From the Photon to Maxwell Equation.
Ponderations on the Concept of Photon
Localizability and Photon Trajectory in a de
Broglie-Bohm Interpretation of Quantum
Mechanics

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Abstract

In this paper using the Clifford bundle formalism we show how starting from the photon concept and its relativistic Hamilton-Jacobi equation (HJE) we immediately get (with a simple hypothesis concerning the form of the photon canonical momentum) Maxwell equation (ME) satisfied by a null 2-form field $F$ which is a plane wave solution (PWS) of ME. Moreover, we show how introducing a potential 1-form $A$ such that $F = dA$ we can see how a duality rotation changed in a spatial rotation transformation besides showing how $i = \sqrt{-1}$ enters Maxwell theory thus permitting the writing of a representative of ME as Schrödinger like equation which plays a key role in answering one of the main questions addressed in this paper, namely: is there any sense in talking about photon trajectories in de Broglie-Bohm like theories? To this end we investigate first the nature of the energy-momentum extensor field of the Maxwell field $T(n)$ in some special situations showing that for some of those cases $T_0 = T(\gamma_0)$ even cannot describe the flow of energy. A proposed solution is offered. We also prove that there exists a pulse reshaping phenomenon even in vacuum. Finally we discuss the Schrödinger equation for a photon that follows from quantum field theory and investigate solutions that some authors think imply in photon localization. We discuss if such an idea is meaningful. Moreover, we show that for such solutions it is possible (once we accept that photon wave functions are extended in the space and also in the time domains) to derive a generalized HJE containing a quantum potential and which may lead to non lightlike photon trajectories in free space. We briefly discuss our findings in relation to a recent experiment.
1 Introduction

In [60] we showed that using the Clifford bundle formalism it is possible to obtain a first order HJE directly from the classical relativistic (quadratic) HJE for a charged spin 1/2 particle in interaction with an external electromagnetic field. It is then shown that the first order HJE is equivalent to a Dirac-Hestenes equation satisfied by a special class of Dirac-Hestenes spinor fields (DHSF) that are characterized by having its Takabayashi angle function equal to 0 or π. It is also shown that the Dirac-Hestenes equation satisfied by a general DHSF produces a generalized relativistic Hamilton-Jacobi equation implying (if this is a licit implication) that the particle follows a trajectory resulting from the action of the external electromagnetic field plus the action of a new potential which seems to be the correct relativistic quantum potential. It is very important to observe that all these results have been obtained without the use of the quantum mechanics formalism. Thus, they suggest of course, a de Broglie-Bohm interpretation of the quantum formalism. Some aspects of the results obtained in [60] and criticisms to some attempts to build a de Broglie-Bohm theory for fermions has been discussed in [42].

Motivated from the above results here (with the aid of the Clifford bundle formalism) we show in Section 2 how starting from the photon concept (a zero mass particle) and its classical relativistic HJE we immediately get the free Maxwell equation (ME) once we postulate that $\partial S = \frac{1}{2} F^0 F$ satisfied by a null 2-form field $F = F_0 e^{\gamma^5 S}$, where $S$ is the classical action and $\gamma^5$ is the volume element of Minkowski spacetime. After getting ME, namely $\partial F = 0$, where $\partial$ is the Dirac operator (see Appendix A) we show that this equation automatically produces a conserved energy momentum extensor field $T : \sec \Lambda^1 T^* M \leftrightarrow \sec \mathcal{Cl}(M, g) \ni n \mapsto T(n) \in \sec \Lambda^1 T^* M \leftrightarrow \sec \mathcal{Cl}(M, g)$. The object $T^0 = T(\gamma^0) = -\frac{1}{2} F^0 F$ is one of the (symmetrical) energy-momentum 1-form fields of the Maxwell field which plays a key role in our considerations concerning the issues about if there is any meaning in the concept of photon trajectories, the localizability of these objects and the description of the energy transport and its velocity for an arbitrary electromagnetic field configuration $F$ (solution of ME). In Section 3 we recall the Hertz potential method to generate solutions of the free ME, recall that it it possess besides the well known luminal (also called null fields, since characterized for satisfying $F^2 = 0$) extraordinary free boundary solutions describing hypothetical subluminal and superluminal electromagnetic field configurations (characterized by having $F^2 \neq 0$). In Section 4 we explicitly discuss the case of the superluminal electromagnetic X-wave, and from that solution

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1 It is to emphasized here that the new quantum potential differs from the ones in the original de Broglie-Bohm theories.

2 Our statement does not mean that we endorse this theory as the correct interpretation of the quantum formalism. Indeed, we leave clear here that we think that the only coherent interpretation of quantum phenomena is the one given by quantum field theory.

3 Note that $T(n)$ differs from the canonical energy-momentum extensor of the electromagnetic field, whose expression is recalled in Appendix B.
we construct a finite aperture approximation to this wave, which has finite energy, and for which its front and rear travels with speed $v = c = 1$, whereas its peak travels for a while with superluminal speed. We show that this notable situation occurs due to the reshaping phenomenon (occurring in vacuum) which we carefully discuss. We discuss the nature of $T^0$ for such finite aperture approximation showing that $T^0$ is a timelike 1-form field and thus cannot describe the propagation of the energy. A eventual solution for this dilemma is proposed. We also analyze in Section 5 the nature of $T^0$ for the case of a static electric plus magnetic field configuration where $T^0$ is non null although nothing “material” is seem to be in motion. In Section 6 we show how introducing a potential 1-form $A$ such that $F = dA$ the duality rotation $e^{i\gamma^5}$ metamorphosis in a rotation transformation thus showing how $i = \sqrt{-1}$ enters Maxwell theory. In Section 6 we show also how to trivially obtain within our formalism a Schrödinger form of ME originally discovered by Riemann and rediscovered by many other authors.

Until Section 6 the analysis is almost classical. So, in Section 7 we investigate the quantum Schrödinger equation for the photon which follows from quantum field theory, concentrating analysis on the solution, one given in [3] and the other in [69]. In [3] the author claims to produce a photon localizability better than earlier attempts. We discuss such a claim and explicitly show that this statement if accepted implies in very odd facts concerning the world we live in. Nevertheless if we are prepared to accept such odd facts and if we suppose that it is meaningful to talk about photon trajectories than we can derive a more general HJE for the photon more general than the one that has been used to derive from the photon concept the ME. This new equation implies that photon trajectories are not lightlike geodesics of Minkowski spacetime (they can be timelike) This produces eventually surreal trajectories, and we will study more this issue in another publication. We also recall in Section 7 the focus wave mode representation of the photon described in [69] and comment on the possible photon trajectories associated with this solution. Section 7 ends with pertinent comments concerning some statements in [20] concerning their Bohm like theory of photon trajectories and the results of a recent experiment [36] that claims to observe photon trajectories. Conclusions are in Section 8.

The paper has several appendices. In Appendix A we show how to nicely obtain the “vector form” of Maxwell equations from the single ME written in the Clifford bundle. The importance of this derivation is the obtain a representation of ME that leads directly to the one originally obtained by Riemann. In Appendix B we recall how to obtain within the Clifford bundle formalism the symmetric energy-momentum extensor $T(n)$ and the angular momentum extensor $J^i(n)$ of the electromagnetic field. We also recall the form of the Poincaré invariants and recall that the spin of the electromagnetic field can only be ob-

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4Which can be launched by a special antenna.
5In this paper we use natural units where the speed of light in vacuum $c$ has the numerical value 1. Also, 1 is the numerical value of the Planck constant $\hbar$.
6Such a form is the one used, e.g., in [27] to discuss what these authors think to be the maximum possible localization of photons.
tained form the Lagrangian formalism through the canonical energy-momentum extensor [58].

2 From Photon to Maxwell Equation

In this paper we suppose that all phenomena take place in Minkowski spacetime, i.e., the structure \((M \simeq \mathbb{R}^4, g, D, \tau_g, \uparrow)\), where \((M \simeq \mathbb{R}^4, g, (\tau_g, \uparrow))\) is an oriented Lorentzian manifold (oriented by \(\tau_g\) in \(\mathbb{R}^4\)) and \(D\) is the Levi-Civita connection of \(g\). \(M\), i.e., the structure \((\ldots)\) is the classical Hamilton-Jacobi equation for a free photon. We shall use in this paper only coordinates in Einstein-Lorentz-Poincaré gauge \(\{x^\nu\}\), for which \(g = \eta_{\mu\nu} \gamma^\mu \otimes \gamma^\nu\), with \(\{\gamma^\mu = dx^\mu\}\) the dual basis of \(\{e_\mu = \partial_{x^\mu}\}\). We denote by \(g = \eta^{\mu\nu} \partial_\mu \otimes \partial_\nu\) the metric on \(T^*M\), such that \(\eta^{\mu\nu} \eta_{\mu\nu} = \delta_\nu^\mu\). We use also in the text the cobasis \(\{\gamma_\mu\}\) which is the reciprocal basis of \(\{\gamma^\mu\}\), i.e., \(g(\gamma^\mu, \gamma_\nu) = \delta_\nu^\mu\). Moreover, we suppose that all fields involved are sections of \(\wedge T^*M \hookrightarrow \mathcal{C}(M, g)\), where \(\wedge T^*M = \bigoplus_{i=0}^4 \wedge^i T^*M\) is the bundle of non homogenous differential forms and \(\mathcal{C}(M, g)\) is the Clifford bundle of differential forms. The typical fiber of \(\mathcal{C}(M, g)\) is \(\mathbb{R}_{1,3}\), the so-called spacetime algebra. The even subalgebra of \(\mathcal{C}(M, g)\) is denoted \(\mathcal{C}^0(M, g)\) and its typical fiber \(\mathbb{R}_{3,0}\) the so-called Pauli algebra. Moreover \(\partial = \gamma^\mu \partial_\mu\) is the Dirac operator acting on sections of the Clifford bundle.

So to start we suppose that photons follows in Minkowski spacetime light like curves. Let then be \(\sigma : \mathbb{R} \rightarrow M\), with \(g(\sigma_*, \sigma_*) = 0\) the worldline of a given photon. Now, imagine a vector field \(V \in \sec TM\) such that \(V|_{\sigma_*} = \sigma_*\) and \(g(V, V) = 0\). Moreover write \(p = \alpha \sigma_*\) (with \(\alpha \in \mathbb{R}\), a constant) the momentum of the photon and put \(P = \alpha V\). Also, define the 1-form fields \(V = g(V, \cdot)\) and \(P = g(P, \cdot)\). Of course, \(P\) must be understood in what follows as a density of momentum.

Next we suppose that \(P\) is exact, i.e., there exists a function \(S \in \sec \wedge^0 T^*M \hookrightarrow \sec \mathcal{C}(M, g)\) such that

\[
P := -dS = -\partial S \in \sec \wedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, g).
\]

Of course, we have

\[
g(\partial S, \partial S) = (\partial S)^2 = P^2 = 0. \tag{2}
\]

where

\[
(\partial S)^2 = \partial S \cdot \partial S = 0 \tag{3}
\]

is the classical Hamilton-Jacobi equation for a free photon.

We now introduce a \(\textbf{F} \in \sec \wedge^2 T^*M \hookrightarrow \sec \mathcal{C}(M, g)\) such that

\[
P = \mu \textbf{F} \gamma^0 \textbf{F} \tag{4}
\]

where \(\mu\) is a constant whose nature will be investigated below.

\(^7\)The matrix with entries \(\eta_{\mu\nu}\) is the diagonal matrix \(\text{diag}(1, -1, -1, -1)\).
Now, write
\[ F = F_0 e^{\gamma^5 S} \in \sec \bigwedge^2 T^* M \hookrightarrow \sec \mathcal{Cl}(M, g) \] (5)
where \( F_0 \in \sec \bigwedge^2 T^* M \hookrightarrow \sec \mathcal{Cl}(M, g) \) is a constant biform. Since \( P^2 = 0 \) we need \( F^2 = 0 \) and since it is also \( P = \mu F_0 \gamma^0 F_0 e^{2\gamma^5 S} \) we need also to impose that \( F_0^2 = 0 \).

Now, free photons are supposed to follow null geodesics of the Minkowski spacetime, so this implies that \( P \) is a constant 1-form field.

From Eq. (5) we get
\[ -\gamma^5 \partial F = \partial SF = PF \] (6)
and using Eq. (4) in Eq. (6) with \( F^2 = 0 \) it follows that
\[ \partial F = 0, \]
(7)
i.e., we have proved that under the above assumptions the Hamilton-Jacobi equation for the photon (Eq. (3)) implies the validity of free Maxwell equation for a special kind of fields, the ones where \( F^2 = 0 \) and \( F_0 \) is a constant 2-form field. Also, since
\[ \partial F = \gamma^5 \partial SF \] (8)
we have on multiplying both members by \( \partial S \) and taking into account Eq. (7) that
\[ 0 = -\gamma^5 \partial S \partial F = \partial S \cdot \partial SF \] (9)
i.e., \( \partial S \cdot \partial S = 0 \). Thus we can state that under the above conditions the Hamilton-Jacobi equation for a photon and the free Maxwell equation are equivalent.

