Floating Bodies of Equilibrium.
Explicit Solution

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Abstract
Explicit solutions of the two-dimensional floating body problem (bodies that can float in all positions) for relative density \( \rho\neq \frac{1}{2} \) and of the tire track problem (tire tracks of a bicycle, which do not allow to determine, which way the bicycle went) are given, which differ from circles. Starting point is the differential equation given in [9, 10].

1 Introduction
In this paper explicit solutions are given to the two-dimensional version of the floating body problem asked by Stanislaw Ulam in the Scottish Book[5] (problem 19): Is a sphere the only solid of uniform density which will float in water in any position? A large class of two-dimensional cross-sections different from the circle were found for bodies of relative density \( \rho = \frac{1}{2} \) by Auerbach[2]. Here we address to the case \( \rho \neq \frac{1}{2} \).

It has been shown[2], that in two dimensions such a body has the property, that any chord dissecting the body in two pieces whose areas are the fractions \( \rho \) and \( 1 - \rho \) of the whole area must have the same constant length \( 2\ell \). Two equivalent properties are: (i) Let us consider two close-by water-lines through the cross-section, \( A_1A_2 \) and \( B_1B_2 \). (We keep the body fixed and assumed the direction of the gravitational force being rotated.) We put the \( x \)-axis parallel to \( A_1A_2 \). Then the vector \( A_1B_1 \) is given by \((dx_1, -l_1d\phi)\) and \( A_2B_2 \) by \((dx_2, l_2d\phi)\). Constant length \( d\ell = 0 \) implies \( dx_2 = dx_1 \). Constant areas implies \( df_1 = \frac{1}{2}l_2^2d\phi = df_2 = \frac{1}{2}l_1^2d\phi \). Thus \( l_1 = l_2 = \ell \). This implies, that the infinitesimal arcs at the perimeter \( du_1 = \sqrt{(dx_1)^2 + l_1^2(d\phi)^2} \) and \( du_2 = \sqrt{(dx_2)^2 + l_2^2(d\phi)^2} \) are equal. Thus the part of the perimeter below the water-line is constant. Of course the same is true for the part above the water-line. One can conclude the other way round: If a curve has the property, that if we move from two fixed points \( A_1 \) and \( A_2 \) by constant arcs \( u \) along the perimeter and the length of the chord remains fixed, then also the areas separated by the chord stay constant. We argue, that since \( \ell \) stays constant we have \( dx_1 = dx_2 \). Since \( du_1 = du_2 \)
also \( l_1 = l_2 \) and thus \( df_1 = df_2 \). In the main part of the paper we will use this property. By measuring the arc from a fixed point of the boundary we introduce the arc parameter \( u \). We will show that for certain differences \( 2\delta u \) of the arc parameter the length of the chord is constant, that is the distance between the point at arc parameter \( u - \delta u \) and at arc parameter \( u + \delta u \) is constant.

(ii) Another interesting property is, that since \( l_1 = l_2 \), one has \( dy_1 = -dy_2 \), which implies, that the angles \( \delta \) between the tangents and the chord are equal, \( \delta_1 = \delta_2 \).

The floating body problem is related to the tire track problem ([7] and papers cited therein). Let me denote the boundary of the body by \( \Gamma \) and the envelope of the waterlines, which consists actually of the midpoints of the chords, by \( \gamma \). Let the distance between the front and the rear wheel of a bicycle be \( \ell \). Then if the front wheel moves along the curve \( \Gamma \) and the rear wheel started on \( \gamma \), then it will stay on \( \gamma \). Since however any tangent through a point on \( \gamma \) meets \( \Gamma \) in the two ends of the chord, the bicycle can move both ways on the same tracks. The tire track problem consists in finding such curves \( \Gamma, \gamma \) different from circles, which thus do not allow to determine which way the bicycle went. It is equivalent to the two-dimensional floating body problem with the exception that the tire tracks need not close or may wind several times around some point.

In a recent paper I suggested that one obtains a solution for the boundary of the floating body from the differential equation

\[
\frac{1}{\sqrt{r'^2 + cr^{-2}}} = ar^2 + b + cr^{-2},
\]

where \( r' = dr/d\psi \), with polar coordinates \((r, \psi)\), provided the resulting curve is sufficiently convex and closed.

By choosing an appropriate condition for the constants \( a, b, \) and \( c \) the curves will be closed. They show dihedral \( D_p \) symmetry. These curves have the remarkable property, that they solve the flotation problem for \( p - 2 \) different densities, which implies that there are \([\frac{p-1}{2}]\) different chord lengths. (If the boundary is the solution for relative density \( \rho \) of the body, then it is also solution for density \( 1 - \rho \) with the same chord length.) These curves show an even more general property: Let us consider two copies of these curves (not necessarily closed). Let us choose an arbitrary point on each curve. Then there exists always an angle by which the two curves can be rotated against each other, so that the length of the chord between the points on the two curves stays constant, if we move from the given points by the same arc. This remarkable property made it possible to obtain the above-mentioned differential equation by choosing the two points infinitesimally close [9, 10].

In [9, 10] I assumed to have proven, that the solutions have this property, but Serge Tabachnikov kindly informed me, that this was not correct. What I had shown, was that if for some chord the angles \( \delta \) between the chord and the tangents on the curve obey \( \delta_1 = \delta_2 \) at \( r_1 \neq r_2 \), that then also the derivative \( \frac{d(\delta_1 - \delta_2)}{du} \) vanishes. This is a necessary condition, but it is not sufficient. That it is not sufficient, can easily be seen, since from this argument one cannot exclude, that \( \delta_1 - \delta_2 \propto u^2 \). Therefore here the differential equation will be
solved explicitly as a function of the arc parameter \( u \), and it will be shown, that the curve has the desired property.

First we will introduce an appropriate parametrization. In sect. 3 we determine the radius \( r \) as a function of the arc \( u \), in the next section the polar angle also as function of \( u \). In section 5 we determine the length of the chord. Considering the chord between points at arc parameter \( u - \delta u \) and \( u + \delta u \) for fixed \( \delta u \), we find that the square of the length of the chord can be expressed as a rational function of the Weierstrass function \( \wp(u/\lambda) \), (\( \lambda \) is a constant scale factor). For special values of \( \delta u \) the residues vanish and thus the chord length \( 2\ell \) becomes independent of \( u \). In section 6 we consider the limit, where the curve oscillates around a straight line. (They do not constitute solutions of the floating body problem, but of the tiretrack problem.) There one finds an infinite set of \( \delta u \), so that again the length of the chord between the points of arc parameter \( u - \delta u \) and \( u + \delta u \) is independent of \( u \). In the last section we shortly discuss various shapes as they appear for example in the papers by Bracho, Montejo and Oliveros \(^6\) assuming they obey eq. \(^1\).

### 2 Parametrization

From equation \(^1\) one obtains

\[
r'^2 = \frac{r^4}{(ar^4 + br^2 + c)^2} - r^2, \tag{2}
\]

\[
\frac{d\psi}{dr} = \frac{ar^4 + br^2 + c}{r\sqrt{r^2 - (ar^4 + br^2 + c)^2}}, \tag{3}
\]

\[
\frac{d\psi}{dq} = \frac{aq^2 + bq + c}{2q\sqrt{q - (aq^2 + bq + c)^2}}. \tag{4}
\]

with \( q = r^2 \). If we denote the arc along the curve by \( u \) then we obtain

\[
\frac{du}{d\psi} = \sqrt{r'^2 + r^2} = \frac{1}{ar^2 + b + cr^{-2}} = \frac{1}{aq + b + cq^{-1}}, \tag{5}
\]

and

\[
\frac{du}{dq} = \frac{1}{2q\sqrt{q - (aq^2 + bq + c)^2}}. \tag{6}
\]

With increasing \( \psi \) the radius \( r \) oscillates periodically between the largest and the smallest radii \( r_\uparrow \) and \( r_\downarrow \), resp. We parametrize these extreme radii by

\[
r_\uparrow = r_0(1 + \epsilon), \quad r_\downarrow = r_0(1 - \epsilon). \tag{7}
\]

We observe, that the polynomial

\[
f(r) := ar^4 + br^2 + c - r = \frac{r^2}{\sqrt{r^2 + r'^2}} - r \tag{8}
\]

vanishes at \( r = r_\uparrow \) and \( r = r_\downarrow \). Moreover the sum of the zeroes of this polynomial vanishes. Therefore we introduce a third parameter \( \mu \) by writing

\[
f(r) = a(r - r_0(1 + \epsilon))(r - r_0(1 - \epsilon))(r + r_0(1 + i\mu))(r + r_0(1 - i\mu)) \tag{9}
\]
In order to determine $a$, $b$, and $c$, we first expand the polynomial
\[ f(r) = a(r^4 + r^2r_0^2(-2 - \epsilon^2 + \mu^2) - 2rr_0^3(\epsilon^2 + \mu^2) + r_0^4(1 - \epsilon^2)(1 + \mu^2)). \] (10)
Comparing the coefficient of $r$, $r^2$, and $r^0$ we obtain
\[ a = \frac{1}{2rr_0^3(\epsilon^2 + \mu^2)}, \] (11)
\[ b = ar_0^2(-2 - \epsilon^2 + \mu^2), \] (12)
\[ c = ar_0^4(1 - \epsilon^2)(1 + \mu^2). \] (13)
In order to estimate $\mu$ we consider an infinitesimal deformation and approximate in lowest order in $\epsilon$
\[ r = r_0(1 + \epsilon \cos(p\psi)) \] (14)
Then one has at $r = r_0$
\[ \frac{1}{\sqrt{r^2 + r'^2}}|_{r=r_0} = \frac{1}{r_0} \left(1 - \frac{\rho^2\epsilon^2}{2}\right) = \frac{1}{r_0} f(r_0) + \frac{1}{r_0}. \] (15)
Substituting $f(r)$ one obtains
\[ \frac{1}{r_0^2} f(r_0) = -ar_0^2\epsilon^2(4 + \mu^2) = -r_0\rho^2\epsilon^2 \] (16)
which yields
\[ \mu^2 = \frac{4}{p^2 - 1}. \] (17)
Since one is interested in solutions with $p > 1$ one expects real $\mu$.

