Conformal flatness, non-Abelian Kaluza-Klein and the Quaternions

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Abstract

Maximally symmetric manifolds with holonomy in the unitary quaternionic group $Sp(d/4)$ emerge from the non-Abelian Kaluza-Klein reduction of conformally flat spaces. Thus, all special manifolds with constant properly ‘holonomy-related’ sectional curvature, are in natural correspondence with conformally flat, possibly non-Abelian, Kaluza-Klein spaces.

Introduction

Maximally symmetric real manifolds, like deSitter and anti-deSitter spacetimes, are conformally flat. Maximally symmetric manifolds with special holonomy are not. When holonomies lie in the unitary group $U(d/2)$, however, the relation between maximal symmetry and conformal flatness is not completely lost. The investigation of the Kaluza-Klein reduction (codimension 1) of geometric tensors characterizing conformal flatness initiated by R. Jackiw and collaborators [1] [2] [3] [4], has shown that complex manifolds with constant holomorphic sectional curvature are naturally associated with conformally flat Kaluza-Klein spacetimes [5]. Here we extend the investigation to the non-Abelian Kaluza-Klein reduction from $D$ to $d = D - c$ dimensions (codimension $c$) of conformal tensors, showing that maximally symmetric manifolds with holonomies in the unitary quaternionic group $Sp(d/4)$, emerge from the dimensional reduction of conformally flat spaces. Thus, all special manifolds with constant properly ‘holonomy-related’ sectional curvature, are in natural correspondence with higher dimensional conformally flat, possibly non-Abelian, Kaluza-Klein spaces.

Our discussion proceeds as follows. In the next section we briefly review quaternionic $Sp(d/4)$ structures on manifolds, with emphasis on the maximally symmetric case. Next, we turn to the non-Abelian dimensional reduction of the tensors characterizing conformal flatness in three and higher dimension and to the equations obtained by imposing their vanishing. The following two sections are devoted to the construction of non-trivial solutions of these equations. These turn out to be non-Abelian Kaluza-Klein spaces with internal 3-dimensional spheres and external maximally symmetric quaternionic Kähler spaces. The last section contains our conclusions, while two Appendices display general non-Abelian reduction formulas for the Cotton and Weyl tensors.
Quaternionic Kähler manifolds

In the last few decades there has been an increasing interest in manifolds with special holonomy [6]. Besides complex manifolds, with holonomy in $U(d/2)$ [7] [8] [9], also quaternionic manifolds, with holonomy in $Sp(d/4)$ [10] [11] [12] [13], have been introduced and investigated. Among these, maximally symmetric spaces, i.e. spaces of constant properly ‘holonomy-related’ sectional curvature, are regarded as the simplest important classes (see also [14] [15] [16] [17]).

In analogy with the complex case, the holonomy restriction to the unitary quaternionic group $Sp(d/4)$ is obtained by imposing an appropriate Clifford-Kähler structure on the base manifold [11]. Namely, one introduces a metric $g_{\mu\nu}$ and three complex structures $J^a_{\nu}\mu$, $a = 1, 2, 3$, closing the quaternionic algebra

\begin{equation}
J^a_{\kappa}\mu J^b_{\nu}\kappa + J^b_{\kappa}\mu J^a_{\nu}\kappa = -2 \delta^{ab} \delta_{\mu\nu},
\end{equation}

(1a)

\begin{equation}
J^a_{\kappa}\nu J^b_{\mu}\kappa - J^b_{\kappa}\nu J^a_{\mu}\kappa = 2 \varepsilon^{abc} J^c_{\mu\nu},
\end{equation}

(1b)

where $\varepsilon_{abc}$ is the completely antisymmetric Levi-Civita symbol. The compatibility between the metric and the quaternionic structures is obtained by requiring the $J^a_{\mu\nu}$ to be isometries

\begin{equation}
J^a_{\mu\nu} g_{\kappa\lambda} = g_{\mu\nu}
\end{equation}

(2a)

(no sum over $a$), parallel with respect to the Levi-Civita connection $\nabla_\kappa$ associated to $g_{\mu\nu}$

\begin{equation}
\nabla_\kappa J^a_{\mu\nu} = \varepsilon_{bc} \theta^b_{\kappa\mu} J^c_{\nu\mu},
\end{equation}

(2b)

where $\theta^a_{\mu}$ are appropriate 1-forms. This last equation is a generalization of the Kähler condition and the resulting geometries are called quaternionic Kähler geometries.

Still in complete analogy with the complex case, one defines quaternionic sectional curvatures along $J^a_{\mu\nu}$-invariant subspaces and finds that the manifolds of constant quaternionic sectional curvature $k$, are characterized by a Riemann curvature tensor of the form

\begin{equation}
R^a_{\mu\nu\kappa\lambda} = \frac{1}{4} k \left( g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa} + J_{\mu\lambda} J^a_{\nu\kappa} - J_{\mu\kappa} J^a_{\nu\lambda} - 2 J_{\mu\nu} J^a_{\kappa\lambda} \right),
\end{equation}

(3a)

with the 2-forms $J^a_{\mu\nu} = J^a_{\mu\nu} g_{\kappa\lambda}$, related to the 1-forms $\theta^a_{\mu}$ by

\begin{equation}
J^a_{\mu\nu} = \frac{1}{k} \left( \partial_{\mu} \theta^a_{\nu} - \partial_{\nu} \theta^a_{\mu} - \varepsilon_{bc} \theta^b_{\mu} \theta^c_{\nu} \right).
\end{equation}

(3b)

In what follows we show that precisely this structure emerges form the non-Abelian Kaluza-Klein reduction of conformally flat spaces.

