Quantum tomography for Dirac spinors

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Abstract

We present a tomographic scheme, based on spacetime symmetries, for the reconstruction of the internal degrees of freedom of a Dirac spinor. We discuss the circumstances under which the tomographic group can be taken as $SU(2)$, and how this crucially depends on the choice of the gamma matrix representation. A tomographic reconstruction process based on discrete rotations is considered, as well as a continuous alternative.

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1 Introduction

There is a long quest on the search of classical-like descriptions of quantum mechanics. As examples, we can mention the approaches of Wigner, Moyal, Feynman and the various tentative hidden variable theories. In the first two cases, a set of (possibly negative) quasiprobability distributions, defined on the phase space, are the basic variables of the theory. In Feynman’s approach, negative probabilities are allowed as a way to avoid the use of (probability) amplitudes. On the other hand, the tomographic formulation of quantum mechanics has received considerable attention in recent years. In such an approach, the dynamical variables of the theory are a set of probability distributions, which have truly classical-like characteristics: they are non-negative, normalized and, in principle, all measurable.

For a review on the principles of the tomographic approach, we refer the reader to. Here we briefly outline its main ideas. Consider a state $|\psi\rangle$ in some Hilbert space $\mathcal{H}$ describing a physical system. Let $\{|v_\alpha\rangle\}$ be an orthonormal basis of $\mathcal{H}$, whose elements are eigenvectors of a commuting set of Hermitian operators. Here, $\alpha$ should be interpreted as a multi-index that might contain discrete and/or continuous indices. Expanding $|\psi\rangle$ in this basis, we have $|\psi\rangle = \sum_\alpha \psi_\alpha |v_\alpha\rangle$, where the complex coefficients $\psi_\alpha$ represent probability amplitudes. The corresponding probabilities $w_\alpha = |\psi_\alpha|^2 = |\langle v_\alpha | \psi \rangle|^2$ are called marginal distributions. Note that $w_\alpha$ are non-negative normalized probabilities which are, in principle, all measurable. The essence of the tomographic approach is to describe the physical state and its dynamics in terms of the marginals.

Of course, the information relative to the phases in $\psi_\alpha$ is lost when we consider the above marginals. Nevertheless, one can consider the action on $\mathcal{H}$ of a family of transformations $U(g)$, labeled by a certain parameter $g$ belonging to a (Lie) group $G$. Defining the “rotated” marginals $w_\alpha(g) = |\langle v_\alpha | U(g) | \psi \rangle|^2$ (which are again measurable in principle) and writing $\psi_\alpha(g) = \sum_\beta U(g)^\dagger_\alpha \beta \psi_\beta$, it follows that the expression of
\[ w_\alpha(g) = |\psi_\alpha(g)|^2 \] carries interference terms among the relative phases of \( \psi_\beta \). As a result, one can find such relative phases in terms of the rotated marginals.

The tomographic schemes are usually written in terms of the density matrix \( \rho \) associated with the physical system. Although in this work we are mainly interested in pure states, this leads to a natural framework to study more general mixed states. Then, the reconstruction process can be implemented by an integral transformation \( \rho = \int \int w_\alpha(g) K(\alpha, g) \), which determines \( \rho \) in terms of the rotated marginals (when \( \alpha \) is a discrete index, the corresponding integral should be replaced by a discrete sum). Some applications of this tomographic scheme can be found in \( [5] \) (optical tomography, with \( G = O(2) \)), \( [7, 8, 13] \) (symplectic tomography, with \( G = Sp(2, \mathbb{R}) \)) and \( [9, 10, 13, 14] \) (spin tomography, with \( G = SU(2) \)). In \( [15] \), the interesting problem of defining a minimum quorum of expectation values for the state reconstruction was addressed. Also, a study of the properties of marginal distributions under relativistic transformations, especially in the context of the relativistic oscillator model, was presented in \( [16] \).

In this work, we present a tomographic scheme for the reconstruction of the internal degrees of freedom of Dirac spinors. These objects are known to describe relativistic spin-1/2 particles, as electrons. More precisely, a Dirac spinor \( |\psi\rangle = (\psi_1 \psi_2 \psi_3 \psi_4)^T \in \mathbb{C}^4 \) is an object carrying the representation \( D^{(1/2, 0)} \oplus D^{(0,1/2)} \) of \( Spin_{\mathbb{C}}^e \cong SL(2, \mathbb{C}) \), the covering group of the restricted Lorentz group. As \( |\psi\rangle \in \mathbb{C}^4 \), a tomographic scheme based on \( SU(4) \) would certainly work for this case. However, to parallel the discussion with the non-relativistic case, and to give a direct physical meaning to the transformations \( U(g) \), we demand that the tomographic group \( G \) be generated by spacetime transformations.

