KAZHDAN–LUSZTIG CELLS AND THE FROBENIUS–SCHUR INDICATOR

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Abstract. Let $W$ be a finite Coxeter group. It is well-known that the number of involutions in $W$ is equal to the sum of the degrees of the irreducible characters of $W$. Following a suggestion of Lusztig, we show that this equality is compatible with the decomposition of $W$ into Kazhdan–Lusztig cells. The proof uses a generalisation of the Frobenius–Schur indicator to symmetric algebras, which may be of independent interest.

1. Introduction

Let $G$ be a finite group and assume that all complex irreducible characters of $G$ can be realised over the real numbers. Then, by a well-known result due to Frobenius and Schur, the number of involutions in $G$ (that is, elements $g \in G$ such that $g^2 = 1$) is equal to the sum of the degrees of the irreducible characters of $G$.

In this note, we consider the case where $G = W$ is a finite Coxeter group. Following a suggestion of Lusztig, we show that the above equality is compatible with the decomposition of $W$ into cells, as defined by Kazhdan and Lusztig [10] (in the equal parameter case) and by Lusztig [11] (in general). The proof relies on two basic ingredients. The first consists of establishing a suitable generalisation of the “Frobenius–Schur indicator” to symmetric algebras. This will be done in Section 2, and may be of independent interest. The second ingredient is the theory around Lusztig’s ring $J$ (originally introduced in [13]) or, rather, its more elementary version constructed in [3]; see Section 3.

To state the main result, let us fix some notation. Let $S$ be a set of simple reflections in $W$. Let $\{c_s \mid s \in S\} \subseteq \mathbb{Z}_{\geq 0}$ be a set of “weights” where $c_s = c_{s'}$ whenever $s, s' \in S$ are conjugate in $W$. This gives rise to a weight function $L : W \to \mathbb{Z}$ in the sense of Lusztig [15]: for $w \in W$, we have $L(w) = c_{s_1} + \cdots + c_{s_k}$ where $w = s_1 \cdots s_k$ ($s_i \in S$) is a reduced expression for $w$. (The original setup in [10] corresponds to the case where $c_s = 1$ for all $s \in S$.) Using the Kazhdan–Lusztig basis of the generic Iwahori–Hecke algebra associated with $W, L$, one can define partitions of $W$ into left, right and two-sided cells. For any such left cell $\Gamma$ of $W$, we have a corresponding left $W$-module $[\Gamma]_1$ with a standard basis indexed by the elements of $\Gamma$; see [10] (equal parameter case) or [11] (in general).

Theorem 1.1. The number of involutions in a left cell $\Gamma$ is equal to the number of terms in a decomposition of $[\Gamma]_1$ as a direct sum of simple $W$-modules.

For $W$ of classical type and the equal parameter case, the above result (in a somewhat more precise form, see Example 3.13 below) was first obtained by Lusztig [12].
12.17, using the representation theory of a finite reductive group with Weyl group \( W \). Our proof works uniformly for all \( W, L \) (including \( W \) of non-crystallographic type). In Corollary \( \text{[3.12]} \), we also obtain a similar result for two-sided cells. Along the way, we establish some properties of left cell modules which previously were only known to hold in the equal parameter case; see Corollaries \( \text{[3.8]} \) and \( \text{[3.9]} \).

2. Symmetric algebras and the Frobenius–Schur indicator

Let \( K \) be a field of characteristic 0 and \( \mathcal{H} \) be a finite-dimensional associative \( K \)-algebra (with 1). We assume that \( \mathcal{H} \) is split semisimple and symmetric, with trace form \( \tau: \mathcal{H} \to K \). Let \( \text{Irr}(\mathcal{H}) \) be the set of simple \( \mathcal{H} \)-modules (up to isomorphism). For \( E \in \text{Irr}(\mathcal{H}) \), let \( \chi_E: \mathcal{H} \to K \) be the corresponding character, \( \chi_E(h) = \text{trace}(h,E) \) for all \( h \in \mathcal{H} \). We have

\[
\tau = \sum_{E \in \text{Irr}(\mathcal{H})} c_E^{-1} \chi_E
\]

where each \( c_E \) is a certain non-zero element of \( K \), called the Schur element associated with \( E \). (We refer to \([8, \text{Chap. 7}]\) for basic facts about symmetric algebras.)

We shall further assume that there is a \( K \)-linear anti-involution \( \dagger: \mathcal{H} \to \mathcal{H}, \ h \mapsto h^\dagger \). This allows us to define, for any finite-dimensional (left) \( \mathcal{H} \)-module \( M \), a corresponding contragredient module \( \hat{M} \). As a \( K \)-vector space, we have \( \hat{M} = \text{Hom}_K(M,K) \); the action of \( h \in \mathcal{H} \) on \( f \in \hat{M} \) is determined by \( (h.f)(m) = f(h^\dagger.m) \) for all \( m \in M \).

Definition 2.1. Let \( M \) be a finite-dimensional (left) \( \mathcal{H} \)-module. We shall say that a bilinear map \( (\cdot,\cdot): M \times M \to K \) is \( \mathcal{H} \)-invariant if

\[
(h.m, m') = (m, h^\dagger.m') \quad \text{for all} \ h \in \mathcal{H} \ \text{and} \ m, m' \in M.
\]

Via the isomorphism \( \text{Hom}_K(M,K) \otimes_K M \cong \text{Hom}_K(M,M) \) (and an identification of \( M \) with \( \text{Hom}_K(M,K) \) using dual bases), an \( \mathcal{H} \)-invariant bilinear form on \( M \) can also be interpreted as an \( \mathcal{H} \)-module homomorphism \( \hat{M} \to M \), and vice versa. In particular, for \( E \in \text{Irr}(\mathcal{H}) \), we have \( E \cong \hat{E} \) if and only if there exists a non-degenerate \( \mathcal{H} \)-invariant bilinear form on \( E \); also note that a non-zero \( \mathcal{H} \)-invariant bilinear form on \( E \) is automatically non-degenerate (by Schur’s Lemma).

Given any basis \( B \) of \( \mathcal{H} \), we denote by \( B^\vee = \{ b^\vee \mid b \in B \} \) the corresponding dual basis, that is, we have

\[
\tau(b'b^\vee) = \begin{cases} 1 & \text{if } b = b', \\ 0 & \text{otherwise}. \end{cases}
\]

Definition 2.2. Let \( B_0 \) be a basis for \( \mathcal{H} \). We say that \( B_0 \) is \( \dagger \)-symmetric if \( b^\dagger = b^\vee \) for all \( b \in B_0 \).

