Revisiting the radiative decays $J/\psi \to \gamma \eta^{(l)}$ in perturbative QCD

Jun-Kang He* and Ya-Dong Yang†

Institute of Particle Physics and Key Laboratory of Quark and Lepton Physics (MOE), Central China Normal University, Wuhan, Hubei 430079, China

Abstract

In the framework of perturbative QCD, the radiative decays $J/\psi \to \gamma \eta^{(l)}$ are revisited in detail, where the involved one-loop integrals are evaluated analytically with the light quark masses kept. We have found that the sum of loop integrals is insensitive to the light quark masses and the branching ratios $B(J/\psi \to \gamma \eta^{(l)})$ barely depend on the shapes of $\eta^{(l)}$ distribution amplitudes. With the parameters of $\eta - \eta'$ mixing extracted from low energy processes and $J/\psi \to \gamma \eta^{(l)}$ by means of nonperturbative matrix elements $\left< 0 | G_{\mu\nu}^{a} \tilde{G}_{a,\mu\nu}^{a} | \eta^{(l)} \right>$ based on $U_{A}(1)$ anomaly dominance argument, we could not give the ratio $R_{J/\psi}$ in agreement with experimental result. However, using the parameters, especially the mixing angle $\phi = 33.5^\circ \pm 0.9^\circ$, extracted from $\gamma^{*}\gamma - \eta'$ transition form factor measured at $q^{2} = 112$ GeV$^{2}$ by BaBar collaboration, we obtain $R_{J/\psi} = 4.70$ in good agreement with $R^{\text{exp}}_{J/\psi} = 4.65 \pm 0.21$. As a crossing check, with $\Gamma^{\text{exp}}(\eta^{(l)} \to \gamma\gamma)$ and our results for $J/\psi \to \gamma \eta^{(l)}$, we get $\phi = 33.9^\circ \pm 0.6^\circ$. The difference between the determinations of $\phi$ is briefly discussed.

*hejk@mails.ccnu.edu.cn
†yangyd@mail.ccsu.edu.cn
1 Introduction

Since its discovery\cite{1, 2}, the $J/\psi$ meson has always been an active topic in particle physics\cite{3–5}. The heavy quarkonium physics might be crucially important to improve our understanding of quantum chromodynamics (QCD), especially, the interplay of perturbative quantum chromodynamics (pQCD) with nonperturbative QCD. The Okubo-Zweig-Iizuka (OZI)-forbidden radiative decays of $J/\psi \rightarrow \gamma \eta^{(0)}$ are expected to proceed predominantly via two virtual gluons which subsequently convert to $\eta^{(0)}$, with the photon emitted from the initial charm quarks. Such decays have been the subject of many experimental and theoretical studies\cite{3–5}. Theoretically, these decays provide a clean environment to study the conversion of gluons into hadrons. In this respect, the radiative decays $J/\psi \rightarrow \gamma \eta^{(0)}$ are of particular interest, since they are also closely related to the issues of $\eta - \eta'$ mixing, which are important ingredients for understanding many interesting phenomena related to the $\eta$ and $\eta'$ mesons.

In the literature, the exclusive radiative decays $J/\psi \rightarrow \gamma \eta^{(0)}$ have been studied in different approaches. The decay widths $\Gamma(J/\psi \rightarrow \gamma \eta^{(0)})$ were calculated by Novikov et al.\cite{6}, with the assumption that these decays occur as a consequence of the $U_A(1)$ anomaly and are, therefore, controlled by the nonperturbative gluonic matrix elements $\langle 0 | G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} | \eta^{(0)} \rangle$, where $G^a_{\mu\nu}$ is the gluon field strength tensor and $\tilde{G}^{a,\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} G^a_{\alpha\beta}$ its dual tensor. Then the ratio of the decay widths takes the form\cite{6}

$$R_{J/\psi} = \frac{\Gamma(J/\psi \rightarrow \gamma \eta')}{\Gamma(J/\psi \rightarrow \gamma \eta)} = \left( \frac{M^2_{J/\psi} - m_{\eta'}^2}{M^2_{J/\psi} - m_{\eta}^2} \right)^3 \frac{\langle 0 | G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} | \eta' \rangle^2}{\langle 0 | G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} | \eta \rangle^2},$$

(1.1)

where the matrix elements $\langle 0 | G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} | \eta^{(0)} \rangle$ can be calculated with the QCD sum rules and other approaches. For example, Chao\cite{7, 8} and Kuang et al.\cite{9} derived the expressions of the matrix elements $\langle 0 | G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} | \eta^{(0)} \rangle$ in the large-$N_c$ approach and QCD multipole expansion, respectively.

In the Feldmann-Kroll-Stech (FKS) scheme for $\eta - \eta'$ mixing system\cite{10}, the nonperturbative matrix elements $\langle 0 | G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} | \eta^{(0)} \rangle$ can also be expressed as function of the phenomenological
parameters $f_q$, $f_s$, $\phi$, and the ratio $R_{J/\psi}$ reads

$$R_{J/\psi} = \frac{\tan^2 \phi}{\cos^2 \phi} \left( \frac{M_{J/\psi}^2 - m_{\eta'}^2}{M_{J/\psi}^2 - m_{\eta}^2} \right)^2,$$

(1.2)

where $\phi$ denotes the mixing angle of $\eta - \eta'$ system. Compared $R_{J/\psi}$ with its experimental value, the mixing angle is found to be $\phi = 39.3^\circ \pm 1.0^\circ$ [10]. However, as it had been pointed out that the above equation was calculated in the approximation with the assumption of ground state dominance and negligence of continuum contributions to the dispersion relations [6, 11]. It is also noticed that the matrix elements $\langle 0| G_{\mu\nu}^{a} \tilde{G}_{a,\mu\nu}^{a} | \eta(\prime) \rangle$ induced by the $U_A(1)$ anomaly are a higher twist effect (twist-4) [12]. The $\langle 0| G_{\mu\nu}^{a} \tilde{G}_{a,\mu\nu}^{a} | \eta(\prime) \rangle$ would give the main contributions to the radiative decays $J/\psi \rightarrow \gamma \eta(\prime)$, only in the case of the leading twist contributions were strongly suppressed. Although the leading twist contributions from the gluonic content of $\eta(\prime)$ are suppressed by a factor of $m_{\eta(\prime)}^2 / M_{J/\psi}^2$ [13], the assumption that the matrix elements $\langle 0| G_{\mu\nu}^{a} \tilde{G}_{a,\mu\nu}^{a} | \eta(\prime) \rangle$ dominate the radiative decays $J/\psi \rightarrow \gamma \eta(\prime)$ could be broken, because the leading twist contributions from the quark-antiquark content of $\eta(\prime)$ are not suppressed much at the energy scale of $M_{J/\psi}$.

The first pQCD investigation of these decays were carried out by Körner et al. [14] about thirty years ago. They took the annihilation of $c\bar{c}$ quarks to be a short-distance process described by pQCD, and nonperturbative dynamics of the bound states factorized to wave functions. It has been argued that pQCD asymptotic behaviors may be expected at the scale of $M_{J/\psi}$ [12, 15–17]. However, in this pioneer work [14], the nonrelativistic quark model with the weak-binding approximation, has been taken for both $J/\psi$ and $\eta(\prime)$. Whereafter, the nonrelativistic approximation for $\eta(\prime)$ wave functions in [14] was improved by Kühn [18] with light-cone expansion. However, in the calculation of the loop integrals, the approximation of $m_{\eta(\prime)}^2 / M_{J/\psi}^2 \approx 0$ was made [18]. In our calculation, we would keep $m_{\eta(\prime)}$ and $m_{u,d,s}$, which result in our final analytical expression of the loop function is much complicated than the ones in Refs. [14, 18]. We notice that our results can reproduce the one in Ref. [18] in the limit of $m_{\eta(\prime)}^2 / M_{J/\psi}^2 \rightarrow 0$, and detail comparison is presented in the Appendix. Recently, several groups have revisited $J/\psi \rightarrow \gamma \eta(\prime)$ in the framework of pQCD [19–23]. In works [20, 21], the light-cone distribution amplitudes (DAs) were adopted for $\eta(\prime)$. However, in Ref. [21], the decay widths
of $J/\psi \to \gamma \eta^{(l)}$ were found to be very sensitive to the light quark masses of $\eta^{(l)}$ involved in the loop integrals, which is needed to be made clear, since the sensitivities of the loop integrals to light quark masses usually point to the possible infra-red (IR) divergences.

