Instantons of Type IIB Supergravity in Ten Dimensions

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A family of $SO(10)$ symmetric instanton solutions in Type IIB supergravity is developed. The instanton of least action is a candidate for the low-energy, semiclassical approximation to the $D=-1$ brane. Unlike a previously published solution, this admits an interpretation as a tunneling amplitude between perturbatively degenerate asymptotic states, but with action twice that found previously. A number of associated issues are discussed such as the relation between the magnetic and electric pictures, an inversion symmetry of the dilaton and the metric, the $R \times S^9$ topology of the background, and some properties of the solution in an “instanton frame” corresponding to a Lagrangian in which the dilaton’s kinetic energy vanishes.

I. INTRODUCTION

Type IIB supergravity (SUGRA) represents the low energy effective field theory describing massless particles with momenta below the scale of massive modes of the superstring. Type IIB SUGRA contains, in addition to the generic fields of the Neveu-Schwarz (NS) sector (the metric tensor $g_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$, and dilaton $\phi$), gauge potentials characteristic of the Ramond-Ramond (RR) sector (a scalar axion $a$, two-form $C_{\mu\nu}$, four-form $C_{\alpha\lambda\mu\nu}$, and their duals). These RR potentials couple locally to charges carried by non-perturbative states called D-branes. D-branes may also be seen as solitonic solutions of the source-free classical field equations of the dual (magnetic) form of SUGRA. The $D=-1$ brane is unique among D-branes since it is localized in time (albeit Euclidean) as well as in space. The interpretation is that it is an instanton, signaling the existence of a non-perturbative transition amplitude of SUGRA. In this note, we present a new class of instanton solutions in Type IIB SUGRA. The solution with least action is a candidate for the SUGRA representation of the $D=-1$ brane. This solution is closely related to, but somewhat different from, one found previously. In fact, the previous solution will be seen to be more like a half-brane, which resolves some of the paradoxes associated therewith. A motivation for seeking a different solution stems in part from difficulties interpreting the latter as a tunneling amplitude between perturbatively degenerate ground states and, therefore, the associated difficulty understanding what the $D=-1$ brane does and how it might affect what the ground states actually are. This is not something that can be easily addressed directly in string theory, since it requires a field theory to discuss the properties of Green’s functions, such as cluster decomposition, a crucial property for an acceptable vacuum state. As cluster decomposition is a property of large spacelike separations, this should not depend crucially on an understanding of the modifications due to massive modes of the superstring. In due course, we wish to clarify several points concerning the relation between the Lorentz and Euclidean actions, between electric and magnetic formulations of the instantons, on supersymmetry in Euclidean spacetime, and the role of the $SL(2,R)$ symmetry for this transition amplitude. This paper is a start which, we hope, will bring new insights into nonperturbative phenomena in string theory.

If the $D=-1$ brane were like other branes, it would appear in both a magnetic formulation, in which it arises as an extended solution of the “source-free” field equations, as well as in an electric formulation, in which it is postulated as an elementary source coupled locally to the dual potential. In this paper, we present a magnetic description in terms of the $C_8$ potential. In a companion paper, we offer an electric description in terms of the axion $a$ potential. In fact, for a complete understanding of the instantons’ properties as well as its interpretation as a tunneling transition in terms of the axion field, we must refer to this dual description. The idea of an electric description of an instanton is a novel construct, to say the least. Heretofore, nonperturbative tunneling amplitudes could be found only when a semiclassical, or WKB-like approximation, was appropriate. The idea that an instanton could be introduced directly as an elementary object opens up a world of new possibilities for field theories, but it may only be in the context of gravity that it becomes relevant because of the associated change in the spacetime background.

The outline of this paper is as follows: In the next section, we review the dual forms of the Lagrangian of interest, paying special attention to the difference between Lorentzian and Euclidean signature. In Sec. II, we present our instanton solutions, and, in Sec. III, we discuss some of the features of the associated tunneling amplitude, as well as the relation of our solutions to the previously discovered. Finally, in Sec. IV, we conclude with a summary and some prospects for further developments.
II. LORENTZIAN VERSUS EUCLIDEAN SIGNATURE

The effective field theory of massless modes of the Type IIB string is of the form

\[ \mathcal{L}_{IIB} = \mathcal{L}_{SUGRA} + \alpha' L_2 + \alpha'^2 L_4 + \ldots, \]  

where the leading term \( \mathcal{L}_{SUGRA} \) is independent of \( \alpha' \)

The dual forms of the SUGRA Lagrangian \( \mathcal{L}_{SUGRA} \) with which we shall be concerned are, in 10 dimensions in the Einstein frame,[1]

\[ \mathcal{L}_0 = -R + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2^2} e^{2\phi} F_{1}^2 \]  

\[ \mathcal{L}_8 = -R + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2 \cdot 9!} e^{-2\phi} F_9^2 \]  

where \( F_1 \equiv da \) and \( F_9 \equiv dC_8 \). Here, we have suppressed all those fields having zero value for the background that we shall be considering, but one must restore them in order to discuss stability, supersymmetry, or to carry out a calculation of the path integral for transition amplitudes. The actions are the integrals over these plus boundary terms involving the extrinsic curvature which, for the asymptotically flat spacetimes and asymptotically trivial field configurations with which we shall be dealing, are of no consequence. The correspondence between the two RR fields is

\[ e^{2\phi} F_1 = * F_9 \]  

where * denotes the Hodge dual. It is generally assumed that either formulation can be used as a starting point for describing the same physics. The corresponding equations of motion (EOM), up to the addition of possible source terms, are

\[ \nabla_{\mu} (e^{2\phi} \nabla_{\mu} a) = 0 \]

\[ -\nabla^2 \phi + e^{2\phi} (\nabla a)^2 = 0 \]  

\[ R_{\mu\nu} = \frac{1}{2} \nabla_{\mu} \phi \nabla_{\nu} \phi + \frac{1}{2} e^{2\phi} \nabla_{\mu} a \nabla_{\nu} a \]  

and

\[ \nabla_{\mu} (e^{-2\phi} F_9^{\mu\nu\ldots}) = 0 \]

\[ \nabla^2 \phi + \frac{1}{2^2} e^{-2\phi} F_9^2 = 0 \]  

\[ R_{\mu\nu} = \frac{1}{2} \nabla_{\mu} \phi \nabla_{\nu} \phi + \frac{1}{2 \cdot 9!} e^{-2\phi} F_{\mu\nu\ldots} F_{\nu\mu\ldots} - g_{\mu\nu} \frac{1}{2 \cdot 9!} e^{-2\phi} F_9^2 \]  

where we have rewritten Einstein’s equations using the form of the scalar curvature in each case,

\[ R = \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} e^{2\phi} (\nabla a)^2 \]  

\[ R = \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2 \cdot 9!} e^{-2\phi} F_9^2 \]  

Up to now, we have not specified whether we interpret the preceding discussion to be in a spacetime with Lorentzian signature (LS) or Euclidean signature (ES). It turns out that there are important differences. With LS, the dual forms of the Lagrangian given in eqs. (2) and (3) do not transform into each other under the formal substitution given in eq. (4). This is because of a minus sign that appears for LS

\[ \frac{1}{2^2} e^{-2\phi} F_9^2 = -e^{2\phi} F_1^2. \]  

However, one may verify that the EOM are interchanged by the duality transformation.[4]

On the other hand, with ES, the correspondence is reversed, viz., the Euclidean actions do transform into each other but the EOM do not, the crucial difference resulting from the sign flip

\[ \frac{1}{2 \cdot 9!} e^{-2\phi} F_9^2 = +e^{2\phi} F_1^2. \]
For all other D-branes, these observations are irrelevant, since they are static solutions of the EOM and, hence, are independent of the sign of the time. However, for the D=–1 brane, these differences are critical. Indeed, one may justifiably wonder whether the presumed equivalence between the dual formulations of the theory should not be reexamined.

