ON DEFORMATIONS OF AFFINE GORENSTEIN TORIC VARIETIES

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ABSTRACT. We consider an affine Gorenstein toric variety $X_P$ given by a lattice polytope $P$. For the Gorenstein lattice degree $R^*$ we construct a miniversal deformation of $X_P$ in all degrees $-kR^*$, $k \in \mathbb{N}$ together. The components of the reduced miniversal deformation correspond to maximal Minkowski decompositions of $P$.

1. INTRODUCTION

The deformation theory of toric varieties is lately an active research area, see [7], [8], [9], [5], [6], [13], [14]. One of the main motivations comes from the classification problem of smooth Fano varieties, cf. [8]. It is expected that low dimensional smooth Fano varieties can be degenerated to (singular) toric Fano varieties.

We study the miniversal deformation of an affine Gorenstein toric variety $X$, which is given by a rational polyhedral cone $\sigma \subset \tilde{N}_{\mathbb{R}}$, where $\tilde{N}$ is a finite dimensional lattice. The Gorenstein assumption means that there is a Gorenstein degree $R^* \in \tilde{M} := \text{Hom}(\tilde{N}, \mathbb{Z})$ such that the integral generators of $\sigma$ lie on an affine hyperplane $[R^* = 1]$ and thus $[R^* = 1] \cap \sigma$ is a lattice polytope that we denote by $P$.

Altmann [1] studied the case when $X$ has an isolated singularity. He described a miniversal deformation and established the correspondence between the components of the reduced miniversal deformation and the maximal Minkowski decompositions of $P$.

Let $A = \mathbb{C}[\sigma^\vee \cap \tilde{M}]$ and thus $X = \text{Spec } A$. The $A$-modules $T_X^1 := \text{Ext}_A^1(\Omega_A|_k, A)$ and $T_X^2 := \text{Ext}_A^2(\Omega_A|_k, A)$ are the tangent and the obstruction space for the deformation functor of $A$. The torus action induces the grading of $T_X^i$ for $i = 1, 2$. The degree $R \in \tilde{M}$ part of $T_X^i$ we denote by $T_X^i(R)$. The most interesting degree from deformation theory point of view is the degree $-R^*$: if $X$ has only an isolated singularity, then $T_X^i(-R^*) = T_X^i$, see [1]. In general $T_X^1$ is spread over infinitely many degrees, cf. [2].

In this paper we consider the tangent space $\oplus_{k \in \mathbb{N}} T_X^1(-kR^*) \subset T_X^1$ and construct a maximal deformation of $X$ with the prescribed tangent space $\oplus_{k \in \mathbb{N}} T_X^1(-kR^*)$. More precisely, we construct a miniversal deformation in degrees $-kR^*$ for $k \in \mathbb{N}$, see Definition 3.1. Besides [1] there were also the papers [4] and [6] where a miniversal deformation of an affine toric variety in a single primitive lattice degree was constructed under some assumptions. For an affine Gorenstein toric variety $X$ all three papers can only describe the miniversal deformation in the degree $-R^*$ under additional assumption that $X$ is smooth in codimension 2. Geometrically the smooth in codimension 2 assumption for affine Gorenstein toric varieties means that every edge of $P$ has lattice length 1.

The idea of how to find a miniversal deformation in the degree $-R^*$ without any additional assumptions is new. Moreover, we will show that it is more natural to consider all the degrees $-kR^*$, $k \in \mathbb{N}$ together. Note that if $X$ is smooth in codimension 2, then $\oplus_{k \geq 2} T_X^1(-kR^*) = 0$. In the present paper we also show that the components of the reduced miniversal deformation correspond to maximal...
Minkowski decompositions of $P$. This was shown in [1] under additional assumption that all edges of $P$ have lattice length 1. In the general case we need to consider all degrees $-kR^*$, $k \in \mathbb{N}$ to obtain this correspondence.

The above correspondence provides a useful information for the problem of finding all the possible smoothings of Gorenstein Fano toric varieties (see also [9]). Note that by the comparison theorem of Kleppe [12] it is enough to understand deformations of affine Gorenstein toric varieties.

The paper is organized as follows: in Section 2 we recall some results from [6] that we also need in this paper. We present them in a slightly different way and provide the important proofs that we need later. The first main section is Section 3, where in the diagram (15) we explain the main idea how to construct the deformation family in all degrees $-kR^*$ together. The next two main sections are Section 4 and Section 6 where we show that the Kodaira-Spencer map of this deformation family is bijective and that the obstruction map is injective, respectively. The computations are different from the ones appearing in [1], [4] and [6] since we are considering all degrees $-kR^*$ at the same time. We conclude the paper by showing that the components of the reduced miniversal deformation correspond to maximal Minkowski decompositions of $P$ in Section 7.

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2. Free pairs

2.1. The setup. We fix $\mathbb{C}$ to be an algebraically closed field of characteristic 0. Let $P$ be a lattice polytope with vertices $v^1, ..., v^p$ in $N$, where $N$ is a lattice. Putting $P$ on height 1 gives us a rational, polyhedral cone

$$\sigma = \langle a^1, ..., a^p \rangle \subset (N \oplus \mathbb{Z}) \otimes \mathbb{R}$$

with $a^i = (v^i, 1)$, $i = 1, ..., p$. Let $M$ denote the dual lattice of $N$ and let us consider the monoid $S = \sigma^\vee \cap (M \oplus \mathbb{Z})$, where $\sigma^\vee := \{ r \in (M \oplus \mathbb{Z}) \otimes \mathbb{R} \mid \langle \sigma, r \rangle \geq 0 \}$. Every affine Gorenstein toric variety is isomorphic to $X := X_P := \text{Spec} \mathbb{C}[S]$ for some lattice polytope $P$. Using the notation from Introduction, we have $\hat{M} = M \oplus \mathbb{Z}$ and $\hat{N} = N \oplus \mathbb{Z}$. Without loss of generality we assume that the vertex $v^1$ of $P$ is equal to 0 $\in N$.

Let $d^1, d^2, ..., d^n$ be the edges of $P$. The lattice length $\ell(d^i) \in \mathbb{N}$ of an edge $d^i$ we also denote by $l_i \in \mathbb{N}$ for $i = 1, ..., n$. Every edge $d^i$ connecting two vertices $v^j$ and $v^k$ we equip with an orientation and present it as a vector $d^i = d^{jk} = v^j - v^k \in \hat{N}$.

2.2. Geometric interpretation of $T^1_X(-kR^*)$. We choose an orientation for every 2-face $\epsilon$ of $P$: let $\delta_\epsilon(d^i) \in \{0, 1, -1\}$ with the property $\delta_\epsilon(d^i) = 0$ if $d^i \notin \epsilon$ and $\delta_\epsilon(d^i) \in \{-1, 1\}$ if $d^i \in \epsilon$ and moreover we require $\sum_{d^i \in \epsilon} \delta_\epsilon(d^i) \cdot d^i = 0$. We set $s_1 = 0$ and for $k \in \mathbb{N}$ we define $V_k(P)$ to be the following vector space:

$$(1) \quad \{ (t_{1k}, ..., t_{nk}, s_k) \in \mathbb{R}^{n+1} \mid \sum_{d^i \in \epsilon} \delta_\epsilon(d^i)t_{ik}d^i = 0 \text{ for every 2-face } \epsilon \text{ in } P, \ t_{ik} = s_k \text{ if } l_i \leq k - 1 \}.$$

Proposition 2.1.

$$T^1(-kR^*) = V_k(P) \otimes \mathbb{C}/(1).$$
Proof. See [2, Theorem 2.5].

The most important vector space is $V_1(P)$ which we also denote by $T(P)$ and use the following coordinates: $t_i := t_{i1}$ for $i = 1, \ldots, n$. Thus

$$T(P) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid \sum_{d' \in \epsilon} \delta_{d'}(d')t_i d^i = 0 \text{ for every 2-face } \epsilon \text{ in } P\}.$$

2.3. The monoid $\tilde{T}$.

Definition 2.2. We define the lattice $T_\mathbb{Z}(P) \subset T(P)$ by

$$(t_1, \ldots, t_n) \in T_\mathbb{Z}(P) : \iff t_i d^i \in \mathbb{N} \text{ for each } i = 1, \ldots, n.$$

Moreover, let us define the semigroup

$$\tilde{T} := \text{Span}_{\mathbb{N}}\{\ell(d^1)t_1, \ldots, \ell(d^n)t_n\} \subset T_\mathbb{Z}(P),$$

where $T_\mathbb{Z}(P)$ is the dual lattice of $T(P)$.

For $c \in M_\mathbb{R}$ we choose a vertex $v(c)$ of $P$ where $\langle c, \cdot \rangle$ becomes minimal.

Definition 2.3. $\eta(c) := -\min_{v \in P} \langle v, c \rangle = -\langle v(c), c \rangle \in \mathbb{Z}$.

The Hilbert basis of $S = \sigma^\vee \cap (M \oplus \mathbb{Z})$ is equal to

$$(2) \quad E := \{s_0 = (c_1, \eta(c_1)), \ldots, s_r = (c_r, \eta(c_r)), R^* = (0, 1)\},$$

with uniquely determined elements $c_i \in M$.

Let $c \in M$ and let us choose a path $v^1 = 0, v^2, \ldots, v^k = v(c)$ along the edges of $P$.

