CALABI-YAU ORBIFOLDS OVER HITCHIN BASES

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ABSTRACT. Any irreducible Dynkin diagram $\Delta$ is obtained from an irreducible Dynkin diagram $\Delta_h$ of type ADE by folding via graph automorphisms. For any simple complex Lie group $G$ with Dynkin diagram $\Delta$ and compact Riemann surface $\Sigma$, we give a Lie-theoretic construction of families of quasi-projective Calabi-Yau threefolds together with an action of graph automorphisms over the Hitchin base associated to the pair $(\Sigma, G)$. These give rise to Calabi-Yau orbifolds over the same base. Their intermediate Jacobian fibration, constructed in terms of equivariant cohomology, is isomorphic to the Hitchin system of the same type away from singular fibers.

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1. Introduction

In recent years, it has been realized that the $G$-Hitchin system ([Hit87a], [Hit87b])

\[ h : \mathcal{M}_H(\Sigma, G) \to \mathcal{B}(\Sigma, G), \]

for a compact Riemann surface $\Sigma$ of genus $\geq 2$ and a simple complex Lie group $G$ of type ADE, are closely related to certain (quasi-projective) Calabi-Yau threefolds ([DDD+06], [DDP07], [AHK14], [AHKS17]). For example, if $G$ is a simple adjoint complex Lie group of type ADE, then there exists a family $\pi : \mathcal{X} \to \mathcal{B}(\Sigma, G)$ of quasi-projective Gorenstein Calabi-Yau threefolds over the Hitchin base $\mathcal{B}(\Sigma, G)$ such that the intermediate Jacobian

\[ J^2(X_b) = H^3(X_b, \mathbb{C}) / \left( F^2 H^3(X_b, \mathbb{C}) + H^3(X_b, \mathbb{Z}) \right), \quad X_b := \pi^{-1}(b), \]

is isomorphic to the Hitchin fiber $h^{-1}(b)$ for generic $b \in \mathcal{B}(\Sigma, G)$ ([DDP07]). In this paper, we extend this to a global result for all simple adjoint and simply-connected complex Lie groups of any Dynkin type by constructing families of Calabi-Yau orbifolds/global orbifold stacks

\[ [\mathcal{X}/\mathcal{C}] \to \mathcal{B}(\Sigma, \Delta) = \mathcal{B}(\Sigma, G), \quad \Delta = \Delta(G). \]

Here $\Delta(G)$ is the irreducible Dynkin diagram associated to $G$ which is obtained by a unique irreducible Dynkin diagram $\Delta_h$ of type ADE by folding via graph automorphisms $\mathcal{C} \subset \text{Aut}(\Delta_h)$, i.e. $\Delta = \Delta_h^{\mathcal{C}}$. Moreover, $\pi : \mathcal{X} \to \mathcal{B}(\Sigma, \Delta)$ is a family of quasi-projective Gorenstein Calabi-Yau threefolds with $\mathcal{C}$-trivial canonical class. More precisely, we prove in Section 4:

**Theorem 1.** Let $[\mathcal{X}/\mathcal{C}] \to \mathcal{B} = \mathcal{B}(\Sigma, \Delta)$ be a family of global Calabi-Yau orbifolds as in (1). Its intermediate Jacobian fibration

\[ J^2([\mathcal{X}/\mathcal{C}]) \to \mathcal{B}, \]

defined in integral equivariant cohomology, is isomorphic to the $G_{\text{ad}}(\Delta)$-Hitchin system over a Zariski-open and dense $\mathcal{B}^0 \subset \mathcal{B}$ where $G_{\text{ad}}(\Delta)$ is the simple adjoint complex Lie group with Dynkin diagram $\Delta$.

The basic idea for constructing $\mathcal{X} \to \mathcal{B}$ (without the $\mathcal{C}$-action) goes back to [Sze04] and [DDP07]. However, we offer two main novelties. The first one is a consequent use of Lie-theoretic methods, namely Slodowy slices, which makes the link to Hitchin systems from the start. For example, we construct nowhere-vanishing sections $s_b \in H^0(X_b, K_{X_b})$ of the canonical class $K_{X_b}$ of $X_b$, $b \in \mathcal{B}$, in terms of the Kostant-Kirillov form. The second one is the construction of a $\mathcal{C}$-action on the family $\mathcal{X} \to \mathcal{B}$ and we prove that the sections $s_b$ are $\mathcal{C}$-invariant. Therefore the global quotient family $[\mathcal{X}/\mathcal{C}] \to \mathcal{B}$ is indeed a family of Calabi-Yau orbifolds.

Another important step in the proof of Theorem 1 is an isomorphism

\[ H^3_b(X_b, \mathbb{Z}) \cong H^3(X_b, \mathbb{Z})^C, \quad b \in \mathcal{B}^0, \]

between Deligne’s $\mathbb{Z}$-MHS on the integral equivariant cohomology $H^3_b(X_b, \mathbb{Z})$ and the $\mathbb{Z}$-MHS on the $\mathcal{C}$-invariant part $H^3(X_b, \mathbb{Z})^C$ of the integral cohomology. Even though (2)

\[ (\text{Since the Hitchin base } \mathcal{B}(\Sigma, G) \text{ only depends on the Dynkin diagram } \Delta(G), \text{ it is sufficient to write } \mathcal{B}(\Sigma, \Delta), \text{ see } 3.1). \]
always holds true over \( \mathbb{Q} \), it is false in general over \( \mathbb{Z} \) due to torsion. For example, it fails for the minimal resolution of the \( \Delta_h \)-singularity (Example 4). Combined with the global isomorphism

\[
\mathcal{J}_C^c(\mathcal{X}^o) \cong \mathcal{M}_H(\Sigma, G_{ad}(\Delta)),
\]

over \( \mathcal{B}_o \) (Corollary 5 in [Bec17]), where \( \mathcal{J}_C^c(\mathcal{X}^o) \to \mathcal{B}_o \) is the intermediate Jacobian fibration defined by \( C \)-invariants in cohomology, we conclude Theorem 1. In fact, the isomorphism (3) gave a hint that we have to use with Calabi-Yau orbifolds if \( C \neq 1 \) instead of ordinary Calabi-Yau threefolds if \( C \neq 1 \). By working with equivariant compactly supported cohomology instead of equivariant cohomology, we further obtain the Langlands dual version of Theorem 1 and hence cover all simple simply-connected complex Lie groups as well.

Besides the \( C \)-action and construction of \( C \)-invariant holomorphic volume forms, the Lie-theoretic construction gives a way to study the smooth locus of the family \( \mathcal{X} \to \mathcal{B} \). As an application, we see in Corollary 2 that if \( \Delta_h \) is of type ADE and admits non-trivial Dynkin graph automorphisms \( C \) (see Remark 7 in Appendix A), then the smooth locus of the family \( \mathcal{X} \to \mathcal{B}(\Sigma, \Delta_h) \) is much larger than the smooth locus of \( \Delta_h \)-cameral curves. It follows that the family \( \mathcal{X} \to \mathcal{B}(\Sigma, \Delta_h) \) does not only geometrically encode the smooth loci of

\[
\mathcal{M}_H(\Sigma, G_{ad}(\Delta)) \quad \mathcal{B}(\Sigma, \Delta) \quad \mathcal{M}_H(\Sigma, G_{sc}(\Delta))
\]

but correspondingly for \( G_{ad}(\Delta) \) and \( G_{sc}(\Delta) \), under the natural inclusion \( \mathcal{B}(\Sigma, \Delta) \subset \mathcal{B}(\Sigma, \Delta_h) \) for \( \Delta = \Delta_h^C \).

1.1. Relation to other works. Our work is intellectually indebted to [Sze04], [DDP07], [DDD\textsuperscript{+}06] and [Slo80b]. We relate our approach to the first three works, for example we give explicit equations in the spirit of [DDD\textsuperscript{+}06] (also see Theorem 3). For that reason, we give two Slodowy slices \( S \subset \mathfrak{so}(5, \mathbb{C}) \) and \( S_h \subset \mathfrak{sl}(4, \mathbb{C}) \) and show by a direct computation that they both realize the semi-universal deformation of the \( B_2 \)-singularity in Appendix B. Moreover, we show that families of [Sze04] provide simultaneous resolutions over a cone in the singular locus of \( \pi : \mathcal{X} \to \mathcal{B} \) (Proposition 3). Graph automorphisms also appear in [Sze04] but we use them differently. In particular, orbifold stacks and their cohomology appear in neither of these works. Finally, it is further crucial for our work [Bec17] that the ‘volume forms’ \( s_b \in H^0(X_b, K_{X_b}) \) induce a period map on \( \mathcal{B}_o \). This period map is a so-called abstract Seiberg-Witten differential which in turn determines the structure of an algebraic integrable system on \( \mathcal{J}_C^c(\mathcal{X}^o/\mathcal{B}^o) \to \mathcal{B}^o \) (cf. Section 4.4 in [Bec17]).

1.2. Outline. Since the key ingredient for the local building blocks of the Calabi-Yau orbifolds (1) are Slodowy slices, we recall and prove important preparatory results on them in Section 2. Section 3 begins with a recap on Hitchin bases and cameral curves and continues with the construction of Calabi-Yau orbifolds and related families of quasi-projective Calabi-Yau threefolds. In Section 4 we study mixed Hodge structures on equivariant cohomology and prove Theorem 1. Appendix A contains our conventions on folding of Dynkin diagrams and Appendix B gives two examples of Slodowy slices that realize the semi-universal deformations of the same surface singularity.
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2. Local preparations

2.1. **∆-singularities and Slodowy slices.** Let ∆ be an irreducible Dynkin diagram. Then there is a unique pair $\Delta_h,AS(\Delta)$ consisting of an ADE-Dynkin diagram and the associated symmetry group $AS(\Delta) \subset Aut(\Delta_h)$ such that $\Delta$ is obtained by folding, i.e. $\Delta = \Delta_h^{AS}$ (see Appendix A for more details). A $\Delta$-singularity $(Y,H)$ ([Slo80b]) consists of a (germ of a) surface singularity $Y = (Y,0)$ of type $\Delta_h$ and a subgroup $H \subset Aut(Y)$ with the following properties:

i) $H \cong AS(\Delta)$;

ii) the action of $H$ on $Y - \{0\}$ is free;

iii) the induced action on the dual resolution graph of the minimal resolution $\hat{Y} \to Y$ coincides with the $AS(\Delta)$-action on $\Delta_h$.