The choice of the gamma matrix representation in the Dirac theory plays a decisive role in this context. In fact, consider the tomographic reconstruction of a generic Dirac spinor \( |\psi\rangle \), with 7 degrees of freedom (discounting a global phase). Let \( \mathcal{L} \) be the restricted Lorentz group and \( \tilde{\mathcal{L}} \) the associated covering group. Then, as we will show later, (i) in the context of the Majorana representation, \( |\psi\rangle \) can be tomographically recovered by taking \( G \) as the \( SU(2) \) rotation subgroup of \( \tilde{\mathcal{L}} \); (ii) in the context of the standard representation, \( |\psi\rangle \) can be tomographically recovered if we take \( G = \tilde{\mathcal{L}} \), but not for \( G = SU(2) \) as in (i); (iii) in the context of the chiral representation, \( |\psi\rangle \) cannot be tomographically reconstructed via spacetime transformations, i.e., even if \( G \) is taken as the whole \( \tilde{\mathcal{L}} \) (unless \( |\psi\rangle \) is a Weyl spinor, corresponding to a massless particle).

It is a well known result that the Lorentz group is a non-compact space which does not admit finite-dimensional unitary representations (except for the trivial one) \( [17] \). This means that boosts inevitably give rise to non-unitary transformations for the spinor space. Although one might live with this situation, it is clearly preferable to work only with rotations, if possible. We see from the discussion above that, among the most common choices for \( \{\gamma^\mu\} \), namely the Majorana, standard and chiral representations, only the first one is compatible with a tomographic procedure based on spatial transformations. In this case, the tomographic group is given by \( SU(2) \). Alternatively, it is also possible to combine the marginals associated with both the standard and chiral representations, so that a tomographic reconstruction based on rotations is similarly achieved.

It should be noted that the discussion above regards the tomographic reconstruction of a full Dirac spinor. If one wants to reconstruct a spinor that is already known to be in the positive energy sector, then clearly less symmetry transformations are required (however, it is well known that this sector is not preserved by time evolution \( [18] \); this is, in fact, a problem of the first quantized Dirac theory).

We observe that a shortcoming of describing a Dirac spinor by means of the marginals \( w_k = |\psi_k|^2 \), when compared to the non-relativistic case, is the lack of a clear interpretation for these objects. In fact, consider the case of the standard representation. The projector associated with positive energy and spin up (in the \( z \)-direction) then reduces to

\[
\frac{1}{2}(I + \gamma_0) \frac{1}{2}(I + i\gamma_1\gamma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

only in the reference frame in which the particle is at rest. When we consider an arbitrary particle in an indefinite state, we must consider the Fourier expansion of \( \psi(x) \) in terms of eigenstates of momentum \( \phi(p) \).
But then, the physical interpretation of the first component of \(\phi(p)\) changes with \(p\). This happens because boosts mix the components \(\psi_k\) of \(|\psi\rangle\). Therefore, the marginals \(w_k = |\psi_k|^2\) do not correspond to an easy-to-describe physical property of the particle.\(^1\) When we consider alternative gamma matrix representations, the interpretation of the marginals are even more unclear.

On the other hand, the bilinear covariants associated with \(|\psi\rangle\) provide another classical-like description of the Dirac theory, in the sense that they are, in principle, measurable tensorial densities.\(^2\) Moreover, it is natural to expect that the manifest covariance of these quantities should somehow favor them over the marginals. For this reason, our plan in this article is to establish a well defined correspondence between the bilinear covariants and the marginals. For this reason, our plan in this article is to establish a well defined correspondence between the bilinear covariants and the marginals. Thus, the above procedure might be useful for obtaining a Dirac equation written in terms of tomographic and classical-like quantities like \(w\) (see also \([24]\)).

This article is organized as follows. In section 2, we review some facts about Dirac spinors, including a reconstruction theorem \([25]\) that allows one to obtain \(|\psi\rangle\) from the bilinear covariants. In section 3, we obtain the aforementioned correspondence between the bilinear covariants and the marginals \(w_k\). This can be considered a generalization of (the non-relativistic) spin tomography techniques presented in \([9, 10, 13, 14]\). The covariance of the bilinear covariants is then explored to tomographically reconstruct the spinor \(|\psi\rangle\) from the marginals. In section 4, we discuss the dependence of this tomographic approach on the choice of the gamma matrix representation. Section 5 is reserved for some final remarks. In what follows, we use natural units \((h = c = 1)\).

