Isomorphisms of type A affine Hecke algebras
and multivariable orthogonal polynomials

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We examine two isomorphisms between affine Hecke algebras of type A associated
with parameters $q^{-1}, t^{-1}$ and $q, t$. One of them maps the non-symmetric Macdonald
polynomials $E_\eta(x; q^{-1}, t^{-1})$ onto $E_\eta(x; q, t)$, while the other maps them onto non-
symmetric analogues of the multivariable Al-Salam & Carlitz polynomials. Using the
properties of $E_\eta(x; q^{-1}, t^{-1})$, the corresponding properties of these latter polynomials
can then be elucidated.

1 Introduction

In several recent works \cite{29}–\cite{31}, \cite{9}–\cite{11}, eigenstates of the rational (type A) Calogero-Sutherland
model have been investigated from an algebraic point of view. In particular it has been shown
that the algebra governing the eigenfunctions of the periodic Calogero-Sutherland model (namely
the type $A$ degenerate affine Hecke algebra augmented by type $A$ Dunkl operators) is isomorphic
to its rational model counterpart. This enables information to be gleaned about the properties
of the eigenfunctions in the rational case (the (non-)symmetric Hermite polynomials) from the
corresponding periodic eigenfunctions (the (non-)symmetric Jack polynomials).

To summarize the argument, consider the type $A$ Dunkl operators

$$d_i := \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{p \neq i} \frac{1 - s_{ip}}{x_i - x_p}$$

which, along with the operators representing multiplication by the variable $x_i$ and the elementary
transpositions $s_{ij}$, satisfy the following commutation relations

$$[d_i, x_j] = \begin{cases} 
-\frac{1}{\alpha} s_{ij} & i \neq j \\
1 + \frac{1}{\alpha} \sum_{p \neq i} s_{ip} & i = j
\end{cases}$$

$$d_is_{ip} = s_{ip}d_p \\
[d_i, s_{jp}] = 0, \quad i \neq j, p$$

(1.1)

It is easily checked that the map $\rho$ defined by

$$\rho(x_i) = \frac{1}{2}(x_i - d_i), \quad \rho(d_i) = d_i, \quad \rho(s_{ij}) = s_{ij}$$

(1.2)

is an isomorphism of the algebra (1.1) \cite{16}.

Now, the non-symmetric Jack polynomials $E_\eta(x)$, indexed by compositions $\eta := (\eta_1, \ldots, \eta_n)$
can be defined \cite{24} as the unique eigenfunctions of the mutually commuting Cherednik operators

$$\xi_i := \alpha x_id_i + \sum_{p > i} s_{ip} - n + 1$$

(1.3)
with a unique expansion of the form
\[ E_\eta(x) = x^\eta + \sum_{\nu < \eta} c_{\eta \nu} x^\nu. \]  

(1.4)

Here, the partial order \(<\) is defined on compositions by: \(\nu < \eta\) iff \(\nu^+ < \eta^+\) with respect to the dominance order (where \(\nu^+\) is the unique partition associated to \(\nu\) etc) or \(\nu^+ = \eta^+, \nu \neq \eta\) and \(\sum_{i=1}^p (\eta_i - \nu_i) \geq 0\), for all \(p = 1, \ldots, n\). The polynomial \(E_\eta(x)\) is an eigenfunction of \(\xi_i\) given by 

\[ \tilde{\eta}_i = \alpha \eta_i - \#\{k < i | \eta_k \geq \eta_i\} - \#\{k > i | \eta_k > \eta_i\} \]  

(1.5)

Using the isomorphism (1.2) it follows that the polynomials \(E_\eta(H)(x) := E_\eta(\rho(x)) \cdot 1\) are eigenfunctions of the operators

\[ h_i = \rho(\xi_i) = \xi_i - \frac{\alpha}{2} d_i^2 \]  

(1.6)

which are precisely the eigenoperators of the non-symmetric Hermite polynomials \(3\). The orthogonality of these latter polynomials with respect to the usual multivariable Hermite inner product then follows from the fact that the operator (1.4) is self-adjoint with respect to the inner product

\[ \langle f, g \rangle := \prod_{i=1}^n \int_{-\infty}^{\infty} dx_i e^{-x_i^2} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2/\alpha} f g \]  

(1.7)

In this work, we provide a similar analysis of the Macdonald case. As such, we introduce an isomorphism of the \(q\)-analogue of the algebra (1.1), namely the (type A) affine Hecke algebra augmented by \(q\)-Dunkl operators. To describe this mapping, we need to introduce some further concepts.

The generalization of the formalism of non-symmetric Jack polynomials to the Macdonald case involves replacing the Cherednik operators (1.3) by their \(q\)-analogues which can be constructed by means of the generators of the affine Hecke algebra \([19]\). In the type A case, one can describe this using the Demazure-Lustig operators

\[ T_i := t + \frac{t x_i - x_{i+1}}{x_i - x_{i+1}} (s_i - 1) \quad i = 1, \ldots, n - 1 \]  

(1.8)

\[ T_0 = t + \frac{t x_n - x_1}{q x_n - x_1} (s_0 - 1) \]  

(1.9)

along with the operator

\[ \omega := s_{n-1} \cdots s_2 s_1 \tau_1 = s_{n-1} \cdots s_i \tau_1 s_i \cdots s_1, \]  

(1.10)

where \(\tau_i\) is the operator which replaces \(x_i\) by \(qx_i\). The affine Hecke algebra is then generated by elements \(T_i, 0 \leq i \leq n - 1\) and \(\omega\), satisfying the relations

\[ (T_i - t) (T_i + 1) = 0 \]  

(1.11)

\[ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \]  

(1.12)

\[ T_i T_j = T_j T_i \quad |i - j| \geq 2 \]  

(1.13)

\[ \omega T_i = T_{i-1} \omega \]  

(1.14)
There is a commutative subalgebra generated by elements of the form \([5, 6]\) which have the following relations with the generators

\[
T_i, \quad E_i, \quad e_i
\]

Using (1.12)–(1.14) it can be shown that the operators \(e_i\) form a set of mutually commuting operators. Our first result is

**Theorem 1.1** We have

\[
E_\eta(e_1, \ldots, e_n; q^{-1}, t^{-1}) \cdot 1 = \alpha_\eta(q, t) E_\eta(x_1, \ldots, x_n; q, t)
\]

where

\[
\alpha_\eta(q, t) = q^{\sum_i (\eta_i^+ - \ell(w_i))}
\]

with \(\ell(w_\eta)\) the length of the (unique) minimal permutation sending \(\eta\) to \(\eta^+\).

The non-symmetric Macdonald polynomials \(E_\eta(x; q, t)\) are defined as the simultaneous eigenfunctions of the commuting operators \(Y_i\) with an expansion of the form [1,4]. The corresponding eigenvalue is \(t^n\) with \(\eta_i\) given in [1,3]. From now on, we drop the dependence on \(q\) and \(t\) and just write \(E_\eta(x) \equiv E_\eta(x; q, t)\) when the meaning is unambiguous.

Define the following degree-raising operator

\[
e_i = t^{i-1}T_i \cdots T_{n-1} x_n \omega T^{-1}_{n-1} \cdots T_{i-1}.
\]

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\]

with \(\ell(w_\eta)\) the length of the (unique) minimal permutation sending \(\eta\) to \(\eta^+\).

