Matrix integrals and generating functions for enumerating rooted hypermaps by vertices, edges and faces for a given number of darts

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Abstract

A recursive method is given for finding generating functions which enumerate rooted hypermaps by number of vertices, edges and faces for any given number of darts. It makes use of matrix-integral expressions arising from the study of bipartite quantum systems. Direct evaluation of these generating functions is then demonstrated through the enumeration of all rooted hypermaps with up to 13 darts.

Keywords: enumeration, rooted hypermap, bipartite quantum system, matrix integral, generating function, divergent power series

1 Introduction

This paper is an extension of work we carried out in a previous paper [1]. In that paper we showed how the mean value of traces of integer powers of the reduced density operator of a finite-dimensional bipartite quantum system are proportional to generating functions for enumerating one-face rooted hypermaps. We then used this relation to derive a matrix integral expression for these generating functions, and found closed form expressions for them.

Matrix integral expressions derived from finding the average of a function of the reduced density operator have been studied for some time [2, 3, 4, 5, 6], so numerous methods for evaluating them have been described. In particular, Lloyd and Pagels were able to reduce the matrix integral to an integral over the space of eigenvalues with the density function [2]

\[ P(p_1, \ldots, p_m) dp_1 \ldots dp_m \propto \delta \left( 1 - \sum_{i=1}^{m} p_i \right) \Delta^2(p_1, \ldots, p_m) \prod_{k=1}^{m} p_k^{n-m} dp_k \]

where \( \Delta(p_1, \ldots, p_m) \) is the Vandermonde determinant of the eigenvalues of the reduced density operator. Using this, in conjunction with the work of Sen [6], we were able to evaluate a closed-form expression for the one-face generating functions.

In this paper we extend these methods to derive expressions for generating functions which enumerate all rooted hypermaps by number of vertices, edges and faces for a given number of
darts (we give a definition of rooted hypermaps in Section 2). These generating functions are
defined recursively in terms of another expression called $F(m, n, \lambda; x)$, which we define in Section
3 and is itself evaluated using a matrix integral as above.

Previous work already exists on enumeration of hypermaps [7, 8, 9], and in particular Walsh
managed to enumerate all rooted hypermaps with up to 12 darts by number of edges, vertices
and darts, and genus [9]. But as far as we are aware this is the first time that generating
functions for enumerating all rooted hypermaps by these properties have been found without
direct computation of the hypermaps themselves. By avoiding having to generate the hypermaps
individually, we are able to vastly speed up the process of enumeration (there are more than $r!$
hypermaps with $r$ darts [1], so generating them all is a very slow process).

We will give an overview of our previous work in Section 2.1 before showing how best to
generalise it to multiple faces in Section 2.2 We will then use this to study the global generating
function for rooted hypermaps $H(m, n, \lambda; x)$ in Sections 3 and 4 before looking at the process
of evaluating these functions and extracting hypermap counts from them in Section 5.

2 Representing rooted hypermaps

A thorough discussion of hypermaps can be found in [10].

A hypermap is a generalisation of a map (a graph embedded on an orientable surface so
that its complement consists only of regions which are homeomorphic to the unit disc) in which
the edges are capable of having any positive number of connections to vertices instead of the
usual two. Hypermaps can be thought of as equivalent to bipartite bicoloured maps on the same
surface (with the two colours of vertices in the map representing the vertices and edges of the
hypermap) [11]. Each edge-vertex connection (the edges in the equivalent bipartite map) is called
a dart, and a rooted hypermap is a hypermap where one of the darts has been labelled as the
root, making it distinct from the others.

The embedding of a hypergraph (the analogue of a graph) on an orientable surface to produce
a hypermap can be represented in other ways which do not require explicit consideration of the
surface involved. These are called combinatorial embeddings, and one such method uses an
object called a 3-constellation:

Definition 1. A 3-constellation is an ordered triple \( \{\xi, \eta, \chi\} \) of permutations acting on some
set \( R \), satisfying the following two properties:

1. The group generated by \( \{\xi, \eta, \chi\} \) acts transitively on \( R \).

2. The product \( \xi\eta\chi \) equals the identity.

A hypermap \( H \) with \( r \) darts can be expressed using a 3-constellation on the set \( R = \{1 \ldots r\} \).
If the elements of \( R \) are associated with the darts in \( H \), then the actions of \( \xi, \eta \) and \( \chi \) are to cycle
the darts around their adjacent faces, edges and vertices respectively. For our purposes here,
the important result is that the number of faces, edges and vertices in a hypermap \( H \equiv \{\xi, \eta, \chi\} \)
are given by the number of cycles in \( \xi, \eta \) and \( \chi \) respectively [10, p 43].

Two 3-constellations \( \{\xi, \eta, \chi\} \) and \( \{\xi', \eta', \chi'\} \) are isomorphic to each other if they are related
by the bijection

\[
\tau : \{\xi, \chi, \eta\} \to \{\xi', \chi', \eta'\} = \{\tau\xi\tau^{-1}, \tau\eta\tau^{-1}, \tau\chi\tau^{-1}\}
\]

(2.1)

for some permutation \( \tau \) [10] p 8, are isomorphic to each other (the action of \( \tau \) on the hypermap
as given above simply involves a reordering of the darts in the set \( R \) without changing the

2
Figure 2.1: A diagrammatic representation of a one-face rooted hypermap $H \equiv \{\xi, \eta, \chi\}$, with $\xi = (12\ldots r)$ and $\eta = (1453)(2)(67)$, referred to as a ladder diagram. Mapping from black to white, the single dashed lines represent $\xi^{-1}$ and the double (solid and dashed) lines represent $\eta$. The number of edges in the hypermap equals the number of closed solid loops, while the number of vertices equals the number of closed dashed loops (the double lines count as either solid or dashed). Also shown are the correspondence between the nodes and the terms in (2.2), which is used in the evaluation of $P_r(m, n)$.

connectivity). With this representation of hypermap isomorphism established, we define rooted hypermaps as hypermaps with the additional property that they are only equivalent under the action of $\tau$ only when $\tau(1) = 1$ (i.e. choosing for the root dart to have the label 1).

2.1 One-face hypermaps

In our previous paper, we used the 3-constellation representation of hypermap embedding to define a diagrammatic representation of rooted hypermaps (see Figure 2.1). If we define

$$\xi = (12\ldots r),$$

then the set of rooted hypermaps with one face is equivalent to the set of permutations $\eta$ on $[1\ldots r]$ (i.e. the symmetric group $Sym_r$) through the then any rooted hypermap with one face is equivalent to, then the set of nonisomorphic rooted hypermaps with $r$ darts is equivalent to the set of permutations all $\eta$ on $[1\ldots r]$ (i.e. the symmetric group $Sym_r$) through the bijection

$$\eta \rightarrow H_\eta \equiv \{\xi, \eta, \eta^{-1}\xi^{-1}\}$$

(as $\xi$ is fixed, no two choices of $\eta$ will result in equivalent rooted hypermaps). The diagrammatic representation in Figure 2.1 (here referred to as a ladder diagram) allows us to quickly count the number of vertices and edges in $H_\eta$ by counting closed loops (the numbers of which are equal to the number permutations in $\eta$ and $\xi\eta$).