Moreover, we have discovered a momentum operator \[ \hat{P} : = -\gamma^5 \partial \]
acting on the sections of the Clifford bundle such that \( \hat{P} F = PF \).

### 2.1 Nature of \( F \) for a Photon Satisfying Eq. (5)

With Eq. (7) the Eq. (6) also implies that
\[ PF = 0 \] (10)

---

8The form of Maxwell equation given by Eq. (7) was first obtained by M. Riez [51] as a simple translation using Clifford algebras of the Maxwell equations written in the vector formalism. It has been divulged by Hestenes in his wonderful book [26] Take notice that here the equation has been obtained from the photon concept.

9In [58] the symbol “” has been used to denote the main involution in Clifford algebras. In this paper a hat over a symbol will denote an operator.
Suppose that the photon moves in the z-direction. Then, if $\omega$ is the frequency of the photon the density of momentum of its “wave” is

$$P = \omega(\gamma^0 + \gamma^3)e^{2\gamma^5 P_{\mu} x^\mu}.$$  \hspace{1cm} (11)

We see that Eq.(10) with a $F$ such $F^2 = 0$ is satisfied if we put, e.g.,

$$F = (\gamma^0 \gamma^1 - \gamma^1 \gamma^3)e^{2\gamma^5 P_{\mu} x^\mu}$$  \hspace{1cm} (12)

i.e., if $F$ is a plane wave with electric and magnetic fields orthogonal to the propagation direction.

Moreover using Eq.(12) in Eq.(4) we get:

$$P = -2\mu(\gamma^0 + \gamma^3)e^{2\gamma^5 P_{\mu} x^\mu}$$  \hspace{1cm} (13)

from where we infer that $\mu = -\frac{1}{2}\omega$.

2.2 The Conserved Energy-Momentum Extensor of the Electromagnetic Field

Let $n$ be a constant 1-form field such that $n^2 = 1$ or $n^2 = -1$. The conserved symmetrical energy-momentum extensor of the electromagnetic field is the mapping (see Appendix B)

$$T : n \mapsto T(n) = -\frac{1}{2}FnF \in \text{sec} \wedge^1 T^* M \mapsto \text{sec} C\ell(M, g)$$  \hspace{1cm} (14)

and we see that

$$P = T(\gamma^0) := T^0 = T^\mu_\mu \gamma^\mu,$$  \hspace{1cm} (15)

i.e., the worldlines of the photons are the integral lines of the vector field $g(T^0, )$ which is as well known the flux of energy-momentum. In classical electromagnetic theory the energy momentum of an electromagnetic field configuration is the 1-form (not a 1-form field)

$$P = \mathcal{E}^\mu P_\mu = \mathcal{E}^\mu \int *T(\gamma_\mu),$$  \hspace{1cm} (16)

which gives divergent values for the $P_\mu$ for a free photon. This shows that although we arrived from the photon concept to the free Maxwell equation, the classical theory cannot deals with the photon concept and as well known a solution requires quantum field where the energy density of a field configuration in a general $n$-photon state is defined by normal order of the creation and destruction operators. We will comment more on this issue below.

10Recall that we’re using natural unities where the numerical values of the speed of light and Planck constant is equal to 1.

11The $\mathcal{E}^\mu$ are 1-forms in a vector space $\mathbb{R}^4$ equipped with a Minkowski scalar product. Details in [59].
3 Extraordinary Solutions of the Free Maxwell Equation

In [60, 42] we arrived at the conclusion that experimental reasons (explanation of the hydrogen atom spectrum) forced us to suppose that the Dirac-Hestenes equation satisfied by general spinor fields is to be taken as a fundamental equation despite the fact that there it was shown that the (linearized) relativistic Hamilton-Jacobi equation is equivalent to the Dirac-Hestenes equation satisfied by a special class of Dirac-Hestenes spinor fields, the ones that have a null Takabayashi angle function.

So, it is appropriate to recall here that long ago it was found that [46, 47, 48, 53, 54, 55, 56] Maxwell equation $\partial F = 0$ as a fundamental equation has extraordinary solutions. Such solutions are characterized by $F \neq 0$ and describe subluminal and superluminal propagating field configurations. We now briefly recall how those extraordinary solutions were found.

The idea [55, 54, 58] is to introduce an object $\Pi \in \text{sec} \bigwedge^2 T^* M \twoheadrightarrow \text{sec} \mathcal{C}(M, g)$, called the Hertz potential, which is supposed to satisfy the wave equation $\partial^2 \Pi = 0$. It generates the electromagnetic potential $A \in \text{sec} \bigwedge^1 T^* M \twoheadrightarrow \text{sec} \mathcal{C}(M, g)$ for which we may evaluate $F = dA$. through the definition

$$A := -\delta \Pi = \partial \cdot \Pi. \quad (17)$$

Before proceeding recall that $\delta A = 0$, i.e., $A$ is a potential (for $F$) in Lorenz gauge.

Now, we have

$$\partial A = (d - \delta)(-\delta \Pi) = dA = F = -d\delta \Pi. \quad (18)$$

Thus,

$$\partial F = -(d - \delta)d\delta \Pi = \delta d\delta \Pi \quad (19)$$

and since

$$\partial^2 \Pi = 0 = -(d\delta + \delta d)\Pi = 0 \quad (20)$$

it follows that

$$d\delta \Pi = -\delta d\Pi \quad (21)$$

Thus substituting Eq. (21) in Eq. (19) we get that

$$\partial F = 0. \quad (22)$$

---

12 In the sense of relativistic quantum mechanics. Of course, as well known, this is still not enough in order to have a coherent theory of interacting charged fermions fields with the quantized electromagnetic field, for which relativistic quantum field theory is necessary.

13 Solutions that have been called in [54] undistorted progressive waves (UPWs). Take notice that all those extraordinary solutions cannot be realized as physical fields. However, finite aperture approximations to these solutions have been launched by special antennas and for a while show peaks travelling at superluminal (or subluminal) speeds while the fronts travel always at the light speed. Thus this phenomena does not implies in any violation of the principle of relativity. All theses issues are discussed with details in the references quoted above.
Now, it is a well fact that the homogeneous wave equation for $\phi \in \sec \bigwedge^0 T^* M \rightarrow \sec \mathcal{O}(M, g)$, i.e.,
\[ \partial^2 \phi = 0 \quad (23) \]
has an infinity number of subluminal and superluminal free boundary solutions \cite{53, 54}. So, if, e.g., $\phi$ is a superluminal solution of Eq. (23) then, e.g., putting $\Pi = \phi \gamma^1 \gamma^2$ it is obvious that $\Pi$ is a superluminal solution of $\partial^2 \Pi$ and thus also a superluminal solution of Maxwell equation.

Remark 1 It is important to emphasize here in order to avoid any possible misunderstanding that most of the extraordinary superluminal and subluminal solutions of Maxwell equations have, as is the case of plane waves of infinity energy besides having no fronts and rears. Moreover, it is a surprising fact that we can solve Maxwell equation with initial conditions for a field configuration moving in the $z$-direction bounded between say $z = -d$ to $z = d$ but with noncompact support in the $xy$ plane such that the front and rear of the field configurations moves at superluminal speed. This result does not violate well known results on the Cauchy problem for hyperbolic differential equations \cite{73} because the initial field configuration is not of compact support. Details may be found in \cite{48}. Also, it is opportune to mention that finite aperture approximations to superluminal (and also subluminal) solutions of Maxwell equation (which, of course, have finite energy) have been launched in several experiments and it has been observed that their peaks travel for a while with superluminal (subluminal) speed. The phenomena ends when the peak meets the front of the wave with always travel at light speed. Some details on this fascinating subject may be found in \cite{46, 56, 47}. An explicit example is given in Section 4.

3.1 $T^0$ Expressed in Terms of the Hertz Potential

Recalling Eq. (13) and Eq. (18) and recalling that the Hertz potential satisfy the wave equation (Eq. (20)) we immediately get
\[ T^0 = -\frac{1}{2} F \gamma^0 F = -\frac{1}{2} d\delta \Pi \gamma^0 d\delta \Pi = -\frac{1}{2} \delta d\Pi \gamma^0 \delta d\Pi \quad (24) \]

Eq. (24) incidentally shows that the Hertz potential cannot be an exact form since it that was the case we would have $T^0 = 0$. Moreover, take into account that form any solution such that any solution of ME such that $F^2 = F \gamma F + F \gamma F \neq 0$ we have (taking into account that $F \gamma F = \gamma^0 L$ where $L$ is a 0-form field that
\[ T_0 \cdot T_0 = \frac{1}{4} [(F \gamma F)^2 - (F \gamma F)^2] > 0, \]
i.e., $T_0$ is a timelike 1-form field.
4 The Superluminal Electromagnetic X-Wave. Pulse Reshaping and the Character of $T^0$

4.1 The X-Wave

Consider first the HWE for $\Phi$ in free space and let $\tilde{\Phi}(\omega, k)$ be the Fourier transform of $\Phi(t, x)$, i.e.,

$$\tilde{\Phi}(\omega, k) = \int_{\mathbb{R}^3} d^3x \int_{-\infty}^{+\infty} dt \Phi(t, x) e^{-i(k\cdot x - \omega t)},$$

(26)

$$\Phi(t, x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3k \int_{-\infty}^{+\infty} d\omega \tilde{\Phi}(\omega, k) e^{i(k\cdot x - \omega t)}.$$  

(27)

Inserting Eq. (27) in the HWE we get

$$(\omega^2 - k^2) \tilde{\Phi}(\omega, k) = 0$$

(28)

and we are going to look for solutions of the HWE and Eq. (28) in the sense of distributions. Then, we rewrite Eq. (28)

$$(\omega^2 - k^2 - \Omega^2) \tilde{\Phi}(\omega, k) = 0.$$  

(29)

where, it is

$$\Omega = k_x e_x + k_y e_y,$$

$$\Omega = |\Omega|^{1/2} = \sqrt{k_x^2 + k_y^2}.$$  

(30)

It is then obvious that any $\tilde{\Phi}(\omega, k)$ of the form

$$\tilde{\Phi}(\omega, k) = \Xi(\Omega, \beta) \delta[\omega - (\beta + \Omega^2/4\beta)] \delta[k_z - (\beta - \Omega^2/4\beta)],$$

(31)

where $\Xi(\Omega, \beta)$ is an arbitrary weighting function is a solution of Eq. (29) since the $\delta$-functions imply that

$$\omega^2 - k^2 = \Omega^2.$$  

(32)

Now, in the dispersion relation given by Eq. (32) define the variables $\overline{k}$ and $\eta$ by

$$k_z = \overline{k}\cos\eta; \quad \cos\eta = k_z/\omega, \quad \omega > 0, \quad -1 < \cos\eta < 1.$$  

(33)

and take moreover

$$\Omega = \overline{k}\sin\eta; \quad \overline{k} > 0 \quad \text{and} \quad \beta = \frac{\overline{k}}{2}(1 + \cos\eta), \quad 0 < \eta < \pi/2.$$  

(34)

Then we have\textsuperscript{14}

\textsuperscript{14}The meaning of the subscript $X$ in Eqs. (35) and (36) will become clear in a while.
\[ \Phi_X(\omega, k) = \Xi_X(\bar{k}, \eta) \delta(k_z - \bar{k}\cos \eta) \delta(\omega - \bar{k}). \tag{35} \]

Now, writing \( \rho = xe_x + ye_y \) and \( \rho = |\rho|^{1/2} \) we have \( \Omega \cdot \rho = \Omega \rho \cos \theta \) and using Eq. (35) in Eq. (27) we have

\[ \Phi_X(t, \mathbf{x}) = \frac{1}{(2\pi)^3} \int_0^\infty d\bar{k} \bar{k} \sin^2 \eta \left[ \int_0^{2\pi} d\theta \Xi_X(\bar{k}, \eta) e^{i\bar{k} \rho \cos \theta} \right] e^{i(\bar{k}\cos \eta z - \bar{k}t)}. \tag{36} \]

Now choose

\[ \Xi(\bar{k}, \eta) = (2\pi)^3 \int_0^\infty d\bar{k} \bar{k} \sin^2 \eta \left[ \int_0^{2\pi} d\theta \Xi_X(\bar{k}, \eta) e^{i\bar{k} \rho \cos \theta} \right] e^{i(\bar{k}\cos \eta z - \bar{k}t)}. \tag{37} \]

where \( z_0 > 0 \) is a constant to get

\[ \Phi_X(t, \mathbf{x}) = \int_0^\infty d\bar{k} \bar{k} \sin^2 \eta \left[ \int_0^{2\pi} d\theta \Xi_X(\bar{k}, \eta) e^{i\bar{k} \rho \cos \theta} \right] e^{i(\bar{k}\cos \eta z - \bar{k}t)}. \tag{38} \]

Calling \( z_0 \sin \eta = a_0 > 0 \), the last equation becomes

\[ \Phi_X(t, \mathbf{x}) = a_0 \int_0^\infty d\bar{k} \bar{k} \sin^2 \eta \left[ \int_0^{2\pi} d\theta \Xi_X(\bar{k}, \eta) e^{i\bar{k} \rho \cos \theta} \right] e^{i(\bar{k}\cos \eta z - \bar{k}t)}. \tag{39} \]

which is the now famous \( X \)-wave solution of the HWE first found (with a different approach in [34]. It is clear that all \( \Phi_X \) propagates with speed \( c_1 = 1/\cos \eta > 1 \) in the \( z \)-direction. This statement is justified for as can be easily seen is no Lorentz frame \( \Phi_X \) is at rest. We can easily construct a superluminal electromagnetic \( X \)-wave using the Hertz potential method. Indeed, following [46] we write the Hertz potential as

\[ H_X = \Phi_X \gamma^{12} \in \sec \Lambda^2 T^* M \rightarrow \sec C \ell(M, g)) \tag{40} \]

The, from Eq. (19) we have with \( \partial_{\mu\nu} := \partial^2 / \partial x^\mu \partial x^\nu \) that

\[ F_X = \partial_{02} \Phi_X \gamma^{01} - \partial_{01} \Phi_X \gamma^{02} + \partial_{32} \Phi_X \gamma^{31} - \partial_{31} \Phi_X \gamma^{32} + (\partial_1^2 + \partial_2^2) \Phi_X \gamma^{12} \tag{41} \]

and since \( F^2 = F \cdot F + F \wedge F \neq 0 \) Eq. (25) says that \( T^0 \) is a timelike 1-form field.

