One-Instanton Test of a Seiberg–Witten Curve from M-theory: 
the Antisymmetric Representation of SU(N)

Stephen G. Naculich 
Department of Physics 
Bowdoin College, Brunswick, ME 04011 

Henric Rhedin[1] 
Martin Fisher School of Physics 
Brandeis University, Waltham, MA 02254 

Howard J. Schnitzer[2] 
Lyman Laboratory of Physics 
Harvard University, Cambridge, MA 02138 
and 
Martin Fisher School of Physics[3] 
Brandeis University, Waltham, MA 02254 

Abstract 

One-instanton predictions are obtained from the Seiberg–Witten curve derived from M-theory by Landsteiner and Lopez for the Coulomb branch of \( N = 2 \) supersymmetric SU(N) gauge theory with a matter hypermultiplet in the antisymmetric representation. Since this cubic curve describes a Riemann surface that is non-hyperelliptic, a systematic perturbation expansion about a hyperelliptic curve is developed, with a comparable expansion for the Seiberg–Witten differential. Calculation of the period integrals of the SW differential by the method of residues of D’Hoker, Krichever, and Phong enables us to compute the prepotential explicitly to one-instanton order. 

It is shown that the one-instanton predictions for SU(2), SU(3), and SU(4) agree with previously available results. For SU(N), \( N \geq 5 \), our analysis provides explicit predictions of a curve derived from M-theory at the one-instanton level in field theory.

---

[1]Supported by the Swedish Natural Science Research Council (NFR) under grant no. F–PD1–883–305.
[2]Research supported in part by the DOE under grant DE–FG02–92ER40706.
[3]Permanent address.
naculich@bowdoin.edu; rhedin.schnitzer@binah.cc.brandeis.edu
1. Introduction

Enormous advances have been made in understanding the exact behavior of low-energy four-dimensional \( N=2 \) supersymmetric gauge theories following the seminal work of Seiberg and Witten [1]. In their program one extracts the physics from a specified Riemann surface particular to the problem, and a preferred meromorphic 1-form, the Seiberg–Witten (SW) differential. This data allows one in principle to reconstruct the prepotential of the Coulomb branch of the theory in the low-energy limit from the period integrals of the SW differential.

For \( N=2 \) gauge theories based on classical groups, either without matter hypermultiplets or with matter hypermultiplets in the defining representation [1]–[3], the associated Riemann surface is hyperelliptic. Such theories have been studied in detail by means of two complementary techniques: the formulation and solution of the coupled set of Picard–Fuchs partial differential equations for the periods [3]–[6], and the direct evaluation of the period integrals by the method of residues developed by D’Hoker, Krichever, and Phong (DKP) [7]–[9]. The Picard–Fuchs approach has the advantage of being able to give global information about the prepotential through explicit solutions to the differential equations, suitably analytically continued [1, 3, 10, 6]. However, the complexity of the set of equations increases rapidly with the rank of the gauge group. The methods of DKP [7]–[9], on the other hand, are not severely limited by the rank of the group, but results are easily obtained only for the first few terms of the instanton expansion of the prepotential.

Not all Seiberg–Witten theories are solved by means of a hyperelliptic surface. String theory has provided new methods of constructing solutions to a wide class of Seiberg–Witten problems. In particular geometric engineering [11] and methods from M-theory [12, 13] have greatly enlarged the class of \( N=2 \) supersymmetric gauge theories that can be studied. These techniques have given rise to Riemann surfaces that are not hyperelliptic [14], and to curves that are not Riemann surfaces.
at all \[1\]. For these new \(N=2\) theories, the explicit formulation and solution of the appropriate Picard–Fuchs equations may be awkward at best. A direct evaluation of the period integrals may be more suitable, but systematic methods for the computation of the period integrals have not yet been developed for non-hyperelliptic surfaces. Attention to this issue is one of the motivations of this paper.

Another issue that must be addressed is the test of the curves predicted by geometric engineering or M-theory against the results of standard \(N=2\) supersymmetric theories. A prediction for a SW curve does not immediately translate into an explicit expression for the prepotential. Although the curves derived by string methods have been subjected to a number of consistency checks, no direct confrontation with \(N=2\) field theory beyond checking the one-loop perturbative prepotential has been presented. In particular, explicit instanton expansions for the prepotential have not been carried out and checked against field theory.

Using M-theory, Landsteiner and Lopez (LL) obtained a non-hyperelliptic curve characterizing the Coulomb phase of \(N = 2\) SU\((N)\) gauge theory with a single matter hypermultiplet in the antisymmetric or symmetric representation of the group \[13\]. LL checked that the one-loop beta function of the theory had the correct coefficient, that their curve had the correct limit as the mass of the multiplet \(m \to \infty\), and that the singular locus of the curve had expected singularities for SU\((2)\) and SU\((3)\). However, instanton predictions from the LL curves are not known.

In this paper and its companion \[15\], we develop methods to extract the instanton predictions of the Landsteiner-Lopez curves \[13\]. We calculate the explicit one-instanton contribution to the prepotential for \(N = 2\) SU\((N)\) gauge theory with matter in the antisymmetric representation. (In ref. \[16\], we do the same for the symmetric representation.) The key idea of this paper is the development of a systematic perturbation scheme about a hyperelliptic approximation to the LL
curves. This induces a perturbative expansion for the SW differential. We apply the method of residues developed by DKP [7]–[9] to each term this expansion, which enables us to calculate the renormalized order parameters of the theory to the one-instanton level. From these, we compute the prepotential to the same order.

Our results provide a test of curves derived from M-theory since there exists independent knowledge of the instanton expansion for the cases of SU(2), SU(3), and SU(4). Specifically, SU(2) with matter in the antisymmetric representation is equivalent to pure gauge theory [1], SU(3) with matter in the antisymmetric representation is equivalent to SU(3) with matter in the defining representation [7], and SU(4) with matter in the antisymmetric representation is equivalent to SO(6) with matter in the defining representation [8, 16]. Happily, our results for the one-instanton predictions of the LL curves coincide with results previously obtained for SU(2), SU(3), and SO(6).

It seems to us that it is extremely important to continue to test the field theoretic predictions of geometric engineering and M-theory. This will require further developments of the methods presented in this paper, as well as new explicit microscopic instanton calculations [17] starting from field theory.

2. The Setup

The formulation of Seiberg–Witten theory has been discussed by many authors, so our setup of the problem will be brief. Consider N=2 supersymmetric SU(N) gauge theory with one matter hypermultiplet in the antisymmetric representation. There is also an N=2 chiral multiplet in the adjoint representation, which contains a complex scalar field φ. This theory is asymptotically free. Along the flat directions of the potential, [φ, φ⁺] = 0, and the symmetry is broken to U(1)ᴺ⁻¹, with an N–1 dimensional moduli space, parametrized classically by the eigenvalues of φ

\[ e_k; \ 1 \leq k \leq N. \] (2.1)
In terms of $N=1$ superfields, the Wilson effective Lagrangian, to lowest order in the momentum expansion, is

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W^i W^j \right]$$  \hspace{1cm} (2.2)$$

where the $A^i$ are $N=1$ chiral superfields. Holomorphy implies that prepotential $\mathcal{F}$ has the form

$$\mathcal{F}(A) = \mathcal{F}_{\text{classical}}(A) + \mathcal{F}_{\text{1-loop}}(A) + \sum_{d=1}^{\infty} \Lambda^{[2N-I(R)]d} \mathcal{F}_{d-\text{inst.}}(A)$$  \hspace{1cm} (2.3)$$

where $I(R) = N - 2$ is the index of the antisymmetric representation, and the summation is over instanton contributions to the prepotential. From perturbation theory one knows that the one-loop prepotential takes the form for massless hypermultiplets

$$\mathcal{F}_{\text{1-loop}} = \frac{i}{4\pi} \sum_{\alpha \in \Delta^+} (\alpha \cdot \alpha)^2 \log \left( \frac{\alpha \cdot \alpha}{\Lambda^2} \right) - \frac{i}{8\pi} \sum_{w \in W_G} (a \cdot w)^2 \log \left( \frac{a \cdot w}{\Lambda^2} \right)$$  \hspace{1cm} (2.4)$$

where $\alpha$ is summed over the positive roots $\Delta^+$ of $G$, $w$ are the weight vectors of the weight system $W_G$ corresponding to the matter representation, and the $a_i$ are the diagonal elements of $\phi$ rotated into the Cartan subalgebra. This becomes

$$\mathcal{F}_{\text{1-loop}} = \frac{i}{8\pi} \left[ \sum_{i,j=1}^{N} (a_i - a_j)^2 \log \left( \frac{a_i - a_j}{\Lambda^2} \right) - \sum_{i<j} (a_i + a_j)^2 \log \left( \frac{a_i + a_j}{\Lambda^2} \right) \right]$$  \hspace{1cm} (2.5)$$

for one massless hypermultiplet in the antisymmetric representation.

