Hilbert Expansion for the Relativistic Vlasov-Maxwell-Landau System

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Abstract

The two-species relativistic Vlasov-Maxwell-Landau system is the most fundamental and complete model describing the dynamics of a dilute plasma, in which particles interact through Coulombic collisions and through their self-consistent electromagnetic field. A rigorous derivation from the two-species relativistic Vlasov-Maxwell-Landau system to the two-fluid relativistic Euler-Maxwell system is a longstanding major open problem in the kinetic theory. Thanks to recent works such as [32, 46, 45], we establish the global-in-time validity of Hilbert expansion for the relativistic Vlasov-Maxwell-Landau system and derive the limiting two-fluid relativistic Euler-Maxwell system as the Knudsen number goes to zero. Inspired by the weighted energy method in the pioneer work of Caflisch [13], our proof is based on a new time-dependent energy method, which is introduced in our first work [74] about the relativistic Landau equation case.

1 Introduction

1.1 Relativistic Vlasov-Maxwell-Landau System

The relativistic Vlasov-Maxwell-Landau system is a fundamental and complete model describing the dynamics of a dilute collisional ionized plasma appearing in nuclear fusion and the interior of stars, etc. Correspondingly, the relativistic Euler-Maxwell system, the foundation of the two-fluid theory in plasma physics, describes the dynamics of two compressible ion and electron fluids interacting with their own self-consistent electromagnetic field. It is also the origin of many celebrated dispersive PDE such as NLS, KP, KdV, Zaharov, etc, as various scaling limits and approximations of such a fundamental model. It has been an important open question if the general relativistic Euler-Maxwell system can be derived rigorously from its kinetic counter-part, the relativistic Vlasov-Maxwell-Landau system, as the Knudsen number goes to zero.

In this paper, we are able to answer this question in the affirmative. Consider the two-species relativistic Vlasov-Landau system for \( (F_+^\varepsilon(t,x,p), F_-^\varepsilon(t,x,p)) \in \mathbb{R} \times \mathbb{R} \):

\[
\begin{align*}
\partial_t F_+^\varepsilon &+ c\hat{p}_+ \cdot \nabla_x F_+^\varepsilon + e_+ (E^\varepsilon + \hat{p}_+ \times B^\varepsilon) \cdot \nabla_p f_+^\varepsilon = \frac{1}{\varepsilon} \big\{ \mathcal{C} [F_+^\varepsilon, F_+^\varepsilon] + \mathcal{C} [F_+^\varepsilon, F_-^\varepsilon] \big\}, \\
\partial_t F_-^\varepsilon &+ c\hat{p}_- \cdot \nabla_x F_-^\varepsilon - e_- (E^\varepsilon + \hat{p}_- \times B^\varepsilon) \cdot \nabla_p F_-^\varepsilon = \frac{1}{\varepsilon} \big\{ \mathcal{C} [F_-^\varepsilon, F_-^\varepsilon] + \mathcal{C} [F_-^\varepsilon, F_+^\varepsilon] \big\},
\end{align*}
\]

(1.1)

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coupled with the Maxwell system for \((E^\varepsilon(t, x), B^\varepsilon(t, x)) \in \mathbb{R}^3 \times \mathbb{R}^3:\)

\[
\begin{aligned}
\partial_t E^\varepsilon - c\nabla_x \times B^\varepsilon &= -4\pi \int_{\mathbb{R}^3} (e_+ \hat{p}_+ F^\varepsilon_+ - e_- \hat{p}_- F^\varepsilon_-) \, dp, \\
\partial_t B^\varepsilon + c\nabla_x \times E^\varepsilon &= 0, \\
\nabla_x \cdot E^\varepsilon &= 4\pi \int_{\mathbb{R}^3} (e_+ F^\varepsilon_+ - e_- F^\varepsilon_-) \, dp, \\
\nabla_x \cdot B^\varepsilon &= 0.
\end{aligned}
\]

(1.2)

Here \(\varepsilon\) is the Knudsen number, \(F^\varepsilon_\pm(t, x, p)\) is the number density function for ions/electrons at time \(t \geq 0\), position \(x = (x_1, x_2, x_3) \in \mathbb{R}^3\) and momentum \(p = (p_1, p_2, p_3) \in \mathbb{R}^3\). \(\hat{p}_\pm = \frac{p}{|p|}\). The constants \(\pm e\) and \(m\) are the ions'/electrons' charges and rest masses, respectively. \(c\) is the speed of light, and \((E^\varepsilon(t, x), B^\varepsilon(t, x))\) are the electromagnetic fields.

Denote the four-momentums \(p^\mu = (p^0, p)\) and \(q^\mu = (q^0, q)\). We use the Einstein convention for indices and summation. The Lorentz inner product is then given by

\[
p^\mu q_\mu := -p^0 q^0 + \sum_{i=1}^3 p_i q_i.
\]

(1.3)

The collision operator \(C\) in the R.H.S. of (1.1), which registers binary collisions between particles, takes the following form:

\[
C[g_+, h_-] := \nabla_p \cdot \left\{ \int_{\mathbb{R}^3} \Phi_{+-}(p, q) \left[ \nabla_p g_+(p) h_-(q) - g_+(p) \nabla_q h_-(q) \right] \, dq \right\},
\]

(1.4)

where the collision kernel \(\Phi_{+-}(p, q)\) is a \(3 \times 3\) non-negative matrix (Here we use \((+, -)\) as an example)

\[
\Phi_{+-}(p, q) := \frac{\Lambda_{+-}(p, q)}{p^0 q^0} S_{+-}(p, q)
\]

(1.5)

with

\[
\Lambda_{+-}(p, q) := \frac{1}{m^2 c^2} \left( \frac{p^\mu q^\mu - \mu}{m^2 c^2} \right)^2 \left( \frac{1}{m^2 c^2} \left( \frac{p^\mu q^\mu - \mu}{m^2 c^2} \right)^2 - 1 \right)^{-\frac{3}{2}},
\]

(1.6)

\[
S_{+-}(p, q) := \left( \frac{1}{m^2 c^2} \left( \frac{p^\mu q^\mu - \mu}{m^2 c^2} \right)^2 - 1 \right)^{\frac{3}{2}} I_3 - (p - q) \otimes (p - q)
\]

\[
- \frac{1}{m^2 c^2} \left( \frac{1}{m^2 c^2} \left( \frac{p^\mu q^\mu - \mu}{m^2 c^2} \right)^2 - 1 \right) (p \otimes q + q \otimes p).
\]

(1.7)

The collision operator \(C\) satisfies the orthogonality property:

\[
\int_{\mathbb{R}^3} \left\{ \left( \frac{p}{p^0_+} \right) C[g_+, h_-](p) \right\} \, dp + \int_{\mathbb{R}^3} \left\{ \left( \frac{p}{p^0_-} \right) C[g_-, h_+](p) \right\} \, dp = 0,
\]

(1.8)

which, combined with (1.1) and (1.2), yields the conservation laws

\[
\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} F^\varepsilon(t, x, p) \, dp \, dx = 0,
\]

\[
\frac{d}{dt} \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} p \left( F^\varepsilon_+(t, x, p) + F^\varepsilon_-(t, x, p) \right) \, dp \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( E^\varepsilon(t, x) \times B^\varepsilon(t, x) \right) \, dx \right\} = 0,
\]

\[
\frac{d}{dt} \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( p^0_+ F^\varepsilon_+(t, x, p) + p^0_- F^\varepsilon_-(t, x, p) \right) \, dp \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( |E^\varepsilon(t, x)|^2 + |B^\varepsilon(t, x)|^2 \right) \, dx \right\} = 0.
\]
1.2 Relativistic Euler-Maxwell System

Corresponding to (1.1)–(1.2), at the hydrodynamic level, the plasma gas obeys the relativistic Euler-Maxwell system for \((n_+(t,x), n_-(t,x), u_+(t,x), u_-(t,x), T(t,x)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}:

\[
\begin{aligned}
\frac{1}{c} \partial_t (n_\pm u_\pm^0) + \nabla_x \cdot (n_\pm u_\pm) = 0, \\
\frac{1}{c} \partial_t \left( (\epsilon_\pm + P_\pm) u_\pm^0 u_\pm \right) + \frac{1}{c^2} \nabla_x \cdot \left( (\epsilon_\pm + P_\pm) u_\pm \otimes u_\pm \right) + \nabla_x P_\pm + \frac{1}{c} \epsilon_\pm n_\pm (u_\pm^0 E + u \times B) = 0, \\
\frac{1}{c} \partial_t \left( (\epsilon_\pm (u_\pm^0)^2 + P_\pm |u_\pm|^2) \right) + \frac{1}{c^2} \nabla_x \cdot \left( (\epsilon_\pm + P_\pm) (u_\pm^0) u_\pm \right) + \frac{1}{c} \epsilon_\pm n_\pm (u_\pm \cdot E) = 0,
\end{aligned}
\]

coupled with the Maxwell system for \((E(t,x), B(t,x)) \in \mathbb{R}^3 \times \mathbb{R}^3:

\[
\begin{aligned}
\partial_t E - c \nabla_x \times B &= -\frac{4\pi}{c} (\epsilon_+ n_+ u_+ - \epsilon_- n_- u_-), \\
\partial_t B + c \nabla_x \times E &= 0, \\
\nabla_x \cdot E &= \frac{4\pi}{c} (\epsilon_+ n_+ u_+^0 - \epsilon_- n_- u_-^0), \\
\nabla_x \cdot B &= 0,
\end{aligned}
\]

where \(n_\pm\) are the ions'/electrons' number densities, \(u_\pm = (u_{\pm,1}, u_{\pm,2}, u_{\pm,3})\), \(u_\pm^0 = \sqrt{|u_\pm|^2 + c^2}\), and \(T\) is the system temperature. In particular, \(\epsilon_\pm(t,x)\) is the total energy (including the rest energy and internal energy) and \(P_\pm(t,x)\) is the pressure given by

\[
P_\pm := \frac{n_\pm u_\pm c^2}{\gamma_\pm} = \frac{k_B}{m_\pm} \rho_\pm T, \\
\epsilon_\pm := \frac{n_\pm m_\pm c^2}{K_2(\gamma_\pm)} \left\{ K_3(\gamma_\pm) - \frac{1}{\gamma_\pm} K_2(\gamma_\pm) \right\},
\]

where \(\rho_\pm := n_\pm m_\pm\) is the mass density, \(\gamma_\pm := m_\pm c^2 (k_B T)^{-1}\) is a dimensionless variable, \(k_B\) is Boltzmann’s constant, \(K_j(\gamma)\) for \(j = 0, 1, 2, \ldots\) are the modified second-order Bessel functions:

\[
K_j(\gamma) := \frac{(2j)!}{(2j)! \gamma^j} \int_\gamma^\infty e^{-\lambda} (\lambda^2 - \gamma^2)^{j-1/2} d\lambda, \quad (j \geq 0).
\]

The system (1.10) has been well-studied in the irrotational context. Denote Faraday’s tensor

\[
\mathcal{F}^{ij} := \begin{pmatrix} 0 & -c^{-1}E_1 & -c^{-1}E_2 & -c^{-1}E_3 \\ c^{-1}E_1 & 0 & -B_3 & B_2 \\ c^{-1}E_2 & B_3 & 0 & -B_1 \\ c^{-1}E_3 & -B_2 & B_1 & 0 \end{pmatrix}.
\]

Let \(h_\pm\) be the specific enthalpy defined by \(h'_\pm(x) = \frac{P_\pm(x)}{x}\) with \(h_\pm > 0\). Then we say the solution to (1.9) and (1.10) is irrotational if

\[
e_- \mathcal{F}_{jk} = -\partial_j (h_- n_- u_- k) + \partial_k (h_- n_- u_- j), \\
e_+ \mathcal{F}_{jk} = -\partial_j (h_+ n_+ u_+ k) + \partial_k (h_+ n_+ u_+ j).
\]
Theorem 1.1 (Theorem 2.3 of [45]). Assume that the initial data \( n_\pm(0, x), u_\pm(0, x), E(0, x), B(0, x) \) satisfies (1.15) and is sufficiently close to the equilibrium \( \left( \frac{e_\pm}{e_\pm}, 0, 0, 0 \right) \) for some constants \( \bar{n} > 0 \). Then there exists a unique global solution \( \left( n_\pm(t, x), u_\pm(t, x), E(t, x), B(t, x) \right) \) to the two-fluid relativistic Euler–Maxwell–Landau system (1.9) and (1.10) that satisfies (1.15) for any \( t > 0 \) and

\[
\sup_{t \in [0, \infty)} \left\| \left( n_\pm(t) - \frac{\epsilon - \epsilon}{\epsilon} \bar{n}, u_\pm(t), E(t), B(t) \right) \right\|_{H^N_0} + \sup_{t \in [0, \infty)} \sup_{|\rho| \leq 4} \left( 1 + t \right)^{\beta_0} \left\| \nabla_x^\rho \left( n_\pm(t) - \frac{\epsilon - \epsilon}{\epsilon} \bar{n}, u_\pm(t), E(t), B(t) \right) \right\|_{L^\infty} \lesssim \bar{\epsilon}_0,
\]

where \( N_0 = 10^4, \beta_0 = \frac{201}{200} \) and \( \bar{\epsilon}_0 \) is a sufficiently small positive constant.

1.3 Main Results

The purpose of our article is to rigorously prove that solutions of the relativistic Vlasov–Maxwell–Landau system (1.1)–(1.2) converge to solutions of the relativistic Euler–Maxwell system (1.9)–(1.10) globally in time, as the Knudsen number \( \epsilon \) tends to zero.

Theorem 1.2. Assume that \( \left( n_{\pm,0}(t, x), u_{\pm}(t, x), E_0(t, x), B_0(t, x) \right) \) is the global solution constructed in Theorem 1.1 and \( M_\pm(t, x, p) = n_{\pm,t}(x, p) \exp \left\{ \frac{u_{\pm}^2 + p}{k_B T_0} \right\} \) is the corresponding local Maxwellian. Then there exists an \( \epsilon_0 > 0 \) such that for any \( 0 \leq \epsilon \leq \epsilon_0, k \geq 3, \) and \( 0 < t \leq \bar{t} \) with \( \bar{t} = e^{-k/(2k-1)} \), the Hilbert expansion (2.3) holds. Moreover, if \( F^{\epsilon}_{\pm}(0, x, p) \geq 0, \) and

\[
\sum_{\pm} \left\| \left( \frac{F^{\epsilon}_{\pm} - M^{\pm}}{\sqrt{M^{\pm}}} \right)(0) \right\|_{H^2 L^2_\rho} + \left\| (E - E_0)(0) \right\|_{H^2} + \left\| (B - B_0)(0) \right\|_{H^2} = O(\epsilon),
\]

then \( F^{\epsilon}_{\pm}(t, x, p) \geq 0 \) and

\[
\sup_{0 \leq t \leq \bar{t}} \left\{ \sum_{\pm} \left\| \left( \frac{F^{\epsilon}_{\pm} - M^{\pm}}{\sqrt{M^{\pm}}} \right)(t) \right\|_{H^2 L^2_\rho} + \left\| (E - E_0)(t) \right\|_{H^2} + \left\| (B - B_0)(t) \right\|_{H^2} \right\} = O(\epsilon).
\]

1.4 Background and Literature

As a key ingredient to attack the well-known Hilbert’s Sixth Problem, the rigorous derivation of fluid equations (Euler equations or Navier-Stokes equations, etc.) from the kinetic equations (Boltzmann equation, Landau equation, etc.) has attracted a lot of attentions since the early twentieth century. The fundamental problem is to justify the asymptotic limits of kinetic solutions as the Knudsen number (which measures the relative mean free path) or the Strouhal number (which measures the relative time-varying speed) shrinks to zero.

There are mainly two genres to study hydrodynamic limits: kinetic-based approach or fluid-based approach. We refer to [60, 64] for more details.

Kinetic-based approach purely relies on the solution theory (well-posedness, regularity, etc.) of the kinetic equations and does NOT assume any a priori properties of the fluid limits. On the one hand, in the context of the renormalized solution and entropy method there are successful applications of this approach (usually referred as BGL Project) to the incompressible Euler/Navier-Stokes limit. We refer to Bardos-Golse [6], Golse-Saint-Raymond [34, 35], Saint-Raymond [76], Masmoudi-Saint-Raymond [72], Arsenio-Saint-Raymond [5], Bardos-Golse-Levermore [7, 8, 9], Lions-Masmoudi [68] and Masmoudi [71]. Interested readers may refer to the books by Saint-Raymond [77] and by Golse [33], and the references
The fluid-based approach DOES assume a priori that we have a well-prepared fluid system, which has a unique smooth solution, and then justify the kinetic solution converging to this fluid solution. In some sense, this is essentially “fluid-to-kinetic” limits and we avoid the complication of possible fluid ill-posedness, like blow-up or shock wave. This approach typically provides hydrodynamic limits in the stronger sense and utilizes the so-called Hilbert expansion techniques. In this paper, we will focus on the fluid-based approach and discuss the progress in detail.

The Hilbert expansion dates back to 1912 by Hilbert [52], who proposed an asymptotic expansion of the distribution function solving the Boltzmann equation with respect to the Knudsen number and formally derived the limiting compressible Euler equations. The similar formal expansion can be naturally extended to treat the Landau equation, and Vlasov systems.

The first rigorous justification of the compressible Euler limit of the Boltzmann equation was due to Caflisch [13]. Later, with the $L^2 - L^\infty$ framework introduced in Guo [41], Guo-Jang-Jiang [48] improved Caflisch’s result and removed the assumption on the initial data $F_R^0(0,x,v) = 0$. This framework was extended to treat the Vlasov-Poisson-Boltzmann (VPB) system in Guo-Jang [47] and the relativistic Boltzmann (r-BOL) equation in Speck-Strain [81]. Recently, this framework was further developed to the investigation of the relativistic Vlasov-Maxwellian-Boltzmann (r-VMB) system in Guo-Xiao [51] and the Boltzmann equation with boundary conditions in half-space in Guo-Huang-Wang [43], Jiang-Luo-Tang [61, 62]. We also refer to Grad [36], Ukai-Asano [86] De Masi-Esposito-Lebowitz [15], and the recent work Jang-Kim [59] for the incompressible Euler limit.

For the convergence of the Boltzmann equation to the basic waves of the Euler equations: the shock waves, rarefaction waves and contact discontinuity, the interested readers may refer to Huang-Wang-Yang [55, 54, 56], Xin-Zeng [96] and Yu [100].

As for the incompressible Navier-Stokes limit of the Boltzmann equation, there are way too many references and we only list some closely related works. The early development tracks back to De-Masi-Esposito-Lebowitz [15] in 2D. Then Guo [40] justified the diffusive limit in the periodic domain via the nonlinear energy method. This result was extended to the whole space in Liu-Zhao [69], to more general initial data with initial layer in Jiang-Xiong [63], and to the Vlasov-Maxwell-Boltzmann (VMB) system in Jang [58]. See also the recent work Gallagher-Tristani [31]. For stationary Boltzmann equation and other settings, we refer to Di-Meo-Esposito [17], Arkeryd-Esposito-Marra-Nouri [4], Esposito-Lebowitz-Marra [29], Esposito-Guo-Marra [28], Esposito-Guo-Kim-Marra [27], Wu [89], Wu-Ouyang [91, 90, 92].

Despite the fruitful progress in the hydrodynamic limits of the Boltzmann type equation, there are very limited works in this direction for relativistic Landau (r-LAN) equation or Landau equation, and relativistic Vlasov-Maxwell-Landau (r-VML) system. For Landau equation, we refer to Guo [40] for the incompressible Navier-Stokes limit, Duan-Yang-Yun [23] for the rarefaction wave limit, and the recent work Rachid [75]. As far as we are aware of, our paper and its sister [74] are the first result to justify the compressible Euler limit for r-LAN equation and r-VML system.

As for the well-posedness issue for fixed Knudsen number and Strouhal number, there are a huge number of literature. We list some closely related to this article. For the r-VML system, we refer to Strain-Guo [83], Yu [99], Yang-Yu [98], Liu-Zhao [70] and Xiao [93]. For the r-LAN equation, we refer to Hsiao-Yu [53] and Yang-Yu [97]. We also mention Guo-Strain [50] and some works in the non-relativistic framework: Villani [87], Guo [38, 39, 42], Strain [82], Duan-Strain [22], Duan [19], Duan-Lei-Yang-Zhao [20], Guo-Hwang-Jang-Ouyang [44], Duan-Liu-Sakamoto-Strain [21] for Landau equation and Dong-Guo-Ouyang [18] for Vlasov-Poisson-Landau (VPL) system.

Finally, we record some significant progress on the compressible fluid system. Sideris [78] justified the classical result on the compressible Euler equation that the solution might blow up even if the initial data is small and irrotational. However, as a key observation, the electric field or the electromagnetic fields might help stabilize the system. Based on the Klein–Gordon effect, Guo [37] and Germain-Masmoudi [32]
constructed global classical solutions to the one-fluid Euler-Poisson system and Euler-Maxwell system, respectively. Using the combination of normal-form method and vector-field method to capture the so-called “null structure”, Guo-Ionescu-Pausader [46] justified the global well-posedness of 3D two-fluid Euler-Maxwell system, and the similar results were extended to treat 3D Euler-Poisson system, and 3D one-fluid/two-fluid relativistic Euler-Maxwell system in Guo-Ionescu-Pausader [45]. The 2D case was justified in Deng-Ionescu-Pausader [16]. More recently, the one-fluid Euler–Maxwell system in 3D with non-vanishing vorticity was studied in Ionescu-Lie [57].

2 Formulation and Discussion

2.1 Hilbert Expansion

We consider the Hilbert expansion with respect to small Knudsen number $\varepsilon$ and $k \geq 2$:

$$F^\varepsilon = \sum_{n=0}^{2k-1} \varepsilon^n F_{\pm,n} + \varepsilon^k F_{\pm,R}, \quad E^\varepsilon = \sum_{n=0}^{2k-1} \varepsilon^n E_n + \varepsilon^k E_{R}, \quad B^\varepsilon = \sum_{n=0}^{2k-1} \varepsilon^n B_n + \varepsilon^k B_{R}. \quad (2.1)$$

To determine the coefficients $F_{\pm,s}(t,x,p), E_{s}(t,x), B_{s}(t,x)$ for $0 \leq s \leq 2k - 1$, we plug the formal expansions (2.1) into equations (1.1)-(1.2) and equate the coefficients on both sides in front of different powers of the parameter $\varepsilon$ to obtain:

$\varepsilon^{-1}$-order:

$$C[F_{\pm,0}, F_{\pm,0}] + C[F_{\pm,0}, F_{\mp,0}] = 0. \quad (2.2)$$

$\varepsilon^0$-order:

$$\partial_t F_{\pm,0} + c\hat{p}_\pm \cdot \nabla_x F_{\pm,0} \pm \varepsilon_\pm \left( E_0 + \hat{p}_\pm \times B_0 \right) \cdot \nabla_p F_{\pm,0}$$

$$= C[F_{\pm,1}, F_{\pm,0}] + C[F_{\pm,1}, F_{\mp,0}] + C[F_{\pm,0}, F_{\pm,1}] + C[F_{\pm,0}, F_{\mp,1}], \quad (2.3)$$

and

$$\begin{align*}
\partial_t E_0 - c\nabla_x \times B_0 &= -4\pi \int_{\mathbb{R}^3} \left( e_+ \hat{p}_+ F_{+,0} - e_- \hat{p}_- F_{-,0} \right) dp, \\
\partial_t B_0 + c\nabla_x \times E_0 &= 0, \\
\nabla_x \cdot E_0 &= 4\pi \int_{\mathbb{R}^3} \left( e_+ F_{+,0} - e_- F_{-,0} \right), \\
\nabla_x \cdot B_0 &= 0. 
\end{align*} \quad (2.4)$$

$\varepsilon^n$-order (1 $n < 2k - 1$):

$$\partial_t F_{\pm,n} + c\hat{p}_\pm \cdot \nabla_x F_{\pm,n} \pm \varepsilon_\pm \left( E_n + \hat{p}_\pm \times B_n \right) \cdot \nabla_p F_{\pm,0} \pm \varepsilon_\pm \left( E_0 + \hat{p}_\pm \times B_0 \right) \cdot \nabla_p F_{\pm,n}$$

$$= \sum_{i+j=n+1 \atop i,j \geq 0} \left\{ C[F_{\pm,i}, F_{\pm,j}] + C[F_{\pm,i}, F_{\mp,j}] \right\} + \varepsilon_\pm \sum_{i+j=n \atop i,j \geq 1} \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \nabla_p F_{\pm,j}, \quad (2.5)$$

and

$$\begin{align*}
\partial_t E_n - c\nabla_x \times B_n &= -4\pi \int_{\mathbb{R}^3} \left( e_+ \hat{p}_+ F_{+,n} - e_- \hat{p}_- F_{-,n} \right) dp, \\
\partial_t B_n + c\nabla_x \times E_n &= 0, \\
\nabla_x \cdot E_n &= 4\pi \int_{\mathbb{R}^3} \left( e_+ F_{+,n} - e_- F_{-,n} \right) dp, \\
\nabla_x \cdot B_n &= 0. 
\end{align*} \quad (2.6)$$
\( \varepsilon^{2k-1} \)-order:

\[
\begin{align*}
\partial_t F_{\pm,2k-1} + c\hat{p}_\pm \cdot \nabla_x F_{\pm,2k-1} & \pm \varepsilon_\pm \left( E_{2k-1} + \hat{p}_\pm \times B_{2k-1} \right) \cdot \nabla_p F_{\pm,0} \pm \varepsilon_\pm \left( E_0 + \hat{p} \times B_0 \right) \cdot \nabla_p F_{\pm,2k-1} \\
= \sum_{i+j=2k, i,j \geq 1} \left\{ C[F_{\pm,i}, F_{\pm,j}] + C[F_{\pm,i}, \nabla F_{\pm,j}] \right\} \pm \varepsilon_\pm \sum_{i+j=2k-1, i,j \geq 1} \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \nabla_p F_{\pm,j},
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
\partial_t E_{2k-1} - c\nabla_x \times B_{2k-1} = -4\pi \int_{\mathbb{R}^3} \left( e_+ \hat{p}_+ F_{+,2k-1} - e_- \hat{p}_- F_{-,2k-1} \right) dp, \\
\partial_t B_{2k-1} + c\nabla_x \times E_{2k-1} = 0, \\
\nabla_x \cdot E_{2k-1} = 4\pi \int_{\mathbb{R}^3} \left( e_+ F_{+,2k-1} - e_- F_{-,2k-1} \right) dp, \\
\nabla_x \cdot B_{2k-1} = 0.
\end{cases}
\end{align*}
\]

The remainder \((F_{\pm,R}^\varepsilon, E_{\pm,R}^\varepsilon, B_{\pm,R}^\varepsilon)\) satisfies

\[
\begin{align*}
\partial_t F_{\pm,R}^\varepsilon + c\hat{p}_\pm \cdot \nabla_x F_{\pm,R}^\varepsilon & \pm \varepsilon_\pm \left( E_{\pm,R}^\varepsilon + \hat{p}_\pm \times B_{\pm,R}^\varepsilon \right) \cdot \nabla_p F_{\pm,0} \pm \varepsilon_\pm \left( E_0 + \hat{p} \times B_0 \right) \cdot \nabla_p F_{\pm,R}^\varepsilon \\
& = \varepsilon^{k-1} \left\{ C[F_{\pm,R}^\varepsilon, F_{\pm,0}] + C[F_{\pm,0}, F_{\pm,R}^\varepsilon] + C[F_{\pm,R}^\varepsilon, F_{\pm,0}] + C[F_{\pm,0}, F_{\pm,R}^\varepsilon] \right\} \\
& + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left( C[F_{\pm,R}^\varepsilon, F_{\pm,i}] + C[F_{\pm,i}, F_{\pm,R}^\varepsilon] + C[F_{\pm,R}^\varepsilon, F_{\pm,i}] + C[F_{\pm,i}, F_{\pm,R}^\varepsilon] \right) \\
& \pm \varepsilon^k \varepsilon_\pm \left( E_{\pm,R}^\varepsilon + \hat{p}_\pm \times B_{\pm,R}^\varepsilon \right) \cdot \nabla_p F_{\pm,R}^\varepsilon \\
& \pm \sum_{i=1}^{2k-1} \varepsilon^i \varepsilon_\pm \left\{ \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \nabla_p F_{\pm,R}^\varepsilon + \left( E_{\pm,R}^\varepsilon + \hat{p}_\pm \times B_{\pm,R}^\varepsilon \right) \cdot \nabla_p F_{\pm,i} \right\} + \varepsilon^k S_\pm,
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
\partial_t E_{\pm,R}^\varepsilon - c\nabla_x \times B_{\pm,R}^\varepsilon = -4\pi \int_{\mathbb{R}^3} \left( e_+ \hat{p}_+ F_{\pm,R}^\varepsilon - e_- \hat{p}_- F_{\pm,R}^\varepsilon \right) dp, \\
\partial_t B_{\pm,R}^\varepsilon + c\nabla_x \times E_{\pm,R}^\varepsilon = 0, \\
\nabla_x \cdot E_{\pm,R}^\varepsilon = 4\pi \int_{\mathbb{R}^3} \left( e_+ F_{\pm,R}^\varepsilon - e_- F_{\pm,R}^\varepsilon \right) dp, \\
\nabla_x \cdot B_{\pm,R}^\varepsilon = 0,
\end{cases}
\end{align*}
\]

where

\[
S_\pm = \sum_{i+j \geq 2k+1, \frac{1}{2} \leq j \leq 2k-1} \varepsilon^{i+j-2k-1} \left\{ C[F_{\pm,i}^\varepsilon, F_{\pm,j}^\varepsilon] + C[F_{\pm,i}^\varepsilon, F_{\pm,j}^\varepsilon] \right\} \\
+ \sum_{i+j \geq 2k, 1 \leq i,j \leq 2k-1} \varepsilon^{i+j-2k} \varepsilon_\pm \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \nabla_p F_{\pm,j}.
\]

From (2.3), we conclude that \( F_{\pm,0} \) should be local Maxwellians:

\[
F_{\pm,0}(t, x, p) = M_{\pm} = \frac{n_{\pm,0}}{4\pi \varepsilon_\pm m_{\pm}^2 c k_B T_0 K_2(\gamma_{\pm})} \exp \left\{ \frac{n_{\pm,0}^\varepsilon p_\mu}{k_B T_0} \right\},
\]
We project the two equations in (2.3) onto the collision invariants,

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} p_i \\ p_i \end{pmatrix}, \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix},
\]

for the relativistic Landau operators to derive the relativistic Euler-Maxwell system (1.9). The higher-order terms \( F_{\pm,k} \) can be derived in a similar fashion (see Appendix).