Definition 2.4. For every $c \in M$ we define

$$\tilde{\eta}(c) := \sum_{j=1}^{k-1} \langle v^j - v^{j+1}, c \rangle \cdot t_{j,j+1} \in T_\mathbb{Z}(P),$$

where $T_\mathbb{Z}(P)$ is the dual lattice of $T_\mathbb{Z}(P)$.

Moreover, for $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$ we define

$$\eta(k) := \eta(k_1 c_1) + \eta(k_2 c_2) + \cdots + \eta(k_r c_r) - \eta(k_1 c_1 + k_2 c_2 + \cdots + k_r c_r) \in \mathbb{Z},$$

$$\tilde{\eta}(k) := \tilde{\eta}(k_1 c_1) + \tilde{\eta}(k_2 c_2) + \cdots + \tilde{\eta}(k_r c_r) - \tilde{\eta}(k_1 c_1 + k_2 c_2 + \cdots + k_r c_r) \in T_\mathbb{Z}(P).$$

To every path between two vertices of $P$ we associate a vector $\mu$ with $n$ components which have values $-1$ or $0$ or $1$. The number $0$ on the $i$-th component means that the edge $d^i$ was not stridden, the number $1$ (resp. $-1$) means that the edge was stridden with the same (resp. opposite) direction compared to the already fixed orientation of $d^i$.

Definition 2.5. For a vertex $v$ of $P$ and $c \in M$ we define the following paths through the 1-skeleton of $P$:

- $\lambda(v) = (\lambda_1(v), \ldots, \lambda_n(v)) := \text{ a path from } 0 \in P \text{ to } v \in P.$
- $\mu^c(v) = (\mu^c_1(v), \ldots, \mu^c_n(v)) := \text{ a path from } v \in P \text{ to } v(v) \in P \text{ such that } \mu^c_1(v)\langle d^i, c \rangle \leq 0 \text{ for each edge } d^i \text{ of } P.$

We define the path $\lambda^c(v) := \lambda(v) + \mu^c(v)$, which is a path from $0 \in P$ to $v(v)$ (that depends on $v$).
Remark 2.6. If \( \mu \) is a closed path, then \( \sum_{j=1}^{n} \mu_j d^j = 0 \). Moreover, every closed path \( \mu \) we can obtain by picking \( m \) 2-faces \( \epsilon_i, i = 1, \ldots, m \) and choosing an orientation \( \epsilon_i \) for each \( \epsilon_i \) such that \( \mu = \sum_{i=1}^{m} \epsilon_i \).

Lemma 2.7. It holds that \( \tilde{\eta}(k) \in \tilde{T} \).

Proof. Let \( c := k_1 c_1 + \cdots + k_r c_r \). We pick the path \( \lambda^c_i(v(c)) \) from 0 to \( v(c_j) \in P \) and we pick the path \( \lambda(v(c)) \) from 0 to \( v(c) \in P \). Now we compute

\[
\tilde{\eta}(k) = \sum_{i=1}^{n} \left( - \sum_{j=1}^{r} (k_j \lambda^c_i(v(c)) \langle d^i, c_j \rangle) + \lambda_i(v(c)) \langle d^i, c \rangle \right) t_i = \sum_{i=1}^{n} \left( - \sum_{j=1}^{r} k_j \mu_i^c(v(c)) \langle d^i, c_j \rangle \right) t_i.
\]

The coefficients before \( t_i \) are either zero or positive multiples of \( \ell(d^i) \), which proves the claim. \( \square \)

2.4. The monoid \( \tilde{S} \). Let us define the monoid

\[
\tilde{S} := \tilde{T} + \text{Span}_\mathbb{N}\{ (c, \tilde{\eta}(c)) \mid c \in M \} \subset M \oplus T^*_Z(P),
\]

where the above sum is considered in the natural way: if \( \tilde{t} \in \tilde{T} \subset T^*_Z(P) \) and \( (c, \tilde{\eta}(c)) \in M \oplus T^*_Z(P) \), then \( \tilde{t} + (c, \tilde{\eta}(c)) := (c, \tilde{t} + \tilde{\eta}(c)) \in M \oplus T^*_Z(P) \).

Definition 2.8. We define the degree map \( \text{deg} : T^*_Z(P) \otimes \mathbb{Z} \rightarrow \mathbb{N} \), which maps all \( t_i \) to 1. Moreover, let us define the map \( \pi := (\text{id}, \text{deg}) : M \oplus T^*_Z(P) \rightarrow M \oplus \mathbb{Z} \).

In particular, for \( c \in M \) we see that

\[
\text{deg}(\tilde{\eta}(c)) = \sum_{j=1}^{k-1} \langle v^j - v^{j+1}, c \rangle = -\langle v(c), c \rangle = \eta(c),
\]

where we used that \( v^1 = 0 \) and \( v^k = v(c) \) as in Definition 2.4. Thus \( \pi(c, \tilde{\eta}(c)) = (c, \eta(c)) \) which implies that \( \pi \) maps the monoid \( \tilde{S} \) to \( S \) and the monoid \( \tilde{T} \) to \( T := \mathbb{N} \).

Recall that we denote the lattice length \( \ell(d^i) \) of the edges \( d^i \) by \( l_i \).

Lemma 2.9. The generators of \( \tilde{S} \) are

\[
\tilde{s}_1 = (c_1, \tilde{\eta}(c_1)), \ldots, \tilde{s}_r = (c_r, \tilde{\eta}(c_r)), \tilde{l}_1 = l_1 t_1, \ldots, \tilde{l}_n = l_n t_n.
\]

Proof. We only need to prove that every \( (c, \tilde{\eta}(c)) \) can be written as a sum of \( \tilde{s}_1, \ldots, \tilde{s}_r \). First we observe the following:

\[
\eta(k) = 0 \implies \tilde{\eta}(k) = 0.
\]

Indeed, from the proof of Lemma 2.7 we see that the coefficients of \( \tilde{\eta}(k) \) in front of \( t_i \) are non-negative and their sum is equal to \( \eta(k) \), which is zero by our assumption and thus \( \tilde{\eta}(k) = 0 \).

By (2) we know that there exist \( a_i \in \mathbb{N} \) such that \( (c, \eta(c)) = \sum_{i=1}^{r} a_i(c_i, \eta(c_i)) \). From this by (5) it follows that \( (c, \tilde{\eta}(c)) = \sum_{i=1}^{r} a_i(c_i, \tilde{\eta}(c_i)) \) and thus we conclude the proof. \( \square \)
2.5. Free pairs. Let us recall the following definition that already appear in [5] and [6].

Definition 2.10. Let $S_1 \subset S_2$ be two sharp monoids, i.e. commutative semigroups with identity satisfying $S_2 \cap (-S_2) = \{0\}$. The boundary of $S_2$ relative to $S_1$ is defined as

$$\partial_{S_1} S_2 = \{ s \in S_2 : (s - S_1) \cap S_2 = \{s\} \}.$$ 

We say that $S_1 \subset S_2$ form a free pair $(S_1, S_2)$ if the addition map $a : (\partial_{S_1} S_2) \times S_1 \rightarrow S_2$ is bijective.

We write a unique decomposition of every element $s \in S_2$ as

$$s = \partial(s) + \lambda(s) \quad \text{with} \quad \partial(s) \in \partial_{S_1} S_2 \quad \text{and} \quad \lambda(s) \in S_1.$$

Proposition 2.11. $(T, S)$ and $(\tilde{T}, \tilde{S})$ are free pairs.

Proof. This was already proven in [6], in our setting the proof is much easier so let us sketch it. Every element of $\tilde{s} \in \tilde{S}$ can be written as $\tilde{s} = \sum_{i=1}^{r} k_i \tilde{s}_i$, where $\tilde{s}_i = (c_i, \tilde{\eta}(c_i))$ as in Lemma 2.9 and $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$. We write $c := \sum_{i=1}^{r} k_i c_i$. It can be easily checked that there exists a unique decomposition:

$$\tilde{s} = \sum_{i=1}^{r} k_i \tilde{s}_i = (0, \sum_{i=1}^{r} k_i \tilde{\eta}(c_i) - \tilde{\eta}(c)) + (c, \tilde{\eta}(c))$$

with

$$(6) \quad \tilde{\partial}(k) := (c, \tilde{\eta}(c)) \in \partial_{\tilde{T}} \tilde{S} \quad \text{and} \quad \tilde{\lambda}(k) := (0, \sum_{i=1}^{r} k_i \tilde{\eta}(c_i) - \tilde{\eta}(c)) = (0, \tilde{\eta}(k)) \in \tilde{T}.$$

Applying the map $\pi$ we get that the unique decomposition of $s := \pi(\tilde{s}) = \lambda(k) + \partial(k) \in S$, with $\lambda(k) := \eta(k) \in \partial_T S$ and $\partial(k) := (c, \eta(c)) \in T = \mathbb{N}$. \hfill $\Box$

Corollary 2.12. We can write

$$(7) \quad \tilde{\lambda}(k) = \sum_{i=1}^{n} \tilde{\lambda}_i(k) t_i,$$

where $\tilde{\lambda}_i(k) := \tilde{\eta}_i(k) := -\sum_{j=1}^{r} k_j \mu_j^T (v(c))(d^i, c_j) \in \mathbb{N}$, see (3). Thus

$$(8) \quad \lambda(k) = \pi(\tilde{\lambda}(k)) = \sum_{i=1}^{n} \tilde{\lambda}_i(k).$$

2.6. Lifting syzygies. Recall the Hilbert basis $\{R^*, s_1, \ldots, s_r\}$ of $S$ from (2). Let us consider the surjection

$$(9) \quad \varphi : \mathbb{C}[t, x_1, \ldots, x_r] \rightarrow \mathbb{C}[S], \quad t \mapsto x^{R^*}, \quad x_i \mapsto x^{s_i}.$$

We fix a representation of $\partial(k)$, i.e. let $\partial(k) = \sum_{j=1}^{r} b_j s_j$ and write

$$x^k = \prod_{j=1}^{r} x_j^{k_j}, \quad x^\partial(k) := \prod_{j=1}^{r} x_j^{b_j}.$$

Lemma 2.13. The binomials $f_k(x, t) := x^k - x^\partial(k) t^{\lambda(k)} \in \mathbb{C}[t, x_1, \ldots, x_r]$ generate

$$I_X = \ker(\varphi : \mathbb{C}[t, x_1, \ldots, x_r] \rightarrow \mathbb{C}[S]).$$
Proof. See [6, Section 5].

