THE ALEKSANDROV-BAKELMAN-PUCCI ESTIMATE
AND THE CALABI-YAU EQUATION

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Abstract. We give two applications of the Aleksandrov-Bakelman-Pucci estimate to
the Calabi-Yau equation on symplectic four-manifolds. The first is solvability of
the equation on the Kodaira-Thurston manifold for certain almost-Kähler
structures assuming $S^1$-invariance, extending a result
of Buzano-Fino-Vezzoni. The second is to reduce the general case of
Donaldson’s conjecture to a bound on the measure of a superlevel
set of a scalar function.

1. Introduction

Yau’s Theorem [28] states that one can prescribe the volume form of a
Kähler metric on a compact Kähler manifold $M^n$ within a given cohomology
class. The proof reduces, via a continuity method, to obtaining uniform $C^\infty$
a priori estimates on a potential function $u$ solving the complex Monge-Ampère
equation

\begin{equation}
(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = e^F \omega^n, \quad \omega + \sqrt{-1} \partial \bar{\partial} u > 0, \quad \sup_M u = 0,
\end{equation}

for a given smooth function $F$. A key step in Yau’s paper [28] was the
acclaimed $L^\infty$ estimate of $u$, which he obtained using a Moser iteration
argument.

There are now alternative proofs of the $L^\infty$ estimate, which have been
used to extend Yau’s Theorem to different settings [14, 3, 20, 21, 19, 24]. In
particular, Cheng and Yau (see [2, p. 75]) used the Aleksandr ov-Bakelman-
Pucci (ABP) estimate to prove an $L^2$ stability result for the complex Monge-
Ampère equation, and later Blocki [3] used this idea to give a new proof of
the $L^\infty$ estimate. Recall that the ABP estimate states, roughly speaking,
that the infimum of a function on a bounded domain can be controlled in
terms of its infimum on the boundary and the integral of the determinant of
its Hessian over the set where the function is convex (see e.g. [13, Lemma
9.2]). Blocki’s argument uses this to reduce the $L^\infty$ estimate of $u$ to an $L^1$
bound of $u$. Recently, Székelyhidi [15] strengthened this method to deal with
equations involving first order derivative terms, and this now has been used

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to establish $L^\infty$ estimates for a large class of Monge-Ampère type equations [17, 7, 8].

The purpose of this note is to apply the ABP estimate to Donaldson’s problem of the Calabi-Yau equation on symplectic 4-manifolds. Donaldson [10] conjectured the following:

**Conjecture 1.1.** Consider a compact symplectic manifold $(M, \omega)$ of real dimension 4, equipped with an almost complex structure $J$ which is tamed by $\omega$, and with $\tilde{\omega}$ another symplectic form compatible with $J$, cohomologous to $\omega$ and solving the Calabi-Yau equation

$$\tilde{\omega}^2 = e^F \omega^2,$$

for some smooth $F$. Then there are $C^\infty$ a priori estimates of $\tilde{\omega}$ depending only on $M, J, \omega$ and $F$.

If true, this would establish an almost-Kähler version of Yau’s Theorem for 4-manifolds, and would also have consequences for Donaldson’s “tamed to compatible” conjecture [10] (partially confirmed by Taubes [18], see also the surveys [11, 22] and the references therein).

While Conjecture 1.1 still remains open in general, we prove two consequences of the ABP estimate. The first is for the Calabi-Yau equation on the Kodaira-Thurston manifold $M = (\text{Nil}^3 / \Gamma) \times S^1$ where $\text{Nil}^3$ is the Heisenberg group of invertible matrices of the form

$$\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}, \quad \text{with } x, y, z \in \mathbb{R},$$

and $\Gamma$ is the subgroup consisting of those elements with integer entries, acting by left multiplication. $M$ admits symplectic forms, but no Kähler metric. The Calabi-Yau equation on $M$ was first considered in [23] where it was shown that equation (1.2) is solvable for certain $(J, \omega)$ assuming a $T^2$ symmetry of the initial data. More cases with $T^2$ symmetry were solved in [12, 5, 26], and this was recently improved by Buzano-Fino-Vezzoni [6] to the case of $S^1$ symmetry, which we now describe.

$M$ has an $S^1$ family of non-integrable almost complex structures $J_\theta$ for $\theta \in [0, 2\pi)$ which are compatible with a symplectic form $\omega_\theta$ on $M$. In addition, there is an $S^1$ action on $M$ given by translation in the $z$ coordinate which preserves $(J_\theta, \omega_\theta)$ for each $\theta$. For details, see Section 3 below.