To prove Theorem 1.2, our main task to solve (2.9)–(2.10). Define \( f_{\pm}^{\varepsilon} \) as

\[
F_{\pm,R}^{\varepsilon} := \sqrt{M_{\pm}}f_{\pm}^{\varepsilon}.
\]

(2.14)

(2.9) and (2.10) can be rewritten as

\[
\begin{align*}
\{ \partial_t + c\hat{p}_\pm \cdot \nabla_x \pm e_\pm (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \} f_{\pm}^{\varepsilon} 
&= e_\pm \frac{u_\pm \sqrt{M_{\pm}}}{k_B T_0} \cdot \left( E_{\pm}^{\varepsilon} + \hat{p}_\pm \times B_{\pm}^{\varepsilon} \right) + \frac{1}{\varepsilon} \mathcal{L}_\pm \left( f_{\pm}^{\varepsilon} \right) \\
&= -\frac{f_{\pm}^{\varepsilon}}{\sqrt{M_{\pm}}} \left\{ \partial_t + c\hat{p}_\pm \cdot \nabla_x \pm e_\pm (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} \sqrt{M_{\pm}} + \varepsilon^{k-1} \Gamma_{\pm} \left[ f_{\pm}^{\varepsilon}, f_{\pm}^{\varepsilon} \right] \\
&+ \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \Gamma_{\pm} \left[ \frac{F_i}{\sqrt{M}}, f_{\pm}^{\varepsilon} \right] + \Gamma_{\pm} \left[ f_{\pm}^{\varepsilon}, \frac{F_i}{\sqrt{M}} \right] \right\} + e_\pm e^{\varepsilon} \left( E_{\pm}^{\varepsilon} + \hat{p}_\pm \times B_{\pm}^{\varepsilon} \right) \cdot \nabla_p f_{\pm}^{\varepsilon} \\
&\pm e_\pm \sum_{i=1}^{2k-1} \varepsilon^{i} \left\{ \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \nabla_p f_{\pm}^{\varepsilon} + \left( E_{R}^{\varepsilon} + \hat{p}_\pm \times B_{R}^{\varepsilon} \right) \cdot \frac{\nabla_p F_{\pm,i}^{\varepsilon}}{\sqrt{M_{\pm}}} \right\} \\
&\pm e_\pm \sum_{i=1}^{2k-1} \varepsilon^{i} \left\{ \frac{u_0^{0,\pm} \nabla_p f_{\pm}^{\varepsilon} - u_\pm}{2k_B T_0} \right\} + \varepsilon^{k} \mathcal{S}_{\pm}
\end{align*}
\]

and

\[
\begin{align*}
\partial_t E_{R}^{\varepsilon} - c\nabla_x \times B_{R}^{\varepsilon} &= -4\pi \int_{\mathbb{R}^3} \left( e_+ \nabla \pm \sqrt{M_{\pm}}f_{\pm}^{\varepsilon} - e_- \nabla \pm \sqrt{M_{\pm}}f_{\pm}^{\varepsilon} \right) dp, \\
\partial_t B_{R}^{\varepsilon} + c\nabla_x \times E_{R}^{\varepsilon} &= 0, \\
\nabla_x \cdot E_{R}^{\varepsilon} &= 4\pi \int_{\mathbb{R}^3} \left( e_+ \sqrt{M_{\pm}}f_{\pm}^{\varepsilon} - e_- \sqrt{M_{\pm}}f_{\pm}^{\varepsilon} \right) dp, \\
\nabla_x \cdot B_{R}^{\varepsilon} &= 0.
\end{align*}
\]

Here the linearized collision operator \( \mathcal{L}[f] \) and nonlinear collision operator \( \Gamma[f, g] \) are defined as follows:

\[
\mathcal{L}[f] = \begin{pmatrix} \mathcal{L}_+ f \\ \mathcal{L}_- f \end{pmatrix} = -\begin{pmatrix} A_+ f + K_+ f \\ A_- f + K_- f \end{pmatrix}
\]

(2.17)

\[
A_\pm f = \frac{1}{\sqrt{M_{\pm}}} \left\{ \mathcal{C} \left[ \sqrt{M_{\pm}} f_{\pm}, M_{\pm} \right] + \mathcal{C} \left[ M_{\pm}, \sqrt{M_{\pm}} f_{\pm} \right] \right\},
\]

(2.18)

\[
K_\pm f = \frac{1}{\sqrt{M_{\pm}}} \left\{ \mathcal{C} \left[ \sqrt{M_{\pm}} f_{\pm}, M_{\pm} \right] + \mathcal{C} \left[ M_{\pm}, \sqrt{M_{\pm}} f_{\pm} \right] \right\},
\]

(2.19)

and

\[
\Gamma[f, g] = \begin{pmatrix} \Gamma_+ [f, g] \\ \Gamma_- [f, g] \end{pmatrix}.
\]

(2.20)
\begin{equation}
\Gamma_\pm[f, g] = \frac{1}{\sqrt{M_\pm}} \left\{ \mathcal{C} \left[ \sqrt{M_\pm} f_\pm, \sqrt{M_\pm} g_\pm \right] + \mathcal{C} \left[ \sqrt{M_\pm} g_\pm, \sqrt{M_\pm} f_\pm \right] \right\}. \tag{2.21}
\end{equation}

Note that the null space of the linearized operator \( \mathcal{L} \) is given by \( 1 \leq i \leq 3 \)
\begin{equation}
\mathcal{N} = \text{span} \left\{ \left( \sqrt{M_+} \right)_0, \left( 0 \sqrt{M_-} \right), p_i \left( \sqrt{M_+} \right), \left( \frac{p_0}{\sqrt{M_+}} \right) \right\}. \tag{2.22}
\end{equation}

Denote \( \mathbf{P} \) as the orthogonal projection from \( L^2_p \) onto \( \mathcal{N} \):
\begin{equation}
\mathbf{P}[f^\varepsilon] = \left( \begin{array}{c} \mathbf{P}_+ [f^\varepsilon] \\ \mathbf{P}_- [f^\varepsilon] \end{array} \right) = \left( \begin{array}{c} \left( a^\varepsilon_+ - \frac{p_{+2} \varepsilon e^2}{\rho_{+1}} \right) M^\frac{3}{2}_+ \\ \left( a^\varepsilon_- - \frac{p_{-2} \varepsilon e^2}{\rho_{-1}} \right) M^\frac{3}{2}_- \\ \frac{p_0}{\rho_0} M^\frac{3}{2}_+ \\ \frac{p_0}{\rho_0} M^\frac{3}{2}_- \end{array} \right), \tag{2.23}
\end{equation}

where
\begin{equation}
\rho_{\pm,1} := \int_{\mathbb{R}^3} M_\pm \, dp = \frac{n_\pm \alpha u_0^0}{c}, \quad \rho_{\pm,2} := \int_{\mathbb{R}^3} p^0_\pm M_\pm \, dp = \frac{e_\pm \alpha (u_0^0)^2 + P_{\pm,0} |u|^2}{c^3}. \tag{2.24}
\end{equation}

## 2.2 Notations and Convention

Throughout the paper, \( C \) denotes a generic positive constant which may change line by line. The notation \( A \lesssim B \) implies that there exists a positive constant \( C \) such that \( A \leq CB \) holds uniformly over the range of parameters. The notation \( A \approx B \) means \( \frac{1}{C} A \leq B \leq CA \) for some constant \( C > 1 \).

Let \( f = \left( f^+_p, f^-_p \right), g = \left( g^+_p, g^-_p \right) \). Let \( \langle \cdot, \cdot \rangle \) denote the \( L^2 \) inner product in \( p \in \mathbb{R}^3 \) and \( \langle \cdot, \cdot \rangle \) the \( L^2 \) inner product in \( (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \):
\begin{equation}
\langle f, g \rangle = \int_{\mathbb{R}^3} \left( f^+_p g^+_p + f^-_p g^-_p \right) \, dp, \tag{2.25}
\end{equation}
\begin{equation}
\langle f, g \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( f^+_p g^+_p + f^-_p g^-_p \right) \, dp \, dx. \tag{2.26}
\end{equation}

Let \( | \cdot |_{L^2} \) denote the \( L^2 \) norm in \( p \in \mathbb{R}^3 \) and \( \| \cdot \| \) the \( L^2 \) norm in \( (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \):
\begin{equation}
|f|^2_{L^2} = \langle f, f \rangle, \quad \|f\|^2 = \langle f, f \rangle. \tag{2.27}
\end{equation}

Note that for quantities related to \( \mathbf{E} \) or \( \mathbf{B} \) which do not depend on \( p \), we also use \( \| \cdot \| \) to denote the \( L^2 \) norm in \( x \in \mathbb{R}^3 \). Similarly, for \( s = 0, 1, 2 \), we define the Sobolev norms
\begin{equation}
\|f\|^2_{H^s} = \|f_0\|^2_{H^s} + \|f_-\|^2_{H^s} = \sum_{|\alpha|=0}^s \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( |\partial_\alpha^p f^+_p|^2 + |\partial_\alpha^p f^-_p|^2 \right) \, dp \, dx. \tag{2.28}
\end{equation}

For \( M_\pm \) given in (2.12), we denote \( (n\pm, 0, u\pm, T_0)(t, x) \) as part of a solution to the relativistic Euler-Maxwell system (1.9)-(1.10), which was constructed in [45], and define the following 3 \times 3 matrix type collision frequency:
\begin{equation}
\sigma_{\pm, \pm}^{ij}(p) = \int_{\mathbb{R}^3} \Phi_{\pm, \pm}^{ij}(q) \, dq.
\end{equation}

To measure the dissipation of the linearized relativistic Landau collision, we define the inner product:
\begin{equation}
(f, g) = \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \left( \sigma^{ij}_{\pm, +} \partial_{p_i} f \partial_{p_j} g + \sigma^{ij}_{\pm, -} \partial_{p_i} f \partial_{p_j} g \right) \, dp.
\end{equation}
\[ + \sum_{i,j=1}^{3} \frac{c^2}{4k_B^2T_0^2} \int_{\mathbb{R}^3} \left( \varphi_{i,j,+}^{ij} + \varphi_{i,j,-}^{ij} \right) \frac{p_i}{p_+^0} \frac{p_j}{p_+^0} f_+ g_+ dp \]
\[ + \sum_{i,j=1}^{3} \frac{c^2}{4k_B^2T_0^2} \int_{\mathbb{R}^3} \left( \varphi_{i,j,-}^{ij} + \varphi_{i,j,+}^{ij} \right) \frac{p_i}{p_-^0} \frac{p_j}{p_-^0} f_- g_- dp. \]

Denote the corresponding \( \sigma \) norms:
\[ |f|_\sigma^2 = (f, f)_\sigma, \quad \|f\|_\sigma^2 = \int_{\mathbb{R}^3} |f(x)|_\sigma^2 dx. \] (2.30)

Similarly, for \( s = 0, 1, 2 \), we define the Sobolev \( \sigma \) norms
\[ \|f\|_{H^s_\sigma} = \sum_{|\alpha|=0}^{s} \int_{\mathbb{R}^3} \left( \|\partial^\alpha f_+\|_\sigma^2 + \|\partial^\alpha f_-\|_\sigma^2 \right) dp dx, \] (2.31)
where \( \partial^\alpha = \partial x_1^\alpha_1 \partial x_2^\alpha_2 \partial x_3^\alpha_3 \) with \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \).

**Remark 2.1.** According to [67], all eigenvalues of \( \varphi_{i,j,+}^{ij}(p) \) are positive and depend on \( p \). Moreover, the eigenvalues converge to positive constants as \( |p| \to \infty \). Then, for the \( | \cdot |_\sigma \) norm defined above, we have
\[ \frac{1}{T_0} |f|_{L^2} + |\nabla_p f|_{L^2} \lesssim |f|_\sigma \lesssim \frac{1}{T_0} |f|_{L^2} + |\nabla_p f|_{L^2} \] (2.32)

Define the weight functions
\[ w_\pm^\ell = (p_\pm^0)^2(N_0 - \ell) \exp \left\{ \frac{p_\pm^0}{5 \ln(e + t) T} \right\}, \quad 0 \leq \ell \leq 2, \] (2.33)
where \( N_0 \) and \( T \) are constants satisfying \( N_0 \geq 3 \) and
\[ T \geq \sup_{t \in [0, e^{-k/(2k-1)]}, x \in \mathbb{R}^3} T_0(t, x). \] (2.34)

It should be pointed out that the integer 5 in the above weight functions is designed to make sure that
\[ (w_\pm^0)^2 \sqrt{\mathbf{M}_\pm} \lesssim e^{-c_0 p_\pm^0}, \] (2.35)
\[ (w_\pm^0)^2 (p_\pm^0)^2 \leq \frac{1}{2} \left( (w_\pm^0)^2 + (w_\pm^0)^2 \right). \]

for some small constant \( c_0 > 0 \).

Correspondingly, define the weighted norms
\[ \|f\|_{w_\ell} := \|w_\ell^\ell f_+\| + \|w_\ell^\ell f_-\|, \quad \|f\|_{H_{w_\ell}} := \sum_{|\alpha|=0}^{s} \|w_\ell^{[\alpha]} \partial^\alpha f\|, \] (2.36)
\[ \|f\|_{w_\ell, \sigma} := \|w_\ell f_+\|_\sigma + \|w_\ell f_-\|_\sigma, \quad \|f\|_{H_{w_\ell, \sigma}} := \sum_{|\alpha|=0}^{s} \|w_\ell^{[\alpha]} \partial^\alpha f\|_\sigma. \] (2.37)

For the solution \( (n_{\pm,0}(t, x), u_{\pm,0}(t, x), E_0(t, x), B_0(t, x)) \), denote
\[ W(t) := \exp \left( \frac{1}{5 \ln(1 + t) T} \right), \quad Y(t) := -\frac{W'}{W} = \frac{1}{5 \ln(1 + t)^2 (1 + t) T}, \] (2.38)
\[ \mathcal{Z}(t) := \sup_{x \in \mathbb{R}^3, 0 \leq \ell \leq 2} \left\{ \left| \nabla^t_{t,x}(n_0, u_0, T_0) \right| + \left| \nabla^t_{t,x}(E_0, B_0) \right| + \left| \nabla^t_{t,x} u_0 \right| \right\}. \]
2.3 Key Proposition

Theorem 1.2 follows naturally from the following proposition.

**Proposition 2.1.** Let $F^\varepsilon_\pm(0, x, p) \geq 0$. Assume that $(n_{\pm, 0}(t, x), u_{\pm}(t, x), E_0(t, x), B_0(t, x))$ is the global solution constructed in Theorem 1.1. Then for $k \geq 3$ in the Hilbert expansion (2.1) and $F^\varepsilon_\pm = M^\frac{1}{2}_\pm F^\varepsilon_{\pm, R}$ defined in (2.14), there exists an $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_0$ and $0 < t \leq \tilde{t}$ with $\tilde{t} = \varepsilon^{-k/(2k-1)}$, if

$$\mathcal{E}(0) \lesssim 1,$$

(2.39) and (2.16) admit a unique solution $(f^\varepsilon, E^\varepsilon_R, B^\varepsilon_R)$ satisfying $F^\varepsilon_\pm(t, x, p) \geq 0$ and

$$\sup_{0 \leq t \leq \tilde{t}} \mathcal{E}(t) + \int_0^\varepsilon \mathcal{D}(s)ds \lesssim \mathcal{E}(0) + 1,$$

(2.40) where

$$\mathcal{E} \simeq \left( \|f^\varepsilon\|^2 + \|E^\varepsilon_R\|^2 + \|B^\varepsilon_R\|^2 + \|(I - P)[f^\varepsilon]\|_{w, 0}^2 \right),$$

(2.41)

$$\mathcal{D} \simeq \left( \varepsilon^{-1} \|(I - P)[f^\varepsilon]\|_{\sigma}^2 + \varepsilon^{-1} \|(I - P)[f^\varepsilon]\|_{w, 0, \sigma}^2 + \sqrt{\|p^\beta(I - P)[f^\varepsilon]\|_{w, 0}^2} \right),$$

(2.42)

**Remark 2.2.** By (2.1), the estimates (A.7) of the coefficients $F_{\pm, i}, E_i, B_i$ in Proposition A.1, and (2.40), we can obtain (1.18).

**Remark 2.3.** In this paper, we will focus on deriving the a priori estimate (2.40) in Proposition 2.1. Then Theorem 1.2 naturally follows from a standard iteration/fixed-point argument. Based on the continuity argument (see [84]), from now on, we will assume that

$$\mathcal{E}(t) \lesssim \varepsilon^{-1/(2k-1)},$$

(2.43)

and try to derive (2.40).

**Remark 2.4.** Our result guarantees that the Hilbert expansion for the relativistic Vlasov-Maxwell-Landau system is valid for any time if $\varepsilon$ is sufficiently small.

**Remark 2.5.** Due to shock formations in the pure compressible Euler flow (even for smooth irrotational initial data with small amplitude), as illustrated in [78, 14], corresponding Hilbert expansion, acoustic limit for the Boltzmann equation or relativistic Boltzmann equation [13, 48, 49, 81] is only valid local in time. However, the electromagnetic interaction in the fluid models [37, 45, 46] could create stronger dispersive effects, enhance linear decay rates, and prevent formation of shock waves with small amplitude.

**Remark 2.6.** The irrotational assumption (1.15) is necessary in the global well-posedness of Euler-Maxwell equation in [46]. Our proof does not rely on the irrotational assumption. Actually, as long as the fluid solution has time decay, our method should be able to justify the convergence.

**Remark 2.7.** Our method doesn’t work for the non-relativistic case since the cubic velocity growth term (2.45) can’t be controlled by the corresponding quadratic velocity dissipation term in (2.46).
2.4 Technical Overview

In our recent work \cite{74} and this paper, we develop a new time-dependent energy method to study the Hilbert expansion of the Landau-type equation in the relativistic framework. This is inspired by Caffisch’s pioneering work \cite{13}. It is well known that in the study of the Hilbert expansion of the Boltzmann/Landau-type equation, the main task is to solve the remainder term $F^\varepsilon_R = \sqrt{M}f^\varepsilon$, and one of the most challenging difficulty is from the linear term with one power moment growth

$$
\frac{f^\varepsilon(t, x, p)}{\sqrt{M}(t, x, p)} \left\{ \partial_t + cp \cdot \nabla_x \right\} \sqrt{M(t, x, p)}
$$

(2.44)

in the relativistic frame, or the linear term with cubic velocity growth

$$
\frac{f^\varepsilon(t, x, v)}{\sqrt{M}(t, x, v)} \left\{ \partial_t + v \cdot \nabla_x \right\} \sqrt{M(t, x, v)}
$$

(2.45)

in the non-relativistic one.

To tame the velocity growth, Caffisch decomposed the remainder $F^\varepsilon_R(t, x, v)$ into low- and high-velocity parts, which satisfy a coupled system and can be separately estimated via a weighted energy method.

This approach motivates us to design a time-dependent weight function $w_e = \exp \left\{ \frac{p^0_0}{\ln(e + t)} \right\}$. Then this exponential momentum function generates an additional dissipation term to control the moment growth terms in (2.44)

$$
w_e^2 f^\varepsilon \partial_t f^\varepsilon = \frac{1}{2} \frac{d}{dt} (w_e f^\varepsilon)^2 + \frac{1}{5(e + t)[\ln(e + t)]^2} p^0(w_e f^\varepsilon)^2.
$$

(2.46)

Correspondingly, the troublesome term in (2.44) is roughly

$$
w_e^2 f^\varepsilon \left\{ \partial_t + cp \cdot \nabla_x \right\} \sqrt{M} \leq \left\{ c |\nabla_{t,x}(n_0, u, T_0)| \left( \frac{1 + T_0}{T_0^2} \right) \right\} p^0(w_e f^\varepsilon)^2 + \text{other terms.}
$$

(2.47)

As long as

$$
\left\{ c |\nabla_{t,x}(n_0, u, T_0)| \left( \frac{1 + T_0}{T_0^2} \right) \right\} < \frac{1}{5(e + t)[\ln(e + t)]^2 T}
$$

(2.48)

holds for $t, x$ under consideration, we can suppress the momentum growth in (2.44). Therefore, if $|\nabla_{t,x}(n_0, u, T_0)|$ is sufficiently small for all $x \in \mathbb{R}^3$, (2.48) holds locally in time; if $|\nabla_{t,x}(n_0, u, T_0)|$ further enjoys suitably fast time decay, (2.48) holds globally in time.

Since 1980s, time-dependent exponential weight functions have been widely used in the study of the collisional kinetic equations. In 1986, Ukai \cite{85} introduced a weight function $w(t, v) \approx \exp \left\{ (\alpha - \kappa t)(1 + |v|^2) \right\}$ with $\alpha, \kappa > 0, t \in [0, \alpha/(2\kappa)]$ to study the local well-posedness of the cutoff Boltzmann equation. Later, this technique was extended by AMUXY \cite{1, 2, 3} for constructing local solutions of the non-cutoff Boltzmann equation in Sobolev spaces. In these works, the weight function provides an extra gain of velocity weight at the expense of the loss of the decay in the time-dependent Maxwellian.

In the exploration of global classical solutions to the one-species VPB system for cutoff hard potentials and moderately soft potentials, to control the large velocity growth in the nonlinear term due to the Coulomb force, Duan-Yang-Zhao \cite{25, 26} introduced another type of weight function $w(t, v) \approx \exp \left\{ \frac{\lambda (1 + |v|^2)}{(1 + t)^{\theta}} \right\}$, where $\lambda, \theta$ are small positive constants. By introducing a new time weighted energy framework, Xiao-Xiong-Zhao \cite{94, 95} removed the so-called neutral condition assumption on the initial datum in previous work \cite{26}, and extended this well-posedness result to the very soft potentials case.

We point out that the nonlinear energy method and macro-micro decomposition technique employed in
[94, 95] play an essential role in the proof of the main results of [74] and this paper. Recently, such techniques was further used in constructing global classical solutions to the cutoff VMB system, non-cutoff VMB system, and VML system [19, 20, 30, 66, 88].

More recently, a new weight function \( w(t, v) \approx \exp \left\{ (q_1 - q_2 \int_0^t q_3(s) \, ds) (1 + |v|)^2 \right\}, \) with constants \( q_1, q_2 > 0 \) and \( q_3 \) being a dissipation energy functional, was used in Duan-Yang-Yu [24] to justify the asymptotic convergence in Landau equation.

The general discussion of our new time-dependent weighted energy method and its application in the Landau equation can be found in the sister paper [74]. Here we will focus on the new ingredients of the VML system.

Firstly, the Hilbert expansion of VML system holds for arbitrary time \([0, t]\) when \( \varepsilon \to 0 \). The mild time decay in Theorem 1.1 ensures the validity of (2.48) as long as \( \varepsilon < \varepsilon_0 \) is sufficiently small. This is a sharp contrast with the Landau case (see [74]), in which (2.48) only holds local in time.

Secondly, we discovered a new phenomenon related to the dissipation of the electromagnetic field. Through an intricate analysis of the macroscopic variables \( \|\nabla_x \mathbf{P}[f^\varepsilon]\|^2 \), we conclude that \( a_{\varepsilon}^x - a_{\varepsilon}^x, \, E_R^\varepsilon \) and \( \nabla_x (a_{\varepsilon}^x - a_{\varepsilon}^x), \, \nabla_x E_R^\varepsilon, \, \nabla_x B_R^\varepsilon \) belongs to the dissipation \( \mathcal{D} \). In the near-global-Maxwellian case (see [19, 98]), these dissipation terms are stronger than the energy \( \mathcal{E} \). However, in our near-local-Maxwellian, due to the Hilbert expansion, these dissipation terms are much weaker

\[
\varepsilon \left( \left\| a_{\varepsilon}^x - a_{\varepsilon}^x \right\|^2 + \left\| E_R^\varepsilon \right\|^2 \right) + \varepsilon^2 \left( \left\| \nabla_x (a_{\varepsilon}^x - a_{\varepsilon}^x) \right\|^2 + \left\| \nabla_x E_R^\varepsilon \right\|^2 + \left\| \nabla_x B_R^\varepsilon \right\|^2 \right) \leq \varepsilon \mathcal{E}, \tag{2.49}
\]

which can be absorbed by \( \mathcal{E} \) after integration w.r.t. time \( t \) for \( t \in [0, \varepsilon^{-k/(2k-1)}] \).

