A PROBABILISTIC APPROACH TO SOME RESULTS BY NIETO AND TRUAX

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1. Introduction

In this paper, we revisit some results by Nieto and Truax about generating functions for arbitrary order coherent and squeezed states. These results were obtained using the exponential of the Laplacian operator; they were later extended by Dattoli et al. [8] using more elaborated operational identities. In this paper, we show that the operational approach can be replaced by a simple probabilistic approach, in the sense that the exponential of derivatives operators can be replaced by equivalent expectation operators. This approach brings new insight about the links between operational and probabilistic calculus.

In the first part, we show that the exponential of the derivation operator of arbitrary integer order can be replaced by an expectation with respect to a carefully chosen random variable.

In the second part, we apply this result to the Gould-Hopper polynomials, which include as special cases the Kampé de Fériet and the Hermite polynomials; this allows us to recover and interpret easily some properties of these polynomials.

The last part is dedicated to the application of these probabilistic representations to the computation of coherent and squeezed states.

2. The operator $\exp \left[ (c \frac{d}{dx})^j \right]$

2.1. Introduction. In [2], Nieto and Truax consider the operators

$$ I_j = \exp \left[ \left( c \frac{d}{dx} \right)^j \right] $$

where $c$ is a constant and $j$ an integer. They remark that, for any well-behaved function $f$, $I_1$ acts as the translation operator

$$ I_1 f(x) = f(x + c). $$

Moreover, $Z_2$ being a Gaussian random variable with variance 2, $I_2$ acts as the Gauss-Weierstrass transform

$$ I_2 f(x) = E_{Z_2} f(x + cZ_2). $$

Since $I_1 f$ can be also written as $E_{Z_1} f(x + Z_1)$ where $Z_1$ is the deterministic variable equal to 1, it is tempting to wonder if the expression

$$ I_j = E_{Z_j} f(x + cZ_j) $$

holds for values of $j \geq 3$ and to study what random variable $Z_j$ possibly comes out in this formula. In [2], the following general expression is proposed

$$ I_j f(x) = \frac{1}{(2\pi)^{\frac{j-1}{2}}} \sqrt{j} \int_0^{+\infty} \frac{dx_1 e^{-x_1}}{x_1^\frac{3}{2}} \int_0^{+\infty} \frac{dx_2 e^{-x_2}}{x_2^\frac{3}{2}} \ldots \int_0^{+\infty} \frac{dx_{j-1} e^{-x_{j-1}}}{x_{j-1}^\frac{3}{2}} \times \sum_{l=1}^{j} f \left( x + jc(x_1 x_2 \ldots x_{j-1} + l) \right) \exp \left( 2i\pi \frac{l}{j} \right) $$

and the authors add: “This result, although in closed form, is usually too complicated to yield results in terms of elementary functions”. The next section is devoted to providing a simple expression for the operator $I_j$. 
2.2. A complex valued stable random variable. We first introduce the following random variables $W_j$ and $Z_j$ where in this whole section, $j$ is an integer such that $j \geq 2$.

**Definition 1.** For $j \in \mathbb{N}$, $j \geq 2$, define the root of the unity

$$\omega_j = \exp\left(\frac{2\pi i}{j}\right)$$

and the random variable $W_j$ as

$$\Pr\{W = \omega_j^l\} = \frac{1}{j}, \quad 0 \leq l \leq j - 1. \quad (2.2)$$

Thus $W_j$ equals equiprobably all values of the $j$–roots of the unity. The random variable $Z_j$ is then defined as

$$Z_j = W_j S_{\frac{1}{j}}^{-1}$$

where $S_{\frac{1}{j}}$ is a stable random variable [6] with characteristic parameter $\frac{1}{j}$, independent of $W_j$.

In the following, when possible, we’ll discard the indices for notational simplicity and write simply

$$Z = WS^{-\frac{1}{j}}. \quad (2.3)$$

We now describe some properties of the random variable $Z$.

