The Semileptonic Lifetime of the D Meson

The full treatment, both nonperturbative - through $O(1/m_c^3)$ - and perturbative - through $O(\alpha_s)$ and BLM type $O(\alpha_s^2)$ - of the semileptonic lifetime of the D meson is given. A dedicated discussion of the numerics involved is made, with the result that the leading order prediction plus corrections fail to match experiment. To explain the difference, a model of duality is invoked, where it is shown that the asymptotic nature of the nonperturbative expansion may be responsible for the theoretical experimental discrepancy. A particularly interesting conclusion which follows from the model is that any attempts to extract $\alpha_s$ from $\tau$ decays are doomed, since the same duality violating mechanism is at work here.
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Richard David Dikeman

and have found that it is complete and satisfactory in all respects and that any and all revisions required by the final examining committee have been made.

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GRADUATE SCHOOL
The Semileptonic Lifetime of the D Meson

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The Semileptonic Lifetime of the D Meson

by Richard David Dikeman

Under the supervision of Mikhail Shifman

ABSTRACT

The full treatment, both nonperturbative - through $O(1/m_c^3)$ - and perturbative - through $O(\alpha_s)$ and BLM type $O(\alpha_s^2)$ - of the semileptonic lifetime of the D meson is given. A dedicated discussion of the numerics involved is made, with the result that the leading order prediction plus corrections fail to match experiment. To explain the difference, a model of duality is invoked, where it is shown that the asymptotic nature of the nonperturbative expansion may be responsible for the theoretical experimental discrepancy. A particularly interesting conclusion which follows from the model is that any attempts to extract $\alpha_s$ from $\tau$ decays are doomed, since the same duality violating mechanism is at work here.
Introduction

“All is mystery; but he is a slave who will not struggle to penetrate the dark veil” - Disraeli

Hopefully, there will be a few people outside the high energy theory community who will at least take a glance at this thesis. This introduction is written for them, in hope that all the complicated equations, diagrams, and verbage following will have some general meaning. I think it was Feynman who said that you should be able to explain your research to a seventh grader, or something like that. How to DO the research, is, of course, another matter all together, but what the problem is, and what its solution is should be understandable to anyone. This is my attempt to explain:

Most people know that there are things called atoms, and have seen simple diagrams (resembling the solar system to some extent) which picture electrons whirling around a nucleus. Although, after a semester of quantum theory, one would most likely cringe at such a diagram, these simplified pictures do in fact illustrate some key features of the atom. The electrons are usually displayed either as clouds, illustrating the wave nature of the very small, or as particles traversing orbits around a center, the nucleus. Usually, the electron, when pictured in the ‘planet’ like orbital picture, is shown as a ‘dot’. In some sense, this is a fair representation. As far as we know at present, the electron is a particle which has no structure - no radius whatsoever, and no constituents. The electron is, as far as we can tell, the perfect physical representation
of a mathematical point. The nucleus, however, is an entirely different story. It has constituents, being made up of protons and neutrons. Surprisingly, although protons are positive in charge (neutrons are neutral!) the nucleus does not ‘fly’ apart (like charges repel). There is another force binding the nucleus together, stronger than the electrostatic repulsion, called the strong force. The modern day theory of the strong force is called QCD - Quantum Chromodynamics, which is the topic of my thesis. The ‘objects’ of QCD, however, are not the proton and neutron. It is known for some decades now that the proton and neutron are not at all like the electron. These particles do have internal structure. Indeed, the proton and neutron (and many other particles like them called either mesons or baryons) are made from particles called quarks and gluons - the true ‘objects’ of QCD. Quarks and gluons themselves are like the electron - seemingly pointlike. Quarks come in different ‘flavors’ (i.e. there are different kinds which differ by their masses). So far, there are six different flavors found: up, down, strange, charm, bottom, and top. Quarks are never found alone, but are confined either in pairs (mesons) or triplets (baryons). As stated above, there is a theory - QCD - which mathematically details the dynamics of how quarks interact. What does it mean to say that there is a ‘theory’ detailing the dynamics? It basically means we should be able to calculate some property of quarks, and compare it to experiment. For example, this thesis is concerned with calculating the lifetime (the strange, charmed, bottom, and top quark are all unstable, and quickly decay when created at a particle accelerator) of a meson called the D meson (the D meson is a pair of quarks - one charmed and one up or down). Specifically, I calculate the decay of the D meson into a pair of leptons (an electron is a member of the lepton ‘family’) and all possible hadrons (a hadron is either a meson or a baryon, but in general I mean here all other possible combinations of quarks). The D meson lifetime is well established experimentally. Calculating it, on the other hand, is absolutely difficult. QCD is a notoriously difficult theory to calculate with. Noble Prize winner Steven Weinberg is quoted as saying:
“There’s a long tradition in theoretical physics, which by no means affected everyone but certainly affected me, that said the strong interactions are too complicated for the human mind.”

Indeed, making an exact calculation of the decay time of the D meson, or any other meson will probably never happen. This is really not such a big deal, however, as this is almost always the case in physics anyway. Hardly anything can be calculated without making some approximations, and when it can, it usually doesn’t correspond to the true physical world anyway. (Of course, nature doesn’t depend on whether we can ‘calculate’ it or not - e.g. atoms do not calculate their own behavior!, still much of the richness of our universe resides in the fact that nature is calculable only in approximation. If nature happened to correspond to a very simple theory which could be used to calculate for all physical phenomenon, we would live in a boring place indeed. (Let me remark that it seems certain that nature abides by some very elegant, and simple theories which lead to, most of the time, very rich phenomena. It is usually very ‘easy’ to write the theory, and most often very difficult to calculate with it - thus the need for approximations!))

The D meson is seemingly an ideal situation for making approximations. The reason is that the charmed quark has a high mass. What can be done computationally, is to express the decay lifetime in a power series:

\[ \Gamma = a_0 + a_1 \left( \frac{\mu}{m_c} \right) + a_2 \left( \frac{\mu}{m_c} \right)^2 + a_3 \left( \frac{\mu}{m_c} \right)^3 + \ldots \]  

(1)

here \( m_c \) is the mass of the charm quark, and \( \mu \) is a tool of the theorist, and is chosen so that the expansion converges (there are subtleties here which even real experts do not get!). Numerically, \( \mu \) is chosen so that \( \mu/m_c = .5 \) or so. Suppose then, that \( a_0, a_1, a_2, \ldots \) are all different - but all close to each other (unless there is some strong reason, this could be expected - a statement of vague intuition to be sure). Then, for each new term in the series, say \( a_4, a_5, \) etc. the new term is also multiplied by increasing powers of \( \mu/m_c = .5 \). Technically, it is impossible to calculate the full series - no one knows how,
but we can approximate the result by calculating the first few terms and then truncating the series when each new term gets insignificant (or too hard!). The calculation of $a_0$, $a_1$, and $a_2$ were all completed before I knew what a heavy quark even was, but not much before! Check out the result of this calculation - equation 57, and in particular note that $m_c$ is in the denominator of the last few terms, in tune with what is written above. Essentially, the first five chapters (aside from the preface) are concerned with this computation. All the details of ‘how to do it’ are here. Chapter 6 and 7 explain the actual numerical values that are used to check and see if equation 5.22 actually matches experiment. This is an interesting area of scientific debate. As it turns out, if one measures numerical values for the charmed quark mass and other parameters from experiments and applies them to the problem of the D lifetime, the theoretical value of the decay time is roughly 50 percent of the experimental value (pretty bad agreement) - check out equation 7.3. The key numerical parameter here is $m_c$, since, as one can see from equation 5.22, it enters into the expression of the lifetime as $m_c^5$. Because $m_c$ is to the fifth power, only a slight increase, roughly 10 percent, of it’s value will compensate the 50 percent deficit between theory and experiment. However, those (Dikeman included) who think that $m_c$ should have a low value and thus spoil the agreement between theory and experiment regarding the D lifetime believe this because it would spoil too many other results which say $m_c$ is low. Thus, there is a problem.

One possible next move is to try and calculate the next term in the series and see if it somehow can raise the decay value a whole 50 percent - an unlikely scenario since remember, each new term should be .5 times smaller than its previous one, and thus numerically - check out equation 88 - we would expect, at best, this ‘cubic’ term in the series to contribute at most 10 percent, but not 50 percent. Still, though, one must try, and so the calculation of the next term in the series, $a_3$, was my first task as a graduate student. It was hoped that this ‘cubic’ term would explain the puzzle. This is the discussion of chapter 8. Well, after a lot of work, no such luck. My advisor Misha
Shifman, his past student Boris Blok, and myself wrote a paper demonstrating that the next term in the series doesn’t help matters, and may even make them worse. At least in this paper we tried to give a number of possible explanations why the puzzle existed - our main guess was that the charmed quark was not quite heavy enough for the power series expansion to work in the first place. Despite the negative results, though, the paper was mildly successful - currently it has been cited in other papers 17 times, which is not at all bad.

With the puzzle deepened, I went on to another project, but then returned to the D meson puzzle a year later. My advisor, Misha Shifman, suggested to me and fellow graduate student and friend Boris Chibisov while in a summer school at Boulder, Colorado that there may be a way of estimating what is ‘leftover’ when truncating the series, i.e. estimating the remaining pieces, $a_5 ... a_{10}$, etc. of the series. This work was the first real attempt at such an estimate concerning the D meson, or any other QCD phenomenon, and so is a much more general paper than the first one, and a bit more successful - it has 21 citations so far, and I expect a lot more in the future. The ideas of this paper are presented in chapter 10. In general, these ideas are interesting, but they don’t exactly solve the remaining puzzle, instead putting it in proper perspective. As I allude to above, it is difficult to solve a hard problem without making approximations, and chapter 10 is loaded with them, but we at least hope to correctly capture some important features of the real, if there ever really is one!, solution to the puzzle. In addition, the model developed in chapter 10 has strong implications for other QCD processes as well, and sheds a lot of doubt on a few particular phenomenological studies which are quite important. And that’s the end of the story.
Dedication

“Born under widespread changes to search for...higher reason. Learning the ropes ok, but fate just runs you around.” Jay Farrar of Son Volt

So many people have been instrumental in my attaining this goal that I shudder to write the usual, brief: e.g. ‘To my parents’. This is a great chance to thank family, and friends, and I want to use it and do a proper job.

Minnesota is an unspeakably cold, dark place, but the opportunities to have a great time still abound. At my side, participating in most of these great times over the last five years has been John Capriotti and so first and foremost, I’d like to thank him. The great music, suds, untold number of laughs, stratocaster jam sessions, and Slim Dunlap shows, all while sorting out the meaning of life, shared with Cap over the last few years have turned what would have been just a remarkable experience in the world of physics, into a rich life experience.

Mostly responsible for my remarkable physics experience noted above is Mikhail Shifman. When I came to Minnesota from Ann Arbor, I laid down to go to sleep on one of my first nights and thought “five years or so from now, I’ll be a Ph.D. in physics!” From my first days of discovering science by myself in popular books by Isaac Asimov, Carl Sagan, etc., I have always loved science, and so the thought was formed not just from the desire to obtain a Ph.D. but to really learn about the universe. After a dissapointing first year in which no experimental condensed matter theorist would
take me (!), Misha, after my successful tour of duty in his Quantum Mechanics class, gave me some problem involving heavy quarks (uhh... what?!). Now, I have some 7 or so papers in the field of QCD, and have to laugh at that memory (I really had no idea just who Misha was: ‘Sum Rules? What are those?’ and what kind of education I had in store for me). The thrill of working with Misha comes from learning what it means to have physical insight into nature. Because of this I was afforded to work on some really neat problems and learn a helluva lot more about science than I had ever dreamed. To return in time to that guy (me!), about to go to sleep and excited about the coming five years and tell him that I’d be working with one of the top theorists in particle physics, eventually traveling the world giving seminars here and there, and producing important work....WOW! - It’s really been an extraordinary experience. Thanks Misha.

I also want to thank the other guys I worked with during my physics career, especially Kolya Uraltsev. I only wish I had half as much passion for physics as Kolya, even though after all our discussions, I found that my own had at least doubled! Kolya was one of those guys so necessary in being able to do any reasonable work if you have modest ability - a guy you can ask absolutely dumb questions to! Thanks Kolya. Also, fellow grad students Boris Chibisov, Jamal Marian, Alejandro Mercado, Dick Madden, Bayram Tekin, Lev Koyrakh, Denne Weslowski, Igor Zutic, Joel Kindem, Jason Sielaff, and Emil Akhmedov were great resources and fun.

I’d also like to thank my high school and college friends: Huey, Voll, Fritz, Bones, Nip, Shankar, Chip, Fuzz, and Klaz. With friends like these, I can confidently say that few have ever had as many laughs and good times as I have over the last dozen years.

Lastly, I’d like to thank my parents and sister who instilled the self confidence and ability in me to take on such a task and complete it successfully. More than anything else, I’m lucky to be in such a happy, loving family.
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Chapter 1

Preface

“The age in which we live is the age in which we are discovering the fundamental laws of nature, and that day will never come again.” - Feynman

“The more the universe seems comprehensible, the more it also seems pointless. But if there is no solace in the fruits of our research, there is at least some consolation in the research itself. Men and women are not content to comfort themselves with tales of gods and giants, or to confine their thoughts to the daily affairs of life; they also build telescopes and satellites and accelerators, and sit at their desks for endless hours working out the meaning of the data they gather. The effort to understand the universe is one of the very few things that lifts human life a little above the level of farce, and gives it some of the grace of tragedy” - Steven Weinberg

In any physical theory, it is often helpful, sometimes crucial and perhaps even mandatory, to identify the simplest problem the theory can solve. Fermi, referring to this feature of theoretical physics, liked to pose the question ‘What is the hydrogen atom of the theory?’ For Quantum Chromodynamics (QCD), the theory of nuclear force, this is not such an easy question to answer. Historically, as a shockingly rich spectra of hadrons was discovered by experimentalists in the 50’s, theorists attacked problems of
strong resonance physics not with powerful analytic results from first principles QCD - QCD did not even exist until 1973, but instead with ideas based on symmetries. The group (flavor) structure of QCD was revealed by the spectroscopy of hadrons. This is the work most associated with Gell-Mann and Ne’eman \[1\]. Gell-Mann realized that the spectra of hadrons - protons, neutrons, and the myriad resonances being discovered - could be explained by introducing \[3\] what he deemed fictitious entities: quarks, which carried a color charge, and some ‘glue’ to hold the quarks together. Aside from Gell-Mann’s ‘Eight-fold way’ which, anyway, dealt with the flavor chiral symmetry \[4\] of QCD, and not its more fundamental color symmetry, the answer to Fermi’s question might be, at least regarding the gauge structure of the theory, the special puzzle of the $\Delta^{++}$ wave function.

