GLOBAL EXISTENCE FOR THE NONLINEAR FRACTIONAL SCHRÖDINGER EQUATION WITH FRACTIONAL DISSIPATION.

MOHAMAD DARWICH.

Wednesday 28th February, 2018

ABSTRACT. We consider the initial value problem for the fractional nonlinear Schrödinger equation with a fractional dissipation. Global existence and scattering are proved depending on the order of the fractional dissipation.

1. INTRODUCTION

Consider the Cauchy problem for the damped fractional nonlinear Schrödinger equation

\[
\begin{cases}
    iu_t - (-\Delta)\alpha u + |u|^{p-1}u + ia(-\Delta)^s u = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, \\
    u(0) = u_0
\end{cases}
\]

(1.1)

where \(a > 0\) is the coefficient of friction, \(d \geq 2\), \(\alpha \in (\frac{d}{2d-1}, 1)\), \(s > 0\) with \(L^2\)-critical nonlinearity i.e \(p = 1 + \frac{4\alpha}{d}\).

In the classical case (\(\alpha = 1\) and \(a = 0\)) equation (1.1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics to describe propagation phenomena in dispersive media.

When \(a = 0\) and \(0 < \alpha < 1\) equation (1.1) (called FNLS: Fractional NLS) can be seen as a canonical model for a nonlocal dispersive PDE with focusing nonlinearity that can exhibit solitary waves, turbulence phenomena which has been studied by many authors [20], [8], [10], [11], [21], [23] and [29] in mathematics, numerics, and physics. The FNLS equation is a fundamental equation of fractional quantum mechanics, which was derived by Laskin [24], [25] as a result of extending the Feynman path integral, from the Brownian-like to Levy-like quantum mechanical paths. The Cauchy problem for FNLS was studied in [14] and [15] and proved that it is well-posed and scatters in the radial energy space and in [16] the author proves that the equation is globally well posed for small data.

In this this paper we complete the \(L^2\)-critical FNLS equation with a fractional laplacian of order \(2s\), \(s > 0\). The fractional laplacian is commonly used to model fractal (anomalous) diffusion related to the Lévy flights (see

Key words and phrases. Damped Fractional Nonlinear Schrödinger Equation, Global existence.
e.g. Stroock [27], Bardos and all [3], Hanyga [18]). It also appears in the physical literature to model attenuation phenomena of acoustic waves in irregular porous random media (cf. Blackstock [4], Gaul [13], Chen-Holm [9]).

Note that for \( s = 0 \), the global existence for (1.1) was proved in [28] for a large damping term (i.e. for large \( a > 0 \)), in this paper we will obtain the global existence result for any damping term \( a > 0 \).

Finally, the case \( \alpha = 1 \) and a nonlinear damping of the type \( ia|u|^p u \), has been studied by Antonelli-Sparber and Antonelli-Carles-Sparber (cf. [1] and [2]). In this case the origin of the nonlinear damping term is multiphoton absorption.

The purpose of this paper is to prove some global well-posedness and scattering results for (1.1) in the radial case and the rest of the paper is organized as follows. Section 2 is devoted to prove the local existence results. In section 3 we will show the main results i.e the global well-posedness of equation (1.1) and the scattering.

Now let us define the following quantities:

**L2-norm**: \( m(u) = \| u \|_{L^2} = \left( \int |u(x)|^2 dx \right)^{1/2} \).

**Energy**: \( E(u) = \frac{1}{2} \| (-\Delta)^{\frac{s}{2}} u \|_{L^2}^2 - \frac{d}{4\alpha + 2d} \| u \|_{L^4}^{\frac{4p}{p-2}} + 2 \).

However, it is easy to prove that if \( u \) is a smooth solution of (1.1) on \([0, T]\), then for all \( t \in [0, T] \) it holds

\[
\frac{d}{dt} (m(u(t))) = -a \| (-\Delta)^{\frac{s}{2}} u \|_{L^2}^2; \quad \tag{1.2}
\]

\[
\frac{d}{dt} (E(u(t))) = -a \int |(-\Delta)^{\frac{s}{2}} u(t)|^2 + a3 \int |(-\Delta)^s u(t)|^p \| u(t) \|_{L^p}^{p-1} \| \sigma(t) \|.
\]

\[
\tag{1.3}
\]

Let us now state our results:

**Theorem 1.1.** Let \( d \geq 2 \), \( \alpha \in \left( \frac{d}{2d-1}, 1 \right) \) and \( 0 < s < \alpha \), such that \( s + \alpha \geq 1 \) then there exists a real number \( \beta > 0 \) such that for any initial datum \( u_0 \in H^\alpha_{rd}(\mathbb{R}^d) \) with \( \| u_0 \|_{L^2} < \beta \), the emanating solution \( u \) is global in \( H^\alpha_{rd}(\mathbb{R}^d) \).

