On consecutive abundant numbers

Yong-Gao Chen, Hui Lv*

School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, P. R. China

Abstract. A positive integer $n$ is called an abundant number if $\sigma(n) \geq 2n$, where $\sigma(n)$ is the sum of all positive divisors of $n$. Let $E(x)$ be the largest number of consecutive abundant numbers not exceeding $x$. In 1935, P. Erdős proved that there are two positive constants $c_1$ and $c_2$ such that $c_1 \log \log \log x \leq E(x) \leq c_2 \log \log \log x$. In this paper, we resolve this old problem by proving that, $E(x)/\log \log \log x$ tends to a limit as $x \to +\infty$, and the limit value has an explicit form which is between 3 and 4.

2010 Mathematics Subject Classification: 11N37, 11N60

Keywords and phrases: abundant numbers; perfect numbers; deficient numbers; sequences.

1 Introduction

Let $\sigma(n)$ is the sum of all positive divisors of $n$. A positive integer $n$ is called an abundant number, a perfect number and a deficient number if $\sigma(n) \geq 2n$, $= 2n$ and $< 2n$, respectively. These numbers have brought extensive research. For example, abundant numbers have been studied in [1-15,17-19]. Let $E(x)$ be the largest number of consecutive abundant numbers not exceeding $x$. In 1935, P. Erdős [13] proved that there are two positive constants $c_1$ and $c_2$ such that $c_1 \log \log \log x \leq E(x) \leq c_2 \log \log \log x$. P. Erdős paid much attention to abundant numbers all his life (see [11-15]).

*ygchen@njmu.edu.cn(Y.-G. Chen)
In this paper, $p$ always denotes a prime and $(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$.

The following result is proved.

**Theorem 1.1.** We have

$$
\lim_{x \to \infty} \frac{E(x)}{\log \log \log x} = (\log \varpi)^{-1},
$$

where

$$
\varpi = \inf \left( \prod_{i=1}^{M} \max \left\{ \frac{2(i, M)}{\sigma((i, M))}, 1 \right\} \right)^{\frac{1}{M}}.
$$

**Remark 1.2.** From the proof of the main theorem, we also have

$$
\varpi = \frac{1}{\beta} \sup \left( \prod_{i=1}^{M} \max \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \right)^{\frac{1}{M}},
$$

where

$$
\beta = \prod_{p} \prod_{t=1}^{\infty} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^t} \right)^{\frac{1}{t(1-\frac{1}{p})}}.
$$

We have $\beta < 1.56635$. For $M = 4840909920000$, we have

$$
\varpi \geq \frac{1}{\beta} \left( \prod_{i=1}^{M} \max \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \right)^{\frac{1}{M}} > 1.3267
$$

and

$$
\varpi \leq \left( \prod_{i=1}^{M} \max \left\{ \frac{2(i, M)}{\sigma((i, M))}, 1 \right\} \right)^{\frac{1}{M}} < 1.3604.
$$

It follows that

$$
3.24 < (\log \varpi)^{-1} < 3.54.
$$

It is not the aim of this paper to find a good numerical result.

Now we give an outline of the proof of Theorem 1.1. One of the main ideas in this paper is to introduce the following sequence:

$$
\{(n, M)\}_{n=1}^{\infty}
$$
for any given positive integer $M$. This is a sequence of period length $M$. If $M$ is
divisible by all “small” prime powers, then two sequences
\[
\left\{ \frac{\sigma(n)}{n} \right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{ \frac{\sigma((n, M))}{(n, M)} \right\}_{n=1}^{\infty}
\]
are “similar” in a sense, but the late one is a sequence of period length $M$. I
believe that the sequence $\{(n, M)\}_{n=1}^{\infty}$ will play an important role in the future
research.

In Section 2, we give an upper bound of $E(x)$:
\[
E(x) \leq (\rho_1(M) + o(1)) \log \log \log x
\]
for a function $\rho_1(M)$ and any given positive integer $M$. In Section 3, we give a
lower bound of $E(x)$:
\[
E(x) \geq (\rho_2(M) + o(1)) \log \log \log x
\]
for a function $\rho_2(M)$ and any given positive integer $M$. In Section 4, it is proved
that
\[
\rho_1(M) = \rho_2(M) + o(1)
\]
for infinitely many positive integers $M$. In Section 5, let $\rho_1 = \inf \rho_1(M)$ and
$\rho_2 = \sup \rho_2(M)$. We finish the proof by proving that $\rho_1 = \rho_2 = (\log \varpi)^{-1}$.