The symmetric Al-Salam & Carlitz (ASC) polynomials were examined in [1] as \(q\)-analogues of multivariable Hermite polynomials. There are two families of ASC polynomials, denoted \(U^{(a)}_\lambda(x; q, t)\) and \(V^{(a)}_\lambda(x; q, t)\), which are simply related by

\[
V^{(a)}_\lambda(x; q^{-1}, t^{-1}) = U^{(a)}_\lambda(x; q, t).
\]

The polynomials \(V^{(a)}_\lambda\) can be defined as the unique polynomials of the form

\[
V^{(a)}_\lambda(x; q, t) = P_\lambda(x; q, t) + \sum_{\mu < \lambda} b_{\lambda\mu} P_\mu(x; q, t)
\]

which are orthogonal with respect to the inner product

\[
\langle f, g \rangle^{(V)} := \int_{[1, \infty]^n} f(x)g(x) d_q\mu^{(V)}(x), \quad d_q\mu^{(V)}(x) := \Delta_q^{(k)}(x) \prod_{l=1}^{\infty} w_V(x; q) d_q x_l.
\]

Here, \(P_\lambda(x; q, t)\) denotes the symmetric Macdonald polynomial [21] and we use the notation for \(q\)-integrals

\[
\int_1^\infty f(x) d_q x := (1 - q) \sum_{n=0}^{\infty} f(q^{-n}) q^{-n}
\]

while

\[
w_V(x; q) = \frac{(q; q)_\infty (q^2; q)_\infty (qa; q)_\infty}{(x; q)_\infty (\frac{x}{a}; q)_\infty},
\]

\[
\Delta_q^{(k)}(x_1, \ldots, x_n) := \prod_{p=-(k-1)}^k \prod_{1 \leq i < j \leq n} (x_i - q^p x_j),
\]
where the dash in \((x; q)_{\infty}'\) denotes that any vanishing factor is to be deleted, and it is assumed \(a < 0\).

The polynomials \(U^{(a)}_\lambda\) are orthogonal with respect to the inner product
\[
\langle f | g \rangle^{(U)} := \int_{[0,1]^n} f(x)g(x) \, dq^U(x), \quad dq^U(x) := \Delta_q^{(k)}(x) \prod_{l=1}^n w_U(x_l; q) \, dq \, x_l
\]
(1.23)

where \(\Delta_q^{(k)}\) is given by (1.22) and
\[
w_U^{(a)}(x; q) := \frac{(qx; q)_\infty (\frac{q^2}{q^2}; q)_\infty}{(q; q)_\infty (a; q)_\infty (\frac{a}{q}; q)_\infty}
\]
(1.24)
\[
\int_a^1 f(x) \, dq x := (1 - q) \left( \sum_{n=0}^{\infty} f(q^n)q^n - a \sum_{n=0}^{\infty} f(aq^n)q^n \right), \quad (a < 0)
\]
(1.25)

This can be regarded as a consequence of (1.19), and the formulas
\[
\frac{1}{1 - q} \int_a^1 w_U^{(a)}(x; q) f(x) \, dq x \bigg|_{q \to q^{-1}} = \frac{1}{1 - q} \int_1^\infty w_U^{(a)}(x; q) f(x) \, dq x
\]
(1.26)
\[
\Delta_q^{(k)}(x) = q^{-kn(n-1)} \Delta_q^{(k)}(x^R)
\]
(1.27)

where \(x^R = (x_n, x_{n-1}, \ldots, x_1)\). The formula (1.26) is established in [1, eq. (2.23)], while (1.27) follows immediately from the definition (1.22).

Non-symmetric analogues of the ASC polynomials can be introduced in the following manner: consider the following \(q\)-analogues of the type \(A\) Dunkl operators [8] examined in [4],
\[
D_i = x_i^{-1} (1 - t^{n-1} T_i^{-1} \cdots T_{n-1}^{-1} \omega T_i^{-1} \cdots T_{i-1}^{-1})
\]
(1.28)

and let
\[
E_i := D_i + (1 + a^{-1}) t^{n-1} Y_i - a^{-1} e_i.
\]
(1.29)

The operators \(E_i\) mutually commute, and our second main result is that

**Theorem 1.2** The polynomials
\[
E^{(V)}_\eta(x; q, t) = \frac{(-a)^{|\eta|}}{\alpha_\eta(q, t)} E_\eta(E; q^{-1}, t^{-1}) \cdot 1
\]
(1.30)

where \(\alpha_\eta(q, t)\) is given by (1.18) are the unique polynomials with an expansion of the form
\[
E^{(V)}_\eta(x; q, t) = E_\eta(x; q, t) + \sum_{\nu < |\eta|} c_{\eta\nu} E_\nu(x; q, t)
\]

which are orthogonal with respect to the inner product (1.24). Furthermore, these polynomials are simultaneous eigenfunctions of the commuting family of eigenoperators
\[
h_i = Y_i + (1 + a) t^{i-n} D_i + a t^{2n} D_i Y_i^{-1} D_i
\]
(1.31)

with eigenvalue \(t^n\).

An immediate consequence of Thm. 1.2, (1.19), and (1.26), (1.27) is
Corollary 1.3 The polynomials
\[ E^{(U)}_n(x; q, t) := E^{(U)}_n(x^R; q^{-1}, t^{-1}) \] (1.32)
are the unique polynomials with an expansion of the form
\[ E^{(U)}_n(x; q, t) = E_n(x^R; q^{-1}, t^{-1}) + \sum_{|\nu|<|\eta|} d_{\eta\nu} E_\nu(x^R; q^{-1}, t^{-1}) \]
which are orthogonal with respect to the inner product (1.23). These polynomials are simultaneous eigenfunctions of the operators \( \hat{h}_i \), where \( \hat{h}_i \) denotes the operator (1.34) modified by the involution \( \hat{\cdot} \), which is defined by the mappings \( q \mapsto q^{-1}, t \mapsto t^{-1} \) and \( x \mapsto x_{n+1-i} \).

In section 2, we examine the various properties of non-symmetric Macdonald polynomials used in subsequent calculations, including raising and lowering operators, and introduce a non-symmetric analogue of Kaneko’s kernel [12]. We finish the section with a proof of Thm. 1.1. An isomorphism between Hecke algebras is introduced in section 3, facilitating a proof of Thm. 1.2.

Various properties of these non-symmetric ASC polynomials are then described including their normalization and a generating function. We conclude by clarifying their relationship to the non-symmetric analogues of the shifted Macdonald polynomials.

2 Non-symmetric Macdonald polynomials

In this section we gather together some (old and new) results concerning non-symmetric Macdonald polynomials \( E_n(x) \) in preparation of the proof of Thm. 1.1, as well as the forthcoming section on the non-symmetric ASC polynomials.

For future reference we note that the operators \( T_i \) and \( \omega \) defined by (1.8) and (1.10) have the properties
\[
\begin{align*}
T_i^{-1} x_{i+1} &= t^{-1} x_i T_i \\
T_i x_i &= t x_{i+1} T_i^{-1} \\
\omega x_i &= q x_i \omega
\end{align*}
\] (2.1)
valid for \( 1 \leq i \leq n-1 \). Also note the following action of \( T_i \) on monomials
\[
T_i^{a_i} x_i^{b_i} x_{i+1}^{c_i+1} = \begin{cases} 
(1-t) x_i^{a_i-1} x_{i+1}^{b_i+1} + \cdots + (1-t) x_i^{b_i+1} x_{i+1}^{a_i-1} + x_i^{b_i} x_{i+1}^{a_i} & a > b \\
t x_i^{a_i} x_{i+1}^{a_i} & a = b \\
(t-1) x_i^{a_i} x_{i+1}^{b_i} + \cdots + (t-1) x_i^{b_i} x_{i+1}^{a_i} + t x_i^{b_i} x_{i+1}^{a_i} & a < b 
\end{cases}
\] (2.2)

There exists a variant of the \( q \)-Dunkl operator (1.28) which is relevant to the forthcoming discussion. With \( \hat{\cdot} \) denoting the involution defined in the statement of Corollary 1.3, this operator is defined as
\[
\hat{D}_i := -q \hat{D}_{n+1-i} = -q x_i^{-1} \left( 1 - t^{-n+1} T_{i-1} \cdots T_1 \omega^{-1} T_{n-1} \cdots T_i \right) = q t^{-2n+i+1} D_i Y_i^{-1} T_i \cdots T_{n-1} T_{n-1} \cdots T_i
\] (2.3)
In obtaining the first equality in (2.3), the facts that
\[
\hat{T}_i = T_{n-i}^{-1} \quad \text{and} \quad \hat{\omega} = \omega^{-1}
\] (2.4)
have been used in applying the operation \( \hat{c} \) to (1.28), while the second equality can be verified by substituting for \( Y_i^{-1} \) using (1.13) and for \( D_i \) using (1.28) and comparing with the first equality.

