Group identities on the units of algebraic algebras with applications to restricted enveloping algebras

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Abstract

An algebra $A$ is called a GI-algebra if its group of units $A^\times$ satisfies a group identity. We provide positive support for the following two open problems.

1. Does every algebraic GI-algebra satisfy a polynomial identity?
2. Is every algebraically generated GI-algebra locally finite?

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1 Introduction

Let $A$ be a unital associative algebra over a field $\mathbb{F}$. We shall denote by $A^\times$ the group of all multiplicative units of $A$. Recall that a group $G$ is said to satisfy a group identity whenever there exists a nontrivial word $w(x_1, \ldots, x_n)$ in the free group generated by \{x_1, x_2, \ldots\} such that $w(g_1, \ldots, g_n) = 1$, for all $g_1, \ldots, g_n \in A^\times$. By way of analogy with the custom of referring to an algebra as a PI-algebra whenever it satisfies a polynomial identity, an algebra $A$ is sometimes called a GI-algebra if $A^\times$ satisfies a group identity. We shall also call a Lie algebra satisfying a (Lie) polynomial identity a PI-algebra.

The class of all GI-algebras has received considerable attention recently. See [8] for a recent and comprehensive overview. In particular, it was shown in [17] that every algebraic GI-algebra is locally finite. Furthermore, if the base field is infinite then such an algebra satisfies a polynomial identity (see also [7]).

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In fact, rather more can be said. But first we need to introduce some
more notation.

Recall that an associative algebra \( A \) can be regarded as a Lie algebra
via \([a, b] = ab - ba\), for every \( a, b \in A \). As is customary, we shall call a Lie
algebra bounded Engel if it satisfies a polynomial identity of the form

\[ [x, y, \ldots, y] = 0. \]

For \( x, y \in A^\times \), we write \((x, y)\) for the group commutator \( x^{-1}y^{-1}xy \). By
analogy, a group is called bounded Engel if it satisfies a group identity of
the form

\( (x, y, \ldots, y) = 1 \).

We shall say that \( A \) is Lie solvable (respectively, bounded Engel or Lie
nilpotent) to mean that \( A \) is solvable as a Lie algebra (respectively, bounded
Engel or nilpotent).

A polynomial identity is called non-matrix if it is not satisfied by the
algebra \( M_2(F) \) of all \( 2 \times 2 \) matrices over \( F \).

For an algebra \( A \), we denote by \( \mathcal{B}(A) \) its prime radical, by \( \mathcal{J}(A) \) its
Jacobson radical, and by \( \mathcal{Z}(A) \) its center. Also, the set of all nilpotent
elements of \( A \) will be denoted by \( \mathcal{N}(A) \).

**Proposition 1.1** Let \( A \) be an algebraic algebra over an infinite field \( F \) of
characteristic \( p \geq 0 \). Then the following conditions are equivalent:

1. The algebra \( A \) is a GI-algebra.

2. The group of units \( A^\times \) is solvable, in the case when \( p = 0 \), while \( A^\times \)
satisfies a group identity of the form \((x, y)^{p^t} = 1\) for some natural
number \( t \), in the case when \( p > 0 \).

3. The algebra \( A \) satisfies a non-matrix polynomial identity.

4. The algebra \( A \) is Lie solvable, in the case when \( p = 0 \), while \( A \) sat-
sifies a polynomial identity of the form \(([x, y]z)^{p^t} = 0\) for some natural
number \( t \), in the case when \( p > 0 \).

Furthermore, in this case, \( \mathcal{N}(A) = \mathcal{B}(A) \) is a locally nilpotent ideal of \( A \) and
\( A/\mathcal{B}(A) \) is both commutative and reduced.

