ON THE EXTERIOR STABILITY OF NONLINEAR WAVE EQUATIONS

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Abstract. We consider a general class of nonlinear wave equations, which admit trivial solutions and not necessarily verify any form of null conditions. They typically include various John’s examples [10, 12], the reduced Einstein equations under wave coordinates and the irrotational fluids. For compactly supported small data, one can only have a semiglobal result [10, 12], which states that the solutions are well-posed up to a finite time-span depending on the size of the Cauchy data. For some of the equations of the class, the solutions blow up within a finite time for the compactly supported data of any size. For data prescribed on \( \mathbb{R}^3 \setminus B_R \) with small weighted energy, without some form of null conditions on the nonlinearity, the exterior stability is not expected to hold in the full domain of dependence, due to the known results of formation of shocks with data on annuli. The classical method can only give the well-posedness up to a finite time.

In this paper, we prove that, there exists a constant \( R(\gamma_0) \geq 2 \), depending on the fixed weight exponent \( \gamma_0 > 1 \) in the weighted energy norm, if the norm of the data are sufficiently small on \( \mathbb{R}^3 \setminus B_R \) with the fixed number \( R \geq R(\gamma_0) \), the solution exists and is unique in the entire exterior of a Schwarzschild cone initiating from \( \{|x| = R\} \) (including the boundary) with small negative mass \(-M_0\). \( M_0 \) is determined according to the size of the initial data. In this exterior region, by constructing the Schwarzschild cone foliation, we can improve the linear behavior of wave equations in particular on the transversal derivative \( \partial_y \). Such improvement enables us to control the nonlinearity violating the null condition without loss, and thus show the solutions converge to the trivial solution. For semi-linear equations, such stability region can be any close to \( \{|x| = r > R\} \) if the weighted norm of the data is sufficiently small on \( \{|x| \geq R\} \).

The other interesting aspect of our method lies in that it treats the massless and massive wave operator in a uniform way. Thus it works for equations with nonnegative variable potentials and an equation system with different potentials. As a quick application, we give the exterior stability result for Einstein (massive and massless) scalar fields. We prove the solution converges to a small static solution, stable in the entire exterior of a Schwarzschild cone with positive mass, which then is patchable to the interior results.

1. Introduction

In this paper, we consider nonlinear wave equations in \( \mathbb{R}^{3+1} \) of the following form

\[
\begin{array}{l}
\{ g^{\alpha\beta}(\phi, \partial \phi) \partial_\alpha \partial_\beta \phi = N^{\alpha\beta}(\phi) \partial_\alpha \phi \partial_\beta \phi + q(x) \phi \\
\phi|_{t=0} = \phi_0, \quad \partial_t \phi|_{t=0} = \phi_1
\end{array}
\tag{1.1}
\]

with the smooth function \( 0 \leq q(x) \leq 1 \). \( g(\phi, \partial \phi) \) is a Lorentzian metric. \( g(y, P) \) is smooth on variables \( y \in \mathbb{R} \) and \( P \in \mathbb{R}^4 \). \( g^{\alpha\beta}(0, 0) = \mathbf{m}^{\alpha\beta} \), where \( \mathbf{m} \) denotes the Minkowski metric. \(^2\) The functions \( N^{\alpha\beta}(y) \) are smooth for \( y \in \mathbb{R} \).

The most important case for us is \( q \equiv 0 \). For convenience, we assume the derivatives of \( q \) satisfy

\[
|r^i \partial^j q| \lesssim r^{-2-\eta} \quad \text{with} \quad i = 1, \cdots, n, \quad r = |x|
\tag{1.2}
\]

where \( \eta > 0 \) is any fixed constant. \(^3\)

\(^1\) We fix the convention that, in the Einstein summation convention, a Greek letter is used for index taking values \( 0, 1, 2, 3 \). \( x^0 = t \) and \( \partial_0 = \partial_t \).

\(^2\) Our proof still works if \( N^{\alpha\beta} \) also smoothly depend on \( \partial \phi \) and \( q \) also depends on \( t \) with nearly no modification. We keep in the simple form for ease of exposition.

\(^3\) For a differential operator \( P \), \( P^{(n)} \) means applying \( P \) to the \( n \)-th order, \( P^{(n)} = \sum_{0 \leq i \leq n} P^{(i)} \), and \( P^{(0)} = id. \)

\(^4\) \( A \lesssim B \) means \( A \leq cB \) with the constant \( c \geq 1 \). \( A \approx B \) means \( A \lesssim B \) and \( B \lesssim A \).
Throughout this paper we set $H^{\alpha \beta} = g^{\alpha \beta} - m^{\alpha \beta}$ and define $g^{\alpha \beta} \partial_\alpha \partial_\beta = \square_g$. In case $g \equiv m$, the quasilinear wave equation (1.1) becomes a semilinear equation.

1.1. Main problem. The discussion in this part will mainly focus on the case that $q \equiv 0$. Incorporating the nontrivial potential in the equation (1.1) is mainly for applying our method to an equation system with various potentials. We emphasize that there is no assumption of any form of special structure on the quadratic nonlinear term on the right-hand side of (1.1). For the general class of wave equations (1.1), we consider to construct the global-in-time classical solution for the generic Cauchy data with small weighted energy on $\{ |x| \geq R \}$.

Throughout this paper, we assume the initial data are not compactly supported in $\mathbb{R}^3$. We first give the definition of the weighted norm for the initial data. Let $1 < \gamma_0 < 2$ be a fixed constant and $q_0 = \sup_{\{|x| \geq R\}} q(x)$. We denote

$$
\mathcal{E}_{k, \gamma_0, R, q_0} = \int_{\Sigma_0 \cap \{ r \geq R \}} (1 + r)^{\gamma_0 - 2} \phi_0^2 dx + \sum_{k \leq K} \int_{\Sigma_0 \cap \{ r \geq R \}} (1 + r)^{\gamma_0 + 2k} (|\partial^k \phi_0|^2 + |\phi_0|^2 + q_0 |\phi_0|^2) dx,
$$

(1.3)

where $\Sigma_0 = \{ t = 0 \}$. The $q_0$ in the subindex may be dropped if we only consider one single equation, instead of an equation system. We may use $\mathcal{E}_{k, \gamma_0}$ as a short-hand notation whenever there occurs no confusion. Here $R$ is a fixed constant and $R \geq 2$.

For initial data with compact support, in either the semilinear or the quasilinear case, there holds only a semi-global existence result, with the time-span of the solution depending on the size of the small data (See [10, 12]). The finite life-span of the solution therein is actually sharp. In [8] and [9], examples of equations of the type (1.1) are constructed which does not admit global solution for data of any size. The potential flow of fluid equations are also typical examples of (1.1) which do not verify null condition. In the work of Christodoulou [3] on the relativistic Euler equation and the work of Speck [30] for the geometric wave equation, when a set of small cauchy data is prescribed in an annulus, the singularity of the characteristic surfaces is formed within finite time. The semi-global well-posedness of the solution holds until the formation of the shock. Based on these well-known facts, one should not expect to have any global-in-time result if the data is compactly supported. For the generic data prescribed on $\{ |x| \geq R \}$, due to the results of [8] and [30], one should not expect a global-in-time result in the full domain of dependence, which is exterior to the outgoing characteristic cone initiated from $\{ |x| = R \}$ up to the boundary, no matter how small the data is, since the weighted norm (1.3) of the small annuli-data can be sufficiently small.

When the data is non-compactly supported, the known energy method can only give the local-in-time result of well-posedness, even if the data are small on $\{ |x| \geq R \}$. See Figure 1 for the regions of well-posedness in the semilinear case by using the time foliation or the double null foliations. The $T(\epsilon)$ and $\nu_\epsilon(\epsilon)$ are both some finite numbers depending on the bound $\mathcal{E}_{N, \gamma_0, R, 0}(\phi[0]) \leq \epsilon$. We raise the question that for the generic data with finite weighted energy, if the blow-up of the solution can only occur in a region interior to certain cone initiated from a sphere $\{ |x| = R \}$. This is trivially true if the datum is compactly supported due to the standard argument of the finite speed of propagation. For the non-compactly supported data, we give an affirmative answer by showing the following result of exterior stability.

Theorem 1.1 (A rough statement of main result). Let $1 < \gamma_0 < 2$ be fixed. Consider (1.1). There exist a universal constant $C \geq 1$, a small constant $\delta_1 > 0$ and a constant $R(\gamma_0, C) \geq 2$ such that, if $\phi[0] = (\phi_0, \phi_1)$ verifies $\mathcal{E}_{3, \gamma_0, R, 0} \leq \delta_1$ with $R \geq R(\gamma_0, C)$, then with $M_0 = C \delta_1^\frac{1}{4}$, exterior to a schwarzschild cone of mass $-M_0$ initiated from $\{ |x| = R \}$ (including the cone itself), there exists a unique global-in-time solution which converges to the trivial solution as

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5 Throughout the paper, a universal constant means a constant that depends only on the constant $\eta$ and the bound in (1.2), the bounds of $|D^{\leq 3}g|$ and $|D^{\leq 3}N|$ on small compact domains.

6See the definition of the metric in (2.1).
$r \to \infty$ for any $t > 0$. The solution not only has the standard asymptotic behavior of the free wave, but also has improved global decay properties.

Figure 1. Illustration of the classical semi-global results

Figure 2. Illustration of the stability region of the main result for the semilinear equations.

One can directly apply the result to the generic data with bounded weighted energy, provided that $R$ is sufficient large. Such schwarzschild cone divides the spacetime into a stability region at its exterior, while all the singularities can only be formed in its open interior, which are compactly supported for all $t > 0$. This gives the affirmative answer to our question.

We have several remarks on the above rough statement of our result.

1. In this result, the mass of the schwarzschild metric is $-M_0$ with $|M_0| \ll \frac{1}{10}$ chosen according to the size of the initial data. The definition of the metric can be found in (2.1). For the general equation (1.1), we choose $M_0 > 0$. Correspondingly, the boundary of the exterior region is slightly spacelike\footnote{Throughout the paper, spacelike, null or timelike are in terms of the Minkowski metric.}. For the semilinear case and Einstein scalar fields, we can have better results than the above statement. For the semilinear case, the stability region can be any close to $\{r > t + R\}$. For Einstein equations, we can choose $M_0 \leq 0$. The corresponding schwarzschild cone is timelike or null. This makes the exterior result patchable with an interior result based on the Minkowskian hyperboloidal foliation if we further assume the smallness of the initial data in $B_R$.\footnote{See a semilinear result \cite{7} for an example of such direct patching.} We refer the readers to the main theorems, Theorem 2.1, Theorem 2.5 and Theorem 2.6 for detailed statements.

2. If the data are compactly supported, the choice of $M_0 = C \delta^\frac{1}{4}_1$ leads to a semi-global result. The life-span coincides with the standard almost global result \cite{10}.
(3) The sharp local-wellposedness result \[28, 31\] implies \((1.1)\) is local-in-time wellposed for data \(\phi[0] = (\phi_0, \phi_1)\) in the normed space \(H^{3+\epsilon}(\mathbb{R}^3) \times H^{2+\epsilon}(\mathbb{R}^3), \epsilon > 0\). In terms of the order of derivative, our data is at the level of \(H^1(\mathbb{R}^3) \times H^0(\mathbb{R}^3)\).

In a standard regime of commuting vector fields approach, it is optimal in terms of regularity.

1.2. **Review of history and inspiration.** In general, in \(\mathbb{R}^{3+1}\), one can construct global-in-time solutions of \((1.1)\) for generic small data only when the quadratic nonlinearity \(N^{\alpha\beta}(\phi)\partial_{\alpha}\phi\partial_{\beta}\phi\) verifies certain null condition. The standard null condition, since it was raised by Klainerman \[11\], has been deeply exploited for proving global existence results for various equations with such structures. The case for the general equation \((1.1)\) with \(q > 0\) being a fixed constant is also studied in \[13\] for small data with compact support. One can refer to \[14, 2\] for the results for quasilinear wave equations verifying null conditions. The null conditions, which are important algebraic cancelation structures can be found in various important geometric or physical field equations, such as wave maps, Maxwell-Klein-Gordon equations, Yang-Mills equations and Einstein equations. For the semilinear case, one can refer to \[22, 32, 34\] for the global results of the massless Maxwell-Klein-Gordon equations, to \[19, 27\] for the massive case, and to \[5, 6\] for the result of Yang-Mills equations. One can find in \[15, 16, 20, 26\] the global well-posedness results with low or optimal regularity with large data for Maxwell-Klein-Gordon equations and Yang-Mills equations.

The Einstein equation system is an important example of the system of quasilinear equations that verifies null conditions in an intrinsic geometric framework, relative to the maximal foliation gauge. Under this gauge, the small data global-existence result was proved by Klainerman and Christodoulou in the monumental work \[3\]. Under the wave coordinate gauge, the reduced Einstein equation system takes the form of \((1.1)\), with the righthand side verifying a so-called weak null condition.

Let us compare the simplest example of equations with the weak null condition,

\[\Box m\phi_1 = -(\partial_t \phi_2)^2, \quad \Box m\phi_2 = 0\]  \(\text{(1.4)}\)

with the example constructed by John in \[8\] which does not have a global solution for compactly supported data of any size

\[\Box m\phi = -(\partial_t \phi)^2.\]  \(\text{(1.5)}\)

With \(L = \partial_t + \partial_\alpha, L_x = \partial_t - \partial_\alpha\), we can decompose \(2\partial_t = L + L_x\). Thus the term of \((\Box f)^2\) appears in the quadratic term of both \((1.4)\) and \((1.5)\). The difference lies in that such bad term in \((1.4)\) can be controlled by the better part of the system, since \(\phi_2\) is actually a free wave solution. Although, one can only obtain the weaker decay property

\[|r + t + 1| |\phi_1| \lesssim \ln t,\]  \(\text{(1.6)}\)

due to the appearance of such bad term, the solution \((\phi_1, \phi_2)\) exists globally. Typically, the weak null system consists of good equations which verify null conditions, and bad equations which formally have the terms of \(\Box f \cdot \Box f\). It is important that the function \(f\) verifies the good equations. Einstein equations under wave coordinates verify such weak null property (see \[21\] and \[24\]). The global results for small data are proved by Lindblad and Rodnianski. Clearly \((1.5)\) gives an example that \(\Box f \cdot \Box f\) appears in the equation of \(f\) itself, which does not satisfy the weak null condition.

For ease of discussion, let us consider the data which have compact support. In the domain of influence, by running a standard energy argument for \((1.5)\), we have

\[\|\partial f^n\phi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \lesssim \|\partial f^n\phi(0, \cdot)\|_{L^2(\mathbb{R}^3)} + \int_0^t \|\partial f^n((\partial_t \phi)^2)(t', \cdot)\|_{L^2(\mathbb{R}^3)} dt'.\]  \(\text{(1.7)}\)

For simplicity we only consider one of the terms in \(\partial f^n((\partial_t \phi)^2) = \sum_{a+b=n} \partial f^a\partial t f^b \partial f \partial t \phi\), which is

\[\partial_t \phi \cdot \partial f^n \partial_t \phi.\]  \(\text{(1.8)}\)

Note the standard decay of the free wave for \(\partial f \phi\) with small data is

\[(|t - r| + 1)^\frac{1^+}{2} (t + r + 1)^{|\partial f \phi|} \lesssim \epsilon^{\frac{1}{2}}.\]
By a direct substitution, \( \| \partial \phi(t) \|_{L^2(\mathbb{R}^3)} \leq t^{C_{\epsilon^2}} \| \partial \phi(0) \|_{L^2(\mathbb{R}^3)} \). However, with such energy growth, one can not recover the linear behavior to \( \partial_t \phi \) without loss of decay in \( t \)-variable. With a weaker decay for \( \partial_t \phi \), we can not achieve the boundedness of energy even allowing growth. The only way to obtain the boundedness of energy is by setting \( t \leq T(\epsilon) < \infty \), which implies the semi-global result.

If the quadratic nonlinearity verifies null conditions such as the null forms

\[
Q_0(\phi, \psi) = \partial^\mu \phi \partial_\mu \psi; \quad Q_{\alpha\beta}(\phi, \psi) = \partial_\alpha \phi \partial_\beta \psi - \partial_\alpha \psi \partial_\beta \phi.
\]

Due to

\[
|Q(\phi, \psi)| \lesssim |\partial \phi| \cdot |\partial \psi| + |\partial \phi||\partial \phi|
\]

where \( \partial = (L, \nabla) \) and \( \nabla_i = \partial_i - \frac{r}{\sqrt{r}} \partial_r \), \( i = 1, 2, 3 \), since the above structure is almost preserved if differentiated by the invariant vector fields of the Minkowski space, and since the decay of \( \partial \phi \) can be improved to \((1 + r + t)^{-\frac{3}{2}}\) by using the commuting vector fields approach, we can obtain an additional \((1 + r + t)^{-1}\) decay in the error integral in (1.7) compared with the case for the equation (1.5). This implies the boundedness of energy easily.

Based on the above examples, clearly the presence of the \( L \phi_1 \cdot L \phi_2 \) type term in the quadratic nonlinearities significantly changes the asymptotic behavior of the solution. It either does not allow the local-in-time solution to be extended, or causes the global solution to lose the sharp decay, which is the case for the simplest system (1.1). In the case that \( \Lambda^{\alpha\beta} = 0 \), \( q = 0 \), Lindblad (23) proves the stability result for (1.1) if the metric \( g \) does not depend on \( \phi \), where the loss of sharp decay occurs for \( \phi \). (See also [1].) So if losing the structure of null conditions either in the semilinear quadratic terms or the quasilinear terms, one should not expect the solution has the standard global linear behavior of a wave equation without loss.

Prescribing data on \( \{ r \geq R \} \) does not improve decay property in terms of \( r \) parameter either. In the standard exterior stability results, [18] for Einstein equations and [19] for the massive Maxwell-Klein-Gordon equation with arbitrary charge, the null conditions of the equations have played a crucial role. If both the quasilinear and semilinear nonlinearity verify the null condition, the solution of (1.1) should be well-posed in the entire domain of dependence of \( \{ r \geq R \} \) up to the characteristic boundary if the data is sufficiently small. To prove this standard result, one may have to check if the extrinsic approach of [23, 21, 24] is strong enough to capture the behavior of the characteristic surfaces in this situation, since it is not in the Einsteinian case. It is possible that one has to adopt the intrinsic approach such as in [18]. Without any assumption on the nonlinear structures in (1.1), one should not expect the solution exists in the entire \( \{ r \geq R + t \} \) for semilinear case, nor in the whole exterior of the global characteristic surfaces (upto the boundary) for the quasilinear case. For the quasilinear equations, the characteristic surface can be singular in finite time. Nevertheless, it is possible that the solution remains regular in the majority of the domain of dependence, meanwhile concentrates in the remaining part. This inspires us to extend the solution in subregions of the domain of dependence. We require such subregion to be global in time. In the semilinear case, we require the region can be any close to \( \{ r \geq t + R \} \).

Note that the John’s example in [7],

\[
\Box_m \phi = -\partial_t \phi \Delta \phi
\]

in the spherical symmetry case can be reduced to a Burgers’ equation. For the latter, we know that shock can be formed once certain monotonicity condition of data is broken and the shock can occur along any characteristic, no matter the support of the data is compact or not. Thus, it seems meaningless to discuss where singularities of the solutions are distributed. However, exactly due to this example, we can imagine that if the classical solution is extended globally in time, their life-span along the regular part of each characteristic surface may still be finite. Geometrically, the lightcone of schwarzschild spacetime intersects with any lightcone of Minkowski space within finite time. So in the semilinear case, if we use a schwarzschild cone initiated from \( \{ |x| = R \} \) as the boundary surface, this region matches the expectation indicated by the Burgers’ equation. Such cone has to be spacelike since we need to obtain the positive
energy flux on the boundary. Note that the region bounded by the schwarzschild cones with the small negative mass \(-M_0\) can exhaust \(\{r > t + R\}\) by letting \(M_0 \to 0\). So such region can be any close to \(\{r > t + R\}\), and identical to \(\{r \geq t + R\}\) if and only if \(M_0 = 0\). (See the second picture in Figure 2.)

Physically, we separate the region where the solution may concentrate the most, along a schwarzschild cone, away from the domain of dependence. We then ask if we can achieve any improvement over the standard linear behavior of wave equations in the remaining region, and if the improvement is strong enough to control the nonlinearities.

1.3. **Strategy of the proofs.** The framework of our approach can be clearly seen in the proof for the semilinear case. The main idea is based on a good combination of the boundedness of the standard energy and \(r\)-weighted energy along the foliations of schwarzschild cones, which leads to an improved set of asymptotic behavior by applying the very flexible version of the weighted Sobolev inequalities in [19]. In this part, we will discuss the following aspects of our approach in this paper.

1. The control of the nonlinearity on the right of (1.1) and the improvements compared with the known standard linear behaviors.
2. The influence of the variable potential \(q\) to our approach.
3. The difficulties in the quasilinear case caused by the nontrivial influence of the metric \(g(\phi, \partial\phi)\).
4. The complexities in the application to Einstein scalar fields.

We first explain how we treat the quadratic nonlinearity to achieve the boundedness of energy. For the free wave equation and data \(\phi[0]\) with \(\mathcal{E}_{2, \gamma_0, R} \leq \varepsilon\), we can apply commuting vector fields to derive the standard decay property in \(\{r \geq t + R\}\),

\[
r|\partial\phi| \lesssim \varepsilon^\frac{1}{2}(r - t + 1)^{-\frac{1}{2} - \frac{1}{r}}.
\]

Under the null frame \(L = \partial_t + \partial_r\), \(\underline{L} = \partial_t - \frac{\gamma}{r}\partial_r\), and with \(\nabla_i = \partial_i - \frac{\gamma}{r}\partial_r\), the Cartesian component of covariant derivative on sphere \(S_{t,r}\), the decay for \(\underline{L}\phi\) and \(\nabla\phi\) can be improved. Nevertheless the decay rate in terms of \(t\) is unimprovable for \(\underline{L}\phi\) in the region \(\{r \geq t + R\}\). If we run the standard energy argument for (1.3), we would still end up with having a finite-in-time result.

We now consider to improve the asymptotic behavior of \(\underline{L}\phi\) exterior to a slightly spacelike schwarzschild cone. Let \(u_0(M_0) = -r_s(M_0, R)\) where \(r_s(M_0, r)\) is defined in (2.2). We may denote \(u_0 = u_0(M_0)\) for convenience. Let \(u = t - r_s(M_0, r)\) and \(\underline{u} = t + r_s(M_0, r)\). In the region with \(\{u \leq u_0\}\), we adopt foliations by level sets of \(u\) and \(\underline{u}\), denoted by \(\mathcal{H}_u\) and \(\mathcal{H}_{\underline{u}}\) respectively. The standard energy for \(\phi\) on \(\mathcal{H}_u\) takes the form of

\[
\int_{\mathcal{H}_u} \frac{M_0}{r} |L\phi|^2 + |L\phi|^2 + |\nabla\phi|^2 \, d\mu_H \lesssim |u|^{-\gamma_0} \mathcal{E}_{0, \gamma_0, R}, \gamma_0 > 1
\]

where the \(d\mu_H\) is the area element of \(\mathcal{H}_u\), comparable to \(r^2 du \, d\mu_S\), and \(\mathcal{E}_{0, \gamma_0, R}\) is defined in (1.3) for the initial data.

Next, we explain how such improvement can essentially help us to control the error integral in the standard energy estimate for (1.3). Let \(E(\partial^{(n)} \phi)(\Sigma)\) denote the standard energy on the hypersurfaces \(\Sigma\). Let \(\underline{u_1} \leq u_1 \leq u_0\) and \(\overline{u_1}\) be arbitrarily large. The energy argument gives

\[
E(\partial^{(n)} \phi)(\mathcal{H}_{\underline{u_1}}) + E(\partial^{(n)} \phi)(\mathcal{H}_{\overline{u_1}}) \\
\lesssim \|\partial\theta^{(n)} \phi(0, \cdot)\|_{L^2(\mathcal{H}_{\underline{u_1}})}^2 + \int_{\underline{u_1} \leq u \leq \overline{u_1}} |\partial^{(n)} \phi(0, \cdot)| d\mu_H(\mathcal{H}_{\overline{u_1}}) + \int_{\underline{u_1} \leq u \leq \overline{u_1}} |\partial^{(n)} \phi(0, \cdot)| d\mu_H(\mathcal{H}_{\overline{u_1}}).
\]

Here the truncated hypersurfaces \(\mathcal{H}_{\underline{u_1}}\) and \(\mathcal{H}_{\overline{u_1}}\) are subsets of \(\mathcal{H}_u\) and \(\mathcal{H}_{\underline{u}}\) respectively, both of which are defined in (2.2) in Section 2.

Again we consider only the term (1.8) in the error terms. If we can recover the linear behavior (1.9) with \(r - t\) replaced by \(|u|\) to \(\partial \phi\) under the assumption that the data verify \(\mathcal{E}_{2, \gamma_0, R} \leq \varepsilon\), we
have
\[
\int_{\{-\omega \leq u \leq u_1\}} |D^a \partial_t \phi \cdot \partial_i \phi \cdot \partial_j \partial^b \phi| \lesssim M_0^{-1} \epsilon \frac{1}{\omega} \int_{-\omega}^{u_1} |u|^{-\frac{\gamma_0}{2} + \frac{\gamma_{ij}}{2}} E [D^a \phi] (H_d^u) du.
\]  
(1.12)

With \( \gamma_0 > 1 \), we can achieve the boundedness of energy by Gronwall's inequality. For the nonlinear problem itself, we certainly can not directly use the linear behavior to control error. The analysis will be based on a bootstrap argument. With \( \epsilon \lesssim CM_0^1 \) and \( C \geq 1 \), we can close the bootstrap argument if \( c_0 R^{1-\gamma} < C^{-1} \), where \( C \) is a fixed constant, \( c_0 \geq 1 \) is a universal constant. This is achievable if \( R(\gamma_0) \), the lower bound of \( R \), satisfies the inequality.

Thus it is crucial for our nonlinear analysis to achieve the linear behavior of (1.9) with \( r - t \) replaced by \( |u| \), without loss of the decay in \( r \)-variable. This requires us to perform our analysis in a no-loss regime. The analysis of the full nonlinearities is more involved than the sample term in (1.12). We explain our basic principle below.

If we write the spacetime error integral, such as the last term of (1.11) symbolically as
\[
\int |B_1||B_2||B_3|dxdt.
\]  
(1.13)

The known procedure for bounding the error integral uniformly in the upper limit of \( t \) is to identify one of the factors \( |B_1| \) as \( |G| \) which has a stronger decay in \( r \) than (1.12), if the standard null conditions are satisfied; or using the better feature of the equation (system) to guarantee \( |B_1| \) can achieve the standard global linear behavior, such as the pointwise decay \((t + r^2)^{-1}\), which implies a control with growth of \( r C t^{\frac{1}{2}} \). The latter occurs when the weak null structure is available. Without any of the extra structure, we rely on the sets of decay estimates in Section 4 and the energy flux of the schwarzschild cone foliation to form the hierarchy of the analysis. For these bad terms \( B_1 \), we manage to recover the standard linear behavior, with the bound comparable to the size of the data, \( \Delta_0 \approx \epsilon \frac{1}{\omega} \). They offer good bounds \( || \cdot ||_G \). We also achieve a set of integrated estimates with the bound \( \Delta_0 M_0^{-\frac{1}{2}} \), denoted by \( || \cdot ||_s \), which are stronger than the standard linear behavior. When applying Hölder's inequality, our basic principle is to bound the worst nonlinearity (1.12) by
\[
\int |B_1||B_2||B_3|dxdt \lesssim ||B_1||_G ||B_2||_s ||B_3||_s,
\]
which allows us to improve the bootstrap argument and achieve the boundedness of the norms \( || \cdot ||_G \) and \( || \cdot ||_s \) for the bad terms.

To derive the set of improved estimates \( || \cdot ||_s \) for bad terms, we note the linear behavior (1.10) shows that once the energy flux on \( H_u \) can be bounded in terms of the initial data, since the weighted energy of data is bounded, the flux automatically decays nicely in \( |u| \). We can combine this property with the weighted Sobolev inequalities in 19 to obtain more improvements, in particular, on the integrated decay estimates. Below we list some of such improved estimates, which are important to the results in this paper.

\[
||r^\frac{1}{2} \partial Z(b\phi)|| L^\infty_{t,x}(u \leq u_1) \lesssim \epsilon M_0^{-1} |u_1|^{-\gamma_0 + 2\zeta(Z(b))}, \quad b \leq n - 1 
\]  
(1.14)

\[
||r^\frac{1}{2} \partial Z(l\phi)|| L^\infty_{t,x}(u \leq u_1) \lesssim \epsilon M_0^{-1} |u_1|^{-\gamma_0 + 2\zeta(Z(l))}, \quad l \leq n - 2 
\]  
(1.15)

\[
||r^\frac{1}{2} Z(a\phi)|| L^\infty_{t,x}(u_1) \lesssim \epsilon M_0^{-1} |u_1|^{-\gamma_0 + 2\zeta(Z(a))}, \quad a \leq n, 
\]  
(1.16)

where \( u_1 \leq u_0 \) and \( E_{\gamma,\gamma_0,R}(\phi(0)) \leq \epsilon, n = 2,3 \). Here we denote the ordered product of vector fields as \( Z^k = Z_1 \cdots Z_k \), with \( Z^k \) the corresponding differential operator of \( k \)-th order, where \( Z_i \in \{\Omega_{ij}, \partial\} \), \( \Omega_{ij} = x_i \partial_j - x_j \partial_i, 1 \leq i < j \leq 3 \). \( Z^0 = \partial_t \). The signature function is defined by
\[
\zeta(Z^k) = \sum_{i=1}^{k} \zeta(Z_i), \quad \zeta(\Omega_{ij}) = 0, \quad \zeta(\partial) = -1.
\]

See Proposition 4.1, Proposition 4.5 and Lemma 3.4 for the proofs of (1.14)-(1.16) and more improved estimates.
\( (1.14) \) are crucial for treating other terms in \((1.11) \) so as to achieve the boundedness of energy without any loss. The other two estimates are important for treating the quasilinear problem without loss. \( (1.15) \) is used to for proving the weighted energy estimate. \( (1.16) \) is used to give an improved Hardy’s inequality \((4.5)\), which is crucial to treat \( [\tilde{g}, Z(n)] \phi \).

Next we comment on the influence of the potential \( q \) to our approach.

1. The scaling vectorfield \( S = t \partial_t + r \partial_r \) can not be used as a commuting vectorfield, which is similar to the massive case i.e. \( q > 0 \) is a fixed constant.

2. The asymptotic behavior of the solution is similar to the massless case i.e. \( q = 0 \). Such set of decay is weaker than the standard decay of the massive case.

The problem with variable potential takes the essential difficulties from both the massive and massless wave equations. To treat the potential term, we take the spirit of the multiplier approach developed in \([19]\), and yet have to make further improvements since the asymptotic behavior of the solution is much weaker. We adopt merely \( \{\Omega_{ij}, \partial\} \) to obtain very good decay property for \( \partial \phi \),

\[
|u^j \tilde{r} |L|\phi| + r^2 |\tilde{\nabla} \phi| + r^2 |L|\phi| \lesssim \epsilon |u|^{-\frac{q_0}{2}}
\]

with decay of higher order derivatives included in Section \([2]\) Section \([4]\) and Section \([6]\). If \( q \equiv q_0 \) where the constant \( q_0 > 0 \), we achieve

\[
q_0 r^\frac{q}{2} |\phi|^2 \lesssim \epsilon u^{-q_0+\frac{q}{2}}.
\]

The sharp decay for \( \tilde{\nabla} \phi \) is achieved when we commute rotation vectorfields \( \Omega_{ij} \) up to the third order. We can obtain sharp decay for \( L\phi \) if we also employ the boost vector field up to the third order, which also improves the decay for \( \phi \) to sharp if \( q_0 > 0 \) is a constant \([4]\). Such improvements are not necessary for proving the main results.

For the quasilinear problems, besides the difficulty caused by the semilinear quadratic error terms, we have to solve the difficulties caused by the metric \( g(\phi, \partial \phi) \). In the sequel, we will explain our approach for solving the following issues.

1. Due to the influence of the metric, how to define the exterior region of the spacetime is actually a fundamental problem if not using the characteristic surfaces.

2. Technically, due to the influence of the metric, we have to modify the energy momentum tensor appropriately.

3. We use a multiplier approach to recover the linear behavior for the nonlinear solution. However it requires stronger decay property on \( H(\phi, \partial \phi) \) than the decay for the free wave, in order to obtain the bounded \( r \)-weighted energy.

None of these issues arises in an intrinsic approach such as \([15]\), which relies on the foliations of characteristic surfaces. They all are shifted to controlling the evolution of the geometry of the characteristic surfaces. As explained, such intrinsic approach is not suitable for our problem.

The first issue is linked to the positivity of the energy flux on the boundary \( \mathcal{H}_{u_0} \). Even for equations verifying the standard null condition, one would encounter the same issue. By using a modified energy momentum tensor, we can compute that the energy density along \( \mathcal{H}_{u} \) takes the form of

\[
\left( \frac{M_0}{r} - H L \right)(L \phi)^2 + H \partial \phi \cdot \tilde{\partial} \phi + |\tilde{\partial} \phi|^2.
\]

(See the calculation in Lemma \([6,3]\).) For Einstein equations, with \( M_0 = 0 \), the positivity of energy flux can be achieved if the data \([13]\) are sufficiently small, since due to the positive mass theorem,

\[
\lim_{|x| \to \infty} r H L (x, 0) = -\frac{1}{2} m_0, \quad m_0 > 0.
\]

In Section \([7]\) we will take advantage of this fact to prove the improved result, Theorem \([2,0]\)

---

\(^9\) It works as well if \( q \geq q_0 \).

\(^{10}\) See Theorem \([4,9]\) and Section \([7]\) for the set-up and the meaning of the data.
In general, (1.18) is not coercively positive along the Minkowski cone, i.e. \( M_0 = 0 \). To achieve the positive energy flux, in Section 3 we choose \( M_0 \) according to the size of the data, so that there exists a small constant \( M \) such that

\[
r(\frac{M_0}{r} - H \frac{L_{\infty}}{L}) > M > 0, \quad \text{and} \quad |M_0| \lesssim M.
\] (1.19)

Other error terms in (1.18) can then be absorbed. This treatment actually needs the smallness of \( |rH \frac{L_{\infty}}{L}| \). Although one can see from (1.3) that even for the system (1.4), which has better structure than the general case (1.1), the solution \( \phi \) does not have the sharp decay. But we manage to achieve it in the region \( \{u \leq u_0\} \). There is a similar issue with the energy density on \( H_{\infty} \), which can be solved in the same way.

Therefore with a suitable choice of the mass \(-M_0\) for the boundary cone, we can gain the control of \( Mr^{-1}|L_0|^2 \) in the energy flux along the level set of \( u(M_0) \), i.e. (1.10) holds with \( M_0 \) replaced by the small constant \( M > 0 \). The choice of \( M_0 \) depends on the bound for the data, so does the size of \( M \). This allows us to follow the treatment for (1.5) to control the quadratic nonlinearity.

In [23], in order to improve the asymptotic behavior for the solution of Einstein equations, the asymptotic schwarzschild coordinates and \( r_* \) were employed in the commuting vector field approach. It was used for better approximating the wave operator, since \( g \) itself approaches a schwarzschild metric. Our foliation is chosen to dominate over the influence of \( H \frac{L_{\infty}}{L} \), thus it is always slightly away from the characteristic surfaces of the asymptotic equations (See the definition in [24]). The schwarzschild cone foliation is used throughout the paper, even for treating the flat wave operator. We use the Minkowskian vector fields and Minkowski metric throughout the paper, so as to take advantage of the difference between the Minkowski geometry and the schwarzschild geometry.

Next, we discuss the other two issues. Schematically, for both the semilinear and quasilinear cases, the main task is to bound the standard energy and \( r \)-weighted energy in terms of initial data. For ease of discussion, we assume \( q \equiv 0 \) in the sequel. In the semilinear case, in terms of the standard energy momentum tensor \( Q_{\alpha\beta} = \partial_\alpha \varphi \cdot \partial_\beta \varphi - \frac{1}{2} \mathfrak{m}_{\alpha\beta} \partial^\rho \varphi \partial_\rho \varphi \), the standard energy is defined by \( \int_{\Sigma} Q_{\alpha\beta} X^\alpha n^\beta d\mu_\Sigma \), where \( X = \partial_t \) and \( n^\alpha \) denotes the surface normal of \( \Sigma \). The \( r \)-weighted energy is defined by using \( X = rL \) with suitable modifications in the energy current. The energy estimates are based on the following calculation

\[
\partial^\alpha (Q_{\alpha\beta} X^\beta) = \Box m_\alpha \varphi X \varphi + Q_{\alpha\beta} \partial^\alpha X^\beta.
\]

For the quasilinear operator, we have to make a modification, otherwise the right hand side contains \( \partial^2 \varphi \). One may adopt the intrinsic version,

\[
\partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi
\]

and lift or lower the indices by the metric \( g \). We construct the energy momentum tensor as follows,

\[
\tilde{\Omega}_{\alpha\beta} [\varphi] = \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} \mathfrak{m}_{\alpha\beta} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + H_{\gamma}^{\rho} \partial_\gamma \varphi \partial_\rho \varphi
\]

which is not symmetric, nevertheless gives nice structures in the energy density under the Minkowski background.

In the quasilinear case, the form of the energy momentum tensor, the choice of multiplier and the modification to the energy current are all very sensitive for proving the \( r \)-weighted energy estimates. Typically, bounding \( r \)-weighted energy requires more decay than a free wave verifies. See [22] Section 1 (3)], where an additional \( r^{-\varepsilon} \) decay is assumed, with \( \varepsilon > 0 \), even for the equations with null condition therein. Our improvement is however weaker than this assumption. Our proof of the inequality for the weighted energy is a very delicate one. Since \( L_\varphi \) term can not take the weight of \( r \), we need to treat terms of \( f(H, \partial H) (L_\varphi)^2 \) carefully. We choose \( X = r(L - H \frac{L_{\infty}}{L}) \), which is influenced by considering the asymptotic equation (see [23 [24 [22]). In Lemma 6.3 it turns out the construction of energy current in (6.18) leads to a good structure in the error terms. Undesirable terms, such as \( \int_{\{u \leq u_0\}} r|LH \frac{L_{\infty}}{L} (L_\varphi)^2|dxdt \), are
We manage to use the estimate of \( \| r^4 \partial H(\phi, \partial \phi) \|_{L^2 L^\infty} \) in (1.15) and the fluxes along schwarzschild cones to cope with error terms.