Dimensional reduction of conformal tensors

A space is said to be conformally flat if a coordinate transformation exists, making its metric tensor proportional to a locally flat metric. All one- and two-dimensional
spaces are conformally flat. In three dimensions conformal flatness is characterized by the vanishing of the Cotton tensor. In dimensions greater than three Cotton flatness reduces to a necessary but non-sufficient condition and the role of probing conformal flatness is taken by the vanishing of the Weyl tensor. We consider a $D$-dimensional pseudo-Riemannian space with $D \geq 3$. The metric tensor $g_{IJ}(x)$ is allowed to carry arbitrary signature, while the corresponding Riemann, Ricci and scalar curvatures are respectively defined as $R_{IJK}^L = \partial_I R_{JK}^L - \ldots$, $R_{IJ} = R_{IJK}^K$ and $R = R_{IJ}$. The Cotton tensor $C_{IJK}$ and the Weyl tensor $C_{IJKL}$ are then conveniently written in terms of the Schouten tensor $S_IJ = R_{IJ} - \frac{1}{2(D-1)} R g_{IJ}$ as

$$C_{IJK} \equiv 2 \nabla [K S_{J[I}], \tag{4}$$

$$C_{IJKL} \equiv R_{IJKL} - \frac{2}{D-2} \left( g_{I[K} S_{L]J} - g_{J[K} S_{L]I} \right). \tag{5}$$

The Cotton tensor is antisymmetric in the last two indices by construction and is traceless in the first two because of the Bianchi identities. The Weyl tensor shares the same symmetries of the Riemann tensor and it is further vanishing under contraction of any pair of indices. In dimension greater than three, Cotton and Weyl tensors are related by the identity

$$(D - 3) C_{IJK} = (D - 2) \nabla_L C_{IJK}^L. \tag{6}$$

The vanishing of the latter implies the vanishing of the former, but the vice versa does not hold.

We reduce geometry to arbitrary dimensions $d$ and codimensions $c \geq 2$, $d + c = D$, by a standard non-Abelian Kaluza-Klein Ansatz $^{[10]}$

$$g_{IJ} = \begin{pmatrix} g_{\mu\nu} + A^2_{\mu} A^b_{\nu} K^a_{\mu} K^b_{\nu}, & A^2_{\mu} K^a_{\nu} K^b_{\mu} \nu_k l_k \mu_i K^i_{\mu} A^0_{\nu}, & A^2_{\mu} K^a_{\nu} K^b_{\mu} \nu_k l_k \mu_i K^i_{\mu} A^0_{\nu} \\ \kappa_{ij} K^i_{\mu} A^0_{\nu}, & \kappa_{ij} K^i_{\mu} A^0_{\nu} \end{pmatrix}. \tag{7}$$

The lower dimensional tensors $g_{\mu\nu}(x)$, $\kappa_{ij}(y)$ are identified with the external $d$-dimensional and internal $c$-dimensional metrics, respectively. The Killing vector fields $K_a^b(y)$ span the internal isometry algebra $\mathcal{I}$, $[K_a, K_b] = c_{abc} K_c^b$, and $A^a_{\mu}(x)$ is an external vector field taking values on $\mathbb{R}^D$. The non-Abelian Kaluza-Klein Ansatz is covariant under the coordinate transformations generated by $\delta x^\mu = \xi^\mu(x)$, $\delta y^i = \xi^i(x) K^i_{\mu}(y)$. The former transformations are identified with external space-time diffeomorphisms. The latter provide the gauge transformation of the gauge potential $A^a_{\mu}$. The corresponding gauge curvature is obtained as $F_{a_{\mu\nu}} = \partial_a A^a_{\nu} - \partial_{\nu} A^a_{\mu} - c_{abc} A^b_{\mu} A^c_{\nu}$. The external

\footnote{Higher $D$-dimensional quantities are denoted in boldface characters with capital Latin indices $I, J, \ldots = 1, \ldots, D$.}$^{[1]}$

\footnote{Here and in what follows, square and round brackets respectively denote anti-symmetrization $t_{IJ} = (t_{IJ} - t_{JI})/2$ and symmetrization $t_{IJ} = (t_{IJ} + t_{JI})/2$.}$^{[2]}$

\footnote{Lower external $d$- and internal $c$-dimensional indices are respectively denoted by small Greek and Latin characters, $\mu, \nu = \ldots = 1, \ldots, d$ and $i, j, \ldots = 1, \ldots, c$. Internal/external coordinates are respectively denoted by $x^a$, $y^i$; while the corresponding curvatures by $R_{\mu\nu\lambda\kappa}$, $R_{\mu\nu}$, $R_{\lambda\kappa}$ and $R_{ijkl}$, $R_{ij}$, $R_{ij}$. The indices $a, b, \ldots$ range over the dimension $n$ of the internal isometry algebra $\mathcal{I}$.}$^{[3]}$
covariant derivative $\nabla_\mu$ associated to $g_{\mu\nu}$, is not covariant under the second group of transformations when acting on internal scalars/tensors. It has to be replaced by

$$\hat{\nabla}_\mu \equiv \nabla_\mu - \mathcal{L}_{A_\mu},$$

with $\mathcal{L}_{A_\mu}$ the Lie derivative with respect to the internal vector field $A_\mu^i \equiv K_\mu^i A_a^i$. When acting on $F^{ij}_{\mu\nu} \equiv K_{ij}^2 F^a_{\mu\nu}$, $\hat{\nabla}_\mu$ reproduces the familiar gauge covariant derivative

$$\hat{\nabla}_{\kappa} F^i_{\mu\nu} = K^j_i (\nabla_{\kappa} F^a_{\mu\nu} - c_{bc}^a A^b_{\kappa} F^c_{\mu\nu}).$$

