Reconstructing Mimetic Cosmology

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We explore the mimetic gravity formulation with the inclusion of a scalar field potential namely, \( V(\phi) \). However, we are not considering any \textit{a priori} specific form this term. By means of the Chevallier-Polarski-Linder parametrization for the parameter state of the fluid we can construct an explicit function for such potential in terms of the cosmological redshift and obtain analytical solutions in the mimetic gravity approach. We revise some cosmological implications of these results and additionally we perform a numerical reconstruction for the potential \( V \) as a function of the mimetic scalar field, \( \phi \). The scenario of gravitational matter production plus reconstructed mimetic gravity is studied in order to evaluate the late times cosmic evolution of the model at an effective level.

I. INTRODUCTION

The evidences collected in the era of big experiments make us confident with respect to the existence of scalar fields in nature \(^1\). A seminal idea considering the use of a scalar field can be found in \(^2\) and the main motivation to do this was based on the incorporation of the Mach’s principle into Einstein’s theory, this is the origin of scalar-tensor models. An interesting description for scalar-tensor theories including more general couplings between gravity and the scalar field was provided by Horndeski in Ref. \(^3\) and consisted in the emulation of a generalization of General Relativity known as Lovelock gravity \(^4\) at dynamical level: the equations of motion governing the physical degrees of freedom are at most of second order. This characteristic keeps the theory free from certain type of instabilities and it is worthwhile to mention that the Horndeski formulation is the most general scalar-tensor framework preserving second order dynamics.

In a more recent context, the \textit{Galileon} field (see for instance \(^5\)), gained the community’s attention on the role of scalar fields in cosmology. In addition to some nice properties inherent to this scalar field its consequences at cosmological level were relevant, for example, accelerated cosmic expansion with no need of exotic components; but also an interesting feature of this scalar field is its second order dynamics besides its geometric origin\(^1\) as the bending modes of the brane in the scheme of extra dimensions for the Universe.

Nowadays we can find the consideration of scalar fields in a wide range of physical phenomena and for several reasons but the use of scalar fields has found a privileged place within cosmology. Our existence itself could be due to a primordial scalar field called \textit{inflaton}. This field was the responsible of driving our primitive Universe into a super-accelerated phase which, after a very short time interval, ended with the decay of the inflaton into the particles of the standard model \(^7\). The switching off mechanism for the inflaton still remains unsolved but some interesting proposals can be found in the literature \(^8\). The inflationary period of our early Universe is a fundamental ingredient in the current understanding we have about the observable Universe.

Scalar-tensor models have proven to be a very prolific theoretical laboratory. It has been shown that in compact objects such as black holes or neutron stars the solutions of General Relativity become unstable when a trivial scalar field is considered, this process is currently known as \textit{spontaneous scalarization} \(^9\). However, the consideration of more general couplings between the metric and the scalar field can lead to stable configurations \(^10\) \(^11\); providing a new wide of solutions for such objects with deviations beyond General Relativity that passes all the solar system tests.

It is a known fact that the conformal transformations, can help to shed light on a vast class of scalar-tensor theories\(^2\). However, when trying to apply the same reasoning

\(^1\) Other modified theories of gravity as the \( f(R) \) theories can also originate a scalar field geometrically \(^6\).

\(^2\) For instance, in Ref. \(^12\) is discussed that the gravitational lensing generated by scalar-tensor gravitational waves is stronger in the Jordan frame than in the Einstein frame; with the recent detection of gravitational waves we could start to get some hints about the equivalence or inequivalence between both frames,
to a more general class of scalar-tensor theories, such as those included in the Horndeski action, it is found that conformal transformations do not work as well as in the standard case because of the kinetic dependence in the free parameters of the Horndeski theory. The conformal transformations are metric transformations consisting on a point-dependent re-scaling of the metric tensor. For dynamical reasons it is usual to assume that the conformal factor has a functional dependence on the scalar field appearing in the theory by means of its first derivatives, i.e.,

\[
\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \Omega^2(\phi, X) \tilde{g}_{\mu\nu} \quad (1)
\]

