polynomial of degree at most $j$ in $Y$ and $h_0(Y) \equiv 1$. Let $H_m(X)$ be the one-variable polynomial $H(X, m)$ for any fixed value $m$. Then $H_m(X) \in \text{GF}(q)[X]$ is a fully reducible polynomial, which reflects some geometric properties of $U$.

Lemma 3.1 (Folklore) Let $H(X, Y)$ be the Rédei polynomial of the pointset $U$, and let $m \in \text{GF}(q)$. Then $X = k$ is a root of $H_m(X)$ with multiplicity $r$ if and only if the line with equation $Y = mX + k$ meets $U$ in exactly $r$ points.

We need another result about polynomials. For $r \in \mathbb{R}$, let $r^+ = \max\{0, r\}$. 
Theorem 3.2 (Szőnyi–Weiner Lemma, [31, Corollary 2.4], [18, Appendix, Result 6]) Let $f$ and $g$ be two-variable polynomials in $GF(q)[X,Y]$. Let $d = \deg f$ and suppose that the coefficient of $X^d$ in $f$ is non-zero. For $y \in GF(q)$, let $h_y = \deg\gcd(f(X,y),g(X,y))$, where $\gcd$ denotes the greatest common divisor of the two polynomials in $GF(q)[X]$. Then for any $y_0 \in GF(q)$,

$$
\sum_{y \in GF(q)} (h_y - h_{y_0})^+ \leq (\deg f(X,Y) - h_{y_0})(\deg g(X,Y) - h_{y_0}).
$$

A partial cover of $PG(2,q)$ with $h > 0$ holes is a set of lines in $PG(2,q)$ such that the union of these lines cover all but $h$ points. We will also use the dual of the following result due to Blokhuis, Brouwer and Szőnyi [8].

Theorem 3.3 ([8, Proposition 1.5]) A partial cover of $PG(2,q)$ with $h > 0$ holes, not all on one line if $h > 2$, has size at least $2q - 1 - h/2$.

Lemma 3.4 Let $S$ be a set of points in $PG(2,q)$, let $\ell$ be a $k$-secant of $S$ with $2 \leq k \leq q$ and let $1 \leq t \leq q - 3$ be an integer. Suppose that through each point of $\ell \cap S$ there pass exactly $t$ tangent lines to $S$. Denote by $s$ the size of $S$ and let $s = k + q - t + \varepsilon$. Let $A(n)$ be the set of those points in $\ell \setminus S$ through which there pass at most $n$ skew lines to $S$. Then the following hold.

- If $t = 1$ and
  1. $\varepsilon < \frac{2}{t} - 1$, then the $k$ tangent lines at the points of $S \cap \ell$ and the skew lines through the points of $A(2)$ belong to a pencil (hence $A(2) \setminus A(1)$ is empty),
  2. if $\varepsilon < \frac{2k}{t} - 2$, then the $k$ tangent lines at the points of $S \cap \ell$ either belong to two pencils or they form a dual $k$-arc. If $k < q$, then the skew lines through the points of $A(2)$ belong to the same pencils or dual $k$-arc.
- If $t \geq 2$ and $k > q - \frac{t}{2} + 1$, then
  3. if $\varepsilon < \frac{k}{t+1} - \frac{1}{2}$, then the $kt$ tangent lines at the points of $S \cap \ell$ and the skew lines through the points of $A(t+1)$ belong to $t$ pencils whose carriers are not on $\ell$ (hence $A(t+1) \setminus A(t)$ is empty),
  4. if $\varepsilon < \frac{k}{t+1} - 1$ and $t \leq \sqrt{q}$, then the skew lines at the points of $S \cap \ell$ belong to $t+1$ pencils whose carriers are not on $\ell$. If $k < q$, then the skew lines through the points of $A(t+1)$ belong to the same pencils.

Proof. First we introduce some notation. Let $\mathcal{H}$ be any subset of points of the line $\ell$. We define the line-set $L_{\mathcal{H}}$ as $L_{\mathcal{H}} = \{r \in PG(2,q) : r \cap \ell \subseteq (S \cap \ell) \cup A(t+1) \setminus \mathcal{H}, r \cap (S \setminus \ell) = \emptyset\}$, that is the set of tangent lines to $S$ at the points of $S \cap \ell$ together with the set of skew lines to $S$ through the points of $A(t+1)$, except those lines that intersect $\ell$ in a point of $\mathcal{H}$. For each point $P \in PG(2,q) \setminus \ell$ we define the $\mathcal{H}$-index of $P$, in notation $\ind_{\mathcal{H}}(P)$, as the number of lines of $L_{\mathcal{H}}$ that pass through $P$. Also, let $k_{\mathcal{H}} = |(\ell \cap S) \setminus \mathcal{H}|$, $a_{\mathcal{H}} = |A(t+1) \setminus \mathcal{H}|$, and let $\delta_{\mathcal{H}}$ be the number of skew lines through the points of $A(t+1) \setminus \mathcal{H}$. If $\mathcal{H} = \emptyset$, we omit the prefix and the subscript $\emptyset$, e.g. we write $\mathcal{L}$ and $\ind(\emptyset)$ instead of $L_{\emptyset}$ and $\ind_{\emptyset}(P)$. If $\mathcal{H} = \{Q\}$ for some $Q \in \ell$, we write $\mathcal{L}$ as prefix or subscript instead of $\{Q\}$. If $P = (m)$, we write e.g. $\ind_{\emptyset}(m)$ instead of $\ind((m))$. Note that if $Q \in \ell \setminus (S \cup A(t+1))$, then $\ind_{Q}(P) = \ind_{\emptyset}(P)$ for all $P \in PG(2,q) \setminus \ell$.