The ingredients for determining the prepotential (2.3) using Seiberg–Witten theory [1] are a Riemann surface (which depends on the moduli) and a preferred meromorphic one-form $\lambda$, the Seiberg–Witten (SW) differential. In terms of these, one may calculate the renormalized order parameters $a_k$ and their duals $a_{D,k}$ using

$$2\pi i a_k = \oint_{A_k} \lambda, \quad 2\pi i a_{D,k} = \oint_{B_k} \lambda$$  \hspace{1cm} (2.6)$$

where $A_k$ and $B_k$ are a canonical basis of homology cycles on the Riemann surface. Given these, the prepotential is determined via

$$a_{D,k} = \frac{\partial \mathcal{F}}{\partial a_k}.$$  \hspace{1cm} (2.7)$$

5
Using arguments from M-theory, Landsteiner and Lopez [13] proposed the curve

\[ y^3 + 2A(x)y^2 + B(x)y + L^6 = 0 \]  \hspace{1cm} (2.8)

and SW differential

\[ \lambda = \frac{dy}{x} \]  \hspace{1cm} (2.9)

where

\[ L^2 = \Lambda^{N+2} \]
\[ A(x) = C(x) + \frac{3}{2}L^2 \]
\[ B(x) = L^2D(x) + 3L^4 \]  \hspace{1cm} (2.10)

and

\[ C(x) = \frac{1}{2}x^2 \prod_{i=1}^{N}(x-e_i) \]
\[ D(x) = (-1)^N x^2 \prod_{i=1}^{N}(x + e_i) \]  \hspace{1cm} (2.11)

Some properties of this curve are described in Appendix B.

The purpose of this paper is to derive the one-instanton contribution \( F_{1-\text{inst.}}(A) \) to the prepotential from the LL curve \( (2.8) \). Since this curve is not hyperelliptic, we have developed an extension of existing methods, suitable at least for the instanton expansion. For quantum scales \( \Lambda \) much smaller than the moduli (the semi-classical limit), one conjectures that the constant term in \( (2.8) \) is negligible relative to the first three terms. The approximate equation

\[ y^2 + 2A(x)y + B(x) \simeq 0 \]  \hspace{1cm} (2.12)

is hyperelliptic, and can be analyzed by previously developed methods [7]–[9]. This hyperelliptic approximation, however, is not sufficiently accurate to compute the one-instanton contribution to the prepotential.
In Appendix A, we solve (2.8) using a systematic perturbation expansion about the solutions of the hyperelliptic approximation (2.12). The first order correction is

\[
\begin{align*}
    y_1 & = (-A - r) - \frac{L^6}{2Br} (A - r) + \ldots \\
    y_2 & = (-A + r) + \frac{L^6}{2Br} (A + r) + \ldots \\
    y_3 & = -\frac{L^6}{B} + \ldots 
\end{align*}
\]

(2.13)

where

\[
r = \sqrt{A^2 - B} .
\]

(2.14)

The approximation (2.13) is sufficiently accurate to compute the one-instanton contribution to \( \mathcal{F}(A) \). To this order in the expansion, the presence of the third sheet \( y_3 \) can be neglected, as it is not connected to the first two sheets and has no branch cuts.

The corrections in (2.13) induce corrections to the SW differential

\[
\lambda = \lambda_I + \lambda_{II} + \cdots
\]

(2.15)

where \( \lambda_I \) is the usual SW differential (C.4) for the hyperelliptic curve (2.12), given by

\[
\lambda_I = \frac{x}{A + r} d(A + r) \simeq \frac{x \left( \frac{A}{A} - \frac{B}{2B} \right)}{\sqrt{1 - \frac{B^2}{A^2}}} dx
\]

(2.16)

and \( \lambda_{II} \) is the first correction to the hyperelliptic approximation. In Appendix C, this correction is shown to be (C.5)

\[
\lambda_{II} = -\frac{L^6 \left( A - \frac{B}{A} \right)}{B^2 \sqrt{1 - \frac{B^2}{A^2}}} dx
\]

(2.17)

The hyperelliptic approximation \( \lambda_I \) is sufficient to obtain \( \mathcal{F}_{1\text{-loop}}(A) \) for the theory, but the correction term \( \lambda_{II} \) is necessary for the computation of \( \mathcal{F}_{1\text{-inst.}}(A) \). Higher order corrections to the hyperelliptic approximation do not contribute at the one-instanton level.
3. The Branch Points

Before beginning the computation of the order parameters $a_k$ and $a_{D,k}$, we need to locate the branch-points $x_k^\pm$, $1 \leq k \leq N$, connecting sheets 1 and 2. (By the involution symmetry described in Appendix B, there are also branch points connecting sheets 2 and 3 at $-x_k^\pm$, but these are not important in the following.) Setting $y_1 = y_2$, we have from (2.13)

$$0 = A^2(x_k^\pm) - B(x_k^\pm) + \frac{L^6 A(x_k^\pm)}{2B(x_k^\pm)} + \ldots .$$

(3.1)

Since $B(x)$ is $\mathcal{O}(L^2)$, the last term in (3.1) is generally $\mathcal{O}(L^4)$ and therefore not important at the one-instanton level. For small $L$, the $x_k^\pm$ are close to $e_k$, and can be Taylor expanded

$$x_k^\pm = e_k + \sum_{m=1}^\infty (\pm 1)^m L^m \delta_k^{(m)}. \quad (3.2)$$

Following DKP, we introduce the residue functions $R_k(x)$, $S_k(x)$, and $S_0(x)$, defined by

$$R_k(x) = \frac{3}{2C(x)} = \frac{R_k(x)}{(x - e_k)},$$

$$D(x) C^2(x) = \frac{S_k(x)}{(x - e_k)^2} = \frac{S_0(x)}{x^2}. \quad (3.3)$$

These may be written explicitly

$$R_k(x) = \frac{3}{x^2 \prod_{i \neq k} (x - e_i)}$$

$$S_k(x) = \frac{4(-1)^N \prod_{i=1}^N (x + e_i)}{x^2 \prod_{i \neq k} (x - e_i)^2}$$

$$S_0(x) = \frac{4(-1)^N \prod_{i=1}^N (x + e_i)}{\prod_i (x - e_i)^2}. \quad (3.4)$$

With these definitions the $\delta_k^{(m)}$ may be computed from (3.1) to be

$$\delta_k^{(1)} = S_k(e_k)^{1/2}$$

$$\delta_k^{(2)} = \frac{1}{2} \frac{\partial S_k}{\partial x}(e_k) - R_k(e_k). \quad (3.5)$$
We end this section by deriving an identity needed below. To the accuracy required, we may write \( A^2(x_k^-) = B(x_k^-) \), that is
\[
\frac{1}{4} (x_k^-)^2 \prod_i (x_k^- - e_i)^2 \left[ 1 + \frac{3L^2}{2C(x_k^-)} \right]^2 = L^2 (-1)^{N+2} \prod_i (x_k^- + e_i) \left[ 1 + \frac{3L^2}{D(x_k^-)} \right].
\] (3.6)

Taking the logarithm of both sides of this equation and expanding, we obtain
\[
0 = 2 \log 2 - 2 \log x_k^- - 2 \sum_i \log (x_k^- - e_i) - \frac{3L^2}{C(x_k^-)} + \frac{9L^4}{4C^2(x_k^-)}
+ 2 \log L + (N + 2) \log(-1) + \sum_i \log (x_k^- + e_i) + \frac{3L^2}{D(x_k^-)}
\] (3.7)
where we need keep the \( L^4 \) term since \( C(x_k^-) \) contains a factor of \( (x_k^- - e_k) \), which by (3.2) is \( O(L) \).

4. The Order Parameters

A canonical homology basis for a hyperelliptic curve is described by DKP [7]. The basis \( A_k, B_k, 2 \leq k \leq N \) is obtained by choosing \( A_k \) to be a simple contour enclosing the slit from \( x_k^- \) to \( x_k^+ \), and \( B_k \) to consist of the curves going from \( x_1^- \) to \( x_k^- \) on the first sheet and from \( x_k^- \) to \( x_1^- \) on the second.

The renormalized order parameters \( a_k \) are given by
\[
2\pi i a_k = \oint_{A_k} \lambda \approx \oint_{A_k} \lambda_I + \lambda_{II}
= \oint_{A_k} dx \left[ \frac{x (A' - B')}{\sqrt{1 - \frac{B}{A}}^2} - L^6 \left( \frac{A - B}{2A} \right) \right]
\] (4.1)

The second term in (4.1) makes no contribution to \( a_k \) to \( O(L^2) \), as it has no poles at \( x = e_k \) to that order. The first term in (4.1) is identical to what one would obtain for a hyperelliptic curve.

A residue calculation essentially identical to eqs. (3.2)-(3.10) of ref.[7] yields
\[
a_k = e_k + L^2 \left[ \frac{1}{4} \frac{\partial S_k}{\partial x} (e_k) - R_k(e_k) \right] + \cdots
\] (4.2)
5. The Dual Order Parameters

The dual order parameters are given by

\[ 2\pi i a_{D,k} = \oint_{B_k} \lambda = \oint_{B_k} \lambda_I + \lambda_{II} \]  

(5.1)

One evaluates the SW differential for the \( B_k \) cycle by means of a contour that goes from \( x_1^- \) to \( x_k^- \) on sheet 1, crosses the branch cut at \( e_k \) to sheet 2, runs back from \( x_k^- \) to \( x_1^- \) on sheet 2, and passes back to sheet 1 through the branch cut at \( e_1 \). From the results of Appendix C, both \( \lambda_I \) and \( \lambda_{II} \) on sheet 2 differ only by a sign from the corresponding SW differentials on sheet 1, so we have

\[ 2\pi i a_{D,k} = 2 \int_{x_1^-}^{x_k^-} dx (\lambda_I + \lambda_{II}) \]

(5.2)

(a) Hyperelliptic Approximation

The dual order parameter in the hyperelliptic approximation is given by

\[ (2\pi i a_{D,k})_I = \oint_{B_k} \lambda_I = \lim_{\xi \to 1} \int_{x_1^-}^{x_k^-} dx \frac{2x (A' - \frac{1}{2} B')}{\sqrt{1 - B^2} (1 - B^2 A^2)} \]  

(5.3)

Following DKP [7], we have introduced a complex parameter \( \xi \) with \( |\xi| < 1 \) so that the denominator can be expanded in a power series in \( \xi^2 \),

\[ (2\pi i a_{D,k})_I = \sum_{m=0}^{\infty} I_m, \]  

