A REMARK ON THE INEQUALITIES OF BERNSTEIN -
MARKOV TYPE IN EXPONENTIAL ORLICZ AND
LORENTZ SPACES.
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Abstract. We prove in this article the generalizations on the expo-
nential Orlicz spaces Markov’s - Bernstein’s inequalities for algebraic
polynomials and rational functions.

Key Words. Polynomials, Exponential Orlicz Spaces, Markov’s
and Bernstein’s Inequalities, equivalent Norms.

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1 Introduction. Statement of problem.

It is well - known the application to the approximation theory
Markov’s (or Bernstein’s) inequalities, for example, for inverse the-
orems of approximation theory; see for instance ( [7], p. 208; [13],
[14]) etc.

Let us denote by $A$ the set of all algebraical polynomials defined
on the $x \in X, X = [-1,1]$; by $T$ the set of all trigonometrical polyno-
mials on the set $X = [0,2\pi]$, and by $R$ the set of all rational functions
defined on the $X = [-1,1]$ without poles on $X$; for $Q \in A \cup T$
the symbol $\deg Q$ will denote the usually degree of $Q$; for irreducible
fraction $Q(x) = Q_1(x)/Q_2(x), Q_2(x) \neq 0, x \in [-1,1], Q_{1,2} \in A$
we define $\deg Q = \max(\deg Q_1, \deg Q_2)$. The space $X$ is equipped usually
normed Lebesgue measure $\mu(dx) = Cdx; \mu(X) = 1$.

$T$. For all $Q \in T$ hold the famous generalized Bernstein’s - Zyg-
mund’s inequalities: for all rearrangement invariant space $S$ on the
set $X$

$$
\|dQ/dx\|S \leq \deg Q \cdot \|Q\|_{S}, \quad (1.1)
$$

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([12], p. 36; [20]) and for \( p \in (0, 1) \)
\[
\|dQ/dx\|_p \leq \deg Q \cdot ||Q||_p,
\]
([7], p. 104), where by definition for all values \( p > 0 \)
\[
\|f\|_p = I^{1/p}(|f|^p), \quad I(f) \overset{\text{def}}{=} \int_X f(x) \mu(dx).
\]

**A.** For all \( Q \in A, \ p > 0 \)
\[
\|dQ/dx\|_p \leq C^{1+1/p} \cdot (\deg Q)^2 \cdot ||Q||_p,
\]
where the symbol \( C \) will denote (here and further) some **absolute** constant ([4], p.406). In the case \( p > 2 \)
\[
\|dQ/dx\|_p \leq K(p) \cdot (\deg Q)^2 \cdot ||Q||_p,
\]
where at \( p \geq 4 \Rightarrow K(p) \leq K(4) = (49\pi)^{1/4} \approx 3.52238228 \ldots \).

**R.** Let now \( r = 1, 2, \ldots, \ p \in (0, \infty), \gamma = \gamma(p,r) = p/(pr + 1), \ Q \in \mathbb{R} \). Then
\[
\|Q(r)\|_{\gamma} \leq D_1(p, r) \cdot (\deg Q)^r \cdot ||Q||_p,
\]
where at \( p \geq 4 \Rightarrow
\]
\[
D_1(p, r) \leq D(r) \overset{\text{def}}{=} \exp(1/e) \cdot r! \cdot (4/3)^{r+1/4} \cdot (r + 1/4)^{r+1/4},
\]
see [13], [14]; in [12], p. 300 - 301 was proved that this estimation (1.3) is exact in different senses.

For the nonnegative General Algebraic Polynomials (GAP), i.e. for the functions of a view
\[
Q(x) = \prod_{j=1}^m |x - z(j)|^{r(j)}, \ r(j) \geq 1, \ z(j) = a(j) + b(j)i, \ i = \sqrt{-1},
\]
\[
\deg Q \overset{\text{def}}{=} \sum_{j=1}^m r(j), \ x \in [-1, 1],
\]
in [4], p.402 - 406 was proved the inequality:
\[ \|dQ/dx\|_p \leq C^{1+1/p} \cdot \deg Q^2 \cdot \|Q\|_p, \quad p > 0. \quad (1.4) \]

There are many generalizations of this inequalities on the Muntz polynomials [10], spline functions [15] and so one ([2], [3], [9], [10], [11], [19]).