At last we comment on the treatment of the Einstein scalar fields. In comparison with Theorem 1.1 (or Theorem 2.5), \( H \) converges to a small static solution instead of 0. The static part slows down the decay properties of \( H \). Fortunately for Theorem 2.5, the derivation of the inequalities of energy and weighted energy relies more on the decay of \( \partial H \), which is barely influenced. The framework of Section 6 still works through. However, borderline terms appear in the commutator \( [\Box_g, Z^{(n)}] \), since \( H \) has less decay in \( |u| \). They are proved to be harmless, when we show the boundedness of energies for \( Z^{(n)}(h^1, \phi) \) with an induction on the signature \( \zeta(Z^n) \) from \(-n\).

As future extensions of this work, we believe the approach can be applied to give the global result for the quasilinear wave systems with weak form of null conditions, if the small weighted data are prescribed throughout the initial slice. We also believe the result of Theorem 2.5 can be generalized to fluids with nontrivial vorticity. It would be also interesting to ask if there is any global-in-time interior stability result for the equation (1.1) with small compactly supported data.

1.4. Structure of the paper. In Section 2 we give the details of the geometric set-up and introduce the main theorems, which are Theorems 2.1, Theorem 2.5 and Theorem 2.6. In Section 3 we introduce the weighted Sobolev inequalities and derive some consequences of bounded standard and \( r \)-weighted energies, including some sharp \( L^p \) type estimates in Lemma 3.4. In Section 4 under the assumption of bounded energies up to \( n \)th-order, with \( n = 2 \) or 3, we derive the full set of decay properties in Proposition 4.1 and Proposition 4.5. In Section 5 we consider the semilinear case of (1.1) and prove Theorem 2.1 and 2.2. This section gives the main framework of our approach. Schematically, we divide it into three steps. We first derive the energy inequalities. Under the bootstrap assumption of the smallness of energies up to \( n = 2 \), we then employ the decay results in Section 4 to analyse the error \((\Box_m - q)Z^{(n)} \phi \). The final step is to achieve the boundedness theorem for the energies by substituting the error estimates into the energy inequalities. In Section 6, we prove Theorem 2.5. Due to the influence of the error, we need to make (1.19) hold in \( \{ u \leq u_0 \} \) which makes the bootstrap argument more involved. We also need to obtain higher order energy control for treating commutators \( [\Box_g, Z^{(n)}] \). We still run the same three steps as for the semilinear case. The analysis is more delicate in each part. In Section 7 we prove Theorem 2.6. We need to show the metric difference, which is one part of the solution is convergent to a small static solution, while the scalar field converges to 0 as \( r \to \infty \). By a simple reduction, we still solve the problem with data convergent to \((0, 0)\) at the spatial infinity. However, the static part in \( H \) has slower decay property. This may change the behavior of the wave operator. In Proposition 7.4 we confirm the inequalities for energy and weighted energy still hold under such background metric. We then analyze commutators in Lemma 7.5 which contain borderline terms. For the error terms not included in (1.1), we treat them in Lemma 7.7. At last we combine the error estimates in Section 6 to complete the proof.

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2. Set-up and main results

We first construct the foliations that will play a very crucial role in improving the asymptotic behavior in this paper. We also need the construction to determine the stability region in the main results.

\textsuperscript{11} Decay in (1.17) for \( LH \) is not strong enough to control this term.

\textsuperscript{12} See Theorem 2.6 for the definitions of \( h^1, \phi \).
Let $|M_0| \ll 1$ be a constant. We set 
\[ h = M_0/r, \quad L' = L - hL, \quad L'' = L - hL, \quad \forall r \geq 1. \]
We first give the optical function of the following metric
\[ g = -\frac{r + M_0}{r - M_0}dt^2 + \frac{r - M_0}{r + M_0}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.1) \]
Note this metric has the same lightcones initiated from $\{x = r, t = 0\}$ as the Schwarzschild metric of the mass $-M_0$
\[ g_s = \frac{r + M_0}{r - M_0}dt^2 + \frac{r - M_0}{r + M_0}dr^2 + (r - M_0)^2(d\theta^2 + \sin^2 \theta d\phi^2). \]
Suppose $u$ is an optical function of the metric $g$,
\[ \partial_t u = \frac{r + M_0}{r - M_0}\partial_r u. \]
Let $t = \gamma(r)$ be the null geodesic initiated from the sphere of radius $R_1$ at $t = 0$. $u(\gamma(r), r) = C$.
\[ \partial_t u\gamma(r) = \partial_r u. \]
Thus
\[ \gamma^\pm(r) = \pm \frac{r - M_0}{r + M_0}. \]
Then $\gamma^\pm(r) = \gamma^\pm(R_1) \pm (r - R_1 - 2M_0 \ln \frac{r + M_0}{r - M_0})$. For convenience, we can regard $R_1 = 1$. Thus by setting
\[ r_*(M_0, r) = r - 2M_0 \ln \frac{r + M_0}{1 + M_0} \quad (2.2) \]
we have $\gamma^+(r) = \gamma^+(1) + r_* - 1$. And we can regard the outgoing lightcone of the metric $(2.1)$, initiated from $\{r = R_1\}$ as a ruled surface generated by $t = \gamma^+(r)$, $\forall \omega \in S^2$. Identically, it also is the level set of $t - \gamma^+(r) = 0$. We can set up the foliation of Schwarzschild lightcones in $(\mathbb{R}^{3+1}, m)$ with the help of the pair of optical functions of $(2.1)$
\[ u = t - r_*, \quad u = t + r*. \]
Similarly, we can check that $\{u = C\}$ is the incoming null cone of $g$, which is a smooth ruled surface by incoming null geodesics. Clearly, $u(0, 1) = -1$ and $-u(0, r) \approx r$ when $r \geq 2$\(^1\) It is direct to compute the generators of the outgoing and incoming null geodesics
\[ -g^{\alpha\beta}\partial_\alpha u\partial_\beta = (1 + h)^{-1}L', \quad -g^{\alpha\beta}\partial_\alpha u\partial_\beta = (1 + h)^{-1}L''. \quad (2.3) \]
which are tangent to $\mathcal{H}_u$ and $\mathcal{H}_u^{-}$ respectively, and coincide with our construction. We denote by $\mathcal{N}$, $\mathcal{N}'$ the surface normals of $\mathcal{H}_u$ and $\mathcal{H}_u^{-}$ in terms of the Minkowski metric, which are normalized in terms of $\langle \mathcal{N}, \partial_t \rangle = -1$ and $\langle \mathcal{N}', \partial_t \rangle = -1$. In view of $(2.3)$, it is easy to compute that
\[ \mathcal{N} = (1 + h)^{-1}(L + hL), \quad \mathcal{N}' = (1 + h)^{-1}(L + hL). \quad (2.4) \]
Let
\[ u_0(M_0) = u_{M_0}(0, R) = -r_*(M_0, R) \quad (2.5) \]
with the fixed constant $R \geq 2$. In case there occurs no confusion, we may use $u_0$ as a shorthand notation. We now consider the region in $(\mathbb{R}^{3+1}, m)$ where $u \leq u_0$. By setting $1 + b^{-1} = L(u)$, we can easily calculate the lapse function
\[ b^{-1} = \frac{1 - h}{1 + h}. \quad (2.6) \]
Instead of using $t$ to parameterize $\mathcal{H}_u$ and $\mathcal{H}_u^{-}$, we will use $\frac{u}{u}$ and $u$. It is straightforward to compute
\[ Lu = \frac{L - L_0}{r + M_0} = 1 - b^{-1} \quad L\frac{u}{u} = L - L_0 = 2 - \frac{2M_0}{r + M_0} = 1 + b^{-1}. \]
\(^1\)We assume $r \geq 2$ throughout the paper if not stated otherwise.
Thus we can obtain for \( u_1 \leq u_0, -u_1 \leq u_0 \) and \( \omega \in S^2 \),
\[
\frac{d}{du} = \frac{r}{2(r - M_0)}L' \text{ on } \mathcal{H}_{u_1}, \quad \frac{d}{du} = \frac{r}{2(r - M_0)}L' \text{ on } \mathcal{H}_u. \tag{2.7}
\]

This implies
\[
\partial_u r = \frac{1}{2} b \text{ on } \mathcal{H}_{u_1}, \quad \partial_u r = -\frac{1}{2} b \text{ on } \mathcal{H}_u. \tag{2.8}
\]

By using \( u, u \) level sets to foliate the spacetime, the standard area element is
\[
dx dt = (2r_s'(r))^{-1}r^2du d\omega = \frac{1}{2}br^2du d\omega,
\]
where \( d\omega \) denotes the standard surface measure on the unit sphere \( S^2 \). Thus in view of (2.6), the area elements of \( \mathcal{H}_u \) and \( \mathcal{H}_u \) are
\[
d\mu_u = \frac{1}{2} br^2du d\omega, \quad d\mu_u = \frac{1}{2} br^2du d\omega. \tag{2.9}
\]

Let \( S_{u,u} = \mathcal{H}_u \cap \mathcal{H}_u \), where \( u \geq -u \). For smooth functions \( f, \int_{S_{u,u}} f = \int_{S_{u,u}} r^2f d\omega \). Clearly, \( b \) is an increasing function of \( h \), \( b = 1 + O(h) \). Thus the area elements in (2.9) are comparable to \( \frac{1}{2}r^2du d\omega \) and \( \frac{1}{2}r^2du d\omega \). Note that \( \partial_u u = -b^{-1} = -\partial_u u \) on \( \Sigma_t \), the area element on \( \Sigma_t \) is \( br^2du d\omega \). Thus on \( \Sigma_t \), \( r^2du d\omega \approx dx \approx r^2du d\omega \).

By the definition of \( u, 1 - \frac{b}{r} = M_0 O(u) \). Thus we can derive
\[
r(S_{u,-u}) = -u(1 + o(M_0)), \quad r(S_{-u,u}) = u(1 + o(M_0)). \tag{2.10}
\]
where the second identity is an application of the first one, based on the fact that \( \mathcal{H}_{-u} \) is initiated from \( S_{-u,u} \).

We also have the basic fact that \( r(u, u, \omega) \) is increasing about \( u \) for fixed \( (u, \omega) \) and decreases as \( u \) increases if \( (u, \omega) \) is fixed. Note also that \( r_s(r) \leq u \leq 2r_s(r) - r_s(R) \). These two facts together with (2.10) imply
\[
-u \leq r_{\min}(H_u), \quad u \approx r \approx r_s(r). \tag{2.11}
\]
(2.10), (2.11) and the fact that \( u \geq -u \) will be frequently used in our analysis, probably without being mentioned.

We use \( \mathcal{H}_u^{\pm} \) and \( \mathcal{H}_u^{\pm} \) to denote the truncated level sets of \( u \) and \( u \) respectively.
\[
\mathcal{H}_u^{\pm} := \{(t, x) : -u \leq u' \leq u \}, \quad \mathcal{H}_u^{\pm} := \{(t, x) : -u \leq u' \leq u \},
\mathcal{D}_u^{\pm} := \{(t, x) : -u \leq u' \leq u \} \tag{2.12}
\]
where \(-u \leq u \leq u_0 \).

We denote
\[
\Sigma^{u_1, u_1} = \{-u_1 \leq u \leq u_1, t = 0\} = \{-u_1 \leq u \leq u_1, t = 0\},
\]
and may drop \( u_1 \) when \( u_1 = \infty \).
We denote by $E[f](\Sigma)$ and $W_1[f](\Sigma)$ the energy (flux) and weighted energy (flux) of the smooth function $f$ on the hypersurface $\Sigma$. For the hypersurfaces of interest to us,

\[
E[f](\Sigma_0^M) = \int_{\Sigma_0^M} |\partial f|^2 + qf^2 \, dx,
\]

\[
E[f](H_0^M) = \int_{H_0^M} \frac{1}{2} r^2 (|Lf|^2 + \frac{M}{r} |L f|^2 + |\nabla f|^2 + qf^2) \, d\omega,
\]

\[
E[f](H_0^M) = \int_{H_0^M} \frac{1}{2} r^2 (|Lf|^2 + \frac{M}{r} |L f|^2 + |\nabla f|^2 + qf2) \, d\omega,
\]

\[
W_1[f](H_0^M) = \int_{H_0^M} \frac{1}{2} r (rL (r f))^2 + r^3 \frac{M}{r} (|\nabla f|^2 + qf2) \, d\omega
d\omega,
\]

\[
W_1[f](H_0^M) = \int_{H_0^M} \frac{1}{2} r^3 (|\nabla f|^2 + \frac{M}{r} |L f|^2 + qf2) \, d\omega,
\]

\[
W_1[f](D_0^M) = \int_{D_0^M} r^2 (|Lf|^2 + |\nabla f|^2) \, dx + dt,
\]

\[
W_1[f](\Sigma_0^M) = \int_{\Sigma_0^M} r^{-1}(L f)^2 + r(|\nabla f|^2 + qf2) \, dx,
\]

where $M > 0$ is a fixed constant to be specified. Throughout the paper, $M$ and $h$ are chosen such that

\[
r|h| \leq M.
\]

Throughout the paper, we set $u_+ = -u$ and let $Z^{(n)}$ be the $n$-th order differential operator $Z_1 \cdots Z_n$, with each $Z_m \in \{\partial, \Omega, 1 \leq i < j \leq 3\}$. We are ready to state the main results of this paper.

**Theorem 2.1.** Consider

\[
\Box_m \phi = N^{\alpha\beta} (\phi) \partial_\alpha \phi \partial_\beta \phi + q(x) \phi
\]

with $0 \leq q \leq 1$ satisfying (1.3) for $n = 2$. Let $1 < \gamma_0 < 2$ and $C > 1$ be fixed constants. There exist a small constant $0 < \delta_1 \ll \frac{1}{100}$ and a universal constant $R(\gamma_0, C) \geq 2$ such that for any $0 < M \leq \delta_1^2$, if the initial data set $\phi(0)$ satisfies $\Box_2 \phi(0) + \frac{1}{2} \Box_1 \phi(0) + \frac{1}{2} \Box_3 \phi(0) \leq R^2$, there exists a unique solution $\phi(t)$ for all $t > 0$.

**Theorem 2.2.** Consider (2.14) with $0 \leq q \leq 1$ satisfying (1.3) for $n = 2$. Let $R \geq 2$ and $1 < \gamma_0 < 2$ be fixed. There exist small constants $0 < \delta_1 \ll \frac{1}{100}$, $\delta_0 > 0$ such that for any $0 < M \leq \delta_1$, if the initial data set verifies

\[
\Box_2 \phi(0) \leq \delta_0 M^2,
\]

there exists a unique solution in the entire region of $\{u(M) \leq u_0(M)\}$ for all $t > 0$. There hold the same set of energy estimates as in (2.10) and the pointwise estimates for any $\underline{u} \leq u \leq u_0(M)$.

**Remark 2.3.** Let $S_M = \{u(M) \leq -r_+(R, M)\}$, which is the exterior region to the schwarzschild outgoing cone initiated from $\{r = R\}$ including the boundary. Clearly $S_{M_1} \subset S_{M_2}$ if $M_1 > M_2$. Note that as $M \to 0$, $u(M) \to t - r$ and $S_M$ approaches the entire open exterior of $\{r = t = R\}$.
Then Theorem 2.1 indicates that, for any \(0 < M \leq \delta_1^2\), the stability result can always holds in \(S_M\) for the set of non-compact supported data with the norm (1.3) bounded by \(M^2\), provided that \(R \geq R(\gamma_0)\). See Figure 2.

Remark 2.4. Theorem 2.2 is a consequence of the proof of Theorem 2.1 under a stronger assumption on data. One can refine it by assuming part of the energy norm of data verifies (2.17).

**Theorem 2.5.** Consider (1.7) which verifies (1.2) for \(n = 3\). There exists a universal constant \(C \geq 1\), a small constant \(0 < \delta_1 < \frac{1}{100}\) and a constant \(R(\gamma_0, C) \geq 2\), such that, if the initial data set \(\phi[0]\) satisfies that

\[
E_{3, \gamma_0, R} \leq \delta_1, \quad \text{with } R \geq R(\gamma_0, C)
\]

(2.18) there exists a unique solution in the entire region \(\{u(M) \leq u_0(M)\}\) with \(M = C\delta_1^2\). The solution verifies the following energy estimates for \(-u \leq u \leq u_0(M)\),

\[
E[Z^{(n)}\phi](H_n^u) + E[Z^{(n)}\phi](H_n^u) \leq E_{3, \gamma_0, R} u_+^{-\gamma_0 + 2\xi(Z)}
\]

\[
W_1[Z^{(n)}\phi](H_n^u) + W_1[Z^{(n)}\phi](H_n^u) + W_1[Z^{(n)}\phi](D_n^u) \leq E_{3, \gamma_0, R} u_+^{-\gamma_0 + 2\xi(Z)}
\]

where \(n \leq 3\), \(Z \in \{\Omega_{ij}, \partial\}\) and verifies the decay estimates

\[
r^3|\nabla Z^{(l)}\phi|^2 + r^2 u_+ |LZ^{(l)}\phi|^2 + r^3 |LZ^{(l)}\phi|^2 \leq E_{3, \gamma_0} u_+^{-\gamma_0 + 2\xi(Z)}, l \leq 1.
\]

As an application, we provide the following exterior stability results for Einstein scalar fields under the wave coordinates.

**Theorem 2.6.** Consider the Einstein scalar field system

\[
\begin{aligned}
\mathbf{R}_{\alpha \beta}(g) &= \partial_\alpha \phi \partial_\beta \phi + \frac{1}{4} q_0 g_{\alpha \beta} \phi^2, \\
\Box_g \phi &= q_0 \phi
\end{aligned}
\]

(2.19)

where the constant \(q_0 \geq 0\). Under the wave coordinate gauge, we set \(h_{\mu \nu} = g_{\mu \nu} - m_{\mu \nu} - \frac{m_0}{r} \delta_{\mu \nu}\). For constants \(1 < \gamma_0 < 2\) and \(C_0 \geq 1\), there exist a small constant \(\delta_1 > 0\) and a constant \(R(\gamma_0, C_0) \geq 2\), such that, if the initial data set \((h^1[0], \phi[0])\) verifies

\[
E_{3, \gamma_0, R_0}(h^1[0]) + E_{3, \gamma_0, R_0}(\phi[0]) \leq C_0 m_0^2, \quad 0 < m_0 < \delta_1
\]

where \(R \geq R(\gamma_0, C_0)\), then there exists a unique solution \(\Psi = (h^1, \phi)\) for (2.17) in the entire region of \(\{u(-m) \leq u_0(-m)\}\), where \(0 \leq m < \frac{1}{4} m_0\) is a fixed constant.\(^{13}\) With \(u, u\) the shorthand notations for \(u(-m)\) and \(u(-m)\), for \(-u \leq u \leq u_0(-m)\), there hold

\[
E[Z^{(n)}\Psi](H_n^u) + E[Z^{(n)}\Psi](H_n^u) \leq m_0^2 u_+^{-\gamma_0 + 2\xi(Z)}
\]

\[
W_1[Z^{(n)}\Psi](H_n^u) + W_1[Z^{(n)}\Psi](H_n^u) + W_1[Z^{(n)}\Psi](D_n^u) \leq m_0^2 u_+^{-\gamma_0 + 2\xi(Z)}
\]

where \(n \leq 3\) and \(Z \in \{\Omega_{ij}, \partial\}\). There also hold the decay estimates,

\[
r^3|\nabla Z^{(l)}\Psi|^2 + r^2 u_+ |LZ^{(l)}\Psi|^2 + r^3 |LZ^{(l)}\Psi|^2 \leq m_0^2 u_+^{-\gamma_0 + 2\xi(Z)}, l \leq 1,
\]

\[
q_0^2 |\nabla Z^{(l)}\phi|^2 \leq m_0^2 u_+^{-\gamma_0 + 2\xi(Z)} + \frac{1}{4}, l \leq 1.
\]

The constants in the above inequalities are independent of \(q_0^{-1}\).

Remark 2.7. The energy estimates and pointwise decay in the above four theorems hold true if \(Z\) belongs to the generators of Poincaré group. If \(q \equiv 0\), one can also extend the result to the set of vector fields \(\{\partial, x^\mu \partial_\mu, \Omega_{\mu \nu}\}\).\(^{13}\)

---

\(^{13}\)We assume they satisfy the constraint equations. See details of the data construction in (2.3)-(2.5),Page 1410.

\(^{16}\)In our proof, we fix \(m = \frac{m_0}{r}\) for convenience.

\(^{17}\)\(\Omega_{\mu \nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad 0 \leq \mu < \nu \leq 3\) where \(x_\mu = m_{\mu \nu} x^\nu\).
Remark 2.8. We do not rely on the weak null condition of the Einstein scalar field equation (2.19) to prove Theorem 2.6. If we further use the precise weak null structure of the reduced Einstein equation, the result can be proved up to $R = 2$ under a weak extra smallness assumption of data.

3. Preliminary estimates

In this section, we adapt the Sobolev inequalities developed for the canonical null hypersurfaces in [19] to $u$ and $\mathbb{u}$ foliations. With the help of this set of Sobolev inequalities, we provide preliminary estimates in the region of $\{u \leq u_0(M_0)\}$ for functions bounded in terms of the energy norms in (2.13). Some of the estimates, such as (3.32) and (3.30) in Lemma 3.4 are stronger than the known estimates for the free wave. They are crucial for the proof of boundedness of energies. We also provide estimates on the initial slice in this section.

For ease of exposition, we denote by $\Omega$ any of the rotation vector fields in $\{\Omega_{ij}, 1 \leq i < j \leq 3\}$ and by $\Omega^{(k)} f$ any of the $k$-th order derivatives by the rotation vector fields. $|\Omega f|^2 = \sum_{1 \leq i < j \leq 3} |\Omega_{ij} f|^2$. $|\Omega^{(k)} f|^2$ is the sum of all the combinations of $k$-th order derivatives by rotation vector fields. The same convention applies to $|P(\Omega f)|$ if $P$ is a linear differential operator.

We adapt from [19, Section 2.1] to obtain the following Sobolev inequalities.

Lemma 3.1 (Sobolev inequalities). For any smooth function $f$ and constants verifying the relation $2\gamma = \gamma_0' + 2\gamma_2$, we have, for all $-\mathcal{M} \leq u \leq u_0(M_0)$,

\[
\sup_{S_{u \mathcal{M}}} |r^\gamma f|^4 \lesssim \sum_{l \leq 1} \int_{S_{u \mathcal{M}}} |r^\gamma \Omega^{(l)} f|^4 r^{-2} + \sum_{k \leq 2} \int_{\mathcal{M}^{u \mathcal{M}}} r^{2\gamma_2} |\Omega^{(k)} f|^2 r^{-2} \cdot \sum_{l \leq 1} \int_{S_{u \mathcal{M}}} r^{\gamma_0} |L'(\Omega^{(l)} (r^\gamma f))|^2 r^{-2},
\]

(3.1)

\[
\int_{S_{u \mathcal{M}}} |r^\gamma f|^4 r^{-2} \lesssim \int_{S_{u \mathcal{M}}} |r^\gamma f|^4 r^{-2} + \int_{\mathcal{M}^{u \mathcal{M}}} r^{\gamma_0} |L'(r^\gamma f)|^2 r^{-2} \cdot \sum_{l \leq 1} \int_{\mathcal{M}^{u \mathcal{M}}} r^{2\gamma_2} |\Omega^{(l)} f|^2 r^{-2}.
\]

(3.2)

The same estimates hold by using the incoming null hypersurface $S_{\mathbb{u} \mathcal{M}}$. In this case $L'$ is replaced by $L'$, and the initial sphere is $S_{-\mathbb{u} \mathcal{M}}$.

This lemma can be proved similarly as in [19] with the help of (2.7) and $|M_0| \ll 1$.

Let (2.14) hold. We first give the following results in the initial slice.

Proposition 3.2. Let $1 < \gamma_0 < 2$ and the constant $R \geq 2$ be fixed. With $n \leq 3$, there hold on $\Sigma_0 \cap \{r \geq R\}$ the following estimates.
Thus, with $F_{16}$ Qian Wang

(1) \[
\int_{S_{r}} r^{1+\gamma_{0} - 2\zeta(Z')} |Z^{(i)}\phi|^{2} \omega \lesssim \mathcal{E}_{i, \gamma_{0}}, \quad i \leq n,
\]
(2) \[
\int_{S_{r}} r^{2+2\gamma_{0} - 4\zeta(Z')} |Z^{(i)}\phi|^{4} \omega \lesssim \mathcal{E}_{i, \gamma_{0}}^{2}, \quad i \leq n,
\]
where $S_{r} = \{|x| = r\}$.

(2) Let $u_1, u_1$, be a pair of fixed numbers verifying $-u_1 \leq u_1 \leq u_0$. There hold

\[
\int_{-u_1}^{u_1} \int_{S_{r}} r^{2-4\zeta(Z')} |Z^{(i)}\phi|^{4} \omega \lesssim u_{1+\gamma_{0} + 1} E_{i, \gamma_{0}}, \quad i \leq n,
\]
\[
\int_{-u_1}^{u_1} \int_{S_{r}} r^{-4\zeta(Z') - 1} |\partial Z^{(i)}\phi|^{4} \omega \lesssim u_{1+\gamma_{0} + 1} E_{i, \gamma_{0}}, \quad i \leq n.
\]
The same estimates hold if the domain of integrals are changed to $\int_{-u_1}^{u_1} \int_{S_{r}} r^{2} |Z^{(i)}\phi|^{4} \omega \lesssim u_{1+\gamma_{0} + 1} E_{i, \gamma_{0}}, \quad i \leq n$.

(3) For $i \leq n$, there holds on $\Sigma_{0} \cap \{r \geq R\}$ that

\[
r^{2} |Z^{(i-1)}\phi|^{2} \omega \lesssim u_{1+\gamma_{0} + 1} E_{i, \gamma_{0}}.
\]

Proof. Note that for $i \leq n$, due to $E_{n, \gamma_{0}, R} < \infty$,

\[
\lim_{r \to \infty} \inf \int_{S_{r}} r^{\gamma_{0} - 2\zeta(Z')} (|Z^{(i)}\phi|^{2} + r^{2} |\partial Z^{(i)}\phi|^{2}) \omega = 0.
\]

By using the Sobolev embedding on $\mathbb{S}^{2}$, we have for any scalar function $F$,

\[
(\int_{S_{r}} r^{4} |F|^{4} \omega)^{\frac{1}{2}} \lesssim \|\Omega F\|_{L^{2}(S_{r})} + \|F\|_{L^{2}(S_{r})}.
\]

Thus, with $F = Z^{(i)}\phi$, $i \leq n$, by using (3.10), we have

\[
\lim_{r \to \infty} \inf \int_{S_{r}} r^{2\gamma_{0} - 4\zeta(Z')} |Z^{(i)}\phi|^{4} \omega = 0.
\]

Now consider with the help of (3.10). By integrating back from the spacelike infinity,

\[
\int_{S_{r}} r^{1+\gamma_{0} - 2\zeta(Z')} |Z^{(i)}\phi|^{2} \omega \leq r^{1+\gamma_{0} - 2\zeta(Z')} \int_{r}^{\infty} |\partial r Z^{(i)}\phi|^{2} \omega \omega' \lesssim |\partial r Z^{(i)}\phi| \cdot r^{1+\gamma_{0} - 2\zeta(Z')} \|L_{r}^{2} L_{r}^{2} \|Z^{(i)}\phi| \cdot r^{2\zeta(Z')} \|L_{r}^{2} L_{r}^{2} \| \lesssim \mathcal{E}_{i, \gamma_{0}}^{2},
\]

which gives (3.3).

By using (3.11) and Hölder’s inequality, we derive

\[
\int_{S_{r}} |Z^{(i)}\phi|^{4} \omega \leq \int_{r}^{\infty} |\partial r Z^{(i)}\phi|^{3} \omega \omega' \lesssim |\partial r Z^{(i)}\phi| \|L_{r}^{2} L_{r}^{2} \|^{2} \|Z^{(i)}\phi|^{3} \|L_{r}^{2} L_{r}^{2} \| \lesssim \mathcal{E}_{i, \gamma_{0}}^{2}.
\]

Note that by using [4] Page 58 (3.2.4a)

\[
\int_{r}^{\infty} |F|^{6} d\omega \lesssim \int_{r}^{\infty} \int_{\mathbb{S}^{2}} |F|^{4} d\omega \int_{\mathbb{S}^{2}} (|F|^{2} + |r\bar{\Psi} F|^{2}) d\omega'.
\]
We then combine the above inequality with (3.12) to obtain
\[
\sup_{r' \geq r} \left( \int_{S_{r'}} |Z(i)\phi|^4 d\omega \right)^{1/4} \lesssim \| \partial Z(i)\phi \|_{L^2L^2(r' \geq r)} \| Z(i)\phi \| + \| r\nabla Z(i)\phi \|_{L^2L^2(r' \geq r)} \lesssim r^{-1}(1-2\gamma(Z')+\gamma_0)\mathcal{E}_{i,\gamma_0}.
\]
This gives (3.3).

Note that \( u_1 \geq u \geq -u_1 \), and (2.10) implies \( r(S_{u_1\omega}) \geq \frac{1}{u} \). By integrating (3.3), we can obtain (3.5). We then combine the above inequality with (3.12) to obtain (3.7), which implies (3.6) immediately. (3.7) follows as a direct integration of (3.3).

Next, we prove (3.9). Let \( r_i \geq R \). We adapt (3.1) to \( \Sigma_0 \cap \{ r \geq R \} \) with \( \gamma'_0 = \gamma = 1 \) and \( \gamma_2 = \frac{1}{2} \). This implies for \( r \geq r_1 \),
\[
\sup_{r'} |rZ(i)\phi|^4 \leq \lim_{r \to \infty} \inf_{r'=\infty} \int_{S_{r'}} |\gamma(\Omega)^{(i-1)}Z(i-1)\phi|^4 r^{2-2} + \int_{\Sigma_0 \cap \{ r \geq r_1 \}} r^{2\gamma_2(\Omega)^{(i-2)}Z(i-1)\phi|^2 r^{2-2}}.
\]
where due to (3.3) and \( \gamma_0 > 1 \), the first term on the right vanished, and we also used the fact that \( |\Omega f| \lesssim |r\partial f| \) to treat the term of \( \Omega^{\leq 2} Z^{(i-1)}\phi \). Thus, in view of (2.10), (3.9) is proved.

The energy or weighted energy norms in (2.13) not only give control on \( \partial \varphi \), they also control \( \varphi \) itself, which is given in the following result.

**Lemma 3.3.** Let \(-u \leq u \leq u_0(M_0)\) and \( \alpha > 0 \) be fixed. There hold the following estimates
\[
\int_{\mathbb{D}_u} r^{2}\varphi^2 d\omega + \int_{\mathbb{D}_{\bar{u}}} (r^2(L\varphi)^2 + \alpha |\frac{u}{r}|^\alpha \varphi^2) d\omega' \lesssim W_1[\varphi](\mathcal{H}_{\omega}) + M \int_{-u}^{u} u^{-1} E[\varphi](\mathcal{H}_{\omega}) du' + \int_{\Sigma_0 \cap \{ r \geq r_1 \}} r^{-1} \varphi^2 dx, (3.13)
\]
\[
\int_{\mathbb{D}_u} r|L'(r\varphi)|^2 d\omega' \lesssim W_1[\varphi](\mathcal{H}_{\omega}) + M E[\varphi](\mathcal{H}_{\omega}) + M^2 \int_{\mathbb{D}_u} r^{-1} |\varphi|^2 d\omega' + E[\varphi](\mathcal{H}_{\omega}), (3.14)
\]
\[
\int_{S_{u-u}} r\varphi^2 d\omega + \int_{S_{u-u}} \varphi^2 d\omega' \lesssim \int_{S_{u-u}} r\varphi^2 d\omega + E[\varphi](\mathcal{H}_{\omega}), (3.15)
\]
\[
\int_{\mathbb{D}_u} \varphi^2 d\omega' \lesssim \int_{S_{u-u}} r\varphi^2 d\omega + \int_{S_{u-u}} r\varphi^2 d\omega' + E[\varphi](\mathcal{H}_{\omega}) + E[\varphi](\mathcal{H}_{\omega}). (3.16)
\]

**Proof.** We first prove
\[
\int_{\mathbb{D}_u} r\varphi^2 d\omega + \int_{\mathbb{D}_{\bar{u}}} \left\{ r^2(L\varphi)^2 + \alpha |\frac{u}{r}|^\alpha \varphi^2 \right\} d\omega' \lesssim \int_{\mathbb{D}_u} |L'(r\varphi)|^2 d\omega' + \int_{\Sigma_0 \cap \{ r \geq r_1 \}} r\varphi^2 d\omega. (3.17)
\]
Due to \( L'r = 1 + h \), (2.7) and (2.8), by directly computing \( |L'(r\varphi)|^2 \), we obtain
\[
\frac{(L'(r\varphi))^2}{2(1-h)(1+h)} = \frac{r^2(L\varphi)^2}{2(1-h)(1+h)} + r\partial_\omega^2(\varphi^2) + \frac{1+h}{2(1-h)} \varphi^2 = \frac{r^2(L\varphi)^2}{2(1-h)(1+h)} + \partial_\omega^2(\varphi^2).
\]
Integrating the above identity in \( \mathcal{D}_{\bar{u}} \) and using the smallness of \(|\bar{u}|\) imply
\[
\int_{\mathbb{D}_u} r\varphi^2 d\omega' + \int_{\mathbb{D}_{\bar{u}}} r^2(L\varphi)^2 d\omega' \lesssim \int_{\Sigma_0 \cap \{ r \geq r_1 \}} r\varphi^2 d\omega + \int_{\mathcal{D}_{\bar{u}}} |L'(r\varphi)|^2 d\omega'. (3.18)
\]
By using $r \approx u$ in (2.11),

$$\int_{D^+} r^{-\alpha} \varphi^2 d\omega d\mu' \lesssim \alpha^{-1} u^\alpha_+ \sup_{-u \leq \nu' \leq u} \int_{H^u_\nu} r\varphi^2 d\omega d\mu'.$$

Since (3.18) holds for any $-u \leq \nu' \leq u$, we can conclude (3.17).

We can prove (3.13) by using (3.17). Note that $L'(r\varphi) = L(r\varphi) + hL(r\varphi)$. We have

$$\int_{D^+} |L'(r\varphi) - L(r\varphi)|^2 d\omega d\mu' \lesssim \int_{D^+} |r hL\varphi - h\varphi|^2 d\omega d\mu'.$$

Due to (2.11), we have

$$\int_{D^+} r^2 h^2 (L\varphi)^2 d\omega d\mu' \lesssim M^2 \int_{D^+} r^{-1}(L\varphi)^2 d\omega d\mu' \lesssim M \int_{-u}^u u^{-1} E[\varphi](H^u_\nu) d\mu'.$$

Due to (2.14), $\int h^2 \varphi^2 d\omega d\mu \lesssim \int M^2 u^{-3} r\varphi^2 d\omega d\mu$. This term can be absorbed by the first term on the left of (3.17) by using Gronwall’s inequality. Thus we can obtain (3.13) by using (3.17).

(3.14) is by direct calculation.

Next we prove (3.15). By using (2.8) we can derive on $H^u_\nu$, $\partial_u (r^{\frac{3}{2}} \varphi) \cdot r^{\frac{1}{2}} \varphi = r \varphi \partial_u \varphi - \frac{1}{4} \varphi b^2$.

By integrating the above identity along $H^u_\nu$ with area element $du d\omega$ and by using (2.7), we can obtain

$$\int_{S_{u\perp}} r\varphi^2 d\omega + \frac{1}{2} \int_{H^u_\nu} b^2 r d\mu' = \int_{S_{\mu\perp}} r\varphi^2 d\omega + \int_{H^u_\nu} \frac{r^2}{L} L' \varphi \cdot \varphi d\mu'. $$

By using Cauchy Schwartz inequality and $b > \frac{3}{4}$ due to the smallness of $|h|$, \n
$$\int_{S_{u\perp}} r\varphi^2 d\omega + \int_{H^u_\nu} \varphi^2 d\omega \lesssim \int_{S_{\mu\perp}} r\varphi^2 d\omega + E[\varphi](H^u_\nu).$$

This proves (3.15). Next, we prove (3.16) in the same fashion. It is direct to derive along $H^u_\nu$

$$\partial_u (r^{\frac{3}{2}} \varphi) r^{\frac{1}{2}} \varphi = r \partial_u \varphi \cdot \varphi + \frac{1}{4} \varphi b^2.$$

In view of the above identity, by using (2.7), we integrate along $H^u_\nu$ with the area element $du d\omega$ to derive

$$\frac{1}{2} \int_{H^u_\nu} b^2 \varphi^2 d\omega d\mu' = \int_{S_{u\perp}} r\varphi^2 d\omega - \int_{S_{\mu\perp}} r\varphi^2 d\omega - \int_{H^u_\nu} \frac{r^2}{r - M_0} L' \varphi \cdot \varphi d\mu' d\omega.$$

We then combine the estimate of (3.15) and Cauchy Schwartz inequality to derive

$$\int_{H^u_\nu} \varphi^2 d\omega d\mu' \lesssim \int_{S_{u\perp}} r\varphi^2 d\omega + \int_{S_{\mu\perp}} r\varphi^2 d\omega + E[\varphi](H^u_\nu) + E[\varphi](H^u_\nu)$$

as desired in (3.16).

### 3.1. Simple facts of vector fields

Before proceeding further, we give basic facts about the vector fields.

