W$^{2,p}$-A PRIORI ESTIMATES FOR THE NEUTRAL POINCARÉ PROBLEM

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To the memory of Filippo Chiarenza

ABSTRACT. A degenerate oblique derivative problem is studied for uniformly elliptic operators with low regular coefficients in the framework of Sobolev’s classes $W^{2,p}(\Omega)$ for arbitrary $p > 1$. The boundary operator is prescribed in terms of a directional derivative with respect to the vector field $\ell$ that becomes tangential to $\partial \Omega$ at the points of some non-empty subset $E \subset \partial \Omega$ and is directed outwards $\Omega$ on $\partial \Omega \setminus E$. Under quite general assumptions of the behaviour of $\ell$, we derive a priori estimates for the $W^{2,p}(\Omega)$-strong solutions for any $p \in (1, \infty)$.

INTRODUCTION

The lecture deals with regularity in Sobolev’s spaces $W^{2,p}(\Omega)$, $\forall p \in (1, \infty)$, of the strong solutions to the oblique derivative problem

$$
\begin{align*}
\mathcal{L}u &= a^{ij}(x)D_{ij}u = f(x) \quad \text{a.e. } \Omega, \\
\mathcal{B}u &= \frac{\partial u}{\partial \ell} = \varphi(x) \quad \text{on } \partial \Omega
\end{align*}
$$

where $\mathcal{L}$ is a uniformly elliptic operator with low regular coefficients and $\mathcal{B}$ is prescribed in terms of a directional derivative with respect to the unit vector field $\ell(x) = (\ell^1(x), \ldots, \ell^n(x))$ defined on $\partial \Omega$, $n \geq 3$. Precisely, we are interested in the Poincaré problem (1) (cf. [19, 22, 18]), that is, a situation when $\ell(x)$ becomes tangential to $\partial \Omega$ at the points of a non-empty subset $E$ of $\partial \Omega$.

From a mathematical point of view, (1) is not an elliptic boundary value problem. In fact, it follows from the general PDEs theory that (1) is a regular (elliptic) problem if and only if the Shapiro–Lopatinskij complementary condition is satisfied which means $\ell$ must be transversal to $\partial \Omega$ when $n \geq 3$ and $|\ell| \neq 0$ as $n = 2$. If $\ell$ is tangent to $\partial \Omega$ then (1) is a degenerate problem and new effects occur in contrast to the regular case. It turns out that the qualitative properties of (1) depend on the behaviour of $\ell$ near the set of tangency $E$ and especially on the way the normal component $\mathcal{N}$ of $\ell$ (with respect to the outward normal $\mathbf{v}$ to $\partial \Omega$) changes or its sign on the trajectories of $\ell$ when these cross $E$. The main results were obtained by Hörmander [6], Egorov and Kondrat’ev [2], Maz’ya [10], Maz’ya and Paneah [11], Melin and Sjöstrand [12], Paneah [17] and good surveys and details can be found in Popivanov and Palagachev [22] and Paneah [18]. The problem (1) has been studied in the framework of Sobolev spaces $H^s(\equiv H^{s,2})$ assuming $C^\infty$-smooth data and this naturally involved techniques from the pseudo-differential calculus.

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The simplest case arises when $\gamma := \ell \cdot \nu$, even if zero on $\mathcal{E}$, conserves the sign on $\partial \Omega$. Then $\mathcal{E}$ and $\ell$ are of neutral type (a terminology coming from the physical interpretation of (1) in the theory of Brownian motion, see [22]) and (1) is a problem of Fredholm type (cf. [2]). Assume now that $\gamma$ changes the sign from “$-$” to “$+$” in positive direction along the $\ell$-integral curves passing through the points of $\mathcal{E}$. Then $\ell$ is of emergent type and $\mathcal{E}$ is called attracting manifold. The new effect appearing now is that the kernel of (1) is infinite-dimensional ([6]) and to get a well-posed problem one has to modify (1) by prescribing the values of $u$ on $\mathcal{E}$ (cf. [2]). Finally, suppose the sign of $\gamma$ changes from “$+$” to “$-$” along the $\ell$-trajectories. Now $\ell$ is of submergent type and $\mathcal{E}$ corresponds to a repellent manifold. The problem (1) has infinite-dimensional cokernel ([6]) and Maz’ya and Paneah [11] were the first to propose a relevant modification of (1) by violating the boundary condition at the points of $\mathcal{E}$. As consequence, a Fredholm problem arises, but the restriction $u|_{\partial \Omega}$ has a finite jump at $\mathcal{E}$. What is the common feature of the degenerate problems, independently of the type of $\ell$, is that the solution “loses regularity” near the set of tangency from the data of (1) in contrast to the non-degenerate case when any solution gains two derivatives from $f$ and one derivative from $\varphi$. Roughly speaking, that loss of smoothness depends on the order of contact between $\ell$ and $\partial \Omega$ and is given by the subelliptic estimates obtained for the solutions of degenerate problems (cf. [4, 5, 6, 11]). Precisely, if $\ell$ has a contact of order $k$ with $\partial \Omega$ then the solution of (1) gains $2-k/(k+1)$ derivatives from $f$ and $1-k/(k+1)$ derivatives from $\varphi$.

For what concerns the geometric structure of $\mathcal{E}$, it was supposed initially to be a submanifold of $\partial \Omega$ of codimension one. Melin and Sjöstrand [12] and Paneah [17] were the first to study the Poincaré problem (1) in a more general situation when $\mathcal{E}$ is a massive subset of $\partial \Omega$ with positive surface measure, allowing $\mathcal{E}$ to contain arcs of $\ell$-trajectories of finite length. Their results were extended by Winzell ([23, 24]) to the framework of Hölder’s spaces who studied (1) assuming $C^{1,\alpha}$-smoothness of the coefficients of $\mathcal{L}$. It is worth noting that $\ell$ has automatically an infinite order of contact with $\partial \Omega$ when $\mathcal{E}$ is a massive subset of the boundary.

To deal with non-linear Poincaré problems, however, we have to dispose of precise information on the linear problem (1) with coefficients less regular than $C^{\infty}$ (see [13, 20, 21, 22]). Indeed, a priori estimates in $W^{2,p}$ for solutions to (1) would imply easily pointwise estimates for $u$ and $Du$ for suitable values of $p > 1$ through the Sobolev imbeddings. This way, we are naturally led to consider the problem (1) in a strong sense,
that is, to searching for solutions lying in $W^{2,p}$ which satisfy $L u = f$ almost everywhere (a.e.) in $\Omega$ and $B u = \varphi$ holds in the sense of trace on $\partial\Omega$.

In the papers [4, 5] by Guan and Sawyer solvability and precise subelliptic estimates have been obtained for (1) in $H^{s,p}$-spaces ($\equiv W^{s,p}$ for integer $s$!). However, [4] treats operators with $C^{\infty}$-coefficients and this determines the technique involved and the results obtained, while in [5] the coefficients are $C^{0,\alpha}$-smooth, but the field $\ell$ is of finite type, that is, it has a finite order of contact with $\partial\Omega$.

