LIMIT CYCLES AND GLOBAL DYNAMIC OF PLANAR CUBIC SEMI-QUASI-HOMOGENEOUS SYSTEMS

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Abstract. Denote by CH, CSH, CQH, and CSQH the planar cubic homogeneous, cubic semi-homogeneous, cubic quasi-homogeneous and cubic semi-quasi-homogeneous differential systems, respectively. The problems on limit cycles and global dynamics of these systems have been solved for CH, and partially for CSH. This paper studies the same problems for CQH and CSQH. We prove that CQH have no limit cycles and CSQH can have at most one limit cycle with the limit cycle realizable. Moreover, we classify all the global phase portraits of CSQH.

1. Introduction. Consider a planar polynomial differential system

\[
\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),
\]

where \(P\) and \(Q\) are real polynomials. As usual, we say that system (1) has degree \(n\), if the maximum degree of \(P\) and \(Q\) is \(n\). In this paper one always assume that \(P\) and \(Q\) are respectively the \((s_1, s_2)\)-quasi-homogeneous real polynomials of weight degrees \(d_1\) and \(d_2\) with \(s_1, s_2, d_1, d_2 \in \mathbb{N}^+\), i.e.,

\[
P(\lambda^{s_1} x, \lambda^{s_2} y) = \lambda^{s_1 - 1 + d_1} P(x, y), \quad Q(\lambda^{s_1} x, \lambda^{s_2} y) = \lambda^{s_2 - 1 + d_2} Q(x, y)
\]

hold for arbitrary \(\lambda \in \mathbb{R}^+\), where \(\mathbb{N}^+\) and \(\mathbb{R}^+\) are respectively the sets of positive integers and positive real numbers.

Observe that system (1) includes the following four special cases.

(i) If \(s_1 = s_2 = 1\) and \(d_1 = d_2\), system (1) is a homogeneous one of degree \(d_1\) [8].

(ii) If \(s_1 = s_2 = 1\) and \(d_1 \neq d_2\), system (1) is a semi-homogeneous one [3, 4, 7].

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If \( s_1 \neq s_2 \) and \( d_1 = d_2 \), system (1) is a quasi-homogeneous one (or weight homogeneous one) with the weight vector \((s_1, s_2, d_1)\) \([1, 2, 5, 10, 13, 14]\).

If \( s_1 \neq s_2 \) and \( d_1 \neq d_2 \), system (1) is a semi-quasi-homogeneous one with the weight vector \((s_1, s_2, d_1, d_2)\) \([16, 18]\).

In the qualitative theory of planar polynomial differential systems there are two classical problems. The first one is on the number of limit cycles, which is proposed by Poincaré in 1881 and later by Hilbert as the second part of his famous 16th problem. The second one is on the global dynamics. We refer readers to the monograph \([9]\) for the detailed introduction of these two problems. Note that any polynomial vector field \((P, Q)\) can be decomposed into the sum of homogeneous or quasi-homogeneous polynomial vector fields. In this sense, to better understand the dynamics of general polynomial systems, one way is to get more information on dynamics of homogeneous and quasi-homogeneous systems. Obviously, the semi-homogeneous systems and the semi-quasi-homogeneous systems are the natural generalizations of the homogeneous and the quasi-homogeneous ones, respectively. During the past three decades, there has been a substantial amount of works devoted to solving the problems on limit cycles and global dynamics for homogeneous, quasi-homogeneous or semi-homogeneous polynomial systems \([3, 4, 6, 7, 8, 11, 12, 14]\). However, there are only a few papers on the same topic of the semi-quasi-homogeneous ones.

Denote by \( \text{CH}, \text{CSH}, \text{CQH}, \) and \( \text{CSQH} \) the planar cubic homogeneous, cubic semi-homogeneous, cubic quasi-homogeneous and cubic semi-quasi-homogeneous systems, respectively. We shall introduce some known results on these systems.

\( \text{CH} \): The global dynamics of \( \text{CH} \) were completely studied in \([8]\). These systems have no limit cycle.

\( \text{CSH} \): The global dynamics of \( \text{CSH} \) with \( \deg P = 1, \deg Q = 3 \) were studied in \([3]\) modulus limit cycles, which also showed that some of these systems can have at least one limit cycle. In \([6]\), the authors proved that the semi-homogeneous polynomial systems have no limit cycle if \( \deg P \) or \( \deg Q \) is even, and the maximum number of limit cycles could be at least \((\deg P + \deg Q)/2\) if both \( \deg P \) and \( \deg Q \) are odd. Hence, the \( \text{CSH} \) can exhibit limit cycles only if \( \{\deg P, \deg Q\} = \{1, 3\} \).