3 The radius as function of the arc

In order to determine the relation between $u$ and $q$ we have to integrate eq. (6). For this purpose we may write
\[ q - (aq^2 + bq + c)^2 = r^2 - (ar^4 + br^2 + c)^2 \]
\[ = (r - ar^4 - br^2 - c)(r + ar^4 + br^2 + c) = -f(r)f(-r) \]
\[ = -a^2 \prod_{i=1}^{4}(r_i^2 - r^2) = -a^2 \prod_{i=1}^{4}(q - r_i^2), \] (18)
where we have denoted the four zeroes of $f(r)$ by $r_i$. We introduce the parametrization
\[ q = r_0^2 \frac{\alpha t + \beta}{t + 1} \] (19)
with constants $\alpha$ and $\beta$. Then the first two factors are rewritten
\[ (q - r_0^2(1 + \epsilon)^2)(q - r_0^2(1 - \epsilon)^2) \]
\[ = r_0^4 \frac{[(\alpha - (1 + \epsilon)^2)t + \beta - (1 - \epsilon)^2][\alpha - (1 - \epsilon)^2)t + \beta - (1 - \epsilon)^2]}{(t + 1)^2}, \] (20)
We require that the linear term in the numerator vanishes, which yields the equation
\[(\alpha - (1 + \epsilon^2))(\beta - (1 + \epsilon^2)) = 4\epsilon^2.\] (21)

Secondly we rewrite
\[\frac{(q - r_0^2(1 + i\mu)^2)(q - r_0^2(1 - i\mu)^2) = (q - r_0^2(1 - \mu^2))^2 + 4r_0^4\mu^2}{(t + 1)^2} = r_0^4[(\alpha - 1 + \mu^2)t + \beta - 1 + \mu^2]^2 + 4\mu^2(1 + t)^2].\] (22)

The term linear in \(t\) in the numerator shall vanish. Thus we require
\[(\alpha - 1 + \mu^2)(\beta - 1 + \mu^2) = -4\mu^2.\] (23)

Equations (21) and (23) have the solutions
\[\alpha = \pm\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}},\] \[\beta = \mp\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}},\] \[\alpha = \frac{\epsilon^2 + \mu^2}{2} \pm 2\epsilon,\] \[\alpha + \alpha = 2\left(\frac{\epsilon^2 + \mu^2}{2}\right)^2 \pm 4\mu^2.\] (26) (27) (28)

Thus we obtain
\[q - (aq^2 + bq + c)^2 = \frac{a^2r_0^8\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}{(1 + t)^4}(A - Bt^2)(C + Dt^2)\] (29)

with
\[A = \frac{(\beta - (1 + \epsilon^2))(\beta - (1 - \epsilon^2))}{\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}} = \mp(\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}},\] \[B = \frac{(\alpha - (1 + \epsilon^2))(\alpha - (1 - \epsilon^2))}{\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}} = \mp(\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}},\] \[C = \frac{(\beta - 1 + \mu^2)^2 + 4\mu^2}{\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}} = 2(\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}),\] \[D = \frac{(\alpha - 1 + \mu^2)^2 + 4\mu^2}{\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}} = 2(\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}).\] (30) (31) (32) (33)

We have incorporated the over-all minus sign in eq. (18) in the coefficients \(A\) and \(B\). In the following we choose the lower signs; then all four coefficients \(A\) to \(D\) are positive. The differential yields
\[dq = r_0^2\frac{\alpha - \beta}{(t + 1)^2}dt = -2r_0^2\sqrt{\alpha + \alpha - 1 + \frac{\epsilon^2 - \mu^2}{2}}dt.\] (34)

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Thus we obtain
\[ du = \frac{-2r_0(\epsilon^2 + \mu^2)dt}{\sqrt{(A - Bt^2)(C + Dt^2)}} \]  
(35)

Then \( t \) runs between \(-\sqrt{A/B}\) and \(+\sqrt{A/B}\). Integration yields

\[ -\frac{u}{\lambda} = F(\sqrt{\frac{A - Bt^2}{A}}; k), \]  
(36)

with the elliptic integral of the first kind

\[ F(\sin \phi; k) := \int_0^{\sin \phi} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}} = F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \]  
(37)

(Note the distinction between \( F(;) \) and \( F(,;) \)) and

\[ k = \sqrt{\frac{AD}{AD + BC}}, \quad \lambda = 2r_0(\epsilon^2 + \mu^2) \sqrt{\frac{A}{AD + BC}} \]  
(38)

\( A - Bt^2 \) can be expressed

\[ A - Bt^2 = \frac{\sqrt{4\alpha_+\alpha_-((1 + \epsilon)^2 - \frac{q}{r_0^2}))(\frac{q}{r_0^2} - (1 - \epsilon^2))}}{\sqrt{\alpha_+\alpha_- - 1 + \frac{\epsilon^2 - \mu^2}{2} + \frac{q}{r_0^2}}}. \]  
(39)

We obtain

\[ \frac{AD}{BC} = 4(\epsilon^2 + \mu^2)\sqrt{\alpha_+\alpha_-} \pm (8(\epsilon^2 - \mu^2) - 2(\epsilon^2 + \mu^2)^2). \]  
(40)

and thus

\[ 1 - 2k^2 = \frac{BC - AD}{BC + AD} = \frac{1}{\sqrt{\alpha_+\alpha_-}} \left( \frac{2(\mu^2 - \epsilon^2)}{\epsilon^2 + \mu^2} + \frac{1}{2}(\epsilon^2 + \mu^2) \right), \]  
(41)

\[ \lambda = \frac{r_0\sqrt{\epsilon^2 + \mu^2}}{\sqrt{2}\sqrt{\alpha_+\alpha_-}}. \]  
(42)

From the solution (36) we obtain

\[ t = \sqrt{\frac{A}{B}} \text{cn}(\tilde{u}, k), \quad \tilde{u} = \frac{u}{\lambda} \]  
(43)

and

\[ q = r_0^2\frac{\alpha\sqrt{A\text{cn}(\tilde{u})} + \beta\sqrt{B}}{\sqrt{A\text{cn}(\tilde{u})} + \sqrt{B}}. \]  
(44)

In the appendix we derive the representation of \( \text{cn} \) in terms of the Weierstrass \( \wp \)-function

\[ \text{cn}(\tilde{u}, k) = \frac{\wp(\tilde{u}, g_2, g_3) - \epsilon_3 - \frac{1}{4}}{\wp(\tilde{u}, g_2, g_3) - \epsilon_3 + \frac{1}{4}}, \]  
(45)

with \( g_2, g_3 \) given in [209, 210] and

\[ \epsilon_3 = \frac{1 - 2k^2}{6}. \]  
(46)
Thus we obtain
\[ \frac{q}{r_0^2} = \frac{2\sqrt{\alpha_+}(1 - \epsilon)^2(\wp(\tilde{u}) - e_3) + \frac{1}{4}\sqrt{\alpha_-}(1 + \epsilon)^2}{2\sqrt{\alpha_+}(\wp(\tilde{u}) - e_3) + \frac{1}{4}\sqrt{\alpha_-}}. \] (47)

The arc is measured from \( \tilde{u} = 0 \) with \( \wp(0) = \infty, \ cn(0) = 1 \) at \( q = r_0^2(1 - \epsilon)^2 \). As \( \tilde{u} \) increases to \( \omega_3 = F(1; k) = F(\frac{\pi}{2}, k) = 2K(k) \) (48)

the minimum of \( \wp(\tilde{u}) \) for real \( \tilde{u} \) is reached, \( \wp(\omega_3) = e_3, \ cn(\omega_3) = -1 \) and \( q = r_0^2(1 + \epsilon)^2 \). Then \( q \) decreases until \( \tilde{u} = 2\omega_3 \), where one period is completed, that is
\[ q(2\omega_3 + \tilde{u}) = q(\tilde{u}). \] (49)

Useful references for elliptic integrals and elliptic functions are for example chapter XIII of [4] and chapters 16 to 18 in [1]. The later reference uses \( m = k^2 \) instead of \( k \). A few relations are listed in the appendix A.1. The elliptic functions are double periodic functions, that is there exist two periods \( 2\omega_i \), the ratio of which is not real, so that
\[ \wp(\tilde{u} + 2\omega_i) = \wp(\tilde{u}). \] (50)

\( 2\omega_3 \) is a real period of the \( \wp \) we use here.

4 The angle as function of the arc
We use the expression for \( q/r_0^2 \) in order to calculate the angle from (5)
\[ \psi(\tilde{u}) = \int du(aq + b + cq^{-1}). \] (51)

Then we have
\[ \frac{q}{r_0^2} = (1 - \epsilon)^2 + \frac{(1 - \epsilon)^2(\wp(v_1) - \wp(v_2))}{\wp(\tilde{u}) - \wp(v_1)}, \] (52)
\[ \frac{r_0^2}{q} = \frac{1}{(1 - \epsilon)^2} + \frac{(1 - \epsilon)^2(\wp(\tilde{u}) - \wp(v_2))}{\wp(v_1) - \wp(v_1)}, \] (53)
\[ \wp(v_1) = e_3 - \frac{1}{4}\sqrt{\alpha_+}. \] (54)
\[ \wp(v_2) = e_3 - \frac{(1 + \epsilon)^2}{4(1 - \epsilon)^2}\sqrt{\alpha_-}. \] (55)

Since \( \wp(v_1) < e_3 \) and \( \wp(v_2) < e_3 \), there are no real solutions \( v_1 \) and \( v_2 \). There are purely imaginary solutions, since \( \wp(z) \) is real along the imaginary axis. \( \wp(z) \)
starts with \( -\infty \) at \( z = 0 \), increases up to \( e_3 \) at a half-period \( \omega' \) and then decreases to \( -\infty \) at \( z = 2\omega' \). We choose the solution in the interval \( 0 < \text{Im} v_i < \omega'/i \). Then \( \wp'(v_i) \) is imaginary and its imaginary part obeys \( \text{Im} \wp'(v_i) < 0 \). Since the invariants \( g_2 \) and \( g_3 \) of \( \wp \) are real, one has \( \wp(\tilde{u}^*) = \wp^*(\tilde{u}) \), where the star indicates the conjugate complex. In particular for real \( \tilde{u} \), there is \( \wp(\tilde{u} - v) = \wp^*(\tilde{u} + v) \), since \( v \) is purely imaginary.
Re z

ω

ω

Im z

0

ω

ω

2ω

Fig. 2. Two elementary cells of periodicity of the function \( \wp(z) \).

The range of the complex plane of \( z \) is plotted for \( 0 \leq \text{Re} \, z \leq 2 \omega_3 \), \( 0 \leq \text{Im} \, z \leq 2 \omega' / i \) in figure 2. It covers two elementary cells for the double-periodic function \( \wp(z) \). The function is real along the straight lines drawn. For constant Im \( z \) (horizontal lines) it varies between \( +\infty \) and \( e_3 \). For constant Re \( z \) (vertical lines) it varies between \( -\infty \) and \( e_3 \). The singularities are at the points indicated by \( \Box \). The derivative of the function \( \wp'(z) \) vanishes at half-periods \( \omega \). \( \omega \) is called a half-period, if it is not a period of \( \wp \), but \( 2 \omega \) is a period. There are three different half-periods which do not differ by periods. They are indicated by \( \Box \), \( + \), and \( - \). Thus one has \( \wp'(\Box) = \wp'(+) = \wp'(-) = 0 \). At these points the function approaches \( \wp(+) = e_1 \), \( \wp(-) = e_2 \), and \( \wp(\Box) = e_3 \). Examples are \( \wp(\omega_1) = \wp(+) = e_1 \) with \( 2\omega_1 = -\omega_3 - \omega' \), \( \wp(\omega_2) = \wp(-) = e_2 \) with \( 2\omega_2 = -\omega_1 + \omega' \), \( \wp(\omega_3) = \wp(\omega') = \wp(\Box) = e_3 \). \( e_1 \) and \( e_2 \) are given in the appendix \[ \text{(200-207)} \]. \( \omega_1 \) and \( \omega_2 \) lie outside the plotted region. The locations of the quarter-periods \( \wp(\omega_3/2) = \wp(\Box) = e_3 + 1/4 \) and \( \wp(\omega'/2) = \wp(\Box) = e_3 - 1/4 \) are also indicated.

