Cosmology in non-local Bopp-Podolski electrodynamics

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Non-local extension of the Bopp-Podolski non-linear electrodynamics is investigated. It is well-known that the theory in flat space time, reduces to the Proca theory. However, it will be shown that in curved space time the resulting theory will differ from the coupled Einstein-Proca system. This theory admits de sitter solution. The cosmological perturbations on top of the de Sitter space-time shows that the tensor and vector modes are healthy. However there is a scalar mode in this model which behaves like the Pais-Uhlenbeck oscillator. This shows that this theory contains an Ostrogradski ghost in the scalar sector. Anisotropic cosmology of the model is also investigated and we will show that the behavior of the universe at late time depends strongly on the initial conditions. However, independent of the parameters of the theory, the model predicts an isotropic universe at late times.

I. INTRODUCTION

The late-time accelerated expansion of our universe is demonstrated by different observational data [1]. The late-time acceleration occurs at very low energy scale $\sim 10^{-4}eV$, so to investigate this phenomenon one can consider a modification to GR as an effective field theory of gravity. There are many attempts in the literature to describe the observational data. The main idea is adding new degrees of freedom to general relativity (GR) in a consistent way so that the resulting theory does not contain instabilities and also satisfy current observational data. Adding new degrees of freedom can be done in different approaches. The first approach is to introduce some matter sources with a large negative pressure. This branch is the well-known as dark energy branch. There are many types of such sources in the literature such as scalar fields, vector fields and etc [2]. The ΛCDM model is the simplest and most successful model to describe current acceleration of the universe. The primary idea to avoid this problem is to add some higher derivative self-interaction terms to the theory. This was first done by Bopp and Podolsky [15] where their Lagrangian contains Ostrogradski instability which shows itself as a massive vector ghost. One should note that both generalizations of the Maxwell theory introduced above, respects U(1) symmetry. Cosmological implications of the Bopp-Podolsky interaction term, the theory contains the massive vector field which has U(1) symmetry and the Proca theory for massive vector field.

The kinetic term of the Maxwell and also Proca theories $\sim 1/4F_{\mu\nu}F^{\mu\nu}$ has a famous problem that it diverges at the point charge. There are some efforts in the literature to avoid this problem. Born and Infeld [13] have considered non-linear terms of the strength tensor of the form

$$\mathcal{L} \propto 1 - \sqrt{\det (\eta_{\mu\nu} + \beta F_{\mu\nu})},$$

to avoid the point charge divergence of the electric charge. The Born-Infeld electrodynamics reduces to the Maxwell’s theory in low energy limits. The cosmological implications of this kind of matter field has been vastly investigated [14].

The other possibility for avoiding the aforementioned divergence is to add some higher derivative self-interaction terms to the theory. This was first done by Bopp and Podolsky [15] where their Lagrangian contains

$$\mathcal{L} \propto -\frac{1}{m^2} \left(\partial_{\nu}F^{\mu\nu}\right)^2,$$

where $m$ is a constant with mass dimension 1. After a field redefinitions it can be shown that the Bopp-Podolsky Lagrangian describes two independent dynamical fields corresponding to one massless and one massive vector field. Because of the higher derivative nature of the Bopp-Podolsky interaction term, the theory contains Ostrogradski instability which shows itself as a massive vector ghost. One should note that both generalizations of the Maxwell theory introduced above, respects U(1) symmetry. Cosmological implications of the Bopp-Podolsky theory and also its ghost instability on top of the de Sitter expanding background has been investigated in [16].
In an effective action the origin of nonlocal terms can be seen in the one-loop approximation via the heat kernel method. Also one can consider the nonlocal contributions to the quantum effective action within the covariant perturbation expansion in the field strengths and spacetime curvatures [17]. From an independent viewpoint, nonlocal terms in classical action may come from integrating out one of the healthy degrees of freedom of the theory. In these cases the resulting nonlocal action is free from pathologies. Also there are special higher dimensional gravity theories, in which reduction to four dimensions gives an action including nonlocal terms, e.g. \( \sqrt{-\Box} \) in DGP model [18]. Nonlocal terms has been used to modify IR and UV limits of GR and nonlocal cosmology is extensively investigated in the literature [19].

Recently, nonlocal terms has been extensively studied in massive field theories and specially massive gravity. In these cases, one can retain the gauge invariance of the theory by introducing some nonlocal terms. Nonlocal terms for massive spin two fields was suggested in the context of degravitation [20]. Also, nonlocal massive gravity was studied in [21]. In this paper, we will consider the effect of nonlocal terms in a vector field theory.

