NEW INEQUALITIES FOR OPERATOR CONCAVE FUNCTIONS INVOLVING POSITIVE LINEAR MAPS

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ABSTRACT. The purpose of this paper is to present some general inequalities for operator concave functions which include some known inequalities as a particular case. Among other things, we prove that if $A \in \mathcal{B}(\mathcal{H})$ is a positive operator such that $mI \leq A \leq MI$ for some scalars $0 < m < M$ and $\Phi$ is a normalized positive linear map on $\mathcal{B}(\mathcal{H})$, then

$$\left(\frac{M + m}{2\sqrt{Mm}}\right)^r \geq \left(\frac{\frac{1}{\sqrt{Mm}} \Phi(A) + \sqrt{Mm} \Phi(A^{-1})}{2}\right)^r \geq \frac{1}{(Mm)^{\frac{r}{2}}} \Phi(A)^r + (Mm)^{\frac{r}{2}} \Phi(A^{-1})^r \geq \Phi(A)^r + (Mm)^{\frac{r}{2}} \Phi(A^{-1})^r,$$

where $0 \leq r \leq 1$, which nicely extend the operator Kantorovich inequality.

1. Introduction and preliminaries

In this paper we consider operator monotone and convex functions defined on the half real line $(0, \infty)$. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space and $I$ denote the identity operator. If $A$ is an operator then we denote $\text{Sp}(A)$ its spectrum. An operator $A$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $A \geq 0$. By $B \geq A$ we mean that $B - A$ is positive, while $B > A$ means that $B - A$ is strictly positive. A mapping $\Phi$ on $\mathcal{B}(\mathcal{H})$ is said to be positive if $\Phi(A) \geq 0$ for each $A \geq 0$ and is called normalized if $\Phi$ preserves the identity operator.

For any strictly positive operator $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0, 1]$, we write

$$A \triangledown_v B := (1 - v) A + vB \quad \text{and} \quad A^{\frac{\#}{v}} B := A^{\frac{1}{v}} A^{-\frac{1}{v}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^v A^{\frac{1}{v}} A^{\frac{\#}{v}}.$$

For the case $v = \frac{1}{2}$, we write $\triangledown$ and $\sharp$, respectively. The operator arithmetic-geometric mean inequality (in short, AM-GM inequality) asserts that $A^{\frac{\#}{v}} B \leq A \triangledown_v B$, for any positive operators $A, B \in \mathcal{B}(\mathcal{H})$ and any $v \in [0, 1]$. A real valued function $f$ defined on an interval $J$ is said to be operator convex (resp. operator concave) if $f(A \triangledown_v B) \leq f(A) \triangledown_v f(B)$ (resp. $f(A \triangledown_v B) \geq f(A) \triangledown_v f(B)$) for all self-adjoint operators $A, B$ with spectra in $J$ and all $v \in [0, 1]$. A continuous real valued function $f$ defined on an interval $J$ is called operator monotone (more precisely, operator monotone increasing) if $B \geq A$.
implies that \( f(B) \geq f(A) \), and operator monotone decreasing if \( B \geq A \) implies \( f(B) \leq f(A) \) for all self-adjoint operators \( A, B \) with spectra in \( J \).

During the past decades several formulations, extensions or refinements of the Kantorovich inequality \([7]\) in various settings have been introduced by many mathematicians; see \([6, 8, 9, 11]\) and references therein.

Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive operator such that \( mI \leq A \leq MI \) for some scalars \( 0 < m < M \) and \( \Phi \) be a normalized positive linear map on \( \mathcal{B}(\mathcal{H}) \), then

\[
\Phi \left( A^{-1} \right) \geq \Phi \left( A \right) \leq \frac{M + m}{2\sqrt{Mm}}.
\]

In addition

\[
\Phi \left( A \right) \geq \frac{M + m}{2\sqrt{Mm}} \Phi \left( A \right),
\]

whenever \( m^2A \leq B \leq M^2A \) and \( 0 < m < M \). The first inequality goes back to Nakamoto and Nakamura in the 1996’s \([12]\), the second is more general and has been proved only in 2009 by Lee \([5]\) (its matrix version).