The $\Delta^{++}$ is a spin 3/2 particle, a fermion, but with entirely symmetric wave-function. As is well known, a half-integer spin particle necessarily has an anti-symmetric wave function, and so there is a puzzle. Instead of appealing to some perverse solution, the elegant notion of a color charge, yielding an antisymmetric, colorless wave function was put forth which solved this and other such problems \[2\]. Thus the idea that quarks carried color, and appeared in nature in only confined combinations allowing a color-singlet wavefunction was born, and so was QCD \[3\], \[5\]. In this sense, i.e. elucidating that there is a color wave function, and thus an $SU(3)$ non-Abelian gauge theory, \[6\] the $\Delta^{++}$ is the hydrogen atom of QCD. But when Fermi asked this question, surely he was referring to a more dynamical problem in the theory - the $\Delta^{++}$ has helped us identify the theory, but now we should calculate the spectrum of the simplest system with it.

Possibly the most genuinely interesting thing about QCD is that there are such a variety of dynamical situations where we in fact have a simple hydrogen atom like problem, which quickly becomes contaminated with the intricacies of QCD. Analyzing phenomena such as $e^+e^-$ annihilation, DIS, heavy quark decays, etc. we, at first, find
quick successes with traditional methods only to later suffer severe impedances where our methods aren’t applicable. At the core of the simplicities of these systems is asymptotic freedom. Asymptotic freedom tells us that we can use our powerful arsenals of perturbative methods, developed already in the study of QED, to attack problems in the ultraviolet (high momentum domain) where the QCD coupling constant is small. In the infrared (soft momentum domain), however, where $\alpha_s \approx 1$ there is what is called infrared slavery, and we must resort to other methods. Essentially, asymptotic freedom creates a calculational boundary, beyond which we must be extremely clever to extract model independant results. Due to the nature of the running of the strong coupling constant, we face a situation where for soft processes, typically at scales around a few GeV, we have no reliable way to calculate in QCD. QCD is a nonlinear system with no analytic solution, and in the infrared, we have no small quantity that we can expand in - a truly great challenge.

And so, as it turns out, the biggest challenge of QCD is to understand its nonperturbative physics. Really, then, if we are to ask Fermi’s question of QCD, cast in the pall of trying to understand the genuine solution to the soft problem, we realize that perhaps we are stuck! Indeed, taking a fairly harsh view of some of the extremely hard, and amazing work in QCD phenomenology done in the last 20 years on the soft problem, e.g. ref. [7], one might say that the problem of the soft, nonperturbative aspects of QCD have, in some sense, merely been swept under the rug! Since the inception of QCD, the primary theoretical tool used to deal with the soft physics of QCD has been Wilson’s Operator Product Expansion [8]. Essentially, the central utility of the OPE is that it allows us to characterize - but not calculate! - the fundamental soft quantities of QCD, which we can then extract from phenomenology. So we give a name for our ignorances, and can even ask for them to be measured by experimentalists, but we have no idea how to calculate these ferocious beasts. With that perspective, perhaps the answer to Fermi’s question will not actually be how to calculate this-or-that process,
but instead how to calculate this-or-that condensate! Who knows.

This thesis will not at all be an attempt to answer the question of how to calculate the condensates in QCD. I, instead, will back down from this Herculean task, and take on a lesser trial. In this thesis, I will look at the lifetime of the D meson. In all hadronic processes there is, perhaps, no system where one can more easily immerse oneself into the intricacies of QCD mentioned above. The D meson, in fact, is much like the hydrogen atom! This meson contains a heavy quark at its center with a light quark bound to it by an infinite number of quark/gluonic degrees of freedom. Because the unstable charm quark at the meson’s core is so heavy, energies involved in its decay will be large enough to invoke asymptotic freedom, and thus traditional perturbative methods. However, since the charmed quark decays in an environment where typical momenta are of the order of $\Lambda_{QCD}$ (the momenta where the strong coupling constant, $\alpha_s \sim 1$), we will also have to deal with nonperturbative effects. By solving the problem of the lifetime of the D meson, hopefully revealing some of nature’s trickeries in the strong interaction section, we also may receive other dividends as well - this is a weak interaction, and so we might get some information on CP violation if we can do the job right.

Not to spoil the story, but as it turns out, we won’t be able to do the job right. No, in fact, we will learn from this case that the answer to Fermi’s question posed above: ‘how do we calculate this-or-that condensate’ still might get us nowhere. Indeed, to really get to the bottom of QCD we will have to know how to calculate the vacuum fluctuations themselves, and how this could ultimately be done, no one knows. Well, actually, it was hoped that instanton physics might pave the way for a complete understanding here. For instance if we knew that the real gauge potential in nature was just the one-instanton gauge potential, and we knew how to calculate the density of states of the instantons, then we could calculate this-or-that condensate, on top of having direct information about the background field of vacuum fluctuations. By
studying the lifetime of the D meson, we will, at least, learn a few things about the character of these fluctuations. The D meson turns out to be an interesting system for the following reason: the charm quark is both heavy and light, so to speak. Heavy because $\alpha_s(m_c) << 1$ and light since in the OPE expansion, $\mu/m_c$ is not small, i.e. $m_c$ is heavy enough for us to use the OPE expansion, $\mu/m_c \sim 0.5$, but it is light enough that this expansion is not so good! Indeed, at the end of the day, we will find that with the constraints imposed on us for $m_c$, and other pertinent parameters, the standard OPE treatment of the D lifetime FAILS! This failure of duality will then be analyzed. In the end, we will find that the D meson is not such a good kitchen for making the dinner of CP Violation, but is a good kitchen for making the breakfast of QCD. So what I will do in this thesis is little more than the standard QCD treatment - with a few bells and whistles which I helped develop over the past few years - of a heavy quark decaying with a light spectator. In the introduction, I will discuss the OPE method, reviewing how to calculate with it, and explaining why it is so useful for QCD, and in particular the decay of a heavy meson. Then, in the following sections, I will discuss perturbative and nonperturbative aspects of the problem of the lifetime of the D meson, and all relevant features and numerics thereof. In the end, we will be faced with the following conclusion: in the semileptonic decay of the D meson, duality fails, i.e. theory and experiment simply do not match. This failure should be taken as a warning sign for all other hadronic processes below $\approx 2$ GeV. For example, we can not trust the low energy $\alpha_s$ extractions from $\tau$ decays.
Chapter 2

Introduction - the OPE

“So, Dave ... you decided to study theoretical particle physics? Its very, very difficult, you know....” - Arkady Vainshtein.

In the treatment of the decay of the D meson, we rely almost exclusively on Wilson’s OPE [8]. Wilson’s OPE is actually nothing more than a method of calculating with a quantum field theory. It is just the method by which a non-local product of fields is expanded into a product of local fields. The utility of the OPE in QCD is almost immediately evident. The OPE provides us with a basis for dividing processes with soft and hard momenta into noncalculable and calculable parts. In QCD it is just what we need! The running of the strong coupling constant allows us to use perturbation theory for ultraviolet parts of diagrams, while we hide our ignorance of infrared dynamics in condensates.

To learn the essentials of the OPE, it is sufficient to consider a simple problem in a simple theory. Everyone and his brother knows how to calculate the Green’s function in scalar $\phi^4$ theory. Below, we will calculate it, and its first correction, with normal perturbation theory, side by side with the same procedure using Wilson’s OPE. The example is stolen straight out of ref. [10]. Basically, Wilson’s OPE is just the following
\[
\begin{align*}
\phi & D(q) \phi = \\
\begin{array}{c}
\phi \\
q \\
\phi
\end{array} & = \\
\begin{array}{c}
\phi \\
\phi
\end{array}
\end{align*}
\]

Figure 2.1: The propagator for \( \phi^4 \) theory to \( O(\lambda) \).

In the example below, we will implement the relation, see how it generalizes the normal perturbation methods, and directly see its beautiful utility in applications to strong coupled theories. For these illustrative purposes, we will consider a simple example: the Higgs model. We can assume the mass squared parameter of the Lagrangian has positive sign - there are no nonperturbative dynamics - and write the Lagrangian as:

\[
L = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4
\]  

where \( m_0 \) and \( \lambda_0 \) are the bare mass and coupling constants, and the field \( \phi \) is real. Our goal is to calculate the Greens function, \( D(q) \), of the field \( \phi \) in the one-loop approximation using traditional methods, \( D^{pert} \) and the OPE, \( D^{OPE} \).

Consider figure 2.1. Using the Pauli-Villars regularization method, \( D(q) \) can be written as,

\[
D^{pert}(q) = \frac{1}{q^2 - m_0^2} \left[ 1 + \frac{\lambda_0}{32 \pi^2} \frac{M^2 - m_0^2 \ln(M^2/m_0^2)}{q^2 - m_0^2} \right].
\]  

Our task is to reproduce this result using the OPE. Essentially, the procedure is the same as the traditional calculation, except that the perturbative integrals - now the
coefficients of given operators - are integrated down to some infrared scale $\mu$, i.e. the perturbative graphs containing lines with momenta less then $\mu$ flowing through them are ‘cut’. These cut graphs are nothing else than the operators. This particular example is an instructive one, since we can calculate the operators explicitly (unlike in QCD).

The expression from the OPE we will need for correspondance is:

$$D^{OPE}(q) = <0|C_1(\mu)1 + C_\phi\phi^2(\mu)|0>$$  \hspace{1cm} (2.4)

Here, and in normal applications, the operator $C_1$ stands for all of perturbation theory - with the caveat that we exclude infrared contributions. The excluded infrared contribution for $C_1$ rears its head in the subsequent operator terms of the OPE expansion, each of which has its own perturbative contribution, ad infinitum. Note another key feature of the OPE - one which we utilize in heavy quark applications. The series of operators is one of increasing dimension, thus our coefficients must come along with decreasing dimension. The utility in QCD applications is that the expansion will be in powers $\sim \Lambda_{QCD}/Q$ where $Q$ is some large external momenta. To wit, we expand our previous result, eq. 2.3, for large $q$ and thus have a prepared result in normal perturbative methods we can compare to our upcoming OPE calculation. Expanding for $q^2 = -Q^2, Q^2 >> m^2$, we get:

$$D^{pert}(q) \rightarrow \frac{1}{Q^2} + \frac{1}{Q^4}[m_0^2 + \frac{\lambda}{32\pi^2}(M^2 - m_0^2 \ln \frac{M^2}{m_0^2})].$$  \hspace{1cm} (2.5)

Now lets get the OPE result. As stated above, the coefficient $C_1$ is nothing but the perturbative result with the infrared cutoff $\mu$ - note here that the ease with which we implement the cutoff $\mu$ in general doesn’t exist, for example a prescription for cutting off infrared dynamics for multi-loop processes would require great care (but is in fact possible [11]). In the case here, the result is easily achieved - just subtract from the previous result, eq. 2.5, the same expression with the ultraviolet regulator $M$ replaced by $\mu$. Then, $C_1$ is just

$$C_1 = -\frac{1}{Q^2} + \frac{1}{Q^4}[m_0^2 + \frac{\lambda}{32\pi^2}(M^2 - \mu^2 - m_0^2 \ln \frac{M^2}{\mu^2})].$$  \hspace{1cm} (2.6)
Figure 2.2: Figures corresponding to the OPE $\mu$ cutoff prescription for the calculation of the propagator in $\phi^4$ theory.

Now we need to calculate the remaining piece. The coefficient, $C_{\phi^2}$ is just figure 2.2a. Calculating the diagram of figure 2.2a yields the result

$$C_{\phi} = \frac{\lambda_0}{2Q^4}. \quad (2.7)$$

where the factor two comes from combinatorics. Then all we need is the vacuum expectation value $<\phi^2>$. It is just the bubble graph of figure 2.2b with the entire contribution coming from perturbation theory. Note, here we remember to only integrate in the soft region from 0 to $\mu$, and so,

$$<0|\phi^2|0> = \frac{1}{16\pi^2} (\mu^2 - m_0^2 \ln \frac{\mu^2}{m_0^2}), \quad (2.8)$$

Combining these ‘OPE’ equations, we see that they give the same result as the standard perturbative methods, i.e.

$$D^{OPE} = D^{pert}. \quad (2.9)$$

Turning to QCD, one quickly sees the utility of the OPE. The OPE allows us to use perturbation theory in the UV - where $\alpha_s$ is small - and get some information from perturbative coefficients, and in addition, allows us to at least give our ignorance of the soft dynamics a name. Here we were able to calculate the operators directly. In a
strongly coupled theory like QCD, we have no idea what the Greens functions of quarks and gluons are for soft momenta and thus have no right to calculate them - instead we *extract* them phenomenologically, as was done first in SVZ sum rule applications. To be a bit more exact about actual practice, what is done in QCD is not exactly what is done in the example above. Technically, the coefficient functions - for infrared safe quantities - are integrated down to \( \mu = 0 \). In the case of the D meson, it is just because we will borrow the the perturbative results of QED (muon decay). This is a blatent double count, but as it turns out a justified numerical approximation that works well in QCD applications. For instance, in the case of the gluon condensate, the true nonperturbative phenomenologically extracted result is numerically much greater than the perturbatively calculated contribution - a sign of the strongly fluctuating vacuum fields. This assumption goes under the name of the practical OPE. The actual, proper, implementation of the OPE would result in the infrared contribution of coefficient functions being excluded, and thus the introduction of terms \((\mu/Q)^n\) (see eq. 2.6). The practical OPE, then, is the approximation wherein these terms are neglected - an approximation to be checked in each and every application.
Chapter 3

The Semileptonic D lifetime - \( \Gamma(D) \) - an Introduction

“It is so easy to calculate it that it is impossible to make a mistake” - Misha Shifman

If the D meson was just a point particle, all c quark, so to speak, then its decay wouldn’t be all that interesting. Aside from its CKM matrix element, in fact, it would be the same calculation as for the decay of the muon, now a textbook favorite, e.g. [12]. Indeed, the semileptonic decay of the muon yields the decay width:

\[
\Gamma(\mu \to e\bar{\nu}_e\nu_\mu) = \frac{G_F^2 m_\mu^5}{192\pi^3} (1 + A^{(1)} \alpha_e) \tag{3.1}
\]

where the first order perturbative correction, \( A^{(1)} \), was first calculated in ref. [13].

The calculation for the free quark decay \( c \to s l \nu \) goes in the same way, since the weak currents governing the decay have the same form. Thus by multiplying the above expression by the CKM element squared, \( |V_{cs}|^2 \), we have the D decay width with one loop corrections. We can play a quick numerical game and see whether the expression agrees with experiment. Setting \( \alpha_s = 0 \) for now,

\[
\Gamma(D) = \frac{G_F^2 |V_{cs}|^2 m_c^5}{192\pi^3} = 1.1 \times 10^{-13} \text{ GeV} \tag{3.2}
\]
(with\footnote{4}) $m_c = 1.4$ GeV which is to be compared to the experimental value:

$$\Gamma_{\text{expt}}(D) = 1.06 \times 10^{-13} \text{ GeV.} \quad (3.3)$$

It looks like a great agreement! We shouldn’t be so hasty, however. Corrections to the width will come both from perturbative and nonperturbative sources. Unfortunately, as we will see, the result gets spoiled by these corrections\footnote{1}.

