VIRTUALLY ABELIAN DIMENSION FOR 3-MANIFOLD GROUPS

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Abstract. For a group $G$, let $F_n$ be the family of all the subgroups of $G$ containing a subgroup isomorphic to $\mathbb{Z}^r$ for some $r = 0, 1, 2, \ldots, n$ of finite index. Joecken, Lafont and Sánchez Saldaña computed the $F_1$-dimension of 3-manifold groups. The goal of this article is to compute the $F_n$-geometric dimension of 3-manifold groups for all $n \geq 2$.

1. Introduction

Given a group $\Gamma$, we say a collection $\mathcal{F}$ of subgroups of $\Gamma$ is a family if it is non-empty, closed under conjugation and under taking subgroups. Fix a group $\Gamma$ and a family $\mathcal{F}$ of subgroups of $\Gamma$. We say that a $\Gamma$-CW-complex $X$ is a model for the classifying space $E_{\mathcal{F}} \Gamma$ if every isotropy group of $X$ belongs to the family $\mathcal{F}$ and the fixed point set $X^H$ is contractible whenever $H$ belongs to $\mathcal{F}$. It can be shown that a model for the classifying space $E_{\mathcal{F}} \Gamma$ always exists and it is unique up to $\Gamma$-homotopy equivalence. We define the $\mathcal{F}$-geometric dimension of $\Gamma$ as

$$gd_{\mathcal{F}}(\Gamma) = \min \{ n \in \mathbb{N} \mid \text{there is a model for } E_{\mathcal{F}} \Gamma \text{ of dimension } n \}.$$ 

Examples of families of subgroups are: the family that only consists of trivial subgroup $\{1\}$, and the family $FIN$ of finite subgroups of $\Gamma$. The $\mathcal{F}$-geometric dimension has been widely studied in the last decades, and the present paper contributes to this topic.

Let $n \geq 0$ be an integer. A group is said to be virtually $\mathbb{Z}^n$ if it contains a subgroup isomorphic to $\mathbb{Z}^n$ of finite index. Define the family

$$F_n = \{ H \leq \Gamma \mid H \text{ is virtually } \mathbb{Z}^r \text{ for some } 0 \leq r \leq n \}.$$ 

The families $\{1\}$, $F_0 = FIN$ and $F_1 = VCYC$ are relevant due to its connection with the Farrell-Jones and Baum-Connes isomorphism conjectures. The families $F_n$ have been recently studied by several people, see for instance [CCMNP17, Pry18, HPa20, SSn20].

In [JLS19] Joecken, Lafont and Sánchez Saldana computed the $F_1$-geometric dimension of 3-manifold groups, that is fundamental groups of connected, closed, orientable 3-manifolds. The main goal of the present article is to set up a natural extension of the latter mentioned result: we explicitly compute the $F_k$-geometric dimension of 3-manifold groups for all $k \geq 2$. Actually for a 3-manifold group we have $F_3 = F_k$ for all $k \geq 4$, see Corollary [7.2] thus our computations have to do only with the families $F_2$ and $F_3$.

To state our first main theorem we use the prime decomposition of a 3-manifold. Recall that a 3-manifold $P$ is prime if $P = P_1 \# P_2$ implies $P_1$ or $P_2$ is the 3-sphere. It is well-known that every connected, closed 3-manifold has a unique decomposition as connected sum of prime 3-manifolds.
Theorem 1.1. Let $M$ be a connected, closed, and oriented 3-manifold. Let $P_1, P_2, \ldots, P_r$ be the pieces in the prime decomposition of $M$. Denote $\Gamma = \pi_1(M)$ and $\Gamma_i = \pi_1(P_i)$. Then, for all $k \geq 2$,

$$
gd_{F_k}(\Gamma) = \begin{cases} 
0 & \text{if } M = \mathbb{R}P^3 \# \mathbb{R}P^3, \\
2 & \text{if } r \geq 2, \Gamma_i \in F_k \text{ for all } 1 \leq i \leq r, \\
\max\{\gd_{F_i}(\Gamma_i) \mid 1 \leq i \leq r\} & \text{otherwise,}
\end{cases}
$$

Moreover

- $\Gamma_i \in F_2$ if and only if $P_i$ is modeled on $S^3$ or $S^2 \times \mathbb{E}$ or $P_i = \mathbb{R}P^3 \# \mathbb{R}P^3$.
- $\Gamma_i \in F_3$ if and only if $\Gamma_i \in F_2$ or $\Gamma_i$ is modelled on $E_S^3$.

In the light of the previous theorem, the task of computing the $F_k$-geometric dimension of a 3-manifold group, it is reduced to computing the $F_k$-geometric dimension of fundamental groups of prime 3-manifolds. If $M$ is a prime manifold, then we can cut-off along certain embedded tori in such a way that the resulting connected components are either hyperbolic or Seifert fiber 3-manifolds. This is the famous JSJ-decomposition after the work of Perelman, see Theorem 2.6.

Theorem 1.2. Let $M$ be a connected, closed, and oriented prime 3-manifold. Let $N_1, N_2, \ldots, N_r$, be the pieces in the minimal JSJ-decomposition of $M$. Denote $\Gamma = \pi_1(M)$ and $\Gamma_i = \pi_1(N_i)$. If $k \geq 2$, then

$$
gd_{F_k}(\Gamma) = \begin{cases} 
2 & \text{if } M \text{ is modelled on Sol}, \\
\max\{\gd_{F_k}(\Gamma_i) \mid 1 \leq i \leq r\} & \text{otherwise.}
\end{cases}
$$

From the previous theorem, it is clear that our next task is to compute the $F_k$-geometric dimension of the fundamental group of all possible JSJ-pieces, that is, the fundamental groups of hyperbolic and Seifert fibered 3-manifolds. We accomplished this and we summarize our results in Table 1. In the $\gd_{F_2}$ column we reference the theorems where the computations were carried out. The last column is justified in Proposition 7.1.

Outline of the paper. In Section 2 we set up the preliminaries such as the definition and some properties of the $F$-geometric dimension, Bass-Serre theory, 3-manifolds, and some push-out type constructions for classifying spaces for families. In Section 3 Section 4 and Section 5 we compute the $F_2$-geometric dimension of hyperbolic, Seifert fiber, and Sol-manifolds respectively. In Section 6 we recall the notion of acylindrical splitting for the fundamental group of a graph of groups, then we prove two results that will be useful to use the computations of the previous sections to prove out main theorems. We can think of the results in Section 6 as the tools that help us to glue together the classifying spaces of the building pieces of a 3-manifold (the prime and JSJ pieces). Finally in Section 7 we prove Theorem 1.1 and Theorem 1.2.

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2. Preliminaries

2.1. Geometric and cohomological dimension for families. Given a group $\Gamma$, we say a collection $\mathcal{F}$ of subgroups of $\Gamma$ is a family if it is non-empty, closed under conjugation and under taking subgroups. Fix a group $\Gamma$ and a family $\mathcal{F}$ of subgroups of $\Gamma$. We say that a $\Gamma$-CW-complex $X$ is a model for the classifying space $E_{\mathcal{F}}\Gamma$ if every isotopy group of $X$ belongs to the family $\mathcal{F}$ and the fixed point set $X^H$ is contractible whenever $H$ belongs to $\mathcal{F}$. It can be shown that a model for the
Type of manifold | $\text{gd}_F(\Gamma)/\text{proved in}$ | $\text{gd}_F(\Gamma)$ with $k \geq 3$
---|---|---
Hyperbolic with empty boundary | 3, Theorem 3.1 | 3
Hyperbolic with non-empty boundary | 3, Theorem 3.1 | 3
Seifert fiber with base orbifold $B$ which is either bad or modelled on $S^2$ | 0, Theorem 4.1 | 0
Seifert fiber with base orbifold $B$ modelled on $\mathbb{H}^2$, and boundary empty or non-empty | 2, Theorem 4.2 | 2
Seifert fiber modeled on $\mathbb{E}^3$ with empty boundary and base orbifold $B$ modelled on $\mathbb{E}^2$ | 5, Theorem 4.3 | 0
Seifert fiber modeled on Nil with empty boundary and base orbifold $B$ modelled on $\mathbb{E}^2$ | 3, Theorem 4.3 | 3
Seifert fibered with non-empty and base orbifold $B$ modelled on $\mathbb{E}^2$ | 0, Theorem 4.4 | 0

Table 1. $F$-geometric dimension of the pieces JSJ.

classifying space $E_F\Gamma$ always exists and it is unique up to $\Gamma$-homotopy equivalence. We define the $F$-geometric dimension of $\Gamma$ as

$$\text{gd}_F(\Gamma) = \min\{n \in \mathbb{N} | \text{there is a model for } E_F\Gamma \text{ of dimension } n\}.$$  

The orbit category $O_F\Gamma$ is the category whose objects are $G$-homogenous spaces $\Gamma/H$ with $H \in F$ and morphisms are $\Gamma$-functions. The category of Bredon modules is the category whose objects are functors $M: O_F\Gamma \rightarrow \text{Ab}$ from the orbit category to the category of abelian group, and morphisms are natural transformation $f: M \rightarrow N$. This is an abelian category with enough projectives. The constant Bredon module $\mathbb{Z}: O_F\Gamma \rightarrow \text{Ab}$ is defined in objects by $\mathbb{Z}(\Gamma/H) = \mathbb{Z}$ and in morphisms by $\mathbb{Z}(\varphi) = \text{id}_{\mathbb{Z}}$. We define the $F$-cohomological dimension of $\Gamma$ as

$$\text{cd}_F(\Gamma) = \min\{n \in \mathbb{N} | \text{there is a projective resolution of } \mathbb{Z} \text{ of dimension } n\}.$$  

The proof of the following proposition is implicit in [Lüc00, Proof of theorem 3.1], see also [MPSSn20, Theorem 2.3]. We include a sketch of proof for the sake of completeness.

