Unitary ambiguity in the extraction of the E2/M1 ratio for the
\[ \gamma N \leftrightarrow \Delta \] transition\(^*\)

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Abstract

The resonant electric quadrupole amplitude in the transition \( \gamma N \leftrightarrow \Delta(1232) \) is of great interest for the understanding of baryon structure. Various dynamical models have been developed to extract it from the corresponding photoproduction multipole of pions on nucleons. It is shown that once such a model is specified, a whole class of unitarily equivalent models can be constructed, all of them providing exactly the same fit to the experimental data. However, they may predict quite different resonant amplitudes. Therefore, the extraction of the E2/M1(\( \gamma N \leftrightarrow \Delta \)) ratio (bare or dressed) which is based on a dynamical model using a largely phenomenological \( \pi N \) interaction is not unique.

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The ratio $R_{EM}$ of the electric quadrupole to the magnetic dipole amplitude of the $\gamma N \leftrightarrow \Delta(1232)$ transition is an important quantity for our understanding of hadronic structure. It provides a powerful test for hadron models since it indicates a deviation from spherical symmetry. For example, in constituent quark models, it is directly related to the tensor interaction between quarks. Consequently, there is considerable experimental effort in measuring the corresponding $E_{1+}$ and $M_{1+}$ isospin $3/2$ multipole amplitudes for photo-production of pions on the nucleon \cite{1,2}. However, all realistic pion photoproduction models show that both multipoles, in particular $E_{1+}^{3/2}$, contain nonnegligible nonresonant background contributions. Unfortunately, their presence complicates the isolation of the resonant parts.

In the literature, there are basically two different approaches in order to extract the $\gamma N \leftrightarrow \Delta$ transition amplitudes. The first one is the Effective-Lagrangian-Approach (ELA) adopted by Olsson and Osypowsky \cite{3} and also used later on by Davidson, Mukhopadhyay and Wittman \cite{4}. In this approach, the $\pi N$ scattering is not treated dynamically and thus unitarity can be implemented only phenomenologically using different unitarization methods (K matrix, Olsson or Noelle prescription) which introduces some model dependence. However, in view of the phenomenological character of these methods the deeper origin of this model dependence remains unclear.

In the second approach, the $\pi N$ interaction is treated dynamically and thus unitarity is respected automatically. Various models of this type have been suggested in the past, e.g., Tanabe and Ohta \cite{5}, Yang \cite{6}, and Nozawa, Blankleider and Lee \cite{7}. However, due to our limited understanding of the dynamics of the $\pi N$ system, all these models are to a large extent phenomenological. Nevertheless, the necessity of such a dynamical treatment has been stressed again by Bernstein, Nozawa and Moinester \cite{8}. Thereby, it is implicitly assumed that the ongoing improvement of the experimental database (for both $\pi N \rightarrow \pi N$ and $\gamma N \rightarrow \pi N$) will finally allow to favor one of the models and thus will lead to a unique $R_{EM}$. In this paper we would like to point out an inherent unitary ambiguity in the latter approach which, to our knowledge, has never been discussed before.

Qualitatively, one may understand this unitary freedom in the following way. First of
all, the separation of a resonant $\Delta$ contribution corresponds to the introduction of a $\Delta$ component into the $\pi N$ scattering state which vanishes in the asymptotic region. The explicit form of a wave function depends on the chosen representation, which can be changed by means of unitary transformations. As a consequence, the probability of a certain wave function component is not an observable, because it depends on the representation. Classical examples are the deuteron $D$ wave or isobaric components in nuclei [9,10]. Introducing a phenomenological $\pi N$ interaction model always implies the choice of a specific representation. However, its relation to other representations and in particular its relationship to hadron models remains unknown. Thus, it is not clear in principle, how the extracted resonant multipoles, which are representation dependent, can be related to the $\gamma N \leftrightarrow \Delta$ transition matrix elements calculated within, e.g., a nonrelativistic quark model.

