Asymmetric wake from two closely located cylinders within the framework of one-dimensional model

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Abstract. Simple oscillatory model for wake hydrodynamics is constructed with the aim of physical modelling of complex object - turbulent plasma wake in glow discharge. A wake from not very closely located parallel cylinders assumed to be of the shape of the Von Karman streets pair, or a modified streets pair. Each street modelled by an oscillator of Van der Pole type. The specific features of our model are the follows: first - account for the nonlinear interaction of Von Karman streets in the area of their formation; second - explicit account for the oscillation frequency dependence on oscillation amplitude of the streets. Several global modes and intermittent regimes of the wake were revealed within the framework of the model. The main accent of this article is a description of two-frequency asymmetric mode of such two-cylinder wake.

1. Introduction

Low temperature plasma flows are of interest for fundamental science as well as in industrial applications such as fast-flow lasers, combustion ignition, electro-chemical reactors [1]. In this paper main interest concentrated around quasi-stationary and quasi-homogeneous flows with high level of heat and mass transfer [2, 3]. One opportunity to realize such flows is the use of a wake of cylinders row with relatively low pitch: \( L/D \leq 3 \); the latter condition is aimed to provide the flow quasi-homogeneity. However, it is well known that such turbulent wakes are very unstable flows, especially in the case of the cylinders close pitch \( 1.5 \leq L/D \leq 2 \) where intermittent regime is observable as a rule [4], when quasi-stationary flow is desirable. Ipso facto some optimization of the wake hydrodynamics is required. And, in parallel with experimental optimization, a theoretical one is useful. For the latter use some wake model preferably simple is desirable.

The object of our main interest is plasma wakes in non-self-sustained glow discharge, which is typical for constant power industrial lasers [5]. In our case using of a certain wake model is not only desirable but is necessary one since flow visualization is unavailable, only velocity signal is acceptable [6, 7]. Accordingly, simple relevant model of the wake hydrodynamic is constructed: [8, 9, 6]. It was found that the model contains five one-frequency modes including asymmetric mode of biased wake with two von Karman streets of different intensities but equal oscillation frequencies. Some wake modes intermits in appropriate range of the model parameters. Furthermore, our model reproduces a two-frequency asymmetric wake mode consisted of two Von Karman streets of distinct intensities and different frequencies as well. This mode is well observable in visualization frames and in velocity...
spectra both, for example [10]. This article mainly concerns characteristic features of this two-frequency mode within the framework of the model.

2. Formulation of one dimensional wake model

Plasma flows of elevated turbulence intensity are used in fast-flow constant-power lasers (see for example [2]). Usually such flows are formed by some network of obstacles that may be a row of cylinders. Here generic flow is a wake produced by solitary cylinder being the Von Karman vortex street for wide range of the flow Reynolds number. The second step is a consideration of a cylinders pair wake, and then adding more cylinders in raw arrangement. If the distance between cylinders axes is intermediate, the whole (laminar) wake expected to be a set of synchronized Von Karman streets that are deformed slightly due to their interaction in the domain of their formation. Within the framework of a reduced-order modeling it is common practice to represent a wake from solitary cylinder by a model of Landau-Stuart type [11] or by a similar model of Van der Pole oscillator [12, 13]. Van der Pole equation is a starting point for our wake model from cylinders group as well:

\[
\left( 1 + \Delta \left( \rho^2 - 4 \right) \right) \frac{d^2 X}{dt^2} + X - \varepsilon \left[ 1 - X^2 - \lambda Y^2 - l XY \right] \frac{d X}{dt} = h_x
\]

(1)

\[
\left( 1 + \Delta \left( r^2 - 4 \right) \right) \frac{d^2 Y}{dt^2} + Y - \varepsilon \left[ 1 - Y^2 - \lambda X^2 - l XY \right] \frac{d Y}{dt} = h_y
\]

(2)