\( \text{CQH} \): In \([11]\), the authors studied the global phase portraits for a subclass of \( \text{CQH} \). In \([2, 10]\), the authors provided the canonical forms of \( \text{CQH} \). But as far as we know, there is no a published paper completing the classification of global dynamics of \( \text{CQH} \), although this problem was solved for planar quasi-homogeneous polynomial systems of degree four and five in \([12]\) and \([14]\), respectively.

\( \text{CSQH} \): To the best of our knowledge, there is no a published paper concerning the global dynamics and the limit cycles of \( \text{CSQH} \), except the one in \([16]\) on its canonical forms.

In short, we have Table 1, which shows what is known and what is unknown on these systems.

Our main objective in this paper is to solve the problems posed in Table 1. Our result is on \( \text{CQH} \) and \( \text{CSQH} \).
Theorem 1.1. The maximum number of limit cycles is 0 for the cubic quasi-homogeneous systems, and is 1 for the cubic semi-quasi-homogeneous systems. Furthermore, any cubic semi-quasi-homogeneous system with a limit cycle can be transformed, after a rescaling of the variables, to

\[
\dot{x} = ay^3 + x, \quad \dot{y} = -y^3 + x,
\]

with \(a < -1\), whose unique limit cycle is stable. And its global phase portrait is shown in Figure 1.

Figure 1. The global phase portraits of system (2) with \(a < -1\)

Remark 1. It was proved in [18, Theorem 13] that the semi-quasi-homogeneous system

\[
\dot{x} = y^{r_1} + \varepsilon \sum_{i=1}^{[r_1/p]} a_i x^{i_q} y^{r_1 - ip}, \quad \dot{y} = -x^{r_2} + \varepsilon \sum_{j=1}^{[r_2/q]} b_j x^{r_2 - jq} y^{jp},
\]

has at least \([([r_1/p] + 1)/2] + ([r_2/q] + 1)/2 - 1\) limit cycles for small \(\varepsilon\) and suitable values of \(a_i\)'s, \(b_j\)'s, where \([s]\) denotes the integer part of the nonnegative number \(s\). This implies that the CSQH could have at least one limit cycle by letting \(r_1 = 3, r_2 = 1\) and \(p = 3, q = 1\). However, such result cannot provide an explicit expressions on the coefficients of the systems due to the perturbation. Theorem 1.1 provides not only the exact number of limit cycles, but also the canonical form of CSQH with the limit cycle. In addition, system (2) can be extended to a class of semi-quasi-homogeneous polynomial systems of odd degree which have a unique limit cycle, as shown in the next result.

Corollary 1. The planar semi-quasi-homogeneous polynomial differential system with odd degree and with the weight vector \((2n + 1, 1, 1, 2n + 1)\) of the form

\[
\dot{x} = ay^{2n+1} + x, \quad \dot{y} = -y^{2n+1} + x,
\]

Table 1. The known results about the CH, CSH, CQH, and CSQH before this paper.
has limit cycles if and only if \( a < -1 \). And under this condition, the limit cycle is unique and stable.

The next result classifies the global dynamics of CSQH.

**Theorem 1.2.** The global phase portrait of any planar cubic semi-quasi-homogeneous polynomial system having finitely many singularities is topologically equivalent to one of the 43 configurations in Figure 2, where the triple \((h, p, e)\) under each of the phase portraits denotes the number of hyperbolic, parabolic and elliptic sectors of the unique finite singularity. Moreover, each of the configurations in Figure 2 is realizable for some cubic semi-quasi-homogeneous polynomial systems.

Theorems 1.1 and 1.2 partially answer the problems posed in Table 1. We now have Table 2.

| Type of cubic systems | The maximum number of limit cycles | The global dynamics |
|-----------------------|------------------------------------|---------------------|
| Homogeneous           | 0                                  | completed           |
| Semi-Homogeneous      | \( \geq 1 \)                        | for some subclasses |
| Quasi-Homogeneous     | 0                                  | for some subclasses |
| Semi-Quasi-Homogeneous| 1                                  | completed           |

**Table 2.** The known results about the CH, CSH, CQH, and CSQH after this paper.

2. **Preliminaries.** In this section, we give some preliminaries which will be used in the proof of the main results.

2.1. **Singularities.** It is well known that the origin is the unique finite isolated singularity of the planar homogeneous, semi-homogeneous and quasi-homogeneous systems. This conclusion is also true for the semi-quasi-homogeneous systems.