The result of Buzano-Fino-Vezzoni [6, Theorem 2] is that one can solve the Calabi-Yau equation for any $S^1$-invariant $F$ with the additional assumption

$$\cos \theta = 0 \quad \text{or} \quad \tan \theta \in \mathbb{Q}. \quad (1.3)$$

We give a new approach using the ABP estimate which removes the assumption (1.3) on $\theta$. More precisely we prove:
Theorem 1.2. Given any $0 \leq \theta < 2\pi$, and any smooth $S^1$-invariant function $F$ on the Kodaira-Thurston manifold $M$ with $\int_M (e^F - 1)\omega^2_\theta = 0$, there is a unique symplectic form $\tilde{\omega}$ on $M$ which is compatible with $J_\theta$, with $[\tilde{\omega}] = [\omega_\theta]$, and solving the Calabi-Yau equation

$$\tilde{\omega}^2 = e^F \omega^2_\theta.$$  

Of course, this gives in particular a proof of Conjecture 1.1 on $(M, \omega_\theta, J_\theta)$ when $F$ is $S^1$ invariant.

Following [6], the equation (1.4) can be written as an elliptic PDE for a scalar function $u$ on the three-torus $T^3$. The ABP estimate shows that the $L^\infty$ estimate reduces to a simple integral bound for $u$ which can be easily established. Given this we can apply the rest of the arguments of [6], which do not need the assumption (1.3) on $\theta$, to obtain all the other estimates.

Our second application of the ABP estimate shows that in the general case of Conjecture 1.1 on a 4-manifold $(M, \omega, J)$, we can reduce all the estimates to an integral bound of a certain potential function. The idea of using the ABP estimate in this context was first pointed out to us by Székelyhidi [16].

We now describe our results more precisely. We apply the ABP estimate to the “almost-Kähler potential” $\varphi$ of [27, 25], defined by

$$\tilde{\omega} = \omega + \frac{1}{2} dJ d\varphi + da, \quad \tilde{\omega} \wedge da = 0, \quad \sup_M \varphi = 0,$$

using the sign convention in [8] and where $a$ is a 1-form (see Section 4 for more details). It was shown in [27, 25] that $C^\infty$ estimates for $\tilde{\omega}$ in (1.2) follow from an $L^\infty$ bound on $\varphi$, and this was reduced in [25] to a bound on $\int_M e^{-\alpha \varphi} \omega^2$ for some $\alpha > 0$. Our result here is that we can further reduce this, via the ABP method, to a much weaker kind of bound. An $L^p$ bound of $\varphi$, for any $0 < p < \infty$, would suffice, but in fact much less than this is needed:

Theorem 1.3. Let $(M, \omega, J)$ be as in Conjecture 1.1 and let $\tilde{\omega}$ solve the Calabi-Yau equation (1.2). Let $F : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be any increasing function with $\lim_{x \to +\infty} F(x) = +\infty$. Then the function $\varphi$, defined by (1.3) satisfies the estimate

$$F(\|\varphi\|_{L^\infty(M)} - 1) \leq C \int_M F(-\varphi) \omega^2,$$

for a uniform constant $C$ depending only on the background data $M, \omega, J, F$. Hence, to prove Conjecture 1.1 it is sufficient to bound $\int_M F(-\varphi) \omega^2$ for some fixed $F$.

Furthermore, if we fix $q > 3/2$, then for a uniform constant $C_q$, the function $\varphi$ satisfies the estimate

$$\|\varphi\|_{L^\infty(M)} \leq C_q \left( \int_{\{\varphi \geq -\lambda\}} \omega^2 \right)^{-q} + \lambda,$$

for any $\lambda > 0$. 

The second estimate (1.7) shows that to prove Conjecture 1.1 it is sufficient to get a uniform positive lower bound of the measure of the set \( \{ \varphi \geq -\lambda \} \) for some (possibly very large) uniform \( \lambda \). To establish (1.7), we prove the strictly stronger result that \( \varphi - \inf_M \varphi \) is uniformly bounded in \( L^p \) for any \( 0 < p < 2/3 \). Note that on the other hand we do have a uniform positive lower bound for the measure of the set \( \{ \varphi \leq \inf_M \varphi + 1 \} \), see (4.4).

