Anomalous Behaviors in Fractional Fokker-Planck Equation

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Abstract

We introduce a fractional Fokker-Planck equation with a temporal power-law dependence on the drift force fields. For this case, the moments of the tracer from the force-force correlation in terms of the time-dependent drift force fields are discussed analytically. The long-time asymptotic behavior of the second moment is determined by the scaling exponent $\xi$ imposed by the drift force fields. In the special case of the space scaling value $\nu = 1$ and the time scaling value $\tau = 1$, our result can be classified according to the temporal scaling of the mean second moment of the tracer for large $t$: $\langle x^2(t) \rangle \propto t$ with $\xi = \frac{1}{4}$ for normal diffusion, and $\langle x^2(t) \rangle \propto t^\eta$ with $\eta > 1$ and $\xi > \frac{1}{4}$ for superdiffusion.

PACS numbers: 89.75.Da, 89.65.Gh, 05.10.-a

Keywords: Fractional, Fokker-Planck equation

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Recently, a lot of interest has been concentrated on nonstationary random processes in
the behavior of disordered systems [1 − 3]. The anomalous feature of diffusive stochastic
processes is known to be qualitatively different from the standard behavior characteristic of
the regular system. Especially, the anomalous diffusive behavior in the disordered system
has mainly been argued from the random walk, a useful theory of classical stochastic pro-
cesses in the past; in fact, the normal diffusion of tracers is well known to be a result of
random walkers characterized by the variance \( < x^2(t) > \propto t \). The random walk theory with
the nearest-neighbor transition probability has been numerically discussed by using the
Green function method. The anomalous transport process of the random walk theory has
been widely extended to a continuous-time random walk theory, which is specified by the
distributions of both the pausing time and the transition probability, which are dependent on
the length between steps. It provides a dynamic framework for a precise connection between
fractional diffusion equations and fractal walks [4, 5] while the transport phenomena for the
motions of tracers have largely been extended to reaction kinetics[6, 7] and strange kinetics
[8].

It is well known that the second moment for the superdiffusive proeess is proportional to
\( t^{\alpha} \) with \( \alpha > 1 \) for superdiffusion and \( \alpha < 1 \) for subdiffusion and to \( log^{3}t \) for strong diffusion.
In the past, the best example of the superdiffusive random proess was a classical paper by
Richardson [9] in which the second moment for a relative separation \( l \) at time \( t \), \( < l^2 > \propto t^3 \), was shown to depend on the motion of two particle in a turbulent fluid. Matheron and
de Marsily [10] showed the anomalous behavior of the longitudinal dispersion, \( < x^2(t) > \propto t^{3/2} \), in a stratified random velocity field. Among other recent examples, we also can
mention anomalous kinetics in chaotic dynamics due to flights and trapping [8], layered
velocity fields[11], anomalous diffusion of amphiphilic molecules [12], anomalous diffusion in
a two-dimensional motion of rotating flow [13], etc. Examples for subdiffusive stochastic
processes are charge transport in amorphous semiconductors [14, 15], NMR diffusometry on
percolation structures [16], and the dynamics of a bead in polymers [17]. Furthermore, the
stochastic process for anomalous diffusion on a fractal structure and Lévy flights has been the subject of several reports in the literature [18–20]. The typical models using Lévy flights include bulk-mediated surface diffusion [21], transport in micelle systems or heterogeneous rocks [22], single molecule spectroscopy [23], and special reaction dynamics [24]. Among the many frameworks for characterizing anomalous diffusion are fractional Brownian motion [25], fractional diffusion equations [26, 27], and generalized Langevin and Fokker-Planck equations [28–31].

The motivation for this paper was to apply the analytical method of a field-theoretical space-time renormalization group to investigate the anomalous behavior of the fractional diffusion equation. The moments for the probability density will be found so that the properties lead to anomalous asymptotics of the diffusion process. The purpose of this paper is to present an anomalous transport process for a test particle described by a fractional Fokker-Planck equation in the presence of a temporal power-law dependence for the force-force correlation of the drift force fields. The resulting formulae for the diffusive behavior of the tracer in the Fourier-Laplace region are discussed analytically, and the anomalously long-time behavior of the moments can be determined by using the scaling exponent $\xi$ imposed by drift force fields.