The integrals over \( [\omega_2, \omega_3] \) are given by

\[
\int \frac{1}{\wp'(-u) - \wp(v)} \, d\tilde{u} = \frac{1}{\wp'(v)} \left( 2\tilde{u} \zeta(v) + \ln \left( \frac{\sigma(\tilde{u} - v)}{\sigma(\tilde{u} + v)} \right) \right). \tag{56}
\]

Thus we obtain

\[
\psi(\tilde{u}) = \int du (aq + b + cq^{-1}) = c_0 u + c_1 \ln \left( \frac{\sigma(\tilde{u} - v_1)}{\sigma(\tilde{u} + v_1)} \right) + c_2 \ln \left( \frac{\sigma(\tilde{u} - v_2)}{\sigma(\tilde{u} + v_2)} \right). \tag{57}
\]

with

\[
c_0 = \frac{1}{r_0(1 - \epsilon)} + \frac{ar_0^2(1 - \epsilon)^2}{r_0^2(1 - \epsilon)^2} \frac{\wp(v_1) - \wp(v_2)}{\wp'(v_1)} 2\zeta(v_1)
+ \frac{c}{r_0^2(1 - \epsilon)^2} \frac{\wp(v_2) - \wp(v_1)}{\wp'(v_2)} 2\zeta(v_2), \tag{58}
\]

\[
c_1 = \frac{ar_0(1 - \epsilon)^2}{r_0^2(1 - \epsilon)^2} \frac{\lambda}{\wp'(v_1)} \frac{\wp(v_1) - \wp(v_2)}{\wp'(v_1)}, \tag{59}
\]

\[
c_2 = \frac{c}{r_0^2(1 - \epsilon)^2} \frac{\lambda}{\wp'(v_2)} \frac{\wp(v_2) - \wp(v_1)}{\wp'(v_2)}. \tag{60}
\]
We evaluate \( \wp'(v_1) \) and \( \wp'(v_2) \) by means of eq. (212)

\[
\wp'(v_1) = -\frac{\sqrt{\alpha - \epsilon^2}}{2\alpha^{3/2}(\epsilon^2 + \mu^2)} \tag{61}
\]

\[
\wp'(v_2) = \frac{(1 + \epsilon)(1 + \mu^2)}{(1 - \epsilon^2)} \wp'(v_1) \tag{62}
\]

The constants \( c_i \) evaluate to

\[
c_1 = \frac{i}{2}, \quad c_2 = -\frac{i}{2} \tag{63}
\]

independent of \( \epsilon \) and \( \mu \), where we have chosen solutions \( v_1 \) and \( v_2 \), for which \( \text{Im} \wp'(v_i) < 0 \). \( c_0 \) reduces to

\[
c_0 = \frac{1}{r_0(1 - \epsilon)} \frac{\zeta(v_1) - \zeta(v_2)}{i\lambda}. \tag{64}
\]

If we increase \( \tilde{u} \) by the period \( 2\omega_3 \), then \( \psi \) increases by

\[
\psi_{\text{per}} = \psi(\tilde{u} + 2\omega_3) - \psi(\tilde{u}) = 2\omega_3 c_0 \lambda - 2i(v_1 - v_2)\zeta(\omega_3) \tag{65}
\]

\[
= \frac{2\omega_3 \lambda}{r_0(1 - \epsilon)} - 2i \left((v_1 - v_2)\zeta(\omega_3) + \omega_3(\zeta(v_2) - \zeta(v_1))\right).
\]

Thus the curve obeys

\[
\begin{align*}
\psi(\tilde{u} + 2\omega_3) &= \psi(\tilde{u}) + \psi_{\text{per}}, \\
r(\tilde{u} + 2\omega_3) &= r(\tilde{u})
\end{align*} \tag{66}
\]

\[
\begin{align*}
r(-\tilde{u}) &= r(\tilde{u}), \\
\psi(-\tilde{u}) &= -\psi(\tilde{u}).
\end{align*} \tag{67}
\]

**Symmetry**

Some of the equations given above are not explicitly symmetric with respect to the change of the sign of \( \epsilon \). We realize, that \( k^2 \) and \( \lambda \) are invariant against such a sign change. However the expression for the angle \( \psi_{\text{per}} \) of the periodicity of \( r(\psi) \) is not obviously invariant. One can observe however, that the reversal of the sign of \( \epsilon \) corresponds to a change from \( v_i \) to \( v_i' = \omega' - v_i \). To see this we write

\[
\varphi(v) = e_3 - \frac{w}{4}, \tag{68}
\]

and determine

\[
\begin{align*}
\varphi(\omega' - v) &= -\varphi(\omega') - \varphi(v) + \frac{(\varphi'(\omega') + \varphi'(v))^2}{(\varphi(\omega') - \varphi(v))^2} \\
&= -2e_3 + \frac{w}{4} \frac{-w(-3e_3 w/4 + w^2/16 + \frac{1}{16})}{w^2/4} = e_3 - \frac{1}{4w}. \tag{69}
\end{align*}
\]

where we use eq. (212) and \( \varphi(\omega') = e_3, \varphi'(\omega') = 0 \). Thus under this transformation \( w \) has simply to be replaced by its inverse \( 1/w \). Comparing with eqs. (54) and (55) we see that reversing the sign of \( \epsilon \) corresponds to replacing \( v_i \) by \( v_i' = \omega' - v_i \). Im \( \wp'(v_i') \) is negative as Im \( \wp'(v_i) \), but the differences \( \wp'(v_1) - \wp'(v_2) \)
and \( \varphi(v'_1) - \varphi(v'_2) \) have now opposite signs. Therefore eq. \ref{eq:65} has to be replaced by

\[
\psi_{\text{per}} = \frac{2\omega_3 \lambda}{r_0(1 + \epsilon)} + 2i \left( (v'_1 - v'_2) \zeta(\omega_3) + \omega_3 (\zeta(v'_2) - \zeta(v'_1)) \right)
\]

\[
= \frac{2\omega_3 \lambda}{r_0(1 + \epsilon)} + 2i \left( (v_1 - v_2) \zeta(\omega_3) + \omega_3 (\zeta(v_2) - \zeta(v_1)) \right).
\] (70)

We use now eq. \ref{eq:194} and obtain

\[
\zeta(\omega' - v_2) - \zeta(\omega' - v_1) = \zeta(v_1) - \zeta(v_2) + \kappa,
\]

\[
\kappa = \frac{1}{2} \left( \frac{\varphi'(v_2) - \varphi'(v_1)}{e_3 - \varphi(v_2)} - \frac{\varphi'(v_1)}{e_3 - \varphi(v_1)} \right).
\] (72)

After some algebra one obtains

\[
\kappa = -\frac{\sqrt{2} \sqrt{e^2 + \mu^2}}{\sqrt{\alpha + \alpha - (1 - e^2)}}.
\] (73)

This yields

\[
\psi_{\text{per}} = \frac{2\omega_3 \lambda}{r_0(1 + \epsilon)} + \frac{4\epsilon \omega_3 \lambda}{r_0(1 - e^2)} - 2i \left( (v_1 - v_2) \zeta(\omega_3) + \omega_3 (\zeta(v_2) - \zeta(v_1)) \right),
\] (74)

which reduces to the expression \ref{eq:65} for \( \psi_{\text{per}} \).

Later we will need \( \varphi(v_1 + v_2) \) and \( \varphi(2v_1) \). By means of eqs. \ref{eq:54} \ref{eq:55} and \ref{eq:61} \ref{eq:62} one finds, that

\[
\varphi(v_1 + v_2) = \varphi(2v_1) = e_3 - \frac{e^2 + \mu^2}{8\sqrt{\alpha + \alpha -}}.
\] (75)

Since \( \varphi(v_1 + v_2) \) and \( \varphi(2v_1) \) are equal, we conclude, that

\[
(v_1 + v_2) + 2v_1 = 3v_1 + v_2 = 2\omega'.
\] (76)

This implies, that we can express all \( v \)'s in terms of a variable \( \tau \),

\[
v_1 = \frac{\omega'}{2} + i\tau, \quad v_2 = \frac{\omega'}{2} - 3i\tau, \quad v'_1 = \frac{\omega'}{2} - i\tau, \quad v'_2 = \frac{\omega'}{2} + 3i\tau.
\] (77)

The sign of \( \epsilon \) agrees with the sign of \( \tau \). Thus for positive \( \epsilon \) one has

\[
\varphi(v_2) < \varphi(v'_1) < e_3 - 1/4 < \varphi(v_1) < \varphi(v'_2) < e_3.
\] (78)

The derivatives \( \varphi'(v) \) are purely imaginary. For positive \( \epsilon \) one has

\[
\text{Im} \varphi'(v_2) < \text{Im} \varphi'(v'_1) < -\frac{k}{2} < \text{Im} \varphi'(v_1) < \text{Im} \varphi'(v'_2) < 0.
\] (79)

5 Constant Chord Length

Starting out from expression \ref{eq:67} and using eq. \ref{eq:104} we obtain for the difference of the angle at \( u + \delta u \) and \( u - \delta u \)

\[
\psi(u + \delta u) - \psi(u - \delta u) = \chi_0 + \sum_i c_i \ln \left( \frac{\varphi(\bar{u}) - \varphi(\delta \bar{u} - v_i)}{\varphi(\bar{u}) - \varphi(\delta \bar{u} + v_i)} \right),
\] (80)
where $\chi_0$ is independent of $u$,
\[
\chi_0 = 2c_0\delta u + 2\sum_i c_i \ln \left( \frac{\sigma(\delta u - v_i)}{\sigma(\delta u + v_i)} \right).
\] (81)