Our goal is generalization of inequalities (1.2) and (1.3) on the exponential Orlicz spaces, i.e. when in the left and right sides of inequalities (1.2), (1.3) instead \( L_p \) norms are the norms on some Orlicz spaces.

2 Description of using Orlicz spaces.

We will consider here a so - called exponential Orlicz spaces on the \( X \) with usually \textit{normed} Lebesgue measure \( \mu \). Recall here that if the function \( N = N(u), u \in R^1 \) is some \( N - \) Orlicz function (even, continuous, downwards convex, \( N(u) \geq 0, \ N(u) = 0 \Leftrightarrow u = 0 \), strictly increasing in the self - line \( R^1_+ \)), then the Orlicz norm \( \|f\|L(N) \) in the Orlicz’s space \( L(N) \) of a (measurable) function \( f : R \rightarrow R \) relative to the \( N – \) Orlicz function \( N = N(u) \) may be defined by the formula

\[ \|f\|L(N) = \inf\{l, \ l > 0, \ I(|f|/l) \leq 1\}. \]

As a particular case, if \( N(u) = |u|^p, \ p = \text{const} \geq 1 \) we obtain the classical \( L_p = L_p(R) \) spaces with norm \( \|f\|_p \). More information about Orlicz spaces see in the books [16], [17].

A very important class of \( N – \) Orlicz functions are the so - called \( EOF = \text{Exponential Orlicz Functions} \) and correspondent \( \text{Exponential Orlicz Spaces} \ EOS \) (like to the terminology of articles [6], [8]) will be considered. We give here a more general definition of this spaces.

Let \( \varphi = \varphi(z), \ z \geq 0 \) be some continuous function such that \( \varphi(z) = 0 \Leftrightarrow z = 0 \), and the function \( h(y) := \varphi(\exp y), \ y \in [-\infty, \infty) \) is strictly increasing, downward convex and

\[ \sum_{k \geq 3} \exp(h(k) - h(k + 1)) < \infty. \]

The set of all those function we will denote \( \Phi; \ \Phi = \{\varphi\} \). Let us define the following \( N – \) Orlicz function \( N = N(u) = N(\varphi; u) : \text{at} |u| \leq C_1 \)

\[ N(\varphi; u) = C_2 |u|, \ C_1, C_2 = C_{1,2}(\varphi(\cdot)) \in (0, \infty) \quad (2.1) \]
and for $|u| > C_1$

$$N(\varphi; u) = \exp \varphi(|u|). \tag{2.2}$$

and will denote the correspondent Orlicz’s space and norm $B(\varphi) \overset{df}{=} L(N(\varphi, \cdot)), || \cdot || B(\varphi)$:

$$||f|| B(\varphi) = ||f|| L(N(\varphi; \cdot)).$$

It is very simple to prove the existence for all $\varphi \in \Phi$ the values $C_1 = C_1(\varphi), C_2 = C_2(\varphi)$ such that $N(\varphi; u)$ is some $N -$ Orlicz’s function.

We denote also $C_3 = C_3(\varphi) = \max(1, C_1, 1/C_2), k_0 = \max\{4 + \max(\log C_1, 1), h^{+/}(1)\}$,

$$C_4(\varphi) = e^2 \left[ N(\varphi; \exp(k_0 - 2)) + \sum_{k \geq k_0} \exp(h(k - 1) - h(k)) \right].$$

Let us consider some examples. Put $\varphi(z) = \varphi_{m,r}(z) = z^m \log^{-m r}[(\exp(m + |r|) + z], z \geq 0, m = const \in (0, \infty); or