Here \(X\) is the simplest coordinate-invariant we can think of using only the metric tensor and the scalar field. In Ref. [15] was found that the so-called “standard scalar-tensor theor” is closed under the conformal transformation \(g_{\mu\nu} = \Omega^2(\phi) \tilde{g}_{\mu\nu}\). The natural question is whether this is also true for the extended conformal transformation between these two metrics is known as “disformal transformation” and is given by

\[
g_{\mu\nu} = A(\phi, X) \tilde{g}_{\mu\nu} + B(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad (3)
\]

where \(X = \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\). Here the functions \(A\) and \(B\) are scalar parameters, called the conformal and disformal factors, respectively. For \(B = 0\) the previous expression reduces to the extended conformal transformation. In general, the functions \(A(\phi, X)\) and \(B(\phi, X)\) are arbitrary, with \(A \neq 0\). In Ref. [22] was shown that, provided the transformation is invertible, the equations of motion for the theory, obtained by the variation of the action with respect to \(g_{\mu\nu}\) and \(\phi\), reduce to those obtained by varying with respect to the metric \(g_{\mu\nu}\). In the same reference [22] and in [23] was shown that if the equations of motion are constrained with \(g_{\mu\nu}\) and \(\phi\), we have the following two equations of motion

\[
\Omega \left( A - X \frac{\partial A}{\partial X} \right) - \omega X \frac{\partial B}{\partial X} = 0,
\]

\[
\Omega X^2 \frac{\partial A}{\partial X} - \omega \left( A - X^2 \frac{\partial B}{\partial X} \right) = 0, \quad (4)
\]

with the following definitions

\[
\Omega := (G^{\mu\nu} - T^{\mu\nu}) \tilde{g}_{\mu\nu}, \quad \omega := (G^{\mu\nu} - T^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi. \quad (5)
\]

The determinant of system [1] is given by

\[
\det = X^2 A \frac{\partial}{\partial X} \left( B + \frac{A}{X} \right). \quad (6)
\]

If this determinant is nonzero, it is trivial to obtain that the resulting set of equations consists of Einstein’s equation \((G_{\mu\nu} = T_{\mu\nu})\) and a second empty equation. Therefore, the theory does not feature new solutions with respect to General Relativity [22]. The situation is quite different if the determinant in (6) is zero, this corresponds to the physical case when the disformal transformation given by (3) is non-invertible or singular. In this case, being the function \(A \neq 0\), it is direct to obtain \(B\), which takes the form

\[
B(\phi, X) = - \frac{A(\phi, X)}{X} + C(\phi) \quad (7)
\]

which has been a subject of controversy among cosmologists for several years. For a review on conformal transformations see also [13]. The conformal transformations with multiple scalar fields can be found in [14].

II. MIMETIC GRAVITY

In Ref. [18] was introduced a class of metric transformations, dubbed as disformal transformations, which are nowadays used in cosmology, e.g. in effective field theories for inflation [19], as well as in models for dark energy and dark matter [20, 21]. Bekenstein [18] considered gravitational theories supplied by two geometries, one for the gravity sector and the other for the matter sector, such as in Brans-Dicke type theories, where the matter metric is related to the gravity metric by a conformal transformation. In this reference Bekenstein finds that because general relativity enjoys invariance under diffeomorphisms, one is free to parametrize the metric \(g_{\mu\nu}\) in terms of an auxiliary metric \(\tilde{g}_{\mu\nu}\) and a scalar field \(\phi\). The transformation between these two metrics is known as “disformal transformation” and is given by

\[
\tilde{g}_{\mu\nu} = \frac{g_{\mu\nu}}{\partial_\mu \phi \partial_\nu \phi} \quad (8)
\]

### References

[1] Reference [15]
[2] Reference [16]
[3] Reference [17]
[4] Reference [18]
[5] Reference [19]
[6] Reference [20]
[7] Reference [21]
[8] Reference [22]
[9] Reference [23]
where $C(\phi) \neq 0$ is an arbitrary function. The corresponding equations of motion differ from those of general relativity, due to the presence of an extra term on the right-hand side of Einstein’s equations. Therefore, when the disformal transformation is singular, we find extra degrees of freedom which result in equations of motion differing from those of general relativity [22].

The parametrization of reference [16] defining mimetic gravity can be identified with a singular disformal transformation, with $A = X$ and $B = 0$ in Eq. (3), and correspondingly $C(\phi) = 1$ in Eq. (7). In general, when the relation defined by (7) exists between the conformal factor $A$ and the disformal factor $B$, the resulting disformal transformation is singular and, as a result, the system possesses additional degrees of freedom, explaining the origin of the extra degree of freedom in mimetic gravity.