**Lemma 2.1.** ([18]) If system (1) is a planar semi-quasi-homogeneous system, and \( P \) and \( Q \) are coprime in \( \mathbb{R}[x, y] \), then the origin is the unique finite singularity of system (1).

As we will see soon, the singularity of the planar semi-quasi-homogeneous system is always nonelementary. In order to study the dynamical behavior of the orbits in a neighborhood of the origin, we shall use some results on the semi-hyperbolic singularity, nilpotent singularity, and the blow-up technique, which are introduced below.

**Lemma 2.2.** (Semi–Hyperbolic Singular Points Theorem, [9, 17]) Consider the planar differential system

\[
\dot{x} = P_2(x, y), \quad \dot{y} = \lambda y + Q_2(x, y),
\]

where the origin is an isolated singularity of the system (4), \( P_2(x, y) \) and \( Q_2(x, y) \) are analytic in the neighborhood of the origin and \( P_2(0, 0) = Q_2(0, 0) = 0, \frac{\partial P_2}{\partial x}(0, 0) = \frac{\partial Q_2}{\partial x}(0, 0) = \frac{\partial P_2}{\partial y}(0, 0) = \frac{\partial Q_2}{\partial y}(0, 0) = 0 \). Given that \( y = \phi(x) \) is the solution of equation \( \lambda y + Q_2(x, y) = 0 \) in the neighborhood of the origin, and \( \phi(0) = \phi'(0) = 0 \). Let \( \psi(x) = P_2(x, \phi(x)) = a_m x^m + o(x^m) \). The following holds.

(i) If \( m \) is odd and \( a_m > 0 \), then \( O(0, 0) \) is an unstable node;
(ii) If \( m \) is odd and \( a_m < 0 \), then \( O(0, 0) \) is a saddle;
(iii) If \( m \) is even, then \( O(0, 0) \) is a saddle-node (see Figure 3).
Figure 2. The topological equivalence classes of the global phase portraits of planar cubic semi-quasi-homogeneous systems.
The above lemma can be applied to the singularity of system (1), whose linear approximation system at this singularity has exactly one zero eigenvalue. When the linearization of (1) at the singularity has two zero eigenvalues but is not identically zero, we will use the following result.

**Lemma 2.3.** ([9]) Consider the system of the form

$$\dot{x} = y + A(x, y), \quad \dot{y} = B(x, y),$$

with the origin $O(0, 0)$ an isolated singularity, where $A(x, y)$ and $B(x, y)$ are analytic in a neighborhood of the origin and $j_1 A(0, 0) = j_1 B(0, 0) = 0$. Let $y = f(x)$ be
the solution of equation \( y + A(x, y) = 0 \) in a neighborhood of the origin, and let \( F(x) = B(x, f(x)), G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x)) \).

(a) Assume that \( F(x) = a_m x^m + o(x^n) \) and \( G(x) = b_n x^n + o(x^n) \) for \( m, n \in \mathbb{N}, m \geq 2, n \geq 1, a_m \neq 0, b_n \neq 0 \).

(a1) If \( m \) is odd and \( a_m > 0 \) then the origin is a saddle (see Figure 4 (a));

(a2) If \( m \) is odd, \( a_m < 0 \) and

(i) either \( m < 2n + 1 \), or \( m = 2n + 1 \) and \( b_n^2 + 4(n+1)a_m < 0 \), then the origin is a center or a focus;

(ii) \( n \) is odd and either \( m > 2n+1 \), or \( m = 2n+1 \) and \( b_n^2 + 4(n+1)a_m \geq 0 \), then the phase portrait of the origin consists of one hyperbolic and one elliptic sector (see Figure 4 (b));

(iii) \( n \) is even and either \( m > 2n+1 \), or \( m = 2n+1 \) and \( b_n^2 + 4(n+1)a_m \geq 0 \), then the origin is a node (see Figure 4 (c));

(a3) If \( m \) is even, and

(i) \( m < 2n + 1 \), then the origin is a cusp (see Figure 4 (d));

(ii) \( m > 2n+1 \), then the origin is a saddle-node (see Figure 4 (e));

(b) Assume that \( F(x) = a_m x^m + o(x^n) \) and \( G(x) = 0 \) for \( m \in \mathbb{N}, a_m \neq 0 \).

(b1) If \( m \) is odd and \( a_m > 0 \), then the origin is a saddle.

(b2) If \( m \) is odd and \( a_m < 0 \), then the origin is a center or a focus.

(c) If \( F(x) = G(x) \equiv 0 \), then the phase portrait of system (5) at the origin is shown in Figure 4 (f).