If $P \in S \setminus \ell$, then the $\mathcal{H}$-index of $P$ is 0 for any $\mathcal{H} \subseteq \ell$. Let $P \in PG(2,q) \setminus (\ell \cup S)$ be an arbitrary point and $Q \in \ell$. Then we use the system of reference so that $P \in Y_\infty \setminus \{Y_\infty\}$, $Q = Y_\infty$ and the points of $(S \cap \ell) \cup A(t+1) \setminus \{Q\}$ are affine points on the line $[1:0:0]$. Then $P = (y_0)$ for some $y_0 \in GF(q)$. Let $\{(0,c_1),\ldots,(0,c_{k+aq})\}$ be the set of points of $(S \cap \ell) \cup A(t+1) \setminus \{Q\}$, let $D = (\ell_\infty \setminus \{Y_\infty\}) \cap S$, $|D| = d$ and let $U = S \setminus (\ell \cup \ell_\infty) = \{(a_1,b_1),\ldots,(a_{q-d-k},b_{q-d-k})\}$. Consider the Rédei polynomials of $(S \cap \ell) \cup A(t+1) \setminus \{Q\}$ and $U$. We denote them by $f(X,Y) = \prod_{j=1}^{k_2+aq}(X - c_j)$ and $g(X,Y) = \prod_{j=1}^{q-d-k}(Ya_j + X - b_j)$, respectively. Let $\overline{D} = \ell_\infty \setminus (D \cup \{Y_\infty\})$. Then for any point $(y) \in \overline{D}$,

$$h_y := \deg\gcd(f(X,y),g(X,y)) = k_Q + a_Q - \ind_{Q}(y).$$

7
Applying the Szőnyi–Weiner Lemma for the polynomials \( f(X,Y) \) and \( g(X,Y) \) we get
\[
\sum_{(y) \in \mathcal{P}} (\text{ind}_Q(y_0) - \text{ind}_Q(y)) \leq \sum_{(y) \in \mathcal{G}(q)} (\text{ind}_Q(y_0) - \text{ind}_Q(y))^+ \leq \text{ind}_Q(y_0)(s - d - k - kQ - aQ + \text{ind}_Q(y_0)).
\]

After rearranging it we obtain
\[
0 \leq \text{ind}_Q(y_0)^2 - \text{ind}_Q(y_0)(q + k + kQ + aQ - s) + \sum_{(y) \in \mathcal{P}} \text{ind}_Q(y). \tag{1}
\]

Because \( \sum_{(y) \in \mathcal{P}} \text{ind}_Q(y) = kQ + \delta_Q \), hence
\[
0 \leq \text{ind}_Q(y_0)^2 - \text{ind}_Q(y_0)(q + k + aQ + t - \varepsilon) + kQt + \delta_Q. \tag{2}
\]

First we prove parts 1, 3 and 4 simultaneously. If we choose \( Q \) so that \( Q \in \ell \setminus \mathcal{S} \), then \( kQ = k \). Thus the condition \( \varepsilon < \frac{k}{(k + 1)} - 1 \) and the obvious fact \( \delta_Q \leq (t + 1)aQ \) imply that (2) gives a contradiction for \( t + 1 \leq \text{ind}_Q(y_0) \leq k + aQ - \varepsilon - 1 \). We say that a point \( P \) has large \( Q \)-index if \( \text{ind}_Q(P) \geq k + aQ - \varepsilon \) holds.

We are going to prove that each line of \( \mathcal{L}_Q \) contains a point with large \( Q \)-index. First let \( \ell' \in \mathcal{L}_Q \) be a tangent to \( \mathcal{S} \) at a point \( T \in \ell \cap \mathcal{S} \). Suppose that each point of \( \ell' \) has index at most \( t \). Then
\[
\sum_{P \in \ell' \setminus T} \text{ind}_Q(P) \leq tq. \tag{3}
\]

On the other hand, the sum on the left-hand side is at least \((k - 1)t + q\), contradicting our assumption on \( k \). Similarly, if \( \ell'' \) is a skew line to \( \mathcal{S} \) passing through a point \( T \in A(t + 1) \setminus \{Q\} \), then the right-hand side of (3) remains the same and the left-hand side is at least \( kt + q \), that is a contradiction, too. Hence there are at least \( t \) points with large \( Q \)-index.

Suppose that there are more than \( t \) points with large \( Q \)-index and let \( R_1, R_2, \ldots, R_{t+1} \) be \( t + 1 \) of them. Then
\[
(t + 1)(k + aQ - \varepsilon) \leq \sum_{j=1}^{t+1} \text{ind}_Q(R_j) \leq \binom{t + 1}{2} + tk + (t + 1)aQ.
\]

This is a contradiction if \( \varepsilon < \frac{k}{(k + 1)} - \frac{t}{7} \), which holds in parts 1 and 3. Regarding part 4, if there would be more than \( t + 1 \) points with large \( Q \)-index, then \( \varepsilon < \frac{k}{(k + 1)} - \frac{t + 1}{7} \) yields a contradiction. The condition on \( k \) and \( t \leq \sqrt{t} \) imply \( \frac{k}{(k + 1)} - 1 \leq \frac{t + 1}{7} - \frac{t}{7} \).

