Necessary and sufficient conditions for the uniform integrability of the stochastic exponential

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Abstract We establish necessary and sufficient conditions for the uniform integrability of the stochastic exponential $\mathcal{E}(M)$.

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1. Introduction. Let us introduce a basic probability space $(\Omega, \mathcal{F}, P)$ and continuous filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$, which means that every local martingale is continuous. Let $\mathcal{F}_\infty$ be the smallest $\sigma-$Algebra containing all $\mathcal{F}_t$ for $t < \infty$. Let $M = (M_t)_{t \geq 0}$ be a local martingale on the stochastic interval $[[0; T]]$, where $T$ is a stopping time. Denote by $\mathcal{E}(M)$ the stochastic exponential of a local martingale $M$:

$$\mathcal{E}_t(M) = \exp\{M_t - \frac{1}{2} \langle M \rangle_t\}.$$
For a given local martingale $M$, the associated stochastic exponential $\mathcal{E}(M)$ is a local martingale, but not necessarily a true martingale. To know whether $\mathcal{E}(M)$ is a true martingale is important for many applications, e.g., when Girsanov's theorem is applied to perform a change of measure.

It is well-known that exponential martingales play an essential role in various questions concerning the absolute continuity of probability laws in stochastic processes. A. A. Novikov [13] showed that $\mathcal{E}(M)$ is a uniformly integrable martingale if $e^{\frac{1}{2}\langle M \rangle_T} \in L_1$ and that the constant $\frac{1}{2}$ can not be improved. In 1979 Kazamaki [10] proved that $\sup_T e^{\frac{1}{2}\langle M \rangle_T} < \infty$ is sufficient for uniform integrability of $\mathcal{E}(M)$. Then in 1994 Kazamaki [11] generalized his assertion introducing mixed Novikov-Kazamaki condition using constant $a \neq 1$ and lower functions (Kazamaki [11], p.19, Theorem 1.12). In 2013 J. Ruf [14] generalized mixed Novikov-Kazamaki criterion introducing general function of local martingale and its quadratic variation. In [4] and [3] the mixed Novikov-Kazamaki criterion is generalized using predictable process $a_s$ instead of the constant $a$. A similar question in the exponential semimartingale framework, in particular, for affine processes, has also attracted attention in Kallsen and Muhle-Kabre [8] and in Kallsen and Shiryaev [9]. In [8], a weak sufficient criterion and in [9] sufficient criterion in terms of cumulant process is given for uniform integrability of $\mathcal{E}(M)$.

The necessary and sufficient conditions for the uniform integrability of $\mathcal{E}(M)$ were provided in Mayerhofer, Muhle-Kabre and Smirnov [12] by considering the case when the initial martingale $M$ represents one component of a multivariate affine process, and in Blei and Engelbert [1] and Engelbert and Senf [5] for the exponential local martingales associated with a strong Markov continuous local martingale. In [12] deterministic necessary and sufficient conditions is provided in terms of the parameters of the initial martingale $M$. In [1], the case of a strong Markov continuous local martingale $M$ is studied and the deterministic criterion is expressed in terms of speed measure of $M$. In [5], the case of a general continuous local martingale $M$ is considered and the condition of uniform integrability of $\mathcal{E}(M)$ is given in terms of time-change that turns $M$ into a (possible stopped) Brownian motion. In [7] Yu. M. Kabanov, R. Sh. Liptser, A. N. Shiryaev showed that if the measure $Q$ is locally absolutely continuous w.r.t. the measure $P$, then for absolute continuity of $Q$ w.r.t. $P$ necessary and sufficient is that $Q\{\langle M \rangle_T < \infty\} = 1$. For the treatment of the related questions of a local absolute continuity of
measures on filtered spaces see also Jacod and Shiryaev [6] and Cheridito, Filipovic and Yor [2]. We establish a necessary and sufficient conditions for the uniform integrability of the stochastic exponential $E(M)$ in terms of the basic measure $P$.

In the next section we formulate the main results of this paper (Theorem 1 and Theorem 2) and then we prove them in the third section. In order to prove theorems we need several Lemmas which are given in Appendix.

