ON THE SOLVABILITY OF THE MATRIX EQUATION

\[(1 + ae^{-\frac{\lambda_1}{b}})X = Y\]

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Abstract. The treated matrix equation \((1 + ae^{-\frac{\lambda_1}{b}})X = Y\) in this short note has its origin in a modelling approach to describe the nonlinear time-dependent mechanical behaviour of rubber. We classify the solvability of \((1 + ae^{-\frac{\lambda_1}{b}})X = Y\) in general normed spaces \((E, \parallel \cdot \parallel)\) w.r.t. the parameters \(a, b \in \mathbb{R}, b \neq 0\), and give an algorithm to numerically compute its solutions in \(E = \mathbb{R}^{m \times n}, m, n \in \mathbb{N}, m, n \geq 2\), equipped with the Frobenius norm.

1. Introduction

In [3] the common approach to extend hyperelastic models by a well-known Prony series is modified. In general, the classic approach using a Prony series for extension results in the need to identify a large number of parameters. The identification is usually an ill-posed problem. Therefore in [3], the authors restrict themselves to a single modified Prony element with a load-dependent relaxation time leading to an approach with only two parameters. Using an implicit Euler-approach (see [3, Eq. (26), p. 7]), solving the underlying matrix differential equation yields to

\[S_{v,k+1} - S_{v,k} = -S_{v,k+1} \frac{\exp(\frac{[S_{v,k+1}]}{\sigma_c})}{\tau_p} + \frac{\Delta S_{el}}{\Delta t}, \quad k \in \mathbb{N}_0,\]

where \(S_v \in \mathbb{R}^{3 \times 3}\) is the deviatoric stress of the modified Prony element, \(S_{el} \in \mathbb{R}^{3 \times 3}\) the elastic Cauchy stress, \(\parallel \cdot \parallel\) the Frobenius norm, \(\sigma_c > 0\) the critical stress, \(\tau_p > 0\) the relaxation timescale in the effective relaxation time and \(\Delta t > 0\) a time step. Equation (1) can be rewritten as

\[(1 + \frac{\Delta t}{\tau_p} \exp(\frac{[S_{v,k+1}]}{\sigma_c})))S_{v,k+1} = \Delta S_{el} + S_{v,k},\]

which has the general form

\[(1 + a \exp(-\frac{\lambda_1}{b})X = Y\]

with \(X := S_{v,k+1}, Y := \Delta S_{el} + S_{v,k}, a := \frac{\Delta t}{\tau_p}\) and \(b := -\sigma_c\).

2. Classification of the solvability in general normed spaces

Let \((E, \parallel \cdot \parallel)\) be a normed space over the field \(K = \mathbb{R}\) or \(\mathbb{C}\), \(a, b \in \mathbb{R}, b \neq 0\), and \(Y \in E\). We are searching for a solution \(X \in E\) of the vector equation

\[(1 + ae^{-\frac{\lambda_1}{b}})X = Y.\]

If we take norms on both sides of (3), then we obtain the scalar equation

\[\|1 + ae^{-\frac{\lambda_1}{b}}\|X\| = \|Y\|\].

The solutions of (3) and (4) are related in the following manner.

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2.1. Proposition. Let \((E, | \cdot |)\) be a normed space, \(Y \in E\) and \(y := |Y|\).

(a) If \(x \in [0, \infty)\) is a solution of \(|1 + ae^{-x/b}|x = y\) and \(1 + ae^{-x/b} \neq 0\), then \(X := (1 + ae^{-x/b})^{-1}Y \in E\)

fulfils \(|X| = x\) and \(X\) is a solution of \((1 + ae^{-|X|/b})X = Y\).

(b) If \(X \in E\) is a solution of \((1 + ae^{-|X|/b})X = Y\), then \(x := |X| \in [0, \infty)\) is a solution of \(|1 + ae^{-x/b}|x = y\).

