About the Linear Complexity of Ding-Helleseth Generalized Cyclotomic Binary Sequences of Any Period

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Abstract. We defined sufficient conditions for designing Ding-Helleseth sequences with arbitrary period and high linear complexity for generalized cyclotomies. Also we discuss the method of computing the linear complexity of Ding-Helleseth sequences in the general case.

Keywords: Generalized cyclotomic sequences, Linear complexity

1 Introduction

For cryptographic applications, the linear complexity ($L$) of a sequence is an important merit factor. It may be defined as the length of the shortest linear feedback shift register that is capable of generating the sequence. The feedback function of this shift register can be deduced from knowledge of just $2L$ consecutive digits of the sequence. Thus, it is reasonable to suggest that "good" sequences have $L > N/2$ (where $N$ denotes the period of the sequence) [1].

Using classical cyclotomic classes and generalized cyclotomic classes to construct binary sequences, which are called classical cyclotomic sequences and generalized cyclotomic sequences respectively, is an important method for sequence design [1]. In their paper [2] C. Ding and T. Helleseth first introduced a new generalized cyclotomy of order 2 with respect to $p_1 \cdots p_t$, which includes classical cyclotomy as a special case and they show how to construct binary sequences based on this new generalized cyclotomy. There are many works devoted to the investigation of the properties of the Ding-Helleseth sequences. In particular, the linear complexity of these sequences is studied in [3-11]. Most of the papers study a special case described in sections 1-5 of [2]. In addition, the linear complexity of the sequences based on cyclotomic classes of higher orders was considered for specific modules ($p^n, pq$) and for a special case [9, 12-14].

The purpose of this paper is to find Ding-Helleseth generalized cyclotomic sequences with arbitrary periods and high linear complexity. In particular, we generalize the result of Tongjiang Yan [15]. Also we discuss the computation of the linear complexity of these sequences in the general case.
2 Basic Definitions and Notations

In this section let us briefly recall the definition of Ding-Helleseth generalized cyclotomic sequences \[2\].

Let \( n = p_1^{e_1} \cdots p_t^{e_t} \), when \( p_1, ..., p_t \) be pairwise distinct odd primes satisfying

\[
gcd(p_i^{e_i-1}(p_i-1), p_j^{e_j-1}(p_j-1)) = 2 \text{ for all } i \neq j
\]

and \( e_1 \geq 1, ..., e_t \geq 1 \) be integers.

Let \( Z_n \) be the ring of residue classes modulo \( n \). According to the Chinese Remainder Theorem

\[
Z_n \cong Z_{p_1^{e_1}} \times \cdots \times Z_{p_t^{e_t}} \tag{1}
\]

relatively to isomorphism \( \varphi(x) = (x \mod p_1^{e_1}, ..., x \mod p_t^{e_t}) \). Here and hereafter \( x \mod n \) denotes the least nonnegative integer that is congruent to \( x \) modulo \( n \).

It is well known that exists a primitive root \( g \) modulo \( p_i^{e_i} \). Let \( D_0^{(p_i^{e_i})} = \{g_j^{(2^i)}| j \in \mathbb{Z}\} \) be the subgroup of \( Z_{p_i^{e_i}}^* \), generated by \( g_j^2 \), and \( D_1^{(p_i^{e_i})} = g_iD_0^{(p_i^{e_i})} \), where the arithmetic is that of \( Z_{p_i^{e_i}}^* \), \( t = 1, 2, ..., t \).

Let \( a = (a_1, ..., a_t) \) be a nonzero vector from \((\mathbb{Z}_2)^t\) and

\[
I_0^{(a,n)} = \left\{ (i_1, ..., i_t) \in (\mathbb{Z}_2)^t \mid \sum_{k=1}^t i_k a_k = 0 \right\}, \quad I_1^{(a,n)} = (\mathbb{Z}_2)^t \setminus I_0^{(a,n)}.
\]

By definition, put \[2\]

\[
E_j^{(a,n)} = \prod_{(i_1, ..., i_t) \in I_j^{(a,n)}} D_i^{(p_i^{e_i})} \quad \text{and} \quad D_j^{(a,n)} = \varphi^{-1}\left(E_j^{(a,n)}\right), \quad j = 0, 1.
\]

From our definition it follows that \[2\]

\[
Z_n^* = D_0^{(a,n)} \cup D_1^{(a,n)}, \quad D_0^{(a,n)} \cap D_1^{(a,n)} = \emptyset. \tag{2}
\]

Clearly there is an element \( b \in Z_n^* \) such that \( D_1^{(a,n)} = bD_0^{(a,n)} \). The \( D_0^{(a,n)} \) and \( D_1^{(a,n)} \) are called generalized cyclotomic classes of order 2 with respect to \( a \) and \( n \). In the following \( D_j^{(a,n)} \) will denote \( D_j^{(a,n)} \mod 2 \).

