ON CONVEX-CYCLIC OPERATORS

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Abstract. We give a Hahn-Banach Characterization for convex-cyclicity. We also obtain an example of a bounded linear operator $S$ on a Banach space with $\sigma_p(S^*) = \emptyset$ such that $S$ is convex-cyclic, but $S$ is not weakly hypercyclic and $S^2$ is not convex-cyclic. This solved two questions of Rezaei in [23] when $\sigma_p(S^*) = \emptyset$. We also characterize the diagonalizable normal operators that are convex-cyclic and give a condition on the eigenvalues of an arbitrary operator for it to be convex-cyclic. We show that certain adjoint multiplication operators are convex-cyclic and show that some are convex-cyclic but no convex polynomial of the operator is hypercyclic. Also some adjoint multiplication operators are convex-cyclic but not 1-weakly hypercyclic.

1. Introduction

Let $X$ be a Banach space and let $L(X)$ denote the algebra of all bounded linear operators on $X$. A bounded linear operator $T$ on $X$ is cyclic if there exists a (cyclic) vector $x$ such that the linear span of the orbit of $x$, $\text{Orb}(T, x) = \{T^n x : n = 0, 1, \cdots \}$, is dense in $X$. An operator $T$ is called convex-cyclic if there exists a vector $x \in X$ such that the convex hull of $\text{Orb}(T, x)$ is dense in $X$ and such a vector $x$ is said to be a convex-cyclic vector for $T$. Clearly all convex-cyclic operators are cyclic. Following Rezaei [23] we will say that a polynomial $p$ is a convex polynomial if it is a (finite) convex combination of monomials $\{1, z, z^2, \ldots \}$. So, $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a convex polynomial if $a_k \geq 0$ for all $k$ and $\sum_{k=0}^{n} a_k = 1$. Then the convex hull of an orbit is $\text{co}(\text{Orb}(T, x)) = \{p(T)x : p \text{ is a convex polynomial}\}$.

A bounded linear operator $T \in L(X)$ is said to be hypercyclic (weakly hypercyclic [11]) if there is a vector $x \in X$ whose orbit is dense in the norm (weak) topology of $X$. An operator $T$ is said to be weakly-mixing if $T \oplus T$ is hypercyclic in $X \oplus X$.

There are certainly examples of convex-cyclic operators that are not hypercyclic. However within certain classes of operators, hypercyclicity and convex-cyclicity are equivalent. This is true for unilateral weighted backward shifts on $\ell^p(\mathbb{N})$ and composition operators on the classical Hardy space, see [23].

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What follows is a list of questions that are answered in this paper. First, notice that every weakly hypercyclic operator is convex-cyclic since the norm and the weak closure of a convex set in a Banach space coincide. In [23] Rezaei asks the following question:

**Question 1.** [23, Question 5.4] Is every convex-cyclic operator acting on an infinite dimensional Banach space weakly hypercyclic?

According to Feldman [16], $T$ is called 1-weakly hypercyclic if there is an $x \in X$ such that $f(\text{Orb}(T, x))$ is dense in $\mathbb{C}$ for each non-zero $f \in X^*$. Every weakly hypercyclic operator is 1-weakly hypercyclic and 1-weakly hypercyclic operators are convex-cyclic. Thus it is also natural to ask if every convex-cyclic operator acting on an infinite dimensional Banach space 1-weakly hypercyclic?

Ansari [2] showed that powers of hypercyclic operators on Banach spaces are hypercyclic operators. The same result was proven for operators on locally convex spaces by Bourdon and Feldman [10]. These results do not have analogues for cyclic operators. The forward unilateral shift $S$ on $\ell^2(\mathbb{N})$ is cyclic but $S^2$ is not cyclic, because the codimension of the range of $S^2$ is two. What about powers of convex-cyclic operators? León and Romero in [22] give examples of convex-cyclic operators where $\sigma_p(S^*)$ is non-empty that have powers that are not convex-cyclic. Thus it is natural to ask:

**Question 2.** [23, Question 5.5] If $S : X \to X$ a convex-cyclic operator on a Banach space $X$ with $\sigma_p(S^*) = \emptyset$, then is $S^n$ convex-cyclic for every integer $n > 1$?

For a positive integer $m$ and a positive real number $p$, an operator $T \in L(X)$ is called an $(m, p)$-isometry if for any $x \in X$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^p = 0 .$$

An operator $T$ is called an $m$-isometry if it is an $(m, p)$-isometry for some $p > 0$. See [1], [4] and [19]. Faghih and Hedayatian proved in [15] that $m$-isometries on a Hilbert space are not weakly hypercyclic. However, there are isometries that are weakly supercyclic [24] (in particular cyclic). Thus a natural question is the following:

**Question 3.** Can an $m$-isometry be convex-cyclic?

In [3], Badea, Grivaux and Müller introduced the concept of an $\varepsilon$-hypercyclic operator.

**Definition 1.1.** Let $\varepsilon \in (0, 1)$ and let $T : X \to X$ be a continuous linear operator. A vector $x \in X$ is called an $\varepsilon$-hypercyclic vector for $T$ if for every non-zero vector $y \in X$ there exists
a non-negative integer $n$ such that
\[ \|T^n x - y\| \leq \varepsilon \|y\|. \]

The operator $T$ is called \(\varepsilon\)-hypercyclic if it has an \(\varepsilon\)-hypercyclic vector.

In [3] it was shown that for every \(\varepsilon \in (0, 1)\), there exists an \(\varepsilon\)-hypercyclic operator on the space \(\ell^1(\mathbb{N})\) which is not hypercyclic. Bayart in [5] extended this result to separable Hilbert spaces. Thus it is natural to ask if:

**Question 4.** Is every \(\varepsilon\)-hypercyclic operator also convex-cyclic?

An operator \(T \in L(X)\) is called hypercyclic with support \(N\) is there exists a vector \(x \in X\) such that the set
\[ \{ T^{k_1}x + T^{k_2}x + \cdots + T^{k_N}x : k_1, \ldots, k_N \in \mathbb{N} \} \]
is dense in \(X\).

**Remark 1.2.** Notice that if \(T\) is hypercyclic with support \(N\), then \(T\) is convex-cyclic. In fact, for any \(y \in X\), there exist \(k_1, \ldots, k_N \in \mathbb{N}\) such that \(T^{k_1}x + T^{k_2}x + \cdots + T^{k_N}x \approx Ny\), thus
\[ \frac{T^{k_1}x + T^{k_2}x + \cdots + T^{k_N}x}{N} \approx y. \]

Any hypercyclic operator with support \(N\) satisfies that \(\sigma_p(T^*)\) is the empty set [6, Proposition 3.1]. However, there are convex-cyclic operators such that \(\sigma_p(T^*)\) is non-empty. So, hypercyclicity with support \(N\) is not equivalent to convex-cyclicity.

