QUANTUM-MECHANICAL SUPERSYMMETRY IN TRAPS

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We discuss the application of quantum-mechanical supersymmetry to particle traps. The supersymmetric-partner wave functions may be used to describe a valence fermion in a trap system with an isotropic harmonic-oscillator potential. Interactions with the core are incorporated analytically. The close similarity of this approach to the application of supersymmetry in atomic systems is made explicit by means of a radial mapping between the two systems.

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Supersymmetry has been an active research area for well over two decades. Despite this, very few physical supersymmetries are experimentally known. One is the appearance of an effective radial potential in the context of atomic systems \cite{14, 15}. We discuss a possible further application in the context of particle traps. More details are given in the two references listed under our names.

The isotropic harmonic oscillator has a radial equation admitting a supersymmetric partner. The physical implications of this mathematical fact may be investigated using trap systems for which an isotropic potential can be established. In the Ioffe-Pritchard trap \cite{11, 12} and the time-averaged orbiting-potential (TOP) trap \cite{23}, this condition may be satisfied. Both are neutral-particle traps that use the interaction of the magnetic dipole moment $\vec{\mu}$ of the particle with a confining magnetic field. The traps select all dipoles aligned opposite to the direction of the magnetic field and draw them into the region of weakest field at the center. The Ioffe-Pritchard trap is purely magnetostatic, with the field provided by two coils and four linear conductors. The TOP trap comprises six coils, four of which have alternating currents, creating a high-frequency rotating magnetic field. An averaging procedure removes the time dependence, yielding an effective magnetostatic potential. With suitable choices of currents in the conductors of these systems \cite{1, 19}, it is possible to ensure isotropy of the potential energy near the center of the trap,

$$U(r) = \mu B_0 \left(1 + r^2/r_0^2\right).$$

Here, $r_0$ is a length characteristic of the trap system, and $B_0 \neq 0$ is the magnitude of the magnetic field at the center of the trap. The radial wave functions describing a single trapped dipole may be expressed in terms of the generalized Laguerre polynomials $L_N^{(a)}(z)$. Ignoring a factor of $r$ that removes the first-order derivative in the differential equation, the functions are

$$W_{N,L}(r) = C_{N,L} \left(r/r_0\right)^{L+1} \exp \left(-r^2/2r_0^2\right) L_{N/2-L/2}^{(L+1/2)} \left(r^2/r_0^2\right).$$

Here, $L = 0, 1, 2, \ldots$ is the angular momentum, and $N = L, L + 2, L + 4, \ldots$ is the principal quantum number. Normalization is ensured via the constant $C_{N,L}$. The full
solutions of the harmonic oscillator $|N, L, M\rangle$ behave as

$$W_{N,L}(r) Y_{L,M}(\theta, \phi),$$

where the azimuthal quantum number $M$ takes the usual values.

Details of supersymmetric quantum mechanics may be found elsewhere [22, 26, 13, 10]. We use the term \textit{bosonic sector} to refer to the given radial system with fixed angular momentum $L$ and spectrum shifted to have zero lowest-state energy. The term \textit{fermionic sector} is used for the partner system.

One of our objectives is to regard the radial-equation fermionic sector as providing an effective potential for an excited valence particle in a trap. It experiences not only the trapping potential, but also interactions with a core of other particles. Before motivating this application of supersymmetry further, let us consider the case of a valence particle with angular momentum $L = 0$ that is excluded from occupying levels below $N = 2$ by a filled core. This exclusion can only hold for fermions, to which we restrict ourselves here. If the interactions between the trapped dipoles are small compared with the natural spacing of the energies in the oscillator, the number of particles in the core may be found by counting the levels in the single-particle bosonic system lying below that of the valence fermion. There are four in this case. One is the ground state, $|N = 0, L = 0, M = 0\rangle$, and the other three can be labelled as $|N = 1, L = 1, M = 0, \pm 1\rangle$. Similarly, the core would have 20 fermions if the $L = 0$ valence fermion was restricted to $N \geq 4$.

