BALANCEDNESS OF SOCIAL CHOICE CORRESPONDENCES

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Abstract. A social choice correspondence satisfies balancedness if, for every pair of alternatives, $x$ and $y$, and every pair of individuals, $i$ and $j$, whenever a profile has $x$ adjacent to but just above $y$ for individual $i$ while individual $j$ has $y$ adjacent to but just above $x$, then only switching $x$ and $y$ in the orderings for both of those two individuals leaves the choice set unchanged. We show how the balancedness condition interacts with other social choice properties, especially tops-only. We also use balancedness to characterize the Borda rule (for a fixed number of voters) within the class of scoring rules.

1. Introduction

We consider the interaction of several social choice properties with a new condition, balancedness. A social choice correspondence satisfies balancedness if, for every pair of alternatives, $x$ and $y$, and every pair of individuals, $i$ and $j$, whenever a profile has $x$ adjacent to but just above $y$ for individual $i$ while individual $j$ has $y$ adjacent to but just above $x$, then only switching $x$ and $y$ in the orderings for both of those two individuals leaves the choice set unchanged.

Social choice theory often considers responsiveness conditions, like monotonicity, but balancedness is a non-responsiveness property. It is a natural equity condition that simultaneously incorporates some equal treatment for individuals short of anonymity, some equal treatment of alternatives short of neutrality, and some equal treatment for differences of position of alternatives in orderings (for example, raising $x$ just above $y$ in the bottom two ranks for individual $j$ exactly offsets lowering $x$ just below $y$ in the top two ranks for $i$).

Let $X$ with cardinality $|X| = m \geq 2$ be the set of alternatives and let $N = \{1, 2, ..., n\}$ with $n \geq 2$ be the set of individuals. A (strong) ordering on $X$ is a complete, asymmetric, transitive relation on $X$ (non-trivial individual indifference is disallowed). The highest ranked element of an ordering $r$ is denoted $r[1]$, the second highest is denoted $r[2]$, etc. And $r[1 : k]$ is the unordered set of alternatives in the top $k$ ranks of $r$. The set of all orderings on $X$ is $L(X)$. A profile $u$ is an element $(u(1), u(2), ..., u(n))$ of the Cartesian product $L(X)^N$.

A social choice correspondence $G$ is a map from the domain $L(X)^N$ to non-empty subsets of $X$. The range of social choice correspondence $G$ is the collection of all sets $S$ such that there exists a profile $u$ with $G(u) = S$.

$G$ satisfies tops-only if for all profiles $u$, $v$, whenever $u(i)[1] = v(i)[1]$ for all $i$, then $G(u) = G(v)$.

Date: March 9, 2018.

Key words and phrases. Social choice correspondence, balancedness, tops-only, unanimity, monotonicity, scoring rules, Borda.
We say profile $v$ is **constructed from profile** $u$ by transposition pair $(x, y)$ **via individuals** $i$ and $j$ if at $u$, $x$ is immediately above $y$ for $i$, and $y$ is immediately above $x$ for $j$, and profile $v$ is just the same as $u$ except that for $i$ and $j$, alternatives $x$ and $y$ are transposed. A social choice correspondence $G$ will be called **balanced** if, for all $x$, $y$, $u$, $v$, $i$, and $j$, whenever profile $v$ is constructed from $u$ by transposition pair $(x, y)$ via individuals $i$ and $j$, then $G(v) = G(u)$. (Otherwise, $G$ will be called **unbalanced**.)

If $m = 2$, balancedness is equivalent to anonymity. For $m > 2$, balancedness holds for the Pareto correspondence, the Borda rule, the Copeland rule, and top-cycle (which selects the maximal set of the transitive closure of simple majority voting). (Thus balancedness is consistent with anonymity, neutrality, and the Pareto condition.) Those rules all fail tops-only. Balancedness fails for the tops-only correspondences that are dictatorship, plurality, and union-of-the-tops. (Thus unbalancedness is also consistent with unanimity, anonymity, neutrality, and the Pareto condition.) In fact, balancedness fails for almost all tops-only correspondences as Section 1 will show. Balancedness also fails for the maximin rule.

To see one way balancedness reflects equal treatment of individuals, first observe again that any dictatorial correspondence (where for some $i$ and all $u$, $G(u) = u(i)[1]$) is unbalanced. This can be extended to cover all correspondences with ineffective individuals. An individual $i$ is **ineffective** for a correspondence $G$ if for all profiles $u$, $u^*$, we have $G(u) = G(u^*)$ whenever $u(j) = u^*(j)$ for all $j \neq i$. For example, if $G$ is dictatorial with dictator $j$, then every other individual is ineffective. We show that balancedness implies every individual is effective.