It is well-known that the massive vector theory can be written in a form that it is explicitly \( U(1) \) symmetric. This can be done by the additional dynamical degree of freedom which behaves correctly under the \( U(1) \) transformation. To illustrate this point further, let us consider a dynamical massive vector field in flat space-time described by the Proca action of the form

\[
S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \right),
\]

where \( F_{\mu\nu} = 2 \partial_{[\mu} A_{\nu]} \) is the field strength of the Proca field \( A_\mu \) with dimension \( M \) and \( m^2 \) is the mass squared. This theory has 3 degrees of freedom on flat space and breaks explicitly the \( U(1) \) symmetry of the Maxwell’s theory due to existence of the mass term. Now, let us Stueckelberg transform the field \( A_\mu \) as

\[
A_\mu \rightarrow A_\mu + \partial_\mu \phi,
\]

where \( \phi \) is a scalar field. The resulting action becomes

\[
S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu 
- m^2 \partial_\mu \phi A^\mu 
- \frac{1}{2} m^2 \partial_\nu \phi \partial^\nu \phi \right),
\]

which has an explicit \( U(1) \) symmetry if both \( A_\mu \) and \( \phi \) fields transform as

\[
A_\mu \rightarrow A_\mu + \partial_\mu \zeta,
\]

\[
\phi \rightarrow \phi - \zeta,
\]

where \( \zeta \) is an arbitrary function of coordinates. One can proceed further and write the action (3) in a form that it contains only the field \( A_\mu \) and also has an explicit \( U(1) \) symmetry. This can be done by obtaining the scalar field \( \phi \) in terms of the vector field \( A_\mu \) from the scalar equation of motion \( \Box \phi = -\partial_\mu A^\mu \), with the result

\[
\phi = -\frac{1}{\Box} (\partial_\mu A^\mu),
\]

and substituting back to the action (3). The result is

\[
S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{4} F_{\mu\nu} \frac{1}{\Box} F^{\mu\nu} \right),
\]

which we have performed integration by parts to simplify the action. One can see that the resulting action has a \( U(1) \) symmetry, but it becomes non-local. As a result, the Proca theory in flat space is equivalent to non-local theory (6).

The generalization of the Proca action has also attracted considerable attention in the literature [22]. In these theories the general kinetic term for a massive vector field beyond Maxwell term is considered which has 3 degrees of freedom. Generalization to the case of non-abelian vector field is also considered in [23].

The line of the paper is as follows. In the next section, we will introduce the model and obtain the field equations and also discuss on the local counterpart of the theory. In section III we will investigate the isotropic cosmology of the model and consider its cosmological perturbation on top of de Sitter solution. Section IV will be devoted to the anisotropic implications of the theory and in the last section we will conclude and discuss on some possible issues and future lines.

II. THE MODEL

In this paper, we will consider the effects of a Proca field in cosmology, but we want to make the theory \( U(1) \) symmetric. This can be achieved by using the non-local counterpart of the Proca action. As a result, consider the following action

\[
S = \int d^4x \sqrt{-g} \left( \kappa^2 (R - 2\Lambda)
- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
+ \sum_n \alpha_n F_{\mu\nu} \frac{1}{\Box^n} F^{\mu\nu} \right),
\]

where we have generalized the non-local term to contain higher order derivatives and \( \Lambda \) is the cosmological constant. Here, \( \alpha_n \) is an arbitrary constant with dimension \( M^{2n} \).

In this paper we will consider the case \( n = 1 \) for the sake of simplicity. As was discussed in Introduction, the above theory with \( n = 1 \), can be considered as a non-local version of the so-called BP electrodynamics [15, 16]. It is well-known that the Einstein-BP theory suffers from Ostrogradski instability in both vector and scalar modes because of its higher order derivative occurring in the action [24]. However, one may expect that the non-local
version is free from instabilities (at least in the vector mode) because it comes from the Proca theory. However, the procedure of obtaining (3) from (2), highly depends on the geometry of space-time. In fact partial derivatives should commute to make these two actions equal. In curved space-times the non-local version of the Proca field (7) is different from the Einstein-Proca system by some non-linear curvature terms which can turn on the Ostrogradski instability of the theory. Another way to see this instability is to rewrite the action (7) with $n = 1$ as

$$S = \int d^4x \sqrt{-g} \left( \kappa^2 (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha F^{\mu\nu} \xi_{\mu\nu} + \lambda^{\mu\nu}(\Box \xi_{\mu\nu} - F_{\mu\nu}) \right),$$

(8)

where $\xi_{\mu\nu}$ and $\lambda_{\mu\nu}$ are two covariant antisymmetric tensors. Obtaining equations of motion of these fields and substituting back to the action (8) one obtains the action (7). Here $\xi_{\mu\nu}$ is dimensionless and $\lambda_{\mu\nu}$ has dimension $M^2$. One can decouple the Maxwell field from the $\xi_{\mu\nu}$ and $\lambda_{\mu\nu}$ fields by applying the field redefinition of the form