2. The main results. In the following theorem we weakened condition $|a_s - 1| \geq \varepsilon > 0$ imposed in [4] and [3], which enable us to obtain new type necessary and sufficient condition:

![Theorem 1.](image)

Notice that because $E(M)$ is a supermartingale, $E\{e^{\frac{1}{2} \langle M \rangle T} \} < \infty$ and Kazamaki’s [10] criterion $\sup_{\tau \leq T} E\{e^{\frac{1}{2} M_\tau} \} < \infty$ are particular cases of Theorem 1 taking $a_s \equiv 0$, $f(x) \equiv 1$ and $a_s \equiv \frac{1}{2}$, $f(x) \equiv \frac{1}{2}$ respectively. Applying Theorem 1 for $a_s \equiv a \neq 1$ and $f(x) \equiv (1 - a)^2$ we obtain the mixed Novikov-Kazamaki’s condition:

$$\sup_{\tau \leq T} E\{e^{a M_\tau + (\frac{1}{2} - a) \langle M \rangle_\tau} \} < \infty.$$ 

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion. Recall that continuous function $\varphi : R_+ \rightarrow R_+$ is said to be a lower function if

$$P\{\omega : \exists t(\omega), \forall t > t(\omega) \Rightarrow B_t < \varphi(t)\} = 0.$$
For example, $\varphi(t) = C\sqrt{t}$ and $\varphi(t) = \sqrt{2t \log \log t}$ are lower functions.

In the next theorem we have no restriction on $a_s$, but we have additional term $f(\int_0^\tau 1_{\{|1-a_s|<\varepsilon\}}d\langle M \rangle_s)$ which will be essential when $a_s$ is close to 1:

**Theorem 2.** For the uniform integrability of the stochastic exponential $\mathcal{E}(M)$, it is necessary and sufficient, that there exists a predictable process $a_s$, positive constant $\varepsilon$, non-decreasing lower function $\varphi$ and a non-decreasing function $f : [0; \infty) \rightarrow (0; \infty)$ with $\lim_{x \to \infty} f(x) = \infty$ such that:

$$
\sup_{\tau \leq T} E\exp\left\{ \int_0^\tau a_s dM_s + \int_0^\tau \left( \frac{1}{2} - a_s \right) d\langle M \rangle_s - \varepsilon \varphi \left( \int_0^\tau 1_{\{|1-a_s|\geq \varepsilon\}}d\langle M \rangle_s \right) \right. \\
\left. + f \left( \int_0^\tau 1_{\{|1-a_s|<\varepsilon\}}d\langle M \rangle_s \right) \right\} < \infty
$$

(1)

where the sup is taken over all stopping times $\tau \leq T$.

**Remark 2.** Novikov’s [13] condition $Ee^{\frac{1}{2}\langle M \rangle_T} < \infty$ and Kazamaki’s [10] criterion $\sup_{\tau} Ee^{\frac{1}{2}\langle M \rangle_T} < \infty$ are particular cases of Theorem 2 taking $a_s \equiv 0$, $\varepsilon = 1$, $\varphi \equiv 0$ and $a_s \equiv \frac{1}{2}$, $\varepsilon = \frac{1}{2}$, $\varphi \equiv 0$ respectively. Applying Theorem 2 for $a_s \equiv a \neq 1$, $\varepsilon = |1-a|$ and $\varphi$ lower function we get the mixed Novikov-Kazamaki’s condition with non-decreasing lower function:

$$
\sup_{\tau \leq T} E\exp\left\{ aM_\tau + \left( \frac{1}{2} - a \right) \langle M \rangle_\tau - |1-a|\varphi(\langle M \rangle_\tau) \right\} < \infty.
$$

Notice that in these cases $f \left( \int_0^\tau 1_{\{|1-a_s|<\varepsilon\}}d\langle M \rangle_s \right) = f(0)$.

It follows from the proof of necessity of Theorem 2 that Ruf’s condition ([14], Corollary 5) is necessary and sufficient:

**Corollary.** For the uniform integrability of the stochastic exponential, it is necessary and sufficient, that there exists a continuous function $h : \mathbb{R}_+ \to \mathbb{R}$ with $\limsup_{x \to \infty} h(x) = \infty$ such that $\sup_{\tau \leq T} E\mathcal{E}_\tau(M)e^{h(\langle M \rangle_\tau)} < \infty$.