**Proposition 2.1.** Let $G$ be a group. Let $F$ and $G$ be families of subgroups of $G$ such that $F \subseteq G$. If $X$ is a model for $E_G\Gamma$, then

$$\text{gd}_F(G) \leq \max\{\text{gd}_{F \cap G\sigma}(G\sigma) + \text{dim}(\sigma) | \sigma \text{ is a cell of } X\}.$$  

**Sketch of proof.** Construct, for each $k \geq 0$ complexes $\tilde{X}_k$ as follows. For each cell $\sigma$ of $X$, fixed a model $X_\sigma$ for $E_{F \cap G\sigma}G\sigma$ of minimal dimension. The construction is by induction, where $\tilde{X}_0$ is obtained by replacing each 0-cell $\sigma$ of $X^{(0)}$ by the chosen model $X_\sigma$. Suppose that $\tilde{X}_k$ has been constructed. Let $\sigma$ be an $k + 1$ dimensional cell of $X$. Then attaching map $\varphi_\sigma: \partial \sigma \rightarrow X^{(n)}$ induces a map $\tilde{\varphi}_\sigma: \partial(\sigma) \times X_\sigma \rightarrow \tilde{X}_k$. Then $\tilde{X}_{k+1}$ is obtained from $\tilde{X}_k$ by attaching the spaces $\sigma \times X_\sigma$ via the attaching maps $\tilde{\varphi}_\sigma$. It turns out the space $X = \bigcup_k \tilde{X}_k$ is a model for $E_F\Gamma$ and its dimension is the number in the left hand side of the inequality of our statement.  

$\square$
Lemma 2.2. Let \( n \geq 1 \) and let \( G \) be a finitely generated group. Assume that for all \( 1 \leq r \leq n \), \( G \) is not virtually \( \mathbb{Z}^r \). Then \( \text{gcd}_{F_n}(G) \geq 2 \).

Proof. It is easy to see that \( \text{gcd}_{F_n}(G) = 0 \) if and only if \( G \in F_n \). Thus by hypothesis \( \text{gcd}_{F_n}(G) \neq 0 \). Now assume \( \text{gcd}_{F_n}(G) = 1 \), then \( G \) acts on a tree with stabilizers in \( F_n \). In particular every element \( g \in G \) has a fixed point. Since \( G \) is finitely generated, by [Ser03, Corollary 3, p.65], \( G \) has a global fixed point in \( X \), but this contradicts our hypothesis that \( \nexists \in F_n \). \( \square \)

2.2. Bass-Serre theory. In this section we remember some results of the theory of Bass-Serre what will use later.

A graph \( Y \) (in the sense of Serre) consists of a set of vertices \( V(Y) \), a set of edges \( E(Y) \) and two functions \( E(Y) \to V(Y) \times V(Y) \), \( y \mapsto (o(y), t(y)) \) and \( E(Y) \to E(Y) \), \( y \mapsto \overline{y} \) satisfying \( \overline{y} = y \) and \( o(y) = t(y) \) for all \( y \in E(Y) \). The geometric realization of \( Y \) is the quotient space, \( V(Y) \bigsqcup (E(Y) \times I) / \sim \), where \( V(Y) \) and \( E(Y) \) have the discrete topology, and the equivalence relation in \( V(Y) \bigsqcup (E(Y) \times I) \) is given as follows: for every \( y \in E(Y) \) and \( t \in I \), \( (y, t) \sim (\overline{y}, 1 - t) \), \( (y, 0) \sim o(y) \) and \( (y, 1) \sim t(y) \).

A graph of groups \( \mathbf{Y} \) consists of a graph \( Y \), a group \( Y_P \) for each \( P \in V(Y) \), and a group \( Y_y \) for each \( y \in E(Y) \), together with monomorphisms \( \phi_y : Y_y \to Y_{t(y)} \). One requires in addition \( Y_{\overline{y}} = Y_y \). Suppose that the group \( G \) acts without inversions on the graph \( Y \), i.e. for every \( g \in G \) and \( y \in E(Y) \), we have \( gy \neq y \). Then we have an induced graph of groups with underlying graph \( Y/G \) by associating to each vertex (resp. edge) the isotropy group of a preimage under the quotient map \( Y \to Y/G \).

Given a graph of groups \( \mathbf{Y} \), one the classic theorems of Bass-Serre theory provides the existence of a group \( G = \pi_1(\mathbf{Y}) \), called the fundamental group of the graph of groups \( \mathbf{Y} \) and the tree \( T \) (a graph with no cycles), called the Bass-Serre tree of \( \mathbf{Y} \), such that \( G \) acts on \( T \) without inversions, and the induced graph of groups is isomorphic to \( \mathbf{Y} \). The identification \( G = \pi_1(\mathbf{Y}) \) is called a splitting of \( G \).

As a direct consequence of [DS99, Lemma 1.1] we get the following result which will be useful later on.

Proposition 2.3. Let \( H \) be a group virtually \( \mathbb{Z}^n \) acting on a tree \( T \). Then exactly one of the following happens:

a) \( H \) fixes a vertex of \( T \).

b) \( H \) acts co-compactly in a unique geodesic line \( \gamma \) of \( T \).

2.3. 3-manifolds and their decompositions. In this section we revisit some results of 3-manifolds that we use later. For more details see [Sco83] and [AFW15].

A Seifert fibered space is a 3-manifold \( M \) with a decomposition of \( M \) into a disjoint union of circles, called fibres, such that each circle has a tubular neighbourhood in \( M \) that is isomorphic to a fibered solid torus or Klein bottle. If we collapse each of these circles we obtain a surface \( B \) that has a natural orbifold structure, we call \( B \) the base orbifold of \( M \). Such an orbifold \( B \) has its orbifold fundamental group \( \pi_1^{orb}(B) \), which is not necessarily the fundamental group of the underlying topological space, but it is related to the fundamental group of \( M \) via the following lemma.

Lemma 2.4. [Sco83, lemma 3.2] Let \( M \) be a Seifert fiber space with base orbifold \( B \). Let \( \Gamma \) be the fundamental group of \( M \). Then there is an exact sequence

\[
1 \to K \to \Gamma \to \pi_1^{orb}(B) \to 1
\]

where \( K \) denotes the cyclic subgroup of \( \Gamma \) generated by a regular fiber. The group \( K \) is infinite except in cases where \( M \) is covered by \( S^3 \).
A 2-orbifold $B$ is of exactly one of the following types depending on the structure of its universal cover: bad, spheric, hyperbolic, flat. In this paper we will divide the computation of the $F_k$-geometric dimension of a Seifert manifold $M$ depending on the type of its base orbifold $B$.

The following is a well-known theorem of Kneser (existence) and Milnor (uniqueness).

**Theorem 2.5 (Prime decomposition).** Let $M$ be a closed, oriented 3-manifold. Then $P_1 \# \cdots \# P_n$ where each $P_i$ is prime. Furthermore, this decomposition is unique up to order and homeomorphism.

Another well-known result we will need is the Jaco-Shalen-Johanson decomposition, after Perelman’s work.

**Theorem 2.6 (JSJ-decomposition).** For a closed, prime, oriented 3-manifold $M$ there is a collection $T \subseteq M$ of disjoint incompressible tori, i.e. two sided property embedded and $\pi_1$-injective, such that each component of $M \setminus T$ is either a hyperbolic or a Seifert fibered manifold. A minimal such collection $T$ is unique up to isotopy.

It is a consequence of the uniformization theorem that compact surfaces (2-manifolds) admit Riemannian metrics with constant curvature; that is, compact surfaces admit geometric structures modeled on $S^2$, $E^2$, or $H^2$. In dimension three, we are not guaranteed constant curvature. Thurston demonstrated that there are eight 3-dimensional maximal geometries up to equivalence ([Sc083, Theorem 5.1]): $S^3$, $E^3$, $H^3$, $S^2 \times E$, $H^2 \times E$, $PSL_2(\mathbb{R})$, Nil, and Sol. A manifold $M$ is called geometric if there is a geometry $X$ and discrete subgroup $\Gamma \leq Isom(X)$ with free $\Gamma$-action on $X$ such that $M$ is diffeomorphic to the quotient $X/\Gamma$; we also say that $M$ admits a geometric structure modeled on $X$.

Similarly, a manifold with nonempty boundary is geometric if its interior is geometric. It is worth saying that every 3-manifold that is not modeled on $H^3$ nor on Sol are Seifert fiber, we will use this fact all along the paper.

### 2.4. Push-out constructions for classifying spaces

In this section we revisit some results that help us to construct classifying spaces using homotopypush-outs of other classifying spaces.

**Definition 2.7.** Let $\Gamma$ be a finitely generated group, and $F \subset F'$ a pair of families of subgroups of $\Gamma$. We say a collection $A = \{A_\alpha\}_{\alpha \in I}$ of subgroups of $\Gamma$ is adapted to the pair $(F, F')$ if the following condition is hold:

a) For all $A, B \in A$, either $A = B$ or $A \cap B \in F$;

b) The collection $A$ is closed under conjugation;

c) Every $A \in A$ is self normalizing, i.e. $N_\Gamma(A) = A$;

d) For all $A \in F \setminus F'$, there is $B \in A$ such that $A \leq B$.

**Theorem 2.8.** [LO09, P. 302] Let $F \subset F'$ families of subgroups of $\Gamma$. Assume that the collection $A = \{A_\alpha\}_{\alpha \in I}$ is adapted to the pair $(F, F')$. Let $\mathcal{H}$ a complete set of representatives of the conjugacy classes within $A$, and consider the cellular $\Gamma$-push-out

$$
\bigsqcup_{H \in \mathcal{H}} \Gamma \times_H E_F H \xrightarrow{g} E_F \Gamma \\
\downarrow \quad \downarrow h \\
\bigsqcup_{H \in \mathcal{H}} \Gamma \times_H E_{F'} H \xrightarrow{\varphi} X
$$
Then $X$ is a model for $E_{F_2} \Gamma$. In the above $\Gamma$-push-out we require either (1) $f$ is the disjoint union of cellular $H$-maps, and $g$ be an inclusion of $\Gamma$-CW-complexes, or (2) $h$ be the disjoint union of inclusion of $H$-CW-complexes, and $g$ is a cellular $\Gamma$-map.

**Theorem 2.9.** [JLS19 Theorem 4.5.] Let $F$ be a family of subgroups of the finitely generated discrete group $\Gamma$. Let $\varphi: \Gamma \rightarrow \Gamma_0$ be a surjective homomorphism. Let $F_0 \subseteq F \subseteq F_0'$ be a nested pair of families of subgroups of $\Gamma_0$ satisfying $\tilde{F}_0 \subseteq F \subseteq \tilde{F}_0'$, and let $A = \{A_\alpha\}_{\alpha \in I}$ be a collection adapted to the pair $F_0 \subseteq F_1$. Let $\mathcal{H}$ be a complete set of representatives of the conjugacy classes in $\tilde{A} = \{\varphi^{-1}(A_\alpha)\}_{\alpha \in I}$, and consider the following cellular $\Gamma$-push-out

\[
\bigsqcup_{H \in \mathcal{H}} \Gamma \times_H E_{\mathcal{H}} H \xrightarrow{g} E_{F_2} \Gamma_0 \xrightarrow{h} \bigsqcup_{H \in \mathcal{H}} \Gamma \times_H E_{F_2} \bar{H} \xrightarrow{\varphi} X
\]

Then $X$ is a model for $E_{F_2} \Gamma$. In the $\Gamma$-push-out above we require either (1) $f$ is the disjoint union of cellular $H$-maps, and $g$ is an inclusion of $\Gamma$-CW-complexes, or (2) $h$ is the disjoint union of inclusion of $H$-CW-complexes, and $g$ is the cellular $\Gamma$-map.