**Theorem 1.** Any non-constant social choice correspondence with an ineffective individual is unbalanced.

**Proof:** Let $i$ be an ineffective individual for non-constant correspondence $G$. Let $u$, $u^*$ be two profiles with $G(u) \neq G(u^*)$. Since $i$ is ineffective, we may assume that $u(i) = u^*(i)$. Now construct a sequence of profiles

$$u = u_1, u_2, ..., u_{T-1}, u_T = u^*$$

from $u$ to $u^*$ such that for any two successive profiles $u_{t-1}$, $u_t$ in the sequence, $u_t$ differs from $u_{t-1}$ only by a transposition of two adjacent alternatives in the ordering of a single individual other than $i$. For some $t$, it must be that $G(u_t) \neq G(u_{t-1})$. Suppose that $u_t$ differs from $u_{t-1}$ because $x$ is transposed with $y$, ranked just below $x$ in $u_{t-1}(j)$.

Construct profile $u_{t-1}'$ from $u_{t-1}$ by moving $y$ adjacent to and just above $x$ in $u_{t-1}(i)$, and construct profile $u_t'$ from $u_t$ by moving $x$ adjacent to and just above $y$ in $u_{t-1}(i)$. Then, by the ineffectiveness of $i$, $G(u_t') = G(u_t)$ and $G(u_{t-1}') = G(u_{t-1})$, so $G(u_t') \neq G(u_{t-1}')$. But $u_t'$ differs from $u_{t-1}'$ by transposition pair $(x, y)$ via individuals $i$ and $j$ and so this constitutes a violation of balancedness. □

In the first half of this paper, we show how the balancedness condition interacts with other social choice properties, especially tops-only. In the second half, we

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1. For definitions and discussions of these social choice rules and the properties mentioned in this paragraph, see Arrow, Sen, and Suzumura (2002) and Heckelman and Miller (2015).
show how balancedness can be used to characterize the Borda rule (for a fixed number of voters) within the class of scoring rules.

2. Tops-only

Balancedness incorporates some equal treatment of positions, so we might expect conflict with the tops-only property. By tops-only, all profiles \( u \) with an alternative \( x \) at everyone’s top must give the same value for \( G(u) \), though it need not be the case that \( G(u) = \{x\} \). Do there exist any social choice correspondences satisfying tops-only and unanimity that are balanced? A positive answer to that question is given by:

**Example 1.** Let \( G(u) \) be the common top at all unanimous profiles and set \( G(u) = X \) (or any other fixed set) on all non-unanimous profiles.

That example illustrates the following result.

**Theorem 2.** For \( m \geq 3, n \geq 3 \), if social choice correspondence \( G \) satisfies tops-only and balancedness, then it is constant on all non-unanimous profiles.

**Proof:** Let \( a \) and \( b \) be two fixed elements of \( X \) and let \( u^* \) be a profile with tops \( abb...bb \). Given any profile \( u \) without a unanimous top, we will show there is a sequence of profiles \( u_1, u_2, ..., u_T \) all without a unanimous top, such that:

1. \( u_1 = u; \)
2. \( u_T = u^*; \)
3. For all \( t \) with \( 1 \leq t < T \), \( u_{t+1} \) can be constructed from \( u_t \) by either a reordering of alternatives below someone’s top or by a paired transposition.

Then \( G(u) = G(u^*) \) for all \( u \) without a unanimous top.

In the following, let \( T(u) \) be the set of all alternatives \( x \) such that \( u(i)[1] = x \) for some \( i \).

**Case 1.** \( T(u) = \{a, b\} \). If \( u(1)[1] = a \), go to the next paragraph. Suppose that \( u(1)[1] = b \). For some \( i > 1 \) with \( a \) in the top rank, construct \( u_2 \) by raising \( a \) to 1’s second rank and \( b \) to \( i \)’s second rank. Then construct \( u_3 \) by transposition pair \((a, b)\) via individuals 1 and \( i \). Now \( u_3(1)[1] = a \) and \( u_3(i)[1] = b \).

If all other tops are now \( b \), we are done. So suppose that some other top, say for individual \( j \), is \( a \). Construct \( u_4 \) by raising a third alternative \( c \) to the second rank for \( j \) and raising \( c \) just above \( a \) for \( i \). Then construct \( u_5 \) by transposition pair \((a, c)\) via \( i \) and \( j \). Now \( c \) is at \( j \)’s top. Recall that at \( u_5 \), we have \( u_5(1)[1] = a \), so we can move \( b \) to the second rank for individual 1, \( c \) to the third rank for individual 1, and for individual \( j \), we move \( b \) to the second rank. Construct the next profile in the sequence by transposition pair \((b, c)\) via 1 and \( j \). Then \( b \) now becomes \( j \)’s top; the \( a \) has been changed to a \( b \). Repeat this until all “a”s have been changed first to “c”s and then to “b”s.

