NAVIER-STOKES EQUATIONS INTERACTING WITH A NONLINEAR ELASTIC SOLID SHELL

C.H. ARTHUR CHENG, DANIEL COUTAND, AND STEVE SHKOLLER

Abstract. We study a moving boundary value problem consisting of a viscous incompressible fluid moving and interacting with a nonlinear elastic solid shell. The fluid motion is governed by the Navier-Stokes equations, while the shell is modeled by the nonlinear Koiter shell model, consisting of both bending and membrane tractions. The fluid is coupled to the solid shell through continuity of displacements and tractions (stresses) along the moving material interface. We prove existence and uniqueness of solutions in Sobolev spaces.

1. Introduction

1.1. The problem statement and background. Fluid-solid interaction problems involving moving material interfaces have been the focus of active research since the nineties. The first problem solved in this area was for the case of a rigid body moving in a viscous fluid (see [11], [18] and also the early works of [22] and [21] for a rigid body moving in a Stokes flow in the full space). The case of an elastic body moving in a viscous fluid was considerably more challenging because of some apparent regularity incompatibilities between the parabolic fluid phase and the hyperbolic solid phase. The first existence results in this area were for regularized elasticity laws, such as in [12] for a finite number of elastic modes, or in [2], [4], and [3] for hyperviscous elasticity laws, or in [20] in which a phase-field regularization “fattens” the sharp interface via a diffuse-interface model.

The treatment of classical elasticity laws for the solid phase, without any regularizing term, was only considered recently in [9] for the three-dimensional linear St. Venant-Kirchhoff constitutive law and in [10] for quasilinear elastodynamics coupled to the Navier-Stokes equations. Some of the basic new ideas introduced in those works concerned a functional framework that scales in a hyperbolic fashion (and is therefore driven by the solid phase), the introduction of approximate problems either penalized with respect to the divergence-free constraint in the moving fluid domain, or smoothed by an appropriate parabolic artificial viscosity in the solid phase (chosen in an asymptotically convergent and consistent fashion), and the tracking of the motion of the interface by difference quotients techniques.

The complimentary fluid-solid interaction problem, studied herein, consists of the motion of a viscous incompressible fluid enclosed by a moving thin nonlinear elastic solid shell. Our companion paper [5] treats the case of a viscous incompressible fluid enclosed by a moving thin nonlinear elastic fluid shell. This is a moving boundary problem that models the motion of a viscous incompressible Newtonian fluid inside of a deformable elastic structure. The main mathematical differences with respect to the previous problem of a deformable solid body moving inside of the fluid is that the shell encloses the fluid and is mathematically the boundary of the fluid. The shell model consists of “elliptic” operators which do not provide the expected regularity associated with the highest order operator coming from
the shell’s bending energy, and, in particular, ellipticity holds only for short time. The only cases considered until now consisted of regularized problems, wherein the elliptic degeneracy occurs along a fixed direction, such as in [13] or [4].

We are concerned here with establishing the existence and uniqueness of solutions to the time-dependent incompressible Navier-Stokes equations interacting with a quasilinear elastic solid shell of Koiter type (see [6] for a detailed account of Koiter shells). The solid shell energy is a nonlinear function of the first and second fundamental forms of the moving boundary.

Let $\Omega \subset \mathbb{R}^n$ denote an open bounded domain with boundary $\Gamma := \partial \Omega$. For each $t \in (0, T]$, we wish to find the domain $\Omega(t)$, a divergence-free velocity field $u(t, \cdot)$, a pressure function $p(t, \cdot)$ on $\Omega(t)$, and a volume-preserving transformation $\eta(t, \cdot) : \Omega \rightarrow \mathbb{R}^n$ such that

\begin{align*}
\Omega(t) &= \eta(t, \Omega), \\
\eta(t, x) &= u(t, \eta(t, x)), \\
u u + \nabla u &= -\nabla p + f & \text{in } \Omega(t), \\
\text{div } u &= 0 & \text{in } \Omega(t), \\
(\nu \text{Def } u - p \text{Id})n &= t_{\text{shell}} & \text{on } \Gamma(t), \\
u(0, x) &= u_0(x) & \forall x \in \Omega, \\
\eta(0, x) &= x & \forall x \in \Omega,
\end{align*}

where $\nu$ is the kinematic viscosity, $n(t, \cdot)$ is the outward pointing unit normal to $\Gamma(t)$, $\Gamma(t) := \partial \Omega(t)$ denotes the boundary of $\Omega(t)$, Def $u$ is twice the rate of deformation tensor of $u$, given in coordinates by $u_{i,j} + u_{j,i}$, where $u_{i,j}$ denotes $\frac{\partial u_i}{\partial x_j}$, and $t_{\text{shell}}$ is the traction imparted onto the fluid by the elastic solid shell, which we describe next.

With $\varepsilon$ denoting the thickness of the Koiter shell, $\lambda/2$ and $\mu/2$ the Lamé constants, the energy stored in the elastic surface has the form

$$E_{\text{mem}} + \varepsilon^3 E_{\text{ben}},$$

where the membrane energy $E_{\text{mem}}$ is

$$E_{\text{mem}} = \int_{\Gamma} \sqrt{aa_{\alpha\beta\gamma\delta}} (g_{\alpha\beta} - g_{0\alpha\beta})(g_{\gamma\delta} - g_{0\gamma\delta}) dS_0$$

and the bending energy $E_{\text{ben}}$ is

$$E_{\text{ben}} = \int_{\Gamma} \sqrt{aa_{\alpha\beta\gamma\delta}} (b_{\alpha\beta} - b_{0\alpha\beta})(b_{\gamma\delta} - b_{0\gamma\delta}) dS_0,$$

where

$$a_{\alpha\beta\gamma\delta} = \frac{4\lambda\mu}{\lambda + 2\mu} g_{\alpha\beta} g_{\gamma\delta} + 2\mu g_{\alpha\gamma} g_{\beta\delta} + g_{\alpha\delta} g_{\beta\gamma}.$$ 

We let $g_{\alpha\beta} = \eta_{,\alpha} \cdot \eta_{,\beta}$ denote the induced metric on $\Gamma(t)$, and $b_{\alpha\beta} = \eta_{,\alpha} \cdot n$ denotes the second fundamental form. $g_0$ and $b_0$ denote the induced metric and second fundamental form of the unstressed initial configuration at $t = 0$. The traction vector

$$t_{\text{shell}} = \varepsilon t_{\text{mem}} + \varepsilon^3 t_{\text{ben}}$$
is computed from the first variation of the energy function $E_{\text{shell}}$, and will be stated in Section 2.

In this paper, we will prove well-posedness for this system in the case that the fluid is two-dimensional and the solid shell is its one-dimensional closed boundary.

2. Formulation of the problem

2.1. Fundamental geometric identities. We use $(\cdot)'$ to denote the derivative of $(\cdot)$ along the boundary, and we use $\delta \eta$ to denote the first variation of $\eta$. The following formulas will be used often:

\begin{align}
\delta n &= -|\eta'|^{-2}(n \cdot \delta \eta')\eta', \\
\eta' &= -|\eta'|^{-2}(\eta'' \cdot n)\eta'' = -g^{-1/2}b\tau, \\
\eta'' &= 3|\eta'|^{-4}(\eta'' \cdot \eta') (\eta'' \cdot n)\eta'' - |\eta'|^{-2}(\eta'' \cdot n)\eta'' - |\eta'|^{-2}(\eta'' \cdot n)\eta'' \\
&= -|\eta'|^{-2}(\eta'' \cdot n)^2 n + |\eta'|^{-4}(\eta'' \cdot \eta') (\eta'' \cdot n)\eta'' - |\eta'|^{-2}(\eta'' \cdot n)\eta'' \\
&= -g^{-1}b_0^2 n + \left[ \frac{g^{-3/2}}{2} \right] g \tau = \frac{g' + g}{2\sqrt{g}} \tau,
\end{align}

where $g = |\eta'|^2$, $\tau = g^{-1/2} \eta'$, and $b = \eta'' \cdot n$.

2.2. The shell traction. The bending energy (1.3) and membrane energy (1.2) are expressed as

\begin{align}
E_{\text{bend}} &= \int \eta''_0^{-3} (b - b_0)^2 dS_0, \\
E_{\text{mem}} &= \int |\eta''_0|^{-3} (g - g_0)^2 dS_0.
\end{align}

Computing the first variation of the bending energy, we find that the bending traction $L_b$ is given by

\begin{align}
L_b(\eta) &= 2 \left[ |\eta''_0|^{-3} (b - b_0) n \right] + \left[ |\eta''_0|^{-3} g^{-1} g' (b - b_0) n \right],
\end{align}

where, once again, $b_0 = \eta''_0 \cdot n_0$ is the second fundamental form of the unstressed initial boundary.

Taking the variation of $E_{\text{mem}}$ we find the membrane traction $L_m$ is

\begin{align}
L_m(\eta) &= -4 \left[ |\eta''_0|^{-3} (|\eta'|^2 - |\eta''_0|^2) \eta' \right].
\end{align}

2.3. Lagrangian formulation. Let $\eta(t, x) = x + \int_0^t u(s, x) ds$ denote the Lagrangian particle placement field, a volume-preserving embedding of $\Omega$ onto $\Omega(t) \subset \mathbb{R}^2$, and denote the inverse matrix of $\nabla \eta(x, t)$ by

\begin{align}
a(x, t) = [\nabla \eta(x, t)]^{-1}.
\end{align}

Let $v = u \circ \eta$ denote the Lagrangian or material velocity field, $q = p \circ \eta$ the Lagrangian pressure function, and $F = f \circ \eta$ the forcing function in the material frame.
The coupled fluid-structure problem has the following Lagrangian description:

\[ v = \eta_t \quad \text{in} \ (0, T) \times \Omega, \]  
\[ v^i_t - \nu (a^j \partial_k v^j) = - (a^j q)_j + F^i \quad \text{in} \ (0, T) \times \Omega, \]  
\[ a^j \partial_k v^j = 0 \quad \text{in} \ (0, T) \times \Omega, \]  
\[ \left[ \nu (a^j v^j_k + a^j_k v^j_k) - q_{\delta k} \right] a^j N^k = \varepsilon L_k (\eta) + \frac{\varepsilon^3}{3} L_m (\eta) \quad \text{on} \ (0, T) \times \Gamma, \]  
\[ v(0, x) = u_0 (x) \quad \text{on} \ \{ t = 0 \} \times \Omega, \]  
\[ \eta = \text{Id} \quad \text{on} \ \{ t = 0 \} \times \Omega. \]

3. Notation and conventions

For \( T > 0 \), we set

\[ \mathcal{V}^1 (T) = \left\{ v \in L^2 (0, T; H^1 (\Omega)) \mid v_t \in L^2 (0, T; H^1 (\Omega)') \right\}; \]  
\[ \mathcal{V}^2 (T) = \left\{ v \in L^2 (0, T; H^2 (\Omega)) \mid v_t \in L^2 (0, T; L^2 (\Omega)) \right\}; \]  
\[ \mathcal{V}^k (T) = \left\{ v \in L^2 (0, T; H^k (\Omega)) \mid v_t \in L^2 (0, T; H^{k-1} (\Omega)) \right\}; \]  
\[ \mathcal{V}^k (T) = \left\{ v \in L^2 (0, T; H^k (\Omega)) \mid v_t \in L^2 (0, T; H^{k-2} (\Omega)) \right\} \text{ for } k \geq 4. \]

We then introduce the space (of “divergence free” vector fields)

\[ \mathcal{V}_v = \left\{ w \in H^1 (\Omega) \mid a^j (t) w^j = 0 \ \forall \ t \in [0, T] \right\} \]

and

\[ \mathcal{V}_v (T) = \left\{ w \in L^2 (0, T; H^1 (\Omega)) \mid a^j (t) w^j = 0 \ \forall \ t \in [0, T] \right\}, \]

where the matrix \( a \) is defined by (2.2). Let \( n(\eta) = (- \eta_2, \eta_1)/|\eta'| \) denote the outward unit normal to \( \Gamma(t) \) at the point \( \eta(x, t) \). We define the space \( E^s_\eta \) as

\[ E^s_\eta = \left\{ \zeta \in H^2 (\Gamma) \mid \zeta'' \cdot n \in H^{s+1} (\Gamma), \zeta' \cdot \eta' \in H^{s+1} (\Gamma) \right\}, \]

with norm

\[ ||\zeta||^2_{E^s_\eta} = \left[ ||\zeta'' \cdot n||^2_{H^{s+1} (\Gamma)} + ||\zeta' \cdot \eta'||^2_{H^{s+1} (\Gamma)} \right]. \]

When then set

\[ E^s_\eta (T) = \left\{ \zeta (t) \in E^s_\eta \ t \ a.e. \mid \int_0^T ||\zeta(s)||^2_{E^s_\eta} ds < \infty \right\} \]

with norm

\[ ||\zeta||^2_{E^s_\eta (T)} = \int_0^T ||\zeta||^2_{E^s_\eta} ds. \]
4. The main theorem

Theorem 4.1. Let $\nu > 0$ be given, and

$$F \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), F_t \in L^2(0, T; L^2(\Omega)), F(0) \in H^1(\Omega).$$

Assume that $T$ is of class $H^{3,5}$ and that the initial data $u_0 \in H^2(\Omega)$ with $\text{div } u_0 = 0$. Then there exists $T > 0$ depending on $u_0$ and $F$ such that there exists a solution $v \in V^3(T)$ of problem (2.3) with $b \in L^2(0, T; H^{2.5}(\Gamma))$ and $g \in L^2(0, T; H^{2.5}(\Gamma))$. Moreover, if the initial data has the regularity $u_0 \in H^4(\Omega)$, then the solution $v \in L^2(0, T; V^3(\Omega))$ is unique.

5. Preliminary results

5.1. Pressure as a Lagrange multiplier. In the following discussion, we use $H^{1,2}(\Omega; \Gamma)$ to denote the space $H^1(\Omega) \cap H^2(\Gamma)$ with norm

$$\|u\|_{H^{1,2}(\Omega; \Gamma)}^2 = \|u\|_{H^1(\Omega)}^2 + \|u\|_{H^2(\Gamma)}^2$$

and $\mathcal{V}_c(\mathcal{V}_c(T))$ to denote the space

$$\{v \in \mathcal{V}_c \mid v \in H^2(\Gamma)\} \left\{v \in \mathcal{V}_c(T) \mid v \in L^2(0, T; H^2(\Gamma))\right\}.$$

Lemma 5.1. For all $p \in L^2(\Omega)$, $t \in [0, T]$, there exists a constant $C > 0$ and $\phi \in H^{1,2}(\Omega; \Gamma)$ such that $a_0^j(t)\phi_j^i = p$ and

$$\|\phi\|_{H^{1,2}(\Omega; \Gamma)} \leq C\|p\|_{L^2(\Omega)}. \quad (5.1)$$

Proof. We solve the following problem on the time-dependent domain $\Omega(t)$:

$$\text{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} \quad \text{in } \eta(t, \Omega) := \Omega(t).$$

The solution to this problem can be written as the sum of the solutions to the following two problems

$$\text{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} - \bar{p}(t) \quad \text{in } \eta(t, \Omega), \quad (5.2)
$$

$$\text{div}(\phi \circ \eta(t)^{-1}) = \bar{p}(t) \quad \text{in } \eta(t, \Omega), \quad (5.3)$$

where $\bar{p}(t) = \frac{1}{|\Omega|} \int_\Omega p(t, x)dx$. The existence of the solution to problem (5.2) with zero boundary condition is standard (see, for example, [15] Chapter 3), and the solution to problem (5.3) can be chosen as a linear function (linear in $x$), for example, $\bar{p}(t)x_1$. The estimate (5.1) follows from the estimates of the solutions to (5.2).