### 4.2 Pulse Reshaping

We now want to solve the following Cauchy problem for the Hertz potential with the following initial conditions
\[ H_X(0, x) = \gamma^{12} \Phi_X = -\sigma^{12} \frac{a_0}{\sqrt{(\rho \sin \eta)^2 + |a_0 - i(z \cos \eta)|^2}} \]

\[ \frac{\partial}{\partial t} H_X(t, x) \bigg|_{t=0} := \frac{\partial}{\partial t} H_X(0, x) = -\sigma^{12} \frac{(a_0^2 - i a_0 \cos \eta)}{[\rho \sin \eta]^2 + |a_0 - i(z \cos \eta)|^2]^{3/2}} \] (42)

As, well known [?] the solution of the proposed Cauchy problem is given by the following formula

\[ H_X(t, x) = -\frac{1}{4\pi} \sigma^{12} \int d^3x \left[ G_R(t', x') \frac{\partial}{\partial t} \Phi_X(t', x') - \Phi_X(t', x') \frac{\partial}{\partial t} G_R(t', x') \right] \bigg|_{t'=0} \] (43)

where

\[ G_R(t - t', x - x') = \frac{\delta(t - t' - R)}{R} \] (44)

\[ R = |x - x'| \] is the retarded Green function and \( R = |x - x'| \).

A simple calculation shows that the integral given by Eq.(43) gives the \( X \)-wave solution of the wave equation given by Eq.(42).

This seems remarkable indeed and at first sight a little bit paradoxical. Indeed, the Green function propagates the field value at any point of the wave in a causal way, i.e., with velocity \( c = 1 \). And yet, e.g., the peak of the \( X \)-wave moves with superluminal velocity. This exercise shows a very important fact. The peaks of the wave at two different instants of time are not causally related! The peak at a given point \( x \) at a given instant of time \( t \) is reconstructed by the points of the wave field which are at a distance \( t - R/c \). This phenomenon is called pulse reshaping (or simply reshaping).

What happens if we cutoff the Cauchy data in Eq.(42) with a window function like \( \Theta(\rho - b) \Theta(|z - \Delta|) \), with \( b, \Delta \in \mathbb{R} \) and \( \Theta \) the Heaviside function? As a tedious calculation can show, such truncated \( F_X \)-wave has finite energy. Then, it may be eventually realized as some real physical phenomenon and indeed experiments realized by [53] shows that this is the case [15].

Now, as well known a classical theorem of the theory of hyperbolic differential equations (see, e.g., [73]) warrants that for an initial field configuration with compact support in space the front and the rear of the wave, localized at \( t = 0 \) respectively at \( z = \Delta \) and \( z = -\Delta \) will travel with maximum velocity \( c = 1 \).

But, does reshaping still occurs in this case, thus allowing the peak of the pulse to travel, at least for a while, with superluminal velocity?

[15] However, take notice that the theoretical analysis of [53] are not correct. A correct analysis is given in [56, 47].

[16] Before proceeding it is important to recall that the lateral spreading of a truncated \( F_X \)-wave happens with a very very small velocity (comparing with the velocity of the front), at least until a distance from the antenna called the depth of the field. For that reason, such waves are (equivocated) called by some authors non-diffracting waves, which of course is not the case, because they spread sensibly after travelling a distance greater than the depth of the field. That such spreading necessarily occurs has been rigorously proved in [15] and it is necessary in order to have energy conservation in certain well defined situations.
The answer is positive. In order to show that statement we need first to recall how we can generate a finite aperture approximation to a finite time pulse $H_X(t,x)$. Let us call such object

\[
H_{fX}(t,x) = -\sigma^{12} \Phi_X(t,x) \Theta(\rho - b) \Theta(|z - \Delta|) = -\sigma^{12} \Phi_{fX}(t,x)
\]

. Now, the finite aperture approximation for the Rayleigh-Sommerfeld of diffraction by a plane screen when $H_X$ is null in all space except on the Hertz radiator in the screen located in the $z = 0$ plane is given by

\[
H_{fX}(t,x,y,z) = -\frac{\sigma^{12}}{2\pi} \int S dS' \left[ \frac{z}{R^2} \frac{\partial}{\partial t'} \Phi_{fX}(t',x',y',0) + \frac{1}{R} \Phi_{fX}(t',x',y',0) \right]_{t' = t - R}
\]

\begin{equation}
(45)
\end{equation}

We see that the peak of $H_{fX}$ at say $z = L$ is produced from rings at the $z = 0$ at different at different retarded times which can be read from Eq. \((45)\) and for a finite time interval $T = 2\Delta$.

One qualitative prediction that can be immediately given in such a case is that the velocity of the peak must slowly as it propagates along the $z$-axis, since it will catch the front which is moving with velocity $c = 1$.

**Remark 2** The experiment described in \([51]\) (and also in \([46]\)) did not look at that effect, since it supposed that the Hertz radiator was on from $-\infty < t < \infty$. However, in that paper it was presented a X-wave solution for Maxwell equations called the SEXW (superluminal electromagnetic X-wave) in \([54]\), and simulations for the motion of finite aperture approximations for that waves (FAASEXW) have been investigated showing that their peaks can propagate (with well designed antennas) for long distances with superluminal velocities.

**Remark 3** Waves of this type (using antennas different from the simple one detailed above) have been produced originally in both the optical \([61]\), as well the microwave regions \([41]\). In that last paper the phenomenon of the slowing of the velocity of the peak was observed. Since the authors of \([41]\) was unaware of the reshaping phenomenon occurring even in vacuum they mislead their readers, claiming to have detected a genuine superluminal motion \([17]\), what has not been the case. We emphasize, what they measured was nothing more than the velocity of the peaks of FAASEXWs while moving along the $z$-axis. The peaks at two different positions (at different times) were not causally connected, something

\[17\text{What has been measured in the experiment \([41]\) was not the velocity of the front of the wave, which must be } c = 1 \text{ according to the the correct theory of the phenomenon. The reason for this claim has to do with the detection threshold of their measurement device. In this respect see \([56, 47]\), where some details of the experiment and its theoretical analysis are presented.}\]
that is obviously if we feed the second member of Eq.\((\ref{eq:42})\) with initial conditions

\[
H_{fX}(0,x) = -\sigma^{12} \Phi X = -\sigma^{12} \frac{a_0 \Theta(\rho - b) \Theta(|z - \Delta|)}{\sqrt{(\rho \sin \eta)^2 + |a_0 - i(z \cos \eta)|^2}},
\]

\[
\frac{\partial}{\partial t} H_{fX}(t,x) \bigg|_{t=0} = \frac{\partial}{\partial t} H_{fX}(0,x) = -\sigma^{12} \frac{(a_0^2 - i2a_0 \cos \eta) \Theta(\rho - b) \Theta(|z - \Delta|)}{[(\rho \sin \eta)^2 + |a_0 - i(z \cos \eta)|^2]^{3/2}},
\]

in order to get \(H_{fX}\) using Eq.\((\ref{eq:43})\).

So, what has been measured in the experiment \([41]\) was not the velocity of the front of the wave, which must be \(c = 1\) according to the correct theory of the phenomenon. The reason for their wrong claim has to do with the detection threshold of their measurement device. In this respect see \([56, 47]\) where some details of the experiment and its theoretical analysis are presented.

Remark 4 It is important to emphasize here that the reshaping phenomena that as we just proved may happen even in vacuum is crucial to explain the existence of superluminal (and even negative) group velocities for electromagnetic waves propagating in a dispersive medium and wave guides.\(^{[18]}\) We refer the reader to the following original papers\([10, 11, 12, 13, 14, 15, 19, 22, 33, 39, 40, 62]\) and for the description of how superluminal (and even negative) group velocities appear\(^{[19]}\), how this is explained by the reshaping phenomenon and how it is possible to prove that even with superluminal velocities the energy in a dispersive medium or wave guide always travel with speed \(v_e \leq 1\).

4.3 Character of \(T_0\) for \(F_{fX}(t,x)\). How Fast Travels the Energy of \(F_{fX}(t,x)\)?

From Eq.\((\ref{eq:31})\) we can easily construct \(F_{fX}(t,x)\) and then verify that

\[
F_{fX}^2 = F_{fX} \cdot F_{fX} + F_{fX} \wedge F_{fX} \neq 0 \tag{47}
\]

implying that energy-momentum 1-form \(T_0 = -1/2 F_{fX} \gamma^0 F_{fX}\) is timelike (as recalled in Section 3). But, of course since the front and the rear of \(F_{fX}(t,x)\) moves with \(c = 1\) it is not reasonable to suppose that the energy flows at

\(^{[18]}\)Like e.g., in the famous Nintz experiment \([44]\) where he claimed to have sent an electromagnetic field configuration through a wave guide with speed greater than the light speed in vacuum. Of course, a careful theoretical analysis of his experiment predicts that he did not measure the velocity of the front of his generated wave. Due to the detection threshold of his device he can only detect the arrival of the peak of his signal which due to the reshaping phenomenon travels at superluminal speed.

\(^{[19]}\)See also \([25]\) for a more recent account of experiments and some theoretical ideas (not all in agreement with our studies on this issue).

\(^{[20]}\)For the case of the tunneling of electrons through a barrier an explanation of the superluminal velocity of the peak of the electron wave function is again the reshaping phenomenon that seems to be universal. A careful analysis is given in \([17]\). These authors show that in this case the integral lines of the Dirac current seems to correspond indeed to the possible electron trajectories.
superluminal speed. Let us them analyze our problem with care and see if we can get a possible solution. To start recall that $T_0$ satisfies the fundamental differential conservation equation

$$
\delta T_0 = 0 \quad (48)
$$

Thus, if we define

$$
T'_0 = T_0 + \delta \mathcal{X} \quad (49)
$$

where $\mathcal{X}$ is a 0-form field then we still have that $\delta T'_0 = 0$;

Thus as well know the energy velocity is supposed to be given by

$$
v_e = \frac{|P|}{u} \quad (50)
$$

where $u$ and $P$ are giving by Eqs. (131), (132) and (133) calculated with $T_0$. It follows from the fact that $T_0$ is timelike $v_e < 1$. Recall now that it is the integral form of the energy-momentum conservation law is

$$
\frac{\partial}{\partial t} \left\{ \int_V d^3 x \frac{1}{2} (E^2 + B^2) \right\} = \oint_S dS \cdot P \quad (51)
$$

that really describes the way in which energy flows. In Eq. (17) it is clear that we can make a choice of $\mathcal{X}$ in Eq. (47) such that the energy density continues to be given by $\frac{1}{2}(E^2 + B^2)$ and such that $P \mapsto P + P'$ with $\nabla \cdot P' = 0$ then there is no alteration of the flow of the energy contained in $V$. We can verify using Eqs. (131), (132) and (133) that we get such a situation if we choose a $\mathcal{X}$ satisfying $\partial_0 \mathcal{X} = 0$ and $\nabla^2 \mathcal{X} = 0$. Indeed, in this case we have the $u$ as in Eq. (17) and $P' = P + \partial \mathcal{X} \sigma_i$. Thus with a thoughtful solution of $\nabla^2 \mathcal{X} = 0$ we can get

$$
v'_e = \frac{|P + P'|}{u} = 1. \quad (52)
$$

**Remark 5** We observe that the $F_{\mathcal{X}}$ that we exhibit above is a classical electromagnetic field configuration, which if described in terms of photons [35] has an undetermined number of them. A such, of course we cannot even in principle to think that for this case the integral lines of $g(T_0, \quad )$ describe any photon trajectory;

