A Variational Principle for the Metric Mean Dimension of Level Sets

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Abstract—We prove a variational principle for the upper and lower metric mean dimension of level sets \( \{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \alpha \} \) associated to continuous potentials \( \varphi : X \to \mathbb{R} \) and continuous dynamics \( f : X \to X \) defined on compact metric spaces and exhibiting the specification property. This result relates the upper and lower metric mean dimension of the above mentioned sets with growth rates of measure-theoretic entropy of partitions decreasing in diameter associated to some special measures. Moreover, we present several examples to which our result may be applied to. Similar results were previously known for the topological entropy and for the topological pressure.

Index Terms—Metric mean dimension, variational principle, level sets.

I. INTRODUCTION

ONE of the most important notions in Dynamical Systems is that of topological entropy. It is a topological invariant and, roughly speaking, measures how chaotic a system is. In particular, it is an effective tool to decide whether two systems are conjugated or not. Nevertheless, there are plenty of systems with infinite topological entropy (for instance, they form a \( C^0 \)-generic set in the space of homeomorphisms of a compact manifold [39] with dimension greater than one) and thus, in this context, the entropy is not useful anymore. Therefore, in order to study these types of systems, new dynamical quantities are required and an example of such a quantity is the metric mean dimension.

The notion of metric mean dimension was introduced by Lindenstrauss and Weiss in [24] as metric-dependent analog of the mean dimension, a topological invariant associated to a dynamical system which was introduced by Gromov [12]. This last notion has several applications, like in the study of embedding problems [18], and the metric mean dimension presents an upper bound to it. But more than that, the metric mean dimension turned out to be useful in several contexts like in the study of compression [15], [16].

In the present paper we give a modest contribution to the study of ergodic theoretical aspects of the metric mean dimension by presenting a variational principle. Previous connections between ergodic theory and metric mean dimension were presented, for instance, by Lindenstrauss and Tsukamoto [23], Velozo and Velozo [35], Tsukamoto [34], Shi [29], Gutman and Śpiewak [17] and Yang, Chen and Zhou [36]. For more on these works, see Section II-H. The main novelty of our work with respect to the previously mentioned ones is that our variational principle holds for special subsets and not only for the whole phase space. More precisely, we consider level sets

\[
K_\alpha = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \alpha \right\}
\]

associated to continuous potentials \( \varphi : X \to \mathbb{R} \) and continuous dynamics \( f : X \to X \) defined on compact metric spaces exhibiting the specification property and present a relation between the upper and lower metric mean dimension of the above mentioned sets and growth rates of measure-theoretic entropy of partitions decreasing in diameter associated to some special measures. This is the content of Theorem A. In Section IV we present several examples to which our result is applicable.

A. Multifractal Analysis

The general idea of multifractal analysis consists in decomposing the phase space into subsets of points with similar dynamical behavior, for instance, in sets of points with the same Birkhoff average, the same Lyapunov exponents or the same local entropies, and to describe the size of each of such subsets from a geometrical or topological viewpoint. The information (collection of numbers) obtained via this procedure for one such decomposition of the phase space is called a multifractal spectrum. Then, in the best-case scenario the idea is that if one knows some of these spectra one could fully recover the dynamics (see for instance [2] and [3]). This phenomenon is sometimes called multifractal rigidity. But even when we are not in such a nice world, we still can get useful information about the dynamics from these various spectra (see for instance [11], [25], [30], and [31]). Our main result, Theorem A, may be seen as a small contribution to the study of one such spectra, namely, the one obtained by measuring the size of level sets of Birkhoff averages with respect to the metric mean dimension. In particular, as a consequence of our result we get that the map \( \alpha \mapsto \mulim_{\mu}(K_\alpha, f, d) \) is concave when restricted to the set of
parameters $\alpha \in \mathbb{R}$ for which $K_\alpha \neq \emptyset$. As far as we know, this is the first time this spectrum was considered and we hope that our results may be of some more help in the study of multifractal analysis of systems with infinite topological entropy.

II. DEFINITIONS AND STATEMENTS

Let $(X, d)$ be a compact metric space and $f : X \to X$ be a continuous map. Given $n \in \mathbb{N}$, we define the dynamical metric $d_n : X \times X \to [0, \infty)$ by

$$d_n(x, z) = \max \left\{ d(f^i(x), f^i(z)) : 0 \leq i \leq n - 1 \right\}. $$

It is easy to see that $d_n$ is indeed a metric and, moreover, generates the same topology as $d$. Furthermore, given $\varepsilon > 0$, $n \in \mathbb{N}$ and a point $x \in X$, we define the open $(n, \varepsilon)$-ball around $x$ by

$$B_n(x, \varepsilon) = \{ y \in X : d_n(x, y) < \varepsilon \}. $$

We sometimes call these $(n, \varepsilon)$-balls dynamical balls of radius $\varepsilon$ and length $n$. We say that a set $E \subset X$ is $(n, \varepsilon)$-separated by $f$ if $d_n(x, z) > \varepsilon$ for every $x, z \in E$.

A. The Metric Mean Dimension

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, let us denote by $s(f, n, \varepsilon)$ the maximal cardinality of all $(n, \varepsilon)$-separated subsets of $X$ by $f$ which, due to the compactness of $X$, is finite.

The upper metric mean dimension of $f$ with respect to $d$ is given by

$$\overline{\text{mdim}}_M (X, f, d) = \limsup_{\varepsilon \to 0} \frac{h(f, \varepsilon)}{\log \varepsilon}, $$

where

$$h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s(f, n, \varepsilon). $$

Similarly, the lower metric mean dimension of $f$ with respect to $d$ is given by

$$\underline{\text{mdim}}_M (X, f, d) = \liminf_{\varepsilon \to 0} \frac{h(f, \varepsilon)}{\log \varepsilon}. $$

In the case when $\overline{\text{mdim}}_M (X, f, d) = \underline{\text{mdim}}_M (X, f, d)$ this common value is called the metric mean dimension of $f$ with respect to $d$ and is denoted simply by $\text{mdim}_M (X, f, d)$.

Recall that the topological entropy of the map $f$ is given by

$$h_{\text{top}}(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon). $$

Consequently, $\overline{\text{mdim}}_M (X, f, d) = \underline{\text{mdim}}_M (X, f, d) = 0$ whenever the topological entropy of $f$ is finite. In particular, the metric mean dimension is a suitable quantity to study systems with infinite topological entropy. For more on these quantities see [23], [24], and [34] and references therein.

B. The Metric Mean Dimension for Non-Compact Subset

We now present the notion of metric mean dimension on non-compact sets introduced in [10]. Given a set $Z \subset X$, let us consider

$$m(Z, s, N, \varepsilon) = \inf_{\Gamma} \left\{ \sum_{i \in I} \exp (-s n_i) \right\}, $$

where the infimum is taken over all covers $\Gamma = \{ B_n(x_i, \varepsilon) \}_{i \in I}$ of $Z$ with $n_i \geq N$. We also consider

$$m(Z, s, \varepsilon) = \lim_{N \to \infty} m(Z, s, N, \varepsilon). $$

One can show (see for instance [26]) that there exists a certain number $s_0 \in [0, \infty)$ such that $m(Z, s, \varepsilon) = 0$ for every $s > s_0$ and $m(Z, s, \varepsilon) = +\infty$ for every $s < s_0$. In particular, we may consider

$$h \left( Z, f, \varepsilon \right) = \inf \{ s : m(Z, s, \varepsilon) = 0 \} = \sup \{ s : m(Z, s, \varepsilon) = +\infty \}. $$

(1)

The upper metric mean dimension of $f$ on $Z$ is then defined as the following limit

$$\overline{\text{mdim}}_M (Z, f, d) = \limsup_{\varepsilon \to 0} \frac{h \left( Z, f, \varepsilon \right)}{\log \varepsilon}. $$

Similarly, the lower metric mean dimension of $f$ on $Z$ is defined as

$$\underline{\text{mdim}}_M (Z, f, d) = \liminf_{\varepsilon \to 0} \frac{h \left( Z, f, \varepsilon \right)}{\log \varepsilon}. $$

In the case when $Z = X$ one can check that the two definitions of upper/lower metric mean dimension given above actually coincide.

C. Level Sets of a Continuous Map

Let $C(X, \mathbb{R})$ denote the set of all continuous maps $\varphi : X \to \mathbb{R}$ and take $\varphi \in C(X, \mathbb{R})$. For $\alpha \in \mathbb{R}$, let

$$K_\alpha = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \alpha \right\}. $$

(2)

We also consider the set

$$\mathcal{L}_\varphi = \{ \alpha \in \mathbb{R} : K_\alpha \neq \emptyset \}. $$

It is easy to see that $\mathcal{L}_\varphi$ is a bounded and non-empty set [32, Lemma 2.1]. Moreover, if $f$ satisfies the so called specification property (see Section II-F) then $\mathcal{L}_\varphi$ is an interval of $\mathbb{R}$ and, moreover, $\mathcal{L}_\varphi = \{ \int \varphi d\mu : \mu \in \mathcal{M}_f(X) \}$ where $\mathcal{M}_f(X)$ stands for the set of all invariant measures (see [33, Lemma 2.5]).