## 2 Bilinear covariants

In order to establish notation, let us briefly review some well known facts about the Dirac theory. Let \(|\psi\rangle = (\psi_1, \psi_2, \psi_3, \psi_4)^T\) be a Dirac spinor (in what follows, \(|\psi\rangle\) always represents a pure state). Under a restricted Lorentz transformation \(\Lambda = (A^\mu_\nu)\), this object transforms as \(|\psi'(x)\rangle = L|\psi(\Lambda^{-1}x)\rangle\), where the matrix \(L\) is related to \(A^\mu_\nu\) by \(L^{-1}\gamma^\mu L = \Lambda^\mu_\nu\gamma^\nu\) \([18]\) \((L\) belongs to the covering space \(\hat{\Lambda}\) of the restricted Lorentz group \([26]\)). Denoting the Dirac conjugate of \(|\psi\rangle\) by \(|\bar{\psi}\rangle = |\psi\rangle\gamma_0\), we then have \(\langle \bar{\psi}' | = \langle \bar{\psi} | L^{-1}\). This immediately yields the following 16 tensorial quantities, known as bilinear covariants:

\[
\begin{align*}
\Omega_1 &= \langle \bar{\psi} | \psi \rangle, \\
J^\mu &= \langle \bar{\psi} | \gamma^\mu | \psi \rangle, \\
S^{\mu\nu} &= \langle \bar{\psi} | \gamma^{\mu\nu} | \psi \rangle, \quad \text{with} \quad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu], \\
K^\mu &= \langle \bar{\psi} | \gamma_{0123}\gamma^\mu | \psi \rangle, \quad \text{with} \quad \gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3, \\
\Omega_2 &= \langle \bar{\psi} | -\gamma_{0123} | \psi \rangle,
\end{align*}
\]

which transform respectively as a scalar, a 4-vector, a tensor of second degree, a pseudo-vector and a pseudo-scalar. These quantities are obviously not independent. The constraint relations among them are given by

\(^1\)We note that this situation does not occur in Pauli theory, for a non-relativistic boost (i.e., a Galilean transformation corresponding to a change of velocity between frames) does not mix the components of a Pauli spinor.

\(^2\)The bilinear covariant \(J^\mu = \langle \bar{\psi} | \gamma^\mu | \psi \rangle\) corresponds to the charge density \((eJ^0)\) and electric current density \((e\gamma^iJ^i)\) associated with a Gibbs ensemble of identical particles; \(S^{\mu\nu} = \langle \bar{\psi} | \gamma^{\mu\nu} | \psi \rangle\) corresponds to magnetic \((\frac{e}{2m}\gamma^0S^{12})\) and electric \((\frac{e}{2m}\gamma^iS^{ij})\) moment densities; \(\frac{1}{2}K^\mu = \frac{1}{2}\langle \bar{\psi} | \gamma_{0123}\gamma^\mu | \psi \rangle\) corresponds to the spin density. The scalar and pseudo-scalar bilinear covariants have a less clear interpretation, but the Fierz identities (see section 4) can be used to express them in terms of \(J^\mu\), \(S^{\mu\nu}\) and \(K^\mu\) \([19, 20]\) (see also \([21]\)).

\(^3\)Note, however, that this would still require the reconstruction of the (position-dependent) global phase of \(|\psi\rangle\).
the Fierz identities (we use the conventions of [26]):
\[
\begin{align*}
J_\mu J^\mu &= \Omega_1^2 + \Omega_2^2, \\
J_\mu J^\nu &= -K_\mu K^\nu, \\
J_\mu K^\nu &= 0, \\
J_\mu K_\nu - K_\mu J_\nu &= -(\Omega_2 S_{\mu\nu} + \Omega_1 \frac{1}{2} \epsilon_{\mu\alpha\beta} S^{\alpha\beta}),
\end{align*}
\]
with $\epsilon_{0123} = 1$. These 9 equations (note the anti-symmetry in $\mu\nu$) reduce the number of independent bilinear covariants to 7, as expected. Indeed, $|\psi\rangle$ has eight real components which reduce to seven independent quantities when a global phase is discarded.

Let us denote a bilinear covariant generically by $\rho^a = \langle \tilde{\psi}|\Gamma^a|\psi\rangle$, where $\Gamma^a$ can be read from eqs. (1) (we note that the 16 matrices $\Gamma^a$ form a basis for the Dirac algebra). As the bilinear covariants are real, the quantity
\[
\rho := \sum_a \rho^a \Gamma_a
\]
(3)
can be thought of as a vector in a 16-dimensional real vector space. In this context, the Fierz identities determine a 7-dimensional submanifold of $\mathbb{R}^{16}$ in which $\rho$ lives in [27]. This submanifold generalizes the Bloch sphere of Pauli theory.