An instanton is used to compute a semiclassical (or WKB) approximation to a tunneling amplitude in field theory. It is usually defined to be a minimum of the Euclidean action, corresponding to a solution of the EOM having finite action. Extending this definition to quantum gravity is complicated by the fact that the Euclidean scalar curvature is not positive semi-definite. Worse, it is not bounded from below, so it would seem to be hopeless to seek the absolute minimum of the Euclidean classical action. It is not our goal to resolve this controversial issue here, but it seems that we must take a position to proceed at all. We will adopt the point of view (advocated by Hartle and Gleichenhagen) that this aspect of the Einstein-Hilbert action, which is associated with the conformal mode, is very likely a kind of gauge artifact and not a physical breakdown in the theory. Regardless of one’s view on the ultimate marriage of gravity with quantum mechanics, it is hard to believe that the leading contribution to the effective field theory at distances large compared to the Planck length is not proportional to the scalar curvature. And even if one does not share the view that the Euclidean formulation is fundamental, any discussion of gravitational instantons requires a resolution of this dilemma. Otherwise, it seems as though there would always be infinite tunneling rates!

In the case at hand, it may be seen that, for any stationary configuration, the source-free EOM yields a non-negative value for the action. For this, one need only consider Einstein’s equations, which imply that the scalar curvature satisfies eq. (8). This implies that the value of the action is

$$S_V = \int d^{10}x \sqrt{g} \frac{1}{16\pi} e^{-2\phi} F_{\mu \nu}^2 \geq 0.$$  

(11)

Thus, one may conclude that any nontrivial solution of Einstein’s equations will yield a strictly positive value for the action.

In the dual formulation in terms of the axion field $a$, the implication of the source-free Einstein equation eq. (8) is that the value of the Lagrangian density eq. (3) is always zero. For static D-brane solitons, a classical source would be introduced, coupled “electrically” to the RR-field in order to reproduce the effects found in the dual, “magnetic” formulation. In the present case, the analog would be a point source located at the site of the instanton (usually chosen to be the origin of coordinates) and coupled locally to the axion field at that point. On the other hand, the introduction of a classical source for a transition amplitude that is supposed to be inherent to the theory seems impermissible. The resolution of this paradox will be seen subsequently to be that, like infinity, the origin is actually not part of the background Euclidean spacetime. Like infinity which, as noted in the next section, cannot be compactified, so also the origin cannot be attached to the classical background as if it were a point. In fact, the roles of the origin and infinity are identical and, because of an isometry of the metric, may be interchanged. Thus, the “source” is transformed into a boundary condition at the origin, as required by current conservation. With the origin removed, the topology of the space is qualitatively different, allowing for nontrivial “windings” about the origin. (See eq. (4) below.)

Because they determine the boundary conditions on the semiclassical solutions, let us reflect on the classical ground states of this theory which are expected to correspond to the approximate ground states in perturbation theory in the quantum theory. In either formulation, these correspond to a flat space, $g_{\mu \nu} = \eta_{\mu \nu}$, and a constant value of the dilaton field $\phi(x) = \phi_0$. In the axion formulation, eq. (3), the axion field also takes an arbitrary, constant value of the field $a(x) = a_0$. The Type IIB SUGRA action (10) possesses an $SL(2, \mathbb{R})$ global symmetry that is explicitly broken by the Type IIB superstring action (11), so $SL(2, \mathbb{R})$ is believed to be an “accidental” symmetry of the SUGRA action and expected to be explicitly broken by higher order terms of the effective field theory (12). Each perturbatively degenerate ground state spontaneously breaks the global $SL(2, \mathbb{R})$ symmetry down to $R_\tau$ where $R_\tau$ denotes those modular transformations that leave $\tau_0 = a_0 + i \exp(\phi_0) \text{ invariant. Thus, two of the three generators of } SL(2, \mathbb{R}) \text{ are spontaneously broken, and the corresponding massless modes (or Goldstone bosons in the quantum theory) are just the fluctuations in } a = a - a_0 \text{ and } \phi = \phi - \phi_0 \text{, i.e., these scalar fields are their own Goldstone bosons. These would therefore remain massless to all orders in perturbation theory, regardless of supersymmetry. However, because such a state is supersymmetric, their masslessness is also protected by the supermultiplet representation. In the dual formulation, eqs. (3) and (10), the ground states have $F_\tau = 0$, so that $C_{\mathcal{S}}$ must be pure gauge, $C_{\mathcal{S}} = dA_T$ (including, possibly, zero). The $SL(2, \mathbb{R})$ symmetry is not a Noether symmetry but nevertheless can be seen to be a symmetry of the source-free EOM. (13) There remains a global $R^8$ scaling symmetry that is spontaneously broken for which, once again, the fluctuations in $\phi = \phi - \phi_0$ form the Goldstone mode. The second scalar mode corresponds to fluctuations in $C_8$, whose masslessness is protected by gauge invariance in this phase and, as in the axion approach, by supersymmetry. In either picture, the natural expectation for the role of an instanton would be to produce tunneling between these distinct ground states, suggesting that the true ground state will be some sort of superposition of these.
III. INSTANTON SOLUTIONS

A. Solving the Euclidean EOM

To solve the field equations eq. (6) in Euclidean space, it is natural to make an ansatz similar to that made for other D-branes. Motivated by the expectation that the minimal action solution occurs for the most symmetric configuration, we seek an SO(10) invariant solution. We make therefore, the ansätze
\[ g_{\mu\nu} = \Omega^2(r) \delta_{\mu\nu}, \quad \phi = \phi(r), \] (12)
where \( r \) is the radial coordinate in ten-dimensions. We further assume that the only non-zero component of \( F_9 \) is the angular component
\[ "F_9" = \frac{qJ}{\Omega_9} \omega_9 \] (13)
where \( \omega_9 \) is the volume nine-form on \( S^9 \), and \( \Omega_9 = 32\pi^4/105 \) is the volume of the unit nine-sphere. This implies that \( F_9 \) is closed, \( dF_9 = 0 \), but not exact, i.e., although \( F_9 = dC_8 \) locally, the potential \( C_8 \) is not globally well-defined. To see this, consider any region \( M \) the includes the origin \( r = 0 \). Then it follows from eq. (13) that
\[ \int_M dF_9 = \int_{\partial M} F_9 = qJ, \] (14)
But if \( F_9 = dC_8 \) for a function \( C_8 \), these integrals would vanish. In fact, \( F_9 \) is singular at the origin, and it is impossible to find a (single-valued) function \( C_8 \) that is nonsingular everywhere on a closed surface (e.g., \( S^9 \)). This sort of situation is familiar, for example, from discussions of the Dirac monopole.†‡ A similar discussion applies in a neighborhood of \( \infty \), where one may think of the opposite charge \( -qJ \) residing. Assuming that, as \( r \to \infty \), the metric becomes flat and \( \phi \) tends to a constant, finiteness of the action eq. (11) requires that \( F_{\mu\nu} \) fall faster than \( O(r^{-5}) \) in spherical coordinates. In fact, by Gauss’s theorem, the ansatz eq. (13) requires that it fall precisely as \( r^{-9} \).

Note that \( qJ \) has dimensions of \( [\text{length}]^8 \) and is the only dimensionful parameter encountered thus far.‡§ Thus, its value is a matter of convention, and we could choose units with \( qJ = \pm 1 \) if we wished. Only its sign is relevant, and it is trivial to transform the solution for one sign into the other. However, for this semiclassical description to be valid, we would expect that the associated length scale be large compared to the scale of the expansion parameter, \( qJ/\alpha'^4 \gg 1 \).