Proof. Statement (b) is obvious, so we only need to prove (a), which follows from

\[\|X\| = (1 + ae^{-|x|/b})^{-1}\|Y\| = (1 + ae^{-|y|/b})^{-1}\|y\| = (1 + ae^{-x/b})^{-1}\|1 + ae^{-x/b}|x\]



\(\square\)

Hence we can use the solutions of the scalar equation \((6)\) to obtain the solutions of the vector equation \((3)\). Lambert’s W function will turn out to be a useful tool to solve the scalar equation \((6)\) with respect to the parameters involved. In the following we denote by \(W_0:[-e^{-1}, \infty) \to [-1, \infty)\) the upper branch and by \(W_{-1}:[-e^{-1}, 0) \to (-\infty, -1)\) the lower branch of Lambert’s W function. These branches are bijective functions and the union of the sets \(\{W_0(z) \mid z \in [-e^{-1}, \infty)\}\) \(\{W_{-1}(z) \mid z \in [-e^{-1}, 0)\}\) is the set of all solutions \(x \in \mathbb{R}\) of the equation \(xe^x = z\) (see e.g. \((1)\)).

2.2. Proposition. Let \(a, b \in \mathbb{R}, b \neq 0,\) and \(y \in [0, \infty)\). The equation

\(|1 + ae^{-x/b}|x = y\) (6)

has in \([0, \infty)\)

(a) a unique solution if \(a \geq 0\) and \(b < 0\),
(b) a unique solution if \(a \leq -1\) and \(b < 0\),
(c) two solutions \(x = 0\) or \(x = \text{b\,ln}\,(|a|)\) if \(-1 < a < 0, b < 0\) and \(y = 0\),
(d) three solutions if \(-1 < a < 0, b < 0\) and \(0 < y < (1 + ae^{W_0(-e/a)})^{-1}b(1 - W_0(-e/a))\),
(e) two solutions if \(-1 < a < 0, b < 0\) and \(y = (1 + ae^{W_0(-e/a)})^{-1}b(1 - W_0(-e/a))\),
(f) a unique solution if \(-1 < a < 0, b < 0\) and \(y > (1 + ae^{W_0(-e/a)})^{-1}b(1 - W_0(-e/a))\),
(g) a unique solution if \(0 \leq a \leq e^2\) and \(b > 0\),
(h) a unique solution if \(a > e^2, b > 0\) and \(y < (1 + ae^{W_{-1}(-e/a)})^{-1}b(1 - W_{-1}(-e/a))\) or \(y > (1 + ae^{W_{-1}(-e/a)})^{-1}b(1 - W_{-1}(-e/a))\),
(i) two solutions if \(a > e^2, b > 0\) and \(y = (1 + ae^{W_{-1}(-e/a)})^{-1}b(1 - W_{-1}(-e/a))\) or \(y = (1 + ae^{W_{-1}(-e/a)})^{-1}b(1 - W_{-1}(-e/a))\),
(j) three solutions if \(a > e^2, b > 0\) and \((1 + ae^{W_{-1}(-e/a)})^{-1}b(1 - W_{-1}(-e/a)) < y < (1 + ae^{W_{-1}(-e/a)})^{-1}b(1 - W_{-1}(-e/a))\),
(k) a unique solution if \(-1 \leq a < 0\) and \(b > 0\),
(l) two solutions \(x = 0\) or \(x = \text{b\,ln}\,(|a|)\) if \(a < -1, b > 0\) and \(y = 0\),
(m) three solutions if \(a < -1, b > 0\) and \(0 < y < -(1 + ae^{W_0(-e/a)})^{-1}b(1 - W_0(-e/a))\),
(n) two solutions if \(a < -1, b > 0\) and \(y = -(1 + ae^{W_0(-e/a)})^{-1}b(1 - W_0(-e/a))\),
(o) a unique solution if \(a < -1, b > 0\) and \(y > -(1 + ae^{W_0(-e/a)})^{-1}b(1 - W_0(-e/a))\).

Proof. Let us define the continuous function

\(f:[0, \infty) \to [0, \infty), f(x) := |1 + ae^{-x/b}|x\),
given by the left-hand side of the equation under consideration. First, we determine the conditions on \(a, b \in \mathbb{R}\) and \(x \geq 0\) such that \(1 + ae^{-x/b} \geq 0\). If \(a \geq 0\), then
$1 + ae^{-x/b} > 0$ and thus $f(x) = (1 + ae^{-x/b})x$ for all $b \in \mathbb{R}$ and $x \geq 0$. Now, let us turn to the case $a < 0$. We have the equivalences

$$1 + ae^{-x/b} \geq 0 \iff e^{-x/b} \leq |a|^{-1} \iff \frac{x}{b} \geq \ln(|a|) \iff \begin{cases} x \geq b\ln(|a|), & b > 0, \\ x \leq b\ln(|a|), & b < 0. \end{cases}$$

