AN ALGEBRAIC APPROACH TO SYMMETRIC EXTENDED FORMULATIONS

GÁBOR BRAUN AND SEBASTIAN POKUTTA

Abstract. Extended formulations are an important tool to obtain small (even compact) formulations of polytopes by representing them as projections of higher dimensional ones. It is an important question whether a polytope admits a small extended formulation, i.e., one involving only a polynomial number of inequalities in its dimension. For the case of symmetric extended formulations (i.e., preserving the symmetries of the polytope) Yannakakis established a powerful technique to derive lower bounds and rule out small formulations. We rephrase the technique of Yannakakis in a group-theoretic framework. This provides a different perspective on symmetric extensions and considerably simplifies several lower bound constructions.

1. Introduction

Extended formulations regained a lot of interest lately (cf., e.g., Conforti et al. [2010], Faenza and Kaibel [2009], Faenza et al. [2012], Fiorini et al. [2011a], Goemans 2009, Kaibel et al. [2010], Kaibel and Pashkovich [2011], Kaibel [2011], Pashkovich [2009]). The main idea behind extended formulations is to represent a given polytope as a projection of a higher dimensional one, which is usually referred to as the extension. Whereas at first this may not seem useful, the higher dimensional polytope might be described by considerably fewer inequalities. Hence it might admit a polynomial time solvable linear program, if not only the number of inequalities is polynomial, but, also the coefficients appearing in the projection and the defining inequalities are appropriately polynomially bounded, e.g., in the dimension. Therefore, we are in particular interested in finding small extended formulations, i.e., whose size (here measured in the number of inequalities only) is polynomial in the dimension of the initial polytope.

Due to its appeal of representing a polytope with an exponential number of inequalities in polynomial size, in the 1980s Swart tried to show $P = \text{NP}$ by devising compact extended formulations for the traveling salesman problem. All these formulations shared the commonality of being symmetric, and it was Yannakakis’s seminal paper (see Yannakakis [1991]) which put an end to this by showing that the traveling salesman polytope does not admit a symmetric extended formulation of polynomial size. In a recent paper (Fiorini et al. [2011b]) it was shown that the requirement for symmetry can be dropped as well and an unconditional super-polynomial lower bound for...
the size of any extended formulation of the traveling salesman polytope was obtained.

At its core Yannakakis’s work provides techniques for computing the size of an extended formulation via decomposing slack matrices as the product of two matrices with non-negative entries. Moreover, his work establishes a method for bounding from below the size of symmetric extended formulations. Using these techniques, he proved, among others, that the perfect matching polytope cannot have a symmetric extended formulation of polynomial size, which was the basis for his impossibility result on the TSP polytope.

This result was later extended by Kaibel et al. [2010] to (weakly-)symmetric extended formulations of cardinality constrained matching which in contrast do possess an asymmetric extended formulation of polynomial size. Similarly, in Goemans [2009] an asymmetric extended formulation of optimal size $O(n \log n)$ for the permutahedron is provided, based on AKS-sorting networks. A symmetric extended formulation for the permutahedron is the Birkhoff polytope with $O(n^2)$ inequalities. This formulation is also optimal in size as established by Pashkovich [2009]; another example for a gap between the best symmetric and asymmetric extension.

A more general framework for constructing (asymmetric) extended formulations by so-called, polyhedral relations was established in Kaibel and Pashkovich [2011]. This quite general method allowed to recast several constructions of asymmetric extended formulations (e.g., the $O(n \log n)$ extended formulation of the permutahedron) in a unified framework.

**Contribution.** We will focus on symmetric extended formulations in this article. We streamline and extend the lower bound estimation technique of Yannakakis [1991] via algebraic arguments with the main structure being a group action expressing the symmetries.

The results of the algebraic recasting are two compact theorems (Theorem 6.1 for general symmetric extended formulations and Theorem 6.2 for super-linear bounds), which virtually encapsulate all the necessary polyhedral and algebraic arguments in black boxes and which provide a uniform view on symmetric extended formulations. From these black boxes many known results follow naturally and shortly (e.g., those in Kaibel et al. [2010], Pashkovich [2009]).

We stress that we do not provide any new or stronger lower bounds but rather a natural algebraic approach to symmetric extensions as a different perspective of known results. We believe that further insights into the underlying mechanics of Yannakakis’s approach can be obtained from this framework and that the algebraic versions are more amendable to SDP extensions. As an indication we formulate Theorem 7.3. However, we were unable to derive new lower bounds for SDP extensions.

As part of streamlining, several technical concepts needed in previous works could be omitted: for example an intermediate extension that has only vertices in $\{0, 1\}$ or indexed families or partitions compatible with sections. Moreover, some restrictions were relaxed at no cost: e.g., the group action can be any affine action and not just coordinate permutation.
In the process of reformulating the technique we also obtain several unnecessary generalizations, i.e., generalizations that do provide further insight into the essence of the problem but do not lead to stronger lower bounds.

Outline. We start with some preliminaries in Section 2 and recall the considered polytopes in Section 3. In Section 4 we study the well-known polytope $A_n$, which is of special importance in the context of cutting-planes and whose face lattice is close to that of the parity polytope. Then we derive the main theorem on lower bounds in Section 5 and reprove Yannakakis’s lower bound for the matching polytope. Next, we conduct a more detailed analysis of polytopes with small extensions in Section 6. We provide significantly shortened proofs for the lower bounds on the symmetric extension complexity of the permutahedron and the cardinality indicating polytope. In Section 7 we provide an SDP version for one of our main theorems (Theorem 5.1).

2. Preliminaries

In the following we briefly recall a few algebraic notions. As usual, we accompany formal definitions with commutative diagrams to give a visual representation. We write maps on the right except for the section map $s$ for reasons of readability. Let $\log(\cdot)$ denote the logarithm to base 2.

2.1. Symmetric extensions. Let $P \subseteq \mathbb{R}^m$ be a polytope. Recall that an extension of $P$ is a polytope $Q \subseteq \mathbb{R}^d$ together with a linear map $p: \mathbb{R}^d \to \mathbb{R}^m$ satisfying $Qp = P$. We use standard notations for group actions as to be found, e.g., in Dixon and Mortimer [1996]: let the group $G$ act on $X$ and let $g \in G$, $x \in X$ be arbitrary elements. The action of $g$ on $x$ is simply $gx$; in particular groups act on the left.

Definition 2.1. Let $G$ be a group with an affine group action on $\mathbb{R}^m$. Then $P \subseteq \mathbb{R}^m$ is a $G$-polytope if $G$ leaves $P$ invariant, i.e., $gP = P$ for all $g \in G$.

The group $G$ will usually be either the symmetric group $S_n$ on $n$ elements or the alternating group $A_n$ on $n$ elements.

We will work with symmetric extensions of a $G$-polytope $P$ defined as follows.

Definition 2.2. A symmetric extension of a $G$-polytope $P$ is an extension $Q$ together with $p: Q \to P$ where $Q$ is a $G$-polytope and $p$ is $G$-invariant, i.e., $g(xp) = (gx)p$ for all $g \in G$ and $x \in Q$.

In order to compare extended formulations we define the following measure.

Definition 2.3. Let $Q$ be an extension of the polytope $P$. Then the size of $Q$ is the number of its facets. The size of the smallest extension of $P$ is denoted by $xc(P)$ and similarly the size of the smallest symmetric extension for a group $G$ is denoted by $xc_G(P)$.

At first glance Definition 2.2 seems more restrictive than Yannakakis’s one. However it turns out that Yannakakis’s seemingly more general definition (and also the generalization given in Kaibel et al., 2010) does not lead to extended formulations of smaller size, as we will see at the end of this section.
We further need the notion of a section which assigns to every vertex in $P$ a pre-image in $Q$ under the projection $p$.

**Definition 2.4.** Let $Q$ and $P$ be $G$-polytopes such that $Q$ is a symmetric extension of $P$. Then $s : \text{vertex}(P) \to Q$ is a section if $s(x)p = x$ for all $x \in \text{vertex}(P)$. Further it is an invariant section if we additionally have $s(gx) = gs(x)$ for all $x \in \text{vertex}(P)$ and $g \in G$.