**Case 2.** Suppose \( T(u) \) contains \( a, b \), and other alternatives. If \( c \) is a top for someone, say \( i \), construct \( u_2 \) by raising \( b \) to the second rank for \( i \) and raise \( b \) just above \( c \) for an individual \( j \) who has \( a \) on top. Then a transposition pair \((b, c)\) via \( i \) and \( j \) reduces by one the number of individuals without \( a \) or \( b \) on top. Continue in this fashion until you reach Case 1.
Case 3. \( T(u) \) contains one of \( a \) and \( b \) but not the other. Suppose that \( u(i)[1] = a \) and \( u(j)[1] = c \). Construct \( u_2 \) by raising \( b \) to second rank for \( j \) and raise \( b \) just above \( c \) for \( i \). Then a transposition of \( b \) and \( c \) for \( i \) and \( j \) creates a profile in either Case 2 or Case 1. A similar analysis holds if it is \( b \) instead of \( a \) at someone's top.

Case 4. \( T(u) \) contains neither of \( a \) or \( b \) but does contain say \( c \) and \( d \). Construct profile \( u_2 \) by raising \( a \) to second rank for some \( i \) with \( c \) on top and \( a \) just above \( c \) for some \( j \) with \( d \) on top. A transposition pair \((a, c)\) via \( i \) and \( j \) yields a profile in Case 3.

A modified version of this argument works also for \( n = 2 \).

But there is a limit to this style of argument; there are not sequences to \( u^* \) from profiles with unanimous tops as balancedness cannot be applied there. This limit on the argument can not be overcome, as seen by the example at the beginning of this section; there, at profiles with a common top, different outcomes may occur.

Corollary. For \( m \geq 3 \), if social choice correspondence \( G \) satisfies tops-only and Pareto, then \( G \) is unbalanced.

Both tops-only and Pareto are needed in the Corollary. If Pareto is not assumed, a constant correspondence satisfies both tops-only and balancedness. If tops-only is not assumed, the correspondence that selects the set of Pareto optimal alternatives satisfies Pareto and balancedness.

3. Top-2

The following social choice correspondence violates tops-only, but outcomes do depend only on which alternatives are ranked first or second by individuals (but not on how the top alternatives are ordered within those two ranks).

Example 2. (Rigid) 2-approval voting is a social choice correspondence \( G \) that is like plurality rule except that instead of selecting the alternatives with the most frequently occurring tops, it selects the alternatives with the most frequent occurrences in the top two ranks for everyone. Like plurality, it is unbalanced.

We now extend Theorem 1 to cover social choice correspondences that, like 2-approval voting, depend only on the top two ranks for every individual. We first need to extend tops-only. \( u(i)[1 : 2] \) is the (unordered) set of alternatives in the top two ranks for individual \( i \) at profile \( u \). Then we say social choice correspondence \( G \) satisfies \textbf{top-2-only} if for all profiles \( u, u^* \), \( G(u) = G(u^*) \) if \( u(i)[1 : 2] = u^*(i)[1 : 2] \) for all individuals \( i \).

Let \( D \) be the subdomain of \( L(X)^N \) consisting of all profiles for which it is \textbf{not} true that \( u(i)[1 : 2] = u(j)[1 : 2] \) for all individuals \( i \) and \( j \), that is, at least three alternatives occur in the top two ranks over all individuals. \( D \) is the analog here of the subdomain of non-unanimous profiles in Section 1.

Theorem 3. Let \( n \geq 3 \) and \( m \geq 4 \). Then any social choice correspondence \( G \) satisfying balancedness and top-2-only must be constant on \( D \).

\footnote{"Rigid" because the number of ranks (here 2) is fixed to be the same for all individuals. For approval voting without rigidity, see Brams and Fishburn (1983). Rigid k-approval voting appeared in Alemante, Campbell, and Kelly (2015) where it was called approval voting (type-k).}
**Proof:** Consider a specific profile $u^*$ with $c$ in everyone’s top rank, $a$ in #1’s second rank, and $b$ in everyone else’s second rank. It will suffice to show that for every profile $u$ in $D$, there is a sequence of profiles in $D$ from $u$ to $u^*$ such that each is obtained from the previous profile by a transposition pair or an application of top-2-only.

We argue by induction on the number $p$ of individuals who, at $u$, do not have $c$ in their top two ranks.

**Basis:** Suppose that we have $p = 0$ at $u$, i.e., every individual has $c$ in their top two ranks. Construct $u'$ from $u$ by (only) raising $c$ to everyone’s top rank. Now consider the subdomain $D'$ of $D$ consisting of all profiles in $D$ where everyone has $c$ top-ranked. Note that for each such profile at least two distinct alternatives must be second-ranked. Let $G'$ be the restriction of $G$ to $D'$. Social choice correspondence $G'$ on $D'$ induces a correspondence on those profiles on $X\setminus\{c\}$ which have non-unanimous tops. Then the analysis in the tops-only section shows that there is a sequence of profiles in $D'$ from $u'$ to $u^*$.