It follows from eq. (3) that
\[
\rho = \Omega_1 + J + i S + i K \gamma_{0123} + \Omega_2 \gamma_{0123},
\]
where $J = J^\mu \gamma_\mu$, $S = \frac{1}{2} S^{\mu\nu} \gamma_{\mu\nu}$ and $K = K^{\mu} \gamma_\mu$. It is also useful to note that there is a natural inner product defined on the Dirac algebra by $(A, B) = (1/4) \text{tr}(AB)$, where $\bar{B} := \gamma_0 B^{\dagger} \gamma_0$ is the Dirac conjugate of $B$. Note that $\rho^a = \langle \tilde{\psi}|\Gamma^a|\psi\rangle = \text{tr}(\Gamma^a |\psi\rangle \langle \tilde{\psi}|)$, so that we can alternatively write
\[
\rho = 4|\psi\rangle \langle \tilde{\psi}|.
\]

A reconstruction theorem [27] can be used to obtain $|\psi\rangle$, apart from a global phase, from the bilinear covariants, i.e., from $\rho$. To see this, consider the action of $\rho$ on a fixed spinor $|\eta\rangle$ (usually taken as $|\eta\rangle = (1 0 0 0)^t$). This gives $\rho|\eta\rangle = 4|\psi\rangle \langle \tilde{\psi}|\eta\rangle$, and so $|\psi\rangle = \frac{1}{4\langle \tilde{\psi}|\eta\rangle} \rho|\eta\rangle$. As $\langle \eta|\rho|\eta\rangle = 4|\psi|\eta\rangle|^2$, we have $|\psi\rangle = e^{i\phi}|\tilde{\psi}|\eta\rangle| = e^{i\phi} \frac{1}{4\langle \eta|\rho|\eta\rangle} |\psi|\eta\rangle|^{1/2}$. Substitution in the expression for $|\psi\rangle$ yields
\[
|\psi\rangle = \frac{e^{-i\phi}}{\sqrt{4\langle \eta|\rho|\eta\rangle}} \rho|\eta\rangle.
\]
This can be brought to a simpler form if we rescale the bilinear covariants by $R = \omega \rho$, with $\omega = (4\langle \eta|\rho|\eta\rangle)^{-1/2}$. As a result, we can write
\[
|\psi\rangle = e^{-i\phi} R|\eta\rangle.
\]

3 Marginal distributions and bilinear covariants

Let us now relate the marginal distributions to the bilinear covariants. To do that, we expand $|\psi\rangle$ in terms of the canonical basis $\{|v_k\rangle\}_{k=1}^4$ of $\mathbb{C}^4$, with $|v_1\rangle = (1 0 0 0)^t$, $|v_2\rangle = (0 1 0 0)^t$ and so on. If $|\psi\rangle = \sum \psi_k |v_k\rangle$, we have
\[
w_k = |\psi_k|^2, \quad k = 1, 2, 3, 4.
\]

Consider the matrices (or projection operators) given by
\[
P_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad P_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad P_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Then
\[
w_k = \langle \psi|P_k|\psi\rangle = \langle \tilde{\psi}|\gamma_0 P_k|\psi\rangle, \quad k = 1, 2, 3, 4.
\]
To go further, we need to choose a specific representation of the gamma matrices to work with. In the Majorana representation \( \mathbf{13} \) (other choices will be discussed shortly)

\[
\gamma_{0}^{m,j} = \begin{pmatrix} 0 & \sigma_2 & 0 \\ \sigma_2 & 0 & 0 \\ 0 & 0 & i\sigma_1 \end{pmatrix}, \quad \gamma_{1}^{m,j} = \begin{pmatrix} -i\sigma_3 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ i\sigma_3 & 0 & 0 \end{pmatrix}, \quad \gamma_{2}^{m,j} = \begin{pmatrix} 0 & \sigma_2 & 0 \\ -\sigma_2 & 0 & 0 \\ 0 & 0 & i\sigma_1 \end{pmatrix}, \quad \gamma_{3}^{m,j} = \begin{pmatrix} i\sigma_1 & 0 & 0 \\ 0 & 0 & i\sigma_1 \\ 0 & i\sigma_1 & 0 \end{pmatrix},
\]

where \( \sigma_k \) are the Pauli matrices, a straightforward calculation yields

\[
P_1 = \frac{1}{2}(1 + \gamma_{20}^{m,j}) \frac{1}{2}(1 + i\gamma_{1}^{m,j}), \quad P_2 = \frac{1}{2}(1 + \gamma_{20}^{m,j}) \frac{1}{2}(1 - i\gamma_{1}^{m,j}),
\]

\[
P_3 = \frac{1}{2}(1 - \gamma_{20}^{m,j}) \frac{1}{2}(1 + i\gamma_{1}^{m,j}), \quad P_4 = \frac{1}{2}(1 - \gamma_{20}^{m,j}) \frac{1}{2}(1 - i\gamma_{1}^{m,j}).
\]

