Letter

Virasoro Symmetry Algebra of Dirac Soliton Hierarchy

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Abstract

A hierarchy of first-degree time-dependent symmetries is proposed for Dirac soliton hierarchy and their commutator relations with time-dependent symmetries are exhibited. Meantime, a hereditary structure of Dirac soliton hierarchy is elucidated and a Lax operator algebra associated with Virasoro symmetry algebra is given.

The main purpose of the present letter is to construct a hierarchy of first-degree time-dependent symmetries for Dirac soliton hierarchy [1]. This kind of symmetries was first proposed in the famous Olver’s paper [2]. They include ones corresponding to Galilean invariance and scalar invariance and they are generally related to the first-degree master symmetries [3]. Afterwards, some theory to describe time-dependent symmetries were developed for various classes of soliton equations [4] [5]. Moreover it is found that for some nonlinear systems, there exist $W_\infty$ symmetry algebras involving arbitrary functions of some independent variables, for example, time variable $t$ [6]. However, for systems of evolution equations, the kind of symmetries involving arbitrary functions of time variable $t$ doesn’t exist [7]. These systems may possess polynomial time-dependent symmetries, which relate to master symmetries of any degree. Usually only first-degree time-dependent symmetries can be found for soliton equations in $1 + 1$ dimensions. Recently, in terms of Lax operators, a simple but systematic scheme for generating first-degree time-dependent symmetries in $1 + 1$ dimensions has been established in [8]. Here we would like to discuss the case of Dirac soliton hierarchy through that trick.

Dirac soliton hierarchy reads as [1] [9]

$$\begin{align*}
\frac{\delta u}{\delta t_m} &= \left( \begin{array}{c} q \\ r \end{array} \right), \\
K_m &= \Phi^m f_0 = \Phi^m J f_0' = J \Phi^m f_0' = J \frac{\delta H_m}{\delta u}, \quad m \geq 0, \quad (1)
\end{align*}$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Phi = \Psi^* = \begin{pmatrix} 2r\partial^{-1}q \\ \frac{1}{2}\partial - 2q\partial^{-1}r \\ -\frac{1}{2}\partial + 2r\partial^{-1}r \\ -2q\partial^{-1}r \end{pmatrix}, \quad f_0 = \begin{pmatrix} 2r \\ -2q \end{pmatrix}. \quad (2)$$

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Here $\partial = \partial / \partial x$ and $\partial^{-1} \partial = \partial \partial^{-1} = 1$. After this hierarchy was presented by Grosse [1], it did not arouse enough attention until its Hamiltonian structure and binary nonlinearization were recently established (e.g. see [3]). It is not difficult to find that $J$ and $M = \Phi J$ constitute a pair of Hamiltonian operators and thus Dirac soliton hierarchy (1) possesses a bi-Hamiltonian structure. It differs from Kaup-Newell hierarchy but assembles AKNS hierarchy. If we move the derivative $\partial$ from off-diagonal to diagonal and interchange the positions of two potentials in $\Phi$, the recursion operator $\Phi$ of Dirac hierarchy will be transformed into one of AKNS hierarchy. The first nonlinear Dirac system in Dirac hierarchy (1) is as follows
\[
\begin{aligned}
q_{t_2} &= -\frac{1}{2} r_{xx} + q^2 r + r^3, \\
r_{t_2} &= \frac{1}{2} q_{xx} - q^3 - qr^2.
\end{aligned}
\]
This system is different from the coupled nonlinear Schrödinger system in AKNS hierarchy because it contains the cubic terms $q^3$, $r^3$.

Dirac soliton hierarchy (1) associates with the following spectral problem
\[
\phi_x = U \phi = U(u, \lambda) \phi, \quad \phi = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right), \quad U = \left( \begin{array}{cc} q & \lambda + r \\ -\lambda + r & -q \end{array} \right).
\]
There have been many results about this Dirac spectral problem. A Gelfand-Levitan-Marchenko equation holds for Dirac spectral problem (3) [3] [4]. Time evolution of scattering data of (3) was discussed in [4] and a detailed analysis on inverse scattering problem was provided by Hinton et al. [4] for a more general spectral problem. Its trace formula has also been carefully investigated in [5]. Its isospectral ($\lambda_{t_m} = 0$) flows are exactly Dirac soliton hierarchy (1). We would like to derive the nonisospectral ($\lambda_{t_n} = \lambda^n, \ n \geq 0$) Dirac flows corresponding to (3) and then present a Virasoro symmetry algebra of (1).