Furthermore, we note that

$$b\ln(|a|) \geq 0 \iff (b > 0, |a| \geq 1) \text{ or } (b < 0, |a| \leq 1).$$

Hence, if $b > 0$ and $-1 \leq a < 0$, then $b\ln(|a|) \leq 0$ and

$$f(x) = (1 + ae^{-x/b})x, \; x \in [0, \infty),$$

if $b > 0$ and $a < -1$, then $b\ln(|a|) > 0$ and

$$f(x) = \begin{cases} -(1 + ae^{-x/b})x, & x \in [0, b\ln(|a|)], \\ (1 + ae^{-x/b})x, & x \in (b\ln(|a|), \infty), \end{cases}$$

if $b < 0$ and $-1 < a < 0$, then $b\ln(|a|) > 0$ and

$$f(x) = \begin{cases} (1 + ae^{-x/b})x, & x \in [0, b\ln(|a|)], \\ -(1 + ae^{-x/b})x, & x \in (b\ln(|a|), \infty), \end{cases}$$

if $b < 0$ and $a \leq -1$, then $b\ln(|a|) \leq 0$ and

$$f(x) = -(1 + ae^{-x/b})x, \; x \in [0, \infty).$$

The set of zeros $\mathcal{N}_f$ of $f$ in $[0, \infty)$ is $\mathcal{N}_f = \{0\}$, if either $a \geq 0$ or $b > 0$ and $-1 \leq a < 0$ or $b < 0$ and $a \leq -1$, and $\mathcal{N}_f = \{0, b\ln(|a|)\}$ if either $b > 0$ and $a < -1$ or $b < 0$ and $-1 < a < 0$. We note that $f$ is infinitely continuously differentiable on $[0, \infty) \setminus \mathcal{N}_f$.

(a) In this case our claim follows from the intermediate value theorem since $f$ is strictly increasing on $[0, \infty)$. $f(0) = 0$ and $\lim_{x \to \infty} f(x) = \infty$.

(b) The first derivative fulfils for all $x \in (0, \infty)$

$$f'(x) = -a \left(1 - \frac{x}{b}\right)e^{-x/b} - 1 > 1 - 1 = 0,$$

implying that $f$ is strictly increasing on $[0, \infty)$. Like in (a) this proves our claim by the intermediate value theorem because $f(0) = 0$ and $\lim_{x \to \infty} f(x) = \infty$ as well.

(c)-(f) We start with the first derivative of $f$. We have

$$f'(x) = \begin{cases} a(1 - \frac{x}{b})e^{-x/b} + 1, & x \in (0, b\ln(|a|)), \\ -a(1 - \frac{x}{b})e^{-x/b} - 1, & x \in (b\ln(|a|), \infty). \end{cases}$$

The function $f$ is strictly increasing on $(b\ln(|a|), \infty)$ since for all $x > b\ln(|a|)$ it holds that

$$f'(x) = -a(1 - \frac{x}{b})e^{-x/b} - 1 > |a|(1 - \frac{b\ln(|a|)}{b})e^{-b\ln(|a|)/b} - 1 = |a|(1 - \ln(|a|))|a|^{-1} - 1 = -\ln(|a|) > 0.$$ 

Next, we compute the local extremum of $f$ on $(0, b\ln(|a|))$. For the second derivative on $(0, b\ln(|a|))$ we remark that

$$f''(x) = -\frac{a}{b} \left(2 - \frac{x}{b}\right)e^{-x/b} < 0, \; x \in (0, b\ln(|a|)),$$
so our local extremum will be a maximum. Setting \( z := 1 - \frac{x}{b} > 0 \) for \( x \in (0, b \ln(|a|)) \), we have the equivalences
\[
0 = f'(x) \iff 0 = ae^{-x} + 1 \iff \frac{e}{a} = ze^x.
\]
Since \( z > 0 \) and \( -\frac{x}{a} > 0 \), the unique solution of the last equation is given by 
\( z = W_0(-e/a) \). This yields by resubstitution the local maximum \( x_0 := b(1 - W_0(-e/a)) \), which is in \((0, b \ln(|a|))\) by Rolle's theorem. Furthermore, \( f \) has two zeros in \([0, \infty)\), namely, \( x_1 := 0 \) and \( x_2 := b \ln(|a|) \), and \( \lim_{x \to \infty} f(x) = \infty \). This implies our claims (c)-(f) using the intermediate value theorem again.