The standard example is the case where \( \mathcal{H} = K[G] \) is the group algebra of a finite group \( G \) over \( K = \mathbb{C} \) and \( \tau \) is the trace form defined by \( \tau(1) = 1 \) and \( \tau(g) = 0 \) for \( g \in G \) such that \( g \neq 1 \). We have an anti-involution \( \dagger: \mathcal{H} \to \mathcal{H} \) given by \( g^\dagger = g^{-1} \); then \( B_0 = G \) is a \( \dagger \)-symmetric basis of \( \mathcal{H} \). Further examples are provided by the algebra \( \bar{J} \) in Section \( \text{[3]} \) and by the “based rings” considered by Lusztig \([14]\).
Indeed, write the identity element of $\tau(a)$ as $\tau(1)$ for all $a \in B_0$. Then a straightforward computation shows that $\tau(b) = \tau(1 \cdot b') = \alpha_b$ for all $b \in B_0$. Now, we certainly have $1 = 1_\mathcal{H} = \sum_{b \in B_0} \alpha_b b'$. Hence, similarly, we also obtain $\tau(b) = \tau(1 \cdot b') = \alpha_b$ for all $b \in B_0$. Thus, (a) holds. Now let $E \in \text{Irr}(\mathcal{H})$. Then, clearly, we have

$$\chi_E(b) = \chi_E(b') = \chi_E(b')$$

for all $b \in B_0$.

This also implies that $c_E = c_E$ since

$$\tau(b) = \tau(b') = \sum_{E \in \text{Irr}(\mathcal{H})} c_E^{-1} \chi_E(b) = \sum_{E \in \text{Irr}(\mathcal{H})} c_E^{-1} \chi_E(b')$$

for all $b \in B_0$.

At first sight, the condition in Definition 2.2 looks rather strong. But the following remark shows that $\dagger$-symmetric bases of $\mathcal{H}$ always exist under some quite natural assumptions.

**Remark 2.4.** There exists a $\dagger$-symmetric basis of $\mathcal{H}$ if the following two conditions are satisfied:

(a) $\tau(h) = \tau(h')$ for all $h \in \mathcal{H}$.

(b) $K$ is sufficiently large (which means here: $K$ contains sufficiently many square roots).

Indeed, consider the bilinear form $\mathcal{H} \times \mathcal{H} \rightarrow K$, $(h, h') \mapsto \tau(h'h')$. By (a), this bilinear form is symmetric; furthermore, one easily sees that it is non-degenerate. Hence, since $\text{char}(K) = 0$, there exists an orthogonal basis of $\mathcal{H}$ with respect to that form. If now $K$ contains sufficiently many square roots, then we can rescale the basis elements and obtain an orthonormal basis of $\mathcal{H}$; any such basis is $\dagger$-symmetric.

We can now state the following two propositions which generalise classical results concerning the Frobenius–Schur indicator for characters of finite groups (see, for example, Etingof et al. [2, §5.1]) to symmetric algebras as above.

**Proposition 2.5.** Assume that $B_0$ is a $\dagger$-symmetric basis of $\mathcal{H}$. Let $E \in \text{Irr}(\mathcal{H})$ and define

$$\nu_E := \frac{1}{c_E \dim E} \sum_{b \in B_0} \chi_E(b^2).$$

Then we have $\nu_E \in \{0, \pm 1\}$; furthermore, the following hold:

(a) $\nu_E = 0$ if and only if $E \ncong \hat{E}$.

(b) $\nu_E = 1$ if and only if $E \cong \hat{E}$ and there exists a non-degenerate, symmetric $\mathcal{H}$-invariant bilinear form on $E$.

(c) $\nu_E = -1$ if and only if $E \cong \hat{E}$ and there exists a non-degenerate, alternating $\mathcal{H}$-invariant bilinear form on $E$.

(In particular, $\nu_E$ does not depend on the choice of $B_0$.)

**Proof.** This very closely follows the original proof of Frobenius and Schur, as presented by Curtis [1, Chap. IV, §3]. We choose a basis of $E$ and obtain a corresponding matrix representation $\rho: \mathcal{H} \rightarrow M_d(K)$ where $d = \dim E$. For $h \in \mathcal{H}$ and $i, j \in \{1, \ldots, d\}$, we denote by $\rho_{ij}(h)$ the $(i,j)$-coefficient of $\rho(h)$. Taking the dual
basis in $\hat{E}$, a matrix representation afforded by $\hat{E}$ is then given by $\hat{\rho}(b) = \rho(b')'$ for all $b \in B_0$, where the prime denotes the transpose matrix.

Assume first that $E \not\cong \hat{E}$. Then the Schur relations in [8, 7.2.2] yield:

$$\sum_{b \in B_0} \rho_{ij}(b) \hat{\rho}_{kl}(b') = 0 \quad \text{for all } i, j, k, l \in \{1, \ldots, d\}.$$  

Using the above description of $\hat{\rho}$, we conclude that

$$\sum_{b \in B_0} \rho_{ij}(b) \rho_{lk}(b) = 0 \quad \text{for all } i, j, k, l \in \{1, \ldots, d\}.$$  

Now let $l = j$ and $k = i$. Then summing over all $i, j$ yields

$$0 = \sum_{1 \leq i, j \leq d} \sum_{b \in B_0} \rho_{ij}(b) \rho_{ji}(b) = \sum_{b \in B_0} \sum_{1 \leq i, j \leq d} \rho_{ii}(b^2) = \sum_{b \in B_0} \chi_E(b^2).$$  

Thus, we have $\nu_E = 0$ in this case, as required.

Now assume that $E \cong \hat{E}$. This means that there exists an invertible matrix $P \in M_d(K)$ such that

$$P \rho(b) = \rho(b')' P \quad \text{for all } b \in B_0.$$  

A standard argument using Schur’s Lemma (see [4, p. 153]) then shows that $P' = \eta P$ where $\eta = \pm 1$. Note that a similar statement is true for any matrix $Q \in M_d(K)$ such that $Q \rho(b) = \rho(b')' Q$ for all $b \in B_0$. Indeed, by Schur’s Lemma, $Q$ will be a scalar multiple of $P$ and so $Q' = \eta Q$, with the same $\eta$ as before. Now our given $P$ defines a bilinear form $(\ , \ ) : E \times E \to K$; the fact that $P \rho(b) = \rho(b')' P$ for all $b \in B_0$ means that $(\ , \ )$ is $\mathcal{H}$-invariant. Thus, we have already shown that if $E \cong \hat{E}$, then there exists a non-degenerate $\mathcal{H}$-invariant bilinear form on $E$ which is either symmetric or alternating. (Conversely, if such a bilinear form exists, then $E \cong \hat{E}$; see Remark [23]) It remains to see how $\eta$ is determined.

