Research Article

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Fractional Powers of Monotone Operators in Hilbert Spaces

https://doi.org/10.1515/ans-2019-2053
Received November 29, 2018; revised June 20, 2019; accepted June 23, 2019

Abstract: The aim of this article is to provide a functional analytical framework for defining the fractional powers $A^s$ for $-1 < s < 1$ of maximal monotone (possibly multivalued and nonlinear) operators $A$ in Hilbert spaces. We investigate the semigroup $\{e^{-A^t}\}_{t \geq 0}$ generated by $-A^s$, prove comparison principles and interpolations properties of $\{e^{-A^s t}\}_{t \geq 0}$ in Lebesgue and Orlicz spaces. We give sufficient conditions implying that $A^s$ has a sub-differential structure. These results extend earlier ones obtained in the case $s = 1/2$ for maximal monotone operators [H. Brézis, Équations d’évolution du second ordre associées à des opérateurs monotones, *Israel J. Math.* **12** (1972), 51–60], [V. Barbu, A class of boundary problems for second order abstract differential equations, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **19** (1972), 295–319], [V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International, Leiden, 1976], [E. I. Poffald and S. Reich, An incomplete Cauchy problem, *J. Math. Anal. Appl.* **113** (1986), no. 2, 514–543], and the recent advances for linear operators obtained in [L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), no. 7–9, 1245–1260], [P. R. Stinga and J. L. Torrea, Extension problem and Harnack’s inequality for some fractional operators, *Comm. Partial Differential Equations* **35** (2010), no. 11, 2092–2122].

Keywords: Monotone Operators, Hilbert Space, Nonlinear Evolution Equations, Bessel Operator, Dirichlet-to-Neumann Operator, Fractional Powers, Nonlinear Semigroups

MSC 2010: 35R11, 47H05, 47H07, 35B65

1 Introduction and Main Results

In the pioneering work [14], Caffarelli and Silvestre showed that, for every $0 < s < 1$, the fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^d$ ($d \geq 1$) coincides up to a multiple constant with the *Dirichlet-to-Neumann operator*

$$
\Lambda_s \varphi := -\lim_{t \to 0^+} t^{1-2s} u'(\cdot, t)
$$

(1.1)

associated with the (negative) Bessel type operator (in divergence form)

$$
\mathcal{B}_{1-2s} := -t^{2s-1} (t^{1-2s} u')' + Au
$$

(1.2)

in the half-space $\mathbb{R}^d \times (0, +\infty)$ for $A = -\Delta$ in $L^2(\mathbb{R}^d)$. More precisely, the Dirichlet-to-Neumann operator $\Lambda_s$ defined on a Hilbert space $H$ assigns to each Dirichlet data $\varphi$ (in $H$) the outer co-normal derivative $\Lambda_s \varphi$ defined

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by (1.1) (in \(H\)) of the unique solution \(u\) to the Dirichlet problem

\[
\begin{align*}
-\frac{1-2s}{t} u' - u'' + Au & \geq 0 \quad \text{in } H_+ , \\
\phi & \in \partial H_+ = H ,
\end{align*}
\]

\((D^s_\phi)\)

in the half-space \(H_+ := H \times (0, +\infty)\). We refer to Definition 1.9 below for the notion of solutions to \((D^s_\phi)\). Since the solution \(u\) of \((D^s_\phi)\) extends \(\phi\) on \(H\) to \(H_+\) via \(\mathbb{B}_{1-2s}u \geq 0\), the identification

\[
c_s A^s = \Lambda_s
\]

is referred to in the literature (for instance, cf. [2, 16, 19]) as a characterization of the fractional power \(A^s\) via the extension problem \((D^s_\phi)\). In (1.3), the constant \(c_s\) is given by \(c_s = 2^{1-2s} \Gamma(1-s)/\Gamma(s)\) for \(0 < s < 1\), where \(\Gamma\) denotes the Gamma function (see, for example, [2, 14, 16, 33]).

Another milestone in this direction was set shortly afterwards by Stinga and Torrea [33]. They showed that (1.3) even holds for general selfadjoint lower bounded operators \(A\) on a Hilbert space \(H\). The approaches in [14] and [33] are quite different. In the paper [14], Caffarelli and Silvestre use the singular integral definition of the fractional power \((-\Delta)^s\) and for proving (1.3) provide two proofs; one is based on classical pde-methods (e.g., construction of a Poisson kernel); the other one relies on the Fourier transform. Stinga and Torrea used in [33] a general Hilbert space framework and use the integrals

\[
A^s = \int_0^\infty \lambda^s \, dE(\lambda) \quad \text{or (equivalently)} \quad A^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tA} - I_H) \frac{dt}{t^{1+s}}
\]

(1.4)

for defining \(A^s\). In the first integral of (1.4), the family \(\{E(\lambda)\}_{\lambda > 0}\) denotes the spectral resolution of \(A\) and \(\{e^{-tA}\}_{t \geq 0}\) the semigroup generated by \(-A\). Thus the method in [33] to obtain (1.3) relies on the spectral decomposition of \(A\), abstract linear semigroup theory and Bessel functions. The techniques in [33] were pushed further by Galé, Miana and Stinga [19] showing that (1.3) even holds for generators \(A\) of integrated linear semigroups in Banach spaces. Recently, Arendt, ter Elst and Warma [2] revisited the characterization (1.3) by focusing on operators \(A\) realized by a sesquilinear form \(a: V \times V \to \mathbb{C}\) defined on another Hilbert space \(V \subseteq H\) and by using classical interpolation theory.

The goal of this paper is to continue the study of fractional powers \(A^s\) in a nonlinear direction. For this, we aim to derive an appropriate definition of fractional powers \(A^s\) of maximal monotone operators \(A\) defined on a Hilbert space \(H\) (see Section 2.1 for a brief review of this theory, or [6, 13]). Because of the lack of linearity, there is not much hope that the fractional power \(A^s\) can be defined via one of the integrals in (1.4) for general monotone operators \(A\) (cf. the comment in [6, Remark 2, p. 338] by Barbu). But, a good definition of \(A^s\) for monotone \(A\) needs to be consistent with the classical one for linear and monotone \(A\). Thus one possibility to define \(A^s\) for maximal monotone \(A\) is offered by the extension problem.

In the nonlinear framework, first results towards fractional powers \(A^s\) were obtained in the case \(s = 1/2\) by Barbu [5] and sharpened by Brezis [12] with a different method. More precisely, they showed that, for maximal monotone \(A\) with 0 in the range \(\text{Rg}(A)\) of \(A\), the Dirichlet problem \((D^s_\phi)\) is well-posed. They introduced a family \(\{T_t\}_{t \geq 0}\) of contractive mappings \(T_t: D(A) \to D(A)\) by setting

\[
T_t \phi = u_\phi(t), \quad t \geq 0, \quad \phi \in D(A),
\]

where \(u_\phi\) is the unique solution of \((D^s_\phi)\) with \(s = 1/2\). Since, for \(s = 1/2\), the differential inclusion in \((D^s_\phi)\) reduces to

\[
- u''(t) + Au(t) \geq 0 \quad \text{on } H_+ ,
\]

the family \(\{T_t\}_{t \geq 0}\) satisfies the semigroup property

\[
T_{t_1 + t_2} = T_{t_1} \circ T_{t_2} \quad (t_1, t_2 \geq 0),
\]

and so \(\{T_t\}_{t \geq 0}\) extends to a strongly continuous semigroup of contractions \(T_t\) on \(D(A)^H\). Thus, by the nonlinear Hille–Yosida theorem [24] (or see Theorem 2.6), there is a unique maximal monotone operator \(A_{1/2}\) such that
$A_{1/2}^s$ is the infinitesimal generator of $\{T_t\}_{t \geq 0}$. Here, $A_{1/2}^s$ denotes the minimal section of $A_{1/2}$ (see Section 2.1 for the definition). Since, for every solution $u$ of the inclusion (1.5), the co-normal derivative associated with $B_0$ is $-u'(0)$, it follows that, for a given $\varphi \in D(A)$, the unique solution $u_\varphi$ of $(D^s_\varphi)_s=1/2$ satisfies

$$A_{1/2}^s \varphi = -\frac{d}{dt}T_t \varphi |_{t=0} = -u'_\varphi(0) = \Lambda_{1/2}^s \varphi.$$ 

Thus, in fact, the operator $A_{1/2}$ is the unique maximal monotone extension of the Dirichlet-to-Neumann operator $\Lambda_{1/2}$. However, neither Barbu [5] nor Brezis [12] identified the infinitesimal generator $A_{1/2}$ with the Dirichlet-to-Neumann operator $\Lambda_{1/2}$. But, since $A_{1/2}$ shares several similar properties with the classical square root $A^{1/2}$ of a positive linear operator $A$ (cf. [5, 12]), the literature refers to the operator $A_{1/2}$ as the square root of a nonlinear maximal monotone operator (see, for instance, [6, Chapter V, Section 2.4, p. 329] or, more recently, [8]). Thus we write $A^{1/2}$ for $A_{1/2}$ even if $A$ is nonlinear. The case of the square root $A^{1/2}$ suggests that, also for nonlinear maximal monotone operators $A$, the extension technique (1.3) is an appropriate way to define fractional powers $A^s$.

Now, our first aim in this paper is to extend the existence and uniqueness results of Dirichlet problem $(D^s_\varphi)_s=1/2$ obtained in [5, 12] to the complete range $0 < s < 1$. But before stating our first theorem, we briefly fix the following notation. For $1 \leq q \leq \infty$, we write $L^q(H)$ instead of $L^q(0, +\infty; H)$, $L^q_{\text{loc}}(H)$ is the abbreviation of the space $L^q_{\text{loc}}((0, +\infty); H)$, $L^2(H)$ denotes the classical $L^2$-space of measurable functions $u : (0, +\infty) \rightarrow H$ equipped with the Haar measure $dt$, $W^{1,1}_1(H)$ is the set of functions $u \in L^1_{\text{loc}}(H)$ with distributional derivative $u' \in L^1_{\text{loc}}(H)$, and $W^{1,1}_{s,1}(H)$ denotes the weighted Sobolev space of all $u \in W^{1,1}_{s,1}(H)$ having $t^s u$ and $t^{s} u'$ in $L^2(H)$ (see Section 2.3). Finally, for an operator $A$ on $H$, the set $\text{Rg}(A) := \bigcup_{x \in H} Ax \subset H$ is called the range of $A$. For more details, we refer to Section 2.1.

**Theorem 1.1.** Let $A$ be a maximal monotone operator on $H$ with $0 \in \text{Rg}(A)$ and $0 < s < 1$. Then, for every $\varphi \in D(A)^H$, there is a unique solution $u \in L^\infty(H)$ of the Dirichlet problem $(D^s_\varphi)$. Moreover, for $y \in A^{-1}(\{0\})$, each solution $u$ of $(D^s_\varphi)$ satisfies

$$||t^{1-2s}u'(t)||_H \leq ||t^{1-2s}u'(t)||_H$$

for all $t \geq 0 > 0$, (1.6)

$$||tu'||_L^2(H) \leq \sqrt{s}||\varphi - y||_H,$$

for all $t > 0$, (1.7)

$$||t^{1-2s}u'(t)||_H \leq \frac{2}{t^{2s}}||\varphi - y||_H$$

for all $t > 0$, (1.8)

$$||t^{1-2s}u'(t)||_H \leq \frac{\sqrt{2(2-s)s}}{2}||tu'||_L^2(H),$$

and if $\varphi \in D(A)$, then $t^{1-2s}u' \in W^{1,1}_{s,1}(H)$,

$$\lim_{t \to 0^+} t^{1-2s}u'(t) ||_H \leq \frac{2(1-s)}{s}||A^0\varphi||_H + ||\varphi - y||_H,$$ (1.10)

$$||u'||_L^2(H) \leq \frac{\sqrt{2(1-s)}}{\sqrt{s}} ||A^0\varphi||_H + ||\varphi - y||_H ||\varphi - y||_H^{1/2},$$ (1.11)

$$||t^{1-2s}u'(t)||_H \leq \frac{\sqrt{2(1-s)}}{\sqrt{s}} ||A^0\varphi||_H + ||A^0\varphi||_H ||\varphi - y||_H^{1/2}.$$ (1.12)

Finally, for every two solutions $u_1$ and $u_2 \in L^\infty(H)$ of $(D^s_\varphi)$, one has

$$||u_1(t) - u_2(t)||_H \leq ||u_1(\hat{t}) - u_2(\hat{t})||_H$$

for all $t \geq \hat{t} \geq 0$. (1.13)

The proof of our first main theorem is based on classical techniques from the theory of maximal monotone operators as used in [12] by Brezis, and combines them with weighted function spaces (as, e.g., $L^2_1(H)$ or $W^{1,1}_{s,1}(H)$) borrowed from linear interpolation theory as in the recent work by Arendt, ter Elst and Warma [2].

It is not difficult to verify that, for $u \in W^{1,1}_{s,1}(H)$ with $u \in L^2_1(H)$ ($0 < s < 1$), the limit $u(0) := \lim_{t \to 0^+} u(t)$ exists in $H$ (see Proposition 2.17 for more details). Thus, by Theorem 1.1, for every $\varphi \in D(A)$, one has that the co-normal derivative associated with $B_{1-2s}$

$$\Lambda_{s} \varphi := -\lim_{t \to 0^+} t^{1-2s}u'(t) \in H.$$ (1.14)
This shows that the Dirichlet-to-Neumann operator \( \Lambda_s : D(A) \to H \) defined by \( (1.14) \) is a well-defined mapping. Moreover, we have the following. Here, \( H_w \) denotes the Hilbert space \( H \) equipped with the weak topology \( \sigma(H, H') \), and for the definition of \( \partial \psi \)-monotone operators and completely accretive operators, we refer to Definition 2.9 in Section 2.1 and Definition 2.14 in Section 2.2.

**Theorem 1.2.** Let \( A \) be a maximal monotone operator on \( H \) with \( 0 \in \text{Rg}(A) \). Then, for every \( 0 < s < 1 \), the Dirichlet-to-Neumann operator

\[
\Lambda_s := \{ (\varphi, w) \in H \times H \mid \text{there exists a solution } u_\varphi \in L^\infty(H) \text{ of Dirichlet problem } (D^s_\varphi) \text{ with } w = - \lim_{t \to 0^+} t^{1-2s} u'(t) \in H \}
\]

is a monotone, well-defined mapping \( \Lambda_s : D(\Lambda_s) \to H \) satisfying the following:

1. \( D(A) \subseteq D(\Lambda_s) \subseteq D(A)^H \).
2. \( D(A) \subseteq \text{Rg}(I_H + \lambda A_s) \) for all \( \lambda > 0 \).
3. For every \( \varphi \in D(A) \) and \( y \in A^{-1}((0)) \), one has that
   \[
   \| \Lambda_s \varphi \|_H \leq \frac{2(1-s)}{s} (\| A^0 \varphi \|_H + \| \varphi - y \|_H). \tag{1.15}
   \]
4. If \( A = \partial \phi \) is the sub-differential operator of a proper, convex and lower semicontinuous function \( \phi : H \to (-\infty, +\infty) \), then \( \Lambda_s \) is cyclically monotone for every \( 0 < s < 1 \), and so there is a proper, convex and lower semicontinuous function \( \phi_s : H \to (-\infty, +\infty) \) such that \( \Lambda_s \subseteq \partial \phi_s \).
5. If \( A = \partial \psi \)-monotone for a given convex, proper and lower semicontinuous function \( \psi : H \to (-\infty, +\infty) \), then also \( \Lambda_s \) is \( \partial \psi \)-monotone for every \( 0 < s < 1 \).
6. Let \( H = L^2(\Sigma, \mu) \) be the Lebesgue space of square-integrable functions defined on a \( \sigma \)-finite measure space \( (\Sigma, \mu) \). Then \( \Lambda_s \) is completely accretive on \( L^2(\Sigma, \mu) \) for every \( 0 < s < 1 \) provided \( A \) is completely accretive on \( L^2(\Sigma, \mu) \).

Further, the closure \( \overline{\Lambda_s}^{H_w} \) of \( \Lambda_s \) in \( H \times H_w \) is characterized by

\[
\overline{\Lambda_s}^{H_w} = \{ (\varphi, w) \in H \times H \mid \text{there exist } (\varphi_n)_{n \geq 1} \subseteq D(\Lambda_s), \text{ solutions } u_\varphi \text{ of } (D^s_\varphi) \text{ and } u_{\varphi_n} \text{ of } (D^s_\varphi)|_{\varphi = \varphi_n} \text{ s.t. } u_{\varphi_n} \to u_\varphi \text{ in } C^0([0, +\infty); H) \text{ and } \Lambda_s \varphi_n \to w \text{ in } H_w \},
\]

and if \( D(A) \) is dense in \( H \), then \( \overline{\Lambda_s}^{H_w} \) is maximal monotone in \( H \).

**Remark 1.3.** The case \( A = \partial \phi \), the sub-differential operator of a proper, convex, lower semicontinuous function \( \phi \) on \( H \), provides more regularity properties to the solution \( u \) of \( (D^s_\varphi) \). In particular, the theory of \( j \)-elliptic functionals developed in [16] can be employed to derive an energy functional of the Dirichlet-to-Neumann operator \( \Lambda_s \). We intend to study this case in full detail in a separate paper.

Thanks to Theorem 1.2 and the theory of maximal monotone operators (see Propositions 2.1 and 2.2 in Section 2.1), for every \( 0 < s < 1 \), there exists a (unique) maximal monotone operator \( M_s \) satisfying

\[
D(\Lambda_s) \subseteq D(M_s) \subseteq D(A)^H \quad \text{and} \quad \frac{1}{c_s} \Lambda_s \varphi = M_s \varphi \quad \text{for every } \varphi \in D(\Lambda_s), \tag{1.16}
\]

where \( c_s = 2^{1-2s} \Gamma(1-s)/\Gamma(s) \). Since, for a (maximal) monotone operator \( A \), also its inverse \( A^{-1} \) defines a (maximal) monotone operator (see \( (2.1) \)), for every \( -1 < s < 0 \), there is a unique maximal monotone operator \( (M_{[-s]})^{-1} \) satisfying

\[
\left( \frac{1}{c_{[-s]}} \Lambda_{[-s]} \right)^{-1} \subseteq (M_{[-s]})^{-1}.
\]

It is worth mentioning that, for \( -1 < s < 0 \), the inverse operator \( (\Lambda_{[-s]})^{-1} \) of \( \Lambda_{[-s]} \) is the Neumann-to-Dirichlet operator associated with the Bessel type operator \( (1.2) \).

**Remark 1.4.** We recall that the operator \( M_s \) is the unique maximal monotone set \( B \) including \( \frac{1}{c_s} \Lambda_s \) (cf. Proposition 2.1) and closed in \( H \times H_w \).

With this in mind, we are now in the position to define for \( -1 < s < 1 \) the \( L^2 \)-fractional power \( A^s \) in \( H \) of a maximal monotone operator \( A \) in \( H \) with \( 0 \in \text{Rg}(A) \).
Definition 1.5. For $0 < s < 1$, the $L^2$-fractional power $A^s$ of a maximal monotone operator $A$ in $H$ with $0 \notin \text{Rg}(A)$ is defined by the maximal monotone extension $M_s$ of the (scaled) Dirichlet-to-Neumann operator $\frac{1}{\epsilon^2} \Lambda_s$ satisfying (1.16). Further, we set $A^0 = I_H$ the identity on $H$. For $-1 < s < 0$, the $L^2$-fractional power $A^s$ of a maximal monotone operator $A$ in $H$ with $0 \notin \text{Rg}(A)$ is defined by the inverse operator $(M_{-s})^{-1}$ of the maximal monotone extension $M_{(-s)}$ of $\frac{1}{\epsilon^2} \Lambda_{(-s)}$.

Remark 1.6. (a) In the linear theory, the fractional power $-A^s$ of a maximal monotone operator $A$ generates a strongly continuous semigroup $\{e^{-A^t}\}_{t \geq 0}$ (see, for instance, [22, 33]). We show in Corollary 1.8 below that this property also holds for the $L^2$-fractional power $A^s$ of a maximal monotone (possibly nonlinear and multivalued) operator $A$. In particular, the recent results by Stinga and Torrea [33] or Arendt, ter Elst and Warma [2] show that the fractional power $A^s$ defined by one of the integrals (1.4) for a linear monotone operator coincides with the $L^2$-fractional power $A^s$ defined in Definition 1.5.

(b) In the nonlinear theory, Definition 1.5 of the $L^2$-fractional power $A^s$ of $A$ extends the case $s = 1/2$ (cf. [5, 12]).

(c) We intentionally call $A^s$ the $L^2$-fractional power of a maximal monotone operator $A$ in $H$ since we already know that the Dirichlet problem $(D^s_\alpha)$ admits an $L^p$ analogue involving an $L^p$-Bessel operator. We conjecture that the Dirichlet-to-Neumann operator $\Lambda_s$ in $L^p$ associated with an $L^p$-Bessel type operator for the classical $p$-Laplace operator $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ then coincides up to a universal constant with the fractional $p$-Laplace operator $(-\Delta_p)^s$ defined by the singular integral

$$
(-\Delta)^s u(\cdot) = \lim_{\epsilon \to 0} \int_{B_\epsilon(y)} \frac{|u(\cdot) - u(y)|^{p-2}(u(\cdot) - u(y))}{|\cdot - y|^{d+sp}} \, dy.
$$

(d) For the class of linear positive maximal monotone operators $A$, the negative power $A^{-s}$ is defined by $(A^s)^{-1}$ (see, for instance, [28, Definition 7.1.1]). In the nonlinear framework, operators $A$ are subsets of $H \times H$, and hence the condition on $A$ of being injective is redundant for defining $A^{-1}$.

(e) In the linear theory, under additional conditions on $A$, the following fundamental properties are known to hold:

$$
A^s A^t = A^{s+t} \quad \text{(additivity),}
$$

$$
(A^s)^t = A^{st} \quad \text{(multiplicativity),}
$$

$$
\lim_{s \to 0^+} A^s = I_H, \quad \lim_{s \to 1^-} A^s = A \quad \text{in the sense of resolvents}
$$

(see, for instance, [22]). However, to the best of our knowledge, all known proofs of these three fundamental properties of $A^s$ employ integrals formulas of $A^s$ similar to (1.4). In the nonlinear theory, the classical definition of the composition $A^s A^t$ in (1.17) does not make much sense for nonlinear operators $A$ and would require a new definition (as, for instance, in [1]). Thus we are not convinced that the properties (1.17) and (1.18) can be established for general maximal monotone operators $A$ without introducing new definitions of $A^s A^t$ and $(A^s)^t$. On the other hand, we conjecture that the limits in (1.19) also hold for (nonlinear) maximal monotone operators $A$ (maybe under additional coercivity/compactness conditions satisfied by the resolvent of $A$).

Due to our analysis of the Dirichlet-to-Neumann operator $\Lambda_s$ stated in Theorem 1.2, the $L^2$-fractional power $A^s$ of $A$ in $H$ admits the following properties. We omit the proofs of these statements since they can be deduced immediately from the previous theorem.

Theorem 1.7. Let $A$ be a maximal monotone operator on $H$ with $0 \notin \text{Rg}(A)$. Then the following statements hold.

1. $D(A^s)$ is dense in $D(A^H)$ for every $0 < s < 1$.