(8)

It is standard matter to obtain reduction formulas for the higher dimensional curvatures $R_{IJKL}, R_{ij}, R$, in terms of the lower dimensional ones $R_{\mu\nu\kappa\lambda}, R_{ijkl}, \ldots$, and of the gauge fields $A^a_{\mu}, F^a_{\mu\nu}$. Substituting these in the higher dimensional Einstein equations produces the standard $d$ dimensional Einstein-Yang-Mills theory. Our interest, instead, is in analogue reduction formulas for the higher dimensional conformal tensors and in the equations obtained by imposing their vanishing. These describe the ‘immersion’ of $d$ dimensional space-time into a $D = d + c$ dimensional conformally flat space, also reproducing the equations of motion of a non-standard $d$ dimensional Einstein-Yang-Mills theory. General reduction formulas for the Cotton and the Weyl tensors are respectively presented in Appendices A and B. Here, we proceed to the analysis of the equations for conformal flatness.

**Equations for conformal flatness**

Demanding conformal flatness in dimensions greater than three, amounts to set all components of the Weyl tensor equal to zero. From the dimensional reduction formulas of Appendix B we obtain the following set of equations

$$C_{\mu\nu\kappa\lambda} + \frac{1}{2} \left( F_{\kappa\mu\nu} F^k_{\kappa\lambda} - F_{\kappa\mu\nu} F^k_{\kappa\lambda} \right) - \frac{3}{2(d-2)} \left( g_{\mu\nu} F_{\kappa\lambda} - g_{\kappa\lambda} F_{\mu\nu} \right) = 0$$

(9a)

$$R_{\mu\nu} - \frac{1}{c} R_{\mu\nu}^e g_{\mu\nu} = -\frac{d+3c-2}{4d} \left( F_{\mu\nu}^2 - \frac{1}{c} F_{\mu\nu}^2 g_{\mu\nu} \right)$$

(9b)

$$\hat{\nabla}_\kappa F^i_{\mu\nu} = 0$$

(9c)

$$C_{ijkl} = 0$$

(9d)

$$R_{ij} - \frac{1}{c} R_{ijkl} K_{ij} = -\frac{c^2-2}{4d} \left( F_{ij}^2 - \frac{1}{c} F_{ij}^2 K_{ij} \right)$$

(9e)

\[\text{In three dimensions conformal flatness requires the Cotton tensor to vanish identically. The only possibility of having } c > 1 \text{ implies a lineal spacetime. The external scalar curvature and the gauge curvature are consequently equal to zero. The formulas of Appendix A show that the only non-identically vanishing Cotton components are } (31) \text{ and } (33). \text{ Setting them equal to zero implies the single equation } \partial_\mu R^{\mu} = 0. \text{ The structure of } D = 3, c = 2 \text{ conformally flat Kaluza-Klein spaces is trivial: vanishing } F^a_{\mu\nu}, \text{ pure gauge } A^a_{\mu}, \text{ vanishing external scalar curvature } R^e, \text{ and constant internal scalar curvature } R^i.\]
\( F_{(i|\mu\kappa}F_{j)\nu}^\kappa = \frac{1}{4}F_{\mu\nu}^2\kappa_{ij} + \frac{1}{2}g_{\mu\nu}F_{ij}^2 - \frac{1}{cd}F_{\mu\nu}^2\kappa_{ij} \quad (9f) \)

\( F_{(i|\mu\kappa}F_{j)\nu}^\kappa = 2\nabla_i F_{j\mu\nu} \quad (9g) \)

\( c(c - 1)R_{ex}^c + d(d - 1)R^{in} + \frac{(c-1)(2d+3c-2)}{4}F^2 = 0 \quad (9h) \)

with

\[
F_{\mu\nu}^2 = F_{\mu\kappa}^k F_{k\nu}^\kappa, \quad F_{ij}^2 = F_{ij\mu} F_{\mu\nu}^\kappa, \quad F^2 = F_{\mu\nu}^\kappa F_{\mu\nu}^\kappa
\]

\( T_{\mu\nu} = F_{\mu\nu}^2 - \frac{1}{2(d-1)}F^2 g_{\mu\nu}. \) The first, second and third equations are the higher co-dimensional generalization of the equations obtained by Grumiller and Jackiw in codimension one (see (17a,b,c) in Ref. [2]). The only remarkable difference is that the whole gauge field is required to be covariantly constant and not only its traceless part. The fourth and fifth equations appear as the internal counterpart of the first and second ones. The sixth and seventh equations fix the internal symmetric and antisymmetric part of the gauge field contraction \( F_{\mu\kappa} F_{\nu}^\kappa. \) Eventually, the eighth equation provides a relation among the gauge squared modulus and the external and internal curvatures. All equations but \( (9c), \) are traceless in all paired internal/external indices.