Given these ansätze, it then follows from the first equation in eq. (6) that
\[ e^{-2\phi}F_9 = *db \] (15)
for some scalar field \( b = b(r) \). Since the classical ground states have \( F_9 = 0 \), \( b(r) \) should tend to a constant value asymptotically. Then, from eqs. (13) and (14), it follows that
\[ J^r = g^{rr}e^{2\phi} \frac{\partial b}{\partial r} = \frac{qJ}{\Omega_9 \sqrt{g}} \] (16)
Nevertheless, eq. (15) together with the Bianci identity, \( dF_9 = 0 \), imply
\[ \nabla_\mu (e^{2\phi} \nabla^\mu b) = 0 \] (17)
except possibly at \( r = 0 \) where \( F_9 \) cannot be defined. Thus, the current \( J^\mu \equiv e^{2\phi} \nabla^\mu b \) is conserved, except possibly at the origin. In fact, in view of eq. (14), it is as if there were a point charge there,
\[ \nabla_\mu J^\mu = \nabla_\mu (e^{2\phi} \nabla^\mu b) = \frac{qJ \delta^{10}(x)}{\sqrt{g}} \] (18)
How is eq. (15) compatible with the view that we seek a solution of the source-free EOM eq. (6)? This question implicitly requires that we specify the region over which we seek such a solution. Although it is conventional to regard such “magnetic” solutions as non-singular, the fact of the matter is that the corresponding RR potentials, in this case \( C_8 \), are simply not well-defined at the “center” of the solution, i.e., at the origin of the transverse coordinates. The behavior of \( C_8 \) at the origin, like its behavior at infinity, is a reflection of the fact that it is necessarily singular on any closed surface surrounding the origin. Thus, the background field is not well-defined at the origin of the D-brane.
In this sense, the source-free EOM hold everywhere except at \( r = 0 \) and \( r = \infty \). This is however, only half the answer, since the origin certainly is a source of RR charge, just as infinity is a complementary sink. The other half of this story will be explained in the next section: the geometry of the background spacetime is not a simply-connected Riemannian surface but is “cylindrical,” \( R \times S^3 \), with the neighborhood of the origin identical to the neighborhood of infinity.\(^{[24]}\) The behaviors at the origin and at infinity correspond to boundary conditions on the fields.

We digress at this point to remark that we are already in a position to determine the value of the action for the instanton, even before having solved the remaining equations! Returning to eq. (11), we reexpress \( F_9 \) in terms of \( b \) and then use eq. (14)

\[
S_V = \int d^{10}x \sqrt{g} e^{2\phi} (\nabla b)^2 = \int d^{10}x \sqrt{g} \nabla_\mu b J^\mu = q_J \int dr \frac{\partial b}{\partial r} = q_J \Delta b, \tag{19}
\]

where \( \Delta b \equiv b(r = \infty) - b(r = 0) \).\(^{[25]}\) By eq. (16), \( b \) is monotonically increasing (decreasing) with \( r \) depending on whether the sign of \( q_J \) is positive (negative). Thus, the sign of \( \Delta b \) is the same as the sign of \( q_J \), and so \( S_V \) is positive, as it must be according to eq. (11). This result eq. (19) depends only on the conservation of the current \( J^\mu \), the presence of the charge \( q_J \), and the net change \( \Delta b \).\(^{[20]}\) It is not at all clear at this point how \( \Delta b \) should be determined, since the asymptotic condition is merely constant \( \phi \) and \( F_9 = 0 \). We shall return to this issue in Section 4.

Next, consider the EOM for the dilaton in eq. (6). If we insert the solution in terms of \( y \), so that, from eq. (23), the relation between \( \Omega^8 \) and \( \phi \) is

\[
\nabla_\mu K^\mu = q_K \frac{\delta^{10}(x)}{\sqrt{g}}. \tag{22}
\]

As remarked earlier, rather than an external source at the origin \( x^\mu = 0 \), we think of the origin as not in the space and the behavior of the fields there as a boundary condition. We have noted previously that the definition of \( K^\mu \) is arbitrary up to a shift by a constant times \( J^\mu \), and, similarly, eq. (21) is valid for any \( b \) satisfying eq. (13). Since \( b \) is arbitrary up to a constant, the value of \( K^\mu \) has no intrinsic physical meaning. For example, the value of the action eq. (13) is clearly independent of \( q_K \). The physical question is how the fields behave in the presence of the non-zero current \( J^\mu \).

The form of eqs. (10) and (21) suggest the introduction of a new coordinate \( y \) defined by

\[
dy \equiv \frac{8 \ell^8 dr}{\sqrt{g} g^{rr}}. \tag{23}
\]

where we have introduced an arbitrary scale \( \ell \) to make \( y \) dimensionless. From the ansatz for the metric, we have \( \sqrt{g} g^{rr} = r^9 \Omega^8 \), so that, from eq. (23), the relation between \( y \) and \( r \) may be expressed as

\[
\Omega^8 dy = d \left( \frac{y}{r} \right)^8. \tag{24}
\]

Consequently, one finds that \( \sqrt{g} g^{yy} = 8 \ell^8 \), a constant.

Returning to the EOM and changing to the coordinate \( y \), eqs. (14) and (21) become

\[
\frac{\partial \hat{b}}{\partial y} = -q_J e^{-2\phi}, \quad \frac{\partial \hat{\phi}}{\partial y} = q_J \hat{b}, \tag{25}
\]
where we defined $\tilde{q}_J \equiv q_J/8\ell^8\Omega_3$, $\tilde{b} \equiv b - k$, and $k \equiv q_K/q_J$. This equation implies

$$\frac{\partial \phi}{\partial \tilde{b}} = -\tilde{b}e^{2\phi}, \text{ so that}$$

$$\tilde{b}^2 - e^{-2\phi} = C^2$$

for some constant $C^2$. The sign of $C^2$ is undetermined at this point, but it turns out that, for $C^2 < 0$, the value of the action is ill-defined, despite eq. (19), because the solution for $b$ has a nonintegrable singularity at some finite radius. This case is discussed in the Appendix, where we show also that the metric in that case is perfectly regular everywhere. On the other hand, for $C^2 > 0$, $b$ undergoes a finite step at some radius $r_s$, but, unlike the case $C^2 < 0$, this occurs in a strong coupling region where string corrections are expected to be important, so one may hope for a resolution of the discontinuity from quantum corrections. The curvature is also singular at $r_s$ in the Einstein frame, although we shall exhibit in subsection III B another frame in which the curvature remains finite everywhere. The case $C^2 = 0$, which was the ansatz considered in ref. [3], will be discussed in due course.

Assuming that $C^2 > 0$, the general solution of eq. (23) can then be found

$$\tilde{b} = C \coth(\omega (y + y_0)), \quad e^{\phi} = C^{-1} [\sinh(\omega(y + y_0))]$$

and $y_0$ is an integration constant. To resolve sign ambiguities, we take $C \geq 0$ and choose the sign of $\omega$ to have the sign of $q_J$. Inasmuch as $q_K$ is not physically observable and is tied to the convention for $b$, there is no loss of generality in choosing $q_K = 0$, so that $\tilde{b} = b - k = b$. We will adopt this convention henceforth.