To calculate the nonperturbative corrections, it will be necessary to introduce an additional formalism, first proposed by Schwinger in QED\footnote{14}, and later discovered independently in QCD in\footnote{16} with the first applications appearing in\footnote{17} and\footnote{18}. For now we focus on the perturbative corrections. $A^{(1)}$, the coefficient of the $O(\alpha_s)$ corrections can be computed by considering diagrams of the type in fig 5.1 with gluon brehmsstrahlung occurring off of the heavy quark line or the intermediate quark. The result is

$$A^{(1)} = -\frac{2}{3\pi}(\pi^2 - \frac{25}{4}) \quad (3.5)$$

If we wish to borrow this result for the QCD decay, we must address two issues. First, since the result was first calculated in QED, there was no reason to introduce the scale $\mu$ separating hard and soft effects, thus we need some theoretical justification for using the QED result in QCD where there is an infrared problem. Second, numerically it is an important issue as to what point we normalize $\alpha_s$.

Formally, using the QED result, we imply the separation scale $\mu = 0$. Thus using the result in QCD, it means literally that everything to $O(\alpha_s)$ has been calculated -

\footnotetext[1]{This value of $m_c$ is taken from the early sum rule estimates\footnote{3}. I will return later to a full discussion of the heavy quark mass.}

\footnotetext[2]{As an aside, note that there is a phase space suppression coming from the fact that the $s$ quark mass in the final state is massive. The suppression to the partonic width is given by}

$$\phi[z] = 1 - 8z + 8z^3 - z^4 + 12z^2 \ln[z] \quad (3.4)$$

Numerically, taking the $s$ quark mass as around 150 MeV, this effect is roughly 5 percent, if we are after an accuracy greater than this, then, of course, these phase space suppressions must be taken into account.
our work is done. Of course, this is nonsense - we have no right to believe that the quark/gluon Green’s functions at low virtuality correspond to the strongly coupled dynamics that actually occur for scales less than $\Lambda_{QCD}$. On the other hand, we don’t want to throw away this useful result. The answer to the puzzle was discussed in section 2: we use what is known as the practical OPE [10]. Proper use of the OPE requires that we cut our integrals off at the infrared scale $\mu$. Doing so would introduce corrections of the order $(\mu/Q)^n$ (see eq.(2.6)). In the practical OPE, these terms are discarded. The assumption is that the perturbative piece that we have no right to include (from virtualities 0 to $\mu$) is numerically much smaller than the phenomenologically extracted operator.

As for the second question, what scale do we use for $\alpha_s$, there seem to be two options. Formally, we are allowed to take whatever scale of $\alpha_s$ we like since the difference is in the next order of $\alpha_s$. By choosing the appropriate scale, however, we minimize the corrections at next order. The most physically reasonable scale is obviously the scale $\mu = \text{const.} \times m_c$. This is the scale involved in the physical decay - any emitted gluon would carry this momenta. To see which scale actually comes into play one has to consider next order corrections. Here, one can use what is called the BLM approach [19]. This generalized approach is applicable when one deals with a single gluon line (dressed by all bubbles). It amounts to inserting the unexpanded expression for the running coupling constant

$$\alpha_s(k^2) \approx \frac{4\pi}{b \ln(k^2 \Lambda_{QCD}^2)}$$

inside the integrand of a one-loop Feynman graph which depends on the gluon momentum $k$, with the subsequent integration over $k$. Letting $\alpha_s$ run we have perturbative integrals like

$$\sim \int \frac{\alpha_s(k^2)}{k^2}.$$  

For the kinematics of the problem at hand, such integrals will be saturated at scales $m_c$. 
Now we would like to include nonperturbative effects. The most efficient way to calculate the nonperturbative corrections is by using the background field technique and the Fock-Schwinger gauge. In the next section, I review both formalisms.
Chapter 4

Calculations in external fields in QCD

“Getting the damn 2’s and π’s in the right place is the whole point!” - Feynman

Here, we review a formalism proposed by Schwinger [14] for the description of motion of particles in external fields. I will rely heavily on the formalism (developed in [20]) to calculate the nonperturbative corrections to the semileptonic D lifetime. The Schwinger operator approach is formulated in coordinate space. Let us introduce a set of states $|x>$ which are the eigenvalues of the coordinate operator $X_\mu$:

$$X_\mu|x>=x_\mu|x>.$$ (4.1)

We also introduce the momentum operator $P_\mu$ which satisfies the commutation relations:

$$[P_\mu,X_\nu]=ig_{\mu\nu}$$ (4.2)

$$[P_\mu,P_\nu]=igT^aG_{\mu\nu}^a$$ (4.3)

where $T^a$ is the color group generator, related to the Gell-Mann matrices by $T^a=\lambda_a/2$.

In addition to the above relations, we will work in a specific gauge - the Fock-Schwinger
gauge. The Fock-Schwinger gauge is the condition

\[ x_\mu A_\mu = 0. \]  \hspace{1cm} (4.4)

There are two conveniences resulting from this gauge choice that we will take advantage of. The first is that the potential is written in terms of the field strength tensor, \( G_{\mu\nu} \), and thus easily yields gauge invariant expressions. The second is that \( A^a_\mu(0) = 0 \), as we will now see.

### 4.1 The gauge potential in the Fock-Schwinger gauge

With our choice of gauge in hand, let’s write out an expansion for the potential, \( A_\mu \). We start with the identity:

\[ A^a_\mu(x) = \int_0^1 \alpha d\alpha G^a_{\rho\mu}(\alpha x)x_\rho. \]  \hspace{1cm} (4.5)

We can prove it, since, from our gauge condition,

\[ A_\mu(y) = \frac{d}{dy_\mu}(A_\rho(y)y_\rho) - y_\rho \frac{dA_\rho(y)}{dy_\mu} = -y_\rho \frac{dA_\rho(y)}{dy_\mu}, \]  \hspace{1cm} (4.6)

then inserting the definition of the field strength tensor,

\[ y_\rho \frac{dA_\rho(y)}{dy_\mu} = y_\rho G_{\mu\rho} + y_\rho d_\rho A_\mu + y_\rho [A_\mu, A_\rho] = y_\rho G_{\mu\rho} + y_\rho d_\rho A_\mu, \]  \hspace{1cm} (4.7)

so then

\[ A_\mu(y) + y_\rho \frac{dA_\mu(y)}{dy_\rho} = y_\rho G_{\rho\mu}(y). \]  \hspace{1cm} (4.8)

Now, reparametrize \( y = \alpha x \), then

\[ \frac{d}{d\alpha}(\alpha A_\mu(\alpha x)) = A_\mu(\alpha x) + \alpha \frac{d}{d\alpha} A_\mu(\alpha x), \]  \hspace{1cm} (4.9)

and so,

\[ \int_0^1 \alpha x_\rho G_{\rho\mu}(\alpha x) d\alpha = \int_0^1 \alpha \frac{d}{d\alpha}(\alpha A_\mu(\alpha x)) d\alpha = x_\mu A_\mu. \]  \hspace{1cm} (4.10)
Now, to get the expansion for the gauge potential, write the field strength tensor as,

\[ G^\alpha_{\rho\mu}(\alpha x) = G_{\rho\mu}(0) + \alpha x \frac{d}{d(\alpha x)} G_{\rho\mu}(0) + \ldots \] (4.11)

insert this into the integral representation and note that since in this gauge, \( d_\alpha = D_\alpha \)

\[ A^\alpha_\mu(x) = \frac{1}{2 \cdot 0!} x_\rho G_{\rho\mu}(0) + \frac{1}{3 \cdot 1!} x_\alpha x_\rho(D_\alpha G_{\rho\mu}(0)) + \ldots \] (4.12)

### 4.2 Calculating propagators in the Fock-Schwinger gauge

As an exercise in this gauge, we can try and calculate propagators using the external field method [17]. We will need such propagators later. The scalar propagator for a massless particle in the Fock Schwinger gauge is written as,

\[ S(q) = \int dx e^{iqx} < x | x \rangle = \int dx < x | (P + q)^2 | 0 >, \] (4.13)

since

\[ e^{iqX} P_\mu = (P_\mu + q_\mu)e^{iqx}. \] (4.14)

Now expand this propagator in powers of \( P/q \). Here we use a simplification resulting in our choice of the Fock-Schwinger gauge referred to above:

\[ A_\mu | 0 > = 0. \] (4.15)

Thus, our main strategy is to expand in \( P/q \) and pull all gauge potentials to the right. Also, since

\[ \int dx < x | p_\mu \cdots = \int dx dy i d_\mu \delta(x - y) < y | \cdots | 0 > = 0 \] (4.16)

we pull all p’s to the left. The only other ingredient is just commutators between p’s and A’s, which are trivial to evaluate in the Fock-Schwinger gauge. Expanding equation (4.13) in \( P/q \),

\[ \frac{1}{(P + q)^2} = \frac{1}{q^2} - \frac{2Pq}{q^4} + \frac{4(Pq)^2}{q^6} - \frac{P^2}{q^4} + \frac{P^2(2Pq)}{q^6} + \frac{(2Pq)^2}{q^8} - \frac{8(Pq)^3}{q^{10}} + O(P^4) \] (4.17)
and performing the procedure described above, we get

\[ D(q) = \frac{1}{q^2} - \frac{g}{3q^6} D_\alpha G_{\alpha \rho} q_\rho \]

\[ -\frac{g^2}{2q^8} (q_\alpha G_{\alpha \rho} G_{\rho \gamma} q_\gamma + \frac{1}{4} q^2 G_{\alpha \rho} G_{\alpha \rho} ) - \frac{ig}{q^8} q_\gamma D_\gamma D_\alpha G_{\alpha \rho} q_\rho + ... \] (4.18)

The case of the scalar propagator is of course easiest since we do not have to worry about the anticommuting \( \gamma \) matrices. We will, however, need the result for spinor particles. The result\(^1\) (in the massless case) of this calculation is, to \( O(DG) \):

\[ S(q) = \frac{1}{q} - \frac{g}{2q^4} q_\alpha \tilde{G}_{\alpha \rho} \gamma_\rho \gamma_5 \]

\[ + \frac{g}{3q^6} [q^2 D_\alpha G_{\alpha \rho} \gamma_\rho - \tilde{q} D_\alpha G_{\alpha \rho} q_\rho - q_\gamma D_\gamma q_\alpha G_{\alpha \rho} \gamma_\rho \]

\[ - 3i q_\gamma D_\gamma q_\alpha \tilde{G}_{\alpha \rho} \gamma_\rho \gamma_5]. \] (4.19)

---

\(^1\)Note, here I use \( \hat{p} \) instead of the Feynman convention, \( \gamma \), to denote the Dirac matrices - Misha Voloshin taught me this, explaining that to him, a slash meant that the given variable was crossed out!
Chapter 5

Nonperturbative corrections to the lifetime of the D

“The heavy quark is heavy - its like a locomotive flying through mosquitoes” - Misha Voloshin

In section 3, we used the textbook calculation for muon decay to get our hands dirty for the case of the D meson. Let me derive this result in a fairly non-standard but equivalent way by using the optical theorem. The technique will be a bit long-winded for the standard partonic calculation, but will simplify life greatly when we are after nonperturbative corrections.

5.1 Derivation of parton result using the optical theorem

The optical theorem relates the imaginary part of a forward scattering amplitude to its observable cut process. The optical relation valid for the total widths is,

$$\Gamma = \frac{1}{M_D} \text{Im} \langle D|\hat{T}|D \rangle,$$

(5.1)
Figure 5.1: The forward scattering diagram for semileptonic decay. The dashed line represents the 'cut', i.e. the imaginary part of the diagram.

(the relation above implies the use of relativistic normalization of states) which is our starting point. Here $\hat{T}$ is the time ordered product of, in this case, the two weak currents governing the semileptonic decay. From figure 5.1, we get the transition operator in configuration space as,

$$\hat{T} = -\frac{G_F^2}{2} \int d^4x \bar{Q}(x) \Gamma_\mu S_q(x,0) l_{\mu\nu}(x) \Gamma_\nu Q(0),$$

where $Q$ represents the heavy quark external line, $S_q$ is the light quark propagator, the $\Gamma$’s are the usual $V \times A$ Dirac matrices ($\Gamma_\mu = \gamma_\mu (1 - \gamma_5)$), and the lepton tensor is,

$$l_{\mu\nu}(x) = \text{Tr}[\Gamma_\mu \frac{-1}{2 \pi^2 x^4} \Gamma_\nu \frac{1}{2 \pi^2 x^4}] = -\frac{2}{\pi^4 x^8} (2x^\mu x^\nu - x^2 g^{\mu\nu}).$$

(5.3)

To get the parton result, take the light quark propagator as free:

$$S_q(x,0) = \frac{\hat{x}}{2\pi^2 x^4}.$$

(5.4)

Upon convolution of Dirac matrices, $\hat{T}$ becomes:

$$\hat{T} = \int d^4x \bar{Q}(x)f(x)(1 + \gamma_5)Q(0),$$

(5.5)

where $f(x)$ is a function of $x$. The interaction of the heavy quark with the light degrees of freedom enters through the Dirac operator, $D_\mu = d_\mu - igA_\mu t^a$. The background
The gluon field $A_\mu$ is small compared to $m_Q$, and thus there is a large ‘mechanical’ part in the x-dependance of $Q(x)$ [23]:

$$Q(x) = e^{im_0t} \tilde{Q}(x).$$

(5.6)

Our transition operator is still a nonlocal object. To write it as a local object we should Taylor expand the heavy quark field in the Fock-Schwinger gauge:

$$\partial \tilde{\bar{Q}}(x) = (1 + x_{a1}\partial_{a1} + \frac{1}{2} x_{a1} x_{a2}\partial_{a1}\partial_{a2} + ...) \tilde{Q}(0)
= (1 + x_{a1}D_{a1} + \frac{1}{2} x_{a1} x_{a2}D_{a1}D_{a2} + ...) \tilde{Q}(0).$$

(5.7)

The above expansion is legal since in the Fock-Schwinger gauge ordinary derivatives may be substituted by full derivatives at the origin. We can get rid of the $\gamma_5$ term in equation (5.5) since, through the equations of motion it appears as a full derivative

$$< D|\frac{\partial}{\partial x_\mu}(\bar{Q}\gamma_\mu\gamma_5 Q)|D > = 2im_Q < D|\bar{Q}\gamma_5 Q|D >,$$

(5.8)
and the derivative of the forward scattering matrix element gives zero (this matrix element and others like it containing $\gamma_5$ are zero from parity arguments anyway). Upon convolution of the lepton tensor with $\Gamma_\mu$’s, etc., we are left with

$$\hat{T} = \text{const.} \int d^4x e^{i p x} \tilde{\bar{Q}}(0) \frac{\hat{x}}{x^{10}} Q(0).$$