(g)-(j) Again, we compute the extrema of \( f \). We have
\[
f'(x) = a(1 - \frac{x}{b})e^{-x/b} + 1 \quad \text{and} \quad f''(x) = -\frac{a}{b}(2 - x/b)e^{-x/b}, \quad x \in (0, \infty).
\]
Defining \( z := 1 - \frac{x}{b} < 0 \) for \( x > b \), we note that
\[
0 = f'(x) \iff -\frac{e}{a} = ze^x.
\]
Since \( z < 0 \) for \( x > b \) and \( -\frac{x}{a} < 0 \), there are two solutions \( z_0 := W_0(-e/a) \) and \( z_1 := W_{-1}(-e/a) \) if \( -\frac{x}{a} \geq e^{-1} \), which is equivalent to \( a \geq e^2 \). We remark that \( z_0 = z_1 \) if \( a = e^2 \). By resubstitution we obtain that \( 0 = f'(x) \) has the solutions \( x_0 := b(1 - W_0(-e/a)) \) and \( x_1 := b(1 - W_{-1}(-e/a)) \) if \( a \geq e^2 \). From \( -1 < W_0(-e/a) < 0 \) and \( -\infty < W_{-1}(-e/a) < -1 \) for \( a > e^2 \) follows \( b < x_0 < 2b \) and \( x_1 > 2b \), which implies
\[
f''(x_0) = -\frac{a}{b}(2 - \frac{x_0}{b})e^{-x_0/b} < 0 \quad \text{and} \quad f''(x_1) = -\frac{a}{b}(2 - \frac{x_1}{b})e^{-x_1/b} > 0.
\]
Hence \( x_0 \) is a local maximum and \( x_1 \) a local minimum if \( a > e^2 \). If \( a = e^2 \), then \( x_0 = x_1 = 2b \) and \( f''(2b) = 0 \) and \( f'''(2b) = a(f'(b))^2 = 0 \), yielding that \( x_0 \) is a saddle point. In addition, we observe that \( f(0) = 0 \) and
\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x + \lim_{x \to \infty} ae^{-x/b}x = \infty + 0 = \infty.
\]
This proves our claims (g)-(j) by the intermediate value theorem, in particular, we deduce that \( f \) is strictly increasing on \([0, \infty)\) if \( 0 \leq a \leq e^2 \).

(k) The first derivative for all \( x \in (0, \infty) \) is given by
\[
f'(x) = a(1 - \frac{x}{b})e^{-x/b} + 1.
\]
As \( f(0) = 0 \) and \( \lim_{x \to \infty} f(x) = \infty \), we only need to show that \( f \) is strictly increasing on \([0, \infty)\) due to the intermediate value theorem. Since
\[
0 < e^{-x/b} < 1 \quad \text{and} \quad 1 - \frac{x}{b} < 1
\]
for all \( x > 0 \), we derive that
\[
f'(x) = a(1 - \frac{x}{b})e^{-x/b} + 1 > a + 1 \geq 0
\]
for all \( x > 0 \), confirming our claim.

(l)-(o) Since \( f(0) = 0 \) and \( \lim_{x \to \infty} f(x) = \infty \), we only need to show that \( f \) is strictly increasing on \((b \ln(|a|), \infty)\) and has a unique local maximum on
Hence \( f'(x) = \begin{cases} -a(1 - \frac{x}{b})e^{-x/b} - 1, & x \in (0, b\ln(|a|)), \\ a(1 - \frac{x}{b})e^{-x/b} + 1, & x \in (b\ln(|a|), \infty). \end{cases} \)

We start with the proof that \( f \) is strictly increasing on \((b\ln(|a|), \infty)\). We have for \( x \in (b\ln(|a|), \infty) \) that

\[
0 < e^{-x/b} < e^{-b\ln(|a|)/b} = |a|^{-1} \quad \text{and} \quad 1 - \frac{x}{b} < 1 - \frac{b\ln(|a|)}{b} = 1 - \ln(|a|),
\]

implying

\[
f'(x) = a(1 - \frac{x}{b})e^{-x/b} + 1 \geq a(1 - \ln(|a|))|a|^{-1} + 1 = -1 + \ln(|a|) + 1 = \ln(|a|) > 0.
\]

Hence \( f \) is strictly increasing on \((b\ln(|a|), \infty)\). Let us turn to the local maximum on \((0, b\ln(|a|))\). Setting \( z := 1 - \frac{b}{a} > 0 \) for \( 0 < x < b \), we have the equivalences

\[
0 = f'(x) \iff 0 = -ae^{-1}e^z - 1 \iff -\frac{e}{a} = ze^z.
\]