$\varphi(z) = \varphi_{\nu}(z) = \log^{1+\nu}(1 + z), z \geq 0, \nu = const > 0$. Then $\varphi_{m,r} \in \Phi, \varphi_{\nu} \in \Phi$. We will denote the norm in this spaces as

$$||f|| B(m, r) = ||f|| B(\varphi_{m,r}(\cdot)), ||f|| B(m) = ||f|| B(m, 0).$$

3 Main results.

We will prove that the direct generalization of inequality (1.2) is true for exponential Orlicz spaces. Denote for $\varphi \in \Phi$

$$W(\varphi; n) = \sup_{Q \in A, Q \neq 0} \frac{||dQ/dx|| B(\varphi)}{||Q|| B(\varphi)}, \tag{3.1}$$

where ”sup” is calculated over all the algebraic polynomials $Q; Q \neq 0, \deg Q = n.$
Theorem 1. For all $\varphi(\cdot) \in \Phi$ there exists $C_5(\varphi) \in (0, \infty)$ such that

$$C_5(\varphi) \ n^2 \leq W(\varphi, n) \leq n^2 \cdot K(4) \cdot \max(1, \psi(4)) \cdot C_4 \cdot C_3. \quad (3.2)$$

Note than since $\mu(X) = 1$, the inequalities (3.2) hold for all the Orlicz’s spaces with the $N$ – functions which are equivalent to $N(\varphi; u)$.

In order to formulate an other result, we introduce some new notations. For $\varphi \in \Phi$ we denote

$$\psi(p) = \psi(\varphi; p) = \exp(h^*(p)/p),$$

where $h^*(p)$ denotes the classical Young - Fenchel, or Legendre transform:

$$h^*(p) = \sup_{y \in (-\infty, \infty)} (py - h(y)).$$

We define for $r = 1, 2, \ldots$, $\varphi \in \Phi$ a new quasinorm

$$\|f\|_{V(\varphi; r)} \overset{\text{def}}{=} \sup_{\beta \in (4/((4r+1),1/r)} \|f\|_{\beta}/\psi(\beta/(1-r\beta))$$

and the correspondent space of measurable functions $V(\varphi; r)$ with finite norm $\|f\|_{V(\varphi; r)} < \infty$.

Theorem 2. \forall \varphi \in \Phi, \forall Q \in \mathbb{R}

$$\|Q(r)\|_{V(\varphi; r)} \leq C_4(\varphi) \cdot D(r) \cdot (\deg Q)^{r} \cdot ||Q||_{B(\varphi)}. \quad (3.3)$$

4 Auxiliary result.

Let us introduce a new Banach space $G(\varphi)$, $\varphi \in \Phi$, as a set of all measurable functions $f : X \rightarrow R$ with finite norm

$$\|f\|_{G(\varphi)} \overset{\text{def}}{=} \sup_{p \geq 1} |f|_p/\psi(p) < \infty.$$ 

Note than by virtue of Iensen - Lyapunov inequality

$$\sup_{p \geq 4} \|f\|_p/\psi(p) \leq \|f\|_{G(\varphi)} \leq \max(1, \psi(4)) \sup_{p \geq 4} ||f||_p/\psi(p).$$
Theorem 3. We propose that the norms $\| \cdot \|_B(\varphi)$ and $\| \cdot \|_G(\varphi)$ are equivalent:

$$C_3^{-1} \| f \|_G(\varphi) \leq \| f \|_B(\varphi) \leq C_4 \| f \|_G(\varphi). \quad (4.1)$$

Proof of theorem 3. Assume at first $\| f \|_B(\varphi) < \infty$. Without loss of generality we can suppose

$$I(N(\varphi; |f|) = 1.$$

Let us introduce the function

$$g(p) = \sup_{z > 0} z^p / N(\varphi; z).$$

We have for all the values $p \geq 1$:

$$g(p) \leq \max \left[ \max_{z \in (0, C_1]} C_2^{-1} z^{p-1}, \sup_{z \geq C_1} z^p \exp(-\varphi(z)) \right] \leq$$

$$\max \left[ C_2^{-1} \max(C_1, 1)^{p-1}, \exp(\sup_{z \geq C_1} (p \log z - \varphi(z))) \right] <$$

$$\max \left[ C_2^{-1} \max(C_1, 1)^{p-1}, \exp(\sup_{v \in (-\infty, \infty)} (pv - h(y))) \right] \leq$$

$$\max [C_3^p, \exp h^*(p)] \leq C_3^p \exp h^*(p).$$

Following, for the values $p \geq 1$ we have: $z \geq 0 \Rightarrow$

$$z^p \leq g(p) \ N(\varphi; z) \leq C_3^p \ \psi^p(p) \ N(\varphi; z).$$

Therefore

$$|f|^p \leq C_3^p \ \psi^p(p) \ N(\varphi; h^*(|f|)), \quad |f|_p \leq C_3(\varphi) \ \psi(p),$$

