Soliton induced critical current oscillations in two-band superconducting bridges

P.M. Marychev and D.Yu. Vodolazov

Institute for Physics of Microstructures, Russian Academy of Sciences, Nizhny Novgorod, 603950 Russia

(Dated: September 18, 2018)

Using time-dependent Ginzburg-Landau theory we find oscillations of critical current density $j_c$ as a function of the length $L$ of the bridge formed from two-band superconductor. We explain this effect by appearance of the phase solitons in the bridge at $j < j_c$, those number changes with change of $L$. In case of sufficiently strong interband coupling oscillations of $j_c$ disappear.

I. INTRODUCTION

The discovery of two-band superconductivity in MgB$_2$ has caused an explosion of research on physical properties of such materials. Later the evidence of two and more superconducting gaps has been observed in many materials, e.g. in OsB$_2$, NbSe$_2$, LiFeAs, FeSe$_{0.9}$, and other iron-based superconductors. Theoretically superconductivity in each band can be described by complex order parameter $\Psi_k = \Delta_k \exp(i\theta_k)$, where $\Delta_k$ and $\theta_k$ are superconducting amplitude and phase in k-th band respectively. In the case of negligible interband impurity scattering, interband interaction can be described by Josephson coupling $\gamma_{ij} \Delta_i \Delta_j \cos(\theta_i - \theta_j)$ in Ginzburg-Landau free energy functional. Interband phase difference $\theta_i - \theta_j$ is determined by the sign of Josephson coupling constant $\gamma_{ij}$ and can be either $0$ or $\pi$ (so called phase-locked state). However, in addition to the phase-locked states, the existence of phase solitons is possible, first discussed by Tanaka.\cite{11} In this states the distribution $\theta_i - \theta_j$ is spatially inhomogeneous and rotates by $2\pi$. Such objects have been observed in artificial multiband structure consisting of mesoscopic aluminium rings\cite{21} and also there is the evidence of their existence, obtained in the experiment with a cuprate film\cite{22}.

The phase solitons generally are not present in the ground state and various methods to create them have been proposed. Gurevich and Vinokur\cite{11} used to excite phase solitons in two-band superconductors by an electric field, where the boundary between superconducting wire and normal lead was the solitons source. It is also possible to create phase solitons at equilibrium when the supercurrent in the weak band (band with smaller superconducting amplitude) exceeds the critical value. It results in phase slippages in weak band, which in turn generate the chain of phase solitons.\cite{15}

In our work in the framework of one-dimensional time-dependent Ginzburg-Landau theory we investigate influence of interband breakdown (transition from phase-locked to soliton state) of critical current density $j_c$, at which stationary superconducting state ceases to exist in a two-band superconducting bridge.\cite{15} Unlike the authors of work\cite{15} we consider influence of strength of Josephson coupling $\gamma$ and bridge length on $j_c$. This breakdown is specific to systems with weak or absent interband Josephson coupling and arises when supervelociry exceeds critical supervelocity $q_c \Delta$ in the weak band. In the case zero and small $\gamma$ for length $L >> \xi_1$ ($\xi_1$ – is the coherence length in the strong band), there are oscillations of critical current as a function of the bridge length, related to the change of number of phase solitons in the system. Also there is a range of parameters, in which the dependence $j_c(L)$ has minimum, corresponding to transition from the phase soliton to the phase-locked state. If interband coupling is sufficiently strong, phase soliton generation is suppressed and above-mentioned features are absent.

II. METHOD

To study the current states and find the critical current of superconducting bridge, we numerically solve one-dimensional time-dependent Ginzburg-Landau equation (TDGL)\cite{15} generalized for a two-band superconductor\cite{2} and coupled with the equation for total current density $j$ in the bridge. In the dimensionless form these equations are written as

\[ u_1 \left( \frac{\partial}{\partial t} + i\varphi \right) \Psi_1 - \Psi_1 + |\Psi_1|^2 \Psi_1 - \frac{\partial^2 \Psi_1}{\partial x^2} - \gamma \Psi_2 = 0, \]

\[ u_2 \left( \frac{\partial}{\partial t} + i\varphi \right) \Psi_2 - \alpha \Psi_2 + \beta |\Psi_2|^2 \Psi_2 - g \frac{\partial^2 \Psi_2}{\partial x^2} - \gamma \Psi_1 = 0, \]

\[ j = -(1 + \sigma) \frac{\partial \varphi}{\partial x} + Im \left( \Psi_1 \frac{\partial \Psi_1}{\partial x} + g \frac{\Omega \Psi_2}{\partial x} \right) , \]

