Cancellation of projective modules over non-Noetherian rings

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Abstract

(i) Let $R$ be a ring of dimension 0 and $A = R[Y, Y^{-1}, f_1 \ldots f_m]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over $R$ and $f_i \in R[Y]$. Then all projective $A$-modules are free and $E_r(A)$ acts transitively on $Um_r(A)$ for $r \geq 3$.

(ii) Let $R$ be a ring of dimension $d$ and $A$ be one of $R[Y]$ or $R[Y, Y^{-1}]$, where $Y$ is a variable over $R$. Let $P$ be a projective $A$-module of rank $\geq d + 1$ satisfying property $\Omega(R)$ (see [4] for definition of property $\Omega(R)$). Then $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$. When $P$ is free, this result is due to Yengui: $A = R[Y]$ and Abedelfatah: $A = R[Y, Y^{-1}]$.

1 Introduction

Rings are assumed to be commutative with unity and modules are finitely generated. The dimension of a ring means its Krull dimension and projective modules are of constant rank.

Let $R$ be a Noetherian ring of dimension $d$ and $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over $R$ and $f_i \in R[Y_i]$. If $P$ is a projective $A$-module of rank $\geq \max\{2, d + 1\}$, then author-Dhorajia ([6], Theorem 3.12) proved that $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$. In particular $P$ is cancellative, i.e. $P \oplus A' \simeq Q \oplus A'$ for some projective $A$-module $Q \Rightarrow P \simeq Q$. The case $n = m = 0$ of this result is due to Bass [1], $n = 1, m = 0$ is due to Plumstead [15], $n = m = 1$ and $f_1 = Y_1$ is due to Mandal [14] (he proved that $P$ is cancellative), $m = 0$ is due to Rao [17] (he proved that $P$ is cancellative) and Laurent polynomial case $f_i = Y_i$ is due to Lindel [13].

Heitmann ([9], Corollary 2.7) generalized Bass’ result to all commutative non-Noetherian rings. It is natural to ask if analog of above results hold for non-Noetherian rings.

Let $R$ be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n]$ be a polynomial ring in $n$ variables $Y_1, \ldots, Y_n$ over $R$. Then Brewer-Costa [5] proved that all projective $A$-modules are free, generalizing the well known Quillen-Suslin theorem [16, 20] (see Ellouz-Lombardi-Yengui [8] for a constructive proof). Abedelfatah [2] generalized Brewer-Costa’s result by proving that $E_r(A)$ acts transitively on $Um_r(A)$ for $r \geq 3$. We generalize these results as follows (see [3, 6, 30]). This is non-Noetherian analog of author-Dhorajia’s result in case $d = 0$.

Theorem 1.1 Let $R$ be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$, where $m \leq n$, $Y_1, \ldots, Y_n$ are variables over $R$ and $f_i \in R[Y_i]$. Then all projective $A$-modules are free and $E_r(A)$ acts transitively on $Um_r(A)$ for $r \geq 3$.

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Let $R$ be a ring of dimension $d$ and $n \geq d + 2$. Then Yengui [24] proved that $E_n(R[Y])$ acts transitively on $Um_n(R[Y])$ which is non-Noetherian analog of Plumstead’s result in free case. Abedelfatah [H] proved that $E_n(R[Y, Y^{-1}])$ acts transitively on $Um_n(R[Y, Y^{-1}])$ which is non-Noetherian analog of Mandal’s result in free case. We generalize both results as follows [11]. See (1.8) for definition of property $\Omega(R)$.

**Theorem 1.2** Let $R$ be a ring of dimension $d$ and $A$ be one of $R[Y]$ or $R[Y, Y^{-1}]$, where $Y$ is a variable over $R$. If $P$ is a projective $A$-module of rank $\geq d + 1$ satisfying property $\Omega(R)$, then $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$. In particular $P$ is cancellative.

We generalize (1.2) for Prüfer domain as follows (see 5.3): Let $R$ be a Prüfer domain of dimension $d$ and $A = R[Y, f^{-1}]$, where $Y$ is a variable over $R$ and $f \in R[Y]$. If $P$ is a projective $A$-module of rank $\geq d + 1$, then $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$.