**Induction step:** Assume now that there is a non-empty set $S$ of alternatives such that for all profiles $v$ in $D$ with the number of individuals who do not have $c$ in their top two ranks being less than $p$ we have the same outcome: $G(v) = S$. Suppose that $u$ in $D$ is a profile where the number of individuals who at $u$ do not have $c$ in their top two ranks is $p > 0$. We will show there is a sequence of profiles (such that each is obtained from the previous profile by a transposition pair or an application of top-2-only) from $u$ to a profile $u'$ in $D$ with $p - 1$ individuals who do not have $c$ in the top two ranks (so $G(u') = S$). Without loss of generality suppose that it is the first $q = n - p$ individuals with $c$ ranked in the top two ranks. We may assume that $c$ has been raised to the top for each of those $q = n - p$ individuals so that $u$ is:

|   | 1 | 2 |   | q | q + 1 |   | n |
|---|---|---|---|---|------|---|---|
|  c | c | c | x |   | z    |   |   |
|   |   |   | y |   | w    |   |   |

Here $x$, $y$, $z$ and $w$ are distinct from $c$ (though these four need not all be distinct from one another).

**Case 1.** At least one of the alternatives, say $x$, in the top two ranks for $q + 1, \ldots, n$ is ranked second by one of $1, \ldots, q$. Without loss of generality, let $q + 1$ be an individual with $x$ in the top two and let $1$ be the individual with $c$ on top and $x$ in the second rank:

|   | 1 | 2 |   | q | q + 1 |   | n |
|---|---|---|---|---|------|---|---|
|  c | c | c | x |   | z    |   |   |
|  x |   |   | y |   | w    |   |   |
|   |   |   |   |   |      |   |   |

Then if necessary, lower $x$ to the second rank for $q + 1$ and raise $c$ to the third rank for that individual.
Now a transposition pair \((c, x)\) via individuals 1 and \(q + 1\) yields a profile in \(D\) with \(q + 1\) individuals with \(c\) in the top two ranks and so only \(p - 1\) individuals with \(c\) not in the top two ranks.

**Case 2.** None of the alternatives in the top two ranks for \(q + 1, \ldots, n\) is ranked second by one of 1, \ldots, \(q\).

**Subcase 2A.** There are two individuals among 1, \ldots, \(q\) with the same alternative \(t\) ranked second:

Now a transposition pair \((t, y)\) via individuals 2 and \(q + 1\) yields a profile in \(D\) still with \(p\) individuals with \(c\) not in the top two ranks but now back in Case 1 (since \(t\) is in the top two ranks for individual \(q + 1\) and is ranked second for individual \#1).

**Subcase 2B.** No element in the top two ranks for \(q + 1, \ldots, n\) is second for any of 1, \ldots, \(q\), and no two of 1, \ldots, \(q\) have the same second element. (Recall \(p = n - q\).)

**Subsubcase 2Bi.** \(p < n - 1\) (and so \(q > 1\)).
where $t$ is $2$'s second and $y$ is in $q+1$'s top two (moved to second rank if necessary).

Now a transposition pair $(t, y)$ via 1 and $q + 1$ takes us back to Case 1 (since $t$ will be in individual $q + 1$'s top two and in #2's second rank).

**Subsubcase 2Bii.** $p = n - 1$ (so $q = 1$).

If not all of 2, ..., $n$ have the same top two alternatives, say $n$ has $z$ in the top two but 2 does not, move $z$ to second place for individual $n$ and raise $c$ to $n$'s third rank while raising $c$ and $z$ to third and fourth rank for #2:

Now a transposition pair $(c, z)$ via 2 and $n$ takes us back to $p = n - 2$ in D.

On the other hand if 2, ..., $n$ all have the same two alternatives in the top two ranks:

Raise $y$ to third rank for #1 and $s$ to third rank for #2

Now a transposition pair $(s, y)$ via individuals 1 and 2 takes us back (in D) to Case 1.

**Case 3.** $p = n$ (so $q = 0$).
Because we are in $D$, there must be an alternative $z$, distinct from $x$ and $y$ (#1’s top two), in the top two ranks for someone, say #2. Move $z$ to second rank for #2,

\[
\begin{array}{cccc}
1 & 2 & \cdots & n \\
x & y & z & \\
\vdots & \vdots & & \\
\end{array}
\]

Then raise $c$ and $z$ to ranks three and four for #1 and $c$ to third rank for #2:

\[
\begin{array}{cccc}
1 & 2 & \cdots & n \\
x & y & c & c \\
z & & & \\
\end{array}
\]

Now transposition pair $(c, z)$ via 1 and 2 takes us back (in $D$) to Case 2Bii. $\Box$

There are straightforward generalizations to top-3-only, etc.

4. Balancedness and Borda

Balancedness is an equity condition that incorporates some equal treatment for differences of position of alternatives in orderings. As observed at the beginning of this paper, raising $x$ just above $y$ in the bottom two ranks for individual $j$ exactly offsets lowering $x$ just below $y$ in the top two ranks for $i$. This equal treatment of differences of position suggests trying to characterize the Borda rule within the class of scoring rules. Let a scoring system be given by weights $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_m$.