Similarly, we consider the surjection

\[ \tilde{\varphi} : \mathbb{C}[u_1, \ldots, u_n, x_1, \ldots, x_r] \to \mathbb{C}[\tilde{S}], \quad u_i \mapsto x_i, \quad x_j \mapsto x_j. \]

As above we can for \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \) define the binomial

\[ F_k(x, u) := x^k - x^{\partial(k)} u \tilde{\lambda}(k) \in \mathbb{C}[u_1, \ldots, u_n, x_1, \ldots, x_r]. \]

Lemma 2.14. The binomials \( F_k \) are liftings of \( f_k \) under the map \( \pi \) and they generate the ideal \( \ker(\tilde{\varphi}) \).

Proof. We see that \( \pi(\tilde{\lambda}(k)) = \lambda(k) \) and \( \pi(\tilde{\partial}(k)) = \partial(k) \) from which it follows that \( F_k \) are liftings of \( f_k \) under the map \( \pi \) defined in Definition 2.8. For the second part of the statement see [6, Section 5].

From the proof of Proposition 2.11 we can also easily check that \( \partial_T \tilde{S} = \{(c, \tilde{\eta}(c)) \mid c \in M\} \) and \( \partial_T S = \{(c, \eta(c)) \mid c \in M\} \), from which it follows that \( \pi \) induces an isomorphism \( \partial_T \tilde{S} \cong \partial_T S \). Thus we will from now on simply write \( \partial(k) \) instead of \( \tilde{\partial}(k) \).

3. Construction of the deformation diagram

Definition 3.1. A deformation of \( X \) is a flat family of schemes \( f : X' \to S \) with \( 0 \in S \) such that \( f^{-1}(0) = X \). We say that a deformation \( f : X' \to S \) of \( X \) is miniversal in degrees \(-nR^s\) for \( n \in \mathbb{N} \) if

1. The Kodaira- Spencer map \( T_0S \to \oplus_{n \in \mathbb{N}} T_X^1(-nR^s) \) is bijective.
2. The obstruction map of \( f \) is injective.

For definitions of the Kodaira-Spencer (resp. obstruction) map we refer the reader to Section 4 and [11, Definition 10.2.17] (resp. Section 5 and [10, Section 4] and [1, Section 7]).

Remark 3.2. Note that by [10, Section 4] the above definition implies that every other deformation \( f' : X' \to S' \), whose Kodaira-Spencer map has its image in \( \oplus_{n \in \mathbb{N}} T_X^1(-nR^s) \), can be induced via base change from a map \( \varphi : S' \to S \). The map \( \varphi \) is not uniquely determined but its induced map on the tangent spaces \( T_0S' \to T_0S \) is.

3.1. Two-dimensional case. Let \( P = [0, m] \subset \mathbb{R} \) be an interval of lattice length \( m \). The deformation theory of \( X_P \) is very well known since

\[ X_P = \text{Spec } \mathbb{C}[S] \cong \text{Spec } \mathbb{C}[x, y, z]/(xy - z^m) \]

is a hypersurface. Taking \( m - 1 \) new variables \( w_2, w_3, \ldots, w_m \), the miniversal deformation of \( X_P \) is given by

\[ X := \text{Spec } \mathbb{C}[x, y, z, w_2, w_3, \ldots, w_m]/(xy - z^m - \sum_{j=2}^{m} w_j z^{m-j}) \to \text{Spec } \mathbb{C}[w_2, w_3, \ldots, w_m]. \]

Using notation of Section 2 we see that the Hilbert basis of \( S \) is \(\{(1, m), (0, 1)\}\). Denoting \( x := \chi(-1, m) \), \( y := \chi(0, 0) \) and \( z := \chi(0, 1) \) we get the relation \( xy - z^m = 0 \).

Moreover, we have \( \bar{T} = \langle mt \rangle \) where \( t := t_0 \) is corresponding to the only edge \( d = P \) of length \( m \). We see that \( \tilde{n}(-1) = mt \) and \( \tilde{n}(1) = 0 \). Thus the generators of \( \tilde{S} \) are \( (0, mt), (-1, mt), (1, 0) \), which yields the relation \( \chi(-1, mt) \cdot \chi(0, 1) = \chi(0, mt) \). In Remark 3.3 we will see how we can obtain the miniversal deformation described in (12) using this notation.
3.2. **General case.** Let $X_P$ be a Gorenstein toric variety. Recall the surjection described in (10). The elements $u_i$ map to $\ell_i = l_it_i = \ell(d^i)t_i$, which are the generators of $\tilde{T}$. For $i = 1, \ldots, n$ we introduce new variables $u_{ij}, \ j = 1, \ldots, l_i - 1$ and define $u_{il_i} := u_i$. Let

$$A_C^N := \text{Spec} \mathbb{C}[u_{ij} \mid i \in \{1, \ldots, n\}, \ (i, j) \in \{(i, 1), \ldots, (i, l_i - 1)\}],$$

where $N = \sum_{i=1}^n (l_i - 1)$. We have the diagram

$$\begin{array}{ccc}
X = \text{Spec} \mathbb{C}[S] & \xrightarrow{\iota} & A_C^N \times \text{Spec} \mathbb{C}[\tilde{S}] \\
\downarrow \pi_1 & & \downarrow \tilde{\pi}_1 \\
A_C^1 = \text{Spec} \mathbb{C}[t] & \xleftarrow{\iota} & A_C^N \times \text{Spec} \mathbb{C}[\tilde{T}],
\end{array}$$

where $\iota$ is defined by the map $u_{ij} \mapsto t^j$. In particular $u_i \mapsto t^1$. The map $\tilde{\iota}$ is defined by the map $u_{ij} \mapsto x^jR^*$ and $x_i \mapsto x_i, \ i = 1, \ldots, r$. The map $\pi_1$ is defined by the map $t \mapsto R^*$ and the map $\tilde{\pi}_1$ is defined by the inclusion map $u_{ij} \mapsto u_{ij}$.

The map $\iota$ defines the degree of $u_{ij}$, which is $j$. Among the degree 1 variables $\{u_{11}, \ldots, u_{n1}\}$ we pick one and denote it by $u_0$. W.l.o.g. we assume $u_0 := u_{11}$ and for each $i = 1, \ldots, n$ let

$$T_{ij} := u_{ij} - u_0u_{i,j-1}, \ i = 1, \ldots, n, \ j \geq 1.$$ 

Since we assume that $u_0 = u_{11}$ we get $T_{11} = 0$. With the above notation we can write

$$u_i = u_0^l + \sum_{j=1}^{l_i} T_{ij}u_0^{l_i-j}.$$

We explain why we choose this notation in Remark 3.3.

Let $\mathbb{C}[T] := \mathbb{C}[T_{ij} \mid i \in \{1, \ldots, n\}, \ (i, j) \in \{(i, 1), \ldots, (i, l_i)\}]$ and consider the following diagram (15).

$$\begin{array}{ccc}
X & \xrightarrow{\pi_1} & A_C^N \times \text{Spec} \mathbb{C}[\tilde{S}] \\
\downarrow & & \downarrow \tilde{\pi}_1 \\
A_C^1 & \xleftarrow{\pi_2} & A_C^N \times \text{Spec} \mathbb{C}[\tilde{T}] \\
\downarrow & & \downarrow \tilde{\pi}_2 \\
0 & \xrightarrow{\pi_2} & \text{Spec} \mathbb{C}[T].
\end{array}$$

The surjective map $\pi_2 : A_C^{N+n} \rightarrow \text{Spec} \mathbb{C}[T]$ is defined by the inclusion

$$\mathbb{C}[T] \subset \mathbb{C}[u] := \mathbb{C}[u_{ij} \mid i \in \{1, \ldots, n\}, \ (i, j) \in \{(i, 1), \ldots, (i, l_i)\}].$$

We define $B$ to be a maximal subscheme in $\text{Spec} \mathbb{C}[T]$ such that $\pi_2^{-1}(B)$ is contained in $A_C^N \times \text{Spec} \mathbb{C}[\tilde{T}]$. Defining $\tilde{X} := \pi_1^{-1}(\pi_2^{-1}(B))$ we see that

$$\tilde{X} \xrightarrow{\pi_2 \circ \tilde{\pi}_1} B.$$
is a deformation of $X$ since both $\hat{\pi}_1$ and $\pi_2$ are flat maps. Indeed, $\pi_2$ is obviously flat and $\hat{\pi}_1$ is flat since $(\hat{T}, \hat{S})$ is a free pair from which it follows that $\mathbb{C}[\hat{S}]$ is a free $\mathbb{C}[\hat{T}]$-module.