If \( k + |A(t + 1)| < q + 1 \), then let \( Q \) be any point of \( \ell \setminus (\mathcal{S} \cup A(t + 1)) \). Thus the lines of \( \mathcal{L}_Q = \mathcal{L} \) are contained in \( t \) pencils (or \( t + 1 \) in part 4) whose carriers have large \( Q \)-index. In this case parts 1, 3 and 4 are proved. So from now on we assume \( k + |A(t + 1)| = q + 1 \). Then we set \( Q \in A(t + 1) \). To finish the proof of parts 1, 3 and 4, we have to show that the lines of \( \mathcal{L} \setminus \mathcal{L}_Q \) also belong to these pencils. Recall that in case of part 4, we assume \( k < q \).

If \( k = q \), then let \( Q \) be the unique point contained in \( A(t + 1) \). The \( kt \) tangents at the points of \( \ell \cap \mathcal{S} \) are contained in \( t \) pencils having carriers with large \( Q \)-index. Denote the set of these carriers by \( \mathcal{P} = \{G_1, G_2, \ldots, G_t\} \). If \( t = 1 \), then through \( G_1 \) there pass \( q \) tangent lines, hence the points of \( \mathcal{S} \setminus \ell \) are contained in the line \( G_1Q \). Thus through \( Q \) there pass only two non-skew lines, \( \ell \) and \( G_1Q \).

The condition \( q - 3 \geq t + 1 = 2 \varepsilon \geq 2 + 2 \), hence \( Q \notin A(2) \), a contradiction. If \( t > 1 \), then \( \mathcal{P} \) is contained in a line through \( Q \) and \( \mathcal{S} \setminus \ell \) is a \((q - t)\)-sectant of \( \mathcal{S} \). Again \( q - 3 \geq t \) implies that through \( Q \) there pass more than \( t + 1 \) skew lines, hence \( Q \notin A(t + 1) \), a contradiction.

If \( k < q \), then let \( Q_1 \) and \( Q_2 \) be two distinct points of \( A(t + 1) \). As seen before, the lines of \( \mathcal{L}_Q \), are blocked by the points with large \( Q_i \)-index for \( i = 1, 2 \), hence, by \( \mathcal{L}_{Q_1} \cup \mathcal{L}_{Q_2} = \mathcal{L} \), it is enough to show that the set of points with large \( Q_1 \)-index is the same as the set of points with large \( Q_2 \)-index. If a point has large \( Q_1 \)-index, then its \( Q_2 \)-index is at least \( k + aQ_1 - \varepsilon = q - \varepsilon \), while the other points have \( Q_2 \)-index at most \( t \) for \( i = 1, 2 \). The inequality \( |\text{ind}_{Q_1}(P) - \text{ind}_{Q_2}(P)| \leq 1 \) obviously holds, thus it is enough to show that \( q - \varepsilon - t > 1 \), which follows from the assumptions \( \varepsilon < \frac{k}{(k + 1)} - 1 \) and \( t \leq q - 3 \).
Finally, we prove part 2. At this part sometimes we will choose \( Q \) from \( \ell \cap S \), so from now on on \( kQ \) is not necessarily equal to \( k \). Let \( P \) be the point of \( PG(2, q) \setminus (\ell \cup S) \) whose index is to be estimated. If \( k + |A(2)| < q + 1 \), then let \( Q \) be any point of \( \ell \setminus (S \cup A(2)) \) and let \( H = \emptyset \). If \( k + |A(2)| = q + 1 \) and \( k = q \), then let \( Q \) be the unique point contained in \( A(2) \) and let \( H = \{Q\} \), otherwise let \( Q \) be any point of \( \ell \) such that \( PQ \) intersects \( S \setminus \ell \) and let \( H = \emptyset \). Note that since \( S \setminus \ell \) is not empty, \( Q \) can be chosen in this way and \( ind_H(P) \) does not depend on the choice of \( Q \). In all cases we investigate the line-set \( L_H \), and we have \( ind_Q(P) = ind_H(P) \). If

\[
\frac{2kQ}{3} - 2 + \frac{aq}{3} > \varepsilon
\]

holds, then \( 2 \) gives a contradiction for \( 3 \leq \text{ind}_H(P) \leq kQ + aQ - 2 - \varepsilon \). In all cases the left-hand side of \( 1 \) is at least \( 2k/3 - 2 \), hence the corresponding lines either form a dual arc or there is a point \( G \) with \( H \)-index at least \( kQ + aQ - 1 - \varepsilon \).

In the latter case let \( B = (\ell \setminus (S \cup A(2))) \cup (S \setminus \ell) \cup G \) and denote by \( h \) the number of lines of \( PG(2, q) \) not blocked by \( B \). It is easy to see that, apart from \( \ell \), \( B \) blocks all but at most \((k + 2|A(2)|) - (kQ + aQ - 1 - \varepsilon)\) lines of \( PG(2, q) \). If \( k + |A(2)| < q + 1 \), then \( B \) blocks \( \ell \) and \( kQ + aQ = k + |A(2)| \), hence \( h \leq |A(2)| + 1 + \varepsilon \). If \( k + |A(2)| = q + 1 \), then \( B \) does not block \( \ell \) and \( kQ + aQ = q \), thus \( h \leq |A(2)| + 3 + \varepsilon \).