The main goal of this lecture is to derive a priori estimates in Sobolev's classes $W^{2,p}(\Omega)$ with any $p \in (1, \infty)$ for the solutions of the Poincaré problem (1), weakening both Winzell's assumptions on $C^{1,\alpha}$-regularity of the coefficients of $L$ and these of Guan and Sawyer on the finite type of $\ell$. We are dealing with the simpler case when $\gamma$ preserves its sign on $\partial\Omega$ which means the field $\ell$ is of neutral type. Of course, the loss of smoothness mentioned, imposes some more regularity of the data near the set $E$. We assume the coefficients of $L$ to be Lipschitz continuous near $E$ while only continuity (and even discontinuity controlled in $VMO$) is allowed away from $E$. Similarly, $\ell$ is a Lipschitz vector field on $\partial\Omega$ with Lipschitz continuous first derivatives near $E$, and no restrictions on the order of contact with $\partial\Omega$ are required. Regarding the tangency set $E$, it may have positive surface measure and is restricted only to a sort of non-trapping condition that all trajectories of $\ell$ through the points of $E$ are non-closed and leave $E$ in a finite time.

The technique adopted is based on a dynamical system approach employing the fact that $\partial u/\partial n$ is a local strong solution, near $E$, to a Dirichlet-type problem with right-hand side depending on the solution $u$ itself. Application of the $L^p$-estimates for such problems leads to the functional inequality (26) for suitable $W^{2,p}$-norms of $u$ on a family of subdomains which, starting away from $E$, evolve along the $\ell$-trajectories and exhaust a sort of their tubular neighbourhoods. Fortunately, that is an inequality with advanced argument and the desired $W^{2,p}$-estimate follows by iteration with respect to the curvilinear parameter on the trajectories of $\ell$. Another advantage of this approach is the improving-of-integrability property obtained for the solutions of (1). Roughly speaking, it asserts that the problem (1), even if a degenerate one, behaves as an elliptic problem for what concerns the degree $p$ of integrability. In other words, the second derivatives of any solution to (1) will have the same rate of integrability as $f$ and $\varphi$. We refer the reader to the paper [16] for outgrowths of the $W^{2,p}$-a priori estimates, such as uniqueness in $W^{2,p}(\Omega)$, $\forall p > 1$, of the strong solutions to (1) as well as its Fredholmness.

Concluding this introduction, we should mention the articles [8, 9, 15] where similar results have been obtained by different technique in the particular case when the tangency set $E$ contains trajectories of $\ell$ with positive, but small enough lengths.

1. Hypotheses and the Main Result

Hereafter $\Omega \subset \mathbb{R}^n$, $n \geq 3$, will be a bounded domain with reasonably smooth boundary and $\nu(x) = (\nu^1(x), \ldots, \nu^n(x))$ stands for the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$. Consider a unit vector field $\ell(x) = (\ell^1(x), \ldots, \ell^n(x))$ on $\partial\Omega$ and let $\ell(x) = \tau(x) + \gamma(x)\nu(x)$, where $\tau: \partial\Omega \rightarrow \mathbb{R}^n$ is the projection of $\ell(x)$ on the hyperplane tangent to $\partial\Omega$ at $x \in \partial\Omega$ and $\gamma: \partial\Omega \rightarrow \mathbb{R}$ is the inner product $\gamma(x) := \ell(x) \cdot \nu(x)$. The set of zeroes of $\gamma$,

$$E := \{ x \in \partial\Omega : \gamma(x) = 0 \},$$

is indeed the subset of $\partial\Omega$ where the field $\ell(x)$ becomes tangent to it.
Fix $\mathcal{N} \subset \overline{\Omega}$ to be a closed neighbourhood of $\mathcal{E}$ in $\overline{\Omega}$. We suppose $\mathcal{L}$ is a uniformly elliptic operator with measurable coefficients, satisfying
\begin{equation}
\lambda^{-1} |\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \lambda |\xi|^2 \quad \text{a.a. } x \in \Omega, \; \forall \xi \in \mathbb{R}^n, \quad a^{ij}(x) = a^{ji}(x)
\end{equation}
for some positive constant $\lambda$. Regarding the regularity of the data, we assume
\begin{equation}
\begin{cases}
a^{ij} \in VMO(\Omega) \cap C^{0,1}(\mathcal{N}), \\
\partial \Omega \in C^{1,1}, \; \partial \Omega \cap \mathcal{N} \in C^{2,1}, \; \ell^i \in C^{0,1}(\partial \Omega) \cap C^{1,1}(\partial \Omega \cap \mathcal{N})
\end{cases}
\end{equation}
with $VMO(\Omega)$ being the Sarason class of functions of vanishing mean oscillation and $C^{k,1}$ denotes the space of functions with Lipschitz continuous $k$-th order derivatives. Let us point out that \eqref{2}, \eqref{3} and the Rademacher theorem give $a^{ij} \in L^\infty(\Omega) \cap W^{1,\infty}(\mathcal{N})$.

For what concerns the boundary operator $\mathcal{B}$, we assume
\begin{equation}
\begin{cases}
\gamma(x) = \ell(x) \cdot \nu(x) \geq 0 \quad \forall x \in \partial \Omega, \quad \text{and} \\
\text{the arcs of the } \ell\text{-trajectories lying in } \mathcal{E} \text{ (which coincide with these of } \tau) \\
\text{are all non-closed and of finite lengths.}
\end{cases}
\end{equation}

The first assumption simply means that $\ell(x)$ is either tangential to $\partial \Omega$ or is directed outwards $\Omega$, that is, the field $\ell$ is of neutral type on $\partial \Omega$, while the second one is a sort of non-trapping condition on the tangency set $\mathcal{E}$. It implies that the $\ell$-integral curves leave $\mathcal{E}$ in a finite time in both directions.

Throughout the text $W^{k,p}$ stands for the Sobolev class of functions with $L^p$-summable weak derivatives up to order $k \in \mathbb{N}$ while $W^{s,p}(\partial \Omega)$ with $s > 0$ non-integer and $p \in (1, +\infty)$, is the Sobolev space of fractional order on $\partial \Omega$. Further, we use the standard parameterization $t \mapsto \psi_L(t; x)$ for the trajectory (equivalently, phase curve, maximal integral curve) of a given vector field $\mathbf{L}$ passing through a point $x$, that is, $\partial_s \psi_L(t; x) = \mathbf{L}(\psi_L(t; x))$ and $\psi_L(0; x) = x$.