The bound (1.6) was established in [25] for the function \( F(x) = e^{ax} \) using an estimate of the trace of \( \tilde{\omega} \) [27, 25]. In fact it is possible to prove (1.6) in general using the arguments of [25, 20] (see in particular Remark 3.1 of [20]). However, the proof we give here using the ABP estimate, which was pointed out to us by Székelyhidi [16], is much simpler and we believe it is more natural.

It is perhaps surprising that the ABP method works here for \( \varphi \), even though the equation (1.2) is not a local scalar PDE involving \( \varphi \). We remark that the proof of Theorem 1.3 does not actually require the assumption that \( [\tilde{\omega}] = [\omega] \), and so this condition (or something similar, see [22, Question 2.1]) must be used in any proof of the missing integral bound of \( \varphi \).

Finally we remark that Theorem 1.3 holds in higher dimensions with almost exactly the same proof, after making a suitable change to the inequality \( q > 3/2 \).

The outline of the paper is as follows. In Section 2 we outline the ABP method of Székelyhidi in a rather general setting. We then use it to prove Theorem 1.2 in Section 3 and Theorem 1.3 in Section 4.

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2. The ABP method of Székelyhidi

Fix a ball \( B = B_r(0) \subset \mathbb{R}^n \) centered at the origin of radius \( r \) with \( 0 < r \leq 1 \). In [15], the following ABP estimate is proved for a smooth function \( v : \overline{B} \rightarrow \mathbb{R} \). (It is stated there for \( r = 1 \) but this can be trivially extended to \( 0 < r \leq 1 \).)

**Proposition 2.1.** Assume that for \( \varepsilon > 0 \), the function \( v : \overline{B} \rightarrow \mathbb{R} \) satisfies \( v(0) + \varepsilon \leq \inf_{\partial B} v \). If we define \( P = \{ x \in B \mid |Dv(x)| < \varepsilon/2, \text{ and } v(y) \geq v(x) + Dv(x) \cdot (y - x) \forall y \in B \} \), then

\[
\varepsilon^n \leq C_0 \int_P \det(D^2v),
\]

for a constant \( C_0 \) depending only on the dimension \( n \).

The difference with the classical ABP estimate [13, Lemma 9.2] is that \( P \) is a subset of those points where \( v \) has small derivative, and this is crucial for our application. We now describe Székelyhidi’s method [15] for applying
this estimate to prove $L^\infty$ bounds (cf. [3]). Let $M$ be a compact manifold of real dimension $n$ with a fixed volume form $d\mu$. Let $u : M \rightarrow \mathbb{R}$ be a smooth function with $\sup_M u = 0$. The function $u$ will satisfy a PDE but to keep the discussion general we will not specify the equation. We wish to obtain an upper bound for $\|u\|_{L^\infty(M)}$ in terms of an integral bound for $u$. We will reduce this to a key pointwise inequality on $P$, for $B$ and $v$ which we will now specify.

Suppose $u$ achieves its infimum at $x_0 \in M$ (assume without loss of generality that $\inf_M u < -1$) and take a coordinate chart centered at $x_0$ which we identify with a ball $B = B_r(0)$ of a fixed radius $0 < r \leq 1$. Consider the function $v$ on $B$ defined by

$$v = u + \frac{\varepsilon}{r^2} \sum_{i=1}^{n} x_i^2,$$

for a small uniform fixed $\varepsilon > 0$. Applying Proposition 2.1 we obtain

$$\varepsilon^n \leq C_0 \int_P \det(D^2 v).$$

The key estimate we need to prove, which will use the PDE satisfied by $u$ and the definition of $v$ and $P$, is the following:

$$\text{(2.1)} \quad \text{at every } x \in P \text{ we have } \det(D^2 v(x)) \leq C,$$

for uniform $C$.

Assume now that (2.1) holds. Then it follows that

$$\varepsilon^n \leq C|P|,$$

up to increasing the uniform constant $C$, where $| \cdot |$ denotes the measure with respect to the volume form $d\mu$.