At profile $u$, the score for an alternative $x$ is the sum

$$S(x, u) = \sum_{i=1}^{n} s(u, i, x)$$

where $s(u, i, x) = s_k$ if $u(i)[k] = x$, i.e., the score for $x$ is the sum of the weights corresponding to the ranks that $x$ occupies in the individual orderings at $u$. The related scoring social choice correspondence $G$ selects at $u$ the alternatives $x$ with lowest $S(x, u)$ values. If

$$s_1 = s_2 = s_3 = \cdots = s_m$$

then $G$ is the constant social choice correspondence with $G(u) = X$ at all $u$, which is not very helpful. Accordingly, we henceforth only consider systems of weights such that at least two weights are distinct.

The Borda correspondence, $G_B$, uses weights $1 < 2 < 3 < \cdots < m$. But of course other weights also generate Borda. If social choice correspondence $G$ is generated by weights
then $G$ is also generated by linearly transformed weights

$$t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_m$$

where $t_i = \alpha + \beta s_i$ for real numbers $\alpha$, $\beta$, with $\beta > 0$.

**Lemma 0.** For $m \geq 3$ and $n \geq 2$: if $G$ is a scoring social choice correspondence for scoring system given by

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_m$$

with at least two weights distinct and if $G$ satisfies balancedness, then $s_1 \neq s_2$.

**Proof:** If $s_1 = s_2$, then there exists a $k$, $2 \leq k < m$, such that $s_1 = s_2 = \cdots = s_k < s_{k+1}$. Construct profile $u$ such that $u(i)[1] = y$ and $u(i)[2] = x$ for all $i < n$ while $u(n)[k] = x$ and $u(n)[k+1] = y$. Then $x \in G(u)$ and $y \notin G(u)$. If profile $u^*$ is constructed from $u$ by transposition pair $(x, y)$ via individuals 1 and $n$, then $x \notin G(u^*)$ and $y \in G(u^*)$. So $G(u^*) \neq G(u)$ and $G$ fails balancedness. \(\square\)

As a consequence of Lemma 0, we assume from now on that the weights have been transformed so that $s_1 = 1$ and $s_2 = 2$.

For profiles of strong orderings, we want to show that generally if a scoring social choice correspondence satisfies balancedness, it must be the Borda rule. However, the next section shows a limitation on this objective.

5. **Borda: $m = n = 3$**

**Example 3:** Let $m = n = 3$ and set scoring weights to be 1, 2, and 3.1.

The related correspondence, $G$, differs from Borda. At profile $u$:

|   | 1 | 2 | 3 |
|---|---|---|---|
| $x$ | $x$ | $y$ |
| $y$ | $y$ | $x$ |
| $z$ | $z$ | $x$ |

the Borda rule has a tie between $x$ and $y$ and $z$ is Pareto-dominated by $y$ so $G_B(u) = \{x, y\}$. But with scoring weights 1, 2, and 3.1, the scores of $x$ and $y$ are 5.1 and 5 respectively: $G(u) = \{y\}$. $G \neq G_B$.

Nevertheless, $G$ is balanced, as can easily be checked. So, for $m = n = 3$, balancedness of a scoring rule does not imply Borda.

This use of small numbers of individuals and alternatives is critical in Example 3, as we will see.
6. Borda: \( n > 3 \)

Our analysis proceeds by induction on \( m \). We begin by looking at \( m = 3 \) and a few small values of \( n > 3 \).

**Lemma 1.** For \( m = 3 \) and \( n = 4, 5, \) or \( 6 \): if \( G \) is a scoring social choice correspondence and \( G \) satisfies balancedness, then \( G \) is the Borda correspondence.

*Proof:* We show the following: if \( G \) is a scoring rule but not the Borda correspondence, then \( G \) violates balancedness.

**Basis case, \( m = 3 \), for Lemma 1**

\( (n = 5) \) We first treat the case \( n = 5 \). Consider the following profile \( v \):

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
x & z & y & x & z \\
y & y & x & z & y \\
z & x & z & y & x \\
\end{array}
\]

Consider a scoring rule \( G \) with \( 1 < 2 \leq s_3 \) (using Lemma 0). We examine the scores for \( x, y, z \):

- For \( x \), \( 4 + 2s_3 \);
- For \( z \), \( 4 + 2s_3 \);
- For \( y \), \( 7 + s_3 \).

So under this scoring rule, either \( x \) and \( y \) are both in \( G(v) \) or neither \( x \) nor \( y \) are in \( G(v) \).