2. Suppose $D(A)$ is dense in $H$. Then, for every $0 < s < 1$, the fractional power $A^s$ of $A$ can be characterized by

$$
c_s A^s = \Lambda_s^{H\times H},
$$

and the following additional properties for $A^s$ hold:

(i) If $A = \partial \phi$ is the sub-differential operator of a proper, convex and lower semicontinuous function $\phi : H \to (-\infty, +\infty]$, then, for every $-1 < s < 1$, there is a proper, convex and lower semicontinuous function $\phi_s : H \to (-\infty, +\infty]$ such that $A^s = \partial \phi_s$. 

(ii) If $A$ is $\partial \psi$-monotone for a given convex, proper, lower semicontinuous function $\psi : H \to (-\infty, +\infty]$, then $A^s$ is $\partial \psi$-monotone for every $0 < s < 1$.

(iii) Let $H = L^2(\Sigma, \mu)$ be the Lebesgue space of square-integrable functions defined on a $\sigma$-finite measure space $(\Sigma, \mu)$, and $A$ is completely accretive on $L^2(\Sigma, \mu)$. Then $A^s$ is completely accretive on $L^2(\Sigma, \mu)$ for every $0 < s < 1$.

Next, we turn to the evolution problem associated with $A^s$. In the case $H = L^2(\Sigma, \mu)$ of a $\sigma$-finite measure space $(\Sigma, \mu)$, the semigroup $\{e^{-tA}\}_{t \geq 0}$ generated by the negative $L^2$-fractional power $A^s$ of a maximal monotone operator $A$ inherits the following interpolation properties. We refer to Section 2.2 and Definition 2.16 for the notion of order-preserving mappings and Orlicz spaces $L^\Psi(\Sigma, \mu)$ with $N$-function $\psi$.

**Corollary 1.8.** Let $A$ be a maximal monotone operator on $H$ with $0 \in \text{Rg}(A)$. Then, for every $−1 < s < 1$, there is a strongly continuous semigroup $\{e^{-tA}\}_{t \geq 0}$ generated by $−A^s$ on $D(A^s)^\ast$. Moreover, if $D(A)$ is dense in $H$, then, for $0 < s < 1$, the following statements hold.

1. For every $\varphi \in H$, there is a mapping $U : [0, +\infty) \times [0, +\infty) \to H$ such that $U(\cdot, t) \in C^b([0, +\infty); H)$ for every $t \geq 0$ is the unique solution of
   \begin{equation}
   \begin{cases}
   -\frac{1}{r} 2 \nu_s U_t(r, t) = U_r(r, t) + AU(r, t) \geq 0 & \text{on } H_+, \\
   U(0, t) = e^{-tA^s} \varphi;
   \end{cases}
   \end{equation}
   if $\varphi \in D(A^s)$, then $U(0, t) = e^{-tA^s} \varphi$ satisfies, in addition,
   \begin{equation}
   \begin{cases}
   \frac{d}{dt} U(0, t) + A^s U(0, t) \geq 0 & \text{for a.e. } t > 0, \\
   U(0, 0) = \varphi;
   \end{cases}
   \end{equation}
   for a.e. $t > 0$, there is a sequence $(\varphi^n)_{n \geq 1} \subseteq D(A_s)$ such that
   \[ c_5 A^s e^{-tA^s} \varphi = \lim_{n \to +\infty} A_s \varphi^n. \]

2. If, in addition, $A = \partial \phi$ is the sub-differential of a proper, convex and lower semicontinuous function $\phi : H \to (-\infty, +\infty]$, then the semigroup $\{e^{-tA}\}_{t \geq 0}$ satisfies the evolution equation (1.21) for all $\varphi \in H$.

3. If $\varphi \in D(A)$, then the unique solution $U$ of (1.20) satisfies the integration by parts rule
   \[ \left( \lim_{t \to 0^+} r^{1-2s} U_t(r, 0) , \xi(0) \right)_H = \int_0^\infty \left( \left\langle r^{1-2s} U_t(r, 0) , \xi(r) \right\rangle_H + \int_0^\infty \left\langle r^{1-2s} U_t(r, 0) , \xi(r) \right\rangle_H \right) dr \]
   for every $\xi \in W^{1,2}_s(H)$.

3. Suppose $H = L^2(\Sigma, \mu)$ of a $\sigma$-finite measure space $(\Sigma, \mu)$, and $A$ is, in addition, completely accretive with some $\psi \in A^{-1}(0) \cap L^2(\Sigma, \mu)$. Then the semigroup $\{e^{-tA}\}_{t \geq 0}$ generated by $-A^s$ on $L^2(\Sigma, \mu)$ has the following additional properties.
   (a) The semigroup $\{e^{-tA}\}_{t \geq 0}$ is order-preserving.
   (b) Every map $e^{-tA^s}$ of $\{e^{-tA}\}_{t \geq 0}$ has a unique contractive extension on $L^q(\Sigma, \mu)$, where $1 \leq q < \infty$, on $L^2 \cap L^{\infty}(\Sigma, \mu)$ and on $L^q(\Sigma, \mu)$ (for every right-continuous $N$-function $\psi$) such that the family $\{e^{-tA}\}_{t \geq 0}$ is again an order-preserving semigroup, and $\{e^{-tA^s}\}_{t \geq 0}$ is strongly continuous on $L^q(\Sigma, \mu)$, where $1 \leq q < \infty$, and on $L^q(\Sigma, \mu)$.

The statements of this corollary are direct consequences of the properties of the $L^2$-fractional power $A^s$ of $A$ obtained in Theorem 1.7, where, for statement (2) in Corollary 1.8, one also uses the smoothing effect of a semigroup generated by a sub-differential operator (see Theorem 2.7 in Section 2.1). We omit the details of this proof.

The theory of fractional powers of operators has a long history, and to the best of our knowledge, it goes back to Abel’s work on the tautochrone, the Riemann–Liouville integral and its generalizations (cf. [22]). In the late 1950s, Hille and Phillips [20] developed the first functional analytical framework of fractional powers $A^s$ for a subclass of positive linear operators $A$ in Banach spaces $X$. Here, a linear operator $A$ on $X$ is
called positive if the interval $(-\infty, 0)$ is contained in the resolvent set $\rho(A)$ of $A$. In 1960, Balakrishnan [4] gave a new definition and extended the theory in [20] to a wider class of positive linear operators; more precisely, for $0 < s < 1$, Balakrishnan proposed to define the fractional operator $A^s$ via the integral

$$A^s x = \frac{\sin s\pi}{\pi} \int_0^\infty t^{s-1}(t + A)^{-1}Ax \, dt \quad \text{for all } x \in D(A),$$

which is later referred to as the Balakrishnan integral (cf. [22, 28] or Galé, Miana and Stinga [19]). In particular, Balakrishnan proved that the identity (1.3) for the square root $A^{1/2}$ holds true (cf. the monograph [28]). Throughout the 1960s, fractional powers of linear operators received a lot of attention and were studied by various celebrated authors (e.g., Krasnosel’skii and Sobolevskii [25], Yosida [35]). To the best of our knowledge, the link between fractional powers $A^s$ of linear operators $A$ and interpolation theory was mainly unveiled by Komatsu [23] and recently used in [2]. One reason for the high popularity of fractional powers of linear operators was certainly Kato’s celebrated square root problem [3, 21].

**Problem.** Given a sesquilinear $\omega$-sectorial form $a : V \times V \to \mathbb{C}$ defined on dense subspace $V$ of a Hilbert space $H$ and the part $A$ of $a$ in $H$, it is always true that the domain

$$D(A^{1/2}) = V \quad \text{with} \quad \|A^{1/2}u\|_H^2 + \|u\|_H^2 \equiv \Re a(u, u) + \|u\|_H^2?$$

Kato’s square root problem is intimately related to bounds on the celebrated Riesz potential and fundamental questions in defining Sobolev norms. Thus it is quite remarkable that Kato’s square root problem could be solved by Brezis [12] for general (possibly nonlinear) sub-differential operators $A = \partial \phi$ of a proper, convex and lower semicontinuous function $\phi : H \to (-\infty, +\infty)$ attaining a global minimum.

We do not intend to introduce the notion of linear operators $A$ in $H$ realized by a sesquilinear $\omega$-sectorial form (for details, see, e.g., [2]). But, for enlightening the link between the two classes of linear and nonlinear operators treated in this paper, it is worth mentioning that, for linear selfadjoint and maximal monotone operators $A$, the linear and nonlinear semigroup theory coincides (cf. [13, Proposition 2.15] or [6, Chapter II, Proposition 2.7]; see also [16]). In other words, both theories coincide in the symmetric case.

In the case $s = 1/2$, Bénilan [9] established existence and uniqueness of solutions to the differential inclusion (1.5) equipped with Dirichlet boundary data for non-autonomous uniformly continuous $A(t)$ and when the Hilbert space $H$ is replaced by a Banach space $X$.

In [34], Véron proved that the following more general (but non-singular) Dirichlet problem

$$\left\{ \begin{array}{ll} -p(t)u''(t) - q(t)u'(t) + Au(t) \geq 0 \quad & \text{in } H_+, \\
 & u(0) = \varphi \end{array} \right.$$

is well-posed for $\varphi \in D(A)$ under the hypothesis that $p \in W^{2,\alpha}(0, +\infty)$, $q \in W^{1,\alpha}(0, +\infty)$ and there is an $a > 0$ satisfying $p(t) \geq a > 0$ for all $t \geq 0$.

Poffald and Reich [29, 30] continued the study by Bénilan [9] on the regularity of solutions to (1.5) in the Banach space case $X$. They established new results on the asymptotic behavior as time $t \to +\infty$ of solutions $u$ to (1.5) and studied existence of solutions to

$$\left\{ \begin{array}{ll} u''(t) + Au(t) \geq f(t) \quad & \text{in } X_+ := X \times (0, +\infty), \\
 & u(0) = \varphi. \end{array} \right.$$

In particular, in [30, Theorem 3.1], Poffald and Reich could sharpen the regularity of solutions to the differential inclusion (1.5) by proving that $T_t$ for every $t > 0$ maps $D(A^{1/2})$ into $D(A^{1/2})$.

Ten years later, Alraabiou and Bénilan [1] introduced the following definition of the square:

$$A^2 := \lim_{\lambda \to 0^+} \frac{A - A_{1/2}}{\lambda}$$

via the Yosida approximation $A_{1/2} := (I - (I + \lambda A)^{-1})/\lambda$ and showed that $A \subseteq (A^{1/2})^2$ provided $A = \partial \phi$ is the sub-differential operator a convex, proper, lower semicontinuous function $\phi$ on $H$. Of course, definition (1.22) of the square power $A^2$ coincides with $A^2 = A$ if $A$ is bounded linear and monotone. For this class
of linear operators $A$, the square root of $A$ is the unique operator $B$ satisfying $A \circ B = B$. Thus the result [1] by Alraabiou and Bénilan supports the point of view (at least for the square root $A^{1/2}$) that the Dirichlet-to-Neumann operator $A_{j}$ is a reasonable candidate for defining the fractional power $A^{s}$ of a maximal monotone operator $A$ in a Hilbert space.

For boundary data $\varphi \in D(A)$, we establish existence and uniqueness of solutions of the more general boundary-value problem

$$\begin{cases}
-\frac{1 - 2s}{t} u'(t) - u''(t) + Au(t) \geq 0 & \text{in } H_+,
\lim_{t \to 0^+} t^{1 - 2s} u'(t) \in \partial j(u(0) - \varphi) & \text{on } \partial H_+,
\end{cases}
$$

(1.23)

where $j: H \to [0, +\infty]$ is a proper, convex, lower semicontinuous function on $H$. For this, we recall that the indicator function $j: H \to [0, +\infty]$ is defined by $j(u) := 0$ if $u = 0$, and otherwise $j(u) := +\infty$ (cf. [7, 13]). Before stating the theorem, we introduce the notion of solutions of boundary-value problem (1.23).

**Definition 1.9.** A function $u: [0, +\infty) \to H$ is called a strong solution of

$$-\frac{1 - 2s}{t} u'(t) - u''(t) + Au(t) \geq 0 \quad \text{in } H_+,
$$

(1.24)

if $u \in W^{1,2}_{loc}((0, +\infty); H)$ and $u(t) \in D(A)$ and $\{t^{1 - 2s} u'(t)\}' \in t^{1 - 2s} A(u(t))$ for almost every $t > 0$. Given $\varphi \in H$, a function $u: [0, +\infty) \to H$ is called a solution of problem (1.23) if $u \in C([0, +\infty); H)$, $u$ is a strong solution of (1.24), $\lim_{t \to 0^+} t^{1 - 2s} u'(t)$ exists in $H$ and $\lim_{t \to 0^+} t^{1 - 2s} u'(t) \in \partial j(u(0) - \varphi)$. In particular, for a given boundary value $\varphi \in H$, a function $u: [0, +\infty) \to H$ is called a solution of Dirichlet problem (1.28) if $u \in C([0, +\infty); H)$, $u(0) = \varphi$ and $u$ is a strong solution of (1.24).

**Remark 1.10.** We note that our definition of strong solutions of the differential inclusion (1.24) is more general than the notion of $(\varphi, s)$-harmonic functions in [2, Definition 3.1] since we do not assume that $u \in W^{1,2}_{loc}(H)$ and $t^{1 - 2s} u' \in W^{1,2}_{loc}(H)$.

**Theorem 1.11.** Let $A$ be a maximal monotone operator on $H$ with $0 \in \text{Rg}(A)$, $j: H \to [0, +\infty]$ a proper, convex, lower semicontinuous function with $j(x_0) = 0$ for some $x_0 \in H$, and suppose that $A$ and $j$ satisfy together at least one of the following assumptions:

1. $j$ is strictly coercive, the subdifferential operator $\partial j: D(\partial j) \to H$ is a weakly continuous mapping, and $A$ is $\partial j$-monotone;
2. $j$ is the indicator function.

Let $0 < s < 1$. For every $\varphi \in D(A)$, there is at least one solution $u$ of problem (1.23) satisfying $u \in C^b([0, +\infty); H)$, $t^{1 - 2s} u' \in W^{1,2}_{loc}(H)$, the estimates (1.6)–(1.9), and for every $y \in A^{-1}([0])$,

$$\|\lim_{t \to 0^+} t^{1 - 2s} u'(t)\|_{H} \leq \frac{2(1 - s)}{s} \|\varphi - y + x_0\|_{H} + \|A^* \varphi\|_{H},
$$

(1.25)

$$\|u'\|_{L^2_{loc}(H)} \leq \frac{\sqrt{2(1 - s)}}{\sqrt{s}} \|\varphi - y + x_0\|_{H} + \|A^* \varphi\|_{H}^{1/2} \|\varphi - y + x_0\|_{H}^{1/2},
$$

(1.26)

$$\|\{t^{1 - 2s} u'(\cdot)\}'\|_{L^2_{loc}(H)} \leq \frac{2(1 - s)}{\sqrt{s}} \|\varphi - y + x_0\|_{H} + \|A^* \varphi\|_{H}^{1/2} \|A^* \varphi\|_{H}^{1/2}.
$$

(1.27)

Moreover, the function $t \mapsto \|u(t) - y\|_{H}^{2}$ is convex, bounded, decreasing on $[0, +\infty)$. Finally, every two strong solutions $u_1$ and $u_2 \in C^b([0, +\infty); H)$ of the differential inclusion (1.24) satisfy (1.13). Finally, if $j$ is strictly convex, then the boundary-value problem (1.23) admits at most one solution $u \in L^{\infty}(H)$.

For a given $\varphi \in D(A)$ and $a > 0$, taking $j(v) = \frac{a}{2} \|v - (1 - a) \varphi\|_{H}^{2}$ ($\varphi \in H$), we obtain well-posedness of the Robin problem

$$\begin{cases}
-\frac{1 - 2s}{t} u'(t) - u''(t) + Au(t) \geq 0 & \text{in } H_+,
\lim_{t \to 0^+} t^{1 - 2s} u'(t) + au(t) = \varphi & \text{on } \partial H_+.
\end{cases}
$$

(1.28)

Before stating the theorem, we fix the following notion of solutions to (1.28).
Definition 1.12. For given boundary value \( \varphi \in H \), a function \( u : [0, +\infty) \to H \) is called a solution of Robin problem (1.28) if \( u \in C([0, +\infty); H) \), \( u \) is a strong solution of (1.24), and \( \lim_{t \to 0^+} t^{1 - 2s}u'(t) \) exists in \( H \) with 
\[
- \lim_{t \to 0^+} t^{1 - 2s}u'(t) + au(0) = \varphi.
\]

Here is our theorem concerning the existence and uniqueness of solutions to the abstract Robin problem (1.28).

Theorem 1.13. Let \( A \) be a maximal monotone operator on \( H \) with \( 0 \in \text{Rg}(A) \), \( 0 < s < 1 \), and \( \alpha > 0 \). Then, for every \( \varphi \in D(A) \), there is a unique solution \( u \) of Robin problem (1.28) satisfying \( u \in L^\infty_0(H) , t^{1 - 2s}u' \in W^{1,2}_s(H) \), estimates (1.6)–(1.9), and for every \( y \in A^{-1}\{0\} \),
\[
\| \lim_{t \to 0^+} t^{1 - 2s}u'(t) \|_H \leq \frac{2(1 - s)}{s} \left\| \frac{1}{\alpha} \varphi - y \right\|_H + \| A^\alpha \varphi \|_H \right) , \tag{1.29}
\]
\[
\| u' \|_{L^2_0(H)} \leq \frac{\sqrt{2(1 - s)}}{\sqrt{s}} \left\| \frac{1}{\alpha} \varphi - y \right\|_H + \| A^\alpha \varphi \|_H \right)^{1/2} \left\| \frac{1}{\alpha} \varphi - y \right\|_H^{1/2} , \tag{1.30}
\]
\[
\| t^{1 - 2s}u' \|_{L^2(H)} \leq \frac{\sqrt{2(1 - s)}}{\sqrt{s}} \left\| \frac{1}{\alpha} \varphi - y \right\|_H + \| A^\alpha \varphi \|_H \right)^{1/2} \left\| \frac{1}{\alpha} \varphi - y \right\|_H^{1/2} . \tag{1.31}
\]

Moreover, the function \( t \mapsto \| u(t) - y \|^2_H \) is convex, bounded, decreasing on \([0, +\infty)\), and every two solutions \( u_1 \) and \( u_2 \in C^0([0, +\infty); H) \) of Robin problem (1.28) satisfy (1.13).

This paper is organized as follows. In the subsequent section, we introduce the functional analytical framework, fix fundamental definitions and notations, and recall a few classical results used throughout this paper. Section 3 is dedicated to the proof of Theorem 1.11. In Section 4, we outline the proof of Theorem 1.1, and in Section 6, we give the proof of Theorem 1.2. We conclude this paper with an application outlined in Section 7 to the classical Leray–Lions operator \( A = -\text{div}(a(x, \nabla u)) \) equipped with either Dirichlet, Neumann or Robin boundary conditions.

2 Preliminaries

Throughout this article, \((H, (\cdot, \cdot)_H)\) denotes a real Hilbert space with inner product \((\cdot, \cdot)_H\), and we write \( \mathbb{R}_+ \) to denote \((0, +\infty)\) and \( \overline{\mathbb{R}}_+ := [0, +\infty] \).

2.1 Nonlinear Semigroups in Hilbert Spaces

Here, we briefly recall some fundamental definitions and important results from the theory of nonlinear semigroup in Hilbert spaces (cf. the standard textbooks [7, 13]).

In this framework, an operator \( A \) on \( H \) is a possibly nonlinear and multivalued mapping \( A : H \to 2^H \). Thus it is standard to identify an operator \( A \) on \( H \) with its graph
\[
A := \{(u, v) \in H \times H \mid v \in Au\} \quad \text{in} \ H \times H
\]
and so, to see \( A \) as a subset of \( H \times H \). The set \( D(A) := \{ u \in H \mid Au \neq \emptyset \} \) is called the domain of \( A \), and \( \text{Rg}(A) := \bigcup_{u \in D(A)} Au \subseteq H \) the range of \( A \). Further, for an operator \( A \) on \( H \), the minimal section \( A^* \) of \( A \) is given by
\[
A^* := \{(u, v) \in A \mid \| v \|_H = \min_{w \in Au} \| w \|_H \}.
\]

An operator \( A \) is called maximal monotone in \( H \) if \( A \) is monotone, that is,
\[
(\tilde{v} - v, \tilde{u} - u)_H \geq 0 \quad \text{for all} \ (u, v), (\tilde{u}, \tilde{v}) \in A,
\]
and \( A \) satisfies the so-called range condition
\[
\text{Rg}(I + \lambda A) = H \quad \text{for one (or equivalently, all)} \ \lambda > 0.
\]
The property that an operator $A$ is monotone is equivalent to the fact that, for every $\lambda > 0$, the resolvent operator $J^A_\lambda := (I + \lambda A)^{-1}$ of $A$ is a contraction on $H$:

$$\|J^A_\lambda u - J^A_\lambda \tilde{u}\|_H \leq \|u - \tilde{u}\|_H$$

for every $u, \tilde{u} \in \text{Rg}(I + \lambda A)$.

Another characterization of maximal monotonicity of $A$, using that $A$ is a subset of $H \times H$, is the following.

**Proposition 2.1** ([13, Définition 2.2 & Proposition 2.2]). An operator $A$ is maximal monotone if it is the largest monotone set in $H \times H$ among other monotone sets $B \subset H \times H$ containing $A$.

Moreover, one has the following extension theorem for monotone operators.

**Proposition 2.2** ([13, Chapter II, Theorem 3.2]). Let $C$ be a closed convex subset of $H$, and let $A$ be a monotone operator in $H$ satisfying $D(A) \subseteq C$. Then there is a maximal monotone operator $\tilde{A}$ on $H$ satisfying

$$D(A) \subseteq D(\tilde{A}) \subseteq C \quad \text{and} \quad Au = \tilde{A}u \quad \text{for every} \ u \in D(A).$$

One of the most important examples of a maximal monotone operator $A$ is given by the sub-differential operator

$$\partial \phi := \{(u, h) \in H \times H | \phi(u + v) - \phi(u) \geq (h, v)_H \text{ for all } v \in H\}$$

of a proper, convex and lower semicontinuous function $\phi: H \to (-\infty, +\infty)$. If $A$ has a sub-differential structure $A = \partial \phi$, then $A$ is cyclically monotone; that is, for every finite cyclic sequence $(u_j)_{j=0}^n \subseteq D(A)$ with $u_0 = u_n$ and every corresponding sequence $(\nu_j)_{j=0}^n$ of points $\nu_j \in Au_j$, one has $\sum_{j=1}^n (u_j - u_{j-1}, \nu_j)_H \geq 0$. But even the reverse statement holds for monotone operators $A$.

**Proposition 2.3** ([13, Théorème 2.5 & Corollaire 2.8]). Let $A$ be a monotone operator on $H$. Then $A$ is cyclically monotone if and only if there is a proper, convex and lower semicontinuous function $\phi: H \to (-\infty, +\infty)$ such that $A \subseteq \partial \phi$. Further, if $A$ is maximal monotone and the minimal section $A^*$ of $A$ is cyclically monotone, then also $A$ is cyclically monotone.

Further, we will employ the notion of lifted operators.

**Proposition 2.4** ([13, Exemples 2.1.3 & 2.3.3]). Let $(\Sigma, \mathcal{B}, \mu)$ be a measure space with positive measure $\mu$. Then, for every monotone operator $A$ on $H$, the lifted operator

$$\mathcal{A} := \{(u, v) \in \mathcal{H} \times \mathcal{H} | v(t) \in Au(t) \text{ for a.e. } t \in \Sigma\}$$

is a monotone operator on the Hilbert space $\mathcal{H} := L^2(\Sigma; \mu; H)$. In addition, if either $\mu(\Sigma) < \infty$ or $0 \in A0$ and $A$ is maximal monotone on $H$, then $\mathcal{A}$ is maximal monotone on $\mathcal{H}$.