**Proof.** We need only collect various known results. First we remark that
every algebraic algebra (or algebra generated by its algebraic elements) over
a field with more than 2 elements is generated by its units. This is a simple
consequence of Wedderburn’s theorem. Consequently, if \( A^\times \) satisfies a group
identity then Proposition 1.2 and Theorem 1.3 in [8] apply yielding the fact that $N(A)$ is a locally nilpotent ideal of $A$ coinciding with the prime radical $B(A)$. By Theorem 1.2 of [6], the subalgebra $F \cdot 1 + N(A)$ satisfies a non-matrix polynomial identity. Since $A/B(A)$ is commutative, by Corollary 1.4 of [8], it is easy to check that $A$ itself satisfies a non-matrix polynomial identity (see [23]). The remaining implications follow as in Theorems 1.3 and 1.4 in [6]. □

In the previous proof we refer to [8] where algebras generated by units are considered. Hence one might maybe expect that Proposition 1.1 is valid in this context as well. However, this is not the case. Indeed, let $FG$ be the group algebra of a relatively free nilpotent group $G$ of class $n > 1$ over a field $F$. By [25] (Chapter V, Corollary 1.7), all units of $FG$ are of the form $\alpha g$, with $\alpha \in F^\times$ and $g \in G$. In particular, the group of units of $FG$ is nilpotent. On the other hand, in view of [15] and [19], $FG$ cannot satisfy any polynomial identity.

Moreover, it is evident that Proposition 1.1 does not hold for finite base fields; however, the following fundamental problem remains open.

**Problem 1.2** Is every algebraic GI-algebra (over a finite field) a PI-algebra?

Problem 1.2 is a broad generalization of a problem first posed by Brian Hartley. Hartley asked if the group algebra $FG$ of a locally finite-$p$ group $G$ over a field $F$ of characteristic $p > 0$ is a PI-algebra whenever it is a GI-algebra. This was subsequently proved true for all periodic groups $G$ and fields $F$ in [11], [12], [16] and [18].

Understanding nilpotent algebras $N$ whose adjoint group $1 + N$ satisfies a group identity is crucial to solving Problem 1.2 in general. Specifically, if such an algebra $N$ satisfies a polynomial identity that depends only on the particular group identity, then Problem 1.2 would have a general positive solution. Indeed, suppose that $A$ is any algebraic algebra over a finite field $F$ such that $A^\times$ satisfies a particular group identity $w = 1$. Let $H$ be an arbitrary finitely generated (unital) subalgebra of $A$. Then, as mentioned earlier, $H$ is finite-dimensional over $F$, and hence actually finite. Thus, by Wedderburn’s theorem, $H/J(H)$ is the direct sum of matrix algebras (over finite fields). So $(H/J(H))^\times$ is the direct product of general linear groups each satisfying $w = 1$. But the degree of a general linear group satisfying $w = 1$ is bounded (see [17]). Hence, $H/J(H)$ satisfies a polynomial identity determined only by $w = 1$. Thus, by our assumption applied to $N = J(H)$, $H$ itself would satisfy a polynomial identity depending only on the group identity $w = 1$. Consequently, $A$ would be a PI-algebra.
Our present focus, however, is to give a positive solution to Problem 1.2 for all algebraic algebras whose units are either solvable or bounded Engel. Furthermore, we intend to characterize these group-theoretic identities in terms of the corresponding Lie polynomial identities.

We shall also address a second fundamental problem. Recall that an algebra is said to be algebraically generated if it is generated by its algebraic elements.

**Problem 1.3** *Is every algebraically generated GI-algebra locally finite?*

Because algebraic GI-algebras are locally finite, in order to solve Problem 1.3, it suffices to show that every algebraically generated GI-algebra is algebraic. Even in the case when the ground field is infinite, this is not known. As evidence for a positive solution, we offer the fact proved in [8] that every group algebra of a periodically generated group over an infinite field is locally finite provided its group of units satisfies a group identity. We shall offer presently some additional support to Problem 1.3 in the case of nilpotently generated algebras. The case when the base field is infinite was settled positively by Theorem 1.2 in [6].

Finally, in the last section, we apply our general results to the special case when the algebra in question is the restricted enveloping algebra of a restricted Lie algebra. Unlike the group algebra case, very little is known about the group of units of a restricted enveloping algebra.

## 2 Algebraic GI-algebras over finite fields

We have seen in the Introduction that Proposition 1.1 answers Problem 1.2 in the case when the ground field is infinite. It is clear, however, that Proposition 1.1 does not extend verbatim to finite ground fields. We are unable to give a complete solution to Problem 1.2 for finite fields, but we will prove the case when the group of units is either solvable or bounded Engel (see Corollary 2.3). In fact, we intend to give complete characterizations of these conditions (for all but the smallest ground fields) in Theorems 2.1 and 2.2.

