PATHWISE EXISTENCE OF SOLUTIONS TO THE IMPLICIT EULER METHOD FOR THE STOCHASTIC CAHN-HILLIARD EQUATION

DAISUKE FURIHATA, FREDRIK LINDGREN, AND SHUJI YOSHIKAWA

Abstract. We consider the implicit Euler approximation of the stochastic Cahn-Hilliard equation driven by additive Gaussian noise in a spatial domain with smooth boundary in dimension \( d \leq 3 \). We show pathwise existence and uniqueness of solutions for the method under a restriction on the step size that is independent of the size of the initial value and of the increments of the Wiener process. This result also relaxes the imposed assumption on the time step for the deterministic Cahn-Hilliard equation assumed in earlier existence proofs.

1. Introduction

Let \( D \subset \mathbb{R}^d, \ d \leq 3, \) be a bounded spatial domain with smooth boundary \( \partial D \) and consider the stochastic Cahn-Hilliard equation written in abstract form,

\[
dX + A(AX + f(X)) \, dt = dW(t), \ t \in (0, T], \quad X(0) = X_0,
\]

where \( A \) is the realisation of the Laplace operator \( -\Delta \) with homogenous Dirichlet boundary conditions in \( H = L^2(D) \) with inner product \( \langle \cdot, \cdot \rangle \) and induced norm \( \|\cdot\| \). The non-linearity \( f \) is given by \( f(s) = s^3 - \beta^2 s \) and \( W \) is an \( H \)-valued \( Q \)-Wiener process.

Existence of solutions to (1) is studied in [1] and spatial semi-discretisation with a finite element method in [7] and [9]. Here, we are interested in existence and uniqueness of the implicit Euler approximation of (1) given by

\[
X^j + kA^2X^j + kAf(X^j) = X^{j-1} + \Delta W^j, \ j \in I_N, \quad X^0 = X_0,
\]

where \( I_N = \{1, \ldots, N\}, N \in \mathbb{N}, k = T/N \) and \( \Delta W^j = W(t_j) - W(t_{j-1}) \) for \( t_j = jk, \ j \in I_N \cup \{0\} \). That is, we study a temporal semi-discretisation.

In the deterministic case, when \( W = 0 \), existence is usually proved [3, 11] by the reformulation of (2) as a fixed point problem in a ball \( \{\|A^{1/2}x\| \leq M\} \). If \( k \leq k_0 \) the constructed mapping in the formulation becomes a contraction and existence and uniqueness follows. However, the constant \( M \) grows and \( k_0 \) shrinks as \( \|A^{1/2}X_0\| \) grows.

---

(Furihata and Lindgren) Cybermedia Center, Osaka University 1-32 Machikaneyama, Toyonaka, Osaka 560-0043, Japan

(Yoshikawa) Graduate School of Science and Engineering, Ehime University 3 Bunkyo-cho, Matsuyama, Ehime 790-8577, Japan

E-mail addresses: furihata@cmc.osaka-u.ac.jp, fredrik.lindgren1979@gmail.com, yoshikawa@ehime-u.ac.jp

2010 Mathematics Subject Classification. 60H35, 35R60, 65J15.

Key words and phrases. Stochastic partial differential equation; Cahn-Hilliard equation; Euler method; numerical approximation; existence proof.
In the present setting this dependence cannot be allowed. At every time step, the right hand side of (2) plays the role of the initial value and, being a Gaussian random variable, \( \Delta W^j \) may be arbitrary large with positive probability. If we would rely on earlier existence results we would be forced to utilise an adaptive time stepping scheme and facing the risk of needing arbitrary small time steps. Instead, we shall prove that the equation
\[
(3) \quad u + kA^2u + kAf(u) = y
\]
has a solution in \( H^2 \cap H_0^1 \) as soon as \( y \in \text{Dom}(A^{-1}) \), the domain of \( A^{-1} \). At each time step, \( u \) corresponds to \( X^j \) and \( y \) to \( X^{j-1} + \Delta W^j \) in (2), so for this assumption to hold it is sufficient that \( X_0, \Delta W^j \in \text{Dom}(A^{-1}), j \in I_N, \) a.s. This holds if, e.g., \( \mathbb{E}\|A^{-1}X_0\|^2 < \infty \) and \( \|A^{-1}Q^{1/2}\|_{\text{HS}} < \infty \), where \( \| \cdot \|_{\text{HS}} \) denotes the Hilbert-Schmidt norm in \( H \) and \( Q \) is the covariance operator of \( W \). More precisely, our main results are the following.