Further, it is important to point out that if $A$ is maximal monotone on $H$, then the inverse operator $A^{-1}$ defined by

$$A^{-1} := \{(u, v) \in H \times H | u \in Av\} \quad (2.1)$$

is also maximal monotone (see [13, Exemple 2.3.2]).

In the next proposition, we summarize some important properties of maximal monotone operators (see [13, Lemme 2.4, Théorème 2.2, Proposition 2.6, Proposition 2.7, Corollaire 2.2 & Proposition 2.11]).

**Proposition 2.5.** Let $A$ be a maximal monotone operator on $H$. Then the following statements hold.

1. For every $\lambda > 0$, the Yosida approximation $A_\lambda := (I_H - J_{\lambda A})/\lambda$ is a maximal monotone and Lipschitz continuous mapping $A_\lambda : H \to H$ with Lipschitz constant $1/\lambda$, $A_\lambda u \in A_\lambda Au$ for every $u \in H$ and for every $u \in D(A)$,

$$\|A_\lambda uu\|_H^2 \leq \|A^* u\|_H^2, \quad A_\lambda u \to A^* u \quad \text{in} \ H \quad \text{as} \ \lambda \downarrow 0$$

with $\|A_\lambda u - A^* u\|_H^2 \leq \|A^* u\|_H^2 - \|A_\lambda^* u\|_H^2.$

2. If $B : H \to H$ is a monotone and Lipschitz continuous mapping, then $A + B$ is maximal monotone.

3. The closure $D(A)^H$ of the domain $D(A)$ of $A$ is a closed convex set.

4. The minimal section $A^*$ of $A$ is a well-defined mapping from $D(A)$ to $H$ with the property of being a principal section of $A$; that is, for every $(u, v) \in \mathcal{D}(A)^H \times H$ satisfying $(A^* \tilde{u} - v, \tilde{u} - u)_H \geq 0$ for all $\tilde{u} \in D(A)$, one has $(u, v) \in A$.
(5) Let $B$ be another maximal monotone operator on $H$. If $D(A) = D(B)$ and $A' = B'$, then $A = B$.

(6) If $\phi : H \to (-\infty, +\infty]$ is a convex, proper, lower semicontinuous function and $A = \partial \phi$, then

$$D(A) \subseteq D(\phi) \subseteq D(A)^H = D(\phi)^H.$$ 

The next classical result is the Hille–Yosida theorem for nonlinear semigroups in Hilbert spaces. For our purposes in this paper, we provide a version of it, which is a summary of [6, Theorem 1.1 & Theorem 1.2] (cf. [18, 24, 32]).

**Theorem 2.6** (Hille–Yosida Theorem for Nonlinear Semigroups). Let $C$ be a nonempty, closed convex subset of $H$, and let $\{T_t\}_{t \geq 0}$ be a semigroup of nonlinear contractions $T_t : C \to C$. Then there is a unique maximal monotone operator $A$ on $H$ such that the negative minimal section $-A^*$ of $A$ is the infinitesimal generator of $\{T_t\}_{t \geq 0}$ and $D(A)^H = C$.

On the other hand, for every maximal monotone operator $A$ on $H$, there is a unique semigroup $\{e^{-At}\}_{t \geq 0}$ on $D(A)^H$ with infinitesimal generator $-A^*$. In particular, for every $u_0 \in D(A)$, the function $u(t) := e^{-At}u_0$ satisfies $\|u'(t)\|_{L^\infty(H)} \leq \|A^* u_0\|_H$; for a.e. $t > 0$, one has $u(t) \in D(A)$ and

$$\begin{cases}
u'(t) + Au(t) \geq 0 & \text{for a.e. } t > 0, \\
u(0) = u_0.
\end{cases}$$

For every $t \geq 0$, the right-hand side derivative $u_{t\nu}(t) := \frac{du}{dt}(t)$ exists, and

$$u_{t\nu}(t) + A' u(t) = 0.$$ 

The semigroup admits a regularization effect if $A$ has a sub-differential structure $A = \partial \phi$ of a convex, proper, lower semicontinuous function $\phi : H \to (-\infty, +\infty]$.

**Theorem 2.7** (Smoothing Effect of Semigroups). Let $\phi : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous function, $A = \partial \phi$, and $\{e^{-At}\}_{t \geq 0}$ the semigroup generated by $-A$. Then, for every $u_0 \in D(A)^H$, one has

1. $e^{-At}u_0 \in D(A)$ for all $t > 0$,
2. $\|A^* e^{-At}\|_H \leq \|A^* v\|_H + \frac{\|v\|_H}{t}$ for all $t > 0$ and $v \in D(A)$,
3. $t \mapsto \phi(e^{-At}u_0)$ is convex, decreasing and Lipschitz on $[t, +\infty)$ ($\delta > 0$),
4. $\frac{d}{dt}\phi(e^{-At}u_0) = -\|\partial_{t\nu} e^{-At}u_0\|_H^2$ for all $t > 0$.

Further, a function $j : H \to (-\infty, +\infty]$ is called strongly coercive if

$$\lim_{\|u\|_H \to +\infty} \frac{j(u)}{\|u\|_H} = +\infty.$$ 

This property of $j$ is equivalent to the following property of $(\partial j)^{-1}$.

**Proposition 2.8** ([13, Proposition 2.14]). Let $j : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous function on $H$. Then $j$ is strongly coercive if and only if the inverse operator $(\partial j)^{-1}$ of $\partial j$ is bounded; that is, for every bounded subset $B$ of $H$, $(\partial j)^{-1}(B)$ is again a bounded subset of $H$.

**Definition 2.9.** Given a convex, proper, lower semicontinuous function $\psi : H \to (-\infty, +\infty]$, an operator $A$ on $H$ is called $\partial \psi$-monotone if, for every $\lambda > 0$,

$$\psi(u - \tilde{u}) \leq \psi(u - \tilde{u} + \lambda(v - \tilde{v})) \quad \text{for every } (u, v), (\tilde{u}, \tilde{v}) \in A.$$ 

Our next proposition summarizes only the equivalent statements from [15, Théorème 3.1] which are relevant for us in this paper.

**Proposition 2.10** ([15, Théorème 3.1]). Let $A$ be a maximal monotone operator on $H$ and $\psi : H \to (-\infty, +\infty]$ a proper, convex, lower semicontinuous function. Then the following statements are equivalent.

(i) $A$ is $\partial \psi$-monotone;
(ii) $A$ is $(\partial \psi)_\mu$-monotone for every $\mu > 0$;
(iii) $A_\lambda$ is $\partial \psi$-monotone for every $\lambda > 0$.

Finally, we complete this subsection with the following definition.
Definition 2.11. A mapping \( T : D(T) \to H \) is called weakly continuous if \( T \) maps weakly convergent sequences to weakly convergent sequences.

2.2 \( T \)-Accretive and Completely Accretive Operators

Let \( (\Sigma, \mu) \) be a \( \sigma \)-finite measure space and \( M(\Sigma, \mu) \) the space (of all classes) of measurable real-valued functions on \( \Sigma \). If \( H \subseteq M(\Sigma, \mu) \) is a Banach lattice, then we shall denote the usual lattice operations \( u \vee \hat{u} \) and \( u \wedge \hat{u} \) to be the almost everywhere point-wise supremum and infimum of \( u \) and \( \hat{u} \in X \). In addition, \( u^+ = u \vee 0 \) is the positive part, \( u^- = (-u) \vee 0 \) the negative part, and \( |u| = u^+ + u^- \) the absolute value of an element \( u \in X \). For every \( u, \hat{u} \in X \), one denotes by \( u \leq \hat{u} \) the usual order relation on \( X \). In this framework, we can now introduce the following definition.

Definition 2.12. A mapping \( S : D(S) \to X \) with domain \( D(S) \subseteq X \) is called order preserving if \( Su \leq S\hat{u} \) for every \( u \leq \hat{u} \), positive if \( Su \geq 0 \) for every \( u \geq 0 \), and a \( T \)-contraction if \( \|Su - S\hat{u}\|_X \leq \|u - \hat{u}\|_X \) for every \( u, \hat{u} \in D(S) \). We shall say that an operator \( A \subseteq X \times X \) is \( T \)-accretive if, for every \( \lambda > 0 \), the resolvent \( J_\lambda \) of \( A \) defines a \( T \)-contraction with domain \( D(J_\lambda) = \text{Rg}(I + \lambda A) \).

Note that if \( S \) is \( T \)-contractive, then it is order-preserving and that the converse holds if \( S \) is contractive and satisfies \( u \vee \hat{u} \) and \( u \wedge \hat{u} \in D(S) \) for every \( u, \hat{u} \in D(S) \) (see [11, Lemma (19.11)]).

The notion of completely accretive operators was introduced in [10] by Crandall and Bénilan and further developed in [17]. Following the same notation as in these two references, \( \mathcal{J}_0 \) denotes the set of all convex, lower semicontinuous functions \( j : \mathbb{R} \to \mathbb{R} \), satisfying \( j(0) = 0 \).

Definition 2.13. A mapping \( S : D(S) \to M(\Sigma, \mu) \) with domain \( D(S) \subseteq M(\Sigma, \mu) \) is called a complete contraction if
\[
\int_\Sigma j(Su - S\hat{u}) \, d\mu \leq \int_\Sigma j(u - \hat{u}) \, d\mu \text{ for all } j \in \mathcal{J}_0 \text{ and every } u, \hat{u} \in D(S).
\]
Choosing \( j(\cdot) = |\cdot|^q \in \mathcal{J}_0 \) if \( 1 \leq q < \infty \) and \( j(\cdot) = |\cdot|^q \in \mathcal{J}_0 \) for \( k \geq 0 \) large enough if \( q = \infty \) shows that each complete contraction \( S \) is \( T \)-contractive in \( L^q(\Sigma, \mu) \) for every \( 1 \leq q \leq \infty \). Choosing \( j(\cdot) = |\cdot|^q \in \mathcal{J}_0 \) for any \( 1 \leq q < \infty \), a complete contraction \( S \) is order preserving; that is, for every \( u, \hat{u} \in D(S) \) satisfying \( u \leq \hat{u} \) a.e. on \( \Sigma \), one has \( Su \leq S\hat{u} \). In fact, the following characterization holds.

Definition 2.14. An operator \( A \) on \( M(\Sigma, \mu) \) is called completely accretive if, for every \( \lambda > 0 \), the resolvent operator \( J_\lambda \) of \( A \) is a complete contraction. Given a linear subspace \( X \) of \( M(\Sigma, \mu) \), an operator \( A \) on \( X \) is called \( m \)-completely accretive in \( X \) if \( A \) is completely accretive and the range condition \( \text{Rg}(I + \lambda A) = X \) holds for all \( \lambda > 0 \). A semigroup \( \{T_t\}_{t \geq 0} \) on a closed subset \( C \) of \( M(\Sigma, \mu) \) is called order preserving if each map \( T_t \) is order preserving.

The next characterization follows immediately from the respective definitions.

Proposition 2.15. For every \( j \in \mathcal{J}_0 \), let \( \phi_j : L^2(\Sigma, \mu) \to \mathbb{R}_+ \) be defined by
\[
\phi_j(u) = \begin{cases} 
\int_\Sigma j(u) \, d\mu & \text{if } j(u) \in L^1(\Sigma; \mu), \\
+\infty & \text{otherwise}
\end{cases}
\]
for every \( u \in L^2(\Sigma, \mu) \). Then an operator \( A \) on \( L^2(\Sigma, \mu) \) is completely accretive if and only if \( A \) is \( \partial \phi_j \)-monotone for every \( j \in \mathcal{J}_0 \).

2.3 Orlicz and Weighted Sobolev Spaces

Next, we first briefly recall the notion of Orlicz spaces. Following [31, Chapter 3], a continuous function \( \psi : [0, +\infty) \to [0, +\infty) \) is an \( N \)-function if it is convex, \( \psi(s) = 0 \) if and only if \( s = 0 \), and \( \lim_{s \to +\infty} \psi(s)/s = 0 \).
Definition 2.16. Given an \( N \)-function \( \psi \), the Orlicz space

\[
L^\psi(\Sigma, \mu) := \left\{ u : \Sigma \to \mathbb{R} \mid u \text{ measurable and } \int_\Sigma \psi\left(\frac{|u|}{\alpha}\right) \, d\mu < \infty \text{ for some } \alpha > 0 \right\}
\]

is equipped with the Orlicz–Minkowski norm

\[
\|u\|_{L^\psi} := \inf\left\{ \alpha > 0 \mid \int_\Sigma \psi\left(\frac{|u|}{\alpha}\right) \, d\mu \leq 1 \right\}.
\]

For \( 1 \leq q \leq \infty \), we write \( L^q_{\text{loc}}(H) \) and \( L^q(H) \) to denote the vector-valued Lebesgue spaces \( L^q_{\text{loc}}(\mathbb{R}_+; H) \) and \( L^q(\mathbb{R}_+; H) \). The derivative \( u' \) of a function \( u \in L^1_{\text{loc}}(H) \) is usually understood in the distributional sense. More precisely, a function \( w \in L^1_{\text{loc}}(H) \) is called the weak derivative of \( u \in L^1_{\text{loc}}(H) \) if

\[
\int_0^{+\infty} u(t) \xi(t) \, dt = - \int_0^{+\infty} w(t) \xi(t) \, dt
\]

for all test functions \( \xi \in C^\infty_c(\mathbb{R}_+) \). A function \( w \in L^1_{\text{loc}}(H) \) satisfying the latter equation for all \( \xi \in C^\infty_c(\mathbb{R}_+) \) is unique, and so one writes \( u' = w \). We denote by \( W^1_{\text{loc}}(H) \) the space of all \( u \in L^1_{\text{loc}}(H) \) with a weak derivative \( u' \in L^1_{\text{loc}}(H) \).

Next, let \( 0 < s < 1 \). Then \( L^2_s(H) \) denotes the Lebesgue space of all \( u \in L^1_{\text{loc}}(H) \) satisfying \( t^s u \in L^2(H) \) which, when equipped with the inner product

\[
(u, v)_{L^2_s(H)} := \int_0^{+\infty} (u(t), v(t))_H t^{s-2} \, dt,
\]

and the induced norm \( \|u\|_{L^2_s(H)} := \sqrt{(u, u)_{L^2_s(H)}} \), is a Hilbert space. We also equip the first-order weighted Sobolev space

\[
W^{1,2}_s(H) = \left\{ u \in W^{1,1}_{\text{loc}}(H) \mid u, u' \in L^2_s(H) \right\}
\]

with the inner product

\[
(u, v)_{W^{1,2}_s(H)} := \int_0^{+\infty} ((u(t), v(t))_H + (u'(t), v'(t))_H) t^{s-2} \, dt.
\]

Then \( W^{1,2}_s(H) \) is a Hilbert space, and we denote by \( \| \cdot \|_{W^{1,2}_s(H)} \) the induced norm of \( W^{1,2}_s(H) \).

2.4 Some Tools from Interpolation Theory

Parts of our next proposition are taken from [27, Proposition 1.2.10].

**Proposition 2.17.** For \( 0 < s < 1 \), let \( u \in W^{1,2}_s(H) \). Then the following statements hold.

1. The limit \( u(0) := \lim_{t \to 0^+} u(t) \) exists in \( H \).
2. The function \( u \) has a continuous representative on \([0, +\infty)\). Moreover, for \( s \geq 1/2 \), this representative is uniformly continuous on \([\delta, +\infty)\) for every \( \delta > 0 \).
3. For \( c_1(s) = \frac{\sqrt{3}}{\sqrt{2(1-s)}} \) and \( c_2(s) = \frac{s+1}{\sqrt{2(s+1)}} \), one has

\[
c_1(s) \|x\|_H \leq \inf_{u \in W^{1,2}_s(H); u(0) = x} \|u\|_{W^{1,2}_s(H)} \leq c_2(s) \|x\|_H \quad \text{for every } x \in H.
\]

4. The trace map \( \text{Tr} : W^{1,2}_s(H) \to H, u \mapsto u(0) \) is surjective, linear and continuous.

We note that the constants \( c_1(s) \) and \( c_2(s) \) in Proposition 2.17 might not be optimal. For later use, we outline the proof of statements (1) and (2) of this proposition.
Proof. Let \( u \in W^{1,1}_{\text{loc}}(H) \) with \( u' \in L^2_s(H) \), and let \( t, \tilde{t} > 0 \). Then
\[
\|u(t) - u(\tilde{t})\|_H \leq \int_t^{\tilde{t}} \|u'(r)\|_H \, dr
\]
\[
= \int_t^{\tilde{t}} \|u'(r)\|_H r^{s-\frac{1}{2}} r^{-\frac{1}{2}} \, dr
\]
\[
\leq \left[ \int_t^{\tilde{t}} \|r^s u'(r)\|_H r \, dr \right]^{1/2} \left[ \int_t^{\tilde{t}} \|r^{-\frac{1}{2}}\|_H r \, dr \right]^{1/2}
\]
\[
\leq \|u'\|_{L^2_s(H)} (2(1-s))^{-\frac{1}{2}} \tilde{t}^{2(1-s)} - t^{2(1-s)} \frac{1}{2}. \tag{2.3}
\]
From this, one sees that \( (u(t_n))_{n \geq 1} \) is a Cauchy sequence in \( H \) for every zero sequence \( (t_n)_{n \geq 1} \) in \( (0, +\infty) \). Thus the limit \( \lim_{t \to 0^+} u(t) \) exists in \( H \). Moreover, inequality (2.3) also shows that if \( s \geq 1/2 \), then \( u \) has a representative, which is uniformly continuous on \([\delta, +\infty)\) for every \( \delta > 0 \).

To conclude this preliminary section, we state the following integration-by-parts rule from [2, Proposition 3.9].

Proposition 2.18. Let \( 0 < s < 1 \). For \( u \in W^{1,2}_s(H) \) and \( \xi \in W^{1,2}_{1-s}(H) \), the functions
\[
t \mapsto (u'(t), \xi(t))_H \quad \text{and} \quad t \mapsto (u(t), \xi(t))_H
\]
belong to \( L^1(0, +\infty) \). Moreover, the following integration-by-parts rule holds:
\[
\int_0^{+\infty} (u'(t), \xi(t))_H \, dt = \int_0^{+\infty} (u(t), \xi'(t))_H \, dt + (u(0), \xi(0))_H.
\]

3 Well-Posedness of Boundary-Value Problems in the Half-Space

This section is concerned with the proof of Theorem 1.11 and thereby, we will establish that the abstract second-order boundary-value problem
\[
\begin{aligned}
&-\frac{1-2s}{t} u'(t) - u''(t) + Au(t) \geq 0 \\
&\lim_{t \to 0^+} t^{1-2s} u'(t) \in \partial j(u(0) - \varphi) \\
&\text{in } H_+,
\end{aligned}
\tag{1.23}
\]
on the half-space \( H_+ \) is well-posed among \( \varphi \in D(A) \) provided \( 0 < s < 1 \), and \( j: H \to \mathbb{R}_+ \) is a proper, convex, lower semicontinuous function such that \( A \) and \( j \) satisfy one of the hypotheses (H.i) or (H.ii).

We begin by focusing on the uniqueness of solutions to problem (1.23) and show that the inequality
\[
\|u(t) - \tilde{u}(t)\|_H \leq \|u(\tilde{t}) - \tilde{u}(\tilde{t})\|_H \quad \text{for all } t \geq \tilde{t} \geq 0. \tag{1.13}
\]
holds among strong solutions \( u \) and \( \tilde{u} \) of the differential inclusion (1.24).

Remark 3.1. Thanks to this inequality, the family \( \{T_t\}_{t \geq 0} \) of mappings
\[
T_t: D(A) \to D(A) \quad \text{defined by} \quad T_t \varphi := u(t) \quad \text{for all } t \geq 0, \varphi \in D(A),
\]
where \( u \) is the unique solution of (1.23) for given \( \varphi \), is contractive and admits a contractive extension on the closure \( \overline{D(A)}^H \) of \( D(A) \) in \( H \). But only in the case \( s = 1/2 \), the uniqueness of solutions to (1.23) (Proposition 3.2 below) will imply that \( \{T_t\}_{t \geq 0} \) is a semigroup. This enlightens the important difference between the two cases \( s = 1/2 \) and \( s \neq 1/2 \).
**Proposition 3.2.** Let $A$ be a monotone operator on $H$ with $0 \in \text{Rg}(A)$ and $j : H \to (-\infty, +\infty]$ a proper, convex, lower semicontinuous function. Then the following statements hold.

(1) For every two strong solutions $u_1, u_2$ of the differential inclusion (1.24) satisfying $u_1, u_2 \in L^{\infty}(H)$, one has
\[
\|u_1(t) - u_2(t)\|^2_H \leq \|u_1(\bar{t}) - u_2(\bar{t})\|^2_H \quad \text{for all } \bar{t} \geq t > 0.
\] (3.1)

(2) Let $u_1, u_2$ be two strong solutions of the differential inclusion (1.24) satisfying $u_1, u_2 \in L^{\infty}(H)$ and
\[
\lim_{t \to 0} t^{1-2s}u_1'(t) \quad \text{and} \quad \lim_{t \to 0} t^{1-2s}u_2'(t).
\] (3.2)

Then one has
\[
\left( -\lim_{t \to 0} t^{1-2s}u_1'(t) - \left( -\lim_{t \to 0} t^{1-2s}u_2'(t), u(0) - \bar{u}(0) \right) \right) \geq 0.
\] (3.3)

(3) If $j$ is strictly convex, then the boundary-value problem (1.23) has at most one solution $u \in L^{\infty}(H)$.

(4) If there is exactly one $y \in H$ such that $A^{-1}(0) = \{y\}$, then problem (1.23) has at most one solution $u$ with the $\sigma(H, H')$-weak $\omega$-limit point $\lim_{z \to +\infty} u(z) = y$ weakly in $H$.

For proving the statements in Proposition 3.2, we use the change of variable
\[
z = \left( \frac{t}{2s} \right)^{2s}.
\] (3.4)

By writing $v(z) = u(t)$, the differential inclusion (1.24) is equivalent to
\[
-z^{1-2s} v''(z) + Av(z) \geq 0,
\] (3.5)
and boundary-value problem (1.23) reduces to
\[
\begin{cases}
  z^{1-2s} v''(z) \in Av(z) & \text{in } H_+,
  \\
v'(0) \in \partial_j(v(0) - \varphi) & \text{on } \partial H_+
\end{cases}
\] (3.6)
for given $\varphi \in H$ and with $j := (2s)^{(1-2s)}$.

**Remark 3.3.** We note that the change of variable (3.4) has been used in past already by several other authors, for instance, by Caffarelli and Silvestre [14].

Before giving the proof of Proposition 3.2, we give some more details about the equivalence between the two differential inclusions (1.24) and (3.5) and the two boundary-value problems (1.23) and (3.6).

**Lemma 3.4.** For $0 < s < 1$ and $u \in W^{2,2}_H((0, +\infty); H)$, let $u(t) = v(z)$ for $z$ given by the change of variable (3.4). Then the following holds.

