Rogue waves in the massive Thirring model

Junchao Chen¹ | Bo Yang² | Bao-Feng Feng³

¹Department of Mathematics, Lishui University, Lishui, China
²School of Mathematics and Statistics, Ningbo University, Ningbo, China
³School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, Edinburg, Texas, USA

Abstract
In this paper, general rogue wave solutions in the massive Thirring (MT) model are derived by using the Kadomtsev–Petviashvili (KP) hierarchy reduction method and these rational solutions are presented explicitly in terms of determinants whose matrix elements are elementary Schur polynomials. In the reduction process, three reduction conditions including one index- and two dimension-ones are proved to be consistent by only one constraint relation on parameters of tau-functions of the KP-Toda hierarchy. It is found that the rogue wave solutions in the MT model depend on two background parameters, which influence their orientation and duration. Differing from many other coupled integrable systems, the MT model only admits the rogue waves of bright-type, and the higher order rogue waves represent the superposition of fundamental ones in which the nonreducible parameters determine the arrangement patterns of fundamental rogue waves. Particularly, the super rogue wave at each order can be achieved simply by setting all internal parameters to be zero, resulting in the amplitude of the sole huge peak of order $N$ being $2N + 1$ times the background. Finally, rogue wave patterns are discussed when one of the internal parameters is large. Similar to other integrable equations, the patterns are shown to be associated with the root structures of the Yablonskii–Vorob’ev polynomial hierarchy through a linear transformation.

Correspondence
Bao-Feng Feng, School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, Edinburg, TX 78541, USA.
Email: baofeng.feng@utrgv.edu

Funding information
U.S. Department of Defense (DoD), Air Force for Scientific Research (AFOSR); National Science Foundation; National Natural Science Foundation of China; Zhejiang Province Natural Science Foundation of China
1 | INTRODUCTION

In recent years, rogue waves appearing in various complex systems have become a fascinating subject of experimental and theoretical studies. They correspond to large-amplitude and spontaneous local excitations with the instability and unpredictability, and hence could cause maritime disasters in oceanography and induce pulse destroys in optics. From the mathematical description, Peregrine soliton characterized by a kind of rational solutions for the focusing nonlinear Schrödinger (NLS) equation was discovered to act as the prototype of realistic rogue waves, as it exhibits the local wave structure in temporal–spatial plane and the height of maximum peak at the center reaches to three times the finite background. Because of the underlying integrability, it was found that Peregrine soliton can be extended to higher order exact rogue wave solutions in many nonlinear wave systems such as the NLS equation, the derivative NLS equation, the Sasa–Satsuma equation, the Manakov equations, the long-wave–short-wave equations, the three-wave resonant interaction equations, the Davey–Stewartson equations and many others. These analytic rational solutions with the higher order polynomials also represent the localized structure in both space and time coordinates, and could possess multiple intensity peaks or higher peak amplitudes. In particular, differing from the scalar system, the coupled and vector integrable systems with the additional degrees of freedom could allow the novel counterpart of rogue wave such as dark and four-petaled types. In addition, more interesting features have been discussed attributed to the explicit expressions for general rogue waves. For instance, a Nth-order rogue wave usually contains N(N + 1)/2 elementary (Peregrine) rogue waves, a super Nth-order rogue wave that all peaks converge to be a sole huge one has the maximum amplitude 2N + 1 times its background. Furthermore, the geometric patterns of elementary rogue waves arrangements have been found to be closely related with the root structures of Yablonskii-Vorob’ev polynomials.

The massive Thirring (MT) model with the two-component couplings, proposed in the context of quantum field theory, represents an exactly solvable example for the nonlinear Dirac equation in one-dimensional space, and is also used to describe nonlinear optical pulse propagation in Bragg nonlinear optical media. The complete integrability of the MT model has been presented by means of the inverse scattering transform method. The orbital stability of Dirac solitons in the MT model has been proved by virtue of conserved quantities. Regarding the rogue wave solutions of the MT model, the first-order rogue wave has been constructed through the Darboux method with a matrix version of the Lagrange interpolation method. Applying this fundamental solution to the coupled mode equations has shown that combining electromagnetically induced transparency with Bragg scattering four-wave mixing may yield rogue waves at low powers. The higher order rogue wave solutions have been investigated by using the n-fold Darboux transformation, and the explicit formulas up to third-order with their patterns have been provided in detail. By virtue of the nonrecursive Darboux transformation method, general super rogue wave solutions have been obtained and their structure analyses reveal that rogue waves properties of both components in the coupled MT model is same as that in scalar nonlinear
systems except for the spatiotemporal distributions. The modulation instability responsible for rogue waves in the MT model has been discussed in.

