Qubit Entanglement Breaking Channels

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Abstract

This paper continues the study of stochastic maps, or channels, for which $(I \otimes \Phi)(\Gamma)$ is always separable in the case of qubits. We give a detailed description of entanglement-breaking qubit channels, and show that such maps are precisely the convex hull of those known as classical-quantum channels. We also review the complete positivity conditions in a canonical parameterization and show how they lead to entanglement-breaking conditions.

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1 Introduction

The preceding paper [11] studied the class of stochastic maps which break entanglement. For a given map $\Phi$ this means that $I \otimes \Phi(\Gamma)$ is separable for any density matrix $\Gamma$ on a tensor product space. It was observed that a map is entanglement breaking if and only if it can be written in either of the following equivalent forms

$$\Phi(\rho) = \sum_k R_k \text{Tr} F_k \rho$$

$$= \sum_k |\psi_k\rangle \langle \psi_k| \langle \phi_k, \rho \phi_k|$$

where each $R_k$ is a density matrix and $F_k$ a positive semi-definite operator. The map $\Phi$ is also trace-preserving if and only if $\sum_k F_k = \sum_k |\phi_k\rangle \langle \phi_k| = I$, in which case the set $\{F_k\}$ form a POVM. Henceforth we will only consider trace-preserving CPT for those which are also completely positive and EBT for those which are also entanglement breaking. An EBT map is called classical-quantum (CQ) if each $F_k = |k\rangle \langle k|$ is a one-dimensional projection; it is quantum-classical (QC) if each density matrix $R_k = |k\rangle \langle k|$ is a one-dimensional projection.

Maps which break entanglement can always be simulated using a classical channel; thus, one is primarily interested in those which preserve entanglement. Nevertheless, it is important to understand the distinction. In this paper we restrict attention to EBT maps on qubits, for which one can obtain a number of results which do not hold for general EBT maps. The main new result, which does not hold in higher dimensions, is that every qubit EBT map can be written as a convex combination of maps in the subclass of CQ maps defined above.

Before proving this result in Section 6, we review parameterizations and complete positivity conditions for qubit maps. We also give a number of more specialized results which use the canonical parameterization and/or the fact that positivity of the partial transpose suffices to test entanglement for states on pairs of qubits.

Recall that any CPT map $\Phi$ on qubits can be represented by a matrix in the canonical basis of $\{I, \sigma_1, \sigma_2, \sigma_3\}$. When $\rho = \frac{1}{2}[I + v \cdot \sigma]$, then $\Phi(\rho) = \frac{1}{2}[I + (t + Tv) \cdot \sigma]$ where $t$ is the vector with elements $t_k = t_{0k}$, $k = 1, 2, 3$ and $T$ is a $3 \times 3$ matrix, i.e., $T = \begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix}$. Moreover, it was shown in [14] that we can assume without loss of generality (i.e., after suitable change of bases) that $T$ is diagonal so that $T$
has the canonical form

\[ T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0
\end{pmatrix}. \tag{3} \]

The conditions for complete positivity in this representation were obtained in [16] and are summarized in Section 4.

In the case of qubits, Theorem 4 of [11] can be extended to give several other equivalent characterizations.

**Theorem 1** For trace-preserving qubit maps, the following are equivalent

A) \( \Phi \) has the Holevo form (7) with \( \{F_k\} \) a POVM.

B) \( \Phi \) is entanglement breaking.

C) \( \Phi \circ T \) is completely positive, where \( T(\rho) = \rho^T \) is the transpose.

D) \( \Phi \) has the “sign-change” property that changing any \( \lambda_k \rightarrow -\lambda_k \) in the canonical form (3) yields another completely positive map.

E) \( \Phi \) is in the convex hull of CQ maps.

Conditions (C) thru (E) are special to qubits. Conditions (C) and (D) use the fact [4, 8, 10, 15] that the PPT (positive partial transpose) condition for separability is also sufficient in the case of qubits.

## 2 Characterizations

In this section, we prove Theorem 1 and provide some results using the canonical parameters. This gives another characterization of qubit EBT maps in the special case of CPT maps which are also unital.

The equivalence (A) \( \Leftrightarrow \) (B) was proved in [11] where it was also shown that both are equivalent to the condition that \( \Upsilon \circ \Phi \) is CPT for all \( \Upsilon \) in a set of entanglement witnesses and that \( \Phi \circ \Upsilon \) is CPT if and only if \( \Upsilon \circ \Phi \) is. In the case of qubits, it is well-known that it suffices to let \( \Upsilon \) be the transpose, which proves the equivalence with (C). Furthermore, changing \( \Phi \rightarrow \Phi \circ T \) is equivalent to changing \( \lambda_2 \rightarrow -\lambda_2 \) in the representation (3), and is unitarily equivalent (via conjugation with a Pauli matrix) to changing the sign of any other \( \lambda_k \) which yields (C) \( \Leftrightarrow \) (D). That (E) \( \Rightarrow \) (A) follows immediately from the facts that CQ maps
are a special type of entanglement-breaking maps and the set of entanglement-breaking maps is convex by Theorem 2 of [11]. The proof that shows \((D) \Rightarrow (E)\) will be given in Section 6.

The proof that \((B) \Rightarrow (A)\) given in [11] relied on the fact that there is a one-to-one correspondence \([5, 10, 12]\) (but not a unitary equivalence) between maps \(\Phi\) and states

\[
\Gamma_{\Phi} = (I \otimes \Phi)(|\beta\rangle\langle \beta|)
\]

where \(|\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\) is one of the maximally entangled Bell states. Moreover, a map is EBT if and only if \(\Gamma_{\Phi}\) is separable since that was shown to be equivalent to writing it in the form (2). One could then apply the reduction criterion for separability \([2, 9, 10]\) to \(\Gamma_{\Phi}\). This condition states that a necessary condition for separability of \(\rho\) is that \(\langle \beta, \rho \beta \rangle \leq \frac{1}{d}\) for all maximally entangled states. In the case of qubits, this criterion is equivalent to the PPT condition, and hence sufficient, and equivalent to \(\rho \leq \frac{1}{2}I\), which gives the following result.

**Theorem 2** A qubit CPT map is EBT if and only if \(\Gamma_{\Phi} \leq \frac{1}{2}I\) with \(\Gamma_{\Phi}\) as in (4).

We now consider entanglement breaking conditions which involve only the parameters \(\lambda_k\).

**Theorem 3** If \(\Phi\) is an entanglement breaking qubit map written in the form (3), then \(\sum_j |\lambda_j| \leq 1\).