Let $\alpha$ be a positive real number. A positive integer $n$ is called an $\alpha$-abundant
number if $\sigma(n) \geq \alpha n$. Let $E_\alpha(x)$ be the largest number of consecutive $\alpha$-abundant
numbers not exceeding $x$.

The method in this paper can be used to prove the following analogous result.

**Theorem 1.3.** We have
\[
\lim_{x \to \infty} \frac{E_\alpha(x)}{\log \log \log x} = \left(\log \varpi_\alpha\right)^{-1},
\]
where
\[
\varpi_\alpha = \inf \left( \prod_{i=1}^{M} \max \left\{ \frac{(i, M)\alpha}{\sigma((i, M))}, 1 \right\} \right)^{\frac{1}{M}}.
\]
2 The upper bound of $E(x)$

In this section we prove that, for any given positive integer $M$,

$$E(x) \leq (\rho_1(M) + o(1)) \log \log \log x$$

for all sufficient large $x$, where

$$\rho_1(M) = (\log \delta_M - \log \beta)^{-1},$$

$$\beta = \prod_p \prod_{t=1}^{\infty} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^t} \right)^{\frac{1}{p^t}(1 - \frac{1}{p})}$$

and

$$\delta_M = \left( \prod_{i=1}^{M} \max \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \right)^{\frac{1}{p^t}}.$$

It is clear that

$$\beta = \prod_p \prod_{t=1}^{\infty} \left( \frac{\sigma(p^t)}{p^t} \right)^{\frac{1}{p^t}(1 - \frac{1}{p})}.$$

Let $m, m+1, \ldots, m+k-1$ be consecutive abundant numbers not exceeding $x$. Let $M$ be a positive integer. For any prime $p$ and any positive integer $t$, let $s_{p^t}$ be the number of integers in $m, m+1, \ldots, m+k-1$ which are divisible by $p^t$. Then

$$s_{p^t} = \frac{k}{p^t} + R_{p^t}, \quad |R_{p^t}| \leq 1. \quad (2.1)$$

If $p^t > m + k - 1$, then $s_{p^t} = 0$.

For any two positive integers $a, b$ with $a \mid b$, we have

$$\frac{\sigma(a)}{a} \leq \frac{\sigma(b)}{b}.$$

Thus

$$\frac{\sigma((m+i, M))}{(m+i, M)} \leq \frac{\sigma(m+i)}{m+i}, \quad i = 0, 1, \ldots, k-1.$$

Since $m+i$ ($1 \leq i \leq k-1$) are abundant numbers, we have

$$2 \leq \frac{\sigma(m+i)}{m+i}, \quad i = 0, 1, \ldots, k-1.$$

It follows that

$$\prod_{i=0}^{k-1} \max \left\{ \frac{\sigma((m+i, M))}{(m+i, M)}, 2 \right\} \leq \prod_{i=0}^{k-1} \frac{\sigma(m+i)}{m+i}.$$
By the definition of $s_{p^t}$, we have
\[
\prod_{i=0}^{k-1} \frac{\sigma(m+i)}{m+i} = \prod_{p|m(m+1)\cdots(m+k-1)} \prod_{t=1}^{\infty} \left( \frac{\sigma(p^t)}{p^t} \right)^{s_{p^t} - s_{p^{t+1}}}. 
\]

If $p^t \leq k$, then $p \mid m(m+1)\cdots(m+k-1)$. Now we split the left product into two parts according to $p^t \leq k$ and $p^t > k$:
\[
\prod_{p} \prod_{t=1, p^t \leq k}^{\infty} \left( \frac{\sigma(p^t)}{p^t} \right)^{s_{p^t} - s_{p^{t+1}}},
\]
and
\[
\prod_{p|m(m+1)\cdots(m+k-1)} \prod_{t=1, p^t > k}^{\infty} \left( \frac{\sigma(p^t)}{p^t} \right)^{s_{p^t} - s_{p^{t+1}}}. 
\]

