A Distributed Observer for a Continuous-Time Linear System with time-varying network

Lili Wang, Ji Liu, and A. Stephen Morse

Abstract—A simply structured distributed observer is described for estimating the state of a continuous-time, jointly observable, input-free, linear system whose sensed outputs are distributed across a time-varying network. It is explained how to design a gain $g$ in the observer so that their state estimation errors all converge exponentially fast to zero at a fixed, but arbitrarily chosen rate provided the network’s graph is strongly connected for all time. A linear inequality for $g$ is provided when the network’s graph is switching according to a switching signal with a dwell time or an average dwell time, respectively. It has also been shown the existence of $g$ when the stochastic matrix of the network’s graph is chosen to be doubly stochastic under arbitrarily switching signals. This is accomplished by exploiting several well-known properties of invariant subspaces and properties of perturbed systems.

I. INTRODUCTION

Distributed state estimation problem has gotten more and more attention in recent years [1]–[13] due to the increasing interest in sensor network and multi-agent systems. The problem is to enable each agent to reconstruct the system state by using its own measurements and communicating with the nearby neighbors in a network. More specifically, for the continuous time case it is to estimate the state of an $m > 0$ channel, $n$-dimensional continuous-time linear system of the form $\dot{x} = Ax, y_i = C_i x, i \in \{1, 2, \ldots, m\}$ under the necessary assumption that the system is “jointly observable”. This problem has been studied in different forms. This problem is originally studied in [1]–[3] through a consensus-based Kalman filter assuming that data fusion can be achieved in finite time. In [4], [5], [11], this problem is solved by recasting this as a decentralized control problem. The method allows to freely assign the spectrum of the estimators under the condition that the network is strongly connected and fixed. The recent work in [8], [9] studies this problem based on the structure of the network. By choosing or constructing a tree in the network, it is able to broadcast the information and do estimation.

In this paper, a simple distributed observer is designed by exploiting several well-known properties of invariant subspaces, i.e., the properties of the unobservable subspaces of each agent. The idea stems from research originally reported in [12], [13] and subsequently extended in [7]. This simplified observer is described and its behaviors is analyzed in [14], and a discrete-time version is studied in [15]. As they stand, those estimators in [6], [7], [12], [13] can only deal with the case when the neighbor graph is fixed. The aims of this paper is to extend the results in [14] to the case when the network’s graph is time-varying.

A. Invariant Subspaces

Throughout this paper certain basic and well-known algebraic properties of invariant subspaces will be exploited. To understand what they are, let $A$ be any square matrix, and suppose $\mathcal{V}$ is an $A$-invariant subspace. Let $Q$ be any full row rank matrix whose kernel is $\mathcal{V}$ and suppose that $V$ is any “basis matrix” for $\mathcal{V}$; i.e., a matrix whose columns form a basis for $\mathcal{V}$. Then the linear equations

$$QA = \tilde{A}_V Q \quad \text{and} \quad AV = VAV$$

have unique solutions $\tilde{A}_V$ and $AV$ respectively. Let $V^{-1}$ be any left inverse of $V$ and let $Q^{-1}$ be that right inverse of $Q$ for which $V^{-1}Q^{-1} = 0$. Then

$$A = H^{-1} \begin{bmatrix} \tilde{A}_V & 0 \\ \tilde{A}_V & AV \end{bmatrix} H,$$

where $H = \begin{bmatrix} Q \\ V^{-1} \end{bmatrix}$ and $\tilde{A}_V = V^{-1}AQ^{-1}$. Use will be made of these simple algebraic facts in the sequel.

II. PROBLEM

We are interested in a network of $m > 1$ agents labeled $1, 2, \ldots, m$ which are able to receive information from their neighbors where by a neighbor of agent $i$ is meant any agent in agent $i$’s reception range. In this paper, the network we consider is time-varying and changes according to a switching signal. Let $\mathcal{P}$ denote a suitably defined set, indexing the set of all possible network. Let $\tau_D > 0$ and let $\sigma : [0, \infty) \to \mathcal{P}$ be a piecewise-constant switching signal whose switching times $t_1, t_2, \ldots$ satisfy $t_{i+1} - t_i \geq \tau_D, i \geq 0$.