**Proof:** It is shown in \([11, 16]\) that a necessary condition for complete positivity is

\[
(\lambda_1 \pm \lambda_2)^2 \leq (1 \pm \lambda_3)^2.
\]

When combined with the sign change condition \((D)\), this yields the requirement \(|\lambda_1| + |\lambda_2| \leq 1 - |\lambda_3|\). QED

For unital qubit channels, the condition in Theorem 3 is also sufficient for entanglement breaking. For unital maps \(t = 0\) and, as observed in \([11, 14, 16]\), the conditions in (5) are also sufficient for complete positivity. Since \(\sum_j |\lambda_j| \leq 1\) implies that (5) holds for any choice of sign in \(\lambda_k = \pm |\lambda_k|\), it follows that any unital CPT map satisfying this condition is also EBT.

**Theorem 4** A unital qubit channel is entanglement breaking if and only if \(\sum_j |\lambda_j| \leq 1\) [after reduction to the form (3)].

Moreover, as will be discussed in section 5 the extreme points of the set of unital entanglement breaking maps are those for which two \(\lambda_k = 0\). Hence these channels are in the convex hull of CQ maps.
For non-unital maps these conditions need not be sufficient. Consider the so-called amplitude damping channel for which \( \lambda_1 = \alpha, \lambda_2 = \alpha, \lambda_3 = \alpha^2, t_1 = t_2 = 0, \) and \( t_3 = 1 - \alpha^2. \) For this map equality holds in the necessary and sufficient conditions
\[
(\lambda_1 \pm \lambda_2)^2 \leq (1 \pm \lambda_3)^2 - t_3^2. \tag{6}
\]
Since the inequalities would be violated if the sign of one \( \lambda_k \) is changed, the amplitude damping maps are never entanglement breaking except for the limiting case \( \alpha = 0. \) Thus there are maps for which \( \sum_j |\lambda_j| = 2\alpha + \alpha^2 \) can be made arbitrarily small (by taking \( \alpha \to 0 \)), but are \textit{not} entanglement-breaking.

3 A product representation

We begin by considering the representation of maps in the basis \( \{ I, \sigma_1, \sigma_2, \sigma_3 \} \).

Let \( \Phi \) have the form (1) and write \( R_k = \frac{1}{2}[I + w^k, \sigma] \) and \( F_k = \frac{1}{2}[u^k_0 + u^k, \sigma] \).

Let \( W, U \) be the \( n \times 4 \) matrices whose rows are \( (1, w^k_1, w^k_2, w^k_3) \) and \( (u^k_0, u^k_1, u^k_2, u^k_3) \) respectively, i.e., \( w_{jk} = w^k_j, \ u_{jk} = u^k_j \ k = 0 \ldots 3 \). Let \( T \) be the matrix \( W^T U \).

Note that the requirement that \( \{ F_k \} \) is a POVM is precisely that the first row of \( T \) is \( (1, 0, 0, 0) \). The matrix \( T = W^T U \) is the representative of \( \Phi \) in the form (3) (albeit not necessarily diagonal). We can summarize this discussion in the following theorem.

**Theorem 5** A qubit channel is entanglement breaking if and only if it can be represented in the form (3) with \( T = W^T U \) where \( W \) and \( U \) are \( n \times 4 \) matrices as above, i.e., the rows satisfy \( (\sum_{k=1}^3 u^2_{jk})^{1/2} \leq u_{0k} \) and \( (\sum_{k=1}^3 w^2_{jk})^{1/2} \leq w_{0k} = 1 \) for all \( k \).

We can use this representation to give alternate proofs of two results of the previous section.

To show that \( (A) \Leftrightarrow (D) \) observe that changing the sign of the \( j \)-th column of \( U \) \( (j = 1, 2, 3) \) is equivalent to replacing \( F_k \) by the POVM with \( u^k_j \to -u^k_j \). The effect on \( T \) is simply to multiply the \( j \)-th column by \(-1\). The critical property about qubits is that the condition \( F_k > 0 \) is equivalent to \( (\sum_j |u^k_j|^2)^{1/2} \leq u^k_0 \) which is unaffected by the replacement \( u^k_j \to -u^k_j \).

Next, we give an alternate proof of Theorem 3 which is of interest because it may be extendable to higher dimensions.

**Proof:** Let \( W, U \) be as in Section 3. Then
\[
\sum_{j=1}^3 |\lambda_j| = \sum_{j=1}^3 \left| \sum_{k=1}^n w^k_j u^k_j \right|
\]
\[
\leq \sum_{j=1}^{3} \sum_{k=1}^{n} |w_j^k u_j^k| = \sum_{k=1}^{n} \sum_{j=1}^{3} |w_j^k u_j^k|
\]
\[
\leq \sum_{k=1}^{n} \left( \sum_{j=1}^{3} |w_j^k|^2 \right)^{1/2} \left( \sum_{j=1}^{3} |u_j^k|^2 \right)^{1/2}
\]
\[
\leq \sum_{k=1}^{n} 1 \cdot u_0^k = 1
\]

where we have used the fact that \(|w_j^k| \leq 1 \) and \(|u_j^k| \leq u_0^k\). That \(\sum_k u_0^k = 1\) is a consequence of the fact that the \(\{F_k\}\) form a POVM. QED

We now consider the decomposition \(T = W^T U\) for the special cases of CQ, QC and point channels. If \(\Phi\) is a CQ channel, we can assume without loss of generality that \(U = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \). Now write \(W = \begin{pmatrix} 1 & w_1^1 \\ 1 & w_2 \end{pmatrix} \). Then

\[
T = W^T U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{w^1 + w^2}{2} & 0 & 0 & \frac{w^1 - w^2}{2}
\end{pmatrix}.
\]

(7)

By acting on the left with a unitary matrix of the form \(\begin{pmatrix} 1 & 0 \\ 0 & \pm R \end{pmatrix}\) where \(R\) is a rotation whose third row is a multiple of \(w^1 - w^2\), this can be reduced to the form \(B\) with \(\lambda_1 = \lambda_2 = 0, |\lambda_3| = \frac{1}{2} |w^1 - w^2|\), and \(t = R w^1 - (0, 0, \lambda_3)^T \) [since \(\frac{1}{2}(w^1 + w^2) = w^1 - \frac{1}{2}(w^1 - w^2)\).] Indeed, it suffices to choose

\[
W = \begin{pmatrix} 1 & t_1 & t_2 & t_3 + \lambda_3 \\ 1 & t_1 & t_2 & t_3 - \lambda_3 \end{pmatrix}.
\]

(8)

Note that the requirement \(|t| \leq 1\) only implies \(t_1^2 + t_2^2 + (t_3 + \lambda_3)^2 \leq 1\); however, the requirement \(|w_j^k| \leq 1\) implies that \(t_1^2 + t_2^2 + (t_3 \pm \lambda_3)^2 \leq 1\) must hold with both signs and this is equivalent to the stronger condition

\[
t_1^2 + t_2^2 + (|t_3| + |\lambda_3|)^2 \leq 1
\]

(9)

which is necessary and sufficient for a CPT map to reduce the Bloch sphere to a line.