Note that a section $s$ is usually non-linear. In fact, as pointed out in Kaibel et al. [2010], if $s$ were affine and $Q$ an extension of $P$, then $Q \cap \text{aff} \{s(x) \mid x \in X\}$ would be isomorphic to $P$. Therefore $Q$ would have at least as many facets as $P$, and so could not have size smaller than $P$.

Recall that a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ is $G$-invariant if it is invariant under the linear part of the action of $G$, i.e., $\langle gx - g0, gy - g0 \rangle = \langle x, y \rangle$ for all $g \in G$ and $x, y \in \mathbb{R}^n$. (The linear part of the $G$-action is $x \mapsto gx - g0$.)

It is easy to see that there always exist an invariant scalar product and an invariant section. In fact the invariant section as well as the invariant scalar product arise from averaging over the group. The proof follows standard arguments; we include it for the sake of completeness in Appendix A.

**Lemma 2.5.** Let $P \subseteq \mathbb{R}^m$ be a $G$-polytope and $Q \subseteq \mathbb{R}^d$ be a $G$-polytope so that $Q$ is a symmetric extension of $P$ with projection $p$ as before. Further let $s : \text{vertex}(P) \to Q$ be a section and $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathbb{R}^d$. Then:

1. There exists an invariant scalar product $\langle \cdot, \cdot \rangle$ defined (via averaging over the linear part) as $\langle x, y \rangle := \frac{1}{|G|} \sum_{g \in G} \langle gx - g0, gy - g0 \rangle$.
2. There exists an invariant section $\bar{s}$ given by $\bar{s}(x) := \frac{1}{|G|} \sum_{g \in G} g^{-1}s(gx)$.

The essence of the proof is the celebrated symmetrizing trick.

### 2.2. Group actions.

Let $G$ act on a set $X$. Recall, that the orbit of an element $x \in X$ under $G$ is defined as $Gx := \{\pi x \mid \pi \in G\}$. The stabilizer of an element $x \in P$ is the subgroup of elements of $G$ that leave $x$ invariant, i.e., $G_x := \{\pi \in G \mid \pi x = x\}$. Recall the following well-known formula for the size of orbits:

**Lemma 2.6 (Orbit-Stabilizer Theorem).** Let $G$ be a finite group acting on a finite set $X$. For any $x \in X$ we have

$$|Gx| = |G : G_x| = |G| / |G_x|.$$  

In particular, if $P$ is a $G$-polytope, then $G$ also acts on the face lattice of $P$. We will be interested in the orbits and stabilizers of faces, for which the following observation and lemma will be helpful. The observation is just a corollary to Lemma 2.6.

**Observation 2.7.** Let $P$ be a $G$-polytope with $d$ facets. Then

$$|G : G_j| \leq d$$  

for any facet $j$ of $P$.  

For a finite set $Y \subseteq X$, we define $A(Y)$ to be the alternating group permuting the elements of $Y$ and leaving $X \setminus Y$ fixed; the ambient set will be clear from the context.

**Lemma 2.8.** [Dixon and Mortimer, 1996, Theorem 5.2A] Let $G \subseteq A_n$ and $n \geq 10$. Then $|A_n : G| < \binom{n}{k}$ with $k \leq \frac{n}{2}$ implies one of the following

1. there is an invariant subset $W$ with $|W| < k$ such that $A([n] \setminus W)$ is a subgroup of $G$;
2. $|A_n : G| = \frac{1}{2}(\frac{n}{2})^n$ with $n$ even, $A_{n/2} \times A_{n/2}$ is a subgroup of $G$, and $k = n/2$.

Note that one can obtain a strengthened version of Lemma 2.8 by iteratively applying it to the obtained subgroup.

**2.3. Weakly symmetric extensions.** We conclude this section by showing how Yannakakis’s concepts fit into our framework. For this we will use the concept of a weakly-symmetric extension, which had been used before in [Kaibel et al. 2010]. We will show that every weakly-symmetric extension (a generalization of, both, our symmetric extensions and Yannakakis’s one) induces a symmetric one of at most the same size. Therefore weakly-symmetric extensions do not provide smaller extended formulations and we maintain full generality by confining ourselves to symmetric extensions while being able to simplify arguments.

**Definition 2.9.** A *weakly-symmetric extension* of a $G$-polytope $P$ is a $\tilde{G}$-polytope $Q$ together with a group epimorphism $\alpha: \tilde{G} \to G$ and a surjective $\alpha$-linear affine map $p: Q \to P$, i.e., $(\tilde{\pi}x)p = (\tilde{\pi}\alpha)(xp)$ for all $\tilde{\pi} \in \tilde{G}$ and $x \in Q$.