**Remark 3.** It is obvious that Ruf’s main result ([14], Theorem 1) is
also necessary and sufficient for the uniform integrability of the stochastic exponential.

3. Proof of the main results.

Proof of Theorem 1.

Sufficiency: According to the condition (ii) from Theorem 1 \( \tilde{f}(x) = 1 + \int_0^x f(t)dt \) is a positive and non-decreasing function with \( \lim_{x \to \infty} \tilde{f}(x) = \infty \).

So using Lemma 1 from appendix there exists absolutely continuous and non-decreasing function \( \tilde{g} : R_+ \to R_+ \) which satisfies conditions (a), (b) and (c) of Lemma 1. Now let us define function \( g(x) = \frac{1}{4} \tilde{g}(\frac{x}{2}) \). Condition (b) from Lemma 1 implies that \( \lim_{x \to \infty} g(x) = \infty \). Then we will have following inequalities:

\[
g\left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) \leq g\left( 2\langle M \rangle_\tau + 2 \int_0^\tau (a_s - 1)^2 d\langle M \rangle_s \right) = \frac{1}{4} \tilde{g}\left( \langle M \rangle_\tau + \int_0^\tau (a_s - 1)^2 d\langle M \rangle_s \right)
\]

Now according to the conditions (c) and (a) of Lemma 1 from (2) we obtain:

\[
g\left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) \leq \frac{1}{4} \tilde{g}\left( \langle M \rangle_\tau \right) + \frac{1}{4} \int_0^\tau (a_s - 1)^2 d\langle M \rangle_s + \frac{1}{2} \leq \frac{1}{4} \int_0^\tau f(\langle M \rangle_s) d\langle M \rangle_s + \frac{1}{4} \int_0^\tau (a_s - 1)^2 d\langle M \rangle_s + \frac{1}{2} \]

In the last inequality we used condition (ii) from Theorem 1. So as a result we obtained inequality \( g\left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) \leq \frac{1}{2} \int_0^\tau (a_s - 1)^2 d\langle M \rangle_s + \frac{1}{2} \), where \( g \) is a continuous, non-decreasing function with \( \lim_{x \to \infty} g(x) = \infty \). Now using this last inequality and condition (i) from Theorem 1 we will have:

\[
E \mathcal{E}_\tau \left( \int adM \right) \exp \left\{ g\left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) \right\} \leq e^{\frac{1}{2}} \exp \left\{ \int_0^\tau a_s dM_s - \frac{1}{2} \int_0^\tau a_s^2 d\langle M \rangle_s + \frac{1}{2} \int_0^\tau (a_s - 1)^2 d\langle M \rangle_s \right\}
\]
\[ e^{\frac{1}{2}} E \exp \left\{ \int_0^T a_s dM_s + \int_0^T \left( \frac{1}{2} - a_s \right) d\langle M \rangle_s \right\} < \infty. \]

According to Ruf’s condition ([14], Corollary 5) this implies that \( \mathcal{E}(\int adM) \) is a uniformly integrable martingale. So we have the equality \( E\mathcal{E}_T(\int adM) = 1. \)

Now define the new probability measure \( \tilde{P} = \mathcal{E}_T(\int adM) dP \) and local martingale \( \tilde{N}_t = \int_0^t (1 - a_s) dM_s. \) According to Girsanov’s theorem

\[ \tilde{N}_t = N_t - \langle N; \int a dM \rangle_t \]

is a \( \tilde{P} \)-local martingale. Let us show that for \( \tilde{N} \) Novikov’s condition is satisfied:

\[ E^\tilde{P} \exp \left\{ \frac{1}{2} \langle \tilde{N} \rangle_T \right\} = E \exp \left\{ \int_0^T a_s dM_s - \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s + \frac{1}{2} \int_0^T (1-a_s)^2 d\langle M \rangle_s \right\} \]

\[ = E \exp \left\{ \int_0^T a_s dM_s + \int_0^T \left( \frac{1}{2} - a_s \right) d\langle M \rangle_s \right\} < \infty \]

