The physical origin of the electron-phonon vertex correction

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\section{I. INTRODUCTION}

In conventional metals, according to Migdal’s theorem\textsuperscript{3} the smallness of the parameter $\lambda \omega_0 / E_F$ where $\lambda$ is the electron-phonon coupling, $\omega_0$ and $E_F$ are typical phonon and electron energies respectively, permits to describe successfully the electron-phonon coupled system by neglecting the vertex corrections in the electronic self-energy. The application of Migdal’s theorem to the superconducting state has led to the Migdal-Eliashberg (ME) theory of superconductivity, which accurately describes the conventional superconductors. However, in recent years, the discovery of complex materials showing high-$T_c$ superconductivity, colossal magnetoresistance etc. has raised doubts concerning the validity of Migdal’s theorem for such class of materials. In fact, the fullerene compounds show vibrational spectra ranging from few meV to about 0.2 eV, while the electronic conduction band has a width of approximately 0.5 eV.\textsuperscript{4} In this situation therefore the adiabatic parameter can be as large as $\lambda \omega_0 / E_F \simeq 0.8$. Also for the cuprates the situation points toward the breakdown of Migdal’s theorem. For example, BSCCO compounds have $\omega_0 / E_F \simeq 0.2$ which is a small but not negligible value.\textsuperscript{5} For these systems therefore the electronic and phononic dynamics have comparable energy scales and there is not an \textit{a priori} justification to neglect the vertex corrections.\textsuperscript{6}

Recently, we have proposed a generalization of the ME theory to include the first nonadiabatic effects.\textsuperscript{7} Our aim was to provide a scenario for the high-$T_c$ superconductivity different from the ones arising from purely electronic or magnetic pairing mechanisms. We were motivated by several clues such as the absence of magnetic ordering in doped fullerenes and the presence of large isotope effects in underdoped cuprates.\textsuperscript{8} Moreover, the recent observation of an ion-mass dependence of the penetration length in underdoped YBa$_2$Cu$_3$O$_{6+x}$ and La$_{2-x}$Sr$_x$CuO$_4$\textsuperscript{9,10} has provided a strong evidence for the breakdown of Migdal’s theorem.

A theoretical framework which goes beyond Migdal’s limit can lead to different situations. For example, by enhancing the coupling to the lattice vibrations one could end to polaron and eventually to bi-polaron formation with the appearance of huge effective masses for the charge carriers. Such a regime is beyond Migdal’s limit, however it unlikely gives rise to high temperature superconductivity and the observation of a Fermi surface contradicts the hypothesis of condensation of preformed bi-polarons in its simplest form. Instead our approach is a perturbative one, where the small parameter is $\lambda \omega_0 / E_F$. In this way the electron-phonon coupled system is away from the polaron formation which is definitely a non-perturbative approach.\textsuperscript{11} One advantage of our method is to have an appreciable enhancement of the effective coupling in the Cooper channel and at the same time reasonable values of charge carrier masses. Moreover, the detailed study of the first nonadiabatic corrections to the ME theory has enlightened the importance of the interplay between the exchanged frequency $\omega$ and the momentum transfer $q$ in the electron-phonon scattering process. The first vertex correction is in fact very sensitive to the ratio $v_F q / \omega$ in such a way that as long as $v_F q / \omega < 1$ the vertex assumes positive values which reflects to an enhancement of the critical temperature $T_c$.\textsuperscript{12} This strong $(\omega, q)$-dependence of the nonadiabatic corrections leads also to interesting effects in the pressure coefficient of the superconducting transition temperature\textsuperscript{13} and in the tunneling current in the superconducting state.\textsuperscript{14}
Another striking feature is certainly the different effective el-ph couplings associated to the normal state and the Cooper channel. In fact the nonadiabatic theory of superconductivity introduces two effective couplings, $\lambda_\Delta$ and $\lambda_Z$, which in the adiabatic limit reduce both to $\lambda$, i.e., the electron-phonon coupling of the ME theory. Such a result opens the possibility of having different effective electron-phonon couplings associated to different physical quantities. One example of such a peculiarity is given by the different dependence of the coefficients of the isotope effects on $T_c$ and on the effective charge carriers mass $m^*$ upon the adiabatic parameter $\omega_F/E_F$.

