Quantum chaos and entanglement in ergodic and non-ergodic systems

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We study entanglement entropy (EE) as a signature of quantum chaos in ergodic and non-ergodic systems. In particular we look at the quantum kicked top and kicked rotor as multi-qubit systems, and investigate the single qubit EE which characterizes bipartite entanglement of this qubit with the rest of the system. We study the correspondence of the Kolmogorov-Sinai entropy of the classical kicked systems with the EE of their quantum counterparts. We find that EE is a signature of global chaos in ergodic systems, and local chaos in non-ergodic systems. In particular, we show that EE can be maximised even when systems are highly non-ergodic, when the corresponding classical system is locally chaotic.

I. INTRODUCTION

Sketched in the 17th century by Newton and others, the deterministic laws of classical mechanics quickly ran into difficulties. Foremost was the fact that equations of Newtonian gravitation resisted analytic solutions for three or more bodies. The struggles of the unsolvability of many classical mechanical equations was further exacerbated when Poincaré proved that perturbation to known integrable solutions in general leads to non-integrability or chaos. This was in contrast to observation, which saw nature as substantially regular, from the periodic movements of planets to the sounds of a piano. This paradox was resolved in the form of KAM theory [1] which formally explains the persistence of quasi-periodic behaviour in chaotic systems.

Chaos in classical physics is characterised by a hypersensitivity of the time evolution of the system to even small changes in the initial conditions. Classically this is well understood in terms of the hypersensitive dependence of the phase space trajectories. Quantum chaos, in contrast, cannot be defined in the same terms, largely due to the fact that there is no general quantum analogue of classical phase space trajectories. To put this into context, the unitary evolution of an initial quantum state $\psi_a(0)$ is

$$\psi_a(t) = U_a(0),$$

where $U = e^{-iHt/\hbar}$ is the unitary time-evolution operator for a system with Hamiltonian $H$. Similarly, starting from a nearby initial state $\psi_b$, yields $\psi_b(t) = U_b(0)$. The scalar product of these states are constant for all time,

$$\langle \psi_a(t), \psi_b(t) \rangle = \langle \psi_a(0), \psi_b(0) \rangle.$$

Therefore, due to the linearity of the Shrödinger equation, differences in initial conditions cannot grow, in stark contrast to the exponential divergence of trajectories of chaotic classical systems. As underlying all classical systems are quantum mechanical ones, the confounding question is: how does chaos arise from quantum systems? This question motivates the search for quantum signatures of chaos.

Approaches to quantum signatures of chaos fall into two categories. One involves investigating quantum variables that distinguish between quantum systems whose classical counterparts are integrable and nonintegrable. These approaches typically look at energy spectra properties [2-5]. A second class of approaches seeks intrinsic quantum definitions of quantum chaos. Examples of these include quantum parallels of the Lyapunov exponents and entropy measures [6-11]. There have also been attempts to develop a quantum analogue of KAM theory [12-15]. In this paper we will look at the linear entanglement entropy (EE) as a signature of quantum chaos.

The connection between EE and chaos was first proposed by Zurek and Paz [16]. Here they studied a classical inverted harmonic oscillator (an unstable but not properly chaotic system) and conjectured that in the corresponding quantum system, weakly coupled to a high temperature bath, the rate of production of the von Neumann entropy equals the sum of the positive Lyapunov exponents. Importantly, this sum is equivalent to the Kolmogorov-Sinai entropy (KSE) [16]. Even though the conjecture is not directly generalizable to less trivial systems [17], Zarum and Sarkar [18] showed a significant correspondence between the entropy contours of the phase space of the classical kicked rotator (CKR) and the quantum kicked rotator (QKR) embedded in a dissipative environment. Subsequent to the Zurek-Paz conjecture, and perhaps motivated by it, Furuya, Nemes, and Pellegrino numerically showed that classical chaos could be related to high EE and classical regular dynamics to low EE in the context of the Jaynes-Cummings model [19]. This result stimulated further studies of the EE as a direct signature of chaos.

More recently bipartite EE as a signature of chaos was
studied in the quantum kicked top (QKT) modelled as a multi-qubit system [20, 22], without the need for an external environment. These authors found that high EE corresponds to chaos in quasi-ergodic systems. Remarkably, it was experimentally observed in a three superconducting qubit system [23]. This correspondence can be intuited when one considers the linear EE measure

$$S = 1 - \text{tr} \rho^2,$$

(3)

where $\rho$ is a reduced density matrix of a bipartition. When $\rho$ is a maximally mixed state, it explores all states equally, i.e., it is ergodic. In this case $S$ is maximised (see Appendix A for proof). For a maximally mixed state, further bipartition of $\rho$ would still result in maximally mixed state, and hence the $S$ would be still maximized.