## 2 Preliminaries

Let $A$ be a ring, $J$ an ideal of $A$ and $M$ an $A$-module. We say that $m \in M$ is unimodular if there exist $\phi \in M^* = Hom_A(M, A)$ such that $\phi(m) = 1$. The set of unimodular elements of $M$ is denoted by $Um(M)$. We write $Um^1(A \oplus M, J)$ for the set of $(a, m) \in Um(A \oplus M)$ such that $a \in J + 1$. We write $Um(A \oplus M, J)$ for the set of $(a, m) \in Um^1(A \oplus M, J)$ such that $m \in JM$. We write $Um_r(A, J)$ for $Um(A \oplus A^{r-1}, J)$.

The group of $A$-automorphism of $M$ is denoted by $Aut_A(M)$. We write $E^1(A \oplus M, J)$ for the subgroup of $Aut_A(A \oplus M)$ generated by automorphisms $\Delta_a \varphi$ and $\Gamma_m$, where

$$
\Delta_a \varphi = \begin{pmatrix} 1 & a \varphi \\ 0 & id_M \end{pmatrix} \quad \text{and} \quad \Gamma_m = \begin{pmatrix} 1 & 0 \\ m & id_M \end{pmatrix} \quad \text{with} \quad a \in J, \varphi \in M^*, m \in M.
$$

We write $E^1(A \oplus M)$ for $E^1(A \oplus M, A)$. Let $E_{r+1}(A)$ denote the subgroup of $SL_{r+1}(A)$ generated by elementary matrices $I + ae_{ij}$, where $a \in A$, $i \neq j$ and $e_{ij}$ is the matrix with only non-zero entry 1 at $(i, j)$-th place. We write $E^1_{r+1}(A, J)$ for the subgroup of $E_{r+1}(A)$ generated by $\Delta_a$ and $\Gamma_b$, where

$$
\Delta_a = \begin{pmatrix} 1 & a \\ 0 & id_F \end{pmatrix} \quad \text{and} \quad \Gamma_b = \begin{pmatrix} 1 & 0 \\ b^t & id_F \end{pmatrix}, \quad \text{where} \quad F = A^r, a \in JF, b \in F.
$$

Let $p \in M$ and $\varphi \in M^*$ be such that $\varphi(p) = 0$. Let $\varphi_p \in End(M)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is an automorphism of $M$. The automorphism $1 + \varphi_p$ of $M$ is called a transvection of $M$ if either $p \in Um(M)$ or $\varphi \in Um(M^*)$. We write $E(M)$ for the subgroup of $Aut(M)$ generated by transvections of $M$.

Due to following result of Bak-Basu-Rao (3, theorem 3.10), we can interchange $E(A \oplus P)$ and $E^1(A \oplus P)$.

**Theorem 2.1** Let $A$ be a ring and $P$ a projective $A$-module of rank $\geq 2$. Then $E^1(A \oplus P) = E(A \oplus P)$. 

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The following result of Heitmann ([9], Corollary 2.7) generalizes Bass’s cancellation [4] to non-Noetherian rings.

**Theorem 2.2** Let $A$ be a ring of dimension $d$ and $P$ a projective $A$-module of rank $\geq d + 1$. Then $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$. In particular $P$ is cancellative.

The following result of Brewer-Costa [5] generalizes Quillen-Suslin theorem [10, 20] to all zero-dimensional rings.

**Theorem 2.3** Let $R$ be a ring of dimension 0 and $A = R[Y_1, \ldots, Y_n]$ a polynomial ring in $n$ variables $Y_1, \ldots, Y_n$ over $R$. Then all projective $A$-modules are free.

We will state five results which are proved with assumption that rings are Noetherian. But the same proof works for non-Noetherian rings.

**Lemma 2.4** ([7], Remark 2.2) Let $A$ be a ring, $I$ an ideal of $A$ and $P$ a projective $A$-module. Then the natural map $E(A \oplus P) \to E(T_N(I(A \oplus P)))$ is surjective.

**Lemma 2.5** ([6], Lemma 3.1) Let $A$ be a ring, $J$ an ideal of $A$ and $P$ a projective $A$-module. Let “bar” denote reduction modulo the nil-radical of $A$. Assume $E^1(A \oplus P,J)$ acts transitively on $Um^1(A \oplus P,J)$. Then $E^1(A \oplus P)$ acts transitively on $Um^1(A \oplus P,J)$.