The features listed above are all consequences of the complex behavior of the first vertex corrections (in the normal and superconducting states). Therefore a detailed analysis of the nonadiabaticity in the electron-phonon problem requires a detailed study of the nonadiabatic diagrams. Other attempts in similar directions have been made also since before the discovery of high-$T_c$ materials. For example in Ref. 14 the Eliashberg equations have been modified by taking into account the first vertex corrections in the context of plasmon-mediated superconductivity. However in this case the vertex was considered only in the $\omega/\nu F < 1$ regime which gives negative nonadiabatic contributions and lowers $T_c$. Other authors have studied the vertex corrections using different approximation schemes which apply only in some particular region of the $(\omega, q)$ plane. In Ref. 19 for example, a strong suppression of $T_c$ contrasts with the strong enhancement found in Ref. 20. All these contradictory results come from particular approximations which predilige different limits of the vertex function. To be more specific, in Refs. 14 18 the $\nu F/\omega > 1$ regime was considered while in Ref. 20 the vertex was evaluated in the $\nu F/\omega < 1$ limit. So far, the most accurate results on vertex corrections are given by numerical calculations which do not privilege any particular region of the $(\omega, q)$ plane.

It is certainly true that the value of $\nu F/\omega$ depends on the characteristic of the specific material one wants to investigate, however it is not completely clear the reason of such a complex behavior of the vertex function. In this paper we try clarify this point by investigating the physics behind the vertex function. Concerning the theory of nonadiabatic superconductivity, the interpretation of the nonadiabatic corrections in terms of physical processes will help to identify the characteristic of the materials which can lead to an enhancement of the critical temperature.

In this paper we first summarize the behavior of the vertex function for different values of the ratio $\omega_0/E_F$ and of the electron-density $n$. In section II we focus on the one-electron case and the anti-adiabatic limit $\omega_0 \rightarrow \infty$. For these two cases the interpretation in terms of physical mechanisms turns out to be particularly straightforward. The last section is devoted to a general discussion and conclusions.

II. BEHAVIOR OF THE VERTEX FUNCTION

In this section we consider the electron-phonon vertex correction and its behavior as a function of the adiabatic parameter $\omega_0/E_F$ and the electron density $n$. In the following analysis, we neglect the Coulomb interaction between electrons. Therefore we consider an Hamiltonian describing electrons with dispersion $\epsilon_k$ interacting with phonons via a momentum dependent electron-phonon matrix element $\gamma_q$:

$$H = \sum_{k,\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \omega_0 \sum_q b_q^\dagger b_q + \sum_{k,\sigma} \gamma_q c_{k\sigma}^\dagger c_{k-q\sigma} \left( b_q + b_q^\dagger \right).$$

(1)

Here, $\omega_0$ is the phonon frequency, assumed to be dispersionless for simplicity, and $c_{k\sigma}^\dagger$ ($c_{k\sigma}$) is the creation (annihilation) operator for an electron with wave number vector $k$ and spin $\sigma$ and $b_q^\dagger$ ($b_q$) is the creation (annihilation) operator for phonons with momentum $q$.

The thermal Green’s function for the electron $G(k)$ satisfies the usual Dyson equation:

$$G^{-1}(k) = G_0^{-1}(k) - \Sigma(k),$$

(2)

where we have used the four-momentum representation $k = (k, i\omega_n)$. $\omega_n = (2n + 1)\pi T$ is the fermionic Matsubara frequency and $T$ is the temperature. In Eq.(3), $G_0^{-1}(k) = i\omega_n - \epsilon_k$ and $\Sigma(k)$ is the electron self-energy due to the electron-phonon coupling:

$$\Sigma(k) = \sum_q g_q^2 \Gamma(k + q, k) D(q) G(k + q),$$

(3)

where $g_q^2 = 2\gamma_q^2/\omega_0$ and

$$D(q) = D(i\omega_q) = \frac{\omega_q^2}{(i\omega_q)^2 - \omega_0^2},$$

(4)

$$\omega_0 = (2n + 1)\pi T$$
The phononic propagator with Matsubara frequencies \( \omega_q = 2n_q \pi T \). In writing Eq. (3), we have used the short notation \( \sum_{q} = -(T/N) \sum_{\omega_q} \sum_q \). We consider the phonons as already renormalized and we interpret Eq. (1) in a phenomenological way.