Now the integral bound for $u$ comes in. Let $F : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be an increasing function with $\lim_{x \rightarrow +\infty} F(x) = +\infty$. By definition, on the set $P$ we have $v \leq v(0) + \varepsilon/2$ and hence $u \leq \inf_M u + \varepsilon/2$. For later use, we note that this with (2.2) implies

$$|\{u \leq \inf_M u + 1\}| \geq \frac{\varepsilon^n}{C}.$$

Since $F$ is increasing, on $P$ we have $F(-u) \geq F(-\inf_M u - \varepsilon/2)$ and so

$$\varepsilon^n \leq C|P| \leq C \frac{\int_M F(-u)d\mu}{F(-\inf_M u - \varepsilon/2)}.$$

Hence we obtain

$$F(\|u\|_{L^\infty} - 1) \leq \frac{C}{\varepsilon^n} \int_M F(-u)d\mu.$$

Since $\varepsilon > 0$ is uniform, it follows that a bound on $\int_M F(-u)d\mu$ gives a bound on $\|u\|_{L^\infty(M)}$. In the special case $F(t) = t^p$ for $p > 0$, an $L^p$ bound of $u$ gives an $L^\infty$ bound of $u$.

In each of the two applications below, we will show that the key estimate (2.1) holds.
3. The Kodaira-Thurston manifold with $S^1$ symmetry

We give the proof of Theorem 1.2. Let $M$ be the Kodaira-Thurston manifold, as described in the introduction. We follow the notation in Buzano-Fino-Vezzoni [6], defining

$$e^1 = dy, \quad e^2 = dx, \quad e^3 = dt, \quad e^4 = dz - xdy,$$

where $t \in [0, 2\pi)$ is the variable in the $S^1$ factor. Consider the almost complex structures $J^{(1)}$ and $J^{(2)}$, defined by

$$J^{(1)}e^1 = e^3, \quad J^{(1)}e^4 = e^2,$$
$$J^{(2)}e^1 = e^4, \quad J^{(2)}e^2 = e^3.$$ 

These give rise to an $S^1$ family $\{J_{\theta}\}_{\theta \in [0, 2\pi)}$ of almost complex structures given by

$$J_{\theta} = \cos \theta J^{(1)} + \sin \theta J^{(2)}.$$ 

Each $J_{\theta}$ is compatible with the symplectic form

$$\omega_{\theta} = (\cos \theta e^1 + \sin \theta e^2) \wedge e^3 + e^4 \wedge (-\sin \theta e^1 + \cos \theta e^2),$$

and the resulting almost-Kähler metric is independent of $\theta$ and equal to

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2.$$ 

$M$ does not admit a Kähler structure, and so none of the $J_{\theta}$ are integrable.

Note that for all $\theta$, the data $(M, \omega_{\theta}, J_{\theta})$ are invariant under the $S^1$ action on $M$ given by translation in the $z$ coordinate.

**Remark 3.1.** From our earlier work [23] one can solve the Calabi-Yau equation (1.2) on $(M, \omega_{\theta}, J_{\theta})$ for any function $F$ which is invariant under the $T^2$ action which translates $z$ and $t$ (in [23] we considered the case $\theta = \pi/2$, but the same argument applies to all values of $\theta$).

**Remark 3.2.** The manifold $M$ also admits the integrable complex structure $J^{(3)} = J^{(1)}J^{(2)}$ given by

$$J^{(3)}e^1 = e^2, \quad J^{(3)}e^3 = e^4,$$

which makes $M$ into a primary Kodaira surface. These three almost complex structure satisfy the quaternionic relations, and all invariant almost complex structures on $M$ which are compatible with $g$ and its orientation are of the form $aJ^{(1)} + bJ^{(2)} + cJ^{(3)}$, with $a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1$; in this paper we are choosing $c = 0$.

The vector fields

$$X = \cos \theta \partial_x - \sin \theta (\partial_y + x \partial_z), \quad Y = \sin \theta \partial_x + \cos \theta (\partial_y + x \partial_z), \quad \partial_t, \partial_z,$$

are the dual frame of the coframe

$$-\sin \theta e^1 + \cos \theta e^2, \quad \cos \theta e^1 + \sin \theta e^2, \quad e^3, \quad e^4.$$ 

When $X$ and $Y$ act on $S^1$-invariant functions they are simply equal to

$$X = \cos \theta \partial_x - \sin \theta \partial_y, \quad Y = \sin \theta \partial_x + \cos \theta \partial_y,$$
and now these commute with each other, and with $\partial_t$. We will use $\partial_X$ and $\partial_Y$ to denote derivatives with respect to these vector fields.