In Sec. 2, we first extend (1.2), then as an application, we obtain a generalization of (1.1). In Sec. 3, we use elementary operations and give some inequalities related to the Bellman type.

### 2. Some operator inequalities involving positive linear map

We prove the following new result, from which (1.2) directly follows:

**Theorem 2.1.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be two strictly positive operators such that \( m_1^2I \leq A \leq M_1^2I \), \( m_2^2I \leq B \leq M_2^2I \) for some positive scalars \( m_1 < M_1, m_2 < M_2 \), and let \( \Phi \) be a normalized positive linear map on \( \mathcal{B}(\mathcal{H}) \). If \( f \) is an operator monotone, then

\[
f \left( \frac{M + m}{2} \right) \Phi \left( A \right) \geq f \left( \frac{Mm \Phi \left( A \right) + \Phi \left( B \right)}{2} \right)
\]

\[
\geq f \left( Mm \Phi \left( A \right) \right) + f \left( \Phi \left( B \right) \right)
\]

\[
\geq f \left( Mm \Phi \left( A \right) \right) \geq f \left( \Phi \left( B \right) \right),
\]

where \( m = \frac{m_2}{M_1} \) and \( M = \frac{M_2}{m_1} \).

**Proof.** According to the assumption, we have

\[
mI \leq \left( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^{\frac{1}{2}} \leq MI,
\]

it follows that

\[
(M + m) \left( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^{\frac{1}{2}} \geq MmI + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}.
\]

The above inequality then implies

\[
\left( \frac{M + m}{2} \right) A \cdot B \geq \frac{MmA + B}{2}.
\]
Using the hypotheses made about $\Phi$,

$$\left(\frac{M + m}{2}\right) \Phi (A \sharp B) \geq \frac{Mm\Phi (A) + \Phi (B)}{2}.$$ 

Thus we have

$$f \left( \left(\frac{M + m}{2}\right) \Phi (A \sharp B) \right) \geq f \left( \frac{Mm\Phi (A) + \Phi (B)}{2} \right) \quad \text{(since $f$ is operator monotone)}$$

$$\geq f (Mm\Phi (A)) + f (\Phi (B)) \quad \text{(by [2, Corollary 1.12])}$$

$$\geq f (Mm\Phi (A)) \sharp f (\Phi (B)) \quad \text{(by AM-GM inequality)},$$

which is the statement of the theorem. □

We complement Theorem 2.1 by proving the following.

**Theorem 2.2.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be two strictly positive operators such that $m_1^2 I \leq A \leq M_1^2 I$, $m_2^2 I \leq B \leq M_2^2 I$ for some scalars $m_1 < M_1$, $m_2 < M_2$, and let $\Phi$ be a normalized positive linear map on $\mathcal{B}(\mathcal{H})$. If $g$ is an operator monotone decreasing, then

$$g \left( \left(\frac{M + m}{2}\right) \Phi (A \sharp B) \right) \leq g \left( \frac{Mm\Phi (A) + \Phi (B)}{2} \right)$$

$$\leq \left\{ \frac{g(Mm\Phi (A))^{-1} + g(\Phi (B))^{-1}}{2} \right\}^{-1}$$

$$\leq g (Mm\Phi (A)) \sharp g (\Phi (B)),$$

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$.