Substitution in eq. \( \text{1} \) leads to

\[
w_{m,j,k} = (\bar{\psi}|\frac{1}{4}(\gamma_{0}^{m,j} \pm \gamma_{2}^{m,j} \pm i\gamma_{01}^{m,j} \pm i\gamma_{12}^{m,j})|\psi) = \frac{1}{4}(J_0 \pm J_2 \pm S_{01} \pm S_{12}),
\]

where the \( \pm \) signs vary with \( k \). More precisely

\[
w_{m,j,1} = \frac{1}{4}(J_0 - J_2 + S_{01} + S_{12}), \quad w_{m,j,2} = \frac{1}{4}(J_0 - J_2 - S_{01} - S_{12}), \quad w_{m,j,3} = \frac{1}{4}(J_0 + J_2 + S_{01} - S_{12}), \quad w_{m,j,4} = \frac{1}{4}(J_0 + J_2 - S_{01} + S_{12}).
\]

After solving for \( J_0, J_2, S_{01} \) and \( S_{12} \), we have

\[
J_0 = w_{m,j,1} + w_{m,j,2} + w_{m,j,3} + w_{m,j,4}, \\
J_2 = -w_{m,j,1} - w_{m,j,2} + w_{m,j,3} + w_{m,j,4}, \\
S_{01} = w_{m,j,1} - w_{m,j,2} - w_{m,j,3} + w_{m,j,4}, \\
S_{12} = w_{m,j,1} - w_{m,j,2} - w_{m,j,3} + w_{m,j,4}.
\]

The result is a partial recovering of \( \rho \) in terms of the marginal distributions. Now we explore the symmetries of the Lorentz group to obtain the full expression for \( \rho \), still in terms of marginal distributions.

Applying a restricted Lorentz transformation \( \Lambda = (\Lambda_{\mu}^{\nu}) \) to our system, we have \( |\psi^\prime\rangle = L|\psi\rangle \). The corresponding \( \Lambda \)-dependent marginal distributions (cf. eq. \( \text{6} \)) are:

\[
w_{k}^{(A)} = (\bar{\psi}|\gamma_{0}^{(A)} P_{k}^{(A)} L|\psi). \tag{9}
\]

On the other hand, the new bilinear covariants are given by \( \Omega_{1}^{(A)} = \Omega_{1}, \quad J_{0}^{(A)} = \Lambda_{0}^{\nu} J_{\nu}, \quad S_{\mu}^{(A)} = \Lambda_{\alpha}^{\mu} \Lambda_{\nu}^{\beta} S_{\alpha\beta}, \quad R_{\mu}^{(A)} = \Lambda_{\mu}^{\nu} R_{\nu}, \quad \Omega_{2}^{(A)} = \Omega_{2} \) (in the above notation, \( w_{m,j,k} = w_{m,j,k}^{(l)} \), \( J_{\mu} = J_{\mu}^{(l)} \), and so on denote the quantities associated with the original frame, before the application of the symmetry transformation). It follows that

\[
w_{m,j,k}^{(A)} = (\bar{\psi}|L^{-1} \frac{1}{4}(\gamma_{0}^{m,j} \pm \gamma_{2}^{m,j} \pm i\gamma_{01}^{m,j} \pm i\gamma_{12}^{m,j}) L|\psi) = \frac{1}{4}(J_{0}^{(A)} \pm J_{2}^{(A)} \pm S_{01}^{(A)} \pm S_{12}^{(A)}),
\]

where the \( \pm \) signs vary with \( k \). More precisely,

\[
J_{0}^{(A)} = w_{m,j,1}^{(A)} + w_{m,j,2}^{(A)} + w_{m,j,3}^{(A)} + w_{m,j,4}^{(A)}, \\
J_{2}^{(A)} = -w_{m,j,1}^{(A)} - w_{m,j,2}^{(A)} + w_{m,j,3}^{(A)} + w_{m,j,4}^{(A)}, \\
S_{01}^{(A)} = w_{m,j,1}^{(A)} - w_{m,j,2}^{(A)} - w_{m,j,3}^{(A)} + w_{m,j,4}^{(A)}, \\
S_{12}^{(A)} = w_{m,j,1}^{(A)} - w_{m,j,2}^{(A)} - w_{m,j,3}^{(A)} + w_{m,j,4}^{(A)}.
\]

Now we can vary \( \Lambda \) in the above expressions to recover all the bilinear covariants:

(a) taking \( \Lambda = I = \text{identity} \) (i.e. no symmetry transformation), we determine (from eqs. \( \text{10} \) or eqs. \( \text{8} \)) \( J_{0}, J_{2}, S_{01} \) and \( S_{12} \) in terms of \( w_{m,j,k} \):