Lemma 2.3 is a part of the Nilpotent Singularity Theorem. The readers are referred to the references [9, p. 116] for more details.
When the linearization of system (1) at the singularity is identically zero, we need blow up with the transformation

\[
\begin{align*}
  x &= \mu^\alpha \bar{x}, \\
  y &= \mu^\beta \bar{y},
\end{align*}
\]

where \(0 \leq \mu \ll 1\). With the suitable choice of positive integers \(\alpha\) and \(\beta\), the degenerate singularity of system (1) can be blown up into several elementary singularities, see [9] for more details and for the method of determining the values of \(\alpha\) and \(\beta\).

2.2. Index and limit cycle. The first result we would like to introduce is the index formula of isolated singularity.

**Lemma 2.4.** (Poincaré Index Formula, [9, 17]) Let \(Q\) be an isolated singularity having the finite sectorial decomposition property. Let \(e, h\) and \(p\) denote the number of elliptic, hyperbolic, and parabolic sectors of \(Q\), respectively, and suppose that \(e + h + p > 0\). Then the index of \(Q\) is \((e - h)/2 + 1\).

**Remark 2.** It is well known that, if a limit cycle surrounds a unique isolate singularity, then the index of this singularity is 1. Thus Lemma 2.4 provides a useful tool for us to exclude the existence of the limit cycles for some CSQH.

The next lemma can be found in [17] which can be employed to determine the uniqueness as well as the stability of the limit cycle for some planar systems.

**Lemma 2.5.** ([17]) Consider the differential equation

\[ \ddot{x} + f(x)\dot{x} + g(x) = 0, \]

or its equivalent system

\[ \frac{dy}{dt} = g(x), \quad \frac{dx}{dt} = -y - F(x), \]

where \(F(x) = \int_0^x f(x)dx\). If the following conditions are satisfied

(a) \(g(x)\) is an odd function, and \(xg(x) > 0\) for \(x \neq 0\);
(b) \(F(x)\) is an odd function, and there exists an \(x_0 > 0\) such that \(F(x) < 0\) for \(0 < x < x_0\), and \(F(x) \geq 0\) is monotonically increasing for \(x \geq x_0\);
(c) \(\int_0^\infty f(x)dx = \int_0^\infty g(x)dx = +\infty\);
(d) \(f(x)\) and \(g(x)\) satisfy the Lipschitz condition on any bounded interval,

then system (7) has a unique limit cycle which is stable.

2.3. Canonical forms. In order to study the dynamics of a class of planar differential systems, it is better to obtain the canonical forms of these systems. The canonical forms of planar quasi-homogeneous systems of degree 2, 3, 4, 5 have been obtained by several authors, see [15] and the references therein. Very recently, the authors in [16] provided the canonical forms of CSQH.

**Lemma 2.6.** ([16]) After appropriate invertible linear transformations, any planar cubic semi-quasi-homogeneous but non-semi-homogeneous systems with the minimal weight vector \(w_m\), can be written as one of the following forms

(A1) \(\dot{x} = y^2 + x, \quad \dot{y} = x^3\), with \(w_m = (2, 1, 1, 6)\).

(B1) \(\dot{x} = y^3 + x^2, \quad \dot{y} = Q(x, y)\), with \(w_m = (3, 2, 4, 2^{k-1}), Q(x, y) \in \{x^2 y, y^2, y\}, \quad \deg(Q(x, y)) = k, \quad k = 1, 2, 3\).

(C1) \(\dot{x} = x^{1+k} y^{2-k}, \quad \dot{y} = y^3 + x^2\), with \(w_m = (3, 2, 5 + k, 5), \quad k = 1, 2\).

(D1) \(\dot{x} = y^2, \quad \dot{y} = y^3 + x^2\), with \(w_m = (3, 2, 2, 5)\).
of system (1) satisfies minimal weight vector while all the systems (3). Note that Proposition 3 of [2] provides the canonical forms of the cubic quasi-homogeneous but non-homogeneous polynomial differential systems, which are denoted by systems (a) – (g). From the proof of Theorem 1 of [2], we find that system (a) has either a center at the origin or an invariant line passing through the origin, while all the systems (b) – (g) have an invariant curve passing through the origin, therefore CQH has no a limit cycle.