Since the \( D_i \) commute, it follows from the definition of \( D_i \) that the \( \{ D_i \} \) also form a commuting set. Moreover, using (2.4), one can check that the operators \( D_i \) possess the same relations with the generators \( T_i, \omega \) as do the \( D_i \), namely

\[
T_i D_{i+1} = t D_i T_i^{-1}, \quad T_i D_i = D_{i+1} T_i + (t - 1) D_i, \quad 1 \leq i \leq n - 1
\]

\[
[T_i, D_j] = 0, \quad j \neq i, i + 1
\]

\[
D_n \omega = q \omega D_1, \quad D_i \omega = \omega D_{i+1} \quad 1 \leq i \leq n - 1
\]

To conclude the preliminaries, we follow Sahi [27] and introduce the generalized arm and leg (co-)lengths for a node \( s \in \eta \) via

\[
a(s) = \eta_i - j \quad l(s) = \#\{k > i| j \leq \eta_k \leq \eta_i\} + \#\{k < i| \eta_k \leq 1 \leq \eta_i\}
\]

\[
a'(s) = j - 1 \quad l'(s) = \#\{k > i| \eta_k > 1 \leq \eta_i\} + \#\{k < i| \eta_k \geq \eta_i\}
\]

and define the quantities

\[
d_{\eta}(q, t) := \prod_{s \in \eta} \left(1 - q^{a(s)+1}t^{l(s)+1}\right) \quad l(\eta) := \sum_{s \in \eta} l(s)
\]

\[
d'_{\eta}(q, t) := \prod_{s \in \eta} \left(1 - q^{a(s)+1}t^{l(s)}\right) \quad l'(\eta) := \sum_{s \in \eta} l'(s)
\]

\[
e_{\eta}(q, t) := \prod_{s \in \eta} \left(1 - q^{a'(s)+1}t^{n-l'(s)}\right) \quad a(\eta) := \sum_{s \in \eta} a(s)
\]

The statistics \( l(\eta), l'(\eta) \) and \( a(\eta) \) generalize the quantity

\[
b(\lambda) := \sum_i (i - 1) \lambda_i = \sum_i \left(\frac{\lambda_i}{2}\right)
\]

from partitions to compositions. From [27] these quantities have the following properties

**Lemma 2.1** Let \( \Phi_\eta := (\eta_2, \ldots, \eta_n, \eta_1 + 1) \). We have

\[
\frac{d_{\Phi_\eta}(q, t)}{d_{\eta}(q, t)} = 1 - qt^{\eta_1}, \quad \frac{d'_{\Phi_\eta}(q, t)}{d'_{\eta}(q, t)} = 1 - qt^{\eta_1 - 1}, \quad e_{s, \eta}(q, t) = e_{\eta}(q, t),
\]

\[
\frac{d_{s, \eta}(q, t)}{d_{\eta}(q, t)} = \frac{1 - t^{\delta_{i, n} + 1}}{1 - t^{\delta_{i, n}}}, \quad \frac{d'_{s, \eta}(q, t)}{d'_{\eta}(q, t)} = \frac{1 - t^{\delta_{i, n} - 1}}{1 - t^{\delta_{i, n}}}
\]

for \( \eta_i > \eta_{i+1} \)

\[
a(\Phi_\eta) = a(\eta) + l(\Phi_\eta) = l(\eta) + \#\{k > 1| \eta_k \leq \eta_1\}
\]

\[
l'(\Phi_\eta) = l'(\eta) + n - 1 - \#\{k > 1| \eta_k \leq \eta_1\}
\]

\[
a(s, \eta) = a(\eta) \quad l'(s, \eta) = l'(\eta) \quad l(s, \eta) = l(\eta) + 1
\]

A consequence of the first two equations in the final line is that

\[
l'(\eta) = l'(\eta^+) = b(\eta^+), \quad a(\eta) = a(\eta^+) = b((\eta^+)^')
\]

where \((\eta^+)')'\) denotes the partition conjugate to \(\eta^+\).
2.1 Raising Operators and Lowering Operators

There are two distinct raising operators which have a very simple action on non-symmetric Macdonald polynomials. Define

\[
\Phi_1 := x_n \omega,
\]
\[
\Phi_2 := x_n T_{n-1} \cdots T_{1}^{-1} \quad \text{(2.10)}
\]

A direct calculation reveals that for \( i = 1, 2 \)

\[
Y_n \Phi_i = q \Phi_i Y_1
\]
\[
Y_j \Phi_i = \Phi_i Y_{j+1} \quad 1 \leq j \leq n - 1
\]

whence \( \Phi_i E_\eta \) is a constant multiple of \( E_{\Phi_\eta} \), where \( \Phi_\eta := (\eta_2, \ldots, \eta_n, \eta_1 + 1) \). This constant is determined by looking at the coefficient of \( x^{\Phi_\eta} \) with the result that

\[
\Phi_1 E_\eta = q^{\eta_n} E_{\Phi_\eta},
\]
\[
\Phi_2 E_\eta = t^{-\# \{ i \mid \eta_i \leq \eta_1 \}} E_{\Phi_\eta}
\]

Remark. These operators are simply related via \( \Phi_1 = t^{n-1} \Phi_2 Y_1 \). Of course any function of the operators \( Y_i \) multiplied by \( \Phi_1 \) will be a raising operator for the non-symmetric Macdonald polynomials but these two are in some sense the simplest.

In a similar manner, one can use the \( q \)-Dunkl operators (1.28) to construct lowering operators as follows,

\[
\Psi_1 := \omega^{-1} D_n,
\]
\[
\Psi_2 := T_1 T_2 \cdots T_{n-1} D_n \quad \text{(2.11)}
\]

\( \Psi_2 \) was introduced previously in [4]. These operators intertwine with the Cherednik operators as

\[
Y_1 \Psi_i = q^{-1} \Psi_i Y_n
\]
\[
Y_j \Psi_i = \Psi_i Y_{j-1} \quad 2 \leq j \leq n
\]

and it is seen that

\[
\Psi_1 E_\eta = q^{-\eta_n+1} (1 - t^{n-1+\eta_n}) E_{\Phi_\eta},
\]
\[
\Psi_2 E_\eta = t^{\# \{ i \mid \eta_i < \eta_n \}} (1 - t^{n-1+\eta_n}) E_{\Phi_\eta}
\]

where \( \Psi_\eta := (\eta_n - 1, \eta_1, \ldots, \eta_{n-1}) \).

2.2 Kernel

Let \( \sim \) denote the involution on the ring of polynomials with coefficients in \( \mathbb{C}(q,t) \), which acts on the the coefficients by sending \( q \mapsto q^{-1} \), \( t \mapsto t^{-1} \), and extend it to act on operators in the obvious way. Define the kernel

\[
K_A(x; y; q, t) = \sum_\eta q^{a(\eta) l(n-1)|\eta|-t'(\eta)} \frac{d_\eta}{d_\eta' e_\eta} E_\eta(x) \bar{E}_\eta(y) \quad \text{(2.12)}
\]

It follows from (2.7) that this kernel is related to the previously introduced kernel [4]

\[
K_A(x; y; q, t) = \sum_\eta \frac{d_\eta}{d_\eta' e_\eta} E_\eta(x) \bar{E}_\eta(y) \quad \text{(2.13)}
\]
(2.13) was denoted by $\mathcal{K}_A$ in \[4\], but for the present purpose it is desirable to use this notation for (2.12) by means of

$$\tilde{\mathcal{K}}_A(x; y; q, t) = K_A(-qy; x; q, t) \tag{2.14}$$

The kernel $\mathcal{K}_A(x; y; q, t)$ satisfies the following properties

**Theorem 2.2**

(a) $(T_i^{\pm 1})^{(x)} \mathcal{K}_A(x; y; q, t) = \left(\tilde{T}_i^{\pm 1}\right)^{(y)} \mathcal{K}_A(x; y; q, t)$

(b) $\Psi_i^{(x)} \mathcal{K}_A(x; y; q, t) = \tilde{\Phi}_2^{(y)} \mathcal{K}_A(x; y; q, t)$

(c) $\mathcal{D}_i^{(x)} \mathcal{K}_A(x; y; q, t) = y_i \mathcal{K}_A(x; y; q, t)$