In fact we have the following commutative diagram for all $\tilde{\pi} \in \tilde{G}$:

```
Q \xrightarrow{\tilde{\pi}} Q
\downarrow p \quad \downarrow p
P \xrightarrow{\tilde{\pi}\alpha} P
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We now show that weakly-symmetric extended formulations do not provide smaller formulations than symmetric extended formulations:

**Proposition 2.10.** For every weakly-symmetric extended formulation $Q$ of $P$ with $Q \subseteq \mathbb{R}^d$ being a $\tilde{G}$-polytope, $P \subseteq \mathbb{R}^m$ being a $G$-polytope, projection $p: Q \to P$, and group epimorphism $\alpha: \tilde{G} \to G$, the restriction to $R := Q^{\ker \alpha}$ is a symmetric extended formulation and $R$ has dimension and facets at most that of $Q$.

**Proof.** As $\ker \alpha$ is a normal subgroup, $R = Q^{\ker \alpha}$ and $X := (\mathbb{R}^d)^{\ker \alpha}$ are invariant under the $\tilde{G}$-action. Since the action is affine, $X$ is an affine subspace. Thus $R$ is the intersection of $Q$ with the affine subspace $X$, and hence it has no higher dimension and no more facets than $Q$.

To make $R$ a $\tilde{G}$-polytope, we define the action of $g \in G$ on an element $x \in R$ via

$$gx := \tilde{g}x, \quad \tilde{g}\alpha = g$$
where $\tilde{g} \in \tilde{G}$ is arbitrary so that $\tilde{g} \alpha = g$ holds. This action is well-defined, because $\ker \alpha$ acts trivially on $R$ by definition, i.e., whenever $\tilde{g} \in \ker \alpha$, then $\tilde{g} x = x$ for all $x \in R$.

It is obvious that the restriction $p: R \to Q$ preserves the $G$-action. Finally we show that $Rp = P$. Let $x \in P$ be arbitrary and choose any $y \in Q$ so that $yp = x$. As a shorthand notation, let $y[H] := \frac{1}{|H|} \sum_{h \in H} hy$ denote the group average of $y$ with respect to any group $H$. Then $y[\ker \alpha] \in R$ and we have

\[(y[\ker \alpha])p = (yp)[(\ker \alpha)\alpha] = yp = x,
\]
and so the claim follows.

3. Considered polytopes

In this section we recall the well-known polytopes that will appear later.

3.1. The cardinality indicating polytope. The cardinality indicating polytope $P_{\text{card}}(n)$ is the convex hull of all vectors $(x, e_{\|x\|_1})$ for $x \in \{0, 1\}^n$ where $e_0, \ldots, e_n$ are linearly independent. The second vector $e_{\|x\|_1}$ indicates the number of 1-entries in $x$.

\[P_{\text{card}}(n) := \text{conv}\left\{(x, e_{\|x\|_1}) \mid x \in \{0, 1\}^n\right\}\]

It can be described by the following system of inequalities (with $z = \sum_{j=0}^n z_j e_j$):

\[
\begin{align*}
\sum_{i \in S} x_i &\leq |S| \sum_{j=0}^n z_j \quad \forall \emptyset \not\subseteq S \subseteq [n] \\
\sum_{i \in [n]} x_i &\leq \sum_{j=0}^n jz_j \\
\sum_{j=0}^n z_j &\leq 1 \\
x_i, z_j &\in [0, 1] \quad \forall i \in [n], j = 0, \ldots, n
\end{align*}
\]

The cardinality indicating polytope has a symmetric extended formulation of size $\Theta(n^2)$ as shown in Köppe et al. [2008].

3.2. The Birkhoff polytope. The Birkhoff polytope $P_{\text{birk}}(n)$ is the convex hull of all doubly stochastic $n \times n$ matrices (or equivalently of all $n \times n$ permutation matrices). It can be described by the following system of inequalities:

\[
\begin{align*}
\sum_{i \in [n]} x_{ij} &\leq 1 \quad \forall j \in [n] \\
\sum_{j \in [n]} x_{ij} &\leq 1 \quad \forall i \in [n] \\
x_{ij} &\in [0, 1] \quad \forall i, j \in [n]
\end{align*}
\]
3.3. The permutahedron. The permutahedron \( P_{\text{perm}}(n) \) is the convex hull of all permutations of the numbers \( 1, \ldots, n \), i.e.,
\[
P_{\text{perm}}(n) := \text{conv} \{\pi(1, \ldots, n) | \pi \in S_n\}.
\]
It can be described by the following system of inequalities:
\[
\sum_{i \in S} x_i \geq \frac{|S|(|S| + 1)}{2} \quad \forall \emptyset \neq S \subseteq [n]
\]
\[
\sum_{i \in [n]} x_i = \frac{n(n+1)}{2}
\]
and it can be obtained by a projection of the Birkhoff polytope, i.e., it has a symmetric extended formulation of size \( O(n^2) \). Also, symmetric extended formulation of the permutahedron needs at least \( \Omega(n^2) \) inequalities by Pashkovich [2009] and so the Birkhoff polytope is an optimal extension. On the other hand there exists an asymmetric extended formulation of the permutahedron of size \( O(n \log n) \) by Goemans [2009] which is optimal.

3.4. The spanning tree polytope. For a graph \( G = (V, E) \) and \( U \subseteq V \) let \( E[U] \) denote the set of edges supported on \( U \). The spanning tree polytope of \( G \) (denoted by: \( P_{\text{STP}}(G) \)) is given by the following system of inequalities:
\[
\sum_{e \in E[U]} x_e \leq |U| - 1 \quad \forall \emptyset \neq U \subseteq V
\]
\[
\sum_{e \in E} x_e = n - 1
\]
\[
x_e \in [0, 1] \quad \forall e \in E.
\]

There exists an extended formulation of size \( O(n^3) \) due to Martin [1991] and a lower bound of \( \Omega(n^2) \) follows from the non-negativity constraints. An interpretation of the associated communication protocol can be found in Fiorini et al. [2011a].

4. The polytope \( \mathcal{A}_n \)

In the following we consider the well-known polytope \( \mathcal{A}_n \), which is of particular interest in the context of cutting-plane procedures. It realizes maximal rank for all known operators and it represents a universal obstruction for any admissible cutting-plane procedure (see Pokutta and Schulz [2010]). Moreover \( \mathcal{A}_n \) will serve as an important example showing that the conditions of Theorem 6.1 are necessary. The polytope \( \mathcal{A}_n \) is given by
\[
\mathcal{A}_n := \left\{ x \in [0,1]^n \left| \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2} \quad \forall I \subseteq [n] \right. \right\}.
\]
With \( F_1^n := \{ x \in \{0, 1/2, 1\}^n | \text{exactly one entry equal to 1/2} \} \) we have \( \mathcal{A}_n = \text{conv} F_1^n \) (see e.g., Pokutta and Schulz [2011]); we drop the index \( n \) if it is clear from the context. For a vector \( v \in F_1 \) let \( \text{supp}_i(v) := \{ j \in [n] | v_j = i \} \).

We provide a symmetric extended formulation of \( \mathcal{A}_n \) of size \( O(n) \).

**Theorem 4.1.** Let \( \mathcal{A}_n \) be defined as above. Then there exists a symmetric extended formulation of \( \mathcal{A}_n \) of size \( O(n) \).
While this formulation is polyhedral it is not given by inequalities. However we can introduce new variables \( L \) and we obtain a new polytope \( L_n \)

\[
L_n := \left\{ (y, z) \in \left[0, \frac{1}{2}\right]^{2n} \middle| y_i + z_i \leq \frac{1}{2}, \sum_{i \in [n]} y_i + z_i = \frac{n-1}{2} \quad \forall i \in [n] \right\}.
\]

Observe that \( L_n \) is given by \( 3n \) inequalities \((n\text{ in the formulation and } y_i, z_i \geq 0 \text{ for all } i \in [n])\) and \( 2n \) variables. Moreover we claim that with the projection \( p \) defined via \((y_i, z_i) \mapsto x_i = y_i - z_i \) for all \( i \in [n] \) we have \( p(L_n) = Q_n \). Clearly \( Q_n \subseteq p(L_n) \). For the inverse inclusion observe that a vertex of \( L_n \) can have only \( \{0, 1/2\}\)-entries.

A larger compact extended formulation of size \( O(n^2) \) can be obtained using Balas’s union of polyhedra (see Balas [1985] and Balas [1998]). This formulation only preserves the symmetries permuting coordinates, however our extension in Theorem 4.1 preserves the full symmetry group \( Z_2 \wr S_n \) of the cube.