### B. Background Geometry

Heretofore, we have not needed the explicit solution for the conformal factor $\Omega(r)$ for the metric. In terms of $y$, the metric becomes

$$ds^2 = \Omega^2[dr^2 + r^2d\omega^2] = \ell^2 \left( \frac{r\Omega}{\ell} \right)^2 \left[ \left( \frac{r\Omega}{\ell} \right)^{16} \frac{dy^2}{64} + d\omega_9^2 \right]$$

As the precise form of the angular measure plays no role in our discussion, we need focus only on the radial dependence. To determine $\Omega(r)$ explicitly, we need to solve Einstein’s equations, eq. (4), which, in terms of our solution for $F_9$, can be shown to reduce to

$$R_{\mu\nu} = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} e^{2\phi} \nabla_\mu b \nabla_\nu b$$

Given our ansatz for the metric, $R_{\mu\nu}$ may be expressed in terms of $\Omega$ as

$$R_{\mu\nu} = -\frac{\delta_{\mu\nu}}{8r^{17}\Omega^8} \frac{\partial}{\partial r} \left( r^{17} \frac{\partial}{\partial r} \Omega^8 \right) + \frac{8\Omega}{r} x_\mu x_\nu \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \Omega^{-1} \right)$$

On the other hand, the right-hand side of eq. (29) clearly is proportional to the tensor $x_\mu x_\nu$ only, so that the coefficient of $\delta_{\mu\nu}$ must vanish. This implies

$$\Omega^8 = 1 - \omega_R^2 \left( \frac{\ell}{r} \right)^{16},$$

where, in accord with our notion of the classical ground states, we imposed the condition that the metric be asymptotically flat as $r \to \infty$. Intuitively, we would expect the integration constant $\omega_R^2 < 0$ to avoid a naked singularity, but, as discussed in the Appendix, this leads to other divergences making the action ill-defined. Therefore, we take $\omega_R^2 > 0$, so this solution eq. (31) apparently holds only beyond the singularity

$$r > r_s \equiv \ell \omega_R^\frac{2}{9}$$

(We shall subsequently discuss extending the solution into the interior region $r < r_s$.) It then follows from eq. (24) that

$$\left( \frac{r}{\ell} \right)^8 = \omega_R \coth(\omega_R y), \quad y \geq 0,$$
where, without loss of generality, we chose \( y = 0 \) to correspond to \( r = \infty \). From eq. (31), the conformal factor is

\[ \Omega^{-4} = \cosh(\omega R y). \]  

(34)  

To satisfy Einstein’s equation, we must also determine that the \( x_\mu x_\nu \) term in eq. (30) agrees with the right-hand-side of eq. (29). This is most simply expressed in terms of the coordinate \( y \) as

\[ R_{yy} = \frac{9}{2} \Omega^4 \frac{\partial^2 \Omega^{-4}}{\partial y^2} = \frac{1}{2} \left( \frac{\partial \phi}{\partial y} \right)^2 - \frac{1}{2} e^{2\phi} \left( \frac{\partial b}{\partial y} \right)^2. \]  

(35)  

The right-hand-side is, according to eqs. (25) and (26), given by

\[ \frac{1}{2} \bar{q}^2 \left( b^2 - e^{-2\phi} \right) = \frac{1}{2} \bar{q}^2 C^2 \equiv \frac{\omega^2}{2} \geq 0. \]  

(36)  

Therefore, eq. (35) reduces to

\[ \frac{\partial^2 \Omega^{-4}}{\partial y^2} = \left( \frac{\omega^2}{9} \right) \Omega^{-4}. \]  

(37)  

Comparing with eq. (34), we see that Einstein’s equations are satisfied provided we take

\[ \omega_R = \frac{|\omega|}{3}. \]  

(38)  

The corresponding scalar curvature takes the form

\[ R = \frac{32 \omega^2}{\ell^2} \left( \frac{\ell}{r \Omega} \right)^8 \geq 0. \]  

(39)  

The nontrivial coordinate dependence in the metric and the curvature involves the single combination \( r \Omega \), which may be expressed as

\[ \left( \frac{\ell}{r \Omega} \right)^8 = \frac{\sinh(2\omega R |y|)}{2\omega R} = \frac{1}{\omega R} \left| \left( \frac{r_s}{r} \right)^8 - \left( \frac{r}{r_s} \right)^8 \right|^{-1}, \]  

(40)  

Thus, in the Einstein frame, the curvature is everywhere nonnegative, and the metric is asymptotically flat as \( r \to \infty \) \((y \to 0+)\). On the other hand, the curvature diverges as \( r \to r_s \) \((y \to +\infty)\). The result in eq. (40) has been derived only for \( y > 0 \) \((r > r_s)\) but will be extended shortly to \( y < 0 \) \((r < r_s)\).

The generic form of the solutions for the other fields is given in eq. (27). To adapt them to the present situation, the subsequent discussion is somewhat simplified if we take \( \omega > 0 \), but it can be easily translated for the case \( \omega < 0 \). To simplify writing, we shall assume \( \omega > 0 \) \((i.e., q_J > 0)\) throughout the remainder of this paper. Then the general solution is

\[ b = C \coth(\omega(y + y_+)), \quad e^\phi = C^{-1} \sinh(\omega(y + y_+)), \quad y \geq 0, \quad \text{with} \quad e^{\phi^+} \equiv C^{-1} \sinh(\omega y_+) > 0. \]  

(41)  

So \( \exp(\phi) \) is finite for \( y \geq 0 \), but diverges as \( y \to +\infty \) \((r \to r_s+)\), with \( \phi \to \omega y \). Note that the position \( r_s \) of the singularity is not a free parameter but is determined by the values of \( q_J \) and \( C \) according to

\[ \frac{r_s^8}{q_J} = \frac{C}{24 \Omega_9}. \]  

(42)  

Inasmuch as the SUGRA action is the leading term in a derivative expansion of the effective action, this divergence of the curvature suggests that these EOM break down as \( r \to r_s \). This is a signal that higher order corrections in \( \alpha' \) must be taken into account. Moreover, the divergence in the dilaton field signifies a region where the local string coupling becomes strong, indicating a breakdown in string perturbation theory. This suggests that the theory would find an expression in a S-dual form which, using the presumed SL(2,Z) symmetry of the Type IIB string, should be a theory of the same form. However, since the metric in the Einstein frame is SL(2,R) invariant, this would not cure the singularity in the curvature. On the other hand, because this is not a purely metric theory of gravity, it sometimes
happens in such cases that the metric is well-behaved in another “frame” associated with a conformal rescaling of the metric by the dilation, i.e.,

\[
\begin{align*}
\tilde{g}_{\mu\nu} &\equiv e^{2\phi} g_{\mu\nu} = e^{2\phi} \Omega^2 \delta_{\mu\nu}, \\
\tilde{ds}^2 &= e^{2\phi} ds^2.
\end{align*}
\]

Since the dilaton approaches a constant as \( r \to 0 \) or \( r \to \infty \), the spacetime remains asymptotically flat in this frame. However, as the singularity is approached, \( \Omega \to \exp(\omega y/12) \) by eq. (34), so that

\[
e^{2\phi} \Omega^2 \to e^{(p-\frac{1}{2})\omega y} \quad \text{as} \quad y \to +\infty.
\]

Therefore, we anticipate that, for \( p = 1/6 \), the background curvature in this frame will be finite as \( r \to r_s \). In this frame, which will be referred to as the instanton frame, the scalar curvature turns out to be

\[
\tilde{R} = \frac{1}{32C^2} e^{-4\phi} R,
\]

from which one finds that \( \tilde{R} \sim |r - r_s| \) as \( r \to r_s^+ \). Because the metric is conformally flat, all information about the curvature is contained in the Ricci tensor. In fact, the Ricci tensor also vanishes at \( r_s \), for example, we find

\[
\tilde{R}_{yy} = R_{yy} \left[ 1 - \coth(2\omega y) \cot\omega (y + y_+) + \frac{3}{2 \sinh^2 \omega (y + y_+)} \right] \sim e^{-4\omega y} \quad \text{as} \quad y \to +\infty,
\]

where \( R_{yy} = \omega^2/2 \), by eqs. (35) and (37). Of course, since in the coordinate \( y \), the singularity is at \( y = +\infty \), this does not necessarily imply that \( \tilde{R}_{rr} \) is finite as \( r \to r_s^+ \). Nevertheless, we find that it vanishes just like the scalar curvature \( \tilde{R} \),

\[
\tilde{R}_{rr} \sim |r - r_s|.
\]

The behavior of the curvature in the instanton frame suggests that it should be possible to connect the exterior solution for \( r > r_s \) to an interior solution for \( r < r_s \). However, because the local string coupling \( \exp(\phi) \) diverges as \( r \to r_s \), the semiclassical approximation breaks down, regardless of frame, so one cannot be sure. Nevertheless, away from the singularity, we can develop an interior solution that is essentially the mirror image of the exterior solution. Returning to the Einstein frame, the analog of eq. (34) is

\[
\Omega^8 = \omega^2_R \left( \frac{\ell}{r} \right)^{10} - 1, \quad r < r_s.
\]

