New conditions for subgeometric rates of convergence in the Wasserstein distance for Markov chains

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Abstract: In this paper, we provide sufficient conditions for the existence of the invariant distribution and subgeometric rates of convergence in the Wasserstein distance for general state-space Markov chains which are not phi-irreducible. Our approach is based on a coupling construction adapted to the Wasserstein distance.

Our results are applied to establish the subgeometric ergodicity in Wasserstein distance of non-linear autoregressive models in R^d and of the preconditioned Crank-Nicolson algorithm MCMC algorithm in a Hilbert space. In particular, for the latter, we show that a simple Hölder condition on the log-density of the target distribution implies the subgeometric ergodicity of the MCMC sampler in a Wasserstein distance.

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1. Introduction

Convergence of general state-space Markov chains in total variation distance (or V-total variation) has been studied by many authors. There is a wealth of contributions establishing explicit rate of convergence under conditions implying geometric ergodicity; see [15, Chapter 16], [16], [1], [5] and the references therein. Subgeometric (or Riemannian) convergence has been more scarcely studied; [18] characterized subgeometric convergence using a sequence of drift conditions, which proved to be difficult to use in practice. [12] have shown that, for polynomial convergence rates, this sequence of drift conditions can be replaced by a single drift conditions, mimicking the classical Foster-Lyapunov approach. This result was later extended by [7] to general subgeometric rate of convergence. Explicit convergence rates were obtained in [19], [9] and [8].

The classical proof of convergence in total variation distance are based either on a regenerative or a coupling construction, which requires the existence of accessible small sets and additional assumptions to control the moments of the
successive return time to these sets. The existence of an accessible small set implies that the chain is $\psi$-irreducible.

In this paper, we establish rate of convergence for general state-space Markov chain which are not $\psi$-irreducible. In such case, the Markov chain does not converge in total variation distance, but nevertheless may converge in a weaker sense; see for example [14]. We study in this paper the convergence in Wasserstein distance, which also implies the weak convergence. The use of the Wasserstein distance to obtain explicit rate of convergence has been considered by several authors, most often under conditions implying geometric ergodicity. A significant breakthrough in this domain has been achieved in [10], which has proposed an extension of small set adapted to the Wasserstein distance. The main motivation of [10] was the convergence of the solutions of stochastic delay differential equations (SDDE) to their invariant measure. Nevertheless, the techniques introduced in this work have found several applications. [11] used these techniques to prove the convergence of Markov chain Monte Carlo method to sample in infinite dimensional Hilbert spaces. An application for switched and piecewise deterministic Markov processes can be found in [6].

[4] generalized the results of [10], and established conditions which imply the existence and uniqueness of the invariant distribution, and subgeometric ergodicity of Markov chain (in discrete time) and Markov processes (in continuous time). [4] used this result to establish subgeometric ergodicity of the solutions of SDDE. It is interesting to note that the rates obtained in [4] do not match the rates established in [7] for the $V$-total variation.

In this paper, we complement and improve the results presented in [4]. The approach developed in this paper is more probabilistic than [4], being extensively based on coupling techniques. We provide a sufficient condition couched in terms of a single drift condition for a coupling kernel outside a appropriately defined coupling set, extending the notion of $d$-small set of [10]. We then show how this single drift condition implies a sequence of drift inequalities from which we deduce an upper bound of some subgeometric moment of the successive return times to the coupling set. The last step is to show that the Wasserstein distance between the distribution of the chain and the invariant probability measure is controlled by these moments. We apply our result to nonlinear autoregressive model with noise whose distribution can be singular with the Lebesgue measure; we also study the convergence of the preconditioned Crank-Nicolson algorithm for a class of target density having density w.r.t. a Gaussian measure on an Hilbert space, under conditions which are weaker than [11].

The paper is organized as follow: in section 2, the main results on the convergence of Markov chains in Wasserstein distance are presented, under different sets of assumptions. In section 3, the applications of these results to nonlinear algorithm and Crank-Nicolson sampling are considered. The proofs are given in section 2 and section 5.
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Notations

Let \((E, d)\) be a Polish space. We denote by \(\mathcal{B}(E)\) the associated Borel \(\sigma\)-algebra and \(\mathcal{P}(E)\) the set of probability measures on \((E, \mathcal{B}(E))\). Let \(\mu, \nu \in \mathcal{P}(E)\); \(\alpha\) is a coupling of \(\mu\) and \(\nu\) if \(\alpha\) is a probability on the product space \((E \times E, \mathcal{B}(E \times E))\), such that \(\alpha(A \times E) = \mu(A)\) and \(\alpha(E \times A) = \nu(A)\) for all \(A \in \mathcal{B}(E)\). The set of couplings of \(\mu, \nu \in \mathcal{P}(E)\) is denoted \(\mathcal{C}(\mu, \nu)\).

The Wasserstein metric associated with \(d\), between two probability measures \(\mu, \nu \in \mathcal{P}(E)\) is defined by:

\[
W_d(\mu, \nu) = \inf_{\gamma \in \mathcal{C}(\mu, \nu)} \int_{E \times E} d(x, y) d\gamma(x, y).
\] (1)

When \(d\) is the trivial metric \(d_0(x, y) = 1_{x \neq y}\), the associated Wasserstein metric is, up to a multiplicative factor, the total variation \(d_{\text{TV}}\) (see ([20, Chapter 6]) defined by:

\[
W_{d_0}(\mu, \nu) = \frac{1}{2} d_{\text{TV}}(\mu, \nu) = \sup_{A \in \mathcal{B}(E)} |\mu(A) - \nu(A)|.
\] (2)

When \(d\) is bounded, the Monge-Kantorovich duality Theorem implies (see [20, Remark 6.5]) that the lower bound in (1) is reached. In addition, \(W_d\) is a metric on \(\mathcal{P}(E)\) and \(\mathcal{P}(E)\) equipped with \(W_d\) is a Polish space; see [20, Theorems 6.8 and 6.16]. Finally, the convergence in \(W_d\) implies the weak convergence (see e.g. [20, Corollary 6.11]).

2. Main results

Let \((E, d_*)\) be a Polish space. Our goal is to provide sufficient conditions for the ergodicity of a Markov kernel \(P\) on \((E, \mathcal{B}(E))\) at a subgeometric rate in the Wasserstein distance.

Definition 1 (Subgeometric functions). The set of measurable functions \(r_0 : \mathbb{R}_+ \to [2, +\infty)\), such that \(r_0\) is non-decreasing, \(x \mapsto \log(r_0(x))/x\) is non-increasing and

\[
\lim_{x \to +\infty} \frac{\log(r_0(x))}{x} = 0
\] (3)

is denoted \(\Lambda_0\). The set of subgeometric functions \(\Lambda\) is the set of positive functions \(r : \mathbb{R}_+ \to (0, +\infty)\), such that there exists \(r_0 \in \Lambda_0\) satisfying:

\[
0 < \liminf_{x \to +\infty} \frac{r(x)}{r_0(x)} \leq \limsup_{x \to +\infty} \frac{r(x)}{r_0(x)} < +\infty.
\]

The set \(\Lambda\) of subgeometric functions contains all the functions on the form

\[
r(x) = (1 + \log(1 + x))^\alpha (1 + x)^\beta \exp(cx^\gamma),\]

with \((\alpha, \beta) \in \mathbb{R}^2\) if \(c > 0\) and \(\gamma \in (0, 1)\), and \((\alpha, \beta) \in (\mathbb{R} \times \mathbb{R}_+^*) \cup (\mathbb{R}_+^* \times \{0\})\) if \(\gamma = 0\).
The key ingredient for the derivation of our bounds is the existence for all \((x,y) \in E \times E\) of a coupling kernel \(Q((x,y),\cdot)\) of the probability measures \(P(x,\cdot), P(y,\cdot)\) such that some iterate \(Q^n\) satisfies a strong contraction property when \((x,y)\) belongs to the coupling set \(\Delta\). This assumption is combined with a condition which implies that in \(n\) iterations of the coupling kernel, the number of visits to \(\Delta\) is large enough so that the Wasserstein distance between \(P^n(x,\cdot)\) and \(P^n(y,\cdot)\) decreases at a subgeometric rate. Let us give a precise definition of such a coupling set.

**Definition 2** (Coupling set). Let \(\Delta \in \mathcal{B}(E \times E)\), \(\ell \in \mathbb{N}^*\), \(\epsilon \in (0,1)\) and \(d\) be a distance on \(E\) topologically equivalent to \(d_\epsilon\). \(\Delta\) is a \((\ell, \epsilon, d)\)-coupling set for the Markov kernel \(P\) on \((E, \mathcal{B}(E))\) if there exists a kernel \(Q\) on \((E \times E, \mathcal{B}(E \times E))\) satisfying the following conditions

(i) for all \(x, y \in E\), \(Q((x,y),\cdot)\) is a coupling of \((P(x,\cdot), P(y,\cdot))\).

(ii) for all \(x, y \in E\), \(Qd(x,y) \leq d(x,y)\).

(iii) for all \((x,y) \in \Delta\), \(Q^\ell d(x,y) \leq (1-\epsilon)d(x,y)\).

A simple way to check that \(\Delta \in \mathcal{B}(E \times E)\) is a coupling set is the following. Let \(d\) be topologically equivalent to \(d_\epsilon\), bounded by \(1\) and let \(\epsilon \in (0,1)\). If for all \((x,y) \in E^2\), \(W_d(P(x,\cdot), P(y,\cdot)) \leq d(x,y)\), and for all \((x,y) \in \Delta\), \(W_d(P(x,\cdot), P(y,\cdot)) \leq (1-\epsilon)d(x,y)\), then [20, corollary 5.22] implies that there exists a Markov kernel \(Q\) on \((E \times E, \mathcal{B}(E \times E))\) which makes \(\Delta\) a \((1, \epsilon, d)\)-coupling set.

We provide sufficient conditions for the existence of an invariant probability measure \(\pi\) for the Markov kernel \(P\) and for subgeometric ergodicity in Wasserstein distance, based on a drift condition on the product space \(E \times E\) outside a coupling set. Let us assume

**H1.** Let \(\ell \in \mathbb{N}^*\), \(\epsilon \in (0,1)\) and \(d\) be a distance on \(E\) topologically equivalent to \(d_\epsilon\) and bounded by \(1\). There exist a \((\ell, \epsilon, d)\)-coupling set \(\Delta\) for \(P\).

**H2.** There exist

- a concave increasing function \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\), continuously differentiable on \([1, +\infty)\), and satisfying \(\phi(0) = 0\), \(\lim_{x \to \infty} \phi(x) = \infty\) and \(\lim_{x \to \infty} \phi'(x) = 0\).

- a constant \(b \geq 0\) and a measurable function \(V : E \to [1, +\infty)\) with \(\sup_{\Delta} \{V(x) + V(y)\} < +\infty\),

such that for all \(x, y \in E\):

\[
PV(x) + PV(y) \leq V(x) + V(y) - \phi(V(x) + V(y)) + b\mathbb{1}_{\Delta}(x,y) .
\] (4)

In **H2**, we can weaken the assumption on \(t \mapsto \phi(t)\) by assuming it is concave increasing and continuously differentiable only for large \(t\) (say \(|t| \geq RV\)). Observe indeed that the function \(\phi\) defined by

\[
\tilde{\phi}(t) = \begin{cases} 
(2\phi'(RV) - \frac{\phi(RV)}{RV})t + \frac{2(\phi'(RV) - RV\phi'(RV))}{\sqrt{RV}} \sqrt{t} & \text{for } 0 \leq t < RV \\
\phi(t) & \text{for } t \geq RV
\end{cases}
\]

for large \(t\).
is concave increasing and continuously differentiable on \([1, +\infty)\), \(\tilde{\phi}(0) = 0\), 
\(\lim_{v \to \infty} \tilde{\phi}(v) = \infty\) and \(\lim_{v \to \infty} \tilde{\phi}'(v) = 0\). The drift inequality (4) implies that 
for all \(x, y \in E\)

\[
P V(x) + P V(y) \leq V(x) + V(y) - \tilde{\phi}(V(x) + V(y)) + \tilde{b}\mathbb{I}_{\Delta \cup \{V \leq R_v\}^2}(x, y),
\]

with \(\tilde{b} = b + \sup_{\{(z,t): V(z) + V(t) \leq R_v\}} \left\{ \tilde{\phi}(V(z) + V(t)) - \phi(V(z) + V(t)) \right\}\). Therefore, since \(\sup_{(x,y) \in \Delta} V(x) + V(y) < \infty\), the set \(\Delta \cup \{V \leq R_v\}^2\) is a coupling set as soon as for any \(v > 0\), \(\{V \leq v\} \times \{V \leq v\}\) are \((\ell, \epsilon, d)\)-coupling sets; then \(H2\) holds with \(\phi\) replaced with \(\tilde{\phi}\).

Examples of functions \(\phi\) satisfying \(H2\) at least for large \(t\) are: \(t \mapsto t^\gamma, \gamma \in (0, 1)\), \(t \mapsto (1 + \log(t))^\alpha, \alpha > 0\), and \(t \mapsto t/(1 + \log(t))^\alpha, \alpha > 0\).

**Theorem 3.** Assume \(H1-H2\). Then, \(P\) admits a unique invariant probability measure \(\pi\) such that \(\pi(\phi \circ V) < \infty\).