**Lemma 2.6** ([13], Lemma 1.1) Let $A$ be a reduced ring and $P$ an $A$-module. Assume $s \in A$ is a non-zerodivisor such that $P_s$ is free of rank $r \geq 1$. Then there exist $p_1, \ldots, p_r \in P$, $\phi_1, \ldots, \phi_r \in P^*$ and $t \in \mathbb{N}$ such that

(i) $s^t P \subset F$ and $s^t P^* \subset G$ with $F = \sum_{1 \leq i \leq r} A p_i$ and $G = \sum_{1 \leq i \leq r} A \phi_i$.

(ii) $(\phi_i(p_j))_{1 \leq i, j \leq r} = \text{diagonal } (s^t, \ldots, s^t)$.

**Lemma 2.7** ([10], Lemma 3.10) Let $A$ be a reduced ring and $P$ a projective $A$-module of rank $r$. Assume there exist a non-zerodivisor $s \in A$ such that $P_s$ is free. Choose $p_1, \ldots, p_r \in P, \phi_1, \ldots, \phi_r \in P^*$ satisfying ([2.2]). Let $(a, p) \in Um(A \oplus P, sA)$ with $p = c_1 p_1 + \ldots + c_r p_r$, where $c_i \in sA$ for all $i$. Assume there exist $\phi \in E_{r+1}^1(A, sA)$ such that $\phi(a, c_1, \ldots, c_r) = (1, 0, \ldots, 0)$. Then there exist $\Phi \in E(A \oplus P)$ such that $\Phi(a, p) = (1, 0)$.

**Lemma 2.8** ([22], Lemma 4.2) Let $A$ be a reduced ring and $P$ an $A$-module. Assume there exist non-zerodivisors $s_1, \ldots, s_r \in A$, $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in P^*$ such that $(\phi_i(p_j))_{r \times r} = \text{diagonal } (s_1, \ldots, s_r) := N$. Let $M$ be the subgroup of $GL_r(A)$ consisting of all matrices of the form $I + TN^2$ for $T \in M_r(A)$. Then the map

$$\Phi : M \to \text{Aut}(P); \quad \Phi(I + TN^2) = id_P + (p_1, \ldots, p_r)TN(\phi_1, \ldots, \phi_r)^t$$

is a group homomorphism.

The following result is from Lam’s book ([11], Proposition VI.1.14).
Proposition 2.9 Let $B$ be a ring and $a, b \in B$ two comaximal elements. Then for any $\sigma \in E_n(B_{ab})$ with $n \geq 3$, there exist $\alpha \in E_n(B_b)$ and $\beta \in E_n(B_a)$ such that $\sigma = (\alpha)_a(\beta)_b$.

We state Quillen-Suslin theorem \[16, 20\]. Note that any commutative ring is a filtered union of Noetherian commutative rings. Hence following result will follow from Noetherian case.

Theorem 2.10 Let $R$ be a ring and $P$ a projective $R[Y]$-module. Let $f \in R[Y]$ be a monic polynomial such that $P_f$ is free. Then $P$ is free.

We state a result of Yengui \[23\] and Abedelfatah \[1\] respectively.

Theorem 2.11 Let $A$ be a ring of dimension $d$, $Y$ a variable over $A$ and $n \geq d + 2$. Then
1. $E_n(A[Y])$ acts transitively on $\text{Um}_n(A[Y])$.
2. $E_n(A[Y,Y^{-1}])$ acts transitively on $\text{Um}_n(A[Y,Y^{-1}])$.

3 Zero dimension case

In this section we prove our first result.

Proposition 3.1 Let $\Sigma(n)$ be set of rings which is closed w.r.t. following properties:
1. If $R \in \Sigma(n)$ and $0 \neq f \in R[Y]$ is non-unit, then $R[Y]_{f(1+fR[Y])} \in \Sigma(n)$.
2. If $R \in \Sigma(n)$, then all projective modules over $R[Y_1,\ldots,Y_n]$ are free, where $Y_1,\ldots,Y_n$ are variables over $R$.

Then, for $R \in \Sigma(n)$, all projective modules over $R[Y_1,\ldots,Y_n, (f_1\ldots f_m)^{-1}]$ are free, where $m \leq n$ and $f_i \in R[Y_i]$.