For this purpose, let $U \in M_d(K)$ be any matrix and define

$$Q_U := \sum_{b \in B_0} \rho(b')' U \rho(b) = \sum_{b \in B_0} \hat{\rho}(b')' U \rho(b).$$

The second equality shows that $Q_U \rho(b) = \rho(b')' Q_U$ for all $b \in B_0$; see [8, 7.1.10]. Hence, as we just remarked, we must have $Q_U' = \eta Q_U$ and so

$$\sum_{1 \leq i, j \leq d} \sum_{b \in B_0} \rho_{il}(b) u_{ij} \rho_{jk}(b) = \eta \sum_{1 \leq i, j \leq d} \sum_{b \in B_0} \rho_{ik}(b) u_{ij} \rho_{jl}(b)$$

for all $k, l \in \{1, \ldots, d\}$, where we write $U = (u_{ij})$. Now take $U$ to be the matrix with coefficient 1 at position $(k, l)$ and coefficient 0, otherwise. Then we obtain

$$\sum_{b \in B_0} \rho_{kl}(b) \rho_{lk}(b) = \eta \sum_{b \in B_0} \rho_{kk}(b) \rho_{ll}(b) \quad \text{for fixed } k, l \in \{1, \ldots, d\}.$$  

Summing over all $k, l$ yields

$$\sum_{b \in B_0} \chi_E(b^2) = \eta \sum_{b \in B_0} \chi_E(b)^2.$$  

Finally, since $E \cong \hat{E}$, we have $\chi_E(b) = \chi_E(b')$. Hence, the right hand side of the above identity equals $\eta \sum_{b \in B_0} \chi_E(b) \chi_E(b')$ which, by the orthogonality relations for the irreducible characters of $\mathcal{H}$ (see [8, 7.2.4]), equals $\eta c_E \dim E$. Thus, $\nu_E = \eta = \pm 1$, as required.
Once the above statements are proved, it follows that for any $E \in \text{Irr}(\mathcal{H})$ we have $\nu_E \in \{0, \pm 1\}$ and the equivalences in (a), (b), (c) hold. \qed

In the standard example where $\mathcal{H} = \mathbb{C}[G]$ for a finite group $G$, we have $c_E = |G|/\dim E$ for all $E \in \text{Irr}(\mathcal{H})$ (see [S 7.2.5]). Hence, in this case, the formula for $\nu_E$ in Proposition 2.5 indeed is the classical formula for the Frobenius–Schur indicator.

**Proposition 2.6.** Assume that there exists a $\dagger$-symmetric basis $B_0$ of $\mathcal{H}$. Then

$$\text{trace}(\dagger: \mathcal{H} \to \mathcal{H}) = \sum_{E \in \text{Irr}(\mathcal{H})} \nu_E \dim E.$$  

In particular, if $B_0 = B_0^\dagger$, then

$$|\{b \in B_0 \mid b^\dagger = b\}| = \sum_{E \in \text{Irr}(\mathcal{H})} \nu_E \dim E.$$

**Proof.** The second equality certainly follows from the first: under the given assumption on $B_0$, we have $\text{trace}(\dagger) = |\{b \in B_0 \mid b^\dagger = b\}|$. In order to prove the first equality, we compute the trace of $\dagger$ with respect to a basis of $\mathcal{H}$ arising from the Wedderburn decomposition. Let $E \in \text{Irr}(\mathcal{H})$. Choosing a basis of $E$, we obtain a corresponding matrix representation $\rho: \mathcal{H} \to M_d(K)$ where $d = \dim E$. We set

$$e_{ij}^E = \frac{1}{c_E} \sum_{b \in B_0} \rho_{ji}(b^\dagger) b \quad \text{for } i, j \in \{1, \ldots, d\}.$$  

Then, by [S 7.2.7], the matrix $\rho(e_{ij}^E)$ has coefficient 1 at position $(i, j)$ and coefficient 0, otherwise; furthermore, $e_{ij}^E$ acts as zero on any simple $\mathcal{H}$-module which is not isomorphic to $E$. The elements

$$\{e_{ij}^E \mid E \in \text{Irr}(\mathcal{H}), 1 \leq i, j \leq \dim E\}$$

form a $K$-basis of $\mathcal{H}$. We shall now compute the trace of $\dagger$ with respect to this basis. First note that, since the dual basis of $B_0^\dagger$ is $B_0$ and since $e_{ij}^E$ is independent of the choice of the basis of $\mathcal{H}$ (see [S §7.2]), we have

$$e_{ij}^E = \frac{1}{c_E} \sum_{b \in B_0} \rho_{ji}(b^\dagger) b = \frac{1}{c_E} \sum_{b \in B_0} \rho_{ji}(b) b^\dagger$$

and so

$$(e_{ij}^E)^\dagger = \frac{1}{c_E} \sum_{b \in B_0} \rho_{ji}(b^\dagger) b = \frac{1}{c_E} \sum_{b \in B_0} \rho_{ij}(b^\dagger) b = e_{ji}^E,$$

where we use the fact that $c_E = c_{\hat{E}}$; see Remark 2.3. This already shows that those $e_{ij}^E$ where $E \neq \hat{E}$ will not contribute to the trace of $\dagger$. So let us now assume that $E \cong \hat{E}$. Let $d = \dim E$. Then there exists an invertible matrix $P \in M_d(K)$ such that $P \rho(b) = \rho(b^\dagger)P$ for all $b \in B_0$. Write $P = (p_{ij})$ and $P^{-1} = (\tilde{p}_{ij})$. Then we have

$$e_{ij}^E = \frac{1}{c_E} \sum_{b \in B_0} \rho_{ji}(b) b = \frac{1}{c_E} \sum_{b \in B_0} \sum_{1 \leq k, l \leq d} \tilde{p}_{jk} p_{li} \rho_{lk}(b^\dagger) b = \sum_{1 \leq k, l \leq d} \tilde{p}_{jk} p_{li} e_{ki}^E.$$  

The coefficient of $e_{ij}^E$ in the expression on the right hand side is $\tilde{p}_{ji}p_{ij}$. The contribution to the trace of $\dagger$ from basis vectors corresponding to $E$ will be the sum of
all these terms. Now, we have $P' = \nu_E P$; see the proof of Proposition \[2.5\]. Hence, the contribution from $E$ is

$$
\sum_{1 \leq i, j \leq d} \hat{p}_{ji} p_{ji} = \nu_E \sum_{1 \leq i, j \leq d} \hat{p}_{ij} p_{ji} = \nu_E \text{trace}(P^{-1} P) = \nu_E \dim E.
$$

Consequently, we have $\text{trace}(\hat{P}) = \sum_E \nu_E \dim E$ where the sum runs over all $E \in \text{Irr}(\mathcal{H})$ such that $E \cong \hat{E}$. Since $\nu_E = 0$ for all $E \in \text{Irr}(\mathcal{H})$ such that $E \not\cong \hat{E}$, this yields the desired formula.

\begin{example}
Let $B_0$ be a \dag-symmetric basis of $\mathcal{H}$ and assume that $K \subseteq \mathbb{R}$. We claim that then $\nu_E = 1$ for all $E \in \text{Irr}(\mathcal{H})$. To see this, we adapt the classical argument for finite groups. Let $E \in \text{Irr}(\mathcal{H})$. Choosing a basis of $E$, we obtain a corresponding matrix representation $\rho : \mathcal{H} \to M_d(K)$ where $d = \dim E$. We set

$$
Q := \sum_{b \in B_0} \rho(b)^t \rho(b) = \sum_{b \in B_0} \hat{\rho}(b) \rho(b).
$$

Clearly, $Q$ is symmetric. As in the proof of Proposition \[2.5\], the second equality shows that $QP(b) = \rho(b)Q$ for all $b \in B_0$, so $Q$ defines a symmetric, $\mathcal{H}$-invariant bilinear form on $E$. Now, the diagonal coefficients of $Q$ are sums of squares of elements of $K$, at least some of which are non-zero (since $\rho(b) \neq 0$ for at least some $b \in B_0$). Hence, since $K \subseteq \mathbb{R}$, these diagonal coefficients are non-zero and so $Q \neq 0$. By Schur’s Lemma, $Q$ is invertible. Thus, we are in case (b) of Proposition \[2.5\].