Define the linear functional on $H^{1,2}(\Omega; \Gamma)$ by $(p, a_0^j(t)\varphi_j^i)_{L^2(\Omega)}$ where $\varphi \in H^{1,2}(\Omega; \Gamma)$. By the Riesz representation theorem, there is a bounded linear operator $Q(t)$:

$$L^2(\Omega) \rightarrow H^{1,2}(\Omega; \Gamma)$$

such that for all $\varphi \in H^{1,2}(\Omega; \Gamma)$,

$$(p, a_0^j(t)\varphi_j^i)_{L^2(\Omega)} = (Q(t)p, \varphi)_{H^{1,2}(\Omega; \Gamma)} := (Q(t)p, \varphi)_{H^1(\Omega)} + (Q(t)p, \varphi)_{H^2(\Gamma)}.$$

Letting $\varphi = Q(t)p$ shows that

$$\|Q(t)p\|_{H^{1,2}(\Omega; \Gamma)} \leq C\|p\|_{L^2(\Omega)}$$

for some constant $C > 0$. By Lemma 5.1,

$$\|p\|_{L^2(\Omega)} \leq \|Q(t)p\|_{H^{1,2}(\Omega; \Gamma)} \|\varphi\|_{H^{1,2}(\Omega; \Gamma)} \leq C\|Q(t)p\|_{H^{1,2}(\Omega; \Gamma)}\|p\|_{L^2(\Omega)}$$
which shows that $R(Q(t))$ is closed in $H^{1/2}(\Omega; \Gamma)$. Since $V_v(t) \subset R(Q(t))^\perp$ and $R(Q(t))^\perp \subset V_v(t)$, it follows that 

$$H^{1/2}(\Omega; \Gamma)(t) = R(Q(t)) \oplus_{H^{1/2}(\Omega; \Gamma)} \overline{V}_v(t).$$ (5.4)

We can now introduce our Lagrange multiplier

**Lemma 5.2.** Let $\mathcal{L}(t) \in H^{1/2}(\Omega; \Gamma)'$ be such that $\mathcal{L}(t)\varphi = 0$ for any $\varphi \in V_v(t)$. Then there exist a unique $q(t) \in L^2(\Omega)$, which is termed the pressure function, satisfying

$$\forall \varphi \in H^{1/2}(\Omega; \Gamma), \quad \mathcal{L}(t)(\varphi) = (q(t), a_1(t)\varphi)_{L^2(\Omega)}.$$ 

Moreover, there is a $C > 0$ (which does not depend on $t \in [0, T]$ and $\epsilon$ and on the choice of $v \in C_T(M)$) such that

$$\|q(t)\|_{L^2(\Omega)} \leq C\|\mathcal{L}(t)\|_{H^{1/2}(\Omega; \Gamma)'}.$$ 

**Proof.** By the decomposition (5.4), for given $\tilde{v}$, let $\varphi = v_1 + v_2$, where $v_1 \in V_v(t)$ and $v_2 \in R(Q(t))$. It follows that

$$\mathcal{L}(t)(\varphi) = \mathcal{L}(t)(v_2) = (\psi(t), v_2)_{H^{1/2}(\Omega; \Gamma)} = (\psi(t), \varphi)_{H^{1/2}(\Omega; \Gamma)}$$ 

for a unique $\psi(t) \in R(Q(t))$. From the definition of $Q(t)$ we then get the existence of a unique $q(t) \in L^2(\Omega)$ such that

$$\forall \varphi \in H^{1/2}(\Omega; \Gamma), \quad \mathcal{L}(t)(\varphi) = (q(t), a_1(t)\varphi)_{L^2(\Omega)}.$$ 

The estimate stated in the lemma is then a simple consequence of (5.1). \qed

5.2. A polynomial-type inequality.

**Lemma 5.3.** Suppose that $x(t)$ is continuous in $[0, T]$, and there are $C_1$, $C_2$ and $\delta \in (0, 1)$ so that

$$x(t) \leq C_1 + C_2t^\delta \mathcal{P}(x(t)) \quad \forall t \in [0, T],$$ 

where $\mathcal{P}$ is a polynomial. Then there $T_1$ (depending only on $C_1$ and $C_2$) such that

$$x(t) \leq 2C_1 \quad \forall t \in [0, T].$$ 

**Proof.** We can assume that $\mathcal{P}(x)$ can be factored as $xQ(x)$ since the constant part can be collected into $C_1$. Therefore, we have

$$x(t) \leq C_1 + C_2t^\delta x(t)Q(x(t)), \quad t \in [0, T]$$

and hence

$$x(t) \left[1 - C_2t^\delta Q(x(t))\right] \leq C_1, \quad t \in [0, T].$$

Let $T_1 > 0$ so that $C_2T_1^\delta Q(2C_1) \leq 1/2$, then

$$\frac{1}{2} x(t) \leq C_1, \quad t \in [0, T].$$ 

\qed
6. Regularized and linearized problem

Given \( \tilde{v} \in \mathcal{V}^3(T) \) with the associated \( \tilde{g}, \tilde{b} \) in \( L^2(0, T; H^{2.5}(\Gamma)) \), set \( \tilde{F} = f \circ \tilde{\eta} \) and

\[
\tilde{L}_b(\eta) = 2\left[|\eta_0|^{-3}(\eta'' \cdot \tilde{n} - b_0)\tilde{n}\right]'' + \left[|\eta_0|^{-3}\tilde{g}'(\eta'' \cdot \tilde{n} - b_0)\tilde{n}\right]',
\]

and

\[
\tilde{L}_m(\eta) = -4\left[(\eta' \cdot \tilde{n} - |\eta_0|\tilde{\eta})\right]',
\]

with \( \tilde{n}(\tilde{\eta}) = (-\tilde{\eta}_2, \tilde{\eta}_1)/|\tilde{\eta}|). \)

The solution \( \tilde{v} \) of (2.3) is found via a limit as \( \kappa \to 0 \) of the fixed-point of the map \( \tilde{v} \mapsto v_\kappa \), where \( v_\kappa \) is the solution of the linearized and \( \kappa \)-regularized problem:

\[
\begin{align*}
v_\kappa &= \eta_{\kappa t} & \text{in } (0, T) \times \Omega, & \text{(6.1a)} \\
v_\kappa^i - \nu(\tilde{a}_i^j \tilde{a}_k^j v_\kappa^k)_{,j} &= -(\tilde{a}_i^j q_{\kappa})_{,j} + \tilde{F}^j & \text{in } (0, T) \times \Omega, & \text{(6.1b)} \\
\tilde{a}_i^j v_\kappa^i_{,j} &= 0 & \text{in } (0, T) \times \Omega, & \text{(6.1c)} \\
\left[\nu(\tilde{a}_i^j v_\kappa^i, k + \tilde{a}_j^k v_\kappa^j, i) - q_{k}\delta_{ij}\right]\tilde{a}_j^j N_\kappa &= \mathcal{L}_m(\eta_{\kappa}) + \tilde{L}_b(\eta) + \kappa \eta^{(4)} & \text{on } (0, T) \times \Gamma, & \text{(6.1d)} \\
v_\kappa(0, x) &= u_0(x) & \text{on } \{t = 0\} \times \Omega, & \text{(6.1e)} \\
\eta_{\kappa} &= \text{Id} & \text{on } \{t = 0\} \times \Omega, & \text{(6.1f)}
\end{align*}
\]

where we set \( \varepsilon = 1 \) and ignore the factor 1/3 in front of \( \mathcal{L}_b \). Note that here we treat the membrane traction as an extra forcing on the boundary. Also note that the time \( T \) a priori depends on \( \kappa \).

Following the same analysis as in [3], we can show that for this regularized problem (for a given and fixed \( \tilde{v} \)), there exists a unique solution \((\eta_{\kappa}, v_\kappa)\) to (6.1) with \( v_\kappa \in \mathcal{V}^3(T) \) and \( \eta_{\kappa} \in L^2(0, T; H^{5.5}(\Gamma)) \).

This follows by first approximating by a penalized problem, and then performing a regularity analysis (energy estimates). By the Tychonoff fixed-point theorem, there exists a fixed point \( \kappa \) in \( \mathcal{V}^3(T_\kappa) \) with \( \int_0^t v_\kappa ds \in L^2(0, T_\kappa; H^{5.5}(\Gamma)) \) and \((a_{\kappa})^i_{,j} v_\kappa^i_{,j} = 0\), and this \( v_\kappa \) and the associated \( \eta_{\kappa} \) satisfy

\[
\begin{align*}
v_\kappa &= \eta_{\kappa t} & \text{in } (0, T_\kappa) \times \Omega, & \text{(6.2a)} \\
v_\kappa^i - \nu([a_{\kappa}])_{,i}^j (a_{\kappa})_{,j}^k v_\kappa^k)_{,j} &= -(a_{\kappa})^i_{,j} q_{\kappa} + \tilde{F}^i & \text{in } (0, T_\kappa) \times \Omega, & \text{(6.2b)} \\
(a_{\kappa})_{,i}^j v_\kappa^i_{,j} &= 0 & \text{in } (0, T_\kappa) \times \Omega, & \text{(6.2c)} \\
\left[\nu(a_{\kappa}^k v_\kappa^j, k + a_{\kappa}^k v_\kappa^j, i) - q_{\kappa}\delta_{ij}\right](a_{\kappa})_{,i}^j N_\kappa &= \mathcal{L}_m(\eta_{\kappa}) + \tilde{L}_b(\eta_{\kappa}) & \text{on } (0, T_\kappa) \times \Gamma, & \text{(6.2d)} \\
&+ \kappa \eta_{\kappa}^{(4)} & & \text{(6.2e)} \\
v_\kappa(0, x) &= u_0(x) & \text{on } \{t = 0\} \times \Omega, & \text{(6.2e)} \\
\eta_{\kappa} &= \text{Id} & \text{on } \{t = 0\} \times \Omega. & \text{(6.2f)}
\end{align*}
\]
7. A priori estimates for $v_\kappa$, $q_\kappa$ and $\eta_\kappa$

7.1. $L^2(\Omega)$-estimate for $q_\kappa$. By (7.2), a solution $v_\kappa$, $q_\kappa$ and $\eta_\kappa$ also satisfy

$$
\langle v_{\kappa t}, \phi \rangle + \nu \frac{\kappa}{2} (a_\kappa)^k_j (v_{\kappa t})^k_j, k + (a_\kappa)^k_j (v_\kappa)^k_j, k \phi^i_k + (a_\kappa)^k_j (\phi_\kappa)^i_j + \langle q_\kappa, (a_\kappa)^i_j \phi_\kappa^i_j \rangle + \int_{\Gamma} \left[ \eta_\kappa^i_j - \eta_\kappa^i_j \right] \nabla_{\kappa}^2 (\phi^i_k \cdot \phi_\kappa^i_j) dS
$$

$$
\int_{\Gamma} \left( \eta_\kappa^i_j - \eta_\kappa^i_j \right) \nabla_{\kappa}^2 (\phi^i_k \cdot \phi_\kappa^i_j) dS + \frac{\kappa}{2} (\eta_\kappa^i_j \cdot \phi_\kappa^i_j) dS = (F, \phi)
$$

for all $\phi \in H^{1,2}(\Omega; \Gamma)$. Therefore, by the Lagrange multiplier lemma,

$$
\|q_\kappa\|^2_{L^2(\Omega)} \leq C \left[ \|v_{\kappa t}\|^2_{L^2(\Omega)} + \|D_{\Omega} v_\kappa\|^2_{L^2(\Omega)} + \|g_\kappa - g_0\|_{\kappa}^2 \right] + \|b_\kappa - b_0\|_{\kappa}^2 \|\eta_\kappa^i_j \cdot \phi_\kappa^i_j\|_{\kappa}^2 \right] \leq C \mathcal{P}(\|\eta_\kappa\|^2_{\kappa} \mathcal{P}(\|\phi_\kappa\|^2_{\kappa} + \|\eta_\kappa\|^2_{\kappa} + \|\phi_\kappa\|^2_{\kappa})
$$

for some constant $C$ independent of $\kappa$.