In order to take advantage of the trick in [8], we need to solve the characteristic operator equation with respect to $\Omega = \Omega(X)$:
\[
[\Omega, U] + \Omega_x = U''[\Phi X] - \lambda U''[X]
\]
with any fixed vector field $X = (X_1, X_2)^T$ and an initial nonisospectral ($\lambda_{t_0} = 1$) key equation with respect to $g_0, B_0$:
\[
U'[g_0] + U_\lambda - B_{0x} + [U, B_0] = 0.
\]
It is easy to work out
\[
U'[X] = \left( \begin{array}{cc} X_1 & X_2 \\ X_2 & -X_1 \end{array} \right), \quad \Phi X = \left( \begin{array}{c} -\frac{1}{2} X_{2x} + 2q \partial^{-1}(q X_1 + r X_2) \\ \frac{1}{2} X_{1x} - 2q \partial^{-1}(q X_1 + r X_2) \end{array} \right).
\]
It follows that
\[
U'[ΦX] − λU''[X] = \left( \begin{array}{cc}
-\frac{1}{2}X_{2x} + 2r\partial^{-1}(qX_1 + rX_2) & \frac{1}{2}X_{1x} - 2q\partial^{-1}(qX_1 + rX_2) \\
\frac{1}{2}X_{1x} - 2q\partial^{-1}(qX_1 + rX_2) & \frac{1}{2}X_{2x} - 2r\partial^{-1}(qX_1 + rX_2)
\end{array} \right) - λ \left( \begin{array}{cc}
X_1 & X_2 \\
X_2 & -X_1
\end{array} \right).
\]  

(6)

If we choose a special form of Ω
\[
Ω = \left( \begin{array}{cc}
c & a + b \\
a - b & -c
\end{array} \right) = aσ_1 + biσ_2 + cσ_3, \quad i = \sqrt{-1},
\]

where σ_i, 1 ≤ i ≤ 3, are Pauli 2×2 matrices, then we have
\[
[Ω, U]+Ω = \left( \begin{array}{cc}
-2λa + 2rb + c_x & 2λc - 2qb + 2rc - 2qa + a_x + b_x \\
2λc - 2qb - 2rc + 2qa + a_x - b_x & 2λa - 2rb - c_x
\end{array} \right).
\]  

(7)

The substitution of (6) and (7) into the characteristic operator equation (4) results in
\[
a = \frac{1}{2}X_1, \quad b = \partial^{-1}(qX_1 + rX_2), \quad c = -\frac{1}{2}X_2.
\]

Thus we obtain an operator solution to the characteristic operator equation (4)
\[
Ω = Ω(X) = \left( \begin{array}{cc}
\frac{1}{2}X_2 & \frac{1}{2}X_1 + \partial^{-1}(qX_1 + rX_2) \\
\frac{1}{2}X_1 - \partial^{-1}(qX_1 + rX_2) & \frac{1}{2}X_2
\end{array} \right).
\]  

(8)

By a similar argument, we may find a pair of solutions to the initial key equation (5)
\[
B_0 = \left( \begin{array}{cc}
0 & x \\
-x & 0
\end{array} \right), \quad g_0 = \left( \begin{array}{cc}
2xr & \\
-2xq &
\end{array} \right).
\]  

(9)

The above results allow us to conclude that
\[
\alpha_{n} = ρ_n = Φ^n g_0, \quad n ≥ 0
\]  

(10)

is just the required hierarchy of nonisospectral (λ_n = λ^n, n ≥ 0) flows and this hierarchy possesses zero curvature representations
\[
U_{tn} - W_{nx} + [U, W_n] = 0, \quad \text{i.e.} \quad U'[ρ_n] + λ^nU_λ - W_{nx} + [U, W_n] = 0
\]  

(11)

with the spectral evolution laws λ_n = λ^n, n ≥ 0, and Lax operators
\[
W_n = \sum_{j=0}^{n} λ^{n-j} B_j = λ^n B_0 + \sum_{j=1}^{n} λ^{n-j} Ω(ρ_{j-1}).
\]  

(12)