The physical interpretation of the fermionic sector is motivated by several observations. It is well known that the fermionic sector is degenerate with the bosonic sector except for the lowest bosonic state, which has no corresponding fermionic state. One may visualize a situation in which \textit{physically} such a corresponding state exists, but is inaccessible to the valence fermion because it is occupied by a core fermion. The absence of a zero-energy state in the fermionic sector of the \textit{mathematical} formalism reflects this physical picture. The core need not be occupied by only one fermion, since there might be others of different angular momenta. Adopting this interpretation, the Pauli principle is seen to underlie radial supersymmetry, and it becomes natural to interpret the fermionic sector as describing a valence fermion in
a multifermion system. For $L = 0$, the effective radial potential for the fermionic sector differs from the corresponding one for the bosonic sector by an expression that includes the term $\hbar^2/mr^2$ where $m$ is the valence fermion mass. This additional repulsion is thus consistent with the meaning of the Pauli principle.

So, the effect of the supersymmetry is to fill an inner core with fermions. This procedure must leave the angular momentum of the valence particle unchanged. We therefore construct the full three-dimensional wave functions from the product of the fermionic radial wave functions $W_{N_s-1,L+1}(r)$ and the same spherical harmonics as in the bosonic sector (3):

$$W_{N_s-1,L+1}(r) Y_{L,M}(\theta, \phi),$$

(4)

where $N_s = L+2, L+4, L+6, \ldots$ is the principal quantum number. Even though the constant $L$ in the radial function appears to have been shifted, the angular momentum of the system is defined by the spherical harmonics and is unchanged. These full wave functions (4) for the fermionic sector differ from the full wave functions (3) for the three-dimensional isotropic harmonic oscillator.

An alternative description for multifermion traps could account for the filled core by using the standard harmonic-oscillator solutions (3) but requiring the values of the principal quantum number for the valence fermion to exclude numbers corresponding to the filled core. However, the valence fermion would then be described by an incomplete set of states and the lowest valence state would have too many nodes. Both of these drawbacks are absent for the fermionic functions obtained via supersymmetry, which form a complete orthonormalizable set of states and for which the lowest state has no nodes. They therefore resemble solutions for other conventional bound systems in quantum mechanics.

The number of fermions in the core of a particular trap described by the functions (3) depends on the angular momentum of the valence fermion. For $L = 0$, the principal quantum number takes values $N_s = 2, 4, 6, \ldots$ and by the reasoning considered above, this trap has four core fermions. The filling of another shell in the core can be accomplished by shifting the fermionic-sector spectrum to have zero lowest-energy state, thereby treating it as a new bosonic sector. A new fermionic sector is then
obtained via the usual supersymmetry procedure. This describes a trap with 20 core fermions and a valence fermion with zero angular momentum. Further iterations of this procedure fills further shells, giving cores with 56, 120, 220, \ldots fermions. For valence fermions with $L = 1$, the concept is the same, and there are 1, 10, 35, 84, \ldots core particles. General formulae for these sequences can be obtained \[19\]. They assume only one spin orientation since the dipoles in the Ioffe-Pritchard and TOP traps are oriented against the magnetic field. The formulae differ for other trap systems.

Although it accounts for the Pauli principle, radial supersymmetry ignores interactions between the valence fermion and the core. We discuss one method of incorporating interactions developed in analogy with analytical supersymmetry-based quantum-defect theory for atomic systems \[16\]. The modifications in the oscillator energy spectrum due to interactions plausibly generate the form

$$E_{N^*} = \mu B_0 + \hbar \omega_0 (N^* + 3/2) ,$$  

(5)

for $\omega_0 = (2\mu B_0/mr_0^2)^{1/2}$. The modified eigenvalues are thus incorporated via a shifted principal quantum number $N^*$. Denoting the shift by $\Delta = \Delta(N, L)$ and including also an integral shift $I = I(L)$, we define $N^* = N + I - \Delta$. Equivalently, with $N_s = N + 2I$, we write $N^* = N_s - I - \Delta$. If in addition we shift the angular momentum, $L^* = L + I - \Delta$, and add the effective potential

$$V_{\text{eff}}(r) = \frac{\hbar^2}{2m} \frac{L^*(L^* + 1) - L(L + 1)}{r^2} + \hbar \omega_0 (N - N^*)$$  

(6)

to the differential operator in the Schrödinger equation for (2), analytical radial wave functions that correspond to the modified eigenspectrum (5) are obtained. They may be expressed in terms of the functional form (2) as $W_{N^*,L^*}(r)$. This analytical defect theory extends the radial supersymmetry quite naturally, and if the defects are switched off appropriately the exact bosonic and noninteracting fermionic sectors are recovered.