We will now derive a lower bound on the extension complexity of \( A_n \).

**Lemma 4.2.** [Goemans, 2005, Theorem 1] Let \( P \) be any polyhedron in \( \mathbb{R}^n \) with \( v(P) \) vertices. Then the number of facets \( t(Q) \) of any extended formulation \( Q \) of \( P \) satisfies

\[
t(Q) \geq \log(v(P)).
\]

Using Lemma 4.2 we obtain the following lower bound on the extension complexity of \( A_n \).

**Lemma 4.3.** Let \( A_n \) be defined as above. Then \( xc(A_n) \in \Omega(n) \).

**Proof.** Observe that \( |F_1| = n2^{n-1} \) and thus by Lemma 4.2 we obtain \( xc(A_n) \geq \log(n) + (n - 1) \in \Omega(n) \).

Combining Lemma 4.3 and Theorem 4.1 we obtain:

**Corollary 4.4.** The symmetric extension complexity \( xc_{A_n}(A_n) = xc(A_n) \) is \( \Theta(n) \).

One can also obtain an extended formulation of size \( O(n) \) using reflections at the hyperplanes \( x_i = \frac{1}{2} \) (see Kaibel and Pashkovich [2011]), however this formulation is asymmetric.

Finally, we would like to point out that all results of this section also apply to the polytope \( B_n \) given by

\[
B_n := \left\{ x \in [0,1]^n \middle| \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq 1 \quad \forall I \subseteq [n] \right\}.
\]
This is of particular interest because the parity polytope given by

\[
\text{Par}_n := \left\{ x \in [0,1]^n \middle| \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq 1 \quad \forall I \subseteq [n], |I| \text{ odd} \right\}
\]

is closely related to \( B_n \) and the cube \([0,1]^n\). In fact, the face lattice of \( \text{Par}_n \) looks very much like \( B_n \) or \([0,1]^n\). By the above results we have \( xc_{A_n}(B_n), xc_{A_n}([0,1]^n) \in O(n) \), even though \( xc_{A_n}(\text{Par}_n) \in \Omega(n \log n) \) by (Pashkovich [2011]).

5. The lower bound black-box for symmetric extended formulations

We will now present the main theorem that we will use in the following to establish lower bounds.

**Theorem 5.1.** Let a \( G \)-polytope \( Q \subseteq \mathbb{R}^d \) be a symmetric extension of a \( G \)-polytope \( P \subseteq \mathbb{R}^m \). For every facet \( j \) of \( Q \) let \( F_j \) be a refinement of the \( G_j \)-orbit partition of the vertex set \( X \) of \( P \). Then for every real solution to the following inequality system in the \( c_x \)

\[
\sum_{x \in X} c_x = 1,
\]

\[
\sum_{x \in F} c_x \geq 0,
\]

\( F \in F_j \), \( j \) facet of \( Q \)

the point \( \sum_{x \in X} c_x x \) lies in \( P \).

**Proof.** Let \( \langle ., . \rangle \) be an invariant scalar product on \( \mathbb{R}^d \). Let \( n_j \) be the normal vector of facet \( j \) pointing inwards. The inequality of the facet \( j \) is thus of the form \( \langle n_j, y \rangle \geq r_j \) for some real \( r_j \). These are clearly invariant: they are permuted together with the facets, i.e., \( n_{gj} = gn_j - g0 \) and \( r_{gj} = r_j \) for all \( g \in G \).

Let \( s : X \to Q \) be an invariant section of \( p \). Via invariance, the value \( \langle n_j, s(x) \rangle - r_j \) is constant as \( x \) runs through a \( G_j \)-orbit. In particular, it is a constant \( A_F \geq 0 \) on every \( F \in F_j \); note that \( F \) is a subset of the vertex set \( X \) of \( P \). Thus

\[
\left\langle n_j, \sum_{x \in X} c_x s(x) \right\rangle - r_j = \sum_{x \in X} c_x \langle n_j, s(x) \rangle - r_j = \sum_{F \in F_j} \sum_{x \in F} c_x A_F \geq 0.
\]

This shows that \( \sum_{x \in X} c_x s(x) \in Q \), hence applying \( p \) we obtain \( \sum_{x \in X} c_x x \in P \).

The result above has a particularly nice interpretation. When considering a symmetric extension we are allowed to consider affine combinations of points, rather than convex combinations, as long as each sum of coefficients along an orbit is non-negative. Put differently, convexity usually requires for a point to be written as a convex combination. In the presence of symmetry this requirement can be relaxed to an affine combination of points that is convex when averaged over the orbits.

**Theorem 5.1** can be used to bound the size of extended formulations as follows.
Theorem 5.3. Let \( \mathcal{Q} \subseteq \mathbb{R}^d \) of a \( G \)-polytope \( P \subseteq \mathbb{R}^m \) with projection \( p \). Then a lower bound on the size of \( Q \) (as the number of facets) can be established in the following way via Theorem 5.1:

1. Choose a subpartition \( \mathcal{F}_j \) of the \( G_j \)-orbit partition of the vertices of \( P \) for all facets \( j \) of a hypothetical \( Q \) of small size.
2. Find a particular solution \( c_x \) with \( x \in X \).
3. Show that \( \sum_{x \in X} c_x x \notin P \).

Steps [2] and [3] are usually performed simultaneously by requiring that a solution to the system in Step [2] violates a valid inequality for \( P \). This roadmap is somewhat similar to Yannakakis’s. However it is more tailored to the requirements of Theorem 5.1. In particular none of the intermediate steps, such as, e.g., subspace extensions (defined by equalities and non-negativity constraints) are needed.

5.1. Applications to the matching polytope. In this section we will simplify and slightly generalize the result of Kaibel et al. [2010], which is itself based on Yannakakis’s technique. We consider the \( \ell \)-matching polytope of the complete graph \( K_n = ([n], E_n) \) with \( n \in \mathbb{N} \). Let \( \mathcal{M}^\ell(n) \) denote the set of all matchings of \( K_n \) of size exactly \( \ell \). The \( \ell \)-matching polytope \( P_{\text{match}}^\ell(n) \) is the convex hull of the characteristic vectors of elements in \( \mathcal{M}^\ell(n) \), i.e.,

\[
P_{\text{match}}^\ell(n) := \left\{ \chi(M) \mid M \in \mathcal{M}^\ell(n) \right\} \subseteq [0, 1]^{E_n}.
\]

With \( S_n \) acting on the vertices of \( K_n \) by permutation, we have that \( P_{\text{match}}^\ell(n) \) is an \( S_n \)-polytope. We will consider \( P_{\text{match}}^\ell(n) \) as an \( A_n \)-polytope, i.e., we require less symmetry for the extension as the \( \ell \)-matching polytope actually possesses. For the size of any symmetric extended formulation of \( P_{\text{match}}^\ell(n) \) we obtain the following lower bound.

Theorem 5.3. Let \( n \in \mathbb{N} \) with \( n \geq 10 \) and let \( Q \subseteq \mathbb{R}^d \) be an \( A_n \)-symmetric extension of \( P_{\text{match}}^\ell(n) \). Then the number of facets of \( Q \) is at least

\[
\left( \frac{n}{\left\lfloor (\ell - 1)/2 \right\rfloor} \right).
\]

The proof is similar to the ones in Yannakakis [1991] and Kaibel et al. [2010] however we can shorten the argument by using Theorem 5.1.

Proof. First we introduce some notation. For readability let \( k := \left\lfloor \frac{\ell - 1}{2} \right\rfloor \).

Let \( V \) and \( E \) be the vertex set and edge set of \( K_n \), respectively. For a set \( M \subseteq E \), let \( V(M) \) denote the support of \( M \), i.e., the set of endpoints of all edges in \( M \). Moreover, for \( V_1, V_2 \subseteq V \) and \( M \subseteq E \) let \( M(V_1 : V_2) \) denote the set of edges in \( M \) with one endpoint in \( V_1 \) and the other endpoint in \( V_2 \).