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + 2\alpha \xi_{\mu\nu} - 2\lambda_{\mu\nu},$$

(9)

which makes the action

$$S = \int d^4x \sqrt{-g} \left( \kappa^2 (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha^2 \xi_{\mu\nu}^{\mu\nu} + \lambda_{\mu\nu}^{\mu\nu} - 2\alpha \lambda_{\mu\nu} \xi_{\mu\nu} + \lambda_{\mu\nu} \Box \xi_{\mu\nu} \right).$$

(10)

The rest of the action containing $\xi_{\mu\nu}$ and $\lambda_{\mu\nu}$ can be further diagonalized by performing the field transformation of the form

$$\xi_{\mu\nu} = \frac{c}{\alpha} A_{\mu\nu} + B_{\mu\nu},$$

$$\lambda_{\mu\nu} = cA_{\mu\nu} + cB_{\mu\nu},$$

(11)

with the result

$$S = \int d^4x \sqrt{-g} \left( \kappa^2 (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha B_{\mu\nu} \Box B^{\mu\nu} - \frac{c^2}{\alpha} (A_{\mu\nu} \Box A^{\mu\nu} - 4\alpha A_{\mu\nu} A^{\mu\nu} B^{\mu\nu}) \right).$$

(12)

One can see from the above action that we have a massless antisymmetric tensor $B_{\mu\nu}$ and a massive one $A_{\mu\nu}$. One should assume that $\alpha > 0$ in order to make the field $B_{\mu\nu}$ healthy. In this case however, one should assume $c^2 < 0$ to make $A_{\mu\nu}$ healthy. This in fact makes the field redefinition (11) complex. The theory then consists of a massless Maxwell field and a massless antisymmetric field $B_{\mu\nu}$ together with a massive complex antisymmetric tensor field $A_{\mu\nu}$. We will not follow this line in this paper, and take all of the fields real. We will show in the following that an instability will be appeared in the scalar sector in the perturbation analysis of the theory on top of the de Sitter solution.

Let us now obtain the equations of motion of the theory (8). The equation of motion for the vector field $A_{\mu}$, the tensor field $\xi_{\mu\nu}$ and the Lagrange multiplier $\lambda_{\mu\nu}$ can be written respectively as

$$\nabla^\alpha (F_{\alpha\beta} - 2\alpha \xi_{\alpha\beta} + 2\lambda_{\alpha\beta}) = 0,$$

(13)

$$\alpha F^{\alpha\beta} + \Box \lambda^{\alpha\beta} = 0,$$

(14)

and

$$\Box \xi_{\alpha\beta} = F_{\alpha\beta}.$$

(15)

As one can see from the above field equations, the Maxwell strength tensor $F_{\mu\nu}$ is the source for both antisymmetric tensors $\xi_{\mu\nu}$ and $\lambda_{\mu\nu}$. Also it is worth mentioning that the theory has a $U(1)$ symmetry by shifting the vector field $A_{\mu} \rightarrow A_{\mu} + \lambda_{\mu} \phi$. This implies that the $A_{\mu}$ field equation satisfies a conservation equation of the form (13). As a result, we have a Noether current associated with the aforementioned symmetry.

The gravitational field equation of motion can be written as

\[
\begin{align*}
\kappa^2 (G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} F_{\alpha\mu} F^{\alpha\nu} + \frac{1}{8} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} - \frac{1}{2} F^{\alpha\beta} g_{\mu\nu} \lambda_{\alpha\beta} - \frac{1}{2} \alpha F^{\alpha\beta} g_{\mu\nu} \xi_{\alpha\beta} + \alpha F^{\alpha\beta} \xi_{\mu\nu}) \\
+ \alpha F^{\alpha\beta} \xi_{\alpha\nu} - \frac{1}{2} \xi^{\alpha\beta} \Box \xi_{\alpha\beta} - \frac{1}{2} \xi_{\mu\nu} \Box \lambda_{\mu\nu} + \frac{1}{2} \nabla_{\alpha} \nabla^{\alpha} \nabla_{\beta} \xi_{\mu\nu} + \frac{1}{2} \nabla_{\alpha} \nabla^{\alpha} \nabla_{\nu} \lambda_{\mu\beta} + \frac{1}{2} \nabla_{\mu} \nabla^{\mu} \nabla_{\nu} \xi_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla^{\mu} \nabla_{\nu} \lambda_{\alpha\beta} \\
- \frac{1}{2} \lambda^{\alpha\beta} \nabla_{\mu} \nabla_{\nu} \xi_{\alpha\beta} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} \\
+ \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \lambda_{\alpha\beta} = 0.
\end{align*}
\]
In the following the cosmological solution of this model is considered.