In Ref. [24] was shown that the two approaches towards mimetic gravity, namely, singular disformal transformation [9] and the so called Lagrange multiplier formulation are equivalent. The idea of the authors of Ref. [16] is to isolate the conformal degree of freedom of gravity by introducing a parametrization of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field $\phi$, dubbed mimetic field, as follows

$$g_{\mu\nu} = (\tilde{-g})^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi) \tilde{g}_{\mu\nu}.$$  \hspace{1cm} (8)

From (8) it is clear that, in such a way, the physical metric is invariant under conformal transformations of the type, $\tilde{g}_{\mu\nu} \rightarrow \Omega(t, x)^2 \tilde{g}_{\mu\nu}$, for the auxiliary metric; being $\Omega(t, x)$ a function of the space-time coordinates. It is also clear that, as a consistency condition, the mimetic field satisfies the following constraint

$$\tilde{g}_{\mu\nu} \partial_{\mu}\phi\partial_{\nu}\phi = -1.$$  \hspace{1cm} (9)

Thus, the gravitational action, taking into account the reparametrization given by (9) now takes the form

$$S = \int_M d^4x \sqrt{-g} \left\{ R(\tilde{g}_{\mu\nu}, \phi) + \mathcal{L}_m \right\},$$  \hspace{1cm} (10)

where $M$ is the spacetime manifold, $R = R(\tilde{g}_{\mu\nu}, \phi)$ is the Ricci scalar, $\mathcal{L}_m$ is the matter Lagrangian and $g = g(\tilde{g}_{\mu\nu}, \phi)$ is the determinant of the physical metric. By varying the action with respect to the physical metric one obtains the equations for the gravitational field [10]

$$G_{\mu\nu} - T_{\mu\nu} = (G - T) (\partial_{\mu}\phi) (\partial_{\nu}\phi) = 0,$$  \hspace{1cm} (11)

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein tensor and the matter energy-momentum tensor, while $G$ and $T$ represent the traces of such tensors, respectively. Notice that the auxiliary metric does not enter the gravitational field equation explicitly, but only through the physical metric, whereas the mimetic field enters explicitly. In fact, the mimetic field contributes to the right hand side of Einstein’s equation through the additional energy-momentum tensor component

$$\tilde{T}_{\mu\nu} = -(G - T) (\partial_{\mu}\phi) (\partial_{\nu}\phi).$$  \hspace{1cm} (12)

We note that both energy-momentum tensors, $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ are covariantly conserved, that is, $\nabla^{\rho} T_{\mu\nu} = \nabla^{\rho} \tilde{T}_{\mu\nu}$, whereas the continuity equation for $\tilde{T}_{\mu\nu}$ with the mimetic constraint [9] leads to

$$\nabla^{\mu} [(G - T) \partial_{\mu}\phi] := \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} (G - T) g^{\mu\nu} \partial_{\nu}\phi \right] = 0.$$  \hspace{1cm} (13)

Finally, the trace of (11) is found to be

$$(G - T) (1 + g^{\mu\nu} \partial_{\mu}\phi\partial_{\nu}\phi) = 0.$$  \hspace{1cm} (14)

In reference [25] was considered an alternative but equivalent formulation for mimetic gravity. The equations of motion obtained from the action written in terms of the auxiliary metric $\tilde{g}_{\mu\nu}$ are equivalent to those that one would conventionally obtain from the action expressed in terms of the physical metric with the imposition of an additional constraint on the mimetic field. In the formulation of the reference [25] the mimetic constraint given by [9] can actually be implemented at the level of the action by using a Lagrange multiplier. That is, the action for mimetic gravity [10] can be written as

$$S = \int_M d^4x \sqrt{-g} \left\{ R(g_{\mu\nu}, \phi) - \frac{1}{2} \left[ g^{\mu\nu}(\partial_{\mu}\phi)(\partial_{\nu}\phi) - 1 \right] + \mathcal{L}_m - V(\phi) \right\},$$  \hspace{1cm} (15)

where we have considered a potential for the mimetic field. The variation of the action [15] with respect to the physical metric, $g_{\mu\nu}$, leads to the following equations of motion [26]

$$G_{\mu\nu} = T_{\mu\nu} + \lambda \partial_{\mu}\phi\partial_{\nu}\phi - g_{\mu\nu} \left[ \frac{\lambda}{2} (\partial^\alpha\phi\partial_{\alpha}\phi - 1) + V(\phi) \right],$$  \hspace{1cm} (16)

