Geometrical and topological aspects of graphene and related materials

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Abstract
Graphene, a two-dimensional crystal made of carbon atoms, provides a new and unexpected bridge between low- and high-energy physics. The field has evolved very quickly and there are already a number of good reviews available in the literature. Graphene constitutes a condensed-matter realization of lower dimensional quantum field theory models that were proposed to confront important—still unresolved—puzzles in the area: chiral symmetry breaking and quark confinement. The new materials named topological insulators, closely related to graphene, are physical realizations of topological field theory. This article reviews some of these topics with the aim of bridging the gap and making these condensed-matter issues accessible to high-energy readers. The electronic interactions in the monolayer are analyzed with special emphasis on the recent experimental confirmation of some theoretical predictions. The issue of spontaneous chiral symmetry breaking in the model materials is also reviewed. Finally we give an extensive description of some recent topological properties of graphene that allow us to understand the main aspects of topological insulators.

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(Some figures may appear in colour only in the online journal)
1. Introduction

The main conceptual advances in modern physics have usually been prompted by almost simultaneous discoveries in different branches. In the past century, statistical physics, quantum field theory (QFT) and condensed matter have had their main developments in parallel with the best physicists (Feynman, Einstein, Landau, Wigner as the most prominent examples) contributing to them all. Cosmology and astrophysics have developed with inputs from particle physics. This fruitful arena has been damaged in the last half of the past century by the fast evolution and specialization of the different fields, and by the excess of information and production of scientific papers which has originated a profound gap between different areas. In 2004, the experimental realization of graphene, a two-dimensional crystal made of carbon atoms, provided a new and unexpected bridge between condensed-matter and high-energy physics which also involves the fields of elasticity, statistical mechanics and cosmology.

Graphene is a two-dimensional crystal of carbon atoms arranged in a honeycomb lattice: a single layer of graphite [1]. Its synthesis [2, 3], amazing properties [4] and potential applications [4, 5], granted the 2010 Nobel Prize in Physics to A Geim and K Novoselov. One of the most interesting aspects of the graphene physics from a theoretical point of view is the deep and fruitful relation that it has with quantum electrodynamics (QED) and other QFT ideas. The connection arises from the fact that the low-energy excitations of the system can be described by the massless Dirac equation in two spatial dimensions. Prior to the synthesis of graphene, a system of electrons in a honeycomb lattice was proposed as a ‘condensed-matter simulation of a three-dimensional anomaly’ [6, 7]. These two works, together with the analysis of the spin–orbit in graphene in [8, 9], constitute the foundation of modern topological insulators [10, 11] which provide a condensed-matter realization of the axion electrodynamics [12, 13]. From the QFT viewpoint, graphene has given rise to other very interesting developments: charge fractionalization [14] has been explored in the honeycomb lattice with special defects [15, 16], and QFT in curved space [17, 18] and cosmological models [19, 20] have been used to study the electronic properties of the corrugated material. A very interesting aspect is associated with the generation of various types of vector fields arising from the elastic properties or from disorder that couples to the electrons in the form of gauge fields [21].

Many good reviews are already available in the literature [21–27], some of them very theoretically oriented [28–30], so we will focus on some particular aspects that either have not been sufficiently clarified or are the basis of subsequent developments. The topics are also chosen to be in the frontier between condensed-matter and high-energy physics. From
a theoretical point of view, the synthesis of graphene has opened a new world where ideas from different branches of physics can be confronted and tested in the laboratory. From the electronic point of view, it can be shown that the low-energy excitations of the neutral system obey a massless Dirac equation in two dimensions. This special behavior originates on the topology of the honeycomb lattice and has profound implications for the transport and optical properties. Although the Fermi velocity is approximately a hundredth of the speed of light, the masslessness of the quasiparticles brings the physics to the domain of relativistic quantum mechanics where phenomena such as the Klein paradox or the Zitterbewegung can be explored [1]. The Schwinger mechanism has also been proposed as experimentally accessible in mono and multilayer graphene [31, 32].

Dirac fermions appeared in condensed matter before the advent of graphene but are ubiquitous nowadays. Again graphene constitutes the cleanest and one of the most robust examples.

Topology is an invaluable mathematical tool in many areas of physics, particularly in high-energy and condensed-matter physics. In the latter case, we can also find apparently disjoint areas where the dynamics of the relevant degrees of freedom in the system are governed by laws based on topological constraints. This is the case, for instance, of critical systems in two dimensions comprising the XY spin model, dislocations and melting of solids, liquid crystals and so on [33]. Also, the concept of the Berry phase will appear linked to the physical properties of crystalline solids, as we will see. In the basis of its well-known properties, we will use graphene as a laboratory to understand most of the properties derived from the topology associated with the Berry phase. Although the field of topological insulators has followed its own path [11], some of the most fundamental aspects of the physical phenomena involved can already be found in graphene. Being a very simple material and model, understanding its topological aspects opens the way to follow the more sophisticated developments.

The organization of the review is summarized in the contents.

2. A summary of the graphene features

The construction of the free action in condensed-matter physics proceeds in a very similar way as in QFT. The non-interacting Hamiltonian is determined by the discrete (crystal) and internal symmetries of the system. The ‘band theory’ provides the dispersion relation of the material and the electronic properties for a given electron occupancy. The low-energy expansion of the dispersion relation arising in most of the usual crystal lattices in two and three dimensions is of the type \( \epsilon(k) = k^2/2m \). This quadratic dispersion implies a finite density of states (DOS) at the Fermi surface that screens the Coulomb interactions and provides a very simple model of Fermi liquids, the Landau Fermi liquid (LFL) theory [34]. This standard model of metals is rooted on the existence of a finite, extended, Fermi surface (line in 2D, point in 1D). The effective description is that of a free Fermi gas with an effective mass that absorbs all the effects of interactions. The effective mass (directly related to the finite DOS at the Fermi level) is the main parameter of the theory that enters in all the physical observables [35]. A very good description of the LFL as a fixed point of the renormalization group (RG) was given in [36, 37] and will be summarized in section 3.1.

In what follows we will see that the special geometry of the honeycomb lattice leads to a very different situation where the Fermi surface is reduced to two points in \( k \) space and the quasiparticles obey a massless Dirac equation in two space dimensions. Besides having a zero effective mass, the DOS at the Fermi level also vanishes, a feature responsible for most of the novel electronic behavior of the material.
2.1. Monolayer graphene

The carbon atom has four external 2s², 2p² orbitals able to form molecular bonds. The crystal structure of graphene consists of a planar honeycomb lattice of carbon atoms shown at the left-hand side of figure 1. In the graphene structure, the in-plane σ bonds are formed from the 2s, 2pₓ, and 2pᵧ orbitals hybridized in a sp² configuration, while the 2pᵦ orbital, perpendicular to the layer, remains decoupled. The σ bonds give rigidity to the structure and the π bonds give rise to the valence and conduction bands. The exotic electronic properties of graphene are based on the construction of a model for the π electrons sitting at the positions of the Honeycomb lattice drawn by the σ bonds. Alternatively, the mechanical properties involve the σ bonds with characteristic energies of the order of 7–10 eV. The low-energy excitations around the Fermi energy will have characteristic energies ranging from a few meV up to 1 eV.

Most of the crystal lattices discussed in textbooks are Bravais lattices. In two dimensions, they can be generated by moving an arbitrary lattice point along two defined vector lattices. This happens in the generalized square and triangular lattices. It is easy to see that this is not the case in the hexagonal lattice. This lattice is very special: it has the lowest coordination in two dimensions (three) and it has two atoms per unit cell. As can be seen in figure 1, the hexagonal lattice can be generated by moving two neighboring atoms along the two vectors defining a triangular sublattice. This is the first distinctive characteristic responsible for the exotic properties of the material.

The dispersion relation of the honeycomb lattice based on a simple tight-binding (TB) calculation is known from the early works in the literature [38, 39]. We will not repeat here the derivation which is very clearly written in any graphene review but will instead highlight the main properties. The first is that two atoms per unit cell imply a two-dimensional wavefunction to describe the electronic properties of the system. The entries of the wavefunction are attached to the probability amplitude for the electron to be in sublattice A or B.

The nearest-neighbor TB approach reduces the problem to the diagonalization of the one-electron Hamiltonian

\[ \mathcal{H} = -t \sum_{\langle i,j \rangle} a_i^+ a_j, \]

where the sum is over pairs of nearest-neighbor atoms i, j on the lattice, and \( a_i \) and \( a_i^+ \) are the usual creation and annihilation operators. The Bloch trial wavefunction has to be built as a superposition of the atomic orbitals from the two atoms forming the primitive cell:

\[ \Psi_k(r) = C_A \phi_A + C_B \phi_B. \]
The eigenfunctions and eigenvalues of the Hamiltonian are obtained from the equation

\[
\begin{pmatrix}
\epsilon \\
-\epsilon \sum_j e^{i a k u_j} - \epsilon \\
\end{pmatrix}
\begin{pmatrix}
C_A \\
C_B \\
\end{pmatrix} = E(\mathbf{k})
\begin{pmatrix}
C_A \\
C_B \\
\end{pmatrix},
\]

(3)

where \(u_j\) is a triad of vectors connecting an A atom with its B nearest neighbors (\(\delta_i\) in figure 1), \(v_j\) is the triad of their respective opposites, \(a\) is the distance between carbon atoms and \(\epsilon\) is the 2p\(_z\) energy level, taken as the origin of the energy. The TB parameter \(t\) estimated to be of the order of 3 eV in graphene sets the bandwidth (6 eV) and is a measure of the kinetic energy of the electrons. The eigenfunctions are determined by equation (3). The eigenvalues of the equation give the energy levels whose dispersion relation is

\[
E(k) = \pm t \sqrt{1 + 4 \cos^2 \frac{\sqrt{3}}{2} ak_x + 4 \cos \frac{\sqrt{3}}{2} ak_x \cos \frac{\sqrt{3}}{2} ak_y}.
\]

(4)

A plot of the dispersion relation can be seen in the right-hand side of figure 1. The neutral system with one electron per lattice site is at half filling. The Fermi surface consists of six Fermi points as can be seen in figure 1 (only two are independent). This is the most important aspect of the system concerning its unusual properties. The existence of a finite Fermi surface (a Fermi line in two dimensions) in metals is at the heart of the LFL standard model. It implies a finite DOS and the screening of the Coulomb interaction. Moreover, it allows the construction of the Landau kinematics leading to the possibility of superconductivity and other collective excitations in the otherwise free electron system [36]. Having the Fermi surface reduced to two points invalidates in principle the LFL construction and is at the basis of the anomalous behavior of the system to be discussed later.

A continuum model can be defined for the low-energy excitations around any of the Fermi points, say \(K_1\), by expanding the dispersion relation around it. Writing \(\mathbf{k} = K_1 + \delta \mathbf{k}\) in (3) gives the low-energy effective Hamiltonian

\[
\mathcal{H} \sim -\frac{3}{2} t a \begin{pmatrix} 0 & \delta k_x + i \delta k_y \\ \delta k_x - i \delta k_y & 0 \end{pmatrix}.
\]

(5)

The limit \(\lim_{a \to 0} \mathcal{H}/a\) defines the continuum Hamiltonian

\[
H = v_F \vec{\sigma} \cdot \vec{k}.
\]

(6)

where \(\sigma\) are the Pauli matrices and the parameter \(v_F\) is the Fermi velocity of the electrons estimated to be \(v_F \sim 3ta/2 \sim c/300\). Hence, the low-energy excitations of the system are massless, charged spinors in two spatial dimensions moving at a speed \(v_F\). We must note that the physical spins of the electrons have been neglected in the analysis, and the spinorial nature of the wavefunction has its origin in the sublattice degrees of freedom and is called pseudospin in the graphene literature. The same expansion around the other Fermi point gives rise to a time-reversed Hamiltonian: \(\mathcal{H}_2 = v_F (-\sigma_x k_x + \sigma_y k_y)\). The degeneracy associated with the Fermi points (valleys in the semiconductor language) is taken as a flavor. Together with the real spin, the total degeneracy of the system is 4.

The two Fermi points can be combined into a four-dimensional Hamiltonian,

\[
H_D = -i v_F h (1 \otimes \sigma_1 \partial_x + \tau^3 \otimes \sigma_2 \partial_y),
\]

(7)

where \(\sigma\) and \(\tau\) matrices are the Pauli matrices acting on the sublattice and valley degree of freedom, respectively. The solutions of the Dirac equation—with positive energy—are of the form

\[
\Psi_{E>0} = \exp(i \mathbf{p} \cdot \mathbf{r}) \begin{pmatrix}
\cos \frac{\mathbf{p} \cdot \mathbf{r}}{2} \\
\sin \frac{\mathbf{p} \cdot \mathbf{r}}{2} \\
\sin \frac{\mathbf{p} \cdot \mathbf{r}}{2} \\
\cos \frac{\mathbf{p} \cdot \mathbf{r}}{2} \\
\end{pmatrix},
\]

(8)
where $\theta$ is the polar angle of the vector $\mathbf{p}$ in real space. The first (second) two components of (8) refer to the bispinor around $K_+$ ($K_-$).

The behavior of (8) under a real space rotation of angle $\alpha$ around the $oz$ axis is

$$\Psi'(r') = \Psi'(R^{-1}r) \equiv T_R \Psi(R^{-1}r).$$

The transformation $\mathbf{p}' = \mathbf{p}R = \mathbf{p}(\cos(\alpha + \theta), \sin(\alpha + \theta))$, determines the $T_R$ matrix to be

$$T_R = \begin{pmatrix} \exp(i \alpha \sigma_3/2) & 0 \\ 0 & \exp(-i \alpha \sigma_3/2) \end{pmatrix},$$

which shows that (8) transforms as a real spinor under spacial rotations of the graphene plane.

The chiral structure of the spectrum described above and the quantum mechanical nature of the condensed-matter system (as opposite to QFT) allow us to test several predictions of the old relativistic quantum mechanics [40]. In particular, electrons in the graphene system will tunnel with transmission probability 1 through a step barrier hit at normal incidence. This phenomenon known as the Klein paradox [41, 42] has been experimentally confirmed [43, 44]. A similar phenomenon is the so-called Zitterbewegung or fast trembling motion of the electrons in external fields [45, 46] whose experimental signature has been proposed in [47]. Another very interesting phenomenon is the supercritical atomic collapse [48, 49] in graphene, a consequence of the large value of the ‘graphene fine-structure constant’ (see section 3.1.2). The former phenomena are not only curious realizations of relativistic quantum mechanics, but they also have profound consequences on the experimental aspects of the material system, most of them not observed before in condensed matter. In particular, the Klein paradox implies that impurities and the other most common sources of disorder will not scatter the electrons in graphene. The graphene system also evades the Anderson localization [53], a very important result establishing that any amount of disorder in free electron systems in two space dimensions will localize the electrons turning the system to an insulator. This gives rise to the high mobility at room temperature and the excellent metallicty of the system. Zitterbewegung was suggested in [45] as an explanation for the observed minimal conductivity of the samples [4], one of the most interesting aspects of graphene whose origin remains uncertain.

2.2. Multilayer graphene

Multilayer graphene consists in a superposition of several graphene layers interconnected by hopping terms. Depending on the relative orientation of the layers, several stackings are possible. The two most common, staggered ($ABA$) and rhombohedral ($ABC$), are sketched in figure 2. The low-energy limit of some multilayer systems also gives rise to interesting QFT models defined in the continuum. Bilayer graphene is one of the best studied cases. It is relatively easy to obtain and manipulate experimentally and can be better than the monolayer for some potential applications [54].
Figure 2. Sketch of the three possible positions of a given layer with respect to the others in a graphene stack. Bernal stacking, 1, 2, 1, 2, ..., is described by two inequivalent planes, while orthorhombic stacking, 1, 2, 3, 1, 2, 3, ..., requires the three inequivalent ones.

We will denote the two inequivalent atoms in the layer $n$ as $(a_n, b_n)$. The simplest model introduces interlayer hoppings $t$ only between nearest neighbors. The resulting Hamiltonian for bilayer graphene in the vicinity of the $K_1$ Fermi point is

$$H(k) = \begin{pmatrix} 0 & k^* & 0 & t \\ k & 0 & 0 & 0 \\ 0 & 0 & 0 & k^* \\ t & 0 & k & 0 \end{pmatrix}$$

and the energy bands are given by

$$\epsilon_k = \pm \frac{t}{2} \pm \sqrt{\frac{t^2}{4} + |k|^2}.$$  

In the limit $E \ll t$, one can obtain an effective Hamiltonian for the lowest energy bands [55]. To this end, it is convenient to reorder the wavefunctions according to $(a_1, b_1, a_2, b_2) \rightarrow (a_2, b_1, a_1, b_2)$, so that in the new basis, the Hamiltonian becomes

$$H(k) = \begin{pmatrix} 0 & 0 & 0 & k^* \\ 0 & 0 & k & 0 \\ k^* & 0 & 0 & t \\ k & 0 & t & 0 \end{pmatrix} \equiv \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where $H_{ij}$ is a $2 \times 2$ block. The identity

$$\text{Det}(H - E) = \text{Det}(H_{11} - H_{12}(H_{22} - E)^{-1}H_{21} - E) \text{Det}(H_{22} - E)$$

shows that, for $E \ll t$, the substitution $H_{22} - E \rightarrow H_{22}$ reduces the computation of the lowest energy bands to the diagonalization of the $2 \times 2$ effective Hamiltonian

$$H_{\text{eff}} = H_{11} - H_{12}H_{22}^{-1}H_{21} = -\frac{1}{t^2} \begin{pmatrix} 0 & k^* \\ k & 0 \end{pmatrix}.$$  

This effective Hamiltonian involves only the atoms $(a_2, b_1)$ which are not linked by $t$. Under a QFT point of view, the Hamiltonian is very exotic. It has a chiral structure and, as the monolayer case, its Fermi surface reduces to a point but the quadratic dispersion implies that it has finite DOS at the Fermi level, which makes it more similar to standard metals.

The former analysis can be easily generalized to multilayer graphene with rhombohedral stacking. This type of staking includes the links $(b_1 - a_2, b_2 - a_3, \ldots, b_{N-1} - a_N)$, and the effective Hamiltonian, which involves only the unlinked atoms $(a_1, b_N)$, is given by

$$H_{\text{eff}} = -\frac{1}{t^{N-1}} \begin{pmatrix} 0 & k^N \\ k^N & 0 \end{pmatrix}.$$
In the next subsection, we will see that these types of multilayer systems have a topological charge associated with the Fermi points that guarantee their topological stability [56].