(5.4)

where

\[ I_m = \frac{2\xi^{2m} \Gamma \left( m + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma (m + 1)} \int_{x_1^-}^{x_k^-} dx \left( \frac{A'}{A} - \frac{B'}{2B} \right) \left( \frac{B}{A^2} \right)^m. \]  

(5.5)

When all the terms have been calculated and resummed, we will set \( \xi \to 1. \)
We begin by computing
\[ I_0 = 2 \int_{x_1}^{x_k} dx \left( \frac{A'}{A} - \frac{B'}{2B} \right). \] (5.6)

Using (2.10), we expand the integrand in powers of \( L \)
\[ \frac{A'}{A} = \frac{C'}{C} + \frac{d}{dx} \left( \frac{3L^2}{2C} - \frac{9L^4}{8C^2} + \cdots \right) \]
\[ \frac{B'}{B} = \frac{D'}{D} + \frac{d}{dx} \left( \frac{3L^2}{D} + \cdots \right) \] (5.7)
keeping only those terms that will contribute at the 1-instanton level, \( \mathcal{O}(L^2) \). (We kept the \( \mathcal{O}(L^4) \) term in the first expression because \( C(x) \) is \( \mathcal{O}(L) \) when \( x \to x_k^- \).) Inserting (5.7) into (5.6) and integrating by parts, we have to that order
\[ I_0 = I_{0a} + I_{0b} + I_{0c} \] (5.8)
where
\[ I_{0a} = 2 \int_{x_1}^{x_k} dx \left[ \frac{C'}{C} - \frac{D'}{2D} \right] \] (5.9)
\[ I_{0b} = \left[ \frac{3xL^2}{C} - \frac{9xL^4}{4C^2} - \frac{3xL^2}{D} \right] \bigg|_{x_1}^{x_k} \] (5.10)
\[ I_{0c} = \int_{x_1}^{x_k} dx \left[ - \frac{3L^2}{C} + \frac{9L^4}{4C^2} + \frac{3L^2}{D} \right]. \] (5.11)
From (2.11), we have
\[ \frac{C'}{C} = \frac{2}{x} + \sum_{i=1}^{N} \frac{1}{x - e_i} \]
\[ \frac{D'}{D} = \frac{2}{x} + \sum_{i=1}^{N} \frac{1}{x + e_i} \] (5.12)
from which we obtain
\[ I_{0a} = (N + 2)x_k^- + 2 \sum_{i=1}^{N} e_i \log(x_k^- - e_i) + \sum_{i=1}^{N} e_i \log(x_k^- + e_i) \] (5.13)
where here, and throughout this section, we will explicitly write only the contribution from the upper limit of integration. The contribution from the lower limit of integration will be identical, but with \( k \) replaced by 1. Combining (5.13) and (5.10), and adding the product of \( x_k^- \) with (3.7) to the result, we find

\[
I_0a + I_0b = [N + 2 + 2 \log 2 + (N + 2) \log(-1) + 2 \log L] x_k^-
\]

\[
-2 \sum_i (x_k^- - e_i) \log(x_k^- - e_i) + \sum_i (x_k^- + e_i) \log(x_k^- + e_i) - 2x_k^- \log x_k^-
\]

Using eqs. (3.2), (3.4), and (3.5), we obtain

\[
I_0a + I_0b = 2x_k^- + [N + 2 \log 2 + (N + 2) \log(-1) + 2 \log L] e_k
\]

\[
-2 \sum_{i \neq k} (e_k - e_i) \log(e_k - e_i) + \sum_i (e_k + e_i) \log(e_k + e_i) - 2e_k \log e_k
\]

\[
-\frac{1}{2} L^2 \frac{\partial S_k}{\partial x}(e_k) + 2L^2 R_k(e_k).
\]

Turning now to the term \( I_{0c} \) (5.11), the middle integral is proportional to \( L^3 \), which can be neglected to 1-instanton accuracy. Using the identities

\[
\frac{1}{x^2 \prod_i (x - e_i)} = \sum_j \frac{e_j^2 \prod_{i \neq j} (e_j - e_i)}{e_j^2 \prod_{i \neq j} (e_j - e_i)} \left[ \frac{1}{x - e_j} - \frac{1}{x} - \frac{e_j}{x^2} \right]
\]

\[
\frac{1}{x^2 \prod_i (x + e_i)} = \sum_j \frac{(-1)^N}{e_j^2 \prod_{i \neq j} (e_j - e_i)} \left[ -\frac{1}{x + e_j} + \frac{1}{x} - \frac{e_j}{x^2} \right]
\]

the first and third terms yield

\[
I_{0c} = \sum_j \frac{L^2}{e_j^2 \prod_{i \neq j} (e_j - e_i)} \left[ -6 \log(x_k^- - e_j) + 9 \log e_k - 3 \log(e_k + e_j) - 3 \frac{e_j}{e_k} \right]
\]

\[
= L^2 \sum_j R_j(e_j) \left[ -2 \log(x_k^- - e_j) + 3 \log e_k - \log(e_k + e_j) - \frac{e_j}{e_k} \right]
\]

\[
= -\frac{3}{4} L^2 \frac{\partial S_0}{\partial x}(0) \log e_k + \frac{3}{4} L^2 \frac{S_0(0)}{e_k} - L^2 \sum_j R_j(e_j) \left[ 2 \log(x_k^- - e_j) + \log(e_k + e_j) \right]
\]
where we have used \( x_k^- = e_k + \mathcal{O}(L) \), (3.2) and the identities (D.1) and (D.2) derived in Appendix D.

Next we compute the \( m \geq 1 \) terms in the series (5.4). Using the identity [7]

\[
\frac{x A'}{A^2} = -\frac{d}{dx} \left[ \frac{x B}{A^2} \right] + \frac{1}{2m} \left( \frac{B}{A^2} \right)^m
\]

(5.18)
together with the result (3.1)

\[
\frac{B(x_k^-)}{A^2(x_k^-)} = 1 + \mathcal{O}(L^3)
\]

(5.19)
we obtain

\[
I_m = \frac{\xi^{2m} \Gamma \left( m + \frac{1}{2} \right)}{m \Gamma \left( \frac{1}{2} \right) \Gamma (m + 1)} \left[ -x_k^- + \int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)^m \right]
\]

(5.20)
(suppressing as usual the contribution from the lower limit of integration). One may expand

\[
\left( \frac{B}{A^2} \right)^m = \left( \frac{L^2 D}{C^2} \right)^m \left(1 + \frac{3L^2}{D} \right)^m \left(1 + \frac{3L^2}{2C} \right)^{-2m}
\]

(5.21)

where

\[
\Gamma_{m,r} = \frac{3^r \Gamma(m + 1)}{\Gamma(m - r + 1) \Gamma(r + 1)}
\]

\[
\tilde{\Gamma}_{m,n} = \frac{(-3/2)^n \Gamma(2m + n)}{\Gamma(2m) \Gamma(n + 1)}
\]

(5.22)

Higher powers of \( L \) must be retained in the expansion, as the integration in (5.20) produces negative powers of \( L \). Next expand the terms in (5.21) in partial fractions

\[
\frac{D^{m-r}(x)}{C^{2m+n}(x)} = \sum_{p=1}^{2m+2n+2r} \frac{Q^{(2m,2n,2r)}_{0,p}}{x^p} + \sum_{i=1}^{N} \sum_{p=1}^{2m+n} \frac{Q^{(2m,2n,2r)}_{i,p}}{(x - e_i)^p}.
\]

(5.23)

We may split

\[
\int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)^m = \int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)_p^m + \int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)_{p>1}^m
\]

(5.24)
The coefficients of the second term in (5.25) are need be evaluated treating the \( p = 1 \) and \( p > 1 \) terms in the partial fraction expansion (5.23) separately. First

\[
\int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)_p^m = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \Gamma_{m,r} \tilde{\Gamma}_{m,n} L^{2m+2n+2r} \left[ Q_{0,1}^{(2m,2n,2r)}(x^-_k) \log x^-_k + \sum_{i=1}^N Q_{i,1}^{(2m,2n,2r)} \log(x^-_k - e_i) \right].
\]

(5.25)

Since \( \log x^-_k = \log e_k + \mathcal{O}(L) \), only the \( m = 1, n = 0, r = 0 \) coefficient for the first term in (5.25) need be evaluated

\[
Q_{0,1}^{(2,0,0)} = \frac{1}{2\pi i} \oint_{x=0} dx \frac{D}{C^2} = \frac{1}{2\pi i} \oint_{x=0} dx \frac{S_0(x)}{x^2} = \frac{\partial S_0}{\partial x}(0).
\]

(5.26)

The coefficients of the second term in (5.23) are

\[
Q_{i,1}^{(2m,2n,2r)} = \frac{1}{2\pi i} \oint_{A_i} dx \frac{D^{m-r}}{C^{2m+n}}
\]

(5.27)

allowing us to resum the series to obtain

\[
\int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)_p^m = L^2 \delta_{m,1} \frac{\partial S_0}{\partial x}(0) \log e_k + \frac{1}{2\pi i} \sum_{i=1}^N \log(x^-_k - e_i) \oint_{A_i} dx \left( \frac{B}{A^2} \right)_p^m.
\]

(5.28)

Next consider the \( p > 1 \) contribution to (5.24). Integration of (5.21)–(5.23) yields

\[
\int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)_p^m = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \Gamma_{m,r} \tilde{\Gamma}_{m,n} L^{2m+2n+2r} \times
\[
\left[ \sum_{p=2}^{2m+2n+2r} Q_{0,p}^{(2m,2n,2r)} \frac{(1-p)(x^-_k)^{p-1}}{p} + \sum_{i=1}^N \sum_{p=2}^{2m+n} Q_{i,p}^{(2m,2n,2r)} \frac{(1-p)(x^-_k - e_i)^{p-1}}{p} \right].
\]