In classical systems, one may have chaos even in non-ergodic systems. In the ergodic systems for example, in the presence of KAM tori the system is far from ergodic, yet local chaos exists. Taking this to the quantum regime, it is not immediately obvious that $\rho$ can be maximised for an analogous non-ergodic $\rho$. We would like to ask: can bipartite EE be a signature of quantum chaos in highly non-ergodic systems?

In the present work we tackle this problem by studying in detail both the top and rotor in the classical and quantum regimes. These prototypical systems have the main advantage that they exhibit the most important features of chaotic dynamics and a rich phase space, despite their relative simplicity. In order to directly compare the QKT and the QKR, we study the latter as a special limit of the former, exploiting the formulation given by Haake and Shepelyansky in [24]. In light of the experimental accessibility of the multi-qubit system, this approach has the further convenience that it allows one to describe the QKT and QKR in the same closed multi-qubit system. Using this system as a case-study, we will show that EE is a signature of quantum chaos even in highly non-ergodic systems. Specifically, in Sec. II we review the CKT and QKR in the quantum regime, we study the latter as a special limit of the QKT. In Sec. III we look at the CKR, the QKR and the QKT. We calculate the KSE of the CKT and the EE of the QKT. In Sec. IV we discuss properties of our quantum system that are reminiscent of KAM theory.

II. QUANTUM KICKED TOP

The Hamiltonian of the QKT is

$$H_{\text{QKT}} = \alpha J_z + \frac{\beta}{2j+1} J_{z}^2 \sum_{n=-\infty}^{\infty} \delta(t - n)$$

(4)

where $J$ is the angular momentum vector that obeys the commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk} J_k.$$ 

(5)

The magnitude $J^2 = j(j + 1)\hbar^2$ is a conserved quantity. The first term in Eq. (4) describes a precession around the $x$-axis with angular frequency $\alpha$. The second term represents a periodic kick. Each kick is an impulsive rotation around the $z$-axis by an angle proportional to $J_z$. For convenience we work in natural units where $\hbar = 1$, whereas the time is counted with the number of kicks. The proportionality factor involves dimensionless coupling constant $\beta/j$, where $\beta$ is known as the torsion strength.

The angular momentum operators at each kick can be obtained from the discrete time evolution of the operators in the Heisenberg picture,

$$J_{n+1} = U_T^\dagger J_n U_T,$$ 

(6)

where $U_T$ is the Floquet operator describing the unitary evolution from kick to kick,

$$U_T = \exp(-i\beta J_z^2) \exp(-i\alpha J_x).$$

(7)

Modelling the system as a $N$-qubit system, the angular momentum operators can be expressed in terms of Pauli operators,

$$J_\gamma = \sum_{i=1}^{N} \frac{\sigma_i \gamma}{2}$$

(8)

where $\gamma = x, y, z$.

We choose the initial pure state to be symmetric under the exchange of any qubits, so that the state vector at an later time is also symmetric. Thus we can write the state of our $N$-qubit system in terms of Dicke states $|j, m\rangle$, where $m = -j, -j + 1, \ldots, j$, with $j = N/2$. To connect the quantum and classical dynamics of the kicked top, we choose the initial state to be the spin coherent state

$$|\Phi_0, \Theta_0\rangle = \exp\{i\Theta_0[J_x \sin \Phi - J_y \cos \Phi_0]\}|j, j\rangle.$$ 

(9)

The state of the system after $n + 1$ kick is

$$|\psi\rangle_{n+1} = U_T |\psi\rangle_n$$

(10)

where $|\psi\rangle_0 = |\Phi_0, \Theta_0\rangle$.

A. Classical kicked top

To obtain the classical limit of the QKT we introduce the normalised vector $X = \langle J \rangle/j$ and take $j \to \infty$ (this is equivalent to taking the thermodynamic limit $N \to \infty$). Substituting this variable in to Eq. (6), we obtain the classical map

$$X_{n+1} = X_n \cos[\beta(Y_n \sin \alpha + Z_n \cos \alpha)] - (Y_n \cos \alpha - Z_n \sin \alpha) \sin[\beta(Y_n \sin \alpha + Z_n \cos \alpha)],$$

(11)

$$Y_{n+1} = X_n \sin[\beta(Y_n \sin \alpha + Z_n \cos \alpha)] + (Y_n \cos \alpha - Z_n \sin \alpha) \cos[\beta(Y_n \sin \alpha + Z_n \cos \alpha)],$$

$$Z_{n+1} = Y_n \sin \alpha + Z_n \cos \alpha.$$
The normalised angular momentum vector can be parameterised in polar coordinates, \( X = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta) \), to give a two-dimensional classical phase space, in the form of a Poincaré map. Fig. 1(a) maps the phase space for \( \alpha = \pi/2, \beta = 3 \).