In terms of Feynmann diagrams the self-energy (3) is shown in Fig. 1a, where the solid and wiggled lines are electronic and phononic propagators, respectively. In Fig. 1b, the open circle is the proper vertex function \( \Gamma(k+q,k) \), which is given by all diagrams which cannot be separated into two different contributions by cutting a single electron or phonon propagator line. In Fig. 1b, we show the expansion of the vertex function up to the first correction: \( \Gamma(k+q,k) = 1 + P(k+q,k) \), where the vertex correction \( P(k+q,k) \) is given by the following expression:

\[
P(k+q,k) = \sum_{k'} g_{k-k'}^2 D(k-k')G(k'+q)G(k').
\]  

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\]  

The aim of the present paper is to provide a physical interpretation of the above vertex correction. In this way, we should be able also to interpret in terms of physical processes its complex behavior already pointed out in Refs. 24 and 25 and that we remind here briefly. First, Migdal’s theorem states that the order of magnitude of the vertex correction (3) is \( P(k+q,k) \sim \lambda \omega_0/(v_P q) \), where \( \lambda \sim \langle g_0^2 \rangle /E_F \) is the electron-phonon coupling and \( v_F \) is the Fermi velocity. Since in conventional metals \( \lambda < 1 \) and the momentum transfer \( q \) is of order of the Debye momentum \( q_D \), \( \omega_0/(v_P q_D) \sim \omega_0/E_F \ll 1 \). The vertex correction \( P \) can be therefore safely neglected and \( \Gamma \sim 1 \).

However, when \( \omega_0 \) is of the same order of \( E_F \) or when \( q \ll q_D \) the vertex correction is no longer negligible. This situation can be outlined by approximating the electron propagators \( G \) with their free electron form \( G_0 \). In this way, the sum over Matsubara frequencies in Eq. (3) can be performed exactly and the vertex correction reduces to the following form:

\[
P(k+q,k; i\omega_k + i\omega_q, i\omega_k) = \frac{\omega_0}{2} \sum_{k'} \frac{g_{k-k'}^2}{\epsilon_{k'}-\epsilon_{k'+q}+i\omega_q} \times \left[ \frac{f(\epsilon_{k'}) + n(-\omega_0)}{\epsilon_{k'}+\omega_0-i\omega_k} - \frac{f(\epsilon_{k'+q}) + n(-\omega_0)}{\epsilon_{k'+q}+\omega_0-i(\omega_k+\omega_q)} + \frac{f(\epsilon_{k'+q}) + n(\omega_0)}{\epsilon_{k'+q}-\omega_0-i(\omega_k+\omega_q)} \right].
\]  

In the above equation, \( f \) and \( n \) are the fermionic and bosonic distribution functions, respectively.

By employing various approximations like for example the ones used in Refs. 24 and 25 (constant electron density of states, structureless electron-phonon coupling and small momentum transfer) it is possible to perform analytically the remaining summation in Eq. (3). The result is shown in Fig. 2, where we plot the vertex correction \( P \) at half filling as a function of the exchanged frequency \( \omega_q \) and for different values of the dimensionless momentum transfer \( Q = q/(2k_F) \). For simplicity, the external electron frequency \( \omega_k \) has been set equal to zero. In the figure we notice that \( P \) can assume positive and negative values depending on the ratio \( v_P q/\omega_q \); for \( v_P q/\omega_q > 1 \) the vertex is positive while \( v_P q/\omega_q < 1 \) the vertex becomes negative. This complex behavior is found also for more realistic band models. and it is also reflected in the different values the vertex assumes in the dynamic and static limits. In fact, in the same approximation scheme used in the calculations reported in Fig. 2, the static limit \( P_s = \lim_{\omega_q \to 0} P(q=0,\omega_q) \) is negative: \( P_s = -\omega_0/(\omega_0 + E_F) \), while the dynamic one \( P_d = \lim_{q \to 0} P(q,\omega_q = 0) \) is instead positive: \( P_d = E_F/(\omega_0 + E_F) \). The vertex correction is therefore non-analytic in \( \omega_q = 0, q = 0 \).

However this non-analyticity is removed when we consider the case of only one electron coupled to the lattice. In this situation the electron distribution functions in (3) are strictly zero and the vertex correction reduces to:

\[
P(k+q,k; i\omega_k + i\omega_q, i\omega_k) = \frac{\omega_0}{2} \sum_{k'} \frac{g_{k-k'}^2}{\epsilon_{k'}-\lambda\epsilon_{k'}-i\omega_k-i\omega_q} \times \left[ \frac{1 + n(\omega_0)}{\epsilon_{k'} - \lambda(\epsilon_{k'}+\omega_0 - i\omega_k - i\omega_q)} + \frac{1 + n(\omega_0)}{\epsilon_{k'} + \omega_0 - i\omega_k - i\omega_q} \right],
\]  

and it is straightforward to realize that the dynamic and static limits of Eq. (7) are in fact equal.