(b) taking \( \Lambda = R_{x} = [\pi/2 \text{-rotation about the } x\text{-axis}], \) we have \( J_{0} = J_{0}^{(R_{x})}, \quad S_{01} = S_{01}^{(R_{x})} \) and \( J_{3} = J_{2}^{(R_{x})}, \quad S_{31} = -S_{12}^{(R_{x})} \). All these quantities are determined by the marginals \( w_{m,j,k}^{(R_{x})} \) from eqs. \( \text{10} \);

(c) taking \( \Lambda = R_{y} = [\pi/2 \text{-rotation about the } y\text{-axis}], \) we analogously obtain \( J_{0} = J_{0}^{(R_{y})}, \quad J_{2} = J_{2}^{(R_{y})} \) and \( S_{03} = -S_{01}^{(R_{y})}, \quad S_{23} = S_{12}^{(R_{y})} \). All these quantities are determined by the marginals \( w_{m,j,k}^{(R_{y})} \) from eqs. \( \text{10} \);

(d) taking \( \Lambda = R_{z} = [\pi/2 \text{-rotation about the } z\text{-axis}], \) we analogously obtain \( J_{0} = J_{0}^{(R_{z})}, \quad S_{12} = S_{12}^{(R_{z})} \) and \( J_{1} = -J_{2}^{(R_{z})}, \quad S_{02} = S_{01}^{(R_{z})} \). All these quantities are determined by the marginals \( w_{m,j,k}^{(R_{z})} \) from eqs. \( \text{10} \).
So far, we have recovered the ten bilinear covariants $J_\mu$ and $S_{\mu\nu}$. The rest of them, namely $\Omega_1$, $\Omega_2$ and $K_\mu$, are easily obtained from the Fierz identities. In fact, eqs. (9) yield the identities (20)

$$\Omega_1 K_\nu = J^\mu (\ast S)_{\mu\nu}, \quad \Omega_2 K_\nu = -J^\mu S_{\mu\nu},$$

where $\ast S_{\mu\nu} = -\frac{1}{8} g_{\mu\nu} \omega_{\alpha\beta} S^{\alpha\beta}$, with $\epsilon_{0123} = 1$. In this way, all the bilinear covariants are obtained from the rotated marginals.

Writing everything in terms of the rotated marginals, it follows from (a)-(d) above and eqs. (10) that:

$$J_0 = w_{m,1} + w_{m,2} + w_{m,3} + w_{m,4},$$
$$J_1 = w_{m,1} + w_{m,2} - w_{m,3} - w_{m,4},$$
$$J_2 = -w_{m,1} - w_{m,2} + w_{m,3} + w_{m,4},$$
$$J_3 = -w_{m,1} - w_{m,2} + w_{m,3} + w_{m,4},$$
$$S_{01} = w_{m,1} - w_{m,2} + w_{m,3} - w_{m,4},$$
$$S_{02} = -w_{m,1} + w_{m,2} - w_{m,3} + w_{m,4},$$
$$S_{03} = -w_{m,1} - w_{m,2} + w_{m,3} + w_{m,4},$$
$$S_{12} = w_{m,1} - w_{m,2} - w_{m,3} - w_{m,4},$$
$$S_{23} = w_{m,1} - w_{m,2} + w_{m,3} - w_{m,4},$$
$$S_{31} = -w_{m,1} + w_{m,2} + w_{m,3} - w_{m,4}. \tag{11}$$

As we mentioned above, these quantities determine all the bilinear covariants, and thus reconstruct the spino $|\psi\rangle$ as in the previous section. Moreover, the 6 relations $J_0 = J_0^{(R_0)} = J_0^{(R_4)} = J_0^{(R_5)}$, $S_{01} = S_{01}^{(R_4)}$, $J_2 = J_2^{(R_5)}$, $S_{12} = S_{12}^{(R_5)}$ in (b)-(d) yield 6 constraint equations among the 16 marginals above. This can be used to reduce the number of marginals in eqs. (11).

It is important to note that, in the above reconstruction of $|\psi\rangle$, we did not employ boosts. Indeed, the relevant tomographic group was generated by the $\Lambda$’s in the rotation subgroup $SO(3)$ of the Lorentz group $\mathcal{L}$. This corresponds to elements $L$ in a $SU(2)$ subgroup of the associated covering group $\tilde{\mathcal{L}} = Spin^+_1(3) \cong SL(2, \mathbb{C})$.