7.2. Interior regularity. Converting the fluid equation (6.2) into Eulerian variables by composing with $\eta_\kappa^{-1}$, we obtain a Stokes problem in the domain $\eta_\kappa(\Omega)$:

$$
-\nu \Delta u_\kappa + \nabla p_\kappa = F \circ \eta_\kappa^{-1} - v_{\kappa t} \circ \eta_\kappa^{-1},
$$

$$
\text{div } u_\kappa = 0,
$$

where $u_\kappa = v_\kappa \circ \eta_\kappa^{-1}$ and $p_\kappa = q_\kappa \circ \eta_\kappa^{-1}$. By the regularity results for the Stokes problem,

$$
\|u_\kappa\|^2_{H^2(\Omega)} + \|p_\kappa\|^2_{H^1(\Omega)} \leq C \left[ \|F\|_{L^2(\kappa)}^2 + \|v_{\kappa t} \circ \eta_\kappa^{-1}\|_{L^2(\kappa)}^2 + \|u_\kappa\|^2_{H^1(\kappa)} \right]
$$

or

$$
\|v_\kappa\|^2_{L^2(\Omega)} \leq C \left[ \|F\|_{L^2(\Omega)}^2 + \|v_{\kappa t}\|_{L^2(\Omega)}^2 \right] + C \mathcal{P}(\|\eta_\kappa\|^2_{\kappa} \mathcal{P}(\|\phi_\kappa\|^2_{\kappa} + \|\eta_\kappa\|^2_{\kappa} + \|\phi_\kappa\|^2_{\kappa})
$$

Similarly,

$$
\|v_\kappa\|^2_{H^2(\Omega)} \leq C \mathcal{P}(\|\eta_\kappa\|^2_{\kappa} \mathcal{P}(\|\phi_\kappa\|^2_{\kappa} + \|\eta_\kappa\|^2_{\kappa} + \|\phi_\kappa\|^2_{\kappa})
$$

7.3. $H^1(\Omega)$-estimate for $v_{\kappa t}$.

$$
\|\nabla v_{\kappa t}\|^2_{L^2(\Omega)} \leq \|\nabla u_\kappa + \frac{1}{t} \nabla v_{\kappa t}\|^2_{L^2(\Omega)} \leq \left[ \|\nabla u_\kappa\|^2_{L^2(\Omega)} + \frac{1}{t} \|\nabla v_{\kappa t}\|^2_{L^2(\Omega)} \right] \leq 2 \left[ \|u_\kappa\|^2_{H^1(\Omega)} + \frac{1}{t} \|v_{\kappa t}\|^2_{L^2(0,T;H^1(\Omega))} \right].
$$
8. Elliptic estimates on the boundary

8.1. Estimates without artificial viscosity. Since \( v_\kappa \in \mathcal{V}^3(T) \), the associated \( \eta_\kappa \) satisfies the boundary condition \((6.2d)\) in the pointwise sense. We start with the estimates without considering the artificial viscosity to illustrate the basic idea; then in the next section we consider the full boundary condition \((6.2d)\) and obtain the desired estimates. By \((8.1)\), we find that

\[
\mathcal{L}_m(\eta) = 4 \left[ \sqrt{\gamma}(g' - g'_0) + \frac{g - g_0}{2\sqrt{\gamma}} g' \right] \tau + 4(g - g_0) b n
\]

and

\[
\mathcal{L}_b(\eta) = 2 \left[ |\eta'_0|^{-3} (b - b_0) \right]'' - 2 |\eta'_0|^{-3} g^{-1} b^2 (b - b_0) + \left( |\eta'_0|^{-3} g^{-1} g'(b - b_0) \right)' n
\]

\[
- \left[ 4 \left( |\eta'_0|^{-3} (b - b_0) \right)' + 2 |\eta'_0|^{-3} g^{-1/2} (b - b_0) b' \right] g^{-1/2} b \tau.
\]

Given \( h \in L^2(0, T; L^{1.5}(\Gamma)) \cap L^\infty(0, T; H^{0.5}(\Gamma)) \), a solution to

\[
\mathcal{L}_m(\eta) + \mathcal{L}_b(\eta) = h \tag{8.1}
\]

satisfies the “normal equation”

\[
2 \left[ |\eta'_0|^{-3} (b - b_0) \right]'' = h \cdot n - 4(g - g_0) b + 2 |\eta'_0|^{-3} g^{-1} b^2 (b - b_0) - \left[ |\eta'_0|^{-3} g^{-1} g'(b - b_0) \right]'.
\tag{8.2}
\]

We also have the “tangential equation”

\[
4 \sqrt{\gamma}(g - g_0)' = h \cdot \tau + g^{-1/2} \left[ 4 \left( |\eta'_0|^{-3} (b - b_0) \right)' b + 2 |\eta'_0|^{-3} g^{-1/2} (b - b_0) b b' - 2(g - g_0) g' \right].
\tag{8.3}
\]

Therefore, by elliptic estimates, a solution to \((8.1)\) satisfies

\[
||b - b_0||_{H^{2.5}(\Gamma)} \leq C \left[ ||h \cdot n||_{H^{0.5}(\Gamma)}^2 + ||(g - g_0) b||_{H^{2.5}(\Gamma)}^2 
\right.
\]

\[
+ ||g^{-1} b^2 (b - b_0)||_{H^{0.5}(\Gamma)}^2 + ||g^{-1} g'(b - b_0)'||_{H^{2.5}(\Gamma)}^2 \bigg],
\tag{8.4a}
\]

\[
||g - g_0||_{H^{2.5}(\Gamma)} \leq C \mathcal{P}(||\eta||_{H^{2.5}(\Gamma)}) \left[ ||h \cdot \tau||_{H^{1.5}(\Gamma)}^2 + ||(b - b_0)' g^{-1/2} b||_{H^{1.5}(\Gamma)}^2 
\right.
\]

\[
+ ||g^{-1} (b - b_0) b b'||_{H^{1.5}(\Gamma)}^2 + ||(g - g_0) g'||_{H^{2.5}(\Gamma)}^2 \bigg], \tag{8.4b}
\]

where \( C \) only depends on \( \Gamma \). Since

\[
||f g||_{H^{0.5}(\Gamma)} \leq C \left[ ||f||_{H^{0.5}(\Gamma)} ||g||_{L^{\infty}(\Gamma)} + ||f||_{H^{0.5}(\Gamma)} ||g||_{L^{\infty}(\Gamma)} \right], \tag{8.5a}
\]

\[
||f g||_{H^{1.5}(\Gamma)} \leq C \left[ ||f||_{H^{1.5}(\Gamma)} ||g||_{H^{1.5}(\Gamma)} + ||f||_{H^{1.5}(\Gamma)} ||g||_{H^{1.5}(\Gamma)} \right], \tag{8.5b}
\]

by the Leibnitz rule, we have

\[
||g^{-1} g'(b - b_0)'||_{H^{0.5}(\Gamma)}^2 \leq C \mathcal{P}(||\eta||_{H^{2.5}(\Gamma)}) \left[ ||g||_{H^{2.5}(\Gamma)}^2 ||b - b_0||_{L^{\infty}(\Gamma)}^2 + ||b - b_0||_{H^{1.5}(\Gamma)}^2 \bigg],
\tag{8.6}
\]

\[
+ \epsilon ||b - b_0||_{H^{2.5}(\Gamma)}^2,
\]
and
\[ \|g^{-1}(b - b_0)b'b\|_{H^{1/2}(\Gamma)}^2 \leq CP(||\eta||_{H^{1/2}(\Gamma)}^2)\|b - b_0\|_{H^{1/2}(\Gamma)}^2 \|b\|_{H^{1/2}(\Gamma)}^2. \]

Let \( X(T) = \|v\|_{L^2(\Omega; T; H^{1/2}(\Gamma))}^2 + \|b\|_{L^2(0, T; H^{1/2}(\Gamma))}^2 + \|g\|_{L^2(0, T; H^{1/2}(\Gamma))}^2 \), then
\[ \|b(t) - b_0\|_{H^{1/2}(\Gamma)}^2 + \|g(t) - g_0\|_{H^{1/2}(\Gamma)}^2 \leq C\sqrt{T}P(X(T)), \]  
(8.7a)
\[ \|b(t) - b_0\|_{H^{1/2}(\Gamma)}^2 + \|g(t) - g_0\|_{H^{1/2}(\Gamma)}^2 \leq CP(X(T)), \]  
(8.7b)
for all \( 0 \leq t \leq T \). Therefore, by choosing \( \epsilon > 0 \) small enough in (8.6),
\[ \|b\|_{H^{1/2}(\Gamma)}^2 \leq CP(||\eta||_{H^{1/2}(\Gamma)}^2)||h||_{L^\infty(\Gamma)}^2 + CP(X(T)) \left[ 1 + \|h\|_{H^{1/2}(\Gamma)}^2 \right] + C\sqrt{T}P(X(T)) \left[ \|b\|_{H^{1/2}(\Gamma)}^2 + \|g\|_{H^{1/2}(\Gamma)}^2 \right], \]
and hence by (8.4) (and also (8.7), (8.8)),
\[ \|g\|_{H^{1/2}(\Gamma)}^2 \leq CP(||\eta||_{H^{1/2}(\Gamma)}^2)||h||_{H^{1/2}(\Gamma)}^2 + CP(X(T)) \left[ 1 + \|h\|_{H^{1/2}(\Gamma)}^2 \right] + C\sqrt{T}P(X(T)) \left[ \|b\|_{H^{1/2}(\Gamma)}^2 + \|g\|_{H^{1/2}(\Gamma)}^2 \right]. \]

With \( h = [\mu(a^k_i v^k_i + a^k_i v^k_i)_i - g_\delta ij]a^k_j N_k \) in mind, we find that
\[ \int_0^t \|b(s)\|_{H^{1/2}(\Gamma)}^2 ds \leq C(t + t^{1/2} + t^{1/4})P(X(T)) \]  
(8.10)
and
\[ \int_0^t \|g(s)\|_{H^{1/2}(\Gamma)}^2 ds \leq C(1 + t^{1/2} + t)P(X(T)). \]  
(8.11)

### 8.2. Estimates with artificial viscosity.
Now we study the full boundary condition
\[ \mathcal{L}_m(\eta) + \mathcal{L}_b(\eta) + \kappa \eta''' = h. \]
(8.12)

By the Leibnitz rule and (2.4),
\[ \eta''' n_k = b''' - g_k^{-1}b_k^3 - \frac{3}{4}g_k^{-2}(g_k')b_k + \frac{1}{2}g_k^{-1}g_k'b_k + g_k^{-1}g''_k b_k, \]  
(8.13a)
\[ \eta''' T = \frac{1}{2}g_k^{-1/2} \left[ g_k''' - 3b_k b_k' - \frac{3}{2}g_k^{-1}g_k'g''_k + \frac{3}{4}g_k^{-2}(g_k')^3 \right], \]  
(8.13b)
where \( g_k = \eta'_k \cdot n_k, \ n_k = g_k^{-1/2} \eta_{k,1} \times \eta_{k,2} \) and \( b_k = \eta''_k \cdot n_k \). Define
\[ X_k(T) = \sup_{0 \leq t \leq T} \left[ \|v_k\|_{L^2(\Omega)}^2 + \|v_k\|_{H^2(\Gamma)}^2 + \|b_k\|_{H^2(\Gamma)}^2 + \|g_k\|_{H^2(\Gamma)}^2 + \|v_k' \cdot n_k\|_{L^2(\Gamma)}^2 \right. \]
\[ + \|v_k''' \cdot n_k\|_{L^2(\Gamma)}^2 + \|b_k\|_{L^2(0, T; H^{1/2}(\Gamma))}^2 + \|g_k\|_{L^2(0, T; H^{1/2}(\Gamma))}^2 \].

By the same technique, we find using (8.2) that
\[ \|b_k\|_{H^{1/2}(\Gamma)}^2 \leq CP(||\eta_k||_{H^{1/2}(\Gamma)}^2)||h||_{L^\infty(\Gamma)}^2 + CP(X_k(T)) \left[ 1 + \|h\|_{H^{1/2}(\Gamma)}^2 \right] + C\sqrt{T}P(X_k(T)) \left[ \|b\|_{H^{1/2}(\Gamma)}^2 + \|g\|_{H^{1/2}(\Gamma)}^2 \right] + C_k \eta P(X_k(T)) \]  
\[ + \frac{\epsilon_k}{P(X_k(T))} \|g_k\|_{H^2(\Gamma)}^2 \]
and from the tangential equation (8.3), we find that
\[\|g_\kappa\|^2_{H^{2.5}(\Gamma)} + \kappa\|g_\kappa\|^2_{H^{1.5}(\Gamma)}\]
\[\leq C\mathcal{P}(\|\eta_\kappa\|^2_{H^{2.5}(\Gamma)}) + C\mathcal{P}(X_\kappa(T))\left[1 + \|\eta\|^2_{H^{0.5}(\Gamma)}\right] + C\sqrt{t}\mathcal{P}(X_\kappa(T))\left[\|b_\kappa\|^2_{H^{2.5}(\Gamma)} + \|g_\kappa\|^2_{H^{2.5}(\Gamma)}\right] + C\kappa\|\eta_\kappa\|^2_{H^{3.5}(\Gamma)},\]
where we use the fact that
\[\|\eta_\kappa''\cdot n_\kappa\|^2_{H^{0.5}(\Gamma)}\]
\[\leq C\kappa\left[\|b_\kappa - b_\kappa'|\|_{L^\infty(\Gamma)}\|b_\kappa\|^2_{H^{2.5}(\Gamma)} + \|g_\kappa\|^2_{H^{2.5}(\Gamma)}\|\eta_\kappa\|^2_{H^{2.5}(\Gamma)} + \|g_\kappa\|^2_{H^{2.5}(\Gamma)}\|g_\kappa\|^2_{H^{2.5}(\Gamma)}\right] + C\kappa\|\eta_\kappa\|^2_{H^{3.5}(\Gamma)} + C\kappa\|\eta_\kappa\|^2_{H^{3.5}(\Gamma)}.
\]
Note that here \(C\) and \(C_\kappa\) are independent of \(\kappa\). Choosing \(\epsilon > 0\) small enough, we find that
\[\left(\int_0^t \|b_\kappa(s)\|^2_{H^{2.5}(\Gamma)} ds\right)^{1/2} \leq C(t + t^{1/2} + t^{1/4})\mathcal{P}(X_\kappa(T)) + \frac{\kappa}{100} \int_0^t \|\eta_\kappa'(s)\|^2_{H^{3.5}(\Gamma)} ds\]
and
\[\int_0^t \left[\|g_\kappa(s)\|^2_{H^{2.5}(\Gamma)} + \kappa\|g_\kappa(s)\|^2_{H^{1.5}(\Gamma)}\right] ds \leq C(1 + t^{1/2} + t)\mathcal{P}(X_\kappa(T)).\]

### 8.3. The estimate of \(n_\kappa\).
By (2.1),
\[\|n_\kappa\|^2_{H^{2.5}(\Gamma)} \leq C\mathcal{P}(X_\kappa(T)),\]
(8.16a)
\[\int_0^t \|n_\kappa(s)\|^2_{H^{3.5}(\Gamma)} ds \leq C(1 + t^{1/2} + t)\mathcal{P}(X_\kappa(T)) ds.\]
(8.16b)

### 8.4. Small time results.
In this section, we rewrite some inequalities in Section 7 that will be used in the later discussion. First of all, note that (7.6) implies that
\[\|v\|^2_{H^{1}(\Omega)} \leq C\left[\|u_0\|^2_{H^{1}(\Omega)} + t\mathcal{P}(X_\kappa(T))\right].\]
(8.17)
Since
\[\|\eta_\kappa\|^2_{H^{2.5}(\Gamma)} \leq 2\left[\|\text{Id}\|^2_{H^{2.5}(\Gamma)} + t\int_0^t \|v_\kappa\|^2_{H^{2.5}(\Omega)} ds\right],\]
(7.2) can be rewritten as
\[\|q_\kappa\|^2_{L^2(\Omega)} \leq C\left[\|v_\kappa\|^2_{L^2(\Omega)} + t\mathcal{P}(X_\kappa(T))\right].\]
(8.18)
Finally, we also have
\[\int_0^t \left[\|v_\kappa\|^2_{H^{2}(\Omega)} + \|g_\kappa\|^2_{H^{1}(\Omega)}\right] ds \leq C\int_0^t \|F\|^2_{L^2(\Omega)} ds + Ct\mathcal{P}(X_\kappa(T)).\]
(8.19)
and
\[
\int_0^t \left[ \| \nu_n \|^2_{H^3(\Omega)} + \| q_n \|^2_{H^2(\Omega)} \right] ds \leq C \int_0^t \left[ \| F \|_{H^1(\Omega)}^2 + \| \nu_{nt} \|^2_{H^1(\Omega)} + \| \nu_n \|^2_{H^2(\Omega)} \right] ds + C(t + \sqrt{t})\mathcal{P}(X_\kappa(T)),
\]
where we have used that
\[
\| \nabla \eta_\kappa(t) - \text{Id} \|_{H^2(\Omega)} + \| a_\kappa(t) - \text{Id} \|_{H^2(\Omega)} \leq C(t + \sqrt{t})\mathcal{P}(X_\kappa(T))
\] (8.21)
to remove the \( a_\kappa \) and \( \eta_\kappa \) dependence from inequalities (7.4) and (7.5).