Here the dependent variables \( X, Y \) represent governing parameters of two Von Karman streets behind the first and the second cylinder, respectively. The interplay between neighboring streets in the area of their formation is accounted for by the term in square brackets (with \( \lambda \) and \( l \) multipliers). With the aim of the model generality the form of interplay term selected is nonlinear one, whereas some linear term in model equations is usually used for the interplay modeling, see for example [11]. (Justification of the latter generally accepted option assumed to be a weakness of the streets interaction for not very closely located cylinders.) The second peculiarity of our model is an explicit account for dependence of oscillation frequency of the street on its amplitude (denoted as \( \rho \) and \( r \), respectively), which is accomplished by the nonlinear term in brackets. This complication of the model was made to account for essential splitting of the oscillation frequencies for different wake modes. Usually this effect of the wake oscillation amplitude on its frequency is neglected within the framework of oscillatory wake modeling [12, 13]. Right-hand sides in equations (1) and (2) represent a source of stochasticity for turbulent wake (a certain analog of Langevin force). This model contains four parameters, namely: \( \varepsilon, \Delta, \lambda, l \). The Van der Pole parameter is small \( \varepsilon \ll 1 \) in agreement with experimental data [14]. The value of quantity \( \Delta \) is also small \( |\Delta| \sim \varepsilon \ll 1 \). The Krylov-Bogolyubov method [15] of slow varying amplitudes and phases of oscillations have to be used for the equations (1) and (2) solving because of parameter \( \varepsilon \) smallness. The substitution (3) into (1), (2) have to be used.

\[
X = \rho \cos P_x; \quad Y = r \cos P_y
\]

(3)

Corresponding truncated (averaged) equations of the model are (4) – (7).

\[
\Phi_{\rho} \frac{d \rho^2}{dt} - \varepsilon \rho^2 \left[ 1 - \rho^2 + \lambda r^2 \left( 2 - \cos(2P) \right) + l \rho r \cos(P) \right] = h_\rho
\]

(4)

\[
\Phi_r \frac{d r^2}{dt} - \varepsilon r^2 \left[ 1 - r^2 + \lambda \rho^2 \left( 2 - \cos(2P) \right) + l \rho r \cos(P) \right] = h_r
\]

(5)
Here notation \( P \) for the independent variable the total phase difference of oscillation in two Von Karman streets was introduced

\[
P = P_x - P_y
\]

(8)

and notation \( \Phi_\rho \), as follows:

\[
\Phi_\rho = 1 + \Delta (0.5 \rho^2 - 4)
\]

(9)

\[
\Phi_y = 1 + \Delta (0.5 r^2 - 4)
\]

(10)

Using independent variable \( P \) the system of four model equations (4), (5), (6), and (7) was reduced to the system of three equations (4), (5), and (11).

\[
\frac{d P_x}{dt} - \frac{1}{\sqrt{1 + \Delta (\rho^2 - 4)}} + \frac{\epsilon}{8 \Phi_\rho} \left[ \lambda r^2 \sin(2P) + l \rho r \sin(P) \right] = h_\theta
\]

(6)

\[
\frac{d P_y}{dt} - \frac{1}{\sqrt{1 + \Delta (r^2 - 4)}} - \frac{\epsilon}{8 \Phi_y} \left[ \lambda \rho^2 \sin(2P) + l \rho r \sin(P) \right] = h_\nu
\]

(7)

Accordingly, the model phase space appears to be a three-dimensional one.

Moreover it turned out that modeling of the wake from solitary cylinder with the help of modified Van der Pole equation

\[
\left( 1 + \Delta (\rho^2 - 4) \right) \frac{d^2 X}{dt^2} + X - \epsilon \left[ 1 - X^2 \right] \frac{d X}{dt} = 0
\]

(12)

(after transformation to truncated equations, in assumption: \( |\Delta| \sim \epsilon \ll 1 \) is equivalent to modeling by means of the Landau-Stuart equation. The latter option has firm ground in the Navier-Stokes system. Accordingly, our wake model has hydrodynamic substantiation as well.

3. Two-frequency wake mode

Our model contains one-frequency global laminar modes of the wake as stationary stable points of equation system (4), (5), (11) with zero right-hand side. There are three symmetric modes among them characterized by equal amplitudes of oscillation in the streets: \( \rho = r \), and two asymmetric modes with \( \rho \neq r \). Among asymmetric one-frequency modes, the first one is a mode with one street of the pair completely damped, while the second global mode with \( \rho \neq r, \rho \neq 0, r \neq 0 \) [6, 9]. From visualization of cylinders close pair wakes it is well known that this wake usually composed of two streets of different strength and different oscillation frequency (see for example [11]). Accordingly, the question arises whether our model contain such two-frequency mode of the form (13) as a solution of the system (4), (5) and (11) with zero right-hand sides.

\[
\rho^2 \approx \text{const}, \quad r^2 \approx \text{const}, \quad P \approx (\text{const})t
\]