To have the singularity occur for the same value of \( r_s \), we must take \( \omega_R^2 \) to have the same value as in the exterior region. (Other, better reasons for this choice, such as current conservation, will be seen below.) Solving eq. (24) once again for the relation between the coordinates \( r \) and \( y \), we find

\[
\left( \frac{r}{\ell} \right)^8 = -\omega_R \tanh(\omega_R y), \quad y \leq 0.
\]

At the risk of some confusion, we have chosen the origin \( r = 0 \) to correspond to \( y \to 0^- \), although we could have chosen any other convenient value as well.\[40\] Thus, \( y \to -\infty \) corresponds to the approach \( r \to r_s^- \) to the singularity from the interior. From eq. (47), the conformal factor turns out to be

\[
\Omega^{-4} = -\sinh(\omega_R y), \quad y \leq 0.
\]

As before, the radial part of Einstein’s equations will then be satisfied provided \( \omega_R \) and \( \omega \) are related by eq. (48). The scalar curvature in the Einstein frame is again given by eqs. (39) and (40). Since the metric in terms of the coordinate \( y \), given in eq. (48), depends only on the combination in eq. (40), the space is asymptotically flat as \( r \to 0 \). Formally, the metrics in the interior and exterior regions are related by the replacement \( y \to -y \) \((r \to r^2/r)\). The metric in the instanton frame also manifests this inversion symmetry (provided also \( y_+ \to y_- \), see immediately below) and is nonsingular everywhere. So it is natural to conjecture that this isometry persists despite the singularity.

Analogous to eq. (41), the corresponding interior solutions for \( b \) and \( \phi \) are

\[
b = -C \coth(\omega (y_- - y)), \quad e^{\phi} = C^{-1} \sinh(\omega (y_- - y)), \quad y \leq 0, \quad \text{with} \quad e^{\phi_-} = C^{-1} \sinh(\omega y_-) > 0.
\]
where, again, $|\omega| = 3\omega_R$. Since the divergences of $\exp(\phi)$ and $\Omega$ at the singularity have the same behavior as in the exterior region, the same transformation eq. (43) from the Einstein frame to the instanton frame removes the singularity. Even though $b$ undergoes a jump from $-C$ to $+C$ in crossing the singularity at $r = r_s$, the value of the action is still given by eq. (19), with

$$\Delta b = C (\coth(\omega y_+) + \coth(\omega y_-)) > 2C. \quad (51)$$

Note that $\nabla_y J^\mu = 0$ at the singularity (most easily seen by noting that $J_y = -q_J$ is constant everywhere), so that current conservation holds across the singularity so long as $q_J$ and $\omega$ are the same in the interior and exterior regions. Therefore, unlike the sources at the origin and at infinity, we do NOT expect this singularity to be resolved through the presence of RR charges; we shall discuss how it might be determined in the next section. The asymptotic values of the dilaton field are related to the string couplings in the initial and final states $g_\mp \equiv \exp(\phi_\mp)$. The value of $C$ is related to the value of $\Delta b$ by eq. (26).

$$C^2 = b_+^2 - e^{-2\phi_+} = b_-^2 - e^{-2\phi_-}. \quad (52)$$

Therefore, we may express $C$ in terms of the string couplings and $\Delta b$ as

$$C = \frac{1}{2|\Delta b|} \Lambda \left( \Delta b^2, e^{-2\phi_+}, e^{-2\phi_-} \right) \frac{1}{\Lambda} = \frac{|\Delta b|}{2} \Lambda \left( \frac{e^{\Delta \phi}}{\Delta b^2}, e^{-\Delta \phi}, 1 \right) \frac{1}{\Delta b^2}, \quad (53)$$

where $\Lambda$ is the “triangle function” defined as $\Lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$, and, in the second expression, we abbreviated $\phi_+ - \phi_- \equiv \Delta \phi$ and $\Delta b^2 \exp(\phi_+ + \phi_-) \equiv \Delta b^2$. We have argued in the preceding that we must have $C^2 \geq 0$, but the function $\Lambda(x, y, z)$ is not positive for all values of its arguments. If we regard the asymptotic values of the string couplings as fixed, this provides a constraint on the allowed range of $\Delta b$. This can be made more transparent by rewriting eq. (53) as

$$4C^2 e^{\phi_+ + \phi_-} \equiv 4C^2 = \frac{1}{\Delta b^2} \left[ \left( \Delta b^2 - 2 \cosh \Delta \phi \right)^2 - 4 \right] = \frac{1}{\Delta b^2} \left[ \left( \Delta b^2 - 4 \cosh^2 \frac{\Delta \phi}{2} \right) \left( \Delta b^2 - 4 \sinh^2 \frac{\Delta \phi}{2} \right) \right]. \quad (54)$$

The requirement that $C^2 > 0$ excludes the range

$$2 \sinh \frac{|\Delta \phi|}{2} < |\Delta b| < 2 \cosh \frac{\Delta \phi}{2}. \quad (55)$$

The relation eq. (54) suggests that there are two possible values of $\Delta b$ corresponding to a given $C$, a small root and a large root. However, in general, the smaller root is spurious and does not obey the constraint eq. (51). Therefore, the correct branch has

$$|\Delta b| \geq 2 \cosh \frac{\Delta \phi}{2}, \quad \text{or equivalently,} \quad |\Delta b| \geq \left( \frac{1}{g_+} + \frac{1}{g_-} \right). \quad (56)$$

Another way to view our construction is in terms of a conformal coordinate $\theta$ defined by

$$\tan(\theta/2) \equiv \left( \frac{r_s}{r} \right)^8. \quad (57)$$

The angle $\theta$ is analogous to the polar angle in the usual projective representation of the plane onto a sphere. The exterior region $r > r_s$ may be viewed as the “upper hemisphere” $0 < \theta < \pi/2$; the interior region $r < r_s$, as the “lower hemisphere” $\pi/2 < \theta < \pi$. The singularity is at $\theta = \pi/2$, and we are patching together the two surfaces across the “equator”. Despite the singularities in the curvature and dilaton fields, the SL(2,R) currents are conserved across the boundary, even though the $b$ field undergoes a jump. We shall address the possible origin of this discontinuity in the next section, but we assume that the behavior across the singularity is such that the value of the action is still given by eq. (19).

The interpretation of the instantons can be facilitated by introducing yet other coordinates, e.g.,

$$ds^2 = a(\eta)^2 \left[ d\eta^2 + d\omega_3^2 \right], \quad \eta \equiv \ln(r/r_s), \quad a(\eta) \equiv r\Omega = r_s \left( 2 \sinh 8|\eta| \right)^{1/4}. \quad (58)$$

$\eta$ is like a conformal Euclidean time coordinate with $\eta \in (-\infty, +\infty)$. Each instanton solution is associated with an asymptotic value of $\phi$ and of the auxiliary field $b$ at $\eta = \pm \infty$. The scale parameter $a(\eta)$ has a singularity at $\eta = 0$. Near the singularity, the curvature becomes large, and the SUGRA approximation breaks down. One must appeal to
the underlying string theory to cut off this divergence, presumably when \( a(\eta) \sim O(\sqrt{\alpha'}) \). One might think that the change \( \Delta b \) should be associated with a property of the asymptotic states, but in fact it is completely independent of the asymptotic configuration \( C_S \) associated with the charge \( q_J \). Indeed, it may be better to simply label the classically degenerate vacua by saying that they correspond to distinct values of the string coupling and a charge \( q_J \).