**Theorem 2.1** *Let $A$ be an algebraic algebra over a field $F$ with $|F| \geq 4$. Then $A^\times$ is solvable if and only if all of the following conditions hold: $A$ is Lie solvable, $A/J(A)$ is commutative, and there exists a chain $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = J(A)$ of ideals of $J(A)$ such that every factor $J_i/J_{i-1}$ is the sum of commutative ideals of $J_m/J_{i-1}$.*
Proof. First we prove necessity. Recall from above that all algebraic GI-algebras are locally finite. So, if $H$ is a finitely generated (unital) subalgebra of $A$ and $\mathbb{F}$ is finite then $H$ is actually finite. By Wedderburn’s Theorem, it follows that $H/\mathcal{J}(H)$ is a direct sum of matrix algebras over field extensions of $\mathbb{F}$. Since $|\mathbb{F}| \geq 4$, the general linear group $\text{GL}_2(\mathbb{F})$ is not solvable (see [14], Kapitel II, Satz 6.10). It follows that $H/\mathcal{J}(H)$ is commutative and therefore that $[H,H]H$ is nilpotent. This proves $[A,A]A$ is locally nilpotent. In particular, $[A,A]A \subseteq \mathcal{J}(A)$, so that $A/\mathcal{J}(A)$ is commutative. Because the adjoint group $1+\mathcal{J}(A)$ is solvable, by [1], $\mathcal{J}(A)$ is Lie solvable and there exists a chain of ideals of $\mathcal{J}(A)$ with the required properties. In particular, it follows that $A$ itself is Lie solvable.

It remains to consider the case when $\mathbb{F}$ is infinite. Then, according to Proposition 1.1, $\mathcal{B}(A) = \mathcal{N}(A)$ is locally nilpotent and $A/\mathcal{B}(A)$ is commutative. In particular, $[A,A]A$ is locally nilpotent and the proof follows as in the finite field case.

Sufficiency follows easily from [1].

Notice that the assumption on the minimal cardinality of the ground field is required. Indeed, if $A = M_2(\mathbb{F}_3)$, where $\mathbb{F}_3$ is the field of 3 elements, then $A^{\times} = \text{GL}_2(\mathbb{F}_3)$ is solvable even though $A$ is not Lie solvable.

We now review some well-known facts for later use in the proof of the next theorem. Let $A$ be a finite-dimensional algebra over a field $\mathbb{F}$. An element $x$ of $A$ is said to be semisimple if the minimal polynomial of $x$ has no multiple roots in any field extension of $\mathbb{F}$. In this case, the adjoint map of $x$ turns out to be a semisimple linear transformation of $A$. In particular, it is clear that semisimple elements are central in a Lie nilpotent algebra. Moreover, if $\mathbb{F}$ is perfect then the Wedderburn-Malcev Theorem implies that for every element $x$ of $A$ there exist $x_s, x_n \in A$ with $x_s$ is semisimple and $x_n$ is nilpotent such that $x = x_s + x_n$ and $[x_s, x_n] = 0$.

We shall also use Lemma 2 from [3]. It can be stated as follows: Let $A$ be a finite-dimensional algebra such that $A/\mathcal{J}(A)$ is commutative and separable. If $G$ is a nilpotent subgroup of $A^{\times}$ then the subalgebra generated by $G$ is Lie nilpotent.

**Theorem 2.2** Let $A$ be an algebraic algebra over a perfect field $\mathbb{F} \neq \mathbb{F}_2$.

1. The group of units $A^{\times}$ is bounded Engel if and only if $A$ is bounded Engel. In this case, $\mathcal{N}(A)$ is a locally nilpotent ideal such that $A = \mathcal{Z}(A) + \mathcal{N}(A)$.