We recall from Buzano-Fino-Vezzoni [6, (61)] that the Calabi-Yau equation on the Kodaira-Thurston manifold with $S^1$ symmetry can be reduced to the following elliptic PDE for a function $u$ on the three-torus $T^3$ (viewed as a quotient $\mathbb{R}^3/\mathbb{Z}^3$ with coordinates $(x, y, t)$):

$$ (u_{XX} + 1)(u_{YY} + u_{tt} + u_t + 1) - u_{XY}^2 - u_{Xt}^2 = e^F, $$

with [6, Proposition 1]

$$ u_{XX} + 1 > 0, \quad u_{YY} + u_{tt} + u_t + 1 > 0. $$

We make the normalization $\sup_M u = 0$. Following [6], it is enough to prove a uniform $L^\infty$ estimate for $u$. Indeed, the key second order estimate $\sup_M |\Delta u| \leq C(1 + \sup_M |\nabla u|)$ [6, Theorem 6] requires only the $L^\infty$ bound for $u$, and the remaining $C^\infty$ estimates are a consequence of this. Note that although this second order estimate is proved by Buzano-Fino-Vezzoni under the assumption $\theta = 0$, the proof for general $\theta \in [0, 2\pi)$ is formally the same, as pointed out in Section 5 of [6].

First note that adding the inequalities (3.2) gives that $\Delta u + u_t > -2$, for $\Delta$ the Laplace operator of the flat metric on the torus. Then a Green’s function argument as in [8, Proposition 2.3] gives a uniform $L^1$ bound on $u$ (or we can also use the weak Harnack inequality as in [15, (43)] to obtain an $L^p$ bound for some $p > 0$).

We now complete the argument for the $L^\infty$ bound of $u$ following the ABP method, as outlined in Section 2. We translate the coordinates so that $\inf_M u$ is achieved at the origin and we work in a ball $B = B_r$ of a fixed radius $r = 1/4$ say, centered at 0. Define

$$ v = u + \frac{\varepsilon}{r^2}(x^2 + y^2 + t^2) $$

for a small $\varepsilon > 0$. We use the terminology of Section 2. It is sufficient to establish (2.1), namely that at every point $x \in P$ we have

$$ \det D^2 v(x) \leq C. $$

By definition of $P$ we may assume $D^2 v \geq 0$ and $|Dv| \leq \varepsilon/2$ at $x$. We can rewrite the PDE (3.1) as

$$ (v_{XX} + 1 - O(\varepsilon))(v_{YY} + v_{tt} + 1 - O(\varepsilon)) - v_{XY}^2 - v_{Xt}^2 = e^F, $$

noting that from $D^2 v \geq 0$ we have $v_{XX}, v_{YY}, v_{tt} \geq 0$. Hence, as long as $\varepsilon$ is sufficiently small,

$$ \frac{1}{2} \left( v_{XX} + v_{YY} + v_{tt} + \frac{1}{2} \right) + v_{XX}v_{YY} + v_{XX}v_{tt} \leq e^F + v_{XX}^2 + v_{XY}^2 + v_{Xt}^2. $$

But since $D^2 v \geq 0$ we have $v_{XY}^2 \leq v_{XX}v_{YY}$ and $v_{Xt}^2 \leq v_{XX}v_{tt}$ and it follows that

$$ v_{YY} + v_{tt} + v_{XX} + \frac{1}{2} \leq 2e^F. $$
Hence $\text{tr}(D^2 v) \leq C$ and by the arithmetic-geometric mean inequality we obtain the required bound $\det D^2 v \leq C$. Since we have already established the $L^1$ bound of $u$, this completes the proof of the $L^\infty$ estimate and Theorem 1.2.

4. The Calabi-Yau equation on symplectic 4-manifolds

We now give the proof of Theorem 1.3, so we assume we are in that setting with $\tilde{\omega}$ solving the Calabi-Yau equation (1.2) and $\varphi$ the almost-Kähler potential given by (1.5). Recall from [27, 25] that to find $\varphi$ given $\omega$, $\tilde{\omega}$, solve the Poisson equation $\tilde{\Delta} \varphi := \text{tr}_{\tilde{\omega}}(\frac{1}{2} dJ d\varphi) = 2 - \text{tr}_{\tilde{\omega}} \omega$ subject to $\sup_M \varphi = 0$, and then define the 1 form $a$ by $da = \tilde{\omega} - \omega - \frac{1}{2} dJ d\varphi$. The function $\varphi$ is unique, while $a$ is unique up to “gauge”. Here $\text{tr}_{\tilde{\omega}} \omega$ is defined to be $2(\tilde{\omega} \wedge \omega)/\tilde{\omega}^2$.