D. The Auxiliary Quantities $\Lambda_\alpha \overline{\text{mdim}}_M (f, \alpha, d)$ and $\Lambda_\alpha \underline{\text{mdim}}_M (f, \alpha, d)$

Fix $\alpha \in \mathbb{R}$ and $\varphi \in C(X, \mathbb{R})$. For $\delta > 0$ and $n \in \mathbb{N}$ define the set

$$P(\alpha, \delta, n) = \left\{ x \in X : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \alpha \right| < \delta \right\}. $$

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Let $N(\alpha, \delta, n, \varepsilon)$ denote the minimal number of $(n, \varepsilon)$-balls needed to cover $P'(\alpha, \delta, n)$. Define

$$
\Lambda_\varphi(\alpha, \varepsilon) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(\alpha, \delta, n, \varepsilon)
$$

and

$$
\Lambda_{\varphi \mathrm{mdim} \! M}(f, \alpha, d) = \limsup_{\varepsilon \to 0} \frac{\Lambda_\varphi(\alpha, \varepsilon)}{\log |\varphi|},
$$

$$
\Lambda_{\varphi \mathrm{mdim} \! M}(f, \alpha, d) = \liminf_{\varepsilon \to 0} \frac{\Lambda_\varphi(\alpha, \varepsilon)}{\log |\varphi|}.
$$

**Remark 1:** Observe that, if $M(\alpha, \delta, n, \varepsilon)$ denotes the maximal cardinality of a $(n, \varepsilon)$-separated set contained in $P'(\alpha, \delta, n)$, then we have that

$$
N(\alpha, \delta, n, \varepsilon) \leq M(\alpha, \delta, n, \varepsilon) \leq N(\alpha, \delta, n, \varepsilon/2).
$$

In particular,

$$
\Lambda_\varphi(\alpha, \varepsilon) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log M(\alpha, \delta, n, \varepsilon).
$$

**E. The Main Quantities** $H_{\varphi \mathrm{mdim} \! M}(f, \alpha, d)$ and $H_{\varphi \mathrm{mdim} \! M}(f, \alpha, d)$

Given $\varphi \in C(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$, let us consider

$$
\mathcal{M}_f(X, \varphi, \alpha) = \left\{ \mu \text{ is } f\text{-invariant and } \int \varphi \, d\mu = \alpha \right\}.
$$

A simple observation is that $\mathcal{M}_f(X, \varphi, \alpha) \neq \emptyset$ for every $\alpha \in \mathcal{L}_\varphi$ (see [32, Lemma 4.1]).

Let $\mu \in \mathcal{M}_f(X)$. We say that $\xi = \{C_1, \ldots, C_k\}$ is a measurable partition of $X$ if every $C_i$ is a measurable set, $\mu(X \setminus \bigcup_{i=1}^k C_i) = 0$ and $\mu(C_i \cap C_j) = 0$ for every $i \neq j$. The **entropy** of $\xi$ with respect to $\mu$ is given by

$$
H_\mu(\xi) = -\sum_{i=1}^k \mu(C_i) \log(\mu(C_i)).
$$

Given a measurable partition $\xi$, we consider $\xi^n = \bigvee_{j=1}^{n-1} f^{-j} \xi$. Then, the **metric entropy** of $(f, \mu)$ with respect to $\xi$ is given by

$$
h_\mu(f, \xi) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\xi^n).
$$

Using this quantity we define

$$
H_{\varphi \mathrm{mdim} \! M}(f, \alpha, d)
$$

$$
= \limsup_{\varepsilon \to 0} \frac{1}{\log |\varphi|} \sup_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \inf_{|\xi| < \varepsilon} h_\mu(f, \xi)
$$

and

$$
H_{\varphi \mathrm{mdim} \! M}(f, \alpha, d)
$$

$$
= \liminf_{\varepsilon \to 0} \frac{1}{\log |\varphi|} \sup_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \inf_{|\xi| < \varepsilon} h_\mu(f, \xi)
$$

where $|\xi|$ denotes the diameter of the partition $\xi$ and the infimum is taken over all finite measurable partitions of $X$ satisfying $|\xi| < \varepsilon$.

We also recall that the **metric entropy** of $(f, \mu)$ is given by

$$
h_\mu(f) = \sup_{\xi} h_\mu(f, \xi)
$$

where the supremum is taken over all finite measurable partitions $\xi$ of $X$.

**F. Specification Property**

We say that $f$ satisfies the **specification property** if for every $\varepsilon > 0$, there exists an integer $m = m(\varepsilon)$ such that for any collection of finite intervals $I_j = [a_j, b_j] \subset \mathbb{N}$, $j = 1, \ldots, k$, satisfying $a_{j+1} - b_j \geq m(\varepsilon)$ for every $j = 1, \ldots, k - 1$ and any $x_1, \ldots, x_k$ in $X$, there exists a point $x \in X$ such that

$$
d(f^{p+\alpha}/x, f^p x_j) < \varepsilon
$$

for all $p = 0, \ldots, b_j - a_j$ and every $j = 1, \ldots, k$.

The specification property is present in many interesting examples. For instance, every topologically mixing locally maximal hyperbolic set has the specification property and factors of systems with specification have specification (see for instance [22]). Other examples of systems satisfying this property which are more adapted to our purposes will appear in Section IV.

**G. Main Result**

Our main result may be seen as an extension of [32, Theorem 5.1] to the infinite entropy setting.

**Theorem A:** Suppose $f : X \to X$ is a continuous transformation with the specification property. Let $\varphi \in C(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$ be such that $K_\alpha \neq \emptyset$. Then

$$
\mathrm{mdim} \! M(K_\alpha, f, d) = \Lambda_{\varphi \mathrm{mdim} \! M}(f, \alpha, d) = \mathrm{mdim} \! M(K_\alpha, f, d).
$$

Similarly,

$$
\mathrm{mdim} \! M(K_\alpha, f, d) = \Lambda_{\varphi \mathrm{mdim} \! M}(f, \alpha, d) = \mathrm{mdim} \! M(K_\alpha, f, d).
$$

We consider the equalities between $\mathrm{mdim} \! M(K_\alpha, f, d)$ and $\mathrm{mdim} \! M(f, \alpha, d)$ and between $\mathrm{mdim} \! M(K_\alpha, f, d)$ and $\mathrm{mdim} \! M(f, \alpha, d)$ to be the most important part of our result because it relates a topological quantity with one that has an ergodic-theoretical flavor. Moreover, in some cases it allows us to obtain some interesting properties about the multifractal spectrum. For instance, will show bellow that

**Proposition 2:** Under the assumptions of Theorem A, the map

$$
\mathcal{L}_\varphi \ni \alpha \mapsto \mathrm{mdim} \! M(f, \alpha, d)
$$

is concave.

Consequently, combining this result with Theorem A we get that

**Corollary 3:** Under the assumptions of Theorem A, the map

$$
\mathcal{L}_\varphi \ni \alpha \mapsto \mathrm{mdim} \! M(K_\alpha, f, d)
$$

is concave.

An interesting question is whether we can change the order between the limit and the supremum in the definition of $H_{\varphi \mathrm{mdim} \! M}(f, \alpha, d)$ and $H_{\varphi \mathrm{mdim} \! M}(f, \alpha, d)$. This would allow, for instance, to talk about the existence of “maximizing measures”: measures that realize the supremum. Such a measure would capture the complexity of the system over...
all scales $\varepsilon > 0$. It was observed in [23, Section VIII] that a similar question involving different ergodic quantities is, in general, false. Nevertheless, under the additional assumption that $f$ has the marker property, one can do such a change (in the setting of [23]) as observed by Yang, Chen and Zhou [36]. As for our hypothesis that $f$ satisfies the specification property, we do not know whether it is actually required for Theorem A to hold or if it is just an artefact of the technique.

**H. Related Results**

As already mentioned, for the topological entropy a result similar to Theorem A was obtained in [32]. In fact, our result was inspired by that one. Moreover, [32] was extended to the framework of topological pressure in [33].

As for variational results involving the upper metric mean dimension, there are several works dealing with this problem. For instance, [23] presented a variational principle relating the metric mean dimension with the supremum of certain rate distortion functions over invariant measures of the system. This was further explored in [35]. More recently, [29] obtained variational principles for the metric mean dimension in terms of Brin-Katok local entropy and Shapira’s entropy of an open cover. One result that is more connected to ours is the one obtained in [17] which says that

$$\overline{\text{mdim}}_M(X, f, d) = \limsup_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \sup_{\mu \in \mathcal{M}_f(X)} \inf_{|t| < \varepsilon} h_\mu(f, \xi)$$

and

$$\underline{\text{mdim}}_M(X, f, d) = \liminf_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \sup_{\mu \in \mathcal{M}_f(X)} \inf_{|t| < \varepsilon} h_\mu(f, \xi).$$

These are variational results for the upper/lower metric mean dimension of the entire space $X$ while Theorem A applies also to level sets of continuous maps $\varphi$. Observe that in the case when $\varphi$ is a constant map equal to $\alpha$, the $\alpha$-level set of it coincides with $X$. In particular, whenever $f$ has the specification property, (6) and (7) may be seen as particular cases of our result. We stress however that the results in [17] do not assume such property.

**III. PROOFS OF THEOREM A AND PROPOSITION 2**

In this section we present the proofs of Theorem A and Proposition 2 starting with the latter one which is much simpler.

**Proof of Proposition 2:** Given measures $\mu_1, \mu_2 \in \mathcal{M}_f(X)$, using that the map $\mu \mapsto H_\mu(\xi)$ is concave for any finite and measurable partition $\xi$ (see [8, Lemma 9.5.1]), it follows that for any $t \in [0, 1]$,

$$th_{\mu_1}(f, \xi) + (1 - t)h_{\mu_2}(f, \xi) \leq h_{t\mu_1 + (1 - t)\mu_2}(f, \xi).$$

In particular,

$$t \inf_{|t| < \varepsilon} h_{\mu_1}(f, \xi) + (1 - t) \inf_{|t| < \varepsilon} h_{\mu_2}(f, \xi) \leq \inf_{|t| < \varepsilon} h_{t\mu_1 + (1 - t)\mu_2}(f, \xi).$$

Now, given $\alpha_1, \alpha_2 \in L_\varphi$, by the comments in Section II-C there exist invariant measures $\mu_1, \mu_2 \in \mathcal{M}_f(X)$ such $\alpha_i = \int \varphi d\mu_i, i = 1, 2$. For any $t \in [0, 1]$, consider $\mu = t\mu_1 + (1 - t)\mu_2$ and $\alpha = t\alpha_1 + (1 - t)\alpha_2$. Then, $\mu \in \mathcal{M}_f(X, \varphi, \alpha)$.

