A UNIFORM MODEL FOR KIRILLOV-RESHETIKHIN CRYSTALS I:
LIFTING THE PARABOLIC QUANTUM BRUHAT GRAPH

CRISTIAN LENART, SATOSHI NAITO, DAISUKE SAGAKI, ANNE SCHILLING, AND MARK SHIMOZONO

Abstract. We lift the parabolic quantum Bruhat graph into the Bruhat order on the affine Weyl
group and into Littelmann’s poset on level-zero weights. We establish a quantum analogue of
Deodhar’s Bruhat-minimum lift from a parabolic quotient of the Weyl group. This result asserts
a remarkable compatibility of the quantum Bruhat graph on the Weyl group, with the cosets for
every parabolic subgroup. Also, we generalize Postnikov’s lemma from the quantum Bruhat graph
to the parabolic one; this lemma compares paths between two vertices in the former graph.
The results in this paper will be applied in a second paper to establish a uniform construction of
tensor products of one-column Kirillov-Reshetikhin (KR) crystals, and the equality, for untwisted
affine root systems, between the Macdonald polynomial with \( t \) set to zero and the graded character
of tensor products of one-column KR modules.

CONTENTS

1. Introduction 2
Acknowledgments 2
2. Notation 3
2.1. Untwisted affine root datum 4
2.2. Affine Weyl group 4
3. Orbits of level-zero weights 5
3.1. \( W_{\text{af}} \)-orbit and \( W \)-orbit 6
3.2. Stabilizers 6
3.3. Affinization of stabilizer 6
3.4. \( J \)-adjusted elements 7
4. Quantum Bruhat graph 9
4.1. Quantum roots 9
4.2. Regular case 9
4.3. Parabolic case 9
4.4. Duality antiautomorphism of \( \text{QB}(W^J) \) 10
5. Quantum Bruhat graph and the affine Bruhat order 11
5.1. Regular case 11
5.2. Embeddings \( \text{QB}(W^J) \hookrightarrow W_{\text{af}} \) 11
5.3. Trichotomy of cosets 11
5.4. Quantum edges induced by left multiplication by suitable reflections 15
5.5. Diamond Lemmas for \( \text{QB}(W^J) \) 16
6. Quantum Bruhat graph and the level-zero weight poset 18
6.1. The level-zero weight poset 18
6.2. Outline of the proof 19
6.3. Results for simple roots 20
6.4. The Diamond Lemma in the level-zero weight poset 21

2000 Mathematics Subject Classification. Primary 05E05. Secondary 33D52, 20G42.
Key words and phrases. Parabolic quantum Bruhat graph, Lakshmibai–Seshadri paths, Littelmann path model,
crystal bases, Deodhar’s lift.
1. Introduction

Our goal in this series of papers is to obtain a uniform construction of tensor products of one-column Kirillov-Reshetikhin (KR) crystals. As a consequence we shall prove the equality \( P_\lambda(q) = X_\lambda(q) \), where \( P_\lambda(q) \) is the Macdonald polynomial \( P_\lambda(q,t) \) specialized at \( t = 0 \) and \( X_\lambda(q) \) is the graded character of a simple Lie algebra coming from tensor products of one-column KR modules. Both the Macdonald polynomials and KR modules are of arbitrary untwisted affine type. The parameter \( \lambda \) is a dominant weight for the simple Lie subalgebra obtained by removing the affine node. Macdonald polynomials and characters of KR modules have been studied extensively in connection with various fields such as statistical mechanics and integrable systems, representation theory of Coxeter groups and Lie algebras (and their quantized analogues given by Hecke algebras and quantized universal enveloping algebras), geometry of singularities of Schubert varieties, and combinatorics.

Our point of departure is a theorem of Ion [15], which asserts that the nonsymmetric Macdonald polynomials at \( t = 0 \) are characters of Demazure submodules of highest weight modules over affine algebras. This applies for the Langlands duals of untwisted affine root systems (and type \( A_{2n}^{(2)} \) in the case of nonsymmetric Koornwinder polynomials). Our results apply to the symmetric Macdonald polynomials for the untwisted affine root systems. The overlapping cases are the simply-laced affine root systems \( A_n^{(1)}, D_n^{(1)} \) and \( E_6^{(1)}, E_7^{(1)}, E_8^{(1)} \).

It is known [8, 9, 11, 17, 38, 39, 43] that certain affine Demazure characters (including those for the simply-laced affine root systems) can be expressed in terms of KR crystals, which motivates the relation between \( P \) and \( X \). For types \( A_n^{(1)} \) and \( C_n^{(1)} \), the above mentioned relation between \( P \) and \( X \) was achieved in [21, 25] by establishing a combinatorial formula for the Macdonald polynomials at \( t = 0 \) from the Ram–Yip formula [40], and by using explicit models for the one-column KR crystals [10]. It should be noted that, in types \( A_n^{(1)} \) and \( C_n^{(1)} \), the one-column KR modules are irreducible when restricted to the canonical simple Lie subalgebra, while in general this is not the case. For the cases considered by Ion [15], the corresponding KR crystals are perfect. This is not necessarily true for the untwisted affine root systems considered in this work, especially for the untwisted non-simply-laced affine root systems.

In this work we provide a type-free approach to the connection between \( P \) and \( X \) for untwisted affine root systems. Lenart’s specialization [21] of the Ram–Yip formula for Macdonald polynomials uses paths in the quantum Bruhat graph (QBG), which was defined and studied in [3] in relation to the quantum cohomology of the flag variety. On the other hand, Naito and Sagaki [32, 33, 34, 35] gave models for tensor products of KR crystals of one-column type in terms of projections of level-zero Lakshmibai–Seshadri (LS) paths to the classical weight lattice. Hence we need to bridge the gap between these two approaches by establishing a bijection between paths in the quantum Bruhat graph and projected level-zero LS paths. For crystal graphs of integrable highest weight modules over quantized universal enveloping algebras of Kac-Moody algebras, Lenart and Postnikov had
already established a bijection between the LS path model and the alcove model [24]. This bijection was refined and reformulated in [28] using Littelmann’s direct characterization of LS paths [20] and Deodhar’s lifting construction for Coxeter groups [5].

In this first paper we set the stage for the connection between the projected level-zero LS path model [32, 33, 34, 35] and the quantum alcove model [22]. We begin by establishing a first lift from the parabolic quantum Bruhat graph (PQBG) to the Bruhat order of the affine Weyl group. This is a parabolic analogue of the fact that the quantum Bruhat graph (QBG) can be lifted to the affine Bruhat order [27], which is the combinatorial structure underlying Peterson’s theorem [38]; the latter equates the Gromov-Witten invariants of finite-dimensional homogeneous spaces $G/P$ with the Pontryagin homology structure constants of Schubert varieties in the affine Grassmannian. We obtain Diamond Lemmas for the PQBG via projection of the standard Diamond Lemmas for the affine Weyl group. We find a second lift of the PQBG into a poset of Littelmann [20] for level-zero weights and characterize its local structure (such as cover relations) in terms of the PQBG. Littelmann’s poset was defined in connection with LS paths for arbitrary (not necessarily dominant) weights, but the local structure was not previously known. Then, we prove the tilted Bruhat theorem, which is a quantum Bruhat graph analogue of the Deodhar lift [5] for Coxeter groups. This will turn out to be important in our second paper [29], where we establish the connection between the LS path model and the quantum alcove model. Our proof uses the novel notion of quantum length, which relies on the fact that the PQBG is strongly connected when using only simple transpositions; see [14]. The theorem ultimately follows from the application of the Diamond Lemmas for the QBG. Finally, we prove the natural generalization from the QBG to the PQBG of Postnikov’s lemma [39, Lemma 1 (2), (3)], which compares the weights of two paths in the former graph (the weight measures the down steps); note that part (1) of Postnikov’s lemma, stating the strong connectivity of the QBG, is generalized earlier in this paper. Besides our second paper on KR crystals, the generalization of Postnikov’s lemma might find applications to parabolic versions of the results in [39] on the quantum cohomology of flag varieties.

The paper is organized as follows. In Section 2 we set up the notation for untwisted affine root systems and affine Weyl groups. In Section 3 we give the definitions of stabilizers of orbits of the affine Weyl group and derive properties of $J$-adjusted elements, where $J$ is the index set of a parabolic subgroup. The PQBG is introduced in Section 4 and the lift to the Bruhat order of the affine Weyl group is given in Section 5 (see Proposition 5.2). This gives rise to the Diamond Lemmas in Section 5.5. In Section 6 we state and prove our characterization of Littelmann’s level-zero weight poset (see Theorem 6.5) and show that the PQBG is strongly connected when using only simple reflections (see Lemma 6.12). In Section 7 we prove the tilted Bruhat Theorem (see Theorem 7.1). Finally, in Section 8 we prove the parabolic generalization of Postnikov’s lemma (see Proposition 8.1).

Note: After this paper was submitted, we learned of two previous appearances of the regular weight poset or nonparabolic QBG; see Remark 6.2. Lusztig’s generic Bruhat order on the affine Weyl group [30, §1.5] (or equivalently, on alcoves), is isomorphic to the weight poset when $\lambda$ is regular. It is shown in [7] that the containment relation for closures of strata in the space of semi-infinite flags (or the space of quasimaps from $\mathbb{P}^1$ to $G/B$) is equivalent to the generic Bruhat order; moreover, the description in [7, Prop. 5.5] is ultimately equivalent to using the QBG, although far less explicit. We presume that analogous results will hold for the PQBG and strata for the space of quasimaps from $\mathbb{P}^1$ to $G/P$.

Acknowledgments. The first two and last two authors would like to thank the Mathematisches Forschungsinstitut Oberwolfach for their support during the Research in Pairs program, where some of the main ideas of this paper were conceived. We would also like to thank Thomas Lam, for helpful discussions during FPSAC 2012 in Nagoya, Japan; Daniel Orr, for his discussions about Ion’s work [15]; and Martina Lanini, for informing us about Lusztig’s order on alcoves. We used
SAGE [41] and SAGE-COMBINAT [42] to discover properties about the level-zero weight poset and to obtain some of the pictures in this paper.

C.L. was partially supported by the NSF grant DMS–1101264. S.N. was supported by Grant-in-Aid for Scientific Research (C), No. 24540010, Japan. D.S. was supported by Grant-in-Aid for Young Scientists (B) No. 23740003, Japan. A.S. was partially supported by the NSF grants DMS–1001256 and OCI–1147247. M.S. was partially supported by the NSF grant DMS–1200804.

2. Notation

2.1. Untwisted affine root datum. Let $I_{af} = I \cup \{0\}$ (resp. $I$) be the Dynkin node set of an untwisted affine algebra $\mathfrak{g}_{af}$ (resp. its canonical subalgebra $\mathfrak{g}$), $(a_{ij} | i, j \in I_{af})$ the affine Cartan matrix, $X_{af} = \mathbb{Z}\delta \oplus \bigoplus_{i \in I_{af}} \mathbb{Z}\Lambda_i$ (resp. $X = \bigoplus_{i \in I} \mathbb{Z}\omega_i$) the affine (resp. finite) weight lattice, $X^\vee_{af} = \text{Hom}_\mathbb{Z}(X_{af}, \mathbb{Z})$ the dual lattice, and $\langle \cdot, \cdot \rangle : X_{af}^\vee \times X_{af} \rightarrow \mathbb{Z}$ the evaluation pairing. Let $X^\vee_{af}$ have dual basis $\{d\} \cup \{\alpha_i^\vee | i \in I_{af}\}$. The natural projection $cl : X_{af} \rightarrow X$ has kernel $\mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\delta$ and sends $\Lambda_i \mapsto \omega_i$ for $i \in I$.

Let $\{\alpha_i | i \in I_{af}\} \subset X_{af}$ be the unique elements such that
\begin{align}
\langle \alpha_i^\vee, \alpha_j \rangle &= a_{ij} \quad \text{for } i, j \in I_{af} \\
(d, \alpha_j) &= \delta_{j, 0}.
\end{align}
The affine (resp. finite) root lattice is defined by $Q_{af} = \bigoplus_{i \in I_{af}} \mathbb{Z}\alpha_i$ (resp. $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$). The set of affine real roots (resp. roots) of $\mathfrak{g}_{af}$ (resp. $\mathfrak{g}$) are defined by $\Phi^\text{af} = W_{af} \{\alpha_i | i \in I_{af}\}$ (resp. $\Phi = W \{\alpha_i | i \in I\}$). The set of positive affine real (resp. positive) roots are the set $\Phi^\text{af+} = \Phi_{af} \cap \bigoplus_{i \in I_{af}} \mathbb{Z}_{\geq 0}\alpha_i$ (resp. $\Phi^+ = \Phi \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$). We have $\Phi^\text{af} = \Phi^\text{af+} \sqcup \Phi^\text{af-}$ where $\Phi^\text{af-} = -\Phi^\text{af+}$ and $\Phi = \Phi^+ \sqcup \Phi^-$ where $\Phi^- = -\Phi^+$.

The null root $\delta$ is the unique element such that $\delta \in \bigoplus_{i \in I_{af}} \mathbb{Z}_{\geq 0}\alpha_i$ which generates the rank 1 sublattice $\{\lambda \in X_{af} | \langle \alpha_i^\vee, \lambda \rangle = 0 \text{ for all } i \in I_{af}\}$. Define $a_i \in \mathbb{Z}_{\geq 0}$ by
\begin{equation}
\delta = \sum_{i \in I_{af}} a_i \alpha_i.
\end{equation}
We have $\delta = \alpha_0 + \theta$, where $\theta$ is the highest root for $\mathfrak{g}$, and
\begin{equation}
\Phi^\text{af+} = \Phi^+ \sqcup (\Phi + \mathbb{Z}_{\geq 0} \delta).
\end{equation}
The canonical central element is the unique element $c \in \bigoplus_{i \in I_{af}} \mathbb{Z}_{\geq 0}\alpha_i^\vee$ which generates the rank 1 sublattice $\{\mu \in X^\vee_{af} | \langle \mu, \alpha_i \rangle = 0 \text{ for all } i \in I_{af}\}$. Define $a_i^\vee \in \mathbb{Z}_{\geq 0}$ by $c = \sum_{i \in I_{af}} a_i^\vee \alpha_i^\vee$. Then $a_0^\vee = 1$ [10]. The level of a weight $\lambda \in X_{af}$ is defined by $\text{level}(\lambda) = \langle c, \lambda \rangle$.

2.2. Affine Weyl group. Let $W_{af}$ (resp. $W$) be the affine (resp. finite) Weyl group with simple reflections $r_i$ for $i \in I_{af}$ (resp. $i \in I$). $W_{af}$ acts on $X_{af}$ and $X^\vee_{af}$ by
\begin{align}
r_i \lambda &= \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \\
r_i \mu &= \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee
\end{align}
for $i \in I_{af}, \lambda \in X_{af}$, and $\mu \in X^\vee_{af}$. The pairing is $W_{af}$-invariant:
\begin{equation}
\langle w\mu, w\lambda \rangle = \langle \mu, \lambda \rangle \quad \text{for } \lambda \in X_{af} \text{ and } \mu \in X^\vee_{af}.
\end{equation}
Since the action of $W_{af}$ on $X_{af}$ is level-preserving, the sublattice $X^0_{af} \subset X_{af}$ of level-zero elements is $W_{af}$-stable. There is a section $X \rightarrow X^0_{af}$ given by $\omega_i \mapsto \Lambda_i - \text{level}(\Lambda_i) \Lambda_0$ for $i \in I$.

For $\beta \in \Phi^\text{af}$ let $w \in W_{af}$ and $i \in I_{af}$ be such that $\beta = w \cdot \alpha_i$. Define the associated reflection $r_\beta \in W_{af}$ and associated coroot $\beta^\vee \in X^\vee_{af}$ by
\begin{align}
r_\beta &= wr_i w^{-1} \\
\beta^\vee &= w\alpha_i^\vee.
\end{align}
Both are independent of \( w \) and \( i \). Of course \( r_{-\beta} = r_\beta \). We have
\[
\begin{align*}
    r_\beta \lambda &= \lambda - \langle \beta^\vee, \lambda \rangle \beta \\
    r_\beta \mu &= \mu - \langle \mu, \beta \rangle \beta^\vee
\end{align*}
\]
for \( \lambda \in X_{af} \) and \( \mu \in X_{af}^\vee \).

There is an isomorphism
\[
W_{af} \cong W \ltimes Q^\vee.
\]
Consider the injective group homomorphism \( Q^\vee := \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee \to W_{af} \) from the finite coroot lattice into \( W_{af} \), denoted by \( \mu \mapsto t_\mu \). Then \( wt_\mu w^{-1} = t_{w\mu} \) for \( w \in W \). Under the map \((2.7)\), for \( \alpha \in \Phi \) and \( n \in \mathbb{Z} \), we have
\[
\begin{align*}
    r_{\alpha+n\delta} &\mapsto r_{\alpha+n\alpha^\vee} \\
    r_0 &\mapsto r_{\theta t_{-\theta^\vee}}
\end{align*}
\]
the latter holding since \( \alpha_0 = \delta - \theta \).

Let \( W_e = W \ltimes X^\vee \) be the extended affine Weyl group where \( X^\vee = \bigoplus_{i \in I} \mathbb{Z} \omega_i^\vee \) is the coweight lattice of \( \mathfrak{g} \). Let \( I^s \subset I_{af} \) be the subset of special or cominuscule nodes, the set of nodes \( i \in I_{af} \) which are the image of 0 under some automorphism of the affine Dynkin diagram. There is a bijection from \( I^s \) to \( X^\vee/Q^\vee \) given by \( i \mapsto \omega_i^\vee + Q^\vee \) where \( \omega_i^\vee := 0 \) and \( Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee \) is the finite coroot lattice. For each \( i \in I^s \) there is a permutation \( \tau_i \) of \( X^\vee/Q^\vee \) (and therefore a permutation of \( I^s \)) defined by adding \(-\omega_i + Q^\vee\). The induced permutation of \( I^s \) extends uniquely to an automorphism \( \tau_i \) of the affine Dynkin diagram. The group \( \text{Aut}^s(I_{af}) \) of special automorphisms is defined to be the group of \( \tau_i \) for \( i \in I^s \). It acts on \( X_{af}, X_{af}^\vee, Q_{af}, Q_{af}^\vee = \bigoplus_{i \in I_{af}} \mathbb{Z} \alpha_i^\vee \), and \( W_{af} \) by permuting \( I_{af} \) on basis elements and for \( W_{af} \), indices of simple reflections.