3. Proofs of the main results. This section is devoted to the proof of the main results of this article.

Proof of Theorem 1.1. Let’s first prove the nonexistence of limit cycles for CQH. Note that Proposition 3 of [2] provides the canonical forms of the cubic quasi-homogeneous but non-homogeneous polynomial differential systems, which are denoted by systems (a) – (g). From the proof of Theorem 1 of [2], we find that system (a) has either a center at the origin or an invariant line passing through the origin, while all the systems (b) – (g) have an invariant curve passing through the origin, therefore CQH has no a limit cycle.
Next consider the existence of limit cycles for CSQH. By Lemma 2.6, we can get the canonical forms of the planar cubic semi-quasi-homogeneous but non-quasi-homogeneous polynomial differential systems. From the proof of Theorem 1.2 soon afterwards, we get that, none of the systems in Lemma 2.6, except system (2) with \( a < -1 \) which is a subclass of system \((A_{2,1})\), can have limit cycles. Now assume that \( a < -1 \). Direct computation shows that system (2) is equivalent to the following Liénard equation
\[
\ddot{y} + (3y^2 - 1)\dot{y} - (a + 1)y^3 = 0.
\]
Let \( F(y) = \int_0^y (3s^2 - 1)ds = y^3 - y \). Lemma 2.5 verifies that system (2) with \( a < -1 \) has a unique limit cycle which is stable.

The global phase portraits of system (2) with \( a < -1 \) will be studied later in the proof of Theorem 1.2.

The proof of of Theorem 1.1 is finished.

Proof of Corollary 1. Consider first the case that \( a \geq -1 \). Making the transformation \( x = \bar{y}, \quad y = \bar{x} + \bar{y} \), we can transform system (3) into
\[
\frac{d\bar{x}}{dt} = -(a + 1)(\bar{x} + \bar{y})^{2n+1}, \quad \frac{d\bar{y}}{dt} = a(\bar{x} + \bar{y})^{2n+1} + \bar{y}.
\]
(8)
Obviously, system (8) has an invariant straight line \( \bar{y} = 0 \) when \( a = 0 \).

If \( a > -1 \) and \( a \neq 0 \), then we get
\[
\bar{y} = \phi(\bar{x}) = -a\bar{x}^{2n+1} + o(\bar{x}^{2n+1})
\]
from \( a(\bar{x} + \bar{y})^{2n+1} + \bar{y} = 0 \). Hence
\[
\psi(\bar{x}) = -(a + 1)(\bar{x} + \phi(\bar{x}))^{2n+1} = -(a + 1)\bar{x}^{2n+1} + o(\bar{x}^{2n+1}).
\]
According to Lemma 2.2, system (8) has a saddle at the origin. This implies that system (8) has no limit cycle since the index of the saddle is -1.

If \( a = -1 \), then system (8) has an invariant straight line \( \bar{x} = 0 \).

Therefore, when \( a \geq -1 \), system (8) (and hence system (3)) has no a limit cycle.

Next, consider the case that \( a < -1 \). By eliminating \( x \) from system (3) we get the Liénard equation
\[
\ddot{y} + ((2n + 1)y^{2n} - 1)\dot{y} - (a + 1)y^{2n+1} = 0.
\]
It follows from Lemma 2.5 that system (3) has a unique limit cycle which is stable. This completes the proof of of Corollary 1.

Proof of Theorem 1.2. The proof is based on the study of the global dynamics of canonical systems in Lemma 2.6 and the classical method of topologically classification of global phase portraits on the Poincaré disk [9]. However, since the discussions of the global dynamics of all the canonical systems are very tedious, here we only provide the details for systems \((A_1)\) and \((B_{1,k}) (k = 1, 2, 3)\). The other ones can be studied in a completely analogous way.

(a) The global dynamics of \((A_1)\).

First study the singularity of system \((A_1)\) at the origin by applying Lemma 2.2 (with the change of coordinates \( x \leftrightarrow y \)). Let \( P_2(x, y) = y^3 \) and \( Q_2(x, y) = x^2 \). By \( y + Q_2(x, y) = 0 \), we get \( y = \phi(x) = -x^2 \), and \( \psi(x) = P_2(x, \phi(x)) = -x^6 \). From Lemma 2.2, we know that system \((A_1)\) has a saddle-node at the origin.
Next consider the singularities of system \((A_1)\) at the infinity. After making the Poincaré transformation \((x,y) \rightarrow (1/z, u/z)\) and the time transformation \(d\tau = dt/z^2\), system \((A_1)\) becomes

\[
\frac{du}{d\tau} = 1 - u^3z - uz^2, \quad \frac{dz}{d\tau} = -u^2z^2 - z^3.
\]

This system has no a singularity at \(z = 0\). In order to study the singularity at the infinity on the \(y\)-axis direction, we make the transformation \((x,y) \rightarrow (v/z, 1/z)\) and \(d\tau = dt/z^2\) for system \((A_1)\), it turns out that

\[
\frac{dv}{d\tau} = z + vz^2 - v^4, \quad \frac{dz}{d\tau} = -zv^3.
\]  