 Compared with the \( L^2-L^\infty \) framework as in [48, 49, 51], our new method has several advantages. Firstly, we don't require an explicit lower bound of the temperature \( T_0 \):

\[
T_M < \max_{t, x} T_0(t, x) < 2T_M \tag{2.50}
\]

for some constant \( T_M > 0 \). This extra restriction on \( T_0 \) is a technical requirement in the \( L^2-L^\infty \) method, and seems artificial from the physical viewpoint. Secondly, our method works for more general settings, including both the Landau type and cutoff/non-cutoff Boltzmann type equations in the relativistic frame. The \( L^2-L^\infty \) framework heavily relies on the analysis of the characteristic, which fails for the presence of the diffusion effect or complicated two-species VMB system.

Finally, we briefly discuss the possible applications of our new method. First, it is hopeful to apply this method to the relativistic non-cutoff Boltzmann equation. Due to the absence of momentum derivative estimate and weak dissipation of \( \mathcal{L} \), the estimation of nonlinear terms related to the electromagnetic field would be critical. Then, we can make use of our time-dependent exponential weight function and ideas in [94, 95]. Second, it is also very interesting to apply our method to deal with the bounded domain problem. Recently, Duan-Liu-Sakamoto-Strain [21] proved the global existence of mild solutions to the non-relativistic Landau equation and non-cutoff Boltzmann equation in solution spaces \( X_T := L_{\bar{k}}^1L_{\bar{\varepsilon}}^\infty L_{x1}^2 \) for \( x \in \mathbb{T}^3 \), and \( X_T := L_{\bar{k}}^1L_{\bar{\varepsilon}}^\infty L_{x1}^2 \) for \( x \in (-1, 1) \times \mathbb{T}^2 \), where \( \bar{k} \) and \( \bar{\varepsilon} \) are the corresponding variables after Fourier transformation w.r.t. \( x = (x_2, x_3) \), respectively. For the case \( x \in \mathbb{T}^3 \), replacing the present \( H_{\bar{k}}^2 \) space by \( L_{\bar{k}}^1 \) space, the extension of our method can be expected. For the case \( x \in (-1, 1) \times \mathbb{T}^2 \), while we are optimistic that the boundary layer analysis in [79, 80] can handle the specular-reflection boundary, the in-flow boundary with possible singularity might be more challenging.

This paper is organized as follows: in Section 3, we will present some preliminary lemmas regarding the linear and nonlinear Landau operators; in Section 4-6, we will prove the a priori estimate for the no-weight and weighted estimates as well as the macroscopic estimates; finally, in Section 7, we justify the main theorem.
3 Preliminary Lemmas

For convenience of later use, we list expressions and some estimates of the operators $A$, $K$ and $\Gamma$ defined in (2.17) and $\Gamma$ in (2.20), which are related to the local Maxwellian in (2.12). The detailed derivation is done in our work [74] about the Hilbert expansion of the relativistic Landau equation.

Corresponding to [83, Lemma 6], we have the following counterpart.

Lemma 3.1. For the operators $A$, $K$ and $\Gamma$, their explicit expressions are as follows:

\[ A f = \partial_{p_i} (\sigma^{ij} \partial_j f) - \frac{\sigma^{ij}}{4k BT_0^2} (u^0 \hat{p}_i - u_i) (u^0 \hat{p}_j - u_j) f + \frac{1}{2k BT_0} \partial_{p_i} \left( \sigma^{ij} (u^0 \hat{p}_j - u_j) \right) f, \]

\[ K f = \left( \frac{1}{1} \right) \left( \partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2k B T_0} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) M^{1/2}(p) M^{1/2}(q) \frac{u^0 q_j + u_j}{2k B T_0} \left( f(q) \cdot \left( \frac{1}{1} \right) \right) dq \quad (3.1) \]

\[ - \left( \frac{1}{1} \right) \left( \partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2k B T_0} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) M^{1/2}(p) M^{1/2}(q) \partial_{q_j} \left( f(q) \cdot \left( \frac{1}{1} \right) \right) dq, \]

and

\[ \Gamma(f, g) = \left( \partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2k B T_0} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) M^{1/2}(q) \partial_{p_j} f(p) \left( g(q) \cdot \left( \frac{1}{1} \right) \right) dq \]

\[ - \left( \partial_{p_i} - \frac{u^0 \hat{p}_i - u_i}{2k B T_0} \right) \int_{\mathbb{R}^3} \Phi^{ij}(p, q) M^{1/2}(q) f(p) \partial_{q_j} \left( g(q) \cdot \left( \frac{1}{1} \right) \right) dq. \]

Here we used $u^0 \hat{p}_i$, $u_i$ and $M^{1/2}$ to denote $\left( \frac{u^0 \hat{p}_i}{u^0 \hat{p} + i}, \frac{u^0 \hat{p} + i}{u^0 \hat{p} - i}, \frac{M^{1/2}_+}{M^{1/2}_-} \right)$, respectively.

Lemma 3.2 (Lemma 2.3 of [74]). The linearized collision operator $\mathcal{L}$ is self-adjoint in $L^2$. It satisfies

\[ (\mathcal{L} f, f) \gtrsim |(I - P)| f|^2 \sigma. \]

Lemma 3.3 (Lemma 2.5 of [74]). For $0 \leq m \leq 2$, one has

\[ (\mathcal{L} f, (w^m)^2 f) \gtrsim |w^m f|^2 \sigma - C |f|^2 \sigma. \]

Lemma 3.4 (Lemma 2.4&2.6 of [74]). For $0 \leq m \leq 2$, one has

\[ |(\Gamma(f, g), h)| \lesssim \left( |f|_{\sigma} |g|_{L^2} + |f|_{L^2} |g|_{\sigma} \right) |h|_{\sigma}, \]

and

\[ |(\Gamma(f, g), (w^m)^2 h)| \lesssim \left( |w^m f|_{\sigma} |g|_{L^2} + |w^m f|_{L^2} |g|_{\sigma} \right) |w^m h|_{\sigma}. \]

Remark 3.1. From the expressions of $\Gamma_+(f, g)$ and $\Gamma_-(f, g)$ in (2.20), functions $M^{1/2}_+ g_+, M^{1/2}_- g_-$ appear together in the convolution part with the collision kernel $\Phi$. Thanks to the exponential decay of $M_\pm$, the weighted estimate of $\Gamma(f, g)$ in (3.7) can be derived in the same way as the relativistic Landau equation in [74].

4 No-Weight Energy Estimates

In this section, we derive the $L^2$ energy estimates for the remainders $(f^\varepsilon, E^\varepsilon_R, B^\varepsilon_R)$. 

4.1 Basic $L^2$ Estimates

We first perform the $L^2$ energy estimate.

**Proposition 4.1.** For the remainders $(f^\varepsilon, E^\varepsilon_R, B^\varepsilon_R)$, it holds that

\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \sqrt{\frac{4\pi k B}{u^0}} f^\varepsilon \right\|^2 + \| E^\varepsilon_R \|^2 + \| B^\varepsilon_R \|^2 \right) + \frac{\delta}{\varepsilon} \|(I - P)[f^\varepsilon]\|_a^2 \\
\lesssim \left( 1 + t \right)^{-\beta_0 + \varepsilon} \mathcal{E} + \varepsilon^2 \mathcal{D} + \varepsilon^{2k+1} (1 + t)^{4k-2} + \varepsilon^k (1 + t)^{2k-2} \sqrt{\mathcal{E}}.
\]

**Proof.** From (2.16), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \sqrt{\frac{4\pi k B T_0}{u^0}} f^\varepsilon \right\|^2 + \| E^\varepsilon_R \|^2 + \| B^\varepsilon_R \|^2 \right) + \frac{\delta}{\varepsilon} \|(I - P)[f^\varepsilon]\|_a^2 \\
\leq \left| \left\langle \frac{f^\varepsilon}{\sqrt{M^\pm}} \left( \partial_t + c \hat{p}_\pm \cdot \nabla x \pm e_\pm (E^\varepsilon_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right) \sqrt{M^\pm}, \frac{4\pi k B T_0}{u^0} f^\varepsilon \right\rangle \right|
\]

\[
+ \frac{1}{2} \left| \left\langle \left( \partial_t + c \hat{p}_\pm \cdot \nabla x \right) \left[ \frac{T_0}{u^0_\pm} \right] f^\varepsilon_\pm, 4\pi k B f^\varepsilon_\pm \right\rangle \right|
\]

\[
+ \left| \left\langle \varepsilon e^\varepsilon \left( u_0^0 \hat{p}_\pm - u_\pm \right) \cdot \left( E^\varepsilon_0 + \hat{p}_\pm \times B^\varepsilon_0 \right) f^\varepsilon_\pm, \frac{2\pi f^\varepsilon_\pm}{u^0_\pm} \right\rangle \right|
\]

\[
+ \sum_{i=1}^{2k-1} \varepsilon^i \left| \left\langle e^\varepsilon \left\{ (E^\varepsilon_i + \hat{p}_\pm \times B_i) \cdot \nabla_p f^\varepsilon_\pm + (E^\varepsilon_R + \hat{p}_\pm \times B^\varepsilon_R) \cdot \frac{\nabla_p F^\varepsilon_\pm}{\sqrt{M^\pm}} \right\}, \frac{4\pi k B T_0}{u^0_\pm} f^\varepsilon_\pm \right\rangle \right|
\]

\[
+ \sum_{i=1}^{2k-1} \varepsilon^i \left| \left\langle e^\varepsilon \left\{ (E^\varepsilon_i + \hat{p}_\pm \times B_i) \cdot \left( u_0^0 \hat{p}_\pm - u_\pm \right) f^\varepsilon_\pm, \frac{2\pi f^\varepsilon_\pm}{u^0_\pm} \right\} \right\rangle \right|
\]

\[
+ \sum_{i=1}^{2k-1} \varepsilon^i \left| \left\langle e^\varepsilon \left\{ (E^\varepsilon_i + \hat{p}_\pm \times B_i) \cdot \left( u_0^0 \hat{p}_\pm - u_\pm \right) f^\varepsilon_\pm, \frac{2\pi f^\varepsilon_\pm}{u^0_\pm} \right\} \right\rangle \right| + \left| \left\langle \vec{S}, \frac{4\pi k B T_0}{u^0} f^\varepsilon \right\rangle \right|.
\]

Now we estimate each term on the R.H.S. of (4.3).

**First Term on the R.H.S. of (4.3):** Note that

\[
\frac{1}{\sqrt{M^\pm}} \left\{ \partial_t + c \hat{p}_\pm \cdot \nabla x \pm e_\pm (E^\varepsilon_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} \sqrt{M^\pm}
\]

are the first-order polynomials of $p$. Then, for a given sufficiently small positive constant $\kappa$, we have

\[
\left| \left\langle \frac{f^\varepsilon_\pm}{\sqrt{M^\pm}} \left[ \partial_t + c \hat{p}_\pm \cdot \nabla x \pm e_\pm (E^\varepsilon_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right] \sqrt{M^\pm}, \frac{4\pi k B T_0}{u^0_\pm} f^\varepsilon_\pm \right\rangle \right|
\]

\[
\lesssim \left( \| E^\varepsilon_0 \|_{L^\infty} + \| B_0 \|_{L^\infty} \right) \| f^\varepsilon \|^2 + \| \nabla x (u_0, u, T_0) \|_{L^\infty} \| \frac{p^0 f^\varepsilon}{u^0} \|^2
\]

\[= (4.1)\]
\[
\lesssim Z \left( \left\| \sqrt{p^0} P [f^\varepsilon] \right\|_2^2 + \int_{\mathbb{R}^3} p^0 |(I - P) [f^\varepsilon]|^2 + \int_{\mathbb{R}^3} \int_{p^0 \leq e^{-1}} p^0 |(I - P) [f^\varepsilon]|^2 \right) \\
\lesssim (1 + t)^{-\beta_0} \left\| f^\varepsilon \right\|^2 + o(1) e^{-1} \left\| (I - P) [f^\varepsilon] \right\|_\sigma^2 + \varepsilon \left\| (I - P) [f^\varepsilon] \right\|_\sigma^2.
\]

Here we have used (1.16) and \( \varepsilon (p^0_+)^2 \gtrsim \varepsilon (p^0_-)^2 \gtrsim p^0_\pm \) for \( p^0_\pm \gtrsim \varepsilon^{-1} \).

**Remark 4.1.** The decay estimate \( Z \lesssim (1 + t)^{-\beta_0} \) is crucial here.

**Second and Third Terms on the R.H.S. of (4.3):** From (1.16) again, we have
\[
\frac{1}{2} \left\{ \left( \partial_t + c \hat{p}_\pm \cdot \nabla x \right) \left( \frac{T_0}{u^0_{\pm}} \right) \left( f^\varepsilon_{\pm}, 4 \pi k_B f^\varepsilon_{\pm} \right) \right\} + \left\{ \mp e^\pm u^\pm \sqrt{M_{\pm}} \cdot \left( E^\varepsilon_{R} + \hat{p}_\pm \times B^\varepsilon_{R} \right) \frac{4 \pi}{u^0_{\pm}} f^\varepsilon_{\pm} \right\} \quad (4.6)
\]
\[
\lesssim \| \nabla_x (n_0, u, T_0) \|_{L^\infty} \left\| f^\varepsilon \right\|_2^2 + \| u \|_{L^\infty} \left\| f^\varepsilon \right\|_2 \left( \| E^\varepsilon_{R} \| + \| B^\varepsilon_{R} \| \right) \\
\lesssim o(1) e^{-1} \left\| (I - P) [f^\varepsilon] \right\|_\sigma^2 + (1 + t)^{-\beta_0} \left( \left\| f^\varepsilon \right\|_2^2 + \| E^\varepsilon_{R} \|_2^2 + \| B^\varepsilon_{R} \|_2^2 \right) .
\]

**Fourth Term on the R.H.S. of (4.3):** We use (3.5) to estimate it as
\[
\varepsilon^{k-1} \left\{ \Gamma \left[ f^\varepsilon, f^\varepsilon \right] , \frac{4 \pi k_B T_0}{u^0} \left( I - P \right) [f^\varepsilon] \right\} = \varepsilon^{k-1} \left\{ \Gamma \left[ f^\varepsilon, f^\varepsilon \right] , \frac{4 \pi k_B T_0}{u^0} \left( I - P \right) [f^\varepsilon] \right\} \quad (4.7)
\]
\[
\lesssim \varepsilon^{k-1} \int_{x \in \mathbb{R}^3} \left\| f^\varepsilon \right\|_2 \left\| f^\varepsilon \right\|_2 \left( \| I - P \| [f^\varepsilon] \right\|_\sigma + \| P [f^\varepsilon] \|_\sigma \right) \left( \| I - P \| [f^\varepsilon] \right\|_\sigma \left\| (I - P) [f^\varepsilon] \right\|_\sigma \\
\lesssim \left\| (I - P) [f^\varepsilon] \right\|_\sigma + \varepsilon \left\| P [f^\varepsilon] \right\|_\sigma \lesssim \left\| (I - P) [f^\varepsilon] \right\|_\sigma^2 + \varepsilon \left\| f^\varepsilon \right\|_2^2 .
\]

**Fifth Term on the R.H.S. of (4.3):** Similarly, considering that \( F_i \) decay fast in \( p \) by (A.7) in Proposition A.1, we have
\[
\left\{ \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\{ \Gamma \left[ \frac{F_i}{\sqrt{M}} , f^\varepsilon \right] + \Gamma \left[ f^\varepsilon , \frac{F_i}{\sqrt{M}} \right] , \frac{4 \pi k_B T_0}{u^0} f^\varepsilon \right\} \right\} \quad (4.8)
\]
\[
\lesssim o(1) e^{-1} \left\| (I - P) [f^\varepsilon] \right\|_\sigma^2 + \varepsilon \left\| f^\varepsilon \right\|_2^2 \lesssim o(1) e^{-1} \left\| (I - P) [f^\varepsilon] \right\|_\sigma^2 + \varepsilon \left\| P [f^\varepsilon] \right\|_\sigma^2 \\
\lesssim o(1) e^{-1} \left\| (I - P) [f^\varepsilon] \right\|_\sigma^2 + \varepsilon \left\| f^\varepsilon \right\|_2^2 .
\]

**Sixth Term on the R.H.S. of (4.3):** According to the assumption (2.43), its upper bound is
\[
\varepsilon^k \left( \| E^\varepsilon_{R} \|_H^2 + \| B^\varepsilon_{R} \|_H^2 \right) \left\| f^\varepsilon \right\|_2^2 \lesssim \varepsilon \left\| f^\varepsilon \right\|_2^2 .
\]

**Seventh Term on the R.H.S. of (4.3):** We use (A.7) in Proposition A.1 to obtain
\[
\left\{ \sum_{i=1}^{2k-1} \varepsilon^i \left\{ e_{\pm} \left( (E_i + \hat{p}_\pm \times B_i) \cdot \nabla_p f^\varepsilon_{\pm} + (E^\varepsilon_{R} + \hat{p}_\pm \times B^\varepsilon_{R}) \cdot \frac{\nabla_p F^\varepsilon_{\pm}}{\sqrt{M_{\pm}}} \right) , \frac{4 \pi k_B T_0}{u^0_{\pm}} f^\varepsilon_{\pm} \right\} \right\} \quad (4.10)
\]
\[
= \sum_{i=1}^{2k-1} \varepsilon^i \left\{ e_{\pm} \left( E^\varepsilon_{R} + \hat{p}_\pm \times B^\varepsilon_{R} \right) \cdot \frac{\nabla_p F^\varepsilon_{\pm} \frac{4 \pi k_B T_0}{u^0_{\pm}} f^\varepsilon_{\pm} \right\} \right\} \\
\lesssim \sum_{i=1}^{2k-1} \varepsilon^i \left\{ \frac{\nabla_p F^\varepsilon_{\pm}}{\sqrt{M_{\pm}}} \right\} \left( \| E^\varepsilon_{R} \| + \| B^\varepsilon_{R} \| \right) \| f^\varepsilon \|_2 \lesssim \sum_{i=1}^{2k-1} \varepsilon^i (1 + t)^{(i-1)} \left( \| f^\varepsilon \|_2^2 + \| E^\varepsilon_{R} \|_2^2 + \| B^\varepsilon_{R} \|_2^2 \right) \\
\lesssim \varepsilon \left( \| f^\varepsilon \|_2^2 + \| E^\varepsilon_{R} \|_2^2 + \| B^\varepsilon_{R} \|_2^2 \right) .
\]
Eighth and Ninth Terms on the R.H.S. of (4.3): Similarly, we estimate the last two terms as
\[
\left| \sum_{i=1}^{2k-1} \varepsilon^i \left( \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \left( u_0^0 \hat{p}_\pm - u_\pm \right) f_\pm^\varepsilon \right) \right| \leq o(1) \varepsilon^{-1} \| (I-P)[f^\varepsilon] \|^2_\sigma + C \varepsilon^2 \sum_{i+j \geq 2k+1} \varepsilon^{2(i+j-2k)} (1+t)^{2(i+j-2k)}
\]
and
\[
\varepsilon^i \left( S, \frac{4\pi k_B T_0}{u_0^0} f_\pm^\varepsilon \right) \leq o(1) \varepsilon^{-1} \| (I-P)[f^\varepsilon] \|^2_\sigma + C \varepsilon^2 \sum_{i+j \geq 2k+1} \varepsilon^{2(i+j-2k)} (1+t)^{2(i+j-2k)}
\]
Summary: We collect these estimates in (4.3), and multiply the resulting inequality to obtain (4.1).

4.2 First-Order Derivatives Estimates
In this part, we continue to perform the \(L^2\) energy estimates for the first-order derivatives of the remainders \(f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon\). To this end, we first apply \(\partial_x^\alpha (1 \leq |\alpha| \leq 2)\) to (2.15) to have
\[
\partial_x^\alpha \left( \left( \partial_t + c \hat{p}_\pm \cdot \nabla_x \pm \varepsilon \left( E_0 + \hat{p}_\pm \times B_0 \right) \cdot \nabla_p \right) \left[ f_\pm^\varepsilon \right] \right) \pm e_\pm \partial_x^\alpha \left( \frac{u_0^0}{k_B T_0} \hat{p}_\pm \sqrt{M} \cdot E_R^\varepsilon \right)
\]
and
\[
= \partial_x^\alpha \left( \frac{f_\pm^\varepsilon}{M} \right) \pm e_\pm \partial_x^\alpha \left( \frac{u_\pm \sqrt{M} \pm \varepsilon \left( E_0 + \hat{p}_\pm \times B_0 \right) \cdot \nabla_p \sqrt{M} \right) + \frac{\partial_x^\alpha L^\pm [f_\pm^\varepsilon]}{\varepsilon}
\]
Proposition 4.2. For the remainders \(f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon\), it holds that
\[
\varepsilon \left| \frac{d}{dt} \left( \left\| \sqrt{\frac{4\pi k_B T_0}{u_0^0}} \nabla_x f^\varepsilon \right\|^2 + \left\| \nabla_x E_R^\varepsilon \right\|^2 + \left\| \nabla_x B_R^\varepsilon \right\|^2 \right) + \frac{\delta}{\varepsilon} \left\| \nabla_x (I-P)[f^\varepsilon] \right\|^2_\sigma \right| \leq \left( 1 + t \right)^{-\beta_0 + \varepsilon} E + \varepsilon_0 \varepsilon^{-1} \| (I-P)[f^\varepsilon] \|^2_\sigma + \varepsilon D + \varepsilon^{2k+2} (1+t)^{4k-2} + \varepsilon^k (1+t)^{2k-2} \sqrt{E}.
\]
Proof. From (2.16), we have

$$
\frac{1}{2} \frac{d}{dt} \left( \|\partial_x E^x_R \| \right)^2 + \|\partial_x B^x_R \|^2 = -4\pi \left\langle \partial_x \left( e_+ \hat{p}_+ \sqrt{M_+} f^x_+ - e_- \hat{p}_- \sqrt{M_-} f^x_- \right), \partial_x E^x_R \right\rangle .
$$

(4.15)

Note that

$$
\mathcal{L}, \partial_x \right] [f^x] = \mathcal{L} \left[ \partial_x f^x \right] - \partial_x \left( \mathcal{L} [f^x] \right) = \mathcal{L} \left[ (I - P) [\partial_x f^x] \right] - \partial_x \left( \mathcal{L} \left[ (I - P) [f^x] \right] \right)
$$

(4.16)

Hence, naturally we have

$$
\varepsilon^{-1} \left\langle \mathcal{L}, \partial_x \right] [f^x], \frac{4\pi k_B T_0}{u^0_\pm} \partial_x f^x \right\rangle \lesssim \varepsilon^{-1} \left\langle \mathcal{L}, \frac{4\pi k_B T_0}{u^0_\pm} \partial_x [f^x], \frac{4\pi k_B T_0}{u^0_\pm} (I - P) [\partial_x f^x] \right\rangle + \varepsilon^{-1} \left\langle \mathcal{L}, \frac{4\pi k_B T_0}{u^0_\pm} (I - P) [f^x], \partial_x f^x \right\rangle .
$$

(4.17)

For the first term, we have

$$
\varepsilon^{-1} \left\langle \mathcal{L}, \partial_x \right] [f^x], \frac{4\pi k_B T_0}{u^0_\pm} \partial_x f^x \right\rangle \leq \varepsilon^{-1} \left\langle \mathcal{L}, \left[ (I - P) [f^x], \frac{4\pi k_B T_0}{u^0_\pm} (I - P) [\partial_x f^x] \right\rangle \right\rangle \lesssim o(1) \varepsilon^{-1} \left\| (I - P) [\partial_x f^x] \right\|^2_\sigma + \varepsilon^{-1} Z \left\| f^x \right\|^2
$$

$$
\lesssim o(1) \varepsilon^{-1} \left\| \partial_x (I - P) [f^x] \right\|^2_\sigma + \varepsilon^{-1} (1 + t)^{-\beta_0} \left\| f^x \right\|^2.
$$

(4.18)

For the second term, since \([\mathcal{L}, \partial_x] \) indicates that \( \partial_x \) only hits the Maxwellian in \( \mathcal{L} \) but not on \((I - P) [f^x] \), we directly bound

$$
\varepsilon^{-1} \left\langle \mathcal{L}, \left[ (I - P) [f^x], \frac{4\pi k_B T_0}{u^0_\pm} (I - P) [\partial_x f^x] \right\rangle \right\rangle \lesssim \varepsilon^{-1} Z \left\| (I - P) [f^x] \right\|_\sigma \left\| \partial_x f^x \right\|_\sigma
$$

(4.19)

\( \lesssim (1 + t)^{-\beta_0} \left\| \nabla_x f^x \right\|^2 + o(1) \varepsilon^{-1} \left\| (I - P) [\partial_x f^x] \right\|^2_\sigma + \varepsilon^{-2} \varepsilon^0 \left\| (I - P) [f^x] \right\|^2_\sigma .

Noting

$$
\left\| (I - P) [\partial_x f^x] \right\|_\sigma \geq \left\| \partial_x (I - P) [f^x] \right\|_\sigma - \left\| [\mathcal{L}, \partial_x] [f^x] \right\|_\sigma \geq \left\| \partial_x (I - P) [f^x] \right\|_\sigma - Z \left\| f^x \right\|,
$$

In total, we have

$$
\varepsilon^{-1} \left\langle \partial_x \mathcal{L} [f^x], \frac{4\pi k_B T_0}{u^0_\pm} \partial_x f^x \right\rangle \geq \varepsilon^{-1} \left\langle \mathcal{L} [\partial_x f^x], \frac{4\pi k_B T_0}{u^0_\pm} \nabla_x f^x \right\rangle - \varepsilon^{-1} \left\langle \left[ \mathcal{L}, \nabla_x \right] [f^x], \frac{4\pi k_B T_0}{u^0_\pm} \nabla_x f^x \right\rangle
$$

(4.20)

$$
\geq \varepsilon^{-1} \left\| (I - P) [\partial_x f^x] \right\|_\sigma^2 - C \varepsilon^0 \varepsilon^{-2} Z \left\| (I - P) [f^x] \right\|_\sigma^2 - C Z \left\| \nabla_x f^x \right\|^2 - \varepsilon^{-1} Z \left\| f^x \right\|^2 .
$$

$$
\geq \varepsilon^{-1} \left\| \partial_x (I - P) [f^x] \right\|_\sigma^2 - C \varepsilon^0 \varepsilon^{-2} \left\| (I - P) [f^x] \right\|_\sigma^2 - C \left( 1 + t \right)^{-\beta_0} \varepsilon^0 \varepsilon^{-1} \left\| \nabla_x f^x \right\|^2 + \varepsilon^{-1} \left\| f^x \right\|^2.
$$

Taking \( |\alpha| = 1 \) in (4.13) and denoting \( \partial_x \) as \( \partial_x, i = 1, 2, 3 \) for convenience, we multiply the equation by \( \frac{4\pi k_B T_0}{u^0_\pm} \partial_x f^x \), and use (4.15), (4.20) to have

$$
\frac{1}{2} \frac{d}{dt} \left( \left\| \frac{4\pi k_B T_0}{u^0_\pm} \partial_x f^x \right\|^2 + \left\| \partial_x E^x_{R} \right\|^2 + \left\| \partial_x B^x_{R} \right\|^2 \right) + \frac{\delta}{\varepsilon} \left\| \partial_x (I - P) [f^x] \right\|^2_\sigma
$$

(4.21)
Now we estimate each term on the R.H.S. of (4.21).