The length $2\ell$ of the chord between these points is given by
\[
(2\ell)^2 = q(u + \delta u) + q(u - \delta u) - 2\sqrt{q(u + \delta u)q(u - \delta u)\cos(\psi(u + \delta u) - \psi(u - \delta u))}
\]
\[
= r_0^2(1 - c)^2(M_1 + M_2 - e^{i\chi_0}M_3 - e^{-i\chi_0}M_4)
\] (82)

with
\[
M_1 = \frac{\Delta(\bar{u} + \delta \bar{u}, v_2)}{\Delta(\bar{u} + \delta \bar{u}, v_1)},
\] (83)
\[
M_2 = \frac{\Delta(\bar{u} - \delta \bar{u}, v_2)}{\Delta(\bar{u} - \delta \bar{u}, v_1)},
\] (84)
\[
M_3 = \sqrt{M_1M_2}e^{i(x - \chi_0)},
\] (85)
\[
M_4 = \sqrt{M_1M_2}e^{-i(x - \chi_0)}
\] (86)

and the abbreviation
\[
\Delta(a, b) := \varphi(a) - \varphi(b).
\] (87)

By means of the identity \textbf{213} derived in the appendix
\[
\Delta(a + b, c)\Delta(a - b, c) = \frac{\Delta^2(a, c)}{\Delta^2(a, b)}\Delta(a + c, b)\Delta(a - c, b)
\] (88)

we obtain
\[
\sqrt{M_1M_2} = \frac{\Delta(\delta \bar{u}, v_2)}{\Delta(\delta \bar{u}, v_1)}\left(\frac{\Delta(\bar{u}, \delta \bar{u} + v_2)\Delta(\bar{u}, \delta \bar{u} - v_2)}{\Delta(\bar{u}, \delta \bar{u} - v_1)\Delta(\bar{u}, \delta \bar{u} + v_1)}\right)^{1/2}.
\] (89)

Together with
\[
e^{i(x - \chi_0)} = \left(\frac{\Delta(\bar{u}, \delta \bar{u} + v_1)\Delta(\bar{u}, \delta \bar{u} - v_2)}{\Delta(\bar{u}, \delta \bar{u} - v_1)\Delta(\bar{u}, \delta \bar{u} + v_2)}\right)^{1/2}
\] (90)

this yields
\[
M_3 = \frac{\Delta(\delta \bar{u}, v_2)\Delta(\bar{u}, \delta \bar{u} - v_2)}{\Delta(\delta \bar{u}, v_1)\Delta(\bar{u}, \delta \bar{u} - v_1)},
\] (91)
\[
M_4 = \frac{\Delta(\delta \bar{u}, v_2)\Delta(\bar{u}, \delta \bar{u} + v_2)}{\Delta(\delta \bar{u}, v_1)\Delta(\bar{u}, \delta \bar{u} + v_1)}
\] (92)

On the other hand we may rewrite
\[
M_1 + M_2 = \frac{\Delta(\bar{u} + \delta \bar{u}, v_2)\Delta(\bar{u} - \delta \bar{u}, v_1) + \Delta(\bar{u} + \delta \bar{u}, v_1)\Delta(\bar{u} - \delta \bar{u}, v_2)}{\Delta(\bar{u} + \delta \bar{u}, v_1)\Delta(\bar{u} - \delta \bar{u}, v_1)}
\]
\[
= 1 + \frac{\Delta(\bar{u} + \delta \bar{u}, v_2)\Delta(\bar{u} - \delta \bar{u}, v_2) - \Delta^2(v_1, v_2)}{\Delta(\bar{u} + \delta \bar{u}, v_1)\Delta(\bar{u} - \delta \bar{u}, v_1)}
\]
\[
= 1 + \frac{\Delta(\bar{u}, \delta \bar{u} + v_2)\Delta(\bar{u}, \delta \bar{u} - v_2)\Delta^2(\delta \bar{u}, v_2)}{\Delta(\bar{u}, \delta \bar{u} + v_1)\Delta(\bar{u}, \delta \bar{u} - v_1)\Delta^2(\delta \bar{u}, v_1)}
\]
\[
= 1 + \frac{\Delta^2(\bar{u}, \delta \bar{u})\Delta^2(v_1, v_2)}{\Delta(\bar{u}, \delta \bar{u} + v_1)\Delta(\bar{u}, \delta \bar{u} - v_1)\Delta^2(\delta \bar{u}, v_1)}.
\] (93)
Thus $(2\ell)^2$ is a rational function of $\varphi(\tilde{u})$ with simple poles at $\varphi(\tilde{u}) = \varphi(\delta \tilde{u} \pm v_1)$,

\[
\frac{4\ell^2}{r_0^2(1 - \ell)^2} = 1 + \frac{\Delta^2(\delta \tilde{u}, v_2) - \Delta^2(v_1, v_2)}{\Delta^2(\delta \tilde{u}, v_1)} - 2\cos(\chi_0) \frac{\Delta(\delta \tilde{u}, v_2)}{\Delta(\delta \tilde{u}, v_1)} + \frac{\nu_1 - \nu_2 e^{-i\chi_0}}{\Delta(\tilde{u}, \delta \tilde{u} + v_1)} + \frac{\nu_2 - \nu_3 e^{i\chi_0}}{\Delta(\tilde{u}, \delta \tilde{u} - v_1)}
\]  

with

\[
\begin{align*}
\nu_1 &= \frac{z_+}{\Delta(\delta \tilde{u} + v_1, \delta \tilde{u} - v_1)\Delta^2(\delta \tilde{u}, v_1)}, \\
\nu_2 &= \frac{z_-}{\Delta(\delta \tilde{u} - v_1, \delta \tilde{u} + v_1)\Delta^2(\delta \tilde{u}, v_1)}, \\
\nu_3 &= \frac{\Delta(\delta \tilde{u}, v_2)\Delta(\delta \tilde{u} - v_1, \delta \tilde{u} - v_2)}{\Delta(\delta \tilde{u}, v_1)}, \\
\nu_4 &= \frac{\Delta(\delta \tilde{u}, v_2)\Delta(\delta \tilde{u} + v_1, \delta \tilde{u} + v_2)}{\Delta(\delta \tilde{u}, v_1)}. \\
\end{align*}
\]

If we can choose $\chi_0$ and $\delta \tilde{u}$, so that the two residua vanish, then the length of the chord does not depend on $u$. It turns out, that the modulus of all four $\nu$s coincide. First we realize, that

\[
\nu_2 = \nu_1^*, \quad \nu_4 = \nu_3^*.
\]

In order to see, that $|\nu_1| = |\nu_2|$ or equivalently $\nu_1\nu_2 = \nu_3\nu_4$ we obtain a simplified form for $z_+$ and $z_-$ by applying in the first line of eq. (93) the identity (88) only to the denominator

\[
M_1 + M_2 = \frac{(\Delta(\tilde{u} + \delta \tilde{u}, v_2)\Delta(\tilde{u} - \delta \tilde{u}, v_1) + \Delta(\tilde{u} + \delta \tilde{u}, v_1)\Delta(\tilde{u} - \delta \tilde{u}, v_2))\Delta^2(\tilde{u}, \delta \tilde{u})}{\Delta(\tilde{u}, \delta \tilde{u} + v_1)\Delta(\tilde{u}, \delta \tilde{u} - v_1)\Delta^2(\delta \tilde{u}, v_1)}
\]

From this expression one reads off the residues $\nu_1$ and $\nu_2$ with

\[
\begin{align*}
\nu_1 &= \Delta(2\delta \tilde{u} + v_1, \delta \tilde{u})\Delta(v_1, v_2)\Delta^2(\delta \tilde{u} + v_1, \delta \tilde{u}), \\
\nu_2 &= \Delta(2\delta \tilde{u} - v_1, \delta \tilde{u})\Delta(v_1, v_2)\Delta^2(\delta \tilde{u} - v_1, \delta \tilde{u}).
\end{align*}
\]

To prove $\nu_1\nu_2 = \nu_3\nu_4$ we recast the expressions by means of eqs. (229) (230)

\[
\begin{align*}
\Delta(2\delta \tilde{u} + v_1, v_1)\Delta(2\delta \tilde{u} - v_1, v_1) &= -\frac{\Delta(2\delta \tilde{u}, 2v_1)\psi^2(v_1)}{\Delta^2(2\delta \tilde{u}, v_1)}, \\
\Delta(\delta \tilde{u} + v_1, \delta \tilde{u})\Delta(\delta \tilde{u} - v_1, \delta \tilde{u}) &= \frac{\Delta(2\delta \tilde{u}, v_1)\psi^2(\delta \tilde{u})}{\Delta^2(\delta \tilde{u}, v_1)}, \\
\Delta(\delta \tilde{u} + v_1, \delta \tilde{u} - v_1) &= -\frac{\psi'(\delta \tilde{u})\psi(v_1)}{\Delta^2(\delta \tilde{u}, v_1)}.
\end{align*}
\]
and obtain
\[ \nu_1 \nu_2 = \frac{\Delta(2\delta \hat{u}, 2v_1)\Delta^2(v_1, v_2)\psi^2(\delta \hat{u})}{\Delta^4(\delta \hat{u}, v_1)}. \] (108)

To evaluate \( \nu_3 \nu_4 \) we use eq. (228)
\[ \Delta(\delta \hat{u} + v_1, \delta \hat{u} + v_2)\Delta(\delta \hat{u} - v_1, \delta \hat{u} - v_2) = \frac{\Delta(2\delta \hat{u}, v_1 + v_2)\Delta^2(v_1, v_2)\psi^2(\delta \hat{u})}{\Delta^2(\delta \hat{u}, v_1)\Delta^2(\delta \hat{u}, v_2)}. \] (109)

and obtain
\[ \nu_3 \nu_4 = \frac{\Delta(2\delta \hat{u}, v_1 + v_2)\Delta^2(v_1, v_2)\psi^2(\delta \hat{u})}{\Delta^4(\delta \hat{u}, v_1)}. \] (110)

By means of the identity \( \varphi(v_1 + v_2) = \varphi(2v_1) \) we find that \( \nu_1 \nu_2 = \nu_3 \nu_4 \) holds.

Thus we define the angle \( \gamma(\delta \hat{u}) \)
\[ e^{i\gamma(\delta \hat{u})} := \frac{\nu_2}{\nu_3} = \frac{\nu_4}{\nu_1}. \] (111)

If now \( \chi_0(\delta \hat{u}) = \gamma(\delta \hat{u}) \) then the length 2\( \ell \) of the chord does not depend on the arc \( u \). If \( \chi_0(\delta \hat{u}) \neq \gamma(\delta \hat{u}) \), then we may introduce two of the curves with equal parameters \( r_0, \epsilon \) and \( \mu \). If we rotate these two curves by the angle \( \gamma - \chi_0 \) against each other, then again the chord connecting the points with arc parameter \( u + \delta u \) on one curve and arc parameter \( u - \delta u \) on the other curve has for fixed \( \delta u \) constant length 2\( \ell \). This property for infinitesimal \( \ell \) allowed initially the derivation of the differential equation (11).