(1) One has
\[
(2s)^{1-2s} v(z) = t^{1-2s} u'(t) \quad \text{for all } z > 0 \quad \text{(respectively, } t > 0),
\] (3.7)
and so $\lim_{t \to 0} t^{1-2s} u'(t)$ exists in $H$ if and only if $v \in C^1((0, +\infty); H)$ with
\[
\lim_{t \to 0} t^{1-2s} u'(t) = (2s)^{1-2s} v'(0).
\]

(2) One has
\[
z^{1-2s} v''(z) = u''(t) + \frac{1-2s}{t} u'(t) = t^{2s-1} (t^{1-2s} u'(t))' \quad \text{for a.e. } z > 0 \quad \text{(or } t > 0).
\] (3.8)

(3) The function $u$ is a solution of (1.23) if and only if
\[
v \in C^1((0, +\infty); H) \cap W^{2,2}_H((0, +\infty); H)
\]
satisfying $v(z) \in D(A)$ and $z^{1-2s} v'(z) \in A(v(z))$ for a.e. $z > 0$, and $v'(0) \in \partial_j(v(0) - \varphi)$.

**Proof.** We note that
\[
z = z(t) = \left( \frac{t}{2s} \right)^{2s} \quad \text{if and only if } \quad t = t(z) = 2sz^{\frac{1}{2s}},
\]
and so \( v(z) := u(t) = u(2sz^{\frac{1}{2s}}) \). Thus the chain rule yields
\[
\frac{d}{dz} v(z) = \frac{du}{dt} \frac{dt}{dz} = u'(t)z^{\frac{1}{2s}} \quad \text{for every } z > 0 \text{ (respectively, } t > 0) .
\]

From this, we can conclude that (3.7) holds and deduce claim (1) of this lemma.

Further, since \( u \in W^{1,2}_{\text{loc}}((0, +\infty); H) \), also \( v \in W^{1,2}_{\text{loc}}((0, +\infty); H) \) and
\[
\frac{d^2}{dz^2} v(z) = \frac{d}{dz} \left( \frac{d}{dz} v(z) \right) = \frac{d}{dz} \left( \frac{d}{dz} u(t)z^{\frac{1}{2s}} \right) + u'(t) \frac{1 - 2s}{2s} z^{\frac{1}{2s} - 1} \quad \text{for a.e. } z > 0 \text{ (or t > 0)} .
\]

Multiplying this equation by \( z^{-\frac{1}{2s}} \), one finds (3.8). Therefore,
\[
t^{2s-1}[t^{1-2s}u'(t)](t) \in Au(t) \quad \text{for a.e. } t > 0
\]
if and only if
\[
z^{-\frac{1}{2s}}v''(z) \in Av(z) \quad \text{for a.e. } z > 0.
\]

Moreover, by claim (1), one has
\[
\lim_{t \to 0} t^{1-2s}u'(t) \in \partial j(u(0) - \varphi)
\]
if and only if \((2s)^{1-2s}v'(0) \in \partial j(v(0) - \varphi)\), which completes the proof of statement (3).

Now, we can outline the proof of Proposition 3.2. Here, we adapt an idea by Brezis [12] to the more general case \(0 < s < 1\).

**Proof of Proposition 3.2.** Let \( u_1 \) and \( u_2 \) be two solutions of the differential inclusion (1.24). We apply the change of variable (3.4) by setting
\[
v_1(z) = u_1(2sz^{1/2s}) \quad \text{and} \quad v_2(z) = u_2(2sz^{1/2s}) \quad \text{for } z \geq 0 .
\]

Then also \( v_1, v_2 \in L^{\infty}(H) \), and thanks to Lemma 3.4, \( v_1 \) and \( v_2 \) are two strong solutions of the differential inclusion (3.5). Thus, by the monotonicity of \( A \) and by (3.5), one finds that the function \( w = v_1 - v_2 \) satisfies
\[
z^{-\frac{1}{2s}}(w''(z), w(z))_H = (z^{-\frac{1}{2s}}v_1''(z) - z^{-\frac{1}{2s}}v_2''(z), v_1(z) - v_2(z))_H \geq 0 \quad \text{for a.e. } z > 0 ,
\]
and in particular, \((w'(z), w(z))_H \geq 0\) for a.e. \( z > 0 \). Therefore,
\[
\frac{d}{dz} \left( \frac{d}{dz} \frac{1}{2} \|w(z)\|_H^2 \right) = \frac{d}{dz} \left( \frac{d}{dz}(w'(z), w(z))_H + \|w'(z)\|_H^2 \right) \geq \|w'(z)\|_H^2 \geq 0 \quad \text{for a.e. } z > 0 ,
\]
which implies that the function \( z \to \|w(z)\|_H^2 \) is convex. Since, by hypothesis, one has \( w \in L^{\infty}(H) \), it follows that the function \( z \to \|w(z)\|_H^2 \) is monotonically decreasing on \((0, +\infty)\). By (3.9), this is equivalent to the fact that \( t \mapsto \|u_1(t) - u_2(t)\|_H^2 \) is monotonically decreasing along \((0, +\infty)\), and hence \( u_1 \) and \( u_2 \) satisfy inequality (3.1). This proves claim (1) of this proposition.

Next, suppose that \( u_1 \) and \( u_2 \) are two strong solutions of the differential inclusion (1.24) satisfying \( u_1, u_2 \in C^1([0, +\infty); H) \), and the two limits (3.2) hold. By Lemma 3.4, one has
\[
(2s)^{1-2s}v_1'(z) = t^{1-2s}u_1'(t) \quad \text{and} \quad (2s)^{1-2s}v_2'(z) = t^{1-2s}u_2'(t) .
\]

Thus the limit (3.9) is equivalent to the fact that \( v_1 \) and \( v_2 \) are in \( C^1([0, +\infty); H) \) with
\[
(2s)^{1-2s}v_1'(0) = \lim_{t \to 0^+} t^{1-2s}u_1'(t) \quad \text{and} \quad (2s)^{1-2s}v_2'(0) = \lim_{t \to 0^+} t^{1-2s}u_2'(t) .
\]

Since \( z \to \|w(z)\|_H^2 \) is monotonically decreasing on \([0, +\infty)\), one has
\[
(w'(z), w(z))_H = \frac{d}{dz} \frac{1}{2} \|w(z)\|_H^2 \leq 0 \quad \text{for every } z \geq 0 ,
\]
or equivalently,
\[
((2s)^{1-2s}v_1'(z) - (2s)^{1-2s}v_2'(z), v_1(z) - v_2(z))_H \geq 0 \quad \text{for every } z \geq 0 .
\]
Evaluating this inequality at \( z = 0 \) and rewriting it by using (3.11) shows that (3.3) holds. Thus claim (2) holds.

Next, suppose that \( u_1 \) and \( u_2 \) are two solutions of boundary-value problem (1.23) for the same \( \varphi \in H \) satisfying \( u_1, u_2 \in L^{\infty}(H) \). Then Lemma 3.4 implies that the two functions \( v_1 \) and \( v_2 \) defined by (3.9) are respectively two solutions of problem (3.6) for the same boundary data \( \varphi \in H \) and \( v_1, v_2 \in L^{\infty}(H) \). Since, for every \( i = 1, 2 \), one has \( v'_i(0) \in \partial j(v_i(0) - \varphi) \), the monotonicity of \( \partial j \) implies

\[
0 \geq (w'(0), w(0))_H = (v'_1(0) - v'_2(0), (v_1(0) - \varphi) - (v_2(0) - \varphi))_H \geq 0.
\]

Combining this with (3.10), one finds

\[
0 \geq (w'(z), w(z))_H = \int_0^z \frac{d}{dr}(w'(r), w(r))_H \,dr \geq \int_0^z \|w'(r)\|^2_H \,dr,
\]

implying \( w'(z) = 0 \) in \( H \) for all \( z \geq 0 \), that is,

\[
v'_1(z) = v'_2(z) \quad \text{for all } z \geq 0.
\]

(3.12)

Thus there is a \( c_0 \in H \) such that \( v_1(z) = v_2(z) + c_0 \) for all \( z \geq 0 \).

Now, if \( v_1 \) and \( v_2 \) have the same \( \sigma(H, H') \)-weak limit as \( z \to +\infty \), then \( c_0 = 0 \). This shows that claim (4) of this proposition holds.

Finally, assume that \( j \) is strictly convex. By (3.12), one has \( v'_1(0) = v'_2(0) \). Since

\[
\hat{j}(v_1(0) - \varphi) - \hat{j}(v_2(0) - \varphi) \geq (v'_2(0), v_1(0) - v_2(0)),
\]

\[
\hat{j}(v_2(0) - \varphi) - \hat{j}(v_1(0) - \varphi) \geq (v'_1(0), v_2(0) - v_1(0)),
\]

it follows that

\[
\hat{j}(v_2(0) - \varphi) - \hat{j}(v_1(0) - \varphi) = (v'_1(0), v_2(0) - v_1(0)).
\]

(3.13)

Now, if \( v_1(0) \neq v_2(0) \), then, by the strict convexity of \( \hat{j} \) and by (3.13),

\[
\frac{1}{2} \hat{j}(v_1(0) - \varphi) + \frac{1}{2} \hat{j}(v_2(0) - \varphi) > \hat{j}\left(\frac{v_1(0) + v_2(0)}{2} - \varphi\right)
\]

\[
\geq \hat{j}(v_1(0) - \varphi) + \left(v'_1(0), \frac{v_2(0) - v_1(0)}{2}\right)
\]

\[
= \frac{1}{2} \hat{j}(v_1(0) - \varphi) + \frac{1}{2} \hat{j}(v_2(0) - \varphi),
\]

which is a contradiction. Therefore, we have again \( v_1(0) = v_2(0) \), implying \( v_1 = v_2 \), or, by (3.9), equivalently, \( u_1 = u_2 \). This shows that also claim (3) holds and completes the proof of this proposition.

Next, we turn to the proof of existence of solutions to problem (1.23). (the first part of Theorem 1.11). First, we outline the idea of the existence proof.

Let \( \varphi \in D(A) \). Then the strategy of proving existence of solutions to (3.6) is in solving the lifted differential inclusion

\[
\{t^{1-2s}u'\}' \in t^{1-2s}A_{loc}u \quad \text{in } L^2_{loc}(H),
\]

(3.14)

equipped with the boundary condition

\[
\lim_{t \to 0^+} t^{1-2s}u'(t) \in \partial j(u(0) - \varphi).
\]

(3.15)

In (3.14), the operator \( A_{loc} \) is given by

\[
A_{loc} := \{(v, w) \in L^2_{loc}(H) \times L^2_{loc}(H) \mid w(t) \in A(u(t)) \text{ for a.e. } t > 0\}.
\]

To achieve this, we proceed in the following steps: first, if \( \mathcal{A} \) is the operator in \( L^2_{1-s}(H) \) given by

\[
\mathcal{A} = \{(v, w) \in L^2_{1-s}(H) \times L^2_{1-s}(H) \mid w(t) \in A(u(t)) \text{ for a.e. } t > 0\},
\]

(3.16)
then we show that, for every $\lambda, \delta > 0$, the regularized differential equation

$$A_\lambda u_\lambda + \delta u_\lambda + \mathcal{B}(u_\lambda) = 0 \quad \text{in } L^2_{1-s}(H)$$

(3.17)

admits a (unique) solution $u_\lambda$. In (3.17), $A_\lambda := \frac{1}{\lambda} J^I_{1-s}(H) - J^I_\lambda$ is the Yosida approximation of $A$ in $L^2_{1-s}(H)$, and

$$\mathcal{B} u := -t^{2-s} (1-t^{2s}) u'$$

for every $u \in D(\mathcal{B}) \subseteq L^2_{1-s}(H)$

(3.18)

is the realization of $\frac{1}{t^{2-s}} u' + u''$ in the weighted Hilbert space $L^2_{1-s}(H)$ equipped with the boundary condition (3.15) (see Proposition 3.5 below). Next, we establish a priori estimates on $(u_\lambda)_{\lambda > 0}$, from which one can conclude that, for every $\delta > 0$, there is a subsequence of $(u_\lambda)_{\lambda > 0}$ converging to a (unique) solution $u_\delta$ of

$$A u_\delta + \delta u_\delta + \mathcal{B}(u_\delta) \geq 0 \quad \text{in } L^2_{1-s}(H).$$

(3.19)

After proving a priori estimates on $(u_\delta)_{\delta \in (0,1)}$, one shows that there is a subsequence of $(u_\delta)_{\delta \in (0,1]}$ converging to a solution $u$ of (3.14) satisfying (3.15). This method generalizes an idea by Brezis [12] to the general fractional power case $0 < s < 1$.

Next, we show that the operator $\mathcal{B}$ defined by (3.18) and equipped with the boundary condition (3.15) is a sub-differential operator on $L^2_{1-s}(H)$.

**Proposition 3.5.** Let $J : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous function on $H$, and $0 < s < 1$. For given $\varphi \in H$, let $\mathcal{E}_1$ and $\mathcal{E}_2 : L^2_{1-s}(H) \to (-\infty, +\infty]$ be defined by

$$\mathcal{E}_1(u) = \begin{cases} \frac{1}{2} \sup_{t \geq 0} \| t^{1-s} u' \|_H^2 + \| \varphi - u(0) \|_H^2 & \text{if } u \in W^{1,2}_{1-s}(H), \\ +\infty & \text{otherwise}, \end{cases}$$

$$\mathcal{E}_2(u) = \begin{cases} \sup_{n \geq 1} \| u_n \|^{1-s}_{L^2_{1-s}(H)} & \text{if } u \in W^{1,2}_{1-s}(H), \\ +\infty & \text{otherwise}. \end{cases}$$

Then the function $\mathcal{E} : L^2_{1-s}(H) \to (-\infty, +\infty]$ defined by $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ is proper, convex and lower semicontinuous on $L^2_{1-s}(H)$. The operator $\mathcal{B}$ defined by (3.18) is the sub-differential $\partial \mathcal{E}$ of $\mathcal{E}$ in $L^2_{1-s}(H)$; more precisely, $\mathcal{B} : D(\mathcal{B}) \to L^2_{1-s}(H)$ is a well-defined mapping, and for every $u \in L^2_{1-s}(H)$, $u$ belongs to $D(\mathcal{B})$ if and only if $t^{1-2s} u' \in W^{1,2}_{2}(H)$, and (3.15) holds.

**Proof.** The function $\mathcal{E}$ is convex as the sum of a squared $L^2$-norm and a convex function. Moreover, $\mathcal{E}$ is proper since $J(x) < +\infty$ for some $x \in H$, and by Proposition 2.17, for every $\varphi \in H$, there is a $u \in W^{1,2}_{1-s}(H)$ satisfying $u(0) - \varphi = x$ and $\mathcal{E}_1(u) < +\infty$. To see that $\mathcal{E}$ is lower semicontinuous on $L^2_{1-s}(H)$, let $c \in \mathbb{R}$ and $(u_n)_{n \geq 1}$ a sequence in $L^2_{1-s}(H)$ converging to some $u$ in $L^2_{1-s}(H)$ and satisfying

$$j(u_n(0) - \varphi) + \frac{1}{2} \| u_n' \|^{1-s}_{L^2_{1-s}(H)} = \mathcal{E}(u_n) \leq c \quad \text{for all } n \geq 1.$$  

(3.20)

Since $j$ is a proper, convex, lower semicontinuous function on $H$, there are $y_0 \in H$ and $b \in \mathbb{R}$ such that $j(v - \varphi) \geq (y_0, v)_H - b$ for all $v \in H$. Applying this to (3.20), one obtains

$$\| u_n' \|^{1-s}_{L^2_{1-s}(H)} \leq c + b - (y_0, u_n(0))_H + (y_0, \varphi)_H \quad \text{for all } n \geq 1.$$  

Thus, by Proposition 2.17, there is a $c_1 > 0$ such that

$$\| u_n(0) \|_H^2 \leq c_1 \sup_{n \geq 1} \| u_n' \|^{1-s}_{L^2_{1-s}(H)} + (y_0, u_n(0))_H \quad \text{for all } n \geq 1.$$

Applying Cauchy–Schwarz’s and a variant of Young’s inequality ($ab \leq \varepsilon a^2 + b^2/\varepsilon b$, $a, b \geq 0$, $\varepsilon > 0$) to $(y_0, u_n(0))_H$, one sees that

$$c_1^2 \| u_n(0) \|_H^2 \leq \sup_{n \geq 1} \| u_n' \|^{1-s}_{L^2_{1-s}(H)} c + b + (y_0, \varphi)_H + \frac{1}{c_1^2} \| u_n(0) \|_H^2 + \frac{c_1^2}{2} \| y_0 \|_H^2 \quad \text{for all } n \geq 1.$$
As \((u_n)_{n \geq 1}\) is bounded in \(L^2_{1-s}(H)\), the last estimate implies that \((u_n(0))_{n \geq 1}\) is bounded in \(H\), and hence, (3.20) implies that \((u_n)_{n \geq 1}\) is bounded in \(W_{1-s}^{1,2}(H)\). \(W_{1-s}^{1,2}(H)\) is reflexive, one can conclude that \(u \in W_{1-s}^{1,2}(H)\), and there is a subsequence of \((u_n)_{n \geq 1}\), which we denote again by \((u_n)_{n \geq 1}\) such that \(u_n \rightharpoonup u^*\) weakly in \(L^2_{1-s}(H)\). From this, one obtains

\[
\|u^*\|^2_{L^2_{1-s}(H)} \leq \liminf_{n \to +\infty} \|u_n\|^2_{L^2_{1-s}(H)}. \tag{3.21}
\]

Since the trace map \(\text{Tr} : W_{1-s}^{1,2}(H) \to H\) maps weakly convergent sequences in \(W_{1-s}^{1,2}(H)\) onto weakly convergent sequence in \(H\), \(u_n(0) \rightharpoonup u(0)\) weakly in \(H\). By hypothesis, \(j\) is convex and lower semicontinuous on \(H\). Then \(j\) is also weakly lower semicontinuous on \(H\). Therefore,

\[
j(u(0) - \varphi) \leq \liminf_{n \to +\infty} j(u_n(0) - \varphi). \tag{3.22}
\]

Combining (3.21) and (3.22), one sees that \(u \in D(\mathcal{E})\) and \(\mathcal{E}(v) \leq c\).

Next, we show that the sub-differential operator

\[
\partial \mathcal{E} := \{(u, w) \in L^2_{1-s}(H) \times L^2_{1-s}(H) \mid \mathcal{E}(u + v) - \mathcal{E}(u) \geq (w, v)_{L^2_{1-s}(H)} \text{ for all } v \in L^2_{1-s}(H)\}
\]

of \(\mathcal{E}\) is a well-defined mapping on \(L^2_{1-s}(H)\); more precisely, for every \(u \in L^2_{1-s}(H)\), \(u\) belongs to \(D(\partial \mathcal{E})\) if and only if \(t^{1-2s}u' \in W_{2}^{1,2}(H)\), (3.15) holds, and

\[
\partial \mathcal{E}(u) = -t^{2s-1}(t^{1-2s}u')'. \tag{3.23}
\]

By the definition of the operator \(\mathcal{B}\) (see (3.18)), this then shows that \(\partial \mathcal{E} = \mathcal{B}\). Now, let \((u, w) \in \partial \mathcal{E}\), and for \(\varepsilon > 0\) and \(\xi \in C^0_c((0, +\infty); H)\), take \(v = \varepsilon\xi\). Then \(u + v \in D(\mathcal{E})\) satisfying \((u + v)(0) = u(0)\) and so,

\[
\mathcal{E}_1(u + \varepsilon\xi) - \mathcal{E}_1(u) = \mathcal{E}(u + \varepsilon\xi) - \mathcal{E}(u) \geq \varepsilon \mathcal{E}(w, \xi)_{L^2_{1-s}(H)}.
\]

Dividing this inequality by \(\varepsilon\) leads to

\[
\int_0^{+\infty} \frac{1}{2}\|u'(t) + \varepsilon\xi'(t)\|^2_H - \frac{1}{2}\|u'(t)\|^2_H t^{2(1-s)}\,dt \geq \int_0^{+\infty} (w(t), \xi(t))_H t^{2(1-s)}\,dt. \tag{3.24}
\]

Since

\[
\int_0^{+\infty} \frac{1}{2}\|u'(t) + \varepsilon\xi'(t)\|^2_H - \frac{1}{2}\|u'(t)\|^2_H t^{2(1-s)}\,dt \geq \frac{1}{\varepsilon} \left( \|u + \varepsilon\xi\|^2_{L^2_{1-s}(H)} - \|u\|^2_{L^2_{1-s}(H)} \right) = (2u + \varepsilon\xi, \xi)_{L^2_{1-s}(H)}, \tag{3.25}
\]

one has

\[
\lim_{\varepsilon \to 0^+} \int_0^{+\infty} \frac{1}{2}\|u'(t) + \varepsilon\xi'(t)\|^2_H - \frac{1}{2}\|u'(t)\|^2_H t^{2(1-s)}\,dt = \int_0^{+\infty} \|u'(t), \xi'(t)\|_H^2 t^{2(1-s)}\,dt. \tag{3.25}
\]

Thus sending \(\varepsilon \to 0^+\) in (3.24) yields

\[
\int_0^{+\infty} (u'(t), \xi'(t))_H t^{2(1-s)}\,dt \geq \int_0^{+\infty} (w(t), \xi(t))_H t^{2(1-s)}\,dt. \tag{3.24}
\]

By proceeding with the same arguments as before, but replacing \(\xi\) by \(-\xi\) gives

\[
\int_0^{+\infty} (u'(t), \xi'(t))_H t^{2(1-s)}\,dt \leq \int_0^{+\infty} (w(t), \xi(t))_H t^{2(1-s)}\,dt. \tag{3.24}
\]

Hence we have shown that

\[
\int_0^{+\infty} (u'(t), \xi'(t))_H t^{2(1-s)}\,dt = \int_0^{+\infty} (w(t), \xi(t))_H t^{2(1-s)}\,dt \quad \text{for every } \xi \in C^0_c((0, +\infty); H),
\]
implying that
\[-t^{1-2s}w(t) = \{t^{1-2s}u\}'(t) \text{ in } H \text{ for a.e. } t \in (0, +\infty). \tag{3.26}\]
Since \( w \in \partial \mathcal{E}(u) \) was arbitrary, the identification (3.26) implies that \( \partial \mathcal{E}(u) \) is single-valued and (3.23) holds. Thus the sub-differential operator \( \partial \mathcal{E} \) of \( \mathcal{E} \) is a well-defined mapping on \( L^2_{-s}(H) \) with domain \( D(\partial \mathcal{E}) \). In addition, since \( w \) and \( u' \) \( \in L^2_{-s}(H) \), (3.26) yields \( t^{1-2s}u' \in W^{1,2}_{-s}(H) \), and so \( \lim_{t \to 0^+} t^{1-2s}u'(t) \) exists in \( H \).