In sum, this construction describes a wormhole to be interpreted as a tunneling amplitude between asymptotically flat spacetimes having definite values for the string coupling. There is a conserved flux of \( F_2 \) between the two spacetimes, but since the construction has Euclidean signature, this is not really a “flow of current.” The “current flux” associated with \( J_\mu \) is analogous to the electric flux emanating from a point charge. At a fixed “time” \( \eta \), the wormhole looks like a sphere \( S^9 \) with radius \( a(\eta) \). Although the radius vanishes \( \eta = 0 \), the semiclassical description breaks down in this region both because the curvature becomes large and because the local string coupling becomes large. As we shall describe further below, there is reason to believe that the throat does not pinch off and that such a nonperturbative amplitude survives in string theory.

For a given charge \( q_J \), the solution has been expressed in terms of three parameters, \( \Delta b \) and \( \phi \pm \) or, alternatively, \( \Delta b \), \( \Delta \phi \), and \( C_8 \). Were it not for the singularity at \( r_s \), the solution should involve only two additional parameters since, after all, the two equations eq. (22) are first-order. Therefore, there should be one additional constraint on these three parameters that depends on connecting the solutions for \( \phi \) and \( b \) across the singularity. There is a natural conjecture for this constraint based on the behavior of the metric in the instanton frame. As the singularity is approached from the interior \((y < 0)\), the behavior of the Ricci tensor is similar to eq. (59)

\[
\tilde{R}_{yy} = R_{yy} \left[ 1 - \coth(2\omega y) \coth \omega (y_- - y) + \frac{3}{2 \sinh^2 \omega (y_- - y)} \right] \sim e^{4\omega y} \text{ as } y \to -\infty.
\]

Although the leading behaviors agree as \( r \to r_s, \pm \), if we want the matching to be smooth, we must take \( y_+ = y_- \), which is to say that we must take the asymptotic values of the string couplings to be identical \( g_+ = g_- = g_S \). This is appealing for two reasons: First, there is no resultant tunneling in the dilaton, so one may assume as usual that the asymptotic states have a fixed value of the string coupling constant. Were \( g_+ \neq g_- \), the correct ground states would necessarily involve a superposition of string coupling constants. Secondly, the isometry of the metric under inversion \((r \to r_s^2/r) \) (or \( y \to -y \)) then extends to the dilaton background, so that this isometry is respected also in the Einstein frame and, in fact, in all frames related by a conformal rescaling of the metric by the dilaton field. Thus, while different p-branes probe local geometries that differ by such conformal transformations, instanton effects may be expected to reflect this symmetry in all cases. The field \( b(r) \) is not invariant under the isometry transformation, but rather is odd \( b(r) \to -b(r_s^2/r) \) (for \( g_K = 0 \)), so that \( \Delta b = 2b_+ \).

Of course, the region near the singularity remains a region where the dilaton diverges, so this symmetry remains only a conjecture. Nevertheless, we will henceforth assume that \( \Delta \phi = 0 \). Then the net effect of an instanton is to produce a change in the asymptotic values of \( b \). For \( \Delta \phi = 0 \), the preceding equations relating \( \Delta b \) to \( C \) simplify considerably,

\[
C^2 = \frac{\Delta b^2}{4} - \frac{1}{g_S}.
\]

The requirement that \( C^2 \geq 0 \) therefore corresponds simply to \( |\Delta b| \geq 2/g_S \), so that the instanton action is bounded below \( S_V \geq 2q_J/g_S \). The lower limit would be expected to correspond to a BPS state or the \( D=–1 \) brane in the string theory. The limit \( C \to 0 \) corresponds to \( \omega \to 0 \). Nearly all of the preceding equations have been written in a form for which they remain finite in this limit, so one may simply carry over the previous formulas to that case. Of course, the physics looks rather different, in that the metric in the Einstein frame becomes perfectly flat (except at the origin where it remains singular). Nevertheless, there remain nontrivial solutions for the dilaton and \( b \) for which \( b = \pm \exp(-\phi) \), in agreement with GGP.

### IV. FURTHER CONSIDERATIONS ABOUT INSTANTON PROPERTIES

#### A. Significance of \( \Delta b \)

Given the asymptotic values of the string couplings, we have found a one-parameter family of instanton solutions, each with action proportional to \( \Delta b \). It is clear that the path integral will be dominated by the smallest allowed value of \( \Delta b \), corresponding formally to \( C = 0 \). Since the position of the singularity is related to \( C \) by eq. (42), this suggests that the singularity collapses to the origin and only the exterior region survives. However, given the inversion symmetry, it is equally correct to say that only the interior survives in this limit. It appears as if the throat of the wormhole is pinched off and two regions become disconnected in this limit. To understand better what is going on,
one must go beyond the lowest order approximation in the effective field theory or appeal to the underlying string theory.

How might this discontinuity be smoothed out in higher order? Further light on the position and nature of the singularity may be shed by reflecting on the symmetry structure. We know that the $SL(2, R)$ symmetry of the solution is explicitly broken in string theory in order $\alpha'^4$, so that the previously conserved currents will develop new source terms unrelated to those RR charges at $\eta = -\infty$ and $\eta = +\infty$. If we consider the EOM for $b$, eq. (25), the derivative of $b$ tends to zero as the singularity is approached, so that the correction terms of order $\alpha'$ will eventually predominate. Since this must be responsible for the jump $2C$ as $b$ crosses the singular region, we anticipate that $2C \sim O(\alpha'/r_s^2)$, where we inserted the appropriate factor of $r_s$ by dimensional analysis, since that is the only scale on which the lowest order solutions depend. By eq. (22), we can estimate

$$r_s \sim (\alpha' q_J)^{1/3}. \quad (61)$$

For $q_J \gg \alpha'^4$, $r_s \gg \sqrt{\alpha'}$. This suggests that the critical radius, while small, can be parametrically large compared to the string scale. Therefore, it may be possible to determine the characteristic features of the instantons from the next term $L_g$ in the effective Lagrangian eq. (1) without having to resort to the full string theory. This possibility would be interesting to explore.

### B. Comparison with the GGP Solution

The instanton of least action has $C = 0$, so that the critical radius $r_s$ collapses to zero in that limit, and the solutions for the metric, dilaton, and $b$ in the exterior region become equivalent to those found in [3]. Nevertheless, a number of paradoxes are resolved by our derivation. In ref. [3], the metric in the Einstein frame was flat, while the metric in the string frame possessed an inversion symmetry similar to the isometry discussed here (but about the radius $(q_J/\alpha_s)^{1/8}$, much different from ours). This suggested to those authors a picture not so different from the one we have found, but, since their dilaton solution not only did not obey that symmetry, but diverged at the origin, it required a flight of fancy to suggest that there was an interior region similar to the exterior region which, in the Einstein frame, was mapped to a point. One of the pretty consequences of our construction is that this intuitively appealing picture can be justified and, since our isometry is exact for both the dilaton and metric, it is a property of every frame. Moreover, our inversion radius does collapse in the limit that they considered, viz., $C \rightarrow 0$. However, the action for the interior region is as large as that in the exterior region, which is why our lower limit, $S_V = 2q_J/\alpha_s$ is twice the action in [3]. The description in the instanton frame strongly suggests that the apparent singularity is an artifact of the approximation and not an insuperable obstacle.

In the limit, $C \rightarrow 0$, our exterior solution becomes the GGP instanton, while the interior solution is the anti-instanton. The persistence of this isometry means that there is not really any difference between the single instanton and anti-instanton. Of course, as the locus of RR-charge is a place where open strings can end, there will be a big difference in whether a string is attached to the source or the sink at infinity. These remarks apply to the interpretation of the single instanton solutions only. Assuming there are multi-instanton solutions, there would of course be a difference between having a identical or opposite charges at two distinct points, but there would be a corresponding modification in the inversion symmetry or isometry of the background.