**Proof:** The proof is postponed to subsection 4.3. \(\square\)

We now derive expressions of the rate of convergence and the dependence upon the initial condition of the chain. The rate of convergence depends on the concave function \(\phi\) and the integrated subgeometric rate \(R_\phi\) defined as follows (see also \([7]\)). For any nondecreasing concave function \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\), continuously differentiable and satisfying \(\phi(1) > 0\) and \(\lim_{t \to \infty} \phi(t) = \infty\), set

\[
H_\phi(t) = \int_1^t \frac{1}{\phi(s)} ds .
\]  

(5)

Since for \(t \geq 1\), \(\phi(t) \leq \phi(1) + \phi'(1)(t-1)\), the function \(H_\phi\) is monotone increasing continuously differentiable, and its inverse, denoted \(H_\phi^{-1}\), is well defined and is continuously differentiable. Define

\[
r_\phi(t) = (H_\phi^{-1})'(t) = \phi(H_\phi^{-1}(t)) ,
\]

(6)

\[
fr_\phi(t) = r_\phi(0) + \int_0^t r_\phi(s) ds .
\]

(7)

**Theorem 4.** Assume \(H1-H2\) and there exists \(C_r\) such that for all \(t_1, t_2 \in \mathbb{R}_+\),

\[
fr_\phi(t_1 t_2) \leq C_r fr_\phi(t_1) fr_\phi(t_2) .
\]  

(8)

Let \(\pi\) be the invariant probability of \(P\). There exists a constant \(C\) such that for all \(x \in E\) and all \(n \geq 1\),

\[
W_d(P^n(x, \cdot), \pi) \leq CV(x)/\phi \circ fr_\phi \{n/\log(fr_\phi(n))\} .
\]

(9)

**Proof:** The proof is postponed to subsection 4.4 \(\square\)
The condition (8) is satisfied for example with $\phi(t) = t^\gamma, \gamma \in (0,1)$; and with $\phi(t) = (1 + \log(t))^\alpha, \alpha > 0$. In these cases, the rate of convergence $\phi \circ f_{\epsilon, n}(n/\log(f_{\epsilon, n}(n)))$ is equivalent when $n \to \infty$ to resp. $(n/\log(n))^\tau, \tau = \gamma/(1 - \gamma)$; and to $\log(n)^\alpha$. However (8) is not satisfied when $\phi(t) = t/(1 + \log(t))^\alpha, \alpha > 0$. The following result is valid without any restriction on the rate function $f_{\epsilon, n}$; when applied to rate functions satisfying (8), the rate given by Theorem 5 is smaller than the rate given by Theorem 4.

**Theorem 5.** Assume H1 H2. Let $\pi$ be the invariant probability of $P$. For all $\delta \in (0,1)$, there exists a constant $C$ such that for all $x \in E$ and all $n \geq 1$

$$W_d(P^n(x,\cdot),\pi) \leq C V(x)/\phi\{f_{\epsilon, n}^\delta(n)\}. \tag{10}$$

**Proof:** The proof is postponed to subsection 4.5 \qed

In the case $\phi(t) = t/(1 + \log(t))^\alpha, \alpha > 0$, which is not covered by Theorem 4, the rate $\phi\left(f_{\epsilon, n}^\delta(n)\right)$ is equivalent when $n \to \infty$ to $n^{-\alpha\delta}\exp(\delta n^\tau)$ with $\tau = 1/(1 + \alpha)$.

We summarize in Table 1 the rates of convergence obtained from Theorem 4 and Theorem 5 for usual concave functions $\phi$.

In practice, it is often easier to establish a drift inequality on $E$ instead of a drift inequality on the product space $E \times E$ as in H2. We show in Proposition 7 that H3 implies H1 and H2.

**H3.** (a) There exist

- a concave increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, continuously differentiable on $[1, +\infty)$ and satisfying $\phi(0) = 0$, $\lim_{x \to \infty} \phi(x) = \infty$ and $\lim_{x \to \infty} \phi'(x) = 0$,

- a measurable function $V : E \to [1, +\infty)$ and a constant $b \geq 0$

such that for all $x \in E$,

$$PV(x) \leq V(x) - \phi \circ V(x) + b. \tag{11}$$

(b) There exists $\nu > \phi^{-1}(2b)$ such that $\{V \leq \nu\} \times \{V \leq \nu\}$ is an $(\ell, \epsilon, d)$-coupling set, where $\ell \in \mathbb{N}^*$, $\epsilon \in (0,1)$ and $d$ is a distance on $E$, bounded by 1, and topologically equivalent to $d_\ast$.

**Remark 6.** Here again, we can assume without loss of generality that $t \mapsto \phi(t)$ is concave increasing and continuously differentiable only for large $t$.

**Proposition 7.** Assume H3. Set $\mathcal{C} = \{V \leq \nu\}$ and $c = 1 - 2b/\phi(v)$. Then, H1 holds with $\Delta = \mathcal{C} \times \mathcal{C}$ and H2 holds with the same function $V$, $\phi \leftarrow c\phi$ and $b \leftarrow 2b$.

**Proof:** The proof of Proposition 7 is postponed to subsection 4.6. \qed

In many applications (see e.g. section 3), we are able to prove a stronger assumption than H3-(b), namely: for any $u > 0$, there exist $\ell \geq 1, \epsilon \in (0,1)$ and a distance $d$ bounded by 1 topologically equivalent to $d_\ast$ such that $\{V \leq u\} \times \{V \leq u\}$
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is a \((\ell, \epsilon, d)\)-coupling set. In this case, we can choose \(\nu\) arbitrary large which yields a constant \(c\) arbitrary close to one.

Our framework and results can be compared to [4] who also addresses the convergence in Wasserstein distance at a subgeometric rate under H3-(a) and the assumption

(B) There exists a distance \(d\) on \(E\), bounded by 1, such that \((E, d)\) is a Polish space and

\[
\begin{align*}
\text{(i) } & \text{the level set } \Delta = \{(x, y) : V(x) + V(y) \leq \phi^{-1}(2b)\} \text{ is } d\text{-small for } P \\
& \text{i.e. there exists } \epsilon \in (0, 1) \text{ such that for all } (x, y) \in \Delta, W_d(P(x, \cdot), P(y, \cdot)) \leq (1 - \epsilon)d(x, y); \\
\text{(ii) } & \text{for all } x, y \in E, W_d(P(x, \cdot), P(y, \cdot)) \leq d(x, y).
\end{align*}
\]

Under these conditions, [4, Theorem 2.1] implies the existence and uniqueness of the stationary distribution \(\pi\) and rates of convergence to stationarity; expressions for these rates are displayed in the last row of Table 1 for various functions \(\phi\). It can be seen that our results always improve on those of [4].

Let us compare our assumption H3-(b) to (B). According to [20, corollary 5.22], (B) implies that there exists \(\epsilon \in (0, 1)\) such that \(\Delta\) is a \((1, \epsilon, d)\)-coupling set. Thus, [4, Theorem 2.1] only covers coupling sets of order 1; this is a serious restriction since in practical examples this order is most likely to be large (see e.g. the examples in Section 3). Checking H3-(b) is easier than checking (B) since allowing the coupling set to be of any order provides far more flexibility.

When some level set \(\{V \leq \nu\}\) is \((\ell, \epsilon, \nu)\)-small, i.e., there exist \(\epsilon \in (0, 1)\) and a probability measure \(\nu \in \mathcal{P}(E)\) such that for any \(x \in \{V \leq \nu\}\), \(P^\ell(x, \cdot) \geq \epsilon \nu\), then H3-(b) is satisfied with \(d = d_0\) the trivial distance. In this case, the distance \(d\) in Theorem 4 and Theorem 5 is the trivial metric and \(W_d\) is the total variation norm (see (2)). Therefore, our results also provide convergence rates in total variation norm and can be compared to the results reported in [7]. In this paper, it is assumed that \(P\) is phi-irreducible, aperiodic, that the drift condition H3-(a) hold and that the level sets \(\{V \leq u\}\) are \((\ell, \nu, \epsilon)\)-small for some \(\ell \in \mathbb{N}^\ast\), \(\epsilon \in (0, 1)\) and a probability \(\nu\) that may depend upon the level set. Under these assumptions, [7, Proposition 2.5] shows that for any \(x \in E\),

\[
\lim_n r_\phi(n) d_{TV}(P^n(x, \cdot), \pi) = 0 .
\]

Table 1 displays the rate \(r_\phi\) obtained in [7] (see penultimate row) and the rates given by Theorem 4 and Theorem 5 (see rows 2 and 3): our results are nearly the same as in [7]. Nevertheless, we would like to stress that our conditions apply in a much more general context.
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Order of the rates of convergence in

| Function          | Parameter | Convergence Rate |
|-------------------|-----------|------------------|
| $\phi(x) = x^\gamma$ for $\gamma \in (0, 1)$ | set $\tau_\star = \gamma/(1 - \gamma)$ | $(\log(n)/n)^{\tau_\star}$ |
| $\phi(x) = x/(1 + \log(x))^\alpha$ for $\alpha > 0$ | set $\tau_\star = 1/(1 + \alpha)$ | $1/\log^{\alpha}(n)$ |
| $\phi(x) = (1 + \log(x))^\alpha$ for $\alpha > 0$ | | $1/\log^{\alpha}(n)$ |

Theorem 4

$1/\log^{\alpha}(n)$

Theorem 5

For all $\delta \in (0, 1)$

| Condition | Exponential Bound |
|-----------|-------------------|
| $1/n^{\tau_\star}$ | $n^{\alpha \tau_\star} \exp(-\delta n^{\tau_\star})$ |
| $n^{\alpha \tau_\star} \exp(-n^{\tau_\star})$ | $1/\log^{\alpha}(n)$ |
| $1/n^{\tau_\star}$ | $n^{\alpha \tau_\star} \exp(-\delta C n^{\tau_\star})$ |

Table 1

Comparison of rates of convergence

3. Application

We illustrate our results by establishing the subgeometric ergodicity in Wasserstein distance of a non linear autoregressive model and a MCMC sampler in an infinite dimensional Hilbert space.

3.1. Non linear autoregressive model

For ease of exposition, we assume in this section that $E = \mathbb{R}^p$ for some $p \in \mathbb{N}^*$. We will denote by $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^p$. The metric $d_*$ is defined by $d_*(x, y) = 1 \wedge \| x - y \|$, so that $(\mathbb{R}^p, d_*)$ is a Polish space. We consider a Markov chain $\{X_n, n \in \mathbb{N}\}$ on $\mathbb{R}^p$, defined by the following non linear autoregressive equation of order 1:

$$X_{n+1} = g(X_n) + \epsilon_{n+1},$$

where

AR.1. $\{\epsilon_n, n \in \mathbb{N}\}$ is an independent and identically distributed (i.i.d.) zero-mean $\mathbb{R}^p$-valued sequence, independent of $X_0$, and satisfying

$$\mathbb{E} \left[ \exp \left( z_0 \| \epsilon_0 \|^\gamma_0 \right) \right] < +\infty$$

for some $z_0 > 0$ and $\gamma_0 \in (0, 1]$.

AR.2. $g : \mathbb{R}^p \to \mathbb{R}^p$ is a measurable function and for all $R > 0$, there exists $k_R \in [0, 1]$ such that $g$ is $k_R$-Lipschitz on $B(0, R)$ with respect to $\| \cdot \|$. Furthermore, there exist positive constants $r, R_0, \alpha$, and $\rho \in [0, 2)$, such that

$$\| g(x) \| \leq \| x \| (1 - r \| x \|^{-\rho}) \quad \text{if} \quad \| x \| \geq R_0.$$

A simple example of function $g$ satisfying AR2 is $x \mapsto x \cdot \max(1/2, |1 - 1/\| x \|^{\rho}|)$ with $\rho \in [0, 2]$.

Proposition 8 (combined with Remark 6) and Proposition 9 establish H3.

Proposition 8. [7, Theorem 3.3] Assume AR1 and AR2, and let $\rho > \gamma_0$. There exist $R_V, M \geq R_0$, $z \in (0, z_0)$ and $c > 0$ such that for all $x \in B(0, R)^c$ the drift
Proposition 9

AR2 implies that \( \text{AR1} \)

Finally, since \( \text{AR2} \) implies that \( g \) is 1-Lipschitz on \( \mathbb{R}^p \), (13) shows that \( \mathbb{E}[d_\eta(g(x) + \epsilon_1, g(y) + \epsilon_1)] \leq d_\eta(x, y) \) for all \( x, y \in \mathbb{R}^p \).

1 We point out that in [7], it is additionally required that the distribution of \( \epsilon_0 \) has a non-trivial absolutely continuous component which is bounded away from zero in a neighborhood of the origin. However, this condition is only required to establish the \( \phi \)-irreducibility of the Markov chain, which is not needed here.
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For all $\eta$, $W_{d_x}$ and $W_{d_n}$ are Lipschitz equivalent. Therefore, by application of Theorem 3, Theorem 5 and Proposition 7, we deduce from Proposition 8 and Proposition 9, the following rate of ergodicity in $d_\ast$-Wasserstein distance.

**Theorem 10.** Assume AR1 and AR2 hold, with $\rho > \gamma_0$. Then $P$ admits an unique invariant probability $\pi$ and there exist two constants $C_1$ and $C_2$ such that for all $x \in E$ and $n \in \mathbb{N}^*$

$$W_{d_x}(P^n(x, \cdot), \pi) \leq C_1 V(x) \exp \left(-C_2 n^{\tau_\ast}\right),$$

where $d_\ast(x, y) = \min\{1, \|x - y\|\}$ and $\tau_\ast = (\gamma_0 \wedge (2 - \rho))/\rho$.

### 3.2. The preconditioned Crank Nicolson algorithm

In this section, we consider the preconditioned Crank-Nicolson algorithm introduced in [2] and analysed in [11] for sampling a distribution in a separable Hilbert $(\mathcal{H}, \|\cdot\|)$ having a density $\pi \propto \exp(-g)$ with respect to a zero-mean Gaussian measure $\gamma$ with covariance operator $C$; see [3]. This algorithm is studied in [11] under conditions which imply the geometric convergence in Wasserstein distance.

**Algorithm 1:** precondioned Crank-Nicolson Algorithm

**Data:** $\rho \in (0, 1]$

**Result:** $\{X_n, n \in \mathbb{N}\}$

**begin**

Initialize $X_0$

for $n \geq 0$ do

Generate $\Xi \sim \gamma$, and set $Z = (\rho X_n + \sqrt{1 - \rho^2} \Xi)$

Generate $U \sim \mathcal{U}([0, 1])$

if $U \leq \alpha(X_n, Z) = 1 \wedge \exp(g(X_n) - g(Z))$ then

$X_{n+1} = Z$

else

$X_{n+1} = X_n$

**end**

We consider the convergence of the Crank-Nicolson scheme when the function $g$ satisfies the following conditions:

**CN1.** The function $g : \mathcal{H} \to \mathbb{R}$ is $\beta$-Hölder for some $\beta \in (0, 1]$ i.e., there exists $C_g$, such that for all $x, y \in \mathcal{H}$, $|g(x) - g(y)| \leq C_g \|x - y\|^\beta$.

Note that under CN1, $\exp(-g)$ is $\gamma$-integrable (see Proposition 24 in section 5). Examples of densities satisfying this assumption are $g(x) = -\|x\|^\beta$ with $\beta \in (0, 1]$. The Crank-Nicolson has been shown to be geometrically ergodic by [11] under the assumptions that $g$ is globally Lipschitz and that there exist positive constants $C, R_1, R_2$ such that for $x \in \mathcal{H}$ with $\|x\| \geq R_1$

$$\inf_{z \in B(\rho x, R_2)} \exp(g(x) - g(z)) \geq C;$$
New conditions for subgeometric convergence rates

see [11, Assumption 2.10-2.11]. Such an assumption implies that the acceptance ratio $\alpha(x, \rho x + \sqrt{1 - \rho^2} \xi)$ is bounded from below as $x \to \infty$ uniformly on the ball $\xi \in \bar{B}(0, R_2/\sqrt{1 - \rho^2})$. In CN1, this condition is weakened in order to address situations when the acceptance-rejection ratio vanishes when $\|x\| \to \infty$: this happens when $\lim_{\|x\| \to +\infty} g(\rho x) - g(x) = +\infty$.