Finally, we remark that there is an extensive literature on further generalisations of the Frobenius–Schur indicator, but usually this is done in the framework of Hopf algebras; see, for example, Guralnick–Montgomery \[9\] and the references there.

3. The ring $\hat{J}$

We shall now apply the results of the previous section to cells in finite Coxeter groups. Let $W$ be a finite Coxeter group and $S$ be a set of simple reflections in $W$. We fix a weight function $L : W \to \mathbb{Z}$ in the sense of Lusztig \[15\], where we assume that $L(s) \geq 0$ for all $s \in S$. Using the Kazhdan–Lusztig basis in the generic Iwahori–Hecke algebra associated with $W, L$, we can define partitions of $W$ into left, right and two-sided cells. (Note that these notions depend on $L$).

The key tool to study these cells will be the theory around Lusztig’s ring $J$, originally introduced in \[13\] in the equal parameter case. Subsequently, Lusztig \[15\] extended the theory to the general case, assuming that certain conjectural properties hold; see \[P1–P15\] in \[15\] 14.2. In order to avoid the dependence on these conjectural properties, we shall work with a version of Lusztig’s ring introduced in \[5\]. Let $\hat{J}$ denote this new version of $J$. The principal advantage of $\hat{J}$ is that it can be constructed without any assumption on $W, L$. On the other hand, the results that are known about $\hat{J}$ are not as strong as those for $J$ but, as we shall see, they are sufficient to deduce Theorem \[11\] (See Remark \[8, 14\] below for some comments on the relation between $J$ and $\hat{J}$.)

We now recall the basic facts about the construction of $\hat{J}$; we use \[7\] §1.5 as a reference. Let $K \subseteq \mathbb{C}$ be any field which is a splitting field for $W$. Let $\text{Irr}_K(W)$ denote the set of simple $K[W]$-modules (up to isomorphism) and write

$$
\text{Irr}_K(W) = \{ E^\lambda \mid \lambda \in \Lambda \} \quad \text{(for some finite indexing set } \Lambda)\text{.}
$$
For each $\lambda \in \Lambda$ let $M(\lambda)$ be a basis of $E^\lambda$. Then, by the construction in [7] §1.4, we obtain corresponding leading matrix coefficients

$$c_{w,\lambda}^{st} \in K \quad \text{where } w \in W, \lambda \in \Lambda \text{ and } s, t \in M(\lambda).$$

(The construction uses the generic Iwahori–Hecke algebra associated with $W, L$ and, hence, the above numbers depend on $L$.) For $x, y, z \in W$, we set

$$\tilde{g}_{x,y,z} = \sum_{\lambda \in \Lambda} \sum_{s,t \in M(\lambda)} f_{\lambda}^{-1} c_{x,\lambda}^{st} t_s u_s c_{y,\lambda}^{st} t_z u_z,$$

where each $f_{\lambda} \in K$ is a non-zero element obtained from the corresponding Schur element of the generic Iwahori–Hecke algebras associated with $W, L$ (see [7, 1.5.3(c)]). Furthermore, the map

$$\tau: \tilde{J} \to \tilde{J}, \quad t_w \mapsto t_{w^{-1}},$$

is an anti-involution of $\tilde{J}$ and $B_0 = \{t_w \mid w \in W\}$ is a $\tau$-symmetric basis of $\tilde{J}$. Finally, $\tilde{J}$ is split semisimple and we have a corresponding labelling

$$\text{Irr}(\tilde{J}) = \{\tilde{E}^\lambda \mid \lambda \in \Lambda\} \quad \text{such that } \dim E^\lambda = \dim \tilde{E}^\lambda \quad \text{for all } \lambda \in \Lambda.$$  

We have $\tau = \sum_{\lambda \in \Lambda} f_{\lambda}^{-1} \chi_{\tilde{E}^\lambda}$, hence the numbers $f_{\lambda} (\lambda \in \Lambda)$ are the Schur elements of $\tilde{J}$. (For all these facts, see [7] §1.5.)

Now, by imitating the original definitions of Kazhdan and Lusztig [10], one can define partitions of $W$ into left, right and two-sided “$\tilde{J}$-cells”; see [7] §1.6. (If we just say “left cell”, “right cell” or “two-sided cell”, then this is always meant to be a cell in the sense of Kazhdan and Lusztig, with respect to the given weight function $L$.) Here are some of the essential properties that we shall need:

1. If $\tilde{g}_{x,y,z} \neq 0$, then $x, y^{-1}$ belong to the same left $\tilde{J}$-cell, $y, z^{-1}$ belong to the same left $\tilde{J}$-cell and $x, z^{-1}$ belong to the same left $\tilde{J}$-cell. (See [7] 1.6.4.)
2. For $\lambda \in \Lambda$, the set of all $w \in W$ such that $c_{w,\lambda}^{st} \neq 0$ for some $s, t \in M(\lambda)$ is contained in a two-sided $\tilde{J}$-cell. (See [7] 1.6.11.)
3. If $\Gamma$ is a left cell of $W$, then $\Gamma$ is a union of left $\tilde{J}$-cells. A similar statement holds for right cells and two-sided cells. (See [7] 2.1.20.)

Now let $C$ be a left $\tilde{J}$-cell or, slightly more generally, a union of left $\tilde{J}$-cells. Then

$$[C]_J = \langle t_x \mid x \in C \rangle_K \subseteq \tilde{J}$$

is a left ideal in $\tilde{J}$ and, thus, can be viewed as a left $\tilde{J}$-module. For $\lambda \in \Lambda$, denote by $\tilde{m}(\mathcal{C}, \lambda)$ the multiplicity of $E^\lambda \in \text{Irr}(\tilde{J})$ as an irreducible constituent of $[\mathcal{C}]_J$. Clearly, if $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are the left $\tilde{J}$-cells contained in $\mathcal{C}$, then

$$[\mathcal{C}]_J = \bigoplus_{1 \leq i \leq r} [\mathcal{C}_i]_J \text{ and } \tilde{m}(\mathcal{C}, \lambda) = \sum_{1 \leq i \leq r} \tilde{m}(\mathcal{C}_i, \lambda) \text{ for all } \lambda \in \Lambda.$$  

Lemma 3.1. Let $\mathcal{C}, \mathcal{C}'$ be left $\tilde{J}$-cells of $W$.