**First Two Terms on the R.H.S. of (4.21):** The two terms can be bounded by

\[
C\|\nabla x(n_0, u, T_0)\|_{L^\infty} \left( \|f^e\|_{H^1}^2 + \|E^e_R\|_{H^1}^2 \right) \lesssim (1 + t)^{-1/2} \left( \|f^e\|_{H^1}^2 + \|E^e_R\|_{H^1}^2 \right). \tag{4.22}
\]

**Third Term on the R.H.S. of (4.21):** For this term, we have

\[
\left| e^{\pm} \left\langle \partial_x \left( \left\{ (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} [f^e_\pm] \right) - \frac{4\pi \kappa B T_0}{u_0^\pm} \partial_x f^e_\pm \right\rangle \right| \lesssim \left( \|\nabla_x E_0\|_{L^\infty} + \|\nabla_x B_0\|_{L^\infty} \right) \|\partial_x f^e\| \|f^e\|_{\sigma} \lesssim (1 + t)^{-1/2} \|\nabla_x E_0\|_{H^1} + \|\nabla_x B_0\|_{H^1}. \tag{4.23}
\]

**Fourth Term on the R.H.S. of (4.21):** Using \( w_0^\pm \geq (p_0^\pm)^3 \), and noticing that for \( p_0^\pm \geq \varepsilon^{-1} \), we have \( \varepsilon (w_0^\pm)^2 \geq \varepsilon (p_0^\pm)^2 \geq p_0^\pm \). Then, for \( \kappa \) sufficiently small, we have

\[
\left| \left\langle \partial_x \left( \frac{f^e_\pm}{\sqrt{M^\pm}} \left\{ \partial_t + c \hat{p}_\pm \cdot \nabla_x \pm \varepsilon (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} \frac{1}{\sqrt{M^\pm}} \right) - \frac{4\pi \kappa B T_0}{u_0^\pm} \partial_x f^e_\pm \right\rangle \right| \lesssim Z \|p^0 \partial_x f^e\| + Z \|\langle p^0 \rangle^2 f^e\|, \tag{4.24}
\]

where

\[
Z \equiv \left( \|\sqrt{p} \partial_x f^e\| \right)^2 + Z \left( \|\sqrt{p^0} F^e\| \right)^2.
\]
\begin{align*}
\lesssim & \mathcal{Z} \left( \left\| \sqrt{p^0} \partial_x P[f^\varepsilon] \right\|^2 + \int_{\mathbb{R}^3} \int_{p^0 \leq \varepsilon^{-1}} p^0 \left\| \partial_x (I - P)[f^\varepsilon] \right\|^2 + \int_{\mathbb{R}^3} \int_{p^0 \geq \varepsilon^{-1}} p^0 \left\| \partial_x (I - P)[f^\varepsilon] \right\|^2 \right) \\
& + \mathcal{Z} \left( \left\| \sqrt{p^0} p^0 P[f^\varepsilon] \right\|^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (p^0)^3 \left\| (I - P)[f^\varepsilon] \right\|^2 \right) \\
\lesssim & \mathcal{Z} \left\| \partial_x f^\varepsilon \right\|^2 + o(1) \varepsilon^{-1} \left\| \partial_x (I - P)[f^\varepsilon] \right\|^2 + \varepsilon \left\| \partial_x (I - P)[f^\varepsilon] \right\|^2 + Z \left\| f^\varepsilon \right\|^2 + Z \left\| (I - P)[f^\varepsilon] \right\|_{\mathfrak{w}_0}^2 \\
\lesssim & o(1) \varepsilon^{-1} \left\| \partial_x (I - P)[f^\varepsilon] \right\|^2 + \varepsilon \left\| \nabla_x (I - P)[f^\varepsilon] \right\|^2_{\mathfrak{w}_0} + \varepsilon_0 \left\| (I - P)[f^\varepsilon] \right\|_{\mathfrak{w}_0}^2 + (1 + t)^{-\beta_0} \left\| f^\varepsilon \right\|_{H^1}^2.
\end{align*}

**Fifth and Sixth Terms on the R.H.S. of (4.21):** The fifth and sixth terms can be bounded by

\begin{align*}
C \mathcal{Z} \left( \left\| \partial_x f^\varepsilon \right\|^2 + \left( \left\| E_R^0 \right\|_{H^1} + \left\| B_R^\varepsilon \right\|_{H^1} \right) \left\| \partial_x f^\varepsilon \right\| \right) 
\leq C(1 + t)^{-\beta_0} \left( \left\| \partial_x f^\varepsilon \right\|^2 + \left\| E_R^0 \right\|_{H^1}^2 + \left\| B_R^\varepsilon \right\|_{H^1}^2 \right). \quad (4.25)
\end{align*}

**Seventh Term on the R.H.S. of (4.21):** Using Lemma 3.4 for \( p \) integral, \((\infty, 2, 2)\) or \((4, 4, 2)\) for \( x \) integral, and Sobolev embedding, we have

\begin{align*}
\left\| \varepsilon^{-k-1} \left\| \partial_x \Gamma f^\varepsilon, f^\varepsilon, \frac{4\pi k_B T_0}{u^0} \partial_x f^\varepsilon \right\| \right\| 
\lesssim & \varepsilon^{-k-1} \int_{x \in \mathbb{R}^3} \left( \left\| f^\varepsilon \right\| L_2 \left\| \nabla_x f^\varepsilon \right\|_{\sigma} + \left\| f^\varepsilon \right\|_{\sigma} \left\| \nabla_x f^\varepsilon \right\|_{L_2} \right) \left\| (I - P)[\nabla_x f^\varepsilon] \right\|_{\sigma} + Z \left\| f^\varepsilon \right\|_{L_2} \left\| f^\varepsilon \right\|_{\sigma} \left\| \nabla_x f^\varepsilon \right\|_{\sigma} \\
\lesssim & \varepsilon^{-k-1} \left\| f^\varepsilon \right\|_{H^2} \left\| f^\varepsilon \right\|_{H^3} \left\| (I - P)[\nabla_x f^\varepsilon] \right\|_{\sigma} + Z \varepsilon^{-k-1} \left\| f^\varepsilon \right\|_{H^2} \left\| f^\varepsilon \right\|_{\sigma} \left\| \nabla_x f^\varepsilon \right\|_{\sigma} \\
\lesssim & \varepsilon^{-k-1} \left\| f^\varepsilon \right\|_{H^3} \left\| (I - P)[\nabla_x f^\varepsilon] \right\|_{\sigma} + \varepsilon^{k+1} Z \left\| f^\varepsilon \right\|_{\sigma} \left\| \nabla_x f^\varepsilon \right\|_{\sigma}.
\end{align*}

Noting

\begin{align*}
\left\| f^\varepsilon \right\|_{H^3} \lesssim & \left\| (I - P)[f^\varepsilon] \right\|_{H^3} + \left\| P[f^\varepsilon] \right\|_{H^3} \lesssim \left\| (I - P)[f^\varepsilon] \right\|_{H^3} + \left\| f^\varepsilon \right\|_{H^1}.
\end{align*}

we use (4.19) to have

\begin{align*}
\left\| \varepsilon^{-k-1} \left\| \nabla_x \Gamma f^\varepsilon, f^\varepsilon, \frac{4\pi k_B T_0}{u^0} \nabla_x f^\varepsilon \right\| \right\| 
\lesssim & \varepsilon^{k+1} \left( \left\| (I - P)[f^\varepsilon] \right\|_{H^3} + \left\| f^\varepsilon \right\|_{H^1} \right) \left( \left\| \partial_x (I - P)[f^\varepsilon] \right\|_{\sigma} + Z \left\| f^\varepsilon \right\| \right) \\
& + \varepsilon^{k+1} Z \left( \left\| (I - P)[f^\varepsilon] \right\|_{\sigma} + \left\| f^\varepsilon \right\| \right) \left( \left\| (I - P)[\nabla_x f^\varepsilon] \right\|_{\sigma} + \left\| \nabla_x f^\varepsilon \right\| \right) \\
\lesssim & \left( \left\| (I - P)[f^\varepsilon] \right\|_{\sigma}^2 + \left\| (I - P)[\nabla_x f^\varepsilon] \right\|_{\sigma}^2 \right) + (1 + t)^{-\beta_0} \varepsilon_0 \left( \left\| f^\varepsilon \right\|^2 + \varepsilon \left\| \nabla_x f^\varepsilon \right\|^2 \right).
\end{align*}

**Eighth Term on the R.H.S. of (4.21):** Similarly, considering that \( F_i \) decay fast in \( p \), we know

\begin{align*}
\sum_{i=1}^{2k-1} \varepsilon^{i-1} \left\| \partial_x \Gamma \left[ M^{-\frac{1}{2}} F_i, f^\varepsilon \right] + \partial_x \Gamma \left[ f^\varepsilon, M^{-\frac{1}{2}} F_i \right], \frac{4\pi k_B T_0}{u^0} \partial_x f^\varepsilon \right\| 
\lesssim & o(1) \varepsilon^{-1} \left\| (I - P)[\nabla_x f^\varepsilon] \right\|_{\sigma}^2 + \left( \varepsilon + Z \right) \left\| f^\varepsilon \right\|_{H^3}^2 \\
\lesssim & o(1) \varepsilon^{-1} \left\| \nabla_x (I - P)[f^\varepsilon] \right\|_{\sigma}^2 + \left( 1 + t \right)^{-\beta_0} \varepsilon_0 + \varepsilon \left( \left\| (I - P)[f^\varepsilon] \right\|_{H^3}^2 + \varepsilon^{-1} \left\| f^\varepsilon \right\|^2 + \left\| f^\varepsilon \right\|_{H^1}^2 \right).
\end{align*}

**Other Terms on the R.H.S. of (4.21):** These terms can be estimated as follows:

\begin{align*}
\varepsilon^k \left\| \varepsilon \partial_x \left( \frac{u_0^0 \hat{p}_+ - u_+}{2k_B T_0} \cdot \left( E_R^0 + \hat{p}_+ \times B_R^0 \right) f^\varepsilon \right) + \frac{4\pi k_B T_0}{u^0} \partial_x f^\varepsilon \right\| 
\lesssim & \varepsilon^k \left\| \varepsilon \partial_x \left( \frac{u_0^0 \hat{p}_+ - u_+}{2k_B T_0} \cdot \left( E_R^0 + \hat{p}_+ \times B_R^0 \right) f^\varepsilon \right) + \frac{4\pi k_B T_0}{u^0} \partial_x f^\varepsilon \right\|.
\end{align*}

(4.30)
\[ \lesssim \varepsilon \left( \|E_R^\varepsilon\|_{L^2} + \|B_R^\varepsilon\|_{H^2} \right) \|f^\varepsilon\|_{H^1} \|\nabla_x f^\sigma\| \lesssim \varepsilon \left( \|f^\varepsilon\|_{H^1}^2 + \|\nabla_x (I - P)[f^\varepsilon]\|_{H^1}^2 \right), \]

\[ \lesssim \varepsilon^k \left( e_{\pm} \partial_x \left( \frac{4\pi k_B T_0}{u_0} \partial_x f^\varepsilon \right) \right) \]

\[ \lesssim \sum_{i=1}^{2k-1} \varepsilon^i \left( \|E_i\|_{L^1} + \|B_i\|_{L^1} \right) \|\nabla_x f^\varepsilon\| \|\nabla_x f^\varepsilon\| \]

\[ \lesssim \sum_{i=1}^{2k-1} \varepsilon^i (1 + t)^{i-1} \left( \|\nabla_x (I - P)[f^\varepsilon]\|_{L^1}^2 + \|f^\varepsilon\|_{H^1}^2 \right) \lesssim \varepsilon \left( \|\nabla_x (I - P)[f^\varepsilon]\|_{L^1}^2 + \|f^\varepsilon\|_{H^1}^2 \right), \]

\[ \lesssim \sum_{i=1}^{2k-1} \varepsilon^i (1 + t)^{i-1} \|f^\varepsilon\|_{H^1}^2 \lesssim \varepsilon \left( \|\nabla_x (I - P)[f^\varepsilon]\|_{L^1}^2 + \|f^\varepsilon\|_{H^1}^2 \right), \]

and

\[ \lesssim o(1) \varepsilon^{-1} \|\partial_x (I - P)[f^\varepsilon]\|_{L^1}^2 + \varepsilon^{2k+1}(1 + t)^{4k-2} + \varepsilon^k (1 + t)^{2k-2} \|\nabla_x f^\varepsilon\|. \]

**Summary:** By collecting all the above estimates in (4.21), we multiply the resulting inequality by \( \varepsilon \) to derive (4.14).

### 4.3 Second-Order Derivatives Estimates

In this subsection, we proceed to the \( L^2 \) estimate of \( \nabla_x^2 \left( f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon \right) \).

**Proposition 4.3.** For the remainders \( (f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon) \), it holds that

\[ \varepsilon^2 \frac{d}{dt} \left( \left\| \sqrt{\frac{4\pi k_B T_0}{u_0}} \nabla_x^2 f^\varepsilon \right\|^2 + \left\| \nabla_x^2 E_R^\varepsilon \right\|^2 + \left\| \nabla_x^2 B_R^\varepsilon \right\|^2 \right) + \frac{\delta}{\varepsilon} \left\| \nabla_x^2 (I - P)[f^\varepsilon] \right\|_{L^1}^2 \]

\[ \lesssim \left[ (1 + t)^{-\beta_0} + \varepsilon \right] \mathcal{E} + \varepsilon^{-1} \mathcal{E}_0 \left( (I - P)[f^\varepsilon] \right)^{2}_{H^1} + \varepsilon \mathcal{D} + \varepsilon^{2k+3}(1 + t)^{4k-2} + \varepsilon^{k+1}(1 + t)^{2k-2} \mathcal{E}. \]
Proof. From (2.16), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \| \partial_x^2 E_R \|^2 + \| \partial_x^2 B_R \|^2 \right) = -4\pi \left( \partial_x^2 \left( e_+ \hat{p}_+ \sqrt{M_+} f_+^e - e_- \hat{p}_- \sqrt{M_-} f_-^e \right), \partial_x^2 E_R \right). \tag{4.37}
\]
Using a similar argument as in Proposition 4.2, we have
\[
\varepsilon^{-1} \left| \left[ \mathcal{L}, \partial_x^2 \right] [f^e], \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right| \leq \varepsilon^{-1} \left| \left[ \mathcal{L}, \left[ P, \partial_x^2 \right] [f^e], \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right] \right| + \varepsilon^{-1} \left| \left[ \mathcal{L}, \partial_x^2 \right] [(I - P)[f^e]], \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right| \tag{4.38}
\]
For the first term in (4.38), we have
\[
\varepsilon^{-1} \left| \left[ \mathcal{L}, \left[ P, \partial_x^2 \right] [f^e], \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right] \right| = \varepsilon^{-1} \left| \left[ \mathcal{L}, \left[ P, \partial_x^2 \right] [f^e], (I - P)[\partial_x^2 f^e] \right] \right|. \tag{4.39}
\]
Note that \([P, \partial_x^2]\) only contains terms that \(\partial_x\) hits \(f^e\) at most once. Hence, we have
\[
\varepsilon^{-1} \left| \left[ \mathcal{L}, \left[ P, \partial_x^2 \right] [f^e], \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right] \right| = \varepsilon^{-1} \left| \mathcal{L} \left[ \left[ P, \partial_x^2 \right] [f^e] \right], (I - P)[\partial_x^2 f^e] \right| \leq o(1) \varepsilon^{-1} \left| (I - P)[\partial_x^2 f^e] \right|_\sigma + \varepsilon^{-1} \mathcal{L} \left( f^e \right) \left( f^e \right)_{H^1} \tag{4.40}
\]
For the second term in (4.38), since \([\mathcal{L}, \partial_x^2]\) indicates that \(\partial_x\) at most hits \((I - P)[f^e]\) once, we directly bound it as follows
\[
\varepsilon^{-1} \left| \left[ \mathcal{L}, \partial_x^2 \right] [(I - P)[f^e]], \partial_x^2 f^e \right| \leq o(1) \varepsilon^{-1} \left| \partial_x^2 (I - P)[f^e] \right|_\sigma + \varepsilon^{-1} \mathcal{L} \left( f^e \right) \left( f^e \right)_{H^1} \tag{4.41}
\]
In total, we have
\[
\varepsilon^{-1} \left| \partial_x^2 \mathcal{L} [f^e], \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right| \leq o(1) \varepsilon^{-1} \left| \partial_x^2 (I - P)[f^e] \right|_\sigma + \varepsilon^{-1} \mathcal{L} \left( f^e \right) \left( f^e \right)_{H^1} \tag{4.43}
\]
Taking \(|\alpha| = 2\) in (4.13) and denoting \(\partial_x^2\) as \(\partial_{x_i} \partial_{x_j}\) with \(i, j = 1, 2, 3\) for convenience, we multiply the equation by \(\frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e\), and further use (4.37), (4.43) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right\|^2 + \left\| \partial_x^2 E_R \right\|^2 + \left\| \partial_x^2 B_R \right\|^2 \right) + \frac{\delta}{\varepsilon} \left| \partial_x^2 (I - P)[f^e] \right|_\sigma \leq \sum_{|\alpha| < 2} \left| e_{+\alpha} \partial_x^2 f^e \right| \left( \frac{u^0}{k_B T_0} \hat{p}_\pm \sqrt{M_\pm} \right) \cdot \partial_x^2 \mathcal{L} \left[ \partial_x^2 \right] [f^e], \frac{4\pi k_B T_0}{u^0} \partial_x^2 f^e \right| \tag{4.44}
\]
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Now we estimate each term on the R.H.S. of (4.44).

First Two Terms on the R.H.S. of (4.44): The two terms can be bounded by

\[ C \mathcal{Z} \left( \|f^\varepsilon\|_{L^2}^2 + \|E_R^\varepsilon\|_{L^2}^2 \right) \lesssim (1 + t)^{-\delta_0} \left( \|f^\varepsilon\|_{L^2}^2 + \|E_R^\varepsilon\|_{L^2}^2 \right). \]  

(4.45)

Third Term on the R.H.S. of (4.44): We can obtain

\[ \left| e_\pm \left\langle \hat{\partial}_x^2 \left( \frac{f^\varepsilon}{\sqrt{M}} \left\{ \partial_t + \hat{c}_\pm \cdot \nabla_x \pm \varepsilon (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} \left[ \sqrt{M} \right] \right) , \frac{4\pi k_B T_0}{u_0} \hat{\partial}_x^2 f^\varepsilon \right\rangle \right| \]

\[ \lesssim \mathcal{Z} \left( \left\| \hat{\partial}_x^2 f^\varepsilon \right\| \left\| \nabla_x f^\varepsilon \right\|_{L^2} \lesssim (1 + t)^{-\delta_0} \left( \left\| \hat{\partial}_x^2 f^\varepsilon \right\|^2 + \| \nabla_x (I - P) [f^\varepsilon] \|^2_{L^2} \right) + \| f^\varepsilon \|^2_{w_{0,\sigma}} + \mathcal{Z} \| \partial_x f^\varepsilon \|^2_{w_{1,\sigma}} \right) \]  

(4.46)

Fourth Term on the R.H.S. of (4.44): Noting that \( \varepsilon (w_\pm^2)^2 \gtrsim \varepsilon (p_\pm^0)^2 \gtrsim p_\pm^0 \) for \( \varepsilon_\pm \gtrsim \varepsilon^{-1} \), we use similar arguments in (4.5) to have

\[ \left| \left\langle \hat{\partial}_x^2 \left( \frac{f^\varepsilon}{\sqrt{M}} \left\{ \partial_t + \hat{c}_\pm \cdot \nabla_x \pm \varepsilon (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} \left[ \sqrt{M} \right] \right) , \frac{4\pi k_B T_0}{u_0} \hat{\partial}_x^2 f^\varepsilon \right\rangle \right| \]

\[ \lesssim \mathcal{Z} \left( \left\| \hat{\partial}_x^2 f^\varepsilon \right\|_{L^2} + \| f^\varepsilon \|_{L^2} \right) + \mathcal{Z} \left( \left\| \hat{\partial}_x^2 f^\varepsilon \right\|_{L^2} + \| f^\varepsilon \|_{L^2} \right) \]  

(4.47)
\[ + Z \| w_0 (I - P)[f^\varepsilon] \|_{\sigma}^2 + Z \| w_1 \partial_x (I - P)[f^\varepsilon] \|_{\sigma}^2 + Z \| f^\varepsilon \|_{H^1}^2 \]
\[ \lesssim o(1) \varepsilon^{-1} \| \partial_x^2 (I - P)[f^\varepsilon] \|_{\sigma}^2 + \varepsilon \varepsilon_0 \| \nabla_x^2 (I - P)[f^\varepsilon] \|_{w^2}^2 + (1 + t)^{-\beta_0} \| \nabla_x^2 f^\varepsilon \|^2 \\
+ \varepsilon \varepsilon_0 \left( \| w_0 (I - P)[f^\varepsilon] \|_{\sigma}^2 + \| w_1 \nabla_x (I - P)[f^\varepsilon] \|_{\sigma}^2 \right) + (1 + t)^{-\beta_0} \| f^\varepsilon \|_{H^1}^2 . \]

**Fifth and Sixth Terms on the R.H.S. of (4.44):** The upper bound of the two terms is
\[
C Z \left( \| \partial_x^2 f^\varepsilon \|^2 + \| E_R^\varepsilon \|_{H^2}^2 + \| B_R^\varepsilon \|_{H^2}^2 \right) \lesssim \left( 1 + t \right)^{-\beta_0} \left( \| \partial_x^2 f^\varepsilon \|^2 + \| E_R^\varepsilon \|_{H^2}^2 + \| B_R^\varepsilon \|_{H^2}^2 \right). \tag{4.48} \]

**Seventh Term on the R.H.S. of (4.44):** Using Lemma 3.4 for \( p \) integral, \((\infty, 2, 2)\) or \((4, 4, 2)\) for \( x \) integral, and Sobolev embedding, we have
\[
\left| \varepsilon^{k-1} \left< \partial_x^2 \Gamma[f^\varepsilon, f^\varepsilon], \frac{4\pi k_B T_0}{u_0} \partial_x^2 f \right> \right| \lesssim \left| \varepsilon^{k-1} \left< \partial_x^2 \Gamma[f^\varepsilon, f^\varepsilon] + \Gamma[f^\varepsilon, \partial_x^2 f^\varepsilon] + \partial_x^2 \Gamma[f^\varepsilon, \partial_x f^\varepsilon], \frac{4\pi k_B T_0}{u_0} (I - P)[\partial_x^2 f] \right> \right| \]
\[ + Z \left| \varepsilon^{k-1} \left< \partial_x^2 \Gamma[f^\varepsilon, f^\varepsilon] + \Gamma[f^\varepsilon, \partial_x f^\varepsilon], \frac{4\pi k_B T_0}{u_0} \partial_x^2 f \right> \right| + Z \left| \varepsilon^{k-1} \left< \Gamma[f^\varepsilon, f^\varepsilon], \frac{4\pi k_B T_0}{u_0} \partial_x^2 f \right> \right| \]
\[ \lesssim \varepsilon^{k-1} \int_{x \in \mathbb{R}^3} \left( \| \nabla_x^2 f^\varepsilon \|_{\sigma} \| f^\varepsilon \|_{L^2} + \| \nabla_x^2 f^\varepsilon \|_{\sigma} \| f^\varepsilon \|_{\sigma} + \| \nabla_x f^\varepsilon \|_{\sigma} \| \nabla_x f^\varepsilon \|_{L^2} \right) \left\| (I - P)[\partial_x^2 f^\varepsilon] \right\|_{\sigma} \]
\[ + Z \left( \| \partial_x f^\varepsilon \|_{\sigma} \| f^\varepsilon \|_{L^2} + \| \partial_x f^\varepsilon \|_{\sigma} \| f^\varepsilon \|_{\sigma} + \| \partial_x f^\varepsilon \|_{\sigma} \| \nabla_x f^\varepsilon \|_{L^2} \right) \left\| \partial_x^2 f^\varepsilon \right\|_{\sigma} \]
\[ \lesssim \varepsilon^{k-1} \| f^\varepsilon \|_{H^2} \| f^\varepsilon \|_{H^2} \| (I - P)[\nabla_x^2 f^\varepsilon] \|_{\sigma} + \varepsilon^{k-1} Z \| f^\varepsilon \|_{H^2} \| f^\varepsilon \|_{H^2} \| \nabla_x^2 f^\varepsilon \|_{\sigma} \]
\[ \lesssim \varepsilon^{k-1} \| f^\varepsilon \|_{H^2} \left( \| \nabla_x^2 (I - P)[f^\varepsilon] \|_{\sigma} + Z \| f^\varepsilon \|_{H^1} \right) + \varepsilon^2 Z \| f^\varepsilon \|_{H^2} \| \partial_x^2 f^\varepsilon \|_{\sigma} \]
\[ \lesssim \varepsilon_0 \| (I - P)[f^\varepsilon] \|_{H^2}^2 + \left[ \varepsilon + (1 + t)^{-\beta_0} + \varepsilon \right] \| f^\varepsilon \|_{H^2}^2 . \]

**Eighth Term on the R.H.S. of (4.44):** Based on a similar argument as the seventh term, we have
\[
\sum_{i=1}^{2k-1} \varepsilon^{i-1} \left< \partial_x^2 \Gamma \left[ \frac{F_i}{\sqrt{M}}, f^\varepsilon \right] + \partial_x^2 \Gamma \left[ \frac{F_i}{\sqrt{M}}, \frac{4\pi k_B T_0}{u_0} \partial_x^2 f^\varepsilon \right] \right> \lesssim o(1) \varepsilon^{-1} \| (I - P)[f^\varepsilon] \|_{H^2}^2 + \left[ \varepsilon + (1 + t)^{-\beta_0} \right] \| f^\varepsilon \|_{H^2}^2 . \tag{4.50} \]

**Other Terms on the R.H.S. of (4.44):** These terms can be estimated as follows:
\[
\varepsilon^k \left| \varepsilon \partial_x^2 \left( \frac{u_0^\varepsilon p_{\varepsilon} - u_0^\varepsilon}{2 k_B T_0} \cdot (E_R^\varepsilon + \hat{p}_{\varepsilon} \times B_R^\varepsilon)[f^\varepsilon] \right), \frac{4\pi k_B T_0}{u_0^\varepsilon} \partial_x^2 f_{\varepsilon}^\varepsilon \right| \lesssim \varepsilon^k \left( \| E_R^\varepsilon \|_{H^2} + \| B_R^\varepsilon \|_{H^2} \right) \| \nabla_x^2 f^\varepsilon \|_{\sigma}^2 \lesssim \varepsilon \left( \| \nabla_x^2 f^\varepsilon \|_{\sigma}^2 + \| E_R^\varepsilon \|_{H^2}^2 + \| B_R^\varepsilon \|_{H^2}^2 \right), \tag{4.51} \]
\[
\varepsilon^k \left| \varepsilon \partial_x^2 \left( (E_R^\varepsilon + \hat{p}_{\varepsilon} \times B_R^\varepsilon) \cdot \nabla f_{\varepsilon}^\varepsilon \right), \frac{4\pi k_B T_0}{u_0^\varepsilon} \partial_x^2 f_{\varepsilon}^\varepsilon \right| \lesssim \varepsilon^k \left( \| E_R^\varepsilon \|_{H^2} + \| B_R^\varepsilon \|_{H^2} \right) \| \nabla_x^2 f^\varepsilon \|_{\sigma} \| \nabla_x f^\varepsilon \|_{H^1} \lesssim \varepsilon \left( \| \nabla_x^2 f^\varepsilon \|_{H^1}^2 + \| \nabla_x (I - P)[f^\varepsilon] \|_{H^3}^2 \right), \tag{4.52} \]
\[
\sum_{i=1}^{2k-1} \varepsilon^i \left| \varepsilon \partial_x^2 \left( (E_i + \hat{p}_i \times B_i) \cdot \nabla f_{\varepsilon}^i \right), \frac{4\pi k_B T_0}{u_0^\varepsilon} \partial_x^2 f_{\varepsilon}^i \right| \lesssim \varepsilon^k \left( \| E_i \|_{H^2} + \| B_i \|_{H^2} \right) \| \nabla_x^2 f^\varepsilon \|_{\sigma} \| \nabla_x f^\varepsilon \|_{H^1} \lesssim \varepsilon \left( \| \nabla_x^2 f^\varepsilon \|_{H^1}^2 + \| \nabla_x (I - P)[f^\varepsilon] \|_{H^3}^2 \right), \tag{4.53} \]
\[
\lesssim \sum_{i=1}^{2k-1} \varepsilon^i \left( \| E_i \|_{W^{1,\infty}} + \| B_i \|_{W^{1,\infty}} \right) \| \nabla_x f^\varepsilon \|_{H^2} \| \nabla_x^2 f^\varepsilon \|
\]
\[
\lesssim \varepsilon \left( \| \nabla_x (1 - P) [f^\varepsilon] \|_{H^2}^2 + \| \nabla_x f^\varepsilon \|_{H^2}^2 \right),
\]
\[
\left| \sum_{i=1}^{2k-1} \varepsilon^i \left( e_\pm \partial_x^2 \left( (E_i^\varepsilon + \hat{p}_\pm \times B_i^\varepsilon) \cdot \frac{\nabla_p F_{\pm,i}}{\sqrt{M_{\pm}}} \right), \frac{4\pi k_B T_0}{u_0^\varepsilon} \partial_x^2 f^\varepsilon \right) \right| \quad (4.54)
\]
\[
\lesssim \sum_{i=1}^{2k-1} \varepsilon^i (1 + t)^{-i-1} \left( \| E_i^\varepsilon \|_{H^2}^2 + \| B_i^\varepsilon \|_{H^2}^2 + \| \nabla_x^2 f^\varepsilon \|_2^2 \right)
\]
\[
\lesssim \varepsilon \left( \| \nabla_x^2 f^\varepsilon \|_2^2 + \| E_i^\varepsilon \|_{H^2}^2 + \| B_i^\varepsilon \|_{H^2}^2 \right),
\]
and
\[
\varepsilon^k \left| \left( \partial_x^2 S, \frac{4\pi k_BT_0}{u_0^\varepsilon} \partial_x^2 f^\varepsilon \right) \right| \quad (4.56)
\]
\[
\leq o(1) \varepsilon^{-1} \left( \| \partial_x^2 (1 - P) [f^\varepsilon] \|_2^2 + C \varepsilon^{2k+1} (1 + t)^{4k-2} + C \varepsilon^k (1 + t)^{2k-2} \left\| \nabla_x^2 f^\varepsilon \right\| .
\]

Summary: We collect the above estimates in (4.44), and multiply the resulting inequality by \( \varepsilon^2 \) to derive (4.36) \( \square \)

5 Weighted Energy Estimates

In this section, we are devoted to the weighted energy estimates of the remainder term \( f^\varepsilon \).