### 3. The hyperbolic case

In this one theorem section we compute the $F_2$-geometric dimension of hyperbolic manifolds with or without boundary.

**Theorem 3.1.** Let $M$ be a hyperbolic 3-manifold of finite volume (possibly with non-empty boundary) and $\Gamma = \pi_1(M, x_0)$. Then $\text{gd}_{F_2}(\Gamma) = 3$.

**Proof.** We have two cases depending on whether the boundary of $M$ is empty or not. First suppose that $M$ has empty boundary. From [Sco83 Corollary 4.6.] $\Gamma$ cannot contain a subgroup isomorphic to $\mathbb{Z}^2$, in consequence $F_1 = F_k$ for all $k \geq 1$. Thus, by [JLS19 Proposition 6.1.], $\text{gd}_{F_1}(\Gamma) = 3$, and the statement follows.

Now suppose that $M$ has non-empty boundary. Define the following collection of subgroups of $\Gamma$

$B = \{H \in F_2 - F_0 | H \text{ there is no } K \in F_2 - F_0 \text{ such that } H < K\}$.

In [LO07 Theorem 2.6] the prove that the collection $B$ is adapted to $(F_0, F_1)$, nevertheless the same proof leads us to the conclusion that $B$ is adapted to $(F_0, F_2)$. Now we are in a good shape to use Theorem 2.8 to construct a model for $E_{F_2} \Gamma$. Let $\mathcal{H}$ be a collection of representatives of conjugacy classes in $B$, then the space $X$ defined by the homotopy $\Gamma$-push-out

\[
\bigsqcup_{H \in \mathcal{H}} \Gamma \times_H E_{\mathcal{H}} H \xrightarrow{g} ET \xrightarrow{h} \bigsqcup_{H \in \mathcal{H}} \Gamma \times_H E_{F_2} H \xrightarrow{\varphi} X
\]

provides a model for $E_{F_2} \Gamma$. We claim that $X$ can be choosen to be of dimension 3. Note that the dimension of $X$ is

\[
\max\{\dim(EH) + 1, \dim(ET), \dim(E_{F_2} H)\}.
\]
By [MT98] Proposition 2.2] $H$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$, therefore $E\mathcal{F}_2 H$ has a model of dimension 1 or 2, and in both cases there is a model for $E\mathcal{F}_2 H$ of dimension 0. The hyperbolic space $\mathbb{H}^3$ is a 3-dimensional model for $E\Gamma$. This proves our claim. As a consequence $\text{gd}_{\mathcal{F}_2}(\Gamma) \leq 3$.

Now we are going to show that $\text{gd}_{\mathcal{F}_2}(\Gamma) \geq 3$. It is well known that $\text{gd}_{\mathcal{F}_2}(\Gamma) \geq \text{cd}_{\mathcal{F}_2}(\Gamma)$, thus it is enough to show that $\text{cd}_{\mathcal{F}_2}(\Gamma) \geq 3$. To show this last inequality, we will prove that $H^3_{\mathcal{F}_2}(\Gamma; \mathbb{Z}) \neq 0$. The Mayer-Vietoris long exact sequence applied to diagram (1) leads to

\[ \cdots \to H^3_{\mathcal{F}_2}(\Gamma; \mathbb{Z}) \to H^3(\Gamma; \mathbb{Z}) \oplus \bigoplus_{H \in \mathcal{H}} H^3_{\mathcal{F}_2}(H; \mathbb{Z}) \to \bigoplus_{H \in \mathcal{H}} H^3(\Gamma; \mathbb{Z}) \to \cdots \]

Since there is a 0-dimensional model for $E\mathcal{F}_2 H$ for all $H \in \mathcal{H}$, $H^3_{\mathcal{F}_2}(H; \mathbb{Z}) = 0$, therefore $\bigoplus_{H \in \mathcal{H}} H^3_{\mathcal{F}_2}(H; \mathbb{Z}) = 0$. By [Mat07] Theorem 1.1.7 there exists a 2-dimensional complex $X \subseteq M$ such that $X$ has the same homotopy type of $M$. Thus the universal cover $\tilde{X}$ of $X$ is a model for $E\Gamma$.

We conclude that $H^3(\Gamma; \mathbb{Z}) = H^3(\Gamma) = H^3(\Gamma; \mathbb{Z}) = 0$.

As a consequence of (2), and the observations in the previous paragraph we get the following exact sequence

\[ \cdots \to H^2(\Gamma; \mathbb{Z}) \xrightarrow{\partial} \bigoplus_{H \in \mathcal{H}} H^2(H; \mathbb{Z}) \to H^3_{\mathcal{F}_2}(\Gamma; \mathbb{Z}) \to 0 \]

In order to show that $H^3_{\mathcal{F}_2}(\Gamma; \mathbb{Z}) \neq 0$, it is enough to show that the map $\varphi$ is not surjective. Since every $\mathbb{Z}^2$ subgroup of $\Gamma$ is conjugated to a unique parabolic subgroup, the map $\varphi$ is the map induced by the inclusion $\partial M \hookrightarrow M$. Using the long exact sequence of the pair $(M, \partial M)$

\[ \cdots \to H^2(M, \partial M) \to H^2(M) \xrightarrow{\partial} H^2(\partial M) \to H^3(M, \partial M) \to 0 \]

we can see that $\varphi$ is not surjective if and only if $H^3(M, \partial M) \neq 0$. By Poincaré duality $H^3(M, \partial M) \cong H_0(M) = \mathbb{Z}$. Therefore $\varphi$ is not surjective. This finishes the proof. \qed

4. The Seifert fiber case

In this section we compute the $\mathcal{F}_2$-dimension of Seifert fiber manifolds with and without boundary. These computations are carried out in Theorem 4.1, Theorem 4.2, Theorem 4.3, and Theorem 4.4.

4.1. Seifert fiber manifolds with bad or spherical base orbifold.

**Theorem 4.1.** [ILST19] Proposition 5.1] Let $M$ be a closed Seifert fiber 3-manifold with base orbifold $B$ and fundamental group $\Gamma$. Assume that $B$ is either a bad orbifold, or a good orbifold modeled on $S^2$. Then $\Gamma$ is virtually cyclic, in particular $\text{gd}_{\mathcal{F}_2}(\Gamma) = 0$ for all $k \geq 1$.

4.2. Seifert fiber manifolds with hyperbolic base orbifold.

**Theorem 4.2.** Let $M$ be a Seifert fiber 3-manifold (possibly with non-empty boundary) with base orbifold $B$ and fundamental group $\Gamma$. Assume that $B$ is modeled on $\mathbb{H}^2$, then $\text{gd}_{\mathcal{F}_2}(\Gamma) = 2$.

**Proof.** First, we are going to show that $\text{gd}_{\mathcal{F}_2}(\Gamma) \leq 2$, for this we will construct a model for $E\mathcal{F}_2 \Gamma$ of dimension 2 using Theorem 2.9. By Lemma 2.4 we obtain an exact sequence of groups

\[ 1 \to K \to \Gamma \xrightarrow{\varphi} \Gamma_0 \to 1 \]

where $\Gamma_0 = \pi_1^{\text{orb}}(B)$ is the orbifold fundamental group of $B$ and $K$ is a cyclic infinite subgroup of $\Gamma$, generated by a regular fiber. Let $\mathcal{F}_n = \{ H \leq \Gamma_0 : H \text{ is virtually } \mathbb{Z}^n \}$ for some $r = 0, 1, \ldots, n$. Note that, since $\Gamma_0$ is a Fuchsian group, it cannot have a subgroup isomorphic to $\mathbb{Z}^2$, therefore $\mathcal{F}_2 = \mathcal{F}_1$. 

\[ 2 \to \Gamma \to \mathcal{F}_2 \xrightarrow{\varphi} \Gamma_0 \to 1 \]
By [LO07] Theorem 2.6 the collection $\mathcal{A} = \{ H < \Gamma_0 : H \) is virtually $\mathbb{Z}$ and maximal in $\mathcal{F}_1 \setminus \mathcal{F}_0 \}$ is adapted to the pair $(\mathcal{F}_0, \mathcal{F}_1)$.

Consider the pulled-back families $\tilde{\mathcal{F}}_0$ and $\tilde{\mathcal{F}}_1$ which by definition are generated by $\{ \varphi^{-1}(L) : L \in \mathcal{F}_0 \}$ and $\{ \varphi^{-1}(L) : L \in \mathcal{F}_1 \}$ respectively. We claim that $\tilde{\mathcal{F}}_0 \subset \tilde{\mathcal{F}}_2 \subset \tilde{\mathcal{F}}_1$. The first inclusion follows from the fact that $\varphi^{-1}(L)$, with $L \in \mathcal{F}_0$, is a virtually cyclic subgroup of $\Gamma$. While the second follows from the following argument. For $L \in \mathcal{F}_2$, we have three options $L$ is finite, $L$ is virtually $\mathbb{Z}$ or $L$ is virtually $\mathbb{Z}^2$. If $L$ is virtually $\mathbb{Z}$, then there is a subgroup $L' < L$ of finite index isomorphic to $\mathbb{Z}^2$, note that $\varphi(L')$ is a cyclic subgroup of $\Gamma_0$ that has finite index in $\varphi(L)$, thus $\varphi(L) \in \mathcal{F}_1$. We conclude that $L \subset \varphi^{-1}(\varphi(L)) \in \tilde{\mathcal{F}}_1$. The other cases follow using a completely analogous argument.

Let $\mathcal{H}$ be a complete set of representatives of conjugacy classes in $\mathcal{A}^* = \{ \varphi^{-1}(H) : H \in \mathcal{A} \}$. Then by Theorem 2.29 we have the following $\Gamma$-push-out gives a model $X$ for $E_{\mathcal{F}_2} \Gamma$

\[
\begin{array}{ccc}
\coprod_{\tilde{H} \in \mathcal{H}} \Gamma \times \tilde{H} E_{\mathcal{F}_0} \tilde{H} & \longrightarrow & E_{\mathcal{F}_0} \Gamma_0 \\
\downarrow & & \downarrow \\
\coprod_{\tilde{H} \in \mathcal{H}} \Gamma \times \tilde{H} E_{\mathcal{F}_2} \tilde{H} & \longrightarrow & X
\end{array}
\]

where $\tilde{H}$ stands for $\varphi^{-1}(H)$.