(a) We have $\text{Hom}_J([\mathcal{C}]_J, [\mathcal{C}']_J) = \{0\}$ unless $\mathcal{C}, \mathcal{C}'$ are contained in the same two-sided $\tilde{J}$-cell.

(b) In general, we have $\dim \text{Hom}_J([\mathcal{C}]_J, [\mathcal{C}']_J) = |\mathcal{C} \cap (\mathcal{C}')^{-1}|$; in particular, $\mathcal{C} \cap (\mathcal{C}')^{-1} = \emptyset$ unless $\mathcal{C}, \mathcal{C}'$ are contained in the same two-sided $\tilde{J}$-cell.

(c) If $\mathcal{C} = \mathcal{C}'$, then the subspace $J_{\mathcal{C}} := \langle t_w \mid w \in \mathcal{C} \cap \mathcal{C}'^{-1} \rangle_K \subseteq \tilde{J}$ is a subalgebra isomorphic to $\text{End}_J([\mathcal{C}]_J)$. Furthermore, $J_{\mathcal{C}}$ is split semisimple and has identity element $1_{\mathcal{C}} := \sum_{w \in \mathcal{C} \cap \mathcal{C}'^{-1}} \tilde{n}_w t_w$.

Proof. (a) Assume that $\text{Hom}_J([\mathcal{C}]_J, [\mathcal{C}']_J) \neq \{0\}$. This means that there is some $\lambda \in \Lambda$ such that $\tilde{m}(\mathcal{C}, \lambda) > 0$ and $\tilde{m}(\mathcal{C}', \lambda) > 0$. By [7, 1.8.1], there exist $w \in \mathcal{C}$ and $w' \in \mathcal{C}'$ such that $c_{w,\lambda}^{\mathcal{C}} \neq 0$ and $c_{w',\lambda}^{\mathcal{C}'} \neq 0$ for some $s, t \in M(\lambda)$. By (J2), $w$ and $w'$ are contained in the same two-sided $\tilde{J}$-cell. Consequently, $\mathcal{C}$ and $\mathcal{C}'$ must be contained in the same two-sided $\tilde{J}$-cell.

(b) This is modelled on the argument of Lusztig [12, 12.15]. First we show that

$$\dim \text{Hom}_J([\mathcal{C}]_J, [\mathcal{C}']_J) \geq |\mathcal{C} \cap (\mathcal{C}')^{-1}|.$$  

If $\mathcal{C} \cap (\mathcal{C}')^{-1} = \emptyset$, this is clear. Now assume that $\mathcal{C} \cap (\mathcal{C}')^{-1} \neq \emptyset$. Let $y \in \mathcal{C} \cap (\mathcal{C}')^{-1}$ and $x \in \mathcal{C}$. Then $t_x t_{y^{-1}} = \sum_{z \in W} \tilde{\gamma}_{x,y^{-1}} z t_{z^{-1}}$. If $\tilde{\gamma}_{x,y^{-1},z} \neq 0$, then $y^{-1}, z^{-1}$ belong to the same left $\tilde{J}$-cell and so $z^{-1} \in \mathcal{C}'$; see (J1). It follows that we have a well-defined left $\tilde{J}$-module homomorphism

$$\varphi_y : [\mathcal{C}]_J \rightarrow [\mathcal{C}']_J, \quad t_x \mapsto t_x t_{y^{-1}} \ (x \in \mathcal{C}).$$

We claim that the collection of maps $\{\varphi_y \mid y \in \mathcal{C} \cap (\mathcal{C}')^{-1}\}$ is linearly independent in $\text{Hom}_K([\mathcal{C}]_J, [\mathcal{C}']_J)$. Indeed, assume that

$$\sum_{y \in \mathcal{C} \cap (\mathcal{C}')^{-1}} \alpha_y \varphi_y = 0 \text{ where } \alpha_y \in K \text{ for all } y \in \mathcal{C} \cap (\mathcal{C}')^{-1}.$$  

Let $x \in \mathcal{C} \cap (\mathcal{C}')^{-1}$. Applying the above linear combination to $t_x$ and then evaluating the trace form $\tau$ on the resulting expression, we obtain

$$0 = \sum_{y \in \mathcal{C} \cap (\mathcal{C}')^{-1}} \alpha_y \tau(\varphi_y(t_x)) = \sum_{y \in \mathcal{C} \cap (\mathcal{C}')^{-1}} \alpha_y \tau(t_x t_{y^{-1}}) = \alpha_x,$$

using the fact that $B_0 = \{t_w \mid w \in W\}$ is a $\dagger$-symmetric basis of $\tilde{J}$. Thus, we have $\alpha_x = 0$ for all $x \in \mathcal{C} \cap (\mathcal{C}')^{-1}$, as required. This certainly implies that (a) holds. We can then complete the proof by a counting argument, exactly as in [12, 12.15]. In particular, this shows that

$$\{\varphi_y \mid y \in \mathcal{C} \cap (\mathcal{C}')^{-1}\}$$

is a vector space basis of $\text{Hom}_J([\mathcal{C}]_J, [\mathcal{C}']_J)$.

(c) Let $\mathcal{C} = \mathcal{C}'$. By (b), we have $\mathcal{C} \cap \mathcal{C}'^{-1} \neq \emptyset$. Let $x, y \in \mathcal{C} \cap \mathcal{C}'^{-1}$ and write $t_x t_y = \sum_{z \in W} \tilde{\gamma}_{x,y,z} t_{z^{-1}}$. If $\tilde{\gamma}_{x,y,z} \neq 0$ then $y, z^{-1}$ belong to the same left $\tilde{J}$-cell and $z, x^{-1}$ belong to the same left $\tilde{J}$-cell; see again (J1). Thus, we must have
$z \in \mathfrak{C} \cap \mathfrak{C}^{-1}$. This shows that $\hat{J}_\mathfrak{C}$ is a subalgebra of $\hat{J}$. Using now the construction in the proof of (b), we obtain an isomorphism of vector spaces

$$\varphi: \hat{J}_\mathfrak{C} \to \text{End}_J([\mathfrak{C}]_J), \quad t_y \mapsto \varphi_y \ (y \in \mathfrak{C} \cap \mathfrak{C}^{-1}).$$

We note that, for any $h \in \hat{J}_\mathfrak{C}$, the map $\varphi(h)$ is given by right multiplication with $h^\dagger$. This certainly implies that $\varphi$ is an algebra homomorphism.