**B. Top Kolmogorov-Sinai entropy**

The Poincaré map provides a pictorial representation of the phase space, through which one can visually distinguish between regular and chaotic regions. However, to have a proper quantitative measure of the degree of chaoticity, we make use of the KSE. The KSE is the rate of change with time of the coarse-grained Gibbs entropy [25, 26] and is calculated as [16]

\[
h_{KS} = \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} \log_2 l_n
\]

where \( l_n = \sqrt{\langle (\delta X_n)^2 + (\delta Y_n)^2 + (\delta Z_n)^2 \rangle} \) is the distance in the phase space between two initially close points after \( n \) kicks. Importantly, Pesin [16] showed that the KSE is equal to the sum of the positive Lyapunov exponents. As the Lyapunov exponents give the rate of separation of two infinitesimally close trajectories, the KSE = 0 for regular regions, and the KSE > 0 for chaotic regions, for times large enough. The KSE therefore is a quantitative measure of the level of chaos.

The procedure to obtain the KSE is as follows. The generalised iterative map \( x_{n+1} = f(x_n) \) is linearised to give its associated tangent map \( \delta x_{n+1} = f(x_n + \delta x_n) - f(x_n) \). The tangent map is rescaled, \( \delta x_n \to \delta x_n/l_n \), before being fed back at each iteration. The tangent map for Eq. (11) is (see Appendix B for derivation)

\[
\begin{align*}
\delta X_{n+1} &= \delta X_n \cos \gamma_n + \delta Y_n \sin \gamma_n - Y_n \beta \sin \alpha \cos \alpha \cos \gamma_n \\
+ &\delta Z_n \left[ -X_n \beta \cos \alpha \sin \gamma_n - Y_n \beta \cos^2 \alpha \cos \gamma_n + Z_n \beta \cos \alpha \sin \alpha \sin \gamma_n \right], \\
\delta Y_{n+1} &= \delta Y_n \sin \gamma_n + \delta Z_n \cos \gamma_n \\
+ &\delta Z_n \left[ -X_n \beta \cos \alpha \sin \gamma_n - Y_n \beta \cos^2 \alpha \sin \gamma_n + Z_n \beta \cos \alpha \sin \gamma_n \right], \\
\delta Z_{n+1} &= \delta Y_n \sin \alpha + \delta Z_n \cos \alpha,
\end{align*}
\]

where \( \gamma_n \equiv Y_n \beta \sin \alpha + Z_n \beta \cos \alpha \).

Fig. 1(b) plots \( h_{KS} \) of the CKT for \( \alpha = \pi/2, \beta = 3 \). \( h_{KS} = 0 \) for regularly regions. For chaotic regions, \( h_{KS} > 0 \), as here the trajectories are divergent.

**C. Top Ergodicity**

Ergodic systems uniformly explore all states over time, such that observables \( O \) averaged over time equals the same observables averaged over all states,

\[
\langle O \rangle_{\text{time}} = \langle O \rangle_{\text{states}}.
\]

The full QKT system is pure, therefore its EE, defined in Eq. (18), is always zero if we don’t take any bipartition of the system. This is not so with its ergodicity. A uniform average over states is given by the microcanonical ensemble \( \rho_{mc} \), therefore the degree to which our system is ergodic is its fidelity with \( \rho_{mc} \),

\[
F(\rho, \rho_{mc}) = \text{tr} \sqrt{\rho \rho_{mc} \rho_{mc} \rho}
\]

where \( \bar{\rho} = \sum_i \rho_i/n \).

In the phase space of the CKT of Fig. 1(a), chaotic initial points explore much of the phase space, in comparison to regular initial points which explore a regular narrow band of the phase space; correspondingly chaotic regions are quasi-ergodic, whereas regular regions are not. The chaotic regions are quasi-ergodic, as there are regular regions which are not visited by initial conditions beginning in the chaotic regions. Does this notion of chaos and ergodicity hold in the quantum case? To answer this question, we calculate the quantum ergodicity at points corresponding to chaotic and regular initial conditions.

We pick two representative initial conditions corresponding to regular and chaos points, and calculate their ergodicity in the QKT: \( (\Phi_0, \Theta_0) = (2.20, 2.25) \) and (3.57, 2.25), labelled respectively as \( T_1 \) and \( T_2 \) in Fig. 1. Fig. 2 plots the ergodicity of these two points: point \( T_2 \) is quasi-ergodic, whereas point \( T_1 \) is far from ergodic. In other words, chaotic regions are ergodic and regular regions are not, in the corresponding quantum system. In the next section, we calculate the EE of these regions.