Define \( v_i \in W \) by the length-additive product
\[
(2.8) \quad w_0 = v_i w_0^j \quad \text{for} \quad i \in I^s
\]
where \( w_0 \in W \) and \( w_0^j \in W_J \) are the longest elements in \( W \) and the subgroup \( W_J \) of \( W \) generated by \( r_j \) for \( j \in J = I \setminus \{i\} \) respectively. In particular \( v_0 = \text{id} \). Then there is an injective group homomorphism
\[
\text{Aut}^s(I_{af}) \to W_e
\]
\[
\tau_i = v_i t_{-\omega_i^\vee} \quad \text{for} \quad i \in I^s.
\]
\( \text{Aut}^s(I_{af}) \) acts on \( W_{af} \) by conjugation. This action may be defined by relabeling indices of simple reflections: \( \tau r_i \tau^{-1} = r_{\tau(i)} \) for all \( \tau \in \text{Aut}^s(I_{af}) \) and \( i \in I_{af} \). As such we have \( W_e \cong \text{Aut}^s(I_{af}) \ltimes W_{af} \).

There is an injective group homomorphism
\[
(2.9) \quad \text{Aut}^s(I_{af}) \to W
\]
\[
\tau_i \mapsto v_i.
\]

**Lemma 2.1.** For every \( i \in I^s \), \( a_i = 1 \) and \( \alpha_i \) occurs in \( \theta \) with coefficient 1 for \( i \in I^s \setminus \{0\} \).

**Proof.** For untwisted affine algebras \( a_0 = 1 [16] \). The lemma follows since \( \text{Aut}^s(I_{af}) \) acts transitively on \( I^s \) and fixes \( \delta \).

**Lemma 2.2.** For every \( i \in I^s \)
\[
(2.10) \quad \ell(v_i) = \langle \omega_i^\vee, 2\rho \rangle.
\]
**Proof.** Fix \( i \in I^s \). Since \( \theta \) is the highest root, it follows from Lemma 2.1 that if \( \alpha_i \) occurs in a positive root then its coefficient is 1. Consequently the right hand side of \((2.10)\) equals the number of positive roots that contain \( \alpha_i \). This is the complement of the number of positive roots in the parabolic subsystem for \( J = I \setminus \{i\} \). But this is equal to \( \ell(w_0) - \ell(w_0^i) = \ell(v_i) \).
3. Orbits of level-zero weights

3.1. \( W_{af}\)-orbit and \( W\)-orbit. The action of \( W_{af} \) on \( X_{af}^0 \) is given by
\[
(3.1) \quad wt_{\mu} \lambda = w \lambda - \langle \mu, \lambda \rangle \delta
\]
for \( w \in W, \mu \in Q^\vee \), and \( \lambda \in X_{af}^0 \).

**Lemma 3.1.** For a dominant weight \( \lambda \in X \cong X_{af}^0 / \mathbb{Z} \delta \) we have \( W_{af} \lambda = W \lambda \) in \( X_{af}^0 / \mathbb{Z} \delta \).

**Proof.** This follows immediately from (3.1). \( \square \)

3.2. Stabilizers. Let \( \lambda \in X \) be a dominant weight, which will be used several times in this paper, so the notation below applies throughout. Let \( W_J \) be the stabilizer of \( \lambda \) in \( W \). It is a parabolic subgroup, being generated by \( r_i \) for \( i \in J \) where
\[
(3.2) \quad J = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}.
\]
Let \( Q_J^\vee = \bigoplus_{i \in J} \mathbb{Z} \alpha_i^\vee \) be the associated coroot lattice, \( W^J \) the set of minimum-length coset representatives in \( W/W_J \), \( \Phi_J = \Phi_J^+ \sqcup \Phi_J^- \) the set of roots and positive/negative roots respectively, and \( \rho_J = (1/2) \sum_{\alpha \in \Phi_J^+} \alpha \).

**Lemma 3.2.** The stabilizer of \( \lambda \) in \( W_{af} \) under its level-zero action on \( X \cong X_{af}^0 / \mathbb{Z} \delta \), is given by the subgroup of elements of the form \( wt_{\mu} \) where \( w \in W_J \) and \( \mu \in Q^\vee \) satisfies \( \langle \mu, \lambda \rangle = 0 \).

**Proof.** This follows immediately from the definitions and Lemma 3.1. \( \square \)

3.3. Affinization of stabilizer. Let \( J = \bigsqcup_{m=1}^{k} I_m \) have connected components with vertex sets \( I_1, I_2, \ldots, I_k \). The coweight lattice \( X_J^\vee \) is the direct sum \( \bigoplus_{m=1}^{k} X_{af}^0 \) where \( X_{af}^0 \) is the coweight lattice for the root system defined by the component \( I_m \). Define \( J_{af} = \bigcup_{m} J_{af}^m \), where \( J_{af}^m = I_m \cup \{0_m\} \) and \( 0_m \) is a separate additional affine node attached to \( I_m \). Define \( (W_J)_{af} = \prod_{m=1}^{k} (W_{af})_{af} \) where \( W_{af} \) and \( (W_{af})_{af} \) are the finite and affine Weyl groups for the root subsystem with Dynkin node set \( I_m \). Under this isomorphism \( r_{0_m} = r_{\theta_m} t_{-\theta_m} \) where \( \theta_m \) is the highest root for \( I_m \).

Define
\[
(3.3) \quad \Phi_J^{af+} = \{\beta \in \Phi_{af}^+ \mid cl(\beta) \in \Phi_J\} = \Phi_J^+ \cup (\mathbb{Z}_{>0} \delta + \Phi_J), \quad \Phi_J^{af-} = -\Phi_J^{af+},
\]
\[
(3.4) \quad (W_J)_{af} = \{x \in W_{af} \mid x\beta > 0 \text{ for all } \beta \in \Phi_J^{af+}\}.
\]

**Lemma 3.3.** [27, Lemma 10.1] \( wt_{\mu} \in (W_J)_{af} \) if and only if, for all \( \alpha \in \Phi_J^+ \), \( \omega \alpha > 0 \) implies that \( \langle \mu, \alpha \rangle = 0 \) and \( \omega \alpha < 0 \) implies that \( \langle \mu, \alpha \rangle = -1 \).

**Proposition 3.4.** [27, Lemma 10.5] [38] Given \( w \in W_{af} \) there exist unique \( w_1 \in (W_J)_{af} \) and \( w_2 \in (W_J)_{af} \) such that \( w = w_1 w_2 \). If \( w \in W \), then \( w_1 \in W^J \) is the minimum-length representative of the coset \( w W_J \).

Define \( \pi_J : W_{af} \to (W_J)_{af} \) by
\[
(3.5) \quad w \mapsto w_1,
\]
with \( w_1 \) as in Proposition 3.4. Note that for \( x \in W_{af}, x \in (W_J)_{af} \) if and only if \( \pi_J(x) = x \).

Let \( W_{af}^- \) be the set of minimum-length coset representatives in \( W_{af}/W \).

**Proposition 3.5.** [27, Proposition 10.8] [38] Let \( x \in W_{af} \) and \( \mu \in Q^\vee \). Then
\begin{enumerate}
  \item \( \pi_J(xv) = \pi_J(x) \) if \( v \in (W_J)_{af} \).
  \item \( \pi_J(W) \subset W^J \subset (W_J)_{af} \).
  \item \( \pi_J(W_{af}^-) \subset W_{af}^- \).
  \item \( \pi_J(xt_{\mu}) = \pi_J(x) \pi_J(t_{\mu}) \).
\end{enumerate}
We shall employ the explicit description of $\pi_J$ in [27, Lemma 10.7]. The element $\mu \in Q^\vee$ can be written uniquely in the form

$$\mu = \sum_{i \in I \setminus J} c_i \omega_i^\vee - \phi_J(\mu) - \sum_{m=1}^k \omega_{j_m}^\vee,$$

where $\phi_J(\mu) \in Q^\vee_J$ and each $j_m \in I_m$ is a cominuscule node. The element $\mu$ is first separated into the part in $X_J^\vee$ and the part not in it, and then one considers the projection of the part in $X_J^\vee$ to $X_J^\vee/Q^\vee_J$, takes a canonical lift (the last sum). Then $\phi_J(\mu) \in Q^\vee_J$ is the correction term. We write $z_\mu = \prod_{m=1}^k v_{j_m}^t$ where $v_{j_m} \in W_{I_m} \subset W_J$ is defined in (2.8). Then for $w \in W$ and $\mu \in Q^\vee$ we have

$$\pi_J(wt_\mu) = \pi_J(w)\pi_J(t_\mu) = \pi_J(w)z_\mu t_\mu + \phi_J(\mu).$$

**Remark 3.6.** By Proposition 3.5 the map

$$Q^\vee \to \text{Aut}^s(J^af) \subset W_J$$

$$\mu \mapsto z_\mu$$

is a group homomorphism.

Denote by $\Sigma_J \subset \text{Aut}^s(J^af) \subset W_J$ the image of the homomorphism (3.8):

$$\Sigma_J = \{ z \in W_J \mid z = z_\mu \text{ for some } \mu \in Q^\vee \}.$$

**3.4. J-adjusted elements.** We say that $\mu \in Q^\vee$ is $J$-adjusted if $\phi_J(\mu) = 0$ or equivalently

$$\pi_J(t_\mu) = z_\mu t_\mu.\tag{3.10}$$

This notion gives a nice parametrization of the set $(W^{-1})_{af}$.

**Lemma 3.7.** Let $w \in W^J$, $z \in W_J$, and $\mu \in Q^\vee$. Then $wzt_\mu \in (W^{-1})_{af}$ if and only if $\mu$ is $J$-adjusted and $z = z_\mu$. In particular every element of $(W^{-1})_{af}$ can be uniquely written as $w\pi_J(t_\mu) = wz_\mu t_\mu$ where $w \in W^J$ and $\mu \in Q^\vee$ is $J$-adjusted.

**Proof.** $wzt_\mu \in (W^{-1})_{af}$ if and only if $wzt_\mu = \pi_J(wzt_\mu) = \pi_J(wz)\pi_J(t_\mu) = w\pi_J(t_\mu)$ from which the result follows. \qed

**Lemma 3.8.** Let $\mu \in Q^\vee$ and consider (3.6). The following are equivalent:

1. $\mu$ is $J$-adjusted.
2. For every component $I_m$ of $J$, either
   a) $\langle \mu, \alpha_i \rangle = 0$ for all $i \in I_m$ (that is, $j_m = 0_m \in I^af_m$), or
   b) there is a unique $j \in I_m$ such that $\langle \mu, \alpha_j \rangle \neq 0$, and in this case $j = j_m$ and $\langle \mu, \alpha_{j_m} \rangle = -1$.
3. $\langle \mu, \alpha \rangle \in \{0, -1\}$ for all $\alpha \in \Phi^+_J$.

**Proof.** Given [27, Lemma 10.7], (1) and (2) are equivalent. Suppose (2) holds. Let $\alpha \in \Phi^+_J$. Then $\alpha$ is a positive root in the subrootsystem $\Phi^+_m$ of $\Phi$ for some component $I_m$ of $J$. Let $\alpha = \sum_{i \in I_m} b_i \alpha_i$. Since $j_m \in I_m$ is cominuscule, $\langle \omega_{j_m}^\vee, \theta_m \rangle = 1$ where $\theta_m \in \Phi^+_m$ is the highest root. Therefore $b_{j_m} \in \{0, 1\}$. Since $b_i = 0$ for $i \in I_m \setminus \{j_m\}$, (3) follows.

Conversely, suppose (3) holds. Let $I_m$ be a component of $J$. Applying (3) to $\theta_m$ and to each of the $\alpha_i$ for $i \in I_m$, we see that (2) must hold. \qed

**Lemma 3.9.** For $\mu \in Q^\vee$, $\mu$ is $W_J$-invariant if and only if $\mu$ is $J$-adjusted and $z_\mu = id$.

**Proof.** The first condition holds if and only if no fundamental coweight $\omega_i^\vee$ occurs in $\mu$ for $i \in J$, which for the expression (3.6) means that $\phi_J(\mu) = 0$ and $j_m = 0_m$ for all $m$. But this holds if and only if $\pi_J(t_\mu) = t_\mu$ by (3.7). \qed
Lemma 3.10. Let $\mu \in Q^\vee$ be $J$-adjusted. Then
\begin{equation}
\ell(z_\mu) = -\langle \mu, 2\rho_J \rangle.
\end{equation}

Proof. The proof reduces to considering each component $I_m$ of $J$. Note that $-\mu$ pairs with roots of $I_m$ like a fundamental cominuscule coweight by Lemma 3.8, and the result follows by Lemma 2.2. \hfill \Box

Lemma 3.11. For every $\mu \in Q^\vee$ and $v \in W_J$, $z_\mu = z_{v\mu}$.

Proof. Using (3.7) with $\ell = 1$, we have $z_\mu t_{\mu+\phi_J(\mu)} = \pi_J(t_\mu) = \pi_J(v_t_\mu v^{-1}) = \pi_J(t_{v\mu})$, which implies the result. \hfill \Box

Lemma 3.12. Given $\alpha \in \Phi^+$ and $x = wt_\mu \in Waf$ with $w \in W$ and $\mu \in Q^\vee$, let $\ell_\alpha(x)$ be the number of roots $\pm \alpha + n\delta \in \Phi^{af}$ with $n \in \mathbb{Z}$, which $x$ sends to $\Phi^{af}$. Then
\begin{equation}
\ell_\alpha(x) = |\chi(w_\alpha \in \Phi^-) + \langle \mu, \alpha \rangle|.
\end{equation}
Here $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ if $S$ is false.

Proof. This follows from $x(\pm \alpha + n\delta) = \pm w_\alpha + (n - \langle \mu, \pm \alpha \rangle)\delta$. \hfill \Box

Lemma 3.13. Let $w \in W^J$, $z \in W_J$, and $\mu \in Q^\vee$ be such that $\langle \mu, \alpha \rangle < 0$ for all $\alpha \in \Phi^+ \setminus \Phi_J^+$ and $x = wz t_\mu \in (W^J)_{af}$. Then $\mu$ is $J$-adjusted, $z = z_\mu$, and
\begin{equation}
\ell(x) = -\langle \mu, 2\rho - 2\rho_J \rangle - \ell(w).
\end{equation}

Proof. By Lemma 3.7 we need only prove the length condition. We have $\ell(x) = \sum_{\alpha \in \Phi^+} \ell_\alpha(x)$. Fix $\alpha \in \Phi^+$. Since $x \in (W^J)_{af}$, if $\alpha \in \Phi_J^+$ then $\ell_\alpha(x) = 0$. Let $\alpha \in \Phi^+ \setminus \Phi_J^+$. By Lemma 3.12 we have $\ell_\alpha(wz t_\mu) = -\chi(w_\alpha \in \Phi^-) - \langle \mu, \alpha \rangle$. Summing this over $\alpha \in \Phi^+ \setminus \Phi_J^+$, we have
\[\ell(x) = -\langle \mu, 2\rho - 2\rho_J \rangle + \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} -\chi(w_\alpha \in \Phi^-).\]
But $z \in W_J$ so it permutes the set $\Phi^+ \setminus \Phi_J^+$. Moreover $w \in W^J$ so $w \Phi_J^+ \subset \Phi^+$. The lemma follows. \hfill \Box

Let $\mu \in Q^\vee$. We say that $\mu$ is antidominant if
\begin{equation}
\langle \mu, \alpha \rangle \leq 0 \quad \text{for all } \alpha \in \Phi^+.
\end{equation}
Say that $\mu$ is strictly $J$-antidominant if it is antidominant and
\begin{equation}
\langle \mu, \alpha \rangle < 0 \quad \text{for all } \alpha \in \Phi^+ \setminus \Phi_J^+.
\end{equation}
Say that $\mu$ is $J$-superantidominant if $\mu$ is antidominant and
\begin{equation}
\langle \mu, \alpha \rangle \ll 0 \quad \text{for } \alpha \in \Phi^+ \setminus \Phi_J^+.
\end{equation}
In the notation of (3.6), the condition (3.16) means that $c_i \ll 0$ for all $i \in I \setminus J$.

Remark 3.14. If $J = \emptyset$, then the $J$-superantidominant property becomes the superantidominant one in [27]. If $\mu$ is superantidominant, then (3.16) and (3.7) show that, in the projection $\pi_J(t_\mu) = z_\mu t_\nu$, the element $\nu$ is $J$-superantidominant.

Lemma 3.15. Let $z \in \Sigma_J$ (see (3.9)). Then there is a $J$-superantidominant, $J$-adjusted element $\mu \in Q^\vee$ such that $z = z_\mu$.

Proof. By assumption there is a $\nu \in Q^\vee$ such that $\pi_J(t_\nu) = z t_{\nu+\phi_J(\nu)}$. Since $\gamma = \phi_J(\nu) \in Q_J^\vee$, by (3.7) we have $\pi_J(t_\gamma) = \id$. We have
\[\pi_J(t_{\nu+\gamma}) = \pi_J(t_\nu) \pi_J(t_\gamma) = z t_{\nu+\gamma} \quad \text{so that } \nu + \gamma \text{ is a } J\text{-adjusted element of } Q^\vee \text{ with } z_{\nu+\gamma} = z.\]
Let $\eta \in Q^\vee$ be $J$-superantidominant and $W_J$-invariant, so that $z_{\eta} = \id$. Then $\nu + \gamma + \eta$ is the required element. \hfill \Box
**Figure 1. Quantum Bruhat graph for $S_3$**

**Lemma 3.16.** Let $w \in W^J$ and let $\mu \in Q^\vee$ be $J$-adjusted and strictly $J$-antidominant. Then $wz_\mu t_\mu \in W^-_a$. 

**Proof.** By [27, Lemma 3.3] $wt_\mu \in W^-_a$. We have $\pi_J(wt_\mu) = \pi_J(w)\pi_J(t_\mu) = wz_\mu t_\mu \in W^-_a$ by Proposition 3.5. 

### 4. Quantum Bruhat graph

The quantum Bruhat graph was first introduced in a paper by Brenti, Fomin and Postnikov [3] and later appeared in connection with the quantum cohomology of flag varieties in a paper by Fulton and Woodward [12]. In this section we define the QBG and its parabolic analogue, and prove some properties we need.

#### 4.1. Quantum roots

Say that $\alpha \in \Phi^+$ is a quantum root if $\ell(r_\alpha) = \langle \alpha^\vee, 2\rho \rangle - 1$.

**Lemma 4.1.** [3, Lemma 4.3] [31, Lemma 3.2] For any positive root $\alpha \in \Phi^+$, we have $\ell(r_\alpha) \leq -1 + \langle \alpha^\vee, 2\rho \rangle$. In simply-laced type all roots are quantum roots.

**Lemma 4.2.** [4] $\alpha \in \Phi^+$ is a quantum root if and only if

1. $\alpha$ is a long root, or
2. $\alpha$ is a short root, and writing $\alpha = \sum c_i \alpha_i^\vee$, we have $c_i = 0$ for all $i$ such that $\alpha_i$ is long.

Here for simply-laced root systems we consider all roots to be long.