We will employ below an extension of the field $\ell$ near $\partial \Omega$ which preserves therein its regularity and geometric properties. All the results and proofs in the sequel work for such an arbitrary $\ell$-extension but, in order to make more evident some geometric constructions, we prefer to introduce a special extension as follows. For each $x \in \mathbb{R}^n$ near $\partial \Omega$ set $d(x) = \text{dist}(x, \partial \Omega)$ and define $\Gamma := \{x \in \mathbb{R}^n : d(x) \leq d_0\}$ with small $d_0 > 0$. Letting $\Omega_0 := \Omega \setminus \Gamma$ and $y(x) \in \partial \Omega$ for the unique point closest to $x \in \Gamma$, we have (see \cite{3} Chapter 14) $y(x) \in C^{0,1}(\Gamma)$ while $y(x) \in C^{1,1}$ near $\mathcal{E}$. Regarding the distance function $d(x) = |x - y(x)|$, it is Lipschitz continuous in $\Gamma$ and inherits the regularity of $\partial \Omega$ at
solution to (1) belongs to $W(5)$ quotients in (1) in the direction of the $L$ problem Theorem 1. Under the hypotheses to (1) for any $p$ equipped with norm at most $\kappa$ normed by $\|E\|$ quantities defined by the data of (1), that is, on $n, p, \lambda$, continuity properties of the lengths of the $\tau$ under the hypotheses get that (see [24, Proposition 3.1] and [22, Proposition 3.2.4]) and $L$ coefficients of $\partial$ transversal to the Banach spaces $\ell$. The normalized representative of $L$ finite jump on $\partial y(x)$ in $\partial \Omega$. Then $u_{\partial u/\partial L}$

Let us point out reader’s attention that the directional derivative $\partial u/\partial L$ of each $W^{2,p}$-solution to (1) belongs to $W^{2,p}(\Omega)$. For, $\partial u/\partial L \in W^{1,p}(\Omega)$ and taking the difference quotients in (1) in the direction of $L$ (cf. [3, Chapter 8 and Lemma 7.24]) gives that $u$ when considered on the parts of $\Gamma$ lying in/out $\Omega$, but its normal derivative has a finite jump on $\partial \Omega$. Anyway, it is a routine to check $(d(x))^2 \in C^{1,1}(\Gamma)$. Setting $L(x)$ for the normalized representative of $\ell(y(x)) + (d(x))^2 \nu(y(x)) \forall x \in \Gamma$, it results $|L(x)| = 1$, $L|_{\partial \Omega} = \ell$, $L_E = \tau$ and $L \in C^{0,1}(\Gamma) \cap C^{1,1}(\Gamma \cap N)$. Moreover, the field $L$ is strictly transversal to $\partial \Omega_0$.

As consequence of the non-trapping condition ([4]), the compactness of $E$ and the semi-continuity properties of the lengths of the $\tau$-maximal integral curves, it is not hard to get that (see [24, Proposition 3.1] and [22, Proposition 3.2.4]) under the hypotheses and (1), there is a finite upper bound $\kappa_0$ for the arclengths of the $\tau$-trajectories lying in $E$. Moreover, each point of $\Gamma$ can be reached from $\partial \Omega$ by an $L$-integral curve of length at most $\kappa = \text{const} > 0$.

In what follows, the letter $C$ will denote a generic constant depending on known quantities defined by the data of (1), that is, on $n, p, \lambda$, the respective norms of the coefficients of $L$ and $B$ in $\Omega$ and $N$, the regularity of $\partial \Omega$ and the constants $\kappa_0$ and $\kappa$.

In order to control precisely the regularity of $u$ near the tangency set $E$, we have to introduce the appropriate functional spaces. For, take an arbitrary $p \in (1, \infty)$ and define the Banach spaces $F^p(\Omega, N) := \{ f \in L^p(\Omega): \partial f/\partial L \in L^p(N) \}$ equipped with norm $\|f\|_{F^p(\Omega, N)} := \|f\|_{L^p(\Omega)} + \|\partial f/\partial L\|_{L^p(N)}$, and $\Phi_p(\partial \Omega, N) := \{ \varphi \in W^{1-1/p, p}(\partial \Omega): \varphi \in W^{2-1/p, p}(\partial \Omega \cap N) \}$ normed by $\|\varphi\|_{\Phi_p(\partial \Omega, N)} := \|\varphi\|_{W^{1-1/p, p}(\partial \Omega)} + \|\varphi\|_{W^{2-1/p, p}(\partial \Omega \cap N)}$.

Our main result asserts that the couple $(L, B)$ improves the integrability of solutions to (1) for any $p$ in the range $(1, \infty)$ and, moreover, provides for an a priori estimate in the $L^p$-Sobolev scales for any such solution.

**Theorem 1.** Under the hypotheses (2) + (4) let $u \in W^{2,p}(\Omega)$ be a strong solution of the problem (1) with $f \in F^p(\Omega, N)$ and $\varphi \in \Phi_p(\partial \Omega, N)$ where $1 < p \leq q < \infty$.

Then $u \in W^{2,q}(\Omega)$ and there is an absolute constant $C$ such that

$$
\|u\|_{W^{2,q}(\Omega)} \leq C \left( \|u\|_{L^q(\Omega)} + \|f\|_{F^p(\Omega, N)} + \|\varphi\|_{\Phi_p(\partial \Omega, N)} \right).
$$

Let us point out reader’s attention that the directional derivative $\partial u/\partial L$ of each $W^{2,p}$-solution to (1) belongs to $W^{2,p}(\Omega)$. For, $\partial u/\partial L \in W^{1,p}(\Omega)$ and taking the difference quotients in (1) in the direction of $L$ (cf. [3, Chapter 8 and Lemma 7.24]) gives that
\( \partial u / \partial L \in W^{2,p}(N) \) is a strong local solution to the Dirichlet problem

\[
\begin{cases}
    L \left( \frac{\partial u}{\partial \xi} \right) = \frac{\partial^2 u}{\partial x^2} + 2a^{ij}D_iL^kD_ku + a^{ij}D_iD_jL^kD_ku - \frac{\partial a^{ij}}{\partial x^2}D_\xi \nabla \phi & \text{a.e. } N, \\
    \frac{\partial u}{\partial \xi} \in \phi & \text{on } \partial \Omega \cap N
\end{cases}
\]

where \( L(x) = (L^1(x), \ldots, L^n(x)) \in C^{1,1}(N) \). Therefore, once having proved \( u \in W^{2,q}(\Omega) \) and the estimate (5), we have

\[
\| \partial u / \partial L \|_{W^{2,q}(N)} \leq C\left( \| u \|_{L^q(\Omega)} + \| f \|_{L^q(\Omega)} + \| \phi \|_{L^q(\partial \Omega, N)} \right)
\]

for any closed neighbourhood \( \tilde{N} \) of \( E \) in \( \bar{\Omega} \), \( \tilde{N} \subset N \), by means of the \( L^p \)-theory of uniformly elliptic equations (see [1] or [3, Chapter 9]). In other words, if a strong solution \( u \) to (1) belongs to \( W^{2,q}(N) \) then \( \partial u / \partial L \in W^{2,q}(N) \) automatically, provided \( f \in F^q(\Omega, N) \) and \( \phi \in \Phi^q(\partial \Omega, N) \).