**Proposition 2.** The random variable $Z$ has moments

$$EZ^k = \begin{cases} 0 & \text{if } k \neq p_j \\ \left(\frac{(pj)!}{p!}\right) & \text{if } k = p_j, \ p \in \mathbb{N}. \end{cases} \quad (2.4)$$

*Proof. The $k$–th order moment of $Z$ is

$$EZ^k = EW^k E\frac{1}{S^\frac{1}{j}}$$

with

$$EW^k = \frac{1}{j} j^{j-1} \sum_{l=0}^{j-1} \omega_j^l = \begin{cases} 0 & \text{if } k \neq p_j \\ 1 & \text{if } k = p_j, \ p \in \mathbb{N} \end{cases}$$

Moreover, with $k = p_j$,

$$E\frac{1}{S^\frac{1}{j}} = E\frac{1}{S^p} = \frac{(pj)!}{p!}$$

(see [3]) so that the result follows. \qed

As a consequence, the generating function $\varphi_Z(u) = E \exp(uZ)$ of $Z$ can be evaluated as follows.

**Proposition 3.** The generating function of $Z$ is

$$\varphi_Z(u) = \exp \left(u^j\right), \ u \geq 0.$$ 

*Proof. A straightforward computation gives

$$\varphi_Z(u) = \sum_{k=0}^{+\infty} \frac{u^k}{k!} EZ^k = \sum_{p=0}^{+\infty} \frac{u^{pj}}{pj! \frac{p!}{p!}} = \exp \left(u^j\right).$$

\qed

From this property, we deduce that the random variable $Z$ exhibits the following stability property.

**Proposition 4.** If $Z_1$ and $Z_2$ are two independent random variables distributed as in (2.3) with parameter $j$ and if $a_1$ and $a_2$ are two real positive numbers then the random variable

$$a_1 Z_1 + a_2 Z_2 \sim \left(a_1^j + a_2^j\right)^{\frac{1}{j}} Z$$

where $Z$ is again distributed as in (2.3) with same parameter $j$ and $\sim$ denotes equality in distribution.

We note that $Z$ is a complex valued random variable; for real valued random variables, the stability property can hold only for values of the parameter $j$ such that $0 < j \leq 2$ (see [6] p. 170).
2.3. **APPLICATION TO THE OPERATOR** $\exp \left[ (c \frac{d}{dx})^j \right]$. We can now give a simplified version of the formula (2.1) as follows.

**Theorem 5.** The operator $I_j = \exp \left[ (c \frac{d}{dx})^j \right]$ acts on a function $f$ as

$$I_j f(x) = E_{Z_j} f(x + cZ_j)$$

where $Z_j$ is defined as in (2.3), for any function $f$ such that the right-hand side of this equality exists.

**Proof.** By definition, with $Z$ defined as in (2.3),

$$I_j f(x) = \exp \left[ (c \frac{d}{dx})^j \right] f(x) = E_Z \exp \left( cZ \frac{d}{dx} \right) f(x) = E_{Z_j} f(x + cZ_j).$$

$\square$

We note that the link between equalities (2.1) and (2.5) can be identified using Williams’ formula [3, end of part 2]: formula (2.1) can indeed be rewritten as

$$I_j f(x) = E_W E_{Y_1, Y_2, \ldots, Y_j} f(x + cW \left(Y_1 Y_2 \ldots Y_{j-1}^{j-1}\right)^{\frac{1}{j}})$$

where $Y_1, Y_2, \ldots, Y_{j-1}$ is the product of $j - 1$ independent Gamma random variables with respective shape parameters $\frac{1}{j}, \frac{2}{j}, \ldots, \frac{j-1}{j}$. But by Williams’ formula,

$$Y_1 Y_2 \ldots Y_{j-1} \sim \left( \frac{1}{j} \right)^j S_{\frac{1}{j}}^{-1}$$

where $S_{\frac{1}{j}}$ is a stable random variable with parameter $\frac{1}{j}$, so that

$$I_j f(x) = E_{W, S_{\frac{1}{j}}} f(x + cW S_{\frac{1}{j}}^{-\frac{1}{j}})$$

which is exactly (2.5).