We claim that $X$ can be choosen to be of dimension 2. Note that the dimension of $X$ is

\[\max\{\dim(E_{\mathcal{F}_0} \tilde{H}) + 1, \dim(E_{\mathcal{F}_0} \Gamma_0), \dim(E_{\mathcal{F}_2} \tilde{H})\}.\]

Since $H \in \mathcal{A}$ we have that $H$ is virtually $\mathbb{Z}$, then a model for $E_{\mathcal{F}_0} \tilde{H}$ is $\mathbb{R}$ for all $\tilde{H} \in \mathcal{H}$. As a consequence of [BH99] corollary 2.8 a model for $E_{\mathcal{F}_0} \Gamma_0$ is $\mathbb{H}^2$. Finally, we show that $\tilde{H}$ is virtually $\mathbb{Z}^2$ for all $\tilde{H} \in \mathcal{H}$, and in consequence the one point space is a model for $E_{\mathcal{F}_2} \tilde{H}$ for all $\tilde{H} \in \mathcal{H}$. Let $\tilde{H} \in \mathcal{H}$, by definition of $\mathcal{H}$, $\tilde{H} \in \mathcal{A}^*$, then $\tilde{H} = \varphi^{-1}(S)$ for some $S \in \mathcal{A}$. The group $S$ is virtually $\mathbb{Z}$, i.e. there is a subgroup $T < S$ such that $T$ is a finite index subgroup of $S$ isomorphic to $\mathbb{Z}$. Thus by Eq. (4) we have the following short exact sequence

\[1 \to K \to \varphi^{-1}(T) \to T \to 1,\]

then $\varphi^{-1}(T)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ which is virtually $\mathbb{Z}^2$. It follows that $\tilde{H}$ is virtually $\mathbb{Z}^2$. Therefore $X$ can be constructed to be a 2-dimensional $\Gamma$-CW-complex.

The group $\Gamma$ is finitely generated as it is the fundamental group of a compact manifold. On the other hand $\Gamma \notin \mathcal{F}_2$ since it has as a quotient the group $\Gamma_0$ [4], which is either virtually non-cyclic free or virtually a surface group. Therefore, by Lemma 2.2, $\text{gd}_{\mathcal{F}_2}(\Gamma) \geq 2$. 

4.3. Seifert fiber manifolds with flat base orbifold.

**Theorem 4.3.** Let $M$ be a Seifert fiber closed 3-manifold (without boundary) with base orbifold $B$ and fundamental group $\Gamma$. Suppose that $B$ is modeled on $\mathbb{E}^2$. Then $M$ is modeled on $\mathbb{E}^3$ or is modeled on Nil. Moreover,

a) If $M$ is modeled on $\mathbb{E}^3$, then $\text{gd}_{\mathcal{F}_3}(\Gamma) = 5$.

b) If $M$ is modeled on Nil, then $\text{gd}_{\mathcal{F}_3}(\Gamma) = 3$.

**Theorem 4.4.** Let $M$ be a compact Seifert fiber 3-manifold, with base orbifold $B$ and fundamental group $\Gamma$. Suppose that $B$ is modeled on $\mathbb{E}^2$ and $M$ has non-empty boundary. Then $M$ is diffeomorphic to $T^2 \times I$ or the twisted I-bundle over Klein bottle. In particular, $\text{gd}_{\mathcal{F}_k}(\Gamma) = 0$ for all $k \geq 2$.

Before proving the above theorems, let us set some notation. For $A \in GL_2(\mathbb{Z})$, we say $A$ is

a) Elliptic, if it has finite order.
b) Parabolic, if it is conjugated to a matrix of the form

\[
\begin{pmatrix}
1 & s \\
0 & 1
\end{pmatrix}
\]

for some \( s \neq 0 \).

c) Hyperbolic, if it is not elliptic nor parabolic.

It is well-known that every matrix in \( A \in \text{GL}_2(\mathbb{Z}) \) is either elliptic, parabolic or hyperbolic. Moreover, \( A \) is elliptic (resp. parabolic, hyperbolic) if and only if \( A' \) is elliptic (resp. parabolic, hyperbolic) for all \( r \geq 1 \).

**Theorem 4.5.** Let \( M \) be a 3-manifold with fundamental group isomorphic to \( \Gamma = \mathbb{Z}^2 \rtimes_{\varphi} \mathbb{Z} \) where \( \varphi: \mathbb{Z} \to \text{Aut}(\mathbb{Z}^2) = \text{GL}_2(\mathbb{Z}) \) is a homomorphism. Then the following statements hold.

a) If \( \varphi(1) \) is elliptic, then \( \text{gd}_{\mathcal{F}}(\Gamma) = 5 \).

b) If \( \varphi(1) \) is parabolic, then \( \text{gd}_{\mathcal{F}}(\Gamma) = 3 \).

c) If \( \varphi(1) \) is hyperbolic, then \( \text{gd}_{\mathcal{F}}(\Gamma) = 2 \).

For the proof of Theorem 4.5 we need the following three lemmas.

**Lemma 4.6.** Let \( \Gamma \) be a virtually \( \mathbb{Z}^2 \)-group, then \( \text{gd}_{\mathcal{F}}(\Gamma) = 5 \).

*Proof.* In [Ono18] Proposition A they show that \( \text{gd}_{\mathcal{F}}(\mathbb{Z}^2) = 5 \), in consequence \( \text{gd}_{\mathcal{F}}(\Gamma) \geq 5 \). On the other hand, in [Pry18] Proposition 1.3. it is proven that \( \text{gd}_{\mathcal{F}}(\Gamma) \leq 5 \). Thus \( \text{gd}_{\mathcal{F}}(\Gamma) = 5 \). \( \square \)

**Lemma 4.7.** Let \( \Gamma = \mathbb{Z}^2 \rtimes_{\varphi} \mathbb{Z} \) with \( \varphi(1) = A \) a parabolic element in \( \text{GL}_2(\mathbb{Z}) \). Then

a) The matrix \( A \) fixes a infinite cyclic subgroup of \( \mathbb{Z}^2 \). Moreover the infinite cyclic maximal subgroup \( N \) fixed by \( A \) is normal in \( \Gamma \) and \( \Gamma/N \) is isomorphic to \( \mathbb{Z}^2 \) or to \( \mathbb{Z} \times \mathbb{Z} \).

b) Consider the homomorphism \( \pi: \Gamma \to \Gamma/N \) and let \( \mathcal{F}'_1 \) be the family of virtually cyclic subgroups of \( \Gamma/N \). Then the pulled-back family \( \mathcal{F}'_1 \) of \( \mathcal{F}'_1 \) is equal to the family \( \mathcal{F}_2 \) of \( \Gamma \).

*Proof.* First we prove part a). Since \( A \) is parabolic, without loss of generality we can assume that \( A \) is a matrix of the form (6). Therefore, \( A \) fixes a maximal infinite cyclic subgroup \( N \) of \( \mathbb{Z}^2 \). It is easy to see that \( N \) lies in the center of \( \Gamma \), and it follows that \( N \) is a normal subgroup of \( \Gamma \). Note that \( \Gamma/N = (\mathbb{Z}^2/N) \rtimes \mathbb{Z} \) is isomorphic to \( \mathbb{Z}^2 \) or to \( \mathbb{Z} \times \mathbb{Z} \).

Now we proof part b). First we show the inclusion \( \mathcal{F}'_1 \subseteq \mathcal{F}_2 \). Note that both \( \Gamma \) and \( \Gamma/N \) are torsion free groups, and in consequence every virtually cyclic subgroup of them is either trivial or infinite cyclic. By definition of \( \mathcal{F}'_1 \) it is enough to prove that \( \pi^{-1}(L) \in \mathcal{F}_2 \) for every infinite cyclic subgroup \( L \) of \( \Gamma/N \). Let \( L \) be an infinite cyclic subgroup of \( \Gamma/N \), hence \( \pi^{-1}(L) \) fits in the following short exact sequence

\[
1 \to \mathbb{Z} \to \pi^{-1}(L) \to \mathbb{Z} \to 1
\]

and we conclude that \( \pi^{-1}(L) \) is isomorphic to \( \mathbb{Z}^2 \) or to \( \mathbb{Z} \times \mathbb{Z} \). In either case \( \pi^{-1}(L) \) is virtually \( \mathbb{Z}^2 \).

Now we show the inclusion \( \mathcal{F}_2 \subseteq \mathcal{F}'_1 \). Let \( L \in \mathcal{F}_2 \), we have three cases: \( L \) is trivial, \( L \) is infinite cyclic, or \( L \) is virtually \( \mathbb{Z}^2 \). If \( L \) trivial, there is nothing to prove. If \( L \) is isomorphic to \( \mathbb{Z} \), then \( \pi(L) \) is trivial or infinite cyclic, and therefore \( \pi(L) \in \mathcal{F}'_1 \). Finally, suppose that \( L \) is virtually \( \mathbb{Z}^2 \), i.e there is a finite index subgroup \( T < L \) such that \( T \) is isomorphic to \( \mathbb{Z}^2 \). We claim that \( T \cap N \) is non-trivial. Suppose \( T \cap N = 1 \), since \( N \) is a subgroup of the center of \( \Gamma \), a generator \( h \) of \( N \) commutes with generators \( \alpha \) and \( \beta \) of \( T \), thus the subgroup generated by \( h, \alpha, \beta \) is isomorphic to \( \mathbb{Z}^3 \). In consequence \( \text{gd}_{\mathcal{F}}(\Gamma) \geq 4 \), but this contradicts the fact that \( \text{gd}_{\mathcal{F}}(\Gamma) = 3 \) (see [JLS19] Proposition 5.4.). From our claim since \( \Gamma/N \) is torsion free, we can see that \( \pi(T) \) is infinite.
cyclic. Since $\pi(T)$ is of finite index in $\pi(L)$, it follows that $\pi(L)$ is virtually $\mathbb{Z}$, then $\pi(L) \in F'_2$. Therefore $L < \pi^{-1}(\pi(L)) \in F'_2$.