In this paper, we derive general rogue wave solutions in the MT model by using the Kadomtsev–Petviashvili (KP) hierarchy reduction method, and these solutions are given explicitly in terms of determinants with the elements given by elementary Schur-polynomial. In the process of constructing rational solutions, two dimension-reduction conditions and one index-reduction one are proved to be consistent. It is found that these rational solutions in the MT model depend on the background parameters that do not affect the height of peak but influence the orientations and durations of rogue waves. Differing from many other coupled systems, there is no additional parameter, thus, the MT model only admits the rogue wave of bright-type. The $N$th-order rogue wave is shown to be the superposition of $N(N + 1)/2$ fundamental ones, and the arrangement pattern depends on the nonreducible parameters. Particularly, the super rogue wave at each order is achieved simply by setting all internal parameters to zero, and the height of the sole huge peak at order $N$ is $2N + 1$ times the background amplitude. Moreover, when one of the internal parameters is extreme larger than other ones, rogue wave patterns with certain arrangement shapes are discussed and these patterns are shown to be associated with the root structure of the Yablonskii–Vorob’ev polynomial hierarchy through a linear transformation.

The remainder of the paper is organized as follows. In Section 2, we present general rogue wave solutions in the MT model, which are given in terms of determinants with Schur-polynomial matrix elements. In Section 3, the rogue wave solutions are derived by using the KP-Toda hierarchy reduction method. In Section 4, the local structures of fundamental and higher order rogue waves are analyzed and illustrated. Section 5 provides the detailed analyses of rogue wave patterns with one large internal parameter and the connection to the root structure of the Yablonskii–Vorob’ev polynomial hierarchy. The paper is concluded in Section 6 by a summary and discussion.

2 | GENERAL ROGUE WAVE SOLUTIONS

The MT model is given in terms of the light-cone coordinates\textsuperscript{43–45}

\begin{align*}
  iu_x + v + \sigma u|v|^2 &= 0, \\
  iv_t + u + \sigma v|u|^2 &= 0,
\end{align*}

with $\sigma = \pm 1$. In the optical context, both components $u$ and $v$ represent envelopes of the forward and backward waves, respectively.\textsuperscript{33} On the left-hand sides of Equations (1) and (2), the second and third terms denote the linear coupling and the cross-phase modulation, inducing the dispersion and nonlinearity effects, and their balance leads to the generation of various soliton solutions.

In this work, rogue waves in the MT model (1)–(2) will be expressed in terms of Schur polynomials. The elementary Schur polynomials $S_j(x)$ are defined via the generating function

\[ \sum_{j=0}^{\infty} S_j(x)\lambda^j = \exp \left( \sum_{j=1}^{\infty} x_j\lambda^j \right), \]
or more explicitly,

$$S_0(x) = 1, \quad S_1(x) = x_1, \quad S_2(x) = \frac{1}{2} x_1^2 + x_2, \quad \cdots, \quad S_j(x) = \sum_{l_1 + 2l_2 + \cdots + ml_m = j} \left( \prod_{j=1}^{m} \frac{x_j^{l_j}}{l_j!} \right),$$

where \( x = (x_1, x_2, \ldots) \).

The main result of this paper, or the general rogue wave solutions in the MT model (1)–(2) are given by the following theorem.

**Theorem 1.** The MT model (1)–(2) possesses rogue wave solutions

$$u = \rho_1 \frac{g}{f} e^{i(1+\sigma_1 \rho_2)} \left( \frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t \right), \quad v = \rho_2 \frac{h}{f} e^{i(1+\sigma_1 \rho_2)} \left( \frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t \right), \quad (4)$$

where

$$f = \sigma_{0,0,0}, \quad f^* = \sigma_{-1,0,0}, \quad g = \sigma_{-1,1,0}, \quad h = \sigma_{-1,0,1}, \quad (5)$$

and the elements in the determinant \( \sigma_{n,k,l} = \det_{1 \leq i,j \leq N} (m^{(n,k,l)}_{2i-1,2j-1}) \) are

$$m^{(n,k,l)}_{1,j} = \sum_{\gamma=0}^{\min(i,j)} \frac{1}{4^\gamma} S_{i-\gamma}(x^+(n, k, l) + \gamma s) S_{j-\gamma}(x^-(n, k, l) + \gamma s), \quad (6)$$

with the vectors \( x^\pm(n, k, l) = (x_1^\pm(n, k, l), x_2^\pm(n, k, l), \ldots) \equiv (x_1^\pm, x_2^\pm, \ldots) \) being defined by

$$x_r^+ = \alpha_r x + \beta_r t + (n + \frac{1}{2}) \vartheta_r + k \vartheta_r + l \vartheta_r + a_r,$n \in \mathbb{N} \}