Finally, being isomorphic to the endomorphism algebra of a module of a split semisimple algebra, $\hat{J}_\mathfrak{C}$ itself has an identity element and is split semisimple. Let $1_{\mathfrak{C}}$ be the identity element and write $1_{\mathfrak{C}} = \sum_{w \in \mathfrak{C} \cap \mathfrak{C}^{-1}} \alpha_w t_w$ where $\alpha_w \in K$. If $x \in \mathfrak{C} \cap \mathfrak{C}^{-1}$, then $t_x \in \hat{J}_\mathfrak{C}$ and so

$$\tilde{n}_x = \tau(t_{x^{-1}}) = \tau(t_{x^{-1}}1_{\mathfrak{C}}) = \sum_{w \in \mathfrak{C} \cap \mathfrak{C}^{-1}} \alpha_w \tau(t_{x^{-1}}t_w) = \alpha_x.$$

Thus, $1_{\mathfrak{C}}$ has the required expression. □

**Remark 3.2.** Let $\mathfrak{C}$ be a left $\hat{J}$-cell. Recall that, by definition, the left $\hat{J}$-module $[\mathfrak{C}]_J = \langle t_w \mid w \in \mathfrak{C} \rangle_k$ is a left ideal in $\hat{J}$. By Lemma 3.1(c), the element $1_{\mathfrak{C}}$ is an idempotent in $\hat{J}$, and it is contained in $[\mathfrak{C}]_J$. In fact, we claim that

$$[\mathfrak{C}]_J = \hat{J}1_{\mathfrak{C}}.$$

Indeed, since $1_{\mathfrak{C}} \in [\mathfrak{C}]_J$, it is clear that $\hat{J}1_{\mathfrak{C}} \subseteq [\mathfrak{C}]_J$. Conversely, we note that right multiplication by $1_j$ is the identity element of $\text{End}_J([\mathfrak{C}]_J)$; see the proof of Lemma 3.1(c). Thus, for any $w \in \mathfrak{C}$, we have $t_w = t_w 1_{\mathfrak{C}} \in \hat{J}1_{\mathfrak{C}}$, as required.

For any subset $X \subseteq W$, we denote by $X_{(2)}$ the set of involutions in $X$.

**Lemma 3.3.** Let $\mathfrak{C}$ be a union of left $\hat{J}$-cells of $W$. Then we have

$$|\mathfrak{C}_{(2)}| = \sum_{\lambda \in \Lambda} \tilde{m}(\mathfrak{C}, \lambda).$$

**Proof.** Since $\mathbb{R}$ is known to be a splitting field for $W$ (see [8, 6.3.8]), we will assume in this proof that $K \subseteq \mathbb{R}$. Let $\mathfrak{C}_1, \ldots, \mathfrak{C}_r$ be the left $\hat{J}$-cells which are contained in $\mathfrak{C}$. Then, clearly, $\mathfrak{C}_{(2)}$ is the union of the sets of involutions in $\mathfrak{C}_1, \ldots, \mathfrak{C}_r$; furthermore, as already noted above, we have $\tilde{m}(\mathfrak{C}, \lambda) = \tilde{m}(\mathfrak{C}_1, \lambda) + \ldots + \tilde{m}(\mathfrak{C}_r, \lambda)$ for all $\lambda \in \Lambda$. Thus, it will be sufficient to deal with the case where $r = 1$ and $\mathfrak{C} = \mathfrak{C}_1$ is just one left $\hat{J}$-cell. In this case, consider the split semisimple algebra $\mathcal{H} := \hat{J}\mathfrak{C}$; see Lemma 3.1(c). We note that $\dagger$ restricts to an anti-involution of $\mathcal{H}$ which we denote by the same symbol. Furthermore, $\tau$ restricts to a trace form on $\mathcal{H}$ where $B_{0,\mathfrak{C}} = \{t_w \mid w \in \mathfrak{C} \cap \mathfrak{C}^{-1}\}$ is a $\dagger$-symmetric basis of $\hat{J}\mathfrak{C}$ such that $B_{0,\mathfrak{C}} = B_{0,\mathfrak{C}}^\dagger$. Thus, we can apply the results in Section 2 to $\mathcal{H}$. Since $K \subseteq \mathbb{R}$ and since $\mathfrak{C}_{(2)} \subseteq \mathfrak{C} \cap \mathfrak{C}^{-1}$, we conclude that

$$|\mathfrak{C}_{(2)}| = \sum_{M \in \text{Irr}(\mathcal{H})} \dim M \quad (\text{see Proposition 2.6 and Example 2.7}).$$

It remains to note that, since $\hat{J}$ is split semisimple and since we have an isomorphism $\mathcal{H} \cong \text{End}_J([\mathfrak{C}]_J)$, there is a bijection between $\text{Irr}(\mathcal{H})$ and the set of simple $\hat{J}$-modules which appear as constituents of $[\mathfrak{C}]_J$; furthermore, if $\tilde{E}^\lambda \in \text{Irr}(\hat{J})$ is such a constituent, then the corresponding simple $\mathcal{H}$-module has dimension $\tilde{m}(\mathfrak{C}, \lambda)$. (This follows from simple facts about $\text{Hom}$ functors; see, e.g., [7, 4.1.3].) □
Corollary 3.4. Let $\mathcal{C}$ be a left $\tilde{J}$-cell. Then $[\mathcal{C}]_j$ is multiplicity-free if and only if $\mathcal{C}(2) = \mathcal{C} \cap \mathcal{C}^{-1}$.

Proof. Recall that $\mathcal{C}(2) \subseteq \mathcal{C} \cap \mathcal{C}^{-1}$. By Lemma 3.3 we have $|\mathcal{C}(2)| = \sum_{\lambda \in \Lambda} \tilde{m}(\mathcal{C}, \lambda)$. On the other hand, by Lemma 3.1(b), we have $|\mathcal{C} \cap \mathcal{C}^{-1}| = \sum_{\lambda \in \Lambda} \tilde{m}(\mathcal{C}, \lambda)^2$. Hence, if $|\mathcal{C}|_j$ is multiplicity-free, then $\tilde{m}(\mathcal{C}, \lambda) = \tilde{m}(\mathcal{C}, \lambda)^2$ for all $\lambda \in \Lambda$ and so $\mathcal{C}(2) = \mathcal{C} \cap \mathcal{C}^{-1}$. On the other hand, if $\mathcal{C}(2) = \mathcal{C} \cap \mathcal{C}^{-1}$, then $\sum_{\lambda \in \Lambda} \tilde{m}(\mathcal{C}, \lambda) = \sum_{\lambda \in \Lambda} \tilde{m}(\mathcal{C}, \lambda)^2$ and so $\tilde{m}(\mathcal{C}, \lambda) \in \{0, 1\}$ for all $\lambda \in \Lambda$. \hfill $\square$

Remark 3.5. Let $\lambda \in \Lambda$ and $w \in W$. By [7, 1.5.7], we have

(a) $c_{w, \lambda} = \text{trace}(t_w, \tilde{E}^{\lambda})$ where $c_{w, \lambda} := \sum_{s \in M(\lambda)} c_{w, s, \lambda}^{ss}$.