Besides the aforementioned QCD approaches, the radiative decays $J/\psi \to \gamma \eta^{(l)}$ have also been studied with phenomenological models, such as the approach with an effective lagrangian [24] and the approach considering the $\eta_c - \eta^{(l)}$ mixings [7, 8, 25]. Generally, predictions compatible with the experimental measurements could be obtained.

In this paper, we present a detail calculation of these decays in the framework of pQCD. The bound-state property of $J/\psi$ is parameterized by its Bethe-Salpeter (B-S) wave function, while the $\eta$ and $\eta'$ are described by their light-cone DAs. Then the loop integrals are evaluated analytically with the light quark masses kept.

In our calculation, the loop function is found to be insensitive to the light quark masses, which differs from the results in Ref. [21]. Moreover, our results of the $B(J/\psi \to \gamma \eta^{(l)})$ are also insensitive to the shapes of the light meson DAs. The theoretical uncertainties due to choices of different $\eta^{(l)}$ DAs available in the literature are negligible in the prediction for the ratio $R_{J/\psi}$, so that, the mixing angle of $\eta - \eta'$ mixing could be reliably extracted. In addition, the corrections from QED processes $J/\psi \to \gamma^* \to \gamma \eta^{(l)}$ are also considered in our calculation.

The paper is organized as follows. In section 2, we present the formalism for calculating the decay amplitudes of $J/\psi \to \gamma \eta^{(l)}$. Numerical results are presented in section 3, and the final part is our summary. The analytical expressions for the dimensionless key function $H_0$ and discussions of its properties are presented in the Appendix.

2 The radiative decays $J/\psi \to \gamma \eta^{(l)}$ in pQCD

2.1 The contributions of the quark-antiquark content of $\eta^{(l)}$

For the quark-antiquark content of $\eta^{(l)}$, the leading order contributions to the radiative decays $J/\psi \to \gamma \eta^{(l)}$ arise from one-loop QCD processes. The corresponding Feynman diagram is illustrated in Fig. 1. There are other five Feynman diagrams from permutations of the photon and gluon legs. Usually, it is convenient to divide the invariant amplitude into two parts. One part describes the effective coupling between $J/\psi$, a real photon and two virtual gluons, the
Figure 1: One typical Feynman diagram for $J/\psi \rightarrow \gamma \eta^{(\prime)}$ with the quark-antiquark content of $\eta^{(\prime)}$. Kinematical variables are labelled.

other part describes the effective coupling between pseudoscalar meson $\eta^{(\prime)}$ and two virtual gluons. To evaluate these effective couplings, we need to deal with the nonperturbative effects of the mesons. Generally, factorization is employed. For the heavy $J/\psi$, we still use the weak-binding approximation, and factorize the nonperturbative bound-state effects into its B-S wave function. For the light mesons $\eta$ and $\eta'$, we use their light-cone DAs.

In the rest frame of $J/\psi$, the amplitude of $J/\psi \rightarrow \gamma g^* g^*$ can be decomposed into hard-scattering part and the B-S wave function of $J/\psi$ [26]

$$A = A^{\alpha\beta\mu\nu} \varepsilon_\alpha(K) \varepsilon^*_\beta(k) \varepsilon^*_\mu(k_1) \varepsilon^*_\nu(k_2) = -i\sqrt{3} \int \frac{d^4q_c}{(2\pi)^4} Tr \left[ \chi(q_c) \hat{O}(q_c) \right], \quad (2.1)$$

where $\chi(q_c)$ is the B-S wave function of $J/\psi$ and $\sqrt{3}$ is the color factor. $K$ and $\varepsilon(K)$ stand for the momentum and the polarization vector of the $J/\psi$, $k$, $k_1$, $k_2$ and $\epsilon(k)$, $\epsilon(k_1)$, $\epsilon(k_2)$ stand for momenta and polarization vectors of the photon and the gluons, respectively. The momenta of the $c$ and $\bar{c}$ quarks are parameterized as

$$f = \frac{K}{2} + q_c, \quad \bar{f} = \frac{K}{2} - q_c, \quad (2.2)$$

where $q_c$ is the relative momentum between the $c$ and $\bar{c}$ quarks. Since the B-S wave function $\chi(q_c)$ is sharply peaked when $q_c \sim 0$ for the heavy $J/\psi$ in the nonrelativistic limit, one may neglect the $q_c$-dependence in the hard-scattering amplitude $\hat{O}(q_c)$ in the leading order.
approximation, then the amplitude can be simplified to

\[
A = -i \sqrt{3} \int \frac{d^4 q_c}{(2\pi)^4} \text{Tr} \left[ \chi(q_c) \hat{\mathcal{O}}(0) \right]
\]

\[
= -\sqrt{3} \int \frac{d^3 q_c}{(2\pi)^3} \text{Tr} \left[ \psi(q_c) \hat{\mathcal{O}}(0) \right],
\]

(2.3)

where the Salpeter function is defined as

\[
\psi(q_c) = \frac{i}{2\pi} \int dq^0_c \chi(q_c).
\]

(2.4)

For the vector meson \(J/\psi\), the Salpeter function with leading order Dirac structures reads\([27, 28]\)

\[
\psi(q_c) = \phi(q^2_c)(M + \not{k})\not{\epsilon}(K),
\]

(2.5)

where \(\phi(q^2_c)\) is a scalar function of \(q^2_c\), and \(M\) represents the mass of \(J/\psi\). With the help of the definition of the \(S\)-wave wave function evaluated at the origin \([29]\)

\[
\int \frac{d^3 q_c}{(2\pi)^3} \phi(q^2_c) = \frac{1}{2} \sqrt{\frac{1}{4\pi M}} R_\psi(0),
\]

(2.6)

one can obtain \([14]\)

\[
A^{\alpha\beta\mu\nu} \varepsilon_\alpha(K) \epsilon^*_\beta(k_2) \epsilon^*_\mu(k_1) = -\frac{1}{2} \sqrt{\frac{3}{4\pi M}} R_\psi(0) \text{Tr} \left[ (M + \not{k})\not{\epsilon}(K) \hat{\mathcal{O}}(0) \right].
\]

(2.7)

And the hard-scattering amplitude \(\hat{\mathcal{O}}(0)\) can be written as

\[
\hat{\mathcal{O}}(0) = i Q_c \bar{c} q^2_c \frac{\delta_{ab}}{6} \not{k} \not{\epsilon}^*(k_2) \frac{-k - k_1 + M}{-2(k + k_1) \cdot k_2} \not{k} \not{\epsilon}^*(k) \frac{-2(k + k_2) \cdot k_1}{-2(k + k_2) \cdot k_1} \not{k} \not{\epsilon}^*(k_1)
\]

\[
+ (5 \text{ permutations of } k_1, k_2 \text{ and } k),
\]

(2.8)

where we have made the nonrelativistic approximation \(M \approx 2m_c\).

The light-cone expansion of the matrix elements of the meson \(\eta^{(\rho)}\) over quark and antiquark fields reads \([12]\)

\[
\langle \eta^{(\rho)}(p) | \bar{q}_\alpha(x) q_\beta(y) | 0 \rangle = \frac{i}{4} f^{\rho}_{\eta^{(\rho)}} (\not{p} \gamma_5)_{\beta \alpha} \int du e^{i (\bar{\Upsilon} p y + p x u)} \phi^\rho(u) + \cdots,
\]

(2.9)
where the high twist terms are omitted. With this definition, one can obtain the coupling of \( g^* g^* - \eta^{(t)} \) [30–32]:

\[
M_{\mu\nu} = -i (4\pi\alpha_s) \delta_{ab} \varepsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma \\
\times \sum_{q=u,d,s} \int_0^1 du \phi^q(u) \left( \frac{1}{\bar{u}k_1^2 + uk_2^2 - \bar{u}m^2 - m_q^2} + (u \leftrightarrow \bar{u}) \right),
\]

(2.10)

where \( u \) is the momentum fraction carried by the quark and \( \bar{u} = 1 - u \), \( m_q \) is the mass of the quark \( (q = u, d, s) \), \( m \) is the mass of \( \eta^{(t)} \). The light-cone DA is [33]

\[
\phi^q(u) = \phi_{AS}(u) \left( 1 + \sum_{n=2,4,\ldots} c_n^q(\mu) C_n^3 (2u - 1) \right)
\]

(2.11)

with the asymptotic form of DA \( \phi_{AS}(u) = 6u(1 - u) \) and \( c_n^q(\mu) \) the Gegenbauer moments. In Table 1, we list three models of the DAs discussed in Ref. [33]. Their shapes are shown in Fig. 2, where \( c_n^q \) are evaluated at \( \mu = m_c \).