(5.29)

Except for the \( i = k \) term in the sum, one only need keep the \( m = 1, n = 0, r = 0 \) term to order \( L^2 \). For the \( i = k \) terms one needs all \( m \) due to the factors of \( (x^-_k - e_k) \) in the denominator, but only \( r = 0 \) and \( n = 0, 1 \) to obtain terms of \( \mathcal{O}(L^2) \). Therefore

\[
\int_{x_1^-}^{x_k^-} dx \left( \frac{B}{A^2} \right)_p^m = -L^2 \delta_{m,1} \left[ \frac{Q_{0,2}^{(2,0,0)}}{x^-_k} + \sum_{i \neq k} \frac{Q_{i,2}^{(2,0,0)}}{(x^-_k - e_i)} \right]
\]

\[
+ L^2 \left[ \frac{Q_{k,2m}^{(2m,0,0)}(x^-_k - e_k)^{2m-1}}{(1-2m)(x^-_k - e_k)^{2m-1}} + \frac{\theta_{m-2} Q_{k,2m-1}^{(2m,0,0)}}{(2-2m)(x^-_k - e_k)^{2m-2}} \right]
\]

\[
- 3m L^{2m+2} \frac{Q_{k,2m+1}^{(2m,2,0)}}{(-2m)(x^-_k - e_k)^{2m}}.
\]

(5.30)
where \( \theta_s = 1 \) for \( s \geq 0 \) and \( \theta_s = 0 \) for \( s < 0 \). From (3.2), we have

\[
\frac{1}{(x_k - e_k)^s} = \frac{(-1)^s}{L^s \left( \frac{\delta(1)}{\delta_k} \right)^s} \left[ 1 + sL \frac{\delta(2)}{\delta_k^{(1)}} + \ldots \right]
\]  
(5.31)

which can be used to simplify (5.30). Combining this result with (5.20) and (5.28), one finds

\[
I_m = \xi^{2m} \frac{\Gamma \left( m + \frac{1}{2} \right)}{m \Gamma \left( \frac{1}{2} \right) \Gamma (m + 1)} \left[ -x_k^+ + \sum_{i=1}^{N} \log (x_k^+ - e_i) \frac{1}{2 \pi i} \oint_{A_i} dx \left( \frac{B}{A^2} \right)^m + L^2 \frac{\delta(1)}{\delta_k^{(1)}} \partial S_0(0) \log e_k - L^2 \delta(1) \right]
\]

\[
\quad + \frac{L}{2(2m-1)} \frac{Q_{k,2m}^{(2m,0,0)}}{\delta_k^{(1)}}^2 + L^2 \frac{Q_{k,2m}^{(2m,0,0)}}{\delta_k^{(1)}}^2 + \frac{3}{2} L^2 \frac{Q_{k,2m+1}^{(2m,2,0)}}{\delta_k^{(1)}}^2 \right].
\]  
(5.32)

The coefficients of the partial fraction expansion may be evaluated by comparing (5.23) with (3.3) and (3.4) to obtain

\[
Q_{k,2m}^{(2m,0,0)} = S_k(e_k)^m
\]

\[
Q_{0,2}^{(2,0,0)} = S_0(0)
\]

\[
Q_{k,2m-1}^{(2m,0,0)} = m S_k(e_k)^{m-1} \frac{\partial S_k(e_k)}{\partial x}
\]

\[
Q_{k,2m+1}^{(2m,2,0)} = \frac{2}{3} S_k(e_k)^m R_k(e_k)
\]  
(5.33)

Using (5.33) and (5.3), equation (5.32) becomes

\[
I_m = \frac{\xi^{2m} \Gamma \left( m + \frac{1}{2} \right)}{m \Gamma \left( \frac{1}{2} \right) \Gamma (m + 1)} \left[ -x_k^+ + \sum_{i} \log (x_k^+ - e_i) \frac{1}{2 \pi i} \oint_{A_i} dx \left( \frac{B}{A^2} \right)^m + L^2 \frac{\delta(1)}{\delta_k^{(1)}} \partial S_0(0) \log e_k - L^2 \delta(1) \right]
\]

\[
\quad + L^2 \delta(1) \frac{\partial S_0(0)}{\partial x} \log e_k - L^2 \delta(1) \left( \frac{S_0(0)}{e_k} + \sum_{i \neq k} \frac{S_i(e_i)}{e_k - e_i} \right)
\]
\[ +L \frac{S_k(e_k)^{\frac{1}{2}}}{(2m-1)} + \frac{1}{2} L^2 \frac{\partial S_k}{\partial x}(e_k) + L^2 \theta_{m-2} \left( \frac{m}{2-2m} \right) \frac{\partial S_k}{\partial x}(e_k) \]  

(5.34)

our final expression for \( I_m, m \geq 1 \).

To carry out the sum of (5.34) over \( m \), several identities found in Appendix A of ref. [7] are useful

\[ \sum_{m=1}^{\infty} \frac{\Gamma \left( m + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma(m+1)} \frac{1}{2m(2m-1)} = 1 - \log 2 \]  

(5.35)

\[ \sum_{m=2}^{\infty} \frac{\Gamma \left( m + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma(m+1)} \frac{1}{2m(2m-2)} = -\frac{1}{4} \log 2 + \frac{1}{4} \]  

(5.36)

\[ \sum_{m=1}^{\infty} \frac{\Gamma \left( m + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma(m+1)m} = 2 \log 2 \]  

(5.37)

First, we separately sum the second term in (5.34), using the identity (5.18) and the fact that a total derivative does not contribute to the \( A \)-cycle

\[ \frac{1}{2\pi i} \oint_{A_i} dx \sum_{m=1}^{\infty} \frac{\Gamma \left( m + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma(m+1)} \frac{1}{m} \left( \frac{B}{A^2} \right)^m \]

\[ = \frac{1}{2\pi i} \oint_{A_i} dx \frac{2x}{A} \left( \frac{A'}{A} - \frac{B'}{2B} \right) \sum_{m=1}^{\infty} \frac{\Gamma \left( m + \frac{1}{2} \right) \xi^{2m}}{\Gamma \left( \frac{1}{2} \right) \Gamma(m+1)} \left( \frac{B}{A^2} \right)^m \]

\[ = \frac{2}{2\pi i} \oint_{A_i} \lambda - \frac{2}{2\pi i} \oint_{A_i} x \left( \frac{C'}{C} + \frac{d}{dx} \frac{3L^2}{2C} + \cdots \right) \]

\[ = 2a_i - 2e_i + 2L^2 R_i(e_i) \]  

(5.38)

The full sum of (5.34) over all \( m \geq 1 \) is then

\[ \sum_{m=1}^{\infty} I_m = -(2 \log 2)x_k^- + 2 \sum_i \left[ a_i - e_i + L^2 R_i(e_i) \right] \log(x_k^- - e_i) \]
\[
+ \frac{1}{2} L^2 \frac{\partial S_0}{\partial x} (0) \log e_k - \frac{1}{2} L^2 \left( \frac{S_0(0)}{e_k} + \sum_{i \neq k} \frac{S_i(e_i)}{e_k - e_i} \right) \\
+ (2 - 2 \log 2) L S_k(e_k)^{\frac{1}{2}} + \left( \frac{1}{2} \log 2 - \frac{1}{4} \right) L^2 \frac{\partial S_k}{\partial x}(e_k) \). \tag{5.39}
\]

We now assemble the contributions (5.15), (5.17), and (5.39) to the hyperelliptic approximation of the dual order parameter and use

\[
2(a_k - e_k) \log(x_k^- - e_k) = (a_k - e_k) \left[ 2 \log L + \log S_k(e_k) \right] \\
= (a_k - e_k)\left[ 2 \log L + 2 \log 2 + (N + 2) \log(-1) \right. \\
\left. + \sum_i \log(e_k + e_i) - 2 \log e_k - 2 \sum_{i \neq k} \log(e_k - e_i) \right] \tag{5.40}
\]

to obtain

\[
(2\pi i a_{D,k})_I = (I_{0a} + I_{0b}) + (I_{0c}) + \left( \sum_{m=1}^{\infty} I_m \right) \tag{5.41}
\]

\[
= (2 - 2 \log 2)x_k^- + [N + 2 \log 2 + (N + 2) \log(-1) + 2 \log L]a_k + [-2 - N](a_k - e_k) \\
-2 \sum_{i \neq k} (a_k - a_i) \log(e_k - e_i) + \sum_i (a_k + e_i) \log(e_k + e_i) - 2a_k \log e_k \\
-\frac{1}{4} L^2 \frac{\partial S_0}{\partial x}(0) \log e_k + \frac{1}{4} L^2 \frac{S_0(0)}{e_k} - L^2 \sum_j R_j(e_j) \log(e_k + e_j) \\
-\frac{1}{4} L^2 \sum_{i \neq k} \frac{S_i(e_i)}{e_k - e_i} + (2 - 2 \log 2) L S_k(e_k)^{\frac{1}{2}} + \left( \frac{1}{2} \log 2 - \frac{1}{4} \right) L^2 \frac{\partial S_k}{\partial x}(e_k) .
\]

Using (3.2), (3.5), and (4.2) to rewrite (5.41) completely in terms of \(a_k\), and using the identities (D.3)–(D.5), we finally obtain the result, accurate to \(O(L^2)\)

\[
(2\pi i a_{D,k})_I = [N + 2 + (N + 2) \log(-1) + 2 \log L]a_k \\
-2 \sum_{i \neq k} (a_k - a_i) \log(a_k - a_i) + \sum_i (a_k + a_i) \log(a_k + a_i) - 2a_k \log a_k
\]
\[-\frac{1}{4} L^2 \partial S_0 (0) \log a_k - \frac{1}{4} L^2 \sum_j \partial S_j (a_j) \log (a_k + a_j) \]

\[+ \frac{1}{4} L^2 \partial S_k (a_k) \quad + \frac{1}{4} L^2 \frac{S_0 (0)}{a_k} - \frac{1}{2} L^2 \sum_{i \neq k} S_i (a_i). \tag{5.42} \]

Although the hyperelliptic approximation (5.42) contains one-instanton \( (O(L^2)) \) contributions to \( a_{D,k} \) it cannot be the complete one-instanton result, because of the presence of unacceptable \( L^2 \log a_k \) and \( L^2 \log (a_k + a_j) \) type terms.