2. Proof of Theorem 1

Fix hereafter \( N' \subset N'' \subset N \) to be closed neighbourhoods of \( E \) in \( \bar{\Omega} \) with \( N'' \) so “narrow” that \( N'' \subset \Omega \setminus \Omega_0 \) (see Figure 3). The next result is an immediate consequence of \( \gamma(x) > 0 \) \( \forall x \in \Omega \setminus N'' \) and the \( L^p \)-theory of regular oblique derivative problems for uniformly elliptic operators with \( V \) as principal coefficients (cf. [2] Theorem 2.3.1]).

**Proposition 2.** Assume (2), (3) and \( \gamma(x) > 0 \) \( \forall x \in \Omega \setminus E \), and let \( u \in W^{2,p}(\Omega) \) be a solution to (1) with \( f \in L^q(\Omega) \) and \( \phi \in W^{1/4,q}(\partial \Omega) \), where \( 1 < q < \infty \).

Then \( u \in W^{2,q}(\Omega \setminus N') \) and there is a constant such that

\[
\| u \|_{W^{2,q}(\Omega \setminus N')} \leq C\left( \| u \|_{L^q(\Omega)} + \| f \|_{L^q(\Omega)} + \| \phi \|_{W^{1/4,q}(\partial \Omega)} \right).
\]

To derive the improving-of-integrability near the tangency set \( E \), consider any solution of the problem (1) for which \( a^{ij}, \partial a^{ij} / \partial L \in L^\infty(N) \) in view of (3) and \( f, \partial f / \partial L \in L^q(\Omega) \) and \( \phi \in W^{2/1+q}(\partial \Omega \cap N) \) by hypotheses.

**Lemma 3.** Under the assumptions of Theorem 1 the solution \( u \) of (1) belongs to \( u \in W^{2,q}(N'') \) and there is a constant such that

\[
\| u \|_{W^{2,q}(N'')} \leq C\left( \| u \|_{L^q(\Omega)} + \| f \|_{L^q(\Omega,N')} + \| \phi \|_{W^{1/4,q}(\partial \Omega,N')} \right).
\]

**Proof.** Take an arbitrary point \( x_0 \in E \). According to (3), the \( L \)-trajectory through \( x_0 \) leaves \( E \) in both directions for a finite time, that is, \( \psi_L(t^-; x_0) \in N'' \setminus N' \), \( \psi_L(t^+; x_0) \in \mathbb{R}^n \setminus \mathbb{P} \) (see Figure 3) for suitable \( t^- < 0 < t^+ \).

Set \( \mathcal{H} \) for the \( (n-1) \)-dimensional hyperplane through \( x_0 \) and orthogonal to \( L(x_0) \), and define

\[
B_r(x_0) := \{ x \in \mathcal{H} : |x - x_0| < r \}
\]

with \( r > 0 \) to be chosen later. It follows from the Picard inequality that if \( r \) is small enough, then the flow of \( B_r(x_0) \) along the \( L \)-trajectories at time \( t^- \),

\[
B'_r(x_0) := \{ \psi_L(t^-; y) : y \in B_r(x_0) \}
\]

is entirely contained in \( N'' \setminus N' \) whence \( B'_r(x_0) \cap E = \emptyset \). The set

\[
\Theta_r := \{ \psi_L(t; z^r) : z^r \in B'_r(x_0), \ t \in (0, t^+ - t^-) \}
\]

It will be clear from the considerations given below that instead of Lipschitz continuity of the coefficients of \( L \) in \( N \) as (3) asks, it suffices to have essentially bounded their directional derivatives with respect to the field \( L \).

\[
\psi_L(t; z^r) \leq 2a^{ij}(N)|z^r - x''| \text{ for all } z^r, x'' \in N.
\]
is an $n$-dimensional neighbourhood of the $L$-trajectory through $x_0$ and defining

$$\mathcal{T}_r := \Theta_r \cap \Omega,$$

the boundary $\partial \mathcal{T}_r$ is composed of the “base” $B_r'(x_0)$ and the “lateral” components $\partial_1 \mathcal{T}_r := \partial \mathcal{T}_r \cap \partial \Omega$ and $\partial_2 \mathcal{T}_r := (\partial \mathcal{T}_r \cap \Omega) \setminus B_r'(x_0)$. Indeed, $\mathcal{T}_r \subset N'$ if $r > 0$ is small enough.

We will derive (8) in $\mathcal{T}_r$ after that the desired estimate will follow by covering the compact $\mathcal{E} \subset \partial \Omega$ by a finite number of sets like $\mathcal{T}_r$. Our strategy is based on a representation of $u(x)$ in $\mathcal{T}_r$ by means of $u(x')$ with $x' = \psi_L(-\xi(x); x) \in B_r'(x_0)$ for some $\xi(x) > 0$, and the integral of $\partial u/\partial L$ along the $L$-trajectory joining $x'$ with $x$. Thus the Sobolev norm of $u$ will be expressed by the respective norm of $\partial u/\partial L$ and that of $u$ itself near $B_r'(x_0)$ where we dispose of (7). Concerning $\partial u/\partial L$, it is a local solution of Dirichlet problem near $\mathcal{E}$ with right-hand side depending on $u$.

Let $\mu : \mathcal{H} \to \mathbb{R}^+$ be a $C^\infty$ cut-off function such that

$$\mu(y) = \begin{cases} 1 & y \in B_{r/2}(x_0), \\ 0 & y \in \mathcal{H} \setminus B_{3r/4}(x_0) \end{cases}$$

and extend it to $\mathbb{R}^n$ as constant on the $L$-trajectory through $y \in \mathcal{H}$. The function $U(x) := \mu(x)u(x)$ is a $W^{2,p}(N)$-solution of

$$\begin{cases} \mathcal{L}U = F(x) := \mu f + 2a^{ij}D_j\mu D_i u + u a^{ij}D_{ij}\mu & \text{a.e. } \mathcal{T}_r, \\ \partial U/\partial L = \Phi := \begin{cases} \mu \varphi & \text{on } \partial_1 \mathcal{T}_r, \\ 0 & \text{near } \partial_2 \mathcal{T}_r, \\ \mu \partial u/\partial L & \text{on } B_r'(x_0) \subset N' \setminus N'' \end{cases}. \end{cases}$$

Indeed, $u \in W^{2,p}(N)$ implies $Du \in L^{np/(n-p)}$ if $p < n$ and $Du \in L^s \forall s > 1$ when $p \geq n$, whence $F \in L^{q'}(N)$ with

$$q' := \begin{cases} \min \left\{ q, \frac{np}{n-p} \right\} & \text{if } p < n, \\ q & \text{if } p \geq n. \end{cases}$$