**D. Top entanglement entropy**

The QKT Hamiltonian acts collectively on all \( N \) qubits, thereby preserving the symmetry of the \( N \) qubit state; this means that the spin expectation value of any single qubit is

\[
\langle s_{\gamma} \rangle = \frac{\langle J_{\gamma} \rangle}{2j}.
\]
FIG. 1. (a) The classical phase space of the CKT, with 500 random initial conditions for a duration of 500 kicks. (b) The KSE of the CKT, calculated on a grid of $200 \times 200$ initial conditions, iterating the linear map for $10^4$ steps. KSE > 0 corresponds to chaotic behaviour, whereas KSE = 0 indicates regular behaviour. Point $T_1, T_2$ marks $(\Phi, \Theta) = (2.20, 2.25)$ and $(3.57, 2.25)$ respectively. (c) The time-averaged EE of the QKT, calculating for a system of $N = 300$ qubits and averaged over $T = 300$ kicks. A comparison of (b) and (c) shows a remarkable correspondence between chaotic (regular) classical behaviour and high (low) EE. However, in the classical case there is a well defined demarcation between chaotic and regular regions, whereas in the quantum case the transition from regions of low to high EE is smooth. (d) plots the time-averaged EE at $\Theta = \pi/2$ for different numbers of qubits $N$. The transition from regions of low EE to high EE becomes more stark with increasing number of qubits, marking the transition to quantum chaos more abruptly, in a similar fashion to classical behaviour. Parameters: $\alpha = \pi/2, \beta = 3$.

The reduced density matrix of a single qubit is

$$\rho_1 = \frac{1}{2} + \langle s \rangle \cdot \sigma.$$  \hfill (17)

In the context of quantum chaos and EE, the choice of bipartition is not well understood; different bipartition choice can lead to different results. In prior work, Ref. [21] bi-partitioned one-particle from the larger systems, Ref. [20] bi-partitioned two-particles from the larger system, and Ref. [28] averages over-all possible partitions. However, only the one-particle bi-partition has been experimentally verified [23]. Here we have chosen the one-particle bi-partition, because it is the simplest. The role that the choice of bipartition plays in quantum chaos would make an interesting future study.

From the definition of linear entropy ($S = 1 - \text{tr}\rho_1^2$), the EE of a single qubit with the rest of the system is [21]

$$S = \frac{1}{2} \left( 1 - \frac{\langle J \rangle \cdot \langle J \rangle}{J^2} \right). \hfill (18)$$

Choosing the initial state to be spin coherent $|\Theta_0, \Phi_0\rangle$ with $0 \leq \Theta_0 < \pi$ and $0 \leq \Phi_0 < 2\pi$, Fig. 1(c) plots the time-averaged EE of the QKT for $\alpha = \pi/2, \beta = 3$. For finite systems, time-averaging is used to estimate the equilibrium value approached by larger systems [23]. Remarkably there is an obvious correspondence between the EE of the QKT and the classical phase space trajectories and KSE, as shown in Fig. 1(a) and (b). Regions of low EE correspond to regular trajectories (KSE = 0), and regions of high EE correspond to chaotic trajectories (KSE > 0).

An important difference between the KSE and EE of the kicked top however, is that in the classical case there is a well defined demarcation between chaotic and regular regions, whereas in the quantum case the transition from regions of low to high EE is smooth. The change in EE becomes greater with increasing number of qubits, and therefore the transition from regions of high EE to low EE occurs more rapidly as shown in Fig. 1(d). From
Fig. 2. Time evolution of the ergodicity for the QKT (dashed lines) and the QKR-limit for $j = 9$ (solid lines) in a regular region (blue) and chaotic regions (red). Points $T_1$, $T_2$ are marked in Fig. 1 and $R_1$, $R_2$ are marked in Fig. 4. The chaotic region in the QKT is quasi-ergodic, but the chaotic region in the QKR-limit is non-ergodic. The number of spins is $N = 500$.

Fig. 1(d) one may conjecture that in the infinite qubit limit, classical chaotic regions correspond to maximum EE = 1/2, and regular regions correspond to minimum EE = 0, with a well defined demarcation between these two EE regions, in the QKT.

The surprising correspondence between EE and KSE is made more stark when one compares the vastly different forms of the KSE Eq. (12) and the EE Eq. (18). Underlying these very different equations however is a commonality in the information that they encapsulate; both are the rate of information production in their relative classical and quantum domains [18].

In the kicked-top, chaotic regions are quasi-ergodic, and these regions are marked by high EE, and non-ergodic regions are marked by low EE. This however is not a general relation, and in the next section we show that non-ergodic regions can also exhibit high EE.