#### 4.2. Regular case

The quantum Bruhat graph $QB(W)$ is a directed graph structure on $W$ that contains two kinds of directed edges. For $w \in W$ there is a directed edge $w \xrightarrow{\alpha} wr_\alpha$ if $\alpha \in \Phi^+$ and one of the following holds.

1. (Bruhat edge) $w \prec wr_\alpha$ is a covering relation in Bruhat order, that is, $\ell(wr_\alpha) = \ell(w) + 1$.
2. (Quantum edge) $\ell(wr_\alpha) = \ell(w) - \ell(r_\alpha)$ and $\alpha$ is a quantum root.

Condition (2) is equivalent to

$$\ell(wr_\alpha) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho \rangle.$$

An example is given in Figure 1 where the quantum edges are drawn in red and $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$. 

---

*$\alpha_1$, $\alpha_2$, $\alpha_3$ denote the simple roots for $S_3$.*
4.3. **Parabolic case.** Let $QB(W^J)$ be the *parabolic quantum Bruhat graph*. Its vertex set is $W^J$. There are two kinds of directed edges. Both are labeled by some $\alpha \in \Phi^+ \setminus \Phi^+_J$. We use the notation $\lfloor w \rfloor$ to indicate the minimum-length coset representative in the coset $wW_J$.

(1) (Bruhat edge) $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$ where $w \lessdot wr_\alpha$. (One may deduce that $wr_\alpha \in W^J$.)

(2) (Quantum edge) $\ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho - 2\rho_J \rangle$.

Condition (2) is equivalent to

(2') $wr_\alpha \xleftarrow{\alpha} w$ is a quantum edge in $QB(W)$ and $wr_\alpha t_\alpha^\vee \in (W^J)_af$.

This equivalence may be deduced from [27, Lemma 10.14] and the proof of [27, Theorem 10.18]. The arguments there rely on geometry, namely, the quantum Chevalley rule and the Peterson-Woodward comparison theorem. An example of a PQBG is given in Figure 2.

We define the *weight* of an edge $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$ in the PQBG to be either $\alpha^\vee$ or 0, depending on whether it is a quantum edge or not, respectively. Then the weight of a directed path $p$, denoted by $\text{wt}(p)$, is defined as the sum of the weights of its edges.

4.4. **Duality anti-automorphism of $QB(W^J)$.** Let $w_0 \in W$ be the longest element. There is an involution on $W$ defined by $w \mapsto w_0w$. It reverses length in that $\ell(w_0w) = \ell(w) - \ell(w)$. It also reverses Bruhat order in $W$: $v < w$ if and only if $w_0v > w_0w$. The map $w \mapsto w_0w$ also has the same properties. In particular $w \mapsto w^* = w_0ww_0$ is a group automorphism of $W$ which preserves length. Define the involution $*$ on the Dynkin diagram $I$ by $w_0r_iw_0 = r_{i^*}^{}$ or equivalently $w_0\alpha_i = -\alpha_{i^*}$. Then $*$ is an automorphism of $I$. The map $w \mapsto w^*$ can be computed on reduced words by replacing each $r_i$ by $r_{i^*}$.

Define the map $w \mapsto w^\circ$ on $W^J$ by $w^\circ = \lfloor w_0w \rfloor$. Let $w_0^J \in W_J$ be the longest element.

**Proposition 4.3.** The map $w \mapsto w^\circ$ is an involution on $W^J$ such that

1. $w^\circ = w_0^Jw_0^J$,
2. $\ell(w^\circ) = \ell(w_0) - \ell(w_0^J) - \ell(w) = |\Phi^+ \setminus \Phi^+_J| - \ell(w)$.
(3) $v \overset{\beta}{\leftrightarrow} w$ is an edge in QB($W^J$) if and only if $w^o \overset{\mu}{\leftrightarrow} v^o$ is an edge in QB($W^J$). Moreover both edges are Bruhat or both are quantum.

In particular this involution reverses arrows in QB($W^J$) and preserves whether an arrow is quantum or not.

**Proof.** For $\alpha \in \Phi^+_J$ we have $w_0^J\alpha \in \Phi^-_J$. Since $w \in W^J$, $ww_0^J\alpha \in \Phi^-$. Then $w_0^Jww_0^J\alpha \in \Phi^+$. Therefore $w_0^Jww_0^J \in W^J$ and $w^o = w_0^Jww_0^J$. This implies (1).

Since elements of $W^J$ permute $\Phi^+_J$, Inv($w$) and Inv($w^o$) are subsets of $\Phi^+ \setminus \Phi^+_J$. Note that for $\alpha \in \Phi^+ \setminus \Phi^+_J$, $\alpha \in$ Inv($w$) if and only if $w\alpha \in \Phi^-$, if and only if $w_0^Jww_0^J\alpha \in \Phi^+$, and if and only if $w_0^J\alpha \in \Phi^+ \setminus \Phi^+_J \setminus$ Inv($w^o$). Therefore the map $\alpha \mapsto w_0^J\alpha$ defines a bijection from Inv($w$) to $(\Phi^+ \setminus \Phi^+_J) \setminus$ Inv($w^o$). This implies (2).

Let $v = wr_\beta$. Then $w^o = w_0^Jww_0^J = w_0^Jww_0^Jr_\beta w_0^J = w^o r_{w_0^J\beta}$, that is, $w^o = w^o r_{w_0^J\beta}$. Since $\beta \in \Phi^+ \setminus \Phi^+_J$, $w_0^J\beta \in \Phi^+ \setminus \Phi^+_J$. Let $\chi$ be 0 or 1 according as the edge $v \overset{\beta}{\leftrightarrow} w$ is Bruhat or quantum. By (2) we have

$$\ell(w^o) = |\Phi^+ \setminus \Phi^+_J| - \ell(w)$$

$$= |\Phi^+ \setminus \Phi^+_J| - (\ell(v) - 1 + \chi(\beta^\vee, 2\rho - 2\rho_J))$$

$$= \ell(v^o) + 1 - \chi(\beta^\vee, 2\rho - 2\rho_J)$$

where the last equality holds by Lemma 4.4. This proves the existence of the required arrow in QB($W^J$).

**Lemma 4.4.** For any $z \in W_J$,

$$z(2\rho - 2\rho_J) = 2\rho - 2\rho_J.$$  

**Proof.** $z \in W_J$ permutes the set $\Phi^+ \setminus \Phi^+_J$, whose sum is $2\rho - 2\rho_J$. 

5. Quantum Bruhat graph and the affine Bruhat order

In this section we consider the lift of the PQBG to the Bruhat order of the affine Weyl group (see Theorem 5.2). This is used in Section 5.5 to establish the Diamond Lemmas for the PQBG.

5.1. Regular case. The following result is [27 Proposition 4.4].

**Proposition 5.1.** Let $\mu \in Q^\vee$ be superantidominant and let $x = wt_{\mu}$ with $w, v \in W$. Then $y = x\tau_{\nu a} < x$ if and only if one of the following hold.

1. $\ell(wv) = \ell(wv\alpha) - 1$ and $n = \langle \mu, \alpha \rangle$, giving $y = wv\tau_{\nu a}v\mu$.
2. $\ell(wv) = \ell(wv\alpha) - 1 + \langle \alpha^\vee, 2\rho \rangle$ and $n = 1 + \langle \mu, \alpha \rangle$, giving $y = wv\tau_{\nu a}(\mu + \alpha^\vee)$.
3. $\ell(v) = \ell(v\alpha) + 1$ and $n = 0$, giving $y = wv\alpha\tau_{\nu a}\mu$.
4. $\ell(v) = \ell(v\alpha) + 1 - \langle \alpha^\vee, 2\rho \rangle$ and $n = -1$ giving $y = wv\alpha\tau_{\nu a}(\mu + \alpha^\vee)$.

Note that if we impose the condition that both $x$ and $y$ are in $W^-_J$ then $v = id$ and only Cases (1) and (2) apply.

5.2. Embeddings QB($W^J$) $\hookrightarrow W^-_af$. We shall give a parabolic analogue (Theorem 5.2 below) of Proposition 5.1 for $W^-_af$. Theorem 5.2 is proved in the same manner as Proposition 5.1 but the latter cannot be directly invoked to prove the former, since $J$-superantidominance does not imply superantidominance.

Let $\Omega_J \subset W^-_af$ be the subset of elements of the form $w_{\pi J}(t_\mu)$ with $w \in W^J$ and $\mu \in Q^\vee$ strictly $J$-antidominant (see (3.15)) and $J$-adjusted. Define $\Omega^\infty_J$ similarly but with strict $J$-antidominance replaced by $J$-superantidominance. We have $\Omega^\infty_J \subset (W^J)_af \cap W^-_af$. Impose the Bruhat covers in $\Omega^\infty_J$
whenever the connecting root has classical part in \( \Phi \setminus \Phi_f \). Then \( \Omega_f^\infty \) is a subposet of the Bruhat poset \( W_{af} \).

**Theorem 5.2.** Every edge in \( \text{QB}(W J) \) lifts to a downward Bruhat cover in \( \Omega_f^\infty \), and every cover in \( \Omega_f^\infty \) projects to an edge in \( \text{QB}(W J) \). More precisely:

1. For any edge \( [w \alpha] \overset{\alpha}{\rightarrow} w \) in \( \text{QB}(W J) \), \( z \in \Sigma_J \) (see (3.9)), and \( \mu \in Q^\vee \) that is \( J \)-superantidominant and \( J \)-adjusted with \( z = z_\mu \) (which exists by Lemma 3.15), there is a covering relation \( y \prec x \) in \( \Omega_f^\infty \) where

\[
x = wzt_\mu, \quad y = xr_\alpha, \quad \alpha = z^{-1} \alpha + (\chi + \langle \mu, z^{-1} \alpha \rangle) \delta \in \Phi_{af}^-,
\]

and \( \chi \) is 0 or 1 according as the arrow in \( \text{QB}(W J) \) is of Bruhat or quantum type respectively.

2. Suppose \( y \triangleleft x \) is an arbitrary covering relation in \( \Omega_f^\infty \). Then we can write \( x = wzt_\mu \) with \( w \in W^J \), \( z = z_\mu \in W_J \), and \( \mu \in Q^J \) \( J \)-superantidominant and \( J \)-adjusted, as well as \( y = x r_\gamma \) with \( \gamma = z^{-1} \alpha + n \delta \in \Phi_{af}^- \), \( \alpha \in \Phi^+ \setminus \Phi_J^+ \), and \( n \in \mathbb{Z} \). With the notation \( \chi := n - \langle \mu, z^{-1} \alpha \rangle \), we have

\[
\chi \in \{0,1\}, \quad \gamma = z^{-1} \alpha + (\chi + \langle \mu, z^{-1} \alpha \rangle) \delta \in \Phi_{af}^-;
\]

furthermore, there is an edge \( wr_\alpha z \overset{z^{-1} \alpha}{\rightarrow} wz \) in \( \text{QB}(W) \) and an edge \( [w \alpha] \overset{\alpha}{\rightarrow} w \) in \( \text{QB}(W J) \), where both edges are of Bruhat type if \( \chi = 0 \) and of quantum type if \( \chi = 1 \).

**Remark 5.3.** The affine Bruhat covering relation considered in part (2) is completely general, subject to both elements being in \( \Omega_f^\infty \) and the transition root having classical part in \( \Phi \setminus \Phi_f \).

**Proof.** (1) Since \( \alpha \in \Phi^+ \setminus \Phi_J^+ \) we have \( z^{-1} \alpha \in \Phi^+ \setminus \Phi_J^+ \). By Lemmas 3.7 and 3.16, \( x \in (W J)_{af} \cap W_{af}^{-} \). We have

\[
y = x r_\alpha = wzt_\mu r_{z^{-1} \alpha} t_{(\chi + \langle \mu, z^{-1} \alpha \rangle) \alpha^\vee}
= wr_\alpha z t_{(\chi + \langle \mu, z^{-1} \alpha \rangle) \alpha^\vee} t_\mu
= wr_\alpha z t_{\chi \alpha^\vee} z t_\mu
\]

where \( \pi_J(y) = \pi_J(wr_\alpha t_{\chi \alpha^\vee} z) \pi_J(t_\mu) = wr_\alpha t_{\chi \alpha^\vee} z t_\mu = y \)

using Proposition 3.5, the assumption on \( \mu \), and (2') of the definition of \( \text{QB}(W J) \) in the case \( \chi = 1 \). We conclude that \( y \in (W J)_{af} \). Let \( i \in I \). We have \( y \alpha_i = wz r_{z^{-1} \alpha} t_{(\chi + \langle \mu, z^{-1} \alpha \rangle) \alpha^\vee} t_{\alpha_i} \). If \( i \notin J \) then the \( J \)-superantidominance of \( \mu \) implies that \( y \alpha_i \in \Phi_{af}^+ \). Suppose \( i \in J \). Then \( \alpha_i \in \Phi_J^+ \) and \( y \alpha_i \in \Phi_{af}^+ \) by the definition of \( y \in (W J)_{af} \). We have shown that \( y \in W_{af}^{-} \). To prove \( x \triangleright y \) we need only show that \( \ell(x) - \ell(y) = 1 \). Suppose \( \chi = 0 \). Since \( y \) and \( x \) are in \( W_{af}^{-} \), by [27, Lemma 3.3] we have

\[
\ell(x) - \ell(y) = \ell(t_\mu) - \ell(wz) - \ell(t_\mu) + \ell(wr_\alpha z)
= -\ell(w) - \ell(z) + \ell(wr_\alpha) + \ell(z)
= 1.
\]

Suppose \( \chi = 1 \). We have \( x = w \pi_J(t_\mu) \) and \( y = [w \alpha] \pi_J(t_{\mu+z^{-1} \alpha^\vee}) \). By Lemma 3.13 we have

\[
\ell(x) - \ell(y) = -\ell(w) - \langle \mu, 2 \rho - 2 \rho_J \rangle + \ell([w \alpha]) + \langle \mu + z^{-1} \alpha, 2 \rho - 2 \rho_J \rangle
= 1 - \langle \alpha^\vee, 2 \rho - 2 \rho_J \rangle + \langle \alpha^\vee, z(2 \rho - 2 \rho_J) \rangle.
\]

by condition (2) of the case \( \chi = 1 \) of the arrow in \( \text{QB}(W J) \). By Lemma 4.4 it follows that \( \ell(x) - \ell(y) = 1 \) as required.
(2) Let \( n = \chi + \langle \mu, z^{-1} \alpha \rangle \) where \( \chi \in \mathbb{Z} \).
We have \( y = wr_\alpha z t_\mu + \chi z^{-1} \alpha^\vee \). Since \( y \in W_{af}^-\), \( \mu + \chi z^{-1} \alpha^\vee \) is antidominant by \cite{27} Lemma 3.3. By \cite{27} Lemma 3.2 we have

\[
1 = \ell(x) - \ell(y)
= (-\langle \mu, 2\rho \rangle - \ell(wz)) - (-\langle \mu + \chi z^{-1} \alpha^\vee, 2\rho \rangle - \ell(wzr_z^{-1} \alpha))
= \ell(wzr_z^{-1} \alpha) - \ell(wz) + \chi \langle z^{-1} \alpha^\vee, 2\rho \rangle
\]

By Lemma 4.1 we deduce that \( \chi \in \{0, 1\} \).
Suppose \( \chi = 0 \). Then \( y = wr_\alpha z t_\mu \) and \( \ell(wzr_z^{-1} \alpha) - \ell(wz) = 1 \), that is, \( wz < wzr_z^{-1} \alpha = wr_\alpha z \).
This gives the required Bruhat cover in QB(\( W \)). Since \( y \in (W^J)_{af} \) we have \( \pi_J(y) = y \) and
\[
wz \alpha t_\mu = [wz \alpha z] \pi_J(t_\mu) = [wr_\alpha z] z t_\mu \text{ using Proposition B.5.}
\]
We deduce that \( wr_\alpha \in W^J \). By length-additivity it follows that \( wr_\alpha \, \alpha \, w \) is a Bruhat arrow in QB(\( W^J \)).
Otherwise we have \( \chi = 1 \). Then \( y = wzr_z^{-1} \alpha ^\vee z t_\mu = wzr_z^{-1} \alpha \) and \( \ell(wzr_z^{-1} \alpha) = \ell(wz) + 1 - \langle z^{-1} \alpha^\vee, 2\rho \rangle \), which yields the required quantum arrow in QB(\( W \)).
Since \( y \in (W^J)_{af} \) we have
\[
wz \alpha t_\mu z = \pi_J(y) = \pi_J(wz \alpha t_\mu z) = \pi_J(wz \alpha z t_\mu z) = \pi_J(wz \alpha z t_\mu z) = \pi_J(wz \alpha z t_\mu z)
from which we deduce that \( wz \alpha \in (W^J)_{af} \) and that \( \alpha^\vee \) is \( J \)-adjusted.

By Remark 3.6 and Lemma 3.11 we have \( z_{\mu+z^{-1} \alpha^\vee} = z_{\mu+z^{-1} \alpha^\vee} = z \mu z_{\alpha^\vee} = z \alpha^\vee \).
Since \( \alpha^\vee \) is \( J \)-adjusted we have \( wz \alpha z = \pi_J(wz \alpha z) = \pi_J(wz \alpha z) = \pi_J(wz \alpha z) = \pi_J(wz \alpha z) = \pi_J(wz \alpha z) \) and the last product is length-additive. Therefore
\[
\ell(wz \alpha z) = \ell(wz \alpha z) - \ell(z_{\mu+z^{-1} \alpha^\vee})
= \ell(wz) + 1 - \langle z^{-1} \alpha^\vee, 2\rho \rangle - \ell(z_{\mu+z^{-1} \alpha^\vee})
= \ell(w) + 1 + \ell(z_{\mu}) - \ell(z_{\mu+z^{-1} \alpha^\vee}) - \langle z^{-1} \alpha^\vee, 2\rho \rangle
= \ell(w) + 1 + \langle z^{-1} \alpha^\vee, 2\rho \rangle - \langle z^{-1} \alpha^\vee, 2\rho \rangle
= \ell(w) + 1 - \langle z^{-1} \alpha^\vee, 2\rho \rangle
\]

using Lemma 3.10 that \( \mu \) and \( \mu + z^{-1} \alpha^\vee \) are \( J \)-adjusted, and Lemma 4.4. This proves the existence of the required edge in QB(\( W^J \)).