2. The group of units $A^{\times}$ is nilpotent if and only if $A$ is Lie nilpotent. Furthermore, the corresponding nilpotency classes coincide.
Proof. (1) Sufficiency holds for all rings (see [22]). For necessity, first recall that $A$ is locally finite. Thus, in the case when $\mathbb{F}$ is finite, if $H$ is a finitely generated (unital) subalgebra of $A$ then $H$ is finite. Thus, by Zorn’s Theorem (see Theorem 12.3.4 in [24]), the finite Engel group $H^\times$ is nilpotent. Another application of Wedderburn’s Theorem easily yields that $H/\mathcal{J}(H)$ is commutative. Since $|\mathbb{F}| \geq 3$, $H$ is generated by units. Also, since $\mathbb{F}$ is perfect, $H/\mathcal{J}(H)$ is a separable algebra. Therefore, Lemma 2 in [3] applies to $H$ and hence $H$ is Lie nilpotent. Next we claim that $\mathcal{N}(H)$ is a locally nilpotent ideal of $H$. Let $x, y \in \mathcal{N}(H)$. Since $H/\mathcal{J}(H)$ is commutative, for every positive integer $t$, we have $(x + y)^{pt} \equiv x^{pt} + y^{pt} \pmod{\mathcal{J}(H)}$, so that, by the nilpotency of $x$ and $y$ and the fact that $\mathcal{J}(H)$ is nilpotent, $x + y \in \mathcal{N}(H)$. Also, if $a \in H$, from $(ax)^{pt} \equiv a^{pt}x^{pt} \pmod{\mathcal{J}(H)}$, it follows that $ax \in \mathcal{N}(H)$, proving the claim. We conclude that the set $\mathcal{N}(A)$ is a locally nilpotent ideal of $A$. Indeed, let $x \in A$. Since $x$ is algebraic and $\mathbb{F}$ is finite (and therefore perfect), by the remarks preceding the theorem (applied to the associative subalgebra generated by $x$) we have $x = x_s + x_n$, where $x_s$ is semisimple, $x_n$ is nilpotent, and $[x_s, x_n] = 0$. Now let $y \in A$ be arbitrary and set $B$ to be the subalgebra generated by $x_s$ and $y$. Since (as seen above) $B$ is a finite Lie nilpotent algebra, the semisimple elements of $B$ are central. In particular, $x_s$ and $y$ commute; hence, $x_s \in \mathcal{Z}(A)$ and $A = \mathcal{Z}(A) + \mathcal{N}(A)$, as required. Finally, since the adjoint group $1 + \mathcal{N}(A)$ is bounded Engel, the locally nilpotent algebra $\mathcal{N}(A)$ is a bounded Engel Lie algebra (by [2]). This proves necessity in the case when $\mathbb{F}$ is finite.

In the case when $\mathbb{F}$ is infinite, Proposition 1.1 informs us that $\mathcal{N}(A)$ is a locally nilpotent ideal of $A$ such that $A/\mathcal{N}(A)$ is both commutative and reduced. Let $H$ be an arbitrary finitely generated (and therefore finite-dimensional) subalgebra of $A$. It follows that $H/\mathcal{N}(H)$ is commutative, and thus $[H, H]/H$ is nilpotent. This implies that $H^\times$ is solvable and so, by a well known result of Gruenberg ([10]), we see that the bounded Engel group $H^\times$ is locally nilpotent. Now, since $\mathbb{F}$ is perfect (by hypothesis) and $H$ is generated by its group of units, Lemma 2 in [3] allows us to conclude that $H$ is Lie nilpotent. Thus, proceeding as above, we obtain $A = \mathcal{Z}(A) + \mathcal{N}(A)$. Applying [2] once more finishes the proof.

(2) Sufficiency holds for all rings (see [13]). To prove necessity, we first infer from part (1) that $\mathcal{N}(A)$ is a locally nilpotent ideal such that $A = \mathcal{Z}(A) + \mathcal{N}(A)$. But, according to a theorem of Du ([9]), $\mathcal{N}(A)$ is Lie nilpotent of class coinciding precisely with the nilpotency class of the adjoint group $1 + \mathcal{N}(A)$. The remaining assertions follow at once. \qed
The ground field $\mathbb{F}_2$ was correctly omitted in Theorem 2.2. Indeed, consider the restricted enveloping algebra $u(L)$ of the restricted Lie algebra $L$ over $\mathbb{F}_2$ with a basis $\{x, y\}$ such that $[x, y] = x, x^2 = 0$, and $y^2 = y$. Then $u(L)^\times$ is isomorphic to the Klein four group. Thus, $u(L)^\times$ is Abelian even though $u(L)$ is not bounded Engel.

One can easily modify the proofs of Theorems 2.1 and 2.2 using the method outlined in the Introduction in order to obtain the following result.

**Corollary 2.3** Let $A$ be an algebraic algebra over an arbitrary field $\mathbb{F}$. If $A^\times$ is either solvable or bounded Engel then $A$ is a PI-algebra.