III. QUANTUM KICKED ROTOR

Another well-known kicked system used in the study of chaos is the kicked rotor. The Hamiltonian of the QKR is

$$H_R = \frac{1}{2I}P^2 + K \cos \Phi \sum_{n=-\infty}^{\infty} \delta(t-n)$$  \hspace{1cm} (19)

where $\Phi$ is the angle operator and $P$ is the angular momentum, canonically conjugate to $\Phi$. $K$ is the kicking strength and $I$ is the moment of inertia. The rotor operators obey the commutation relation

$$[P, \Phi] = -i$$  \hspace{1cm} (20)

The angular momentum and angle operator at each kick can be obtained from the discrete time evolution of the operators in the Heisenberg picture,

$$P_{n+1} = U_R^\dagger P_n U_R$$,

$$\Phi_{n+1} = U_R^\dagger \Phi_n U_R$$

where the Floquet operator $U_R$ is

$$U_R = \exp\left(-i\frac{P^2}{2I}\right) \exp\left(-iK \cos \Phi\right).$$ \hspace{1cm} (22)

This produces the stroboscopic equations

$$P_{n+1} = P_n + K \sin \Phi_n,$$

$$\Phi_{n+1} = \Phi_n + P_{n+1}/I.$$ \hspace{1cm} (23)

As there are no products of $P$ and $\Phi$ terms, this equation is also valid classically.

The phase space of the rotor is a cylinder, $-\infty < P < \infty$, $0 \leq \Phi < 2\pi$. This is topologically different than the spherical phase space of the top. Although the rotor is unbounded in $P$, the stroboscopic equations show that the system is invariant under $2\pi I$ translations in $P$ and $2\pi$ in $\Phi$. Fig. 4(a) plots the classical rotor phase space for $K = 0.9, I = 1$.

A. Classical rotor-limit

Rotor dynamics may be derived from the top if we confine the top to an equatorial waistband as depicted in Fig. 3 [24]. This is achieved by reducing the precession frequency about the $x$-axis and increasing the torsion strength about the $z$-axis through the rescaling

$$\alpha = K/j, \hspace{0.5cm} \beta = j/I,$$  \hspace{1cm} (24)

where $j \to \infty$. We call this substitution the rotor-limit of the top, or simply the rotor-limit.

If one begins in the equatorial waistband, this rescaling confines the angular momentum to (Fig. 3)

$$X = \cos \Phi, \hspace{0.5cm} Y = \sin \Phi, \hspace{0.5cm} Z = P/j.$$ \hspace{1cm} (25)

Substitution of Eq. (24) and (25) into the kicked-top map of Eq. (11), takes one to the kicked-rotor map of Eq. (23).

The rotor may be approximated by the top even for relatively modest values of $j$. Fig. 4(c) plots the top phase space with the rescaled $\alpha$ and $\beta$ for $j = 9$, in between $P = 0$ and $2\pi$. A comparison with the rotor phase space of Fig. 4(a), shows that rotor characteristics are clearly seen in the rotor-limit of the top phase space of Fig. 4(c).

B. Rotor Kolmogorov-Sinai entropy

To further quantify the similarities of the rotor and the rotor-limit, we compare the KSE of the two. From
the whole phase space; here the system is highly non-ergodic. These chaotic regions are localised, as opposed to the global chaos exhibited in the CKT.

We calculate the ergodicity corresponding to a point in one of these regular region and and also in a local chaos region: \((\Theta_0, \Phi_0) = (\pi, 0)\) and \((\pi, \pi/2)\), respectively. These points are marked by \(R_1\) and \(R_2\) in Fig.4(d). Fig.2 shows that the ergodicity of the regular region is low, but that point \(R_2\) is also highly non-ergodic. We would like to know, whether EE can still be a signature of quantum chaos in these highly non-ergodic regions.

### E. Rotor entanglement entropy

As the classical rotor may be extracted from the top for modest values of \(j\), quantum rotor physics may also be extracted from the quantum top. However in the quantum case we will need a large number of qubits to identify the correspondence between EE and chaos in the rotor, as we will show.

We begin with the rotor-limit with \(j = 9\), as with the classical example. For the range \(0 \leq \langle \hat{P} \rangle < 2\pi\), \(\Theta\) is restricted to \(\pi/2 \leq \Theta < \arccos(2\pi/j)\) or \(-\pi/2 \geq \Theta > -\arccos(2\pi/j)\), since \(\hat{P} = Zj = j\cos\Theta\). We choose the latter range for \(\Theta\), as this will correspond to the rotor map of Eq. (23). For large \(j\), this range is a small strip in the equatorial waistband of the top phase space.

