Time crystal phase in a superconducting ring

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We demonstrate a possible setup to exhibit the spontaneous symmetry breaking of the time translation symmetry. Here we consider a quasi-one-dimensional superconducting ring with a static Zeeman magnetic field applied along the ring and static Aharonov-Bohm magnetic flux penetrating the ring. The superconducting ring with magnetic flux produces a persistent current, whereas the Zeeman split of Fermi energy results in the spatial modulation of the pair potential. We show that these two magnetic fields stabilize the twisted kink crystal (Fulde-Ferrel-Larkin-Ovchinnikov) phase, in which both the phase and amplitude have spatial modulations. In this phase, the time translation symmetry is spontaneously broken.

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Spontaneous symmetry breaking is one of the most important notions in physics. Recently, the spontaneous breaking of the time translation symmetry has been proposed 1,2 (see also Refs. 3, 4). If the time translation symmetry is spontaneously broken, a ground state in which the order parameter enjoys only discrete translation invariance in time is expected to emerge, analogous to a crystal phase which stems from the spontaneous symmetry breaking of spatial translation symmetry. This emergent phase is called a time crystal. In Ref. 1, the superconducting ring with attractive self-interaction threaded by the magnetic flux is considered as a possible realization of the time crystal. The author of Ref. 1 concludes that the pair potential forms the soliton with stationary current in the ground state. An experimental setup for the time crystal has also been discussed 5, where the authors consider ions trapped in a ring threaded by the magnetic flux, with repulsive interaction. In the case of repulsive interaction, if the magnetic flux is zero, the Wigner crystal is formed as the ground state. Thus they conclude that the ground state becomes the Wigner crystal with stationary current 5. The proposals in 1, 5 give rise to a controversy. The papers 5-8 indicate that the time crystal is not realized by the above setups. In those papers it is shown that the ground state for the setup given in Ref. 1 has no stationary current. Moreover, the insensitivity of the Wigner crystal against the Aharanov-Bohm (AB) effect 4 excludes the possibility of time crystal phase in the setup considered in Ref. 5. In Ref. 10, the author of Ref. 1 tried to refute statements in Refs. 5-8, but the question of existence or non-existence of the time crystal does not possess a conclusive answer yet.

In this Letter, we propose an alternative setup which has the time crystal phase as the ground state. We consider a superconducting ring with the AB magnetic flux penetrating the ring and the static Zeeman magnetic field along the ring. The finite AB flux results in the persistent current along the ring, and the resulting order parameter becomes a plane wave like \( \Delta \propto e^{imx} \) with a constant \( m \); we refer to this state as a Fulde-Ferrel (FF) state 11. On the other hand, the Zeeman field along the ring separates the energy spectrum for particles with different spins (Zeeman split). Thus Cooper pairs with nonzero momentum are formed. In the case of Zeeman field, the current is not generated and the resulting order parameter has sine-like shape \( \Delta \propto \text{sn}(x, \nu) \) with the elliptic parameter \( \nu \); we refer to this state as a Larkin-Ovchinnikov (LO) state 12, 13. In the presence of these two magnetic fields, the phase transition between FF state and LO state was reported in Ref. 14. Another group suggested the existence of the half-vortex state in the above setup 15. In this Letter, we demonstrate that a novel phase, the so-called twisted kink (complex kink or grey soliton) crystal, is stabilized, in which both amplitude and phase of the order parameter are spatially modulated. A twisted kink crystal in an infinite system has been proposed in the context of high energy physics 16, but it is not yet observed in condensed matter physics. We refer to this state as Fulde-Ferrel-Larkin-Ovchinnikov (FFLO) state as in the terminology of condensed matter physics. In the FFLO state on a ring, twisted kinks rotate in time in the presence of a superflow, giving rise to a time crystal.

Analytic solution of time crystal- We study a quasi-one-dimensional superconducting ring under a magnetic...
fields by the mean-field Bogoliubov-de Gennes (BdG) equation (Fig. 1) [17]. Although the mean-field approximation is not valid in strictly one dimension, we assume a quasi-one-dimensional system, which is more relevant to experiments and can be well described by the BdG equation for quasiparticles $u(x)$ and $v(x)$ (we adopt the units $\hbar = 1, \epsilon = 1, e = 1$)

$$
\begin{bmatrix}
H_\uparrow & \Delta(x) \\
\Delta^*(x) & -H_\uparrow
\end{bmatrix}
\begin{bmatrix}
u(x) \\
v(x)
\end{bmatrix}
= E
\begin{bmatrix}
u(x) \\
v(x)
\end{bmatrix},
$$