We write $\mathcal{N}_i(\sigma(t))$ for the set of labels of agent i’s neighbors at time $t$ and take agent $i$ to be a neighbor of itself. Relations between neighbors are characterized by a directed graph $\mathcal{N}(\sigma(t))$ with $m$ vertices and a set of arcs defined so that there is an arc from vertex $j$ to vertex $i$ whenever agent $j$ is a neighbor of agent $i$. Each agent $i$ can sense a signal $y_i \in \mathbb{R}^{\xi_i}$ where

$$\dot{x} = Ax, \quad y_i = C_i x, \quad i \in \mathcal{M} = \{1, 2, \ldots, m\},$$

and $x \in \mathbb{R}^n$. We assume throughout that $\mathcal{N}(\sigma(t))$ is strongly connected and that the system defined by (1) is jointly observable; i.e., with $C = [C_1’ C_2’ \cdots C_m’]$, the matrix
pair \((C, A)\) is observable. Joint observability is equivalent to the requirement that

\[
\bigcap_{i \in \mathbb{m}} V_i = 0
\]

where \(V_i\) is the unobservable space of \((C_i, A)\). As is well known, \(V_i\) is the largest \(A\)-invariant subspace contained in the kernel of \(C_i\).

Each agent \(i\) is to estimate \(x\) using an \(n\)-dimensional linear system with state \(x_i(t) \in \mathbb{R}^n\) and we assume that the information agent \(i\) can receive from neighbor \(j\) at time \(t\) is \(x_j(t)\). The problem of interest is to construct a suitably defined family of linear estimators in such a way so that under switching neighbor graphs no matter what the estimators’ initial states are, for each \(i \in \mathbb{m}\), \(x_i(t)\) is an asymptotically correct estimate of \(x(t)\) in the sense that the estimation error \(x_i(t) - x(t)\) converges to zero as fast as \(e^{-\lambda t}\) does, where \(\lambda\) is an arbitrarily chosen but fixed positive number.

III. THE OBSERVER

The observer to be considered consists of \(m\) private estimators of the form

\[
\dot{x}_i = (A + K_i C_i) x_i - K_i y_i
\]

\[
-g P_i \left( x_i - \frac{1}{m_i(\sigma(t))} \sum_{j \in N_i(\sigma(t))} x_j \right), \quad i \in \mathbb{m}(2)
\]

where \(m_i(\sigma(t))\) is the number of labels in \(N_i(\sigma(t))\), \(g\) is a suitably defined positive gain, each \(K_i\) is a suitably defined matrix, and for each \(i \in \mathbb{m}\), \(P_i\) is the orthogonal projection on the unobservable space of \((C_i, A)\).

To begin with, each matrix \(K_i\) is defined as follows. For each fixed \(i \in \mathbb{m}\), write \(Q_i\) for any full rank matrix whose kernel is the unobservable space of \((C_i, A)\) and let \(C_i\) and \(A_i\) be the unique solutions to \(C_i Q_i = C_i\) and \(Q_i A_i = A_i Q_i\) respectively. Then the matrix pair \((C_i, A_i)\) is observable. Thus by using a standard spectrum assignment algorithm, a matrix \(K_i\) can be chosen to ensure that the convergence of \(e^{(A + K_i C_i) t}\) to zero is as fast as the convergence of \(e^{-\lambda t}\) to zero is. Here \(\lambda\) is a positive number which is greater than \(\lambda\). Having chosen such \(K_i, K_i\) is then chosen to be \(K_i = Q_i^{-1} K_i\) where \(Q_i^{-1}\) is a right inverse for \(Q_i\). The definition implies that \(Q_i (A + K_i C_i) = (A_i + K_i C_i) Q_i\) and that \((A + K_i C_i) V_i \subset V_i\). The latter, in turn, implies that there is a unique matrix \(A_i\) which satisfies \((A + K_i C_i) V_i = V_i A_i\)

where \(V_i\) is a basis matrix for \(V_i\). To understand what needs to be considered in choosing \(g\) it is necessary to delve more deeply into the structure of the overall observer. This will be done next.