where $\alpha = |\alpha_2|/|\alpha_1|$, $\beta = \beta_2/\beta_1$, $g = g_2/g_1$ and $\sigma = \sigma_n^2/\sigma_n^1$. Here $\varphi$ is the electric potential, $\sigma_n$ is
normal conductivity in k-th band. \( \alpha_k, \beta_k \) and \( g_k \) are the GL expansion coefficients, \( \gamma \) is the interband Josephson coupling constant. \( \alpha_k, \beta_k, g_k \) and \( \gamma \) was derived within the BCS theory through microscopic parameters in Refs. 4 and 13 for clean and dirty superconductors respectively. Coordinate \( x \) is measured in units of the coherence length in the first (strong) band \( \xi_1 = \sqrt{g_1 \hbar^2/|\alpha_1|} \), order parameters are measured in units of \( |\Psi_{10}| = |\alpha_1|/\beta_1 \), time is measured in units of the current relaxation time in the first band \( t_1 = \beta_1 |\sigma_n|/8e^2 g_1 |\alpha_1| \). Josephson coupling constant \( \gamma \) is in units of \( |\alpha_1| \), and the current density is measured in units of \( j_0 = 4e \hbar g_1 |\Psi_{10}|^2/\xi_1 \). In the phase-locked state interband phase difference \( \theta = \theta_1 - \theta_2 \) is determined by the sign of interband coupling constant \( \gamma \): for \( \gamma > 0 \) phase difference \( \theta = 0 \) and \( \theta = \pi \) for \( \gamma < 0 \). Further we consider case \( \gamma > 0 \).

Parameters \( u_k \) in Eqs. 1, 2 describe the ratio of relaxation times of the order parameter in the corresponding band to the current relaxation time in the first band. In the case of gapless superconductors parameter \( u_{1,2} = 5.79 \). For finite gap superconductors one still can use Eqs. 1, 2 but with larger value of \( u_k \) to take into account relatively large relaxation time of \( |\Psi_k| \) due to inelastic electron-phonon scattering. As we show later large \( u_k \) could lead to smaller value of the critical current but our main result - oscillations of the critical current with changing length of the bridge does not depend on specific value of \( u_k \). Therefore in calculations we use as small \( u_k < 1 \) (providing largest value of \( j_c \)) as large \( u_k >> 1 \). For simplicity and in order to not multiply the value of independent parameters we choose \( u_1 = u_2 \).

Equations 1, 2, 3 are solved for bridges with superconducting and normal leads. We assume that the cross section of superconducting leads is much larger than the cross section of the bridge. Then the equations 1, 2, 3 are solved with the boundary condition

\[
q^{pl} = 2 \sqrt{1 + \alpha} \cos \left[ \frac{1}{3} \arccos \left( \frac{j}{\sqrt{6}} \frac{6 + (1 + \alpha)^{3/2}}{1 + \alpha} \right) - \frac{2\pi}{3} \right],
\]

\[
q_c^{pl} = \sqrt{\frac{1 + \alpha}{6}},
\]

\[
j_c^{pl} = \frac{2}{3} \sqrt{\frac{(1 + \alpha)^3}{6}}.
\]

In the case of different order parameters, i.e. \( \alpha < 1 \), the critical supervelocity \( q_c^{pl} \) is larger than the critical supervelocity in the second band \( q_{c2}^{pl} = \sqrt{\alpha/3} \). Then, at \( q^{pl} > q_{c2}^{pl} \) superconductivity is destroyed in the weak

\[
\Psi_k(0, t + \Delta t) = \Psi_k(0, t) = |\Psi_k^0|,
\]

\[
\Psi_k(L, t + \Delta t) = \Psi_k(L, t) e^{-i\varphi_L \Delta t},
\]

where \( \Psi_k^0 \) are bulk order parameters at zero current, \( L \) is the bridge length, \( \varphi_L \) is the electric potential at the point \( x = L \) and \( \Delta t \) is the time step. To find electric potential \( \varphi \) the equation 3 is solved for each time step with given total current and the boundary condition \( \varphi(0) = 0 \). We use following criterion for reaching the stationary state: \( \varphi(L) - \varphi(0) = 0 \).

In the case of normal leads we use boundary conditions

\[
\Psi_k(0, t) = \Psi_k(L, t) = 0.
\]

with following criterion for the stationary state: \( \partial |\Psi_k|/\partial t = 0 \).