### 5 Character of $T^0$ for a Static Electric Plus Magnetic Field Configuration

Consider a spherical conductor in electrostatic equilibrium with uniform superficial charge density (total charge $Q$) and with a dipole magnetic moment of radius $R$. Then, we have (with $e_r := x/|x|$) for $r = |x| > R$

$$
E = \frac{Q}{r^2} e_r \quad ; \quad B = \frac{C}{r^3} (2 \cos \theta \ e_r + \sin \theta \ e_\theta) \quad (53)
$$
which of course, for $r > R$ satisfy the free Maxwell equations. We have

$$ P = E \times B = \frac{CQ}{r^6} \sin \theta \varphi , \quad u = \frac{1}{2} (Q^2 + C^2 (3 \cos^2 \theta + 1)) \tag{54}$$

Thus

$$ \frac{|P|}{u} = \frac{2CQ \sin \theta}{Q^2 + C^2 (3 \cos^2 \theta + 1)} \leq 1 \tag{55}$$

and since the fields are static the conservation law Eq. (51) continues to hold true, and the first member is obviously null and thus for any closed surface containing the spherical conductor we have

$$ \oint_S dS \cdot P = 0. \tag{56}$$

So, we arrive at the conclusion that in static electric plus magnetic field something carrying energy and momentum is in motion around the charged magnetized sphere. If this is true then there must be an angular momentum stocked in the field. And in fact this angular momentum vector is [52]:

$$ \int dV (\mathbf{x} \times \mathbf{P}) = \frac{1}{3} \frac{CQ}{4 \pi R^2} e_z. \tag{57}$$

It has been shown in [64] that if the sphere is slowly discharged (in order to not radiate) through its south pole it acquires mechanical angular momentum exactly equal to the electromagnetic angular momentum [21]. The fact that static electric plus magnetic fields stocked electromagnetic angular momentum has been verified experimentally by Lahoz and Graham in 1989 [31].

With the results of this and the previous section we think it is time to quote Stratton [70] that in section 2.19 of his classical book said:

"By this standard there is every reason to retain the Poynting-Heaviside viewpoint until a clash with new experimental evidence shall call for its revision."

Well, here we present two cases where a revision is necessary. But with this revision the idea of associating photon trajectories to the integral lines of $\mathbf{g}(\mathbf{T}_0, \ )$ becomes of course meaningless. It is necessary to check if association of photon trajectories as in the Bohm like theory presented in [20] can predict correct the photon trajectories for the electromagnetic field configuration $\mathbf{F}_{fX}$. We also suggest here that an experiment should be done to clarify which kind of trajectories can be determined in this case using, e.g., the techniques of the Toronto experiment [36].

[21] A more general and important case showing how to get energy from an external magnetic field in order to put a motor in motion has been discussed in [57].
6 How \( i = \sqrt{-1} \) Enters Maxwell Theory

6.1 Helicity States

Starting from ME the most simple solution are PWS. Choose a solution, moving, e.g., in the \( z \)-direction, given by

\[
F = f e^{r^5 k \cdot x} \tag{58}
\]

with \( k = k_0 \gamma^\mu, \ k_1 = k_2 = 0, \ x = x^\mu \gamma_\mu \), and \( F \) a constant 2-form. It is immediately to conclude that \( kF = 0 \) and \( k \cdot k = 0 \) and then

\[
k \cdot k = 0 \Rightarrow k_0 = \pm |k|. \tag{59}
\]

It is interesting to understand the fundamental role of the volume element \( \gamma^5 \) (duality operator) in electromagnetic theory. In particular since \( e^{r^5 k \cdot x} = \cos k \cdot x - \gamma_5 \sin k \cdot x \), we see that

\[
F = f \cos k x - \gamma_5 f \sin k x. \tag{60}
\]

Writing \( F = E + iB \), with

\[
i \equiv \gamma_5 = e_1 e_2 e_3, \\
e_i = \gamma_i \gamma_0 \tag{61}
\]

and choosing, e.g., \( f = e_1 + ie_2, \ e_1 \cdot e_2 = 0, \ e_1, \ e_2 \) constant 2-forms to be treated in calculations below (with the appropriate symbols) as 1-forms in the Pauli subbundle \( \mathcal{Cl}^0(M, g) \) of \( \mathcal{Cl}(M, g) \)

\[
(e + ib) = e_1 \cos k \cdot x + e_2 \sin k x + i(-e_1 \sin k \cdot x + e_2 \cos k \cdot x). \tag{62}
\]

This equation is important because it shows that we must take care with the \( i = \sqrt{-1} \) that appears in usual formulations of Maxwell Theory using complex electric and magnetic fields. The \( i = \sqrt{-1} \) in many cases unfolds a secret that can only be known through Eq.(62). From the fact that \( kF = 0 \) we can also easily show (see below) that \( k \cdot E = k \cdot B = 0 \), i.e., PWS of ME are transverse waves. From ME satisfied by \( F \) we have immediately that

\[
k_0 F = kF.
\]

On the other hand we can write \( kF = 0 \) as

\[
k \gamma_0 \gamma_0 F \gamma_0 = 0 \tag{63}
\]

and since \( k \gamma_0 = k_0 + k \), defining \( F^* = \gamma_0 F \gamma_0 = -E + iB \) we have

\[
k_0 F^* = -kF^*. \tag{64}
\]
So, we see that \( * \) plays the role of the operator of space conjugation \( [26] \). Of course, for \( f = e + ib \) we have

\[
f^* = -e + ib; \quad k_0^* = k_0; \quad k^* = -k.
\]

We can now interpret the two solutions of \( k^2 = 0 \), i.e., \( k_0 = |k| \) and \( k_0 = -|k| \) as corresponding to the solutions

\[
(a) \ k_0 f = kf, \quad (b) \ k_0 f^* = -kf^*
\]

where \( f \) and \( f^* \) correspond in quantum theory to “photons” which are of positive or negative helicities. This will be important for our discussion in Section 4.

Observe that from Eq. (66(a)) we can write

\[
k_0(e + ib) = k(e + ib) = k_0 e + k \wedge e + k \cdot (ib) + k \wedge (ib)
\]

and comparing the grades in both members, taking into account that \( k \wedge (ib) = i(k \cdot b) \) and \( k \cdot (ib) = i(k \wedge b) \) we get that

\[
k_0 e = k_0 b = 0
\]

and

\[
k_0 (e + ib) = k \wedge e + i(k \wedge b)
\]

and interpreting the objects in Eq. (69) as objects with values in the Pauli algebra it is \( k \times e := -i(k \wedge e) \) and we can write

\[
-ik_0 f = (k \times e) + i(k \times b) = k \times f.
\]

**Remark 6** In what follows we will impose that

\[
|f|^2 = f \cdot f^* = 1
\]

and write

\[
f_+ := f, \quad f_- := f^*
\]

corresponding respectively to right and left hand photons.

### 6.2 How a Duality Rotations Turns Up in a Spatial Rotation

The plane wave solutions (PWS) of ME are usually obtained by looking for solutions to these equations such that the potential \( A = A_\mu \gamma^\mu \) is in the Coulomb (or radiation) gauge, i.e.,

\[
A_0 = 0, \quad \partial_i A^i := \nabla \cdot A = 0
\]

Eq. (73), of course, implies that \( \partial \cdot A = -\delta A = 0 \), i.e., the Lorenz gauge condition is also satisfied. Now, in the absence of sources, \( A \) and \( F = \partial \wedge A = \)
$dA$ satisfy, respectively the homogeneous wave equation (HWE) and the free ME

$$\Box A = 0, \quad (74)$$
$$\partial F = 0 \quad (75)$$

The PWS can be obtained directly from the free ME ($\partial F = 0$) once we suppose that $F$ is a null field, i.e., $F^2 = 0$. However, for the purposes we have in mind we think more interesting to find these solutions by solving $\Box A = 0$ with the subsidiary condition given by Eq. (73).

In order to do that we introduce besides $\{x^\mu\}$ another set of coordinate $\{x^\nu\}$ also in the Einstein-Lorentz-Poincaré gauge which are also a $(n\mathcal{A}cs|\partial/\partial x^0)$, such that, $x^0 = x^0, x^i = R^i_j x^j$. Putting $\varepsilon^\mu = dx^\mu$ we then have

$$\varepsilon^\mu = R^\mu_\nu \bar{R}; \quad \varepsilon^0 = \gamma^0, \quad \varepsilon^i \neq \gamma^i, \quad (76)$$

where the (constant) $R \in \sec SP_{\text{In}(M)}$ generates a global rotation of the space axes. We also write

$$\varepsilon_i \varepsilon_j \equiv \varepsilon_{ij}, \quad \varepsilon_i \varepsilon_0 = e_i, \quad e_i e_j \equiv \varepsilon_{ij} = -\varepsilon_{ij}, \quad i = e_1 e_2 e_3, \quad i, j = 1, 2, 3 \text{ and } i \neq j. \quad (77)$$

We consider next the following two linearly independent monochromatic plane solutions $A^{(i)}$, $i = 1, 2$, of Eq. (74) satisfying the subsidiary condition giving by Eq. (73) and moving in the $e_3$ direction,

$$A^{(i)} = \exp \left[ \frac{(-1)^{i+1} \varepsilon_{12} \bar{\phi}_i}{2} \right] \varepsilon_1 \exp \left[ \frac{(-1)^{i+1} \varepsilon_{12} \bar{\phi}_i}{2} \right] = \exp \left[ \varepsilon_{12} (-1)^{i+1} \phi_i \right] e_1,$$

$$\bar{\phi}_i : M \to \mathbb{R}, \varepsilon \mapsto \bar{\phi}_i(\varepsilon) = k'_\mu x'^\mu + \bar{\varphi}_i = \omega T - k' \cdot x' + \bar{\varphi}_i, \omega = |k'|,$$

$$\mathbf{k}' = \omega e_3. \quad (78)$$

where the $\bar{\varphi}_i$ are real constants, called the initial phase. Since $A^{(i)}_0 = 0$ we write,

$$A^{(i)} = A^{(i)} \varepsilon_0 = \exp \left[ (-1)^{i+1} e_2 e_1 \bar{\phi}_i \right] e_1 = e_1 \exp \left[ (-1)^{i+1} e_2 e_1 \bar{\phi}_i \right], \quad (79)$$

Now,

$$F^{(i)} = \partial \wedge A^{(i)} = \partial \wedge A^{(i)} = \partial \varepsilon_0 e_0 A^{(i)}$$

$$= (\partial_t - \nabla)(-A^{(i)}). \quad (80)$$

We calculate in details $F^{(1)}$ in order for the reader to see explicitly how $i = e_1 e_2 e_3$ enters in the classical formulation of the electromagnetic field. We have,

$$F^{(1)} = -\omega \left[ e_1 e_2 e_1 \exp(e_{21} \bar{\phi}_1) - e_3 e_1 e_2 e_1 \exp(e_{21} \bar{\phi}_1) \right]$$

$$= \omega \left[ e_2 - ie_1 \right] \exp(-e_2 e_1 \bar{\phi}_1)$$

$$= \omega \left[ e_1 - ie_2 \right] \exp(-e_1 e_2 \bar{\phi}_1)$$

$$\phi_1 = \omega t - \mathbf{k}' \cdot x' + \varphi_1, \quad \varphi_1 = \bar{\varphi}_1 - \frac{\pi}{2} \quad (81)$$
This formula shows how a duality rotation originally appearing in the formula for $F$ turns up into spatial rotation in that formula a really non trivial result. Besides that, Eq. (81) shows that representation of electromagnetic fields through complex fields is a simple representation of what has been just found.

