Some limit theorems for rescaled Wick powers

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Abstract

We establish the strong $L^2(P)$-convergence of properly rescaled Wick powers as the power index tends to infinity. The explicit representation of such limit will also provide the convergence in distribution to normal and log-normal random variables. The proofs rely on some estimates for the $L^2(P)$-norm of Wick products and on the properties of second quantization operators.

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1 Introduction

In the last decade several authors have identified the Wick product as a necessary tool for the study of certain types of stochastic partial differential equations (SPDEs) or for the solution to some related problems. This is motivated by the crucial features of the Wick product: firstly it represents a bridge between stochastic and classical integration theories; secondly it provides an efficient way of multiplying infinite dimensional distributions. Since several SPDEs of interest do not admit classical solutions, the possibility of treating nonlinearities becomes fundamental. Important examples in these regards are the stochastic quantization equation, which was studied among others in [3], [4] and [5], and the KPZ equation, studied for instance in [1] and [2]. Also the problem of finding Itô’s type formulas for SPDEs leads in a natural way to the use of Wick powers as a renormalization technique. This is shown in [12] and [9]. We also mention the book [6] which proposes a systematic use of Wick products for the formulation and the study of a variety of SPDEs.
Let us briefly introduce the Wick product (see the next section for precise definitions). Consider two multiple Itô integrals $I_n(h_n)$ and $I_m(g_m)$ where $h_n \in \mathcal{L}^2([0,T]^n)$ and $g_m \in \mathcal{L}^2([0,T]^m)$. The well-known Hu-Meyer formula establishes that

\[ I_n(h_n) \cdot I_m(g_m) = \sum_{r=0}^{n\wedge m} \frac{n!}{r!} \binom{n}{r} \binom{m}{r} I_{n+m-2r}(h_n \otimes_r g_m), \quad (1.1) \]

where

\[ (h_n \otimes_r g_m)(t_1, \ldots, t_{n+m-2r}) := \int_{[0,T]^r} h_n(t_1, \ldots, t_{n-r}, s) g_m(t_{n-r+1}, \ldots, t_{n+m-2r}, s) ds, \quad (1.2) \]

and $\otimes$ stands for symmetrization. If $h_n$ and $g_m$ are only tempered distributions then $I_n(h_n)$ and $I_m(g_m)$ become generalized random variables. In this case the pointwise product (1.1) is not anymore well defined due to the presence of the trace terms (1.2) that do not make sense in this new situation. To overcome this problem one can drop these problematic terms and define a new product obtained keeping only the term with $r = 0$ in the sum (1.1), namely:

\[ I_n(h_n) \cdot I_m(g_m) := I_{n+m}(h_n \hat{\otimes} g_m). \]

This is called the Wick product of $I_n(h_n)$ and $I_m(g_m)$. If for example we denote by $W_t$ the time derivative of a Brownian motion $B_t$ then we can write $W_t = I_1(\delta_t)$ ($\delta_t$ stands for the Dirac's delta function concentrated in $t$) and obtain

\[ W_t \cdot W_t = I_2(\delta_t \otimes \delta_t). \]

Observe that applying formally (1.1) we get

\[ W_t \cdot W_t = "W_t^2 - \int_0^T \delta_t^2(s) ds". \]

The above mentioned bridge between stochastic and classical integration theories can be now formalized precisely as

\[ \int_0^T \xi_t dB_t = \int_0^T \xi_t \cdot W_t dt, \]

where the left hand side denotes the Itô integral of the stochastic process $\xi_t$.

The aim of the present paper is the investigation of the limiting behavior of the sequence

\[ X^{\otimes n} := X \cdot \cdots \cdot X, \quad n \geq 1, \quad (1.3) \]

as $n$ goes to infinity. The motivation for doing this is twofold: on one hand Wick powers of the type (1.3) appear in the formulation of the stochastic quantization equation (see [3]);
on the other hand the Wick product, as suggested in [8], can be viewed as a convolution between (generalized) random variables. Therefore a theorem about the limiting behavior of the sequence in (1.3) constitutes a result in the spirit of the central limit theorem for the convolution \( \diamond \).