We showed that these diagrams also arise in the evaluation of the function

$$P_r(m, n) = \frac{\partial}{\partial \alpha_{a_1 b_1}} \cdots \frac{\partial}{\partial \alpha_{a_r b_r}} (\alpha_{a_1 b_2} \cdots \alpha_{a_r b_1}) \bigg|_{\alpha = 0},$$

(2.2)

where $\alpha$ is an $m \times n$ real matrix: when the multiderivative in (2.2) is fully expanded out, it has the form

$$P_r(m, n) = \sum_{\eta \in Sym_r} \prod_{i=1}^r \delta[a_i, a_{\eta(i)}] \delta[b_i, b_{\eta(i)}] = \sum_{\eta \in Sym_r} n^{cyc(\eta)} n^{cyc(\xi\eta)}$$

3
where \( \text{cyc}(\sigma) \) is the number of cycles in the permutation \( \sigma \). As \( \text{cyc}(\eta) \) and \( \text{cyc}(\xi \eta) \) are respectively the number of edges and vertices in the rooted hypermap \( H_\eta = \{ \xi, \eta, \eta^{-1} \xi^{-1} \} \), and the number of faces in \( H_\eta \) is \( \text{cyc}(\xi) = 1 \), \( P_r \) is therefore the generating function for enumerating one-face rooted hypermaps with \( r \) darts by number of edges and vertices. \( P_r \) can also be computed using ladder diagrams as above, then, each diagram contributing a single \( m^{\text{cyc}(\eta)} n^{\text{cyc}(\xi \eta)} \) term.

We also showed, through Gaussian integration, that

\[
P_r(m, n) = \int_{\mathbb{C}^{mn}} d^{2mn} x e^{-x^* x} x^{a_1 b_1} x^{b_2} \ldots x^{a_r b_r} \]

\[
= \frac{\Gamma(mn + r)}{\Gamma(mn)} \langle \text{Tr}[\hat{\rho}^A]^r \rangle,
\]

where \( \hat{\rho}^A \) is the reduced density operator of an \( m \)-dimensional subsystem of an \( mn \)-dimensional bipartite quantum system, and the mean is being taken over all possible pure states of the overall bipartite system. What this means is explained in more detail in [1], but the facts of most relevance here are that [2, 12] is symmetric in \( m \) and \( n \), and, when \( n \geq m \), the mean can be represented as an integral over the eigenvalues \( (p_1, \ldots, p_m) \) of \( \hat{\rho}^A \) with the density function

\[
P(p_1, \ldots, p_m) dp_1 \ldots dp_m \propto \delta \left( 1 - \sum_{i=1}^{m} p_i \right) \Delta^2(p_1, \ldots, p_m) \prod_{k=1}^{m} p_k^{n-m} dp_k,
\]

where \( \Delta^2(p_1, \ldots, p_m) \) is the Vandermonde discriminant of the eigenvalues, giving

\[
\langle \text{Tr}[\hat{\rho}^A]^r \rangle \propto \int \delta \left( 1 - \sum_{i=1}^{m} p_i \right) \Delta^2(p_1, \ldots, p_m) \prod_{k=1}^{m} (p_k^{n-m} dp_k) \sum_{j=1}^{m} p_j^r.
\]

Using a coordinate substitution given in [2], we multiply this by the factor

\[
\frac{1}{\Gamma(mn + r)} \int_{0}^{\infty} \lambda^{mn+r-1} e^{-\lambda} d\lambda
\]

and define \( q_i = \lambda p_i \), integrating over \( \lambda \) in order to remove the \( \delta \) function, giving

\[
\langle \text{Tr}[\hat{\rho}^A]^r \rangle \propto \frac{1}{\Gamma(mn + r)} \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} (e^{-q_k} q_k^{n-m} dq_k) \sum_{j=1}^{m} q_j^r.
\]

Finally, we normalise this by using the fact that, as \( n \geq m \), \( \langle \text{Tr}[\hat{\rho}^A]^0 \rangle = \langle \text{Tr}[I_m] \rangle = m \), giving

\[
\langle \text{Tr}[\hat{\rho}^A]^r \rangle = \frac{\Gamma(mn)}{\Lambda_{mn} \Gamma(mn + r)} \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} (e^{-q_k} q_k^{n-m} dq_k) \sum_{j=1}^{m} q_j^r
\]

where

\[
\Lambda_{mn} = \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} (e^{-q_k} q_k^{n-m} dq_k).
\]

We now need to generalise both the concept of ladder diagrams and the closely connected \( P_r(m, n) \) functions in order to enumerate hypermaps with more than one face, and we will define these generalisations in the next section.
Figure 2.2: A ladder diagram of a three-face rooted hypermap $H \equiv \{\xi, \eta, \chi\}$. As in Figure 2.1, $\eta = (1453)(2)(67)$, but in this case $\xi = (12)(3456)(7)$. The double edges have been greyed out for clarity. Summation over possible permutations $\eta$ in this case generates $P_{2,4,1}(m,n)$

2.2 Multiple faces

In Figure 2.1, there was just one loop consisting only of single lines (i.e. single solid lines and single dotted lines, but not the solid/dotted paired lines), which corresponded to the single cycle in $\xi$, and therefore to the face in the associated rooted hypermap. It follows that a hypermap with multiple faces would have a diagram with multiple such loops (i.e. $\xi$ has multiple cycles, one for each face). An example of such a diagram is shown in Figure 2.2.

In these diagrams, the solid lines in combination with the dotted lines defined by $\xi$ can be thought of as a fixed backbone, on which the double lines given by $\eta$ are superimposed. We described in Section 2.1 how, when $\xi = (12...r)$, we can sum over all possible $\eta$ and in each case count the solid and dotted loops in order to get a generating function for enumerating rooted one-face hypermaps with $r$ darts. We also showed that this function was equivalent to (2.2) and (2.3).

We can apply the same procedure to diagrams with other backbones. Looking at (2.2) and (2.3), we can see that the single cycle of length $r$ in $\xi$ corresponds to a term $\text{Tr}[(\hat{\rho}^A)^r]$ in the quantum expression for $P_r$. By extension it follows that if consists of $N$ cycles with lengths $r_1, r_2, \ldots, r_N$ (e.g. the $\xi$ used in Figure 2.2 corresponds to $N = 3$, $\{r_1, r_2, r_3\} = \{2, 4, 1\}$), summing over all ladder diagrams with such a backbone and following the same procedure as in section 2.1 we get the function

$$P_{r_1, r_2...r_N}(m, n) = \frac{\Gamma(mn + \Sigma_{i=1}^{N} r_i)}{\Gamma(mn)} \left\langle \prod_{j=1}^{N} \text{Tr}[(\hat{\rho}^A)^{r_j}] \right\rangle$$

$$= \frac{1}{\Lambda_{mn}} \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} (e^{-q_k n - m} dq_k)$$

$$\times \prod_{i=1}^{N} m \sum_{j=1}^{m} q_j^{r_i},$$

(2.4)

again valid when $m \leq n$.