2.3. Discrete symmetries and topological stability of the Fermi points

The existence of the Fermi points is crucial for the special electronic properties of graphene. Being two single points in a two-dimensional space, it could be thought that they would be unstable and a gap could open under mild perturbations. It can be shown that elastic deformations of the lattice change the position of the Fermi points but do not open a gap unless the two points merge to the same position in momentum space, a situation that requires very strong deformations [21]. The Fermi points are also very robust against interactions and disorder of various kinds.

It was proven in [56] that each of the individual Fermi points is topologically stable against deformations of the Hamiltonian preserving the product of the discrete symmetries time-reversal and spacial inversion. The topological stability is based on the non-trivial momentum space topology of the Fermi points [57] and their associated topological charge.

Let us first analyze the discrete symmetries of the low-dimensional model and the relation between the two low-energy Hamiltonians defined around the Fermi points. If the Fermi points are chosen to be \( K_1 \) and \( K_2 = -K_1 \), the respective low-energy Hamiltonians are

\[
H_1 \equiv H(\vec{K}_1 + \vec{k}) \sim k_x \sigma_x + k_y \sigma_y, \quad H_2 \equiv H(-\vec{K}_1 + \vec{k}) \sim -k_x \sigma_x + k_y \sigma_y. \tag{17}
\]

The most relevant discrete symmetries of the system are time-reversal \( T : t \rightarrow -t \) and spatial inversion \( I : (x, y) \rightarrow (-x, -y) \). Because the \( \pi \) orbitals are real, time reversal only reverses \( \vec{k} \) when acting on the Bloch wavefunction (3):

\[
T \Phi_i(\vec{k}) = \Phi^*_i(-\vec{k}). \tag{18}
\]

Spatial inversion reverses \( \vec{k} \) and also exchanges the two sublattices:

\[
I \Phi_A(\vec{k}) = \Phi_B(-\vec{k}), \quad I \Phi_B(\vec{k}) = \Phi_A(-\vec{k}). \tag{19}
\]

The individual Hamiltonians \( H_{1,2} \) given in (17) are not time-reversal invariant but they transform into one another under time reversal maintaining the invariance of the whole system. This fact will be important in the topological insulator construction of section 4.5. The product of time-reversal and spatial inversion acts on each \( 2 \times 2 \) Hamiltonian as

\[
TI : H(\vec{k}) \equiv \sigma_i H^*(\vec{k}) \sigma_i \tag{20}
\]

implying \( H_{11}(\vec{k}) = H_{22}(\vec{k}) \). This enforces the diagonal components of \( H_i \) to be zero and the \( 2 \times 2 \) matrices have only two real degrees of freedom.

With this result we can demonstrate the topological stability against gap opening of \( H_i \). Consider

\[
H = v_F \vec{\sigma} \cdot \vec{k} \tag{21}
\]

as a map from the circle \( k_x^2 + k_y^2 = R^2 \) to the space of \( 2 \times 2 \) Hamiltonians \( H = \vec{\sigma} \cdot \vec{r} \):

\[
k = Re^{i\theta} \rightarrow (h_x, h_y, h_z) = R(\cos \theta, \sin \theta, 0). \tag{22}
\]

The Fermi points correspond to zeros of the determinant \( -\text{Det}(H) = h_x^2 + h_y^2 + h_z^2 \). A perturbation will be able to create a gap only if the loop represented by map (22) is contractible in the space of Hamiltonians with nonvanishing determinants. These maps are classified by the first homotopy class. In the most general case, the space of Hamiltonians will be \( \mathbb{R}^3 - \{0\} \) and a gap will open since \( \pi_1(\mathbb{R}^3 - \{0\}) = \pi_1(S^2) = 0 \). The discrete invariance \( TI \) forces the
diagonal components of (21) to be zero and the Hamiltonians are represented by points in $R^2$. Since
\[ \pi_1(R^2 - \{0\}) = \pi_1(S^1) = Z, \]
the Fermi points will be stable by smooth deformations of the Hamiltonian preserving TI. The topological charge associated with the topological stability of each individual Fermi point in graphene is simply the winding number which can be computed from
\[ N = \frac{1}{4\pi i} \int_0^{2\pi} d\theta \text{Tr}(\sigma_z H^{-1} \partial_\theta H). \]
Since the two Fermi points have opposite winding numbers, a perturbation involving scattering between them or short-range disorder will be able to open a gap.

The former analysis can be extended to multilayer graphene with rhombohedral stacking (the Bernal bilayer also belongs to this class). As we saw in the previous section, these types of compounds admit a low-energy description around the Fermi points given by
\[ H_{\text{eff}} = -\frac{1}{2} t N - \frac{1}{2} (0 \ k^N \ k^N \ 0). \]
The analysis goes through for these $2 \times 2$ matrices and the topological charges of the $N$-layer compounds around the Fermi points are $\pm N$. Note however that there is no topological stability for the multilayer Bernal stacking with $N > 2$.

3. QFT aspects of graphene

In this section, we will review the models for interactions in the graphene system. The monolayer case is particularly interesting under the QFT point of view as a realization of ‘brane QED’, a three-dimensional system of charges confined to a two-dimensional plane [58].

3.1. Electron–electron interactions in monolayer graphene

The standard model of metals (LFL) is rooted on the existence of a finite, extended, Fermi surface (line in 2D, point in 1D). The low-energy expansion of the dispersion relation arising in most of the usual crystal lattices in two and three dimensions is of the type $\varepsilon(k) = k^2/2m$. This quadratic dispersion implies a finite DOS at the Fermi surface that screens the Coulomb interactions and provides a very simple continuum description of the Fermi liquid as a non-interacting system with dispersion $\varepsilon(k) = k^2/2m^*$.

The effective mass $m^*$ is the only parameter in the LFL that enters in all the phenomenological expressions for the LFL observables. The analysis of the LFL as an infrared fixed point of RG can be found in [36, 37].

As we have seen in section 2.1, the situation in the honeycomb lattice is very different. The Fermi surface is reduced to two points and the dispersion relation around them is linear which implies a vanishing effective mass and DOS. This in turn implies that the Coulomb interaction between the quasiparticles will not be screened and we have to deal with a singular long-range interaction.

The effects of the electron–electron interactions in graphene have recently become a hot and highly debated topic. In the early times of the graphene boom, it was widely assumed that the main features of graphene could be reasonably studied using independent particle models. Theoretical works which did not include interactions were highly successful in explaining the anomalous features observed in the integer quantum Hall regime, the dependence on carrier concentration of the electron mobility, transport in p–n junctions (the Klein paradox) or the
transparency at optical frequencies. The lack of interaction effects seemed to be confirmed by careful measurements of the electron compressibility, well explained by independent particle models. The advent of new samples with greatly enhanced homogeneity, and with carrier mobilities higher by about two orders of magnitude, has changed completely the perception about the role of interactions.

The situation was settled after the observation of the fractional quantum Hall effect (QHE) in high-mobility suspended samples. This phenomenon is a definite proof of the presence of interactions in clean, low-temperature samples. Although it proved very elusive, when found, the associated energy gaps (proportional to the strength of the interaction) turned out to be higher than those in other 2D electron liquids \[59, 60\]. Recently, the interaction effects have been observed in angle-resolved photoemission experiments (ARPES) \[61, 62\], scanning tunneling microscopy (STM) \[63, 64\] and transport experiments \[65, 66\]. Most remarkably, a recent experiment has proven the renormalization of the Fermi velocity and the absence of chiral symmetry breaking (CSB) at the incredibly low energies of a fraction of meV \[67\].

### 3.1.1. Static long-range Coulomb interactions: graphene versus QED

The QFT modeling of Coulomb interactions in graphene has been explained in several review articles \[1, 22, 25, 68\]. We will here summarize its main aspects.

As described in the previous section, the standard non-interacting model for the electronic excitations around a single Fermi point in graphene (in units \(\hbar = 1\)) is given by

\[
\mathcal{H} = v_F \int d^2r \bar{\psi}(r) \gamma^i \partial_i \psi(r),
\]

where \(i = 1, 2\), \(\bar{\psi}(r) = \psi^+(r) \gamma^0\), and the gamma matrices can be chosen as \(\gamma_5 = \sigma_2\), \(\gamma_3 = -\sigma_1\), \(\gamma^0 = \sigma_1\), \(\sigma_i\) are the Pauli matrices and \(v_F\) is the Fermi velocity, the only free parameter in the Hamiltonian. The standard condensed-matter form of the Coulomb interaction is

\[
\mathcal{H}_{\text{int}} = \frac{e^2}{2\pi} \int d^2r_1 \int d^2r_2 \frac{\bar{\psi}(r_1) \psi(r_1) \bar{\psi}(r_2) \psi(r_2)}{|r_1 - r_2|}.
\]

This interaction can be described as an effective four-Fermi interaction coming from a standard QED-type Lagrangian,

\[
\mathcal{L} = \int d^2x \left[ \bar{\Psi} \gamma^\mu \partial_\mu \Psi + e j^\mu A_\mu \right],
\]

where \(A_\mu\) is the gauge potential \(\mu = 0, 1, 2\) and the electronic current in our case is defined as

\[
j^\mu = \left( \bar{\Psi} \gamma^0 \Psi, \frac{v_F}{e} \bar{\Psi} \gamma^\mu \Psi \right).
\]

The treatment of the Coulomb interaction is what makes a difference between the graphene model and the usual QED(2+1). In the last model, all the fields, including the electromagnetic potential \(A_\mu\), are defined in two space dimensions. In QFT, massless vector fields are defined by the Lagrangian

\[
L = \int d^{D+1}x \ F_{\mu \nu} F^{\mu \nu}, \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

which has two space derivatives. The effective potential in Fourier space is \(V(k) \sim 1/k^2\). To see the spatial dependence of the interaction, we must Fourier transform \(V(k)\) and obtain the known result

\[
V(r - r') \sim \begin{cases} 
\frac{1}{|r - r'|} & D = 3 \\
\log(|r - r'|) & D = 2 \\
|r - r'| & D = 1 
\end{cases}.
\]
In two spatial dimensions, the Coulomb interaction in real space is logarithmic. A naive scale dimension analysis shows that in this case, the interaction $e$ in (28) has dimensions of energy and the theory is super-renormalizable (see the next section).

The condensed-matter situation that we are discussing is more interesting: only the charges are confined to the graphene plane while the interaction propagates in three dimensions. The reduced gauge field propagator in the 2D model in the static limit can be computed from

$$
\Pi_{\mu\nu}^0(t, r; t', r') = -i\delta_{\mu\nu}\int \frac{d^d k}{(2\pi)^d} \frac{e^{i k (r-r')}}{\omega^2 + k^2 - i\epsilon}.
$$

Note that the static limit is equivalent to considering $\lim v_F/c \to 0$. To be consistent, the model of static Coulomb interaction has to be supplemented by reducing the interaction (28) to the time component of current (29). The photon propagator for the scalar (density–density) Coulomb interaction is then

$$
\Pi_0(k) = \frac{1}{2} \frac{1}{|k|}.
$$

It is easy to see that with this dependence, the interaction term in (28) is dimensionless similar to the case of QED(3+1). The graphene model with the static scalar Coulomb interaction set in this section is scale invariant and describes a (quantum) critical point.

We must note that despite the formal identification of the graphene Hamiltonian with QED(2+1), the different photon propagators produce different results even if we assume a ‘retarded’ Coulomb interaction from the beginning. This will be of special relevance for the discussion of the possibility of getting CSB (see section 3.3).

**3.1.2. Perturbative renormalization.** Ultraviolet divergences arise in quantum field theories due to the singular behavior of the fields at very short distances in real space—or at very large energies in Fourier space. The existence of an underlying lattice in most condensed-matter theories provides a natural cutoff and the issue of divergences can usually be ignored. Infrared divergences appear also in massless theories—and hence they are also present in the graphene case—and can be treated with different techniques. Here, condensed matter again resorts to the finite size of the samples to avoid them. The actual philosophy of renormalization after the RG evolution is that all theories are effective and their validity is restricted to a certain energy range. Irrespective of weather the cutoff is finite or not, it cannot enter into the definition of physical observables of the low-energy theory. Renormalization [69, 70] is a prescription to get rid of ultraviolet divergences and construct sensible models where physical quantities can be accurately computed. The idea is that the ultraviolet divergences can be canceled by a redefinition of the parameters (mass, coupling constant, wavefunction) of the theory by adding counterterms to the Lagrangian. The process is usually done order by order in perturbation theory. If done appropriately, at each order, one finds finite results independent of the cutoff in the computation of physical observables.

Standard QED(3+1) is a strictly renormalizable theory and the perturbative series can be organized as a series in powers of the logarithms of the cutoff. The coupling constant is dimensionless and, in the massless case, the model is scale invariant. The three primitively divergent diagrams shown in figure 4 are associated with the three free parameters in the theory (the coupling constant and the electron and photon wavefunctions). QED(2+1) is super-renormalizable. The coupling constant has a positive dimension of mass ($\sqrt{M}$) which improves
the convergence of the perturbative series: higher loops are less divergent and become finite at a given order. Massless QED (2+1) is ultraviolet finite although it has infrared divergences [71]. As we will see the standard model for graphene sits in between QED(3) and QED(4). It has a dimensionless coupling but only the electron self-energy has divergences.

3.1.3. Green’s functions and observable quantities in graphene. Before entering into the renormalization of the graphene model, let us say a few words on the observables of the system and their relation to its Green’s functions. The building blocks of the perturbative theory are the electron and photon propagators and the vertex function.

The electron self-energy \( \Sigma(\omega, k) \) is defined from the electron Green function \( G(\omega, k) \) by

\[
G^{-1} = G^{-1}_0 - \Sigma.
\]

The diagrams contributing to its renormalization up to two-loop level are shown in figure 3. It contains the following physical information.

- The DOS \( n(\omega) = \text{Im} \int d^2k \text{ tr}[G(\omega, k)\sigma_3] \): it is renormalized by the diagram in figure 2(a). The computation of this diagram is zero if the chemical potential is put to zero as in the neutral case but it will renormalize the DOS in the doped case. The DOS can be measured with STM probes.
- The Fermi velocity renormalization: it is already obtained at the one-loop level from the diagram in figure 2(b). We will describe it at length in section 3.1.5. It defines the shape of the dispersion relation, a quantity that can be observed with ARPES and other techniques.
- The quasiparticle lifetime \( \tau^{-1} \sim \lim_{\omega \to 0} \text{Im} \Sigma(\omega, k) \) (inverse of the scattering rate): this is an observable that can also be measured with ARPES and STM probes. Its behavior with the energy characterizes the system: in a Fermi liquid, it goes like \( \tau^{-1}(\omega) \sim \omega^2 \), and its fast decay is related to the robustness of the quasiparticle description. The imaginary part of the electron self-energy also gives the various scattering rates in the case of adding disorder to the model.
- The wavefunction renormalization \( Z_{\psi} \sim \frac{\partial \Sigma(\omega, k)}{\partial \omega} \big|_{\omega=0} \) defines the anomalous dimension of the field \( \gamma = \partial \log Z_{\psi} / \partial l \) (\( l \) is the RG parameter) and, hence, of the fermion propagator: \( G(\omega, k) \sim \omega^{-\gamma} \). It is a critical exponent that determines the universality class of the given model. From the physical point of view, it affects the interlayer tunneling [72] and other transport properties. A Fermi liquid is characterized by having a finite value of \( Z_{\psi} \) at the Fermi surface.

The photon propagator \( \Pi(q, \omega) \) is directly related to the optical properties of the system. It enters in the definition of the dielectric function \( \epsilon(q, \omega) \) as

\[
\frac{1}{\epsilon(q, \omega)} = 1 + V_0(q)\Pi(q, \omega),
\]

which, in the limit \( \omega \to 0 \), gives the static screening properties of the system and is purely real. It renormalizes the effective Coulomb interaction. The imaginary part of the photon self-energy is related to the frequency-dependent conductivity through

\[
\text{Re} \sigma(\omega) = \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} \Pi(\omega, q),
\]

Figure 3. Feynman diagrams renormalizing the electron self-energy up to two-loop order.
from which the optical response of the system can also be obtained. In the case of having a general current–current interaction with a generic vertex \( \Gamma^{\mu} = e\gamma^{\mu} \), a Kubo formula relates the generalized conductivity tensor \( \sigma_{\mu\nu}(\omega) \) to the corresponding \( \Pi_{\mu\nu}(q, \omega) \). In particular, the Hall conductivity that will be discussed at length in section 4 can be computed in this way.

The imaginary part of the photon self-energy gives the density of electron–hole excitations of the system.

The vertex corrections renormalize the electric charge as in QED. If the gauge invariance is maintained, they are related to the electron self-energy by a Ward identity.

### 3.1.4. Renormalization of the graphene model

The three primitively divergent Feynman graphs of QED(3+1) are shown in figure 4. In the graphene case, only the electron self-energy (figure 4(a)) diverges. The spatial part has a logarithmic divergence leading to the renormalization of the velocity parameter.

As we have seen in section 3.1.1, the model of graphene with static Coulomb interactions is described by the Lagrangian

\[
L = \int d^3k \left[ \bar{\psi}(\gamma^0 k^0 + v \mathbf{\cdot} \mathbf{k})\psi - e\bar{\psi}\gamma^0 \psi A_0 + A_0 |\mathbf{k}| A_0 \right].
\]  

(36)

The Feynman diagrams of the model are built with the electron and photon propagators in momentum space:

\[
G_0(k^0, \mathbf{k}) = \frac{i \gamma^0 k^0 + v \mathbf{\cdot} \mathbf{k}}{-(k^0)^2 + v^2 |\mathbf{k}|^2},
\]

(37)

\[
\Pi_0(\mathbf{k}) = \frac{1}{2} \frac{1}{|\mathbf{k}|},
\]

(38)

and the tree-level vertex: \( \Gamma = e\gamma^0 \).

The renormalization functions associated with the electron self-energy are defined as

\[
G_0^{-1} - \Sigma(k^0, \mathbf{k}) = \frac{1}{2} \gamma^0 \mathbf{\cdot} \mathbf{k} \left[ k^0 \gamma^0 - Z_e(k^0, \mathbf{k}) v \mathbf{\cdot} \mathbf{k} \right].
\]

(39)

The extra parameter \( v \) appearing in (37) in the graphene case has an associated renormalization function \( Z_v \) which is a new feature characteristic of graphene. In the Lorentz-invariant massless QED in any dimensions, only the electron wavefunction renormalization is associated with the electron self-energy.