First, the fractional diffusion equation for the probability density $P(r, t)$ on the $d$-dimensional fractal structure can be expressed in terms of

$$\frac{\partial^\beta}{\partial t^\beta} P(r, t) = \frac{1}{r^{d-1}} \nabla_\alpha D(r) r^{d-1} \nabla_\alpha P(r, t),$$

(1)

where $\nabla_\alpha = \frac{\partial^\alpha}{\partial (\frac{r^\alpha}{r^\alpha})}$ and $D(r)$ is the anomalous diffusivity and depends only on the position. The normalization condition for the probability density is given by

$$\int_0^\infty Dr^{d-1} P(r, t) dr = 1.$$

(2)

If $\alpha = \beta = 1$, the form of Eq. (1) transforms to a spherically symmetric diffusion equation in $d$-dimensional Euclidean space, showing that this equation has analytical solutions for a diffusive behavior characteristic of fractal systems. Let us now introduce the anomalous
diffusivity given by

\[ D(r) = Dr^\theta, \]  

(3)

where \( D \) is the diffusion constant and \( \theta \) is the scaling exponent and can be determined from the fractal structure. The feature of this diffusivity describes the statistical property of diffusion on fractals, and even though it extends to a number of \( \theta \) and \( \alpha \) we here consider the special case of \( \theta = \alpha \) and \( 0 < \alpha < 1 \) in order to find the moments. After multiplying Eq. (1) by \( r^\alpha \) and \( r^{2\alpha} \), respectively, and integrating, we obtain the asymptotic result

\[ < r^2(t) > \sim t^{2\beta/\alpha}. \]  

(4)

This represents the behavior for a superdiffusive particle for the scaling exponent relation \( 2\beta > \alpha \) and a subdiffusive particle for \( 0 < 2\beta < \alpha \). From Eq. (4), the value of the scaling exponent, \( 2\beta = \alpha \) is equal to one for normal diffusion.

From now, we introduce the equation of motion in the presence of a spatiotemporal power-law dependence of both the drift force and an external harmonic potential. Since we take into account an overdamped particle of mass \( m \) moving in a viscous medium. The equation of motion can be expressed in terms of

\[ m\ddot{r} = f(\vec{v}, t) - V'(\vec{r}) + \Gamma(t), \]  

(5)

where \( f(\vec{v}, t) = -\gamma(t)\vec{v} \) is a time-dependent drift force field, and a harmonic force is given by \( F(r) = -V'(\vec{r}) \). The term \( \Gamma(t) \) is a fluctuating force with \( < \Gamma(t) > = 0 \) and \( < \Gamma(t_1)\Gamma(t_2) > = 2D\delta(t_1 - t_2) \). From Eq. (5) we can find the dispersive behavior of a test particle, and that mathematical technique leads us to a more general result.

In the three-dimensional case of an external harmonic potential \( V(r) = \frac{1}{2}ar^2 \), the formal solution of Eq. (5) is

\[ \vec{r}(t) = \vec{r}(0)Y(t) + Y(t) \int_0^t \frac{\Gamma(t')}{Y(t')} dt', \]  

(6)

where \( Y(t) = exp[- \int_0^t \dot{\gamma}(t')dt'] \) and \( \dot{\gamma}(t') = \gamma(t') + a \). Then, one can show that a direct
calculation of the variance gives

\[ \sigma^2(t) = De^{-2at}Y^2(t) \int_0^t e^{-2a t'} Y^2(t') dt' \]  

(7)

From Refs. 32 and 33, the quantity $\sigma^2(t)$ is easily compared with that of other results. If $\gamma(t)$ in terms of the time-dependent drift force field is given by a general form, we can discuss extensively the anomalous diffusive behavior of the tracer. Therefore, the Fokker-Planck equation associated with Eq. (5) is

\[
\frac{\partial}{\partial t} P(\vec{r}, t) = - \nabla \cdot [\gamma(t)\vec{r} + \vec{F}(\vec{r})] P(\vec{r}, t) + D \nabla^2 P(\vec{r}, t). 
\]  

(8)

Especially, we will focus on two fractional Fokker-Planck equations:

\[
\frac{\partial}{\partial t} P(\vec{r}, t) = - \nabla \cdot [\vec{F}(\vec{r}, t) \partial^{1-\tau}_{t^{1-\tau}} P(\vec{r}, t)] + D \nabla^2 \partial^{1-\tau}_{t^{1-\tau}} P(\vec{r}, t)
\]  

(9)

and

\[
\frac{\partial}{\partial t} P(\vec{r}, t) = - \nabla \cdot [\vec{F}(\vec{r}, t) P(\vec{r}, t)] + D \nabla^\nu P(\vec{r}, t)
\]  

(10)

with $\frac{1}{2} < \tau < 1$ and $1 < \nu \leq 2$, where $\tau$ and $\nu$ are, respectively, the time and the space scaling exponents. In Eq. (10), the Fourier-Laplace transform of the probability density function is described as

\[ P(\vec{k}, \omega) = \tilde{L} \tilde{F}[P(\vec{r}, t) \Theta(t)], \]  