These functions are not yet useful generating functions, however, for two reasons: the sum over diagrams used to calculate them can include disconnected diagrams (hypermaps are necessarily connected, so cannot correspond to disconnected ladder diagrams), and any two hypermaps which are related through cyclic permutation of one of the cycles in $\xi$ are equivalent, producing a degeneracy. We will overcome these issues in the following sections; first we will define some additional functions in terms of the various $P_{r_1...}$ in Section 3 which will account for the presence of disconnected diagrams, and then we will use these to construct global generating functions for counting rooted hypermaps in Section 4.
### 3 Connected diagrams

As stated in the previous section, the functions $P_{r_1...}$ defined in (2.4) are generating functions each of which count over a set of ladder diagrams. As defined, however, they include disconnected diagrams in this count, whereas we require generating functions which count only over connected diagrams. In this section we will define such functions.

For any given $P_{r_1...r_N}$, define $\bar{P}_{r_1...r_N}$ to be a generating function defined as a summation over the same set of diagrams as $P_{r_1...r_N}$ except with any disconnected diagrams excluded. In the one-loop case, $P_r = \bar{P}_r$ as all one-loop ladder diagrams are connected. When there is more than one loop present, $P_{r_1...r_N}$ may be factorised in terms of $\bar{P}_{r_1...r_N}$ using the fact that any disconnected ladder diagram can be split into a number of disjoint connected subdiagrams. We write this factorisation

\[
P_{rr_1...r_N} = \bar{P}_{r_1...r_N} + \bar{P}_r \bar{P}_{r_1...r_N} + \sum_{u,v \neq \emptyset} \bar{P}_{ru_1...v_1...r_N},
\]

where the summation is over all partitions of the ordered multiset $\{r_1,...,r_N\}$ into two disjoint non-empty subfamilies. This factorisation works by breaking up each diagram in the summation into its disjoint connected subdiagrams and considering which subdiagram the loop of length $r$ is in. This loop is factored out in a $\bar{P}$ term. As an example, when $N = 3$,

\[
P_{rab} = \bar{P}_{rab} + \bar{P}_r P_{abc} + \bar{P}_r P_{ab} + \bar{P}_r P_{bc} + \bar{P}_r P_{ac} + \bar{P}_r P_{ab}.
\]

If the definition of $P_{r_1...r_N}$ is extended to include

\[
P(m,n) = 1,
\]

which is consistent with (2.4), then (3.1) can be written more simply as

\[
P_{rr_1...r_N} = \sum_{u,v \neq \emptyset} \bar{P}_{ru_1...v_1...r_N},
\]

where $u$ and $v$ are now allowed to be empty.

(3.1) and (2.4) can be used recursively to construct integral expressions for any given $\bar{P}_{r_1...r_N}$. However, when constructing the global generating function in Section 4 it will be more useful to work with the functions

\[
\Pi^{(N)}(m,n;x) = \sum_{r_1=1}^{\infty} \frac{x^{r_1}}{r_1} \cdots \sum_{r_N=1}^{\infty} \frac{x^{r_N}}{r_N} P_{r_1...r_N}(m,n)
\]

(3.3)

\[
\bar{\Pi}^{(N)}(m,n;x) = \sum_{r_1=1}^{\infty} \frac{x^{r_1}}{r_1} \cdots \sum_{r_N=1}^{\infty} \frac{x^{r_N}}{r_N} \bar{P}_{r_1...r_N}(m,n)
\]

(3.4)

\[
\Sigma^{(N)}(m,n;x) = \sum_{r_1=1}^{\infty} \frac{x^{r_1}}{r_1} \cdots \sum_{r_N=1}^{\infty} \frac{x^{r_N}}{r_N} P_{r_1...r_N}(m,n).
\]

(3.5)

These are specifically defined as formal power series in $x$; in general these series will be divergent if treated as functions of a finite parameter $x$. Noting the special cases $\Pi^{(0)}(m,n;x) = $
\[ \Pi_r^{(0)}(m, n; x) = P_r(m, n) \text{ and } \Sigma^{(0)}(m, n; x) = 1, \]

we use \( (3.2) \) to derive the recursion relation

\[
\Pi_r^{(N)}(m, n; x) = \sum_{r_1=1}^{\infty} \frac{x^{r_1}}{r_1} \cdots \sum_{r_N=1}^{\infty} \frac{x^{r_N}}{r_N} P_{r_1 \ldots r_N}(m, n)
= \sum_{r_1=1}^{\infty} \frac{x^{r_1}}{r_1} \cdots \sum_{r_N=1}^{\infty} \frac{x^{r_N}}{r_N} \sum_{u \cup v = \{r_1 \ldots r_N\}} P_{ru_1 \ldots}(m, n) P_{v_{\ldots}}(m, n)
= \sum_{k=0}^{N} \binom{N}{k} \Pi_r^{(k)}(m, n; x) \Sigma^{(N-k)}(m, n; x),
\]

(3.6)

with the sum over partitions in \( (3.2) \) becoming a sum over the different possible sizes of the partitions instead.

In addition to these three sets of series, we will need one more series to be defined:

\[ F(m, n, \lambda; x) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \Sigma^{(N)}(m, n; x). \]

(3.7)

This series’ derivative satisfies

\[
x \frac{\partial}{\partial x} F(m, n, \lambda; x) = x \frac{\partial}{\partial x} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{r_1=1}^{\infty} \frac{x^{r_1}}{r_1} \cdots \sum_{r_N=1}^{\infty} \frac{x^{r_N}}{r_N} P_{r_1 \ldots r_N}(m, n)
= \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{r_1=1}^{\infty} x^{r_1} \sum_{r_2=1}^{\infty} \frac{x^{r_2}}{r_2} \cdots \sum_{r_N=1}^{\infty} \frac{x^{r_N}}{r_N} P_{r_1 \ldots r_N}(m, n)
= \sum_{N=1}^{\infty} \frac{\lambda^N}{(N-1)!} \sum_{r=1}^{\infty} x^{r} \Pi_r^{(N-1)}(m, n; x)
= \sum_{N=0}^{\infty} \frac{\lambda^{N+1}}{N!} \sum_{r=1}^{\infty} x^{r} \Pi_r^{(N)}(m, n; x),
\]

(3.8)

and when \( x \) is set to zero,

\[
F(m, n, \lambda; 0) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \Sigma^{(N)}(m, n; 0)
= \Sigma^{(0)}(m, n; 0)
= 1.
\]

(3.9)

With these various functions and series defined, we can proceed to define the global generating function for enumerating rooted hypermaps in terms \( F \). After doing this in the next section, we will return to \( F \) in Section \( \text{[4]} \) and discuss methods for evaluating it.