The function \( Z_v \) defines the running of the coupling \( v \) whose beta function is defined as

\[
\beta_v = \partial Z_v / \partial l.
\]

(40)

The fixed points of the system are determined by the zeros of this beta function and the running of the velocity is given by the local behavior of the beta function around the fixed point [70]. This will be discussed at length in the following section.
3.1.5. Velocity renormalization. The computation of the graph in figure 4(a) gives [73]

$$\Sigma_\Lambda^{(1)}(k) = -\frac{g}{4} v \gamma \cdot k \left( -\log \frac{k^2}{\Lambda^2} + 4 \log 2 \right),$$

(41)

where $\Lambda$ is an ultraviolet cutoff. The electron self-energy can be made finite at this order in perturbation theory by including at tree level a counterterm of the form of a tree-level fermion propagator with the associated Feynman rule:

$$\Sigma_{ct,\Lambda}^{(1)}(k) = \frac{g}{4} v \gamma \cdot k (-\log \Lambda^2 + 4 \log 2 + F_\Lambda).$$

(42)

Since the only requirement to impose on it is to cancel the divergent part of the given diagram, it contains a momentum-independent arbitrary finite part $F_\Lambda/\Lambda$ to be fixed by the renormalization condition, i.e. by an experimental measure that allows us to extract the value of the two-point function at a given momentum $k_R$.

Summing up the contributions of the tree level plus the two Feynman graphs of order $g$ (figure 4(a)) and the counterterm, the two-point function is

$$\Sigma_{\Lambda}^{(1)}(k) = -i \left( \gamma^0 k_0 + v \gamma \cdot k \left[ 1 - \frac{g}{4} \log \frac{k^2}{k_R^2} + F_\Lambda \right] \right),$$

(43)

which can be written as

$$\Sigma_{\Lambda}^{(1)}(k) = -i (\gamma^0 k_0 + v(k) \gamma \cdot k),$$

(44)

with $v(k) = v \left[ 1 - \frac{g}{4} \log \frac{k^2}{k_R^2} + F_\Lambda \right]$. (45)

The two-point function (44) has the same form as the free one with a $k$-dependent arbitrary parameter $v(k)$. The last step of the renormalization program is to fix the arbitrariness with a renormalization condition. For this, we need an experimental measure of the Fermi velocity at a given value of the momentum $k_R$. We can take as an example the experimental value of the Fermi velocity given in [74],

$$v(125 \text{ meV}) = 1.093 \times 10^6 \text{ m s}^{-1} \equiv v_F.$$  

(46)

With this condition, we fix the value of $F_\Lambda$ choosing the bare velocity to be $v = v_F$. The physical Fermi velocity will depend on the energy at which it is measured and on the renormalization point $k_R$ (125 meV in this case):

$$v_R(k) = v_F \left[ 1 - \frac{g}{4} \log \left( \frac{k^2}{k_R^2} \right) \right].$$

(47)

The definition of the running velocity defines the running coupling constant:

$$g_R(k) = \frac{e^2}{4\pi v_R(k)}. \quad (48)$$

Two different renormalization prescriptions for the Fermi velocity measured at the points $k_A$ and $k_B$ are related by the RG equation

$$\frac{v(k_A)}{v(k_B)} = 1 - \frac{g_B}{4} \log \left( \frac{k_A}{k_B} \right). \quad (49)$$

Hence, the RG ensures that the exact theory is independent of the experimental point chosen in (46). This in turn guarantees the consistency of the renormalization procedure.

At this point, it is worth mentioning that diagram 4(a) in massless QED(2+1) does not have a logarithmic divergence because the photon propagator has an extra inverse power of momentum. In QED(4), this diagram induces a wavefunction renormalization. It is interesting to note that the electron charge in the graphene model is not renormalized (we will discuss this in section 3.1.8) but the velocity renormalization induces a renormalization of the graphene structure constant similar to the one obtained in QED(4). This issue will be explored further when discussing the experimental observations in section 3.2.2.
3.1.6. The coupling constant. Before continuing with the perturbative renormalization of the model, we must say a few words about the actual value of the coupling constant. It can be seen that in the static model, the perturbative series is organized in terms of a coupling constant $\alpha_G = \frac{e^2}{4\pi v_F^2}$. Plugging in the Fermi velocity measured in the nanotube experiments $v_F \sim c/300$ [75], and the bare electron charge, we obtain $\alpha_G \sim 2.3-2.5$. Although this value puts the model in the strong coupling regime, the results obtained in perturbation theory agree with the experimental findings. A possible explanation lies in the fact that the effective constant in a material has an extra parameter: the dielectric constant $\varepsilon_G$, taking the value $\varepsilon_G = 1$ in the vacuum. In samples on a substrate, the dielectric constant reduces the effective coupling constant: $\alpha_G = \frac{e^2}{4\pi \varepsilon_G v_F^2}$, where $\varepsilon_G$ includes intrinsic contributions and effects due to the environment in which graphene is immersed. A simple estimate gives $\varepsilon_G \sim (\varepsilon_{\text{air}} + \varepsilon_{\text{subs}})/2 \sim 2-3$ for typical substrates. The actual value of the intrinsic contribution to the dielectric constant due to the graphene layer itself is still object of vivid controversies. By extrapolating measurements of the excitation spectra in graphite, a large intrinsic dielectric constant in single-layer graphene of the order of $\varepsilon_G \sim 13$ was proposed in [76]. An independent estimate of $\alpha$ in graphene has been obtained from measurements of the carrier-plasmon interaction in samples with a finite carrier concentration [61]. The result $\alpha \approx 2.2$ is consistent with previous analysis [77] and from recent numerical calculations [78]. Theoretical values of the order of $\varepsilon_G \sim 4$ have also been reported using various approximations [79, 80].

3.1.7. $1/N$ result for the velocity renormalization. Since the bare coupling constant of graphene can be large at the experimentally accessible energies, the former renormalization scheme can be improved by performing a $1/N$ expansion [81] where $N$ is the number of fermionic species. In the case of graphene with the spin and Fermi point degeneracies, the physical value is $N = 4$. This procedure was followed in the early publications [82, 83] and was later retaken in [84–86].

The simplest non-perturbative calculation amounts to computing the electron self-energy graph in figure 4(a) with an effective propagator obtained from the resummation of the planar diagrams dominant in the $1/N$ approximation. The $\Pi_{00}$ component of the one-loop photon self-energy in figure 4(b) is given by [87]

$$\Pi(k, \omega) = \frac{e^2}{8} \frac{k^2}{\sqrt{v_F^2 k^2 - \omega^2}}; \quad (50)$$

hence, the effective Coulomb potential obtained by summing the geometric Dyson series for the photon self-energy in the so-called random phase approximation (RPA) is

$$\Pi(\omega, k) = \frac{-i}{2|k| + \frac{e^2}{8} \frac{k^2}{\sqrt{v_F^2 k^2 - \omega}}} \quad (51)$$

The electron self-energy computed with this expression gives the following $\beta$ function for the Fermi velocity [83, 84]:

$$\beta_v = -\frac{8}{\pi^2 N} \left(1 + \frac{\arccos g}{g \sqrt{1 - g^2}}\right) + \frac{4}{\pi g} \quad (52)$$

where $g = \frac{N\varepsilon^2}{32N_T}$. The large $N$ limit amounts to taking the limit $N \rightarrow \infty$ keeping $g$ fixed. As we see, the dependence on $g$ is non-perturbative and the growth of the velocity at low energies is slightly different than that set in (49).
3.1.8. Physical predictions for the rest of the observables. The photon polarization of figure 4(c) is finite in two dimensions. In the most general case of a full vertex interaction, it is given by

\[ \Pi_{\mu\nu}(\omega, p) = \frac{e^2}{8} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{p^2}{\sqrt{v_F^2 p^2 - \omega^2}}. \]  

(53)

From this result and using expression (35), we can readily compute the conductivity of graphene that, after restoring the constants, is

\[ \sigma_0(\omega) = 4 \frac{e^2 \pi}{\hbar}, \]  

(54)

the factor 4 is due to the degeneracy of the system discussed in section 2. This is a very peculiar result that indicates that the dynamical conductivity of graphene is a constant independent of the frequency, a result that has been ascertained in experiments [88]. Moreover, it implies that graphene has a finite conductivity at zero frequency, a very puzzling situation that was already found in the first experimental papers and that remains one of the most mysterious aspects of graphene [89]. The influence of Coulomb interactions on the dynamical conductivity has been analyzed in [90–92].

The finiteness of the photon self-energy has been proven at the two-loop level and in the RPA approximation and conjectured to remain true to all orders in perturbation theory. This implies that in undoped graphene, the electric charge is not renormalized. Moreover, no finite screening length is generated and the effective Coulomb potential remains long ranged.

Another very interesting prediction of the model of graphene with static Coulomb interactions is the linear dependence of the quasiparticle lifetime on the energy found in [82]. The first contribution to \( \tau^{-1} \) arises at the two-loop level from diagrams (c) and (d) in figure 3. Experimental evidence of this anomalous behavior has been reported in [93–96].

As for the electron wavefunction renormalization, in the static model it begins at the two-loop level. The computation of the graphs in figures 3(c) and (d) gives

\[ Z_\Psi = 1 - \frac{c_1}{16\pi^2} \frac{e^4}{v^2} + O(e^6), \]  

(55)

where \( c_1 \) is a constant of order 1. It can be demonstrated that despite this result, the residue of the quasiparticle pole given by the value of \( Z_\Psi(\omega) \) at the Fermi surface is finite in the static model, an issue that has been discussed in [83, 97].

Since the electron–photon vertex that we are considering corresponds to the time component of the current and the model respects gauge invariance, the Ward identity ensures that the vertex remains finite at the one-loop level. This can also be seen by a direct computation of the diagram in figure 4(b).

3.2. RG features

3.2.1. The structure of the perturbative series in graphene. The perturbative series in QED, \( \lambda, \Phi^4 \) and other standard renormalizable QFT in \( D = 4 \) can be written as a power series in the logarithm of the cutoff \( \Lambda \) (or of \( 1/\epsilon \) in dimensional regularization). It means that divergent Feynman graphs at \( n \) loops have leading divergences of the type \( (\log \Lambda)^n \). We have seen that in the static model for the Coulomb interactions in graphene, the divergence of the two-loop diagrams in the electron propagator has a single log \( \Lambda \). In this section, we will make a few comments on the origin and implications of such a behavior. We will follow closely [98].
In a renormalizable theory, the renormalized Green functions are reparametrization invariant, a fact expressed by a Callan–Symanzik equation. For the graphene model, this equation was derived in [87]:

$$\left[ \frac{\partial}{\partial \mu} + \mu \frac{\partial e}{\partial \mu} + \mu \frac{\partial v}{\partial \mu} - \frac{n}{2} \frac{\partial \log Z}{\partial v} \right] \Gamma^{(n)} = 0, \quad \text{(56)}$$

with the standard definitions of the $\beta$ functions,

$$\beta_v = \mu \frac{\partial v}{\partial \mu}, \quad \beta_e = \mu \frac{\partial e}{\partial \mu}, \quad \gamma \psi = \mu \frac{\partial \log Z}{\partial \mu}. \quad \text{(57)}$$

The one-loop $\beta$ function for $v$ derived in section 3.1.5,

$$\beta_v = \frac{e^2}{16\pi}, \quad \text{(58)}$$

can be integrated to obtain

$$v(\mu') = v(\mu) + e^2 \frac{1}{16\pi} \log \left( \frac{\mu}{\mu'} \right). \quad \text{(59)}$$

This equation allows us to see that in the model with instantaneous Coulomb interaction, $v_F$ behaves in a rather unusual way. In standard QFT, marginal parameters are corrected by a series of logarithms that can usually be resummed: the leading log resummation corresponds to integrating the one-loop beta function; the next leading log sum corresponds to the two-loop result and so on. In our case, we see that integrating the one-loop beta function produces only one log, and this is all we have for the leading log series. This unusual behavior occurs because the beta function at one loop is independent of $v$. This fact can also be seen more explicitly by deriving the recursion relations among the leading log coefficients. Consider the bare Fermi velocity in dimensional regularization [99]. It can be written as a Laurent series in $\epsilon$,

$$v_0 = \mu^{2\epsilon} \left( v + \sum_{k=1}^{\infty} \frac{a_k(v, e)}{\epsilon^k} \right), \quad \text{(60)}$$

where $\mu$ is the regularization scale. Taking the derivative with respect to $\mu$ and since $e$ is not renormalized, this yields

$$0 = 2e \left( v + \sum_{k=1}^{\infty} \frac{a_k(v, e)}{\epsilon^k} \right) + \mu \frac{\partial v}{\partial \mu} \left( 1 + \sum_{k=1}^{\infty} \frac{a'_k(v, e)}{\epsilon^k} \right). \quad \text{(61)}$$

This is the usual equation that gives recursion relations among the coefficients in the logs, by matching powers of $\epsilon$ in the Laurent series, and which allows the resummation of the leading logs. (It is an equivalent statement of renormalizability.)

The matching of terms to order $\epsilon^{-1}$ can be evaluated to order $\epsilon^4$ with the one-loop calculations, yielding

$$2(a_1^2 + e^2 a_2^2 + e^4 a_3^2) = -\frac{2e^2}{32\pi} \frac{\partial}{\partial v} (a_0^2 + e^2 a_1^2), \quad \text{(62)}$$

where the superindex denotes the order in $\epsilon$. The computation to one loop shows that $a_2^2 = a_3^2 = 0$, and that the right-hand side is independent of $v$, so this implies that

$$a_4^2 = 0, \quad \text{(63)}$$

which in turn implies that there is no squared log term in the two-loop self-energy. This is a non-trivial implication of the renormalizability of the theory. This argument can be carried out iteratively to show that there is no log$^n$ term at $n$ loops. This statement is another perspective of the fact that integrating the one-loop $\beta$ function gives just (59).
3.2.2. Experimental confirmation of the Fermi velocity renormalization. The RG prediction of the velocity renormalization has very recently been confirmed in several experimental reports [67, 63, 62] following earlier more indirect pieces of evidence [100, 65]. The experimental evidence more directly related to the physics discussed in this review is that in [67]. The experiment measures the effective mass of graphene at different carrier densities in high-mobility suspended samples and also in graphene on a BN substrate. The experiment probes the suspended graphene samples at low energies in a range from 100 to 0.2 meV never reached before. They use the temperature dependence of Shubnikov–de Haas oscillations to infer the dependence of the Fermi energy, $\epsilon_F$, on the area of the Fermi surface, $S_F$. The effective mass is defined as

$$m_{\text{eff}} = \frac{\hbar^2}{2\pi} \frac{dS_F}{d\epsilon_F},$$

so that for graphene

$$m_{\text{eff}} = \frac{\hbar k_F}{v_F}.$$  \hfill (65)

A comparison of experimental results and fits based in equation (52) is shown in the left-hand side of figure 5.

The behavior of the inverse running coupling constant in graphene taken from the data in [67] is shown in the right-hand side of figure 5.

The difficulty in achieving these results is similar to that in going several orders of magnitude higher in energy to confirm the fine-structure constant renormalization [68]. It is interesting to compare with the case of QED(3+1). The fine-structure constant is defined by $\alpha_{\text{QED}} = e^2/(4\pi\epsilon_0c)$ where $e$ is the electron charge, $c$ the speed of light and $\epsilon_0$ the dielectric constant of the vacuum. As such, it is a dimensionless quantity whose universally known value is $\alpha^{-1} = 137$. From its definition, measuring the fine-structure constant amounts to measuring the electron charge with the higher possible accuracy since the other quantities in the definition are true constants. The actual value obtained from the magnetic moment of the electron has an accuracy of 0.7 parts per billion [101]. This allows a determination of $\alpha_{\text{QED}}$ of similar accuracy. The calculation includes contributions from 891 Feynman diagrams and is one of the most demanding comparisons of any calculation and experiment ever performed.
The impressive accuracy reached at the solid-state energies (around 100 mK considered as zero for the high-energy running coupling constant) is spoiled at energies of the order of the proton mass (1 GeV) where strong-interaction diagrams contribute to the determination of $\alpha_{\text{QED}}$. A relatively recent measurement made at the large electron–positron (LEP) collider at CERN provides a value of $\alpha^{-1}(M_Z) = 128.9460.030$ \cite{103, 104}. From this, we can see that the slope of the variation in two orders of magnitude is similar to the one obtained in graphene.

### 3.2.3. Inclusion of disorder

In graphene, many classes of lattice defects can be described by gauge fields coupled to the two-dimensional Dirac equation \cite{21}. The standard techniques of disordered electrons \cite{105} can be applied to graphene by averaging over the random effective gauge fields. A random distribution of defects can be modeled as a random gauge field, with a variance related to the type of defect and its concentration. These random fields when treated perturbatively lead to a renormalization of the Fermi velocity which makes it decrease at lower energies opposing the upward renormalization induced by the long-range Coulomb interaction. The simultaneous presence of interaction and disorder gives rise to new interesting fixed points. The issue was analyzed in \cite{106}. The most interesting case arises when considering a random gauge potential which models elastic distortions and some topological defects. There is a line of fixed points with Luttinger-like behavior for each disorder correlation strength $\Delta$ given by $v_F^* = \frac{e^2}{\Delta}$. An extensive analysis of the issue in the large $N$ limit is done in \cite{84}.

### 3.2.4. Retarded Coulomb interactions

The success of the effective description of the long-range Coulomb interaction in graphene by means of a static Coulomb potential described above and the observation of the Fermi velocity renormalization means that retardation effects of the order of $v_F/c \sim 1/300$ are ineffective in today’s experimentally accessible region. An RG estimation of the energy scale $\delta$ at which the value of $v_F$ approaches $c$ gives

$$\delta = k_R \exp \left[ \frac{-16\pi (c - v(k_R))}{e^2} \right],$$

where $k_R$ and $v_R$ are the renormalization conditions chosen to define the theory. A typical estimate will of course produce an energy of approximately 100 orders of magnitude below $k_R$. This discussion is very similar to that leading to the Landau pole of QED \cite{107} and simply points to the incompleteness of the theory. In our case, there are physical low-energy bounds much less stringent than that, preventing to access the infrared region. But even if this does not impose any real bound on the validity of the model from an experimental point of view, it is interesting to know that a more complete theory exists of which the static limit is only an approximation. The retarded model has very interesting properties that we will revisit here.

The inclusion of a covariant photon propagator

$$\Pi_0(\omega, k) = \frac{1}{2} \frac{1}{\sqrt{\omega^2 - |k|^2}},$$

instead of (33) leads to important conceptual differences in the infrared behavior of the model. The most important difference concerns the running of the Fermi velocity.