Combining this observation with (8) we get that

$$t \sup_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \inf_{|t| < \varepsilon} h_\mu(f, \xi) + (1 - t) \sup_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \inf_{|t| < \varepsilon} h_\mu(f, \xi) \leq \inf_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \sup_{|t| < \varepsilon} h_{t\mu_1 + (1 - t)\mu_2}(f, \xi).$$

Then, dividing everything by $\log \varepsilon$, taking “lim inf $\varepsilon \to 0$” and using that $\lim \inf (a) + \lim \inf (b) \leq \lim \inf (a + b)$ we get that

$$\frac{t}{\log \varepsilon} \text{mdim}_M(f, \alpha_1, d) + \frac{(1 - t)}{\log \varepsilon} \text{mdim}_M(f, \alpha_2, d) \leq \frac{1}{\log \varepsilon} \text{mdim}_M(f, \alpha, d).$$

Consequently, the map $\mathcal{L}_\alpha: \alpha \mapsto \text{mdim}_M(f, \alpha, d)$ is concave concluding the proof of the proposition.

**Remark 4:** It is not clear to us whether a version of Proposition 2 holds for the map $\mathcal{L}_\alpha: \alpha \mapsto \text{mdim}_M(f, \alpha, d)$. In fact, in order to get the desired conclusion in the aforementioned proposition we have used the property that $\lim \inf (a) + \lim \inf (b) \leq \lim \inf (a + b)$ which obviously does not hold for the lim sup.

We now present the proof of Theorem A. We start considering the first claim of the theorem and, for the sake of clarity of the presentation, we split it into three main propositions. We emphasize that this proof is an adaptation of the proof of Theorem 5.1 of [32] to our setting. Fix $\varphi \in C(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$ such that $\Lambda_\alpha \neq 0$. Moreover, assume initially that all the quantities $\text{mdim}_M(K_\alpha, f, d)$, $\text{mdim}_M(f, \alpha, d)$ and $\text{mdim}_M(f, \alpha, d)$ are finite.

**Proposition 5:** Under the hypotheses of Theorem A we have that

$$\text{mdim}_M(K_\alpha, f, d) \leq \Lambda_\alpha \text{mdim}_M(f, \alpha, d).$$

**Proof:** Let $\{\varepsilon_j\}_{j \in \mathbb{N}}$ be a sequence of positive numbers converging to zero such that

$$\text{mdim}_M(K_\alpha, f, d) = \lim_{j \to \infty} \frac{h(K_\alpha, f, \varepsilon_j)}{\log \varepsilon_j}.$$
Now, given \( k \in \mathbb{N} \), since \( G(\alpha, \delta, k) \subseteq P(\alpha, \delta, n) \) for \( n \geq k \), it follows that \( G(\alpha, \delta, k) \) may be covered by \( N(\alpha, \delta, n, \varepsilon_j) \) dynamical balls of radius \( \varepsilon_j \) and length \( n \). Thus, for every \( s \geq 0 \) and \( n \geq k \) we have

\[
m(\alpha, \delta, k), s, \varepsilon_j) \leq N(\alpha, \delta, n, \varepsilon_j) \exp(-ns).
\]

Let \( s = s(\varepsilon_j) > \Lambda_\varphi(\alpha, \varepsilon_j) \) and \( \gamma(\varepsilon_j) = (s - \Lambda_\varphi(\alpha, \varepsilon_j))/2 \). Then, if \( \delta_j > 0 \) is small enough, there exists an increasing sequence \( \{n_k\} \subseteq \mathbb{N} \) such that

\[
N(\alpha, \delta_j, n_k, \varepsilon_j) \leq \exp(n_k(\Lambda_\varphi(\alpha, \varepsilon_j) + \gamma(\varepsilon_j))).
\]

Thus, assuming without lost of generality that \( n_1 \geq k \) and combining the previous observations we conclude that

\[
m(\alpha, k, \delta_j), s(\varepsilon_j), \varepsilon_j) \leq \exp(-n_k\gamma(\varepsilon_j)).
\]

In particular, as \( \gamma(\varepsilon_j) > 0 \), letting \( n_k \to \infty \) we obtain

\[
h(\alpha, k, \delta_j), s(\varepsilon_j), \varepsilon_j) = 0.
\]

Consequently,

\[
h(\alpha, k, \delta_j), f, \varepsilon_j) \leq s(\varepsilon_j)
\]

which implies that

\[
h(K_\alpha, f, \varepsilon_j) \leq \sup_k h(G(\alpha, k, \delta_j), f, \varepsilon_j) \leq s(\varepsilon_j).
\]

Hence,

\[
\overline{\text{mdim}}_M(K_\alpha, f, d) = \limsup_{j \to \infty} \frac{h(K_\alpha, f, \varepsilon_j)}{|\log \varepsilon_j|}
\]

\[
\leq \limsup_{j \to \infty} \frac{s(\varepsilon_j)}{|\log \varepsilon_j|}
\]

\[
\leq \limsup_{j \to \infty} \frac{2\gamma(\varepsilon_j)}{|\log \varepsilon_j|} + \limsup_{j \to \infty} \frac{\Lambda_\varphi(\alpha, \varepsilon_j)}{|\log \varepsilon_j|}
\]

\[
\leq \limsup_{j \to \infty} \frac{2\gamma(\varepsilon_j)}{|\log \varepsilon_j|} + \Lambda_\varphi(\alpha, d, \odot).
\]

Therefore, as we can choose \( s(\varepsilon_j) \) arbitrarily close to \( \Lambda_\varphi(\alpha, \varepsilon_j) \), the limsup in the last step is zero for an adequate choice of \( s(\varepsilon_j) \). Then,

\[
\overline{\text{mdim}}_M(K_\alpha, f, d) \leq \Lambda_\varphi(\alpha, d, \odot)
\]

completing the proof of the proposition.

**Proposition 6:** Under the hypotheses of Theorem A we have that

\[
H_\varphi(\text{mdim})_M(f, \alpha, d) \leq \overline{\text{mdim}}_M(K_\alpha, f, d).
\]

The strategy of the proof consists in constructing a fractal set \( F \) contained in \( K_\alpha \) and a special probability measure \( \eta \) supported on \( F \) that satisfies the hypothesis of the so called Entropy Distribution Principle (see Lemma 13). This will be enough to get the desired inequality. As a step towards the definition of \( F \), we introduce a family of finite sets \( S_k \) which play a major role in the construction.

In order to prove Proposition 6 we will need the following auxiliary quantity. For \( \mu \in \mathcal{M}_f(X) \), \( \delta > 0 \) and \( n \in \mathbb{N} \), let us denote by \( N_\mu(\delta, \varepsilon, n) \) the minimal number \( (n, \varepsilon) \)-balls needed to cover a set of \( \mu \)-measure bigger than \( 1 - \delta \). Then, we define

\[
h_\mu(f, \varepsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log N_\mu(\delta, \varepsilon, n).
\]

**Proof of Proposition 6:** Fix \( \gamma > 0 \) and let \( \{\delta_k\}_{k \in \mathbb{N}} \) be a decreasing sequence converging to 0. Take \( \varepsilon = \varepsilon(\gamma) > 0 \) and let \( \mu \in \mathcal{M}_f(X, \varphi, \alpha) \) so that

\[
\inf_{\varepsilon \leq 2\varepsilon} \frac{h_\mu(f, \varepsilon)}{|\log \varepsilon|} \geq H_\varphi(\text{mdim})_M(f, \alpha, d) - \gamma/2
\]

and

\[
\frac{h(K_\alpha, f, \varepsilon/2)}{|\log \varepsilon/2|} \leq \text{mdim}_M(K_\alpha, f, d + \gamma).
\]

Let \( \mathcal{U} \) be a finite open cover of \( X \) with diameter \( \text{diam}(\mathcal{U}) \leq 5\varepsilon \) and Lebesgue number \( \text{Leb}(\mathcal{U}) \geq \varepsilon \). We now construct an auxiliary measure which is a finite combination of ergodic measures and “approximates” \( \mu \). To prove this lemma we follow the idea from [37, p. 535]. In what follows, \( \partial \xi \) will denote the boundary of the partition \( \xi \) which is just the union of the boundaries of all the elements of the partition and \( \xi \supseteq \mathcal{U} \) means that \( \xi \) refines \( \mathcal{U} \), that is, each element of \( \xi \) is contained in an element of \( \mathcal{U} \).

**Lemma 7:** For each \( k \in \mathbb{N} \), there exists a measure \( \nu_k \in \mathcal{M}_f(X) \) satisfying

(a) \( \nu_k = \sum_{i=1}^{j(k)} \lambda_i \nu_{i,k} \), where \( \lambda_i > 0 \), \( \sum_{i=1}^{j(k)} \lambda_i = 1 \) and \( \nu_{i,k} \in \mathcal{M}_f^{\text{erg}}(X) \);

(b) \( \inf_{\xi \supseteq \mathcal{U}} h_\mu(f, \xi) \leq \inf_{\xi \supseteq \mathcal{U}} h_{\nu_k}(f, \xi) + \delta_k/2 \);

(c) \( \left| \int_X \varphi \ d\nu_k - \int_X \varphi \ d\mu \right| < \delta_k \).

