Initial problem for heat equation with multisoliton inhomogeneity and one-loop quantum corrections

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Abstract

The generalized zeta-function is built by a dressing method based on the Darboux covariance of the heat equation and used to evaluate the correspondent functional integral in quasiclassical approximation. Quantum corrections to a kink-like solutions of Landau-Ginzburg model are calculated.

1 Introduction

In the paper of V.Konoplich [1] quantum corrections to a few classical solutions by means of Riemann zeta-function are calculated. Most interesting of them are the corrections to the kink - the separatrix solution of field $\phi^4$ model. The method of [1] is rather complicated and it could be useful to simplify it. We use the dressing technique based on classical Darboux transformations (DT) with a new applications to Green function construction [2]. It is the main aim of this note with eventual possibility to generalize the result due to universality of the technique when a link to integrable (soliton, SUSY)
systems is established. The suggested approach opens new possibilities; for example it allows to show the way to calculate the quantum corrections to Q-balls and periodic solutions of the models. The last problem is posed in the useful review.

2 Heat equation Cauchy problem

We will base on the DT-covariance of the heat equation for the function 
\[ \rho(\tau, x, y) = -\rho_\tau + \rho_{xx} + u(x)\rho = 0, \]
that means the form-invariance of (11) with respect to iterated DT, defined by the Wronskian \[ W[\phi_1, ..., \phi_N] \]
of the solutions of (11)
\[ \rho \to \rho[N] = \frac{W[\phi_1, ..., \phi_N, \rho]}{W[\phi_1, ..., \phi_N]}, \]
\[ u \to u[N] = u + 2ln W[\phi_1, ..., \phi_N]. \]

Consider now a Cauchy problem for the equation (11), where \[ u(x) \] represents the reflectionless potential in a sense that it could be produced by the DT and the initial condition is
\[ \rho(0, x, y) = \delta(x - y). \]

The problem is formulated for a Green function: it is rather general and may be applied as a model of classical diffusion or heat conductivity. We, however, would follow other applications in the theory of quasiclassical quantization, where the function \[ \rho \] is treated as density matrix whence \[ \tau \] stands for inverse temperature [?].

The algorithm of such problem solution is the dressing procedure organized by a sequence of DTs defined by (2):
\[ \left( \frac{\partial}{\partial x} - \ln_x \phi_1(x, y) \right) \rho_0(0, x, y) = g_1(x, y), \]
\[ \left( \frac{\partial}{\partial x} - \ln_x \phi_2[1](x, y) \right) g_1(x, y) = g_2(x, y), ..., \]
\[ \left( \frac{\partial}{\partial x} - \ln_x \phi_k[k-1](x, y) \right) g_{k-1} = g_k x, y, \]
\[ g_N(x, y) = \delta(x, y), 2 \leq k \leq N. \]

and the following theorem

**Theorem** The function \[ \rho[N] \] being built by (2) will be a solution of the problem (11) with the potential \[ u[N] \], if \[ \rho(\tau, x, y) \] is a solution of the (1) with the initial condition \[ \rho_0(0, x, y) \].

The result is used when static solutions of \[ \phi^4 \] model are quantized by means of Riemann function \[ \zeta(s) \] expressed via the Green functions of the
The one-loop quantum correction to action is evaluated directly as
\[ S_q = -\zeta'(0). \]

3 Example of kink

Most popular example of the kink is obtained in this scheme by means of DT over zero seed \( u = 0 \). The solution \( \rho \) of (1) with \( \rho_0 \) as initial condition for this case is a simple heat equation solution
\[ \rho(\tau, x, y) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \rho_0(z, y) \exp\left[-\frac{(x-z)^2}{4\tau}\right] dz. \]

The initial condition \( \rho_0 \) is evaluated by direct integration in (1):
\[ \rho_0(x, y) = \phi_1(x) \begin{cases} \phi_1^{-1}(y), & x > y \\ 0, & x < y \end{cases} \]

The Green function \( \rho[2] \) (density matrix) for the kink solution as the potential is built by the two-fold DT by the Wronskian formula (2) that results in
\[ \rho[2](\tau, x, y) = \exp\left[-\frac{(x-y)^2}{4\sqrt{\tau}}\right] + \frac{1}{2} \sum_{m=1}^{2} \rho_m \psi_m(x) \psi_m(y) \left[ \text{Erf}\left[\frac{(x-y+2b_m\tau)}{2\sqrt{\tau}}\right] - \text{Erf}\left[\frac{(x-y-2b_m\tau)}{2\sqrt{\tau}}\right] \right], \]

where \( b_k = km/\sqrt{2}, \rho_k = ||\psi||^{-2}, k=1,2. \) After multiplication of the Green function by \( \exp[-4m^2\tau] \):
\[ \rho \rightarrow \rho \exp[-4m^2\tau], \]

the first term of the Green function leads to a divergent integral. This divergence is well-known, its origin is a zero vacuum oscillations. In our approach this fact has transparent explanation, because the divergent term is simply a solution of heat equation with constant coefficients, that appear when the self-action of scalar field is neglected. Such divergence is usually compensated by addition of contra terms of normal order.

Our procedure deletes all ultraviolet divergencies of 1+1 \( \phi^4 \) model including energy of zero oscillations and one-meson states if one evaluates the generalized zeta-function by the formula
\[ \zeta_D(s) = M^{2s} \int_0^\infty \gamma(t)t^{s-1}dt / \Gamma(s) \]

\( \Gamma(s) \) is the Euler gamma function and \( M \) is a mass scale. The function \( \gamma(t) \) in the integrand of (3) is expressed via the Green functions \( G(x, y, \tau) \) and \( G_0(x, y, \tau) \) difference. The result coincides with one from [1].
4 Conclusion

As a conclusion let us note that this approach is elaborated in [6] (published in a local conference abstract book) and allows to calculate one-loop corrections to the N-level reflectionless potential and, very similarly, solitons of SG. Some eventual applications are visible in the case studied at [8].

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