Now unlike the classical case, where the demarcation between regular and chaotic regions are well defined, in the quantum regime the transition between corresponding regions of low and high EE is gradual. This means that deeper in the quantum regime, features corresponding to the classical features may be washed out. Fig. 6 shows that points \(R_1\) and \(R_2\) correspond to classical regular behaviour, and points \(R_2\) and \(R_3\) correspond to chaos. A comparison of Fig 6(a) and (b) shows that as one increases the number of qubits these regions become more distinguishable, in that there is less overlap of the EE marking each region. This is also reflected in Fig. 4(f) which plots the EE at \(\Phi = \pi\) for various \(N\), where the difference between EE of chaotic and regular regions becomes greater with increasing \(N\).

Fig. 6 shows that points \(R_2\) and \(R_3\) are different, revealing an asymmetry in the rotor-limit, that is not present in the rotor phase space. This asymmetry is also clearly evident in Fig. 4(f). In the classical case, the KSE plot of the rotor-limit [Fig. 4(d)] is also asymmetric, whereas the KSE plot of the rotor is not [Fig. 4(b)]. The root of the asymmetry lies in the fact that we have used a finite value of \(j\), whereas the rotor is reached from the top only in the limit of \(j \to \infty\).

Fig. 5 plots the time-averaged EE at \(\Phi = \pi\) for various \(j\) with constant \(N = 500\). Now two operational properties of the rotor-limit is revealed here. Firstly, insulating...
FIG. 4. (a) and (c) show the classical phase space of the CKR and the CKR-limit \((j = 9)\), with 500 random initial conditions for a duration of 500 kicks. (b) and (d) show the KSE of the CKR and CKR-limit \((j = 9)\), calculated on a grid of 200 \(\times\) 200 initial conditions, iterating the linear map for \(10^4\) steps. The points \(R_1, R_2, R_3, R_4, R_5, R_6\) mark \((\Phi, \Theta) = (\pi, 0), (\pi, \pi/2), (\pi, 3\pi/4), (\pi, 2\pi), (0, 0), (0, 2\pi)\). (e) plots the time-averaged EE of the QKR-limit, for \(N = 300\) qubits and a duration of \(T = 300\) kicks. (f) plots the time-averaged EE of the QKR-limit at \(\Phi = \pi\) for various \(N\); the transition from regions of low EE to high EE becomes more stark with increasing number of qubits. Parameters: \(K = 0.9, I = 1\).

Increasing \(j\) means that the behaviour of the system approaches that of the quantum rotor, thereby reducing the aforementioned asymmetry. Secondly, increasing \(j\) for a constant \(N\) presents a trade-off: although the system approaches the quantum rotor, for larger values of \(j\) one requires more qubits to achieve the correspondence between EE and the classical features of the phase space; i.e. the difference between the EE of chaotic and regular regions is reduced. The intuitive reason for this is that, larger \(j\) means that we are working in a narrower equatorial waistband. In the parameters of the top, this means that \(\Theta\) is confined to the range \(\{\pi/2, \arccos(2\pi/j)\}\). As we are working in an increasingly narrower region as \(j\) increases, one requires a larger number of qubits to be able to distinguish the corresponding classical features, as exemplified in Fig. 6.

Finally we plot the EE corresponding to the entire classical phase space for the QKR-limit in Fig. 4(e). We see a remarkable correspondence with the KSE of the CKR-limit in Fig. 4(d). Importantly, in contrast to the kicked
top, here the system is far from ergodic. Therefore EE can be a signature of quantum chaos even in non-ergodic systems.

To intuit how this is so, recall Eq. (18) which gives the EE of the one-qubit bi-partition. For maximally mixed states, the state space is explored uniformly, so that \(\langle J \rangle = 0\). One may roughly consider this to be the case for the quasi-ergodic QKT in Fig. 1. However, one should only consider this as an intuitive explanation, as our system is in fact not maximally mixed; it is the time-averaging of procedure that gives rise to these effects. In the rotor-limit however, exploration in the \(J_z\) direction is suppressed, so that \(J_z \to 0\), for states beginning in the equatorial waistband. This suppression means that not all state space are uniformly explored, and therefore the system is far from being ergodic. This means that, \(\langle J \rangle = 0\) under the conditions that \(\langle J \rangle_x = \langle J \rangle_y = 0\); under these conditions EE is maximised even though the system is non-ergodic conditions. In the next section, we provide an alternative explanation which is reminiscent of KAM theory.

IV. QUANTUM KOLMOGOROV-ARNOL’D-NOSER THEORY

An integrable Hamiltonian \(H_0\) in the presence of perturbation is written as

\[
H = H_0(\kappa) + \epsilon V(\kappa, \lambda)
\]

(31)

where \(\kappa\) and \(\lambda\) are the action variables with total \(D\) dimension, and \(\epsilon\) is a small perturbation parameter. Integrable \(H_0\) generates periodic phase-space trajectories that lie on \(D\)-dimensional tori surfaces. The KAM theorem states that for sufficiently small \(\epsilon\), the tori of \(H_0\) do not vanish but are deformed, so that the trajectories generated by \(H\) are conditionally periodic [1]. What is the quantum analogue of KAM theory?