On the other hand, the variation with respect to the Lagrange multiplier field $\lambda$ provides the condition [9] while the variation with respect to the mimetic field $\phi$ reads

$$\nabla^{\mu}(\lambda \partial_{\mu}\phi) - \frac{dV(\phi)}{d\phi} = 0,$$  \hspace{1cm} (17)

which corresponds to a generalization of the Klein-Gordon equation. If we take the trace of Eq. (16) and consider the condition given in [16] we can obtain an explicit expression for the Lagrange multiplier

$$\lambda = G - T + 4V,$$  \hspace{1cm} (18)

substituting the previous expression for the Lagrange multiplier in (16) and considering $V = 0$ together with the condition [9], we recover the equation (11).
Now, we consider a flat FLRW metric of the form 
\[ ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \]
where \( a(t) \) is the scale factor. We must take into account that in order to preserve the homogeneity and isotropy of spacetime we must have for the scalar field, \( \phi = \phi(t) \). From Eq. (9) we obtain, \( \dot{\phi}^2 = 1 \), which leads immediately to \( \dot{\phi} = \dot{t} \).

For convenience we have chosen a null value for the integration constant. The Friedmann constraint and acceleration equation follows from (16) and can be written as
\[ 3H^2 = \rho + \lambda + V, \] (19)
\[ 2\dot{H} + 3H^2 = V - p \Rightarrow \dot{H} + H^2 = -\frac{1}{6}(\rho + 3p + \lambda - 2V), \] (20)
where the energy momentum tensor of the matter content is modeled by a perfect fluid, being \( \rho \) and \( p \) its density and pressure, respectively. In our description, we will consider a barotropic fluid, therefore the equation state to consider will have the form, \( p_x = \omega_x \rho_x \). Note that the term \( \lambda + V \) in the Friedmann constraint (19) is the contribution coming from mimetic gravity to the energy density of the fluid, for simplicity we will call it \( \rho_{MG} \).

From the acceleration equation (20) we can identify the pressure added by mimetic gravity, \( \rho_{MG} = -V \). Using the Eqs. (19) and (20) we can obtain for both energy densities
\[ \dot{\rho} + 3H\rho(1 + \omega) + \dot{\rho}_{MG} + 3H\rho_{MG}(1 + \omega_{MG}) = 0, \] (21)
for a barotropic fluid. If we consider the Bianchi identity in Eq. (16) we have for the matter sector, \( \nabla^\mu T_{\mu\nu} = 0 \), which takes its standard form \( \dot{\rho} + 3H\rho(1 + \omega) = 0 \), therefore from Eq. (21) we have that the energy density, \( \rho_{MG} \), is also conserved. From the acceleration equation (20) we can obtain the deceleration parameter after a straightforward calculation, yielding
\[ q = \frac{1}{2} + \frac{3}{2} \left( \frac{\omega + \omega_{MG}\rho_{MG}}{\rho + \rho_{MG}} \right). \] (22)

Using the fact that \( \lambda = \rho_{MG} - V \), the equation (17) can be integrated to obtain
\[ \rho_{MG} = \frac{c}{a^3} + \frac{3}{a^3} \int_0^t a^3(t') H(t') V(t') dt', \] (23)
\[ = \frac{c}{a^3} + \frac{3}{a^3} \int_0^t a^2 V da, \] (24)
being \( c \) an integration constant. It is worthy to mention that for \( V = 0 \), the mimetic scalar field resembles the dark matter behavior since \( \rho_{MG} = G - T \propto a^{-3} \) and \( \rho_{MG} = 0 \). Typically one can have explicit solutions for \( \rho_{MG} \) until a specific Ansatz for \( V(\phi) \) is assumed, an extensive literature can be found on this topic following a route of this kind.

### III. CPL PARAMETRIZATION: RECONSTRUCTING THE POTENTIAL

Instead of choosing a specific form for the scalar field potential, let us consider the case where the corresponding equation of state parameter for the mimetic fluid is assumed to take the form of the CPL parametrization [29]
\[ \omega_{MG} = \omega_0 + \omega_a \frac{z}{1 + z}, \] (25)
where \( \omega_0 \) and \( \omega_a \) are constants, therefore from the previous expression and the conservation condition for \( \rho_{MG} \), the energy density evolves as a function of the redshift as follows
\[ \rho_{MG}(z) = \rho_0(1 + z)^3(1 + \omega_0 + \omega_a) \exp \left( -3\omega_a \frac{z}{1 + z} \right), \] (26)
where we have considered the standard relation between the scale factor and the redshift, \( 1 + z = a^{-1} \). On the other hand, if we equate the previous expression for the energy density with the Eq. (24) one gets an explicit form of the potential given as a function of the redshift
\[ V(z) = -\rho_0 \left[ \omega_0 + (\omega_0 + \omega_a)z \right] (1 + z)^2(1 + \omega_0 + \omega_a) \times \exp \left( -3\omega_a \frac{z}{1 + z} \right). \] (27)

Note that at present time \( z = 0 \) the potential takes the constant value \( V = -\rho_0 \omega_0 \) and at the far future, i.e., in the limit \( z \to -1 \) we have \( V(z \to -1) \to 0 \) only if \( \omega_a < 0 \). We display the potential \( V(z) \) in Fig. 1.