We will prove that for any square integrable \( X \) a properly rescaled version of the sequence in (1.3) converges in the strong topology of \( L^2(\mathcal{P}) \) to a so-called stochastic exponential; this result will imply the convergence in distribution to log-normal random variables. We will also show that under the assumption of the positivity of \( X \) the logarithm of the above mentioned sequence converges in distribution to a normal random variable.

The paper is organized as follows: Section 2 recalls some classical background information and introduce the necessary definitions. We refer the reader to the book [10] for more detailed material. Section 3 presents the main results of the paper together with some important corollaries concerning convergence in distribution.

# 2 Preliminaries

Let \( (\Omega, \mathcal{F}, \mathcal{P}) \) be the classical Wiener space over the time interval \([0, T]\) and denote by \( B_t(\omega) := \omega(t), t \in [0, T] \) the coordinate process which is a Brownian motion under the measure \( \mathcal{P} \). Set as usual

\[
\mathcal{L}^2(\mathcal{P}) := \left\{ X : \Omega \to \mathbb{R} \text{ measurable s.t. } E[|X|^2] := \int_{\Omega} |X(\omega)|^2 d\mathcal{P}(\omega) < +\infty \right\},
\]

and

\[
\|X\| := (E[|X|^2])^{\frac{1}{2}}.
\]

According to the Wiener-Itô chaos representation theorem any \( X \in \mathcal{L}^2(\mathcal{P}) \) can be written uniquely as

\[
X = \sum_{n \geq 0} I_n(h_n),
\]

where \( I_0(h_0) = E[X] \) and for \( n \geq 1 \), \( h_n \in \mathcal{L}^2([0, T]^n) \) is a symmetric deterministic function which is called the \( n \)-th order kernel of \( X \). Moreover for \( n \geq 1 \), \( I_n(h_n) \) stands for the \( n \)-th order multiple Itô integral of \( h_n \) w.r.t. the Brownian motion \( \{B_t\}_{0 \leq t \leq T} \).

By means of this representation the \( \mathcal{L}^2(\mathcal{P}) \)-norm of \( X \) takes the following form:

\[
\|X\|^2 = \sum_{n \geq 0} n!|h_n|^2_{\mathcal{L}^2([0, T]^n)}.
\]

Given two square integrable random variables \( X \) and \( Y \) with chaotic representations:

\[
X = \sum_{n \geq 0} I_n(h_n) \text{ and } Y = \sum_{n \geq 0} I_n(g_n),
\]
we call Wick product of $X$ and $Y$ the following quantity:

$$X \diamond Y := \sum_{n \geq 0} I_n(k_n), \text{ with } k_n := \sum_{j=0}^{n} h_j \hat{\otimes} g_{n-j},$$

where $\hat{\otimes}$ denotes the symmetric tensor product. We also denote

$$X^{\otimes n} := X \diamond \cdots \diamond X \text{ (n-times)}.$$

In general $X \diamond Y$ does not belong to $L^2(\mathcal{P})$ since it may happen that

$$\|X \diamond Y\|^2 = \sum_{n \geq 0} n!|k_n|^2_{L^2([0,\tau])} = +\infty.$$

The next inequality is a straightforward generalization of Theorem 9 in [8] where the Wick product of two random variables is considered. This result, that will be of crucial importance in our proofs, provides a sufficient condition for the Wick product of random variables to be square integrable.