III. ISOTROPIC COSMOLOGY

Let us now assume that the universe can be described by a flat FRW line element of the form
\[
ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2),
\]  
where \(a\) is the scale factor. The homogeneity and isotropy of the FRW metric implies that the vector field \(A_\mu\) should have a form
\[
A_\mu = (A_0(t), 0, 0, 0).
\] This form is the only choice which preserve isotropy and homogeneity of the universe. The antisymmetric tensor fields \(\lambda_{\mu \nu}\) and \(\xi_{\mu \nu}\) has six degrees of freedom consists of a 3-vector corresponding to the \((0i)\) components and a pseudo 3-vector field corresponding to \((ij)\) components [25]. However, in an isotropic and homogeneous FRW universe these tensors should vanish, because otherwise there is a non-zero 3-vector field which breaks the isotropy of the space-time. So, in the FRW universe, we have \(\lambda_{\mu \nu} = 0 = \xi_{\mu \nu}\). Also note that the form (18) for the vector field implies that the field strength \(F_{\mu \nu}\) vanishes.

As a result, equations (13)-(15) are trivially satisfied and equation (16) reduces to the Einstein’s equation in the presence of the cosmological constant which has a de Sitter solution with the Hubble parameter of the form
\[
H = \frac{\sqrt{\Lambda}}{3}.
\] Note that in an isotropic universe, the non-local BP interaction does not contribute to the evolution of the universe. In the following, we will see that the non-local BP affects the perturbations around de Sitter space time. Also, in the next section, we will consider the anisotropic universe in which the BP interaction term will affect the evolution of the universe.

A. Cosmological perturbations

Let us consider the perturbation of the action (8) on top of the de Sitter solution obtained in the previous section. The metric perturbation can be written as
\[
ds^2 = -(1 + 2\varphi)\, dt^2 + 2a(S_i + \partial_i B)dx^i\, dt + a^2((1 + 2\psi)\delta_{ij} + \partial_i \partial_j E + \partial_i F_{ij})\, dx^i dx^j,
\]
where \(h_{i j}\) is the traceless and transverse tensor mode with
\[h_{i i} = 0 = \partial_i h_{i j}, \quad F_{ij}\] and \(S_i\) are transverse vector modes with
\[\partial_i F_{i j} = 0 = \partial_i S_i, \quad \psi, \varphi, B\] and \(E\) are four scalar modes. Note that the special indices \((i, j = 1, 2, 3)\) are raised and lowered with \(\delta_{ij}\).

The vector field \(A_\mu\) can be decomposed as
\[
A_\mu = (A_0 + \delta A_0, A_\perp^i + \partial_i \delta A_i),
\] where \(\delta A_0\) and \(\delta A\) are scalar modes and \(A_\perp^i\) is the vector mode with \(\partial_i A_\perp^i = 0\). Note that the background value \(A_0\) is an arbitrary function of time since it is not contribute to the background cosmology. However in this section we will assume that \(A_0\) is constant for the sake of simplicity. For the antisymmetric tensor fields \(\xi_{\mu \nu}\) and \(\lambda_{\mu \nu}\), the background values are zero and one has a decomposition
\[
\xi_{00} = \xi^\perp_0 + \partial_0 \xi, \quad \xi_{ij} = \epsilon_{ijk}(\chi^k_\perp + \partial_k \chi),
\]
and
\[
\lambda_{00} = \lambda^\perp_0 + \partial_0 \lambda, \quad \lambda_{ij} = \epsilon_{ijk}(\rho^k_\perp + \partial_k \rho),
\]
where \(\xi^\perp_i, \chi^\perp_i, \lambda^\perp_i\) and \(\rho^\perp_i\) are vector modes with vanishing divergence, and \(\xi, \chi, \lambda\) and \(\rho\) are scalar modes. Also, \(\epsilon_{ijk}\) is the 3-dimensional Levi-Civita symbol. In summary, we have one tensor mode associated to the metric perturbation, 7 vector modes and 10 scalar modes. Now, let us define the gauge invariant quantities associated to above perturbation variables. For the metric perturbation, one can define two gauge invariant scalar perturbations of the form
\[
\Phi = \varphi + \partial_0 \left( aB - \frac{a^2}{2} \partial_0 E \right), \quad \Psi = \psi + H \left( aB - \frac{a^2}{2} \partial_0 E \right),
\]
and one gauge invariant vector perturbation
\[
\Pi_i = S_i - \frac{1}{2} a \partial_0 F_i.
\]

Also, the tensor perturbation \(h_{ij}\) is gauge invariant. For the vector field \(A_\mu\), one can define two gauge invariant scalar perturbations of the form
\[
\mathcal{Y} = \delta A_0 + A_0 \partial_i \left( aB - \frac{a^2}{2} \partial_0 E \right), \quad \mathcal{Z} = \delta A + A_0 \left( aB - \frac{a^2}{2} \partial_0 E \right).
\] The vector mode \(A_\perp^i\) is gauge invariant. Finally, one can easily check that because of the zero background values for the antisymmetric tensor fields \(\xi_{\mu \nu}\) and \(\lambda_{\mu \nu}\), all vector and scalar modes associated to these fields are gauge invariant.