Further, by \[2\] we have a partition

\[
Z_n \setminus \{0\} = \bigcup_{d|n, d > 1} \frac{n}{d} Z_d^*
\]

Let \( d > 1 \) be a positive integer and \( d | n \), and the nonzero vector \( a_d = (a_1^{(d)}, ..., a_n^{(d)}) \in (\mathbb{Z}_2)^n \), where \( m \) is a number of different prime numbers participating in the factorization \( d \). By \[2\] and \[3\] we obtain

\[
Z_n \setminus \{0\} = \bigcup_{d|n, d > 1} \frac{n}{d} \left(D_0^{(a_d,d)} \cup D_1^{(a_d,d)}\right).
\]
Let
\[ C_0 = \bigcup_{d|n,d>1} \frac{n}{d} D_0^{(a_d,d)} \quad \text{and} \quad C_1 = \bigcup_{d|n,d>1} \frac{n}{d} D_1^{(a_d,d)} \cup \{0\}. \]

Then \( \{C_0, C_1\} \) is a partition of \( \mathbb{Z}_n \), i.e. \( \mathbb{Z}_n = C_0 \cup C_1 \) and \( C_0 \cap C_1 = \emptyset \).

In accordance with \([2]\), the binary sequence \( s_\infty \) is then defined by
\[ s_i = j \text{ if and only if } j \pmod{n} \in C_j. \]

The \( s_\infty \) is called Ding-Helleseth sequence, with the most frequently discussed options when \( a_d = (0, \ldots, 0, 1) \).

### 3 Evaluation of the Linear Complexity of Ding-Helleseth Sequences

In this section we find sufficient conditions for Ding-Helleseth sequences to have high linear complexity. Define \( g \) to be the unique solution of the following set of congruences
\[ g \equiv g_j \pmod{p_{l_1}^{j_1} \ldots p_{l_m}^{j_m}}, \quad j = 1, 2, \ldots, t. \]

**LEMMA 1.** If \( \sum_{k=1}^{m} a_k^{(d)} \) is an odd number, then
\[ gD_j^{(a_d,d)} = D_{j+1}^{(a_d,d)} \]
for \( j = 0, 1 \), where the arithmetic is that of \( \mathbb{Z}_d \).

**PROOF.** Let \( d = p_{l_1}^{j_1} \ldots p_{l_m}^{j_m} \), where \( j_k \in \{1, 2, \ldots, t\}, k = 1, 2, \ldots, m \) and integers \( l_k \) satisfy the set of inequalities \( 1 \leq l_k \leq e_{j_k}, k = 1, 2, \ldots, m \). By the definitions of \( D_j^{(a_d,d)} \) and \( g \) we obtain
\[ \phi \left( gD_j^{(a_d,d)} \right) = \prod_{(j_1, \ldots, j_m) \in I_0^{(a_d,d)}} g_{j_1} D_{j_1}^{(p_{l_1}^{j_1})} \times \ldots \times g_{j_m} D_{j_m}^{(p_{l_m}^{j_m})}, \]
where \( \phi(x) = (x \pmod{p_{l_1}^{j_1}}, \ldots, x \pmod{p_{l_m}^{j_m}}) \) or
\[ \phi \left( gD_0^{(a_d,d)} \right) = \prod_{(j_1, \ldots, j_m) \in I_0^{(a_d,d)}} D_{j_1+1}^{(p_{l_1}^{j_1})} \times \ldots \times D_{j_m+1}^{(p_{l_m}^{j_m})}. \]

The sum \( \sum_{k=1}^{m} (j_k + 1) a_k^{(d)} = \sum_{k=1}^{m} a_k^{(d)} \) for \( (j_1, \ldots, j_m) \in I_0^{(a_d,d)} \), therefore, by the condition of Lemma \( (j_1 + 1, \ldots, j_m + 1) \in I_1^{(a_d,d)} \) and by (2) we have
\[ D_{j_1+1}^{(p_{l_1}^{j_1})} \times \ldots \times D_{j_m+1}^{(p_{l_m}^{j_m})} \in E_1^{(a_d,d)}. \]
Then \( gD_0^{(d,d)} \subset D_1^{(d,d)} \), but as their orders are equal we obtain \( gD_0^{(d,d)} = D_1^{(d,d)} \).