**Remark 3.3.** In the two-dimensional case, cf. Subsection 3.1, we see that the procedure described in this subsection yields $\tilde{X} \cong \text{Spec } \mathbb{C}[x, y, z, T_{12}, T_{13}, \ldots, T_{1m}]/(xy - z^m - \sum_{j=2}^{m} T_{1j}z^{m-j})$, since $T_{11} = 0$. We get the miniversal deformation from (12) by inserting $T_{1j} = w_j$ for $j = 2, ..., n$. Note that this also gives us an idea in the general case, cf. (14). In the upcoming sections we will show that this choice was indeed natural.

### 3.3. The Dimension of $B$

For an integer $z \in \mathbb{Z}$ we define

$$z^+ := \begin{cases} z & \text{if } z \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad z^- := \begin{cases} -z & \text{if } z \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let us consider the ideal $I_{\hat{T}} := \ker(\mathbb{C}[u_1, ..., u_n] \rightarrow \hat{T})$. It is clear that

$$I_{\hat{T}} = \left( \prod_{i=1}^{n} u_i^{-d_i^+} - \prod_{i=1}^{n} u_i^{-d_i^-} \mid d \in \mathbb{T}_N(P) \cap \mathbb{T}(P)^\perp \right) \subset \mathbb{C}[u_1, ..., u_n]$$

with

$$\mathbb{T}(P)^\perp = \text{Span}_\mathbb{R} \left[ \left\{ \delta_c(d^1)(d^1, c), ..., \delta_c(d^n)(d^n, c) \right\} \mid c \in M_\mathbb{R}, \epsilon \text{ a 2-face in } P \right].$$

The equations of $B$ are obtained as follows: let $d \in \mathbb{T}_N(P) \cap \mathbb{T}(P)^\perp$ (as in the equation (17)) and let

$$p_d(u_1, ..., u_n) := \prod_{i=1}^{n} u_i^{-d_i^+} - \prod_{i=1}^{n} u_i^{-d_i^-} \in I_{\hat{T}} \subset \mathbb{C}[u_1, ..., u_n].$$

We insert $u_i = u_i^l + \sum_{j=1}^{l_i} T_{ij}u_{ij}^{l_i-j}$, cf. (14), into $p_d(u_1, ..., u_n)$ for $i = 1, ..., n$. Since $p_d(u_1, ..., u_n)$ is homogenous (say of degree $g_d \in \mathbb{N}$) we can write in a unique way

$$p_d(u_1, ..., u_n) = \sum_{i=1}^{g_d} p_d^{(i)}(T)u_0^{g_d-i},$$

where $p_d^{(i)}(T) \in \mathbb{C}[T]$ are homogenous of degree $i$. Immediately we see that $p_d^{(g_d)}(T) = 0$ and that $p_d^{(i)}(c) \in (T)$, where $(T)$ stands for the ideal generated by $T_{ij}$, i.e. $(T) = (T_{11}, ..., T_{1l_1}, ..., T_{n1}, ..., T_{nl_n})$. By definition of $B$ given below the diagram (15) we see that the ideal $\mathcal{I}_B$ of $B \subset \text{Spec } k[T]$ is generated by the polynomials $p_d^{(i)}(T)$ appearing in (19) for all $d \in \mathbb{T}_N(P) \cap \mathbb{T}(P)^\perp$. In particular, we see that $\mathcal{I}_B$ is a homogenous ideal and the tangent space $T_0B$ of $B \subset \text{Spec } k[T]$ at 0 is

$$\left\{ (T_{11}, ..., T_{1l_1}, ..., T_{n1}, ..., T_{nl_n}) \in \mathbb{C}^{N+n} \mid T_{11} = 0, \sum_{d^i, l_i \geq k} \frac{\delta_c(d^i)}{l_i} d^i T_{ik} = 0, \text{ for each } k \in \mathbb{N} \text{ and 2-face } \epsilon \right\}.$$

**Proposition 3.4.** $\dim_\mathbb{C} T_0B = \dim_\mathbb{C} \oplus_{n \in \mathbb{N}} T_X^{1}(-nR^*)$. 
Proof. For a natural number \( k \geq 2 \) let us denote
\[
T_0B(k) := \{(T_{1k}, \ldots, T_{nk}) \in \mathbb{C}^n \mid T_{1k} = 0 \text{ if } l_i < k, \quad \sum_{d^i: l_i \geq k} \frac{\delta_i(d^i)}{l_i} d^i T_{1k} = 0, \text{ for each 2-face } \epsilon \}.
\]
Clearly, \( \dim_{\mathbb{C}} T_0B(k) = \dim_{\mathbb{C}} V_k - 1 = \dim_{\mathbb{C}} T_X^1(-kR^*) \), see Proposition 2.1. Set
\[
T_0B(1) := \{(T_{11}, \ldots, T_{n1}) \in \mathbb{C}^n \mid T_{11} = 0, \quad \sum_{d^i} \frac{\delta_i(d^i)}{l_i} d^i T_{11} = 0, \text{ for each 2-face } \epsilon \},
\]
which gives us \( \dim_{\mathbb{C}} T_0B(1) = \dim_{\mathbb{C}} V_1 - 1 = \dim T_X^1(-R^*) \). From the equation (20) the proof follows.

Example 3.5. Consider the polytope \( P = \text{conv}\{(0,0), (2,0), (2,2), (1,2), (0,2)\} \), where \( \text{conv} \) denotes the convex hull. We have two non-trivial lattice Minkowski decompositions of \( P \).

\[
\begin{array}{c}
\bullet + \bullet + \bullet = d_3 \\
\bullet \quad \bullet \quad \bullet = d_5
\end{array}
\]

By Proposition 2.1 we see that
\[
\dim_{\mathbb{C}} T_X^1(-R^*) = 2, \quad \dim_{\mathbb{C}} T_X^1(-2R^*) = 1, \quad \dim_{\mathbb{C}} T_X^1(-kR^*) = 0, \text{ for } k \geq 3.
\]

We denote
\[
d^1 = (-1, 1), \quad d^2 = (-1, -1), \quad d^3 = (0, -2), \quad d^4 = (2, 0), \quad d^5 = (0, 2).
\]
As in Subsection 3.2 we denote \( u_0 = u_1 = u_{11} \). The ideal \( \mathcal{I}_T \) defined in (17) is in this case generated by
\[
\mathcal{I}_T = \langle u_4 - u_1 u_2, \quad u_5 u_1 - u_2 u_3 \rangle.
\]
The two generators are obtained from
\[
((d^1, (1, 0)), \ldots, (d^5, (1, 0))), \quad ((d^1, (0, 1)), \ldots, (d^5, (0, 1))) \in T_\mathcal{Z}(P) \cap \mathcal{T}(P)^\perp.
\]
After inserting (14) into the two generators of \( \mathcal{I}_T \) we get
\[
\begin{align*}
(u_0^2 + T_{12} + u_0 T_{41}) - u_0 (u_0 + T_{21}) &= u_0 (T_{41} - T_{21}) + T_{42}, \\
(u_0^2 + T_{52} + u_0 T_{51}) u_0 - (u_0 + T_{21}) (u_0^2 + T_{32} + u_0 T_{31}) &= u_0^2 (T_{51} - T_{21} - T_{31}) + u_0 (T_{52} - T_{32} - T_{21} T_{31}) - T_{21} T_{32}.
\end{align*}
\]
Thus the ideal \( \mathcal{I}_B \) of \( B \subset \text{Spec} \mathbb{C}[T_{21}, T_{31}, T_{32}, T_{41}, T_{42}, T_{51}, T_{52}] \) is given by
\[
\mathcal{I}_B = (T_{41} - T_{21}, T_{42}, T_{51} - T_{21} - T_{31}, T_{52} - T_{32} - T_{21} T_{31}, T_{21} T_{32}),
\]
from which we see that
\[
B \cong \text{Spec} \mathbb{C}[T_{21}, T_{31}, T_{32}] / (T_{21} T_{32}).
\]
The two irreducible components given by \( T_{21} = 0 \) and \( T_{32} = 0 \) correspond to the two Minkowski decompositions of \( P \) above as we will see in Section 7.
4. The Kodaira-Spencer Map

Recall that \( I_R \) is the ideal of \( B \subset \text{Spec}[T] \) and recall the surjection (9), where \( t \) is mapped to \( x^R \). Moreover, recall \( F_k(x, u) = F_k(x, u_1, \ldots, u_n) \) from (11). We write

\[
F_k(x, t, \mathbf{T}) := F_k(x, t^{i_1} + \sum_{j=1}^{l_1} t^{l_1-j} T_{i_1}, \ldots, t^{i_n} + \sum_{j=1}^{l_n} t^{l_n-j} T_{i_n}),
\]

i.e. in \( F_k(x, u) \) we insert \( u_i = t^{i_1} + \sum_{j=1}^{l_i} t^{l_i-j} T_{i_1} \), which is the equation (14) with \( t = u_0 \). Our flat family (16) corresponds to the flat \( \mathbb{C}[T]/I_R \)-module \( \mathbb{C}[x, t, \mathbf{T}]/(I_R, F_k(x, t, \mathbf{T}) \mid k \in \mathbb{N}^r) \).