We begin with the proof of (1.6). As noted in the introduction, this argument is due to Székelyhidi. Following again the method of Section 2, we work in a coordinate chart, identified with the unit ball $B$, in which $\varphi$ achieves its inf $M \varphi$ at the origin. Define $v = \varphi + \varepsilon |x|^2$. We need to prove (2.1). Let $x$ be a point where $D^2 v \geq 0$ and $|Dv| \leq \varepsilon/2$. It suffices to show that $\det D^2 v$ is bounded from above at this point.

Note that, writing $(D^2 v)_J$ for the $J$-invariant part of $D^2 v$, given by

$$ (D^2 v)_J = \frac{1}{2}(D^2 v + J^T (D^2 v) J), $$

we have that, at $x$,

$$ (dJ d\varphi)^{(1,1)} = (D^2 v)_J + O(\varepsilon) \geq O(\varepsilon), $$

e.g. thanks to [19] p.443. Note that here we are using the condition $|Dv| \leq \varepsilon/2$. It follows that

$$ \omega^{(1,1)} + \frac{1}{2}(dJ d\varphi)^{(1,1)} \geq \frac{\omega^{(1,1)}}{2}, $$

provided we choose $\varepsilon$ sufficiently small (but uniform). If we wedge this with $\tilde{\omega}$ we get

$$ \tilde{\omega}^2 = \tilde{\omega} \wedge \left( \omega^{(1,1)} + \frac{1}{2}(dJ d\varphi)^{(1,1)} \right) \geq \frac{\tilde{\omega} \wedge \omega}{2}, $$

since $\tilde{\omega} \wedge da = 0$ and $\tilde{\omega}$ is of type (1, 1), and so

$$ \text{tr}_{\tilde{\omega}} \omega \leq 4. $$

By the Calabi-Yau equation (1.2), this implies that $\omega^{(1,1)}$ and $\tilde{\omega}$ are uniformly equivalent. But from (4.2) and

$$ \tilde{\Delta} \varphi = \frac{1}{2} \text{tr}_{\tilde{\omega}}(dJ d\varphi)^{(1,1)} = 2 - \text{tr}_{\tilde{\omega}} \omega \leq 2, $$

we see that $(dJ d\varphi)^{(1,1)}$ is uniformly bounded. We can then argue exactly as in [8] to obtain an upper bound on $\det D^2 v$ (see also [3, 15]). Indeed, using
and the inequality $\det(A + B) \geq \det A + \det B$ for nonnegative matrices $A, B$, we have

$$\det D^2 v \leq 8 \det((D^2 v)^J) \leq C,$$

as required. This completes the proof of (1.6).

Furthermore, recalling the estimate (2.3) of Section 2, we have

$$(4.4) \quad |\{\varphi \leq \inf_M \varphi + 1\}| \geq \delta,$$

for a uniform $\delta > 0$, where $| \cdot |$ is the measure of the set with respect to the volume form $\omega^2$.

**Remark 4.1.** In fact, a simple modification of these arguments shows that (1.6) holds for the “almost-Kähler potentials” $\varphi_t$ for $\frac{1}{2} < t \leq 1$ (as defined in [27]), when $\omega$ is also assumed to be compatible with $J$. On the other hand $\varphi_{\frac{1}{2}}$ is uniformly bounded, as observed by Donaldson using Moser iteration (see [27, Remark 6.1]).

To finish the proof of Theorem 1.3 it suffices to prove the following:

**Proposition 4.2.** Given any $0 < p < 2/3$, the function $\varphi$ satisfies the estimate

$$(4.5) \quad \|\varphi - \inf_M \varphi\|_{L^p(M)} \leq C_p,$$

for a uniform constant $C_p$ which depends only on the background data and on $p$.

Indeed, suppose that Proposition 4.2 holds, and let $\lambda > 0$. We may assume without loss of generality that $\lambda < -\inf_M \varphi$. Then for any $0 < p < 2/3$,

$$(\lambda - \inf_M \varphi)^p|\{\varphi \geq \lambda\}| \leq \int_{\{\varphi \geq \lambda\}} (\varphi - \inf_M \varphi)^p \omega^2 \leq C_p,$$

and then (1.7) holds with $q = 1/p > 3/2$ and $C_q = C_p^{1/p}$.

**Proof of Proposition 4.2.** Using the elementary formula

$$\int_M u^p \omega^2 = p \int_0^\infty |\{u \geq s\}|s^{p-1}ds,$$

applied to $u = \varphi - \inf_M \varphi$, it is enough to show that

$$(4.6) \quad |\{\varphi \geq \inf_M \varphi + s\}| \leq Cs^{-\frac{2}{5}},$$

for a uniform constant $C > 0$ and for all $s \geq 2$.