Therefore \( P(k+q,k) \) has a strong dependence on the exchanged frequency and momentum, and in particular is non-analytic in \( (\omega_q = 0, q = 0) \), only when the electron density is finite. In fact, like in other response functions, the non-analyticity of the vertex correction is given by hole-particle excitations which are present in Eq. (3). These hole-particle processes can be explicitly singled out by rearranging Eq. (3) in the following way:

\[
P(k+q,k; i\omega_k + i\omega_q, i\omega_k) = P_{pol}(k+q,k; i\omega_k + i\omega_q, i\omega_k) + P_{ex}(k+q,k; i\omega_k + i\omega_q, i\omega_k) \]
basically a single electron process. The second term of Eq. (8),

\[ P \]

is part of the effective potential, so that we can interpret the vertex correction in terms of the physically mediated electron-electron interaction.

When the electron density is finite, a as it is clear from the expression in Eq. (8), the particle-hole excitations vanish linearly with the momentum transfer \( q \). On the other hand, when \( \omega \) is nonzero, the particle-hole excitations vanish linearly with the momentum transfer \( q \) when \( q \to 0 \). In the following, we show that the physical process which leads to particle-hole excitations in \( P_{\text{ex}} \) is given by the exchange part of the phonon mediated electron-electron interaction.

From the above discussion, we have seen that the two contributions \( P_{\text{pol}} \) and \( P_{\text{ex}} \) of the vertex function \( P \) have different behaviors. Therefore the problem of finding the physical interpretation of the vertex \( P \) can be simplified by looking separately at \( P_{\text{pol}} \) and \( P_{\text{ex}} \). This can be done by considering particular limiting cases. For example, in the limit of one electron in the system, the particle-hole contributions vanish and the second term of Eq. (8), \( P_{\text{ex}} \), is zero. Therefore the vertex is determined entirely by \( P_{\text{pol}} \), and we are left only with \( P_{\text{pol}} \) in the form of Eq. (8). On the other hand, as it is clear from Eq. (8), when we employ \( \omega_0 = \infty \) limit (non-retarded phonon propagator) for a finite electron density, the polarization part \( P_{\text{pol}} \) vanishes and the vertex is determined purely by \( \lim_{\omega_0 \to \infty} P_{\text{ex}} \).

The physical interpretation of the one electron and \( \omega_0 = \infty \) limits becomes straightforward if we introduce an external potential \( U_{\text{ext}} \) which couples to the electron density. In fact, when the coupling to the lattice is absent, this potential modifies the electron distribution in a way which depends on the explicit form of \( U_{\text{ext}} \). However, when the electrons interact with the phonons, the response of the electrons to the external potential changes because of the electron-phonon coupling. This situation can be described in terms of an effective potential \( U_{\text{eff}} \). Actually, the vertex function is part of the effective potential, so that we can interpret the vertex correction in terms of the physically more intuitive \( U_{\text{eff}} \).

In the next section therefore we study the response of the electrons to the external potential \( U_{\text{ext}} \) for both the one electron case and the \( \omega_0 = \infty \) limit.

### III. Limiting Cases

#### A. One-electron case: \( P_{\text{pol}} \)

First we consider the case of only one electron in the system. In this situation, the particle-hole contributions of the vertex function vanish and the second term of Eq. (8), \( P_{\text{ex}} \), is zero. Therefore the vertex is given by the one electron limit of \( P_{\text{pol}} \), Eq. (8), which at zero temperature reduces to:

\[ P_{\text{pol}}(k + q, k; i\omega_k + i\omega_q, i\omega_k) = \frac{\omega_0}{2} \sum_{k'} \left( \frac{\delta_{k-k'}^2}{(\epsilon_{k'} + \omega_0 - i\omega_k)(\epsilon_{k'} + q + \omega_0 - i\omega_k - i\omega_q)} \right). \]

As already pointed out before, the dynamic and static limits of \( P_{\text{pol}} \) coincide and, by employing the same approximation scheme of Refs. 6, 7, the \( q = 0 \), \( \omega_q = 0 \) limit becomes:
where we have neglected the external electron frequency $\omega_k$ and $E$ is the electronic bandwidth.

Our aim is to find the physical origin of Eq.\((11)\) and to explain the reason why the limit in Eq.\((11)\) is positive. To this end, we approach the problem by reasoning in terms of the electron response to an external potential $U_{\text{ext}}$.\(\square\)

Let us for the moment neglect the electron-phonon interaction. For simplicity, we assume also that, in the absence of the external perturbation $U_{\text{ext}}$, the electron wavefunction for the state $k$ and energy $\epsilon_k$ is well described by a simple plane-wave $\psi^0_k(r) = \exp(\imath k \cdot r)/\sqrt{V}$, where $V$ is the volume.