Thus, up to signs, the numbers $c_{w, \lambda}$ are the leading coefficients of character values as defined by Lusztig [14]. We claim that

(b) $c_{w, \lambda} = 0$ unless $w, w^{-1}$ belong to the same left $\tilde{J}$-cell.

Indeed, assume that $c_{w, \lambda} \neq 0$. Using (a), we conclude that $t_w$ can not be nilpotent. Consequently, $t_w^2 \neq 0$ and so $\gamma_{w, w, x} \neq 0$ for some $x \in W$. Hence, by (J1), the elements $w, w^{-1}$ must belong to the same left $\tilde{J}$-cell, as claimed. (In the equal parameter case, this argument is due to Lusztig [14, 3.5].)

Example 3.6. Recall that $1_j = \sum_{w \in W} \tilde{n}_w t_w$. Let $\mathcal{D} = \{w \in W \mid \tilde{n}_w \neq 0\}$. If P1-P15 in [15, 14.2] were known to hold for $W, L$, then we could deduce that every element of $\mathcal{D}$ is an involution. In the present context, we can at least show that $w, w^{-1}$ belong to the same left $\tilde{J}$-cell. Indeed, if $\tilde{n}_w \neq 0$, then the defining equation shows that $c_{w, \lambda} \neq 0$ for some $\lambda \in M(\lambda)$. So Remark 3.3(b) implies that $w, w^{-1}$ belong to the same left $\tilde{J}$-cell, as claimed. In particular, if we are in a case where all $\tilde{J}$-modules $[\mathcal{C}]_j$ are multiplicity-free (for any left $\tilde{J}$-cell $\mathcal{C}$ of $W$), then $w^2 = 1$ for all $w \in \mathcal{D}$ (see Corollary 3.4).

Now let $\Gamma$ be a left cell of $W$. Recall that we have a corresponding left $K[W]$-module $[\Gamma]_1$. For any $\lambda \in \Lambda$, let $m(\Gamma, \lambda)$ denote the multiplicity of $E^\lambda \in \text{Irr}_K(W)$ as an irreducible constituent of $[\Gamma]_1$. Now, $\Gamma$ is a union of left $\tilde{J}$-cells; see (J3). Thus, in order to complete the proof of Theorem 1.1, we need to compare the multiplicities $m(\Gamma, \lambda)$ and $\tilde{m}(\Gamma, \lambda)$.

Lemma 3.7. With the above notation, we have $\tilde{m}(\Gamma, \lambda) = m(\Gamma, \lambda)$ for any $\lambda \in \Lambda$. Consequently, Theorem 1.1 holds.

Proof. Let $\lambda \in \Lambda$. Using [3, Prop. 4.7] (see also the proof of [7, 2.2.4]), we have

$$\sum_{s, t \in M(\lambda)} \sum_{w \in \Gamma} c_{w, t, \lambda}^{st} c_{w^{-1}, \lambda}^{ts} = m(\Gamma, \lambda) f_\lambda |M(\lambda)|.$$ 

On the other hand, let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the left $\tilde{J}$-cells which are contained in $\Gamma$. Then, using [7, 1.8.1], we have

$$\sum_{s, t \in \tilde{M}(\lambda)} \sum_{w \in \mathcal{C}_i} c_{w, t, \lambda}^{st} c_{w^{-1}, \lambda}^{ts} = \tilde{m}(\mathcal{C}_i, \lambda) f_\lambda |M(\lambda)| \quad \text{for } i = 1, \ldots, r.$$
Summing these identities over $i = 1, \ldots, r$ and using the fact that $\tilde{m}(\Gamma, \lambda) = \tilde{m}(\mathcal{C}_1, \lambda) + \ldots + \tilde{m}(\mathcal{C}_r, \lambda)$, we obtain

$$\sum_{s, t \in M(\lambda)} \sum_{w \in \Gamma} e_{w, s}^{lt} e_{w^{-1}, \lambda} = \tilde{m}(\Gamma, \lambda) f_{\lambda} |M(\lambda)|.$$ 

We conclude that $m(\Gamma, \lambda) = \tilde{m}(\Gamma, \lambda)$, as required. In combination with Lemma 3.8 this yields that $|\Gamma_{(2)}| = \sum_{\lambda \in \Lambda} m(\Gamma, \lambda)$. Note that the right hand side is just the number of terms in a decomposition of $[\Gamma]$ as a direct sum of simple $K[W]$-modules. Thus, Theorem 1.1 is proved.

Corollary 3.8 (See Lusztig [12, 5.8] in the equal parameter case). Let $\Gamma$ be a left cell of $W$ and $\lambda, \mu \in \Lambda$. Then

$$\sum_{w \in \Gamma} c_{w, \lambda} c_{w^{-1}, \mu} = \begin{cases} m(\Gamma, \lambda) f_{\lambda} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the left $J$-cells which are contained in $\Gamma$. By [7, 1.8.1], it is already known that, for any $i \in \{1, \ldots, r\}$, we have

$$\sum_{w \in \mathcal{C}_i} c_{w, \lambda} c_{w^{-1}, \mu} = \begin{cases} \tilde{m}(\mathcal{C}_i, \lambda) f_{\lambda} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

We sum these identities over all $i = 1, \ldots, r$. Then it remains to use Lemma 3.7 and the fact that $\tilde{m}(\Gamma, \lambda) = \tilde{m}(\mathcal{C}_1, \lambda) + \ldots + \tilde{m}(\mathcal{C}_r, \lambda)$.

Corollary 3.9 (See Lusztig [12, 12.15] in the equal parameter case). Let $\Gamma, \Gamma'$ be left cells of $W$. Then

$$\dim \text{Hom}_{K[W]}([\Gamma]_1, [\Gamma']_1) = |\Gamma \cap (\Gamma')^{-1}|.$$ 

Furthermore, we have $\text{Hom}_{K[W]}([\Gamma]_1, [\Gamma']_1) = \{0\}$ unless $\Gamma, \Gamma'$ are contained in the same two-sided cell of $W$.

