On a conjecture by Haipeng Qu

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November 19, 2018

Abstract

In this note, we prove that $D_8 \times C_2^{n-3}$ is the non-elementary abelian 2-group of order $2^n$, $n \geq 3$, whose number of subgroups of possible orders is maximal. This solves a conjecture by Haipeng Qu [7]. A formula for counting the subgroups of an (almost) extraspecial 2-group is also presented.

MSC2000: Primary 20D30; Secondary 20D60, 20D99.

Key words: subgroup lattice, (elementary) abelian 2-group, (almost/generalized) extraspecial 2-group, Goursat’s lemma, central product.

1 Introduction

Let $G$ be a finite $p$-group of order $p^n$. For $k = 0, 1, ..., n$, we denote by $s_k(G)$ the number of subgroups of order $p^k$ of $G$. The starting point for our discussion is given by Theorem 1.4 of [7] which proves that if $p$ is odd and $G$ is non-elementary abelian then

$$s_k(G) \leq s_k(M_p \times C_p^{m-3}), \forall 0 \leq k \leq n,$$

where $M_p = \langle a, b | a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$ is the non-abelian group of order $p^3$ and exponent $p$. Moreover, in [7] it is conjectured that the inequalities (1) also hold for $p = 2$. Note that in this case there is no direct analogue of $M_p$, as a group of exponent 2 is abelian.
But the dihedral group of order 8 is close, as it is at least generated by elements of order 2.

In the current note we prove the inequalities (1) for \( p = 2 \) by replacing \( M_p \) with \( D_8 \). This completes the work of Haipeng Qu. We also give a formula for the number of subgroups of an (almost) extraspecial 2-group depending on the number of elementary abelian subgroups of possible orders. Our main result is the following.

**Theorem 1.1.** Let \( G \) be a finite non-elementary abelian 2-group of order \( 2^n \), \( n \geq 3 \). Then

\[
s_k(G) \leq s_k(D_8 \times C_2^{n-3}), \forall 0 \leq k \leq n.
\]

Throughout this paper, we will use \( A \ast B \) to denote the (amalgamated) central product of the groups \( A \) and \( B \) having isomorphic centres, and \( X^{sr} \) to denote the central product of \( r \) copies of the group \( X \). Also, we will denote by \( \binom{n}{k}_p \) the number of subgroups of order \( p^k \) in an elementary abelian \( p \)-group of order \( p^n \).

We recall several basic definitions and results on 2-groups that will be useful to us. A finite 2-group is called:

- *extraspecial* if \( Z(G) = G' = \Phi(G) \) has order 2;
- *almost extraspecial* if \( G' = \Phi(G) \) has order 2 and \( Z(G) \cong C_4 \);
- *generalized extraspecial* if \( G' = \Phi(G) \) has order 2 and \( G' \subseteq Z(G) \).

The structure of these groups is well-known (see e.g. Theorem 2.3 of \[2\] and Lemma 3.2 of \[8\]):

**Lemma 1.2.** Let \( G \) be a finite 2-group.

a) If \( G \) is extraspecial, then \( |G| = 2^{2r+1} \) for some positive integer \( r \) and either \( G \cong D_8^{sr} \) or \( G \cong Q_8 \ast D_8^s(r-1) \).

b) If \( G \) is almost extraspecial, then \( |G| = 2^{2r+2} \) for some positive integer \( r \) and \( G \cong D_8^{sr} \ast C_4 \).

c) If \( G \) is generalized extraspecial, then either \( G \cong E \times A \) or \( G \cong (E \ast C_4) \times A \), where \( E \) is an extraspecial 2-group and \( A \) is an elementary abelian 2-group.

The key result of \[7\] is Theorem 1.3. For \( p = 2 \), it states the following.
Lemma 1.3. Let $G$ be a finite 2-group of order $2^n$ and $M$ be a normal subgroup of order 2 of $G$. Then

$$s_k(G) \leq s_k(G/M \times C_2), \forall 0 \leq k \leq n.$$  

We also present a result that follows from Corollary 2 of [3].