As an example, we consider the case $j = 2$ for which $W$ is Bernoulli distributed with parameter $\frac{1}{2}$ and since $S_{\frac{1}{2}}$ is Lévy distributed, according to [3],

$$\frac{1}{\sqrt{S_{\frac{1}{2}}}} \sim \sqrt{2|N|}$$

where $N$ is Gaussian with variance $\sigma_N^2 = 1$ so that $Z_2 = W_2 S_{\frac{1}{2}}^{-\frac{1}{2}}$ is itself Gaussian with variance $\sigma_Z^2 = 2$ and we recover the Gauss-Weierstrass transform operator

$$I_2 f(x) = Ef(x + cZ_2).$$

3. **PROPERTIES OF THE GOULD HOPPER POLYNOMIALS**

As an application of the former result, we consider the particular case $f(x) = x^n$. The polynomials

$$g_m^n(x, h) = \exp \left[ h \left( \frac{d}{dx} \right)^m \right] x^n, \quad (m, n) \in \mathbb{N}^2$$

are known as the Gould-Hopper polynomials, as introduced in [1, p.58]. The particular case $m = 2$ corresponds to the Kampé de Fériet polynomials, and the polynomials $g_2^n(2x, -1)$ coincide with the classical Hermite polynomials. From the results of section 2 and with $h = c^m$, we deduce

$$g_m^n(x, h) = E_Z \left(x + h^\frac{1}{m} Z\right)^n$$

where $Z$ is distributed as in (2.3) with $j = m$. $^1$ We note that this representation is different from the one given by [5, Thm.1], which seems of little use since it involves integer moments of stable random variables.

We show now that the representation (3.2) allows to recover and extend easily some well-known properties of the GH polynomials.

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$^1$we replace here $j$ by $m$ to stick to the notation introduced by Gould
(1) the generating function of the GH polynominal is \[ 1 \] (6.3)
\[
\sum_{n=0}^{+\infty} g^m_n (x, h) \frac{t^n}{n!} = E_Z \sum_{n=0}^{+\infty} \left( x + h \frac{t}{Z} \right)^n \frac{t^n}{n!} = E_Z \exp \left( t \left( x + h \frac{t}{Z} \right) \right) = \exp \left( tx + ht^m \right);
\]
(2) the derivative of a GH polynomial \[ 1 \] (6.4) is easily computed as
\[
\frac{d}{dx} g^m_n (x, h) = n E_Z \left( x + h \frac{t}{Z} \right)^{n-1} = n g^m_{n-1} (x, h).
\]
(3) the GH polynomials satisfy the following addition theorem
\[
\sum_{n=0}^{+\infty} g^m_n \left( \sum_{i=1}^{r} x_i, \sum_{i=1}^{r} h_i \right) = \sum_{n_1, \ldots, n_r = 0}^{\infty} \prod_{k=1}^{r} g^m_{n_k} (x_k, h_k),
\]
the proof of which is straightforward using the representation (3.2). This result generalizes the case \( r = 2 \) obtained by Gould and Hopper under the form \[ 1 \] (6.19)
\[
\sum_{k=0}^{n} \binom{n}{k} g^m_k (x, h) g^m_{n-k} (y, h) = g^m_n (x + y, 2h).
\]
Moreover, the classical addition theorem for Hermite polynomials \[ 7 \] (40) p. 196
\[
\left( \sum_{k=1}^{r} a_k x_k \right)^2 \sum_{n=0}^{\infty} \frac{g^m_n \left( \sum_{k=1}^{r} \frac{a_k x_k}{(a_k^2)^{1/2}} \right)}{n!} H_n \left( \sum_{k=1}^{r} \frac{a_k x_k}{(a_k^2)^{1/2}} \right) = \sum_{n_1, \ldots, n_r = 0}^{\infty} \prod_{k=1}^{r} \frac{g^m_{n_k}}{n_k!} H_{n_k} (x_k)
\]
is easily deduced from (3.3) using the link between Hermite polynomials and Gould polynomials mentioned above.
(4) Another result from \[ 1 \] (6.9), namely
\[
\exp \left( h \frac{d^j}{dx^j} \right) (x^n \exp (tx)) = \frac{d^n}{dx^n} \left( \exp \left( tx + ht^j \right) \right)
\]
can be recovered easily using (2.5) since
\[
\exp \left( h \frac{d^j}{dx^j} \right) (x^n \exp (tx)) = E_Z \left( x + h \frac{t}{Z} \right)^n \exp \left( t \left( x + h \frac{t}{Z} \right) \right) = \frac{d^n}{dx^n} E_Z \exp \left( t \left( x + h \frac{t}{Z} \right) \right) = \frac{d^n}{dx^n} \left( \exp \left( tx + ht^j \right) \right).
\]