The one-loop computation of the electron propagator in this case gives for the velocity beta function the result \cite{87}

$$\beta_v = \frac{1}{v} \left( 1 - 2v^2 + 4v^4 \right) \arccos v + \frac{1}{v} \left( 1 - 4v^2 \right),$$

(68)

where the speed of light has been put to 1. The equation $\beta_v = 0$ has now a non-trivial solution $v = 1$ which in our units means that we have a fixed point of RG where the electronic velocity
equals the speed of light. Since the electric charge $e$ is not renormalized, the value of the coupling constant at the fixed point $g^*$ is the fine-structure constant of QED:

$$g^* = \alpha_{\text{QED}} = \frac{1}{137}.$$ (69)

This issue initially studied in [87] was later confirmed in a non-perturbative approach in [108]. Hence, the graphene system with retarded Coulomb interactions is a non-trivial interacting infrared fixed point of RG with the fixed point value of $\alpha_G$ equal to the fine-structure constant and with emergent Lorentz covariance. Under the QFT point of view, finding a non-trivial infrared fixed point is already very interesting. From the condensed-matter side, the system departs from the Fermi liquid behavior and shows emergent Lorentz covariance.

Moreover, in the retarded model with the full vertex interaction, the time component of the one-loop diagram in figure 4(a) diverges (and so does this in 4(b) related by the Ward identity). The wavefunction renormalization at the one-loop level gives rise to an anomalous dimension:

$$\eta = \frac{e^2}{12\pi^2},$$ (70)

which implies the departure of the system from the Fermi liquid behavior.

### 3.3. CSB or gap opening in monolayer graphene

Opening a gap (or generating a mass) in graphene is crucial for electronic applications. From the QFT point of view, the electron mass (gap) is protected by the 3D version of chiral symmetry and hence a gap will not open by radiative corrections at any order in perturbation theory. As discussed in [56], interactions or disorder can open a gap in the sample when the product of time reversal and inversion symmetry is broken (staggered potential, external magnetic field), or when the two Fermi points are involved (Kekul’e distortion). A discussion of the various ‘masses’ that can be generated in graphene when the valley and spin degrees of freedom are taken into account will be given in section 4.5 when discussing the topological insulator aspects of graphene. Even if the two Fermi points are taken into account in a four-dimensional formalism, that a gap will be spontaneously generated by Coulomb interactions in monolayer graphene is very unlikely and there is so far no experimental evidence of it.

#### 3.3.1. Long-range Coulomb interactions.

One of the interesting features of QED(3) is that a fermion mass can be generated dynamically, breaking chiral symmetry. Starting with massless bare fermions, they can acquire a dynamical mass through non-perturbative effects. Writing the full fermion propagator as

$$S^{-1}(p) = A(p)\gamma \cdot p + B(p),$$

a nonzero solution for $B(p)$ implies a nonzero condensate, and signals dynamical mass generation. The infrared value of the dynamical mass function defined by

$$m(p) = \frac{B(p)}{A(p)},$$

$m(0) = B(0)/A(0)$ can be used as an order parameter.

The standard non-perturbative approach to study dynamical fermion mass generation is to solve the Dyson–Schwinger (DS) equation

$$G^{-1}(p) = G_0^{-1}(p) - \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu \Gamma_\nu(p,k)D_{\mu\nu}(p-k),$$ (71)
where $\Gamma_\nu (p, k)$ is the vertex function and $D_{\mu\nu}$ is the photon propagator. Equation (71) can be decomposed into a couple of equations for $A(k^2)$ and $B(k^2)$:

$$A(p^2) = 1 + \frac{1}{4p^2} \int \frac{d^3k}{(2\pi)^3} \text{tr}[\gamma \cdot k \gamma_\nu G(k) \Gamma_\nu D_{\mu\nu}(p - k)],$$

(72)

$$B(p^2) = -\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \text{tr}[\gamma_\mu G(k) \Gamma_\nu D_{\mu\nu}(p - k)].$$

(73)

If the DS equation for $B(p^2)$ has only a vanishing solution, the fermions remain massless and are stable against gauge fluctuations.

The issue of CSB in massless fermion models in QFT is far from being settled. Lower dimensional QED, in particular QED(3), was proposed in the early times as a toy model for CSB [109–112] and confinement [113, 114]. It is clear that no mass will be generated in the weak coupling limit. But, as we described in section 3.1.6, the bare coupling constant of graphene is not small.

In the strong coupling regime, the best studied approximation is based on a large number of fermionic species, the $1/N_f$ expansion (not to be confused with the $1/N$ methods used in QCD where $N$ is the number of colors of an $SU(N)$ gauge theory). The crucial difference between the perturbative and non-perturbative approaches is that the DS equation is nonlinear which makes it possible for the chiral phase transition to occur at the bifurcation point. Spontaneous CSB in QED(3) in the $1/N$ approximation usually requires an unphysical number of fermionic species $N \lesssim 2–3$ [115]. The situation with graphene is even worse since there is no mass parameter to begin with. The gauge propagator is crucial in these approaches and hence it is important to distinguish between QED(3) and graphene. This problem has been addressed in [58] with the conclusion that for small $g$, there is no solution with spontaneous CSB. A very good updated review with a fair list of references can be found in [116].

The simplest non-perturbative calculation that can be performed to study the issue of the gap opening in graphene is the RPA type described in section 3.1.7 [83]. The absence of a constant term in the inverse electron propagator ensures that no mass is generated in this approximation. The $1/N$ expansion approach to the problem has recently been revised in [117] with the conclusion that a gap will not open for the physical values of the electronic degeneracy in graphene ($N = 4$). A variational approach to the excitonic phase transition in graphene including the renormalization of the Fermi velocity [118] also produces quite negative results. At finite temperature, the critical $N_c$ is temperature dependent [119], while a finite chemical potential leads to strong suppression of the critical fermion flavor $N_c$ and the dynamical fermion mass in the symmetry broken phase [120]. Volume effects have been analyzed in [121] to try to explain the discrepancies between the continuum and lattice results.

The situation can be improved by the presence of an external magnetic field giving rise to the so-called magnetic catalysis [122]. This possibility was put forward in the early papers [123–125] and is discussed at length in [23].

3.3.2. Four-Fermi local interactions. Short-range interactions, such as an onsite Hubbard term $U$, are irrelevant in the RG sense [68] in the weak limit. This is due to the vanishing DOS at the Fermi level that occurs at Fermi points, a characteristic of graphene. This is a very important aspect since short-range interactions are responsible for the Fermi liquid properties of usual metals (having a finite Fermi surface) in two space dimensions. Nevertheless, the DOS at low energies can be increased by the presence of disorder and a finite temperature which, in turn, enhances the effect of short-range interactions [126]. These interactions can be relevant in the strong coupling regime of the hexagonal electronic and optical lattices [127, 128].
Four-Fermi interactions were considered in low-dimensional models in high-energy physics again searching for CSB and confinement. One of the most popular models is the Nambu–Jona-Lasinio (NJL) in two space dimensions [129]. The original NJL model was an extension of the BCS theory of superconductivity to the domain of spontaneous symmetry breaking in the strong interaction, one of the best examples of hybridization between condensed-matter and high-energy physics. The Lagrangian density is given by

$$L = -i \bar{\Psi} \gamma^\mu \partial_\mu \Psi + g [ (\bar{\Psi} \Psi)^2 - (\bar{\Psi} \gamma_5 \Psi)^2 ],$$  \hspace{1cm} (74)

The four-fermion interaction is attractive for fermions and antifermions of opposite chirality. Fermion–antifermion pairs form bound states, which form a condensate:

$$\langle \bar{\Psi} \Psi \rangle \neq 0$$

that changes the nature of the vacuum of the theory.

In the condensed-matter language, the NJL model is similar to the Hubbard model that has been extensively studied in lattice systems. These types of interactions have been used in the honeycomb lattice searching for the possibility of gap opening in graphene [85, 130–134]. More recently, lattice gauge field theory techniques are being used to explore this possibility. An interesting recent analysis of the possible phases arisen in the Jona–Lasinio model in three dimensions is done in [135]. Non-perturbative constructive field theory techniques have also been used and are described in [136]. Although the issue remains controversial, it is fair to say that there is no conclusive evidence up to now that the physical parameters of graphene lie in the region where CSB will occur.

A gap can open in the sample by various extrinsic means although without altering the basic properties of graphene, the produced gaps are very small and not entirely controllable. Small gaps have been experimentally reported in samples on a substrate whose lattice is commensurate with that of graphene [137]. Graphene, a hydrogenated compound having hydrogen atoms forming sp³ bonds, has a reasonable gap and is an interesting material on its own grounds [138].

3.4. Electronic interactions in multilayer systems

The situation concerning electron–electron interactions is much more interesting in bilayer graphene. The quadratic dispersion relation and a finite density of states at the Fermi surface enhance the role of four-Fermi interaction, and a variety of broken symmetry phases can arise similarly to what happens in the one-dimensional case [139].

New experiments in high-mobility samples hint to the existence of novel phases at zero carrier concentration and low temperatures. The first indication came from the electron compressibility measurements reported in [140]. These experiments showed a rise in resistivity near the neutrality point, consistent with a tendency toward an insulating state, although the resistivity never went above a value of a few thousand ohms. The existence of a gapped insulating phase was reinforced by extrapolations from measurements made at finite magnetic fields [141], which suggest a gap of about $\Delta \approx 2$ meV.

The experiments reported in [142] present a different picture. The DOS at the Fermi energy is inferred from careful measurements of the carrier density and temperature dependence of the resistivity in a number of high-mobility suspended samples. The results show a crossover to a low-temperature regime where the DOS is significantly reduced. The band structure of bilayer graphene changes from parabolic to four Dirac cones, due to trigonal warping. The crossover found experimentally occurs at an energy of about 6 meV, which is substantially larger than the crossover energy related to trigonal warping, about 1 meV. Hence, the results suggest a spontaneous symmetry breaking associated with interactions. This explanation is
supported by measurements performed at finite magnetic fields, which imply that the lowest Landau level is fourfold degenerate, while independent electron calculations give an eightfold degenerate \( n = 0 \) Landau level. In any case, the results reported in [142] do not show a finite gap at any concentration.

A last batch of recent experiments is discussed in [143]. The conductivity of a suspended sample at the neutrality point was measured as function of bias voltage, magnetic field and perpendicular electric field. The results indicate a gap of bias voltages between about \(-3\) and 3 meV at zero magnetic field, zero electric field and zero carrier concentration. The differential conductance shows peaks at voltages above and below this gap, while they tend toward a constant value at larger bias voltages. Peaks adjacent to a gap are a hallmark of tunneling experiments into superconductors, although there is no apparent reason for these peaks to show up in a dc transport experiment.

All the experiments described above coincide in that a broken symmetry phase due to electron–electron interactions seems likely in bilayer graphene near the neutrality point and in the absence of a perpendicular electric field. They differ in many details, however, and the experiments in [142] suggest a gapless phase, while those in [140], and especially those in [143], seem to imply the existence of a gap. The experiments analyze suspended samples with high electron mobility, although the mobilities differ, being highest in [142] and lowest in [140, 141], while the samples in [143] show an intermediate value. The experimental setups also differ: the measurements in [140, 141] give the values of the compressibility in samples with one back gate, while the experiments in [142] and [143] report the dc conductivity in samples with no and with two gates, respectively.

On the theory side, it was soon realized that the combination of parabolic bands and short-range screened interactions leads to logarithmically divergent susceptibilities. The variety of electronic degrees of freedom in bilayer graphene allows for many possible broken symmetry ground states, which, in turn, can be induced by appropriately tuned interactions [144–151, 97]. There is a gapless, nematic phase consistent with the experiments in [142], and different gapped phases could explain the results in [140], while the interpretation of the experiments in [143] remains less clear. Some of the proposed phases break time-reversal symmetry, leading to states with similar properties to a 2D electron gas in the integer quantum Hall regime [7]. Recent calculations suggest that many possible phases are almost degenerate, with energy differences per atom below 1 meV [152].

Interactions in a graphene bilayer are screened, so that they decay faster than the inverse of the separation beyond some distance. The nature of the broken symmetry phase may depend on the value of the screening length [153]. This leads to the intriguing possibility that the two phases which look more stable, a gapped layer antiferromagnet and a metallic nematic phase, could have been observed in different experiments, depending on the number and position of gates. Further complications are introduced by strains, which are probably unavoidable in suspended samples. Disorder can induce, in some circumstances, local gaps [154]. This situation is reminiscent of other materials with many competing interactions, such as the cuprate superconductors or the manganites, where the interpretation of the low-temperature phases is still debated. At least, graphene has a simple, stochiometric composition.

4. Topological aspects

The classification of the different phases of electronic systems has traditionally relied on two big pillars: the Landau theory of the Fermi liquid and band structure theories and the Landau theory of phase transitions and the spontaneous breakdown of symmetries. The discovery of the QHE was one of the first indications that the classification of quantum phases in terms of
symmetry was incomplete. The new body of knowledge to incorporate into the aforementioned paradigm has to do with topological ideas.

The QHE occurs when an external homogeneous magnetic field is applied in the transverse direction to a two-dimensional electronic system. In the presence of the magnetic field, the electronic spectrum is reorganized into discrete Landau levels separated by forbidden energy gaps \[155\]. The system is in a very special type of insulating state characterized by a vanishing longitudinal conductivity like most insulators, but it displays a transverse quantized conductivity—the Hall conductivity \(\sigma_H\) – not present in the other types of insulators. The Hall conductivity is found to be

\[
\sigma_H = ne^2/h, \tag{75}
\]

where \(n\) takes the integer values \(n = 1, \ldots, N\). Because of the robustness of the effect and the capability to make very accurate measurements of the quantized Hall resistance, the QHE has been used to measure the fine-structure constant with enormous accuracy \[101\]. In a standard 2D system, the energies of the Landau levels are

\[
E_n = \hbar \omega_c (n + 1/2)
\]

and there is no mode at zero energy. Another important aspect of the QHE is that the system supports a number of conducting states at the edges of the sample that are ‘chiral’ in the sense that spin-up (-down) electrons travel in opposite directions. The number of edge states coincides with the numerical value \(n\) of the quantized transverse conductivity (75).

Nowadays, it is well understood that the presence of this quantized transverse conductivity makes the systems in the quantum Hall regime be a type of insulator topologically distinct from other standard insulators which can be understood in terms of the conventional band theory. Soon after the discovery of the QHE, it was shown that the elementary excitations of the QHE system are well described by a Chern–Simons topological action \[156\].

In the following sections, by using graphene as a laboratory, we will study the topological concepts that arise in order to account for the special properties of the QHE, and more generally, for the understanding of a wider class of topological insulating phases of matter.

4.1. QHE in graphene

As was discussed in section 2, the band structure of graphene possesses two distinctive features when compared with other realizations of a two-dimensional electron gas: the linear dispersion at low energies and the presence of an extra \(SU(2)\) quantum number different from the spin, coming from the two Fermi points (valley pseudospin). These two distinctions of graphene are the key features when considering the topological properties of this system related to its Berry phase. We will review the salient features of the QHE in graphene and see how some topological concepts came into play. We will find the low-energy effective topological field theory describing the QHE phase in graphene by using the knowledge of the Landau level structure of the system \[157\].

Let us consider a single layer of graphene under the effect of an external homogeneous magnetic field, described by the vector potential \(A = B(-y, 0, 0)\). In the basic description of the QHE in graphene, we can restrict ourselves to one of the two species (valleys) of Dirac fermions and take into account the fact that, because the external magnetic field breaks time-reversal symmetry, both species will contribute equally to the Hall conductivity.

The low-energy Hamiltonian of the system can be written as \((\hbar = 1)\)

\[
H = v\sigma \Pi = v(\sigma_x \Pi_x + \sigma_y \Pi_y), \tag{76}
\]
with \( \Pi_j = -i \partial_j + e A_j \). In terms of the ladder operators \( a = \frac{i}{\sqrt{\hbar}} (\Pi_x - i \Pi_y) \) and \( a^\dagger = \frac{i}{\sqrt{\hbar}} (\Pi_x + i \Pi_y) \) (\( l_B = \frac{1}{\sqrt{\varepsilon_B}} \)), satisfying \( [a, a^\dagger] = 1 \), the Hamiltonian \( H \) reads

\[
H = \frac{\sqrt{2} e}{l_B} \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix}.
\] (77)

It is important to note that here we are considering the situation of a perfect crystalline layer of graphene with inversion symmetry, so there is no term proportional to \( \sigma_z \). The presence of this and similar terms will be considered later on. The Schrödinger equation with Hamiltonian (77) can be easily solved in terms of the solutions of the harmonic oscillator. The important difference here with the standard two-dimensional electron gas is that Hamiltonian (77) is a two-dimensional matrix, and the eigenvectors are two-dimensional spinors of the form \( \psi_n = (u_n, v_n)^T \). In terms of the second component \( v_n \), the eigenvalue equation is

\[
\frac{2\pi^2}{l_B^2} \hat{n} v_n = E^n v_n = n v_n,
\] (78)

with \( n \) being an integer. For each value of \( n \), there are two eigenvalues

\[
E_{\pm,n} = \pm \sqrt{2|n|/l_B} \equiv \pm \sqrt{\varepsilon_n} |n|.
\] (79)

An important observation from this set of solutions is that there is an eigenstate corresponding to the value \( n = 0 \) with no analog in the standard non-relativistic two-dimensional case [158].