In [23], Rezaei characterizes which diagonal matrices on \(\mathbb{C}^n\) are convex-cyclic as those whose eigenvalues are distinct and belong to the set \(\mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})\). This naturally leads to the question about infinite diagonal matrices and even the following more general question.

**Question 5.** If \(T\) is a continuous linear operator on a complex Banach space \(X\) and \(T\) has a complete set of eigenvectors whose eigenvalues are distinct, and belong to the set \(\mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})\), then is \(T\) convex-cyclic?

In this paper, we answer these five questions and also give some examples. The paper is organized as follows. In Section 2, we give the Hahn-Banach characterization for convex-cyclicity. In Section 3, we give an example of an operator \(S\) that is convex-cyclic but \(S^2\) is not convex-cyclic and thus \(S\) is not weakly hypercyclic, this answers Questions 1 and 2 when \(\sigma_p(S^*) = \emptyset\). In Section 4, we prove that \(m\)-isometries are not convex-cyclic, answering Question 3. In Section 5 we prove that any \(\varepsilon\)-hypercyclic operator is convex-cyclic. In fact, every \(\varepsilon\)-hypercyclic vector is a convex-cyclic vector. Finally, in Section 6 we answer Question
5 affirmatively and give examples of such operators including diagonal operators and adjoints of multiplication operators.

![Implications between different definitions related with hypercyclicity and cyclicity.](image)

**2. The Hahn-Banach Characterization for Convex-Cyclicity**

Rezaei gave a (universality) criterion for an operator to be convex-cyclic [23, Theorem 3.10]. In the following result, using the Hahn-Banach Separation Theorem, we give a necessary and sufficient condition for a set to have a dense convex hull, as a result we get a criterion for a vector to be a convex-cyclic vector for an operator.

**Proposition 2.1.** Let $X$ be a locally convex space over the real or complex numbers and let $E$ be a nonempty subset of $X$. The following are equivalent:

1. The convex hull of $E$ is dense in $X$.
2. For every nonzero continuous linear functional $f$ on $X$ we have that the convex hull of $\text{Re}(f(E))$ is dense in $\mathbb{R}$.
3. For every nonzero continuous linear functional $f$ on $X$ we have that
   \[ \sup \text{Re}(f(E)) = \infty \text{ and } \inf \text{Re}(f(E)) = -\infty. \]
4. For every nonzero continuous linear functional $f$ on $X$ we have that
   \[ \sup \text{Re}(f(E)) = \infty. \]

**Proof.** Let $\mathbb{F}$ denote either the real or complex numbers. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) holds. Now assume that (4) holds and by way of contradiction, assume that $\text{co}(E)$ is not
dense in $X$. Then there exists a point $p \in X$ that is not in the closure of $\text{co}(E)$. So, by the Hahn-Banach Separation Theorem ([12, Theorem 3.13]), there exists a continuous linear functional $f$ on $X$ so that $\text{Re}(f(x)) < \text{Re}(f(p))$ for all $x \in \text{co}(E)$. It follows that $\text{Re}(f(E))$ is bounded from above and thus $\sup \text{Re}(f(E)) \neq \infty$. This contradicts our assumption that $(4)$ is true. Thus it must be the case that if $(4)$ holds, then $(1)$ does also. Hence all four conditions are equivalent. □

**Corollary 2.2** (The Hahn-Banach Characterization for Convex-Cyclicity). Let $X$ be a locally convex space over the real or complex numbers, $T : X \to X$ a continuous linear operator, and $x \in X$. Then the following are equivalent:

1. The convex hull of the orbit of $x$ under $T$ is dense in $X$.
2. For every non-zero continuous linear functional $f$ on $X$ we have 
   $$\sup \text{Re}(f(\text{Orb}(T,x))) = \infty.$$

Below are some simple consequences of the Hahn-Banach characterization for convex-cyclic vectors.

As it was pointed in the Introduction the range of a cyclic operator may not be dense. For example, the range of the unilateral shift has codimension one. However, the closure of the range of a cyclic operator has codimension at most one. Notice that the range of hypercyclic operator is always dense. The Hahn-Banach characterization of convex-cyclicity easily shows that convex-cyclic operators must also have dense range, see the following result.

**Proposition 2.3.** If $T$ is a convex-cyclic operator on a locally convex space $X$, then $T$ has dense range.

**Proof.** Suppose that $T$ is a convex-cyclic operator and let $x$ be a convex-cyclic vector for $T$, and by way of contradiction, suppose that $T$ does not have dense range. Then there exists a continuous linear functional $f$ such that $f(R(T)) = \{0\}$, where $R(T)$ denotes the range of $T$. By the Hahn-Banach characterization, Corollary 2.2, we must have that $\sup \text{Re}(f(\text{Orb}(T,x))) = \infty$. However, since $T^n x \in R(T)$ for all $n \geq 1$ it follows that $f(T^n x) = 0$ for all $n \geq 1$. So, $\sup \text{Re}(f(\text{Orb}(T,x))) = \sup \text{Re}(\{ f(T^0 x), 0 \}) < \infty$. It follows from Corollary 2.2 that $x$ is not a convex-cyclic vector, a contradiction. Thus, $T$ must have dense range. □

In general, if $T$ is hypercyclic and $c > 1$, then $cT$ may not be hypercyclic. However, León-Saavedra and Müller [21] proved that if $T$ is hypercyclic and $\alpha$ is a unimodular complex number, then $\alpha T$ is hypercyclic. The same property is also true for weak hypercyclic operators [13, Theorem 2.8]. Next we present a similar result for convex-cyclic operators, that follows from the Hahn-Banach characterization of convex-cyclic vectors.
Proposition 2.4. If $T$ is a convex-cyclic operator on a real or complex locally convex space $X$, and if $c > 1$, then $cT$ is also convex-cyclic. Furthermore, every convex-cyclic vector for $T$ is also a convex-cyclic vector for $cT$.

Proof. Suppose that $x$ is a convex-cyclic vector for $T$, and we will show that $x$ is also a convex-cyclic vector for $cT$, by using the Hahn-Banach characterization (Corollary 2.2). Let $f$ be any non-zero continuous linear functional on $X$. Since $x$ is a convex-cyclic vector for $T$, then $\sup \Re(f(T^n x)) = \infty$. Since $c > 1$, then we have that $\sup c^n \Re(f((cT)^n x)) \geq \sup \Re(f(T^n x)) = \infty$. So, by the Hahn-Banach characterization, $x$ is a convex-cyclic vector for $cT$. □

Corollary 2.5. If $|c| \geq 1$ and $T$ is weakly hypercyclic, then $cT$ is convex-cyclic.