In the following, we prove that the conditions of H3 are satisfied.

**Proposition 11.** Assume CN1, and let $\rho \in [0, 1)$. Set $V(x) = \exp(\|x\|)$. Then there exist $c \in (0, 1)$, $\kappa > 0$, $b, u \in \mathbb{R}_+$ such that for all $x \in \mathcal{H}$

$$PV(x) \leq V(x) - \phi \circ V(x) + b\mathbb{I}_{\{V \leq u\}}(x),$$

where $\phi$ satisfies the condition H3-(a) and $\phi(t) = ct \exp(-\kappa \log(t)^{\beta})$ for large enough $t$.

**Proof:** The proof is postponed to subsection 5.1. \qed

We now deal with showing H3-(b). To that goal, we introduce the distance

$$d_\tau(x, y) = 1 \wedge \tau^{-1}\|x - y\|^\beta,$$  \hspace{1cm} (14)

for any $\tau > 0$ and for $x, y \in E$, the basic coupling between $P(x, \cdot)$ and $P(y, \cdot)$: the same Gaussian variable $\Xi$ and the same uniform variable $U$ are generated to build $X_1$ and $Y_1$, with initial conditions $x, y$. It defines a Markov kernel $Q_{pCN}$ on $E \times E$. Define $\Lambda^\rho_{(x, y)}(z) = (\rho x + \sqrt{1 - \rho^2} z, \rho y + \sqrt{1 - \rho^2} z)$ and $\tilde{\gamma}$ the pushforward of $\gamma$ by $\Lambda^\rho_{(x, y)}$. Then an explicit form of $Q_{pCN}$ is given by the following expression:

$$Q_{pCN}((x, y), \Delta) = \int_{\Delta} \alpha(x, z) \wedge \alpha(y, t) d\tilde{\gamma}(z, t) \hspace{1cm} (15)$$

$$+ \int_{\mathcal{H} \times \mathcal{H}} (\alpha(y, t) - \alpha(x, z))_+ \mathbb{I}_{\Delta}(x, t) d\tilde{\gamma}(z, t)$$

$$+ \int_{\mathcal{H} \times \mathcal{H}} (\alpha(x, z) - \alpha(y, t))_+ \mathbb{I}_{\Delta}(z, y) d\tilde{\gamma}(z, t)$$

$$+ \delta_{(x, y)}(\Delta) \int_{\mathcal{H} \times \mathcal{H}} (1 - \alpha(x, z) \vee \alpha(y, t)) d\tilde{\gamma}(z, t)$$

where for $r \in \mathbb{R}$, $r_+ = \max(r, 0)$. In Proposition 12, we prove that there exists $\tau > 0$ such that for any level set $\mathcal{C} = \{V \leq u\}$, $\mathcal{C} \times \mathcal{C}$ is a $(\ell, \epsilon, d_\tau)$-coupling set for some $\ell \in \mathbb{N}^*$ and $\epsilon \in (0, 1)$ (the coupling may chosen to be $Q_{pCN}$), showing H3-(b). Note that for all $\tau > 0$, $d_\tau$ is topologically equivalent to $\|\cdot\|$.

**Proposition 12.** Assume CN1. Set $V(x) = \exp(\|x\|)$. Let $\tau > 0$ be given by Lemma 25. For every $u > 1$, there exist $\ell \in \mathbb{N}^*$ and $\epsilon \in (0, 1)$ such that $\{V \leq u\}^2$ is $(\ell, \epsilon, d_\tau)$-coupling set.

**Proof:** See subsection 5.2. \qed
As a consequence of Proposition 11, Proposition 12 and Theorem 5, Proposition 7, we have

**Theorem 13.** Let $P$ be the kernel of the preconditioned Crank-Nicolson algorithm with target density $d\pi \propto \exp(-g)\,d\gamma$ and design parameter $\rho \in (0,1]$. Assume CN1. Then $P$ admits $\pi$ as an unique invariant probability measure and for $\tau > 0$ sufficiently small and $\delta \in (0,1)$, there exists $C_\delta$ such that for all $n \in \mathbb{N}^*$ and $x \in \mathcal{H}$

$$W_d(P^n(x,\cdot),\pi) \leq C_\delta \exp(\|x\|) \phi(f_{\rho}(n)^{\beta})$$

with $\phi(t) = ct \exp(-\kappa \log(t)^{\beta})$ for large enough $t$ and $\kappa > 0$, $\phi, f_{\rho}$ are given by (6) and (45) and $d_\tau(x,y) = \tau^{-1}\|x - y\|^\beta \land 1$.

We did not find an analytic expression of the rate of convergence in Theorem 13. But it is clear that $t^n = o(\phi(t))$ for $a \in (0,1)$, and $\phi(t) = o(t/(1+\log(t))^a)$ for $a \in (0, +\infty)$. Therefore, the rate of convergence given by Theorem 13 is between the polynomial case and the subexponential one; see Table 1 for details.

4. Proofs of section 2

Before proceeding to the actual derivation of the proof, we present the roadmap of the proofs. The key step is given by Lemma 19 which provides an explicit expression of $B(n,m)$ such that for any $x, y \in E$

$$W_d(P^n(x,\cdot), P^n(y,\cdot)) \leq B(n,m) \cdot (V(x) + V(y)) \cdot \phi(f_{\rho}(n)^{\beta}).$$

(16)

First, this inequality will imply that $P$ admits at most one invariant probability. By applying (16) with $n \leftarrow n + m$, and $y \leftarrow x$, we then show that $(P^n(x,\cdot))_{n \geq 0}$ is a Cauchy sequence in $(\mathcal{P}(E),W_d)$ and therefore converges in $W_d$ to some probability measure $\pi_x$ which may be shown to be invariant for $P$. Using that $P$ admits one invariant probability measure will imply that $\pi_x$ does not depend on $x$, (see subsection 4.3).

The proof of Theorem 4 and Theorem 5 also follow from (16) this time taking $n = m$, and integrating this inequality w.r.t. the unique invariant distribution $\pi$. The only difficulties to be dealt with stem from the fact that the right hand side of the inequality is not integrable; a truncation is therefore required to conclude the proof.

Let us now explain the computation of the upper bound (16). Let $Q$ be the coupling kernel under which $\Delta$ is a $(\ell, \epsilon, d)$-coupling set. Note that this implies that for any $n \in \mathbb{N}^*$ and $x, y \in E$, $Q^n((x,y),\cdot)$ is a coupling of $(P^n(x,\cdot), P^n(y,\cdot))$. Therefore, by (1),

$$W_d(P^n(x,\cdot), P^n(y,\cdot)) \leq \widetilde{E}_{x,y}[d(X_n,Y_n)]$$

where $((X_n,Y_n),n \geq 0)$ is a Markov chain on the product space $E \times E$ with Markov kernel $Q$ and $\widetilde{E}_{x,y}$ is the associated canonical expectation when the initial distribution is the Dirac mass at $(x,y)$.
The contraction property of $Q$ (see Definition 2)
\[ \overline{d}_{x,y}[d(X_1, Y_1)] \leq d(x, y) \quad \text{for all } (x, y) \in E \times E , \]  
(17)
combined with the Markov property of $((X_n, Y_n), n \geq 0)$ imply that $(d(X_n, Y_n), n \geq 0)$ is a supermartingale with respect to the filtration $\mathcal{F}_n = \sigma(X_0, Y_0, \ldots, X_n, Y_n)$.

The next step of the proof is to show that this supermartingale property implies that for any $n, m \geq 1$, (see Proposition 17)
\[ \overline{d}_{x,y}[d(X_n, Y_n)] \leq (1 - \epsilon)^{m-1} + \overline{d}_{x,y}[T_m \geq n] \]
where $(T_m, m \geq 1)$ are the successive return times to $\Delta$. More precisely, set $\tau_\Delta = \inf \{n > 0 | (X_n, Y_n) \in \Delta \}$, $T_0 = \tau_\Delta \circ \theta^{\ell-1} + \ell - 1$ where $\ell$ is given by H1; and for any $j \geq 1$, define the successive return-times to $\Delta$ after $\ell - 1$ steps by
\[ T_j = \tau_\Delta \circ \theta^{T_j-1+\ell-1} + T_{j-1} + \ell - 1 , \]  
(18)
where $\theta$ is the shift operator.

By the Markov inequality, for any increasing rate function $R$, it holds
\[ \overline{d}_{x,y}[d(X_n, Y_n)] \leq (1 - \epsilon)^{m-1} + \frac{\overline{d}_{x,y}[R(T_m)]}{R(n)} . \]  
(19)

The last step of the proof is to compute an upper bound for the moment $\overline{d}_{x,y}[R(T_m)]$. Then $m$ is chosen in order to balance the two terms in the RHS of (19).

To get precise estimate of subgeometric moments of the return times, we introduce, similarly to [18] a sequence of drift conditions; in our setting, it is convenient to formulate this condition on the product space $E \times E$.

**H4. There exist**

- a sequence of measurable functions $(V_n)_{n \geq 0}$, $V_n : E \times E \rightarrow \mathbb{R}_+$,
- a set $\Delta \in \mathcal{B}(E \times E)$, a constant $b < \infty$ and a sequence $r \in \Lambda$

such that for all $x, y \in E$ and for every coupling $\alpha \in C(P(x, \cdot), P(y, \cdot))$:
\[ \int_{E \times E} V_{n+1}(z, t) d\alpha(z, t) \leq V_n(x, y) - r(n) + b \chi(\Delta, x, y) . \]  
(20)

Moreover, there exist measurable functions $(V_n)_{n \geq 0}$, $V_n : E \rightarrow \mathbb{R}_+$ such that for all $x, y \in E$ and any $n \geq 0$:
\[ V_n(x, y) \leq V_n(x) + V_n(y) \quad \text{and} \quad P V_{n+1}(x) \leq V_n(x) + b \chi(n) . \]  
(21)

Finally, for all $k \geq 0$,
\[ \sup_{(x,y) \in \Delta} \{ P^k V_0(x) + P^k V_0(y) \} < +\infty \quad \text{and, for all } x \in E, P^k V_0(x) < +\infty . \]  
(22)
Under \( H4 \), we will get some bounds on the moments \( \tilde{E}_{x,y}[R(T_0)] \) for \( x, y \in E \) (see Proposition 17), where

\[
R(n) = \sum_{k=0}^{n-1} r(k) \quad \text{for} \quad n \geq 1 \quad R(0) = 1.
\]

(23)

We will then distinguish two cases: these bounds on \( R(T_0) \) will provide bounds on the moments \( R(T_{m/m}) \) and \( R(T_m) \). To that goal, in the first case \( R \) is approximated by a convex function; while in the second case \( R \) is approximated by some geometric sequence. This second approach, despite it provides a tight bound when the sequence \( (R(n))_n \) is of subexponential order \( \exp(cn^\alpha) \), for \( c > 0 \) and \( \alpha \in (0, 1) \), is not appropriate when the sequence is of polynomial or logarithmic order. This is the reason why our convergence results will always be split into two parts (one applicable to polynomial or logarithmic sequences and the other to truly subgeometric sequences). The above discussion is formalized in Lemma 18.

Finally, in Proposition 22, we check that \( H4 \) is implied by \( H2 \).

### 4.1. Convergence results under a sequence of drift conditions

**Proposition 14.** Assume \( H1 \). Then, for all \( x, y \in E \), and \( n, m \in \mathbb{N}, m \geq 1 \):

\[
\tilde{E}_{x,y}[d(X_n, Y_n)] \leq (1 - \epsilon)^{m-1} + \tilde{P}_{x,y}[T_m \geq n].
\]

(24)

We preface the proof by stating the following Lemma, which is a restatement of [13, Lemma 3.1].

**Lemma 15.** Let \( (Z_n)_{n \geq 0} \) be a nonnegative \( \mathcal{F}_n \)-supermartingale upper bounded by \( K \). Let \( (\tau_n)_n \) be a sequence of increasing stopping times with respect to \( \mathcal{F}_n \), with \( \tau_0 = 0 \). Assume there exists \( \epsilon \in (0, 1) \) such that for every \( n \geq 1 \)

\[
\mathbb{E}[Z_{\tau_{n+1}}|\mathcal{F}_n] \leq (1 - \epsilon)Z_{\tau_n}.
\]

Then, for all \( n, m \in \mathbb{N}, m \geq 1 \),

\[
\mathbb{E}[Z_n] \leq K \left( (1 - \epsilon)^{m-1} + \mathbb{P}[^{\tau_m \geq n}] \right).
\]

**Proof of Proposition 14:** Set \( Z_n = d(X_n, Y_n) \); under \( H1 \), \( (Z_n, \tilde{F}_n)_{n \geq 0} \) is a bounded non-negative supermartingale and for all \( (x, y) \in \Delta, \tilde{E}_{x,y}[Z_\ell] \leq (1 - \epsilon)d(x, y) \). Denote by \( Z_\infty \) its \( \tilde{P}_{x,y}\text{-a.s.} \) limit. By the optional stopping theorem, we have for every \( m \geq 0 \):

\[
\tilde{E}_{x,y}[Z_{T_{m+1}}|\tilde{F}_{T_m + \ell}] \leq Z_{T_m + \ell}.
\]

(25)

On the other hand, the strong Markov property imply for every \( m \geq 0 \)

\[
\tilde{E}_{x,y}[Z_{T_m + \ell}|\tilde{F}_{T_m}] \leq (1 - \epsilon)Z_{T_m}.
\]

(26)

By (25) and (26), it yields \( \tilde{E}_{x,y}[Z_{T_{m+1}}|\tilde{F}_{T_m}] \leq (1 - \epsilon)Z_{T_m} \). Under \( H1 \), \( Z_n \) is upper bounded by 1 and the proof follows from Lemma 15. \( \square \)
To get an estimate of \( \bar{P}_{x,y} [T_m \geq n] \) for \( x, y \in E \) and \( n, m \in \mathbb{N} \), we derive from H4 some bound on \( \bar{E}_{x,y} [R(T_0)] \), where \( R \) is given by (23).