**Increase of \( \gamma \) with \( \delta \hat{u} \)**

To investigate the variation of \( \gamma \) let us factor \( e^{i\gamma} \)
\[ e^{i\gamma(\delta \hat{u})} = \frac{\nu_4}{\nu_1} = \frac{n_0 n_1}{n_2 n_3}, \] (112)
\[ n_0 = \frac{\Delta(\delta \hat{u}, v_1)\Delta(\delta \hat{u}, v_2)}{\psi^2(\delta \hat{u})\Delta(v_1, v_2)}; \] (113)
\[ n_1 = \frac{\Delta(\delta \hat{u} + v_1, \delta \hat{u} + v_2)}{\varphi(\delta \hat{u})}; \] (114)
\[ n_2 = \frac{\Delta(\delta \hat{u} + v_1, \delta \hat{u})}{\varphi(\delta \hat{u})}; \] (115)
\[ n_3 = \frac{\Delta(2\delta \hat{u} + v_1, v_1)}{\Delta(\delta \hat{u} + v_1, \delta \hat{u} - v_1)}. \] (116)

The first factor \( n_0 \) is positive and finite. Thus it does not contribute to the variation of \( \gamma \).

The factor \( n_1 \) varies from \( n_1(0) = \varphi(v_1) - \varphi(v_2) > 0 \) to \( n_1(\omega_3) = \varphi(v_1 + \omega_3) - \varphi(v_2 + \omega_3) = \varphi(v_1 + \omega') - \varphi(v_2 + \omega') = \varphi(v'_1) - \varphi(v'_2) < 0 \). The imaginary part of \( n_1 \) is given by
\[ \text{Im } n_1 = -\frac{\varphi'(\delta \hat{u})}{2i} \left( \frac{\varphi'(v_1)}{(\varphi(\delta \hat{u}) - \varphi(v_1))^2} - \frac{\varphi'(v_2)}{(\varphi(\delta \hat{u}) - \varphi(v_2))^2} \right). \] (117)

For large \( \varphi(\delta \hat{u}) \) the second term in the large parenthesis is larger than the first term, since \( |\varphi'(v_2)| > |\varphi'(v_1)| \). Then \( \text{Im } n_1 > 0 \). The large parenthesis in (114)
vanishes only for
\[ \frac{\varphi(\delta \tilde{u}) - \varphi(v_2)}{\varphi(\delta \tilde{u}) - \varphi(v_1)} = \pm \sqrt{\frac{\varphi'(v_2)}{\varphi'(v_1)}} \] (118)

The left hand side increases from 1 at \( \delta \tilde{u} = 0 \) to \( (\epsilon_3 - \varphi(v_2))/(\epsilon_3 - \varphi(v_1)) \) at \( \delta \tilde{u} = \omega_3 \). Thus, if
\[ \frac{\epsilon_3 - \varphi(v_2)}{\epsilon_3 - \varphi(v_1)} = \frac{(1 + \epsilon)^2}{(1 - \epsilon)^2} < \sqrt{\frac{\varphi'(v_2)}{\varphi'(v_1)}} = \sqrt{\frac{(1 + \epsilon)(1 + \mu^2)}{(1 - \epsilon)^3}}, \] (119)

which is the case for curves which obey
\[ \frac{(1 + \epsilon)^3}{1 - \epsilon} < 1 + \mu^2, \] (120)

then \( \Im n_1 > 0 \) in the whole interval. However, if \( \epsilon > 0 \) and \( \mu \) are varied, then \( \Im n_1(\delta \tilde{u}) > 0 \) will still hold at \( \Re n_1(\delta \tilde{u}) = 0 \), since there is no real solution \( \delta \tilde{u} \) of \( \varphi(\delta \tilde{u} + v_1) = \varphi(\delta \tilde{u} + v_2) \). The solution of this equation reads \( \delta \tilde{u} = \omega - (v_1 + v_2)/2 = \omega - v_1', \) where \( \omega \) is a period or half-period. All these solutions have an imaginary part. Therefore variation of \( \epsilon \) and \( \mu \) will never yield a vanishing \( \Delta(\delta \tilde{u} + v_1, \delta \tilde{u} + v_2) \) for real \( \delta \tilde{u} \). Thus \( n_1 \) contributes a change \( +\pi \) to \( \gamma(\omega_3) - \gamma(0) \).

Next we consider the factor \( n_2(\delta \tilde{u}) \). We find
\[ n_2(0) = -1, \quad n_2(\omega_3) = \frac{\varphi(\omega_3 + v_1) - \varphi(\omega_3)}{\epsilon_3} = \frac{\varphi(v_1') - \epsilon_3}{\epsilon_3} < 0. \] (121)

Here we have used \( \varphi(\omega_3 + v_1) = \varphi(\omega' + v_1) = \varphi(\omega' - v_1) = \varphi(v_1') < \epsilon_3. \) The imaginary part of \( n_2 \) is given by
\[ \Im n_2(\delta \tilde{u}) = -\frac{\varphi'(\delta \tilde{u})\varphi'(v_1)}{2\varphi(\delta \tilde{u})(\varphi(\delta \tilde{u}) - \varphi(v_1))}. \] (122)

It is positive in the whole interval. Since \( n_2 \) has the same sign at both endpoints, it does not contribute to \( \gamma(\omega_3) - \gamma(0). \)

Finally we consider the factor \( n_3(\delta \tilde{u}) \). At the endpoints of the interval numerator and denominator vanish. The limits are
\[ n_3(0) = \frac{2\varphi'(v_1)}{2\varphi'(v_1)} = 1, \quad n_3(\omega_3) = \frac{2\varphi'(2\omega_3 + v_1)}{2\varphi'(\omega_3 + v_1)} = \frac{\varphi'(v_1)}{\varphi'(\omega' + v_1)} = -\frac{\varphi'(v_1)}{\varphi'(v_1')} < 0. \] (123)

The denominator of \( n_3 \) is purely imaginary
\[ \Delta(\delta \tilde{u} + v_1, \delta \tilde{u} - v_1) = -\frac{\varphi'(\delta \tilde{u})\varphi'(v_1)}{\Delta^2(\delta \tilde{u}, v_1)}. \] (124)

The numerator yields
\[ \Delta(2\delta \tilde{u} + v_1, v_1) = (R + iI), \] (125)
\[ R = -\varphi(2\delta \tilde{u}) - 2\varphi(v_1) + \frac{\varphi^2(2\delta \tilde{u}) + \varphi^2(v_1)}{4\Delta^2(2\delta \tilde{u}, v_1)}, \] (126)
\[ I = i\frac{\varphi'(2\delta \tilde{u})\varphi'(v_1)}{2\Delta^2(2\delta \tilde{u}, v_1)}. \] (127)
Thus the real part of $n_3$ is
\[ \Re n_3 = \frac{\psi'(2\delta \bar{u}) \Delta^2(\delta \bar{u}, v_1)}{2\psi'(\delta \bar{u}) \Delta^2(2\delta \bar{u}, v_1)}. \] (128)

The second fraction is always positive. The first fraction is positive for $0 < \delta \bar{u} < \omega_3/2$ and negative for $\omega_3/2 < \delta \bar{u} < \omega_3$. Thus the real part turns from positive to negative at $\delta \bar{u} = \omega_3$. The imaginary part at $\delta \bar{u} = \omega_3$ is given by
\[ \Im n_3 = \frac{R(\omega_3/2)(\psi(\omega_3/2) - \psi(v_1))^2}{\psi'(\omega_3/2) \psi'(v_1)} \] (129)

Since $R(\omega_3/2)$ is negative ($\psi^2(v_1)$ is negative) and $\psi'(\omega_3/2)$ is also negative, the sign of $\Im n_3$ is given by the sign of $i/\psi'(v_1)$ which is negative. Thus $n_3$ contributes $\pi$ to $\gamma(\omega_3) - \gamma(0)$ since $n_3$ appears in the denominator of (112).

In total we have
\[ \gamma(\omega_3) - \gamma(0) = 2\pi. \] (130)

The condition for a constant length $\ell$ independent of $\tilde{u}$ is according to (111)
\[ \nu = \nu_4 e^{-i\chi_0}. \] (131)

Thus the condition is
\[ \gamma(\delta \tilde{u}) - \chi_0(\delta \tilde{u}) \equiv 0 \mod 2\pi. \] (132)

Thus any time $\gamma(\delta \tilde{u}) - \chi_0(\delta \tilde{u})$ increases by $2\pi$ we have again a chord of constant length $2\ell$. If one of the endpoints of the chord goes around a closed curve of dihedral symmetry group $D_p$, then $\delta \tilde{u}$ increases by $p\omega_3$. Then $\gamma(p\omega_3) - \chi_0(p\omega_3) = 2(p - 1)\pi$. Thus we find $p - 1$ solutions $\delta \tilde{u}$, for which $\ell$ is constant. However, one solution is the trivial solution $\delta \tilde{u} = 0, \ell = 0$. Thus we have $p - 2$ non-trivial solutions in agreement with the argument in [8, 9, 10]. We cannot exclude, that there are more than $p - 2$ solutions, since we have not shown, that $\gamma(\delta \tilde{u}) - \chi_0(\delta \tilde{u})$ is a monotonously increasing function of $\delta \tilde{u}$.

### 6 The Linear Case

#### 6.1 The curve

In the limit $r_0 \to \infty$ one obtains the linear case, that is a curve oscillating around a straight line. In this limit $\epsilon$ and $\mu$ approach 0 and we require for fixed $\xi$ and $\eta$
\[ r_0 \to \infty, \quad \epsilon = \frac{\eta}{r_0}, \quad \mu = \frac{\xi}{r_0}, \quad y = r - r_0, \quad x = r_0\psi. \] (133)

$AD$ and $BC$ become infinitesimal with leading contributions
\[ AD = 16\epsilon^2, \quad BC = 16\mu^2. \] (134)

Then the constants approach
\[ \lambda = \frac{\sqrt{\xi^2 + \eta^2}}{2}, \quad k = \frac{\eta}{2\lambda}, \quad c_3 = \frac{\xi^2 - \eta^2}{24\lambda^2}. \] (135)
The differential equations for the curve read

\[
\frac{1}{\sqrt{1 + \left(\frac{du}{dx}\right)^2}} = \frac{2y^2 + \xi^2 - \eta^2}{4\lambda^2}, \quad (136)
\]

\[
\frac{du}{dy} = \frac{2\lambda^2}{\sqrt{(\eta^2 - y^2)(\xi^2 + y^2)}}, \quad (137)
\]

\[
\frac{dx}{du} = \frac{2y^2 - \eta^2 + \xi^2}{4\lambda^2}, \quad (138)
\]

One obtains \( y \) as a function of the length of the arc \( u = \lambda \tilde{u} \)

\[
y = \eta \frac{-\phi(\tilde{u}) + e_3 + \frac{1}{4}}{\phi(\tilde{u}) - e_3 + \frac{1}{4}}. \quad (139)
\]