$$\| f \|_G(\varphi) \leq C_3(\varphi(\cdot)) < \infty.$$

Inverse, suppose

$$|f|^p_p \leq \exp(h^*(p)), \quad p \geq 1.$$

We have by virtue of Chebyshev’s inequality for the values $w \geq e^2$:
\( T(|f|, w) \overset{\text{def}}{=} \mu \{ x : |f(x)| > w \} \leq \exp (h^*(p) - p \log w) \),
and after the minimization over \( p \):
\[
T(|f|, w) \leq \exp \left( - \sup_{p \geq 1} (p \log w - h^*(p)) \right) \leq \\
\exp \left( - \sup_p (p \log w - h^*(p)) \right) = \\
\exp (-h^{**}(\log w)) = \exp (-h(\log w))
\]
by virtue of theorem Fenchel - Moraux.

We conclude for the value \( \varepsilon = \exp(-2) \), choosing \( W(k) = \exp(k) \) and denoting
\[
U(k) = U(|f|, k) = \{ x : W(k) \leq |f(x)| < W(k + 1) \} :
\]
\[
I(\exp(N(\varphi; \varepsilon |f|))) = \int_{\{x:|f(x)| \leq \exp(k_0)\}} N(\varepsilon |f(x)|) \, d\mu + \\
\int_{\{x:|f(x)| > \exp(k_0)\}} N(\varphi; \varepsilon |f(x)|) \, d\mu \leq N(\varphi; \exp(k_0 - 2)) + \\
\sum_{k \geq k_0} \int_{U(k)} \exp(\varphi(\varepsilon |f(x)|)) \, dx \leq \\
N(\varphi; \exp(k_0 - 2) + \sum_{k \geq k_0} \exp[(h(\varepsilon W(k + 1))] \cdot [T(|f|, W(k))] \leq \\
N(\varphi; \exp(k_0 - 2)) + \sum_{k \geq k_0} \exp(h(k) - h(k + 1)) = C_4 \, e^{-2} < \infty.
\]
This completes the proof of theorem 3.

Note than this result is some generalization of [5], p. 309 - 314, [6], [8], [17], p. 305.
For example, let $N(u) = N_{m,r}(u) = N(\varphi_{m,r}(\cdot); u)$. It follows from theorem 3 that
\[
||f||L(B(m, r)) < \infty \iff \sup_{p\geq 4} |f|_p \cdot (p^{-1/m} \log^{mr} p) < \infty,
\]
or equally
\[
\exists \varepsilon > 0, \ I(N(\varphi_{m,r}; \varepsilon|f|) < \infty \iff \sup_{p\geq 4} |f|_p \cdot (p^{-1/m} \log^{mr} p) < \infty.
\]

**Notice.** Let us introduce the weight Lorentz norm:
\[
||f||_b G(\varphi) = \sup_{p \geq 1} ||f||_{p,b} / \psi(p),
\]
where $||f||_{p,b}$ is the Lorentz norm (more exactly, seminorm):
\[
||f||_{p,b} = \left[ \int_0^\infty T^{p/b}(|f|, x) \, dx \right]^{1/b},
\]
$p \in [1, \infty)$, $b \in [1, \infty]$, where in the case $b = +\infty$
\[
||f||_{p,\infty} = \sup_{x \geq 0} \left( x \, T^{1/p}(|f|, x) \right).
\]

Using the embedding theorem for Lorentz spaces it is easy to prove as well as by proving of theorem 4 that all the following norms are equivalent
\[
|| \cdot || B(\varphi) \sim || \cdot || G(\varphi) \sim || \cdot ||_b^{*} G(\varphi)
\]
with constants does not depending on $b$. Therefore, it is easy to formulate theorems 1, 2 in the terms of those spaces.