If \(\Phi\) is a QC channel, we can assume without loss of generality that \(W = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \) and \(U = \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ 1 - u_0 & -u_1 & -u_2 & -u_3 \end{pmatrix} \), from which one easily finds that the second and third rows of \(T = W^T U\) are identically zero and the
fourth row is \((2u_0 - 1 \quad 2u_1 \quad 2u_2 \quad 2u_3)\). One then easily verifies that multiplication on the right by a matrix as above with \(R\) a rotation whose third column is a multiple of \((u_1 \quad u_2 \quad u_3)\) reduces \(T = W^T U\) to the canonical form (3) with 
\[
\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 2\sqrt{u_1^2 + u_2^2 + u_3^2} = |u| \leq \min\{2u_0, 2(1 - u_0)\} \leq 1, \quad \text{and} \quad t_3 = 2u_0 - 1.
\]
(Not that \(t_3 + \lambda_3 \leq |2u_0 - 1| + \min\{2u_0, 2(1 - u_0)\} \leq 1\) with equality if and only if the image reaches the Bloch sphere.)

It is interesting to note that for qubits channels, every QC channel is unitarily equivalent to a CQ channel. Indeed, a channel which, after reduction to canonical form has non-zero elements \(\lambda_3\) and \(t_3\) with \(|\lambda_3| + |t_3| \leq 1\) and \(|t_3| < 1\) can be written as either a QC channel with
\[
W = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad U = \frac{1}{2} \begin{pmatrix} 1 + t_3 & 0 & 0 & \lambda_3 \\ 1 - t_3 & 0 & 0 & -\lambda_3 \end{pmatrix}
\]
or as a CQ channel with
\[
W = \begin{pmatrix} 1 & 0 & 0 & t_3 + \lambda_3 \\ 1 & 0 & 0 & t_3 - \lambda_3 \end{pmatrix} \quad U = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.
\]
For point channels \(W = (1 \quad t_1 \quad t_2 \quad t_3)\) and \(U = \frac{1}{2} (1 \quad 0 \quad 0 \quad 0)\).

We conclude this section with an example of map of the form (11) with an extreme POVM, for which the corresponding map \(\Phi\) is not extreme. Let \(E_k = \frac{1}{2} [I + w^k \sigma]\) with \(w^1 = (1, 0, 0), w^2 = (-\frac{1}{2}, 0, \sqrt{3}/2), w^3 = (-\frac{1}{2}, 0, -\sqrt{3}/2)\). Then, irrespective of the choice of \(R_k\), the third column of \(T = W^T U\) is identically zero, which implies that, after reduction to canonical form, one of the parameters \(\lambda_k = 0\). However, it is easy to find density matrices, e.g., \(R_k = \frac{1}{2} [I + \sigma_k]\), for which the resulting map \(\Phi\) is not CQ or point. But by Theorem [III] \(\Phi\) is a convex combination of CQ maps and hence, not extreme.

## 4 Complete positivity conditions revisited

Not only is the set of CPT maps convex, in a fixed basis corresponding to the canonical form (3) the set of \(\lambda_k\) corresponding to any fixed choice of \(t = (t_1, t_2, t_3)\) is also a convex set which we denote \(\Lambda_t\). We will also be interested in the convex subset \(\Lambda_{t, \lambda_3}\) of the \(\lambda_1 - \lambda_2\) plane for fixed \(t, \lambda_3\), and in the convex set \(\Xi_{t_3, \lambda_3}\) of points \((t_1, t_2, \lambda_1, \lambda_2)\) corresponding to fixed \(t_3, \lambda_3\). Although stated somewhat differently, the following result was proved in [16].

**Theorem 6** Let \(t\) and \(\lambda_3\) be fixed with \(|t_3| + |\lambda_3| < 1\). Then the convex set \(\Lambda_{t, \lambda_3}\) consists of the points \((\lambda_1, \lambda_2)\) for which \(I - R_{\Phi}^T R_{\Phi}\) (or, equivalently \(I - R_{\Phi} R_{\Phi}^T\)) is positive semi-definite, where
\[
R_{\Phi} = \begin{pmatrix}
\frac{t_1 + it_2}{(1 + t_3 + \lambda_3)^{1/2}(1 - t_3 - \lambda_3)^{1/2}} & \frac{\lambda_1 + \lambda_2}{(1 + t_3 + \lambda_3)^{1/2}(1 - t_3 - \lambda_3)^{1/2}} \\
\frac{\lambda_1 - \lambda_2}{(1 + t_3 - \lambda_3)^{1/2}(1 - t_3 + \lambda_3)^{1/2}} & \frac{t_1 + it_2}{(1 + t_3 - \lambda_3)^{1/2}(1 - t_3 + \lambda_3)^{1/2}}
\end{pmatrix}.
\]
Similarly, $\Xi_{t_3, \lambda_3}$ also consists of the points $(t_1, t_2, \lambda_1, \lambda_2)$ for which $I - R_\Phi^\dagger R_\Phi \geq 0$. Moreover, the extreme points of $\Lambda_{t_3, \lambda_3}$ are those for which $R_\Phi^\dagger R_\Phi = I$.

Although this result is stated in a form in which $t_3$ and $\lambda_3$ play a special role and does not appear to be symmetric with respect to interchange of indices, the conditions which result are, in fact, invariant under permutations of 1, 2, 3.

Theorem 6 follows from Choi’s theorem \[5\] that $\Phi$ is completely positive if and only if $\Gamma_\Phi$, given by (4), is positive semi-definite. As noted in \[16\], this implies that it can be written in the form

$$\Gamma_\Phi = \begin{pmatrix} \hat{\Phi}(E_{11}) & \sqrt{\hat{\Phi}(E_{11})} R_\Phi \sqrt{\hat{\Phi}(E_{22})} \\ \sqrt{\hat{\Phi}(E_{22})} R_\Phi^\dagger \sqrt{\hat{\Phi}(E_{11})} & \hat{\Phi}(E_{22}) \end{pmatrix} \quad (11)$$

where $R_\Phi$ is a contraction. (Note, however, that the expression for $R_\Phi$ given in (10) was obtained by applying this result to the adjoint $\hat{\Phi}$, i.e., to $(I \otimes \hat{\Phi})(|\beta\rangle\langle\beta|)$.