For $\Delta = \Delta_h$, these are just ADE-surface singularities. Every $\Delta$-singularity $(Y,H)$ is quasi-homogeneous (cf. Table 1 in Appendix A), in particular, $Y$ carries a $\mathbb{C}^*$-action that commutes with the $H$-action. A $\mathbb{C}^*$-deformation of $(Y,H)$ is therefore a $\mathbb{C}^* \times H$-deformation $\mathcal{Y} \to B$ such that $H$ acts trivially on the base. Each $(Y,H)$ has a semi-universal $\mathbb{C}^*$-deformation, see Section 2 in [Slo80b].

The relation to the simple complex Lie algebra $\mathfrak{g} = \mathfrak{g}(\Delta)$ of type $\Delta$ is as follows: Let $x \in \mathfrak{g}$ be a subregular nilpotent element and $(x,y,h)$ an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with semisimple $h$. Then the Slodowy slice through $x$ (associated to the triple $(x,y,h)$) is given by

$$S = S(\Delta) := x + \ker\text{ad}(y).$$

It carries a non-trivial action by the group $\mathbb{C}^* \times AS(\Delta)$ (cf. Section 6 in [Slo80b]). Here the $AS(\Delta)$-action is defined by the action (via adjunction) of the group

$$\mathcal{C} = C(x,h)/C(x,h)^0 \cong AS(\Delta) \quad (4)$$

which is the group of connected components of $C(x,h) = \{g \in G_{ad} \mid g \cdot x = x, g \cdot h = h\}$. For a fixed Cartan subalgebra $t \subset \mathfrak{g}$, denote by $\chi : \mathfrak{g} \to t/W$ the adjoint quotient and

$$\sigma := \chi_{|S} : S \to t/W$$

its restriction to the Slodowy slice $S$. If we let $\mathbb{C}^*$ act on $t/W$ with twice the usual weight and $\mathcal{C}$ on $t/W$ trivially, then $\sigma$ is $\mathbb{C}^* \times \mathcal{C}$-equivariant. Slodowy has shown that $\sigma : S \to t/W$ with the $\mathbb{C}^* \times \mathcal{C}$-action is a $\mathbb{C}^*$-semi-universal deformation of the $\Delta$-singularity $(\sigma^{-1}(0), x)$ ([Slo80b], Section 8.7).

---

2The subindex of $\Delta_h$ stands for homogeneous.

3We often use $\mathcal{C}$ to emphasize how the $AS(\Delta)$-action is realized.
Finally, we recall Grothendieck’s simultaneous resolution of \( \chi : g \to t/W \),

\[
\begin{array}{ccc}
\tilde{g} & \xrightarrow{\psi} & g \\
\downarrow & & \downarrow \chi \\
t & \xrightarrow{\theta} & t/W.
\end{array}
\]  

If \( S = x + \ker \text{ad}(y) \) is a Slodowy slice, then (5) restricts to the simultaneous resolution \( \psi : \tilde{S} \to S \) of \( \sigma : S \to t/W \) (see Chapter 5 in [Slo80b]). If \((x, y, h)\) is the \( \mathfrak{sl}_2\)-triple defining \( S \), then the resulting square is a square of \( \mathbb{C}^*\)-spaces if \( h \in t \) (see Remark 1.53 in [Bec] for details). Since any two Cartan subalgebras are conjugate to each other, we will assume \( h \in t \) from now on.

2.2. Relative symplectic form. It is well-known (e.g. Chapter 1, [CG10]) that the Kostant-Kirillov form \( \nu = \omega_{KK} \) restricts to a symplectic form on each adjoint orbit \( O \subset g \). Consequently, the relative differential form \( \hat{\nu} \in \Omega^2(\chi) \) induced by \( \nu \) restricts to a relative symplectic form on the locus \( g_{\text{reg}} \) of regular elements in \( g \). This holds true for the restriction \( \hat{\nu}_{|S_{\text{reg}}} \in \Omega^2(\sigma) \) to \( S_{\text{reg}} := S \cap g_{\text{reg}} \) as well (cf. [GG02], Appendix 7). Since \( S - S_{\text{reg}} \) is of codimension at least 2, \( \hat{\nu}_{|S_{\text{reg}}} \) extends to a global section of \( K_{\sigma} \) which we also denote by \( \hat{\nu} \in \Gamma(S, K_{\sigma}) \).

**Proposition 1.** Let \( S \subset g \) be a Slodowy slice and \( \hat{\nu} \in \Gamma(S, K_{\sigma}) \) be the previously constructed section. Then \( \hat{\nu} \) is nowhere vanishing, in particular \( K_{\sigma} \) is trivializable. Moreover, it is \( \mathbb{C}^*\)-equivariant,

\[ \lambda^* \hat{\nu} = \lambda^2 \hat{\nu}, \quad \lambda \in \mathbb{C}^*, \]

as well as \( \mathbb{C} \)-invariant.

**Proof.** The proof proceeds in two steps. In the first step, we use Yamada’s realization ([Yam95]) of \( \theta : \tilde{g} \to t \) as a relative symplectic reduction to relate the corresponding relative symplectic form \( \hat{\omega} \in \Omega^2(\chi) \) to \( \hat{\nu} \in \Omega^2(g) \) over the locus \( g_{\text{reg}} \subset g \) of regular elements. In the second step, we restrict to \( S_{\text{reg}} \) and extend over the non-regular locus.

For the first step, fix a maximal torus \( T \) and a Borel subgroup \( B \supset T \) in the adjoint group \( G = G_{\text{ad}} \) and denote by \( N \subset B \) the nilradical of \( B \). Then the symplectic manifold \((M := T^*(G/N), \omega_c)\), with the canonical symplectic structure \( \omega_c \), carries a natural Hamiltonian \( G \)- and \( T \)-action induced by left and right multiplication on \( G/N \) respectively. The corresponding moment maps descend to \( \hat{\mu}_G : M/T \to g^* \) and \( \hat{\mu}_T : M/T \to t^* \). Yamada ([Yam95]) identifies the square (5) with the diagram

\[
\begin{array}{ccc}
M/T & \xrightarrow{\hat{\mu}_G} & g \\
\downarrow \hat{\mu}_T & & \downarrow \chi \\
t & \xrightarrow{\psi} & t/W
\end{array}
\]

after \( g^* \cong g \) and \( t^* \cong t \) via the Killing form (which we will do from now on). Under this identification, the symplectic form \( \omega_c \) descends and induces the relative symplectic form \( \hat{\omega} \in \Omega^2(\chi) \). Moreover, for each \( t \in t \) the induced \( G \)-action on \((\theta^{-1}(t), \hat{\omega}_t)\) is Hamiltonian and the restriction \( \psi_t : \theta^{-1}(t) \to g \) of \( \psi : \tilde{g} \to g \) is the corresponding moment map (Theorem 2.5...
in [Yam95]). Now let $\psi_{\text{reg}} : \tilde{g}_{\text{reg}} \rightarrow g_{\text{reg}}$ be the restriction of $\psi : \tilde{g} \rightarrow g$ to the regular locus. Note that

\begin{equation}
\psi_{\text{reg}}^*: \tilde{g}_{\text{reg}} \cap \theta^{-1}(t) \rightarrow g_{\text{reg}} \cap \chi^{-1}(\tilde{t}), \quad q(t) = \tilde{t},
\end{equation}

is an isomorphism for each $t \in t$. In fact, it is not difficult to see that $\psi_{\text{reg}}^*$ is a symplectomorphism if the left-hand side is endowed with the symplectic form $\tilde{\omega}$ and the right-hand side with the Kostant-Kirillov form $\nu$. Since $\psi_{\text{reg}}$ is smooth, $(\psi_{\text{reg}}^*)^* \Omega^2_x \simeq \Omega^2_\sigma$ naturally. Under this isomorphism, we have

$$
\psi^* \tilde{\nu} = \tilde{\omega} \in \Gamma(\tilde{g}_{\text{reg}}, \Omega^2_\sigma)
$$

because $\psi_{\text{reg}}^*$ is a symplectomorphism for each $t \in t$.

For the second step, we note that the restriction of $\tilde{\omega}$ to $\tilde{S} \subset \tilde{g}$ is again a relative symplectic form (Theorem 4.5 in [Yam95]), again denoted by $\tilde{\omega} \in \Omega^2_\sigma(\tilde{S})$. Again we have a natural isomorphism

$$
\Phi_{\text{reg}} : \psi^* K_{\sigma_{\text{reg}}} = \psi^* \Omega^2_{\sigma_{\text{reg}}} \rightarrow \Omega^2_{\tilde{g}_{\text{reg}}}
$$

over $\tilde{S}_{\text{reg}}$ with $\Phi_{\text{reg}}(\psi^* \tilde{\nu}) = \tilde{\omega}$. We claim that $\Phi_{\text{reg}}$ extends to an isomorphism $\Phi : \psi^* K_\sigma \rightarrow K_\tilde{\sigma}$ satisfying

\begin{equation}
\Phi(\psi^* \tilde{\nu}) = \tilde{\omega}.
\end{equation}

Since $K_\sigma$ and $K_\tilde{\sigma}$ are reflexive, it is sufficient to prove codim $\tilde{T} \geq 2$ for $\tilde{T} := \tilde{S} - \tilde{S}_{\text{reg}}$. The components of $\tilde{T}$ of highest dimension lie over the hypersurfaces $t_\alpha - \cap_{\beta \neq \alpha} t_\beta \cap t_\alpha$. If $t$ lies in such a hypersurface, then $\tilde{\sigma}^{-1}(\tilde{t}) \cap \tilde{T}$ consists of the exceptional divisor of $\psi_1 : \tilde{S}_t \rightarrow S_t$ for $\tilde{t} = q(t)$. Hence these components of $\tilde{T}$ have dimension $(r - 1) + 1 = r$ which is of codimension 2 in $\tilde{S}$. Equation (7) implies that $\tilde{\nu}$ is nowhere vanishing because $\tilde{\omega}$ is and $\psi$ is surjective.

Finally, the $C^*$-equivariance follows from that of $\tilde{\omega}$ (Corollary 4.6 in [Yam95]) whereas the $C$-invariance is a consequence of the Aut$(g)$-invariance of the Kostant-Kirillov form $\nu = \omega_{KK}$.

The morphism $\tilde{\sigma} : \tilde{S} \rightarrow t$ is topologically trivial (cf. [Slo80a]). Hence parallel transport defines canonical isomorphisms $P_t : H^2(\tilde{S}_t, \mathbb{C}) \rightarrow H^2(\tilde{S}_0, \mathbb{C})$, $t \in t$.

**Corollary 1.** The period map

\begin{equation}
P_{\tilde{S}} : t \rightarrow H^2(\tilde{S}_0, \mathbb{C})^C, \quad t \mapsto P_t([\tilde{\omega}_t]),
\end{equation}

is a $W$-equivariant isomorphism.