**Proof of Lemma 7:** Given \( k \in \mathbb{N} \), let \( \beta_k > 0 \) be such that for every \( \tau_1, \tau_2 \in \mathcal{M}_f(X) \),

\[
d_{\mathcal{M}_f(X)}(\tau_1, \tau_2) < \beta_k \implies \left| \int \varphi d\tau_1 - \int \varphi d\tau_1 \right| < \delta_k
\]

where \( d_{\mathcal{M}_f(X)}(\tau) \) is a metric in \( \mathcal{M}_f(X) \). Let \( \mathcal{P} = \{P_1, \ldots, P_{j(k)}\} \) be a partition of \( \mathcal{M}_f(X) \) whose diameter with respect to \( d_{\mathcal{M}_f(X)} \) is smaller than \( \beta_k \). By the Ergodic Decomposition Theorem there exists a measure \( \hat{\mu} \) on \( \mathcal{M}_f(X) \) satisfying \( \hat{\mu}(\mathcal{M}_f^{\text{erg}}(X)) = 1 \) such that

\[
\int \psi(x) \ d\mu(x) = \int_{\mathcal{M}_f(X)} \left( \int_X \psi(x) d\tau(x) \right) d\hat{\mu}(\tau)
\]

for every \( \psi \in C(X, \mathbb{R}) \). Let us consider now \( \lambda_i = \hat{\mu}(P_i) \) and take \( \nu_{i,k} \in \mathcal{P} \cap \mathcal{M}_f^{\text{erg}}(X) \) such that

\[
\inf_{\xi \supseteq \mathcal{U}} h_{\nu_{i,k}}(f, \xi) \geq \inf_{\xi \supseteq \mathcal{U}} h_\mu(f, \xi) - \delta/2
\]

for \( \mu \)-almost every \( \tau \in \mathcal{P} \cap \mathcal{M}_f^{\text{erg}}(X) \). Observe that such a measure \( \nu_{i,k} \) exists because \( \sup_{\tau \in \mathcal{M}_f^{\text{erg}}(X)} \inf_{\xi \supseteq \mathcal{U}} h_\mu(f, \xi) < +\infty \). This latter fact follows from Lemma 3 and Theorem 5 of [29] and the fact that the upper metric mean dimension is finite. Finally, define \( \nu_k = \sum_{i=1}^{j(k)} \lambda_i \nu_{i,k} \). It is easy to see that \( \nu_k \) satisfies properties a) and c) from the statement. Let us now check that it also satisfies b). By [19, Proposition 5] we know that

\[
\inf_{\xi \supseteq \mathcal{U}} h_\mu(f, \xi) \leq \int_{\mathcal{M}_f(X)} \inf_{\xi \supseteq \mathcal{U}} h_\tau(f, \xi) d\hat{\mu}(\tau).
\]
Thus, by our choice of the measures $\nu^k_i$ it follows that
\[
\inf_{\xi \in \mathcal{U}} h_{\mu_i} (f, \xi) = \int_{\mathcal{Y}_i} \inf_{\tau \in \mathcal{U}} h_{\tau} (f, \xi) d\mu_i (\tau) \\
\leq \sum_{i=1}^{j(k)} \lambda_i \inf_{\xi \in \mathcal{U}} h_{\nu_i^k} (f, \xi) + \delta_k / 2 \\
\leq \inf_{\xi \in \mathcal{U}} h_{\nu_k} (f, \xi) + \delta_k / 2
\]
completing the proof of the lemma.

Let $\nu_k$ be as in the previous lemma. Using the fact that each measure $\nu^k_i$ is ergodic, by the proof of [29, Theorem 9] there exists a finite Borel measurable partition $\xi_k$ which refines $\mathcal{U}$ so that
\[
h_{\nu_i^k} (f, 5\epsilon, \gamma) \leq h_{\nu_i^k} (f, \xi_k) \leq h_{\nu_i^k} (f, 5\epsilon/4, \gamma) + \delta_k.
\]
Now, take a finite Borel partition $\xi$ refining $\mathcal{U}$ with $\mu(\partial \xi) = 0$ such that
\[
h_{\mu} (f, \xi) - \delta_k \leq \inf_{\xi \in \mathcal{U}} h_{\nu_k} (f, \xi).
\]
In particular, since $\xi_k \supset \mathcal{U}$,
\[
h_{\mu} (f, \xi) - \delta_k \leq h_{\nu_k} (f, \xi_k).
\]
Moreover, since $\xi \supset \mathcal{U}$ it follows that $|\xi| < 5\epsilon$ and thus
\[
\frac{h_{\mu} (f, \xi)}{\log 5\epsilon} \geq H_c \inf \dim_M (f, \alpha, d) - \gamma.
\]
Again, since each $\nu^k_i$ is ergodic, there exists $\ell_k \in \mathbb{N}$ large enough for which the set
\[
Y_i(k) = \left\{ x \in X : \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int_X \varphi \, d\nu^k \right\} < \delta_k \text{ and } n \geq \ell_k \right\}
\]
has $\nu^k$-measure bigger than $1 - 4\gamma$ for every $k \in \mathcal{N}$ and $i \in \{1, \ldots, j(k)\}$.

By [33, Lemma 3.6], there exists $\hat{n}_k \rightarrow \infty$ with $[\lambda_i \hat{n}_k] \geq \ell_k$ so that the maximal cardinality of an $([\lambda_i \hat{n}_k], 5\epsilon/4)$-separated set in $Y_i(k)$, denoted by $M_{k,i}$, satisfies
\[
M_{k,i} \geq \exp \left( [\lambda_i \hat{n}_k] \left( h_{\nu_i^k} (f, 5\epsilon/4, \gamma) - 4\gamma / j(k) \right) \right).
\]
Furthermore, the sequence $\hat{n}_k$ can be chosen such that $\hat{n}_k \geq 2^{m_k}$ where $m_k = m(\epsilon/2^{k+1})$ is as in the definition of the specification property. Let $n_k := m_k (j(k) - 1) + \sum_i [\lambda_i \hat{n}_k]$. Observe that $n_k / \hat{n}_k \rightarrow 1$.

Denote by $E_{1,k}(\lambda_i \hat{n}_k, 5\epsilon/4)$ a maximal $([\lambda_i \hat{n}_k], 5\epsilon/4)$-separated set in $Y_i(k)$. By the specification property, for each $x_1 \in E_{1,k}(n_1, 5\epsilon/4)$, $x_2 \in E_{2,k}(n_2, 5\epsilon/4)$, $\ldots$, $x_{j(k)} \in E_{j(k),k}(n_{j(k)}, 5\epsilon/4)$, there exists $y = y(x_1, \ldots, x_{j(k)}) \in X$ so that the pieces of orbits
\[
\{x_i, f(x_1), \ldots, f([\lambda_i \hat{n}_k]^{-1} x_i) : i = 1, \ldots, j(k)\}
\]
are $\epsilon/2^{k}$-shadowed by $y$ with gap $m_k$. We claim that if $(x_1, \ldots, x_{j(k)}) \neq (x'_1, \ldots, x'_{j(k)})$ then $y(x_1, \ldots, x_{j(k)}) \neq y(x'_1, \ldots, x'_{j(k)})$. Indeed, if $x_i \neq x'_i$,
\[
\frac{5\epsilon}{4} < d_{[\lambda_i \hat{n}_k]} (x_i, x'_i) \\
\leq d_{[\lambda_i \hat{n}_k]} (x_i, f([\lambda_i \hat{n}_k]+[\lambda_i \hat{n}_k]+(i-1)m_k) (y)) \\
+ d_{[\lambda_i \hat{n}_k]} (x'_i, f([\lambda_i \hat{n}_k]+[\lambda_i \hat{n}_k]+(i-1)m_k) (y')) \\
+ d_{[\lambda_i \hat{n}_k]} (f([\lambda_i \hat{n}_k]+[\lambda_i \hat{n}_k]+(i-1)m_k) (y), f([\lambda_i \hat{n}_k]+[\lambda_i \hat{n}_k]+(i-1)m_k) (y')) \\
< \frac{2\epsilon}{2^{k+2}} + d_{[\lambda_i \hat{n}_k]} (f([\lambda_i \hat{n}_k]+[\lambda_i \hat{n}_k]+(i-1)m_k) (y), f([\lambda_i \hat{n}_k]+[\lambda_i \hat{n}_k]+(i-1)m_k) (y'))
\]
which implies that $d_{n_k} (y, y') > 9\epsilon/8$ proving our claim. Moreover, as a by-product of this observation we get that
\[
S_k = \{ y(x_1, \ldots, x_{j(k)}) : x_i \in E_{i,k}([\lambda_i \hat{n}_k], \frac{5\epsilon}{4}), i = 1, \ldots, j(k) \}
\]
is a $(n_k, 9\epsilon/8)$-separated set with cardinality $M_k := \prod_{i=1}^{j(k)} M_{k,i}$. Combining (11), (12) and (14) with the choices of $\epsilon$, $\gamma$ and $n_k$ and recalling that $n_k / \hat{n}_k \rightarrow 1$ we get that for $k$ sufficiently large
\[
M_k = \prod_{i=1}^{j(k)} \# E_{i,k}([\lambda_i \hat{n}_k], 9\epsilon/8) \\
\geq \exp \left( \sum_{i=1}^{j(k)} [\lambda_i \hat{n}_k] \left( h_{\nu_i^k} (f, 5\epsilon/4, \gamma) - 4\gamma / j(k) \right) \right) \\
\geq \exp \left( \hat{n}_k \sum_{i=1}^{j(k)} \lambda_i h_{\nu_i^k} (f, 5\epsilon/4, \gamma) - 4\hat{n}_k \gamma \right) \\
\geq \exp \left( \hat{n}_k \sum_{i=1}^{j(k)} \lambda_i h_{\nu_i^k} (f, \xi_k, \gamma) - 4\hat{n}_k \gamma - \hat{n}_k \delta_k \right) \\
\geq \exp \left( \hat{n}_k (h_{\nu_k} (f, \xi_k, \gamma) - 4\gamma - \delta_k) \right) \\
\geq \exp \left( R_k n_k (h_{\nu_k} (f, \xi_k, \gamma) - 4\gamma - \delta_k) \right) \\
\geq \exp \left( R_k n_k h_{\nu_k} (f, \xi_k, \gamma) - 2\delta_k \right) \\
\geq \exp \left( R_k n_k h_{\nu_k} (f, \xi, \gamma) - 5\gamma \right)
\]
for some $R_k \in (0, 1)$. Let $y = y(x_1, \ldots, x_k) \in S_k$. Then,
\[
|S_{n_k} \varphi (y) - n_k \alpha| \\
\leq \left| S_{n_k} \varphi (x) - n_k \left( \int \varphi d\nu_k - \delta_k \right) \right| \\
\leq \sum_{i=1}^{j(k)-1} \left| S_{[\lambda_i \hat{n}_k]} \varphi (f \sum_{i=1}^{[\lambda_i \hat{n}_k]}(i-1)m_k) (y) - n_k \lambda_i \left( \int \varphi d\nu_k^i \right) + n_k \delta_k + m_k (j(k) - 1) \right| \| \varphi \| \\
\leq \sum_{i=1}^{j(k)-1} \left| S_{[\lambda_i \hat{n}_k]} \varphi (x_i) - [\lambda_i \hat{n}_k] \left( \int \varphi d\nu_k \right) + n_k \delta_k + m_k (j(k) - 1) \right| \| \varphi \| \\
\leq \delta_k \sum_{i=1}^{j(k)-1} [\lambda_i \hat{n}_k] + m_k j(k) \| \varphi \| + n_k \delta_k + n_k \Var (\varphi, \epsilon/2^k) \]
Thus, for sufficiently large $k$,
\[\left| \frac{1}{n_k} S_{n_k} \varphi(y) - \alpha \right| \leq \delta_k + \text{Var}(\varphi, \varepsilon/2^k) + \frac{1}{k}. \tag{16}\]