IV. ANALYSIS

For each \(i \in \mathbb{m}\), write \(e_i\) for the state estimation error \(e_i = x_i - x\). In view of (1) and (2),

\[
\dot{e}_i = (A + K_i C_i) e_i - g P_i \left( e_i - \frac{1}{m_i(\sigma(t))} \sum_{j \in N_i(\sigma(t))} e_j \right)
\]

(3)

It is possible to combine these \(m\) error equations into a single equation with state \(e = \text{column } \{e_1, e_2, \ldots, e_m\}\). For this let \(\bar{A} = \text{block diagonal } \{A + K_1 C_1, A + K_2 C_2, \ldots, A + K_m C_m\}\), \(P = \text{block diagonal } \{P_1, P_2, \ldots, P_m\}\) and write \(S(\sigma(t))\) for the stochastic matrix \(S(\sigma(t)) = D^{-1}(\sigma(t)) A^{\dagger}(\sigma(t))\) where \(A^{\dagger}(\sigma(t))\) is the adjacency matrix of \(\mathbb{N}(\sigma(t))\) and \(D(\sigma(t))\) is the diagonal matrix whose \(i\)th diagonal entry is the in-degree of \(\mathbb{N}(\sigma(t))\)’s \(i\)th vertex. The error model is then

\[
\dot{e} = (\bar{A} - g P ((I_m - S(\sigma(t))) \otimes I_n)) e
\]

(4)

where \(\otimes\) denotes the Kronecker product.

As a first step towards this end, note that for any value of \(g\), the direct sum \(V = V_1 \oplus V_2 \oplus \cdots \oplus V_m\) is \(\bar{A} - g P ((I_m - S(\sigma(t))) \otimes I_n)\) invariant. This is because \((A + K_i C_i) V_i \subset V_i\), \(i \in \mathbb{m}\) and because \(V = \text{column span of } P\). Let \(Q = \text{block diagonal } \{Q_1, Q_2, \ldots, Q_m\}\) and \(V = \text{block diagonal } \{V_1, V_2, \ldots, V_m\}\) in which case \(Q\) is a full rank matrix whose kernel is \(V\) and \(V\) is a basis matrix for \(V\) whose columns form an orthonormal set. It follows that \(P = V V'\), that \((\bar{A} - g P ((I_m - S(\sigma(t))) \otimes I_n)) V = V (A - g V' (I_m - S(\sigma(t))) \otimes I_n) V\) where \(A\) is the unique solution to \(AV = VA\). Let \(V^{-1}\) be any left inverse of \(V\) and let \(Q^{-1}\) be that right inverse of \(Q\) for which \(V^{-1} Q^{-1} = 0\) and \(V' = V^{-1}\). Then

\[
\bar{A} - g P ((I_m - S(\sigma(t))) \otimes I_n) = H^{-1} \begin{bmatrix} \bar{A}_V & 0 \\ \bar{A}_V (\sigma(t)) & A_V (\sigma(t)) \end{bmatrix} H,
\]

where

\[
H = \begin{bmatrix} Q' \\ V' \end{bmatrix},
\]

\[
\bar{A}_V = \begin{bmatrix} A_1 + K_1 C_1 & 0 & \ldots & 0 \\ 0 & A_2 + K_2 C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_m + K_m C_m \end{bmatrix},
\]

\[
A_V (\sigma(t)) = \bar{A} - g V' ((I_m - S(\sigma(t))) \otimes I_n) V
\]

and \(\bar{A}_V (\sigma(t)) = V' A_V (\sigma(t)) \otimes I_n\).

In order to show exponential convergence of \(x_i(t) - x(t)\), i.e., (3), it is equivalent to look at the stability of system

\[
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_V & 0 \\ \bar{A}_V (\sigma(t)) & A_V (\sigma(t)) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\]

(5)
where $e = H [z_1' \ z_2']$. System (5) can be written as
\[ \dot{z}_1 = A_V z_1 \]  
and
\[ \dot{z}_2 = A_V (\sigma(t)) z_2 + A_V (\sigma(t)) z_1 \] 
A. Stability of a switching system with dwell time $\tau_D$

Notice that in practical applications, the neighbor graph usually does not switch arbitrarily fast. In other words, the change of neighbor graphs must satisfy a dwell time constraint or an average dwell time constraint. Given a positive constant $\tau_D$, let $S(\tau_D)$ denote the set of all switching signals with interval between consecutive discontinuities no smaller than $\tau_D$. The constant $\tau_D$ is called the (fixed) dwell-time.