Since \( z > 0 \) and \(-\frac{e}{a} > 0\), the unique solution of the last equation is given by \( z = W_0(-e/a) \) This yields by resubstitution \( x_0 := b(1 - W_0(-e/a)) \) and \( x_0 \in (0, b\ln(|a|)) \) by Rolle’s theorem. In addition, we observe that \( x_0 = b(1 - W_0(-e/a)) < b \), giving

\[
f''(x_0) = \frac{a}{b} \left(2 - \frac{x_0}{b}\right)e^{-x_0/b} < 0.
\]

Therefore \( x_0 \) is a local maximum on \((0, b\ln(|a|))\).

**Figure 1.** Graph of \( f \) with (a) \( a = 1 \) and \( b = -1 \), (b) \( a = -1 \) and \( b = -5 \), (c)-(f) \( a = -1/2 \) and \( b = -10 \), (g) \( a = e^2 \) and \( b = 1 \), (h)-(j) \( a = 15 \) and \( b = 1 \), (k) \( a = -1/2 \) and \( b = 10 \), (l)-(o) \( a = -2 \) and \( b = 10 \).
Due to the preceding propositions we obtain a solution of our problem.

2.3. Theorem. Let \((E, \| \cdot \|)\) be a normed space, \(E \neq \{0\}\), \(a, b \in \mathbb{R}\), \(b \neq 0\), and \(Y \in E\). Then the equation

\[
(1 + ae^{\frac{X_1}{a}})X = Y
\]

has in \(E\)

(a) a unique solution if \(a \geq 0\) and \(b < 0\),

(b) a unique solution if \(a \leq -1\) and \(b < 0\),

(c) the set of solutions \(\{0\} \cup \{X \in E \mid \|X\| = b \ln(|a|)\}\) if \(-1 < a < 0\), \(b \leq 0\) and \(Y = 0\),

(d) three solutions if \(-1 < a < 0\), \(b < 0\) and \(0 < \|Y\| < (1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(e) two solutions if \(-1 < a < 0\), \(b < 0\) and \(\|Y\| = (1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(f) a unique solution if \(-1 < a < 0\), \(b < 0\) and \(\|Y\| > (1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(g) a unique solution if \(0 \leq a \leq e^2\) and \(b > 0\),

(h) a unique solution if \(a > e^2\), \(b > 0\) and \(\|Y\| < (1 + ae^{W_1(-e/a)})b(1 - W_1(-e/a))\) or \(\|Y\| > (1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(i) two solutions if \(a > e^2\), \(b > 0\) and \(\|Y\| = (1 + ae^{W_1(-e/a)})b(1 - W_1(-e/a))\) or \(\|Y\| = (1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(j) three solutions if \(a > e^2\), \(b > 0\) and \((1 + ae^{W_1(-e/a)})b(1 - W_1(-e/a)) < \|Y\| < (1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(k) a unique solution if \(-1 \leq a < 0\) and \(b > 0\),

(l) the set of solutions \(\{0\} \cup \{X \in E \mid \|X\| = b \ln(|a|)\}\) if \(a < -1\), \(b > 0\) and \(Y = 0\),

(m) three solutions if \(a < -1\), \(b > 0\) and \(0 < \|Y\| < (1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(n) two solutions if \(a < -1\), \(b > 0\) and \(\|Y\| = -(1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\),

(o) a unique solution if \(a < -1\), \(b > 0\) and \(\|Y\| > -(1 + ae^{W_0(-e/a)})b(1 - W_0(-e/a))\).

Proof. All cases except for (c) and (l) are a direct consequence of our Proposition 2.2 and Proposition 2.4. Now, we turn to the cases (c) and (l). First, we observe that \(b \ln(|a|) > 0\). Now, we only need to remark that \((1 + ae^{\frac{-X_1}{a}})X = 0\) for some \(X \in E\) if and only if \(X = 0\) or \(\|X\| = b \ln(|a|)\). \(\Box\)

Part (a) implies that our motivating equation (2) from the introduction is uniquely solvable.