Now we turn to Problem 1.3. An affirmative solution (first proved in [6]) follows from Proposition 1.1 in the case when $A$ is generated by nilpotent elements over an infinite field. A complete solution will likely be difficult. However, for GI-algebras generated by nilpotent elements over a finite field, we offer the following result. Its proof uses Theorem A from [20], which is essentially a corollary of a deep theorem due to Zelmanov. Theorem A can be stated as follows: Let $A$ be an associative algebra generated by a finite subset $X$. Suppose that the Lie subalgebra $L$ of $A$ generated by $X$ is a PI-algebra and every Lie commutator (of length one or more) in $X$ is nilpotent. Then $A$ is a nilpotent algebra.

**Theorem 2.4** Let $A$ be an algebra over a (finite) field of characteristic $p > 0$. Suppose $L \subseteq A$ is a Lie subalgebra consisting of nilpotent elements. If either $A^\times$ is solvable or bounded Engel then the associative subalgebra $S$ generated by $L$ is locally nilpotent.

Proof. We can assume $L$ is finitely generated. By Corollary 2 of [3], there exists a positive integer $d$ such that $x^d \in \mathcal{B}(A)$ for every $x \in \mathcal{N}(A)$. Clearly we can assume $d = p^t$ for some $t$. Our hypothesis now implies that every element in the Lie subalgebra $L + \mathcal{B}(A)/\mathcal{B}(A)$ of $A/\mathcal{B}(A)$ is nilpotent of index at most $p^t$. In particular, $L + \mathcal{B}(A)/\mathcal{B}(A)$ is bounded Engel (since $(\text{ad } x)^{p^t} = \text{ad}(x^{p^t})$, for each $x \in L$) and thus $L + \mathcal{B}(A)/\mathcal{B}(A)$ satisfies a polynomial identity. But $\mathcal{B}(A) \subseteq \mathcal{J}(A)$ is either Lie solvable or bounded Engel (by [1] and [2], respectively), and so $L$ itself satisfies a polynomial identity. Consequently, $S$ is nilpotent by Theorem A described above.  

3 Applications to restricted enveloping algebras

Let $L$ be a restricted Lie algebra over a field of characteristic $p > 0$. We recall that an element $x$ of $L$ is said to be $p$-nilpotent if there exists a positive
integer \( n \) such that \( x^{[p]^n} = 0 \). The set of all \( p \)-nilpotent elements of \( L \) will be denoted by \( \mathcal{P}(L) \). In particular, \( L \) is called \( p \)-nil if \( L = \mathcal{P}(L) \). We say that \( L \) is algebraically generated if it is generated as a restricted subalgebra by elements which are algebraic with respect to the \( p \)-map of \( L \). It follows from the PBW theorem (see [26], for example) that the minimal polynomial of an element \( x \) in \( L \) coincides precisely with the minimal polynomial of \( x \) when viewed as an element in the restricted enveloping algebra \( u(L) \) of \( L \).

The main result of this section is the following. It lends more support for the existence of an affirmative solution to Problem 1.2.

**Theorem 3.1** Let \( L \) be a restricted Lie algebra over an infinite perfect field of characteristic \( p > 0 \). If \( L \) is algebraically generated and \( u(L) \) is a GI-algebra then \( u(L) \) is locally finite.

We split the proof into several lemmas.

**Lemma 3.2** Suppose that \( L \) is a finite-dimensional restricted Lie algebra over a perfect field \( F \) such that \([L, L] \subseteq \mathcal{P}(L)\). Then the ideal \( \mathcal{P}(L)u(L) \) coincides precisely with the set \( \mathcal{N}(u(L)) \). In particular, \( B(u(L)) = \mathcal{P}(L)u(L) \).