We will consider the situation of zero temperature and finite chemical potential \( \mu \). The action \( S \) of the system, written in the LL basis, reads

\[
S = \int \frac{dk}{2\pi} \sum_n \psi^{\dagger n} (i \partial_t - \mu E_n) \psi_n.
\] (80)

Within a path integral approach, we can compute an effective action for the electromagnetic field by integrating out the fermion fields in (80),

\[
S_{\text{eff}} = \int d\omega dk \sum_{n=\infty}^{\infty} \ln(\omega - \mu - E_n).
\] (81)

Because \( E_n \) does not depend on the momentum \( k \), each Landau level is (highly) degenerate. The degeneracy of each Landau level was first estimated by Landau himself to be \( \frac{eB}{2\pi} \). By adding a small parameter \( \eta \to 0^+ \) sign(\( \omega \)) to the argument of the logarithm, the integration over the \( \omega \) variable can be performed, leading to the following effective Lagrangian:

\[
\mathcal{L}_{\text{eff}} = \frac{|eB|}{4\pi} \sum_{n=-\infty}^{\infty} |E_n - \mu|.
\] (82)

Splitting the contributions from \( n = 0 \) and \( n \neq 0 \), we obtain

\[
\mathcal{L}_{\text{eff}} = \frac{|eB|}{4\pi} |\mu| + \frac{|eB|}{4\pi} \sum_{n=1}^{\infty} (|\mu + \omega \sqrt{|n|}| + |\mu - \omega \sqrt{|n|}|).
\] (83)

The averaged particle number \( \langle n \rangle \) can be calculated from (83) by taking the derivative of \( \mathcal{L}_{\text{eff}} \) with respect to its conjugate variable \( \mu \):

\[
\langle n \rangle = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \mu} = \frac{|eB|}{4\pi} \sum_{n=1}^{\infty} (\text{sign}(\mu + \omega \sqrt{|n|}) + \text{sign}(\mu - \omega \sqrt{|n|})) + \frac{|eB|}{4\pi} \text{sign}(\mu).
\] (84)

Before entering into the details of equation (84), this equation tells us that the effect of an external magnetic field is to induce a nonzero average charge in the system: \( \langle n \rangle = \sigma_H B \) (we will call \( \sigma_H \) the Hall conductivity). By noting that the average particle number \( \langle n \rangle \) is the
temporal component of the gauge-invariant electronic current, \( \langle n \rangle \equiv J^0 \), and that the magnetic field \( B \) can be written in terms of the vector potential \( A_\mu \) as \( B = B_z = \partial_2 A_1 - \partial_1 A_2 \), we can write the relation \( \langle n \rangle = \sigma_H B \) in the following gauge-invariant way:

\[
J^\mu = \sigma_H \epsilon^{\mu\nu\rho} \partial_\nu A_\rho.
\]  
(85)

Equation (85) can be read as the equation of motion for the electromagnetic gauge field \( A_\mu \) deriving from the well-known Chern–Simons action:

\[
S_{CS} = \frac{\sigma_H}{4\pi} \int d^3 x [\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - J^\mu A_\mu].
\]  
(86)

So from the knowledge of the Landau levels, we have derived the CS action as the effective Lagrangian describing the physics of the integer QHE. The behavior of the Hall conductivity \( \sigma_H \) can be obtained by inspection of equation (84) and it is plotted in figure 6. The first two terms on the right-hand side of (84) have the form of a staircase changing its value by integers when the chemical potential crosses the energy of each Landau level. This is exactly the behavior found in the standard two-dimensional electron gas, plotted also in figure 6. The difference in the case of graphene lies in the last term proportional to sign(\( \mu \)), which adds a remarkable change of one-half to the entire Hall conductivity. It was shown by Thouless and coworkers [159] that \( \sigma_H \) must take integer values. The last term in (84) actually does not contradict the result of Thouless et al just because we have two Dirac points in the band structure of graphene and we have to multiply the entire result by two Dirac species and by two spin orientations, so the actual value taken by the Hall conductivity is (\( N \in \mathbb{Z} \))

\[
\sigma_H = \frac{4e^2}{h} \left( N + \frac{1}{2} \right).
\]  
(87)
4.2. The Callan–Harvey mechanism and the edge states in the QHE

We will next comment on two aspects concerning the effective Chern–Simons Lagrangian: one quite general, which relates the gauge structure of action (86) to the existence of chiral states when boundaries are considered, and the second, more specific to graphene, regarding the appearance of the parity anomaly when a mass term is added to Hamiltonian (76).

The first of the two aspects is usually known in the high-energy community as the Callan–Harvey mechanism [160]. It is easy to check that, when dealing with ideally infinite systems, action (86) is gauge invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda,$$

where \(\Lambda\) is some well-behaved function in the entire space. However, when a bounded region is considered, (86) is no longer gauge invariant under the previous gauge transformation and an extra surface term appears of the form

$$\delta S_{CS} = \frac{\sigma_H}{4\pi} \int_{\partial \Omega} d^2x \Lambda(x) \epsilon^{\mu\nu\rho} \hat{n}_\mu(x) \partial_\nu A_\rho,$$

(88)

where \(\hat{n}_\mu(x)\) is the unit vector perpendicular to the boundary \(\partial \Omega\) at each point. Without loss of generality, we can choose the boundary to lie on the line \(y = 0\) and thus the vector \(\hat{n}_\mu\) to point in the \((0, 0, 1)\) direction. Then, the second term on the right-hand side of (88) becomes

$$\delta S = \frac{\sigma_H}{4\pi} \int d^2x \Lambda(x) \epsilon^{ij} \partial_i A_j = \frac{\sigma_H}{4\pi} \int d^3x \Lambda(x) \epsilon^{ij} F_{ij}.$$

(89)

For a generic regular function \(\Lambda\) inside the bounded region \(\Omega\), the right-hand side of (89) is nothing but the expression for the \((1 + 1)\) chiral anomaly, so through the noninvariance of the Chern–Simons action in a region with boundaries, we reach the interesting conclusion that we can recover the gauge invariance of the system in a bounded region if we add to the system a chiral fermion whose anomalous contribution to the action exactly cancels the noninvariant piece of (89). So through this mechanism, we can rationalize the existence of chiral edge states at the boundaries of a system in the quantum Hall regime. Because \(\sigma_H\) is an integer, we see that the value of the quantized Hall conductance tells us the number of chiral edge states we have in our system. Let us stress that this argument lies only on the gauge noninvariance of the Chern–Simons action and is applicable to any two-dimensional fermionic system, be it graphene or the standard two-dimensional electron gas.

4.3. Parity anomaly and QHE

In QED(2+1), an interesting phenomenon emerges when a single species of massive Dirac fermions in the continuum is considered [161]. The action for such a system coupled to an Abelian gauge field \(A_\mu\) is (in units \(\hbar = c = 1\))

$$S = \int d^3x \bar{\psi} \left[ (\gamma^\mu (i\partial_\mu + eA_\mu) - m) \right] \psi.$$

(90)

The mass term in action (90) breaks time-reversal symmetry and parity and allows the emergence of a Chern–Simons term in the one-loop effective action for the field \(A_\mu\).

The one-loop contribution to the polarization function is given by the bubble diagram in figure 4(c) that, in terms of the Feynman propagator for the Dirac field, takes the form

$$\Pi^\mu_{\nu}(k) = e^2 \text{Tr} \int \frac{d^3q}{8\pi^3} \gamma^\mu \gamma^\nu \frac{1}{q^\rho (q_\sigma + k_\rho) - m} \gamma^\sigma q_\sigma - m.$$

(91)

Instead of evaluating (91) in its full glory, we will take a shortcut to see how the Chern–Simons term emerges. We see from (86) that the Chern–Simons term is linear in derivatives, so in terms of momenta, it will be linear in \(k\). We can then expand (91) in powers of \(k\) and calculate...
the linear term. Differentiating (91) w.r.t. \( k \) and setting \( k = 0 \), we see that the coefficient is given by the following integral:

\[
e^{2\text{Tr}} \int \frac{d^3q}{8\pi^3} \gamma^\mu q_\sigma - m \gamma^\sigma q_\mu - m \gamma^\rho q_\sigma - m.
\] (92)

Among the terms one obtains after performing the trace over the \( \gamma \) matrices in (92), there is one proportional to \( m^3 \epsilon_{\mu\nu\rho} \) that has the form

\[
\epsilon_{\mu\nu\rho} e^{2m^3} \int \frac{d^3p}{8\pi^3} \frac{1}{(p^2 - m^2)^3}.
\] (93)

This integral can be easily evaluated using the fact that it depends on \( m^2 \) and it is insensitive to the sign of the mass (also, \( m^3 = m^2 m = |m|^2 m \)). The final result is that the one-loop effective action for the gauge field contains a Chern–Simons term of the form

\[
L_{\text{cs}} = \frac{e^2}{4\pi} \frac{m}{|m|} \epsilon_{\mu\nu\rho} A_\mu k_\nu A_\rho.
\] (94)

We must stress that this expression has been calculated by expanding the polarization bubble to first order in \( k \), so it is not exact. Nevertheless, it can be proven that the full dependence of this term with \( k \) is on the form \( f(k)\epsilon_{\mu\nu\rho} k_\nu \), where \( f(k) \) is a function only on the modulus of \( k \), so the algebraic structure of the Chern–Simons term is maintained but with a more subtle dependence on the momentum. Moreover, it was proven by Coleman and Hill [162] that the Chern–Simons part of the polarization operator given by the one-loop contribution (91) does not get renormalized by higher loop contributions, so the coefficient depending on \( m \) in (94) is exact in this model.

A natural question is if we can understand all the phenomenology of the QHE coming from the Chern–Simons Lagrangian (86) by just simply studying QED. For this, let us set aside this result for a while and go back to the original problem of graphene in the single-cone and low-energy approximation under the effect of an external magnetic field, but this time adding to the original Hamiltonian (76) a mass term. We can play the same game as we did before, that is, calculate the new Landau level spectrum, project the effective action onto this Landau level basis for a finite chemical potential if necessary, calculate the average particle number as the derivative of the effective Lagrangian with respect to the chemical potential and directly read the expression for the Hall conductivity.

In this case, the Landau level energies are given by the following expression:

\[
E_n = \pm \frac{e}{l_B} \sqrt{2|n| + m^2},
\] (95)

but now this relation holds for \( n \neq 0 \) only. From the pair of solutions that naively would lead the value \( n = 0 \), only one of them is actually normalizable (the sign of \( B \) determines which solution is normalizable). This means that, as in the case of \( m = 0 \), there are pairs of Landau levels related by \( E_{-n} = -E_n \) except for the case \( n = 0 \), which gives an unpaired state. In this case, the average number of particles \( \langle n \rangle \) is given by the rather cumbersome expression [158]

\[
\langle n \rangle = -\frac{eB}{4\pi} \frac{m}{|m|} \theta(|m| - \mu) + \frac{|eB|}{2\pi} I(\mu, m, B) + \frac{1}{2} \theta(\mu - |m|),
\] (96)

where \( I \) is an integer-valued function of the absolute value of the mass, the field and the chemical potential. The limit \( \mu \to 0 \) in this expression gives the contribution to the average number of particles coming from the unpaired Landau level:

\[
\langle n \rangle = -\frac{eB}{4\pi} \frac{m}{|m|},
\] (97)
which of course can be written in a gauge-invariant phason through a Chern–Simons Lagrangian (86) with Hall conductivity $\sigma_H$ given by

$$\sigma_H = \frac{e^2 m}{4\pi |m|}.$$  

(98)

At this time, it can be shown that the gauge-invariant current is parity odd (recall the difference with the contribution of the zero mode in the massless case: there the magnetic field entered through its absolute value. This is another way of seeing that there is no parity anomaly in the massless limit, although the time-reversal symmetry is already broken and a genuine Chern–Simons term appears) [158, 163].

Despite the superficial resemblance between the two theories (86) and (94), few differences are apparent. In the case of QHE, both in graphene and in the standard two-dimensional electron gas, the coefficient in front of the Chern–Simons action changes its value when the chemical potential eventually crosses a new Landau level, while in QED$_{2+1}$ the Hall conductivity is fixed. Also, in the case of the QHE, the expression for the Chern–Simons Lagrangian is exact, while in QED$_{2+1}$, as we said, the momentum dependence in expression (94) is valid up to first order in $k$ [162]. The third and definite difference is that in QED$_{2+1}$, the Chern–Simons term is absent if we have pairs of Dirac species related by time-reversal symmetry (or parity). This is what happens in the case of graphene with a mass, where we do have two species of Dirac fermions related by time-reversal symmetry, so in the absence of any time-reversal breaking perturbation, the two copies would lead to corresponding Chern–Simons copies, but with opposite sign, giving a vanishing Hall conductivity. In the presence of the external magnetic field, both species contribute with the same coefficient to the Hall conductivity, so we have a nonzero $\sigma_H$.

In the next section, we will see that all the systems showing a Chern–Simons term in their effective action have a non-trivial topology coming from the Berry phase. We will also see that playing with the Berry phase [164], we can find time-reversal symmetric systems with a topological term in their effective actions similar to some extent to the Chern–Simons term.

4.4. The Berry phase and the integer QHE

We have seen in the previous section that the low-energy effective action governing the physics of an electronic system (both graphene and the standard two-dimensional electron gas) under the effect of an external homogeneous magnetic field is the Chern–Simons Lagrangian. We obtained this term under the assumption that the system was in the quantum Hall regime, that is, when the magnetic field was strong enough to induce the presence of Landau levels. The quantization of the Hall conductivity seemed to follow from the existence of such quantized levels. Although this picture is true, the presence of Landau levels is not the ultimate reason for the quantized nature of the Hall conductivity (although extremely hard to find in condensed-matter physics, the example of a single species of planar Dirac fermions is the proof of that). Also, in the absence of interactions, it is hard to find crystalline solids where time-inversion symmetry is broken so we cannot expect to find any term similar to (94) in any way by looking for situations where the one-particle spectrum is described by Landau levels, that is, having flat bands in the spectrum does not necessarily mean that those bands will display a Chern–Simons-like term in the effective Lagrangian.

In order to understand the ultimate reason of the quantization of the Hall conductivity, and envisage any possibility of finding similar physics in condensed-matter systems, we have to go to the lattice and analyze the same problem, but this time not relying on a continuum approximation. We will see that the concept of the Berry phase is behind this quantization, that
it has a topological nature and that this is the element that eventually will allow us to extend these concepts to time-reversal-invariant systems.

4.4.1. Berry phase and the quantization of the monopole charge. Let us begin this section with a brief review of the concept of the Berry phase. We will follow a rather pedagogical way and introduce the concept of geometrical phase following the line of thinking from Berry’s original work [164].

Let us consider a quantum one-particle system described by the Hamiltonian $H(s)$ depending on the set of parameters $s = (s_1, \ldots, s_n)$. We will assume that the Hamiltonian is time dependent through the parameters $s(t)$. If the parameters $s$ change slowly with time, the Schrödinger equation can be solved in the adiabatic limit. If the system of eigenstates in principle is not degenerate, at any given $t$, we have

$$|n(t)\rangle = e^{-i\int_0^t dt E_n(t) + i\gamma_n(t)}|n(0)\rangle. \quad (99)$$

The first term in the exponential is the standard dynamical phase, while the second term is a yet undetermined phase allowed by the adiabatic theorem. If (99) is a solution of the Schrödinger equation $H(s(t))|n(t)\rangle = E_n(t)|n(t)\rangle$, then the phase factor $\gamma_n$ can be calculated as

$$\gamma_n(t) = i \int_0^t dt \langle n(\tau) | \frac{\partial}{\partial \tau} |n(\tau)\rangle, \quad (100)$$

or, in terms of the parameters $s$,

$$\gamma_n(t) = i \int_{s(0)}^{s(t)} ds \langle n(s) | \nabla_s | n(s) \rangle = i \int_{s(0)}^{s(t)} ds A_n, \quad (101)$$

with the obvious definition $A_n = \langle n | \nabla_s | n \rangle$. The vector $A_n$ is usually termed the Berry connection.

The interesting situation arises when we consider the particular case of a system with a multidimensional parameter set $s$ and a time evolution following a closed path $\Gamma$ in parameter space such that $s(T) = s(0)$. In this situation, the Berry phase $\gamma_n$ reads (after using the Stokes theorem)

$$\gamma_n = i \oint_\Gamma ds A_n = i \int_S dS \nabla_s \times A_n = i \int_S dS \cdot B_n. \quad (102)$$

In the absence of any singularity in the parameter space or in the vector field $A_n(s)$, any loop $\Gamma$ can be continuously deformed to zero and there is no Berry phase effect in the wavefunction. The situation changes when the parameter space is itself non-trivial or the field $A_n$ has some singularities. Let us note that although we have made use of the adiabatic hypothesis, it is actually not needed since the time $t$ (or $\tau$) disappears from expressions (100)–(102) and the final expressions are written in terms of geometrical quantities defined in the parameter space.

We can follow the original work of Berry and transform (102) a little bit more:

$$\gamma_n = i \int dS \nabla_s \times \langle n | \nabla_s n \rangle = i \int dS \langle \nabla_s n | \cdot | \nabla_s n \rangle$$

$$= i \int dS \sum_{n \neq m} \langle \nabla_s n | m \rangle \times \langle m | \nabla_s n \rangle. \quad (103)$$

If we consider the situation where the eigenvalues $E_n$ are not degenerate, for $n \neq m$, we have $(E_n - E_m) \langle m | \nabla_s H | n \rangle = \langle m | \nabla_s H | n \rangle$ and we find the important final relation

$$\gamma_n = -i \int_S dS \sum_{n \neq m} \frac{\langle n | \nabla_s H | m \rangle \times \langle m | \nabla_s H | n \rangle}{(E_n - E_m)^2}. \quad (104)$$
Although in expression (104) we have written the Berry phase $\gamma_n$ making use of the non-degeneracy of the eigenstates of the system, the previous expressions can be used to calculate this phase in the situations when degeneracies in the parameter space exist. We will use the low-energy Hamiltonian in graphene as a toy model to show that the Dirac quantization of the monopole charge is actually a topological quantization due to the existence of a non-trivial Berry phase in that system.

This time we will take into account both species of massless Dirac fermions in the low-energy sector of the spectrum in graphene. Our parameter space here will be the momentum space. As described in previous sections, the effective Hamiltonian around the Fermi points $K$ and $K'$ can be written as

$$H_{K,K'} = v(\tau_3 \sigma_y k_x + 1_\tau \sigma_y k_y).$$

The eigenvalues around both Fermi points are $E(k) = \pm v |k|$, and the eigenstates are collectively written as

$$\Psi_{s,\tau} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \tau e^{i\theta_k} \end{array} \right),$$

where $s = \pm 1$ and $\tau = \pm 1$ stand for the band index and Fermi points, respectively, and $\theta_k$ is the angle defined by the wave vector $k$ with the horizontal axis. The Berry connection in (101) is easily calculated to be

$$A_{s,\tau}^i(k) = \frac{\tau}{2 |k|^2} (e^{i/k}).$$

The magnetic field associated with the Berry connection is

$$B_{s,\tau}^i = \frac{\tau}{2} g^{ij}(k),$$

which corresponds to the magnetic field of a magnetic monopole of charge $\pm 1/2$. In this particular case, the total magnetic charge is zero as dictated by the Nielsen–Ninomiya theorem [165]. This case exemplifies how a non-trivial structure in the wavefunctions leads to a vector field (Berry curvature) which is singular at some points of the parameter space, and the total integral of the curvature associated with this singular connection over a closed surface (in this case, any two disjoint spheres containing separately the points $K$ and $K'$) leads to a quantized value.