Figure 3. $\mathcal{T}_r$ is the dotted set, while the double-dotted one is $P_{r, T}$.
Further, $\partial F/\partial L \in L^q(N'')$ as consequence of (9), $\partial u/\partial L \in W^{2,q}(N'' \setminus N')$ by Proposition 2 whence $\Phi \in W^{2-1/q,q}(\partial T_r)$. Thus (2), (3). $T_r \subset N''$ and (9) give that

$$V(x) := \partial U/\partial L$$

is a $W^{2,p}(T_r)$-solution of the Dirichlet problem

$$\begin{cases}
L V = \partial F/\partial L + 2a^{ij}D_jD_kD_iD_kU + a^{ij}D_iD_jL^kD_kU - \frac{\partial a^{ij}}{\partial x}D_iU & \text{a.e. } T_r, \\
V = \Phi & \text{on } \partial T_r.
\end{cases}$$

(12)

Now we pass from $x \in \Theta_r$ into the new variables $(x', \xi)$ with $x' = \psi_L(-\xi(x); x) \in B'_r(x_0)$ and $\xi : \Theta_r \to (0, t^+ - t^-)$, $(\xi(x) \in C^{1,1}(\Theta_r)$. The transform $x \mapsto (x', \xi)$ defines a $C^{1,1}$-diffeomorphism because the field $L$ is transversal to $B'_r(x_0)$. Moreover, $\partial/\partial L \equiv \partial/\partial \xi$, $\psi_L(t; x') = (x', t)$ and $V(x', \xi) = \partial U(x', \xi)/\partial \xi$ as $(x', \xi) \in T_r$. Since $V(x', \xi)$ is an absolutely continuous function in $\xi$ for a.a. $x' \in B'_r(x_0))$ (after redefining it, if necessary, on a set of zero measure) we get

$$U(x', \xi) = U(x', 0) + \int_0^\xi V(x', t)dt \quad \text{for a.a. } (x', \xi) \in T_r,$$

where the point $(x', 0) \in B'_r(x_0)$ lies in $N'' \setminus N'$ and $U(x', 0) \in W^{2,q}$ there by Proposition 2 the Fubini theorem and [13], Remark 2.1. Passing to the new variables $(x', \xi)$ in (12), taking the derivatives of (13) up to second order and substituting them into the right-hand side of (12), this last reads

$$\begin{cases}
\mathcal{L}' V = F_1(x', \xi) + \int_0^\xi \mathcal{D}_2(\xi)V(x', t)dt & \text{a.e. } T_r, \\
V = \Phi & \text{on } \partial T_r.
\end{cases}$$

(14)

where $\mathcal{L}'$ is the operator $\mathcal{L}$ in terms of $(x', \xi) = (x_1', \ldots, x_{n-1}')$, $\xi$,

$$F_1(x', \xi) := \partial F/\partial L + \mathcal{D}_1V(x', \xi) + \mathcal{D}_1'U(x', \xi) + \mathcal{D}_2'U(x', 0),$$

$$\mathcal{D}_2(\xi)V(x', t) := \sum_{i,j=1}^{n-1} A^{ij}(x', \xi)D_{x_i'}x_j'V(x', t), \quad A^{ij} \in L^\infty,$$

$\mathcal{D}_1, \mathcal{D}_1', \mathcal{D}_2$ are linear differential operators with $L^\infty$-coefficients, ord $\mathcal{D}_1 = \text{ord } \mathcal{D}_1' = 1$, ord $\mathcal{D}_2' = 2$. The Sobolev imbedding theorem implies $F_1 \in L^q(T_r)$ with $q'$ by (11) as consequence of $\partial F/\partial L \in L^q(N'')$, $U(x', 0) \in W^{2,q}(B'_r(x_0))$ and $U, V \in W^{2,p}(N'')$. Nevertheless the second-order operator $\mathcal{D}_2(\xi)$ has a quite rough characteristic form which is neither symmetric nor sign-definite, the improving-of-integrability holds for (13) thanks to the particular structure of $T_r$ as union of $L$-trajectories through $B'_r(x_0)$. Actually, we will show that if $V \in W^{2,q'}$ on a subset of $T_r$ with $\xi < T$, then $V$ remains a $W^{2,q'}$-function on a larger subset with $\xi < T + r$ for small enough $r$, after that the higher integrability of $U$ will follow from Proposition 2 and (13). For, take an arbitrary $T \in (0, t^+ - t^-)$ and define

$$\mathcal{P}_{r,T} := \{(x', \xi) \in T_r : \xi < T\}.$$

For a fixed $r > 0$, $\{\mathcal{P}_{r,T}\}_{T \geq 0}$ is a non-decreasing family of domains exhausting $T_r$ and $\mathcal{P}_{r,T} \equiv T_r$ for values of $T$ greater than the maximal exit-time

$$T_{\text{max}} := \sup_{x' \in B'_r(x_0)} \sup \{t > 0 : \psi_L(t; x') \in \Omega, x' \in B'_r(x_0)\}.$$

Proposition 4. Let $T \in (0, t^+ - t^-)$ and consider the solution $V \in W^{2,q}(T_r)$ of the problem (11). Suppose $V \in W^{2,q}(\mathcal{P}_{r,T})$ where $q'$ is given by (11).

There exists an $r_0 > 0$ such that $V \in W^{2,q}(\mathcal{P}_{r,T+r})$ for all $r < r_0$. 


Proof. There are three possible cases to be distinguished.

Case A: $T + 3r < T_{\text{max}}$. We have $\mathcal{P}_{r,T} \subset \mathcal{P}_{r,T+3r} \subset \mathcal{T}_r = \mathcal{P}_{r,T_{\text{max}}}$ and consider the $C^\infty$-function $\eta: \mathbb{R} \to [0,1]$ such that

$$\eta(\xi) = \begin{cases} 1 & \text{as } \xi \in (-\infty,T+r], \\ \text{strictly decreases} & \text{as } \xi \in (T+r,T+2r), \\ 0 & \text{as } \xi \geq T+2r. \end{cases}$$

Setting $\tilde{\mathcal{V}}(x',\xi) := \eta(\xi)V(x',\xi)$, it follows $\mathcal{L}'\tilde{\mathcal{V}} = \eta(\mathcal{L}' \mathcal{V}) + \mathcal{L}_1 \mathcal{V}$ where $\mathcal{L}_1$ is a first-order differential operator with $L^\infty$-coefficients depending on these of $\mathcal{L}'$ and on the derivatives of $\eta$. Therefore,

$$\mathcal{L}'\tilde{\mathcal{V}} = \eta F_1 + \mathcal{L}_1 \mathcal{V} + \eta(\xi) \int_0^T \mathcal{D}_2(\xi) \tilde{\mathcal{V}}(x',t) dt$$

which is a second-order operator acting in the $x'$-variables only.