by condition \((i)\) from Theorem 1. This means that \( E^\tilde{P} \mathcal{E}_T(\tilde{N}) = 1. \) Finally we get

\[ 1 = E^\tilde{P} \mathcal{E}_T(\tilde{N}) = E \exp \left\{ \int_0^T a_s dM_s - \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s \right\} \times \exp \left\{ \int_0^T (1 - a_s) dM_s - \int_0^T a_s(1 - a_s) d\langle M \rangle_s - \frac{1}{2} \int_0^T (a_s - 1)^2 d\langle M \rangle_s \right\} \]

\[ = E \exp \left\{ M_T - \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s - \int_0^T a_s d\langle M \rangle_s + \int_0^T a_s^2 d\langle M \rangle_s \right\} \]

\[ - \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s + \int_0^T a_s d\langle M \rangle_s - \frac{1}{2} \langle M \rangle_T \right\} = E\mathcal{E}_T(M). \]

Thus \( E\mathcal{E}_T(M) = 1, \) which implies that \( \mathcal{E}(M) \) is a uniformly integrable martingale.
Necessity: Now we know that $\mathcal{E}(M)$ is a uniformly integrable martingale which implies that $E\mathcal{E}_T(M) = 1$. So we can define new probability measure $dQ = \mathcal{E}_T(M)dP$ and a $Q$–local martingale $\tilde{M} = M - \langle M \rangle$. It follows from [7] that $Q(\langle \tilde{M} \rangle_T < \infty) = 1$, so according to Lemma 2 there exists absolutely continuous and non-decreasing function $h$ with $\lim_{x \to \infty} h(x) = \infty$ such that $E^Q e^{h(\langle \tilde{M} \rangle_T)} < \infty$. Because $h$ is non-decreasing and absolutely continuous, there exists non-negative function $f$ such that $h(x) = h(0) + \int_0^x f(t)dt$ and therefore $h(\langle \tilde{M} \rangle_T) = h(0) + \int_0^T f(\langle \tilde{M} \rangle_s)d\langle \tilde{M} \rangle_s$. So if we define $a_s$ such that $2(a_s - 1)^2 = f(\langle \tilde{M} \rangle_s)$ then the condition (ii) of Theorem 1 will be satisfied.

Now let us check condition (i):

\[
E \exp \left\{ \int_0^T a_s dM_s + \int_0^T \left( \frac{1}{2} - a_s \right) d\langle M \rangle_s \right\} = E^Q \exp \left\{ \int_0^T (a_s - 1)d\tilde{M}_s \right\}
\]

\[
= E^Q \exp \left\{ \int_0^T (a_s - 1)d\tilde{M}_s - \int_0^T (a_s - 1)^2 d\langle \tilde{M} \rangle_s \right\} \times \exp \left\{ \int_0^T (a_s - 1)^2 d\langle \tilde{M} \rangle_s \right\}
\]

\[
\leq \sqrt{E^Q \mathcal{E}_T \left( 2 \int (a - 1)d\tilde{M} \right)} \times \sqrt{E^Q \exp \left\{ 2 \int_0^T (a_s - 1)^2 d\langle \tilde{M} \rangle_s \right\}}
\]

\[
\leq \sqrt{E^Q \exp \left\{ 2 \int_0^T (a_s - 1)^2 d\langle \tilde{M} \rangle_s \right\}} = \sqrt{E^Q \exp \left\{ h(\langle \tilde{M} \rangle_T) - h(0) \right\}} < \infty.
\]

Proof of Theorem 1 is completed. 

Proof of Theorem 2.

Sufficiency: Let us first show that $E\mathcal{E}_T(\int a dM) = 1$. It follows from Ruf’s condition ([14], Corollary 5) that for this it is sufficient to show that

\[
\sup_{\tau \leq T} E\mathcal{E}_\tau \left( \int a dM \right) \exp \left\{ h \left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) \right\} < \infty
\]