6.3 Schrödinger Form of Maxwell Equation

Here we derive a three dimensional representation of the free ME, first presented by Riemann\(^{22}\) and rediscovered by Silberstein\(^{65}\), Bateman\(^2\), Majorana\(^23\) but obtained in a completely different way from the one given below. Our starting point is Maxwell equation written in the Clifford bundle, i.e., \(\partial F = 0\) which since it is also satisfied by \(\gamma_5 F\) can be rewritten (in the case \(J = 0\)) in the following equivalent ways in the even subbundle \(Cl^0(M, \eta)\) of \(Cl(M, \eta)\) once we write \(\sigma^i = \gamma^i \gamma^0\) and \(\sigma^0 = 1\)

\[
\partial (\gamma_5 F) = 0, \\
i\sigma^\mu \partial_\mu F = 0, \\
(\frac{i\sigma^0}{2}) \frac{\partial}{\partial T} F = -\frac{i}{2} \sigma^i \partial_i F. 
\]

Now, we recall that \([\sigma^i/2, \sigma^j/2] = i\epsilon^{ijk} \sigma^k/2\), i.e., the set \(\{\sigma^i/2\}\) is a basis for any \(e \in M\) of the Lie algebra su(2) of SU(2), the universal covering group of SO\(_3\), the special rotation group in three dimensions. A three dimensional representation of su(2) is given by the Hermitian matrices

\[
\hat{\Sigma}^p = \begin{bmatrix}
0 & -i\delta^p_3 & i\delta^p_2 \\
i\delta^p_3 & 0 & -i\delta^p_1 \\
i\delta^p_2 & i\delta^p_1 & 0
\end{bmatrix}
\]

and

\[
[\hat{\Sigma}^p, \hat{\Sigma}^q] = i\epsilon^{pqr} \hat{\Sigma}^r.
\]

Writing moreover \(\hat{\Sigma}^0 = I_3\) for the three dimensional unitary matrix and defining

\[
F = \begin{bmatrix}
E_1 + iB_1 \\
E_2 + iB_2 \\
E_3 + iB_3
\end{bmatrix},
\]

we can obtain ME in three dimensional form with the substitutions

\[
\frac{1}{2} \sigma^\mu \mapsto \hat{\Sigma}^\mu, \quad i \mapsto i = \sqrt{-1}, \quad F \mapsto F.
\]

in Eq. (82). We get

\[
i \frac{\partial}{\partial T} F = -i \hat{\Sigma} \cdot \nabla F
\]

\(^{22}\)See details in \[74\].
\(^{23}\)See the article \[37\].
Note the doubling of the representative of the unity element of $\mathcal{C}L^0(M, \eta)$ when
go to the three dimensional representation. This corresponds to the fact that in relativistic quantum field theory, the $2 \times 2$ matrix representation of Eq. (82) (projected in the idempotent $\frac{1}{2}(1 + \sigma^3)$) represents the wave equation for a single quantum of a massless spin 1/2 field, whereas Eq. (87) represents the wave equation for a single quantum of a massless spin 1 field.

**Remark 7** Eq. (87) has been used in several papers that discusses the possibility of writing photon wave functions [3] and the question of the maximum localizability of photons, e.g., in [1, 27, 63]. We are not going to discuss these papers here, for instead we shall present an alternative path to the photon wave function based on a paper by Bialynicki-Birula [4], but using Maxwell equation in its form given in the Clifford bundle $\mathcal{C}L(M, g)$ by Eq. (7) or its form as given in the Clifford bundle $\mathcal{C}L^0(M, g)$ by Eq. (82).

### 7 Enter the Quantum Schrödinger Equation for the Photon

Once we have discovered that there are two different polarizations associated to a PWS of ME we can write (once we fix an inertial reference frame and use coordinates in ELP gauge) a general solution of Eq. (7) as

$$F(t, x) = \frac{1}{(2\pi)^4} \int d^4k \delta(k^2) f(k) U(k) e^{-\gamma^5 k \cdot x}.$$  (88)

We choose the 2-forms $f$ such that $f(k_0, \pm k) = f_{\pm}(k)$ describe the two possible polarizations discussed above. Thus, we can write

$$F(t, x) = F_+(t, x) + F_-(t, x)$$

$$= \frac{1}{(2\pi)^3} \int d^3k \frac{1}{2\omega_k} \{ f_+(k) u_+(k) + f_-(k) u_-(k) \} e^{-i\omega_k t + k \cdot x}.$$  (89)

with $\omega_k = k_0 = |k|$ and where $u_\pm : \mathbb{R}^3 \rightarrow \mathbb{R} \oplus i\mathbb{R}$ are “complex” functions conveniently chosen and whose nature investigate in what follows. In order to do that we recall once again that the classical energy density is given by

$$\hat{T}_{00}(t, x) = F(t, x)^* \cdot F(t, x) = |F(t, x)|^2 = \frac{1}{2} (E^2 + B^2)$$  (90)

---

24 Of course, it is necessary for the quantum mechanical interpretation to multiply both sides of eq. (87) by $\hbar$, the Planck constant.
25 Indeed in quantum mechanics the Pauli matrices $\sigma_i$ and the matrices $\Sigma_i$ are the quantum mechanical spin operators and

$$\sum_{i=1}^{3} (\sigma_i)^2 = \frac{1}{2} (1 + \frac{1}{2}) = \frac{3}{4}, \sum_{i=1}^{3} (\Sigma_i)^2 = 1.(1 + 1) = 2.$$
and as we already saw, for a monochromatic PWS its energy \( \int d^3x T_{00}(t, x) \) diverges. So, if we are looking at \( \mathbf{F} \) giving by Eq.\( \text{(88)} \) to have finite energy it is necessary to impose that

\[
\int d^3k [ |u_+(k)|^2 + |u_-(k)|^2 ] < \infty. \tag{91}
\]

But, what to do with the energy of a monochromatic photon? As already mentioned in order to give meaning to the energy of a monochromatic photon we need to go to the formalism of quantum field theory where the classical field \( \mathbf{F}(t, x) \mapsto \hat{\mathbf{F}}(t, x) \), an operator valued distribution acting on the appropriated (Fock) Hilbert space \( \mathcal{H} \) for the photon field. As well know this is done by introducing creation and destruction operators for the two kind of polarized photons, i.e., we write \[3\]

\[
\hat{\mathbf{F}}(t, x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \frac{1}{2\omega_k} \left[ f_+(k) a(k) e^{-ik_0 t + k \cdot x} + f_-(k) b^\dagger(k) e^{ik_0 t - k \cdot x} \right]
\]

\[
+ \frac{1}{(2\pi)^{3/2}} \int d^3k \frac{1}{2\omega_k} \left[ f_-(k) b(k) e^{-ik_0 t + k \cdot x} + f_+(k) a^\dagger(k) e^{ik_0 t - k \cdot x} \right] \tag{92}
\]

where \( a^\dagger(k) \) and \( b^\dagger(k) \) are the creation operators corresponding to left and right polarizations and \( a(k) \) and \( b(k) \) the respective annihilation operators. We have

\[
[a(k), a^\dagger(k')] = \delta(k - k'),
\]

\[
[b(k), b^\dagger(k')] = \delta(k - k'), \tag{93}
\]

with all commutators which includes one operator corresponding to left polarization and other corresponding to right polarization being null.

A one photon state \(|\text{ph}\rangle \in \mathcal{H} \) is then written as

\[
|\text{ph}\rangle = \int d^3k (a^\dagger(k) f_+(k) + f_-(k) b^\dagger(k)) |0\rangle \tag{94}
\]

and the the quantum energy density (when Eq.\( \text{(91)} \) is normalized to 1) of the state \(|\text{ph}\rangle \) is given by

\[
\hat{T}^{\eta}_{00}(t, x) = \langle \text{ph}| :\hat{\mathbf{F}}^\dagger(t, x) \hat{\mathbf{F}}(t, x): |\text{ph}\rangle \tag{95}
\]

where as usual :\( \hat{\mathbf{F}}^\dagger(t, x) \hat{\mathbf{F}}(t, x) : \) denotes the normal ordering of the operators. We have

\[
\hat{T}^{\eta}_{00}(t, x) = |\mathbf{F}_+(t, x)|^2 + |\mathbf{F}_-(t, x)|^2 \tag{96}
\]

Since Eq.\( \text{(96)} \) shows \( \mathbf{F}_+ \) and \( \mathbf{F}_- \) contributes independently in what follows we choose to work with only one polarization, and we construct following \[4\] a localized photon wave function using the Hertz potential method. describe in
previous section. Since the Hertz potential $H \in \sec \wedge^2 T^* M$ satisfy the wave equation we write a general free boundary solution as

$$H(t, x) = \frac{1}{(2\pi)^4} \int d^4 k \delta(k^2) h(k) e^{-\gamma k \cdot x}$$  \hspace{1cm} (97)

where $h(k) = h_1 + i h_2$ is an arbitrary 2-form. Thus, it is

$$H(t, x) = \int d^3 k \{ h_+(k) e^{-i(\omega_k t - k \cdot x)} + h^*_-(k) e^{i(\omega_k t - k \cdot x)} \}$$  \hspace{1cm} (98)

and the positive frequency part of $F_+ = -d\delta H_+$ is

$$F_+(t, x) = \int d^3 k \{ i |k| h_+(k) - k \times h_+(k) \} e^{-i(\omega_k t + k \cdot x)}.$$  \hspace{1cm} (99)

With the choice

$$h_+(k) = m \sqrt{|k|} \frac{2}{|k|} e^{-\frac{\sqrt{1+1^2}}{|k|}}$$  \hspace{1cm} (100)

with $m$ a constant 2-form it is

$$H_+(t, x) = m \frac{2\pi^{3/2}}{i r} \left( e^{-2 \sqrt{1+1^2}} + e^{-2 \sqrt{1+1^2}} \right).$$  \hspace{1cm} (101)

Thus, $F_+$ represents the wave function of a photon having the form of a spherical shell converging and diverging respectively for positive and negative values of $t$ and having maximum localizability at $t = 0$. Moreover, $H_+$ and also its time derivative at $t = 0$ goes as $e^{-\sqrt{2r/l}}$ multiplied by some positive power of $r$. This implies that the photon energy density (Eq.(96)) also decreases exponentially.

We can also show that $T_0 = T_0 e^{-\gamma}$ has an exponential decrease.

### 7.1 Can Relativistic Particles be Localized?

Recall that a single relativistic particle is described by a one particle wave function obtained from its standard quantum field. As such it satisfy an hyperbolic equation\textsuperscript{26}. As such, a well result (already quoted \textsuperscript{27}) from the theory of hyperbolic equations establishes that in solving the Cauchy problem (in a well defined inertial reference frame) any solution\textsuperscript{27} resulting from an initial field configuration having compact support in space is such that its front and rear will travel with the light speed $c = 1$. For massive particles this immediately leads to a contradiction with the fundamental assumption of quantum theory that for any time $t$ the wave function describes the probability amplitude for finding the particle at a position $x$ in space\textsuperscript{28}.

\textsuperscript{26}E.g., Klein-Gordon, Dirac and Maxwell equations, respectively for scalar, spin 1/2 fermions and the photon field.

\textsuperscript{27}And in particular for solutions that are elements of of a Hilbert space, as required by quantum theory.

\textsuperscript{28}We could think that for photons such a problem does not exist, but that is not the case, see below.
Indeed, as the particle is supposed to travel at group speed \( v_g = d\omega/d|k| < 1 \) if its probability amplitude is described by a wave of compact support in a while the particle and its wave will uncouples, an absurd.

We can easily prove that localizability for relativistic particles in the sense of \( \delta \)-functions is forbidden for this would result in violation of Einstein causality. See e.g., [72].

More realistic, as proved in a series of papers by Hegerfeldt (see [?] and references therein) the following fundamental result:

**Proposition 8** Any solution of a relativistic wave equation in a Hilbert space \( \mathcal{H} \) supposed to represent the motion of a particle (and written in such a way that its Hamiltonian is restricted to be a self-adjoint operator, positive and bounded from below) is such that if a particle is strictly localized in a bounded region of space \( V \) (as determined in a inertial reference frame) it happens that for any finite time interval thereafter the particle localization develops infinite tails.

This, in particular implies (see, [71]) that, e.g., positive energy-solutions of the Dirac-equation always are noncompact support in space (have infinity support) which must be the case as we showed above.

Hegerfeldt theorem immediately implies that it is impossible to localize photons with compact wave functions and the question arises if it is possible to design localized wave functions for photons which are better localized than the ones reported, e.g., in [1, 43, 63, 27]?

Some people think that an yes can be given to that question once one builds (as it was the case of the above solution) a wave function with an infinity tail that is exponentially localized at \( t = 0 \). However, for the above solution and others that already have been reported (see, e.g., [69]) the following criticisms seems to us to invalidate such a claim.

(i) Take into account that the wave function given by Eq.(99) 29 exists for all time from \( t = -\infty \) up to \( t = +\infty \) and thus can only describe (as it is the case of the PWS that we arrived from the photon concept following a lightlike worldline in Minkowski spacetime) an *eternally* propagating photon.

(ii) Note that it seems physically impossible to create a photon described by a wave function like the one in Eq.(99) since all photons are produced in nature in a *finite time*, say \( t = T \) in any emission process and thus must have a front and a rear.

Hegerfeldt theorem says that such a wave function of compact support (whatever it may be) immediately develops tails in space, but of course, it is hard to suppose that it develops tails in *time* for if this was the case we arrive at the conclusion that the photon wave function start to be create before it was born!

Anyway, some one may claim that this is only a new non intuitive aspect of quantum theory.