Clearly, system \((9)\) has a nilpotent singularity at the origin. Let \(A(v,z) = vz^2 - v^4, B(v,z) = -zv^3\). By \(z + A(v,z) = 0\), we get \(z = f(v) = v^4 - v^9 + \cdots\), and hence

\[
F(v) = B(v, f(v)) = -v^7 + o(v^8),
\]

\[
G(v) = (\partial A/\partial v + \partial B/\partial z)(v, f(v)) = -5v^3 + o(v^4).
\]

From Lemma 2.3, the neighbourhood of the origin consists of a hyperbolic sector and an elliptic sector, where the former is below the latter. Furthermore, it follows from Lemma 2.4 and Remark 2 that system \((A_1)\) has no limit cycle. These verify easily that the global phase portrait of \((A_1)\) is that shown in Figure 6.

(b) The global dynamics of \((B_{1,k})\) \((k = 1, 2, 3)\).

Notice that for every \(k = 1, 2, 3\), systems \((B_{1,k})\) has an invariant straight line \(y = 0\) which excludes the existence of limit cycles.

First consider the system \((B_{1,1})\) : \(\dot{x} = y^3 + x^2, \quad \dot{y} = y\). By the Semi-Hyperbolic Singular Points Theorem, it yields that system \((B_{1,1})\) has a saddle-node at the origin.

Next, we make the Poincaré transformation \((x,y) \rightarrow (1/z, u/z)\) and \(d\tau = dt/z^2\), it changes system \((B_{1,1})\) into

\[
\frac{du}{d\tau} = -uz + uz^2 - u^4, \quad \frac{dz}{d\tau} = -zu^3 - z^2.
\]  

On the \(u\)-axis, system \((10)\) has a unique singularity at \(A(0,0)\) which is degenerate. We will blow-up it by using the transformation \((6)\) with \((\alpha, \beta) = (1,3)\).

In the positive \(u\)-direction, we consider the transformation \((u,z) = (\mu, \mu^3z)\) and time transformation \(\mu^3d\tau = d\bar{\tau}\), system \((10)\) becomes

\[
\frac{d\mu}{d\bar{\tau}} = -\mu - \mu \bar{z} + \mu^4 \bar{z}^2, \quad \frac{d\bar{z}}{d\bar{\tau}} = 2\bar{z} + 2\bar{z}^2 - 3\mu^3 \bar{z}^3.
\]  

On the straight line \(\{\mu = 0\}\), system \((11)\) has two singularities at \(B(0,0)\) and \(C(0,-1)\), respectively. It is easy to check that \(B\) is a saddle. In order to discuss the singularity at \(C\), we make the transformation \(z_1 = \bar{z} + 1, \quad d\bar{\tau} = -2d\tau_1\), and it changes system \((11)\) into

\[
\begin{align*}
\frac{d\mu}{d\tau_1} &= -\frac{1}{2}\mu^4 + \frac{3}{2}\mu z_1 + \mu^4 z_1 - \frac{3}{2}\mu^4 z_1^2, \\
\frac{dz_1}{d\tau_1} &= z_1 + \frac{9}{2}\mu^3 z_1^2 - \frac{3}{2}\mu^3 z_1^2 - \frac{9}{2}\mu^3 z_1^2 + \frac{3}{2}\mu^3 z_1.
\end{align*}
\]

Using Lemma 2.2, we get that \((0,0)\) is a saddle-node of system \((12)\). Thus, system \((11)\) has a saddle-node at \(C\).
In the negative \(u\)-direction, we consider the transformation \((u, z) = (-\mu, \mu^3 z)\) and \(\mu^3 d\tau = d\tilde{\tau}\). System (10) becomes

\[
\begin{align*}
\frac{d\mu}{d\tilde{\tau}} &= -\mu - \mu\bar{z} + \mu^4 z^2, \\
\frac{d\bar{z}}{d\tilde{\tau}} &= -2\bar{z} + 2z^2 - 3\mu^3 z^3. 
\end{align*}
\]

System (13) has two singularities at \(B'(0, 0)\) and at \(C'(0, 1)\) respectively on the straight line \(\{\mu = 0\}\). It is easy to check that \(B'\) is a saddle. Let \(z_1 = \bar{z} - 1\) and \(d\tau = 2d\tau_1\), system (13) becomes

\[
\begin{align*}
\frac{d\mu}{d\tau_1} &= \frac{1}{2}\mu^4 - \frac{1}{2}\mu z_1 + \mu^4 z_1 + \frac{1}{2}\mu^4 z_1^2, \\
\frac{d\bar{z}}{d\tau_1} &= z_1 - \frac{9}{2}\mu^3 z_1 - \frac{3}{2}\mu^3 + z_1^2 - \frac{9}{2}\mu^3 z_1^2 - \frac{3}{2}\mu^3 z_1^3. 
\end{align*}
\]

By applying Lemma 2.2, we get that system (14) has a saddle-node at the origin, which implies that system (13) has a saddle-node at \(C'\).