In fact, we can calculate that
\[
W_{nx} - [U, W_n] 
= \sum_{j=0}^{n} λ^{n-j} (B_{jx} - [U, B_j]) 
= λ^n U'[g_0] + λ^n U_λ + \sum_{j=1}^{n} (U'[ρ_j] - λU'[ρ_{j-1}]) 
= λ^n U_λ + U'[ρ_n], \quad n ≥ 1.
\]
and thus zero curvature equations (11) for \( n \geq 0 \) hold. Completely similar to the deduction of the nonisospectral case [13], we can obtain the following isospectral \((\lambda_{tm} = 0)\) Lax operators associated with Dirac soliton hierarchy (1)

\[
V_m = \sum_{i=0}^{m} \lambda_i A_i = \lambda^m A_0 + \sum_{i=1}^{m} \lambda_i \Omega(K_{i-1}), \quad A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]  

(13)

It is known that IST technique can be applied to not only isospectral soliton equations, but also nonisospectral ones (e.g. see [14]). The above neat forms of isospectral and nonisospectral Lax operators may provide us with some help to carry out IST technique. The nonisospectral Lax operators \( V_m, m \geq 0 \), are all local. But the nonisospectral Lax operators \( W_n, n \geq 0 \), are nonlocal except the first two ones \( W_0, W_1 \) since \( \rho_0 = (2xx, -2xq)^T \), \( \rho_1 = (q + xq, r + xr)^T \) are local and the other nonisospectral vector fields \( \rho_n, n \geq 2 \), are nonlocal.

In what follows, we want to show that the nonisospectral flows (10) are all the first-degree master symmetries of Dirac hierarchy (1) and simultaneously to establish a Virasoro symmetry algebra for Dirac hierarchy (1).

Towards this end, first we directly prove the operator \( \Phi \) in (2) is a hereditary operator [15], which also shows that Dirac soliton hierarchy possesses a kind of nice structure, i.e. hereditary structure. The hereditariness of \( \Phi \) means that \( \Phi \) satisfies the following equation

\[
\Phi'[\Phi X]Y - \Phi'[\Phi Y]X - \Phi\{\Phi'[X]Y - \Phi'[Y]X\} = 0
\]  

(14)

for any two vector fields \( X, Y \), which is exactly the same as an operator identity that is proposed for analyzing the reason why soliton equations come in hierarchies in [16]. For ease of writing, we introduce an equivalent relation:

\[
K \cong S \iff (K - S)(X, Y) = (K - S)(Y, X)
\]

for two expressions \( K(X, Y), S(X, Y) \) depending on vector fields \( X, Y \). By this equivalent relation, the equality (14) becomes

\[
\Phi'[\Phi X]Y - \Phi'[\Phi X]Y = (\cdots, (\Phi'[\Phi X]Y)_i, \cdots)^T - (\cdots, (\Phi'[\Phi X]Y)_i, \cdots)^T \cong 0.
\]

Let \( P = qX_1 + rX_2, Q = qY_1 + rY_2 \) for \( X = (X_1, X_2)^T, Y = (Y_1, Y_2)^T \). A direct
calculation can give rise to

\[
\Phi'[\Phi X] = \begin{pmatrix}
[X_{1x} - 4q(\partial^{-1}P)]\partial^{-1}q + r\partial^{-1}[-X_{2x} + 4r(\partial^{-1}P)], \\
[X_{2x} - 4r(\partial^{-1}P)]\partial^{-1}q + q\partial^{-1}[-X_{2x} - 4r(\partial^{-1}P)], \\
[X_{1x} - 4q(\partial^{-1}P)]\partial^{-1}r + r\partial^{-1}[-X_{1x} - 4q(\partial^{-1}P)] \\
[X_{2x} - 4r(\partial^{-1}P)]\partial^{-1}r + q\partial^{-1}[-X_{1x} + 4q(\partial^{-1}P)]
\end{pmatrix},
\]

\[
\Phi'[X]Y \cong \begin{pmatrix}
2X_2(\partial^{-1}Q) \\
-2X_1(\partial^{-1}Q)
\end{pmatrix}.
\]