Now we consider a transposition pair \( (x, y) \) via individuals 1 and 2 that yields the profile \( u \):

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
y & z & y & x & z \\
x & x & x & z & y \\
z & y & z & y & x \\
\end{array}
\]

We examine again the scores for \( x, y, z \):

- For \( x \), \( 7 + s_3 \);
- For \( y \), \( 4 + 2s_3 \);
- For \( z \), \( 4 + 2s_3 \).

This time, either \( y \) and \( z \) are both in \( G(u) \) or both are out of \( G(u) \). But by balancedness, \( G(u) = G(v) \), so at each profile all scores must be the same. Therefore \( 7 + s_3 = 4 + 2s_3 \), which has the unique solution \( s_3 = 3 \), i.e., \( G \) is the Borda correspondence.

\( (n = 4) \) Next, we consider \( n = 4 \). We examine the following profile \( v \):

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
x & z & x & z \\
y & y & y & y \\
z & x & z & x \\
\end{array}
\]

Consider a scoring rule \( G \) with weights: \( 1 < 2 \leq s_3 \) (using Lemma 0). We examine the scores for \( x, y, z \):

- For \( x \), \( 2 + 2s_3 \);
For \( z \), \( 2 + 2s_3 \);
For \( y \), \( 8 \).

Under this scoring rule, either \( x \) and \( z \) are both in \( G(v) \) or neither \( x \) nor \( z \) is in \( G(v) \).

Now we consider a sequence of transposition pairs \((x, y)\) first via individuals 1 and 2, and then via 3 and 4, which yields the profile \( u \):

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
y & z & y & z \\
x & x & x & x \\
z & y & z & y \\
\end{array}
\]

We examine again the scores for \( x, y, z \):
For \( x \), \( 8 \);
For \( y \), \( 2 + 2s_3 \);
For \( z \), \( 2 + 2s_3 \).

This time, either \( y \) and \( z \) are both in \( G(v) \) or neither \( y \) nor \( z \) is in \( G(v) \). But \( u \) is obtained from \( v \) by a sequence of transposition pairs, and balancedness implies \( G(u) = G(v) \), which means both must be \( \{x, y, z\} \), which, in turn, means at each profile, all three scores must be the same. Therefore, \( 2 + 2s_3 = 8 \), i.e., \( G \) is the Borda correspondence.

\((n = 6)\) Finally, the analysis for \( n = 6 \) proceeds just as for \( n = 4 \), but starting from the profile \( v \):

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
y & z & y & z & y & z \\
x & x & x & x & x & x \\
z & y & z & y & z & y \\
\end{array}
\]

and this time doing three transposition pairs (for 1 and 2, then 3 and 4, and then 5 and 6).

For \( n > 6 \), merely observe that any such \( n \) is equal to one of 4, 5, or 6 plus some multiple of 3. Using profiles above that are expanded by that many multiples of a voting paradox profile (on which all alternatives have the same score regardless of the weighting scheme) provides profiles showing \( s_3 = 3 \).

**Lemma 2.** For \( m \geq 3 \) and \( n = 4, 5, \) or \( 6 \): if \( G \) is a scoring social choice correspondence and \( G \) satisfies balancedness, then \( G \) is the Borda correspondence.

**Proof:** We prove by induction on \( m \). The basis step is given by Lemma 1. For \( M \geq 4 \), suppose that the lemma holds for all \( m < M \) and we have linearly transformed weights
\[ 1 \leq s_3 \leq \cdots \leq s_M \]
such that for some \( j > 3 \), it is the case that \( s_j \neq j \). Let \( G \) be the corresponding scoring social choice correspondence. We show that there is a profile \( u \) such that \( G(u) \neq G_B(u) \) and that \( G \) must fail balancedness. Consider the smallest integer \( j \) such that \( s_j \neq j \). If \( j < M \), just take a profile \( v \) that works for \( j \) alternatives (using the induction hypothesis) and append \( M - j \) additional alternatives to everyone’s bottom.
So we need only consider the case where the first \( j \) such that \( s_j \neq j \) is \( j = M \). The weights are:

\[
1 < 2 < 3 < \cdots < M - 1 \leq w \text{ where } w \neq M.
\]

Analysis here is for \( w > M \). The same profile applies to the case \( M - 1 \leq w < M \). We treat \( n = 5 \), but the same construction works for 4 or 6 individuals. (See the footnote below, for example, for \( n = 4 \).) For \( n = 5 \) we previously looked at profile:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 \\
x & z & y & x & z \\
y & y & x & z & y \\
z & x & z & y & x \\
\end{array}
\]

Now at each stage of the construction, we add another alternative, placing it just above \( y \) and \( x \) for \#2, just below \( x \) and \( y \) for \#1, and at the bottom for everyone else. At the first stage we insert \( a \):

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 \\
x & z & y & x & z \\
y & a & x & z & y \\
a & y & z & y & x \\
z & x & a & a & a \\
\end{array}
\]

(after which \#1 and \#2 have opposite rankings)\(^3\) The Borda scores are 11 for \( x \), \( y \), and \( z \) and 17 for \( a \): \( G_B(u) = \{x, y, z\} \). For the correspondence \( G \) with weights 1, 2, 3, and \( w_4 > 4 \), the scores are \( 7 + w_4 \) for \( x \) and \( z \), 11 for \( y \) and \( 5 + 3w_4 \) for \( a \). Because \( w_4 > 4 \), the smallest of these is 11 and \( G(u) = \{y\} \). Construct \( v \) by transposing \( x \) and \( y \) for \#1 and \#2; this yields score 11 for \( x \), and \( 7 + w_4 \) for both \( y \) and \( z \) and \( 5 + 3w_4 \) for \( a \). Again the smallest of these is 11, so \( G(v) = \{x\} \), a failure of balancedness.

