About a Class of Analytic Functions Defined by Noor-Sălăgean Integral Operator

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Abstract: In this paper we introduce a new integral operator as the convolution of the Noor and Sălăgean integral operators. With this integral operator we define the class $C_{NS}(\alpha)$, where $\alpha \in [0,1)$ and we study some properties of this class.

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1. Introduction

Let $U = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(U)$ denote the set of holomorphic (analytic) functions in $U$. We denote by

$$A = \{ f \in H(U) : f(0) = f'(0) - 1 = 0 \}$$

and

$$S = \{ f \in A : f \text{ is univalent in } U \}.$$

We say that $f$ is starlike in $U$ if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a starlike domain in $\mathbb{C}$ with respect to origin. It is well-known that $f \in A$ is starlike in $U$ if and only if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \text{ for all } z \in U.$$

The class of starlike functions with respect to origin is denoted by $S^*$. Let $T$ denote a subclass of $A$ consisting of functions $f$ of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,$$  \hspace{1cm} (1.1)
where $a_j \geq 0$, $j = 2, 3, \ldots$ and $z \in U$. A function $f \in T$ is called a function with negative coefficients. For the class $T$, the followings are equivalent [6]:

(i) $\sum_{j=2}^{\infty} ja_j \leq 1$,

(ii) $f \in T \cap S$,

(iii) $f \in T^*$, where $T^* = T \cap S^*$.

Let $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \ldots$

and $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, \ldots$

then the convolution or the Hadamard product is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), \quad z \in U.$$ 

The study of operators plays an important role in geometric function theory. For $f \in H(U)$, $f(0) = 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the $I^n_S$ Sălăgean integral operator is defined as follows [7]:

(i) $I^n_S f(z) = f(z)$,

(ii) $I^n_S f(z) = I f(z) = \int_0^z f(t)t^{-1}dt$,

(iii) $I^n_S f(z) = I_S(I^n_S f(z))$.

We remark that if $f$ has the form (1.1), then

$$I^n_S f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{j^n} z^j,$$

where $n \in \mathbb{N}_0$.

In [5] Noor defined an integral operator $I^n_N : \mathcal{A} \to \mathcal{A}$ as follows

$$I^n_N f(z) = \frac{n+1}{z^n} \int_0^z t^{n-1} I^n_N (f(t)) dt,$$

where $n \in \mathbb{N}_0$.

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$ and let $f_n^{(-1)}(z)$ be defined such that
We note that
\[ I_n^R f(z) = f_n^{(-1)}(z) * f(z) = \left[ \frac{z}{(1-z)^{n+1}} \right]^{(-1)} * f(z). \]

We remark that if \( f \) has the form (1.1), then
\[
I_n^R f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{C(n,j)} z^j,
\] (1.4)
where \( C(n,j) = \frac{(n+j-1)!}{n!(j-1)!} \).

2. Preliminaries

The following definitions and lemmas will be required in the sequel.

**Definition 2.1.** [2, 3] Let \( f \) and \( g \) be analytic functions in \( U \). We say that the function \( f \) is subordinate to the function \( g \), if there exist a function \( w \), which is analytic in \( U \) and for which \( w(0) = 0, |w(z)| < 1 \) for \( z \in U \), such that \( f(z) = g(w(z)) \), for all \( z \in U \). We denote by \( \prec \) the subordination relation.

**Definition 2.2.** [3] Let \( Q \) be the class of analytic functions \( q \) in \( U \) which has the property that are analytic and injective on \( U \setminus E(q) \), where \( E(q) = \{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \} \), and are such that \( q'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(q) \).

**Lemma 2.1.** [2, 3] Let \( q \in Q \), with \( q(0) = a \), and let \( p(z) = a + a_n z^n + \ldots \) be analytic in \( U \) with \( p(z) \neq a \) and \( n \geq 1 \). If \( p \not\prec q \), then there are two points \( z_0 = r_0 e^{i\theta_0} \in U \), and \( \zeta_0 \in \partial U \setminus E(q) \) and a number \( m \geq n \geq 1 \) for which \( p(U_{r_0}) \subset q(U) \),

(i) \( p(z_0) = q(\zeta_0) \)
(ii) \( z_0 p'(z_0) = m \zeta_0 q'(\zeta_0) \)
(iii) \( \Re \frac{z_0 p''(z_0) + 1}{p(z_0)} + 1 \geq m \Re \left( \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right). \)

The following result is a particular case of Lemma 2.1.