A nonzero external perturbation $U_{\text{ext}}$ modifies the electronic wavefunction which, to the first order of the time-independent perturbation theory, is given by:

$$\psi_k(r) = \psi^0_k(r) + \sum_q \frac{U_{\text{ext}}(q)}{\epsilon_k - \epsilon_{k+q}} \psi^0_{k+q}(r),$$

where $U_{\text{ext}}(q) = \langle \psi^0_{k+q} | U_{\text{ext}}(r) | \psi^0_k \rangle$. If we consider $U_{\text{ext}}(r)$ to be given by a potential well of the form:

$$U_{\text{ext}}(r) = \begin{cases} -U_0 & r \leq R \\ 0 & r > R \end{cases}$$

then the density of probability $|\psi_k(r)|^2$ of finding a slow electron at position $r$ is enhanced inside the potential well and lowered outside the region $r \leq R$. This is shown in Fig.\((3)\) where we plot $|\psi_k(r)|^2$ for $k = 0$ by a dashed line. In the lower panel of the same figure we also plot the potential well $U_{\text{ext}}(r)$ of Eq.\((13)\) (dashed line).

Now we study how the above picture is modified when the electron is weakly coupled to the lattice vibrations. Intuitively, we expect that the lattice is more polarized where the probability of finding the electron is large, that is, inside the potential well (for $k$ small). The lattice polarization, in turns, provides an attractive potential which is added to the external one. We can study this situation by replacing $\psi^0_k(r)$ in Eq.\((12)\) by the wavefunction modified by the coupling to the lattice vibrations:

$$\phi^0_k(r) = \psi^0_k(r) + \sqrt{\frac{\omega_0}{2}} \sum_{q'} \frac{g_{k-k'}}{\epsilon_k - \epsilon_{k'} - \omega_0} \hat{b}^\dagger_{k'-k} \psi^0_k(r).$$

In this way, the new electron wavefunction in the presence of the external potential reduces to

$$\phi_k(r) = \phi^0_k(r) + \sum_q \frac{U_{\text{eff}}(k + q, k)}{\epsilon(k) - \epsilon(k + q)} \phi^0_{k+q}(r),$$

where $\epsilon(k)$ is the electron dispersion modified by the electron-phonon coupling and $U_{\text{eff}}(k + q, k)$ is the effective external potential which results from the lattice polarization. From the above discussion, the lattice polarization should in general amplify the potential seen by the electron. This is confirmed by the numerical results reported in Fig.\((4)\), where in the lower panel we plot the Fourier transform of the effective potential \((16)\) for $k = 0$ (solid line). Moreover, the enhanced potential leads to an enhanced probability of finding the electron in the vicinity of the potential well (solid line in the upper panel of Fig.\((4)\)).

At this point it is straightforward to understand why the $q \to 0$ limit of the vertex function for one electron is positive (see Eq.\((11)\)). In fact, for $k = 0$, $\lim_{q \to 0} P_{\text{pol}}(q) = \langle P_{\text{pol}}(r) \rangle$, where $\langle \cdots \rangle$ means the average over the volume of the system, and

$$\langle P_{\text{pol}}(r) \rangle = \frac{\langle U_{\text{eff}}(r) \rangle - \langle U_{\text{ext}}(r) \rangle}{\langle U_{\text{ext}}(r) \rangle} > 0.$$  

Of course the treatment of the problem followed in this section does not consider the effect of the other electrons when the electron density $n$ is finite. This effect is partially contained in the general expression of $P_{\text{pol}}$ given by Eq.\((12)\).
which in fact can be interpreted along the same lines followed in this section by taking into account the fermionic statistics of the electrons. However, $P_{\text{pol}}$ is basically a single electron process and it belongs to the class of processes for which the same electron can generate a phonon at a certain time $t$ and then absorb it at a later time $t'$. Such kind of processes are consequence of the retarded phonon propagation. On the other hand, for many electrons systems, a different kind of processes are those for which an electron generates a phonon which is absorbed by a different electron. This is a many-electrons process which is not contained in $P_{\text{pol}}$ but it is given by $P_{\text{ex}}$, which is in fact determined by particle-hole excitations.

In the next section we provide an interpretation for the physical origin of $P_{\text{ex}}$. As anticipated before, we will introduce an external potential and we will consider the resulting effective potential by employing the $\omega_0 \rightarrow \infty$ limit for which $P_{\text{pol}}$ vanishes and the interpretation of $P_{\text{ex}}$ is particularly simple.