Proof. Let $c := e^{i\theta} \beta$, where $\theta \in \mathbb{R}$ and $\beta \geq 1$. Then by de la Rosa [13, Theorem 2.8] we obtain that $e^{i\theta}T$ is weakly hypercyclic, hence $e^{i\theta}T$ is convex-cyclic. Thus, $cT = \beta(e^{i\theta}T)$ is convex cyclic by Proposition 2.4. □

Let us define the following convex polynomials

$$p^c_k(t) := \begin{cases} \frac{1 + t + \cdots + t^{k-1}}{k} & \text{if } c = 1 \\ \frac{c - 1}{c^k - 1} \left( c^{k-1} + c^{k-2}t + \cdots + t^{k-1} \right) & \text{if } c > 1 \end{cases}$$

Definition 2.6. Let $X$ and $Y$ be topological spaces. A family of continuous operators $T_i : X \rightarrow Y$ ($i \in I$) is universal if there exists an $x \in X$ such that $\{T_i x : i \in I\}$ is dense in $Y$.

Let $T \in L(X)$. Denotes $M_n(T)$ the arithmetic means given by

$$M_n(T) := \frac{I + T + \cdots + T^{n-1}}{n}.$$

Recall that an operator $T$ is Cesàro hypercyclic if there exists $x \in X$ such that $\{M_n(T)x : n \in \mathbb{N}\}$ is dense in $X$. See [20].

In [20, Theorem 2.4] it is proved that $T$ is Cesàro hypercyclic if and only if $(\frac{T^k}{k})_{k=1}^{\infty}$ is universal.

Proposition 2.7. Let $X$ be a Banach space, $c > 1$ and $T \in L(X)$ such that $cI - T$ has dense range. Then the following are equivalent:

1. $\frac{T}{c}$ is hypercyclic
2. $(p^c_k(T))_{k \in \mathbb{N}}$ is universal.
Proof. Notice that if \( c > 1 \),

\[
p_k^c(T)(cI - T)x = (cI - T)p_k^c(T)x = (c - 1)\frac{c^k}{c^k - 1} \left( x - \left( \frac{T}{c} \right)^k x \right).
\]

\( \square \)

**Proposition 2.8.** If \( T \) is Cesàro hypercyclic or \( \frac{T}{c} \) is hypercyclic for some \( c \geq 1 \), then \( T \) is convex-cyclic.

Notice that the proof of the sufficient condition for a bilateral weighted backward shift on \( \ell^p(\mathbb{Z}) \) to be convex-cyclic given in [23, Theorem 4.2] is not correct.

3. Convex-cyclic operators whose squares are not convex-cyclic

As noted in the Introduction, powers of hypercyclic and weakly hypercyclic operators remain hypercyclic and weakly hypercyclic, respectively. In this section, we give an example of a convex-cyclic operator \( S \) with \( \sigma_p(S^*) = \emptyset \) such that \( S^2 \) is not convex-cyclic. Moreover, the same example gives an operator that is convex-cyclic with \( \sigma_p(S^*) = \emptyset \) that is not weakly hypercyclic.

Recall that León-Saavedra and Romero de la Rosa [22] provide an example of a convex-cyclic operator \( S \) with \( \sigma_p(S^*) \neq \emptyset \) such that \( S^n \) fails to be convex-cyclic. Also, a \( 2 \times 2 \) diagonal matrix \( D \) with eigenvalues \( 2i \) and \( -2i \) is convex-cyclic, but \( D^2 \) has a real eigenvalue and thus is not convex-cyclic.

**Theorem 3.1.** [17] Let \( T \) be a hypercyclic operator on an infinite dimensional separable Banach space. The following assertions are equivalent:

(1) \( T \oplus T \) is hypercyclic.

(2) \( T \oplus T \) is cyclic.

**Theorem 3.2.** ([6, Proposition 2.3] \& [28, Corollary 5.2]) Let \( T \) be a hypercyclic operator on a separable Banach space. Then \( T \oplus -T \) is hypercyclic with support 2 and 1-weakly hypercyclic.

**Corollary 3.1.** If \( T \) is a hypercyclic operator on an infinite dimensional Banach space such that \( T \oplus T \) is not hypercyclic, then \( T \oplus -T \) is convex-cyclic, but not weakly hypercyclic and \( (T \oplus -T)^2 \) is not cyclic.

Proof. Suppose that \( T \) is a hypercyclic operator such that \( T \oplus T \) is not hypercyclic. Then by Theorem 3.1, \( T \oplus T \) is not cyclic. Thus \( (T \oplus T)^2 \) is not cyclic, hence \((T \oplus -T)^2 = (T \oplus T)^2\) is not cyclic. It follows that \( T \oplus -T \) is not weakly hypercyclic, for if it was, then \((T \oplus -T)^2 = \ldots \)
\( (T \oplus T)^2 \) would be weakly hypercyclic, and hence cyclic, a contradiction. Thus \( T \oplus -T \) is convex-cyclic but not weakly hypercyclic, and \( (T \oplus -T)^2 \) is not cyclic. \( \square \)

Examples of operators satisfying that \( T \) is hypercyclic but \( T \oplus T \) is not hypercyclic are given in [14], [8, Corollary 4.15] and [7]. Using these examples we have the following result.

**Theorem 3.3.** There exists an operator \( S \) on \( c_0(\mathbb{N}) \oplus c_0(\mathbb{N}) \) or on \( \ell^p(\mathbb{N}) \oplus \ell^p(\mathbb{N}) \) with \( p \geq 1 \) that is convex-cyclic, but not weakly hypercyclic, and \( S^2 \) is not convex-cyclic.

Using similar ideas of Shkarin [27, Lemma 6.5] we obtain the following result.