![Fig. 1: V(z) using the best fit values for \( \omega_0 \) and \( \omega_a \). Case (a): \( \omega_0 = -1.53 \), \( \omega_a = 0.26 \), case (b) corresponds to \( \omega_0 = -1.15 \), \( \omega_a = 1.41 \) of Ref. 30 and case (c) \( \omega_0 = -0.96 \), \( \omega_a = -0.29 \) from Planck 31.](image-url)

Notice that for case (a) the potential has a minimum around \( z \simeq 0.6 \) and start to increase to the past (large \( z \)). In the case (b) the potential decrease rapidly as redshift increase and there seems to reach a minimum around \( z \simeq 0.7 \) and after a small increase reaching a maximum around \( z \simeq 1.5 \). In the case (c), the potential
does not have a minimum and decrease monotonically as redshift increases.

In Fig. 2 we show the behavior of the energy density \( \rho(z) \) and \( \rho_{MG}(z) \). where we have considered the value \( \rho_0 = 1 \) in the three plots and we have used the observational values obtained in Ref. [30] for cases (a) and (b) for \( \omega_0 \) and \( \omega_a \) and the latest results from Planck for case (c) [31]. It is worthy to mention that in the cases (a) and (b) the energy density exhibits a growing behavior from the present time to the far future. This feature is interesting since resembles the expected behavior for dark energy, because at some point of the cosmic evolution this must become dominant over other components of the Universe. However, for the case (c) we observe that the energy density dilutes as the Universeexpand.

\[
\lambda(z) = \rho_0 (1+z)^2 + 3(\omega_0 + \omega_a) [1 + \omega_0 + (\omega_0 + \omega_a) z] \times \\
\exp \left( -3\omega_a \frac{z}{1+z} \right).
\]

At present time the Lagrange multiplier is only a constant given by \( \lambda_0 = \rho_0 (1 + \omega_0) \), which lies in the interval \( \rho_0 [-0.1, 0.04] \) in the cases (a), (b) and (c) commented before for the pair of values \( \omega_0 \) and \( \omega_a \). It is worthwhile to mention that in the absence of scalar field potential we have, \( \lambda = \rho_{MG} \), and from Eq. (24) we observe that the Lagrange multiplier can be used to model standard dark matter since it decays as \( a^{-3} \). Then, the responsible of having a late time accelerated expansion for the Universe is \( V(\phi) \) or \( V(t) \), because for \( V \neq 0 \) the \( a^{-3} \) behavior for the Lagrange multiplier is modified as can be seen in expression (28). This deviation from the \( a^{-3} \) behavior is characteristic of models in which interaction is allowed between dark matter and dark energy, represented in our case by \( \lambda \) and \( \phi \), respectively.

\[ E^2(z) := \frac{H^2(z)}{H_0^2} = \Omega_{m,0}(1+z)^3 + \]
\[ + \Omega_{MG,0}(1+z)^3(1+\omega_0+\omega_a) \exp \left( -3\omega_a \frac{z}{1+z} \right), \]

where we have considered the usual definition for the density parameters, \( \Omega_{x,0} := \rho_{x,0}/3H_0^2 \). Note that in this case we must have the normalization condition \( \Omega_m,0 + \Omega_{MG,0} = 1 \). In Fig. 3 we show the behavior of the Normalized Hubble parameter \( E(z) = H(z)/H_0 \) [29]. For the density parameter of the matter sector we considered the values \( 0.315 \pm 0.007 \) [31] and as in the previous plot, we take the same values for \( \omega_0 \) and \( \omega_a \) obtained in Ref. [30] for (a) and (b) and the values from [31] for (c). As can be observed, the behavior for the normalized Hubble parameter at late times with the three different pairs of values for the constants \( \omega_0 \) and \( \omega_a \) differs from the \( \Lambda \)CDM model evolution. For the cases (a) and (b) the normalized Hubble parameter diverges as we approach the far future \( (z = -1) \), this kind of future singularity corresponds to a little big rip [33]. On the other hand, if we consider a Universe filled with...
matter and dark energy which is modeled by the mimetic energy density, then we can write the coincidence parameter from Eq. (29) which is given as the quotient between the dark matter and dark energy densities as follows

\[ r(z) = r_0 (1 + z)^{-3(\omega_0 + \omega_a)} \exp\left(3\omega_a \frac{z}{1 + z}\right), \]  