**Proof.** The proof of this result follows the same line of reason as in \[4\, Thm 5.2], using the facts that

$$x_i = t^{-n+i}\widetilde{T}_i^{-1} \cdots \widetilde{T}_{n-1}^{-1}\Phi_2 \cdots \tilde{T}_{i-1}^{-1} \tag{2.15}$$

$$\mathcal{D}_i = t^{-n+i}\tilde{T}_{i-1}^{-1} \cdots \tilde{T}_i^{-1} \Psi_1 \tilde{T}_{n-1} \cdots \tilde{T}_i \tag{2.16}$$

We recall from \[4\] that the analogue of property (c) for the kernel $K_A(x; y; q, t)$ is

$$\mathcal{D}_i^{(x)} K_A(x; y; q, t) = y_i K_A(x; y; q, t). \tag{2.17}$$

A feature of both property (c) and (2.17) is that the $q$-Dunkl operator $\mathcal{D}_i$ (resp. $\mathcal{D}_i$) act on the left set of variables only. However, by applying the operation $\sim$ and using (2.14), we can form similar identities where they act on the right set of variables, namely

**Corollary 2.3**

$$(\tilde{\mathcal{D}}_i)^{(x)} K_A(z; x; q, t) = -q^{-1}z_i K_A(z; x; q, t) \tag{2.18}$$

$$(\tilde{\mathcal{D}}_i)^{(y)} K_A(x; y; q, t) = -q x_i K_A(x; y; q, t) \tag{2.19}$$

### 2.3 First isomorphism

Returning to the proof of Thm. \[4\], we claim that it follows from the subsequent

**Proposition 2.4** The map $\phi$ defined by

$$\phi(Y_i^{-1}) = Y_i, \quad \phi(x_i) = e_i, \quad \phi(T_i^{\pm 1}) = T_i^{\pm 1}. \tag{2.20}$$

is an algebra isomorphism.

**Proof.** As the relations between $\tilde{Y}_i^{-1}$ and $x_j$ are somewhat unsightly, it is more convenient to consider the operator $\tilde{\omega}^{-1}$ (from which the $\tilde{Y}_i^{-1}$ can be constructed) which possess simpler relations with the $x_j$ (c.f. \[2.14\]). Indeed by defining

$$\phi(\tilde{\omega}^{-1}) = T_1 \cdots T_{n-1} \omega T_1^{-1} \cdots T_{n-1}^{-1} \tag{2.21}$$

we obtain the relation $\phi(\tilde{Y}_i^{-1}) = Y_i$ as a consequence.

The explicit relations satisfied by the algebra $\{\omega^{-1}, T_i^{\pm 1}, x_i\}$ can be obtained by applying $\sim$ to the relations \[1.1\]--\[1.14\], \[2.1\]. The fact that the map $\phi$ as defined by (2.20), (2.21) is an isomorphism then follows by a standard calculation.  \[\Box\]
**Proof of Thm. 1.1.** We know that $E_{\eta}(x; q^{-1}, t^{-1})$ is an eigenfunction of $\widetilde{Y}_i^{-1}$. In principle this can be shown by utilizing the commutation relations amongst the operators $\{\widetilde{Y}_i^{-1}, \widetilde{T}_i^{\pm 1}, x_i\}$ to move the operators $\widetilde{Y}_i^{-1}$ through the terms in $E_{\eta}(x; q^{-1}, t^{-1})$, until obtaining $\widetilde{Y}_i^{-1}.1 = t^{-n+1}.1$. By adopting this viewpoint in the eigenvalue equation (considered as an operator equation)

$$\widetilde{Y}_i^{-1} E_{\eta}(x; q^{-1}, t^{-1}).1 = t^{\eta_0} E_{\eta}(x; q^{-1}, t^{-1}).1$$

and applying the map $\phi$ to both sides it then follows from Lemma 2.4 that $E_{\eta}(e; q^{-1}, t^{-1}).1$ is an eigenfunction of $\phi(\widetilde{Y}_i^{-1}) = Y_i$, with leading order term $x^{\eta}$ and hence must be proportional to $E_{\eta}(x; q, t)$.

To determine the proportionality constant $\alpha_{\eta}(q, t)$ say, it follows from the action of $T_i$ given by (2.2) that

$$e_1^{\eta_1} e_2^{\eta_2} \cdots e_n^{\eta_n} . 1 = q^{f(\eta)} t^{g(\eta)} x^{\eta} + \sum_{\nu < \eta} b_{\eta\nu} x^{\nu}$$

where $f(\eta) = \sum_i l(\eta_i)$ and

$$g(q) = \sum_{i=1}^n (\eta_i - 1)(i - 1) + \sum_{i=0}^{\eta_{n-1}} \chi(\eta_i \leq i) + \sum_{i=0}^{\eta_{n-2}} \chi(\eta_{n-1} \leq i) + \sum_{i=0}^{\eta_n} \chi(\eta_n \leq i) + \sum_{i=0}^{\eta_{n-3}} \chi(\eta_{n-2} \leq i) + \cdots + \sum_{i=0}^{\eta_2} \chi(\eta_2 \leq i)$$

where $\chi(P) = 1$ if $P$ is true, and zero otherwise. The simplification $g(q) = \sum_i (n - i) \eta_i^+ - \ell(w_\eta)$ then follows from the above expression by induction on $\ell(w_\eta)$. □

### 3 Al-Salam&Carlitz polynomials

The isomorphism $\phi$ introduced in the previous section can be generalized to another isomorphism $\psi_\alpha$ such that $\psi_\alpha(x_i)$ includes not just degree-raising parts, but degree-preserving and lowering parts as well. It will turn out that this isomorphism is precisely what is needed to obtain non-symmetric analogues of the Al-Salam&Carlitz polynomials in the same way as was done for the Hermite case.

As previously mentioned, the symmetric ASC polynomials $V^{(a)}_\lambda$ can be defined via their orthogonality with respect to the inner product $\prod_{i=1}^n \rho_{a}(t^{-\ell(n-1); x_i}; q)$. We remark that under this inner product we have the important result that the adjoint operators of $\widetilde{T}_i^{\pm 1}$, $\omega$ are given by

$$(T_i^{\pm 1})^* = T_i^{\pm 1}, \quad (\omega^{-1})^* = \frac{t^{n-1}}{aq} \omega(x_1 - q)(x_1 - aq)$$

(3.1)

The ASC polynomials $V^{(a)}_\lambda$ can equivalently be defined by means of the generating function

$$\prod_{i=1}^n \frac{1}{\rho_{a}(t^{-\ell(n-1); x_i}; q) \omega^{a_0}(x; y; q, t)} = \sum_{\lambda} \frac{(-1)^{\lambda_1} q^{b(\lambda)} V^{(a)}_\lambda(y; q, t) P_\lambda(x; q, t)}{d^0_\lambda(q, t) P_\lambda(1, t, \ldots, t^{n-1}; q, t)}.$$  

Here, $\rho_{a}(x) := (x; q)_\infty(a x; q)_\infty$, $b(\lambda)$ is defined by (2.8) and

$$P_\lambda(1, t, \ldots, t^{n-1}; q, t) = t^{\ell(\lambda)} \prod_{s \in \lambda} (1 - q^{a_0(s) - l(s)})(1 - q^{a(s) - \ell(s) + 1})$$

$$\omega^{a_0}(x; y; q, t) := \sum_{\lambda} \frac{(-1)^{\lambda_1} q^{b(\lambda)} P_\lambda(x; q, t) P_\lambda(y; q, t)}{d^0_\lambda(q, t) P_\lambda(1, t, \ldots, t^{n-1}; q, t) P_\lambda(x; q, t) P_\lambda(y; q, t)}.$$  

(3.2)
This latter kernel was previously introduced by Kaneko \cite{12} in connection with hypergeometric solutions of systems of $q$-difference equations.