Example 5.4. Let \( g \) be of type \( A_2 \) and \( J = \{1\} \). Then QB(\( W^J \)) is given by

\[
\begin{array}{c}
\alpha_1 + \alpha_2 \\
\downarrow \alpha_2 \\
\downarrow r_2 \\
\downarrow id \\
\end{array}
\]

\[
\begin{array}{c}
r_1 r_2 \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_2 \\
\downarrow \alpha_2 \\
\end{array}
\]

\[
\begin{array}{c}
r_2 \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_1 + \alpha_2 \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_2 \\
\end{array}
\]
where the quantum arrow is dotted. In $\Omega_J \subset W_{af}$, let $\mu = -6\omega_2^\vee$ and $\nu = -3\omega_2^\vee - \theta^\vee$. We have

$$t_\mu = t_{-6\omega_2^\vee} \quad \text{and} \quad x = r_1t_\nu$$

We have a single chain running from $t_{-6\omega_2^\vee}$ down to $t_{-3\omega_2^\vee}$. The diagram is broken at $t_{-\nu}$, which appears at the bottom on the left and the top on the right. If the bottom element is removed from each side then one obtains an upside-down copy of QB($W^J$). In this case the quantum arrows transition to a different copy of QB($W^J$). The left hand copy has $z = \text{id}$ and the right hand copy has $z = r_1$ where in this situation $\Sigma_J$ is generated by $r_1$. The poset $\Omega_J^\infty$ is an infinite chain that wraps down onto the 3-cycle given by QB($W^J$) with two flavors of lifts, one for $z = \text{id}$ and the other for $z = r_1$.

**Warning:** generally not every affine cover is produced by left multiplication by a simple reflection, nor is a general quantum cover always induced by left multiplication by $r_0$ (although we shall see that left multiplication by $r_0$ always induces a quantum arrow).

We say that a walk in the directed graph QB($W^J$) is locally-shortest if any segment of the walk not containing a repeated vertex is a shortest path.

**Corollary 5.5.** Downward saturated chains in $\Omega_J^\infty$ project to locally-shortest walks in QB($W^J$). Conversely, shortest paths in QB($W^J$) are projections of downward saturated chains in $\Omega_J^\infty$.

**Proof.** Say $x_0 > x_1 > \cdots > x_N$ is a saturated Bruhat chain in $\Omega_J^\infty$. Let $\pi_J(x_i) = w_iz_it_\mu$ where $w_i \in W_J$, $z_i \in W_J$, and $\mu_i \in Q^\vee$. Then Theorem 5.2 asserts that $w_0 \to w_1 \to \cdots \to w_N$ is a locally-shortest walk in QB($W^J$).

Now let $u = u_0 \to u_1 \to \cdots \to u_N = u'$ be a shortest path in QB($W^J$). We apply Theorem 5.2 to the edge $u_0 \to u_1$ with $\mu = \mu_0$ $J$-superantidominant and $W_J$-invariant. The element $x_0 = u_0t_\mu$ lifts $u_0$ since $\pi_J(t_\mu) = t_\mu$. Then the Proposition produces a cocover $x_1 = u_1z_1t_\mu_1$ of $x_0$ with $z_1 \in \Sigma_J$. In general we have a descending Bruhat chain $x_0 > x_1 > \cdots > x_{i-1} = u_{i-1}z_{i-1}t_{\mu_{i-1}}$ with $z_{i-1} \in \Sigma_J$ and we apply the Proposition to obtain a cocover $x_i = u_iz_it_{\mu_i}$ of $x_{i-1}$ with $z_i \in \Sigma_J$ and by induction the required affine chain is produced.

**Corollary 5.6.** For each $z \in \Sigma_J$ there is a copy of QB($W^J$) inside QB($W$), embedded by $w \mapsto wz$ such that the edge label $\alpha$ is sent to the root $\alpha^{-1}$, and Bruhat and quantum edges are sent to the same kind of edge.

**Proof.** For every $z \in \Sigma_J$, we take an edge $[wr_\alpha] \xleftarrow{\alpha} w$ in QB($W^J$), lift it to $wzt_\mu \mapsto wr_\alpha zt_\nu$ for some $\nu$, and project to an edge $w_\alpha z \xleftarrow{\alpha^{-1}} wz$ in QB($W$); the lift is based on Theorem 5.2 (1), and the projection on Theorem 5.2 (2).
Remark 5.7. Lifting quantum edges causes a “phase shift” by an element \( z \in \Sigma_J \). Theorem 5.2 is just general enough to lift in the presence of such a shift. If one tries to twist by a \( z \in W_J \) that is not in \( \Sigma_J \) then the affine element of the form \( x = wz\mu \) no longer lies in the set \((W^J)_\mathrm{af}\) and lifting the edge of \( \text{QB}(W^J) \) starting from \( x \) is not possible in general.

5.3. Trichotomy of cosets.

Lemma 5.8. [5] Let \( W \) be a Weyl group, \( W_J \subset W \) a parabolic subgroup, \( v \in W^J \) and \( r \in W \) a simple reflection. Then one of the following holds.

1. If \( rv < v \) then \( rv \in W^J \) and \( rvW_J < vW_J \).
2. If \( rv > v \) and \( v^{-1}rv \in W_J \) then \( rvW_J = vW_J \).
3. If \( rv > v \) and \( v^{-1}rv \notin W_J \) then \( rv \in W^J \) and \( rvW_J > vW_J \).

Lemma 5.9. Let \( v \in W \) and \( \alpha \in \Phi^+ \). Let \( \lambda \in X \) be a dominant weight (cf. Section 3.2 and the notation thereof, e.g., \( W_J \) is the stabilizer of \( \lambda \)).

1. Let \( \langle \alpha^\vee, v\lambda \rangle < 0 \). Then \( v^{-1}\alpha \in \Phi^- \setminus \Phi^-_J \) and \( r_{\alpha \lambda}vW_J < vW_J \).
2. Let \( \langle \alpha^\vee, v\lambda \rangle = 0 \). Then \( v^{-1}\alpha \in \Phi_J \) and \( r_{\alpha \lambda}vW_J = vW_J \).
3. Let \( \langle \alpha^\vee, v\lambda \rangle > 0 \). Then \( v^{-1}\alpha \in \Phi^+ \setminus \Phi^+_J \) and \( r_{\alpha \lambda}vW_J > vW_J \).

The proof of the above lemma is easy using standard techniques for Weyl groups (see for example [2] Proposition 2.5.1).

5.4. Quantum edges induced by left multiplication by suitable reflections.

Proposition 5.10. [5] Let \( w \in W^J \) and \( j \in I \). Then exactly one of the following holds.

1. \( w^{-1}\alpha_j \in \Phi^- \setminus \Phi^-_J \). In this case \( r_jw \in W^J \) and there is a Bruhat edge \( w \leftarrow r_jw \) in \( \text{QB}(W^J) \).
2. \( w^{-1}\alpha_j \in \Phi_J \). In this case \( w^{-1}\alpha_j \in \Phi_J^+ \) and \( [r_jw] = w \).
3. \( w^{-1}\alpha_j \in \Phi^+ \setminus \Phi^+_J \). In this case \( r_jw \in W^J \) and there is a Bruhat edge \( r_jw \leftarrow w \) in \( \text{QB}(W^J) \).

Proposition 5.11. Let \( w \in W^J \). Then exactly one of the following holds.

1. \( w^{-1}\theta \in \Phi^- \setminus \Phi^-_J \). In this case there is an edge \( [r_\theta w] \leftarrow [r_\theta w] \) of quantum type in \( \text{QB}(W^J) \).
2. \( w^{-1}\theta \in \Phi_J \). In this case \( w^{-1}\theta \in \Phi_J^+ \) and \( [r_\theta w] = w \).
3. \( w^{-1}\theta \in \Phi^+ \setminus \Phi^+_J \). In this case there is an edge \( w \leftarrow [r_\theta w] \) of quantum type in \( \text{QB}(W^J) \),

where \( z \in W_J \) is defined by \( r_\theta w = [r_\theta w]z \), so that \( z = (z_{w^{-1}\theta})^{-1} \).

Proof. The three cases correspond to those in Lemma 5.9 with \( w = v \) and \( \alpha = \theta \). The conclusion in Case (2) is immediate from the definition of \( W \in W^J \). By exchanging the roles of \( w \) and \( [r_\theta w] \), it suffices to prove the existence of the edge in (1). Let \( w^{-1}\theta \in \Phi^- \setminus \Phi^-_J \).

Choose any \( \mu \in Q^\vee \) that is \( J \)-superantidominant and \( W_J \)-invariant. We have \( \pi_J(t_\mu) = t_\mu \) and \( x := wt_\mu \in W_\mathrm{af} \cap (W^J)_\mathrm{af} \). We have \( (5.1) \)

\[
\begin{align*}
x^{-1}\alpha_0 &= -w^{-1}\theta + (1 + \langle \mu, -w^{-1}\theta \rangle) \delta \in \Phi^\mathrm{af}_-,
\end{align*}
\]

since \( w^{-1}\theta \in \Phi^- \). We conclude that \( x > y := r_\theta x = r_\theta wt_\mu - w^{-1}\theta \). Since \( x \in W^-_\mathrm{af} \) it follows that \( y \in W^-_\mathrm{af} \) as well. Let \( \beta \in \Phi^\mathrm{af}_+ \). Suppose \( y\beta \in \Phi^\mathrm{af}_- \). Since \( x \in (W^J)_\mathrm{af} \) we have \( x\beta = r_\theta y\beta \in \Phi^\mathrm{af}_+ \). Since \( r_\theta \) has the sole inversion \( \alpha_0 \), it follows that \( x\beta = \alpha_0 \) or \( x^{-1}\alpha_0 = \beta \in \Phi^\mathrm{af}_- \). But this contradicts [5.1]. Therefore \( y \in (W^J)_\mathrm{af} \).

By Theorem 5.2 the required quantum edge exists in \( \text{QB}(W^J) \).
Corollary 5.12. For every $\gamma = w^{-1} \theta \in W \cap (\Phi^+ \setminus \Phi^+_J)$, $z\gamma^\vee$ is $J$-adjusted, where $z \in W_J$ is defined, as in part (3) of Proposition 5.11, by $r_\theta w = [r_\theta w]z$. In addition, if $w \in W^J$, then $z = (z_{w^{-1} \theta\vee})^{-1}$. Also, for every $\gamma = w^{-1} \theta \in W \cap (\Phi^- \setminus \Phi^-_J)$, $-z'\gamma^\vee$ is $J$-adjusted, where $z'$ is the element of $W_J$ defined by $w = [w]z'$. (Obviously, if $w \in W^J$, then $z' = 1$.)

Proof. follows from the existence of the edges in $QB(W^J)$. □

5.5. Diamond Lemmas for $QB(W^J)$. We recall the Diamond Lemma for Coxeter groups and the Bruhat order.

Lemma 5.13. [13] Let $W$ be any Coxeter group, $v, w \in W$, and $r$ a simple reflection.

(1) Suppose $v < w$, $rw < w$ and $v \neq rw$. Then $rv < v$ and $rv < rw$.

(2) Suppose $v > w$, $rw > w$ and $v \neq rw$. Then $rv > v$ and $rv > rw$.

In the following diagrams, a dotted (resp. plain) edge represents a quantum (resp. Bruhat) edge in $QB(W^J)$. We always refer to the PQBG on $W^J$. Given $w \in W$ and $\gamma \in \Phi^+$, define $z, z' \in W_J$ by

$$r_\theta[w] = [r_\theta w]z, \quad r_\theta[wr_\gamma] = [r_\theta[wr_\gamma]]z' = [r_\theta wr_\gamma]z'.$$

We are now ready to state the Diamond Lemmas for the PQBG.

Lemma 5.14. Let $\alpha$ be a simple root in $\Phi$, $\gamma \in \Phi^+ \setminus \Phi^+_J$, and $w \in W^J$. Then we have the following cases, in each of which the bottom two edges imply the top two edges in the left diagram, and the top two edges imply the bottom two edges in the right diagram. Moreover, in each diagram the weights of the two directed paths (on the left side and the right side) are congruent modulo $Q^J$.

(1) In both cases we assume $\gamma \neq w^{-1} \alpha$ and have $r_\alpha[wr_\gamma] = r_\alpha wr_\gamma = [r_\alpha wr_\gamma]$.

(2) Here we have $r_\alpha[wr_\gamma] = [r_\alpha wr_\gamma]$ in both cases.

(3) Here $z, z'$ are defined as in (5.2). In subcase (5.5) (resp. (5.6)) we assume that $(\gamma^\vee, w^{-1} \theta)$ is nonzero (resp. zero). In all cases, we have $wr_\gamma = [wr_\gamma]$.
can obtain a right diagram by labeling $w$ diagrams in (5.3), (5.4), (5.7), (5.6), (5.5), and (5.8), respectively, and vice versa. For instance, we follow that $y = \emptyset$ to an affine Bruhat cover $W$. The affine diamond is pushed down to $QB(W)$.

There the diamond is completed using the usual Diamond Lemma 5.13 for the affine Weyl group.

Proof of Lemma 5.14. By Proposition 4.3 only the left diagrams need to be established. In all cases, the bottom half of a diamond in $QB(W^J)$ is lifted to the affine Bruhat order using Theorem 5.2 (1).

Consider the left diagram in (5.4). By Theorem 5.2 (1), the quantum edge \( [ wr_\gamma ] \rightarrow w \) lifts to an affine Bruhat cover $y < x$ in $\Omega^\infty_J$ where $x = wt_\mu$, $\mu$ is $J$-superantidominant with $z_\mu = \text{id}$, $y = wr_\gamma t_{\gamma^+ + \mu} = x r_\gamma$, and $\gamma = \gamma + (1 + \langle \mu, \gamma \rangle)\delta \in \Phi^\text{af}^-$. Since $r_\alpha w > w$ and $r_\alpha w \in W^J$, it follows that $r_\alpha x < x$. Moreover this covering relation is the affine lift into $\Omega^\infty_J$ of the Bruhat edge $r_\alpha w \leftarrow w^{-1}\gamma$. The elements $r_\alpha x$ and $y$ are distinct since they have different translation components. By the Diamond Lemma 5.13 for the affine Weyl group, we have $r_\alpha x > r_\alpha y$ and $y > r_\alpha y$. The latter cover implies that $r_\alpha y \in \Omega^\infty_J$. Theorem 5.2 (2) yields the top half of the left diagram in (5.4).

Consider the bottom half of the left diagram in (5.5) (which is also the same half diagram in (5.6)). The quantum edge \( [ r_\theta w ] \rightarrow w^{-1}\theta \) lifts to the affine cover in $\Omega^\infty_J$ given by $r_0 x < x$ where $x = wt_\mu$ and $r_0 x = r_\theta w t_{\mu - w^{-1}\theta}$. The Bruhat edge $wr_\gamma \leftarrow w$ lifts to the affine cover in $\Omega^\infty_J$ given by $wr_\gamma t_\mu = x r_\gamma < wt_\mu$ where $\gamma = \gamma + \langle \mu, \gamma \rangle\delta$. One may verify that $r_0 x \neq x r_\gamma$. By the Diamond Lemma 5.13 for the affine Weyl group, we have $r_0 x > r_0 x r_\gamma$ and $x r_\gamma > r_0 x r_\gamma$. Arguing as in the proof of Proposition 5.11 and using that $x r_\gamma \in \Omega^\infty_J$, one may show that $r_0 x r_\gamma \in \Omega^\infty_J$. By Theorem 5.2 (2)
we obtain edges in QB($W^J$) which complete the diamond, with the only remaining issue being the type of the edge $[r_0w] \rightarrow [r_0wr_\gamma]$. It is quantum or Bruhat depending on whether the translation elements in the affine lift $r_0x \geq r_0x r_\gamma$ are different or the same. Since $r_0x r_\gamma = r_\thetawr_\gamma t_{\mu - r_\gamma w = 1(\theta^\vee)}$ we see that the translation element changes in passing from $r_0x$ to $r_0x r_\gamma$ if and only if $\langle w^{-1}\theta^\vee, \gamma \rangle \neq 0$, as required.

The cases for the diagrams (5.7) and (5.8) are similar to those for (5.5) and (5.6).

Let us now prove the congruence of the weights of the two paths, by focusing, as an example, on the left diagram in (5.5); the proofs for the other diagrams are similar. The weight of the directed path on the left side of the mentioned diagram is equal to $-w^{-1}\theta^\vee + z\gamma^\vee$, and hence congruent to $-w^{-1}\theta^\vee + \gamma^\vee \mod Q^\vee_J$.

The weight of the directed path on the right side of the diagram is equal to $-[wr_\gamma]^{-1}\theta^\vee$, and hence is congruent to

$$-r_\gamma w^{-1}\theta^\vee = -w^{-1}\theta^\vee + \langle w^{-1}\theta^\vee, \gamma \rangle \gamma^\vee \mod Q^\vee_J.$$ 

Because $w \rightarrow [wr_\gamma]$ is a Bruhat edge, we see that $w\gamma$ is a positive root. Notice that $w\gamma \neq \theta$ since $w^{-1}\theta$ is a negative root. Also, since $\langle \gamma^\vee, w^{-1}\theta \rangle \neq 0$ by the assumption of part (3), we see that $\langle w^{-1}\theta^\vee, \gamma \rangle \neq 0$. Recalling the well-known fact that $\langle \theta^\vee, \beta \rangle$ can only be 0 or 1 for each $\beta \in \Phi^+$ with $\beta \neq \theta$, it follows that $\langle w^{-1}\theta^\vee, \gamma \rangle = \langle \theta^\vee, w\gamma \rangle = 1$. Therefore we obtain

$$-w^{-1}\theta^\vee + \langle w^{-1}\theta^\vee, \gamma \rangle \gamma^\vee = -w^{-1}\theta^\vee + \gamma^\vee,$$

as desired. \hfill \Box

6. **Quantum Bruhat graph and the level-zero weight poset**

In [20], Littelmann introduced a poset related to Lakshmibai–Seshadri (LS) paths for arbitrary (not necessarily dominant) integral weights. We consider this poset for level-zero weights. Littelmann did not give a precise local description of it. Our main result in this section is a characterization of its cover relations in terms of the PQBG.

6.1. **The level-zero weight poset.** Fix a dominant weight \(\lambda\) in the finite weight lattice \(X\) (cf. Section 3.2 and the notation thereof, e.g., \(W_J\) is the stabilizer of \(\lambda\)). We view \(X\) as a sublattice of \(X^0_{af}\). Let \(X^0_{af}(\lambda)\) be the orbit of \(\lambda\) under the action of the affine Weyl group \(W_{af}\).

**Definition 6.1. (Level-zero weight poset [20])** A poset structure is defined on \(X^0_{af}(\lambda)\) as the transitive closure of the relation

\[
\mu < r_\beta \mu \iff \langle \beta^\vee, \mu \rangle > 0,
\]

where $\beta \in \Phi^0_{af}$. This poset is called the level-zero weight poset for \(\lambda\).

**Remarks 6.2.**

1. Assume that \(W_J\) is trivial, and we set $\mu = w\lambda$ for $w \in W_{af}$. Then, for $\beta \in \Phi^0_{af}$, we have $\mu < r_\beta \mu$ in the level-zero weight poset if and only if $w^{-1}r_\beta \prec w^{-1}$ in the generic Bruhat order $\prec$ on $W_{af}$ introduced by Lusztig [30]. Indeed, this equivalence follows from the definitions of these partial orders by using [44, Claim 4.14, page 96]. The generic Bruhat order also recently appeared in [19].