(30)

where we have defined \( r_0 := \Omega_{m,0}/(1 - \Omega_{m,0}) \). The behavior of the coincidence parameter is shown in Fig. 4 with the cases (a), (b) and (c) for \( \omega_0 \) and \( \omega_a \), as discussed before and 0.315 ± 0.007 [31] for \( \Omega_{m,0} \). For the cases (a) and (b), the coincidence parameter decreases as the Universe expands. However, for the case (c) we have that around \( z \approx -0.8 \), this parameter changes its tendency and starts to increase. In order to be in agreement with observations we must have, \( r(z) \rightarrow -1 \rightarrow 0 \) [34], therefore the behavior for the coincidence parameter in case (c) is not the desired one. Then under the appropriate election of the parameters involved in the model, the cosmological coincidence problem can be alleviated if mimetic gravity models the dark energy content of the Universe with a CPL parametrization for the parameter state.

In terms of the normalized Hubble parameter we can write the deceleration parameter as follows

\[ q(z) = -1 + (1 + z) \frac{d \ln E(z)}{dz}. \]  

(31)

In Fig. 5 we depict the previous expression for the deceleration parameter with the Eq. (29) for \( E(z) \). We have used the same values for the constant parameters as was done in the previous plots for the cases (a), (b) and (c). As can be seen, \( q(z = 0) < 0 \) in the three cases, however only the case (a) and (b) maintain negative values (and increasing) as the Universe expands. At some stage of the cosmic evolution these models, the cases (a) and (b), could mimic the ΛCDM model \( (q = -1) \), but as can be seeing as we move to the future they reach a behavior where \( q < -1 \), which represents a phantom regime. Finally, for the case (c) we can observe that the cosmic evolution moves from a quintessence dark energy behavior to a decelerated expansion, i.e., in this case we can have a Milne Universe characterized by, \( q(z) = 0 \), at some stage of the cosmic evolution.

![Fig. 4: Coincidence parameter, \( r(z) \).](image)

![Fig. 5: Behavior of the deceleration parameter.](image)

Now we want to reconstruct \( V(\phi) \) numerically. For this task, we will use \( V(z) \) from (27) and the field \( \phi \) is obtained from the Hubble function definition \( H = a^2/a \) which implies

\[ \frac{dt}{aH} = \frac{dz}{(1 + z)H(z)}. \]  

(32)

where we have used that \( a = (1 + z)^{-1} \). This means that \( t(z) \) satisfy the equation

\[ \frac{dt}{dz} + \frac{1}{(1 + z)H(z)} = 0. \]  

(33)

Before use this numerically, let us divide by \( t_0 = t(z = 0) \) then defining the new variable \( \tau = t/t_0 \) the equation leads to

\[ \frac{d\tau}{dz} + \frac{1}{t_0 H_0 (1 + z)E(z)} = 0, \]  

(34)

where now \( \tau \) evolve in the range \( (0, 1) \) and \( E(z) \) is defined by the Eq. (29). For the ΛCDM model the quantity \( t_0 H_0 \) is given by

\[ t_0 H_0 = \frac{2}{3\sqrt{\Omega_V}} \ln \frac{1 + \sqrt{\Omega_V}}{\sqrt{1 + \Omega_V}} \approx 0.964 \]  

(35)

which the numerical value applies for \( \Omega_V = 0.7 \).
We display three realizations of \( V(\phi) \) for different pairs of CPL parameters: (a) \( w_0 = -1.03, w_a = 0.26 \), (b) \( w_0 = -1.1, w_a = 1.41 \) from [30] and (c) \( w_0 = -0.961 \) and \( w_a = -0.29 \) from Planck [31]. Using (34) we can integrate \( f(z) \) for each one of our three cases. In Fig. (6) we display the results of the integration. Notice that we have written \( \dot{\phi}/\phi_0 \) in the vertical axis, understanding for \( \phi_0 \) as the value of the field today.