**Lemma 2.2.** [2, 3] Let \( p(z) = 1 + a_n z^n + \ldots \) be analytic in \( U \) with \( p(z) \neq 1 \) and \( n \geq 1 \). If \( \Re p(z) \neq 0, z \in U \), then there is a point \( z_0 \in U \), and there are two real numbers \( x, y \in \mathbb{R} \) such that

(i) \( p(z_0) = ix \)
(ii) \( z_0 p'(z_0) = y \leq -\frac{n(x^2+1)}{2} \),
(iii) \( \Re z_0^2 p''(z_0) + z_0 p'(z_0) \leq 0. \)
If \( f(z) = z - \sum_{j=2}^{\infty} a_j z^j \), using the Noor and Salagean integral operators we define a new operator as follows:

\[
I_{NS}^n f(z) = I_N^nf(z) * I_S^nf(z) = z - \sum_{j=2}^{\infty} \frac{a_j^2}{j^nC(n,j)} z^j, \tag{2.1}
\]

where \( C(n,j) = \frac{(n + j - 1)!}{n!(j - 1)!} \) and \( n \in \mathbb{N}_0 \).

**Remark 2.1.** Differentiate the relation (2.1), we get

\[
[I_{NS}^n f(z)]' = 1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n-1C(n,j)} z^{j-1}. \tag{2.2}
\]

Multiplying the equality (2.2) with \( z/n \) we obtain

\[
\frac{z}{n}[I_{NS}^n f(z)]' = \frac{z}{n} - \sum_{j=2}^{\infty} \frac{a_j^2}{nj^{n-1}C(n,j)} z^j,
\]

which is equivalent to

\[
\frac{z}{n}[I_{NS}^n f(z)]' + \frac{z}{n} (n-1) = z - \sum_{j=2}^{\infty} \frac{a_j^2}{nj^{n-1}C(n,j)} z^j. \tag{2.3}
\]

Now let \( g \in T \) and \( g(z) = z - \sum_{j=2}^{\infty} (n + j - 1) z^j \). Then from (2.3), we obtain the following relation between \( I_{NS}^{n-1} f(z) \) and \( I_{NS}^n f(z) \) operators:

\[
I_{NS}^{n-1} f(z) = \frac{z}{n}[I_{NS}^n f(z)]' * g(z) + \frac{n-1}{n} z * g(z). \tag{2.4}
\]

Using the Noor-Salagean integral operator, we define the following class of analytic functions:

**Definition 2.3.** A function \( f \in T \) belongs to the class \( C_{NS}(\alpha) \) if

\[
\text{Re} \left\{ \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} \right\} > \alpha, \tag{2.5}
\]

where \( \alpha \in [0,1] \) and \( z \in U \).

### 3. Main Results

**Theorem 3.1.** Let \( f(z) = z - \sum_{j=2}^{\infty} a_j z^j \). Then \( f \in C_{NS}(\alpha) \) if and only if

\[
\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[ 1 - \frac{\alpha}{j} \right] < 1 - \alpha. \tag{3.1}
\]
Proof. Let \( f \in C_{NS}(\alpha) \), then we have
\[
\text{Re} \left( \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} \right) > \alpha, \quad z \in U.
\]
If \( z \in [0, 1) \), we obtain
\[
z - \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} z^j > \alpha.
\]
(3.2)

Since the denominator of (3.2) is positive, the relation (3.2) is equivalent with
\[
\alpha - 1 \leq \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} z^{j-1} - \sum_{j=2}^{\infty} \frac{a_j^2}{j^nC(n,j)} z^{j-1}.
\]
and finally we get
\[
\alpha - 1 \leq \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left( \frac{\alpha}{j} - 1 \right).
\]
Considering \( z \to 1^- \) along to the real axis, we get:
\[
\alpha - 1 \leq \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left( \frac{\alpha}{j} - 1 \right).
\]

To prove the reciprocal implication we consider \( f \) with the form (1.1) and for which the (3.1) inequality holds.