2. We can define the poset $X^0_{af}(-\lambda)$ on the orbit of the antidominant weight $-\lambda$ in the same way, using (6.1). The posets $X^0_{af}(\lambda)$ and $X^0_{af}(-\lambda)$ are dual isomorphic, in the sense that, for $\mu, \nu \in X^0_{af}(\lambda)$, we have

$$\mu < \nu \iff -\mu > -\nu.$$ 

Therefore, all the statements in this section can be easily rephrased for $X^0_{af}(-\lambda)$. 

An example of $X_0^0(\lambda)$ is given in Figure 6.1. As we can see from this example, $X_0^0(\lambda)$ is not a graded poset in general.

Littelmann [20] introduced a distance function on the level-zero weight poset. Namely, if $\mu \leq \nu$ in $X_0^0(\lambda)$, then $\text{dist}(\mu, \nu)$ is the maximum length of a chain from $\mu$ to $\nu$. Clearly, covers correspond to elements at distance 1.

**Lemma 6.3.** [20] Lemma 4.1] Let $\mu, \nu \in X_0^0(\lambda)$.

1. If $\mu \leq \nu$ and $\alpha$ is a simple root in $\Phi^0$ such that $\langle \alpha^\vee, \mu \rangle \geq 0$ but $\langle \alpha^\vee, \nu \rangle < 0$, then $\mu \leq r_\alpha \nu$ and $\text{dist}(\mu, r_\alpha \nu) < \text{dist}(\mu, \nu)$.

2. If $\mu \leq \nu$ and $\alpha$ is a simple root in $\Phi^0$ such that $\langle \alpha^\vee, \mu \rangle > 0$ but $\langle \alpha^\vee, \nu \rangle \leq 0$, then $\nu = r_\alpha \nu$ and $\text{dist}(r_\alpha \mu, \nu) < \text{dist}(\mu, \nu)$.

3. If $\mu \leq \nu$ and $\alpha$ is a simple root in $\Phi^0$ such that $\langle \alpha^\vee, \mu \rangle, \langle \alpha^\vee, \nu \rangle > 0$ (respectively $\langle \alpha^\vee, \mu \rangle, \langle \alpha^\vee, \nu \rangle < 0$), then $\text{dist}(\mu, \nu) = \text{dist}(r_\alpha \mu, r_\alpha \nu)$.

We label a cover $\mu \lessdot \nu = r_\beta \mu$ of $X_0^0(\lambda)$ by the corresponding positive real root $\beta$. Preliminary results about the covers of $X_0^0(\lambda)$ were obtained by Naito and Sagaki.

**Lemma 6.4.**

1. [35] Remark 2.10 and Lemma 2.11] For untwisted types, a necessary condition for $\mu \lessdot \nu$ to be a cover in $X_0^0(\lambda)$ is that $\nu = r_\beta \mu$ with $\beta \in \Phi^+$ or $\beta \in \delta - \Phi^+$.

2. [35] Remark 2.10 (2)] Let $\mu, \nu \in X_0^0(\lambda)$ be such that $\nu = r_\alpha \mu$ for a simple root $\alpha$ in $\Phi^0$ such that $\langle \alpha^\vee, \mu \rangle > 0$. Then $\text{dist}(\mu, \nu) = 1$.

We consider the standard projection map $\text{cl}$ from $X_0^0(\lambda)$ to the orbit of $\lambda$ under the finite Weyl group (by factoring out the $\delta$ part). We identify $W \lambda \simeq W/W_J \simeq W^J$, and consider on $W^J$ the PQBG structure. Note that, by contrast with $X_0^0(\lambda)$, the edges of the latter are labeled by positive roots $\gamma \in \Phi^+$ (of the finite root system) corresponding to right multiplication by $r_\gamma$. We use solid arrows to denote covers in the Bruhat order, whereas dotted arrows denote quantum edges in the PQBG on $W^J$.

Our main result is that the level-zero weight poset is an affine lift of the corresponding parabolic quantum Bruhat graph. This is illustrated in Figure 6.1, where the edges of the (parabolic) Bruhat graph (i.e., the slice of the level-zero weight poset with no $\delta$, onto which we project) are shown in red. Projecting all vertices onto the red part, one obtains the QBG of Figure 6.1.

**Theorem 6.5.** Let $\mu \in X_0^0(\lambda)$ and $w := \text{cl}(\mu) \in W^J$. If $\mu \lessdot \nu$ is a cover in $X_0^0(\lambda)$ labeled by $\beta \in \Phi^0$, then $w \rightarrow \text{cl}(\nu)$ is a Bruhat (respectively quantum) edge in the PQBG on $W^J$ labeled by $w^{-1}\beta \in \Phi^+ \setminus \Phi^+_J$ (respectively $w^{-1}(\beta - \delta)$), depending on $\beta \in \Phi^+$ (respectively $\beta \in \delta - \Phi^+$).

Conversely, if $w \xrightarrow{\gamma} \text{wr}_\gamma = w'$ (respectively $w \cdots \xrightarrow{\gamma^0} \text{wr}_\gamma = w'$) in the PQBG for $\gamma \in \Phi^+ \setminus \Phi^+_J$, then there exists a cover $\mu \lessdot \nu$ in $X_0^0(\lambda)$ labeled by $w_\gamma$ (respectively $\delta + w_\gamma$) with $\text{cl}(\nu) = w'$.

The proof of Theorem 6.5 is given in the remainder of this section.

### 6.2. Outline of the proof

Let us begin by giving a brief outline of the proof. To relate the cover relations in the level-zero weight poset $X_0^0(\lambda)$ and the edges in the PQBG, we use the so-called Diamond Lemma on $X_0^0(\lambda)$ to successively move a cover $\mu \lessdot r_\beta \mu$ “closer” to a cover $\mu \lessdot r_\alpha \mu$ for a simple root $\alpha$ in $\Phi^0$. For simple roots, the statement of Theorem 6.5 is proved in Section 6.3.

The Diamond Lemma in the level-zero weight poset is the subject of Section 6.4. Recall that the Diamond Lemmas for the PQBG were treated in Section 5.5. In Section 6.5 we prove some further statements related to the Diamond Lemmas for the PQBG that we need for our arguments. We conclude in Section 6.6 with the main argument, based on matching the diamond reductions in the level-zero weight poset with those in the PQBG.

---

1The notation in [20] is $\text{dist}(\nu, \mu)$. 
6.3. Results for simple roots. In this section, we characterize a cover relation $\mu \prec \nu$ in $X^0_{af}(\lambda)$ when $\mu$ and $\nu$ are related by an affine simple reflection.

We start with a simple lemma. Since some versions of it will be needed beyond this section, we collect all of them here.

**Lemma 6.6.** Let $\alpha$ be a simple root, $\beta$ a positive root (both in $\Phi_{af}$), and $\mu = wt_{\tau}\lambda$ with $w \in W$ and $\tau \in Q^\vee$. Let $\gamma \in \Phi^+$ be given by $\beta = \pm w\gamma + k\delta$.

1. We have
   \[ \text{cl}(\mu) = [w], \quad \text{cl}(r_{\beta}\mu) = [wr_{\gamma}]. \]

2. If $\alpha \neq \alpha_0$, assume that $r_{\alpha}[w] \in W^J$, i.e., cl($\mu$) $\neq$ cl($r_{\alpha}\mu$). Then we have
   \[ \text{cl}(r_{\alpha}\mu) = \begin{cases} r_{\alpha}[w] = [r_{\alpha}w] & \text{if } \alpha \neq \alpha_0 \\ [r_{\beta}w] & \text{if } \alpha = \alpha_0. \end{cases} \]

3. If $\alpha \neq \alpha_0$, assume that $r_{\alpha}[wr_{\gamma}] \in W^J$, i.e., cl($r_{\beta}\mu$) $\neq$ cl($r_{\alpha}r_{\beta}\mu$). Then we have
   \[ \text{cl}(r_{\alpha}r_{\beta}\mu) = \begin{cases} r_{\alpha}[wr_{\gamma}] = [r_{\alpha}[wr_{\gamma}]] & \text{if } \alpha \neq \alpha_0 \\ [r_{\beta}wr_{\gamma}] & \text{if } \alpha = \alpha_0. \end{cases} \]

**Proof.** We have
\[ \mu = wt_{\tau}\lambda = w\lambda - \langle \tau, \lambda \rangle \delta. \]
So cl($\mu$) = [w]. Similarly, we have
\[ \text{cl}(r_{\beta}\mu) = \text{cl}(r_{\gamma}t_{\pm kw_{\gamma}w}t_{\tau}(\lambda)) = [wr_{\gamma}], \quad \text{and} \quad \text{cl}(r_{\alpha}\mu) = \begin{cases} [r_{\alpha}w] & \text{if } \alpha \neq \alpha_0 \\ [r_{\beta}w] & \text{if } \alpha = \alpha_0. \end{cases} \]
In addition, if $\alpha \neq \alpha_0$, then $[r_{\alpha}w]$ can only be $[w]$ or $r_{\alpha}[w]$, by Lemma 5.8 but the first case cannot happen by the assumptions of the lemma. The calculation of cl($r_{\alpha}r_{\beta}\mu$) is similar, by also noting that, if $\alpha \neq \alpha_0$, then $[r_{\alpha}wr_{\gamma}] = [r_{\alpha}[wr_{\gamma}]].$}
PQBG on $W^J$ labeled by $w^{-1}\alpha$ (respectively $-w^{-1}\theta$), where $w = \text{cl}(\mu) \in W^J$, depending on $\alpha \neq \alpha_0$ (respectively $\alpha = \alpha_0$).

**Proof.** Since $\alpha$ is a simple root, we have by Lemma 6.4 (2) that $\text{dist}(\mu, \nu) = 1$ if $\mu < \nu$. So in this case $\mu < \nu$ is equivalent to $\mu < \nu$. Letting $\mu = w\tau(\lambda)$ with $w \in W^J$ and $\tau \in Q^\vee$, we have $\text{cl}(\mu) = w$, by Lemma 6.6 (1). Let us first assume that $\alpha \neq \alpha_0$. Then

$$\langle \alpha^\vee, \mu \rangle = \langle \alpha^\vee, w\lambda \rangle = \langle w^{-1}\alpha^\vee, \lambda \rangle,$$

where for the first equality we used (6.2). Hence

$$\mu < r_\alpha \mu \iff \langle \alpha^\vee, \mu \rangle > 0 \iff w^{-1}\alpha \in \Phi^+ \setminus \Phi^+_J \iff w < r_\alpha w \text{ in } W^J,$$

where the last equivalence is based on Lemma 5.9. The last condition is equivalent to $\text{cl}(\mu) \rightarrow \text{cl}(\nu)$ being a Bruhat edge in the PQBG, by Lemma 6.6 (2). This proves the claim for $\alpha \neq \alpha_0$.

Now assume $\alpha = \alpha_0$. Similarly to before

$$\langle \alpha^\vee, \mu \rangle = \langle \alpha^\vee, w\lambda \rangle = \langle -\theta^\vee, w\lambda \rangle = -\langle w^{-1}\theta^\vee, \lambda \rangle,$$

where we used $\alpha_0 = -\theta + \delta$, or $\alpha_0^\vee + \theta^\vee = c$. Hence

$$\mu > r_\alpha \mu \iff \langle \alpha^\vee, \mu \rangle > 0 \iff w^{-1}\theta \in \Phi^- \setminus \Phi^-_J.$$

By Proposition 5.11, the last condition is equivalent to the fact that $w \cdot w^{-1}\theta \cdot r_\theta w$ is a quantum edge in the PQBG. Also note that $\text{cl}(r_\alpha \mu) = [r_\theta w]$, by Lemma 6.6 (2). This proves the claim for $\alpha = \alpha_0$. \hfill \Box

### 6.4. The Diamond Lemma in the level-zero weight poset

In this section we investigate the Diamond Lemma in the level-zero weight poset $X^0_{af}(\lambda)$.

**Lemma 6.8.** Let $\mu \in X^0_{af}(\lambda)$ and $\mu < r_\beta \mu$ in $X^0_{af}(\lambda)$, where $\beta \in \Phi_{af}^+$. Then there exists a simple root $\alpha$ in $\Phi_{af}^+$ such that $\langle \alpha^\vee, \beta \rangle > 0$, and either

1. $\mu < r_\alpha \mu$ or
2. $r_\alpha r_\beta \mu < r_\beta \mu$

is a cover in $X^0_{af}(\lambda)$.

**Proof.** We pick a simple root $\alpha$ in the decomposition of $\beta$ such that $\langle \alpha^\vee, \beta \rangle > 0$. This clearly exists, and in fact $\alpha \neq \alpha_0$ if $\beta \in \Phi$.

By Definition 6.1, we have $\langle \beta^\vee, \mu \rangle > 0$. We claim that either

$$\langle \alpha^\vee, \mu \rangle > 0 \quad \text{or} \quad \langle -r_\beta \alpha^\vee, \mu \rangle = -\langle \alpha^\vee, r_\beta \mu \rangle > 0.$$

Indeed, the reflection formula

$$r_\beta \alpha^\vee = \alpha^\vee - \langle \alpha^\vee, \beta \rangle \beta^\vee$$

implies that $\alpha^\vee - r_\beta \alpha^\vee$ is a positive multiple of $\beta^\vee$. Now (6.3) follows since $\langle \beta^\vee, \mu \rangle > 0$. We conclude the proof by combining (6.3) with Lemma 6.4 (2). \hfill \Box

Next we state the Diamond Lemma for the level-zero weight poset.

**Lemma 6.9.** Let $\alpha$ be a simple root, $\beta \neq \alpha$ a positive root (both in $\Phi_{af}^+$), and $\mu \in X^0_{af}(\lambda)$. In the left diagram, the bottom two covers imply the top two covers, while the top two covers imply the bottom two covers in the right diagram.

![Diagram](Image)

(6.4)
Remark 6.11

1 = \text{dist}(z, z)

More on the Diamond Lemmas for the PQBG.

Lemma 6.5. Note that

Proof.

22 C. LENART, S. NAITO, D. SAGAKI, A. SCHILLING, AND M. SHIMOZONO

Let

for the left one, and then Lemma 6.3 (3) for the right one.

\[ \nu := \nu \]

Proof.

Turning to the remaining edge of the diamond, we clearly have

\[ r_\alpha \nu = r_\alpha \beta (r_\alpha \mu) \quad \text{and} \quad \langle r_\alpha \beta \nu, r_\alpha \mu \rangle = \langle \beta \nu, \mu \rangle > 0; \]

note that \( r_\alpha \beta \) is a positive root, as \( \alpha \neq \beta \). The hypotheses of Lemma 6.3 (3) apply, so we have

1 = \text{dist}(r_\alpha \mu, \nu) = \text{dist}(r_\alpha \mu, r_\alpha \nu). \]

We conclude that we have the cover \( r_\alpha \mu \prec r_\alpha \nu \).

The proof for the right diagram is similar, where we now assume that the top two arrows are covers. More precisely, in order to prove that the bottom arrows are covers, we use Lemma 6.3 (1) for the left one, and then Lemma 6.3 (3) for the right one.

6.5. More on the Diamond Lemmas for the PQBG. Recall the Diamond Lemmas for the PQBG on \( W_J \) from Section 5.5. Recall that, given \( w \in W \) and \( \gamma \in \Phi^+ \), in (5.2) we defined

\[ r_\theta[w] = [r_\theta w] z, \quad r_\theta[w \gamma] = [r_\theta[w \gamma]] z'. \]

We need an analogue of Lemma 6.8.

Lemma 6.10. Let \( w \in W \), and let \( \gamma \in \Phi^+ \setminus \Phi_f^+ \). Define \( \beta \in \Phi^{af+} \) by

\[
(6.5) \quad \beta := \begin{cases} 
\gamma & \text{if } w \gamma \in \Phi^+ \\
\delta + \gamma & \text{if } w \gamma \in \Phi^-.
\end{cases}
\]

There exists an affine simple root \( \alpha \) (in fact, \( \alpha \neq \alpha_0 \) if \( w \gamma \in \Phi^+ \)) such that \( \langle \alpha \gamma, \beta \rangle > 0 \), and we have the edge in the PQBG indicated either in case (1) or (2) below:

\[
(1) \quad \begin{cases} 
[w] \overset{[w]^{-1}}{\longrightarrow} \overset{r_\alpha[w]}{\longrightarrow} \text{if } \alpha \neq \alpha_0 \\
[w] \overset{[w]^{-1}}{\longrightarrow} \overset{r_\theta[w]}{\longrightarrow} \text{if } \alpha = \alpha_0,
\end{cases}
\]

\[
(2) \quad \begin{cases} 
\overset{r_\alpha[w \gamma]}{\longrightarrow} \overset{[w \gamma]^{-1}}{\longrightarrow} \overset{r_\alpha[w \gamma]}{\longrightarrow} \text{if } \alpha \neq \alpha_0 \\
[r_\theta[w \gamma]] \overset{[w \gamma]^{-1}}{\longrightarrow} \overset{[w \gamma]}{\longrightarrow} \text{if } \alpha = \alpha_0.
\end{cases}
\]

Remark 6.11. If \( w \overset{\gamma}{\longrightarrow} [w \gamma] \), then \( w \gamma \in \Phi^- \) for the following reason. Observe that \( \ell([w \gamma]) \leq \ell(w) - 1 \) since \( \gamma \in \Phi^+ \setminus \Phi_f^+ \). Now suppose, by contradiction, that \( w \gamma \in \Phi^+ \). Then, we have \( w \gamma > w \) in the usual Bruhat order on \( W \). Therefore, by [2 Proposition 2.5.1], we obtain \( |w \gamma| \geq |w| = w \), which implies that \( \ell([w \gamma]) \geq \ell(w) \). This is a contradiction. This proves that \( w \gamma \in \Phi^- \).

Proof. Let \( \mu := w \lambda \), where \( \lambda \in X_0^{af} \) is the fixed dominant element in the finite weight lattice whose stabilizer is \( W_J \). We claim that \( \mu > r_\beta \mu \in X_0^{af}(\lambda) \), which means that \( \langle \beta \gamma, \mu \rangle > 0 \). Indeed, since \( \gamma \in \Phi^+ \setminus \Phi_f^+ \), it follows from (6.5) that in both cases we have

\[ \langle \beta \gamma, \mu \rangle = \langle w \gamma \gamma, w \lambda \rangle = \langle \gamma \gamma, \lambda \rangle > 0. \]

We now apply Lemma 6.8 to deduce the existence of a simple root \( \alpha \) in \( \Phi^{af} \) (in fact, \( \alpha \neq \alpha_0 \) if \( w \overset{\gamma}{\longrightarrow} w \gamma \)) such that \( \langle \alpha \gamma, \beta \rangle > 0 \), and either

\[ \begin{align*}
(1) & \quad \mu \prec r_\alpha \mu \quad \text{or} \quad (2) \quad r_\alpha r_\beta \mu \prec r_\beta \mu
\end{align*} \]

in \( X_0^{af}(\lambda) \). By Lemma 6.7 and Lemma 6.6, cases (1) and (2) can be rephrased as cases (1) and (2) in the lemma to be proved, respectively.