**Theorem 1.2.** Let \( d \geq 2 \), \( \alpha \in \left( \frac{d}{2d-1}, 1 \right) \) and \( s \geq \alpha \). Then the Cauchy problem (1.1) is globally well-posed in \( H^\alpha_{rd}(\mathbb{R}^d) \).

**Theorem 1.3.** Let \( \alpha \in \left( \frac{d}{2d-1}, 1 \right) \), \( s = \alpha \), \( u_0 \in H^\alpha_{rd}(\mathbb{R}^d) \) and \( u \in C(\mathbb{R}^+, H^\alpha_{rd}) \) be the global solution to (1.1). Then:

1. There exists \( u_+ \in L^2 \) such that \( \|(u - S_{\alpha,\alpha,s}(\cdot))u_+)(t)\|_{L^2} \to 0 \), as \( t \to +\infty \).
2. \( \|u\|_{L^{\frac{4p}{p-2}}(\mathbb{R}^d \times \mathbb{R}^d)} \to 0 \), when \( a \to +\infty \).

Acknowledgements: The author thanks Luc Molinet for his valuable remarks and comments in this paper.

2. LOCAL EXISTENCE RESULT

Recall that the main tools to prove the local existence results for the FNLS equation are the Strichartz estimates for the associated linear propagator
\[ e^{i(-\Delta)^s t}. \] Let us mention that in the case \( a > 0 \) the same results on the local Cauchy problem for (1.1) can be established in exactly the same way as in the case \( a = 0 \), since the same Strichartz estimates hold.

### 2.1. Strichartz estimate.

**Definition 2.1.** A pair \((q, r)\), \(q, r \geq 2\) is said to be admissible if:

\[
\frac{4d + 2}{2d - 1} \leq q \leq \infty, \quad \frac{2}{q} + \frac{2d - 1}{r} \leq d - \frac{1}{2},
\]

or

\[
2 \leq \frac{4d + 2}{2d - 1}; \quad \frac{2}{q} + \frac{2d - 1}{r} < d - \frac{1}{2}.
\]

These Strichartz estimates read in the following proposition see [17]:

**Proposition 2.1.** Suppose \( d \geq 2 \), \( a = 0 \) and \( u \) be a radial solution of (1.1), then for every admissible pair \((q, r)\) satisfy the following condition:

\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \gamma, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2} + \gamma
\]

it holds:

\[
\|u\|_{(L^q_t L^r_x \cap L^\infty_t H^\gamma)} \leq \|u_0\|_{(L^q_t L^r_x \cap L^\infty_t H^\gamma)} + \|u\|^p_{(L^{q'}_t L^{r'}_x)}
\]

For \( a \geq 0 \), \( \alpha \geq 0 \) and \( s \geq 0 \) we denote by \( S_{a,\alpha,s} \) the linear semi-group associated with (1.1), i.e. \( S_{a,\alpha,s}(t) = e^{i(-\Delta)^s t-a(-\Delta)^s t} \). It is worth noticing that \( S_{a,\alpha,s} \) is irreversible.

We will see in the following proposition that that the linear semi-group \( S_{a,\alpha,s} \) enjoys the same Strichartz estimates as \( e^{i(-\Delta)^s t} \).

