Solution of Integral Equations Arising in Mathematical Problems of Construction Science, using the Bogolyubov – Krylov method

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Abstract. The possibility of a numerical solution of construction problems leading to Fredholm integral equations of the first and second kinds is considered. The following problems of building sciences, for example, lead to such equations: in the theory of elasticity – elastoplastic torsion of bodies, stress concentration in a body during its bending; in the theory of vibrations, in the mechanics of structural failure – in the formation of cracks; when studying non-stationary phenomena in solids – non-stationary heat transfer, quasistatic viscoelasticity, propagation of longitudinal and transverse waves and other processes. According to the formulas of numerical integration, the problem can be reduced to a system of linear algebraic equations with a matrix of coefficients, which can be ill-conditioned, and the task is classically incorrect. This circumstance complicates its decision by traditional methods. To solve ill-posed problems similar to the problem in question, there are classical regularization methods. However, the numerical implementation of regularizing algorithms is not always easy to implement. In this regard, the goal of our study was to develop a numerical method for solving the Fredholm integral equations of the first and second kinds, which allow solving such problems without using regularization methods. The authors propose to solve problems of this type on the basis of the Bogolyubov – Krylov formula for representing certain integrals, in the form of finite-dimensional sums. At the same time, it becomes possible to use a non-uniform grid of nodes for splitting the integration interval. To efficiently define the set of nodes for splitting the integration interval, it is proposed to use a priori information about the properties of the solution of the problem. For example, the number of split points of the integration interval increases where the intended solution undergoes the most rapid changes, and decreases in areas where the intended solution is close to linear. The optimal arrangement of the splitting nodes of the integration interval makes it possible to increase the conditionality of the system of linear equations corresponding to the difference analogue of the problem and, thereby, to prevent the divergence of the iterative process. The article provides an example of calculating the optimal arrangement of the nodes splitting the integration interval when solving a test problem.

1. Introduction

The methods of mathematical and computer modeling are actively used in solving fundamental and applied problems in various fields of construction science. The implementation of mathematical
models involves the development of effective numerical procedures for solving computational problems necessary for carrying out numerical experiments and calculations.

By virtue of the diversity of processes and objects considered in construction science, mathematical models in this field are various mathematical objects: from simple linear and nonlinear algebraic equations to complex systems of integro-differential equations and optimization and optimal control problems [1].

In particular, the set of mathematical construction problems are formulated in the form of various Fredholm and Voltaire integral equations of the first and second kinds, which are the classical incorrect problems of mathematical physics.

Problems that lead to integral equations in construction science arise when solving various problems in the theory of elasticity (for example, elastoplastic torsion of bodies, stress concentration in a body during its bending), in the theory of oscillations, in structural fracture mechanics (for example, cracking), in the study of nonstationary phenomena in solids (for example, such as nonstationary heat transfer, quasistatic viscoelasticity, propagation of longitudinal and transverse waves) and in the study of many other processes.

Many mathematical methods used in solving problems of calculating building structures also lead to integral equations. For example, a common method is the boundary integral equations, which is based on the use of the Green function on an extended domain, including the region occupied by the construction. Another possible way to obtain boundary integral equations is to apply the Somiliana formula in combination with Green's formulas on an unbounded domain [2]. The convenience of the method lies in the fact that the functions of the integral equations only on the boundary of the domain are included as unknown, while in solving the boundary value problem by difference methods it is necessary to find the functions in the entire domain. The main difference between the method of boundary integral equations and the finite element method during approximation is the discretization method (in the case of integral equations, it is carried out not inside the region, but on the border) [3]. The method is more accurate than other numerical methods due to the fact that the solution of the problem depends only on the values of stresses and displacements on the boundary, precisely satisfying the equations inside the region.

In [4], the application of the method of nonlinear boundary integral equations for the approximate solution of a three-dimensional quasistatic contact problem of the interaction of linearly elastic bodies in the presence of Coulomb friction between them is shown.

The papers [5, 6] are devoted to the original analytical solutions of contact problems and methods for their numerical implementation.

In [7], an analytical solution is given to the axisymmetric problem of the action of a distributed load on a half space under given mixed conditions on its boundary, obtained using the integral Hankel transform. When constructing solutions of mixed problems, separate statements of the theory of Fredholm integral equations of the second kind, Neumann series, Bessel, Neumann, Struve special functions, gamma functions, and elliptic integrals of the first and second kind were used. The numerical solution of integral equations is carried out using the method of successive approximations.