(13)
Putting $\varepsilon = 0$ as a first step in solving the system (4), (5), (11) one may deduce: such mode exists in line \( \lambda = 0.5 \) under the condition: \( \rho^2 + r^2 = 4 \). Furthermore, this mode is marginally stable along the direction: \( \rho^2 + r^2 = 4 \) and is stable in opposite direction. For the second step, taking into account first non vanishing terms to small parameters $\varepsilon$ and $\Delta$, a complicated system of nonlinear equations may be deduced for desired configuration of the mode. Its approximate solution is as follows

\[
R^2 \approx 4
\]

\[
N \approx \pm \frac{1}{2} \sqrt{\frac{l^2 + \lambda^2}{l^2 + 128[(1/\lambda) - 2](\Delta/\varepsilon)^2}}
\]

\[
\Omega \approx \pm 2\Delta \sqrt{\frac{l^2 + \lambda^2}{l^2 + 128[(1/\lambda) - 2](\Delta/\varepsilon)^2}}
\]

Here collective variables [6, 7] of the wake were used, namely: \( R^2 = \rho^2 + r^2 \) for entire oscillation intensity, and \( N = \frac{1}{2} \left( \rho^2 - r^2 \right) / \left( \rho^2 + r^2 \right) \) for asymmetry of the wake. Quantity $\Omega$ is a mean splitting of oscillation frequencies of the two Von Karman streets, which made up the total wake. According to the solution (13), (14), and (15) this frequency splitting $\Omega$ is proportional to product of the model parameter $\Delta$ and the variable $N$. High values of the frequency splitting correspond to high values of the wake asymmetry: $\Omega \approx 4\Delta N$, accordingly maximum splitting value is restricted as $\Omega < 2\Delta$. Maximum values of $\Omega$ are achievable in the process: $\lambda \to 0.5; \lambda < 0.5$.

An example of the wake stabilization process on two-frequency mode is presented in figure 1.

![Figure 1](image1)

**Figure 1.** Time dependence of collective variables of the wake relaxing to two-frequency asymmetric mode in the case $\varepsilon = 0.2; \Delta = -0.12; \lambda = 0.49; l = 3$: the entire intensity $R^2$; the asymmetry of the wake $N$; the total phase shift for oscillations in two streets $P$.

The stability of this mode with respect to small perturbations was studied directly, by numerical solving the equations system (4), (5), and (11) with initial conditions according to (13), (14), and arbitrary $P_0$. It was revealed that two-frequency mode is stable to small perturbations in some aria of
the model parameters. It covers narrow interval $\lambda - 0.5 << 1$ and some segment of quantity $|l|$ while within the framework of the above rough analysis, this mode is marginally stable one.

It is easily seen from the wake parameters dynamics illustrated in figure 1 that all the characteristics of the wake are far from simple constants or linear functions of time as in rough approximation (13), but mean values of collective variables are in good agreement with approximate prediction (14), (15), and (16). Relaxation time, being proportional to $\varepsilon^{-1}$, is not long for this choice of quasi-equilibrium initial conditions.

4. Discussion and brief conclusion

Developed one-dimensional model contains some symmetric and asymmetric wake global modes (laminar modes). In studying turbulent wakes the main question is how to distinguish wake relaxing modes if one disposes only the velocity oscillograms, moreover turbulent – “noisy” oscillograms. For the case of two closely-located cylinders the main question is how to distinguish between one- and two-frequency asymmetric wake modes. Here the key feature of any mode is its phase trajectory. Below relevant examples of trajectory projection on $P, N$ plane are shown in figure 2. Accordingly, qualitative difference of the $P, N$ dependence for one- and two-frequency asymmetric wake modes is evident.

**Figure 2.** Comparison of the wake relaxation dynamics: towards one-frequency asymmetric mode in the case:
(a) $-\varepsilon = 0.12; \Delta = 0.12; \lambda = 0.34; l = 3$;
(b) towards two-frequency asymmetric mode in the case $\varepsilon = 0.12; \Delta = 0.12; \lambda = 0.49; l = 3$;
(c) towards two-frequency asymmetric mode in the case $\varepsilon = 0.12; \Delta = -0.12; \lambda = 0.49; l = 3$.

1. Summarizing, our one-dimensional model of cylinders pair wake is comparatively rich in different wake regimes. Respectively, one may hope of the model capability to reproduce the observed
features of gas and plasma wakes in our experiment. Moreover, the model is consistent with Landau-Stuart model of solitary cylinder laminar wake, which has firm ground in Navier-Stokes equations.

2. Two circumstances have to be taken into account while estimating the model parameters from experimental (or simulation) data. Magnitudes of the first pair of the model parameters ($\varepsilon, \Delta$) are conveniently assessable from data for solitary cylinder. Information on phase difference of oscillation in the two Von Karman streets ($P_{(i)}$) composed cylinders pair wake would be useful for evaluating the parameters second pair ($\lambda, l$).

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