Note that we do not need all the cases of the diamond Lemma 5.14 for the PQBG, for instance the one where all four edges are quantum edges. By stating that we have a certain edge in the PQBG, we implicitly assume that both its vertices are in \( W_J \).
6.6. Main argument. We address separately the direct (⇒) and the converse (⇐) statements. Recall that the height of a root is the sum of the coefficients in its expansion in the basis of simple roots.

Proof of (⇒) in Theorem 6.5. Consider the cover µ < ν = rβµ in X_{af}^0(λ) labeled by β, and let w := cl(µ). We proceed by induction on the height of β. If β is a simple root, the conclusion follows directly from Lemma 6.7. If β is not a simple root, we apply Lemma 6.8; this gives an affine simple root α ≠ β with ⟨α, β⟩ > 0, which also satisfies condition (1) or (2) in the mentioned lemma. Depending on these two cases, by Lemma 6.9, we have one of the two diamonds in (6.4) (in X_{af}^0(λ)). Let β := rαβ. We will need the fact that β and β' are in Φ⁺ or δ − Φ⁺ (not necessarily both in the same set), by Lemma 6.4 (1).

Assume that we have the left diamond in (6.4), as the reasoning is completely similar for the right diamond (we simply interchange the statements of the form “bottom implies top” and “top implies bottom” provided by Lemmas 6.9 and 5.14). Lemma 6.7 tells us that, by projecting its edges pointing northwest (labeled by the simple root α) via the map cl, we obtain two Bruhat edges or two quantum edges in the PQBG (depending on α ≠ α₀ or α = α₀, respectively). Moreover, by Lemma 6.6, the four vertices of the projected diamond and its top left edge are labeled as in left diamond in (5.3) (or (5.4), which has the same labels), and (5.7), respectively, where γ is defined as in Lemma 6.6: indeed, if γ' is defined with respect to β' and rαw as γ is defined with respect to β and w in Lemma 6.6, then γ' = γ in the first case, and γ' = z(γ) in the second case. Since ⟨α, β⟩ > 0, the height of β' is strictly smaller than the height of β; so by induction we know that the top left edge of the projected diamond is a Bruhat or quantum edge in the PQBG, depending on β' ∈ Φ⁺ or β' ∈ δ − Φ⁺, respectively.

By Lemma 6.8 we have one of the following three cases:

(6.6) \( β ∈ Φ⁺, α ≠ α₀ \), \( β ∈ δ − Φ⁺, α ≠ α₀ \), \( β ∈ δ − Φ⁺, α = α₀ \).

By calculating β' = rαβ, we deduce that, in the mentioned three cases, we have

(6.7) \( β' ∈ Φ⁺ \), \( β' ∈ δ − Φ⁺ \), \( β' ∈ Φ⁺ \),

respectively. For the last computation, let β = δ − θβ̄ and write

(6.8) \( β' = rα(δ − θβ̄) = rαt − t − (δ − θβ̄) = −rαθβ̄ + (1 − ⟨θγ, θβ̄⟩)θβ̄ \).

here the coefficient of δ needs to be 0 or 1, as noted above, but the second case cannot happen since

(6.9) \( ⟨θγ, θβ̄⟩ = ⟨αγ, β⟩ ≠ 0 \).

Hence, in the three cases in (6.6) and (6.7), the top two edges of the projected diamond (and their vertices) are as in the left diamonds in (5.3), (5.4), and (5.5), respectively. By Remark 5.15, these three diamonds coincide, up to relabeling, with the right diamonds in (5.3), (5.4), and (5.5), respectively. Therefore, we can apply the statements in Lemma 5.14 associated with the latter diamonds (stating that their top two edges imply their bottom two edges) to deduce that the condition of the edge µ < ν is as claimed, namely a Bruhat edge in the first case, and a quantum edge in the last two cases (in the PQBG). Note that the condition γ ≠ |w⁻¹α| needed in the first case is satisfied since β = |wγ| in this case and β ≠ α; here |α| = ±α depending on whether α is positive or negative. In addition, the condition ⟨γγ, w⁻¹θ⟩ = ⟨wγγ, θ⟩ ≠ 0 needed in the third case is precisely (6.9). This concludes the induction step.

Now let us turn to the converse statement.

Proof of (⇐) in Theorem 6.5. Assume that cl(µ) = w and we have the edge in the PQBG w ⇒ wrγ = w' or w ⇀ [wrγ] = w'. Defining β as in (6.5), we claim that ν := rβµ satisfies the conditions in the theorem. Indeed, note first that cl(ν) = w', by Lemma 6.6 (1). We now proceed by induction
on the height of $\beta$. If $\beta$ is an affine simple root, the conclusion follows directly from Lemma \[6.14\].

If $\beta$ is not a simple root, we apply Lemma \[6.10\] this gives an affine simple root $\alpha \neq \beta$ satisfying $\langle \alpha^\vee, \beta \rangle > 0$ and either condition (1) or (2) in the mentioned lemma. Assume that condition (1) holds, as the reasoning is completely similar if condition (2) holds (we simply interchange the statements of the form "bottom implies top" and "top implies bottom" provided by Lemmas \[5.14\] and \[6.9\]).

By Lemma \[6.10\] we have one of the following three cases:

\[(\beta \in \Phi^+, \alpha \neq \alpha_0), \quad (\beta \in \delta - \Phi^+ \setminus \Phi_f^+, \alpha \neq \alpha_0), \quad (\beta \in \delta - \Phi^+, \alpha = \alpha_0).\]

By Lemma \[5.14\] we have the left diamonds in \[5.3\], \[5.4\], and \[5.7\], respectively. Note that the conditions $\gamma \neq w^{-1} \alpha$ and $\gamma \neq -w^{-1} \theta$ needed in the first and third cases, respectively, are satisfied since $\beta \neq \alpha$, where we recall the definition of $\beta$ in \[6.5\]; in addition, the condition $\langle \gamma^\vee, w^{-1} \theta \rangle \neq 0$ needed in the third case follows from $\langle \alpha^\vee, \beta \rangle > 0$, cf. \[6.9\] above. Let $\beta'$ be defined as in \[6.5\] for the top left edge of these diamonds. It is not hard to check that in all cases $\beta' = r_\alpha \beta$. For instance, letting $\beta = \delta - \beta$ the third case (where $\beta = -w \gamma \in \Phi^+$), we have

$$\beta' = [r_\theta w] z(\gamma) = r_\theta w \gamma = -r_\theta \beta = r_\alpha (\delta - \beta);$$

here the last equality follows from \[6.8\] and \[6.9\] above, as well as the well-known fact that $\langle \theta^\vee, \beta \rangle$ can only be 0 or 1 if $\beta \neq \theta$ (which is clearly true).

Since $\langle \alpha^\vee, \beta \rangle > 0$, the height of $\beta'$ is strictly smaller than the height of $\beta$. Therefore, we can use induction (together with the calculation of $\text{cl}(r_\alpha \mu)$ from Lemma \[6.6\] (2)) to deduce that we have a cover

$$r_\alpha \mu \xrightarrow{\beta'} r_{\beta'} r_\alpha \mu = r_\alpha r_{\beta} \mu$$

in $X^0_\text{af}(\lambda)$. On the other hand, by Lemma \[6.6\] we can see that $r_{\beta} \mu$ and $r_\alpha r_{\beta} \mu$ project to the vertices of the top right edge of the left diamonds in \[5.3\], \[5.4\], and \[5.7\], depending on the case. Therefore, by Lemma \[6.7\] we also have the cover

$$r_{\beta} \mu \xrightarrow{\alpha} r_\alpha r_{\beta} \mu$$

in $X^0_\text{af}(\lambda)$. We now proved that we have the top two edges in the left diamond in \[6.4\]. As $\beta \neq \alpha$, we can now apply the statement of Lemma \[6.9\] corresponding to the right diamond in \[6.4\] (which is just a relabeling of the left one) to deduce that we have the cover $\mu \leq r_\beta \mu$ labeled by $\beta$ in $X^0_\text{af}(\lambda)$. This concludes the induction step. \(\square\)

### 6.7. Connectivity of the parabolic quantum Bruhat graph and quantum length.

In this subsection we show that the PQBG is strongly connected when using only simple reflections. For the QBG, this result is [14 Theorem 4.2].

We use the following notation:

$$\overline{\alpha_i} := \begin{cases} \alpha_i & \text{if } i \neq 0, \\ -\theta & \text{if } i = 0, \end{cases}, \quad s_i := \begin{cases} r_i & \text{if } i \neq 0, \\ r_\theta & \text{if } i = 0. \end{cases}$$

Also, in this subsection we do not draw quantum edges in the PQBG by dotted lines.

**Lemma 6.12.** For each $u, v \in W^J$, there exist a sequence $u = x_0, x_1, \ldots, x_n = v$ of elements of $W^J$ and a sequence $i_1, i_2, \ldots, i_n \in I_\text{af}$ such that $x_{k+1} = [s_{i_{k+1}} x_k]$ with $x_k^{-1} \overline{\alpha_{i_{k+1}}} \in \Phi^+ \setminus \Phi_f^+$ for each $0 \leq k \leq n - 1$.

**Remark 6.13.** Keep the notation in the lemma above. We see from Lemma \[6.7\] and Lemma \[6.4\] (2) that

$$u = x_0 x_0^{-1} \overline{\alpha_1} x_1 x_1^{-1} \overline{\alpha_2} \cdots x_{n-2}^{-1} \overline{\alpha_{n-1}} x_{n-1} x_{n-1}^{-1} \overline{\alpha_n} x_n = v$$
in the PQBG. In particular, the PQBG is strongly connected when using only simple reflections (i.e., for each \( u, v \in W^J \), there exists a directed path from \( u \) to \( v \) in the PQBG, where the edges correspond to multiplying on the left by simple reflections). Note that a similar result for the QBG is stated in [39] Lemma 1 (1).

We are now ready to define the notion of quantum length of an element in \( W^J \). This will be used in the proofs of the tilted Bruhat Theorem 7.1 and the generalization of Postnikov’s lemma (Proposition 8.1).

**Definition 6.14.** Let \( u \in W^J \). We see from Lemma 6.12 (with \( v = e \), where \( e \) is the identity in \( W_J \)) and Remark 6.13 that there exist a sequence \( u = x_0, x_1, \ldots, x_n = v \) of elements of \( W^J \) and a sequence \( i_1, i_2, \ldots, i_n \in I_{af} \) such that

\[
u = x_0 \xrightarrow{\alpha_{i_1}^{-1}} x_1 \xrightarrow{\alpha_{i_2}^{-1}} \cdots \xrightarrow{\alpha_{i_{n-2}}^{-1}} x_{n-1} \xrightarrow{\alpha_{i_{n-1}}^{-1}} x_n = e
\]

in the PQBG. We define the quantum length \( q\ell_J(u) \) of \( u \) to be the minimal of the length \( n \) of such sequences.

When \( J = \emptyset \), we denote the quantum length (in the QBG) by \( q\ell(u) \).

**Proof of Lemma 6.12** Let \( \lambda \) be a dominant weight such that \( \{ j \mid \langle \alpha_j^\vee, \lambda \rangle = 0 \} = J \); note that the stabilizer of \( \lambda \) in \( W \) is identical to \( W_J \), and hence \( W\lambda \cong W/W_J = W^J \). Set \( \mu := u\lambda \) and \( \nu := v\lambda \). We see from [1] Lemma 1.4 that there exists \( i_1, i_2, \ldots, i_n \in I_{af} \) such that

\[
u = s_{i_1} \cdots s_{i_n} \mu = \nu,
\]

\[
\langle \alpha_{i_{k+1}}^\vee, s_{i_k} \cdots s_{i_1} \mu \rangle > 0 \quad \text{for all } 0 \leq k \leq n - 1.
\]

For each \( 0 \leq k \leq n \), we define \( x_k \in W^J \) to be the minimal coset representative for the coset containing \( s_{i_k} \cdots s_{i_1} \); note that \( x_0 = u \) and \( x_n = v \). It is obvious that \( x_{k+1} = [s_{i_{k+1}} x_k] \) for every \( 0 \leq k \leq n - 1 \). Also, because

\[
\langle \alpha_{i_{k+1}}^\vee, x_k \lambda \rangle = \langle \alpha_{i_{k+1}}^\vee, s_{i_k} \cdots s_{i_1} \mu \rangle > 0,
\]

it follows immediately that \( x_k^{-1} \alpha_{i_{k+1}} \in \Phi^+ \setminus \Phi_J^+ \). Thus we have proved the lemma.

7. **Tilted Bruhat theorem**

7.1. **Tilted Bruhat order.** Given \( u \in W \) the \( u \)-tilted Bruhat order on \( W \) [3] is defined by \( w_1 \preceq_w w_2 \) if there is a shortest path in the quantum Bruhat graph \( QB(W) \) from \( u \) to \( w_2 \) that passes through \( w_1 \). More precisely, if we denote by \( \ell(u \Rightarrow w_2) \) the length of a shortest directed path from \( w_1 \) to \( w_2 \) in the quantum Bruhat graph \( QB(W) \), then for \( u, w_1, w_2 \in W \),

\[
w_1 \preceq_w w_2 \iff \ell(u \Rightarrow w_2) = \ell(u \Rightarrow w_1) + \ell(w_1 \Rightarrow w_2).
\]

It was shown in [3] that this is a partial order. In [27] Theorem 4.8 it was reproved by showing that \((W, \preceq_w)\) is (dual to) an induced subposet of the affine Bruhat order.

Here we prove a property of the \( u \)-tilted Bruhat order with respect to any parabolic subgroup \( W_J \subset W \) of the finite Weyl group.

**Theorem 7.1** (Tilted Bruhat Theorem). For every \( u, z \in W \) and any parabolic subgroup \( W_J \subset W \), the coset \( zW_J \) contains a unique \( \preceq_w \)-minimal element.

The tilted Bruhat theorem is a QBG analogue of the Deodhar lift [5] (see also [28] Proposition 3.1]), which states that if \( \tau \in W/W_J \) and \( v \in W \) such that \( vW_J \leq \tau \) in \( W/W_J \), then the set

\[
\{w \in W \mid v \leq w \text{ and } wW_J = \tau\}
\]

has a Bruhat-minimum.

We start by stating a weaker version of Theorem 7.1, which is easily proved.
Proposition 7.2. Fix \( u, z \in W \). There exists a unique element \( x \in zW_J \) such that the distance \( \ell(u \Rightarrow x) \) attains its minimum value.

Proposition 7.2 suffices for our main application in [29], namely for bijecting the models for KR crystals based on projected LS-path and quantum Bruhat chains. However, an explicit construction of this bijection depends on an algorithm for determining \( x = x_0 \in zW_J \) minimizing \( \ell(u \Rightarrow x) \); such an algorithm is given in the proof of Theorem 7.3. The proof of Proposition 7.2 relies on the shellability of the QBG with respect to a reflection ordering on the positive roots [6], which we now recall.

Theorem 7.3. \([3]\) Fix a reflection ordering on \( \Phi^+ \).

1. For any pair of elements \( v, w \in W \), there is a unique path from \( v \) to \( w \) in the quantum Bruhat graph \( \text{QB}(W) \) such that its sequence of edge labels is strictly increasing (resp., decreasing) with respect to the reflection ordering.
2. The path in (1) has the smallest possible length \( \ell(v \Rightarrow w) \) and is lexicographically minimal (resp., maximal) among all shortest paths from \( v \) to \( w \).

The proof of Proposition 7.2 is immediate once we have the following two easy lemmas. These are in terms of a reflection ordering whose top (also called an initial section) consists of the roots in \( \Phi^+ \setminus \Phi_J^+ \), while its bottom is a reflection ordering on \( \Phi_J^+ \). Such an order was constructed in [28, Section 4.3] in terms of a dominant weight \( \lambda \) whose stabilizer is \( W_J \). The roots in \( \Phi^+ \setminus \Phi_J^+ \) are ordered according to the lexicographic order on \( \mathbb{Q}^r \) via the injective map

\[
\alpha \mapsto \frac{1}{\langle \alpha^\vee, \lambda \rangle} (c_1, \ldots, c_r),
\]

where \( \alpha^\vee = c_1\alpha_1^\vee + \cdots + c_r\alpha_r^\vee \) expresses \( \alpha^\vee \) in the basis of simple coroots (on which we fix an order). For the roots in \( \Phi_J^+ \), we choose any reflection ordering.

Lemma 7.4. Assume that \( \ell(u \Rightarrow x) \), as a function of \( x \in zW_J \), has a minimum at \( x = x_0 \). Then the path from \( u \) to \( x_0 \) with increasing edge labels has all its labels in \( \Phi^+ \setminus \Phi_J^+ \).

Proof. The mentioned path has length \( \ell(u \Rightarrow x_0) \), by Theorem 7.3 (2). Assume that it has at least one label in \( \Phi_J^+ \). By the structure of our particular reflection ordering, all of these labels must be at the end of the path. This means that the tail of the path starting with some \( x_1 \neq x_0 \) consists entirely of elements in \( zW_J \). Since \( \ell(u \Rightarrow x_1) < \ell(u \Rightarrow x_0) \), we reached a contradiction.

Lemma 7.5. Assume that the paths with increasing edge labels from \( u \) to two elements \( x_0, x_1 \) in \( zW_J \) have all labels in \( \Phi^+ \setminus \Phi_J^+ \). Then \( x_0 = x_1 \).

Proof. Assume \( x_0 \neq x_1 \). The induced subgraph of \( \text{QB}(W) \) on \( zW_J \), to be denoted \( \text{QB}(zW_J) \), is isomorphic to \( \text{QB}(W_J) \) under the map \( w \mapsto [z]w \) for \( w \in W_J \) (this is immediate from definitions and the length-additive factorization of the elements in \( zW_J \)). Thus, by Theorem 7.3 (1), we can consider the path from \( x_0 \) to \( x_1 \) in \( \text{QB}(zW_J) \) with increasing edge labels (in \( \Phi_J^+ \)). By concatenating this path with the one from \( u \) to \( x_0 \) in the hypothesis (whose labels are in \( \Phi^+ \setminus \Phi_J^+ \)), we obtain a path with increasing edge labels from \( u \) to \( x_1 \). But this path is clearly different from the one in the hypothesis between the same vertices. This contradicts the uniqueness statement in Theorem 7.3 (1).

Proof of Proposition 7.2. This is immediate by combining Lemmas 7.4 and 7.5.

Next we prepare for the proof of the tilted Bruhat Theorem 7.1.
7.2. Preliminaries. We use the same notation for $\tilde{\alpha}_i$ and $s_i$ as in Section 6.7. In addition, we denote the identity of $W$ by $e$.