**Lemma 16.** Assume H4 holds. Then, for all \( x, y \in E \) and all \( k \geq 0 \)

\[
\sup_{(x,y) \in \Delta} Q^k V_0(x,y) < +\infty \quad Q^k V_0(x,y) < +\infty ,
\]

Proof: By (21) and definition of \( Q \), \( Q^k V_0(x,y) \leq P^k V_0(x) + P^k V_0(y) \), for all \( k \geq 0 \). Eq. (22) concludes the proof. \( \square \)

**Proposition 17.** Assume H4 holds. Let \( R \) be the sequence defined by (23). Then,

\[
\bar{E}_{x,y} [R(T_\Delta)] \leq \begin{cases} V_0(x,y), & (x,y) \notin \Delta \\ r(0) + c Q V_0(x,y), & (x,y) \in E \times E . \end{cases} \tag{27}
\]

where \( c = \sup_{k \in \mathbb{N}} (r(k+1)/r(k)) \) is finite, and

\[
\sup_{(x,y) \in \Delta} \bar{E}_{x,y} [R(T_0)] < +\infty . \tag{28}
\]

In addition, for all \( j \geq 0 \) and \( (x,y) \in E \times E \),

\[
\bar{P}_{x,y} [T_j < \infty] = 1 . \tag{29}
\]

Proof: Since \( r \in \Lambda \), Lemma 36 shows that the constant \( c \) is finite. (27) follows from [15, proposition 11.3.3]. The second statement follows from (27), Lemma 16 and the Markov property. Finally, (28) shows that for any \( x,y \), \( \bar{P}_{x,y}(T_0 < \infty) = 1 ; (29) \) now follows by a straightforward induction. \( \square \)

**Lemma 18.** Assume H1 and H4. Let \( R \) be the sequence defined by (23). Then,

\begin{itemize}
  \item There exists a constant \( C \) such that for all \( x, y \in E \) and for all \( n, m \in \mathbb{N} \),

\[
\bar{P}_{x,y} [T_m \geq n] \leq \frac{C}{R([n/(m+1)])} (1 + P^n V_0(x) + P^n V_0(y)) . \tag{30}
\]

  \item For all \( \alpha > 0 \), there exists a constant \( C_\alpha \) satisfying for all \( x, y \in E \) and for all \( n, m \in \mathbb{N} \),

\[
\bar{P}_{x,y} [T_m \geq n] \leq \frac{C_\alpha}{R(n)} (1 + P^n V_0(x) + P^n V_0(y)) (1 + \alpha)^m . \tag{31}
\]
\end{itemize}

Proof: Set \( C_\Delta = \sup_{(z,t) \in \Delta} \bar{E}_{z,t} [R(T_0)] \), finite by Proposition 17. We first establish (30). Let \( \psi_r \) be the increasing convex function given by Lemma 37 such that there exist positive constants \( C_i \), \( i \in \{1, 2\} \), for every \( n \in \mathbb{N}^* \),

\[
C_1 \psi_r(n) \leq R(n) \leq C_2 \psi_r(n) . \tag{32}
\]

By the Markov inequality, since \( \psi_r \) is increasing,

\[
\bar{P}_{x,y} [T_m \geq n] \leq \psi_r(n/(m+1))^{-1} \bar{E}_{x,y} [\psi_r(T_m/(m+1))] \\
\leq C_2 R([n/(m+1)])^{-1} \bar{E}_{x,y} [\psi_r(T_m/(m+1))] . \tag{33}
\]
By construction,

\[ T_m = T_0 + \sum_{k=0}^{m-1} \{ \tau_{\Delta} \circ \theta^{T_k+\ell-1} + \ell - 1 \}, \]

with the convention that \( \sum_{k=0}^{-1} = 0 \). Since \( \psi_r \) is convex it follows from (32), that

\[ \mathbb{E}_{x,y}[\psi_r(T_m/(m+1))] \leq \mathbb{E}_{x,y}\left[ \frac{1}{m+1} \left( \psi_r(T_0) + \sum_{k=0}^{m-1} \psi_r(\tau_{\Delta} \circ \theta^{T_k+\ell-1} + \ell - 1) \right) \right] \]

\[ \leq \frac{1}{C_1(m+1)} \mathbb{E}_{x,y}\left[ \left( R(T_0) + \sum_{k=0}^{m-1} R(\tau_{\Delta} \circ \theta^{T_k+\ell-1} + \ell - 1) \right) \right]. \]

Using Proposition 17 and the strong Markov property, there exists \( C > 0 \) such that for any \( x, y \in E \) and \( m \geq 0 \),

\[ \mathbb{E}_{x,y}[\psi_r(T_m/(m+1))] \leq \frac{C}{C_1(m+1)} (Q^t \nu_0(x,y) + mC_\Delta + 1). \tag{34} \]

It remains to use (21) and plug (34) in (33) to get the first upper bound.

We now consider (31). Again by the Markov inequality, since \( R \) is increasing,

\[ \mathbb{P}_{x,y}[T_m \geq n] \leq R^{-1}(n) \mathbb{E}_{x,y}[R(T_m)]. \tag{35} \]

If \( m = 0 \), the result follows from Proposition 17. If \( m \geq 1 \), using the definitions of \( T_m \) and \( R \), given respectively in (18) and (23), and the assertion Lemma 36-(iv)

\[ \mathbb{E}_{x,y}[R(T_m)] \leq \mathbb{E}_{x,y}[R(T_{m-1})] + C_1 \mathbb{E}_{x,y}[r(T_{m-1})R(\tau_{\Delta} \circ \theta^{T_{m-1}+\ell-1} + \ell - 1)], \]

for a constant \( C_1 > 0 \). Thus, by the strong Markov property

\[ \mathbb{E}_{x,y}[R(T_m)] \leq \mathbb{E}_{x,y}[R(T_{m-1})] + C_2 \mathbb{E}_{x,y}[r(T_{m-1})], \tag{36} \]

where \( C_2 = C_1 C_\Delta \). Let \( \alpha > 0 \). According to Lemma 35-(iv), there exists \( N_\alpha \) such that for any \( n \geq N_\alpha \), \( r(n) \leq \alpha R(n) \). Then

\[ \mathbb{E}_{x,y}[r(T_{m-1})] \leq r(N_\alpha) + \alpha \mathbb{E}_{x,y}[R(T_{m-1})], \]

so that (36) becomes

\[ \mathbb{E}_{x,y}[R(T_m)] \leq (1 + C_2 \alpha) \mathbb{E}_{x,y}[R(T_{m-1})] + C_2 r(N_\alpha). \]

By a straightforward induction and definition of \( N_\alpha \), we get,

\[ \mathbb{E}_{x,y}[R(T_m)] \leq C_\alpha (1 + C_2 \alpha)^m (\mathbb{E}_{x,y}[R(T_0)] + 1), \]

for some constant \( C_\alpha > 0 \). Plugging this result in (35) and using Proposition 17 concludes the proof. \( \square \)
Lemma 19. Assume $H1$ and $H4$. Let $R$ be the sequence defined by (23). Then,

- There exists a constant $C$ such that for all $x, y \in E$, all $n, m \in \mathbb{N}$,
  \[ W_d(P^n(x, \cdot), P^{n+m}(y, \cdot)) \leq C \frac{1 + P^nV_0(x) + P^{m+\ell}V_0(y)}{R(\lceil -n \log(1 - \epsilon)/\log(R(n)) \rceil)}. \] (37)

- For all $\delta \in (0, 1)$, there exists a constant $C_\delta$ such that for all $x, y \in E$ and $n, m \in \mathbb{N}$,
  \[ W_d(P^n(x, \cdot), P^{n+m}(y, \cdot)) \leq C_\delta \frac{1 + P^nV_0(x) + P^{m+\ell}V_0(y)}{R^\delta(n)}. \] (38)

Proof: We first establish (37). Lemma 28 implies

\[ W_d(P^n(x, \cdot), P^{n+m}(y, \cdot)) \leq \inf_{\alpha \in C(\delta, \delta, P^m)} \int_{E \times E} W_d(P^n(z, \cdot), P^n(t, \cdot))dz(t). \]

Since $Q((z, t), \cdot)$ is a coupling of $(P(z, \cdot), P(t, \cdot))$ then for any $n \geq 1$, $Q^n((z, t), \cdot)$ is a coupling of $(P^n(z, \cdot), P^n(t, \cdot))$. Therefore,

\[ W_d(P^n(z, \cdot), P^n(t, \cdot)) \leq \bar{E}_{z, t}[d(X_n, Y_n)]. \]

The next step is to upper bound the RHS. By Proposition 14 and Lemma 18-(30), there exists $C$ such that for all $x, y \in E$ and for all $n \geq 0$ and $m \geq 1$

\[ \bar{E}_{x, y}[d(X_n, Y_n)] \leq (1 - \epsilon)^{m-1} + \bar{P}_{x, y}[T_m \geq n] \]

\[ \leq (1 - \epsilon)^{m-1} + C \frac{1 + P^nV_0(x) + P^\ell V_0(y)}{R(\lceil n/(m+1) \rceil)}. \]

Using this inequality with $m = \lceil -\log(R(n))/\log(1 - \epsilon) \rceil - 1$ and since $R$ is increasing, there exists a constant $C_1$ such that for all $z, t \in E$,

\[ \bar{E}_{z, t}[d(X_n, Y_n)] \leq C_1 \frac{1 + P^nV_0(z) + P^\ell V_0(t)}{R(\lceil -n \log(1 - \epsilon)/\log(R(n)) \rceil)}. \] (39)

The result now follows easily.

The proof of (38) is along the same lines, using Lemma 18-(31) instead of Lemma 18-(30). In this case, for some fixed $\delta \in (0, 1)$, we choose $m$ such that $(1 - \epsilon)^{m-1} = R^{-\delta}(n)$; and in Lemma 18-(31), we choose $\alpha > 0$ such that

\[ \log(1 + \alpha) \leq \frac{1 - \delta}{\delta} \log(1 - \epsilon). \]

\[ \Box \]

Proposition 20. Assume $H1$ and $H4$ hold. Then $P$ admits at most one invariant probability measure.
Proof: Under H1, \((\mathcal{P}(E), W_d)\) is a Polish space, and \(W_d\) is continuous on \(\mathcal{P}(E) \times \mathcal{P}(E)\); see [20, Theorem 6.16]. Therefore, \((x, y) \mapsto W_d(P^n(x, \cdot), P^n(y, \cdot))\) is measurable.

Assume that there exist two invariant distributions \(\pi\) and \(\nu\), and let \(\alpha\) be a coupling of \(\pi\) and \(\nu\). According to Lemma 28, we have for every integer \(n\):

\[
W_d(\pi, \nu) = W_d(\pi P^n, \nu P^n) \leq \int_{E \times E} W_d(P^n(x, \cdot), P^n(y, \cdot)) \alpha(dx, dy) .
\]

By (37), there exists a constant \(C\) such that for all \(x, y \in E\) and \(n \geq 0\),

\[
g_n(x, y) \overset{\Delta}{=} W_d(P^n(x, \cdot), P^n(y, \cdot)) \leq C \frac{1 + P^t V_0(x) + P^t V_0(y)}{R([-n \log(1 - \epsilon)/\log(R(n))])} . \tag{40}
\]

Since \(r \in \Lambda\), Lemma 36-(ii) and (v) shows that

\[
\lim_{n \to +\infty} R([-n \log(1 - \epsilon)/\log(R(n))]) = +\infty .
\]

Eq. (40) shows that the sequence of functions \(\{g_n, n \in \mathbb{N}\}\) converges pointwise to 0 and is bounded by 1 since, by assumption, the distance \(d\) is bounded by one. Therefore, by the Lebesgue dominated convergence theorem, we have:

\[
\int_{E \times E} W_d(P^n(x, \cdot), P^n(y, \cdot)) \alpha(dx, dy) \xrightarrow{n \to +\infty} 0 ,
\]

showing that \(W_d(\pi, \nu) = 0\), or equivalently \(\nu = \pi\) since \(W_d\) is a distance on \(\mathcal{P}(E)\).

\(\square\)

4.2. From the drift condition H2 to the sequence of drifts H4

Throughout this section, H2 is assumed to hold. Define for \(k \geq 0\), \(H_k : [1, \infty) \to \mathbb{R}^+\) and \(V_k : E \times E \to \mathbb{R}^+\) by

\[
H_k(x) = \int_0^{H_\phi(x)} r_\phi(t + k)dt = H_\phi^{-1}(H_\phi(x) + k) - H_\phi^{-1}(k) , \tag{41}
\]

\[
V_k(x, y) = H_k(V(x) + V(y)) , \tag{42}
\]

where \(H_\phi\) and \(r_\phi\) are respectively given by (5) and (6). Note that \(H_\phi(x) \leq x\) so \(V_0(x, y) \leq V(x) + V(y)\). The proof of the following lemma is adapted from [7, Proposition 2.1].

Lemma 21. Under the condition H2, for all \(x, y \in E\) and every coupling \(\alpha \in C(P(x, \cdot), P(y, \cdot))\), we have:

\[
\int_{E \times E} V_{k+1}(z, t)\alpha(z, t) \leq V_k(x, y) - r_\phi(k) + b \frac{r_\phi(k + 1)}{r_\phi(0)} \mathbb{1}_\Delta(x, y) ,
\]

where \(r_\phi\) and \(V_k\) are defined in (6) and (42) respectively.
Proof: Set $V(x, y) = V(x) + V(y)$. First, note $H_k$ that is twice continuously differentiable on $[1, +\infty)$ and concave for all $k \geq 0$ (see Lemma 32 and Proposition 33-(ii)). This implies that for all $u \geq 1$ and $x \in \mathbb{R}$ such that $x + u \geq 1$, we have

$$H_{k+1}(u + x) - H_{k+1}(u) \leq H'_{k+1}(u)x.$$ \hspace{1cm} (43)

In addition, according to the proof of [7, Proposition 2.1]: for every $u \geq 1$

$$H_{k+1}(u) - \phi(u)H'_{k+1}(u) \leq H_k(u) - r_\phi(k).$$ \hspace{1cm} (44)

Therefore, since $H_{k+1}$ is concave, the Jensen inequality and (4) imply

$$\int_{E \times E} V_{k+1}(z, t) d\alpha(z, t) \leq H_{k+1} \left( \int_{E \times E} V(z, t) d\alpha(z, t) \right) \leq H_{k+1}(V(x, y) - \phi(V(x, y)) + b \mathbb{1}_\Delta(x, y)).$$

Using (43), (44) and the inequality $H'_{k+1}(V(x, y)) \leq H'_{k+1}(1)$ we get that

$$\int_{E \times E} V_{k+1}(z, t) d\alpha(z, t) \leq H_{k+1}(V(x, y) - \phi(V(x, y))H'_{k+1}(V(x, y)) + bH'_{k+1}(1) \mathbb{1}_\Delta(x, y) \leq H_k(V(x, y) - r_\phi(k) + bH'_{k+1}(1) \mathbb{1}_\Delta(x, y)).$$

The proof is concluded upon noting that $H'_{k+1}(1) = r_\phi(k + 1)/r_\phi(0)$. \hfill \Box

Proposition 22. Assume $H^2$ and let $x_0 \in E$. Then $H_4$ holds with the same set $\Delta$, $r \leftarrow r_\phi$,

$$b \leftarrow \left( \frac{b + V(x_0)}{r_\phi(0)} + 1 \right) \sup_{k \geq 1} \frac{r_\phi(k + 1)}{r_\phi(k)},$$

$V_n(x, y) \leftarrow H_n(V(x) + V(y))$ and $V_n \leftarrow H_n \circ V + r_\phi(n)$ where $H_n$ is given by (41).