## 4 Hypermap generating functions

Let us define \( H(m, n, \lambda; x) \) as the generating function for enumerating all rooted hypermaps in the form

\[ H(m, n, \lambda; x) = \sum_{e, v, f, r} H_{e, v, f, r} m^e n^r \lambda^f x^r, \]

(4.1)
Figure 4.1: A cyclic permutation one place to the left applied to the second loop in a ladder diagram. As the two diagrams are isomorphic they are equivalent to the same hypermap, but they contribute separately to the function $\bar{P}_{241}$ as the diagrams themselves are distinct. The total degeneracy in this case is $r_2 \cdot r_3 = 4$ (there is no degeneracy associated with the first loop as it contains the root, and is therefore fixed against permutation).

where $H_{vefr}$ is the number of rooted hypermaps with $v$ edges, $e$ vertices, $f$ faces and $r$ darts. As with the expressions used in Section 2, this generating function is strictly speaking a formal power series in $x$ which will be divergent in general. However, if we write

$$H(m, n, \lambda; x) = \sum_{r=0}^{\infty} H_r(m, n, \lambda)x^r,$$

then the individual $H_r$ will be well-behaved polynomial functions enumerating all rooted hypermaps with $r$ darts. Ultimately our aim will be to compute these.

It is worth noting the symmetry properties of these functions:

**Theorem 1.** Each $H_r$ is completely symmetric in its three parameters, or, equivalently,

$$H_r(m, n, \lambda) = H_r(n, m, \lambda) = H_r(m, \lambda, n).$$

*Proof.* This result follows easily from considering a rooted hypermap as a 3-constellation $\{\xi, \chi, \eta\}$. The mapping

$$T_{ef} : \{\xi, \chi, \eta\} \rightarrow \{\chi^{-1}, \xi^{-1}, \eta^{-1}\}$$

maps rooted hypermaps with $r$ darts onto each other, and specifically maps a rooted hypermap with $v$ vertices, $e$ edges and $f$ faces onto one with $v$ vertices, $f$ faces and $e$ edges. As $T_{ef}$ is bijective (it is its own inverse), this means that $H_{vefr} = H_{efr}$, and so

$$H_r(m, n, \lambda) = \sum_{v, e, f} H_{vefr}m^vn^r\lambda^f = H_r(m, \lambda, n).$$

Similarly, the mapping

$$T_{ve} : \{\xi, \chi, \eta\} \rightarrow \{\xi^{-1}, \eta^{-1}, \chi^{-1}\}$$

is a bijection which swaps the number of edges and vertices in each rooted hypermaps, meaning $H_{vefr} = H_{efr}$ and

$$H_r(m, n, \lambda) = H_r(n, m, \lambda).\qed$$

While we cannot evaluate $H$ directly, we are able to define it in relation to the series $F$ defined previously:
Theorem 2. The generating function $H$ satisfies the relation

$$H(m, n, \lambda; x)F(m, n, \lambda; x) = x \frac{\partial}{\partial x} F(m, n, \lambda; x).$$

Proof. Each $\overline{P}_{r_1 \ldots r_N}$ is a generating function for a set of rooted hypermaps with $N$ faces (one for each of the loops in the associated ladder diagrams), and if all possible $\overline{P}_{r_1 \ldots r_N}$ for fixed $r_1 + \ldots + r_N = r$ are summed over, then the resulting function will include terms for every rooted hypermap with $N$ faces and $r$ darts, as any such hypermap has at least one associated ladder diagram which contributes a term to one of the $\overline{P}_{r_1 \ldots r_N}$. However, each such hypermap with have a total of $(N - 1)!r_2r_3 \ldots r_N$ such ladder diagrams (the $N - 1$ loops of length $r_2$ through $r_N$ can be put in any order to get distinct diagrams to get a degeneracy of $(N - 1)!$, and each of these loops can have its nodes permuted cyclically – see Figure 4.1, giving a degeneracy of $r_2r_3 \ldots r_N$; in both cases the $r_1$ loop is fixed because it is associated with the root), so in order to get a generating function which only counts each rooted hypermap once, each $\overline{P}_{r_1 \ldots r_N}$ must be divided by this degeneracy.

We therefore write out $H$ explicitly by summing over all $P_{r_1 \ldots r_N}$, dividing each by $(N - 1)!r_2r_3 \ldots r_N$, and multiplying each by $\lambda^N x^{r_1 + \ldots + r_N}$ in order to index the enumeration by number of faces and darts as well. The resulting expression is

$$H(m, n, \lambda; x) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \ldots \sum_{r_N=1}^{\infty} x^{r_1} \frac{r_1!}{r_2!} \frac{r_2!}{r_3!} \ldots \frac{r_N!}{r_1!} P_{r_1 r_2 \ldots r_N}(m, n).$$

We simplify this by substituting in (3.4):

$$H(m, n, \lambda; x) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{r=1}^{\infty} x^{r} \Pi_r^{(N)}(m, n; x). \tag{4.2}$$

Now, multiplying this by $F$ as defined in (3.7), we get

$$H(m, n, \lambda; x)F(m, n, \lambda; x) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{r=1}^{\infty} x^{r} \Pi_r^{(N)}(m, n; x) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \Sigma^{(k)}(m, n; x)$$

$$= \sum_{r=1}^{\infty} x^{r} \sum_{k=0}^{\infty} \frac{\lambda^{N+k+1}}{N!} \Pi_r^{(N)}(m, n; x) \Sigma^{(k)}(m, n; x)$$

$$= \sum_{r=1}^{\infty} x^{r} \sum_{k=0}^{\infty} \frac{\lambda^{N+1}}{(N-k)!} \Pi_r^{(N-k)}(m, n; x) \Sigma^{(k)}(m, n; x)$$

$$= \sum_{r=1}^{\infty} x^{r} \sum_{N=0}^{\infty} \frac{\lambda^{N+1}}{(N-k)!} \Pi_r^{(N-k)}(m, n; x) \Sigma^{(k)}(m, n; x).$$
This has a clear similarity to (3.6), so we substitute in (3.6) and (3.8), giving

\[ H(m, n, \lambda; x)F(m, n, \lambda; x) = \sum_{N=0}^{\infty} \frac{\lambda^{N+1}}{N!} \sum_{r=1}^{\infty} x^r \Pi_r(m, n; x) \]

\[ = x \frac{\partial}{\partial x} F(m, n, \lambda; x). \tag{4.3} \]

(4.3)