**Proof.** Since $g$ is operator monotone decreasing on $(0, \infty)$, so $\frac{1}{g}$ is operator monotone on $(0, \infty)$. Now by applying Theorem 2.1 for $f = \frac{1}{g}$, we have

$$g \left( \left(\frac{M + m}{2}\right) \Phi (A \sharp B) \right)^{-1} \geq g \left( \frac{Mm\Phi (A) + \Phi (B)}{2} \right)^{-1}$$

$$\geq g (Mm\Phi (A))^{-1} + g(\Phi (B))^{-1}$$

$$\geq g (Mm\Phi (A))^{-1} \sharp g(\Phi (B))^{-1}.$$ 

Taking the inverse, we get

$$g \left( \left(\frac{M + m}{2}\right) \Phi (A \sharp B) \right) \leq g \left( \frac{Mm\Phi (A) + \Phi (B)}{2} \right)$$

$$\leq \left\{ \frac{g(Mm\Phi (A))^{-1} + g(\Phi (B))^{-1}}{2} \right\}^{-1}$$

$$\leq \left\{ g(Mm\Phi (A))^{-1} \sharp g(\Phi (B))^{-1} \right\}^{-1}$$

$$= g (Mm\Phi (A)) \sharp g (\Phi (B)),$$

proving the main assertion of the theorem. □
As a byproduct of Theorems 2.1 and 2.2, we have the following result.

**Corollary 2.1.** Under the assumptions of Theorem 2.1.

(i) If $0 \leq r \leq 1$, then

$$
\left( \frac{M + m}{2\sqrt{Mm}} \right)^r \Phi(A^\#B)^r \geq \left( \frac{Mm\Phi(A) + \Phi(B)}{2\sqrt{Mm}} \right)^r
$$

$$
\geq \frac{(Mm)^r\Phi(A)^r + \Phi(B)^r}{2(Mm)^{\frac{r}{2}}}
$$

$$
\geq \Phi(A)^r \Phi(B)^r.
$$

The important special case

$$
\frac{M + m}{2\sqrt{Mm}} \Phi(A^\#B) \geq \frac{Mm\Phi(A) + \Phi(B)}{2\sqrt{Mm}} \geq \Phi(A^\#B),
$$

was observed by Moslehian et al. [11] (see [9, Theorem 2.5] for much stronger result).

(ii) If $-1 \leq r \leq 0$, then

$$
\left( \frac{M + m}{2\sqrt{Mm}} \right)^r \Phi(A^\#B)^r \leq \left( \frac{Mm\Phi(A) + \Phi(B)}{2\sqrt{Mm}} \right)^r
$$

$$
\leq \frac{1}{(Mm)^{\frac{r}{2}}} \left\{ \frac{(Mm)^{-r}\Phi(A)^{-r} + \Phi(B)^{-r}}{2} \right\}^{-1}
$$

$$
\leq \Phi(A)^r \Phi(B)^r.
$$

Our next result is a straightforward application of Theorems 2.1 and 2.2.

**Corollary 2.2.** Let $A \in \mathcal{B}(\mathcal{H})$ be positive operator such that $mI \leq A \leq MI$ for some scalars $0 < m < M$ and $\Phi$ be a normalized positive linear map on $\mathcal{B}(\mathcal{H})$.

(i) If $f$ is an operator monotone, then

$$
f \left( \frac{M + m}{2Mm} \right) \geq f \left( \frac{1}{Mm}\Phi(A) + \Phi(A^{-1}) \right)
$$

$$
\geq \frac{f \left( \frac{1}{Mm}\Phi(A) \right) + f \left( \Phi(A^{-1}) \right)}{2}
$$

$$
\geq f \left( \frac{1}{Mm}\Phi(A) \right)^\# f \left( \Phi(A^{-1}) \right).
$$

(ii) If $g$ is an operator monotone decreasing, then

$$
g \left( \frac{M + m}{2Mm} \right) \leq g \left( \frac{1}{Mm}\Phi(A) + \Phi(A^{-1}) \right)
$$

$$
\leq \left\{ g \left( \frac{1}{Mm}\Phi(A) \right)^{-1} + g \left( \Phi(A^{-1}) \right)^{-1} \right\}^{-1}
$$

$$
\leq g \left( \frac{1}{Mm}\Phi(A) \right)^\# g \left( \Phi(A^{-1}) \right).
$$

In the same vein as in Corollary 2.1, we have the following consequences.
Corollary 2.3. Under the assumptions of Corollary 2.2.