III. CURRENT STATES IN TWO-BAND SUPERCONDUCTOR WITHOUT INTERBAND JOSEPHSON COUPLING

First we consider the case of zero interband Josephson coupling \( \gamma = 0 \). Such two-band system corresponds to either liquid metallic hydrogen15 or superconductor with negligible interband coupling \( \gamma << \alpha \). In the ground state interband phase difference \( \theta = 0 \) which is ensured by boundary conditions and the absence of electric field in the bridge. For long bridge Eqs. 1, 2, 3 can be written in the stationary case as

\[
|\Psi_1|((|q^{pl}|^2 - 1) + |\Psi_1|^3) = 0,
\]

\[
|\Psi_2|((|q^{pl}|^2 - \alpha) + \beta|\Psi_2|^3) = 0,
\]

\[
j = q^{pl} (|\Psi_1|^2 + g|\Psi_2|^2),
\]

where \( q^{pl} = \partial \theta_1/\partial x = \partial \theta_2/\partial x \) is supervelocity in the phase-locked state. For simplicity we assume \( \beta = g = 1 \). Using these equations, one can obtain expressions for the supervelocity, critical supervelocity and supercurrent in the phase-locked state, when supervelocities in both bands are the same.
band, and maximum supercurrent in phase-locked state is limited by \( j(q_{c2}) \). Arising phase slip process and electric field redistributes supervelocities, which in turn causes current redistribution between bands. As a result, the system goes into stationary, uniform state with linear interband phase difference \( \theta = 2\pi m x/L \) (soliton-like state with size of soliton core equal to length of the bridge), where \( m \) is an integer number which is equal to difference between number of phase slips in the strong and weak bands (it is equal to number of solitons in the bridge). If we choose \( \theta(x = 0) = 0 \) then at \( x = L \) interband phase difference \( \theta = 2\pi m \), i.e. the phase-locked state in the leads is preserved. In this soliton state the order parameters are defined by usual formulas \( |\Psi_1| = \sqrt{1 - q_1^2}, |\Psi_2| = \sqrt{\alpha - q_2^2} \), and supervelocities \( q_{1,2} \) with supercurrent are determined by following expressions

\[
q_{ps}^1 = 2\sqrt{\frac{1 + \alpha}{6} - \frac{\pi^2 m^2}{L^2}} \cos \left( \frac{1}{3} \arccos \left( -\frac{\left( \frac{\pi m}{L} (\alpha - 1) + j \right) \left( \frac{1 + \alpha}{6} - \frac{\pi^2 m^2}{L^2} \right)^{-3/2}}{4} \right) \right) - \frac{2\pi}{3} + \frac{\pi m}{L}, \tag{12}
\]
\[
q_{ps}^2 = 2\sqrt{\frac{1 + \alpha}{6} - \frac{\pi^2 m^2}{L^2}} \cos \left( \frac{1}{3} \arccos \left( -\frac{\left( \frac{\pi m}{L} (\alpha - 1) + j \right) \left( \frac{1 + \alpha}{6} - \frac{\pi^2 m^2}{L^2} \right)^{-3/2}}{4} \right) \right) - \frac{2\pi}{3} - \frac{\pi m}{L}, \tag{13}
\]
\[
q_{c1}^m = \sqrt{\frac{1 + \alpha}{6} - \frac{\pi^2 m^2}{L^2}} + \frac{\pi m}{L}, \quad q_{c2}^m = \sqrt{\frac{1 + \alpha}{6} - \frac{\pi^2 m^2}{L^2}} - \frac{\pi m}{L}, \quad j_{c}^m = q_{c1}^m (1 - (q_{c1}^m)^2) + q_{c2}^m (\alpha - (q_{c2}^m)^2). \tag{14}
\]

The value of \( m \) is determined by conditions \( q_{ps}^1 < q_{c1}, q_{ps}^2 < q_{c2} \) in the stationary state and length of the bridge because each phase slip changes the phase difference in each band along the bridge by \( 2\pi \) leading to corresponding change of supervelocities \( q_{c1,2} \) by \( \sim \pm 2\pi / L \). Since the expressions \( 13 \) depend on the bridge length too, we can expect oscillations of critical current with increasing the length of the bridge, which corresponds to change in the number of phase slips and phase solitons.

Fig. 1 shows the comparison of numerically calculated critical current \( j_c \) with analytical expression \( j_{c}^m(q_{c1}^m, q_{c2}^m) \) (see Eq. (14)). We choose the ratio of critical currents of different bands (superconductors) \( R = j_{c2}/j_{c1} \) as a parameter reflecting experimentally observed difference between the band characteristics (or, in the case of artificial structure, characteristics of superconductors). Minimums of the dependence \( j_c(L) \) correspond to crossover between regions with different number of solitons.