If \( F \) and \( H \) were both well-behaved functions, this expression would be sufficient to evaluate \( H \) given \( F \). As both are formal power series, however, it is only meaningful to consider this expression in terms of the terms in these series. Defining the functions \( F_r(m, n, \lambda) \) such that

\[ F(m, n, \lambda; x) = \sum_{r=0}^{\infty} F_r(m, n, \lambda)x^r, \]

(4.3) becomes

\[ \sum_{k=0}^{r} H_{r-k}(m, n, \lambda)F_k(m, n, \lambda) = rF_r(m, n, \lambda). \tag{4.4} \]

When \( r = 0 \) this simply gives

\[ H_0(m, n, \lambda) = 0, \]

and then we can recursively construct other \( H_r \) for \( r > 0 \). For example, the first few are

\[ H_1(m, n, \lambda) = F_1(m, n, \lambda) \]
\[ H_2(m, n, \lambda) = 2F_2(m, n, \lambda) - [F_1(m, n, \lambda)]^2 \]
\[ H_3(m, n, \lambda) = 3F_3(m, n, \lambda) - 3F_1(m, n, \lambda)F_2(m, n, \lambda) + [F_1(m, n, \lambda)]^2, \]

where we have made use of the fact that \( F_0(m, n, \lambda) = F(m, n, \lambda; 0) = 0 \) as shown in (3.9).

All that remains, then, is to evaluate the various \( F_r \). We will do this in the next section.

5 Evaluating \( F_r \)

We now have the generating function \( H \) defined in terms of the series \( F \). The problem of evaluating terms in the \( x \)-series expansion of \( H \) is therefore equivalent to the problem of evaluating the terms in \( F \). In this section we will establish an integral representation of \( F \) and then discuss the use of this to explicitly evaluate the terms \( F_r \) in \( F \).

**Theorem 3.** The series \( F \) has the integral representation

\[ F(m, n, \lambda; x) = \frac{1}{\Lambda_{mn}} \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} e^{-q_k} q_k^{n-m} dq_k \sum_{a=0}^{\infty} \frac{\Gamma(\lambda + a)}{a! \Gamma(\lambda)} q_k^a x^a \] \hspace{1cm} (5.1)

for positive integers \( m, n, \lambda \) and \( x \) satisfying \( m \leq n \), where \( \Delta(q_1, \ldots, q_m) \) is the Vandermonde determinant, the integral is over the range \( 0 \leq q_k < \infty \) for all \( 1 \leq k \leq m \),

\[ \Lambda_{mn} = \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} e^{-q_k} q_k^{n-m} dq_k. \]
Proof. From (3.7) and (3.5) we have that
\[ F(m, n, \lambda; x) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \prod_{r_1=1}^{x^{r_1}} \cdots \prod_{r_N=1}^{x^{r_N}} P_{r_1 \cdots r_N} (m, n). \] (5.2)

If we then substitute (2.4) into this, we get that, when \( m \leq n \),
\[ F(m, n, \lambda; x) = \frac{1}{A_{mn}} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \prod_{r_1=1}^{x^{r_1}} \cdots \prod_{r_N=1}^{x^{r_N}} \int \Delta^2 (q_1, \ldots, q_m) \]
\[ \times \prod_{k=1}^{m} (e^{-q_k n^m} dq_k) \prod_{i=1}^{N} \sum_{j=1}^{m} q_j^{r_i} \]
\[ = \frac{1}{A_{mn}} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int \Delta^2 (q_1, \ldots, q_m) \prod_{k=1}^{m} (e^{-q_k n^m} dq_k) \]
\[ \times \prod_{i=1}^{N} \sum_{j=1}^{m} \sum_{r_i=1}^{x^{r_i}} q_j^{-r_i} \]
\[ = \frac{1}{A_{mn}} \int \Delta^2 (q_1, \ldots, q_m) \prod_{k=1}^{m} (e^{-q_k n^m} dq_k) \]
\[ \times \prod_{N=0}^{\infty} \frac{\lambda^N}{N!} \left( \prod_{j=1}^{\infty} \sum_{r=1}^{x^r} \right)^N. \] (5.3)

This expression is divergent for any given non-zero \( x \), as almost all of the domain of integration has at least one \( q_j \) such that \(|q_j x| > 1\), making
\[ \sum_{r=1}^{\infty} q_j^{-r} \]
diverge. However, we are still able to make more progress by considering (5.3) as a formal power series in \( x \) again. we have the identity
\[ \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \left( \prod_{j=1}^{\infty} \sum_{r=1}^{x^r} \right)^N = \prod_{j=1}^{m} \sum_{a_j=0}^{\infty} \frac{\Gamma (\lambda + a_j)}{a_j \Gamma (\lambda)} q_j^{-a_j} \]
for \( \lambda > 0 \) (see Theorem 6 in Appendix 1), so we rewrite (5.3) as
\[ F(m, n, \lambda; x) = \frac{1}{A_{mn}} \int \Delta^2 (q_1, \ldots, q_m) \prod_{k=1}^{m} (e^{-q_k n^m} dq_k) \sum_{a_k=0}^{\infty} \frac{\Gamma (\lambda + a_k)}{a_k \Gamma (\lambda)} q_k^{-a_k} x^{a_k}. \]

This expression still bears similarities to expressions used in past work [3, 4, 5, 6, 1]. To evaluate the integral, we will use a method similar to that used by Foong [4].
Theorem 4.

\[ F(m, n, \lambda; x) = \sum_{a_0=0}^{\infty} \cdots \sum_{a_{m-1}=0}^{\infty} \prod_{0 \leq i < j \leq m} \left( a_i - a_j \right) \prod_{s=0}^{m-1} \frac{\Gamma(\lambda + a_s)}{\Gamma(\lambda)} \frac{\Gamma(n - s + a_s)}{\Gamma(n - s)} \frac{x^{a_s}}{a_s!} \]  

for positive integers \( m, n, \lambda \) and \( x \) satisfying \( m \leq n \).

