Conformal Curves on WO₃ Surface

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We have studied the iso-height lines on the WO₃ surface as a physical candidate for conformally invariant curves. We have shown that these lines are conformally invariant with the same statistics of domain walls in the critical Ising model. They belong to the family of conformal invariant curves called Schramm-Loewner evolution (or SLEκ), with diffusivity of κ ∼ 3. This can be regarded as the first experimental observation of SLE curves. We have also argued that Ballistic Deposition (BD) can serve as a growth model giving rise to contours with similar statistics at large scales.

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The study of random surfaces, especially their statistical properties as well as growth and evolution dynamics, has been growing over the last two decades. They describe many important problems of real surfaces appearing in condensed matter physics such as deposited metal. In addition, this kind of problems is closely related to other problems in physics such as frustrated surfaces, string theory, phase transitions in two dimensions etc [1, 2]. The aim of this paper is to show that conformally invariant curves appear on the iso-height contours of deposited films of WO₃. This is the first observation of SLE in real physical system. We believe that the same result may be observed for other surfaces.

Metal oxides are a large family of materials possessing various interesting properties. One of the most interesting metal oxides is tungsten oxide, WO₃, which has been investigated extensively because of its distinctive applications such as electrochromic [3, 4, 5, 6, 7], photochromic [8], gas sensor [9, 10, 11], photo-catalyst [12], and photo-luminescence properties [13]. Many properties of tungsten oxide are related to its surface structure (e.g. porosity, surface-to-volume ratio) and surface morphology and statistics such as grain size and height distribution of the sample. These properties are also affected by conditions during the growth process such as deposition method. One can change the statistics of the growth surface by imposing external parameters such as annealing temperature which can even cause phase transition in the sample [14].

On the other hand, calculation of various geometrical exponents of a random Gaussian surface is of interest to theoretical physics. From this point of view, we want to study the morphology and geometrical statistics of experimental grown surfaces of WO₃. We consider contour lines (the nonintersecting iso-height lines) on the WO₃ samples and show that they are conformally invariant. This analysis when applied to the contour lines of the ballistic disposition (BD) model shows similar conformally invariant surface in large scales. The scaling behavior of contour lines in the growth models are widely investigated. For example in [15] the authors derived some of the universal exponents of contour lines, especially their fractal dimension D₀ and its relation to the roughness exponent α. Moreover, Schramm and Sheffield recently showed that the contour lines in a two-dimensional discrete Gaussian free field are conformally invariant and belong to a wide class of conformally invariant curves called Schramm-Loewner evolution (SLEκ) curves, whose diffusivity κ converges to 4 [16] - every conformal curve in two dimensional plane can be studied by SLE [17] (to review about SLE see [18]). Moreover, it has been proved by Smirnov that some domain walls in the critical Ising model, in the scaling limit, can be described by SLE [19].

In addition some physical systems have been studied recently using some numerical methods. It has been shown that the statistics of zero-vorticity lines in the inverse cascade of 2D Navier-Stokes turbulence displays conformal invariance and can be described by SLEκ which is in the universality class of percolation [20]. Similar studies have been done for inverse cascade of surface quasigeostrophic turbulence [21], domain walls of spin glasses [22] and the nodal lines of random wave functions [23].

To study the geometrical statistics of the surface of WO₃, 34 samples were independently deposited on glass microscope slides with the area 2.5cm × 2.5cm using a thermal evaporation method in same conditions. The
deposition system was evacuated to a base pressure of \( \sim 4 \times 10^{-3} \) Pa. The thickness of the deposited films was considered to be about 200 nm, measured by the stylus and optical techniques. Using the atomic force microscopy (AFM) techniques we have obtained the height profiles \( h(x) \) of the rough surfaces with the resolution of 1/256 \( \mu m \) in the scale of 1 \( \mu m \times 1 \mu m \). The AFM scans in this scale were performed in the various non-overlapping domains (10 profiles for each sample) from the centric region of the deposited samples (Fig. 1)–to ensure that the boundary effects are negligible for the AFM profiles. First we have determined the clusters by considering the connected domains with like-sign heights (mean height is set to zero for a range of heights around the mean value much less than \( h_{\text{rms}} \) (Fig. 2), then the cluster boundaries (contour lines) have been specified (Fig. 2). We have obtained an ensemble of such contour lines using MATLAB algorithm for contour plot. The calculation of the fractal dimension of these contour lines yields \( D_0 = 1.38 \pm 0.02 \) using the box-counting method. The roughness exponent for the samples is calculated by using the second-order structure function analysis, which gives \( \alpha = 0.33 \pm 0.03 \).