The following result is developed in the paper.

**Theorem 1**: For any switching signal $\sigma(t) \in S(\tau_D) : \mathbb{R} \to \mathcal{P}$ with any dwell time $\tau_D > 0$, and any given positive number $\lambda$, if the neighbor graph $N(\sigma(t))$ is strongly connected and the system defined by (1) is jointly observable, there are matrices $K_i, i \in \mathbb{N}$ such that for $g$ sufficiently large, each state estimation error $x_i(t) - x(t)$ of the distributed observer defined by (2), converges to zero as $t \to \infty$ as fast as $e^{-\lambda t}$ converges to zero.

By [14] if $\sigma(t)$ is a fixed constant, the estimation error $e(t)$ of (2) can converge to zero as fast as $e^{-\lambda t}$ by choosing large enough $g$. However, the stability of all subsystems for each fixed value of $\sigma(t)$ does not ensure the stability of the switching system. Two lemmas about switching system are needed in order to prove theorem 1

**Lemma 1**: Given a set of matrices \{$(M(1), 0, \ldots, M(p))$\} where each $M(i) \in \mathbb{R}^{n \times n}, i \in \mathcal{P}$ is exponentially stable. For any $\tau_D > 0$, and $\sigma(t) \in S(\tau_D) : \mathbb{R} \to \mathcal{P}$, there exists a positive number $g$ so that $x$ converges to zero as fast as a preassigned convergence rate $\lambda$ under the switching system
\[ \dot{x} = gM(\sigma(t))x \]

**Proof** Since $M(i) i \in \mathcal{P}$ is exponentially stable, there is are positive constants $c_i$ and $\lambda_i$ such that
\[ \|e^{M(i)t}\| \leq c_i e^{-\lambda_i t} \]
where $\| \cdot \|$ is any norm on $\mathbb{R}^{n \times n}$ for which the submultiplicative property holds. Each $c_i$ is chosen to be larger than 1. Thus,
\[ \|e^{M(i)t}\| = \|e^{M(i)t}\| \leq c_i e^{-\lambda_i t} \]
By [16, Lemma 2], for $g\lambda_i \geq \lambda$ if
\[ \tau_D \geq \frac{\ln c_i}{g\lambda_i - \lambda} \quad \forall i \in \mathcal{P} \]
x(t) \leq ce^{-\lambda t }x(0) where $c = \max_{i \in \mathcal{P}} c_i$. Therefore, by choosing $g$ so that $g\lambda_i \geq \lambda$, and $g \geq \frac{\ln c_i + \lambda \tau_D}{n - \lambda_i} \forall i \in \mathcal{P}$, for any given $\sigma(t)$ with dwell time $\tau_D$ there exists $g$ so that $x$ converges to zero with convergence rate $\lambda$.

**Lemma 2**: Given a set of matrices \{$(M(1), 0, \ldots, M(p))$\} where each $M(i) \in \mathbb{R}^{n \times n}, i \in \mathcal{P}$ is exponentially stable, and a bounded matrix $N \in \mathbb{R}^{n \times n}$. For any $\tau_D > 0$, and $\sigma(t) \in S(\tau_D) : \mathbb{R} \to \mathcal{P}$, there exists a positive number $g$ so that $x$ converges to zero as fast as a preassigned convergence rate $\lambda$ under the switching system
\[ \dot{x} = (N + gM(\sigma(t)))x \]