Recall that \( A_n \) acts on \( V \), \( E \) and the set of facets of \( Q \).

The proof is by contradiction following the roadmap in Remark 5.2 so we suppose that \( Q \) has less than \( \binom{n}{k} \) facets.

Second we define a subpartition \( \mathcal{F}_j \) of the \((A_n)_j\)-orbit partition of the vertex set of \( P_{\text{match}}^\ell(n) \) for all facets \( j \). Let \( j \) be a fixed facet. Since the number of facets is less than \( \binom{n}{k} \) we have \( |A_n : (A_n)_j| \leq \binom{n}{k} \) by Lemma 2.7.
We apply Lemma 2.8 to obtain a set $V_j \subseteq V$ of size at most $k$ for any facet $j$ of $Q$ so that $H_j := A(V \setminus V_j) \subseteq (A_n)_j$. Let us define for all matching $W \subseteq E(V_j : V)$ with $|W| \leq \ell$

$$F_W := \{ M \ \ell\text{-matching} \mid M(V_j : V) = W \}.$$ 

The family $F_j$ is chosen to be the collection of the non-empty $F_W$, which is easily seen to refine the orbit partition of $H_j$ and hence form a subpartition of the $(A_n)_j$-orbit partition of vertex($P^\ell_{\text{match}}(n)$).

Next we find a solution to the system in Theorem 5.1. Let $V_*$ and $V^*$ be arbitrary disjoint subsets of $V$ of size $l_*$ and $l^*$, respectively, with $l_* + l^* = 2\ell$. When $\ell$ is odd, we select $l_* = l^* = \ell$, and when $\ell$ is even, we choose $l_* = \ell - 1$ and $l^* = \ell + 1$. Thus $l_*$ and $l^*$ are always odd.

Let $\mathcal{M}$ denote the set of matchings supported on $V_* \cup V^*$. These matchings are all the vertices of a face of $P^\ell_{\text{match}}(n)$ (defined by $x_e = 0$ for all $e \notin E(V_* \cup V^*)$). Since $l_*$ and $l^*$ are odd, every such matching must have an odd number of edges between $V_*$ and $V^*$, so $|M(V_* : V^*)| \geq 1$ is valid for the face. We select an affine combination $\sum_{M \in \mathcal{M}} c_M M$ to violate this inequality. All other $c_M$ with $\ell$-matching $M \notin \mathcal{M}$ are set to 0. All in all, we need to choose the $c_M$ to satisfy

$$\sum_{M \in \mathcal{M}} c_M = 1,$$
$$\sum_{M \in F_W \cap \mathcal{M}} c_M \geq 0, \quad \forall W \subseteq E(V_j : V) \text{ matching}, j \text{ facet of } Q,$$
$$\sum_{M \in \mathcal{M}} c_M |M(V_* : V^*)| = 0.$$

In fact, the chosen $c_M$ will only depend on $|M(V_* : V^*)|$, so we will set

$$b_i = c_M \cdot |\{ M : |M(V_* : V^*)| = i \}|,$$

and let $I$ denote the set of encountered values $|M(V_* : V^*)|$. We can simplify the system to

$$\sum_{i \in I} b_i = 1,$$
$$\sum_{i \in I} b_i \frac{|\{ M \in F_W \cap \mathcal{M} : |M(V_* : V^*)| = i \}|}{|\{ M \in \mathcal{M} : |M(V_* : V^*)| = i \}|} \geq 0, \quad \forall W \text{ as above},$$
$$\sum_{i \in I} b_i i = 0.$$

Now we determine the coefficients in (5.1). For this we compute the number of matchings $M$ with $|M(V_* : V^*)| = i$. Note that $S(V_*) \times S(V^*)$ acts transitively on these matchings, so the number is the index of the stabilizer of any such matching by Lemma 2.6. The stabilizer consists of the permutations permuting the edges between $V_*$ and $V^*$, the edges lying completely in $V_*$, and the edges lying completely in $V^*$. Also endpoints of the latter two kinds of edges can be flipped independently, however not
those of the edges between \(V_s\) and \(V^*\). So the stabilizer is
\[
(S(V_s) \times S(V^*))_M = S_i \times (\mathbb{Z}_2 \wr S_{i-1}) \times (\mathbb{Z}_2 \wr S_{i-1}),
\]
and its index (by Lemma 2.6) is
\[
|\{M \in \mathcal{M} : |M(V_s : V^*)| = i\}| = \frac{l_s! \cdot l^*!}{i! \cdot 2^{l_s-i} \cdot l^* - i \cdot l_s - i - 1 \cdot 2^{l^* - i} - 1}.
\]

Next we compute the number of matchings \(M \in F_W \cap \mathcal{M}\) for which
\[
|M(V_s : V^*)| = i
\]
provided that such matchings exist. Let
\[
a_s := |W(V_s)|, \quad a^* := |W(V^*)|, \quad a^*_s := |W(V_s : V^*)|,
\]
where \(W(V_s) = W(V^*_s : V^*_s)\) is the set of edges in the matching \(W\) supported on \(V_s\), the set \(W(V^*)\) is similarly defined, and \(W(V_s : V^*)\) is the set of edges with one endpoint in \(V_s\) and the other one in \(V^*\). This is essentially the same problem as above with different parameters. We conclude
\[
|\{M \in F_W \cap \mathcal{M} : |M(V_s : V^*)| = i\}| = \frac{(l_s - 2a_s - a^*_s)! \cdot (l^* - 2a^* - a^*_s)!}{(i - a^*_s)! \cdot 2^{l_s - 2a_s - i - 1} \cdot l_s - 2a_s - i \cdot 2^{l^* - 2a^* - i - 1} \cdot l^* - 2a^* - i}.
\]

All in all, (5.1) expands to
\[
\sum_{i \in \mathcal{I}} b_i \frac{2a_s + a^*}{l_s - i} \cdot (l_s - 2a_s - a^*_s)! \cdot (l^* - 2a^* - a^*_s)! \cdot i(i-1) \ldots (i-a^*_s + 1)
\]
\[
\cdot \frac{l_s - i}{2} \left(\frac{l_s - i}{2} - 1\right) \ldots \left(\frac{l_s - i}{2} - a_s + 1\right)
\]
\[
\cdot \frac{l^* - i}{2} \left(\frac{l^* - i}{2} - 1\right) \ldots \left(\frac{l^* - i}{2} - a^*_s + 1\right) \geq 0.
\]

Observe that this is a polynomial in \(i\) of degree \(a_s + a^* + a^*_s \leq |V_j| \leq k\) with a non-negative constant term. Furthermore \(|\mathcal{I}| \leq k + 1\), as \(\min(l_s, l^*) = 2k + 1\) and \(\mathcal{I}\) contains only odd numbers. Hence to satisfy all the inequalities, we can choose the \(b_i\) such that
\[
\sum_{i \in \mathcal{I}} b_i f(i) = f(0) \quad \text{deg}\ f \leq k
\]
for every polynomial \(f\) of degree at most \(k\).

\(\square\)

6. Establishing Quadratic Lower Bounds

We will now present a technique to establish super linear lower bounds on the size of symmetric extended formulations. The technique is based on Pashkovich [2009], however we generalize previous constructions and provide a uniform, algebraic framework. In fact it suffices to check few conditions to establish super linear lower bounds.

The following theorem will be central to our following discussion. A similar result had been already established in Pashkovich [2009] in a combinatorial fashion. We provide a new, significantly shorter, algebraic proof.
Theorem 6.1. Let \( Q \subseteq \mathbb{R}^d \) be a symmetric extension of an \( A_n \)-polytope \( P \subseteq \mathbb{R}^m \). Assume that the number \( N \) of facets of \( Q \) is less than \( n(n-1)/2 \). If \( j \) is a facet of \( Q \), then either \( A_n j \cong \{n\} \) or \( A_n j \cong \{1\} \). In particular, the orbits of the facets of \( Q \) decompose \( [N] \) into sets of sizes \( n \) and \( 1 \).

Proof. Let \( j \) be a facet of \( Q \). As \( N < \frac{n(n-1)}{2} \) we obtain \( [A_n : \{j\}] < \frac{n(n-1)}{2} \), where \( \{j\} \) is the stabilizer of \( j \) in \( A_n \). Applying Lemma 2.8 yields that there exists an \( A_n \)-invariant subset \( W_j \) with \( |W_j| \leq 1 \) such that \( A([n] \setminus W_j) \) is a subgroup of \( \{j\} \).

Since \( |W_j| \leq 1 \), there does not exist a non-identical permutation of \( W_j \), hence the subgroup \( A([n] \setminus W_j) \) is maximal with the property of leaving \( W_j \) invariant, so, in fact, \( \{j\} = A([n] \setminus W_j) \). It follows that either \( A_n j \cong \{n\} \) (when \( |W_j| = 1 \)) or \( A_n j \cong \{1\} \) (when \( W_j = \emptyset \)). This proves the first part of the claim. The second part follows immediately as the orbits induce a partition of \( [N] \). \( \Box \)