**Theorem 1.1.** Assume \( y \in \text{Dom}(A^{-1}) \) and \( k < 4/\beta^4 \), then (3) has a unique solution \( x \in H^2 \cap H_0^1 \).

**Corollary 1.2.** If, a.s., \( X_0, \Delta W^j \in \text{Dom}(A^{-1}), j \in I_N \) and \( k < 4/\beta^4 \), then there is an a.s. unique solution to (2) with \( X^j \in H^2 \cap H_0^1 \) for \( j \in I_N \).

We shall prove the existence part of Theorem 1.1 by applying Schaefer’s fixed point theorem to the mapping \( z = T_y(x) \) given by
\[
(4) \quad z + kA^2z + kAx^2 = y + k\beta^2Ax.
\]
Clearly, a fixed point, \( z = x \), of (4) is a solution of (3).

The outline is as follows. In Section 2 we give some necessary definitions and state Schaefer’s fixed point theorem and other required results. In Section 3 we give the mapping \( T_y \) a rigorous meaning and show that it fulfills the assumptions of Schaefer’s fixed point theorem.

### 2. Preliminaries

We shall use the abbreviation \( L_y = L_y(D) \) for the standard function spaces on \( D \) and \( H^r = H^r(D) \) refers to the usual Sobolev spaces with all partial derivatives of order \( \leq r \) being square integrable. The space \( H_0^1 \) is the completion of \( C_0^\infty(D) \) in \( H^1 \). It hold that the operator \( A \) with \( \text{Dom}(A) = H_0^1 \cap H^2 \) has strictly positive eigenvalues \( \lambda_1 < \lambda_2 < \ldots \) diverging to infinity so any real power \( A^s \) may be defined and \( A^s \) is positive definite and self-adjoint with \( \text{Dom}(A^{s/2}) =: H^s \). If \( s_1 \geq s_2 \) then \( H^{s_1} \subset H^{s_2} \) and
\[
(5) \quad \|A^{s/2}x\| \leq \lambda_1^{(s_1-s_2)/2}\|A^{s_1/2}x\|.
\]
In particular, \( H_0^1 = H^1 \) and \( H^{-1} := (H_0^1)^* = H^{-1} \). The space \( H_0^1 \) is a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_1 := \langle A^{1/2}, A^{1/2} \cdot \rangle \). More generally, we have the family of inner products \( \langle \cdot, \cdot \rangle_s := \langle A^{s/2}, A^{s/2} \cdot \rangle \) and induced norms \( \| \cdot \|_s = \langle \cdot, \cdot \rangle_1^{1/2} \) on \( H^s \). We shall use \( \langle \cdot, \cdot \rangle \) also for the duality pairing of \( H^s \) and \( H^{-s} \).

We will frequently utilise the embeddings \( H_0^1 \subset L_6 \subset L_3 \) with
\[
(6) \quad c\|u\|_{L_3} \leq \|u\|_{L_6} \leq C\|u\|_1,
\]
and the resulting inequality
\[
(7) \quad \|u\|_{-1} \leq C\|u\|_{L_6/5}.
\]
Theorem 2.1. The third, (10), is a consequence of (7) and Hölder’s inequality. The proof in our case is almost identical. We also have
\[ (8) \]
\[ k \]
In particular, (12) has a unique weak solution \( w \) if \( x \in L_3 \) and \( y \in \dot{H}^{-3/2} \). This is of the form \( B_x(z,v) = L_{y,x}(v) \) where \( B_x \) is an inner product and \( L_{y,x} \) is a bounded linear functional on \( H^1_0 \) if \( x \in L_3 \) and \( y \in \dot{H}^{-3} \). That \( B_x \) has the claimed domain follows from (8) and that \( H^1_0 \subset L_6 \subset L_3 \). From (8) and (6), we get that \( k\|u\|^2 \leq B_x(u,u) \leq (\lambda^{-2} + k + C\|x\|^2) \|u\|^2 \). Thus, \( (H^1_0, B_x) \) is a Hilbert space. Lemma 3.1 is then immediate from Riesz representation theorem.