First we need to recall that for $\lambda \in \mathbb{R}$ we denote by $\Gamma(\lambda)$ the following operator:

$$\Gamma(\lambda)X = \Gamma(\lambda) \sum_{n \geq 0} I_n(h_n) := \sum_{n \geq 0} \lambda^n I_n(h_n).$$

**Theorem 2.1** Let $X_1, X_2, \ldots, X_n \in L^2(\mathcal{P})$. Then

$$\|X_1 \diamond \cdots \diamond X_n\| \leq \|\Gamma(\sqrt{n})X_1\| \cdots \|\Gamma(\sqrt{n})X_n\|,$$

or equivalently

$$\left\| \Gamma\left(\frac{1}{\sqrt{n}}\right) \left(X_1 \diamond \cdots \diamond X_n\right) \right\| \leq \|X_1\| \cdots \|X_n\|.$$

In particular for $X_1 = \cdots = X_n = X$, we get

$$\|X^{\otimes n}\| \leq \|\Gamma(\sqrt{n})X\|^n,$$

or equivalently

$$\left\| \Gamma\left(\frac{1}{\sqrt{n}}\right) X^{\otimes n} \right\| \leq \|X\|^n.$$

**Remark 2.2** If $\lambda \in [0,1]$, the operator $\Gamma(\lambda)$ can be expressed in terms of the Ornstein-Uhlenbeck semigroup. In fact if we write for $t \geq 0$,

$$(P_tX)(\omega) := \int_{\Omega} X(e^{-t}\omega + \sqrt{1-e^{-2t}}\tilde{\omega}) d\mathcal{P}(\tilde{\omega}),$$
then
\[ \Gamma(\lambda) = \Gamma(e^{\log \lambda}) = P_{-\log \lambda}. \]

In particular
\[ \Gamma\left(\frac{1}{\sqrt{n}}\right) = P_{\frac{1}{2} \log n}. \]

We conclude this section observing that for \( \lambda, \mu \in \mathbb{R} \),
\[ \Gamma(\mu) \Gamma(\lambda) = \Gamma(\mu \lambda), \quad (2.1) \]
and that for \( X, Y \in L^2(\mathcal{P}) \),
\[ \Gamma(\lambda)(X \circ Y) = \Gamma(\lambda)X \circ \Gamma(\lambda)Y. \quad (2.2) \]

3 Main results

We are now ready to state one of the main results of this paper.

**Theorem 3.1** Let \( X \in L^2(\mathcal{P}) \) with \( E[X] \neq 0 \) and denote by \( h_1 \in L^2([0, T]) \) the first-order kernel in the chaos decomposition of \( X \). Then
\[ \Gamma\left(\frac{1}{n}\right)X_{\circ n} \in L^2(\mathcal{P}) \text{ for any } n \geq 1, \]
and
\[ \lim_{n \to \infty} \frac{\Gamma\left(\frac{1}{n}\right)X_{\circ n}}{E[X]^n} = \exp \left\{ \int_0^T h_1(s)dB_s - \frac{1}{2} \int_0^T h_1^2(s)ds \right\}, \quad (3.1) \]
where the convergence is in the strong topology of \( L^2(\mathcal{P}) \).

To ease the notation for \( h \in L^2([0, T]) \) we set
\[ \mathcal{E}(h) := \exp \left\{ \int_0^T h(s)dB_s - \frac{1}{2} \int_0^T h^2(s)ds \right\}. \]

The random variable \( \mathcal{E}(h) \) belongs to \( L^2(\mathcal{P}) \) and its chaotic representation is
\[ \mathcal{E}(h) = \sum_{n \geq 0} I_n \left( \frac{h_{\circ n}}{n!} \right). \]
From this identity one can easily derive the following properties:

\[
\|\mathcal{E}(h)\| = \exp\left\{ \frac{1}{2}|h|_2^2 \mathbb{E}^2([0,T]) \right\}; \quad (3.2)
\]

\[
\Gamma(\lambda)\mathcal{E}(h) = \mathcal{E}(\lambda h); \quad (3.3)
\]

\[
\mathcal{E}(h) \circ \mathcal{E}(g) = \mathcal{E}(h + g). \quad (3.4)
\]

**Proof.** First of all note that since \(E[X]\) is a constant we can write

\[
\Gamma\left(\frac{1}{n}\right)X^{\circ n} = \Gamma\left(\frac{1}{n}\right)\left(\frac{X}{E[X]}\right)^{\circ n};
\]

Therefore we can assume without loss of generality that \(E[X] = 1\) and prove that

\[
\lim_{n \to \infty} \Gamma\left(\frac{1}{n}\right)X^{\circ n} = \mathcal{E}(h_1). \quad (3.5)
\]