2.4. Remark. a) Proposition 2.2 and Theorem 2.3 remain valid if \(\| \cdot \|\) is replaced by an absolutely \(\mathbb{R}\)-homogeneous function \(p : E \to [0, \infty)\), i.e. a function \(p\) such that \(p(\lambda X) = |\lambda|p(X)\) for all \(\lambda \in \mathbb{R}\).

b) There are at least three solutions in (c) and (l) if \(E \neq \{0\}\). Namely, choosing \(\hat{X} \in E\) with \(\|\hat{X}\| = 1\), we always have the solutions \(X = 0\), \(X = b \ln(|a|)\hat{X}\) or \(X = -b \ln(|a|)\hat{X}\). If \(E = \mathbb{C}\), then there are infinitely many \(\hat{X} \in E\), \(\|\hat{X}\| = 1\), and so there are infinitely many solutions in (c) and (l).

c) Let \(E := \mathbb{R}^{m \times n}\), \(m, n \in \mathbb{N}\), and denote by \(\| \cdot \|_p\) with \(p \in [1, \infty]\) the \(p\)-norm and by \(\| \cdot \|_F\) the Frobenius norm on \(E\). If \(\| \cdot \| \in \{\| \cdot \|_1, \| \cdot \|_2, \| \cdot \|_{-\infty}, \| \cdot \|_F\}\) and \(m, n \in \mathbb{N}\), \(m, n \geq 2\), then there are infinitely many solutions in case (c) and (l). It suffices to show that there are infinitely many matrices \(X \in \mathbb{R}^{m \times n}\) with \(\|X\| = b \ln(|a|)\). We choose \(c \in \mathbb{R}\) with \(0 \leq c \leq b \ln(|a|)\). If \(\| \cdot \| = \| \cdot \|_F\), we define \(X := (x_{ij}) \in \mathbb{R}^{m \times n}\) by \(x_{11} := \sqrt{b^2 \ln^2(|a|) - c^2}\), \(x_{12} := c\) and \(x_{ij} := 0\).
else. Then we have $\|X\|_F = \sqrt{x_{11}^2 + x_{12}^2} = b \ln(|a|)$, which implies

$$(1 + ae^{-\frac{1}{|X|}})X = 0 \cdot X = 0.$$  

Since the maximal singular value of $X$ is $\sqrt{x_{11}^2 + x_{12}^2} = b \ln(|a|)$, we have $\|X\| = b \ln(|a|) = \|X\|_F$. Thus we have infinitely many solutions if $\|\cdot\|$ is the 2-norm or the Frobenius norm. If $\|\cdot\| = \|\cdot\|_1$, we define $X := (x_{ij}) \in \mathbb{R}^{m \times n}$ by $x_{11} := b \ln(|a|) - c$, $x_{21} := c$ and $x_{ij} := 0$ else. If $\|\cdot\| = \|\cdot\|_\infty$, we define $X := (x_{ij}) \in \mathbb{R}^{m \times n}$ by $x_{11} := b \ln(|a|) - c$, $x_{12} := c$ and $x_{ij} := 0$ else. Then $\|X\| = b \ln(|a|)$ if $\|\cdot\|$ is the 1-norm or $\infty$-norm, which again implies that there are infinitely many solutions.

If $n = m = 1$, then there are only the three solutions $X = 0$, $X = b \ln(|a|)$ or $X = -b \ln(|a|)$ from b).

d) Let $E \neq \{0\}$, $Y \in E$ and $y := \|Y\|$. If $(x_k)_{k \in \mathbb{N}}$ is a sequence in $[0, \infty)$ which converges to a solution $x \in [0, \infty)$ of $[1 + ae^{-x}]b = y$ with $1 + ae^{-x}b \neq 0$, then the sequence given by $X_k := (1 + ae^{-x_k}b)^{-1}Y \in E$ is well-defined if $k$ is big enough and converges to the solution $X := (1 + ae^{-x}b)^{-1}Y \in E$ of $(1 + ae^{-1/b}X)X = Y$ w.r.t. $\|\cdot\|$ because

$$|X_k - X| = \|(1 + ae^{-x_k}b)^{-1}Y - (1 + ae^{-x}b)^{-1}Y\|
= \|(1 + ae^{-x_k}b)^{-1} - (1 + ae^{-x}b)^{-1}\|\|Y\|$$

and $\lim_{k \to \infty} (1 + ae^{-x_k}b)^{-1} = (1 + ae^{-x}b)^{-1}$. Thus, if we use a numerical method which produces a sequence $(x_k)_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} x_k = x$, we obtain by $(X_k)_{k \in \mathbb{N}}$ a sequence of matrices which converges to the solution $X$.