5.1 Weighted Estimate

Apply microscopic projection \( (I - P_\pm) \) onto (2.15) to have

\[
\{ \partial_t + c_\pm \cdot \nabla_x \pm e_\pm (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \} \left( f^\varepsilon_\pm \right) \quad (5.1)
\]
\[
\pm e_\pm (I - P_\pm) \left[ \left( E_{R}^\varepsilon + \hat{p}_\pm \times B_{R}^\varepsilon \right) \cdot \frac{-u_0^\varepsilon \hat{p}_\pm + u_\pm \sqrt{M_{\pm}}}{k_BT_0} \right] + \mathcal{L}_\pm [f^\varepsilon_\pm] = - \frac{(I - P_\pm) [f^\varepsilon_\pm]}{\sqrt{M_{\pm}}} \left\{ \partial_t + c_\pm \cdot \nabla_x \pm e_\pm (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} \sqrt{M_{\pm}} + \varepsilon^{k-1} \Gamma_\pm [f^\varepsilon, f^\varepsilon]
\]
\[
+ \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left( \Gamma_\left[ \left[ F_i / \sqrt{M}, f^\varepsilon \right] + \Gamma_\left[ f^\varepsilon, F_i / \sqrt{M} \right] \right) \mp e_\pm \varepsilon^k (E_{R}^\varepsilon + \hat{p}_\pm \times B_{R}^\varepsilon) \cdot \nabla_p (I - P_\pm) \left[ f^\varepsilon_\pm \right]
\]
\[
\pm e_\pm \varepsilon^k \frac{1}{2k_BT_0} (u_0^\varepsilon \hat{p}_\pm - u_\pm) \cdot (E_{R}^\varepsilon + \hat{p}_\pm \times B_{R}^\varepsilon) (I - P_\pm) \left[ f^\varepsilon_\pm \right].
\]
\[ e_{\pm} \sum_{i=1}^{2k-1} \varepsilon^i (E_i + \hat{p}_{\pm} \times B_i) \cdot \nabla_p (I - P_{\pm}) [f_{\pm}^\varepsilon] \]
\[ + e_{\pm} \sum_{i=1}^{2k-1} \varepsilon^i (I - P_{\pm}) \left( (E_R^\varepsilon + \hat{p}_{\pm} \times B_R^\varepsilon) \cdot \frac{\nabla_p F_{\pm, i}}{\sqrt{M_{\pm}}} \right) \]
\[ \pm e_{\pm} \sum_{i=1}^{2k-1} \varepsilon^i \left( (E_i + \hat{p}_{\pm} \times B_i) \cdot \frac{1}{2k_B T_0} (u_0^0 \hat{p}_{\pm} - u_{\pm}) (I - P_{\pm}) [f_{\pm}] \right) \]
\[ + e_k (I - P) [\hat{S}]_\pm + \|[P, \tau_{\pm, B}] f_{\pm}^\varepsilon, \]

where \([P, \tau_{\pm, B}] = P\tau_{\pm, B} - \tau_{\pm, B} P\) denotes the commutator of two operators \(P\) and \(\tau_{\pm, B}\) where

\[ \tau_{\pm, B} := \partial_t + c \hat{p}_{\pm} \cdot \nabla_x \pm e_{\pm} (E_0 + \hat{p}_{\pm} \times B_0) \cdot \nabla_p \]
\[ + \frac{1}{\sqrt{M_{\pm}}} \left\{ \partial_t + c \hat{p}_{\pm} \cdot \nabla_x \pm e_{\pm} (E_0 + \hat{p}_{\pm} \times B_0) \cdot \nabla_p \right\} \sqrt{M_{\pm}} \]
\[ \pm e_{\pm} \varepsilon^k \frac{1}{2k_B T_0} \left( u_0^0 \hat{p}_{\pm} - u_{\pm} \right) (E_R^\varepsilon + \hat{p}_{\pm} \times B_R^\varepsilon) \pm e_{\pm} (E_R^\varepsilon + \hat{p}_{\pm} \times B_R^\varepsilon) \cdot \nabla_p \]
\[ \pm e_{\pm} \sum_{i=1}^{2k-1} \varepsilon^i \left( (E_i + \hat{p}_{\pm} \times B_i) \cdot \left[ \nabla_p - \frac{1}{2k_B T_0} (u_0^0 \hat{p}_{\pm} - u_{\pm}) \right] \right). \]

**Proposition 5.1.** For the remainders \((f_{\pm}^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)\), it holds that

\[ \frac{d}{dt} \| (I - P) [f_{\pm}^\varepsilon] \|^2_{w_0} + \frac{\delta}{\varepsilon} \| (I - P) [f_{\pm}^\varepsilon] \|^2_{w_0, \sigma} + Y \left\| \sqrt{p^0} (I - P) [f_{\pm}^\varepsilon] \right\|^2_{w_0} \]
\[ \lesssim \frac{1}{\varepsilon} \| (I - P) [f_{\pm}^\varepsilon] \|_{w_0}^2 + \varepsilon \| \nabla_x f_{\pm}^\varepsilon \|^2 + \varepsilon (E + D) + \varepsilon^{2k+1}(1 + t)^{2k-2}. \]

**Proof.** Noting

\[ (w_0^0)^2 (I - P_{\pm}) [f_{\pm}^\varepsilon] \cdot \partial_t \{ (I - P_{\pm}) [f_{\pm}^\varepsilon] \} = \frac{1}{2} \partial_t |w_0^0 (I - P_{\pm}) [f_{\pm}^\varepsilon]|^2 + p_0^2 + \varepsilon (w_0^0)^2 |(I - P_{\pm}) [f_{\pm}^\varepsilon]|^2, \]

we take the \(L^2\) inner product of (5.1) with \((w_0^0)^2 (I - P_{\pm}) [f_{\pm}^\varepsilon]\) and use (3.3) to have

\[ \frac{1}{2} \frac{d}{dt} \| (I - P) [f_{\pm}^\varepsilon] \|^2_{w_0} + \frac{\delta}{\varepsilon} \| (I - P) [f_{\pm}^\varepsilon] \|^2_{w_0, \sigma} + Y \left\| \sqrt{p^0} (I - P) [f_{\pm}^\varepsilon] \right\|^2_{w_0} \]
\[ \lesssim \frac{C}{\varepsilon} \| (I - P) [f_{\pm}^\varepsilon] \|_{w_0}^2 + \left| \langle e_{\pm} (E_0 \mp \hat{p}_{\pm} \times B_0) \cdot \nabla_p (I - P_{\pm}) [f_{\pm}^\varepsilon], (w_0^0)^2 (I - P_{\pm}) [f_{\pm}^\varepsilon] \rangle \right| \]
\[ + \left| \langle e_{\pm} (I - P_{\pm}) \left[ (E_R^\varepsilon + \hat{p}_{\pm} \times B_R^\varepsilon) \cdot \frac{-u_0^0 \hat{p}_{\pm} + u_{\pm}}{k_B T_0} \sqrt{M_{\pm}}, (w_0^0)^2 (I - P_{\pm}) [f_{\pm}^\varepsilon] \right) \rangle \right| \]
\[ + \left| \left\langle \left\{ \partial_t + c \hat{p}_{\pm} \cdot \nabla_x \pm e_{\pm} (E_0 + \hat{p}_{\pm} \times B_0) \cdot \nabla_p \right\} \left[ \sqrt{M_{\pm}}, (w_0^0)^2 (I - P_{\pm}) [f_{\pm}^\varepsilon] \right] \right\rangle \right| \]
\[ + \varepsilon^{k-1} \left| \langle \Gamma [f_{\pm}^\varepsilon, f_{\pm}^\varepsilon], (w_0^0)^2 (I - P) [f_{\pm}^\varepsilon] \rangle \right| \]
\[ + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \left| \langle \Gamma \left[ \frac{F_i}{\sqrt{M}}, f_{\pm}^\varepsilon \right] + \Gamma \left[ f_{\pm}^\varepsilon, \frac{F_i}{\sqrt{M}} \right], (w_0^0)^2 (I - P) [f_{\pm}^\varepsilon] \right| \]
\[ + \varepsilon^k \left| \varepsilon^i \left( \frac{u_0^0 \hat{p}_{\pm} - u_{\pm}}{2k_B T_0} \cdot (E_R^\varepsilon + \hat{p}_{\pm} \times B_R^\varepsilon), (w_0^0)^2 (I - P_{\pm}) [f_{\pm}^\varepsilon] \right| \right| \]
and Sobolev embedding, it can be controlled by
\[
\left\langle \left( \partial_t + c \hat{p}_\pm \cdot \nabla_x \pm \epsilon_0 + (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right) \left( \sqrt{\frac{M_+}{M_\pm}} \right), \right| \left| w_0^0 (I - P) \left[ f^{\frac{\epsilon}{\omega}} \right] \right| \right\rangle
\]
(5.8)

Then, we have
\[
\left\langle \left\{ \partial_t + c \hat{p}_\pm \cdot \nabla_x \pm \epsilon_\pm (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla_p \right\} \left( \sqrt{\frac{M_+}{M_\pm}} \right), \right| \left| w_0^0 (I - P) \left[ f^{\frac{\epsilon}{\omega}} \right] \right| \right\rangle
\]
(5.9)

Fifth Term on the R.H.S. of (5.5): Using (3.7) in Lemma 3.4 and Sobolev embedding, it can be controlled by
\[
\mathcal{Z} \lesssim (1 + t)^{-\beta_0} \ll Y.
\]
(5.7)

Second Term on the R.H.S. of (5.5): It is bounded by
\[
C \left( \| E_0 \|_{L^\infty} + \| B_0 \|_{L^\infty} \right) \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0}^2 \lesssim (1 + t)^{-\beta_0} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0}^2.
\]
(5.6)

Third Term on the R.H.S. of (5.5): Noting (2.35), we bound it by
\[
C \left( \| E_0^R \| + \| B_0^R \| \right) \| (I - P) [f^{\frac{\epsilon}{\omega}}] \| \lesssim \frac{1}{\epsilon} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|^2 + \epsilon \left( \| E_0^R \|^2 + \| B_0^R \|^2 \right).
\]

Fourth Term on the R.H.S. of (5.5): By the smallness of \( \bar{\epsilon}_0 \), it holds that
\[
\mathcal{Z} \lesssim (1 + t)^{-\beta_0} \ll Y.
\]

Sixth Term on the R.H.S. of (5.5): Similarly, we use (2.35), (3.7) and (A.7) in Proposition A.1 to obtain
\[
\lesssim o(1) \epsilon^{-1} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2 + \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma}^2 - \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{H^2_{\omega, \sigma}}^2 + \epsilon^2 \| f^{\frac{\epsilon}{\omega}} \|_{H^2_{\omega, \sigma}}^2.
\]

\[
\lesssim \left( \frac{F_i}{\sqrt{M}} \right) \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma} + \left( \frac{F_i}{\sqrt{M}} \right) \| f^{\frac{\epsilon}{\omega}} \|_{H^2_{\omega, \sigma}} \right) \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2
\]
(5.9)

\[
\lesssim o(1) \epsilon^{-1} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2 + \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma}^2 + o(1) \epsilon^{-1} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2 + \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma}^2.
\]

\[
\lesssim o(1) \epsilon^{-1} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2 + \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma}^2 + o(1) \epsilon^{-1} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2 + \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma}^2.
\]

\[
\lesssim o(1) \epsilon^{-1} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2 + \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma}^2 + o(1) \epsilon^{-1} \| (I - P) [f^{\frac{\epsilon}{\omega}}] \|_{w^0, \sigma}^2 + \epsilon \| f^{\frac{\epsilon}{\omega}} \|_{w^0, \sigma}^2.
\]
Seventh and eighth Terms on the R.H.S. of (5.5): Its upper bound is
\[
C\varepsilon^k \left( \|E_R\| + \|B_R\| \right)_{H^2} \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma} \lesssim \varepsilon \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma}. \tag{5.10}
\]

Other Terms on the R.H.S. of (5.5): For these terms, we have
\[
\sum_{i=1}^{2k-1} \varepsilon^{i} \left| \left\langle e_{\pm} \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \nabla_p (I - P_{\pm}) \left[ f^\varepsilon \right] , (w_0^0)^2 (I - P_{\pm}) \left[ f^\varepsilon \right] \right\rangle \right| \\lesssim \sum_{i=1}^{2k-1} \varepsilon^{i} \left( \|E_i\|_{L^\infty} + \|B_i\|_{L^\infty} \right) \|(I - P)[f^\varepsilon]\|^2_{w^0} \tag{5.11}
\]
\[
\lesssim \sum_{i=1}^{2k-1} \varepsilon^{i} \|(1 + t)^{i-1} \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma} \lesssim \varepsilon \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma},
\]
for sufficiently small \(\varepsilon\),
\[
\sum_{i=1}^{2k-1} \varepsilon^{i} \left| \left\langle e_{\pm} \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \left( \frac{\nabla_p F_{\pm,i}}{\sqrt{M}_\pm} \right) , (w_0^0)^2 (I - P_{\pm}) \left[ f^\varepsilon \right] \right\rangle \right| \\lesssim \sum_{i=1}^{2k-1} \varepsilon^{i} \|(1 + t)^{i-1} \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma} \lesssim \varepsilon \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma}, \tag{5.12}
\]
\[
\sum_{i=1}^{2k-1} \varepsilon^{i} \left| \left\langle e_{\pm} \left( E_i + \hat{p}_\pm \times B_i \right) \cdot \left( \frac{u_\pm - u_\pm}{2kB_0} \right) , (w_0^0)^2 (I - P_{\pm}) \left[ f^\varepsilon \right] \right\rangle \right| \\lesssim \sum_{i=1}^{2k-1} \varepsilon^{i} \|(1 + t)^{i-1} \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma} \lesssim \varepsilon \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma}, \tag{5.13}
\]
\[
\varepsilon^k \left| \left\langle (I - P)[\tilde{S}] , (w_0^0)^2 (I - P)[f^\varepsilon] \right\rangle \right| \\lesssim \frac{O(1)}{\varepsilon} \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma} + C \varepsilon^{2k+1} \sum_{2 \leq i,j \leq 2k-1} \varepsilon^{2(i+j-2k-1)}(1 + t)^{2(i+j-2)} \tag{5.14}
\]
\[
+ C \sum_{1 \leq i,j \leq 2k-1} \varepsilon^{2(i+j-k)}(1 + t)^{2(i+j-2)} \lesssim \frac{O(1)}{\varepsilon} \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma} + C \varepsilon^{2k+1}(1 + t)^{4k-2},
\]
and
\[
\left| \left\langle [P, \tau_{\pm,B}] \left[ f^\varepsilon \right] , (w_0^0)^2 (I - P) \left[ f^\varepsilon \right] \right\rangle \right| \lesssim \frac{1}{\varepsilon} \|(I - P)[f^\varepsilon]\|^2_{w^0,\sigma} + C \left( \|E_R\|^2 + \|B_R\|^2 + \|f^\varepsilon\|^2_{H^1} \right). \tag{5.15}
\]

Summary: We collect the above estimates in (5.5) to derive (5.3).
5.2 Weighted First-Order Derivatives Estimates

In this subsection, we proceed to the weighted $L^2$ estimate of $\nabla_x f^{\varepsilon}$.

**Proposition 5.2.** For the remainders $\left(f^{\varepsilon}, E_R^{\varepsilon}, B_R^{\varepsilon}\right)$, it holds that

$$
\varepsilon \left( \frac{d}{dt} \| \nabla_x (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} + \frac{\delta}{\varepsilon} \| \nabla_x (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} + Y \| \nabla (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} \right)
\leq \left( (1 + t)^{-\beta_0} + \varepsilon \right) \varepsilon^2 \| \nabla_x^2 f^{\varepsilon} \|^2 + \varepsilon^{-1} \| \nabla_x (I - P)[f^{\varepsilon}] \|^2_{\sigma} + \varepsilon D + C \varepsilon^{2k+2}(1 + t)^{4k-2}.
$$

**(5.16)**

**Proof.** Noting that $[\mathcal{L}, \nabla_x]$ only contains terms that hit $\mathcal{L}$, it holds that

$$
\varepsilon^{-1} \left\langle \partial_x \mathcal{L} (I - P)[f^{\varepsilon}], w^2 \partial_x (I - P)[f^{\varepsilon}] \right\rangle
\geq \varepsilon^{-1} \left\langle \mathcal{L} \partial_x (I - P)[f^{\varepsilon}], w^2 \partial_x (I - P)[f^{\varepsilon}] \right\rangle - \varepsilon^{-1} \left\langle \partial_x \mathcal{L} (I - P)[f^{\varepsilon}], w^2 \partial_x (I - P)[f^{\varepsilon}] \right\rangle
\geq \delta \varepsilon^{-1} \| \partial_x (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} - C \varepsilon^{-1} \| \partial_x (I - P)[f^{\varepsilon}] \|^2_{\sigma} - C \varepsilon^{-1} \varepsilon_0 \| (I - P)[f^{\varepsilon}] \|^2_{w^{0, \sigma}}.
$$

**(5.17)**

Then, by Lemma 3.3 and (4.19), we have

$$
\varepsilon^{-1} \left\langle \partial_x \mathcal{L} (I - P)[f^{\varepsilon}], w^2 \partial_x (I - P)[f^{\varepsilon}] \right\rangle
\geq \varepsilon^{-1} \left\langle \mathcal{L} \partial_x (I - P)[f^{\varepsilon}], w^2 \partial_x (I - P)[f^{\varepsilon}] \right\rangle - \varepsilon^{-1} \left\langle \partial_x \mathcal{L} (I - P)[f^{\varepsilon}], w^2 \partial_x (I - P)[f^{\varepsilon}] \right\rangle
\geq \delta \varepsilon^{-1} \| \partial_x (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} - C \varepsilon^{-1} \| \partial_x (I - P)[f^{\varepsilon}] \|^2_{\sigma} - C \varepsilon^{-1} \varepsilon_0 \| (I - P)[f^{\varepsilon}] \|^2_{w^{0, \sigma}}.
$$

Apply $\partial_x$ to (5.1) and take the $L^2$ inner product of the resulting equation with $(w_1^1)^2 \partial_x (I - P)[f^{\varepsilon}]$. Then, by similar arguments as in (5.4), we use (5.17) to have

$$
\left\| \frac{d}{dt} \| \partial_x (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} + \frac{\delta}{\varepsilon} \| \partial_x (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} + Y \| \nabla (I - P)[f^{\varepsilon}] \|^2_{w^{1, \sigma}} \right\|
\leq C \varepsilon^{-1} \left\| \partial_x (I - P)[f^{\varepsilon}] \|^2_{\sigma} + \varepsilon_0 \| (I - P)[f^{\varepsilon}] \|^2_{w^{0, \sigma}} \right\|
$$

**(5.18)**
\[ + \left| \sum_{i=1}^{2k-1} \varepsilon^i \left( \varepsilon \partial_x \left( \left(E_i + \hat{p}_z \times B_i \right) \frac{w_0^0 \hat{p}_z - w_0^1}{2k B} (I - \mathbf{P})_{\pm} \right) \left[ f^\varepsilon \right] + (w_1^1) \partial_x (I - \mathbf{P})_{\pm} \right) \right|^2 \]
\[ + \varepsilon^k \left| \langle \partial_x (I - \mathbf{P}) \hat{S}, (w_1^1)^2 \partial_x (I - \mathbf{P}) [f^\varepsilon] \rangle \right| + \left| \langle \partial_x [\mathbf{P}, \tau_B] f^\varepsilon, (w_1^1)^2 \partial_x (I - \mathbf{P}) [f^\varepsilon] \rangle \right|. \]

Now we estimate each term on the R.H.S. of (5.18).

Second Term on the R.H.S. of (5.18): It can be bounded by
\[
C \left( \| \nabla_x E_0 \|_{W^{1,\infty}} + \| \nabla_x B_0 \|_{W^{1,\infty}} \right) \left( \| \partial_x (I - \mathbf{P}) [f^\varepsilon] \|_{w^1}^2 + \| (I - \mathbf{P}) [f^\varepsilon] \|_{w^0,\sigma}^2 \right) \leq (1 + t)^{-\beta_0} \varepsilon_0 \left( \| \partial_x (I - \mathbf{P}) [f^\varepsilon] \|_{w^1}^2 + \| (I - \mathbf{P}) [f^\varepsilon] \|_{w^0,\sigma}^2 \right). \tag{5.19} \]

Third Term on the R.H.S. of (5.18): It can be controlled by
\[
\frac{C}{\varepsilon} \| \partial_x (I - \mathbf{P}) [f^\varepsilon] \|_{w^1}^2 + C \varepsilon \left( \| E^\varepsilon_{R} \|_{H^1}^2 + \| D^\varepsilon_{R} \|_{H^1}^2 \right). \tag{5.20} \]

Fourth Term on the R.H.S. of (5.18): Similar to (5.8), we bound it by
\[
C Z \left\| \sqrt{p} \partial_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1}^2 + C Z \| (I - \mathbf{P}) [f^\varepsilon] \|_{w^0}^2 \leq \frac{Y}{2} \left\| \sqrt{p} \partial_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1}^2 + (1 + t)^{-\beta_0} \varepsilon_0 \| (I - \mathbf{P}) [f^\varepsilon] \|_{w^0,\sigma}^2. \]

Fifth Term on the R.H.S. of (5.18): Using Lemma 3.4 for \( p \) integral, \((\infty, 2, 2)\) or \((4, 4, 2)\) for \( x \) integral, Sobolev embedding and (2.35), we have
\[
\left| \langle \varepsilon^{k-1} \partial_x \Gamma [f^\varepsilon, f^\varepsilon], (w_1^1)^2 \partial_x (I - \mathbf{P}) [f^\varepsilon] \rangle \right| \leq \varepsilon^{k-1} \int_{x \in \mathbb{R}^3} \left[ \| w_1^1 \partial_x f^\varepsilon \|_{L^2} \| f^\varepsilon \|_{\sigma} + \| w_1^1 \partial_x f^\varepsilon \|_{\sigma} \| f^\varepsilon \|_{L^2} + \| w_1^1 f^\varepsilon \|_{L^2} \| \partial_x f^\varepsilon \|_{\sigma} + \| w_1^1 f^\varepsilon \|_{\sigma} \| \partial_x f^\varepsilon \|_{L^2} \right. \]
\[ + Z \left( \| w_1^1 f^\varepsilon \|_{L^2} \| f^\varepsilon \|_{\sigma} + \| w_1^1 f^\varepsilon \|_{\sigma} \| f^\varepsilon \|_{L^2} \right) \| w_1^1 \partial_x (I - \mathbf{P}) [f^\varepsilon] \|_{\sigma} \]
\[ \leq \varepsilon^{k-1} \left( \| f^\varepsilon \|_{H^2} \| w_1^1 f^\varepsilon \|_{H^2} + \| f^\varepsilon \|_{H^2} \| w_1^1 f^\varepsilon \|_{H^1} \right) \| \partial_x (I - \mathbf{P}) [f^\varepsilon] \|_{w^1,\sigma} \]
\[ \leq \varepsilon^{k-1} \left( \| w_1^1 f^\varepsilon \|_{H^2} + \varepsilon \| f^\varepsilon \|_{H^2} \right) \left\| \partial_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1,\sigma} \]
\[ \leq o(1) \varepsilon^{-1} \left\| \partial_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1,\sigma}^2 + \varepsilon^2 \left\| w_1^1 f^\varepsilon \right\|_{H^2}^2 + \varepsilon^3 \left\| f^\varepsilon \right\|_{H^2}^2 \]
\[ \leq o(1) \varepsilon^{-1} \left\| \partial_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1,\sigma}^2 + \varepsilon^2 \left( \| w_1^1 \mathbf{P} [f^\varepsilon] \right\|_{H^2}^2 + \| w_1^1 (I - \mathbf{P}) [f^\varepsilon] \|_{H^2}^2 \right) + \varepsilon^3 \left( \| \mathbf{P} [f^\varepsilon] \right\|_{H^2}^2 + \left\| (I - \mathbf{P}) [f^\varepsilon] \right\|_{H^2}^2 \]
\[ \leq o(1) \varepsilon^{-1} \left\| \nabla_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1,\sigma}^2 + \varepsilon^2 \left\| (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^0,\sigma}^2 + \varepsilon^3 \left\| (I - \mathbf{P}) [f^\varepsilon] \right\|_{H^2}^2 + \varepsilon^3 \left\| f^\varepsilon \right\|_{H^2}^2 + \varepsilon^2 \left\| f^\varepsilon \right\|_{H^1}^2. \]