Eventually the profile looks like \( v^* \):

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 \\
x & z & y & x & z \\
y & a_{M-3} & x & z & y \\
a_1 & : & z & y & x \\
a_2 & a_2 & a_1 & a_1 & a_1 \\
: & a_1 & a_2 & a_2 & a_2 \\
a_{M-3} & y & : & : & : \\
z & x & a_{M-3} & a_{M-3} & a_{M-3} \\
\end{array}
\]

(after which \#1 and \#2 have opposite rankings). It is easy to check that: \( G_B(v^*) = \{x, y, z\} \). For the correspondence \( G \) with weights 1, 2, 3, ..., \( w > M \), the scores are \( 7 + w \) for \( x \) and \( z \), \( 7 + M \) for \( y \), and the rest \( a_1, a_2, ... \) have higher scores. Because \( w > M \), the smallest of these is \( 7 + M \) and \( G(v^*) = \{y\} \).

\(^3\)So for \( n = 4 \), we also look at the profile for \( n = 4 \) in the basis step in Lemma 1, then insert \( a \) just above \( y \) and \( x \) for \#2, just below \( x \) and \( y \) for \#1, and at the bottom for \#3 and \#4.
Construct \( u^* \) by transposing \( x \) and \( y \) for \#1 and \#2. This yields scores \( 7 + M \) for \( x \), and \( 7 + w \) for both \( y \) and \( z \), with the remaining scores unchanged. Since \( w > M \), we now have \( G(u^*) = \{x\} \), a failure of balancedness. \( \square \)

7. Borda: \( n = 3, m > 3 \)

Let’s return to the case \( n = 3 \), where we learned in Example 1 that for \( m = 3 \) not every scoring social choice correspondence satisfying balancedness is the Borda rule. For the case \( m = 4 \) and \( n = 3 \), a natural analog of Example 1 has scoring weights 1, 2, 3, and 4.1. But that rule is unbalanced, as can be seen at profile \( u \):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
 a & b & x \\
y & a & y \\
x & x & b \\
b & y & a \\
\end{array}
\]

Here the scores are \( S(a) = 7.1 \), \( S(b) = 8.1 \), \( S(x) = 7 \), \( S(y) = 8.1 \). The smallest of these scores is 7 and so \( G(u) = \{x\} \). If \( v \) is constructed from profile \( u \) by transposition pair \((x, y)\) via individuals 1 and 2, then the scores become \( S(a) = 7.1 \), \( S(b) = 8.1 \), \( S(x) = 7.1 \), \( S(y) = 8 \). So \( G(v) = \{x, a\} \), a failure of balancedness.

**Theorem 5.** For \( n = 3 \) and \( m > 3 \): if \( G \) is a scoring social choice correspondence and \( G \) satisfies balancedness, then \( G \) is the Borda correspondence.

**Proof:** As in the proof of Lemma 2, we argue by induction on \( m \). Starting with a basis case, new alternatives are inserted into a profile to unbalancedness of a scoring rule with weights different from those of the Borda rule. For the basis case here, with \( m = 4 \), the scoring rule has weights 1, 2, \( s_3 \), \( s_4 \).

**Basis case, \( m = 4 \), for Theorem 5**

**Proof.** Let’s consider a scoring rule \( G \) but not the Borda correspondence: \( 1 < 2 \leq s_3 \leq s_4 \). We consider the following profile \( u \):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
 a & b & x \\
y & a & y \\
x & x & b \\
b & y & a \\
\end{array}
\]

We examine the scores for \( a, b, x, y \):

For \( a \), \( 3 + s_4 \);
For \( x \), \( 1 + 2s_3 \);
For \( b \), \( 1 + s_3 + s_4 \);
For \( y \), \( 4 + s_4 \).

Since \( S(a) < S(y) \) and \( S(x) < S(b) \) under this scoring rule, at least \( a \in G(u) \) or \( x \in G(u) \) (or both).

Now we consider a transposition pair \((x, y)\) via individual 1 and 2 that yields the profile \( v \):
We examine the scores for $a, b, x, y$:

For $a$, $3 + s_4$;
For $x$, $3 + s_4$;
For $b$, $1 + s_3 + s_4$;
For $y$, $1 + s_3 + s_4$.