**Lemma 4.8.** Let $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$ with $\varphi(1) = A$ a hyperbolic element in $GL_2(\mathbb{Z})$. Let $H$ be the subgroup $\mathbb{Z}^2 \rtimes \{0\}$ of $\Gamma$. Then

a) Every subgroup of $\Gamma$ isomorphic to $\mathbb{Z}^2$ is subgroup of $H$. In particular, the family $F_2 = F_1 \cup \text{SUB}(H)$, where $\text{SUB}(H)$ is the family of all subgroups of $H$.

b) Let $C$ be an infinite cyclic subgroup of $\Gamma$, then

$$N_{\Gamma}C \cong \begin{cases} \mathbb{Z} & \text{if } C \not\leq H \\ \mathbb{Z}^2 & \text{if } C \leq H \end{cases}$$

**Proof.** First we proof the part a) by contradiction. Suppose that there is $L$ a subgroup of $\Gamma$ isomorphic to $\mathbb{Z}^2$ that is not subgroup of $H$. From the short exact sequence

$$1 \to H \to \Gamma \xrightarrow{\varphi} \mathbb{Z} \to 1$$

we obtain the following short exact sequence

$$1 \to L \cap H \to L \to \psi(L) \to 1 \tag{8}$$

Since that $L$ is not subgroup of $H = \ker \varphi$, we have that $\psi(L)$ is non-trivial, then $\psi(L) = \langle r \mathbb{Z} \rangle$ with $r \neq 0$. Since $L$ is isomorphic to $\mathbb{Z}^2$ and $\psi(L) = \langle r \mathbb{Z} \rangle$, it follows that $L \cap H$ is isomorphic to $\mathbb{Z}$, in consequence $L = (L \cap H) \times \langle r \mathbb{Z} \rangle$. Thus $A^r$ fixes an infinite cyclic subgroup of $H \cong \mathbb{Z}^2$, and therefore $A^r$ is elliptic, which implies that $A$ is elliptic. This is a contradiction since we were assuming that $A$ is hyperbolic.

Now we going to show that $F_2 = F_1 \cup \text{SUB}(H)$. It is clear that $F_1 \cup \text{SUB}(H) \subseteq F_2$. It only remains to prove that $F_2 \subseteq F_1 \cup \text{SUB}(H)$. Let $L \in F_2$ then $L$ is either virtually cyclic or $L$ is virtually $\mathbb{Z}^2$. If $L$ is virtually cyclic, by definition $L \in F_1 \subseteq F_1 \cup \text{SUB}(H)$. Now suppose that $L$ is virtually $\mathbb{Z}^2$. From $[S]$ we get the following short exact sequence

$$1 \to L \cap H \to L \to \psi(L) \to 1 \tag{9}$$

Since $L$ is virtually $\mathbb{Z}^2$ and, by part a), $H$ contains all the subgroups of $\Gamma$ isomorphic to $\mathbb{Z}^2$ we have that $L \cap \mathbb{Z}^2$ is isomorphic to $\mathbb{Z}^2$. Then $[10]$ is equivalent to

$$1 \to \mathbb{Z}^2 \to L \to \psi(L) \to 1 \tag{10}$$

Once more, since $L$ is virtually $\mathbb{Z}^2$ we conclude that $\psi(L)$ must be trivial. It follows that $L \in \text{SUB}(H) \subseteq F_1 \cup \text{SUB}(H)$.

Now we prove part b). First suppose $C \not\leq H$, then the elements of $C$ are the form $((0,0), l)$. Let $((x,y), w) \in N_{\Gamma}C$, then

$$(x,y), w)((0,0), l)((x,y), w)^{-1} = ((x,y) + A^w(-x, -y), 1)$$

It follows that $A^w(-x, -y) = (-x, -y)$. By hypothesis $A$ is hyperbolic, thus $A^w$ is hyperbolic, therefore $(-x, -y) = (0,0)$. We conclude that $N_{\Gamma}C = C \cong \mathbb{Z}$.

Now suppose that $C \leq H$. From $[S]$ we have the following short exact sequence

$$1 \to N_{\Gamma}C \cap H \to N_{\Gamma}C \to \psi(N_{\Gamma}C) \to 1 \tag{12}$$

by hypothesis $C \leq H$, then $H < N_{\Gamma}C$, thus the sequence above is equivalent to

$$1 \to H \to N_{\Gamma}C \to \psi(N_{\Gamma}C) \to 1 \tag{13}$$
We are going to show that $\psi(N_1 C) = 0$, for this is enough to see that $N_1 C$ does not contain elements of the form $((a, b), l)$ with $l \neq 0$. Suppose that $C$ is generated by $((x, y), 0)$, then

$$((a, b), l)((x, y), 0)((a, b), l)^{-1} = (A^l(x, y), 0)$$

It follows that $A^l(x, y) = \pm (x, y)$. But we are assuming that $A$ is hyperbolic which implies that $A^l$ is hyperbolic. Therefore $\psi(N_1 C) = 0$. We conclude of (13) that $N_1 C \cong \mathbb{Z}^2$.

**Proof of Theorem 4.5.**

a) We claim that $\Gamma$ is virtually $\mathbb{Z}^3$. By hypothesis $A$ is elliptic, then $A$ has finite order. Let $n$ the order of $A$, then the subgroup $\langle 0 \rangle \times \langle n \mathbb{Z} \rangle$ acts trivially on the factor $\mathbb{Z}^2$. Therefore $\mathbb{Z}^2 \rtimes \langle n \mathbb{Z} \rangle = \mathbb{Z}^2 \rtimes \mathbb{Z}$ is a finite index subgroup of $\mathbb{Z}^2 \rtimes \mathbb{Z}$ isomorphic to $\mathbb{Z}^3$. Now the claim follows from Lemma 4.6.

b) First we construct a model for $E_{F_{\mathbb{Z}^2}} \Gamma$ of dimension 3, and as a consequence we get $\text{gd}_{E_{F_{\mathbb{Z}^2}}} (\Gamma) \leq 3$. By Lemma 4.7 part a) the matrix $A$ fixes a maximal infinite cyclic subgroup $N$ and $\Gamma/N = (\mathbb{Z}^2 / N) \rtimes \mathbb{Z}$ is isomorphic to either $\mathbb{Z}^2$ or $\mathbb{Z} \rtimes \mathbb{Z}$. In both cases $\Gamma/N$ is 2-crystallographic group, then by [CFH06] there is a 3-dimensional model $Y$ for $E_{F_1}(\Gamma/N)$ where $F_1$ is the family of virtually cyclic subgroups of $\Gamma/N$. By Lemma 4.7 part b) we have $F_1 \cap F_2 = F_2$, then $Y$ is also a model for $E_{F_{\mathbb{Z}^2}} \Gamma$ with the $\Gamma$-action induced by $\pi: \Gamma \to \Gamma/N$.

Now we show $\text{gd}_{E_{F_{\mathbb{Z}^2}}} (\Gamma) \geq 3$. It is enough to prove that $H^3_{F_{\mathbb{Z}^2}}(\Gamma, \mathbb{Z}) \neq 0$. Let $Y$ be the model for $E_{F_{\mathbb{Z}^2}} \Gamma$ that we mentioned in the previous paragraph, then

$$H^3_{F_{\mathbb{Z}^2}}(\Gamma; \mathbb{Z}) = H^3(Y/\Gamma; \mathbb{Z}) = H^3(Y/(\Gamma/N); \mathbb{Z}).$$

The latter homology group was shown to be nonzero in [CFH06] p.8 proof Theorem 1.1].

c) In order to prove $\text{gd}_{E_{F_{\mathbb{Z}^2}}} (\Gamma) \leq 2$, we construct a model for $E_{F_{\mathbb{Z}^2}} \Gamma$ of dimension 2. Let $H$ denote $\mathbb{Z}^2 \rtimes \{0\}$. By Lemma 4.8 part a) the family $F_2$ of $\Gamma$ is the union $F_2 = F_1 \cup \text{SUB}(H)$ where $\text{SUB}(H)$ is the family of all subgroups of $H$. By [DQR11] Lemma 4.4] the space $X$ given by the following homotopy $\Gamma$-push-out is a model for $E_{F_1} \Gamma$.

\[
\begin{array}{ccc}
E_{F_1(H)} \Gamma & \xrightarrow{f} & E_{F_1} \Gamma \\
\downarrow & & \downarrow \\
E_{\text{SUB}(H)} \Gamma & \xrightarrow{\iota} & X \\
\end{array}
\]

where $F_1(H) = \text{SUB}(H) \cap F_1$. We claim that, with suitable choices, $X$ is 2-dimensional. Since $H$ is a normal subgroup of $\Gamma$, $E(\Gamma/N) = E \mathbb{Z} = \mathbb{R}$ is a model for $E_{\text{SUB}(H)} \Gamma$. Now we are going to show that there exist models for $E_{F_1(H)} \Gamma$ and $E_{F_1} \Gamma$ such that the function $f$ in (14) is an inclusion. Since $\Gamma$ is a torsion free poly-$\mathbb{Z}$ group, by [LW12] Lemma 5.15, Theorem 2.3] we get the following $\Gamma$-push-out

\[
\begin{array}{ccc}
\bigsqcup_{C \in \Gamma} \Gamma \times_{N_1 C} E N_1 C & \xrightarrow{\iota} & E \Gamma \\
\downarrow & & \downarrow \\
\bigsqcup_{C \in \Gamma} \Gamma \times_{N_1 C} E W_1 C & \xrightarrow{\iota} & E_{F_1} \Gamma \\
\end{array}
\]
where $I$ is a set of representatives of commensuration classes in $F_1 - F_0$. By Lemma 4.5 part b) the set $I$ is the disjoint union of $I_1 = \{ C \in I | N_T C \cong \mathbb{Z} \} = \{ C \in I | C \not\leq \mathbb{Z}^2 \times \{0\} \}$ and $I_2 = \{ C \in I | C \leq \mathbb{Z}^2 \times \{0\} \}$. Therefore, the $\Gamma$-push-out (15) can be written as

$$
\begin{align*}
\bigcup_{C \in I_1} \Gamma \times_{N_T C} EN_T C & \quad \bigcup_{C \in I_2} \Gamma \times_{N_T C} EN_T C \quad \longrightarrow \quad E\Gamma \\
\bigcup_{C \in I_2} \Gamma \times_{N_T C} EW_T C & \quad \bigcup_{C \in I_2} \Gamma \times_{N_T C} EW_T C \quad \longrightarrow \quad E_{F_1} \Gamma
\end{align*}
$$

We note the following:

i) If $C \in I_1$ then a model for $EN_T C \cong \mathbb{R}$, thus it contributes to $E_{F_1} \Gamma$ with a subspace of dimension 2 in the homotopy push-out.

ii) If $C \in I_2$ then a model for $EN_T C \cong \mathbb{R}^2$, thus it contributes to $E_{F_1} \Gamma$ with a subspace of dimension 3 in the homotopy push-out.

iii) By [LW12] Theorem 5.13 $E\Gamma$ has a model of dimension 3.