Next, let \( u \in D(\partial \mathcal{E}) \) and \( \xi_0 \in H \). By Proposition 2.17, there is a \( \xi \in W^{1,2}_{-s}(H) \) satisfying \( \xi(0) = \xi_0 \) in \( H \). Suppose \( u(0) + \varepsilon \xi \in D(\tilde{\mathcal{E}}_2) \) (otherwise, inequality (3.28) below always holds). Then, by the definition of the sub-differential \( \partial \mathcal{E} \), by (3.23), and by (3.25) applied to \( \varepsilon = 1 \), one gets
\[
\int_0^{+\infty} (u'(t), \xi'(t)) t^{2(1-s)} \frac{dt}{t} + j(u(0) + \xi_0 - \varphi) - j(u(0) - \varphi) \geq -\int_0^{+\infty} \langle t^{1-2s}u', \xi(t) \rangle_H dt. \tag{3.27}\]
Integrating by parts (Proposition 2.18) on the right-hand side of (3.27) gives
\[
\int_0^{+\infty} (u'(t), \xi'(t)) t^{2(1-s)} \frac{dt}{t} + j(u(0) + \xi_0 - \varphi) - j(u(0) - \varphi) \geq \left( \lim_{t \to 0^+} t^{1-2s}u'(t), \xi_0 \right)_H + \int_0^{+\infty} (u'(t), \xi'(t)) t^{2(1-s)} \frac{dt}{t}
\]
or, equivalently,
\[
j(u(0) + \xi_0 - \varphi) - j(u(0) - \varphi) \geq \left( \lim_{t \to 0^+} t^{1-2s}u'(t), \xi_0 \right)_H. \tag{3.28}\]
Since \( \xi_0 \in H \) was arbitrary, we have thereby shown that, for every \( u \in D(\partial \mathcal{E}) \), (3.15) holds, and therefore \( \partial \mathcal{E}(u) = \mathcal{B}(u) \) for every \( u \in D(\partial \mathcal{E}) \).

On the other hand, if \( u \in W^{1,2}_{1-s}(H) \) such that \( t^{1-2s}u' \in W^{1,2}_{-s}(H) \) and (3.15) holds, then for every \( \xi \in W^{1,2}_{-s}(H) \), adding \( \int_0^{+\infty} (u'(t), \xi'(t)) t^{2(1-s)} \frac{dt}{t} \) on both sides of (3.28) (with \( \xi_0 := \xi(0) \)) and subsequently integrating by parts the integral term on the right-hand side yields (3.27). Since
\[
\int_0^{+\infty} (u'(t), \xi'(t)) t^{2(1-s)} \frac{dt}{t} \leq \frac{1}{2} \|u' + \xi'\|^2_{L^2_{-s}(H)} - \frac{1}{2} \|u'\|^2_{L^2_{-s}(H)}
\]
and since \( \xi \in W^{1,2}_{-s}(H) \) was arbitrary, (3.27) implies \(-t^{2s-1}(t^{1-2s}u')' = \partial \mathcal{E}(u) \). This completes the proof of this proposition.

With these preliminaries, we are now in the position to give the details of the proof of existence to boundary value problem (1.23).

**Proof of Theorem 1.11 (Existence of Solutions).** Let \( A \) be a maximal monotone operator on \( H \) with \( 0 \in \text{Rg}(A) \). Then there is a \( y \in D(A) \) such that \( 0 \in Ay \). The translation \( \tilde{A}(x) := A(x + y) \) with domain \( D(\tilde{A}) = D(A) - y \) is then also maximal monotone on \( H \) and satisfies \( (0, 0) \in \tilde{A} \). Moreover, for every given \( \varphi \in D(A) \), one has \( \varphi_y := \varphi - y \in D(\tilde{A}) \), and \( u_y \) is a solution of the boundary-value problem (1.23) with \( A \) replaced by \( \tilde{A} \) and boundary value \( \varphi_y \) if and only if \( u(t) := u_y(t) + y \) is a solution of (1.23) with boundary value \( \varphi \). Since \( \tilde{A}\varphi_y = A\varphi \), either both functions \( u \) and \( u_y \) satisfy the inequalities (1.25)–(1.27) with \( A' \) or none of them. Thus, after possibly translating \( A \), we may assume without loss of generality that \( (0, 0) \in \text{Rg}(A) \). Then the lifted operator \( \tilde{A} \) on \( L^2_{-s}(H) \) defined by (3.16) is maximal monotone (Proposition 2.4) and \( (0, 0) \in \tilde{A} \). In particular, the Yosida approximation \( A_\lambda \) of \( A \) is maximal monotone and Lipschitz continuous on \( L^2_{-s}(H) \). We fix a boundary value \( \varphi \in D(\tilde{A}) \). Then, by Proposition 3.5, the operator \( \mathcal{B} \) defined in (3.18) is maximal monotone on \( L^2_{-s}(H) \). Now, it follows from claim (2) of Proposition 2.5 that, for every \( \lambda \) and \( \varepsilon > 0 \), there is a \( u_{\lambda \varepsilon} \in L^2_{-s}(H) \) satisfying equation (3.17). Since \( u_{\lambda \varepsilon} \in D(\mathcal{B}) \), we have \( u_{\lambda \varepsilon} \in W^{1,2}_{1-s}(H) \) such that \( t^{1-s}u' \in W^{1,2}_{-s}(H) \). This, together with the Lipschitz continuity of the Yosida approximation \( A_\lambda \) of \( A \), implies \( u_{\lambda \varepsilon} \in C^2((0, +\infty); H) \). In addition, the following *a priori estimates* hold.
(1) The following a priori estimates hold uniformly for all \( \lambda > 0 \) and \( 0 < \delta \leq 1 \):

\[
\|u_A(t)\|_H \leq \|u_A(\tilde{t})\|_H \quad \text{for all } t \geq \tilde{t} \geq 0,
\]

\[
\|t^{1-2s}u_A'(t)\|_H \leq \|\tilde{t}^{1-2s}u_A'(\tilde{t})\|_H \quad \text{for all } t \geq \tilde{t} \geq 0,
\]

\[
\|A_hu_A\|_{L^2_2(H)} \leq \|(t^{1-2s}u_A')'\|_{L^2_2(H)}
\]

\[
\lim_{t \to +0} t^{1-2s}u_A'(t) = c_1 \begin{cases} \sqrt{s}/(\sqrt{2(1-s)}), & \text{if } s \neq 1/2 \\ 0, & \text{if } s = 1/2 \end{cases}
\]

\[
\|u_A'\|_{L^2_2(H)} \leq c_1 \begin{cases} \sqrt{s}/(\sqrt{2(1-s)}), & \text{if } s \neq 1/2 \\ 0, & \text{if } s = 1/2 \end{cases}
\]

\[
\|\{(t^{1-2s}u_A')'\} \|_{L^2_2(H)} \leq c_1 \begin{cases} \sqrt{s}/(\sqrt{2(1-s)}), & \text{if } s \neq 1/2 \\ 0, & \text{if } s = 1/2 \end{cases}
\]

\[
\text{where } c_1 := \sqrt{s}/(\sqrt{2(1-s)}) > 0 \text{ is the constant provided by Proposition 2.17 (in Section 2.4) and } x_0 \in H \text{ is such that } j(x_0) = 0. \text{ Further, there is a constant } C > 0 \text{ such that}
\]

\[
\|u_A(0)\| \leq C \quad \text{for all } \lambda > 0.
\]

We begin by noting that, thanks to (3.18), equation (3.17) is equivalent to

\[
\{t^{1-2s}u_A'(t)\} = t^{1-2s}A_hu_A(t) + t^{1-2s} \delta u_A(t) \quad \text{for every } t > 0.
\]

Multiplying (3.36) by \( u_A \) with respect to the \( H \)-inner product, applying the monotonicity of \( A_h \), and since \( 0 \in A_h0 \), one sees that

\[
(t^{1-2s}u_A'(t), u_A(t))_H = (t^{1-2s}(A_hu_A), u_A(t))_H + t^{1-2s} \delta \|u_A(t)\|_H^2.
\]

Thus

\[
\int_0^\infty \{t^{1-2s}u_A'(r), u_A(r)\}_H \, dr \geq \delta \|t^{1-2s}u_A(0)\|_H^2 - \int_0^\infty t^{1-2s}u_A(r)_H^2 \, dr \geq 0 \quad \text{for every } t > 0.
\]

Next, by Cauchy–Schwarz’s inequality,

\[
\int_0^\infty \{t^{1-2s}u_A'(t), u_A(t)\}_H \, dt \leq \|u_A'\|_{L^2_2(H)} \|u_A\|_{L^2_2(H)}.
\]

For

\[
\frac{d}{dt}\|(t^{1-2s}u_A')', u_A(t)\|_H = \left(\|t^{1-2s}u_A'(t), u_A(t)\|_H + \|t^{1-2s}u_A'(t)\|_H^2\right) \frac{1}{t}, \quad t > 0,
\]

by Cauchy–Schwarz’s inequality, for the first term, we have

\[
\int_0^\infty \{t^{1-2s}u_A'(t), u_A(t)\}_H \, dt \leq \|\{(t^{1-2s}u_A')'\} \|_{L^2_2(H)} \|u_A\|_{L^2_2(H)}.
\]

and since \( u_A' \in L^2_1(H) \), also the second term belongs to \( L^1(H) \). Thus the function \( t \mapsto (t^{1-2s}u_A'(t), u_A(t))_H \) belongs to \( W^{1,1}(0, +\infty) \) and so admits a uniformly continuous representative on \( [0, +\infty) \), and

\[
\lim_{t \to +\infty} (t^{1-2s}u_A'(t), u_A(t))_H = 0.
\]

Thus and by (3.37),

\[
-(t^{1-2s}u_A'(t), u_A(t))_H = \int_0^\infty \{t^{1-2s}u_A'(r), u_A(r)\}_H \, dr + \int_0^\infty t^{1-2s}u_A'(r)_H^2 \, dr \geq 0 \quad \text{for every } t \geq 0.
\]

Here, we recall that the limit \( \lim_{t \to +0} t^{1-2s}u_A'(t) \) exists in \( H \) since \( t^{1-2s}u_A' \in W^{1,2}_s(H) \). In particular, this implies

\[
(t^{1-2s}u_A', u_A(t))_H \leq 0 \quad \text{for all } t \geq 0,
\]
Thus the function $t \mapsto \frac{1}{2} \|u_A(t)\|_H^2$ is monotonically decreasing along $[0, +\infty)$. Thus (3.29) holds.

By the Lipschitz continuity of $A_\lambda : H \to H$, the function

$$w_A(t) := A_\lambda u_A(t) + \delta u_A(t) \quad (t \in [0, +\infty))$$

(3.40)

is differentiable at a.e. $t \in \mathbb{R}_+$, with weak derivative

$$w_A'(t) = \frac{d}{dt} A_\lambda(u_A(t)) + \delta u_A'(t) \quad \text{for a.e. } t \in \mathbb{R}_+.$$  

On the other hand, by (3.36), $w_A = t^{2s-1} |t|^{1-2s} u_A'$. Hence

$$\frac{d}{dt} (w_A(t), t^{1-2s} u_A'(t))_H = (w_A'(t), t^{1-2s} u_A'(t))_H + \|t^{\delta} |t|^{1-2s} u_A'\|_H^2 \frac{1}{t} \quad \text{for a.e. } t > 0.$$  

(3.41)

By claim (1) of Proposition 2.5, one has

$$\left\| \frac{d}{dt} A_\lambda(v_A(t)) \right\|_H \leq \frac{1}{\lambda} \|v_A'(t)\|_H \quad \text{for a.e. } t > 0,$$

and so

$$|(w_A'(t), t^{1-2s} u_A'(t))_H| \leq \frac{1}{\lambda} \|t^{1-2s} u_A'\|_H^2 \frac{1}{t} \quad \text{for a.e. } t > 0.$$  

Therefore and since $t^{1-2s} u_A' \in W^{1,2}(H)$, (3.41) yields that the function $t \mapsto (w_A(t), t^{1-2s} u_A'(t))_H$ belongs to $W^{1,1}(0, +\infty)$ and hence admits a uniformly continuous representative on $[0, +\infty)$, and

$$\lim_{t \to +\infty} (w_A(t), t^{1-2s} u_A'(t))_H = 0,$$

from where we can conclude that

$$- (w_A(t), t^{1-2s} u_A'(t))_H = \int_{t_0}^{+\infty} \frac{d}{dt} (w_A(t), t^{1-2s} u_A'(t))_H \, dt \quad \text{for every } t \geq 0.$$  

(3.42)

Moreover, by the monotonicity of $A_\lambda$,

$$\left( \frac{d}{dt} A_\lambda(u_A(t), u_A'(t)) \right)_H = \lim_{h \to 0} \left( \frac{A_\lambda(u_A(t+h)) - A_\lambda(u_A(t))}{h}, \frac{u_A(t+h) - u_A(t)}{h} \right)_H \geq 0 \quad \text{for a.e. } t > 0.$$  

Therefore, one has

$$(w_A'(t), t^{1-2s} u_A'(t))_H = \left( \frac{d}{dz} A_\lambda(u_A(t), t^{1-2s} u_A'(t)) \right)_H + \delta \|t^{1-2s} u_A'\|_H^2 \frac{1}{t} \geq 0 \quad \text{for a.e. } z \in \mathbb{R}_+.$$  

(3.43)

Applying this inequality to (3.41), one sees that

$$\frac{d}{dt} (w_A(t), t^{1-2s} u_A'(t))_H \geq 0 \quad \text{for a.e. } t > 0,$$

and hence (3.42) implies

$$(w_A(t), t^{1-2s} u_A'(t))_H \leq 0 \quad \text{for every } t \geq 0.$$  

From this, it follows that

$$\frac{d}{dt} \frac{1}{2} \|t^{1-2s} u_A'(t)\|_H^2 = (t^{1-2s} u_A'(t))' \cdot t^{1-2s} u_A'(t) = t^{1-2s} (w_A(t), t^{1-2s} u_A'(t))_H \leq 0 \quad \text{for a.e. } t > 0,$$

showing that $t \mapsto \|t^{1-2s} u_A'(t)\|_H^2$ is monotonically decreasing on $[0, +\infty)$. In particular, (3.30) holds.
To see that inequality (3.31) holds, we multiply (3.17) by $A_\lambda u_\lambda$ with respect to the $L^2_{1-s}(H)$-inner product. Then, by the monotonicity of $A_\lambda$ and by (3.18), one sees that

$$
\|A_\lambda u_\lambda\|_{L^2_{1-s}(H)}^2 + \delta(A_\lambda u_\lambda, A_\lambda u_\lambda)_{L^2_{1-s}(H)} = (t^{2s-1}t^{1-2s}u_\lambda'(t), A_\lambda u_\lambda)_{L^2_{1-s}(H)} \\
\leq \|t^{1-2s}u_\lambda'(t)\|_{L^2_{1-s}(H)}^2 \|A_\lambda u_\lambda\|_{L^2_{1-s}(H)},
$$

from where we see that (3.31) holds.

Next, applying (3.43) to (3.41) gives

$$
\frac{d}{dt}(w_\lambda(t), t^{1-2s}u_\lambda'(t))_H \geq \|t^{1-2s}u_\lambda'(t)\|_H^2 \frac{1}{t} \quad \text{for a.e. } t > 0.
$$

Integrating this inequality over $(0, +\infty)$, one finds that

$$
\|t^{1-2s}u_\lambda'(t)\|_{L^2_{1-s}(H)}^2 \leq -(w_\lambda(0), \lim_{t \to 0^+} t^{1-2s}u_\lambda'(t))_H
$$

By hypothesis, $A$ is $\partial j$-monotone, and since $u_\lambda$ satisfies (3.15), it follows from Proposition 2.10 and by (3.40) that

$$
\|t^{1-2s}u_\lambda'(t)\|_{L^2_{1-s}(H)}^2 \leq -(w_\lambda(0), \lim_{t \to 0^+} t^{1-2s}u_\lambda'(t))_H = -(A_\lambda u_\lambda(0) + \delta u_\lambda(0), \lim_{t \to 0^+} t^{1-2s}u_\lambda'(t))_H \\
\leq -(A_\lambda \varphi + \delta \varphi, \lim_{t \to 0^+} t^{1-2s}u_\lambda'(t))_H,
$$

and since $\varphi \in D(A)$,

$$
\|t^{1-2s}u_\lambda'(t)\|_{L^2_{1-s}(H)}^2 \leq \lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)\|_{H}^2 \|A^* \varphi\|_H + \delta\|\varphi\|_H \right)^{1/2},
$$

where $A^*$ is the minimal section of $A$. Next, by taking $t = 0$ in (3.38) and applying (3.37), one sees that

$$
\left(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t), u_\lambda(0)\right)_H + \int_0^{+\infty} \|r^{-s}u_\lambda'(r)\|_H^2 \frac{dr}{r} = -\int_0^{+\infty} \left(\|r^{-2s}u_\lambda'(r)\|_H^2, u_\lambda(r)\right)_H \, dr \leq 0.
$$

Using this together with the fact that $u_j$ satisfies (3.15), and since $\partial j$ is monotone with $0 \in \partial j(x_0)$, one gets

$$
\|u_\lambda'(t)\|_{L^2_{1-s}(H)}^2 \leq -(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t), u_\lambda(0))_H \\
= -(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t), u_\lambda(0) - \varphi - x_0)_H - \left(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t), \varphi + x_0\right)_H \\
\leq -(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t), \varphi + x_0)_H,
$$

and hence Cauchy–Schwarz’s inequality gives that

$$
\|u_\lambda'(t)\|_{L^2_{1-s}(H)}^2 \leq \lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)\|_H\|\varphi + x_0\|_H.
$$

Since $t^{1-2s}u_\lambda' \in W^s_{1,2}(H)$, we can apply Proposition 2.17. Then there is a constant $c_1(s) > 0$ such that

$$
\|\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)\|_{H}^2 \leq c_1^2(s)\|t^{1-2s}u_\lambda'(t)\|_{W^s_{1,2}(H)}^2.
$$

We apply (3.44) and (3.45) to the right-hand side of the latter inequality. Then

$$
\|\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)\|_{H}^2 \leq c_1^2(s)\left(\|u_\lambda'(t)\|_{L^2_{1-s}(H)}^2 + \|t^{1-2s}u_\lambda'(t)\|_{L^2_{1-s}(H)}^2\right) \\
\leq c_1^2(s)\left(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)\|H\|\varphi + x_0\|_H + \|\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)\|_H\|A^* \varphi\|_H + \delta\|\varphi\|_H\right) \\
= c_1^2(s)\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)\|_{H}^2(\|\varphi + x_0\|_H + \delta\|\varphi\|_H + \|A^* \varphi\|_H),
$$

from where we can conclude that (3.32) holds. Now, inserting (3.32) into (3.44), one obtains (3.34), and inserting (3.32) into (3.45), one sees that (3.33) holds. Finally, we recall that $j$ is assumed to be strongly coercive, which, by Proposition 2.8, is equivalent to the fact that $(\partial j)^{-1}$ maps bounded sets into bounded sets. (cf. [13, Proposition 2.14]). Since

$$
u_\lambda(0) \in (\partial j)^{-1}(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t)) + \varphi \quad \text{for every } \lambda > 0,$$

and since the sequence $(\lim_{t \to 0^+} t^{1-2s}u_\lambda'(t))_{\lambda>0}$ is bounded in $H$ by (3.32), there is a constant $C > 0$ such that (3.35) holds.
(2) To establish the existence of a solution $u_\delta$ of (3.19), we begin with the following convergence result. Here, $(u_k)_{k \geq 1}$ represents a sequence $(u_{\lambda_k})_{k \geq 1}$ for a zero sequence $(\lambda_k)_{k \geq 1} \subseteq (0, +\infty)$. But since, at the end, the limit function $u_\delta$ is identified as the unique solution of (3.19), the limit does not depend on the choice of $(\lambda_k)_{k \geq 1}$, and hence, for simplicity, we consider $(u_k)_{k \geq 1}$ as a sequence.

**Lemma 3.6.** For every $\delta > 0$, the sequence $(u_k)_{k \geq 1}$ of solutions $u_\delta$ of (3.17) is a Cauchy sequence in $L^2_{1-s}(H)$. In particular, there is a $u_\delta \in L^2_{1-s}(H)$ such that

$$\lim_{\lambda \to 0^+} u_\lambda = u_\delta \quad \text{in } L^2_{1-s}(H).$$ (3.46)

**Proof of Lemma 3.6.** For $\lambda, \tilde{\lambda} > 0$, let $u_\lambda$ and $u_{\tilde{\lambda}}$ be two solutions of (3.17). Then multiplying

$$\delta(u_\lambda - u_{\tilde{\lambda}}) = -((A_\lambda u_\lambda - A_{\tilde{\lambda}} u_{\tilde{\lambda}}) - (B(u_\lambda) - B(u_{\tilde{\lambda}})))$$

by $u_\lambda - u_{\tilde{\lambda}}$ with respect to the $L^2_{1-s}(H)$-inner product and using that $B$ is monotone yields

$$\delta \|u_\lambda - u_{\tilde{\lambda}}\|^2_{L^2_{1-s}(H)} \leq -((A_\lambda u_\lambda - A_{\tilde{\lambda}} u_{\tilde{\lambda}}, u_\lambda - u_{\tilde{\lambda}}))_{L^2_{1-s}(H)}$$

We recall from [13, p. 28] that, for the resolvent operator $J_{\lambda}^A$ of $A$, one has $A_\lambda u \in A J_{\lambda}^A u$. Thus, by the monotonicity of $A$, one has

$$\delta \|u_\lambda - u_{\tilde{\lambda}}\|^2_{L^2_{1-s}(H)} \leq -((A_\lambda u_\lambda - A_{\tilde{\lambda}} u_{\tilde{\lambda}}, u_\lambda - u_{\tilde{\lambda}}))_{L^2_{1-s}(H)}$$

By Cauchy–Schwarz’s and Young’s inequality,

$$(\lambda + \tilde{\lambda}) \|A_\lambda u_\lambda, A_{\tilde{\lambda}} u_{\tilde{\lambda}}\|_{L^2_{1-s}(H)} \leq (\lambda + \tilde{\lambda}) \|A_\lambda u_\lambda\|_{L^2_{1-s}(H)} \|A_{\tilde{\lambda}} u_{\tilde{\lambda}}\|_{L^2_{1-s}(H)}$$

$$\quad \leq \lambda \|A_\lambda u_\lambda\|^2_{L^2_{1-s}(H)} + \tilde{\lambda} \|A_{\tilde{\lambda}} u_{\tilde{\lambda}}\|^2_{L^2_{1-s}(H)} + \frac{\lambda}{4} \|A_\lambda u_\lambda\|^2_{L^2_{1-s}(H)} + \frac{\tilde{\lambda}}{4} \|A_{\tilde{\lambda}} u_{\tilde{\lambda}}\|^2_{L^2_{1-s}(H)}.$$ 

Hence

$$\delta \|u_\lambda - u_{\tilde{\lambda}}\|^2_{L^2_{1-s}(H)} \leq \frac{\lambda}{4} \|A_\lambda u_\lambda\|^2_{L^2_{1-s}(H)} + \frac{\tilde{\lambda}}{4} \|A_{\tilde{\lambda}} u_{\tilde{\lambda}}\|^2_{L^2_{1-s}(H)},$$

which, by (3.31) and (3.34), gives

$$\delta \|u_\lambda - u_{\tilde{\lambda}}\|^2_{L^2_{1-s}(H)} \leq \frac{\lambda + \tilde{\lambda}}{4} c_1^2(s) \|A' \varphi\|^2_{\dot{H}} + (1 + 2\delta) \|A' \varphi\|_{\dot{H}} + (\delta + \delta^2) \|\varphi\|^2_{\dot{H}}.$$ 

Therefore $(u_k)_{k \geq 1}$ is a Cauchy sequence in $L^2_{1-s}(H)$ for every $\delta > 0$. This proves the claim of this lemma. □

**Continuation of the proof of Theorem 1.11.** By Lemma 3.6, there is a $u_\delta \in L^2_{1-s}(H)$ such that (3.46) holds. Thus, by the a priori estimates (3.33) and (3.34), one has $u_\delta \in W^{1,2}(H), t^{-1-2s}u_\delta' \in W^{1,2}_{1-s}(H)$, and after possibly passing to a subsequence of $(u_k)_{k \geq 1}$, one has

$$\lim_{\lambda \to 0^+} u_\lambda' = u_\delta' \quad \text{weakly in } L^2_{1-s}(H),$$ (3.47)

$$\lim_{\lambda \to 0^+} (t^{1-2s} u_\lambda')' = [t^{1-2s} u_\delta']' \quad \text{weakly in } L^2_{2-s}(H).$$ (3.48)

Since $B(u_\lambda) = -t^{2s-1}(t^{1-2s} u_\lambda')'$ and by [13, Proposition 2.5], the limits (3.46) and (3.48) imply that $u_\delta \in D(B)$ and $-t^{2s-1}(t^{1-2s} u_\delta')' = B(u_\delta)$. Next, by (3.31), there is a $\varphi \in L^2_{1-s}(H)$ and another subsequence of $(u_k)_{k \geq 1}$, which we denote again by $(u_k)_{k \geq 1}$, such that

$$\lim_{\lambda \to 0^+} A_\lambda u_\lambda = \varphi \quad \text{weakly in } L^2_{1-s}(H).$$ (3.49)
Then, by (3.46),
\[
\lim_{\lambda \to 0^+} (\lambda u_\lambda, u_\delta)_{L^2_{\lambda}(\mathcal{H})} = \lim_{\lambda \to 0^+} (\lambda u_\lambda, u_\lambda - u_\delta)_{L^2_{\lambda}(\mathcal{H})} + \lim_{\lambda \to 0^+} (\lambda u_\lambda, u_\delta)_{L^2_{\lambda}(\mathcal{H})} = (\chi, u_\delta)_{L^2_0(\mathcal{H})}
\]
On the other hand, due to (3.46), \( u_\delta \in \overline{D(\mathcal{A})} \). Thus \( f^\lambda_\delta u_\delta \to u_\delta \) in \( L^2_{\lambda-\delta}(\mathcal{H}) \) as \( \lambda \to 0^+ \), and since
\[
\|f^\lambda_\delta u_\lambda - u_\delta\|_{L^2_{\lambda}(\mathcal{H})} \leq \|f^\lambda_\delta u_\lambda - f^\lambda_\delta u_\delta\|_{L^2_{\lambda}(\mathcal{H})} + \|f^\lambda_\delta u_\delta - u_\delta\|_{L^2_{\lambda}(\mathcal{H})}
\]
\[
\leq \|u_\lambda - u_\delta\|_{L^2_{\lambda}(\mathcal{H})} + \|f^\lambda u_\delta - u_\delta\|_{L^2_{\lambda}(\mathcal{H})},
\]
o one has
\[
\lim_{\lambda \to 0^+} f^\lambda_\delta u_\lambda = u_\delta \quad \text{in} \quad L^2_0(\mathcal{H}).
\]
Therefore and since \( \mathcal{A}_\lambda u_\lambda \in \mathcal{A} f^\lambda_\delta u_\lambda \), [13, Proposition 2.5] implies \( u_\delta \in \overline{D(\mathcal{A})} \) and \( \chi \in \mathcal{A} u_\delta \). Now, by (3.46), (3.48) and (3.49), taking the \( L^2_{\lambda-\delta}(\mathcal{H}) \)-weak limit in (3.17) yields that \( u_\delta \) is a solution of (3.19) for every \( \delta > 0 \).