We now choose a sequence $\{N_k\}_{k \in \mathbb{N}}$ of positive integers such that $N_1 = 1$

1. $[R_k N_k] \geq 2^{k+1+m_k+1}$, for $k \geq 2$;
2. $[R_k+1 N_k+1] \geq 2^m [R_k N_k (n_k + m_k)]$, for $k \geq 1$.

Observe that this sequence $\{N_k\}_{k \in \mathbb{N}}$ grows very fast and
\[
\lim_{k \to \infty} \frac{n_k + m_k + 1}{R_k N_k} = 0.
\]
and
\[
\lim_{k \to \infty} \frac{R_1 N_1 n_1 + \cdots + R_k N_k (n_k + m_k)}{R_k N_k} = 0. \tag{17}
\]

Moreover, we enumerate the points in $S_k$ as
\[S_k = \{x_i^k : i = 1, \ldots, M_k\}.
\]

For any $(i_1, \ldots, i_{N_k}) \in \{1, 2, \ldots, M_k\}^{[R_k N_k]}$, let $y(i_1, \ldots, i_{[R_k N_k]}) \in X$ be given by the specification property so that its orbit $\varepsilon/2^k$-shadows, with gap $m_k$, the pieces of orbits $\{x_i^k, f(x_i^k), \ldots, f^{n_k-1}(x_i^k)\}$, $j = 1, 2, \ldots, [R_k N_k]$. Then, define
\[C_k = \{y(i_1, \ldots, i_{[R_k N_k]} : (i_1, \ldots, i_{[R_k N_k]}) \in \{1, 2, \ldots, M_k\}^{[R_k N_k]}\}.
\]

Moreover, consider
\[c_k = [R_k N_k] n_k + ([R_k N_k] - 1) m_k.
\]

We now observe that different sequences in $\{1, 2, \ldots, M_k\}^{[R_k N_k]}$ give rise to different points in $C_k$ and that such points are uniformly separated with respect to $d_{ck}$.

**Lemma 8 (Lemma 5.1 of [32]):** If $(i_1, \ldots, i_{[R_k N_k]}) \neq (j_1, \ldots, j_{[R_k N_k]})$, then
\[d_{ck}(y(i_1, \ldots, i_{[R_k N_k]}), y(j_1, \ldots, j_{[R_k N_k]})) > \varepsilon.
\]

In particular $\sharp C_k = M_k^{[R_k N_k]}$.

Our next step is to construct inductively an auxiliary sequence of finite sets $T_k$. Let $T_1 = C_1$ and $t_1 = c_1$. Now, suppose that we have already constructed the set $T_k$ and we will describe how to construct $T_{k+1}$. Consider
\[t_{k+1} = t_k + m_k + 1 + c_{k+1}\]
\[= [R_1 N_1] n_1 + [R_2 N_2] (n_2 + m_2) + \cdots + [R_k N_k] (n_k + m_k + 1). \tag{18}\]

For $x \in T_k$ and $y \in C_{k+1}$, let $z = z(x, y)$ be some point such that
\[d_{tk}(x, z) < \frac{\varepsilon}{2^{k+1}} \text{ and } d_{ck+1}(y, f^{t_k + m_k + 1}(z)) < \frac{\varepsilon}{2^{k+1}}. \tag{19}\]

Observe that the existence of such a point is guaranteed by the specification property of $f$. Then, let us consider
\[T_{k+1} = \{z(x, y) : x \in T_k, y \in C_{k+1}\}.
\]

By proceeding as in the proof of the Lemma 8 we can see that different pairs $(x, y)$, $x \in T_k$, $y \in C_{k+1}$, produce different points $z = z(x, y)$. In particular, $\sharp T_{k+1} = \sharp T_k \cdot \sharp C_{k+1}$. Therefore, proceeding inductively,
\[\sharp T_k = \sharp C_1 \cdots \sharp C_k = M_1^{[R_1 N_1]} \cdots M_k^{[R_k N_k]}.
\]

In particular, by Lemma 8 and (19) we have that for every $x \in T_k$ and $y, y' \in C_{k+1}$ with $y \neq y'$,
\[d_{tk}(x, y) < \frac{\varepsilon}{2^{k+1}} \]
\[\text{and}
\]
\[d_{tk+1}(z, y) > \frac{3\varepsilon}{4} \tag{20}\]

For every $k \in \mathbb{N}$ let us consider
\[F_k := \bigcup_{x \in T_k} B_{t_k(x, \varepsilon/2^{k+1})},
\]

where $B_{t_k(x, \varepsilon/2^{k+1})}$ denotes the closure of the open ball $B_{t_k(x, \varepsilon/2^{k+1})}$. As a simple consequence of (20) we have the following observation.

**Lemma 9 (Lemma 5.2 of [32]):** For every $k$ the following is satisfied:

1. for any $x, x' \in T_k$, $x \neq x'$, the sets $B_{t_k(x, \varepsilon/2^{k+1})}$ and $B_{t_k(x', \varepsilon/2^{k+1})}$ are disjoint;
2. if $z \in T_{k+1}$ is such that $z = z(x, y)$ for some $x \in T_k$ and $y \in C_{k+1}$, then
\[B_{t_{k+1}}(z, \varepsilon, \varepsilon/2^{k+1}) \subset B_{t_k}(x, \varepsilon, \varepsilon/2^{k+1}).
\]

Hence, $F_{k+1} \subseteq F_k$.

Consider
\[F := \bigcap_{k \in \mathbb{N}} F_k.
\]

Observe that, since each $F_k$ is a closed and non-empty set and, moreover, $F_{k+1} \subseteq F_k$, the set $F$ is a non-empty and closed set too. Furthermore, using (16) we may prove that

**Lemma 10 (Lemma 5.3 of [32]):** Under the above conditions,
\[F \subseteq K_{\alpha}.
\]

Now, for every $k \geq 1$, let us consider the probability measure $\eta_k$ given by
\[\eta_k = \frac{1}{\sharp T_k} \sum_{z \in T_k} \delta_z.
\]

Observe that, as $T_k \subseteq F_k$, $\eta_k(F_k) = 1$. Moreover,

**Lemma 11 (Lemma 5.4 of [32]):** The sequence of probability measures $(\eta_k)_{k \in \mathbb{N}}$ converges in the weak*-topology to some probability measure $\eta$. Furthermore, the limiting measure $\eta$ satisfies $\eta(F) = 1$.

An important feature of the measure $\eta$ that can be obtained by exploring its definition and (15) is that the $\eta$-measure of some appropriate dynamical balls decay exponentially fast. More precisely,

**Lemma 12 (Lemma 5.5 of [32]):** For every $n$ sufficiently large and $q \in X$ so that $B_n(q, \varepsilon/2) \cap F \neq \emptyset$ one has
\[\eta \left( B_n \left( q, \frac{\varepsilon}{2} \right) \right) \leq \exp \left( -n(h_{\mu}(f, \xi) - 8\gamma) \right).
\]
In order to conclude our proof we need a simple yet
interesting fact whose proof we include for the sake of
completeness. This is a version of the Entropy Distribution
Principle of [32] (see [32, Theorem 3.6]). Observe that for
this result, the measure involved does not need to be invariant, as it
is the case of the measure \( \eta \) obtained in the previous lemmas.