In order to investigate the scale dependency of the contours we have analyzed their multifractal behavior. The generalized fractal dimensions are defined by \( D_q = \frac{1}{q-1} \lim_{a \to 0} \frac{\log Z_q(a)}{\log a} \) where \( Z_q(a) = \sum \rho_i(a)^q \), and the sum runs over all the non-overlapping boxes with size \( a \) which cover the curve. \( \rho_i(a) \) is the mass of the curve in the \( i \)th box. The multifractal spectrum of the contours is independent of \( q \) with the value of 1.38 \( \pm 0.03 \). It shows that the curves are scale invariant. To investigate more the scale dependency of contour lines, one can check some of the exponents associated with clusters and loops. Fig. 4a shows the scaling of the average area of clusters \( M \) with their radius of gyration \( R \), \( M \sim R^D \) which yields the fractal dimension of clusters \( D = 2 - (8 - \kappa)(3\kappa - 8)/32\kappa \) [24]. Fractal dimension of loops relates their length \( s \) to radius of gyration \( R \) as \( s \sim R^{D_0} \). In our case, it is in good agreement with both the fractal dimension obtained using the box-counting method and that of domain walls of critical Ising model (Fig. 4b). We have also checked the other scaling relations, the average number of loops with radius \( R \), \( n(R) \sim R^{-2+\alpha} \) (Fig. 4c) and that with perimeter \( s \), \( n(s) \sim s^{(-2+\alpha)/D_0} \) where \( \alpha \) is the roughness exponent (Fig. 4d). Another is the scaling relation for the number of loops with the area greater than \( A \), \( n(A) \sim A^{-1+\alpha/2} \).

Consistency of the results for the contour lines of \( WO_3 \) samples and Ising cluster boundaries suggests that a contour line of these two belongs to the same universality class. However assumption of translational and rotational symmetry within contour ensembles has been made, which seems right. Therefore the iso-height contours of \( WO_3 \) may be conformally invariant and belong to the \( SLE_\kappa \) curves with \( \kappa = 3 \). The fractal dimension \( D_0 = 1 + \kappa/8 \) is also consistent with our finding provided \( \kappa = 3 \).

To examine this hypothesis directly, we first consider an arbitrary placed horizontal line representing the real axis in the complex plane across the AFM profiles. Then we cut the portion of each curve \( \gamma_t \) above the real line as it is in the upper half plane \( \mathbb{H} \).

Now let us to consider the chordal SLE in the \( \mathbb{H} \) from 0 to \( \infty \) which is described by the Loewner equation \( \partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t} \). If we parameterize the contour lines in the \( \mathbb{H} \) with the dimensionless parameter \( t \), \( g_t(z) \) maps the upper half plane except the curve up to time \( t \), to \( \mathbb{H} \) itself. For any self avoiding curve \( \gamma_t \) the driving function \( \xi_t \) is a continuous real function with the correspondence of \( g_t(\gamma_t(z)) = \xi_t \). In the other words, \( g_t(z) \) in each time, maps the tip of the curve to a real point \( \xi_t \). Random curves whose driving functions are proportional to the Brownian walk \( B_t \) are conformally invariant.