After expanding the action (8) up to second order in perturbation parameters around the de Sitter background (19), one can see that tensor, vector and scalar
modes are decoupled from each other and as a result, in the following we will consider these modes independently.

1. Tensor perturbations

Expanding the action (8) up to second order in the tensor mode $h_{ij}$, one obtains

$$S^{(2)}_{\text{tensor}} = \frac{1}{2} \sum_{+, \times} \int d^3 k \, dt \, a^3 \left[ \ddot{h}_{ij} - \frac{\ddot{k}^2}{a^2} h_{ij}^2 \right],$$

(27)

where we have performed Fourier transformation and sum over two helicity degrees of freedom.

The above result shows that the tensor mode in Non-local BP theory is equivalent to the standard Einstein’s theory with a cosmological constant. This is however not surprising since we do not have a source for a tensor mode from the non-linear BP interaction. As a result the speed of the propagation of tensor mode in this theory is equal to the speed of light, satisfying recent gravitational wave observations [26].

2. Vector perturbation

Expanding the action (8) up to second order in gauge invariant perturbation vector modes and performing Fourier transformation, one obtains

$$S^{(2)}_{\text{vector}} = \int d^3 k \, dt \, a \left[ \dot{\tilde{A}} - \frac{\ddot{k}^2}{a^2} \tilde{A}^2 + \sqrt{\frac{16 \Lambda}{3}} (\tilde{A} \cdot \tilde{\xi}^\perp + \tilde{\chi}^\perp \cdot \tilde{\rho}^\perp) + 4 \left( \frac{\ddot{k}^2}{a} - \frac{4 \Lambda}{a^3} \right) \left( \tilde{A} \cdot \tilde{\xi}^\perp - \tilde{\rho}^\perp \cdot \tilde{\xi}^\perp \right) - 4 \alpha \tilde{A} \cdot \tilde{\xi}^\perp + 4 \alpha \left( \tilde{\chi} \cdot \tilde{\rho}^\perp - \tilde{\rho} \cdot \tilde{\chi} \right) \right].$$

(28)

where $\tilde{\xi}^\perp$ and $\tilde{\chi}^\perp$ are known functions of $\tilde{A}^\perp$ from equations (29).

We should note that in the standard BP electrodynamics, there are two vector modes on top of the de Sitter solution, one of which is an Ostrogradski ghost. By promoting the BP interaction to a non-local term, the Ostrogradski ghost becomes a Lagrange multiplier and vanishes from the action.

3. Scalar perturbations

Let us now consider the scalar perturbation of the theory (8). As is obtained from the previous section, the theory has 8 gauge invariant scalar degrees of freedom. After Fourier transforming the second order action in scalar modes, one obtains

$$S^{(2)}_{\text{vector}} = \int d^3 k \, dt \, a \left[ \dot{\tilde{A}}^\perp - \frac{\ddot{k}^2}{a^2} \tilde{A}^\perp + 4 \alpha \tilde{A}^\perp \cdot \tilde{\xi}^\perp + 4 \alpha \tilde{A}^\perp \cdot \tilde{\rho}^\perp \cdot \tilde{\xi}^\perp + 4 \alpha \tilde{A}^\perp \cdot \tilde{\rho}^\perp \cdot \tilde{\chi}^\perp + 4 \alpha \tilde{A}^\perp \cdot \tilde{\chi}^\perp \cdot \tilde{\rho}^\perp \cdot \tilde{\chi}^\perp \right] + 4 \left( \frac{\ddot{k}^2}{a} - \frac{4 \Lambda}{a^3} \right) \left( \tilde{A} \cdot \tilde{\xi}^\perp - \tilde{\rho}^\perp \cdot \tilde{\xi}^\perp \right) + 4 \alpha \tilde{A} \cdot \tilde{\xi}^\perp + 4 \alpha \left( \tilde{\chi} \cdot \tilde{\rho}^\perp - \tilde{\rho} \cdot \tilde{\chi} \right) \right].$$