Given the complexity of the problem, it is useful to establish integrability conditions for \( (9). \) In view of \( (6), \) these are obtained by imposing the vanishing of the Cotton tensor. From the formulas of Appendix A and by taking into account \( (9b), (9c), (9d) \) and \( (9h), \) we find that the only non-trivially satisfied conditions are

\[
F_{\kappa k\mu} R_{\kappa\nu} + F_{\mu\nu}^l R_{lk} + \frac{1}{2} F_{\kappa k\mu} F_{\kappa\nu}^2 - \frac{1}{4} F_{\mu\nu}^l F_{\mu\nu}^l = 0, \quad (11a)
\]

\[
\partial_i \left( R^{in} - \frac{3d+4c-4}{4d} F^2 \right) = 0. \quad (11b)
\]

Equations \( (9), \) together with the integrability conditions \( (11), \) are solved by vanishing \( F_{\mu\nu}^2, \) pure gauge \( A_\mu^a \) and maximally symmetric external and internal spaces with (constant) scalar curvatures related by

\( c(c - 1)R_{ex}^c + d(d - 1)R^{in} = 0. \quad (12) \)

The question is whether less trivial solutions exist, carrying non-vanishing non-Abelian gauge configurations endowing the external space with interesting geometrical structures, like in codimension one \([5].\)

**Curvatures**

We start by analyzing the constraints imposed by equations \( (9) \) on curvatures. First, we prove that the scalars \( R_{ex}^c, R^{in} \) and \( F^2 \) have to be constant. By considering the
internal derivative of (9h) together with (11b), one obtains the simultaneous linear equations
\[\begin{align*}
4d(d-1) \partial_i R^\text{in} + (c-1)(2d+3c-2) \partial_i F^2 &= 0, \\
4d \partial_i R^\text{in} - (3d+4c-4) \partial_i F^2 &= 0,
\end{align*}\] (13)
that impose the vanishing of $\partial_i R^\text{in}$ and $\partial_i F^2$ for $D \neq 1$ and $D \neq 2$. It follows that the internal curvature $R^\text{in}$ is constant, while the squared modulus of the gauge curvature $F^2$ is only allowed to depend on external coordinates. On the other hand, (9c) implies
\[\partial_\kappa F^2 = \hat{\nabla}_\kappa F^\mu_\nu F^\mu_\nu + F^i_\mu \nabla_\kappa F^\mu_\nu = 0,
\] requiring $F^2$ not to depend on external coordinates, hence its constancy. Eventually, (9h) requires the constancy of the external curvature $R^\text{ex}$, fixing its value to
\[R^\text{ex} = -\frac{d}{c(c-1)} R^\text{in} - \frac{2d+3c-2}{4c} F^2.
\] (14)
Next, we consider the curvature tensors. Along internal directions, equation (9d) together with the constancy of the internal scalar curvature $R^\text{in}$, immediately implies that the internal space is maximally symmetric. The internal Riemann and Ricci curvatures rewritten in terms of the constant $R^\text{in}$ and of the internal metric $\kappa_{ij}$ are then given by
\[R^\text{ijkl} = \frac{1}{c(c-1)} R^\text{in} (\kappa_{ik} \kappa_{lj} - \kappa_{il} \kappa_{kj}),
\] (15a)
\[R_{ij} = \frac{1}{c} R^\text{in} \kappa_{ij}.
\] (15b)
The internal space is isomorphic to the pseudo-Euclidean space $\mathbb{R}^c_s$ for $R^\text{in} = 0$, to the pseudo-projective space $\mathbb{P}^c_s$ (the pseudo-sphere $S^c_s$) for $R^\text{in} > 0$ and to the pseudo-hyperbolic space $\mathbb{H}^c_s$ for $R^\text{in} < 0$, $s = 0, \ldots, c$ (see e.g. §8 of Ref. [19]). In particular, the internal space is Einstein, so that for $c > 2$ equation (9e) imposes the vanishing of the traceless part of the symmetric tensor $F^2_{ij}$.
Along external directions, we solve equation (9a) by the substitution
\[R^\mu_{\nu\kappa\lambda} = r^\mu_{\nu\kappa\lambda} - \frac{1}{2} \left( F^\mu_{\rho\nu} F^k_{\kappa\lambda} - F^k_{\rho\nu} F^\mu_{\kappa\lambda} \right),
\] (16)
with $r^\mu_{\nu\kappa\lambda}$ a tensor sharing the algebraic symmetries of a Riemann tensor and further satisfying the conditions
\[r^\mu_{\nu\kappa\lambda} - \frac{2}{2d-2} \left( g^\mu_{[\nu} r_{\lambda]\nu} + g^\nu_{[\lambda} r_{\nu]\mu} \right) - \frac{2}{(d-1)(d-2)} r g^\mu_{[\nu} g^\lambda_{\mu]} = 0,
\] (17)
with $r^\mu_{\nu} = r^\mu_{\nu\kappa}$ and $r = r^\mu_{\mu}$. These are $\frac{1}{12} d(d+1)(d+2)(d-3)$ simultaneous linear equations in $\frac{1}{12} d^2(d^2-1)$ variables, with coefficients only depending on the external spacetime metric $g_{\mu\nu}$. The general solution depends on $\frac{1}{2} d(d+1)$ arbitrary functions and is obtained as
\[r^\mu_{\nu\kappa\lambda} = 2 \left( g^\mu_{[\nu} \rho_{\kappa]\nu} - g^\nu_{[\lambda} \rho_{\kappa]\mu} \right) + 2 \rho g^\mu_{[\nu} g^\lambda_{\mu]}.
\] (18)
with \( \rho_{\mu\nu} \) a traceless symmetric tensor and \( \rho \) a scalar. The quantities \( \rho_{\mu\nu} \) and \( \rho \) are determined by equations (9b) and (14) as follows. From (16) and (18) we obtain
\[
R_{\mu\nu} = \left( d - 1 \right) \rho_{\mu\nu} + \left( d - 2 \right) \mu - \frac{1}{4} F_{\mu\nu} F^{\kappa} \quad \text{and} \quad R^{ex} = d(d - 1) \rho - \frac{3}{4} F^2,
\]
which substituted back in (9b) and (14) yields
\[
\rho_{\mu\nu} = -\frac{1}{4c} \left( F_{\mu\nu}^2 - \frac{1}{d} F^2 g_{\mu\nu} \right), \quad \rho = -\frac{1}{c(c - 1)} R^{in} - \frac{1}{2c} F^2 g_{\mu\nu}. \tag{19}
\]
Eventually, by substituting (19) back in (18) and this in (16), the external Riemann and Ricci tensors are obtained in terms of the external metric \( g_{\mu\nu} \), the internal metric \( \kappa_{ij} \), the gauge field \( F_{k\mu\nu} \) and the constant \( R^{in} \) as
\[
R_{\mu\nu\kappa\lambda} = -\frac{1}{c(c - 1)} R^{in} \left( g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\kappa\nu} \right) - \frac{1}{2c} \left( F_{\mu\nu} F_{\kappa\lambda} - F_{\kappa\mu} F_{\lambda\nu} \right), \tag{20a}
\]
\[
R_{\mu\nu} = -\frac{d - 1}{c(c - 1)} R^{in} g_{\mu\nu} - \frac{d + c - 2}{4c} F_{\mu\nu}^2 - \frac{1}{4c} F^2 g_{\mu\nu}. \tag{20b}
\]
Equation (9c), together with the constancy of \( R^{in} \), guarantee that the Bianchi integrability conditions are satisfied.