## 5 Proof of the main results.

**Proof of theorem 1.** The low bound in (3.2) is attained, for instance, on the so-called Jacobi ultraspherical polynomials $Q = P_{n}^{2,2}$:
\[
P_{n}^{2,2}(x) = \frac{(-1)^n (1 - x^2)^{-2}}{2^n \, n!} \left( \frac{d}{dx} \right)^n \left[ (1 - x^2)^{n+2} \right]:
\]
\[
|| (d/dx) P_{n}^{2,2} ||_p \geq C \, n^2 \, || P_{n}^{2,2} ||_p, \ n = \deg P_{n}^{2,2}.
\]
see [18], p. 66, 165; [11]. The upper bound may be simple prove by virtue of theorem 3. Namely, suppose $Q \in A$, $Q \neq 0$, $\deg Q = n \geq 2$, $\|Q\|B(\varphi) = 1$. Then

$$\|Q\|G(\varphi) \leq C_3\|Q\|B(\varphi) = C_3 < \infty;$$

$$\sup_{p \geq 4} (\|Q\|/\psi(p)) \leq C_3; \Rightarrow \|Q\|_p \leq C_3\psi(p)$$

for all the values $p \geq 4$. It follows from [1] that

$$\|dQ/dx\|_p \leq n^2 \cdot K(4) \cdot \psi(4) \cdot C_3.$$

Therefore

$$\|dQ/dx\|G(\psi) \leq n^2 \cdot K(4) \cdot \max(1, \psi(4)) \cdot C_3,$$

and finally

$$\|dQ/dx\|B(\varphi) \leq n^2 \cdot K(4) \cdot C_3 \cdot C_4 \cdot \max(1, \psi(4)) \cdot \|Q\|B(\varphi).$$

**Proof of theorem 2.** Let $Q \in R$, $\deg Q = n$, $\|Q\|B(\varphi) = 1$. By virtue of theorem 3 we receive:

$$\|Q\|G(\varphi) \leq C_4; \Rightarrow \|Q\|_p \leq C_4\psi(p), p \geq 4.$$

From (1.3) it follows

$$\|Q^{(r)}\|_{\gamma} \leq n^r \cdot D(r) \cdot C_4(\varphi) \cdot \psi(p).$$

Since $p \geq 4$, $\gamma \in [4/(4r + 1), 1/r)$. After the substitution $\beta = p/(pr + 1) \in [4/(4r + 1), 1/r)$, or $p = \beta/(1 - \beta r)$, we obtain that for all $\beta \in [4/(4r + 1), 1/r)$

$$\|Q^{(r)}\|_{\beta} \leq n^r \cdot C_4(\varphi) \cdot D(r) \cdot \psi(\beta/(1 - \beta r)),$$

which is equivalent to the statement (3.3) of theorem 2.
6 Concluding Remark.

The spaces $V(\varphi, r)$ are only Fréchet spaces. Now we investigate the connection between the norm $||f||V(\varphi_m, 0; r)$ in those spaces and tail behavior of distribution $T(|f|, u)$. At first assume that $||f||V(\varphi_m, 0) = 1$ for some $m > 0$. then for all values $\beta \in [4/(4r + 1), 1/r)$

$$||f||_{\beta} \leq \left(\frac{\beta}{1 - r\beta}\right)^{1/m}, \quad I\left(||f||_{\beta}\right) \leq \left(\frac{\beta}{1 - r\beta}\right)^{\beta/m}.$$  

We obtain using the Chebyshev’s inequality for sufficiently large values $u \geq 3$:

$$T(|f|, u) \leq u^{-\beta} \left(\frac{\beta}{1 - r\beta}\right)^{\beta/m}.$$  

We conclude after the minimization over $\beta$:

$$T(|f|, u) \leq C_9(m, r) u^{-1/r} [\log u]^{1/(mr)}, \quad u \geq 3. \quad (6.1)$$  

Inverse, if inequality (6.1) holds, then

$$I\left(||f||_{\beta}\right) \leq C_{11}(\beta, r, m) \int_{3}^{\infty} x^{\beta - 1 - 1/r} (\log x)^{1/mr} \, dx \leq C_{12}(1/r - \beta)^{-(mr+1)/(mr)}; \quad ||f||_{\beta} \leq C_{13}(1/r - \beta)^{(mr+1)/m};$$  

$$||f||V(\varphi_{m/(mr+1)}, 0; r) \leq C_{14}(m, r) < \infty.$$  

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