**Proposition 2.2.** Let \( d \geq 2, u_0 \in H^\gamma(\mathbb{R}^d), \gamma \in \mathbb{R}, s \geq 0 \) and \( s > \frac{d}{2d-1} \). Then for every admissible pair \((q, r)\) satisfy the following condition:

\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \gamma, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2} + \gamma
\]

it holds:

\[
\|u\|_{(L^q_t L^r_x \cap L^\infty_t H^\gamma)} \leq \|u_0\|_{(L^q_t L^r_x \cap L^\infty_t H^\gamma)} + \|u\|^p_{(L^{q'}_t L^{r'}_x)}
\]

**Proof.** Let for any \( t \geq 0 \), \( H_{a,s}(t, x) = \int e^{-\imath \xi \cdot x} e^{-at |\xi|^2 s} d\xi \), it holds

\[
S_{a,\alpha,s}(t) \varphi = H_{a,s}(t, \cdot) * e^{i(-\Delta)^s \varphi}, \quad \forall t \geq 0.
\]

Noticing that for \( s > 0 \), \( \|H_{a,s}(t, \cdot)\|_{L^1} = \|H_{1,s}(1, \cdot)\|_{L^1} \) and that, according to Lemma 2.1 in [26], \( H_{1,s}(1, \cdot) \in L^1(\mathbb{R}^d) \) for \( s > 0 \). Now let \( g = |u|^p \) we
have that:

\[
\|u\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} = \|S_{a,a,s}(t)(u_0) - i \int_0^t S_{a,a,s}(t-t')f(t')dt'\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} \leq \|S_{a,a,s}(t)(u_0)\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} + \| \int_0^t S_{a,a,s}(t-t')f(t')dt'\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} \nonumber \\
= \|H_{a,s}(t_0) * e^{i(t-t')^a} (u_0)\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} + \| \int_0^t H_{a,s}(t-t') * e^{i(t-t')^a} (f(t'))dt'\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} \nonumber \\
\leq \|H_{a,s}(t_0)\|_{L^1} \|e^{i(t-t')^a} (u_0)\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} + \| \int_0^t H_{a,s}(t-t') * e^{i(t-t')^a} (f(t'))dt'\|_{(L^r_t L^2_x \cap L^{r'}_x H^s)} \nonumber \\
\leq \|u_0\|_{H^s} + \|f\|_{L^q_t L^r_x} \nonumber \\
\]

With Proposition 2.2 in the hand, it is not too hard to check that the local existence results for equation (1.1) (see Theorem 4.2 in [17]). More precisely, we have the following statement:

**Proposition 2.3.** Let \( a \geq 0, \alpha \in \left( \frac{d}{2d-1}, 1 \right), s > 0 \) and \( u_0 \in H^a_{rd}(\mathbb{R}^d) \) with \( d \geq 2 \). There exists \( T > 0 \) and a unique solution \( u \in C([0,T]; H^a_{rd}) \cap L^{\frac{2a}{p}+2}_T L^{\frac{2a}{q}+2} \) to (1.1) emanating from \( u_0 \).

3. GLOBAL EXISTENCE RESULTS

In this section, we will prove the global existence results. Let us start by the second theorem:

### 3.1. Proof of theorem 1.2

To prove theorem 1.2 we will establish an a priori estimate on the Strichartz norm.

**Proposition 3.1.** Suppose that \( a > 0, \alpha > 0 \) and \( s > 0 \). Then there exists \( \epsilon > 0 \) such that if \( u_0 \in H^a_{rd}(\mathbb{R}^d) \) and \( \|S_{a,a,s}(\cdot)u_0\|_{L^{\frac{2a}{p}+2}_T L^{\frac{2a}{q}+2}} \leq \epsilon \), then the maximal time of the existence \( T^* \) of the solution emanating from \( u_0 \) equal to \( +\infty \).

To prove this claim, we will use the following proposition:

**Proposition 3.2.** There exists \( \delta > 0 \) with the following property. If \( u_0 \in L^2(\mathbb{R}^d) \) and \( T \in (0, \infty) \) are such that \( \|S_{a,a,s}(\cdot)u_0\|_{L^{p+1}([0,T], L^{p+1})} < \delta \), there exists a unique solution \( u \in C([0,T], L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T], L^{p+1}(\mathbb{R}^d)) \) of (1.1). In addition, \( u \in L^q([0,T], L^r(\mathbb{R}^d)) \) for every admissible pair \((q,r)\), for \( t \in [0,T] \). Finally, \( u \) depends continuously in \( C([0,T], L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T], L^{p+1}(\mathbb{R}^d)) \) on \( u_0 \in L^2(\mathbb{R}^d) \). If \( u_0 \in H^a(\mathbb{R}^d) \), then \( u \in C([0,T], H^a(\mathbb{R}^d)) \).