and

$$
\hat{\Delta}(x) = -2g^2 \sum_{E_n < 0} \hat{u}_n(x)\hat{v}_n(x)^*.
$$

The effect of the AB phase appears only as a new boundary condition

$$
\hat{\Delta}(x + L) = e^{2i\phi} \hat{\Delta}(x).
$$

If the attractive interaction is smaller than the Fermi energy $E_F = \mu_\sigma$, fermions near the Fermi surfaces form Cooper pairs. In this case, we may adopt the the Andreev approximation [18]. Let $\hat{u}(x) = e^{ik_F x}\hat{u}_0(x)$ and $\hat{v}(x) = e^{-ik_F x}\hat{v}_0(x)$, where $k_F = \sqrt{2M\varepsilon_F}$, $\hat{u}_0(x)$ and $\hat{v}_0(x)$ vary slowly on the length scale of $1/\kappa_F$. Neglecting the second derivative term of $\hat{u}_0(x)$ and $\hat{v}_0(x)$, the BdG equation reduces to

$$
\begin{bmatrix}
-ivF_\sigma \frac{\partial}{\partial x} & \hat{\Delta}_0^*(x) \\
ivF_\sigma \frac{\partial}{\partial x} & -\hat{\Delta}_0(x)
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0(x) \\
\hat{v}_0(x)
\end{bmatrix}
\approx E
\begin{bmatrix}
\hat{u}_0(x) \\
\hat{v}_0(x)
\end{bmatrix},
$$

where $v_{F\sigma} = k_{F\sigma}/M$ is the Fermi velocity and $\hat{\Delta}_0 = e^{-ik_F x}\Delta$. This approximated BdG equation [8] and the gap equation [4] are used in Ref. 19, 20 except for the boundary condition. Thus the method used there can be applied to the present problem. It is known that the general solution for the gap function $\hat{\Delta}_0(x)$ is [16, 19, 20]

$$
\hat{\Delta}_0(x) = -\alpha A \frac{\sigma(Ax + iK') - i\theta/2}{\sigma(Ax + iK')\sigma(i\theta/2)}
\times \exp \left[ iAx(-i\zeta(i\theta/2) + \text{ns}(i\theta/2)) + i\theta\eta_3/2 \right],
$$

where $\alpha$, $\zeta$, and $\eta_3 = 1/3n$ are, respectively, the Weierstrass sigma, zeta functions and Jacobi elliptic functions, characterized by the elliptic parameter $\nu$ and the half-periods $\omega_1$ and $\omega_3$ for real and imaginary direction, respectively. We set the half-periods to $\omega_1 = K$ and $\omega_3 = iK'$, with $K(\nu) = \int_0^{\pi/2} dt(1 - \nu \sin^2 t)^{-1/2}$ and $K' \equiv K(1 - \nu)$. The constant $\eta_3$ is defined by $\zeta(iK')$. The parameter $A$ represents the scale of the condensate as $A = -2i\text{sc}(i\theta/4)\text{nd}(i\theta/4)$. Here $\text{sc} = \text{sn}/cn$ and $\text{nd} = 1/\text{dn}$ are Jacobi elliptic functions, and $m$, $\theta$ are related to the amplitude and the phase modulation, respectively. In addition, we have introduced the imbalance parameter as $\alpha = \sqrt{v_{F\uparrow}v_{F\downarrow}}/v_F$ ($0 \leq \alpha \leq 1$) with $v_F = (v_{F\uparrow} + v_{F\downarrow})/2$. In this solution, the amplitude is periodic and the phase wins a certain angle over each period [10]. Furthermore, it is known that this solution includes all previously known solutions as special cases, such as the constant condensation, the FF state, the LO state, the complex (twisted) kink [21], and the real kink. We again note that the effect of AB flux enters via the uniform phase modulation $\Delta(x) = \exp(2i\phi x)\exp[i(k_{F\uparrow} + k_{F\downarrow})x]\Delta_0(x)$. This solution obviously belongs to the time crystal phase. We plot

FIG. 1: Schematic picture of our setup. The superconducting ring is penetrated by the magnetic flux $\Phi$ and another magnetic field is applied along the ring. One edge of the strip represents the condensate with varying amplitude and winding complex phase.
Energy in our system for small finite temperature. In our numerical calculations, we discretize Eq. (1) and solve it self-consistently together with Eq. (3). The discretized BdG equation becomes

$$\sum_j \left[ H_{i,j,\sigma} \Delta_i \delta_{i,j} - H_{i,j,\sigma}^* \right] \begin{bmatrix} u^\nu_{i\sigma} \\ v^\nu_{i\sigma} \end{bmatrix} = E^\nu_{i\sigma} \begin{bmatrix} u^\nu_{i\sigma} \\ v^\nu_{i\sigma} \end{bmatrix},$$  

(10)

where \( H_{i,j,\sigma} = -t_{i,j} - \mu \delta_{i,j} + \sigma \hbar \delta_{i,j} \) and \( i, \sigma, \nu \) label the site, the spin of the particle, and eigenenergy, respectively. We treat the effect of the AB flux penetrating the ring by using the Peierls phase, \( t_{i,i+1} = t \exp(i\phi/N) \), \( t_{i,i+1} = t \exp(-i\phi/N) \) with transfer integral \( t \) and the total site number \( N \), we only consider the nearest neighbor hopping. The gap equation is almost identical to Eq. (3), except that we consider the finite temperature \( T \) here.

$$\Delta_i = \frac{\alpha^2}{2T} \sum_{\nu=1}^{2N} u^\nu_{i\uparrow} v^\nu_{i\downarrow} \exp(E^\nu_{i\sigma}/T).$$  