9. NONLINEAR ESTIMATES

In the following discussion, we will always assume that \( T \leq 1 \). Therefore, all the time dependent functions appearing in the previous section, such as \( t, (t + \sqrt{t}), (t + t^{1/2} + t^{1/4}) \), etc., can be replaced by \( t^\delta \) for some fixed \( \delta \in (0, 1) \).

9.1. Partition of unity. Since \( \Omega \) is compact, by partition of unity, we can choose two non-negative smooth functions \( \zeta_0 \) and \( \zeta_1 \) so that
\[
\zeta_0 + \zeta_1 = 1 \quad \text{in} \quad \Omega; \quad \text{supp}(\zeta_0) \subset \subset \Omega; \quad \text{supp}(\zeta_1) \subset \subset \Gamma \times (-\epsilon, \epsilon) := \Omega_1.
\]
We will assume that \( \zeta_1 = 1 \) inside the region \( \Omega'_1 \subset \Omega_1 \) and \( \zeta_0 = 1 \) inside the region \( \Omega' \subset \Omega \). Note that then \( \zeta_1 = 1 \) while \( \zeta_0 = 0 \) on \( \Gamma \).

9.2. Energy estimates for \( \nu''_n \) near the boundary. Use \( (\zeta^2_1 \nu''_n)'' \) as a test function in (7.1), we find that
\[
\frac{1}{2} \frac{d}{dt} \left[ \| \zeta_1 \nu''_n \|^2_{L^2(\Omega)} + 2 \int_{\Gamma} |\eta_\kappa|^{-3} \left[ \| (g_\kappa - g_0)'' \|^2 + \| (b_\kappa - b_0)'' \|^2 \right] dS + \kappa \int_{\Gamma} |\eta_\kappa''|^2 dS \right] + \nu' \| \zeta_1 D_{\eta_\kappa} \nu''_n \|_{L^2(\Omega)}^2 = \langle F, (\zeta^2_1 \nu''_n)'' \rangle + (K_1 + K_2 + \cdots + K_7),
\]
where \( (D_{\eta_\kappa} \omega)_i^j = (a_\kappa)_i^k (w^j_k)_k + (a_\kappa)_i^k (w^j_k)'_k \) is the nonlinear version of the rate of deformation tensor, and \( K'_s \)s are defined by

\[
K_1 = -2\nu \int_{\Omega} \zeta_1 \left[ (a^\kappa)_i^k (v^j_k)'_k + (a'^\kappa)_i^k (v'^j_k)'_k \right] (a_\kappa)_i^k \zeta_1 (v''_k)_k dx
- \nu \int_{\Omega} \zeta_1 \left[ (a''^\kappa)_i^k (v''_k)_k + (a'^\kappa)_i^k (v'_k)_k \right] (a_\kappa)_i^k \zeta_1 (v''_k)_k dx
- 2\nu \int_{\Omega} \zeta_1 \left[ (a''^\kappa)_i^k (v''_k)_k + (a'^\kappa)_i^k (v'_k)_k \right] \left( a'_\kappa \right)_i^k \zeta_1 (v''_k)_k dx
- 2\nu \int_{\Omega} \zeta_1 \left[ (a''^\kappa)_i^k (v''_k)_k + (a'^\kappa)_i^k (v'_k)_k \right] \left( a''_\kappa \right)_i^k \zeta_1 (v''_k)_k dx.
\]
\[ K_2 = 4 \int_\Gamma |\eta_0^{(5)}(g_0 - g_0)|^2 \left[ \eta_k^{(5)} \cdot v_\kappa + 4 \eta_k^{(4)} \cdot v_\kappa'' + 6 \eta_k'' \cdot v_\kappa'' + 4 n_k'' \cdot v_\kappa'^{(4)} \right] ds, \]

\[ K_3 = -8 \int_\Gamma (|\eta_0^{(3)}|'(g_0 - g_0)'(v_\kappa' \cdot v_\kappa''')) ds - 4 \int_\Gamma (|\eta_0^{(3)}|''(g_0 - g_0)(\eta_k' \cdot v_\kappa''') ds - 4 \int_\Gamma (|\eta_0^{(3)}|''(g_0 - g_0))'' g_0'' ds, \]

\[ K_4 = 2 \int_\Gamma |\eta_0^{(3)}(b_\kappa - b_0)| \left[ (\eta_\kappa'' \cdot n_\kappa)(v_\kappa'' + 4 \eta_k'' \cdot v_\kappa'' + 6 n_k'' \cdot v_\kappa'^{(4)}) \right. \]
\[ + \left. 4 n_k' \cdot v_\kappa^{(5)} \right] ds, \]

\[ K_5 = -4 \int_\Gamma (|\eta_0^{(3)}|'(b_\kappa - b_0)'(v_\kappa'' \cdot n_\kappa''')) ds - 2 \int_\Gamma (|\eta_0^{(3)}|''(b_\kappa - b_0)(v_\kappa'' \cdot n_\kappa''')) ds, \]

\[ K_6 = \int_\Gamma |\eta_0^{(3)}g_0^{-1}(b_\kappa - b_0)g_0'(v_\kappa^{(5)} \cdot n_\kappa) ds, \]

and

\[ K_7 = -\int_{\Omega} q_\kappa(a_\kappa) \left[ (C^2 v_\kappa''')'' \right] dx. \]

By (8.21),

\[ |K_1| \leq C(1 + t^\delta P(X_\kappa(T))) \|v_\kappa\|_{H^2(\Omega)} \|v_\kappa\|_{H^3(\Omega)} \]

and hence by Young’s inequality and (8.19),

\[ \left| \int_0^t K_1 ds \right| \leq C t^\delta \left[ \|F\|^2_{L^\infty(0,T;L^2(\Omega))} + P(X_\kappa(T)) \right] + \epsilon \int_0^t \|v_\kappa\|^2_{H^3(\Omega)} ds \]

\[ \leq C t^\delta \left[ M_0 + P(X_\kappa(T)) \right] + \epsilon \int_0^t \|v_\kappa\|^2_{H^3(\Omega)} ds. \quad (9.2) \]

Integrating by parts in space, we find that

\[ |K_2| \leq C \left[ \|g_\kappa - g_0\|_{H^2(\Gamma)} \|v_\kappa\|_{H^{1.5}(\Gamma)} + \|g_\kappa - g_0\|_{H^{1.5}(\Gamma)} \|v_\kappa\|_{H^{2.5}(\Gamma)} \right] \|\eta_\kappa\|_{H^{3.5}(\Gamma)} + C \left[ \|g_\kappa - g_0\|_{H^{1.5}(\Gamma)} \|\eta_\kappa\|_{H^{3.5}(\Gamma)} + \|g_\kappa - g_0\|_{H^{1.5}(\Gamma)} \|\eta_\kappa\|_{H^{3.5}(\Gamma)} \right] \|v_\kappa\|_{H^{2.5}(\Gamma)} \]

and hence

\[ \left| \int_0^t K_2 ds \right| \leq C t^\delta P(X_\kappa(T)) + \epsilon \int_0^t \left[ \|v_\kappa\|^2_{H^2(\Gamma)} + \|g_\kappa\|^2_{H^{2.5}(\Gamma)} \right] ds \quad (9.3) \]

here we use (8.74) to estimate \( \|g_\kappa - g_0\|_{L^\infty(\Gamma)} \). As for \( K_3 \), integrating by parts for the first two integrals, we find that

\[ |K_3| \leq C \|P(X_\kappa(T))\| \|g_\kappa - g_0\|_{H^2(\Gamma)} \|v_\kappa\|_{H^2(\Gamma)}. \]

Therefore,

\[ \left| \int_0^t K_3 ds \right| \leq C t^\delta P(X_\kappa(T)) + \epsilon \int_0^t \|v_\kappa\|^2_{H^3(\Omega)} ds. \quad (9.4) \]

Similarly, integrating by parts and \( H^{1.5}(\Gamma) - H^{-1.5}(\Gamma) \) or \( H^{0.5}(\Gamma) - H^{-0.5}(\Gamma) \) duality pairing lead to

\[ |K_4| + |K_5| \leq C \|P(X_\kappa(T))\| \left[ \|b_\kappa - b_0\|_{H^{1.5}(\Gamma)} \|n_\kappa\|_{H^{3.5}(\Gamma)} + \|b_\kappa - b_0\|_{H^{1.5}(\Gamma)} \|n_\kappa\|_{H^{2.5}(\Gamma)} \right] \|v_\kappa\|_{H^{2.5}(\Gamma)} \]
and hence
\[
\left| \int_0^t (K_4 + K_5) ds \right| \leq C t^\beta \mathcal{P}(X_\kappa(T)) + \frac{\kappa}{100} \int_0^t \| \eta'_\kappa(s) \|^2_{\mathcal{H}^3(\Gamma)} ds , \tag{9.5}
\]
here we use (S.7k) and (S.14). By the Leibnitz rule,
\[
K_6 = -\int_\Gamma \left[ |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \right] b_{\kappa}'' ds - \int_\Gamma \left[ |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \right] n_{\kappa}' \cdot v_{\kappa}''' ds \\
+ \int_\Gamma \left[ |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \right] \left[ (n_{\kappa} \cdot n_{\kappa})'' + v_{\kappa}'' \cdot n_{\kappa}' + 2v_{\kappa}''' \cdot n_{\kappa}' \right] ds.
\]

The worst situation for the last integral is when the derivative outside the bracket is put on \( g_{\kappa}' \). In this case, since
\[
n_{\kappa} = -g_{\kappa}^{-1}(v_{\kappa}' \cdot n_{\kappa})\eta'_{\kappa} \quad \text{or} \quad \eta'_{\kappa} \cdot n_{\kappa} = -\frac{1}{2} g_{\kappa}^{-1} g_{\kappa}'(v_{\kappa}' \cdot n_{\kappa}) , \tag{9.6}
\]
the worst term will be
\[
\int_\Gamma |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \left[ -\frac{1}{2} g_{\kappa}^{-1} g_{\kappa}'''(v_{\kappa}' \cdot n_{\kappa}) \right] ds .
\]

For the first term, since \( g_{\kappa}''' g_{\kappa}'' \) forms a perfect derivative, after integrating by parts we have
\[
\int_\Gamma |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \left[ -\frac{1}{2} g_{\kappa}^{-1} g_{\kappa}'''(v_{\kappa}' \cdot n_{\kappa}) \right] ds \\
\leq C \mathcal{P}(\| \eta_{\kappa} \|^2_{H^2(\Gamma)}) \| b_\kappa - b_0 \|_{H^{1.5}(\Gamma)} \| g_{\kappa} \|^2_{H^2(\Gamma)} \| v_{\kappa} \|_{H^{1.5}(\Gamma)} \\
+ C \mathcal{P}(\| \eta_{\kappa} \|^2_{H^2(\Gamma)}) \| b_\kappa - b_0 \|_{L^\infty(\Gamma)} \| g_{\kappa} \|^2_{H^2(\Gamma)} \| v_{\kappa}' \cdot n_{\kappa} \|_{L^2(\Gamma)} \\
+ C \mathcal{P}(\| \eta_{\kappa} \|^2_{H^2(\Gamma)}) \| b_\kappa - b_0 \|_{L^\infty(\Gamma)} \| g_{\kappa} \|_{H^2(\Gamma)} \| g_{\kappa} \|_{H^{1.5}(\Gamma)} \| v_{\kappa} \|_{H^2(\Gamma)} .
\]

For the remaining two terms, we use \( H^{0.5}(\Gamma) - H^{-0.5}(\Gamma) \) duality pairing and obtain
\[
\int_\Gamma |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \left[ -\frac{1}{2} g_{\kappa}^{-1} g_{\kappa}'''(v_{\kappa}' \cdot n_{\kappa}) + 2v_{\kappa}''' \cdot n_{\kappa}' \right] ds \\
\leq C \mathcal{P}(\| \eta_{\kappa} \|^2_{H^2(\Gamma)}) \| b_\kappa - b_0 \|_{H^1(\Gamma)} \| g_{\kappa} \|_{H^2(\Gamma)} \| g_{\kappa} \|_{H^{1.5}(\Gamma)} \| v_{\kappa} \|_{H^2(\Gamma)} .
\]

Therefore, by (S.7k) and (S.14), we find that
\[
\int_0^t \int_\Gamma \left[ |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \right] \left[ (n_{\kappa}' \cdot n_{\kappa})'' + v_{\kappa}'' \cdot n_{\kappa}' + 2v_{\kappa}''' \cdot n_{\kappa}' \right] ds ds \\
\leq C t^\beta \mathcal{P}(X_\kappa(T)) + \frac{\kappa}{100} \int_0^t \| \eta'_\kappa(s) \|^2_{\mathcal{H}^3(\Gamma)} ds + \epsilon \int_0^t \| v_{\kappa} \|^2_{\mathcal{H}^3(\Omega)} ds .
\]

For the second integral of \( K_6 \), integrating by parts and using \( H^{0.5}(\Gamma) - H^{-0.5}(\Gamma) \) duality pairing, we find that
\[
\int_0^t \int_\Gamma \left[ |\eta'_\kappa|^{-3} g_{\kappa}^{-1}(b_\kappa - b_0) g_{\kappa}' \right] n_{\kappa}' \cdot v_{\kappa}''' ds ds \\
\leq C t^\beta \mathcal{P}(X_\kappa(T)) + \frac{\kappa}{100} \int_0^t \| \eta'_\kappa(s) \|^2_{\mathcal{H}^3(\Gamma)} ds + \epsilon \int_0^t \| v_{\kappa} \|^2_{\mathcal{H}^3(\Omega)} ds .
\]
Finally, for the first integral of $K_6$, we time integrate it first and then integrate by parts in both space and time to obtain

\[
\int_0^t \int_{\Gamma} |\eta_0'|^{-3} g_1^{-1} (b_\kappa - b_0) g'_k b''_{\kappa t} dS ds \n
= - \int_0^t \int_{\Gamma} \left[ |\eta_0'|^{-3} g_1^{-1} g'_k (b_\kappa - b_0) \right] b''_k dS ds
\]

\[
= - \left[ \int_{\Gamma} \left( |\eta_0'|^{-3} g_1^{-1} g'_k (b_\kappa - b_0) \right) b''_k dS \right] (t)
\]

\[
+ \int_0^t \int_{\Gamma} \left[ |\eta_0'|^{-3} g_1^{-1} g'_k (b_\kappa - b_0) \right] b''_k dS ds .
\]

Therefore,

\[
\left| \int_0^t \int_{\Gamma} |\eta_0'|^{-3} g_1^{-1} (b_\kappa - b_0) g'_k b''_{\kappa t} dS ds \right|
\]

\[
\leq \left| \int_{\Gamma} \left( |\eta_0'|^{-3} g_1^{-1} g'_k (b_\kappa - b_0) \right) b''_k dS \right| (t)
\]

\[
+ \left. \left| \int_0^t \int_{\Gamma} \left[ |\eta_0'|^{-3} g_1^{-1} g'_k (b_\kappa - b_0) \right] b''_k dS ds \right| \right|
\]

\[
\leq C t^5 P(X_\kappa (T)) + \frac{\kappa}{100} \int_0^t \left| \eta'_k (s) \right|_{H^3 (\Gamma)}^2 dS .
\]

Combining the estimates above,

\[
\left| \int_0^t K_6 ds \right| \leq C t^5 P(X_\kappa (T)) + \frac{\kappa}{100} \int_0^t \left| \eta'_k (s) \right|_{H^3 (\Gamma)}^2 ds + \epsilon \int_0^t \left| \eta_{\kappa} \right|_{H^3 (\Omega)}^2 ds .
\]

**Remark 1.** In the estimate of $K_6$, the fact that $n = 2$ is necessary to use the Sobolev embedding $L^4 (\Gamma) \subset H^{0.25} (\Gamma)$ with

\[
\left| h \right|_{H^{0.25} (\Gamma)} \leq C \left| h \right|_{L^4 (\Gamma)}
\]

and

\[
\left| \langle f, g \rangle \right|_{\Gamma} \leq C \left| f \right|_{H^1 (\Gamma)} \left| g \right|_{H^{0.5} (\Gamma)} \left| h \right|_{H^{-0.5} (\Gamma)}
\]

for some constant $C$. These inequalities no longer holds if $n = 3$.