So, to investigate the conformal invariancy of the contour lines we should explicitly show that their driving function \( \xi_t = \sqrt{\kappa} B_t \), i.e., its increments are identically and independently distributed and \( \langle \xi_t^2 \rangle = \kappa t \), where \( \kappa \) is diffusion
To investigate the behavior of the contour lines of BD growth surface, we simulated $10^2$ independent samples with the size of $1024 \times 1024$. The roughness exponent of the samples is $0.36 \pm 0.02$. As shown in Fig. 6h, the statistics of the heights are self-similar and the height increments have approximately Gaussian statistics. The multifractal spectrum of the BD contour lines has been numerically obtained which shows that the contour lines seem to be scale invariant. The fractal dimension of the lines is $1.39 \pm 0.03$ (Fig. 6b). In the first view, these results show that the BD’s contour lines may belong statistically to the same universality class as the domain walls in the critical Ising model. Applying the Inverse Loewner equation, we have analyzed the driving function considered by one element shorter. The algorithm is repeated until the whole curve is transformed. The final output of the process is a series of driving functions obtained from the contour ensemble of the WO3 samples. The accuracy of the code has been checked on an ensemble of SLE traces with known $\kappa$ where it yielded the correct value with an error of $\sim 3\%$.

As shown in Fig. 5 the statistics of the driving function $\langle \xi(t)^2 \rangle$ converges to a Gaussian process with variance $\langle \xi(t)^2 \rangle = \kappa t$ and $\kappa = 3 \pm 0.2$ and uncorrelated increments. It confirms that the contour lines of the WO3 samples are locally $SLE_3$ curves i.e., conformally invariant. To investigate the effect of the scale of AFM images on the statistics of the contour lines, we took over $10^2$ AFM profiles from samples at each of three scales $5\mu m \times 5 \mu m$, $10 \mu m \times 10 \mu m$ and $20 \mu m \times 20 \mu m$ with a fixed number of lattice points $256^2$. After rescaling of size to one, we observe no conclusive differences within numerical errors (Fig. 5).

In the following, due to some similarities between some of the results for WO3 surface and BD model, we wish to briefly discuss this model.

There are many numerical algorithms for extracting the driving function of a given random curve by inversion of loewner equation. We have used the successive discrete, conformal slit maps – based on the piece-wise constant approximation of the driving function – that swallow one segment of the curve at each time step.

To do this at first, we have determined all of the contour lines $\gamma$’s of the WO3 samples as described above by sequences of the points $\{z_0, z_1, \cdots, z_N\}$, we obtained 2319 such curves with the average number of points about 1210. After that all of the contour lines were mapped by $\varphi(z) = z_N z/(z_N - z)$. To avoid numerical errors only the part of the curves corresponding to capacity $t \leq 0.25$ were used.

Then, by setting the starting point $z_0$ in $(0,0)$, all of the points except the first one have been mapped with $g_t(z) = \sqrt{(z - \xi_t)^2 + 4t} + \xi_t$ with $t = \frac{1}{2}|m(z_1)|^2$ and $\xi_t = Re(z_1)$ and the resulting points have been renum-
of the iso-height curves. This analysis shows that in some large scales the increments of the driving function are independently distributed with Gaussian statistics of variance $\kappa = 3.1 \pm 0.2$.

A point which remains is the difference between the standard way of generating an interface in the Ising model and the one considered in this paper. To generate chordal SLE in the half plane for the Ising model, one usually takes fixed boundary conditions on either side of the origin. Whereas in the procedure proposed here, we choose an arbitrary horizontal line which clearly has more complex boundary conditions on it. Certainly different boundary conditions can introduce different drift terms in the driving function. Question is which boundary condition with the above procedure leads to chordal SLE in the Ising model. We have run simulations on the two dimensional critical Ising model with periodic boundary conditions on the square lattice of size 1024$^2$, and found that the resulting driving function has leading term (considering finite size effects) with the same statistics of Brownian motion with diffusivity of $\kappa = 2.9 \pm 0.2$. Details will appear in a forthcoming work.

In conclusion, we have found an experimental example, the iso-height lines of the WO$_3$ surface, for SLE curves. It has been numerically shown that the statistics of the contour lines are conformally invariant and it is very similar to the statistics of the domain walls in the critical Ising model. However, this approach can be used for other experimental samples to know more about the relation of the statistics of the contour lines to the various geometrical and physical properties of the surfaces. Moreover, we have proposed a model of growth, ballistic deposition model whose contour lines seem to belong to the same universality class as the Ising model at large scales.

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