It is clear that
\[
\frac{\sigma(p^t)}{p^t} = 1 + \frac{1}{p} + \cdots + \frac{1}{p^t} < 1 + \frac{1}{p-1}. 
\]
Let $T = T_p$ be the integer with $p^{T_p} \leq k < p^{T_p+1}$. Since there is at most one integer in $m, m+1, \ldots, m+k-1$ which is divisible by $p^{T_p+1}$, it follows that $s_{p^{T_p+1}} \leq 1$. Thus
\[
\prod_{p|m(m+1)\cdots(m+k-1)} \prod_{t=1, p^t > k}^{\infty} \left( 1 + \frac{1}{p-1} \right)^{s_{p^t} - s_{p^{t+1}}} 
\leq \prod_{p|m(m+1)\cdots(m+k-1)} \left( 1 + \frac{1}{p-1} \right)^{s_{p^{T_p+1}}}
\leq \prod_{p|m(m+1)\cdots(m+k-1)} \frac{m(m+1)\cdots(m+k-1)}{\phi(m(m+1)\cdots(m+k-1))}. 
\]

Noting that (see [16, Theorem 328]),
\[
\phi(n) \gg \frac{n}{\log \log n}, 
\]
where \( \gg \) is the Vinogradov symbol, we have

\[
\prod_{p \mid m+1 \cdots (m+k-1), \text{ } t=1, p^t > k} \prod_{t=1}^{\infty} \left( \frac{\sigma(p^t)}{p^t} \right)^{s_{p^t} - s_{p^t+1}} \\
\leq \frac{m(m+1) \cdots (m+k-1)}{\phi(m(m+1) \cdots (m+k-1))} \\
\ll \log \log (m(m+1) \cdots (m+k-1)) \\
\ll \log \log (x^k) \\
\ll \log k + \log \log x \\
\ll (\log k) \log \log x,
\]

where \( \ll \) is also the Vinogradov symbol.

Now we deal with the product (2.2).

By (2.1), we have

\[
s_{p^t} - s_{p^t+1} = \frac{1}{p^t} \left( 1 - \frac{1}{p} \right) k + R_{p^t} - R_{p^t+1} \\
\leq \frac{1}{p^t} \left( 1 - \frac{1}{p} \right) k + 2.
\]

Noting that \( T_p = 0 \) for \( p > k \) and \( T_p \leq 2 \log k \) for any prime \( p \), we have

\[
\prod_{p \leq k} \prod_{t=1, p^t \leq k} \left( \frac{\sigma(p^t)}{p^t} \right)^{s_{p^t} - s_{p^t+1}} \\
\leq \prod_{p \leq k} \prod_{t=1}^{T_p} \left( \frac{\sigma(p^t)}{p^t} \right)^{\frac{1}{p^t} \left( 1 - \frac{1}{p} \right) k + 2} \\
\leq \prod_{p \leq k} \prod_{t=1}^{\infty} \left( \frac{\sigma(p^t)}{p^t} \right)^{\frac{1}{p^t} \left( 1 - \frac{1}{p} \right) k} \cdot \prod_{p \leq k} \prod_{t=1}^{T_p} \left( \frac{\sigma(p^t)}{p^t} \right)^{2} \\
\leq \prod_{p \leq k} \prod_{t=1}^{\infty} \left( \frac{\sigma(p^t)}{p^t} \right)^{\frac{1}{p^t} \left( 1 - \frac{1}{p} \right) k} \cdot \prod_{p \leq k} \left( 1 + \frac{1}{p - 1} \right)^{4 \log k} \\
= \beta^k \prod_{p \leq k} \left( 1 + \frac{1}{p - 1} \right)^{4 \log k} \\
\leq \beta^k (c \log k)^{4 \log k},
\]

where the last inequality is due to Mertens’ theorem (see [16, Theorem 429]) and
c is a positive constant. Hence
\[
\prod_{i=0}^{k-1} \max \left\{ \frac{\sigma((m+i,M))}{(m+i,M)}, 2 \right\} \ll \beta^k (c \log k)^{4 \log k + 1} \log \log x.
\]

Recall that
\[
\delta_M = \left( \prod_{i=1}^M \max \left\{ \frac{\sigma((i,M))}{(i,M)}, 2 \right\} \right)^\frac{1}{M}.
\]

Since
\[
\prod_{i=0}^{k-1} \max \left\{ \frac{\sigma((m+i,M))}{(m+i,M)}, 2 \right\} \geq \left( \prod_{i=1}^M \max \left\{ \frac{\sigma((i,M))}{(i,M)}, 2 \right\} \right)^{k/M-1} \gg_M \delta_M^k,
\]

it follows that
\[
\delta_M^k \ll_M \beta^k (c \log k)^{4 \log k + 1} \log \log x.
\]