Sixth Term on the R.H.S. of (5.18): Similarly, we use (3.7) and (A.7) in Proposition A.1 to bound it by
\[
\sum_{i=1}^{2k-1} \varepsilon^{i-1} (1 + t)^{i-1} \left\| \partial_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1,\sigma} \left\| f^\varepsilon \right\|_{H^2,\sigma} \]
\[ \leq o(1) \varepsilon^{-1} \left\| \partial_x (I - \mathbf{P}) [f^\varepsilon] \right\|_{w^1,\sigma}^2 + C \sum_{i=1}^{2k-1} \varepsilon^{2i-1} (1 + t)^{2(i-1)} \| f^\varepsilon \|_{w^1,\sigma}^2 \]
\[ \leq o(1) \varepsilon^{-1} \left\| (I - \mathbf{P}) [f^\varepsilon] \right\|_{H^2,\sigma}^2 + \varepsilon \left\| f^\varepsilon \right\|_{H^1}^2. \]
Seventh and eighth Terms on the R.H.S. of (5.18): The two terms can be bounded by

\[ C \varepsilon^k \left( \| E_R^{\varepsilon} \|_{H^2} + \| B_R^{\varepsilon} \|_{H^1} \right) \| \partial_x (I - P) [f^\varepsilon] \|_{w^1,\sigma}^2 \lesssim \varepsilon \| \partial_x (I - P) [f^\varepsilon] \|_{w^1,\sigma}^2. \tag{5.23} \]

Other Terms on the R.H.S. of (5.18): For these terms, we have

\[ \sum_{i=1}^{2k-1} \varepsilon^i \left( \| E_i \|_{W^{1,\infty}} + \| B_i \|_{W^{1,\infty}} \right) \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right) \lesssim \varepsilon \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right), \tag{5.24} \]

\[ \sum_{i=1}^{2k-1} \varepsilon^i \left( \| E_i \|_{W^{1,\infty}} + \| B_i \|_{W^{1,\infty}} \right) \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right) \lesssim \varepsilon \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right), \tag{5.25} \]

\[ \sum_{i=1}^{2k-1} \varepsilon^i \left( \| E_i \|_{W^{1,\infty}} + \| B_i \|_{W^{1,\infty}} \right) \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right) \lesssim \varepsilon \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right), \tag{5.26} \]

\[ \sum_{i=1}^{2k-1} \varepsilon^i \left( \| E_i \|_{W^{1,\infty}} + \| B_i \|_{W^{1,\infty}} \right) \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right) \lesssim \varepsilon \left( \| (I - P) [f^\varepsilon] \|_{w^0,\sigma}^2 + \| \partial_x (I - P) [f^\varepsilon] \|_{w^1}^2 \right), \tag{5.27} \]

\[ \varepsilon^k \left| \langle \partial_x (I - P) [S], (w^1)^2 \partial_x (I - P) [f^\varepsilon] \rangle \right| \lesssim o(1) \varepsilon^{k-1} \| \partial_x (I - P) [f^\varepsilon] \|_{w^1,\sigma}^2 + \varepsilon^{2k+1} (1 + t)^{4k-2}, \tag{5.28} \]

and

\[ \left| \langle \partial_x [P, \tau_B] f^\varepsilon, (w^1)^2 \partial_x (I - P) [f^\varepsilon] \rangle \right| \leq o(1) \varepsilon^{k-1} \| \partial_x (I - P) [f^\varepsilon] \|_{w^1,\sigma}^2 + C \varepsilon \left( \| E_R^{\varepsilon} \|_{H^1}^2 + \| B_R^{\varepsilon} \|_{H^1}^2 + \| f^\varepsilon \|_{H^2}^2 \right). \tag{5.29} \]

**Summary:** We collect the above estimates in (5.18) and multiply it by \( \varepsilon \) to obtain (5.16).

\[ \square \]

### 5.3 Weighted Second-Order Derivatives Estimates

In this subsection, we proceed the weighted \( L^2 \) estimate of \( \nabla^2 f^\varepsilon \).
Proposition 5.3. For the remainders \((f^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)\), it holds that
\[
\varepsilon^3 \left( \frac{d}{dt} \left\| \nabla f^\varepsilon \right\|_{w_2}^2 + Y \left\| \sqrt{\rho} \nabla_x f^\varepsilon \right\|_{w_2}^2 + \frac{\delta}{\varepsilon} \left\| \nabla_x (I - P) [f^\varepsilon] \right\|_{w_2, \sigma}^2 \right) \leq \varepsilon^2 \left\| \partial \partial_x f^\varepsilon \right\|_{w_2, \sigma}^2 + \varepsilon (\mathcal{E} + \mathcal{D}) + \varepsilon^{2k+4}(1 + t)^{4k-2}. \tag{5.29}
\]
\[
\text{Proof. Noting that } |\mathcal{L}, \nabla_x^2|\left[f^\varepsilon\right] \text{ contains terms that } \nabla_x \text{ hits } f^\varepsilon \text{ at most once, we have}
\varepsilon^{-1} \left| \left\langle [\mathcal{L}, \partial \partial_x] \left[f^\varepsilon\right], (w_2^2 \partial \partial_x f^\varepsilon) \right\rangle \right|
\lesssim o(1) \varepsilon^{-1} \left\| \partial \partial_x (I - P) [f^\varepsilon] \right\|_{w_2, \sigma}^2 + \varepsilon_0 \varepsilon^{-1} \left\| (I - P) [f^\varepsilon] \right\|_{H_{w_2, \sigma}}^2 + (1 + t)^{-\beta_0} \varepsilon^{-1} \left\| f^\varepsilon \right\|_{H_2}^2. \tag{5.30}
\]
Then we use (3.5) to further obtain
\[
\varepsilon^{-1} \langle \partial \partial_x \mathcal{L} [f^\varepsilon], (w_2^2 \partial \partial_x f^\varepsilon) \rangle \geq \varepsilon^{-1} \langle \mathcal{L} [\partial \partial_x f^\varepsilon], (w_2^2 \partial \partial_x f^\varepsilon) \rangle - \varepsilon^{-1} \left| \left\langle [\mathcal{L}, \partial \partial_x] \left[f^\varepsilon\right], (w_2^2 \partial \partial_x f^\varepsilon) \right\rangle \right|
\geq \varepsilon^{-1} \left\| \partial \partial_x (I - P) [f^\varepsilon] \right\|_{w_2, \sigma}^2 - C\varepsilon^{-1} \varepsilon_0 \left\| (I - P) [f^\varepsilon] \right\|_{H_{w_2, \sigma}}^2 - C\varepsilon^{-1} \left\| f^\varepsilon \right\|_{H_2}^2. \tag{5.31}
\]
Now we take \(|\alpha| = 2\) in (4.13) and multiply the equation by \((w_2^2) \partial \partial_x f^\varepsilon\) to get
\[
\frac{1}{2} \frac{d}{dt} \left\| \partial \partial_x f^\varepsilon \right\|_{w_2}^2 + Y \left\| \sqrt{\rho} \partial \partial_x f^\varepsilon \right\|_{w_2}^2 + \frac{\delta}{\varepsilon} \left\| \partial \partial_x (I - P) [f^\varepsilon] \right\|_{w_2, \sigma}^2
\leq \left| \left\langle e \partial \partial_x \left( \sqrt{\mathcal{M}} \left( E_R + \hat{p}_x \times B_R \right) \cdot \frac{n_0 \hat{p}_x - u_\perp}{k_B T_0} \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \left| \left\langle e \partial \partial_x \left( (E_0 + \hat{p}_x \times B_0) \cdot \nabla_p f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \left| \left\langle \partial \partial_x \left( \frac{f^\varepsilon}{\sqrt{\mathcal{M}}} \left( \delta_t + c \hat{p}_x \cdot \nabla_x \pm e \left( E_0 + \hat{p}_x \times B_0 \right) \cdot \nabla_p \right) \left\{ \sqrt{\mathcal{M}} \right\} \left( \sqrt{\mathcal{M}} \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \varepsilon^{-1} \left| \left\langle \partial \partial_x \left[ f^\varepsilon, \frac{E_0 + \hat{p}_x \times B_0}{\sqrt{\mathcal{M}}} \right], (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \sum_{i=1}^{2k-1} e^{-i-1} \left| \left\langle \partial \partial_x \left[ \frac{E^\varepsilon}{\sqrt{\mathcal{M}}}, f^\varepsilon \right], (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \varepsilon^k \left| \left\langle e \partial \partial_x \left( \frac{u_0^0 \hat{p}_x - u_\perp}{2k_B T_0} \cdot \left( E_R + \hat{p}_x \times B_R \right) f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \varepsilon^k \left| \left\langle e \partial \partial_x \left( (E^\varepsilon \hat{p}_x \times B_R) \cdot \nabla_p f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \sum_{i=1}^{2k-1} e^i \left| \left\langle e \partial \partial_x \left( E_i + \hat{p}_x \times B_i \right) \cdot \nabla_p f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \sum_{i=1}^{2k-1} e^i \left| \left\langle e \partial \partial_x \left( (E_R + \hat{p}_x \times B_R) \cdot \nabla_p f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \sum_{i=1}^{2k-1} e^i \left| \left\langle e \partial \partial_x \left( E_i + \hat{p}_x \times B_i \right) \cdot \nabla_p f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \sum_{i=1}^{2k-1} e^i \left| \left\langle e \partial \partial_x \left( (E_R + \hat{p}_x \times B_R) \cdot \nabla_p f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \sum_{i=1}^{2k-1} e^i \left| \left\langle e \partial \partial_x \left( (E_i + \hat{p}_x \times B_i) \cdot \nabla_p f^\varepsilon \right), (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right|
+ \left| \left\langle \partial \partial_x S, (w_2^2) \partial \partial_x f^\varepsilon \right\rangle \right| + C\varepsilon_0 \varepsilon^{-1} \left\| (I - P) [f^\varepsilon] \right\|_{H_{w_2, \sigma}}^2 + C\varepsilon^{-1} \left\| f^\varepsilon \right\|_{H_2}. \tag{5.32}
\]
Now we treat the terms in the R.H.S. of (5.32).

First Term on the R.H.S. of (5.32): By (2.35), it can be bounded by
\[
C \left( \left\| f^\varepsilon \right\|_{H_2}^2 + \left\| E_R^\varepsilon \right\|_{H_2}^2 + \left\| B_R^\varepsilon \right\|_{H_2}^2 \right). \tag{5.33}
\]
Second Term on the R.H.S. of (5.32): It follows that

\[
| \varepsilon_{\pm} \left( \partial^2_x \left( (E_0 + \tilde{\rho}_\pm \times B_0) \cdot \nabla_p f_{\pm}^\varepsilon \right), (w_{\pm}^2)^2 \partial^2_x f_{\pm}^\varepsilon \right) | \geq \mathcal{Z} \| \mathcal{F} \|_{H^{2,\sigma}}^2 \leq \frac{1}{4} \left( 1 + \frac{1}{2} \right)^{-\beta_0} \varepsilon_0 \left( \| (I - P) [\mathcal{F}] \|_{H^{2,\sigma}}^2 + \| \mathcal{F} \|_{H^2}^2 \right).
\]

Third Term on the R.H.S. of (5.32): Similar to (4.5), we estimate it as follows

\[
\left| \partial^2_x \left( \frac{f_{\pm}^\varepsilon}{\sqrt{M}} \left\{ \partial_t + c \tilde{\rho}_\pm \cdot \nabla_x \pm \varepsilon (E_0 + \tilde{\rho}_\pm \times B_0) \cdot \nabla_p \right\} \right| \right| \left( \sqrt{M} \right) \geq \left| (w_{\pm}^2)^2 \partial^2_x f_{\pm}^\varepsilon \right| \leq \mathcal{Z} \left| \sqrt{p^0 \partial^2_x f_{\pm}^\varepsilon} \right|_{w_2}^2 + \mathcal{Z} \| \mathcal{F} \|_{H^{2,\sigma}}^2 \leq \frac{Y}{2} \left| \sqrt{p^0 \partial^2_x f_{\pm}^\varepsilon} \right|_{w_2}^2 + C(1 + \frac{1}{2})^{-\beta_0} \varepsilon_0 \left( \| (I - P) [\mathcal{F}] \|_{H^{2,\sigma}}^2 + \| \mathcal{F} \|_{H^2}^2 \right).
\]

Fourth Term on the R.H.S. of (5.32): We use (2.35), (3.7) in Lemma 3.4 and Sobolev’s inequalities to have

\[
\left| \varepsilon^{k-1} \partial^2_x \Gamma \left[ f_{\pm}^\varepsilon, f_{\mp}^\varepsilon \right], (w_{\pm}^2)^2 \partial^2_x f_{\pm}^\varepsilon \right| \leq \varepsilon^{k-1} \left( \frac{\varepsilon}{\mathcal{Z}} \right) \left( \| f_{\pm}^\varepsilon \|_{H^2} + \| f_{\mp}^\varepsilon \|_{H^2} \right) \left| \nabla^2 f_{\pm}^\varepsilon \right|_{w_2} \leq \frac{1}{2} \left( \| f_{\pm}^\varepsilon \|_{H^2} + \| f_{\mp}^\varepsilon \|_{H^2} \right) \left| \nabla^2 f_{\pm}^\varepsilon \right|_{w_2}
\]

Fifth Term on the R.H.S. of (5.32): Similarly, we have

\[
\left| \sum_{i=1}^{2k-1} \varepsilon^{i-1} \partial^2_x \Gamma \left[ \frac{F_i}{\sqrt{M}}, f_{\pm}^\varepsilon \right] \right| \leq \frac{1}{2} \left( \| (I - P) [\mathcal{F}] \|_{H^{2,\sigma}}^2 + \| \mathcal{F} \|_{H^2}^2 \right).
\]

Other Terms on the R.H.S. of (5.32): These terms can be estimated as follows:

\[
\varepsilon^k \left| \varepsilon_{\pm} \partial^2_x \left( \frac{(u_{\pm}^0 \tilde{\rho}_\pm - u_{\pm})}{2k_B T_0} \cdot (E_{\pm}^R + \tilde{\rho}_\pm \times B_{\pm}^R) f_{\pm}^\varepsilon \right), (w_{\pm}^2)^2 \partial^2_x f_{\pm}^\varepsilon \right| \leq \varepsilon^k \left( \| E_{\pm}^R \|_{H^2} + \| B_{\pm}^R \|_{H^2} \right) \left| \nabla^2 f_{\pm}^\varepsilon \right|_{w_2} \leq \varepsilon \left( \| (I - P) [\mathcal{F}] \|_{H^{2,\sigma}}^2 + \| \mathcal{F} \|_{H^2}^2 \right),
\]

\[
\varepsilon^k \left| \varepsilon_{\pm} \partial^2_x \left( (E_{\pm}^R + \tilde{\rho}_\pm \times B_{\pm}^R) \cdot \nabla_p f_{\pm}^\varepsilon \right), (w_{\pm}^2)^2 \partial^2_x f_{\pm}^\varepsilon \right| \leq \varepsilon^k \left( \| (I - P) [\mathcal{F}] \|_{H^{2,\sigma}}^2 + \| \mathcal{F} \|_{H^2}^2 \right).
\]
\[
\lesssim \varepsilon^k \left( \| P[\varepsilon] [f^\varepsilon] \|_{H^2_\omega} + \| B_R^\varepsilon \|_{H^2_\omega} \right) \| f^\varepsilon \|_{H^2_\omega}^2 \lesssim \varepsilon \left( \| (I - P)[f^\varepsilon] \|_{H^2_\omega}^2 + \| f^\varepsilon \|_{H^2_\omega}^2 + \| P[\varepsilon] [f^\varepsilon] \|_{H^2_\omega} + \| B_R^\varepsilon \|_{H^2_\omega} \right)
\]

\[
\sum_{i=1}^{2k-1} \varepsilon^i \left( \| E_i \|_{W^{2,\infty}} + \| B_i \|_{W^{2,\infty}} \right) \| f^\varepsilon \|_{H^2_\omega}^2 \lesssim \varepsilon \left( \| (I - P)[f^\varepsilon] \|_{H^2_\omega}^2 + \| f^\varepsilon \|_{H^2_\omega} \right)
\]

\[
\lesssim \varepsilon \left( \| P[\varepsilon] [f^\varepsilon] \|_{H^2_\omega} + \| B_R^\varepsilon \|_{H^2_\omega} + \| f^\varepsilon \|_{H^2_\omega} \right),
\]

\[
\sum_{i=1}^{2k-1} \varepsilon^i \left( \| E_i \|_{W^{2,\infty}} + \| B_i \|_{W^{2,\infty}} \right) \| f^\varepsilon \|_{H^2_\omega}^2 \lesssim \varepsilon \left( \| f^\varepsilon \|_{H^2_\omega} \right),
\]

and

\[
\varepsilon^k \left| \langle \partial_x^2 S, (w^2)^2 \partial_x^2 f^\varepsilon \rangle \right| \lesssim o(1) \varepsilon^{-1} \left\| \partial_x^2 (I - P)[f^\varepsilon] \right\|_{w^{2,\sigma}}^2 + \varepsilon^{-1} \left\| \nabla_x^2 f^\varepsilon \right\|^2 + C \varepsilon^{2k+1} (1 + t)^{4k-2}.
\]

**Summary:** We collect the above estimates in (5.32) and multiply the resulting inequality by \( \varepsilon^3 \) to derive (5.29).

## 6 Macroscopic Estimates and Electromagnetic Dissipation

In this section, we study the macroscopic estimates of \( f^\varepsilon \) and electromagnetic dissipation. With these estimates and the estimates obtained in the previous two Sections, we can finally close the whole energy estimate.

### 6.1 Macroscopic Estimates

To capture the dissipation of the macroscopic part of the \( f^\varepsilon \), which can be seen as a perturbation around a local Maxwellian. As in (2.23), we write \( P[\varepsilon] [f^\varepsilon] \) as

\[
P[\varepsilon] [f^\varepsilon] = \left( \begin{pmatrix} a^\varepsilon + \frac{\rho_{\varepsilon,2}^2}{\rho_{\varepsilon,1}} \varepsilon^2 \\ 0 \end{pmatrix} \mathbf{M}_+^2 \right) + \left( \begin{pmatrix} 0 \\ a^\varepsilon - \frac{\rho_{\varepsilon,2}^2}{\rho_{\varepsilon,1}} \varepsilon^2 \end{pmatrix} \mathbf{M}_-^2 \right) + b_i p_{\varepsilon}^0 \left( \begin{pmatrix} \mathbf{M}_+^2 \\ \mathbf{M}_-^2 \end{pmatrix} \right) + \varepsilon^2 \left( \begin{pmatrix} p_{\varepsilon}^0 \mathbf{M}_+^2 \\ p_{\varepsilon}^0 \mathbf{M}_-^2 \end{pmatrix} \right),
\]

where \( \rho_{\varepsilon,1} \) and \( \rho_{\varepsilon,2} \) are defined in (2.24).

**Proposition 6.1.** There are two functionals \( \mathcal{E}_i^{mac} \) for \( i = 1, 2 \) satisfying

\[
\mathcal{E}_i^{mac} \lesssim \| \nabla_x^{-1} f^\varepsilon \| \| \nabla_x f^\varepsilon \|,
\]

(6.1)
such that Thus, we have
\begin{equation}
- \frac{d}{dt}(\varepsilon \mathcal{E}^{\text{mac}} + \varepsilon^2 \mathcal{E}_2) + \varepsilon \left( \| \nabla \cdot \mathbf{P}[f^\varepsilon] \|^2 + \| (a^\varepsilon_+ - a^\varepsilon_-) \|^2 + \| (\nabla \times \mathbf{E}_R^\varepsilon) \|^2 \right) + \varepsilon^2 \left( \| \nabla \cdot \mathbf{P}[f^\varepsilon] \|^2 + \| \nabla \cdot (a^\varepsilon_+ - a^\varepsilon_-) \|^2 + \| \nabla \times (\nabla \cdot \mathbf{E}_R^\varepsilon) \|^2 \right) \lesssim \varepsilon^{-1} \| (\mathbf{I} - \mathbf{P})[f^\varepsilon] \|_{\sigma}^2 + \| \nabla \times (\mathbf{I} - \mathbf{P})[f^\varepsilon] \|_{\sigma}^2 + \varepsilon \| \nabla \times (\mathbf{I} - \mathbf{P})[f^\varepsilon] \|_{\sigma}^2 + \varepsilon (\mathcal{E} + \mathcal{D}) + \varepsilon^k (1 + t)^{4k-1}.
\end{equation}

**Proof.** Motivated by [40, Lemma 6.1], we will prove this proposition by two key ingredients: local conservation laws and the macroscopic equations of \( f^\varepsilon \). For convenience, we write (2.15) as
\begin{equation}
\partial_t f^\varepsilon_{\pm} + cp_{\pm} \cdot \nabla f^\varepsilon_{\pm} = \pm \frac{n_0}{k_B T_0} p_{\pm} \cdot \hat{M}^\varepsilon_{\pm} \cdot E_R^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}^\varepsilon [f^\varepsilon_{\pm}] = \tilde{h}^\varepsilon_{\pm},
\end{equation}

where
\begin{equation}
\tilde{h}^\varepsilon_{\pm} = \pm \varepsilon \left( E_0 + \hat{p}_{\pm} \times B_0 \right) \cdot \nabla \cdot \pm \varepsilon \frac{n_0}{k_B T_0} p_{\pm} \cdot \hat{M}^\varepsilon_{\pm} \cdot \left( E_R^\varepsilon + \hat{p}_{\pm} \times B_R^\varepsilon \right)
\end{equation}

Local conservation laws: Firstly, we derive the local conservation laws of \( a^\varepsilon_{\pm}, b^\varepsilon, c^\varepsilon \). Note that
\begin{equation}
\frac{\rho_{\pm,2}}{\rho_{\pm,1}} = \frac{\varepsilon_{\pm,0}(u^0_\pm)^2 + P_{\pm,0}|u|^2}{c^2}, \quad \frac{c}{n_{\pm,0} u^0_\pm} = \frac{\varepsilon_{\pm,0}(u^0_\pm)^2 + P_{\pm,0}|u|^2}{n_{\pm,0} c^2 u^0_\pm}.
\end{equation}

Projecting (6.3) onto the null space \( \mathcal{N} \), similar to the derivation of (A.3), (A.4) and (A.5), we can obtain
\begin{equation}
\frac{n_{\pm,0} u^0_\pm}{c} \partial_t a_\pm^\varepsilon + P_{\pm,0} \nabla \times b^\varepsilon
\end{equation}

\begin{equation}
= \Xi_{\pm,1} \left[ \langle a^\varepsilon_{\pm}, b^\varepsilon, c^\varepsilon \rangle - c \nabla \cdot \int_{\mathbb{R}^3} \hat{p}_{\pm} \hat{M}^\varepsilon_{\pm} (\mathbf{I} - \mathbf{P}) [f^\varepsilon_{\pm}] dp + \int_{\mathbb{R}^3} \hat{M}^\varepsilon_{\pm} \bar{h}^\varepsilon_{\pm} dp, \right. \nabla \times \left. \frac{n_{\pm,0} m_0^2 u^0_\pm K_3(\gamma_{\pm})}{\gamma_{\pm} K_2(\gamma_{\pm})} \partial_t b^\varepsilon + P_{\pm,0} \nabla \left( a^\varepsilon - \frac{\varepsilon_{\pm,0} u^0_\pm}{n_{\pm,0} c^2} c^\varepsilon \right) + \frac{n_{\pm,0} m_0^2 u^0_\pm K_3(\gamma_{\pm})}{\gamma_{\pm} K_2(\gamma_{\pm})} \nabla \cdot c^\varepsilon + e \frac{n_{\pm,0} u^0_\pm}{c} E_R^\varepsilon \right] \end{equation}

\begin{equation}
= \Xi_{\pm,2} \left[ \langle a^\varepsilon_{\pm}, b^\varepsilon, c^\varepsilon \rangle - c \nabla \cdot \int_{\mathbb{R}^3} \hat{p}_{\pm} \hat{M}^\varepsilon_{\pm} (\mathbf{I} - \mathbf{P}) [f^\varepsilon_{\pm}] dp + \int_{\mathbb{R}^3} p \hat{M}^\varepsilon_{\pm} \bar{h}^\varepsilon_{\pm} dp, \right. \nabla \times \left. \frac{\varepsilon_{\pm,0} u^0_\pm}{c^3} \partial_t \left( a^\varepsilon - \frac{\varepsilon_{\pm,0} u^0_\pm}{n_{\pm,0} c^2} c^\varepsilon \right) + \frac{n_{\pm,0} m_0^2 (u^0_\pm)^2 [3 K_3(\gamma_{\pm}) + \gamma K_2(\gamma_{\pm})]}{\gamma_{\pm} K_2(\gamma_{\pm})} \partial_t c^\varepsilon + \frac{n_{\pm,0} m_0^2 u^0_\pm c^2 K_3(\gamma_{\pm})}{\gamma_{\pm} K_2(\gamma_{\pm})} \nabla \cdot b^\varepsilon \right] \end{equation}

\begin{equation}
= \Xi_{\pm,3} \left[ \langle a^\varepsilon_{\pm}, b^\varepsilon, c^\varepsilon \rangle + \int_{\mathbb{R}^3} p^0 \hat{M}^\varepsilon_{\pm} \bar{h}^\varepsilon_{\pm} dp, \right. \nabla \times \left. \frac{\varepsilon_{\pm,0} u^0_\pm}{c^3} \partial_t \left( a^\varepsilon - \frac{\varepsilon_{\pm,0} u^0_\pm}{n_{\pm,0} c^2} c^\varepsilon \right) + \frac{n_{\pm,0} m_0^2 (u^0_\pm)^2 [3 K_3(\gamma_{\pm}) + \gamma K_2(\gamma_{\pm})]}{\gamma_{\pm} K_2(\gamma_{\pm})} \partial_t c^\varepsilon + \frac{n_{\pm,0} m_0^2 u^0_\pm c^2 K_3(\gamma_{\pm})}{\gamma_{\pm} K_2(\gamma_{\pm})} \nabla \cdot b^\varepsilon \right]
\end{equation}
where \( \Xi_{\pm,j}(a_{\pm}^\varepsilon, b_{\varepsilon}, c_{\varepsilon}) \) for \( j = 1, 2, 3 \) denotes a combination of linear terms of \( a_{\pm}^\varepsilon, b_{\varepsilon}, c_{\varepsilon} \) with coefficients
\( \nabla_{t,x}(n_0, u, T_0) \), and derivatives of \( a_{\pm}^\varepsilon, b_{\varepsilon}, c_{\varepsilon} \) with coefficient \( u \). Since they are small perturbations and thus
will not affect the estimates, we will ignore the details for clarity.