Let $\psi_s = \max(-\varphi + \inf_M \varphi + s, 0)$. Then $\psi_s$ is a Lipschitz function that satisfies $0 \leq \psi_s \leq s$. It is a well-known fact that on the set $\{\psi_s = 0\}$ the gradient of $\psi_s$ is zero a.e. (see e.g. [1, Theorem 3.2.6]). We have an obvious pointwise bound

$$|\partial \psi_s|^2 \leq tr_\omega |\partial \psi_s|^2_\omega,$$
where we define
\[ |\partial \psi_s|_\omega^2 = \frac{\omega \wedge d\psi_s \wedge Jd\psi_s}{\omega^2}. \]

We also have the \( L^1 \) bound \( \int_M (\text{tr}_\omega \bar{\omega}) \omega^2 \leq C \). Then, using in addition the Calabi-Yau equation (1.2),
\[
\int_M |\partial \psi_s|_\omega \omega^2 \leq C \left( \int_M (\text{tr}_\omega \bar{\omega}) \omega^2 \right)^{\frac{1}{2}} \left( \int_M |\partial \psi_s|_\omega^2 \omega^2 \right)^{\frac{1}{2}}
\]
(4.7)
\[
= C \left( \int_M |\partial \psi_s|_\omega^2 \omega^2 \right)^{\frac{1}{2}}.
\]

Since \( \psi_s \) is smooth on the open set \( \{ \psi_s > 0 \} \) and vanishes on its boundary, we can integrate by parts (this is justified by exhausting this set by smooth domains) and get, using (4.3) and again (1.2),
\[
\int_M |\partial \psi_s|_\omega \omega^2 \leq C \left( \int_{\{ \psi_s > 0 \}} \psi_s \Delta (-\psi_s) \omega^2 \right)^{1/2} \leq C \left( \int_{\{ \psi_s > 0 \}} \psi_s \omega^2 \right)^{1/2} \leq C s^{\frac{1}{2}}.
\]

Now the Sobolev-Poincaré inequality gives
\[
\| \psi_s - \bar{\psi}_s \|_{L^4(M)} \leq C \| \partial \psi_s \|_{L^1(M)} \leq C s^{\frac{1}{2}},
\]
where we define \( \bar{\psi}_s = \frac{\int_M \psi_s \omega^2}{\int_M \omega^2} \). Let
\[ \gamma(s) = |\{ \psi_s = 0 \}| = |\{ \varphi \geq \inf_M \varphi + s \}|, \]
so that
\[
\| \psi_s - \bar{\psi}_s \|_{L^4(M)} \geq \left( \int_{\{ \psi_s = 0 \}} |\psi_s - \bar{\psi}_s|^{\frac{4}{3}} \omega^2 \right)^{\frac{3}{4}} = \gamma(s)^{\frac{3}{4}} \psi_s.
\]
(4.9)

Then from (4.8) and (4.9) we get
\[
\gamma(s)^{\frac{3}{4}} \psi_s \leq C s^{\frac{1}{4}}.
\]
(4.10)

Now we look at the set \( \{ \varphi \leq \inf_M \varphi + 1 \} \) which by (4.4) has volume at least \( \delta \). On this set we have that \( \psi_s \geq s - 1 \geq s/2 \), as long as \( s \geq 2 \). Then we have
\[
\psi_s \int_M \omega^2 = \int_M \psi_s \omega^2 \geq \int_{\{ \varphi \leq \inf_M \varphi + 1 \}} \psi_s \omega^2 \geq \frac{\delta s}{2}.
\]
Combining this with (4.10) we finally get
\[ \gamma(s) \leq C s^{-\frac{2}{3}}, \]
completing the proof of the proposition. \[ \square \]
THE ABP ESTIMATE AND THE CALABI-YAU EQUATION

References

[1] L. Ambrosio, P. Tilli Topics on analysis in metric spaces, Oxford Lecture Series in Mathematics and its Applications, 25. Oxford University Press, Oxford, 2004.

[2] E. Bedford Survey of pluri-potential theory, in Several complex variables (Stockholm, 1987/1988), 48–97, Math. Notes 38, Princeton Univ. Press, Princeton, NJ, 1993.

[3] Z. Blocki On uniform estimate in Calabi-Yau theorem, Sci. China Ser. A 48 (2005), suppl., 244–247.