The solution for \( x(\tilde{u}) \) is obtained from the solution for \( \psi \). It turns out, that \( \tau \) becomes infinitesimal

\[
\tau = \frac{\epsilon}{2k}. \quad (140)
\]

Therefore we can put

\[
v_i = v'_i = \frac{\omega'}{2}, \quad \phi(v_i) = \phi(v'_i) = e_3 - \frac{1}{4}, \quad \phi'(v_i) = \phi'(v'_i) = -\frac{ik}{2}. \quad (141)
\]

unless we have to evaluate the difference of expressions, which differ only in arguments \( v \)s and \( v' \)s, resp. From (137, 138, 141) we obtain

\[
x = \hat{c}_0 \tilde{u} + 2\lambda(\zeta(\tilde{u} + v) + \zeta(\tilde{u} - v)) \quad (142)
\]

with

\[
\hat{c}_0 = \frac{\xi^2 + 2\eta^2}{3\lambda}. \quad (143)
\]

By means of (139) we may rewrite

\[
\zeta(\tilde{u} + v) + \zeta(\tilde{u} - v) = 2\zeta(\tilde{u}) + \frac{\phi'(\tilde{u})}{\phi(\tilde{u}) - e_3 + \frac{1}{4}}. \quad (144)
\]

During one period \( x \) increases by

\[
x_{\text{per}} = x(\tilde{u} + 2\omega_3) - x(\tilde{u}) = 2\hat{c}_0 \omega_3 + 8\lambda \zeta(\omega_3). \quad (145)
\]

Let us consider a few symmetries. First one sees immediately, that since we start at the minimum \( y(0) = -\eta \), \( x \) is an odd and \( y \) is an even function of \( \tilde{u} \),

\[
x(-\tilde{u}) = -x(\tilde{u}), \quad y(-\tilde{u}) = y(\tilde{u}). \quad (146)
\]

Secondly if one increases \( \tilde{u} \) by the half-period \( \omega_3 \), then \( x \) is shifted by \( x_{\text{per}}/2 \) and \( y \) changes sign,

\[
x(\tilde{u} + \omega_3) = x(\tilde{u}) + \frac{x_{\text{per}}}{2}, \quad y(\tilde{u} + \omega_3) = -y(\tilde{u}). \quad (147)
\]

The result for \( y \) can be seen by substituting

\[
\phi(\tilde{u} + \omega_3) = e_3 + \frac{1}{16(\phi(\tilde{u}) - e_3)}. \quad (148)
\]
The result for $x$ is obtained by starting from (142) and observing
\[
\begin{align*}
\zeta(\hat{u} + v + \omega_3) &= \zeta(\hat{u} - v + (\omega_3 + \omega')) = \zeta(\hat{u} - v) - 2\zeta(\omega_1), \\
\zeta(\hat{u} - v + \omega_3) &= \zeta(\hat{u} + v) - 2\zeta(\omega_2), \\
-2\zeta(\omega_1) - 2\zeta(\omega_2) &= 2\zeta(\omega_3). 
\end{align*}
\] (149)(150)(151)

6.2 Length of the chord

Again we consider the chord length $2\ell$ between the two points at $\hat{u} + \delta\hat{u}$ and $\hat{u} - \delta\hat{u}$. We are interested in the $\delta\hat{u}$ for which the chord length becomes independent of $\hat{u}$. Therefore we determine the differences of the coordinates. One obtains for the difference of the $x$-coordinates
\[
\dot{x} := x(\hat{u} + \delta\hat{u}) - x(\hat{u} - \delta\hat{u}) = 2c_0\delta\hat{u} + 2\lambda(\zeta(\hat{u} + \delta\hat{u} + v) + \zeta(\hat{u} + \delta\hat{u} - v) + \zeta(\delta\hat{u} + v - \hat{u}) + \zeta(\delta\hat{u} - v - \hat{u})).
\] (152)

This can be rearranged to
\[
\begin{align*}
\dot{x} &= x_0(\delta\hat{u}) + 2\lambda \left( \frac{\varphi'(\hat{u} + v)}{\Delta(\delta\hat{u} + v, \hat{u})} + \frac{\varphi'(\delta\hat{u} - v)}{\Delta(\delta\hat{u} - v, \hat{u})} \right), \\
x_0(\delta\hat{u}) &= 2c_0\delta\hat{u} + 4\lambda(\zeta(\delta\hat{u} + v) + \zeta(\delta\hat{u} - v)).
\end{align*}
\] (153)(154)

We bring the difference of the $y$-coordinates to a common denominator
\[
\begin{align*}
\dot{y} := y(\hat{u} + \delta\hat{u}) - y(\hat{u} - \delta\hat{u}) &= -\frac{\eta}{2} \frac{\Delta(\hat{u} + \delta\hat{u}, \hat{u} - \delta\hat{u})}{\Delta(\hat{u} + \delta\hat{u}, v)\Delta(\hat{u} - \delta\hat{u}, v)}.
\end{align*}
\] (155)

We rewrite numerator and denominator
\[
\begin{align*}
\Delta(\hat{u} + \delta\hat{u}, \hat{u} - \delta\hat{u}) &= -\frac{\varphi'(\hat{u})\varphi'(\delta\hat{u})}{\Delta^2(\hat{u}, \delta\hat{u})}, \\
\Delta(\hat{u} + \delta\hat{u}, v)\Delta(\hat{u} - \delta\hat{u}, v) &= \frac{\Delta(\hat{u}, \delta\hat{u} + v)\Delta(\hat{u}, \delta\hat{u} - v)\Delta^2(\delta\hat{u}, v)}{\Delta^2(\hat{u}, \delta\hat{u})}.
\end{align*}
\] (156)(157)

and thus obtain
\[
\dot{y} = \frac{\eta\varphi'(\hat{u})}{2\varphi'(v)} \left( \frac{1}{\Delta(\hat{u}, \delta\hat{u} + v)} - \frac{1}{\Delta(\hat{u}, \delta\hat{u} - v)} \right).
\] (158)

Performing a partial fraction decomposition with respect $\varphi'(\hat{u})$ yields
\[
\dot{y} = \frac{\eta\varphi'(\hat{u})}{2\varphi'(v)} \left( \frac{1}{\Delta(\hat{u}, \delta\hat{u} + v)} + \frac{1}{\Delta(\hat{u}, \delta\hat{u} - v)} \right).
\] (159)

In total we have
\[
\begin{align*}
\dot{x} &= x_0 + \frac{x_1}{\Delta(\hat{u}, \delta\hat{u} + v)} + \frac{x_2}{\Delta(\hat{u}, \delta\hat{u} - v)}, \\
x_1 &= -2\lambda\varphi'(\delta\hat{u} + v), \quad x_2 = -2\lambda\varphi'(\delta\hat{u} - v), \\
\dot{y} &= \frac{y_1\varphi'(\hat{u})}{\Delta(\hat{u}, \delta\hat{u} + v)} + \frac{y_2\varphi'(\hat{u})}{\Delta(\hat{u}, \delta\hat{u} - v)}, \\
y_1 &= \frac{\eta}{2\varphi'(v)}, \quad y_2 = -\frac{\eta}{2\varphi'(v)}. 
\end{align*}
\] (160)(161)(162)(163)
Thus the square of the length $2\ell$ of the chord can be written

$$4\ell^2 = x^2 + y^2 = x_0^2 + \frac{2x_0x_1}{\Delta(u, \delta u + v)} + \frac{2x_0x_2}{\Delta(u, \delta u - v)} + \frac{x_1^2 + y_1^2\nu^2(\bar{u})}{\Delta^2(\delta u + v)} + \frac{x_2^2 + y_2\nu^2(\bar{u})}{\Delta^2(\delta u - v)} + \frac{2x_1x_2 + 2y_1y_2\nu^2(\bar{u})}{\Delta(u, \delta u + v)\Delta(u, \delta u - v)}$$  \hspace{1cm} (164)

With

$$x_1^2 + y_1^2\nu^2(\bar{u}) = 4\lambda^2(\nu^2(\delta u + v) - \nu^2(\bar{u}))$$  \hspace{1cm} (165)

one obtains

$$\frac{x_1^2 + y_1^2\nu^2(\bar{u})}{\Delta^2(\delta u + v)} = -\frac{4\lambda^2(4\nu^2(\delta u + v) + 4(\delta u + v)\nu(\bar{u}) + 4\nu^2(\bar{u}) - g_2)}{\Delta(\delta u + v)}$$

$$= -\frac{4\lambda^2(12\nu^2(\delta u + v) + g_2 - 16\lambda^2(\nu(\bar{u}) + 2\nu(\delta u + v)))}{\Delta(\delta u + v)\Delta(\delta u - v)}.$$  \hspace{1cm} (166)

A similar result holds for $(x_2^2 + y_2\nu^2(\bar{u}))/\Delta^2(\delta u - v)$. Further we obtain

$$x_1x_2 + y_1y_2\nu^2(\bar{u}) = 4\lambda^2(\nu^2(\delta u + v) + \nu^2(\bar{u})),$$  \hspace{1cm} (167)

which yields

$$\frac{2x_1x_2 + 2y_1y_2\nu^2(\bar{u})}{\Delta(\delta u + v)\Delta(\delta u - v)} = 32\lambda^2(\nu(\bar{u}) + \nu(\delta u + v) + \nu(\delta u - v))$$

$$+ 8\lambda^2\nu(\delta u + v)(\nu(\delta u + v) + \nu(\delta u - v))$$

$$- 8\lambda^2\nu(\delta u - v)(\nu(\delta u + v) + \nu(\delta u - v))$$

$$= 32\lambda^2(\nu(\bar{u}) + \nu(\delta u + v) + \nu(\delta u - v))$$

$$+ 8\lambda^2\nu(\delta u + v)(\nu(\delta u + v) + \nu(\delta u - v))$$

$$- 8\lambda^2\nu(\delta u - v)(\nu(\delta u + v) + \nu(\delta u - v)).$$  \hspace{1cm} (168)

Collecting all terms $4\ell^2$ can be brought to the form

$$4\ell^2 = \dot{x}_0^2 + \frac{\dot{\bar{\nu}}}{\Delta(\delta u + v)} + \frac{\dot{\bar{\nu}}}{\Delta(\delta u - v)}.$$  \hspace{1cm} (169)

$$\dot{\bar{\nu}} = 2x_0x_1 - 4\lambda^2(12\nu^2(\delta u + v) + g_2)$$

$$+ 8\lambda^2\nu(\delta u + v)(\nu(\delta u + v) + \nu(\delta u - v))$$  \hspace{1cm} (170)

$$\dot{\bar{\nu}} = 2x_0x_2 - 4\lambda^2(12\nu^2(\delta u - v) + g_2)$$

$$- 8\lambda^2\nu(\delta u - v)(\nu(\delta u + v) + \nu(\delta u - v)).$$  \hspace{1cm} (171)