Remark 7.6. Let $w \in W$, and $i \in I_{af}$. If $w^{-1}\tilde{\alpha}_i$ is positive, then we have

$$w \xrightarrow{w^{-1}\tilde{\alpha}_i} s_i w$$

in the QBG by Theorem 6.5. Here, this arrow is a Bruhat arrow (resp., quantum arrow) if $i \neq 0$ (resp., $i = 0$).

The following lemma will be needed in the proof of the tilted Bruhat Theorem 7.1 in generalizing Postnikov’s lemma (in Section 8), as well as in our second paper. Only certain parts of the lemma are needed in each of the mentioned proofs; for instance, in this section we only need weaker versions of parts (1) and (3), and no reference to the weights of the considered paths. In the sequel, the symbol $\equiv$ means equivalence modulo $Q_j^\gamma$.

Lemma 7.7. Let $w_1, w_2 \in W^I$, and let $j \in I_{af}$. Let

$$(7.1) \quad p : w_1 = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_n} x_n = w_2$$

be a directed path from $w_1$ to $w_2$ of length $n$ in the PQBG. In addition, $\lambda$ is a dominant weight with stabilizer $W_j$.

1. If $\langle \tilde{\alpha}_j^\gamma, w_2 \lambda \rangle < 0$, and there exists $0 \leq k \leq n$ such that $\langle \tilde{\alpha}_j^\gamma, x_k \lambda \rangle \geq 0$, then there exists a directed path $p'$ from $w_1$ to $[s_j w_2]$ of length $n - 1$ in the PQBG such that

$$\text{wt}(p') \equiv \text{wt}(p) + \delta_{j,0} w_2^{-1} \tilde{\alpha}_j^\gamma.$$ 

2. If $\langle \tilde{\alpha}_j^\gamma, w_2 \lambda \rangle < 0$ and $\langle \tilde{\alpha}_j^\gamma, w_1 \lambda \rangle < 0$, then there exists a directed path $p'$ from $[s_j w_1]$ to $[s_j w_2]$ of length $n$ in the PQBG such that

$$\text{wt}(p') \equiv \text{wt}(p) - \delta_{j,0} w_1^{-1} \tilde{\alpha}_j^\gamma.$$ 

3. If $\langle \tilde{\alpha}_j^\gamma, w_1 \lambda \rangle > 0$, and there exists $0 \leq k \leq n$ such that $\langle \tilde{\alpha}_j^\gamma, x_k \lambda \rangle \leq 0$, then there exists a directed path $p'$ from $[s_j w_1]$ to $w_2$ of length $n - 1$ in the PQBG such that

$$\text{wt}(p') \equiv \text{wt}(p) - \delta_{j,0} w_1^{-1} \tilde{\alpha}_j^\gamma.$$ 

4. If $\langle \tilde{\alpha}_j^\gamma, w_1 \lambda \rangle > 0$, and $\langle \tilde{\alpha}_j^\gamma, w_2 \lambda \rangle > 0$, then there exists a directed path $p'$ from $[s_j w_1]$ to $[s_j w_2]$ of length $n$ in the PQBG such that

$$\text{wt}(p') \equiv \text{wt}(p) - \delta_{j,0} w_1^{-1} \tilde{\alpha}_j^\gamma.$$ 

5. In each of parts above, if the directed path $p$ is shortest in the PQBG, then the directed path $p'$ is also shortest in the PQBG.

Proof. We will omit the proofs of parts (3) and (4), since they are similar to those of parts (1) and (2), respectively; alternatively, we can reduce the former to the latter by using Proposition 4.3.

1. Since $\langle \tilde{\alpha}_j^\gamma, w_2 \lambda \rangle < 0$, we have $w_2^{-1} \tilde{\alpha}_j \in \Phi^- \setminus \Phi_j$. Thus it follows from Propositions 5.10 (1) and 5.11 (3) that

$$[s_j w_2]$$

$$\xrightarrow{\gamma_n} x_{n-1} \xrightarrow{\gamma_n} w_2$$

If $\langle \tilde{\alpha}_j^\gamma, x_{n-1} \lambda \rangle < 0$, then we can apply the assertion for the right diagram in Lemma 5.14 (the diamond lemma) to this diagram; choose a suitable right diagram in Lemma 5.14 depending on
the types (Bruhat or quantum) of the edges \( x_{n-1} \xrightarrow{\gamma_n} w_2 \) and \([s_j w_2] \rightarrow w_2\), and also on the value of \( (\gamma_n^\vee, x_{n-1}^\vee) \) if \( j = 0 \). Thus we obtain

\[
\begin{array}{c}
[s_j x_{n-1}] \xrightarrow{z_n \gamma_n} [s_j w_2] \\
x_{n-2} \xrightarrow{\gamma_{n-1}} x_{n-1} \xrightarrow{\gamma_n} w_2
\end{array}
\]

for some \( z_n \in W_J \); by Lemma 5.14, the weights of the directed paths from \([s_j x_{n-1}]\) to \( w_2 \) appearing in the diagram above are all congruent modulo \( Q'_J \). Next, if \( (\tilde{\alpha}_j^\vee, x_{n-2} \lambda) < 0 \), then by the same reasoning as above, we obtain

\[
\begin{array}{c}
[s_j x_{n-2}] \xrightarrow{z_{n-1} \gamma_{n-1}} [s_j x_{n-1}] \xrightarrow{z_n \gamma_n} [s_j w_2] \\
x_{n-3} \xrightarrow{\gamma_{n-2}} x_{n-2} \xrightarrow{\gamma_{n-1}} x_{n-1} \xrightarrow{\gamma_n} w_2
\end{array}
\]

for some \( z_{n-1} \in W_J \); by Lemma 5.14, the weights of the directed paths from \([s_j x_{n-2}]\) to \( w_2 \) appearing in the diagram above are all congruent modulo \( Q'_J \). Continue this procedure until \( (\tilde{\alpha}_j^\vee, x_k \lambda) \geq 0 \) for the first time. Then we have

\[
\begin{array}{c}
[s_j x_{k+1}] \xrightarrow{z_{k+2} \gamma_{k+2}} \cdots \xrightarrow{z_n \gamma_n} [s_j w_2] \\
x_k \xrightarrow{\gamma_{k+1}} x_{k+1} \xrightarrow{\gamma_{k+2}} \cdots \xrightarrow{\gamma_n} w_2
\end{array}
\]

where \( z_{k+2}, z_{k+3}, \ldots, z_n \in W_J \); by Lemma 5.14, the weights of the directed paths from \([s_j x_{k+1}]\) to \( w_2 \) appearing in the diagram above are all congruent modulo \( Q'_J \). Because \( (\tilde{\alpha}_j^\vee, x_{k+1} \lambda) < 0 \) and \( (\tilde{\alpha}_j^\vee, x_k \lambda) \geq 0 \), we deduce from Lemma 6.3 (1) and Theorem 6.5 that \([s_j x_{k+1}] = x_k\); in this case, the type (Bruhat or quantum) of the edges \( x_k \xrightarrow{\gamma_{k+1}} x_{k+1} \) and \([s_j x_{k+1}] \rightarrow x_{k+1}\) are the same, and their weights are congruent modulo \( Q'_J \). Concatenating with the remaining edges in \( p \), we obtain

\[
\begin{array}{c}
[s_j x_{k+1}] \xrightarrow{z_{k+2} \gamma_{k+2}} \cdots \xrightarrow{z_n \gamma_n} [s_j w_2] \\
x_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_k} x_k \xrightarrow{\gamma_{k+1}} x_{k+1} \xrightarrow{\gamma_{k+2}} \cdots \xrightarrow{\gamma_n} w_2
\end{array}
\]

Set

\[
(7.3) \quad p' : w_1 = x_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_k} x_k = [s_j x_{k+1}] \xrightarrow{z_{k+2} \gamma_{k+2}} \cdots \xrightarrow{z_n \gamma_n} [s_j w_2] = w_2,
\]

the length of \( p' \) is equal to \( n - 1 \). Also, in the diagram \(7.2\), we set

\[
\begin{align*}
q_0 & : w_1 = x_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_k} x_k \\
q_1 & : [s_j x_{k+1}] \xrightarrow{z_{k+2} \gamma_{k+2}} \cdots \xrightarrow{z_n \gamma_n} [s_j w_2] \\
q_2 & : x_{k+1} \xrightarrow{\gamma_{k+2}} \cdots \xrightarrow{\gamma_n} w_2
\end{align*}
\]

Then we have

\[
\text{wt}(q_1) + \text{wt}([s_j w_2] \rightarrow w_2) \equiv \text{wt}([s_j x_{k+1}] \rightarrow x_{k+1}) + \text{wt}(q_2).
\]
Here, recall that \( \operatorname{wt}([s_j x_{k+1}] \rightarrow x_{k+1}) = \operatorname{wt}(x_k^{\gamma_{k+1}} x_{k+1}) \). Therefore we obtain
\[
\operatorname{wt}(p') = \operatorname{wt}(q_0) + \operatorname{wt}(q_1)
\equiv \operatorname{wt}(q_0) + \operatorname{wt}([s_j x_{k+1}] \rightarrow x_{k+1}) + \operatorname{wt}(q_2) - \operatorname{wt}([s_j w_2] \rightarrow w_2)
\equiv \operatorname{wt}(q_0) + \operatorname{wt}(x_k^{\gamma_{k+1}} x_{k+1}) + \operatorname{wt}(q_2) - \operatorname{wt}([s_j w_2] \rightarrow w_2)
= \operatorname{wt}(p) - \operatorname{wt}([s_j w_2] \rightarrow w_2) \equiv \operatorname{wt}(p) + \delta_{j,0} w_2^{-1} \alpha_j^\vee,
\]
as desired.

(2) Assume first that \( \langle \alpha_j^\vee, x_k \lambda \rangle < 0 \) for all \( 0 \leq k \leq n \). Continuing the procedure in the proof of part (1) above, we finally obtain
\[
[s_j w_1] \xrightarrow{z_1 \gamma_1} [s_j x_1] \xrightarrow{z_2 \gamma_2} [s_j x_2] \xrightarrow{z_3 \gamma_3} \cdots \xrightarrow{z_n \gamma_n} [s_j w_2]
\]
for some \( z_1, z_2, \ldots, z_n \in W_j \); by Lemma 5.14, the weights of the directed paths from \([s_j w_1]\) to \(w_2\) appearing in the diagram above are all congruent modulo \( Q^\vee_j \). Set
\[
(7.4)\quad p': [s_j w_1] = [s_j x_0] \xrightarrow{z_1 \gamma_1} [s_j x_1] \xrightarrow{z_2 \gamma_2} \cdots \xrightarrow{z_n \gamma_n} [s_j x_n] = [s_j w_2];
\]
the length of \( p' \) is equal to \( n \). Furthermore, we obtain
\[
\operatorname{wt}(p') + \operatorname{wt}([s_j w_2] \rightarrow w_2) \equiv \operatorname{wt}([s_j w_1] \rightarrow w_1) + \operatorname{wt}(p),
\]
and hence
\[
\operatorname{wt}(p') \equiv \operatorname{wt}(p) + \operatorname{wt}([s_j w_1] \rightarrow w_1) - \operatorname{wt}([s_j w_2] \rightarrow w_2)
\equiv \operatorname{wt}(p) - \delta_{j,0} w_1^{-1} \alpha_j^\vee + \delta_{j,0} w_2^{-1} \alpha_j^\vee,
\]
as desired.

Now assume that there exists \( 0 < k < n \) such that \( \langle \alpha_j^\vee, x_k \lambda \rangle \geq 0 \). By part (1), there exists a directed path \( p'' \) from \( w_1 \) to \([s_j w_2]\) of length \( n-1 \) in the PQBG such that
\[
\operatorname{wt}(p'') \equiv \operatorname{wt}(p) + \delta_{j,0} w_2^{-1} \alpha_j^\vee.
\]
By concatenating this directed path \( p'' \) and the edge \([s_j w_1] \rightarrow w_1\), we obtain a directed path \( p' \) from \([s_j w_1]\) to \([s_j w_2]\) of length \( n-1+1=n \) in the PQBG such that
\[
\operatorname{wt}(p') \equiv \operatorname{wt}(p'') + \operatorname{wt}([s_j w_1] \rightarrow w_1) \equiv \operatorname{wt}(p) - \delta_{j,0} w_1^{-1} \alpha_j^\vee + \delta_{j,0} w_2^{-1} \alpha_j^\vee,
\]
as desired.

(5) We give the proofs only for parts (1) and (2); the proofs for the other cases are similar. Suppose that in part (1), \( p \) is shortest in the PQBG, but \( p' \) is not shortest in the PQBG. Concatenating a shortest directed path from \( w_1 \) to \([s_j w_2]\) in the PQBG (note that its length is less than \( n-1 \)) and \([s_j w_2] \rightarrow w_2\), we obtain a directed path from \( w_1 \) to \( w_2 \) whose length is less than \( n \). This contradicts the assumption that \( p \) is shortest.

Suppose that in part (2), \( p \) is shortest in the PQBG, but \( p' \) is not shortest in the PQBG. Concatenating a shortest directed path from \([s_j w_1]\) to \([s_j w_2]\) in the PQBG (note that its length is less than \( n \)) and \([s_j w_2] \rightarrow w_2\), we obtain a directed path from \([s_j w_1]\) to \( w_2 \) whose length is less than \( n+1 \). By the assumption of part (2), \( \langle \alpha_j^\vee, s_j w_1 \lambda \rangle > 0 \) and \( \langle \alpha_j^\vee, w_2 \lambda \rangle < 0 \). Therefore it
follows from part (3) that there exists a directed path from \( w_1 \) to \( w_2 \) whose length is less than \( n \), which contradicts the assumption that \( p \) is shortest in the PQBG.

This completes the proof of the lemma. \( \square \)

7.3. Proof of the tilted Bruhat Theorem [7.1]

Proof of Theorem [7.1] The proof proceeds by induction on \( q\ell(u) \). If \( q\ell(u) = 0 \), then \( u = e \). We know from \([3, p. 435]\) that the \( e \)-tilted Bruhat order \( \preceq_e \) on \( W \) is just the Bruhat order on \( W \). Hence, for each \( z \in W \), the minimal coset representative in \( zW_J \) is the unique \( \preceq_e \)-minimal element. Therefore the assertion holds.

Assume that \( q\ell(u) > 0 \). Let \( u = x_0, x_1, \ldots, x_n = e \) be a sequence of elements in \( W \) satisfying the condition in Lemma 6.12 with \( n = q\ell(u) \). Put \( v := x_1 \); note that \( q\ell(v) = q\ell(u) - 1 \). Thus the inductive assumption is:

Theorem [7.1] is true for this \( v \) (and arbitrary \( z \in W \)).

Assume that \( v = s_iz \) for some \( i \in I_{af} \). Since \( u^{-1} \alpha_i \) is positive, it follows from Remark 7.6 that

\[
(7.5) \quad u \xrightarrow{u^{-1} \alpha_i} v = s_iz,
\]

where this arrow is an Bruhat arrow (resp., a quantum arrow) if \( i \neq 0 \) (resp., \( i = 0 \)).

Case 1. Assume that \( z^{-1} \alpha_i \in \Delta^- \setminus \Delta^-_J \); note that \((zy)^{-1} \alpha_i \) is negative for all \( y \in W_J \).

By the inductive assumption, there exists a unique minimal element in the coset \( zW_J \) with respect to \( \preceq_v \), which we denote by \( \text{min}(zW_J, \preceq_v) \). Let \( x \in W_J \) be such that \( \text{min}(zW_J, \preceq_v) = zx \).

Let us show that \( zx \in zW_J \) is a unique minimal element in the coset \( zW_J \) with respect to \( \preceq_u \), that is,

\[
\text{min}(zW_J, \preceq_u) = zx.
\]

Let \( y \in W_J \) be an arbitrary element in \( W_J \). There exists a shortest directed path from \( v \) to \( zy \) that passes through \( zx \):

\[
v \to \cdots \to zx \to \cdots \to zy.
\]

Concatenating \( u \to v \) of (7.5) and this directed path, we obtain a directed path

\[
(7.6) \quad u \to v \to \cdots \to zx \to \cdots \to zy
\]

of length \( \ell(v \Rightarrow zy) + 1 \). Let us show that this directed path is shortest. Suppose that \( \ell(u \Rightarrow zy) < \ell(v \Rightarrow zy) + 1 \). Recall that \( u^{-1} \alpha_i \) is positive, and \((zy)^{-1} \alpha_i \) is negative. By Lemma 7.7(3), we obtain a directed path from \( s_iu = v \) to \( zy \) whose length is equal to \( \ell(u \Rightarrow zy) - 1 \). Hence

\[
\ell(v \Rightarrow zy) \leq \ell(u \Rightarrow zy) - 1 < \ell(v \Rightarrow zy) + 1 - 1 = \ell(v \Rightarrow zy),
\]

which is a contradiction. Therefore, the directed path (7.6) is shortest.

Case 2. Assume that \( z^{-1} \alpha_i \in \Delta^+ \setminus \Delta^+_J \); note that \((zy)^{-1} \alpha_i \) is positive for all \( y \in W_J \), which implies that \( zy \to s_izy \) by Remark 7.6.

By the inductive assumption, there exists a unique minimal element in the coset \( s_izW_J \) with respect to \( \preceq_v \), which we denote by \( \text{min}(s_izW_J, \preceq_v) \). Let \( x \in W_J \) be such that

\[
\text{min}(s_izW_J, \preceq_v) = s_izx.
\]

Let us show that \( zx \in zW_J \) is a unique minimal element in the coset \( zW_J \) with respect to \( \preceq_u \),

\[
\text{min}(zW_J, \preceq_u) = zx.
\]

Let \( y \in W_J \) be an arbitrary element in \( W_J \). We construct a directed path from \( u \) to \( zy \) that passes through \( zx \) as follows: First, we construct a directed path from \( u \) to \( zx \). Concatenating \( u \to v \).
of (7.5) and a shortest directed path from \( v \) to \( s_i z x \), we obtain a directed path from \( u \) to \( s_i z x \) of length \( \ell(v \Rightarrow s_i z x) + 1 \):

\[
u \rightarrow v \rightarrow \cdots \rightarrow s_i z x
\]

Because \( u^{-1} \bar{\alpha}_i \) is positive and \( (s_i z x)^{-1} \bar{\alpha}_i \) is negative, it follows from Lemma 7.7 (1) that there exists a directed path from \( u \) to \( z x \) of length \( \ell(v \Rightarrow s_i z x) + 1 - 1 = \ell(v \Rightarrow s_i z x) \):

\[
\begin{array}{c}
u \\
\rightarrow \\
\cdots \\
\rightarrow \\
s_i z x
\end{array}
\]

Next, we construct a directed path from \( z x \) to \( z y \). Concatenating \( z x \rightarrow s_i z x \) and a shortest directed path from \( s_i z x \) to \( s_i z y \), we obtain a directed path from \( z x \) to \( s_i z y \) of length \( \ell(s_i z x \Rightarrow s_i z y) + 1 \):

\[
\begin{array}{c}
u \\
\rightarrow \\
\cdots \\
\rightarrow \\
\rightarrow \\
s_i z x \\
\rightarrow \\
\cdots \\
\rightarrow \\
s_i z y
\end{array}
\]

Because \( (z x)^{-1} \bar{\alpha}_i \) is positive and \( (s_i z y)^{-1} \bar{\alpha}_i \) is negative, it follows from Lemma 7.7 (1) that there exists a directed path from \( z x \) to \( z y \) of length \( \ell(s_i z x \Rightarrow s_i z y) + 1 - 1 = \ell(s_i z x \Rightarrow s_i z y) \):

\[
\begin{array}{c}
u \\
\rightarrow \\
\cdots \\
\rightarrow \\
\rightarrow \\
s_i z x \\
\rightarrow \\
\cdots \\
\rightarrow \\
s_i z y
\end{array}
\]

Concatenating the directed paths above, we obtain a directed path from \( u \) to \( z y \) of length \( \ell(v \Rightarrow s_i z x) + \ell(s_i z x \Rightarrow s_i z y) = \ell(v \Rightarrow s_i z y) \) (recall that \( s_i z x \preceq_v s_i z y \) by the definition of \( x \in W_f \)) that passes through \( z x \).