We set $\Omega_r \subset \mathcal{P}_{r,T+3r} \setminus \mathcal{P}_{r,T-3r}$ for a $C^{1,1}$-smooth domain containing $\mathcal{P}_{3r/4,T+2r} \setminus \mathcal{P}_{3r/4,T-2r}$ and such that

$$r^{-1} \Omega_r := \left\{ (\tilde{y}',\tilde{\xi}) : \tilde{y}' = x'/r, \tilde{\xi} = (\xi - T)/r, (x',\xi) \in \Omega_r \right\} \in C^{1,1}$$

uniformly in $r$. The boundary of $\Omega_r$ consists of the “lateral” parts $\partial_1 \Omega_r := \partial_1 \Omega_r \cap \partial \Omega$ and $\partial_2 \Omega_r := \partial_2 \Omega_r \cap \Omega \cap \{ \xi \in (T-2r,T+2r) \} \subset (\mathcal{P}_{r,T+2r} \setminus \mathcal{P}_{r,T-2r}) \setminus (\mathcal{P}_{3r/4,T+2r} \setminus \mathcal{P}_{3r/4,T-2r})$, and of two $C^{1,1}$-smooth components $\partial_1 \Omega^\pm_r$ lying in $\mathcal{P}_{r,T+3r} \setminus \mathcal{P}_{r,T+2r}$ and $\mathcal{P}_{r,T-3r} \setminus \mathcal{P}_{r,T-2r}$, respectively. The properties of $\mu$ (cf. [9]) ensure $\mathcal{U} \equiv 0$, $\mathcal{V} \equiv 0$, $\tilde{\mathcal{V}} \equiv 0$ on $\mathcal{T}_r \setminus \mathcal{T}_{3r/4}$ whence $\tilde{\mathcal{V}} \equiv 0$ near $\partial_2 \Omega_r$.

For an arbitrary $(x',\xi) \in \Omega_r$, the factor $\eta(\xi)/\eta(t)$ in (17) vanishes when $\xi \geq T+2r$ while $\eta(\xi)/\eta(t) \leq 1$ because $\eta$ decreases in $(T+r,T+2r)$. Moreover, $|\xi - T| < 3r$ for $(x',\xi) \in \Omega_r$ and

$$\int_0^T \eta(t) \mathcal{D}_2(\xi) \tilde{\mathcal{V}}(x',t) dt = \int_0^T \eta(t) \mathcal{D}_2(\xi) \tilde{\mathcal{V}}(x',t) dt + \int_T^\infty \eta(t) \mathcal{D}_2(\xi) \tilde{\mathcal{V}}(x',t) dt$$

by means of (15) and since $\eta(t) = \eta(T) = 1$ as $t \leq T$.

We get from (12) and (17) that $\tilde{\mathcal{V}} \in W^{2,p}(\Omega_r)$ solves the Dirichlet problem

$$\begin{cases} \mathcal{L}'\tilde{\mathcal{V}} = F_2(x',\xi) + \int_T^\infty \eta(t) \mathcal{D}_2(\xi) \tilde{\mathcal{V}}(x',t) dt & \text{a.e. } (x',\xi) \in \Omega_r, \\ \eta \mu \varphi \in W^{2-1/q,q} & \text{on } \partial_1 \Omega_r \quad \text{(by (10)),} \\ \eta \mu \varphi = 0 & \text{on } \partial_2 \Omega_r \quad \text{(by (10)),} \\ \eta \mu \varphi = 0 & \text{on } \partial_1 \Omega^+_r \quad \text{(by (16),)} \\ V \in W^{2-1/q',q'} & \text{on } \partial_1 \Omega^- \quad \text{(since } \xi < T-2r \text{ and } V \in W^{2,q'}(\mathcal{P}_{r,T})).} 
\end{cases}$$

where, recalling $V \in W^{2,q'}(\mathcal{P}_{r,T})$, we have

$$F_2(x',\xi) := \eta F_1 + \mathcal{L}_1 V + \eta(\xi) \int_0^T \mathcal{D}_2(\xi) V(x',t) dt \in L^{q'}(\Omega_r).$$
We are going to prove now that \( \tilde{V} \in W^{2,q}(\Omega_r) \) for small enough \( r > 0 \), whence it will follow \( V \in W^{2,q}(\mathcal{P}_{r,T+r}) \) in view of (10) and \( V \equiv 0 \) near \( \partial_2 \Omega_r \). The claim is obvious if \( q' = p \) because \( V \in W^{2,p}(\mathcal{P}_r) \). Otherwise, take an arbitrary \( s \in [p,q'] \) and denote by \( W^{2,s}_r(\Omega_r) \) the Sobolev space normed with
\[
\|u\|_{W^{2,s}_r(\Omega_r)} := \|u\|_{L^r(\Omega_r)} + \|Du\|_{L^r(\Omega_r)} + r^2\|D^2u\|_{L^r(\Omega_r)}.
\]
Define now the operator \( \Phi : W^{2,s}_r(\Omega_r) \to W^{2,s}_r(\Omega_r) \) as follows: for any \( w \in W^{2,s}_r(\Omega_r) \) the image \( \Phi w \in W^{2,s}_r(\Omega_r) \) normed with
\[
\|w\|_{W^{2,s}_r(\Omega_r)} := \|w\|_{L^r(\Omega_r)} + \|Dw\|_{L^r(\Omega_r)} + r^2\|D^2w\|_{L^r(\Omega_r)}.
\]
In order to apply the \( L^s \)-a priori estimates from [14] or [3] for the solutions of (21), we have to control the dependence on \( r \) therein. For, we recall that \( r^{-1}\Omega_r \in C^{1,1} \) uniformly in \( r \) and apply a standard approach consisting of dilation of \( \Omega_r \) onto \( r^{-1}\Omega_r \), reduction of the problem (21) to a new one in variables \( (\tilde{\eta}', \xi) \in r^{-1}\Omega_r \), application of the \( L^s \)-estimates from [3] Theorem 9.17 and finally turning back to (21) (see the Proof of Lemma 2.2, Eq. (2.12) in [14]). This way, one gets
\[
\|\Phi w_1 - \Phi w_2\|_{W^{2,s}_r(\Omega_r)} \leq C r^2 \left\| \int_T^T \frac{\eta(\xi)}{\eta(t)} D_2(\xi)(w_1 - w_2)(x', t) dt \right\|_{L^r(\Omega_r)}
\]
where the constant \( C \) is independent of \( r \). Jensen’s integral inequality yields
\[
r^2 \left\| \int_T^T \frac{\eta(\xi)}{\eta(t)} D_2(\xi)(w_1 - w_2)(x', t) dt \right\|_{L^r(\Omega_r)} \leq C \max_{(x', \xi) \in \Omega_r} |\xi - T|\|w_1 - w_2\|_{W^{2,s}_r(\Omega_r)}
\]
and thus (22) rewrites into
\[
\|\Phi w_1 - \Phi w_2\|_{W^{2,s}_r(\Omega_r)} \leq C \max_{(x', \xi) \in \Omega_r} |\xi - T|\|w_1 - w_2\|_{W^{2,s}_r(\Omega_r)}.
\]
We have \( \max_{(x', \xi) \in \Omega_r} |\xi - T| < 3r \), \( C \) is independent of \( r \) and therefore \( \Phi \) will be really a contraction from \( W^{2,s}_r(\Omega_r) \) into itself for any \( s \in [p,q'] \) if \( r \leq r_0 \) with \( r_0 \) under control and small enough. Fixing \( r = r_0/2 \), there is a unique fixed point of \( \Phi \) in \( W^{2,s}_r(\Omega_r) \) for all \( s \in [p,q'] \). However, \( \tilde{V} \in W^{2,p}(\Omega_r) \) is already a fixed point of \( \Phi \) since it solves (18) and therefore \( \tilde{V} \in W^{2,q'}(\Omega_r) \). It follows \( V \in W^{2,q'}(\mathcal{P}_{r,T+r}) \) by means of \( V \in W^{2,q'}(\mathcal{P}_{r,T}) \), \( \bar{V} \equiv 0 \) on \( \mathcal{T}_r \setminus \mathcal{T}_{3r/4} \) and the properties of \( \eta(\xi) \).