Lemma 1.4. Let $G$ be a finite non-elementary abelian 2-group of order $2^n$, $n \geq 3$, and $L_1(G)$ be the set of cyclic subgroups of $G$. Then

$$|L_1(G)| \leq 7 \cdot 2^{n-3} = |L_1(D_8 \times C_2^{n-3})|.$$

2 Proofs of the main results

First of all, we recall the well-known Goursats lemma (see e.g. (4.19) of [9], I) that will be intensively used in our proofs.

Theorem 2.1. Let $A$ and $B$ be two finite groups. Then every subgroup $H$ of the direct product $A \times B$ is completely determined by a quintuple $(A_1, A_2, B_1, B_2, \varphi)$, where $A_1 \leq A_2 \leq A$, $B_1 \leq B_2 \leq B$ and $\varphi : A_2/A_1 \rightarrow B_2/B_1$ is an isomorphism, more exactly $H = \{(a, b) \in A_2 \times B_2 \mid \varphi(aA_1) = bB_1\}$. Moreover, we have $|H| = |A_1||B_2| = |A_2||B_1|$.

Next we remark that Lemma 1.3 can be easily generalized in the following way.

Lemma 2.2. Let $G$ be a finite 2-group of order $2^n$ and $M$ be a normal subgroup of order $2^r$ of $G$. Then

$$s_k(G) \leq s_k(G/M \times C_2^r), \forall 0 \leq k \leq n.$$  

Proof. Take a chief series of $G$ containing $M$ and use induction on $r$. \qed

The following lemma shows that the inequalities (2) hold for finite abelian 2-groups.

Lemma 2.3. Let $G$ be a finite abelian 2-group of order $2^n$, $n \geq 3$. If $G$ is non-elementary abelian, then

$$s_k(G) \leq s_k(D_8 \times C_2^{n-3}), \forall 0 \leq k \leq n.$$
Proof. Since \( G \) is not elementary abelian, we infer that it has a subgroup \( M \) of order \( 2^{n-2} \) such that \( G/M \cong C_4 \). Then Lemma 2.2 leads to
\[
s_k(G) \leq s_k(C_4 \times C_2^{m-2}), \forall 0 \leq k \leq n,
\]
and so it suffices to prove that
\[
s_k(G_1 \times C_2^{m-3}) \leq s_k(D_8 \times C_2^{m-3}), \forall 0 \leq k \leq n,
\]
where \( G_1 = C_4 \times C_2 \). By Theorem 2.1, we know that a subgroup of order \( 2^k \) of \( G_1 \times C_2^{m-3} \) is completely determined by a quintuple \( (A_1, A_2, B_1, B_2, \varphi) \), where \( A_1 \trianglelefteq A_2 \leq G_1, B_1 \trianglelefteq B_2 \leq C_2^{m-3}, \varphi : A_2/A_1 \longrightarrow B_2/B_1 \) is an isomorphism, and \( |A_2||B_1| = 2^k \). Note that \( \varphi \) can be chosen in a unique way for the elementary abelian sections of orders 1 and 2 of \( G_1 \), and in \( 6 = |\text{Aut}(C_2^2)| \) ways for the elementary abelian sections of order 4 of \( G_1 \). We distinguish the following cases:

a) \( A_2 = 1 \).

Then \( A_1 = 1 \) and \( B_1 = B_2 \) is one of the \( \binom{n-3}{k} \) subgroups of order \( 2^k \) of \( C_2^{m-3} \). Clearly, these determine \( \binom{n-3}{k} \) distinct subgroups of \( G_1 \times C_2^{m-3} \).

b) \( A_2 \) is one of the three subgroups of order 2 of \( G_1 \).