The assertion \( gD_1^{(d,d)} = D_0^{(d,d)} \) can be proven similarly.

Let \( \alpha \) be a primitive \( n \)-th root of unity in the extension of field \( GF(2) \). Then by Blahut’s theorem for the linear complexity \( L \) of the sequence \( s^\infty \) we have

\[
L = n - \left| \{ v \mid S(\alpha^v) = 0, v = 0, 1, \ldots, n - 1 \} \right|,
\]

where \( S(x) \) is defined by \( S(x) = \sum_{i \in C_1} x^i \).

In order to investigate the values of \( S(\alpha^v) \), let us introduce subsidiary polynomials. Let \( S_A(x) = \sum_{i \in A} x^i \), where \( A \) is a subset of \( Z_n \). Then for any \( v = 1, 2, \ldots, n - 1 \) we obtain

\[
S_{C_0}(\alpha^v) + S_{C_1}(\alpha^v) = 0.
\]

LEMMA 2. If \( \sum_{k=1}^{m} a_k^{(d)} \) is an odd number, then for any \( v = 1, 2, \ldots, n - 1 \) we have

\[
S_{\#D_1^{(d,d)}}(\alpha^v) = S_{\#D_0^{(d,d)}}(\alpha^v).
\]

PROOF. By Lemma 1 \( gD_1^{(d,d)} = D_0^{(d,d)} \) in the ring \( Z_d \), then \( \#D_1^{(d,d)} = \#gD_0^{(d,d)} \) in the ring \( Z_n \) and the statement of Lemma follows from the definition of auxiliary polynomial.

Let \( \delta = \begin{cases} 1, & \text{if } S(1) = 0, \\ 0, & \text{otherwise.} \end{cases} \) Therefore \( \delta = \begin{cases} 1, & \text{if } n \equiv 3 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases} \) because by definition of sequence \( S(1) = (n + 1)/2 \).

THEOREM 1. Suppose that for any integer \( d > 1 \) with \( d \mid n \) sum \( \sum_{k=1}^{m} a_k^{(d)} \) is an odd number; then for the linear complexity \( L \) of the sequence \( s^\infty \) we have

\[
L \geq (n + 1)/2 - \delta.
\]

PROOF. By definition of the sequence \( s^\infty \) for all \( v = 1, \ldots, n - 1 \) we obtain

\[
S(\alpha^v) = \sum_{d \mid n, d > 1} S_{\#D_1^{(d,d)}}(\alpha^v) + 1.
\]

Then, by Lemma 2

\[
S(\alpha^v) = \sum_{d \mid n, d > 1} S_{\#D_0^{(d,d)}}(\alpha^v) + 1,
\]

therefore \( S(\alpha^v) = S_{C_0}(\alpha^v) + 1. \) Hence, by (5) we get \( S(\alpha^v) + S(\alpha^{dv}) = 1 \) for all \( v = 1, \ldots, n - 1 \). So, the order of the set

\[
\left| \{ v \mid S(\alpha^v) = 0, v = 1, 2, \ldots, n - 1 \} \right| \leq (n - 1)/2,
\]

then by (4) we have, \( L \geq (n + 1)/2 - \delta \), which was to be demonstrated.
COROLLARY. If \( 2 \equiv g \pmod{n} \) under conditions of Theorem 1, then \( L = n - \delta \). Indeed, in this case \( S(\alpha^v) + S^2(\alpha^v) = 1 \) and \( S(\alpha^v) \neq 0 \) for all \( v = 1, 2, \ldots, n-1 \).

Let us make some more remarks on Theorem 1. When \( a_d = (0, \ldots, 0, 1) \), i.e. in the special case of Ding-Helleseth generalized cyclotomic sequence, the condition of the Theorem 1 is automatically satisfied, so for this kind of sequences the evaluation of the linear complexity \( L \geq \frac{n+1}{2} - \delta \) is always valid. It is easy to see that this is in accord with already known results about the linear complexity of the sequences of periods \( p^n, pq \) [3-12]. It should be noted that not all the sequences examined in [3-12] can be defined as \( s^\infty \).