Now we turn to the estimate of $K_7$. By the identity

\[
(a_\kappa)^\dagger (\xi^e v''_{\kappa} v''_{\kappa})^\dagger = - \left[ (a_\kappa)^\dagger (\xi^e v''_{\kappa})^\dagger - 2(a_\kappa)^\dagger (\xi^e v''_{\kappa})^\dagger - 2(a_\kappa)^\dagger (\xi^e v''_{\kappa})^\dagger - 2(a_\kappa)^\dagger (\xi^e v''_{\kappa})^\dagger \right],
\]

and inequality (8.21), we find that

\[
\left| \int_\Omega g_\kappa (a_\kappa)^\dagger (\xi^e v''_{\kappa})^\dagger \right|
\]

\[
\leq C |a_\kappa|_{H^2 (\Omega)} \left[ \left| g_\kappa \right|_{H^2 (\Omega)} \left| v_\kappa \right|_{H^2 (\Omega)} \left| \xi_1 v''_{\kappa} \right|_{L^2 (\Omega)} \right]
\]

\[
\leq C \left( 1 + t^5 P(X_\kappa (T)) \left( \left| v_\kappa \right|_{H^1 (\Omega)}^2 \right) + \epsilon \left[ \left| \eta_{\kappa} \right|_{H^3 (\Omega)}^2 \right] \right)
\]

Therefore, by (8.19) and (8.20),

\[
\left| \int_0^t K_7 ds \right| \leq C \left( M_0 + P(X_\kappa (T)) \right) + \epsilon \int_0^t \left[ \left| v_{\kappa t} \right|_{H^1 (\Omega)}^2 + \left| v_\kappa \right|_{H^2 (\Omega)}^2 \right] ds .
\]
Since
$$\int_0^t \langle F, (\zeta^n_\kappa \eta^n_\kappa)'' \rangle dt \leq C \varepsilon \int_0^t \| F \|_{H^1(\Omega)}^2 dt + \varepsilon \int_0^t \| v \|_{H^3(\Omega)}^2 dt,$$
time integrating by parts (if necessary)\footnote{9.10}, we find that
$$\sup_{0 \leq t \leq T} \int_\Gamma \| G^n_\kappa \|_{L^2(\Gamma)}^2 dt = C_t \int_\Gamma \| G^n_\kappa \|_{L^2(\Gamma)}^2 dt + \varepsilon \int_\Gamma \| v \|_{H^3(\Omega)}^2 dt + \varepsilon \int_0^t \| v \|_{H^3(\Omega)}^2 dt,$$
and hence by (8.7),
$$\int_0^t \| F_0 \|_{H^1(\Omega)}^2 dt \leq C \varepsilon \int_0^t \| v \|_{H^3(\Omega)}^2 dt + \varepsilon \int_0^t \| v \|_{H^3(\Omega)}^2 dt.$$

9.3. Energy estimates for $v_{\kappa t}$. Time differentiate (7.1) and then use $v_{\kappa t}$ as the test function, we find that
$$\frac{1}{2} \frac{d}{dt} \| v_{\kappa t} \|^2_{L^2(\Omega)} + \int_\Gamma |\eta_0^n|^{-3} |\eta^n_\kappa \cdot \eta^n_\kappa|^2 + 2 \| \kappa' \cdot n_{\kappa} \|^2_{L^2(\Gamma)} dt + \int_\Gamma \| v_{\kappa t} \|^2_{L^2(\Gamma)} dt + \varepsilon \int_\Gamma \| v \|_{H^3(\Omega)}^2 dt,$$
and
$$\frac{1}{2} \frac{d}{dt} \| v_{\kappa t} \|^2_{L^2(\Omega)} + \int_\Gamma |\eta_0^n|^{-3} |\kappa' \cdot \kappa^n_\kappa|^2 + 2 \| \kappa' \cdot n_{\kappa} \|^2_{L^2(\Gamma)} dt + \int_\Gamma \| v_{\kappa t} \|^2_{L^2(\Gamma)} dt + \varepsilon \int_\Gamma \| v \|_{H^3(\Omega)}^2 dt,$$
where
$$L_1 = - \int \left( D_{\kappa \eta} v_0 (a_{\kappa})_{\kappa} (\eta_0^n)_{\kappa} \cdot (v_{\kappa t} \cdot (a_{\kappa})_{\kappa} v^2_{\kappa t}) - \int \left( (a_{\kappa})_{\kappa} v^2_{\kappa t} v^n_{\kappa} + (a_{\kappa})_{\kappa} v_{\kappa t} v^n_{\kappa} \right) dx,$$
$$L_2 = - \int \left( g_\kappa - g_0 \right) (v_{\kappa t} \cdot v_\kappa) dx + \int \left( |\eta_0^n|^{-3} |\eta_0^n |^{-3} |\eta_0^n \cdot \eta_0^n |^2 + 2 \| \kappa \cdot n_{\kappa} \|^2_{L^2(\Gamma)} dx,$$
$$L_3 = - \int \left( |\eta_0^n|^{-3} |\eta_0^n \cdot \eta_0^n |^2 + 2 \| \kappa' \cdot n_{\kappa} \|^2_{L^2(\Gamma)} dx,$$
$$L_4 = - \int \left( |\eta_0^n|^{-3} \left| \left( g_\kappa - g_0 \right) (v_{\kappa t} \cdot v_\kappa) \right| \right) dx,$$
$$L_5 = - \int \left( |\eta_0^n|^{-3} \left| \left( g_\kappa - g_0 \right) (v_{\kappa t} \cdot n_{\kappa}) \right| \right) dx.$$

Since $a_{\kappa} = (\nabla \eta_\kappa)^{-1}, (a_{\kappa})_{\kappa} = -a_{\kappa} (v_{\kappa t} \cdot a_{\kappa} v_{\kappa t})$ and $\| a_{\kappa} \|_{L^\infty(\Omega)} \leq \mathcal{P}(\| \eta \|_{L^2(\Gamma)}).$

Therefore,
$$\int_0^t L_1 dt \leq C_t \mathcal{P}(X_\kappa(T)) + \varepsilon \int_0^t \| v_{\kappa t} \|^2_{H^1(\Omega)} dt.$$
while for the third integral of $L_3$, by integrating by parts,
\[
\int_{\Gamma} |\eta_0''|^{-3}(b_{\kappa} - b_0)(v''_{ct} \cdot n_{ct})dS = -\int_{\Gamma} \left[ |\eta_0''|^{-3}(b_{\kappa} - b_0)n_{ct} \right]' \cdot v'_{ct}dS
\]
\[
= -\int_{\Gamma} \left[ |\eta_0''|^{-3}(b_{\kappa} - b_0) \right]'n_{ct} \cdot v'_{ct}dS - \int_{\Gamma} |\eta_0''|^{-3}(b_{\kappa} - b_0)n'_{ct} \cdot v'_{ct}dS
\]
and hence by (9.9) and (8.7a),
\[
\left| \int_{\Gamma} |\eta_0''|^{-3}(b_{\kappa} - b_0)(v''_{ct} \cdot n_{ct})dS \right| \leq C_{T}P(X_{\kappa}(T)) \left[ 1 + t^\delta \|v''_{\kappa}\|^2_{H^1(\Omega')} \right] + \epsilon \|v_{ct}\|^2_{H^1(\Omega)}.
\]
For the second integral of $L_3$,
\[
\int_{\Gamma} |\eta_0''|^{-3}(\eta''_{ct} \cdot n_{ct})(v''_{ct} \cdot n_{ct})dS = \int_{\Gamma} |\eta_0''|^{-3}(\mathbf{1} \cdot n_{ct})(v''_{ct} \cdot n_{ct})dS \quad (\equiv L_{31})
\]
\[
+ \int_{\Gamma} \left[ |\eta_0''|^{-3} \left( \int_0^t \eta''_{ds} \cdot n_{ct} \right)(v''_{ct} \cdot n_{ct})dS \quad (\equiv L_{32})
\]
\[
- \int_{\Gamma} \left[ |\eta_0''|^{-3} \left( \int_0^t \eta''_{ds} \cdot n_{ct} \right)(v''_{ct} \cdot n_{ct})dS. \quad (\equiv L_{33})
\]
By $H^{1.5}(\Gamma) - H^{-1.5}(\Gamma)$ duality pairing,
\[
|L_{31}| \leq C_{T}P(X_{\kappa}(T))\|v''_{\kappa}\|^2_{H^1(\Omega')} + \epsilon \|v_{ct}\|^2_{H^1(\Omega)}
\]
and standard Hölder’s inequality implies
\[
|L_{33}| \leq C_{T}P(X_{\kappa}(T)) \|v''_{\kappa}\|_{H^1(\Omega')} \int_0^t \|v_{\kappa}\|_{H^1(\Omega')}dS \leq C\sqrt{T}P(X_{\kappa}(T)) \|v''_{\kappa}\|_{H^1(\Omega')}.
\]
For $L_{32}$, integrating in time and integrating by parts in time leads to
\[
\int_0^t L_{32}ds = \int_{\Gamma} |\eta_0''|^{-3} \left( \int_0^t \eta''_{ds} \cdot n_{ct} \right)(v''_{ct} \cdot n_{ct})dS - \int_0^t \left[ \int_{\Gamma} |\eta_0''|^{-3}(v''_{ct} \cdot n_{ct})dS \right] ds
\]
\[
- \int_0^t \left[ \int_{\Gamma} \left[ \int_0^t \eta''_{ds} \cdot n_{ct} \right](v''_{ct} \cdot n_{ct})dS \right] \cdot ds.
\]
The first two integrals can be bounded by
\[
C\sqrt{T}P(X_{\kappa}(T)) + CP(X_{\kappa}(T)) \int_0^t \|v''_{\kappa}\|^2_{H^1(\Omega')}dS.
\]
The worst term in the last integral is
\[
\int_0^t \left[ \int_{\Gamma} |\eta_0''|^{-3} \left( \int_0^t \eta''_{ds} \cdot \eta_0'' \right) g_{\kappa}^{-1}(v''_{ct} \cdot n_{ct})(v''_{ct} \cdot n_{ct})dS \right] ds,
\]
and integrating by parts, the worst term becomes
\[
\int_0^t \left[ \int_{\Gamma} |\eta_0''|^{-3} \left( \int_0^t \eta''_{ds} \cdot \eta_0'' \right) g_{\kappa}^{-1}(v''_{ct} \cdot n_{ct})(v''_{ct} \cdot n_{ct})dS \right] ds
\]
\[
= \int_0^t \left[ \int_{\Gamma} |\eta_0''|^{-3} \left( (\eta''_{ct} - \mathbf{1}) \cdot \eta_0'' \right) g_{\kappa}^{-1}(v''_{ct} \cdot n_{ct})(v''_{ct} \cdot n_{ct})dS \right] \cdot ds. \quad (9.15)
\]
Standard Hölder’s inequality and interpolation inequalities show that
\[
\left| \int_0^t \int_{\Gamma} |\eta'_0|^{-3} (\eta'''_{\kappa} \cdot \eta''_{\kappa} | g^{-1}_{\kappa} (v_{\kappa t} \cdot n_{\kappa}) (v''_{\kappa} \cdot n_{\kappa}) | dS ds \right|
\leq \mathcal{P}(X_{\kappa}(T)) \int_0^t (1 + \|\eta_{\kappa}\|_{H^{1/2}(\Gamma)}) \|v_{\kappa t}\|_{H^{1/2}(\Gamma)} ds
\leq C_t \epsilon^b \mathcal{P}(X_{\kappa}(T)) + \epsilon \int_0^t \|v_{\kappa t}\|_{H^1(\Omega)}^2 ds.
\]
Estimates for \(L_{31}, L_{32}\) and \(L_{33}\) leads to
\[
\left| \int_0^t \int_{\Gamma} |\eta'_0|^{-3} (\eta'''_{\kappa} \cdot v_{\kappa t}) (v_{\kappa t} \cdot n_{\kappa}) ds \right|
\leq C_t \epsilon^b \mathcal{P}(X_{\kappa}(T)) + C_t \mathcal{P}(X_{\kappa}(T)) \int_0^t \|v'''_{\kappa}\|_{H^1(\Omega')}^2 ds + \int_0^t \|v_{\kappa t}\|_{H^1(\Omega)}^2 ds.
\]
Therefore,
\[
\left| \int_0^t L_{33} ds \right| \leq C_t \epsilon^b \mathcal{P}(X_{\kappa}(T)) + C_t \mathcal{P}(X_{\kappa}(T)) \int_0^t \|v'''_{\kappa}\|_{H^1(\Omega')}^2 ds + \epsilon \int_0^t \|v_{\kappa t}\|_{H^1(\Omega)}^2 ds + \frac{\kappa}{100} \int_0^t \|\eta''_{\kappa}\|_{H^{1/2}(\Gamma)}^2 ds.
\] (9.16)
\(L_4\) can be rewritten as
\[
L_4 = - \int_0^t \|\eta'_0|^{-3} (b_{\kappa} - b_0) (g_{\kappa}^{-1} g_{\kappa}'(n_{\kappa} \cdot v_{\kappa t}) ds - \int_0^t \|\eta'_0|^{-3} (b_{\kappa} - b_0) g_{\kappa}^{-1} g_{\kappa}' n_{\kappa} \cdot v_{\kappa t} ds.
\]
The presence of \((b_{\kappa} - b_0)\) and \(H^{0.5}(\Gamma)-H^{-0.5}(\Gamma)\) duality pairing imply that
\[
\left| \int_0^t \int_{\Gamma} |\eta'_0|^{-3} (b_{\kappa} - b_0) (g_{\kappa}^{-1} g_{\kappa}'(n_{\kappa} \cdot v_{\kappa t}) ds \right| \leq C_t \epsilon^b \mathcal{P}(X_{\kappa}(T)).
\] (9.17)
For the second integral, since \(b_{\kappa t} = v''_{\kappa} \cdot n_{\kappa} + \eta''_{\kappa} \cdot n_{\kappa t},\) by \(H^{0.5}(\Gamma)-H^{-0.5}(\Gamma)\) duality pairing,
\[
\left| \int_0^t \int_{\Gamma} |\eta'_0|^{-3} (b_{\kappa} - b_0) g_{\kappa}^{-1} g_{\kappa}'(n_{\kappa} \cdot v_{\kappa t}) ds \right|
\leq C \int_0^t \mathcal{P}(\|\eta''_{\kappa}\|_{H^{1/2}(\Gamma)})(1 + \|g_{\kappa}\|_{H^{1/2}(\Gamma)}) \|v''_{\kappa}\|_{H^1(\Omega')}) \|v_{\kappa t}\|_{H^1(\Omega)} ds
\leq C_t \mathcal{P}(X_{\kappa}(T)) \int_0^t \|v''_{\kappa}\|_{H^1(\Omega')}^2 ds + \epsilon \int_0^t \|v_{\kappa t}\|_{H^1(\Omega)}^2 ds.
\] (9.18)
Combining (9.17) and (9.18), we have
\[
\left| \int_0^t L_4 ds \right| \leq C t \epsilon^b \mathcal{P}(X_{\kappa}(T)) + C_t \mathcal{P}(X_{\kappa}(T)) \int_0^t \|v''_{\kappa}\|_{H^1(\Omega')}^2 ds + \epsilon \int_0^t \|v_{\kappa t}\|_{H^1(\Omega)}^2 ds.
\] (9.19)
For \(L_5,\) first note that by \((a_{\kappa t})^t = -(a_{\kappa})^t v_{\kappa t}^k(a_{\kappa})^t,\) we find that
\[
\left| \int_{\Omega} g_{\kappa} (a_{\kappa})^t v_{\kappa t}^k dx \right| \leq \|g_{\kappa}\|_{H^{0.5}(\Omega)} \|a_{\kappa}\|_{L^{\infty}(\Omega)} \|v_{\kappa t}\|_{H^{1/2}(\Omega)} \|v_{\kappa t}\|_{H^{1/2}(\Omega)}
\leq C_t \mathcal{P}(\|\eta''_{\kappa}\|_{H^{1/2}(\Gamma)}) \|g_{\kappa}\|^2_{L^2(\Omega)} + \epsilon \left[ \|g_{\kappa}\|^2_{H^1(\Omega)} + \|v_{\kappa t}\|^2_{H^1(\Omega)} \right].
\] (9.20)
By the “divergence free” condition, \((a_\kappa)_{l}^j v_{\kappa,l,j}^i = -(a_\kappa)_{l}^j v_{\kappa,j}^i\). Time integrating the first integral,