**Lemma 13**: Let \( f : X \to X \) be a continuous transformation
and \( \varepsilon > 0 \). Given a set \( Z \subset X \) and a constant \( s > 0 \), suppose
there exist a constant \( C > 0 \) and a Borel probability measure \( \eta \)
satisfying:

(i) \( \eta(Z) > 0 \);
(ii) \( \eta(B_n(x, \varepsilon)) \leq C e^{-n s} \) for every ball \( B_n(x, \varepsilon) \) such that
\( B_n(x, \varepsilon) \cap Z \neq \emptyset \).

Then \( h(Z, f, \varepsilon) \geq s \).

**Proof of Lemma 13**: Let \( \Gamma = \{ B_n(x_i, \varepsilon) \}_{i} \) be some
cover of \( Z \). Without loss of generality we may assume that
\( B_n(x_i, \varepsilon) \cap Z \neq \emptyset \) for every \( i \). In such case we have that
\[
\sum_i \exp(-s \eta_i) \geq C^{-1} \sum_i \eta(B_n(x, \varepsilon)) \\
\geq C^{-1} \eta \left( \bigcup_i B_n(x, \varepsilon) \right) \\
\geq C^{-1} \eta(Z) > 0.
\]

Therefore, \( m(Z, s, \varepsilon) > 0 \) and hence \( h(Z, f, \varepsilon) \geq s \).

By Lemma 10 we have that \( h(K_\alpha, f, \varepsilon/2) \geq h(F, f, \varepsilon/2) \).

and Lemmas 12 and 13 gives us that \( h(F, f, \varepsilon/2) \geq \mu(F, \xi) - 8 \gamma \).

Consequently,
\[
h(K_\alpha, f, \varepsilon/2) \geq \mu(F, \xi) - 8 \gamma.
\]

Thus, combining this observation with (10) and (13) we get that
\[
\Lambda_{\varphi} \text{indim}_M(f, \alpha, d) - 9 \gamma \leq \frac{\mu(F, \xi) - 8 \gamma}{\log \varepsilon} \\
\leq \frac{h(K_\alpha, f, \varepsilon/2)}{\log \varepsilon/2} + \log 10 \\
\leq \text{indim}_M(K_\alpha, f, d) + \gamma.
\]

Thus, since \( \gamma > 0 \) is arbitrary, the proof of the proposition is
complete. \( \square \)

**Proposition 14**: Under the hypotheses of Theorem A we have that
\[
\Lambda_{\varphi} \text{indim}_M(f, \alpha, d) \geq \Lambda_{\varphi} \text{indim}_M(f, \alpha, d).
\]

**Proof**: Fix \( \gamma > 0 \). Let \( \{\varepsilon_j\}_j \in \mathbb{N} \) be a sequence of positive
numbers which converges to zero and satisfies
\[
\Lambda_{\varphi} \text{indim}_M(f, \alpha, d) = \lim_{j \to \infty} \frac{\Lambda_{\varphi}(\alpha, \varepsilon_j)}{\log \varepsilon_j}.
\]

Then, there exists \( \varepsilon_0 > 0 \) so that for all \( \varepsilon_j \in (0, \varepsilon_0] \) we have
\[
\Lambda_{\varphi}(\alpha, \varepsilon_j) > \Lambda_{\varphi} \text{indim}_M(f, \alpha, d) - \frac{1}{3} \gamma.
\]

In particular, for every \( \varepsilon_j \in (0, \varepsilon_0] \),
\[
\Lambda_{\varphi}(\alpha, \varepsilon_j) > \left( \Lambda_{\varphi} \text{indim}_M(f, \alpha, d) - \frac{1}{3} \gamma \right) \cdot \log \varepsilon_j.
\]

Thus,
\[
\left| \int_X \varphi \, d\mu^{(j)} - \alpha \right| \leq \frac{1}{M_{j, k}} \sum_{x \in C_{j, k}} \left| \int_{n_{j, k}^{-1}} \varphi(f^i(x)) \right| - \alpha \\
\leq \delta_{j, k}.
\]

Fix \( j \in \mathbb{N} \) such that \( \varepsilon_j \in (0, \varepsilon_0] \). By the alternative
expression of \( \Lambda_{\varphi}(\alpha, \varepsilon_j) \) given in (4) it follows that there exists
a sequence of positive numbers \( (\delta_{j, k})_{k \in \mathbb{N}} \) converging to zero and such that
for every \( k \in \mathbb{N} \),
\[
\liminf_{n \to \infty} \frac{1}{n} \log M(\alpha, \delta_{j, k}, n, \varepsilon_j) \\
> \Lambda_{\varphi}(\alpha, \varepsilon_j) - \frac{2}{3} \gamma \\
> \left( \Lambda_{\varphi} \text{indim}_M(f, \alpha, d) - \frac{1}{3} \gamma \right) \cdot \log \varepsilon_j - \frac{2}{3} \gamma.
\]

Similarly, there exists a sequence \( (n_{j, k})_{k \in \mathbb{N}} \) in \( \mathbb{N} \) satisfying
\( \lim_{k \to \infty} n_{j, k} = +\infty \) and
\[
M_{j, k} := M(\alpha, \delta_{j, k}, n_{j, k}, \varepsilon_j) \\
> \exp \left( n_{j, k} \left( \left( \Lambda_{\varphi} \text{indim}_M(f, \alpha, d) - \frac{1}{3} \gamma \right) \cdot \log \varepsilon_j - \gamma \right) \right).
\]

(21)

Consider a maximal \( (n_{j, k}, \varepsilon_j) \)-separated set \( C_{j, k} \) of
\( P(\alpha, \delta_{j, k}, n_{j, k}) \). For each \( j, k \in \mathbb{N} \) consider
\[
\alpha_k^{(j)} = \frac{1}{M_{j, k}} \sum_{x \in C_{j, k}} \delta_x,
\]
\[
\mu_k^{(j)} = \frac{1}{n_{j, k}} \sum_{i=0}^{n_{j, k}-1} \left( f^i \right)_*(\delta_x).
\]

It is not difficult to see that any accumulation point of
\( \{\mu_k^{(j)}\}_{k \in \mathbb{N}} \), say \( \mu^{(j)} \), is \( f \)-invariant (see [38, Theorem 6.9]).
Moreover, \( \int_X \varphi \, d\mu^{(j)} = \alpha \) for every \( j \in \mathbb{N} \). Indeed, we may
assume without loss of generality that \( \lim_{k \to \infty} \mu_k^{(j)} = \mu^{(j)} \).
Then, for every \( j \) and \( k \in \mathbb{N} \) we have
\[
\left| \int_X \varphi \, d\mu_k^{(j)} - \alpha \right| \\
\leq \frac{1}{M_{j, k}} \sum_{x \in C_{j, k}} \left| \int_{n_{j, k}^{-1}} \varphi(f^i(x)) - \alpha \right| - \alpha \\
\leq \delta_{j, k},
\]

Thus,
\[
\left| \int_X \varphi \, d\mu^{(j)} - \alpha \right| \\
\leq \int_X \varphi \, d\mu^{(j)} - \int_X \varphi \, d\mu_k^{(j)} \\
+ \int_X \varphi \, d\mu_k^{(j)} - \alpha \\
\leq \int_X \varphi \, d\mu^{(j)} - \int_X \varphi \, d\mu_k^{(j)} + \delta_{j, k}.
\]

Consequently, making \( k \to +\infty \) we conclude that
\( \int_X \varphi \, d\mu^{(j)} = \alpha \) for every \( j \in \mathbb{N} \) as claimed.

For every \( j \in \mathbb{N} \) choose a Borel partition \( \xi(j) = \{A_1, \ldots, A_t\} \) of \( X \) so that \( \text{diam}(\xi(j)) < \varepsilon_j \) and
\( \mu^{(j)}(\partial A_i) = 0 \) for \( 0 \leq i \leq \ell \) (see [38, Lemma 8.5(ii)]). Then,

\[
H_{\sigma_k^{(j)}} \left( \sum_{i=0}^{n_{j,k}^{-1}} f^{-i} \xi(j) \right) = \log M(\alpha, \delta_{j,k}, n_{j,k}, \epsilon_j).
\]

Indeed, observe that if \( x \) and \( y \) belong to the same element of \( \bigvee_{i=0}^{n_{j,k}^{-1}} f^{-i} \xi(j) \) then \( d_{\sigma_k^{(j)}}(x, y) < \epsilon_j \). In particular, no element of \( \bigvee_{i=0}^{n_{j,k}^{-1}} f^{-i} \xi(j) \) can contain more than one point of a maximal \((n_{j,k}, \epsilon_j)\)-separated set. Thus, exactly \( M(\alpha, \delta_{j,k}, n_{j,k}, \epsilon_j) \) elements of \( \bigvee_{i=0}^{n_{j,k}^{-1}} f^{-i} \xi(j) \) have \( \sigma_k^{(j)} \)-measure equal to \( \frac{1}{M(\alpha, \delta_{j,k}, n_{j,k}, \epsilon_j)} \). All others have zero \( \sigma_k^{(j)} \)-measure.