Since $v = \omega^2/2$ one finds

$$\dot{\bar{\nu}} = 2x_1(x_0(\delta u) - d(\delta u)), \quad \dot{\bar{\nu}} = 2x_2(x_0(\delta u) - d(\delta u))$$  \hspace{1cm} (172)

with

$$d(\delta u) = -\lambda k^2 \frac{\nu(\delta u)}{4} \frac{\nu(\delta u) - e_3 - \frac{1}{2}}{\nu(\delta u) + e_3 + \frac{1}{2}}.$$  \hspace{1cm} (173)
The function \( d(\delta \tilde{u}) \) has period \( \omega_3 \),
\[
d(\delta \tilde{u} + \omega_3) = d(\delta \tilde{u}).
\] (174)

One can see this by means of eq. (148) and
\[
\wp'(\delta \tilde{u} + \omega_3) = -\frac{\wp'(\delta \tilde{u})}{16(\wp(\delta \tilde{u}) - e_3)^2}
\] (175)

Referring to figure 2 the zeroes of \( d \) are given by
\[
d(\circ) = d(\square) = d(+) = d(-) = 0
\] (176)
and the poles by
\[
d(\mathbb{0}) = d(\infty) = \infty.
\] (177)

Thus along the real axis \( d(\delta \tilde{u}) \) has poles at \( \delta \tilde{u} = \omega_3(n + \frac{1}{2}) \) with integer \( n \). With increasing argument it runs from \(+\infty\) to \(-\infty\). Since \( x_0(\delta \tilde{u}) \) is a slowly varying function without poles at finite \( \delta \tilde{u} \), there is in each interval \( \omega_3(n - \frac{1}{2})...\omega_3(n + \frac{1}{2}) \) at least one solution \( \delta \tilde{u} \) for \( x_0(\delta \tilde{u}) - d(\delta \tilde{u}) = 0 \), for which one obtains a chord of constant length \( x_0 \). Only the solution \( \delta \tilde{u} = 0 \) is a trivial solution. Similarly as for the case of a curve winding around the origin one can for given \( \delta \tilde{u} \) find two copies of the curve shifted by the distance \( x_0 - d \) along the x-axis, so that the chord between these two curves stays constant, as one increases the arc parameter on both curves by the same \( u \).

7 Shapes

7.1 Convexity

If the curve bounds a convex region, then any chord between two points of the curve will lie completely inside the region. Such a convexity is guaranteed, if even at \( r = r_0(1 - \epsilon) \) the curvature is oriented to the center. Therefore in the vicinity of this point one should have \( r = r_0(1 - \epsilon)(1 + \frac{h\psi^2}{2}) \) with \( h < \frac{1}{2} \). One obtains
\[
f(r) = -\frac{\epsilon(1 - \epsilon)((2 - \epsilon)^2 + \mu^2)}{\epsilon^2 + \mu^2} \frac{h\psi^2}{2} = -\frac{\epsilon^2}{\sqrt{\epsilon^2 + \mu^2}} = -\frac{h^2}{2}(1 - \epsilon)\psi^2
\] (178)
from which we conclude the condition for convexity
\[
h = \frac{2\epsilon((2 - \epsilon)^2 + \mu^2)}{\epsilon^2 + \mu^2} \leq \frac{1}{2}.
\] (179)

This condition is sufficient for a floating body, however, violation of convexity is allowed for floating bodies, as long as the chord stays completely inside the body.

7.2 Radial and perpendicular tangents

If \( \epsilon \) (or \( \eta \)) is not too large, then \( \psi \) (or \( x \)), resp. will be a monotonously increasing functions of \( \tilde{u} \). For sufficiently large \( \epsilon \) however, the curves will bend over. Suppose this happens in the case of a curve winding around the origin at \( r = r_0\kappa \). Then the function \( f(r) \), eq. (9) obeys
\[
f(r_0\kappa) = -r_0\kappa,
\] (180)
from which we conclude

\[(\kappa^2 - 1)^2 + (\kappa^2 + 1)\mu^2 - (\kappa^2 - 1)\epsilon^2 - \epsilon^2\mu^2 = 0 \quad (181)\]

with the solution

\[\kappa^2 = 1 + \frac{\epsilon^2 - \mu^2}{2} \pm \sqrt{2\epsilon^2 - 2\mu^2 + \left(\frac{\epsilon^2 + \mu^2}{2}\right)^2}. \quad (182)\]

Thus the condition

\[2\epsilon^2 - 2\mu^2 + \left(\frac{\epsilon^2 + \mu^2}{2}\right)^2 \geq 0 \quad (183)\]

has to be fulfilled in order to obtain a curve with radial tangent. We note, that in this case there are two solutions \(\kappa\) with

\[\kappa_1\kappa_2 = \sqrt{(1 - \epsilon^2)(1 + \mu^2)}. \quad (184)\]

In the case of the linear curve one obtains perpendicular tangents at

\[y_\perp = \pm \sqrt{\eta^2 - \xi^2} \quad (185)\]

provided the condition \(\eta^2 \geq \xi^2\) is met.

Bracho, Montejano and Oliveros [6, 3] have determined tire track curves with the special restriction, that the arc between the two end points of the chord is one fifth of the perimeter by introducing a carousel consisting of an equilateral pentagon with the property, that the midpoints of the sides move parallel to them. The midpoints describe the tracks \(\gamma\) of the rear wheels. The corners of the pentagon describe the tracks \(\Gamma\) of the front wheels. They require that all five midpoints move on the same curve and consequently the corners describe the same curves. They have found closed curves for \(p = 7\) (fig. 3 of [3] and fig. 9 of [6]) and for \(p = 12\) (fig. 1 of [3]). I do not know, whether these curves belong to the solutions given here, but I will assume this in the following discussion. Consequently the curve with \(p = 7\) is simultaneously a tire track curve \(\Gamma\) for two other chord lengths. Since \(p\) is odd, one of the chord lengths belongs to perimeter ratio 1/2. It is a special Auerbach curve[2], although it is not convex. Similarly I expect that the curve for \(p = 12\) is a tire track curve for five further chord lengths. We note, that the requirement of finding a closed curve which winds once around the center and the requirement of spanning one fifth of the perimeter by the endpoints of the chord yields only discrete solutions for \(\epsilon\) and \(\mu\), so that one cannot require simultaneously convexity. Since in our approach the fraction of the perimeter covered by the arc between the endpoints of the chord can be varied continously, we can obtain convex solutions. A solution, which is not convex, but 'suffiently convex' to serve for a floating body for \(p = 7\) is shown in fig. 4 of [9] and fig. 1 of [10].

Fig. 2 of [3] shows a set of five curves. They have identical shape, but are shifted against each other by equal distances. They can be numbered 1 to 5 and have the property, that any pair of curves \(i\) and \(i + 1\) \((i + 5 \equiv i)\) can be connected by a chord of constant length, as the ends of the chord move by equal arc lengths. This is in agreement with the behavior we described in the linear case. Here however, the authors required, that after traversing through five
chords, one does not only return to the initial curve but even to the same point. (The corners of the equilateral pentagons can be well seen at the left ends and at the right ends of the depicted parts of the curves.) From the discussion in this paper I conclude, that there is an infinity of chord lengths with this property, if one waives the requirement, that after traversing through five chords one returns to the same point. Taking only one of these curves, it is also a tire track curve for an infinity of chord lengths.

7.3 Eights

An extreme case is the situation, where a curve with tangential or perpendicular slopes returns to the initial point after an increase of \( \tilde{u} \) by \( 2\omega_3 \). Then such a curve has the shape of an eight and \( \psi_{\text{per}} = 0 \) (eq. 65) and \( x_{\text{per}} = 0 \) (eq. 143), resp., have to be fulfilled. An example of an eight can be found in fig. 8 of [6].

7.4 Other shapes

In this paper I have assumed during the discussion of the curves, that \( \mu \) is real and tacitely \(-1 < \epsilon < 1\). Obviously there are also other values of \( \epsilon \) and \( \mu \) which yield real polar coordinates \( r \) and \( \psi \). The properties of the corresponding curves will be investigated elsewhere.

Acknowledgment I am indebted to Serge Tabachnikov for useful correspondence.

A Some Formulae

A.1 Some formulae for the Weierstrass function and its integrals

The Weierstrass function \( \wp \) is defined by

\[
\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)
\]

(186)

with the requirement that one of the singularities is at \( z = 0 \) and

\[
e_1 + e_2 + e_3 = 0.
\]

(187)

Commonly the two integrals are defined

\[
\zeta(a) = \frac{1}{a} - \int_0^a (\wp(z) - \frac{1}{z^2})dz, \quad (188)
\]

\[
\sigma(a) = a \exp \left( \int_0^a (\zeta(z) - \frac{1}{z})dz \right), \quad (189)
\]

The function \( \wp \) is an even function of its argument, \( \zeta \) and \( \sigma \) are odd functions. The Laurent and Taylor expansions start with

\[
\wp(a) = \frac{1}{a^2} + \frac{g_2}{20}a^2 + \frac{g_3}{28}a^4 + \ldots
\]

(190)
\[ \zeta(a) = \frac{1}{a} - \frac{g_2}{60} a^3 - \frac{g_3}{140} a^5 - \ldots \]  
\[ \sigma(a) = a - \frac{g_2}{240} a^5 - \frac{g_3}{840} a^7 - \ldots \]  

(191)  
(192)

There exist addition theorems

\[ \varphi(a + b) = -\varphi(a) - \varphi(b) + \frac{\left(\varphi'(a) - \varphi'(b)\right)^2}{4(\varphi(a) - \varphi(b))^2}, \]  
\[ \zeta(a + b) = \zeta(a) + \zeta(b) + \frac{1}{2} \frac{\varphi'(a) - \varphi'(b)}{\varphi(a) - \varphi(b)}. \]  
\[ \sigma(a + b) \sigma(a - b) = -\sigma^2(a) \sigma^2(b) (\varphi(a) - \varphi(b)), \]  

(193)  
(194)  
(195)

If \( \omega \) is a half-period, that is \( \omega \) itself is not a period of \( \wp \), but \( 2\omega \) is, then the following relations hold for integer \( n \)

\[ \varphi(a + 2n\omega) = \varphi(a), \]  
\[ \zeta(a + 2n\omega) = \zeta(a) + 2n\zeta(\omega), \]  
\[ \sigma(a + 2n\omega) = (-)^n \sigma(a) e^{2n(a+n\omega) \zeta(\omega)}. \]  

(196)  
(197)  
(198)

From (194) one obtains

\[ \zeta(a + b) + \zeta(a - b) = 2\zeta(a) + \frac{\varphi'(a)}{\varphi(a) - \varphi(b)}. \]  

(199)