(11)

The iterative calculations of Eqs. (10) and (11) yields the order parameters, eigenspinors, and eigenenergies. In order to find which is the ground state, we calculate and compare the total free energy

$$F = -T \sum_{\nu} \ln(1 + e^{-E^\nu_{i\sigma}/T}) + \sum_i \frac{|\Delta_i|^2}{2g^2} - \sum_i (\mu + h).$$  

(12)

We set the chemical potential, the attractive potential, and the temperature to be \( \mu = -0.5, g = 1.0, T = 0.005 \), respectively in the unit of the transfer integral. Here the temperature is chosen to be much smaller than the critical temperature. The size of the ring is \( N = 50 \), which is supposed to be sufficiently large to reach the thermodynamic limit [14]. The corresponding model has been already used in Refs. [14] and [15]. In Ref. [14], the phase transition between LO phase and FF phase was discussed. The existence of an additional phase called half-vortex phase was mentioned in Ref. [15]. The order parameter of the half-vortex phase proposed in Ref. [15] has the form \( \Delta \propto \cos(m\pi x/L) \times \exp(i\pi n x/L) \), with half-integers \( m, n \). We show that this competing phase between the LO and FF phases is not the half-vortex phase but the FFLO phase. We plot typical profiles of FF and LO phases in Fig. 3 and an FFLO phase in Fig. 4. In the case of FFLO phase, both the amplitude and the phase of the order parameter have spatial modulation. Moreover, the amplitude does not vanish in the whole region. We also show the phase diagram in Fig. 5. The shaded circles should be categorized as time crystals for continuous limit. The discretization of the original model brings the difficulty to categorize the phase in a certain region (white circles and gray circles). We refer the other parts of those phases as LO* , which should be categorized in the LO phase. We can see that the LO
phase and FF phase are realized for $\phi = 0$ and $h > 0.2$, and for $h = 0$ and $\phi > \pi/2$ (or equivalently $2\phi > \pi$), respectively. These results verify the naive discussion made above; the order parameter tends to rotate if the AB flux penetrates the ring, whereas it tends to have a spatial modulation in the presence of Zeeman field. The most remarkable result is that the time crystal phase appears in a wide range of the parameters.

Reference [9] suggests a no-go theorem excluding the time crystal state for the setup in Ref. [1]. On the other hand, we show that the FFLO state can be the lowest energy state. This contradiction is explained as follows. In Ref. [9], the free energy is assumed to be differentiable as a function of the angular momentum. For the setup considered in Ref. [1], these conditions are satisfied, where the order parameter in the presence of the flux is given as a sinusoidal function multiplied by the phase factor $e^{i\phi x}$. In the Ref. [9], the author concludes that the energy for rotating condensates is always higher than that without rotation from the following perturbative analysis:

$$E_{\phi,n}^{(i)} - E_{\phi,n}^{(0)} = I_{\phi,n}\Omega^2/2 + O(\Omega^2),$$

where

$$I_{\phi,n} = 2\sum_{m\neq n} \left(\left|\langle \phi_{\phi,n}^{0} | \hat{L}_z | \phi_{\phi,m}^{0}\rangle\right|^2\right) / (E_{\phi,n}^{(0)} - E_{\phi,n}^{(0)}).$$

Here $E_{\phi,n}^{(i)}$ and $E_{\phi,n}^{(0)}$ are the eigenenergies in the rotating frame and that in the static frame, respectively. The state $|\phi_{\phi,n}^{0}\rangle$ is the eigenstate corresponding to $E_{\phi,n}^{(0)}$, and $L_z$ is the $z$-component of an angular momentum operator. It should be noted that those equations are only valid for the case without a phase transition. In our setup, however, our
numerical calculation suggests that there is a first order phase transition between the FFLO and LO phases. Our results can also answer a question in Ref. [7]; the amplitude of the wave function near the antipode of the soliton is exponentially small and thus the sensitivity to the AB flux should also be exponentially small. For our solution (9), there is no antipode. Moreover, our results are consistent with the physical intuition that the rotating motion always raise the energy. In our case, the energy loss due to the rotation of the condensate is compensated by the softening of the spatial modulation of the condensate.

While we exhausted the phase diagram with the number of sites equal to 50, we have also confirmed that the configurations remain qualitatively the same even if we increase the number of sites to be 59, 73, and 100, for some choices of the parameters.

**Conclusion**- In conclusion, we have demonstrated that the FFLO phase can be realized as the ground state of a superconducting ring threaded by the AB flux and with the Zeeman field along the ring, which provides an example of time crystals. As for the observability of the time crystal, it was noted in Refs. [22, 23] that in systems with off-diagonal long-range order a small amount of dissipation, namely the non-conservation of a certain charge over time scales much larger than the energy relaxation time, should be necessary. In Refs. [24, 25], exact self-consistent solutions were found in quasi-one dimension, in which twisted kinks with arbitrary phase shifts are separated at arbitrary distances. A ring version of this case may be stabilized by non-uniform magnetic fields.

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