Since $F_1(H)$ is the family of all virtually cyclic subgroups of $H$, we can use a straightforward variation of [LW12] Lemma 5.15, Theorem 2.3 to prove that we have the following $\Gamma$-push-out

$$
\begin{align*}
\bigcup_{C \in I_2} \Gamma \times_{N_T C} EN_T C & \quad \longrightarrow \quad E\Gamma \\
\bigcup_{C \in I_2} \Gamma \times_{N_T C} EW_T C & \quad \longrightarrow \quad E_{F_1(H)} \Gamma
\end{align*}
$$

With the models constructed for $E_{F_1} \Gamma$ and $E_{F_1(H)} \Gamma$ we see that $f$ in (14) is an inclusion, and therefore (14) can be taken to be just a $\Gamma$-pushout instead of a homotopic one. Therefore in the $\Gamma$-push-out (14) we are collapsing $E_{F_1(H)} \Gamma \subseteq E_{F_1} \Gamma$ down to $E_{\text{SUB}(H)} \Gamma = \mathbb{R}$ that has dimension 1. Therefore, by observations i)-iii) above, the push-out (14) gives a model for $E_{F_1} \Gamma$ of dimension 2.

Clearly $\Gamma$ is finitely generated and $\Gamma \not\subseteq F_2$, therefore by Lemma 2.2 $\text{gd}_{F_2}(\Gamma) \geq 2$. 

**Proof of Theorem 4.3.** The first conclusion follows from [Sco83] Theorem 5.3. For part a), we have that $\Gamma$ is 3-crystallographic, hence $\Gamma$ is virtually $\mathbb{Z}^3$. Hence part a) follows from Lemma 4.6.

For part b), we have that $\Gamma$ is free torsion as it is the fundamental group of a spherical manifold, then by [Tho68] Theorem N we obtain a short exact sequence

$$
1 \rightarrow \mathbb{Z}^2 \rightarrow \Gamma \rightarrow \mathbb{Z} \rightarrow 1
$$

where the morphism $\varphi: \mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}^2)$ is such that $\varphi(1) = A$ is parabolic. Hence by the Theorem 4.5 part b) we have $\text{gd}_{F_2}(\Gamma) = 3$. 

**Proof of Theorem 4.4.** By [Mor05] Theorem 1.2.2 $M$ is modeled on $\mathbb{R}^3$, Nil or $M$ is diffeomorphic to $T^2 \times I$ or the twisted $I$-bundle over the Klein bottle. The 3-manifold $M$ cannot be model on $\mathbb{R}^3$ and Nil as any such manifold must have empty boundary, see [Mor05] p. 60, paragraph 2 and last paragraph. We conclude that $M$ is diffeomorphic to $T^2 \times I$ or to the twisted $I$-bundle over the Klein bottle. The fundamental groups of $T^2 \times I$ and the twisted $I$-bundle over the Klein
bottle are isomorphic to \( Z^2 \) and \( Z \times Z \) respectively. In either case \( \Gamma \) is virtually \( Z^2 \) and therefore \( \text{gd}_{\mathcal{F}_2}(\Gamma) = 0 \).

5. The Sol case

In this section we compute the \( \mathcal{F}_2 \)-dimension of manifolds modeled on Sol. This computation is relevant in the statement and proof of Theorem 1.2.

**Proposition 5.1.** Let \( M \) be a connected closed 3-manifold modeled on Sol with fundamental group \( \Gamma \). Then \( \text{gd}_{\mathcal{F}_2}(\Gamma) = 2 \).

In order to prove Proposition 5.1 we need the following lemma. Let \( \mathcal{K} \) be the fundamental group of the Klein bottle.

**Lemma 5.2.** Let \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) be copies of \( \mathcal{K} \), and let \( \mathcal{A} \) be the index two \( \mathbb{Z}^2 \)-subgroup of \( \mathcal{K} \). Assume \( \varphi: \mathcal{A} \to \mathcal{A} \) is a hyperbolic isomorphism. Consider \( \Gamma = \mathcal{K}_1 * \mathcal{K}_2 \) the amalgamated product associated to \( \varphi \). Then

a) \( \mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{F}(\mathcal{K}_1, \mathcal{K}_2) \), where \( \mathcal{F}(\mathcal{K}_1, \mathcal{K}_2) \) is the smallest family of \( \Gamma \) containing \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \).

b) Let \( \mathcal{C} \) be an infinite cyclic subgroup of \( \Gamma \) then

\[
N_r C \cong \begin{cases} 
\text{is virtually } \mathbb{Z}^2 & \text{if } C < \mathcal{A} \\
\text{is virtually } \mathbb{Z} & \text{otherwise.} 
\end{cases}
\]

**Proof.** a) We show the inclusion \( \mathcal{F}_2 \subseteq \mathcal{F}_1 \cup \mathcal{F}(\mathcal{K}_1, \mathcal{K}_2) \). Let \( \mathcal{S} \in \mathcal{F}_2 \) then \( \mathcal{S} \) is virtually cyclic or virtually \( \mathbb{Z}^2 \). If \( \mathcal{S} \) is virtually cyclic by definition \( \mathcal{S} \in \mathcal{F}_1 \subseteq \mathcal{F}_1 \cup \mathcal{F}(\mathcal{K}_1, \mathcal{K}_2) \). Now, let \( \mathcal{S} \) be virtually \( \mathbb{Z}^2 \). We have the following short exact sequence

\[
1 \to \mathcal{A} \to \Gamma \overset{\psi}{\to} D_\infty \to 1
\]

Let \( L \) be the subgroup isomorphic to \( \mathcal{Z} \) of \( D_\infty \) of index 2. Then the subgroup \( \Gamma' = \psi^{-1}(L) \cong \mathcal{A} \rtimes \varphi L \leq \Gamma \) is also of index 2. It follows that \( \Gamma' \cap \mathcal{S} \) is finite index in \( \mathcal{S} \). Since \( \mathcal{S} \leq \Gamma \) is virtually \( \mathbb{Z}^2 \), then \( \Gamma' \cap \mathcal{S} \leq \Gamma' \) is also virtually \( \mathbb{Z}^2 \). By Lemma 4.8, \( \Gamma' \cap \mathcal{S} \leq \mathcal{A} \). Therefore in the next short exact sequence

\[
1 \to \mathcal{A} \cap \mathcal{S} \to \mathcal{S} \to \psi(\mathcal{S}) \to 1
\]

we have that \( \psi(\mathcal{S}) \) is finite. If we think of \( D_\infty \) as the free product of two cyclic groups of order two, we can easily see that every finite subgroup of \( D_\infty \) is either trivial or a conjugate of one of the free factors. It follows that \( \mathcal{S} \) is subconjugated to \( \mathcal{K}_1 \) or \( \mathcal{K}_2 \). The other inclusion \( \mathcal{F}_1 \cup \mathcal{F}(\mathcal{K}_1, \mathcal{K}_2) \subseteq \mathcal{F}_2 \) is clear.

b) First suppose \( C \leq \mathcal{A} \). Then by Lemma 4.8 \( N_r C \cap \Gamma' \) is isomorphic to \( \mathbb{Z}^2 \). Now \( \Gamma' \) is of index 2 in \( \Gamma \), then \( [N_r C : N_r C \cap \Gamma'] \leq 2 \). It follows that \( N_r C \) is virtually to \( \mathbb{Z}^2 \). Now suppose that \( C \not\leq \mathcal{A} \). By Lemma 4.8 we have \( N_r C \cap \Gamma' \) is isomorphic to \( \mathcal{Z} \). Now \( \Gamma' \) is of index 2 in \( \Gamma \), then \( [N_r C : N_r C \cap \Gamma'] \leq 2 \). It follows that \( N_r C \) is virtually to \( \mathcal{Z} \).

**Proof of Proposition 5.1.** First we show \( \text{gd}_{\mathcal{F}_2}(\Gamma) = 2 \). In [AFW15b, Theorem 1.8.2, p.17] they show that if \( M \) is modeled on Sol, then either \( M \) is the mapping torus of \( (T^2, A) \) with \( A \) Anosov or \( M \) is a double of \( K \times I \), the twisted \( I \)-bundle over the Klein bottle. In the first case we have that \( \Gamma = \mathbb{Z}^2 \times \varphi \mathcal{Z} \) with \( \varphi(1) = A \) hyperbolic. By Theorem 1.8 part c) \( \text{gd}_{\mathcal{F}_2}(\Gamma) = 2 \).

Suppose now that \( M \) is a double of \( K \times I \). In this case, by Seifert-Van Kampen theorem, \( \Gamma = \mathcal{K}_1 *_{\mathbb{Z}_2} \mathcal{K}_2 \) where \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are copies of \( \mathcal{K} \). In the proof of Lemma 5.4 we proved that \( \Gamma \) contains an index 2 subgroup isomorphic to \( \mathbb{Z}^2 \times \varphi \mathcal{Z} \) with \( A \) a hyperbolic isomorphism. It follows, from our previous case \( \text{gd}_{\mathcal{F}_2}(\Gamma) \geq \text{gd}_{\mathcal{F}_2}(\mathbb{Z}^2 \times \varphi \mathcal{Z}) = 2 \).
Now we construct a model for $E_{F_2} \Gamma$ of dimension 2, and as a consequence $\text{gd}_{F_2}(\Gamma) \leq 2$. By Lemma 6.2, $F_2 = F_1 \cup F(K_1, K_2)$ where $F(K_1, K_2)$ is the smallest family of $\Gamma$ containing $K_1$ and $K_2$. Then, by [DQR11] Lemma 4.4 the following $\Gamma$-push-out give a model for $E_{F_2} \Gamma$

\[
\begin{array}{ccc}
E_{F_1(K_1,K_2)^{\Gamma}} & \longrightarrow & E_{F_1}^{\Gamma} \\
\downarrow & & \downarrow \\
E_{F(K_1,K_2)^{\Gamma}} & \longrightarrow & X
\end{array}
\]

where $F_1(K_1,K_2) = F(K_1,K_2) \cap F_1$. The rest of the construction is a minor variation of the one described in the proof of Theorem 4.5 part c), and the details are left to the reader. □

6. A SMALL DETOUR ON ACYCLINDRICITY OF GROUPS ACTING ON TREES

In this section we prove Lemma 6.2 and Theorem 6.3. These results will be relevant in the next section to compute the $F_2$-dimension of fundamental group of prime manifolds from the $F_2$-dimension of the JSJ-pieces, and to compute the $F_2$-dimension of a 3-manifold group from the $F_2$-dimensions of the prime pieces.