**Proof.** The case of ADE-Dynkin diagrams, i.e. $C = 1$, goes back to [Yam95]. For $C \neq 1$, it has been checked in [Bec], Corollary 1.98. \qed
2.3. Derivative of the adjoint quotient. Let \( b : U \to t/W \) be a morphism, where \( U \subset \mathbb{C} \) is a Zariski-open subset. Then we define \( \tilde{U}_b \) and \( Y_b \) by the fiber products

\[
\begin{array}{c}
\tilde{U}_b \\
p_b \\
U \\
p_b \\
Y_b \\
\end{array} \quad \begin{array}{c}
t \\
b \\
\sigma \\
\sigma \\
S. \\
\end{array}
\]

(9)

In Section 3 it will become clear that \( \tilde{U}_b \) is a local model for a cameral curve and \( Y_b \) is a local model for the quasi-projective Calabi-Yau threefolds that we construct in loc. cit. Both \( \tilde{U}_b \) and \( Y_b \) are non-singular if \( b \) is transversal to \( q \) and \( \sigma \). This condition in particular implies that the rank of \( dq \) may not be less than \( r - 1 \) for \( r = \text{rk}(g) \).

Example 1. We consider the simplest example, \( g = \mathfrak{sl}_2(\mathbb{C}) \). Of course, a Slodowy slice \( S \) has to be all of \( g \), \( t = \mathbb{C} \cong t/W \) and

\[
q : \mathbb{C} \to \mathbb{C}, \quad z \mapsto z^2,
\]

\[
\sigma : S \to \mathbb{C}, \quad A \mapsto \det(A).
\]

A morphism \( b : U \to t/W \) is transversal to \( q \) iff it has only zeros of multiplicity 1. If this is the case \( \tilde{U}_b = \{ (x, y) \in U \times \mathbb{C} \mid y^2 - b(x) = 0 \} \) is non-singular and a branched double covering of \( U \). It branches precisely over the zeros of \( b \).

Identifying \( \mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3 \), we write \( \sigma(u, v, w) = -u^2 - vw \). Again using that \( b \) has simple zeros only, we see that \( Y_b = \{ ((u, v, w), x) \in \mathbb{C}^3 \times U \mid -u^2 - vw - b(x) = 0 \} \) is non-singular. Note however, that if \( b(x) = 0 \), then \( \pi^{-1}_b(x) \) is an \( A_1 \)-singularity.

Lemma 1. Let \( x = h \) be semisimple and \( t \in t \) with \( \chi(h) = q(t) \). Then \( \text{im}(dq_t) \subset \text{im}(d\chi_h) \) and

\[
\text{rk}(dq_t) = \dim \bigcap_{\alpha \in R, \alpha(t) = 0} \ker \alpha = \dim C(Z_g(h))
\]

where \( R \) are the roots corresponding to \( t \subset g \).

Proof. Let \( t(h) \subset g \) be a Cartan subalgebra that contains \( h \). Since \( t(h) \) is conjugate to \( t \) we may assume that \( h \in t \) and in fact even \( t = h \) by the Ad-invariance of \( \chi \) and \( q \). The first claim is obvious because \( \chi|_t = q \).

For the second claim we may assume \( t = h \) as before. Then the first equality is proven in [Ste]. For the second equality we claim

\[
\bigcap_{\alpha \in R, \alpha(t) = 0} \ker \alpha = C(Z_g(h))
\]

(10)

which can be seen as follows: Let \( g = t \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \) be the root space decomposition with respect to \( t \). Then we get

\[
Z_g(h) = t \oplus \bigoplus_{\alpha \in R, \alpha(h) = 0} \mathfrak{g}_\alpha.
\]

Since \([t, \mathfrak{g}_\alpha] \neq \{0\} \) and \([h, \mathfrak{g}_\alpha] = 0 \) iff \( \alpha(h) = 0 \), the equality (10) follows. \( \square \)
Lemma 2. If \( x = h \) is semisimple and \( t \in \mathfrak{t} \) with \( q(t) = \chi(h) \), then \( \dim(\text{im}(dq_t)) = \dim(\text{d}\chi_x) \).

Proof. If \( x = h + v \) is the Jordan decomposition of \( x \in \mathfrak{g} \), for \( h \) semisimple and \( v \) nilpotent, then (see [Ric87])

\[
\text{rk}(\text{d}\chi_x) = \dim(C(Z_\mathfrak{g}(h))) + \text{rk}(\text{d}\chi_{1,v}).
\]

Here \( \chi_1 : \mathfrak{g}_1 := [Z_\mathfrak{g}(h), Z_\mathfrak{g}(h)] \to \mathfrak{t}_1/W_1 \) is the induced adjoint quotient of the semisimple Lie algebra \( \mathfrak{g}_1 \) and \( C(Z_\mathfrak{g}(h)) \) is the ceter of the centralizer of \( h \) in \( \mathfrak{g} \). By the previous lemma, it remains to show that \( d\chi_0 = 0 \) for any semisimple adjoint quotient \( \chi : \mathfrak{g} \to \mathfrak{t}/W \). This follows from the fact that the degrees \( d_i \) of any basis \( \chi_i \) of \( G_{ad} \)-invariant polynomials are greater or equal 2 because \( \mathfrak{g} \) is semisimple ([Bou02]). \( \square \)

Proposition 2. Let \( b : U \to \mathfrak{t}/W \) be a morphism from an open \( U \subset \mathbb{C} \) which is transversal to \( q : \mathfrak{t} \to \mathfrak{t}/W \). Then it is also transversal to \( \chi : \mathfrak{g} \to \mathfrak{t}/W \) and \( \sigma : S \to \mathfrak{t}/W \).

Proof. Let \( x = h + v \in \mathfrak{g} \) and \( t \in \mathfrak{t} \) such that \( \chi(x) = q(t) \). The previous corollary together with (11) implies that
\[
r \geq \text{rk}(\text{d}\chi_x) \geq \text{rk}(\text{d}q_t) \geq r - 1.
\]
If \( x = h \) is semisimple, then \( \chi \) is also transversal to \( b \) at \( x \) by the previous lemma. So we are left with the case \( x = h + v \) and \( \text{rk}(\text{d}\chi_x) = r - 1 = \text{rk}(\text{d}q_t) \). We claim that \( v = 0 \) and \( x = h \) must be semisimple which concludes the proof. Without loss of generality we assume again that \( x \in \mathfrak{t} \). Since \( \dim(C(Z_\mathfrak{g}(h))) = \text{rk}(\text{d}q_t) = r - 1 \) by Lemma 1, it follows that \( Z_\mathfrak{g}(h) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \) for a root \( \alpha \) with respect to \( \mathfrak{t} \). Therefore the derived algebra is
\[
[Z_\mathfrak{g}(h), Z_\mathfrak{g}(h)] = \langle h_\alpha, \mathfrak{g}_{\pm \alpha} \rangle \cong \mathfrak{sl}_2(\mathbb{C}),
\]
where \( h_\alpha \) generates the commutator \( [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t} \). As a consequence, \( v \) can be considered as a nilpotent element in \( \mathfrak{sl}_2(\mathbb{C}) \) because \( v \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \subset Z_\mathfrak{g}(h) \). By formula (11) and \( \text{rk}(\text{d}q_t) = r - 1 \) we must have \( \text{rk}(\text{d}\chi_{1,v}) = 0 \) for the adjoint quotient \( \chi_1 = \det : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{t}_1/W_1 \). But \( d_A \det = (-2a, -c, -b) \) for \( A = aH + bX + cY \in \mathfrak{sl}_2(\mathbb{C}) \) in the standard basis \( H, X, Y \) of \( \mathfrak{sl}_2(\mathbb{C}) \). Hence we must have \( v = 0 \), i.e. \( x = h \) is semisimple (and subregular). \( \square \)

3. Calabi-Yau orbifolds over Hitchin bases

3.1. Hitchin bases and cameral curves. This subsection mainly fixes notation and we refer to [DG02], [DP12] for more details. As before, we let \( G \) be a simple complex Lie group with Dynkin diagram \( \Delta = \Delta(G) \) and \( \mathfrak{t} \subset \mathfrak{g} = \mathfrak{g}(\Delta) \) a Cartan subalgebra in \( \mathfrak{g} = \text{Lie}(G) \). If \( \Sigma \) is a compact connected Riemann surface of genus \( \geq 2 \), we denote by \( \mathcal{M}_H(\Sigma, G) \) the moduli space of semistable \( G \)-Higgs bundles of degree 1 in \( \pi_1(\Sigma) \) and by
\[
\mathfrak{h} : \mathcal{M}_H(\Sigma, G) \to \mathcal{B}(\Sigma, G)
\]
the Hitchin map. It maps to the Hitchin base
\[
\mathcal{B} : = \mathcal{B}(\Sigma, G) = H^0(\Sigma, \mathcal{U})
\]
for the bundle \( u : \mathcal{U} = K_{\Sigma} \times_{\mathbb{C}^*} \mathfrak{t}/W \to \Sigma \) of cones where \( \mathbb{C}^* \) acts by the standard action on \( \mathfrak{t}/W \). Choosing generators \( \chi_1, \ldots, \chi_\tau \in \mathbb{C}[\mathfrak{t}]/W \) of Weyl group invariants gives \( \mathcal{U} \) the structure
of a vector bundle via the isomorphism
\[ U \cong \bigoplus_{j=1}^{r} K_{\Sigma}^{d_j} \]
for the degrees \( d_j = \text{deg}(\chi_j) \). In particular, the Hitchin base \( B \) inherits a vector space structure. By its construction, \( B \) only depends on the Lie algebra \( g \) of \( G \) so that the notation
\[ B(\Sigma, \Delta) := B(\Sigma, G) \]
is justified. The total space of the vector bundle \( \tilde{u}: \tilde{U} = K_{\Sigma} \otimes t \to \Sigma \) maps to \( U \) via the natural quotient map \( q: \tilde{U} \to U \). The fibers of the universal cameral curve
\[ p: \tilde{\Sigma} := ev^* \tilde{U} \to B, \]
for the evaluation map \( ev: \Sigma \times B \to U \), are the cameral curves \( \tilde{\Sigma}_b := p^{-1}(b) \). These are smooth over the Zariski-open and dense
\[ B^\circ := \{ b \in B \mid b \text{ intersects } \text{disc}(q) \text{ transversally} \} \]
and the generic Hitchin fibers \( h^{-1}(b), b \in B^\circ \), are generalized Prym varieties defined in terms of \( \tilde{\Sigma}_b \).