4. Application to Coherent and Squeezed States

4.1. Multisection of series. In calculations of higher-order and squeezed states quantities, the following generating functions of Hermite polynomials appear - we follow here the notation by Nieto and Truax:

\[
S (j, k, z) = \sum_{n=0}^{+\infty} \frac{z^{jn+k}}{(jn+k)!}
\]
\[
G (j, k, x, z) = \sum_{n=0}^{+\infty} \frac{z^{jn+k} H_{jn+k} (x)}{(jn+k)!}
\]
and
\[
g (j, k, x, y, z) = \sum_{n=0}^{+\infty} \frac{z^{jn+k} H_{jn+k} (x) H_{jn+k} (y)}{(jn+k)!}
\]
where \( H_n (x) \) is the Hermite polynomial of degree \( n \), and \( j \) and \( k \) are integers.

These quantities are all computed in \[ 2 \] using an operational calculus approach, but we show here that the probabilistic approach introduced in Section \[ 2 \] gives a new insight on these formulas and allows to extend them to other classes of polynomials.

We introduce the following theorem.
Theorem 6. If
\[ \phi(t, x) = \sum_{n=0}^{+\infty} t^n f_n(x) \]
is the generating function of the sequence of functions \( \{f_n(x)\} \) then\(^2\)
\[ \sum_{n=0}^{+\infty} t^{jn+k} f_{jn+k}(x) = E_{W_j} W^{-k} \phi(tW, x) \]
where \( W \) is distributed as in (2.2).

Proof. We compute
\[ E_{W_j} W^{-k} \phi(tW, x) = \sum_{n=0}^{+\infty} E_{W_j} W^{-k} t^n f_n(x) \]
and use the property (2.4) to deduce the result. \( \square \)

We note that this theorem is the probabilistic formulation of the technique of “multisection of series” described by Riordan [9, section 4.3] as a “process of ancient vintage”. This process is also characterized in [8, (19)] as a consequence of the sieving principle.

4.2. Applications. As an application of this theorem, we recover easily the results in [2]:
1. choosing \( f_n(x) = \frac{H_n(x)}{n!} \) \( \forall x \in \mathbb{R}, \forall n \in \mathbb{N} \) we deduce
\[ \phi(t, x) = \exp(t) \]
and
\[ S(j, k, z) = E_{W_j} W^{-k} \exp(zW) \]
2. choosing \( f_n(x) = \frac{H_n(x)}{n!} \), we deduce the well-known generating function of the Hermite polynomials
\[ \phi(t, x) = \exp(2tx - t^2) \]
so that
\[ G(j, k, x, z) = E_{W_j} W^{-k} \exp(2Wx - z^2W^2) \]
3. the more general case
\[ \sum_{n=0}^{+\infty} \frac{t^{jn+k}}{(jn+k)!} H_{jn+k+m}(x) \]
can also be easily derived: choosing \( f_n(x) = \frac{H_{n+m}(x)}{n!} \), we deduce the generating function [11, (1) p.197]
\[ \phi(t, x) = \exp(2tx - t^2) H_m(x - t) \]
so that
\[ \sum_{n=0}^{+\infty} \frac{t^{jn+k}}{(jn+k)!} H_{jn+k+m}(x) = E_{W_j} W^{-k} \exp(2tWx - t^2W^2) H_m(x - tW) \]
which coincides with [8, (21)].
4. remarking that the variable \( x \) in theorem [6] may be multidimensional, choosing \( x = (x_1, x_2) \) and
\[ f(x) = \frac{H_n(x_1) H_n(x_2)}{n!} \]
we deduce, by the Mehler formula [7, (22) p.194],
\[ \phi(t, x_1, x_2) = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(-\frac{4t^2x^2 + t^2y^2 - txy}{1 - 4t^2}\right) \]
so that
\[ g(j, k, x, y, z) = E_{W_j} W^{-k} \frac{1}{\sqrt{1 - 4tW^2}} \exp\left(-\frac{4t^2W^2x^2 + t^2W^2y^2 - tWxy}{1 - 4t^2W^2}\right) \].