Proof. We have

$$\dim \text{Hom}_{K[W]}([\Gamma]_1, [\Gamma']_1) = \sum_{\lambda \in \Lambda} m(\Gamma, \lambda) m(\Gamma', \lambda)$$

$$= \sum_{\lambda \in \Lambda} \tilde{m}(\Gamma, \lambda) \tilde{m}(\Gamma', \lambda) \quad (\text{see Lemma 3.3})$$

$$= \dim \text{Hom}_J([\Gamma]_j, [\Gamma']_j).$$

Thus, it is sufficient to show that $\dim \text{Hom}_J([\Gamma]_j, [\Gamma']_j) = |\Gamma \cap (\Gamma')^{-1}|$. To see this, let $\mathcal{C}_1, \ldots, \mathcal{C}_s$ be the left $J$-cells which are contained in $\Gamma$ and let $\mathcal{C}'_1, \ldots, \mathcal{C}'_s$ be the left $J$-cells which are contained in $\Gamma'$. Then we have

$$\dim \text{Hom}_J([\Gamma]_j, [\Gamma']_j) = \sum_{1 \leq i \leq r} \sum_{1 \leq j \leq s} \dim \text{Hom}_J([\mathcal{C}_i]_j, [\mathcal{C}'_j]_j),$$

and so the desired equality immediately follows from Lemma 3.1(b). Finally, if $\Gamma, \Gamma'$ are not contained in the same two-sided cell, then $\mathcal{C}_i, \mathcal{C}_j$ (for any $i, j$) are not contained in the same two-sided $J$-cell; see (J3). Thus, Lemma 3.1(a) and the above formula show that $\text{Hom}_{K[W]}([\Gamma]_1, [\Gamma']_1) = \{0\}$ in this case.
**Example 3.10.** Let $\Gamma, \Gamma'$ be left cells of $W$ which are contained in the same two-sided cell. If $L(s) = 1$ for all $s \in S$, it is known that we always have $\Gamma \cap (\Gamma')^{-1} \neq \emptyset$; see Lusztig [12, 12.16]. For general $L$, it can happen that $\Gamma \cap (\Gamma')^{-1} = \emptyset$; see [4] Cor. 4.8 (case "b = 2a" in Table 2) for an example in type $F_4$.

**Corollary 3.11.** Let $\Gamma$ be a left cell of $W$. Then $[\Gamma]_1$ is multiplicity-free if and only if $\Gamma(2) = \Gamma \cap \Gamma^{-1}$.

**Proof.** Once Theorem 1.1 and Corollary 3.9 are established, this follows by an argument entirely analogous to that in Corollary 3.4. $\square$

Now let $\mathfrak{c}$ be a two-sided cell of $W$. Given $E^\lambda \in \text{Irr}_K(W)$, we write $E^\lambda \sim_L \mathfrak{c}$ and say that $E^\lambda$ belongs to $\mathfrak{c}$ if $E^\lambda$ is a constituent of some $[\Gamma]_1$ where $\Gamma$ is a left cell which is contained in $\mathfrak{c}$.

**Corollary 3.12.** The number of involutions in a two-sided cell $\mathfrak{c}$ of $W$ is equal to

$$\sum_{\lambda \in \Lambda} \dim E^\lambda \text{ where the sum runs over all } \lambda \in \Lambda \text{ such that } E^\lambda \sim_L \mathfrak{c}.$$ 

**Proof.** Let $\Gamma_1, \ldots, \Gamma_m$ be all the left cells of $W$ (with respect to the given $L$). Then the direct sum $\bigoplus_{1 \leq i \leq m} [\Gamma_i]_1$ is isomorphic to the regular representation of $W$ and so each $E^\lambda \in \text{Irr}_K(W)$ appears with multiplicity equal to $\dim E^\lambda$ in that direct sum. Now, by Corollary 3.9 we have $\text{Hom}_K(W) ([\Gamma_i]_1, [\Gamma_j]_1) = \{0\}$ whenever $\Gamma_i, \Gamma_j$ are not contained in the same two-sided cell of $W$. Consequently, if $I$ denotes the set of all $i \in \{1, \ldots, m\}$ such that $\Gamma_i \subseteq \mathfrak{c}$, then we have

$$\sum_{i \in I} [\Gamma_i]_1 = \sum_{\lambda \in \Lambda : E^\lambda \sim_L \mathfrak{c}} (\dim E^\lambda) E^\lambda$$

(in the appropriate Grothendieck group of representations). Thus, the number of terms in a decomposition of $\bigoplus_{i \in I} [\Gamma_i]_1$ as a direct sum of irreducible representations is equal to $\sum_{\lambda} \dim E^\lambda$ where the sum runs over all $\lambda \in \Lambda$ such that $E^\lambda \sim_L \mathfrak{c}$. On the other hand, by Theorem 1.1 this number is also equal to the sum $\sum_{i \in I} |\Gamma_i(2)|_1$ which is just the number of involutions in $\mathfrak{c}$. $\square$

**Example 3.13** (Lusztig). Assume that we are in the equal parameter case where $L(s) = 1$ for all $s \in S$. Let $\mathfrak{c}$ be a two-sided cell of $W$. By [12] Chap. 4, one can attach a certain finite group $\mathcal{G}_\mathfrak{c}$ to $\mathfrak{c}$ (or the corresponding family of $\text{Irr}_K(W)$). Assume now that $W$ is of classical type. Then $|\mathcal{G}_\mathfrak{c}| = 2^d$ for some $d \geq 0$ and, by [12] 12.17, it is known that $[\Gamma]_1$ is multiplicity-free with exactly $2^d$ irreducible constituents, for every left cell $\Gamma \subseteq \mathfrak{c}$. Hence, $\Gamma(2) = \Gamma \cap \Gamma^{-1}$ also has cardinality $2^d$ for any left cell $\Gamma \subseteq \mathfrak{c}$. Now let $E_\mathfrak{c}$ be the unique special representation which belongs to $\mathfrak{c}$ (see [12] 4.14, 5.25). Since $E_\mathfrak{c}$ occurs with multiplicity 1 in $[\Gamma]_1$ for every left cell $\Gamma \subseteq \mathfrak{c}$, we conclude that $|\mathfrak{c}(2)| = 2^d \dim E_\mathfrak{c}$. Combining this with Corollary 3.12 we obtain the following identity:

$$2^d \dim E_\mathfrak{c} = \sum_{\lambda \in \Lambda : E^\lambda \sim_L \mathfrak{c}} \dim E^\lambda,$$

which shows that the order of the group $\mathcal{G}_\mathfrak{c}$ is determined by the set of all $E^\lambda \in \text{Irr}_K(W)$ which belong to $\mathfrak{c}$. (If $W$ is of exceptional type, then such an identity will not hold in general.)

**Remark 3.14.** Assume that the conjectural properties P1–P15 in [15] 14.2 hold for $W, L$. Then we do have $J = \tilde{J}$; see [7] 2.3.16. In particular, this implies that:
\[ \tilde{\gamma}_{x,y,z} \in \mathbb{Z} \text{ and } \tilde{n}_w = \pm 1 \text{ for all } x, y, z, w \in W; \]

- every left \( J \)-cell contains a unique element of \( D = \{ w \in W \mid \tilde{n}_w \neq 0 \} \);
- the left cells of \( W \) are precisely the left \( J \)-cells.

(See [7, 2.5.3].) It would be highly interesting to prove these statements directly, without reference to \( P1-P15 \); at present, we do not see any way of doing this. In [6], we have formulated a somewhat different set of conjectural properties which, in some cases, are easier to verify than \( P1-P15 \). However, the case where \( W \) is of type \( B_n \) and \( L \) is a general weight function remains completely open.

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