**Case B:** \( T < T_{\max} \leq T + 3r \). We have \( \mathcal{T}_r \setminus \mathcal{P}_{r,T} \neq \emptyset, \mathcal{P}_{r,T+3r} \equiv \mathcal{T}_r \), now and we do not need anymore the cut-off function \( \eta \) because \( V = \partial U/\partial L \equiv 0 \) near the points of \( \partial_2 \mathcal{T}_r \) where \( \xi > T \) (cf. (19)). Thus, it suffices to repeat the above arguments with \( \eta(\xi) \equiv 1 \forall \xi \in \mathbb{R} \) and \( \Omega_r \in C^{1,1} \) defined as before when \( \xi \leq T \) while \( \mathcal{T}_{3r/4} \setminus \mathcal{P}_{3r/4,T} \subset (\Omega_r \cap \{\xi > T\}) \subset \mathcal{T}_r \setminus \mathcal{P}_{r,T} \).
(cf. (9)). We have anyway a problem like (18) for \( V \equiv \tilde{V} \) with boundary condition

\[
V = \partial U / \partial L = \begin{cases}
\mu \varphi \in W^{2-1/q,q} & \text{on } \partial_1 \Omega_r = \partial_\Omega \cap \partial \Omega, \\
0 & \text{on } \partial_2 \Omega_r = \partial_\Omega \cap \Omega \setminus \{ \xi > T - 3r \}, \\
V \in W^{2-1/q',q'} & \text{on } \partial \Omega^-_r \quad \text{(by hypothesis)}.
\end{cases}
\]

Therefore, the procedure from Case A gives \( V \in W^{2,q'}(\mathcal{P}_{r,T+3r}) \).

**Case C**: \( T_{\text{max}} \leq T \). We have \( \mathcal{P}_{r,T+r} = \mathcal{P}_{r,T} \equiv \mathcal{T}_r \) now and thus the claim.

\[\Box\]

**Proposition 5.** Suppose \( r < r_0 \) with \( r_0 \) given in Proposition 4. Then the solution \( V \) of the problem (14) lies in \( W^{2,q}(\mathcal{T}_r) \) and satisfies the estimate

\[
\| V \|_{W^{2,q}(\mathcal{T}_r)} \leq C \left( \| u \|_{L^q(\Omega)} + \| f \|_{L^q(\Omega,\mathcal{N})} + \| \varphi \|_{L^q(\partial_\Omega,\mathcal{N})} + \| \partial u / \partial L \|_{W^{1,q}(\mathcal{T}_r)} \right),
\]

where \( \mathcal{N} \) is the Neumann boundary condition.

**Proof.** We note that \( V \in W^{2,q} \subseteq W^{2,q'} \) near \( B'_r(x_0) \) in view of \( B'_r(x_0) \subset N^m \setminus N^p \), Proposition 2 and 6. Therefore, successive applications of Proposition 4 with increasing values of \( T \) will give \( V \in W^{2,q'}(\mathcal{T}_r) \), \( q' > p \). After that, in order to get \( V \in W^{2,q}(\mathcal{T}_r) \), it suffices to put \( q' \) in the place of \( p \) in (11) and to repeat finitely many times the above arguments until \( q' = q \).

To obtain (23), we take \( T \in (0,t^+-t^-) \) to be arbitrary, fix \( r = r_0/2 \), and consider the domains \( \Omega_r \) defined in the proof of Proposition 4. Let \( \tilde{V} = \eta V \in W^{2,q}(\mathcal{T}_r) \) solve (18) with \( \eta \) given by (10) in Case A and \( \eta \equiv 1 \) in Case B. Since \( \tilde{V} \) is a fixed point of the mapping \( \mathcal{F} : W^{2,q}(\Omega_r) \to W^{2,q}(\Omega_r) \), \( \mathcal{F} \tilde{V} = \tilde{V} \), we get

\[
\| D^2 \tilde{V} \|_{L^q(\Omega_r)} = \| D^2(\mathcal{F} \tilde{V}) \|_{L^q(\Omega_r)} \leq \| D^2(\mathcal{F} \tilde{V} - \mathcal{F} 0) \|_{L^q(\Omega_r)} + \| D^2(\mathcal{F} 0) \|_{L^q(\Omega_r)},
\]

while

\[
\| D^2(\mathcal{F} w_1 - \mathcal{F} w_2) \|_{L^q(\Omega_r)} \leq \theta \| D^2(w_1 - w_2) \|_{L^q(\Omega_r)} \quad \forall w_1, w_2 \in W^{2,q}(\Omega_r), \quad \theta < 1
\]

because \( \mathcal{F} \) is a contraction, (22) and the fact that \( D_2(\xi) \) is a homogeneous second-order operator (cf. (14)). This way, \( \| D^2(\mathcal{F} \tilde{V} - \mathcal{F} 0) \|_{L^q(\Omega_r)} \leq \theta \| D^2(\tilde{V} - 0) \|_{L^q(\Omega_r)} = \theta \| D^2 \tilde{V} \|_{L^q(\Omega_r)} \) and therefore

\[
\| D^2 \tilde{V} \|_{L^q(\Omega_r)} \leq C \| D^2(\mathcal{F} 0) \|_{L^q(\Omega_r)}
\]

with \( \mathcal{F} 0 \in W^{2,q}(\Omega_r) \) being the unique solution of the Dirichlet problem

\[
\left \{ \begin{array}{l}
\mathcal{L}'(\mathcal{F} 0) = F_2 \quad \text{a.e. } \Omega_r, \\
\mathcal{F} 0 = \tilde{\Phi} \quad \text{on } \partial \Omega_r
\end{array} \right.
\]

(see (20)), for which the \( L^p \)-theory (cf. [3, Chapter 9]) gives

\[
\| D^2(\mathcal{F} 0) \|_{L^q(\Omega_r)} \leq \| \mathcal{F} 0 \|_{W^{2,q}(\Omega_r)} \leq C \left( \| F_2 \|_{L^q(\Omega_r)} + \| \tilde{\Phi} \|_{W^{2-1/q',q}(\partial_\Omega)} \right).
\]
Direct applications, based on (19) and (15), yield
\[
\|F_2\|_{L^q(\Omega_r)} \leq C \left( \|\partial F/\partial L\|_{L^q(\Omega_r)} + \|U\|_{W^2,q(N'' \setminus N')} + \|U\|_{W^1,q(\Omega_r)} + \|V\|_{W^1,q(\Omega_r)} + \|D^2V\|_{L^q(\Omega_r)} \right)
\]
\[
\leq C \left( \|\partial f/\partial L\|_{L^q(\Omega_r)} + \|u\|_{W^2,q(N'' \setminus N')} + \|u\|_{W^1,q(\Omega_r)} + \|\partial u/\partial L\|_{W^1,q(\Omega_r)} + \|D^2V\|_{L^q(\Omega_r)} \right)
\]
in view of (7), (10), (26).