Conversely, given a CPT map $\Phi$ and any contraction $U$ on $\mathbb{C}^2$, one can define a $4 \times 4$ matrix in block form,

$$M = \begin{pmatrix} \hat{\Phi}(E_{11}) & \sqrt{\hat{\Phi}(E_{11})} U \sqrt{\hat{\Phi}(E_{22})} \\ \sqrt{\hat{\Phi}(E_{22})} U^\dagger \sqrt{\hat{\Phi}(E_{11})} & \hat{\Phi}(E_{22}) \end{pmatrix} \quad (12)$$

It then follows that there is another CPT map which (with a slight abuse of notation) we denote $\Phi_U$ for which $(I \otimes \hat{\Phi}_U)(|\beta\rangle\langle\beta|) = M$. However, (12) need not, in general, correspond to a map $\Phi_U$ which has the canonical form (3) since that requires $\hat{\Phi}(E_{12}) = \sqrt{\hat{\Phi}(E_{11})} U \sqrt{\hat{\Phi}(E_{22})} = (t_1 + it_2)I + \lambda_1 \sigma_x + i\lambda_2 \sigma_y$. For $U$ an arbitrary unitary or contraction, we can only conclude that

$$\hat{\Phi}(\sigma_x) = \sqrt{\hat{\Phi}(E_{11})} U \sqrt{\hat{\Phi}(E_{22})} + \sqrt{\hat{\Phi}(E_{22})} U^\dagger \sqrt{\hat{\Phi}(E_{22})} = \sum_{k=0}^{3} t_{1k} \sigma_k$$

$$\hat{\Phi}(\sigma_y) = \sqrt{\hat{\Phi}(E_{11})} U \sqrt{\hat{\Phi}(E_{22})} - \sqrt{\hat{\Phi}(E_{22})} U^\dagger \sqrt{\hat{\Phi}(E_{22})} = \sum_{k=0}^{3} t_{2k} \sigma_k$$

so that the map $\Phi_U$ corresponds to a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ t_{10} & t_{11} & t_{12} & t_{13} \\ t_{20} & t_{21} & t_{22} & t_{23} \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}$$

with $t_{jk}$ real.

In order to study the general case of non-zero $t_k$, it is convenient to rewrite (10) in the following form (using notation similar to that introduced in \[13\]).

$$R_\Phi = \begin{pmatrix} \tau & \lambda_+ & \lambda_+ \\ \sqrt{c_{++} c_{--}} & \sqrt{c_{++} c_{--}} & \sqrt{c_{++} c_{--}} \\ \sqrt{c_{--} c_{++}} & \sqrt{c_{--} c_{++}} & \sqrt{c_{--} c_{++}} \end{pmatrix} \quad (13)$$
where \( \lambda_\pm = \lambda_1 \pm \lambda_2 \), \( \tau = t_1 + it_2 \), and \( c_{\pm\pm} = 1 \pm \lambda_3 \pm t_3 \), e.g., \( c_{++} = 1 + \lambda_3 - t_3 \).

Then

\[
I - R_\Phi^\dagger R \Phi \equiv M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}
\]

(14)

with

\[
m_{11} = 1 - \frac{|\tau|^2}{c_{++} c_{--} - c_{--} c_{++}} - \frac{|\lambda_+|^2}{c_{--} c_{++}}
\]

(15)

\[
m_{22} = 1 - \frac{|\tau|^2}{c_{++} c_{--} - c_{--} c_{++}} - \frac{|\lambda_-|^2}{c_{++} c_{--}}
\]

(16)

\[
m_{12} = \frac{\tau \lambda_+}{c_{++} \sqrt{c_{--} c_{++}}} + \frac{\tau \lambda_-}{c_{++} \sqrt{c_{--} c_{--}}}
\]

(17)

Note that the denominators, although somewhat messy, are essentially constants depending only on \( t_3 \) and \( \lambda_3 \). Considering \( \tau \) as also a fixed constant it suffices to rotate (and dilate) the \( \lambda_1 - \lambda_2 \) plane by \( \pi/4 \) and work instead with the variables \( \lambda_\pm \).

The diagonal conditions \( m_{11} \geq 0 \) and \( m_{22} \geq 0 \) define a rectangle in the \( \lambda_+ - \lambda_- \) plane, namely

\[
|\lambda_-|^2 \leq c_{--} c_{++} - \frac{c_{++}}{c_{--}} |\tau|^2 = (1 - \lambda_3)^2 - t_3^2 - \frac{1 - \lambda_3 + t_3}{1 + \lambda_3 + t_3} |\tau|^2
\]

(18)

\[
|\lambda_+|^2 \leq c_{++} c_{--} - \frac{c_{++}}{c_{--}} |\tau|^2 = (1 + \lambda_3)^2 - t_3^2 - \frac{1 + \lambda_3 + t_3}{1 - \lambda_3 + t_3} |\tau|^2
\]

(19)

These diagonal conditions imply the necessary conditions

\[
|\lambda_\pm|^2 \leq (1 \pm \lambda_3)^2 - t_3^2
\]

(20)

for complete positivity, which also become sufficient when \( \tau = 0 \). The determinant condition \( m_{11} m_{22} \geq |m_{12}|^2 \) is more complicated, but basically has the form

\[
[a - b\lambda_+^2] [c - d\lambda_-^2] \geq e\lambda_+^2 + f\lambda_-^2 + g\lambda_+ \lambda_-
\]

(21)

In particular, we would like to know if the values of \( (\lambda_+, \lambda_-) \) satisfying (21) necessarily lie within the rectangle defined by (18) and (19). Extending the lines bounding this rectangle, i.e., \( m_{11} = 0 \) and \( m_{22} = 0 \) one sees that the \( \lambda_+ - \lambda_- \) plane is divided into 9 regions, as shown in Figure 1 and described below.

- the rectangle in the center which we denote ++,
- four (4) outer corners which we denote -- since both \( m_{11} < 0 \) and \( m_{22} < 0 \),
• the four (4) remaining regions (directly above, below and to the left and right of the center rectangle) which we denote as $+$-- or --$+$ according to the signs of $m_{11}$ and $m_{22}$.