Understanding the crossover behaviour arising from the integrability breaking in quantum systems have been pursued through indirect measures such as level statistics [29–32] and in the quasi-classical limit of systems using semi-classical eigenfunction hypothesis [33–35]. In more direct analogy with KAM theory, an existence conditions for localisation in non-integrable quantum systems has also been developed [36]. We do not give a quantum KAM theory here, but simply show properties in our quantum system that are reminiscent of KAM theory. Our motivation is that this may lead to a robust quantum KAM theory in future work.

A. Entanglement entropy and quantum KAM tori

An alternative perspective on why regions with maximum EE may not be ergodic, can be found by considering the CKR in the context of KAM theory [37, 38]. For very small kicking strength trajectories are regular, for very large kicking strength trajectories are chaotic. In between these two extremes, both types of trajectories exist in the phase space, with islands of chaotic regions separated by regular torus regions or KAM trajectories. This means that these islands of chaotic trajectories are bounded and do not explore the whole phase space. The critical value of \(K\) where the last of the KAM trajectories disappears is \(K_C \approx 0.971635\) \((I = 1) [38]\).

One may consider torus regions of low EE as the quantum counterpart of classical KAM trajectories, separating islands of high EE. By analogy to the classical case, we conjecture that the presence of torus regions of low EE indicate that states beginning in different islands of high EE will explore mutually exclusive states, i.e. low EE tori separate orthogonal states. This conjecture is numerically supported in Fig. 7(a)-(d) which shows the time-averaged density matrices of the full system with initial conditions \((\Phi_0, \Theta_0) = (3.57, 2.25)\) marked by \(T_2\) in Fig. 7(b), and \((\Phi_0, \Theta_0) = (0, 0), (\pi, \pi/2), (\pi, \pi/2)\) marked...
Appendix A: Proof that ergodic system maximise entanglement entropy

The linear entanglement entropy defined as $S = 1 - \text{tr} \rho^2$ is maximized when $\text{tr} \rho^2$ is minimized. The density matrix $\rho$, is an Hermitian operator of unitary trace. We want to prove that $\text{tr} \rho^2$ is minimised if all the diagonal elements $\rho_{11} = \rho_{11} = \cdots = \rho_{NN} = 1/N$, i.e. if the system is ergodic. This is a constrained optimization problem, solvable with the Lagrange multipliers method. The constraint is $\text{tr} \rho = 1$. The quantity to minimize is $\text{tr} \rho^2 = \rho_{11}^2 + \rho_{22}^2 + \cdots + \rho_{NN}^2$. We define the Lagrangian as:

$$\mathcal{L} = \rho_{11}^2 + \rho_{22}^2 + \cdots + \rho_{NN}^2 - \lambda (\rho_{11} + \rho_{22} + \cdots + \rho_{NN} - 1).$$  \hspace{1cm} (A1)
FIG. 7. (a)-(d) shows the time-averaged density matrices for point $T_1$ in Fig. 1(c) and points $R_5, R_2, R_3$ in Fig. 4(d), respectively. $R_2$ and $R_5$ belong to the same island of high EE, whereas $R_3$ belongs to a different high EE island. $T_1$ uniformly explores the full Hilbert space and therefore is ergodic. $R_2, R_3, R_5$ explore a subset of the full Hilbert space, and therefore is not ergodic. $R_2$ and $R_3$ explore different regions of the subspace, whilst $R_2$ and $R_5$ explore the same subspace, leading to the notion that evolution from initial conditions belonging to the same island will explore the same subspace, but different islands explore different subspaces.

FIG. 8. The time averaged fidelity of points $R_5 = (0, 0)$ and $R_6 = (0, 2\pi)$, $F(\hat{\rho}_{R_5}, \hat{\rho}_{R_6})$, as a function of $K$ for $N = 500, 1000, 2000, 3000$. The fidelity of points $R_5$ and $R_6$ begin to increase not at $K_C \approx 0.971635$, but just after, supporting the idea that cantori correspond to an impenetrable barrier in the quantum case.
We want to minimize $\mathcal{L}$ with respect to $\rho_i$ and $\lambda$, $\frac{\partial \mathcal{L}}{\partial \rho_i} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$. The system to solve is:

$$
\begin{align*}
2\rho_{11} - \lambda &= 0; \\
2\rho_{22} - \lambda &= 0; \\
\vdots \\
2\rho_{NN} - \lambda &= 0; \\
\rho_{11} + \rho_{22} + \cdots + \rho_{NN} - 1 &= 0.
\end{align*}
$$

(A2)

The solution of this system is $\rho_{11} = \rho_{11} = \cdots = \rho_{NN} = 1/N$.