Proof: Since $H_k$ is twice continuously differentiable and $V$ is measurable, $V_n$ is measurable for all $n \in \mathbb{N}$. By Lemma 21,

$$\int_{E \times E} V_{k+1}(z, t) d\alpha(z, t) \leq V_k(x, y) - r_\phi(k) + b \frac{r_\phi(k + 1)}{r_\phi(0)} \mathbb{1}_\Delta(x, y).$$

By Proposition 33(i), $r_\phi \in \Lambda$ and Lemma 36-(iv) shows that there exists a constant $C$ such that $\sup_k r_\phi(k + 1)/r_\phi(k) \leq C$. Therefore

$$\int_{E \times E} V_{k+1}(z, t) d\alpha(z, t) \leq V_k(x, y) - r_\phi(k) + \frac{bC}{r_\phi(0)} r_\phi(k) \mathbb{1}_\Delta(x, y).$$

By Lemma 34-(ii), for any $k \geq 0$,

$$V_k(x, y) \leq H_k(V(x)) + H_k(V(y)) + 2r_\phi(k) = V_k(x) + V_k(y).$$
By Proposition 33-(iii), for all \( x \in E \) and \( k \geq 0 \),
\[
PV_{k+1}(x) \leq V_k(x) - 2r_\phi(k) + \frac{(b + V(x_0))r_\phi(k + 1)}{r_\phi(0)} + r_\phi(k + 1)
\]
\[
\leq V_k(x) + r_\phi(k) \left( \frac{b + V(x_0)}{r_\phi(0)} C + C - 2 \right).
\]
Finally \( V_0(x) \leq V(x) + r_\phi(0) \), and by (4), \( P^k V(x) + P^k V(y) \leq V(x) + V(y) + kb \) for \( k \in \mathbb{N} \). Therefore under \( H_2 \), \( \sup_{(x,y) \in \Delta}(P^k V(x) + P^k V(y)) < +\infty \) for all \( k \in \mathbb{N} \); and \( P^k V_0(x) < \infty \) for any \( x \in E \) and \( k \in \mathbb{N} \). □

By Proposition 22, \( H_2 \) implies \( H_4 \) with \( V_0 \leq V + r_\phi(0) \) and \( r \leftarrow r_\phi \), where \( r_\phi \) is given by (6). Thus Lemma 19 and the results of subsection 4.1 apply with \( R \leftarrow R_\phi \) where
\[
R_\phi(n) = \sum_{k=0}^{n-1} r_\phi(k) \quad \text{for } n \geq 1 \quad \text{and } R_\phi(0) = 1. \tag{45}
\]
Note that by iterating the drift inequality (4) applied with \( x = y \), it holds
\[
P^\ell V(x) \leq V(x) + \frac{b}{2} \ell, \quad \forall \ell \geq 1, \forall x \in E. \tag{46}
\]

4.3. Proof of Theorem 3

By Proposition 20, if an invariant probability measure exists, it is unique. Let us prove such a measure exists.

Let \( x_0 \in E \). We first show there exists \( (m_k)_k \) such that \( (P^{m_k}(x_0, \cdot))_k \) is a Cauchy sequence for \( W_d \). By \( H_2 \),
\[
PV(x) \leq PV(x) + PV(x_0) \leq V(x) + V(x_0) - \phi \circ V(x_0) + b
\]
where we used that \( \phi(V(x) + V(x_0)) \geq \phi(V(x)) \). This implies, by Lemma 31, that \( \lim_n n^{-1} \sum_{k=0}^{n-1} P^k(\phi \circ V)(x_0) \leq b + V(x_0) \). Fix \( M_\phi > b + V(x_0) \); there exists an increasing sequence \( (n_k)_k \) such that \( \lim_k n_k = +\infty \) and
\[
P^{n_k}(\phi \circ V)(x_0) \leq M_\phi, \quad \text{for all } k \in \mathbb{N}. \tag{47}
\]

Let \( n, k \in \mathbb{N}^* \) and choose \( M_V > 0 \). By Lemma 28:
\[
W_d(P^n(x_0, \cdot), P^{n+n_k}(x_0, \cdot))
\leq \inf_{\alpha \in \mathcal{C}(\delta_{x_0}, P^{n_k}(x_0, \cdot))} \int_{E \times E} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt)
\leq \inf_{\alpha \in \mathcal{C}(\delta_{x_0}, P^{n_k}(x_0, \cdot))} \left\{ \int_{E \times E} 1_{\{V(t) \geq M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt)ight.
\leq \int_{E \times E} 1_{\{V(t) < M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) \right\}. \tag{48}
\]
We consider the two terms in turn. Let $\alpha \in \mathcal{C}(\delta x_0, P^n(x_0, \cdot))$. Since $W_d$ is bounded by 1,
\[
\int_{E \times E} \mathbb{1}_{\{V(t) \geq M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) \leq P^{n_k}(x_0, \{V \geq M_V\})
\leq P^{n_k}(x_0, \{\phi \circ V \geq \phi(M_V)\}) \leq \frac{P^{n_k}(\phi \circ V)(x_0)}{\phi(M_V)} \leq \frac{M_\phi}{\phi(M_V)}, \tag{49}
\]
where we used (47) and the Markov inequality. In addition, by Lemma 19-(38) applied with $\delta = 1/2$, there exists $C > 0$ such that:
\[
\int_{E \times E} \mathbb{1}_{\{V(t) < M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt)
\leq \frac{C}{\sqrt{R_\phi(n)}} \int_{E \times E} \mathbb{1}_{\{V(t) < M_V\}} \left( P^n t V(z) + P^n t V(t) \right) \alpha(dz, dt)
\leq \frac{C}{\sqrt{R_\phi(n)}} \int_{E \times E} \mathbb{1}_{\{V(t) < M_V\}} \left( V(z) + V(t) + b \right) \alpha(dz, dt),
\]
where we used (46) in the last inequality. Furthermore, $x \mapsto \phi(x)/x$ is non-increasing so that $V(t) \leq M_V \phi(V(t))/\phi(M_V)$ on $\{V \leq M_V\}$. This implies
\[
\int_{E \times E} \mathbb{1}_{\{V(t) < M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt)
\leq \frac{C}{\sqrt{R_\phi(n)}} \left( V(x_0) + b + \frac{M_\phi M_V}{\phi(M_V)} \right), \tag{50}
\]
Combining (49) and (50) in (48), we have for every $M_V > 0$, $n, k \in \mathbb{N}^*$
\[
W_d(P^n(x_0, \cdot), P^{n+n_k}(x_0, \cdot)) \leq \frac{M_\phi}{\phi(M_V)} + \frac{C}{\sqrt{R_\phi(n)}} \left( V(x_0) + b + \frac{M_\phi M_V}{\phi(M_V)} \right).
\]
Setting $M_V = \sqrt{R_\phi(n)}$, this equation shows there exists a constant $C'$ such that for all $n, k \in \mathbb{N}^*$
\[
W_d(P^n(x_0, \cdot), P^{n+n_k}(x_0, \cdot)) \leq C' \frac{V(x_0)}{\phi(\sqrt{R_\phi(n)})}. \tag{51}
\]
Let us define the sequence $(m_k)_k$. By Lemma 36-(ii), $\lim_{x \to +\infty} \phi(x) = +\infty$ and by definition, $\lim_{k \to +\infty} n_k = +\infty$; hence there exists $(u_k)_k$ such that $u_0 = 1$ and
\[
u_{k+1} = \inf \left\{ n_l \mid l \in \mathbb{N}; \phi \left( \sqrt{R_\phi(n_l)} \right)^{-1} \leq 2^{-k-1} \right\}. \tag{52}
\]
Set $m_k = \sum_{l=0}^{k} u_i$. Since for all $k \in \mathbb{N}^*$, $m_{k+1} = m_k + u_{k+1}$, by (51) and (52) $W_d(P^{m_k}(x_0, \cdot), P^{m_{k+1}}(x_0, \cdot)) \leq C'2^{-k}V(x_0)$, which implies that the series $\sum_k W_d(P^{m_k}(x_0, \cdot), P^{m_{k+1}}(x_0, \cdot))$ converges and $(P^{m_k}(x_0, \cdot))_k$ is Cauchy for $W_d$. 

Since under H1, (\(\mathcal{P}(E), W_d\)) is Polish, there exists \(\pi \in \mathcal{P}(E)\) such that \(\lim_{k \to +\infty} W_d(P^{m_k}(x_0, \cdot), \pi) = 0\). The second step is to prove that \(\pi\) is invariant. As \(W_d\) is continuous on \(\mathcal{P}(E) \times \mathcal{P}(E)\), \(W_d(\pi, \pi P) = \lim_{k \to +\infty} W_d(P^{m_k}(x_0, \cdot), \pi P)\).

By the triangular inequality, it holds

\[
W_d(\pi, \pi P) \leq \lim_{k \to +\infty} W_d(P^{m_k}(x_0, \cdot), \delta_{x_0} P^{m_k} P) + \lim_{k \to +\infty} W_d(\delta_{x_0} P^{m_k} P, \pi P).
\]

(53)

By Lemma 19-(38) and Lemma 28, there exists \(C\) such that for any \(k \geq 1\),

\[
W_d(P^{m_k}(x_0, \cdot), \delta_{x_0} P^{m_k+1} P) \leq \inf_{\alpha \in C(\delta_{x_0} P^{m_k})} \int_{E \times E} W_d(P^{m_k}(z, \cdot), P^{m_k}(t, \cdot)) d\alpha(z, t) \leq \frac{C}{\sqrt{R_\phi(m_k)}} (P^t V(x_0) + P^{t+1} V(x_0)).
\]

By (46), \(P^t V(x_0) + P^{t+1} V(x_0)\) is finite. By definition, \(\lim_k m_k = +\infty\) so that by Lemma 36-(ii), the RHS converges to 0 when \(k \to +\infty\). In addition, by Lemma 30, \(W_d(\delta_{x_0} P^{m_k} P, \pi P) \leq W_d(P^{m_k}(x_0, \cdot), \pi)\), and this RHS converges to 0 by definition of \(\pi\). Plugging these results in (53) yields \(W_d(\pi, \pi P) = 0\), and therefore \(\pi P = \pi\).

Finally, Lemma 31 implies that \(\pi(\phi \circ V) < \infty\).

4.4. Proof of Theorem 4

Fix \(M_V > 0\) such that \(\pi(V \leq M_V) \geq 1/2\); such a constant exists since \(\pi(E) = 1\) and \(E = \bigcup_{k \in \mathbb{N}} \{V \leq k\}\). Note that \(\pi(V \leq M) \geq 1/2\) for any \(M \geq M_V\). Fix \(M > M_V\) and denote by \(\pi_M\) the probability in \(\mathcal{P}(E)\) defined by \(\pi_M(A) = \pi(A \cap \{V \leq M\})/\pi(\{V \leq M\})\).

Since \(\pi\) is invariant for \(P\), \(W_d(P^n(x, \cdot), \pi) = W_d(P^n(x, \cdot), \pi P^n)\) and the triangular inequality implies for all \(n \geq 1:\)

\[
W_d(P^n(x, \cdot), \pi) \leq W_d(P^n(x, \cdot), \pi M P^n) + W_d(\pi_M P^n, \pi P^n).
\]

(54)

Consider the first term in (54). By Lemma 28, for all \(x \in E\) and \(n \geq 1:\)

\[
W_d(P^n(x, \cdot), \pi M P^n) \leq \inf_{\alpha \in C(\delta_{x}, \pi M)} \int_{E \times E} W_d(P^n(z, \cdot), P^n(t, \cdot)) d\alpha(z, t).
\]

By Lemma 19-(37), there exists \(C_1 > 0\) such that for all \(x \in E\) and \(n \geq 1:\)

\[
W_d(P^n(x, \cdot), \pi M P^n) \leq \frac{C_1}{R_\phi \left(\left\lfloor \frac{-n \log(1-\epsilon)}{\log(R_\phi(V))} \right\rfloor\right)} \inf_{\alpha \in C(\delta_{x}, \pi M)} \int_{E \times E} (P^t V(z) + P^t V(t)) d\alpha(z, t)
\]

\[
\leq \frac{C_1}{R_\phi \left(\left\lfloor \frac{-n \log(1-\epsilon)}{\log(R_\phi(V))} \right\rfloor\right)} (V(x) + \pi_M(V) + b).\]

(55)
where we used (46) in the last inequality. Finally, since \( x \mapsto \phi(x)/x \) is non-increasing and \( V(t) \leq M\phi(V(t))/\phi(M) \) on \( \{V \leq M\} \). It yields:

\[
\pi_M(V) \leq \frac{\pi(\phi \circ V)}{\pi(\{V \leq M\})} \frac{M}{\phi(M)} \leq 2\pi(\phi \circ V)M/\phi(M) .
\] (56)

We deduce from Proposition 33-(i), Lemma 36-(i) and (8), applied twice, that there exists \( C_2 > 0 \) such that

\[
R_\phi([-n \log(1 - \epsilon)/\log(R_\phi(n))] \geq C_2 f_\phi \{n/\log(f_\phi(n))\} .
\] (57)

Using (56) and (57), (55) becomes:

\[
W_d(P^n(x, \cdot), \pi_M P^n) \leq \frac{C_1 (lb + V(x) + 2\pi(\phi \circ V)M/\phi(M))}{C_2 f_\phi \{n/\log(f_\phi(n))\}} .
\] (58)

Consider the second term in (54). Since \( d \) is bounded by 1, Lemma 30 and Lemma 26 imply \( W_d(\pi_M P^n, \pi P^n) \leq W_d(\pi_M, \pi) \leq d_{TV}(\pi_M, \pi) \). Since by definition of \( \pi_M \) it holds for any measurable set \( A \)

\[
|\pi_M(A) - \pi(A)| = |\pi_M(A)(1 - \pi(\{V \leq M\})) + \pi_M(A)\pi(V \leq M) - \pi(A)| \\
\leq \pi(\{V > M\}) + \pi_M(A)\pi(\{V > M\}) \leq 2\pi(\{V > M\}) ,
\]

then by (2),

\[
W_d(\pi_M P^n, \pi P^n) \leq 2\pi(\{V > M\}) = 2\pi(\{\phi(V) > \phi(M)\}) \leq \frac{2\pi(\phi \circ V)}{\phi(M)} .
\] (59)

Combining (59) and (58) in (54), we have for all \( M > M_V \) and \( n \geq 1 \),

\[
W_d(P^n(x, \cdot), \pi) \leq \frac{C_1 (lb + V(x) + 2\pi(\phi \circ V)M/\phi(M))}{C_2 f_\phi \{n/\log(f_\phi(n))\}} + \frac{2\pi(\phi \circ V)}{\phi(M)} .
\]

For all \( n \) large enough, we choose \( M = f_\phi \{n/\log(f_\phi(n))\} \) (note that by Lemma 36-(i)-(ii) and (v), \( \lim_{n \to +\infty} f_\phi \{n/\log(f_\phi(n))\} = +\infty \) so that \( M > M_V \) for all \( n \) large enough). The proof is concluded upon noting that \( \lim_{t \to +\infty} \phi(t)/t = 0 \).