Noting \( \beta < 2 \leq \delta_M \), we have
\[
k \leq (\log \delta_M - \log \beta + o(1))^{-1} \log \log x = (\rho_1(M) + o(1)) \log \log \log x.
\]

### 3 The lower bound of \( E(x) \)

In this section we prove that, for any given positive integer \( M \),
\[
E(x) \geq (\rho_2(M) + o(1)) \log \log \log x
\]
for all sufficient large \( x \), where
\[
\rho_2(M) = (\log 2 - \log \tau_M)^{-1},
\]
\[
\tau_M = \left( \prod_{i=1}^M \min \left\{ \frac{\sigma((i,M))}{(i,M)}, 2 \right\} \right)^{1/M}.
\]
Let $M$ be any given positive integer and let $q_1, q_2, \ldots$ be all primes in increasing order which are greater than $M$. Let

$$A = \prod_{M < p < \frac{x}{2} \log x} p.$$ 

By Mertens’ theorem (see [16, Theorem 429]), we have

$$\frac{\sigma(A)}{A} = \prod_{M < p < \frac{x}{2} \log x} \left(1 + \frac{1}{p}\right) \geq c_M \log \log x,$$

where $c_M$ is a positive constant depending only on $M$.

Let $j_0 = 0$. For any integer $l \geq 1$, let

$$a_l = (l, M)q_{j_l-1+1} \cdots q_{j_l} := (l, M)b_l,$$

where $j_l$ is the least integer with $j_l \geq j_{l-1} + 1$ and $\sigma(a_l) \geq 2a_l$.

Since $((l, M), b_l) = 1$, it follows that

$$\frac{\sigma(a_l)}{a_l} = \frac{\sigma(b_l) \sigma((l, M))}{b_l (l, M)}$$

It is clear that

$$\frac{\sigma(a_l)}{a_l} \leq \max \left\{ 2, \frac{\sigma((l, M))}{(l, M)} \right\} \left(1 + \frac{1}{q_{j_l}}\right) \leq \max \left\{ 2, \frac{\sigma((l, M))}{(l, M)} \right\} \left(1 + \frac{1}{l}\right).$$

Let $k$ be the integer with

$$b_1 b_2 \cdots b_k \leq A < b_1 b_2 \cdots b_{k+1}.$$

Then

$$\frac{\sigma(A)}{A} < \frac{\sigma(b_1) \cdots \sigma(b_{k+1})}{b_1 \cdots b_{k+1}} \leq \frac{\sigma(a_1) \cdots \sigma(a_{k+1})}{a_1 \cdots a_{k+1}} \frac{(1, M) \cdots (k + 1, M)}{\sigma((1, M)) \cdots \sigma((k + 1, M))} \leq \prod_{i=1}^{k+1} \max \left\{ 2, \frac{\sigma((i, M))}{(i, M)} \right\} \cdot \prod_{i=1}^{k+1} \left(1 + \frac{1}{i}\right) \cdot \prod_{i=1}^{k+1} \frac{(i, M)}{\sigma((i, M))}$$

$$= 2^{k+1}(k + 2) \prod_{i=1}^{k+1} \max \left\{ \frac{(i, M)}{\sigma((i, M))}, \frac{1}{2} \right\} \leq 2^{k+1}(k + 2) \left( \prod_{i=1}^{M} \max \left\{ \frac{(i, M)}{\sigma((i, M))}, \frac{1}{2} \right\} \right)^{(k+1)/M-1}.$$
Recall that
\[
\tau_M = \left( \prod_{i=1}^{M} \min \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \right)^{1/M},
\]
we have
\[
\frac{\sigma(A)}{A} \leq 2^{k+1}(k+2) \left( \prod_{i=1}^{M} \max \left\{ \frac{(i, M)}{\sigma((i, M))}, \frac{1}{2} \right\} \right)^{(k+1)/M-1}
\ll M^{-1} 2^{k+1}k\tau^{-k}_M.
\]
It follows from (3.1) that
\[
c_M \log \log x \leq \frac{\sigma(A)}{A} \ll M^{-1} 2^{k+1}k\tau^{-k}_M.
\]
Noting that \(\tau_M < 2\), we have
\[
k \geq (\log 2 - \log \tau_M + o(1))^{-1} \log \log \log x
\]
\[
= (\rho_2(M) + o(1)) \log \log \log x.
\]
Although we can prove that \(k \ll \log \log \log x\), in order to avoid unnecessary arguments, we prefer to write
\[
k' = \min \{k, 2[\rho_2(M) \log \log \log x]\},
\]
where \([z]\) denotes the integral part of real number \(z\).