In the following we will compute the Kodaira-Spencer map of this flat family. Let

\[
t = (t_{11}, \ldots, t_{1t}, \ldots, t_{nt}, \ldots, t_{ntn}) \subset \mathbb{C}^{N+n}
\]

be an element of \( T_0 B \subset \mathbb{C}^{N+n} \), cf. (20). It induces the flat \( \mathbb{C}[\epsilon]/\epsilon^2 \)-module

\[
\mathbb{C}[S]_t := \mathbb{C}[x, t, \epsilon]/(\epsilon^2, F_k(x, t, \epsilon t) \mid k \in \mathbb{N}^r),
\]

where

\[
F_k(x, t, \epsilon t) = x^k - x^{\partial(k)} \prod_{i=1}^{n} \left( t^{i_i} + \sum_{j=1}^{l_i} t^{l_i-j} \epsilon t_{ij} \right)^{\tilde{\lambda}_i(k)_{i_j}},
\]

with \( \tilde{\lambda}_i(k) \) defined in (7).

Let us write \( A := \mathbb{C}[S] \) and recall the surjective map \( \mathbb{C}[x_1, \ldots, x_r, t] \to A \) with the kernel \( I_X \). The following exact sequence is well known:

\[
0 \to \text{Der}_k(A, A) \to A^{r+1} \to \text{Hom}_A(I_X/I_X^2, A) \xrightarrow{\psi} T_X^1(A) \to 0.
\]

From (24) we will now construct the associated element in \( \text{Hom}_A(I_X/I_X^2, A) \), which is the most important step for constructing the Kodaira-Spencer map, see e.g. [11, Definition 10.2.17]. By (25) we get that

\[
F_k(x, t, \epsilon t) = x^k - x^{\partial(k)} \left( t^{\sum_{i=1}^{n} \tilde{\lambda}_i(k)_{i_j}} + \epsilon \sum_{i=1}^{n} \sum_{j=1}^{l_i} \tilde{\lambda}_i(k)_{i_j} \cdot t_{ij} \cdot t^{(\sum_{i=1}^{n} \tilde{\lambda}_i(k)_{i_j})-j} \right) \in \mathbb{C}[x, t, \epsilon]/(\epsilon^2).
\]

Since \( \lambda(k) = \sum_{i=1}^{n} \tilde{\lambda}_i(k) \) by (8) we see that

\[
F_k(x, t, \epsilon t) = f_k(t, x) + \epsilon \sum_{i=1}^{n} \sum_{j=1}^{l_i} \tilde{\lambda}_i(k)_{i_j} \cdot t_{ij} \cdot x^{\partial(k)} t^{\lambda(k)-j} \in \mathbb{C}[x, t, \epsilon]/(\epsilon^2).
\]

Thus the Kodaira-Spencer map is given by

\[
T_0 B \xrightarrow{\varphi} \text{Hom}_A(I_X/I_X^2, A) \xrightarrow{\psi} T_X^1(A),
\]

where \( \varphi(t) = \left( f_k \mapsto \sum_{i=1}^{n} \sum_{j=1}^{l_i} \tilde{\lambda}_i(k)_{i_j} \cdot t_{ij} \cdot x^{\partial(k)} t^{\lambda(k)-j} \in A \right) \).

Let us now look which elements \( t \in T_0 B \) map via \( \psi \circ \varphi \) to \( T_X^1(-kR^*) \subset T_X^1 \). For those elements it holds that the exponent of \( t \) in \( \varphi(t) \) is equal to \( \lambda(k) - k \). This means that \( T_0 B(k) \) surjects to \( T_X^1(-kR^*) \).
for all $k \in \mathbb{N}$ and since $T_0 B(k)$ and $T_X^1(-kR^*)$ have the same dimension, cf. Proposition 3.4, we prove the following theorem.

**Theorem 4.1.** The Kodaira-Spencer map $T_0 B \to \oplus_{k \in \mathbb{N}} T_X^1(-kR^*)$ of the flat family (16) is bijective.

5. THE OBSTRUCTION MAP

Let us pick a basis $T_b \subset T$ of $T_0 B$ and an ideal $I_b \subset C[T_b]$ such that

$$B = \text{Spec } k[T]/I_B \cong \text{Spec } C[T_b]/I_b.$$  

**Example 5.1.** In Example 3.5, cf. (23), we pick $T_b = \{T_{21}, T_{31}, T_{32}\}$ and $I_b = (T_{21}T_{32}).$

We consider the ideal $J_b := I_b \cdot (T_b),$ where $(T_b)$ stands for the ideal generated by those $T_{ij}$ that appears in $T_b.$ In the obstruction theory the vector space $W := W_b := I_b/J_b$ is important, see [11, Section 4]. Clearly the vector space $W$ is independent of the choice of the basis $T_b$ (a different choice of the basis $T_b$ provides the vector space $W_b$ that is isomorphic to $W$).

We have a natural $\mathbb{N}$-grading on $W$ given by the degrees of the homogenous polynomials and thus we write $W = \oplus_{k \in \mathbb{N}} W_k.$ The obstruction map of the flat family (16) is an element of

$$\oplus_{k \in \mathbb{N}} (T_X^2(-kR^*) \otimes C W_k),$$

see [1, Section 7] and [11, Section 4] for details. The description of this obstruction map is very similar as in [1, Section 7] or [6, Section 7.3] which we will see in Proposition 5.4 below. Since the techniques for proving this are the same as in the previously mentioned papers [6, Section 7.3] and [1, Section 7.1], we only state the results that we will use in the upcoming section to prove the injectivity of the obstruction map.

The following definitions already appeared in [1, Section 6]. Recall the Hilbert basis $E$ of $S = \sigma^\vee \cap (M \oplus \mathbb{Z})$ from the equation (2) and for $R \in M \oplus \mathbb{Z}$ we consider

$$E_i^R := \{e \in E \mid \langle a^i, e \rangle < \langle a^i, R \rangle \}.$$  

For a subface $\tau$ of $\sigma$ (denoted $\tau \leq \sigma$) let $E_{\tau}^R := \bigcap_{a^i \in \tau} E_i^R.$ The $\mathbb{Z}$-module of all linear relations among elements in $E_{\tau}^R$ we denote by $L(E_{\tau}^R).$

**Proposition 5.2.**

$$T_X^2(-R)^* \cong \ker \left( \bigoplus_i L_{C}(E_i^R) \rightarrow L_{C}(E) \right) \left/ \text{image} \left( \bigoplus_{(a^i, ak) \leq \sigma} L_{C}(E_i^R \cap E_k^R) \rightarrow \bigoplus_i L_{C}(E_i^R) \right) \right..$$

**Proof.** See [3, Propositions 5.4, 5.5]. \hfill \qed

For a closed path $\mu$ along the edges of $P$ and $c \in M$ we denote

$$d(\mu, c) := (\langle \mu_1, d^i \rangle, ..., \langle \mu_n, d_i \rangle) \in T_X^*(P) \cap T_{\perp}(P),$$

with $T_{\perp}(P)$ defined in (18) and thus $p_{d(\mu, c)}(u) \in I_{\perp}.$ We use the following notation:

$$p(\mu, c) := p_{d(\mu, c)}(u) \in I_{\perp} \subset C[u].$$
We fix a map \( h : \mathbb{C}[T] \rightarrow \mathbb{C}[T_b] \) that induces an isomorphism \( \tilde{h} : \mathbb{C}[T]/\mathcal{I}_B \rightarrow \mathbb{C}[T_b]/\mathcal{I}_b \), cf. (26). For \( p^{(k)}_{d(M,c)}(T) \in \mathcal{I}_B \subset \mathbb{C}[T] \) we define

\[
p^{(k)}(u,c) := h(p^{(k)}_{d(M,c)}(T)) \in \mathcal{I}_b \subset \mathbb{C}[T_b]. \tag{29}
\]

Recall the paths given in Definition 2.5 and the elements \( c_1, \ldots, c_r \) appearing in \( E \) from (2). We define the map

\[
\psi^{(k)}_i : L_C(E_{a^kR^*}^*) \rightarrow W_k,
\]

\[
q \mapsto \sum_{j=1}^r q_j p^{(k)}(\lambda^c_j(v^i) - \lambda(v(c_j)), c_j).
\]

Note that the \( q \)-coordinate corresponding to \( R^* \), which is an element of \( E_{a^kR^*}^* \) if \( k \geq 2 \), is not used in the definition of \( \psi^{(k)}_i \).