---

29 And this is the case also for the solution reported in [69].
7.2 The Quantum Potential

If we can accept that a photon wave function \( F \) (solution of the free ME) may be a function that is extended not only in space but also in time and we moreover think that photons trajectories is a meaningful concept they may be evaluated as follows:

(i) First from \( F \) we evaluate \( T_0 \) and from our fundamental assumption
\[-\partial S = T_0 \]
we have
\[ S = -\int T^0 = \frac{1}{2} \int F \gamma^0 \dot{F} = \frac{1}{2} \int d\delta \Pi \gamma^0 d\delta \Pi \quad (102) \]

(ii) Using Eq. (102) we can write \( F \) as
\[ F = F e^{\gamma^5 S} \in \wedge^2 T^* M \mapsto \mathcal{C}(M, g) \quad (103) \]

where \( F \in \wedge^2 T^* M \mapsto \mathcal{C}(M, g) \) is not of course, for a general solution a constant biform. From ME (Eq. (7)) we have
\[ \partial S F + \gamma^5 \partial \ln F F = 0. \quad (104) \]

Multiplying this equation on the right by \(-1/2\gamma^0 F\) and using Eqs. (1) and (4) we get
\[ \partial S \partial S + \gamma^5 \partial \ln F \partial S = 0 \quad (105) \]

which means that the “generalized” HJE for the photon is
\[ \partial S \cdot \partial S = -\gamma^5 \partial \ln F \partial S = -\langle \gamma^5 \partial \ln F \partial S \rangle_0 \quad (106) \]

and
\[ Q_F = \langle \gamma^5 \partial \ln F \partial S \rangle_0 \quad (107) \]

and of course it must be the case that the following constraints must be satisfied
\[ \langle \gamma^5 \partial \ln F \partial S \rangle_2 = \langle \gamma^5 \partial \ln F \partial S \rangle_4 = 0. \quad (108) \]

**Remark 9** Now, for a given solution such that \( \gamma^5 \partial \ln F \partial S \neq 0 \) in all \( M \) we conclude that \( \partial S \) is not a lightlike vector field, so it has an inverse \( \partial S^{-1} = \partial S/|\partial S|^2 \). Thus, multiplying Eq. (108) on the right by \( \partial S^{-1} \) we get
\[ \partial S + \gamma^5 \partial \ln F = 0 \quad (109) \]

and we may name
\[ Q_F = \gamma^5 \partial \ln F \quad (110) \]

the “linearized” quantum potential;

\( \partial \ln F := \partial F F^{-1} \).
Remark 10 Eq. (106) or its linearized version (Eq. 109) shows that the generalized HJE for the photon implies that the “would be” quantum trajectories are not in general lightlike geodesics of the Minkowski spacetime. At first sight this seems very odd, but a simple look at a double slit interferometric experiment suggests that indeed photons are not travelling in lightlike geodesics. Moreover, observe that in principle $Q_F$ can be positive or or null (in some regions, e.g., besides the double slit screen) implying the existence of timelike and lightlike world lines. The existence of such non intuitive paths for photons may eventually lead one to think that eventually such kind of trajectories may give an answer (at least for the case of photons) for the pertinent analysis of Chen and Kleinert [9] that Bohmian trajectories are deficient in view of the basic quantum mechanics principles.

7.3 The Focus Wave Mode Hertz Potential as a Model of a Photon

Returning to Eq. (31) we can show (see, e.g., [54]) that if we choose

$$
\Xi(\Omega, \beta) = \frac{\pi^2}{12} \exp(-\Omega^2 z_0/4\beta),
$$

we get, assuming $\beta > 0$ and $z_0 > 0$, the following Hertz potential

$$
H_{fwm}(t, x) = \gamma^{21} \Phi_{fwm}(t, x) = \gamma^{21} e^{i\beta(z+t)} \frac{\exp\left(-\rho^2 \beta/|z_0 + i(z-t)|\right)}{4\pi i |z_0 + i(z-t)|}.
$$

where the function $\Phi_{fwm}$ is called the focus wave mode. Function $\Phi_{fwm}$ has very interesting properties, as discussed in details in [69]. For appropriate choice of parameters it may describe a wave moving in the positive $z$-direction with a maximum concentrate in a small region, and that solution is indeed an improvement over the solution reported above (found in [3]). Namely, $\Phi_{fwm}$ is a nondiffracting wave! In this notable paper authors study in detail the diffraction of $\Phi_{fwm}$ in a modelled double slit experiment. They show (with appropriate choice of parameters) that after the screen with the holes the solution of the wave equation generated by the arrival of $\Phi_{fwm}$ continues to be a function with a well localized maximum concentrated in a very small region which then hits the detection screen. Although the classical electromagnetic energy evaluated (trough Eq. (24)) using $H_{fwm}$ diverges, it may be interesting to investigated how we possibly could renormalize such energy function in order for it to represent indeed a photon of finite energy. We will return to this issue in another paper.

31For the case of electrons the real trajectories must be calculated using the new generalized Hamilton-Jacobi equation found in [60, 42].

32According to [5].

33See also the detailed analysis of Jung [30] showing the calculated trajectories by Philippidis, Dewdney and Hiley [49] using the nonrelativistic Bohm theory does not agree with experiment.

34It is a special case of Brittingham’s wave focus modus solutions of the wave equation.
7.4 Some Additional and Pertinent Comments

(i) Despite the facts presented in the last two sections some authors, e.g., Flack and Hiley [20] believe that photon trajectories can only be described by their version of a Bohm like theory for the photon field. Moreover, they claim that these trajectories can be revealed in weak measurements of the field momentum. Moreover, they claim that in the double slit experiment of, e.g., the Toronto experiment [36] where their authors claim to have measured the trajectories of photons (although they based their claim using the nonrelativistic Bohm theory, a nonsequitur in our opinion) what they indeed measured has been the integral lines of the Pointing vector evaluated from $T^0$.

Well, we cannot leave out to emphasize that this is what the theory presented in this paper predicts from the fundamental postulated equation $\partial S = \frac{1}{2} F \gamma^0 F$ once $F$ is evaluated as a solution of Eq.(7) with the appropriate boundary and initial conditions, a result distinct from the Bohmian version of the electromagnetic theory according to [20].

(ii) Moreover, we must also recall that the formula $\partial S = \frac{1}{2} F \gamma^0 F$ fits well the experimental data on the tunneling of electromagnetic waves in experiments where the peak of the field configuration seems to go through the barrier with a “superluminal speed” due to the reshaping phenomenon that we described above [19].

(ii) Finally, we must also mention that the statement in [20] that in Dirac theory the path of electrons correspond to the integral lines of the $(0\bar{r})$-components of Tetrode energy-momentum tensor is not justified by the results of (see [60, 42]). In fact, the study of the tunneling of electrons through a barrier are well explained (through the reshaping phenomenon) supposing that electron follows one of the integral lines of the current $J = e \psi \gamma^0 \bar{\psi}$ where $\psi$ is a Dirac-Hestenes spinor field solution of the Dirac-Hestenes equation for the the conditions of the experiment [17].

8 Conclusions

We showed how starting from the photon concept and its relativistic HJE we immediately get (with a simple hypothesis concerning the form of the photon canonical momentum) ME satisfied by a null 2-form field $F$ which is a plane wave solution (PWS) of ME. Also, it was shown how introducing a potential 1-form $A$ such that $F = dA$ we can see a duality rotation to change in a spatial rotation with the bonus of also showing how $i = \sqrt{-1}$ enters Maxwell theory. This permits the writing of a representative of ME as Schrödinger like equation which plays a key role in answering one of the main questions addressed in this paper, namely:is there any sense in talking about photon trajectories in de Broglie-Bohm like theories? To this end we investigate the nature of the energy-momentum extensor field of the Maxwell field $T(n)$ in some special situations which explicitly shows in which sense it seems licit (in the spirit of de Broglie-Bohm theory) to associate worldlines for photons as being the integral lines of
g(T₀, ). We also discuss if there is any meaning in saying that photons can be said to be localizable due to the simple fact that there exists free boundary solutions of ME which is Gaussian concentrated (at least for \( t = 0 \)) in the sense of showing an exponential decay or for the case of nondiffracting solutions like \( F_{fwm} \). Acceptance of such functions as describing photons implies that we need to accept an odd fact, namely a non localizability in time.

## A Maxwell Equations in Vector Formalism

Let \( \{ x^\mu \} \) be global coordinates for Minkowski spacetime in Einstein-Lorentz-Poincaré gauge and \( e_\mu = \partial/\partial x^\mu \in \text{sec}TM, (\mu, \nu = 0, 1, 2, 3), \) be an orthogonal basis \( g(e_\mu, e_\nu) = \eta_{\mu\nu} \). The global field \( e_0 \) determines an inertial reference frame \[ 58 \]. Let \( \gamma^\mu = dx^\mu \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} Cl(M, \eta) \) be the dual basis of \( \{ e_\mu \} \) and let \( \gamma_\mu = \eta_{\mu\nu} \gamma^\nu \) be the reciprocal basis to \( \{ \gamma^\mu \} \), i.e., \( \gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu \).

The electromagnetic field is represented by a two-form \( F \in \text{sec} \bigwedge^2 T^*M \hookrightarrow \text{sec} Cl(M, \eta) \) We have

\[
F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = F^{\mu\nu} \gamma_\mu \gamma_\nu, \quad F^{\mu\nu} = \begin{pmatrix}
0 & -E^1 & -E^2 & -E^3 \\
E^1 & 0 & -B^3 & B^2 \\
E^2 & B^3 & 0 & -B^1 \\
E^3 & -B^2 & B^1 & 0
\end{pmatrix}, \quad (113)
\]

where \( (E^1, E^2, E^3) \) and \( (B^1, B^2, B^3) \) are respectively the Cartesian components of the electric and magnetic fields in the reference frame \( e_0 \).

Let \( J \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} Cl(M, \eta) \) be such that

\[
J = J^\mu \gamma_\mu = \rho \gamma_0 + J^1 \gamma_1 + J^2 \gamma_2 + J^3 \gamma_3, \quad (114)
\]

where \( \rho \) and \( (J^1, J^2, J^3) \) are the Cartesian components of the charge and (3-dimensional) current densities. Maxwell equation in the language of differential forms reads

\[
dF = 0, \quad \delta F = -J, \quad (115)
\]

where \( d \) is the differential and \( \delta \) is the Hodge codifferential operator.

Since \( dF \) and \( \delta F \) are sections of \( Cl(M, \eta) \) we can add the two equations in Eq.(114) and get

\[
(d - \delta) F = J.
\]

Now, recalling that \( d - \delta = \partial \) is the Dirac operator \[ 58 \] acting on sections of \( Cl(M, \eta) \), and we get

\[
\partial F = J \quad (116)
\]

\[35\] The operator \( \partial \) is not to be confused with the Dirac operator acting on sections of spin-Clifford bundles. See \[ 58 \].
which may now be called *Maxwell equation*, instead of Maxwell equations.

We now write Maxwell equation in $\mathcal{C}^0(M, \eta)$, the even sub-algebra of $\mathcal{C}(M, \eta)$. The typical fiber of $\mathcal{C}^0(M, \eta)$, which is a vector bundle, is isomorphic to the Pauli algebra (see, e.g., [58]).

We put
\[
\sigma_i = \gamma_i \gamma_0; \quad i = \sigma_1 \sigma_2 \sigma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_5.
\]
(117)

Recall that $i$ commutes with bivectors and since $i^2 = -1$ it acts like the imaginary unit $i = \sqrt{-1}$ in $\mathcal{C}^0(M, \eta)$. We now may easily verify that we can write
\[
F = E + iB
\]
(118)
with $E = E^i \sigma_i$, $B = B^j \sigma_j$, $i, j = 1, 2, 3$.

Now, since $\partial = \gamma_\mu \partial^\mu$ we get $\partial \gamma_0 = \partial/\partial x^0 + \bar{\sigma}_i \gamma_i = \partial/\partial x^0 - \nabla$. Multiplying Eq. (116) on the right by $\gamma_0$ we have
\[
\partial \gamma_0 \gamma_0 F \gamma_0 = J \gamma_0,
\]
(119)
and then
\[
(\partial/\partial x^0 - \nabla)(-E + iB) = \rho + J,
\]
(120)
where we used $\gamma_0 F \gamma_0 = -E + iB$ and $J := J^i \sigma_i$.

From Eq. (120) we have
\[
- \partial_0 E + i \partial_0 B + \nabla E - i \nabla B = \rho + J
\]
\[
- \partial_0 E + i \partial_0 B + \nabla \cdot E + \nabla \wedge E - i \nabla \cdot B - i \nabla \wedge B = \rho + J
\]
(121)

Now we put (details, e.g., in [58]) for any 2-form $A$ in $\mathcal{C}(M, g)$ (which can be identified with a Euclidean vector field)
\[
\nabla \times A := -i \nabla \wedge A
\]
(122)
since the usual vector product between two “vectors” $A = A^i \sigma_i$, $B = B^j \sigma_j$ can be identified with the dual of the bivector $A \wedge B$ through the formula $A \times B = -i(A \wedge B)$. Observe that in this formalism $A \times B$ is a true vector and not the nonsense pseudo vector of the Gibbs vector calculus. Using Eq. (122) and equating the terms with the same grade we have
\[
\nabla \cdot E = \rho; \quad \nabla \times B + \partial_0 E = J;
\]
\[
\nabla \times E + \partial_0 B = 0; \quad \nabla \cdot B = 0;
\]
(123)
which are Maxwell equations in the usual vector notation.