**Theorem 3.4.** Let \( T \in L(X) \). If \( T^2 \) is convex-cyclic, then \( T \oplus -T \) is convex-cyclic.

**Proof.** Let \( x \) be a convex-cyclic vector for \( T^2 \) and let \( S := T \oplus -T \). Then for all \( y \in X \) there exists a sequence \( (p_k) \) of convex polynomials such that \( p_k(T^2)x \) converges to \( y \) as \( k \) tends to infinity. Thus

\[
p_k(S^2)(x,x) \to (y,y),
\]

and

\[
S p_k(S^2)(x,x) \to (Ty,-Ty).
\]

Since \( T^2 \) is convex-cyclic, \( T \) is convex-cyclic. By Proposition 2.3, the range of \( T \) is dense. By other hand, \( p_k(x^2) \) and \( xp_k(x^2) \) are convex polynomials. Thus the closed convex hull of \( \text{Orb}(S,(x,x)) \) contains the spaces \( L_0 := \{(u,u): u \in X\} \) and \( L_1 := \{(u,-u): u \in X\} \). So, if we are given \( (y,z) \in X \times X \), then let \( (q_k) \) and \( (h_k) \) be sequences of convex polynomials such that

\[
q_k(S^2)(x,x) \to (y+z,y+z)
\]

and

\[
S h_k(S^2)(x,x) \to (y-z,z-y).
\]

Then \( p_k(t) := \frac{1}{2}q_k(t^2) + \frac{1}{2}h_k(t^2) \) is a sequence of convex polynomials and

\[
p_k(S)(x,x) \to (y,z).
\]

Thus \( S = T \oplus -T \) is convex-cyclic. \( \square \)

Ansari [2] proved that an operator \( T \) is hypercyclic if and only if \( T^n \) is hypercyclic. In fact \( T \) and \( T^n \) have the same set of hypercyclic vectors for any positive integer \( n \). This property is also true for weakly hypercyclic vectors (see [10, Theorem 2.4]), thus we get the following corollary.

**Corollary 3.2.** If \( T \) is weakly-hypercyclic, then \( T \oplus -T \) is convex-cyclic.
In the following result we obtain that if \( T \) and \( T^n \) are convex-cyclic operators, the set of convex-cyclic vectors could be different.

**Proposition 3.3.** There are hypercyclic operators such that \( T \) and \( T^2 \) do not have the same convex-cyclic vectors.

**Proof.** Let \( T \) be twice the backward shift, \( T := 2B \), on \( \ell^2(\mathbb{N}) \) and let \( D \) be the doubling map on \( \ell^2(\mathbb{N}) \), given by \( D(x_0, x_1, x_2, \ldots) = (x_0, x_0, x_1, x_2, \ldots) \). By [16, Theorem 5.3] there exists an \( x \in \ell^2(\mathbb{N}) \) such that \( x \) is a 1-weakly hypercyclic vector for \( T \) (and hence a convex-cyclic vector for \( T \)) and \( \text{Orb}(T^2, x) \subseteq D(\ell^2(\mathbb{N})) \). Thus,

\[
\text{co}(\text{Orb}(T^2, x)) \subseteq \text{span}[\text{Orb}(T^2, x)] \subseteq D(\ell^2(\mathbb{N})) \neq \ell^2(\mathbb{N}).
\]

Since \( D(\ell^2(\mathbb{N})) \) is a proper closed subspace of \( \ell^2(\mathbb{N}) \), this complete the proof. \(\square\)

4. **M-isometries are not convex-cyclic**

Bayart proved the following spectral result for \( m \)-isometries on Banach spaces.

**Proposition 4.1.** [4, Proposition 2.3] Let \( T \in L(X) \) be an \( m \)-isometry. Then its approximate point spectrum lies in the unit circle. In particular, \( T \) is one-to-one, \( T \) has closed range and either \( \sigma(T) \subseteq \mathbb{T} \) or \( \sigma(T) = \mathbb{D} \).

On the other hand, Rezaei proved the following properties for convex-cyclic operators.

**Proposition 4.2.** [23, Propositions 3.2 and 3.3] Let \( T \in L(X) \). If \( T \) is convex-cyclic, then

1. \( \|T\| > 1 \).
2. \( \sigma_p(T^*) \subset \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{R}) \).

**Theorem 4.1.** An \( m \)-isometry on a Banach space \( X \) is not convex-cyclic.

**Proof.** If \( m = 1 \) and \( T \) is an \( m \)-isometry, then \( T \) is actually an isometry, thus \( \|T\| = 1 \) and thus by part (1) of Proposition 12 \( T \) cannot be convex-cyclic.

Assume that \( m \geq 2 \) and that \( T \) is convex-cyclic and a strict \((m, p)\)-isometry for some \( p > 0 \). We will use an argument similar to the proof of [4, Theorem 3.3]. Let

\[
|x| := \lim_{n \to \infty} \frac{\|T^nx\|}{n^{-p}}.
\]

By [4, Proposition 2.2] we have that \(|.| \) is a semi-norm on \( X \) and \( T(\text{Ker}(|.|)) \subseteq \text{Ker}(|.|) \), where \( \text{Ker}(T) \) denotes the kernel of \( T \). Also the codimension of \( \text{Ker}(|.|) \) is positive, because \( T \) is not a \((m - 1)\)-isometry. Moreover, for each \( x \in X \), \( |Tx| = |x| \) and there exists \( C > 0 \) such that \( |x| \leq C\|x\| \) for all \( x \in X \).
Let $Y := X/Ker(\|\cdot\|)$ and $\overline{T}$ be the operator induced by $T$ on $Y$. Then $|\overline{T}x| = |x|$ for all $x \in Y$. So, $\overline{T}$ is an isometry on $Y$.

Since $T$ is convex-cyclic there exists a vector $x \in X$ such that the convex hull generated by $\text{Orb}(T, x)$ is dense in $X$. Given $y \in X$ and $\varepsilon > 0$ there exists a convex polynomial such that $\|y - p_n(T)x\| < \frac{\varepsilon}{C}$. Thus $|y - p_n(T)x| \leq C\|y - p_n(T)x\| < \varepsilon$. Then $|\overline{y} - p_n(\overline{T})\overline{x}| < \varepsilon$ and we obtain that $\overline{T}$ is convex-cyclic in $Y$.

Thus the extension of $\overline{T}$ to the completion of $Y$ is a convex-cyclic isometry on a Banach space; which is a contradiction. 

\[ \square \]

**Corollary 4.3.** An $m$-isometry on a Banach space is not $1$-weakly hypercyclic.

5. **$\varepsilon$-HYPERCYCLIC OPERATORS VERSUS CONVEX-CYCLIC OPERATORS**

Let us now exhibit the relation between $\varepsilon$-hypercyclic and convex-cyclic operators.