\[
\int_0^t \int_\Omega q_\kappa(t)(a_\kappa)_{l}^j v_{\kappa,l,j}^i dx ds = -\int_0^t \int_\Omega q_\kappa[(a_\kappa)_{k}^j v_{\kappa,k}^j(a_\kappa)_{l}^j v_{\kappa,j}^i]_t dx ds + \int_\Omega q_\kappa(s)(a_\kappa(s))_{l}^j v_{\kappa,j}^i dx \bigg|_{s=0}^t .
\]

Similar to (9.20), we have

\[
\left| \int_0^t \int_\Omega q_\kappa[(a_\kappa)_{k}^j v_{\kappa,k}^j(a_\kappa)_{l}^j v_{\kappa,j}^i]_t dx ds \right| \leq C_\epsilon \mathcal{P}(\|\eta_\kappa\|_{H^2(\Gamma)}) \int_0^t \|q_\kappa\|_{L^2(\Omega)} ds \quad (9.21)
\]

\[+ \epsilon \int_0^t \left[\|q_\kappa\|_{H^1(\Omega)}^2 + \|v_{\kappa,l}\|_{H^1(\Omega)}^2\right] ds.
\]

For the boundary term, when \(s = 0\), it is bounded by a constant independent of \(\kappa\), say \(M_0\). When \(s = t\), first note that by \((a_\kappa)_{l}^j = -a_{\kappa l}^j v_{\kappa,l}^j a_{\kappa l}^j\),

\[\|(a_\kappa)_{l}^j(t) - (a_\kappa)_{l}^j(0)\|_{L^2(\Omega)}^2 \leq \|v_{\kappa,l}^j - u_{\kappa,l}^j\|_{L^2(\Omega)}^2 + \|a_{\kappa l}^j - \delta_{\kappa l}^j\|_{L^2(\Omega)}^2 \leq C(t + \sqrt{t})\mathcal{P}(X_\kappa(T)).
\]

Therefore, by adding and subtracting \(\int_\Omega q_\kappa(t)(a_\kappa(t))_{l}^j v_{\kappa,j}^i(t) dx\), we find that

\[
\left| \int_\Omega q_\kappa(t)(a_\kappa(t))_{l}^j v_{\kappa,j}^i(t) dx \right| \leq \left| \int_\Omega q_\kappa(t)(a_\kappa(t) - a_\kappa(t))_{l}^j v_{\kappa,j}^i(t) dx \right| + \left| \int_\Omega q_\kappa(t)u_{\kappa,l}^j v_{\kappa,j}^i dx \right|
\]

\[\leq C\epsilon \mathcal{P}(X_\kappa(T)) + C\epsilon \left[\|u_0\|_{H^1(\Omega)}^2 + \sqrt{t}\mathcal{P}(X_\kappa(T))\right] + \epsilon \|v\|_{L^2(\Omega)}^2 ,
\]

where (8.17) is used to estimate \(\|v\|_{H^1(\Omega)}^2\). Combining (9.20), (9.21) and (9.22), by (8.18) and (8.19) we obtain

\[
\int_0^t L_5 ds \leq C\epsilon \mathcal{P}(X_\kappa(T)) + \epsilon \int_0^t \|v_{\kappa,l}\|_{H^1(\Omega)}^2 ds + \epsilon \|v_{\kappa,l}\|_{L^2(\Omega)}^2 .
\]

Time integrating (9.12), choosing \(\epsilon > 0\) small enough together with inequalities (9.13), (9.14), (9.16), (9.19) and (9.23), we find that

\[
\sup_{0 \leq t \leq T} \left[\|v_{\kappa,l}\|_{L^2(\Omega)}^2 + \|v_{\kappa,l}^\prime \cdot \eta_\kappa\|_{L^2(\Gamma)}^2 + \|v_{\kappa,l}'' \cdot n_\kappa\|_{L^2(\Gamma)}^2 + \kappa \|v_{\kappa,l}\|_{H^1(\Omega)}^2 \right] \leq M_0 + C\epsilon \mathcal{P}(X_\kappa(T)) + C\mathcal{P}(X_\kappa(T)) \int_0^t \|v_{\kappa,l}''\|_{H^1(\Omega)}^2 ds
\]

\[+ \frac{\kappa}{100} \int_0^T \|\eta_\kappa''\|_{H^2(\Gamma)}^2 ds,
\]

for some constant \(M_0\) depending on \(\|u_0\|_{H^2(\Omega)}^2\), \(\|\text{Id}\|_{H^2(\Gamma)}^2\), \(\|F\|_{L^\infty(0,T;H^1(\Omega))}^2\) and \(\|F_\kappa\|_{L^2(0,T;L^2(\Omega))}^2\).
9.4. $\kappa$-independent estimates. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the left-hand side quantities of (9.11) and (9.24), respectively. Then

$$\mathcal{E}_1 \leq CT^d \left[ M_0 + \mathcal{P}(X_\kappa(T)) \right] + \epsilon \mathcal{E}_2 + \epsilon \int_0^t \| v_\kappa \|_{H^3(\Omega)}^2 ds, \quad (9.25a)$$

$$\mathcal{E}_2 \leq M_0 + CT^d \mathcal{P}(X_\kappa(T)) + C \mathcal{P}(X_\kappa(T)) \mathcal{E}_1 + \frac{\kappa}{100} \mathcal{E}_1. \quad (9.25b)$$

By (9.25a), for $\epsilon > 0$ small enough (but not fixed yet), say $C \epsilon \mathcal{P}(X_\kappa(T)) \leq 0.5$, we have

$$\mathcal{E}_2 \leq M_0 + CT^d \mathcal{P}(X_\kappa(T)) + \epsilon \int_0^t \| v_\kappa \|_{H^3(\Omega)}^2 ds + \frac{\kappa}{100} \mathcal{E}_1. \quad (9.26)$$

Finally, since

$$\| v_\kappa \|_{H^3(\Omega)} \leq C \epsilon \| v_\kappa \|_{H^3(\Omega)}^2 + \epsilon \int_0^t \| v \|_{H^3(\Omega)}^2 ds \quad (9.27)$$

combining (8.14), (8.15), (8.20), (9.25a), (9.26b) and (9.27), and choosing $\epsilon > 0$ small enough, we have

$$X_\kappa(T) \leq C \left[ M_0 + T^d \mathcal{P}(X_\kappa(T)) \right]. \quad (9.28)$$

$X_\kappa$ is clearly continuous in its variable. By Lemma 5.3 there is a constant $M$ independent of $\kappa$ and $T_1 \leq T_\kappa$ so that

$$X_\kappa(t) \leq M \quad \forall \ t \in [0, T_1]. \quad (9.29)$$

Without loss of generality, we may assume that $T_1 = T_\kappa$ (by setting $T_\kappa$ equaling $T_1$). Let $X_{0.5}(t) \leq M$ for $t \in [0, T_{0.5})$. For $\kappa < 0.5$, say, $\kappa = 0.1$, $X_{0.1}(t) \leq M$ for $t \in [0, T_{0.1})$ where $T_{0.1}$ is in general smaller than $T_{0.5}$. Since this estimate is independent of $\kappa$, we are able to extend the time interval $[0, T_{0.1})$ in which the fixed point $\bar{v}_{0.1}, \bar{\eta}_{0.1}, \bar{g}_{0.1}$ and $\bar{b}_{0.1}$ exist. This extension will proceed until $T_{0.1}$ hits $T_{0.5}$, and hence $X_{0.1}(t) \leq M$ for $t \in [0, T_{0.5})$. This argument holds for all $\kappa < 0.5$, so we conclude that (with $T = T_{0.5}$)

$$X_\kappa(t) \leq M \quad \forall \ t \in [0, T], \ \kappa \in (0, 0.5]. \quad (9.29)$$

Remark 2. By (5.20), we can also include $\| g_\kappa \|_{L^2(0, T; H^2(\Omega))}^2$ in $X_\kappa(T)$.

9.5. Weak limits of $v_\kappa$ as $\kappa \to 0$. By (9.29), there exist $v$ (and $v_1$) so that

$$v_\kappa \rightharpoonup v \quad \text{in} \ L^2(0, T; H^3(\Omega)), \quad (9.30a)$$

$$v_\kappa \to v \quad \text{in} \ L^2(0, T; H^2(\Omega)), \quad (9.30b)$$

$$v_{\kappa,t} \to v_t \quad \text{in} \ L^2(0, T; H^1(\Omega)), \quad (9.30c)$$

$$v_{\kappa,t} \to v_t \quad \text{in} \ L^2(0, T; L^2(\Omega)), \quad (9.30d)$$

for some subsequence $v_{\kappa,i}$. Also, there exists $\eta$ (the associated Lagrangian variable of $v$), $g$, $b$ and $n$ so that

$$\eta_{\kappa,i} \to \eta \quad \text{in} \ L^2(0, T; H^3(\Gamma)) \cap L^2(0, T; H^2(\Omega)), \quad (9.31a)$$

$$g_{\kappa,i} \to g \quad \text{in} \ L^2(0, T; H^2(\Gamma)), \quad (9.31b)$$

$$b_{\kappa,i} \to b \quad \text{in} \ L^2(0, T; H^2(\Gamma)), \quad (9.31c)$$

$$n_{\kappa,i} \to n \quad \text{in} \ L^2(0, T; H^2(\Gamma)). \quad (9.31d)$$
Since $\eta_\omega$ converges a.e. to $\eta$ in $H^3(\Gamma)$, we have that $g = |\eta'|^2$, $b = \eta'' \cdot n$, and

$$n = \left(-\eta'_\omega, \eta''_\omega/\eta'\right).$$

Also, since $a_{\omega} \to a$ strongly in $L^2(0, T; H^4(\Omega))$, by (7.1) we conclude that $v, q, \eta, g, b$ satisfy (2.3).