We will illustrate our arguments quantitatively by means of a simple model [11] whose main features are taken from Ref. [5]. It assumes as Hilbert space $\Delta \oplus \pi N \oplus \gamma N$ with corresponding projectors $P_\Delta, P_{\pi N}, P_{\gamma N}$, and a Hamiltonian of the form

$$h = t(m_\Delta) + v^B_{\pi\pi} + v_\pi + v^\dagger_{\pi\gamma} + v^B_{\pi\gamma} + v_\gamma,$$

(1)

with the background $\pi N$ interaction $v^B_{\pi\pi} = P_{\pi N}(h - t)P_{\pi N}$, the $\pi N\Delta$ vertex $v_\pi = P_\Delta h P_{\pi N}$, the nonresonant $\gamma N \rightarrow \pi N$ driving term $v^B_{\pi\gamma} = P_{\pi\gamma} h P_{\gamma N}$, and the $\gamma N\Delta$ vertex $v_\gamma = P_\Delta h P_{\gamma N}$. The kinetic energy $t$ in the $\Delta$ sector depends on the bare resonance mass $m_\Delta$ which is a model parameter. The pure hadronic sector $(v^B_{\pi\pi}, v_\pi, m_\Delta)$ of our model is identical to model B of [5] and thus yields a good fit of the $\pi N$ scattering phase shift in the $P_{33}$ channel. The electromagnetic background $v^B_{\pi\gamma}$ is modeled differently in order to guarantee gauge invariance (for details see [11]). For such a dynamical model, the general structure of one of the total pion production multipoles $M (M_1^{3/2} \text{ or } E_1^{3/2})$ is shown diagrammatically in Fig. [4]. It consists of three parts, namely in the notation of [8], the background $M_B$, the bare resonant multipole $M_\Delta$, and the vertex renormalization part $M_{VR}$. The sum $M_R = M_\Delta + M_{VR}$ is referred to as the dressed resonant multipole. Formally, one has

$$M = \langle \pi N^{(-)} | v_\gamma + v^B_{\pi\gamma} | \gamma N \rangle_M,$$

(2)
with the decomposition $M_{\Delta} = \langle \pi N(-)|v\gamma|\gamma N\rangle_M$ and $M_B + M_{VR} = \langle \pi N(-)|v_{\pi\gamma}^B|\gamma N\rangle_M$, where $|\pi N(-)\rangle$ denotes the $\pi N$ scattering state. The index $M$ on the r.h.s. indicates the angular momentum configuration for the magnetic dipole or the electric quadrupole absorption of the photon.

Any unitary transformation can be written as $U(\alpha) = e^{i\alpha\chi}$, with a generator $\chi = \chi^\dagger$ and an arbitrary real number $\alpha$. Clearly, only generators which are nondiagonal with respect to $\Delta \oplus \pi N$ have to be considered here. Keeping in mind that $\chi$ has to be odd under time reversal, a prototype is given by

$$\chi = i \left[ v_{\pi} + v_{\pi}^\dagger, v_{\pi\gamma}^B \right].$$

(3)

It obviously mixes resonant and background $\pi N$ interactions and leaves the $\gamma N$ sector unchanged. Assuming the background interaction to be of separable form, i.e., $v_{\pi\gamma}^B = \lambda|b\rangle\langle b|$ with $\langle b|b\rangle = 1$, as was actually done in [5–7], $U(\alpha)$ can be evaluated without a perturbative expansion. Even though the total pion production multipole $M$ remains invariant under $U(\alpha)$, its decomposition changes according to

$$M_{\Delta}(\alpha) = \langle \pi N(-)|U(-\alpha)P_{\Delta}U(\alpha)(v\gamma + v_{\pi\gamma}^B)|\gamma N\rangle_M, \quad (4)$$

$$M_B(\alpha) + M_{VR}(\alpha) = \langle \pi N(-)|U(-\alpha)P_{\pi N}U(\alpha)(v\gamma + v_{\pi\gamma}^B)|\gamma N\rangle_M. \quad (5)$$