Further, if

\[
gcd\left(p_i^{e_i-1}(p_i - 1), p_j^{e_j-1}(p_j - 1)\right) = r \text{ for all } i \neq j,
\]

then like in [9, 12-14] for \( p_i^{e_i} \) we can define the generalized cyclotomic classes \( H_k(p_i^{e_i}), k = 0, 1, \ldots, r - 1 \) of order \( r \). Now, we suppose \( D_0(p_i^{e_i}) = \bigcup_{k=0}^{r/2-1} H_k(p_i^{e_i}) \) and \( D_1(p_i^{e_i}) = \bigcup_{k=r/2}^{d-1} H_k(p_i^{e_i}) \). Then, for \( a_d = (0, \ldots, 0, 1) \) the evaluation of the linear complexity given in Theorem 1 is valid also for generalized cyclotomic sequences built on new classes. We can prove it by just replacing the element \( g \) with \( g^{r/2} \) in Lemmas 1 an 2. This is consistent with the results from [9, 12-14].

So, Theorem 1 establishes sufficient conditions for existence of Ding-Helleseth sequences with high linear complexity.

In the next section we discuss how the linear complexity of the generalized cyclotomic sequences could be computed in the general case. In particular, we show that if amongst vectors \( a_d \) there exists one with the even sum of coordinates, then there exists some \( n \) for which the statement of Theorem 1 is not true.

### 4 About Computing of the Linear Complexity of Ding-Helleseth Sequences

Now let us generalize the method of computing the linear complexity of the generalized cyclotomic sequences with period \( pq \) proposed in [14]. By construction we see that

\[
S(\alpha^v) = \sum_{d|n, d > 1} S_{D_1(p_i^{e_i}, \alpha^v)}(\alpha^v) + 1, \tag{6}
\]

First, we examine the items of this sum.

If \( d = \prod_{i=1}^{m} p_i^{e_i} \), then, by (14) there exist integers \( b_i, i = 1, \ldots, m \) such that

\[
b_1 \frac{n}{p_{j_1}^{e_1}} + \cdots + b_m \frac{n}{p_{j_m}^{e_m}} = \frac{n}{d},
\]

where each \( b_i, i = 1, \ldots, m \) is uniquely determined modulo \( p_i^{e_i} \) and \( b_i \not\equiv 0 \pmod{p_i^{e_i}} \).
Let $\beta_k = \alpha_{b_k n/p_k}, k = 1, \ldots, m$, then $\beta_k$ is a primitive $p_k^{1k}$th root of unity in the extension of the field $GF(2)$ and

$$\alpha = \beta_1 \ldots \beta_m.$$  

Generalizing Theorem 1 from [10] we obtain the following.

LEMMA 3. If $d = p_1^{1} \ldots p_m^{1}$, then for $v = 1, \ldots, n - 1$ we have

$$S_{\pi D}^{(a_v,d)} (\alpha^v) = \sum_{(i_1, \ldots, i_m) \in I_1^{(a_v,d)}} S_{i_1}^{(\beta_1^v)} \ldots S_{i_m}^{(\beta_m^v)},$$

where $S_{i_k}^{(\beta_k^v)} (x) = \sum_{i \in D_k} (\beta_k^v)^i x^i$.

The method of computation of $S_{\pi D}^{(p_k^{1k})} (\beta_k^v)$ over the values of the classical cyclotomic sequences polynomial was proposed in [7-9]. Thus, formula (5), Lemma 3 and the results from [10] allow us to compute the values of the polynomial $S(\alpha^v)$, and consequently the linear complexity of Ding-Helleseth sequence if we know the kind of factorization of $n$. By example let us look on the case when the conditions of Theorem 1 does not hold.

Let $n = p_1 p_2$ and $a_{p_1 p_2} = (1,1)$ then

$$I_1^{(a_{p_1 p_2},p_1 p_2)} = \{(0,1), (1,0)\} \quad \text{and} \quad I_1^{(a_{p_1},p_1)} = I_1^{(a_{p_2},p_2)} = \{(1)\},$$

because by definition $a_{p_1} = a_{p_2} = (1)$. If $d = p_1 p_2$, then $\beta_1 = \alpha_{b_1 p_2}$ and $\beta_2 = \alpha_{b_2 p_1}$, where $b_1 p_1 + b_2 p_2 = 1$. Therefore $\alpha p_1 = \beta_2^{p_1}, \alpha p_2 = \beta_1^{p_2}$.