Appendix B: Derivation of the tangent space of the CKT

The generalised iterative map $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ of Eq. (11), is linearised to give its associated tangent map $\delta \mathbf{x}_{n+1} = f(\mathbf{x}_n + \delta \mathbf{x}_n) - f(\mathbf{x}_n)$, where $\mathbf{x}_n = (X_n, Y_n, Z_n)$. We explicitly do the calculation in detail for the coordinate $X$ ($Y$ and $Z$ follow in a similar fashion).

$$
dX_{n+1} = f(X_n + \delta X_n) - f(X_n)
$$

$$
= (X_n + dX_n) \cos \left\{ \beta \left[ (Y_n + dY_n) \sin \alpha + (Z_n + dZ_n) \cos \alpha \right] \right\} \\
- \left[ (Y_n + dY_n) \cos \alpha - (Z_n + dZ_n) \sin \alpha \right] \sin \left\{ \beta \left[ (Y_n + dY_n) \sin \alpha + (Z_n + dZ_n) \cos \alpha \right] \right\} \\
- X_n \cos \left[ \beta (Y_n \sin \alpha + Z_n \cos \alpha) \right] + (Y_n \cos \alpha - Z_n \sin \alpha) \sin \left[ \beta (Y_n \sin \alpha + Z_n \cos \alpha) \right].
$$

Expanding the expression and keeping only the terms up to the first order in $dX_n$, $dY_n$ and $dZ_n$, we obtain,

$$
dX_{n+1} = X_n \cos \left[ (\beta Y_n \sin \alpha + \beta Z_n \cos \alpha) + (\beta dY_n \sin \alpha + \beta dZ_n \cos \alpha) \right] + dX_n \cos \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right]
$$

$$
- Y_n \cos \alpha \sin \left[ (\beta Y_n \sin \alpha + \beta Z_n \cos \alpha) + (\beta dY_n \sin \alpha + \beta dZ_n \cos \alpha) \right] - dY_n \cos \alpha \sin \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right]
$$

$$
+ Z_n \sin \alpha \sin \left[ (\beta Y_n \sin \alpha + \beta Z_n \cos \alpha) + (\beta dY_n \sin \alpha + \beta dZ_n \cos \alpha) \right] + dZ_n \sin \alpha \sin \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right]
$$

$$
- X_n \cos \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right] + Y_n \cos \alpha \sin \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right]
$$

$$
- Z_n \sin \alpha \sin \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right].
$$

Using the relation $\cos(x + dx) - \cos(dx) = -\sin(x)dx$ we get

$$
dX_{n+1} = -X_n \sin \left( \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right) d(\beta Y_n \sin \alpha + \beta Z_n \cos \alpha)
$$

$$
- Y_n \cos \alpha \cos \left( \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right) d(\beta Y_n \sin \alpha + \beta Z_n \cos \alpha)
$$

$$
+ Z_n \sin \alpha \cos \left( \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right) d(\beta Y_n \sin \alpha + \beta Z_n \cos \alpha)
$$

$$
+ dX_n \cos \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right] - dY_n \cos \alpha \sin \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right] + dZ_n \sin \alpha \sin \left[ \beta Y_n \sin \alpha + \beta Z_n \cos \alpha \right].
$$

Finally, grouping the $dX_n$, $dY_n$ and $dZ_n$ terms, we write down the tangent map in its standard linear form $d\mathbf{x}_{n+1} = L(x_n) d\mathbf{x}_n$ in Eq. (13).

Appendix C: Derivation of the tangent space of the CKR

The generalised iterative map $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ of Eq. (23), is linearised to give its associated tangent map $\delta \mathbf{x}_{n+1} = f(\mathbf{x}_n + \delta \mathbf{x}_n) - f(\mathbf{x}_n)$, where $\mathbf{x}_n = (\Phi_n, P_n)$.

$$
\delta P_{n+1} = P_n + \delta P_n + K \sin(\Phi_n + \delta \Phi_n) - (P_n + K \sin \Phi_n)
$$

$$
= \delta P_n + K \cos(\Phi_n) \delta \Phi_n.
$$

(C1)

$$
\delta \Phi_{n+1} = \Phi_n + \delta \Phi_n + \frac{P_{n+1} + \delta P_{n+1}}{I} - \Phi_n - \frac{P_n}{I}
$$

$$
= \delta \Phi_n + \frac{\delta P_{n+1}}{I} = \delta \Phi_n + \frac{\delta P_n + K \cos(\Phi_n) \delta \Phi_n}{I}
$$

$$
= (1 + \frac{K}{I} \cos \Phi_n) \delta \Phi_n + \frac{\delta P_n}{I}.
$$

(C2)
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