Similarly the ASC polynomials $U^{(a)}_\lambda$ can be defined by the generating function \cite{11}
\[ \rho_a(x_1; q) \cdots \rho_a(x_n; q) \mathcal{F}_0(x; y; q, t) = \sum_\kappa \frac{t^{b(\kappa)} U^{(a)}_\lambda(y; q, t) P_\kappa(x; q, t)}{b'_\kappa(q, t) P_\kappa(1, t, \ldots, t^{n-1}; q, t)} \] (3.3)
where the hypergeometric function $\mathcal{F}_0$ is defined by
\[ \mathcal{F}_0(x; y; q, t) := \sum_\kappa \frac{t^{b(\kappa)}}{d'_\kappa(q, t) P(1, t, \ldots, t^{n-1}; q, t)} P_\kappa(x; q, t) P_\kappa(y; q, t). \] (3.4)

3.1 Second isomorphism

Consider the involution $\circ$ on polynomials and operators defined in the statement of Corollary 1.3. The operator $E_i$ introduced in (1.29) has its origins in this involution, namely,
\[ E_i := (\hat{D}_{n+1-i})^* := (-\frac{1}{q} D_i)^*. \] (3.5)

The form (1.29) follows from (3.3) by making use of the adjoint formulae (3.1). Regarding the algebra satisfied by the $E_i$ and operators such as the $T_i$, note that application of the adjoint operation $^*$ to the relations involving $D_i, T_i$ (2.3) gives, in place of the first relation in (2.3) for example,
\[ T_i^{-1} E_i T_i^{-1} = t^{-1} E_{i+1}. \] (3.6)

Now consider the mapping $\psi_a : \{\omega^{-1}, T_i, x_i, D_i\} \rightarrow \{\omega, T_i, x_i, D_i\}$ defined by
\[ \psi_a(x_i) = E_i, \]
\[ \psi_a(\omega^{-1}) = T_1 \cdots T_{n-1} (Y_n + (1 + a)t^{1-n}D_n + at^{2-2n}D_nY_nD_n), \]
\[ \psi_a(T_i^{-1}) = T_i, \]
\[ \psi_a(D_i) = -at^{n+1-2i}T_{i-1} \cdots T_1 T_i \cdots T_{i-1} E_i^* T_i^{-1} \cdots T_{n-1} T_{n-1} \cdots T_i^{-1} \] (3.7)

The proof of this result consists of checking that the operators $\psi_a(u)$ given in (3.7) satisfy the same relations as the original operators $u$, given by (1.11)–(1.14), (2.1) and (2.3), (after application of the involution $\circ$). For example, the first formula in (2.1), after application of the involution $\circ$, reads
\[ \hat{T}_i^{-1} x_{i+1} = t x_i \hat{T}_i. \]

Now applying the mapping $\psi_a$ gives
\[ T_i E_{i+1} = t E_i T_i^{-1}. \]

But this is equivalent to (3.6) so the algebra is indeed preserved. The calculations involved in checking the other relations are typically more tedious; however they are similar to those undertaken in \cite{3}, and so for brevity will be omitted.

As with the relationship between Prop. 2.4 and the proof of Thm. 1.1 we are in a position to complete the
Proof of Thm. 1.2  From Thm 1.1, and the definition (1.29) of the operators $E_i$ it follows that $E_i(V)$ has leading term $E_\eta(x; q, t)$. In addition, it follows from (3.7) that

$$\psi_a(Y_i^{-1}) = Y_i + (1 + a)t^{1-n}D_i + at^{2-2n}D_iY_i^{-1}D_i$$

(3.8)

and from Prop. 3.1 that these are eigenoperators for the non-symmetric ASC polynomials defined by (1.30). The corresponding eigenvalue is simply $t^n$. By writing these operators out explicitly, it is seen that they are self-adjoint w.r.t. the inner product (1.20). Hence by standard arguments, the polynomials (1.30) are orthogonal w.r.t. (1.20).

3.2 Normalization

The images of the raising and lowering operators (2.10), (2.11) (after application of $\tilde{\phi}$) under the map $\psi_a$ are guaranteed, by virtue of Prop. 3.1, to be raising and lowering operators for the polynomials $E_\eta^{(V)}(x)$. In particular, using (2.16) and (3.7) we see that

$$\psi_a(\tilde{\Psi}_1) = aq^{-1}t^{1-n} \Psi_1$$

so that $\Psi_1$ remains a raising operator for the polynomials $E_\eta^{(V)}$. By examination of the leading terms, we must have

$$\Psi_1 E_\eta^{(V)} = q^{n+1} \frac{d\eta}{d\phi_\eta} E_\eta^{(V)}$$

(3.9)

Also, use of (2.15) and (3.7) gives

$$\psi_a(\tilde{\Phi}_2) = -q^{-1} \Psi_1^*$$

so that $\Psi_1^*$ is a raising operator for $E_\eta^{(V)}$. Indeed,

$$\Psi_1^* E_\eta^{(V)} = a^{-1}t^{n-1}q^{n+1} E_\phi^{(V)}$$

(3.10)

By an argument similar to that used in [2, Prop. 3.6] it follows from (3.3) and (3.10) that

$$\langle E_\phi^{(V)}, E_\phi^{(V)} \rangle^{(V)} = at^{1-n}q^{2m-1} \frac{d\phi_\eta}{d\phi_\eta} \langle E_\eta^{(V)}, E_\eta^{(V)} \rangle^{(V)}$$

(3.11)

Also, we have

$$\langle E_{s_1}^{(V)}, E_{s_2}^{(V)} \rangle^{(V)} = \frac{(1 - t^\delta a^{-1})(1 - t^\delta a^{-1})}{t(1 - t^\delta a)^2} \langle E_\eta^{(V)}, E_\eta^{(V)} \rangle^{(V)}$$

(3.12)

The solution of the recurrence relations (3.11), (3.12) gives

Proposition 3.2

$$\mathcal{N}_\eta^{(V)} := \langle E_\eta^{(V)}, E_\eta^{(V)} \rangle^{(V)} = (aq^{-1}t^{2-2n})^{n} q^{-2a(\eta)} t(\eta + t(\eta)) \frac{d\eta}{d\eta} N_0^{(V)}$$

(3.13)

where for $t = q^k$,

$$N_0^{(V)} = (1 - q^n) a^{kn(n-1)/2} t^{-2k(\frac{n}{2}) - k(\frac{n}{2})} \prod_{l=1}^{n} \frac{(q; q)_k}{(q; q)_k}.$$
By using the formulas (1.32), (1.26) and (1.27) we see that the norm $N_U^\eta(U)$ of the non-symmetric ASC polynomials $E_U^\eta(U)$ with respect to the inner product (1.23) is given by simply replacing $q,t$ by $q^{-1},t^{-1}$ in (3.13). Use of (2.7) then gives

**Corollary 3.3**

\[
N_U^\eta(U) := \left\langle E_U^\eta(U), E_U^\eta(U) \right\rangle^U = (a^n t^{-l(\eta)}) \frac{d^n e_\eta}{d_\eta} N_0^U
\]

where for $t = q^k$, \[N_0^U = (1 - q)^n (-a)^{kn(n-1)/2} k^n \left( \frac{q}{a} \right) \prod_{l=1}^{n} \left( \frac{q}{a} \right)^{kl} \frac{q}{a} \right.^{k}.
\]

### 3.3 Generating function

The raising operator expression (1.30) facilitates the derivation of the generating function for the non-symmetric ASC polynomials. Also required will be the $q$-symmetrization of (2.12).

**Proposition 3.4** Let $U^+ = \sum_\sigma T_\sigma$ where $T_\sigma := T_{i_1} \cdots T_{i_p}$ for a reduced word decomposition $\sigma = s_{i_1} \cdots s_{i_p}$. We have

\[
(U^+)^{(x)} K_A(x; y; q, t) = [n]_1 t_0 \psi_0(x; -t^{n-1} y; q, t)
\]

where $\psi_0$ is defined by (3.2).