4.4.2. QHE on the lattice and the magnetic Brillouin zone. In the previous sections, we have solved the Schrödinger equation for an electronic system in the presence of a magnetic field in the continuous approximation and we have computed the value for the Hall conductivity $\sigma_H$. We have seen that the presence of a discrete spectrum (Landau levels) leads to the quantization of $\sigma_H$. One might think that the quantized character of the Hall conductivity is due to the discreteness of the spectrum. In this section, we will see that this is not true and, in fact, one needs to understand the problem on the lattice to fully grasp the topological reasons behind the quantization of the Hall conductance.

Let us consider, for this time, fermions defined on the lattice whose dynamics under the effect of a magnetic field is described by a nearest-neighbor TB Hamiltonian. In what concerns the basics, there is no difference if we study the problem in the square or in the honeycomb lattice [166]. We will follow the original work of Hofstadter [167] and use the square lattice. The TB Hamiltonian reads

$$H = \sum_{(i,j)} \ell(R_i, R_j) C^+ (R_i) C(R_j),$$

(109)
where \( \mathbf{R}_i \) labels the sites of the square lattice, and the four nearest neighbors are \( \mathbf{R}_i = (\mathbf{R}_i \pm \alpha \mathbf{e}_x, \mathbf{R}_i \pm \alpha \mathbf{e}_y) \). The presence of the magnetic field will be accounted for by the standard Peierls substitution into the hopping integral \( t(\mathbf{R}_i, \mathbf{R}_j) \):

\[
t(\mathbf{R}_i, \mathbf{R}_j) = t e^{i \theta_{k_i} A(r) \cdot d r}.
\]

In the Landau gauge \( A = B(0, x, 0) \), the phase of the hopping integral is just \( eBax \), where \( x = am \) labels the \( x \)-component of the position \( \mathbf{R}_m \) and \( a \) is the lattice spacing. Due to the translational invariance along the \( y \) direction, we can rewrite the Hamiltonian \( H \) as

\[
H = t \sum_m \sum_{k_y} C_m^{+}(k_y)C_{m+1}(k_y) + C_m(k_y)C_{m-1}(k_y) + 2 \cos(2\pi \phi m + k_x a)C_m^+(k_y)C_m(k_y),
\]

where \( \phi \) is the magnetic flux within each plaquette. When this magnetic flux per plaquette is a rational number, \( \phi = p/q \), the cosine term in (111) is invariant under the change \( m \rightarrow m + q \) and so is the Hamiltonian. The Bloch theorem tells us that \( C_{m+q} = e^{i \theta m q} C_m \) with \( q \) fixed by the magnetic flux. Our Schrödinger equation is a \( q \times q \) matrix problem, with \( q \) eigenvalues \( E = E_j(k) \) with \( j = 1, ..., q \).

We can fix the periodicity of \( E_j(k) \) by noting that the points \( k_x \) and \( k_x + \frac{2\pi}{m} \) are equivalent in (111) so \( 0 < k_x < \frac{2\pi}{m}, 0 < k_x < \frac{2\pi}{m} \). We can define a Brillouin zone, which has the form of a torus (that the Brillouin zone is a torus and thus a closed surface is one of the key ingredients for the quantization of the Hall conductivity, as we will see). The important message here is that although the presence of an external magnetic field breaks the original translational symmetry of the system at \( B = 0 \), we still have a reduced translational symmetry, which can be seen as if the original Brillouin zone were split up into \( q \) copies. Also we have enlarged the number of bands and the new eigenstates are eigenvectors of \( q \) components.

Instead of studying the consequences of this effective folding by the general properties of the solutions of (111), for our purposes it is enough to analyze two specific simple examples for two concrete values of the pair \( (p, q) \).

Let us start with the pair \( (p, q) = (1, 2) \). In this case, the Schrödinger equation is a \( 2 \times 2 \) matrix equation, which takes the particularly simple form

\[
H = 2t \begin{pmatrix} -\cos(ak_x) & \cos(ak_y) \\ \cos(ak_x) & \cos(ak_y) \end{pmatrix}.
\]

It is worth mentioning that for the situation with half of the flux per plaquette (known as the ‘pi-flux phase’), the system is actually time-reversal invariant. As we said before, there is a periodicity in \( \phi \) and the system with \( \phi = \frac{1}{2} \) is equivalent to the system with \( \phi = -\frac{1}{2} \). Nevertheless, the example \( q = 2 \) is interesting because the spectrum takes the form \( E_{\pm}(k) = \pm 2t \sqrt{ \cos(ak_x)^2 + \cos(ak_y)^2 } \). The spectrum consists of two bands that are degenerate at the points \( (\pi/2, \pm \pi/2) \). Around these two points, Hamiltonian (112) takes the familiar form

\[
H = v \tau_x \sigma_y \delta k_x + v \tau_y \sigma_x \delta k_y,
\]

that is, the low-energy Hamiltonian of the system consists of two species of massless Dirac fermions, which are, interestingly, related by parity. This example illustrates a general property of (111) for even values of \( p \): the band structure consists of gapped bands except for the two central bands that are degenerate at \( q \) isolated points in the Brillouin zone (as in graphene, \( \tau_z \) labels the two Dirac species). Around those points, the low-energy system can be described by \( q \) species of massless Dirac fermions [168].
Figure 7. Band structure of the pi-flux phase corresponding to the value \( q = 2 \) (left) and \( q = 3 \) (right).

For odd values of \( q \), the bands are globally separated by gaps in the entire Brillouin zone. We will exemplify this situation with the values \((p, q) = (1, 3)\). The Hamiltonian is the following \( 3 \times 3 \) matrix:

\[
H = t \begin{pmatrix}
\cos(ak_y + 2\pi/3) & 1 & e^{3iak_x} \\
1 & \cos(ak_y + 4\pi/3) & 1 \\
e^{-3iak_x} & 1 & \cos(ak_y + 6\pi/3)
\end{pmatrix}.
\] (114)

The eigenvalues \( E(k) \) are the solutions of the secular equation

\[
E^3 - 6t^2E = 2t^3(\cos(3k_x) + \cos(3k_y))
\]

and they are depicted in figure 7 (right) \([168, 169]\).

These two examples illustrate what happens for electrons living on a lattice under the influence of an external homogeneous magnetic field. The original Brillouin zone is split into \( q \) copies of a new magnetic Brillouin zone, which is still a torus. Also, the new wavefunctions become multicomponent Bloch functions \(|u_k\rangle\). This fact seems to be rather paradoxical if we again compare with the problem of Landau levels. In the previous sections, we found that the spectrum of electrons in two dimensions under the effect of an external magnetic field consisted of discrete Landau levels, and, due to the explicit dependence of the vector potential \( A \) with the spatial coordinates, the translational invariance was lost. A deeper look at the problem will tell us that this is not the case. Effectively, the physical momenta \( \Pi \) satisfy \([\Pi_x, \Pi_y] = -ieB\), so two elements of the translation group do not commute in general. If \( T(R) = e^{i\Phi R} \), then \([167]\)

\[
T(R_1)T(R_2) = T(R_2)T(R_1)e^{ieB(R_1 \times R_2)}.
\] (115)

They do commute, however, when the phase \( \Phi = eB \cdot (R_1 \times R_2) \) is an integer number, that is, when the flux passing through the area defined by \( R_1 \times R_2 \) is an integer. In terms of the original primitive lattice vectors written in the original lattice basis, \( R_1 = pa \) and \( R_2 = qb \),
this condition means that the flux per unit cell must be a rational number, \( pq\phi_{\text{plaq}} = n \), for some integer \( n \). This condition is precisely the one that we found at the beginning of this section, when we considered the eigenstates of (111). We then find that even when the original translational invariance is lost, we can still label the eigenstates within each Landau level in terms of a wave vector \( \mathbf{k} \) when the magnetic flux per unit cell is a rational number.

4.4.3. The Berry phase, Chern number and Hall conductivity. In this section, we will close the circle and see the explicit connection between the Hall conductivity and the Berry phase constructed through the Bloch states corresponding to the new magnetic subbands \( E_i(\mathbf{k}) \). Our tasks are, first, to find an expression that relates the Hall conductivity \( \sigma_H \) and the Berry phase \( \gamma_n \), and, second, to demonstrate that the Berry connection defined by the Bloch eigenstates of (111) is a singular vector field at some points of the Brillouin zone, and thus the total monopole charge enclosed by this surface is an integer.

We will employ linear response theory to find the desired relation between \( \sigma_H \) and \( \gamma_n \) [159] and assume that, together with the external homogeneous magnetic field \( \mathbf{B} \) applied perpendicular to the system, we apply a weak electric field \( \mathbf{E} \) pointing, say, in the \( x \) direction. The corresponding vector field thus takes the form \( A_0 = E_0x \) and \( \mathbf{A} = (0, Bx, 0) \). We shall assume for simplicity that the magnetic subbands are globally separated by a gap in the entire Brillouin zone, so \( E_i(\mathbf{k}) \neq E_j(\mathbf{k}) \) for every momentum \( \mathbf{k} \) and \( i \neq j \). Using the magnetic Bloch functions in the real space representation \( |\psi_n(\mathbf{r})\rangle = e^{ikr}|u_n(\mathbf{k})\rangle \) as the basis of unperturbed wavefunctions, simple perturbation theory gives the first correction for the perturbed wavefunctions:

\[
|u_n(\mathbf{k})\rangle^{(1)} = \langle u_m|E_j|\langle u_m(\mathbf{k})|u_m(\mathbf{k})\rangle \frac{\partial}{\partial k_j}|u_n(\mathbf{k})\rangle \bigg|_{k=0}.
\]

(116)

In single-particle band Hamiltonians, the velocity operator is defined as the gradient of the Hamiltonian, \( \mathbf{v}(\mathbf{k}) = \nabla_\mathbf{k} H(\mathbf{k}) \), and the averaged electronic current is just the averaged velocity times the electric charge: \( \langle \mathbf{j} \rangle = e \langle \mathbf{v} \rangle \). We then can use the expression for \( \mathbf{v}(\mathbf{k}) \) and the perturbed eigenstates (116) to calculate the first-order correction to the \( y \) component of the electronic current of the state \( n \) (as long as \( \langle v_y \rangle \) is a real function, we must include also the term given by the complex conjugate of (116):

\[
\langle v_{n,j}\rangle^{(1)} = \langle u_n|v_j|u_n\rangle + i e E_x \int \frac{d^2k}{4\pi^2} \sum_{n\neq m} \frac{\langle u_n|v_j|u_n\rangle \langle u_m|\partial_{k_j} u_n\rangle}{E_m - E_n} + \text{c.c.}
\]

(117)

We now make use here of the identity \((E_m - E_n)|n\rangle\langle k| = (|m\rangle\nabla_k|n\rangle)\) and plug it in (117) to obtain

\[
\langle v_{n,y}\rangle^{(1)} = \langle u_n|v_y|u_n\rangle + i e E_x \int \frac{d^2k}{4\pi^2} \sum_{n\neq m} \langle u_m|\partial_{k_y} u_n\rangle \langle u_m|\partial_{k_x} u_n\rangle - \langle u_m|\partial_{k_x} u_n\rangle \langle u_m|\partial_{k_y} u_n\rangle.
\]

(118)

We readily read in (118) a term similar to the one in the last part of expression (103), so the second term of (118) can be written in terms of a Berry connection [164],

\[
\langle v_{n,y}\rangle^{(1)} = \langle u_n|v_y|u_n\rangle + \frac{e E_x}{4\pi^2} \sum_n \int d^2k \nabla_k \times A_n(\mathbf{k}).
\]

(119)

If the Fermi level lies on a gap above the band \( E_n(\mathbf{k}) \), the first term on the right-hand side of (119) does not contribute to the total current. Because we have the relation \( \langle j^y \rangle = e \langle v^y \rangle \) and \( \langle j^y \rangle = \sigma_{xy} E^y \), we read from (119) the value of the Hall conductivity:

\[
\sigma_H = \sigma_{xy} = \frac{e^2}{4\pi^2} \sum_n \int d^2k \nabla_k \times A_n(\mathbf{k}) = \frac{e^2}{4\pi^2} \oint_{\Gamma} A_n(\mathbf{k}),
\]

(120)
where $\Gamma$ is the loop defined by the corners of the magnetic Brillouin zone. This is the desired relation to prove: the Hall contribution to the conductivity is the total flux passing through the Brillouin zone of a Berry curvature. When we discussed the quantization of the charge of a magnetic monopole in the context of graphene, we found that if the Berry connection was singular at some points inside some closed surface, the monopole charge was quantized. Here we must show that the Berry connection in (120) defined through the magnetic Bloch eigenstates $|\psi_n(r)\rangle$ is singular at some point within the Brillouin zone.

The proof that there are points on the magnetic Brillouin zone where the Berry connection is singular is rather involved and we will only sketch it, referring to the original work for details [170, 171].

We have stressed in the previous section that the magnetic Bloch wavefunctions that are solutions of (111) are no longer eigenstates of the translations of the original lattice, but they are of the new enlarged unit cell, which contains an integer value of magnetic flux passing through each new magnetic plaquette. If we consider a loop consisting in one of these magnetic plaquettes, the phase of the magnetic Bloch wavefunction $|\psi_k(r)\rangle = \rho_k(r)e^{i\phi_k(r)}$ will change by $2\pi p$: $\int \nabla \phi_k(r) = 2\pi p$. This can be read as an Aharonov–Bohm effect of a state encircling a region of the space where there is a nonzero magnetic flux. It means that the wavefunction phase has a vortex structure at some point of the magnetic unit cell in order to accommodate the nonzero phase in a closed loop. As long as the wavefunction is a non-singular function of its arguments, the wavefunction must be zero at the point where the vortex is in the magnetic Brillouin zone, as long as the phase $\phi_k(r)$ is a function of $k$. Because the Bloch wavefunction has zeros on the torus defined by the magnetic Brillouin zone, the Berry phase will have points where it is ill-defined.

Let us now assume that we can divide the magnetic Brillouin zone into two regions $RI$ and $RII$ separated by a loop $\gamma$, one of them containing the zero of the wavefunction. In the region that does not contain the zero, the phase of the wavefunction is well defined globally, but in the region that contains the zero, the phase might pick up any arbitrary value. Once chosen, we can always choose the phase continuously in a small neighborhood around the zero with the condition that the phase cannot be defined globally. When we compare the wavefunctions away from the zero, the two choices of phase must be consistent through a gauge change, that is, $\Psi_I = e^{if(k)}\Psi_{II}$, or, equivalently

$$A_{n,I}(k) = A_{n,II}(k) + \nabla_k f(k). \quad (121)$$

Using the Stokes theorem, we can apply the definition of Hall conductivity (120) and write each contribution for the two regions:

$$\sigma_H = \frac{ie^2}{4\pi^2} \int_{RI} d^2k \nabla_k \times A_{n,I} + \frac{e^2}{4\pi^2} \int_{RII} d^2k \nabla_k \times A_{n,II}. \quad (122)$$

and then, because $RI$ and $RII$ have opposite orientation,

$$\sigma_H = \frac{e^2}{4\pi^2} \int_\gamma (A_{n,I} - A_{n,II}) \, dk = \frac{e^2}{4\pi^2} \int_\gamma \nabla_k f(k) \, dk. \quad (123)$$

Expression (123) tells us that the Hall conductivity counts the number of times the function $f(k)$ winds around $2\pi$ as $k$ travels along the loop $\gamma$. Since this winding number does not change when the loop $\gamma$ is deformed without passing through the zero of the wavefunction, this winding number is a topological invariant, known as the first Chern number, and it takes integer values, which is the result we wanted to prove. We have written the Hall conductivity as the surface integral of a Berry curvature, which is nothing but the total Berry phase calculated along a loop corresponding to the ‘boundary’ of the Brillouin zone. In order to have a nonzero $\sigma_H$, the wavefunction $|\psi_n(k)\rangle$ must pick up a nonzero Berry phase when circulating along the
mentioned loop. If the wavefunctions have no Berry phase, or equivalently, the Stokes theorem applied to this situation tells us that if $A_n$ is non-singular and we perform the integration over a surface without boundaries, the integral is zero. This is what happens with most of the band insulators. However, there is an intermediate case, which is when the system has an even number of singular points in the Berry connection, by requirements of time-reversal symmetry, and also the winding numbers of those points have opposite values (we saw this in the case of the low-energy Hamiltonian of graphene). The total Berry phase is zero (and so does $\sigma_H$) but we are still able to find a modified topological quantity which takes into account the value of each singular point separately. We will analyze this situation in the next section.

Another interesting observation concerning (120) is that when in normal band insulators the Fermi level lies on the gap, the Bloch wavefunction of the ground state is made of linear combination of (maximally) localized Wannier functions, that is, the wavefunction of a state at the point $r_i$ decays at large distances as (or faster than) $|r - r_i|^{-2}$ [172]. It means that the Bloch wavefunction $|\psi(k)\rangle = e^{ik}\mu(k)$ in the momentum space is an analytic function of $k$. A nonvanishing Hall conductivity is equivalent to the fact that we cannot define globally the phase of the Bloch wavefunction, so it will not be analytic [173]. It thus implies that the ground state of a Hall system cannot be described in terms of maximally localized Wannier functions. This observation has interesting consequences in the modern geometrical theory of the electric polarization based on the Berry phase of maximally localized Wannier functions [174].

4.5. Topological insulating phases in graphene

In the previous sections, we have found a relation between the Hall conductivity $\sigma_H$ and the Berry phase for the QHE. We have also seen that despite the presence of a homogeneous magnetic field, we can still define a Hamiltonian which enjoys translational invariance (this translational invariance is not the same as that of the original system) and with the magnetic Bloch wavefunctions we can define a Berry connection, whose circulation around the perimeter of the Brillouin zone gives the quantized value of $\sigma_H$. However, at the beginning of the previous section, we worked out the QHE with a single species of massive Dirac fermion in two spatial dimensions finding in the two cases a Chern–Simons term in the effective electromagnetic field theory, with the Hall conductivity being the coefficient of the CS term. We will now see how we get the same conclusion for QED$_{(2+1)}$ by using the Berry phase (104). We will make extensive use of this calculation in the subsequent sections.