(5.9)

To do the integral (see the Appendix in [20]), we use the result

$$I = \int d^4x \frac{e^{ipx}}{(x^2)^n} = \frac{i(-1)^n 2^{(4-2n)}/\pi^2}{\Gamma(n-1)\Gamma(n)} (p^2)^{n-2} \ln[-p^2],$$

(5.10)
which gives, in our case,

$$\hat{T} = -i\gamma_\mu \frac{d}{dp_\mu} I = -i\gamma_\mu \frac{d}{dp_\mu} \frac{i(-1)^5 2^{-6} \pi^2 p^6 (\ln|p^2| + i\pi)}{4! 3!}$$

(5.11)
and so finally we get

$$\hat{T} = i\frac{G_F^2 V_{e3}^2 \bar{Q}(0) p^4 \hat{p} Q(0)}{384\pi^3}. $$

(5.12)

1Note, this, the near end result for the parton decay width, is the beginning of where we look for the nonperturbative $1/m^2_c$ pieces.
To leading order, $p^4 = \hat{P}^4$ (we will see it later). Substituting this result, and using the Dirac equation, we are left with

$$\Gamma_0(D) = \frac{1}{M_D} \text{Im}(\frac{iG_F^2 V_{cs}^2 m_c^5}{384\pi^3}) < D|\bar{Q}(0)Q(0)|D >$$

(5.13)

Also to leading order (we will see it explicitly later (equation (5.14)),

$$< D|\bar{Q}(0)Q(0)|D > \sim < D|\bar{Q}(0)\gamma_0 Q(0)|D > = 2M_D,$$

(5.14)

i.e. this operator merely counts the number of $c$ quarks in the D meson - it is basically the number density. Here we get into points concerning $\mu$ dependence of operators involved in our expression. Technically, $\bar{Q}Q$ carries $\mu$ dependence. Moreover, if $\mu$ is large, say a few GeV, then we are not entitled to say that there is only one $c$ quark in the D - we must evolve the operator down to a few hundred MeV - a typical hadronic scale. Later we will see that the heavy quark mass should be at a normalization point $\mu = m_c$. If so, then the operator $\bar{Q}Q$ is also at $\mu = m_c$. We must evolve this operator down to a lower normalization point where we can say that all fluctuations are soft, and thus do not involve the creation of heavy quark anti-heavy quark pairs. Evolving this operator down is equivalent to pumping the fluctuation contained in the operator into the coefficient function. These fluctuations are actually already contained in the perturbative correction, $A^{(1)}$ anyway, so we can forget about them. Now, upon combining the relations above, we finally get the expected result:

$$\Gamma_0(D) = \frac{G_F^2 V_{cs}^2 m_c^5}{192\pi^3}.$$  

(5.15)

### 5.2 $O(1/m_Q^2)$ corrections

To get the $O(1/m_Q^2)$ corrections, we will need one additional piece of machinery, the expansion

$$\bar{Q}Q = \bar{Q}\gamma_0 Q - \frac{\bar{Q}[(i\not{D})^2 - (i/2)\sigma G]Q}{2m_Q^2} + O(1/m_Q^4) + \text{total derivatives.}$$

(5.16)
The $O(1/m_Q^2)$ result in the above equation was first derived in [21]. The relation itself can be easily established from the equations of motion. Since,

$$\frac{1 - \gamma_0}{2} c = \frac{1}{2m_c} \hat{\pi} c,$$

(5.17)

then,

$$\bar{c} \frac{1 - \gamma_0}{2} c = \bar{c} \frac{1 - \gamma_0}{2} \frac{1 - \gamma_0}{2} c = \frac{1}{4m_c^2} \bar{c} \hat{\pi} \hat{\pi} c,$$

(5.18)

which implies that

$$\bar{c} (1 - \gamma_0) c = \frac{1}{2m_c^2} \bar{c} (\pi^2 + \frac{i}{2} \sigma G) c.$$  
(5.19)

Then, using the equations of motion again, we see that $\bar{c} \pi^2 c$ actually reduces to an operator of dimension 7, since

$$\pi_0 Q = -\frac{\pi^2 + (i/2) \sigma G}{2m_Q} Q$$

(5.20)

and so we arrive at the result above.

There are various possible sources for $1/m_c^2$ corrections to the parton result. First, recall that the first term of the gauge potential in the Fock-Schwinger gauge is $\sim G$. This term actually gives a contribution of $O(1/m_Q^3)$ (we will see these terms later). There are however two other sources of $1/m_c^2$ terms. The first is from equation (5.16) above in the expansion of $\bar{Q}Q$, and the second is from the identity

$$P^2 = \hat{P}^2 - \frac{i}{2} \sigma G$$

(5.21)

which we plug into eq. 5.12 (the identity above is obvious since $\gamma_\mu \gamma_\nu = g_{\mu\nu} + \sigma_{\mu\nu}$).

Using these two results, we then plug $\hat{T}$ into the optical relation. The result is

$$\Gamma(D) = \frac{G_F^2 V_{cs}^2}{192\pi^3 m_c^5} \times [1 + A^{(1)} \alpha_s - \frac{3}{2} \frac{\mu_c^2}{m_c^2} - \frac{1}{2} \frac{\mu_\pi^2}{m_\pi^2}]$$

(5.22)

Two points are in order here. First, notice that there is no contribution coming from operators $O(1/m_c)$, i.e. there is no operator of dimension 4 [22]. Second, since $\Gamma$ is a Lorentz scalar, we should not be surprised to see that the only operators appearing
in $\Gamma$ are Lorentz scalars ($\mu^2$ enters only in combination with $Q\gamma_0Q$ - from the original Lorentz invariant operator $Q\bar{Q}$).

Our task is now somewhat complete - we have the partonic width, and the leading perturbative and nonperturbative corrections. Now we would like to do some numerics and see whether we match experiment. This is an area of hot debate for sure, but as we will see, the failure of the perturbative and nonperturbative corrections to give the experimental result is fairly insensitive to most of the numerical debates. Clearly the most important numerical issue will be the mass of the $c$ quark, since it appears as the fifth power in the parton result. We now turn to a complete discussion of numerics.
Chapter 6

Theoretical meaning and numerics of nonperturbative parameters

“...the set of values to be passed from the elders to the young generations included the idea that high energy physics is an experimental science that must be very closely related to phenomena taking place in nature....” - Misha Shifman

“So Dave, you would like a problem to work on? Well, I must warn you that I will not give you something very mathematical, but instead more phenomenological - any results you will obtain will probably have direct relevance to current experiments...” - Misha Shifman, much to my excitement!

In this section, I discuss the meaning and numerical values of the heavy quark mass $m_c$, $\mu^2_\pi$, and $\mu^2_G$. 
6.1 The meaning of the heavy quark mass

The issue of the heavy quark mass can easily be anticipated to be a sticky one. As everyone knows, quarks are not color singlets and thus do not appear isolated in nature (quarks are not asymptotic states, if you like). And yet, the theory of QCD is built upon quarks and gluons. Clearly, a key parameter in any QCD calculation will then be the quark mass - it enters in the QCD Lagrangian. In the calculation of the D meson lifetime, we will be borrowing perturbative results from QED: the one loop correction to muon decay. In the QED calculation, the muon mass used is just the pole mass. In QCD, this becomes a problem since by using the pole mass of the heavy quark, we are unjustifiably including long distance dynamics. Using the pole mass, we are essentially assuming that we have knowledge of gluonic Greens functions at all scales - a clearly unjustified assumption. In fact, it can be shown that the pole mass of the heavy quark is defined through an asymptotic series in $\alpha_s$, leading to an ambiguity of order $\Lambda_{QCD}$. This is a strong statement, and with the propagation of the words ‘pole mass’ in almost every paper on QCD, one gets the feeling that this fact is somehow avoided, or misunderstood. The misunderstanding comes from whether we are talking about the ‘pole mass’ or the ‘one loop pole mass’. Although, as we will see below, the pole mass is ill-defined, one can rightfully use the one loop pole mass, as I will below, in any typical problem. In the problem of the D meson decay, use of the one-loop pole mass really helps simplify life - we can just use the well-known QED result for the muon decay, without including in it some scale dependance (i.e. cut off loop integrations at some scale). Then, use of the pole mass is proper, since we have, in effect, set the scale - at zero virtuality - by using the QED result. Here, we see the conceptual difference between the $\phi^4$ exercise offered earlier, and what is called the practical OPE by QCD practitioners. Let me then, in turn, review two main points concerning the heavy quark mass:
6.1.1 Illegitimacy of the pole mass

Although commonly used in practical applications, it should be understood that the pole mass of the quark is an ill-defined object. The subject is thoroughly discussed in [26], and I only briefly touch upon some essentials here. To see that the pole-mass is ill-defined, consider the relationship between the pole mass and the running mass,

$$m_{\text{pole}}^{(k)} = m_Q(\mu) \sum_{n=0}^{k} C_n \left( \frac{\mu}{m} \right)^n \left( \frac{\alpha_s(\mu)}{\pi} \right)^n$$  \hspace{1cm} (6.1)

The above series is not well behaved, diverging factorially. Such a series is an asymptotic series, and must be truncated at a critical order in the expansion. After truncation, the remainder of the series shows up as an uncertainty (knowing the exact remainder would constitute having knowledge of the vacuum fluctuations).

One particular source of the factorial divergence is the infrared renormalon. The infrared renormalon is the effect of the insertion of the bubble chain in gluon lines, i.e. allowing the running of the strong coupling constant, $\alpha_s \rightarrow \alpha_s(k^2)$, see figure 6.1. For nonrelativistic momenta, $k^2 \ll \mu^2$, the expression for the mass correction is

$$\delta m_Q \sim \int \frac{d^4k}{(2\pi)^4 i k_o} \frac{4\pi \alpha_s(-k^2)}{k^2} \sim \int \frac{d^3k}{4\pi^2} \frac{\alpha_s(k^2)}{k^2}.$$  \hspace{1cm} (6.2)

Expressing the running $\alpha_s(k^2)$ in terms of $\alpha_s(\mu^2)$, $k^2 < \mu^2$,

$$\alpha_s(k^2) = \alpha_s(\mu^2) \left( 1 + \frac{\alpha_s(\mu^2)}{4\pi} b \ln \frac{k^2}{\mu^2} \right)^{-1} , \quad b = \frac{11}{3} N_c - \frac{2}{3} N_f$$  \hspace{1cm} (6.3)
one can expand $\alpha_s(k^2)$ in a power series of $\alpha_s(\mu^2)$. One then finds for the $(n+1)$-th order contribution,

$$
\frac{\delta m_Q^{(n+1)}}{m_Q} \sim \alpha_s(\mu^2)\pi n!(\frac{b \alpha_s(\mu^2)}{2\pi})^n.
$$

(6.4)

Observe that the coefficients grow factorially and contribute with the same sign. Therefore one cannot define the sum unambiguously. An optimal truncation leaves one with an irreducible error of $O(\Lambda_{QCD})$. To see this, note that truncating the factorial growing series at optimal order (see the section on duality violations in chapter 10) leaves an exponential uncertainty,

$$
\Delta(m_Q^{pole} - m_Q(\mu_0)) \sim \mu_0 \exp(-\frac{2\pi}{b\alpha_s(\mu_0)}) \sim \Lambda_{QCD}.
$$

(6.5)

6.1.2 Irrelevance of the pole mass

Anyway, in applications, it is the running mass and not the pole mass which enters into our theoretical expressions. As explained above, the perturbative series for $m_Q^{pole}$ diverges factorially. In our expression for the width, a similar divergence occurs in the $\alpha_s$ expansion due to renormalons. If the running mass is used, however, both effects combine to cancel each other. This point is investigated thoroughly in [26]. The result is really neat. It tells us, basically, to perform the following procedure when extracting quantities like the mass from total widths [33]. First, write the total width as an expansion in $\alpha_s$ and $1/m_c$. The $\alpha_s$ series is, as in this case, to be taken at $\mu = 0$, i.e., the QED results can be borrowed. Next, one can numerically investigate the coefficients of the $\alpha_s$ series. In the BLM procedure, the corrections at two-loops are easily calculated, and give large coefficients. Ref. [26] tells us that these coefficients are nothing more than the signal of the factorially diverging perturbation series. Not to worry, since, by inserting the running mass, $m_Q(m_Q)$ we can ‘cancel’ the infrared renormalon effect responsible for these large coefficients. After this procedure, aside

\footnote{Same sign series can not be summed since the pole of such a series will lie on the real axis, and the contour around such a pole when Borel summing cannot be unambiguously defined.}
from issues of duality violation, it is perfectly alright to extract the running mass from the theoretical expression for the width. The running mass itself is easily calculated (see figure 6.2): to one loop the result is

\[ m_Q(\mu) = m_Q(\mu_0) + \frac{2\alpha_s}{3\pi}(\mu_0 - \mu). \]  

(6.6)

6.2 The value of \( m_c \)

In this subsection we address the question ‘what is the value of \( m_c \)’. \( m_c \) was first extracted from the QCD sum rule application to charmonium, and so we review the sum rules here. As a cross check, I also present an estimate of \( m_c \) from \( m_b \).

6.2.1 \( m_c \) from sum rules

Here, I derive one of the first OPE type applications in QCD - charmonium sum rules, \cite{7}. The derivation relies on the standard OPE treatment found in sum rules, or heavy quark applications. The starting point is quark-hadron duality: on the left-hand side of the equation, there is a theoretical construction in terms of quarks and gluons, on the
right hand side is a quantity extracted from some hadronic experiment. The assumption of quark-hadron duality is built in from the first step. As we will see later, this is a well justified assumption in applications where we rely on dispersion relations like the sum rules\(^2\), but we must take care in making the assumption for the D meson. The reason for this is thoroughly discussed in chapter 10, and resides in the fact that sum rules rely on Euclidean kinematics, while the lifetime of the D meson is of Minkowskian kinematics.