See [3] for the proof

**Proposition 3.3.** Let \( u_0 \in H^a \) and \( u \) be the solution of (1.2). Let \( T^* \) be the maximal time of the existence of \( u \) such that \( \|u\|_{L^{\frac{2a}{p}+2}_{\infty,T^*}[0,T^*) L^{\frac{2a}{q}+2}} < +\infty \), then \( T^* = +\infty \).
Proof. Observe that $\| S_{\alpha, \theta}(.)u_0 \|_{L_{\alpha}^{\frac{4\alpha}{d}+2}(0, T; L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2})} \to 0$ as $T \to 0$. Thus for sufficiently small $T$, the hypotheses of Proposition 3.2 are satisfied. Applying iteratively this proposition, we can construct the maximal solution $u \in C([0, T^*), H^\alpha(\mathbb{R}^d)) \cap L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}([0, T^*), L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}(\mathbb{R}^d))$ of (1.1). We proceed by contradiction, assuming that $T^* < \infty$, and $\| u \|_{L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}(0, T; L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2})} < \infty$.

Let $t \in [0, T^* - t)$. For every $s \in [0, T^* - t)$ we have

$$S(.)u(t) = u(t + \tau) + i \int_0^t \frac{\partial}{\partial \tau} S_{\alpha, \theta}(t - \tau)(|u|^{p-1} u) d\tau.$$ 

Then by Strichartz estimate, there exists $K > 0$ such that:

$$\| S_{\alpha, \theta}(.)u(t) \|_{L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}([0, T^* - t), L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}(\mathbb{R}^d))} \leq \| u \|_{L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}([0, T^*), L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}(\mathbb{R}^d))} + K(\| u \|_{L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}([0, T^*), L_{\frac{4\alpha}{d}+2}^{\frac{4\alpha}{d}+2}(\mathbb{R}^d))}^{\frac{4\alpha}{d}+1}$$

Therefore, for every $s$ close enough to $T^*$, it follows that

$$\| S_{\alpha, \theta}(.)u(t) \|_{L^\infty([0, T^* - t), L^\infty(\mathbb{R}^d))} \leq \delta.$$ 

Applying Proposition 3.2 we find that $u$ can be extended after $T^*$, which contradicts the maximality.

Now let us return to the proof of proposition 3.1.

Let $q = \frac{4\alpha(1+p)}{d(p-1)} = \frac{4\alpha}{d} + 2$ and $q' = \frac{4\alpha + \theta}{4\alpha + 2d}$, then $q'$ verifies: $\frac{1}{q'} = \frac{1}{q} + \frac{p-1}{\theta}$, by Holder inequalities and the Strichartz estimate we obtain:

$$\| u \|_{L_\theta^q L_p^{p+1}} \lesssim \| u_0 \|_{L_2} + \| u \|_{L_\theta^q L_p^{p+1}}^{p-1} \| u \|_{L_\theta^q L_p^{p+1}} \lesssim \| u_0 \|_{L_2} + \| u \|_{L_\theta^q L_p^{p+1}}^p \| u \|_{L_\theta^q L_p^{p+1}}$$

Remark that:

$$\frac{2\alpha}{\theta} = d\left(\frac{1}{2} - \frac{1}{r}\right), \quad \frac{2\alpha}{\theta} = d\left(\frac{1}{2} - \frac{1}{r}\right)$$

Now $\frac{2\alpha}{\theta} + 2\frac{d-1}{r} < d - \frac{1}{r}$ if $\alpha > \frac{d}{2d+1}$.

If we take $\gamma = 0$ in the Strichartz estimate this gives with Holder inequality:

$$\| u \|_{L_\theta^q L_r^r} \lesssim \| S_{\alpha, \theta}(.)u_0 \|_{L_\theta^q L_r^r} + \| u \|_{L_\theta^q L_p^{p+1}}^p \| u \|_{L_\theta^q L_p^{p+1}} \lesssim \epsilon + \| u \|_{L_\theta^q L_r^r}^p \| u \|_{L_\theta^q L_r^r} \lesssim \epsilon$$