Now we prove that there are \(k'\) consecutive abundant numbers not exceeding \(x\). It follows that
\[
E(x) \geq (\rho_2(M) + o(1)) \log \log \log x.
\]
By the Chinese remainder theorem (see [16, Theorem 121]), there exists a positive integer \(m \leq Mb_1 \cdots b_{k'}\) such that \(m \equiv 0 \pmod{M}\) and
\[
m + i \equiv 0 \pmod{b_i}, \quad i = 1, 2, \ldots, k'.
\]
Now we prove that \(m+1, m+2, \ldots, m+k'\) are consecutive abundant numbers which do not exceed \(x\).

By the prime number theorem (see [16, Theorem 6]), we have
\[
\log(k'MA) < k'M + \log A < k'M + \frac{2}{3} \log x < \log x
\]
for all sufficiently large \( x \). It follows that

\[
m + k' \leq mk' \leq k'Mb_1 \cdots b_{k'} \leq k'MA < x
\]

for all sufficiently large \( x \).

For \( 1 \leq i \leq k' \), by \( m \equiv 0 \pmod{M} \) we have \((i, M) | m \) and then \((i, M) | m+i \). Noting that \( b_i | m+i \) and \(((i, M), b_i) = 1 \), we have \((i, M)b_i | m+i \). It follows that \( a_i | m+i \). Since \( a_i (1 \leq i \leq k') \) are abundant numbers, it follows that

\[
m + 1, m + 2, \ldots, m + k'
\]

are consecutive abundant numbers.

4 \quad \rho_1(M) = \rho_2(M) + o(1) \text{ for infinitely many } M

Let \( U \) be a large integer and let

\[
M_U = \prod_{p < U} p^{v_p U}. 
\]

In this section we prove that

\[
\rho_1(M_U) = \rho_2(M_U) + o(1). \tag{4.1}
\]

Recall that

\[
\rho_1(M) = (\log \delta_M - \log \beta)^{-1}
\]

and

\[
\rho_2(M) = (\log 2 - \log \tau_M)^{-1},
\]

it is enough to prove that

\[
\delta_{M_U} \tau_{M_U} = 2\beta + o_U(1). \tag{4.2}
\]

Let \( M = M_U \). For any prime \( p < U \) and any positive integer \( t \leq U \), let \( v_{p^t} \) be the number of integers in \( 1, 2, \ldots, M \) which are divisible by \( p^t \). Then

\[
v_{p^t} = \frac{M}{p^t}.
\]
Let \( v_{pU+1} = 0 \) for any prime \( p < U \). Thus

\[
\delta_{M_U} T_{M_U} = \left( \prod_{i=1}^{M} \max \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \right)^{\frac{1}{\psi}} \left( \prod_{i=1}^{M} \min \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \right)^{\frac{1}{\psi}}
\]

\[
= \left( \prod_{i=1}^{M} \max \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \min \left\{ \frac{\sigma((i, M))}{(i, M)}, 2 \right\} \right)^{\frac{1}{\psi}}
\]

\[
= \left( \prod_{i=1}^{M} \left( 2 \frac{\sigma((i, M))}{(i, M)} \right) \right)^{\frac{1}{\psi}}
\]

\[
= 2 \left( \prod_{i=1}^{M} \frac{\sigma((i, M))}{(i, M)} \right)^{\frac{1}{\psi}}
\]

\[
= 2 \left( \prod_{p < U} \prod_{t=1}^{p} \left( \frac{\sigma(p^t)}{p^t} \right)^{v_{p^t} - v_{p^t+1}} \right)^{\frac{1}{\psi}}
\]

\[
= 2 \prod_{p < U} \prod_{t=1}^{p} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^t} \right)^{\frac{1}{\psi}} (1 - \frac{1}{p})
\]

\[
\cdot \prod_{p < U} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{U+1}} \right)^{\frac{1}{\psi}}
\]

\[
= 2^\beta + o_U(1),
\]

where

\[
\log \prod_{p < U} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{U+1}} \right)^{\frac{1}{p^{U+1}}}
\]

\[
< \sum_{p < U} \frac{1}{p^{U+1}} \log \left( 1 + \frac{1}{p - 1} \right)
\]

\[
< \sum_{p < U} \frac{1}{p^{U+1} (p - 1)}
\]

\[
< \frac{1}{p^U} \sum_{p} \frac{1}{p(p - 1)}
\]

\[
= o_U(1)
\]

and then

\[
\prod_{p < U} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{U+1}} \right)^{\frac{1}{p^{U+1}}} = 1 + o_U(1).
\]