Noting that
\[
P_{\pm,0} = \frac{n_{\pm,0}m_{\pm}c^2}{\gamma_{\pm}}, \quad \epsilon_{\pm,0} = n_{\pm,0}m_{\pm}c^2 \left( \frac{K_3(\gamma_{\pm})}{K_2(\gamma_{\pm})} - \frac{1}{\gamma_{\pm}} \right) = n_{\pm,0}m_{\pm}c^2 \left( \frac{K_1(\gamma_{\pm})}{K_2(\gamma_{\pm})} + \frac{3}{\gamma_{\pm}} \right),
\]
we can further write the above system as
\[
\begin{align*}
\frac{n_{\pm,0}u_{\pm}^0}{c} \partial_t a_{\pm}^\varepsilon + n_{\pm,0}m_{\pm}c^2 \nabla_x \cdot b_{\varepsilon} &= \Xi_{\pm,1}(a_{\pm}^\varepsilon, b_{\varepsilon}, c_{\varepsilon}) - c \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p}_{\pm} M_{\pm}^2 (I - P) [f_{\pm}] \, dp + \int_{\mathbb{R}^3} \frac{1}{\gamma_{\pm}} \hat{h}_{\pm} \, dp, \\
n_{\pm,0}m_{\pm}c^2 \left( \frac{K_1(\gamma_{\pm})}{\gamma_{\pm} K_2(\gamma_{\pm})} + \frac{4}{\gamma_{\pm}} \right) \partial_t b_{\varepsilon} + \frac{n_{\pm,0}m_{\pm}c^2}{\gamma_{\pm}} \nabla_x a_{\varepsilon} + \frac{n_{\pm,0}m_{\pm}^2c^2}{\gamma_{\pm}^2} \nabla_x c_{\varepsilon} + \epsilon_{\pm} n_{\pm,0}u_{\pm}^0 E_R^e \\
&= \Xi_{\pm,2}(a_{\pm}^\varepsilon, b_{\varepsilon}, c_{\varepsilon}) - c \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p}_{\pm} p M_{\pm}^2 (I - P) [f_{\pm}] \, dp + \int_{\mathbb{R}^3} p M_{\pm}^2 \hat{h}_{\pm} \, dp, \\
n_{\pm,0}m_{\pm}^2c^2 \left( -\frac{K_2(\gamma_{\pm})}{K_2(\gamma_{\pm})} - \frac{3}{\gamma_{\pm} K_2(\gamma_{\pm})} \right) \partial_t c_{\varepsilon} + \frac{n_{\pm,0}m_{\pm}^2c^2}{\gamma_{\pm}^2} \nabla_x \cdot b_{\varepsilon} \\
&= \Xi_{\pm,3}(a_{\pm}^\varepsilon, b_{\varepsilon}, c_{\varepsilon}) + \int_{\mathbb{R}^3} \hat{p}_{\pm} M_{\pm}^2 \hat{h}_{\pm} \, dp - m_{\pm}u_{\pm}^0 \left( \frac{K_1(\gamma_{\pm})}{K_2(\gamma_{\pm})} + \frac{3}{\gamma_{\pm}} \right) \left( -c \nabla_x \cdot \int_{\mathbb{R}^3} \hat{p}_{\pm} M_{\pm}^2 (I - P) [f_{\pm}] \, dp + \int_{\mathbb{R}^3} \frac{1}{\gamma_{\pm}} \hat{h}_{\pm} \, dp \right).
\end{align*}
\]
This system fully describes the evolution of \( a_{\pm}^\varepsilon, b_{\varepsilon} \) and \( c_{\varepsilon} \).

**Macroscopic equations:** Secondly, we turn to the macroscopic equations of \( f_{\varepsilon} \). Splitting \( f_{\varepsilon} \) as the macroscopic part \( P[f_{\varepsilon}] \) and the microscopic \( (I - P)[f_{\varepsilon}] \) part in (6.3), we have
\[
\left\{ \partial_t \left( a_{\pm}^\varepsilon - \frac{\rho_{\pm,2}}{\rho_{\pm,1}} c_{\varepsilon} \right) + p \cdot \partial_t b_{\varepsilon} + p_{\pm,0} \partial_t c_{\varepsilon} \right\} M_{\pm}^1 \\
+ c \hat{p}_{\pm} \cdot \left( \nabla_x \left( a_{\pm}^\varepsilon - \frac{\rho_{\pm,2}}{\rho_{\pm,1}} c_{\varepsilon} \right) \right) + \nabla_x b_{\varepsilon} \cdot p + p_{\pm} \nabla_x c_{\varepsilon} \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \right) E_R = \ell_{\pm}^e + h_{\pm}^e,
\]
where
\[
\ell_{\pm}^e := - \left( \partial_t + \hat{c}_\pm \cdot \nabla_x \right) \left[ (I - P) [f_{\pm}] \right] - \frac{1}{\varepsilon} \mathcal{L} [f_{\pm}],
\]
\[
h_{\pm}^e := - \left\{ \left( a_{\pm}^\varepsilon - \frac{\rho_{\pm,2}}{\rho_{\pm,1}} c_{\varepsilon} \right) + p \cdot b_{\varepsilon} + p_{\pm,0} c_{\varepsilon} \left( \partial_t + \hat{c}_\pm \cdot \nabla_x \right) M_{\pm}^1 + c_{\varepsilon} \left( \partial_t + \hat{c}_\pm \cdot \nabla_x \right) \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \right) + \hat{h}_{\pm} \right\}.
\]
For fixed \( t, x \), we compare the coefficients in front of
\[
\left\{ \left( \begin{array}{c} M_{\pm}^1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ M_{\pm}^2 \end{array} \right), p_{\pm} \left( \begin{array}{c} M_{\pm}^1 \\ 0 \end{array} \right), \left( p_{\pm} M_{\pm}^2 \right), \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} M_{\pm}^2 \right), \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} M_{\pm}^2 \right), \left( \frac{p_{\pm}}{\rho_{\pm}} M_{\pm}^2 \right) \right\}
\]
on both sides of (6.13) and get the following macroscopic equations:
\[
\partial_t a_{\pm}^\varepsilon - \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \partial_t c_{\varepsilon} = \ell_{\pm,a}^e + h_{\pm,a}^e + \partial_t \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \right) c_{\varepsilon},
\]
(6.17)
\[ \partial_t b_\pm^\varepsilon + c \partial_t c_\pm^\varepsilon = \ell_\pm^\varepsilon + h_\pm^\varepsilon + \frac{\rho_{\pm,2}}{\rho_{\pm,1}} E_{R,i}^\varepsilon, \]
\[ \partial_t c_\pm^\varepsilon = \ell_{\pm,c}^\varepsilon + h_\pm^\varepsilon, \]
\[ c \partial_t a_\pm^\varepsilon - \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \partial_t c_\pm^\varepsilon \pm e_\pm^0 \frac{u_\pm^0}{k_B T_0} E_{R,i}^\varepsilon = \ell_{\pm,ai}^\varepsilon + h_\pm^\varepsilon + c \partial_t \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \right) c_\pm^\varepsilon, \]
\[ c \partial_t b_i^\varepsilon = \ell_{\pm,ii}^\varepsilon + h_\pm^\varepsilon, \]
\[ c \partial_t b_\pm^\varepsilon + c \partial_t b_i^\varepsilon = \ell_{\pm,ij}^\varepsilon + h_\pm^\varepsilon. \quad i \neq j. \]

Here \( \ell_a^\varepsilon, h_a^\varepsilon, \ell_{bi}^\varepsilon, h_{bi}^\varepsilon, \ell_c^\varepsilon, h_c^\varepsilon, \ell_{ai}^\varepsilon, h_{ai}^\varepsilon, \ell_{ii}^\varepsilon, h_{ii}^\varepsilon \) and \( \ell_{ij}^\varepsilon, h_{ij}^\varepsilon \) take the form
\[ (\ell^\varepsilon, \zeta) \quad \text{and} \quad (h^\varepsilon, \zeta), \]
where \( \zeta \) is linear combinations of vectors in (6.16). Combing (6.17) and (6.19) yields
\[ \partial_t a_\pm^\varepsilon = \ell_{\pm,a}^\varepsilon + h_{\pm,a}^\varepsilon + \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \left( \ell_{\pm,c}^\varepsilon + h_\pm^\varepsilon \right) + \partial_t \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \right) c_\pm^\varepsilon. \]

For \( j = 0, 1 \), we have the following estimates:
\[ \left\| \nabla_x^j h_\pm^\varepsilon \right\| + \left\| \nabla_x^j h_\pm^\varepsilon \right\| + \left\| \nabla_x^j h_\varepsilon \right\| + \left\| \nabla_x^j h_\varepsilon \right\| + \left\| \nabla_x^j h_\varepsilon \right\| \]
\[ \lesssim Z \left( \left\| \nabla_x^j E_{R}^\varepsilon \right\| + \left\| \nabla_x^j B_{R}^\varepsilon \right\| \right) + \left( Z + e^{\frac{1}{2}} \right) \left\| \nabla_x^j f_\varepsilon \right\| \]
\[ + \sum_{l=1}^{2k-1} \left( \left\| M^{-\frac{1}{2}} F_1 \right\|_{H^3} \left\| \nabla_x^j F_\varepsilon \right\|_{\sigma} + \left\| M^{-\frac{1}{2}} F_1 \right\|_{H^3} \left\| \nabla_x^j F_\varepsilon \right\| \right) \]
\[ + \sum_{l=1}^{2k-1} \varepsilon^l (1 + t)^{-l-1} \left( \left\| \nabla_x^j f_\varepsilon \right\|_{\sigma} + \left\| \nabla_x^j E_{R}^\varepsilon \right\| + \left\| \nabla_x^j B_{R}^\varepsilon \right\| \right) + Z \left\| \nabla_x^j f_\varepsilon \right\| + e^k (1 + t)^{4k-2} \]
\[ \lesssim \left\| \nabla_x^j (I - P) [f_\varepsilon] \right\|_{\sigma} + \left\| \nabla_x^j f_\varepsilon \right\| + \left[ (1 + t)^{-\beta_0} + \varepsilon \right] \left( \left\| \nabla_x^j E_{R}^\varepsilon \right\| + \left\| \nabla_x^j B_{R}^\varepsilon \right\| \right) + e^k (1 + t)^{4k-2}. \]

The estimates w.r.t. \( b_\varepsilon \) and \( c_\varepsilon \) in (6.2) can be derived similarly as in [40, Lemma 6.1] and [74]. For brevity, we only give the estimate of \( \left\| \nabla_x a_\varepsilon \right\| \) and \( \left\| a_\varepsilon - a_\varepsilon \right\| \). From (6.20), we have
\[ -c \Delta a_\varepsilon^\pm + \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \Delta c_\varepsilon^\pm \pm e_\pm^0 \frac{u_\pm^0}{k_B T_0} \nabla_x \cdot E_{R}^\varepsilon \]
\[ = -3 \sum_{i=1}^{3} \partial_i \left( \ell_{\pm,ai}^\varepsilon + h_{\pm,ai}^\varepsilon \right) - c \nabla_x \left( \frac{\rho_{\pm,2}}{\rho_{\pm,1}} \right) \cdot \nabla_x c_\varepsilon^\pm \pm \nabla_x \left( e_\pm^0 \frac{u_\pm^0}{k_B T_0} \right) \cdot E_{R}^\varepsilon. \]

On the other hand, by (2.16) and \( \bar{n} = e_{\pm} n_\pm \), we have
\[ \nabla_x \cdot E_{R}^\varepsilon = 4\pi \int_{\mathbb{R}^3} \left( e_{+} \sqrt{M} f_+^\varepsilon - e_{-} \sqrt{M} f_-^\varepsilon \right) dp \]
\[ = 4\pi \left[ e_{+} n_{+,0} a_+^\varepsilon (u_+^0 - c) - e_{-} n_{-,0} a_-^\varepsilon (u_-^0 - c) \right] + \frac{4\pi}{c} \left[ e_{+} n_{+,0} a_+^\varepsilon (u_+^0 - c) - e_{-} n_{-,0} a_-^\varepsilon (u_-^0 - c) \right]. \]

Collecting (6.27) in (6.26), we have
\[ -c \Delta a_\varepsilon^\pm + \frac{e_\pm^0}{n_{\pm,0}} \Delta c_\varepsilon^\pm \pm e_\pm^0 \frac{4\pi \bar{n}}{k_B T_0} \left( a_+^\varepsilon - a_-^\varepsilon \right) = -3 \sum_{i=1}^{3} \partial_i \left( \ell_{\pm,ai}^\varepsilon + h_{\pm,ai}^\varepsilon \right) + \Xi_{\pm,5} \left[ a_\varepsilon^\pm, b_\varepsilon, c_\varepsilon \right], \]
where
\[
\Xi_{\pm,5}[a^\pm, b^\pm, c^\pm] = -c^{-\varepsilon}(u^0_\pm - c) \Delta c^\pm \mp \sum_{i=1}^\varepsilon \frac{4\pi \bar{n}(u^0_\pm - c)}{ck_BT_i_0}(a_+^\varepsilon - a_-^\varepsilon) \tag{6.29}
\]
\[-c'\nabla_x \left( \frac{P_{\pm,2}}{P_{\pm,1}} \right) \cdot \nabla_x c^\pm \mp \nabla_x \left( \varepsilon \frac{u^0_\pm}{k_BT_i_0} \right) \cdot E_R^T.\]

Note that
\[
\left| \left\langle c \frac{\varepsilon_{\pm,0}}{n_{\pm,0} \Delta c^\pm}, n_{\pm,0}a_+^\pm \right\rangle \right| = c \left| \left\langle \frac{\varepsilon_{\pm,0}}{n_{\pm,0}} \nabla_x c^\pm, \nabla_x (n_{\pm,0}a_+^\pm) \right\rangle \right| \tag{6.30}
\leq o(1) \| \nabla_x a_+^\pm \|^2 + C \| \nabla x c^\pm \|^2 + \left| \left\langle \Xi_{\pm,5}[a^\pm, b^\pm, c^\pm], n_{\pm,0}a_+^\pm \right\rangle \right|.
\]

We multiply (6.28) by \(n_{\pm,0}a_+^\pm\) and add the resulting equalities to have
\[
\frac{3c}{4} \left( \| n_{+0} \nabla_x a_+^\pm \|^2 + \| n_{-0} \nabla x a_+^\pm \|^2 \right) + \left\| \frac{4\pi}{k_BT_i_0} \bar{n}(a_+^\varepsilon - a_-^\varepsilon) \right\|^2 \tag{6.31}
\leq \sum_{\pm} \sum_{i=1}^3 \left\langle \ell_{\pm,ai}^\varepsilon, n_{\pm,0} \partial_t a_+^\pm \right\rangle + \sum_{\pm} \sum_{i=1}^3 \left\langle h_{\pm,ai}^\varepsilon, n_{\pm,0} \partial_t a_+^\pm \right\rangle + \sum_{\pm} \left| \left\langle \Xi_{\pm,5}[a^\pm, b^\pm, c^\pm], n_{\pm,0}a_+^\pm \right\rangle \right| + C \| \nabla_x c^\pm \|^2.
\]

For the first term in (6.31), we have
\[
\left\langle \ell_{\pm,ai}^\varepsilon, n_{\pm,0} \partial_t a_+^\pm \right\rangle = \left\langle \left( - \partial_t + cP_{\pm} \cdot \nabla_x \left[ (I - P_{\pm})[f_+^\pm] \right] - \frac{1}{\varepsilon} \mathcal{L}[f_+^\pm], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle \tag{6.32}
\leq \left\langle \left( - \partial_t \left[ (I - P_{\pm})[f_+^\pm] \right], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle + \left\langle \left( cP_{\pm} \cdot \nabla_x \left[ (I - P_{\pm})[f_+^\pm] \right] - \frac{1}{\varepsilon} \mathcal{L}[f_+^\pm], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle \leq - \frac{d}{dt} \left\langle \left( \left[ (I - P_{\pm})[f_+^\pm], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle + \left\langle \left( \left[ (I - P_{\pm})[f_+^\pm], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle \right\rangle \right\rangle + o(1) \| \nabla_x a_+^\pm \|^2 + C \left( \| \nabla_x (I - P_{\pm})[f_+^\pm] \|^2 + \varepsilon^{-2} \| (I - P_{\pm})[f_+^\pm] \|^2 \right).
\]

By (6.24) and (6.25), we have
\[
\left\langle \left( \left[ (I - P_{\pm})[f_+^\pm], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle \right\rangle \leq \left\langle \left( \partial_t \left[ (I - P_{\pm})[f_+^\pm] \right], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle + \left\langle \left( \partial_t n_{\pm,0}(I - P_{\pm})[f_+^\pm], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle \right\rangle \left\| \partial_t a_+^\pm \right\|^2 + \left\| (I - P_{\pm})[f_+^\pm] \right\|^2 + \| \nabla_x (I - P_{\pm})[f_+^\pm] \|^2 \leq \| (I - P_{\pm})[f_+^\pm] \|^2 + \| \nabla_x (I - P_{\pm})[f_+^\pm] \|^2 + \left( 1 + t \right)^{-\beta_0} + \varepsilon \left( \| E_R^\varepsilon \|^2 + \| B_R^\varepsilon \|^2 \right) + \varepsilon^k (1 + t)^{4k-2}.
\]

Therefore,
\[
\sum_{\pm} \sum_{i=1}^3 \langle \ell_{\pm,ai}^\varepsilon, n_{\pm,0} \partial_t a_+^\pm \rangle \leq - \frac{d}{dt} \left\langle \left( \left[ (I - P_{\pm})[f_+^\pm], \zeta_{\pm,ai} \right), n_{\pm,0} \partial_t a_+^\pm \right\rangle + o(1) \| \nabla x a_+^\varepsilon \|^2 + C \| \nabla x (I - P_{\pm})[f_+^\varepsilon] \|^2 \right\rangle \left( \| E_R^\varepsilon \|^2 + \| B_R^\varepsilon \|^2 \right) + \varepsilon^k (1 + t)^{4k-2}. \tag{6.34}
\]
For the second and third terms in (6.31), we use (6.25) to have
\[
\sum_{\pm} \sum_{i=1}^{3} \left\langle \left\langle n_{\pm, ai}^{\xi}, n_{\pm, 0} a_{\pm}^{\xi} \right\rangle \right\rangle \lesssim o(1) \| \nabla_x a^\xi \|^2 + \| (I - P) [f^\xi] \|_\sigma^2 + \| \nabla_x (I - P) [f^\xi] \|_\sigma^2 + \| f^\xi \|^2 \] (6.35)
\[+ (1 + t)^{-\beta_0} \left( \| E_R^\xi \|^2 + \| B_R^\xi \|^2 \right) + \varepsilon^k (1 + t)^{4k - 2}.
\]
and
\[
\sum_{\pm} \left| \left\langle \Xi_{\pm, 5} [a_{\pm}^{\xi}, b^\xi, c^\xi], n_{\pm, 0} a_{\pm}^{\xi} \right\rangle \right| \lesssim o(1) \| \nabla_x P [f^\xi] \|^2 + C(1 + t)^{-\beta_0} \left( \| f^\xi \|^2 + \| E_R^\xi \|^2 \right). \] (6.36)
Collecting the estimates (6.34), (6.35) and (6.36) in (6.31), we obtain
\[
\frac{C}{2} \left( \| n_{\pm, 0} a_{\pm}^{\xi} \|^2 + \frac{1}{2} \| \sqrt{n_{\pm, 0} a_{\pm}^{\xi}} \|^2 \right) + \left\| \sqrt{\frac{4\pi}{kBT_0}} \tilde{n} (a_{+}^{\xi} - a_{-}^{\xi}) \right\|^2 \] (6.37)
\[\leq - \frac{d}{dt} \left( \langle (I - P) [f^\xi] \rangle, \frac{\partial}{\partial x_i} \right) + C \| \nabla_x (I - P) [f^\xi] \|_\sigma^2 \]
\[+ C \left( \varepsilon^{-2} \| (I - P) [f^\xi] \|_\sigma^2 + \| f^\xi \|^2 + \left( (1 + t)^{-\beta_0} + \varepsilon \right) \left( \| E_R^\xi \|^2 + \| B_R^\xi \|^2 \right) + \varepsilon^k (1 + t)^{4k - 2} \right). \]

\[\square\]

**Remark 6.1.** The explicit form of $P [f^\xi]$ is designed to make sure that
\[
\int_{\mathbb{R}^3} M_{\pm}^i \xi f_{ij}^\xi dp = a_{\pm}^{\xi}.
\]
With this expression, we can derive $\nabla_x \cdot E_R^\xi \sim a_{+}^{\xi} - a_{-}^{\xi}$ as in (6.27). Then we can further obtain the dissipation of $a_{+}^{\xi} - a_{-}^{\xi}$ and $\nabla_x a_{\pm}^{\xi}$ at the same time.

### 6.2 Electromagnetic Dissipation

In this subsection, we derive the dissipation of the electromagnetic field $(E_R^\xi, B_R^\xi)$.

**Proposition 6.2.** It holds that, for $i = 0, 1$
\[
\frac{1}{2} \varepsilon^{i+1} \left\| \frac{e_+ u_0^0}{kBT_0} \nabla_x E_R^\xi \right\|^2 \] (6.38)
\[\leq - \sum_{j=1}^{3} \varepsilon^{i+1} \frac{d}{dt} \left( \left\langle \left( \nabla_x (I - P) [f^\xi] \right), \nabla_x E_{R,j}^\xi \right\rangle \right) + C \varepsilon^{i+1} \| (I - P) [f^\xi] \|_\sigma^2 \]
\[+ C \left[ \varepsilon^{i+1} \| \nabla_x P [f^\xi] \|_\sigma^2 + \varepsilon (E + D) + \varepsilon^k (1 + t)^{4k+i-1} \right].
\]

**Proof.** For brevity, we only prove (6.38) for $i = 0$ since the case $i = 1$ can be proved in the same way. From the third equality in (6.17), we have
\[
c_\beta a_{\pm}^{\xi} - c_{\rho+1} \frac{\rho_+ + 2}{\rho_+} \partial_t a_\xi + \frac{e_+ u_0^0}{kBT_0} E_{R,i}^\xi = \ell_{\pm, ai}^{\xi} + h_{\pm, ai}^{\xi}.
\] (6.39)
We multiply this equation by $E_{R,i}^\varepsilon$ to have

$$
\left\| \sqrt{\frac{e_+ u_+^0}{k_B T_0}} E_{R}^\varepsilon \right\|^2 \leq \sum_{i=1}^3 \left\langle \ell_{+,ai}^\varepsilon, E_{R,i}^\varepsilon \right\rangle + \sum_{i=1}^3 \left| \left\langle c\partial_i a_+^\varepsilon c - c\frac{\rho_{+,2}}{\rho_{+,1}} \partial_i E_{R,i}^\varepsilon \right\rangle \right| \right| + \sum_{i=1}^3 \left| \left\langle h_{+,ai}^\varepsilon, E_{R,i}^\varepsilon \right\rangle \right|. \tag{6.40}
$$

For the first term in (6.40), we use (2.16) to have

$$
\left\langle \ell_{+,ai}^\varepsilon, E_{R,i}^\varepsilon \right\rangle \left( - \partial_i (I - P_+) [f_+^\varepsilon], \zeta_{+,ai} \right) \left( - c\hat{p}_+ \cdot \nabla_x (I - P_+) [f_+^\varepsilon] - \frac{1}{\varepsilon} \mathcal{L} [f_+^\varepsilon], \zeta_{+,ai} \right) \geq \frac{d}{dt} \left\langle \left[ (I - P_+) [f_+^\varepsilon], \zeta_{+,ai} \right], E_{R,i}^\varepsilon \right\rangle + \frac{d}{dt} \left\langle \left[ (I - P_+) [f_+^\varepsilon], \zeta_{+,ai} \right], \partial_i E_{R,i}^\varepsilon \right\rangle + o(1) \| E_{R}^\varepsilon \|^2 + C \left\langle \left\| \nabla_x (I - P) [f_+^\varepsilon] \right\|_{\|}^2 + \varepsilon^{-2} \| (I - P) [f_+^\varepsilon] \|_{\|}^2 \right\rangle ^2
$$

For the second and third terms in (6.40), we bound them as follows:

$$
\sum_{i=1}^3 \left( c\partial_i a_+^\varepsilon c - c\frac{\rho_{+,2}}{\rho_{+,1}} \partial_i E_{R,i}^\varepsilon \right) \lesssim o(1) \| E_{R}^\varepsilon \|^2 + \| \nabla_x P [f_+^\varepsilon] \|^2 \tag{6.42}
$$

$$
\sum_{i=1}^3 \left( h_{+,ai}^\varepsilon, E_{R,i}^\varepsilon \right) \lesssim o(1) \| E_{R}^\varepsilon \|^2 + \| (I - P) [f_+^\varepsilon] \|_{\|}^2 + \| \nabla_x (I - P) [f_+^\varepsilon] \|_{\|}^2 + \varepsilon^2 \| \nabla_x B_{R}^\varepsilon \|^2 + \| f_+^\varepsilon \|^2
$$

We collect the above estimates in (6.40) to obtain

$$
\frac{1}{2} \left\| \sqrt{\frac{e_+ u_+^0}{k_B T_0}} E_{R}^\varepsilon \right\|^2 \leq - \sum_{i=1}^3 \frac{d}{dt} \left\langle \left[ (I - P_+) [f_+^\varepsilon], \zeta_{+,ai} \right], E_{R,i}^\varepsilon \right\rangle + C \left\langle \left\| \nabla_x (I - P) [f_+^\varepsilon] \right\|_{\|}^2 + \varepsilon^{-2} \| (I - P) [f_+^\varepsilon] \|_{\|}^2 \right\rangle
$$

This verifies (6.38) for $i = 0$.

**Proposition 6.3.** It holds that

$$
\frac{c}{2} \varepsilon^2 \| \nabla_x B_{R}^\varepsilon \|^2 \leq \varepsilon^2 \frac{d}{dt} \langle E_{R}^\varepsilon, \nabla_x \times B_{R}^\varepsilon \rangle + c \varepsilon^2 \| \nabla_x E_{R}^\varepsilon \|^2 + C \varepsilon^2 \mathcal{E}. \tag{6.44}
$$

**Proof.** From (2.16), we have

$$
\langle \nabla_x \times B_{R}^\varepsilon, \partial_t E_{R}^\varepsilon \rangle \leq \frac{d}{dt} \langle E_{R}^\varepsilon, \nabla_x \times B_{R}^\varepsilon \rangle - \langle E_{R}^\varepsilon, \nabla_x \times \partial_t B_{R}^\varepsilon \rangle + C \| f_+^\varepsilon \| \| \nabla_x \times B_{R}^\varepsilon \| \tag{6.45}
$$
\[ \leq \frac{d}{dt} \langle E^e_R, \nabla_x \times B^e_R \rangle + c \langle E^e_R, \nabla_x \times (\nabla_x \times E^e_R) \rangle + o(1) \| \nabla_x \times B^e_R \|^2 + C \| f^e \|^2. \]

Noting
\[ c \langle E^e_R, \nabla_x \times (\nabla_x \times E^e_R) \rangle = c \| \nabla_x \times E^e_R \|^2, \] (6.46)
we have
\[ \frac{c}{2} \| \nabla_x \times B^e_R \|^2 \leq \frac{d}{dt} \langle E^e_R, \nabla_x \times B^e_R \rangle + c \| \nabla_x \times E^e_R \|^2 + C \| f^e \|^2. \] (6.47)

Noting \( \nabla \cdot B^e_R = 0 \), this further implies (6.44).

**Remark 6.2.** By proper linear combination of (6.2), (6.38) and (6.44), we can obtain the macroscopic dissipation and the electromagnetic field dissipation together. However, the dissipation of the electromagnetic field are too weak to be necessarily included in \( D \).