### 4.5. Proof of Theorem 5

The proof is along the same lines as the proof of Theorem 4: the upper bound Lemma 19-(38) is used instead of Lemma 19-(37). Details are omitted.

### 4.6. Proof of Proposition 7

**Proof:** Note under \( H_3 \), \( c \in (0, 1) \). It is sufficient to prove that for all \( x, y \in E \),

\[
PV(x) + PV(y) \leq V(x) + V(y) - c \phi(V(x) + V(y)) + 2b \mathbb{1}_{c \times c}(x, y) .
\] (60)
By (11),
\[ PV(x) + PV(y) \leq V(x) + V(y) - c\phi (V(x) + V(y)) + 2b \mathbb{1}_{C \times C} + \Omega(x, y) \]
where
\[ \Omega(x, y) = c\phi (V(x) + V(y)) - \phi(V(x)) - \phi(V(y)) + 2b \{ \mathbb{1}_{E \times C^c}(x, y) + \mathbb{1}_{C^c \times E}(x, y) \} . \]
Let us show that for all \( x, y \in E \), \( \Omega(x, y) \leq 0 \). Since \( \phi \) is sub-additive, for all \( x, y \in E \)
\[ \Omega(x, y) \leq -(1 - c) (\phi(V(x)) + \phi(V(y))) + 2b \{ \mathbb{1}_{E \times C^c}(x, y) + \mathbb{1}_{C^c \times E}(x, y) \} . \]
On \( (E \times C^c) \cup (C^c \times E) \), \( \phi(V(x)) + \phi(V(y)) \geq \phi(v) \). The definition of \( c \) implies that \( \Omega(x, y) \leq 0 \). Then, (60) holds and the proof of the proposition follows. \( \square \)

5. Proofs of section 3

We will use the following results in the proof.

**Lemma 23.** Assume CN1 and set \( r(x) = (1 - \rho)/2^{1/\beta} \|x\| \). Then, for all \( x \in \mathcal{H} \),
\[ \inf_{z \in \mathbb{B}(r(x))} \exp(g(x) - g(z)) \geq \exp(-C_g(3/2)(1 - \rho)^\beta \|x\|^\beta) . \] (61)

**Proposition 24.** Let \( (\mathcal{H}, \| \cdot \|) \) be a separable Hilbert space and \( \gamma \) be a Gaussian measure on \( \mathcal{H} \).

1. There exist \( \theta \in \mathbb{R}_+ \) and a constant \( C_\theta \) such that
\[ \int_{\mathcal{H}} \exp(\theta \|\xi\|^2) d\gamma(\xi) \leq C_\theta . \]
2. There exists a constant \( C_a \) such that for all \( K > a/(2\theta) \),
\[ \int_{\|\xi\| \geq K} \exp(a \|\xi\|) d\gamma(\xi) \leq C_a \exp(-\theta K^2 + aK) . \]

**Proof:** (1) is Fernique’s theorem; see [3, Theorem 2.8.5].
(2) follows from [11, Proposition A.1]. \( \square \)

5.1. Proof of Proposition 11

Set \( r(x) = (1 - \rho)/2^{1/\beta} \|x\| \). Since \( \lim_{x \to +\infty} r(x) = +\infty \), there exists \( R \geq 1 \), such that for
\[ \frac{r(x)}{\sqrt{1 - \rho^2}} > \frac{\sqrt{1 - \rho^2}}{2\theta}, \quad x \notin B(0, R) , \] (62)
where \( \theta \) is given by Proposition 24-(1). Using the definition of the Crank-Nicolson kernel,

\[
\sup_{x \in B(0,R)} PV(x) \leq \sup_{x \in B(0,R)} \int \mathcal{H} \exp \left( \| x \| + \sqrt{1 - \rho^2} \| \xi \| \right) d\gamma(\xi) ,
\]

and Proposition 24-(1) implies that the RHS is finite.

Let \( x \in \mathcal{H} \). Define the events \( \mathcal{F}(x) = \left\{ \| \Xi \| \leq r(x) / \sqrt{1 - \rho^2} \right\} \), \( \mathcal{A}(x) = \left\{ \alpha(x, \rho x + \sqrt{1 - \rho^2} \Xi) > U \right\} \), and \( \mathcal{R}(x) = \left\{ \alpha(x, \rho x + \sqrt{1 - \rho^2} \Xi) < U \right\} \), where \( U \sim \mathcal{U}([0,1]) \), \( \Xi \sim \gamma \), and \( U \) and \( \Xi \) are independent. It holds

\[
PV(x) \leq E_x \left[ V(X_1) \mathbb{1}_{\mathcal{F}(x)} \right] + E_x \left[ V(X_1) \mathbb{1}_{\mathcal{A}(x)} + \mathbb{1}_{\mathcal{R}(x)} \right] .
\]

For the first term in the RHS, using again \( V(X_1) \leq \max(V(x), V(\rho x + \sqrt{1 - \rho^2} \Xi)) \)

\[
E_x \left[ V(X_1) \mathbb{1}_{\mathcal{F}(x)} \right] \leq \exp(\| x \|) \int \sqrt{1 - \rho^2} \| \xi \| \geq r(x) \exp \left( \sqrt{1 - \rho^2} \| x \| \right) d\gamma(\xi) .
\]

By definition of \( R \) (see (62)) and by Proposition 24-(2), there exist constants \( C_1 \in \mathbb{R}_+^* \) such that for any \( x \notin B(0, R) \),

\[
E_x \left[ V(X_1) \mathbb{1}_{\mathcal{F}(x)} \right] \leq C_1 \exp \left( -C_2 r(x)^2 + r(x) + \| x \| \right) .
\]

The RHS is uniformly bounded on \( \mathcal{H} \) since \( -C_2 r(x)^2 + r(x) + \| x \| \xrightarrow{||x|| \rightarrow +\infty} -C_2 r(x)^2 \) when \( ||x|| \) tends to infinity. Hence, there exists a constant \( b < \infty \) such that

\[
\sup_{x \notin B(0,R)} E_x \left[ V(X_1) \mathbb{1}_{\mathcal{F}(x)} \right] \leq b .
\]

Consider the second term in the RHS of (64) for \( x \notin B(0, R) \). On the event \( \mathcal{A}(x) \cap \mathcal{R}(x) \), the move is accepted and \( \| X_1 - \rho x \| \leq r(x) \). On \( \mathcal{R}(x) \), the move is rejected and \( X_1 = x \). Hence,

\[
E_x \left[ V(X_1) \mathbb{1}_{\mathcal{A}(x)} + \mathbb{1}_{\mathcal{R}(x)} \right] \leq E_x \left[ \sup_{z \in B(\rho x, r(x))} V(z) \mathbb{1}_{\mathcal{A}(x)} \right] + E_x \left[ V(x) \mathbb{1}_{\mathcal{A}(x)} \mathbb{1}_{\mathcal{R}(x)} \right] .
\]

For \( z \in B(\rho x, r(x)), V(z) \leq \exp(\rho \| x \| + (1 - \rho) / 2^{1/\beta} \| x \|) = \exp(C \| x \|) \) where \( C \in (0, 1) \) since \( \beta \in (0, 1) \) and \( \rho \in (0, 1) \). Therefore for any \( x \notin B(0,R) \),

\[
sup_{z \in B(\rho x, r(x))} V(z) \leq lV(x) ,
\]

with \( l = \exp((C - 1)R) < 1 \). This yields

\[
E_x \left[ V(X_1) \mathbb{1}_{\mathcal{A}(x)} + \mathbb{1}_{\mathcal{R}(x)} \right] \leq lV(x) \mathbb{P} \left[ \mathcal{A}(x) \right] + V(x) \mathbb{P} \left[ \mathcal{R}(x) \right] \leq V(x) \mathbb{P} \left[ \mathcal{A}(x) \right] - (1 - l)V(x) \mathbb{P} \left[ \mathcal{A}(x) \right] .
\]

Since \( U \) and \( \Xi \) are independent,

\[
\mathbb{P} \left[ \mathcal{A}(x) \right] = \mathbb{E} \left[ 1 \wedge \exp(\rho x + \sqrt{1 - \rho^2} \Xi) \right] \mathbb{1}_{\mathcal{A}(x)} .
\]
By definition of the set $\mathcal{I}(x)$ and Lemma 23, there exists $\kappa > 0$ such that
\[
\mathbb{P}[\mathcal{A}(x) \cap \mathcal{I}(x)] \geq e^{-\kappa \|z\|^{\beta}} \mathbb{P}[\mathcal{I}(x)] = \exp(-\kappa \log^{\beta} V(x)) \mathbb{P}[\mathcal{I}(x)].
\]
Hence, for any $x \notin B(0,R)$,
\[
\mathbb{E}_{x} \left[ V(X_{1}) 1_{\mathcal{I}(x)} (1_{\mathcal{A}(x)} + 1_{\mathcal{I}(x)}) \right] \leq V(x) - (1 - l) V(x) \exp(-\kappa \log^{\beta} V(x)).
\] (66)

Combining (65) and (66) in (64) and using (63), it follows that there exists $\tilde{b} > 0$ such that
\[
PV(x) \leq V(x) - (1 - l) V(x) \exp(-\kappa \log^{\beta} V(x)) + \tilde{b}, \quad \forall x \in \mathcal{H}.
\]
The proof is then concluded by Remark 6.

5.2. Proof of Proposition 12

We preface the proof of Proposition 12 by a Lemma establishing a first step to the contracting property of $Q_{pCN}$. Roughly, the idea of the proof is that the probability the two moves of the basic coupling are accepted can control the probability that only one is.

Lemma 25. Assume CN1. There exists $\tau > 0$ and for any $L > 0$ there exists $k_{L} \in (0,1)$ such that

- for all $x, y \in B(0,L)$ and $d_{\tau}(x, y) < 1$,
  \[
  Q_{pCN} d_{\tau}(x, y) \leq k_{L} d_{\tau}(x, y).
  \] (67)

- for all $x, y \in E$,
  \[
  Q_{pCN} d_{\tau}(x, y) \leq d_{\tau}(x, y).
  \] (68)

Proof: Let $\tau \in (0,1)$; for ease of notation, we simply write $Q$ for $Q_{pCN}$. Let $L > 0$ and $x, y \in B(0,L)$ such that $d_{\tau}(x, y) < 1$. Let $(X_{1}, Y_{1})$ be the basic coupling between $P(x, \cdot)$ and $P(y, \cdot)$; let $\Xi, U$ be resp. the Gaussian variable and the uniform variable used for the basic coupling. Set

\[
\mathcal{I} = \left\{ \sqrt{1 - \rho^{2}} \|z\| \leq 1 \right\},
\]

\[
\mathcal{A}(x, y) = \left\{ \alpha(x, \rho x + \sqrt{1 - \rho^{2}} \Xi) \land \alpha(y, \rho y + \sqrt{1 - \rho^{2}} \Xi) > U \right\},
\]

\[
\mathcal{R}(x, y) = \left\{ \alpha(x, \rho x + \sqrt{1 - \rho^{2}} \Xi) \lor \alpha(y, \rho y + \sqrt{1 - \rho^{2}} \Xi) < U \right\}.
\]

On the event $\mathcal{A}(x, y)$, the moves are both accepted so that $X_{1} = \rho x + \sqrt{1 - \rho^{2}} \Xi$ and $Y_{1} = \rho y + \sqrt{1 - \rho^{2}} \Xi$; On the event $\mathcal{R}(x, y)$, the moves are both rejected so that $X_{1} = x$ and $Y_{1} = y$. It holds,
\[
Q d_{\tau}(x, y) \leq \mathbb{E}_{x, y} [d_{\tau}(X_{1}, Y_{1})]
\]
\[
\leq \mathbb{E}_{x, y} [d_{\tau}(X_{1}, Y_{1}) (1_{\mathcal{A}(x, y)} \lor \mathcal{R}(x, y))] + \mathbb{E} \left[ 1_{\mathcal{A}(x, y)} \lor \mathcal{R}(x, y) \right],
\] (69)
where we have used $d_\tau$ is bounded by 1. Since $d_\tau(X_1, Y_1) = \rho^\beta d_\tau(x, y)$, on $\mathcal{A}(x, y)$, and $d_\tau(X_1, Y_1) = d_\tau(x, y)$, on $\mathcal{B}(x, y)$. Then,

$$
\tilde{E}_{x,y} [d_\tau(X_1, Y_1)(\mathbb{1}_{\mathcal{A}(x, y) \cup \mathcal{B}(x, y)})] \leq \rho^\beta d_\tau(x, y) P[\mathcal{A}(x, y)] + d_\tau(x, y) P[\mathcal{B}(x, y)].
$$