for some continuous function $h$ with $\limsup_{x \to \infty} h(x) = \infty$. According to the conditions of Theorem 2 function $f$ from (1) is non-decreasing, so using Lemma 1 there exists positive, non-decreasing and absolutely continuous
function $g$ which satisfies conditions (a), (b) and (c) of Lemma 1. Let us define function $h$ as $h(x) = \delta g \left( \frac{1}{(1+\varepsilon)x} \right)$ where $0 < \delta < 1$ is a constant sufficiently close to $0$. It is well known that $\limsup_{x \to \infty} \frac{\varphi(x)}{x} = 0$ for any non-decreasing lower function $\varphi$, which implies inequality $\varphi(x) \leq \delta x + C_\delta$ for any $\delta > 0$. Now using the definition of $h$ and inequality $\varphi(x) \leq \delta x + C_\delta$ we get:

$$
\begin{align*}
&h \left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) + \varepsilon \varphi \left( \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s \right) \\
&\leq \delta g \left( \frac{1}{(1+\varepsilon)^2} \int_0^\tau a_s^2 d\langle M \rangle_s \right) + \varepsilon \delta \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s + \varepsilon C_\delta \\
&= \delta g \left( \frac{1}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 < \varepsilon\}} d\langle M \rangle_s \right) + \frac{1}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 \geq \varepsilon\}} d\langle M \rangle_s \\
&\quad + \varepsilon \delta \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s + \varepsilon C_\delta \\
\end{align*}
$$

(3)

Now using the inequality $g(x+y) \leq g(x) + y + 2$ we will have from (3):

$$
\begin{align*}
&h \left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) + \varepsilon \varphi \left( \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s \right) \\
&\leq \delta g \left( \frac{1}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 < \varepsilon\}} d\langle M \rangle_s \right) + \frac{\delta}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 \geq \varepsilon\}} d\langle M \rangle_s \\
&\quad + 2\delta + \varepsilon \delta \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s + \varepsilon C_\delta \\
\end{align*}
$$

(4)

It is clear that $\frac{1}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 < \varepsilon\}} d\langle M \rangle_s \leq \int_0^\tau 1_{\{a_s-1 < \varepsilon\}} d\langle M \rangle_s$ and

$$
\frac{\delta}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 \geq \varepsilon\}} d\langle M \rangle_s + \varepsilon \delta \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s \\
$$

$$
\leq \int_0^\tau (\delta a_s^2 + \varepsilon \delta) \mathbb{1}_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s.
$$

So if we use inequality $g(x) \leq f(x)$ and non-decreasing property of $f$ we obtain from (4):

$$
\begin{align*}
&h \left( \int_0^\tau a_s^2 d\langle M \rangle_s \right) + \varepsilon \varphi \left( \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s \right) \\
&\leq \delta g \left( \frac{1}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 < \varepsilon\}} d\langle M \rangle_s \right) + \frac{\delta}{(1+\varepsilon)^2} \int_0^\tau a_s^2 1_{\{a_s-1 \geq \varepsilon\}} d\langle M \rangle_s \\
&\quad + 2\delta + \varepsilon \delta \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s + \varepsilon C_\delta \\
\end{align*}
$$
\[ f(\int_0^\tau 1_{\{|a_s-1|<\epsilon\}} d\langle M\rangle_s) + \int_0^\tau (\delta a^2 + \delta \epsilon) 1_{\{|a_s-1|\geq \epsilon\}} d\langle M\rangle_s + 2\delta + \epsilon C_\delta \]
\[ \leq f(\int_0^\tau 1_{\{|a_s-1|<\epsilon\}} d\langle M\rangle_s) + 1/2 \int_0^\tau (a_s - 1)^2 d\langle M\rangle_s + 2\delta + \epsilon C_\delta. \]  

(5)

In the last inequality we used Lemma 4 to obtain estimation \((\delta a^2 + \delta \epsilon) 1_{\{|a_s-1|\geq \epsilon\}} \leq \frac{1}{2}(a_s - 1)^2\) for \(\delta > 0\) sufficiently close to 0. So in (5) we got the inequality

\[ h(\int_0^\tau a^2 d\langle M\rangle_s) + \epsilon \varphi(\int_0^\tau 1_{\{|1-a_s|\geq \epsilon\}} d\langle M\rangle_s) \]

\[ \leq f(\int_0^\tau 1_{\{|a_s-1|<\epsilon\}} d\langle M\rangle_s) + 1/2 \int_0^\tau (a_s - 1)^2 d\langle M\rangle_s + 2\delta + C_\delta \]

which is equivalent to the following

\[ -\frac{1}{2} \int_0^\tau a^2 d\langle M\rangle_s + h(\int_0^\tau a^2 d\langle M\rangle_s) \leq \int_0^\tau (\frac{1}{2} - a_s) d\langle M\rangle_s \]

\[ -\epsilon \varphi(\int_0^\tau 1_{\{|1-a_s|\geq \epsilon\}} d\langle M\rangle_s) + f(\int_0^\tau 1_{\{|a_s-1|<\epsilon\}} d\langle M\rangle_s) + 2 + \epsilon C_\delta. \]  