### B. Anti-adiabatic limit: $P_{\text{ex}}$

Let us consider the electron-phonon coupled system in the limit $\omega_0 \rightarrow \infty$. In order to have non-trivial results, we perform this limit in such a way that the quantity $g_\text{pol}^2$ remains finite. From Eq.\((\ref{eq:8})\), $\lim_{\omega_0 \rightarrow \infty} P_{\text{pol}} = 0$ and the vertex function reduces to:

$$
\text{lim}_{\omega_0 \rightarrow \infty} P(k + q, k) = \text{lim}_{\omega_0 \rightarrow \infty} P_{\text{ex}}(k + q, k) = \sum_{k'} g_{k-k'}^2 \frac{f(\epsilon_{k'}) - f(\epsilon_{k'+q})}{\epsilon_{k'} - \epsilon_{k'+q} + i\omega_q}.
$$

As we have seen before, this term has different dynamic and static limits, and in particular we have:

$$
\text{lim}_{\omega_0 \rightarrow 0, q \rightarrow 0} P_{\text{ex}}(k + q, k) = 0,
$$

$$
\text{lim}_{q \rightarrow 0, \omega_0 \rightarrow 0} P_{\text{ex}}(k + q, k) = \sum_{k'} g_{k-k'}^2 \frac{df(\epsilon_{k'})}{d\epsilon_{k'}} \simeq -\lambda,
$$

where the last equality holds true at small temperatures and $\lambda$ is the electron-phonon coupling constant. Although we can interpret the zero value of the dynamic limit as due to the vanishing contribution of the hole-particle excitation contribution, the reason why the static limit is negative remains unclear. In this section we try to clarify this point by considering the problem of the electron response to an external potential which couples to the electron density.

The anti-adiabatic limit $\omega_0 \rightarrow \infty$ transforms the electron-phonon interaction into an effective non-retarded electron-electron interaction. The Hamiltonian can be obtained by integrating out the phononic degrees of freedom and then taking the $\omega_0 \rightarrow \infty$ limit. The result is:

$$
H = \sum_{k,\sigma} \epsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + \sum_{q} U_{\text{ext}}(q)n(q) - \frac{1}{2} \sum_{q} g_q^2 n(q)n(-q),
$$

where $n(q)$ is the electron density operator:

$$
n(q) = \sum_{k,\sigma} c_{k+q,\sigma}^\dagger c_{k,\sigma}.
$$

Now let us consider the response of the system to the external potential $U_{\text{ext}}$. Since the electrons are interacting through $g_q^2$, the response depends in general on the whole electron configuration. The simplest way in order to deal with this problem is to perform the Hartree-Fock approximation in the four-operator term in Eq.\((\ref{eq:21})\). This approach is equivalent to consider anti-symmetrized states of $N$ independent one-electron wavefunctions and leads to the random phase approximation for the effective potential seen by the electrons. When we apply the Hartree-Fock approximation, the interaction term in Eq.\((\ref{eq:21})\) becomes:

$$
-\frac{1}{2} \sum_{q} g_q^2 n(-q) + \sum_{q} g_q^2 n(q) - \sum_{q} \sum_{k,\sigma} g_q^2 (c_{k+q,\sigma}^\dagger c_{k,\sigma}) (c_{k-q,\sigma}^\dagger c_{k,\sigma}),
$$

and, after a manipulation of the momenta indexes, the Hamiltonian \((\ref{eq:21})\) can be rewritten as follows:

$$
H = \sum_{k,\sigma} \epsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + \sum_{q} U_{\text{ext}}(q)n(q) + \sum_{q} V_{\text{HF}}(k+q, q)c_{k+q,\sigma}^\dagger c_{k,\sigma},
$$
where $V_{\text{HF}}$ is the Hartree-Fock potential in the presence of the external perturbation:

$$V_{\text{HF}}(k + q, k) = -g_q^2 \langle n(-q) \rangle + \sum_{k'} g_{k-k'}^2 \langle c_{k',\sigma}^\dagger c_{k+q,\sigma} \rangle. \tag{25}$$

The first term in the right hand side is the Hartree contribution which results from the potential generated by the electrons regardless the specific electronic configuration. The last term of Eq. (23) is instead the Fock contribution which treats the electrons as being dressed by the exchange hole and it comes from the last term of Eq. (24). Below we show that the vertex function $P_{\text{ex}}$ originates from the Fock contribution of Eq. (23), i.e., from the exchange term of the phonon mediated electron-electron interaction.