Using Theorem 6.1 we will now derive a sufficient condition for an \( A_n \)-polytope to admit only symmetric extensions of size \( \Omega(n^2) \): in fact the condition can be applied more widely and \( \binom{n}{2} \) is the limiting case. The main idea is that a small symmetric extended formulation has to average combinatorial properties of the polytope. The smaller the required size, the more the formulation averages. As a consequence, highly asymmetric combinatorial properties are obstructions to small formulations. In a slightly more abstract framework, we can say that the language defined by the vertices of, say, a 0/1-polytope is too complex to be decided by a small symmetric extension.

We would like to stress that the dimension of the polytope in the next theorem is irrelevant.

Theorem 6.2. Let \( P \) be an \( A_n \)-polytope. Let \( J \subseteq [n-1] \) be a non-empty subset of size \( k \). For all \( j \in J \), let \( H_j \subseteq A_n \) be a subgroup with orbits \( \{1,2,\ldots,j\} \) and \( \{j+1,\ldots,n\} \) in \( [n] \). Then \( \min h \geq \frac{n^2}{4k} \) if there exist

1. a family \( \{F_j \mid j \in J\} \) of faces of \( P \) such that \( F_j \) is invariant under \( H_j \);
2. a permutation \( \zeta_j \in A_n \) for all \( j \in J \) such that \( \zeta_j^{-1}([j]) = [j-1] \cup [j+1] \) and vertices \( \{v_j \mid j \in J\} \) such that each \( v_j \) belongs to all the faces \( F_i \) with \( i \in J \) and \( \zeta_j v_j \notin F_j \).

Remark 6.3. The above formulation of Theorem 6.2 is tailored towards deriving lower bounds: for specific polytopes it is particularly easy to check the existence of the \( v_j \). A more theoretical approach is that instead of the vertices \( v_j \) we require equivalently \( \zeta_j F \not\subset F_j \) where \( F := \bigcap_{j \in [n-1]} F_j \). (In particular, \( F := \bigcap_{j \in J} F_j \neq \emptyset \) is a face.) This rephrases the condition completely in the language of the face lattice of the polytope.

Proof of Theorem 6.2. Let \( F := \bigcap_{j \in [n-1]} F_j \). Then \( v_j \in F \) and hence \( \zeta_j F \not\subset F_j \) for all \( j \in J \). In particular, \( F \) is a non-empty face, so there exists \( v \in \relint(F) \).

First observe that \( \zeta_j v \notin F_j \) for all \( j \in J \): we have \( \zeta_j v \in \relint(\zeta_j F) \), and hence \( \zeta_j F \) is the smallest face containing \( \zeta_j v \). Therefore \( \zeta_j v \in F_j \).
would imply \( \zeta F \subseteq F_j \), which contradicts our assumption. We introduce the following notation for symmetrization: let \( v[G] := \frac{1}{|G|} \sum_{g \in G} g v \) the group average of \( v \) with respect to the group \( G \).

Second we define points \( v_{\epsilon,j} \) for \( j \in J \) and \( \epsilon > 0 \) as follows:

\[
v_{\epsilon,j} := (1 + \epsilon)v[H_j] - \epsilon(\zeta_j v)[H_j].
\]

Observe that \( v[H_j], (\zeta_j)v[H_j] \in P \). We claim that \( v_{\epsilon,j} \notin P \) for all \( j \in J \) and \( \epsilon > 0 \). As \( F_j \) is \( H_j \)-invariant we obtain that \( v[H_j] \in F_j \). Similarly, we have that \( (\zeta_j v)[H_j] \notin F_j \) as \( \zeta_j v \notin F_j \). For any \( \epsilon > 0 \) the point \( v_{\epsilon,j} \) lies on the line of \( v[H_j], (\zeta_j v)[H_j] \) with \( v[H_j] \) separating \( (\zeta_j v)[H_j] \) and \( v_{\epsilon,j} \). In particular, \( v_{\epsilon,j} \) is on the wrong side of \( F_j \) (more precisely, it is on the wrong side of any hyperplane cutting out \( F_j \) from \( P \)), so \( v_{\epsilon,j} \notin P \). The points \( v_{\epsilon,j} \) will serve as those that any symmetric extension of size less than \( nk/2 \) fails to cut off.

Now let \( Q \subseteq \mathbb{R}^d \) be a symmetric extension of \( P \), i.e., \( Q \) is itself an \( A_n \)-polytope and let \( p \) be the associated projection. We choose \( w \in Q \) such that \( wp = v \). We define points \( w_{\epsilon,j} \) as follows

\[
w_{\epsilon,j} := (1 + \epsilon)w[H_j] - \epsilon(\zeta_j w)[H_j].
\]

As before we have \( w[H_j], (\zeta_j w)[H_j] \in Q \). Now that \( p \) is invariant, we obtain that \( w_{\epsilon,j}p = v_{\epsilon,j} \) for any \( j \in J \) and \( \epsilon > 0 \). However, \( v_{\epsilon,j} \notin P \) and therefore \( w_{\epsilon,j} \notin Q \) for any \( j \in J \) and \( \epsilon > 0 \). We will count how many facets \( Q \) has to have in order to ensure this.

For contradiction, suppose that \( Q \) is given by less than \( nk/2 \leq n(n-1)/2 \) inequalities, hence Theorem 6.1 applies and we obtain that the orbits of facets under \( A_n \) are isomorphic either to \([1]\) (fixed point) or to \([n]\). Let \( T \) be any facet of \( Q \). If \( w[H_j] \notin T \) then \( v_{\epsilon,j} \) is on the side of \( T \) pointing inwards for \( \epsilon \) small enough, as then \( w_{\epsilon,j} \) is close to \( w[H_j] \). Hence the point could not be separated and therefore we only have to consider the other case: \( w[H_j] \in T \), i.e., for all \( h \in H_j \) we have \( hw \in T \) and equivalently \( w \in hT \). Now \( T \) cuts off \( w_{\epsilon,j} \) if and only if \( (\zeta_j w)[H_j] \notin T \). In other words, there exists \( h \in H_j \) such that \( w \notin \zeta_j^{-1}hT \). This is not possible if the orbit of \( T \) is a fixed point, as it requires both \( w \in T \) and \( w \notin T \); a contradiction.

If the orbit of \( T \) is isomorphic to \([n]\), let \( T_i \) denote the face in the orbit corresponding to \( i \in [n] \). If \( T \) lies in the \( H_j \)-orbit \( \{T_1, \ldots, T_j\} \) then the above conditions state that \( w \) is contained in \( T_1, \ldots, T_j \) but not in at least one of \( T_1, \ldots, T_{j-1}, T_{j+1} \) (using the condition \( \zeta_j^{-1}(\{j\}) = [j-1] \cup \{j+1\} \)), which is only possible if \( w \) is not contained in \( T_{j+1} \). Similarly, if \( T \) lies in the \( H_j \)-orbit \( \{T_{j+1}, \ldots, T_n\} \) then the above conditions say that \( w \) is contained in \( T_{j+1}, \ldots, T_n \) but not in \( T_j \).

All in all, an orbit of facets cuts off \( w_{\epsilon,j} \) for small \( \epsilon > 0 \) if and only if it is isomorphic to \([n]\), and

1. \( w \in T_i \) for all \( i \leq j \) but \( w \notin T_{j+1} \), or
2. \( w \in T_i \) for all \( i \geq j + 1 \) but \( w \notin T_j \).

Observe that either case is satisfied by at most one \( j \in [n-1] \) for a given orbit. Therefore every orbit can cut off \( w_{\epsilon,j} \) for small \( \epsilon \) for at most two \( j \). Hence we need at least \( k/2 \) orbits of size \( n \), so altogether at least \( \frac{nk}{2} \) facets; a contradiction. \( \square \)
Observe that property \([2]\) from above is very similar to the \textit{basis exchange property} of matroids. In fact the functions \(\zeta_j\) perform such a basis exchange (and possibly more); see Corollary 6.10.

\textbf{Remark 6.4.} Observe that Theorem 6.2 is only about a \textit{linear} number of faces of \(P\). It is natural to wonder why one cannot just \textit{add} these additional constraints. It turns out that this is not possible due to the \(A_n\)-symmetry of \(P\). In fact, we would have to add a linear number of cosets of facets, each of which is of linear size.