Table 1: Gegenbauer coefficients of three sample models at the scale of \( \mu_0 = 1 \) GeV.

| Model | \( c_2^q(\mu_0) \) | \( c_4^q(\mu_0) \) | \( c_2^g(\mu_0) \) |
|-------|-------------------|-------------------|-------------------|
| I     | 0.10              | 0.10              | -0.26            |
| II    | 0.20              | 0.00              | -0.31            |
| III   | 0.25              | -0.10             | -0.25            |

Figure 2: The shapes of the corresponding DAs at the scale of \( \mu = m_c \)

The decay amplitude of \( J/\psi \to \gamma\eta^{(t)} \) can be obtained by multiplying the above two cou-
pling, inserting the gluon propagators and performing the loop integrations

\[ M_T = T^{\alpha\beta} \varepsilon_\alpha(K) \varepsilon_\beta^*(k) = \frac{1}{2} \int \frac{d^4k_1}{(2\pi)^4} A^{\alpha\beta\mu\nu} M_{\mu\nu} \varepsilon_\alpha(K) \varepsilon_\beta^*(k) \frac{i}{k_1^2} + i \epsilon \frac{i}{k_2^2} + i \epsilon, \quad (2.12) \]

where the factor \(1/2\) takes into account that the two gluons have already been interchanged both in \(A^{\alpha\beta\mu\nu}\) and \(M_{\mu\nu}\). Using parity conservation, Lorentz invariance and gauge invariance, it can be proved that

\[ T^{\alpha\beta} \sim \epsilon^{\alpha\beta\mu\nu} p_\mu k_\nu, \quad (2.13) \]

so there is only one independent helicity amplitude \(H^q_{QCD}\) [14]

\[ T^{\alpha\beta} \varepsilon_\alpha(K) \varepsilon_\beta^*(k) = H^q_{QCD} h^{\alpha\beta} \varepsilon_\alpha(K) \varepsilon_\beta^*(k), \quad (2.14) \]

where

\[ h^{\alpha\beta} = \frac{i}{p \cdot k} \epsilon^{\alpha\beta\mu\nu} p_\mu k_\nu. \quad (2.15) \]

With the help of the helicity projector [14]

\[ \Pi^{\alpha\beta} = \frac{1}{2} h^{\alpha\beta'} \left( -g^{\alpha\alpha'} + \frac{K^\alpha K^{\alpha'}}{M^2} \right) (-g^{\beta\beta'}) = -\frac{i}{2p \cdot k} \epsilon^{\alpha\beta\mu\nu} p_\mu k_\nu, \quad (2.16) \]

one can obtain the helicity amplitude

\[ H^q_{QCD} = T^{\alpha\beta} \Pi^{\alpha\beta} = \frac{2Q_c}{9} \sqrt{4\pi \alpha_e (4\pi \alpha_s)^2} \sqrt{\frac{3}{\pi M}} R_\psi(0) \sum_{q=u,d,s} \frac{f_{\eta(c)}^q}{M} H_q. \quad (2.17) \]

The dimensionless function \(H_q\) reads

\[ H_q = -\frac{1}{16\pi^2} \frac{2}{1 - x} \int du \phi^q(u) I(u) \quad (2.18) \]
with $x = m^2/M^2$. $I(u)$ is the sum of the loop integrals of the six Feynman diagrams for the decays

$$I(u) = \frac{1}{i\pi^2} \int \frac{d^4q}{16} \left( \frac{N_1}{C_1D_2D_3D_4D_5} + \frac{N_2}{C_1D_1D_3D_4D_5} + \frac{N_3}{D_1D_2D_3D_4D_5} \right)
+ \frac{N_4}{C_1D_1D_3D_4D_5} + \frac{N_5}{C_1D_2D_3D_4D_5} + \frac{N_6}{D_1D_2D_3D_4D_5} + (u \leftrightarrow \bar{u}) \quad \text{(2.19)}$$

with $q = k_1 - k_2$. Here the expressions of the denominators are given by

$$C_1 = \frac{1}{4} \left[ (p - k)^2 - M^2 \right] + i\epsilon,$$

$$D_1 = \frac{1}{4} \left[ (q - k)^2 - M^2 \right] + i\epsilon,$$

$$D_2 = \frac{1}{4} \left[ (q + k)^2 - M^2 \right] + i\epsilon,$$

$$D_3 = \frac{1}{4} \left[ (q + (\bar{u} - u)p)^2 - 4m^2q \right] + i\epsilon,$$

$$D_4 = \frac{1}{4} (q + p)^2 + i\epsilon,$$

$$D_5 = \frac{1}{4} (q - p)^2 + i\epsilon \quad \text{(2.20)}$$

and the six numerators $N_i$ read

$$N_1 = N_5 = \frac{1}{4} k \cdot p \left[ k \cdot q \left( m^2 - p \cdot q \right) + k \cdot p \left( q^2 - p \cdot q \right) \right],$$

$$N_2 = N_4 = \frac{1}{4} k \cdot p \left[ k \cdot p \left( q \cdot q + q^2 \right) - k \cdot q \left( m^2 + p \cdot q \right) \right],$$

$$N_3 = N_6 = \frac{1}{4} \left[ 2k \cdot pk \cdot qp \cdot q - m^2k \cdot q^2 - q^2k \cdot p^2 \right] \quad \text{(2.21)}$$

The third and the sixth terms in $I(u)$ are five-point loop integrals, and the other terms are four-point loop integrals since the denominator $C_1$ is independent of the loop momentum $q$. Before going to calculate the integral, we would present a short analysis of its IR properties. When one of the gluons goes on-shell, i.e., $q \to p$, the denominators $D_2$, $D_3$(with $u = 1 - m_q/m$), $D_5$ tend to zero. Following to the procedure in Ref. [34], we can find that the one-loop integrals in the individual Feynman diagram for $J/\psi \to \gamma\eta(\prime)$ have soft singularities, which depend on the momentum fraction $u$, and can make the convolution integral over $u$ become sensitive to the
shapes of the $\eta^{(t)}$ DAs. Moreover, the divergent term due to light quark pole would result in the final numerical results strongly dependent on the light quark masses. However, summing up the six Feynman diagrams, one can obtain

$$I(u) = \frac{1}{i\pi^2} \int \frac{d^4q}{16} \left( \frac{(m^2 - q^2)(k \cdot pq^2 - k \cdot qp \cdot q)}{4D_1 D_2 D_3 D_4 D_5} \right) + (u \leftrightarrow \bar{u}).$$

(2.22)

When $q = p + \lambda$ ($\lambda$ is a small quantity), the numerator, which arises from the summation of all Feynman diagrams,

$$(m^2 - q^2)(k \cdot pq^2 - k \cdot qp \cdot q) \sim \lambda^2,$$

(2.23)

and the on-shell propagators

$$D_2 \simeq \frac{1}{2} K \cdot \lambda \sim \lambda, \quad D_3 \simeq \frac{m_q}{m} p \cdot \lambda \sim \lambda, \quad D_5 \simeq \frac{1}{4} \lambda^2 \sim \lambda^2.$$

(2.24)

For the ultrasoft gluon ($\lambda \to 0$), the contributions to the loop integral have the form

$$\int \frac{d^4q}{d_{p+\lambda}} \left( \frac{(m^2 - q^2)(k \cdot pq^2 - k \cdot qp \cdot q)}{D_1 D_2 D_3 D_4 D_5} \right) \sim \int d^4\lambda \frac{\lambda^2}{\lambda^4} \to 0.$$

(2.25)

It means the sum of the loop integrals is IR safe.