**Theorem 5.1.** Every $\varepsilon$-hypercyclic vector is a convex-cyclic vector.

**Proof.** Let $x$ be an $\varepsilon$-hypercyclic vector for an operator $T$ and we will prove that for a non-zero vector $y \in X$ and $\delta > 0$, there exists a convex polynomial $p$ such that

$$\|p(T)x - y\| < \delta.$$ 

Since $\varepsilon \in (0, 1)$, there exists $N \in \mathbb{N}$ such that $2\varepsilon^N\|y\| < \delta$. As $x$ is an $\varepsilon$-hypercyclic vector for $T$, there exists a positive integer $k_1$ such that

$$\|T^{k_1}x - Ny\| \leq \varepsilon\|Ny\| = \varepsilon N\|y\|.$$ 

If $T^{k_1}x - Ny = 0$, we choose $l_2$ such that

$$\left\| T^{l_2}x - \frac{N}{N-1}\varepsilon^Ny \right\| \leq \varepsilon^{N+1}\frac{N}{N-1}\|y\|.$$ 

Thus

$$\left\| \frac{N-1}{N}T^{l_2}x - \varepsilon^Ny \right\| \leq \varepsilon^{N+1}\|y\|.$$ 

Hence

$$\left\| \frac{1}{N}T^{k_1}x + \frac{N-1}{N}T^{l_2}x - y \right\| = \left\| \frac{N-1}{N}T^{l_2}x \right\| \leq 2\varepsilon^N\|y\| < \delta$$

and the proof ends by letting $p(z) = \frac{1}{N}z^{k_1} + \frac{N-1}{N}z^{l_2}$.

If $T^{k_1}x - Ny \neq 0$, there exists a positive integer $k_2$ such that

$$\|T^{k_1}x + T^{k_2}x - Ny\| = \|T^{k_2}x - (Ny - T^{k_1}x)\| \leq \varepsilon\|Ny - T^{k_1}x\| \leq \varepsilon^2 N\|y\|.$$
If \( T^{k_1}x + T^{k_2}x - Ny = 0 \), analogously to the above situation we choose \( l_3 \) such that
\[
\left\| \frac{1}{N}T^{k_1}x + \frac{1}{N}T^{k_2}x + \frac{N-2}{N}T^{l_3}x - y \right\| = \left\| \frac{N-2}{N}T^{l_3}x \right\| \leq \frac{2\epsilon N}{N}\|y\| < \delta
\]
and the proof ends.

If \( T^{k_1}x + T^{k_2}x - Ny \neq 0 \), there exists a positive integer \( k_3 \) such that
\[
\left\| T^{k_1}x + T^{k_2}x + T^{k_3}x - Ny \right\| \leq \epsilon^3 N\|y\|.
\]

By induction, in the step \( N \), if \( T^{k_1}x + T^{k_2}x + \cdots + T^{k_{N-1}}x - Ny = 0 \), we choose \( l_N \) such that
\[
\left\| \frac{1}{N}T^{k_1}x + \frac{1}{N}T^{k_2}x + \cdots + \frac{1}{N}T^{k_{N-1}}x + \frac{1}{N}T^{l_N}x - y \right\| \leq \frac{2\epsilon N}{N}\|y\| < \delta
\]
and the proof ends.

If \( T^{k_1}x + T^{k_2}x + \cdots + T^{k_{N-1}}x - Ny \neq 0 \), there exists a positive integer \( k_N \) such that
\[
\left\| T^{k_1}x + T^{k_2}x + \cdots + T^{k_{N-1}}x + T^{k_N}x - Ny \right\| \leq \epsilon^3 N\|y\|
\]
Thus
\[
\left\| \frac{T^{k_1}x + \cdots + T^{k_N}x}{N} - y \right\| \leq \epsilon^3 N\|y\| < \delta
\]
Ending completely the proof. \( \square \)

6. Diagonal Operators and Adjoint Multiplication Operators

By a Fréchet space we mean a locally convex space that is complete with respect to a translation invariant metric.

If \( A \) is a nonempty collection of polynomials and \( T \) is an operator on a space \( X \), then \( T \) is said to be \( A \)-cyclic and \( x \in X \) is said to be an \( A \)-cyclic vector for \( T \) if \( \{p(T)x : p \in A\} \) is dense in \( X \). Furthermore, \( T \) is said to be \( A \)-transitive if for any two nonempty open sets \( U \) and \( V \) in \( X \), there exists a \( p \in A \) such that \( p(T)U \cap V \neq \emptyset \). Since the set of all polynomials with the topology of uniform convergence on compact sets in the complex plane forms a separable metric space, then any set of polynomials is also separable, hence the following result is routine (see for example the Universality Criterion in [18, Theorem 1.57]).

**Proposition 6.1.** Suppose that \( T : X \to X \) is a continuous linear operator on a real or complex Fréchet space and \( A \) is a nonempty set of polynomials. Then the following are equivalent:

1. \( T \) has a dense set of \( A \)-cyclic vectors.
2. \( T \) is \( A \)-transitive. That is, for any two nonempty open sets \( U, V \) in \( X \), there is a polynomial \( p \in A \) such that \( p(T)U \cap V \neq \emptyset \).
3. \( T \) has a dense \( G_\delta \) set of \( A \)-cyclic vectors.
By choosing various sets of polynomials for $A$, we can get results for hypercyclic and supercyclic operators, as well as cyclic operators that have a dense set of cyclic vectors. If $A$ is the set of all convex polynomials, then we get the following immediate corollary.

**Corollary 6.2.** Let $T : X \to X$ be a continuous linear operator on a real or complex Fréchet space, then the following are equivalent.

1. $T$ has a dense set of convex-cyclic vectors.
2. $T$ is convex-transitive. That is, for any two nonempty open sets $U, V$ in $X$, there is a convex polynomial $p$ such that $p(T)U \cap V \neq \emptyset$.
3. $T$ has a dense $G_δ$ set of convex-cyclic vectors.

**Proposition 6.3.** Let $A$ be a nonempty set of polynomials and let $\{T_k : X_k \to X_k\}_{k=1}^{\infty}$ be a uniformly bounded sequence of linear operators on a sequence of Banach spaces $\{X_k\}_{k=1}^{\infty}$ such that for every $n \geq 1$, the operator $S_n = \bigoplus_{k=1}^{n} T_k$ on $X^{(n)} = \bigoplus_{k=1}^{n} X_k$ has a dense set of $A$-cyclic vectors. Then $T = \bigoplus_{k=1}^{\infty} T_k$ is $A$-cyclic on $X^{(\infty)} = \bigoplus_{k=1}^{\infty} X_k$ and $T$ has a dense set of $A$-cyclic vectors.