We shall now provide simplified proofs for known lower bounds using Theorem 6.2. The first two results already appeared in Pashkovich [2009].

The polytopes we will consider can be found in Kaibel et al. [2010], Pashkovich 2009, and Fiorini et al. [2011a] (see also Appendix 3).

For simplicity, in the examples we specify explicitly neither the permutahedron \(S_{\text{perm}}\) nor the groups \(H_j\). In fact, the actual choice of \(\zeta_j\) does not matter: a canonical choice is the transposition \(\zeta_j = (j, j + 1)\). Moreover, we can always choose \(H_j := A_n \cap (S_{\text{perm}} \times S_{[n]\setminus[j]})\).

\textbf{Corollary 6.5 (Permutahedron).} Let \(P_{\text{perm}}(n) \subseteq \mathbb{R}^n\) be the permutahedron on \([n]\). Then \(\text{xc}_{A_n}(P_{\text{perm}}(n)) \geq \frac{n(n-1)}{2}\).

\textit{Proof.} Let \(F_j := \left\{ \sum_{i=1}^{j} x_i = \frac{j(j+1)}{2} \right\}\) for \(j \in [n-1]\) and \(v_j = v := (1, 2, \ldots, n)\). Observe that \(v\) is contained in all the \(F_j\) (in fact, \(\bigcap_{j=1}^{n-1} F_j = \{v\}\)). Clearly, \(F_j\) is invariant under \(H_j\) and we can also verify that \(\zeta_j v \notin F_j\). The result now follows from Theorem 6.2. \(\Box\)

With the remark in Section 3.3 this yields \(\text{xc}_{A_n}(P_{\text{perm}}(n)) = \Theta(n^2)\).

\textbf{Corollary 6.6 (Cardinality indicating polytope).} Let \(P_{\text{card}}(n) \subseteq \mathbb{R}^n\) be the cardinality indicating polytope. Then \(\text{xc}_{A_n}(P_{\text{card}}(n)) \geq \frac{n(n-1)}{2}\).

\textit{Proof.} Let \(F_j := \left\{ \sum_{i=1}^{j} x_i = \sum_{i=1}^{j} i z_i + \sum_{i=j+1}^{n} j z_i \right\}\) and choose the \(x\)-part of \(v_j\) to be \((1, 1, \ldots, 1, 0, 0, \ldots, 0)\) with 1 appearing \(j\) times for \(j \in [n-1]\). We observe that \(v_j \in F_i\) for all \(i\) and, as before, \(\zeta_j v_j \notin F_j\). The result follows from Theorem 6.2. \(\Box\)

Note that the \(A_n\)-symmetry of \(P_{\text{card}}(n)\) permutes only the entries of \(x\) but leaves the entries of \(z\) unchanged. Together with the remark in Section 3.1 we obtain that \(\text{xc}_{A_n}(P_{\text{card}}(n)) = \Theta(n^2)\).

Observe that we can obtain a uniform \(v\), i.e., \(v \in F\) such that \(\zeta_j v \notin F_j\) for all \(j \in [n-1]\); e.g., \(v := \frac{1}{n-1} \sum_{j=1}^{n-1} v_j\). In fact, any convex combination of the \(v_j\) (with all coefficients non-zero) is sufficient. Such an averaged point is not a vertex however and might be harder to identify right away.

Often it suffices to identify an ascending chain of subsets \(S_1 \subseteq \cdots \subseteq S_{n-1} \subseteq [n-1]\) and derive the \(F_j\) from those. We will demonstrate this for the case of the spanning tree polytope.

\textbf{Corollary 6.7 (Spanning tree polytope).} Let \(P_{\text{STP}}(K_n)\) be the spanning tree polytope of the complete graph \(K_n\) on \(n\) vertices. Then \(\text{xc}_{A_n}(P) \geq \frac{n(n-1)}{2}\).
Proof. Let \( S_j := [j] \) and

\[
F_j := \left\{ \sum_{e \in E(S_j)} x_e = |S_j| - 1 \right\},
\]

where \( E(S_j) \) denotes the set of edges between the vertices in \( S_j \). Now let \( v := (1, 2, \ldots, n) \) be the path from 1 to \( n \). Observe that \( v \) is a vertex of all the \( F_j \). Moreover, we have \( \zeta_jv \notin F_j \) as \( \zeta_jv \) restricted to \( S_j \) is not a connected graph and hence does not lie on the facet \( F_j \). Again we can apply Theorem 6.2 and the claim follows. \( \square \)

As mentioned earlier, a lower bound \( \Omega(n^2) \) for the extension complexity of the spanning tree polytope follows directly from the non-negativity constraints and Corollary 6.7 highlights that an \( \Omega(n^2) \) lower bound would also follow from solely examining the remaining constraints; i.e., considering a different part of the slack matrix.

We will now show that the Birkhoff polytope is an optimal symmetric extension of itself. This has been also shown in [Fiorini et al. 2011a], even for non-symmetric extended formulation. Whereas the proof for the general case is based on combinatorial rectangle coverings of the support of the slack matrices, for the symmetric case the reason for the lower bound is of an algebraic nature and follows naturally from Theorem 6.2.

Corollary 6.8 (Birkhoff polytope). Let \( P_{\text{birk}}(n) \subseteq \mathbb{R}^{n^2} \) be the Birkhoff polytope of \( n \times n \) permutation matrices. Let \( A_n \) act on \( P_{\text{birk}}(n) \) via permuting the columns of matrices. Then \( xc_{A_n}(P) \geq \frac{n(n-1)}{2} \).

Proof. Let \( F_j := \left\{ \sum_{i=1}^{j} x_{j+1,i} = 0 \right\} \), which is the intersection of \( x_{j+1,i} \geq 0 \) for \( i \in [j] \). Then \( \bigcap_{j=1}^{n-1} F_j \) is just the vertex \( v \) with \( v_{i,j} = 1 \) for all \( i \). It is easy to see that \( \zeta_jv \notin F_j \) and clearly \( F_j \) is invariant under \( H_j \). The result follows with Theorem 6.2. \( \square \)

We will now provide an example showing that the conditions specified in Theorem 6.2 are necessary. In particular we show why Theorem 6.2 fails for \( k \geq 5 \) when applied to \( [0, 1]^n \); the standard formulation of the cube has \( 2n \) inequalities and \( k \geq 5 \) would imply a lower bound of \( \frac{5}{2}n > 2n \). In fact Theorem 6.2 fails already for \( k \geq 3 \).

Example 6.9 (Applying Theorem 6.2 to \( [0, 1]^n \)). Contrary to intuition, the cube \( [0, 1]^n \) has only small families \( J \) of faces satisfying the condition of Theorem 6.2. In particular, all the families contain at most two faces. We are now providing a direct proof.

The proper faces \( F_j \) with stabilizer orbits \( \{1, 2, \ldots, j\} \) and \( \{j+1, \ldots, n\} \) are only

\[
\begin{align*}
x_1 &= x_2 = \cdots = x_j = 0, \\
x_1 &= x_2 = \cdots = x_j = 1, \\
x_{j+1} &= \cdots = x_n = 0, \\
x_{j+1} &= \cdots = x_n = 1.
\end{align*}
\]
Note that the family of faces cannot include, e.g., $F_j = \{x_1 = \cdots = x_j = 0\}$ and $F_k = \{x_1 = \cdots = x_k = 0\}$ for $j < k$. Otherwise 
\[ \zeta_j v_j \in \zeta_j F_k = \left\{ x_{\zeta_j(1)} = \cdots = x_{\zeta_j(k)} = 0 \right\} \subseteq F_j, \]
as 
\[ [j] = \zeta_j([j - 1] \cup \{j + 1\}) \subseteq \zeta_j([k]). \]
Moreover, as $\{x_1 = \cdots = x_j = 0\}$ and $\{x_1 = \cdots = x_k = 1\}$ are disjoint, they cannot be both contained in the family.

Therefore the family can contain at most one of the faces of the form $\{x_1 = \cdots = x_j = 0\}$ and $\{x_1 = \cdots = x_k = 1\}$. Similarly, it contains at most one of the other faces: $\{x_{j+1} = \cdots = x_n = 0\}$ and $\{x_{j+1} = \cdots = x_n = 1\}$. This implies a total of 2 faces at most.

We conclude this section with a matroid version of Theorem 6.2. In this case Condition (2) asks for (repeated) failure of the basis-exchange property.

A matroid $\mathcal{M} = (E, \mathcal{F})$ is a $G$-matroid for some group $G$, if $G$ acts on $E$ preserving the independent sets, i.e., $\pi F \in \mathcal{F}$ for all $\pi \in G$ and $F \in \mathcal{F}$.