Now, let \( x \in \mathcal{H} \). Then, by Proposition 2.17, there is a \( \xi \in W^{1,2}_{\lambda-\delta}(\mathcal{H}) \) with trace \( \xi(0) = x \), and since
\[
t^{1-2s} u^\lambda_\delta \in W^{1,2}_{\delta}(\mathcal{H}),
\]
integrating by parts (Proposition 2.18) yields
\[
\left( \lim_{t \to 0^+} t^{1-2s} u^\lambda_\delta(t), x \right)_\mathcal{H} = - \int_0^{+\infty} \left( \left( |t^{1-2s} u^\lambda_\delta(t)|^2 \right)(t), \xi(t) \right)_\mathcal{H} dt - \int_0^{+\infty} \left( t^{1-2s} u^\lambda_\delta(t), \xi'(t) \right)_\mathcal{H} dt.
\]
Thanks to (3.47) and (3.48), we can send \( \lambda \to 0^+ \) in the two integrals on the right-hand side of the previous equality and apply again the integration by parts on \( u_\delta \). This shows that
\[
\left( \lim_{t \to 0^+} t^{1-2s} u^\lambda_\delta(t), x \right)_\mathcal{H} = - \int_0^{+\infty} \left( \left( |t^{1-2s} u^\lambda_\delta(t)|^2 \right)(t), \xi(t) \right)_\mathcal{H} dt - \int_0^{+\infty} \left( t^{1-2s} u^\lambda_\delta(t), \xi'(t) \right)_\mathcal{H} dt
\]
\[
= \left( \lim_{t \to 0^+} t^{1-2s} u^\lambda_\delta(t), x \right)_\mathcal{H} \quad \text{as} \quad \lambda \to 0^+.
\]
Since \( x \in \mathcal{H} \) was arbitrary, we have thereby shown that
\[
\lim_{t \to 0^+} t^{1-2s} u^\lambda_\delta(t) \to \lim_{t \to 0^+} t^{1-2s} u^\lambda_\delta(t) \quad \text{weakly in} \quad \mathcal{H} \quad \text{as} \quad \lambda \to 0^+.
\] (3) A priori estimates on \( (u^\delta)_{\delta>0} \). One has
\[
\|t^{1-2s} u^\delta(t)\|_\mathcal{H} \leq c^{1/2}_1(s) [\|A^* \varphi\|_\mathcal{H} + \|\varphi + x_0\|_\mathcal{H} + \delta]\|\varphi\|_\mathcal{H}, \quad (3.51)
\]
\[
\|t^{1-2s} u^\delta(t)\|_{L^2_{\lambda-\delta}(\mathcal{H})} \leq c^{1/2}_1(s) [\|A^* \varphi\|_\mathcal{H} + \|\varphi + x_0\|_\mathcal{H} + \delta]\|\varphi\|_\mathcal{H}]^{1/2} \|\varphi + x_0\|_\mathcal{H}^{1/2}, \quad (3.52)
\]
\[
\|t^{1-2s} u^\delta(t)\|_{L^2(\mathcal{H})} \leq c^{1/2}_1(s) [\|A^* \varphi\|_\mathcal{H}^2 + \|\varphi + x_0\|_H A^* \varphi\|_H + \delta\|\varphi\|_H (2\|A^* \varphi\|_H + \|\varphi + x_0\|_H + \delta\|\varphi\|_H)]^{1/2} \quad (3.53)
\]
for \( x_0 = \arg \min_{\mathcal{H}} f(u) \) and
\[
(t^{1-2s} u^\delta(t), u^\delta(t))_\mathcal{H} \leq 0 \quad \text{for all} \quad t \geq 0, \quad (3.54)
\]
\[
\|t^{1-2s} u^\delta(t)\|_\mathcal{H} \leq \|t^{1-2s} u^\delta(t)\|_\mathcal{H} \quad \text{for all} \quad t \geq \hat{t} \geq 0, \quad (3.55)
\]
\[
\|t u^\delta_\hat{t}\|_{L^2_0(\mathcal{H})} \leq \sqrt{\delta} \|u_\delta(0)\|_\mathcal{H}, \quad (3.56)
\]
\[
\|u^\delta_\hat{t}\|_\mathcal{H} \leq \|u_\delta(t)\|_\mathcal{H} \leq \|u_\delta(0)\|_\mathcal{H} \leq C \quad \text{for all} \quad \hat{t} \geq t \geq 0, \quad (3.57)
\]
for all \( \delta > 0 \), where the constant \( C \) is independent of \( \delta \).

Due to the limits (3.50), (3.47) and (3.48), sending \( \lambda \to 0^+ \) in (3.32), (3.33) and (3.34) yields (3.51), (3.52) and (3.53). Next, multiplying (3.19) by \( t^{1-2s} u_\delta \) with respect to the \( H \)-inner product, subsequently applying the monotonicity of \( A \) and that \( 0 \in AO \) gives
\[
\left( t^{1-2s} u^\delta(t), u^\delta(t) \right)_H \geq t^{1-2s} \delta \|u^\delta(t)\|_H^2 \quad \text{for a.e.} \quad t > 0. \quad (3.58)
\]
By continuing with the same reasoning as in step (1), one finds that inequalities (3.37) and (3.38) hold for $u_\lambda$ replaced by $u_\delta$. From this, we deduce that (3.54) and (3.55) hold (cf. (3.39) and (3.55) with $u_\lambda$ replaced by $u_\delta$), and

$$\frac{d}{dt} \frac{1}{2} \|u_\delta(t)\|^2_H = (u_\delta'(t), u_\delta(t))_H \leq 0 \quad \text{for all } t > 0.$$

Thus the first inequality from the left in (3.57) holds.

To see that (3.56) holds, we note that, by (3.58),

$$\int_0^t r^{2s} (|r^{1-2s} u_\delta'(r)|, u_\delta(r))_H \, dr \geq 0 \quad \text{for every } t \geq 0.$$

Thus and by (3.54),

$$0 \geq t^{2s} (t^{1-2s} u_\delta'(t), u_\delta(t))_H$$

$$= \int_0^t \frac{d}{dr} (r^{2s} (t^{1-2s} u_\delta'(r), u_\delta(r))_H) \, dr$$

$$= 2s \int_0^t r^{2s-1} (t^{1-2s} u_\delta'(r), u_\delta(r))_H \, dr + \int_0^t r^{2s} (|r^{1-2s} u_\delta'(r)|, u_\delta(r))_H \, dr + \int_0^t r \|u_\delta'(r)\|^2_H \, dr$$

$$\geq 2s \int_0^t \frac{d}{dr} \|u_\delta(r)\|^2_H \, dr + \int_0^t r \|u_\delta'(r)\|^2_H \, dr$$

$$\geq s \|u_\delta(t)\|^2_H - s \|u_\delta(0)\|^2_H + \int_0^t r \|u_\delta'(r)\|^2_H \, dr \quad \text{for all } t \geq 0.$$

From this, we deduce that (3.56) and the second inequality from the left in (3.57) holds.

(4) We show that there is a function

$$u \in C([-\infty, +\infty); H) \cap C^1((0, +\infty); H) \cap L^\infty(H)$$

satisfying $t^{1-2s} u' \in W^{1,2}_\delta(H)$ and

$$\lim_{\delta \to 0^+} u_\delta' = u' \quad \text{strongly in } L^{2}_{1-s}(H),$$

$$\lim_{\delta \to 0^+} u_\delta'(t) = u'(t) \quad \text{strongly in } H \text{ for all } t > 0,$$

$$\lim_{\delta \to 0^+} (t^{1-2s} u_\delta') = (t^{1-2s} u') \quad \text{weakly in } L^{2}_2(H),$$

$$\lim_{\delta \to 0^+} (t^{1-2s} u_\delta(t)) = \lim_{t \to 0^+} t^{1-2s} u'(t) \quad \text{weakly in } H,$$

$$\lim_{\delta \to 0^+} u_\delta = u \quad \text{weakly in } L^{2}(\mathcal{K}; H),$$

$$\lim_{\delta \to 0^+} u_\delta(t) = u(t) \quad \text{weakly in } H \text{ for every } t \geq 0,$$

for every compact sub-interval $\mathcal{K} \subseteq [0, +\infty)$. We begin by proving (3.59). For this, let $\delta, \hat{\delta} \in (0, 1]$; let $u_\delta$ and $u_\hat{\delta}$ be two strong solutions of (3.19), and set $w = u_\delta - u_\hat{\delta}$. Then

$$t^{2s-1} [t^{1-2s} w']'(t) = Au_\delta(t) - Au_\hat{\delta}(t) + \hat{\delta} u_\hat{\delta}(t) - \delta u_\delta(t) \quad \text{for almost every } z > 0.$$

By the monotonicity of $A$, one has

$$t^{2s-1} ([t^{1-2s} w']'(t), w(t))_H \geq (\delta u_\delta(t) - \hat{\delta} u_\hat{\delta}(t), u_\delta(t) - u_\hat{\delta}(t))_H$$

$$= \delta \|u_\delta(t)\|^2_H - \delta (u_\delta(t), u_\hat{\delta}(t))_H - \hat{\delta} \|u_\delta(t)\|^2_H + \hat{\delta} \|u_\delta(t)\|^2_H.$$
By Young’s inequality,
\[ \delta(u_0(t), u_0^\delta(t))_H \leq \delta\|u_0(t)\|_H^2 + \delta\|u_0^\delta(t)\|_H^2, \]
\[ \delta(u_0^\delta(t), u_0(t))_H \leq \delta\|u_0^\delta(t)\|_H^2 + \delta\|u_0(t)\|_H^2. \]

Therefore and by (3.57),
\[ t^{2s-1}(t^{1-2s}w')'(t), w(t))_H \geq \frac{-\delta}{4}\|u_0^\delta(t)\|_H^2 + \frac{\delta}{4}\|u_0(t)\|_H^2 \geq -\frac{\delta + \delta}{4}C^2. \]

Dividing this inequality by \( t^{2s-1} \) and then integrating over \((0, t)\) gives
\[ \int_0^t \frac{d}{dr}(r^{1-2s}w'(r), w(r))_H \, dr = \int_0^t r^{1-2s}\|w'(r)\|_H^2 \, dr \]
\[ = \int_0^t ((r^{1-2s}w')'(r), w(r))_H \, dr \]
\[ \geq -\frac{(\delta + \delta)}{4}C^2 \int_0^t r^{1-2s} \, dr \quad \text{for every } t > 0. \]

Since
\[ \lim_{t \to 0^+} t^{1-2s}u'_0(t) \in \partial j(u_0(0) - \varphi) \quad \text{and} \quad \lim_{t \to 0^+} t^{1-2s}u'_0(t) \in \partial j(u_3(0) - \varphi), \]
the monotonicity of \( \partial j \) implies
\[ (\lim_{t \to 0^+} t^{1-2s}w'(t), w(0))_H \geq 0. \]

Thus
\[ \int_0^t \frac{d}{dr}(r^{1-2s}w'(r), w(r))_H \, dr \leq (t^{1-2s}w'(t), w(t))_H + \frac{(\delta + \delta)}{4}C^2 \int_0^t r^{1-2s} \, dr, \]
and by Cauchy–Schwarz and inequality (3.57),
\[ \int_0^t \|r^{1-2s}w'(r)\|_H^2 \, dr \leq \|t^{1-2s}w'(t)\|_H C + \frac{(\delta + \delta)}{4}C^2 \frac{t^{2(1-s)}}{2(1-s)}. \] (3.65)

On the one hand,
\[ t^{2s} \int_0^{+\infty} \|r^{1-2s}u'_0(r)\|_H^2 \, dr \leq \int_0^{+\infty} \|ru'_0(r)\|_H^2 \, dr, \]
and on the other hand, due to (3.55),
\[ \int_0^{+\infty} \|ru'_0(r)\|_H^2 \, dr = \int_0^{+\infty} \|r^{1-2s}u'_0(r)\|_H^{2(s)} \, dr \geq \|t^{1-2s}u'_0(t)\|_H^2 \int_0^{+\infty} r^{4s-1} \, dr = \|t^{1-2s}u'_0(t)\|_H^2 \frac{t^{4s}}{4s}. \]

Thus and by (3.6) and (3.57),
\[ \|t^{1-2s}u'_0(t)\|_H^2 \frac{t^{4s}}{4s} + t^{2s} \int_0^{+\infty} \|r^{1-2s}u'_0(r)\|_H^2 \, dr \leq \|ru'_0\|_{L^2_1(H)}^2 \leq s\|u_0(0)\| = sC^2. \]
Hence

\[
\int_{t}^{+\infty} \|r^{1-s}u_\delta'(r)\|_H^2 \, dr \leq \frac{4s^2 C^2}{t^{2s}} \quad \text{for every } \delta, t > 0, \quad (3.66)
\]

\[
\|r^{1-2s}u_\delta'(t)\|_H \leq \frac{2sC}{t^{2s}} \quad \text{for every } \delta, t > 0.
\]

Applying the last estimate to (3.65) gives

\[
\int_{0}^{t} \|r^{1-s}w(r)\|_H^2 \, dr \leq \frac{2sC^2}{t^{2s}} + \frac{(\delta + \tilde{\delta})^2}{4} \frac{t^{2(1-s)}}{2(1-s)},
\]

Now, using this one and (3.66) together with the elementary inequality \((a + b)^{1/2} \leq a^{1/2} + b^{1/2}\) (holding for all \(a, b \geq 0\)), one sees that

\[
\|w'\|_{L^1_{-1-s}(H)} = \left( \int_{0}^{t} \|r^{1-s}w'(r)\|_H^2 \, dr \right)^{1/2} \leq \left( \int_{0}^{t} \|r^{1-s}w'(r)\|_H^2 \, dr \right)^{1/2} + \left( \int_{t}^{\infty} \|r^{1-s}w'(r)\|_H^2 \, dr \right)^{1/2}
\]

\[
\leq \left( \int_{0}^{t} \|r^{1-s}w'(r)\|_H^2 \, dr \right)^{1/2} + \left( \int_{t}^{\infty} \|r^{1-s}u_\delta'(r)\|_H^2 \, dr \right)^{1/2} + \left( \int_{t}^{\infty} \|r^{1-s}u_\delta'(r)\|_H^2 \, dr \right)^{1/2}
\]

\[
\leq \left( \int_{0}^{t} \|r^{1-s}w'(r)\|_H^2 \, dr \right)^{1/2} + \left( \frac{2sC^2}{t^{2s}} + \frac{(\delta + \tilde{\delta})^2}{4} \frac{t^{2(1-s)}}{2(1-s)} \right)^{1/2} + \frac{2sC}{t^{2s}} \quad \text{for every } t, \delta, \tilde{\delta} > 0.
\]

Choosing \(t = (\delta + \tilde{\delta})^{-1/2}\) and inserting \(w = u\delta - u\tilde{\delta}\), then

\[
\|u\delta' - u\tilde{\delta}'\|_{L^1_{-1-s}(H)} \leq \left( \int_{0}^{\infty} \frac{sC^2 + \frac{C^2}{8(1-s)}}{t^{2s}} \, dt \right)^{1/2} (\delta + \tilde{\delta})^{1/2} + \frac{4sC}{t^{2s}} \frac{(\delta + \tilde{\delta})^2}{4} \frac{t^{2(1-s)}}{2(1-s)},
\]

showing \((u\delta)'_{\delta > 0}\) is a Cauchy sequence in \(L^2_{-1-s}(H)\). First, we denote the limit of \((u\delta)'_{\delta > 0}\) in \(L^2_{-1-s}(H)\) by \(\hat{u}\).

On the other hand, by (3.57), \((u\delta)_{\delta > 0}\) is bounded in \(L^2(0, T; H)\) for every \(T > 0\). Thus there is a subsequence of \((u\delta)_{\delta > 0}\) and a function \(u \in L^2_{0\infty}([0, +\infty); H)\) such that the weak limit (3.64) holds.

It is worth mentioning that, in step (5) below, we show that \(u\) is a solution of boundary-value problem (1.23). But without the strict convexity condition on \(j\) (cf. Proposition 3.2), the limit \(u\) depends, actually, on the choice of the zero sequence \((\delta_n)_{n \geq 1}\) in \((u\delta)_{\delta > 0}\). Since, for the completeness of this theorem, the existence of one solution to (1.23) is sufficient, we omit, for simplicity, the notation of a zero sequence \((\delta_n)_{n \geq 1}\), but we are aware of it and only write \(\delta \to 0\) even though we might speak from subsequences.

Then, by the latter two limits, we can conclude that \(\hat{u} = u'\), and hence (3.59) holds. Further, since \(u' \in L^2_{-1-s}(H)\), Proposition 2.17 says that \(u\) admits a continuous representative on \([0, +\infty)\).

Next, let \(\delta, t > 0\). Then one easily sees that

\[
\int_{t}^{\infty} r^{2s-1} \|r^{1-2s}u\delta'(r) - r^{1-2s}u'(r)\|_H^2 \, dr \leq \|u\delta' - u'\|_{L^1_{-1-s}(H)}^2
\]

for every \(t \geq 0\).

Further,

\[
\frac{d}{dr} \left( \int_{t}^{\infty} r^{2s-1} \|r^{1-2s}u\delta'(r) - r^{1-2s}u'(r)\|_H^2 \, dr \right)
\]

\[
= (2s - 1) r^{2s-2} \|r^{1-2s}u\delta'(r) - r^{1-2s}u'(r)\|_H^2
\]

\[
+ r^{2s-1} \left( \{r^{1-2s}u\delta'(r) - \{r^{1-2s}u'(r)\} \right) \right) \quad \text{for a.e. } r \in (0, +\infty)
\]

and

\[
\int_{t}^{\infty} r^{2s-2} \|r^{1-2s}u\delta'(r) - r^{1-2s}u'(r)\|_H^2 \, dr \leq r^{-1} \|u\delta' - u'\|_{L^2_{-1-s}(H)}^2
\]
and by Cauchy–Schwarz’s inequality,
\[
\int_1^\infty r^{2s-1} \left| \left( \{ r^{1-2s} u_\delta'(r) \} \right) - \{ r^{1-2s} u'(r) \} \right| \, dr
\]
\[
= \int_1^\infty \left| r^{(1-2s)}u_\delta'(r) - \{ r^{1-2s} u'(r) \} \right|^2 \, dr
\]
\[
\leq \| \{ t^{1-2s} u_\delta'(t) \} \|_{L^1([t, \infty))}^2 \left( \| r^{1-2s} u'(r) \|_{L^1([t, \infty))}^2 \right)
\]
\[
\leq 2C\| u_\delta - u' \|_{L^1([t, \infty))}^2.
\]
where the constant $C$ is independent of $\delta$ and given thanks to (3.53) and the weak limit (3.61). Thus, for every $\varepsilon > 0$, the real-valued function
\[
t \mapsto t^{2s-1} \| t^{1-2s} u_\delta'(t) - t^{1-2s} u'(t) \|_{H^1}
\]
elongs to $W^{1,1}((\varepsilon, +\infty))$ and is therefore uniformly continuous on $(\varepsilon, +\infty)$, and
\[
\lim_{t \to +\infty} t^{2s-1} \| t^{1-2s} u_\delta'(t) - t^{1-2s} u'(t) \|_{H^1} = 0.
\]
Furthermore, by the previous estimates, one sees that
\[
t^{2s-1} \| t^{1-2s} u_\delta'(t) - t^{1-2s} u'(t) \|_{H^1}^2 = \int_1^\infty \frac{d}{dr} \left( t^{2s-2} \| r^{1-2s} u_\delta'(r) - r^{1-2s} u'(r) \|_{H^1}^2 \right) \, dr
\]
\[
= (2s - 1) \int_1^\infty t^{2s-2} \left( \| r^{1-2s} u_\delta'(r) - r^{1-2s} u'(r) \|_{H^1}^2 \right) \, dr
\]
\[
+ \int_1^\infty t^{2s-1} \left( \{ r^{1-2s} u_\delta'(r) \} - \{ r^{1-2s} u'(r) \} \right)^2 \, dr
\]
\[
\leq |2s - 1| t^{1-1} \| u_\delta' - u' \|_{L^1([t, \infty))}^2 + 2C\| u_\delta - u' \|_{L^1([t, \infty))}^2 
\]
for all $t > 0$.

Thus and by (3.59), we obtain that the strong limit (3.60) holds.