\section*{A.2 \ cn in terms of the Weierstrass \( \wp \) function}

We perform the transformations of Jacobi’s elliptic functions \( cn, \ dn \) and \( sn \)

\[ cn(\tilde{u}, k) = \ dn(k\tilde{u}, \frac{1}{k}), \]  
\[ \dn((1 + k_1)u', \frac{2\sqrt{k_1}}{1 + k_1}) = \frac{1 - k_1 \sn^2(u', k_1)}{1 + k_1 \sn^2(u', k_1)} \]  

(200)  
(201)

with

\[ \frac{2\sqrt{k_1}}{1 + k_1} = \frac{1}{k}, \quad \sqrt{k_1} = k + i\sqrt{1 - k^2}, \]  
\[ k^{1/2} = 2k^2 - 1 \pm 2ik\sqrt{1 - k^2}, \quad u' = \frac{k u}{1 + k}, \]  

(202)  
(203)

Note, that \( k_1 \) and \( u' \) are complex. We express \( sn \) by the Weierstrass \( \wp \) function

\[ \sn(u', k_1) = \sn\left(\frac{\tilde{u}}{2\sqrt{k_1}}, k_1\right) = \frac{\sqrt{e_1 - e_3}}{\sqrt{\wp(\tilde{u}) - e_3}}. \]  
\[ k_1^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad \sqrt{e_1 - e_3} = \frac{1}{2\sqrt{k_1}}, \quad e_1 + e_2 + e_3 = 0. \]  

(204)  
(205)

Then we obtain

\[ e_1 = \frac{1}{12k_1}(2 - k_1^2) = \frac{1}{12}(2k^2 - 1 - 6ik\sqrt{1 - k^2}), \]  

(206)
\[ c_2 = \frac{1}{12k_1}(-1 + 2k_1^2) = \frac{1}{12}(2k^2 - 1 + 6ik\sqrt{1 - k^2}), \]  
(207)

\[ e_3 = \frac{1}{12k_1}(-1 - k_1^2) = \frac{1}{6}(1 - 2k^2), \]  
(208)

\[ g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = -\frac{1}{4} + \frac{1}{3}(1 - 2k^2)^2 = -\frac{1}{4} + 12e_3^2, \]  
(209)

\[ g_3 = 4e_1e_2e_3 = \frac{1}{24}(1 - 2k^2) - \frac{1}{27}(1 - 2k^2)^3 = \frac{1}{4}e_3 - 8e_3^3. \]  
(210)

Thus we obtain finally for \( cn \)

\[ cn(\tilde{u}, k) = \frac{\varphi(\tilde{u}, g_2, g_3) - e_3}{\varphi(\tilde{u}, g_2, g_3)} \]  
(211)

and for the derivative of \( \varphi \) from (186)

\[ \varphi'^2(\tilde{u}) = 4(\varphi(\tilde{u}) - e_3)(\varphi(\tilde{u})^2 + e_3\varphi(\tilde{u}) + \frac{1}{16} - 2e_3^2). \]  
(212)

### A.3 Three-Arguments Addition Theorem

Using the addition theorem (193) one finds an addition theorem including three arguments

\[(\varphi(a + b) - \varphi(c))(\varphi(a - b) - \varphi(c))(\varphi(a) - \varphi(b))^2\]

\[= (\varphi(a + c) - \varphi(b))(\varphi(a - c) - \varphi(b))(\varphi(a) - \varphi(c))^2\]

\[=(\varphi(b + c) - \varphi(a))(\varphi(b - c) - \varphi(a))(\varphi(b) - \varphi(c))^2.\]  
(213)

In the following its derivation is sketched. Using (193) one obtains

\[ (\varphi(a + b) - \varphi(c))(\varphi(a - b) - \varphi(c))(\varphi(a) - \varphi(b))^2 \]

\[= (\varphi(a + c) - \varphi(b))(\varphi(a - c) - \varphi(b))(\varphi(a) - \varphi(c))^2 \]

\[= S^2(\varphi(a) - \varphi(b))^2 - \frac{S}{2}(\varphi'^2(a) + \varphi'^2(b)) + \frac{1}{16}(\varphi'^2(a) - \varphi'^2(b))^2 \]  
(214)

with

\[ S = \varphi(a) + \varphi(b) + \varphi(c). \]  
(215)

One obtains the expressions for \( \varphi'^2 \) and obtains for the last term

\[ \frac{(\varphi'^2(a) - \varphi'^2(b))^2}{16(\varphi(a) - \varphi(b))^2} = (\varphi'^2(a) + \varphi(a)\varphi(b) + \varphi'^2(b) - \frac{1}{4}g_2)^2. \]  
(216)

Now all contributions one obtains

\[ (\varphi(a + b) - \varphi(c))(\varphi(a - b) - \varphi(c))(\varphi(a) - \varphi(b))^2 \]

\[= \varphi'^2(a)\varphi'^2(b) + \varphi'^2(a)\varphi'^2(c) + \varphi'^2(b)\varphi'^2(c) \]

\[- 2(\varphi(a) + \varphi(b) + \varphi(c))\varphi(a)\varphi(b)\varphi(c) \]

\[+ \frac{g_2}{2}(\varphi(a)\varphi(b) + \varphi(a)\varphi(c) + \varphi(b)\varphi(c)) \]

\[+ g_3(\varphi(a) + \varphi(b) + \varphi(c)) + \frac{g_2^2}{16}. \]  
(217)

This expression is invariant under any permutations of \( a, b, \) and \( c \). Thus (218) follows.
A.4 Another Three-Arguments Addition Theorem

Starting from the addition theorem (105) we obtain

\[ \Delta(a \pm b, a \pm c) = \Delta(c, b) + \frac{1}{4} \left( \frac{\psi'(a) \mp \psi'(b)}{\Delta(a, b)} \right)^{2} - \frac{1}{4} \left( \frac{\psi'(a) \mp \psi'(c)}{\Delta(a, c)} \right)^{2}. \]  

(218)

Thus we obtain

\[ \Delta(a + b, a + c)\Delta(a - b, a - c) \]

\[ = \left( \Delta(c, b) + \frac{1}{4} \frac{\psi'^{2}(a) + \psi'^{2}(b)}{\Delta^{2}(a, b)} - \frac{1}{4} \frac{\psi'^{2}(a) + \psi'^{2}(c)}{\Delta^{2}(a, c)} \right)^{2} \]

\[ - \frac{1}{4} \left( \frac{\psi'(a)\psi'(b)}{\Delta^{2}(a, b)} - \frac{\psi'(a)\psi'(c)}{\Delta^{2}(a, c)} \right)^{2}. \]  

(219)

Some rearrangement yields

\[ \Delta(a + b, a + c)\Delta(a - b, a - c) = A^{2} + \psi'^{2}(a)B, \]  

(220)

\[ A = \Delta(c, b) - \frac{1}{4} \frac{\psi'^{2}(a) - \psi'^{2}(b)}{\Delta^{2}(a, b)} + \frac{1}{4} \frac{\psi'^{2}(a) - \psi'^{2}(c)}{\Delta^{2}(a, c)}, \]  

(221)

\[ B = \Delta(c, b) \left( \frac{1}{\Delta^{2}(a, b)} - \frac{1}{\Delta^{2}(a, c)} \right) - \frac{1}{4} \frac{(\psi'(b) - \psi'(c))^{2}}{\Delta^{2}(a, b)\Delta^{2}(a, c)}. \]  

(222)

Using

\[ \frac{\psi'^{2}(a) - \psi'^{2}(b)}{\Delta(a, b)} = 4\psi^{2}(a) + 4\psi(a)\psi(b) + 4\psi^{2}(b) - g_{2}, \]  

(223)

we obtain

\[ A = \frac{\Delta(c, b)}{\Delta(a, b)\Delta(a, c)} \left( 3\psi^{2}(a) - \frac{g_{2}}{4} \right). \]  

(224)

For B we obtain

\[ B = \frac{\Delta^{2}(c, b)}{\Delta^{2}(a, b)\Delta^{2}(a, c)} (2\psi(a) + \psi(b + c)). \]  

(225)

Substitution of A and B into (220) yields

\[ \Delta(a + b, a + c)\Delta(a - b, a - c) = \] 

\[ \frac{\Delta^{2}(c, b)}{\Delta^{2}(a, b)\Delta^{2}(a, c)} \left( \psi^{4}(a) + \frac{g_{2}}{2} \psi^{2}(a) + 2g_{3}\psi(a) + \frac{g_{2}^{2}}{16} - \psi'^{2}(a)\psi(b + c) \right). \]  

(226)

Since

\[ \psi^{4}(a) + \frac{g_{2}}{2} \psi^{2}(a) + 2g_{3}\psi(a) + \frac{g_{2}^{2}}{16} = \psi(2a)\psi'^{2}(a), \]  

(227)

we obtain finally

\[ \Delta(a + b, a + c)\Delta(a - b, a - c) = \frac{\Delta^{2}(c, b)\Delta(2a, b + c)\psi'^{2}(a)}{\Delta^{2}(a, b)\Delta^{2}(a, c)}. \]  

(228)

In the limit \( c \to 0 \) we obtain

\[ \Delta(a + b, a)\Delta(a - b, a) = \frac{\Delta(2a, b)\psi'^{2}(a)}{\Delta^{2}(a, b)}. \]  

(229)
Further we obtain
\[\Delta(a+b,a-b) = \frac{1}{4} \frac{(\psi'(a)-\psi'(b))^2}{\Delta^2(a,b)} - \frac{1}{4} \frac{(\psi'(a)+\psi'(b))^2}{\Delta^2(a,b)} = -\frac{\psi'(a)\psi'(b)}{\Delta^2(a,b)}. \quad (230)\]

References

[1] M. Abramowitz and I. A. Stegun, ed. Handbook of Mathematical Functions
Dover Publ., New York

[2] H. Auerbach, Sur un probleme de M. Ulam concernant l’equilibre des corps
flottant, Studia Math. 7 (1938) 121-142

[3] J. Bracho, L. Montejano, D. Oliveros, Carousels, Zindler curves and the
floating body problem, Per. Math. Hung. 49 (2004) 9-23

[4] A. Erdelyi, ed. Higher Transcendental Functions. Bateman Manuscript
Project Mc-Graw Hill 1955 New York vol. 2

[5] R.D. Mauldin (ed.), The Scottish Book, Birkhäuser Boston 1981

[6] D. Oliveros and L. Montejano, De volantines, espirógraphos y la flotación
de los cuerpos, Revista Ciencias 55-56 (1999) 46-53

[7] S. Tabachnikov Tire track geometry: variations on a theme Israel J. of
Math. 151 (2006) 1-28 archive math.DG/0405445

[8] F. Wegner, Floating Bodies of Equilibrium I, e-Print archive
physics/0203061

[9] F. Wegner, Floating Bodies of Equilibrium II, e-Print archive
physics/0205059

[10] F. Wegner, Floating Bodies of Equilibrium Studies in Appl. Math. 111
(2003) 167-183