Then \( B_1 \) is of order \( 2^{k-1} \) and can be chosen in \( \binom{n-3}{k-1} \) ways. If \( A_1 = A_2 \) then \( B_2 = B_1 \), while if \( A_1 = 1 \) then \( B_2 \) is one of the \( 2^{n-k-2} - 1 \) subgroups of order \( 2^k \) of \( C_2^{m-3} \) containing \( B_1 \). So, in this case we have
\[
3 \cdot \frac{(n-3)!}{(k-1)!} + 3 \cdot (2^{n-k-2} - 1) \binom{n-3}{k-1} = 3 \cdot 2^{n-k} \binom{n-3}{k-1}
\]
distinct subgroups of \( G_1 \times C_2^{m-3} \).

c) \( A_2 \) is one of the two cyclic subgroups of order 4 of \( G_1 \).

Then \( B_1 \) is of order \( 2^{k-2} \) and can be chosen in \( \binom{n-3}{k-2} \) ways. If \( A_1 = A_2 \) then \( B_2 = B_1 \), while if \( |A_1| = 2 \) then \( B_2 \) is one of the \( 2^{n-k-1} - 1 \) subgroups of order \( 2^{k-1} \) of \( C_2^{m-3} \) containing \( B_1 \). So, in this case we have
\[
2 \cdot \frac{(n-3)!}{(k-2)!} + 2 \cdot (2^{n-k-1} - 1) \binom{n-3}{k-2} = 2^{n-k} \binom{n-3}{k-2}
\]
distinct subgroups of \( G_1 \times C_2^{m-3} \).
d) $A_2$ is the unique subgroup isomorphic to $C_2^2$ of $G_1$. Again, $B_1$ is of order $2^{k-2}$ and can be chosen in $\binom{n-3}{k-2}$ ways. If $A_1 = A_2$ then $B_2 = B_1$; if $|A_1| = 2$ then $B_2$ is one of the $2^{n-k-1} - 1$ subgroups of order $2^{k-1}$ of $C_2^{n-3}$ containing $B_1$; if $A_1 = 1$ then $B_2$ is one of the $\binom{n-k}{2}$ subgroups of order $2^k$ of $C_2^{n-3}$ containing $B_1$. One obtains

$$\left(\frac{n-3}{k-2}\right)_2 + 3(2^{n-k-1} - 1)\binom{n-3}{k-2} + 6\binom{n-3}{2}\left(\frac{n-1}{k-2}\right)_2 = 2^{2n-2k-2}\binom{n-3}{k-2}$$

distinct subgroups of $G_1 \times C_2^{n-3}$.

e) $A_2 = G_1$.

In this case $B_1$ is of order $2^{k-3}$ and can be chosen in $\binom{n-3}{k-3}$ ways. If $A_1 = A_2$ then $B_2 = B_1$; if $|A_1| = 4$ then $B_2$ is one of the $2^{n-k} - 1$ subgroups of order $2^{k-2}$ of $C_2^{n-3}$ containing $B_1$; if $A_1$ is the unique subgroup of order 2 of $G_1$ such that $A_2/A_1 \cong C_2^2$ then $B_2$ is one of the $\binom{n-k}{2}$ subgroups of order $2^{k-1}$ of $C_2^{n-3}$ containing $B_1$. One obtains

$$\left(\frac{n-3}{k-3}\right)_2 + 3(2^{n-k} - 1)\binom{n-3}{k-3} + 6\binom{n-3}{2}\left(\frac{n-2}{k-3}\right)_2 = 2^{2n-2k}\binom{n-3}{k-3}$$

distinct subgroups of $G_1 \times C_2^{n-3}$.