Suppose to the contrary that these \( h \) lines do not pass through one point. Then from Theorem 5.3 we have \( |B| \geq 2q - h/2 \) or, equivalently,

\[
q + 1 - (k + |A(2)|) + (q - 1 + \varepsilon) + 1 \geq 2q - h/2.
\]

Rearranging it we obtain \( \varepsilon \geq k + |A(2)| - 2 - h/2 \). If \( k + |A(2)| < q + 1 \), then this would imply \( \varepsilon \geq 2k/3 - 5/3 + |A(2)|/3 \). If \( k + |A(2)| = q + 1 \), then \( \varepsilon \geq q + k)/3 - 2 \) would follow. Both cases yield a contradiction because of our assumption on \( \varepsilon \). Hence the corresponding lines can be blocked by \( G \) and one more point.

\[\square\]

**Corollary 3.5** Let \( S_1 \) be a semioval in \( PG(2, q) \) and let \( \ell \) be a \( k \)-secant of \( S_1 \). If \( |S_1| < q + \frac{3k}{2} - 2 \), then the \( k \) tangent lines at the points of \( S_1 \setminus \ell \) belong to a pencil. If \( |S_1| < q + \frac{3k}{2} - 3 \), then the \( k \) tangent lines at the points of \( S_1 \setminus \ell \) either belong to two pencils or they form a dual \( k \)-arc.

If \( k = q - 1 \), then we get a stronger result than the previous characterization of Kiss [21, Corollary 3.1].

**Corollary 3.6** Let \( S_1 \) be a semioval in \( PG(2, q) \). If \( S_1 \) has a \((q - 1)\)-secant \( \ell \) and \( |S_1| < \frac{kQ}{2} - \frac{q}{2} \) holds, then \( S_1 \) is contained in a vertexless triangle and it has two \((q - 1)\)-secants.

**Proof.** Let \( \ell \setminus S_1 = \{A, B\} \). It follows from Corollary 3.5 that the tangents at the points of \( S_1 \setminus \ell \) are contained in a pencil with carrier \( C \). Thus \( S_1 \) is contained in the sides of the triangle \( ABC \). Suppose to the contrary that each of \( AC \) and \( BC \) intersects \( S_1 \) in less than \( q - 1 \) points. Then there exist \( P, Q \) such that \( P \in AC \setminus (S_1 \cup \{A, C\}) \) and \( Q \in BC \setminus (S_1 \cup \{B, C\}) \). The point \( E := PQ \cap AB \) is in \( S_1 \) and \( PQ \) is a tangent to \( E \). This is a contradiction since \( C \notin PQ \).

\[\square\]

Since \( t < q \) implies \( k \leq q + 1 - t \), the assumption \( q - \frac{2}{t} + 1 < k \) in Lemma 3.3 can hold only if \( t < \sqrt{q} \).

**Corollary 3.7** Let \( S_1 \) be a \( t \)-semiarc in \( PG(2, q) \), \( q \geq 7 \), with \( 1 < t < \sqrt{q} \). Suppose that \( S_1 \) has a \( k \)-secant \( \ell \) and \( q - \frac{2}{t} + 1 < k \) holds. If \( |S_1| < (q - t + k) + \frac{kQ}{t} - 1 \), then the \( kt \) tangent lines at the points of \( S_1 \setminus \ell \) belong to \( t + 1 \) pencils. If \( |S_1| < (q - t + k) + \frac{kQ}{t} - \frac{q}{2} \), then the \( kt \) tangent lines at the points of \( S_1 \setminus \ell \) belong to \( t \) pencils.

**Remark 3.8** Theorem 2 follows from Lemma 3.4 with \( t = 1 \) and \( \varepsilon = 0 \). To see this, let \( S = U \cup (\ell_\infty \setminus D_\infty) \). Then through each point of \( \ell_\infty \cap S \), there passes a unique tangent to \( S \). According to Lemma 3.4, these tangent lines are contained in a pencil, whose carrier can be added to \( U \).
Example 3.9 It follows from Theorem 3.3 that a cover of the complement of a conic in PG(2, q), q odd, by external lines, contains at least 3(q−1)/2 lines, see [8, Proposition 1.6]. Blokhuis et al. also remark that this bound can be reached for q = 3, 5, 7, 11 and there is no other example of this size for q < 25, q odd. Now, let ℓ be a tangent to a conic C at the point P ∈ C and let U be a set of 3(q−1)/2 interior points of the conic such that these points block each non-tangent line. From the dual of the above result we know that such set of interior points exists at least when q = 3, 5, 7, 11. Let S = U ∪ ℓ \ {P}. Then the tangents to S at the points of ℓ ∩ S obviously do not pass through one point and this shows that part 1 of Lemma 3.3 is sharp if k = q and q = 5, 7, 11.

Example 3.10 ([23 Theorem 3.2]) In PG(2, 8) there exists a semioval S1 of size 15, contained in a triangle without two of its vertices. The side opposite to the one vertex contained in S1 is a 6-secant and the other two sides are 5-secants. The tangents at the points of the 6-secant do not pass through one point. Hence Corollary 3.2 is sharp at least for q = 8.

In the following we give some examples for small t-semiarcs with long secants in the cases t = 1, 2, 3 such that the tangents at the points of a long secant do not belong to t pencils. These assertions can easily be proved using Menelaus’ Theorem. Denote by GF(q)⁺ and GF(q)× the additive and multiplicative groups of the field GF(q), q = p^k, p prime, respectively, and by A ⊕ B the direct sum of the groups A and B.

Example 3.11 ([23 p. 104]) Consider GF(q), q square, as the quadratic extension of GF(√q) by i. Then the pointset S₁ = {1 : 0 : 0} ∪ {1 : 0 : 1} ∪ {0 : 0 : 1} \ {Y_∞ : (0 : 1 : 1), (1 : 1 : 1), (1 : s + i : 0) : s ∈ GF(√q)} is a semioval in PG(2, q) with three (q - √q)-secants.