**Proof.** We remark that this is the analogue of the result [4, Prop. 5.4]

\[
(U^+)^{(x)} K_A(x; y; q, t) = [n]_1 t_0 F_0(x; y; q, t)
\]

In fact in our proof of (3.15) we will use the formula

\[
U^+ E_\eta(x) = [n]_1 t^{l(\eta)} \frac{e_\eta}{P_\lambda(t^l \eta)d_\eta} P_\lambda(x), \quad \lambda = \eta^+
\]

which was deduced [4, eqs. (5.8)&(5.18)] as a corollary of (3.16). Thus we apply $U^+$ to (2.12) and use (3.17) to compute its action. Simplifying the result using the first equation in (2.9) and the formula [19]

\[
P_\lambda(y) = \sum_{\eta; \eta^+ = \lambda} \frac{d^n \eta}{d_\eta} E_\eta(y),
\]

the result then follows.

\[\square\]

Consider now the generating function

\[
F_1(y; z) = \sum_\nu A_\nu E_\nu^{(y)}(y) \tilde{E}_\nu^{(z)}
\]

where

\[
A_\nu = (a/q)^{\lvert \nu \rvert} \frac{N_0^{(y)}}{\alpha_\nu(q, t) N^{(y)}_\nu} = q^{a(\nu)} t^{(n-1)\lvert \nu \rvert - l(\nu)} \frac{d_\nu}{d_\eta} e_\nu
\]

Here we have used the fact that $l(\eta) = l(\eta^+) + \ell(\nu)$ to rewrite $\alpha_\eta(q, t)$ as defined by (1.18) as

\[
\alpha_\eta(q, t) = q^{a(\eta)} t^{(n-1)\lvert \eta \rvert - l(\eta)}.
\]
Clearly
\[ \left\langle F_1(y; z), E_\eta^{(V)}(y) \right\rangle_{y}^{(V)} = (a/q)^{|\eta|} \frac{N_0^{(V)}}{\alpha_\eta(q, t)} \tilde{E}_\eta(z) \]

Next note the integration formula
\[ \left\langle \mathcal{K}_A(y; z), 1 \right\rangle_{y}^{(V)} = \frac{1}{[n]!} \left\langle U_y^+ \mathcal{K}_A(y; z), 1 \right\rangle_{y}^{(V)} \]
\[ = \left\langle 0 \psi_0(y; -t^{n-1}z), 1 \right\rangle_{y}^{(V)} = N_0^{(V)} \prod_{i=1}^{n} \rho_a(-z_i) \]

which follows from the symmetrization formula (3.13), the fact that \( U_y^+ \) is self adjoint w.r.t. \( \left\langle \cdot, \cdot \right\rangle_{y}^{(V)} \) and an integral formula for the kernel \( 0 \psi_0(y; z) \) given in \([\text{P}], \text{Prop 4.8}\), and consider the generating function
\[ F_2(y; z) = \prod_{i=1}^{n} \frac{1}{\rho_a(-z_i)} \mathcal{K}(y; z) \]

We have
\[ \left\langle F_2(y; z), E_\eta^{(V)}(y) \right\rangle_{y}^{(V)} \]
\[ = \frac{(-a)^{|\eta|}}{\alpha_\eta(q, t)} \prod_{i} \frac{1}{\rho_a(-z_i)} \left\langle \mathcal{K}(y; z), \tilde{E}_\eta(E^{(y)}(y)) \right\rangle_{y}^{(V)} \]
\[ = \frac{(a/q)^{|\eta|}}{\alpha_\eta(q, t)} \prod_{i} \frac{1}{\rho_a(-z_i)} \left\langle \tilde{E}_\eta(D^{(y)}) \mathcal{K}(y; z), 1 \right\rangle_{y}^{(V)} \]
\[ = \frac{(a/q)^{|\eta|}}{\alpha_\eta(q, t)} \prod_{i} \frac{1}{\rho_a(-z_i)} \tilde{E}_\eta(z) \left\langle \mathcal{K}(y; z), 1 \right\rangle_{y}^{(V)} \]
\[ = \frac{(a/q)^{|\eta|}}{\alpha_\eta(q, t)} N_0^{(V)} \tilde{E}_\eta(z) \]

In the above chain of equalities, we have used (1.30), (3.5), the kernel property Thm. 2.2 (c) and (3.20) respectively. The non-symmetric ASC polynomials \( E_\eta^{(V)}(y) \) are a complete basis for polynomials in \( y \) and hence from above we have \( F_1 = F_2 \). That is, we have the generating function for non-symmetric ASC polynomials \( E_\eta^{(V)} \).

**Proposition 3.5** With \( A_\nu \) given by (3.13)
\[ \prod_{i=1}^{n} \frac{1}{\rho_a(-z_i)} \mathcal{K}_A(y; z) = \sum_{\nu} A_\nu E_\nu^{(V)}(y) \tilde{E}_\nu(z). \]

We remark that this generating function could also be derived in a manner similar to that done in the symmetric case \([\text{P}], \text{namely by applying the operator} \ (Y_i^{-1})^z \text{ to both sides of} \ (3.21) \text{ and deducing that} \ E_\eta^{(V)}(y) \text{ is an eigenfunction of} \]
\[ h_i = \psi_a(Y_i^{-1}) = Y_i T_{i-1} \cdots T_1 (1 + D_1) (1 + a D_1) T_{i-1}^{-1} \cdots T_{i-1}^{-1} \]
with leading term \( E_\eta(0) \) (some manipulation using (2.3) and (2.9) cast this into the form given in (1.31)). Note also that by applying the operation \( \cdot \) with the respect to the \( y \)-variables in (3.21) and using the formula (2.14) as well as
\[ \frac{1}{\rho_a(x; q)} \bigg|_{q=q_{-1}} = \rho_a(q x; q), \]
(see e.g. \([\text{P}]\)) we deduce the generating function formula for the polynomials \( E_\nu^{(U)} \).
Corollary 3.6
\[
\prod_{i=1}^{n} \rho_a(z_i) K_A(z; y^R; q, t) = \sum_{\nu} \frac{d_{\nu}}{d_{\nu} e_{\nu}} E_{\nu}^{(U)}(y) E_{\nu}(z) \tag{3.23}
\]

The generating function formulas in turn imply a further class of operator formulas relating the ASC polynomials and the non-symmetric Jack polynomials (c.f. [1] eqs. (3.9)&(3.10)).

Corollary 3.7 We have
\[
E_{\nu}^{(V)}(y) = \prod_{i=1}^{n} \frac{1}{\rho_a(-D_i^{(y)})} E_{\nu}(y) \tag{3.24}
\]
\[
E_{\nu}^{(U)}(y) = \prod_{i=1}^{n} \rho_a \left( -q D_i^{(y)} \right) \tilde{E}_{\nu}(y). \tag{3.25}
\]

Proof. The first identity follows from (3.24) by using Thm. 2.2 (c) and comparing coefficients of \( \tilde{E}_{\nu}(y) \), while the second identity follows similarly from (3.24) and (2.18).

As further applications of the generating functions we will present some evaluation formulas for \( E_{\nu}^{(V)} \) at the special points \( t^{\tilde{\delta}-n+1} \) and \( at^{\tilde{\delta}-n+1} \), where \( t^{\tilde{\delta}} := (1, t, t^2, \ldots, t^{n-1}) \).

Proposition 3.8 We have
\[
E_{\nu}^{(V)}(t^{\tilde{\delta}-n+1}) = (-a)^{|\nu|} q^{-a(\nu) - (n-1)|\nu|} E_{\nu}(t^{\tilde{\delta}}) \tag{3.26}
\]
\[
E_{\nu}^{(V)}(at^{\tilde{\delta}-n+1}) = (-1)^{|\nu|} q^{-a(\nu) - (n-1)|\nu|} E_{\nu}(t^{\tilde{\delta}}) \tag{3.27}
\]
where
\[
E_{\nu}(t^{\tilde{\delta}}) = t^{l(\nu)} \frac{e_{\nu}}{d_{\nu}}. \tag{3.28}
\]