For later reference, we further need the cone
\[ (12) \quad \tilde{B}/W \hookrightarrow B, \quad \tilde{B} := H^0(\Sigma, \tilde{U}), \]
where the Weyl group \( W \) acts pointwise on sections. Away from \( 0 \in B \), this is the locus of completely reducible but reduced cameral curves, i.e.
\[ \tilde{\Sigma}_b = \prod_{w \in W} \tilde{\Sigma}_{b,w}, \quad \tilde{\Sigma}_{b,w} \cong \Sigma. \]
The irreducible components \( \tilde{\Sigma}_{b,w} \) intersect over the points of the divisor of the section
\[ \prod_{\alpha \in R} \alpha(b) \in H^0(\Sigma, K_{\Sigma}), \]
where \( R = R(\Delta) \) is the corresponding root system (which makes sense because \( \prod_{\alpha \in R} \alpha \in \mathbb{C}[t]^{W} \)).

### 3.2. Surfaces

For the following constructions, we fix a spin bundle \( L \in \text{Pic}^{g-1}(\Sigma), L^2 = K_{\Sigma} \) and denote by \( \alpha = \alpha_L \in H^1(\Sigma, \mathcal{O}^*) \) its cohomology class. Moreover, let \( \Delta \) be an irreducible Dynkin diagram with associated symmetry group \( C = AS(\Delta) \) and \( S = S(\Delta) \subset g(\Delta) \) a Slodowy slice. Twisting the \( C^* \)-spaces \( S \) and \( t/W \) by \( L \), we obtain
\[ (13) \quad S := L \times_{C^*} S, \quad L \times_{C^*} t/W \cong U. \]
\[ ^{4}\text{Recall that we usually let } C^* \text{ act on } t/W \text{ by twice the natural weights.} \]
By its $\mathbb{C}^*$-equivariance, the morphism $\sigma : S \to t/W$ and its simultaneous resolution $\tilde{\sigma} : \tilde{S} \to t$ glue to give the following commutative diagram

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\psi} & S \\
\downarrow{\tilde{\sigma}} & & \downarrow{\sigma} \\
\tilde{U} & \xrightarrow{q} & U
\end{array}
$$

of $\mathbb{C}$-spaces where $\mathbb{C}$ acts trivially on $\tilde{U}$ and $U$. However, it takes more care to glue the sections $\hat{\omega} \in \Gamma(\tilde{S}, \Omega^2)_{\tilde{\sigma}}$ and $\hat{\nu} \in \Gamma(S, K_{\sigma})$.

**Lemma 3.** With the notation of (13) and (14), the following holds:

i) The section $\hat{\omega}$ glues to a section $\hat{\omega} \in \Gamma(\tilde{S}, \Omega^2 \otimes (\tilde{u} \circ \tilde{\sigma})^* K_\Sigma)$ which is $\mathbb{C}$-invariant. It induces a fiberwise period map

$$
\eta : \tilde{U} \to \tilde{u}^* \tilde{U}
$$

which coincides with the tautological section $\tau \in \Gamma(\tilde{U}, \tilde{u}^* \tilde{U})$.

ii) Likewise, the section $\hat{\nu}$ glues to a $\mathbb{C}$-invariant section $\hat{\nu} \in \Gamma(S, K_{\sigma} \otimes (u \circ \sigma)^* K_\Sigma)$ such that

$$
\psi^* \hat{\nu} = \hat{\omega}
$$

under the natural isomorphism $\psi^* K_{\sigma} \cong \Omega^2_{\sigma}$.

In particular, the sections $\tilde{\sigma}$ and $\hat{\nu}$ give respective isomorphisms

$$
\Omega^2_{\sigma} \cong (\tilde{u} \circ \tilde{\sigma})^* K^{-1}_\Sigma \quad \text{and} \quad K_{\sigma} \cong (u \circ \sigma)^* K^{-1}_\Sigma.
$$

**Proof.** To construct the section $\hat{\omega}$, we need the gluing data for the sheaf $\Omega^2_{\sigma}$. Let $\alpha = \alpha_{ij}$ be the cocycle corresponding to $L$ and denote by $(\alpha_{ij})$ a Čech representative for a fixed open covering $(D_{ij})$ of $\Sigma$. In particular, $(\beta_{ij}) = (\alpha_{ij}^2)$ is a cocycle for $K_\Sigma$. Trivializing $\tilde{S}$ over each $D_i$ gives rise to the commutative diagram

$$
\begin{array}{ccc}
\tilde{S}_{ij} & \xrightarrow{\psi_i} & D_{ij} \times \tilde{S} \\
\downarrow{\tilde{\sigma}_{ij}} & & \downarrow{\text{id} \times \tilde{\sigma}} \\
\tilde{U} & \xrightarrow{\text{id} \times \tilde{\sigma}} & \tilde{U}
\end{array}
$$

over $D_{ij} = D_i \cap D_j$ with

$$
g_{ij}(x, s) = (x, \alpha_{ij}(x) \cdot s) = (x, \mu(\alpha_{ij}(x), s)),
$$

$$
h_{ij}(x, t) = (x, \alpha_{ij}(x) \cdot t) = (x, \beta_{ij}(x)t).
$$

for $(x, s) \in D_{ij} \times \tilde{S}$ and the action map $\mu : \mathbb{C}^* \times \tilde{S} \to \tilde{S}$. On each $D_i \times \tilde{S}$ we have the sheaves $\mathcal{E}_i := \Omega^2_{\text{id} \times \tilde{\sigma}} \cong \text{pr}^*_S \Omega^2_{\sigma}$ together with the sections $\text{pr}^*_S \hat{\omega}$. Clearly, $\mathcal{E}_i$ and $\mathcal{E}_j$ are canonically...
isomorphic over $D_{ij}$. Now $\Omega^2_{\mathcal{S}}$ is glued from $^5\psi^*_{\mathcal{E}_i}$ on $D_{ij}$ via the isomorphisms

$$\varphi_{ij} := \psi^*_i dg^i_{ji} : \psi^*_j \mathcal{E}_j = \psi^*_i g^i_{ji} \mathcal{E}_j \to \psi^*_i \mathcal{E}_i$$

over $D_{ij}$. Here we denote by $dg^i_{ji} : g^i_{ji} \mathcal{E}_j \to \mathcal{E}_i = \mathcal{E}_j$ the natural morphism (over $D_{ij}$). Observe that we have $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ and $(\varphi_{ij})$ is the gluing (or descent) datum for $\Omega^2_{\mathcal{S}}$. Indeed, we can write this composition as

$$\begin{array}{cccc}
\psi^*_i \mathcal{E}_k & \xrightarrow{\varphi_{kj}} & \psi^*_j \mathcal{E}_j & \xrightarrow{\varphi_{ji}} & \psi^*_i \mathcal{E}_i \\
\psi^*_i (g^i_{ji}) g^k_{kj} \mathcal{E}_j & \xrightarrow{\psi^*_i g^i_{ji} dg^i_{kj}} & \psi^*_i g^i_{ji} \mathcal{E}_j & \xrightarrow{\psi^*_i dg^i_{ji}} & \psi^*_i \mathcal{E}_j.
\end{array}$$

The lower line is $\varphi_{ik}$ by the chain rule, showing the cocycle condition for $(\varphi_{ij})$.

**Claim.** Define the local sections

$$\hat{\omega}_i := \psi^*_i \text{pr}_{2,i} \hat{\omega} \in \Gamma(S_i, \psi^*_i \mathcal{E}_i).$$

On the overlaps $\tilde{S}_{ij}$, they transform as follows

$$\varphi_{ij}(\hat{\omega}_j) = ((\text{pr}_{1,i} \circ \psi_i)^* \beta_{ji}) \hat{\omega}_i. \tag{15}$$

Before we prove this claim, let us see how it yields the desired section. Observe that $(\text{pr}_{1,i} \circ \psi_i)^* \beta_{ji}$ is a cocycle for $(\bar{u} \circ \bar{\sigma})^* K^{-1}_\Sigma$. Hence in order to obtain a well-defined global section on $\tilde{S}$, we have to tensor with $(\bar{u} \circ \bar{\sigma})^* K_{\Sigma}$. More precisely, let $\zeta_i \in \Gamma(D_i, K_{\Sigma})$ be the local frames of $K_{\Sigma}$ over $D_i$ so that $\zeta_i = \beta_{ij} \zeta_j$ on $D_{ij}$. Letting $\hat{\zeta}_i := \psi^*_i \text{pr}_{1,i} \zeta_i$, we see that the local sections

$$\hat{\omega}_i \otimes \hat{\zeta}_i \in \Gamma(\tilde{S}_i, \Omega^2_{\mathcal{S}} \otimes (\bar{u} \circ \bar{\sigma})^* K_{\Sigma})$$

glue to give the global section $\hat{\omega} \in \Gamma(\tilde{S}, \Omega^2_{\mathcal{S}} \otimes (\bar{u} \circ \bar{\sigma})^* K_{\Sigma})$ as claimed.

We still have to give a proof of (15). To simplify notation, we drop the subscript $ij$ if not necessary and only write $g : D \times \tilde{S} \to D \times \tilde{S}$ etc. Then the second component of $dg : TD \oplus T\tilde{S} \to TD \oplus T\tilde{S}$ at $(x, s) \in D \times \tilde{S}$ is given by

$$\begin{align*}
d\mu_{\alpha(x), s}(d\alpha_x(v), w) &= d\mu_{\alpha(x), s}(d\alpha_x(v), 0) + d\mu_{\alpha(x), s}(0, w). \tag{16}
\end{align*}$$

Note that $d\mu_{\alpha(x), s}(0, w) = d(\mu_{\alpha(x)})_s(w)$ where $\mu_{\alpha(x)} = \mu(\alpha(x), -)$. Now let $p \in \tilde{S}_{ij}$ and $\psi_i(p) = (x, s) \in D_{ij} \times \tilde{S}$. Then we clearly have

$$\ker d(x, s)(\text{id} \times \bar{\sigma}) = 0 \oplus \ker d_x \bar{\sigma} \subset T_x D_{ij} \oplus T_s \tilde{S}.$$
In particular, the first summand in (16) plays no role for our discussion. For \( w_k \in \ker d_\sigma \tilde{\sigma} (k = 1, 2) \) one computes

\[
\varphi_{ij}(\hat{\omega}_j)_p((0, w_1), (0, w_2)) = \text{pr}_{2,j}^* \hat{\omega}_{g_{ji}(x,s)} \circ dg_{ji}(x,s)((0, w_1), (0, w_2)) (\psi_j \circ \psi_i^{-1} = g_{ji})
\]

\[
= \beta_{ji}(x) \hat{\omega}_s(w_1, w_2) \quad \text{(C*-equivariance and (16))}
\]

\[
= (\text{pr}_{1,i} \circ \psi_i)^* \beta_{ji}(p) (\hat{\omega}_i)_p((0, w_1), (0, w_2)).
\]

Here we have used the C*-equivariance of the relative form \( \hat{\omega} \) showing (15). To construct the period map \( \eta \) out of \( \hat{\omega} \), observe that it induces the morphism

\[
(x, t) \mapsto \tilde{\mathcal{P}}(t) \otimes ((x, t), \zeta_i(x))
\]

in each trivialization \( \tilde{U}_{|D_i} \cong D_i \times t \) where \( \tilde{\mathcal{P}} \) is as in (8). These morphisms glue to give the morphism

\[
\eta: \tilde{U} \to t \otimes \tilde{u}^* K_{\Sigma}
\]

which coincides with the tautological section \( \tau \in H^0(\tilde{U}, \tilde{u}^* \tilde{U}) \) by construction.