Let $\nabla$ be the covariant derivative on $S_{u\perp}$. Its component under the Cartesian frame $\bar{\partial}_i, i = 1, 2, 3$ takes the form of $\nabla_i = \partial_i - \omega^j \partial_j, \omega^i = x^i / r$. We set $\bar{\nabla} = (\nabla, L), \bar{\partial} = (\nabla, L)$. For smooth functions $f$, we have

1. By direct calculation, there hold

$$[\partial_{\rho}, \Omega_{\mu\nu}] = m_{\mu\rho} \partial_{\nu} - m_{\rho\nu} \partial_{\mu}, \quad 0 \leq \mu < \nu \leq 3, \quad (3.19)$$

$$[\Omega_{ij}, \nabla_i] f = -\delta^k_j \nabla_k f + \delta^k_i \nabla_k f, \quad (3.20)$$

$$[L, \Omega_{ij}] = 0 = [L, \Omega_{ij}] = [\partial_{\rho}, \Omega_{ij}], \quad [L, L] = 0, \quad (3.21)$$

$$[L, \nabla] f = -r^{-1} \nabla f, \quad [L, \nabla] f = r^{-1} \nabla f, \quad [\partial_{\rho}, \nabla] f = -r^{-1} \nabla f. \quad (3.22)$$
(2) For $X = L_{\Omega}, L', L''$, due to $X\omega^i = 0$, there hold
\[ |X(rLf)| + |X(rL'f)| \lesssim |X(r\partial f)|; \quad |XLf| + |X\underline{L}f| \lesssim |X\partial f|. \tag{3.23} \]

(3)
\[
|\nabla Lf| + |\nabla Lf| + |\nabla \partial_r f| \lesssim |\nabla \partial f| + r^{-1}|\nabla f|, \tag{3.24}
\]
\[
|\partial f|^2 + |\partial \Omega f|^2 \simeq |\partial f|^2 + |\partial \Omega f|^2; \quad \mu = 0, \ldots, 3, \tag{3.25}
\]
\[
\Omega^\ell \partial_r f = D_{\cdot} \Omega^\ell \partial_r f, \quad \text{if } D_{\cdot} \in \{\nabla, \partial_r \partial_r \bar{\partial}\} \quad \ell \in \mathbb{N}, \tag{3.26}
\]

where $\Omega$ means one of $\{\Omega_{ij}, 1 \leq i < j \leq 3\}$. Indeed, it is direct to check
\[
\nabla \omega^i = r^{-1}(\delta^i - \omega^i); \quad \Omega_{ij} \omega^i = r^{-1}(x^i \delta^j_j - x^j \delta^i_i), \quad 1 \leq i < j \leq 3. \tag{3.27}
\]

3.2. $L^4$ type estimates. In order to give the product estimates for the nonlinear terms of (1.1), we will rely on $L^4$ type estimates. They can be derived by using the Sobolev inequality (3.24) and the energies in (2.13).

**Lemma 3.4.** Let $-\underline{u}_1 \leq u_1 \leq u_0(M_0)$ and $\alpha > 0$ be fixed. For smooth functions $F$ and $\psi$, there hold the following estimates,
\[
\|r^\frac{1}{2} F\|^2_{L^2_{\mathbb{R}}L^2_{\mathbb{R}}(\mathcal{D}_{\mathbb{R}})} \lesssim \mathcal{W}_1[F](\mathcal{D}_{\mathbb{R}}) + \mathcal{W}_1[F](\mathcal{D}_{\mathbb{R}}^u) + \|r^{-\frac{1}{2}} F\|^2_{L^2(\mathcal{D}_{\mathbb{R}}^u)} + M \int_{\mathbb{R}^u} u_{1}^{-1} E[F](\mathcal{H}^u_{\partial_r} ) du,
\]
\[
\alpha \|r^\frac{1}{2} F\|^2_{L^2_{\mathbb{R}}L^2_{\mathbb{R}}(\mathcal{D}_{\mathbb{R}})} \lesssim \mathcal{W}_1[F](\mathcal{D}_{\mathbb{R}}) + \|r^{-\frac{1}{2}} F\|^2_{L^2(\mathcal{D}_{\mathbb{R}}^u)} + M \int_{\mathbb{R}^u} u_{1}^{-1} E[F](\mathcal{H}^u_{\partial_r} ) du,
\tag{3.29}
\]
\[
\int_{S_{u_1}} r^2 |\partial \psi| \omega \omega \lesssim \int_{S_{u_1}} r^2 |\partial \psi| \omega \omega + (E[\partial \psi](\mathcal{H}^u_{\partial_r} )) + \int_{S_{u_1}} |F|^2 d\omega d\mu_1 ^2, \tag{3.31}
\]
\[
\int_{S_{u_1}} r^2 |\partial \psi| \omega \omega \lesssim \int_{S_{u_1}} r^2 |\partial \psi| \omega \omega + (E[\partial \psi](\mathcal{H}^u_{\partial_r} )) + \int_{S_{u_1}} |F|^2 d\omega d\mu_1 ^2, \tag{3.31}
\]
\[
\left( M^{-1} \int_{\mathbb{R}^u} E[|\partial \psi|](\mathcal{H}^u_{\partial_r} ) du + u_{1}^{-1} (\sup_{u \leq u_1} E[\psi](\mathcal{H}^u_{\partial_r} )) \right)^{\frac{1}{2}} \tag{3.32}
\]
\[
\int_{S_{u_1}} r^2 |\partial \psi| \omega \omega \lesssim \int_{S_{u_1}} r^2 |\partial \psi| \omega \omega + (E[\partial \psi](\mathcal{H}^u_{\partial_r} )) + u_{1}^{-2} E[\psi](\mathcal{H}^u_{\partial_r} ) \tag{3.33}
\]

\[
\int_{S_{u_1, u}} |rL\psi|^4 \, d\omega \lesssim \int_{S_{u_1, u}} |rL\psi|^4 \, d\omega + (E[\partial\psi](\mathcal{H}^{u_1}_{u_1}) + u_1^{-2}E[\psi](\mathcal{H}_{u_1}))
\]

\[
\cdot E[(\Omega \leq 1)^n](\mathcal{H}_{u_1}^{u_1})) \tag{3.34}
\]

\[
\|r\partial^2\psi\|_{L^2_{x,S}(\mathcal{H}_{u_1}^{u_1})}^2 \lesssim \ E[(\Omega \leq 1)^n](\mathcal{H}_{u_1}^{u_1}) \tag{3.35}
\]

\[
\|rL\psi\|_{L^2_{x,S}(\mathcal{H}_{u_1}^{u_1})} \lesssim \int_{-u_1}^{u_1} (\int_{S_{-u_1, u}} |rL\psi|^4 \, d\omega)^{\frac{1}{2}} \, du + \left( \int_{-u_1}^{u_1} E[(\Omega \leq 1)^n](\mathcal{H}_{u_1}^{u_1}) \, du \right)^{\frac{1}{2}}
\]

\[
\cdot \left( \int_{-u_1}^{u_1} (E[\partial\psi](\mathcal{H}_{u_1}^{u_1}) + u_1^{-2}E[\psi](\mathcal{H}_{u_1}^{u_1})) \, du \right)^{\frac{1}{2}} \tag{3.36}
\]

**Proof.** We first derive directly from the Sobolev inequality on unit sphere for \(-u \leq u \leq 0\),

\[
\|r\frac{1}{2}F\|_{L^2_{x}(S_{-u, 0})} \lesssim \|r\frac{1}{2}\nabla F\|_{L^2_{x}(S_{-u, 0})} + \|r^{-1/2}F\|_{L^2_{x}(S_{-u, 0})}. 
\]

Integrating the above inequality in \(u\) variable along \(\mathcal{H}_{u_1}\) implies

\[
\|r\frac{1}{2}F\|_{L^2_{x,S}(\mathcal{H}_{u_1}^{u_1})} \lesssim W_1[F]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{u_1}) + \|r^{-1/2}F\|_{L^2_{x}(\mathcal{H}_{u_1}^{u_1})}. 
\]

We then use (3.13) to obtain (3.28).

By the Sobolev embedding on the unit sphere

\[
\|F\|_{L^2_{x,L^2_{x}(\mathcal{H}_{u_1}^{u_1})}} \lesssim \|\nabla F\|_{L^2_{x}(\mathcal{D}_{u_1}^{u_1})} + \|r^{-1}F\|_{L^2_{x}(\mathcal{D}_{u_1}^{u_1})}
\]

and (3.13), we can obtain (3.29).

Similarly, by using the Sobolev inequality that

\[
\|r\frac{1}{2}\partial\psi\|_{L^2_{x,L^2_{x}(\mathcal{H}_{u_1}^{u_1})}} \lesssim \|r\frac{1}{2}\nabla \partial\psi\|_{L^2_{x}(\mathcal{D}_{u_1}^{u_1})} + \|r^{-1}\partial\psi\|_{L^2_{x}(\mathcal{D}_{u_1}^{u_1})},
\]

\(|\nabla f| \approx r^{-1}|\Omega f|\) and (3.25), we can obtain (3.30).

To prove (3.31), we set in (3.2) \(\gamma_0 = 1\), \(\gamma = \frac{1}{2}\), \(\gamma_2 = 0\).

\[
\int_{S_{u_1, u}} \frac{r^2}{2} |F|^4 \, d\omega \tag{3.37}
\]

\[
\lesssim \int_{S_{u_1, u}} r^2 |F|^4 \, d\omega + \int_{S_{u_1, u}} |L' r^{1/2}F| \cdot (|F|^2 + r^2 |\nabla F|^2) \, d\omega. 
\]

Note that \(r^{1/2} L'(r^{1/2}F) = rL'F + \frac{1}{2} (1 + h)F\). We can derive in view of the smallness of \(|h|\) in (2.14) that

\[
\int_{\mathcal{H}_{u_1}^{u_1}} \frac{1}{2} |L' r^{1/2}F|^2 \, d\mu \lesssim E[F](\mathcal{H}_{u_1}^{u_1}) + \int_{\mathcal{H}_{u_1}^{u_1}} |F|^2 \, d\mu. 
\]

Substituting the above inequality into (3.37) implies (3.31).

Now we prove (3.32). Note that by taking \(\gamma_0 = 0\) and \(\gamma_2 = \frac{1}{2} = \gamma\) in (3.32), we can derive for any smooth scalar function \(F\) and \(-u_1 \leq u \leq u_1\) that

\[
\|r^{1/2}F\|_{L^{\infty}_{x,L^2_{x}(\mathcal{H}_{u_1}^{u_1})}} \lesssim \left( \int_{S_{u, -u}} r^2 |F|^4 \, d\omega \right)^{\frac{1}{2}} + \left( \int_{\mathcal{H}_{u_1}^{u_1}} |L'(r^{1/2}F)|^2 \, d\mu \right)^{\frac{1}{2}}
\]

\[
\cdot \left( \int_{S_{u, -u}} (|F|^2 + |\Omega F|^2) \, r \, d\mu \right)^{\frac{1}{2}}. 
\]

We then apply the above inequality to \(F = \partial\psi\), followed with integrating in \(u\) variable.

\[
\|r^{1/2} \partial\psi\|_{L^{\infty}_{x,L^2_{x}(\mathcal{D}_{u_1}^{u_1})}} \lesssim \left( \int_{-u_1}^{u_1} \int_{S_{-u, -u}} r^2 |\partial\psi|^4 \, d\omega \right)^{\frac{1}{2}} \, du
\]

\[
+ \left( \int_{-u_1}^{u_1} \int_{\mathcal{H}_{u_1}^{u_1}} |L'(r^{1/2} \partial\psi)|^2 \, d\mu \right)^{\frac{1}{2}} 
\]

\[
\cdot \left( \int_{-u_1}^{u_1} \int_{\mathcal{H}_{u_1}^{u_1}} (|\partial\psi|^2 + |\Omega \partial\psi|^2) \, r \, d\mu \right)^{\frac{1}{2}}. \tag{3.38}
\]
Note also that \( |L'(r^+ \partial \psi)| \lesssim r^+ |L' \partial \psi| + r^{-\frac{3}{2}} (1 + h) |\partial \psi| \) and the smallness of \(|h|\) imply
\[
\int_{D^0_{\omega_1}} \frac{1}{|L'(r^+ \partial \psi)|^2} \operatorname{d}ud\omega \frac{du}{|L'(r^+ \partial \psi)|} \lesssim M^{-1} \int_{-\omega_1}^\omega E[\partial \psi](H^u_\omega) \frac{du}{|L'(r^+ \partial \psi)|} \\
+ u_{1+}^{-2} \sup_{-\omega_1 \leq u \leq \omega_1} (E[\psi](H^u_\omega)) + \sup_{-\omega_1 \leq u \leq \omega_1} (E[\psi](H^u_\omega)),
\]
where the last line is the bound for \( \int_{D^0_{\omega_1}} r^{-3} |\partial \psi|^2 \). It is achieved by using \( |\partial \psi| \lesssim |\bar{\partial} \psi| + |\bar{\psi}| \), \( 2.13 \) and \( 2.11 \), followed with integrating in \( u \) or \( \omega \) variable. We then substitute \( 3.25 \) and \( 3.39 \) to \( 4.36 \), which implies \( 4.32 \).

Noting \( |L'(r \bar{\partial} \psi)| \lesssim |r L \bar{\partial} \psi| + |\bar{\partial} \psi| \), \( 3.33 \) can be proved by using \( 3.2 \) for \( \bar{\partial} \psi \) along \( H^u_\omega \) with \( \gamma_0 = 0, \gamma_2 = \gamma = 1 \), with the help of \( 3.20 \).

Note \( 3.26 \) and the smallness of \(|h|\) imply \( |L'(r L \psi)| \lesssim |L \psi| + r |L \partial \psi| \). Applying \( 3.2 \) to \( L \psi \) along \( H^u_\omega \) with the same combination of weight exponents, we can similarly obtain \( 3.34 \) with the help of \( 4.20 \).

The Sobolev embedding on \( S^2 \) gives
\[
||r \bar{\partial} \psi||_{L^2_{\omega_1} (H^u_\omega)} \lesssim ||r \partial \psi||_{L^2_{\omega_1} (H^u_\omega)} + ||r \bar{\partial} \psi||_{L^2_{\omega_1} (H^u_\omega)}.
\]
Then with a direct substitution of \( 3.26 \), we can obtain \( 3.35 \).

\( 3.36 \) follows by a direct integration of \( 3.34 \) in \(-\omega_1 \leq u \leq \omega_1\).

\( \square \)

4. Decay estimates

In this section, we provide in Proposition 4.1 and Proposition 4.3 the decay properties for a smooth function \( \phi \in \mathbb{R}^{3+1} \) with bounded energies.

To be more precise, let \( n = 2 \) or \( 3 \) be fixed. We suppose \( \phi \) verifies \( \mathcal{E}_{\omega_1, \gamma_0, R}(\phi(0)) < \infty \) for a fixed \( 1 < \gamma_0 < 2 \) and the following assumptions:

- Let \( u_\omega > -u_0(M_0) \) be a fixed number and \( \Delta_0 > 0 \) be a fixed small constant. There hold with 0 \( \leq l \leq n-2 \),
  \[
  E[Z^0(\phi)](H^u_\omega) + E[Z^0(\phi)](L^u_\omega) \leq 2 \Delta_0 u_+^{-\gamma_0 + 2 \zeta(Z^0)},
  \]
  \[
  W_1[Z^0(\phi)](H^u_\omega) + W_1[Z^0(\phi)](L^u_\omega) \leq 2 \Delta_0 u_+^{-\gamma_0 + 1 + 2 \zeta(Z^0)},
  \]
  (BA)  

(2) \( \omega_1, u_1 \) be any pair of numbers verifying \(-u_1 \leq -u_1 \leq u_1 \leq u_0 \). With \( a \leq n \) and \( l \leq n-2 \), there hold
  \[
  ||r^{\frac{l}{2}} L Z^0(\phi) ||_{L^2_{\omega_1} (H^u_\omega)} \leq (\mathcal{E}_{\omega_1, \gamma_0} + \Delta_0 M^1) u_1^{-\gamma_0 + 2 \zeta(Z^0)},
  \]
  \[
  ||r^{\frac{l}{2}} Z^0(\phi) ||_{L^2_{\omega_1} (H^u_\omega)} \leq u_1^{-\gamma_0 + 2 + 2 \zeta(Z^0)} (\Delta_0 M^1 + \mathcal{E}_{\omega_1, \gamma_0}).
  \]
With \( p > -\frac{2}{2} + \frac{3}{2} \), there holds for \( a \leq n \)
  \[
  \| u_+^P r^{-\frac{3}{2}} Z^0(\phi) \|_{L^2_{\omega_1}(H^u_\omega)} \leq u_1^{-\frac{2m}{2} + \frac{3}{2} - \gamma_0 + 2 \zeta(Z^0)} (\Delta_0 M^{-\frac{1}{2}} + \mathcal{E}_{\omega_1, \gamma_0}).
  \]
Remark 4.2. The estimates with $M^{-1}$ appeared in the bound are stronger than the standard estimates for the free wave in the region $\{r \geq t + R\}$. (4.3)-(4.10) are not used for the proofs for Theorem 2.1 and 2.2. They will be used in Section 6. In particular, (4.3.1) is an improved Hardy’s inequality, which is proved by using the sharp improved estimate (4.3). The estimate (4.5) takes a weight of $r^{\frac{1}{2}}$ up to the top order derivative, which is much stronger than the standard Hardy’s type inequality. Such type of estimates will be crucial for the result for the quasilinear equations.

Remark 4.3. We emphasize that we only need $(BA_2)$ and $\mathcal{E}_{2,\gamma_0}(\phi[0])$ to be bounded in order to obtain the above results with $n = 2$. There involves neither the third order control from $(BA_3)$ nor the bound of $\mathcal{E}_{3,\gamma_0}$.

Remark 4.4. We can also prove

$$\| (\frac{L}{u_+})(r^{\frac{1}{2}}L(r^Z)) \|_{L^2(Z; T^\infty)}^2 \lesssim (\mathcal{E}_{l+2,\gamma_0} + \Delta_0)u_{1+}^{-\gamma_0+2\zeta(Z)}, \ l \leq n-2$$

which is not used for the proofs in this paper.

Proof. We first consider the inequality for $LZ^l(\phi)$ in (4.1). With $\gamma_2 = 1 = \gamma$, $\gamma_0 = 0$ in (3.1) and $f = LZ^l(\phi)$, we have

$$\sup_{S_{\gamma_+}} \ |r^\gamma f|^4 \lesssim \int_{S_{\gamma_+}} |r^\gamma \Omega^{1 \leq l} f|^4 r^{-2} + \int_{H^m_{\gamma_+}} |r^\gamma \Omega^{1 \leq l} Z^l(\phi)|^2 r^{-2}$$

Note that $L'(\Omega^m(r^\alpha f)) = L'(\alpha L\Omega^m f) = -\alpha(1+h)\alpha^{-1}L\Omega^m f + \alpha L'_{\Omega} \Omega^m f$ holds for any smooth function $f$. Due to the smallness of $|h|$ and (3.23),

$$\int L'_{\Omega^m}(r^\alpha f) \lesssim r^{-1} |L\Omega^m f| + r^\alpha |L\partial^\Omega^m f|, \ \alpha \geq 0.$$  \hfill (4.6)

With the help of (3.21), we can derive in view of (4.6) with $\alpha = 1$,

$$\sup_{S_{\gamma_+}} \ |rLZ^l(\phi)|^4 \lesssim \int_{S_{\gamma_+}} |rL\Omega^{1 \leq l} Z^l(\phi)|^4 r^{-2} + \int_{H^m_{\gamma_+}} |rL\Omega^{1 \leq l} Z^l(\phi)|^2 r^{-2}$$

$$\lesssim \int_{S_{\gamma_+}} |r\Omega^{1 \leq l} Z^l(\phi)|^4 r^{-2} + (u_{l+}^{-2}E[\Omega_{l+} \Omega^l(\phi)]H^m + E[\partial^\Omega^l \Omega^l(\phi)]H^m)$$

where we employed (3.4) with $Z^l = \partial^\Omega^l \Omega^l$ for $l \leq n-2$. (2.11) and (BA). Applying (3.1) along $H^m_{\gamma_+}$ with $\gamma_2 = \gamma = \frac{1}{2}$ and $\gamma'_0 = 0$, to $f = \nabla Z^l(\phi)$ implies

$$\sup_{S_{\gamma_+}} \ |r^{\frac{1}{2}} \nabla Z^l(\phi)|^4 \lesssim \int_{S_{\gamma_+}} |r^{\frac{1}{2}} \Omega^{1 \leq l} \nabla Z^l(\phi)|^4 r^{-2} + \int_{H^m_{\gamma_+}} r^3 |\Omega^{1 \leq l} \nabla Z^l(\phi)|^2 r^{-2}$$

$$\lesssim \int_{S_{\gamma_+}} |r^{\frac{1}{2}} \Omega^{1 \leq l} \nabla Z^l(\phi)|^2 r^{-2}.$$  \hfill (4.7)

By using (3.20) and (3.22), symbolically, for $m = 0, 1$, in view of $L'(r) = -1 - h$,

$$|L'(r^{\frac{1}{2}}\Omega^m \nabla f)| \lesssim r^{\frac{1}{2}} |\nabla \Omega^m f| + r^{\frac{1}{2}} |L'(\nabla \Omega^m f) + [\Omega^m, \nabla] f)|$$

$$\lesssim r^{\frac{1}{2}} |\nabla \Omega^m f| + r^{\frac{1}{2}} |\nabla \Omega^m f| + r^{\frac{1}{2}} |\nabla \Omega^m f| + r^{\frac{1}{2}} |\nabla \Omega^m f|$$

$$\lesssim r^{\frac{1}{2}} |\nabla \Omega^m f| + r^{\frac{1}{2}} |\nabla \Omega^m f|.$$
where we used the smallness of $|h|$, \eqref{3.22} and \eqref{3.23}. By using the above calculation for $f = Z^{(l)}\phi$ and \eqref{3.22}, we deduce from \eqref{4.7} and \eqref{3.4} that

$$
\sup_{S_{u,-}} |r^\frac{1}{2}D^rZ^{(l)}\phi|^4 \lesssim \int_{S_{u,-}} |r^\frac{1}{2}D^r\Omega^{(\leq 1)}Z^{(l)}\phi|^4 r^{-2} + W_1[\Omega^{(\leq 2)}Z^{(l)}\phi](\mathcal{H}_u^Z) \\
\quad \cdot \left( W_1[\partial\Omega^{(\leq 1)}Z^{(l)}\phi](\mathcal{H}_u^Z) + u_+^{-1}E[\Omega^{(\leq 1)}Z^{(l)}\phi](\mathcal{H}_u^Z) \right) \\
\quad \lesssim u_+^{-2\gamma_0 + 2\gamma(Z')} \left( \Delta_0^2 + E_{l+2,\gamma_0}^l \right),
$$

where we also used \eqref{2.11}. Thus the first two estimates in \eqref{4.1} are proved.

In view of the above two inequalities, the smallness of $\gamma_0 = \gamma = \frac{3}{2}$ and $\gamma_0 = 0$.

$$
\sup_{S_{u,-}} |r^\frac{1}{2} \cdot r^{-1} L(rZ^{(l)}\phi)|^4 \lesssim \int_{S_{u,-}} |r^\frac{1}{2} \Omega^{(\leq 1)}L(rZ^{(l)}\phi)|^4 r^{-2} + \int_{\mathcal{H}_u^Z} r^\frac{1}{2} \Omega^{(\leq 2)} \left( r^{-1} L(rZ^{(l)}\phi) \right)^2 r^{-2}.
$$

Note that due to \eqref{3.21} and the smallness of $|h|$, \eqref{4.8},

$$
|L'(\Omega_{ij} r^\frac{1}{2} L(rZ^{(l)}\phi))| \lesssim \frac{r}{2} |L(r\Omega_{ij} Z^{(l)}\phi)| + \frac{r}{2} |L'(L(r\Omega_{ij} Z^{(l)}\phi))|.
$$

By \eqref{3.22}, we have

$$
|L(\Omega_{ij} L(rZ^{(l)}\phi))| \lesssim |L \Omega_{ij} Z^{(l)}\phi| + |L(r\partial \Omega_{ij} Z^{(l)}\phi)| \lesssim |L(\partial \Omega_{ij} Z^{(l)}\phi)| + |L \Omega_{ij} Z^{(l)}\phi|,
$$

and due to $[L, L] = 0$ and \eqref{3.23}

$$
|h| |L L (r \Omega_{ij} Z^{(l)}\phi)| \lesssim |h| |L \Omega_{ij} Z^{(l)}\phi| + |L (r \Omega_{ij} Z^{(l)}\phi)| \lesssim |h| |L (r \partial \Omega_{ij} Z^{(l)}\phi)| + |L \Omega_{ij} Z^{(l)}\phi|.
$$

In view of the above two inequalities, the smallness of $|h|$, \eqref{4.10} and \eqref{2.11} we have

$$
\int_{\mathcal{H}_u^Z} \left| L'(\Omega^{(\leq 1)}(\frac{1}{2} L(rZ^{(l)}\phi)))^2 r^{-2} \lesssim W_1[\partial \Omega Z^{(l)}\phi](\mathcal{H}_u^Z) + E[\Omega Z^{(l)}\phi](\mathcal{H}_u^Z)u_+^{-1} \right.

\lesssim \Delta_0 u_+^{-\gamma_0 - 1 + 2\gamma(Z')},
$$

Thus we derive from \eqref{4.9} and \eqref{3.1} that

$$
\sup_{S_{u,-}} |r^\frac{1}{2} \cdot r^{-1} L(rZ^{(l)}\phi)|^4 \lesssim \int_{S_{u,-}} |r^\frac{1}{2} \Omega^{(\leq 1)}L(rZ^{(l)}\phi)|^4 r^{-2} + W_1[\Omega^{(\leq 2)}Z^{(l)}\phi](\mathcal{H}_u^Z) \Delta_0 u_+^{-\gamma_0 - 1 + 2\gamma(Z')} \lesssim (E_{l+2,\gamma_0}^l + \Delta_0^2)u_+^{-2\gamma_0 + 4\gamma(Z')},
$$

as desired in \eqref{4.3}.

Next we prove \eqref{4.2}. For any fixed point $(u, u, \omega)$, we integrate the estimate of $LZ^{(l)}\phi$ in \eqref{4.1} along the ingoing integral curve of $\partial u'$ along $\mathcal{H}_u^Z$ from $t = 0$. For $l \leq n - 2$, we derive in view of \eqref{2.7} that

$$
|Z^{(l)}\phi(u, u, \omega) - Z^{(l)}\phi(-u, u, \omega)| \lesssim \int_{-\frac{1}{2}}^{u} \frac{1}{2} (1 - h)^{-1} |L'Z^{(l)}\phi(u', u, \omega) du' \\
\lesssim \int_{-\frac{1}{2}}^{u} |LZ^{(l)}\phi(u', u, \omega) + r^{-1} |h| |L(rZ^{(l)}\phi)| + |Z^{(l)}\phi|(u', u, \omega)) du'.
$$
We then bound the first term on the right with the help of (3.9) and (2.11) and
\( L \) to \( E \). We apply (3.6) to \( u \) and (4.8). The last term on the right can be treated by using Gronwall’s inequality by using \( u \approx r \), the second estimate of (4.1) and (4.8). Thus by using \( u \approx r \) again, we can derive
\[
|ru^{Z(\phi)}(u, u, \omega)| \lesssim \|ru^{Z(\phi)}(-u, u, \omega)| + (\Delta_0 + \varepsilon_{i+2, \gamma_0})\gamma |ru^{Z(\phi)}(-u, u, \omega)|.
\]
We then bound the first term on the right with the help of (3.9) and \( r \approx u \), which implies
\[
|ru^{Z(\phi)}| \lesssim (\varepsilon_{i+2, \gamma_0} + \Delta_0 \gamma)u^{\gamma + 1} + \varepsilon(\gamma Z).
\]
The proof of (4.2) is then completed. The last estimate in (4.1) can be obtained by using (4.1) can be obtained by using (4.1) and (4.8).

Next, we prove (4.3). It suffices to consider the estimate for \( Lu^{Z(\phi)} \), other estimates follow from integrating the better estimates in (4.1). We first apply the Sobolev inequality (3.1) for \( \mathcal{H}_u^u \) to \( Lu^{Z(\phi)} \) with \( \gamma = \gamma = \frac{1}{2} \), \( \gamma_0 = 0 \), which yields
\[
\sup_{S_{2, \gamma}} u^2 \|Lu^{Z(\phi)}\| \lesssim \int \int |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega + \int \int |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega.
\]
We then derive
\[
\int \int_{-\infty}^{\infty} \left( \sup_{u \leq u \leq u_1} \sup_{S_{2, \gamma}} u^2 \|Lu^{Z(\phi)}\|^2 d\omega \right) \lesssim \int \int |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega + \int \int |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega.
\]
We apply (3.6) to \( Lu^{Z(\phi)} \), which then bounds the first term on the righthand side of the inequality by \( u_1 \gamma - \gamma_0 + 2(\gamma \gamma_0 + E_{i+2, \gamma_0}) \). For the second term, due to (3.21) we derive
\[
\int \int_{-\infty}^{\infty} |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega \lesssim M^{-1} \int \int |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega d\omega.
\]
By using (3.23) and in view of (4.6) with \( \alpha = \frac{1}{2} \), \( f = Lu^{Z(\phi)} \), we have
\[
\int \int_{-\infty}^{\infty} |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega \lesssim \int \int |\Omega| \|Lu^{Z(\phi)}\|^2 d\omega d\omega.
\]
By substituting the above two estimates to (4.11), (4.3) is proved.

Next, we prove (4.5) and (4.7) follows as an immediate consequence by integrating in \( u \)-variable.
By applying $L'(rf^2) = (-1 - h)f^2 + rL'(f^2)$ to $f = Z^{(a)}\phi$, and in view of $1 + h > 0$ due to the smallness of $h$, we integrate the result in $D_{a1}^2$, with the area element $\frac{1}{2}(1 - h)^{-1}du_1 du_2 d\omega$, which yields,

$$\int_{D_{a1}^2} r|Z^{(a)}\phi|^2(u_1, u_2, \omega) dud\omega$$

$$\lesssim \int_{-u_1}^{u_1} \int_{S^2} r|Z^{(a)}\phi|^2(-u_1, u_2, \omega) dud\omega + \int_{u_1}^{u_1} \|r^{-\frac{1}{2}}L^2Z^{(a)}\phi\|_{L^2(H_{a1}^2)}\|r^{-\frac{1}{2}}Z^{(a)}\phi\|_{L^2(H_{a1}^2)} du,$$

where we dropped the integral of $(1 + h)|Z^{(a)}\phi|^2$ due to the positivity. Note that

$$\|r^{-\frac{1}{2}}L^2Z^{(a)}\phi\|_{L^2(H_{a1}^2)} \lesssim M^{-\frac{1}{2}}E[Z^{(a)}\phi]^\frac{1}{2} \lesssim u_1^{\frac{-2\gamma+\gamma(Z^n)}{2}}, \quad a \leq n$$

and the first term on the right of (12) can be bounded by $u_1^{-1-\gamma+2\gamma(Z^n)}E_{a,\gamma_0}$ by using (3.7).

By multiplying both sides by $u_1^p$ with $p \geq 1$, followed by applying Gronwall's inequality,

$$u_1^p \left( \int_{D_{a1}^2} r|Z^{(a)}\phi|^2(u_1, u_2, \omega) dud\omega \right)^\frac{1}{2}$$

$$\lesssim u_1^{\frac{-1-\gamma_0+\gamma(Z^n)}{2}+\xi(Z^n)}E_{a,\gamma_0}^\frac{1}{2} + u_1^p \int_{-u_1}^{u_1} \|r^{-\frac{1}{2}}L^2(Z^{(a)}\phi)\|_{L^2(H_{a1}^2)} u_1^{-\frac{1}{2}} du.$$

We can obtain (4.4) for $a \leq n$ by using (13).

With the help of the assumption of (BA$_n$) with $n = 2, 3$, we can derive the following estimates of mixed $L^p$ norms.

**Proposition 4.5.** Let $-u_1 \leq u_0 \leq u_1 \leq 0$, $b \leq n - 1$, $a \leq n$, $\gamma > \frac{1}{2}$ and $\alpha > 0$. There hold the following estimates

$$\|r^{-\frac{1}{2}}Z^{(a)}\phi\|_{L^2(D_{a1}^2)} \lesssim u_1^{-\frac{1-\gamma}{2}+\gamma(Z^n)}M^{\frac{1}{2}}$$

(4.14)

$$\|r^{-\frac{1}{2}}Z^{(a)}\phi\|_{L^2(D_{a1}^2)} \lesssim u_1^{-\frac{1-\gamma_0}{2}+\gamma(Z^n)} \Delta_0^{\frac{1}{2}}$$

(4.15)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} + \alpha\|\frac{u_1^{\frac{1}{2}}}{r}\|_{H^\alpha(D_{a1}^2)}^2 \lesssim u_1^{-1-\gamma_0+2\gamma(Z^n)}(\Delta_0 + E_{a,\gamma_0})$$

(4.16)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-1-\gamma_0+2\gamma(Z^n)}(\Delta_0 + E_{a,\gamma_0})$$

(4.17)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-1-\gamma_0+2\gamma(Z^n)}(\Delta_0 + E_{a,\gamma_0})$$

(4.18)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-1-\gamma_0+2\gamma(Z^n)}(\Delta_0 + E_{a,\gamma_0})$$

(4.19)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-1-\gamma_0+2\gamma(Z^n)}(\Delta_0 + E_{a,\gamma_0})$$

(4.20)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-1-\gamma_0+2\gamma(Z^n)}(\Delta_0 + E_{a,\gamma_0})$$

(4.21)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-1-\gamma_0+2\gamma(Z^n)}(\Delta_0 + E_{a,\gamma_0})$$

(4.22)

If the constant $p > -\frac{1-\gamma}{2} + \frac{1}{2}$, there hold

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-\gamma(Z^n)-\frac{1}{2}+\frac{1}{2}-p}(\Delta_0 + E_{a,\gamma_0})$$

(4.23)

$$\|r^\frac{1}{2}Z^{(a)}\phi\|^2_{L^2_{\alpha}L^2_{\alpha}D_{a1}^2} \lesssim u_1^{-\gamma(Z^n)-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}(\Delta_0 + E_{a,\gamma_0})$$

(4.24)

**Proof.** We first derive by using (BA$_n$) that

$$\|r^{-\frac{1}{2}}Z^{(a)}\phi\|_{L^2(D_{a1}^2)} \lesssim M^{-\frac{1}{2}}(\int_{-u_1}^{u_1} E[Z^n\phi](H_{a1}^2) du)^\frac{1}{2} \lesssim M^{-\frac{1}{2}}u_1^{-\frac{1}{2}}u_1^{\frac{-2\gamma+\gamma(Z^n)}{2}} \Delta_0^{\frac{1}{2}}, \quad a \leq n.$$
By using (BA$_n$), (4.13) is a direct consequence of $|\partial Z^a(\phi)| \lesssim |\partial Z^a(\phi)| + |\partial Z^a(\phi)|$; (4.16) is a consequence of (3.13) and (3.7).

Note that by using (3.29), (3.7) and (BA$_n$), we can derive

$$\alpha \|\left(\frac{u_1 + r}{r}\right)^\gamma Z^a(\phi)\|^2_{L^2_{\Omega} L^2_{\partial(\Omega)^n}} \lesssim \mathcal{W}_{1}(Z^a(\phi))(\mathcal{D}_{\partial}^a) + \|r^{-\frac{1}{2}} Z^a(\phi)\|^2_{L^2(T_{\Omega}^2, \omega)}$$

$$+ M \int_{-u_1}^{u_1} u'_+ E[Z^a(\phi)(H^\mu_{\omega})] du'$$

$$\lesssim u_1^{-\gamma_0 + 1 + 2\zeta(Z^a)}(\Delta_0 + \mathcal{E}_{a,\gamma_0}),$$

as desired in (4.17).

Next we consider (4.18). For the estimate of $LZ(b)\phi$, we apply (3.32) to $\psi = Z(b)\phi$, which yields for $\mathcal{I} = \{(u, \mu) : -u_1 \leq -\mu \leq u \leq u_1\}$ that

$$\int_{S_{\mathcal{I}}} |rLZ(b)\phi|^4 d\omega \lesssim \int_{S_{\mathcal{I}}} |rLZ(b)\phi|^4 d\omega + (E[\partial Z^b(\phi)(H^\mu_{\omega}) + u_+^2 E[Z^b(\phi)(H^\nu_{\mu})])$$

$$\cdot E[\Omega^{\leq 1}Z^b(\phi)(H^\mu_{\omega})].$$

The first term on the right can be bounded by applying (3.3) to $\partial Z^b(\phi)$, which is bounded by $u_+^{-2 - 2\gamma_0 + 4\zeta(Z^a)}c_{b, 1 + \gamma_0}^2$. We then use (BA$_n$) to obtain

$$\int_{S_{\mathcal{I}}} |rLZ(b)\phi|^4 d\omega \lesssim u_+^{-2 - 2\gamma_0 + 4\zeta(Z^a)}(\mathcal{E}_{b, 1 + \gamma_0}^2 + \Delta_0^2).$$

By repeating the same procedure for $\partial \psi$ in view of (3.32), we can get the same estimate with $L$ replaced by $\partial$. This implies (4.18).

Integrating (4.18) along $H^\mu_{\omega}$ for any $-u_1 \leq u \leq u_1$ implies (4.19) directly.

We note also that (4.18) is independent of $\mu$. For $(u, \mu) \in \mathcal{I}$, we can take supremum of (4.18) for $-u \leq \mu \leq u_1$, followed with integrating $u$ from $-u_1$ to $u_1$. Thus (4.20) is proved.

Next, we apply (3.32) to $\psi = Z(b)\phi$ for $b \leq n - 1$ and also by using (BA$_n$),

$$\|r^{-\frac{1}{2}} \partial Z^b(\phi)\|^2_{L^2_{\Omega} L^2_{\partial(\Omega)^n}} \lesssim \int_{-u_1}^{u_1} \int_{S_{\mathcal{I}}} r^2|\partial Z^b(\phi)|^4 d\omega d\mu + M^{-1} u_1^{-\Delta_0} u_2^2 \zeta(Z^a)\gamma_0.$$

We then apply (3.3) to $\partial Z^b(\phi)$, which bounds the first term on the right by $\mathcal{E}_{b, 1 + \gamma_0} u_1^{-\gamma_0 + 1 + 2\zeta(Z^a)}$.