**Lemma 5.3.** Let

\[
\xi \in T_2(P) \cap \text{Span}_R \left\{ \left( \delta^a(d^1)(d^1, c), \ldots, \delta^a(d^n)(d^n, c) \right) ; c \in M_\epsilon, \epsilon \text{ 2-face in } P \right\},
\]

which means that \( \xi \) lies in the same space as \( d \) from the equation (17). It holds that

\[
p^{(k)}_{d+\xi}(T_b) = p^{(k)}_{d}(T_b) + p^{(k)}_{\xi}(T_b) \in W_k. \tag{30}
\]

**Proof.** Straightforward computation shows that

\[
\frac{1}{2} p_{d+\xi}(u) \left( \prod_{i=1}^n u_i^{d^+} + \prod_{i=1}^n u_i^{d^-} \right) + \frac{1}{2} p_{d}(u) \left( \prod_{i=1}^n u_i^{d^+} + \prod_{i=1}^n u_i^{d^-} \right) = \prod_{i=1}^n u_i^{d^+} \prod_{i=1}^n u_i^{d^-} \prod_{i=1}^n u_i^{d^-} = \prod_{i=1}^n u_i^{d^+} \prod_{i=1}^n u_i^{d^-} \prod_{i=1}^n u_i^{d^-},
\]

where

\[
S_1 = \{ i \in \{1, \ldots, n\} \mid d_i > 0, e_i < 0, d_i + e_i > 0 \},
\]

\[
S_2 = \{ i \in \{1, \ldots, n\} \mid d_i < 0, e_i > 0, d_i + e_i > 0 \},
\]

\[
S_3 = \{ i \in \{1, \ldots, n\} \mid d_i > 0, e_i < 0, d_i + e_i < 0 \},
\]

\[
S_4 = \{ i \in \{1, \ldots, n\} \mid d_i < 0, e_i > 0, d_i + e_i < 0 \}.
\]

After inserting \( u_i = u_0^{d_i} + \sum_{j=1}^{l_i} T_{ij} u_0^{l_i-j} \), cf. (14), we see that in \( W_k \) the equation (30) holds. \hfill \Box

**Proposition 5.4.** \( \psi^{(k)}_i \) induce the linear map \( \psi^{(k)} : T_X^2(-k R^*)^* \rightarrow W_k \) and the map

\[
\psi = \sum_{k \in \mathbb{N}} \psi^{(k)} : \bigoplus_{k \in \mathbb{N}} T_X^2(-k R^*)^* \rightarrow W
\]

is the adjoint of the obstruction map of the flat family (16).
Proof. The idea of the proof is similar to [1, Lemma 7.7]. Let \( \rho^j \) denote the path consisting of the single edge running from \( v^i \) to \( v^j \). For \( q \in L(E_{a_0}^{kR^*} \cap E_{a_0}^{kR^*}) \) we see by Lemma 5.3 that

\[
\psi^{(k)}_i(q) - \psi^{(k)}_j(q) = \sum_{l=1}^{r} qp^{(k)}(\Delta(a^i) - \Delta(a^j) + \rho^j, c_l) + \sum_{l=1}^{r} qp^{(k)}(\mu^i(a^i) - \mu^i(a^j) - \rho^j, c_l).
\]

We want to show that the above expression is equal to 0. The first sum is zero by Lemma 5.3 using \( \sum_{l=1}^{r} q c_l = 0 \). For the second sum we observe that for \( q \in L(E_{a_0}^{kR^*} \cap E_{a_0}^{kR^*}) \) the following holds: if \( q \neq \emptyset \), then \( (c_1, c_1) \in E \) satisfies \( \langle c_1, c_1 \rangle < \langle kR^*, a^i \rangle = k \), from which it follows, using \( a_i = (v_i, 1) \), that \( \langle c_1, v_i \rangle - \langle c_1, v(c_1) \rangle < k \) and similarly we see that \( \langle c_1, v_j \rangle - \langle c_1, v(c_1) \rangle < k \). From this it follows that the degree of \( p(q)^i(a^i) - \mu^i(a^j) - \rho^j, c_l \) is strictly smaller than \( k \) and thus \( p(q)^i(a^i) - \mu^i(a^j) - \rho^j, c_l = 0 \), which concludes the proof that \( \psi^{(k)}_i \) induce the linear map \( \psi^{(k)}: T^2_k(-kR^*) \to W_k \).

The proof that \( \psi \) is the adjoint of the obstruction map is similar to [6, Proposition 7.6] or [1, Proposition 7.8] so we omit the proof. \( \square \)

6. Injectivity of the obstruction map

In this section we prove the surjectivity of the map \( \psi \), which is the adjoint of the obstruction map and thus we prove that the obstruction map is injective. The idea of the proof is new, with the previous techniques we were not able to obtain the injectivity of the obstruction map in the degree \(-R\) if \( P \) has at least one edge of lattice length \( \geq 2 \), cf. [6, Example 6.5, Remark 7.9].

6.1. Three-dimensional case. A polygon \( P \) induces a three-dimensional Gorenstein toric variety \( X_P \) whose cone \( \sigma = \text{cone}(P) \) has cyclically ordered generators \( a^1, ..., a^n, a^{n+1} := a^1 \) with \( a^i = (v_i^i, 1) \in N \oplus \mathbb{Z} \). Denoting \( d^i := v^{i+1} - v^i \) gives us \( \sum_{i=1}^{n} d^i = 0 \). In this case we have the following formula for \( T^2_X \): for \( R \in M \) we denote \( K_{R,a_i} := K_{R,i} := \{ r \in S \mid \langle a^i, r \rangle < \langle a^i, R \rangle \} \) and \( K_{R,i,i+1} := K_{R,a_i} \cap K_{R,a_{i+1}} \).

It holds that

\[
T^2_k(-R)^* \cong \bigcap_i (\text{Span}_C K_{R,i,i+1}^* / \text{Span}_C (\bigcap_i K_{R,i,i+1}^*)),
\]

which was proven in [3, Corollary 5.4].

Let \( \varphi := \sum_{k \in \mathbb{N}} \varphi^{(k)} \), where

\[
\varphi^{(k)} : \left( \bigcap_i (\text{Span}_Z K_{R,i,i+1}^*) / \text{Span}_Z (\bigcap_i K_{R,i,i+1}^*) \right) \to W_k \]

\[
(c, m) \in M \oplus \mathbb{Z} \mapsto p^{(k)}(c) := p^{(k)}(1, c).
\]

Note that we have already oriented \( P \), which is a 2-face, and thus we can simply take \( \mu := 1 \). Let us check that the map \( \varphi^{(k)} \) is well defined: we need to show that

\[
\varphi^{(k)}(c) = 0 \in W_k \text{ for } c \in \bigcap_i K_{R,i,i+1}^*.
\]

For \( c \in M \) we denote

\[
d(c) := \max \{ \langle v^i, c \rangle \mid i = 1, ..., n \} - \min \{ \langle v^j, c \rangle \mid i = 1, ..., n \}.
\]
Let us define \( p(c) := p(1, c) \in \mathcal{T}_p \). We immediately see that the degree of the homogenous polynomial \( p(c) \) is equal to \( d(c) \). Thus (32) follows from the following lemma.

**Lemma 6.1.** There exists \( m \in \mathbb{Z} \) such that \( (c, m) \in \bigcap_i K_{i,i+1}^{kR^*} \) if and only if \( d(c) \leq k - 1 \).

**Proof.** It follows immediately by definitions: note that \( a^i = (v^i, 1) \) and that \( r \in \bigcap_i K_{i,i+1}^{kR^*} = \bigcap_i K_{a^i}^{kR^*} \) if and only if \( 0 \leq \langle a^i, r \rangle \leq k - 1 \) for every \( i = 1, \ldots, n \).

**Corollary 6.2.** The map \( \varphi \) is well defined.

**Proposition 6.3.** \( \mathbb{C} \)-linear extension of \( \varphi \) equals the map \( \psi \) from Proposition 5.4.

**Proof.** Let us connect the two descriptions of \( T^\mathbb{Z}_R (-kR^*) \) given in (31) and (27). Starting from \( c \in \bigcap_i (\text{Span}_\mathbb{Z} K_{i,i+1}^{kR^*}) \) we obtain the corresponding element

\[
L(c) \in \ker \left( \bigoplus_i L_C(E_i^{kR^*}) \to L_C(E) \right)
\]

as follows: we can write

\[
c = \sum_{j=1}^r q_{i,j}c_j + q_i(\underline{0}, 1),
\]

where \( q_{i,j} \neq 0 \) implies that \( (c_j, \eta(c_j)) \in E_{a^i_1}^{kR^*} \cap E_{a^i_1+1}^{kR^*} \). Let

\[
L(c)_i := \sum_j (q_{i,j} - q_{i-1,j})(c_j, \eta(c_j)) + (q_i - q_{i-1})(\underline{0}, 1) = 0
\]

be an element in \( L(E_{a^i_1}^{kR^*}) \), which defines \( L(c) := \sum_i L(c)_i \in \bigoplus_i L(E_i^{kR^*}) \). The element \( L(c) \) clearly lies in the kernel of the map \( \bigoplus_i L(E_i^{kR^*}) \to L(E) \) since this map sums the given summands. Thus we can easily see that the map

\[
\frac{\bigcap_i (\text{Span}_\mathbb{Z} K_{i,i+1}^{kR^*})}{\text{Span}_\mathbb{Z} (\bigcap_i K_{i,i+1}^{kR^*})} \to \frac{\ker \left( \bigoplus_i L(E_i^{kR^*}) \to L(E) \right)}{\text{Image} \left( \bigoplus_i L(E_i^{kR^*} \cap E_{i+1}^{kR^*}) \to \bigoplus_i L(E_i^{kR^*}) \right)}
\]

is an isomorphism of \( \mathbb{Z} \)-modules.