With algebraic identities

$$q^2 = 2 \left( D_4 + D_5 - \frac{m^2}{2} \right) = 2 \left( D_1 + D_2 + \frac{M^2}{2} \right),$$

$$k \cdot pq^2 - k \cdot qp \cdot q = \frac{2k \cdot pm^2}{M^2 + m^2} (D_4 + D_2) + \frac{2k \cdot p M^2}{M^2 + m^2} (D_4 + D_5) - k \cdot q (D_4 - D_5),$$

(2.26)

the loop function $I(u)$ can be decomposed into a sum of four- and three-point one-loop integrals

$$I(u) = \frac{1}{i\pi^2} \int \frac{d^4q}{16} \left( \frac{m^4(M^2 - m^2)}{M^2 + m^2} \frac{1}{D_1 D_2 D_3 D_4 D_5} - \frac{M^2(M^2 - m^2)^2}{2(M^2 + m^2)} \frac{1}{D_1 D_2 D_3 D_4 D_5} \right) + (u \leftrightarrow \bar{u}),$$

(2.27)
which can be analytically calculated with the technique proposed in Ref. [35] or the computer program Package − X [36, 37]. Performing the convolution integral between the loop function $I(u)$ and the DA $\phi^q(u)$, we find that the function $H_q$ in Eq. (2.18) is very insensitive to the light quark masses. Specifically, the change of the absolute value of the function $H_q$ does not exceed 3% when the value of $m_q$ goes from 0 to 100 MeV for all the three kinds of $\eta^{(')}$ DAs in Fig. 2. Actually, when the light quark propagator is near its mass shell, i.e., $q \sim \pm(u - \bar{u})p$, the factor $(k \cdot pq^2 - k \cdot qp \cdot q)$ in the numerator of Eq. (2.22) tends to zero and cancels the light quark propagator pole. In Fig. 3, we show the dependence on $m_q$ of the functions $H^\eta_q = H_q|_{m = m_\eta}$ and $H_q'^\eta = H_q|_{m = m_\eta'}$ with a asymptotic DA. It is noticed that the one-loop QCD contribution to the decay $J/\psi \to \gamma \pi^0$ vanishes as a consequence of the antisymmetrical flavor wave function of $\pi^0$, even for $m_u \neq m_d$, which disagrees with the result in Ref. [21].

For simplicity, we can take the following limit safely

$$I_0(u) = \lim_{m_q \to 0} I(u),$$

$$H_0 = -\frac{1}{16\pi^2} \frac{2}{1 - x} \int du \phi^q(u) I_0(u), \quad (2.28)$$

that is $H_q(q = u, d, s) = H_0$. The expression of $I_0(u)$ is presented in the Appendix. Numerically, we find that the loop function $I_0(u)$ is quite steady over the most region of $u$. In Fig. 4, we show the $u$ dependence of $I_0^\eta(u) = I_0(u)|_{m = m_\eta}$ and $I_0'^\eta(u) = I_0(u)|_{m = m_\eta'}$ with the range $u \in (0, 1)$. Unlike the result in the limit of $m^2/M^2 \to 0$ [18], the loop functions $I_0^\eta(u)$ and $I_0'^\eta(u)$ change

Figure 3: The $m_q$ dependence of real and imaginary parts of the dimensionless functions $H^\eta_q$ and $H_q'^\eta$. 
Figure 4: The $u$ dependence of real and imaginary parts of the loop functions $I_0^0(u)$ and $I_0^0'(u)$.

slowly over the momentum fraction $u$ near the endpoints. As a consequence, the convolution integral between the loop function $I_0(u)$ and the DA becomes insensitive to the shapes of $\eta^{(t)}$ DAs. For example, the difference between the convolution integral of $I_0(u)$ with a “narrow” DA (model I in Fig. 2) and the one with a “broad” DA (model II and III in Fig. 2) is less than 2%.

After these analyses, we return to the remaining calculations. The helicity amplitude $H_{QCD}^g$ in Eq. (2.17) can be simplified to

$$H_{QCD}^g = \frac{2Q_c}{9}\sqrt{4\pi\alpha_\text{e}(4\pi\alpha_s)^2}\sqrt{\frac{3}{\pi M}} R_\psi(0) \frac{f_\eta^{(t)}}{M} H_0$$

with the effective decay constants

$$f_\eta' = f_\eta'^u + f_\eta'^d + f_\eta'^s, \quad f_\eta = f_\eta^u + f_\eta^d + f_\eta^s.$$  

(2.30)

Besides the one-loop QCD contributions, QED processes $J/\psi \to \gamma^* \to \gamma\eta^{(t)}$ can also contribute to the decays $J/\psi \to \gamma\eta^{(t)}$. The corresponding Feynman diagrams are depicted in Fig. 5, and the contribution reads

$$H_{QED} = -Q_c(4\pi\alpha_\text{e})^2 \sqrt{\frac{3}{\pi M}} R_\psi(0) \sum_{q=u,d,s} \frac{Q_q^2 f_\eta^{(t)}}{M} h_q.$$  

(2.31)
where the dimensionless function is

\[ h_q = \frac{1 - x}{2} \int du \phi^q(u) \left( \frac{1}{u - u\bar{x} - \frac{m_q^2}{M^2} + i\epsilon} + (u \leftrightarrow \bar{u}) \right) \]  

(2.32)

and the \( Q_q \) represents the light quark charge.

### 2.2 The contributions of the gluonic content of \( \eta^{(l)} \)

The gluonic content of \( \eta^{(l)} \) can contribute to the \( J/\psi \rightarrow \gamma \eta^{(l)} \) at the tree level. However, such contributions are suppressed by a factor of \( m^2/M^2 \) [19, 23]. The corresponding Feynman diagram is shown in Fig. 6. There are other two Feynman diagrams from permutations of the photon and the gluon legs. The leading twist in the light-cone expansion of the matrix elements

of the meson \( \eta^{(l)} \) over two-gluon fields is [33, 38, 39]:

\[
\langle \eta^{(l)}(p)|A^a(x)A^b(y)\rangle 0 = \frac{1}{4} \epsilon_{\alpha\beta\rho\sigma} \frac{k^\alpha p^\beta}{p \cdot k} \frac{C_F}{\sqrt{3}} \delta^{ab} f_{\eta^{(l)}}^{\rho} \int du e^{i(u p \cdot x + \bar{u} p \cdot y)} \frac{\phi^q(u)}{u(1-u)} \]  

(2.33)
with the effective decay constant $f_{\eta(0)}^g = \frac{1}{\sqrt{3}} (f_{\eta(0)}^u + f_{\eta(0)}^d + f_{\eta(0)}^s)$ and the gluonic twist-2 DA $[33, 39, 40]$

$$\phi^g(u) = 30u^2(1-u)^2 \sum_{n=2,4\ldots} e_n^g(\mu)C_{n-1}^g/\sqrt{\pi M f_{\eta(0)}^g}(2u-1). \quad (2.34)$$

The corresponding contributions can be expressed as

$$H_{QCD}^g = \frac{2Q_c}{9} \sqrt{4\pi \alpha_e(4\pi \alpha_s)} \frac{R_\psi(0) f_{\eta(0)}^g}{\sqrt{\pi M} M} H_g \quad (2.35)$$

with

$$H_g = \int \frac{d\phi_g(u)}{u(1-u)} \frac{2x(2u-1)}{1-x^2(1-2u)^2}. \quad (2.36)$$

From the Eq. (2.36), one can find that $H_g$ is proportional to a suppression factor of $x = m^2/M^2$. Actually, the leading twist gluonic content contributions are almost two on-shell gluons contributions, which are suppressed by the factor $m^2/M^2$ due to the special form of the Ore-Powell matrix elements as found in Refs. [41, 42] years ago.