We know that the determinant condition (21) is never satisfied in the $+$-- or --$+$ regions since $m_{11}m_{22} - |m_{12}|^2 < 0$ when $m_{11}$ and $m_{22}$ have opposite signs. This implies that equality in (21) defines a curve which bounds a convex region lying entirely within the $++$ rectangle. Although (21) also has solutions in the ---- regions as shown in Figure 1, one expects that these will typically lie outside the region for which $|\lambda_+ + \lambda_-| \leq 2(1 - |t_1|)$ and $|\lambda_+ - \lambda_-| \leq 2(1 - |t_2|)$. However, John Cortese [6] has shown that this need not necessarily be the case. Nevertheless, one need only check one of the two conditions $m_{11} > 0$, $m_{22} > 0$, and might substitute a weaker condition, such as $\text{Tr} M > 0$, to exclude points in the ---- regions.

For example, one could substitute for the diagonal conditions,

$$c_{--}m_{11} + c_{+-}m_{22} \geq 0$$

which is equivalent to

$$(\lambda_1^2 + \lambda_2^2)(1 + t_3) + \lambda_3^2(1 - t_3) \leq (1 + t_3)(1 - |t|)^2 + 2\lambda_1\lambda_2\lambda_3. \tag{22}$$

Thus, strict inequality in both (21) and (22) suffice to ensure complete positivity.

In general, when $t \neq 0$, the convex set $\Lambda_{t,\lambda_3}$ is determined by (21), i.e., by the closed curve for which equality holds and its interior. Since changing the sign of $\lambda_1$ or $\lambda_2$ is equivalent to changing $\lambda_+ \leftrightarrow \lambda_-$, the corresponding set of entanglement breaking maps is given by the intersection of this region with the corresponding one with $\lambda_+$ and $\lambda_- switched, as shown in Figure 2.

Remark: If, instead of looking at $I - R_\Phi^\dagger R_\Phi$, we had considered $I - R_\Phi R_\Phi^\dagger$, the matrix $M$ would change slightly and the conditions (18) or (19) would be modified accordingly. (In fact, the only change would be to replace $+t_3$ by $-t_3$ in the fraction multiplying $|\tau|^2$.) However, the determinant condition (21) would not change. Since $R_\Phi^\dagger R_\Phi$ and $R_\Phi R_\Phi^\dagger$, are unitarily equivalent,

$$\text{det}[I - R_\Phi^\dagger R_\Phi] = \text{det} \left( U[I - R_\Phi R_\Phi^\dagger] U^\dagger \right) = \text{det}[I - R_\Phi R_\Phi^\dagger].$$

It is worth noting that whether or not $R_\Phi$ is a contraction is not affected by the signs of the $t_k$. (In particular, changing $t_2 \leftrightarrow -t_2$ takes $R_\Phi \mapsto T_\Phi$, changing $t_3 \leftrightarrow -t_3$ takes $R_\Phi \mapsto \sigma_x R_\Phi^\dagger \sigma_x$, and changing $t_1 \leftrightarrow -t_1$ takes $R_\Phi \mapsto -\sigma_z R_\Phi \sigma_z$.)

Therefore, one can change the sign of any one of the $t_k$ without affecting completely positivity.

By contrast, one can not, in general, change $\lambda_k \rightarrow -\lambda_k$ without affecting the complete positivity conditions. (Note, however, that one can always change the signs of any two of the $\lambda_k$ since this is equivalent to conjugation with a Pauli
matrix on either the domain or range. The latter will also change the signs of two of the \( t_k \).

Changing the sign of \( \lambda_2 \) is equivalent to composing \( \Phi \) with the transpose, so that changing the sign of one of the \( \lambda_k \) is equivalent to composing \( \Phi \) with the transpose and conjugation with one of the Pauli matrices. Furthermore, if changing the sign of one particular \( \lambda_k \) does not affect complete positivity, then one can change the sign of any of the \( \lambda_k \) without affecting complete positivity.

In view of the role of the sign change condition it is worth summarizing these remarks.

**Proposition 7** Let \( \Phi \) be a CPT map in canonical form (3) and let \( T(\rho) = \rho^T \) denote the transpose. Then

(i) \( T \circ \Phi \circ T \) is also completely positive, i.e., changing \( t_k \to -t_k \) does not affect complete positivity.

(ii) \( \Phi \circ T \) is completely positive if and only if changing any \( \lambda_k \to -\lambda_k \) does not affect complete positivity.

(iii) \( \Phi \circ T \) is completely positive if and only if \( T \circ \Phi \) is.

The only difference between \( \Phi \circ T \) and \( T \circ \Phi \) is that the former changes the sign of \( \lambda_2 \) while the latter changes the signs of both \( t_2 \) and \( \lambda_2 \).

5 Geometry

**Image of the Bloch sphere**

We first consider the geometry of entanglement breaking channels in terms of their effect on the Bloch sphere. It follows from the equivalence with the sign change condition in Theorem 1 that any CPT map with some \( \lambda_k = 0 \) is entanglement breaking. We call such channels **planar** since the image lies in a plane within the Bloch sphere. Similarly, we call a channel with two \( \lambda_k = 0 \) **linear**. If all three \( \lambda_k = 0 \), the Bloch sphere is mapped into a point. Note that the subsets of channels whose images lie within points, lines, and planes respectively are not convex. However, they are well-defined and useful classes to consider.

**Points:** A channel which maps the Bloch sphere to a point has the Holevo form (1) in which the sum reduces to a single term with \( R = \frac{1}{2} \left[ I + t \cdot \sigma \right] \) and \( E = I \). Then \( \Phi(\rho) = R \text{Tr}(E\rho) = R \forall \rho \) and \( T = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix} \). when \( |t| = 1 \), \( R \) is a pure state and the map is extreme. It is also a special case of the so-called amplitude damping channels, and (as noted at the end of section 2) these are the only amplitude damping channels which break entanglement.
**Lines:** When two of the $\lambda_k = 0$ so that the image of the Bloch sphere is a line, the conditions for complete positivity reduce to a single inequality, which becomes (9) in the case $\lambda_1 = \lambda_2 = 0$. Moreover, it is straightforward to verify that any such channel can be realized as a CQ channel. Indeed, it suffices to choose $W$ as in (8).

**Planar channels:** The image of a map with exactly one $\lambda_k = 0$ lies in a plane. When this is $\lambda_3$, the condition $I - R_k^R R_k \geq 0$ becomes

\[
\begin{pmatrix}
1 - |t|^2 - (\lambda_1 - \lambda_2)^2 & 2(t_1 \lambda_1 + it_2 \lambda_2) \\
2(t_1 \lambda_1 - it_2 \lambda_2) & 1 - |t|^2 - (\lambda_1 + \lambda_2)^2
\end{pmatrix} \geq 0,
\]

where $|t|^2 = t_1^2 + t_2^2 + t_3^2$, and the condition on the diagonal becomes

\[
(|\lambda_1| + |\lambda_2|)^2 + |t|^2 \leq 1. \tag{23}
\]

Now, if either diagonal element is identically zero, then one must have $t_1 \lambda_1 = t_2 \lambda_2 = 0$. Thus, if both $\lambda_1, \lambda_2 \neq 0$ and equality holds in the necessary condition (23), one must have $t_1 = t_2 = 0$, in which case it reduces to $(|\lambda_1| + |\lambda_2|)^2 + t_3^2 = 1$. This implies that a truly planar channel can not touch the Bloch sphere, unless it reduces to a point or a line.