**(b) Corrections to the hyperelliptic approximation**

From the results of appendix C, eq. (C.7), the \( O(L^2) \) correction to the hyperelliptic approximation is

\[
(2 \pi i a_{D,k})_{II} = \oint_{B_k} \lambda_{II}
\]

\[
= -2 L^2 \int_{x_i^-}^{x_k^-} dx \frac{C(x)}{D^2(x)}
\]

\[
= -L^2 \int_{x_i^-}^{x_k^-} dx \frac{\prod_i (x - e_i)}{x^2 \prod_i (x + e_i)^2}
\]

\[
= \frac{1}{4} (-1)^N L^2 \int_{x_i^-}^{x_k^-} dx \frac{\bar{D}(x)}{\bar{C}^2(x)}. \tag{5.43} \]

where \( \bar{C}(x) \) and \( \bar{D}(x) \) are obtained from \( C(x) \) and \( D(x) \) respectively by letting \( e_i \to -e_i \).

\[
\bar{C}(x) = \frac{1}{2} x^2 \prod_{i=1}^N (x + e_i)
\]

\[
\bar{D}(x) = (-1)^N x^2 \prod_{i=1}^N (x - e_i). \tag{5.44} \]

We expand the integrand of (5.43) in partial fractions

\[
\frac{\bar{D}(x)}{\bar{C}^2(x)} = \sum_{p=1}^2 \frac{\bar{Q}^{(2,0,0)}_{0,p}}{x^p} + \sum_{j=1}^N \sum_{p=1}^2 \frac{\bar{Q}^{(2,0,0)}_{j,p}}{(x + e_j)^p}. \tag{5.45} \]
which results in

\[
(2\pi i a_{D,k})_{II} = -\frac{1}{4}(-1)^N L^2 \left[ \frac{Q_{0,1}^{(2,0,0)}}{Q_{0,2}^{(2,0,0)}} \log e_k + \sum_{j=1}^{N} \frac{Q_{j,1}^{(2,0,0)}}{Q_{j,2}^{(2,0,0)}} \log(e_k + e_j) - \frac{Q_{0,2}^{(2,0,0)}}{e_k} - \sum_{j=1}^{N} \frac{Q_{j,2}^{(2,0,0)}}{e_k + e_j} \right].
\]

(5.46)

As before, we introduce residue functions \( \bar{S}_j(x) \) and \( \bar{S}_0(x) \), defined by

\[
\bar{D}(x) \quad \bar{C}^2(x) = \frac{\bar{S}_j(x)}{(x + e_j)^2} = \frac{\bar{S}_0(x)}{x^2}
\]

(5.47)

in terms of which the partial fraction coefficients may be expressed

\[
Q_{0,1}^{(2,0,0)} = \frac{\partial \bar{S}_0}{\partial x}(0)
\]

\[
Q_{j,1}^{(2,0,0)} = \frac{\partial \bar{S}_j}{\partial x}(-e_j)
\]

\[
Q_{0,2}^{(2,0,0)} = \bar{S}_0(0)
\]

\[
Q_{j,2}^{(2,0,0)} = \bar{S}_j(-e_j).
\]

(5.48)

From the explicit expressions for the residue functions

\[
\bar{S}_0(x) = \frac{4(-1)^N \prod_{i=1}^{N}(x - e_i)}{\prod_{i}(x + e_i)^2}
\]

\[
\bar{S}_j(x) = \frac{4(-1)^N \prod_{i=1}^{N}(x - e_i)}{x^2 \prod_{i \neq j}(x + e_i)^2}
\]

(5.49)

and eq. (5.4), we ascertain

\[
\frac{\partial S_0}{\partial x}(0) = -(-1)^N \frac{\partial S_0}{\partial x}(0)
\]

\[
\frac{\partial S_j}{\partial x}(-e_j) = -(-1)^N \frac{\partial S_j}{\partial x}(e_j)
\]

\[
\bar{S}_0(0) = (-1)^N S_0(0)
\]

\[
\bar{S}_j(-e_j) = (-1)^N S_j(e_j)
\]

(5.50)
Combining (5.46) with (5.48) and (5.50), we obtain the contribution to the dual order parameters from the correction $\lambda_{II}$ to the hyperelliptic SW differential, accurate to one-instanton order. Since the entire correction is $O(L^2)$, we may replace $e_i$ by $a_i$ throughout, resulting in

$$
(2\pi i a_{D,k})_{II} = \frac{1}{4}L^2 \left[ \frac{\partial S_0}{\partial x}(0) \log a_k + \sum_{j=1}^{N} \frac{\partial S_j}{\partial x}(a_j) \log(a_k + a_j) + \frac{S_0(0)}{a_k} + \sum_{j=1}^{N} \frac{S_j(a_j)}{a_k + a_j} \right].
$$

(5.51)

6. The Prepotential

Combining the results (5.42) and (5.51) of the preceding section, one obtains the following expression for the dual order parameters, accurate to the one-instanton level:

$$
2\pi i a_{D,k} = [N + 2 + (N + 2) \log(-1) + 2 \log L]a_k
$$

$$
-2 \sum_{i \neq k} (a_k - a_i) \log(a_k - a_i) + \sum_{i} (a_k + a_i) \log(a_k + a_i) - 2a_k \log a_k
$$

$$
+ L^2 \left[ \frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) + \frac{1}{2} \frac{S_0(0)}{a_k} - \frac{1}{2} \sum_{i \neq k} \frac{S_i(a_i)}{a_k - a_i} + \frac{1}{4} \sum_{j=1}^{N} \frac{S_j(a_j)}{a_k + a_j} \right]
$$

$$
-(k \to 1)
$$

(6.1)

where we have restored the dependence on the lower integration limit that was suppressed throughout the calculation, and where

$$
S_k(x) = \frac{4(-1)^N \prod_{i=1}^{N}(x + a_i)}{x^2 \prod_{i \neq k}(x - a_i)^2}
$$

$$
S_0(x) = \frac{4(-1)^N \prod_{i=1}^{N}(x + a_i)}{\prod_{i}(x - a_i)^2}.
$$

(6.2)

Note that since the residue functions $S_k(x)$ and $S_0(x)$ appear in (6.1) multiplied by $L^2$, we have replaced the unrenormalized order parameters $e_i$ in the original definitions (3.4) with renormalized order parameters $a_i$ in (6.2).
The prepotential $F(a)$ is found via (2.7) when written in terms of the independent variables $a_2, \ldots, a_k$. Since, using (D.4) and (D.5), $a_1$ obeys the constraint

$$\sum_{j=1}^{N} a_j = 0 \quad (6.3)$$

for a massless hypermultiplet, if $F(a)$ is written in terms of all the variables $a_j$, (2.7) becomes

$$a_{D,k} = \frac{\partial F}{\partial a_k} - \frac{\partial F}{\partial a_1}. \quad (6.4)$$

To one-instanton order this becomes

$$a_{D,k} = \frac{\partial}{\partial a_k} \left[ F_{\text{classical}} + F_{1-\text{loop}} + \Lambda^{N+2} F_{1-\text{inst.}} + \cdots \right] - (k \to 1). \quad (6.5)$$

Integrating (6.1), we obtain

$$F_{\text{classical}} + F_{1-\text{loop}} = \frac{1}{4\pi i} \left[ \frac{3}{2} (N + 2) + (N + 2) \log(-1) + 2 \log 2 \right] \sum_j a_j^2 \quad (6.6)$$

$$+ \frac{i}{8\pi} \left[ \frac{N}{2} (a_i - a_j)^2 \log \left( \frac{a_i - a_j}{\Lambda^2} \right) - \sum_{i<j} (a_i + a_j)^2 \log \left( \frac{a_i + a_j}{\Lambda^2} \right) \right]$$

and

$$F_{1-\text{inst.}} = \frac{1}{2\pi i} \left[ -\frac{1}{2} S_0(0) + \frac{1}{4} \sum_k S_k(a_k) \right] \quad (6.7)$$

a beautiful, concise result in view of the lengthy calculations required. To arrive at (6.7), we employed the identities

$$\frac{\partial}{\partial a_k} \left[ S_k(a_k) \right] = \frac{\partial S_k}{\partial x}(a_k) + \frac{S_k(a_k)}{2a_k}$$

$$\frac{\partial}{\partial a_k} \left[ \sum_{i \neq k} S_i(a_i) \right] = -2 \sum_{i \neq k} \frac{S_i(a_i)}{a_k - a_i} + \sum_{i \neq k} \frac{S_i(a_i)}{a_k + a_i}$$

$$\frac{\partial}{\partial a_k} \left[ S_0(0) \right] = -\frac{S_0(0)}{a_k} \quad (6.8)$$
which follow directly from (6.2). Note that the results (6.6) and (6.7) are invariant under permutation of the $a_k$, and hence the Weyl group of SU(N).