(i) If $0 \leq r \leq 1$, then

$$
\left( \frac{M + m}{2\sqrt{Mm}} \right)^r \geq \left( \frac{\frac{1}{\sqrt{Mm}} \Phi(A) + \sqrt{Mm} \Phi(A^{-1})}{2} \right)^r
$$

$$
\geq \frac{(Mm)^{\frac{r}{2}} \Phi(A)^r + (Mm)^{\frac{r}{2}} \Phi(A^{-1})^r}{2}
$$

$$
\geq \Phi(A)^r \# \Phi(A^{-1})^r.
$$

For the special case in which $r = 1$, we have

$$
\frac{M + m}{2\sqrt{Mm}} \geq \frac{\frac{1}{\sqrt{Mm}} \Phi(A) + \sqrt{Mm} \Phi(A^{-1})}{2} \geq \Phi(A) \# \Phi(A^{-1}).
$$

(ii) If $-1 \leq r \leq 0$, then

$$
\left( \frac{M + m}{2\sqrt{Mm}} \right)^r \leq \left( \frac{\frac{1}{\sqrt{Mm}} \Phi(A) + \sqrt{Mm} \Phi(A^{-1})}{2} \right)^r
$$

$$
\leq \left\{ \frac{(Mm)^{\frac{r}{2}} \Phi(A)^{-r} + \Phi(A^{-1})^{-r}}{2(Mm)^{\frac{r}{2}}} \right\}^{-1}
$$

$$
\leq \Phi(A)^r \# \Phi(A^{-1})^r.
$$

3. Operator Bellman inequality with negative parameter

Let $A, B \in \mathcal{B} (\mathcal{H})$ be two strictly positive operators and $\Phi$ be a normalized positive linear map on $\mathcal{B} (\mathcal{H})$. If $f$ is an operator concave, then for any $v \in [0, 1]$, the following inequality obtained in [10, Theorem 2.1]:

$$
\Phi (f (A)) \nabla_v \Phi (f (B)) \leq f (\Phi (A \nabla_v B)).
$$

In the same paper, as an operator version of Bellman inequality [3], the authors showed that

$$
\Phi ((I - A)^r \nabla_v (I - B)^r) \leq \Phi (I - A \nabla_v B)^r,
$$

where $A, B$ are two operator contractions (in the sense that $\|A\|, \|B\| \leq 1$) and $r, v \in [0, 1]$.

Under the convexity assumption on $f$, (3.2) can be reversed:

**Theorem 3.1.** Let $A, B \in \mathcal{B} (\mathcal{H})$ be two contraction operators and $\Phi$ be a normalized positive linear map on $\mathcal{B} (\mathcal{H})$. Then

$$
\Phi (I - A \nabla_v B)^r \leq \Phi ((I - A)^r \nabla_v (I - B)^r),
$$

for any $v \in [0, 1]$ and $r \in [-1, 0] \cup [1, 2]$. 
Proof. If $f$ is operator convex, we have
\[ f(\Phi(A\nabla_v B)) \leq \Phi(f(A\nabla_v B)) \]
(by Choi-Davis-Jensen inequality [4, p. 62])
\[ \leq \Phi(f(A)\nabla_v f(B)) \]
(by operator convexity of $f$).

The function $f(t) = t^r$ is operator convex on $(0, \infty)$ for $r \in [-1, 0] \cup [1, 2]$ (see [4, Chapter 1]). It can be verified that $f(t) = (1 - t)^r$ is operator convex on $(0, 1)$ for $r \in [-1, 0] \cup [1, 2]$. This implies the desired result (3.3). \qed

However, we are looking for something stronger than (3.3). The principal object of this section is to prove the following:

**Theorem 3.2.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be two contraction operators and $\Phi$ be a normalized positive linear map on $\mathcal{B}(\mathcal{H})$. Then
\[ \Phi(I - A\nabla_v B)^r \leq \Phi((I - A)^r\#_v (I - B)^r) \]
\[ \leq \Phi((I - A)^r\nabla_v (I - B)^r), \]
where $v \in [0, 1]$ and $r \in [-1, 0]$.