At \( m = 1 \) critical current \( j_c \) decreases with decreasing length \( L \), according to formulas \( 13 \) \( 14 \). For relatively small bridge length \( L \leq 10\xi_1 \) supervelocity redistribution caused by phase slips is too large to keep the system in stationary state and up to critical current the bridge stays in the phase-locked state. For such a small lengths \( j_c \sim 1 / L \) as in single band superconducting bridge.20 Therefore, the dependence \( j_c(L) \) has a minimum at \( L \sim 10 - 20\xi_1 \), which corresponds to the crossover between the phase-locked state (small lengths) and one soliton state (large lengths). Because position of the minima is associated with value \( j_{c}^m \), it is very sensitive to values of \( \alpha \) and \( j_{c2} \), see Eq. \( 11 \). This feature is confirmed by numerical calculations (see Fig. 2).

We also did calculations for bridge with normal leads. In this case oscillations \( j_c(L) \) and the minimum at \( L \sim 10 - 20\xi_1 \) are absent. We explain it by absence of rigid boundary conditions, i.e. there is no fixed interband phase difference \( \theta(L) = 2\pi m \).
with increasing of $R$ we expect oscillations of $j_{\text{super current}}$ on because each phase slip decreases the supervelocity and phase slips depends on the radius of the ring (its length), vorticity $N$. Hence density in the upper arc exceeds current of superconducting ring formed from two arcs.

IV. CASE OF FINITE INTERBAND COUPLING

Let us consider now how finite interband Josephson coupling influences the critical current of superconducting bridge. As it is shown in Ref. [10], presence of weak Josephson coupling leads to appearance of the phase solitons at $q^B > q_c$, Additionally, in the work [21], it is shown that phase solitons are stable only for small interband coupling constant $\gamma$. Therefore one could expect that the phase solitons can affect the critical current only for small $\gamma < \alpha_1, \alpha_2$.

To find the critical current $j_c$, we numerically solve Eqs. [1—3] for different bridge lengths, different parameters $\gamma$ and we mainly use $R = 0.15$. The results for $\gamma = 0.005$ and $\gamma = 0.1$ are shown in Fig. 4. When $\gamma$ is small the phase-locked state is destroyed in a way similar to the case $\gamma = 0$. The only difference is that the presence of finite $\gamma$ gives rise to nonlinear dependence $\theta(x)$ along the bridge which also lead to small variations of $|\Psi_k(x)|$ (see Fig. 5). As one can see from inset to Fig. 5 oscillations of critical current density $j_c(L)$ corresponds to changing the number of phase solitons $m$ as length of the bridge changes. Crossover between regions with different $m$ are responsible for the minimums of $j_c(L)$.

This behavior is similar to dependence of the critical current of superconducting ring formed from two arcs with different critical currents (see Fig. 5). When current density in the upper arc exceeds $j_{c1}$ the phase slip process starts in this arc and the current density redistributes, leading to $j < j_{c1}$ in the upper arc and nonzero vorticity $N = \oint \varphi ds/2\pi$ in the ring. The number of phase slips depends on the radius of the ring (its length), because each phase slip decreases the supervelocity and supercurrent on $\sim 1/R \sim 1/L$. In this system one could expect oscillations of $j_c$ as function of radius, because with increasing of $R$ vorticity increases discretely. So the vorticity in such a ring is the analog of phase soliton in the two-band superconducting bridge.

The amplitude of oscillations of $j_c(L)$ decreases with increasing constant $\gamma$ and they completely disappear for sufficiently strong interband coupling (Fig. 5). This occurs due to the suppression of the independent phase slippage in different bands and absence of soliton states. But for not very large $\gamma$ finite $\theta(x)$ still exists at the current just below $j_c$ which originates from the nonuniform distribution of the order parameter along the bridge. This distribution $\theta(x)$ (so called "phase texture") has small

FIG. 2. The critical current density $j_c(L)$ in the absence of interband Josephson coupling at different values of parameter $R$ and $u_1 = u_2 = 0.2$. Solid curve corresponds to the sum of intraband critical currents $j_{c1} + j_{c2}$.

FIG. 3. The critical current density $j_c(L)$ for the interband Josephson coupling $\gamma = 0.005$ (see inset) and $\gamma = 0.1$ with $R = 0.15$. Calculations are done for the value of parameter $u = u_1 = u_2 = 0.2$. Solid curve is Eq. (14).