For any \(n \geq 1\),

\[
\Gamma\left(\frac{1}{n}\right)X^{\circ n} \in \mathcal{L}^2(P).
\]

In fact according to Theorem 2.1,

\[
\|\Gamma\left(\frac{1}{n}\right)X^{\circ n}\| = \|\Gamma\left(\frac{1}{\sqrt{n}}\right)\Gamma\left(\frac{1}{\sqrt{n}}\right)X^{\circ n}\|
\]

\[
\leq \|\Gamma\left(\frac{1}{\sqrt{n}}\right)X^{\circ n}\|
\]

\[
\leq \|X\|^n.
\]

Moreover for \(n \geq 1\),

\[
\mathcal{E}(h_1) = \mathcal{E}\left(\frac{h_1}{n} + \cdots + \frac{h_1}{n}\right)_{n \text{ times}}
\]

\[
= \mathcal{E}\left(\frac{h_1}{n}\right) \circ \cdots \circ \mathcal{E}\left(\frac{h_1}{n}\right)_{n \text{ times}}
\]

\[
= \mathcal{E}\left(\frac{h_1}{n}\right)^{\circ n}.
\]

We have to prove that

\[
\lim_{n \to \infty} \|\Gamma\left(\frac{1}{n}\right)X^{\circ n} - \mathcal{E}(h_1)\| = 0.
\]
Since the Wick product is commutative, associative and distributive with respect to the sum, the following identity holds:

\[ Y^o n - Z^o n = (Y - Z) \circ \left( \sum_{j=0}^{n-1} Y^{o j} \circ Z^{o(n-1-j)} \right). \]

Therefore from Theorem 2.1 and properties (3.2)-(3.4) we obtain

\[
\| \Gamma \left( \frac{1}{n} \right) X^{o n} - \mathcal{E}(h_1) \| = \| \Gamma \left( \frac{1}{n} \right) X^{o n} - \mathcal{E}(\frac{h_1}{n})^{o n} \|
\]

\[
\leq \| \Gamma(\sqrt{2}) \left( \Gamma \left( \frac{1}{n} \right) X - \mathcal{E}(\frac{h_1}{n}) \right) \| \times \| \Gamma(\sqrt{2}) \left( \sum_{j=0}^{n-1} \Gamma \left( \frac{1}{n} \right) X^{o j} \circ \mathcal{E}(\frac{h_1}{n})^{o(n-1-j)} \right) \|
\]

\[
\leq \| \Gamma \left( \frac{\sqrt{2}}{n} \right) X - \mathcal{E}(\frac{\sqrt{2}h_1}{n}) \|
\times \| \sum_{j=0}^{n-1} \Gamma \left( \frac{\sqrt{2}}{n} \right) X^{o j} \circ \mathcal{E}(\frac{\sqrt{2}h_1}{n})^{o(n-1-j)} \|
\]

\[
\leq \| \Gamma \left( \frac{\sqrt{2}}{n} \right) X - \mathcal{E}(\frac{\sqrt{2}h_1}{n}) \|
\times \sum_{j=0}^{n-1} \| \Gamma(\sqrt{n-1}) \Gamma \left( \frac{\sqrt{2}}{n} \right) X \|^j \| \Gamma(\sqrt{n-1}) \mathcal{E}(\frac{\sqrt{2}h_1}{n}) \|^{n-1-j}
\]

\[
= \| \Gamma \left( \frac{\sqrt{2}}{n} \right) X - \mathcal{E}(\frac{\sqrt{2}h_1}{n}) \|
\times \sum_{j=0}^{n-1} \| \Gamma \left( \frac{\sqrt{2(n-1)}}{n} \right) X \|^j \| \mathcal{E}(\frac{{\sqrt{2(n-1)h_1}}}{n}) \|^{n-1-j}.
\]