Further information on curvatures can be obtained from the integrability condition (11a). While the analysis of this equation is possible in general, for simplicity of presentation we restrict to \( c > 2 \), where equation (9e) imposes the vanishing of the traceless part of \( F_{ij}^2 \). By inserting (15b) and (20b) in (11a) we obtain
\[
F_{i\lambda \kappa} F_{j\kappa}^{\prime 2
\]
\[
F_{\mu\nu} = -\frac{4}{c - 1} R^{in} g_{\mu\nu}. \tag{22}
\]
One more contraction yields a relation expressing the gauge field modulus in terms of the internal curvature
\[
F^2 = -\frac{4d}{c - 1} R^{in}. \tag{23}
\]
This last two equations, together with (11b) imply the external space is Einstein. By inserting them in (20) we eventually obtain the external Riemann and Ricci curvatures as
\[
R_{\mu\nu\kappa\lambda} = -\frac{1}{4c} F_{\mu\nu} g_{\kappa\lambda} - \frac{1}{2} \left( F_{k\mu\nu} F_{\kappa\lambda} - F_{k\mu\kappa} F_{\lambda\nu} \right), \tag{24a}
\]
\[
R_{\mu\nu} = -\frac{d + c - 1}{4c} F^2 g_{\mu\nu}. \tag{24b}
\]
Absolute parallelism and Clifford structures

Next, we turn our attention to equations (9f) and (9g). Contracting both equations with $K^c_a K^1_b$, taking into account (22) and the commutation relations $[K_a, K_b] = c_{ac}^d K^d_c$, we respectively rewrite them as

$$g_{ac} g_{bd} \left( F^c_{\mu} F^d_{\nu} + F^d_{\mu} F^c_{\nu} \right) = -\frac{2}{d} g_{ac} g_{bd} F^c_{\mu} F^{d\lambda\nu} g^\nu_{\mu}, \quad (25a)$$

$$g_{ac} g_{bd} \left( F^c_{\mu} F^d_{\nu} - F^d_{\mu} F^c_{\nu} \right) = 2 \left( K^i_a \partial_i g_{bc} - K^i_b \partial_i g_{ac} - c_{ab}^d \partial_d g_{cd} \right) F^\nu_{\mu}, \quad (25b)$$

where $g_{ab} = K^i_a K^i_b \delta_{ij}$ is the matrix implicitly appearing in all gauge field internal contractions. The similarity of equations (25) with (1) for constant $g_{ab}$ is striking. We therefore restrict attention to solution with constant $g_{ab}$. This assumption has strong implications. Proceeding by a Gram-Schmidt process, applied to $K^1_a$, ..., $K^7_a$ at one point, we obtain orthonormal Killing vector fields globally. On the other hand, the number of these can not exceed the internal dimension $c$. On the other hand, for $c > 2$ equation (9e) requires

$$K_{ai} F^a_{\mu} F^{b\nu}_{\mu} K^i_b = \frac{1}{c} F^2 \delta_{ij}. \quad (27a)$$