Lemma 3.1. If \( x \in L_3 \) and \( y \in \dot{H}^{-3/2} \) then (11) has a unique solution \( z \in H^1_0 \). In particular, \( T_y \) is well defined as a mapping on \( H^1_0 \).

We now let \( z \) be this solution and consider the system of equations
\[ (12) \quad kAw = -z + y \]
\[ (13) \quad Au = w - x^2z + \beta^2x. \]
By standard elliptic theory, (12) has a unique weak solution \( w \in H \) as soon as \( y \in \dot{H}^{-2} \). We then get a unique weak solution \( u \in H^1_0 \) to (13) if also \( x \in L_3 \). We leave to the reader to check that \( u = z \). From (13) and (4), Theorem 4, Section 6.3]
\[ (14) \quad \|z\|_{H^2} \leq C\|w - x^2z + \beta^2x\| \leq C (\|w\| + \|x^2z\| + \|x\|). \]
Taking \( v = z \) in (11), using the positivity of the third term in the left hand side, the self-adjointness of \( A^{1/2} \) and Hölder’s and Cauchy’s inequalities we compute
\[ \|z\|^2_1 + k\|z\|^2 + k\|xz\|^2 = k\beta^2(x,z) + (A^{-1}y,z) \]
\[ = k\beta^2(A^{-1/2}x, A^{1/2}z) + (A^{-3/2}y, A^{1/2}z) \leq Ck\|x\|^2_1 + C \left( k\|x\|^2_1 + k^{-1}\|y\|^2_3 \right) \]
whith \( C = C_\epsilon \). Clearly, there is an \( \epsilon > 0 \) such that
\[ (15) \quad \|z\|^2_1 + k\|z\|^2 \leq C \left( k\|x\|^2_1 + k^{-1}\|A^{-3/2}y\|^2 \right). \]
It follows from (12), the properties of $A$ and (15) that
\[
k\|w\| \leq \|A^{-1}z\| + \|A^{-1}y\| \leq \lambda_1^{-1/2}\|z\|_{-1} + \|A^{-1}y\|
\]
(16)
\[
\leq C \left( k^{1/2}\|x\|_{-1} + k^{-1/2}\|A^{-3/2}y\| + \|A^{-1}y\| \right).
\]
From (9), (6) and (15) we also get
\[
\|z\|_{H^2} \leq C \left( k^{-1/2}\|x\|_{-1} + k^{-3/2}\|A^{-3/2}y\| + k^{-1}\|A^{-1}y\| + \|x\|_{L_1}^2 \right)
\]
(17)
\[
+ \|x\|_{L_1}^2\left( \|x\|_{-1} + k^{-1}\|A^{-3/2}y\| \right) \leq C_k \left( \|x\|_{L_1}^2 + \|A^{-1}y\|_{L_1}^2 \right).
\]
Compactness of $T_y$ then follows from Kondrachov-Rellich’s compactness theorem [4, Theorem 1, Section 5.7]. We have the following lemma.

**Lemma 3.2.** If $y \in \dot{H}^{-2}$, then $T_y$ is a compact mapping from $H_0^1$ to $H_0^1$.

We now want to verify that $T_y$ is continuous.