Note that by construction $U(\alpha)$ does not modify the initial state $|\gamma N\rangle$. Actually, it would be of interest to find out the representation dependence of both the bare and the dressed resonant multipole because, as pointed out in [5], it is intuitive to compare predictions of nucleon models without and with a pion cloud to the bare and dressed resonant multipoles, respectively. In this paper we focus on the bare multipoles. One finds

$$M_{\Delta}(\alpha) = M_{\Delta}(0) \left[ 1 - \frac{1}{2} (1 - r g_M) (1 - \cos 2\tilde{\alpha}) + \frac{1}{2} (r + g_M) \sin 2\tilde{\alpha} \right] \quad (6)$$

with $g_M = \langle b|v_{\pi\gamma}^B|\gamma N\rangle_M/\langle \Delta|v\gamma|\gamma N\rangle_M$ and $r = \langle \pi N(-)|b\rangle/\langle \pi N(-)|\Delta\rangle$, where $|\Delta\rangle$ is the bare $\Delta$ state, i.e., $P_{\Delta} = |\Delta\rangle\langle \Delta|$. Moreover, we have introduced a dimensionless parameter $\tilde{\alpha}$ which is proportional to $\alpha$ (for details see [11]). Note, that $M_{\Delta}(\alpha)$ still carries the $P_{33}$ phase
shift. It is easily verified that, irrespective of the model quantities \( r \) and \( g_M \), Eq. (1) implies that \( M_\Delta(\alpha) \) always goes through zero for a certain value of \( \alpha \). Consequently, the ratio of the bare multipoles \( R_{EM}^\Delta = E^{3/2}_{1+,\Delta}/M^{3/2}_{1+,\Delta} \) as a function of \( \alpha \) is in principle unbound.

The representation dependence of both bare multipoles is plotted in Fig. 2. It is already sufficient to consider only transformations close to the identity. Even then, the bare amplitudes change substantially, as can be seen by comparing the dotted (\( \tilde{\alpha} = 10^\circ \)) and dash-dotted curves (\( \tilde{\alpha} = -10^\circ \)) with the dashed one (\( \tilde{\alpha} = 0 \)). For positive \( \tilde{\alpha} \) also the bare electric multipole exhibits a more pronounced resonance behavior. For negative \( \tilde{\alpha} \), the bare multipoles, in particular \( E^{3/2}_{1+,\Delta} \), come closer to those of Nozawa et al. (see Fig. 2 of [8]). For completeness we note that the predicted total multipoles are in satisfactory agreement with experimental results. Even though we have demonstrated the representation dependence for the bare multipoles, a similar though weaker dependence occurs also for the dressed multipoles.

The ratio \( R_{EM}^\Delta \) is plotted for \( \tilde{\alpha} = 0^\circ, \pm 5^\circ, \pm 10^\circ \) in Fig. 3. At resonance, it varies strongly between \(-1.5\% \) and \(-5\% \). The ratio predicted by our original model is \(-3.1\% \) [12], which is identical to the result of [7]. The transformed ratios exhibit a slight energy dependence whereas the original one is energy independent, which is just a consequence of the simple ansatz for \( v_\gamma \) in [5] and does not have a deeper physical origin. Moreover, the generated energy dependence is weak compared to the dependence on \( \tilde{\alpha} \).

Now it remains to check whether the transformed Hamiltonian \( h(\alpha) = U(\alpha) h U(-\alpha) \) corresponds to a “physically reasonable” interaction model. Therefore, we write it in the following form

\[
h(\alpha) = t(m_\Delta(\alpha)) + v_{\pi\pi}^B(\alpha) + v_\pi(\alpha) + v_\pi^I(\alpha) + v_{\pi\gamma}^B(\alpha) + v_\gamma(\alpha),
\]

(7)

where \( m_\Delta(\alpha) = \langle \Delta | h(\alpha) | \Delta \rangle \). The interaction pieces are defined completely analogous to Eq. (1), e.g., \( v_{\pi\pi}^B(\alpha) = P_{\pi N} (h(\alpha) - t) P_{\pi N} \). The explicit \( \alpha \)-dependence of the various terms is rather lengthy and will be reported elsewhere [11]. Since one deals with semiphenomenological interactions here, the only criterion whether Eq. (7) is “physically resonable” will
be the shape of the transformed form factors. We will demonstrate this by considering, for example, the $\pi N\Delta$ form factor. Suppressing the isospin structure, it reads

$$\langle \Delta | v_\pi(\alpha) | \vec{q} \rangle = i \vec{S} \cdot \vec{q} v_\pi(q; \tilde{\alpha}) ,$$