The effective potential given by the redistribution of the electrons is readily obtained by the linear response theory. In fact, for small values of $U_{\text{ext}}$, the expectation values appearing in Eq. (25) can be rewritten as:

$$\langle n(-q) \rangle = \langle n(-q) \rangle_0 + \delta \langle n(-q) \rangle \tag{26}$$

$$\langle c_{k',\sigma}^\dagger c_{k+q,\sigma} \rangle = \langle c_{k',\sigma}^\dagger c_{k+q,\sigma} \rangle_0 + \delta \langle c_{k',\sigma}^\dagger c_{k+q,\sigma} \rangle \tag{27}$$

where the first and the last terms in the right hand sides are the expectation values in the absence and in the presence of the external potential, respectively. From Eqs. (25-27), $V_{\text{HF}}$ can be rewritten as $V_{\text{HF}}^0 + \delta V_{\text{HF}}$ and the hamiltonian (24) becomes:

$$H = H_{\text{HF}}^0 + \sum_{q,k,\sigma} U_{\text{eff}}(k + q, k)c_{k+q,\sigma}^\dagger c_{k,\sigma}, \tag{28}$$

where $H_{\text{HF}}^0$ is the hamiltonian in the Hartree-Fock approximation for $U_{\text{ext}} = 0$ and

$$U_{\text{eff}}(k + q, k) = U_{\text{ext}}(q) + \delta V_{\text{HF}}(k + q, k). \tag{29}$$

The above expression is a self-consistent equation because $\delta V_{\text{HF}}$ depends implicitly on $U_{\text{eff}}$. We can repeat the calculations by assuming a time-dependent external potential. In this case, by employing the linear response theory applied to the Hamiltonian (28), the self-consistent equation for $U_{\text{eff}}$ is given by:

$$U_{\text{eff}}(k + q, k, i\omega_q) = U_{\text{ext}}(q, i\omega_q) - 2g_q^2 \sum_{k'} f(\epsilon_{k'}) - f(\epsilon_{k'+q} + i\omega_q) U_{\text{eff}}(k' + q, k', i\omega_q)$$

$$+ \sum_{k'} g_{k-k'}^2 \frac{f(\epsilon_{k'}) - f(\epsilon_{k'+q} + i\omega_q)}{\epsilon_{k'} - \epsilon_{k'+q} + i\omega_q} U_{\text{eff}}(k' + q, k', i\omega_q), \tag{30}$$

where $\omega_q$ is the Matsubara frequency provided by the time-dependence of the external potential and $\epsilon_k$ is now the electron dispersion in the Hartree-Fock approximation.

Equation (30) is just the random phase approximation applied to the electron-phonon coupled system in the anti-adiabatic limit $\omega_0 \to \infty$. The second term of Eq. (34) represents the electron response to $U_{\text{ext}}$ governed by the Hartree potential while the last term is the correction due to the exchange potential. To the first order in $U_{\text{ext}}$, the last term of Eq. (31) can be rewritten as $P_{\text{ex}} U_{\text{ext}}$ where $P_{\text{ex}}$ is the vertex correction in the anti-adiabatic limit [13]. Therefore $P_{\text{ex}}$ is just the Fock contribution to the electronic response to the external potential when $\omega_0 \to \infty$.

We show in Fig. 3 the diagramatic represenatation of the self-consistent equation (31). The Hartree term is represented by the set of bubble diagrams while the Fock contribution is given by the ladder contribution which, to the first order in $U_{\text{ext}}$, is the vertex diagram.

At this point we can explain the behavior of the vertex function in the anti-adiabatic limit already outlined in Eqs. (19-20). In fact, the negative static limit of equation (31) can be understood by the following reasoning. If we consider a static potential, then we must set $\omega_q = 0$ in Eq. (30). For simplicity let us also consider an attractive potential like for example the one in Eq. (13). If we neglect the Fock contribution, the electrons will tend to form a cloud around the potential well and an added electron will experience an effective potential given by the bare well plus the electron cloud. Since the phonon mediated el-el interaction is attractive such an effective potential will be stronger than the bare one. However, when we consider the effect of Pauli principle, the electron becomes dressed by the exchange hole which repels the other electrons. Therefore the net effect of the Pauli principle is to weaken the electron-electron attraction and this weakening is reflected in the negative sign of the exchange term. In conclusion, in the anti-adiabatic limit, the negative sign of the vertex function for $\omega_q = 0$, Eq. (24), is exclusively due to the exchange effect which lower the phonon mediated electron-electron interaction.