Fix natural numbers \( q \) and \( n_{j,k} \) with \( 1 < q < n_{j,k} \) and define, for \( 0 \leq s \leq q-1 \), \( a(s) = [(n_{j,k} - s)/q] \) where \([p]\) denotes the integer part of \( p \). Fix \( 0 \leq s \leq q-1 \). Then, by [38, Remark 2(ii), p. 188] we have that

\[
\sum_{i=0}^{n_{j,k}^{-1}} f^{-i} \xi(j) = \sum_{r=0}^{a(s)-1} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right)
\]

where \( L \) is a set with cardinality at most \( 2q \). Therefore, using [38, Theorem 4.3(viii)] and [38, Corollary 4.2.1],

\[
\log M(\alpha, \delta_{j,k}, n_{j,k}, \epsilon_j) \leq H_{\sigma_k^{(j)}} \left( \sum_{i=0}^{n_{j,k}^{-1}} f^{-i} \xi(j) \right)
\]

\[
\leq \sum_{i=0}^{a(s)-1} H_{\sigma_k^{(j)}} f^{-q+s} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right) + \sum_{t \in L} H_{\sigma_k^{(j)}} f^{-t} \xi(j)
\]

\[
\leq \sum_{i=0}^{a(s)-1} H_{\sigma_k^{(j)}} f^{-q+s} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right) + 2q \log(\ell).
\]

Summing the previous inequality over \( s \) from 0 to \( q-1 \) and using [38, Remark 2(iii), p. 188], we get that

\[
q \log M(\alpha, \delta_{j,k}, n_{j,k}, \epsilon_j) \leq \sum_{p=\infty}^{n_{j,k}^{-1}} H_{\sigma_k^{(j)}} f^{-p} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right) + 2q \log(\ell).
\]

Thus, dividing everything by \( n_{j,k} \) in the above inequality and using (21) and the concavity of the map \( M \rightarrow H_\mu(\xi) \) we obtain

\[
q \left( \frac{\Lambda_{\mu} \text{mdim}_M(f, \alpha, d)}{\text{mdim}_M(f, \alpha, d)} \right) \cdot \left| \log \epsilon_j \right| - \gamma
\]

\[
q \left( \frac{\Lambda_{\mu} \text{mdim}_M(f, \alpha, d)}{\text{mdim}_M(f, \alpha, d)} \right) \geq \frac{q}{n_{j,k} \log M(\alpha, \delta_{j,k}, n_{j,k}, \epsilon_j)}
\]

\[
\leq \frac{H_{\mu_k^{(j)}} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right) + 2q \log(\ell)}{n_{j,k}}.
\]

Now, since the elements of \( \bigvee_{i=0}^{n_{j,k}^{-1}} f^{-i} \xi(j) \) have boundaries of \( \mu^{(j)} \)-measure zero, it follows from the weak convergence of the measures \( \mu_k^{(j)} \) to \( \mu^{(j)} \) that \( \lim_{k \to \infty} H_{\mu_k^{(j)}} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right) = H_{\mu^{(j)}} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right) \). Thus, by (22) we have that

\[
\leq \frac{H_{\mu_k^{(j)}} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right)}{n_{j,k}}.
\]

Dividing both sides of the previous inequality by \( q \) and letting \( q \) go to \( +\infty \) we obtain

\[
\left( \Lambda_{\mu} \text{mdim}_M(f, \alpha, d) \right) \cdot \left| \log \epsilon_j \right| - \frac{2\gamma}{3} \leq \frac{H_{\mu} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right)}{n_{j,k}}.
\]

for all \( j \in \mathbb{N} \), which implies that

\[
\Lambda_{\mu} \text{mdim}_M(f, \alpha, d) \leq \frac{H_{\mu} \left( \sum_{i=0}^{q^{-1} r-q+s} \sum_{t \in L} f^{-t} \xi(j) \right)}{n_{j,k}}.
\]

for all \( j \in \mathbb{N} \). Therefore,\n
\[
\Lambda_{\mu} \text{mdim}_M(f, \alpha, d) \leq \inf_{\epsilon \leq \epsilon_j} \frac{h_{\mu}(f, \xi) + \gamma}{\log \epsilon_j}
\]

for all \( j \in \mathbb{N} \) and consequently,

\[
\Lambda_{\mu} \text{mdim}_M(f, \alpha, d) \leq \lim_{q \to +\infty} \sup_{D \in M_f(X, \alpha, d)} \inf_{\epsilon \leq \epsilon_j} \frac{h_{\mu}(f, \xi)}{\log \epsilon_j}
\]

completing the proof of the proposition. \( \square \)

Finally, the first claim of Theorem A follows directly by combining Propositions 5, 6 and 14. For the general case, that is, without the assumption that the quantities \( \text{mdim}_M(K_{\alpha}, f, d) \). \( \Lambda_{\mu} \text{mdim}_M(f, \alpha, d) \) and \( \text{mdim}_M(f, \alpha, d) \) are finite, we observe that a simple modification of our proof show us that if one of the quantities is infinite then the other two must also be infinite and, therefore, the first claim of Theorem A is still true. As for the second claim in Theorem A, again one can easily see that simple adaptations of the previous proof yield the desired conclusion. The proof of Theorem A is complete.

IV. EXAMPLES

In this section we present some examples of settings with positive upper/lower metric mean dimension where our results may be applied. Moreover, we also present a simple application of Theorem A to calculate \( \text{mdim}_M(K_{\alpha}, f, d) \).

Example 15: Let \((Z, D)\) be a compact metric space with upper box-counting dimension \( \text{dim}_B Z < \infty \). Let us consider \( X = Z^{[n]} \) endowed with the metric

\[
d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} \frac{1}{2^n} D(x_n, y_n)
\]

and let \( \sigma : X \to X \) be the shift map. It is well known that \( \sigma \) has the specification property and \( \text{mdim}_M(X, \sigma, d) = \text{dim}_B Z \) and \( \text{mdim}_M(X, \sigma, d) = \text{dim}_B Z \) (see for
instance [28]). In particular, we may apply Theorem A to it getting, for instance, that for any \( \varphi \in C^0(X, \mathbb{R}) \) and \( \alpha \in \mathbb{R} \),

\[
\text{mdim}_M \left( K_\alpha, \sigma, d \right) = \Lambda_\varphi \text{mdim}_M (\sigma, \alpha, d) = \text{mdim}_M (\sigma, \alpha, d).
\]

**Example 16:** Let \( X = [0, 1]^\mathbb{N} \) be endowed with the metric induced by the Euclidean distance in \([0, 1]\) as in the previous example and consider the set

\[
E = \left\{ \left( x^{(i,j)} \right)_{i,j \in \mathbb{N}} \in X : x_n^{(i,j)} = \frac{1}{2^i} \text{ if } i = n \text{ and } x_n^{(i,j)} = 0 \text{ if } i \neq n \right\} \cup \{ e \},
\]

where \( e = (0, 0, \ldots) \), which is closed and shift invariant. If \( 2^E \) denotes the space of subsets of \( X \) endowed with the Hausdorff distance \( d_H \), by [20, Proposition 3.6] we have that

\[
\text{mdim}_M \left( E, \sigma, d \right) = 0 \text{ and } \text{mdim}_M \left( 2^E, \sigma_x, d_H \right) = 1,
\]

where \( \sigma_x \) is the induced map by \( \sigma \) on the hyperspace \( 2^E \). By [6, Proposition 4] we have that \( \sigma_x \) has the specification property and then Theorem A may be applied.

**Example 17:** It was proved in [1] and [9] that for \( C^0 \)-generic homeomorphisms acting on a compact and smooth manifold \( X \) with dimension greater than one, the upper metric dimension with respect to the smooth metric coincides with the dimension of the manifold. Moreover, they also proved that the set of homeomorphisms with positive lower metric mean dimension is \( C^0 \) dense in the set of homeomorphisms of \( X \). Now, in order to be able to apply Theorem A to elements of those sets, we need to guarantee that they have the specification property. For this purpose we restrict ourselves to the set of conservative homeomorphisms, where the specification property holds \( C^0 \)-generically.

We fix a good Borel probability measure \( \mu \in \mathcal{M}(X) \), i.e., a probability measure that satisfies the following conditions:

\[
\begin{align*}
(1) & \quad \text{[Non-atomic]} \text{ For every } x \in X \text{ one has } \mu(\{x\}) = 0; \\
(2) & \quad \text{[Full support]} \text{ For every nonempty open set } U \subset X \text{ one has } \mu(U) > 0; \\
(3) & \quad \text{[Boundary with zero measure]} \mu(\partial X) = 0.
\end{align*}
\]

In a forthcoming paper by S. Romáñ and G. Lacorda it is proved that there exists a Baire generic subset of \( \text{Homeo}_\mu(X, d) \) (the set of conservative homeomorphisms on \( X \)) with metric mean dimension equal to the dimension of \( X \). Consequently, since according to [14] the specification property is a Baire generic property in \( \text{Homeo}_\mu(X, d) \), there exists a \( C^0 \)-open and dense subset of \( \text{Homeo}_\mu(X, d) \) whose elements have positive upper metric mean dimension and the specification property and, in particular, Theorem A may be applied to those elements.

In the next two examples we consider the specification property for linear operators acting on Banach spaces and we start by recalling the appropriate definition for this setting. Let \( B \) be a Banach space over \( \mathbb{K} (= \mathbb{R} \text{ or } \mathbb{C}) \) and \( T : B \to B \) be a linear operator. We say that \( T \) has the **operator specification property** if there exists a sequence of \( T \)-invariant sets \( \{ K_m \}_{m \in \mathbb{N}} \) with \( B = \bigcup_{m \in \mathbb{N}} K_m \) for which \( T|_{K_m} : K_m \to K_m \) satisfies the specification property. We emphasize that the sets \( K_m \) do not need be compact, although in all known examples we have compactness for such sets.