Let us show that this directed path is shortest. Suppose that \( \ell(u \Rightarrow z y) < \ell(v \Rightarrow s_i z y) \). Concatenating a shortest directed path from \( u \) to \( z y \) and the directed path \( z y \rightarrow s_i z y \), we obtain a directed path from \( u \) to \( s_i z y \) of the form:

\[
u \rightarrow \cdots \rightarrow z y \rightarrow s_i z y;
\]

note that its length is \( \ell(u \Rightarrow z y) + 1 \). Because \( u^{-1} \bar{\alpha}_i \) is positive, and \( (s_i z y)^{-1} \bar{\alpha}_i \) is negative, it follows from Lemma 7.7 (3) that there exists a directed path from \( s_i z y \) to \( s_i z y \) of length \( \ell(u \Rightarrow z y) + 1 - 1 = \ell(u \Rightarrow z y) \). Since \( \ell(u \Rightarrow z y) < \ell(v \Rightarrow s_i z y) \), this is a contradiction.

**Case 3.** Assume that \( z^{-1} \bar{\alpha}_i \in \Delta_f \); note that \( s_i z W_f = z W_f \).

By the inductive assumption, there exists a unique minimal element in the coset \( z W_f \) with respect to \( \preceq_v \), which we denote by \( \min(z W_f, \preceq_v) \). Let \( x \in W_f \) be such that

\[
\min(z W_f, \preceq_v) = z x.
\]

**Subcase 3.1.** Assume that \( (z x)^{-1} \bar{\alpha}_i \in \Delta_f \). Let us show that

\[
\min(z W_f, \preceq_u) = z x.
\]

Take an arbitrary \( y \in W_f \).
3.1.1. Assume first that \((zy)^{-1} \alpha_i \in \Delta^+_J\). Then we can check in exactly the same way as in Case 1 that concatenating \(u \to v\) of (7.5) and a shortest directed path from \(v\) to \(zy\) that passes through \(zx\) gives a shortest directed path from \(u\) to \(zy\):

\[
\begin{align*}
\text{shortest} & \quad u \to v \to \cdots \to zx \to \cdots \to zy. \\
\text{(7.5)} & \\
\end{align*}
\]

3.1.2. Assume next that \((zy)^{-1} \alpha_i \in \Delta^+_J\). Concatenating \(u \to v\) of (7.5) and a shortest directed path from \(v\) to \(zx\), we obtain a directed path from \(u\) to \(zx\) of length \(\ell(v \Rightarrow zx) + 1\):

\[
\begin{align*}
\text{shortest} & \quad u \to v \to \cdots \to zx. \\
\text{(7.5)} & \\
\end{align*}
\]

Because \((zx)^{-1} \alpha_i\) is positive, and \((s_i zy)^{-1} \alpha_i\) is negative, we see by applying Lemma 7.7(1) to a shortest directed path from \(zx\) to \(s_i zy\) that there exists a directed path from \(zx\) to \(zy\) of length \(\ell(zx \Rightarrow s_i zy) - 1\):

\[
\begin{align*}
\text{3 directed path} & \quad \text{of length } \ell(zx \Rightarrow s_i zy) - 1 \\
& \quad zy \\
& \quad \text{\&} \\
& \quad u \to v \to \cdots \to zx \to \cdots \to s_i zy. \\
\end{align*}
\]

Concatenating these directed paths, we obtain a directed path from \(u\) to \(zy\) that passes through \(zx\); its length is equal to

\[
\begin{align*}
(\ell(v \Rightarrow zx) + 1) + (\ell(zx \Rightarrow s_i zy) - 1) &= \ell(v \Rightarrow zx) + \ell(zx \Rightarrow s_i zy) \\
&= \ell(v \Rightarrow s_i zy);
\end{align*}
\]

recall that \(zx \prec_v s_i zy\). We can show in exactly the same way as in Case 2 that this directed path is shortest.

**Subcase 3.2.** Assume that \((zx)^{-1} \alpha_i \in \Delta^-_J\). Let us show that

\[
\min(zW_J, \preceq_u) = s_i zx.
\]

Take an arbitrary \(y \in W_J\).

3.2.1. Assume that \((zy)^{-1} \alpha_i \in \Delta^-_J\). Concatenating \(u \to v\) of (7.5) and a shortest directed path from \(v\) to \(zx\), we obtain a directed path from \(u\) to \(zx\) of length \(\ell(v \Rightarrow zx) + 1\):

\[
\begin{align*}
\text{shortest} & \quad u \to v \to \cdots \to zx. \\
\text{(7.5)} & \\
\end{align*}
\]

Because \(u^{-1} \alpha_i\) is positive, and \((zx)^{-1} \alpha_i\) is negative, it follows from Lemma 7.7(1) that there exists a directed path from \(u\) to \(s_i zx\) of length \(\ell(v \Rightarrow zx) + 1 - 1 = \ell(v \Rightarrow zx)\):

\[
\begin{align*}
\text{3 directed path} & \quad \text{of length } \ell(v \Rightarrow zx) \\
& \quad \text{\&} \\
& \quad u \to \cdots \to zx \to s_i zx.
\end{align*}
\]
Concatenating this directed path, $s_i zx \rightarrow zx$, and a shortest directed path from $zx$ to $zy$, we obtain a directed path from $u$ to $zy$ that passes through $s_i zx$:

The length of this directed path is equal to

$$\ell(v \Rightarrow zx) + 1 + \ell(zx \Rightarrow zy) = \ell(v \Rightarrow zy) + 1;$$

recall that $zx \preceq_v zy$. We can show in exactly the same way as the argument in Case 1 that this directed path is shortest.

3.2.2. Assume that $(zy)^{-1} \tilde{\alpha}_i \in \Delta^+_J$. By the same argument as in 3.2.1, we have

Concatenating $s_i zx \rightarrow zx$ and a shortest directed path from $zx$ to $s_i zy$, we obtain a directed path from $s_i zx$ to $s_i zy$ of length $\ell(zx \Rightarrow s_i zy) + 1$:

Since $(s_i zx)^{-1} \tilde{\alpha}_i$ is positive, and $(s_i zy)^{-1} \tilde{\alpha}_i$ is negative, it follows from Lemma 7.7 (1) that there exists a directed path from $s_i zx$ to $zy$ of length $\ell(zx \Rightarrow s_i zy) + 1 - 1 = \ell(zx \Rightarrow s_i zy)$:

Concatenating these directed paths, we obtain a directed path from $u$ to $zy$ that passes through $s_i zx$; its length is equal to

$$\ell(v \Rightarrow zx) + \ell(zx \Rightarrow s_i zy) = \ell(v \Rightarrow s_i zy) \quad (\because \; zx \preceq_v s_i zy).$$

We can show in exactly the same way as the argument in Case 2 that this directed path is shortest. Thus we have proved Theorem 7.1. □

8. Postnikov’s lemma

We now prove a generalization to the PQBG of a lemma due to Postnikov [39].

**Proposition 8.1** (cf. [39] Lemma 1 (2), (3)). Let $x, y \in W^J$. Let $p$ and $q$ be a shortest and an arbitrary directed path from $x$ to $y$ in QB($W^J$), respectively. Then there exists $h \in Q_+^J$ such that

$$\text{wt}(q) - \text{wt}(p) \equiv h \mod Q_+^J.$$
In particular, if \( q \) is also shortest, then \( \text{wt}(q) \equiv \text{wt}(p) \mod Q_J^\vee \).

Proof. We prove the first assertion of the proposition (for general \( q \)) by induction on \( q \ell_J(x) \), where \( q \ell_J(x) \) is the quantum length of Definition 6.14. If \( q \ell_J(x) = 0 \), then \( x = e \). It is easy to see that the \( e \)-tilted parabolic Bruhat order (cf. Section 7.1) is just the usual parabolic Bruhat order on \( W_J \), cf. \[1\]. (Indeed, a path from \( e \) to \( y \) contains at least \( \ell(y) \) edges, because an edge either increases length by 1 or decreases length; so the path consisting of covers in the parabolic Bruhat order is shortest.) Thus \( p \) contains no quantum edges in this case, so \( \text{wt}(p) = 0 \), which makes the statement to prove trivial.

Now assume that \( q \ell_J(x) > 0 \), and take \( j \in I_{af} \) such that \( q \ell_J([s_j x]) = q \ell_J(x) - 1 \); note that \( x^{-1} \hat{\alpha}_j \in \Phi^+ \setminus \Phi_J^+ \), or equivalently, \( (\hat{\alpha}_j^\vee, x \lambda) > 0 \), where \( \lambda \) is a dominant weight with stabilizer \( W_J \).

**Case 1.** Assume that \( (\hat{\alpha}_j^\vee, y \lambda) > 0 \). Applying Lemma 7.7 (4) and (5) to \( p \) (resp., \( q \)), we obtain a shortest directed path \( p' \) (resp., a directed path \( q' \)) from \([s_j x]\) to \([s_j y]\) such that
\[
\text{wt}(p') \equiv \text{wt}(p) - \delta_{j,0}x^{-1}\hat{\alpha}_j^\vee + \delta_{j,0}y^{-1}\hat{\alpha}_j^\vee,
\]
\[
\text{wt}(q') \equiv \text{wt}(q) - \delta_{j,0}x^{-1}\hat{\alpha}_j^\vee + \delta_{j,0}y^{-1}\hat{\alpha}_j^\vee.
\]
By the induction hypothesis, \( \text{wt}(p') - \text{wt}(q') \equiv h' \) for some \( h' \in Q_J^\vee \). Then we have \( \text{wt}(p) - \text{wt}(q) \equiv h' \in Q_J^\vee \), as desired.

**Case 2.** Assume that \( (\hat{\alpha}_j^\vee, y \lambda) \leq 0 \). Applying Lemma 7.7 (3) and (5) to \( p \) (resp., \( q \)), we obtain a shortest directed path \( p' \) (resp., a directed path \( q' \)) from \([s_j x]\) to \( y \) such that
\[
\text{wt}(p') \equiv \text{wt}(p) - \delta_{j,0}x^{-1}\hat{\alpha}_j^\vee,
\]
\[
\text{wt}(q') \equiv \text{wt}(q) - \delta_{j,0}x^{-1}\hat{\alpha}_j^\vee.
\]
By the induction hypothesis, \( \text{wt}(p') - \text{wt}(q') \equiv h' \) for some \( h' \in Q_J^\vee \). Then we have \( \text{wt}(p) - \text{wt}(q) \equiv h' \in Q_J^\vee \), as desired. Thus we have proved the first assertion.

Next, assume that \( q \) is also shortest. By interchanging \( p \) and \( q \) in the first assertion, we see that there exists \( h'' \in Q_J^\vee \) such that \( \text{wt}(p) - \text{wt}(q) \equiv h'' \mod Q_J^\vee \), which implies that \( \text{wt}(q) \equiv \text{wt}(p) \mod Q_J^\vee \). Thus we have proved the proposition. \( \square \)

**References**

[1] Akasaka, T., and M. Kashiwara. “Finite-dimensional representations of quantum affine algebras.” *Publ. Res. Inst. Math. Sci.*, 33, (1997): 839–867.

[2] Björner, A., and F. Brenti. *Combinatorics of Coxeter groups*. Graduate Texts in Mathematics Vol. 231. New York: Springer, 2005.

[3] Brenti, F., S. Fomin, and A. Postnikov. “Mixed Bruhat operators and Yang-Baxter equations for Weyl groups.” *Int. Math. Res. Not.*, no. 8 (1999): 419–441.

[4] Braverman, A., D. Maulik, and A. Okounkov. “Quantum cohomology of the Springer resolution.” *Adv. Math.* 227 (2011): 421–458.

[5] Deodhar, V. “A splitting criterion for the Bruhat orderings on Coxeter groups.” *Comm. Algebra* 15 (1987): 1889–1894.

[6] Dyer, M. J. “Hecke algebras and shellings of Bruhat intervals.” *Compositio Math.* 89 (1993): 91–115.

[7] Feigin, B., M. Finkelberg, A. Kuznetsov, and I. Mirković. Semi-infinite flags. II. Local and global intersection cohomology of quasimaps’ spaces. Differential topology, infinite-dimensional Lie algebras, and applications, 113–148, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999.

[8] Fourier, G., and P. Littelmann. “Tensor product structure of affine Demazure modules and limit constructions.” *Nagoya Math. J.* 182 (2006): 171–198.

[9] Fourier, G., and P. Littelmann. “Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions.” *Adv. Math.* 211 (2007): 566–593.

[10] Fourier, G., M. Okado, and A. Schilling. “Kirillov–Reshetikhin crystals for nonexceptional types.” *Adv. Math.* 222 (2009): 1080–1116.

[11] Fourier, G., A. Schilling, and M. Shimozono. “Demazure structure inside Kirillov–Reshetikhin crystals.” *J. Algebra* 309 (2007): 386–404.

[12] Fulton, W., and C. Woodward. “On the quantum product of Schubert classes.” *J. Algebraic Geom.* 13 (2004): 641–661.
[13] Humphreys, J. E. Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics Vol. 29, Cambridge: Cambridge University Press, 1990.

[14] Hivert, F., A. Schilling, and N. M. Thiéry. “Hecke group algebras as quotients of affine Hecke algebras at level 0.” J. Combin. Theory Ser. A 116 (2009): 844–863.

[15] Ion, B. “Nonsymmetric Macdonald polynomials and Demazure characters.” Duke Math. J. 116 (2003): 299–318.

[16] Kac, V. Infinite dimensional Lie algebras, 3rd ed. Cambridge: Cambridge University Press, 1990.

[17] Kuniba, A., K. .C. Misra, M. Okado, and J. Uchiyama. “Demazure modules and perfect crystals.” Comm. Math. Phys. 192 (1998): 555–567.

[18] Kuniba, A., K. .C. Misra, M. Okado, T. Takagi, and J. Uchiyama. “Crystals for Demazure modules of classical affine Lie algebras.” J. Algebra 208 (1998): 185–215.

[19] Lanini, M. “On the stable moment graph of an affine Kac–Moody algebra.” preprint arXiv:1210.3218.

[20] Littelmann, P. “Paths and root operators in representation theory.” Ann. of Math. (2) 142 (1995): 499–525.

[21] Lenart, C. “From Macdonald polynomials to a charge statistic beyond type A.” J. Combin. Theory Ser. A 116 (2009): 844–863.

[22] Lenart, C., and A. Schilling. “Demazure crystals and tensor products of perfect Kirillov-Reshetikhin crystals with various levels.” Adv. Math. 229 (2012): 875–934.

[23] Lenart, C., and A. Postnikov. “Affine Weyl groups in K-theory and representation theory.” Int. Math. Res. Not. no. 12 (2007): 1–65. Art. ID rnm038.

[24] Lenart, C., and A. Postnikov. “A combinatorial model for crystals of Kac-Moody algebras.” Trans. Amer. Math. Soc. 360 (2008): 4349–4381.

[25] Lenart, C., A. Schilling. “Crystal energy via the charge in types A and C.” Math. Zeitschrift, 273 (2013): 401–426.

[26] Lakshmibai, V., and C. S. Seshadri. “Standard monomial theory.” Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 279–322, Manoj Prakashan, Madras, 1991.

[27] Lam, T., and M. Shimozono. “Quantum cohomology of G/P and homology of affine Grassmannian.” Acta Math. 204 (2010): 49–90.

[28] Lenart, C., and M. Shimozono. “Equivariant K-Tate’s rule for Kac-Moody flag manifolds.” preprint arXiv:1203.3237.

[29] Lenart, C., S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. “A uniform model for Kirillov-Reshetikhin crystals II.” in preparation.

[30] Lusztig, G. “Hecke algebras and Jantzen’s generic decomposition patterns.” Adv. Math. 37 (1980): 121–164.

[31] Mare, A.-L. “Polynomial representatives of Schubert classes in QH*(G/B).” Math. Res. Lett. 9 (2002): 757–769.

[32] Naito, S., and D. Sagaki. “Path model for a level-zero extremal weight module over a quantum affine algebra.” Int. Math. Res. Not. no. 32 (2003): 1731–1754.

[33] Naito, S., and D. Sagaki. “Path model for a level-zero extremal weight module over a quantum affine algebra.” Adv. Math. 200 (2006): 102–124.

[34] Naito, S., and D. Sagaki. “Lakshmibai-Seshadri paths of level-zero shape and one-dimensional sums associated to level-zero fundamental representations.” Compos. Math. 144 (2008): 1525–1556.

[35] Naito, S., and D. Sagaki. “Crystal structure on the set of Lakshmibai-Seshadri paths of an arbitrary level-zero shape.” Proc. Lond. Math. Soc. 96 (2008): 582–622.

[36] Naoi, K. “Weyl modules, Demazure modules and finite crystals for non-simply laced type.” Adv. Math. 229 (2012): 875–934.

[37] Naoi, K. “Demazure crystals and tensor products of perfect Kirillov-Reshetikhin crystals with various levels.” J. Algebra 374 (2013): 1–26.

[38] Peterson, B. “Quantum cohomology of G/P.” Lecture notes, Massachusetts Institute of Technology, Cambridge, MA, Spring 1997.

[39] Postnikov, A. “Quantum Bruhat graph and Schubert polynomials.” J. Combin. Theory Ser. A 116 (2009): 699–709.

[40] Stein, W. A., and others. Sage Mathematics Software (Version 5.4). The Sage Development Team, 2012. http://www.sagemath.org.

[41] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008-2012. http://sage-combinat.org.