## 7 Proof of Main Theorem

**Proof of Proposition 2.1.**

**Proof of Energy Estimates:** We multiply (6.2) by a sufficiently small constant \( \kappa_1 \) and collect the resulting inequality, (4.1), (4.14) and (4.36) to have
\[ \frac{d}{dt} \left\{ \sum_{i=0}^{2} \varepsilon^i \left( \| \nabla_x f^e \|^2 + \| \nabla_x E^e_R \|^2 + \| \nabla_x B^e_R \|^2 \right) \right\} - \kappa_1 \left( \varepsilon \mathcal{E}^1_{mac} + \varepsilon^2 \mathcal{E}^2_{mac} \right) \]
\[ + \delta \varepsilon \| \nabla_x P[f^e] \|^2 + \delta \varepsilon^2 \| \nabla_x^2 P[f^e] \|^2 \]
\[ + \left( \varepsilon^{-1} \| (I - P)[f^e] \|_{\sigma}^2 + \| \nabla_x (I - P)[f^e] \|_{\sigma}^2 + \varepsilon \| \nabla_x^2 (I - P)[f^e] \|_{\sigma}^2 \right) \]
\[ \lesssim \left[ 1 + (1-t)^{-\beta_0} + \varepsilon \right] E + \varepsilon(\mathcal{E} + D) + \varepsilon^{2k+1} (1 + t)^{4k-2} + \varepsilon^k (1 + t)^{2k-2} \sqrt{E}. \] (7.1)

Multiplying (7.1) by a large constant \( C_1 \) and adding it to the sum of (5.3), (5.16), and (5.29), we have
\[ \frac{d}{dt} \mathcal{E} + \frac{3}{2} D \lesssim \varepsilon \mathcal{E} + \varepsilon D + \varepsilon^{2k+1} (1 + t)^{4k-2} + \varepsilon^k (1 + t)^{2k-2} \sqrt{E}, \] (7.2)
where \( D \) is given in (2.42), and
\[
\mathcal{E} = C_1 \left[ \sum_{i=0}^{2} \varepsilon^i \left( \| \nabla_x^i f^e \|^2 + \| \nabla_x^i B^e_R \|^2 \right) \right] - \kappa_1 \left( \varepsilon \mathcal{E}^1_{mac} + \varepsilon^2 \mathcal{E}^2_{mac} \right) \]
\[ + \left( \| (I - P)[f^e] \|_{w,a}^2 + \varepsilon \| \nabla_x f^e \|_{w,a}^2 + \| \nabla_x^2 (I - P)[f^e] \|_{w,a}^2 \right) \].
(7.3)

Note that
\[
\varepsilon \mathcal{E}^1_{mac} + \varepsilon^2 \mathcal{E}^2_{mac} \lesssim \varepsilon \| f^e \| \| \nabla_x f^e \| + \varepsilon^2 \| \nabla_x f^e \| \| \nabla_x^2 f^e \| \]
\[ \lesssim \varepsilon^\frac{1}{2} \left( \| f^e \|^2 + \varepsilon \| f^e \| \| \nabla_x f^e \| + \varepsilon^2 \| \nabla_x f^e \| \right) \] (7.4)
by (6.1). This verifies (2.41). Then for sufficiently small constant \( \varepsilon > 0 \), we use (7.2) to have
\[ \frac{d}{dt} \mathcal{E} + D \lesssim (\varepsilon + Z) \mathcal{E} + \varepsilon^{2k+1} (1 + t)^{4k-2} + \varepsilon^k (1 + t)^{2k-2} \sqrt{E}. \] (7.5)
By the fact $\frac{k}{2k-1} < 1$, and
\[
\int_0^{\varepsilon^{-k/(2k-1)}} \varepsilon^k (1+t)^{2k-2} dt \lesssim 1
\]
for $k \geq 3$, we apply Gronwall’s inequality to the above inequality to have
\[
\sup_{s \in [0, \bar{t}]} \mathcal{E}(s) \lesssim \mathcal{E}(0) + 1.
\]
This verifies (2.40).

**Proof of Positivity:** First we show that there exists $F^\varepsilon_{\pm R}(0, x, p)$ such that $F^\varepsilon_{\pm R}(0, x, p) \geq 0$. The procedure is motivated by the analysis in [40, Lemma A.2]. We first estimate the microscopic part of the coefficients $(I - P) \left[ \frac{F_{\pm,1}}{\sqrt{M_{\pm}}} \right]$. By (2.3) and the definition of $\mathcal{L}$ in (2.17), we have
\[
\mathcal{L}_{\pm} \left[ (I - P) \left[ \frac{F_{\pm,1}}{\sqrt{M_{\pm}}} \right] \right] = -\frac{1}{\sqrt{M_{\pm}}} \left[ \partial_t M_{\pm} + c_\pm \cdot \nabla x M_{\pm} \pm e_\pm (E_0 + \hat{p}_\pm \times B_0) \cdot \nabla p M_{\pm} \right] \tag{7.6}
\]
Then, we use Lemma 3.2 to have
\[
\left\| (I - P) \left[ \frac{F_{\pm,1}}{\sqrt{M_{\pm}}} \right] \right\| \lesssim \left\| \nabla_x (n_{\pm,0}, u_{\pm}, T_0) \right\| + |E_0| + |B_0|. \tag{7.7}
\]
By similar arguments in the proof of Lemma 3.3, we can obtain that for any $\kappa < 1$,
\[
\left( M_{\pm}^{-\kappa} \mathcal{L} \left[ (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right], (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right) \geq M_{\pm}^{-\frac{\kappa}{2}} \left[ (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right]^2 - C \left\| (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right\|^2 |E_0|^2 + |B_0|^2. \tag{7.8}
\]
Here we used $F_1$ and $M$ to denote $\left( \frac{F_{+,1}}{F_{-,1}} \right)$ and $\left( \frac{M_{+,1/2}}{M_{-,1/2}} \right)$, respectively for simplicity. Now we combine (7.6), (7.7) and (7.8) to have
\[
\left\| M_{\pm}^{-\frac{\kappa}{2}} (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right\|^2 - C \left\| (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right\|^2 \lesssim o(1) \left\| M_{\pm}^{-\frac{\kappa}{2}} (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right\|^2 + C \left\| \nabla_x (n_0, u, T_0) \right\|^2 + |E_0|^2 + |B_0|^2. \tag{7.9}
\]
Namely,
\[
\left\| M_{\pm}^{-\frac{\kappa}{2}} (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right\| \lesssim \left\| \nabla_x (n_0, u, T_0) \right\| + |E_0| + |B_0|. \tag{7.10}
\]
Similarly, we can obtain that
\[
\sum_{0 \leq j \leq 2} \left\| \nabla_p^j \left( M_{\pm}^{-\frac{\kappa}{2}} (I - P) \left[ \frac{F_1}{\sqrt{M}} \right] \right) \right\| \lesssim \left\| \nabla_x (n_0, u, T_0) \right\| + |E_0| + |B_0|. \tag{7.11}
\]
By the Sobolev imbedding, this implies
\[
(I - P) \left[ \frac{F_i}{\sqrt{M}} \right] \lesssim M^{-\frac{3}{2}} \left[ |\nabla_x(n_0, u, T_0)| + |E_0| + |B_0| \right].
\] (7.12)

By induction, we can use the equations (A.3), (A.4) and (A.5) in the appendix to obtain
\[
(I - P) \left[ \frac{F_i}{\sqrt{M}} \right] \lesssim M^{-\frac{3}{2}} \left[ |\nabla_x^i(n_0, u, T_0)| + |\nabla_x^{i-1}E_0| + |\nabla_x^{i-1}B_0| \right)
+ \sum_{1 \leq j \leq i} \left( |\nabla_x^{i-j}(a_j, b_j, c_j)| + |\nabla_x^{i-j-1}E_j| + |\nabla_x^{i-j-1}B_j| \right)
\] (7.13)

for all $\kappa < 1$ and $2 \leq i \leq 2k - 1$. Note that here $a_j = \left( \begin{array}{c} a_i \\ a_{-i} \end{array} \right), b_j, c_j$, the coefficients of the macroscopic part of $P\left( \frac{F_i}{\sqrt{M}} \right)$, are defined in (A.1). Then we use (7.13) to have
\[
F_{\pm,i}(0, x, p) \lesssim M_{\pm}^k \left[ |\nabla_x^i(n_{\pm,0}, u_{\pm, T_0})| + |\nabla_x^{i-1}E_0| + |\nabla_x^{i-1}B_0| \right) + |(a_{\pm,j}, b_j, c_j)|
+ \sum_{1 \leq j \leq i} \left( |\nabla_x^{i-j}(a_{\pm,j}, b_j, c_j)| + |\nabla_x^{i-j-1}E_j| + |\nabla_x^{i-j-1}B_j| \right)
\] (7.14)

Now we choose $F_{\pm,R}^i(0, x, p)$ in the following form
\[
F_{\pm,R}^i(0, x, p) = M_{\pm}^k(0, x, p) \left[ \sum_{j=1}^{2k-1} \left( |\nabla_x^j(n_{\pm,0}, u_{\pm, T_0})| + |\nabla_x^{j-1}E_j| + |\nabla_x^{j-1}B_j| \right) + \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-1-i} \left( |\nabla_x^j(a_{\pm,i}, b_i, c_i)| + |\nabla_x^{j-1}E_i| + |\nabla_x^{j-1}B_i| \right) \right]
\] (7.15)

with $0 < \tau < 1$. We choose $\kappa < 1$ such that
\[
k(1 - \kappa) + \tau < \kappa.
\] (7.16)

From (7.14), we have
\[
\sum_{i=1}^{2k-1} \varepsilon^i F_{\pm,i}(0, x, p) \leq C\varepsilon M^\kappa(0, x, p) \left[ \sum_{j=1}^{2k-1} \left( |\nabla_x^j(n_{\pm,0}, u_{\pm, T_0})(0, x)| + |\nabla_x^{j-1}(E_j, B_j)(0, x)| \right) + \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-1-i} \left( |\nabla_x^j(a_{\pm,i}, b_i, c_i)| + |\nabla_x^{j-1}(E_i, B_i)| \right) \right]
\] (7.17)

for some uniform constant $C_0 \geq 1$. We discuss the positivity of $F_{\pm,R}^i(0, x, p)$ in two domains in $\mathbb{R}_x^3 \times \mathbb{R}_p^3$:
\[
A_\pm := \left\{ (x, p) : M_{\pm}(0, x, p) \geq C_0 \varepsilon M_{\pm}^\kappa(0, x, p) \right\},
\] (7.18)
\[
B_\pm := \left\{ (x, p) : M_{\pm}(0, x, p) < C_0 \varepsilon M_{\pm}^\kappa(0, x, p) \right\}.
\] (7.19)

In the domain $A_\pm$, by the expression of the Hilbert expansion (2.1), we have $F_{\pm,R}^i(0, x, p) \geq 0$. In the domain $B_\pm$, for the chosen $\kappa$, we have
\[
\varepsilon^k M_{\pm}^\kappa(0, x, p) > C_0^{k+1} \varepsilon^{k+1} M_{\pm}^\kappa(0, x, p) \geq C_0 \varepsilon M_{\pm}^{k(1-\kappa)}(0, x, p) M_{\pm}^\kappa(0, x, p) \geq C_0 \varepsilon M_{\pm}^\kappa(0, x, p).
\] (7.20)
This implies that the remainder term is the dominant term in (2.1) and \( F^\varepsilon_\pm (0, x, p) \geq 0 \) for \( \varepsilon \) small enough. Therefore we have \( F^\varepsilon_\pm (0, x, p) \geq 0 \) for all \( (x, p) \).

Based on the proof of [83, Lemma 9, Page 307–308], we may rearrange the equation (1.1) as

\[
\begin{align*}
\partial_t F^\varepsilon_\pm + \bar{p} \cdot \nabla_x F^\varepsilon_\pm + e_\pm \left( E^\varepsilon + \bar{p} \times b^\varepsilon \right) \nabla_p F^\varepsilon_\pm &= \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^3} \Phi^{ij}(p, q)(F^\varepsilon_+(q) + F^\varepsilon_-(q)) dq \right) \partial_{p_i} \partial_{p_j} F^\varepsilon_\pm \\
&+ \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^3} \partial_{p_i} \Phi^{ij}(p, q)(F^\varepsilon_+(q) + F^\varepsilon_-(q)) dq \right) \partial_{p_j} F^\varepsilon_\pm - \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \partial_{q_j} (F^\varepsilon_+(q) + F^\varepsilon_-(q)) dq \right) \partial_{p_i} F^\varepsilon_\pm \\
&- \frac{1}{\varepsilon} \partial_{p_i} \left( \int_{\mathbb{R}^3} \Phi^{ij}(p, q) \partial_{q_j} (F^\varepsilon_+(q) + F^\varepsilon_-(q)) dq \right) F^\varepsilon_\pm. 
\end{align*}
\]

Then clearly, there is an elliptic structure on the R.H.S. of (7.21). Therefore, using the maximum principle (see the proof of [83, Lemma 9, Page 308] and [65, Theorem 1.1, Page 201]), we have for \( F = \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \)

\[
\min_{t,x,p} \{ F^\varepsilon \} = \min_{x,p} \{ F^0_0 \} \geq 0. 
\]

Then for sufficiently smooth \( F^\varepsilon \), as long as the initial data \( F^\varepsilon_\pm (0, x, p) \geq 0 \), we naturally have \( \min_{t,x,p} \{ F^\varepsilon \} \geq 0 \). For general \( F^\varepsilon \), a standard mollification and approximation argument leads to the desired result. \( \square \)

A Appendix

In this part, we list our result about the construction and regularity estimates of the coefficients in the Hilbert expansion (2.1). For \( n \in [1, 2k - 1] \), we decompose \( \frac{F_{\pm, n}}{\sqrt{M_{\pm}}} \) as the sum of macroscopic and microscopic parts:

\[
\frac{F_{\pm, n}}{\sqrt{M_{\pm}}} = P_{\pm} \left[ \frac{F_{\pm, n}}{\sqrt{M_{\pm}}} \right] + (I - P_{\pm}) \left[ \frac{F_{\pm, n}}{\sqrt{M_{\pm}}} \right] = \left( a_{\pm, n}(t, x) + b_{\pm, n}(t, x) \cdot p + c_{n}(t, x)p^0_{\pm} \right) \sqrt{M_{\pm}} + (I - P_{\pm}) \left[ \frac{F_{\pm, n}}{\sqrt{M_{\pm}}} \right].
\]

**Proposition A.1.** For any \( n \in [0, 2k - 2] \), assume that \( (F_i, E_i, B_i) \) have been constructed for all \( 0 \leq i \leq n \). Then the microscopic part \( (I - P_{\pm}) \left[ \frac{F_{\pm, n+1}}{\sqrt{M_{\pm}}} \right] \) can be written as:

\[
(I - P_{\pm}) \left[ \frac{F_{\pm, n+1}}{\sqrt{M_{\pm}}} \right] = \mathcal{L}^{-1}_{\pm} \left[ - \frac{1}{\sqrt{M_{\pm}}} \left( \partial_t F_{\pm, n} + c\bar{p}_{\pm} \cdot \nabla_x F_{\pm, n} - \frac{1}{\varepsilon} \sum_{i+j=n+1, i,j \geq 1} \mathcal{C}(F^\varepsilon_{\pm, i}, F^\varepsilon_{\pm, j}) \right) \right. \\
+ \mathcal{C}(F^\varepsilon_{\pm, i}, F^\varepsilon_{\pm, j}) \right] + \sum_{i+j=n, i,j \geq 0} e_{\pm} \left( E_i + c\bar{p}_{\pm} \times B_i \right) \cdot \nabla_p F_{\pm, j}. 
\]

And \( a_{\pm, n+1}(t, x), b_{n+1}(t, x), c_{n+1}(t, x), E_{n+1}(t, x), B_{n+1}(t, x) \) satisfy the following system:

\[
\begin{align*}
\frac{1}{c} \partial_t \left( n_{\pm, 0} u_{\pm, n+1} + \frac{e_{\pm, 0} + P_{\pm, 0}}{c^2} u_{\pm, n+1} \right) + \nabla_x \left( n_{\pm, 0} u_{\pm, n+1} + \frac{e_{\pm, 0} + P_{\pm, 0}}{c^2} u_{\pm, n+1} \right) + \frac{e_{\pm, 0} + P_{\pm, 0}}{c^2} u_{\pm, n+1} + P_{\pm, 0} b_{n+1} + \frac{e_{\pm, 0} + P_{\pm, 0}}{c^2} u_{\pm, n+1} \\
+ \nabla_x \cdot \int_{\mathbb{R}^3} c\bar{p}_{\pm} \sqrt{M_{\pm}} (I - P_{\pm}) \left[ \frac{F_{\pm, n+1}}{\sqrt{M_{\pm}}} \right] dp = 0, 
\end{align*}
\]

\[
\text{with} \quad n_{\pm, 0} = n_{\pm, 0} \quad \text{and} \quad b_{n+1} = b_{n+1}.
\]
\[ \frac{1}{c} \partial_t \left\{ \frac{\epsilon_{\pm,0} + P_{\pm,0}}{c^2} u_{\pm,0}^0 \cdot a_{\pm,n+1} + \frac{n_{\pm,0} m_{\pm}^2}{\gamma_{\pm} K_2(\gamma_{\pm})} \left( 6 K_3(\gamma_{\pm}) + \gamma_{\pm} K_2(\gamma_{\pm}) \right) u_{\pm,0}^0 \cdot \left( u_{\pm} \cdot b_{n+1} \right) + c^2 K_3(\gamma_{\pm}) u_{\pm,0}^0 b_{n+1,1} \right\} \\
+ \frac{n_{\pm,0} m_{\pm}^2}{\gamma_{\pm} K_2(\gamma_{\pm})} \left( 5 K_3(\gamma_{\pm}) + \gamma_{\pm} K_2(\gamma_{\pm}) \right) u_{\pm,0}^0 \cdot u_{\pm}(u_{\pm} \cdot b_{n+1}) + u_{\pm,0}^0 c_{n+1} \right\} \\
\n+ \nabla_x \cdot \left( \frac{\epsilon_{\pm,0} + P_{\pm,0}}{c^2} u_{\pm,0}^0 u_{\pm,0} a_{\pm,n+1} + \frac{n_{\pm,0} m_{\pm}^2}{\gamma_{\pm} K_2(\gamma_{\pm})} \left( 6 K_3(\gamma_{\pm}) + \gamma_{\pm} K_2(\gamma_{\pm}) \right) u_{\pm,0}^0 \cdot \left( u_{\pm} \cdot b_{n+1,1} \right) + u_{\pm,0}^0 c_{n+1} \right) \\
+ \partial_x \left( P_{\pm,0} a_{\pm,n+1} \right) + \nabla_x \cdot \left( \frac{n_{\pm,0} m_{\pm}^2 c^2 K_3(\gamma_{\pm})}{\gamma_{\pm} K_2(\gamma_{\pm})} u_{\pm,0}^0 \cdot \left( u_{\pm} \cdot b_{n+1} + u_{\pm,0} b_{n+1} \right) \right) \right( A.4 \right) \\
+ \partial_x \left( n_{\pm,0} m_{\pm}^2 c^2 K_3(\gamma_{\pm}) \right) u_{\pm,0}^0 \cdot \left( u_{\pm} \cdot b_{n+1} + u_{\pm,0} b_{n+1} \right) \right) \\
+ e_{\pm} \frac{E_{0,j}}{c} \left( n_{\pm,0} u_{\pm,0}^0 a_{\pm,n+1} + \frac{\epsilon_{\pm,0} + P_{\pm,0}}{c^2} u_{\pm,0}^0 \cdot \left( u_{\pm} \cdot b_{n+1} \right) + \frac{\epsilon_{\pm,0} u_{\pm,0}^0}{c^2} \right) \right) \\
+ e_{\pm} \left( n_{\pm,0} u_{\pm,0}^0 \cdot E_{n+1,j} + \frac{n_{\pm,0} u_{\pm,0}^0}{c^2} \right) \right) \\
+ e_{\pm} \sum_{k+l=n+1, k \geq 1} \frac{E_{k,l}}{c} \left( n_{\pm,0} u_{\pm,0}^0 k a_{\pm,l} + \frac{\epsilon_{\pm,k} + P_{\pm,k}}{c^2} u_{\pm,0}^0 \cdot \left( u_{\pm,k} \cdot b_{l} \right) \right) \right) \\
+ e_{\pm} \sum_{k+l=n+1, k \geq 1} \left( \left( n_{\pm,0} u_{\pm,0}^0 k a_{\pm,l} + \frac{\epsilon_{\pm,k} + P_{\pm,k}}{c^2} u_{\pm,0}^0 \cdot \left( u_{\pm,k} \cdot b_{l} \right) \right) + \frac{\epsilon_{\pm,k} + P_{\pm,k}}{c^2} u_{\pm,0}^0 \cdot \left( u_{\pm,k} \cdot b_{l} \right) \right) \right) \right) \\
+ c \nabla_x \cdot \left( \int_{\mathbb{R}^3} \frac{p_j P}{P_{\pm}^0} \sqrt{M_{\pm}(1 - P_{\pm})} \left[ \frac{F_{\pm,n+1}}{\sqrt{M_{\pm}}} \right] dp + e_{\pm} \left( \int_{\mathbb{R}^3} \hat{p}_j \times B_0 \sqrt{M_{\pm}(1 - P_{\pm})} \left[ \frac{F_{\pm,n+1}}{\sqrt{M_{\pm}}} \right] dp \right) \right) \\
+ e_{\pm} \sum_{k+l=n+1, k \geq 1} \left( \int_{\mathbb{R}^3} \hat{p}_j \times B_k \sqrt{M_{\pm}(1 - P_{\pm})} \left[ \frac{F_{\pm,j}}{\sqrt{M_{\pm}}} \right] dp \right) = 0, \right. \\
for j = 1, 2, 3 with b_{n+1} = (b_{n+1,1, b_{n+1,2, b_{n+1,3}}}, E_{n+1} = (E_{n+1,1, E_{n+1,2, E_{n+1,3}}}, \right. \\
\]
\[ P_{\pm,0} E_0 \cdot b_{n+1} + \frac{\epsilon_{\pm,0} + P_{\pm,0} u_0^0 (u_{\pm} \cdot E_0) c_{n+1}}{c^2} \]
\[ \pm e_\pm \int_{\mathbb{R}^4} \hat{p}_\pm \sqrt{M_\pm} (I - P_\pm) \left[ \frac{F_{\pm,n+1}}{\sqrt{M_\pm}} \right] dp \cdot E_0 \]
\[ \pm \frac{\epsilon_{\pm}}{c} \sum_{k+l=n+1} \left( n_{\pm,k} u_{\pm,k} \cdot E_k a_{\pm,l} + \frac{\epsilon_{\pm,k} + P_{\pm,k} u_{\pm,k} \cdot E_k}{c^2} (u_{\pm,k} \cdot b_l) (u_{\pm,k} \cdot E_k) + P_{\pm,k} E_k \cdot b_l + \frac{\epsilon_{\pm,k} + P_{\pm,k} u_0^0 (u_{\pm,k} \cdot E_k) c_l}{c^2} \right) \]
\[ \pm \frac{\epsilon_{\pm}}{c} \int_{\mathbb{R}^4} c\hat{p}_\pm \sqrt{M_\pm} (I - P_\pm) \left[ \frac{F_{\pm,l}}{\sqrt{M_\pm}} \right] dp \cdot E_k = 0, \]

\[
\partial_t E_{n+1} - c \nabla_x \times B_{n+1} = -4\pi e_+ \left( \frac{n_{+0} u_{a_{n+1}} + P_{+0} b_{n+1}}{c} + \frac{\epsilon_{+0} + P_{+0} u_0^0 (u_+ \cdot b_{n+1})}{c^2} \right) + 4\pi e_- \left( \frac{n_{-0} u_{a_{n+1}} + P_{-0} b_{n+1}}{c} + \frac{\epsilon_{-0} + P_{-0} u_0^0 (u_- \cdot b_{n+1})}{c^2} \right) \]
\[-4\pi \int_{\mathbb{R}^3} \left( e_+ \hat{p}_+ \sqrt{M_+} (I - P_+) \left[ \frac{F_{+,n+1}}{\sqrt{M_+}} \right] - e_- \hat{p}_- \sqrt{M_-} (I - P_-) \left[ \frac{F_{-,n+1}}{\sqrt{M_-}} \right] \right) dp, \]

\[
\partial_t B_{n+1} + c \nabla_x \times E_{n+1} = 0, \]
\[ \nabla_x \cdot E_{n+1} = \frac{4\pi}{c} e_+ \left( \frac{n_{+0} u_{0}^0 a_{n+1} + \epsilon_{+0} + P_{+0} u_0^0 (u_+ \cdot b_{n+1})}{c^2} \right) + 4\pi e_- \left( \frac{n_{-0} u_{0}^0 a_{n+1} + \epsilon_{-0} + P_{-0} u_0^0 (u_- \cdot b_{n+1})}{c^2} \right) \]
\[-4\pi \nabla_x \cdot B_{n+1} = 0. \]

Furthermore, assume \( a_{\pm,n+1}(0,x), b_{\pm,n+1}(0,x), c_{\pm,n+1}(0,x), E_{n+1}(0,x), B_{n+1}(0,x) \in H^{N}, N \geq 0 \) be given initial data to the system consisted of equations (A.3), (A.4), (A.5) and (A.6). Then the linear system is well-posed in \( C^0([0,\infty); H^{N}) \). Moreover, it holds that

\[
|F_{\pm,n+1}| \lesssim (1 + t)^n M_{\pm}^L, \quad |\nabla_p F_{\pm,n+1}| \lesssim (1 + t)^n M_{\pm}^L, \]
\[
|\nabla_x F_{\pm,n+1}| \lesssim (1 + t)^n M_{\pm}^L, \quad |\nabla_x \nabla_p F_{\pm,n+1}| \lesssim (1 + t)^{n} M_{\pm}^L, \]
\[
|\nabla_x^2 F_{\pm,n+1}| \lesssim (1 + t)^n M_{\pm}^L, \quad |\nabla_x^2 \nabla_p F_{\pm,n+1}| \lesssim (1 + t)^{n} M_{\pm}^L, \]
\[
|E_{n+1}| + |B_{n+1}| + |\nabla_x E_{n+1}| + |\nabla_x B_{n+1}| + |\nabla^2 E_{n+1}| + |\nabla^2 B_{n+1}| \lesssim (1 + t)^n. \quad \text{(A.7)} \]

**Proof.** The whole proof follows from analogous arguments as in [51, Appendix 3]. Here we just mention some key differences. [45] justified the time decay rate \( \beta_0 = \frac{2m}{2m+1} \) of \( (n_{\pm}(t) - \epsilon_n \bar{n}, u_{\pm}(t), E(t), B(t)) \) up to fourth-order derivatives. In fact, by interpolating between \( L^{\infty} \) norm of lower-order derivatives and \( H^N \) norm, we can easily verify that the higher-order derivatives also enjoy time decay rate (the decay rate is larger than 1 up to \( 53^{th} \) order derivatives). Note that systems of \( a_{+,n+1}(t,x), b_{+,n+1}(t,x), c_{+,n+1}(t,x) \) and \( a_{-,n+1}(t,x), b_{-,n+1}(t,x), c_{-,n+1}(t,x) \) take the same form as the corresponding system \( a_{n+1}(t,x), b_{n+1}(t,x), c_{n+1}(t,x) \) in [51], which is symmetric and positive definite. Therefore, we can first proceed the \( H^N \) energy estimates for the Euler Maxwell system (A.6), and the two system separately. Then we can combine these estimates together and to obtain the total \( H^N \) energy estimates for \( a_{\pm,n+1}(t,x), b_{\pm,n+1}(t,x), c_{\pm,n+1}(t,x) \), \( E_{n+1}(t,x), B_{n+1}(t,x) \). The momentum decay of \( F_{\pm,n+1} \) and its derivatives can be established as the derivation of (7.13). \( \square \)
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