The application of supersymmetry in traps closely follows the application of supersymmetry in multi-electron atoms and in ions \[14, 15\]. The similarity in the two applications is more than a mathematical parallel and can be made explicit in the form of a mapping. The existence of a natural correspondence between the radial
three-dimensional Coulomb problem and the radial harmonic oscillator in two or four
dimensions was first noted by Schrödinger more than fifty years ago [29], and since
then has received much attention [2, 4, 7, 13]. We consider Coulomb dimensions
d > 1, to avoid normalization issues associated with the one-dimensional case, and
oscillator dimensions D ≥ 1. For these arbitrary-dimensional cases, the radial equa-
tions may still be separated [21], and we write the radial solutions as W_{D,N,L}(r)
for the oscillator and w_{d,n,l}(r) for the Coulomb case. Details of these functions
may be found elsewhere [18]. As an example, W_{D=3,N,L}(r) is identical to (2).
We adopt lower-case symbols for the Coulomb system and upper-case symbols for
the oscillator systems, with an exception made for the oscillator radial variable r.
In the Coulomb system, this convention gives angular momentum l and principal quantum
number n.

A natural mapping between these two radial systems exists subject to certain con-
ditions on the dimensions, the angular momenta, and the principal quantum numbers.
The relationship between the radial wave functions is [17]

\[ W_{D,N,L}(r) = K_{d,n,\lambda} r^{-1/2} \left( (n + \gamma) r^2 \right), \]

where γ = (d − 3)/2 is a dimension parameter for the Coulomb system that van-
ishes in the three-dimensional case. The constant K_{d,n,\lambda} is selected to preserve
the normalization. The restrictions on this correspondence may be expressed as

\[ D = 2d - 2 - 2\lambda, \]
\[ N = 2n - 2 + \lambda, \]
\[ L = 2l + \lambda, \]

where it can be seen from (10) that λ, which gives an extra degree of freedom in the
mapping, has to be integer valued. For d = 3, λ may equal zero or one, yielding
oscillator dimensions of D = 4, 2. This is Schrödinger’s original result. Equation (8)
shows there is no such correspondence between the physically interesting cases of
the D = 3 oscillator system and the d = 3 Coulomb system. The oscillator is limited to
even dimensions D only.

To circumvent these dimensional restrictions on the correspondence between the
exact systems, we broaden the class of systems considered to include ones with ana-
lytical modifications of the type introduced above for interactions between particles in a trap. So, whereas there is no natural mapping between the single-particle Coulomb and oscillator systems, a map may exist between a trap with several fermions and the exact Coulomb system.

We allow for an integral shift $J$ in the oscillator dimension via the definition and requirement $D^* = D + J \geq 1$. If we also define a dimension parameter $\Gamma^* = (D^* - 3)/2$, which vanishes for $D^* = 3$, then with the choice of effective potential

$$V_{\text{eff}}(r) = \frac{\hbar^2}{2m} \frac{(L^* + \Gamma^*)(L^* + \Gamma^* + 1) - (L + \Gamma)(L + \Gamma + 1)}{r^2} + \hbar \omega_0 (N - N^* + \Gamma - \Gamma^*)$$

we obtain a differential equation with analytical solutions $W_{D^*,N^*,L^*}(r)$. The values of the parameters $\Delta$, $I$, and $J$ are restricted if normalizability and orthogonality are desired \[18\]. With this broader class of oscillator radial systems, the case of a mapping from the $D^* = 3$ oscillator to the $d = 3$ Coulomb system becomes possible. It is obtained by setting

$$\Delta - I = \lambda - 1/2 \ ,$$

and since $\lambda$ can only take values zero or one, the mapping requires a nonzero defect $\Delta$ in the oscillator. Explicitly, the relationship and its constraints are

$$W_{3,N^*,L^*}(r) = K_{3,n,1/2} r^{-1/2} w_{3,n,l} \left( n \right r^2 \right) ,$$

$$N^* = 2n - 3/2 \ ,$$

$$L^* = 2l + 1/2 \ .$$

Condition (14) ensures that the entire stack of states of the one system maps across to the entire stack for the other, with the lowest states in the stacks identified with each other, the second lowest states with each other, and so on.