Proof. The formula (3.28) is a special case of a result of Cherednik [7] (see also [21]). For the derivation of (3.26) and (3.27) we follow the strategy of the proof of the analogous result in the symmetric case [1, Prop. 4.3]. First, note from the definition (1.8) that in general
\[
T_{\delta} f(t^{\tilde{\delta}}) = tf(t^{\tilde{\delta}}),
\]
and so
\[
U^{+} f(t^{\tilde{\delta}}) = (U^{+} 1) f(t^{\tilde{\delta}}) = [n]! f(t^{\tilde{\delta}}).
\]
Use of this latter formula in (3.13) with \( y = t^{\tilde{\delta}} \) gives
\[
K_A(t^{\tilde{\delta}}; z; q, t) = 0 \psi_0(t^{\tilde{\delta}}; -t^{n-1} z; q, t) = \prod_{i=1}^{n} (-t^{n-1} z_i; q)_{\infty}, \tag{3.29}
\]
and similarly, from (3.16)
\[
K_A(t^{\tilde{\delta}}; z; q, t) = 0 F_0(t^{\tilde{\delta}}; z; q, t) = \frac{1}{\prod_{i=1}^{n} (z_i; q)_{\infty}}, \tag{3.30}
\]
where the final equalities in (3.29) and (3.30) are known results [18, 13]. Now set \( y = t^{\tilde{\delta}-n+1} \) in the generating function (3.13). Use of (3.29) with \( z \) replaced by \( t^{-n+1} z \), and then use of (3.30) allows the l.h.s. of the resulting expression to be written
\[
\frac{1}{\prod_{i=1}^{n} (-az_i; q)_{\infty}} = K_A(t^{\tilde{\delta}}; -az; q, t) = \sum_{\nu} \frac{(-a)^{|\nu|} d_{\nu}}{d_{\nu} e_{\nu}} E_{\nu}(t^{\tilde{\delta}}) \tilde{E}_{\nu}(z).
\]
Equating with \( \tilde{E}_{\nu}(z) \) on the r.h.s. of the resulting expression gives (3.26). The formula (3.27) follows similarly, by substituting \( y = at^{\tilde{\delta}-n+1} \) in (3.13). \( \square \)
3.4 Relationship to the symmetric ASC polynomials

The non-symmetric ASC polynomials are related to the corresponding symmetric ASC polynomials in an analogous way to the relationship (3.17) between the non-symmetric and symmetric Macdonald polynomials.

**Proposition 3.9** Let

\[ a_\eta(q, t) = \frac{e_\eta}{P_{\eta}(t^\delta)d_\eta}. \]

We have

\[ U^+ E^{(V)}_\eta(y) = a_\eta(q, t)V^{(a)}_{\eta^+}(y; q, t) \]

(3.31)

\[ U^+ E^{(U)}_\eta(y) = a_\eta(q, t)U^{(a)}_{\eta^+}(y; q, t). \]

(3.32)

**Proof.** Consider the action of the \( U^+ \) operator on \((3.24)\) and \((3.25)\). From the first three equations of \((2.3)\) one can check that \( T_i \) commutes with any symmetric function of the \( D_i \). Thus the action of \( U^+ \) can be commuted to act to the right of \( \prod_i \rho_a(-\frac{1}{q}D_i) \) and \( 1/ \prod_i \rho_a(-D_i) \). Use of \((3.17)\) then gives

\[ U^+ E^{(V)}_\eta(y) = a_\eta(q, t) \frac{1}{\prod_i \rho_a(-D_i)} P_{\eta^+}(y) = a_\eta(q, t) \frac{1}{\prod_i \rho_a(qD_i)} P_{\eta^+}(y) \]

\[ U^+ E^{(U)}_\eta(y) = a_\eta(q, t) \prod_i \rho_a(-qD_i) P_{\eta^+}(y) = a_\eta(q, t) \prod_i \rho_a(D_i) P_{\eta^+}(y), \]

where in obtaining the first equality in the second formula we have used the fact that \( \tilde{P}_\eta(y^R) = P_{\eta}(y) \), while the second equalities in both formulas make use of \((2.3)\) and the fact that \( P_{\eta^+} \) is symmetrical. But the resulting operator formulas are precisely representations obtained in \([1, \text{eq. (3.9)} & (3.10)]\) for the symmetric ASC polynomials.

We can also relate the eigenoperators \( h_i \) for the non-symmetric ASC polynomials \( E^{(V)}_\eta \) to the eigenoperator \([1, \text{eq. (3.28)}]\)

\[ \mathcal{H} = t^{1-n} \sum_{i=1}^n Y_i^{1-1} - (1 + a) \sum_{i=1}^n t^{1-i} \sum_{i=1}^n t^{1-i} D_i Y_i^{1-1} + a \sum_{1 \leq i < j \leq n} t^{1-i} D_j D_i Y_i^{1-1} \]

(3.33)

for the symmetric ASC polynomials \( U^{(a)}_\lambda \).

**Proposition 3.10** Let \( h_i \) be given by \((1.34)\) and \( \mathcal{H} \) by \((3.33)\). When acting on symmetric functions

\[ \sum_{i=1}^n h_i = t^{1-n} \tilde{\mathcal{H}}. \]

**Proof.** From Theorem \([1.2, \text{by summing over } i \text{ in (1.31)}]\) we have

\[ \sum_{i=1}^n h_i E^{(V)}(x; q, t) = t^{1-n} e(\eta^+) E^{(V)}(x; q, t), \]

where \( e(\eta^+) = \sum_{i=1}^n t^\eta_i = \sum_{i=1}^n q^\eta^+_i t^{n-i} \). We would next like to apply the operator \( U^+ \) to both sides of this eigenvalue equation. For this purpose we require the fact that \( T_i \) commutes with
\sum_{i=1}^{n} h_i$ (this follows from (1.16), and the fact that these same equations apply with the $Y_i$ replaced by $D_i$). Thus, making use of (3.31), this operation gives

$$
\sum_{i=1}^{n} h_i V_{\eta_i}^{(a)}(x; q, t) = t^{1-n} e(\eta^+) V_{\eta_i}^{(a)}(x; q, t).
$$

But from [1] we know that this same eigenvalue equation applies with $\sum_{i=1}^{n} h_i$ replaced by $t^{1-n}\tilde{R}$. The result now follows from the fact that $\{V_{\eta_i}^{(a)}\}$ are a basis for symmetric functions.

We remark that an alternative proof is to establish directly that when acting on symmetric functions

$$
\sum_{i=1}^{n} \tilde{Y}_i^{-1} = \sum_{i=1}^{n} Y_i \quad \text{(3.34)}
$$

$$
- \sum_{i=1}^{n} t^{-1+i} \tilde{D}_i \tilde{Y}_i^{-1} = \sum_{i=1}^{n} D_i \quad \text{(3.35)}
$$

$$
\sum_{i=1}^{n} t^{-1+i} \tilde{D}_i^2 \tilde{Y}_i^{-1} + (1 - t) \sum_{1 \leq i < j \leq n} t^{-1+i} \tilde{D}_j \tilde{D}_i \tilde{Y}_i^{-1} = t^{1-n} \sum_{i=1}^{n} D_i Y_i^{-1} D_i \quad \text{(3.36)}
$$

\[ \square \]

### 3.5 Non-symmetric shifted Macdonald polynomials

In [1] it was observed that the symmetric ASC polynomials $V_{\lambda}^{(a)}(x)$ coincide (up to a factor and change of variables) with the shifted Macdonald polynomials when $a = 0$. We show now that this behaviour carries over to the non-symmetric case.

Following Knop and Sahi [14, 15, 26], the non-symmetric shifted Macdonald polynomials $G_\eta(z)$ are defined as the unique polynomial with expansion

$$
G_\eta(z; q, t) = \tilde{E}_\eta(z) + \sum_{|\nu| < |\eta|} b_{\nu\eta} \tilde{E}_\nu(z)
$$

(in [26] what we denote $\tilde{E}_\eta(z)$ is called the non-symmetric Jack polynomial), which vanishes at the points $z = t^\xi$ for all compositions $\xi$ such that $|\xi| \leq |\eta|$. Here $t^\xi$ is given by (1.3). Equivalently [14, 23] they can be defined as eigenfunctions of the “inhomogeneous” Cherednik operators

$$
\Xi_i = \tilde{Y}_i + \tilde{D}_i
$$

where the operators are defined with the variables $z_i$. For such polynomials, Knop and Sahi defined a raising operator $\Phi_{KS} = (z_n - t^{1-n})\omega^{-1}$ with a simple action on $G_\eta(z; q, t)$. It is easily seen that

$$
\lim_{a \to 0} \frac{-a}{q} \Psi_1^a = \Phi_{KS} \bigg|_{z_i = t^{n-1} x_i}, \quad \lim_{a \to 0} h_i = \Xi_i \bigg|_{z_i = t^{n-1} x_i}
$$

which immediately implies the sought relationship between $G_\eta$ and $E_\eta^{(V)}$.