**Geometry of $\lambda_k$ space**

We now consider, instead of the geometry of the images of entanglement-breaking maps, the geometry of the allowed set of maps in $\lambda_k$ space. After reduction to the canonical form (3) it is often useful to look at the subset of $[\lambda_1, \lambda_2, \lambda_3]$ which correspond to a particular class of maps. We first consider maps for which $t = 0$.

**Theorem 8** In a fixed (diagonal) basis, the set of unital entanglement breaking maps on qubits corresponds to the octahedron whose extreme points correspond to the channels for which $[\lambda_1, \lambda_2, \lambda_3]$ is a permutation of $[\pm 1, 0, 0]$.

Since this octahedron is precisely the subset with $\sum_j |\lambda_j| \leq 1$ the result follows immediately from Theorem 4. Alternatively, one could use Theorem 10 and the fact that the unital CQ maps must have the form above.

**Remarks:**

1. The channels corresponding to a permutation of $[\pm 1, 0, 0]$ belong to the subclass known as CQ channels. Hence, the set of unital entanglement breaking maps is the convex hull of unital CQ maps.
2. This octahedron in Theorem 8 is precisely the intersection of the tetrahedron with corners \([1, 1, 1], [1, -1, -1], \) \([-1, 1, -1], [-1, -1, 1] \) with its inversion through the origin, as shown in Figure 3 (A similar picture arises in studies of entanglement and Bell inequalities. See, e.g., Figure 3 in [18] or Fig. 2 in [3]).

3. The tetrahedron of unital maps is precisely the intersection of the four planes of the form \(n \cdot [\lambda_1, \lambda_2, \lambda_3] = 1\) with \(n = [\pm 1, \pm 1, \pm 1] \) and an odd number of negative signs, i.e., \(n_1n_2n_3 = -1\). The octahedron of unital EBT maps is precisely the intersection of all eight planes of this form.

4. If the octahedron of unital entanglement breaking maps is removed from the tetrahedron of unital maps, one is left with four disjoint tetrahedrons whose sides are half the length of the original. Each of these defines a region of “entanglement-preserving” unital channels with fixed sign. For example, the tetrahedron with corners, \([1, 1, 1], [1, 0, 0], [0, 1, 0], [1, 0, 0] \); this is the interior of the intersection of the plane \([-1, -1, -1] \cdot [\lambda_1, \lambda_2, \lambda_3] = -1\) and the three planes of the form \(n \cdot [\lambda_1, \lambda_2, \lambda_3] = 1\) with \(n = [1, 1, -1], [1, -1, 1], [-1, 1, 1] \). For many purposes, e.g., consideration of additivity questions, it suffices to confine attention to one of these four corner tetrahedrons. Indeed, conjugation with one of the Pauli matrices, transforms the corner above into one of the other four.

We next consider non-unital maps, for which one finds the following analogue of Theorem 8.

**Theorem 9** Let \(t = (t_1, t_2, t_3) \) be a fixed vector in \(\mathbb{R}^3\) and let \(\Lambda_t\) denote the convex subset of \(\mathbb{R}^3\) corresponding to the vectors \([\lambda_1, \lambda_2, \lambda_3] \) for which the canonical map with these parameters is completely positive. Then the intersection of \(\Lambda_t\) with its inversion through the origin (i.e., \(\lambda_j \rightarrow -\lambda_j\)) is the subset of EBT maps with translation \(t\).

**Remark:** The effect of changing the sign of \(\lambda_2\) is \(\lambda_+ \leftrightarrow \lambda_-\) and of changing the sign of \(\lambda_1\) is \(\lambda_+ \leftrightarrow -\lambda_-\). In either case, the effect on the determinant condition (21) is simply to switch \(\lambda_+ \leftrightarrow \lambda_-\), i.e., to reflect the boundary across the \(\lambda_+ = \lambda_-\) line. Thus, the intersection of these two regions will correspond to entanglement breaking channels. The remainder will, typically, consist of 4 disjoint (non-convex) regions, corresponding to the four corners remaining after the “rounded octahedron” of Theorem 9 is removed from the “rounded tetrahedron”.
6 Convex hull of qubit CQ maps

In [16] we found it useful to generalize the extreme points of the set of CPT maps \( S \) to include all maps for which \( R_\Phi \) is unitary, which is equivalent to the statement that both singular values of \( R_\Phi \) are 1. In addition to true extreme points, this includes “quasi-extreme” points which correspond to the edges of the tetrahedron of unital maps. Some of these quasi-extreme points are true extreme points for the set of entanglement-breaking maps. However, there are no extreme points of the latter which are not generalized extreme points of \( S \). This will allow us to conclude the following.

**Theorem 10** Every extreme point of the set of entanglement-breaking qubit maps is a CQ map. Hence, the set of entanglement-breaking qubit maps is the convex hull of qubit CQ maps.

The goal of the section is to prove this result. Because our argument is somewhat subtle, we also include, at the end of this section a direct proof of some special cases.

First we note that the following was shown in [16]. After reduction to canonical form (3), for any map which is a generalized extreme point, the parameters \( \lambda_k \) must satisfy (up to permutation) \( \lambda_3 = \lambda_1 \lambda_2 \). This is compatible with the sign change condition if and only if at least two of the \( \lambda_k = 0 \), which implies that \( \Phi \) be a CQ map.

We now wish to examine in more detail those maps for which \( R_\Phi \) is not unitary. We can assume, without loss of generality, that the singular values of \( R_\Phi \) can be written as \( \cos \theta_1 \) and \( \cos \theta_2 \), that \( \cos \theta_1 \geq \cos \theta_2 \), and that \( 0 \leq \cos \theta_2 < 1 \). Recall that we showed in Lemma 15 of [16] that one can use the singular value decomposition of \( R_\Phi \) to write

\[
R_\Phi = V \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix} W^\dagger = \frac{1}{2} U_+ + \frac{1}{2} U_-	ag{24}
\]

where \( U_\pm = V \begin{pmatrix} e^{i \theta_1} & 0 \\ 0 & e^{i \theta_2} \end{pmatrix} W^\dagger \). and \( V, W \) are unitary. Thus, \( \Phi \) is the midpoint of a line segment in \( S \) and can be written as

\[
\Phi = \frac{1}{2} \Phi_{U+} + \frac{1}{2} \Phi_{U-}
\]

with \( \Phi_{U\pm} \) defined as in (12). Although \( \Phi_{U\pm} \) need not have the canonical form (3), they are related so that their sum does.