36The symbol $\cdot$ denotes the Euclidean scalar product. Details in [58] and is not to be confused with the symbol $\cdot$ that denotes the Minkowskian scalar product.
B The Symmetrical Energy-Momentum Extensor of the Electromagnetic Field

B.1 $\partial_n \cdot \partial T(n) = J \cdot F$

We now introduce the energy momentum extensor of the electromagnetic field and the energy-momentum 1-forms of stress-energy. Since $\partial F = J$ we have $\partial F = J$. Multiplying the first equation on the left by $F$ and the second on the right by $F$ and summing we have:

$$\frac{1}{2}(\partial F + F \partial F) = J \cdot F,$$

(124)

Now, let $n$ be a 1-form field and $\gamma^\mu \cdot \partial_n = \eta^\mu_\nu \frac{\partial}{\partial n^\nu}$. Then we can write

$$\frac{1}{2}(F \partial F + F \partial F)$$

$$= \frac{1}{2}(\partial \mu F \gamma^\mu F + F \gamma^\mu \partial_\mu F)$$

$$= \frac{1}{2} \gamma^\mu \cdot \partial_n (\partial_\muFnF + F \partial_\mu nF + F n \partial_\mu F)$$

(125)

where we have used that $\gamma^\mu \cdot \partial_n \partial_\mu n = \eta^\mu_\nu \frac{\partial}{\partial n^\nu} \partial_\mu n^\alpha \gamma^\alpha = \eta^\mu_\nu \delta^\alpha_\nu \partial_\mu \gamma^\alpha = 0$. Then we have that

$$\gamma^\mu \cdot \partial_n \partial_\mu \left(\frac{1}{2} F n F\right) = J \cdot F$$

(126)

Eq. (126) means that there exists a differential operator and a $(1, 1)$ extensor field defined by

$$\partial_n \cdot \partial = \gamma^\mu \cdot \partial_n \partial_\mu$$

and

$$T^+ : \sec \wedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, \eta) \hookrightarrow \wedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, \eta),$$

such that

$$\partial_n \cdot \partial T^+(n) = J \cdot F.$$  

(128)

B.1.1 $T(n) = T^+(n)$

Before going on observe that if $b$ is a 1-form field it is

$$T(n) = \partial_b n \cdot T^+(b) = \partial_b n \cdot \frac{1}{2} F b F$$

$$= \partial_b T^+(n) \cdot b = T^+(n),$$

(129)

37See [58] for details on the theory of extensor fields.

38If $M$ is a $(1, 1)$ extensor field we denote its adjunt by $M^+$ and if $n, m$ are 1-form fields it is: $M(n) \cdot m = n \cdot M^+(m) = M^+(m) \cdot n$. 

30
i.e., \( T \) is symmetric.

We call the objects \( T^\mu := T(\gamma^\mu) = T^\mu_\nu \gamma^\nu \) the energy momentum 1-form fields. It is clear that we have

\[
\partial \cdot T^\mu = (J \cdot F) \cdot \gamma^\mu \quad (130)
\]

Of course, if \( J = 0 \) the energy momentum 1-form fields of the electromagnetic field is conserved, i.e.,

\[
\partial \cdot T^\mu = 0 \iff \partial_\mu T^\mu = 0.
\]

We now define the energy density \( u \) and the Pointing vector \( P \) by

\[
T(\gamma_0)\gamma_0 = u + P, \\
\gamma_0 T(\gamma_0) = u - P,
\]

where

\[
\begin{align*}
u &= \frac{1}{2} (T(\gamma_0)\gamma_0 + \gamma_0 T(\gamma_0)) = \frac{1}{2} (F\gamma_0 F\gamma_0 + \gamma_0 F\gamma_0 F) \\
&= -\frac{1}{2} [\Gamma + \frac{1}{4} \Gamma + \frac{1}{4} \Gamma (\gamma_0 F - \gamma_0 F)] \\
&= \frac{1}{2} (E^2 + B^2)
\end{align*}
\]

and

\[
\begin{align*}
P &= \frac{1}{2} (T(\gamma_0)\gamma_0 - \gamma_0 T(\gamma_0)) = \frac{1}{2} (F\gamma_0 F\gamma_0 + \gamma_0 F\gamma_0 F) \\
&= -\frac{i}{2} (\nabla \times \Gamma) =: E \times B.
\end{align*}
\]

From Eq. (130) we can write

\[
\partial \cdot T_0 = (J \cdot F) \cdot \gamma_0
\]

which we rewrite as

\[
\begin{align*}
\frac{1}{2} (\partial T_0 + \bar{\partial} T_0) &= \frac{1}{4} (\gamma_0 F\gamma_0 F - \gamma_0 F\gamma_0 F - \gamma_0 F\gamma_0 F - \gamma_0 F\gamma_0 F), \\
(\partial \gamma_0 \gamma_0 T_0 + \bar{\partial} \gamma_0 \gamma_0 T_0) &= \frac{1}{2} [(\gamma_0 F\gamma_0 F - \gamma_0 F\gamma_0 F - \gamma_0 F\gamma_0 F - \gamma_0 F\gamma_0 F)].
\end{align*}
\]

Now,

\[
\begin{align*}
\partial_\gamma_0 \gamma_0 T_0 &= (\partial_\gamma - \nabla) (u - P) \\
&= \partial_\gamma u - \partial_\gamma P - \nabla u + \nabla P \\
&= \partial_\gamma u - \nabla u - \partial_\gamma P + \nabla \cdot P + i \nabla \times P
\end{align*}
\]

(136)
and then
\[ \frac{1}{2} (\partial T_0 + \tilde{\partial} T_0) = 2 (\partial_0 u + \nabla \cdot P). \] (137)

Also
\[ J_\gamma_0 \gamma_0 F_\gamma_0 - FJ_\gamma_0 = -2 (\rho E + J \cdot E + J \times B) \] (138)
and thus
\[ \frac{1}{2} [J_\gamma_0 \gamma_0 F_\gamma_0 - FJ_\gamma_0 + (\gamma_0 J F - \gamma_0 F \gamma_0 J)] = -4J \cdot E. \] (139)

Using this results we can rewrite Eq. (134) as
\[ \partial u / \partial T = - (\nabla \cdot P + J \cdot E), \] (140)
a result which is known as Poynting theorem.

B.2 Explicit Form of the Components \( T^{\mu\nu} \)

Since \( F_\gamma \cdot F \in \text{sec} \bigwedge^1 T^* M \rightarrow \text{sec} C\ell(M, \eta) \) we can write
\[ T^{\mu\nu} = - \frac{1}{2} (F_{\gamma^\mu} F) \cdot \gamma^\nu = - \frac{1}{2} (F_{\gamma^\mu} F \gamma^\nu)_0 \] (141)

Since \( \gamma^\mu \cdot F = \frac{1}{2} (\gamma^\mu F - F \gamma^\mu) = - F \cdot \gamma^\mu \), we have
\[ \frac{1}{2} (F_{\gamma^\mu} F \gamma^\nu)_0 = - ( (\gamma^\mu \cdot F) F \gamma^\nu)_0 + \frac{1}{2} (\gamma^\mu \cdot F \gamma^\nu)_0 \]
\[ = + (\gamma^\mu \cdot F) \cdot (\gamma^\nu \cdot F) + \frac{1}{2} (F \cdot F) \gamma^\mu \cdot \gamma^\nu \]
\[ = \frac{1}{4} F_{\mu\nu} F_{\lambda\lambda} - \frac{1}{2} (F \cdot F) \gamma^\mu \cdot \gamma^\nu \]
\[ = \frac{1}{4} F_{\mu\nu} F_{\lambda\lambda} - \frac{1}{8} F_{\alpha\beta} F_{\alpha\beta} \eta_{\mu\nu}. \] (142)

Thus,
\[ T^{\mu\nu} = \frac{1}{4} \left( - F_{\mu\nu} F_{\lambda\lambda} + \frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} \eta_{\mu\nu} \right) \] (143)

B.3 Angular Momentum Extensor

We now define the density of angular momentum extensor. Take a point event \( o \in M \) such that its coordinates are \( (0, 0, 0, 0) \) and define 1-form \( x = x^\mu \gamma_\mu = x_\mu \gamma^\mu \). The angular momentum extensor is the mapping
\[ M^1 : \text{sec} \bigwedge^1 T^* M \rightarrow \text{sec} C\ell(M, \eta) \rightarrow \bigwedge^2 T^* M \rightarrow \text{sec} C\ell(M, \eta), \]
\[ M^1(n) = T(n) \wedge x = \frac{1}{2} (T(n) x - x T(n)) \] (144)

In particular we define the angular momentum 2-form fields by
\[ M^\dagger_\mu = M^\dagger(\gamma_\mu) = T_\mu \wedge x = \frac{1}{2}(x_\alpha T_{\mu\nu} - x_\nu T_{\alpha\mu}) \gamma^\nu \wedge \gamma^\alpha. \]  
(145)

We immediately get that
\[ \partial_n \cdot \partial M^\dagger(n) = -x \wedge (J \mathcal{J} F) \]  
(146)

and in particular it is
\[ \partial^\mu M^\dagger_\mu = -\frac{1}{2}(x^\alpha J_\mu F^{\mu\nu} - x^\nu J_\mu F^{\mu\alpha}) \gamma_\alpha \wedge \gamma_\nu. \]  
(147)

**Remark 11** It is important to emphasize here that without the Lagrangian formalism it is not possible to identify the spin extensor of the electromagnetic field [58].

### B.4 Poincaré Invariants

The Poincaré invariants of the electromagnetic field \( F \) are \( F \mathcal{J} F \) and \( F \wedge F \),
\[ F^2 = F \mathcal{J} F + F \wedge F; \]
\[ F \mathcal{J} F = -\frac{1}{2} F^{\mu\nu} F_{\mu\nu}; \quad F \wedge F = \frac{1}{4} F_{\mu\nu} F_{\alpha\beta} \varepsilon^{\mu\nu\alpha\beta} \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{1}{4} \gamma^{\mu
u\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \]  
(148)

Writing as before \( F = E + iB \) we have
\[ F^2 = (E^2 - B^2) - 2iE \times B = F \mathcal{J} F + F \wedge F. \]  
(149)

### B.5 The Canonical Energy-Momentum Extensor of the Electromagnetic Field

The action for the free Maxwell field \( F = dA, A \in \sec \wedge^1 T^* M \hookrightarrow \sec Cl(M, \eta) \) is
\[ S = \int d^4x \mathcal{L}(A, \partial \wedge A), \]
\[ \mathcal{L}(A, \partial \wedge A) = -\frac{1}{2} (\partial \wedge A) \cdot (\partial \wedge A) \]  
(150)

and the canonical energy-momentum tensor is [58]
\[ T^\dagger_\nu(n) = T^\dagger(n) - (n \mathcal{J} F) \cdot \partial A. \]  
(151)

The canonical formalism naturally gives a conserved total angular momentum \((1, 2)\)-extensor field for any field theory. For our case, we have that
\[ J^\dagger : \sec \wedge^1 T^* M \hookrightarrow \sec Cl(M, \eta) \rightarrow \wedge^2 T^* M \hookrightarrow \sec Cl(M, \eta), \]
\[ J^\dagger_\nu(n) = T^\dagger_\nu(n) \wedge x + S^\dagger(n), \]  
(152)
where [39]

\[ S^I(n) = \langle A \times \partial_{\theta} \wedge A \Sigma (A, \partial \wedge A) n \rangle_2 \]

\[ = (n \cdot F) \wedge A = n \wedge (A \wedge F) - (n \cdot A) F \] (153)

is the spin extensor of the electromagnetic field which is a gauge dependent quantity.[16][23].

References

[1] Acharya, M.A., and Sudarshan, E.C., “Front” Description Relativistic Quantum Mechanics, J. Math. Phys. 1, 532-536 (1960).

[2] Bateman, H., The Mathematical Analysis of Electrical and Wave Motion on the Basis of Maxwell Equations, Cambridge Univ. Press, Cambridge, 1915.

[3] Bialynicki-Birula, I., Photon Wave Function, in Wolf, E. (editor), Progress in Optics XXXVI, pp. 245-294, Elsevier, Amsterdam, 1996.

[4] Bialynicki-Birula, I., Exponential Localization of Photons, Phys. Rev. Lett. 24, 5247-5250 (1998).

[5] Bohm, D. and Hilley, B. J., Undivided Universe, Routledge, London and New York 1993.

[6] Bolda, E. L., Garrison, J. C. and Chiao, R. Y., Optical Pulse Propagation at Negative Group Velocities due to Nearby Gain Line, Phys. Rev. A 49, 2938-2947 (1994).

[7] Brittingham, J. N, Focus Wave Modes in Homogeneous Maxwell Equations: Transverse Electric Mode, J. Appl. Phys. 54, 403-428 (1994).

[8] Brillouin, L., Wave Propagation and Group Velocity, Academic Press, New York 1960.