### 3 Numerical results

The decay widths of $J/\psi \rightarrow \gamma\eta(0)$ can be expressed as

$$\Gamma(J/\psi \rightarrow \gamma\eta(0)) = \frac{2}{3} \frac{1-x}{16\pi} \frac{1}{M} |H_{QCD}^g + H_{QCD}^g + H_{QED}|^2. \quad (3.1)$$

In order to remove the uncertainties from $R_\psi(0)$, we relate the decay widths $\Gamma(J/\psi \rightarrow \gamma\eta(0))$ to $\Gamma(J/\psi \rightarrow e^+e^-)$

$$\mathcal{B}(J/\psi \rightarrow \gamma\eta(0)) = \frac{\Gamma(J/\psi \rightarrow \gamma\eta(0))}{\Gamma(J/\psi \rightarrow e^+e^-)} \mathcal{B}^{exp}(J/\psi \rightarrow e^+e^-) \quad (3.2)$$

with the leptonic decay width $[43]$

$$\Gamma(J/\psi \rightarrow e^+e^-) = \frac{4\alpha_e^2 Q_c^2}{M^2} |R_\psi(0)|^2 \left(1 - \frac{16}{3} \alpha_s \right). \quad (3.3)$$
\[ B^{\exp}(J/\psi \rightarrow e^+e^-) = (5.971 \pm 0.032) \times 10^{-2}. \] (3.4)

For numerical calculations, all the values of meson masses, \( \Gamma_{J/\psi} \) and \( f_\pi \) are quoted from the PDG [44]. We use the FKS scheme for the \( \eta - \eta' \) mixing [10], and then the effective decay constants \( f_{\eta(\eta')}^q \) can be parameterized as

\[
\begin{align*}
    f_{\eta}^{u(d)} &= \frac{f_q}{\sqrt{2}} \cos \phi, & f_{\eta}^s = -f_s \sin \phi, \\
    f_{\eta'}^{u(d)} &= \frac{f_q}{\sqrt{2}} \sin \phi, & f_{\eta'}^s = f_s \cos \phi.
\end{align*}
\] (3.5)

For the three phenomenological parameters, i.e., the mixing angle \( \phi \) and the decay constants \( f_q, f_s \), they have been determined in different methods [10, 45–48]. Here we take the up-to-date values from Refs. [45, 48] which are recapitulated in Table 2. The values in the first line are extracted from the low energy processes (LEPs) \( V \to \eta^{(\prime)} \gamma \), \( \eta^{(\prime)} \to V \gamma (V = \rho, \omega, \phi) \). The second and the third lines are the results extracted with rational approximations for the \( \eta^{(\prime)} \) transition form factor (TFF) \( F_{\gamma^* \gamma \eta^{(\prime)}}(Q^2) \). The results of both the first and the second lines are generally consistent with the known FKS results [10] where the ratio \( R_{J/\psi} \) with the approximation of nonperturbative matrix elements \( \langle 0|G_{\mu\nu}^{\alpha} \tilde{G}^{\alpha,\mu\nu}\eta^{(\prime)}|0 \rangle \) was adopted. While in the third line, the parameters are extracted with rational approximations for the \( \eta' \) TFF \( F_{\gamma^* \gamma \eta'}(Q^2) \), which is in accord with the BaBar measurement in the timelike region at \( q^2 = 112 \text{ GeV}^2 \) [49].

The Gegenbauer moments \( c_1^q(\mu), c_3^q(\mu) \) still have large uncertainty as depicted in Table 1. Fortunately, the dimensionless function \( H_q \) in Eq. (2.18) is insensitive to the shapes of the \( \eta^{(\prime)} \) DAs as we have shown in section 2.1. Furthermore, both \( H_g \) and \( h_q \) are an order of magnitude smaller than the \( H_q \), therefore, the uncertainty of the Gegenbauer moments impacts
our numerical calculations lightly (less than 2% among the results with three different models in Table 1). So in the following numerical calculations, we choose the Model I with
\[ c_q^2(\mu_0) = 0.10, \quad c_q^4(\mu_0) = 0.10, \quad c_g^2(\mu_0) = -0.26. \] (3.6)

As known, it is very hard to give precise predictions for individual decay width. With the nonperturbative matrix elements \( \langle 0|G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}|\eta^{(0)} \rangle \), the decay widths of \( J/\psi \rightarrow \gamma\eta^{(0)} \) have been given in Ref. [6]
\[
\Gamma(J/\psi \rightarrow \gamma\eta^{(0)}) = \frac{2^3}{5^23^{10}} \frac{\alpha_s^3\alpha_s^2}{\pi} \left( \frac{M^4}{m_c^2} \right) \left( 1 - \frac{m_{\eta^{(0)}}^2}{M^2} \right)^3 \frac{|\langle 0|G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}|\eta^{(0)} \rangle|^2}{\Gamma(J/\psi \rightarrow e^+e^-)}, \] (3.7)

where the factor \( m_c^{-8} \) will bring very large uncertainty. While, in the pQCD approach employed in this paper, there also exist large uncertainty due to the factor \( \alpha_s^4(\mu) \).

For comparison, in Table 3, we present results of these two methods with \( m_c = 1.5 \) GeV and \( \alpha_s = 0.34 \) as benchmarks. For the results in the second column, the matrix elements \( \langle 0|G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}|\eta^{(0)} \rangle \) are evaluated with the updated values of \( \phi, f_q \) and \( f_s \) in the first line of Table 2. Generally, one may expect the order of magnitude could be correctly predicted by the both approaches. However, we find the pQCD estimation of \( B(J/\psi \rightarrow \gamma\eta) \) is too small to be comparable to its experimental one. The reason may be due to our choice of the inputs \( \phi, f_q \) and \( f_s \) extracted from low energy processes, or our understanding of \( \eta - \eta' \) mixing scheme is incomplete which is beyond the scope of this paper.

Table 3: The branching ratios \( B(J/\psi \rightarrow \gamma\eta^{(0)}) \) obtained with nonperturbative gluonic matrix elements and pQCD approaches.

| \( B(J/\psi \rightarrow \gamma\eta^{(0)}) \) | \( \langle 0|G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}|\eta^{(0)} \rangle \) [6] | pQCD | Exp. [44, 50–52] |
|----------------------------------|-------------------------------|-------|-------------------|
| \( B(J/\psi \rightarrow \gamma\eta') \) | \( 1.96 \times 10^{-3} \left( \frac{1.5 \text{ GeV}}{m_c} \right)^8 \) | \( 4.91 \times 10^{-3} \left( \frac{\alpha_s(\mu)}{0.34} \right)^4 \) | \( (5.13 \pm 0.17) \times 10^{-3} \) |
| \( B(J/\psi \rightarrow \gamma\eta) \) | \( 3.99 \times 10^{-4} \left( \frac{1.5 \text{ GeV}}{m_c} \right)^8 \) | \( 9.84 \times 10^{-6} \left( \frac{\alpha_s(\mu)}{0.34} \right)^4 \) | \( (11.04 \pm 0.34) \times 10^{-4} \) |

Recently, the BaBar collaboration [49] has made the measurements of the \( \eta \) and \( \eta' \) TFFs at \( q^2 = 112 \) GeV\(^2\), which have challenged the theoretical prediction very much. It is true that precise theoretical estimation of the TFFs is hard, due to uncertainties in the \( \phi, f_q, f_s \), the DAs of \( \eta^{(0)} \) and even the mixing scheme at high energy. In the literature, there are extensive
discussions of this hot topic [33]. In Ref. [48], Escribano et al. have presented a treatment of \( \eta \) and \( \eta' \) TFFs with rational approximations and extracted these parameters in Table 2. With these inputs and \( \alpha_s = 0.34 \), we show our results in Table 4. From the Table, we find \( R_{J/\psi} \) is in good agreement with its experimental value \( R_{J/\psi}^{exp} = 4.65 \pm 0.21 \) [44], only when we adopt the set of parameter values of \( \eta' \) TFF. For other two sets of parameter values, \( B(J/\psi \rightarrow \gamma \eta) \) is estimated to be too small, which results in \( R_{J/\psi} \) two orders higher than \( R_{J/\psi}^{exp} \).

Table 4: The branching ratios \( B(J/\psi \rightarrow \gamma \eta^{(0)}) \) with three sets of inputs obtained in Refs. [45, 48].

|                | LEPs         | \( \eta \) TFF | \( \eta' \) TFF | Exp. [44, 50–52] |
|----------------|--------------|-----------------|-----------------|------------------|
| \( B(J/\psi \rightarrow \gamma \eta') \) | \( 4.91 \times 10^{-3} \) | \( 4.43 \times 10^{-3} \) | \( 2.59 \times 10^{-3} \) | \( (5.13 \pm 0.17) \times 10^{-3} \) |
| \( B(J/\psi \rightarrow \gamma \eta) \)  | \( 9.84 \times 10^{-6} \) | \( 1.74 \times 10^{-5} \) | \( 5.52 \times 10^{-4} \) | \( (11.04 \pm 0.34) \times 10^{-4} \) |
| \( R_{J/\psi} \) | \( 499.19 \) | \( 254.18 \) | \( 4.70 \) | \( 4.65 \pm 0.21 \) |

In Table 5, we present results with the contributions due to the gluonic content of \( \eta^{(0)} \) and the one from QED processes \( J/\psi \rightarrow \gamma^* \rightarrow \gamma \eta^{(0)} \). We can find that such contributions enhance \( B(J/\psi \rightarrow \gamma \eta) \) much. However, \( B(J/\psi \rightarrow \gamma \eta) \) is still far from its experimental value for both LEPs and \( \eta \) TFF values of \( \phi, f_q \) and \( f_s \). Only \( \eta' \) TFF set of parameter values can give \( B(J/\psi \rightarrow \gamma \eta^{(0)}) \) and \( R_{J/\psi} \) comparable with their experimental data.