e) In the MATLAB [2] m-file named msolve (see Appendix A) our propositions and theorem are used to solve the norm equation (6) for given inputs $a, b \in \mathbb{R}$, $b \neq 0$, and $Y \in \mathbb{R}^{m \times n} := E$ w.r.t. to the Frobenius norm. The MATLAB function fzero is applied to compute the roots of $g_y(x) := [1 + ae^{-x}b]x - y$ in $[0, \infty)$. Then these roots are used to obtain the solutions of the matrix equation (3) via formula (5).

f) Instead of fzero one might use the Newton-Raphson method to compute the roots of $g_y$, at least, in some of our cases. One of the difficulties of the Newton-Raphson method is the choice of suitable initial values $x_0 \in [0, \infty)$ for the Newton iteration. For example in case (a) this can be solved since $g_y'(x) = f'(x) > 0$ and $g_y''(x) = f''(x) > 0$ for all $x \in [0, \infty)$, which implies that the Newton-Raphson method converges by [4], Satz 30, p. 229 for every initial value $x_0 \in [0, \infty)$ with $x_0 > x^*$ where $x^*$ is the root of $g_y$ in $[0, \infty)$. Using that

$$g_y(y) := (1 + ae^{-y}b)y - y = ae^{-y}b > 0$$

and that $g_y$ is strictly increasing in case (a), we can choose $x_0 := y$ as a suitable initial value.

APPENDIX A. m-file to compute the solutions of $(1 + ae^{-\frac{1}{|X|}})X = Y$

As mentioned in Remark 2.4c) the solutions of $(1 + ae^{-\frac{1}{|X|}})X = Y$ in $E = \mathbb{R}^{m \times n}$, $m, n \in \mathbb{N}$, $m, n \geq 2$, for the Frobenius norm $\|\cdot\| = \|\cdot\|_F$ are computed in MATLAB using the following m-file named msolve.

```matlab
function X=msolve(Y,a,b)
y-norm(Y,'inf'); % other norms like norm(Y,1), norm(Y,2)
if nargin "= 3 || b==0 || min(size(Y))<2 > error('Inputs have the wrong dimensions! ');
end
```
% Simple cases where no calculation is needed
if (y==0) & & (1 < a & & a<0 & & b<0) | | (a<0 & & b >0)) % cases (c) and (1)
    fprintf('Infinitely many solutions!');
else if (y==0) % special case of the other cases
    X = Y;
else if (y>0 & & a==0) % special case of (a) and (g)
    X = Y;
end

% Simple cases with unique solutions
if (y>0) & & (0<a & & b<0)
    x = fzero(@(x) (1+a*exp(-x/b))*x-y, y);
else if (y>0) & & (a<=0 & & b<0)
    x = fzero(@(x) -(1+a*exp(-x/b))*x-y, y-b*log(2));
else if (y>0) & & (a>0 & & b<0)
    x = fzero(@(x) (1+a*exp(-x/b))*x-y,y);
else (y>0) & & (-1 < a & & a<0 & & b>0)
    x = fzero(@(x) (1+a*exp(-x/b))*x-y, 2*y+b*log(-2*a));
end

% Lambert W function for the more complicated cases
if (y>0) & & ((-1 < a & & a<0 & & b<0) | | (a<0 & & b >0))% for cases (d)–(f)
    z0 = lambertw(0,-exp(1)/a); % z0 = W_0(-e/a)
    fx0 = (1+a*exp(x0-1))*b*(1-z0); % fx0 = f(b(1-W_0(-e/a))
else if (y>0) & & (a>exp(2) & & b>0)
    z0 = lambertw(0,-exp(1)/a); % z0 = W_0(-e/a)
    x1 = lambertw(-1,-exp(1)/a); % x1 = W_{-1}(-e/a)
    fx0 = (1+a*exp(x0-1))*b*(1-z0); % fx0 = f(b(1-W_{-1}(-e/a)))
    fx1 = (1+a*exp(x1-1))*b*(1-z1); % fx1 = f(b(1-W_{-1}(-e/a)))
end