**Remark 3.7.** In the case of Dirichlet boundary conditions $u_\delta(0) = \varphi$ (for all $\delta > 0$), we have
\[
\lim_{\delta \to 0^+} u_\delta = u \quad \text{strongly in } C(\mathcal{K}; H)
\]
for every compact subset $\mathcal{K}$ of $[0, +\infty)$. To see this, note that, by (2.3) applied to $(u_\delta' - u_\lambda')$ and with $s$ replaced by $1 - s$, one has
\[
\| u_\delta(t) - u_\delta(t) - (u_\delta(t) - u_\delta(t)) \|_H \leq \| u_\delta' - u_\delta' \|_{L^1([t, \infty))}^2 (2s)^{-\frac{s}{2}} \| t^{2s} - t^{2s} \|_{1/2} = 0.
\]
Then, due to $u_\delta(0) = \varphi$ for all $\delta > 0$, sending $\delta \to 0$ in the last inequality gives
\[
\sup_{t \in \mathcal{K}} \| u_\delta(t) - u_\delta(t) \|_H \leq \| u_\delta' - u_\delta' \|_{L^1([t, \infty))} (2s)^{-\frac{s}{2}} \| t^{2s} - t^{2s} \|_{1/2}
\]
for every compact $\mathcal{K} \subset [0, +\infty)$, $\delta, \delta > 0$. Thus limit (3.59) implies that $(u_\delta)_{\delta > 0}$ is a Cauchy sequence in $C(\mathcal{K}; H)$ for every compact $\mathcal{K} \subset [0, +\infty)$, implying that (3.67) holds.

**Continuation of step (4).** By (3.52) and (3.53), the sequence $(t^{1-2s} u_\delta')_{\delta > 0}$ is bounded in $W^{1,2}_s((\varepsilon, +\infty))$. Thus $u$ has a second weak derivative $u'' \in L^2_{\text{weak}}(H)$ such that $t^{1-2s} u'' \in W^{1,2}_s((\varepsilon, +\infty))$ and (3.61) holds. In particular, by Proposition 2.17, we get that the limit $\lim_{\delta \to 0^+} t^{1-2s} u'(t)$ exists in $H$. Since the trace operator $\text{Tr}: W^{1,2}_s((\varepsilon, +\infty)) \to H$ is linear and continuous, (3.59) and (3.61) imply that (3.62) holds.

In order to show that (3.64) holds, we use the substitution
\[
v_\delta(z) := u_\delta(2sz^{1/2s}) \quad \text{and} \quad v(z) := u(2sz^{1/2s}) \quad \text{for all } z \geq 0.
\]
Then, by Lemma 3.4, \( v'_\delta \in L^2(H) \) with \( v'_\delta(z) = (2s)^{2s-1} t^{1-2s} u'_\delta(t) \), and by (3.59),
\[
\lim_{\delta \to 0^+} v'_\delta = v \quad \text{in } L^2(H).
\] (3.68)

Now, let \( x \in H \), and for \( \rho \in C^\infty((0, +\infty)) \) satisfying \( 0 \leq \rho \leq 1, \rho \equiv 1 \) on \([0, 1]\) and \( \rho \equiv 0 \) on \([2, +\infty)\), set \( \xi(z) = \rho(z)x \) for every \( z \geq 0 \). Then, by (3.63) and (3.68), one has
\[
(u_\delta(0), x)_H = (v_\delta(0), x)_H = -\int_0^2 \frac{d}{dz}(v_\delta(z), \xi(z))_H \, dz
\]
\[
= -\int_0^2 (v'_\delta(z), \xi(z))_H \, dz - \int_0^2 (v_\delta(z), \xi'(z))_H \, dz
\]
\[
= -\int_0^2 (v'_\delta(z), \xi(z))_H \, dz - \int_0^2 (u(2sz^{1/2s}), \xi'(z))_H \, dz
\]
\[
= (v(0), x)_H = (u(0), x)_H \quad \text{as } \delta \to 0^+.
\]

Since \( x \in H \) was arbitrary, this shows that (3.64) holds for \( t = 0 \). Moreover, for every \( t > 0 \) with \( t = 2sz^{1/2s} \), one has
\[
u_\delta(t) = v_\delta(z) = v_\delta(0) + \int_0^z v'_\delta(r) \, dr \to v(0) + \int_0^z v'(r) \, dr = v(z) = u(t) \quad \text{weakly in } H \quad \text{as } \delta \to 0^+.
\]

This completes the proof of (3.64). By (3.57) and (3.64), we can conclude that \( u \in L^\infty(H) \).

(5) Let \( K := [a, b] \) with \( 0 < a < b < +\infty \) be a compact interval. Then we write \( L^2(K; H) \) to denote the space \( L^2(K; H) \) of all \( L^2 \)-integrable and \( H \)-valued functions \( v \) on \( K \). Further, let \( \mathcal{A}_K \) denote the restriction of the operator \( \mathcal{A} \) on \( K \) given by
\[
\mathcal{A}_K = \{(u, w) \in L^2(K; H) \times L^2(K; H) \mid w(t) \in Au(t) \text{ for a.e. } t \in K \}.
\]

Then, by the maximal monotonicity of \( A \) in \( H \) and since \( K \) has finite measure, one has that \( \mathcal{A}_K \) is maximal monotone on \( L^2(K; H) \). Moreover, since each function \( u \in L^2(K; H) \) restricted on \( K \) belongs to \( L^2(K; H) \), we set
\[
f_\delta := t^{2s-1/2} u'_\delta - \delta u_\delta \quad \text{for every } \delta > 0.
\]

Then, by (3.19), we have \( f_\delta \in \mathcal{A}_K u_\delta \). The two limits (3.57) and (3.61) imply
\[
\lim_{\delta \to 0^+} f_\delta = t^{2s-1/2} u'_\delta \quad \text{weakly in } L^2(K; H).
\] (3.69)

Thus and by (3.63), if
\[
\lim_{\delta \to 0^+} (u_\delta, f_\delta)_{L^2(K; H)} \leq (u, \xi'')_{L^2(K; H)},
\] (3.70)

then [13, Proposition 2.15] yields \( u \in D(\mathcal{A}_K) \) and \( t^{2s-1/2} u'_\delta \) \((t^{2s-1/2} u'_\delta)' \in \mathcal{A}_K u \). To see that (3.70) holds, we write
\[
(u_\delta, f_\delta)_{L^2(K; H)} = (u_\delta - u, f_\delta)_{L^2(K; H)} + (u, f_\delta - t^{2s-1/2} u'_\delta)'_{L^2(K; H)}
\]
and note that, by (3.69),
\[
\lim_{\delta \to 0^+} (u, f_\delta - t^{2s-1/2} u'_\delta)'_{L^2(K; H)} = 0.
\]
Further,
\[
(u_\delta - u, f_\delta)_{L_K^2(H)} = (u_\delta - u, t^{2s-1}(t^{1-2s}u_\delta')' - \delta u_\delta)_{L_K^2(H)}
\]
\[
= (u_\delta - u, t^{2s-1}(t^{1-2s}u_\delta')' - (t^{1-2s}u')')_{L_K^2(H)}
\]
\[
+ (u_\delta - u, t^{2s-1}(t^{1-2s}u')' - \delta u_\delta)_{L_K^2(H)}
\]
\[
= \int_a^b (u_\delta(t) - u(t)) (t^{1-2s}u_\delta' - t^{1-2s}u')_H t^{2s-1} dt
\]
\[
+ (u_\delta - u, t^{2s-1}(t^{1-2s}u')' - \delta u_\delta)_{L_K^2(H)}
\]
\[
= (u_\delta(t) - u(t), u_\delta'(t) - u'(t))_H t^{1-2s} dt
\]
\[
- \int_a^b (u_\delta'(t) - u'(t), t^{1-2s}u_\delta' - t^{1-2s}u')_H t^{2s-1} dt
\]
\[
+(u_\delta - u, t^{2s-1}(t^{1-2s}u')' - \delta u_\delta)_{L_K^2(H)}
\]
\[
= (u_\delta(t) - u(t), u_\delta'(t) - u'(t))_H t^{1-2s} dt
\]
\[
+ (u_\delta - u, t^{2s-1}(t^{1-2s}u')' - \delta u_\delta)_{L_K^2(H)}.
\]

Now, on $K$, the limit (3.59) implies $u_\delta' \to u'$ strongly in $L_K^2(H)$, and by (3.60), we have $u_\delta'(t) \to u'(t)$ strongly in $H$ for every $t \in K$. Thus and by (3.64), we see that (3.70) holds for $K$. Since the compact sub-interval $K = [a, b]$ of $(0, +\infty)$ was arbitrary, we have thereby shown that $u$ is a solution of (3.14).

It remains to show that $u$ satisfies boundary condition (3.15). To see this, note first that if $j$ is the indicator function, then (3.15) reduces to the Dirichlet boundary condition $u(0) = \varphi$. Then, for all $\delta > 0$, the solution $u_\delta$ of (3.19) satisfies $u_\delta(0) = \varphi$, and since (3.67), we get $u(0) = \varphi$. Now, suppose $\partial j: D(\delta) \to H$ is a weakly continuous mapping. Since, for every $\delta \in (0, 1)$, one has
\[
\lim_{t \to 0^+} t^{1-2s}u_\delta'(t) = \partial j(u_\delta(0) - \varphi),
\]
the weak continuity of $\partial j$ together with the two limits (3.64) and (3.62) imply that (3.15) holds.

(6) By the limits (3.59), (3.61) and (3.62), sending $\delta \to 0^+$ in the a priori estimates (3.51), (3.52) and (3.53), one finds that $u$ satisfies (1.25), (1.26) and (1.27) with $y = 0$. Further, by (3.60), letting $\delta \to 0^+$ in (3.55), one sees that $u$ satisfies (1.6).

By the first inequality on the right-hand side of (3.57) and by (3.64), we can conclude that $u \in L^{\infty}(H)$. By step (5), this means that $u$ is a bounded strong solution of differential inclusion (1.24). Now, let $u_1$ and $u_2$ be two strong solutions of (1.24), and suppose that $u_1, u_2 \in L^{\infty}(H) \cap C([0, +\infty); H)$. Then, by claim (1) of Proposition 3.2, one has that $u_1$ and $u_2$ satisfy (1.13). Thus, by choosing $u_2(t) \equiv 0$, one sees that, for our limit function $u$, one has that $t \mapsto \|u(t)\|_H$ is monotonically decreasing along $[0, +\infty)$. Hence
\[
(u'(t), u(t))_H = \frac{d}{dt} \frac{1}{2} \|u(t)\|_H^2 \leq 0 \quad \text{for all } t > 0.
\]
(3.71)

Since $u$ is a strong solution of the differential inclusion (1.24) and $0 \in A_0$, the monotonicity of $A$ implies
\[
(u^{1-2s}u')' = (u(t))_H = \frac{d}{dt} \left( u(t) \right) \geq 0 \quad \text{a.e. } t > 0.
\]

Using this inequality, one sees that, for a.e. $t > 0$,
\[
\frac{d}{dt} (u'(t), u(t))_H = \frac{d}{dt} t^{2s}(u^{1-2s}u'(t), u(t))_H
\]
\[
= 2st^{2s-1}(1-2s)u'(t), u(t))_H + t^{2s}(u^{1-2s}u'(t), u(t))_H + t\|u'(t)\|_H^2
\]
\[
\geq 2st^{2s-1}(1-2s)u'(t), u(t))_H + t\|u'(t)\|_H^2
\]
\[
= s \frac{d}{dt} \frac{1}{2} \|u(t)\|_H^2 + t\|u'(t)\|_H^2.
\]
For given $t > \varepsilon > 0$, integrating the last estimate over $(\varepsilon, t)$ gives
\[
\int_{\varepsilon}^{t} r ||u'(r)||_{H}^{2} \, dr. \]

Since $t^{1-2s}u' \in W_{s}^{1,2}(H)$, we have that $\lim_{t \to 0^{+}} t^{1-2s}u'(t)$ exists in $H$. Thus taking $\varepsilon \to 0^{+}$ in the previous inequality gives
\[
\int_{0}^{t} r ||u'(r)||_{H}^{2} \, dr, \]
and by (3.71),
\[
s||u(0)||_{H}^{2} \geq \int_{0}^{t} \frac{||ru'(r)||_{H}^{2}}{r} \, dr. \quad (3.72)
\]
Sending here $t \to +\infty$ and taking square roots on both sides of the resulting inequality shows that $u$ satisfies (1.7).

By applying (1.6) to (3.72), one sees that
\[
s||u(0)||_{H}^{2} \geq \int_{0}^{t^{4s-1}} t^{1-2s} ||u'(r)||_{H}^{2} \, dr \geq \int_{0}^{t^{4s-1}} \frac{t^{1-2s} ||u'(t)||_{H}^{2}}{4s} = \frac{t^{4s}}{4s} ||t^{1-2s}u'(t)||_{H}^{2}.
\]
Rearranging this estimate gives
\[
||t^{1-2s}u'(t)||_{H}^{2} \leq 4s \frac{||u(0)||_{H}^{2}}{t^{4s}} \quad \text{for all } t > 0,
\]
from where we can deduce (1.8).

Finally, we show that $u$ satisfies (1.9). For this, fix $h > 0$ and $\beta > 2 - 2s$. Then, by the monotonicity of $A$, we have
\[
((t + h)^{2s-1}[t^{1-2s}u]'(t + h) - t^{2s-1}[t^{1-2s}u]'(t), u(t + h) - u(t))_{H} \geq 0 \quad \text{for a.e. } t > 0.
\]
Thus
\[
(t + h)^{2s-1}([t^{1-2s}u]'(t + h) - [t^{1-2s}u]'(t), u(t + h) - u(t))_{H} + [(t + h)^{2s-1} - t^{2s-1}][([t^{1-2s}u]'(t), u(t + h) - u(t))_{H} \geq 0 \quad \text{for a.e. } t > 0,
\]
and so, for
\[
\psi(t) := ((t + h)^{1-2s}u'(t + h) - t^{1-2s}u'(t), u(t + h) - u(t))_{H} \quad (t > 0),
\]
one has
\[
\frac{d}{dt} [t^{\beta}(t^{1-2s}u'(t + h) - t^{1-2s}u'(t), u(t + h) - u(t))_{H} + t^{\beta}((t + h)^{1-2s}u'(t), u' - u')_{H} \geq -t^{\beta}(t + h)^{2s-1} - t^{2s-1}([t^{1-2s}u]'(t), u(t + h) - u(t))_{H} + t^{\beta}((t + h)^{1-2s}u'(t), u'(t + h) - u'(t))_{H} - t^{\beta}(t + h)^{1-2s}[([t^{1-2s}u]'(t), u(t + h) - u(t))_{H} + t^{\beta}(t + h)^{2s-1}[([t^{1-2s}u]'(t), u'(t + h) - u'(t))_{H} + t^{\beta}(t + h)^{2s-1} [([t^{1-2s}(t + h)^{1-2s}] - (t + h)^{1-2s} \times ((t + h)^{1-2s}u'(t + h) - t^{1-2s}u'(t), u'(t))_{H}.\]
Integrating this inequality over \((0, t)\) for \(t > 0\), dividing by \(h^2\) and letting \(h \to 0^+\), one finds

\[
t^\beta \langle (t^{1-2s} u')', (t, u'(t)) \rangle_H \geq (\beta - (4s - 2)) \int_0^t \frac{d}{dr} \frac{1}{2} r^{1-2s} u'^2_H + \int_0^t r^{\beta - 2s - 1} \|r^{1-2s} u'\|_H^2 \ dr.
\]

By (1.6), an integration by parts shows that

\[
0 \geq t^\beta \langle (t^{1-2s} u')', (t, u'(t)) \rangle_H \geq \frac{\beta - (4s - 2)}{2} t^{\beta - 2s + 2s} \|t^{1-2s} u'(t)\|_H^2 - \frac{(\beta - (4s - 2))(\beta - 2 + 2s)}{2} \int_0^t r^{\beta - 3 + 2s} \|r^{1-2s} u'(r)\|_H^2 \ dr + \int_0^t r^{\beta - 2s - 1} \|r^{1-2s} u'(r)\|_H^2 \ dr,
\]

where we used \(\beta > 2-2s\). Now, choosing \(\beta = 2 + 2s\) in the previous inequality leads to

\[
(2 - s)4s \int_0^t r u'(r) \|u\|_H \ dr \geq (2 - s) t^{2s} \|t^{1-2s} u'(t)\|_H^2 + \int_0^t t^{1+2s} \|t^{1-2s} u'\|_H^2 \ dr.
\]

By (1.7), we can send \(t \to +\infty\) in this inequality. Then one sees that (1.9) holds. This completes the proof of this theorem. \(\square\)

### 4 Well-Posedness of the Dirichlet Problem in the Half-Space

In the previous section, we outlined the proof of existence and uniqueness of boundary-value problem (1.23) for boundary data \(\varphi \in D(A)\). If \(j\) is the indicator function, then problem (1.23) reduces to Dirichlet problem \((D_0^\varphi)\). Now, we outline the proof of existence and uniqueness of solutions to \((D_0^\varphi)\) for boundary data \(\varphi \in D(A)^H\). This completes the proof of the statements in Theorem 1.1.

**Proof of Theorem 1.1.** Due to Theorem 1.11 applied to the indicator function \(j\), for every \(0 < s < 1\), and \(\varphi \in D(A)\), there is a unique solution \(u\) of \((D_0^\varphi)\) satisfying \(u \in L^\infty(H)\), \(t^{1-2s} u' \in W_s^{1,2}(H)\), and for every \(y \in A^{-1}([0, \infty))\), the estimates (1.6)–(1.9). Moreover, we deduce from (1.25)–(1.27) the three estimates (1.10), (1.11), and (1.12). In addition, by Proposition 3.2, we have that every two solutions \(u_1, u_2 \in C([0, +\infty); H)\) of \((D_0^\varphi)\) satisfy (1.13). Next, we take \(\varphi \in D(A)^H\). Then, by Theorem 1.11, there are sequences \((\varphi_n)_{n \geq 1} \subseteq D(A)\) and \((u_n)_{n \geq 1} \subseteq L^\infty(H) \cap C([0, +\infty); H)\) of solutions \(u_n\) of \((D_0^\varphi)\) with boundary value \(u_n(0) = \varphi_n\) and satisfying \(\varphi_n \to \varphi\) in \(H\) as \(n \to +\infty\). Applying (1.13) to \(u_n\) and \(u_m\) gives

\[
\sup_{t \geq 0} \|u_n(t) - u_m(t)\|_H \leq \|\varphi_n - \varphi_m\|_H
\]

for every \(n, m \geq 0\).

This shows that \((u_n)_{n \geq 1}\) is a Cauchy sequence in \(C^b((0, +\infty); H)\). Hence there is a function \(u \in C^b((0, +\infty); H)\) such that

\[
\lim_{n \to +\infty} u_n = u \quad \text{in } C^b((0, +\infty); H).
\]

From this limit, we can conclude that \(u\) satisfies (1.13), and since \(u_n(0) = \varphi_n \to \varphi\) in \(H\), one has \(u(0) = \varphi\) in \(H\). By (1.7) and (1.9) applied to \(u_n\), we can conclude that \(u \in W_{loc}^{1,2}((0, +\infty); H)\), and after possibly passing to a subsequence of \((u_n)_{n \geq 1}\) and taking the limit-inferior as \(n \to +\infty\) in (1.7) and (1.9), one finds that \(u\) satisfies (1.7) and (1.9). Next, one proceeds as in step (5) of the proof of Theorem 1.11 to show that \(u\) is a strong solution of (1.24) and, by the same argument as in step (6) of the proof of Theorem 1.11, to prove that \(u\) satisfies (1.9). This proves the statement of Theorem 1.1. \(\square\)
5 Well-Posedness of the Robin Problem in the Half-Space

In this section, we outline the proof of existence and uniqueness of solutions to Robin problem (1.28).

Proof of Theorem 1.13. For given \( \varphi \in D(A) \) and \( \alpha > 0 \), let \( j : H \to [0, +\infty) \) be defined by

\[
j(u) = \frac{\alpha}{2} \left\| u - \left( \frac{1}{\alpha} - 1 \right) \varphi \right\|^2_H \quad (u \in H).
\]

Then \( j \) is convex, proper, strongly coercive and continuously differentiable. Thus the sub-differential \( \partial j \) is a well-defined mapping and coincides with the Fréchet derivative \( j' \) of \( j \) which is given by

\[
j'(u) = au - (1 - a) \varphi \quad \text{for every } u \in H.
\]

Since \( j' \) is affine on \( H \), we have that \( j' \) is a weak-to-weak continuous mapping. Moreover, \( j(x_0) = 0 \) if and only if \( x_0 = (\frac{1}{\alpha} - 1)\varphi \). Thus, by Theorem 1.11, for every \( \varphi \in D(A) \), there exists a unique solution \( u \) to Robin problem (1.28) satisfying \( u \in L^{\infty}(H) \), \( t^{1-2s}u' \in W^{1,2}_p(H) \), the estimates (1.29)–(1.31) and (1.6)–(1.9). \( \square \)

6 The Dirichlet-to-Neumann Operator

In this part of the paper, we outline the proofs of the statements in Theorem 1.2.

Proof of Theorem 1.2. Let \( \Lambda_s \) be the operator defined by the set of all pairs \( (\varphi, w) \in H \times H \) with the property that there is a solution \( u_{\varphi} \in L^{\infty}(H) \) of Dirichlet problem \( (D^s_\varphi) \) such that \( w = -\lim_{r \to 0^+} t^{1-2s}u'_{\varphi}(t) \) exists in \( H \). Then, by claim (2) of Proposition 3.2, \( \Lambda_s \) is monotone.

To see that \( \Lambda_s \) is a well-defined mapping, let \( (\varphi, w) \) and \( (\varphi, \dot{w}) \in \Lambda_s \). By definition of \( \Lambda_s \), there are solutions \( u_{\varphi}, \dot{u}_{\varphi} \in L^{\infty}(H) \) of Dirichlet problem \( (D^s_\varphi) \) such that \( w = \lim_{r \to 0^+} t^{1-2s}u'_{\varphi}(t) \) and \( \dot{w} = \lim_{r \to 0^+} t^{1-2s}\dot{u}'_{\varphi}(t) \). Since both \( u_{\varphi} \) and \( \dot{u}_{\varphi} \) exist in \( H \), Proposition 3.2 implies that \( u_{\varphi} = \dot{u}_{\varphi} \), and hence one has \( w = \dot{w} \), showing that \( \Lambda_s \) is a single-valued mapping from \( D(\Lambda_s) \) into \( H \).

According to Theorem 1.1, for every \( \varphi \in D(A) \), there is a unique solution \( u_{\varphi} \in L^{\infty}(H) \) of Dirichlet problem \( (D^s_\varphi) \) such that \( w := \lim_{r \to 0^+} t^{1-2s}u'_{\varphi}(t) \) exists in \( H \). Thus \( D(A) \subseteq D(\Lambda_s) \). Moreover, \( D(\Lambda_s) \subseteq \overline{D(A)} \) since, for every \( \varphi \in D(\Lambda_s) \), there is a function \( u_{\varphi} \in C((0, +\infty)) \) satisfying \( u_{\varphi}(0) = 0 \) and \( u_{\varphi}(t) \in D(A) \) for a.e. \( t > 0 \). This completes the proofs to the first part of Theorem 1.2.