**Example 18:** Fix \( \nu = (\nu_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \) so that \( \nu_n > 0 \) for all \( n \in \mathbb{N} \) and \( \sum_{n=1}^{\infty} \nu_n < \infty \). Given \( 1 \leq p < +\infty \), consider

\[
\ell^p(\nu) = \left\{ (x_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}} : \| (x_n)_{n \in \mathbb{N}} \|_{\ell^p(\nu)} < \infty \right\},
\]

where \( \| (x_n)_{n \in \mathbb{N}} \|_{\ell^p(\nu)} := \left( \sum_{n=1}^{\infty} |x_n|^p \nu_n^p \right)^{\frac{1}{p}} \), which is a Banach space, and the shift map \( \sigma : \ell^p(\nu) \to \ell^p(\nu) \). By [5, Theorem 2.1] we have that \( \sigma : \ell^p(\nu) \to \ell^p(\nu) \) has the operator specification property with \( K_m = mK, m \in \mathbb{N} \), where \( K \) is the compact set \( K = \{ (x_n)_{n \in \mathbb{N}} \in \ell^p(\nu) : \| x_n \| \leq 1 \text{ for all } n \in \mathbb{N} \} \). We now observe that \( T|_K : K \to K \) has positive metric mean dimension. More precisely,

**Lemma 19:**

\[
\text{mdim}_M \left( K, \sigma, \| \cdot \|_{\ell^p(\nu)} \right) = \text{mdim}_M \left( K, \sigma, \| \cdot \|_{\ell^p(\nu)} \right) = 1.
\]

**Proof:** Given \( \varepsilon > 0 \) and \( n \in \mathbb{N} \), we observe that

\[
\left\{ x \in K : x_{1} \in \left\{ 0, \varepsilon \sqrt{1/p_1}, 2\varepsilon \sqrt{1/p_1}, \ldots, \left\lfloor \frac{1}{\varepsilon \sqrt{1/p_1}} \right\rfloor \varepsilon \sqrt{1/p_1} \right\} \right. \\
\left. \quad \text{for all } 1 \leq i \leq n \right\}
\]

is a \((n, \varepsilon)\)-separated set in \( K \). In particular,

\[
\text{mdim}_M \left( K, \sigma, \| \cdot \|_{\ell^p(\nu)} \right) = \lim_{\varepsilon \to 0} \frac{h(\sigma, \varepsilon)}{|\log \varepsilon|} \geq \lim_{\varepsilon \to 0} \frac{\text{lim sup}_{n \to \infty} \frac{1}{n} \log \left( \left\lfloor \frac{1}{\varepsilon \sqrt{1/p_1}} \right\rfloor \right)}{|\log \varepsilon|} = 1.
\]

In order to get the reverse inequality, let \( \ell \in \mathbb{N} \) be so that

\[
\sum_{n \geq \ell} \nu_n < \frac{\varepsilon}{2}
\]

and define \( M = \left( \sum_{k \in \mathbb{N}} \nu_k \right)^{1/p} > 0 \). We consider an open cover of \([-1, 1] \) by

\[
I_k = \left[ \frac{(k - 1)\varepsilon}{2M}, \frac{(k + 1)\varepsilon}{2M} \right],
\]

for \( k \in \mathbb{N} \). Note that each \( I_k \) has length \( \frac{\varepsilon}{2M} \). Given \( n \geq 1 \), let us consider the following open cover of \( K \):

\[
\{ x : x_{1} \in I_{k_1}, x_{2} \in I_{k_2}, \ldots, x_{n+\ell} \in I_{k_{n+\ell}} \},
\]

where \( |2M/\varepsilon| \leq k_1, \ldots, k_{n+\ell} \leq |2M/\varepsilon| \). Observe that each element of this open cover has diameter less than \( \varepsilon \) with respect to the metric \( d_n \) (induced by \( \| \cdot \|_{\ell^p(\nu)} \)). So,

\[
\lim_{\varepsilon \to 0} \frac{h(\sigma, \varepsilon)}{|\log \varepsilon|} \leq \lim_{\varepsilon \to 0} \frac{\text{lim sup}_{n \to \infty} \frac{1}{n} \log \left( 2 \cdot |12M/\varepsilon| \right)^{n+\ell+1}}{|\log \varepsilon|} = 1.
\]
Hence,
\[ \overline{\text{mdim}}_{\overline{M}} \left( K, \sigma, \| \cdot \|_{\ell_p(\nu)} \right) \leq 1. \]  
(24)

Finally, combining (23) and (24) we get the desired result. \( \square \)

As a consequence of the previous proof we also get that
\[ \overline{\text{mdim}}_{\overline{M}} \left( K_m, \sigma, \| \cdot \|_{\ell_p(\nu)} \right) = 1 \]
for all \( m \in \mathbb{N} \). In particular, we may apply Theorem A to \( \sigma[K_m : K_m \rightarrow K_m] \) for every \( m \in \mathbb{N} \).

**Example 20:** Another class of examples is given by the weighted shift \( \ell^n \). We let \( \nu = (\nu_n)_{n \in \mathbb{N}} \) be a weight sequence and define the weighted shift on \( \ell^n \) as \( B_w((x_n)_{n \in \mathbb{N}}) = (w_{n+1}x_{n+1})_{n \in \mathbb{N}} \).

It was observed in [5, p. 602] that if one considers \( a = (a_n)_{n \in \mathbb{N}} \) given by
\[ a_1 = 1 \text{ and } a_n := w_2 \ldots w_n, \text{ for all } n > 1, \]
and \( \bar{\nu} = (\bar{\nu}_n)_{n \in \mathbb{N}} \) given by
\[ \bar{\nu}_n = \frac{1}{\prod_{j=2}^{n} |w_j|^p}, \text{ for all } n \in \mathbb{N}, \]
then
\[ \phi_\nu^{-1}(x)_{n \in \mathbb{N}} \in \ell^n \iff \phi_\bar{\nu}(x)_{n \in \mathbb{N}} = (a_1x_1, a_2x_2, \ldots) \in \ell^n(\bar{\nu}) \]
defines a topological conjugacy between the weighted shift and the backward shift given in the previous example. Moreover, they observed that this topological conjugacy is also an isometry, which implies that
\[ \overline{\text{mdim}}_{\overline{M}} \left( \phi_\nu^{-1}(K_m), B_w, \| \cdot \|_{\ell_p} \right) = \overline{\text{mdim}}_{\overline{M}} \left( \phi_\bar{\nu}^{-1}(K_m), B_w, \| \cdot \|_{\ell_p} \right) = 1 \]
for all \( m \in \mathbb{N} \). Furthermore, if \( \sum_{n=1}^{\infty} \bar{\nu}_n < \infty \) we have that
\( B_w \) has the operator specification property (see [5, Theorem 2.3]) and then we are in the context of Theorem A.

**Example 21:** Let us consider \( X = [0, 1]^{\mathbb{N}} \) endowed with the metric
\[ d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} |x_n - y_n| \]
and let \( \sigma : X \rightarrow X \) be the left shift map. Similarly to Example 15, \( \sigma \) has the specification property and, moreover, \( \overline{\text{mdim}}_{\overline{M}} (X, \sigma, d) = 1 \). In particular, Theorem A may be applied in this setting. Let \( \lambda \) be the Lebesgue measure on \([0, 1]\) and consider \( \mu = \lambda^2 \). Then, it is a well known fact that \( \mu \) is ergodic. Given \( \varphi \in C^0(X, \mathbb{R}) \), take \( \alpha = \int \varphi d\mu \). We will show that \( \overline{\text{mdim}}_{\overline{M}} (K, \sigma, d) = 1 \). In order to do it we recall the definition of Brin-Katok local entropy for an ergodic measure \( \mu \in \mathcal{M}_\sigma(X), \varepsilon > 0 \) and a point \( x \in X \), we consider
\[ h^{BK}_\mu (\varepsilon, x) = \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)), \]
where \( B_n(x, \varepsilon) \) is defined as in Section II. Since \( \mu \) is ergodic, the map \( x \mapsto h^{BK}_\mu (\varepsilon, x) \) is constant \( \mu \)-almost everywhere. Denote this constant by \( h^{BK}_\mu (\varepsilon) \). Then, we have the following observation.

**Lemma 22** (See [17]): For any ergodic measures \( \mu \in \mathcal{M}_\sigma(X) \) and any \( \varepsilon > 0 \),
\[ h^{BK}_\mu (\varepsilon, x) = \inf_{|\xi| < \varepsilon} h_\mu(\sigma, \xi), \]
where the infimum is taken over all finite measurable partitions of \( X \) with diameter smaller than \( \varepsilon \).

Therefore, considering the measure \( \mu = \lambda^2 \) given above and using Theorem A we get that
\[ \overline{\text{mdim}}_{\overline{M}} \left( K, \sigma, d = \text{H}_{\mu} \overline{\text{mdim}}_{\overline{M}} (f, \alpha, d) \geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \left( \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} h^{BK}_\mu (\varepsilon) \right). \]

Now, in [29, Example 12] it is proved that
\[ 1 = \overline{\text{mdim}}_{\overline{M}} (X, \sigma, d) = \overline{\text{mdim}}_{\overline{M}} (X, \sigma, d) \geq \overline{\text{mdim}}_{\overline{M}} (K, \sigma, d) \geq 1 \]
and thus, \( \overline{\text{mdim}}_{\overline{M}} (K, \sigma, d) = \overline{\text{mdim}}_{\overline{M}} (K, \sigma, d) = \overline{\text{mdim}}_{\overline{M}} (K, \sigma, d) = 1 \) as claimed. We observe that our Theorem A combined with Lemma 22 may be very useful for giving lower bounds for \( \overline{\text{mdim}}_{\overline{M}} \left( K, \sigma, d \right) \) and \( \overline{\text{mdim}}_{\overline{M}} \left( K, \sigma, d \right) \). In fact, it is enough to take an ergodic measure \( \mu \) satisfying \( \alpha = \int \varphi d\mu \) and estimate \( h^{BK}_\mu (\varepsilon) \) which, in general, may be easier than estimating \( \inf_{|\xi| < \varepsilon} h_\mu(\sigma, \xi) \).

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