The result of (4.21) follows by a direct substitution.

(4.22) follows by applying (3.30) to $\psi = Z(b)\phi$ and also using (BA$_n$).

We can obtain (4.23) by applying the sobolev embedding on unit sphere

$$\|r^{-\frac{1}{2}} u_+^p Z^b(\phi)\|_{L^2_{\Omega} L^2_{\partial(\Omega)^n}} \lesssim \|r^{-\frac{1}{2}} u_+^{-p \Omega^{\leq 1}} Z^b(\phi)\|_{L^2_{\Omega} L^2_{\partial(\Omega)^n}},$$

followed by applying (4.15) to $\Omega^{\leq 1}Z^b(\phi)$ with $p > -\frac{\kappa_0}{2} + \frac{3}{2}$.

We deduce by applying (3.28) to $F = Z^a(\phi)$ followed with using (3.7) and (BA$_n$) that

$$\|r^{-\frac{1}{2}} Z^a(\phi)\|_{L^2_{\Omega} L^2_{\partial(\Omega)^n}} \lesssim \mathcal{W}_{1}(Z^a(\phi))(\mathcal{D}_{\partial}^a) + \mathcal{W}_{1}(Z^a(\phi))(H^\mu_{\omega}) + \|r^{-\frac{1}{2}} Z^a(\phi)\|^2_{L^2(T_{\Omega}^2, \omega)}$$

$$+ M^{-\frac{1}{2}} \int_{-u_1}^{u_1} u'_+ E[Z^a(\phi)(H^\mu_{\omega})] du' \lesssim u_1^{-\zeta(Z^a) - \Delta_0 + \Delta_0} + \mathcal{E}_{a, \gamma_0}^2,$$

as desired in (4.24).

\[ \square \]

5. Semilinear Wave Equations

In this section we consider the equation of (2.15) in $\mathbb{R}^{3+1}$, i.e.

$$\Box m\phi = N^{\alpha^\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi + q(x)\phi,$$

where $0 \leq q(x) \leq 1$ satisfies (1.2) with $n = 2$. We will prove Theorem 2.1 and Theorem 2.2.

In Section 3 and 4 for functions with (1.3) bounded for $k = 2$ or 3, under the assumption of (BA$_k$), we have obtained decay properties and a set of Sobolev inequalities. In this section, by
a bootstrap argument, we will prove (BA) with $\Delta_0$ comparable with $\mathcal{E}_{2,\gamma_0,R}$, provided that the latter is sufficiently small. The analysis in Sections 3 and 4 will play a crucial role for achieving the boundedness of the sets of energies.

We will give the fundamental standard and weighted energy inequalities for the $u$ and $\overline{u}$ foliations in Proposition 5.2 and Proposition 5.4. The goal is to justify the energy norms given in (2.13) can be achieved along the foliations $\mathcal{H}_u$, $\mathcal{H}_{\overline{u}}$ and $\mathcal{D}_h$ provided that $(\Box - q)f$ can be bounded as desired.

Throughout this section, $M > 0$ is a fixed small number, with the upper bound determined during the proofs of Theorem 2.1 and 2.2. Let

$$h = \frac{M}{r}, \quad u_0 = u_0(M, R),$$

where $R \geq 2$, whose lower bound will be finally determined at the end of the proof of Theorem 2.1.

For the wave equation $\Box \varphi - q \varphi = F$, we define the energy momentum tensor,

$$Q_{\alpha \beta}[\varphi] = \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} m_{\alpha \beta}(\partial^\alpha \varphi \partial_\alpha \varphi + q \varphi^2).$$

(5.1)

Recall $\mathcal{N}$, $\overline{\mathcal{N}}$ from (2.3). It is straightforward to obtain the energy densities on $\mathcal{H}_u$ and $\mathcal{H}_{\overline{u}}$

$$Q[\varphi](\partial_t, \mathcal{N}) = \frac{1}{2}(1 + h)^{-1}\{(L \varphi)^2 + h(L \varphi)^2 + (1 + h)(\nabla \varphi^2 + q \varphi^2)\},$$

$$Q[\varphi](\partial_t, \overline{\mathcal{N}}) = \frac{1}{2}(1 + h)^{-1}\{(L \varphi)^2 + h(L \varphi)^2 + (1 + h)(\nabla \varphi^2 + q \varphi^2)\}.$$ (5.2)

Next, we give the fundamental energy inequality in $\{u \leq u_0\}$.

**Lemma 5.1.** Consider the equation $\Box \varphi - q \varphi = F$. There holds for a smooth vectorfield $X$ that

$$\partial^\alpha(Q_{\alpha \beta}[\varphi]X^\beta) = Q_{\alpha}^{(X)}(5.3) + \Box \varphi + (q \varphi^2)X \varphi - \frac{1}{2}X q \cdot \varphi^2,$$

where $(5.3)_{\pi_{\alpha \beta}} = \frac{1}{2}\left(\langle \partial_\alpha X, \partial_\beta \rangle + \langle \partial_\beta X, \partial_\alpha \rangle\right)$.

There also holds the following energy estimate,

$$E[\varphi](\mathcal{H}_{\mathcal{N}}^+) + E[\varphi](\mathcal{H}_{\overline{\mathcal{N}}}^+) \lesssim \int_{\mathcal{D}_h^+} F \cdot \partial_t \varphi + E[\varphi](\mathcal{D}_h^+)^0$$

(5.4)

for all $-\overline{u} \leq u \leq u_0$.

**Proof.** For $Q_{\alpha \beta}[\varphi]$ in (5.1), it is straightforward to derive

$$\partial^\alpha Q_{\alpha \beta}[\varphi] = (\Box \varphi - q \varphi)\partial_\beta \varphi - \frac{1}{2} \partial_\beta \varphi q \cdot \varphi^2.$$

(5.3) follows as a consequence.

Applying (5.3) with $X = \partial_t$ gives $\partial^\alpha(Q_{\alpha \beta}[\partial_t]) = (\Box \varphi - q \varphi)\partial_t \varphi$. We then integrate the identity in $\mathcal{D}_h^+$ to obtain

$$\int_{\mathcal{H}_{\mathcal{N}}^+} Q(\mathcal{N}, \partial_t) d\mu_H + \int_{\mathcal{H}_{\overline{\mathcal{N}}}} Q(\overline{\mathcal{N}}, \partial_t) d\mu_{\overline{H}} + \int_{\mathcal{D}_h^+} F \cdot \partial_t \varphi dx dt = \int_{\mathcal{D}_h^+} Q(\partial_t, \partial_t) dx.$$

(5.4) is proved in view of (5.2).

As a consequence, we derive the following energy estimate.

**Proposition 5.2.** Let $\mathcal{I} = \{(u_1, u_2) : -\overline{u} \leq u \leq u_1 \leq u_2 \leq u_0\}$, where $u_2$ and $u_0$ are fixed. There holds the following energy inequality for $(u, \overline{u}) \in \mathcal{I}$,

$$u_1^{-2p+\gamma_0}(E[\varphi](\mathcal{H}_{\mathcal{N}}^+) + E[\varphi](\mathcal{H}_{\overline{\mathcal{N}}}^+)) \lesssim \sup_{(u_1, u_2) \in \mathcal{I}} \left\{u_1^{-2p+\gamma_0} E[\varphi](\mathcal{D}_h^+)^0_0 + u_1^{-2p+\gamma_0}(\|F\|_2^2)_{L_1^1 L_2^1 L_2^2(\mathcal{D}_h^+)} + \|F\|_{L_1^1 L_2^1 L_2^2(\mathcal{D}_h^+)} + u_1^{-2p+\gamma_0} M^{-1}\|F\|_{L_1^1(\mathcal{D}_h^+)}\right\},$$
where $p \leq 0$ is any constant \[5\], $F = F^8 + F^9$, and the two smooth functions $F^8$ and $F^9$ are in the corresponding normed spaces.

**Proof.** We apply Lemma 5.1 with $(u, u) \in I$. This implies

$$u_+^{-2p^+\gamma_0}(E[\varphi](H^u_\gamma) + E[\varphi](\mathcal{H}^u_\gamma)) \lesssim \sup_{(u, u) \in I} \left\{ u_+^{-2p^+\gamma_0} \int_{D^u_{\mathcal{M}}} F \cdot \partial_1 \varphi dxdt \right\} + u_+^{-2p^+\gamma_0} E[\varphi](\Sigma^u_1, \mathcal{M}). \quad (5.5)$$

To control the first term on the right, we first consider the $F^9$ term in the decomposition $F = F^7 + F^9$.

$$|\int_{D^u_{\mathcal{M}}} F^9 \cdot \partial_1 \varphi dxdt| \lesssim \int_{D^u_{\mathcal{M}}} F^9 \cdot L \varphi dxdt + \int_{D^u_{\mathcal{M}}} |F^9 \cdot L \varphi| dxdt = I + II.$$

We estimate $I$ and $II$ in the following way,

$$I \lesssim \|u_+^{-\frac{p+1}{2}} r F^9\|_{L^1_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \|u_+^{-\frac{1}{2}} r L \varphi\|_{L^\infty_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \lesssim u_+^{-\frac{3}{2}} r F^9\|_{L^1_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \sup_{-u_1 \leq u \leq u_1} u_+^{\frac{p+1}{2}} E[\varphi](\mathcal{H}^u_\gamma),$$

$$II \lesssim \|r F^9\|_{L^1_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \|r L \varphi\|_{L^\infty_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \lesssim u_+^{-\frac{3}{2}} r F^9\|_{L^1_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \sup_{-u_1 \leq u \leq u_1} u_+^{\frac{p+1}{2}} E[\varphi](\mathcal{H}^u_\gamma).$$

Similarly, we can derive

$$\int_{D^u_{\mathcal{M}}} |F^7 \cdot \partial_1 \varphi| dxdt \leq M^{-\frac{1}{2}} \|F^7\|_{L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \left( \int_{-u_1}^{u_1} E[\varphi](\mathcal{H}^u_\gamma) du \right)^{\frac{1}{2}} \lesssim M^{-\frac{1}{2}} \|F^7\|_{L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \left( \int_{-u_1}^{u_1} E[\varphi](\mathcal{H}^u_\gamma) du \right)^{\frac{1}{2}}$$

By multiplying the weight $u_+^{-2p+\gamma_0}$ to the three inequalities, followed by using Cauchy-Schwartz inequality, we can derive

$$u_+^{-2p+\gamma_0} |\int_{D^u_{\mathcal{M}}} F \cdot \partial_1 \varphi dxdt| \lesssim u_+^{-1} \left( u_+^{-2p+\gamma_0} \|F^8\|_{L^1_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})} \right)_1 + u_+^{-2p+\gamma_0} \|F^9\|_{L^1_{\mathcal{M}} L^2_{\mathcal{M}}(D^u_{\mathcal{M}})}^2 + \frac{\|F^7\|_{L^2_{\mathcal{M}}(D^u_{\mathcal{M}})}^2}{u_1^{-\gamma_0 (p+1)}} \left( \sum_{-u_1 \leq u \leq u_1} u_+^{-2p+\gamma_0} E[\varphi](\mathcal{H}^u_\gamma) + \sup_{-u_1 \leq u \leq u_1} u_+^{-2p+\gamma_0} E[\varphi](\mathcal{H}^u_\gamma) \right). \quad (5.6)$$

By substituting the above inequality to (5.5), followed with taking supremum for $(u, u) \in I$, the last line of (5.6) can be absorbed. Thus Proposition 5.2 is proved. \[\square\]

Next, we give the weighted energy estimate.

---

\[\text{The inequality holds uniformly for any } p \leq 0.\]
Lemma 5.3. For any $\underline{u} \leq u \leq u_0$, there holds the following inequality,

\[
\int_{\Sigma_+} r(L(r\varphi))^2 + hr^3(|\nabla \varphi|^2 + q\varphi^2)du'd\omega + \int_{\Sigma^-} r^3(h(L\varphi)^2 + |\nabla \varphi|^2 + q\varphi^2)du'd\omega
\]

\[
+ \frac{1}{2} \int_{\Sigma_+} \{L(r\varphi)]^2 + r^3|\nabla \varphi|^2\} du'd\omega
\]

\[
\lesssim \bigg(|\nabla_1[\varphi](\Sigma_0^{\alpha\beta}) + \int_{\Sigma_+} |F \cdot L(r\varphi)| dxdt + \int_{\Sigma^-} E[\varphi](H^0_\alpha du' + E[\varphi](H^0_\alpha)
\]

\[
+ \|r^{-1/2}\varphi\|^2_{L^2(\Sigma_+^{\alpha\beta})} + \int_{\Sigma_+^{\alpha\beta}} r\varphi^2 d\omega,\]

where $q(x) \geq 0$ verifies (1.2) with $n = 2$.

Proof. We define

\[
(\pi^{(x)}_\alpha = Q_{\alpha\beta}X^\beta + \frac{1}{2}\partial_\alpha(\varphi^2) + Y_\alpha,
\]

where

\[
X = rL, \quad Y = \frac{1}{2}r^{-1}\varphi^2L.
\]

Similar to the calculation in [19], we have

\[
\partial^{\alpha(x)}\varphi_\alpha = (\Box m\varphi - q\varphi)(X\varphi + \varphi) + \frac{1}{2}(r^{-2}(L(r\varphi))^2 + |\nabla \varphi|^2) - \frac{1}{2}(Xq + q\varphi^2). \tag{5.7}
\]

Indeed, by using (5.3), we derive

\[
\partial^{\alpha(x)}\varphi_\alpha = (\Box m\varphi - q\varphi)X\varphi - \frac{1}{2}Xq \cdot \varphi^2 + Q_{\alpha\beta}(\pi^{(x)}_{\alpha\beta} + \frac{1}{2}\partial_\alpha(\varphi^2) + \partial_\alpha Y_\alpha. \tag{5.8}
\]

We can check the nonvanishing components of $(\pi^{(x)})_{\alpha\beta}$ for $X = rL$ are

\[
(\pi^{(x)}_{LL}) = 2, \quad (\pi^{(x)}_{LL}) = -1, \quad (\pi^{(x)}_{eAeB}) = \delta_{AB}, \quad A, B = 1, 2
\]

and

\[
Q_{LL} = (L\varphi)^2, \quad Q_{LL} = (L\varphi)^2, \quad Q_{LL} = |\nabla \varphi|^2 + q\varphi^2
\]

\[
Q_{AB} = \Box A\varphi\Box B\varphi - \frac{1}{2}m_{AB}(L\varphi|L\varphi + |\nabla \varphi|^2 + q\varphi^2).
\]

By combining the calculations of $Q_{\alpha\beta}$ and $(\pi^{(x)}_{\alpha\beta})$, we derive

\[
Q_{LL}(\pi^{(x)}_{LL}) = \frac{1}{2}(L\varphi)^2, \quad Q_{LL}(\pi^{(x)}_{LL}) = \frac{1}{2}|(\nabla \varphi|^2 + q\varphi^2
\]

\[
Q_{AB}(\pi^{(x)}_{AB}) = L\varphi L\varphi - q\varphi^2.
\]

Thus

\[
Q_{\alpha\beta}(\pi^{(x)}_{\alpha\beta}) = \frac{1}{2}((L\varphi)^2 - |\nabla \varphi|^2 + L\varphi L\varphi - \frac{3}{2}q\varphi^2.
\]

It is straightforward to have

\[
\frac{1}{2}\partial_\alpha(\varphi^2) = \partial_\alpha \varphi \partial_\alpha \varphi + \Box m\varphi \cdot \varphi = -L\varphi L\varphi + |\nabla \varphi|^2 + \Box m\varphi \cdot \varphi.
\]

Note that $\partial_\alpha L_\alpha = 2/r$. We have

\[
\partial_\alpha Y_\alpha = \frac{1}{2}\partial_\alpha(r^{-1}\varphi^2L_\alpha) = \frac{1}{2}\{\varphi^2L(r^{-1}) + r^{-1}L(\varphi^2) + r^{-1}\varphi^2 \partial_\alpha L_\alpha\}
\]

\[
= \frac{1}{2}(r^{-1}L(\varphi^2) + r^{-2}\varphi^2).
\]

Combining the above calculations with (5.8) implies (5.7).

On the other hand, by divergence theorem, we have

\[
\int_{\Sigma_{0+}} N^{\alpha(x)}\partial_\alpha d\mu_\Sigma + \int_{\Sigma_{0-}} N^{\alpha(x)}\partial_\alpha d\mu_\Sigma = \int_{\Sigma_{0-}} (\pi^{(x)}_\alpha(\partial_\alpha)^2 dx - \int_{D^{0+}} \partial^{\alpha(x)}\partial_\alpha dxdt \tag{5.9}
\]
where the area elements $d\mu_\mathcal{H}$ and $d\mu_{\mathcal{H}r}$ can be found in (2.9). A direct substitution implies

$$r^2(1+h)\mathcal{N}^{\alpha(X)}\mathcal{P}_a = r\{(L(r\varphi)^2 + r^2h(|\nabla\varphi|^2 + q\varphi^2) - r^{-\frac{1}{2}}L'(r^2\varphi^2)\},$$

$$r^2(1+h)\mathcal{N}^{\alpha(X)}\mathcal{P}_a = r^3(h(Lr)^2 + |\nabla r|^2 + q\varphi^2) + \frac{1}{2}L'(r^2\varphi^2) + rh(Lr^2 + \varphi^2), \quad (5.10)$$

$$r^2\partial_1^{\alpha(X)}\mathcal{P}_a = \frac{1}{2}r^3(r^{-2}|L(r\varphi)|^2 + |\nabla \varphi|^2 + q\varphi^2) - r^{-\frac{1}{2}}\partial_1(r^2\varphi^2).$$

Note that, due to (1.2), the following energy inequality holds

$$\exists Q, \forall \alpha_1 \leq u_1 \leq u_0, \text{ there holds}$$

$$\int_{d\mu_{\mathcal{H}r}} \frac{d}{du}(r^2\varphi^2)d\omega du - \int_{d\mu_{\mathcal{H}r}} \frac{d}{du}(r^2\varphi^2)d\omega du - \int_{\Sigma_{1}^{u} \mathcal{H}_{r}} \partial_1(r^2\varphi^2)d\omega dr = 0. \quad (5.11)$$

By adding this identity to (5.9), in view of (5.10) and (2.7), we can obtain

$$\int_{d\mu_{\mathcal{H}r}} (r(L(r\varphi))^2 + h^{-1}b(\varphi^2)) (1 + h)^{-1}b^2d'u'd\omega + \int_{d\mu_{\mathcal{H}r}} r^3(h|L\varphi|^2$$

$$+ |\nabla \varphi|^2 + q\varphi^2) (1 + h)^{-1}b^2d'u'd\omega + \frac{1}{2} \int_{d\mu_{\mathcal{H}r}} \{(L(r\varphi))^2 + r^2|\nabla \varphi|^2\}bdu'd'\omega$$

$$\lesssim \mathcal{W}_1[\varphi](\Sigma_{1}^{u} \mathcal{H}_{r}) + \int_{d\mu_{\mathcal{H}r}} |\nabla x|^2 + q\varphi^2 \}bdu'd'\omega$$

$$+ \int_{d\mu_{\mathcal{H}r}} r|\nabla \varphi|^2 + rL(\varphi)^2\}d\omega.$$ 

The last term can be treated by using (3.15) and Cauchy-Schwartz inequality,

$$\int_{d\mu_{\mathcal{H}r}} |h[(r|L(\varphi)^2)| + \varphi^2]d\omega \leq \int_{S_{-\mathcal{H}r}} r\varphi^2d\omega + E[\varphi](\mathcal{H}_{\mathcal{H}r}) + \int_{d\mu_{\mathcal{H}r}} |h||L\varphi|^2d\omega'$$

$$\lesssim \int_{S_{-\mathcal{H}r}} r\varphi^2d\omega + E[\varphi](\mathcal{H}_{\mathcal{H}r}).$$

Note that, due to (1.2), $|\nabla x| \lesssim (1 + r)^{-2-\eta}$. We then have by using (3.13) that

$$\int_{d\mu_{\mathcal{H}r}} |\nabla x|^2 \lesssim \int_{-\mathcal{H}r} (1 + u)^{-1-\eta} \mathcal{W}_1[\varphi](\mathcal{D}_{\mathcal{H}r})d'\omega + \int_{\Sigma_{1}^{u} \mathcal{H}_{r}} r^{-1}\varphi^2 + M \int_{S_{-\mathcal{H}r}} u^{-1}E[\varphi](\mathcal{H}_{\mathcal{H}r})d\omega'. \quad (5.12)$$

The first term can be absorbed by using Gronwall’s inequality, other terms can be derived by direct integration. Thus, Lemma 5.3 is proved. □

We then can derive the following result.

**Proposition 5.4.** Let $p \leq 0$ be any fixed number. With $I = \{(u, \mathcal{H}r) : -u \leq u_1 \leq u_0\}$, the following energy inequality holds

$$\mathcal{W}_1[\varphi](\mathcal{H}_{\mathcal{H}r}) + \mathcal{W}_1[\varphi](\mathcal{D}_{\mathcal{H}r}) + \mathcal{W}_1[\varphi](\mathcal{D}_{\mathcal{H}r})$$

$$\lesssim \mathcal{W}_1[\varphi](\Sigma_{0}^{u} \mathcal{H}_{r}) + \|r^2F\|_{L^2}^2 + u_1^{-\gamma_0 + 2p + 1} \sup_{-\mathcal{H}r \leq u \leq u_1} u_1^{-2p + \gamma_0} E[\varphi](\mathcal{H}_{\mathcal{H}r})$$

$$+ E[\varphi](\mathcal{H}_{\mathcal{H}r}) + \|r^{-1/2}\varphi\|_{L^2(\Sigma_{1}^{u} \mathcal{H}_{r})} + \int_{S_{-\mathcal{H}r}} r\varphi^2d\omega.$$ 

**5.1 Preliminaries.** The proof is based on a bootstrap argument, with the assumption of (BA2) and $\Delta_0 = C_1E_2\gamma_0$, with $C_1 > 1$ to be determined. We recast the assumption as follows.

Let $u_\star > -u_0$ be any fixed large number. For $0 \leq n \leq 2$, we suppose

$$E[Z^{(n)}\phi](\mathcal{H}_{\mathcal{H}r}) + E[Z^{(n)}\phi](\mathcal{D}_{\mathcal{H}r}) \leq 2\Delta_0 u_+^{\gamma_0 + 2\gamma(Z^\infty)}, \quad (5.13)$$

$$\mathcal{W}_1[Z^{(n)}\phi](\mathcal{H}_{\mathcal{H}r}) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{D}_{\mathcal{H}r}) \leq 2\Delta_0 u_+^{\gamma_0 + 1 + 2\gamma(Z^\infty)} \quad (5.14)$$

hold for all $-u_\star \leq -u \leq u \leq u_0$. 

The local well-posedness result, in \{-\mu < u \leq u_0\} with \(\mu\) finite, can follow by running a standard iteration argument (see [23]), or by using the standard local existence result up to the characteristic boundary, i.e. \(\{r \geq t + R\}\). Thus the above assumptions hold for some \(\mu > -u_0\). Our task is to show that the estimates in the assumption hold for any \(\mu > -u_0\), with the bound improved to be \(< 2\Delta_0\). \(^{19}\)

As a direct consequence of the bootstrap assumption, we have

**Lemma 5.5.** For \(Z = \Omega \) or \(\partial\), there holds

\[
|Z\phi| \lesssim u_+^{-\frac{1}{2} - \frac{\Delta_0}{2}} \Delta_0^{\frac{3}{2}} r^c(Z).
\] (5.15)

**Proof.** It follows by using (4.1) with \(l = 0\) that

\[
 r|\nabla \phi| \lesssim r^{-\frac{1}{2}} u_+^{-\frac{\Delta_0}{2}} \Delta_0^{\frac{3}{2}}, \quad r|\partial \phi| \lesssim u_+^{-\frac{1}{2} - \frac{\Delta_0}{2}} \Delta_0^{\frac{3}{2}}
\]

which gives (5.15). \(\Box\)

Let \(0 \leq m \leq n\). For the ordered product of vector fields, \(Z^n = Z_1 \cdots Z_n\), we denote by \(Z^m \subset Z^n\) if \(Z^n = Z_{k_1} \cdots Z_{k_m}\) with \(k_1 < k_2 < \cdots < k_m\). By \(Z^a \cup Z^b = Z^n\), we denote a decomposition of \(Z^n\) into \(Z^a\) and \(Z^b\). It means, \(Z^a, Z^b \subset Z^n\) with \(Z^a = Z_{k_1} \cdots Z_{k_a}\) and \(Z^b = Z_{m_1} \cdots Z_{m_b}\), none of the subindices among \((k_1, \ldots, k_a)\) and \((m_1, \ldots, m_b)\) are equal, i.e., \(k_i \neq m_j\) and \(a + b = n\). \(Z^n \cup \cdots \cup Z^m\) can be understood inductively.

If \(Z_1 Z_2 \cdots Z_n\) is regarded as a differential operator, we denote it as \(Z^{(n)}\). We set \(Z^{(n-1)} = Z_1 \cdots Z_{n-1} Z_{n+1} \cdots Z_n\), for \(i = 1, \ldots, n\). \(Z^{(n-1)}\) represents the corresponding product of vector fields.

**Lemma 5.6.** For each killing vector field \(Z\), \([Z, \partial]\) = \(C Z_\alpha \gamma \partial_\gamma\), where \(C Z_\alpha \gamma = -\partial_\alpha Z_\gamma\) is a \((1,1)\) tensor. \(C Z_\alpha \gamma = 0\) if \(Z = 0\). Due to (3.19), the components of \(C Z\) are \(1\) or \(-1\) if \(Z = \Omega_{\mu\nu}\).

Thus, symbolically, we may ignore the tensorial feature of \(C Z\), and regard \(C Z\) as constants. The tensor products \(C Z^m = C Z_1 \cdots C Z_n\) may be regarded as a set of product of constants, since \(C Z_i\) is understood as a set of constants with \(|C Z_i| = 1\) if \(Z_i = \Omega\); and \(C Z_i = 0\) if \(Z_i = \partial\). \(^{20}\)

1. For \(n = 1, 2, 3\), there holds the symbolic identity

\[
[\partial, Z^{(n)}] f = \sum_{Z^n \cup Z^b = Z^n, a \geq 1} C_{Z^a} \partial Z^{(b)} f.
\] (5.16)

Thus, if \(\zeta(Z^n) = -n\), \([\partial, Z^{(n)}] f = 0\).

2. For \(n = 1, 2, 3\)

\[
|Z^{(n)} \partial f| \lesssim \sum_{Z^n \cup Z^b = Z^n} r^c(Z^n)|\partial Z^{(b)} f|.
\] (5.17)

**Remark 5.7.** In application, most of the time we will replace \(r^c(Z^n)\) by \(u_+^{\zeta(Z^n)}\) which is a weaker version of the result.

**Proof.** (5.16) follows by direct calculation. It follows directly from (1) in view of the definition of \(\zeta(\cdot)\) that \(|[\partial, Z^{(n)}] f| \lesssim \sum_{Z^n \cup Z^b = Z^n, a \geq 1} r^c(Z^n)|\partial Z^{(b)} f|\). (5.17) follows as its consequence. \(\Box\)

**Lemma 5.8.** Let \(\phi\) be a smooth function and \(n = 1, 2, 3\). Under the assumption that \(|\lambda^{(i)}(\phi)| \leq C\) with \(i = 1 \cdots n\), there holds

\[
|Z^{(n)}(\lambda(\phi))| \lesssim |Z^{(n)}\phi| + \sum_{i=1}^n |Z^{(n-1)} \phi \cdot Z_i \phi| + |\Pi_{i=1}^n Z_i \phi|
\] (5.18)

and consequently

\[
|Z^{(n)}(\lambda(\phi))| \lesssim \sum_{Z^n \cup Z^b = Z^n, a \geq 1} |Z^{(a)} \phi| \Delta_0^{\frac{1}{2} - \frac{\Delta_0}{2}} u_+^{\zeta(Z^n)}.
\] (5.19)

\(^{19}\)The same argument is employed for setting up the bootstrap assumptions in Section 6 and Section 7 which will not be repeated in later sections.

\(^{20}\)C_{id} = 0
Remark 5.9. Under the bootstrap assumption $\text{(BA}_2\text{)}, (5.12)$ holds, which imply $|\phi| \lesssim r^{-1}(\xi_{\gamma_0}^2 + \Delta_0^{-1}) \lesssim 1$. Since $N(y)$ is smooth, we can obtain $|N_y^{(i)}(\phi)| \lesssim 1$, $i \leq k$ for any fixed $k \in \mathbb{N}$. So the assumption holds for $\varphi = \phi$. We also remark that we only used $|Z\phi| \lesssim \Delta_0^2 u_{\gamma_0}^{(2)}$ to prove the above result.

**Proof.** It is straightforward to derive

$$
Z(1)(\varphi) = N'(\varphi)Z(1)\varphi,
$$

$$
Z(2)(\varphi) = N'(\varphi)Z(2)\varphi + N''(\varphi)Z_2\varphi Z_1\varphi,
$$

$$
Z(3)(\varphi) = N'(\varphi)Z(3)\varphi + N''(\varphi)\sum_{i=1}^{3} Z^{(n-i)}\varphi \cdot Z_i\varphi + N'''(\varphi)\Pi_{i=1}^{3} Z_i\varphi.
$$

We then can derive $(5.18)$ for $n = 1, 2, 3$. $(5.19)$ follows by using $(5.15)$. $\square$

5.2. **Error estimates.** We will improve the bootstrap assumptions $(5.13)$ and $(5.14)$ by deriving energy estimates, with the help of Proposition 5.2 and Proposition 5.3. For deriving both types of estimates for $Z^{(i)}\phi$, the main task is to obtain the error estimates on $(\bigcup_{m=q} Z^{(i)}\phi$ with $i \leq 2$. We analyze in the following result these major error terms.

**Lemma 5.10.** Let $\mathcal{F} = N(\phi)\partial\phi \cdot \partial\phi$, $\mathcal{F} = Z^{(n)}\mathcal{F}$ and $\mathcal{F} = \hat{\mathcal{F}}$. For $n = 0, \cdots, 3$,

(1)

$$
|Z^{(n)}(\partial\phi \cdot \partial\phi)| \lesssim \sum_{Z = \bigcup Z^a} u_+^{(Z_1)}|\partial Z^{(a)}\phi||\partial Z^{(a)}\phi|.
$$

(2)

$$
|\mathcal{F}| \lesssim |\mathcal{F}_Q| + |\mathcal{F}_C|
$$

where the quadratic part and the cubic part are

$$
\mathcal{F}_Q = \sum_{Z = \bigcup Z^a} u_+^{(Z_1)}|\partial Z^{(a)}\phi||\partial Z^{(a)}\phi|,
$$

$$
\mathcal{F}_C = \sum_{Z = \bigcup Z^a} |Z^{(n)}\phi||\partial Z^{(a)}\phi||\partial Z^{(a)}\phi|u_+^{(Z_1)}.
$$

**Remark 5.11.** The result with $n = 3$ will be used in Section 6 and 7.

**Proof.** We first can obtain $(5.20)$ by using $(5.17)$ and

$$
Z^{(n)}(\partial\phi \cdot \partial\phi) = \sum_{Z = \bigcup Z^b} Z^{(a)}\partial\phi \cdot Z^{(b)}\partial\phi.
$$

Next we derive the estimates of $\mathcal{F}$ in view of

$$
\mathcal{F} = \left(\sum_{a \geq 1} + \sum_{a = 0}\right) \sum_{Z = \bigcup Z^a} Z^{(a)}(N(\phi))Z^{(b)}(\partial\phi \cdot \partial\phi).
$$

The $a = 0$ term can be bounded by using $(5.20)$ directly. For the terms with $a \geq 1$ in $(5.21)$, we can apply $(5.19)$ with $n = a$ and $(5.20)$ with $n = b$ to derive the cubic type of terms. We then combine the estimates for $0 \leq a \leq n$ to obtain

$$
|\mathcal{F}| \lesssim \sum_{Z = \bigcup Z^a} \sum_{1 \leq a \leq n} \Delta_0^{-\frac{1}{2}}(a_{a+1})|Z^{(a+1)}\phi||\partial Z^{(a)}\phi||\partial Z^{(a)}\phi|u_+^{(Z_1)} + \sum_{Z = \bigcup Z^a} u_+^{(Z_1)}|\partial Z^{(a)}\phi||\partial Z^{(a)}\phi|.
$$

The second line is the quadratic term $\mathcal{F}_Q$. The first line on the righthand side is a sum of cubic terms of $\phi$. By the boundedness of $\Delta_0^{-\frac{1}{2}}$, we can obtain the formula for $\mathcal{F}_C$ in $(5.21)$. $\square$
As an important remark, we can write according to Lemma 5.10 that
\[
|F| \lesssim (|\partial Z\phi| + u_+^{c(Z)}|\partial \phi|)|\partial \phi|,
\]
for which the cubic term is already controlled by using (6.16). Thus, symbolically,
\[
|F| \lesssim (F_Q).
\]

(5.23)

**Proposition 5.12.** For \(n \leq 2\) and \(-u_1 \leq -u_1 \leq u_1 \leq u_0\), the following estimates hold
\[
u_1^{1/2} + \frac{1}{2} - \gamma_0 < (Z^n) \|r_+^{1/2} \langle F_Q \rangle_{L^2(P_{D_{Q}})} \lesssim u_1^{1/2} + \frac{1}{2} - \gamma_0 \Delta_0 M^{-1/2},
\]
(5.24)
\[
u_1^{1/2} + \frac{3}{2} - \gamma_0 < (Z^n) \|r_+^{1/2} \langle F_Q \rangle_{L^2(P_{D_{Q}})} \lesssim u_1^{1/2} + \frac{3}{2} - \gamma_0 \Delta_0 M^{-1/2},
\]
(5.25)
\[
u_1^{1/2} + \frac{1}{2} - \gamma_0 < (Z^n) \|r_+^{1/2} \langle F_Q \rangle_{L^2(P_{D_{Q}})} \lesssim u_1^{1/2} + \frac{1}{2} - \gamma_0 \Delta_0 M^{-1/2},
\]
(5.26)
\[
u_1^{1/2} + \frac{3}{2} - \gamma_0 < (Z^n) \|r_+^{1/2} \langle F_Q \rangle_{L^2(P_{D_{Q}})} \lesssim u_1^{1/2} + \frac{3}{2} - \gamma_0 \Delta_0 M^{-1/2}.
\]
(5.27)

**Proof.** We first decompose \(\{n\} F_Q\) as below
\[
\{n\} F_Q, 1 = \sum_{Z^n \cup Z^n \cup Z^n = Z^n, b = 0} u_1^{c(Z^n)} |\partial Z^{b} \phi| |\partial Z^{a} \phi|,
\]
(5.28)
\[
\{n\} F_Q, 2 = \sum_{Z^n \cup Z^n \cup Z^n = Z^n, b \geq 1} u_1^{c(Z^n)} |\partial Z^{b} \phi| |\partial Z^{a} \phi|,
\]
where we assume \(b \leq a\) without loss of generality. We will frequently use (5.13) and (5.14) in the sequel.

Note that with \(a \leq n\), we can apply (4.11) and (4.14) to derive
\[
\|r_+^{1/2} \langle F_Q \rangle_{L^2(P_{D_{Q}})} \lesssim \sum_{Z^n \cup Z^n \cup Z^n = Z^n, b = 0} u_1^{c(Z^n)} \|r \partial Z^{b} \phi \|_{L^1_{n}(P_{D_{Q}})} \|r^{-1/2} \partial Z^{a} \phi \|_{L^2_{n}(P_{D_{Q}})} \|
\]
(5.29)
\[
\lesssim \Delta_0 M^{-1/2} u_1^{1/2 - \gamma_0 + \gamma(Z^n)}.
\]

We note that by (5.23) in the case of \(n \leq 1\), \(\{n\} F_Q, 2\) vanishes. If \(n = 2\), \(a = b = 1 \leq n - 1\). In view of (4.19) and (4.21), we deduce for \(\{n\} F_Q, 2\) that
\[
\|r_+^{1/2} \langle F_Q \rangle_{L^2(P_{D_{Q}})} \lesssim \sum_{Z^n \cup Z^n \cup Z^n = Z^n, b \geq 1} u_1^{c(Z^n)} \|r \partial Z^{b} \phi \|_{L^1_{n}(P_{D_{Q}})} \|r^{-1/2} \partial Z^{a} \phi \|_{L^2_{n}(P_{D_{Q}})} \|
\]
\[
\lesssim \Delta_0 M^{-1/2} u_1^{1/2 - \gamma_0 + \gamma(Z^n)}.
\]

(5.24) follows by combining the estimates of \(\{n\} F_Q, 1\) and \(\{n\} F_Q, 2\).

Next, we prove (5.25). Using (4.11) and (4.14) implies
\[
\|r_+^{1/2} \langle F_Q \rangle_{L^2_{n}(P_{D_{Q}})} \lesssim \sum_{Z^n \cup Z^n \cup Z^n = Z^n} u_1^{c(Z^n)} \|r \partial Z^{a} \phi \|_{L^1_{n}(P_{D_{Q}})} \|r^{1/2} \partial Z^{a} \phi \|_{L^2_{n}(P_{D_{Q}})} \|
\]
\[
\lesssim \Delta_0 M^{-1/2} u_1^{1/2 - \gamma_0 + \gamma(Z^n)}.
\]

(5.26)
In the case that $1 \leq b \leq a$, again $a \leq n - 1$. By using (4.20) and (4.22), we have
\[
\| r^{\frac{3}{2}} \mathcal{F} c \|_{L^1_{\gamma} L^2_{\gamma} L^2_{\gamma} (\mathbb{D}_{\gamma}^n)} \lesssim \sum_{Z^a \cup Z^b \cup Z^c = Z^a, b \geq 1} u_{1+}^{\zeta(Z^a)} \| r \partial Z(b) \phi \|_{L^\infty_{\gamma} L^\infty_{\gamma} L^1_{\gamma} (\mathbb{D}_{\gamma}^n)} \cdot \| r^{\frac{1}{2}} \partial Z(b) \phi \|_{L^2_{\gamma} L^2_{\gamma} L^1_{\gamma} (\mathbb{D}_{\gamma}^n)} \lesssim \Delta_0 M^{-\frac{\beta}{2}} u_{1+}^{\zeta(Z^a) - \gamma_0 + \frac{1}{2}}.
\]
By combining the above two estimates, we can obtain (5.25).