FIG. 4. The amplitude of second gap $|\Psi_2|$ (black triangles) and interband phase difference $\theta$ (white triangles) for the interband Josephson coupling $\gamma = 0.01$, the parameter $u = u_1 = u_2 = 0.5$, $R = 0.15$ and the current densities $j = 0.37j_0$ (down triangles) and $j = 0.4j_0$ (up triangles). For illustrative purposes the interband phase difference for the current density $j = 0.37j_0$ is multiplied by five and the interband phase difference for the current density $j = 0.4j_0$ is shifted down by $2\pi$. 

FIG. 5. Diagrams for the phase-locked state $\theta(x)$ at $\gamma = 0$. The solid curve corresponds to the sum of intraband critical currents $j_{c1} + j_{c2}$.
amplitude and does not create finite interband phase difference between the leads. Similar interband phase distribution also exists for \( \gamma \neq 0 \) below the crossover to the soliton state (see Fig. 4).

Dependence of \( j_c(\gamma) \) is shown for the bridge with \( L = 60\xi_1 \) in Fig. 6. At small \( \gamma \) critical current decreases with increasing \( \gamma \) because order parameter is stronger suppressed in both bands in place of location of phase solitons\(^{21}\) (see inset in Fig. 4), which makes superconducting stationary state unstable at smaller current. The minimum of dependence \( j_c(\gamma) \) is reached at \( \gamma = 0.022 \) (for chosen parameters) which corresponds to transition from phase soliton state to the phase texture state. With further increase of \( \gamma \) critical current increases because the weaker band is strengthened. In Fig. 6 we also plot critical current \( j^{pl}_c \) which is found using the assumption that the transition to the resistive state occurs from soliton state (see phase-slippage process). Therefore we calculate \( j^{pl}_c \) for sufficiently large \( \gamma \) and parameters \( u = u_1 = u_2 = 0.2 \) and \( R = 0.15 \) (squares). Obtained results are compared with calculated critical current \( j^{pl}_c \) in the phase-locked state (triangles).

We also consider two-band bridge with normal leads. Despite the presence of phase solitons, the critical current \( j_c \) is practically unaffected by the bridge length, similarly to the case of zero interband Josephson coupling.

**V. INFLUENCE OF LARGE RELAXATION TIME OF \( |\Psi_k| \) ON CRITICAL CURRENT OSCILLATIONS**

In our previous calculations we use small value \( u_k \). In the case of superconductors with strong inelastic electron-phonon scattering \( u_k \gg 1 \) (Ref.\(^{17}\)) which affects the phase-slippage process. Therefore we calculate \( j_c \) for different lengths, two values of parameter \( u = 5.79 \) and \( u = 50 \) and compare results with case of small \( u \) - see Fig. 7. In the absence of interband Josephson coupling we find that the critical current depends on parameter \( u \) for sufficiently large lengths, when transition to resistive state occurs from soliton state. The found effect is connected with large relaxation time of magnitude of the order parameter which is proportional to \( u \). For large \( u \) magnitude of the order parameter \( |\Psi_{1,2}| \) does not have time to recover after phase slip event and it favors transition to the resistive state instead of transition to the soliton state with larger \( m \). At finite interband Josephson coupling similar effect of large \( u \) also can be observed, except that the difference starts at larger \( L \) and \( m \).

**VI. CONCLUSION**

In the framework of time-dependent Ginzburg-Landau theory we study critical current of quasi-one-dimensional bridge formed from two-band superconductor. We found oscillatory dependence of \( j_c \) on the length of the bridge.
which is explained by formation of phase solitons at $j < j_c$ for relatively long bridges. This effect is noticeable for relatively small ratio between critical currents of weak and strong bands (which they would have in absence of Josephson coupling) and relatively weak Josephson coupling $\gamma$ between bands.

Most suitable for observation of predicted effect is FeSe$_{0.94}$ ($\gamma = 0.01$, $T_c = 8.3$ K, $T_{c2} = 3.1$ K\footnote{H. J. Choi, D. Roundy, H. Sun, M. L. Cohen and S. G. Louie, The origin of the anomalous superconducting properties of MgB$_2$, Nature, 418, 758 (2002).}) or artificial structure, which is consisted from two Josephson coupled superconducting layers where large difference in critical currents could be realized via creation of artificial defects in the layers. Nonmonotonous dependence $j_c(L)$ could be seen either via variation of the length of the bridge or, at fixed length, via variation of the temperature which leads to change of ratio $L / \xi_1(T)$ (we expect in this case some kind of kinks on dependence $j_c(T)$).

**ACKNOWLEDGMENTS**

D.Yu. V. acknowledges support from the Russian Scientific Foundation for Basic Research, grant No 15-42-02365, and P.M. M. by grant No. 15-41-02519.