Example 3.12 Let GF(q)⁺ = A ⊕ B, where A and B are proper subgroups of GF(q)⁺ and let X = A ∪ B. The pointset S₂ = {(0 : s : 1), (1 : s : 1), (1 : s : 0) : s ∈ GF(q) \ X}, is a 2-semiarc in PG(2, q) with three (q + 1 - |A| - |B|)-secants. Note that 2√q ≤ |A| + |B| ≤ q/p + p.

Similarly, let GF(q)× = A ⊕ B and X = A ∪ B, where A and B are proper subgroups of GF(q)×. The pointset S₃ = {(0 : s : 1), (s : 0 : 1), (1 : s : 0) : s ∈ GF(q) \ (X ∪ {0})}, is a 3-semiarc in PG(2, q) with three (q - |A| - |B|)-secants. Note that 2√q ≤ |A| + |B| ≤ (q + 3)/2.

4 Semiarc and blocking sets

First we associate a blocking set to each semiarc.

Lemma 4.1 Let Π_q be a projective plane of order q, let k ≤ q and 1 ≤ t ≤ q - 3 be integers. Let S be a set of k + q - t + ε points in Π_q such that the line ℓ is a k-semiarc of S. Let A(n) be the set of those points in ℓ ∩ S through which there pass at most n skew lines to S. Suppose that through each of the k points of ℓ ∩ S there pass exactly t tangent lines to S, and also suppose that these kt tangent lines and the skew lines through the points of A(n) belong to n pencils. Let P be the set of carriers of these pencils and assume that P ∩ ℓ = ∅. Define the pointset B_n(S, ℓ) in the following way:

\[ B_n(S, ℓ) := (ℓ \ A(n) ∪ S) \cup (S \ \ell) ∪ P. \]

Then B_n(S, ℓ) has size 2q + 1 + ε - n - t - k - |A(n)|. If ℓ ∩ B_n(S, ℓ) = ∅ (that is ℓ ⊆ A(n) ∪ S), then B_n(S, ℓ) is an affine blocking set in the affine plane Π_q \ ℓ; otherwise B_n(S, ℓ) is a blocking set in Π_q. In the latter case the points of ℓ ∩ B_n(S, ℓ) are essential points.

Proof. Let ℓ′ ≠ ℓ be any line in Π_q and let E be the point ℓ ∩ ℓ′. If ℓ′ meets (ℓ \ (A(n) ∪ S)) ∪ (S \ ℓ), then ℓ′ is blocked by B_n(S, ℓ). Otherwise ℓ′ is a tangent to S at a point of ℓ ∩ S or ℓ′ is a skew line to S that intersects A(n). In both cases ℓ′ is blocked by P, hence it is also blocked by B_n(S, ℓ).

If ℓ ⊆ A(n) ∪ S, then B_n(S, ℓ) is an affine blocking set in the affine plane Π_q \ ℓ. Otherwise ℓ is blocked by ℓ \ (A(n) ∪ S) and hence B_n(S, ℓ) is a blocking set in Π_q. In the latter case through each point of
Hence if we consider $PG(2, q)$ with $k = \max(4, \frac{n}{2} - 1)$, then $S$ is an affine blocking set if and only if $\ell \in B_n(S, \ell)$. Now we are ready to prove our main characterization theorems for small semiarcs with a long secant. We distinguish two cases, as the results on blocking sets in $PG(2, q)$ were given in [12]. Here we cite just a particular case. 

**Theorem 4.6** Let $S_t$ be a $t$-semiarc in $PG(2, p)$, $p$ prime, and let $\ell$ be a $k$-secant of $S_t$. 

**Example 4.2** If $S_1$ is the semioval described in Example 2.7 and $\ell$ is one of the $(q-1)/2$-secants of $S_1$, then $S_1$ and $\ell$ satisfy the conditions of Lemma 4.1 with $n = 1$ and the obtained blocking set $B_1(S_1, \ell)$ is a minimal blocking set called the projective triangle (see e.g. [19, Lemma 13.6]). 

**Lemma 4.3** Let $S_t$ be a $t$-semiarc in $PG(2, q)$, $q = p^h$, $p$ prime, with $t \leq \sqrt{2q/3}$. Let $\ell$ be a $k$-secant of $S_t$, and suppose that $S_t$ and $\ell$ satisfy the conditions of Lemma 4.1 with $n = t$. If $p = 2$ and $\varepsilon < k - \frac{1}{3}(q-1)$, or $p$ is odd and $\varepsilon < k - \frac{1}{2}(q-1)$, then $|A(t)| \geq t$. 

**Proof.** In both cases we have $|B_t(S_t, \ell)| = 2q + \varepsilon - k - |A(t)| < 3(q+1)/2$, hence $B_t(S_t, \ell)$ is a small blocking set. Let $B$ be the unique (cf. Theorem 1.2) minimal blocking set contained in $S_t$ and let $\ell$ be the exponent of $B$ (cf. Theorem 1.2). Note that if $\varepsilon < k - \frac{1}{3}(q-1)$, then $p^\varepsilon \geq 8$ follows from Theorem 1.2. Also $p^\varepsilon \geq 3$ holds when $p$ is odd. 