Proof. From Theorem 3 we have

\[ F(m, n, \lambda; x) = \frac{1}{\Lambda_{mn}} \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} \left( e^{-q_k} q_k^{n-m} dq_k \sum_{a_k=0}^{\infty} \frac{\Gamma(\lambda + a_k)}{a_k! \Gamma(\lambda)} q_k^{a_k} x^{a_k} \right). \]

As in [4], we multiply the integrand by a “damping factor” \( \exp(-\sum_{k=1}^{m} \epsilon_k q_k) \), and then replace the \( q_i \) in the Vandermonde discriminant \( \Delta^2(q_1, \ldots, q_m) \) by \( D_i = -\partial/\partial \epsilon_i \):

\[ F(m, n, \lambda; x) = \lim_{\epsilon \to 0} \frac{\Delta^2(D_1, \ldots, D_m)}{\Lambda_{mn}} \prod_{k=1}^{m} \int_0^{\infty} e^{-(1+\epsilon_k)q_k} q_k^{n-m} dq_k \]

\[ \times \sum_{a_k=0}^{\infty} \frac{\Gamma(\lambda + a_k)}{a_k! \Gamma(\lambda)} q_k^{a_k} x^{a_k} \bigg|_{\epsilon_i = \epsilon}. \]  

(5.5)

Foong then notes that

\[ \Delta^2(D_1, \ldots, D_m) = |D D^T|, \]

where

\[ D = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ D_1 & D_2 & \cdots & D_m \\ \vdots & \vdots & \ddots & \vdots \\ D_1^{m-1} & D_2^{m-1} & \cdots & D_m^{m-1} \end{bmatrix}, \]

and that

\[ |D D^T| f(\{\epsilon_i\})|_{\epsilon_i = \epsilon} = m! |D_2 D_3^2 \cdots D_m^{m-1} f(\{\epsilon_i\})|_{\epsilon_i = \epsilon} \]
when \( f(\{\epsilon_i\}) \) is a symmetric function of \( \epsilon_i \). Given this, we rewrite 5.5 as

\[
F(m, n, \lambda; x) = \lim_{\epsilon \to 0} \frac{m!}{\Lambda_{mn}} |D| \left\{ \prod_{k=1}^{m} \int_0^\infty D_k \, e^{-(1+\epsilon_k)x} q_k^n d\epsilon_k \right\}
\times \sum_{a_k=0}^\infty \frac{\Gamma(\lambda + a_k)}{a_k ! \Gamma(\lambda)} q_k^a x^a |_{\epsilon_i = \epsilon}
\]

\[
= \lim_{\epsilon \to 0} \frac{m!}{\Lambda_{mn}} |D| \left\{ \prod_{k=1}^{m} \int_0^\infty e^{-(1+\epsilon_k)x} q_k^n d\epsilon_k \right\}
\times \sum_{a_k=0}^\infty \frac{\Gamma(\lambda + a_k)}{a_k ! \Gamma(\lambda)} q_k^a x^a |_{\epsilon_i = \epsilon}
\]

\[
= \frac{m! \Lambda_{mn}}{\Lambda_{mn}} \sum_{a_1=0}^\infty \frac{\Gamma(\lambda + a_1)}{a_1 ! \Gamma(\lambda)} q_1^a \cdots \sum_{a_m=0}^\infty \frac{\Gamma(\lambda + a_m)}{a_m ! \Gamma(\lambda)} q_m^a \times \lim_{\epsilon \to 0} \frac{m!}{\Lambda_{mn}} |D| \left\{ \prod_{k=1}^{m} \frac{\Gamma(n - m + k + a_k)}{(1 + \epsilon_k)^{n-m+k+a_k}} \right\} |_{\epsilon_i = \epsilon}
\]

(5.6)

Let

\[
Q = \lim_{\epsilon \to 0} \frac{m!}{\Lambda_{mn}} |D| \left\{ \prod_{k=1}^{m} \frac{\Gamma(n - m + k + a_k)}{(1 + \epsilon_k)^{n-m+k+a_k}} \right\} |_{\epsilon_i = \epsilon}
\]

Expanding the determinant out explicitly in terms of the Levi-Civita symbol, we get

\[
Q = \lim_{\epsilon \to 0} \frac{m!}{\Lambda_{mn}} |D| \left\{ \prod_{k=1}^{m} \frac{\Gamma(n - m + k + i_k - 1 + a_k)}{(1 + \epsilon_k)^{n-m+k+i_k-1+a_k}} \right\} |_{\epsilon_i = \epsilon}
\]

\[
= \lim_{\epsilon \to 0} \frac{m!}{\Lambda_{mn}} \Gamma(n - m + k + i_k - 1 + a_k) \left| \begin{array}{cccc}
\Gamma(n - m + 1 + a_1) & \Gamma(n - m + 2 + a_1) & \cdots & \Gamma(n + a_m) \\
\Gamma(n - m + 1 + a_2) & \Gamma(n - m + 2 + a_2) & \cdots & \Gamma(n + 1 + a_m) \\
\Gamma(n - m + 1 + a_3) & \Gamma(n - m + 2 + a_3) & \cdots & \Gamma(n + 2 + a_m) \\
\cdots & \cdots & \cdots & \cdots \\
\Gamma(n + a_1) & \Gamma(n + 1 + a_2) & \cdots & \Gamma(n + m + 1 + a_m) \\
\end{array} \right|
\]

We then simplify this determinant by a process of subtracting multiples of different rows of the matrix from each other as follows:

1. Subtract \((n - m)\) times the first row from the second, \((n - m + 1)\) times the second from the third etc. to give

\[
Q = \left| \begin{array}{cccc}
\Gamma(n - m + 1 + a_1) & \Gamma(n - m + 2 + a_1) & \cdots & \Gamma(n + a_m) \\
(a_1 + 1) \Gamma(n - m + 1 + a_1) & (a_2 + 2) \Gamma(n - m + 2 + a_1) & \cdots & (a_m + m) \Gamma(n + a_m) \\
\cdots & \cdots & \cdots & \cdots \\
(a_1 + 1) \Gamma(n - m + 1 + a_1) & (a_2 + 2) \Gamma(n + a_2) & \cdots & (a_m + m) \Gamma(n + m - 2 + a_m) \\
\end{array} \right|
\]
2. Subtract \((n - m)\) times the second row from the third, \((n - m + 1)\) times the third row from the fourth etc. to give

\[
Q = \begin{vmatrix}
\Gamma(n - m + 1 + a_1) & \Gamma(n - m + 2 + a_2) & \cdots & \Gamma(n + a_m) \\
(a_1 + 1)^2 \Gamma(n - m + 1 + a_1) & (a_2 + 2)^2 \Gamma(n - m + 2 + a_2) & \cdots & (a_m + m)^2 \Gamma(n + a_m) \\
\vdots & \vdots & \ddots & \vdots \\
(a_1 + 1)^2 \Gamma(n - 2 + a_1) & (a_2 + 2)^2 \Gamma(n - 1 + a_2) & \cdots & (a_m + m)^2 \Gamma(n + m - 3 + a_m)
\end{vmatrix}.
\]

3. Continue to repeat this process, starting a row further down each time, until

\[
Q = \Delta(a_1 + 1, \ldots, a_m + m) \prod_{k=1}^{m} \Gamma(n - m + a_k + k) = \prod_{0 < i < j \leq m} (a_j - a_i + j - i) \prod_{k=1}^{m} \Gamma(n - m + a_k + k).
\]