### 10. Uniqueness

Let $v$ and $\tilde{v}$ in $Y^5(T')$ be two solutions to (2.3) ($q$ and $\tilde{q} \in L^2(0, T; H^4(\Omega)) \cap L^\infty(0, T; H^3(\Omega))$, $q_0$ and $\tilde{q}_0 \in L^2(0, T; H^2(\Omega))$, $g, \tilde{g}, b, \tilde{b} \in L^2(0, T; H^{4.5}(\Gamma)) \cap L^\infty(0, T; H^4(\Gamma))$), and $w = v - \tilde{v}$, $r = q - \tilde{q}$, $E = \eta - \tilde{\eta}$. Then $w, r, E$ satisfy

\[

dF + \nu(a^i_k a^k_i w^i_k)_j = -a^i_j r_j + (\delta F)^i \quad \text{in } (0, T) \times \Omega, \tag{10.1a}
\]

\[
\delta L_1 = -\nu \left( (a^i_k - \tilde{a}^i_k) \tilde{v}^i_k \right) a^j_i N_k - \nu \left( \tilde{a}^i_j \tilde{v}^j_k + a^i_j \tilde{v}^j_k \right) (a^j_k - \tilde{a}^j_k) N_k
\]

\[
\delta L_2 = 4 \left| \eta_0' \right|^{-3} (E' \cdot \eta') \eta' + (\tilde{g} - g_0) E' \tag{10.2a}
\]

\[
\delta L_3 = 2 \left| \eta_0'' \right|^{-3} (\eta'' \cdot (n - \tilde{n})) (n - \tilde{n}) \tag{10.2b}
\]

\[
\delta L_4 = \left| \eta_0'' \right|^{-3} (g^{-1} - \tilde{g}^{-1}) (b - \tilde{b}) g' n + \tilde{g}^{-1} (b - \tilde{b}) g' n + \tilde{g} (\tilde{b} - b_0) (g - \tilde{g})' n
\]

\[
\quad + \tilde{g}^{-1} (\tilde{b} - b_0) \tilde{g}' (n - \tilde{n}) \tag{10.2c}
\]

with the following inequalities from [5]:

\[
\|\delta a\|_{H^k(\Omega)}^2 + \|\delta F\|_{H^k(\Omega)}^2 \leq Ct \int_0^t \|w\|_{H^{k+1}(\Omega)}^2 ds, \quad \text{for } k = 0, 1, 2, \tag{10.2a}
\]

\[
\|\delta a_0\|_{L^2(\Omega)}^2 + \|\delta F_0\|_{L^2(\Omega)}^2 \leq C\sqrt{t} \int_0^t \|w\|_{L^2(\Omega)}^2 ds, \tag{10.2b}
\]

\[
\|\delta L_1\|_{H^{4.5}(\Gamma)}^2 + \|\delta L_1\|_{L^2(\Gamma)}^2 \leq Ct \int_0^t \|w\|_{H^{4.5}(\Omega)}^2 ds. \tag{10.2c}
\]
Furthermore, \( w, \ r \) \ and \( \mathcal{E} \) satisfy the following variational form:

\[
\langle w, \phi \rangle + \frac{\nu}{2}(a^k w^j_k + a^j w^k_j, a^k \phi^j_k + a^j \phi^k_j) + (r, a^j \phi^i_j) + \int_\Gamma (\delta L_1) \cdot \phi dS
\]

\[
+ \int_\Gamma |\eta_0|^{-3} \left[ 4(g - \tilde{g})(\eta' \cdot \phi') + 2(b - \tilde{b})(\phi'' \cdot n) \right] dS
\]

\[
+ \int_\Gamma |\eta_0|^{-3} \left[ 4(\tilde{g} - g_0)(\xi' \cdot \phi') + 2(\tilde{b} - b_0)(\phi'' \cdot (n - \tilde{n})) \right] dS
\]

\[
(10.3)
\]

\[
- \int_\Gamma |\eta_0|^{-3}(g^{-1} - \tilde{g}^{-1})(b - b_0)g'(\phi' \cdot n) dS - \int_\Gamma |\eta_0|^{-3}\tilde{g}^{-1}(b - b_0)g'(\phi' \cdot n) dS
\]

\[
(10.4)
\]

and

\[
\|w\|_{H^2(\Omega)}^2 + \|r\|_{H^2(\Omega)}^2 \leq C \left[ \|\delta F\|_{L^2(\Omega)}^2 + \|w_t\|_{L^2(\Omega)}^2 + \|\delta a\|_{H^1(\Omega)}^2 + \|w\|_{H^{1.5}(\Gamma)}^2 \right]
\]

\[
+C \left[ \|w_t\|_{H^1(\Omega)}^2 + \|w\|_{H^{1.5}(\Gamma)}^2 + t \int_0^t \|w\|_{H^2(\Omega)}^2 d\tau \right].
\]

For \( T \) small enough, (10.5) implies that

\[
\int_0^t \left[ \|w\|_{H^2(\Omega)}^2 + \|r\|_{H^2(\Omega)}^2 \right] d\tau \leq C \int_0^t \left[ \|w_t\|_{H^1(\Omega)}^2 + \|w\|_{H^{1.5}(\Gamma)}^2 \right] d\tau.
\]

We can also setup elliptic equations for \( b - \tilde{b} \) \ and \( g - \tilde{g} \) and obtain the following elliptic estimates for \( b - \tilde{b} \) \ and \( g - \tilde{g} \) (where we use (10.4) \ and (10.5) \ to estimate the norm of \( r \):

\[
\|b - \tilde{b}\|_{H^{1.5}(\Gamma)}^2 \leq C t \int_0^t \|w_t\|_{H^1(\Omega)}^2 d\tau + C \left[ \|w_t\|_{L^2(\Omega)}^2 + \|w\|_{H^{1.5}(\Gamma)}^2 \right],
\]

\[
(10.7a)
\]

\[
\|b - \tilde{b}\|_{H^2(\Gamma)}^2 \leq C \left[ \int_0^t \|w\|_{H^2(\Omega)}^2 d\tau + \|w\|_{H^2(\Gamma)}^2 + \|g - \tilde{g}\|_{H^{1.5}(\Gamma)}^2 \right]
\]

\[
+C \left[ \|w_t\|_{L^2(\Omega)}^2 + \|w\|_{H^{1.5}(\Gamma)}^2 \right],
\]

\[
(10.7b)
\]

\[
\|g - \tilde{g}\|_{H^{2.5}(\Gamma)}^2 \leq C \left[ \|w\|_{H^2(\Omega)}^2 + \|b - \tilde{b}\|_{H^{2.5}(\Gamma)}^2 + \|\tilde{b} - b_0\|_{H^{2.5}(\Gamma)}^2 \right] \|b - \tilde{b}\|_{H^{1.5}(\Gamma)}^2
\]

\[
+C t \int_0^t \|w\|_{H^2(\Omega)}^2 d\tau.
\]

Using (10.7b) in (10.7), for \( T \) small enough, \( \|g - \tilde{g}\|_{H^{2.5}(\Gamma)}^2 \) dependence on the right-hand side can be absorbed by the left-hand side of (10.7), so we conclude
that

\[ \int_0^t \|g - \tilde{g}\|_{H^{2.5}(\Gamma)}^2 \, ds \leq C \int_0^t \left[ \|w_t\|_{H^1(\Omega)}^2 + \|w\|_{H^{2.5}(\Gamma)}^2 \right] \, ds , \]

\[ \int_0^t \|b - \tilde{b}\|_{H^{2.5}(\Gamma)}^2 \, ds \leq C \int_0^t \left[ t\|w_t\|_{H^1(\Omega)}^2 + t\|w\|_{H^{2.5}(\Gamma)}^2 + \|w_t\|_{L^2(\Omega)}^2 + \|w\|_{H^{1.5}(\Gamma)}^2 \right] \, ds . \]

By the identity \( \eta'' = g^{-1}(\eta'' \cdot \eta')n + (\eta'' \cdot n)\eta = \frac{1}{2}g^{-1}g'\eta + bn \), we find that

\[ \|E\|_{H^{3.5}(\Gamma)} \leq C \left[ \|g - \tilde{g}\|_{H^{2.5}(\Gamma)} + \|E\|_{H^{2.5}(\Gamma)} + \|b - \tilde{b}\|_{H^{1.5}(\Gamma)} \right] \]

and hence

\[ \int_0^t \|E\|_{H^{3.5}(\Gamma)}^2 \, ds \leq C \int_0^t \left[ \|w\|_{H^{2.5}(\Gamma)}^2 + \|w_t\|_{H^1(\Omega)}^2 \right] \, ds . \] (10.8)

By (2.1b),

\[ \|n - \tilde{n}\|_{H^{s}(\Gamma)} \leq C \left[ \|E\|_{H^{s}(\Gamma)}^2 + \|b - \tilde{b}\|_{H^{s-1}(\Gamma)}^2 \right] \text{ for } s > 1.5 . \]

Since \( E'' \cdot n = (b - \tilde{b}) + \tilde{\eta}'' \cdot (n - \tilde{n}) \) and \( \tilde{\eta} \in L^\infty(0, T; H^{4.5}(\Gamma)) \) by assumption, we find that

\[ \int_0^t \|E'' \cdot n\|_{H^{2.5}(\Gamma)}^2 \, ds \leq C \int_0^t \left[ \|b - \tilde{b}\|_{H^{2.5}(\Gamma)}^2 + \|n - \tilde{n}\|_{H^{2.5}(\Gamma)}^2 \right] \, ds \]

\[ \leq Ct \int_0^t \left[ \|w_t\|_{H^1(\Omega)}^2 + \|w\|_{H^{2.5}(\Gamma)}^2 \right] \, ds + C \int_0^t \left[ \|w_t\|_{L^2(\Omega)}^2 + \|w\|_{H^{1.5}(\Gamma)}^2 \right] \, ds . \] (10.9)
10.2. Estimates for $w_t$. We study the time differentiated problem first. Time differentiating (10.3) and then use $w_t$ as a test function, we find that

$$
\frac{d}{dt} \left[ \frac{1}{2} \|w_t\|_{L^2(\Omega)}^2 + 2 \|\eta_0^{\prime\prime\prime\prime}w_t^{\prime\prime} \cdot \eta_0^{\prime\prime}w_t^{\prime\prime} \|_{L^2(\Gamma)}^2 + \|\eta_0^{\prime\prime\prime\prime}w_t^{\prime\prime} \cdot n\|_{L^2(\Gamma)}^2 + \|D_n w_t\|_{L^2(\Omega)}^2 \right] + \frac{\nu}{2} \|D_n w_t\|_{L^2(\Omega)}^2
\]

$$
= \frac{\nu}{2} ((a_t^\dagger)_{e,j} w_{j,k} + (a_t^\dagger)_{e,k} w_{j,k}, (\text{Def}_n w_t)_{i,k}) - \frac{\nu}{2} ((\text{Def}_n w_t)_{i,k}, (a_t^\dagger_{e,j} w_{j,k} + (a_t^\dagger)_{e,k} w_{j,k}))
$$

$$
- \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ (\tilde{v}^\prime \cdot E)(\tilde{n}^\prime \cdot w_t)\right) - (w_t^{\prime\prime} \cdot \eta_0^{\prime\prime}) (v_0^{\prime\prime} \cdot w_t) - (v_t^{\prime\prime} \cdot E)(v_0^{\prime\prime} \cdot w_t) \right] ds
$$

$$
- \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ -(w_0^{\prime\prime\prime\prime} \cdot n)(w_0^{\prime\prime\prime\prime} \cdot n_i) + [\tilde{v}_0^{\prime\prime\prime\prime} \cdot (n - \tilde{n}) + (\eta_0^{\prime\prime\prime\prime} \cdot n_i) - (\tilde{n}_t^{\prime\prime\prime\prime} \cdot \tilde{n}_i)](w_0^{\prime\prime\prime\prime} \cdot n) \right] ds
$$

$$
- \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ 4(g - \tilde{g})(v_0^{\prime\prime} \cdot w_t^0) + 2(b - \tilde{b})(w_0^{\prime\prime} \cdot n_i) \right] ds
$$

$$
- \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ 8(\tilde{v}^{\prime\prime} \cdot \tilde{n})(E^{\prime\prime} \cdot w_t^0) + 2(\tilde{v}_0^{\prime\prime\prime\prime} \cdot \tilde{n}_i)(w_0^{\prime\prime\prime\prime} \cdot (n - \tilde{n}) \right] \right] ds
$$

$$
- \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ 4(\tilde{g} - g_0)(w_0^{\prime\prime} \cdot w_t^0) + 2(\tilde{b} - b_0)(w_0^{\prime\prime} \cdot (n_i - \tilde{n}_i)) \right] ds
$$

$$
+ \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ (g_0^{\prime\prime\prime\prime} - \tilde{g}_0^{\prime\prime\prime\prime})(b - b_0) + (g_0^{\prime\prime\prime\prime} - \tilde{g}_0^{\prime\prime\prime\prime})b_t \right] g_t^{\prime\prime\prime\prime}(w_t^{\prime\prime\prime\prime} \cdot n) ds \right] \right] 10.10
$$

$$
+ \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ (g_0^{\prime\prime\prime\prime} - \tilde{g}_0^{\prime\prime\prime\prime})(b - b_0) \right] g_t^{\prime\prime\prime\prime}(w_t^{\prime\prime\prime\prime} \cdot n) ds \right] \right] \right] 10.10
$$

$$
+ \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ (g_0^{\prime\prime\prime\prime} - \tilde{g}_0^{\prime\prime\prime\prime})(b - b_0) \right] g_t^{\prime\prime\prime\prime}(w_t^{\prime\prime\prime\prime} \cdot n) ds \right] \right] \right] 10.10
$$

$$
+ \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ (\tilde{g}_0^{\prime\prime\prime\prime} - \tilde{g}_0^{\prime\prime\prime\prime})(\tilde{b} - b_0) \right] g_t^{\prime\prime\prime\prime}(w_t^{\prime\prime\prime\prime} \cdot n) ds \right] \right] \right] 10.10
$$

$$
+ \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ (\tilde{g}_0^{\prime\prime\prime\prime} - \tilde{g}_0^{\prime\prime\prime\prime})(\tilde{b} - b_0) \right] g_t^{\prime\prime\prime\prime}(w_t^{\prime\prime\prime\prime} \cdot n) ds \right] \right] \right] 10.10
$$

$$
+ \int \left[ \eta_0^{\prime\prime\prime\prime} \left[ (\tilde{g}_0^{\prime\prime\prime\prime} - \tilde{g}_0^{\prime\prime\prime\prime})(\tilde{b} - b_0) \right] g_t^{\prime\prime\prime\prime}(w_t^{\prime\prime\prime\prime} \cdot n) ds \right] \right] \right] 10.10
$$

With $v_t \in L^2(0, T; H^l(\Omega))$ (so that $v_t$ has trace on the boundary), similar to the computation of estimating $E_1$ in page 48 of [5], we find that

$$
\left| \langle r_t, a_t^{\prime\prime\prime\prime} w_t^{\prime\prime\prime\prime} \rangle + \langle r, (a_t^{\prime\prime\prime\prime})^{\prime\prime\prime\prime} w_t^{\prime\prime\prime\prime} \rangle \right| \leq C_\epsilon \|r\|_{L^2(\Omega)}^2 + \epsilon \|w_t\|_{H^l(\Omega)}^2 + \epsilon \|w_t\|_{H^l(\Omega)}^2
$$

For $v$ and $\tilde{v}$ and the associated metric tensor, the second fundamental form and the pressure in the space described in the beginning of this section, by (10.2),

$$
\left| \langle \delta F, w_t \rangle \right| \leq C \int_0^T ||w_t||_{H^l(\Omega)}^2 ds + \epsilon \|w_t\|_{H^l(\Omega)}^2 + \epsilon \|w_t\|_{H^l(\Omega)}^2
$$

By (10.2) and interpolation inequalities,

$$
\left| \int \delta L_t \cdot w_t ds \right| \leq Ct \int_0^T ||w_t||_{H^l(\Omega)}^2 ds + C_\epsilon \|w_t\|_{L^2(\Omega)}^2 + \epsilon \|w_t\|_{H^l(\Omega)}^2
$$