This mapping is not the only one possible between the three-dimensional systems. An alternative method involves allowing for an analytical modification in the Coulomb system instead of in the oscillator and then following methods similar to those leading to (13). Such an analytical quantum defect may be introduced by shifting the principal quantum number to give eigenenergies according to the well-known
Rydberg formula $E_0/(n^*)^2$ [24], where $E_0$ is the ground state energy of the Coulomb system and $n^* = n - \delta$, with $\delta = \delta(n, l)$. Analytical solutions exist for this system, and their applications include the study of highly excited valence electrons [3, 4, 5, 6, 7]. Since the Rydberg formula models the spectra of multi-electron atoms, this option establishes a mapping between a single-particle trap and a multiparticle atom such as an alkali-metal atom. A more general third option involves analytical defects in both systems and provides a mapping from a multiparticle trap to a multi-electron atom. This most general form of the mapping incorporates in special cases the bosonic sectors, the fermionic sectors, and the quantum-defect sectors for various dimensions. For example, in a mapping between the three-dimensional cases, the generalization of (12) can take the form

$$\Delta - I = 2(\delta - i) + \lambda - 1/2 ,$$

(16)

where $i$ is the analogue for the Coulomb problem of $I$ for the oscillator.

Unlike the defects $\Delta$ for traps, the defects $\delta$ have been measured for many atoms [20]. They depend on $l$ but are asymptotically independent of $n$ for large $n$. This feature is attractive as it allows approximate orthogonality of the solutions. It would be equally attractive if it could be established for the oscillator system. Indeed, if the oscillator defects $\Delta$ could be measured, it would be amusing to know if the constraint (16) on the radial mappings between the three-dimensional systems is consistent with the known atomic defects $\delta$.

1. T. Bergeman, G. Erez and H.J. Metcalf (1987): Phys. Rev. A 35, 1535 .
2. D. Bergmann and Y. Frishman (1965): J. Math. Phys. 6, 1855 .
3. R. Bluhm and V.A. Kostelecký (1994): Phys. Rev. A 49, 4628 [quant-ph/9508020].
4. R. Bluhm and V.A. Kostelecký (1994): ibid., 50, R4445 [hep-ph/9410325].
5. R. Bluhm and V.A. Kostelecký (1995): ibid., 51, 4767 [quant-ph/9506008].
6. R. Bluhm, V.A. Kostelecký and B. Tudose (1995): Phys. Rev. A 52, 2234 [quant-ph/9509010].
7. R. Bluhm, V.A. Kostelecký and B. Tudose (1996): ibid., 53, 937 (quant-ph/9510023).
8. E. Chacón, D. Levi and M. Moshinsky (1976): J. Math. Phys. 17, 1919.
9. J. Čížek and J. Paldus (1977): Internat. J. Quantum Chem. 12, 875.
10. F. Cooper, A. Khare and U. Sukhatme (1995): Phys. Rep. 251, 267.
11. Y.V. Gott, M.S. Ioffe and V.G. Tel’kovskii (1962): Nucl. Fusion, Suppl. Pt. 3, 1045.
12. D.E. Pritchard (1983): Phys. Rev. Lett. 51, 1336.
13. V.A. Kostelecký (1994): Symmetries in Science VII, eds. B. Gruber and T. Otsuka (Plenum, New York), 295 (quant-ph/9508013).
14. V.A. Kostelecký and M.M. Nieto (1984): Phys. Rev. Lett. 53, 2285.
15. V.A. Kostelecký and M.M. Nieto (1985): Phys. Rev. A 32, 1293.
16. V.A. Kostelecký and M.M. Nieto (1985): ibid., 32, 3243.
17. V.A. Kostelecký, M.M. Nieto and D.R. Truax (1985): Phys. Rev. D 32, 2627.
18. V.A. Kostelecký and N.E. Russell (1996): J. Math. Phys. 37, 2166 (quant-ph/9602007).
19. V.A. Kostelecký and N.E. Russell (1997): Phys. Lett. A 235, 305 (quant-ph/9708020).
20. H.G. Kuhn (1969): Atomic Spectra (Academic, New York).
21. J.D. Louck (1960): J. Mol. Spectr. 4, 298.
22. H. Nicolai (1976): J. Phys. A 9, 1497.
23. W. Petrich, M.H. Anderson, J.R. Ensher and E.A. Cornell (1995): Phys. Rev. Lett. 74, 3352.
24. J.R. Rydberg (1890): Kongl. Sven. vetensk.-akad. hand. 23, no. 11.
25. E. Schrödinger (1941): Proc. Roy. Irish Acad. Sect. A 46, 183.
26. E. Witten (1981): Nucl. Phys. B185, 513.