(6)

Now from (6) we obtain

\[ \sup_{\tau \leq T} E\mathcal{E}_\tau \left( \int_0^\tau adM \right) \exp \left\{ h(\int_0^\tau a^2 d\langle M\rangle_s) \right\} \]

\[ \leq e^{2+\epsilon C_\delta} \sup_{\tau \leq T} E \exp \left\{ \int_0^\tau a_s dM_s + \int_0^\tau \left( \frac{1}{2} - a_s \right) d\langle M\rangle_s \right\} \]

\[ -\epsilon \varphi(\int_0^\tau 1_{\{|1-a_s|\geq \epsilon\}} d\langle M\rangle_s) + f(\int_0^\tau 1_{\{|a_s-1|<\epsilon\}} d\langle M\rangle_s) \}

< \infty. \]

According to Ruf’s condition (14, Corollary 5) this means that \(\mathcal{E}(\int adM)\) is a uniformly integrable martingale which implies that \(E\mathcal{E}_T(\int adM) = 1\).

Now define the new probability measure \(d\tilde{P} = \mathcal{E}_T(\int adM) dP\) and local martingale \(N_t = \int_0^t (1 - a_s) dM_s\). According to Girsanov’s theorem

\[ \tilde{N}_t = N_t - \langle N; \int adM \rangle_t \]
\[ \int_0^t (1 - a_s) dM_s - \int_0^t a_s (1 - a_s) d\langle M \rangle_s \]

is a $\tilde{P}$–local martingale. Define function $\psi(x) = \varepsilon \varphi\left( \frac{x}{\varepsilon} \right)$ which is lower function according to Lemma 3. Let us show that for $\tilde{N}$ Novikov’s condition with lower function is satisfied:

\[
E^{\tilde{P}} \exp \left\{ \frac{1}{2} \langle \tilde{N} \rangle_T - \psi(\langle \tilde{N} \rangle_T) \right\} = E \exp \left\{ \int_0^T a_s dM_s - \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s \right\} \\
+ \frac{1}{2} \int_0^T (1 - a_s)^2 d\langle M \rangle_s - \varepsilon \varphi\left( \frac{1}{\varepsilon^2} \int_0^T (1 - a_s)^2 d\langle M \rangle_s \right) \\
= E \exp \left\{ \int_0^T a_s dM_s + \int_0^T \left( \frac{1}{2} - a_s \right) d\langle M \rangle_s - \varepsilon \varphi\left( \frac{1}{\varepsilon^2} \int_0^T (1 - a_s)^2 d\langle M \rangle_s \right) \right\} \\
\leq E \exp \left\{ \int_0^T a_s dM_s + \int_0^T \left( \frac{1}{2} - a_s \right) d\langle M \rangle_s \\
- \varepsilon \varphi\left( \int_0^T 1_{\{1 - a_s \geq \varepsilon\}} d\langle M \rangle_s \right) \right\} < \infty. 
\tag{7}
\]

Here we used the inequality

\[
\varepsilon \varphi\left( \int_0^T 1_{\{1 - a_s \geq \varepsilon\}} d\langle M \rangle_s \right) \leq \varepsilon \varphi\left( \frac{1}{\varepsilon^2} \int_0^T (1 - a_s)^2 d\langle M \rangle_s \right)
\]

which follows from non-decreasing property of lower function $\varphi$.