### 3.1 A continuous alternative

From our previous discussion, we see that a crucial step to the tomographic recovering process is to reconstruct a vector $v \in \mathbb{R}^3$ if one of its components is known in all frames. Let us fix a reference frame $K$ and let us denote the $K$-components of $v$ by $(v', v'', v^3)$. Suppose we know the third component of $v$ in all frames. Given another reference frame $K'$, if $\theta$ and $\varphi$ are the polar and azimuthal angles of $v'$ in relation to $K$, we have

$$e'_3(\theta, \varphi) = \sin \theta \cos \varphi \ e_1 + \sin \theta \sin \varphi \ e_2 + \cos \theta \ e_3.$$  

Let $\nu(\theta, \varphi)$ be the third $K'$-component of $v$, i.e. $\nu(\theta, \varphi) = v \cdot e'_3$. Then, we can reconstruct $v$ from $\nu(\theta, \varphi)$ by at least two procedures:

- **(I) Discrete method** As $e_3'(\pi/2, 0) = e_1$, $e_3'(\pi/2, \pi/2) = e_2$ and $e_3'(0, 0) = e_3$ we have $v^1 = \nu(\pi/2, 0)$, $v^2 = \nu(\pi/2, \pi/2)$ and $v^3 = \nu(0, 0)$. Thus, $v = \nu(\pi/2, 0)e_1 + \nu(\pi/2, \pi/2)e_2 + \nu(0, 0)e_3$. This is the reconstruction method we used in the previous section.

- **(II) Continuous method** This method goes along the lines of (9, 10), in which all the directions $(\theta, \varphi)$ are considered. The idea is to recover $v$ by an integral transformation of $\nu(\theta, \varphi) : v = \int\! dt\! d\Omega \mathbf{A}(\theta, \varphi) \nu(\theta, \varphi)$, where $d\Omega = \sin \theta d\theta d\varphi$ is the solid angle element on the sphere. There is a lot of ambiguity in choosing the kernel $\mathbf{A}(\theta, \varphi)$, but a simple choice is given by $\mathbf{A}(\theta, \varphi) = \left( \frac{2}{\pi} \cos \phi, \frac{2}{\pi} \sin \phi, \frac{3}{\pi} \cos \theta \right)$.

Each of the methods above lead to a different set of tomographic quantities describing the spinor. In the previous section, we employed a discrete method. On the other hand, if the continuous method is employed, the spinor would be described in terms of continuous variables analogous to $\nu(\theta, \varphi)$.
4 On the choice of the gamma matrix representation

In this section, we discuss the dependence of the tomographic approach developed above on the choice of the gamma matrix representation. Consider the expression \( w_k^{(A)} \) for \( w_k^{(A)} \) in terms of the projection operators in eqs. (12):

\[
w_k^{(A)} = \langle \bar{\psi} | L^{-1} \gamma_0 P_k L | \psi \rangle.
\]

Of course, the functional dependence of \( P_k \) in terms of \( \gamma_\mu, \mu = 0, 1, 2, 3 \), depends on the particular choice of the gamma matrix representation. For the Majorana representation, this is given by eqs. (14). A straightforward calculation shows that the analogous expressions for the standard and chiral representations are

\[
P_k^{st} = \frac{1}{2} (1 \pm \gamma_0) \frac{1}{2} (1 \pm i \gamma_{12}),
\]

\[
P_k^{ch} = \frac{1}{2} (1 \pm \gamma_0) \frac{1}{2} (1 \pm i \gamma_{0123}),
\]

where the \( \pm \) signs vary with \( k \). It follows from eq. (12) that

\[
w_{st, k}^{(A)} = \frac{1}{4} \left[ J_0^{(A)} \pm J_0^{(A)} \pm S_{12}^{(A)} \pm K_3^{(A)} \right],
\]

\[
w_{ch, k}^{(A)} = \frac{1}{4} \left[ J_0^{(A)} \pm J_3^{(A)} \pm K_0^{(A)} \pm K_3^{(A)} \right].
\]

For the standard representation, we see from eq. (14a) that, by performing rotations, we can recover \( \Omega_1, J_0, S_{23}, S_{31}, S_{12}, K_1, K_2, K_3, K_3 \) from the marginals. Unfortunately, these bilinear covariants apparently do not suffice to entirely recover a generic Dirac spinor, with 7 degrees of freedom (discounting a global phase). This is more easily seen from the form of the generators of rotations (associated with the standard representation):

\[
i \gamma_{jk}^{st} = \frac{i}{2} \delta_{jl} \left( \begin{array}{cc} \sigma_l & 0 \\ 0 & \sigma_l \end{array} \right), \quad (jkl) \text{ cyclic}, \ l = 1, 2, 3.
\]

It follows that rotations do not mix the first two components (i.e. \( \psi_1 \) and \( \psi_2 \)) of \( |\psi\rangle = \langle \psi_1 | \psi_2 | \psi_3 | \psi_4 \rangle \) with the last ones (i.e. \( \psi_3 \) and \( \psi_4 \)). In this way, the relative phase between the first and the last set of components of \( |\psi\rangle \) cannot be recovered solely with rotations. On the other hand, if boosts were allowed to reconstruct the spinor, then we could also obtain \( J_1, J_2, J_3 \) (from \( J_0 \)), \( S_{01}, S_{02}, S_{03} \) (from \( S_{23}, S_{31}, S_{12} \)) and \( K_0 \) (from \( K_3 \)). This would certainly reconstruct the spinor. Therefore, we have shown that, in the standard representation, the state can be reconstructed by means of restricted Lorentz transformations, but not through rotations (cf. introduction).