**Proof** Since $M(i) i \in \mathcal{P}$ is exponentially stable, there is are positive constants $c_i$ and $\lambda_i$ such that
\[ \|e^{M(i)t}\| \leq c_i e^{-\lambda_i t} \]
where $\| \cdot \|$ is any norm or induced norm on $\mathbb{R}^{n \times n}$ for which the submultiplicative property holds. Each $c_i$ is chosen to be larger than 1. Let $\Phi_{gM}(t, \tau)$ be the transition matrix of $gM(\sigma(t))$. By Lemma 1 for any given positive number $\lambda$, and $\sigma(t)$ with dwell time $\tau_D$ by choosing $g$ so that $g\lambda_i \geq \lambda$, and $g \geq \frac{\ln c_i + \lambda \tau_D}{n - \lambda_i} \forall i \in \mathcal{P}$, there exists $g$ so that $\Phi_{gM}(t, \tau) \leq ce^{-\lambda(t-\tau)}$ where $c = \max_{i \in \mathcal{P}} c_i$.

Since $N$ is bounded, there exists $b$ so that $\|N\| \leq b$.
Now look at system $\dot{x} = gM(\sigma(t))x + Nx$. By viewing $Nx$ as a forcing function in the preceding, one may write the variation of constants formula
\[ x(t) = \Phi_{gM}(t, 0)x(0) + \int_0^t \Phi_{gM}(t, \mu)Nx(\mu)d\mu \]
Therefore
\[ \|x(t)\| \leq \|\Phi_{gM}(t, 0)x(0)\| + \int_0^t \|\Phi_{gM}(t, \mu)Nx(\mu)\|d\mu \]
\[ \leq ce^{-\lambda t} \|x(0)\| + \int_0^t c e^{-\lambda(t-\mu)} \|N\| \|x(\mu)\|d\mu \]
That is
\[ e^{\lambda t} \|x(t)\| \leq c \|x(0)\| + \int_0^t e^{\lambda t} \|x(\mu)\|d\mu \]
By the Bellman-Gronwall Lemma,
\[ e^{\lambda t} \|x(t)\| \leq c \|x(0)\| e^{\lambda t} c e^{\lambda t} = c \|x(0)\| e^{\lambda t} \]
Therefore
\[ \|x(t)\| \leq c \|x(0)\| e^{-\lambda t} \]
If $\overline{\lambda}$ is chosen so $\overline{\lambda} \geq \lambda + bc$, and $g$ is chosen so that $g\lambda_i \geq \overline{\lambda}$, and $g \geq \frac{\ln c_i + \lambda \tau_D}{n - \lambda_i} \forall i \in \mathcal{P}$,
\[ \|x(t)\| \leq c \|x(0)\| e^{-\lambda t} \]

**Proof of Theorem 1** is provided in the following.

**Proof of Theorem 1**: In order to show exponential convergence of $x_1(t) - x(t)$, i.e., (4), it is equivalent to look at the stability of system (5), i.e., (6) and (7). First, since the spectrum of $\overline{A}_1 + \overline{K}_i \overline{C}_i, i \in \mathbb{N}$ is assignable with $\overline{K}_i$, there are $\overline{K}_i$s so that $\|z_1(t)\| \leq e^{-\lambda t} \|z_1(0)\|$ where $\lambda > \overline{\lambda}$. It is left to show that for $g$ sufficiently large, $z_2$ of (3) converges to zero with a prescribed convergence rate as large as $\lambda$.
Consider $A_V(\sigma(t))z_1$ of (7) as a forcing function, and let $\Phi_V(t, \tau)$ be the transition matrix of $A_V(\sigma(t))$ for any $t \geq \tau \geq 0$. Thus,
\[ z_2(t) = \Phi_V(t, 0)z_2(0) + \int_0^t \Phi_V(t, \mu)A_V(\mu)z_1(\mu)d\mu \]
Recall that $A_V(\sigma(t)) = \tilde{A} - gV'((I_m - S(\sigma(t))) \otimes I_n)V$. By [14, Proposition 1], for any $i \in \mathcal{P}$, $-V'((I_m - S(i)) \otimes I_n)V$ is exponentially stable. Let $c_i$ and $\lambda_i$ be two positive constants such that