Hence by Lemma 3 and (10) for $n = p_1 p_2$ and $a_{p_1 p_2} = (1,1)$ we obtain

$$S(\alpha^v) = S_{j}^{(p_1)}(\beta_1^v)S_{1}^{(p_2)}(\beta_2^v) + S_{j}^{(p_1)}(\beta_1^v)S_{0}^{(p_2)}(\beta_2^v) + S_{j}^{(p_1)}(\beta_1^v)S_{1}^{(p_2)}(\beta_2^v) + 1.$$  

The properties of the polynomials $S_{j}^{(p_1)}(x), j, i = 0, 1$ were examined in [17].

LEMMA 4. If $v \in Z_n^*$ then

$$S(\alpha^v) = \begin{cases} 0, & \text{if } p_1 \equiv 3 \pmod{4} \text{ and } p_2 \equiv 3 \pmod{4}, \\ 1, & \text{otherwise}. \end{cases}$$

PROOF. If $v \in Z_n^*$, then from (7), we obtain

$$S(\alpha^v) = S_{j}^{(p_1)}(\beta_1^v) + S_{0}^{(p_2)}(\beta_2^v) + S_{1}^{(p_1)}(\beta_1^{p_2 v}) + S_{1}^{(p_2)}(\beta_2^{p_1 v}).$$

(8)

Let us consider two cases

1) If $p_1 \equiv p_2 \equiv 3 \pmod{4}$, then in accordance with the law of quadratic reciprocity the Legendre symbols $\left(\frac{p_1}{p_2}\right)$ and $\left(\frac{p_2}{p_1}\right)$ are different. Without loss of generality, we can assume that $\left(\frac{p_1}{p_2}\right) = 1$. Then

$$S_{j}^{(p_1)}(\beta_1^{p_2 v}) = S_{0}^{(p_1)}(\beta_1^v) \quad \text{and} \quad S_{1}^{(p_2)}(\beta_2^{p_1 v}) = S_{1}^{(p_2)}(\beta_2^v).$$  

6
By means of these relations we replace the last two terms in and now obtain $S(\alpha^v) = 0$ for all $v \in \mathbb{Z}_n^*$.

2) If $p_1 \equiv 1 \pmod{4}$ or $p_2 \equiv 1 \pmod{4}$ then $\left( \frac{p_2}{p_1} \right) = \left( \frac{p_1}{p_2} \right)$. In this case without loss of generality, we can assume that

$$S_{1}^{(p_1)}(\beta_{1}^{p_{2}^v}) = S_{1}^{(p_1)}(\beta_{1}^{v}) \quad \text{and} \quad S_{1}^{(p_2)}(\beta_{2}^{p_{1}^v}) = S_{1}^{(p_2)}(\beta_{2}^{v})$$

Then by \ref{17} and \ref{8} we get $S(\alpha^v) = 1$ for all $v \in \mathbb{Z}_n^*$.

Lemma 4 makes it clear that the statement of Theorem 1 is not true for $n = p_1 p_2$ and $a_{p_1 p_2} = (1, 1)$.

With Lemma 4 we can conclude the computation of the linear complexity. If $n = p_1 p_2$ and $a_{p_1 p_2} = (1, 1)$, then for the sequence $s^\infty$ we have

1. $L = p_1 + p_2 - 1$, if $p_1 \equiv 3 \pmod{8}$ and $p_2 \equiv 3 \pmod{8}$;
2. $L = p_1 + (p_2 - 1)/2$, if $p_1 \equiv 3 \pmod{8}$ and $p_2 \equiv 7 \pmod{8}$;
3. $L = p_2 + (p_1 - 1)/2$, if $p_1 \equiv 7 \pmod{8}$ and $p_2 \equiv 3 \pmod{8}$;
4. $L = (p_1 + p_2)/2$, if $p_1 \equiv 7 \pmod{8}$ and $p_2 \equiv 7 \pmod{8}$.

5 Conclusions

In the paper we defined sufficient conditions for designing Ding-Helleseth sequences with arbitrary period and high linear complexity for generalized cyclotomies. In particular, we have that the most frequently considered variant of Ding-Helleseth sequences possesses high linear complexity for any period. Also we discuss the method of computing the linear complexity of Ding-Helleseth sequences in the general case.

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