The points of $\ell \cap B_t(S_t, \ell)$ are essential points of $B_t(S_t, \ell)$ hence $\ell \cap B_t(S_t, \ell) = \ell \cap B$. The size of $B \cap (S_t \setminus \ell)$ is at least $q - t$; let $U$ be $q - t$ points from this pointset. Consider $\ell$ as the line at infinity and apply Lemma 2.12 with $E = A(t)$, $F = \ell \setminus S_t$, $z = p^\varepsilon$ and with $P$ defined as in Lemma 4.1. Note that $t \leq \sqrt{2q/3} \leq \sqrt{q(z-1)/z}$. Through each point of $U$ there pass $t$ tangents to $S_t$. These lines are also tangents to $U$ and they have direction in $F$. If $\ell'$ is one of these tangents, then $B \cap \ell' \equiv 1 \pmod z$ thus if $\ell'$ has direction in $F \setminus E$, then $(P \cup U) \cap \ell' \equiv 0 \pmod z$. Hence the two required properties of Lemma 2.12 hold, thus $|A(t)| \geq t$. 

Semiars with two long secants were investigated by Csajbók. He proved the following. 

**Lemma 4.4** ([12, Theorem 13]) Let $\Pi_q$ be a projective plane of order $q$, $1 < t < q$ an integer and $S_t$ be a $t$-semiarc in $\Pi_q$. Suppose that there exist two lines $\ell_1$ and $\ell_2$ such that $|\ell_1 \setminus (S_t \cup \ell_2)| = n$ and $|\ell_2 \setminus (S_t \cup \ell_1)| = m$. If $\ell_1 \cap \ell_2 \notin S_t$, then $n = m = t$ or $q \leq \min \{n, m\} + 2nm/(t-1)$.

The complete characterization of $t$-semiars in $PG(2, q)$ with two $(q - t)$-secants whose common point is not in the semiarc was also given in [12]. Here we cite just a particular case.

**Theorem 4.5** ([12, Theorem 22]) Let $S_t$ be a $t$-semiarc in $PG(2, q)$, $q = p^h$, $p$ prime, with two $(q - t)$-secants such that the point of intersection of these secants is not contained in $S_t$, and let $t \leq q - 2$. Then the following hold.

1. If $\gcd(q, t) = 1$, then $S_t$ is contained in a vertexless triangle.
2. If $\gcd(q, t) = 1$ and $\gcd(q - 1, t - 1) = 1$, then $S_t$ consists of the symmetric difference of two lines with $t$ further points removed from each line.
3. If $\gcd(q - 1, t) = 1$, then $S_t$ is contained either in a vertexless triangle, or in the union of three concurrent lines with their common point removed.

Now we are ready to prove our main characterization theorems for small semiars with a long secant. We distinguish two cases, as the results on blocking sets in $PG(2, q)$ are stronger if $q$ is a prime. 

**Theorem 4.6** Let $S_t$ be a $t$-semiarc in $PG(2, p)$, $p$ prime, and let $\ell$ be a $k$-secant of $S_t$. 

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1. If $t = 1$, $p \geq 5$ and $k \geq \frac{p-1}{2}$, then
   - $S_1$ is contained in a vertexless triangle and has two $(p-1)$-secants, or
   - $S_1$ is projectively equivalent to Example 2.4, or
   - $|S_1| \geq \min\left\{\frac{3k}{t} + p - 2, 2k + \frac{p+1}{2}\right\}$.

2. If $t = 2$, $p \geq 7$ and $k \geq \frac{p+3}{2}$, then
   - $S_2$ consists of the symmetric difference of two lines with two further points removed from each line, or
   - $|S_2| \geq \min\left\{\frac{4k}{t} + p - 3, 2k + \frac{p+1}{2}\right\}$.

3. If $3 \leq t < \sqrt{p}$, $p \geq 23$ and $k > p - \frac{t}{2} + 1$, then
   - $S_t$ is contained in a vertexless triangle and has two $(p-t)$-secants, or
   - $|S_t| \geq k \frac{t+2}{t} + p - t - 1$.

**Proof of part 1.** Assume that $|S_1| < \min\left\{\frac{3k}{t} + p - 2, 2k + \frac{p+1}{2}\right\}$. If $|S_1| = k + p - 1 + \varepsilon$, then we have
   - $\varepsilon < \min\left\{\frac{k}{t} - 1, k - \frac{p-3}{2}\right\}$, hence Lemma 3.3 implies that the tangents at the points of $\ell \cap S_1$ and the skew lines through the points of $A(1)$ are contained in a pencil with carrier $P$. Construct the small blocking set $B_1(S_1, \ell)$ as in Lemma 1.1 with $n = 1$. The size of $B_1(S_1, \ell)$ is $2p + 1 + \varepsilon - k - |A(1)| < 3(p+1)/2 + 1$, thus Theorem 1.3 implies that $B_1(S_1, \ell)$ either contains a line, or it is a minimal blocking set of size $3(p+1)/2$, each of its points has exactly $(p-1)/2$ tangents, and $\varepsilon = k - \frac{p-1}{2}$.

In the first case, let $\ell_1$ be the line contained in $B_1(S_1, \ell)$. Since no $p$ points of $S_1$ can be collinear by Theorem 2.1, we have that $\ell_1$ is a $(p-1)$-secant of $S_1$. The assertion now follows from Corollary 3.6. In the latter case, since the point $P \in B_1(S_1, \ell)$ has at least $k$ tangents, we have $k = (p-1)/2$ and hence $\varepsilon = 0$. It follows from Corollary 2.14 that $S_1$ is projectively equivalent to the projective triangle.

**Proof of part 2.** Assume that $|S_2| < \min\left\{\frac{4k}{t} + p - 3, 2k + \frac{p+1}{2}\right\}$. If $|S_2| = k + p - 2 + \varepsilon$, then we have $\varepsilon < \min\left\{\frac{k}{t} - 1, k - \frac{p-3}{2}\right\}$, hence Lemma 3.4 implies that the tangents at the points of $\ell \cap S_2$ and the skew lines through the points of $A(2)$ are contained in two pencils with carriers $P_1$ and $P_2$. Construct the blocking set $B_2(S_2, \ell)$ as in Lemma 1.1. Theorem 1.3 implies that $B_2(S_2, \ell)$ either contains a line or it is a minimal blocking set of size $3(p+1)/2$ and each of its points has exactly $(p-1)/2$ tangents.