We remark that Theorem [6] is not restricted to Hermite polynomials; let us give another few examples:

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\( ^2 \)we recall the notation \( E_{W_j} f(W) = \frac{1}{z} \sum_{j=0}^{+\infty} f(\omega_j) \)
(1) for the Gould Hopper polynomials \( g_n^m(x,h) \), choosing \( f_n(x) = \frac{g_n^m(x,h)}{n!} \) as defined in (4.1), we deduce
\[
\sum_{n=0}^{\infty} \frac{t^{jn+k}}{(jn+k)!} g_{jn+k}(x,h) = E_W W^{-k} \exp (txW + ht^mW^m)
\]

(2) for the Laguerre polynomials, with \( f_n(x) = L_n^\alpha(x) \) we have \( \phi(t,x) = (1-t)^{-\alpha-1} \exp \left( \frac{x}{1-t} \right) \) so that
\[
\sum_{n=0}^{\infty} \frac{t^{jn+k}}{(jn+k)!} L_n^\alpha(x) = E_W W^{-k} (1-tW)^{-\alpha-1} \exp \left( \frac{xtW}{tW-1} \right).
\]

4.3. Further sums. In their study, Nieto and Truax consider also the sums

\[ K(j,k,p,q,x,z) = \sum_{n=0}^{\infty} \frac{z^{jn+k} H_{jn+k}(x)}{(pn+q)!} \]

for which they explicit some specific cases. We give here the general result only in the cases \( j = p \) and \( j = 2p \) by looking first at the simple sum

\[ k(j,k,p,q,z) = \sum_{n=0}^{\infty} \frac{z^{jn+k}}{(pn+q)!}. \]

A straightforward computation shows that this sum is related to the sum \( S(j,k,z) \) in (4.1) as

\[ k(j,k,p,q,z) = z^{j-k} \frac{S(p,q,z)}{\beta}. \]

As a generalization of this result, we obtain the following

**Theorem.** If the sequence of functions \( \{h_n(x)\} \) can be expressed as a sequence of moments
\[ h_n(x) = E_U \varphi^n(x,U) \]
for some function \( \varphi \), then

\[ \sum_{n=0}^{\infty} \frac{z^{jn+k}}{(pn+q)!} h_{jn+k}(x) = E_W, \varphi \left( \left( z \varphi(x,U) \right)^{k-j} \frac{\rho}{\beta} \right) \exp \left( \left( z \varphi(x,U) \right)^{k-j} \frac{\rho}{\beta} \right). \]

**Proof.** The proof is straightforward replacing \( z \) by \( z \varphi(x,U) \) in (4.6) and taking expectation over \( U \). \( \square \)

As a consequence, using the moment representation of the Hermite polynomials
\[ H_n(x) = 2^n E_N (x+iN)^n \]
where \( N \) is a Gaussian random variable with variance \( \sigma_N^2 = \frac{1}{2} \), we deduce

\[ K(j,k,p,q,x,z) = (2z)^{j-k} \frac{\rho}{\beta} E_{W,N} W^{-q}(x+iN)^{k-j} \frac{\rho}{\beta} \exp \left( (2z(x+iN))^{k-j} \frac{\rho}{\beta} \right) \]

The expectation with respect to the Gaussian variable \( N \) is difficult to obtain for an arbitrary value of the ratio \( \frac{\rho}{\beta} \), so that in the following, we give explicit expressions only in the cases \( j = p \) and \( j = 2p \).