**Lemma 3.3.** The mapping $T_y$ is continuous on $H_0^1$ if $y \in \dot{H}^{-3}$.

**Proof.** Take $x_1$ and $x_2$ in $\dot{H}^1$ and let $z_1 = T_y(x_1)$ and $z_2 = T_y(x_2)$. Consider these equations of the form (11) and subtract the latter from the former, using $v = z_1 - z_2$. We then arrive at
\[
\|z_1 - z_2\|_{-1}^2 + k\|z_1 - z_2\|_{L_1}^2 + k\langle z_1 - z_2, z_1 - z_2 \rangle = \|x_1 - x_2\|_{L_1}^2 + \frac{1}{2}\|z_1 - z_2\|_{L_1}^2
\]
(19)
and thus
\[
\|z_1 - z_2\|_{-1}^2 + k\|z_1 - z_2\|_{L_1}^2 + k\langle z_1 - z_2, z_1 - z_2 \rangle = \|x_1 - x_2\|_{L_1}^2 + \frac{1}{2}\|z_1 - z_2\|_{L_1}^2
\]
(20)
Further, using Hölder’s and Cauchy’s inequalities and (10) we get
\[
|\langle x_1 - x_2, z_2, z_1 - z_2 \rangle| = |\langle A^{-1/2}(x_1 - x_2)z_2, z_1 - z_2 \rangle|
\leq \frac{1}{2}\|\langle x_1 - x_2, z_2, z_1 - z_2 \rangle\|^2_{-1} + \frac{1}{2}\|z_1 - z_2\|_{L_1}^2
\]
\[
\leq C\|x_1 - x_2\|_{L_0}(\|z_2\|_{L_0}^2 + \|x_1\|_{L_0}^2 + \|x_2\|_{L_0}^2) + \frac{1}{2}\|z_1 - z_2\|_{L_1}^2.
\]
Inserting (20) into (19), rearranging and applying (21) we find that
\[
\frac{1}{2}\|z_1 - z_2\|_{-1}^2 + k\|z_1 - z_2\|_{L_1}^2 + k\|z_1(z_1 - z_2)\|^2
\leq C\|x_1 - x_2\|_{L_0}(\|z_2\|_{L_0}^2 + \|x_1\|_{L_0}^2 + \|x_2\|_{L_0}^2) + \frac{1}{2}\|z_1 - z_2\|_{L_1}^2
\]
(22)
Subtracting $k\|z_1 - z_2\|_{L_1}^2$ from both sides and multiplying by 2 we conclude that
\[
k\|z_1(z_1 - z_2)\|^2 \leq C\left( \|x_1 - x_2\|_{L_0}(\|z_2\|_{L_0}^2 + \|x_1\|_{L_0}^2 + \|x_2\|_{L_0}^2) + \frac{1}{2}\|z_1 - z_2\|_{L_1}^2 \right)
\]
after also dropping redundant terms in the left hand side. As \(x_1, x_2\) are in \(H^1_0\) by assumption and \(z_2\) is in \(H^1_0\) by \([15]\), it follows from \([\ref{9}]\) that \(T_y\) is continuous on \(H^1_0\).

\(\square\)

**Lemma 3.4.** Assume that \(4k\beta^4 < 1\) and that \(y \in \text{Dom}(A^{-3/2})\). If \(\zeta \in [0, 1]\) and \(u = \zeta T_y(u)\), then for some \(M > 0\) it must hold that \(\|u\|_1 \leq M\|A^{-3/2}y\|\).

**Proof.** It is trivial for \(\zeta = 0\) so assume \(0 < \zeta \leq 1\) and write \(u = \zeta \frac{y}{\zeta} = T(y)\) and substitute \(z = \frac{y}{\zeta}\) and \(x = \frac{x}{\zeta}\) in \((11)\) and take \(v = u\). Then,

\[
\frac{1}{\zeta} (\|u\|_1^2 - 1 + k\|u\|_1^2 + k\|u\|_4^2) = \langle y, u \rangle - 1 + k\beta^2\|u\|^2.
\]