To see that claim (2) of Theorem 1.2 holds, we apply Theorem 1.11 to the function \( j(u) := \frac{1}{2\lambda} \| u \|^2_H, u \in H \), for given \( \lambda > 0 \). It is not hard to verify that \( j \) is convex, strictly convex, strongly coercive and continuously differentiable on \( H \) with derivative \( j'(u) = \frac{1}{\lambda} u, u \in H \). Thus, for every \( \varphi \in D(A) \), there is a unique solution \( u \in C^h((0, +\infty); H) \) of the boundary-value problem

\[
\begin{cases}
\frac{1 - 2s}{t} u'(t) - u''(t) + Au(t) \geq 0 & \text{in } H_+, \\
-\lambda \lim_{t \to 0^+} t^{1-2s} u'(t) + u(0) = \varphi & \text{on } \partial H_+.
\end{cases}
\]

Since \( t^{1-2s} u' \in W^{1,2}_p(H) \), one has \( u(0) \in D(\Lambda_s) \) with \( \Lambda_s u(0) = \frac{1}{\lambda}(\varphi - u(0)) \). This shows that, for every \( \varphi \in D(A) \) and \( \lambda > 0 \), there is a \( u(0) \in D(\Lambda_s) \) satisfying \( (I_H + \lambda \Lambda_s) u(0) = \varphi \). Since \( \varphi \in D(A) \) was arbitrary, this proves claim (2).

Inequality (1.15) follows immediately from (1.10) of Theorem 1.1. Thus claim (3) holds.

Next, we show that \( \Lambda_s \) is cyclically monotone provided \( A = \partial \varphi \) is the sub-differential operator of a proper, convex, lower semicontinuous function \( \varphi \) on \( H \). For this, let \( \varphi_i \}_{i=0}^n \) be a finite sequence in \( D(\Lambda_s) \) satisfying \( \varphi_n = \varphi_0 \); there is a sequence \( (u_{\varphi_i})_{i=0}^n \) of solutions \( u_{\varphi_i} \in L^{\infty}(H) \) of Dirichlet problem \( (D^s_\varphi) \) with \( u_{\varphi_i}(0) = \varphi_i \). By the uniqueness of Dirichlet problem \( (D^s_\varphi) \), we have \( u_{\varphi_i} = u_{\varphi_0} \). By hypothesis, one has \( A = \partial \varphi \), and so, for every \( i = 1, \ldots, n \), we have \( t^{2s-1} \{ t^{1-2s} u_{\varphi_i}'(t) \} \in \partial \varphi(u(t)) \) for a.e. \( t > 0 \). Hence, by definition of the
sub-differential $\partial \phi$, we have

$$\sum_{i=0}^{n-1} \left( \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H + \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i}}(t) \rangle_H \right)$$

$$= \sum_{i=0}^{n-1} \left( t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right) + \sum_{i=0}^{n-1} \left( t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i}}(t) \rangle_H \right)$$

$$\geq \sum_{i=0}^{n-1} \left( \phi(u_{\phi_i}(t)) - \phi(u_{\phi_{i+1}}(t)) = 0 \right) \text{ for a.e. } t > 0.$$

Dividing this inequality by $t^{2s-1}$ gives

$$\sum_{i=0}^{n-1} \left( \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H + \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i}}(t) \rangle_H \right) \geq 0 \text{ for a.e. } t > 0.$$

Then, by the product rule, one finds that

$$\frac{d}{dt} \left[ \sum_{i=0}^{n-1} \left( \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right) - \sum_{i=0}^{n-1} \left( t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right) \right]$$

$$= \frac{d}{dt} \left( t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right) + \frac{d}{dt} \left( t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right)$$

$$\geq \sum_{i=0}^{n-1} \left( t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right) - \sum_{i=0}^{n-1} \left( t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right)$$

$$\geq t^{1-2s} \sum_{i=0}^{n-1} \| u_{\phi_i}'(t) \|_H^2 - t^{1-2s} \frac{n-1}{2} \sum_{i=0}^{n-1} \| u_{\phi_i}'(t) \|_H^2 \geq 0,$$

where we used $u_{\phi_i}' = u_{\phi_{i+1}}'$. Therefore, the function

$$t \mapsto \sum_{i=0}^{n-1} \left( \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H + \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right)$$

is monotonically increasing along $[0, +\infty)$. By (1.8) in Theorem 1.1, one has $\lim_{t \to +\infty} t^{1-2s} u_{\phi_i}'(t) = 0$, and since $u_{\phi_i} \in L^\infty(H)$, it follows that

$$\sum_{i=0}^{n-1} \left( \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H + \langle t^{2s-1} \partial \phi(t), u_{\phi_i}(t) - u_{\phi_{i+1}}(t) \rangle_H \right) \leq 0 \text{ for all } t > 0.$$

Since $t^{1-2s} u_{\phi_i}' \in W_{2s}^1(H)$ and $u_{\phi_i} \in C([0, +\infty); H)$, we take the limit $t \to 0^+$ in the above inequality and multiply it by $(-1)$. From this, we obtain

$$\sum_{i=0}^{n-1} \left( \Lambda_s \phi_i, \phi_i - \phi_{i+1} \right)_H + \left( \Lambda_s \phi_n, \phi_n - \phi_0 \right)_H \geq 0.$$

Since $(\phi_i)_{i=0}^n$ was an arbitrary, finite sequence in $D(\Lambda_s)$ satisfying $\phi_n = \phi_0$, we have thereby shown that $\Lambda_s$ is cyclically monotone. The rest of claim (4) of this theorem follows from Proposition 2.3.
Now, let $A$ be $\partial \psi$-monotone for a given proper, convex and lower semicontinuous function

$$\psi: \mathcal{H} \to (-\infty, +\infty].$$

Then we intend to show that, for every $0 < s < 1$, the Dirichlet-to-Neumann operator $\Lambda_s$ is $\partial \psi$-monotone. To do this, for $\mu > 0$, let $\psi_\mu: \mathcal{H} \to \mathbb{R}$ be the inf-convolution of $\psi$, which is defined by

$$\psi_\mu(u) = \min_{v \in \mathcal{H}} \left\{ \frac{1}{2\mu} \| v - u \|^2_H + \psi(v) \right\} \quad \text{for all } u \in \mathcal{H}.$$

Let $\varphi_1, \varphi_2 \in D(\Lambda_s)$ and $u_{\varphi_1}, u_{\varphi_2} \in L^\infty(\mathcal{H})$ be two solutions of Dirichlet problem $D^\mu_s$ for the boundary value $\varphi = \varphi_1$ and $\varphi = \varphi_2$. By (3.5) and since $A$ is $\partial \psi$-monotone (see Proposition 2.10),

$$v_1(z) = u_{\varphi_1}(2sz^{1/2s}) \quad \text{and} \quad v_2(z) = u_{\varphi_2}(2sz^{1/2s}) \quad \text{for } z \geq 0. \tag{6.1}$$

Then also $v_1, v_2 \in L^\infty(\mathcal{H})$, and thanks to Lemma 3.4, $v_1$ and $v_2$ are two strong solutions of the differential inclusion (3.5) satisfying $v_1(0) = \varphi_1$ and $v_2(0) = \varphi_2$. By (3.5) and since $A$ is $\partial \psi$-monotone (see Proposition 2.10),

$$z^{-\frac{2s}{s-1}}(v''_1(z) - v''_2(z), \psi_\mu(v_1(z) - v_2(z)))_H \geq 0 \quad \text{for a.e. } z > 0.$$

Further, according to [13, Proposition 2.11], $\psi_\mu \in C^1(\mathcal{H}; \mathbb{R})$ and the Fréchet derivative $\psi_\mu' = (\partial \psi)_\mu$, the Yosida approximation of the sub-differential $\partial \psi$. Since $\psi_\mu'$ is Lipschitz continuous on $\mathcal{H}$, one has $\psi_\mu' + w \in W^{1,1}_0(\mathcal{H})$ for every $w \in W^{1,1}_0(\mathcal{H})$, and so the monotonicity of $\psi_\mu'$ implies

$$\left( \frac{d}{dz} (\psi_\mu'(v_1(z) - v_2(z))), v'_1(z) - v'_2(z) \right)_H = (\psi_\mu''(v_1(z) - v_2(z))(v'_1(z) - v'_2(z)), v'_1(z) - v'_2(z))_H \geq 0 \quad \text{for a.e. } t > 0.$$

Therefore,

$$\frac{1}{2} \frac{d^2}{dz^2} \psi_\mu(v_1(z) - v_2(z)) = \frac{d}{dz} (\psi_\mu'(v_1(z) - v_2(z)), v'_1(z) - v'_2(z))_H$$

$$= \left( \frac{d}{dz} (\psi_\mu'(v_1(z) - v_2(z))), v'_1(z) - v'_2(z) \right)_H$$

$$+ (v'_1(z) - v'_2(z), \psi_\mu'(v_1(z) - v_2(z)))_H \geq 0 \quad \text{for a.e. } z > 0,$$

implying that the composed function $\psi_\mu(v_1(\cdot) - v_2(\cdot))$ is convex. By hypothesis, $v_1 - v_2 \in C_b([0, +\infty); \mathcal{H})$, and since $\psi_\mu \in C(\mathcal{H}; \mathbb{R})$, it follows that $\psi_\mu'(v_1(\cdot) - v_2(\cdot))$ is monotonically decreasing on $[0, +\infty)$, and so

$$(\psi_\mu'(v_1(z) - v_2(z)), v'_1(z) - v'_2(z))_H = \frac{d}{dz} \psi_\mu(v_1(z) - v_2(z)) \leq 0 \quad \text{for every } z \geq 0.$$

Multiplying this inequality by $-(2s)^{1-2s}$, evaluating it at $z = 0$, replacing $(2s)^{1-2s}v'_1(0)$ by $\lim_{t \to 0^+} t^{1-2s}u'_\varphi(t)$ and using (6.1), one finds

$$\left( \psi_\mu'(\varphi_1 - \varphi_2), (- \lim_{t \to 0^+} t^{1-2s}u'_\varphi(t)) - (- \lim_{t \to 0^+} t^{1-2s}u'_\varphi(t)) \right)_H \geq 0.$$

Since $\varphi_1, \varphi_2 \in D(\Lambda_s)$ were arbitrary, Proposition 2.10 implies that $A_s$ is $\partial \psi$-monotone, which is claim (5).

To see that claim (6) of this theorem holds, we note that if $A$ is completely accretive on $H = L^2(\Sigma, \mu)$, then, by the definition, for every $\lambda > 0$, the resolvent $A^\lambda$ of $A$ is a complete contraction. By Proposition 2.15, this means that $A$ is $\partial \psi$-monotone for every convex function $j \in J_0$, for $\varphi_j$ defined by (2.2). Now, by claim (5), this yields that, for every $0 < s < 1$, the Dirichlet-to-Neumann operator $A_s$ is $\partial \psi$-monotone for every function $j \in J_0$, proving that $A_s$ is completely accretive on $H = L^2(\Sigma, \mu)$.

Finally, we show that the closure $\overline{\Lambda_s H^s H_w}$ of $\Lambda_s$ in $H \times H_w$ is characterized by the operator

$$B := \{(\varphi, w) \in H \times H \mid \text{there exist } (\varphi_n)_{n \in \mathbb{N}} \subseteq D(\Lambda_s), \text{ solutions } u_{\varphi_n} \text{ of } D^\mu_s(\varphi_n) \text{ and } u_{\varphi_n}, \text{ of } (D^\mu_\varphi)_{\varphi=\varphi_n}$$

$$\text{s.t. } u_{\varphi_n} \to u_{\varphi_n} \text{ in } C^0([0, +\infty); \mathcal{H}) \text{ and } \Lambda_s \varphi_n \to w \text{ in } H_w\}$$
Obviously, the set \( B \) is contained in the closure \( \overline{\Lambda}^{H \times H_w} \) of \( \Lambda_{\delta} \) in \( H \times H_w \). Thus it remains to show that the inclusion \( \overline{\Lambda}^{H \times H_w} \subseteq B \) holds. For this, let \( (\varphi, w) \in \overline{\Lambda}^{H \times H_w} \). Then there is a sequence \((\varphi_n, w_n)\) s.t. \( (\varphi_n, w_n) \to (\varphi, w) \) in \( H \times H_w \) as \( n \to +\infty \). By definition of \( \Lambda_{\delta} \), for every \( \varphi_n \), there is a unique solution \( u_{\varphi_n} \in C^b([0, +\infty); H) \) of Dirichlet problem \((D_{\varphi}^s)_{\varphi=\varphi_n}\) satisfying \( \Lambda_{\delta} u_{\varphi_n} = w_n \). By (1.13) in Theorem 1.1, one sees that \( u_{\varphi_n} \) is a Cauchy sequence in \( C^b([0, +\infty); H) \). Thus there is a function \( u \in C^b([0, +\infty); H) \) such that \( u_{\varphi_n} \to u \) in \( C^b([0, +\infty); H) \). Moreover, by inequalities (1.7) and (1.9) in Theorem 1.1 and by the same arguments as in step (5) of the proof of Theorem 1.11, one finds that \( u = u_{\varphi} \) is the unique solution of Dirichlet problem \((D_{\varphi}^s)\), proving that \( (\varphi, w) \in B \).

It remains to show that \( \overline{\Lambda}^{H \times H_w} \) is maximal monotone provided \( D(A) \) is dense in \( H \). First, we note that since \( \Lambda_{\delta} \subseteq \overline{\Lambda}^{H \times H_w} \) and since \( \Lambda_{\delta} \) is monotone, we also have that \( \overline{\Lambda}^{H \times H_w} \) is monotone. Further, the domain \( D(\overline{\Lambda}^{H \times H_w}) \) is contained in \( D(A)^H \) and by claim (2) of this theorem, we know that \( D(A) \subseteq \text{Rg}(I_H + \lambda \overline{\Lambda}^{H \times H_w}) \) for every \( \lambda > 0 \). To see that also \( D(A)^H \subseteq \text{Rg}(I_H + \lambda \overline{\Lambda}^{H \times H_w}) \), let \( \varphi \in D(A)^H \) and \( \lambda > 0 \). Then there is a sequence \((\varphi_n)\) s.t. \( D(A)^H \subseteq \text{Rg}(I_H + \lambda \overline{\Lambda}^{H \times H_w}) \). By claim (2), for each \( \varphi_n \), there is a unique solution \( u_n \in L^{\infty}(H) \cap C([0, +\infty); H) \) of Dirichlet problem \((D_{\varphi_n}^s)_{\varphi=\varphi_n}\) satisfying

\[
 u_n(0) + \lambda \Lambda_{\delta} u_n(0) = \varphi_n.
\]

By this equation and the monotonicity of \( \Lambda_{\delta} \), one gets for the difference \( u_n(0) - u_m(0) \) that

\[
 \|u_n(0) - u_m(0)\|^2_H = \left( \varphi_n - \varphi_m, u_n(0) - u_m(0) \right)_H - \lambda \left( \Lambda_{\delta} u_n(0) - \Lambda_{\delta} u_m(0), u_n(0) - u_m(0) \right)_H \\
\leq \|\varphi_n - \varphi_m\|_H \|u_n(0) - u_m(0)\|_H,
\]

and so

\[
\|u_n(0) - u_m(0)\|_H \leq \|\varphi_n - \varphi_m\|_H.
\]

By (1.13), \((u_n)\) is a Cauchy sequence in \( C^b([0, +\infty); H) \), and hence there is a \( u \in C^b([0, +\infty); H) \) such that \( u_n \to u \) in \( C^b([0, +\infty); H) \) as \( n \to +\infty \). Due to estimates (1.7) and (1.9), one can proceed as in step (5) of the proof of Theorem 1.1 to conclude that \( u \in C^b([0, +\infty); H) \) is the unique solution of Dirichlet problem \((D_{\varphi}^s)_{\varphi=\varphi(0)}\). Moreover, we have

\[
w := \lim_{n \to +\infty} \Lambda_{\delta} u_n(0) = \lim_{n \to +\infty} \frac{1}{\lambda} (\varphi_n - u_n(0)) = \frac{1}{\lambda} (\varphi - u(0)) \text{ exists in } H,
\]

showing that \((u(0), w) \in \overline{\Lambda}^{H \times H_w} \) and \( \varphi \in \text{Rg}(I_H + \lambda \overline{\Lambda}^{H \times H_w}) \). Thus \( D(A)^H \subseteq \text{Rg}(I_H + \lambda \overline{\Lambda}^{H \times H_w}) \), and therefore, if \( D(A)^H = H \), then \( \overline{\Lambda}^{H \times H_w} \) is maximal monotone on \( H \). This completes the proof of this theorem. \( \square \)

### 7 Applications

In this section, we outline an example for the theory developed in the previous sections.

Throughout this final section, we use the following notation and setting. Let \( \Sigma \) be an open subset of \( \mathbb{R}^d \) \((d \geq 1)\), and let \( \mu = L^d \) be the \( d \)-dimensional Lebesgue measure. Then we write \( L^d(\Sigma) \) instead of \( L^d(\Sigma, \mu) \) and denote by \( L^2_0(\Sigma) \) the closed subspace of all \( u \in L^d(\Sigma) \) with mean \( \int_\Sigma u \, dx = 0 \).

For \( 1 < p < +\infty \), suppose that \( a : \Sigma \times \mathbb{R}^d \to \mathbb{R}^d \) is a Carathéodory function satisfying the conditions

\[
 a(x, \xi) \xi \geq \eta |\xi|^p \quad \text{\((p\text{-coercivity)}\)},
\]

\[
 |a(x, \xi)| \leq c_1 |\xi|^{p-1} + h(x) \quad \text{\((\text{growth condition)}\)},
\]

\[
 a(x, \xi_1) - a(x, \xi_2) (\xi_1 - \xi_2) > 0 \quad \text{\((\text{strict monotonicity)}\)}
\]

for a.e. \( x \in \Sigma \) and all \( \xi_1, \xi_2 \in \mathbb{R}^d \) with \( \xi_1 \neq \xi_2 \), where \( h \in L^p(\Sigma) \) and \( c_1, \eta > 0 \) are constants independent of \( x \in \Sigma \) and \( \xi \in \mathbb{R}^d \).

Under the hypotheses (7.1)–(7.3), the second-order differential operator

\[
 \mathcal{B} u := -\text{div}(a(x, \nabla u)) \quad \text{in } D'(\Sigma) \quad (u \in W^{1,p}_0(\Omega))
\]
belongs to the class of Leray–Lions operators of which the classical $p$-Laplace operator $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$ is an important prototype (see [26] or also [17, Section 6]). We denote by $A$ the operator $\mathcal{B}$ equipped with either homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial \Sigma \times (0, \infty) \quad \text{if } \Sigma \subseteq \mathbb{R}^d,$$

homogeneous Neumann boundary conditions

$$a(x, \nabla u) \cdot v = 0 \quad \text{on } \partial \Sigma \times (0, \infty) \quad \text{if } |\Sigma| < \infty,$$

or homogeneous Robin boundary conditions (with $b \in L^\infty(\partial \Omega)$ and $b \geq \alpha > 0$)

$$a(x, \nabla u) \cdot v + b(x)|u|^{p-1}u = 0 \quad \text{on } \partial \Sigma \times (0, \infty) \quad \text{if } |\Sigma| < \infty.$$ (R0)

Then it is not difficult to verify that $A$ is a maximal monotone, completely accretive and single-valued operator on $L^2(\Sigma)$ (respectively, on $L^2_0(\Sigma)$ if $\mathcal{B}$ is equipped with (N0)). In particular, $A$ has a dense domain $D(A)$ in $L^2(\Sigma)$ (respectively, in $L^2_0(\Sigma)$), and $A0 = 0$.

Due to Theorem 1.1, for every $0 < s < 1$, the following Dirichlet problem on the cylinder $\Sigma_+ := \Sigma \times (0, +\infty)$ is well-posed. For every $\varphi \in L^2(\Sigma)$ (respectively, for every $\varphi \in L^2_0(\Sigma)$), there is a unique solution

$$u_\varphi \in C^b((0, +\infty); L^2(\Sigma))$$

of

$$\left\{ \begin{array}{ll}
\frac{1-2s}{t} u'_\varphi - u''_\varphi - \text{div}(a(x, \nabla u_\varphi)) = 0 & \text{on } \Sigma \times \mathbb{R}_+, \\
\text{ } & \text{ } \\
\varphi \text{ satisfies } (D_0) ((N_0) \text{ if } \varphi \in L^2_0(\Sigma)), \text{ or } (R_0) \text{ on } \partial \Sigma \times \mathbb{R}_+, \\
\ varphi(\cdot, 0) = \varphi(\cdot) & \text{on } \Sigma,
\end{array} \right.$$ (7.4)

and the solutions $u_\varphi$ are continuously dependent in $L^2(\Sigma)$ on the boundary data $\varphi \in L^2(\Sigma)$ (respectively, for every $\varphi \in L^2_0(\Sigma)$).

Thanks to the well-posedness of Dirichlet problem (7.4) and the regularity of solutions $u_\varphi$ to (7.4), Theorem 1.2 yields that the Dirichlet-to-Neumann operator $\Lambda_s : D(\Lambda_s) \to L^2(\Sigma)$ (respectively, $\Lambda_s : D(\Lambda_s) \to L^2_0(\Sigma)$ if $A$ satisfies (N0)) is given by

$$\Lambda_s \varphi := - \lim_{t \to 0^+} t^{1-2s} u'_\varphi(t) \quad \text{in } L^2(\Sigma) \quad \text{(respectively, in } L^2_0(\Sigma)).$$

Now, by Theorem 1.7, for every $0 < s < 1$, the $L^2$-fractional power $A^s$ of $A$ in $L^2(\Sigma)$ (respectively, $L^2_0(\Sigma)$ if $A$ satisfies (N0)) is given by

$$A^s := \left\{ (\varphi, w) \in L^2(\Sigma) \times L^2(\Sigma) \mid \text{there exist } (\varphi_n)_{n \geq 1} \subseteq D(\Lambda_s), \text{ solutions } u_{\varphi_n} \text{ of } (D_s^\varphi) \text{ and } u_{\varphi_n} \text{ of } (D_s^\varphi)_{\varphi = \varphi_n} \right. \\
\text{s.t. } u_{\varphi_n} \to u_{\varphi} \text{ in } C^b \text{ and } \Lambda_s \varphi_n \to w \text{ in } \sigma(L^2(\Sigma), (L^2(\Sigma))^*)$$

where $L^2(\Sigma)$ is replaced by $L^2_0(\Sigma)$ if $A$ satisfies (N0), and by Corollary 1.8, $-A^s$ generates a strongly continuous semigroup $\{e^{-At}\}_{t \geq 0}$ on $L^2(\Sigma)$ (respectively, on $L^2_0(\Sigma)$). The semigroup $\{e^{-At}\}_{t \geq 0}$ is order-preserving, and each $e^{-At}$ has a unique contractive extension on $L^p(\Sigma)$ (respectively, on $L^p_0(\Sigma)$) for any $N$-function, on $L^1(\Sigma)$ and on $L^2 \cap L^\infty(\Sigma)^{\text{loc}}$.

**Acknowledgment:** The results presented in this paper are the outgrowths of a summer vacation research project at the University of Sydney under the supervision of D. Hauer. Ms Y. He and Mr D. Liu were undergraduate students who participated and actively contributed at this research project. The first author wants to express his warm gratitude towards Professor Haim Brezis, Professor Viorel Barbu, Professor Laurent Véron, and Professor Simeon Reich for the helpful discussions about the historical development of the square root of maximal monotone operators. Moreover, the authors want to thank the anonymous referee for his careful reading and the important criticisms. They were warmly welcome!
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