Suppose that $B_2(S_2, \ell)$ contains a line $\ell_1$. Since $S_2$ cannot have more than $p-2$ collinear points, we have that $\ell_1$ is a $(p-2)$-secant of $S_2$. Similarly we can construct $B_2(S_2, \ell_1)$ and get that there is a line $\ell_2 \neq \ell_1$ and $\ell_2 \cap \ell_1 \notin S_2$, which is also a $(p-2)$-secant, or $B_2(S_2, \ell_1)$ is a minimal blocking set of size $3(p+1)/2$.

In the first case Theorem 1.3 implies that $S_2$ consists of the symmetric difference of two lines with two further points removed from each line. If this is not the case, then $B_2(S_2, \ell)$ or $B_2(S_2, \ell_1)$ is a minimal blocking set of size $3(p+1)/2$. We may suppose that $B_2(S_2, \ell)$ is such a blocking set, hence both $P_1$ and $P_2$ have exactly $(p-1)/2$ tangent lines. But this is a contradiction since these two points together have at least $2k$ tangents, which is greater than $p$.

**Proof of part 3.** Assume that $|S_t| < k \frac{t+2}{t} + p - t - 1$. Then $|S_t| = k + p - t + \varepsilon$, where $\varepsilon < \frac{1}{t+1} - 1 < k - \frac{p-1}{2}$, hence Lemma 3.3 implies that the tangents at the points of $\ell \cap S_t$ are contained in $t+1$ pencils. Construct the small blocking set $B_{t+1}(S_t, \ell)$ as in Lemma 1.1. Theorem 1.3 implies that $B_{t+1}(S_t, \ell)$ contains a line $\ell_1$. Note that $\ell_1 \cap \ell \notin S_t$. Since $S_t$ cannot have more than $p-t$ collinear points we have that $\ell_1$ is a $(p-t)$-secant or a $(p-t-1)$-secant of $S_t$.

If $\ell$ were a $(p-t)$-secant or a $(p-t-1)$-secant, then Lemma 1.3 would imply that both $\ell$ and $\ell_1$ are $(p-t)$-secants. Otherwise $\ell \cap S_t < |\ell| \cap S_t$ and hence the conditions hold also with $\ell_1$ instead of $\ell$. Constructing $B_{t+1}(S_t, \ell_1)$ we get that there is a line $\ell_2 \neq \ell_1$ such that $\ell_2 \cap \ell_1 \notin S_t$ and $\ell_2$ is either a $(p-t)$-secant or a $(p-t-1)$-secant. Again Lemma 1.3 implies that both $\ell_1$ and $\ell_2$ are $(p-t)$-secants. Since $\gcd(t, p) = 1$, Theorem 1.3 implies that $S_t$ is contained in a vertexless triangle.

If the projective plane $\Pi_q$ contains a Baer subplane, then there exist $t$-semiarcs of size $(q - \sqrt{q} - t) + (q - t)$ with a $(q - \sqrt{q} - t)$-secant, see Example 2.6. The first part of the following theorem states that if a line
\( \ell \) intersects a \( t \)-semiarc \( S_t \) in \( \text{PG}(2,q) \), \( q \) square, in at least \( k \geq q - \sqrt{q} - t \) points, \( t \) is not too large and the size of \( S_t \) is close to \( k + q - t \), then either \( S_t \) is the semiarc described in Example 2.6 or \( S_t \) has two \((q-t)\)-secants.

**Theorem 4.7** Let \( S_t \) be a \( t \)-semiarc in \( \text{PG}(2,q) \), \( q = p^h \), \( h > 1 \) if \( p \) is an odd prime and \( h \geq 6 \) if \( p = 2 \). Suppose that \( S_t \) has a \( k \)-secant \( \ell \) with

\[
k \geq \begin{cases} 
q - \sqrt{q} - t & \text{if } h \text{ is even}, \\
q - c_p q^{2/3} - t & \text{if } h \text{ is odd},
\end{cases}
\]

where \( c_p = 2^{-1/3} \) for \( p = 2, 3 \) and \( c_p = 1 \) for \( p > 3 \) (cf. Theorem 1.3). Then the following hold.

1. In case of \( h = 2d \) and \( t < \left( \frac{3}{2} - 1 \right) k/(t+1) - t/2 \) for \( t \geq 2 \),
   - if \( |S_t| < 2k + \sqrt{q} \), then \( S_t \) has two \((q-t)\)-secants whose point of intersection is not in \( S_t \);
   - if \( |S_t| = 2k + \sqrt{q} \) and \( q > 9 \), then either \( S_t \) has two \((q-t)\)-secants whose point of intersection is not in \( S_t \), or \( S_t \) is as in Example 2.6.

2. If \( h = 2d + 1 \), \( t < q + c_p q^{2/3} \) and \( t < q^{1/3} - 3/2 \) (or \( t < (2q)^{1/3} - 2 \) when \( p = 2, 3 \)), then \( S_t \) has two \((q-t)\)-secants whose point of intersection is not in \( S_t \).