The right hand side term of this equation is a non-singular $c \times c$ matrix. Hence, the $c \times n$, $n \times n$ and $n \times c$ matrices appearing in the left hand side term, should have at least rank $c$. This implies the presence of at least $c$ linearly independent Killing vector fields. Hence, the number of orthonormal Killing vector fields $K^i_a$ corresponding to non-vanishing $F^a_{\mu\nu}$ has to be exactly equal to $c$. That is, the internal space has to support a global Killing parallelization. The only manifolds of constant curvature admitting a Killing parallelization are isomorphic to the $c$-dimensional pseudo-Euclidean space $\mathbb{R}^c_s, s = 0, ..., c$, the 1-dimensional sphere $S^1$, the 3-dimensional (pseudo-)spheres $S^3$, $S^4$, the 7-dimensional (pseudo-)spheres $S^7$, $S^8$. The special dimensions of the (pseudo-)spheres depend essentially on the existence of a multiplication in $\mathbb{R}^2$ (complex numbers $\mathbb{C}$), $\mathbb{R}^4$ (quaternions $\mathbb{H}$), and $\mathbb{R}^8$ (octonions $\mathbb{O}$) [20, 21]. The isometry algebras of $\mathbb{R}^c_s$ and $S^1$ are Abelian, taking the analysis back to [5]. In the non-Abelian case, the internal space is isomorphic to $S^3$, $S^4$, $S^7$ or $S^8$. For simplicity of the presentation we restrict our analysis to $S^3$ and $S^7$. The remaining two cases are treated along the very same lines. After the orthonormalization process, $g_{ab}$ takes form of a 3- or 7-dimensional Euclidean diagonal metric $\delta_{ab}$. Introducing

$$J^\nu_{\mu} = \sqrt{cd} F^a_{\mu}, \quad (26)$$

equations (25) take the form

$$J^a_{\mu} J^b_{\nu} + J^b_{\mu} J^a_{\nu} = -2 \delta^{ab} \delta^\nu_{\mu}, \quad (27a)$$

$$J^a_{\mu} J^b_{\nu} - J^b_{\mu} J^a_{\nu} = 2 \delta^{ab} \epsilon^\nu_{\mu}, \quad (27b)$$

\footnote{We suspect these to be the only possible solutions, but we could not prove the statement.}
with $k^{ab}_c = \sqrt{\frac{3}{2}} c^{ab}$. This is enough to exclude $S^7$ (as well as $S^3_7$) as a possible internal space, because there are not seven roots of unit closing any associative (matrix) algebra. We are left with the sole $c = 3$ case. A direct inspection of the isometry algebra of $S^3$, shows that the three Killing vector fields parallelizing the 3-sphere and setting $g_{ab} = \delta_{ab}$, close the $su(2)$ algebra with structure constants $c^{ab}_c = \sqrt{\frac{2|R_{\text{in}}|^3}{3}} \varepsilon^{abc}$.

Taking into account (23), we therefore obtain

$$k^{ab}_c = \varepsilon^{abc}.$$  

The rescaled gauge fields $J^a_{\mu\nu}$ close the quaternionic algebra (1). To conclude that $g_{\mu\nu}$, $J^a_{\mu\nu}$ actually define a quaternionic Kähler structure of constant quaternionic sectional curvature, we just have to bring together a few formulas scattered around the paper. By their very definition (26), the three complex structures $J^a_{\mu\nu}$ are isometries of the external space

$$J^a_{\mu\nu} g_{\kappa\lambda} = g_{\mu\nu},$$  

(no sum over $a$), while (2c) and (8) guarantee the $J^a_{\mu\nu}$ to be parallel with respect to the Levi-Civita connection associated to $g_{\mu\nu}$

$$\nabla_{\kappa} J^a_{\mu\nu} = \varepsilon_{bc} A^b_{a\kappa} J^c_{\mu\nu}$$  

with $A^a_{a\kappa} = \sqrt{\frac{2|R_{\text{in}}|^3}{3}} \kappa$. The compatibility conditions (2) are therefore satisfied. Eventually, taking again into account (23), equation (24a) fixes the external Riemann tensor to

$$R_{\mu\nu\kappa\lambda} = \frac{1}{3} \frac{2|R_{\text{in}}|^3}{3} (g_{\mu\lambda} g_{\kappa\nu} - g_{\mu\kappa} g_{\nu\lambda} + J^a_{\mu\lambda} J^a_{\nu\kappa} - J^a_{\mu\kappa} J^a_{\nu\lambda} - 2 J^a_{\mu\nu} J^a_{\kappa\lambda}),$$  

while the identity

$$J^a_{\mu\nu} = \frac{3}{2|R_{\text{in}}|^3} \left( \partial_{\mu} \theta^a_{\nu} - \partial_{\nu} \theta^a_{\mu} - \varepsilon_{bc} a^{bc} \theta^b_{a\mu} \theta^c_{\nu} \right),$$  

follows from the very definition of $F^a_{\mu\nu}$ as curvature associated to $A^a_{\mu}$. Thus, (3) are satisfied.