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For $0 \leq j \leq n-1$,

$$
\left\| \mathcal{E}\left(\frac{\sqrt{2(n-1)}}{n} h_1 \right) \right\|^{n-1-j} = \exp\left\{ \frac{n-1}{n^2} (n-1-j) |h_1|^2 \right\} \\
\leq \exp\left\{ \frac{(n-1)^2}{n^2} |h_1|^2 \right\} \\
\leq \exp\{|h_1|^2\}.
$$

Therefore

$$
\left\| \Gamma\left(\frac{1}{n}\right) X^{\circ n} - \mathcal{E}(h_1) \right\| \leq e^{|h_1|^2} \left\| \Gamma\left(\frac{\sqrt{2}}{n}\right) X - \mathcal{E}\left(\frac{\sqrt{2} h_1}{n}\right) \right\| \cdot \sum_{j=0}^{n-1} \left\| \Gamma\left(\frac{\sqrt{2(n-1)}}{n} \right) X \right\|^j \\
= e^{|h_1|^2} \left\| \Gamma\left(\frac{\sqrt{2}}{n}\right) X - \mathcal{E}\left(\frac{\sqrt{2} h_1}{n}\right) \right\| \left\| \Gamma\left(\frac{\sqrt{2(n-1)}}{n} \right) X \right\|^{n-1}.
$$

Now,

$$
\lim_{n \to \infty} \left\| \Gamma\left(\frac{\sqrt{2(n-1)}}{n} \right) X \right\|^n = \lim_{n \to \infty} \left( 1 + \frac{2(n-1)}{n^2} |h_1|^2 + o\left(\frac{1}{n}\right) \right)^\frac{n}{2} \\
= \lim_{n \to \infty} \left( 1 + \frac{2}{n} |h_1|^2 + o\left(\frac{1}{n}\right) \right)^\frac{n}{2} \\
= \exp\{|h_1|^2\},
$$

and

$$
\lim_{n \to \infty} \frac{\left\| \Gamma\left(\frac{\sqrt{2}}{n}\right) X - \mathcal{E}\left(\frac{\sqrt{2} h_1}{n}\right) \right\|}{\left\| \Gamma\left(\frac{\sqrt{2(n-1)}}{n} \right) X \right\| - 1} = \lim_{n \to \infty} \frac{\left( 2! \frac{4}{n^4} |h_2 - h_1^\circ|^2 + o\left(\frac{1}{n}\right) \right)^\frac{1}{2}}{\left( 1 + \frac{2(n-1)}{n^2} |h_1|^2 + o\left(\frac{1}{n}\right) \right)^\frac{1}{2} - 1} \\
= 0.
$$

Hence we can conclude that

$$
\lim_{n \to \infty} \left\| \Gamma\left(\frac{1}{n}\right) X^{\circ n} - \mathcal{E}(h_1) \right\| = 0.
$$

\[\square\]

**Remark 3.2** According to the observation of Remark 2.2 the statement of Theorem 3.1 can be formulated as follows:

$$
\lim_{n \to \infty} \frac{P_{\log n} X^{\circ n}}{E[X]^n} = \exp\left\{ \int_0^T h_1(s) dB_s - \frac{1}{2} \int_0^T h_1^2(s) ds \right\}.
$$
Theorem 3.1 assumes that $E[X] \neq 0$. The case of zero mean random variables is treated in the following theorem.

**Theorem 3.3** Let $X \in \mathcal{L}^2(\mathcal{P})$ with $E[X] = 0$. Assume that we can find a sequence of real numbers $\{a_n\}_{n \geq 1}$ such that

$$\Gamma(a_n)X^{\circ n} \in \mathcal{L}^2(\mathcal{P}) \text{ for any } n \geq 1,$$

and

$$\lim_{n \to \infty} \Gamma(a_n)X^{\circ n}$$

exists in the strong topology of $\mathcal{L}^2(\mathcal{P})$. Then the limit must be zero.