\]
It is also clear that the first two terms (due to viscosity) on the right-hand side is bounded by $C\|w\|_{H^1(\Omega)}\|w_t\|_{H^1(\Omega)}$, and by Young’s inequality,
\[
\left\|(a^k)w^{i,k} + (a^j)w^{j,k} + a^k w^{i,k} + a^j w^{j,k} + (a^k)w^{i,k} + (a^j)w^{j,k}\right\|
\leq C\varepsilon \int_0^t \|w_t\|^2_{H^1(\Omega)} ds + \varepsilon \|w_t\|_{H^1(\Omega)}^2.
\]
For those terms having $w''_t$ in the integrands, following the same procedure of estimating $L_4$, we find that those terms are bounded by
\[
C\varepsilon \left[ \int_0^t \|w\|^2_{H^2(\Omega)} ds + \|w''_t\|^2_{H^1(\Omega)} \right] + \varepsilon \|w_t\|_{H^1(\Omega)}^2.
\]
The terms having $w''_t$ inside the integrands are
\[
M_1 = \int_{\Gamma} |\eta_0|^{-3}(b - \tilde{b})(w''_t \cdot n_t) dS,
\]
\[
M_2 = \int_{\Gamma} |\eta_0|^{-3}(\eta'' \cdot n_t - (\tilde{\eta}'') \cdot \tilde{n}_t)(w''_t \cdot n_t) dS,
\]
\[
M_3 = \int_{\Gamma} \left[ F_1(\eta_0, \eta, \tilde{\eta})(n - \tilde{n})w''_t + F_2(\eta_0, \eta, \tilde{\eta})(\tilde{b} - b_0)(n - \tilde{n})w''_t \right] dS.
\]
Following the same procedure of estimating $L_3$, we find that
\[
|M_1| + |M_3| \leq C\varepsilon \left[ \int_0^t \|w\|^2_{H^2(\Omega)} ds + \|w''_t\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)} \right] + \varepsilon \|w_t\|_{H^1(\Omega)}^2,
\]
and hence
\[
\left| \int_0^t (M_1 + M_3) ds \right| \leq C\varepsilon \left[ \int_0^t \|w\|^2_{H^2(\Omega)} ds + \|w''_t\|^2_{H^1(\Omega)} \right].
\]
For $M_2$, by interpolations and (10.8), its time integral satisfies
\[
\left| \int_0^t M_2 ds \right| \leq C\varepsilon \int_0^t \left[\|\xi\|^2_{H^2(\Gamma)} + \|w''_t\|^2_{H^1(\Omega)}\right] ds + \varepsilon \int_0^t \|w_t\|_{H^1(\Omega)}^2 ds
\leq C\varepsilon \int_0^t \|\xi\|^2_{H^2(\Gamma)} ds + C\varepsilon \int_0^t \|w''_t\|^2_{H^1(\Omega)} ds
\leq C\varepsilon \int_0^t \|w''_t\|^2_{H^1(\Omega)} ds + \varepsilon \int_0^t \|w_t\|_{H^1(\Omega)}^2 ds.
\]
All the remaining terms can be bounded by
\[
C\left[\|w\|^2_{H^2(\Gamma)} + \int_0^t \|w''_t\|^2_{H^1(\Omega)} ds\right].
\]
Time integrating (10.10), choosing $\varepsilon > 0$ and $T > 0$ small enough, by (10.3) and (10.6) we find that
\[
Y(T) + \int_0^T \|w_t\|^2_{H^1(\Omega)} ds \leq C\int_0^T Y(T) ds + C\int_0^T \|w''_t\|^2_{H^1(\Omega)} ds \quad (10.11)
\]
where

\[ Y(T) = \sup_{0 \leq t \leq T} \left[ \|w_t\|^2_{L^2(\Omega)} + \|w' \|^2_{L^2(\Gamma)} + \|w'' \cdot n\|^2_{L^2(\Gamma)} \right]. \]

10.3. Estimates for \( w'' \). Let \( \phi = (\xi^2 w'')'' \) in (10.3), then

\[
\frac{d}{dt} \int_\Omega \|\xi_1 w''\|^2_{L^2(\Omega)} + 2\|\eta_0^{-1/2}(\xi' \cdot \eta')''\|^2_{L^2(\Gamma)} + 2\|\eta_0^{-3/2}(\xi'' \cdot \eta)''\|^2_{L^2(\Gamma)} + \frac{\nu}{2} \|\xi D_t w''\|^2_{L^2(\Omega)}
\]

\[
= -\nu \int_\Omega \xi_1 \left[ 2 \left( a_{i,j}^{k,l} w_k'' + a_{j}^{k,l} w_i'' \right) + \left( a_i^{k,l} w_k'' + a_j^{k,l} w_i'' \right) \right] a_{i,j}^{k,l} \xi_1 w'' dx
\]

\[
- 2\nu \int_\Omega \xi_1 \left[ \left( a_i^{k,l} w_k'' + a_j^{k,l} w_i'' \right) \right] a_{i,j}^{k,l} \xi_1 w'' dx
\]

\[
+ 4 \int_\Omega \left| \eta_0^{-3}(g - \tilde{g}) \right| \left[ (\eta'' \cdot \xi')' + 4(\eta'' \cdot \xi')'' \right] \right\| (\eta'' \cdot \xi')'' dS
\]

\[
- 4 \int_\Gamma \left[ \left| \eta_0^{-3}(g - \tilde{g}) \right| \left( \eta'' \cdot (n - \tilde{n}) \right)'' \right] (\eta'' \cdot \xi')'' dS
\]

\[
- 4 \int_\Gamma \left| \eta_0^{-3}(g'' - \tilde{g}'') \right| (\eta'' \cdot \xi')'' (\eta'' \cdot \xi')'' dS - 2 \int_\Gamma \left| \eta_0^{-3}(\tilde{b} - b_0)(n - \tilde{n}) \right|'' (\eta'' \cdot \xi')'' dS
\]

\[
- \int_\Gamma \left[ \left| \eta_0^{-3}(\tilde{b} - b_0) \right|'' \right] (\eta'' \cdot (n - \tilde{n}))'' \right\| u''(x^4) dS
\]

\[
+ \int_\Gamma \left| \eta_0^{-3}(\tilde{b} - b_0)' \right|'' (\eta'' \cdot (n - \tilde{n}))'' \right\| u''(x^4) dS
\]

\[
- 4 \int_\Gamma \left| \eta_0^{-3}(\tilde{b} - b_0)' \right|'' \left[ \left( \xi'' \cdot \eta' + \eta' \cdot \xi'' \right) g^{-1} \right]'' (\tilde{b} - b_0) \right\| (\eta'' \cdot \xi')'' dS
\]

\[
+ \int_\Gamma \left| \eta_0^{-3}(\tilde{b} - b_0) \right|'' \left( \xi'' \cdot \eta' + \eta' \cdot \xi'' \right) \right\| (\eta'' \cdot \xi')'' dS
\]

As the estimate of \( K_1 \) in Section 9.1, the first two integrals (due to viscosity) on the right-hand side can be bounded by \( C\|w\|_{H^2(\Omega)}\|w\|_{H^4(\Omega)} \), and by Young’s inequality,

\[
\left| \int_\Omega \xi_1 \left[ 2 \left( a_{i,j}^{k,l} w_k'' + a_{j}^{k,l} w_i'' \right) + \left( a_i^{k,l} w_k'' + a_j^{k,l} w_i'' \right) \right] a_{i,j}^{k,l} \xi_1 w'' dx \right|
\]

\[
+ \left| \int_\Omega \xi_1 \left[ \left( a_i^{k,l} w_k'' + a_j^{k,l} w_i'' \right) \right] a_{i,j}^{k,l} \xi_1 w'' dx \right|
\]

\[
\leq C_* \|w\|_{H^2(\Omega)}^2 + \epsilon \|w\|_{H^4(\Omega)}^2.
\]
Similar to the computation of estimating $D_1$ in page 48 of [3], we find that

$$|(r, a^j_1 (\zeta^2 w^m)^n_j)| \leq C rt \int_0^t \|w\|_{H^2(\Omega)}^2 + \epsilon \|r\|_{H^2(\Omega)}^2.$$  

By (10.24), (10.25) and interpolation inequalities,

$$|(\delta F, (\zeta^2 w^m)^n_j)| + \left| \int_\Gamma \langle \delta L, v \rangle w^m dS \right| \leq C rt \int_0^t \|w\|_{H^3(\Omega)}^2 + \epsilon \|w\|_{H^3(\Omega)}^2.$$  

By $H^{1.5} - H^{-1.5}$ or $H^{0.5} - H^{-0.5}$, we find that

$$\left| \int_\Gamma |\eta|^{-3} (g - \tilde{g}) \left[ (\eta^{(1)} \cdot \mathcal{E}^r) + 4(\eta^{(1)} \cdot \mathcal{E}^r) + 6(\eta^{(2)} \cdot \mathcal{E}^s) + 4(\eta^{(3)} \cdot \mathcal{E}^s) \right] dS \right| \leq C \|g - \tilde{g}\|_{H^1(\Gamma)} \left[ \|w\|_{H^2(\Gamma)} + \|\mathcal{E}\|_{H^2(\Gamma)} \right]$$

and

$$\left| \int_\Gamma \left[ (\eta_0)^{-3} (\mathcal{E}^r \cdot \eta')' + (\eta_0)^{-3} (\mathcal{E}^s \cdot \eta')' \right] dS \right| \leq C \left[ \|\mathcal{E}^r\|_{H^1(\Gamma)} + \|\mathcal{E}^s\|_{H^1(\Gamma)} \right] \left[ \|\mathcal{E}\|_{H^2(\Gamma)} + \|w\|_{H^2(\Gamma)} \right].$$

Standard Hölder’s inequality shows that

$$\left| \int_\Gamma |\eta|^{-3} (\mathcal{E}^r \cdot \eta')'' (\mathcal{E}^r \cdot \eta')' dS \right| \leq C \left[ \|\mathcal{E}^r\|_{H^2(\Gamma)} + \|\mathcal{E}\|_{H^2(\Gamma)} \right].$$

Similar to the estimates of $K_3$, $K_4$ and $K_5$, by (8.7), (8.10) and (10.4), we find that the time integral of the remaining terms can be bounded by

$$(C rt + \epsilon) \int_0^t \left[ \|w\|_{H^2(\Omega)}^2 + \|w\|_{H^2 s(\Gamma)}^2 \right] ds + C e \int_0^t \left[ \|w\|_{H^2 s(\Gamma)}^2 \right] ds.$$  

Time integrating (10.12), we find that

$$Z(T) + \int_0^T \|w''\|_{H^1(\Omega_t)}^2 dS \leq \left[ (C r + \epsilon) \int_0^T \left[ \|w\|_{H^2 s(\Gamma)}^2 \right] ds \right. \left. + C \int_0^T (Y(t) + Z(t)) dt \right],$$

where

$$Z(T) = \sup_{0 \leq T \leq T} \left[ \|w''\|_{L^2(\Omega_T)} + \|\mathcal{E}^r \cdot \eta\|_{L^2(\Gamma)}^2 + \|\mathcal{E}^s \cdot \eta\|_{L^2(\Gamma)}^2 + \|\mathcal{E}^r \cdot n\|_{L^2(\Gamma)}^2 \right].$$

Combining (10.11) and (10.13), choosing $\epsilon > 0$ small enough and then $T > 0$ small enough, we find that

$$Y(T) + Z(T) + \int_0^T \left[ \|w\|_{H^2(\Omega_t)}^2 + \|w''\|_{H^2(\Omega_t)}^2 \right] ds \leq C \int_0^T (Y(t) + Z(t)) dt.$$
References

[1] F. Auricchio, L. Beirão da Veiga and C. Lovadina, Remarks on the asymptotic behaviour of Koiter shells, Computers and Structures, 80 (2002), 735-745.

[2] H. Beirão da Veiga, On the existence of strong solutions to a coupled fluid-structure evolution problem, J. Math. Fluid Mech., 6 2004, 21–52.

[3] M. Boulakia, Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid, J. Math. Pures Appl. (9) 84 (2005), no. 11, 1515-1554.

[4] A. Chambolle, B. Desjardins, M.J. Esteban, C. Grandmont, Existence of weak solutions for an unsteady fluid-plate interaction problem, Preprint.

[5] C.H. Cheng, D. Coutand, S. Shkoller, Navier-Stokes equations interacting with a nonlinear elastic shell, Preprint.

[6] P.G. Ciarlet, An Introduction to Differential Geometry with Applications to Elasticity, Springer 2005

[7] P.G. Ciarlet, Introduction to Linear Shell Theory, Series in Applied Mathematics (Paris), vol. 1, Gauthier-Villars, Editions Scientifiques et Médicales Elsevier, Paris, 1998.

[8] D. Coutand and S. Shkoller, On the motion of an elastic solid inside of an incompressible viscous fluid, to appear in Arch. Rational Mech. Anal.

[9] D. Coutand and S. Shkoller, On the interaction between quasilinear elastodynamics and the Navier-Stokes equations.

[10] B. Desjardins, M.J. Esteban, Existence of weak solutions for the motion of rigid bodies in a viscous fluid, Arch. Rational Mech. Anal., 146 (1999), 59-71.

[11] B. Desjardins, M.J. Esteban, C. Grandmont, P. Le Tallec, Weak solutions for a fluid-structure interaction problem, Rev. Mat. Complut., 14 (2001), 523-538.

[12] L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, 19 American Mathematical Society, Providence, RI, 1998.

[13] F. Flori, P. Orenga, Fluid-structure interaction: analysis of a 3-D compressible model, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), 753-777.

[14] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations Volume I, Springer Tracts in Natural Philosophy, Vol 38.

[15] Z. Ge, H.P. Kruse and J.E. Marsden, The limits of Hamiltonian structures in three-dimensional elasticity, shells, and rods, J. Nonlinear Sci., Vol. 6 (1996), 19-57.

[16] E. Givelberg, Modeling Elastic Shells Immered in Fluid, Comm. Pure Appl. Math., 57 (2004), no. 3, 283-309.

[17] C. Grandmont, Y. Maday, Existence for unsteady fluid-structure interaction problem, Math. Model. Numer. Anal., 34 (2000), 699-636.

[18] R.J. Leveque, C.S. Peskin and P.D. Lax, Solution of a two-dimensional cochlea model with fluid viscosity, SIAM J. Appl. Math., 45 (1985), no. 3, 450-464.

[19] C. Liu, N.J. Walkington, An Eulerian description of fluids containing visco-elastic particles, Arch. Rational Mech. Anal., 159 (2001), 229-252.

[20] D. Sette, Chute libre d’un solide dans un fluide visqueux incompressible: existence, Japan J. Appl. Math., 4 (1987), 33–73.

[21] H.F. Weinberger, Variational properties of steady fall in Stokes flow, J. Fluid Mech., 52 1972, 321-344.

E-mail address: cchsiao@math.ucdavis.edu
E-mail address: coutand@math.ucdavis.edu
E-mail address: shkoller@math.ucdavis.edu

Department of Mathematics, University of California, Davis, CA 95616