We then substitute this expression into (5.6):

\[
F(m, n, \lambda; x) = \frac{m!}{\Lambda_{mn}} \sum_{a_1=0}^{\infty} \frac{\Gamma(\lambda + a_1)}{a_1! \Gamma(\lambda)} x^{a_1} \cdots \sum_{a_m=0}^{\infty} \frac{\Gamma(\lambda + a_m)}{a_m! \Gamma(\lambda)} x^{a_m} \\
\times \lim_{\epsilon \to 0} \left| \text{det} \left[ \prod_{k=1}^{m} \frac{\Gamma(n - m + k + a_k)}{(1 + \epsilon_k)^{n-m+k+a_k}} \right] \right|_{\epsilon_i = \epsilon} = \frac{m!}{\Lambda_{mn}} \sum_{a_1=0}^{\infty} \frac{\Gamma(\lambda + a_1)}{a_1! \Gamma(\lambda)} x^{a_1} \cdots \sum_{a_m=0}^{\infty} \frac{\Gamma(\lambda + a_m)}{a_m! \Gamma(\lambda)} x^{a_m} \\
\times \prod_{0 < i < j \leq m} (a_j - a_i + j - i) \prod_{k=1}^{m} \Gamma(n - m + a_k + k),
\]

and simplify this by making the substitutions \(i \to m - i, j \to m - j, k \to m - k\) and \(a_s \to a_{m-s}\) such that

\[
F(m, n, \lambda; x) = \frac{m!}{\Lambda_{mn}} \sum_{a_0=0}^{\infty} \frac{\Gamma(\lambda + a_0)}{a_0! \Gamma(\lambda)} x^{a_0} \cdots \sum_{a_{m-1}=0}^{\infty} \frac{\Gamma(\lambda + a_{m-1})}{a_{m-1}! \Gamma(\lambda)} x^{a_{m-1}} \\
\times \prod_{0 \leq i < j \leq m} (a_i - a_j + j - i) \prod_{s=0}^{m-1} \Gamma(n - s + a_s).
\]

We know from (3.9) that \(F(m, n, \lambda; 0) = 1\), which means we can now fix the value of the normalisation constant \(\Lambda_{mn}\), as

\[
F(m, n, \lambda; 0) = \frac{m!}{\Lambda_{mn}} \prod_{0 \leq i < j \leq m} (j - i) \prod_{s=0}^{m-1} \Gamma(n - s) = 1.
\]

Therefore,

\[
F(m, n, \lambda; x) = \sum_{a_0=0}^{\infty} \cdots \sum_{a_{m-1}=0}^{\infty} \prod_{0 \leq i < j \leq m} \left( \frac{a_i - a_j}{j - i} + 1 \right) \prod_{s=0}^{m-1} \frac{\Gamma(\lambda + a_s)}{\Gamma(\lambda)} \frac{\Gamma(n - s + a_s)}{\Gamma(n - s - a_s)} x^{a_s}.
\]
This expression can now be used to evaluate any given $F_r$, by summing only over the cases \{a_0, \ldots, a_{m-1}\} for which $a_0 + \ldots + a_{m-1} = r$. Unlike the closed-form expressions we derived previously in [1], however, this expression cannot be used directly to find polynomial expansions for $F_r(m,n,\lambda)$, due to the dependence on $m$ of the number of summations and the ranges of the products.

However, we know that each $H_r$ must be a symmetric polynomial (Theorem 1) of order at $r$ in each of its parameters (a hypermap with $r$ darts can have at most $r$ edges), and that $H_r(m,n,\lambda) = 0$ if any of its parameters are zero (all hypermaps must have at least one each of vertices, edges and faces). Given that $F_0(m,n,\lambda) = 1$, it follows from (4.4) that $F_r(m,n,\lambda)$ for any $r > 0$ is also symmetric, order $r$ in each parameter, and zero when $m$, $n$ or $\lambda$ are zero. Therefore, we can compute the polynomial coefficients of $H_r$ (and therefore enumerate rooted hypermaps) by evaluating $F_r(m,n,\lambda)$ — and by extension $H_r(m,n,\lambda)$ — using (5.4) and (4.4) at all $1 \leq m \leq n \leq \lambda \leq r$ and using polynomial interpolation.

We used this method to compute the coefficients of $H_r$ for all $1 \leq r \leq 13$. Some of the output is given in Appendix 2, and the results agree exactly with past computations, in particular Walsh’s enumeration of all rooted hypermaps up to $r = 12$ [9]. Running on a 2012 Dell XPS 12 these calculations took 107 minutes, in comparison to the few days taken by Walsh’s algorithm.

5.1 Special cases

While no simple closed-form polynomial expressions are available for $H_r(m,n,\lambda)$, there are a few special cases in which we can get more useful results.

Consider the function

$$H_r(1,m,n) = \sum_{v,e,f} H_{vefr} m^e n^f.$$  \hfill (5.7)

This is the generating function for enumerating rooted hypermaps with $r$ darts by number of edges and faces (with all possible numbers of vertices summed over). By the symmetry of $H_r$, (5.7) could also be used to enumerate by number of vertices and edges, summing over all numbers of faces etc.

**Theorem 5.** For all $r > 0$,

$$H_r(1,m,n) = \frac{1}{(r-1)!} \frac{\Gamma(m+r)}{\Gamma(m)} \frac{\Gamma(n+r)}{\Gamma(n)} - \sum_{k=1}^{r-1} \frac{1}{k!} \frac{\Gamma(m+k)}{\Gamma(m)} \frac{\Gamma(n+k)}{\Gamma(n)} H_{r-k}(1,m,n).$$  \hfill (5.8)

**Proof.** From (5.4) we have that

$$F(1,m,n;x) = \sum_{a=0}^{\infty} \frac{\Gamma(m+a)}{\Gamma(m)} \frac{\Gamma(n+a)}{\Gamma(n)} \frac{x^a}{a!},$$

so

$$F_r(1,m,n) = \frac{1}{r!} \frac{\Gamma(m+r)}{\Gamma(m)} \frac{\Gamma(n+r)}{\Gamma(n)}.$$
We substitute this into (4.4) and rearrange to get

\[
H_r(1, m, n) = rF_r(1, m, n) - \sum_{k=1}^{r-1} F_k(1, m, n)H_{r-k}(1, m, n)
\]

\[
= \frac{1}{(r-1)!} \frac{\Gamma(m + r) \Gamma(n + r)}{\Gamma(m) \Gamma(n)}
- \sum_{k=1}^{r-1} \frac{1}{k!} \frac{\Gamma(m + k) \Gamma(n + k)}{\Gamma(m) \Gamma(n)} H_{r-k}(1, m, n).
\]

In contrast to expressions such as (5.1) and (5.4), this expression obviously gives rise to symmetric polynomial functions.

Given (5.8), the following two results follow trivially:

\[\square\]

**Corollary 1.** For all \( r > 0 \),

\[
H_r(1, 1, m) = r \frac{\Gamma(m + r)}{\Gamma(m)} - \sum_{k=1}^{r-1} \frac{\Gamma(m + k)}{\Gamma(m)} H_{r-k}(1, 1, m).
\]

**Corollary 2.** For all \( r > 0 \),

\[
H_r(1, 1, 1) = r \cdot r! - \sum_{k=1}^{r-1} k! H_{r-k}(1, 1, 1).
\]

The second in particular allows us to count how many rooted hypermaps there are in total with \( r \) darts. The first few values are 1, 3, 13, 71, 461...