**Proof.** Notice that since \( \mathcal{P}(L) \) is a finite-dimensional \( p \)-nil restricted ideal of \( L \), it follows that \( \mathcal{P}(L)u(L) \) is a nilpotent ideal of \( u(L) \) (see [21], for example). As \( u(L/\mathcal{P}(L)) \cong u(L)/\mathcal{P}(L)u(L) \), to complete the proof it suffices to show that \( u(L) \) is reduced whenever \( \mathcal{P}(L) = 0 \); consequently, we may also assume that \( L \) is Abelian. Suppose now, to the contrary, that \( y \) is a nonzero nilpotent element in \( u(L) \). Fix an ordered basis \( \{x_1, \ldots, x_n\} \) of \( L \) and write \( y \) in its standard PBW representation:

\[
y = \sum \alpha x_1^{a_1} \cdots x_n^{a_n}.
\]

Then, by the commutativity of \( u(L) \), there exists a positive integer \( t \) such that

\[
\sum \alpha x_1^{p^t a_1} \cdots x_n^{p^t a_n} = y^{p^t} = 0.
\]

It follows from the PBW theorem that \( \{x_1^{p^t}, \ldots, x_n^{p^t}\} \) is a linearly dependent set. Therefore, since \( F \) is perfect, there exist \( \beta_1, \ldots, \beta_n \) in \( F \), not all zero, such that

\[
\left( \sum_{i=1}^n \beta_i x_i \right)^{p^t} = \sum_{i=1}^n \beta_i^{p^t} x_i^{p^t} = 0.
\]

This contradicts our assumption that \( L \) has no nonzero \( p \)-nilpotent elements.

\( \square \)
Lemma 3.2 does not extend to arbitrary ground fields. Indeed, let $\mathbb{F}$ be a field with positive characteristic $p > 0$ containing an element $\alpha$ with no $p$-th root in $\mathbb{F}$ and consider the Abelian restricted Lie algebra $L = \mathbb{F}x + \mathbb{F}y$ with $x^{[p]} = x$ and $y^{[p]} = \alpha x$. Then it is easy to check that $\mathcal{P}(L) = 0$ while $0 \neq x^{p-1}y - y \in \mathcal{N}(u(L))$.

**Lemma 3.3** Suppose that $L$ is a finite-dimensional restricted Lie algebra over an infinite field $\mathbb{F}$. If $u(L)$ is a GI-algebra then $[L, L] \subseteq \mathcal{P}(L)$.

Proof. First note that since $L$ is finite-dimensional, so is $u(L)$. Thus, by Proposition 1.1, $u(L)/\mathcal{J}(u(L))$ is commutative. Since $\mathcal{J}(u(L))$ is nilpotent, it follows that every element in $[L, L]$ is $p$-nilpotent. \hfill \Box

**Lemma 3.4** Let $L$ be an algebraically generated restricted Lie algebra over an infinite perfect field $\mathbb{F}$ of characteristic $p > 0$ and suppose that $u(L)$ is a GI-algebra then $u(L)/\mathcal{B}(u(L))$ is reduced. If $u(L)$ is semiprime, then $u(L)/[u(L), u(L)] = u(L)/\mathcal{P}(L)$ is reduced.

Proof. As remarked in the proof of Proposition 1.1, such an algebra $u(L)$ is generated by units. Hence, by Theorem 1.3 in [3], $\mathcal{B}(u(L)) = \mathcal{N}(u(L))$ is a locally nilpotent ideal of $u(L)$. It follows that $\mathcal{P}(L) = L \cap \mathcal{B}(u(L))$ is a restricted ideal in $L$ such that $\mathcal{P}(L)u(L) \subseteq \mathcal{B}(u(L))$. In order to prove the reverse inclusion, it suffices to assume $\mathcal{P}(L) = 0$. Let $\mathcal{D}(L)$ be the sum of all finite-dimensional restricted ideals in $L$. Then, according to Corollary 6.4 of [3], $u(L)$ is semiprime if and only if $u(\mathcal{D}(L))$ is $L$-semiprime. But, under our assumption, Lemma 3.2 implies that $\mathcal{B}(u(I)) = 0$ for every finite-dimensional restricted ideal $I$ of $L$. Thus, $u(L)$ is semiprime, as required. \hfill \Box

**Lemma 3.5** Let $L$ be an algebraically generated restricted Lie algebra over an infinite perfect field $\mathbb{F}$ of characteristic $p > 0$ and $u(L)$ is a GI-algebra then $u(L)/\mathcal{B}(u(L))$ is commutative.

Proof. We first apply the previous lemma in order to pass to the case when $u(L)$ is reduced. Now let $x$ and $y$ be nonzero elements of $L$ with $x$ algebraic. We intend to show that necessarily $[x, y] = 0$.