Since $P[\mathcal{A}(x, y)] + P[\mathcal{B}(x, y)] \leq 1$, we have

$$
\tilde{E}_{x,y} [d_\tau(X_1, Y_1)(\mathbb{1}_{\mathcal{A}(x, y) \cup \mathcal{B}(x, y)})]
\leq d_\tau(x, y) - (1 - \rho^\beta) d_\tau(x, y) P[\mathcal{A}(x, y)]
\leq d_\tau(x, y) - (1 - \rho^\beta) d_\tau(x, y) P[\mathcal{A}(x, y) \cap \mathcal{F}].
$$

Set

$$
\Delta(x, y, \xi) = \left| \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) - \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) \right|.
$$

Since $d_\tau$ is bounded by 1 and $\Xi$ and $U$ are independent, it follows

$$
P[\{(\mathcal{A}(x, y) \cup \mathcal{B}(x, y))^c\}] \leq \int_\mathcal{H} \Delta(x, y, \xi) d\gamma(\xi).
$$

Plugging (70) and (72) in (69) yields

$$
Q d_\tau(x, y) \leq d_\tau(x, y)
- (1 - \rho^\beta) d_\tau(x, y) \tilde{E}_{x,y} [\mathcal{A}(x, y) \cap \mathcal{F}]
+ \int_\mathcal{H} \Delta(x, y, \xi) d\gamma(\xi).
$$

Let us now define $h: \mathcal{H} \rightarrow \mathbb{R}$ by

$$
h(z) = g(z) - g(\rho z).
$$

We bound from below $P[\mathcal{A}(x, y) \cap \mathcal{F}]$. Since $U$ is independent of $\Xi$, it follows

$$
P[\mathcal{A}(x, y) \cap \mathcal{F}] \geq E \left[ \left( \alpha(x, \rho x + \sqrt{1 - \rho^2} \Xi) \wedge \alpha(y, \rho y + \sqrt{1 - \rho^2} \Xi) \right) \mathbb{1}_\mathcal{F} \right].
$$

By CN1, for all $\xi$ such that $\sqrt{1 - \rho^2} \|\xi\| \leq 1$, it holds for $z \in \mathcal{H}$

$$
g(z) - g(\rho z + \sqrt{1 - \rho^2} \xi) \geq h(z) - C_g.
$$

Then,

$$
\alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) \wedge \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi)
\geq 1 \wedge (e^{-C_g e^{h(x)}}) \wedge (e^{-C_g e^{h(y)})} \geq e^{-C_g} \left[ 1 \wedge e^{h(x) \wedge h(y)} \right].
$$

Therefore,

$$
P[\mathcal{A}(x, y) \cap \mathcal{F}] \geq e^{-C_g} \left[ 1 \wedge e^{h(x) \wedge h(y)} \right] P[\mathcal{F}].
$$
We now upper bound the integral term in (73). Define the partition of \( \mathcal{H} \),

\[
\begin{align*}
\mathcal{K}_1(x, y) &= \{ \xi \in \mathcal{H} : \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) = \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) = 1 \} \\
\mathcal{K}_2(x, y) &= \{ \xi \in \mathcal{H} : \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) = 1 > \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) \} \\
\mathcal{K}_3(x, y) &= \{ \xi \in \mathcal{H} : \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) = 1 > \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) \} \\
\mathcal{K}_4(x, y) &= \{ \xi \in \mathcal{H} : \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) < 1 \text{ and } \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) < 1 \}.
\end{align*}
\]

Since on \( \mathcal{K}_1(x, y) \), \( \Delta(x, y, \xi) = 0 \),

\[
\int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) = \sum_{j=2}^{4} \int_{\mathcal{K}_j(x, y)} \Delta(x, y, \xi) d\gamma(\xi). \tag{76}
\]

For any \( a, b > 0 \), we have \( |a - b| = (a \vee b) [1 - ((a/b) \wedge (b/a))] \). Set

\[
S(x, y, \xi) = \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) \vee \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi).
\]

Upon noting that \( 1 - e^{-t} \leq t \) for any \( t \geq 0 \), we have

\[
\Delta(x, y, \xi) \leq S(x, y, \xi) \left| g(y) - g(x) - g(\rho y + \sqrt{1 - \rho^2} \xi) \right| + g(\rho x + \sqrt{1 - \rho^2} \xi) \mathbb{1}_{\mathcal{K}_2(x, y) \cup \mathcal{K}_3(x, y) \cup \mathcal{K}_4(x, y)}(\xi).
\]

By CN1, this yields

\[
\Delta(x, y, \xi) \leq 2C_\rho \| y - x \|^\beta S(x, y, \xi) \leq 2C_\rho \tau d_\tau(x, y) S(x, y, \xi).
\]

On \( \mathcal{K}_2(x, y) \), (74) \( h(x) \geq g(\rho x + \sqrt{1 - \rho^2} \xi) - g(\rho x) \), and by CN1, \( h(x) \geq -C_\rho (1 - \rho^2)^{\beta/2} \| \xi \|^\beta \). Then,

\[
\int_{\mathcal{K}_2(x, y)} \Delta(x, y, \xi) d\gamma(\xi) \leq 2C_\rho \tau d_\tau(x, y) \int_{\mathcal{K}_2(x, y)} d\gamma(\xi) \leq 2C_\rho \tau d_\tau(x, y) \left\{ e^{h(x)} \int_{\mathcal{K}_2(x, y)} e^{C_\rho (1 - \rho^2)^{\beta/2} \| \xi \|^\beta} d\gamma(\xi) \right\} \wedge 1 \leq C_\tau d_\tau(x, y) \left\{ e^{h(x)} \wedge 1 \right\}, \tag{77}
\]

for a constant \( C_\tau \), which is finite according to Proposition 24-(1). By symmetry, on \( \mathcal{K}_3(x, y) \),

\[
\int_{\mathcal{K}_3(x, y)} \Delta(x, y, \xi) d\gamma(\xi) \leq C_\tau d_\tau(x, y) \left\{ e^{h(y)} \wedge 1 \right\}. \tag{78}
\]

On \( \mathcal{K}_4(x, y) \), using CN1,

\[
\alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) = e^{g(x)} - g(\rho x + \sqrt{1 - \rho^2} \xi) \wedge 1 \leq \left( e^{h(x)} e^{C_\rho (1 - \rho^2)^{\beta/2} \| \xi \|^\beta} \right) \wedge 1;
\]
and by symmetry, we obtain a similar upper bound for $\alpha(y, \rho y \sqrt{1-\rho^2})$. It follows

$$S(x, y, \xi) \leq e^{C_\alpha(1-\rho^2)\beta/2\|\xi\|^2} (e^{h(x)\wedge h(y)} \wedge 1).$$

Hence, using again Proposition 24-(1), there exists $C_I < +\infty$ such that

$$\int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) \leq C_I \tau d_{\tau}(x, y) \left[e^{h(x)\wedge h(y)} \wedge 1 \right]. \quad (79)$$

Plugging (77), (78), (79) in (76), it follows

$$\int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) \leq 3C_I \tau d_{\tau}(x, y) \left[e^{h(x)\wedge h(y)} \wedge 1 \right].$$

Finally by CN1 and since $d_{\tau}(x, y) < 1$, $|h(x) - h(y)| \leq 2C_g \|x - y\|^\beta \leq 2C_g \tau^\beta$. Therefore $e^{h(x)\wedge h(y)} \wedge 1 \leq 2e^{2C_g \tau^\beta} \left[e^{h(x)\wedge h(y)} \wedge 1 \right]$, and

$$\int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) \leq \tilde{C}_I \tau d_{\tau}(x, y) \left[e^{h(x)\wedge h(y)} \wedge 1 \right]. \quad (80)$$

for $\tilde{C}_I = 3C_I e^{2C_g}$. Plugging (75) and (80) in (73) yields

$$Qd_{\tau}(x, y) \leq d_{\tau}(x, y) \left(1 - \left(1 - \rho^\beta\right)e^{-C_\beta \mathbb{P}[\mathcal{F}] - \tilde{C}_I \tau} \left[e^{h(x)\wedge h(y)} \wedge 1 \right] \right).$$

Therefore, we can choose $\tau$ such that there exists $\delta \in (0, 1)$ and

$$Qd_{\tau}(x, y) \leq d_{\tau}(x, y) \left(1 - \delta \left[e^{h(x)\wedge h(y)} \wedge 1 \right] \right). \quad (81)$$

(81) implies (67) upon noting that by definition, $\inf_{B(0,L)} h > -\infty$.

We now consider (68), let $x, y \in \mathcal{H}$. If $d_{\tau}(x, y) = 1$, $Qd_{\tau}(x, y) \leq 1$ since $d_{\tau}$ is bounded by 1. If $d_{\tau}(x, y) < 1$, there exists $L \geq 0$ such that $x, y \in B(0, L)$, and (67) implies $Qd_{\tau}(x, y) \leq d_{\tau}(x, y)$.

**Proof of Proposition 12** Let $(X_n, Y_n)$, $n \in \mathbb{N}$ be a Markov chain with Markov kernel $Q$ given by (15). We denote for all $n \in \mathbb{N}^*$, $\Xi_n$ and $U_n$, respectively the common gaussian variable and uniform variable, sampled to build $(X_n, Y_n)$. Note that by definition the variables $\{\Xi_n, U_n; n \in \mathbb{N}\}$ are independent.

For ease of notation, we simply write $d_{\ell}$ instead of $d_{\tau}$, and $Q$ for $Q_{\ell}CN$. Since $\{x : V(x) \leq u\} = \{x : \|x\| \leq \log(u)\}$, for $u \geq 1$, we only prove that for all $L > 0$, there exist $\ell \in \mathbb{N}^*$ and $\epsilon > 0$ such that $\overline{B(0,L)}^2$ is a $(\ell, \epsilon, d_{\tau})$-coupling set; see Definition 2 By definition of $Q$ and Lemma 25 the condition (i) and (ii) of Definition 2 are satisfied. Let $L > 0$, and $x, y$ be in $\overline{B(0,L)}$. Assume first $d_{\tau}(x, y) < 1$. Then by Lemma 25, there exists $k_L \in (0, 1)$, independent of $x, y$, such that $Qd_{\tau}(x, y) \leq k_L d_{\tau}(x, y)$. Then by Lemma 30, for every $n \in \mathbb{N}^*$,

$$Q^n d_{\tau}(x, y) \leq Q^{n-1} d_{\tau}(x, y) \leq \cdots \leq k_L d_{\tau}(x, y). \quad (82)$$
Consider now the case \( d_r(x, y) = 1 \). Let \( \{(X_n, Y_n), n \in \mathbb{N}\} \) be the Markov chain with Markov kernel \( Q \) starting in \((x, y)\). Let \( n \in \mathbb{N}^\ast \) and denote for all \( 1 \leq i \leq n \)
\[
\mathcal{A}_i(x, y) = \{U_i \leq \Psi(X_{i-1}, Y_{i-1}, \Xi_i)\}
\]
\[
\tilde{\mathcal{A}}^\tau(x, y) = \bigcap_{1 \leq j \leq i} \left\{ \{1 - \rho^2\|\Xi\| \leq L/n\} \cap \mathcal{A}_j(x, y) \right\},
\]
where \( \Psi(X_{i-1}, Y_{i-1}, \Xi_i) = \alpha(X_{i-1}, \rho X_{i-1} + \sqrt{1 - \rho^2} \Xi_i) / \omega(Y_{i-1}, \rho Y_{i-1} + \sqrt{1 - \rho^2} \Xi_i) \).

On the set \( \tilde{\mathcal{A}}^\tau(x, y) \), \( X_j = \rho X_{j-1} + \sqrt{1 - \rho^2} \Xi_j \) and \( Y_j = \rho Y_{j-1} + \sqrt{1 - \rho^2} \Xi_j \) for all \( 1 \leq j \leq i \). Then, since \( d_r(X_n, Y_n) \leq \tau^{-1} \|X_n - Y_n\|^\beta \), on \( \tilde{\mathcal{A}}^\tau(x, y) \) it holds
\[
d_r(X_n, Y_n) \leq \tau^{-1} \rho^\beta \|x - y\|^\beta.
\]
This inequality and \( d_r(z, u) \leq 1 \) yield
\[
Q^n d_r(x, y) = \tilde{E}_{x, y} \left[ d_r(X_n, Y_n) \left( \mathbb{1}_{\tilde{\mathcal{A}}^\tau(x, y)} + \mathbb{1}_{\tilde{\mathcal{A}}^\tau(x, y)^c} \right) \right]
\leq \rho^\beta \|x - y\|^\beta \tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y) \right] + \tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y)^c \right]
\leq \rho^\beta \|2L\|^\beta \tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y) \right] + \tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y)^c \right]
\leq 1 + \left( \rho^\beta \|2L\|^\beta - 1 \right) \tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y) \right].
\]
(83)

As \( \rho \in (0, 1) \), there exists \( \ell \) such that, \( \rho^\beta \|2L\|^\beta < 1 \). It remains to lower bound \( \tilde{P}_{x, y} \left[ \tilde{\mathcal{A}}^\tau(x, y) \right] \) by a positive constant to conclude, which is done by the following inequalities, where we use the independence of the random variables \( \{\Xi_i, U_i; i \in \mathbb{N}^\ast\} \).

\[
\tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y) \right]
= \tilde{P} \left[ \left( \tilde{\mathcal{A}}^{\tau-1}(x, y) \cap \left\{ \sqrt{1 - \rho^2}\|\Xi_\ell\| \leq L/\ell \right\} \right) \right]
\times \tilde{E}_{x, y} \left[ \Psi(X_{\ell-1}, Y_{\ell-1}, \Xi) \left| \tilde{\mathcal{A}}^{\tau-1}(x, y) \cap \left\{ \sqrt{1 - \rho^2}\|\Xi_\ell\| \leq L/\ell \right\} \right. \right].
\]

For all \( 1 \leq i \leq \ell \), on the event \( \bigcap_{j \leq i} \left\{ \sqrt{1 - \rho^2}\|\Xi_j\| \leq L/\ell \right\} \), it holds
\[
\Psi(X_{i-1}, Y_{i-1}, \Xi_i) \geq \exp \left( - \sup_{z \in B(0, 2L)} g(z) + \inf_{z \in B(0, 2L)} g(z) \right) = \delta,
\]
where \( \delta \in (0, 1) \). Therefore, since \( \Xi_i \) is independent of \( \tilde{\mathcal{A}}^{i-1}(x, y) \), we have
\[
\tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y) \right] \geq \delta \tilde{P} \left[ \tilde{\mathcal{A}}^{\tau-1}(x, y) \right] \tilde{P} \left[ \left\{ \sqrt{1 - \rho^2}\|\Xi_\ell\| \leq L/\ell \right\} \right].
\]
An immediate induction leads to
\[
\tilde{P} \left[ \tilde{\mathcal{A}}^\tau(x, y) \right] \geq \left( \tilde{P} \left[ \sqrt{1 - \rho^2}\|\Xi_1\| \leq \frac{L}{\ell} \right] \right)^\ell \delta^\ell.
\]
Plugging this result in (83) and (82) implies there exists \( s \in (0, 1) \) such that for all \( x, y \in \overline{B}(0, L) \), \( Q^\ell d_r(x, y) \leq s d_r(x, y) \). \( \square \)
Appendix A: Wasserstein distance: some useful properties

Lemma 26 ([20, Particular Case 6.16]). Let \((E, d)\) be a Polish space. Then, for all \(\mu, \nu \in \mathcal{P}(E)\):

\[
W_d(\mu, \nu) \leq \sup_{x, y \in E} d_*(x, y) \ d_{TV}(\mu, \nu).
\]

Hence, when \(d\) is bounded, the convergence in total variation distance implies the convergence in the Wasserstein metric.