To finish the proof we need to show that \( \psi^{(k)}(L(c)) = \varphi^{(k)}(c) = p^{(k)}(c) \), which means that we need to show that

\[
\sum_{i=1}^n \sum_{j=1}^r (q_{i,j} - q_{i-1,j})p^{(k)}(\chi c_j + \Lambda(v(c_j)), c_j) = p^{(k)}(c).
\]

Using Lemma 5.3 this is a straightforward check, see also [1, Section 7.9 (iii)].

**Lemma 6.4.** For an edge \( d^i = v^{i+1} - v^i \) it holds that

\[
c \in (d^i)^\perp
\]

if and only if there exists \( m \in \mathbb{N} \) such that

\[
(c, m) \in (a^i)^\perp \cap (a^{i+1})^\perp
\]

**Proof.** Recall that \( a^i = (v^i, 1) \in N \oplus \mathbb{Z} \) and thus (36) follows from (35) by picking \( m := -\langle c, v^i \rangle = -\langle c, v^{i+1} \rangle \). From (36) it follows that \( \langle c, v^i \rangle = \langle c, v^{i+1} \rangle \), from which (35) follows.
Proposition 6.5. \( \psi \) is surjective in the three-dimensional case.

Proof. By Proposition 6.3 and the definition of \( \varphi \) it is enough to show that if
\begin{equation}
\text{for each } n \in \mathbb{Z} \text{ it holds that } (c, n) \notin \bigcap_i \text{Span}_\mathbb{Z} K^k_{i,i+1},
\end{equation}
then \( p^k(c) = 0 \in W_k \). For \( k \geq 2 \) we immediately see that
\begin{equation}
\text{Span}_\mathbb{Z} K^k_{i,i+1} \cong \begin{cases}
\text{Span}_\mathbb{Z} \left( (M \oplus \mathbb{Z}) \cap (a^i)^\perp \cap (a^{i+1})^\perp, R^* \right) & \text{if } \ell(d^i) \geq k,
M \oplus \mathbb{Z} & \text{if } \ell(d^i) < k.
\end{cases}
\end{equation}

Let \( m \) be a maximal number of linearly independent edges of \( P \) that have lattice length \( \geq k \) and let \( d^{\nu_1}, \ldots, d^{\nu_m} \) be those edges. Since \( P \) is a polygon we have \( m \in \{0, 1, 2\} \). We define the index set
\[ I_k := \{ i \in \{1, \ldots, n\} \mid \ell(d^i) \geq k, i \notin \{\nu_1, \ldots, \nu_m\} \}. \]

Note that \( \{T_{i,k} \mid i \in I_k\} \) is a basis of \( T_0 B(k) \cong T^1_X(-kR^*) \), cf. (21). We fix a basis
\[ T_b := \bigcup_{k \in \mathbb{N}} \{T_{i,k} \mid i \in I_k\} \subset T \]
of \( T_0 B \cong \bigoplus_{k \in \mathbb{N}} T^1_X(-kR^*) \).

For \( c \neq 0 \) we see by Lemma 6.4 and (38) that if (37) holds, then \( c \notin (d^{\nu_1})^\perp \cap \cdots \cap (d^{\nu_m})^\perp \). Thus \( \langle c, d^\nu \rangle \neq 0 \) for some \( i \in \{1, \ldots, m\} \), from which it follows that \( p^k(c) = 0 \in W_k \) since the coefficient in front of \( T_{\nu,k} \) in \( p^k_{\langle 1, c \rangle} \) is non-zero and \( T_{\nu,k} \notin T_b \).

\begin{corollary}
The obstruction map of the flat family (16) is injective in the three-dimensional case.
\end{corollary}

Corollary 6.7. The flat family (16) is miniversal in the three-dimensional case.

6.2. Equations of the miniversal base space in the three-dimensional case. In this subsection we compute \( \dim C T^2_X(-kR^*) \) and analyse the minimal equations of the miniversal base space \( B \) in the three-dimensional case. The results of this subsection will not be used later.

Recall \( d(c) \) from (33).

Definition 6.8. Let \( n_1 := \min \{d(c) \mid c \in M \setminus 0\} \) and
\[ n_2 := \min \{\max \{d(c), d(c')\} \mid c, c' \in M \setminus 0 \text{ linearly independent}\}. \]

We define \( \ell_1 := \max \{\ell(d^i) \mid i = 1, \ldots, n\} \) to be the maximum length of edges and let
\[ \ell_2 := \max \{\min \{\ell(d^i), \ell(d^j)\} \mid d^i, d^j \text{ linearly independent}\}. \]

We have \( n_1 \leq n_2 \) and \( \ell_1 \geq \ell_2 \).

Proposition 6.9. It holds that
\[ \dim C T^2_X(-kR^*) = \begin{cases}
1 & \text{if } \ell_2 < k \leq \ell_1 \text{ and } k \leq n_1 \\
1 & \text{if } k > \ell_1 \text{ and } n_1 < k \leq n_2 \\
2 & \text{if } k > \ell_1 \text{ and } k \leq n_1 \\
0 & \text{otherwise}.
\end{cases} \]

Proof. It follows immediately from (31), Lemma 6.1 and (38). \( \square \)
We pick two edges \( d^{i_1} \) and \( d^{i_2} \) that have lattice length \( \ell_1 \) and \( \ell_2 \), respectively. We then choose \( T_b := \mathcal{T} \setminus \{ T_{1,k}, T_{2,k} \mid k \in \mathbb{N} \} \) for the basis of \( T_0 \mathcal{B} \cong \oplus_{k \in \mathbb{N}} T_k^1(−kR*) \). Since \( \mathcal{B} \) is the miniversal base space we know that the minimal number of degree \( k \) equations of \( \mathcal{B} \) is less or equal than \( \dim_{\mathbb{C}} T_X^2(−kR*) \), cf. [10, Section 4]. By a minimal number of degree \( k \) equations we mean the minimal number of degree \( k \) generators of the homogenous ideal \( \mathcal{I}_b \).

**Example 6.10.** In Example 3.5 we pick \( d^{i_1} = d^5 \) and \( d^{i_2} = d^4 \) (\( \ell_1 = \ell_2 = 2 \) and \( n_1 = 2, n_2 = 3 \) in this case). Thus we have \( T_b = \{ T_{21}, T_{31}, T_{32} \} \) and \( \mathcal{I}_b = (T_{21}T_{32}) \). By Proposition 6.9 we see that \( \dim_{\mathbb{C}} T_X^2(−3R*) = 1 \) and \( \dim_{\mathbb{C}} T_X^2(−kR*) = 0 \) for \( k \neq 3 \). We indeed have one generator of \( \mathcal{I}_b \) that is of degree 3: it is equal to \( T_{21}T_{32}(\deg T_{21} = 1 \text{ and } \deg T_{32} = 2) \), cf. (23).

**Remark 6.11.** In this remark we sketch the proof that the number of degree \( k \) equations of \( \mathcal{B} \) is indeed less or equal than \( \dim_{\mathbb{C}} T_X^2(−kR*) \). Let us fix two linearly independent vectors \( b_1, b_2 \in M \cong \mathbb{Z}^2 \) such that \( d(b_1) = n_1 \) and \( d(b_2) = n_2 \). We know that \( \mathcal{I}_B = (p_{d,\mu\in\mathbb{C}}(\mathcal{T}) \mid k \in \mathbb{N}, \mu \in \mathbb{M}) \). From the equality appearing in Lemma 5.3 we can see that only two linearly independent elements \( M \cong \mathbb{Z}^2 \) are enough to generate this ideal and thus

\[
\mathcal{I}_B = (p_{d,\mu\in\mathbb{C}}(\mathcal{T}), p_{d,\mu\in\mathbb{C}}(\mathcal{T}) \mid k \in \mathbb{N} \subset \mathbb{C}[\mathcal{T}], \text{ for } k \in \mathbb{N}.
\]

Since we pick \( b_1, b_2 \in M \cong \mathbb{Z}^2 \) such that \( d(b_1) = n_1 \) and \( d(b_2) = n_2 \) we can easily see that our claim on the minimal number of equations of \( \mathcal{B} \) follows.

6.3. **Higher-dimensional case.** Recall the notation \( p^{(k)}(\mu, c) \in \mathcal{I}_b \) from (29). Note that by the construction the polynomials \( p^{(k)}(\mu, c) \) are generating \( \mathcal{I}_b \) for all \( k \in \mathbb{N} \), closed paths \( \mu \) and \( c \in \mathbb{M} \).

**Proposition 6.12.** For \( k \in \mathbb{N} \) and a closed path \( \mu \) let

\[
c \in \bigcap_{d^{i}:\ell(d^{i}) \geq k, \mu_i \neq 0} (d^{i})^\perp,
\]

where the intersection is taken over those edges of lattice length \( \geq k \) that lie on the path \( \mu \). It holds that \( p^{(k)}(\mu, c) \) is in the image of the map \( \psi^{(k)} \) from Proposition 5.4. Moreover, if

\[
c \notin \bigcap_{d^{i}:\ell(d^{i}) \geq k, \mu_i \neq 0} (d^{i})^\perp,
\]

then \( p^{(k)}(\mu, c) = 0 \in \mathcal{W}_k \). In particular, the map \( \psi^{(k)} \) is surjective.