Noticing that the fractional Leibniz rule (see [22]) and by Holder inequality, leads to

$$\| u \|_{L_\theta^q L_r^r} \lesssim \| u \|_{L_\theta^q L_r^r} + \| u \|_{L_\theta^q L_r^r}^p \| u \|_{L_\theta^q L_r^r} \lesssim \| u \|_{L_\theta^q L_r^r} \| u \|_{L_\theta^q L_r^r}^p \| u \|_{L_\theta^q L_r^r} \lesssim \epsilon \| u \|_{L_\theta^q L_r^r}$$

This implies, with Strichartz estimates in the hand, that:

$$\| u \|_{L_\theta^q H_\alpha^r} \lesssim \| u_0 \|_{H_\alpha^r} + \epsilon \| u \|_{L_\theta^q H_\alpha^r}$$

this gives that:

$$\| u \|_{L_\theta^q H_\alpha^r} \lesssim \| u_0 \|_{H_\alpha^r} + \frac{\epsilon}{1 - \epsilon} \| u \|_{L_\theta^q H_\alpha^r}$$
Now, with Strichartz estimate:

\[
\|u\|_{L^\infty H^\alpha} \lesssim \|u_0\|_{H^\alpha} + \|u|u|^{p-1}\|_{L^{p'} H^{\alpha',r}} \\
\lesssim \|u_0\|_{H^\alpha} + e^{p-1}\frac{\|u_0\|_{H^\alpha}}{1-e^{p-1}} \\
\lesssim 1 + e^{p-1}\frac{\|u_0\|_{H^\alpha}}{1-e^{p-1}}
\]

Then for \(\epsilon\) small we obtain

\[
\|u\|_{L^\infty T H^\alpha} < \infty
\]

this gives that \(T = \infty\).

Now we are ready to prove Theorem 1.2

Let \(u \in C([0,T];H^\alpha_{rd}(\mathbb{R}^d))\) be the solution emanating from some initial datum \(u_0 \in H^\alpha_{rd}(\mathbb{R}^d)\). We have the following a priori estimates:

**Lemma 3.1.** Let \(u \in C([0,T];H^\alpha_{rd}(\mathbb{R}^d))\) be the solution of (1.1) emanating from \(u_0 \in H^\alpha_{rd}(\mathbb{R}^d)\). Then

\[
\|u\|_{L^\infty T L^2} \leq \|u_0\|_{L^2} \quad \text{and} \quad \|(-\Delta)^{\frac{s}{2}} u\|_{L^2_{t}L^2} \leq \frac{1}{\sqrt{2a}} \|u_0\|_{L^2}.
\]

**Proof.** Assume first that \(u_0 \in H^\infty(\mathbb{R}^d)\). Then \(\text{(L.1)}\) ensures that the mass is decreasing as soon as \(u\) is not the null solution and \(\text{(L.2)}\) leads to

\[
\int_0^T \|(-\Delta)^{\frac{s}{2}} u(t)\|^2_{L^2} dt = -\frac{1}{2a} (\|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2}) \leq \frac{1}{2a} \|u_0\|^2_{L^2}.
\]

This proves \(\text{(3.1)}\) for smooth solutions. The result for \(u_0 \in H^\alpha(\mathbb{R}^d)\) follows by approximating \(u_0\) in \(H^\alpha\) by a smooth sequence \((u_0^n) \subset H^\infty(\mathbb{R}^d)\). \(\square\)

From the first estimate in \(\text{(3.1)}\) we can obtain that:

\[
\|u\|_{L^2_{t}L^2} \leq T^{\frac{1}{2}} \|u\|_{L^\infty L^2} \leq T^{1/2} \|u_0\|_{L^2}
\]

and thus by interpolation:

\[
\|(-\Delta)^{\frac{s}{2}} u\|_{L^2_{t}L^2} \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{L^4_{t}L^\infty} \|u\|_{L^2_{t}L^2} \lesssim \frac{1}{a^\frac{\theta}{d}} T^{\frac{\theta}{2}(1-\frac{\theta}{d})}
\]

Interpolating now between \(\text{(3.2)}\) and the first estimate of \(\text{(3.1)}\) we get

\[
\|u\|_{L^2_{t}H^{\alpha}_{rd}} \lesssim \frac{1}{a^\frac{\theta}{d}} T^{\frac{\theta}{2}(1-\frac{\theta}{d})} \quad \text{where} \quad \theta = \frac{4a}{d} + 2
\]

and the embedding \(H^{\frac{\theta}{2}}(\mathbb{R}^d) \hookrightarrow L^{\frac{4a}{d}+2}(\mathbb{R}^d)\) ensures that

\[
\|u\|_{L^\infty T L^{\frac{4a}{d}+2}} \lesssim \frac{1}{a^\frac{\theta}{d}} T^{\frac{\theta}{2}(1-\frac{\theta}{d})}.
\]
Denoting by $T^*$ the maximal time of existence of $u$ and letting $T$ tends to $T^*$, this contradicts proposition (3.3) whenever $T^*$ is finite. This proves that the solutions are global in $H^\alpha(\mathbb{R}^d)$.