Table 5: The same as the caption of Table 4, but including contributions of the QED and the gluonic content of \( \eta^{(0)} \).

|                | LEPs         | \( \eta \) TFF | \( \eta' \) TFF | Exp. [44, 50–52] |
|----------------|--------------|-----------------|-----------------|------------------|
| \( B(J/\psi \rightarrow \gamma \eta') \) | \( 6.01 \times 10^{-3} \) | \( 5.43 \times 10^{-3} \) | \( 3.19 \times 10^{-3} \) | \( (5.13\pm0.17) \times 10^{-3} \) |
| \( B(J/\psi \rightarrow \gamma \eta) \)  | \( 3.02 \times 10^{-5} \) | \( 4.25 \times 10^{-5} \) | \( 7.40 \times 10^{-4} \) | \( (11.04\pm0.34) \times 10^{-4} \) |
| \( R_{J/\psi} \) | \( 198.91 \) | \( 127.60 \) | \( 4.31 \) | \( 4.65 \pm 0.21 \) |

In the remaining part of this section, we would present a determination of \( \phi \) without the input values in Table 2. The ratio \( R_{J/\psi} \) in our calculation is

\[
R_{J/\psi} = \frac{M^2 - m_{\eta'}^2 |H_{QCD}^q + H_{QCD}^g + H_{QED}^q|_{m=m_{\eta'}}^2}{M^2 - m_{\eta}^2 |H_{QCD}^q + H_{QCD}^g + H_{QED}^q|_{m=m_{\eta}}^2}.
\]

Since the scalar functions are insensitive to the shapes of the \( \eta^{(0)} \) DAs, the ratio \( R_{J/\psi} \) mainly
depends on the angle $\phi$ and the ratio $f_s/f_q$. However, with the help of the ratio

$$
\frac{\Gamma(\eta \to \gamma\gamma)}{\Gamma(\eta' \to \gamma\gamma)} = \frac{m_3^2}{m_3'^2} \left( \frac{5\sqrt{2}f_s - 2\tan\phi}{5\sqrt{2}f_q\tan\phi} \right)^2
$$

(3.9)

and the experimental values [44, 53]

$$
\Gamma^{\text{exp}}(\eta' \to \gamma\gamma) = 4.36(14) \text{ KeV}, \quad \Gamma^{\text{exp}}(\eta \to \gamma\gamma) = 0.516(18) \text{ KeV},
$$

(3.10)

the ratio $R_{J/\psi}$ becomes a function which only depends on the mixing angle $\phi$. In Fig. 7, we show the dependence of the ratio $R_{J/\psi}$ on the mixing angle $\phi$. As displayed by the horizontal dashed lines in Fig. 7 of the experimental measurement $R^{\text{exp}}_{J/\psi} = 4.65 \pm 0.21$ [44], one can find

$$
\phi = 33.9^\circ \pm 0.6^\circ,
$$

(3.11)

which is in good agreement with $\eta'$TFF result $\phi = 33.5^\circ \pm 0.9^\circ$ [48][1], but in clear disagreement.

---

[1] It is noticed that the predicted $\eta$ TFF is $\lim_{Q^2 \to +\infty} Q^2 F_{\eta \gamma^* \gamma}(Q^2) = (0.160 \pm 0.024) \text{ GeV}$, which is not in line with the BaBar measurement $q^2|F_{\eta \gamma^* \gamma}(q^2)|_{q^2=112 \text{ GeV}^2} = (0.229 \pm 0.031) \text{ GeV}$, while the predicted $\eta'$ TFF, $\lim_{Q^2 \to +\infty} Q^2 F_{\eta' \gamma^* \gamma}(Q^2) = (0.255 \pm 0.004) \text{ GeV}$, is in accord with the BaBar measurement $q^2|F_{\eta' \gamma^* \gamma}(q^2)|_{q^2=112 \text{ GeV}^2} = (0.251 \pm 0.021) \text{ GeV}$. More discussions could be found in Ref. [48].

---

Figure 7: The dependence of $R_{J/\psi}$ on the mixing angle $\phi$. The blue band is our calculated results with the uncertainties of the $\Gamma^{\text{exp}}(\eta\to \gamma\gamma)$ included. The hatched band indicates the experimental value of $R_{J/\psi}$ with $1\sigma$ uncertainty.
with the $\phi = 40.6^\circ \pm 0.9^\circ$ \cite{45} extracted from low energy processes with nonperturbative methods. It is noticed that the lattice calculation of the UKQCD collaboration \cite{54} indicates a value $\phi^{\mu \nu} \sim 34^\circ$, while the ETM collaboration \cite{55–57} gives $\phi$ in the range of $40^\circ \sim 46^\circ$.

Last but not least, we would compare our result of the loop function with the one obtained by Kühn \cite{18}. Since, besides the approximation $M \approx 2m_c$, no further approximation is made, so our loop function $I_0(u)$, as shown in the Appendix, is much more complicated than the one in Ref. \cite{18}. However, in the limit of $m^2/M^2 \rightarrow 0$, our $I_0(u)$ is reduced to the weight function $W(u)$ in the Eq. (11) of Ref. \cite{18}.

### 4 Summary

In this paper, we have revisited the radiative decays $J/\psi \rightarrow \gamma\eta^{(0)}$ in detail in the framework of pQCD. Comparing to the pioneer work \cite{14}, we do not take the weak-binding approximation for the final mesons and evaluate the involved one-loop integrals analytically with the light quark masses kept. Moreover, we also consider the contributions from the QED processes and the gluonic contents of $\eta^{(0)}$. Different from the results obtained in Ref. \cite{21}, our numerical results are insensitive to the light quark masses. In addition, we find that the $\mathcal{B}(J/\psi \rightarrow \gamma\eta^{(0)})$ are insensitive to the shapes of the $\eta^{(0)}$ DAs.

Using three sets of values for $\phi$, $f_q$ and $f_s$, namely LEPs \cite{45}, $\eta$TFF \cite{48} and $\eta'$TFF \cite{48}, we have presented our numerical results in Table 4 and 5, where $\mathcal{B}(J/\psi \rightarrow \gamma\eta)$ is too small to be comparable with the experimental one for the parameters extracted from LEPs and $\eta$ TFF. Only with the values extracted from $\eta'$ TFF, which is in accord with the BaBar measurement in the timelike region at $q^2 = 112$ GeV$^2$ \cite{49}, our results of $\mathcal{B}(J/\psi \rightarrow \gamma\eta^{(0)})$ and $R_{J/\psi}$ are in good agreement with the experimental data.

As a crossing check, we use our calculation of $R_{J/\psi}$ and $\Gamma^{\exp}(\eta^{(0)} \rightarrow \gamma\gamma)$ as inputs to extract the value of $\phi$, and find $\phi = 33.9^\circ \pm 0.6^\circ$, which is in good agreement with the $\eta'$TFF one $\phi = 33.5^\circ \pm 0.9^\circ$ remarkably. However, such a small $\phi$ differs too much from the $\phi = 40.6^\circ \pm 0.9^\circ$ extracted from low energy processes and $J/\psi \rightarrow \gamma\eta^{(0)}$ with nonperturbative matrix elements $\langle 0 | G^{\mu \nu} \tilde{G}^{\alpha \beta} | \eta^{(0)} \rangle$ due to $U_A(1)$ anomaly dominance argument. The difference may arise from $g^* g^* - \eta^{(0)}$ TFF used in our calculation, in like manner, $\gamma^* \gamma - \eta'$ TFF in the extraction of $\phi$ from
the BaBar measurement of \( q^2 |F_{\gamma'\gamma}(q^2)|_{q^2=112 \text{ GeV}^2} \). Anyhow, the physics under the difference is interesting and certainly worth further investigations.

**Acknowledgements**

This work is supported by the National Natural Science Foundation of China under Grant Nos. 11675061, 11775092 and 11435003.