[9] Chen, P. and Kleinert, H., Deficiencies of Bohmian Trajectories in View of Basic Quantum Principles, Electronic Journal of Theoretical Physics 12, 1-11 (2016).

[10] Chiao, R. Y., Superluminal (but causal) Propagation of Wave Packets in Transparent Media with Inverted Populations. Phys. Rev. A 48, R34-R37 (1993).

[11] Chiao, R. Y., Kozhekin, A. E. and Kurizki, G., Tachyonlike Excitations in Inverted Two-Level Media, Phys. Rev. Lett. 77, 1254-1257 (1996).

[12] Deutch, J. M. and Low, F. E., Barrier Penetration and Superluminal Velocity, Ann. Phys. 228, 184-201 (1993).

[13] Diener, G., Energy Transport in Dispersive Media and Superluminal Group Velocities. Phys. Lett. A 235, 118-124 (1997).

\[ ^{39} \text{With } \times \text{ in Eq. (153) denoting the commutator product in the Clifford bundle, i.e., for } K, L \in \text{sec} \mathcal{C}(M, \eta) \text{ it is } K \times L = 1/2(KL - LK). \]
Diener, G., Superluminal Group Velocities and Information Transfer, *Phys. Lett. A* **223**, 327-331 (1996).

Diener, G., Energy Balance and Energy Transport in Dispersive Media, *Ann. der Physik* 7, 639-644 (1998).

de Vries, H., On the Electromagnetic Chern Simons Spin Density as a Hidden Variable and EPR Correlations. [http://physics-quest.org/ChernSimonsSpinDensity.pdf](http://physics-quest.org/ChernSimonsSpinDensity.pdf) 2012.

Doran, C., Lasenby, A., and Gull, S., STA and the Interpretation of Quantum Mechanics, Chapter 11 in Baylis, W. E. (ed.), *Clifford (Geometric) Algebras with Applications in Physics, Mathematics and Engineering*, Birkhäuser, Basel, 1996.

Donnelly, R. and Ziolkowski, R. W., A Method for Construction Solutions of Homogeneous Partial Differential Equations: Localized Waves, *Proc. R. Soc. London A* **437**, 673-692 (1992).

Emig, T., Propagation of an Electromagnetic Pulse Through a Wave Guide With a Barrier: A Time Domain Solution within Classical Electrodynamics, *Phys. Rev. E* **54**, 5780-5787 (1996).

Flack, R. and Hiley, B. J., *Weak Values of Momentum of the Electromagnetic Field. Average Momentum Flow Lines, Not Photons Trajectories* [arXiv:1611.06510v1 [quant-ph]]

Fleming, G. N., Nonlocal Properties of Stable Particles, *Phys. Rev.* **139**, B963-B968 (1965).

Garrett, C. G. B. and McCumber, D. E., Propagation of a Gaussian Pulse Through an Anomalous Dispersion Medium, *Phys. Rev. A* **1**, 305–313 (1970).

Giglio, J. F. T. and Rodrigues, W. A. Jr., Gravitation and Electromagnetism as a Geometrical Objects of a Riemann-Cartan Spacetime Structure, *Adv. Appl. Clifford Algebras* **22**, 649-664 (2012).

Hegerfeldt, G. C., Instantaneous Spreading and Einstein Causality in Quantum Theory, *Ann. Phys.* (Leipzig), **7-8**, 716-725 (1998).

Hernandez-Figueroa, H. E., Zamboni-Rached, M. and Recami, E., *Localized Waves*, Localized Waves, John Wiley & Sons, Inc., Hoboken, New Jersey, 2008.

Hestenes, D., *Space-Time Algebra* (second edition), Birkhäuser, Springer Int. Publ. Switzerland, 2015. Originally published by Gordon and Breach Sci. Pub., New York, 1966.

Jadczyk, A. Z. and Jancewicz, B., Maximal Localizability of Photons, *Bull. de l’Academie Polonaise de Sciences, Serie de Sciences Math. Astr et Phys.* **XXI**, 477-483 (1972).

Jancewicz, B., *Multivectors and Clifford Algebras in Electrodynamics*, World Sci. Publ. Co., Singapore, 1988.
[29] Jancewicz, B., A Hilbert Space for the Classical Electromagnetic Field, *Found. Phys.* 23, 1405-1421 (1993).

[30] Jung, K., Is the de Broglie-Bohm Interpretation of Quantum Mechanics Really Plausible?, *J. Physics: Conference Proceedings Series* 422, 012060 (2013).

[31] Lahoz, D. G. and Graham, G. M., Observation of Static Electromagnetic Angular Momentum in Vacua, *Nature* 285, 154-155 (1980).

[32] Kurşunoğlu, B., *Modern Quantum Theory*, W. Freeman and Co., San Francisco and London, 1962.

[33] Low, F. E., Comments on Apparent Superluminal Propagation, *Ann. der Physik* 7, 660-661 (1998).

[34] Lu, J.-Y. and Greenleaf, J. F., Nondiffracting X-Waves-Exact Solutions to Free Space Scalar Equation and their Finite Aperture Realizations, *IEEE Transact. Ultrason. Ferroelec. Freq. Contr.* 39, 441-446 (1992).

[35] Mandel, L. and Wolf, E., Optical Coherence and Quantum Physics, Cambridge Univ. Press, Cambridge, 1995.

[36] Mahler, D. H., Rozema, L., Fisher, K., Vermeyden, L., Resch, K. J., Wiseman, H. M. and Steinberg, A., Experimental Nonlocal and Surreal Bohmian Trajectories, *Sci. Adv.* 2, e1501466 (2016).

[37] Mignani, R., Recami, E., and Baldo, M., About a Dirac-like Equation for the Photon According to Ettore Majorana, *Lett. Nuovo Cimento* 11, 568- (1974).

[38] Miller, W. Jr., *Symmetry Principles and their Applications*, Academic Press, New York, 1972.

[39] Mitchell, M. W., Chiao, R. Y., Negative Group Delay and “Fronts” in a Causal System: An Experiment with Very Low Frequency Bandpass Amplifiers, *Phys. Lett. A* 230, 133–138 (1997).

[40] Mende, P. F., Comments on Apparent Superluminal Propagation, *Ann. Phys.* 210, 380-387 (1991).

[41] Mugnai, D., Ranfagni, A. and Ruggieri, R., Observation of Superluminal Behaviors in Wave Propagation, *Phys. Rev. Lett* 84, 4830 (2000).

[42] Moya, A. M., Rodrigues, W. A., Jr. and Wainer, S. A., *The Dirac-Hestenes Equation and its Relation to the Relativistic de Broglie-Bohm Theory*, [arXiv:1610:09655v1[math-ph]]

[43] Newton, T. D. and Wigner, E. P., Localized States for Elementary Systems, *Rev. Mod. Phys.* 21, 400-406 (1949).

[44] Nimtz, G., New Knowledge from Photonic Experiments in Mugnai, D., Ranfagni, A. and Schulman, L. S. (eds.), *Proc. of the Adriatico Conference: Tunneling and its Application* (30 July-02—August 1996), pp. 223-237, World Sci. Publ. Co., Singapore, 1997
[45] Notte-Cuello, E. and Rodrigues, W. A. Jr., Superposition Principle and the Problem of the Additivity of the Energies and Momenta of Distinct Electromagnetic Fields, *Rep. Math. Phys.* **62**, 91-101 (2008).

[46] Oliveira, E. C., Rodrigues Jr., W. A., Superluminal Electromagnetic Waves in Free Space, *Ann. der Physik* **7**, 654-659 (1998).

[47] Oliveira, E. C., Rodrigues Jr., W. A., Thober, D. S., Xavier, A. L., Jr., Thoughtful Comments on ‘Bessel Beams and Signal Propagation’, *Phys. Lett. A* **284**, 296-303 (2001).

[48] Oliveira, E. C., Rodrigues Jr., W. A., Finite Energy Superluminal Solutions of Maxwell Equations, *Phys. Lett. A* **296**, 367-370 (2001).

[49] Philippidis, C., Dewdney, C. and Hiley, B. J., Quantum Interference and Quantum Potential, *Nuovo Cimento B* **52**, 15028 (1979).

[50] Raymer, M. G. and Smith, Brian J, The Maxwell Wave Function of the Photon, *SPIE Conference Optics and Photonics, Conference* **5866**, *The Nature of Light: What is a Photon?* San Diego, 2005.

[51] Riez, M., *Clifford Algebras and Spinors*, Lecture Notes 38, The Institute of Fluid Dynamics and Applied Mathematics, Univ. of Maryland (1958).

[52] Rodrigues, W. A. Jr., Vaz, J. Jr. and Recami, E., A Generalization of Dirac Nonlinear Electrodynamics, and Spinning Charged Particles, *Found. Phys.* **23**, 469-485 (1993).

[53] Rodrigues, W. A. Jr. and Maiorino, J. E., A Unified Theory for Construction of Arbitrary Speeds (0 ≤ v < ∞) Solutions of the Relativistic Wave Equations, *Random Oper. Stochastic Equ.* **4**, 355–400 (1996).

[54] Rodrigues, W. A. Jr. and Lu, J.-Y., On the Existence of Undistorted Progressive Waves (UPWs) of Arbitrary Speeds 0 ≤ v < ∞ in Nature. *Found. Phys.* **27**, 435–508 (1997).

[55] Rodrigues, W. A. Jr., Vaz, J. Jr., Subluminal and Superluminal Solutions in Vacuum of the Maxwell Equations and the Massless Dirac Equation. Talk presented at the International Conference on the Theory of the Electron, Mexico City, 1995, *Advances in Appl. Clifford Algebras* **7** (Sup.), 453–462 (1997).

[56] Rodrigues, W. A. Jr.,Thober,D. S. and Xavier, A. L., Causal Explanation for Observed Superluminal Behavior of Microwave Propagation in Free Space, *Phys. Lett. A* **284**, 217-224 (2001).

[57] Rodrigues, W. A., Jr. and Capelas de Oliveira, E., Extracting Energy from an External Magnetic Field, in V. Dvoeglazov (ed.), *Einstein and Others: Unification*, Contemporary Fundamental Physics Series, Chapter 6, Nova Publ., New York, 2014.

[58] Rodrigues, W. A., Jr. and Capelas de Oliveira, E., *The Many Faces of Maxwell, Dirac and Einstein Equations. A Clifford Bundle Approach*. Lecture Notes in Physics **922** (second revised and enlarged edition), Springer, Heidelberg, 2016.
Rodrigues, W. A., Jr. and Wainer, S. A., Notes on Conservation Laws, Equations of Motion of Matter, and Particle Fields in Lorentzian and Teleparallel de Sitter Space-Time Structures, Adv. Math. Phys. 2016, 5465263 (2016).

Rodrigues, W. A., Jr. and Wainer, S. A., The Relativistic Hamilton-Jacobi Equation for a Massive, Charged and Spinning Particle, its Equivalent Dirac Equation and the de Broglie-Bohm Theory, [arXiv:1610.03310v2 [math-ph]]

Saari, P. and Reivelt, K., Evidence of a X-Shaped Propagation-Invariant Localized Light Waves, Phys. Rev. Lett. 79, 4135-4138 (1997).

Steinberg, A. M., and Chiao, R. Y., Dispersionless, Highly Superluminal Propagation in a Medium with a Gain Doublet, Phys. Rev. A 49, 2071-2075 (1994).

Schwinger, J., Particles, Sources and Fields (vol 1), Addison-Wesley Publ. Co., Reading, MA, 1970.

Sharam, N. L., Field Versus Action-at-a Distance in a Static Situation, Am. J. Phys. 56, 420-424 (1988).

Silberstein, L, Electromagnetische Grundgleichungen in Bivectorieller Behandlung, Ann. der Physik 22, 579- (1907)

Silberstein, L., Nachtrag zur Abhandlung über “Electromagnetische Grundgleichungen in Bivectorieller Behandlung, Ann. der Physik 24, 783- (1907)

Silberstein, L., The Theory of Relativity (second enlarged version), MacMillan, London, 1924.

Steinberg, A. M., Chiao, R. Y., Dispersionless, Highly Superluminal Propagation in a Medium with a Gain, Phys. Rev. A 49, 2071-2075 (1994).

Shaarawi, A., Besieris, I. and Ziolkowski, R. W., Diffraction of a Nondispersive Wave Packet in the Two Slit Interference Experiment, Phys. Lett. A 188, 218-224 (1994).

Stratton, J. A., Electromagnetic Theory, McGraw-Hill, New York 1941.

Thaller, H., The Dirac Equation, Springer, Berlin, 1992.

Ticiatti, R., Quantum Field Theory for Mathematicians, Encyclopedia of Mathematics and its Applications 72, Cambridge Univ. Press, Cambridge, 1999.

Vladimirov, V. S., Equations of Mathematical Physics, Marcel Decker Inc., New York, 1971.

Weber, H., Die Partiellen Differential-Gleichungen der Mathematischen Physik nach Riemann’s Vorlesungen, p.347, Friedrich Vieg unw Sohn, Braunschweig, 1901.