By summing all above quantities, we get

$$s_k(G_1 \times C_2^{n-3}) = \left(\frac{n-3}{k}\right)_2 + 3 \cdot 2^{n-k-2}\left(\frac{n-3}{k-1}\right)_2 + 2^{n-k}(2^{n-k-2} + 1)\left(\frac{n-3}{k-2}\right)_2 + 2^{2n-2k}\left(\frac{n-3}{k-3}\right)_2.$$ 

A similar computation leads to

$$s_k(D_8 \times C_2^{n-3}) = \left(\frac{n-3}{k}\right)_2 + 5 \cdot 2^{n-k-2}\left(\frac{n-3}{k-1}\right)_2 + 2^{n-k-1}(2^{n-k} + 1)\left(\frac{n-3}{k-2}\right)_2 + 2^{2n-2k}\left(\frac{n-3}{k-3}\right)_2.$$ 

It is now clear that the inequalities (3) are true, completing the proof. □
In what follows we will focus on describing the subgroup lattice of an (almost) extraspecial 2-group $G$. This can be easily made by using Lemma 2.6 of [2].

**Lemma 2.4.** If $G$ is an (almost) extraspecial 2-group, then $L(G)$ consists of:

a) the trivial subgroup;

b) all subgroups containing $\Phi(G)$; moreover, these are the normal non-trivial subgroups of $G$;

c) all complements of $\Phi(G)$ in the elementary abelian subgroups of order $\geq 4$ containing $\Phi(G)$; moreover,

- $\Phi(G)$ has $2^i$ complements in an elementary abelian subgroup of order $2^{i+1}$ containing $\Phi(G)$;

- two non-normal subgroups $H$ and $K$ of $G$ are conjugate if and only if $H\Phi(G) = K\Phi(G)$;

- given two non-normal subgroups $H$ and $K$ of $G$, if $H^x \subseteq K$ for some $x \in [G/N_G(H)]$, then $x$ is the unique element of $[G/N_G(H)]$ with this property.

It is well-known that the order of maximal elementary abelian subgroups of an extraspecial 2-group of order $2^{2r+1}$ or of an almost extraspecial 2-group of order $2^{2r+2}$ is $2^{r+1}$. Let us denote by $e_i(G)$ the number of elementary abelian subgroups of order $2^i$ containing $\Phi(G)$, $i = 2, 3, ..., r+1$. Under this notation, a formula for the number of subgroups of $G$ can be inferred from Lemma 2.4.

**Corollary 2.5.** If $G$ is an extraspecial 2-group of order $2^{2r+1}$, then

$$|L(G)| = 1 + \sum_{i=0}^{2r} \binom{2r}{i} + \sum_{i=1}^{r} e_{i+1}(G)2^i, \quad (4)$$

while if $G$ is an almost extraspecial 2-group of order $2^{2r+2}$, then

$$|L(G)| = 1 + \sum_{i=0}^{2r+1} \binom{2r+1}{i} + \sum_{i=1}^{r} e_{i+1}(G)2^i. \quad (5)$$
In the particular cases \( r = 2 \) and \( r = 1 \) the equalities (4) and (5), respectively, lead to some known results (see e.g. Chapter 3.3 of [5] and Example 4.5 of [6]).

**Examples.**

a) For \( G = D_8 \ast D_8 \) we obtain \( e_2(G) = 9 \) and \( e_3(G) = 6 \), implying that

\[
|L(G)| = 1 + \sum_{i=0}^{4} \binom{4}{i} \cdot 2^i 9 + 4 \cdot 6 = 110.
\]

b) For \( G = Q_8 \ast D_8 \) we obtain \( e_2(G) = 5 \) and \( e_3(G) = 0 \), implying that

\[
|L(G)| = 1 + \sum_{i=0}^{4} \binom{4}{i} \cdot 2^i 5 = 78.
\]

c) For \( G = D_8 \ast C_4 \) we obtain \( e_2(G) = 3 \), and so

\[
|L(G)| = 1 + \sum_{i=0}^{3} \binom{3}{i} \cdot 2^i 3 = 23.
\]

**Remark.** As we have seen above, the subgroup structure of an (almost) extraspecial 2-group \( G \) depends on its elementary abelian subgroups containing \( \Phi(G) = \langle x \rangle \). We remark that these are the totally singular subspaces of \( G/\Phi(G) \) with respect to the quadratic form \( q : G/\Phi(G) \rightarrow \mathbb{F}_2 \), where \( q(\bar{v}) \) is the element \( a \in \mathbb{F}_2 \) such that \( v^2 = x^a, \forall \bar{v} \in G/\Phi(G) \).