The condition \( \text{Re} \left( \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} \right) > \alpha \) is equivalent to
\[
\alpha - \text{Re} \left( \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right) < 1.
\]

We have
\[
\alpha - \text{Re} \left( \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right) \leq \alpha + \left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right|
\]
\[
= \alpha + \left| \sum_{j=2}^{\infty} \frac{a_j^2}{j^{nC(n,j)}} z^j - \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} z^j \right|
\]
\[
= \alpha + \left| \sum_{j=2}^{\infty} \frac{a_j^2}{j^{nC(n,j)}} \frac{a_j^2}{j^{n-1}C(n,j)} z^j \right|
\]
\[
= \alpha + \left| \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left( \frac{1}{j} - 1 \right) \right|
\]
\[
\leq \alpha + \frac{\sum_{j=2}^{\infty} \frac{a_j^2}{j^{nC(n,j)}} |z|^{j-1} \left| \frac{1}{j} - 1 \right|}{1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^{nC(n,j)}} |z|^{j-1}} < \alpha + \frac{\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left( 1 - \frac{1}{j} \right)}{1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^{nC(n,j)}} |z|^{j-1}}.
\]
\[ \alpha + \sum_{j=2}^{\infty} \frac{a_j}{j^{\alpha} C(n,j)} \left[ 1 - \frac{1}{j} - \frac{\alpha}{j} \right] \]

To finish our proof, we need to show
\[ \alpha + \sum_{j=2}^{\infty} \frac{a_j}{j^{\alpha} C(n,j)} \left[ 1 - \frac{1}{j} - \frac{\alpha}{j} \right] \frac{1}{1 - \sum_{j=2}^{\infty} \frac{a_j}{j^{\alpha} C(n,j)}} < 1. \] \quad (3.3)

The (3.3) inequality is equivalent to
\[ \sum_{j=2}^{\infty} \frac{a_j}{j^{\alpha-1} C(n,j)} \left[ 1 - \frac{\alpha}{j} \right] < 1 - \alpha, \] \quad (3.4)

which is the (3.1) condition. \qed

Let \( E_{NS}(\alpha) \) be a subclass of \( C_{NS}(\alpha) \). The class is defined as follows:
\[ E_{NS}(\alpha) = \left\{ f \in T : \left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| < 1 - 2\alpha \text{ and } \alpha \in \left(0, \frac{1}{2}\right) \right\}. \] \quad (3.5)

**Theorem 3.2.** Let \( f \in T \) of the form (1.1). If \( f \in E_{NS}(\alpha) \), then \( \text{Re} \frac{I_{NS}^n f(z)}{z} > 0 \).

**Proof.** Suppose \( f \in E_{NS}(\alpha) \). Then
\[ \left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| < 1 - 2\alpha. \] \quad (3.6)

Let
\[ I_{NS}^n f(z) = zp(z). \] \quad (3.7)

Differentiate (3.7), we obtain
\[ [I_{NS}^n f(z)]' = zp'(z) + p(z). \] \quad (3.8)

Then (3.6) is equivalent to
\[ \left| \frac{zp'(z)}{p(z)} \right| < 1 - 2\alpha. \]

If the condition \( \text{Re} p(z) = \text{Re} \frac{I_{NS}^n f(z)}{z} > 0 \) does not hold, then according to Lemma 2.2, there is a point \( z_0 \in U \), and there are two real numbers \( x, y \in \mathbb{R} \) such that
\[ p(z_0) = ix \]
and
\[ z_0p'(z_0) = y \leq \frac{1 + x^2}{2}. \]
These inequalities imply
\[ \left| \frac{z_0p'(z_0)}{p(z_0)} \right| = \left| \frac{y}{ix} \right| \geq \left| \frac{\frac{1}{2}(1 + x^2)}{x} \right| = \left| \frac{1}{2} \left( x + \frac{1}{x} \right) \right| \geq 1 - 2\alpha. \]

The above inequality contradicts (3.6) and consequently
\[ \Re p(z) = \Re \frac{p_{NS}(z)}{z} > 0, \]
where \( z \in U \).