This picture is still valid when we take into account the retardation of the phonon-mediated electron-electron interaction ($\omega_0 < \infty$). In fact the last term of Eq. (31) can be interpreted as the Fock-like contribution for a retarded potential.
IV. DISCUSSION AND CONCLUSIONS

From the above analysis, we have seen that the vertex function results from electron-phonon processes of different origins. In fact, we have shown that the vertex $P$ can be decomposed into two different contributions: $P_{\text{pol}}$ and $P_{\text{ex}}$. The first one is the result of the lattice polarization as induced by the electron motion. At low frequencies, this term leads to a positive contribution to the electron-phonon interaction and tends to enhance the pairing. Moreover, $P_{\text{pol}}$ is basically a single electron process. The second term, $P_{\text{ex}}$, is instead due to the exchange effect of the phonon mediated electron-electron interaction. Since $P_{\text{ex}}$ is basically an exchange, it tends to reduce the electron-phonon effective pairing. The presence of $P_{\text{ex}}$ is equivalent to have a lattice polarization effect due to an electron which is moving with its exchange hole. Moreover, since the exchange term gives rise to particle-hole excitations, $P_{\text{ex}}$ is sensitive to the momentum transfer and the exchanged frequency of the electron-phonon scattering process and it gives rise to the different values of the dynamic and static limits of the vertex function. In particular, in the dynamic limit the hole-particle excitation vanish and the vertex is given only by its polarization part $P_{\text{pol}}$, while in the static limit the particle-hole excitations give rise to the exchange part which lower the effective electron-phonon scattering.

From the above results and discussion, it is therefore straightforward to identify scenarios in which the vertex correction can give rise to an enhancement of the effective electron-phonon coupling. In fact, an enhancement can be automatically obtained if the exchange effects become less important. This can be achieved, for example, when the charge carrier density is low, so that the average distance between electrons can exceed the size of the exchange hole leading to a negligible $P_{\text{ex}}$. In this way we can understand the results reported in Ref. 17 where it is shown that, by reducing the charge carrier density, the vertex correction enhances the effective coupling.

A more interesting situation in favour of an enhancement of the vertex corrected electron-phonon coupling is given by considering only small $q$ scattering in the electron-phonon interaction. In fact, for a small enough momentum transfer, say $v_F |q| \ll \omega_0$, the particle-hole contributions have little weight and the negative exchange term $P_{\text{ex}}$ becomes negligible. This result therefore clarifies on physical grounds why the effective nonadiabatic electron-phonon interaction and so the superconducting critical temperature $T_c$ is enhanced by the vertex corrections when the electron-phonon interaction is only via small momentum transfer.

It is important to stress that several works have shown how strong electronic correlations can lead to an electron-phonon interaction peaked at small momentum transfer. Therefore strong electronic correlations can have a positive role in enhancing the nonadiabatic effective electron-phonon coupling by introducing a small upper cut-off on the momentum transfer. However, in the light of the results presented in this work, we can more generally say that strong electronic correlations tend to suppress the particle-hole excitations consequently reducing the negative contribution of the exchange term $P_{\text{ex}}$ in the vertex.

In conclusion, the two situations listed above, electron-phonon interaction with small momentum transfer (or more generally the suppression of the particle-hole excitations due to strong electronic correlations) and small carrier concentration, are both present in high-$T_c$ materials suggesting that the vertex corrections can enhance the pairing and give rise to high-$T_c$ superconductivity.

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Since in the one-electron case the static and dynamic limits of the vertex function are equal, we can consider a static potential \( \omega_0 = 0 \) and no ambiguity is found in the \( q = 0 \) limit.

FIG. 1. (a): electron self-energy. The open circle represents the set \( \Gamma \) of all irreducible vertex diagrams. (b): expansion of \( \Gamma \). The first diagram represents the bare electron-phonon interaction \( g(q) \) while the second one is the first vertex correction \( g(q)P(k + q, k) \) which in the adiabatic limit gives a negligible contribution to \( \Sigma \).
FIG. 2. Vertex function $P$ as a function of the exchanged frequency and for different values of the dimensionless momentum transfer $Q = q/(2k_F)$.

FIG. 3. Representation of the particle-hole phase space. The shaded area is the Fermi see, the open circle an hole and the filled circle an electron. For zero exchanged frequencies, particle-hole excitations can be obtained by connecting the hole with the electron by the momentum transfer $q$ such that $q \cdot k' = 0$. 
FIG. 4. Upper panel: density of probability of one electron in the presence of an external potential (plotted in the lower panel). Solid (dashed) lines: case with (without) electron-phonon interaction. The potentials are plotted in units of $U_0$ and the densities of probability are properly normalized. In order to make clear the effect of the electron-phonon coupling we have used suitable values of the parameters entering in Eqs. (15, 16).

FIG. 5. Diagramatic representation of the self-consistent equation (30). The dashed circles represent the effective potential $U_{\text{eff}}$. The second diagram comes from the Hartree interaction while the last one from the Fock term. The wiggled lines represent the phonon mediated electron-electron interaction in the antiadiabatic limit.