As we can see all the three cases behaves similarly. Actually we have extended the plot until redshift \( z \simeq 12 \) to notice appreciable differences among the three cases. In what follows, we combine both previous results to numerically reconstruct the potential as a function of the field \( V(\phi) \) for our three cases. This is displayed in Fig. (7). As we can see, for case (a) the potential has a stable minimum around \( \phi \simeq 0.6 \phi_0 \) and start to increase slowly until today. The potential for case (b) shows a local minimum around \( \phi \simeq 0.5 \phi_0 \) and its value is increasing as the field \( \phi \) approach its current value. The case (c), which is the based on the Planck best fit, implies the potential is convex, with a maximum around \( \phi \simeq 0.6 \phi_0 \) after that its value decreases as the field approach its current value.

We would like to emphasize that the numerical reconstruction for \( V(\phi) \) coming from the expression [27] will represent in this case the appropriate scalar field potential for which the mimetic gravity will be in good agreement with the cosmological observations under the CPL parametrization, i.e., given a set of values fitted by the observations for \( \omega_0 \) and \( \omega_a \), we constructed the potential \( V(\phi) \). Reconstructions for the scalar field potential can be also found in the context of inflation, see for instance the Refs. [35, 36].

A. Gravitational matter production

If we consider that only matter production exists, then for a FLRW spacetime we have

\[
\dot{n} + 3Hn = n\Gamma, \quad \dot{\rho} + 3H(\rho + P) = 0, \quad (36)
\]

where \( \Gamma > 0, \Gamma < 0 \) acts like a source or sink of particles, respectively; \( n \) is the particle number density and \( P = p + p_c \). Here \( p_c \) accounts for the pressure from matter creation and is defined as

\[
p_c = -\frac{\rho + p}{3H}\Gamma, \quad (37)
\]

where we have considered an adiabatic expansion for the Universe, \( \dot{S} = 0 \). In general from the Gibbs law and Eqs. (36) we can write \( \dot{n}T \dot{S} = -3H\rho - (\rho + p)\Gamma \), in this case the Friedmann constraint reads, \( 3H^2 = \rho \), and \( T \) represents the temperature of the fluid which is definite positive. If we restrict ourselves to the case where the created matter behaves as dark matter, we have \( p = 0 \), therefore the continuity equation (36) takes the from \( \dot{\rho} + 3H\rho(1 + \omega_{\text{eff}}) = 0 \), where the effective parameter state is defined as

\[
\omega_{\text{eff}} = -\frac{\Gamma}{3H}, \quad (38)
\]

this form for the effective parameter state was also studied in Ref. [37]. We will focus on the following form for the particle production rate

\[
\Gamma = 3\xi H_0 \left( \frac{H}{H_0} \right)^\delta, \quad (39)
\]

being \( \xi \) and \( \delta \) dimensionless constants and \( H_0 \) represents the Hubble constant. This model was proposed in Refs. [38, 39] and contains most of the Ansätze found in the literature for \( \Gamma \) if we consider some specific values for \( \xi \)
and \(\delta\). By means of the Friedmann constraint and \((36)\), the solution for the Hubble parameter is given as follows:

\[
E(z) = \begin{cases} 
(1 + (1 - \xi)(1 + z)^{3(1-\delta)/2})^{1/\delta}, & \delta \neq 1, \\
(1 + z)^{3(1-\xi)/2}, & \delta = 1,
\end{cases} \tag{40}
\]

where, \(E(z) := H(z)/H_0\), it is known normalized Hubble parameter. If we consider only the first case of the previous solution we have

\[
\frac{H^2(z)}{H_0^2} = \left[ \xi + (1 - \xi)(1 + z)^{3(1-\delta)/2} \right]^{2/\delta}, \tag{41}
\]

therefore

\[
\rho(z) = \rho_0 \left[ \xi + (1 - \xi)(1 + z)^{3(1-\delta)/2} \right]^{2/\delta}. \tag{42}
\]

where \(\rho_0\) is the value of the density at present time. For this model we have the following expression for the effective parameter state given in \((38)\)

\[
\omega_{\text{eff}} = -\xi E(z)^{\delta-1}. \tag{43}
\]