[4] Z. Blocki On the uniform estimate in the Calabi-Yau theorem, II, Sci. China Math. 54 (2011), no. 7, 1375–1377.

[5] E. Buzano, A. Fino, L. Vezzoni The Calabi-Yau equation for $T^2$-bundles over $T^2$: the non-Lagrangian case, Rend. Semin. Mat. Univ. Politec. Torino 69 (2011), no. 3, 281–298.

[6] E. Buzano, A. Fino, L. Vezzoni The Calabi-Yau equation on the Kodaira-Thurston manifold, viewed as an $S^1$-bundle over a 3-torus, J. Differential Geom. 101 (2015), no. 2, 175–195.

[7] T.C. Collins, A. Jacob, S.-T. Yau (1, 1) forms with specified Lagrangian phase: a priori estimates and algebraic obstructions, preprint. [arXiv:1508.01934]

[8] J. Chu, V. Tosatti, B. Weinkove The Monge-Ampère equation for non-integrable almost complex structures, preprint, [arXiv:1603.00706]

[9] S. Dinew, S. Kołodziej Pluripotential estimates on compact Hermitian manifolds, Advances in geometric analysis, 69–86, Adv. Lect. Math. (ALM), 21, Int. Press, Somerville, MA, 2012.

[10] S.K. Donaldson, Two-forms on four-manifolds and elliptic equations, in Inspired by S. S. Chern, 153–172, Nankai Tracts Math., 11, World Sci. Publ., Hackensack, NJ, 2006.

[11] T. Draghici, T.-J. Li, W. Zhang Geometry of tamed almost complex structures on 4-dimensional manifolds, in Fifth International Congress of Chinese Mathematicians. Part 1, 2, 233–251, AMS/IP Stud. Adv. Math., 51, pt. 1, 2, Amer. Math. Soc., Providence, RI, 2012.

[12] A. Fino, Y.Y. Li, S. Salamon, L. Vezzoni The Calabi-Yau equation on 4-manifolds over 2-tori, Trans. Amer. Math. Soc. 365 (2013), no. 3, 1551–1575.

[13] D. Gilbarg, N.S. Trudinger Elliptic partial differential equations of second order, Springer-Verlag, 1983.

[14] S. Kołodziej, The complex Monge-Ampère equation, Acta Math. 180 (1998), no. 1, 69–117.

[15] G. Székelyhidi Fully non-linear elliptic equations on compact Hermitian manifolds, preprint, [arXiv:1501.02762]

[16] G. Székelyhidi Private communication.

[17] G. Székelyhidi, V. Tosatti, B. Weinkove Gauduchon metrics with prescribed volume form, preprint, [arXiv:1503.04491]

[18] C.H. Taubes Tamed to compatible: symplectic forms via moduli space integration., J. Symplectic Geom. 9 (2011), no. 2, 161–250.

[19] V. Tosatti, Y. Wang, B. Weinkove, X. Yang $C^{2,\alpha}$ estimates for non-linear elliptic equations in complex and almost complex geometry, Calc. Var. Partial Differential Equations 54 (2015), no.1, 431–453.

[20] V. Tosatti, B. Weinkove Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds, Asian J. Math. 14 (2010), no.1, 19–40.

[21] V. Tosatti, B. Weinkove The complex Monge-Ampère equation on compact Hermitian manifolds, J. Amer. Math. Soc. 23 (2010), no. 4, 1187–1195.

[22] V. Tosatti, B. Weinkove The Calabi-Yau equation, symplectic forms and almost complex structures, in Geometry and Analysis, No. 1, 475–493, Advanced Lectures in Math. 17, International Press, 2011.
[23] V. Tosatti, B. Weinkove *The Calabi-Yau equation on the Kodaira-Thurston manifold*, J. Inst. Math. Jussieu **10** (2011), no.2, 437–447.

[24] V. Tosatti, B. Weinkove *On the evolution of a Hermitian metric by its Chern-Ricci form*, J. Differential Geom. **99** (2015), no.1, 125–163.

[25] V. Tosatti, B. Weinkove, S.-T. Yau *Taming symplectic forms and the Calabi-Yau equation*, Proc. London Math. Soc. **97** (2008), no.2, 401–424.

[26] L. Vezzoni *On the Calabi-Yau equation in the Kodaira-Thurston manifold*, preprint.

[27] B. Weinkove *The Calabi-Yau equation on almost-Kähler four-manifolds*, J. Differential Geom. **76** (2007), no. 2, 317–349.

[28] S.-T. Yau *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

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