The Hamiltonian corresponding to action (90) takes the following form in the momentum space:

$$ H = v\sigma k + m\sigma_z. $$

(124)

($v$ is the Fermi velocity and $\sigma = (\sigma_x, \sigma_y)$). The eigenvalues of this Hamiltonian are, of course, $E_+ = -E_- = \sqrt{v^2|k|^2 + m^2} \equiv E_k$, with the corresponding eigenvectors

$$ |+\rangle = \sqrt{m + E_k \over 2E_k} \left( 1, {vk \over m + E_k} \right)^T, $$

$$ |-\rangle = \sqrt{m + E_k \over 2E_k} \left( -vk^* \over m + E_k, 1 \right)^T. $$

(125)
We have used the notations \( k = k_x + ik_y \) and \( k^* = k_x - ik_y \). Because we have two bands, the Berry curvature for the positive band takes the simple form

\[
B_{z,+} = i \sum_{nl} \frac{e^{i \theta_l} |+| \partial_{k_l} H (-)|-| \partial_{k_l} H (+)|}{4E_k^2}.
\]  

(126)

After some simple algebra, we obtain

\[
B_{z,+} = -\frac{m}{2} \left( \frac{m}{(\nu^2 |k|^2 + m^2)^{\frac{3}{2}}} \right).
\]  

(127)

From (126), it is easy to see that \( B_{z,+} = -B_{z,-} \). We then end up with the expected result (94) (as usual \( \hbar = 1 \)):

\[
\sigma_H \equiv e^2 \gamma_e = \frac{e^2 \nu^2}{2} \int \frac{d^2k}{4\pi^2} \frac{m}{(\nu^2 |k|^2 + m^2)^{\frac{3}{2}}} = \frac{e^2}{4\pi} \frac{m}{|m|}.
\]  

(128)

We have closed the circle and demonstrated that the common origin of the Hall conductance in the case of the QHE and QED\(_{2+1}\) is the non-trivial Berry phase in the momentum space acquired by the eigenstates of the system. To end this introduction, we can comment on the properties of \( B_z(k) \) under the effect of discrete symmetries, such as time-reversal symmetry and parity. If the system is time-reversal invariant, one can show that \( B_z(-k) = -B_z(k) \), while if the system is parity invariant, the Berry curvature satisfies \( B_z(-k) = B_z(k) \). These two relations imply that if the system is time-reversal invariant, the Berry connection does not need to be zero, but the integral of the Berry curvature, i.e. the Hall conductivity, vanishes. Also, if the system is both time-reversal invariant and parity invariant, \(-B_z(k) = B_z(-k) = B_z(k)\), so the Berry curvature is zero.

4.5.1. The Haldane model and the valley Hall effect. Now it is time to look for situations different from the already analyzed QHE but where the system displays similar physics. The condition of vanishing Hall conductivity for time-reversal-invariant systems might seem so strong that one might not expect the physics described by the Chern–Simons term in any time-invariant system. However, with the aid of graphene as an ideal laboratory to play with, we will see that this is actually not the case and we can still find time-reversal-invariant systems displaying non-trivial topological features. For the details and for an exhaustive list of references, we refer to the excellent reviews on the subject [11, 175, 176].

As we have seen, a necessary condition for having a nonvanishing \( \sigma_H \) is the appearance of mass terms in the low-energy Hamiltonian for graphene, so we have to consider the ways in which we can modify the lattice Hamiltonian describing the low-energy spectrum in graphene in order to find the proper masses. Leaving aside the real spin, we can add to (105) the following matrices: \( M_1 = 1_z \sigma_x, M_2 = \tau_z \sigma_z \), and the two matrices, \( M_{3,4} = \tau_x \sigma_z \). Physically, \( M_1 \) represents an atomic energy imbalance between the atoms belonging to the two sublattices in the honeycomb lattice. In graphene, it might physically arise due to a mismatch between the graphene sheet and the substrate lattice as in the boron nitride. \( M_{3,4} \) physically correspond to two degenerate realizations of the coupling between the electrons and a Kekulé distortion (a kind of optical phonon in the honeycomb lattice). The Kekulé distortion is a quite interesting coupling on its own right. In its vector form \( H_{\text{Kekulé}} = \Delta(r) \tau \sigma_z \), it constitutes a vector coupling with the electronic structure at low energies. This coupling is invariant under chiral rotations of the form \( U(\theta) = e^{i \theta / 2} \) if \( \Delta(r) \) rotates as a vector in two dimensions. If the coupling \( H_{\text{Kekulé}} \) is generated by any form of interaction, the particular choice of \( \Delta(r) \) implies a spontaneous CSB in graphene. At finite temperatures, increasing \( T \) would eventually lead to a chiral symmetry restoring by the appearance of defects in the vector field \( \Delta(r) \), through a Kosterlitz–Thouless
The matrices $M_{1,3,4}$ are time-reversal invariant, as long as the time-reversal operation acts in the following way on the TB Hamiltonian: $\Theta H(K) \Theta^{-1} = H'(-K)$, that is, it interchanges the two Dirac species and takes the complex conjugate. The only matrix that breaks time-reversal symmetry is $M_2$. Haldane [7] devised a TB Hamiltonian where a mass like $M_2$ appears in the low-energy sector by adding complex second neighbor hoppings with opposite value in the two sublattices and an unbalanced site energy between the two sublattices of the honeycomb lattice.

We will make extensive use of the calculation performed in the previous section and compute $\sigma_H$ for this model. The Haldane model in the low-energy sector reads

$$H = v_1 \tau_x \sigma_x + v_2 \tau_y \sigma_y + m \tau_z \sigma_z.$$  \hfill (129)

Because there is no mixing between the two Dirac species, we can compute the Berry phase by applying formula (126) to each species separately. For the first species, we obtain (127). The important part comes now. The eigenstates for the two species are related by

$$|K \sigma \rangle = \psi_1^{(1)}(\gamma_0 \sigma_0 + v_1 \gamma_1 \sigma_1 + v_2 \gamma_2 \sigma_2) + \psi_1^{(2)}(\gamma_0 \sigma_0 + v_1 \gamma_1 \sigma_1 + v_2 \gamma_2 \sigma_2) + \theta_1(\gamma_0 \sigma_0 + v_1 \gamma_1 \sigma_1 + v_2 \gamma_2 \sigma_2) \langle K \sigma |,$$

where $\theta_1$ is a solution for the other species, respectively. Now, in terms of the momentum variable, both species equally contribute to the Hall conductivity, and we obtain $\sigma_H = 1/2 \text{sign}(m)$. This is consistent with the fact that the system is not time-reversal invariant.

What if we consider now the same model, but with $M_1$ instead of $M_2$? This time the eigenstates for the two species satisfy $|2, + \rangle = \sigma_1 |1, + \rangle$ and $|2, - \rangle = -i \sigma_y |1, - \rangle$ while the velocity operators satisfy $v_{1,2}^{(1,2)} = \sigma_1 \nabla \sigma_1 H^{(1,2)}(k) \sigma_1$, where the superscript labels the two species. Taking into account the relations between the eigenstates and the velocity operators, one finds $B_+^{(1)}(k) = B_+^{(2)}(k)$. It means that after integrating over the momentum variable, both species equally contribute to the Hall conductivity, and we obtain $\sigma_H = 1/2 \text{sign}(m)$. This is consistent with the fact that the system is not time-reversal invariant.

It is interesting to note that, despite the total Berry curvature and $\sigma_H$ being zero, the contribution of each species to $B_{\gamma,+}$ is not. So the two species have a non-trivial topological structure separately represented by a nonzero Berry curvature, as long as there is no coupling between them. Then the question is: is there any physical observable depending on the separate Berry phase for each species? In order to address this question, we will make use of the Chern–Simons effective action formalism.

Let us denote the Hall conductivity of each species by $\sigma_K$ and $\sigma_K'$. We know that $\sigma_K + \sigma_K' = 0$, so $\sigma_K' = -\sigma_K \equiv -\sigma$. Let us postulate the existence of two independent $U(1)$ gauge fields, one being the usual electromagnetic field $A_\mu$ and a second one $V_\mu$. The gauge field $A_\mu$ will be coupled to the standard fermionic current $J_\mu^a$, while the vector $V_\mu$ will couple to a chiral current $J_\mu^a$, which couples with opposite sign to each Dirac species. The gauged Dirac Lagrangian now reads

$$\mathcal{L} = \bar{\psi}_k (\gamma_0 \omega - v_1 k - m) \psi_k - e J_\mu^a A_\mu - e J_\mu^a V_\mu,$$ \hfill (130)

with the obvious definition of the $\gamma$-matrices and $\tilde{\psi} = \psi^+ y^0 \equiv \psi^+ M_1$. We can play the game played previously in the context of the QED$_{(2+1)}$ and calculate the effective one-loop action for the two $U(1)$ gauge fields integrating out the fermion fields. Among all the terms in the effective action, there are two nonvanishing terms that have the structure of (pseudo) Chern–Simons terms:

$$S_{\text{pcs}} = \frac{1}{4\pi} (\sigma_K - \sigma_K') \int d^4 x e^{i\mu\nu} (V_\mu \partial_\nu A_\nu + A_\mu \partial_\nu V_\nu) + e J_\mu^a A_\mu + e J_\mu^a V_\mu,$$ \hfill (131)
where the couplings with the fermion currents have been added by hand. The average of the
quiral current is found by taking the functional derivative of (131) with respect to the chiral
field \( V_\mu \):

\[
\langle J_\mu \rangle \equiv \frac{\delta S_{cs}}{\delta V_\mu} = \frac{e}{\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho.
\]

(132)

This expression means that there is a nonzero transverse current reacting to an external
electromagnetic field, which is actually an imbalance between currents belonging to different
Dirac species. In order to make this statement more precise, let us employ the Callan–
Harvey mechanism discussed above. Let us take linear combinations of the fields
\( a_\mu = A_\mu + V_\mu \) and \( b_\mu = A_\mu - V_\mu \), so
\( A_\mu = \frac{1}{2} (a_\mu + b_\mu) \) and \( V_\mu = \frac{1}{2} (a_\mu - b_\mu) \), and we do
the same for the currents: \( J^\mu = J^\mu_K + J^\mu_K' \) and \( J^\mu = J^\mu_K - J^\mu_K' \). Writing (131) in terms of
\( a_\mu \) and \( b_\mu \), we have

\[
S_{pcs} = \frac{1}{4\pi} \int d^3 x \epsilon^{\mu\nu\rho} (a_\mu \partial_\nu a_\rho - b_\mu \partial_\nu b_\rho) + e J^\mu_K a_\mu + e J^\mu_K' b_\mu;
\]

(133)
we obtain two independent Chern–Simons terms with opposite signs. We note that the
corresponding Hall conductivity is the Berry phase for each single Dirac species. Now put the
system described by (133) in a finite region. Applying the Callan–Harvey mechanism, we find
that the system develops two counter-propagating chiral currents belonging to different Dirac
species. This is known in the context of graphene as the valley Hall effect.

In the case of the QHE, we found that the system developed chiral one-dimensional
propagating states at the boundaries in a number given by the value of \( \sigma_H \). Because the Hall
conductivity can have any integer value, we can have the situation of obtaining any integer
value of chiral states. In the case of QED\(_{2+1}\) and the valley Hall effect, this is not possible,
because \( \sigma_H \) can only take two values \( \pm 1 \) in appropriate units. This is another important
difference between the QHE and QED\(_{2+1}\) and time-reversal-invariant topological insulators:
the topological invariant in the two later cases is a \( \mathbb{Z}_2 \) invariant. We note by passing that in the
Haldane model, the topological invariant is also a \( \mathbb{Z}_2 \) invariant, and we have one chiral state
propagating along the same direction at the boundaries.

We can make a comment concerning the robustness of the edge states in these systems. In
the case of the QHE, all the edge states present in the system propagate in the same direction,
independently of their number. It means that those states cannot scatter backward, so they
are dissipationless and the Hall conductivity remains quantized even in disordered realistic
samples. The same happens in the Haldane model. In the case of the valley Hall effect, however,
we have pairs of counter-propagating edge states, but belonging to different Dirac species, so
in principle the edge states can backscatter if there is any perturbation or interaction coupling
the two Dirac species, so the system is not protected against all kinds of perturbations, as
it happens in the QHE. However, it turns out that the backscattering rate between species is
extremely small as can be seen in the case of impurities [177] and interactions [178]. We then
conclude that we can find the way to unravel the non-trivial topological structure of some time-
reversal invariants by using the Berry phase. Here we took the route of analyzing the system
in terms of an effective pseudo Chern–Simons theory, but, as with the QHE, we can study the
topological structure of the system by means of the behavior of the Bloch wavefunctions in
the Brillouin zone. We will not pursue this line here and we will refer to [11] for a complete
list of references.

### 4.5.2. Quantum spin Hall effect

In graphene, when the real spin degree of freedom is taken
into account, the possibilities of mass-like couplings notably increase. Among all of these
possible couplings, there is one that is of paramount importance, since it is the one that will be
generally responsible for the QHE in general time-reversal-invariant materials, different from graphene. It is the spin–orbit coupling. The spin–orbit coupling couples the spin momentum with the orbital motion of the electrons in the lattice. It takes the general form of $H_{so} = \lambda \mathbf{S} \cdot \mathbf{L}$, with $\lambda$ some constant depending on the microscopic details of the atomic potential. It is easy to see that the term $H_{so}$ respects time-reversal symmetry because both the operators $\mathbf{S}$ and $\mathbf{L}$ flip their directions under reversing the time. The low-energy effective spin–orbit coupling in graphene takes the form [179]

$$H_{so} = \Delta_{so}\sigma_z\tau_zs_z,$$

(134)

It is interesting to note that the matrix structure of (134) is similar to $M_2$ in the Haldane model. In the Haldane model, the mass does not respect time-reversal symmetry, but, because the spin changes its sign under time inversion, the product $M_2s_z = \sigma_z\tau_zs_z$ as a whole is time-reversal invariant. It is then clear that the low-energy Hamiltonian for graphene in addition to term (134) consists of two time-reversal related copies of the Haldane model. From the discussion of the Haldane model, we infer that term (134) opens a gap for the two spin components, and, because the effective field theory in the case of the Haldane model possesses a genuine $\mathbb{Z}_2$ Chern–Simons term, the low-energy effective field theory in the case of graphene with a spin–orbit term will have a term similar to (133), but this time each current does not correspond to different Dirac species, but they belong to different spin projection. It means that in a confined geometry, the spin–orbit term will induce at each boundary two counter-propagating edge states with opposite spin projection, so if one has a wide sample and is capable of manipulating the system at the scale of the edges, under the effect of an appropriate bias voltage, one will get a spin filtered current, which constitutes the holy grail in the field of spintronics.

Despite this appealing property, the spin–orbit constant $\Delta_{so}$ in graphene turns out to be extremely small. In units of temperature, $\Delta_{so}$ is estimated to be of the order of $0.01$ K, so any other effect in graphene is likely to have a bigger characteristic scale and be dominant compared with $H_{so}$. Also, in graphene there is another coupling that mixes spin and orbital movement. It is known as the extrinsic or Rashba spin–orbit coupling, which has the form $H_R = \lambda_R(\sigma \times p) \cdot \hat{e}_z$. In graphene, the Rashba coupling reads

$$H_R = \lambda_R(\tau_\sigma\tau_\sigma - 1\tau\sigma\tau_\sigma).$$

(135)

The Rashba term, depending on $s_x$ and $s_y$, competes with the spin–orbit term (134), and it can be found that the system remains to be an insulator if $\lambda_R < \Delta_{so}$, so while this condition is fulfilled, the spin Hall conductivity remains nonzero and the system is a topological insulator. We have to mention that in this case, although the spin Hall conductivity is nonzero, it does not generally take quantized values [180]:

$$\sigma_{sH} = \frac{e^2}{4\pi} \ln \left| \frac{\lambda_R + \Delta_{so}}{\lambda_R - \Delta_{so}} \right|.$$

(136)

It is interesting to note that in order to induce backscattering between the two spin filtered edge states at the boundary, we must have a perturbation that couples the two spin polarizations, that is, a time-reversal breaking perturbation. This is a manifestation of the fact that the edge states are protected against backscattering by time-reversal invariance.

4.5.3. Topological insulating phases in bilayer graphene. Hamiltonian (134) together with the continuum effective Dirac Hamiltonian for graphene is useful for describing the physics we are interested in in a simple way, but of course not all the two-dimensional topological insulators can be described at the lattice level by this Hamiltonian, although its effective field theory always contains a Chern–Simons term. We saw an example of this in the lattice
description of the QHE. Indeed, in the case of graphene the discussion made so far remains valid as long as the value of the gap is not too big compared with the bandwidth, and the Berry curvature concentrates around the Dirac points. For these situations where the Dirac approximation is not the correct one, one may take two routes: to construct a fairly good effective Hamiltonian for the electrons that captures the topological features of the system, or to develop a general theory for the $\mathbb{Z}_2$ topological invariant in time-reversal-invariant systems taking into account all the information contained in the Brillouin zone.

The second route is the subject of current research and several methods have been proposed to construct such a topological invariant. We will not review them here and instead refer to the literature \cite{8, 181–184}. Concerning the first route, we will use bilayer graphene as an example.

In the case of Bernal stacking bilayer graphene, the unit cell is made of two pairs of carbon atoms, each pair belonging to a different graphene layer. The simplest lattice Hamiltonian that shows non-trivial topological features is the one where only hopping amplitudes are allowed between the most adjacent atoms in the two layers. If we define the wavefunction of the system for the low-energy effective model around the $K$ point by $\Psi_k = (a_{1,k}, b_{1,k}, a_{2,k}, b_{2,k})^T$, with the amplitudes in the sublattices $a$ and $b$ being the components of the two layers, this simplest model reads $H = \sum_k \mathcal{H}_k \Psi_k^\dagger \Psi_k$ with

$$\mathcal{H}_k = \begin{pmatrix} \sigma k & t_\perp \sigma_+ \\ t_\perp \sigma_+ & -\sigma k \end{pmatrix},$$

(137)

where we have used the standard definition, $\sigma_k = (\sigma z \pm i \sigma y)/2$, and $t_\perp$ is the hopping amplitude between the sublattices $a_2$ and $b_1$. The effective model for the other $K$ point is found from (137) by substituting $k_x$ by $-k_x$. In what follows, we will use the following convention for the different Pauli matrices appearing in the model. The $\sigma$ matrices stand for sublattice, $\tau$ stand for the $K, K'$ points, $\mu$ stands for the layer index and $s$ refers to the spin quantum number. The system consists then of four bands around each $K$ point and are degenerate in spin, i.e. 16 bands for the whole effective low-energy model.

In the absence of any other coupling or perturbation and for energies much smaller than $t_\perp$, the model can be simplified further to an effective two-band model for each $K$ point. As described in section 2.2, this effective model for the amplitudes $(a_{2,k}, b_{1,k})^T$ is

$$\mathcal{H}_k^{\text{eff}} = \frac{v^2}{t_\perp} \sigma k \sigma, \sigma k = \frac{v^2}{t_\perp} \begin{pmatrix} 0 & (k_x - i k_y)^2 \\ (k_x + i k_y)^2 & 0 \end{pmatrix},$$

(138)

In some cases, model (138) is enough to analyze the topological structure of the system. This is the case, for instance, when an external homogeneous magnetic field is applied. The low-energy part of the Landau spectrum of the system can be derived directly from (138) to obtain $E_n = \pm \hbar v F \sqrt{n(n-1)}$ \cite{55}, that is, each valley and spin projection contributes with two zero modes instead of one, as it happened for monolayer graphene. This change can be experimentally seen in the value of the Hall conductivity $\sigma_H$ for bilayer graphene at zero carrier concentration, with the value of $\sigma_H$ being twice the value in monolayer graphene \cite{185}.