The proof is at the end of this section. The following lemma will play an important role in our proof.

**Lemma 3.1.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be two strictly positive operators and $\Phi$ be a normalized positive linear map on $\mathcal{B}(\mathcal{H})$. If $f$ is an operator monotone decreasing, then for any $v \in [0, 1]$
\[ f(\Phi(A\nabla_v B)) \leq f(\Phi(A))\#_v f(\Phi(B)) \leq \Phi(f(A)\nabla_v f(B)), \]
and
\[ f(\Phi(A\nabla_v B)) \leq \Phi(f(A))\#_v f(B) \leq \Phi(f(A)\nabla_v f(B)). \]

More precisely,
\[ f(\Phi(A\nabla_v B)) \leq f(\Phi(A))\#_v f(\Phi(B)) \leq \Phi(f(A))\#_v f(B) \leq \Phi(f(A)\nabla_v f(B)). \]

**Proof.** As Ando and Hiai mentioned in [2, (2.16)], the function $f$ is an operator monotone decreasing if and only if
\[ f(A\nabla_v B) \leq f(A)\#_v f(B). \]

We emphasize here that if $f$ satisfies in (3.7), then is operator convex (this class of functions is called operator log-convex). It is easily verified that if $Sp(A), Sp(B) \subseteq J$, then $Sp(\Phi(A)), Sp(\Phi(B)) \subseteq J$. So we can replace $A, B$ by $\Phi(A), \Phi(B)$ in (3.7), respectively. Therefore we can write
\[ f(\Phi(A\nabla_v B)) \]
\[ \leq f(\Phi(A))\#_v f(\Phi(B)) \]
\[ \leq \Phi(f(A))\#_v f(B) \]
(by Choi-Davis-Jensen inequality and monotonicity property of mean)
\[ \leq \Phi(f(A)\nabla_v f(B)) \]
(by AM-GM inequality).
This completes the proof of the inequality (3.4). To prove the inequality (3.5), note that if $Sp(A), Sp(B) \subseteq J$, then $Sp(A \nabla_v B) \subseteq J$. By computation
\[
f(\Phi(A \nabla_v B)) \leq \Phi(f(A \nabla_v B)) \quad \text{(by Choi-Davis-Jensen inequality)}
\]
\[
\leq \Phi(f(A)^{\#} f(B)) \quad \text{(by (3.7))}
\]
\[
\leq \Phi(f(A))^{\#} \Phi(f(B)) \quad \text{(by Ando’s inequality [1, Theorem 3])}
\]
\[
\leq \Phi(f(A) \nabla_v f(B)) \quad \text{(by AM-GM inequality)},
\]
proving the inequality (3.5). We know that if $g$ is operator monotone on $(0, \infty)$, then $g$ is operator concave. As before, it can be shown that
\[
g(\Phi(A))^{\#} g(\Phi(B)) \geq \Phi(g(A))^{\#} \Phi(g(B)) \geq \Phi(g(A)^{\#} g(B)).
\]
Taking the inverse, we get
\[
g(\Phi(A))^{-1} g(\Phi(B))^{-1} \leq \Phi(g(A))^{-1} g(\Phi(B))^{-1} \leq \Phi\left(\frac{g(A) - 1}{g(A)}^{\#} g(B) - 1\right).
\]
If $g$ is operator monotone, then $f = \frac{1}{g}$ is operator monotone decreasing, we conclude
\[
f(\Phi(A))^{\#} f(\Phi(B)) \leq \Phi(f(A))^{\#} f(B).
\]
This proves (3.6). \hfill \Box

We are now in a position to present a proof of Theorem 3.2.

Proof of Theorem 3.2. It is well-known that the function $f(t) = t^r$ on $(0, \infty)$ is operator monotone decreasing for $r \in [-1, 0]$. It implies that the function $f(t) = (1 - t)^r$ on $(0, 1)$ is operator monotone decreasing too. By applying Lemma 3.1, we get the desired result. \hfill \Box

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