By balancedness, $G(u) = G(v)$. Since at profile $v$, alternatives $a$ and $x$ have the same score, and at $u$, at least $a \in G(u)$ or $x \in G(u)$, we have $a$ and $x$ are both in $G(u)$ and $G(v)$. So (from $u$), $S(a) = 3 + s_4 = S(x) = 1 + 2s_3$, i.e.,

$$2s_3 - s_4 = 2 \quad (1)$$

Again, consider a transposition pair $(x, y)$ via individual 2 and 3 that yields the profile $v'$:

We examine the scores for $a, b, x, y$:

For $a$, $3 + s_4$;
For $x$, $4 + s_3$;
For $b$, $1 + s_3 + s_4$;
For $y$, $1 + s_3 + s_4$.

By balancedness, $a, x \in G(v')$, so $4 + s_3 = 3 + s_4$, or

$$s_3 - s_4 = -1 \quad (2)$$

Equations (1) and (2) have the unique solution $s_3 = 3, s_4 = 4$, i.e., $G$ is the Borda correspondence.

Now suppose the Theorem holds for all $m < M$ and we have weights

$$1 < 2 \leq s_3 \leq \cdots \leq s_M$$

such that for some $j > 3$, it is the case that $s_j \neq j$. Let $G$ be the corresponding scoring social choice correspondence. We show that there is a profile $u$ such that $G(u) \neq G_B(u)$ and that $G$ must fail balancedness. Consider the first $j$ such that $s_j \neq j$. If $j < M$, just take a profile $v$ that works for $j$ alternatives (using the induction hypothesis) and append $M - j$ additional alternatives to everyone’s bottom.

So we need only consider the case where the first $j$ such that $s_j \neq j$ is $j = M$. The weights are:

$$1 < 2 < 3 < \cdots < M - 1 \leq w \text{ where } w \neq M.$$

Analysis here is for $w > M$. The same profile works for $w < M$. 

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
a & b & x \\
x & a & y \\
y & y & b \\
b & x & a
\end{array}
\]
We first show one stage. Look at the profile $u$ just above. Now insert alternative $c$ just above $b$ for $#1$ and just below $b$ for $#2$. The third individual has $c$ at the bottom:

|   | 1  | 2  | 3   |
|---|----|----|-----|
| $a$| $b$| $x$|     |
| $x$| $c$| $y$|     |
| $y$| $a$| $b$|     |
| $c$| $y$| $a$|     |
| $b$| $x$| $c$|     |

The Borda scores for $a$ and $x$ are the lowest, so $G_B(u) = \{a, x\}$. With weights 1, 2, 3, 4, $w_5$, with $w_5 > 5$, the scores at $u$ are: $S(a) = 8$, $S(b) = 4 + w_5$, $S(c) = 6 + w_5$, $S(x) = 3 + w_5$, and $S(y) = 9$, so $G(u) = \{a\}$. After transposing $x$ and $y$ for individuals 1 and 2 to create profile $v$, the scores become: $S(a) = 8$, $S(b) = 4 + w_5$, $S(c) = 6 + w_5$, $S(x) = 8$, and $S(y) = 4 + w_5$, so $G(v) = \{a, x\}$ and balancedness is violated.

Eventually the profile looks like $u^*$:

|   | 1  | 2  | 3   |
|---|----|----|-----|
| $a$| $b$| $x$|     |
| $x$| $c_{M-4}$| $y$|     |
| $y$| $a$| $c_2$|     |
| $c_1$| $c_2$| $a$|     |
| $c_2$| $c_1$| $c_1$|     |
| $c_{M-4}$| $y$| $a$|     |
| $a$| $x$| $c_{M-4}$|     |

With weights 1, 2, ..., $M - 1$, $w$, with $w > M$, the scores at $u^*$ are: $S(a) = M + 3$, $S(b) = w + 4$, $S(x) = w + 3$, $S(y) = M + 4$, and the rest have higher scores. So $G(u^*) = \{a\}$ given $w > M$. After transposing $x$ and $y$ for individuals 1 and 2 to create profile $v^*$, the scores become: $S(a) = M + 3$, $S(b) = w + 4$, $S(x) = M + 3$, $S(y) = w + 4$, and the remaining scores are unchanged, so $G(v^*) = \{a, x\}$ and balancedness is violated. □

We obtain from this result a new characterization of the Borda correspondence. While Young (1974), Hansson and Sahlquist (1976), Coughlin (1979/80), Nitzan and Rubinstein (1981), and Debord (1992) have characterizations of Borda’s rule, they work in a different context than ours; for these authors, a rule has to work for a variable number of individuals and they use variable population properties like that of separability introduced by Smith (1973). Here we can get a characterization of Borda’s rule with a fixed set of individuals by appending balancedness to a characterization of scoring rules for fixed populations [see Fishburn (1973a, 1973b)].

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