For any measurable function \(l : E \times E \to \mathbb{R}_+\), we define the optimal transportation for \(\mu, \nu \in \mathcal{P}(E)\) by:

\[
W_l(\mu, \nu) = \inf_{\alpha \in C(\mu, \nu)} \int_{E \times E} l(x, y) d\alpha(x, y).
\]

(84)

Note that we may have \(W_l(\mu, \nu) = +\infty\), and for all \(x, y \in E \times E\), \(W_l(\delta_x, \delta_y) = l(x, y)\). We consider the case when the function \(l\) is a distance-like function (see also [10])

Definition 27. A function \(l : E \times E \to \mathbb{R}_+\) is said to be a distance-like if

1. For all \((x, y)\) in \(E^2\), \(l(x, y) = 0\) if and only if \(x = y\).
2. \(l\) is lower semicontinuous.
3. For all \((x, y)\) in \(E^2\), \(l(x, y) = l(y, x)\).

The following lemma establishes the convexity of \(W_l\), when \(l\) is a distance-like function.

Lemma 28. Let \((E, d)\) be a Polish space. Let \(P\) be a Markov kernel on \((E, \mathcal{B}(E))\) and \(l : E \times E \to \mathbb{R}_+\) be a distance-like function. For any \(\mu, \nu \in \mathcal{P}(E)\)

\[
W_l(\mu P, \nu P) \leq \inf_{\alpha \in C(\mu, \nu)} \int_{E \times E} W_l(P(x, \cdot), P(y, \cdot)) \alpha(dx, dy).
\]

Proof: Let \(\alpha\) be a coupling of \(\mu\) and \(\nu\). We get

\[
\mu P(dz) = \int_E P(x, dz) \mu(dx) = \int_{E \times E} P(x, dz) \alpha(dx, dy),
\]

\[
\nu P(dz) = \int_{E \times E} P(y, dz) \alpha(dx, dy).
\]

Therefore,

\[
W_l(\mu P, \nu P) = W_l\left(\int_{E \times E} P(x, \cdot) \alpha(dx, dy), \int_{E \times E} P(y, \cdot) \alpha(dx, dy)\right).
\]

Since \(l\) is lower semicontinuous and \(l \geq 0\), by [20, Theorem 4.8]

\[
W_l(\mu P, \nu P) \leq \int_{E \times E} W_l(P(x, \cdot), P(y, \cdot)) \alpha(dx, dy).
\]

The proof is concluded since this inequality holds for all coupling \(\alpha\). \qed
Lemma 29. Let $(E, d)$ be a Polish space and let $P$ be Markov kernel on $(E, \mathcal{B}(E))$. Assume that $d$ is weakly contracting for $P$, i.e.

$$ W_d(P(x, \cdot), P(y, \cdot)) \leq d(x, y), \quad \forall (x, y) \in E^2. \quad (85) $$

Then for all $\mu, \nu \in \mathcal{P}(E)$,

$$ W_d(\mu P, \nu P) \leq W_d(\mu, \nu). \quad (86) $$

Proof: According to Lemma 28 and using the stated assumptions,

$$ W_d(\mu P, \nu P) \leq \inf_{\alpha \in \mathcal{C}(\mu, \nu)} \int_{E \times E} W_d(P(x, \cdot), P(y, \cdot)) \alpha(dx, dy) \leq \inf_{\alpha \in \mathcal{C}(\mu, \nu)} \int_{E \times E} d(x, y) \alpha(dx, dy) = W_d(\mu, \nu). $$

□

Lemma 30. Let $(E, d)$ be a Polish space and let $P$ be a Markov kernel on $(E, \mathcal{B}(E))$. Assume there exists a Markov kernel $Q$ on $(E \times E, \mathcal{B}(E \times E))$ satisfying:

(i) for all $x, y \in E$, $Q((x, y), \cdot)$ is a coupling of $(P(x, \cdot), P(y, \cdot))$.

(ii) for all $x, y \in E$, $Qd(x, y) \leq d(x, y)$.

Then for all $x, y \in E$, $W_d(P(x, \cdot), P(y, \cdot)) \leq d(x, y)$ and for all probability measures $\mu, \nu \in \mathcal{P}(E)$,

$$ W_d(\mu P, \nu P) \leq W_d(\mu, \nu). \quad (87) $$

Proof: By assumption and the definition of the Wasserstein distance (1), we have for all $x, y$, $W_d(P(x, \cdot), P(y, \cdot)) \leq Qd(x, y) \leq d(x, y)$. The second statement is a consequence of Lemma 29. □

Appendix B

Lemma 31 ([4, lemma 4.1]). Assume that there exist a measurable function $V : E \to \mathbb{R}_+$, a nonnegative constant $b$ and a measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$ PV + \phi \circ V \leq V + b. \quad (88) $$

Then for every $n \geq 1$,

$$ \sum_{i=0}^{n-1} P^i(\phi \circ V) \leq V + nb. \quad (89) $$

If $\pi$ is an invariant probability measure for $P$, then $\pi(\phi \circ V) \leq b$.

We remind that for $\phi$ given by $H2$, $H_\phi$ and $r_\phi$ are respectively given by (5) and (6). Here are some results about $H_\phi$. 

Lemma 32. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing, concave, and differentiable function satisfying $\lim_{+\infty} \phi = +\infty$. Then,

(i) $H_\phi$ given by (5) is concave, increasing, $C^2$ on $[1, +\infty)$ and $\lim_{+\infty} H_\phi = +\infty$.

(ii) $H_\phi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is $C^2$, increasing, convex and $\lim_{+\infty} H_\phi^{-1} = +\infty$.

Proof: (i) is trivial. For (ii), note that $(H_\phi^{-1})' = \phi \circ H_\phi^{-1}$ and since both $y \mapsto H_\phi^{-1}(y)$ and $y \mapsto \phi(y)$ are non-decreasing, $y \mapsto (H_\phi^{-1}(y))'$ is non-decreasing showing that $H_\phi^{-1}$ is convex. \(\square\)

Let $H_k$ be given by (41) and set

$$\widetilde{V}_k = H_k \circ V.$$ (90)

Proposition 33 ([7, Lemma 2.3 and Proposition 2.1]). Assume H2 holds.

(i) $r_\phi \in \Lambda$ and is log-concave.

(ii) for every $k \geq 0$, $H_k$ is concave.

(iii) for all $x_0 \in E$ and $k \geq 0$

$$PV_{k+1} \leq \widetilde{V}_k - r_\phi(k) + (b + V(x_0)) \frac{r_\phi(k + 1)}{r_\phi(0)}.$$ 

Here are some additional properties on the functions $H_k$ and $\widetilde{V}_k$.

Lemma 34. Assume H2. Let $r_\phi$, $H_k$ and $\widetilde{V}_k$ be given by (6), (41) and (90).

(i) There exists some nonnegative constant $C$ such that for every $x \in E$

$$\sup_{k \geq 0} \frac{\widetilde{V}_k(x)}{r_\phi(k)} \leq C \tilde{V}_0(x) \leq CV(x).$$

(ii) For all $x, y \in [1, +\infty[$, and every integer $k \geq 1$

$$H_k(x + y) \leq H_k(x) + H_k(y) + 2r_\phi(k).$$

Proof: (i): By definition for every $x \in \mathbb{R}_+$,

$$\frac{H_k(x)}{r_\phi(k)} = \frac{1}{r_\phi(k)} \int_0^{H_\phi(x)} r_\phi(t + k)dt.$$ 

Since by Proposition 33-(i) $r_\phi \in \Lambda$, Lemma 36-(iv) shows that there exists a constant $C$ such that for any $t, k \geq 0$, $r_\phi(k + t) \leq Cr_\phi(k)r_\phi(t)$. Then

$$\frac{H_k(x)}{r_\phi(k)} \leq C \int_0^{H_\phi(x)} r_\phi(t)dt = CH_0(x) \leq Cx.$$ 

Applying this inequality with $x \leftarrow V(x)$ concludes the proof.

(ii): by Proposition 33 $z \mapsto H_k(z + 1)$ defined on $\mathbb{R}_+$ is concave, and $H_k(1) = 0$; thus it is sub-additive. Then, since $H_k$ is nondecreasing, it yields

$$H_k(x + y) \leq H_k(x + y + 1) \leq H_k(x + 1) + H_k(y + 1).$$
Since $H_k$ is concave, for every $z \geq 1$
\[ H_k(z + 1) - H_k(z) \leq H'_k(z) \leq H'_k(1) = r_\phi(k). \]
These two inequalities imply that for all $x, y \geq 1$,
\[ H_k(x + y) \leq H_k(x) + H_k(y) + 2r_\phi(k). \]

\[\square\]

Appendix C: Subgeometric functions and sequences

For $r \in \Lambda$, we denote by $t \mapsto f_r(t)$ the function
\[ f_r(t) = r(0) + \int_0^t r(u)du. \] (91)

**Lemma 35.** Let $r \in \Lambda_0$, $R$ and $f_r$ be resp. given by (23) and (91).

(i) \cite[Lemma 1]{17}: For all $t, u \in \mathbb{R}_+$,
\[ r(t + u) \leq r(t)r(u). \] (92)

(ii) $f_r$ is convex, increasing to $+\infty$, and there exists $C > 1$ such that for all $t \geq 0$:
\[ f_r(t + 1) \leq Cf_r(t). \] (93)

(iii) There exists a constant $C \in (0, 1)$ such that
\[ C f_r(n) \leq R(n) \leq f_r(n) \quad \forall n \in \mathbb{N}, \] (94)
\[ C f_r(t) \leq R([t]) \leq f_r(t) \quad \forall t \geq 0. \]

(iv) $\lim_{n \to \infty} r(n)/R(n) = 0$.

**Proof:** (ii) By Definition 1 $r$ is non-decreasing, thus is bounded on every compact set; then, $f_r$ is continuous. Moreover, it is differentiable and its derivative is $r$, which is non-decreasing. Then $f_r$ is convex. In addition $r(0) > 2$, thus $f_r$ is increasing to $+\infty$. Let us show (93). By (91), for all $t \geq 0$ $f_r(t + 1) = r(0) + \int_0^t r(u)du + \int_0^t r(1 + u)du$. Then by (92), and since $f_r$ is increasing, for all $t \geq 0$
\[ f_r(t + 1) \leq f_r(1) + r(1)f_r(t). \]

The proof is concluded since $\lim_{t \to +\infty} f_r(t) = +\infty$.

(iii) Since $r$ is non-decreasing, by (91) and an integral test we get for all $n \geq 1$,
\[ f_r(n - 1) \leq R(n) \leq f_r(n). \]

This inequality combined with (93) implies (94). The upper bound in the second inequality is a consequence of the first one and the monotonicity of $f_r$. For the lower bound, by (94) and the monotonicity of $f_r$ there exists $C_1$ such that
\[ C_1 f_r(t - 1) \leq R([t]); \]
by (93) there exists $C_2 > 0$ such that $f_r(t)/C_2 \leq f_r(t - 1)$. The monotonicity of $f_r$ concludes the proof.

(iv) Set $u_n := \log(r(n))/n$. By Definition 1 $u_n$ is decreasing, then

$$
\log \left(1 + \frac{r(n + 1) - r(n)}{r(n)}\right) = \log \left(\frac{r(n + 1)}{r(n)}\right) = n(u_{n+1} - u_n) + u_{n+1} \leq u_{n+1}.
$$

In addition, by (3) $\lim_{n \to +\infty} u_n = 0$, so $\lim_{n \to +\infty} (r(n + 1) - r(n))/r(n) = 0$.
Therefore, for all $\varepsilon > 0$, there exists $N$ such that for all $n \geq N$,

$$
(r(n + 1) - r(n)) \leq \varepsilon r(n).
$$

This result implies that for $n \geq N$, $\sum_{k=N}^{n} (r(k + 1) - r(k)) \leq \varepsilon \sum_{k=N}^{n} r(n)$, so that

$$
r(n)/R(n) \leq \varepsilon + r(N)/R(n) \quad \forall n \geq N.
$$

Since $R(n) \geq nr(0)$, $\lim_{n \to +\infty} R(n) = +\infty$. This concludes the proof. $\square$

The following lemma is a trivial consequence of Definition 1 and Lemma 35. The proof is omitted.

**Lemma 36.** Let $r \in \Lambda$; and let $(R(n))_n$ and $f_r$ be resp. given by (23) and (91).

(i) There exist two positive constants $C_1, C_2$ such that for all $t \geq 0$,

$$
C_1 f_r(t) \leq R(|t|) \leq C_2 f_r(t).
$$

(ii) $\lim_{n \to +\infty} R(n) = +\infty$.

(iii) $\lim_{n \to +\infty} r(n)/R(n) = 0$.

(iv) There exists a non-negative constant $C$ such that for all $x, y \geq 0$, $r(x+y) \leq Cr(x)r(y)$. In particular, $\limsup_{x \to +\infty} r(x + 1)/r(x) < +\infty$.

(v) $\lim_{n \to +\infty} \log(R(n))/n = 0$.

**Lemma 37.** Let $r \in \Lambda$. There exist a measurable, increasing and convex function $\psi_r$, and two positive constants $C_1, C_2$ such that for every integer $n \geq 0$

$$
C_1 \psi_r(n) \leq R(n) \leq C_2 \psi_r(n),
$$

where the sequence $(R_n)_n$ is defined by (23).

**Proof:** Since $r \in \Lambda$, there exist $r_0 \in \Lambda$ and positive constants $c_1, c_2$ such that for any $n \geq 0$, $c_1 r_0(n) \leq r(n) \leq c_2 r_0(n)$. The result now follows from Lemma 35. $\square$

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