**Conclusions**

We proved that the equations for conformal flatness (9) of arbitrary non-Abelian Kaluza-Klein spaces are solved by a set of lower dimensional Kaluza-Klein functions $\kappa_{ij}$, $K^i_{a}$, $g_{\mu\nu}$, $F^a_{\mu\nu} = \sqrt{\frac{2|R_{\text{in}}|^3}{3}} J^a_{\mu\nu}$ where:

1. $\kappa_{ij}$ is the metric of a 3-sphere of arbitrary constant curvature $R_{\text{in}}$
2. $K^i_{a}$ are three orthonormal Killing vector fields parallelizing the 3-sphere
3. $g_{\mu\nu}$, $J^a_{\mu\nu}$ define a quaternionic Kähler structure of constant quaternionic sectional curvature $\frac{2|R_{\text{in}}|^3}{3}$ on the external space
Proceeding in the very same way, it is immediate to prove that also pseudo-quaternionic and para-quaternionic (corresponding to the internal space $S^3$) Kähler structures of constant pseudo-/para-quaternionic sectional curvature are solutions. We suspect this to be the only possible non-trivial solution, but we could not prove the statement.

Vice versa, given any $d$-dimensional (pseudo-/para-)quaternionic Kähler manifold of constant (pseudo-/para-)quaternionic sectional curvature, \[C\] defines a $d+3$ dimensional conformally flat manifold. Recalling that a similar statement holds for (pseudo-/para-)Kähler manifolds, we conclude that all maximally symmetric special dimensional conformally flat manifold. Recalling that a similar statement holds for conformally flat, possibly non-Abelian, Kaluza-Klein spaces.

**Appendices**

**A Dimensional reduction of the Cotton tensor**

The Cotton tensor is the relevant conformal tensor in three dimensions. However, its vanishing is a necessary—though not sufficient—condition for conformal flatness in any dimensions. For completeness and because some of the following identities simplify the analysis of conformal flatness in arbitrary dimensions, we present the general Cotton reduction formulas. The Cotton non-trivial property $C'_{j,l} = C''_{j',l} + C^{j,l} = 0$, makes it convenient to define the quantities

$$C_I \equiv C''_{j,l} = -C^j_{,l}.$$ 

These are expressed in terms of non-Abelian Kaluza-Klein functions as

$$C^\mu = \frac{1}{2(d-1)} \nabla^\mu \left( c R_{\kappa\eta} + \frac{2(d+3-e-2)}{4} F^2 \right) + \frac{1}{2} F^{\mu\kappa} \nabla^\kappa F_{\kappa\eta},$$

$$C_I = -\frac{1}{2(d-1)} \nabla^I \left( d R_{\kappa\eta} - \frac{2(d+3-e-4)}{4} F^2 \right),$$

where here and in the following we take advantage of the fact that only contravariant external and covariant internal components of higher dimensional tensor properly transform as lower dimensional tensors. In terms of these quantities Cotton reduction formulas take the reasonably compact form

$$C^{\mu\nu\kappa} = \frac{2}{d-1} C^{\mu|\nu\kappa} + C^{\mu\nu\kappa} - \frac{1}{2} \nabla^\lambda (F^\lambda_{\mu\nu} F^{\mu\nu\kappa}) + F_{\lambda\kappa} [\kappa \nabla^\nu] F^{\mu\lambda} - \frac{1}{2} F^{\mu\nu[\kappa \nabla^\lambda]} F^{\mu\nu\kappa} F_{\lambda\rho} + \frac{3}{4(d-1)} g^{\mu[\nu} \tilde{F}_{\kappa]} F^2,$$

$$C_{ijk} = -\frac{2}{d-1} C_{[k\nu\eta]i} + C_{ijk} + \frac{3}{4(d-1)} \nabla^I \left[k \nabla^j F^2 \eta_{ji} \right] - \frac{1}{2} \nabla^I \left[k \nabla^j F^2 \eta_{ji} \right],$$

$$C^{\mu\nu}_{\kappa} = \frac{1}{4} g^{\mu\nu} C_{\kappa} + \frac{1}{4} \nabla^\nu \nabla^\kappa F^\mu_k - \frac{1}{4} \nabla^\kappa \left(F^\mu_{\nu\kappa} - \frac{1}{2} F^2 g^{\mu\nu} \right) + \frac{1}{4} F^{\mu\nu} R^\kappa_{\nu k} + \frac{1}{4} F^{\mu\nu} R^\kappa_{\nu k} - \frac{1}{8} F^{\mu\nu} F^2_{\kappa k},$$

$$C^\nu_{\kappa j} = -\frac{1}{2} \nabla_j \nabla^\nu \nabla^\kappa C_{j} - \frac{1}{4} \nabla^\nu \left(F^\mu_{\kappa j} - \frac{1}{2} F^2_{\kappa ij} \right) + \frac{1}{4} F^{\mu\nu} R^\kappa_{\nu j} + \frac{1}{4} F^{\mu\nu} R^\kappa_{\nu j} F^{\nu\mu}_{\kappa ij},$$

$$C^\nu_{\nu j} = -\frac{1}{2} \nabla_j \nabla^\nu \nabla^\kappa C_{j} - \frac{1}{4} \nabla^\nu \left(F^\mu_{\kappa j} - \frac{1}{2} F^2_{\kappa ij} \right) + \frac{1}{4} F^{\mu\nu} R^\kappa_{\nu j} + \frac{1}{4} F^{\mu\nu} R^\kappa_{\nu j} F^{\nu\mu}_{\kappa ij},$$

10
\[ C_{\mu jk} = 2C_{[kj]\mu}, \quad (36) \]

\[ C_{\alpha \nu} = 2C^{[\alpha \nu]}_i, \quad (37) \]

where the contracted expressions \( F^{\mu \nu \kappa} F_{\kappa \nu}^{\nu}, F_{\mu \nu} F_{\mu \nu}, F_{\mu \nu}^2 \) have been shortened to \( F^2\mu \nu, F^2_{ij} \) and \( F^2 \), respectively. Relations (32), (37) and (33), (36) only hold in \( d > 1 \) and \( c > 1 \) respectively.