where $| \vec{q} \rangle$ denotes a plane wave $\pi N$ state with relative momentum $\vec{q}$ and $\vec{S}$ the $N \rightarrow \Delta$ transition spin operator. In Fig. 4, we have plotted $q v_\pi(q; \tilde{\alpha})$ for various values of $\tilde{\alpha}$. Apparently, none of them can be ruled out. Here, we just mention that the modifications of the remaining parts of the transformed Hamiltonian do not change this conclusion [11]. Incidentally, the high momentum components become more and more suppressed when going from $\tilde{\alpha} = +10^\circ$ to $-10^\circ$ with a corresponding increase of $R_{EM}^\Delta$ from $-5\%$ to $-1.5\%$.

In summary, we have demonstrated that any extraction of resonant (bare or dressed) and nonresonant contributions from the experimental $E_{1+}^{3/2}$ and $M_{1+}^{3/2}$ multipoles for photoproduction of pions from nucleons which is based on a dynamical treatment of a phenomenological $\pi N$ interaction model suffers from inherent ambiguities. More precisely, once a phenomenological model is specified, a whole class of completely equivalent models can be constructed by means of unitary transformations, all of them providing exactly the same fit to the experimental data while predicting different ratios $R_{EM}^\Delta$. Incidentally, also the K-matrix residues in the $\Delta$ region which have been extracted by Davidson and Mukhopadhyay [14] are not affected by the unitary freedom.

Thus we have to conclude that even with a perfectly accurate data base one will not be able to discriminate between any of these models, which actually are merely different representations. Moreover, those representations which are sufficiently close to the original one, are not at all less “physically reasonable” because they cannot be excluded by arguments based on physical intuition, say, what form factors should look like. However, even those models close to the original one predict significantly different resonant amplitudes. With respect to our example, none of the different representations, say for $|\tilde{\alpha}| \leq 10^\circ$, can be favored, although the resonant multipoles differ considerably, in particular the ratio $R_{EM}^\Delta$ varies from $-1.5\%$ to $-5\%$. But one has to keep in mind, that this variation may be even
larger if one considers other choices for $\chi$ than the one in Eq. (3). However, $\chi$ should not contain any quantity which is completely unrelated to the original Hamiltonian.

With respect to the model dependence in the ELA, mentioned in the introduction, it remains to be clarified in the future whether it can be traced back to unitary transformations relating the different unitarization procedures. The arguments presented here may also affect the separation of resonant and nonresonant amplitudes in other reactions like, e.g., the particularly interesting $S_{11}(1535)$ in the photoproduction of $\eta$ mesons on nucleons. Notwithstanding this unitary ambiguity in the extraction of resonance properties, we would like to stress the urgent need for more precise data on pion photoproduction in the $\Delta$ region providing the necessary basis for an accurate multipole analysis. However, the challenge is on the theoretical side because for a clean test of any microscopic hadron model, the amplitude for the complete process $\gamma N \rightarrow \pi N$ including background contributions rather than the $\gamma N \leftrightarrow \Delta$ transition alone, has to be calculated dynamically within the same model.
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FIGURES

FIG. 1. The pion photoproduction multipole $M_{1+}^{3/2}$ or $E_{1+}^{3/2}$.

FIG. 2. The multipoles $M_{1+}^{3/2}$ and $E_{1+}^{3/2}$ as functions of the photon laboratory energy $E_{\gamma}$. Dashed, dotted and dash-dotted curves are bare multipoles corresponding to transformation angles $\tilde{\alpha} = 0^\circ$, $10^\circ$ and $-10^\circ$, respectively. The solid curves show the total multipoles which are representation independent. The data are taken from [13].
FIG. 3. The ratio $R_{EM}^{\Delta}(\tilde{\alpha})$ for $\tilde{\alpha} = 0^\circ, \pm 5^\circ, \pm 10^\circ$ as function of the photon laboratory energy $E_\gamma$.

FIG. 4. The $\pi N\Delta$ form factor $q\psi_\pi(q; \tilde{\alpha})$ from Eq. (8) for $\tilde{\alpha} = 0^\circ, \pm 5^\circ, \pm 10^\circ$. 