(7) implies that $E^{\tilde{P}} \mathcal{E}_T(\tilde{N}) = 1$ which is equivalent to the $E \mathcal{E}_T(M) = 1$:

\[
1 = E^{\tilde{P}} \mathcal{E}_T(\tilde{N}) = E \exp \left\{ \int_0^T a_s dM_s - \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s \right\} \\
\times \exp \left\{ \int_0^T (1 - a_s) dM_s - \int_0^T a_s (1 - a_s) d\langle M \rangle_s - \frac{1}{2} \int_0^T (a_s - 1)^2 d\langle M \rangle_s \right\} \\
= E \exp \left\{ MT - \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s - \int_0^T a_s d\langle M \rangle_s + \int_0^T a_s^2 d\langle M \rangle_s \\
- \frac{1}{2} \int_0^T a_s^2 d\langle M \rangle_s + \int_0^T a_s d\langle M \rangle_s - \frac{1}{2} \langle M \rangle_T \right\} = E \mathcal{E}_T(M).
\]

Thus $E \mathcal{E}_T(M) = 1$, which implies that $\mathcal{E}(M)$ is a uniformly integrable martingale.
Necessity: Now we know that $\mathcal{E}(M)$ is a uniformly integrable martingale. So, we can define the new probability measure $dQ = \mathcal{E}_T(M)dP$. It follows from [7] that $Q(\langle M \rangle_T < \infty) = 1$, so we can apply Lemma 2 to find non-decreasing and absolutely continuous function $f$ with $\lim_{x \to \infty} f(x) = \infty$ such that $E^Qe^{f(\langle M \rangle_T)} < \infty$. Now if we insert $a_s \equiv 1$, $\varphi \equiv 0$, $\varepsilon > 0$ and $f$ in (1) we obtain:

$$\sup_{\tau \leq T} E \exp \left\{ \int_0^\tau a_s dM_s + \int_0^\tau \left( \frac{1}{2} - a_s \right) d\langle M \rangle_s - \varepsilon \varphi \left( \int_0^\tau 1_{\{1-a_s \geq \varepsilon\}} d\langle M \rangle_s \right) \right\} + f \left( \int_0^\tau 1_{\{1-a_s < \varepsilon\}} d\langle M \rangle_s \right) = \sup_{\tau \leq T} E \exp \left\{ M_\tau - \frac{1}{2} \langle M \rangle_\tau - \varepsilon \varphi(0) + f(\langle M \rangle_\tau) \right\} \leq \sup_{\tau \leq T} E^Qe^{f(\langle M \rangle_\tau)} = E^Qe^{f(\langle M \rangle_T)} < \infty.$$

Proof of Theorem 2 is completed.

4. Appendix.

Lemma 1. Let $f : [0; +\infty) \to (0; +\infty)$ be a non-decreasing function with $\lim_{x \to \infty} f(x) = \infty$. Then a non-decreasing, absolutely continuous function $g : [0; +\infty) \to [0; +\infty)$ exists which satisfies the following conditions:

(a) $g(x) \leq f(x)$;
(b) $\lim_{x \to \infty} g(x) = \infty$;
(c) $g(x + y) \leq g(x) + y + 2$.

proof. Define function $F$: $F(x) = \sum_{k=1}^{\infty} f(k - 1)1_{[k-1,k]}(x)$. It is obvious that $F(x) \leq f(x)$. Let us denote by $\Delta F(k) = F(k) - F(k - 1)$ jumps of $F$. Because $f$ is non-decreasing, the jumps of $F$ will be non negative. Now define non-decreasing sequence $(g_k)_{k \geq 1}$ with recurrence relationship:

$g_0 = 0; \quad g_1 = 1 \wedge f(0); \quad g_2 = g_1 + 1 \wedge \Delta F(2); \quad g_k = g_{k-1} + 1 \wedge \Delta F(k) \quad k \geq 2.$

As a result we have points $(k; g_k) \quad k \geq 0$. Define function $g$ by connecting points $(k; g_k)$ with straight lines. It follows from the definition that $g$ is absolutely continuous, non-decreasing, $g(x) \leq F(x) \leq f(x)$ and $\lim_{x \to \infty} g(x) = \infty$. It remains to show that $g(x + y) \leq g(x) + y + 2$. Let us
take \( x \in [k - 1; k] \) and \( y \in [n - 1; n] \). It is clear that \( x + y \leq k + n \). Using the definition of function \( g \) we obtain:

\[
g(x + y) - g(x) \leq g(k + n) - g(k - 1) = \sum_{i=1}^{k+n} 1 \wedge \Delta F(i) - \sum_{i=1}^{k-1} 1 \wedge \Delta F(i) =
\]

\[
= \sum_{i=k}^{k+n} 1 \wedge \Delta F(i) \leq n + 1 \leq y + 2.
\]