Next we will compare the subgroup lattices of an (almost) extraspecial 2-group \( G \) of order \( 2^n \), \( n \geq 3 \), and of \( D_8 \times C_2^{n-3} \). Since \( \Phi(D_8 \times C_2^{n-3}) \) is of order 2, \( D_8 \times C_2^{n-3}/\Phi(D_8 \times C_2^{n-3}) \) and \( G/\Phi(G) \) have the same dimension over \( \mathbb{F}_2 \), namely \( n - 1 \). Also, we note that the order of maximal elementary abelian subgroups of \( D_8 \times C_2^{n-3} \) is \( 2^{n-1} \), which is greater or equal to the order of maximal elementary abelian subgroups of \( G \). We infer that

\[
|L(D_8 \times C_2^{m-3})| = 1 + \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot 2^i + \sum_{i=1}^{n-2} \epsilon_{i+1}(D_8 \times C_2^{m-3}) 2^i.
\]

The following lemma will be crucial in our proof.
Lemma 2.6. Under the above notation, we have $e_2(G) \leq e_2(D_8 \times C_2^{n-3})$.

Proof. By Lemma 1.4, we know that

$$|L_1(G)| \leq |L_1(D_8 \times C_2^{n-3})|.$$ 

Let $c_i(G)$ be the number of cyclic subgroups of order $2^i$ of $G$. Since both $G$ and $D_8 \times C_2^{n-3}$ are of exponent 4, the above inequality can be rewritten as

$$1 + c_2(G) + c_4(G) \leq 1 + c_2(D_8 \times C_2^{n-3}) + c_4(D_8 \times C_2^{n-3}).$$

On the other hand, we have

$$2^n = 1 + c_2(G) + 2c_4(G) = 1 + c_2(D_8 \times C_2^{n-3}) + 2c_4(D_8 \times C_2^{n-3})$$

and consequently

$$2^n - c_4(G) \leq 2^n - c_4(D_8 \times C_2^{n-3}),$$

i.e.

$$c_4(D_8 \times C_2^{n-3}) \leq c_4(G).$$

(6)

It is clear that the cyclic subgroups of order 4 of these groups contain the Frattini subgroup. Thus (6) leads to

$$e_2(G) \leq e_2(D_8 \times C_2^{n-3}),$$

as desired. \qed

Since any elementary abelian subgroup of $G$ containing $\Phi(G)$ is a direct sum of elementary abelian subgroups of order 4, from Lemma 2.6 we infer that

$$e_i(G) \leq e_i(D_8 \times C_2^{n-3}), \text{ for all } i.$$ 

An immediate consequence of this fact is that the inequalities (2) also hold for (almost) extraspecial 2-groups.

Corollary 2.7. If $G$ is an (almost) extraspecial 2-group of order $2^n$, $n \geq 3$, then

$$s_k(G) \leq s_k(D_8 \times C_2^{n-3}), \forall 0 \leq k \leq n.$$ 

A similar thing can be said about elementary abelian sections of $G$ and of $D_8 \times C_2^{n-3}$.  

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Corollary 2.8. Given a 2-group $G$, we denote by $S_{(\alpha,\beta)}(G)$ the set of all elementary abelian sections $H_2/H_1 \cong C_2^n$ of $G$ with $|H_1| = 2^\beta$. If $G$ is (almost) extraspecial of order $2^n$, then

$$|S_{(\alpha,\beta)}(G)| \leq |S_{(\alpha,\beta)}(D_8 \times C_2^{m-3})|, \text{ for all } \alpha \text{ and } \beta.$$ 