Proof Let $P$ be a projective $A = R[Y_1,\ldots,Y_n, (f_1\ldots f_m)^{-1}]$-module of rank $r$. If $m = 0$, then $P$ is free by assumption (ii). Assume $m > 0$ and use induction on $m$. Write $C = R[Y_1,\ldots,Y_n, (f_1\ldots f_{m-1})^{-1}]$, $S = 1 + f_m R[Y_m]$ and $B = R[Y_m]_{f_m S}$. Then $A = C_{f_m}$, $B \in \Sigma(n)$ by assumption (i) and $S^{-1} A = B[Y_1,\ldots,Y_{m-1}, Y_{m+1},\ldots,Y_n, (f_1\ldots f_{m-1})^{-1}]$. By induction on $m$, $S^{-1} P$ is free. Since $P$ is finitely generated, we can find $g \in S$ such that $P_g$ is free. Note that $f_m$ and $g$ are comaximal elements of $R[Y_m]$. Consider the fiber product diagram

\[
\begin{array}{ccc}
C & \rightarrow & C_{f_m} = A \\
\downarrow & & \downarrow \\
C_g & \rightarrow & C_{f_m g} = A_g
\end{array}
\]

Patching projective modules $P$ over $C_{f_m}$ and $(C_g)^r$ over $C_g$, we get $P \rightarrow Q_{f_m}$, where $Q$ is a projective $C$-module of rank $r$. By induction on $m$, projective modules over $C$ are free. Hence $Q$ is free and therefore $P$ is free. \[\square\]
Proposition 3.2 Let \( R \) be a ring of dimension 0 and \( A = \mathbb{R}[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}] \), where \( m \leq n \), \( Y_1, \ldots, Y_n \) are variables over \( R \) and \( f_i \in \mathbb{R}[Y_i] \). Then all projective \( A \)-modules are free.

Theorem 3.3 Let \( R \) be a ring of dimension 0 and \( A = \mathbb{R}[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}] \), where \( m \leq n \), \( Y_1, \ldots, Y_n \) are variables over \( R \) and \( f_i \in \mathbb{R}[Y_i] \). Then \( E_r(A) \) acts transitively on \( \text{Um}_r(A) \) for \( r \geq 3 \).

Proof The case \( m = 0 \) is due to Abedelfatah [2]. Assume \( m > 0 \) and use induction on \( m \). Let \( v \in E_r(A) \). Write \( C = \mathbb{R}[Y_1, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}] \), \( S = 1 + f_m \mathbb{R}[Y_m] \) and \( B = \mathbb{R}[Y_m]/S \). Then \( B \) is 0 dimensional, \( A = C_f \) and \( S^{-1}A = B[Y_1, \ldots, Y_{m-1}, Y_{m+1}, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}] \). By induction on \( m \), \( E_r(S^{-1}A) \) acts transitively on \( \text{Um}_r(S^{-1}A) \). Hence there exist \( \sigma \in E_r(S^{-1}A) \) such that \( \sigma(v) = e_1 = (1, 0, \ldots, 0) \). We can find \( g \in S \) and \( \tilde{\sigma} \in E_r(C_{f_m}) \) such that \( \tilde{\sigma}(v) = e_1 \). Note that \( f_m \) and \( g \) are comaximal elements of \( R[Y_m] \). Consider the fiber product diagram

\[
\begin{array}{ccc}
C & \rightarrow & C_{f_m} = A \\
\downarrow & & \downarrow \\
C_g & \rightarrow & C_{f_m} g = A_g
\end{array}
\]

By [24], \( \tilde{\sigma} \) has a splitting \( \tilde{\sigma} = (\alpha)_{f_m}(\beta)_g \), where \( \alpha \in E_r(C_g) \) and \( \beta \in E_r(C_{f_m}) \). We have unimodular elements \( \beta(v) \in \text{Um}_r(C_{f_m}) \) and \( \alpha^{-1}(e_1) \in \text{Um}_r(C_g) \) whose images in \( C_{f_m} g \) are same. Hence patching \( \beta(v) \) and \( \alpha^{-1}(e_1) \), we get \( w \in \text{Um}_r(C) \) such that its image in \( C_{f_m} \) is \( \beta(v) \). By induction on \( m \), \( E_r(C) \) acts transitively on \( \text{Um}_r(C) \). Hence there exist \( \phi \in E_r(C) \) such that \( \phi(w) = e_1 \). If \( \Phi_1 \in E_r(C_{f_m}) \) is the image of \( \phi \), then \( \Phi_1(\alpha(v)) = e_1 \). Write \( \Phi = \Phi_1 \alpha \in E_r(A) \), we are done.