After multiplication with \(\zeta\) and similar arguments as above we get

\[
\|u\|_1^2 + k\|u\|_1^2 \leq C \epsilon^{2/3} \|y\|^2_3 + \epsilon\zeta^2\|u\|_1^2 + \|u\|_1^2 + \frac{(\zeta k)^2\beta^4}{4}\|u\|^2_4 := G(\zeta).
\]

It holds that \(\sup_{0 < \zeta \leq 1} G(\zeta) = G(1)\) so with \(\zeta = 1\) we see that under the assumption on \(k\) we may pick \(0 < \epsilon < k(1 - k\beta^4/4)\) to achieve the desired result. \(\square\)

**Proof of Theorem 1.1.** Existence in \(H^1_0\) follows immediately from Lemmata 3.2 – 3.4 and uniqueness from \((22)\). That the solution is in \(H^2\) is a result of \((18)\). \(\square\)

4. Extensions and future work

The method above generalises to e.g. homogeneous Neumann boundary conditions as in \([3]\) and to arbitrary odd order polynomial \(f\) with positive leading coefficient, cf. \([1]\), but the target non-linearity in the Cahn-Hilliard context, the logarithmic potential \(f(s) = \log((1 + s)/(1 - s)) - \beta^2 s\) remains a challenge.

Error analysis for the stochastic Cahn-Hilliard equation is performed in \([5]\). A proof of strong convergence inspired by \([8]\), where the stochastic Allen-Cahn (SAC) equation is treated, is given. To show the rate of convergence remains a challenge (see \([6]\) for the SAC equation). So does fully discrete schemes.

A drawback with the proof in this paper is that it does not come with a constructive algorithm to find a solution. When Banach’s fixed point theorem is utilised fixed point iteration comes for free. With Schaefer’s fixed point theorem this is no longer the case and a numerical method must be given and analysed.

**Acknowledgements**

F. Lindgren was supported by JSPS KAKENHI Grant Number 15K45678.

**References**

[1] G. Da Prato and A. Debussche. “Stochastic Cahn-Hilliard equation”. In: *Nonlinear Anal.* 26 (1996), pp. 241–263. doi: 10.1016/0362-546X(94)00277-0.

[2] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Vol. 44. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 1992, pp. xviii+454.

[3] C. M. Elliott and S. Larsson. “Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation”. In: *Math. Comp.* 58 (1992), 603–630, S33–S36. doi: 10.2307/2153205.
[4] L. C. Evans. *Partial differential equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. doi: [10.1090/gsm/019].

[5] D. Furihata et al. “Strong convergence of numerical approximation of the stochastic Cahn-Hilliard equation”. In preparation. 2016.

[6] M. Kovács, S. Larsson, and F. Lindgren. “On the discretization in time of the stochastic Allen-Cahn equation”. In: *ArXiv e-prints* (Oct. 2015). arXiv: [1510.03684](http://arxiv.org/abs/1510.03684) (math.NA).

[7] M. Kovács, S. Larsson, and A. Mesforush. “Finite element approximation of the Cahn-Hilliard-Cook equation”. In: *SIAM J. Numer. Anal.* 49 (2011), pp. 2407–2429. doi: [10.1137/110828150](http://dx.doi.org/10.1137/110828150).

[8] M. Kovács, S. Larsson, and F. Lindgren. “On the backward Euler approximation of the stochastic Allen-Cahn equation”. In: *J. Appl. Probab.* 52.2 (2015), pp. 323–338. doi: [10.1239/jap/1437658601](http://dx.doi.org/10.1239/jap/1437658601).

[9] M. Kovács, S. Larsson, and A. Mesforush. “Erratum: Finite element approximation of the Cahn-Hilliard-Cook equation [MR2854602]”. In: *SIAM J. Numer. Anal.* 52.5 (2014), pp. 2594–2597. doi: [10.1137/140968161](http://dx.doi.org/10.1137/140968161).

[10] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations*. Vol. 1905. Lecture Notes in Mathematics. Berlin: Springer, 2007, pp. vi+144.

[11] S. Yoshikawa. “Energy method for structure-preserving finite difference schemes and some properties of difference quotient”. Submitted. 2015.