**Proof.** Suppose that for every $n \geq 1$ the operators $S_n$ are $A$-cyclic and have a dense set of $A$-cyclic vectors. We will show that $T$ is $A$-transitive. Let $U$ and $V$ be two nonempty open sets in $X^{(\infty)}$. Since the vectors in $X$ with only finitely many non-zero coordinates are dense in $X$, then we may choose vectors $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ in $X^{(\infty)}$ such that $x_k = 0$ and $y_k = 0$ for all sufficiently large $k$, say $x_k = 0$ and $y_k = 0$ for all $k \geq N$, and such that $x \in U$ and $y \in V$. Since $S_N$ is $A$-cyclic and has a dense set of $A$-cyclic vectors in $X^{(N)}$, there exists a vector $u = (u_1, u_2, \ldots, u_N) \in X^{(N)}$ such that $u$ is an $A$-cyclic vector for $S_N$ and so that $(u_1, u_2, \ldots, u_N)$ is close enough to $(x_1, x_2, \ldots, x_N)$ so that the infinite vector $\hat{u} = (u_1, u_2, \ldots, u_N, 0, 0, \ldots) \in U$. Since $S_N$ is $A$-cyclic, there is a polynomial $p \in A$ such that $p(S_N)(u_1, u_2, \ldots, u_N)$ is close enough to $(y_1, y_2, \ldots, y_N)$ such that $p(T)\hat{u} \in V$. Thus, $T$ is $A$-transitive on $X^{(\infty)}$, and thus by Proposition 6.1 we have that $T$ has a dense set of $A$-cyclic vectors. 

We next apply the previous proposition to infinite diagonal operators where $A$ is the set of all convex polynomials. This extends the finite dimensional matrix result given by Rezaei [23, Corollary 2.7] to infinite dimensional diagonal matrices.

**Theorem 6.1.** Suppose that $T$ is a diagonalizable normal operator on a separable (real or complex) Hilbert space with eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. 
(a) If the Hilbert space is complex, then $T$ is convex-cyclic if and only if we have that the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are distinct and for every $k \geq 1$, $|\lambda_k| > 1$ and $\text{Im}(\lambda_k) \neq 0$.

(b) If the Hilbert space is real, then $T$ is convex-cyclic if and only if the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are distinct and for every $k \geq 1$ we have that $\lambda_k < -1$.

Proof. By the spectral theorem we may assume that $T = \text{diag}(\lambda_1, \lambda_2, \ldots)$ is an infinite diagonal matrix acting on $\ell^2_{\mathbb{C}}(\mathbb{N})$ and let $\{e_k\}_{k=1}^{\infty}$ be the canonical unit vector basis where $e_k$ has a one in its $k^{th}$ coordinate and zeros elsewhere.

(a) If $T$ is convex-cyclic with convex-cyclic vector $x = (x_n)_{n=1}^{\infty} \in \ell^2_{\mathbb{C}}(\mathbb{N})$, then by Corollary 2.2 we must have for every $k \geq 1$ that $\infty = \sup_{n \geq 1} \text{Re}(\langle T^n x, e_k \rangle) = \sup_{n \geq 1} \text{Re}(\lambda^n_k x_k)$. This implies that $x_k \neq 0$ and that $|\lambda_k| > 1$ for each $k \geq 1$. Likewise, since the Hilbert space is complex in this case, we must have

$$\infty = \sup_{n \geq 1} \text{Re} \left( \langle T^n x, \frac{-i}{x_k} e_k \rangle \right) = \sup_{n \geq 1} \text{Re} \left( \lambda^n_k x_k \frac{i}{x_k} \right) = \sup_{n \geq 1} \text{Re}(i\lambda^n_k) .$$

This implies that $\lambda_k$ cannot be real, hence $\text{Im}(\lambda_k) \neq 0$ for all $k \geq 1$.

Conversely, suppose that for every $k \geq 1$ we have that $|\lambda_k| > 1$ and $\text{Im}(\lambda_k) \neq 0$. Then for $n \geq 1$, let $T_n := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the diagonal matrix on $\mathbb{C}^n$ where $\lambda_k$ is the $k^{th}$ diagonal entry. Since the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are distinct and $|\lambda_k| > 1$ and $\text{Im}(\lambda_k) \neq 0$ for $1 \leq k \leq n$, then we know from Rezaei [23] that $T_n$ is convex-cyclic on $\mathbb{C}^n$ and that every vector all of whose coordinates are non-zero is a convex-cyclic vector for $T_n$. Since such vectors are dense in $\mathbb{C}^n$ for every $n \geq 1$, then it follows from Proposition 6.3 that $T$ is also convex-cyclic and has a dense set of convex-cyclic vectors. (b) The proof of the real case is similar to that above.

The next theorem says that if an operator has a complete set of eigenvectors whose eigenvalues are distinct, not real, and lie outside of the closed unit disk, then the operator is convex-cyclic.

**Theorem 6.2.** Let $S := \{re^{i\theta} : r > 1 \text{ and } 0 < |\theta| < \pi\} = \mathbb{C} \setminus (\overline{D} \cup \mathbb{R})$. Suppose that $T$ is a bounded linear operator on a complex Banach space $X$ and that $T$ has a countable linearly independent set of eigenvectors with dense linear span in $X$ such that the corresponding eigenvalues are distinct and are contained in the set $S$. Then $T$ is convex-cyclic and has a dense set of convex-cyclic vectors.

**Proof.** Suppose that $\{v_n\}_{n=1}^{\infty}$ is a linearly independent set of eigenvectors for $T$ that have dense linear span in $X$ and such that the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are distinct and contained in the set $S$. By replacing each eigenvector $v_n$ with a constant multiple of itself we may assume that $\sum_{n=1}^{\infty} \|v_n\|^2 < \infty$. Let $D$ be the diagonal normal matrix on $\ell^2(\mathbb{N})$ whose $n^{th}$
diagonal entry is $\lambda_n$. Then define a linear map $A : \ell^2(\mathbb{N}) \to X$ by $A(\{a_n\}_{n=1}^\infty) = \sum_{n=1}^\infty a_n v_n$. Notice that since $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$, then we have that

$$\|A(\{a_n\}_{n=1}^\infty)\| = \left\| \sum_{n=1}^\infty a_n v_n \right\| \leq \left( \sum_{n=1}^\infty |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^\infty \|v_n\|^2 \right)^{1/2} = C\|\{a_n\}_{n=1}^\infty\|_{\ell^2(\mathbb{N})}$$

where $C := (\sum_{n=1}^\infty \|v_n\|^2)^{1/2}$, which is finite. The above inequality implies that $A$ is a well defined continuous linear map from $\ell^2(\mathbb{N})$ to $X$. It follows that since the eigenvectors $\{v_n\}_{n=1}^\infty$ have dense linear span in $X$, that $A$ has dense range. Also, if $\{e_n\}_{n=1}^\infty$ is the standard unit vector basis in $\ell^2(\mathbb{N})$, then clearly $A(e_n) = v_n$ for all $n \geq 1$ and thus $A$ intertwines $D$ with $T$. To see this notice that $AD(e_n) = A(\lambda_n e_n) = \lambda_n v_n = T(v_n) = TA(e_n)$. Thus $AD(e_n) = TA(e_n)$ for all $n \geq 1$, thus $AD = TA$. Finally, since $D$ has distinct eigenvalues that all lie in the set $S$, it follows from Proposition 6.1 that $D$ is convex-cyclic and has a dense set of convex-cyclic vectors. Since $A$ intertwines $D$ and $T$ and $A$ has dense range, then $A$ will map convex-cyclic vectors for $D$ to convex-cyclic vectors for $T$. Thus, $T$ is convex-cyclic and has a dense set of convex-cyclic vectors. \hfill \Box