**Remark 3.1.** Note that for $s = \alpha$, we have that $\|u\|_{L^{4\alpha+2}_T L^{4\alpha+2}(\mathbb{R}^d)} \lesssim 1$ for any $T > 0$ which show that 

$$
\|u\|_{L^{4\alpha+2}(\mathbb{R}^d)} < +\infty
$$

In plus, 

$$
\|u\|_{L^{4\alpha+2}(\mathbb{R}^d)} \to 0, \quad \text{when} \quad a \to +\infty
$$

### 3.2. Proof Theorem 1.1

Now we will prove the global existence for small data, to do this we will use the following fractional Gagliardo-Niremberq inequalities (see [19]):

**Lemma 3.2.** Let $q, r$ be any real numbers satisfying $1 \leq q, r \leq \infty$, and $s, s_1$ be two reals numbers. If $u$ is any functions in $C_0^\infty(\mathbb{R}^d)$, then

$$
\|D^s u\|_{L^p} \leq C \|D^{s_1} u\|_{L^r}^{1-a} \|u\|_{L^r}^{1-a}
$$

where

$$
\frac{1}{p} = \frac{s}{d} + a(\frac{1}{r} - \frac{s_1}{d}) + (1 - a)\frac{1}{q},
$$

for all $a$ in the interval

$$
\frac{s}{s_1} \leq a \leq 1,
$$

where $C$ is a constant depending only on $d, s, s_1, q, r$ and $a$.

Now we have the following one:

**Proposition 3.4.** Let $\alpha > 0$ and $u \in H^\alpha(\mathbb{R}^d)$. Then there exists $C = C(d, \alpha)$ such that:

$$
\|(-\Delta)^{\alpha/2} u\|^2 \left(\frac{1}{2} - C \|u\|_{L^2}^{2\alpha} \right) \leq E(u(t))
$$

**Proof of proposition 3.4**

We have that:

$$
E(u(t)) = \frac{1}{2} \|(-\Delta)^{\alpha/2} u\|_{L^2}^2 - \frac{d}{4\alpha + 2d} \|u\|_{L^{4\alpha+2}}^{4\alpha+2}
$$

But by the fractional Gagliardo-Niremberq inequality there exists $A = A(\alpha, d)$ such that:

$$
\int |u|^{4\alpha+2} \, dx \leq A \left( \int \|(-\Delta)^{\alpha/2} u\|^2 \, dx \right) \left( \int |u|^2 \, dx \right)^{2\alpha}
$$

then

$$
E(u(t)) \geq \frac{1}{2} \|(-\Delta)^{\alpha/2} u\|_{L^2}^2 - A \frac{d}{4\alpha + 2d} \left( \int \|(-\Delta)^{\alpha/2} u\|^2 \, dx \right) \left( \int |u|^2 \, dx \right)^{2\alpha}
$$