In the following definition we state the notion of acylindricity, which will be key to understand the abelian subgroups of a 3-manifold group.

Definition 6.1. Let $Y$ be a graph of groups with fundamental group $G$. The splitting $G = \pi_1(Y)$ is acylindrical if there is an integer $k$ such that, for every path $\gamma$ of length $k$ in the Bass-Serre tree $T$ of $Y$, the stabilizer of $\gamma$ is finite.

Lemma 6.2. Let $Y$ be a graph of groups with fundamental group $\Gamma$, and Bass-Serre tree $T$. Suppose that the splitting of $\Gamma$ is acylindrical. Then

a) the setwise stabilizer of every geodesic line in $T$ is virtually cyclic.

b) every virtually $\mathbb{Z}^n$ subgroup of $\Gamma$ with $n \geq 2$ fixes a vertex of $T$.

Proof. a) Let $L$ be a geodesic line of $T$. Denote by $\text{Fix}_T(L)$ (resp. $\text{Stab}_T(L)$) the set of all elements of $\Gamma$ that fix $L$ pointwise (resp. setwise). Note that the group of simplicial automorphisms of $L$, denoted $\text{Aut}(L)$, is isomorphic to $D_{\infty}$, in particular, $D$ is virtually cyclic. Since $\text{Stab}_T(L)$ acts by simplicial automorphisms on $L$ with kernel $\text{Fix}_T(L)$, we have the following short exact sequence

$$1 \rightarrow \text{Fix}_T(L) \rightarrow \text{Stab}_T(L) \rightarrow D \rightarrow 1$$

where $D$ is isomorphic to a subgroup of $D_{\infty}$. Since $L$ contains arbitrarily long paths of $T$, the acylindricity hypothesis implies that $\text{Fix}_T(L)$ is finite. Therefore $\text{Stab}_T(L)$ is virtually cyclic.

b) Let $H$ be a virtually $\mathbb{Z}^n$ subgroup of $\Gamma$ with $n \geq 2$, then by Proposition 2.3 it happens exactly one of the following: either $H$ fixes a vertex of $T$ or $H$ acts co-compactly in a unique geodesic line $L$ of $T$. It remains to rule out the second possibility. Suppose that $H$ acts co-compactly in a unique geodesic line $L$ of $T$. Let $S$ be a finite index subgroup of $H$ isomorphic to $\mathbb{Z}^n$. Note that $S$ acts by restriction on $L$, i.e. we have a homomorphism $S \xrightarrow{\varphi} \text{Aut}(L)$ whose image contains an infinite subgroup of translations. Since $\text{Aut}(L) \cong D_{\infty}$, we have $\varphi(S)$ either is isomorphic to $\mathbb{Z}$ or to $D_{\infty}$. But $\varphi(S)$ is not isomorphic $D_{\infty}$, otherwise the abelian group $S/\text{ker}(\varphi)$ would be isomorphic to the non-abelian group $D_{\infty}$. The homomorphism $L \xrightarrow{\varphi} \text{Aut}(\gamma)$ induces the following short exact sequence

$$1 \rightarrow \ker(\varphi) \rightarrow S \rightarrow \varphi(L) \rightarrow 1.$$ 

Since $\ker(\varphi) \subseteq \text{Fix}(L)$, and the latter is finite by acylindricity, we conclude that $S \cong \mathbb{Z}^n$ embeds in $\mathbb{Z}$ which is a contradiction. □
In the following theorem we describe a 2-dimensional model $\tilde{T}$ for a classifying space with respect to a family that contains the family $F_n$. This $\tilde{T}$ will be used in the next section to compute the $F_2$ dimension of a 3-manifold group using the prime splitting and the JSJ decomposition.

**Theorem 6.3.** Let $Y$ be a graph of groups with fundamental group $\Gamma$ finitely generated and Bass-Serre tree $T$. Consider the collection $A$ of all the geodesics of $T$ that admit a co-compact action of an infinite virtually cyclic subgroup of $\Gamma$. Then the space $\tilde{T}$ given by the following homotopy $\Gamma$-push-out

$$
\begin{array}{ccc}
\bigcup_{\gamma \in A} \gamma & \longrightarrow & T \\
\downarrow & & \downarrow \\
\bigcup_{\gamma \in A} \{\gamma\} & \longrightarrow & \tilde{T}
\end{array}
$$

is a model $\tilde{T}$ for $E_{\text{Iso}(\Gamma)}$, where $\text{Iso}(\tilde{T})$ is the family generated by the isotropy groups of $\tilde{T}$, i.e. by coning-off on $T$ the geodesics in $A$ we obtain a model for $E_{\text{Iso}(\tilde{T})}$. Moreover, if the splitting $\Gamma = \pi_1(Y)$ is acylindrical, then the family $\text{Iso}(\tilde{T})$ contains the family $F_n$ of $\Gamma$ for all $n \geq 0$.

**Proof.** If $H \leq \Gamma$ acts cocompactly on the geodesic line $\gamma$ of $T$ and $g \in \Gamma$, then $gHg^{-1}$ acts cocompactly on $g\gamma$. It follows that both $\bigcup_{\gamma \in A} \gamma$ and $\bigcup_{\gamma \in A} \{\gamma\}$ are $\Gamma$-CW-complexes, and therefore the space $\tilde{T}$ is a $\Gamma$-CW-complex.

Clearly $\tilde{T}^K$ is nonempty if an only if $K \in \text{Iso}(\tilde{T})$. We will see that for $K \in \text{Iso}(\tilde{T})$ the fixed point set $\tilde{T}^K$ is contractible. We have two cases: $T^K \neq \emptyset$ or $T^K = \emptyset$. In the first case we have that $T^K$ is a sub-tree of $T$. Thus $\tilde{T}^K$ is obtained from $T^K$ by coning-off some geodesic segments, then the space $\tilde{T}^K$ which is contractible, it follows that $\tilde{T}^K$ is contractible. In the second case we have that $\tilde{T}^K$ consists of a union of some cone points. Note that $\gamma \in \tilde{T}^K$ if and only if $K \leq \text{Stab}_\Gamma(\gamma)$. By hypothesis $T^K = \emptyset$, then from [Ser03, Corollary 3] we have there is a hyperbolic element $h \in K$ that acts co-compact on $\gamma$. Since $h$ can only act cocompactly on a unique geodesic of $T$, we conclude $\tilde{T}^K = \star$, therefore it is contractible.

We show that the family $F_n$ of $\Gamma$ is contained in $\text{Iso}(\tilde{T})$. Let $K \in F_n$ then we have three cases: $K$ is finite, $K$ is virtually $\mathbb{Z}$ or $K$ is virtually $\mathbb{Z}^k$ with $k \geq 2$. If $K$ is finite, then it is well-known that $K$ has a point fix in $T$, thus $K \in \text{Iso}(\tilde{T})$. If $T$ is virtually $\mathbb{Z}$ then, by Proposition 2.3 $K$ fixes a vertex in $T$ or it acts cocompactly on a unique geodesic $\gamma$. In the first case that it is clear that $K \in \text{Iso}(\tilde{T})$, while for the second case $K$ fixes $\star$, and therefore $K \in \text{Iso}(\tilde{T})$. Finally, if $K$ is virtually $\mathbb{Z}^k$ with $k \geq 2$, we have by Lemma 6.2 that $K$ fixes a point in $T$, and therefore $K \in \text{Iso}(\tilde{T})$.

7. **Proofs of Theorem 1.1 and Theorem 1.2**

In this section we prove the main theorems of the present paper, but before we need some preliminary results.

**Proposition 7.1.** Let $\Gamma$ be fundamental group of a 3-manifold $M$ that it is either hyperbolic, Seifert fiber possibly with non-empty boundary, or modeled on $\text{Sol}$. The following statements hold

1. If $M$ is not modeled on $\mathbb{E}^3$, then $F_2 = F_k$ for all $k \geq 3$. In particular $\text{gd}_{F_2}(\Gamma) = \text{gd}_{F_k}(\Gamma)$ for all $k \geq 3$. 


(2) If $M$ is modeled on $E^3$, then $F_3 = F_k$ for all $k \geq 4$. Moreover, $gd_{F_k}(\Gamma) = 0$ for all $k \geq 3$.

**Proof.** By [Ono18, Proposition A], $gd_{F_3}(\mathbb{Z}^3) = 5$. As a consequence, if $\Gamma$ contains a subgroup isomorphic to $\mathbb{Z}^3$, then $gd_{F_3}(\Gamma) \geq 5$. By the second column of Table 1 and Proposition 5.1 we conclude that $\Gamma$ does not contain a $\mathbb{Z}^3$-subgroup unless $M$ is modeled on $E^3$. This proves the first item. The second item follows by noticing that if $M$ is modeled on $E^3$, then $\Gamma$ is virtually $\mathbb{Z}^3$.

**Corollary 7.2.** Let $\Gamma$ be the fundamental group of the 3-manifold $M$. Let $H$ be a $\mathbb{Z}^n$-subgroup of $\Gamma$, then $n \leq 3$. Moreover $\Gamma$ contains a $\mathbb{Z}^3$-subgroup if and only if one of the prime pieces of $M$ is modeled on $E^3$.

**Proof.** Let $n \geq 2$. Let $\Gamma = \Gamma_1 \ast \cdots \ast \Gamma_r$ be the splitting of $\Gamma$ associated to the prime decomposition of $M$, and let $H \leq \Gamma$ be a $\mathbb{Z}^n$-subgroup of $\Gamma$. Then by Kurosh subgroup theorem $H$, with out loss of generality, is a subgroup of $\Gamma_1$. Next we look at the graph of groups $Y$ given by the JSJ decomposition of $\Gamma_1$. Then by Theorem 7.3 the splitting $\Gamma_1 = \pi_1(Y)$ is acylindrical, and by Lemma 6.2 $H$ is a subgroup of the fundamental group of a JSJ piece $N$ of $\Gamma_1$. By Proposition 7.4 we conclude $n \leq 3$. Moreover, if we assume $n = 3$, by Proposition 7.1 such a piece must be modeled on $E^3$. On the other hand every manifold modeled on $E^3$ has empty boundary, thus $N$ is a prime piece of $\Gamma$. Finally, if $M$ has a prime piece modeled on $E^3$ it is clear that $\Gamma$ contains a $\mathbb{Z}^3$-subgroup.