Moreover, if the \( \{T^j\} \) and therefore on the point \( x_0 \in \mathcal{E} \), it is clear that \( T_{\text{max}} \leq m r \) and iterate (26) in order to obtain
\[
\|D^2V\|_{L^q(\Omega_r)} \leq \|D^2V\|_{L^q(\Omega_r)} + \|D^2\tilde{V}\|_{L^q(\Omega_r)}.
\]

To get (23), we let \( m \) to be the least integer such that \( T_{\text{max}} \leq m r \) and iterate (26) in order to obtain
\[
\|D^2V\|_{L^q(\Omega_r)} = \|D^2V\|_{L^q(\Omega_r)} = \zeta(T_{\text{max}}) = \zeta(m r) = \zeta((m-1)r + r)
\]
\[
\leq C(K + \zeta((m-1)r)) = C(K + \zeta((m-2)r + r))
\]
\[
\leq K(C + C^2 + C^2\zeta((m-2)r))
\]
\[
\ldots
\]
\[
\leq K \sum_{j=1}^m C^j + C^m \zeta(0) = K \sum_{j=1}^m C^j
\]

This proves (23). □

**Remark 6.** It is important to note that the constant \( C \) in Proposition 5 depends on \( m \) through \( T_{\text{max}} \), and therefore on the point \( x_0 \in \mathcal{E} \). Actually, that constant will have the very same value for each other point of \( \mathcal{E} \) lying on the same \( L \)-trajectory as \( x_0 \).

Moreover, if the improving-of-integrability property asserted in Propositions 4 and 5 holds on a set \( S \subset \Omega \) then it is guaranteed, on the base of (13), on any other set which can be reached from \( S \) along \( L \)-trajectories.

To complete the proof of Lemma 8, we select a finite set \( \{T^j\}_{j=1}^N \) of neighbourhoods covering the compact \( \mathcal{E} \), each of the type \( T_r \) above with \( r = r_0/2 \), and such that \( T := \text{closure} \left( \bigcup_{j=1}^N T^j \right) \subset N'' \) is a closed neighbourhood of \( \mathcal{E} \) in \( \Omega \). It is clear that Proposition 2 remains true with \( T \) instead of \( N' \) and then (7) rewrites into
\[
\|u\|_{W^2,q(\Omega \setminus T)} \leq C \left( \|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\partial u/\partial L\|_{W^1,q(\Omega)} \right).
\]
The *improving-of-integrability* claimed in Lemma 3 then follows from (13), Proposition 5 and (27) (recall $U = u$ on $\mathcal{T}_{r/2}$). Similarly, (13), (24) and (23) yield

\begin{equation}
\|u\|_{W^{2,q}(\mathcal{N}''')} \leq \|u\|_{W^{2,q}(\mathcal{T})} + \|u\|_{W^{2,q}(\mathcal{N}'\setminus\mathcal{T})}
\leq C\left(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega,\mathcal{N})} + \|\varphi\|_{L^q(\partial\Omega,\mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{N}''')}\right).
\end{equation}

Later on, $\mathcal{N} \setminus \mathcal{N}'' \subset \Omega \setminus \mathcal{N}'$ and

\begin{align*}
\|u\|_{W^{1,q}(\mathcal{N}')} &\leq \|u\|_{W^{1,q}(\mathcal{N}''')} + \|u\|_{W^{1,q}(\mathcal{N}\setminus\mathcal{N}'')} \\
&\leq \varepsilon \|u\|_{W^{2,q}(\mathcal{N}'')} + C(\varepsilon)\left(\|u\|_{L^q(\Omega)} + \|u\|_{W^{2,q}(\Omega,\mathcal{N}'')}\right),
\end{align*}

in view of the interpolation inequality for the $W^{2,q}(\mathcal{N}'')$-norms with $\varepsilon > 0$ under control. In the same manner,

\begin{align*}
\|\partial u/\partial L\|_{W^{1,q}(\mathcal{N}')} &\leq \|\partial u/\partial L\|_{W^{1,q}(\mathcal{N}''')} + \|\partial u/\partial L\|_{W^{1,q}(\mathcal{N}\setminus\mathcal{N}'')} \\
&\leq \varepsilon \|\partial u/\partial L\|_{W^{2,q}(\mathcal{N}'')} + C(\varepsilon)\left(\|\partial u/\partial L\|_{L^q(\mathcal{N}'')} + \|u\|_{W^{2,q}(\Omega,\mathcal{N}'')}\right),
\end{align*}

while

\begin{equation}
\|\partial u/\partial L\|_{W^{2,q}(\mathcal{N}'')} \leq C\left(\|u\|_{W^{2,q}(\mathcal{N}'')} + \|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega,\mathcal{N})} + \|\varphi\|_{L^q(\partial\Omega,\mathcal{N})}\right)
\end{equation}

by means of the local *a priori* estimates (3 Theorem 9.11) for the problem (6).

A substitution of the above expressions into (28) and (7) give

\begin{align*}
\|u\|_{W^{2,q}(\mathcal{N}'')} &\leq C\left(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega,\mathcal{N})} + \|\varphi\|_{L^q(\partial\Omega,\mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{N}'')}\right) \\
&+ \varepsilon \|u\|_{W^{2,q}(\mathcal{N}'')} + C(\varepsilon)\left(\|\partial u/\partial L\|_{L^q(\mathcal{N}'')}\right)
\end{align*}

whence, choosing $\varepsilon > 0$ small enough, we get

\begin{equation}
\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C\left(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega,\mathcal{N})} + \|\varphi\|_{L^q(\partial\Omega,\mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{N}'')}\right).
\end{equation}

Similarly, another application of the interpolation inequality yields

\begin{equation}
\|u\|_{W^{1,q}(\mathcal{N}'')} \leq \|u\|_{W^{1,q}(\mathcal{N}'')} \leq \delta \|u\|_{W^{2,q}(\mathcal{N}'')} + C(\delta)\|u\|_{L^q(\mathcal{N}'')}
\end{equation}

and thus

\begin{equation}
\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C\left(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega,\mathcal{N})} + \|\varphi\|_{L^q(\partial\Omega,\mathcal{N})}\right).
\end{equation}

for small $\delta > 0$. The proof of Lemma 3 is completed. \hfill $\square$

The statement of Theorem 1 follows from Proposition 2 and Lemma 3.

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