Next we consider the estimates of $\{2\} c$ by using (4.1), (4.21) and (4.24),
\[
\| r^{\frac{1}{2}} \mathcal{F} c \|_{L^2_{\gamma} (\mathbb{D}_{\gamma}^n)} \lesssim \sum_{Z^a \cup Z^b \cup Z^c = Z^a, b \geq 1, b \leq c} \| Z(a) \phi \|_{L^\infty_{\gamma} L^\infty_{\gamma} L^1_{\gamma} (\mathbb{D}_{\gamma}^n)} \| r \partial Z(b) \phi \|_{L^\infty_{\gamma} (\mathbb{D}_{\gamma}^n)} \| r^{\frac{1}{2}} \partial Z(c) \phi \|_{L^2_{\gamma} L^2_{\gamma} L^1_{\gamma} (\mathbb{D}_{\gamma}^n)} \lesssim \Delta_0 M^{-\frac{\beta}{2}} u_{1+}^{\zeta(Z^a) - \frac{3}{2} \gamma_0 - \frac{1}{2} + \zeta(Z^b)}.
\]
Here we assumed $b \leq c$ without loss of generality, which implies $b = 0$. By using (4.1), (4.20) and (4.17), we have
\[
\| r^{\frac{3}{2}} \mathcal{F} c \|_{L^1_{\gamma} L^2_{\gamma} L^2_{\gamma} (\mathbb{D}_{\gamma}^n)} \lesssim \sum_{Z^a \cup Z^b \cup Z^c = Z^a, b \geq 1, b \leq c} u_{1+}^{\zeta(Z^a)} \| r^{\frac{3}{2}} Z(a) \phi \|_{L^2_{\gamma} L^2_{\gamma} L^1_{\gamma} (\mathbb{D}_{\gamma}^n)} \| r \partial Z(b) \phi \|_{L^\infty_{\gamma} (\mathbb{D}_{\gamma}^n)} \| r^{\frac{1}{2}} \partial Z(c) \phi \|_{L^2_{\gamma} L^2_{\gamma} L^1_{\gamma} (\mathbb{D}_{\gamma}^n)} \lesssim \Delta_0 M^{-\frac{\beta}{2}} u_{1+}^{\zeta(Z^a) - \frac{1}{2} \gamma_0 - \frac{1}{2}}.
\]
Thus (5.26) and (5.27) are both proved. \(\square\)

**Lemma 5.13.** Under the assumption of (1.2), there hold for $n \leq 3$ and $-u_a \leq -u_1 \leq u_1 \leq u_0$ that
\[
u_{1+}^{-\zeta(Z^a) + \frac{1}{2} \gamma_0} \left( \| r^{\frac{3}{2}} |Z(n)|, q| \phi \|_{L^1_{\gamma} L^2_{\gamma} L^2_{\gamma} (\mathbb{D}_{\gamma}^n)} + \| r^{\frac{3}{2}} |Z(n)|, q| \phi \|_{L^2_{\gamma} L^2_{\gamma} L^1_{\gamma} (\mathbb{D}_{\gamma}^n)} \right) \lesssim \mathcal{E}_{n, \gamma_0}^{1/2} \sup_{-u_1 \leq \omega \leq u_1} \sum_{Z^a \subseteq Z^a} \left( E[Z^a(m) \phi] \right)^{1/2} (H^{\omega}_{\gamma_0}) u_{1+}^{\zeta(Z^a) - \frac{1}{2} \gamma_0 - \frac{1}{2}}.
\]
\[
u_{1+}^{-\zeta(Z^a) + \frac{1}{2} \gamma_0} \left( \| r^{\frac{3}{2}} |Z(n)|, q| \phi \|_{L^1_{\gamma} L^2_{\gamma} L^2_{\gamma} (\mathbb{D}_{\gamma}^n)} \right) \lesssim \mathcal{E}_{n, \gamma_0}^{1/2} \sum_{Z^a \subseteq Z^a} u_{1+}^{-\zeta(Z^a) + \frac{1}{2} \gamma_0 - \frac{1}{2} \gamma_0 - \frac{1}{2}} \left( W_1[Z^a(m) \phi] (H^{\omega}_{\gamma_0}) \right) + M \int_{-u_1}^{u_1} E[Z^a(m) \phi] (H^{\omega}_{\gamma_0}) du \right)^{1/2}.
\]

**Proof.** It is direct to obtain
\[
[Z^n, q| \phi = \sum_{i=1}^n Z(i) q Z(n-i) \phi,
\]
where all $Z^{n-i} \subseteq Z^n$. Note that the assumption (1.2) on $q$ implies
\[
\| r^{\frac{1}{2} + \frac{1}{2} \eta} Z(i)^n q \|_{L^2_{\gamma} L^\infty (\mathbb{D}_{\gamma}^n)} + \| r^{\frac{1}{2} + \frac{1}{2} \eta} Z(i)^n q \|_{L^2_{\gamma} L^\infty (\mathbb{D}_{\gamma}^n)} \lesssim u_{1+}^{\zeta(Z^a)}.
\]
Thus
\[ \| r[Z^n], q \phi \|_{L^1 L^2 L^\infty(D^n)} + \| r[Z^n], q \phi \|_{L^1 L^2 L^\infty(D^n)} \lesssim \sum_{1 \leq i \leq n} u_{1+}^{\zeta(Z^n)} \| r^{-\frac{3}{2} + \gamma} Z^{(n-i)} \phi \|_{L^2(D^{3n})}. \]

By using (3.13), we can derive
\[ \| r^{-\frac{3}{2} + \gamma} Z^{(n-i)} \phi \|_{L^2(D^{3n})} \lesssim \sup_{u_{1+} \leq u \leq u_0} \| r^{-1} Z^{(n-i)} \phi \|_{L^2(D^{3n})} \]
\[ \lesssim \sup_{u_{1+} \leq u \leq u_0} \left( \int_{S_{u_{1+}}} r(Z^{(n-i)} \phi)^2 d\omega \right)^{\frac{1}{2}} + E[Z^{(n-i)} \phi]^{\frac{1}{2}}(H^{u_0}). \]

We can apply (3.3) to \( Z^{(n-i)} \phi \) to control the first term. The first inequality of Lemma 5.13 holds by combining the above two estimates.

To see the second inequality, we have by using (5.30) that
\[ \| r^{\frac{3}{2}} [Z^n], q \phi \|_{L^1 L^2 L^\infty(D^{3n})} \lesssim \sum_{1 \leq i \leq n} u_{1+}^{\zeta(Z^n)} \| r^{-1 - \frac{3}{2} + \gamma} Z^{(n-i)} \phi \|_{L^2(D^{3n})}. \]

It follows by using (3.13) that
\[ \| r^{-1} Z^{(n-i)} \phi \|_{L^2(D^{3n})}^2 \lesssim W_1[Z^{(n-i)} \phi](D^{3n}) + M \int_{u_{1+}}^{u_{1-}} u^{-1} E[Z^{(n-i)} \phi](H^{u_0}) du \]
\[ + \int_{\Sigma_{u_{1+}}} r(Z^{(n-i)} \phi)^2 d\omega. \]

By combining the above two inequalities and applying (5.7) to \( Z^{(n-i)} \phi \), we can obtain Lemma 5.13.

5.3. Boundedness of energies. Next, we will use the fact
\[ \square_m Z^{(n)} \phi - q Z^{(n)} \phi = (n) \phi \]
Proposition 5.12, Lemma 5.13, Proposition 5.2, and Proposition 5.4 to prove the boundedness of energies in (5.10).

Proposition 5.14. Let \( \mathcal{I} = \{(u, u_i), -u_i \leq u \leq u_0\} \) and \( n \leq 2 \). For \( (u, u_i) \in \mathcal{I} \) with \( u \leq u_i \leq u_0 \), there hold
\[ u_{1+}^{\gamma_0 - 2 \zeta(Z^n)} (E[Z^{(n)} \phi](H^{u_0}) + E[Z^{(n)} \phi](H^{u_0})) \lesssim \mathcal{E}_{n, \gamma_0} + M^{2} \Delta_0^2 u_{1+}^{\gamma_0}, \]
\[ u_{1+}^{\gamma_0 - 2 \zeta(Z^n) - 1 + \gamma_0} \left( W_1[Z^{(n)} \phi](H^{u_0}) + W_1[Z^{(n)} \phi](H^{u_0}) + W_1[Z^{(n)} \phi](D^{3n}) \right) \lesssim \mathcal{E}_{n, \gamma_0} + M^{2} \Delta_0^2 u_{1+}^{\gamma_0}. \]

Proof. In view of (5.31), we will use Proposition 5.2 with \( \mathcal{F} = \mathcal{F}^c \), \( \mathcal{F} \), and \( \mathcal{F}^b = [Z^{(n)}, q] \phi \).

By using (5.24), (5.26) and Lemma 5.13, we have
\[ u_{1+}^{\gamma_0 - 2 \zeta(Z^n)} (E[Z^{(n)} \phi](H^{u_0}) + E[Z^{(n)} \phi](H^{u_0})) \]
\[ \lesssim \mathcal{E}_{n, \gamma_0} + M^{2} \Delta_0^2 u_{1+}^{\gamma_0} + \sup_{-u_i \leq u \leq u_0} \sum_{i=1}^{n} E[Z^{(n-i)} \phi](H^{u_0}), \]
where the last term vanishes when \( n = 0 \). This implies the first estimate in Proposition 5.14 by induction.

In view of (5.24) and (5.26) in Proposition 5.12 for \( \mathcal{F} \), \( \mathcal{F}^c \), we can derive for \( (u_1, u_k) \in \mathcal{I}, \)
\[ u_{1+}^{\gamma_0 - \zeta(Z^n) + \gamma_0 - \frac{1}{2}} \| r^{\frac{3}{2}} f \|_{L^1 L^2 L^\infty(D_{3n})} \lesssim \Delta_0 M^{-\frac{1}{2}}. \]
By using the above estimate, Proposition 5.1, (3.3), (3.7), (3.8), the second estimate in Lemma 5.13 and (5.32), we can derive for \((u_1, u_1) \in \mathcal{I},\)
\[
u_1^{-2K(Z^n) - 1 + \gamma_0} \left( W_1[Z^n] \phi([H^2_1]) + W_1[Z^n] \phi([H^2_1]) + W_1[Z^n] \phi([D^2_1]) \right)
\leq \epsilon_{n, \gamma_0} + \Delta_0^2 u_1^{1+\gamma_0} (M^{-1} + M^{-2} u_1^2) + \sum_{i=1}^n \nu_1^{-2K(Z^{n-i}) - 1 + \gamma_0} W_1[Z^{n-i}] \phi([D^2_i]).
\]
Note that when \(n = 0\) the last term on the right vanishes. The weighted energy estimate can then be derived by induction.

**Improvement on the bootstrap assumption** Let us denote the universal constant in \(\epsilon \) in the estimates of Proposition 5.14 as \(C_3 > 0\). We need to show that
\[
C_3(\epsilon_{2, \gamma_0} + M^{-2} \Delta_0^2 u_1^{1+\gamma_0}) < 2\Delta_0, \forall u \leq u_0
\]
Recall that \(\Delta_0 = C_1 \epsilon_{2, \gamma_0}\) with \(C_1\) to be chosen, and \(\epsilon_{2, \gamma_0} \leq CM^2\) with \(C \geq 1\) a fixed constant. Thus we need to choose \(C_1\) such that
\[
C_3(C_1^{-1} + M^{-2} \Delta_0 u_1^{1+\gamma_0}) < 2.
\]
Let \(C_1 = 4C_3\). It is reduced to \(M^{-2} \Delta_0 u_1(R) (R^{1-\gamma_0} < \frac{7}{16C_3^2}\). Since \(\delta_1\) in Theorem 2.1 can be sufficiently small, \(M < \frac{1}{16}\) can be achieved. Thus \(u_1(R) > \frac{1}{2} R\) can be guaranteed. Thus we need
\[
\left(\frac{R}{2}\right)^{1-\gamma_0} < \frac{7}{16C_3^2}.
\]
\(R\) is fixed to satisfy the inequality of (5.33). Thus the proof of Theorem 2.1 is completed. If \(R = 2\) but \(C\) is allowed to be chosen, with \(C \leq \delta_0\) such that \(\frac{16}{C_3^2} \delta_0 < (\frac{7}{2})^{1-\gamma_0} = 1\), (5.33) can also be achieved. This proves Theorem 2.2

6. Quasilinear equations

In this section, we consider the general quasilinear equations (1.1) in \(\mathbb{R}^{3+1}\) which verifies (1.2) for \(n = 3\). \(g(\phi, \partial \phi)\) and \(N^{\alpha \beta} (\phi)\) are both smooth functions of their arguments, \(g(0,0) = m\). For convenience, we set \(\Phi = (\phi, \partial \phi)\), which is a 5- vector valued function, and \(|\Phi| = |\phi| + \sum_{\mu=0}^3 |\partial \phi|\). We will prove Theorem 2.5 in this section.

6.1. **Bootstrap assumptions.** Due to the influence of the metric \(g^{\alpha \beta}\), the bootstrap assumptions are more delicate than (5.13) and (5.14) in Section 5. We first need to fix the constant \(M_0\) since the region where the stability result holds is determined by \(u_0(M_0)\). By using \(H^{\alpha \beta}(0,0) = 0\) and the fact that \(H^{\alpha \beta}\) are smooth functions of the arguments, we can derive
\[
\sup_{0 \leq \alpha, \beta \leq n} |H^{\alpha \beta}(\phi, \partial \phi)| \leq |\phi| + |\partial \phi| \text{ on } \Sigma_0 \cap \{r \geq R\}.
\]
By using the above estimate and (3.3), there holds
\[
\sup_{\alpha, \beta} r |H^{\alpha \beta}| \leq C u_+^{\frac{2n+\frac{1}{2}}{2}} \epsilon_{2, \gamma_0, R}^\frac{1}{2} \text{ on } \Sigma_0 \cap \{r \geq R\}. \tag{6.1}
\]
We can choose \(M_0 = 3C_1^\frac{1}{2}\) in the definition of (2.1) and \(h = \frac{M_0}{\nu_1}\). By this choice and (8.5) on \(\Sigma_0 \cap \{r \geq R\}\) there holds
\[
\sup_{\alpha, \beta} r (h - H^{\alpha \beta}) > C(3C_1^\frac{1}{2} - u_+^{\frac{2n+\frac{1}{2}}{2}} \epsilon_{2, \gamma_0, R}). \tag{6.2}
\]
Since \(u_+(R) \geq \frac{1}{2} R\), with \((\frac{5}{2})^{1-\gamma_0} < \frac{1}{2}\), on \(\Sigma_0 \cap \{r \geq R\}\),
\[
\sup_{\alpha, \beta} r (h - H^{\alpha \beta}) > \frac{5}{2} C_1^\frac{1}{2} = \frac{5}{2} M_1 = M_0.
\]

The bootstrap assumption for proving Theorem 2.5 consists of the control of the metric and the boundedness of energies.
Let $\mathbf{u} > -u_0$ be a fixed number and let $\mathcal{I} = \{(u, \mathbf{u}) : -u_0 \leq -\mathbf{u} \leq u \leq u_0\}$. We suppose there hold on $\mathcal{I}$ that
\[
\|\mathbf{u} - H \mathbf{a} \| \geq \frac{M}{3}, \quad \|\mathbf{u} - H \mathbf{L} \| \geq \frac{M}{3},
\] (A1)
and for any $(\mathbf{u}, u_1) \in \mathcal{I}, n \leq 3,$
\[
E[Z^{(n)}] (H^{\mathbf{u}}) + E[Z^{(n)}] (H^{u_1}) \leq 2 \Delta u_1^{-\gamma_0 + 2Z^{(n)}},
\] (6.3)
where \(\Delta u_1 \leq C_1 M^2\) with \(C_1 > 0\) to be chosen later and \(Z \in \{\Omega_{ij}, \partial\}\).

As a consequence of the above assumptions, all the estimates in Section 4 hold. We can first summarize some of the decay estimates that will be frequently used in this section.

Proposition 6.1 (Decay estimates). There hold the following decay estimates\(^{21}\) in $\mathcal{I}$
\[
\begin{align*}
\|Z^{(n)} \| \leq & \Delta u_1^{-\gamma_0 + 2Z^{(n)}}, \quad l, n \leq 1 \quad (6.5) \\
\|Z^{(n)} H \| \leq & \Delta u_1^{-\gamma_0 + 2Z^{(n)}}, \quad n \leq 1, \quad (6.6) \\
\|Z^{(n)} H \| \leq & \Delta u_1^{-\gamma_0 + 2Z^{(n)}}, \quad n \leq 1, \quad (6.7) \\
\|Z^{(n)} H \| \leq & \Delta u_1^{-\gamma_0 + 2Z^{(n)}}, \quad n \leq 1, \quad (6.8)
\end{align*}
\]
where \((u, \mathbf{u}, u_1) \in \mathcal{I}.

Proof. If \(l = 1, (6.5)\) is a consequence of (4.1); if \(l = 0\), it is the estimate of (3.2). The result of (6.5) with \(n = 0\), \(H(0, 0) = 0\) and the fact that \(H\) is smooth imply there hold for all \((u, \mathbf{u}) \in \mathcal{I}\) that
\[
\|H(\Phi)\| \leq |\Phi| + |\partial \phi|; \quad (D^{(l)} H)(\Phi) \| \leq 1, \quad i \geq 1.
\] (6.9)

The \(n = 0\) case in (6.6) can then be derived by using (6.5) with \((n, l) = (0, 1), (0, 0)\). Due to (6.9), \(|Z^{(1)}(H(\Phi))\| \leq |Z^{(1)} \| + |Z^{(1)} \partial \phi|\), by also using (5.17) and (6.5) we have
\[
\|Z^{(1)} \partial \phi\| \leq u_1^{-\gamma_0 + 2Z^{(1)}}. \Delta u_1^{-\gamma_0 + 2Z^{(1)}}.
\] (6.10)
The estimate for \(Z^{(1)} \Phi\) follows from (6.5). Thus \(n = 0\) case in (6.6) is treated. (6.7) follows by using (6.9) and (4.1); similarly, (6.8) can be proved by using (4.3).

6.2. Energy and weighted energy inequalities. In this subsection, we derive the fundamental energy estimate and \(\tau\)-weighted energy estimate for (1.1).

We will always use the Minkowski metric to lift and lower the indices. For example \(H^\beta_\alpha := m_{\alpha\gamma} H^{\alpha\gamma}\). We define the following \((0, 2)\) tensor, which is not necessarily symmetric,
\[
\tilde{\omega}_{\alpha\beta}[\varphi] = \omega_{\alpha\beta}[\varphi] + H^\gamma_\alpha \partial_\gamma \varphi \partial_\beta \varphi,
\]
where
\[
\omega_{\alpha\beta}[\varphi] = \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} m_{\alpha\beta}(g^{\gamma\rho} \partial_\gamma \varphi \partial_\rho \varphi + q \varphi^2).
\]

Lemma 6.2. Let \(X\) be a smooth vector field. There holds
\[
\begin{align*}
\partial^\alpha (\tilde{\omega}_{\alpha\beta} X^\beta) = & X^\beta \{(\tilde{g} \varphi - q \varphi + \partial^\alpha H^\gamma_\alpha \partial_\gamma \varphi) \cdot \partial_\beta \varphi - \frac{1}{2} \partial_\beta q \cdot \varphi^2 \\
& - \frac{1}{2} \partial_\beta H^\gamma_\alpha \partial_\gamma \varphi \cdot \partial^\alpha \varphi \} + \omega_{\alpha\beta}(X)^{\alpha\beta} + X^\beta \partial_\beta \varphi \partial_\alpha \varphi \partial^\alpha X^\beta.
\end{align*}
\] (6.10)

Proof. We calculate \(\partial^\alpha \tilde{\omega}_{\alpha\beta}\) below,
\[
\begin{align*}
\partial^\alpha (\tilde{\omega}_{\alpha\beta} X^\beta) = & \partial^\alpha \omega_{\alpha\beta} X^\beta + \omega_{\alpha\beta}(X)^{\alpha\beta} + X^\beta \partial_\beta \varphi \partial_\alpha \varphi \partial^\alpha X^\beta.
\end{align*}
\]
\(^{21}\)\(Z^{(n)} H\) is understood as \(Z^{(n)}(H(\Phi))\).
For the first term, we proceed as follows

\[
\partial^\alpha \tilde{\mathcal{D}}_{\alpha\beta} X^\beta = (\Box_m \varphi \cdot \partial_\beta \varphi - \frac{1}{2} \partial_\beta (H^\alpha \varphi \partial_\alpha \varphi) + \partial^\alpha (H^\alpha \varphi \partial_\alpha \varphi)) X^\beta \\
= X^\beta \{ ((\Box_m - q) \varphi + H^\alpha \varphi \partial_\alpha \varphi + \partial^\alpha H^\alpha \varphi \partial_\alpha \varphi - \frac{1}{2} \partial_\beta H^\alpha \varphi \partial_\alpha \varphi - \frac{1}{2} \partial_\beta q \varphi^2 \}
\]

as desired. □

We now give the energy density on \( H_{\alpha}^\mu, H_{\alpha}^m \) and \( \Sigma_t \).

**Lemma 6.3.** There hold the following identities for energy densities,

\[
\tilde{\mathcal{D}}_{\alpha\beta} \partial_t^\alpha (L^\alpha + hL^\alpha) = \frac{1}{2} \langle L\varphi \rangle^2 (h + (h - 1)H^L - 2hH^L) \\
+ \frac{1}{2} (|L\varphi|^2 + q\varphi^2) + (H + hH) \partial_\alpha \varphi \partial^\alpha \varphi,
\]

\[
\tilde{\mathcal{D}}_{\alpha\beta} \partial_t^\alpha (\bar{L}^\alpha + hL^\alpha) = \frac{1}{2} \langle L\varphi \rangle^2 (h + (h - 1)H^L - 2hH^L) \\
+ \frac{1}{2} (|L\varphi|^2 + q\varphi^2) + (H + hH) \bar{\partial}_\alpha \varphi \partial^\alpha \bar{\varphi},
\]

and

\[
\tilde{\mathcal{D}}_{\alpha\beta} \partial_t^\alpha \partial_\beta = \frac{1}{2} \{ -g^{00}(\partial_\alpha \varphi)^2 + g^{ij} \partial_\alpha \varphi \partial_j \varphi + q\varphi^2 \},
\]

where \( \bar{\partial} = (L, \bar{\nabla}), \bar{2} = (\bar{L}, \bar{\nabla}) \).

**Proof.** For the energy density on \( H_{\alpha}^\mu \), we derive

\[
\tilde{\mathcal{D}}_{\alpha\beta} \partial_t^\alpha (L^\alpha + hL^\alpha) \\
= \frac{1}{2} (L\varphi + hL\varphi)(L\varphi + L\varphi) + \frac{1}{2} (h + 1) ( -L\varphi L\varphi + |\nabla \varphi|^2 + q\varphi^2 + H^L (L\varphi)^2 \\
+ H \partial_\alpha \varphi \partial^\alpha \varphi) + ( -2H^L \partial_\alpha \varphi \partial^\alpha \varphi + hH^L \partial_\alpha \varphi \partial^\alpha \varphi) \\
= \frac{1}{2} (L\varphi)^2 + h(L\varphi)^2 + \frac{1}{2} (h + 1) ( |\nabla \varphi|^2 + q\varphi^2 + H^L (L\varphi)^2 + H \partial_\alpha \varphi \partial^\alpha \varphi) \\
- (H^L (L\varphi)^2 + H \partial_\alpha \varphi \partial^\alpha \varphi) + h \frac{1}{2} H^L (\partial_\alpha \varphi)^2 + H \partial_\alpha \varphi \partial^\alpha \varphi).
\]

Thus, (6.11) is proved. The energy density (6.12) on \( H_{\alpha}^m \) can be derived by directly swapping \( L \) and \( \bar{L} \) in the above calculation.

The energy density on \( \Sigma_t \) can be derived by

\[
\tilde{\mathcal{D}}_{\alpha\beta} \partial_t^\alpha \partial_\beta = (\partial_\alpha \varphi)^2 + \frac{1}{2} q\varphi^2 + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + H^\alpha \partial_\alpha \varphi \partial^\alpha \varphi \\
= \frac{1}{2} \{ -g^{00}(\partial_\alpha \varphi)^2 + g^{ij} \partial_\alpha \varphi \partial_j \varphi + q\varphi^2 \},
\]

where all other terms have been cancelled. This gives (6.13). □

With the help of Lemma 6.3 we give the fundamental energy estimates.

**Proposition 6.4 (Energy inequality).** Suppose there hold on \( \mathcal{I} \) the assumptions (A1), (6.3), (6.4) and

\[
\tilde{C} M^{-1} \Delta_0 \left( \frac{R}{2} \right)^{\frac{1}{2} - \frac{n}{p}} \leq \frac{1}{6}
\]

(A2)

with the universal constant \( \tilde{C} \geq 1 \) specified in the proof.
Let \((u_2, \mu_2) \in \mathcal{I}\). For \((u, \mu) \in \mathcal{I}\) with \(u \leq u_2\), there holds the following energy inequality for any constant \(p \leq 0\)

\[
u_1^{2p+\gamma_0}(E[\varphi](H_u^\mu) + E[\varphi](L_u^\mu)) \lesssim \sup_{(u_1, \mu_1) \in \mathcal{I}, u_1 \leq u_2} \{u_1^{-2p+\gamma_0}E[\varphi](\Sigma_0^{u_1, \mu_1}) + u_1^{-2p+\gamma_0}(\|rF\|^2_{L^1_x L^2_t(D_u^{\mu_1})} + \|\nabla_x F\|^2_{L^2_x L^1_t(D_u^{\mu_1})}) + u_1^{-2p+\gamma_0}M^{-1}\|r^{1/2}F\|^2_{L^2_t(D_u^{\mu_1})}\},
\]

where \(\nabla \varphi - q\varphi = F + \mathcal{F}\).

**Proof.** For convenience, we denote \(\mathcal{F} = \nabla \varphi - q\varphi\). We first show that, in \(D_u^\mu\) with \((u, \mu) \in \mathcal{I}\) and \(u \leq u_2\), there holds

\[
E[\varphi](H_u^\mu) + E[\varphi](H_u^\mu) \lesssim E[\varphi](\Sigma_0^{u, \mu}) + \int_{D_u^\mu} \{(|\mathcal{F} + \partial^\alpha H \partial_\beta \varphi| \partial_\gamma \varphi - \frac{1}{2} \partial_\beta H \partial_\gamma \varphi \partial^\alpha \varphi\}.
\]

(6.14)

If \(X = \partial_\beta\) in (6.10), the last two terms vanish due to \(\partial^\alpha \partial_\beta = 0\). By divergence theorem, we have

\[
\int_{\mathcal{D}_u^\mu} \frac{1}{2} ((L\varphi\partial_\beta)^2 + (L^\varphi)^2(h - H^L)) + (1 + h)(|\nabla \varphi|^2 + q\varphi^2) + (H + hH)\partial_\beta \partial_\varphi |(1 + h)^{-1}d\mu_u
\]

\[
+ \int_{\mathcal{H}_u^\mu} \frac{1}{2} ((L\varphi)^2 + (L^\varphi)^2((h - H^L)) + (1 + h)(|\nabla \varphi|^2 + q\varphi^2) + (H + hH)\partial_\beta \partial_\varphi |(1 + h)^{-1}d\mu_u
\]

\[
= \frac{1}{2} \int_{\mathcal{H}_u^\mu} \left(\frac{\partial_\beta}{\nabla \varphi}(\partial_\beta \partial_\varphi)^2 + g^{ij} \partial_\beta \varphi \partial_\gamma \varphi + q\varphi^2\right)dx - \int_{\mathcal{D}_u^\mu} \{(|\mathcal{F} + \partial^\alpha H \partial_\beta \varphi| \partial_\gamma \varphi - \frac{1}{2} \partial_\beta H \partial_\gamma \varphi \partial^\alpha \varphi\} \partial_\beta.
\]

We note that by using (6.3) and the smallness of \(|h|\),

\[
\int_{\mathcal{H}_u^\mu} \frac{1}{2} |(H + hH)\partial_\beta \partial_\varphi |(1 + h)^{-1}d\mu_u \lesssim \int_{\mathcal{H}_u^\mu} |H|(|\partial \varphi \partial_\varphi| + |\partial_\varphi|^2)d\mu_u
\]

\[
\lesssim \Delta_0^{1/2} |\partial_\varphi |_{L^2(\mathcal{H}_u^\mu)} (M^{-1/2} + u_+^{-1/2} |\partial_\varphi|_{L^2(\mathcal{H}_u^\mu)} + u_+^{-1/2} |\partial_\varphi|^2 d\mu_u)
\]

\[
\lesssim \Delta_0^{1/2} (M^{-1/2} + u_+^{-1/2} \Delta_0^{1/2} u_+^{-2/3} E[\varphi](H_u^\mu)).
\]

Similarly,

\[
\int_{\mathcal{H}_u^\mu} |(H + hH)\partial_\beta \partial_\varphi |(1 + h)^{-1}d\mu_u \lesssim \Delta_0^{1/2} (M^{-1/2} + u_+^{-1/2} \Delta_0^{1/2} u_+^{-2/3} E[\varphi](H_u^\mu)).
\]

Since \(\Delta_0^{1/2} M^{-1/2} \lesssim M^{1/2}\), the coefficient can be sufficiently small. Due to (A1), this pair of error terms will be absorbed by the leading positive terms on the lefthand side.
With the help of (6.11), we can derive
\[
\int_{\mathbb{S}_0^m} (-g^{00}(\partial_t \varphi)^2 + g^{ij} \partial_i \varphi \partial_j \varphi + q \varphi^2) dx \lesssim E[\varphi](\Sigma_0^{u \varphi}).
\]

Thus
\[
\int_{\mathbb{B}_0^m} ((L \varphi)^2 + |\nabla \varphi|^2 + q \varphi^2 + \frac{M}{r} (L \varphi)^2) + \int_{\mathbb{M}_0^m} ((L \varphi)^2 + \frac{M}{r} (L \varphi)^2 + |\nabla \varphi|^2 + q \varphi^2)
\leq C(E[\varphi](\Sigma_0^{u \varphi}) + \int_{\mathbb{P}_0^m} (|\mathcal{F} + \partial^a H^\alpha \gamma \partial_\gamma \varphi \partial^a \varphi| - \frac{1}{2} \partial_i H^\alpha \gamma \partial_\gamma \varphi \partial^a \varphi)).
\]
(6.15)

Thus (6.14) is proved.

We remark that
\[
\text{Tr}[\varphi] = \partial^a H^\alpha \gamma \partial_\gamma \varphi \partial^a \varphi \leq \frac{1}{2} \partial_i H^\alpha \gamma \partial_\gamma \varphi \partial^a \varphi = \frac{1}{4} L H^\alpha L^a (L \varphi)^2 + \text{Tr}(\partial H, \partial \varphi, \partial \varphi)
\]
(6.16)
where the trilinear term Tr means that, in the product of the three terms, at least one of the derivatives is $\partial_{ij}$.

Symbolically, $\text{Tr}[\varphi] = \partial H \cdot \partial \varphi \cdot \partial \varphi$. Let $\alpha = -p + \frac{m}{2}$ with the constant $p \leq 0$.

We calculate for any $(u_2, u_\varphi) \in \mathcal{I}$ by using (6.8)
\[
u_2^{2+} \int_{\mathcal{F}_2} |\partial H \cdot \partial \varphi | \partial \varphi |
\lesssim \nu_2^{2+} ||r^\frac{1}{4} \partial H||_{L^2 \mathcal{L}^{m} \varphi} (||r \partial \varphi||_{L^2 \mathcal{L}^{m} \varphi} + ||r \partial \varphi||_{L^2 \mathcal{L}^{m} \varphi} ||r^\frac{1}{4} \partial \varphi||_{L^2 \mathcal{L}^{m} \varphi})
\lesssim \nu_2^{2+} ||r^\frac{1}{4} \partial H||_{L^2 \mathcal{L}^{m} \varphi} M^{-\frac{1}{2}} (\int_{\mathcal{F}_2} E[\varphi](\mathcal{H}_{u \varphi}^\alpha) du) \frac{1}{2} (\sup_{-u_2 \leq u \leq u_2} u_2^\alpha E[\varphi](\mathcal{H}_{u \varphi}^\alpha)
\]
\begin{equation*}
+ \sup_{-u_2 \leq u \leq u_2} u_2^\alpha E[\varphi](\mathcal{H}_{u \varphi}^\alpha))
\end{equation*}
\begin{equation*}
\lesssim M^{-1} \Delta_0 \frac{1}{2} \nu_2^{2+} (\sup_{-u_2 \leq u \leq u_2} u_2^\alpha E[\varphi](\mathcal{H}_{u \varphi}^\alpha) + \sup_{-u_2 \leq u \leq u_2} E[\varphi](\mathcal{H}_{u \varphi}^\alpha) u_2^{2+}).
\end{equation*}

To treat the integral of $\mathcal{F} \cdot \partial \varphi$ in (6.14), we repeat the derivation in (6.5). Thus with $\mathcal{F} = \mathcal{F}_2 + \mathcal{F}_1$, we have
\[
u_2^{2+} \int_{\mathcal{F}_2} (|\mathcal{F} \cdot \partial \varphi | + |\partial H \cdot \partial \varphi |)
\leq C(\epsilon_1) \nu_2^{2+} ||r \mathcal{F}_2||_{L^2 \mathcal{L}^{m} \varphi}^2 ||r \mathcal{F}_2||_{L^2 \mathcal{L}^{m} \varphi} ||r \mathcal{F}_2||_{L^2 \mathcal{L}^{m} \varphi}^2
\]
\begin{equation*}
+ \nu_2^{2+} \int_{\mathcal{F}_2} (|\mathcal{F}_2|_{L^2 \mathcal{L}^{m} \varphi}) + (\epsilon_1 + \tilde{C} M^{-1} \Delta_0 \frac{1}{2} \nu_2^{2+} (\sup_{-u_2 \leq u \leq u_2} u_2^\alpha E[\varphi](\mathcal{H}_{u \varphi}^\alpha) + \sup_{-u_2 \leq u \leq u_2} u_2^{2+} E[\varphi](\mathcal{H}_{u \varphi}^\alpha)))
\end{equation*}
\begin{equation*}
\end{equation*}

We fix the constant in (A2) by $\tilde{C} = C \cdot \tilde{C}$. By using (A2) and choosing $\epsilon_1$ sufficiently small, the last term can be absorbed. Substituting the above estimate to (6.15) implies Proposition 6.4.

Next we establish the $r$-weighted energy inequality by first giving the following result.

$^{22}$In Section 7, we will take advantage of this structure when the wave coordinates condition is available.
Lemma 6.5. Let \(-u_\epsilon \leq -u_{\pm} \leq u_1 \leq u_0\). Under the assumptions of (6.3), (6.4) and (A7), there holds the following estimate for any constant \(p \leq 0\)

\[
\int_{\mathbb{R}^{n+1}} \left| \left( \Box_{\gamma} \varphi - q \varphi \right) (X \varphi + \varphi) + \frac{1}{2} \left( r^{-2} (L(r \varphi))^2 + |\nabla \varphi|^2 \right) - (\partial^\alpha X \partial_\alpha + \frac{1}{2} (Xq + q) \varphi^2) \right| \leq \Delta_0^{-\frac{1}{2}} M^{-\frac{1}{2}} \left\{ u_1 \left| u_{\pm}^{-2p} \sup_{-u_{\pm} \leq u \leq u_1} \varphi^2 \right| (H_{\varphi}^2) + u_1 \left| u_{\pm}^{-2p} \sup_{-u_1 \leq u \leq u_1} \varphi^2 \right| (H_{\varphi}^2) \right\} + \int_{-u_1}^{u_1} u_1^{-\frac{1}{2}} W_1 [\varphi] (H_{\varphi}^2) du, \tag{6.17}
\]

where the energy current is defined by

\[
^{(X)} \mathcal{P}_\alpha = \mathcal{Q}_{\alpha \beta} X^\beta + \frac{1}{2} m_{\alpha \beta \gamma} g^{\alpha \gamma} \partial_\gamma (\varphi^2) + Y_\alpha \tag{6.18}
\]

with

\[
X = r (L - H \tilde{L}) \quad Y = \frac{1}{2} r^{-1} \varphi^2 L.
\]

The proof of (6.17) relies on an important cancelation thanks to the construction of \(\mathcal{Q}_{\alpha \beta}\) and the choice of the multiplier \(X\). The cancelation will be seen in the following proof. For convenience, we denote by \(\mathcal{P}\) the current \(^{(X)} \mathcal{P}_\alpha\) from now on.