In the positive \(z\)-direction, we consider the transformation \((u, z) = (\mu\bar{u}, \mu^3)\) and \(\mu^3 d\tau = d\tilde{\tau}\), system (10) becomes

\[
\begin{align*}
\frac{d\mu}{d\tilde{\tau}} &= -\frac{1}{3}\mu - \frac{1}{3}\mu\bar{u}, \\
\frac{d\bar{u}}{d\tilde{\tau}} &= -\frac{2}{3}\bar{u} - \frac{2}{3}u^4 + \mu^3 \bar{u}. 
\end{align*}
\]

Clearly, system (15) has a stable node at the origin.

In the negative \(z\)-direction, we change system (10) into

\[
\begin{align*}
\frac{d\mu}{d\tilde{\tau}} &= \frac{1}{3}\mu - \frac{1}{3}\mu\bar{u}, \\
\frac{d\bar{u}}{d\tilde{\tau}} &= \frac{2}{3}\bar{u} - \frac{2}{3}u^4 + \mu^3 \bar{u}, 
\end{align*}
\]

by taking the transformation \((u, z) = (\mu\bar{u}, -\mu^3)\) and \(\mu^3 d\tau = d\tilde{\tau}\). Clearly, system (16) has an unstable node at the origin.

Blowing down together with all of the above information gives the local structure of system (10) at \(A\), see Figure 5.

Now, let’s study the singularity of system (10) at the infinity on the \(y\)-axis direction. By the transformation \((x, y) \rightarrow (v/z, 1/z)\) and \(d\tau = d\tau/z^2\), system \((B_{1,1})\) is changed into

\[
\begin{align*}
\frac{dz}{d\tilde{\tau}} &= 1 + zv^2 - vz^2, \\
\frac{dv}{d\tilde{\tau}} &= -z^3. 
\end{align*}
\]

Thus system (10) has no a singularity at the infinity on the \(y\)-axis direction.

Combining all of the above information, we can obtain the four global phase portraits of system \((B_{1,1})\) which are shown in Figure 6.

For the phase portrait of system \((B_{1,2})\) : \(\dot{x} = y^3 + x^2, \ \dot{y} = y^2\), we can use the same transformation of coordinates in system \((B_{1,1})\) to discuss the trajectory structure both at the origin and at the infinity, and then get the global phase portraits of system \((B_{1,2})\), as shown in Figure 6.

Next consider the phase portraits of system \((B_{1,3})\) : \(\dot{x} = y^3 + x^2, \ \dot{y} = x^2 y\).

Since the origin is a degenerate singularity, we will blow-up it by using the transformation (6) with \(\alpha = 3, \beta = 2\). On the other hand, we can make the same transformation as did for \((A_1)\) to discuss the singularity at the infinity. We omit the details for the sake of brevity.

Taking advantage of these information, we can get also the global phase portraits of system \((B_{1,3})\), see Figure 6.

(c) The global dynamics of systems \((M_{1,k,l}^\pm)\) \((k, l = 1, 2)\).

First consider system \((M_{1,1,1}^\pm)\) : \(\dot{x} = y^3 + xy, \ \dot{y} = \pm x^3\). Since this system has a degenerate singularity at the origin, we shall blow-up it by using the
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Figure 5. Local phase portrait of system (10) at the origin

Figure 6. The global phase portraits of systems \((A_1)\) and \((B_{1,k}) (k = 1, 2, 3)\)

transformation (6) with \(\alpha = 2, \beta = 1\). The local phase portrait of system \((M_{1,1,1}^\pm)\) at the origin is obtained by discussing the four blow-up systems along the \(x\)-direction and \(y\)-direction, respectively.

Let’s study the singularities of system \((M_{1,1,1}^\pm)\) at the origin. For system \((M_{1,1,1}^+)\), after making the Poincaré transformation \((x, y) \rightarrow (1/z, u/z)\) and the time rescale \(d\tau = dt/z^2\), the obtained system has a stable node at \((1, 0)\) and an unstable node at \((-1, 0)\) respectively on the \(u\)-axis. Moreover, by direct calculations we find that system \((M_{1,1,1}^-)\) has no singularity at the origin.