% The more complicated cases
if (-1 < a & & a<0 & & b<0) & & (y>0 & & y<fx0)
    xl = fzero(@(x) (1+a*exp(-x/b))*x-y,[0,b*(1-z0)]);
    xl = (1/(1+a*exp(-x/l/b)))*Y;
    zm = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z0),b*log(-a)]);
    zm = (1/(1+a*exp(-zm/b)))*Y;
    xr = fzero(@(x) -(1+a*exp(-x/b))*x-y+b*log(-a/2));
    xr = (1/(1+a*exp(-x/r/b)))*Y;
    X = [xl,xm,xr]; % three solutions stored in one matrix
else if (-1 < a & & a<0 & & b<0) & & (y>0 & & y==fx0)
    Xl = (1/(1+a*exp(x0-1)))*Y;
    Xr = (1/(1+a*exp(-x/r/b)))*Y;
    X = [Xl,Xr]; % two solutions stored in one matrix
else if (-1 < a & & a<0 & & b<0) & & (y>0 & & y>fx0) % case (f)
    x = fzero(@(x) -(1+a*exp(-x/b))*x-y+b*log(-a/2));
    x = (1/(1+a*exp(-x/r/b)))*Y;
else (a>exp(2) & & b>0) & & (y>0 & & y>fx0)
    x = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z1),y]);
    x = (1/(1+a*exp(-x/b)))*Y;
else (a>exp(2) & & b>0) & & (y>0 & & y<fx0)
    x = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z1),y]);
    x = (1/(1+a*exp(-x/b)))*Y;
x = fzero(@(x) (1+a*exp(-x/b))*x-y,[0,b*(1-z0)]); 
X = (1/(1+a*exp(-x/b)))*Y; 
elseif (a>exp(2) && b>0 && y>=fz0) \% case (i)  
XI = (1/(1+a*exp(z0-1)))*Y; 
xr = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z1),y]); 
Xr = (1/(1+a*exp(-xr/b)))*Y; 
X = [XI,Xr]; \% two solutions stored in one matrix 
elseif (a>exp(2) && b>0 && (y==fz0) \% case (i)  
Xr = (1/(1+a*exp(-x/b)))*Y; 
xl = fzero(@(x) (1+a*exp(-x/b))*x-y,[0,b*(1-z0)]); 
Xl = (1/(1+a*exp(-xl/b)))*Y; 
X = [XI,Xr]; \% two solutions stored in one matrix 
end 
elseif (a>exp(2) && b>0 && (y>0) && (y==fz1) \% case (j)  
xr = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z1),y]); 
Xr = (1/(1+a*exp(-x/b)))*Y; 
xm = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z0),b*(1-z1)]); 
Xm = (1/(1+a*exp(-xm/b)))*Y; 
xl = fzero(@(x) (1+a*exp(-x/b))*x-y,[0,b*(1-z0)]); 
Xl = (1/(1+a*exp(-xl/b)))*Y; 
X = [XI,Xm,Xr]; \% three solutions stored in one matrix 
elseif (a<-1 && b>0 && (y>0) && y<fx0) \% case (m)  
xl = fzero(@(x) (1+a*exp(-x/b))*x-y,[0,b*(1-z0)]); 
xm = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z0),b*(1-z1)]); 
Xm = (1/(1+a*exp(-xm/b)))*Y; 
xr = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z1),y]); 
Xr = (1/(1+a*exp(-xr/b)))*Y; 
X = [XI,Xm,Xr]; \% three solutions stored in one matrix 
elseif (a<-1 && b>0 && y==fz0) \% case (n)  
xl = fzero(@(x) (1+a*exp(-x/b))*x-y,[0,b*(1-z0)]); 
xm = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z0),b*log(-a)]); 
Xm = (1/(1+a*exp(-xm/b)))*Y; 
xr = fzero(@(x) (1+a*exp(-x/b))*x-y,[2*y+b*log(-2*a)]); 
Xr = (1/(1+a*exp(-xr/b)))*Y; 
X = [XI,Xm,Xr]; \% three solutions stored in one matrix 
elseif (a<-1 && b>0 && y=-fz0) \% case (o)  
xl = fzero(@(x) (1+a*exp(-x/b))*x-y,[0,b*(1-z0)]); 
xm = fzero(@(x) (1+a*exp(-x/b))*x-y,[2*y+b*log(-2*a)]); 
Xm = (1/(1+a*exp(-xm/b)))*Y; 
xr = fzero(@(x) (1+a*exp(-x/b))*x-y,[b*(1-z1),y]); 
Xr = (1/(1+a*exp(-xr/b)))*Y; 
X = [XI,Xm,Xr]; \% three solutions stored in one matrix 
end 
end 

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