4.3.1. the case \( j = p \). In this case we obtain the following result.

**Theorem 7.** If \( k-q \in \mathbb{N} \), the function \( K(j,k,j,q,x,z) \) reads

\[ K(j,k,j,q,x,z) = z^{j-k} E_W W^{-q} \exp (-zW(zW-2x)) H_{k-q}(x-zW) \]

**Proof.** The proof is immediate remarking that in (4.8) the Gaussian expectation

\[ E_N (x+iN)^{k-q} \exp (2zW(x+iN)) = (2W)^{q-k} \frac{d^{k-q}}{dz^{k-q}} \exp (2xzW) E_N \exp (i2zWN) \]

involves the Gaussian characteristic function \( E_N \exp (i2zWN) = \exp (-z^2W^2) \) so that it is equal to

\[ (2W)^{q-k} \exp (x^2) \frac{d^{k-q}}{dz^{k-q}} \exp \left( -(zW-x)^2 \right) \]

which, using Rodriguez formula for the Hermite polynomials, can be expressed as

\[ 2^{q-k} \exp (x^2) \exp \left( -(zW-x)^2 \right) (-1)^{k-q} H_{k-q}(zW-x) \]
from which we deduce

\[
K(j, k, q, x, z) = z^{k-q}E_{W}W^{-q} \exp (-zW(zW-2x)) H_{k-q}(x-zW)
\]

4.3.2. case \( j = 2p \). This case can also be solved as follows.

**Theorem 8.** If \( k-2q \in \mathbb{N} \), the function \( K(2p, k, p, q, x, z) \) reads

\[
K(2p, k, p, q, x, z) = z^{k-2q}E_{W_p}W^{-q} \frac{\exp \left( \frac{4z^2W}{1+4z^2W} \right)}{(1+4z^2W)^{\frac{k-2q+1}{2}}} H_{k-2q} \left( \frac{x}{\sqrt{1+4z^2W}} \right)
\]

**Proof.** With \( j = 2p \), replacing \( x \) by \( ix \), we need to compute the Gaussian expectation

\[
E_{N}(x+N)^{k-2q} \exp \left( -4z^2W(x+N)^2 \right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (x+N)^{k-2q} \exp \left( -4z^2W(x+N)^2 \right) \exp (-N^2) dN
\]

which, after change of variable \( M = x+N \) reads

\[
\frac{1}{\sqrt{\pi}} \exp \left( -x^2 \right) \int_{-\infty}^{+\infty} M^{k-2q} \exp \left( -M^2 \left( 1+4z^2W \right) \right) \exp (2xM) dM.
\]

This classical Gaussian integral is equal to [10, 2.3.15.10]

\[
\left( \frac{i}{2} \right)^{k-2q} \exp \left( -x^2 \right) \frac{\exp \left( \frac{x^2}{1+4z^2W} \right)}{(1+4z^2W)^{\frac{k-2q+1}{2}}} H_{k-2q} \left( \frac{ix}{\sqrt{1+4z^2W}} \right)
\]

and we deduce, replacing \( x \) by \( (-ix) \),

\[
K(2p, k, p, q, x, z) = z^{k-2q}E_{W_p}W^{-q} \frac{\exp \left( \frac{4z^2W}{1+4z^2W}x^2 \right)}{(1+4z^2W)^{\frac{k-2q+1}{2}}} H_{k-2q} \left( \frac{x}{\sqrt{1+4z^2W}} \right).
\]

5. Conclusion

In this paper, we have shown that there exists a probabilistic counterpart to the classical operational calculus. A further direction of research is the extension of this approach to the multivariate case.

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