Ahmadi S.F., Eskandari M. [8] investigated the indentation of an axisymmetric hard stamp into an isotropic half-space covered with an elastic thin film. With the help of potential functions of transfer and transformation of Hankel, the problem was reduced to the Fredholm integral equation of the second kind.

Taking into account the layering of a half-space when solving contact problems is especially important in structural mechanics. Modern approaches to solving contact problems of this class can be illustrated by the example of [9], in which the axisymmetric contact problem for a three-layer elastic half-space is reduced to the Fredholm integral equation of the second kind, its approximate solution obtained by the collocation method on nodes of the Legendre polynomial.

In this paper, we consider the possibility of numerical solution of construction problems leading to the Fredholm equations of the I-st and II-nd kind by the Bogolyubov-Krylov method with an uneven grid of nodes splitting the integration interval.
2. Algorithm Description

Integral equations are a special case of operator equations:

\[ y = A y + f. \]  \hspace{1cm} (1)

Suppose that the operator \( A \) can be approximated by a continuous finite-dimensional operator. Then, instead of equation (1), we can consider its approximation:

\[ y = \bar{A} \bar{y} + f. \]

We assume that the operator exists and is bounded. In this case, there is an approximation for which the condition is satisfied

\[ \| A - \bar{A} \| \cdot \| (I - \bar{A})^{-1} \| < 1. \]

In \([10]\), an estimate is given:

\[ \| y - \bar{y} \| \cdot \| (I - \bar{A})^{-1} \| < \frac{\| A - \bar{A} \| \cdot \| (I - \bar{A})^{-1} \| \cdot \| f \|}{1 - \| A - \bar{A} \| \cdot \| (I - \bar{A})^{-1} \|}. \]  \hspace{1cm} (2)

Relation (2) suggests that the error of the solution is the smaller, the higher the degree of approximation of the operator \( A \) by the operator \( \bar{A} \). Improving the accuracy of approximation of the operator, an indicator of which is the value \( \| A - \bar{A} \| \), can be achieved either by increasing the dimension of the operator \( \bar{A} \), or by special selection of the basic elements that define it. If the operator equation is the Fredholm equation of the first or second kind, then when solving it by the Bogolyubov – Krylov method \([11]\), the optimal choice of basic elements can be made by correctly arrangeing the nodes \( \{x_i, i = 0, \ldots, n\} \) of the integration interval \([a, b]\) kernels of an integral equation for a given number of them.

Consider the Fredholm integral equation – type II

\[ \zeta \omega(x) = \int_a^b K(x, y) \omega(y)dy + C \]  \hspace{1cm} (3)

with some additional condition

\[ \int_a^b \omega(y)dy = 1. \]  \hspace{1cm} (4)

Here \( \zeta \) is the parameter, \( C \) is some unknown constant, \( x, y \subset [a, b] \) are variables, \( \omega(x) \) is the desired function, \( K(x,y) \) is the kernel of the integral equation.

The system of integral equations (3) - (4) can be reduced by quadrature formulas to a system of linear algebraic equations (SLAE). The solution of the system for \( \zeta \), greater than some \( \zeta_{cr} \), does not cause difficulties. However, the problem is classically incorrect for \( \zeta = 0 \), and its solution is difficult by conventional methods for \( 0 \leq \zeta \leq \zeta_{cr} \). Incorrect problems, similar to problem (3) - (4), are solved by classical regularization methods. However, the numerical implementation of regularizing algorithms is not always easy to implement. Therefore, the problem of finding quadrature formulas that allow solving such problems without the use of regularization is relevant.

For the basis of the method of solving problems of type (3) - (4), we take the well-known Bogolyubov – Krylov formula for calculating a definite integral. After applying this formula, problem (3) - (4) is reduced to a system of linear algebraic equations:

\[
\begin{align*}
 z_i &= \zeta \omega_i - \sum_{j=1}^{n} a_{ij} \omega_j - C = 0, \\
 z_{n+1} &= \frac{1}{n} \sum_{j=1}^{n} \omega_j - 1 = 0.
\end{align*}
\]  \hspace{1cm} (5)
where \( \omega_i = \omega(x_i), \ a_{ij} = \int_{x_i}^{x_j} K(x_i, y) \, dy \); \{x_i, i = 1, \ldots, n\} – nodes split the interval of integration \([a, b]\).

Using the Bogolyubov–Krylov formula allows you to plan the placement of split nodes of the interval \([a, b]\). based on a priori information about the solution.

For example, the number of split points of the integration interval should be increased where the proposed solution undergoes the most rapid changes, and decreases in areas where the proposed solution is close to linear.