**B  Dimensional reduction of the Weyl tensor**

The relevant conformal tensor in dimensions greater than three is the Weyl tensor. Its property \( C_{IJK} = C_{I\kappa J^\kappa} + C_{IkJ^k} = 0 \) makes it convenient to introduce the quantities

\[ C_{IJ} = C_{I\kappa J^\kappa} = -C_{IkJ^k} \quad \text{and} \quad C_{\mu \nu} = -C_{i}^{i}. \]

The lower dimensional scalar \( C \) can be rewritten in terms of \( R^{ex}, R^{in} \) and \( F^2 \) as

\[ C = \frac{1}{(d-1)(d-2)} \left[ d(c-1)R^{ex} + d(d-1)R^{in} + \frac{(c-1)(2d+3c-2)}{4} F^2 \right], \quad (38) \]

while the symmetric quantities \( C_{IJ} \) can be rewritten in terms of \( C \) and lower dimensional tensors as

\[ C^{\mu \nu} = \frac{1}{d} Cg^{\mu \nu} + \frac{c}{d-2} \left( R^{\mu \nu} - \frac{1}{d} R^{ex} g^{\mu \nu} \right) + \frac{d+3c-2}{4(d-2)} \left( F^2 g^{\mu \nu} - \frac{1}{d} F^2 g^{\mu \nu} \right), \quad (39) \]

\[ C_{ij} = -\frac{1}{c} C\kappa_{ij} - \frac{d}{d-2} \left( R_{ij} - \frac{1}{c} R^{in} \kappa_{ij} \right) - \frac{c-2}{4(d-2)} \left( F^2_{ij} - \frac{1}{c} F^2 \kappa_{ij} \right), \quad (40) \]

\[ C_{\mu j} = \frac{c-1}{2(d-2)} \nabla_\nu F_{\mu \nu}^{j}. \quad (41) \]

These quantities allow to express Weyl reduction formulas in the reasonably compact form

\[ C^{\mu \nu \kappa \lambda} = -\frac{2}{d-2} \left( g^{\mu |\kappa} C_{l}^{l \lambda} |\nu - g^{\nu |\kappa} C_{l}^{l \lambda} |\mu \right) + \frac{2}{(d-1)(d-2)} Cg^{\mu |\kappa} C_{l}^{l \lambda} |\nu \right) + \frac{2}{(d-1)(d-2)} Cg^{\mu |\kappa} C_{l}^{l \lambda} |\nu \right) - \frac{3}{2(d-2)} \left( g^{\mu |\kappa} T^{\lambda} |\nu - g^{\nu |\kappa} T^{\lambda} |\mu \right) \quad (42) \]

with \( T^{\mu \nu} = F^{2\mu \nu} - \frac{1}{2(d-1)} F^2 g^{\mu \nu} \),

\[ C_{ijkl} = -\frac{2}{c-2} \left( \kappa_{i[k} C_{l]j} - \kappa_{j[k} C_{l]i} \right) - \frac{2}{(c-1)(c-2)} C\kappa_{i[k} C_{l]j} + C_{ijkl} \quad (43) \]

\[ C_{i}^{\nu \kappa \lambda} = \frac{2}{c-2} g^{\nu |\kappa} C_{l}^{l \lambda} - \frac{1}{2} \nabla_\nu F_{l}^{\lambda} \quad (44) \]

\[ C_{\mu jkl} = -\frac{2}{c-1} C^{\mu |k} C_{l]j} \quad (45) \]
\[ C_{\mu \nu \kappa \lambda} = \nabla_k F^\mu_{[\nu} + \frac{1}{2} F^\mu_{[k} F^\nu_{\kappa]} \]  

(46)

\[ C_{\mu \nu}^\kappa = \frac{1}{\sqrt{2}} g^\mu_{(\kappa} C_{\nu)\kappa} + \frac{1}{\sqrt{2}} g^\nu_{(\kappa} C_{\mu)\kappa} + \frac{1}{\sqrt{2}} g^\nu_{(\kappa} C_{\mu)\kappa} - \frac{1}{4} F^2 g^\mu_{(\kappa} F^\nu_{\kappa)} - \frac{1}{4} F^2 g^\mu_{(\kappa} F^\nu_{\kappa)} \]  

(47)

Relations (42) and (43) only hold in \( d > 2 \) and \( c > 2 \) respectively. In \( d = 2 \) equation (42) has to be replaced by \( C_{\mu \nu \kappa \lambda} = C_{\mu \nu}^\kappa \), while in \( c = 2 \) equation (43) is substituted by \( C_{ijkl} = C_{\kappa[i} g_{k\lambda]\nu]} \).

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