The calculation above is for a massless hypermultiplet in the antisymmetric representation. Shifting

$$a_i \longrightarrow a_i + \frac{m}{2} \quad (6.9)$$

in (6.6) and (6.7) gives a result for a hypermultiplet with mass $m$ that is consistent with the known cases (see next section).

7. Tests of the One-Instanton Predictions

Comparison of (2.5) with (6.6) shows that the Landsteiner-Lopez curve correctly predicts the perturbative one-loop prepotential. Different curves, however, can provide the same predictions to one-loop order, $G_2$ [18] and $E_6$ [19] being well-known examples. Therefore, one needs to compare (at least) the one-instanton predictions of the curve with field theoretic results before one can be certain that the curve correctly describes the low-energy field theory. One should hold M-theory to this standard.

The one-instanton contribution to the prepotential for a hypermultiplet in the antisymmetric representation is already known for SU(2), SU(3), and SU(4) from other considerations. The antisymmetric representation of SU(2) is the trivial representation so the case of SU(2) corresponds to pure SU(2) gauge theory. We compare (6.7) for $N = 2$ with the results of DKP [7], eq. (4.33b) for $N_c = 2$ and $N_f = 0$, finding agreement with the change of scale $L^2 = \frac{1}{16} \tilde{\Lambda}_{DKP}^2$.

The antisymmetric representation of SU(3) should give the same result as the defining representation. Comparing (6.7) (with the shift (6.9)) for $N = 3$ with the results of DKP [7], eq. (4.33b), for $N_c = 3$ and $N_f = 1$, and using $a_1 + a_2 + a_3 = 0$, we find agreement, again with a change of the quantum scale.
The antisymmetric representation of SU(4) is equivalent to the defining representation of SO(6). In particular, the weights of the antisymmetric representation of SU(4) are

$$\pm (a_1 + a_2), \pm (a_2 + a_3), \pm (a_3 + a_1).$$  \hfill (7.1)

The weights of the fundamental representation of SO(6) are

$$\pm d_i \quad (i = 1, 2, 3).$$ \hfill (7.2)

These weights are identified as follows:

$$d_1 = a_1 + a_2 \quad d_2 = a_2 + a_3 \quad d_3 = a_3 + a_1.$$ \hfill (7.3)

DKP [8] (and also ref. [16]) give the one-instanton contribution for SO(6) with one massless hypermultiplet in the defining representation

$$\mathcal{F}_{1-\text{inst.}} = \frac{1}{4\pi^2} \Lambda_{\text{DKP}}^6 \sum_{k=1}^{3} \Sigma_k(d_k)$$ \hfill (7.4)

where

$$\Sigma_k(x) = \frac{x^6}{(x + d_k)^2 \prod_{j \neq k} (x^2 - d_j^2)^2}.$$ \hfill (7.5)

Using $$a_4 \equiv -(a_1 + a_2 + a_3)$$, we find that (6.7) for $$N = 4$$ is equivalent to (7.4), with a change of the quantum scale. Thus, we find agreement for this case as well.

8. Concluding Remarks

In this paper we derived the one-instanton contribution to the prepotential for $$N=2$$ supersymmetric SU(N) gauge theory with a matter hypermultiplet in the antisymmetric approximation, using the non-hyperelliptic curve obtained from M-theory by Landsteiner and Lopez. To carry out
this calculation, we developed a systematic perturbation theory for the curve and Seiberg-Witten
differential, where the zeroth order curve is hyperelliptic. Our results for the Landsteiner-Lopez
curve agree with known results for SU(2), SU(3), and SU(4), and provide predictions for SU(N),
$N \geq 5$, which could be checked against future “microscopic” instanton calculations \[17\] in $N=2$
supersymmetric gauge theories.

A companion \[15\] to this paper describes a similar calculation for the symmetric representa-
tion of SU(N). It is extremely important to continue to develop the bridge between string theory
and field theory, and in particular to verify the predictions of geometric engineering and M-theory
for $N=2$ supersymmetric gauge theories, so as to gain confidence in the validity of string theory
predictions of field theoretic phenomena. We believe that the methods of this paper will prove
useful for that purpose.

**Acknowledgements**

We wish to thank Isabel Ennes, José Isidro, Michael Mattis, João Nunes, and Özgür Sarıoğlu
for discussions on various aspects of this work. HJS wishes to thank the Physics Department of
Harvard University for their hospitality during the spring semester of 1998.

Finally, we wish to express our appreciation for the beautiful work of D’Hoker, Krichever,
and Phong, which greatly influenced this paper.
Appendix A: Hyperelliptic Perturbation Theory

Consider the cubic curve

\[ y^3 + 2A(x)y^2 + B(x)y + \epsilon(x) = 0. \quad (A.1) \]

We may eliminate the quadratic term by changing variables

\[ w = y + \frac{2}{3} A, \quad (A.2) \]

yielding

\[ w^3 + \left( B - \frac{4}{3} A^2 \right) w + \left( \frac{16}{27} A^3 - \frac{2}{3} AB + \epsilon \right) = 0. \quad (A.3) \]

The solutions of (A.3) satisfy

\[ w_1 + w_2 + w_3 = 0 \]

\[ w_1 w_2 + w_1 w_3 + w_2 w_3 = B - \frac{4}{3} A^2 \]

\[ w_1 w_2 w_3 = -\frac{16}{27} A^3 + \frac{2}{3} AB - \epsilon. \quad (A.4) \]

When \( \epsilon \) vanishes, the curve (A.1) factors into \( y = 0 \) and a hyperelliptic curve \( y^2 + 2A(x)y + B(x) = 0 \). We will use the solutions of this as a starting point for a perturbative expansion in \( \epsilon \). We remark that though \( \epsilon \) is proportional to (some power of) \( \Lambda \), our perturbative expansion is \textit{not} equivalent to an expansion in \( \Lambda \). In particular, since \( A(x) \) and \( B(x) \) have \( \Lambda \) dependence, the hyperelliptic approximation, which is zeroth order in \( \epsilon \), contains all orders in \( \Lambda \).

For \( \epsilon = 0 \), the solutions of (A.2) are

\[ \bar{y}_1 = -A - r \]

\[ \bar{y}_2 = -A + r \]

\[ \bar{y}_3 = 0 \quad (A.5) \]
where $r = \sqrt{A^2 - B}$. Equivalently, the solutions of (A.3) are

$$
\begin{align*}
\tilde{w}_1 & = \frac{1}{3} A - r \\
\tilde{w}_2 & = \frac{1}{3} A + r \\
\tilde{w}_3 & = \frac{2}{3} A.
\end{align*}
$$

(A.6)

When $\epsilon \neq 0$, the solutions to (A.3) can be written as expansions around the hyperelliptic solutions (A.6),

$$
\begin{align*}
    w_i & = \tilde{w}_i + \delta w_i = \tilde{w}_i + \alpha_i \epsilon + \beta_i \epsilon^2 + \cdots 
\end{align*}
$$

(A.7)

Substituting this into (A.4), we obtain

$$
\begin{align*}
    \delta(w_1 + w_2 + w_3) & = 0 \\
    \delta(w_1 w_2 + w_2 w_3 + w_3 w_1) & = 0 \\
    \delta(w_1 w_2 w_3) & = -\epsilon
\end{align*}
$$

(A.8)

To first order in $\epsilon$, we have

$$
\begin{align*}
    \alpha_1 + \alpha_2 + \alpha_3 & = 0 \\
    \tilde{w}_1 \alpha_1 + \tilde{w}_2 \alpha_2 + \tilde{w}_3 \alpha_3 & = 0 \\
    \tilde{w}_2 \tilde{w}_3 \alpha_1 + \tilde{w}_1 \tilde{w}_3 \alpha_2 + \tilde{w}_1 \tilde{w}_2 \alpha_3 & = -1.
\end{align*}
$$

(A.9)

which can be solved to give

$$
\begin{align*}
    \alpha_1 & = \frac{1}{(\tilde{w}_1 - \tilde{w}_2)(\tilde{w}_3 - \tilde{w}_1)} = -\frac{1}{2r(A + r)} = -\frac{A - r}{2Br} \\
    \alpha_2 & = \frac{1}{(\tilde{w}_2 - \tilde{w}_3)(\tilde{w}_1 - \tilde{w}_2)} = \frac{1}{2r(A - r)} = \frac{A + r}{2Br} \\
    \alpha_3 & = \frac{1}{(\tilde{w}_3 - \tilde{w}_1)(\tilde{w}_2 - \tilde{w}_3)} = \frac{1}{B}.
\end{align*}
$$

(A.10)
The solutions of (A.1) are therefore

\[ y_1 = -A - r - \epsilon \frac{A - r}{2Br} + \mathcal{O}(\epsilon^2) \]

\[ y_2 = -A + r + \epsilon \frac{A + r}{2Br} + \mathcal{O}(\epsilon^2) \]

\[ y_3 = -\epsilon \frac{1}{B} + \mathcal{O}(\epsilon^2) \]  
(A.11)

Note that, to this order, \( y_3 \) does not exhibit a branch-cut, while sheets labelled by \( y_1 \) and \( y_2 \) are linked by the branch-cuts that arise from \( r = \sqrt{A^2 - B} \).