If $G$ is an open set in the complex plane, then by a reproducing kernel Hilbert space $\mathcal{H}$ of analytic functions on $G$ we mean a vector space of analytic functions on $G$ that is complete with respect to a norm given by an inner product and such that point evaluations at all points in $G$ are continuous linear functionals on $\mathcal{H}$. Naturally we also require that $f = 0$ in $\mathcal{H}$ if and only if $f(z) = 0$ for all $z \in G$. This is equivalent to the reproducing kernels having dense linear span in $\mathcal{H}$. Given such a space $\mathcal{H}$, a multiplier of $\mathcal{H}$ is an analytic function $\varphi$ on $G$ so that $\varphi f \in \mathcal{H}$ for every $f \in \mathcal{H}$. In this case, the closed graph theorem implies that the multiplication operator $M_\varphi : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator.

**Corollary 6.4.** Suppose that $G$ is an open set in $\mathbb{C}$ with components $\{G_n\}_{n \in J}$ and $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions on $G$, and that $\varphi$ is a multiplier of $\mathcal{H}$. If $\varphi$ is non-constant on every component of $G$ and $\varphi(G_n) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$ for every $n \in J$, then the operator $M_\varphi^* \mathcal{H}^*$ is convex-cyclic on $\mathcal{H}$ and has a dense set of convex-cyclic vectors.

**Proof.** We will show that the eigenvectors for $M_\varphi^*$ with eigenvalues in the set $S = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ have dense linear span in $\mathcal{H}$. It will then follow from Theorem 5.2 that $M_\varphi^*$ is convex-cyclic.

Every reproducing kernel for $\mathcal{H}$ is an eigenvector for $M_\varphi^*$. In fact, if $\lambda \in G$, then $M_\varphi^* K_\lambda = \varphi(\lambda) K_\lambda$, where $K_\lambda$ denotes the reproducing kernel for $\mathcal{H}$ at the point $\lambda \in G$. By assumption, for every component $G_n$ of $G$, $\varphi$ is non-constant on $G_n$, thus the set $\{\lambda \in G_n : |\varphi(\lambda)| > 1\}$ is a nonempty open subset of $G_n$. Also since $\varphi$ is an open map on $G_n$, $\varphi$ cannot map the open set $\{\lambda \in G_n : |\varphi(\lambda)| > 1\}$ into $\mathbb{R}$. Thus, for all $n \in J$, $E_n = \{\lambda \in G_n : |\varphi(\lambda)| > 1$ and $\varphi(\lambda) \notin \mathbb{R}\}$
Proposition 6.7. If such that \( p \) by Corollary 6.4.

Suppose also that \( H \) is the unilateral backward shift, is not 1-weakly-hypercyclic, however \( B \) follows from Theorem 6.2 that \( M_\varphi \) is convex-cyclic and has a dense set of convex-cyclic vectors.

Remark 6.5. In the previous corollary, if \( G \) is an open connected set, \( \varphi \) is a non-constant-multiplier of \( \mathcal{H} \) and if the norm of \( M_\varphi \) is equal to its spectral radius, then \( M_\varphi^* \) is convex-cyclic if and only if \( \varphi(G) \cap \{ z \in \mathbb{C} : |z| > 1 \} \neq \emptyset \). This is the case if \( \mathcal{H} \) is equal to \( H^2(G) \) or \( L^2_\alpha(G) \), the Hardy space or Bergman space on \( G \) or if \( M_\varphi \) is hyponormal.

Next we give an example of a convex-cyclic operator that is not 1-weakly hypercyclic.

Example 6.6. Let \( M_{2+z}^\ast \) be the adjoint of the multiplication operator associated to the multiplier \( \varphi(z) := 2 + z \) on \( H^2(\mathbb{D}) \). By [28, Theorem 5.5] the operator \( M_{2+z}^\ast = 2I + B \), where \( B \) is the unilateral backward shift, is not 1-weakly-hypercyclic, however \( M_{2+z}^\ast \) is convex-cyclic by Corollary 6.4.

The following result is true since powers of convex polynomials are also convex polynomials.

Proposition 6.7. If \( T \) is an operator on a Banach space and there exists a convex polynomial \( p \) such that \( p(T) \) is hypercyclic, then \( T \) is convex-cyclic.

By a region in \( \mathbb{C} \) we mean an open connected set in \( \mathbb{C} \). In the following theorem, we consider the operator which is the adjoint of multiplication by \( z \), the independent variable.

Theorem 6.3. Suppose that \( G \) is a bounded region in \( \mathbb{C} \) and \( G \cap \{ z : |z| > 1 \} \neq \emptyset \). Suppose also that \( \mathcal{H} \) is a reproducing kernel Hilbert space of analytic functions on \( G \), then \( M_\varphi^* \) is convex-cyclic on \( \mathcal{H} \). In fact, there exists a convex polynomial \( p \) such that \( p(M_\varphi^*) \) is hypercyclic on \( \mathcal{H} \).

Proof. Choose \( n \geq 1 \) such that \( G^n := \{ z^n : z \in G \} \) satisfies \( G^n \cap \{ z \in \mathbb{C} : Re(z) < 1 \} \neq \emptyset \). To see how to do this, choose a polar rectangle \( R = \{ re^{i\theta} : r_1 < r < r_2 \) and \( \alpha < \theta < \beta \) such that \( R \subseteq G \). Then simply choose a positive integer \( n \) such that \( n(\beta - \alpha) > 2\pi \). Then \( R^n \subseteq G^n \) and \( R^n \) will contain the annulus \( \{ re^{i\theta} : r_1^n < r < r_2^n \} \), so certainly \( G^n \cap \{ z \in \mathbb{C} : Re(z) < 1 \} \neq \emptyset \). Now if \( 0 < a \leq 1 \), then the convex polynomial \( p_a(z) = az + (1 - a) \) maps the disk \( B(\frac{a-1}{a}, \frac{1}{a}) \)
onto the unit disk. Notice that the family of disks \( \{ B(\frac{n-1}{a}, \frac{1}{a}) : 0 < a < 1 \} \) is the family of all disks that are centered on the negative real axis and pass through the point \( z = 1 \). Thus it follows that \( \{ z \in \mathbb{C} : Re(z) < 1 \} = \bigcup_{0 < a < 1} B(\frac{n-1}{a}, \frac{1}{a}) \). So we can choose an \( a \in (0, 1) \) such that \( G^{n} \cap \partial B(\frac{n-1}{a}, \frac{1}{a}) \neq \emptyset \). It follows that the polynomial \( p(z) = p_{a}(z^{n}) \) is a convex polynomial and furthermore it satisfies \( p(G) \cap \partial \mathbb{D} \neq \emptyset \).