(30)
\[
S^{(2)}_{\text{scalar}} = \int d^3k dt \left[ \frac{4a(k^2 \chi + \Lambda \xi)}{\sqrt{3}} + 4a\Lambda k^2 \xi - \frac{4ak^2 \chi \rho_k}{\sqrt{3}} - 4ak^2 \rho_k - 2ak^2(2\alpha \xi - 2\lambda + \gamma)\ddot{Z} + ak^2 \dot{Z}^2 \right.
+ 8\sqrt{3}a^3\kappa^2 \Phi \dot{\psi} - 12a^3\kappa^2 \dot{\psi}^2 - 4a^3\kappa^2 \Lambda \Phi^2 + \frac{4\kappa^4 \xi}{a} - \frac{4k^4 \rho_k}{a} + 8\alpha \kappa^2 k^2 \Phi \Psi + 4\alpha k^2 \dot{\psi}^2
- \frac{16}{3}\kappa^2 \Lambda \xi + \frac{16}{3} ak^2 \lambda \rho_k + ak^2 \gamma^2 + 4a\alpha k^2 \xi \gamma - 4a\alpha k^2 \gamma ] .
\]

(31)

One can see from the above action that the scalar perturbations \( \Phi, \gamma, \lambda \) and \( \rho \) are non-dynamical, with equations of motion
\[
\Lambda \Phi = \frac{k^2}{a^2} \Psi + \sqrt{3} \Lambda \dot{\psi},
\]
\[
\gamma = 2\alpha + 2\alpha \xi + \ddot{\zeta},
\]
\[
2\lambda = \ddot{\xi} + \sqrt{\frac{\Lambda}{3}} \xi - \left( \frac{k^2}{a^2} - 4\alpha - 2\alpha \right) \xi,
\]
and
\[
\left( 3\frac{k^2}{a^2} - 4\alpha \right) \chi + \sqrt{3} \Lambda \ddot{\chi} + 3\ddot{\chi} = 0.
\]

(32)

(33)

It is worth mentioning that the scalar mode \( \rho \) is a Lagrange multiplier and the corresponding equation of motion determine the evolution of the perturbed field \( \chi \). Also the scalar mode \( \lambda \) is an auxiliary field.

Substituting the above solutions back to the action (31), one can see that the scalar mode \( \Psi \) becomes non-dynamical with equation of motion \( \Psi = 0 \). This is similar to the Einstein-Hilbert theory since there is no non-minimal interactions between the curvature tensor and the fields \( \xi_{\mu\nu} \) and \( A_{\mu} \). The resulting action becomes
\[
S^{(2)}_{\text{scalar}} = \int d^3k dt \frac{a k^2}{a^2} \left[ \dot{\xi}^2 - \left( \frac{k^2}{a^2} - \frac{4}{3}(2\alpha - 3\alpha) \right) \dot{\xi}^2 \right.
+ \left( \frac{k^4}{a^4} + 2(2\alpha - \Lambda) \frac{k^2}{a^2} + \frac{16}{9} \Lambda(\Lambda - 3\alpha) \right) \dot{\xi}^2 \right].
\]

(34)

The above theory is equivalent to the Pais-Uhlenbeck oscillator [27] with angular momentums
\[
\omega_{1,2}^2 = \frac{k^2}{a^2} + 2\alpha - \frac{4\Lambda}{3} \pm \sqrt{4\alpha^2 - \frac{2\Lambda k^2}{3} \frac{a^2}{a^2}},
\]

(35)

which is well-known to suffer from an Ostrogradski instability. This shows that the scalar sector of the theory has a ghost degree of freedom. This is in fact similar to the standard Bopp-Podolski electrodynamics [24], so the non-local theory can not remove the ghost in the scalar sector. In order to cure this problem one can add a constraint term to reduce the dimensionality of the phase space and then remove the Ostrogradski instability from the theory [28]. It may be done by adding another higher derivative Maxwell term to the action as a Lagrange multiplier which will be the scope of our future works. From another point of view, the action (7) with \( n = 1 \) in Minkowski space time, is the non local version of the Proca action. So it does not have any pathology. However, as mentioned above it is not the case for curved space time.

It should be noted that the \( \xi \) mode corresponds to the \( A_{\mu} \) field of section II. This can be seen in the subhorizon limit of the theory where the above action reduces to
\[
S^{(2)}_{\text{scalar}} = \int d^3k dt \frac{a k^2}{a^2} \left[ \dot{\xi}^2 - \frac{1}{2} \dot{\xi}^2 \right],
\]

(36)

where \( \ddot{\xi} = i \xi \). So, the \( \xi \) field corresponds to the scalar part of the complex tensor field \( A_{\mu\nu} \) as was obtained in section II.