Since $x$ is an algebraic element of $L$, there exists a minimal positive integer $n$ with the property that there exist $\alpha_0, \ldots, \alpha_n$ in $\mathbb{F}$, not all zero, such that

$$\sum_{i=0}^{n} \alpha_i x^{p^i} = 0.$$
Clearly we may also assume that $\alpha_n = 1$. As explained above, this is precisely the minimal polynomial of $x$ when viewed as an algebraic element of $u(L)$. Since the element

$$[x, y] \sum_{i=0}^{n} \alpha_i x^{p^i - 1}$$

has square zero, it must be zero by assumption. Applying the PBW theorem again yields that $\{[x, y], x, x^p, \ldots, x^{p^{n-1}}\}$ is a linearly dependent set in $L$. Consequently, the minimality of $n$ implies $[x, y] \in \langle x \rangle_p$, where $\langle x \rangle_p$ denotes the restricted subalgebra generated by $x$. This implies $[x, y, x] = 0$ and hence $[y, x^p] = 0$. Thus, we may assume $\alpha_0 = 0$ for otherwise $x \in \langle x^p \rangle_p$ commutes with $y$. In this case, the element

$$\sum_{i=1}^{n} \alpha_i x^{p^i - 1}$$

has square zero and hence, in fact, is zero because $u(L)$ is reduced. However, this violates our choice of $n$, and hence proves the lemma. $\square$

Theorem 3.1 follows from a combination of Lemmas 3.4 and 3.5 and the fact that algebraic GI-algebras are locally finite mentioned in the Section 1.

We conclude this section with the following corollaries, each of them is a consequence of statements proved above and results from [21] and [23].

**Corollary 3.6** Let $L$ be a restricted Lie algebra over a field $F$ of characteristic $p > 0$. Then $u(L)$ is locally finite provided any one of the following conditions hold.

1. The restricted Lie algebra $L$ is locally finite.

2. The field $F$ is both infinite and perfect, $L$ is algebraically generated, and $u(L)$ is a GI-algebra.

3. The field $F$ is infinite, $L$ is generated by $p$-nilpotent elements, and $u(L)$ is a GI-algebra.

4. The field $F$ is finite, $L$ is $p$-nil, and $u(L)^{\times}$ is solvable or bounded Engel.

**Corollary 3.7** Let $L$ be a restricted Lie algebra over an infinite field of characteristic $p > 0$ and suppose that $u(L)$ is algebraic. Then the following conditions are equivalent.

1. The algebra $u(L)$ is a GI-algebra.
2. The algebra $u(L)$ satisfies a non-matrix polynomial identity.

3. The restricted Lie algebra $L$ contains a restricted ideal $I$ such that $L/I$ and $[I, I]$ are finite-dimensional and $[L, L]$ is $p$-nil of bounded index.

Corollary 3.8 Let $L$ be a restricted Lie algebra over a field of odd characteristic $p$ with at least 5 elements. If $u(L)$ is algebraic then the following conditions are equivalent.

1. The group of units $u(L)^\times$ is solvable.

2. The algebra $u(L)$ is Lie solvable.

3. The derived subalgebra $[L, L]$ is both finite-dimensional and $p$-nilpotent.

Corollary 3.9 Let $L$ be a restricted Lie algebra over a perfect field of positive characteristic $p$ with at least 3 elements. If $u(L)$ is algebraic then the following conditions are equivalent.

1. The group of units $u(L)^\times$ is bounded Engel.

2. The algebra $u(L)$ is bounded Engel.

3. The restricted Lie algebra $L$ is nilpotent, $L$ contains a restricted ideal $I$ such that $L/I$ and $[I, I]$ are finite-dimensional, and $[L, L]$ is $p$-nil of bounded index.

Corollary 3.10 Let $L$ be a restricted Lie algebra over a perfect field of positive characteristic $p$ with at least 3 elements. If $u(L)$ is algebraic then the following conditions are equivalent.

1. The group of units $u(L)^\times$ is nilpotent.

2. The algebra $u(L)$ is Lie nilpotent.

3. The restricted Lie algebra $L$ is nilpotent and $[L, L]$ is both finite-dimensional and $p$-nilpotent.

Finally, we would like to mention that the Kurosh problem for restricted enveloping algebras remains open:

**Problem 3.11** Is every algebraic restricted enveloping algebra $u(L)$ locally finite?

In other words, must the underlying restricted Lie algebra $L$ be locally finite?
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