**Proof.** The proof is based on the following identity on \(\mathcal{D}_{u_1}\).

\[
\partial^\alpha \mathcal{P}_\alpha + \frac{1}{2} (Xq + q) \varphi^2 = \left( \Box_{\gamma} \varphi - q \varphi \right) (X \varphi + \varphi) + \frac{1}{2} \left( r^{-2} (L(r \varphi))^2 + |\nabla \varphi|^2 \right) + I + II + III \tag{6.19}
\]

where the error terms \(I, II\) and \(III\) are

\[
I = \partial^\alpha H_\alpha \gamma \partial_\gamma \varphi (X \varphi + \varphi) - \frac{1}{2} \left( X \partial^\alpha H_\alpha \gamma \partial_\gamma \varphi + \frac{1}{2} r L H \tilde{L} (L \varphi) \right) \tag{6.20}
\]

To show (6.19), by using (6.10), we first derive

\[
\partial^\alpha \mathcal{P}_\alpha = X^\beta \left( \left( \Box_{\gamma} \varphi - q \varphi + \partial^\alpha H_\alpha \gamma \partial_\gamma \varphi \right) \partial_\beta \varphi - \frac{1}{2} \partial_\beta q \varphi^2 - \frac{1}{2} \partial_\beta H_\alpha \gamma \partial_\gamma \varphi + \partial^\alpha \varphi \right) + \mathcal{E}
\]

where

\[
\mathcal{E} = \mathcal{Q}_{\alpha \beta} \partial^\alpha X^\beta + \partial^\alpha \left( \frac{1}{2} m_{\alpha \beta \gamma} g^{\alpha \gamma} \partial_\gamma (\varphi^2) + Y_\alpha \right).
\]

It is straightforward to check that

\[
^{(X)} \pi_{AB} = \delta_{AB} (1 + \frac{LH\tilde{L}}{r}) \quad ^{(X)} \pi_{LL} = -1 - \frac{LH\tilde{L}}{r} + rLH\tilde{L}
\]

It is easy to check in view of the definition of \(\mathcal{Q}_{\alpha \beta}\) that

\[
\mathcal{Q}_{LL} = (L \varphi)^2 \quad \mathcal{Q}_{LL} = (L \varphi)^2
\]

\[
\mathcal{Q}_{AB} = \nabla_A \varphi \nabla_B \varphi - \frac{1}{2} m_{AB} (L \varphi L \varphi + |\nabla \varphi|^2 + q \varphi^2 + H \partial_\rho \varphi \partial_\sigma \varphi)
\]

\[
\mathcal{Q}_{L\varphi} = |\nabla \varphi|^2 + q \varphi^2 + H \partial_\rho \varphi \partial_\sigma \varphi.
\]
By combining the lists of $\mathcal{D}_{\alpha\beta}$ and $(X)_{\pi}^{\alpha\beta}$, we derive

$$\mathcal{D}_{LL}^{(X)\pi} = \frac{1}{2} (L^2 \varphi)^2$$

$$\mathcal{D}_{LL}^{(X)\pi} = \frac{1}{2} (L^2 \varphi)^2 (H_{LL} + rL^2 H_{LL})$$

$$\mathcal{D}_{LL}^{(X)\pi} = \frac{1}{4} (-1 - H_{LL} + rL^2 H_{LL}) (|\nabla \varphi|^2 + q\varphi^2 + H^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi)$$

$$\mathcal{D}_{AB}^{(X)\pi} = (1 + H_{LL})(L^2 \varphi \varphi - q\varphi^2 - H^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi).$$

Note that

$$\frac{1}{2} \partial^\alpha (m_{\alpha\beta} g^{\rho\gamma} \partial_\gamma (\varphi^2)) = \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2) + g^{\alpha\gamma} \partial_{\alpha'} \varphi \partial_{\gamma} \varphi + \Box g \varphi \cdot \varphi$$

$$= \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2) + \partial^\alpha \varphi \cdot \partial_\alpha \varphi + H^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi + \Box g \varphi \cdot \varphi.$$

Since $\partial^\alpha L_\alpha = 2/r$ and $\partial^\alpha L_{\alpha} = -2/r$, we have

$$\partial^\alpha Y_\alpha = \frac{1}{2} \partial^\alpha (r^{-2} \varphi^2 L_\alpha) = \frac{1}{2} \left\{ \varphi^2 L (r^{-1}) + r^{-1} L (\varphi^2) + r^{-1} \varphi^2 \partial^\alpha L_\alpha \right\}$$

$$= \frac{1}{2} \left( r^{-1} L (\varphi^2) + r^{-2} \varphi^2 \right).$$

Thus

$$\mathcal{D} = \Box g \varphi \cdot \varphi + \frac{1}{2} (r^{-2} (L (r \varphi)))^2 + |

\nabla \varphi|^2 + H_{LL} (L \varphi)^2 \right) + (- \frac{1}{2} - \frac{3}{2} H_{LL})$$

$$+ \frac{1}{2} rLH_{LL} H^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi + \frac{1}{2} (L \varphi)^2 rLH_{LL} + \frac{1}{2} L (rH_{LL}) \varphi^2$$

$$+ H_{LL} L \varphi + q\varphi^2 (- \frac{3}{2} (1 + H_{LL}) + \frac{1}{2} rLH_{LL}) + \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2).$$

Note that

$$-H_{LL} (L \varphi)^2 + H^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi = H \tilde{\varphi} \partial \varphi.$$

Hence we conclude that

$$\mathcal{D} = \Box g \varphi \cdot \varphi + \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2) + (r^{-2} (L (r \varphi)))^2 + |

\nabla \varphi|^2 \right) + (H + r \partial H) (H^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi + q\varphi^2)$$

$$+ \frac{1}{2} rLH_{LL} (L \varphi)^2 + \frac{1}{2} L (rH_{LL}) \varphi^2 + H \partial \varphi \cdot \tilde{\varphi} - \frac{3}{2} q\varphi^2.$$

Substituting the above formula to (6.21) yields (6.19).

Next we prove (6.17) by controlling the error terms in the identity. We claim

$$I = r LH \partial \varphi \cdot \tilde{\varphi} + \partial H \cdot \partial \varphi \cdot L (r \varphi) + rH \cdot \partial H \cdot (\partial \varphi)^2,$$

$$II = r \partial H \partial \varphi \cdot \tilde{\varphi}^2 + H \partial \varphi \cdot \tilde{\varphi}.$$

Indeed, the last term of $I$ cancels completely the bad component of the second term, which can be seen below

$$-\frac{1}{2} X H \partial \varphi \partial \varphi + \frac{1}{2} rLH_{LL} (L \varphi)^2$$

$$= \frac{1}{2} r (-LH \partial \varphi \partial \varphi + LH_{LL} (L \varphi)^2) + \frac{1}{2} rH_{LL} L H \partial \varphi \partial \varphi$$

$$= \frac{1}{2} rLH \partial \varphi \partial \varphi + \frac{1}{2} rH \partial H \partial \varphi \cdot \tilde{\varphi}^2.$$

By direct calculation, we can obtain the symbolic formula for $I$. The formula of $II$ is a simple recast. It remains to consider the error term $III$ in (6.19). For the first term, note that

$$\partial^\alpha X^\beta = \partial^\alpha r (L - H_{LL} L)^\beta + r (\partial^\alpha L^\beta - \partial^\alpha H_{LL} L^\beta - H_{LL} \partial^\alpha L^\beta)$$
and the nontrivial components of $\partial^a L^b$ and $\partial^a L^a$ are $\partial_A L_B = r^{-1} \delta_{AB}$ and $\partial_A L_B = -r^{-1} \delta_{AB}$.

By a direct substitution, we can obtain

$$|H^a \partial_a \varphi \partial b \varphi \partial^a X^b| \lesssim |H \partial \varphi \cdot \tilde{\partial} \varphi| + (r|\partial H| + |H|) \cdot |H|(|\partial \varphi|^2 + q\varphi^2)|$$

and then

$$|III| \lesssim |H \partial \varphi \cdot \tilde{\partial} \varphi| + (r|\partial H| + |H|) \cdot (|H|(|\partial \varphi|^2 + q\varphi^2)). \quad (6.23)$$

Next we prove the following error estimate

$$\int_{D^+} |I| + |II| + |III| \lesssim \Delta_0 \frac{1}{2} M^{-\frac{1}{2}} u_1^{\frac{1}{2}} + \frac{\sup_{u \leq u_1} E[\varphi(\mathcal{H}_u)] u_1^0}{u_1} + \frac{\sup_{u \leq u_1} E[|\mathcal{H}_u|] u_1^0}{u_1} + \frac{\sup_{u \leq u_1} E[\varphi(\mathcal{H}_u)] u_1^0}{u_1}.$$ \quad (6.24)

Let us consider the error terms in $I$ in (6.22). By using (6.6) and (6.8), we have

$$\int_{D^+} r|H||\partial H|(\tilde{\partial} \varphi)^2$$

$$\lesssim (r \tilde{\partial} \varphi L_2 L^2 L^2(\mathcal{D}_u^+)) \||r^{-\frac{1}{2}} \tilde{\partial} \varphi||L^2 L^2(\mathcal{D}_u^+)) \|r^{-\frac{1}{2}} \partial H||L^2 L^2(\mathcal{D}_u^+)) \|r H||L^2(\mathcal{D}_u^+))$$

$$\lesssim M^{-1} \Delta_0 u_1^{-2\gamma + \delta} \sup_{u \leq u_1} E[\varphi(\mathcal{H}_u)] u_1^0.$$ \quad (6.25)

Similarly, by using (6.6) and (6.7), we can obtain

$$\int_{D^+} r(|L| + |H||\partial H|)|\partial \varphi \cdot \partial \varphi| \lesssim \int_{-\Delta_1} u_1^{\frac{1}{2}} \||\partial \varphi||L^2(\mathcal{H}_u^+)\||r^{-\frac{1}{2}} \partial \varphi||L^2(\mathcal{H}_u^+)\|u_1^{\frac{1}{2}} \Delta_0^{\frac{1}{2}} du$$

$$\lesssim \Delta_0 \frac{1}{2} M^{-\frac{1}{2}} u_1^{\frac{1}{2}} + \frac{\sup_{u \leq u_1} E[\varphi(\mathcal{H}_u)] u_1^0}{u_1}.$$ \quad (6.26)

Thus the error terms in $I$ are all treated.

It again follows by using (6.6) that

$$\int_{D^+} \bar{L} \tilde{\partial} \varphi \cdot \tilde{\partial} \varphi \cdot \partial H \lesssim \int_{-\Delta_1} u_1^{\frac{1}{2}} \bar{L} \tilde{\partial} \varphi \cdot \tilde{\partial} \varphi \cdot \partial H$$

$$\lesssim \Delta_0 \frac{1}{2} \int_{-\Delta_1} u_1^{\frac{1}{2}} \frac{\sup_{u \leq u_1} E[\varphi(\mathcal{H}_u)] u_1^0}{u_1}.$$\quad (6.27)

Thus the terms of $II$ are all treated.

Next we estimate $III$ in view of (6.23). The first term on the right of (6.23) has been estimated in $II$. Note that by using (6.6),

$$r|\partial H| + \frac{r |H|}{u_1} \lesssim \Delta_0 u_1^{-\gamma_0}.$$
which, in view of $\gamma_0 > 1$, imply
\[
\int_{D_{u_1}^0} (r|\partial H| + |H|)|H|(|H(\partial \varphi)|^2 + q\varphi^2) \lesssim \Delta_0 \int_{D_{u_1}^0} u_+^{-\gamma_0} \left\| r^{-\frac{1}{2}} \partial \varphi \right\|^2_{L^2(u_1^0)} du + \Delta_0 \frac{1}{2} \int_{D_{u_1}^0} u_+^{-\frac{1}{2}} r \varphi^2 \\
\lesssim (\Delta_0 M^{-1} + \Delta_0 \frac{1}{2}) u_+^{-\frac{1}{2} - \frac{1}{2} \gamma_0} \sup_{-\frac{1}{2} u_1 \leq u \leq u_1} E[\varphi](H^u_\varphi) u_+^{\gamma_0}.
\]
By combining the estimates for $I$, $II$ and $III$, $(6.23)$ is proved since $\Delta_0 M^{-1} \leq (\Delta_0 M^{-1})^{\frac{1}{2}}$. Thus we can conclude the inequality in $(6.17)$. \(\square\)

Next, we give the $r$-weighted energy estimate.

**Proposition 6.6.** Under the assumptions of $(6.6)$, $(6.7)$ and $(6.11)$, with $-u_1 \leq -u \leq u_1 \leq u_0$, there holds the following weighted energy estimate for $\partial_\alpha \mathcal{P}_\alpha$ in $D_{u_1}^0$. We first confirm the boundary terms give the desired weighted energy.

Let us first compute $(1 + h)\mathcal{P}_\alpha N^\alpha$. Recall $X = r(L - H^L L)$ and $N$ from $(2.1)$,
\[
(1 + h)\mathcal{P}_\alpha X_\beta N^\alpha \\
= (L + hL_\alpha^0)\varphi X_\beta^\gamma N^\gamma + \frac{1}{2} r(L + hL_\alpha^0) (L - H^L L)^{\alpha} X_\beta^\gamma N^\gamma \\
+ r H_\alpha^\gamma \partial_\gamma \varphi \partial_\alpha \varphi (L + hL_\alpha^0) (L - H^L L)^{\alpha} \\
+ r \{ (L \varphi - H^L L \varphi)(L \varphi + hL_\alpha^0) 2(H \varphi + lH^L \varphi)\partial_\alpha \varphi + (h - H^L L) \}
\cdot (-L \varphi L_\alpha^0 + |\varphi|^2 + q\varphi^2 + H^\alpha \partial^{\alpha} \varphi \partial_\alpha \varphi) \\
= (L \varphi - H^L L \varphi)(L \varphi + hL_\alpha^0 - 2(H \varphi + lH^L \varphi)\partial_\alpha \varphi + (h - H^L L)^{\alpha} \\
\cdot (-L \varphi L_\alpha^0 + |\varphi|^2 + q\varphi^2 + H^\alpha \partial^{\alpha} \varphi \partial_\alpha \varphi) \\
= r \{ (L \varphi - H^L L \varphi)(L \varphi + hL_\alpha^0 - 2(hL^L + hH^L L)\partial_\alpha \varphi + (h - H^L L)^{\alpha} \\
\cdot (-2hL^L L_\beta \varphi + (h + hL_\alpha^0) \partial_\alpha \varphi + (h - H^L L)^{\alpha} H \partial_\alpha \varphi \partial_\alpha \varphi).\}
\]
We can cancel the first term in the last line by noting that
\[
L \varphi + (h - 2(hL^L + hH^L L))\varphi = (h - H^L L) L \varphi + L \varphi - H^L L \varphi - 2hL^L L_\beta \varphi,
\]
where the first term on the right gives the cancelation after substitution. Hence
\[
(1 + h)\mathcal{P}_\alpha X_\beta N^\alpha \\
= r \{ (L \varphi - H^L L \varphi)^2 + (h - H^L L)(|\varphi|^2 + q\varphi^2) + (L \varphi - H^L L \varphi)^2 \\
\cdot (-2hL^L L_\beta \varphi + (h + hL_\alpha^0) \partial_\alpha \varphi + (h - H^L L)^{\alpha} H \partial_\alpha \varphi \partial_\alpha \varphi)\}.
\]
We remark that only the first term on the right hand side is involved with the further cancelations with the next two identities.

Note also that
\[
\frac{1}{2} (1 + h) N^\alpha m_\alpha^\beta g^{\alpha\gamma} \partial_\gamma (\varphi^2) = \frac{1}{2} m_\alpha^\beta (H^{\alpha\gamma} + m^{\alpha\gamma}) \partial_\gamma (\varphi^2) (L^\alpha + hL^\alpha_\alpha) \\
= \varphi \{(H_{L}^\alpha + hH^\alpha L)_\gamma \partial_\gamma \varphi + (L^\alpha + hL_\alpha^0)\},
\]
and
\[
(1 + h) N^\alpha Y_\alpha = -r^{-1} \varphi^2 h.
\]
In view of the definition of $\mathcal{E}^\alpha$ in (6.18) and the above three identities, we can derive that
\[
\mathcal{E} = r^2(1 + h)\mathcal{E}^\alpha \quad (6.27)
\]
\[
= r\{(L(\varphi) - rH^LL\varphi)^2 + r^2(h - H^LL)(|\nabla\varphi|^2 + q\varphi^2) - r^{-1}\frac{1}{2}L(r^2\varphi^2)
+ r[2hH^LL\varphi + 2(hH^L + hH^L)\varphi\delta\varphi + r^{-1}h\varphi L(r\varphi)]
+ r^2[(L\varphi - H^LL\varphi) - 2hH^LL\varphi + (H + hH)\partial\varphi] + (h - H^LL)H\partial\varphi\partial\varphi\}.
\]

Thus the calculation of the energy density on $\mathcal{H}_{\mathfrak{u}}$ is completed.

Next we consider the weighted energy on $\mathcal{H}_{\mathfrak{u}}$. We first compute
\[
(1 + h)\mathcal{E}^\alpha \quad (6.28)
\]
\[
= r\{(L\varphi + hL\varphi)(L\varphi - H^LL\varphi) - \frac{1}{2}(L - H^LL)\varphi L + hL)((H^{\mu\nu} + m^{\mu\nu})\partial_{\mu}\varphi\partial_{\nu}\varphi) + q\varphi^2\}
= r\{(L\varphi L\varphi - hH^LL) + h(L\varphi) - H^LL(L\varphi)^2 - \frac{1}{2}(-2 + 2hH^LL)
\cdot (H^LL(L\varphi)^2 + H^LL(L\varphi)^2 + H\partial_{\varphi}\partial\varphi - L\varphi L\varphi + |\nabla\varphi|^2 + q\varphi^2)\}
= r\{(L\varphi)^2(h + 1 - H^LLh)(H^LL) - (L\varphi)^2 H^LL + (1 - H^LLh)(|\nabla\varphi|^2 + q\varphi^2)
+ (1 - H^LLh)H\partial\varphi\partial\varphi\}.
\]

and
\[
(1 + h)H^\gamma_{\alpha\beta} \partial_{\gamma} \partial_{\beta} \varphi\mathcal{N}^\alpha X^\beta \quad (6.29)
\]
\[
= r\{(L\varphi L\varphi(-2H^LL - 2hH^LL + 2H^LL + hH^LL)) + 2(L\varphi)^2 H^LL(H^LL + hH^LL)
- 2(L\varphi)^2(H^LL + hH^LL) + (H + hH)\partial\varphi\partial\varphi\}.
\]

Note that
\[
\frac{1}{2}(1 + h)m_{\alpha\gamma}\gamma_{\alpha\gamma}\partial_{\gamma}(\varphi^2)\mathcal{N}^\alpha = \frac{1}{2}m_{\alpha\gamma}(H^\alpha_{\gamma} + m^\alpha_{\gamma})\partial_{\gamma}(\varphi^2)(L^\alpha + hL^\alpha)
= \frac{1}{2}\{(L + hL)(\varphi^2) + (H^\gamma_{\gamma} + hH^\gamma)\partial_{\gamma}(\varphi^2)\}
\]
and
\[
(1 + h)\mathcal{N}^\alpha Y_{\alpha} = \frac{1}{2}r^{-1}\varphi^2(L, L + hL) = -r^{-1}\varphi^2.
\]

We then combine the above calculations for the two terms to obtain
\[
r^2(1 + h)\mathcal{E}^\alpha \left(\frac{1}{2}m_{\alpha\gamma}\gamma_{\alpha\gamma}\partial_{\gamma}(\varphi^2) + Y_{\alpha}\right) \quad (6.30)
\]
\[
= \frac{1}{2}r\{(L + hL)(\varphi^2) + r^2(H^\gamma_{\gamma} + hH^\gamma)\partial_{\gamma}(\varphi^2)\} - rh\varphi^2.
\]

Thus, by combining (6.28)-(6.30) and in view of the definition (6.18), we can obtain
\[
(1 + h)r^2\mathcal{E}^\alpha \quad (6.31)
\]
\[
= r^3\{(L\varphi)^2(h - H^LL + hH^2 + hH) + (L\varphi)^2(hH^2 + H) + (|\nabla\varphi|^2 + q\varphi^2)(1 - H^LLh)
+ H\partial\varphi\partial\varphi\}
+ \frac{1}{2}\{(L + hL)(r\varphi^2) + r^2(H + hH)\partial(\varphi^2)\} - rh\varphi^2,
\]
where the term $H(1 + h + hH)\partial\varphi\partial\varphi$ is simplified to be $H\partial\varphi\partial\varphi$ due to $|h| + |H| \leq 1$. 

In the sequel, we will constantly use the fact that $|h| + |H| \leq 1$ to shorten the symbolic formula. Recall from (2.9) for the area elements $d\mu_H$ and $d\mu_L$. By (6.27), we have

$$I = \int_{\mathcal{H}_1} (1 + h)\mathcal{N}^{\alpha} \mathcal{P}_\alpha (1 + h)^{-1} d\mu_H$$

$$= \int_{\mathcal{E}_1} \left( r(L(\varphi) - rH_{L}L \varphi)^2 + r^3(h - H_{L}L)(|\nabla \varphi|^2 + q \varphi^2) - \frac{1}{2}(L - hL)(r^2 \varphi^2) \right) \frac{r}{2(r - M_0)} d\omega d\mu + Er_1,$$

$$Er_1 = \int_{\mathcal{E}_1} \left\{ r^2(L(r \varphi) - rH_{L}L \varphi) \cdot (H \partial \varphi + hHL \varphi) + r^3(h - H_{L}L)H \partial \varphi \tilde{\partial} \varphi + r^2 \varphi H(\tilde{\partial} \varphi + h\partial \varphi) \right\} \frac{r}{2(r - M_0)} d\omega d\mu.$$ (6.32)

On $\mathcal{H}_1$, by using (6.31), we have

$$II = \int_{\mathcal{H}_1} (1 + h)\mathcal{N}^{\alpha} \mathcal{P}_\alpha (1 + h)^{-1} d\mu_H$$

$$= \int_{\mathcal{E}_1} \left( r^3((L \varphi)^2(h - H^{LL} + hH) + (q \varphi^2 + |\nabla \varphi|^2)(1 - Hh) + (L \varphi)^2HH + H\partial \varphi \tilde{\partial} \varphi] + \frac{1}{2}(L - hL)(r^2 \varphi^2) + rH(2r^2 \varphi^2 - rh \varphi^2) \right) \frac{r}{2(r - M_0)} d\omega d\mu$$

$$= \int_{\mathcal{E}_1} r^3((L \varphi)^2(h - H^{LL}) + |\nabla \varphi|^2 + q \varphi^2) + \frac{1}{2}(L - hL)(r^2 \varphi^2) \frac{r}{2(r - M_0)} d\omega d\mu + Er_2,$$

where

$$Er_2 = \int_{\mathcal{E}_1} \left\{ r^2(H + h)\partial(\varphi^2) + rh \varphi^2 + r^3H(h(L \varphi)^2 + (L \varphi)^2 + \partial \varphi \cdot \tilde{\partial} \varphi + q h \varphi^2) \right\} \frac{r}{2(r - M_0)} d\omega d\mu.$$ (6.33)

In $I$ and $II$, the coefficients of leading terms are precise, while $Er_1$ and $Er_2$ are symbolic formulas for the error terms. Note that applying divergence theorem to $\mathcal{E}_1$ implies

$$I + II - \int_{\Sigma_0} \mathcal{P}_\alpha \partial_t^{\alpha} dx = - \int_{\mathcal{D}_{E}^{\perp}} \partial^{\alpha} \mathcal{P}_\alpha.$$

For convenience, we set

$$\tilde{I} = I + \int_{\mathcal{H}_1} \frac{1}{2}(L - hL)(r^2 \varphi^2) \frac{r}{2(r - M_0)} d\omega d\mu;$$

$$\tilde{II} = II - \int_{\mathcal{E}_1} \frac{1}{2}(L - hL)(r^2 \varphi^2) \frac{r}{2(r - M_0)} d\omega d\mu;$$

$$\tilde{III} = \int_{\Sigma_0} \{ \mathcal{P}_\alpha \partial_t^{\alpha} r^2 + \frac{1}{2} \partial_t(r^2 \varphi^2) \} d\omega d\mu.$$ (6.11)

Then, in view of (6.11) and (2.7), we have

$$\tilde{I} + \tilde{II} - \tilde{III} = - \int_{\mathcal{D}_{E}} \partial^{\alpha} \mathcal{P}_\alpha.$$
Now combining (6.17) with the above identity, in view of the definitions of \( \tilde{I}, \tilde{II} \), we derive

\[
\int_{\mathbb{R}^n_+} \frac{1}{2} (r^{-2} |L(\varphi)|^2 + |\nabla \varphi|^2) + \int_{\mathbb{R}^n_+} \left\{ r(L(\varphi) - rH_{LL}L(\varphi))^2 + r^3 (h - H_{LL})(|\nabla \varphi|^2 + q\varphi^2) \right\} \frac{r}{2(r - M_0)} \, d\omega \, du
\]
\[
+ \int_{\mathbb{R}^n_+} r^3 ((L(\varphi))^2 (h - H)_{L\varphi}) + |\nabla \varphi|^2 + q\varphi^2 \right\} \frac{r}{2(r - M_0)} \, d\omega \, du
\]
\[
\leq |\tilde{II}| + |\mathbb{E}_1(1) + |\mathbb{E}_2| + \int_{\mathbb{R}^n_+} |\nabla (X \varphi + \varphi)| + \frac{g + |Xq|}{2} \varphi^2
\]
\[
+ C_\Delta \Delta_0 \frac{1}{2} M^{-\frac{1}{2}} \left\{ u_1 + \sup_{-u_1 \leq u \leq u_1} E[\varphi] \right\} (H_{\mathbb{U}}) u_1^{-\gamma_0-2p} + u_1^{-\gamma_0-2p} \sup_{-u_1 \leq u \leq u_1} E[\varphi] (H_{\mathbb{U}}) \}
\]
\[
\int_{-u_1}^{u_1} \left\{ u_1 - \gamma_0+1 \right\} W_1 \left[ \varphi \right] (H_{\mathbb{U}}) \, du \right\}
\]

Moreover, by using (6.10), it is straightforward to obtain

\[
\int_{\mathbb{R}^n_+} r^3 |H|^2 (\nabla \varphi)^2 \, d\omega \, du \leq E[\varphi] (H_{\mathbb{U}}) u_1^{-\gamma_0+1} \Delta_0 \Delta^{-1},
\]

which leads to

\[
\left| \int_{\mathbb{R}^n_+} r(L(\varphi) - rH_{LL}L(\varphi))^2 \frac{r}{2(r - M_0)} \, d\omega \, du - \int_{\mathbb{R}^n_+} r(L(\varphi))^2 \frac{r}{2(r - M_0)} \, d\omega \, du \right| \leq E[\varphi] (H_{\mathbb{U}}) (u_1)^{-\gamma_0+1} \Delta_0 \Delta^{-1}. \tag{6.35}
\]

Also by using (X1), we can derive

\[
\int_{\mathbb{R}^n_+} \frac{1}{2} (r^{-2} |L(\varphi)|^2 + |\nabla \varphi|^2) + \int_{\mathbb{R}^n_+} \left\{ r(L(\varphi))^2 + \frac{1}{3} r^2 M(|\nabla \varphi|^2 + q\varphi^2) \right\} \frac{r}{2(r - M_0)} \, d\omega \, du
\]
\[
+ \int_{\mathbb{R}^n_+} r^3 \left\{ \frac{M}{3r} (L(\varphi))^2 + |\nabla \varphi|^2 + q\varphi^2 \right\} \frac{r}{2(r - M_0)} \, d\omega \, du
\]
\[
\leq |\tilde{II}| + |\mathbb{E}_1(1) + |\mathbb{E}_2| + \int_{\mathbb{R}^n_+} |\nabla (X \varphi + \varphi)| + \frac{g + |Xq|}{2} \varphi^2
\]
\[
+ C_\Delta \Delta_0 \frac{1}{2} M^{-\frac{1}{2}} \left\{ u_1 + \sup_{-u_1 \leq u \leq u_1} E[\varphi] \right\} (H_{\mathbb{U}}) u_1^{-\gamma_0-2p} + u_1^{-\gamma_0-2p} \sup_{-u_1 \leq u \leq u_1} E[\varphi] (H_{\mathbb{U}}) \}
\]
\[
+ \int_{-u_1}^{u_1} \left\{ u_1 - \gamma_0+1 \right\} W_1 \left[ \varphi \right] (H_{\mathbb{U}}) \, du \right\}. \tag{6.36}
\]

It is straightforward to derive

\[
r^2 \partial_{\alpha} \partial_{\beta} L(\varphi) + \frac{1}{2} \partial_{\alpha} (r^2 \partial_{\beta}) = \frac{1}{2} (r(L(\varphi)))^2 + r^3 (|\nabla \varphi|^2 + q\varphi^2) + r^3 H ((\varphi)^2 + q\varphi^2 + \partial_{\alpha} \varphi). \]

Note that with the help of (X1),

\[
|\tilde{II}| \lesssim \| r^\frac{1}{2} \partial_{\alpha} \varphi \|^2_{L^2(\Sigma_0^1, \omega)} + \| r^{-\frac{1}{2}} \partial_{\beta} \varphi \|^2_{L^2(\Sigma_0^1, \omega)} + q_0 \| r^\frac{1}{2} \varphi \|^2_{L^2(\Sigma_0^1, \omega)} + \| r^\frac{1}{2} \varphi \|^2_{L^2(\Sigma_0^1, \omega)}. \tag{6.37}
\]

It remains to estimate \( \mathbb{E}_1 \) and \( \mathbb{E}_2 \). We first estimate \( \mathbb{E}_1 \) in view of

\[
| r^\frac{1}{2} (H \partial_{\alpha} \varphi + hH L(\varphi)) | \lesssim (|rH L(\varphi)| + r |\partial_{\beta} \varphi|) r^\frac{1}{2} H
\]
\[
r^3 (h - H) H \partial_{\alpha} \partial_{\beta} \varphi | \lesssim |r \partial_{\beta} \varphi| r^\frac{1}{2} \partial_{\alpha} \varphi |r (h - H)| r^\frac{1}{2} H
\]
\[
r^2 |\varphi H (\partial_{\alpha} \varphi + h \partial_{\beta} \varphi) | \lesssim (|r \partial_{\beta} \varphi| + |r H \partial_{\beta} \varphi|) |\varphi| |H|. \]
By Hölder’s inequality, (6.6) and $|h| \leq M$, we have
\[
\|r^{\frac{3}{2}}(H \tilde{\phi} + hHL\phi)\|_{L^2(U_2^\varepsilon(H_{u_{1+}^\varepsilon}))} \lesssim u_{1+}^{-\frac{29}{2}} \Delta_0^{-\varepsilon} E[\phi]^{\frac{1}{2}}(H_{u_{1+}^\varepsilon})
\]
\[
\int_{H_{u_{1+}^\varepsilon}} r^3|h-H|H\partial_{\phi}\tilde{\phi} d\omega d\mu \lesssim E[\phi](H_{u_{1+}^\varepsilon}) \Delta_0^{-\varepsilon} u_{1+}^{-\frac{29}{2}} (M^{-\varepsilon} \Delta_0^{-\varepsilon} + M^\varepsilon),
\]
\[
\int_{H_{u_{1+}^\varepsilon}} r^2|\phi H|(|\tilde{\phi} + |h\partial_{\phi}||) d\omega d\mu \lesssim E[\phi](H_{u_{1+}^\varepsilon})(E[\phi](H_{u_{1+}^\varepsilon}) + E[\phi](H_{u_{1+}^\varepsilon})
\]
\[
+ \int_{S_{u_{1+}^\varepsilon}} r\phi^2 d\omega + \int_{S_{u_{1+}^\varepsilon}} r\phi^2 d\omega \Delta_0^\varepsilon u_{1+}^{-\frac{29}{2}} ,
\]
where we employed (8.10) to derive the last inequality. Thus, noting that $\frac{r}{r-M_0} \lesssim 1$ and $\Delta_0 M^{-\varepsilon} < 1$ we conclude that
\[
|E_{11}| \lesssim u_{1+}^{-\frac{29}{2}} \Delta_0^{-\varepsilon} \left( u_{1+}^{-\varepsilon} \int_{H_{u_{1+}^\varepsilon}} r(L(r\phi) - rHLL\phi)^2 d\omega d\mu \right) E[\phi]^{\frac{1}{2}}(H_{u_{1+}^\varepsilon})
\]
\[
+ E[\phi](H_{u_{1+}^\varepsilon}) + E[\phi](H_{u_{1+}^\varepsilon}) + \sup_{-u_{1+} \leq u \leq u_{1+}} \int_{S_{u_{1+}}} r\phi^2 d\omega,
\]
(6.38)
\[
\lesssim u_{1+}^{-\frac{29}{2}} \Delta_0^{-\varepsilon} \left( u_{1+}^{-\varepsilon} \int_{H_{u_{1+}^\varepsilon}} r(L(r\phi))^2 d\omega d\mu + E[\phi](H_{u_{1+}^\varepsilon}) + E[\phi](H_{u_{1+}^\varepsilon})
\]
\[
+ \sup_{-u_{1+} \leq u \leq u_{1+}} \int_{S_{u_{1+}}} r\phi^2 d\omega,
\]
where we employed (8.10) to derive the last inequality. We remark that the term of $L(r\phi)$ can be absorbed when $E_{11}$ is substituted back to (6.39).

Next we control the error term in $E_{21}$ in a similar fashion. By using (6.6) and $|h| \leq M$, we derive
\[
\int_{H_{u_{1+}^\varepsilon}} r^3(L\phi)^2 |hH| d\omega d\mu \lesssim \Delta_0^{-\varepsilon} u_{1+}^{-\frac{29}{2}} \int_{H_{u_{1+}^\varepsilon}} M r(L\phi)^2 d\omega d\mu \lesssim \Delta_0^{-\varepsilon} u_{1+}^{-\frac{29}{2}} E[\phi](H_{u_{1+}^\varepsilon}),
\]
\[
\int_{H_{u_{1+}^\varepsilon}} r^3|H|(|L\phi|^2 + q\phi^2) d\omega d\mu \lesssim \Delta_0^{-\varepsilon} u_{1+}^{-\frac{29}{2}} E[\phi](H_{u_{1+}^\varepsilon}),
\]
\[
\int_{H_{u_{1+}^\varepsilon}} r^3|H|L\phi|\partial_{\phi}\tilde{\phi}| d\omega d\mu \lesssim \Delta_0^{-\varepsilon} u_{1+}^{-\frac{29}{2}} E[\phi](H_{u_{1+}^\varepsilon})(M^{-\varepsilon} \left( \int_{H_{u_{1+}^\varepsilon}} M r^2(L\phi)^2 d\omega d\mu \right)^\varepsilon
\]
\[
+ E[\phi](H_{u_{1+}^\varepsilon}),
\]
\[
\int_{H_{u_{1+}^\varepsilon}} r^2(|H| + |h|)|\partial_{\phi}\tilde{\phi}|^2 d\omega d\mu \lesssim (\Delta_0^{-\varepsilon} u_{1+}^{-\frac{29}{2}} + M)\|r^{-1}\phi\|^2_{L^2(H_{u_{1+}^\varepsilon})}
\]
\[
\times (\|\partial_{\phi}\|_{L^2(H_{u_{1+}^\varepsilon})} + M^{-\varepsilon} \|\Delta_0^{-\varepsilon} L\phi\|^2_{L^2(H_{u_{1+}^\varepsilon})}),
\]
\[
\int_{H_{u_{1+}^\varepsilon}} r|h\phi|^2 d\omega d\mu \lesssim M\|r^{-1}\phi\|^2_{L^2(H_{u_{1+}^\varepsilon})},
\]
Thus by combining the above estimates, and also by using (8.10) to treat the term of $\|r^{-1}\phi\|^2_{L^2(H_{u_{1+}^\varepsilon})}$ we can derive
\[
|E_{21}| \lesssim (\Delta_0^{-\varepsilon} M^{-\varepsilon} u_{1+}^{-\frac{29}{2}} + M^\varepsilon) \left( \sup_{-u_{1+} \leq u \leq u_{1+}} \|r^{-\varepsilon}\phi\|^2_{L^2(S_{u_{1+}})} + E[\phi](H_{u_{1+}^\varepsilon}) + W_1[\phi](H_{u_{1+}^\varepsilon}) \right)
\]
\[
\lesssim M^\varepsilon (W_1[\phi](H_{u_{1+}^\varepsilon}) + E[\phi](H_{u_{1+}^\varepsilon}) + \sup_{-u_{1+} \leq u \leq u_{1+}} \|r^{-\varepsilon}\phi\|^2_{L^2(S_{u_{1+}})}),
\]
where the first term on the righthand side of the last inequality will be absorbed due to the smallness of $M$. 