The existence of the global section \( \tilde{\nu} \in H^0(S, K_\sigma \otimes (u \circ \sigma)^* K_{\Sigma}) \) works similarly: There exists a section \( \tilde{\nu} \in \Gamma(S, K_\sigma) \), which is constructed as \( \hat{\omega} \) by replacing \( \hat{\omega} \) with \( \tilde{\nu} \). Here \( S_{\text{reg}} \subset S \) is the locus which is glued from \( S_{\text{reg}} \subset S \). Using a codimension argument as in the proof of Proposition 1, we see that it uniquely extends to a section \( \tilde{\nu} \in \Gamma(S, K_{\Sigma}) \). It satisfies \( \psi^* \tilde{\nu} = \hat{\omega} \) under the isomorphism \( \psi^* K_{\Sigma} \cong K_{\tilde{\sigma}} \) by construction, since this holds for the corresponding local sections.

\[\square\]

3.3. Calabi-Yau threefolds. The family \( \sigma: S \to U \) of surfaces over \( U \) pulls back to the product \( \Sigma \times B \), \( B = B(\Sigma, \Delta) \) via the evaluation map \( \text{ev}: \Sigma \times B(\Sigma, \Delta) \to \tilde{U} \). Projecting to the second factor yields the family

\[
\mathcal{X}_L(\Delta) \xrightarrow{\pi_1 L} \Sigma \times B \xrightarrow{\pi_2 L} B
\]

of threefolds with a C-action over the Hitchin base \( B \). Analogously, pulling back \( \tilde{\mathcal{S}} \) along the natural map \( \tilde{\Sigma} \to \tilde{U} \) gives the family

\[
\tilde{\mathcal{X}}_L(\Delta) \xrightarrow{\tilde{\pi}_1 L} \tilde{\Sigma} \xrightarrow{\tilde{\pi}_2 L} B
\]

of threefolds with C-action over \( B \).

Theorem 2. Let \( \Delta \) be an irreducible Dynkin diagram and \( L \) be a spin bundle of \( \Sigma \). Then

\[
\mathcal{X}_L(\Delta) \longrightarrow B(\Sigma, \Delta) \leftarrow \tilde{\mathcal{X}}_L(\Delta)
\]

are algebraic families of quasi-projective Gorenstein threefolds with C-trivial canonical class. They are smooth over the locus \( B^\circ(\Sigma, \Delta) \subset B(\Sigma, \Delta) \) of smooth cameral curves.
Proof. The projections \( U \rightarrow \Sigma \) and \( S \rightarrow \Sigma \) are quasi-projective morphisms because they are (affine) bundles over the projective curve \( \Sigma \). Therefore the morphism \( \sigma : S \rightarrow U \) which factorizes as

\[
\begin{array}{ccc}
S & \xrightarrow{\sigma} & \Sigma \\
\downarrow & & \\
U & \rightarrow & \Sigma
\end{array}
\]

is quasi-projective as well. Hence \( \pi : X \rightarrow B \) is quasi-projective. Using the fact that \( \sigma : S \rightarrow U \) is a Gorenstein morphism, a similar argument shows that each \( X_b = \pi^{-1}(b) \), \( b \in B \), is Gorenstein.

To see the statement about the canonical class, let \( j : \mathcal{X} \rightarrow S \) be the natural morphism obtained from base change. Then the section \( \tilde{\nu} \) of Lemma 3 pulls back to (dropping \( L \) from the notation)

\[
s := j^* \tilde{\nu} \in H^0(\mathcal{X}, K_{\pi_1} \otimes (pr_1 \circ \pi_1)^* K_{\Sigma}).
\]

Base change and the adjunction formula yields the isomorphism

\[
(K_{\pi_1} \otimes (pr_1 \circ \pi_1)^* K_{\Sigma})|_{X_b} \cong K_{\pi_b} \otimes \pi_b^* K_{\Sigma} \cong K_{X_b}.
\]

Hence each restriction \( s_b := s|_{X_b} \) is a nowhere vanishing section of the locally free sheaf \( X_b \).

Since the \( C \)-action on \( X_b \) is pulled back from the \( C \)-action on \( S \), \( s_b \) is \( C \)-invariant by Lemma 3.

Finally, the statement about smoothness follows from Proposition 2.

The proof for \( \tilde{X}_{L}(\Delta) \) works in complete analogy by replacing \( s \) with

\[
\tilde{s} := j^* \tilde{\omega} \in H^0(\tilde{\mathcal{X}}, K_{\tilde{\pi}_1} \otimes (pr_1 \circ \tilde{p}_1)^* K_{\Sigma})
\]

for the natural morphism \( \tilde{j} : \tilde{\mathcal{X}} \rightarrow \tilde{S} \).

We close this subsection with the deformation-theoretic meaning of the Hitchin base \( B = B(\Sigma, \Delta) \) for the family \( \mathcal{X} \rightarrow B \) (see [DDD+06] for the \( A_1 \)-case). To this end, observe that the central fiber

\[
X_0 = L \times_{\mathbb{C}^*} S_0
\]

has a curve of \( \Delta \)-singularities along \( \Sigma \hookrightarrow X_0 \). It is easily seen from the construction that \( X_b, b \neq 0 \), only has isolated singularities. In particular, the deformation \( \mathcal{X} \rightarrow B \) of \( X_0 \) is not a locally trivial one. More precisely, we have:

**Theorem 3.** Let \( X = X_0 \) be the central fiber of a family \( \mathcal{X} \rightarrow B \) of quasi-projective Gorenstein Calabi-Yau with \( C \)-trivial canonical class. Then

\[
\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)/H^1(X, T_X) \cong B,
\]

i.e. the space of \( C \)-deformations of \( X \) modulo locally trivial \( C \)-deformations is isomorphic to the corresponding Hitchin base.

Proof. We first consider the case \( C = 1 \) and set \( r := \text{rk}(g) \). Since \( \pi : X \rightarrow \Sigma \) is affine, the Leray spectral sequence implies that \( H^2(X, T_X) = 0 \). Therefore the local-to-global spectral sequence implies

\[
\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)/H^1(X, T_X) \cong H^0(X, \mathcal{E}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)).
\]
If $k : \Sigma \hookrightarrow X$ is the closed imbedding, then we claim
\[ (19) \quad \mathcal{E} := \mathcal{E}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \cong k_* \mathcal{O}. \]

As a first step, we realize $X$ as a regular imbedding. Let $\pi_S : S \to \Sigma$ be the natural projection and $\pi_S^* S \to S$ the induced vector bundle. Then there exists a unique section $\tau : S \to \pi_S^* S$ such that $pr_U \circ \tau = \sigma$ for the natural projection $pr_U : \pi_S^* S \to U$. In a trivialization of $S_{|D} \cong D \times S$ for an affine $D \subset \Sigma$, the section is given by
\[ D \times S \to (D \times S) \times r/W, \quad (x, s) \mapsto ((x, s), \sigma(s)). \]

Therefore the vanishing locus $Z(\tau) \subset S$ coincides with $i : X \hookrightarrow S$ and is a regular imbedding. The latter implies that the normal sheaf satisfies
\[ (20) \quad \mathcal{N}_{X/S} \cong \pi_S^* S_{|X}, \]
cf. [FL85], Chapter IV. In order to compute $\mathcal{E}$, let $J$ be the ideal sheaf defining $X \subset S$. Since the latter is a regular imbedding, we have the locally free resolution
\[ 0 \to J/J^2 \to i^* \Omega^1_S \to 0 \]
of $\Omega^1_X$. Dualizing gives the sequence
\[ 0 \to \mathcal{H}om_{\mathcal{O}_X}(i^* \Omega^1_S, \mathcal{O}_X) \to \mathcal{N}_{X/S} \to 0 \]
so that $\operatorname{coker}(\varphi) = \mathcal{E}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ which is supported on $\Sigma \subset X$. In order to prove (19) it therefore suffices to show that $\mathcal{E}$ has constant rank $r$ along $\Sigma$ in light of (20).

Let $U := D \times S_0 \subset X$ and $D \subset \Sigma$ be an affine open in $X$ and $\Sigma$ respectively. Hence
\[ U = \operatorname{Spec}(B), \quad B = \mathbb{C}[t, x, y, z]/(f(x, y, z)), \quad D = \operatorname{Spec}(C), \quad C = \mathbb{C}[t, x, y, z], \]
where $f$ defines the $\Delta$-singularity $S_0$. The sheaf $\mathcal{E}$ corresponds to the $B$-module
\[ \operatorname{Ext}^1_B(\Omega^1_B, B) \cong \mathbb{C}[t, x, y, z]/(f, \partial f). \]

As a $C$-module, it is isomorphic to $\mathbb{C}[t]^r$. This concludes the proof for $C = 1$. If $C \neq 1$, the same proof goes through if we work with $C$-invariants and recall that $B^C = B$. \hfill \Box

3.4. Relation between $X_L(\Delta)$ and $X_L(\Delta_h)$. We next compare the two families
\[ X = X_L(\Delta) \to B = B(\Sigma, \Delta) \]
\[ X_h = X_L(\Delta_h) \to B_h = B(\Sigma, \Delta_h) \]
in case $\Delta \neq \Delta_h$. As a first step, we construct a non-trivial $AS(\Delta)$-action on $X_h$.

Let $S_h := S(\Delta_h) = x + \ker(\operatorname{ad}(y)) \subset g_h := g(\Delta_h)$ be a Slodowy slice. Then the group
\[ (21) \quad CA(x, h) := \{ \phi \in \operatorname{Aut}(g_h) \mid \phi(x) = x, \phi(h) = h \} \]
acts on $S_h$. There is a subgroup $CA \subset CA(x, h)$ which is isomorphic to $AS(\Delta)$ (see [Slo80b], 7.5). Of course, the definition of $CA$ and $C$ makes sense for $S$ and $S_h$ respectively. Then $CA \cong C$ in the former and $C = 1$ in the latter case.