We will focus in the interesting case given by \(\delta = -1\), known as creation of cold dark matter that was discussed extensively in Refs. \([38,40]\). Now, by considering the contribution of mimetic gravity plus matter production effects, the Friedmann constraint turns to \(3H^2 = \rho_{\text{MG}} + \rho\). Then the effective parameter state given in \((43)\) takes the following form if we consider the expressions \((26)\) for the mimetic gravity sector and \((12)\) for the created matter,

\[
\omega_{\text{eff}} = -\frac{\xi}{\Omega_{\rho,0}[\xi - (1 - \xi)(1 + z)^3] + (1 - \Omega_{\rho,0})(1 + z)^3(1 + \omega_0 + \omega_a) \exp \left[-3\omega_a \frac{z}{1+z}\right]}.
\]

Notice that in the previous expression we have considered the normalization condition, \(\Omega_{\rho,0} + \Omega_{\text{MG},0} = 1\). In Fig. \((4)\) we show the parameter state \((44)\) by considering the three pairs of values for \(\omega_0\) and \(\omega_a\) discussed in the previous section for the CPL parametrization and \(\xi = 0.72 \pm 0.05\), which was obtained in Ref. \([37]\) with the use of recent cosmological observations. As can be seen, in this approach the phantom scenario at effective level is provided only for the case (c) around \(z \approx -0.6\). For the cases (a) and (b) we have \(\omega_{\text{eff}}(z \to -1) \to 0\), i.e., the model transient from a quintessence scenario to a dark matter one type. This behavior for the effective parameter state is obtained since \(\omega_{\text{eff}} \propto (\rho_{\text{MG}})^{-1}\) and as can be seen in Fig. \((2)\), for the case (c) we have a decreasing behavior for \(\rho_{\text{MG}}\) which leads effectively to a phantom scenario and to a dark matter behavior for the cases (a) and (b).

**IV. FINAL REMARKS**

In this work we explored the mimetic gravity approach from a different perspective, i.e., we did not adopt any specific Ansatz for the scalar field potential, \(V(\phi)\). As discussed before, this potential is the responsible of the dark energy behavior at late times for this kind of Universe. This can be seen in the expression \((28)\) for the Lagrange multiplier, which deviates from the typical \(a^{-3}\) decay obtained in the mimetic description. The construction of the potential as a function of the cosmological redshift and its posterior numerical reconstruction as a function of the scalar field were possible with the assumption of the CPL parametrization for the parameter state of the fluid. In the discussion of our results we considered three cases labeled as (a), (b) and (c); which correspond to the best fit for the constants \(\omega_0\) and \(\omega_a\) of the CPL parametrization obtained with different sets of cosmological data, the case (c) was taken from the latest results of the Planck collaboration \([31]\).
Then, in each case we obtain a different shape for the potential; according to its functional form, we will have that $V(z \rightarrow -1) \gg 1$ for $\omega_a > 0$ and $V(z \rightarrow -1) \rightarrow 0$ for $\omega_a < 0$. This latter case is present in the pair of values (c). On the other hand, from Eq. (24) we can see that the potential determines the behavior of the energy density $\rho_{\text{MG}}$ and of course also the behavior of the Hubble parameter. This is the reason why the energy density (and the normalized Hubble parameter termed as $E(z)$) visualized in Figs. 2 and 3 tend to zero as we approach the far future for the case (c). Besides, for the cases (a) and (b) we have a future singularity at $z = -1$ for $E(z)$, therefore the model admits a little big rip.

It is worthy to mention that for the cases (a) and (b) we obtained desirable scenarios when the coincidence and deceleration parameters are computed. By assuming the standard form for the matter content and the dark energy sector described by mimetic gravity, we observed that the coincidence parameter is less than unity at present time and as we approach the far future this parameter decays to zero. This late behavior is not observed for the case (c), on contrary, the coincidence parameter starts to grow close to the far future. For the deceleration parameter we observe accelerated expansion for the cases (a) and (b) along cosmic evolution while in the case (c) the accelerated expansion is only a transient behavior, eventually the model has a smooth transition to a decelerated phase around $z \simeq -0.7$. In these scenarios, the Planck results are not favored by the nature of the observable Universe. However, if we include the particle production effects and construct the effective parameter state resulting from the combination with the mimetic gravity approach, we obtained that in this scheme the case (c) drives the cosmic evolution from quintessence to phantom scenario. For the three case we have that the effective parameter state at present time takes the constant value $\omega_{\text{eff,0}} = -\xi/[1-2\Omega_{\text{a},0}(1-\xi)] \simeq -0.87$, being $\xi$ a constant parameter characterizing the matter production effects. Finally, for the cases (a) and (b) the model evolves from quintessence to a dark matter type behavior at effective level. This new framework for mimetic gravity deserves a deeper investigation, we will discuss this topic elsewhere.

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