$$
= \|(-\Delta)^{\alpha/2} u\|_{L^2}^2 \left( \frac{1}{2} - A \frac{d}{4\alpha + 2d} \left( \int |u|^2 \, dx \right)^{2\alpha} \right).
$$
Now let us return to the proof of theorem 1.1.
Let \( u \in C([0, T]; H^\infty(\mathbb{R}^d)) \) be a solution to (1.1) emanating from \( u_0 \in H^\infty(\mathbb{R}^d) \). Then it holds
\[
\frac{d}{dt} E(u(t)) = -a \int (\langle -\Delta \rangle^{\frac{s}{d}} u(t))^2 + a Im \int ((\langle -\Delta \rangle^{\ast} u(t)) |u(t)|^{p-1} u(t)
\]
and Hölder inequalities in physical space and in Fourier space lead to
\[
\int ((\langle -\Delta \rangle^{\ast} u) |u|^{p+1}) \leq \| (\langle -\Delta \rangle^{\ast} u) \|_{L^2} \| u \|_{L^{p+1}}^{p+1+\alpha}
\]
with
\[
\| (\langle -\Delta \rangle^{\ast} u) \|_{L^2} \leq \| (\langle -\Delta \rangle^{\ast}^{\frac{s+\alpha}{d}} u) \|_{L^{\infty}} \| u \|_{L^{\alpha+\beta}}.
\]
Using Gagliardo-Nirenberg inequality, we obtain
\[
\| u \|_{L^{\infty}}^{\frac{\alpha+\beta}{d} + 2} \leq C_d^\frac{\alpha+\beta}{d} \| \nabla u \|_{L^2} \| u \|_{L^\infty}^{\frac{\alpha+\beta}{d} + 2 - \alpha}.
\]
This estimate together with Cauchy-Schwarz inequality (in Fourier space)
\[
\| \nabla u \|_{L^2} \leq \| (\langle -\Delta \rangle^{\ast}^{\frac{s+\alpha}{d}} u) \|_{L^2} \| u \|_{L^{\infty}}^{\frac{s+\alpha}{d} + 1} \| u \|_{L^{\infty}}^{\frac{s+\alpha}{d} - \frac{2}{d}}.
\]
lead to
\[
\| u \|_{L^{\infty}}^{\frac{\alpha+\beta}{d} + 1} \leq C_d^\frac{\alpha+\beta}{d} \| (\langle -\Delta \rangle^{\ast}^{\frac{s+\alpha}{d}} u) \|_{L^2} \| u \|_{L^{\infty}}^{\frac{s+\alpha}{d} + 1 - \frac{2}{d}}.
\]
Combining the above estimates we eventually obtain
\[
\frac{d}{dt} E(u(t)) \leq a \| (\langle -\Delta \rangle^{\ast}^{\frac{s+\alpha}{d}} u) \|_{L^2}^2 (C_d^\frac{\alpha+\beta}{d} \| u \|_{L^\infty}^{\frac{2}{d}} - 1)
\]
which together with (3.4) implies that \( E(u(t)) \) is not increasing for \( \| u \|_{L^2} \leq \frac{1}{C_d^\frac{\alpha+\beta}{d}} \) implies \( E(u(t)) \leq E(u_0) \) for all \( t \geq 0 \).
Now with proposition 3.4 in the hand we obtain that \( \| (\langle -\Delta \rangle^{\ast} u) \|_{L^2} \leq E(u_0), \) for \( \| u_0 \|_{L^2} \) small enough. This finishes the proof.

Proof of theorem 1.3: The second part of this theorem was proved previously (see remark 3.1). Let us prove the scattering: Let \( v(t) := S_{-a, -\alpha, -s}(t) u(t) \) then
\[
v(t) = u_0 + i \int_0^t S_{a, \alpha, s}(s)(\| u \|_{L^{\frac{2d}{d+\alpha}}}^{\frac{2d}{d+\alpha}} u(s)) ds.
\]
Therefore for \( 0 < t < \tau \),
\[
v(t) - v(\tau) = i \int_\tau^t S_{a, \alpha, s}(-t')(\| u \|_{L^{\frac{2d}{d+\alpha}}}^{\frac{2d}{d+\alpha}} u) dt'.
\]
It follows from Strichartz’s estimates (as previously) that:
\[
\| v(t) - v(\tau) \|_{L^2} = \| i \int_\tau^t S_{a, \alpha, s}(-t')(\| u \|_{L^{\frac{2d}{d+\alpha}}}^{\frac{2d}{d+\alpha}} u) \|_{L^2} \leq \| u \|_{L^{\frac{2d}{d+\alpha} + 2}(t, \tau)}^{\frac{2d}{d+\alpha} + 2}
\]
But by remark 3.1, for \( s = \alpha \) we have that \( u \in L^{\frac{2d}{d+\alpha} + 2}((0, \infty), L^{\frac{4d}{d+2}}) \), then the right hand side goes to zero when \( t, \tau \to +\infty \). The scattering follows from the Cauchy criterion.
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Mohamad Darwich: Faculty of Sciences, Laboratory of Mathematics, Doctoral School of Sciences and Technology, Lebanese University Hadat, Lebanon.