4 Main Theorem

The following result is proved in ([10], Lemma 3.3) with the assumption that ring is Noetherian. Using [28], same proof works for non-Noetherian ring. Hence we omit the proof.

Lemma 4.1 Let \( A \) be a reduced ring and \( P \) a projective \( A \)-module of rank \( r \). Assume there exist a non-zerodivisor \( s \in A \) such that [2.6] holds. Assume \( R^s \) is cancellative, where \( R = A[X]/(X^2 - s^2X) \). Then any element of \( \text{Um}^1(A \oplus P, s^2A) \) can be taken to \( (1, 0) \) by some element of \( \text{Aut}(A \oplus P, sA) \).

An immediate consequence of [4.1] is the following result. Its proof is same as of ([10], Corollary 3.5) using [2.2].
Corollary 4.2 Let $A$ be a reduced ring of dimension $d$ and $P$ a projective $A$-module of rank $d$. Assume there exist a non-zerodivisor $s \in A$ such that (2.6) holds. Assume $R^d$ is cancellative, where $R = A[X]/(X^2 - s^2X)$. Then $P$ is cancellative.

Let $R$ be a ring and $I$ an ideal of $R$. For $n \geq 3$, let $E_n(I)$ be the subgroup of $E_n(R)$ generated by $E_{ij}(a) = I + ae_{ij}$ with $a \in I$ and $1 \leq i \neq j \leq n$. Let $E_n(R, I)$ denote the normal closure of $E_n(I)$ in $E_n(R)$. We have two characterisation of $E_n(R, I)$ due to Suslin-Vaserstein [21] and Stein [19] respectively.

Proposition 4.3 The kernel of the natural map $E_n(R) \to E_n(R/I)$ is isomorphic to $E_n(R, I)$.

Proposition 4.4 Consider the following fiber product diagram

\[
\begin{array}{ccc}
R(I) & \xrightarrow{p_1} & R \\
\downarrow{p_2} & & \downarrow{j_1} \\
R & \xrightarrow{j_2} & R/I
\end{array}
\]

Then $E_n(R, I)$ is kernel of the natural surjection $E_n(p_1) : E_n(R(I)) \to E_n(R)$.

Using [4.3, 4.4, 2.7] and following the proof of [7, Lemma 3.3], we get the following result. In [7], it is proved for Noetherian ring.

Lemma 4.5 Let $A$ be a reduced ring and $P$ a projective $A$-module of rank $r$. Assume there exist a non-zerodivisor $s \in A$ such that (2.6) holds. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$, where $B = A[X]/(X^2 - s^2X)$. Then any element of $\text{Um}(A \oplus P, s^2A)$ can be taken to $(1, 0)$ by some element of $E(A \oplus P)$.

The proof of the following result is same as of [7, Theorem 3.4] using [4.3, 2.7].

Proposition 4.6 Let $A$ be a reduced ring of dimension $d$ and $P$ a projective $A$-module of rank $r \geq d$. Assume there exist a non-zerodivisor $s \in A$ such that (2.6) holds. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$, where $B = A[X]/(X^2 - s^2X)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.

Remark 4.7 By [11, Exercise 2.34], any reduced ring $R$ can be embedded in a reduced non-Noetherian ring $S$ such that $S$ equals the total quotient ring $Q(S)$ of $S$ and $R$ is a retract of $S$. In particular, if $P$ is a non-free projective $R$-module, then $P \otimes_R S$ is a non-free projective $S$-module. Hence, if $R$ is a reduced non-Noetherian ring and $P$ a projective $R$-module, then we can not say that $P_\tau$ is free, for some non-zerodivisor $s \in R$.

Definition 4.8 Let $R \subset S$ be rings and $P$ a projective $S$-module. We say that $P$ satisfies property $\Omega(R)$ if for any ideal $I$ of $R$ and $\overline{P} = P/IP$, there exist a non-zerodivisor $\overline{t} \in R/I$ such that $\overline{P}_{\overline{t}}$ is free. The property $\Omega(R)$ avoids situation [4.7].
The following result generalises (2.11).

**Theorem 4.9** Let $R$ be a ring of dimension $d$ and $A$ is one of $R[Y]$ or $R[Y,Y^{-1}]$, where $Y$ is a variable over $R$. Let $P$ be a projective $A$-module of rank $r \geq d+1$ which satisfies property $\Omega(R)$. Then $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$.