Remark 4. Even though $C \cong AS(\Delta) \cong CA$, we often write $C$ and $CA$ to emphasize how the $AS(\Delta)$-action is realized. To illustrate the relation between $S$ and $S_h$ together with the corresponding $AS(\Delta)$-actions, we give a worked out example in Appendix B.
The CA-action on $S_h$ induces a CA-action on $t_h/W_h$ such that $\sigma_h : S_h \to t_h/W_h$ is equivariant. Since the CA-action commutes with the $\mathbb{C}^*$-action on $S_h$, we obtain a CA-action on $X_h$. Lemma 3 and Theorem 2 analogously hold for the CA-action (cf. [Bec] for details). However, CA acts non-trivially on the base so that a general member $X_{h,b}$ does not have CA-trivial canonical bundle.

**Corollary 2.** With the previous notation,

$$B \cong B_h^{CA} \hookrightarrow B_h$$

but $B^o \cap B_h^o = \emptyset$. The family $X \to B$ is the restriction of the family $X_h \to B_h$ under the inclusion (22) and the $AS(\Delta)$-action on $X$ is induced by the $AS(\Delta)$-action on $X_h$. In particular, $X_h$ is smooth over a Zariski open and dense subset containing $B^o \cup B_h^o \subset B_h$.

**Proof.** It is not difficult to see that $t/W \cong (t_h/W_h)^{CA}$ which gives the inclusion $i : t/W \hookrightarrow t_h/W_h$ and hence (22). To see that $B^o \cap B_h^o = \emptyset$, let $\alpha \in \Delta$ be a long root. Under folding, it corresponds to an $AS(\Delta)$-orbit $O(\beta)$ of length $\geq 2$ for some $\beta \in \Delta_h$. If $t_\alpha \subset t$ is the fixed point locus of $s_\alpha \in W$ (and analogously for $t_h$), then

$$t_\alpha = \bigcap_{\beta' \in O(\beta)} t_{h,\beta'} \subset t_h$$

under the inclusion $t \subset t_h$. If $b \in B^o$, then $b$ necessarily maps to the smooth locus $\text{disc}(q)^{sm}$ of the discriminant of $q : U \to U$. Since $\Sigma_b$ necessarily intersects $K_\Sigma \otimes t_\alpha \subset U$ and because of (23), $b$ cannot map to $\text{disc}(q_h)^{sm}$ under the inclusion $U \subset U_h$. Hence $B^o \cap B_h^o = \emptyset$.

Slodowy has proven (Chapter 8.8 in [Slo80b]) that $i^*S_h \cong S$ over $t/W$ as $\mathbb{C}^*$-deformations of the $\Delta$-singularity over $\tilde{0} \in t/W$. By the $\mathbb{C}^*$-equivariance, this isomorphism induces the isomorphism $\tilde{X} \cong \tilde{X}_h$ over $B \subset B_h$. \hfill \Box

### 3.5. Equations.

To connect to similar constructions in the physical mathematics literature ([DDD+06], [AHK14], [AHKS17]), we give explicit equations of the families $X \to B$ in appropriate bundles over $\Sigma$.

If we identify $S = \cong \mathbb{C}^{r+2}$ appropriately, then the weights of the $\mathbb{C}^*$-action are given by $w_1 = 2d_1, \ldots, w_r = 2d_r, w_{r+1}, w_{r+2}, w_{r+3}$ for the degrees $d_j = \deg(\chi_j)$ as in Section 3.1 and weight $w_{r+1}, w_{r+2}, w_{r+3}$ as in Table 1 in Appendix A. It follows that

$$S = L \times_{\mathbb{C}^*} S \cong \bigoplus_{j=1}^{r} K^{d_j} \oplus \bigoplus_{j=1}^{3} L^{w_{r+k}}.$$  

Let $f \in \mathbb{C}[x,y,z]$ define the $\Delta$-singularity and let $g_1, \ldots, g_r \in \mathbb{C}[x,y,z]^\mathbb{C}$ be representatives of generators of the $\mathbb{C}$-invariant Jacobi ring $\mathbb{C}[x,y,z]^\mathbb{C}/(f, \partial f)$. Then the family $X \to B$ reads as

$$X = \{ (x,y,z, b) : f(x,y,z) + \sum_{i=1}^{r} b_i g_i(x,y,z) = 0 \in \text{tot}(K^d) \}$$

in $\text{tot}(\bigoplus_{i=1}^{3} L^{w_{r+k}}) \times B$ with the obvious projection and where $d$ is again as in Table 1.
Example 2 ($\Delta = B_2$). Let us give at least one simple example (also compare with Appendix B). In this case $f(x, y, z) = x^4 - yz$ and $C = \mathbb{Z}/2\mathbb{Z}$ acts as $(x, y, z) \mapsto (-x, z, y)$. Therefore (24) reads as

$$X = \{(x, y, z, (b_1, b_2)) \mid x^4 - yz + b_1x^2 + b_2 = 0 \in \text{tot}(K^4)\}$$

in $\text{tot}(K \oplus K^2 \oplus K^2) \times B$ with $B = H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^4)$. 

3.6. Simultaneous resolutions. The families $\pi : \mathcal{X} \to B$ and $\tilde{\pi} : \mathcal{X} \to B$ share many features of the Slodowy slice $S$. It seems unlikely though that our methods provide an explicit description of a simultaneous resolution for these families. However, we succeed over the locus $\tilde{B}/W \subset B$ (see (12)) which is in the singular locus of $\pi$. 

To see this, let $\tilde{\text{ev}} : \Sigma \times \tilde{B} \to B$ be the evaluation map which gives rise to the family

$$\tilde{\text{ev}}^* \tilde{S} = \tilde{X}_L \xrightarrow{\tilde{\pi}_L} \Sigma \times \tilde{B} \xrightarrow{\pi_L} B$$

of threefolds with a non-trivial $C$-action. With the methods of the proof of Theorem 2, one proves that $\tilde{\pi}_L : \tilde{X}_L \to \tilde{B}$ is a smooth family of quasi-projective threefolds with $C$-trivial canonical class.

Proposition 3. The family $\tilde{\pi} : \tilde{X} \to \tilde{B}$ is a simultaneous $C$-resolution of $p^* \mathcal{X}$ where $p : \tilde{B} \to \tilde{B}/W \subset B$ is the natural projection. It descends to a simultaneous small $C$-resolution over $\tilde{B}/W - \{0\} \subset B$.

Proof. Using the fiber product property, we obtain a unique $C$-equivariant morphism $\tilde{X} \to p^* \mathcal{X}$ over $\Sigma \times B$ which makes the obvious diagrams commute.

For the second claim, observe that the $C$-family $\tilde{X}/W \to \tilde{B}/W$ is still smooth over $B^* := \tilde{B}/W - \{0\} \subset B$. Then the morphisms $\tilde{X} \to p^* \mathcal{X}$ descends to the $C$-morphism

$$\tilde{X}/W \to p^* \mathcal{X}/W \cong \mathcal{X}$$

over $B^*$. It remains to show that $(\tilde{X}/W)_b \to X_b$ is a small resolution for $b \in B^*$. Let $\tilde{b}$ with $p(\tilde{b}) = b$. Then $X_b$ is only singular at the singularities of the fibers $p_b^{-1}(x)$ for

$$x \in \text{Div}(\prod_{\alpha \in R} \alpha(b))$$

and the projection $p_b : X_b \to \Sigma$. By assumption $\prod_{\alpha \in R} \alpha(b) \neq 0$, i.e. the singularities are isolated, so that $(\tilde{X}/W)_b \to X_b$ is a small resolution.

Remark 5. The last statement is false if we include $0 \in B$: Then $X_0 \cong L \times_C Y$ for the $\Delta$-singularity $Y$ and $\tilde{X}_0 \cong L \times_C \tilde{Y}$ for its minimal resolution $\tilde{Y} \to Y$. Hence it is not a small resolution.
Example 3. The singularities of \( X_b, b \in B^* \), are precisely the Gorenstein threefold singularities studied in [KM92]. The simplest (local) example is

\[
\hat{X} = \{(x, y, z, t, [u : v]) \in \mathbb{C}^4 \times \mathbb{P}^1 \mid x^2 + y^2 + z^2 - t^2 = 0, \ xv = u(z + t)\}
\]

\[
X = \{(x, y, z, s) \in \mathbb{C}^4 \mid x^2 + y^2 + z^2 - s = 0\}
\]

for the small resolution is \((x, y, z, t, [u : v]) \mapsto (x, y, z, t^2)\).

4. Calabi-Yau orbifolds and Hitchin systems

Let \( X \to B \) be a family of quasi-projective Calabi-Yau threefolds with \( \mathbb{C} \)-trivial canonical class as in the previous section. For each \( b \in B \), we denote by \([X_b/\mathbb{C}]\) the corresponding quotient stack and refer to it as a Calabi-Yau orbifold. By construction, they fit into the family \( \pi^{\circ} : [X/\mathbb{C}] \to B \) of Calabi-Yau orbifolds. Before we study the integral (equivariant) cohomology groups \( H^3([X_b/\mathbb{C}], \mathbb{Z}) = H^3_C(X_b, \mathbb{Z}), \ b \in B^* \), we need a general result.

4.1. MHS on equivariant cohomology. Let \( G \) be a finite group acting on a locally compact topological space \( X \). We replace the orbifold stack \([X/G]\) by the simplicial space

\[
[X/G]_\bullet = ((G^{p+1} \times X)/G)_{p \geq 0}
\]

where \( G \) acts on \( G^{p+1} \times X \) via

\[
g \cdot (g_0, \ldots, g_p, x) = (g_0 g^{-1}, \ldots, g_p g^{-1}, g \cdot x).
\]

The simplicial structure maps of \([X/G]_\bullet\) are induced by the standard simplicial structure on \( G^\bullet \). The importance of \([X/G]_\bullet\) is that it is a simplicial model for \( X_G := X \times_G EG \) which defines equivariant cohomology

\[
H^k_G(X, R) = H^k(X_G, R), \ R = \mathbb{Z}, \mathbb{Q}.
\]

More precisely, we have

\[
H^k([X/G]_\bullet, R) \cong H^k([X/G], R) = H^k_G(X, R), \ R = \mathbb{Z}, \mathbb{Q}.
\]

An important tool to compute the left-hand side is the spectral sequence

\[
E^q_\infty = H^q([X/G]_p, R) \Rightarrow E^k_\infty = H^k([X/G]_\bullet, R) = H^k_G(X, R), \ R = \mathbb{Z}, \mathbb{Q}.
\]

By [Del74], this spectral sequence is a spectral sequence of MHS if \( X \) is a complex algebraic variety (considered in the analytic topology) on which \( G \) acts algebraically. In particular, the equivariant cohomology groups \( H^k_G(X, R) \) carry natural MHS.