### 6 Conclusions

We have demonstrated a method for computing generating functions to enumerate rooted hypermaps by number of vertices, edges and faces for any given number of darts. This is an extension of previous work where we derived closed form generating functions for counting enumerating rooted hypermaps with one face \([1]\), but in contrast to that case the method shown here defines the generating function \( H_r \) for \( r \) darts recursively in terms of \( H_1, \ldots, H_{r-1} \), and it only allows \( H_r \) to be evaluated numerically, not expanded directly as a polynomial. We were still able to obtain a polynomial expansion, however, by using polynomial interpolation.

This work is a further demonstration of the use of matrix integration as a tool for finding generating functions for enumerating sets of combinatoric objects. It specifically demonstrates the link, first discussed in \([1]\), between rooted hypermaps and the ensemble of reduced density operators on random states of a bipartite quantum system.

We also discussed a number of related results. First we showed the symmetry of the generating functions \( H_r \), arising from the symmetry of 3-constellations, and used this to speed up computation of \( H_r \) by reducing the range over which \( H_r \) needed to be evaluated to fix the polynomial expansion. Then we looked at cases where one or more parameters in \( H_r \) were set to unity, giving generating functions for enumerating larger sets of rooted hypermaps (such as all those with \( r \) darts and \( f \) faces, summing over all possible numbers of edges and vertices). In particular, this allowed us to easily count all rooted hypermaps with \( r \) darts and any number of edges, vertices and faces.
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Appendix 1

Theorem 6. For positive integer $m$ and $\lambda > 0$, the formal power series

$$\sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \left( \sum_{j=1}^{m} \sum_{r=1}^{\infty} \frac{q_j^r x^r}{r} \right)^N = \prod_{j=1}^{m} \sum_{a_j=0}^{\infty} \frac{\Gamma(\lambda + a_j)}{a_j! \Gamma(\lambda)} q_j^{a_j} x^{a_j},$$

where $q_j$ are components of an $m$-dimensional real vector.

Proof. Let

$$L_{\lambda,q}(x) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \left( \sum_{j=1}^{m} \sum_{r=1}^{\infty} \frac{q_j^r x^r}{r} \right)^N.$$  \hspace{1cm} (7.1)

For any given positive integer $a$, we see by inspection that this series contains only a finite number of terms of order $x^a$, as such terms can only come from cases where $1 \leq N \leq a$. In addition, there is only one constant term: the $N = 0$ case which equals unity. Therefore, $L_{\lambda,q}(x)$ can be written in the form

$$L_{\lambda,q}(x) = \sum_{a=0}^{\infty} f_a(\lambda, q)x^a$$  \hspace{1cm} (7.2)

where each $f_a(\lambda, q)$ is a polynomial in $\lambda$ and $q$.

$L_{\lambda,q}(x)$ converges when $(|q_j x|) < 1$ for all $j$ to

$$L_{\lambda}(x) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \left( -\sum_{j=1}^{m} \ln(1 - q_j x) \right) = \exp[-\lambda \sum_{j=1}^{m} \ln(1 - q_j x)] = \prod_{j=1}^{m} \frac{1}{(1 - q_j x)^{\lambda}}.$$  \hspace{1cm} (7.3)

This has a series expansion in $x$, also valid when $(|q_j x|) < 1$ for all $j$, of

$$\prod_{j=1}^{m} \sum_{a_j=0}^{\infty} \frac{\Gamma(\lambda + a_j)}{a_j! \Gamma(\lambda)} q_j^{a_j} x^{a_j}.$$  \hspace{1cm} (7.3)

This can also be rearranged into the form (7.2). (7.1) and (7.3) are therefore both Taylor series with the same radius of convergence, and they are equal to each other everywhere within it, so it follows from the uniqueness of Taylor series expansions of smooth functions that they are equivalent, i.e.

$$\sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \left( \sum_{j=1}^{m} \sum_{r=1}^{\infty} \frac{q_j^r x^r}{r} \right)^N = \prod_{j=1}^{m} \sum_{a_j=0}^{\infty} \frac{\Gamma(\lambda + a_j)}{a_j! \Gamma(\lambda)} q_j^{a_j} x^{a_j}.$$
Appendix 2

Numbers of rooted hypermaps with \(v\) vertices, \(e\) edges, \(f\) faces and \(r\) darts, calculated by computing the generating functions \(H_r\). Only the cases with \(v \leq e \leq f\) are given, as the rest follow from the symmetry of \(H_r\). The cases \(1 \leq r \leq 7\) are included for comparison with Walsh’s previous computation \[9\], with all cases up to \(r = 12\) agreeing with his computation. The new case \(r = 13\) is also shown.

\[
\begin{array}{c|c|c|c}
 r = 1 & v & e & f \\
 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 r = 2 & v & e & f \\
 1 & 2 & 2 & 2 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 r = 3 & v & e & f \\
 1 & 3 & 3 & 3 \\
 1 & 2 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 r = 4 & v & e & f \\
 1 & 4 & 4 & 4 \\
 1 & 3 & 4 & 4 \\
 1 & 2 & 4 & 4 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 r = 5 & v & e & f \\
 1 & 5 & 5 & 5 \\
 1 & 4 & 5 & 5 \\
 1 & 3 & 5 & 5 \\
 1 & 2 & 5 & 5 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 r = 6 & v & e & f \\
 1 & 6 & 6 & 6 \\
 1 & 5 & 6 & 6 \\
 1 & 4 & 6 & 6 \\
 1 & 3 & 6 & 6 \\
 1 & 2 & 6 & 6 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 r = 7 & v & e & f \\
 1 & 7 & 7 & 7 \\
 1 & 6 & 7 & 7 \\
 1 & 5 & 7 & 7 \\
 1 & 4 & 7 & 7 \\
 1 & 3 & 7 & 7 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 r = 13 & v & e & f & N \\
 1 & 13 & 13 & 13 & 88428800 \\
 1 & 12 & 12 & 12 & 686597184 \\
 1 & 11 & 11 & 11 & 292274616 \\
 1 & 10 & 10 & 10 & 292051496 \\
 1 & 9 & 9 & 9 & 119473744 \\
 1 & 8 & 8 & 8 & 462307084 \\
 1 & 7 & 7 & 7 & 66728552 \\
 1 & 6 & 6 & 6 & 264642472 \\
 1 & 5 & 5 & 5 & 636184120 \\
 1 & 4 & 4 & 4 & 223926420 \\
 1 & 3 & 3 & 3 & 109425316 \\
 1 & 2 & 2 & 2 & 988043771 \\
\end{array}
\]

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