IV. ANISOTROPIC COSMOLOGY

Let us now consider the anisotropic cosmology of the non-local BP electrodynamics. Assuming that the universe can be described by the Bianchi type-I space-time of the form
\[
ds^2 = -dt^2 + a^2 dx^2 + b^2(dy^2 + dz^2),
\]

(37)

where \( a(t) \) and \( b(t) \) are the scale factors corresponding to spacial directions \( x, y \) and \( z \). Let us define the following quantities which are very useful in the analysis of anisotropic cosmology
\[
H_i = \frac{\dot{a}}{a}, \quad H_2 = \frac{\dot{b}}{b}, \quad \Delta H_i = H_i - H, \quad 3H = \sum_i H_i = H_1 + 2H_2, \quad 3A = \sum_i \left( \frac{\Delta H_i}{H} \right)^2, \quad q = -1 + \frac{d}{dt} \left( \frac{1}{H} \right).
\]

(38)

Here, \( H \) is the mean Hubble parameter, \( A \) is the anisotropic parameter and \( q \) is the deceleration parameter.
In order to solve the set of equations (13)-(16), let us perform some assumptions. From equation (13), one obtains
\[ \nabla^\beta (F_{\alpha \beta} - 2\alpha \xi_{\alpha \beta} + 2\lambda_{\alpha \beta}) = 0, \] (39)
which has a solution
\[ \lambda_{\alpha \beta} = \alpha \xi_{\alpha \beta} - \frac{1}{2} F_{\alpha \beta}, \] (40)
where we have set the integration constant to zero. Note that \( F_{\mu \nu} \) is the field strength of the vector field \( A_\mu \). Inspecting from solution (40), one can assume that \( \xi_{\alpha \beta} \) and \( \lambda_{\alpha \beta} \) are also the field strength of some vector fields \( \xi_\mu \) and \( \lambda_\mu \) respectively
\[ \begin{align*}
\xi_{\alpha \beta} &= \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha, \\
\lambda_{\alpha \beta} &= \nabla_\alpha \lambda_\beta - \nabla_\beta \lambda_\alpha.
\end{align*} \] (41)
Substituting back to (40), one obtains
\[ \lambda_\mu = \alpha \xi_\mu - \frac{1}{2} A_\mu. \] (42)
Now, let us assume that the vectors \( A_\mu \) and \( \xi_\mu \) can be decomposed as
\[ \begin{align*}
A_\mu &= \kappa \left( 0, \int a U_1 dt, \int b U_2 dt, \int b U_2 dt \right), \\
\xi_\mu &= \kappa \left( 0, \int a X_1 dt, \int b X_2 dt, \int b X_2 dt \right),
\end{align*} \] (43)
where \( H_0 \) is the current Hubble parameter and \( U_i \) and \( X_i \) with \( i = 1, 2 \) are some arbitrary dimensionless functions of time. The time component of the above vectors are set to zero, because they do not contribute to the field equations. Also, the complicated form of the above assumption will help us to write the field equations in a more compact form.

Now, let us define the following dimensionless parameters
\[ \tau = H_0 t, \quad H = H_0 h, \quad \alpha = \beta H_0^2, \quad \Lambda = H_0^2 \lambda. \] (44)
With the above assumptions, equations of motion (13)-(16) reduces to

\[ h^2 \left[ 3(\ddot{A}^2 - 1) + 2(\ddot{A}^2 + 1)S_1 X_1 + 2(2 - 2\ddot{A} + 5\dddot{A})S_2 X_2 \right] + \lambda + \sum_{j=1}^2 \left[ j \left( S_j' X_j' - X_j S_j'' + S_j X_j'' - \frac{1}{4} U_j^2 \right) \right] = 0, \] (45)

\[ h' \left[ -2(\ddot{A} + 1) + 4(1 - 2\ddot{A})S_2 X_2 \right] - h^2 \left[ (\dddot{A} + 1)^2(3 + 2S_1 X_1) + 2(-4 + 10\ddot{A} + 5\dddot{A})S_2 X_2 \right] - h \left[ 2(1 + 4S_2 X_2)\dddot{A} + 3X_1^2 \left( \frac{S_1}{X_1} \right)' + 4(2\ddot{A} - 1) (S_2 X_2)' \right] + \lambda + 2U_2 (S_2 + \beta X_2) + S_1 X_1'' - X_1 S_1'' \
- \sum_{j=1}^2 \left[ j \left( S_j' X_j' + \frac{1}{4} U_j^2 \right) \right] = 0, \] (46)

\[ h^2 \left[ 2Z_1 + 2S_1 X_1 - 3 + \dddot{A}(1 - \dddot{A})(2Z_1 + 8S_2 X_2 + 3) \right] + h' \left[ 2Z_1 - 2 + \dddot{A}(2Z_1 + 1) \right] + h \left[ \frac{1}{2} \dddot{A}' (1 + 2Z_1) + 2(1 + \dddot{A})Z_1' \
- 3X_2^2 \left( \frac{S_2}{X_2} \right)' \right] + \lambda - \frac{1}{4} U_1^2 + 2S_2 U_2 + U_1 (S_1 + \beta X_1) - S_1' X_1' - 2S_2' X_2' - X_2 S_2'' + S_2' X_2'' = 0, \] (47)