(11)

where $\Theta(t)$ is a step function, and $\tilde{F}$ and $\tilde{L}$ are, respectively, the Fourier and the Laplace transform operators. In the Fourier-Laplace transform domain, Eqs. (9) and (10) become

\[
[-i\omega + D|\vec{k}|^2(-i\omega)^{1-\tau}]P(\vec{k}, \omega) =
\]

\[ 1 - \frac{i\vec{k}(-i\omega)^{1-\tau}}{(2\pi)^{d+1}} \int d\vec{k}' d\omega' \tilde{F}(\vec{k} - \vec{k}', \omega - \omega') P(\vec{k}', \omega') \]

(12)

and

\[
[-i\omega + D|\vec{k}|^\nu]P(\vec{k}, \omega) =
\]
\[
1 - \frac{i\vec{k}}{(2\pi)^{d+1}} \int d\vec{k}' d\omega' \vec{F}(\vec{k} - \vec{k}', \omega - \omega') P(\vec{k}', \omega').
\]

(13)

Since \(\Lambda(\vec{k}, \omega)\) and \(\Xi(\vec{k}, \omega)\) in the two cases where Eqs. (2) and (3) are, respectively, defined by

\[
\Lambda(\vec{k}, \omega) = \frac{(-i\omega)^{-1}}{(-i\omega)^{r} + D|\vec{k}|^2}, \Xi(\vec{k}, \omega) = \frac{1}{(-i\omega)^{r} + D|\vec{k}|^2}
\]

(14)

and

\[
\Lambda(\vec{k}, \omega) = \frac{1}{-i\omega + D|\vec{k}|^{\nu}}, \Xi(\vec{k}, \omega) = \frac{1}{-i\omega + D|\vec{k}|^{\nu}},
\]

(15)

Eqs. (12) and (13) can be put into a unitary form:

\[
P(\vec{k}, \omega) = \Lambda(\vec{k}, \omega) - \frac{i}{(2\pi)^{d+1}} \times \int d\vec{k}' d\omega' G(\vec{k}, \vec{k} - \vec{k}', \omega - \omega') P(\vec{k}', \omega'),
\]

(16)

where

\[
G(\vec{k}, \vec{k} - \vec{k}', \omega - \omega') = \Xi(\vec{k}, \omega)\vec{k} \cdot F(\vec{k} - \vec{k}', \omega - \omega').
\]

(17)

For the special case of a tracer in the presence of a time-dependent force, the force-force correlation can be described as

\[
< \vec{F}(\vec{r}, t) \vec{F}(\vec{r}', t') >= b^2 (tt')^{\xi} \delta(\vec{r} - \vec{r}') \delta(t - t')
\]

(18)

and

\[
< \vec{F}(\vec{k}, \omega) \vec{F}(\vec{k}', \omega') >= 2\pi b^2 \Gamma(2\beta + 1)
\]

\[
\times (i\omega - i\omega')^{-(2\xi + 1)} \delta(\vec{k}_1 + \vec{k}_2).
\]

(19)

Hence, from the inverse Fourier-Laplace transform, the \(n\)-th moment of the tracer can be given by

\[
\overline{x^n(t)} = \mathcal{L}^{-1} [i^n \frac{\partial^n}{\partial \vec{k}^n} P(\vec{k}, \omega)]_{\vec{k}=0},
\]

(20)
where $\tilde{L}^{-1}$ is the inverse Laplace-transform operator. In order to investigate the relaxation dynamics of the probability density toward the stationary solution, we now restrict our attention to the special class with the space scaling value $\xi = 1$ and the time scaling value $\tau = 2$. We find the statistical behavior of a tracer from the moments of the probability density. From our fractional Fokker-Planck equation via Eqs. (12) – (19), the mean second moment for large $t$ is

$$<x^2(t)> \simeq 2Dt + \frac{4\pi^{1/2}b^2}{(2\xi + 1)D^{1/2}}t^{2\xi + 1/2}. \quad (21)$$

From Eq. (21), $<x^2(t)> \propto t$ with $\xi = \frac{1}{4}$ for normal diffusion and $<x^2(t)> \propto t^\eta$ with $\eta > 1$ and $\xi > \frac{1}{4}$ for superdiffusion.

In conclusion, from a fractional Fokker-Planck equation, the tracer dispersion for an anomalous transport process has been discussed in the presence of a temporal power-law dependence of the drift and an external harmonic potential. The mean moments of the tracer were analyzed for the subdiffusive or superdiffusive behavior for several values of the exponent $\xi$ in the temporal power-law dependence of the drift. We think that for arbitrary time and space scaling exponents, it may not be easy to discuss the diffusive behaviors of the fractional Fokker-Planck equation. We also will present the results for non-Fickian diffusion for arbitrary time and space scaling exponents in other journals. We expect the detailed description of the anomalous behavior to be used to study extensions of numerical stratified models and fractal lattice models.

ACKNOWLEDGMENT

This work was supported by grant No. 2000-2-133300-001-3 from the Basic Research Program of the Korea Science and Engineering Foundation.
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