$$||e^{-V'((I_m - S(i)) \otimes I_n)V}|| \leq c_i e^{-\lambda_i t}$$

Each $c_i$ is chosen to be larger than 1. Let $c = \max_{i \in \mathcal{P}} c_i$. Since $\tilde{A}$ is fixed, let $||\tilde{A}|| \leq b$. Moreover according to Lemma 2 if for $\hat{\lambda} \geq \lambda + bc$, $g$ is chosen so that $g\lambda_i \geq \hat{\lambda}$, and $g \geq \frac{\ln c + \lambda_i \tau_D}{\lambda_i} \forall i \in \mathcal{P}$,

$$||\Phi_V(t, \tau)|| \leq ce^{-\lambda(t-\tau)}; \forall \tau \geq 0$$

Once $g$ is fixed, there exists a positive number $\hat{e}$ so that $||A_V(\sigma(\mu))|| \leq \hat{e}$. Therefore,

$$||z_2(t)|| \leq ||\Phi_V(t, 0)|| ||z_2(0)|| + \int_0^t ||\Phi_V(t, \mu)|| ||\dot{A}_V(\sigma(\mu))|| ||z_1(\mu)|| d\mu$$

$$\leq ce^{-\lambda t} ||z_2(0)|| + \int_0^t ce^{-\lambda(t-\tau)} c e^{-\lambda t} ||z_1(0)|| d\mu$$

$$= ce^{-\lambda t} ||z_2(0)|| + ce^{-\lambda t} ||z_1(0)|| \int_0^t e^{(\lambda-\lambda)\mu} d\mu$$

$$= ce^{-\lambda t} ||z_2(0)|| + ce^{-\lambda t} ||z_1(0)|| \left[ \frac{e^{\lambda \mu}}{\lambda - \lambda} \right]_0^t$$

$$\leq ce^{-\lambda t} ||z_2(0)|| + ce^{-\lambda t} ||z_1(0)|| \left[ \frac{e^{\lambda \mu}}{\lambda - \lambda} \right]_0^t$$

Thus

$$\left[ \begin{array}{c} ||z_1(t)|| \\ ||z_2(t)|| \end{array} \right] \leq e^{-\lambda t} \left[ \begin{array}{cc} 1 & 0 \\ c \frac{1}{\lambda - \lambda} \end{array} \right] \left[ \begin{array}{c} ||z_1(0)|| \\ ||z_2(0)|| \end{array} \right]$$

According to the proof of Theorem 1 the bound of $g$ can be derived. Recall that $c_i$ and $\lambda_i$ are two positive constants such that

$$||e^{-V'((I_m - S(i)) \otimes I_n)V}|| \leq c_i e^{-\lambda_i t}$$

and $c = \max_{i \in \mathcal{P}} c_i$. $b$ is the constant so that $||\tilde{A}|| \leq b$. Let $\lambda^* = \min_{i \in \mathcal{P}} \lambda_i$.

$$g \geq \max \left\{ \frac{\lambda + bc}{\lambda^*}, \frac{\lambda + bc}{\lambda^*} + \frac{\ln c}{\lambda_i \tau_D} \right\}$$

(8)

B. Stability of a switching system with average dwell time $\tau_D$

However, in certain situations, the switching signals may occasionally have consecutive discontinuities separated by less than $\tau_D$, but for which the average interval between consecutive discontinuities is no less than $\tau_D$. This leads to the concept of average dwell time. For each switching signal $\sigma(t)$ and each $t \geq \tau \geq 0$, let $N_\sigma(t, \tau)$ denote the number of discontinuities of $\sigma(t)$ in the open interval $(\tau, t)$. For given $N_0, \tau_D > 0$, we denote by $S_{ave}(\tau_D, N_0)$ the set of all switching signals for which

$$S_{ave}(\tau_D, N_0) = \{ \sigma(t) : N_\sigma(t_0, t) \leq N_0 + \frac{t-t_0}{\tau_D} \}.$$
Let $N_0$, there exists a positive number $g$ so that $x$ converges to zero as fast as a preassigned convergence rate $\lambda$ under the switching system
\[
\dot{x} = (N + gM(\sigma(t)))x
\]
By using the result of Lemma \ref{lemma3}, the proof of Lemma \ref{lemma4} is similar to the proof of Lemma \ref{lemma2} which is omitted here.