**Appendix: The dimensionless function \( H_0 \)**

The dimensionless function \( H_0 \) reads

\[
H_0 = -\frac{1}{16\pi^2} \frac{2}{1 - x} \int_0^1 du \phi^a(u) I_0(u). \tag{1}
\]

Since the \( H_0 \) is insensitive to the shapes of the \( \eta^{(n)} \) DAs, we simply take the asymptotic DA \( \phi^a(u) = 6u(1-u) \) in the following discussion. The analytical expression of the one-loop function \( I_0(u) \) can be expressed as

\[
I_0(u) = g_1(\xi) + g_2(\xi) + g_3(\xi) + g_4(\xi) + g_5(\xi) + g_6(\xi) + (\xi \rightarrow -\xi) \tag{2}
\]

with \( \xi = 1 - 2u \) and

\[
g_1(\xi) = \frac{2x \left[ 1 + 3(1-x)\xi - x\xi^2 \right]}{(1-x)(1+\xi)(1-x\xi)} \left[ \text{Li}_2 \left( \frac{1 - x\xi}{2 - x - x\xi} \right) + \text{Li}_2 \left( \frac{1 - x\xi}{x(1+\xi - 2x\xi)} \right) \right] + \text{Li}_2 \left( \frac{1 + \xi + 2x^2\xi(1+\xi) - x(1 + 4\xi + \xi^2)}{x(1+\xi - 2x\xi)} - i\xi \right) - \text{Li}_2 \left( \frac{1 - x\xi}{1 + \xi - 2x\xi} \right) \\
+ \text{Li}_2 \left( \frac{(1 - 2x)(2 - x - x\xi)}{x(1+\xi - 2x\xi)} \right) - \text{Li}_2 \left( \frac{1 + x^2\xi(1+\xi) - x(1 + 2\xi)}{x\xi(-2 + x + x\xi)} - i\xi \right) \\
- \text{Li}_2 \left( \frac{1 + \xi - 2x\xi}{2 - x - x\xi} \right) + \text{Li}_2 \left( \frac{2 - x - x\xi}{1 + \xi - 2x\xi} \right)
\]

\[
g_2(\xi) = -\frac{2(1 - x)}{(1 + x)(1 - \xi)(1 - x\xi)} \left[ -4\text{Li}_2(x) + 2\text{Li}_2(2x - 1) - 2\text{Li}_2(-\xi + x\xi + x\xi^2) \right. \\
+ 2\text{Li}_2 \left( \xi - x\xi + x\xi^2 \right) - \ln^2 x - 2i\pi \ln x + 4i\pi \ln \left( \frac{2 - x - x\xi}{1 + \xi - 2x\xi} \right) \bigg],
\]
In the above expressions, the dilogarithm function $\text{Li}_2(x)$ and the logarithm function $\ln(x)$ without “$i\epsilon$” prescription are defined as

\[
\text{Li}_2(x) = \lim_{\epsilon \to 0^+} \ln(x - i\epsilon),
\]

\[
\ln(x) = \lim_{\epsilon \to 0^+} \ln(x + i\epsilon).
\]
and
\[ I_0(u) \to \frac{8}{1 - \xi^2} \left( \frac{\pi^2}{4} - \ln 2 - \frac{\xi}{2} \ln \left( \frac{1 - \xi}{1 + \xi} \right) + \frac{1}{2} \ln (1 - \xi^2) + \xi \text{Li}_2(-\xi) - \xi \text{Li}_2(\xi) \right), \quad (6) \]

which reproduces the weight function \( \mathcal{W}(u) \) in the Eq. (11) of the work by Kühn [18]. Performing the convolution integral between the loop function \( I_0(u) \) and the asymptotic DA, we display the mass dependence of \( H_0 \) in Fig. 8. For comparison, we also present the dimensionless function \( H^K \) obtained in Ref. [14]. As known, the contribution of on-shell gluons in the partial wave \( 0^{-+} \) is suppressed by a factor of \( m^2/M^2 \) [41, 42], and our results for the absorptive part of the function \( H_0 \), which represents the contribution of on-shell gluons, indeed follow this character in the limit of \( m^2/M^2 \to 0 \). However, the dispersive part of the function \( H_0 \), which represents the contribution of virtual gluons, is not suppressed in the limit of \( m^2/M^2 \to 0 \). While the matrix elements \( \langle 0 | G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} | \eta(0) \rangle \) are twist-4 effects which are suppressed by the factor of \( m^2/M^2 \) relative to the leading twist terms. That is the reason why the mass dependence of the decay widths obtained in this work differs from that obtained in Ref. [6] by an additional factor \( M^4/m^4 \). In additions, when the factor \( m^2/M^2 \) is not very small, the suppression for the absorptive part is no longer operative. For example, the dimensionless function \( H^0_0 = H_0 |_{m=m_q} \) is dominated by its dispersive part, while the dispersive part and the absorptive part of the

![Figure 8: The mass dependence of the dimensionless function \( H_0 \). The function \( H^k \) was obtained in Ref. [14].](image)
\[ H_0' = H_0|_{m=m_0'} \] are comparable.

References

[1] E598 Collaboration, J. J. Aubert et al., *Experimental Observation of a Heavy Particle J*, Phys. Rev. Lett. **33** (1974) 1404.

[2] SLAC-SP-017 Collaboration, J. E. Augustin et al., *Discovery of a Narrow Resonance in $e^+e^-$ Annihilation*, Phys. Rev. Lett. **33** (1974) 1406. [Adv. Exp. Phys.5,141(1976)].

[3] Quarkonium Working Group Collaboration, N. Brambilla et al., *Heavy quarkonium physics*, hep-ph/0412158.

[4] N. Brambilla et al., *Heavy quarkonium: progress, puzzles, and opportunities*, Eur. Phys. J. **C71** (2011) 1534, [arXiv:1010.5827].

[5] M. B. Voloshin, *Charmonium*, Prog. Part. Nucl. Phys. **61** (2008) 455, [arXiv:0711.4556].

[6] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *A Theory of the $J/\psi \rightarrow \eta(\eta')\gamma$ Decays*, Nucl. Phys. **B165** (1980) 55.

[7] K.-T. Chao, *Issue of $\psi \rightarrow \gamma\eta, \gamma\eta'$ Decays and $\eta-\eta'$ Mixing*, Phys. Rev. **D39** (1989) 1353.

[8] K.-T. Chao, *Mixing of $\eta, \eta'$ with $c\bar{c}, b\bar{b}$ states and their radiative decays*, Nucl. Phys. **B335** (1990) 101.

[9] Y.-P. Kuang, Y.-P. Yi, and B. Fu, *Multipole Expansion in Quantum Chromodynamics and the Radiative Decays $J/\psi \rightarrow \gamma + \eta$ and $J/\psi \rightarrow \gamma + \pi^0$*, Phys. Rev. **D42** (1990) 2300.

[10] T. Feldmann, P. Kroll, and B. Stech, *Mixing and decay constants of pseudoscalar mesons*, Phys. Rev. **D58** (1998) 114006, [hep-ph/9802409].

[11] P. Ball, J. M. Frère, and M. Tytgat, *Phenomenological evidence for the gluon content of $\eta$ and $\eta'$*, Phys. Lett. **B365** (1996) 367, [hep-ph/9508359].

[12] V. L. Chernyak and A. R. Zhitnitsky, *Asymptotic Behavior of Exclusive Processes in QCD*, Phys. Rept. **112** (1984) 173.
[13] V. N. Baier and A. G. Grozin, *Meson Wave Functions With Two Gluon States*, Nucl. Phys. **B192** (1981) 476.

[14] J. G. Körner, J. H. Kühn, M. Krammer, and H. Schneider, *Zweig Forbidden Radiative Orthoquarkonium Decays in Perturbative QCD*, Nucl. Phys. **B229** (1983) 115.

[15] A. Duncan and A. H. Mueller, *Heavy Quarkonium Decays and the Renormalization Group*, Phys. Lett. **B93** (1980) 119.

[16] S. J. Brodsky and G. P. Lepage, *Helicity Selection Rules and Tests of Gluon Spin in Exclusive QCD Processes*, Phys. Rev. **D24** (1981) 2848.

[17] V. L. Chernyak and A. R. Zhitnitsky, *Exclusive Decays of Heavy Mesons*, Nucl. Phys. **B201** (1982) 492. [Erratum: Nucl. Phys.B214,547(1983)].

[18] J. H. Kühn, *Light Cone Expansion and Scaling Laws for Radiative Orthoquarkonium Decays*, Phys. Lett. **B127** (1983) 257.