**Theorem 3.3.** Let
\[ F(z) = L_c f(z) = \frac{c + 1}{z^c} \int_0^z f(t)t^{c-1}dt, c \in \mathbb{N}. \]
If \( f \in C_{NS}(\alpha) \), then \( F = L_c f \in C_{NS}(\beta) \), where
\[ \beta = \beta(\alpha, 2) = 1 - \frac{(1 - \alpha)(c + 1)^2}{(c + 2)^2(2 - \alpha) - (c + 1)^2(1 - \alpha)} \quad (3.9) \]
and \( \beta > \alpha, \alpha \in [0, 1) \).

**Proof.** Suppose \( f \in C_{NS}(\alpha) \). Then by Theorem 3.1 we have
\[ \sum_{j=2}^{\infty} \frac{a_j^2(j - \alpha)}{j^n C(n, j)(1 - \alpha)} < 1. \]
We know that if \( f \) has the form (1.1), then
\[ F(z) = \frac{c + 1}{z^c} \int_0^z f(t)t^{c-1}dt = z - \sum_{j=2}^{\infty} \frac{c + 1}{c + j} a_j z^j, \]
and to prove that \( F \in C_{NS}(\beta) \) is sufficient to have
\[ \sum_{j=2}^{\infty} \frac{j - \beta}{j^n C(n, j)(1 - \beta)} \left( \frac{c + 1}{c + j} \right)^2 a_j^2 < 1. \]
This last inequality is implied by
\[ \frac{j - \beta}{1 - \beta} \frac{(c + 1)^2 a_j^2}{j^n C(n, j)(c + j)^2} \leq \frac{j - \alpha}{1 - \alpha} \frac{a_j^2}{j^n C(n, j)}. \quad (3.10) \]
for all \( j \in \mathbb{N} \) and \( j \geq 2 \).

From (3.10) we deduce that
\[
\beta \leq 1 - \frac{(1 - \alpha)(c + 1)^2(j - 1)}{(c + j)^2(j - \alpha) - (c + 1)^2(1 - \alpha)} = \beta(\alpha, j),
\]
\( j \in \mathbb{N}, j \geq 2 \). We will prove that
\[
\beta(\alpha, j) \geq \beta(\alpha, 2), \ j \in \mathbb{N}, \ j \geq 2.
\]

Let consider the function \( \varphi : [2, \infty) \to \mathbb{R} \),
\[
\varphi(x) = \frac{x - 1}{(x + c)^2(x - \alpha) - (c + 1)^2(1 - \alpha)}, \ x \in [2, \infty).
\]
Then
\[
\varphi'(x) = \frac{g(x)}{((x + c)^2(x - \alpha) - (c + 1)^2(1 - \alpha))^2},
\]
where \( g(x) = -2x^3 + (3 - 2c - \alpha)x^2 + (4c - 2\alpha)x - 2c - (1 - \alpha) \).

We have
\[
g'(x) = -6x^2 + 2(3 - 2c - \alpha)x + 4c - 2\alpha,
g''(x) = -12x + 6 - 4c - 2\alpha < 0,
\]
\( x \in [2, \infty) \). Then
\[
g'(x) \leq g'(2) = -12 - 4c - 6\alpha < 0, \ x \in [2, \infty)
\]
and
\[
g(x) \leq g(2) = -4 - 8\alpha - 2c - (1 - \alpha) < 0, \ x \in [2, \infty).
\]

We obtain \( \varphi'(x) < 0, \ x \in [2, \infty) \) and from this
\[
\beta(\alpha, j) = 1 - \varphi(j)(1 - \alpha)(c + 1)^2 \geq 1 - \varphi(2)(1 - \alpha)(c + 1)^2 = \beta(\alpha, 2),
\]
where \( \beta(\alpha, 2) \) is given by (3.9). Finally \( \beta > \alpha \) is equivalent to
\[
1 - \alpha > \frac{(1 - \alpha)(c + 1)^2}{(c + 2)^2(2 - \alpha) - (c + 1)^2(1 - \alpha)}.
\]

\( \Box \)

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