Our starting point in the investigation is the T product of two electromagnetic currents, (see figure 6.3):

\[
i < 0 | \int dx e^{iqx} T[j_\mu(x) j_\nu(0)]|0> = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(Q^2),
\]

(6.7)

where \(j_\mu = \bar{c}\gamma_\mu c\), and \(Q^2 = -q^2\). The polarization operator, \(\Pi\) satisfies dispersion relations:

\[
-\frac{d}{dQ^2} \Pi = \frac{1}{12\pi^2 Q^2} \int \frac{R_c(s) ds}{(s + Q^2)^2}, \quad R_c = \frac{3\sigma_c}{4\pi^2}.
\]

(6.8)

We will especially be interested in the moments of \(\Pi\),

\[
M_n = \frac{1}{12\pi^2 Q^2} \int \frac{R_c(s) ds}{s^{(n+1)}} = \frac{1}{n!} \left(-\frac{d}{dQ^2}\right)^n \Pi|_{Q^2 \rightarrow 0}.
\]

(6.9)

The simplest quark loop, is easily shown to be:

\[
M_n^{(0)} = \frac{3}{4\pi^2} \frac{2^n(n+1)(n-1)!}{(2n+3)!!} \frac{1}{(4m_c^2)^n}.
\]

(6.10)

Of course, the point of the sum rules was to go beyond such a simple expression, and include not only \(\alpha_s\) corrections, but also new power corrections like those we have been discussing above. In the case of charmonium, the first power correction is the gluon condensate. As it turns out, however, the contribution of the gluon condensate is highly suppressed for low moments, as are perturbative corrections - and so the key parameter

\(^2\)It is another reason I review charmonium sum rules.
for low moments is just $m_c$. Extracting it from the data, and ascribing a rather large uncertainty \[7\],
\[
m_c(m_c) = 1.26 \pm 0.10 \text{ GeV} \tag{6.11}
\]

### 6.2.2 $m_c$ from $m_b$

As a check of this result, we can examine another source where $m_c$ can be extracted. In \[24\] the following result was first derived,
\[
M_B - M_D = m_b - m_c + \frac{\mu_\pi^2 - \mu_G^2}{2m_b} - \frac{\mu_\pi^2 - \mu_G^2}{2m_c} + O(m_c^2, m_b^2). \tag{6.12}
\]

The meson masses are measured accurately, ($M_B = 5.287$ GeV and $M_D = 1.864$ GeV), and the value of $m_b$ is known with high accuracy from upsilon sum rules \[32\] $m_b(1\text{GeV}) = 4.64 \pm 0.05$ GeV. Plugging in these values, we get $m_c = 1.2 \pm 0.05$ GeV where the uncertainty comes mainly from the uncertainty in $\mu_\pi^2$ (see below).

### 6.2.3 Comments on the literature

Despite the above arguments, there are still claims in the literature that one can take the pole mass of the charmed quark to be very high, $m_c = 1.65$ GeV. A typical argument can go as follows. First, expressions for $\Gamma(D)$, and $m_c^{pole}$ are given, both with full one-loop and BLM type two-loop corrections. The explicit expression for the mass is
\[
m_c^{pole} = \bar{m}_c[1 + \frac{4}{3} \frac{\alpha_s(m_c)}{\pi}(1 + 1.04)] \tag{6.13}
\]
where ‘1.04’ is due to the BLM type two-loop correction. The high value $m_c = 1.65$ GeV is obtained by directly accomodating the experimental measurement of $\Gamma(D)$ with the full theoretical expression, including nonperturbative effects. Then one uses the above equation to run $m_c^{pole}$ to $\bar{m}_c$, and obtain $\bar{m}_c \sim 1.34$ GeV which is claimed to be close to the sum rule prediction.
A clear signal that this analysis is in trouble is that the value of the pole mass of the b quark extracted is $m_{b}^{\text{pole}} \sim 5.0 \text{ GeV}$ in disagreement with Voloshin’s extraction, $m_{b}^{\text{pole}} \sim 4.8 \text{ GeV}$. What is wrong with the above analysis? The error is nothing more than a manifestation of the problems occurring when using the pole mass. Clearly, the above expansion for $m_{c}$ is not converging well. This fact is actually to the benefit of those who would like to fit $m_{c}^{\text{pole}}$ from $\Gamma(D)_{\text{expt}}$ and then run down to $\bar{m}_{c}$ - the factor ‘1.04’ allows such a large running. The problem is that it is not the pole-mass which really enters theoretical expressions, but the running mass.

Above, it was discussed that the proper mass to use in expressions for the full width is the running mass. In ref. [33] it was explicitly shown that using the proper running mass, we reduce large second order corrections to the width in heavy quark decay. Explicitly, ref. [33] showed that for the semileptonic width we have large two-loop corrections,

$$\frac{\Gamma(B)}{\Gamma_{0}(B)} = 1 + a_{1}\left(\frac{\alpha_{s}}{\pi}\right) + a_{2}\left(\frac{\alpha_{s}}{\pi}\right)^{2}$$

(6.14)

where $a_{2}$ is the BLM type correction, and $a_{1} = -2.41$, and $a_{2} = -19.7$ but upon insertion of the proper OPE scale, say, $\mu \sim 400 \text{ MeV}$, $a_{1} \rightarrow -0.5$ and $a_{2} \rightarrow -0.4$, and thus upon insertion of the running mass, the second order corrections become small. Upon insertion of these corrections to the total width, we obtain

$$\Gamma(D) = \Gamma_{0}(1 - 0.07 - 0.008 - 0.27 - 0.09) = \Gamma_{0}(0.55),$$

(6.15)

where $-0.07, -0.008, -0.27,$ and $-0.09$ are from $\alpha_{s}, \alpha_{s}^{2}, \mu_{G}^{2},$ and $\mu_{\pi}^{2}$ corrections respectively (I will discuss these numerics in full in the next section).

Now, extracting the mass from this result, the running mass mind you, we get

$$m_{c}(400\text{MeV}) = 1.57\text{GeV}$$

(6.16)

\footnote{the result for $a_{2}$ is presented in the V scheme, a value around $a_{2} \sim -30$ would be obtained for the MS scheme used in some papers, allowing an even GREATER value of $m_{c}$.}
which is in disagreement with the sum rules estimate, a result which I discuss again in section 7. When using the proper OPE procedure, outlined above, the extraction of $m_c$ from semileptonic D decay and charmonium sum rules disagree.

6.3 About $\mu_G^2$

The mesonic matrix elements of the chromomagnetic operator, $\sigma G$ can be extracted from the hyperfine splitting. First, note that

$$\vec{\sigma} \cdot \vec{G} = \vec{\sigma} \cdot \vec{B} + O(1/m_c). \quad (6.17)$$

Now, $\vec{B}$ is an axial vector and is determined by the dynamics of the light spectator cloud, thus we have only one choice, $\vec{B} = \text{const} \times \vec{S}_q$ and so the matrix element must be:

$$< Q\bar{q} | \frac{g\vec{\sigma} \cdot \vec{B}}{2m_Q} | Q\bar{q} > = C < Q\bar{q} | \vec{S}_Q \cdot \vec{S}_\bar{q} | Q\bar{q} >. \quad (6.18)$$

Now,

$$\vec{S}_Q \cdot \vec{S}_\bar{q} = \frac{1}{2} (\vec{S}_Q^2 + \vec{S}_\bar{q}^2) - \frac{1}{2} (\vec{S}_Q^2 + \vec{S}_\bar{q}^2) = \frac{1}{2} S_{TOT} (S_{TOT} + 1) - \frac{3}{4}, \quad (6.19)$$

and,

$$< 0^- | \vec{\sigma} \cdot \vec{B} | 0^- > = -3 < 1^- | \vec{\sigma} \cdot \vec{B} | 1^- >, \quad (6.20)$$

since for an uncorrelated spin state, the matrix element is zero. Then, we find simply that,

$$M_{(0^-)} - M_{(1^-)} = < 0^- | \vec{\sigma} \cdot \vec{B} | 0^- > - < 1^- | \vec{\sigma} \cdot \vec{B} | 1^- > = \frac{4}{3} < 0^- | \vec{\sigma} \cdot \vec{B} | 0^- > \quad (6.21)$$

and so,

$$< \mu_G^2 >_D = D |(\vec{Q}^2/2 \vec{\sigma} \cdot \vec{G})Q|D >= \frac{3}{2} m_c (M_{D^*} - M_D) = \frac{3}{4} (M_{D^*}^2 - M_D^2). \quad (6.22)$$

Numerically, taking this mass splitting from the D and B system respectively, we get:

$$< \mu_G^2 >_D = 0.41 \text{ GeV}^2, < \mu_G^2 >_B = 0.37 \text{ GeV}^2. \quad (6.23)$$
A measure of the reliability of the nonperturbative expansion is then provided by \( \sqrt{<\mu_G^2>_D/m_c^2} = 0.46 \). And thus, right off the bat there is good reason to pursue higher order nonperturbative terms, \( O(1/m_c^3) \), in the expansion.

### 6.4 About \( \mu_{\pi}^2 \)

The mesonic matrix element of the time dilation operator, \( <D|\bar{Q}(i\vec{D})^2Q|D> = \mu_{\pi}^2 \) is not known accurately. An analysis based on sum rules yields \( 31 \)

\[
\mu_{\pi}^2 = 0.5 \pm 0.1 \text{ GeV}^2.
\] (6.24)

This result is in agreement by a bound from \( 27, 28, 29, 34 \),

\[
\mu_{\pi}^2 \geq \mu_G^2 \geq 0.37 \text{ GeV}^2
\] (6.25)

This inequality is of huge importance, and much controversy \( 31 \). The derivation of this inequality using quantum mechanics, taught to me by Misha Voloshin in his excellent class on phenomenology, is simple. Consider the absolute positive quantity,

\[
(\vec{s} \cdot \vec{\pi})^2 = \vec{\pi}^2 - g\vec{s} \cdot \vec{B},
\] (6.26)

then, obviously, the above inequality holds. The field-theoretic inequality was later derived in \( 34 \). There, the sum rule for the pseudoscalar weak current, \( J_5 \) was written at zero-recoil,

\[
\frac{1}{2\pi} \int_0^\mu w^{(5)}(\epsilon) d\epsilon = \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 (\mu_{\pi}^2(\mu) - \mu_G^2(\mu)).
\] (6.27)

Since the structure function \( w^{(5)} \) is non-negative, one arrives at the conclusion that \( \mu_{\pi}^2(\mu) \geq \mu_G^2(\mu) \).

Still, there are arguments in the literature that this inequality is broken by the perturbative evolution of \( \mu_{\pi}^2 \). Consider figure 6.4 below The figure represents the issue of concern regarding the impact of the perturbative evolution on the numerical value
Figure 6.4: The diagram corresponding to the perturbative evolution of $\mu_\pi^2$.

of $\mu_\pi^2$. I bring up the issue not only due to its possible numerical relevance, but also because it sheds light on the actual meaning of both $\mu_\pi^2$ and $m_Q$, and their relation to each other. Recall that in the case of the operator $\bar{Q}Q$ (i.e. just the heavy quark mass diagram) we had a result (for the running of $m_Q$) which had no logarithmic dependance. In such a case, it is tempting to - due to the safe infrared limit - run $\mu$ to zero. For instance, this is what we have done in the case of $\bar{Q}Q$. As we have discussed above, however, this is an illegal procedure. As stated above, however, what is done in practice is the following: one takes known expressions, in our case one-loop QED corrections, and adds to these perturbative terms nonperturbative corrections expressed via certain matrix elements. What is done in the case at hand is the following.

From figure 6.4 it is easy to see that we can define the one-loop perturbative $\mu_\pi^2$ as just the infrared piece of the operator $\bar{Q}Q$, i.e.

$$\bar{Q}\pi^2 Q_{\text{one-loop}} = \frac{4\alpha_s}{3\pi} \mu^2 \bar{Q}Q$$  \quad (6.28)

Now, as we know, this quantity should be subtracted from the actual $\mu_\pi^2$ (we cannot double count this piece!) Numerically, one usually considers a value of $\mu$ relatively high so that $\alpha_s$ is small, but yet small enough so that the expansion in $\mu/m_c$ is good. To estimate the perturbative evolution, I will choose $\mu = 1$ GeV. Doing so, the correction to $\mu_\pi^2$ is quite sizable - 0.15 GeV$^2$. The inequality stated above, $\mu_\pi^2 > \mu_G^2$ still is safe, however, as can be seen from eq. 6.27, at any scale.
For our purposes concerning the D lifetime, we realize, however, that the theoretical value of $\mu_n^2$ may not be completely crucial, for even if we take $\mu_n^2$ to have a very large perturbative evolution, so that $\mu_n^2 = 0.15 \text{ GeV}^2$, and so assuming the inequality held, $\mu_{\tilde{n}}^2 = 0.15 \text{ GeV}^2$ still we would have trouble fitting the lifetime, as can be seen in the next chapter.
Chapter 7

Putting it all together- The numerical prediction for $\Gamma(D)$ with leading order perturbative and nonperturbative corrections

With the issues of numerics under our belts, we can finally check to see how our formula for $\Gamma_D$,

$$\Gamma(D) = \Gamma_0 \left( 1 + A_1 \alpha_s - \frac{3 \mu_G^2}{2 m_c^2} - \frac{\mu_\pi^2}{2 m_c^2} \right), \quad (7.1)$$

works. We take the mindset that we push everything in the direction of agreement. This corresponds to a high $m_c = 1.4 \text{ GeV}$, low $\mu_\pi^2 = 0.5 \text{ GeV}^2 - 0.15 \text{ GeV}^2 = 0.35 \text{ GeV}^2$ and low $\mu_G^2 = 0.35 \text{ GeV}^2$ - I slightly lower the value of $\mu_G^2$ to reflect the fact that the inequality stated above still holds (due to the perturbative evolution mentioned in chapter 6). Plugging everything in to the lifetime, then, we get:

$$\Gamma(D) = \Gamma_0 (1 - 0.24 - 0.27 - 0.09) = \Gamma_0 (0.40) \quad (7.2)$$
Recall that for $m_c = 1.4$ GeV we had perfect agreement with the lifetime without corrections, thus the corrections spoil agreement, lowering the lifetime about by half. To alleviate such a situation, we have to really push our numbers: one choice is to make $m_c = 1.6$ GeV $(1.6/1.4)^5 \approx 2)$. This raises the rate by about half, and solves our problem. However, doing so really kills any agreement with the $m_c$ extraction from the QCD sum rules. Under our philosophy, we wanted to take the sum rule determination as bedrock - especially, since as we will see there are exponentially small duality violation possibilities for sum rules, but not for decay widths.

Another option is to try and drastically lower $\mu^2$. For instance, it might be that the extracted value of $\mu^2 = 0.35$ GeV$^2$ then the perturbative evolution could knock the value down by say 0.15 GeV$^2$, and assuming that the inequality still holds, we still get a large theoretical deficit:

$$\Gamma(D) = \Gamma_0(1.00 - 0.24 - 0.153 - 0.05) = \Gamma_0(0.55)$$  \hspace{1cm} (7.3)

And so it seems that even really pushing our nonperturbative corrections to their lowest possible values, we are still in trouble.
Chapter 8

$1/m_c^3$ Corrections

“How tragic is wisdom when it brings no profit to the wise”

Now, it looks like we are stuck. How can we accommodate such a large width? A fairly non-imaginative option is that for some reason the - it is always an option - perturbative, or nonperturbative corrections are enhanced at the next order. Thus, for instance, someone could try to go farther and calculate the $1/m_c^3$ corrections. This attempt was made in [35], and in this section we review the calculation of these cubic terms. As we saw in the last section in the numerical discussion, the hope is that the $1/m_c^3$ corrections can give us an increase in the width around 30% to 50%. Clearly, such a contribution will be due to an unsuspected enhancement in the coefficient of the $1/m_c^3$ operator.