**Proof.** To apply Lemma 3.1, we need \( k > q - \frac{t}{2} + 1 \); furthermore, \( \varepsilon < k/2 - 1 \) for \( t = 1 \) and \( \varepsilon < k/(t+1) - t/2 \) for \( t \geq 2 \). Let us consider the condition on \( k \); that on \( \varepsilon \) we treat later. If \( q \) is a square, then \( k \geq q - \sqrt{q} - t > q - \frac{t}{2} + 1 \) holds if \( t < \Phi(\sqrt{q} - 1) \), where \( \Phi = \sqrt[2]{q} - 1 \approx 0.618034 \). Note that if \( t < \Phi(\sqrt{q} - 1) \), then Theorem 2.1 implies that \( S_t \) cannot have more than \( q - t \) collinear points. If \( q \) is not a square, then \( t < q^{1/3} - 3/2 \) (or \( t < (2q)^{1/3} - 2 \) when \( p = 2, 3 \)) and \( k \geq q - c_p q^{2/3} - t \) imply \( k > q - \frac{t}{2} + 1 \).

Next we define \( b(q) \) as follows.

\[
b(q) := \begin{cases} 
\sqrt{q} & \text{if } h \text{ is even}, \\
c_p q^{2/3} & \text{if } h \text{ is odd}.
\end{cases}
\]

For \( |S_t| < 2k + b(q) \), we prove the \( h \) even and \( h \) odd cases of the theorem simultaneously. Let us verify the condition of Lemma 3.4 on \( \varepsilon \). If \( |S_t| = k + q - t + \varepsilon \), then \( |S_t| < 2k + b(q) \) implies \( \varepsilon < k - q + b(q) + t \). If \( t = 1 \), then the upper bounds on \( t \) imply \( q \geq 9 \) for \( h = 2d \), and \( q \geq 27 \) for \( h = 2d + 1 \). From these lower bounds on \( q \) and from \( k \leq q - 1 \) it follows that \( k/2 \leq (q-1)/2 \leq q - b(q) - 2 \), thus \( \varepsilon < k - q + b(q) + 1 \leq t - 1 \), so the conditions of Lemma 3.4 hold if \( t = 1 \).

Now suppose that \( t \geq 2 \). As \( \varepsilon \leq k - q + b(q) + t \leq k - t - 1 \), it is enough to prove \( k - q + b(q) + t < \frac{k - t}{t} - \frac{t}{2} \).

After rearranging we get that this is equivalent to

\[
k < (q - t) + \left( \frac{q - b(q)}{t} - \frac{t}{2} - b(q) - \frac{3}{2} \right).
\]

thus it is enough to see (as \( k \leq q - t \) holds automatically) that

\[
\frac{q - b(q)}{t} - \frac{t}{2} - b(q) - \frac{3}{2} > 0.
\]

As a function of \( t \) the left hand side decreases monotonically. It is positive when \( t \) is maximal (under the respective assumptions), hence the conditions of Lemma 3.4 are satisfied.

Construct the blocking set \( B_t(S_t, \ell) \) as in Lemma 3.1. The conditions in Lemma 3.4 hold, hence the size of \( A(t) \) is at least \( t \). The size of \( B_t(S_t, \ell) \) is \( 2q + 1 + \varepsilon - k - |A(t)| < q + b(q) + 1 \), thus Theorem 1.3 implies that \( B_t(S_t, \ell) \) contains a \( \ell \)-secant. Since \( S_t \) cannot have more than \( q - t \) collinear points, we get that \( \ell \)-secant of \( S_t \) contains a \( \{\ell \cap S_t\} \leq |\ell \cap S_t| \) and hence the conditions in Lemmas 3.4 and 3.5 hold also with \( \ell \) instead of \( \ell \). Constructing \( B_t(S_t, \ell_1) \) we get that there exists another \((q-t)\)-secant of \( S_t \), having no common point with \( \ell \cap S_t \).

Now consider the case \( h = 2d \), \( |S_t| = 2k + \sqrt{q} \) and suppose that \( S_t \) does not have two \((q-t)\)-secants. We can repeat the above arguing and get that \( B_t(S_t, \ell) \) or \( B_t(S_t, \ell_1) \) is a Baer subplane because of Theorem 3.5.
Among the lines of the Baer subplane \(B_1(S_t, \ell)\) is a Baer subplane and hence \(\varepsilon < k/2 - 1\) in case of \(t = 1\), we use \(q > 9\). We may suppose that \(B_1(S_t, \ell)\) is a Baer subplane and hence \(\varepsilon = k - q +\sqrt{q} + t\) and \(|A(t)| = t\). The size of \(\ell \cap B_1(S_t, \ell)\) is either 1 or \(\sqrt{q} + 1\). In the latter case \(k = q - \sqrt{q} - t\) and we get Example 2.20. In the first case \(k = q - t\). We show that this cannot occur. Denote by \(R\) the common point of \(\ell\) and \(B_1(S_t, \ell)\) and let \(P\) be any point of \(B_1(S_t, \ell) \setminus (\ell \cup S_t)\). Among the lines of the Baer subplane \(B_1(S_t, \ell)\) there are \(\sqrt{q} + 1\) lines incident with \(P\), of one of them is \(PR\), which meets \(S_t\) in at least \(\sqrt{q} - t > 1\) points. Each of the other \(\sqrt{q}\) lines of the subplane intersects \(S_t\) in at least \(\sqrt{q} + 1 - t > 1\) points, thus these lines cannot be tangents to \(S_t\). But the pencil of lines through \(P\) contains \(k = q - t\) tangents to \(S_t\), one at each point of \(\ell \cap S_t\), too. Thus the total number of lines through \(P\) is at least \(1 + \sqrt{q} + q - t > q + 1\), this is a contradiction. \(\square\)

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