**Theorem 7.3.** [JLS93] Proposition 8.2 Let $M$ be a closed, oriented, connected, prime 3-manifold which is not geometric. Let $Y$ be the graph of groups associated to its minimal JSJ decomposition. Then the splitting of $G = \pi_1(Y)$ as the fundamental group of $Y$ is acylindrical.

**Proposition 7.4.** Let $M$ be a closed, oriented, connected, prime 3-manifold which is not geometric. Let $N_1, N_2, \ldots, N_r$ be the pieces of the minimal JSJ decomposition of $M$. Denote $\Gamma = \pi_1(M)$ and $\Gamma_i = \pi_1(N_i)$. If $k \geq 2$, then

$$\max\{gd_{F_k}(\Gamma_i)|1 \leq i \leq r\} \leq gd_{F_k}(\Gamma) \leq \max\{2, gd_{F_k}(\Gamma_i)|1 \leq i \leq r\}.$$

**Proof.** For all $1 \leq i \leq r$, the group $\Gamma_i$ is a subgroup of $\Gamma$, then $gd_{F_k}(\Gamma_i) \leq gd_{F_k}(\Gamma)$, and the first inequality follows.

Now we show the second inequality. Let $Y$ the graph of groups associated to the JSJ decomposition of $M$ with Bass-Serre tree $T$. By Theorem 7.3 the decomposition of $\pi_1(M) = \Gamma$ is acylindrical. Therefore we can use Theorem 6.3 to obtain a 2-dimensional space $\tilde{T}$ what is obtained from $T$ coning-off some geodesics of $T$, the space $\tilde{T}$ is a model for $E_{\text{Isor}(\tilde{T})}(\Gamma)$ where $\text{Isor}(\tilde{T})$ is the family generated by the isotropy groups of $\tilde{T}$, and $F_k \subset \text{Isor}(\tilde{T})$. We have everything set up to apply Proposition 2.1 that is we only have to compute $gd_{F_k \cap \Gamma_{\sigma}}(\Gamma_{\sigma}) + \dim(\sigma)$ for each cell $\sigma$ of $\tilde{T}$. Once done this the proof will be complete.

- If the 0-cell $\sigma$ of $\tilde{T}$ belongs to $T$, then $\Gamma_{\sigma} = \Gamma_i$ for some $1 \leq i \leq r$. If the 0-cell $\sigma$ belongs to $\tilde{T} \setminus T$, then $\Gamma_{\sigma}$ is the setwise stabilizer of a geodesic of $\tilde{T}$, and therefore is virtually cyclic by Lemma 5.2. Hence in this case $gd_{F_k \cap \Gamma_{\sigma}}(\Gamma_{\sigma}) + \dim(\sigma) = gd_{F_k}(\Gamma_i)$ or 0.

- If the 1-cell $\sigma$ of $\tilde{T}$ belongs to $T$, then $\Gamma_{\sigma}$ is isomorphic to $\mathbb{Z}^2$. If the 1-cell $\sigma$ has a vertex in $\tilde{T} \setminus T$, then $\Gamma_{\sigma}$ is virtually cyclic as in the previous item. Hence in this case $gd_{F_k \cap \Gamma_{\sigma}}(\Gamma_{\sigma}) + \dim(\sigma) = 1$.

- If $\sigma$ is a 2-cell of $\tilde{T}$, then it always contain a vertex of $\tilde{T} \setminus T$, and therefore $\Gamma_{\sigma}$ is virtually cyclic. Hence in this case $gd_{F_k \cap \Gamma_{\sigma}}(\Gamma_{\sigma}) + \dim(\sigma) = 2$.

**Proof of Theorem 7.3** If the minimal JSJ decomposition of $M$ has only one piece, then the manifold cannot be modeled on Sol, since such manifolds are neither Seifert nor hyperbolic. Hence if $M$ has
only one JSJ piece the theorem follows. From now on, suppose that the minimal JSJ decomposition of \( M \) has more than one piece. We have two cases: \( M \) is geometric or not. If \( M \) is not geometric we claim that
\[
gd_{\mathcal{F}_k}(\Gamma) = \max\{\gcd(F_i(\Gamma))|1 \leq i \leq r\}.
\]
By Proposition 7.1 it is enough to see that there is a piece \( N_i \) in the minimal JSJ decomposition of \( M \) such that \( \gcd(F_i(\Gamma)) \geq 2 \). By definition, the pieces in the JSJ decomposition of \( M \) are either hyperbolic with boundary or Seifert fiber with boundary. If the JSJ decomposition of \( M \) has a hyperbolic piece or a Seifert fiber with base orbifold \( B \) modeled on \( \mathbb{H}^2 \), then we are done, since by the Table 1 the fundamental groups of these pieces have \( \gcd \) equal to 3 and 2 respectively. It remains to see what happens if we only have Seifert fiber pieces with base orbifold \( B \) modeled on \( \mathbb{E}^2 \). By Theorem 7.4 these pieces either are diffeomorphic to \( T^2 \times I \) or to twisted \( I \)-bundle over the Klein bottle. If we have a piece of the form \( T^2 \times I \), then minimal the JSJ decomposition of \( M \) would have only one piece, otherwise we will contradict the minimality of the JSJ decomposition. Then \( M \) would be the mapping torus of a self-diffeomorphism of \( T \), would be isomorphic to \( \mathbb{R} \). Therefore the factors of \( G \) are asomorphic to \( T \) and \( \mathbb{R} \). By Lemma 7.6 we have two cases: the group \( \Gamma \) is isomorphic to \( \mathbb{Z} \) and \( \mathbb{Z} \). Since the edge stabilizers of \( T \) are trivial, the splitting of \( \Gamma \) is acylindrical. Now the proof is completely analogous to the proof of Proposition 7.2 and the details are left to the reader.

**Theorem 7.5.** Let \( M \) be a closed, connected, oriented 3-manifold. Let \( P_1, P_2, \ldots, P_r \) be the pieces of the prime decomposition of \( M \). Denote \( \Gamma = \pi_1(M) \) and \( \Gamma_i = \pi_1(P_i) \). If \( k \geq 2 \), then
\[
\max\{\gcd(F_i(\Gamma))|1 \leq i \leq r\} \leq \gcd(F_k(\Gamma)) \leq \max\{2, \gcd(F_i(\Gamma))|1 \leq i \leq r\}.
\]

**Proof.** Let \( T \) be the Bass-Serre tree of the splitting \( \Gamma = \Gamma_1 \ast \cdots \ast \Gamma_r \). Since the edge stabilizers of \( T \) are trivial, the splitting of \( \Gamma \) is acylindrical. Now the proof is completely analogous to the proof of Proposition 7.2 and the details are left to the reader.

**Lemma 7.6.** Let \( G = H_1 \ast \cdots \ast H_k \) with \( k \geq 2 \) and \( H_i \neq 1 \) for all \( i \). Then exactly one of the following hold:
\begin{enumerate}[a)]
  \item \( G \) is isomorphic to \( D_\infty \) with \( k = 2 \) and \( H_1, H_2 \) isomorphic to \( \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \)
  \item \( G \) contains a non-cyclic free subgroup.
\end{enumerate}

**Proof.** Consider \( G \) as the following two fold free product \( H_1 \ast (H_2 \ast \cdots \ast H_k) \). By Lemma 1.11.2, p.24] this free product contains a non-cyclic free subgroup unless the factors are isomorphic to \( \mathbb{Z}_2 \). By hypothesis \( H_i \neq 1 \) for all \( i \), then \( H_2 \ast \cdots \ast H_k \) is isomorphic to \( \mathbb{Z}_2 \) if and only if it has only one factor isomorphic to \( \mathbb{Z}_2 \). Therefore the factors of \( G = H_1 \ast (H_2 \ast \cdots \ast H_k) \) are isomorphic to \( \mathbb{Z}_2 \) if and only if \( k = 2 \) and \( H_1, H_2 \) are isomorphic to \( \mathbb{Z}_2 \).

**Proof of Theorem 7.4.** If we only have one piece the theorem follows as we are necessarily in the third case of our conclusion. From now on suppose that we have at least two pieces in the prime decomposition. Then \( \Gamma = \pi_1(P_1) \ast \pi_1(P_2) \ast \cdots \ast \pi_1(P_r) \) with \( r \geq 2 \). By Lemma 7.3 we have two cases: the group \( \Gamma \) is isomorphic to \( D_\infty \) with \( r = 2 \) and \( \pi_1(P_1), \pi_1(P_2) \) are isomorphic to \( \mathbb{Z}_2 \), or \( \Gamma \) contains a non-cyclic free subgroup. In the first case we are done, since \( \pi_1(P_1) = \mathbb{Z}_2 \) implies that \( P_1 \) is homeomorphic to \( \mathbb{R}P^3 \).
From now on suppose also that $\Gamma$ contains a non-cyclic free subgroup. Then $\Gamma$ is not virtually abelian, and by Lemma 2.2 we get $\text{gd}_{\mathbb{F}_i}(\Gamma) \geq 2$. Next we consider two cases for fix a $k$: $\text{gd}_{\mathbb{F}_i}(\Gamma_i) = 0$ for all $i$, or not. In the first case, by Theorem 7.5 $\text{gd}_{\mathbb{F}_i}(\Gamma) \leq 2$, and therefore $\text{gd}_{\mathbb{F}_i}(\Gamma) = 2$, hence we are done in this case. In the second case we have that there is a $\Gamma_s$ such that $\text{gd}_{\mathbb{F}_i}(\Gamma_s) \neq 0$, and by Lemma 2.2 we have $\text{gd}_{\mathbb{F}_i}(\Gamma_s) \geq 2$. Therefore by Theorem 7.5

$$\text{gd}_{\mathbb{F}_s} = \max\{\text{gd}_{\mathbb{F}_i}(\Gamma_i) | 1 \leq i \leq r\},$$

and we are done in this final case.

To finish, we prove the moreover part of the statement. Assume $\Gamma = \pi_1(M)$ is virtually abelian. By [AFW15b] Theorem 1.11.1, $M$ is spherical, $\mathbb{R}P^3 \# \mathbb{R}P^5$, $S^1 \times S^2$, or is covered by a torus bundle. If $\Gamma$ is covered by a torus bundle, then it has a finite index subgroup isomorphic to $K = \mathbb{Z}^2 \rtimes \mathbb{Z}$, and therefore $K$ must be also virtually abelian. Since $K$ is poly-$\mathbb{Z}$ of rank 3, then $K$ must be virtually $\mathbb{Z}^3$, thus $K$ and $\Gamma$ are modeled on $\mathbb{E}^3$. Now the moreover part follows easily. \qed

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