We then substitute the estimates of $E_1$, $E_2$, (6.36) and (6.37) to derive

$$
W_1[\varphi](\mathcal{H}_{\theta}^u) + W_1[\varphi](\mathcal{H}_{\theta}^{u_1}) + W_1[\varphi](\mathcal{H}_{\theta}^{u_1})
\leq ||r^{-\frac{1}{2}}\partial \varphi| + r^{-\frac{1}{2}}|\varphi| + 96r^{-\frac{1}{2}}|\varphi||^2_{L^2(S_{01}, \omega_{11})} + \int_{\Omega} |\mathcal{F}(X\varphi + \varphi)| + \frac{q + |X\varphi|}{2} \varphi^2
+ M^{\frac{1}{2}}E[\varphi](\mathcal{H}_{\theta}^{u_1}) + E[\varphi](\mathcal{H}_{\theta}^{u_1}) + \sup_{-u_{11} \leq u \leq u_{11}} \int_{S_{u_{11} - u}} r\varphi^2 d\omega
+ \Delta_0^{\frac{1}{2}}M^{-\frac{1}{2}}\left\{ u_1^{1-\frac{2p+1}{2} + 2p} \sup_{-u_{11} \leq u \leq u_{11}} E[\varphi](\mathcal{H}_{\theta}^{u_1})u_1^{-2p} \right\}
+ u_1^{\gamma - 2p} \sup_{-u_{11} \leq u \leq u_{11}} E[\varphi](\mathcal{H}_{\theta}^{u_1}) + \int_{-u_{11}}^{u_{11}} u_{11}^{-\frac{2p+1}{2} - 2p} W_1[\varphi](\mathcal{H}_{\theta}^{u_1}) du).
$$

Note that $\int_{-u_{11}}^{u_{11}} q\varphi^2 \leq \int_{-u_{11}}^{u_{11}} E[\varphi](\mathcal{H}_{\theta}^{u_1}) du$. The term $\int_{\Omega} |X\varphi|^2 dxdt$ can be treated exactly as in (5.12). The term of $||r^{-\frac{1}{2}}(X\varphi + \varphi)||_{L^2(\Omega_{01})}$ can treated by applying (6.35) on each $\mathcal{H}_{\theta}^{u_1}$, with the bound of error included in the first term in the line of (6.39). By using Gronwall’s inequality (see [19] Section 2.3, Lemma 3), the last term on the right-hand side of the inequality and the first term on the right of (5.12) can both be absorbed. We summarize the result after the above treatments as below

$$
W_1[\varphi](\mathcal{H}_{\theta}^{u_1}) + W_1[\varphi](\mathcal{H}_{\theta}^{u_1}) + W_1[\varphi](\mathcal{H}_{\theta}^{u_1})
\leq C(\varepsilon_1)||r^{-\frac{1}{2}}\partial \varphi| + r^{-\frac{1}{2}}|\varphi| + 96r^{-\frac{1}{2}}|\varphi||^2_{L^2(S_{01}, \omega_{11})} + u_1^{1-\gamma_0 + 2p + 1 + 2p + 1} \varepsilon_1 \sup_{-u_{11} \leq u \leq u_{11}} u_1^{-2p - 1} W_1[\varphi](\mathcal{H}_{\theta}^{u_1})
+ u_1^{1-\gamma_0 + 2p} \sup_{-u_{11} \leq u \leq u_{11}} E[\varphi](\mathcal{H}_{\theta}^{u_1})u_1^{-2p} + ||r^{-\frac{1}{2}}|\partial \varphi| + r^{-\frac{1}{2}}|\varphi||^2_{L^2(S_{01}, \omega_{11})} + q_0||2^l \varphi||_{L^2(S_{01}, \omega_{11})}
+ (M^{\frac{1}{2}} + \Delta_0^{\frac{1}{2}}M^{l/2})\left\{ u_1^{1-\frac{2p+1}{2} + 2p} \sup_{-u_{11} \leq u \leq u_{11}} E[\varphi](\mathcal{H}_{\theta}^{u_1}) + \sup_{-u_{11} \leq u \leq u_{11}} \int_{S_{u_{11} - u}} r\varphi^2 d\omega\right\},
$$

where $p \leq 0$ is any fixed constant. We then multiply both sides by $u_1^{\gamma_0 - 1 - 2p}$ followed by taking supremum on $u_1$ in $-u_{11} \leq u_1 \leq u_{11}$ for a fixed $u_2$. In view of the assumption that $\Delta_0 \lesssim M^2$, by choosing the constant $\varepsilon_1 > 0$ sufficiently small, Proposition 6.3 can be proved. \hfill \square

6.3 Error estimates. The main estimates of this part are the error estimates for controlling $(\mathbb{D}_g - q\mathcal{Z}^n)\phi$ in Proposition 6.4. Comparing this result with Proposition 5.12 for the semilinear equation (2.15), one difference lies in that Proposition 6.4 copes with the term of $|\mathbb{D}_g\mathcal{Z}^n|\phi$, with $n \leq 3$. Such error terms arise due to the nontrivial influence of the metric $g(\phi, \partial \phi)$ and vanish in the semilinear case. The treatment needs a sharp decay property for the term $ZH$, particularly for $Z = \Omega_{ij}$, which requires us to bound energies for $\mathcal{Z}^3 H$. Therefore in Proposition 6.9 we treat $\mathcal{Z}^n$ with $n \leq 3$ for the solution of quasilinear equation (1.1), while in Proposition 5.12 we only need to bound the terms of $\mathcal{Z}^n(\square \phi - q\phi)$ with $n \leq 2$.

Lemma 6.7. Let $\Phi = (\phi, \partial \phi)$. For $n \leq 3$,

$$
|\mathcal{Z}^n|H(\Phi)| \lesssim \sum_{Z^{n1}Z^n = Z^n} |\mathcal{Z}^n|\Phi|_{u_+^{(Z^n)}}. \tag{6.40}
$$

Note that due to (6.5) and (6.17), (6.18) holds for $\partial \phi$ as well. Thus, with the help of (6.9), we can repeat the proof of (5.12) with $\mathcal{N}(\phi)$ replaced by $H(\Phi)$ to obtain (6.40).
Proposition 6.8.

\[ |[\Box_g, Z^n]| \phi \lesssim \sum_{Z^n \supset Z^{n-1} = Z^{n-1}} (u_+^{(Z^{1})}|H| + |Z^{1}|H|)|\partial^2 Z^{(n-1)}| \phi| + \sum_{Z^n \supset Z^{n-1} = Z^{n-1}} u_+^{(Z^{1})}|Z^{n}|H|\partial^2 Z^{(n-1)}| \phi|, \quad 1 \leq n \leq 3 \]  

(6.41)

where the last term on the righthand side vanishes if \( n = 1 \).

Proof. Since \( Z \in \{\Omega, \partial\} \) are killing vector fields, \( [\Box_g, Z] = [H^{\mu \nu} \partial_\mu \partial_\nu, Z] \). We can derive

\[ [\Box_g, Z]| \psi = -ZH^{\alpha \beta} \partial_\alpha \partial_\beta | \psi + 2H^{\alpha \beta} C_{Z^\gamma} \gamma \partial_\alpha \partial_\beta | \psi, \]

(6.42)

since

\[ H^{\alpha \beta} |Z, \partial_\alpha \partial_\beta | \psi = H^{\alpha \beta} |Z, \partial_\alpha \partial_\beta | \psi = H^{\alpha \beta} C_{Z^\gamma} \gamma \partial_\alpha \partial_\beta | \psi, \]

where \( C_{Z^\alpha} \gamma \) has been defined in Lemma 5.6. For convenience we will drop the coefficient 2 in the calculation, and we will adopt the convention in Lemma 5.6. Similarly,

\[ \Box_g Z^{(2)} | \psi = Z^{(2)} \Box_g | \psi + Z^{(2)} Z_1 \Box_g | \psi + [Z_2, \Box_g] Z_1 | \psi \]

\[ = Z^{(2)} \Box_g | \psi + Z^{(2)} Z_1, H^{\alpha \beta} \partial_\alpha \partial_\beta | \psi + [Z_2, H^{\alpha \beta} \partial_\alpha \partial_\beta ] Z_1 | \psi. \]

By using (6.42), we also give the following commutation identities

\[ \Box_g Z^{(2)} | \psi = Z^{(2)} \Box_g | \psi + \sum_{X \cup Y = Z^2} \{ Z^{(2)} H^{\mu \nu} \partial_\mu \partial_\nu Y \psi + X H^{\mu \nu} \partial_\mu \partial_\nu Y \psi + X H^{\mu \nu} (C_X \cdot \partial^2 Y \psi)_{\mu \nu} \}

\[ + H^{\mu \nu} C_{X \parallel} \parallel \partial_\mu \partial_\nu Y \psi + H^{\mu \nu} (C_X C_Y \partial^2 Y \psi)_{\mu \nu} \}, \]

(6.43)

\[ \Box_g Z^{(3)} | \psi = Z^{(3)} \Box_g | \psi + Z^{(3)} H^{\mu \nu} \partial^2 Y \psi + \sum_{a=1}^3 Z^{(3\alpha)} H^{\mu \nu} (\partial_\mu \partial_\nu Z_\alpha \phi + (C_{Z_\alpha} \cdot \partial^2 Y \psi)_{\mu \nu})

\[ + H^{\mu \nu} \sum_{a=1}^3 (C_{Z_\alpha} \gamma \partial_\gamma \partial_\alpha Z^{(3\alpha)} \psi + (C_{Z^{(3\alpha)}} \partial^2 Y \psi)_{\mu \nu}) + (C_{Z^{(3\alpha)}} \partial^2 Y \psi)_{\mu \nu} \}. \]

(6.44)

We then summarize the terms in (6.42) - (6.44) into

\[ |[\Box_g, Z^{(n)}]| \phi \lesssim \sum_{Z^{1} \cup Z^{n-1} = Z^n} (u_+^{(Z^{1})}|H| + |Z^{1}|H|)|\partial^2 Z^{(n-1)}| \phi| + \sum_{Z^{n} \cup Z^{n-1} = Z^{n}, b \leq n-2} u_+^{(Z^{1})}|Z^{n}|H|\partial^2 Z^{(n-1)}| \phi|, \]

(6.45)

where the last term on the righthand side vanishes if \( n = 1 \). By using (6.40) to treat the term \( Z^{(a)} H \), the last term on the righthand side of (6.45) can be bounded by

\[ \sum_{Z^{n+1} \cup Z^{n} = Z^n, b \leq n-2} u_+^{(Z^{1})}|Z^{(a)}|H|\partial^2 Z^{(n-1)}| \phi|. \]

We then combine the above estimates to conclude Proposition 6.8 \( \Box_g \)

In view of Proposition 6.8 and (2) in Lemma 5.10, we will prove the following result.
Proposition 6.9. For $0 \leq n \leq 3$, there hold for $(u_1, u_1) \in \mathcal{I}$ that

$$u_1^{-\frac{\gamma_0}{2} + \frac{1}{2}} ||r_1 \hat{Z} G_n^1, u_1^9||_{L^2(D_{M}^1)} \lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.46)

$$u_1^{-\frac{\gamma_0}{2} + \frac{1}{2}} ||r_1 \hat{Z} G_n^2, u_1^9||_{L^2(D_{M}^2)} \lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.47)

$$u_1^{-\frac{\gamma_0}{2} + \frac{1}{2}} ||r_1 \hat{Z} G_n^3, u_1^9||_{L^2(D_{M}^3)} \lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.48)

$$u_1^{-\frac{\gamma_0}{2} + \frac{1}{2}} ||r_1 \hat{Z} G_n^4, u_1^9||_{L^2(D_{M}^4)} \lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.49)

$$u_1^{-\frac{\gamma_0}{2} + \frac{1}{2}} ||r_1 \hat{Z} G_n^5, u_1^9||_{L^2(D_{M}^5)} \lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.50)

$$u_1^{-\frac{\gamma_0}{2} + \frac{1}{2}} ||r_1 \hat{Z} G_n^6, u_1^9||_{L^2(D_{M}^6)} \lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.51)

Proof. If $n = 0$, the commutator is identically 0. Thus the corresponding estimates are trivially true. Thus for the commutator estimates (6.46) and (6.47), we only need to consider the cases $1 \leq n \leq 3$. We first prove (6.46). Denote by $I_n$ and $H_n$ the terms on the right of (6.41) respectively.

We apply (6.46) to $Z^{(i)} H_1, i \leq 1$ and bound $\partial^2 Z^{(n-1)} \phi$ by using (4.14). Thus, we can directly obtain

$$\|r_1^\frac{1}{2} I_n \|_{L^2(D_{M}^1)} \lesssim \sum_{Z^{a_1} \subset V \subset Z_n} || \frac{r_1^\frac{1}{2} Z^{(1)} H_1 + u_1^\frac{1}{2} Z^{(1)} |H_1|} {L^\infty(D_{M}^1)} ||_{L^\infty(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)}$$

(6.48)

$$\lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.49)

For the term $I_n = \sum_{Z^{a_1} \subset V \subset Z_n} u_1^\frac{1}{2} \| Z^{(a_1)} \|_{L^2(D_{M}^1)} \| \partial^2 Z^{(b)} \|_{L^\infty(D_{M}^1)}$, we first derive

$$\| r_1^\frac{1}{2} I_n \|_{L^2(D_{M}^1)} \lesssim \sum_{1 \leq b \leq n-2} \sum_{Z^{a_1} \subset V \subset Z_n} u_1^\frac{1}{2} \| Z^{(a_1)} \|_{L^2(D_{M}^1)} \| \partial^2 Z^{(b)} \|_{L^\infty(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)}$$

(6.50)

With the following estimates

$$\| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)} \| u_1^{-\frac{1}{2}} \|_{L^2(D_{M}^1)}$$

(6.51)

we can directly obtain

$$\| r_1^\frac{1}{2} I_n \|_{L^2(D_{M}^1)} \lesssim \Delta_0 M^{-\frac{1}{2}} u_1^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{4}}$$

(6.52)

Now we derive the estimates (6.52) and (6.53). We first apply (4.18) and (6.46) to obtain (6.53) and (6.55). By using (6.23) and (4.14), we can obtain (6.52) and (6.54) if $\Phi$ is simply $\phi$. By using (6.17) and (4.22), we can obtain (6.52) for $\Phi$; by using (6.17) and (4.14), (6.54) is proved for $\Phi$.

By combining the estimates of $I_n$ and $H_n$, (6.46) is proved.
Next we prove (6.47). By using (6.6) and (6.3), we derive
\[ u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \| r_{1+} F_n \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
\[ \lesssim u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \| r_{1+} \frac{1}{2} \phi^2 Z^{-1} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
\[ \lesssim u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \| \phi \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
\[ \lesssim u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \| \phi \|_{L^\infty_{\Delta} L^\infty(D_{\infty}^n)} \]
\[ \lesssim u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \| \phi \|_{L^\infty_{\Delta} L^\infty(D_{\infty}^n)} \]
which holds for any \( Z^{-1} \subset Z^n \).
To estimate \( I_{1n} \), by repeating the derivation of (6.52) and (6.54), we have
\[ \| u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \lesssim u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \]
\[ \| u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \lesssim u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \]
Combining (6.52), (6.55) with the above two estimates, by a standard H"older's inequality, we can derive
\[ \| \| I_{1n} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \lesssim \| r_{1+} F_n \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \| u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \lesssim u_{1+}^{-\zeta(Z^n) + \frac{1}{2}u_{1+} + \frac{1}{2}r} \]
(6.47) follows by combining the estimates for \( I_n \) and \( I_{1n} \).
We now consider \( F_{Q,2} \). Recall from (6.28) in the proof of Proposition 6.12 that \( F_{Q}=F_{Q,1}+F_{Q,2} \). We will control the terms of \( F_{Q,1} \) and \( F_{Q,2} \) in a similar way.
Similar to the estimate of \( I_n \), by using (6.5) and using (4.14) due to \( b \leq n \), we can derive
\[ \| \| F_{Q,1} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \lesssim \| \| F_{Q,2} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \lesssim \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
\[ \lesssim \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
Note that \( 1 \leq b \leq a \leq n - 1 \) in \( F_{Q,2} \). We can employ (4.22) for the term \( \partial Z^{(b)} \phi \) and apply (4.13) to \( \partial Z^{(a)} \phi \)
\[ \| \| F_{Q,2} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \lesssim \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
\[ \lesssim \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
By combining the above two estimates, we can derive (6.48).
Next we prove (6.49). We first consider \( F_{Q,1} \). By using (6.5) and (4.14), we have
\[ \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
\[ \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
\[ = \| \| r_{1+} \|_{L^2_{\Delta} L^2(D_{\infty}^n)} \]
Due to $1 \leq b \leq a \leq n - 1$ in $(n)^{Q,2}$, we can derive by using (6.20) and (6.22) that

$$\|r^{\frac{1}{2}}(n)^{Q,2}\|_{L_{t}^{1}L_{x}^{2}(\mathbb{R}^{n}_{+})} \lesssim \sum_{Z^{n} \cup Z^{a} \cup Z^{c} = Z^{n}} u^{(Z^{c})}_{+}\|r \partial Z^{(b)} \phi\|_{L_{t}^{\infty}L_{x}^{\infty}(\mathbb{R}^{n}_{+})} \|r^{\frac{1}{2}} \partial Z^{(c)} \phi\|_{L_{t}^{2}L_{x}^{2}(\mathbb{R}^{n}_{+})}$$

$$\lesssim \Delta_{0} M^{-\frac{1}{2}} \sum_{Z^{n} \cup Z^{a} \cup Z^{c} = Z^{n}} u^{(Z^{c})}_{1+} - \frac{2a}{2} + \zeta(Z^{b}) + \zeta(Z^{c}) - \frac{2a}{2} + \frac{1}{2} u^{(Z^{a})}_{1+} - \gamma_{0} + \frac{1}{2} + \zeta(Z^{n}).$$

By combining the estimates for $(n)^{Q,1}$ and $(n)^{Q,2}$, we can obtain (6.49).

At last we consider the term $(n)^{C}$. We recall the definition of $(n)^{C}$ from Lemma 5.10 and will first show (6.50). Note that at least one of $b$ or $c$ is $\leq 1$. Assume, without loss of generality, $b \leq c$. Since $a \geq 1$ in $(n)^{C}$, $b + c \leq n - 1$. Thus $0 \leq b \leq c \leq n - 1$ and $b \leq 1$. In view of (6.5), (4.21) and (4.24), we have

$$\|r^{\frac{1}{2}}(n)^{C} \|_{L_{t}^{2}(\mathbb{R}^{n}_{+})} \lesssim \sum_{Z^{n} \cup Z^{a} \cup Z^{c} = Z^{n}, a \geq 1, b \leq c} u^{(Z^{a})}_{+} \|r \partial Z^{(b)} \phi\|_{L_{t}^{\infty}(\mathbb{R}^{n}_{+})} \|r^{\frac{1}{2}} \partial Z^{(c)} \phi\|_{L_{t}^{2}L_{x}^{2}(\mathbb{R}^{n}_{+})}$$

$$\cdot \|\phi\|_{L_{t}^{\infty}L_{x}^{\infty}(\mathbb{R}^{n}_{+})} \lesssim \Delta_{0} M^{-\frac{1}{2}} u^{(Z^{a})}_{1+} - \frac{2a}{2} + \gamma_{0} + \frac{1}{2} + \zeta(Z^{n}),$$

which gives (6.50).

By using (6.5), (4.20) and (4.17), we have

$$\|r^{\frac{1}{2}}(n)^{C} \|_{L_{t}^{1}L_{x}^{2}(\mathbb{R}^{n}_{+})} \lesssim \sum_{Z^{n} \cup Z^{a} \cup Z^{c} = Z^{n}, a \geq 1, b \leq c} u^{(Z^{a})}_{+} \|r \partial Z^{(b)} \phi\|_{L_{t}^{\infty}(\mathbb{R}^{n}_{+})} \|r^{\frac{1}{2}} \partial Z^{(c)} \phi\|_{L_{t}^{2}L_{x}^{2}(\mathbb{R}^{n}_{+})}$$

$$\cdot \|r^{\frac{1}{2}} Z^{a} \phi\|_{L_{t}^{2}L_{x}^{1}(\mathbb{R}^{n}_{+})} \lesssim \Delta_{0} M^{-\frac{a}{2}} u^{(Z^{a})}_{1+} - \frac{2a}{2} + \gamma_{0} - \frac{1}{2} + \zeta(Z^{n}).$$

Thus (6.51) is proved.

6.4. Boundedness theorem. We now combine the energy and weighted energy inequalities in Section 6.2 and the error estimates in Section 6.3 to give the boundedness of energy and the weighted energy. The proof follows similarly as for Proposition 5.14.

Theorem 6.10 (Boundedness of energies). For $n \leq 3$, under the assumptions (A1) and (A2), there hold for $(u, \bar{u}) \in \mathcal{I}$ that

$$\sup_{-2 \leq u' \leq u} u^{\gamma_{0} - 2\zeta(Z^{n})}(E[Z^{(n)}] \phi)(\mathcal{H}_{u'}) + E[Z^{(n)}] \phi(\mathcal{H}_{u'})' \lesssim \mathcal{E}_{n, \gamma_{0}} + M^{-2} \Delta_{0}^{2} u^{1-\gamma_{0}},$$

(6.56)

$$\sup_{-2 \leq u' \leq u} u^{\gamma_{0} - 2\zeta(Z^{n}) - 1 + \gamma_{0}} (W_{1}[Z^{(n)}] \phi)(\mathcal{H}_{u'}) + W_{1}[Z^{(n)}] \phi(\mathcal{H}_{u'})' + W_{1}[Z^{(n)}] \phi(\mathcal{H}_{u'}) \lesssim \mathcal{E}_{n, \gamma_{0}} + M^{-2} \Delta_{0}^{2} u^{1+\gamma_{0}}.$$  (6.57)

Remark 6.11. We then can find a universal constant $C_{3} \geq 1$ such that both of the inequalities are bounded by

$$C_{3}(\mathcal{E}_{n, \gamma_{0}} + M^{-2} \Delta_{0}^{2} u^{1-\gamma_{0}}).$$  (6.58)

Proof. We first consider (6.50). Similar to (6.31), we have $\Box \phi - q \phi = \mathcal{F}$, where

$$\mathcal{F}^{c} = [\Box \phi, Z^{(n)}] \phi + (n)^{Q} + (n)^{C}, \quad \mathcal{F}^{s} = \phi.$$
Then we apply the first inequality in Lemma 5.13 to treat $\mathcal{F}$, and apply (6.46), (6.48) and (6.51) to treat $\mathcal{F}^\theta$. By using Proposition 6.4, we have due to $\Delta_0 M^{-1} \leq 1$ that
\[
\sup_{-u \leq u \leq u} u^{\gamma_0 - 2c(Z^n)}(E[Z^n]\phi)(H^n) + E[Z^n]\phi(H^n) \\
\leq \mathcal{E}_{n,\gamma_0} + M^{-2}\Delta_0^2(u^{\gamma_0 + 2\gamma} + \sum_{n=1}^\infty E[Z^{n-i}])(H^n)u^{\gamma_0 - 2c(Z^n)},
\]
where the last term vanishes when $n = 0$. Thus under the assumptions (A1) and (A2), (6.50) holds true by induction.

Let $(u_1, u_2) \in \mathcal{I}$ be fixed. To see the weighted energy estimate for $Z^n\phi$, by using Proposition 6.34 and the fact that $||r^2 \partial Z^n\phi||_{L^2(S_0^\alpha, \omega)} \lesssim u_1^{\gamma_0 + 1} \mathcal{E}_{n,\gamma_0}$ for $a \leq n$, we derive
\[
\sup_{-u \leq u \leq u} u^{\gamma_0 - 2c(Z^n)}(W_1[Z^n]\phi)(D^n) + W_1[Z^n]\phi(H^n) \\
\lesssim u_1^{\gamma_0 - 2c(Z^n)}(M^{-1} + 1) + \sum_{n=1}^\infty u_1^{\gamma_0 - 2c(Z^n)}(M^{-1} + 1) + \sum_{n=1}^\infty \mathcal{E}_{n,\gamma_0},
\]
where we also employed the second inequality in Lemma 5.13 and (6.47), (6.49) and (6.51). We then substitute the result of (6.57) followed with an induction argument to derive (6.58).

Thus (6.57) is proved. \qed

6.5. **Proof of Theorem 2.5** With $\Delta_0 = C_1 \mathcal{E}_{3,\gamma_0}$ and $C_1 = 4C_3C$, in view of (6.50), (6.57) and $M = 3\delta_1^2$, to improve the bootstrap assumptions (6.3) and (6.4), we need to have
\[
C_3(C_1^{-1} \Delta_0 + (3C)^{-2}\delta_1^{-1} \Delta_0 C_1 \mathcal{E}_{3,\gamma_0} u_1^{1-\gamma_0}) < 2\Delta_0,
\]
where $C, C_1, C_3 > 1$. Identically,
\[
C_3(C_1^{-1} + (3C)^{-2}C_1 u_1^{1-\gamma_0}) < 2,
\]
which requires
\[
\frac{4}{9} C^2 C_1^{-1} u_1^{1-\gamma_0} \leq \frac{4}{9} C_3 C_1^{-1} u_0(R) u_1^{1-\gamma_0} < \frac{7}{4}.
\]
Next we determine $R$ so that (A1) can be improved and (A2) can be satisfied. Note that by using (6.6), for $(u, \omega) \in \mathcal{I},$
\[
R|H^n\beta(u, \omega)| \leq C_2 \Delta_0^{-\frac{1}{2}} u_1^{-\gamma_0 + \frac{1}{2}}.
\]
Thus, in view of (6.2),
\[
\sup_{\alpha, \beta} r(H^n\beta(u, \omega)) \leq C_2 \Delta_0^{-\frac{1}{2}} u_1^{-\gamma_0 + \frac{1}{2}}.
\]
With
\[
u_1^{-\frac{2}{2} + \frac{1}{2} C_1^2} C_2 \leq u_0(R) u_1^{-\frac{2}{2} + \frac{1}{2} C_1^2} C_2 < 2C,
\]
the strict inequality in (A1) holds true.

By a direct substitution, to make (A2) hold, we need
\[
C_1^2 \frac{R}{2} \frac{1}{2} < \frac{1}{2} C C^{-1}.
\]
Thus we require $u_0(R)$ to satisfy the second inequalities in (6.59), (6.61) and the above inequality. With the help of $u_0(R) > \frac{1}{2} R$, we can fix $R(\gamma_0, C)$, the lower bound of $R$, such that these inequalities hold.
7. Einstein scalar fields

In this section, we apply the approach in Section 6 to prove the nonlinear stability result for Einstein scalar fields, exterior to a Schwarzschild cone with small positive mass, which is stated in Theorem 2.6.

Under the wave coordinates \( h_{\mu \nu} = g_{\mu \nu} - m_{\mu \nu} \). The Einstein equation with scalar fields takes the form of

\[
\square_g h_{\mu \nu} = (A^{\alpha \beta}_\mu + G^{\alpha \beta}_{\mu \nu}(h)) \partial_\alpha h_{\beta \nu} + \partial_\alpha \phi \cdot \partial_\beta \phi + g_{\mu \nu} q_0 \phi^2,
\]

where we assume the constant \( 0 \leq q_0 \leq 1 \) without loss of generality. For each fixed \((\mu, \nu)\), \( A_{\mu \nu} \) is a matrix of constant components. For each fixed \((\mu, \nu)\), \( G^{\alpha \beta}_{\mu \nu}(h) \) are smooth functions of \( h \).

They represent the product \( h_{\alpha \beta} \) or \( H^{\alpha \beta} \) with components of \( g \). We will symbolically represent all such functions as \( \mathcal{G}(h) \). Such \( \mathcal{G}(h) \) vanishes at \((h_{\alpha \beta}) = (0)\). Other constant coefficients on the righthand side of (7.1) have been simplified to be 1. \(^{24}\)

Let \( m_0 > 0 \) be a fixed small number. Denote \( \hat{h}_{\mu \nu} = \frac{m_0}{r} \delta_{\mu \nu} \). We decompose

\[
h_{\mu \nu} = h^1_{\mu \nu} + \hat{h}_{\mu \nu}.
\]

This reduces (7.1) to the equations for \((h^1, \phi)\),

\[
\begin{align*}
\square_g h^1_{\mu \nu} &= \mathcal{N}(h)(\partial_\alpha \hat{h} \cdot \partial_\beta h^1 + \partial_\alpha \phi \cdot \partial_\beta \phi + g_{\mu \nu} q_0 \phi^2 + S_{\mu \nu}, \quad \text{(ES)} \\
\square_g \phi - q_0 \phi &= 0,
\end{align*}
\]

where \( \mathcal{N}(h) = 1 + \mathcal{G}(h) \) symbolically and

\[
S_{\mu \nu} = -\square_g \hat{h}_{\mu \nu} + \mathcal{N}(h) \partial_\alpha \hat{h} \partial_\beta \hat{h}.
\]

We remark that the structure of wave coordinates implies (see \(^{[24]}\) Lemma 8.1)

\[
|LH| \leq |\partial H| + |H \cdot \partial H|
\]

which can provide some convenience to the proof of boundedness of energy. This will be shown shortly. In this section, we prove Theorem 2.6 by applying our approach in Section 6 to \((h^1, \phi)\) with potentials \((0, q_0)\).

Let \( 1 < \gamma_0 < 2 \) be fixed. We define for the initial data \( h^1(0) = (h^1(0), \partial_\mu h^1(0)) \) and \( \phi(0) = (\phi(0), \partial_\mu \phi(0)) \) the weighted norm

\[
\mathcal{E}_{3, \gamma_0, R}(h^1, \phi) = \mathcal{E}_{3, \gamma_0, R, 0}(h^1(0)) + \mathcal{E}_{3, \gamma_0, R, q_0}(\phi(0)).
\]

The extra subindex \( C \) of \( \mathcal{E}_{3, \gamma_0, R, C} \) on the right denotes the constant potential function of each equation. In this section, we assume

\[
\mathcal{E}_{3, \gamma_0, R}(h^1, \phi) \leq C_0 m_0^2
\]

where \( C_0 \geq 1 \) is a fixed constant, \( R \geq 2 \) with the lower bound determined later.

\[
\mathcal{E}_{3, \gamma_0, R, 0}(h^1(0)) < \infty \text{ implies } \liminf_{|x| \to \infty} g_{\mu \nu}(0, x) = \overset{\circ}{g}_{\mu \nu} \text{. where } \overset{\circ}{g}_{\mu \nu} = m_{\mu \nu} + \frac{m_0}{r} \delta_{\mu \nu} \text{. It is direct to compute}
\]

\[
\overset{\circ}{g}^{-1} = \overset{\circ}{g} \overset{\circ}{g}^{-1} < -\frac{m_0}{2r}.
\]

To prove Theorem 2.6 we fix

\[
h = -\frac{m_0}{20r}, \text{ i.e. } M_0 = -\frac{m_0}{20}
\]

and show that with \( M = m_0 \) and the constant potentials \((0, q_0)\), all the norms in (2.13) for \( Z^{0}(h^1, \phi) \), \( i \leq 3 \) remain small in the region \( \{u \leq u_0(M_0)\} \) provided that \( m_0 \) in (7.4) is

\(^{24}\)The wave coordinates \( \{x^\mu\}_{\mu=0}^{\infty} \) are required to be the solution of \( \square_g x^\mu = 0 \) where \( \square_g \) is the Laplace-Beltrami operator of the Einstein metric \( g \).

\(^{25}\)See \(^{[24]}\) for the more detailed structure. We do not need the weak null structure to prove the result of Theorem 2.6.
sufficiently small and $R \geq R_0(\gamma_0, C_0)$. The constant $R_0(\gamma_0, C_0)$ will be specified at the end of the proof.

7.1. Preliminaries.

Lemma 7.1. If $E_{5, \gamma_0, R}(h^1[0]) \leq C_0m_0^2$, $C_0 \geq 1$, there exists a constant $C_1 \geq 1$ depending on $C_0$ and $\gamma_0$, if $C_1R^{-\frac{m_0^2}{2}+\frac{1}{2}} \leq 1$, there hold

$$r(h - H^{LL}) > \frac{m_0}{3}, \quad r(h - H^{LL}) > \frac{m_0}{3}, \quad \text{on } \Sigma_0 \cap \{r \geq R\}. \quad (7.6)$$

Proof. It follows from (5.9) that

$$r|\tilde{h}^0h^1(u, \omega, \omega)| \lesssim u^{1+\frac{1}{2}(\frac{m_0^2}{2}+\frac{1}{2})} \rho_{2, \gamma_0}, \quad l \leq 1 \quad (7.7)$$

Thus $r|\tilde{h}^0| \lesssim (\rho_{2, \gamma_0} + m_0)$.

For small $h$, $H^{\mu\nu} = -h^{\mu\nu} + O(\nu^2(h^2)$ where $h^{\mu\nu} = \gamma^{\mu\nu} \gamma^{\rho\sigma}h_\gamma_{\rho,\sigma}$ and $O(\nu^2(h^2)$ vanishes to second order at $h = 0$. Thus

$$|H^{\mu\nu} - \tilde{h}^{\mu\nu}| \lesssim |h|^2 \quad (7.8)$$

where $\tilde{h}^{\mu\nu} = \tilde{g}^{\mu\nu} - \gamma^{\mu\nu}$. By using the above two inequalities, (7.8) and (7.5), we can derive

$$r(h - H^{LL}) \geq \frac{9}{20}m_0 - C(u_+^{-\frac{m_0^2}{2}+\frac{1}{2}} \rho_{2, \gamma_0} + C_0^2 + (m_0 + \rho_{2, \gamma_0})^2)$$

where the universal constant $C \geq 1$. Due to $|u(R)| > R^2$ and $m_0 < 1$, if we choose $R$ such that

$$C((\frac{R}{2})^{-\frac{m_0^2}{2}+\frac{1}{2}} C_0 + R^{-1}(1 + C_0^2)^2) < \frac{1}{10}$$

Therefore we may require $\frac{1}{4}C((\frac{R}{2})^{-\frac{m_0^2}{2}+\frac{1}{2}} C_0 + 2)^2 \lesssim \frac{1}{m_0}$, then (7.6) holds. The same treatment works the same for $r(h - H^{LL})$. The proof is completed. \(\square\)

The above result shows that the assumption of (A1) holds true on $\Sigma_0 \cap \{r \geq R\}$ with $M = m_0$ if $R^{\frac{m_0^2}{2}+\frac{1}{2}} \leq C_1^{-1}$. We will further specify the lower bound $R_0(\gamma_0, C_0)$ during the proof of Theorem 2.6.

Bootstrap assumptions To prove Theorem 2.6, we make the following bootstrap assumptions.

Let $u_+ > u_0$ be any fixed number, where $u_0 = -r_0(-\frac{m_0^2}{2}, R)$. In $I = \{(u, u), -u_+ \leq -u \leq u \leq u_0\}$, suppose (A1) holds, and the assumption (BA3) holds for $(h^1, \phi)$ with $\Delta_0 = C_1m_0^2$, $C_1 \geq 1$ to be chosen. (BA3) is restated as below for $n \leq 3$ and $Z \in \{\Omega, \tilde{\Omega}\}$,

$$E[Z^{(n)}(h^1, \phi)](H^{\mu\nu}) + E[Z^{(n)}(h^1, \phi)](H^{\mu\nu}) \leq 2\Delta_0u_+^{-\gamma_0+2\zeta(Z^n)}$$

$$\mathcal{W}_1[Z^{(n)}(h^1, \phi)](H^{\mu\nu}) + \mathcal{W}_1[Z^{(n)}(h^1, \phi)](H^{\mu\nu}) \leq 2\Delta_0u_+^{-\gamma_0+1+2\zeta(Z^n)}$$

for all $(u, u) \in I$.

Thus the full set of decay estimates in Section 3 hold for $(h^1, \phi)$. We will quote the results in the proofs whenever necessary.

Before starting to prove Theorem 2.6, we highlight the difference in analysis in comparison with Section 6

- In the Einsteinian case, there holds $H^{\mu\nu} = -h^{\mu\nu} + O(h^2)$. Thus $H$ does not depend on $\tilde{h}(h^1, \phi)$. In terms of regularity it seems better than in Section 6. Nevertheless, the small static part $h^0$ in $h$ slows down the decay of $\tilde{h}$. In the proofs of Theorem 2.1 and Theorem 2.3 we take advantage of the fact that $R^{\frac{m_0^2}{2}+\frac{1}{2}}$ can be small so as to improve the bootstrap assumption, while in this section, we lose such extra smallness for the critical terms. This requires us to separate carefully the evolutionary part of the metric $h^1$ from $h^0$ in analysis. For the borderline terms appeared in the error estimates, we bound them by energies which have been controlled in the induction. Although these
Proof. To begin with, noting that 0 ≤ \( q_0 \leq 1 \) is now a fixed constant, (3.4) can be improved to be

\[
q_0 \int_{\Omega_{1+2,\gamma_0}} (\mathcal{E}_1) u_+^{-\gamma_0 + 2\gamma(Z') + \frac{1}{2}}, \quad l \leq 1.
\]

(7.10)

**Proof.**

To begin with, noting that 0 ≤ q_0 ≤ 1 is now a fixed constant, (3.4) can be improved to be

\[
q_0 \int_{\Omega_{1+2,\gamma_0}} (\mathcal{E}_1) u_+^{-\gamma_0 + 2\gamma(Z') + \frac{1}{2}}, \quad l \leq 1.
\]

(7.10)

We first give an improve decay estimate of \( \phi \) compared with (4.2) in Proposition 4.1.

**Proposition 7.2.** For \( (u, u') \in \mathcal{I} \), there holds

\[
q_0 (\mathcal{E}_1) u_+^{-\gamma_0 + 2\gamma(Z') + \frac{1}{2}}, \quad l \leq 1.
\]

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**Proof.**

To begin with, noting that 0 ≤ q_0 ≤ 1 is now a fixed constant, (3.4) can be improved to be

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\]

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**Proof.**

To begin with, noting that 0 ≤ q_0 ≤ 1 is now a fixed constant, (3.4) can be improved to be

\[
q_0 \int_{\Omega_{1+2,\gamma_0}} (\mathcal{E}_1) u_+^{-\gamma_0 + 2\gamma(Z') + \frac{1}{2}}, \quad l \leq 1.
\]

(7.10)

We first give an improve decay estimate of \( \phi \) compared with (4.2) in Proposition 4.1.
\[ |Z^{(n)}H| \lesssim \sum_{Z=aL^b=Z, a \geq 1} u^{c(Z^b)} (|Z^a|H^1 + m_0r^{-1-a}), \quad 1 \leq n \leq 3. \] (7.18)

**Proof.** The proofs of (7.14)-(7.16) can follow directly from the property of \( H^{\mu\nu}(h) = h^{\mu\nu} + h_{\mu\nu} + m_{\mu\nu} \) and applying (4.1) and (4.2) to \( h^1 \). (7.17) can be similarly proved as Lemma 5.8. (7.18) is an improved version, which relies on (7.13) instead of (7.12).

**Proposition 7.4.** Consider

\[ \Box_h \varphi = \mathcal{F} + q \varphi \]

where \( q \geq 0 \) is a constant, \( h^1 \) in the Lorentzian metric \( g_{\mu\nu} = h_{\mu\nu} + h_{\mu\nu} + m_{\mu\nu} \) satisfy the bootstrap assumptions (7.9) and (7.4). By assuming (7.4), Proposition 6.6 holds true, and Proposition 6.7 holds if assuming (7.3) instead of (A2).

**Proof.** We first confirm that if the background metric is the Einstein metric \( g(h) \), the weighted energy estimate in Proposition 6.6 holds. The proof is carried out in two steps.

**Step 1** For the terms \( I, II \) and \( III \) defined in (6.20), we will show there holds for \((u_1, u_\gamma) \in \mathcal{I}\) that

\[
\int_{D_{u_1}} |I| + |II| + |III| \lesssim (\Delta_1^\frac{1}{2} m_0^{-\frac{1}{2}} + m_1^\frac{1}{2})(u_1^{-\frac{2m_0}{m_1}+1} + \sup_{-u_1 \leq u \leq u_1} E[\varphi](H_u u^\gamma) u_1^\gamma + \sup_{-u_1 \leq u \leq u_1} E[\varphi](H_u u^\gamma) u_1^\gamma + \int_{-u_1}^{u_1} (u_1^{-\frac{2m_0}{m_1}+1} + u_1^{-\frac{2m_0}{m_1}+2})W_1[\varphi](H_u u^\gamma) du). \]

(7.19)

This implies (6.17) holds with the bound replaced by the righthand side of (7.19).