Proof. Write

$$S_{(\alpha,\beta)}(G) = S_{(\alpha,\beta)}^1(G) \cup S_{(\alpha,\beta)}^2(G) \cup S_{(\alpha,\beta)}^3(G) \cup S_{(\alpha,\beta)}^4(G),$$

where

- $S_{(\alpha,\beta)}^1(G) = \{H_2/H_1 \in S_{(\alpha,\beta)}(G) \mid \Phi(G) \subseteq H_1\}$,
- $S_{(\alpha,\beta)}^2(G) = \{H_2/H_1 \in S_{(\alpha,\beta)}(G) \mid H_1 = 1\}$,
- $S_{(\alpha,\beta)}^3(G) = \{H_2/H_1 \in S_{(\alpha,\beta)}(G) \mid \Phi(G) \nsubseteq H_2, H_1 \neq 1\}$,
- $S_{(\alpha,\beta)}^4(G) = \{H_2/H_1 \in S_{(\alpha,\beta)}(G) \mid \Phi(G) \subseteq H_2, \Phi(G) \nsubseteq H_1, H_1 \neq 1\}$.

Since $G/\Phi(G) \cong D_8 \times C_2^{m-3}/\Phi(D_8 \times C_2^{m-3})$, we have

$$|S_{(\alpha,\beta)}^1(G)| = |S_{(\alpha,\beta)}^1(D_8 \times C_2^{m-3})|. \quad (7)$$

Clearly, $S_{(\alpha,\beta)}^2(G) = S_{(\alpha,0)}(G)$ is the number of elementary abelian subgroups of order $2^\alpha$ of $G$, and so

$$|S_{(\alpha,\beta)}^2(G)| = e_\alpha(G) + e_{\alpha+1}(G)2^\alpha,$$

implying that

$$|S_{(\alpha,\beta)}^2(G)| \leq |S_{(\alpha,\beta)}^2(D_8 \times C_2^{m-3})|. \quad (8)$$

We observe that every section $H_2/H_1 \in S_{(\alpha,\beta)}^3(G)$ determines a section $H_2\Phi(G)/H_1\Phi(G) \in S_{(\alpha,\beta+1)}^1(G)$ with $H_2\Phi(G) \cong C_2^{\alpha+\beta+1}$ and $H_1\Phi(G) \cong C_2^{\beta+1}$. Conversely, every section $A/B \in S_{(\alpha,\beta+1)}^1(G)$ with $A \cong C_2^{\alpha+\beta+1}$ and $B \cong C_2^{\beta+1}$ determines $2^\beta$ sections $H_2/H_1 \in S_{(\alpha,\beta)}^3(G)$. Since there are $e_{\alpha+\beta+1}(G)(\alpha+\beta+1)\beta+1$ such sections $A/B \in S_{(\alpha,\beta+1)}^1(G)$, we infer that

$$|S_{(\alpha,\beta)}^3(G)| = e_{\alpha+\beta+1}(G)\left(\frac{\alpha + \beta + 1}{\beta + 1}\right)2^\beta \leq |S_{(\alpha,\beta)}^3(D_8 \times C_2^{m-3})|. \quad (9)$$
Let $H_2/H_1 \in S_{(\alpha,\beta)}^1(G)$. Then $\Phi(H_2) \subseteq H_1$. On the other hand, we have $\Phi(H_2) \subseteq \Phi(G)$ because $H_2$ is normal in $G$. Thus $\Phi(H_2) = 1$, i.e. $H_2$ is elementary abelian, and it can be chosen in $e_{\alpha+\beta}(G)$ ways. Also, $H_1$ is a complement of $\Phi(G)$ in one of the $\binom{\alpha+\beta}{\beta+1}2^\beta$ subgroups of order $2^{\beta+1}$ of $H_2$, and it can be chosen in $\binom{\alpha+\beta}{\beta+1}2^\beta$ ways. Hence

$$|S_{(\alpha,\beta)}^1(G)| = e_{\alpha+\beta}(G)\binom{\alpha + \beta}{\beta + 1}2^\beta \leq |S_{(\alpha,\beta)}^1(D_8 \times C_2^{m-3})|. \quad (10)$$

Obviously, the relations (7), (8), (9) and (10) lead to

$$|S_{(\alpha,\beta)}(G)| \leq |S_{(\alpha,\beta)}(D_8 \times C_2^{m-3})|,$$

as desired. \qed

We are now able to prove our main result.