Further one may acquire

\[
(\Phi'[\Phi X]Y)_1 = [X_{1x} - 4q(\partial^{-1}P)](\partial^{-1}Q)
+ r\partial^{-1}(X_{1x}Y_2 - X_{2x}Y_1) + 4r\partial^{-1}[(rY_1 - qY_2)(\partial^{-1}P)], 
\tag{15}
\]

\[
(\Phi'[\Phi X]Y)_2 = [X_{2x} - 4r(\partial^{-1}P)](\partial^{-1}Q)
+ q\partial^{-1}(X_{2x}Y_1 - X_{1x}Y_2) + 4q\partial^{-1}[(qY_2 - rY_1)(\partial^{-1}P)];
\tag{16}
\]

\[
(\Phi\Phi'[X]Y)_1 \cong 4r\partial^{-1}[(qX_2 - rX_1)(\partial^{-1}Q)] + [X_1(\partial^{-1}Q)]_x,
\tag{17}
\]

\[
(\Phi\Phi'[X]Y)_2 \cong -4q\partial^{-1}[(qX_2 - rX_1)(\partial^{-1}Q)] + [X_2(\partial^{-1}Q)]_x. 
\tag{18}
\]

Therefore by \[(15)\] and \[(17)\], one has

\[
(\Phi\Phi'[X]Y)_1 - (\Phi'[\Phi X]Y)_1 
\cong 4r\{\partial^{-1}[(qX_2 - rX_1)(\partial^{-1}Q)] - \partial^{-1}[(rY_1 - qY_2)(\partial^{-1}P)]
-X_1(\partial^{-1}Q)_x - X_{1x}(\partial^{-1}Q) - r\partial^{-1}(X_{1x}Y_2 - X_{2x}Y_1)
\}
\cong \{X_1(\partial^{-1}Q)_x - X_{1x}(\partial^{-1}Q) - r\partial^{-1}(X_{1x}Y_2 - X_{2x}Y_1)\} \cong 0.
\]

With the same argument by \[(16)\] and \[(18)\], one can verify that

\[
(\Phi\Phi'[X]Y)_2 - (\Phi'[\Phi X]Y)_2 \cong 0.
\]

Therefore \(\Phi\) defined by \[(4)\] is a hereditary symmetry, indeed. We may also show that the operator \(\Phi(\alpha)\) with arbitrary constant coefficient \(\alpha\) of \(\partial\)

\[
\Phi(\alpha) = \begin{pmatrix}
2r\partial^{-1}q & -\alpha\partial + 2r\partial^{-1}r \\
\alpha\partial - 2q\partial^{-1}q & -2q\partial^{-1}r
\end{pmatrix} := \begin{pmatrix}
0 & -\alpha \\
\alpha & 0
\end{pmatrix} \partial + \Phi_0
\]

is still hereditary and that there exists only this sort of hereditary operators among \(\Lambda\partial + \Phi_0\), where \(\Lambda\) is any constant matrix.

Secondly, we need the following relation on the triple \((\Phi, f_0, g_0)\), which may directly be shown,

\[
L_{f_0}\Phi = 0, \ L_{g_0}\Phi = 1, \ \Phi[f_0, g_0] = [f_0, \Phi g_0] = 0, \tag{19}
\]
where the Lie derivative $L_X \Phi$ with respect to $X$ is defined by

$$L_X \Phi = \Phi' [X] - [X', \Phi] = \Phi'[X] - X'\Phi + \Phi X'$$  \hspace{1cm} (20)$$

and the commutator $[X,Y]$ of two vector fields, by $[X,Y] = X'[Y] - Y'[X]$.

In view of the known results given in \[5\] in the case of (14) and (19), one easily obtains a vector field Lie algebra

\[
\begin{align*}
[K_m, K_n] &= [\Phi^m f_0, \Phi^n g_0] = 0, \quad m, n \geq 0, \\
[K_m, \rho_n] &= [\Phi^m f_0, \Phi^n g_0] = mK_{m+n-1}, \quad K_{-1} = 0, \quad m, n \geq 0, \\
[\rho_m, \rho_n] &= [\Phi^m f_0, \Phi^n g_0] = (m-n)\rho_{m+n-1}, \quad \rho_{-1} = 0, \quad m, n \geq 0.
\end{align*}
\]  \hspace{1cm} (21)