\[ 2(\ddot{A} - 2) \left[ h' Z_2 + 3h^2 \left( S_2 X_2 - S_1 X_1 + \dddot{A} Z_2 \right) \right] + 2h \left[ \dddot{A} Z_2 + \dddot{A} (4Z_2' - 6(S_1' X_2' + X_1 S_2')) + Z_2 - 6(X_1 S_2' + X_2 S_1') \right] \
- 4\beta U_2 X_1 + U_1 (2U_2 - 4\beta X_2) + 2(S_1 X_1'' + S_2 X_2'' - X_2 S_2'' - S_2' X_2') = 0, \] (48)

\[ -2(\ddot{A} + 1) [3h^2 + 2h'] S_2 X_2 + h \left[ 5X_2 S_2' + S_2 \left( 2\dddot{A} X_2 - X_2' \right) + 2\dddot{A} (S_2 X_2) \right] + \frac{1}{2} U_2 (U_2 - 4\beta X_2) \
- X_2 S_2'' + S_2' X_2'' = 0, \] (49)

\[ U_1'' + 3h U_1' + U_1 \left( 4\beta - 2h^2 (1 + \dddot{A})^2 \right) = 0, \] (50)
where prime represents derivative with respect to $\tau$ and we have defined $\tilde{A} = \sqrt{A/2}$ and we have defined

$$S_j = \beta X_j - \frac{1}{2} U_j,$$
$$Z_1 = S_1 X_1 + S_2 X_2,$$
$$Z_2 = S_1 X_2 + S_2 X_1.$$  

In figures (1) and (2) we have shown the numerical solution of the above equations for deceleration parameter $q$ and anisotropy parameter $A$ in terms of the dimensionless time parameter $\tau$ for three different values of $\beta = 1.1, 1, 0.4$ and $\lambda = 1.9, 1, 0.8$ respectively. These figures show that the late time behavior of the theory is consistent with the late time dynamics of the universe; the dynamic of universe begins from a highly anisotropic state and at the late time the anisotropy parameter goes to zero which dictated that universe becomes isotropic. Also, the study of the deceleration parameter (1), shows that the universe enters to the accelerated expansion phase. This shows that the non-linear BP interaction affects the IR behavior of the space-time making the universe to accelerate at late times.

\[ U''_2 + 3h U'_2 + U_2 \left( 4\beta - h^2 (2 - 2\tilde{A} + 5\tilde{A}^2) \right) = 0, \]  

\section*{V. CONCLUSIONS}

In this paper we have considered the non-local modification of the non-linear Bopp-Podolski electrodynamics. The Bopp-Podolski term is a higher derivative electromagnetic self-interaction term which can solve the problem of the divergence of the electric potential at the source point. Because of the higher derivative nature of the interaction, the theory suffers from Ostrogradski ghost instability. The cosmological implications of this higher derivative self-interaction term was investigated extensively in [16] where the authors shows that the vector and scalar modes suffer from instability on top of de Sitter background. In the present paper we have considered the possibility that whether the promotion of Bopp-Podolski interaction to a non-local term can cure the theory from instability. The non-local version of the Bopp-Podolski interaction term in flat space time is reduced to the co-called Proca action, describing a massive spin-1 field. However, in curved space-times, the action differs from Einstein-Proca theory. In fact one can transform the action to a local theory with an additional Lagrange multiplier, and the resulting action reduces to Einstein-Maxwell theory with two additional antisymmetric tensor fields one massive and one massless.

The theory admits a de Sitter solution on the FRW background trivially, since the vector field can not contribute to an isotropic and homogeneous universe. Cosmological perturbations on top of this de Sitter solution, reveals that the tensor mode is identical to ΛCDM model. This shows that the propagation speed of the gravitational waves in this theory is equal to the speed of light, satisfying recent gravitational wave observations [26]. The vector mode however, differs from Bopp-Podolski theory since there exists an additional Lagrange multiplier which kills the ghost degree of freedom in the vector mode. As a result we have shown that promoting to non-local theory, makes the vector sector of the Bopp-Podolski theory healthy. The scalar sector however has a mode which behaves like the Pais-Uhlenbeck oscillator. Because of higher derivative nature of the Pais-Uhlenbeck
Lagrangian, the scalar sector suffers from Ostrogradski instability. This instability can be cured by reducing the dimensionality of the phase space, as is done [28]. It is worth mentioning that in [29] the authors have proven that there is no local healthy theory with $U(1)$ symmetry. However, our theory is non-local and such a theorem does not apply to our case. As a result such an interaction can in principle be constructed which will be the scope of our future works.

As we have discussed above, the Bopp-Podolski interaction term does not contribute to the homogeneous and isotropic FRW universe. So, we have investigated the dynamics of the universe in anisotropic Bianchi-I universe. We have seen that the universe starts from a highly anisotropic and decelerating phase and at late times the universe becomes isotropic and accelerating. As a result the model satisfies observational data on the late time behavior of the universe.

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