Assuming our conjecture that $\Delta \phi = 0$, the transition amplitude in the $C_8$ formulation does not correspond to a tunneling between different perturbative vacuum states. In perturbation theory, the vacuum-to-vacuum amplitude $+\langle 0|0^- \rangle$ receives corrections from vacuum bubbles. In the presence of instantons, it also receive nonperturbative contributions. Although the instantons do not cause a qualitative change in the ground state, instanton effects on Green’s functions may modify the effective Lagrangian or scattering amplitudes. Many of the applications cited do not depend on the detailed character of the supergravity solution, only on the dimensionality of the $D=1$ brane, its characteristic dependence of the action on the charge $q_J$ and string coupling $\alpha_s$, and on the $SL(2, Z)$ symmetry of the Type IIB superstring. Thus, at least at first sight, these results would not be changed by the alternate picture of the $D=1$ brane developed here.

However, there does seem to be some confusion in the description of the instanton found in GGP, to which has been attributed some of the properties of the electric picture and some of the properties of the magnetic picture. It has been variously described as if the axion $a$ becomes imaginary under a Wick rotation to Euclidean space. This is suggestive, because the equations of motion for the auxiliary field $b$ in the magnetic formalism are the same as if you replaced $a$ by $ib$ in the electric formalism. As a consequence, that instanton is sometimes described as a “saddle point” of the action, whereas there is every reason to believe that it is a bona fide local minimum in the magnetic picture. Moreover, the Euclidean path integral over the axion field would diverge under such a replacement. There are other ways to see that this replacement is simply not correct. Starting from the action in terms of $C_8$, this...
replacement is valid only for the first variation and not for the value of the action nor for its second variation in the magnetic picture. The idea that a pseudoscalar becomes imaginary upon a Wick rotation is no more true for the axion than it is for the pion.

It has even been suggested that the axion field $a$ is imaginary everywhere except on the boundary where it remains real, a kind of schizophrenic axion.\cite{6, 22} The discussion of the instanton’s supersymmetric properties also manifests a kind of split personality when it comes to dealing with the supersymmetry algebra for Euclidean signature.\cite{5, 14} For a chiral theory, at best, the definition is arbitrary and, at worst, it is inconsistent.\cite{8} Regarding the supersymmetric properties of the instanton, there can be no doubt that the associated transition amplitude will remain supersymmetric, regardless of whether the instanton is BPS or not. The reason is that supersymmetry is gauged and, therefore, must not be anomalous.

The GGP instanton has been described as a flow of RR-charge through a wormhole, which is to be “interpreted as a violation of the conservation of global charge in physical processes.”\cite{3} On the contrary, there is no violation of global charge associated with our instanton, although, assuming $\Delta \phi = 0$, there remains spontaneous symmetry breaking of SL(2,R) charge associated with the dilaton having a nonzero vacuum expectation value. The interpretation of charge conservation is rather different for Euclidean and Lorentz signatures. The charge conservation associated with our instanton, although, assuming $\Delta \phi = 0$, there remains spontaneous symmetry breaking of SL(2,R) charge associated with the dilaton having a nonzero vacuum expectation value. The interpretation of charge conservation is rather different for Euclidean and Lorentz signatures. The charge conservation associated with the current $J_\mu$ is like the conservation of electric flux of the Coulomb field of a static point charge; electric flux is conserved but there is no “flow of charge.” It is true that it appears as if there is a RR-charge $q_f$ located at the origin and charge $-q_f$ located at infinity, but, paradoxically, the associated asymptotic states in the Hilbert space involve the same charge. The point is that, upon Wick rotation, the former is to be identified with an in-state, and the latter, with an out-state. The inversion symmetry is like time-reversal, with the direction of the outward-directed normal flipping. In other words, the instanton appears to be the same in coordinates $r' = r^2/r$ as it was in the coordinate system $r$.

More can undoubtedly be learned about the stringy aspects of the D=−1 brane through the use of various dualities, such as T-duality in 9-dimensions. However, we are skeptical about conclusions\cite{4} based upon compactification of the time coordinate ($r$ or $\eta$). This is transparently inconsistent for toroidal compactification, because periodicity in Euclidean time corresponds to a thermal state with temperature $T_H = 1/L$, ($L$ is the compactification radius) rather than a zero temperature ground state. We do not understand how a compactified time coordinate is to be identified with a matrix element in the Hilbert space rather than with a density matrix.

We hope the preceding treatment indicates what is correct for the magnetic picture and refer to \cite{5} for a corresponding electric description.

\section{SUMMARY AND CONCLUSIONS}

To summarize the picture that has emerged, the coordinate $r$ (or $\eta$) is like a Euclidean time in which $r \rightarrow 0 (\eta \rightarrow -\infty)$ corresponds to the distant past and $r \rightarrow \infty (\eta \rightarrow +\infty)$, to the distant future. The semiclassical approximation for the instanton solutions breaks down at some intermediate “time” $r_s$ ($\eta = 0$.) Nevertheless, arguments from string theory suggest the existence of such a D = −1 brane, so there is reason to hope that this naked singularity will be cured by stringy corrections. The value of the instanton action eq. (\ref{eq1}) is independent of the metric singularity, but complicated by the discontinuity in $b$, since $b \rightarrow \pm C$ as $r \rightarrow r_s \pm$. Even though $b$ undergoes a step at $r_s$, it is natural to expect that the effect of higher order corrections will be to smooth out the jump so that $b$ will vary smoothly from $-C$ to $+C$ in passing through the region near $r_s$. Since $\Delta b = b_+ - b_- > 2C$, the naive value for the action in eq. (\ref{eq11}) makes sense despite this discontinuity in $b$. In fact, $\partial b/\partial r \sim (r - r_s)^2$ near the singularity, so there is no problem formally integrating across the singularity to obtain the value of the action given by eq. (\ref{eq13})\cite{10}.

Near the singularity, the curvature is large in the Einstein frame but vanishes in the instanton frame. This latter property is only suggestive, since exp($\phi$) is large near the singularity so, from the point of view of string theory, this remains a strong coupling regime regardless of frame. However, we know of no argument suggesting that the apparent naked singularity in the Einstein frame metric is anything more than a breakdown in the perturbative, weak-coupling approximation to the underlying string theory. It might be amusing to determine the $O(a')$ corrections to the SUGRA Lagrangian, regardless of the magnitude of exp($\phi$), to see what effect they have on the singularity. The effects could be quite dramatic, since such corrections are expected to explicitly break the SL(2,R) symmetry. This breaking of current conservation may account for the change in the scalar field $b$ near the singularity.

We have not attempted the calculation of the full transition amplitude by carrying out the appropriate path integration. We expect that it would be futile to try because of the breakdown in the semiclassical approximation near $r_s$.

Generically, our instanton family would cause tunneling between asymptotically flat spacetimes with distinct values of the dilaton field $\phi_\pm$. If so, this would require the ground state to be a superposition of such states corresponding to another SL(2,R) charge. Since there is no evidence for this in superstring theory, we have assumed that, when the
apparent singularity in the dilaton at \( r_s \) is resolved, the only consistent solution in fact has \( \Delta \phi = 0 \). Alternatively, this could be seen as a consequence of an exact inversion symmetry, \( r \to r^2/r \), of the dilaton-metric background, an isometry that seems quite natural in the instanton frame. In that case, the effect of the instantons on the ground state is rather benign in the magnetic picture; however, in the electric picture, they do lead to tunneling between states associated with the axion \( a \).

We have noted earlier that the instanton frame is one in which the kinetic energy of the dilaton vanishes in the SUGRA action. Since that is a property of the NS-sector only, this is common to all SUGRA theories. The dilation EOM becomes a constraint equation in this frame, from which the dilaton field may be expressed in terms of the other fields of the theory. However, second time derivatives of the dilaton field enter Einstein’s equation in frames other than the Einstein frame, so it is not so clear what is accomplished by going to this frame. More work on sorting out the dynamics of this frame may be revealing, quite independent of the issues under discussion here.

Finally, even though the presentation here is given in 10-dimensions, the construction presented in this paper will work in dimensions other than 10 in which the background field content is similar to an axion-dilaton-metric theory, whether in compactified supergravity models[28] or in other interesting field theories.[6]

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VI. APPENDIX – THE CASE \( \omega^2 < 0 \).