Since the total width is a Lorentz scalar, the only new operators relevant at the level of dimension 6 are the four-fermion operators of the type:

$$O_6 = c\Gamma q\bar{q}c$$

(8.1)

where $q$ stands for the light quark, and $\Gamma$ stands for Lorentz, and color structures. There
are two distinct sources of the $1/m_c^3$ corrections, just as there were for the $1/m_c^2$ corrections: operators of dimension 6 arising from the expansion of $\hat{T}$, and $1/m_c$ corrections in the D meson matrix elements of the operators $\bar{c}\sigma Gc$ and $\bar{c}c$.

### 8.1 The four-fermion operators at $O(\alpha_s)^0$ and in LLA

A four-fermion operator appears in $\hat{T}$ in the zeroeth order in $\alpha_s$ in figure 8.1. The corresponding result can be read from Eq. (17c) of ref [37],

$$\text{Im } \hat{T}^0 = -\frac{G_F^2 m_c^2}{8\pi} [\bar{c}_i \Gamma_\mu c_k - (2/3)\bar{c}_i \gamma_\mu \gamma_5 c_k] [\bar{s}_k \Gamma_\mu s_i],$$

(8.2)

here $\Gamma_\mu = \gamma_\mu (1 - \gamma_5)$. The expression above, however, yields zero for two reasons. First, in the factorization approximation, $\hat{T}^0$ corresponds to the annihilation contribution $c\bar{s} \rightarrow l\nu$, and our spectator here is not an s quark. Second, even if we deal with the $D_s$, the chiral structure yields a vanishing contribution when factorization is invoked, since under factorization $< D|\hat{T}^0|D > = 0$. Thus this is not the contribution that we are looking for. But let’s not be hasty. The effective Lagrangian given above is at the virtuality scale, $\mu = m_c$. The type of pre-asymptotic effect we are after is at a different scale, $\mu = \Lambda_{QCD}$. Under the rules of the OPE, we therefore must ‘evolve’ the operator down to this infrared scale (‘evolving’ corresponds to sloshing some of the operator at scale $m_c$ into part of the coefficient at the scale $\Lambda_{QCD}$). Physically, this corresponds to
including gluonic degrees of freedom, and thus, we can expect that the chirality, and hence vanishing character of the contribution which we obtained at scale $m_c$ will be different at the scale $\Lambda_{QCD}$. This contribution is calculated in ref. [37], and is

$$\text{Im} \hat{T}^1 = -\frac{G_F^2 m_c^2}{8\pi} |V_{cs}|^2 \left[ -\frac{\alpha_s}{3\pi} \ln \frac{m_c^2}{\mu^2} \left( \frac{2}{3} \epsilon \Gamma_\mu t^a c + \frac{1}{3} \tilde{\epsilon} \Gamma_\mu t^a c \right) \sum_q \bar{q} \gamma_\mu t^a q \right], \quad (8.3)$$

where $\Gamma_\mu = \gamma_\mu (1 + \gamma_5)$.

Notice that now, the light-quark current runs over all flavors and both left-handed AND right-handed fields appear, leading to the non-vanishing contribution of this result after factorization. This is a typical situation with the penguin contribution, and arises from the simple fact that the gluonic correction connects the $V \times A$ strange quark current to the pure vector background current. Above we have calculated the $1/m_c^3$ contribution arising from the logarithmic mixing of penguin operators and operators built from the light quark currents. Keeping only logarithms, however, may be an unjustified assumption, since $m_c/\mu$ is not large. Thus, in the next sections, we calculate the full $O(\alpha_s)$ contribution.

### 8.2 Full $O(\alpha_s)$ calculation

Here we divide the calculation of the full $O(\alpha_s)$ contribution into three separate pieces.
8.2.1 The ‘free’ piece

To start with, we calculate $1/m^3_Q$ pieces coming without any logarithms. We get terms here in the same we got $1/m^2_c$ terms when we examined $\bar{Q}p^4\bar{p}Q$. And so, we can just start with the object:

$$\text{Im} \hat{T}_0 = \frac{G_F^2|V_{cd}|^2\bar{Q}(0)p^4\bar{p}Q(0)}{384\pi^3},$$  \hspace{1cm} (8.4)

where

$$p_\mu = iD_\mu - gA_\mu.$$ \hspace{1cm} (8.5)

There are two possible contributions coming from the first two terms of the expansion for $A_\mu$, recall that

$$A_\mu = \frac{1}{2}x_\rho G_{\rho\mu} + \frac{1}{3}x_\rho x_\alpha(D_\alpha G_{\rho\mu}) + ...$$ \hspace{1cm} (8.6)

(higher order terms will not contribute dimension 6 operators). First, lets focus on the first term of the expansion of $A_\mu$, and so consider

$$\bar{Q}\hat{p}p^4Q = \bar{Q}(\hat{P} - \hat{A})(P - A)^4Q$$

As before, the strategy is to drag all $A_\mu$’s to the left, and $P_\mu$’s to the right. When $A_\mu$ stands to the left it gives zero (Q is taken at the origin, so since $A \sim x$, $AQ \to 0$), and when P stands to the right, the Dirac equation can be invoked. Thus, we can already write

$$\bar{Q}(\hat{P})(P - \frac{1}{2}x_\rho G_{\rho\mu})^4Q.$$ \hspace{1cm} (8.8)

Continuing the procedure of pulling all gauge potentials to the left, we will find ourselves in need of a few identities:

$$[P_\mu, A_\mu] = iD_\mu A_\mu = iD_\mu(\frac{1}{2}x_\rho G_{\rho\mu}) = \frac{i}{2}x_\rho D_\mu G_{\rho\mu},$$ \hspace{1cm} (8.9)

\footnote{In ref. [35] this contribution was left out, but has minimal numerical effect anyway. The discrepancy between [35] and [36] still exists even after this left out contribution is included}
\[ [P_\nu, A_\mu] = i D_\nu A_\mu = \frac{i}{2} (G_{\nu\mu} + x_\rho(D_\nu G_{\rho\mu})) \]  
(8.10)

\[ \gamma_\mu G_{\mu\nu} P_\nu = -\frac{1}{4}(\hat{P}\sigma G - \sigma G\hat{P}). \]  
(8.11)

Using these identities, the final result for the first term of the gauge potential expansion is

\[ \bar{Q} \gamma_\rho D_\alpha G_{\alpha\rho} Q. \]  
(8.12)

The second term in the expansion of the gauge potential also will give us contributions \( O(1/m_Q^3) \). Again, consider

\[ \bar{Q} p^4 \hat{p} Q = \bar{Q}(P - A)^4(\hat{P} - \hat{A})Q \]

\[ = \bar{Q}(P^4 - P^2P\cdot A - P^2A\cdot P - A\cdot PP^2 - P\cdot AP^2)(\hat{P} - \hat{A})Q \]  
(8.13)

We drop the term \( \sim P^4\hat{P} \) since it gives no terms \( O(1/m_Q^3) \), and also drop the term \( \sim A\cdot PP^2 \) since \( A \) here is completely to the left. Thus, there are three terms which we need to examine which will give \( O(1/m_Q^3) \) contributions:

\begin{align*}
\text{Term 1} & : \bar{Q}(-P^4\hat{A})Q \\
\text{Term 2} & : \bar{Q}(-P^2[P_\mu, A_\mu]\hat{P})Q \\
\text{Term 3} & : \bar{Q}(-2P^2A\cdot P\hat{P})Q \\
& \ldots.
\end{align*}  
(8.14)

To get the results of each term, one merely needs to know the few simple commutators, easily worked out above in the Fock-Schwinger gauge. The results are, term by term,

\begin{align*}
\text{Term 1} & : \frac{4}{3} \bar{Q}(D_\gamma G_{\gamma\mu}\gamma_\mu)Qm_Q^2 \\
\text{Term 2} & : -\frac{2}{3} \bar{Q}(D_\gamma G_{\gamma\mu}\gamma_\mu)Qm_Q^2 \\
\text{Term 3} & : \frac{4}{3} \bar{Q}(D_\gamma G_{\gamma\mu}\gamma_\mu)Qm_Q^2.
\end{align*}  
(8.15)
Adding the results from both the first and second terms of the expansion of the gauge potential we get the transition operator:

$$\hat{T}_{\text{free}} = -iG^2_F|V_{cs}|^2 \frac{\alpha_s}{32\pi^2} m_c^2 \bar{c} \gamma_\rho t^\alpha \bar{c} \gamma_\rho t^\alpha q,$$

where we have used the equation of motion,

$$D^\alpha G^\alpha_{\alpha\mu} = -g \bar{q} \gamma_\mu t^\alpha q.$$  

(8.17)

This piece contributes, upon factorization (see later in this section)

$$\frac{\Delta \Gamma}{\Gamma_0} = \frac{4\alpha_s \pi f_D^2 M_D}{3m_c^3}.$$  

(8.18)

Now we calculate the full logarithmic contribution referred to above. It will carry with it the piece (infrared divergent) proportional to log that we treat with the OPE cutoff prescription, and will also carry a finite piece.

### 8.2.2 The ‘Log’ piece

There is another contribution in the expansion of $\hat{T}$ arising from $O(DG)$ terms in the expansion of the light quark propagator. The origin of the term proportional to $D_\alpha G_{\alpha\mu}$ was made evident in section 4. Here we need to keep the mass of the $s$ quark finite, however, since we will use it as an infrared regulator (we will perform the proper OPE, subtracting out from our coefficient the soft contribution form 0 to $\mu$). Keeping the mass of the $s$-quark finite results in the introduction of McDonald functions into the propagator. The result is:

$$S_q(x,0) = \frac{-im^2 K_1(m\sqrt{-x^2})}{4\pi^2 \sqrt{-x^2}} - \frac{m^2 \dot{x}^2}{8\pi^2} K_2(m\sqrt{-x^2}) + \frac{G_{\rho\lambda} mK_1(m\sqrt{-x^2})}{8\pi^2 \sqrt{-x^2}} (x_\rho \gamma_\lambda \gamma_5) + \frac{G_{\rho\lambda}}{16\pi^2} mK_0(m\sqrt{-x^2})\sigma_{\rho\lambda} + \frac{g}{32\pi^2} \left(2K_0(m\sqrt{-x^2})D^\alpha G_{\alpha\gamma} - (D^\alpha G_{\alpha\rho}) \dot{x}^\rho \right) x^\gamma + \bar{x}^\gamma x^\alpha D_\gamma G_{\alpha\gamma} \gamma - 3i \bar{x}^\gamma x^\alpha D_\gamma \tilde{G}_{\alpha\gamma} \gamma_5 \gamma_5 \frac{mK_1(m\sqrt{-x^2})}{\sqrt{-x^2}}) + \cdots$$ 

(8.19)
Figure 8.3: The diagram with the $DG$ term in the $s$-quark line.

Due to the introduction of the McDonalds functions, we will have integrals of the type

$$\int \left[ K_n(m\sqrt{(-x^2)}) \right] \frac{e^{ipx}}{(-x^2)^p} d^4x$$

which can be found in ref. [38].

To get the $O(1/m^3_Q)$ pieces, we isolate the piece of the propagator proportional to $DG$ and one gamma matrice, see figure 8.2. The singular nature of the contribution can be understood, since $K_n(m_s(\sqrt{-x^2})) \sim \ln[m_s(\sqrt{-x^2})].$

To avoid the infrared singularity, which shouldn’t be included in the proper OPE calculation anyway, we calculate the graph of figure 8.3b, and subtract it from the graph of figure 8.3a. In this graph (10b), the lepton pair is hard-pointlike - and the $s$ quark is soft. After subtracting the OPE infrared piece, we get the result,

$$\hat{T}_{\ln} = iG_F^2|V_{cs}|^2 \left( \frac{\alpha_s}{\pi} m^2_c (\ln \frac{m^2_c}{\mu^2} + \frac{2}{3}) \right)$$

$$+ (2\bar{c} \Gamma_{\mu} t^a c + \bar{c} \tilde{\Gamma}_{\mu} t^a c) \sum \bar{q} \gamma_{\mu} t^a q$$

which gives a contribution after factorization,

$$\frac{\Delta \Gamma}{\Gamma} = -\frac{16\pi\alpha_s}{9m^3_c} \left( \ln \frac{m^2_c}{\mu^2} + \frac{2}{3} \right) f_D^2 M_D.$$  

Lastly, we cannot forget to include pieces proportional to $DG$ coming from the expansion of $1/m^2$ operators. (Note, it is exactly how we picked up a piece of the $1/m^2$ contribution but there are no $1/m^3$ terms coming from $\bar{c}c$).
8.2.3 The ‘$1/m^2$’ piece

Just as there were contributions of $1/m^2$ pieces coming from leading order terms so also are there $1/m^3$ pieces originating from next to leading order terms. Previously, we expressed $< D|\bar{c}\gamma_\mu c G|D >$ in terms of the $D^*D$ mass splitting. This result is only valid to the leading order in $1/m_c$. Let us observe that the spin splitting yielding $M_{D^*}^2 - M_D^2$ is determined by the following terms in the heavy quark Hamiltonian:

$$\Delta H = \frac{1}{2m_c} \vec{\sigma} \cdot \vec{B} + \frac{1}{4m_c^2} \vec{\sigma} \cdot \vec{E} \times \vec{\pi},$$  \hspace{1cm} (8.23)$$

where $\vec{B}$ and $\vec{E}$ are the chromomagnetic and chromoelectric fields respectively ($\vec{B} = g\vec{B}^a t^a$ and $\vec{E} = g\vec{E}^a t^a$.) To leading order

$$\Delta_D = \frac{3}{4}(M_{D^*}^2 - M_D^2) = -< \vec{\sigma} \cdot \vec{B} > = 0.405 \text{ GeV}^2.$$  \hspace{1cm} (8.24)$$

At the level of $1/m_c$ the second term in the heavy quark Hamiltonian becomes important in $\Delta_D$, as well as the second-order iteration in $(2m_c)^{-1} < \vec{\sigma} \cdot \vec{B} >$. Assuming that both effects are of the same order of magnitude, we can roughly estimate the matrix element $(2m_c)^{-1} < \vec{\sigma} \cdot \vec{E} \times \vec{\pi} >$ as the difference between $\Delta_D$ and $\Delta_B$:

$$| (2m_c)^{-1} < \vec{\sigma} \cdot \vec{E} \times \vec{\pi} > | \leq \Delta_D - \Delta_B \sim 0.04 \text{ GeV}. $$  \hspace{1cm} (8.25)$$

Next observe that

$$\frac{i}{2} \bar{c} \sigma_{\mu\nu} G_{\mu\nu} c = -\bar{c} \sigma \cdot \vec{B} c - \frac{1}{m_c} \bar{c} \sigma \cdot \vec{E} \times \vec{\pi} c - \frac{1}{2m_c} \bar{c}(D_i E_i)c.$$  \hspace{1cm} (8.26)$$

The last term in the above equation reduces to the four-fermion operator which we take into account explicitly. The second term will be estimated as an uncertainty in the expression relating $< \frac{i}{2} \sigma G >$ to $\Delta_D$:

$$\mu_G^2 = < \frac{i}{2} \sigma G > = \Delta_D \pm 2(\Delta_B - \Delta_D) - (2m_c)^{-1} 4\pi \alpha_s < c\gamma_\mu t^a c\bar{q} \gamma_\mu t^a q >.$$  \hspace{1cm} (8.27)$$

Using factorization for the $O_6$ term above, and the same values for the parameters as above, we get $+0.01$ for the contribution for $O_6$, so that

$$\mu_G^2 = \Delta_D \pm 2(\Delta_B - \Delta_D) = 0.42 \pm 0.08 \text{ GeV}^2.$$  \hspace{1cm} (8.28)$$
As for $\mu_\pi^2$, it was shown in [25] that the sign of the $1/m_c$ correction is negative. We will assume that the error bars in the numerical value of $\mu_\pi^2$ itself give the estimate of its $1/m_c$ contribution.