**Proof.** If \( c \in \bigcap_{d^{i}:\ell(d^{i}) \geq k, \mu_i \neq 0} (d^{i})^\perp \), we can write

\[
c = \sum_{j=1}^{r} q_{i,j} c_j + q_i(0,1),
\]

where \( q_{i,j} \neq 0 \) implies that \( (c_j, \eta(c_j)) \in \mathcal{E}^{kR*} \) (as in (34)). Then the same procedure as in Subsection 6.1 implies that \( p^{(k)}(\mu, c) \) is in the image of \( \psi^{(k)} \).

Fix a basis \( T_b \subset \mathcal{T} \) of \( T_0 \mathcal{B} \). If

\[
c \notin \bigcap_{d^{i}:\ell(d^{i}) \geq k, \mu_i \neq 0} (d^{i})^\perp,
\]
then denote by $d_{\nu_1}, \ldots, d_{\nu_m}$ linearly independent edges of lattice length $\geq k$ such that $T_{\nu_1 k}, \ldots, T_{\nu_m k} \not\in T_b$ and that they lie on the path $\mu$, i.e. $\mu_{\nu_i} \neq 0$ for $i = 1, \ldots, m$. Clearly such edges of $P$ exist by definition of $T_b$. By our assumption, there exists $d_{\nu_i}$ such that $\langle c, d_{\nu_i} \rangle \neq 0$. Thus $p^{(k)}(\mu, c) = 0 \in W_k$ since the coefficient in front of $T_{\nu_i k}$ in $p^{(k)}(\mu, c)(T)$ is non-zero. 

**Corollary 6.13.** The obstruction map is injective since $\psi$ is its adjoint map.

7. Reduced miniversal base space and Minkowski decompositions of $P$

By construction it is natural to expect that the irreducible components of the reduced base space $B_{\text{red}}$ correspond to maximal Minkowski decompositions of $P$.

The computations are straightforward so we omit some computations and only give the main ideas and compute some examples. For each Minkowski decomposition $P = P_0 + \cdots + P_m$ we first define the map

$$f : \mathbb{C}[u_1, \ldots, u_n] / \mathcal{I}_B \rightarrow \mathbb{C}[K_0, \ldots, K_m]$$

in the following way. The Minkowski decomposition induces the decomposition of each edge: $\ell(d^i) = l_i = \sum_{k=0}^{m} n_{ik}$, where $n_{ik} \in \mathbb{N}$ is the length of the part of the edge $d^i$ that lies in $P_k$. We define

$$u_i \mapsto \prod_{k=0}^{m} K_{ik}^{\nu_{ik}}$$

for which it holds that $f(\mathcal{I}_B) = 0$ and thus $f$ is well defined.

Recall that $\mathcal{I}_B$ is the ideal of $B \subset \text{Spec}[T]$ and let $p_{ik} \in \{0, 1, \ldots, n_{ik}\}$. We define the map

$$g : \mathbb{C}[T] / \mathcal{I}_B \rightarrow \mathbb{C}[K_i - K_j \mid i, j \in \{0, 1, \ldots, m\}]$$

by

$$T_{ij} \mapsto \sum_{p_{ik} : \sum_{k=0}^{m} p_{ik} = j} \binom{n_{i0}}{p_{i0}} \cdots \binom{n_{im}}{p_{im}} (K_0 - f(u_0))^{p_{i0}} \cdots (K_m - f(u_0))^{p_{im}},$$

see also Example 7.2. We need to show that $g$ is well defined: it is enough to show that

$$f(u_i) = f(u_0)^{l_i} + \sum_{j=1}^{l_i} f(u_0)^{l_i-j} g(T_{ij}).$$

The monomial $f(u_i) = \prod_{k=0}^{m} K_{ik}^{\nu_{ik}}$ appears on the right hand side of (41) if we pick $p_{ik} = n_{ik}$ for all $k = 0, \ldots, m$. We need to show that everything else sums to zero. First, observe that

$$\sum_{p_{ik} : \sum_{k=0}^{m} p_{ik} = j} \binom{n_{i0}}{p_{i0}} \cdots \binom{n_{im}}{p_{im}} = \binom{l_i}{j}.$$

Next, we observe that

$$f(u_0)^{l_i} + \sum_{j=1}^{l_i} f(u_0)^{l_i-j} \binom{l_i}{j} (-f(u_0))^j = 0$$

and similarly we check that the other terms also sums in zero, which proves (41). Thus $g$ is well defined and this map also establish the correspondence between the maximal Minkowski decompositions of $P$ and the irreducible components of $B_{\text{red}}$. 

Remark 7.1. By [2] we know that the Minkowski decomposition \( P = P_0 + \cdots + P_m \) induces \( m \)-parameter flat family \( \mathcal{M} \to \text{Spec} \, \mathbb{C}[K_i - K_j \mid i, j \in \{0, 1, \ldots, m\}] \). This family is induced via base change from our miniversal deformation \( \bar{X} \to \mathcal{B} \) using the map \( g \).

Example 7.2. Let us verify that the maps \( f \) and \( g \) are well defined for the following decomposition, cf. Example 3.5.

\[
\begin{align*}
\begin{array}{ccc}
\text{d}_1 & \rightarrow & \text{d}_2 \\
\downarrow & & \downarrow \\
\text{d}_5 & \rightarrow & \text{d}_3 \\
\end{array}
\end{align*}
\]

The map \( f \) is described by:

\[
\begin{align*}
&u_1 \mapsto K_0, \quad u_2 \mapsto K_1, \quad u_3 \mapsto K_0K_2, \quad u_4 \mapsto K_0K_1, \quad u_5 \mapsto K_1K_2. \\
\end{align*}
\]

Let us check that \( f(I_0) = 0 \):

\[
\begin{align*}
&u_4 - u_1u_2 \mapsto K_0K_1 - K_0K_1 = 0, \quad u_5u_4 - u_2u_3 \mapsto (K_1K_2)K_0 - K_1(K_0K_2) = 0. \\
\end{align*}
\]

We choose \( u_0 = u_{11} = u_1 \) as usual. We know that \( f(u_1) = K_0 \) and thus the map \( g \) is equal to the following (see (40)):

\[
\begin{align*}
&\begin{array}{cccc}
T_{11} & \mapsto & K_0 - f(u_1) = 0, & T_{21} \mapsto K_1 - K_0, & T_{31} \mapsto (K_0 - f(u_1)) + (K_2 - f(u_1)) = K_2 - K_0, & T_{32} \mapsto 0, \\
T_{41} & \mapsto & K_1 - K_0, & T_{32} \mapsto 0, & T_{51} \mapsto (K_1 - K_0) + (K_2 - K_0), & T_{52} \mapsto (K_1 - K_0)(K_2 - K_0). \\
\end{array}
\end{align*}
\]

We need to show that \( g \) is well defined, which means that the equations of \( \mathcal{B} \) in (22) needs to be 0 after mapping \( T \to K \):

\[
\begin{align*}
&T_{41} - T_{21} \mapsto (K_0 - K_1) - (K_0 - K_1) = 0, \\
&T_{42} \mapsto 0, \\
&T_{51} - T_{21} - T_{31} \mapsto (K_1 - K_0) + (K_2 - K_0) - (K_1 - K_0) - (K_2 - K_0) = 0, \\
&T_{52} - T_{32} - T_{21}T_{31} \mapsto (K_1 - K_0)(K_2 - K_0) - 0 - (K_1 - K_0)(K_2 - K_0) = 0, \\
&T_{21}T_{32} \mapsto 0.
\end{align*}
\]

Let us consider the other Minkowski decomposition:

\[
\begin{align*}
\begin{array}{ccc}
\text{d}_1 & \rightarrow & \text{d}_2 \\
\downarrow & & \downarrow \\
\text{d}_5 & \rightarrow & \text{d}_3 \\
\end{array}
\end{align*}
\]

The map \( f \) is in this case equal to the following:

\[
\begin{align*}
&u_1 \mapsto K_0, \quad u_2 \mapsto K_0, \quad u_3 \mapsto K_1K_2, \quad u_4 \mapsto K_0^2, \quad u_5 \mapsto K_1K_2. \\
\end{align*}
\]

Let us check that \( f(I_0) = 0 \):

\[
\begin{align*}
&u_4 - u_1u_2 \mapsto K_0^2 - K_0K_0 = 0, \quad u_5u_4 - u_2u_3 \mapsto (K_1K_2)K_0 - K_0(K_1K_2) = 0.
\end{align*}
\]
The map \( g \) is in this case equal to the following:

\[
\begin{align*}
T_{11} &\mapsto 0, \\
T_{21} &\mapsto 0, \\
T_{31} &\mapsto (K_1 - K_0) + (K_2 - K_0), \\
T_{32} &\mapsto (K_1 - K_0)(K_2 - K_0), \\
T_{41} &\mapsto 0, \\
T_{42} &\mapsto 0, \\
T_{51} &\mapsto (K_1 - K_0) + (K_2 - K_0), \\
T_{52} &\mapsto (K_1 - K_0)(K_2 - K_0)
\end{align*}
\]

and as above we can check that \( g \) is well defined.

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