When solving Einstein’s equations, we chose the asymptotically-flat solution with a naked singularity eq. (31) rather than the apparently more natural solution

\[
\Omega^8 = 1 + \omega_R^2 \left( \frac{\ell}{r} \right)^{16}.
\]

To treat this case, many of the formulas in the body of the text can be carried over simply by replacing \( \omega_R \to i\omega_R \) (and \( \omega \to i\omega \)). However, the associated replacement of hyperbolic with circular trigonometric functions leads to some dramatic changes. For example, eq. (33) becomes

\[
\left( \frac{\ell}{r} \right)^8 = \frac{\tan(\omega_R y)}{\omega_R},
\]

so the range of \( y \) is over one period, e.g., \( 0 < \omega_R y < \pi/2 \), corresponding to \( \infty > r > 0 \). Because \( |\omega| = 3\omega_R \), this corresponds to \( 0 < |\omega|y < 3\pi/2 \). While the background geometry is perfectly regular, this leads to problems for the solutions for \( b \) and \( \phi \),

\[
\hat{b} = C \cot(\omega y + \theta), \quad C e^\phi = |\sin(\omega y + \theta)|
\]

where \( \theta \) is an integration constant (defined modulo \( \pi \)). Because of the range spanned by \( \omega y \), it is unavoidable that \( \hat{b} \) and \( \exp(-\phi) \) have a singularity at some radius (cf. eq. (27)). In contrast to the case treated in text, this singularity corresponds to a place where the curvature is finite and where the string coupling vanishes. As a result, there is no reason to expect the leading SUGRA Lagrangian or the semiclassical approximation to break down here. Unfortunately, this singularity is non-integrable and renders the action integral eq. (19) ill-defined. One may attempt to define this by analytic continuation in \( \theta \), but we have been unable to convince ourselves that this is a sensible procedure.

Another reason to be skeptical about the case \( \omega^2 < 0 \) is presented in Ref. [5], where it can be seen that the only nonzero component of the Ricci tensor \( R_{yy} \) in the Einstein frame is necessarily nonnegative. Thus, the dual description requires \( \omega^2 > 0 \) and does not permit \( \omega^2 < 0 \), as considered in this Appendix. Therefore, if there is any chance for a sensible field theoretic treatment of the D=1 brane of string theory, it must be in the case treated in text, despite its apparent singularity and breakdown in the semiclassical approximation.

[1] J. Polchinski, “String Theory. Vol. 2: Superstring Theory and Beyond,” Cambridge, UK: Univ. Pr. (1998).
Note, for later reference, that this result holds even if this assumes that the naked singularity discussed in the next subsection is somehow smoothed over. This result is therefore identical to that obtained in [3] even though our solution to the EOM is different. That the value of this observation is in accord with Dirac's original discussion of duality in electrodynamics in four-dimensions. Reference [3] will be referred to as GGP.

Unlike monopoles in broken GUTs, in this case, there is no corresponding homotopic argument requiring b parameter does not enter the classical EOM.

More generally, expressed in terms of parameter does not enter the classical EOM. For spinor fields and supersymmetry in Euclidean space, "A continuous Wick rotation for spinor fields and supersymmetry in Euclidean space," ibid. "Continuous Wick rotation for spinor fields... Four-Dimensions." 2. Regularization Of The Model," 3. On The Normalization Of Schwinger Functions," "On A New Characterization Of Scalar Supersymmetric Theories," Phys. Lett. B 89, 341 (1980). B. Zumino, "Euclidean Supersymmetry And The Many-Instanton Problem," Phys. Lett. B 69, 369 (1977).

M. A. Shifman and A. I. Vainshtein, “Instantons versus supersymmetry: Fifteen years later,” in ITEP Lectures in Particle Physics and Field Theory, vol. 2, M. Shifman (ed.). Singapore: World Scientific, 1999 arXiv:hep-th/9902018.

A. V. Belitsky, S. Vandoren and P. van Nieuwenhuizen, “Instantons, Euclidean supersymmetry and Wick rotations,” Phys. Lett. B 477, 345 (2000) arXiv:hep-th/0001010.

S. Weinberg, “The Quantum Theory Of Fields. Vol. 1: Foundations,” Cambridge, UK: Univ. Pr. (1995).

M. B. Einhorn, “Instantons and the Ground State of Type IIB Supergravity,” in preparation.

M. B. Green, Phys. Lett. B 354, 271 (1995) arXiv:hep-th/9504108.

M. Gutperle, “Aspects of D-instantons,” arXiv:hep-th/9712150.

M. B. Green and M. Gutperle, “Configurations of two D-instantons,” Phys. Lett. B 398, 69 (1997) arXiv:hep-th/9612127.

M. B. Green and P. Vanhove, Phys. Lett. B 408, 122 (1997) arXiv:hep-th/9704143.

M. Gutperle, “Contact terms, symmetries and D-instantons,” Nucl. Phys. B 508, 133 (1997) arXiv:hep-th/9703023.

M. B. Green and M. Gutperle, “D-particle bound states and the D-instanton measure,” JHEP 9801, 005 (1998) arXiv:hep-th/9711077.

M. B. Green and M. Gutperle, “D-instanton partition functions,” Phys. Rev. D 58, 046007 (1998) arXiv:hep-th/9804123.

M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, JHEP 9808, 013 (1998) arXiv:hep-th/9807031.

M. B. Green and M. Gutperle, “D-instanton induced interactions on a D3-brane,” JHEP 0002, 014 (2000) arXiv:hep-th/0002011.

Martin B. Einhorn, unpublished.

Reference [5] will be referred to as GGP. This observation is in accord with Dirac’s original discussion of duality in electrodynamics in four-dimensions. It has been conjectured that there remains an exact SL(2,Z) discrete gauge symmetry. Unlike monopoles in broken GUTs, in this case, there is no corresponding homotopic argument requiring qµ to be quantized. Of course, the quantum of action in ten-dimensions, κ2 = entries, the calculation of the transition amplitude, but this parameter does not enter the classical EOM. This assumes that the naked singularity discussed in the next subsection is somehow smoothed over. This result is therefore identical to that obtained in arXiv:hep-th/9412184 even though our solution to the EOM is different. That the value of the action depends only on the boundary conditions is a necessary but not sufficient condition for a BPS-like solution. Note, for later reference, that this result holds even if b is only piecewise continuous. Since b(r) is arbitrary up to a constant, the definition of Kµ may correspondingly be shifted by a constant times Jµ. More generally, expressed in terms of Cµ, one can show that Kµ is not invariant under gauge transformations of Cµ, and so is not a physically observable current density.

Note that the form of the solution cannot be guessed by naively extrapolating from higher D-branes.Independent of the form of the classical solution, the Lagrangian in the “instanton frame” has the property that the kinetic energy for the dilaton vanishes, so its EOM becomes an equation of constraint, an interesting frame in its own right.
Since \( y \to 0^+ \) corresponds to \( r \to \infty \), these two limits \( y \to 0^\pm \) correspond to opposite ends of the space.

We shall shortly argue that \( g_+ = g_- \), but since this is not rigorous, we postpone that argument temporarily. The reader willing to assume this may wish to skip to eq. (57) below.

Its explicit relation to \( y \) is \( \eta = (\epsilon(y)/8) \ln \coth(\omega Ry) \).

Recall that we chose to set the fourth parameter \( q_k = 0 \).

As we shall discuss in the next section, the change \( \Delta b \) is not really an observable property of the asymptotic states.

This lower limit is, for obvious reasons, twice the value obtained by GGP.

In the dual picture in terms of the axion, however, the instantons have a much more dramatic effect on the ground state.

See [5] for further discussion.

We do not understand the “hyperbolic complex number formalism” employed in [3].

This is in contrast to the case discussed in the Appendix in which, although the spacetime is nonsingular, the action integrand is non-integrable.