We now use the singular value decomposition of \( R_\Phi \) to decompose it into unitary maps in another way.

\[
R_\Phi = V \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix} W^\dagger
\]

14
\[
\begin{align*}
&= V \left[ \frac{\cos \theta_1 + \cos \theta_2}{2} I + \frac{\cos \theta_1 - \cos \theta_2}{2} \sigma_z \right] W^\dagger \\
&= \frac{\cos \theta_1 + \cos \theta_2}{2} V W^\dagger + \frac{\cos \theta_1 - \cos \theta_2}{2} V \sigma_z W^\dagger.
\end{align*}
\] (27)

Moreover, it follows from (27) that

\[
\Phi = \frac{\cos \theta_1 + \cos \theta_2}{2} \Phi_{VW^\dagger} + \frac{\cos \theta_1 - \cos \theta_2}{2} \Phi_{V\sigma_zW^\dagger} + (1 - \cos \theta_1) \Phi_0
\] (28)

where \(\Phi_0\) is the QC map corresponding to \(M = \begin{pmatrix} \hat{\Phi}(E_{11}) & 0 \\ 0 & \hat{\Phi}(E_{22}) \end{pmatrix}\). Since we have assumed that we do not have \(\cos \theta_1 = \cos \theta_2 = 1\), equation (28) represents \(\Phi\) as a non-trivial convex combination of at least two distinct CPT maps, the first two of which are generalized extreme points. (Unless \(\cos \theta_1 = 1\) or \(\cos \theta_1 = \cos \theta_2\), we will have three distinct points, and can already conclude that \(\Phi\) lies in the interior of a segment of a plane within \(S\).) Now, the assumption that \(\cos \theta_2 \neq 1\) suffices to show that the decompositions (28) and (25) involve different sets of extreme points and, hence, that \(\Phi\) can be written as a point on two distinct line segments in \(S\). Therefore, there is a segment of a plane in \(S\) which contains \(\Phi\) and for which \(\Phi\) does not lie on the boundary of the plane (although the plane might be on the boundary of \(S\)). Thus we have proved the following.

**Lemma 11** Every map \(\Phi\) in \(S\) lies in one of two disjoint sets which allows it to be characterized as follows. Either

1) \(\Phi\) is a generalized extreme point of \(S\), or

2) \(\Phi\) is in the interior of a segment of a plane in \(S\).

Now let \(T\) denote the set of maps for which \(\Phi \circ T\) or, equivalently \((-I) \circ \Phi\), is in \(S\). Since \(T\) is a convex set isomorphic to \(S\), its elements can also be broken into two classes as above. The set of entanglement breaking maps is precisely \(S \cap T\). We can now prove Theorem 5 by showing that the convex hull of CQ maps is \(S \cap T\).

**Proof:** Let \(\Phi\) be in \(S \cap T\) which is also a convex set. If \(\Phi\) is a generalized extreme point of either \(S\) or \(T\), then the only possibility consistent with \(\Phi\) being entanglement-breaking is that it is CQ. Thus we suppose that \(\Phi\) belongs to class II for both \(S\) and \(T\). Then \(\Phi\) lies within a plane in \(S\) and within a plane in \(T\). The intersection of these two planes is non-empty (since it contains \(\Phi\)) and its intersection must contain a line segment in \(S \cap T\) which contains \(\Phi\) and for which \(\Phi\) is not an endpoint. Therefore, \(\Phi\) is not an extreme point of \(S \cap T\). Thus all possible extreme points of \(S \cap T\) must be generalized extreme points of \(S\) or \(T\), in which case they are CQ. QED
Remark: Although this shows that all extreme points of \( S \cap T \) are CQ maps, this need not hold for the various convex subsets, corresponding to allowed values of \( \lambda_k, t_k \) in a fixed basis, discussed at the start of Section 4. The following remark shows that “most” points in the convex subset \( \Lambda_{t,\lambda_3} \) of the \( \lambda_1, \lambda_2 \) plane can, in fact, be written as a convex combination of CQ maps in canonical form in the same basis. It also shows why it is necessary to go outside this region for those points close to the boundary.

a) First consider the set of entanglement-breaking maps with \( \lambda_3 = 0 \), which is the convex set \( \bigcup_{t_3} \Xi_{t_3,0} \). Every extreme point must be an extreme point of the convex set \( \Xi_{t_3,0} \) for some \( t_3 \). By Theorem 6, these are the maps for which

\[
\frac{1}{\sqrt{1-t_3^2}} \begin{pmatrix} \tau & \lambda_+ \\ \lambda_- & \tau \end{pmatrix}
\]

is unitary, which implies that either

(i) \( t_1 = t_2 = 0 \) and \( (\lambda_1 \pm \lambda_2)^2 = 1 - t_3^2 \) which implies that either \( \lambda_1 = 0 \) or \( \lambda_2 = 0 \) with \( t_3^2 + \lambda_j^2 = 1 \) for \( j = 1 \) or \( 2 \), or

(ii) \( \lambda_1 = \lambda_2 = 0 \) and \( |t|^2 = 1 \).

The first type of extreme point is obviously a CQ map; the second is a “point” channel which, as noted before, is a special case of a CQ map. Thus any map in \( \Xi_{t_3,0} \) can be written as a convex combination of CQ maps in \( \Xi_{t_3,0} \).

Similar results hold if \( \lambda_1 = 0 \) or \( \lambda_2 = 0 \). Therefore, any entanglement breaking channel with some \( \lambda_k = 0 \), can be written as a convex combination of CQ channels with at most one non-zero \( \lambda_k \) in the same basis. Thus any planar channel can be written as a convex combination of CQ channels in the same plane.

b) Next consider entanglement-breaking maps with at most one non-zero \( t_k \).