**Corollary 6.10.** Let $\mathcal{M} = (E, \mathcal{F})$ be an $A_n$-matroid with rank function $r$. Furthermore, let $J \subseteq [n-1]$ be a non-empty subset of size $k$. For all $j \in J$, let $H_j \subseteq A_n$ be a subgroup with orbits $\{1, 2, \ldots, j\}$ and $\{j + 1, \ldots, n\}$. Let $P := \left\{ x \in [0,1]^E \bigg| \sum_{e \in F} x_e \leq r(F) \right\}$ be the independent set polytope associated with $\mathcal{M}$. Then $x e_{A_n}(P) \geq \frac{n}{2}$ if there exist

1. a family $\{F_j \mid j \in J\}$ of flats of $\mathcal{M}$ such that $F_j$ is invariant under $H_j$;
2. a permutation $\zeta_j \in A_n$ and $S_j \in \mathcal{F}$ for all $j \in J$ so that $\zeta_{j}^{-1}[j] = [j - 1] \cup \{j + 1\}$ and $|S_j \cap F_i| = r(F_i)$ for all $i \in J$, but $|\zeta_j S_j \cap F_j| < r(F_j)$.

**Proof.** Follows immediately from Theorem 6.2 with faces $\left\{ \sum_{e \in F_j} x_e = r(F_j) \right\}$ for $j \in J$.

### 7. SDP-version of Theorem 5.1

In Section 5 we established the key result for bounding the size of symmetric extended formulations where the extension is a polytope. We will now extend Theorem 5.1 to the case where the extension is a semidefinite program (SDP).

Given two square matrices $A, B \in \mathbb{R}^{n \times n}$ with $n \in \mathbb{N}$, the (standard) Frobenius (inner-) product of $A$ and $B$ is defined as
\[ A \bullet B := \sum_{i,j \in [n]} A_{ij} B_{ij}. \]
If a square matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite, we write $A \succeq 0$ as usual. An SDP is an optimization problem of
\[
\begin{align*}
\min & \quad C \bullet X \\
\text{s.t.} & \quad A_j \bullet X = b_j, \\ & \quad X \succeq 0,
\end{align*}
\]
where \( f \in \mathbb{N} \) and \( A_j, C, X \in R^{m \times m} \) are symmetric square matrices with \( j \in [f] \). Slightly abusing notions we will use the term SDP to refer to the feasible region of an SDP; we are not interested in any particular objective function. Given a group \( G \), a feasible region of an SDP \( Q \) is a \( G \)-SDP if \( gQ = Q \) and \( gX \succeq 0 \) whenever \( X \succeq 0 \) for all \( g \in G \). Note that the second requirement ensures that the action of \( G \) preserves the positive semidefinite cone. In a first step we will establish the existence of a \( G \)-invariant Frobenius product, i.e., for \( A, B \in R^{m \times m} \) we have \( A \bullet B = gA \bullet gB \). The following lemma is the analog of Lemma 2.5.

**Lemma 7.1.** Let \( G \) be a group acting linearly and faithfully on \( R^{m \times m} \). Then there exists a \( G \)-invariant Frobenius product defined as

\[
A \bar{\bullet} B := \frac{1}{|G|} \sum_{g \in G} gA \bullet gB
\]

with \( A, B \in R^{m \times m} \).

**Proof.** Let \( \pi \in G \) and \( A, B \in R^{m \times m} \). As before we have

\[
\pi A \bar{\bullet} \pi B = \frac{1}{|G|} \sum_{g \in G} \pi gA \bullet \pi gB = \frac{1}{|G|} \sum_{g \in G} gA \bullet gB = A \bullet B.
\]

\( \square \)

**Definition 7.2.** A symmetric SDP-extension of a \( G \)-polytope \( P \) is a \( \tilde{G} \)-SDP \( Q \) together with a group epimorphism \( \alpha: \tilde{G} \to G \) and linear map \( p: \mathbb{R}^{d \times d} \to \mathbb{R}^m \) that is also \( \alpha \)-linear, i.e., \( p \) has to satisfy \( Qp = P \) and \( (\tilde{\pi}Q)p = (\tilde{\pi}\alpha)(Qp) \) for all \( \tilde{\pi} \in \tilde{G} \).

We are ready to prove the SDP-variant of Theorem 5.1.

**Theorem 7.3.** Let a \( \tilde{G} \)-SDP \( Q \subseteq \mathbb{R}^{d \times d} \) be a symmetric SDP-extension of a \( G \)-polytope \( P \subseteq \mathbb{R}^m \) via \( \alpha: \tilde{G} \to G \) and an \( \alpha \)-linear map \( p: \mathbb{R}^{d \times d} \to \mathbb{R}^m \). For every facet \( j \) of \( Q \) let \( F_j \) be a refinement of the \( \tilde{G}_j \)-orbit partition of the vertex set \( V \) of \( P \) and let \( s: V \to Q \) be a section. Then for every real solution to the following inequality system in the \( c_v \)

\[
\sum_{v \in V} c_v = 1,
\]

\[
\sum_{v \in F} c_v s(v) \succeq 0,
\]

\( F \in F_j, \ j \) facet of \( Q \)

the point \( \sum_{v \in V} c_v v \) lies in \( P \).

**Proof.** Let \( \bullet \) be a \( \tilde{G} \)-invariant Frobenius product on \( \mathbb{R}^{d \times d} \) and let \( Q \) be given with respect to that product in the form

\[
Q = \left\{ X \in \mathbb{R}^{d \times d} \mid A_j \bullet X = b_j \ \forall j \in [f], X \succeq 0 \right\},
\]

with \( f \in \mathbb{N} \) and \( A_j \in \mathbb{R}^{d \times d} \) symmetric for all \( j \in [f] \). Obviously,

\[
A_j \bullet \left( \sum_{v \in V} c_v s(v) \right) - b_j = \sum_{v \in V} c_v (A_j \bullet s(v) - b_j) = 0.
\]
Moreover we have that
\[ \sum_{v \in V} c_v s(v) = \sum_{F \in \mathcal{F}} \sum_{v \in F} c_v s(v) \geq 0. \]

This shows that \( \sum_{v \in V} c_v s(v) \in Q \), hence applying \( p \) we obtain \( \sum_{v \in V} c_v v \in P \). \( \square \)

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**Appendix A. Invariant scalar products and sections**

**Lemma 2.5.** Let $P \subseteq \mathbb{R}^m$ be a $G$-polytope and $Q \subseteq \mathbb{R}^d$ be a $G$-polytope so that $Q$ is a symmetric extension of $P$ with projection $p$ as before. Further let $s : \text{vertex}(P) \to Q$ be a section and $(\langle \cdot, \cdot \rangle)$ be a scalar product on $\mathbb{R}^d$. Then:

1. There exists an invariant scalar product $\overline{\langle \cdot, \cdot \rangle}$ defined as

   $$\overline{\langle x, y \rangle} := \frac{1}{|G|} \sum_{g \in G} \langle gx - g0, gy - g0 \rangle,$$

2. There exists an invariant section $\overline{s}$ given by

   $$\overline{s}(x) := \frac{1}{|G|} \sum_{g \in G} g^{-1}s((gα)x).$$

**Proof.** To simplify calculations for the scalar product, we confine ourselves to linear group actions as it suffices to consider the linear part of an action. We therefore assume that $g0 = 0$ for $g \in G$; note that we can do this without loss of generality. Let $(\langle \cdot, \cdot \rangle)$ be defined as above. We claim that $(\langle \cdot, \cdot \rangle)$ is a well-defined scalar product such that

   $$\overline{\langle gx, gy \rangle} = \overline{\langle x, y \rangle}$$

for all $x, y \in \mathbb{R}^d$ and $g \in G$. Observe that $(\langle \cdot, \cdot \rangle)$ is a symmetric bilinear function. Moreover, $\overline{\langle x, x \rangle} = \frac{1}{|G|} \sum_{g \in G, i \in [n]} \langle gx, gx \rangle > 0$ for $x \neq 0$. Therefore $(\langle \cdot, \cdot \rangle)$ is a well-defined scalar product. In order to show that it is invariant under the action of $G$, let $\pi \in G$ and observe

   $$\frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle = \frac{1}{|G|} \sum_{g \in G} \langle gπx, gπy \rangle = \overline{\langle πx, πy \rangle},$$

as $gπ$ runs through $G$, when $g$ does so, because $G$ is a group.

Now consider $\overline{s}(x)$, let $x \in \text{vertex}(P)$, and let $\pi \in G$. The map $\overline{s}(x)$ is indeed a section, as

   $$\overline{s}(x)p = \frac{1}{|G|} \sum_{g \in G} g^{-1}s(gx)p = \frac{1}{|G|} \sum_{g \in G} g^{-1}(gx) = x.$$
For $\pi \in G$ we have

$$
\pi \bar{s}(x) = \pi \left( \frac{1}{|G|} \sum_{g \in G} g^{-1} s(gx) \right) = \frac{1}{|G|} \sum_{g \in G} \pi g^{-1} s(gx)
= \frac{1}{|G|} \sum_{g \in G} g^{-1} s(g\pi x) = \bar{s}(\pi x)
$$

\[\square\]