So, by arbitrariness of \( k \) and \( n \), \( g(x + y) \leq g(x) + y + 2 \). \( \square \)

**Lemma 2.** For any random variable \( \xi \) such that \( P(0 \leq \xi < \infty) = 1 \) there exists a positive, non-decreasing and continuous function \( g \) with \( \lim_{x \to \infty} g(x) = \infty \), such that \( Eg(\xi) < \infty \).

**Proof.** Let \( F_\xi(x) = P(\xi \leq x) \) be the probability distribution function of \( \xi \). Let us take \( f(x) = \frac{1}{\sqrt{1 - F_\xi(x)}} \). Then we will have:

\[
Ef(\xi) = \int_0^\infty \frac{1}{\sqrt{1 - F_\xi(x)}} \, dF_\xi(x) = \left[ -2\sqrt{1 - F_\xi(x)} \right]_0^\infty
\]

\[
- \sum_{0 < x < \infty} \left[ -2\sqrt{1 - F_\xi(x)} + 2\sqrt{1 - F_\xi(x-)} - \frac{\Delta F_\xi(x)}{\sqrt{1 - F_\xi(x-)}} \right] \leq \left[ -2\sqrt{1 - F_\xi(x)} \right]_0^\infty = 2\sqrt{1 - F_\xi(0)} < \infty.
\]

Here we have used inequality \(-2\sqrt{1 - F_\xi(x)} + 2\sqrt{1 - F_\xi(x-)} - \frac{\Delta F_\xi(x)}{\sqrt{1 - F_\xi(x-)}} \geq 0\) which follows from the convexity of the function \( y \to -2\sqrt{1 - y} \). Now if we use Lemma 1 we can find absolutely continuous, positive and non-decreasing function \( g \) with \( \lim_{x \to \infty} g(x) = \infty \), such that \( g(x) \leq f(x) \). Inequalities \( g(x) \leq f(x) \) and \( Ef(\xi) < \infty \) implies that \( Eg(\xi) < \infty \). \( \square \)

**Lemma 3.** Let \( \varphi \) be a lower function. Then the function \( \psi(x) = \varepsilon \varphi(\frac{x}{\varepsilon^2}) \) also will be a lower function.

**Proof.** It is well known that if \( B_t \) is a Brownian motion, then \( W_t = \varepsilon B_{t/\varepsilon^2} \) will be Brownian motion too. With this if we take \( s = t/\varepsilon^2 \) then we will have:

\[
P\left\{ \omega : \exists t(\omega), \forall t > t(\omega) \Rightarrow W_t < \psi(t) \right\} =
\]
\[ P \{ \omega : \exists s(\omega), \forall s > s(\omega) \Rightarrow B_s < \varphi(s) \} = 0. \]

**Lemma 4.** For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) sufficiently close to 0 such that inequality \( \delta x^2 + \delta \varepsilon \leq \frac{1}{2}(x - 1)^2 \) holds true for any \( x \notin (1 - \varepsilon; 1 + \varepsilon) \).

**Proof.** It is clear that we can take \( \delta > 0 \) sufficiently close to 0 such that

\[
1 - \varepsilon < \frac{1 - \sqrt{2\delta(1 + \varepsilon - 2\delta\varepsilon)}}{1 - 2\delta} < \frac{1 + \sqrt{2\delta(1 + \varepsilon - 2\delta\varepsilon)}}{1 - 2\delta} < 1 + \varepsilon.
\]

It is easy to check that for such \( \delta \) condition \( |x - 1| \geq \varepsilon \) implies inequality

\[
(1 - 2\delta)x^2 - 2x + 1 - 2\delta\varepsilon \geq 0
\]

which is equivalent to the following \( \delta x^2 + \delta \varepsilon \leq \frac{1}{2}(x - 1)^2 \).
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