Thus \( M^{*}_{p} \) is hypercyclic on \( \mathcal{H} \). However, \( M^{*}_{p} = p^{\#}(M^{*}_{z}) \) where \( p^{\#}(z) = \overline{p(z)} \). Also, since \( p \) is a convex polynomial, all of its coefficients are real, thus \( p^{\#} = p \). Thus, \( p(M^{*}_{z}) = p^{\#}(M^{*}_{z}) = M^{*}_{p} \) is hypercyclic on \( \mathcal{H} \). \( \square \)

In the next result we give an example of an operator that is convex-cyclic but no convex polynomial of the operator is hypercyclic. In other words, the operator is purely convex-cyclic.

**Example 6.8.** Let \( \{ \alpha_{n} \}_{n=1}^{\infty} \) and \( \{ \beta_{n} \}_{n=1}^{\infty} \) be two strictly decreasing sequences of positive numbers that are interlaced and converging to zero. In other words, \( 0 < \alpha_{n+1} < \beta_{n+1} < \alpha_{n} \) for all \( n \geq 1 \) and \( \alpha_{n} \to 0 \) (and hence \( \beta_{n} \to 0 \)). For each \( n \geq 1 \), let

\[
G_{n} := \{ re^{i\theta} : 2 < r < 2 + \frac{1}{n} \text{ and } \alpha_{n} < \theta < \beta_{n} \}.
\]

Let \( G := \bigcup_{n=1}^{\infty} G_{n} \) and let \( L^{2}_{a}(G) \) be the Bergman space of all analytic functions on \( G \) that are square integrable with respect to area measure on \( G \). Then the operator \( M^{*}_{z} \) is purely convex-cyclic on \( L^{2}_{a}(G) \); meaning that \( M^{*}_{z} \) is convex-cyclic on \( L^{2}_{a}(G) \), but \( p(M^{*}_{z}) \) is not hypercyclic on \( L^{2}_{a}(G) \) for any convex polynomial \( p \).

**Proof.** By Corollary [6.4] we know that \( M^{*}_{z} \) is convex-cyclic on \( L^{2}_{a}(G) \). In order to show that no convex polynomial of \( M^{*}_{z} \) is hypercyclic, suppose, by way of contradiction, that there exists a convex polynomial \( p \) such that \( p(M^{*}_{z}) \) is hypercyclic. Since \( p \) is a convex polynomial it has real coefficients thus \( p^{\#}(z) = p(z) \) where \( p^{\#}(z) := \overline{p(z)} \). Thus \( p(M^{*}_{z}) = M^{\#}_{p} = M^{*}_{p} \) and it follows that \( M^{*}_{p} \) is hypercyclic on \( L^{2}_{a}(G) \). Thus it follows that every component \( G_{n} \) of \( G \) must satisfy that \( p(G_{n}) \cap \partial \mathbb{D} \neq \emptyset \). However since \( p \) is a convex polynomial, \( p \) is (strictly) increasing on the interval \([0, \infty)\). Thus, \( p(2) > p(1) = 1 \). Choose an \( \varepsilon > 0 \) such that \( \varepsilon < p(2) - 1 \). Since \( p \) is continuous at \( z = 2 \), and since we have an \( \varepsilon > 0 \), then there exists a \( \delta > 0 \) such that if \( |z - 2| < \delta \), then \( |p(z) - p(2)| < \varepsilon \). Notice that for \( n \) sufficiently large we have that \( G_{n} \subseteq B(2, \delta) \), thus, \( p(G_{n}) \subseteq B(p(2), \varepsilon) \subseteq \{ z \in \mathbb{C} : Re(z) > 1 \} \). Thus, \( p(G_{n}) \cap \partial \mathbb{D} = \emptyset \) for all large \( n \), a contradiction. It follows that no convex polynomial of \( M^{*}_{z} \) is hypercyclic, hence \( M^{*}_{z} \) is purely convex-cyclic. \( \square \)
7. Open Questions

It is well known that hypercyclic operators have a dense set of hypercyclic vectors. In fact, the set of hypercyclic vectors is a dense $G_\delta$ set.

**Question 1.** If $T$ is convex-cyclic, then does $T$ have a dense set of convex-cyclic vectors?

Sanders [24] proved that if $T : H \to H$ is a hyponormal operator on a Hilbert space $H$, then $T$ is not weakly hypercyclic. A hyponormal operator is pure if its restriction to any of its reducing subspaces is not normal. That is, a hyponormal operator $T$ is pure if $T$ cannot be written in the form $T = S \oplus N$ where $N$ is a normal operator.

**Question 2.** Are there pure hyponormal operators or continuous normal operators that are convex-cyclic?

**Question 3.** If $T$ is convex-cyclic on a complex Hilbert space, then is $(-1)T$ also convex-cyclic?

The above question is true for diagonal normal operators/matrices and the other examples in this paper and also whenever $T^2$ is convex-cyclic.

**Question 4.** If $T$ is a convex-cyclic operator, then how big can the point spectrum of $T^*$ be? Can it have non-empty interior?

Bourdon and Feldman [10] showed that if a vector $x \in X$ has a somewhere dense orbit under a bounded linear operator $T$, then the orbit of $x$ under $T$ must be everywhere dense in $X$. A similar question was posed for convex-cyclicity by Rezaei. Recently, León-Saavedra and Romero de la Rosa provide an example where Bourdon and Feldman’s result fails for convex-cyclic operators $T$ such that $\sigma_p(T^*) \neq \emptyset$.

**Question 5.** [23, Question 5.5] Let $X$ be a Banach space and $T \in L(X)$ where $\sigma_p(T^*) = \emptyset$. If $x \in X$ and $co(Orb(T, x))$ is somewhere dense in $X$, then is $co(Orb(T, x))$ dense in $X$?

Since it is unknown if there exists a Banach space on which every hypercyclic operator is weakly mixing, we ask:

**Question 6.** Given a separable Banach space $X$, is there a convex-cyclic operator $S$ on $X$ such that $S^2$ is not convex-cyclic?

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