The Leray (or Serre) spectral sequence for the fibration \( X_G \to BG \) implies that \( H^k_G(X, \mathbb{Q}) \cong H^k(X, \mathbb{Q})^G \) as abelian groups. Of course, both sides carry MHS and the next lemma shows that they agree. This is well-known to experts\(^6\) but for lack of a reference, we give a proof here for completeness.

\[^6\text{We kindly acknowledge the help of Donu Arapura via MathOverflow.}\]
Lemma 4. Let $G$ be a finite group acting on a complex algebraic variety $X$. Then the spectral sequence (27) degenerates on the $E_2$-page and yields an isomorphism

$$H^k_G(X, \mathbb{Q}) \cong H^k(X, \mathbb{Q})^G$$

of $\mathbb{Q}$-MHS for each $k \geq 0$.

Proof. We express $E_1^{pq} = H^q((G^{p+1} \times X)/G, \mathbb{Z})$ of (27) as follows:

$$E_1^{pq} = H^q(G^{p+1} \times X, \mathbb{Z})^G$$

(freeness of action)

$$= \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, H^q(G^{p+1} \times X, \mathbb{Z}))$$

$$= \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, H^0(G^{p+1}, \mathbb{Z}) \otimes \mathbb{Z} H^q(X, \mathbb{Z}))$$

(Künneth formula)

$$= \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \text{Hom}_{\mathbb{Z}}(H_0(G^{p+1}, \mathbb{Z}), H^q(X, \mathbb{Z})))$$

(freeness of $H_0(G^{p+1}, \mathbb{Z})$)

$$= \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z} \otimes \mathbb{Z}[G] H_0(G^{p+1}, \mathbb{Z}), H^q(X, \mathbb{Z}))$$

(adjunction)

$$= \text{Hom}_{\mathbb{Z}[G]}(H_0(G^{p+1}, \mathbb{Z}), H^q(X, \mathbb{Z})).$$

In the last line we have used the adjunction between the trivial module functor and $- \otimes \mathbb{Z}[G]$ ([Wei94]). Since the complex $H_0(G^{*+1}, \mathbb{Z})$ coincides with the bar resolution $B_* \to \mathbb{Z}$, we see that

$$E_1^{pq} = \text{Hom}_{\mathbb{Z}[G]}(B_*, H^q(X, \mathbb{Q}))$$

as complexes for each $q \geq 0$. In particular, $E_1^{pq}$ computes group cohomology $H^k(G, H^q(X, \mathbb{Z}))$ for each $q \geq 0$ so that $E_2^{pq} = H^p(G, H^q(X, \mathbb{Q}))$. But $G$ is finite and we work over $\mathbb{Q}$ so that $H^p(G, H^q(X, \mathbb{Q})) = 0$ for all $p \geq 1$. Hence the spectral sequence (27) of MHS degenerates on the $E_2$-page to give isomorphisms

$$H^k(X, \mathbb{Q})^G \cong H^k([X/G], \mathbb{Q}) = H^k_G(X, \mathbb{Q})$$

of MHS.

\[\square\]

4.2. CY orbifolds and Hitchin systems. We begin with a general result on the Leray spectral sequence for equivariant maps.

Lemma 5. Let $G$ be a discrete group acting on topological spaces $X,Y,Z$. Further let $f : X \to Y$, $h : Y \to Z$ be $G$-equivariant morphisms. Then the Leray spectral sequence

$$R^p g_* f^q_* A_X \Rightarrow R^{p+q}(g \circ f)_* A_X$$

for any constant sheaf $A_X$ of an abelian group $A$ on $X$ lifts via the forgetful functor $\text{For} : \text{Sh}_G(Z) \to \text{Sh}(Z)$ to $G$-equivariant abelian sheaves $\text{Sh}_G(Z)$ on $Z$.

Proof. First of all, the exact forgetful functor $\text{For} : \text{Sh}_G(W) \to \text{Sh}(W)$ for $W = X,Y,Z$ commutes with the equivariant direct image functor $f_*^G$, e.g. For $\circ f_*^G \simeq f_* \circ \text{For}$. Since $\text{Sh}_G(W)$ has enough injectives, we can form derived direct image functors, e.g.

$$Rf_*^G : D^b_G(X) \simeq D^b(\text{Sh}_G(X)) \to D^b_G(Y) \simeq D^b(\text{Sh}_G(Y)).$$

Again they commute with the exact forgetful functors which implies the claim. \[\square\]
Proposition 4. Let $X = X_b$, $b ∈ B^o$, be a smooth quasi-projective Calabi-Yau threefold with $C$-action as in Section 3. Then there is a natural isomorphism

$$H^3([X/C], Z) = H^3_C(X, Z) \cong H^3(X, Z)^C$$

of $Z$-MHS.

We emphasize that we work over the integers giving a stronger result as Lemma 4. In general, such a result is false due to torsion, cf. Example 4 below.

Proof. As in the proof of Lemma 4, we employ the spectral sequence (27). In that proof we have seen that its $E_2$-page is of the form

$$E_2^{pq} = H^p(C, H^q(X, Z)).$$

We show that it degenerates on the $E_3$-page in this situation by proving

$$H^p(C, H^q(X, Z)) = 0, \ \forall p \geq 1, q \notin \{3, 4\}.$$  (30)

Of course this is automatic if we worked over $Q$. To show it over the integers, we observe that

$$H^0(X, Z) \cong H^0(Σ, π^*Z), \quad H^1(X, Z) \cong H^1(Σ, π^*Z), \quad H^2(X, Z) \cong H^2(Σ, π^*Z),$$

$$H^3(X, Z) \cong H^1(Σ, R^2π^*Z), \quad H^4(X, Z) \cong H^2(Σ, R^2π^*Z), \quad H^q(X, Z) = 0 \ \forall q ≥ 5$$

for the projection $π : X → Σ$. This is seen by using the Leray spectral sequence (cf. [DDP07], [Bec17]). Since $C$ acts trivially on $π^*Z$, it follows that $C$ acts trivially on $H^q(X, Z)$ for $q \notin \{3, 4\}$ by Lemma 5. This yields (30) so that the spectral sequence (27) gives the isomorphism

$$E_3^{0,3} = H^3(X, Z)^C \cong E_∞^{3,0} = H^3_C(X, Z)$$

of Z-MHS.

Example 4. The analogous statement is false for the minimal resolution $Ŷ → Y$ of the $Δ$-singularity $Y$ (assuming $Δ = Δ_b$): A group cohomology computation shows

$$H^3_C(Ŷ, Z) \cong H^2(Ŷ, Z)^C ⊕ Z/2Z.$$  (31)

Since $H^3(X_b, Z)$, $b ∈ B^o$, is up to a Tate twist a polarizable $Z$-HS (see [Bec17], Lemma 5) of weight 1, it follows that the orbifold intermediate Jacobian

$$J^2([X_b/C]) = H^3([X_b/C], C)/ (F^2H^3([X_b/C], C) + H^3([X_b/C], Z))$$

of the Calabi-Yau orbifold $[X_b/C]$ is an abelian variety. To globalize the previous discussion, we consider the augmented simplicial morphism associated to the family $π_C : [X/C] → B$ of Calabi-Yau orbifolds. By abuse of notation, it is again denoted by

$$π_C : [X/C], \quad \pi^*_C \hookrightarrow B, \quad α \rightarrow B.$$  

where the last arrow is the augmentation. In accordance with (26), we set

$$R^kπ_{C,*}Z = sR^k(π_{C,*})Z,$$

compare [Del74], Section 5.2, where $s$ stands for the total complex.
**Proposition 5.** Let $\pi_C^0 : [X^0/C] \to B^0$ be the family of orbifolds associated with the smooth family $\pi : X^0 \to B^0$. Then $R^3\pi^0_{C,*}\mathbb{Z}$ carries the structure of a polarizable $\mathbb{Z}$-VMHS such that

$$(R^3\pi^0_{C,*}\mathbb{Z})_b \cong H^3_c(X_b, \mathbb{Z}), \quad b \in B^0,$$

naturally as $\mathbb{Z}$-MHS.

**Proof.** This is a consequence of the more general treatment in [Bec18] on VMHS for simplicial smooth and quasi-projective morphisms (which are topologically locally trivial on each level). Alternatively, the V(M)HS-structure can be constructed directly in this present case by using an approximation of the classifying space $BC$ of $C$ by finite-dimensional smooth projective varieties. □

Therefore the orbifold intermediate Jacobians $J^2(\mathcal{X}^0/C)$ fit into the family $J^2([X^0/C]) \to B^0$ of abelian varieties. Using the methods of [Bec17], it is seen to be an algebraic integrable system.

**Theorem 6.** Let $\Delta$ be an irreducible Dynkin diagram, $C = AS(\Delta)$ and $G = G(\Delta)$ the corresponding simple adjoint complex Lie group. Further let $X \to B(\Sigma, \Delta)$ be one of the families of Calabi-Yau threefolds of Section 3 with $C$-action. Then

$$J^2([X^0/C]) \cong \text{Higgs}^0_1(\Sigma, G)$$
$$J^2([X^0/C]) \cong \text{Higgs}^0(\Sigma, L G),$$

as algebraic integrable systems over $B^0(\Sigma, \Delta)$.

**Proof.** The spectral sequence (27) globalizes to a spectral sequence of polarizable $\mathbb{Z}$-VMHS. Hence Proposition 4 implies

$$R^3\pi^0_{C,*}\mathbb{Z} \cong (R^3\pi_*\mathbb{Z})^C \cong (R^3\pi^0_{C,*}\mathbb{Z})^C$$

as polarizable $\mathbb{Z}$-VHS of weight 1 (up to a Tate twist). In particular,

$$J^2([X^0/C]) \cong J^2_c([X^0])$$

over $B^0$ where the right-hand side is the intermediate Jacobian fibration defined by $C$-invariants in cohomology. Now the claim follows from [Bec17], Theorem 6.