Then based on Lemma \ref{lemma4} the proof of Corollary \ref{corollary1} is almost the same as the proof of Theorem \ref{theorem2} which is omitted here. The linear inequality which $g$ has to satisfy to ensure the convergence rate of \ref{corollary1} is still \ref{corollary1} where $\tau_D$ will be the average dwell time instead of dwell time.

C. Stability of a switching system with arbitrary switching

In this section, a special case is studied. It can be shown that when the stochastic matrix $S(\sigma(t))$ of each neighbor graph $\mathbb{N}(\sigma(t))$ is doubly stochastic, state estimation can be achieved under arbitrary switching.

**Theorem 2:** For any switching signal $\sigma(t) : \mathbb{R} \to \mathbb{P}$, and any given positive number $\lambda$, if the neighbor graph $\mathbb{N}(\sigma(t))$ is strongly connected, the stochastic matrix $S(\sigma(t))$ of graph $\mathbb{N}(\sigma(t))$ is doubly stochastic, and the system defined by \ref{corollary1} is jointly observable, there are matrices $K_i, i \in \mathbb{m}$ such that for $g$ sufficiently large, each state estimation error $x_i(t) - x(t)$ of the distributed observer defined by \ref{corollary1}, converges to zero as $t \to \infty$ as fast as $e^{-\lambda t}$ converges to zero.

**Proof:** Recall $\Phi_V(t, \tau)$ is the transition matrix of $A_V(\sigma(t))$ for any $t, \tau \geq 0$. If we can show that there exist a constant $c$ so that

\[
\|\Phi_V(t, \tau)\| \leq ce^{-\lambda(t-\tau)}, \quad \forall t, \tau \geq 0
\]

the remaining proof is exactly the same as the proof of Theorem \ref{theorem2} which is omitted here.

It is left to show that $\|\Phi_V(t, \tau)\| \leq ce^{-\lambda(t-\tau)}, \quad \forall t, \tau \geq 0$ by choosing $g$ sufficiently large. We exploit matrix $A_V(\sigma(t))$. Recall that $A_V(\sigma(t)) = \tilde{A} - gV((2I_m - S(\sigma(t))) \otimes I_n)V$.

In particular,
\[
(\lambda I + A_V(\sigma(t))) + (\lambda I + A_V(\sigma(t)))' = (\lambda I + \tilde{A}) + (\lambda I + \tilde{A})' - gV((2I_m - S(\sigma(t)) - S'(\sigma(t))) \otimes I_n)V
\]

Since each $S(\sigma(t))$ is doubly stochastic, $2I_m - S(\sigma(t)) - S'(\sigma(t))$ has row sum 0, all its off-diagonal entries are non-positive, and all its diagonal entries are positive. That is this matrix can be seen as a generalized Laplacian matrix of a strongly connected graph. By \cite[Proposition 1]{14}, for any $t$, $-V((2I_m - S(\sigma(t)) - S'(\sigma(t))) \otimes I_n)V$ is negative definite.

Thus by picking $g$ sufficiently large, $(\lambda I + A_V(\sigma(t))) + (\lambda I + A_V(\sigma(t)))'$ will be negative definite for any time $t$.

Consider system
\[
\dot{z} = A_V(\sigma(t))z
\]
Let $V = \bar{z}'\bar{z}$.

\[
\dot{V} = \bar{z}'(A_V(\sigma(t))' + A_V(\sigma(t)))\bar{z} \leq -2\lambda \bar{z}'\bar{z}
\]

Therefore, $\Phi_V(t, \tau)$ converges to zero as fast as $e^{-\lambda(t-\tau)}$ does, i.e.,

\[
\|\Phi_V(t, \tau)\| \leq ce^{-\lambda(t-\tau)}, \quad \forall t, \tau \geq 0
\]

V. Conclusion

This paper studies the distributed observer problem when the neighbor graph is time-varying but always strongly connected. It has been shown that for any switching signal with a dwell time or an average dwell time, for $g$ large enough, each agent can estimate the state exponentially fast with a pre-assigned convergence rate. Study the distributed observer problem when the neighbor graph is not always strongly connected would be future work.

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