**Proof.** Suppose there exist a sequence of real numbers $\{a_n\}_{n \geq 1}$ and $Z \in \mathcal{L}^2(\mathcal{P})$, $Z \neq 0$ such that

$$\Gamma(a_n)X^{\circ n} \in \mathcal{L}^2(\mathcal{P}) \text{ for any } n \geq 1,$$

and

$$\lim_{n \to \infty} \left\| \Gamma(a_n)X^{\circ n} - Z \right\| = 0.$$

Define

$$n_0 := \min\{n \geq 0 : z_n \neq 0\},$$

where $z_0 = E[X]$ and for $n \geq 1$, $z_n \in \mathcal{L}^2([0, T])$ is the $n$-th order kernel in the Wiener-Itô chaos decomposition of $Z$.

Since $Z$ is the strong $\mathcal{L}^2(\mathcal{P})$-limit of the sequence $\Gamma(a_n)X^{\circ n}$, it is also its weak limit, that means

$$\lim_{n \to \infty} E[\Gamma(a_n)X^{\circ n}U] = E[ZU],$$

for all $U \in \mathcal{L}^2(\mathcal{P})$.

Since $E[X] = 0$ the first non zero term in the Wiener-Itô chaos decomposition of $\Gamma(a_n)X^{\circ n}$ is at least of order $n$ and therefore for any $n > n_0$ we have

$$E[\Gamma(a_n)X^{\circ n}I_{n_0}(z_{n_0})] = 0,$$

(by the orthogonality of homogenous chaos of different orders) and hence

$$\lim_{n \to \infty} E[\Gamma(a_n)X^{\circ n}I_{n_0}(z_{n_0})] = 0.$$
On the other hand

\[ E[ZI_{n_0}(z_{n_0})] = n_0!|z_{n_0}|^2 > 0, \]

by the definition of \( n_0 \). This means that \( Z \) is not the weak limit of the sequence \( \Gamma(a_n)X^{on} \). This contradiction completes the proof.

\[ \square \]

If \( \{X_n\}_{n \geq 1} \) is a sequence of random variables, by the symbol

\[ X_n \Rightarrow X \text{ as } n \to \infty, \]

we mean that the sequence \( \{X_n\}_{n \geq 1} \) converges in distribution as \( n \) goes to infinity to the random variable \( X \).

Since convergence in \( L^2(\mathcal{P}) \) is stronger than convergence in distribution we have the following result.

**Corollary 3.4** Let \( X \in L^2(\mathcal{P}) \) with \( E[X] \neq 0 \) and denote by \( h_1 \in L^2([0,T]) \) the first-order kernel in the chaos decomposition of \( X \). Then for any \( n \geq 2 \) the distribution of the random variable

\[ \Gamma \left( \frac{1}{n} \right) X^{on}, \]

is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \). Moreover

\[ \frac{\Gamma \left( \frac{1}{n} \right) X^{on}}{E[X]^{n}} \Rightarrow Y \text{ as } n \to \infty, \]

where \( Y \) is either a log-normal random variable, more precisely \( \ln Y \) is a normal random variable with mean \(-\frac{1}{2}|h_1|^2_{L^2([0,T])}\) and variance \(|h_1|^2_{L^2([0,T])}\) or \( Y = 1 \).

**Proof.** Observe that

\[ \Gamma \left( \frac{1}{n} \right) X^{on} = \Gamma \left( \frac{1}{\sqrt{n}} \right) \Gamma \left( \frac{1}{\sqrt{n}} \right) X^{on} \]

\[ = \Gamma \left( \frac{1}{\sqrt{n}} \right) Z \]

where we set \( Z := \Gamma \left( \frac{1}{\sqrt{n}} \right) X^{on} \). From Theorem 2.1 we know that \( Z \in L^2(\mathcal{P}) \) and therefore \( \Gamma \left( \frac{1}{n} \right) X^{on} \) can be written as the image through the operator \( \Gamma \left( \frac{1}{\sqrt{n}} \right) \) of a square integrable random variable. By Theorem 4.24 in [7] this implies the absolute continuity of the distribution of \( \Gamma \left( \frac{1}{n} \right) X^{on} \).
The rest of the proof follows from Theorem 3.1 and the fact that if \( h_1 \neq 0 \) then \( \int_0^T h_1(s) dB_s \) is a zero mean gaussian random variable with variance \( |h_1|^2_{L^2([0,T])} \).