Thus we have proved that (4.2) holds and so does (4.1).
Proof of Theorem 1.1

We have proved that, for any given positive integer $M$,

$$E(x) \leq (\rho_1(M) + o(1)) \log \log \log x$$

and

$$E(x) \geq (\rho_2(M) + o(1)) \log \log \log x$$

for all sufficient large $x$. In order to obtain the optimal upper bound and the optimal lower bound of $E(x)$, we should choose $M_1$ and $M_2$ such that $\rho_1(M_1)$ is as small as possible and $\rho_2(M_2)$ is as large as possible. Let

$$\rho_1 = \inf \rho_1(M), \quad \rho_2 = \sup \rho_2(M).$$

Then

$$\rho_2 + o(1) \leq \frac{E(x)}{\log \log \log x} \leq \rho_1 + o(1).$$

So $\rho_2 \leq \rho_1$.

Now we prove that $\rho_2 \geq \rho_1$.

Let $U$ be a large integer and $M_U$ be as in the previous section. Then

$$\rho_1 \leq \rho_1(M_U), \quad \rho_2 \geq \rho_2(M_U).$$

Since

$$\rho_1(M_U) = \rho_2(M_U) + o(1),$$

it follows that

$$\rho_1 \leq \rho_1(M_U) = \rho_2(M_U) + o(1) \leq \rho_2 + o(1).$$

This implies that $\rho_1 \leq \rho_2$. Therefore, $\rho_2 = \rho_1$ and then

$$\frac{E(x)}{\log \log \log x} = \rho_2 + o(1) = (\log \pi)^{-1} + o(1).$$

This completes the proof of our main theorem.

Acknowledgments. The first author is supported by the National Natural Science Foundation of China, Grant No. 11371195 and a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.
References

[1] L. Alaoglu and P. Erdős, On highly composite and similar numbers, Trans. Amer. Math. Soc. 56, (1944), 448–469.

[2] M. R. Avidon, On the distribution of primitive abundant numbers, Acta Arith. 77 (1996), 195–205.

[3] F. Behrend, Über numeri abundantes, Preuss. Akad. Wiss. Sitzungsber 21/23 (1932), 322–328.

[4] F. Behrend, Über numeri abundantes, II, Preuss. Akad. Wiss. Sitzungsber 6 (1933), 280–293.

[5] K. Briggs, Abundant numbers and the Riemann hypothesis, Experiment. Math. 15 (2006), 251–256.

[6] S. Chowla, On abundant numbers, J. Indian Math. Soc. 1 (1934), 41–44.

[7] G. L. Cohen, Primitive $\alpha$-abundant numbers, Math. Comp. 43 (1984), 263–270.

[8] H. Davenport, über numeri abundantes, S.-Ber. Preu. Akad. Wiss., math.-nat. Kl. (1933), 830–837.

[9] L. E. Dickson, Finiteness of the odd perfect and primitive abundant numbers with $n$ distinct prime factors, Amer. J. Math. 35 (1913), 413–422.

[10] L. E. Dickson, Even abundant numbers, Amer. J. Math. 35 (1913), 423–426.

[11] P. Erdős, On the density of the abundant numbers, J. London Math. Soc. 9 (1934), 278–282.

[12] P. Erdős, On primitive abundant numbers, J. London Math. Soc. 10 (1935), 49–58.

[13] P. Erdős, Note on consecutive abundant numbers, J. London Math. Soc. 10 (1935), 128–131.

[14] P. Erdős, Remarks on number theory. I. On primitive $\alpha$-abundant numbers, Acta Arith. 5 (1958), 25–33.
[15] P. Erdős, On abundant-like numbers, Canad. Math. Bull. 17 (1974), 599–602.

[16] G. H. Hardy and E. M. Wright, An Introduction to the theory of numbers, Oxford Univ. Press 1979.

[17] M. Kobayashi, A new series for the density of abundant numbers, Int. J. Number Theory 10 (2014), 73–84

[18] M. Kobayashi and P. Pollack, The error term in the count of abundant numbers, Mathematika 60 (2014), 43–65.

[19] H. N. Shapiro, On primitive abundant numbers, Comm. Pure Appl. Math. 21 (1968), 111–118.