The optimal arrangement of the splitting nodes of the integration interval makes it possible to increase the conditionality of the system of linear equations corresponding to the difference analogue of the problem and, thereby, to prevent the divergence of the iterative process.

3. The results of numerical experiments

We will demonstrate the applicability of the proposed method with an example of solving a problem (3) - (4) with the kernel of an integral equation \( K(x, y) = \ln|x^2 - y^2| \).

At the beginning of the numerical solution, the integration interval \([0, 1]\) is split evenly by eleven nodes \( x_i \). In further calculations, the above-noted merit of the Bogolyubov–Krylov formula is used, namely, the possibility of choosing \( x_i \) based on a priori information about the solution of the problem.

At small values of \( \zeta \), which causes the incorrectness of the system (3) - (4), this information is as follows. The distribution of partitioning nodes should sharply increase to the right edge of the interval \([0, 1]\) with \( x \) close to unity, as can be seen from the well-known formula for the analytical solution of a problem with:

\[
\omega(x) = \frac{2}{\pi \sqrt{1 - x^2}}.
\]  

The dependence of the conditional number of the SLAE, which is formed during the discretization of the problem, on the layout plan of the nodes \{\( x_i \)\} on the interval \([0, 1]\) is shown in Table 1.

**Table 1.** The dependence of the condition number (cond) and the accuracy of the solution (discrepancy) on the arrangement of the splitting nodes of the integration interval.

| Split Nodes | cond | Discrepancy |
|-------------|------|-------------|
| 0.00 0.10 0.20 0.30 0.40 0.50 0.60 0.70 0.80 0.90 1.00 | 14.80 0.52 |
| 0.00 0.05 0.11 0.19 0.36 0.51 0.64 0.75 0.84 0.91 1.00 | 21.90 0.57 |
| 0.05 0.14 0.22 0.39 0.54 0.67 0.78 0.87 0.94 0.95 1.00 | 69.20 1.41 |
| 0.00 0.50 0.70 0.90 0.93 0.95 0.96 0.97 0.98 0.99 1.00 | 153.90 5.20 |

The best distribution found by us from the condition of minimization of the condition number is given in Table 2 (cond = 8.35).

**Table 2.** The optimal distribution of nodes \{\( x_i \)\} in the interval \([0, 1]\).

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) | \( x_9 \) | \( x_{10} \) |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.095     | 0.275     | 0.435     | 0.575     | 0.695     | 0.795     | 0.875     | 0.935     | 0.975     | 0.995     |

Table 3 shows the results of numerical experiments for various sequences \{\( x_i \)\} and the exact solution of the problem by formula (6).
Table 3. Results of calculations and numerical experiments.

| \( \omega(x_1) \) | \( \omega(x_2) \) | \( \omega(x_3) \) | \( \omega(x_4) \) | \( \omega(x_5) \) | \( \omega(x_6) \) | \( \omega(x_7) \) | \( \omega(x_8) \) | \( \omega(x_9) \) | \( \omega(x_{10}) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.637           | 0.644           | 0.657           | 0.680           | 0.713           | 0.762           | 0.838           | 0.962           | 1.208           | 2.033           |
| 0.648           | 0.655           | 0.669           | 0.692           | 0.727           | 0.780           | 0.860           | 0.996           | 1.212           | 2.759           |
| 0.639           | 0.643           | 0.658           | 0.679           | 0.713           | 0.763           | 0.837           | 0.957           | 1.190           | 2.280           |

The first line in Table 3 contains the results of solving the problem (3) - (4) on the basis of formula (6). The second line contains the numerical solution of the same problem, obtained by the Bogolyubov – Krylov formula with a uniform arrangement of the split nodes of the interval [0, 1]. The third line contains the results of applying the Bogolyubov – Krylov formula with the nodes of nonuniform partitioning given above. In this case, the solution is interpolated into the midpoints of the segments of the uniform partition.

As can be seen, with the same number of split nodes of the interval [0,1], the accuracy of the solution increases significantly from their planning.

Thus, it is shown that the incorrectness of the problem (3) with \( \zeta = 0 \) entails the poor conditionality of the coefficient matrix of the linear system (5).

4. Conclusions

The arrangement of nodes \( x_j, j = 0, n \) allows, in some cases, to increase the conditionality of the system, which can be used with the optimal choice \( x_j \) along with a priori information about the decision.

In addition, the ability to arrange the integration nodes in the solution of integral equations allows the use of a priori information about the solution, such as information about the rate of change of the function.

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