### 8.3 Tallying the result

In estimating the numerical value of the $1/m_c^3$ contribution we choose $f_D = 170$ GeV, $\alpha_s = .3$, and $m_c = 1.4$GeV. Adding up all contributions, we get

$$\frac{\Delta \Gamma}{\Gamma} = -0.06 \pm 0.06 + 0.03$$  \hspace{1cm} (8.29)

here the first number is due to the four-quark terms in the transition operator, the second is due to the uncertainty of $O_G$, and the third is due to $O_\pi$. Unfortunately, the result doesn’t help our situation. Here we hardly have the 50 percent contribution we were looking for.

So it seems that again we are stuck - is there no way out?
Chapter 9

Ways out

“Deny Everything” - Hunter S. Thompson

It appears that, in the case of the semileptonic decay of the D meson, all of our methods of attack have failed. What could have gone wrong? In this section, we review the case up to now, focusing on possible solutions. Some possible solutions were reviewed in previous sections, and relied on increasing or decreasing relevant numerical parameters. There, we found it is clear that we cannot increase or decrease the numerical value of any parameters in the expression for $\Gamma(D)$ to obtain agreement with experiment - at least not at the cost of creating even deeper puzzles in other charm phenomena. Also, it seems clear that the way out of this problem is not due to the perturbative series. In previous sections we saw quite clearly that the next order perturbative corrections make minimal impact. What other possibilities are left?

One approximation that was made in chapter 8 was the factorization approximation in the evaluation of the dimension 6 matrix elements. Corrections to the factorization approximation were reviewed in [39]. There, corrections were shown to be proportional to $1/N_c$ and seem to be small. Remember, we would need for the corrections to be quite large.

The next possibility might be that dimension 7 operators could save the day. In
principle, this may happen since the expansion parameter is \( \approx \sqrt{\mu_G^2/m_c} \approx 0.5 \) and is of order unity. However, since the correction due to \( 1/m_c^3 \) terms is of order 10 percent, this seems unlikely - not to mention, it would involve a very painstaking calculation!

The most likely solution, however, does lie in the fact that our nonperturbative expansion parameter is large. This idea is tied conceptually to the fact that the kinematics of the problem at hand are essentially Minkowskian. One justifies an OPE based procedure by keeping in mind an analytical continuation. In the problem of the semileptonic width this may be a continuation of the lepton pair - one considers the transition operator at such momenta where one is actually off the cuts corresponding to production of the hadronic states - in the Euclidean domain. The prediction on the cuts is made by invoking dispersion relations, in full analogy with what is usually done in the problem of the total \( e^+ e^- \) annihilation. In general, one can analytically continue in some auxiliary momenta which has nothing to do with any of the physical momenta.

Whatever analytic continuation is done, the prediction for each given term in the \( 1/m_c \) expansion refers to the Euclidean domain and is translated to the Minkowski domain only in the sense of averaging which occurs automatically through the dispersion relations. If the integrand is smooth, however, we can forget about the averaging because in this case smearing is not needed. This is what happens, in particular, with the total hadronic cross section in \( e^+ e^- \) annihilation at high energies - quark-hadron duality sets in and the OPE-based consideration yields the value of the cross section at a given energy, locally (without smearing). At what energy release is the integrand smooth and can the terms in the \( 1/m_c \) expansion be predicted locally? The existing theory gives no answer to the question, but the problem at hand seems to suggest that the duality limit sets in well above \( 1.4 \text{ GeV} \)

In the next section, I review the work done in [40] regarding duality violations. This work was the first real attempt to try and capture the essence of QCD violations, and provides probably the best framework with which to escape from the puzzle at hand.
Chapter 10

Duality violations

“Dave - you can’t just learn this subject by reading papers about it, you have to solve some problems by hand!” - Misha Shifman

“You don’t get it? Sit down with the book - AND A PENCIL IN HAND! - and work out every intermediate step, and you’ll get it. Good for the soul you know....” - Serge Rudaz

Depressing as our failure may seem at first, we should not look at this result as a failure, but instead as an opportunity. In fact, the D is an ideal testing ground for duality violations. Our QCD based calculations have failed to describe the hadronic dynamics, lending evidence to the idea that there must be a contribution which we have left out. In this section, I discuss the concept of duality, its violations, and its relation to heavy quark physics - in particular the problem of the semileptonic D decay.

The concept of duality was introduced long ago [45]. Essentially, its meaning is this: we calculate amplitudes with quarks and gluons, and compare our results to hadronic observables. If duality holds, then the complicated hadronic dynamics involved in a given process yield essentially little effect - the most important effects come just from the simple point-like interaction of the quarks and gluons. In accounting, say, for
the effect of Fermi motion - including $\mu^2$, or in general any other operators, we take a stab at including some of these hadronization effects - from the quark-gluon side. Thus, each time we include some new operator, etc. into our calculation we redefine what we mean by duality. In the present problem of D decay, our definition of duality consists of the expansion in $\alpha_s$ and $1/m_c$ corresponding to the hadronic lifetime. We prayed that this quark-gluon calculation was dual to the hadronic decay width. Well, it looks like its not! Thus, we need to stretch ourselves, and find some contribution which we have left out. Clearly it is a difficult task. Since we have the first few terms in both the perturbative and nonperturbative expansions, the effect that we are looking for will have to come from somewhere deeper. The perturbative expansion goes like $\sim \alpha_s / \pi$ which is numerically $\sim 0.1$. The nonperturbative series goes something like $\sqrt{< \mu^2 > / m_c^2} \sim 0.5$, so the nonperturbative series seems to be most suspect if our duality analysis targets poor convergence properties of our expansions. The first suggestion of this type of contribution came in ref. [47].

10.1 Exponential terms and asymptotic series

Consider the series $\sum_n a_n n!$. This is an example of an asymptotic series: for a few first terms the series homes in on a limiting value, but then proceeds to skate away with the inclusion of higher order terms. Of course, including all the terms (for the problem at hand) we get a finite result. In the perturbative and nonperturbative expansions we deal with this type of series. Actually, it is not a scary thing that we deal with asymptotic series: hopefully, it can be the case in some situations that the impossible task of calculating higher order terms in a given expansion is indeed a groundless task. Perhaps in a given case the optimal order is $n = 2$ or $3$, and we can forget about the higher order terms - the series should be truncated here. This might be just the case with the D meson - not a bad hunch, since we have experience leading us to believe $1/m_c$ is a large expansion parameter.
Truncating the series at a finite order, we introduce an exponential error. This can be seen in the following way. Suppose we have some function, $f(x)$ represented as the expansion,

$$f(x) = a_0 + a_1 x + a_2 x^2 + .... \quad (10.1)$$

if the expansion is factorially growing, the coefficients can be written like $a_n \sim c^n n!$ (in QCD, $c$ is roughly $\mu$). Now, as stated above, the series converges, but then starts to diverge again at an optimal $n$. Roughly this $n$ occurs when

$$x \sim (c^n n!)^{1/n}. \quad (10.2)$$

We want to take into account the size of the last term neglected. Using Stirling’s formula,

$$n! \sim \sqrt{2\pi n} n^n e^{-n}, \quad (10.3)$$

it is easy to see that the last term neglected is $\sim \exp(-x/c)$. Thus, truncating the series in $\mu/m_Q$ we get an exponential accuracy $\sim \exp(-\rho m_Q)$ where $\rho \sim 1/\Lambda_{QCD}$ is some infrared distance.

### 10.2 Physical picture of the exponential terms

Surprisingly, there is a clear physical picture of the exponential error [41]. Consider figure 10.1. In discussing the figures, let me reemphasize some of the points I have made earlier regarding the OPE. First, consider figure 10.1a. Figure 10.1a is a usual perturbative diagram where the quark emits and reabsorbs a gluon. As is typical of perturbative graphs, a large energy release is carried by one or at most two quanta. Figure 10.1b is the nonperturbative ‘cut’ diagram of figure 10.1a necessitated by the OPE. The cut line (cut since we have integrated down to our infrared point $\mu$) now becomes an operator, in particular, $\mu_z^2$. Figure 10.1b. represents the diagrams which we normally take into account, and have, in the case of the semileptonic D lifetime,
Figure 10.1: Possible diagrams contributing to heavy quark decay in a toy model.

where one or at most two lines become soft. Figure 10.1c represents an additional contribution, the kind which gives us the exponential effect that we are looking for. Here, we have neither one or two hard lines nor one or two soft lines but rather, a hard contribution transmitted by many lines, so that each line is soft. The situation is one where a hard momenta is shared by a coherent, possibly classic field configuration.

As we will see, by using instantons - just a solution to the classic equations of motion - we will be able to model the exponential contribution which is conceptually related to figure 10.1c. Actually, these type of exponential terms were known long ago since the early days of QCD [7]. There, they were essentially disregarded as most of the early QCD application, e.g. sum rules, are of Euclidean nature (dispersion relations are used). In typical heavy quark cases we deal with Minkowski type kinematics (no dispersion relations, but instead a direct analytic continuation to the physical cut), and here the nature of the truncation error differs drastically - the exponentially decaying truncation error in the Euclidean domain becomes oscillatory in the Minkowski domain, suppressed only by powers of our expansion parameter.

Our task then is simply to develop some methods to generate this type of exponential term. In fact, the exponential terms appear in the same way as the power terms before them. A concrete example is given in [26]. There, it is shown that the truncation of
the factorially diverging series in $\alpha_s$ leads to an exponential error

$$\exp\left(-\frac{8\pi}{b_0\alpha_s(Q^2)}\right) \sim \frac{\Lambda_{\text{QCD}}^4}{Q^4}.$$  \tag{10.4}$$

Thus, from the exponential error involved in truncation of the perturbative series, we can see the presence of power terms. Here, we consider the series of power like terms. This too is an asymptotic series [11], and its truncation will give an exponential contribution of the type we are interested.

### 10.3 Generating the exponential contribution

To illustrate the main details of the instanton calculations, let's outline the practical motivation for the inclusion of the corresponding effects from the general perspective of the short distance expansion. Consider a generic two point function $\Pi(Q^2)$, the polarization operator of two vector currents (the Lorentz structure of the currents is irrelevant).

$$\Pi(Q) = \int d^4x e^{iqx} \Pi(x) = -\int d^4x e^{iqx} <G(x,0)G(0,x)>$$ \tag{10.5}

where $G(x,y)$ is the quark Green’s function in an external gauge field and averaging over the field configurations is implied - we work in Euclidean space. The normal power type corrections occur when we consider the expansion of Green’s functions near $x=0$ where the Green’s function is singular.

We now examine the question ‘what happens when the propagator has a pole not near $x=0$?’ For example, consider the simplest finite-$x$ singularity for the two-point function,

$$\Pi(x) = \frac{1}{(x^2 + \rho^2)^\nu}, \tag{10.6}$$

then,

$$\Pi(Q^2) = \int d^4xe^{iqx}\Pi(x) \sim e^{-Q\rho}. \tag{10.7}$$
These types of contributions are not seen in the normal OPE, and thus represent a violation of duality. Moreover, they mimic the type of behavior we expect from the higher order terms left out in the truncated series.

Unfortunately, the physics of these duality violating terms brings us, again, to the main difficulty of QCD - we would like to know the background field fluctuations of the vacuum, but we don't. The problem enters since, although we do now the explicit form of the light quark propagator in the background of a distinct type of vacuum fluctuation - instantons (and this propagator has the form we want - poles off the origin), we don't know the density of instanton fluctuations in the infrared domain (i.e. where the poles are). Although, we know that instantons are not the dominant background fluctuation, we are not down on our luck completely, however, we will simply have to develop a model of the background fluctuations and fit it phenomenologically from known examples of possible duality violations. In the original paper [40], a two parameter model function was introduced, and fitted from two sources of duality violations: the semileptonic lifetime of the D, and the invariant hadronic mass distribution of $\tau$. The model was then used to explore possible duality violations elsewhere, e.g. B decays, $\alpha_s$ extraction from $\tau$ decays, and more. In the next sections, we review this model.

### 10.4 Instantons and the OPE

Before calculating the actual exponential terms in the real QCD case of D decay, it will be useful to review certain universal aspects of the calculation in a toy model. The toy model generalizes the actual QCD calculation. Calculating the instanton contribution with the toy model, we clearly see that the instanton calculation can give three different types of contributions to the transition operator:

(i) Small-size instantons affect the coefficient functions of the OPE. We are not interested in these terms. They do not appear in our calculations below explicitly
since our instanton density function excludes them. I will return to this issue when we consider the question of the instanton density function in real QCD.

(ii) Terms proportional to \(1/(m_Q \rho)\). They represent the instanton contributions to the matrix elements of various finite dimension operators that are present in the OPE. In principle, one could calculate each matrix element, pretending that the only contribution is from the one-instanton background. Such an attempt leads to results grossly violated in nature.

(iii) The exponential terms. These terms, \(\exp(-m_Q \rho)\) are what we are after.

Finding the exponential terms in the case of the heavy quark decay goes just like our example of the two-point function above, but with one subtlety - the external lines are colored objects, and thus feel the influence of the instanton. This could lead to considerable calculational difficulty. Fortunately there is a simplification. To see it, it is simplest to consider a toy model where all spins are neglected. The relevant features of the toy model translate to the case of real QCD.

### 10.5 Instanton Calculation for a Toy Model

The Lagrangian of the toy model we consider has the form

\[
L_W = hQ\bar{q}\phi + h.c. \tag{10.8}
\]

which describes the decay of a heavy scalar quark \(Q\) into a massless quark \(q\) and a scalar photon \(\phi\); the coupling \(h\) has dimensions of mass. Both quarks are in the spinor representation of the color group (SU(2) here). The basic strategy of the instanton calculation has been outlined above, here I work out details specific to the heavy quark case.