If \( X \) is assumed to be non negative then Corollary 3.4 can be reformulated as follows.

**Corollary 3.5** Let \( X \in \mathcal{L}^2(\mathcal{P}) \) be a non negative random variable with \( \mathcal{P}(X > 0) > 0 \) and denote by \( h_1 \in \mathcal{L}^2([0,T]) \) the first-order kernel in the chaos decomposition of \( X \). Then for any \( n \geq 2 \),

\[
\mathcal{P}\left( \Gamma\left(\frac{1}{n}\right) X^{\otimes n} > 0 \right) = 1,
\]

and

\[
\ln \Gamma\left(\frac{1}{n}\right) X^{\otimes n} - n \ln E[X] \Rightarrow Z \text{ as } n \to \infty,
\]

where \( Z \) is either a normal random variable with mean \(-\frac{1}{2}|h_1|^2_{L^2([0,T])}\) and variance \(|h_1|^2_{L^2([0,T])}\) or \( Z = 0 \).

**Proof.** The second assertion is a straightforward consequence of Corollary 3.4 since convergence in distribution is preserved under the action of continuous functions.

We have to prove that for any \( n \geq 2 \),

\[
\mathcal{P}\left( \Gamma\left(\frac{1}{n}\right) X^{\otimes n} > 0 \right) = 1.
\]

Let us write as before

\[
\Gamma\left(\frac{1}{n}\right) X^{\otimes n} = \Gamma\left(\frac{1}{\sqrt{n}}\right) Z,
\]

where

\[
Z = \Gamma\left(\frac{1}{\sqrt{n}}\right) X^{\otimes n}.
\]

We want to prove that \( Z \) is non negative; according to Theorem 4.1 in [11] this is equivalent to prove that the function

\[
h \in \mathcal{L}^2([0,T]) \mapsto E[Z \mathcal{E}(ih)] e^{-\frac{i}{2}|h|^2_{L^2([0,T])}} \in \mathbb{C},
\]

where \( i \) is the imaginary unit, is positive definite. Since \( \Gamma\left(\frac{1}{\sqrt{n}}\right) \) is self-adjoint in \( \mathcal{L}^2(\mathcal{P}) \) and for any \( h \in \mathcal{L}^2([0,T]), \)

\[
E[(X \diamond Y) \mathcal{E}(h)] = E[X \mathcal{E}(h)] E[Y \mathcal{E}(h)],
\]

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we get

\[
E[Z\mathcal{E}(ih)]e^{-\frac{1}{2}\frac{|h|^2}{L^2([0,T])}} = E\left[\Gamma\left(\frac{1}{\sqrt{n}}\right)X^{\square n}\mathcal{E}(ih)\right]e^{-\frac{1}{2}\frac{|h|^2}{L^2([0,T])}}
\]

\[
= E\left[X^{\square n}\mathcal{E}\left(\frac{ih}{\sqrt{n}}\right)\right]e^{-\frac{1}{2}\frac{|h|^2}{L^2([0,T])}}
\]

\[
= \left(E\left[X\mathcal{E}\left(\frac{ih}{\sqrt{n}}\right)\right]\right)^n e^{-\frac{1}{2}\frac{|h|^2}{L^2([0,T])}}
\]

Since \(X\) is by assumption non negative, the function

\[
E\left[X\mathcal{E}\left(\frac{ih}{\sqrt{n}}\right)\right] e^{-\frac{1}{2n}\frac{|h|^2}{L^2([0,T])}}
\]

is positive definite as well as its \(n\)-th power, proving the non negativity of \(Z\).

The image through the operator \(\Gamma\left(\frac{1}{\sqrt{n}}\right)\) of a non negative random variable is in virtue of Corollary 4.29 in [7] a \(\mathcal{P}\)-a.s. positive random variable. Since \(X = \Gamma\left(\frac{1}{\sqrt{n}}\right)Z\) the previous assertion applies to \(X\) and completes the proof.

\[\Box\]

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