**Proposition 3.11**

$$
E_\eta^{(V)}(x; q, t) \bigg|_{a=0} = t^{-(n-1)|\eta|} G_\eta(t^{n-1} x; q^{-1}, t^{-1}) \quad \text{(3.37)}
$$

or equivalently

$$
E_\eta^{(U)}(x; q, t) \bigg|_{a=0} = t^{(n-1)|\eta|} G_\eta(t^{1-n} x; q, t) \quad \text{(3.38)}
$$
One immediate application of \( \text{(3.37)} \) is the evaluation of \( G_\eta(0; q, t) \), which follows from \( \text{(3.27)} \). This is a special case of a result of Sahi [26, Th. 1.1], in which an evaluation formula is given for \( G(\alpha t^\delta; q, t) \), for a general scalar \( \alpha \). In fact use of (3.37) also allows this more general evaluation formula to be deduced.

**Proposition 3.12** With \( (\alpha)_\lambda^{(q,t)} := \prod_{s \in \lambda} (t^{\ell(s)} - q^{\alpha(s)} \alpha) \) we have

\[
G_\eta(t^{-\delta} \alpha; q, t) = \alpha^{\eta / \eta^t} (1/\alpha)^{(q,t)} t^{-(n-1)|\eta|} \frac{E_\eta}{d_\eta}. \]

**Proof.** Choosing \( a = 0 \) and \( y = t^{n-1-\delta} \alpha \) in \( \text{(3.23)} \), and using \( \text{(3.30)} \) and \( \text{(3.38)} \), we see that

\[
\sum_\eta \alpha^{-|\eta|} t^{\ell(n-1)|\eta|} \frac{d_\eta}{d_\eta'} G_\eta(t^{-\delta} \alpha; q, t) E_\eta(z) = \prod_{i=1}^n \frac{(z_i/\alpha; q)_\infty}{(z_i; q)_\infty}. \]

But we know that \( \text{[8, 13]} \)

\[
\prod_{i=1}^n \frac{(z_i/\alpha; q)_\infty}{(z_i; q)_\infty} = \sum_\lambda \frac{(1/\alpha)_\lambda^{(q,t)}}{d_\kappa^{(q,t)}} P_\kappa(z; q, t) = \sum_\eta \frac{(1/\alpha)^{(q,t)}_\eta}{d_\eta} E_\eta(z). \]

The result follows by equating coefficients of \( E_\eta(z) \). \( \square \)

### 3.6 q-binomial coefficients

Sahi [26] uses the polynomials \( G_\eta \) to introduce non-symmetric q-binomial coefficients \( \binom{\eta}{\nu}_{q,t} \) according to

\[
\binom{\eta}{\nu}_{q,t} := \frac{G_\nu(t^{\eta})}{G_\nu(t^{\nu})} \quad (3.39) \]

(\( \tilde{\nu} \) is defined by \( \text{(1.3)} \)). Our generating function characterization of the ASC polynomials, and thus by Proposition \( \text{(3.11)} \) of the polynomials \( G_\eta \), makes it natural to extend Lassalle’s [17] definition of the symmetric q-binomial coefficients to the non-symmetric case by defining the non-symmetric q-binomial coefficients \( \binom{\eta}{\nu}_{q,t} \) according to the generating function formula

\[
\bar{E}_\nu(x) \prod_{i=1}^n \frac{1}{(x_i; q)_\infty} = \sum_\eta \binom{\eta}{\nu}_{q,t} t^{l(\eta)-l(\nu)} \frac{d_\nu'}{d_\eta'} \bar{E}_\eta(x) \quad (3.40) \]

We can then use the generating function \( \text{(3.13)} \) to relate these binomial coefficients to the polynomials \( G_\eta \).

**Proposition 3.13** With \( \binom{\eta}{\nu}_{q,t} \), defined by \( \text{(3.41)} \), we have

\[
\frac{G_\eta(x)}{G_\nu(0)} = \sum_\nu \binom{\eta}{\nu}_{q^{-1},t^{-1}} \frac{\bar{E}_\nu(x)}{G_\nu(0)} \quad (3.41) \]

**Proof.** Multiply both sides of (3.40) by \( q^{a(\nu) t^{\ell(n-1)|\nu| - l(\nu)} \frac{d_\nu}{d_\nu'} E_\nu(y) \) and sum over \( \nu \), rewriting the l.h.s. according to (3.15). Now equate coefficients of \( \bar{E}_\nu(x) \) on both sides. The result then follows upon using (3.28) and (3.37).

Since (3.41) is a formula satisfied by the non-symmetric q-binomial coefficients of Sahi [26, Cor. 1.3], and this formula suffices to implicitly define these coefficients, we have that

\[
\binom{\eta}{\nu}_{q,t} = \binom{\eta}{\nu}_{q,t}. \quad (3.42) \]
Finally, let us present some formulas relating the coefficients \( \left( \begin{array}{c} \eta \\ \nu \end{array} \right)_{q,t} \) to their symmetric counterparts \( \left( \begin{array}{c} \kappa \\ \mu \end{array} \right)_{q,t} \), which can be characterized by either of the formulas \[ 3.43 \]

\[ P_{\mu}(x; q, t) \prod_{i=1}^{n} \frac{1}{(x_i; q)_{\infty}} = \sum_{\lambda} \left( \begin{array}{c} \lambda \\ \mu \end{array} \right)_{q,t} \frac{b^{(\lambda)-b(\mu)}}{d^{\lambda}_{\lambda}} P_{\lambda}(x; q, t). \]

Here \( P_{\lambda}^{\kappa} \) is the shifted Macdonald polynomial, which is related to the symmetric ASC polynomial \( V_{\lambda}^{(0)} \) by \[ 3.46 \]

\[ P_{\lambda}^{\kappa}(yt^{-\delta+n-1}; q^{-1}, t^{-1}) = t^{(n-1)|\lambda|} V_{\lambda}^{(0)}(y; q, t). \]

**Proposition 3.14** With \( \eta^+ = \kappa, \nu^+ = \mu, \)

\[ \sum_{\nu \mid \nu^+ = \mu} \left( \begin{array}{c} \eta \\ \nu \end{array} \right)_{q,t} = \left( \begin{array}{c} \kappa \\ \mu \end{array} \right)_{q,t} \]

\[ \frac{d^\eta_{\lambda}}{d^{\lambda}_{\mu}} P_{\lambda}(t^\delta) E_{\nu}(t^\delta) \sum_{\eta \mid \eta^+ = \kappa} \left( \begin{array}{c} \eta \\ \nu \end{array} \right)_{q,t} \frac{E_{\eta}(t^\delta)}{d^\eta_{\eta}} = \left( \begin{array}{c} \kappa \\ \mu \end{array} \right)_{q,t} \]

**Proof.** The proof follows the strategy given in \[ 3.46 \] for the proof of the corresponding results in the \( q = t^{\alpha}, q \to 1 \) limit (binomial coefficients associated with non-symmetric Jack polynomials). For \( 3.46 \) we apply the \( U^+ \) operator to \( 3.41 \), making use of \( 3.17 \) and \( 3.31 \). Use of the fact that

\[ \frac{a_{\nu}}{E_{\nu}(V_{\eta^+}(0; q, t))} = \frac{[\eta]!}{V_{\eta^+}^{(0)}(0; q, t)} \]

and \( 3.45 \) then gives

\[ \frac{P_{\lambda}^{\kappa}(xt^{-\delta}; q^{-1}, t^{-1})}{P_{\lambda}^{\kappa}(0; q^{-1}, t^{-1})} = \sum_{\nu} \left( \begin{array}{c} \eta \\ \nu \end{array} \right)_{q,t} \frac{P_{\nu^+}(x; q, t)}{P_{\nu^+}^{\kappa}(0; q^{-1}, t^{-1})}. \]

Comparison with \( 3.44 \) implies \( 3.46 \) The identity \( 3.47 \) follows similarly, by applying \( U^+ \) to \( 3.40 \) and comparing with \( 3.43 \).

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