Consider the transition amplitude:

\[
T = \frac{1}{2M_D} < D|\hat{T}|D > = \frac{1}{2M_D} < D|\int d^4x iT[L_W(x)L_W(0)]|D >, \tag{10.9}
\]
Figure 10.2: The diagram representing heavy quark decay in a toy model in the instanton background field.

For our choice of the toy model Lagrangian,

\[ \hat{T} = i \int \bar{Q}(x)S(x,0)Q(0)G_\phi(x^2)e^{i m_v x}d^4x, \]  

(10.10)

where \( G_\phi \) is the propagator of the scalar photon, and \( S(x,y) \) is the propagator of the massless scalar quark in the external (instanton) field. The propagator of the massless scalar particle in the instanton background is known exactly: \[ S(x,y) = \frac{1}{4\pi^2(x-y)^2}(1 + \rho^2/x^2)^{-1/2}(1 + \frac{\rho^2(\tau^+ x)(\tau y)}{x^2 y^2})(1 + \rho^2/y^2)^{-1/2}, \]  

(10.11)

where we have fixed the instanton center at \( z = 0 \), and

\[ \tau = (\vec{\tau}, i); \quad \tau^+ = (\vec{\tau}, -i); \quad \tau_\alpha^+ \tau_\beta = \delta_{\alpha\beta} + \eta_{\alpha\beta\gamma} \tau_\gamma, \]  

(10.12)

where \( \vec{\tau} \) are the Pauli matrices acting in the color subgroup.

We now take a closer look at the final state quark propagator, rewriting it using the Feynman parametrization,

\[ S(x,y) = \frac{1}{\pi} \int_0^1 d\alpha [\alpha(1-\alpha)]^{-1/2} \frac{1}{\alpha(1-\alpha)(x-y)^2 + \rho^2 + \tilde{z}^2} \times \frac{1}{4\pi^2(x-y)^2}(1 + \frac{\rho^2(\tau^+ x)(\tau y)}{x^2 y^2})(1 + \frac{1}{x^2 y^2}) \]  

(10.13)

where

\[ \tilde{z} = z - x\alpha - (1-\alpha)y. \]  

(10.14)
In this form, the analytic structure of the propagator is clearer - we can now easily pick up the pole off the origin in our integrations, and avoid singularities at the origin that give contributions which we are not interested in. We want the pole:

\[ x^2 = \frac{\rho^2 + z^2}{\alpha(1 - \alpha)}. \]  

(10.15)

After doing the integration over time, at \( m_c \rho \gg 1 \), the remaining integrations are nearly Gaussian, and run over narrow intervals,

\[ x^2 \sim \frac{\rho}{m_c}; \quad (\alpha - \frac{1}{2})^2 \sim \frac{1}{m_c \rho}; \quad (z - \frac{x}{2})^2 \sim \frac{\rho}{m_c}. \]  

(10.16)

Thus, one performs the remaining integrations merely by evaluating all of the pre-exponential factors at the saddle point. In particular, consider the heavy quark external line. The heavy field \( \tilde{Q}(x_0, \vec{x}) \) can be written in the leading order as

\[ \tilde{Q}(x_0, \vec{x}) = T e^i \int_0^{\infty} A_0(r, \vec{x}) \, dr \tilde{Q}(0, \vec{x}) + O(1/(m_c \rho)) = U(\vec{x}) \tilde{Q}(0, \vec{x}) + O(1/(m_c \rho)), \]  

(10.17)

where \( U(\vec{x}) \) is some complicated function (just the exponentiated, integrated instanton gauge potential). In principle, the complexity of the function \( U(\vec{x}) \) makes for an extremely difficult job of evaluating the transition operator. Fortunately, we are lucky, and in the saddle point approximation, \( U(\vec{x}) \) is just equal to 1! The heavy quark field enters at distances \( \vec{x} \sim \sqrt{\rho/m_c} \ll \rho \) and, therefore the transition operator is finally proportional to \( \bar{Q}(0)Q(0) \). Collecting all remaining factors,

\[ \hat{T}(k_0) = h^2 \tilde{Q}(0) \left( G_\phi(-4\rho^2) \right) \int d\alpha d^4x d^4ze^{-k_0 \sqrt{(\rho^2 + z^2)/(\alpha(1 - \alpha)) + \vec{x}^2}} Q(0) \]  

(10.18)

where

\[ G_\phi(x^2) = \frac{1}{4\pi^2 x^2} \]  

(10.19)

is the free scalar propagator. Performing the remaining Gaussian integrations, and taking the matrix element (note for scalar particles, \( < H_Q | \bar{Q}Q | H_Q > = M_{H_Q}/m_Q \)) we
finally arrive at
\[ T = -\hbar^2 \frac{e^{(2im_0n)}}{16m_Q(m_Q^4)} , \quad (10.20) \]
which gives us a contribution to the total width,
\[ \Gamma_{scal}(\rho) = -\hbar^2 \frac{\sin(2m_0n\rho)}{8m_Q^5\rho^4}. \quad (10.21) \]

Note the oscillatory nature of the result upon continuation to Minkowski space. In applications, we set \( \sin(m_0n\rho) = 1 \), since we use our model only as an approximate estimate of duality violations, and since, anyhow, the suspicion is that the strong dependance of \( \sin(m_0n\rho) \) on the heavy quark mass, or position of the fixed-size instanton is somewhat artificial. The issue of the the remaining \( \rho \) dependance in the pre-exponential will be dealt with in the next section.

### 10.6 The exponential contribution for \( \Gamma_D \)

We now proceed to the actual calculation of the case of real D decay in the instanton model. Let me outline the treatment. First, we write the transition operator in the instanton background field. It has the same form as equation (5.2).
\[ \hat{T} = G_{F|V_{es}}^2 \int \bar{Q}(x)\Omega_{\mu,\nu}(x)\sum(x)\nu(x)\Gamma_{\mu,\nu}(x)e^{(mn\nu)}d^4x \quad (10.22) \]

The Green function of the light quark, \( S((x,0), z, \rho) \) is expanded in powers of \( m_q \), and has the form:
\[ S(x,y) = -\frac{1}{m_q}P_0(x,y) + G(x,y) + m_q\tilde{\Delta}(x,y) + O(m_q^2). \quad (10.23) \]

Technically, difficulties with the instanton model develops at the first step: the Green’s function of the light quark in the background of one-instanton is not defined since the Dirac operator has a zero mode (zero modes are denoted above by \( P_0 \), and regulated by \( m_q \)). However, since the weak amplitude we consider contains left-handed
quarks only, the problem of zero modes disappears completely, albeit somewhat artificially. Hopefully, this not fully self-consistent procedure will work satisfactorily enough for our purposes. Of course, we have no right, in general, to believe that the one-instanton contribution will give us good phenomenology, still, since we scale our model from other duality violations, we can hope then that the duality violation hierarchy from decay to decay is captured correctly.

Proceeding with the rest of the calculation see ref. \[40\] for an exhaustive discussion, We arrive at the result:

\[
\Gamma_{sl}^I = -\frac{2}{3} \Gamma_0 \frac{96\pi}{(m_Q\rho)^8} \sin(2m_Q\rho)D(\rho). \tag{10.24}
\]

Now, the real question of QCD dynamics enters in full. We need to integrate over the instanton size \(\rho\), and an explicit representation of \(D(\rho)\) is required. The calculation of \(D(\rho)\) was first undertaken by t’Hooft \[48\]. This instanton density function, however, is not what we want. In t’Hooft’s calculation, only small-size instantons can be considered. His instanton density function took the form:

\[
D(\rho) = \text{const.}(\rho\Lambda_{\text{QCD}})^b \tag{10.25}
\]

where \(b\) is the first coefficient in the Gell-Mann Low function. Integrating over this instanton density function leaves us not with the exponential quantity we are after, but instead power like contributions. This contribution was discussed above. Small size instantons are hard fluctuations. Taking them into account should in no way reflect the contribution of the series of higher order operators - yielding an exponential term. Indeed, considering the small-size instantons we are outside the validity of the standard heavy quark expansion (HQE). The standard HQE requires the decomposition of the heavy quark field in the form \(Q(x) = \exp(im_Q v_\mu x_\mu) \tilde{Q}(x)\) which in the hard instanton background becomes inapplicable, as well as the statement that heavy quark spin effects are suppressed by \(1/m_Q\), and so on.

To get the exponential contribution coming from instantons We need the density
function in the infrared but, of course, it cannot be calculated there. We thus model $D(\rho)$ with the simplest idea we can think of: a fixed size instanton:

$$D(\rho) = N\delta(\rho - \rho_0).$$  \hspace{1cm} (10.26)

Actually, there is some reason to believe that this density function at least represents the character of the the soft instanton fluctuations. Results from the instanton liquid model \cite{44} show that the density function should have a sharp rise and steep fall-off. Thus, we can hope at least that our model correctly captures the essence of the real QCD instanton density function.

### 10.7 Numerics of the instanton model

After integrating over the instanton density function, we have the results of our duality model for the semileptonic decay expressed in terms of two free parameters:

$$\Gamma_{sl}^I = -\Gamma_0 \frac{2}{3} N \frac{96\pi}{(m_c\rho_0)^8} \sin(2m_c\rho_0) = \Gamma_0 \frac{2}{3} N \frac{96\pi}{(m_c\rho_0)^8}. \hspace{1cm} (10.27)$$

Let me repeat that since the value of $\sin(2m_c\rho_0)$ is sensitive to how close the argument is to $n\pi$, a very model-dependant feature, we thus set $\sin(2m_c\rho_0) = 1$, and take the absolute value of the expression, taking the conservative point of view that our model at best determines an uncertainty due to duality violations.

Now, we are at a crossroads. Unfortunately we cannot test whether the model we have developed is capable of giving us a phenomenologically acceptable result. The dependance of the result on two unfixed parameters requires us to extract some knowledge of duality violations from some other processes. At this stage of development of both theory, and experiment, examples are rare. An obvious place to go hunting is where hadronic processes have small energy releases. In fact, there is one place - the invariant mass spectrum of hadronic $\tau$ decays - where we actually do see some preliminary signs of duality violations. The issue is considered in detail in ref. \cite{40}, here I just sketch the results.
Figure 10.3: Plot of $R^{(V-A)}(E)$ with Euclidean and Minkowski truncation errors sketched on top of the data.
Figure 10.3 is the experimental plot of the quantity \( R^{(V-A)}(E) \). From the plot, it can be seen that there is, at the tail of the distribution starting at about 2 GeV, evidence of some kind of oscillation \( \sim 10\% \). This oscillation is exactly the behavior predicted by the Euclidean exponential error when continued to the Minkowski domain. In ref. [40], this oscillation was used as an input to determine one of the two free parameters of the instanton density function.

Because other sources of duality violations are scarce, we can fit the remaining parameter of the model to the \( \sim 50\% \) duality violating contribution required in the semileptonic D lifetime, and then use the model to predict violations in other processes. For example, calculating a smattering of B lifetimes we end up with the result:

\[
\frac{|\Gamma_I(b \to c\bar{c}s)|}{\Gamma_0(b \to c\bar{c}s)} \sim 0.006
\]  
\[
(10.28)
\]

and

\[
\frac{\Delta \Gamma_I}{\Gamma_0}(b \to c\bar{c}s) \sim 2\frac{\Delta \Gamma_I}{\Gamma_0}(b \to cud) \sim 5\frac{\Delta \Gamma_I}{\Gamma_0}(b \to u\bar{u}d)
\]

\[
\sim 16\frac{\Delta \Gamma_I}{\Gamma_0}(b \to c\ell\nu) \sim 75\frac{\Delta \Gamma_I}{\Gamma_0}(b \to s\gamma)
\]

\[
\sim 300\frac{\Delta \Gamma_I}{\Gamma_0}(b \to ul\nu),
\]  
\[
(10.29)
\]

thus, the deviations from duality in B decays are negligible.

Let me briefly mention one other prediction the instanton model makes with, unlike in the case of the B decays, more observable implications. If the approach above is applied to the hadronic \( \tau \) width, the deviations from duality are estimated as

\[
\frac{\Delta \Gamma_I(\tau \to \text{hadrons})}{\Gamma_0(\tau \to \text{hadrons})} \sim 0.05
\]  
\[
(10.30)
\]

while this seems like a relatively small result, this 5\% uncertainty in the width translates into a \( \sim 30\% \) uncertainty in the value of \( \alpha_s(m_\tau) \). This is an interesting result in that the low energy determinations of \( \alpha_s \) currently disagree with measurements at the Z peak - except for the case of \( \alpha_s \) extraction from \( \tau \) decays. Shifman [42] points out that
this mismatch between low and high energy $\alpha_s$ extraction could be a prompt of new physics. The model outlined above should at the very least make those who extract $\alpha_s$ from $\tau$ widths a bit nervous, and thus raise an eyebrow at the possibility of new physics being seen. At the very least, here the strong case for large duality violations in $m_c$ where the energy release is $\sim 1.40$ GeV should make one suspect at least minimal violations in $\tau$ where the energy release is $\sim 1.77$ GeV.

Let me make one more closing remark concerning the semileptonic decay $b \to c\tau\nu$. This decay was recently measured by [49]. The theoretical prediction, including $1/m^2$ effects is found in [43]. In the decay, there is an energy release of $\sim 1.6$ GeV, thus, it is tempting to consider that there is perhaps a duality violation on the order $5-10\%$. Currently, the numerics on both the theoretical and experimental side are not trustworthy enough to make an observation of duality violation, but it very well may be a place to hunt for duality violations in the future, and thus possibly supply evidence for the utility of the instanton model.
Chapter 11

Conclusion

“Physics is rich, but life is richer” - Emil Akhmedov

We have seen that the physics of the D meson decay is rich indeed! Calculation of the total decay width requires all of our theoretical might, and then some. OPE power corrections, normal perturbative terms, duality fluctuations - what a problem - and we are still evaded. In the end, we found that, at least at present, the semileptonic width of the D meson is at best a theoretical kitchen for the study of duality violations. This in itself is not a complete disappointment. Many predictions from other hadronic decays concerning CP violation, $\alpha_s$ extraction, etc. will come in the future, and almost all, due to their Minkowskian nature, will come with the built in assumption of duality. To wit, if we want to go from predictions made by QCD without some flip assumption, and instead a well investigated one, we need to study $D$ and $\tau$ decays to better understand the assumption of duality. It is not ironic, but rather somewhat expected that in our duality investigations, we here too come up against the problem of understanding QCD dynamics in the infrared. Unfortunately, it seems that this is a problem which will continue to defy solution for some time, and thus, it makes the job of approaching QCD physics from the phenomenological side outlined in this thesis all the more important. It seems that in QCD, the answer to Fermi’s question is still yet to come.
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