**Proof** By (2.5), we may assume $R$ is reduced. If $d = 0$, then $P$ is free by (3.2) and we can use (2.11). Hence assume $d \geq 1$ and use induction on $d$. Since $P$ satisfies property $\Omega(R)$, we can find a non-zerodivisor $s \in R$ such that $P_s$ is free and (2.6) holds. If $R' = R[X]/(X^2 - s^2X)$, then dim $R' = d$. Write $B = A[X]/(X^2 - s^2X)$. Then $B$ is one of $R'[Y]$ or $R'[Y,Y^{-1}]$. By (2.11), $E_{r+1}(B)$ acts transitively on $Um_{r+1}(B)$. Applying (2.5), we get every element of $Um(A \oplus P, s^2A)$ can be taken to $(1,0)$ by some element of $E(A \oplus P)$. Therefore it is enough to show that every element of $Um(A \oplus P)$ can be taken to an element of $Um(A \oplus P, s^2A)$ by some element of $E(A \oplus P)$.

Let “bar” denote reduction modulo $s^2A$. Then dim $R/s^2 \cdot d < d$. By assumption, $P/s^2 \cdot d$ satisfies property $\Omega(R/s^2 \cdot d)$. Hence by induction on $d$, $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$. Using (2.4), any element of $Um(A \oplus P)$ can be taken to an element of $Um(A \oplus P, s^2A)$ by $E(A \oplus P)$. This completes the proof.

## 5 Some Auxiliary results

**Lemma 5.1** Let $R$ be a ring of dimension $d$ such that dimension of the polynomial ring $A = R[Y_1, \ldots, Y_n]$ is $d + n$. Then every stably free $A$-module $P$ of rank $\geq d + 1$ is free.

**Proof** The case $n = 0$ is due to Heitmann (2.2). Assume $n > 0$ and use induction on $n$. Let $S$ be the set of all monic polynomials in $R[Y_n]$. Then dim $R[Y_n]_S = d$ and dim $R[Y_n]_S[Y_1, \ldots, Y_{n-1}] = d + n - 1$. Hence by induction on $n$, $S^{-1}P$ is free. By (2.10), $P$ is free.

**Proposition 5.2** Let $R$ be a ring of dimension $d$ such that dimension of the polynomial ring $R[Y_1, \ldots, Y_n]$ is $d + n$. Let $A = R[Y_1, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}]$ with $m \leq n$ and $f_i \in R[Y_i]$ a monic polynomial for all $i$. Then every stably free $A$-module $P$ of rank $r \geq d + 1$ is free.

**Proof** The case $m = 0$ follows from (5.1). Assume $m > 0$ and use induction on $m$. Let $C = R[Y_1, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}]$. If $S = 1 + f_m R[Y_m]$, then dim $R[Y_m]_{f_mS} = d$ (since dim $R[Y_m] = d + 1$) and $S^{-1}A = R[Y_m]_{f_mS}[Y_1, \ldots, Y_{m-1}, Y_{m+1}, \ldots, Y_n, (f_1 \ldots f_{m-1})^{-1}]$. By induction on $m$, $S^{-1}P$ is free. Choose $g \in S$ such that $P_g$ is free. Patching projective modules $P$ and $C_g$ over $C_{f_mg}$, we get a projective $C$-module $Q$ such that $Q_{f_m} = P$. Since $P$ is stably free, $(Q \oplus C')_{f_m}$ is free for some $t$. By (2.10), $Q \oplus C'$ is free, i.e., $Q$ is stably free. By induction on $m$, $Q$ and hence $P$ is free.

It is natural to ask if all projective $A$-modules of rank $\geq d + 1$ in (5.2) are cancellative. We give a partial answer.
**Theorem 5.3** Let $R$ be an integral domain of dimension $d$ such that $\dim R[Y] = d + 1$. Let $A = R[Y, f^{-1}]$ with $f \in R[Y]$ and $P$ a projective $A$-module of rank $r \geq \max\{2, d + 1\}$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.