**Proof of Theorem 1.1.** Let $|G'| = 2^m$. If $m = 0$, then $G$ is abelian and the conclusion follows from Lemma 2.3. Assume that $m \geq 1$.

If $G/G'$ is not elementary abelian, then

$$s_k(G) \leq s_k(G/G' \times C_2^m) \leq s_k(D_8 \times C_2^{m-3}), \forall 0 \leq k \leq n,$$

where the first inequality is obtained by Lemma 2.2, while the second one by Lemma 2.3.

If $G/G'$ is elementary abelian, then $G' = \Phi(G)$. Let $M$ be a normal subgroup of $G$ such that $M \subseteq G'$ and $[G': M] = 2$. Then

$$s_k(G) \leq s_k(G_1 \times C_2^{m-1}), \forall 0 \leq k \leq n,$$

where $G_1 = G/M$ satisfies the conditions

$$G_1' = \Phi(G_1), |G_1'| = 2 \text{ and } G_1' \subseteq Z(G_1),$$

i.e. it is a generalized extraspecial 2-group. Then either

$$G_1 \cong E \times A \text{ or } G_1 \cong (E \ast C_4) \times A,$$

where $E$ is an extraspecial 2-group and $A$ is an elementary abelian 2-group, by Lemma 1.2. In other words, $G_1$ is a direct product of an (almost) extraspecial
2-group and an elementary abelian 2-group. So, it suffices to prove that if \( G_2 \) is an (almost) extraspecial 2-group of order \( 2^q \), then

\[
s_k(G_2 \times C_2^{n-q}) \leq s_k((D_8 \times C_2^{q-3}) \times C_2^{n-q}), \forall 0 \leq k \leq n.
\]

(11)

Let \( A \) be one of the groups \( G_2 \) or \( D_8 \times C_2^{q-3} \). From Theorem 2.1 it follows that a subgroup \( H \leq A \times C_2^{n-q} \) of order \( 2^k \) is completely determined by a quintuple \((A_1, A_2, B_1, B_2, \varphi)\), where \( A_1 \leq A_2 \leq A \), \( B_1 \leq B_2 \leq C_2^{n-q} \), \( \varphi : A_2/A_1 \rightarrow B_2/B_1 \) is an isomorphism, and \(|A_2||B_1| = 2^k\). By fixing the section \( B_2/B_1 \) of \( C_2^{n-q} \) and \( \varphi \in \text{Aut}(B_2/B_1) \), we infer that \( H \) depends only on the choice of the section \( A_2/A_1 \in \mathcal{S}_{(\alpha,\beta)}(A) \), where \((\alpha,\beta) \in \{(i,j) \mid i = 0, 1, ..., k, j = 0, 1, ..., k - i\}\). So, to prove the inequalities (11) it suffices to compare the numbers of elementary abelian sections \( A_2/A_1 \) of the two groups \( G_2 \) and \( D_8 \times C_2^{q-3} \), where \(|A_1|\) and \(|A_2|\) are arbitrary. It is now clear that the conclusion follows from Corollary 2.8, completing the proof. \( \square \)

Finally, we mention that a result similar with Lemma 1.4 can be obtained from Theorem 1.1.

**Corollary 2.9.** Let \( G \) be a finite non-elementary abelian 2-group of order \( 2^n \), \( n \geq 3 \). Then

\[
|L(G)| \leq |L(D_8 \times C_2^{n-3})|.
\]

**Acknowledgements.** The author is grateful to the reviewer for its remarks which improve the previous version of the paper.

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