We first give the following straightforward calculations.

\[
\int_{D_{u_1}} r^{-2} (\partial^2 \varphi)^2 \lesssim u_1^{-1} \sup_{-u_1 \leq u \leq u_1} E[\varphi](H_u u^\gamma), \]

(7.20)

\[
\int_{D_{u_1}} r^{-1} |\partial \varphi \cdot \partial \varphi| \lesssim \int_{D_{u_1}} r^{-1} (|\partial \varphi||\partial \varphi| + |\partial \varphi|^2) \lesssim (r^{-1} |\partial \varphi| L^2(D_{u_1}))|\partial \varphi| L^2(D_{u_1}) + \int_{-u_1}^{u_1} u_1^{-1} E[\varphi](H_u u^\gamma) du \lesssim u_1^{-\gamma_0} \sup_{-u_1 \leq u \leq u_1} E[\varphi](H_u u^\gamma) u_1^\gamma + \sup_{-u_1 \leq u \leq u_1} E[\varphi](H_u u^\gamma) u_1^\gamma, \]

(7.21)

\[
\int_{D_{u_1}} r^{-2} |L(r\varphi)\partial \varphi| \lesssim \int_{-u_1}^{u_1} ||r^{-\frac{1}{2}} L(r\varphi)|| L^2(D_{u_1}) ||r^{-\frac{1}{2}} \partial \varphi|| L^2(D_{u_1}) r^{-1} du, \]

(7.22)

\[
\lesssim m_0^{-\frac{1}{2}} \int_{-u_1}^{u_1} \{W_1[\varphi](H_u u^\gamma) u_1^\gamma + u_1^{-\frac{3m_0}{2m_1}+2} E[\varphi](H_u u^\gamma) u_1^\gamma \} du. \]

We also note that by using (7.14) and (7.15)

\[
|H||\partial H| \lesssim r^{-1}(m_0 + \Delta_0^\frac{1}{2}) (r^{-2} m_0 + |\partial h^1|), \]

(7.23)

\[
r(|LH| + |H||\partial H|) \lesssim r |Lh^1| + (m_0 + \Delta_0^\frac{1}{2}) |\partial h^1| + r^{-1}(m_0 + \Delta_0^\frac{1}{2}) \]

(7.24)
The parts of \( \partial h^1 \) will be treated similar to Section 6. We first control \( I \) and \( II \) in view of (6.22). By using (7.20), (7.23) and (4.3), we have
\[
\int_{D_{u_+}} r|H| |\partial H| |\bar{\partial} \varphi|^2 \lesssim (m_2 + \Delta_0 \frac{\varphi}{2}) (\int_{D_{u_+}} |\partial h^1| |\bar{\partial} \varphi|^2 + m_0 r^{-2} |\partial \varphi|^2)
\]
\[
\lesssim (m_2 + \Delta_0 \frac{\varphi}{2}) \{(r^{\frac{\varphi}{2}} h^1)_{L^2}^2 + \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}^u_{\Delta_0}) \}
\]
\[
\lesssim (m_2 + \Delta_0 \frac{\varphi}{2}) (m_0) u_1^{\frac{\varphi}{2} + 1} (\sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}^u_{\Delta_0}) u_1^{70} + \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}^u_{\Delta_0}) u_1^{70}).
\]

Next by using (7.21), (7.24), (7.21), and (7.14), we can derive
\[
\int_{D_{u_+}} r|Lh| + |H| |\partial H| + |H||\partial \varphi \cdot \bar{\partial} \varphi|
\]
\[
\lesssim \int_{D_{u_+}} (r|Lh| + r^{-1}(m_2 + \Delta_0 \frac{\varphi}{2})) |\partial \varphi \cdot \bar{\partial} \varphi|
\]
\[
\lesssim (\Delta_0 \frac{\varphi}{2} m_2 + m_0) u_1^{\frac{\varphi}{2} + 1} (\sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}^u_{\Delta_0}) u_1^{70} + \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}^u_{\Delta_0}) u_1^{70}),
\]
where we used the estimate (6.26) to treat the product term with \( Lh^1 \).

By using (7.10) and (7.19), we have
\[
\int_{D_{u_+}} |L(r \varphi)| |\partial \varphi \cdot \partial H| \lesssim \int_{D_{u_+}} |L(r \varphi)| |\partial \varphi \cdot \partial h| + \int_{D_{u_+}} r^{-2} m_0 |L(r \varphi)| |\partial \varphi|.
\]

Note that the first term on the right can be treated as (6.26) and the second term is treated by using (7.22). Thus
\[
\int_{D_{u_+}} |L(r \varphi)| |\partial \varphi \cdot \partial H| \lesssim (\Delta_0 \frac{\varphi}{2} m_2 + m_0) (\int_{-u_1} u_1^{\frac{\varphi}{2} + 1} + \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}^u_{\Delta_0}) u_1^{70}.)
\]

Hence we completed the estimates for \( I \) and \( II \).

Next we consider the term \( III \) defined in (6.20) with the bound given in (6.23). The first term on the right of (6.23) has been treated. We will treat the remaining terms in the sequel. We note that by using Proposition 7.6
\[
(r |\partial H| + |H|) |\partial H| \lesssim r^{-2} (m_0 + \Delta_0 \frac{\varphi}{2})^2 + (m_0 + \Delta_0 \frac{\varphi}{2}) |\partial h^1|.
\]

Thus by using (4.11) for \( \partial h^1 \)
\[
\int_{D_{u_+}} (r |\partial H| + |H|) |\partial \varphi|^2 \lesssim (m_0 + \Delta_0 \frac{\varphi}{2}) \int_{D_{u_+}} \left(r^{-1}(m_0 + \Delta_0 \frac{\varphi}{2}) + \Delta_0 \frac{\varphi}{2} u_1^{\frac{\varphi}{2} + 1} \right) \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}^u_{\Delta_0}) u_1^{70}.
\]
Similarly,
\[
\int_{\mathbb{R}^n_+} (r|\partial H| + |H|) q \varphi^2 \lesssim (m_0 + \Delta_0 \frac{1}{2}) \int_{\mathbb{R}^n_+} u_+^{-1} q \varphi^2 
\]
\[
\lesssim (m_0 + \Delta_0 \frac{1}{2}) u_1^{-\gamma_0} \sup_{-u \leq u \leq u_1} E[\varphi] (H^{\alpha_1}_{u_1}) u_+^{\gamma_0}.
\]

Thus the estimate for III is completed and (7.19) is proved.

**Step 2** We give the estimates of $E_{r_1}$ defined in (6.32) and $E_{r_2}$ defined in (6.33).

Noting that (7.14) implies $r|H| \lesssim m_0 + \Delta_0 \frac{1}{2}$, we can follow the same treatment as in the proof of Proposition 6.6 to obtain
\[
|E_{r_1}| \lesssim (\Delta_0 \frac{1}{2} + m_0) (u_1^{-1} \int_{\mathbb{R}^n_+} r(L(r\varphi) - H_{u_1} L\varphi)^2 dud\omega + E[\varphi] (H^{\alpha_1}_{u_1}) + E[\varphi] (H_{u_1}^{\alpha_1}))
\]
\[
+ \sup_{-u \leq u \leq u_1} \int_{S_{u-u}} r \varphi^2 d\omega,
\]
\[
|E_{r_2}| \lesssim (\Delta_0 \frac{1}{2} m_0 \frac{1}{2} + m_0 \frac{1}{2}) (W_1[\varphi] (H^{\alpha_1}_{u_1}) + E[\varphi] (H^{\alpha_1}_{u_1})) + \sup_{-u \leq u \leq u_1} \|r^{-\frac{1}{4}} \varphi\|_{L^2(S_u-u)}^2.
\]

By substituting these two estimates in to (6.34) and noting that
\[
\int_{\mathbb{R}^n_+} r^3 |H|^2 |L\varphi|^2 dud\omega \lesssim \int_{\mathbb{R}^n_+} r(\Delta_0 \frac{1}{2} + m_0)^2 |L\varphi|^2 dud\omega
\]
\[
\lesssim (\Delta_0 \frac{1}{2} + m_0)^2 m_0^{-1} E[\varphi] (H^{\alpha_1}_{u_1}),
\]
we have
\[
W_1[\varphi] (D_{u_1}^{\alpha_1}) + W_1[\varphi] (H^{\alpha_1}_{u_1}) + W_1[\varphi] (H^{\alpha_1}_{u_1})
\]
\[
\lesssim \|r^{\frac{1}{2}} \partial \varphi + r^{-\frac{1}{4}} \varphi + q_0 r^{\frac{3}{4}} \varphi \|_{L^2(S_{u_1} \omega)}^2 + \int_{\mathbb{R}^n_+} |F(X \varphi + \varphi)| + \frac{q}{2} \varphi^2
\]
\[
+ (m_0 \frac{1}{2} + \Delta_0 \frac{1}{2} m_0 \frac{1}{2}) (E[\varphi] (H^{\alpha_1}_{u_1}) + E[\varphi] (H^{\alpha_1}_{u_1}) + \sup_{-u \leq u \leq u_1} \int_{S_{u-u}} r \varphi^2 d\omega)
\]
\[
+ (\Delta_0 \frac{1}{2} m_0 \frac{1}{2} + m_0 \frac{1}{2}) (u_1^{-1} \int_{\mathbb{R}^n_+} r^{\frac{3}{4}} |\varphi|^2 + \sup_{-u \leq u \leq u_1} E[\varphi] (H^{\alpha_1}_{u_1}) u_+^{-2p})
\]
\[
+ u_1^{-\gamma_0-2p} \sup_{-u \leq u \leq u_1} E[\varphi] (H^{\alpha_1}_{u_1})) + \int_{-u_1}^{u_1} (u_+^{\frac{3}{4}} u_+^{\frac{1}{2}} + u_+^{\frac{3}{4}} - 2p) W_1[\varphi] (H^{\alpha_1}_{u_1}) du,
\]
where $p \leq 0$ is any constant. By repeating the same treatment on (6.39), we can derive the same inequality in Proposition 6.6.

Next we show Proposition 6.4 holds without the assumption of (A2). Indeed, note that due to the wave coordinate condition (7.23), we can derive in view of (7.14), (7.15), (7.16) and (6.16) that
\[
|\text{Tr}[\varphi]| \lesssim (\Delta_0 \frac{1}{2} + m_0) r^{-1} u_+^{-\gamma_0} ((H^{\alpha_1}_{u_1})^2 |\partial \varphi|^2 + |\partial \varphi| |\partial \varphi|).
\]

Substituting the above inequality to (6.11) followed with using Gronwall inequality, we can obtain Proposition 6.3 without the assumption of (A2).

**7.2. Error estimates.** The commutator $[\Box_2, Z^{(n)}] \varphi$ with $\varphi \in \{h^1, \phi\}$ contains borderline terms, which are treated in the following result.
Lemma 7.5. For \( \varphi \in \{ \psi, \phi \} \) and \( n \leq 3 \), there hold for \((u_1, \omega_1) \in I\) that
\[ u_1 + \zeta(\eta^n) + \frac{1}{2} \gamma_0 + \frac{1}{2} \| r \cdot [\partial G, \langle \eta^n \rangle \varphi] \|_{L^2(D_{r})} \lesssim (\Delta_0 m_0^{-\frac{1}{2}} + m_0) u_1^{-\frac{1}{2} \gamma_0 + \frac{1}{2}} + m_0^{\frac{3}{8} - \frac{1}{n} \leq u_1 \leq 1, \eta^0 \leq Z^n \leq Z^0 \leq Z^n \) \]
\[ \lesssim (\Delta_0 m_0^{-\frac{1}{2}} + m_0) u_1^{-\frac{1}{2} \gamma_0} + \Delta_0 \frac{1}{2} m_0^{\frac{1}{2}} u_1^{-\frac{1}{2}}, \] (7.28)
where the last term in (7.27) vanishes if \( Z^n = \partial^n \).

Proof. We first rewrite the terms in (6.22) into
\[ \| [\partial G, \langle \eta^n \rangle \varphi] \|_{L^2(D_{r})} \] (7.29)
where the first term on the right vanishes completely if \( Z^n = \partial^n \).

By using (7.15), we have
\[ \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} |Z^n H| \| \partial^2 Z^n \varphi \| \lesssim \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} u_1^{\zeta(Z^n)} \| |Z^n H| \| \partial^2 Z^n \varphi \|, \] (7.30)
where \( a \geq 1 \) and \( b \leq n - 1 \). Thus
\[ \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} u_1^{\zeta(Z^n)} \| \partial^2 Z^n \varphi \| \lesssim \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} u_1^{\zeta(Z^n)} \| |Z^n H| \| \partial^2 Z^n \varphi \|, \] (7.31)

\[ \lesssim \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} u_1^{\zeta(Z^n)} \| |Z^n H| \| \partial^2 Z^n \varphi \|, \] (7.32)

Since \( h_1 \) verifies (6.5) which is stronger than the estimate (6.6) of \( H \) for (1.1), moreover \( \varphi \in (h_1, \phi) \) verifies all the estimates in Section 3 by repeating the same procedure for proving (6.46) and (6.47) in Proposition 6.9, we have
\[ \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} \| u_1^{\zeta(Z^n)} \| r \cdot [\partial G, \langle \eta^n \rangle \varphi] \|_{L^2(D_{r})} \]
\[ \lesssim \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} u_1^{\zeta(Z^n)} \| |Z^n H| \| \partial^2 Z^n \varphi \|, \] (7.33)

By combining the above two sets of estimates and noting that \(-a \leq \zeta(Z^n)\), the terms contributed by \( h_1 \) in (7.33) decay better since \( 1 < \gamma_0 \leq 2 \). Thus we obtain
\[ \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} \| u_1^{\zeta(Z^n)} \| r \cdot [\partial G, \langle \eta^n \rangle \varphi] \|_{L^2(D_{r})} \]
\[ \lesssim \sum_{Z^n \cup Z^0 \cup Z^0 = Z^n, a \geq 1} u_1^{\zeta(Z^n)} \| |Z^n H| \| \partial^2 Z^n \varphi \|, \] (7.34)

\[ \lesssim (m_0^{-\frac{3}{8}} \Delta_0 + m_0) u_1^{-\gamma_0 + \zeta(Z^n)}. \]
Next we consider the borderline terms in (7.29). By using (7.14), we can derive

\[
\sum_{Z^n \cup Z^b = Z^n, b \leq n - 1} \| u^c(\varphi) H \partial^2 Z^b \varphi \|_{L^2(D^n_H)}
\]

\[
\lesssim \sum_{Z^n \cup Z^b = Z^n, b \leq n - 1} \| u^c(\varphi) H \partial^2 Z^b \varphi \|_{L^2(D^n_H)}
\]

\[
\lesssim u^c(\varphi)(m_0 \frac{b}{u_1 + \frac{1}{2} - b} + \Delta_0 \frac{b}{u_1 - \frac{1}{2} + m_0}) \sup_{-m_0 \leq u \leq u_1, Z^b \subset Z^n} E[\partial Z^b(\varphi)] H_{\partial^2 Z^b} u^c(\varphi) + b + 1.
\]

We then substitute (7.9) into the inequality if the product contains \( \Delta_0 \frac{b}{u_1 - \frac{1}{2} + m_0} \). Combining the results with (7.32) implies (7.27). Similarly,

\[
\sum_{Z^n \cup Z^b = Z^n, b \leq n - 1} \| u^c(\varphi) H \partial^2 Z^b \varphi \|_{L^2(D^n_H)}
\]

\[
\lesssim \sum_{Z^n \cup Z^b = Z^n, b \leq n - 1} \| u^c(\varphi) H \partial^2 Z^b \varphi \|_{L^2(D^n_H)}
\]

\[
\lesssim u^c(\varphi)(m_0 \frac{b}{u_1 + \frac{1}{2} - b} + \Delta_0 \frac{b}{u_1 - \frac{1}{2} + m_0}) \sup_{-m_0 \leq u \leq u_1, Z^b \subset Z^n} E[\partial Z^b(\varphi)] H_{\partial^2 Z^b} u^c(\varphi) + b + 1.
\]

(7.28) then follows with the substitution of (7.9). □

Remark 7.6. If \( Z \in \{ \partial, \Omega_{\mu \nu}, 0 \leq \mu < \nu \leq 3 \} \), the result in this lemma still holds. For the proof, we only need to modify (7.30) by separating the case that \( Z^n = \partial^n \) from the general case.

Lemma 7.7. For \((u_1, u_0) \in I \) and \( n \leq 3 \),

\[
\| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} + \| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} \lesssim u_1^{\alpha + 1} + \zeta(\varphi)(\Delta_0 + m_0^2),
\]

(7.33)

\[
\| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} \lesssim u_1^{\alpha + 1} + \zeta(\varphi)(\Delta_0 + m_0^2),
\]

(7.34)

\[
\| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} \lesssim \Delta_0 u_1^{\gamma_0 + 1},
\]

(7.35)

\[
\| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} \lesssim \Delta_0 u_1^{\gamma_0 + 1},
\]

(7.36)

For \( \varphi \in \mathbf{\partial^n} \),

\[
\| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} + \| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} \lesssim (u_1 + 1 + m_0 + u_1^{\frac{1}{2} - \frac{b}{2}} + \Delta_0^{\frac{1}{2}}) u_1^{\frac{1}{2} + \zeta(\varphi)},
\]

(7.37)

\[
\| r^{\alpha}(\varphi) \|_{L^1_{\partial^n} L^1_\mathbf{\partial^n}(D^n_H)} \lesssim (u_1 + 1 + m_0 + u_1^{\frac{1}{2} - \frac{b}{2}} + \Delta_0^{\frac{1}{2}}) u_1^{\frac{1}{2} + \zeta(\varphi)},
\]

(7.38)

Proof. We first analyse \( S \) which is defined in (7.2), by using (7.11) and (7.14).

\[
\| Z^n(H^{\mu \nu} \partial \mathbf{\partial \hat{h}}) \| \lesssim m_0 \sum_{Z^n \cup Z^d \cup Z^c = Z^n} | Z^{a}(\varphi) h | + m_0 r^{-1} z^{(b)^\gamma} u_+^{\zeta(\varphi)}
\]

\[
\lesssim \sum_{Z^n \cup Z^d \cup Z^c = Z^n} m_0 (| Z^{a}(\varphi) h | + m_0 r^{-1} z^{(b)^\gamma} u_+^{\zeta(\varphi)}),
\]

(7.39)

\[
\| Z^n(G(\varphi) \partial \mathbf{\partial \hat{h}}) \| \lesssim m_0 \sum_{Z^n \cup Z^d \cup Z^c = Z^n} | Z^{a}(\varphi) h | + m_0 r^{-1} z^{(b)^\gamma} u_+^{\zeta(\varphi)}
\]

\[
\lesssim \sum_{Z^n \cup Z^d \cup Z^c = Z^n} m_0 (| Z^{a}(\varphi) h | + m_0 r^{-1} z^{(b)^\gamma} u_+^{\zeta(\varphi)}),
\]

(7.40)
We then combine the above two inequalities and integrate in view of (7.33) and (7.34) for $\mathcal{S}$ defined in (7.32).

Next, by using (7.12), we derive

\begin{align*}
\| r^{\frac{1}{4}} Z^{(c)} \phi \|_{L^2(\mathbb{R}^d_+)} &\lesssim \| r^{\frac{1}{4}} u_{1+}^{\frac{1}{2} + \gamma_0 + \frac{3}{2} \zeta} | Z^{(c)} \phi \|_{L^2(\mathbb{R}^d_+)} \\
&\lesssim \| r^{\frac{1}{4}} u_{1+}^{\frac{1}{2} + \gamma_0 + \frac{3}{2} \zeta} | Z^{(c)} \phi \|_{L^2(\mathbb{R}^d_+)} \\
&\lesssim \Delta_0 u_{1+}^{-\gamma_0 - \frac{1}{2} - \frac{3}{2} \zeta}.
\end{align*}

In view of $|r^{\zeta} Z^{(c)} \phi| \lesssim u_{1+}^{\frac{1}{2} + \gamma_0 + \frac{3}{2} \zeta}$, we can obtain the estimates for the first term on the right of (7.40). Next we consider the other term. Since at least two of the indices $a, b, c$ are $\leq 1$, and for convenience, we denote by $\Psi = (\mathbf{h}^1, \phi)$ and assume $a \leq b \leq c$. By using (3.10) and (3.3), for any $(u, \mathbf{h}) \in \mathcal{I}$ there holds

\begin{align*}
\| r^{-\frac{1}{2}} Z^{(c)} \Psi \|_{L^2(\mathbb{R}^d_+)} &\lesssim u_{1+}^{\frac{3}{2} \zeta} (\mathcal{C}_1 \gamma_0 + \Delta_0 \frac{1}{2}).
\end{align*}
We also note that in the symbolic notation $Z^{(a)}\Psi$, $Z^{(b)}\Psi$, at least one of the functions $\Psi$ is actually $\phi$. In this case, by using (1.22), (7.10) and the above inequality, we have
\[
\sum_{Z=\cup Z^{(a)\cup Z^{(c)}}=Z^n} q_0 \| r Z^{(a)}\Psi \cdot Z^{(b)}\Psi \cdot Z^{(c)}\Psi \|_{L^2(D^n_1)} \leq u_1 Z^{(a)} + \frac{1}{2} \Delta_0^{\frac{1}{2}}.
\]
and
\[
\sum_{Z=\cup Z^{(a)\cup Z^{(c)}}=Z^n} q_0 \| r Z^{(a)}\Psi \cdot Z^{(b)}\Psi \cdot Z^{(c)}\Psi \|_{L_1^1 L_2^\infty L_2^\infty(D^n_1)} \lesssim u_1 (Z^n) + 1 - \frac{3}{2} \gamma_0 \Delta_0^{\frac{1}{2}}.
\]
Thus we completed the proof of (7.35) and (7.36).

At last we prove (7.37) and (7.38). By using (5.17) and (7.12), we derive
\[
|Z^{(n)}(\partial_\Psi \cdot \partial_\phi)| \lesssim \sum_{Z^{(a)\cup Z^{(c)}}=Z^n} m_0 u_1 - 2 + \zeta(Z^{(c)}) |\partial Z^{(c)}\phi|.
\]
For the cubic part $Z^{(a)}(G(h)\partial_\Psi \cdot \partial_\phi)$, we first derive by using (7.17), (7.14) and (7.12)
\[
|Z^{(n)}(G(h)\partial_\Psi \cdot \partial_\phi)| \lesssim \sum_{Z^{(a)\cup Z^{(c)}}=Z^n} m_0 u_1 - 2 + \zeta(Z^{(c)}) (|Z^{(a)}h| + m_0 r - 1 + \zeta(Z^{(c)})) |\partial Z^{(c)}\phi|.
\]
Thus
\[
|Z^{(n)}(\nabla(h)\partial_\Psi \cdot \partial_\phi)| \lesssim \sum_{Z^{(a)\cup Z^{(c)}}=Z^n} m_0 u_1 - 2 + \zeta(Z^{(c)}) (|Z^{(a)}h| + m_0 r - 1 + \zeta(Z^{(c)})) |\partial Z^{(c)}\phi|. \tag{7.41}
\]
We first note that with $0 \leq \alpha < \frac{1}{2}$ by (4.15),
\[
\| r^{-1 + \alpha} \partial Z^{(c)}\phi \|_{L^2(D_1^n)} \lesssim u_1 - \frac{1}{2} - \frac{3}{2r} + \zeta(Z^{(c)}) \Delta_0^{\frac{1}{2}}, \quad c \leq 3. \tag{7.42}
\]
For simplicity, we denote $\| \cdot \|_b$ either the norm $\| \cdot \|_{L_1^1 L_2^\infty L_2^\infty(D^n_1)}$ or $\| \cdot \|_{L_1^1 L_2^\infty L_2^\infty(D^n_1)}$. Thus, by using the above inequality with $\alpha = 0$,
\[
\sum_{Z^{(a)\cup Z^{(c)}}=Z^n} \| r u_1^{(c)} m_0 r^{-2 + \zeta(Z^{(c)})} \partial Z^{(c)}\phi \|_b \lesssim \sum_{Z^{(a)\cup Z^{(c)}}=Z^n} m_0 u_1 \| r^{-1} \partial Z^{(c)}\phi \|_{L^2(D_1^n)} (\| r^{-1} \|_{L_1^1 L_2^\infty(D^n_1)} + \| r^{-1} \|_{L_2^1 L_2^\infty(D^n_1)}) \lesssim u_1^{-1 + \frac{1}{2}} + \zeta(Z^{(c)}) m_0 \Delta_0^{\frac{1}{2}}. \tag{7.43}
\]
With $0 < \alpha < \frac{1}{2}$ in (7.42) and $u_+ \lesssim r$
\[
\sum_{Z^{(a)\cup Z^{(c)}}=Z^n} \| r u_1^{(c)} m_0 r^{-2 + \zeta(Z^{(c)})} \partial Z^{(c)}\phi \|_{L_1^1 L_2^\infty L_2^\infty(D^n_1)} \lesssim m_0 \sum_{Z^{(a)\cup Z^{(c)}}=Z^n} u_1 \| u_1^{-1} r^{-1} \partial Z^{(c)}\phi \|_{L^2(D_1^n)} (\| r^{-1} u_+^{-\alpha} \|_{L_2^1 L_2^\infty(D^n_1)}) \lesssim u_1^{-1 + \frac{1}{2}} + \zeta(Z^{(c)}) m_0 \Delta_0^{\frac{1}{2}}. \tag{7.44}
\]
By the Sobolev embedding on the unit sphere and (7.42), we have for $c \leq 2$ and $0 \leq \alpha < \frac{1}{2}$ that
\[
\|r^\alpha \partial (Z^c) \varphi\|_{L^2(Z) L^2(D^m)} \lesssim \|r^{\alpha - \frac{1}{2} - \frac{2a}{z} + \zeta(Z^c)} \partial (Z^c) \varphi\|_{L^2(D^m)} \lesssim u_1^{\alpha - \frac{1}{2} - \frac{2a}{z} + \zeta(Z^c)} \Delta_0^{\frac{1}{2}}. \tag{7.45}
\]
We can also use (3.31), (3.34) and the bootstrap assumption to obtain for $0 \leq 3$ that
\[
\|r^{\frac{1}{2}} \partial (Z^a) h^1\|_{L^2(S^m_0)} \lesssim u_1^{\frac{2a}{z} + \zeta(Z^a)} \Delta_0^{\frac{1}{2}}, \quad (u, \mu) \in \mathcal{T}. \tag{7.46}
\]
With $0 < \alpha < \frac{1}{2}$, we then employ (7.45) and (7.46) to treat the first term on the right if $c \leq n - 1$
\[
\sum_{Z = Z^c \cup Z^a \cup Z^n = Z} u_1 (\zeta(Z^a)) m_0 \|r^{-\frac{1}{2} + \alpha} u_1^{\frac{1}{2}} \partial (Z^c) \varphi \cdot Z^a) h^1\|_{L^2(D^m)} \lesssim u_1^{\alpha - \gamma \zeta(Z^c)} m_0 \Delta_0.
\]
When $c = n$, by using (7.42) and (4.2) for $h^1$, we derive
\[
m_0 \|r^{\frac{1}{2} + \alpha} u_1^{\frac{1}{2}} \partial (Z^a) \varphi \cdot Z^a) h^1\|_{L^2(D^m)} \lesssim m_0 \|r^{-1 + \alpha} \partial (Z^a) \varphi\|_{L^2(D^m)} \|u_1^{\frac{1}{2}} r^{\frac{1}{2}} h^1\|_{L^2(D^m)} \lesssim u_1^{\alpha - \gamma \zeta(Z^c)} m_0 \Delta_0.
\]
Thus
\[
\sum_{Z = Z^c \cup Z^d \cup Z^n = Z} u_1 (\zeta(Z^a)) m_0 \|r^{-\frac{1}{2} + \alpha} u_1^{\frac{1}{2}} \partial (Z^c) \varphi \cdot Z^a) h^1\|_{L^2(D^m)} \lesssim u_1^{\alpha - \gamma \zeta(Z^n)} m_0 \Delta_0.
\]
Also by using Hölder’s inequality in $D^m_n$,
\[
\sum_{Z = Z^c \cup Z^d \cup Z^n = Z} u_1 (\zeta(Z^a)) m_0 \|r^{-1} \partial (Z^c) \varphi \cdot Z^a) h^1\| \lesssim u_1^{1 - \gamma \zeta(Z^n)} m_0 \Delta_0
\]
and
\[
\sum_{Z = Z^c \cup Z^d \cup Z^n = Z} u_1 (\zeta(Z^a)) m_0 \|r^{-\frac{1}{2}} \partial (Z^c) \varphi \cdot Z^a) h^1\|_{L^2(D^m)} \lesssim u_1^{\gamma \zeta(Z^n)} m_0 \Delta_0.
\]
In view of (7.41), we combine these two estimates with (7.43) and (7.44) to obtain (7.37) and (7.38).

7.3. Boundedness of energy. Note that, according to the notation in Lemma 5.10 for the equation (ES), we can set
\[
\mathcal{F}[h^1] = \mathcal{N}(h) \partial h^1 \partial h^1 + \partial \varphi \cdot \partial \varphi \quad \mathcal{F}[\varphi] = 0.
\]

When consider the energy of $h^1$, we decompose $\mathcal{F}[h^1]$ into two parts,
\[
\mathcal{F}_h^i = [\partial, Z^n] h^1 + \mathcal{F}(h^1) + q_0 Z^n(g_{\mu} \varphi^2); \quad \mathcal{F}_h^i = Z^n S + Z^n (\mathcal{N}(h) \partial \varphi \partial h^1)
\]
and for the energy of the scalar field $\varphi$, we have
\[
\mathcal{F}_\varphi = \mathcal{F}_\varphi^i = [\partial, Z^n] \varphi.
\]
Hence for $h^1$, we can treat $\mathcal{F}_h^i$ by using (7.37) and (7.38). To treat $\mathcal{F}_h^i$, we can apply (6.48) - (6.51) to treat $\mathcal{F}(h^1)$ and treat the remaining terms by using (7.35), (7.36) and Lemma
We also have the bound for $r$ Let $C_66 QIAN WANG$

\[ \| r \mathcal{F}_h \|_{L^1_t L^2_x(D_{\mathbb{R}^1})} + \| r \mathcal{F}_h \|_{L^2_t L^2_x(D_{\mathbb{R}^1})} \lesssim u_1^{-\frac{1}{2} + \varepsilon(Z^n)} (\Delta_0 + m_0^2) \]

\[ \| r \mathcal{F}_h \|_{L^1_t L^2_x(D_{\mathbb{R}^1})} + \| r \mathcal{F}_h \|_{L^2_t L^2_x(D_{\mathbb{R}^1})} \lesssim (\Delta_0 m_0^{-\frac{1}{2}} u_{1+}^{1-\gamma_0 + \varepsilon(Z^n)} + \Delta_0 u_{1+}^{-1}) u_{1+}^{-\varepsilon(Z^n)} + u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} m_0^{-\frac{1}{2}} \Delta_0 m_0^{-\frac{1}{2}} \]

\[ \lesssim (\Delta_0 m_0^{-\frac{1}{2}} + \Delta_0 m_0^{-\frac{1}{2}}) u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} \]  

(7.47)

\[ \| r \mathcal{F}_h \|_{L^1_t L^2_x(D_{\mathbb{R}^1})} \lesssim \Delta_0 m_0^{-\frac{1}{2}} u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} + u_{1+}^{-\gamma_0 - \gamma_0 + \varepsilon(Z^n)} m_0^{-\frac{1}{2}} \]

\[ + u_{1+}^{-\gamma_0 - \gamma_0 + \varepsilon(Z^n)} m_0^{-\frac{1}{2}} \sup_{-u \leq u \leq u_1} E[\partial Z^{(b)} \mathcal{F}_h] (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} + u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} \]

where the last term in the last inequality vanishes if $\varepsilon(Z^n) = -n$.

We use Lemma 7.6 to treat $\mathcal{F}_h$,

\[ \| r \mathcal{F}_h \|_{L^1_t L^2_x(D_{\mathbb{R}^1})} \lesssim u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} (m_0^{\frac{1}{2}} \Delta_0 m_0^{-\frac{1}{2}} + \Delta_0 m_0^{-\frac{1}{2}}) \]

\[ \| r \mathcal{F}_h \|_{L^1_t L^2_x(D_{\mathbb{R}^1})} \lesssim (\Delta_0 m_0^{-\frac{1}{2}} u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} + \Delta_0 u_{1+}^{-1}) u_{1+}^{-\varepsilon(Z^n)} + u_{1+}^{-\gamma_0 - \gamma_0 + \varepsilon(Z^n)} m_0^{-\frac{1}{2}} \]

\[ + u_{1+}^{-\gamma_0 - \gamma_0 + \varepsilon(Z^n)} m_0^{-\frac{1}{2}} \sup_{-u \leq u \leq u_1, Z^n \subset Z^{-n-1} \subset Z^n} E[\partial Z^{(b)} \mathcal{F}_h] (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} + u_{1+}^{-\gamma_0 + \varepsilon(Z^n)} \]

(7.48)

where the last terms in the above inequalities vanish if $\varepsilon(Z^n) = -n$.

By a substitution of (7.47) and (7.48) to the energy inequalities in Proposition 6.4 and Proposition 6.6, we can obtain the boundedness of energies for $\Psi = (h^1, \phi)$ on $\mathcal{I} = \{(u, \omega) : -u_1 \leq -u \leq u \leq u_1\}$

\[ \sup_{u_1 \leq u \leq u_1} u_{1+}^{-2\varepsilon(Z^n) + \gamma_0} (E[Z^n] \Psi) (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) + E[Z^n] \Psi) (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) \]

\[ \lesssim \varepsilon_{n, \gamma_0} (h^1, \phi) + u_{1+}^{-\varepsilon(Z^n) + \gamma_0} (\Delta_0^2 + m_0^2) + u_{1+}^{-1} m_0^{-2} \Delta_0^2 \]

\[ + \sup_{-u \leq u \leq u_1, Z^n \subset Z^{-n-1} \subset Z^n} E[\partial Z^{(b)} \Psi] (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) u_{1+}^{-2\varepsilon(Z^n) + \gamma_0 + 2} \]

(7.49)

where the last term in the above inequality vanishes if $Z^n = \partial^n$. We then run the induction on the signature $\varepsilon(Z^n)$. When $\varepsilon(Z^n) = -n + l$, $1 \leq l \leq n$, we substitute the estimates for $Z^n \Psi$ with $\varepsilon(Z^n) = -n + l - 1$. This procedure allows us to conclude that (7.49) can be bounded by

\[ C_3 (m_0^{\frac{1}{2}} u_{1+}^{-\gamma_0} + u_{1+}^{-1} m_0^{-2} \Delta_0^2 + \varepsilon_{n, \gamma_0} (h^1, \phi)). \]

(7.50)

We also have the bound for $r$-weighted energy,

\[ \sup_{\Delta_0 \leq u \leq u_1} u_{1+}^{-2\varepsilon(Z^n) + \gamma_0} (W_1[Z^n] \Psi) (\mathcal{D}_{m_0^{\frac{1}{2}}}^n) + W_1[Z^n] \Psi) (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) + W_1[Z^n] \Psi) (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) \]

\[ \lesssim u_{1+}^{-1} (\Delta_0 m_0^2 \Delta_0^2 m_0^{-1}) + \varepsilon_{n, \gamma_0} (h^1, \phi) \]

\[ + \sup_{-u \leq u \leq u_1, Z^n \subset Z^{-n-1} \subset Z^n} \{ u_{1+}^{-2\varepsilon(Z^n)} (E[Z^n] \Psi) (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) + E[Z^n] \Psi) (\mathcal{H}_{m_0^{\frac{1}{2}}}^n) \} \]

(7.51)

For the last line, we directly substitute the bound (7.50), thus we conclude that (7.51) is bounded by (7.50) with a larger constant $C_3$.

Since $\varepsilon_{n, \gamma_0} (h^1, \phi) \leq C_0 m_0^{\frac{1}{2}}$ with $\Delta_0 = C_1 m_0^2$ and $C_0, C_1 \geq 1$ we need

\[ C_3(C_0 C_1^{-1} \Delta_0 + u_{1+}^{-\gamma_0 - 2} \Delta_0^2 C_1^{-2} + u_{1+}^{-1} \gamma_0 \Delta_0 C_1) < 2 \Delta_0. \]

Let $C_1 = 4C_0 C_3$ and $\Delta_0 < 1$, due to $u_{1+}^{-\gamma_0 + 1} \leq (\frac{R}{2})^{1-\gamma_0}$, as long as

\[ 2C_3 C_1 (\frac{R}{2})^{1-\gamma_0} < \frac{7}{4}, \]

(7.52)
we can obtain for $n \leq 3$ and $-u_* \leq u \leq u_0$ that
\[
E[Z^{(n)}(h_1, \phi)](|H_+^{n/2}| + E[Z^{(n)}(h_1, \phi)]|H_+^{n/2}| < 2\Delta_0 u_+^{-\gamma_0 + 2\zeta(Z^n)},
\]
\[
W_1[Z^{(n)}(h_1, \phi)]|H_+^{n/2}| + W_1[Z^{(n)}(h_1, \phi)]|D_+^{n/2}| < 2\Delta_0 u_+^{-\gamma_0 + 2\zeta(Z^n)}.
\]
It remains to improve (A1) with $\geq$ replaced by $>$. Due to (7.8), (4.2) and (7.14),
\[
r |H_{\mu\nu}\frac{\partial}{\partial H_{\mu\nu}}| \leq C(\Delta_0 \bar{R}^{-\frac{1}{2}} + r^{-1}(m_0 + \Delta_0 \bar{R})^2).
\]
Similar to Lemma 7.1 with $0 < m_0 < 1$, we choose
\[
C(C_1 (R_0^2)^{1-\eta} + R^{-1}(1 + C_1^2))^2 < \frac{1}{10}
\]
which implies
\[
r(h - H^{\mu\nu}), \quad r(h - H^{\mu\nu}) > \frac{7m_0}{20}
\]
which improves (A1). By choosing the lower bound $R(\gamma_0, C_0)$ of $R$ such that (7.53), (7.52) and $C_1 R^{-\frac{1}{2}} \leq 1$ as requested in Lemma 7.1 hold, the proof of Theorem 2.6 is completed.

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