**Proof** If $d = 0$, then $P$ is free and we are done by (3.3). Assume $d \geq 1$. Choose $0 \neq s \in R$ such that (2.6) holds. Write $R' = R[X]/(X^2 - s^2X)$ and $B = R'[Y, f^{-1}]$. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$. By (1.3), any $(a, p) \in \text{Um}(A \oplus P, s^2A)$ can be taken to $(1, 0)$ by some element in $E(A \oplus P)$. Let “bar” denote reduction modulo $s^2A$. Then $\dim \overline{A} = d$ and rank $\overline{P} \geq d + 1$. Applying (2.2), we get $E(\overline{A} \oplus \overline{P})$ acts transitively on $\text{Um}(\overline{A} \oplus \overline{P})$. Using (2.3), every $V \in \text{Um}(A \oplus P)$ can be taken to $W \in \text{Um}(A \oplus P, s^2A)$ by some element of $E(A \oplus P)$. Therefore, it is enough to show that $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$.

Let $v \in \text{Um}_{r+1}(B)$. If $C = R'[Y]$, then $B = C_f$. Since $R'$ is an integral extension of $R$, $\dim R'[Y] = d + 1 = \dim R[Y]$. Hence $\dim C_{f(1+f^2Y')} = d$. Applying (2.2), we get $\sigma \in E_{r+1}(C_{f(1+f^2Y')})$ such that $\sigma(v) = (1, \ldots, 0)$. We can find $g \in 1 + fR'[Y]$ and $\overline{\sigma} \in E_{r+1}(C_{f\overline{g}})$ such that $\overline{\sigma}(v) = (1, 0, \ldots, 0)$.

By (2.3), $\overline{\sigma}$ has a splitting $\overline{\sigma} = (\alpha f)(\beta)g$, where $\alpha \in E_{r+1}(C_g)$ and $\beta \in E_{r+1}(C_f)$. We have two unimodular elements $\beta(v) \in \text{Um}_{r+1}(C_f)$ and $\alpha^{-1}(1, 0, \ldots, 0) \in \text{Um}_{r+1}(C_g)$ whose images in $C_{f\overline{g}}$ are same. Hence, patching $\beta(v)$ and $\alpha^{-1}(1, 0, \ldots, 0)$, we get $w \in \text{Um}_{r+1}(C)$ whose image in $C_f$ is $\beta(v)$. By Yengui (2.11), $E_{r+1}(C_f)$ acts transitively on $\text{Um}_{r+1}(C)$. Hence, we can find $\phi \in E_{r+1}(C)$ such that $\phi(w) = (1, 0, \ldots, 0)$. If $\Phi_1$ is the image of $\phi$ in $C_f$, then $\Phi_1(\alpha(v)) = (1, 0, \ldots, 0)$ and $\Phi_1 \alpha \in E_{r+1}(B)$. This completes the proof. ■

**Remark 5.4** (1) By a result of Seidenberg (15, Theorem 4), if $R$ is a Prüfer domain, then $\dim R[Y_1, \ldots, Y_n] = \dim R + n$. Hence (5.2) holds for a Prüfer domain $R$ and generalizes (4.9).

(2) Lequain-Simis have shown [12] that if $R$ is a Prüfer domain, then projective modules over $R[Y_1, \ldots, Y_n]$ are extended from $R$. In particular, if $R$ is a valuation domain (local Prüfer domain), then projective $R[Y_1, \ldots, Y_n]$-modules are free. It is natural to ask if projective modules over $R[Y_1, \ldots, Y_n, (f_1 \ldots f_m)^{-1}]$ are free, where $R$ is a valuation domain, $m \leq n$ and $f_i \in R[Y_i]$. If each $f_i$ is a monic polynomial, then (5.2) gives a partial answer.

**Proposition 5.5** Let $R$ be a valuation domain of dimension $d$ and $A = R[X, Y_1, \ldots, Y_n, f^{-1}]$ with $f \in R[X]$. Then every stably free $A$-module $P$ of rank $\geq d + 1$ is free.

**Proof** If $d = 0$, then $P$ is free, by (5.2). Assume $d \geq 1$. Let $C = R[X, Y_1, \ldots, Y_n]$ and $S = 1 + fR[X]$. Since $\dim R[X] = d + 1$ by Seidenberg [13], $\dim R[X]_{fS} = d$ and $\dim R[X]_{fS}[Y_1, \ldots, Y_n] = d + n$. By (6.1), $S^{-1}P$ being stably free, is free. Choose $g \in S$ such that $P_g$ is free. Patching projective modules $P$ and $(C_g)^\ast$ over $C_{fS}$, we get a projective $C$-module $Q$ such that $P \sim Q_f$. By Lequain-Simis [12], every projective $C$-module is free. Therefore $Q$ and hence $P$ is free. ■
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