Another interesting effect is the application of electrostatic potential difference between the two layers. This effect is modeled by the term $V = V_0 \tau_1 \mu \tau_1 \sigma_0$ in (137) or $V = V_0 \sigma_0$ in the effective model (138) \cite{186}. By making use of (101) and the eigenstates of $\mathcal{H}_k + V$, it can be shown that the Hall conductivity of bilayer graphene in the presence of a voltage difference is zero as long as this voltage difference respects the time-reversal-inversion symmetry, but as with monolayer graphene, we can still have a valley Hall effect with the valley Hall conductivity being $\sigma_K = \frac{e^2}{2h} \text{sign}(V_0)$. This means that after applying a voltage difference between the layers, a pair of counterpropagating edge states polarized in valley appear at the edge of the sample. This is quite appealing because it is by far easier for experimentalists to apply an external voltage difference between the layers in bilayer graphene than to open
a gap of the type $M_1$ in monolayer graphene. Concerning spin, we expect to find a similar topological structure to that in monolayer graphene when the intrinsic and Rashba spin–orbit terms are considered. The most general term involving both effects in bilayer graphene can be found to be \[ H_{so} = \lambda_1 \tau_z s_x \sigma_z + \lambda_2 \mu_Z \tau_z s_x \sigma_z + \lambda_3 (\mu_Z \tau_z s_y \sigma_y - \mu_4 \tau_z s_y \sigma_x) + \lambda_4 (\mu_4 \tau_z s_x \sigma_y - \mu_4 \tau_z s_x \sigma_x). \tag{139} \]

In Hamiltonian (139), the term with $\lambda_1$ is the equivalent version of the spin–orbit term in monolayer graphene, while the term with $\lambda_2$ is proper from bilayer graphene. Both terms open non-trivial gaps in the bilayer spectrum and lead to a quantized value for the spin Hall coefficient, $\sigma_s = \frac{e^2}{2\pi} \text{sign}(\lambda_1 + \lambda_2)$. The terms associated with $\lambda_3$ and $\lambda_4$ are Rashba terms which destroy the quantization of the spin Hall response in a similar manner to what happens in monolayer graphene.

An interesting situation appears when the intrinsic spin–orbit terms and the $V$ term are considered simultaneously. The system displays both spin and valley Hall effects, where both Hall conductances are quantized,

$$\sigma_s = \frac{e^2}{2\pi} (\text{sign}(\lambda_1 + \lambda_2 + V_0) + \text{sign}(\lambda_1 + \lambda_2 - V_0)),$$

and

$$\sigma_K = \frac{e^2}{2\pi} (\text{sign}(\lambda_1 + \lambda_2 + V_0) + \text{sign}(V_0 - \lambda_1 - \lambda_2)).$$

As usually happens, the quantized values for the distinct Hall responses are associated with the fact that a gap is opened in the system. Of course, bilayer graphene is not an exception and the presence of a finite Fermi surface makes any Hall response to be nonquantized.

To conclude this section, we can comment on the fate of the edge states when a gated graphene bilayer is considered. As we have said, the system shows a nonzero valley Hall conductivity $\sigma_K$ so if we apply the Callan–Harvey mechanism to this situation, we will find two counterpropagating edge states (without taking into account the real spin) filtered in the valley number [188]. As in monolayer graphene or in the QHE, the topological number gives us only the number of propagating chiral states, but not their velocities, which depend on the microscopic details of the model. It might happen that both counterpropagating chiral states possess different velocities. This fact has important consequences for the transport properties, because the backscattering rates when disorder or interactions are considered strongly depend on these values [178].

5. Summary and future directions

One of the purposes of this review, besides presenting some QFT aspects of condensed matter in a language available to high-energy colleagues, was to explore the theoretical aspects of the problem that can still be developed in the near future. The giant explosion of graphene publications after its synthesis have explored all possible aspects of it and the latest activity is mostly centered on the applications, devices and exploring standard condensed-matter phenomena that are not very different in graphene than in a usual electron gas.

It is fair to say that most of the special properties of the material originate at the Fermi points and as such are difficult to access. New suspended samples allow for the measurement of the electronic properties within a range of $\rho \approx 10^8$ cm$^{-2}$ or $\epsilon_F \approx 1$ meV in single-layer graphene. The density dependence of the effective mass and Fermi velocity at low carrier concentrations showed an enhancement of the Fermi velocity of almost a factor of 3. This
effect is consistent with the expected renormalization of the Fermi velocity in the absence of screening, given graphene’s ‘fine-structure constant’ $\alpha = e^2/v_F \approx 2.3–2.5$. Similar estimates for $\alpha$ have been used to explain the observation of plasmaron satellites in samples with high carrier concentrations. This value of $\alpha$ puts single-layer graphene into the intermediate to strong coupling regime, although its flow toward zero takes the system to the weak coupling regime at low energies.

The issue of opening a gap in monolayer graphene is one of the few fundamental questions that still remain controversial in the topic. Some recent lattice gauge theory calculations indicate that the physical parameters of graphene can be at the edge of the symmetry broken region. Given the interest of the subject on the theoretical side and even more for the practical applications, a special effort has to be devoted to developing more sophisticated calculations in graphene. We must although warn that there is at present no experimental evidence favoring a gap opening from interactions in the intrinsic monolayer system that has been tested at incredibly low energies [67].

There are some QFT aspects of graphene of current interest that have been left from this review not because they are not interesting and worth pursuing, but for the lack of space/time, or expertise or because we consider that the field is not yet mature. A very interesting aspect that is the subject of great attention and will probably be developed in the near future is that of the ADS/CFT correspondence [189] which has been studied in the context of graphene in [190]. The search for charge fractionalization both for purely theoretical reasons and for potential application in the field of quantum computing has also used graphene as a simple example to show the main features [191]. The same can be said on the search for Majorana fermions, another growing subject in condensed matter [192] with a clear relation to its high-energy counterpart. We have also left aside the connections of graphene with general relativity that started when exploring the influence of the curvature of the samples on the electronic properties. In this respect, there have been very recent developments trying to go beyond the approach of QFT in curved background [193, 194].

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References

[1] Katsnelson M and Novoselov K 2007 Solid State Commun. 143 3
[2] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katsnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Nature 438 197
[3] Zhang Y, Tan Y W, Stormer H L and Kim P 2005 Nature 438 201
[4] Geim A K 2009 Science 324 1530
[5] Soldano C, Mahmood A and Dujardin E 2010 arXiv:1002.0370
[6] Semenoff G V 1984 Phys. Rev. Lett. 53 2449
[7] Haldane F D M 1988 Phys. Rev. Lett. 61 2015
[8] Kane C and Mele E 2005 Phys. Rev. Lett. 95 226801
[9] Kane C and Mele E 2005 Phys. Rev. Lett. 95 146802
[10] Bernevig B A, Hughes T L and Zhang S 2006 Science 314 1757
[11] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[12] Wilczek F 1987 Phys. Rev. Lett. 58 1799
[13] Li R, Wang J, Qi X-L and Zhang S-C 2010 Nature Phys. 6 284
[14] Niemi A J and Semenoff G W 1986 Phys. Rep. 135 99
[15] Hou C, Chamon C and Mudry C 2007 Phys. Rev. Lett. 98 186809
[16] Jackiw R and Pi S 2007 Phys. Rev. Lett. 98 266402
[17] González J, Guinea F and Vozmediano M A H 1992 Phys. Rev. Lett. 69 172
[18] de Juan F, Cortijo A and Vozmediano M A H 2007 Phys. Rev. B 76 165409
[19] Cortijo A and Vozmediano M A H 2007 Eur. Phys. Lett. 77 47002
[20] Cortijo A and Vozmediano M A H 2007 Nucl. Phys. B 763 293
[21] Vozmediano M A H, Katsnelson M I and Guinea F 2010 Phys. Rep. 496 109
[22] Castro Neto A H, Guinea F, Peres N R M, Novoselov K S and Geim A K 2009 Rev. Mod. Phys. 81 109
[23] Gusynin V P, Sharapov S G and Carbotte J P 2007 Int. J. Mod. Phys. B 21 4611
[24] Abegel C, Apalkov V, Berashevich J, Ziegler K and Chakraborty T 2010 Adv. Phys. 59 261
[25] Kotov V N, Uchoa B, Pereira V M, Castro Neto A H and Guinea F 2012 Rev. Mod. Phys. 84 1067
[26] Cooper D R et al 2011 arXiv:1110.6557
[27] Das Sarma S, Adam S, Hwang E H and Rossi E 2011 Rev. Mod. Phys. 83 407
[28] Hatsugai Y, Fukui T and Aoki H 2007 Eur. Phys. J. Spec. Top. 148 133
[29] Pachos J K 2009 Contemp. Phys. 50 375
[30] Fialkovsky I V and Vassilevich D V 2011 arXiv:1111.3017
[31] Allor D, Cohen T D and McGady D A 2008 Phys. Rev. D 78 096009
[32] Katsnelson M I and Vozlovik G E 2011 arXiv:1203.1578
[33] Kosterlitz J M and Thouless D J 1973 J. Phys. C: Solid State Phys. 6 1181
[34] Landau L 1957 Sov. Phys.—JETP 3 920
[35] Abrikosov A A, Gorkov L P and Dzyaloshinski I E 1975 Methods of Quantum Field Theory in Statistical Physics (New York: Dover)
[36] Polchinski J 1993 Effective field theory and the Fermi surface Proc. 1992 Theoretical Advanced Institute in Elementary Particle Physics ed J Harvey and J Polchinski (Singapore: World Scientific)
[37] Shankar R 1994 Rev. Mod. Phys. 66 129
[38] Wallace P R 1947 Phys. Rev. 71 622
[39] Slonczewski J C and Weiss P R 1958 Phys. Rev. 109 272
[40] Bjorken J D and Drell S D 1965 Relativistic Quantum Mechanics (New York: McGraw-Hill)
[41] Katsnelson M I, Novoselov K S and Geim A K 2006 Nature Phys. 2 620
[42] Dombey N and Calogeracos A 1999 Phys. Rep. 315 41
[43] Young A F and Kim P 2009 Nature Phys. 5 222
[44] Standl N, Haard B and Goldhaber-Gordon D 1999 Phys. Rev. Lett. 102 026807
[45] Winkler R, Zulicke U and Bolte J 2007 Phys. Rev. B 75 205314
[46] Rusin T M and Zawadzki W 2009 Phys. Rev. B 80 045416
[47] Shtov A V, Katsnelson M I and Levitov L S 2007 Phys. Rev. Lett. 98 236801
[48] Pereira V M, Nilsson J and Neto A H C 2007 Phys. Rev. Lett. 99 166802
[49] Schwinger J 1951 Phys. Rev. 82 664
[50] Allor D and Cohen T D 2008 Phys. Rev. D 78 096009
[51] Dora B and Moessner R 2010 Phys. Rev. B 81 165431
[52] Anderson P W 1958 Phys. Rev. 109 1492
[53] Castro E V, Novoselov K S, Morozov S V, Peres N R M, Lopes dos Santos J, Nilsson J, Guinea F, Geim A K and Neto A H C 2006 Phys. Rev. Lett. 99 216802
[54] McCann E and Fal’ko V I 2006 Phys. Rev. Lett. 96 086805
[55] Mañes, J L, Guinea F and Vozmediano M A H 2007 Phys. Rev. B 75 155424
[56] Volovik G E 2011 arXiv:1111.4627
[57] Gorbar E V, Gusynin V P and Miransky V 2002 Phys. Rev. D 64 105028
[58] Du X, Skachko I, Duerr F, Luican A and Andrei E 2009 Nature 462 192
[59] Bolotin K I, Ghahari F, Shulman M D, Stormer H L and Kim P 2009 Nature 462 196
[60] Bostwick A, Speck F, Seyller T, Horn K, Polini M, Asgari R, MacDonald A H and Rotenberg E 2010 Science 328 999
[61] Siegel D A, Park C, Hwang C, Deslippe J, Fedorov A, Louie S G and Lanzara A 2011 Proc. Natl Acad. Sci. 45 1081365
[62] Luican A, Li G and Andrei E 2011 Phys. Rev. B 83 041405
[63] Decker R, Wang Y, Brar V W, Regan W, Tsai H, Wu Q, Gannett W, Zettl A and Crommie M F 2011 Nano Lett. 11 4631
[64] Li Z Q, Henriksenand E A, Jiang Z, Ha Z, Martin M C, Kim P, Stormer H L and Basov D N 2008 Nature Phys. 4 532
[66] Peres N M R 2010 Rev. Mod. Phys. 82 2673
[67] Elias D C et al 2011 Nature Phys. 7 701
[68] Vozmediano M A H 2011 Phil. Trans. R. Soc. A 369 2625
[69] Nash C 1978 Relativistic Quantum Fields (New York: Academic)
[70] Collins J 1984 Renormalization (Cambridge: Cambridge University Press)
[71] Mitra I, Ratabolea R and Sharatchandraa H 2005 Phys. Lett. B 611 289
[72] González J, Guinea F and Vozmediano M A H 2002 Phys. Rev. Lett. 89 166401
[73] de Juan F, Grushin A G and Vozmediano M A H 2010 Phys. Rev. B 82 125409
[74] Droniou R, Chuang K C, Nicholas R J, Novoselov K and Geim A 2007 Phys. Rev. B 76 081406
[75] Liang W, Bockrath M, Bozovic D, Hafner J H, Tinkham M and Park H 2001 Nature 411 665
[76] Reed J P, Uchoa B, Joe Y I, Gan Y, Casa D, Fradkin E and Abamonte P 2010 Science 330 805
[77] Ando T 2006 J. Phys. Japan 75 074716
[78] Wehling T O, Sasoglu E, Friedrich C, Lichtenstein A I, Katsnelson M I and Blugel S 2011 Phys. Rev. Lett. 106 236805
[79] Hwang E H and Das Sarma S 2007 Phys. Rev. B 75 205418
[80] van Schilfgaarde M and Katsnelson M I 2011 Phys. Rev. B 83 081409
[81] ’t Hooft G 1974 Nucl. Phys. B 72 461
[82] González J, Guinea F and Vozmediano M A H 1996 Phys. Rev. Lett. 77 3589
[83] González J, Guinea F and Vozmediano M A H 1999 Phys. Rev. B 59 2474R
[84] Foster M S and Aleiner I L 2008 Phys. Rev. B 77 195413
[85] Drut J and Lahde T A 2009 Phys. Rev. B 79 165425
[86] Gamayun O V, Gorbar E V and Gusynin V P 2010 Phys. Rev. B 81 075429
[87] González J, Guinea F and Vozmediano M A H 1994 Nucl. Phys. B 424 [FS] 595
[88] Nair R R, Blake P, Grigorenko A N, Novoselov K S, Booth T J, Stauber T, Peres N M R and Geim A 2008 Science 320 1308
[89] Geim A K 2007 Nature Mater. 6 183
[90] Mishchenko E G 2008 Europhys. Lett. 83 17005
[91] Grushin A G, Valenzuela B and Vozmediano M A H 2009 Phys. Rev. B 80 155417
[92] Juricic V, Vafek O and Herbut I F 2010 Phys. Rev. B 82 235402
[93] Zhou S, Qwen G H, Graf J, Fedorov A, Spataru C, Diehl R, Lee D H, Louie S G and Lanzara A 2006 Nature Phys. 2 595
[94] Jiang Z, Hemmiksen E A, Tung L C, Wang Y J, Schwartz M E, Han M, Kim P and Stormer H L 2007 Phys. Rev. Lett. 98 197403
[95] Li G, Luican A and Andrei E 2009 Phys. Rev. Lett. 102 176804
[96] Zhou S, Gweon G H, Graf J, Fedorov A, Spataru C, Diehl R, Kopelevich Y, Lee D H, Louie S G and Lanzara A 2006 Nature Phys. 2 595
[97] Li G, Luican A and Andrei E 2009 Phys. Rev. Lett. 102 176804
[98] Appelquist T, Bowick M J, Karabali D and Wijewardhana L C R 1986 Phys. Rev. D 33 3774
[99] Semenoff G 2012 Phys. Scr. T146 014016
[100] Guinea F 2010 New J. Phys. 12 083063
J. Phys. A: Math. Theor. 45 (2012) 383001

175] Xiao D, Chang M C and Niu Q 2010 Rev. Mod. Phys. 82 1959
176] Qi X L and Zhang S C 2011 Rev. Mod. Phys. 83 1057
177] Prada E, San-Jose P, Leon G, Fogler M M and Guinea F 2010 Phys. Rev. B 81 161402
178] Cortijo A, Oroszlany L and Schomerus H 2010 Phys. Rev. B 81 235422
179] Dresselhaus G and Dresselhaus M S 1965 Phys. Rev. 140 A401
180] Cortijo A, Grushin A G and Vozmediano M A H 2010 Phys. Rev. B 82 195438
181] Moore J E and Balents L 2007 Phys. Rev. B 75 121306
182] Fukui T and Hatsugai Y 2007 J. Phys. Soc. Japan 76 053702
183] Qi X L, Hughes T L and Zhang S C 2008 Phys. Rev. B 78 195424
184] Roy R 2009 Phys. Rev. B 79 195321
185] Novoselov K,McCann E, Morozov S V, Fal’ko V I, Katsnelson M I, Zeitler U, Jiang D, Schedin F and Geim A K 2006 Nature Phys. 2 177
186] McCann E 2006 Phys. Rev. B 74 161403
187] McCann E and Koshino M 2010 Phys. Rev. B 81 241409
188] Castro E V, Peres N M R, dos Santos J M B L, Neto A H C and Guinea F 2007 Phys. Rev. Lett. 100 026802
189] Sachdev S 2010 arXiv:1002.2947
190] Muller M, Schmalian J and Fritz L 2009 Phys. Rev. Lett 103 025301
191] Jackiw R 2007 AIP Conf. Proc. 939 6341
192] Wilczek F 2009 Nature Phys. 5 614
193] Iorio A and Lambiase G 2011 arXiv:1108.2340
194] Cvetic M and Gibbons G 2012 arXiv:1202.2938