We can assume, without loss of generality, that \( t_1 = t_2 = 0 \) in which case the conditions for complete positivity reduce to (20). Combining this with the sign change condition yields

\[
(|\lambda_1| + |\lambda_2|)^2 \leq (1 - |\lambda_3|)^2 - t_3^2.
\]  

(29)

It follows that for each fixed value of \( \lambda_3 \) the set of allowable \( (\lambda_1, \lambda_2) \) form a square with corners \( (0, \pm A_3), (\pm A_3, 0) \) where \( A_3 = \sqrt{(1 - |\lambda_3|)^2 - t_3^2} \). Thus, the extreme points of \( \Lambda_{(0,0,t_3),\lambda_3} \) are planar channels which, by part(a) are in the convex hull of CQ channels. In particular, a map with \( \lambda_1 = 0, \lambda_2 = \pm A_3 \), can be written as a convex combination of CQ maps with either \( \lambda_2 = 0 \) or \( \lambda_3 = 0 \). However, these maps need not necessarily lie in \( \Lambda_{(0,0,t_3),\lambda_3} \); we can only be sure that \( \lambda_1 = 0 \) and \( t_1 = 0 \), but not that \( t_2 = 0 \). Thus we can only
state that $\Lambda_{(0,0,t_3),\lambda_3}$ is in the convex hull of those CQ maps with $\lambda_j = 0$ and $t_j = 0$ for either $j = 1$ or $2$. Although it may be necessary to enlarge the set $\Lambda_{(0,0,t_3),\lambda_3}$ in order to ensure that it is in the convex hull of some subset of CQ maps, these CQ maps will have the canonical form in the same basis, and the same value for $\lambda_3$ in that basis.

c) Now consider the convex subset $\Lambda_{t,\lambda_3} \cap \Lambda_{t,-\lambda_3}$ of the $\lambda_1-\lambda_2$ plane corresponding to entanglement breaking maps with $t,|\lambda_3|$ fixed. These two regions intersect when either $\lambda_1 = 0$ or $\lambda_2 = 0$ (or, equivalently, $|\lambda_+| = |\lambda_-|$ where $\lambda_\pm = \lambda_1 \pm \lambda_2$). One can again use part (a) to see that these intersection points can be written as convex combinations of CQ maps in canonical form in the same basis. Since their convex hull has the same property, the resulting parallelogram, as shown in Figure 4, is also a convex combination of CQ maps of the same type. Only for those points in the strip between the parallelogram and the boundary might one need to make a change of basis in order to write the maps as a convex combination of CQ maps.

d) Now suppose $|\lambda_1| = |\lambda_2| = |\lambda_3| = \lambda > 0$. Since any two signs can be changed by conjugation with a Pauli matrix, $\Phi$ is unitarily equivalent to a map with $\lambda_1 = \lambda_2 = \lambda_3 = \pm \lambda$. One can then conjugate with another unitary matrix (corresponding to a rotation on the Bloch sphere) to conclude that $\Phi$ is unitarily equivalent to a channel $\Phi'$ with $t_1 = |t| = \sqrt{\sum_k t_k^2}$, and $t_2 = t_3 = 0$. It then follows from part (b) that $\Phi'$, and thus also $\Phi$, can be written as a convex combination of CQ channels which have the form described above in the rotated basis. However, these maps need not necessarily have the canonical form in the original basis.

Consider the region $\Lambda_{t,\lambda_3}$ with $0 < \lambda_3 = \lambda < \frac{1}{3}$ and $|t|^2 = 1 - 2\lambda + 3\lambda^2$. The maps with $|\lambda_1| = |\lambda_2| = \lambda$ lie on the boundary of this region (in fact, at the intersection of the boundary with the $\lambda_\pm$ axes, as shown in Figure 5). Since these maps have the form considered in part (d) they can be written as a convex combination of CQ maps; however, those CQ maps need not have the canonical form in the original basis. Nevertheless, every point in the octagon formed from the convex hull of the intersection points of the lines $|\lambda_+| = |\lambda_-|$, $|\lambda_+| = 0$, and $|\lambda_-| = 0$ with the boundary, as shown in Figure 5, can be written as a convex combination of CQ maps as described above.

As another example, consider the set of entanglement breaking maps with $t = (0,0,t_3)$ fixed. For any fixed $\lambda_1$, the set $\Lambda_{t,\lambda_1} \cap \Lambda_{t,-\lambda_1}$ is a convex subset of the $\lambda_2-\lambda_3$ plane. Let $(\lambda_2, \lambda_3)$ be a point in this subset that lies between the boundary and a parallelogram as described in (c) above. By considering the associated map as a point in the set $\Lambda_{t,\lambda_3} \cap \Lambda_{t,-\lambda_3}$ instead, one can be sure that it can be written as a convex combination of CQ maps since this subset of the $\lambda_1-\lambda_2$ plane is of
the type described in (b). Moreover, these boundary points can be added to the convex hull of CQ maps without need for a change of basis.

One might expect that additional boundary points could be added in various ways with additional ingenuity and bases changes. That this is always true, is the essence of Theorem 10. Only for points near the boundary with two $t_k$ non-zero is it necessary to actually make the change of basis used in the proof of this theorem. In other cases, the necessary convex combinations (which are not unique) can be formed using the strategies outlined above.

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Figure 1: The $\lambda_+ - \lambda_-$ plane showing the regions described by the diagonal conditions (dotted lines) and the curves corresponding to $\det(I - R_{\Phi}^\dagger R_{\Phi}) = 0$ for $t = (0.2, 0.3, 0)$ and $\lambda_3 = 0.35$. The closed curve and its interior describes the parameters for which the corresponding map is completely positive.
Figure 2: The $\lambda_+-\lambda_-$ plane showing the region determined by determinant condition when $t = (0.4, 0.3, 0.0)$ and $\lambda_3 = 0.15$ and the corresponding region with $\lambda_+$ and $\lambda_-$ interchanged. Their intersection corresponds to the entanglement breaking maps with the indicated parameters.

Figure 3: The tetrahedron of bistochastic maps and its inversion through the origin (left); their intersection gives the octahedron of unital entanglement breaking maps (right). (Figures by K. Durstberger appeared in \cite{3}. )
Figure 4: The region of the $\lambda_+ - \lambda_-$ plane corresponding to entanglement breaking maps with $t = (0.4, 0.3, 0.0)$ and $\lambda_3 = 0.15$. The dotted lines show the convex hull of the intersection points, which are planar maps.

Figure 5: The region of the $\lambda_+ - \lambda_-$ plane corresponding to entanglement breaking maps with $t = (0.4, 0.3, 0.3742)$ and $\lambda_3 = 0.20$, Because the intersections of the axes with the boundary (at $\lambda_\pm = \pm 0.4$, for which all $|\lambda_k| = 0.2$) correspond to maps known to be in the convex hull of CQ maps, one can enlarge the convex hull of such maps from the dotted line to the octagon shown by the dashed line.