For example, one can also directly show that

\[
\begin{align*}
[K_m, \rho_n] &= [\Phi^m f_0, \Phi^n g_0] \\
&= \Phi^n [\Phi^m f_0, g_0] \quad \text{(due to } L_{f_0} \Phi = 0) \\
&= \Phi^{n+1} [\Phi^{m-1} f_0, g_0] + \Phi^{m+n-1} f_0 \\
&= \Phi^{n+2} [\Phi^{m-2} f_0, g_0] + 2\Phi^{m+n-1} f_0 \\
&= \ldots \ldots \\
&= \Phi^{m+n} [f_0, g_0] + m\Phi^{m+n-1} f_0 \\
&= mK_{m+n-1}.
\end{align*}
\]

Here from the third step, we have used the equality $[X, \Phi Y] = \Phi [X, Y] - (L_X \Phi) Y$, induced by (20), $m$ times. Those relations imply that the vector fields $\rho_n$, $n \geq 0$, are all common master symmetries of the first degree for the whole hierarchy (I) and thus one sees that the $l$th Dirac system $u_{tl} = K_l$ in Dirac soliton hierarchy (I) has infinitely many first-degree time-dependent symmetries

$$\tau^{(l)}_n = t[K_l, \rho_n] + \rho_n = ltK_{n+l-1} + \rho_n, \quad n \geq 0,$$  \hspace{1cm} (22)

besides infinitely many time-independent symmetries $K_m, m \geq 0$. The symmetries $\tau^{(l)}_0$ and $\tau^{(l)}_1$ correspond to Galilean transformation group and scalar transformation group, respectively. Furthermore these two hierarchies of symmetries constitute a semi-product of a Kac-Moody algebra and a centerless Virasoro algebra. More precisely, they possess the following commutator relations

\[
\begin{align*}
[K_m, K_n] &= 0, \quad m, n \geq 0, \\
[K_m, \tau^{(l)}_n] &= mK_{m+n-1}, \quad m, n \geq 0, \\
[\tau^{(l)}_m, \tau^{(l)}_n] &= (m-n)\tau^{(l)}_{m+n-1}, \quad \tau^{(l)}_{-1} = 0, \quad m, n \geq 0.
\end{align*}
\]  \hspace{1cm} (23)

It is referred to as a Virasoro symmetry algebra or hereditary algebra \[17\] of symmetries.
The above symmetry algebraic structure may also be derived from a Lax operator algebra of $V_m, W_n$ determined by (13) and (12)

$$
\begin{align*}
[V_m, V_n] &:= V_m'[K_n] - V_n'[K_m] + [V_m, V_n] = 0, \quad m, n \geq 0, \\
[V_m, W_n] &:= V_m' [\rho_n] - W_n' [K_m] + [V_m, W_n] + \lambda^n V_m \lambda \\
&= m V_{m+n-1}, \quad V_{-1} = 0, \quad m, n \geq 0, \\
[W_m, W_n] &:= W_m' [\rho_n] - W_n' [\rho_m] + [W_m, W_n] + \lambda^n W_{m+n} - \lambda^m W_{n\lambda} \\
&= (m - n) W_{m+n-1}, \quad W_{-1} = 0, \quad m, n \geq 0,
\end{align*}
$$

which can be given in a similar way to [18] or by some direct deduction. A similar Lie algebraic structure for AKNS hierarchy has been shown in [19]. We have known that if the equalities

$$
\begin{align*}
U'[K] + f(\lambda) U_\lambda - V_x + [U, V] &= 0, \\
U'[S] + g(\lambda) U_\lambda - W_x + [U, W] &= 0
\end{align*}
$$

hold, then we have [18]

$$
U'[[K, S]] + [f, g] U_\lambda - [V, W]_x + [U, [V, W]] = 0,
$$

where $[K, S]$ is a commutator of $K, S$, and $[f, g]$ and $[V, W]$ are defined by

$$
[f, g](\lambda) = f'(\lambda) g(\lambda) - f(\lambda) g'(\lambda), \quad [V, W] = V'[S] - W'[K] + [V, W] + g V_\lambda - f W_\lambda.
$$

By using this result and the injective property of the Gateaux derivative operator $U' : K \mapsto U'[K]$, we right now obtain the vector field Lie algebra (21) and further Virasoro symmetry algebra (23) from the above Lax operator algebra.

To summarize, we have constructed a hierarchy of first-degree time-dependent symmetries and have given the commutator relations of the resulting time-dependent symmetries and the original time-independent symmetries. This kind of symmetry algebras is also a common property enjoyed by soliton equations.

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