Replacing $\pi_{C,*}$ with $\pi_{C,1}$ and using Theorem 8 of [Bec17] gives the second isomorphism. □

**Appendix A. Folding**

Let $\Delta$ be an irreducible Dynkin diagram. We follow [Slo80b] and define the associated symmetry group of $\Delta$ via

$$AS(\Delta) := \begin{cases} 1, & \text{\Delta is of type ADE}, \\ \mathbb{Z}/2\mathbb{Z}, & \text{\Delta is of type B}_k, C_k, F_4, \\ S_3, & \text{\Delta is of type G}_2, \end{cases}$$

for $k \geq 2$. There is a unique irreducible ADE-Dynkin diagram $\Delta_h$ such that $AS = AS(\Delta) \subset \text{Aut}(\Delta_h)$ and $\Delta = \Delta_h^{AS}$. Here $\Delta_h^{AS}$ stands for the Dynkin diagram which is obtained by taking $AS(\Delta)$-invariants $R_h^{AS}$ in the root space $R_h = R(\Delta_h)$ associated with $\Delta_h$ (cf. [Bec], Chapter 1.2, or [Spr09]).
Remark 7. To obtain a reasonable notion of folding on the level of root spaces, it is important that we only work with Dynkin graph automorphisms $\text{Aut}_D(\Delta_h) \subset \text{Aut}(\Delta_h)$. These are graph automorphisms $a \in \text{Aut}(\Delta_h)$ such that $a(v)$ and $v$ are not direct neighbors for each vertex $v \in \Delta_h$. If $\Delta_h \neq A_{2n}$, then $\text{Aut}_D(\Delta_h) = \text{Aut}(\Delta_h)$ and $\text{Aut}_D(\Delta_h) = 1$ if $\Delta_h = A_{2n}$.

Restricted to Dynkin diagrams of type $B_k, C_k, F_4, G_2$ (BCFG-Dynkin diagrams for short), we obtain a bijection

$$\{\Delta \text{ of type BCFG}\} \rightarrow \{ (\Delta_h, C) \mid \Delta_h \text{ ADE, } 1 \neq C \subset \text{Aut}_D(\Delta_h) \}$$

$$\Delta \mapsto (\Delta_h, \text{AS}(\Delta))$$

We say that $\Delta = \Delta_h^C$ is obtained from $(\Delta_h, C)$ (or simply $\Delta_h$) by folding. For convenience we summarize the corresponding types in the following table

| $\Delta$ | $\Delta_h$ | $\text{AS}(\Delta)$ |
|----------|-----------|---------------------|
| $B_{k+1}$ | $A_{2k+1}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $C_k$ | $D_{k+1}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $F_4$ | $E_6$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $G_2$ | $D_4$ | $S_3$ |

(34)

Finally, we collect the weights of the ADE-singularities that are naturally induced by their Lie-theoretic realization.

| Dynkin type of $\Gamma$ and equation | $(w_{r+1}, w_{r+2}, w_{r+3}; d)$ |
|-------------------------------------|----------------------------------|
| $A_k$: $x^{k+1} - yz = 0$ | $(2, k + 1, k + 1; 2(k + 1))$ |
| $D_k$: $x^{k-2} - y^2 - z^2 = 0$ | $(2, k - 2, k - 1; 2k - 2)$ |
| $E_6$: $x^4 + y^3 + z^2 = 0$ | $(6, 8, 12; 24)$ |
| $E_7$: $x^3 y + y^3 + z^2 = 0$ | $(8, 12, 18; 36)$ |
| $E_8$: $x^5 + y^3 + z^2 = 0$ | $(12, 20, 30; 60)$ |

(33)

Table 1. ADE-singularities together with their natural weights

Here $r$ is the rank of the corresponding simple complex Lie algebra. Note that by definition, this table contains the natural weights of all $\Delta$-singularities.

Appendix B. Slodowy Slices for $B_2$-singularities

Let $\Delta = B_2$ and $^7(\Delta_0, C) = (A_3, \mathbb{Z}/2\mathbb{Z})$ be the associated pair. Further we let

$$g_0 = g(\Delta_0) = \mathfrak{sl}(4, \mathbb{C}), \quad g = g(\Delta) = \mathfrak{so}(5, \mathbb{C})$$

be the corresponding simple complex Lie algebras. We give two Slodowy slices $S_0 \subset g_0$ and $S \subset g$ together with the CA- and C-action respectively (see (4) and (21)) and directly compute that they both realize a semi-universal deformation of the $B_2$-singularity.

$^7$Here we use the notation $\Delta_0$ instead of $\Delta_h$ in the main text for notational reasons.
B.1. **Extrinsic case** \((g_0)\). We consider the following \(\mathfrak{sl}_2\)-triple in \(g_0\):

\[
(35) \quad x_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\]

The Slodowy slice with respect to \((x_0, y_0, h_0)\) is given by

\[
(36) \quad S_0 = x_0 + \ker \text{ad}(y_0) = \begin{cases} 
-3a & b & 0 & 0 \\
0 & a & 1 & 0 \\
0 & c & a & 1 \\
d & e & c & a 
\end{cases} : a, b, c, d, e \in \mathbb{C}
\]

Let \(\phi \in \text{Aut}(g_0)\) be given by \(\phi(A) = -A^t\) and

\[
(37) \quad g_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \end{pmatrix}
\]

(actually the class in \(G_0 = PGL(4, \mathbb{C})\)). Then it is immediate to check that

\[
\tau := \text{Ad}(g_0) \circ \phi \in CA(x, h)
\]

and \(\tau^2 = 1\). It generates the subgroup \(CA \subset CA(x, h)\) which is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\). Its action on \(S_0\) is given by

\[
(38) \quad \tau \cdot \begin{pmatrix} -3a & b & 0 & 0 \\
0 & a & 1 & 0 \\
0 & c & a & 1 \\
d & e & c & a \end{pmatrix} = \begin{pmatrix} 3a & d & 0 & 0 \\
0 & -a & 1 & 0 \\
0 & c & -a & 1 \\
b & -e & c & -a \end{pmatrix}.
\]

To determine its action on the base, we compute \(\sigma_0 : S_0 \to t_0/W_0\). We take as homogeneous generators the coefficients \(\chi_2, \chi_3, \chi_4\) of degree 2, 3, 4 of the characteristic polynomial. If \(B \in S_0\) is as in (36), then

\[
(39) \quad \sigma_0(B) = (\chi_2(B), \chi_3(B), \chi_4(B)) = (-6a^2 - 2c, 8a^3 - 4ac - e, -3a^4 + 6a^2c - bd - 3ae).
\]

It can be readily checked that \(\sigma^{-1}(0, 0, 0)\) is an \(A_3\)-singularity given by

\[
(40) \quad c = -3a^2, \quad e = 20a^3, \quad -81a^4 - bd = 0.
\]

in the notation of (36). Moreover, we have

\[
(41) \quad \sigma_0(\tau \cdot B) = (\chi_2(B), -\chi_3(B), \chi_4(B)).
\]

Hence if we restrict to the subspace

\[
(t_0/W_0)^{CA} = \{(s, 0, t) \in \mathbb{C}^3 \cong t_0/W_0 \mid s, t \in \mathbb{C}\} \xrightarrow{i} t_0/W_0,
\]

then \(CA\) preserves the fibers of \(\sigma_0\). Since \(i^*S_0 \subset S_0\) is defined by \(e = 4ac - 8a^3\), we obtain

\[
(42) \quad \sigma_0(B) = (-6a^2 - 2c, 0, -27a^4 + 18a^2c - bd), \quad B \in i^*S_0.
\]

Finally, we see that \(CA\) acts on the \(A_3\)-singularity \(\sigma_0^{-1}(0, 0, 0)\) as described in Example 2 by setting \(x = -3a, y = b, z = d\) in the notation of (40).
B.2. Intrinsic case ($\mathfrak{g}$). For this example, we use the conventions and methods, e.g. the classification of nilpotent orbits in terms of weighted Dynkin diagrams, of [CM93], Chapter 5. First of all, let

$$\mathfrak{g} = \mathfrak{so}(5, \mathbb{C}) \cong \begin{cases} \begin{pmatrix} 0 & u_1 & u_2 & v_1 & v_2 \\ -v_1 & a_1 & a_2 & 0 & b \\ -v_2 & a_3 & a_4 & -b & 0 \\ -u_1 & 0 & c & -a_1 & -a_3 \\ -u_2 & -c & 0 & -a_2 & -a_4 \end{pmatrix} : u_i, v_j, a_k, b, c \in \mathbb{C} \end{cases}$$

This form of $\mathfrak{so}(5, \mathbb{C})$ has the advantage that diagonal matrices give a Cartan $t \subset \mathfrak{g}$. Then we consider the following $\mathfrak{sl}_2$-triplet $(x, y, h)$ where $x \in \mathfrak{g}$ is a subregular nilpotent element:

$$x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. $$

The corresponding Slodowy slice $S = x + \ker \text{ad} y \subset \mathfrak{g}$ is given by

$$S = \begin{cases} C = \begin{pmatrix} 0 & -a & -b & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & c & -b & 0 & 0 \\ a & 0 & d & 0 & -c \\ b & -d & 0 & -1 & b \end{pmatrix} : a, b, c, d \in \mathbb{C} \end{cases} $$

Again using the coefficients $\chi_2, \chi_4$ of the characteristic polynomial of degree 2 and 4 respectively, the adjoint quotient restricted to $S$ is given by

$$\sigma(C) = (\chi_2(C), \chi_4(C)) = (-b^2 - 2a - 2c, 2ab^2 + 2b^2c + 2ac + c^2 + 2cd).$$

In the notation of (45), we see that $\sigma^{-1}(0, 0, 0)$ is given by

$$a = -c - \frac{b^2}{2}, \quad b^4 + c(b^2 - 2d + c) = 0$$

which is immediately seen to be an $A_3$-singularity.

Using the fact that $Z_G(h)$ is the subgroup in $G \cong PSO(5, \mathbb{C})$ generated by the diagonal maximal torus $T$ and the unipotent subgroup $U_{\alpha_2}$ corresponding to the short root $\alpha_2 \in \Delta$, we compute

$$C(x, h) = Z_G(x) \cap Z_G(h) = \begin{cases} D^k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}^k : k = 0, 1 \end{cases} $$
(actually the class in $G$). Observe that in this case $C = C(x, h)$, cf. (4). Finally, the action of $C$ on $S$ is given by

$$
D \cdot C = \begin{pmatrix}
0 & a + 2d & b & 1 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 2a + c + 2d & b & 0 & 0 \\
-a - 2d & 0 & d & 0 & -2a - c - 2d \\
-b & -d & 0 & -1 & -b
\end{pmatrix}.
$$

It is readily seen from (47) that $(\sigma^{-1}(0), C)$ is indeed a $B_2$-singularity.

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