The second order corrections in (A.7) must obey

\[ \beta_1 + \beta_2 + \beta_3 = 0 \]

\[ \bar{w}_1 \beta_1 + \bar{w}_2 \beta_2 + \bar{w}_3 \beta_3 = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 \]

\[ \bar{w}_2 \bar{w}_3 \beta_1 + \bar{w}_3 \bar{w}_1 \beta_2 + \bar{w}_1 \bar{w}_2 \beta_3 = -\alpha_1 \alpha_2 \bar{w}_3 - \alpha_2 \alpha_3 \bar{w}_1 - \alpha_3 \alpha_1 \bar{w}_2 \]  
(A.12)

which have the solution

\[ \beta_1 = \frac{3 \bar{w}_1}{(\bar{w}_1 - \bar{w}_2)^3(\bar{w}_3 - \bar{w}_1)^3} \]

\[ \beta_2 = \frac{3 \bar{w}_2}{(\bar{w}_2 - \bar{w}_3)^3(\bar{w}_1 - \bar{w}_2)^3} \]

\[ \beta_3 = \frac{3 \bar{w}_3}{(\bar{w}_3 - \bar{w}_1)^3(\bar{w}_2 - \bar{w}_3)^3} \]  
(A.13)

In particular, this implies

\[ y_3 = -\epsilon \frac{1}{B} - \epsilon^2 \frac{2A}{B^3} + \mathcal{O}(\epsilon^3) \]  
(A.14)

so that branch cuts in \( y_3 \) do not appear to second order in \( \epsilon \).

We have verified that the \( \mathcal{O}(\epsilon^2) \) terms of (A.11) will not contribute to the one-instanton correction to the prepotential; the first order solutions suffice.
Appendix B: The Landsteiner-Lopez Curve

The spectral curve proposed by Landsteiner and Lopez [13] for a hypermultiplet in the anti-symmetric representation is

\[ y^3 + \left[ f(x) + 3 \Lambda^{N+2} \right] y^2 + \Lambda^{N+2} \left[ g(x) + 3 \Lambda^{N+2} \right] y + \Lambda^{3N+6} = 0 , \]  \hfill (B.1)

where

\[ f(x) = x^2 \prod_{i=1}^{N} (x - e_i) \]
\[ g(x) = (-1)^{N} x^2 \prod_{i=1}^{N} (x + e_i) \]  \hfill (B.2)

Since \( f(x) = g(-x) \), the curve (B.1) is invariant under the involution [13]

\[
\begin{cases}
y \rightarrow \Lambda^{2N+4}/y \\
x \rightarrow -x
\end{cases}
\]  \hfill (B.3)

Consequently, if \( y(x) \) is a solution of (B.1), then \( \tilde{y}(x) \equiv \Lambda^{2N+4}/y(-x) \) is also a solution.

Solutions of (B.1) in the hyperelliptic expansion introduced in Appendix A are

\[ y_1(x) = -A - r - \frac{L^6(A - r)}{2Br} + \cdots \]
\[ y_2(x) = -A + r + \frac{L^6(A + r)}{2Br} + \cdots \]
\[ y_3(x) = -\frac{L^6}{B} + \cdots \]  \hfill (B.4)

where

\[ L^2 = \Lambda^{N+2} \]
\[ A = \frac{1}{2} f(x) + \frac{3}{2} L^2 \]
\[ B = L^2 g(x) + 3L^4 \]
\[ r = \sqrt{A^2 - B} \]  \hfill (B.5)
It may be verified to the order we are working that the involution \( (B.3) \) permutes these solutions as follows:

\[
\begin{align*}
\tilde{y}_1(x) &= y_3(x) \\
\tilde{y}_2(x) &= y_2(x) \\
\tilde{y}_3(x) &= y_1(x)
\end{align*}
\]  \hspace{1cm} (B.6)

where \( \tilde{y}_i(x) \equiv L^4/y_i(-x) \).

From (B.6), as well as explicit analysis, one deduces the following structure for the three-fold branched covering of the sphere.

1) Sheets corresponding to \( y_1 \) and \( y_2 \) are connected by \( N \) square-root branch-cuts centered about \( x = e_i \) \((i = 1 \text{ to } N)\).

2) Sheets corresponding to \( y_2 \) and \( y_3 \) are connected by \( N \) square-root branch-cuts centered about \( x = -e_i \) \((i = 1 \text{ to } N)\).

3) From (B.1) and (B.2), sheets \( y_1, y_2, y_3 \) coincide at \( x = 0 \), where

\[
y_{1,2,3} = -L^2 + \mathcal{O}(x). \hspace{1cm} (B.7)
\]

There are, however, no branch-cuts at \( x = 0 \).

This three-sheeted branched covering of the sphere is a Riemann surface of genus \( 2N - 2 \).

From (B.7), one sees that the hyperelliptic perturbation theory breaks down for \( x \ll L \). This does not, however, change the dual order parameters \( a_{D,k} \) or, therefore, the prepotential.
Appendix C: The Seiberg-Witten Differential

The Seiberg–Witten (SW) differential for the curve (2.8) is

$$\lambda = \frac{dy}{y}$$ \hspace{1cm} (C.1)

which takes a different value on each of the sheets labeled by the solutions (2.13). To the order we are working, sheet 3 is disconnected from sheets 1 and 2, so we only consider $y_1$ and $y_2$. We write these solutions in the hyperelliptic expansion (2.13) as

$$y_1 = -(A + r) \left[ 1 + \frac{L^6(A - r)}{2Br(A + r)} + \ldots \right]$$

$$= -(A + r) \left[ 1 + \frac{L^6(2A^2 - B - 2Ar)}{2B^2r} + \ldots \right]$$ \hspace{1cm} (C.2)

with $y_2$ obtained from this by letting $r \to -r$. The SW differential on sheet 1 is

$$\lambda_1 = x \frac{dy_1}{y_1} = \frac{x}{A + r} d(A + r) + x \cdot d \left[ \frac{L^6(2A^2 - B - 2Ar)}{2B^2r} \right] + \ldots$$ \hspace{1cm} (C.3)

where the first term is

$$\lambda_I = \frac{x}{A + r} d(A + r)$$

$$\simeq \frac{x}{r} \left( A' - \frac{AB'}{2B} \right) dx$$

$$= \frac{x \left( \frac{A'}{A} - \frac{B'}{2B} \right)}{\sqrt{1 - \frac{B}{A}}} dx$$ \hspace{1cm} (C.4)

the usual hyperelliptic form of the SW differential, while the second term is the correction to the differential

$$\lambda_{II} = x \cdot d \left[ \frac{L^6(2A^2 - B - 2Ar)}{2B^2r} \right]$$

$$\simeq - \left[ \frac{L^6(2A^2 - B)}{2B^2r} \right] dx$$

$$= - \frac{L^6 \left( A - \frac{B}{2A} \right)}{B^2 \sqrt{1 - \frac{B}{A}}} dx$$ \hspace{1cm} (C.5)
where $\simeq$ in both (C.4) and (C.5) means we have dropped terms that do not contribute to period integrals around the $A$ or $B$ cycles. The SW differential on the second sheet, $\lambda_2$, is obtained by taking the negative sign for the square root in both (C.4) and (C.5).

For the Landsteiner-Lopez curve, we have (2.10)

\[
A(x) = C(x) + O(L^2)
\]
\[
B(x) = L^2 D(x) + O(L^4)
\]

so the $O(L^2)$ correction to the SW differential becomes simply

\[
\lambda_{II} = -L^2 \frac{C}{D^2} \, dx
\]

Appendix D: Identities

In this Appendix we derive three identities used in the paper, namely

\[
\sum_{k=1}^{N} e_k R_k(e_k) + \frac{3}{4} S_0(0) = 0
\]

(D.1)

\[
\sum_{k=1}^{N} R_k(e_k) + \frac{1}{4} \frac{\partial S_0}{\partial x}(0) = 0
\]

(D.2)

and

\[
\sum_{k=1}^{N} \left[ R_k(e_k) - \frac{1}{4} \frac{\partial S_k}{\partial x}(e_k) \right] = 0.
\]

(D.3)

These identities allow us to eliminate all reference to the residue function $R_k$ in the final expressions for the prepotential. Also recall that by (4.2)

\[
a_k = e_k + L^2 \left[ \frac{1}{4} \frac{\partial S_k}{\partial x}(e_k) - R_k(e_k) \right] + \ldots
\]

(D.4)

so (D.3) implies that

\[
\sum_{k=1}^{N} a_k = \sum_{k=1}^{N} e_k
\]

(D.5)

to $O(L^2)$. 

31
To prove the identities above, we define

\[ F(x) = \frac{3}{x \prod_{i=1}^{N} (x - e_i)}. \]  

(D.6)

Then

\[ (x - e_k)F(x) = \frac{3}{x \prod_{i \neq k} (x - e_i)}. \]  

(D.7)

and

\[ \sum_{k=1}^{N} [(x - e_k)F(x)]_{x=e_k} = \sum_{k} \frac{3}{e_k \prod_{i \neq k} (e_k - e_i)} = \sum_{k} e_k R_k(e_k). \]  

(D.8)

Also

\[ [xF(x)]_{x=0} = \frac{3(-1)^N}{\prod_{i=1}^{N} e_i} = \frac{3}{4} S_0(0). \]  

(D.9)

Thus

\[ \sum_{k} e_k R_k(e_k) + \frac{3}{4} S_0(0) = \sum_{k=1}^{N} [(x - e_k)F(x)]_{x=e_k} + [xF(x)]_{x=0} \]  

(D.10)

The right hand side is the sum of the residues of \( F(x) \). This vanishes, since \( F(x) \) has no poles at infinity, thus proving \( \text{(D.1)} \).