On the other hand, it is easy to see that both systems \((M_{1,1,1}^\pm)\) have no limit cycle because of the symmetry \(P(x, -y) = -P(x, y), Q(x, -y) = Q(x, y)\).

Combining all of the above information, we can obtain the eight global phase portraits of systems \((M_{1,1,1}^\pm)\) which are shown in Figure 7, with four topologically different.

Next consider the system \((M_{1,1,2}^+)\) : \(\dot{x} = -y^3 + xy, \quad \dot{y} = \pm x^3\).

We can use the same changes as in system \((M_{1,1,1}^+)\) to analyse the types of both the singularities at the origin and at the infinity, respectively. After the concrete calculations, we find that, the cases of system \((M_{1,1,2}^+)\) and system \((M_{1,1,2}^-)\) concerning the singularities at the infinity and limit cycles, are just
the same with the cases of system \((M_{1,1,1}^-)\) and system \((M_{1,1,1}^+)\), respectively. Take advantage of all the obtained results, we can easily get the global phase portraits of \((M_{1,1,1}^+)^2\). See the third and fourth pictures in Figure 7.

Analogously, we can get the global phase portraits of systems \((M_{1,2,1}^+)\) and systems \((M_{1,2,2}^+)\). See also Figure 7.

\[ \text{Figure 7. The phase portrait of systems } (M_{1,k,l}^+) \]

(d) The global dynamics of system \((A_{2,1}^-)\) : \( \dot{x} = ay^3 + x, \dot{y} = -y^3 + x \) with \( a(a + 1) \neq 0 \).

The study of the singularities at both of the origin and the infinity is completely analogous to the discussion on systems \((A_1)\), \((B_{1,k})\) and \((M_{1,k,l}^+)\). We omit the details for the sake of brevity. It is found that system \((A_{2,1}^-)\) has a saddle and a node at the origin for \( a \in (-1, 0) \cup (0, \infty) \) and for \( a < -1 \), respectively. Therefore, when \( a \in (-1, 0) \cup (0, \infty) \), system \((A_{2,1}^-)\) has no a limit cycle because the index of the origin is \(-1\). When \( a < -1 \), by Theorem 1.1, we know that system \((A_{2,1}^-)\) has a unique stable limit cycle. Taking into account all of the discussions above, we can easily get the global phase portraits of \((A_{2,1}^-)\), as shown in Figure 8, where topologically different ones are only six.

\[ \text{Figure 8. The global phase portraits of systems } (A_{2,k}) \]
Figure 8. Continued

Figure 9. The global phase portraits of systems \((C_{1,k}), (D_1), (E_{1,k})\) and \((F_1)\)
Figure 9. Continued

Figure 10. The global phase portraits of systems \((G_{1,k,l}^+)\)

Figure 11. The global phase portraits of systems \((H_{1,k}^\pm)\) and \((I_{1,k}^\pm)\)
Figure 11. Continued

Figure 12. The phase portraits of systems $(J^{\pm}_{1,k,l})$ and $L^{\pm}_{1,k,l}$

$(B_{1})$: $a < 1$ and $a \neq 0$  
$(B_{2})$: $a > 1$  
$(B_{3})$: $a > -1$ and $a \neq 0$  
$(B_{4})$: $a < -1$
Figure 13. The global phase portraits of systems $(B_{2,k}^\pm)$

Figure 14. The global phase portraits of systems $(C_{2,1})$ and $(C_{2,2}^{\pm})$

Figure 15. The global phase portraits of systems $(D_{2,k}^{\pm})$
**Figure 15.** Continued

**Figure 16.** The global phase portraits of systems $(E_{2,k})$
Figure 17. The global phase portraits of systems \((F_{2,k})\)

The global dynamics of the other systems in Lemma 2.6 can be studied in a similar way. All the global phase portraits obtained are shown in Figure 6 to Figure 17. Finally, by utilizing the Markus–Neumann–Peixoto Theorem (see the first chapter of [9]), which says that two continuous flows with only isolated singularities are topologically equivalent if and only if their completed separatrices are equivalent, we can classify all the global phase portraits of CSQH having finitely many singularities into 43 topologically equivalent classes, as shown in Figure 2, where the number \((i)\) under every figure means that this figure is topologically equivalent to all the global phase portraits in Figure 6 to Figure 17 with the same number \((i)\) in the upper right corner of these figures.

It completes the proof of Theorem 1.2.

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