For the chiral representation, we see from eq. (14b) that even if the whole Lorentz group is used, we can only recover \( J_\mu \) and \( K_\mu, \mu = 0, 1, 2, 3, \) from the marginals. This is not sufficient to reconstruct all the bilinear covariants for a generic Dirac spinor, with 7 degrees of freedom (discounting a global phase). In fact, we know from the Fierz identities that \( J_\mu J_\mu = -K^K_\mu K_\mu \) and \( J_\mu K_\mu = 0 \), and thus there are only 6 independent quantities in \( J_\mu \) and \( K_\mu \). This situation changes if \( |\psi\rangle \) is a Weyl spinor, corresponding to a massless spin-1/2 particle. In that case, we always have \( \Omega_1 = \Omega_2 = 0 \) and \( S = 0 \) and then the reconstruction process could proceed as before. \(^4\) Note that the marginals \( w_{ch, k} \) are associated with the probability of finding the particle with positive/negative chirality and spin up/down (in the z-direction). As a massless particle has definite chirality, the knowledge of the \( w_{ch, k}, k = 1, 2, 3, 4 \), would be enough information to recover its state. But, as we have seen, this is not true for a massive particle. This shows our claim (see introduction) that, in the chiral representation, the state of a massive particle cannot be reconstructed from spacetime symmetries, i.e., with \( G \) contained in \( \mathcal{L} \).

Therefore, unlike the Majorana representation, neither the standard nor the chiral representations can be isolatedly used in such a tomographic scheme, based on rotations, for a generic Dirac spinor. A way out of this difficulty is to combine the marginals coming from both the standard and the chiral transformations. We see from the above discussion that, by performing rotations, we can then recover \( \Omega_1, J_\mu, S_{23}, S_{31}, S_{12}, S_{01} \) and \( K_\mu \). These 12 bilinear covariants determine the rest of them through the Fierz identities, and we can proceed as before.

\(^4\)We also note that \( G \) (i) for a Dirac spinor, \( \Omega_1 \) and \( \Omega_2 \) are not both zero, (ii) for a Weyl spinor, \( \Omega_1 = \Omega_2 = 0 \) and \( S_{\mu \nu} = 0 \) and (iii) for a Majorana spinor, \( \Omega_1 = \Omega_2 = 0 \) and \( K_\mu = 0 \).
The general case

It is well known that an arbitrary representation \( \{ \gamma_\mu \} \) of the gamma matrices can be written as \( \gamma_\mu = U \gamma^m_\mu U^{-1} \), where \( \gamma^m_\mu \) corresponds to the standard representation and \( U \) is an unitary matrix. It follows from eq. (13a) that, in terms of the new gamma matrices \( \{ \gamma_\mu \} \):

\[
P_k = \frac{1}{2} (1 \pm u) \frac{1}{2} (1 \pm i \sigma),
\]

where \( u = U^{-1} \gamma_0 U \) and \( \sigma = U^{-1} \gamma_12 U \). The discussion above shows that a tomographic scheme, based on spacetime transformations, works for the representation \( \{ \gamma_\mu \} \) only if the bilinear covariants associated with \( \gamma_0, \gamma_0 u, \gamma_0 \sigma \) and \( \gamma_0 u \sigma \) are independent enough to reconstruct the spinor. It may happen that such a reconstruction is possible using only spatial rotations (as in the Majorana representation), or using necessarily boosts and rotations (as in the standard representation), or even not possible inside the Lorentz group (as in the chiral representation, with massive particles).

5 Conclusion remarks

We have presented a tomographic scheme, based on spacetime transformations, for the reconstruction of the internal degrees of freedom of a Dirac spinor. The assumption that the tomographic group \( G \) is generated by spacetime transformations was shown to restrict the choice of the gamma matrices. The cases of standard, chiral and Majorana representations were studied in detail. We also analyzed under what conditions \( G \) can be taken as \( SU(2) \). A direct tomographic process based on discrete rotations was considered, as well as a continuous alternative. Finally, as we mentioned in the introduction, the method considered here might be useful for obtaining an analogue of the Dirac equation in terms of tomographic quantities.

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