Next define

\[ H(x) = \frac{3}{x^2 \prod_{i=1}^{N} (x - e_i)}. \]  

(D.11)

The sum of residues at \( x = e_k \) is

\[ \sum_{k=1}^{N} [(x - e_k)H(x)]_{x=e_k} = \sum_{k=1}^{N} R_k(e_k) \]  

(D.12)

\( H(x) \) also has a double pole at \( x=0 \), with residue

\[ \left. \frac{\partial}{\partial x} \frac{3}{\prod_{i} (x - e_i)} \right|_{x=0} = - \sum_{k} \left. \frac{3}{(x - e_k) \prod_{i} (x - e_i)} \right|_{x=0} \]

\[ = (-1)^N \sum_{k} \frac{3}{e_k \prod_{i} e_i} \]

\[ = \frac{3}{4} S_0(0) \sum_{k} \frac{1}{e_k}. \]

\[ = \frac{1}{4} \frac{\partial S_0}{\partial x}(0) \]  

(D.13)
where the last step follows directly from (3.4). Thus, the vanishing of the sum of residues of $H(x)$ implies the identity (D.2).

Finally, define

$$K(x) = \frac{D(x)}{C^2(x)} = \frac{4(-1)^N \prod_{i=1}^{N} (x + e_i)}{x^2 \prod_{i=1}^{N} (x - e_i)^2}. \quad (D.14)$$

Then residues of $K(x)$ are given by

$$\text{Res} \ K(x)|_{x=e_k} = \left[ \frac{\partial}{\partial x} S_k(x) \right]_{x=e_k}$$

$$\text{Res} \ K(x)|_{x=0} = \left[ \frac{\partial}{\partial x} S_0(x) \right]_{x=0}. \quad (D.15)$$

The vanishing of the sum of residues of $K(x)$ implies

$$\frac{\partial S_0}{\partial x}(0) + \sum_k \frac{\partial S_k}{\partial x}(e_k) = 0 \quad (D.16)$$

which together with (D.2) implies the identity (D.3).
References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, erratum, ibid B430 (1994) 485 (hep-th/9407087); Nucl. Phys. B431 (1994) 484 (hep-th/9408099).

[2] A. Klemm, W. Lerche, S. Theisen, and S. Yankielowicz, Phys. Lett. B344 (1995) 169 (hep-th/9411048);
P.C. Argyres and A.E. Faraggi, Phys. Rev. Lett. 74 (1995) 3931 (hep-th/9411057);
A. Brandhuber and K. Landsteiner, Phys. Lett. B358 (1995) 73 (hep-th/9507008);
A. Hanany and Y. Oz, Nucl. Phys. B452 (1995) 283 (hep-th/9505073);
A. Hanany, Nucl. Phys. B466 (1996) 85 (hep-th/9509176);
P.C. Argyres and A. Shapere, Nucl. Phys. B461 (1996) 437 (hep-th/9509175);
M.R. Abolhasani, M. Alishahiha, and A.M. Ghezelbash, Nucl. Phys. B480 (1996) 279 (hep-th/9606043);
U.H. Danielsson and B. Sundborg, Phys. Lett. B358 (1995) 273 (hep-th/9504102);
P.C. Argyres and M. Douglas, Nucl. Phys. B448 (1995) 93 (hep-th/9505062);
P.C. Argyres, M.R. Plesser, and A.D. Shapere, Phys. Rev. Lett. 75 (1995) 1699 (hep-th/9505100);
T. Eguchi, K. Hori, K. Ito, and S.-K. Yang, Nucl. Phys. B471 (1996) 430 (hep-th/9603002);
U.H. Danielsson and B. Sundborg, Phys. Lett. B370 (1996) 83 (hep-th/9511180);
P.C. Argyres, M.R. Plesser, and N. Seiberg, Nucl. Phys. B471 (1996) 159 (hep-th/9603042);
J.A. Minahan and D. Nemeshansky, Nucl. Phys. B464 (1996) 3 (hep-th/9507032).

[3] A. Klemm, W. Lerche, and S. Theisen, Int. J. Mod. Phys. A11 (1996) 1929 (hep-th/9506150).
[4] J. Isidro, A. Mukherjee, J. Nunes, and H. Schnitzer, Nucl. Phys. B492 (1997) 647 (hep-th/9609116); Int. Jour. Mod. Phys. A13 (1998) 233 (hep-th/9703176); Nucl. Phys. B502 (1997) 363 (hep-th/9704174).

[5] M. Alishahiha, Phys. Lett. B398 (1997) 100 (hep-th/9609157); (hep-th/9703186).

[6] K. Ito and S.-K. Yang (hep-th/9603073); Phys. Lett. B366 (1996) 165 (hep-th/9507144);
K. Ito and N. Sasakura, Nucl. Phys. B484 (1997) 141 (hep-th/9608054);
S. Ryang, Phys. Lett. B365 (1996) 113 (hep-th/9508163);
A. Bilal (hep-th/9601007);
A. Bilal and F. Ferrari, Nucl. Phys. B480 (1996) 589 (hep-th/9605101); Nucl. Phys. B469 (1996) 387 (hep-th/9606111);
H. Ewen, K. Foerger, and S. Theisen, Nucl. Phys. B485 (1997) 63 (hep-th/9609062);
H. Ewen and K. Foerger, Int. Journ. Mod. Phys. A12 (1997) 475 (hep-th/9610049);
Y. Ohta, Journ. Math. Phys. 37 (1996) 6074 (hep-th/9604051); Journ. Math. Phys. 38 (1997) 687 (hep-th/9604059).

[7] E. D’Hoker, I. Krichever, and D. Phong, Nucl. Phys. B489 (1997) 179 (hep-th/9609041).

[8] E. D’Hoker, I. Krichever, and D. Phong, Nucl. Phys. B489 (1997) 211 (hep-th/9609147).

[9] E. D’Hoker, I. Krichever, and D. Phong, Nucl. Phys. B494 (1997) 89 (hep-th/9610156);
E. D’Hoker and D. Phong, Phys. Lett. B397 (1997) 94 (hep-th/9701053); (hep-th/9709053).

[10] M. Douglas and S.H. Shenker, Nucl. Phys. B447 (1995) 271 (hep-th/9503163).

[11] S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69 (hep-th/9505105);
S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, Nucl. Phys. **B459** (1996) 537 (hep-th/9508155);  
A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. Warner, Nucl. Phys. **B477** (1996) 746 (hep-th/9604034);  
S. Katz, A. Klemm, and C. Vafa, Nucl. Phys. **B497** (1997) 173 (hep-th/9609239);  
M. Bershadsky, V. Sadov, and C. Vafa, Nucl. Phys. **B463** (1996) 420 (hep-th/9511222);  
A. Klemm and P. Mayr, Nucl. Phys. **B469** (1996) 37 (hep-th/9601014);  
S. Katz, D. Morrison, and M. Plesser, Nucl. Phys. **B477** (1996) 105 (hep-th/9601108);  
P. Berghund, S. Katz, A. Klemm, and P. Mayr, Nucl. Phys. **B483** (1997) 209 (hep-th/9605154);  
M. Aganagic and M. Gremm (hep-th/9712011);  
S. Katz, P. Mayr, and C. Vafa, Adv. in Theor. and Math. Phys. **1** (1998) 53 (hep-th/9706110).  
For reviews see: W. Lerche, Nucl. Phys. Proc. Suppl. **55B** (1997) 81 (hep-th/9611191);  
A. Klemm (hep-th/9705131).  

[12] A. Hanany and E. Witten, Nucl. Phys. **B492** (1997) 152 (hep-th/9611230);  
E. Witten, Nucl. Phys. **B500** (1997) 2 (hep-th/9703168);  
K. Landsteiner, E. Lopez, and D. Lowe, Nucl. Phys. **B507** (1997) 197 (hep-th/9705199);  
A. Brandhuber, J. Sonnenschein, S. Theisen, and S. Yankielowicz, Nucl. Phys. **B504** (1997) 175 (hep-th/9705232); (hep-th/9704044);  
N. Evans, C. Johnson, and A. Shapere, Nucl. Phys. **B505** (1997) 251 (hep-th/9703210);  
S. Terashima and S.-K. Yang (hep-th/9803014).
[13] K. Landsteiner and E. Lopez (hep-th/9708118).

[14] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299 (hep-th/9510101); 
E. Martinec and N. Warner, Nucl. Phys. B459 (1996) 97 (hep-th/9509161); 
E. Martinec, Phys. Lett. B367 (1996) 91 (hep-th/9510204).

[15] I. Ennes, S. Naculich, H. Rhedin, and H. Schnitzer, BRX-TH-433, BOW-PH-111, HUTP-98/A39.

[16] T. Masuda and H. Suzuki (hep-th/9609063).

[17] M. Slater (hep-th/9701170); 
N. Dorey, V. Khoze, and M. Mattis, Phys. Rev. D54 (1996) 2921 (hep-th/9603136); Phys. Lett. B388 (1996) 324 (hep-th/9607066); Phys. Rev. D54 (1996) 7832 (hep-th/9607202); 
Phys. Lett. B390 (1997) 205 (hep-th/9606199); 
K. Ito and N. Sasakura, Phys. Lett. B382 (1996) 95 (hep-th/9602073); (hep-th/9609104); 
H. Aoyama, T. Harano, M. Sato, and S. Wada, Phys. Lett. 388 (1996) 331 (hep-th/9607076); 
T. Harano and M. Sato, Nucl. Phys. B484 (1997) 167 (hep-th/9608066); 
Y. Yoshida (hep-th/9610211); 
V. Khoze, M. Mattis, and M. Slater (hep-th/9804009)

[18] K. Landsteiner, J. Pierre, and S. Giddings, Phys. Rev. D55 (1997) 2367 (hep-th/9609059); 
K. Ito, Phys. Lett. B406 (1997) 54 (hep-th/9703180).

[19] A. Ghezelbash (hep-th/9710068)