[19] J. P. Ma, *Reexamining radiative decays of 1−− quarkonium into η′ and η*, Phys. Rev. **D65** (2002) 097506, [hep-ph/0202256].

[20] Y.-D. Yang, *Radiative decays J/ψ → η(′)γ in perturbative QCD*, hep-ph/0404018.

[21] G. Li, T. Li, X.-Q. Li, W.-G. Ma, and S.-M. Zhao, *Revisiting the OZI-forbidden radiative decays of orthoquarkonia*, Nucl. Phys. **B727** (2005) 301, [hep-ph/0505158].

[22] Y.-J. Gao, Y.-J. Zhang, and K.-T. Chao, *Radiative decays of charmonium into light mesons*, Chin. Phys. Lett. **23** (2006) 2376.

[23] B. A. Li, Υ(1s) → γ(η′, η) decays, Phys. Rev. **D77** (2008) 097502, [arXiv:0712.4246].

[24] J.-M. Gérard and A. Martini, *Ultimate survival in anomalous ψ(2S) decays*, Phys. Lett. **B730** (2014) 264, [arXiv:1312.3081].

[25] Q. Zhao, *Understanding the radiative decays of vector charmonia to light pseudoscalar mesons*, Phys. Lett. **B697** (2011) 52, [arXiv:1012.1165].
[26] J. Resag, C. R. Munz, B. C. Metsch, and H. R. Petry, Analysis of the instantaneous Bethe-Salpeter equation for $q\bar{q}$ bound states, Nucl. Phys. A578 (1994) 397, [nucl-th/9307026].

[27] G.-L. Wang, Decay constants of heavy vector mesons in relativistic Bethe-Salpeter method, Phys. Lett. B633 (2006) 492, [math-ph/0512009].

[28] H. Negash and S. Bhatnagar, Spectroscopy of ground and excited states of pseudoscalar and vector charmonium and bottomonium, Int. J. Mod. Phys. E25 (2016) 1650059, [arXiv:1508.06131].

[29] B. Guberina, J. H. K"uhn, R. D. Peccei, and R. R"uckl, Rare Decays of the $Z^0$, Nucl. Phys. B174 (1980) 317.

[30] T. Muta and M.-Z. Yang, $\eta' - g^* - g$ transition form factor with gluon content contribution tested, Phys. Rev. D61 (2000) 054007, [hep-ph/9909484].

[31] M.-Z. Yang and Y.-D. Yang, Revisiting charmless two-body $B$ decays involving $\eta'$ and $\eta$, Nucl. Phys. B609 (2001) 469, [hep-ph/0012208].

[32] A. Ali and Ya. Parkhomenko, The $\eta'g^*g^*$ vertex with arbitrary gluon virtualities in the perturbative QCD hard scattering approach, Phys. Rev. D65 (2002) 074020, [hep-ph/0012212].

[33] S. S. Agaev, V. M. Braun, N. Offen, F. A. Porkert, and A. Schäfer, Transition form factors $\gamma^*\gamma \to \eta$ and $\gamma^*\gamma \to \eta'$ in QCD, Phys. Rev. D90 (2014) 074019, [arXiv:1409.4311].

[34] S. Dittmaier, Separation of soft and collinear singularities from one loop $N$ point integrals, Nucl. Phys. B675 (2003) 447, [hep-ph/0308246].

[35] A. Denner, U. Nierste, and R. Scharf, A compact expression for the scalar one loop four point function, Nucl. Phys. B367 (1991) 637.

[36] H. H. Patel, Package-X: A Mathematica package for the analytic calculation of one-loop integrals, Comput. Phys. Commun. 197 (2015) 276, [arXiv:1503.01469].

[37] H. H. Patel, Package-X 2.0: A Mathematica package for the analytic calculation of one-loop integrals, Comput. Phys. Commun. 218 (2017) 66, [arXiv:1612.00009].
[38] P. Kroll and K. Passek-Kumerički, *The two gluon components of the $\eta$ and $\eta'$ mesons to leading twist accuracy*, Phys. Rev. D67 (2003) 054017, [hep-ph/0210045].

[39] P. Ball and G. W. Jones, $B \rightarrow \eta^{(p)}$ Form Factors in QCD, JHEP 08 (2007) 025, [arXiv:0706.3628].

[40] S. Alte, M. König, and M. Neubert, *Exclusive Radiative Z-Boson Decays to Mesons with Flavor-Singlet Components*, JHEP 02 (2016) 162, [arXiv:1512.09135].

[41] M. Krammer, *A Polarization Prediction From Two Gluon Exchange for $1^{--}$ ($Q\bar{Q}
\rightarrow \gamma \gamma$), Phys. Lett. B74 (1978) 361.

[42] A. Billoire, R. Lacaze, A. Morel, and H. Navelet, *The Use of QCD in OZI Violating Radiative Decays of Vector Mesons*, Phys. Lett. B80 (1979) 381.

[43] P. B. Mackenzie and G. P. Lepage, *QCD Corrections to the Gluonic Width of the YM Meson*, Phys. Rev. Lett. 47 (1981) 1244.

[44] Particle Data Group Collaboration, M. Tanabashi et al., *Review of Particle Physics*, Phys. Rev. D98 (2018) 030001.

[45] R. Escribano and J.-M. Frère, *Study of the $\eta - \eta'$ system in the two mixing angle scheme*, JHEP 06 (2005) 029, [hep-ph/0501072].

[46] R. Escribano and J. Nadal, *On the gluon content of the $\eta$ and $\eta'$ mesons*, JHEP 05 (2007) 006, [hep-ph/0703187].

[47] F.-G. Cao, *Determination of the $\eta$-$\eta'$ mixing angle*, Phys. Rev. D85 (2012) 057501, [arXiv:1202.6075].

[48] R. Escribano, P. Masjuan, and P. Sanchez-Puertas, *$\eta$ and $\eta'$ transition form factors from rational approximants*, Phys. Rev. D89 (2014) 034014, [arXiv:1307.2061].

[49] BaBar Collaboration, B. Aubert et al., *Measurement of the $\eta$ and $\eta'$ transition form-factors at q$^2 = 112$GeV$^2$, Phys. Rev. D74 (2006) 012002, [hep-ex/0605018].

[50] BES Collaboration, M. Ablikim et al., *Measurement of the branching fractions for J/ψ → γπ$^0$, γ$\eta$ and γ$\eta'$*, Phys. Rev. D73 (2006) 052008, [hep-ex/0510066].
[51] CLEO Collaboration, T. K. Pedlar et al., *Charmonium decays to $\gamma\pi^0$, $\gamma\eta$ and $\gamma\eta'$*, Phys. Rev. D79 (2009) 111101, [arXiv:0904.1394].

[52] BESIII Collaboration, M. Ablikim et al., *Measurement of the Matrix Element for the Decay $\eta' \to \eta\pi^+\pi^-$*, Phys. Rev. D83 (2011) 012003, [arXiv:1012.1117].

[53] KLOE-2 Collaboration, D. Babusci et al., *Measurement of $\eta$ meson production in $\gamma\gamma$ interactions and $\Gamma(\eta \to \gamma\gamma)$ with the KLOE detector*, JHEP 01 (2013) 119, [arXiv:1211.1845].

[54] UKQCD Collaboration, E. B. Gregory, A. C. Irving, C. M. Richards, and C. McNeile, *A study of the $\eta$ and $\eta'$ mesons with improved staggered fermions*, Phys. Rev. D86 (2012) 014504, [arXiv:1112.4384].

[55] ETM Collaboration, K. Ottnad, C. Michael, S. Reker, C. Urbach, C. Michael, S. Reker, and C. Urbach, *$\eta$ and $\eta'$ mesons from $N_f = 2 + 1 + 1$ twisted mass lattice QCD*, JHEP 11 (2012) 048, [arXiv:1206.6719].

[56] ETM Collaboration, C. Michael, K. Ottnad, and C. Urbach, *$\eta$ and $\eta'$ mixing from Lattice QCD*, Phys. Rev. Lett. 111 (2013) 181602, [arXiv:1310.1207].

[57] ETM Collaboration, K. Ottnad and C. Urbach, *Flavor-singlet meson decay constants from $N_f = 2 + 1 + 1$ twisted mass lattice QCD*, Phys. Rev. D97 (2018) 054508, [arXiv:1710.07986].