The parameters \( \alpha_r, \beta_r, \vartheta_r, \vartheta_r, \zeta_r \) and \( s = (s_1, s_2, \ldots) \) are the coefficients from the following expansions

$$\frac{\rho_2}{\rho_1} [\beta(e^\rho - 1)] = \sum_{r=1}^{\infty} \alpha_r \kappa^r, \quad \frac{\rho_1 \rho}{\rho_2} \left[ \frac{1}{\hat{\rho} + i \rho} - \frac{1}{e^\rho \hat{\rho} + i \rho} \right] = \sum_{r=1}^{\infty} \beta_r \kappa^r, \quad \ln \frac{e^\rho \hat{\rho} + i \rho}{\hat{\rho} + i \rho} = \sum_{r=1}^{\infty} \alpha_r \kappa^r, \quad \ln \frac{e^\rho \hat{\rho} + i \sigma_1 \rho}{\hat{\rho} + i \sigma_1 \rho} = \sum_{r=1}^{\infty} \beta_r \kappa^r, \quad \ln \left( \frac{2 e^\rho - 1}{\kappa e^\rho + 1} \right) = \sum_{r=1}^{\infty} s_r \kappa^r,$$

with \( \zeta_1 = 1, \quad \zeta_r = 0 \ (r = 2, 3, \ldots), \quad \hat{\rho} = \sqrt{-\sigma_1 \rho_2 \rho} \) and \( \rho = 1 + \sigma_1 \rho_2 \). Here \( \rho_1 \) and \( \rho_2 \) are real parameters that satisfy the conditions: \(-1 < \rho_1 \rho_2 < 0 \) for \( \sigma = 1 \) or \(0 < \rho_1 \rho_2 < 1 \) for \( \sigma = -1 \), and \( a_r \ (r = 1, 2, \ldots) \) are arbitrary complex parameters.

We make three remarks in a row. First, the rogue wave of order \( N \) has \( N - 1 \) free irreducible complex parameters, \( a_3, a_5, \cdots, a_{2N-1} \), which is the same as that in the NLS equation,\(^{13}\) the DNLS equation\(^{18}\) and the three-wave resonant interaction system.\(^{34}\) The rogue waves of order \( N \) contain \( 2N - 1 \) complex parameters \( a_1, a_2, \cdots, a_{2N-1} \). However, as the tau function \( \sigma_{n,k,l} \) can be rewritten as the summation formula in Ref.\(^{13}\) that can be treated by the technique in Ref.\(^{18}\), one finds that the rogue wave is independent of all even-indexed parameters \( a_{\text{even}} \). This enables us to take these
dummy parameters as zeros, that is, \( a_2 = a_4 = \cdots = a_{\text{even}} = 0 \). Of the remaining parameters, \( a_1 \) can be normalized into zero through a shift of variables \( x \) and \( t \). Thus, there only exist \( N - 1 \) free irreducible complex parameters \((a_3, a_5, \cdots, a_{2N-1})\) in the rogue wave of order \( N \).

Second, the rogue wave in the MT model is independent of the parameter \( \alpha \) though it is introduced in the process of constructing rogue wave solutions and appears in Theorem 2 where solutions are taken in differential operator form. This parameter is finally removed when rogue wave solutions are expressed by elementary Schur polynomials. This implies that \( \alpha \) is a reducible parameter. In spite of the fact that the DNLS equation and the MT model belong to the same integrable hierarchy, the parameter \( \alpha \) is irreducible in rogue wave solutions of the DNLS equation and it controls the orientation and duration of rogue waves.\(^{18}\) However, for the MT model, as analyzed in Section 4, two free amplitude parameters \( \rho_1 \) and \( \rho_2 \) are shown to affect these features. Differing from other coupled systems in which abundant patterns of rogue wave such as dark and four-petaled flower structures could usually appear, as analyzed in Subsection 4.1, the MT model only allows the fundamental rogue wave of bright-type that possesses three critical points (one maximum and two minima) and the fixed height of peak.

Third, the rogue wave solutions of the MT model in Theorem 1 are presented in the light cone coordinates, one can rewrite them in the laboratory coordinates straightforwardly through the independent variables transformations.\(^{43,45,60}\) The higher order rogue wave solutions in Ref. [44] can be expressed explicitly via the scaling transformations:

\[
\begin{align*}
u &= \pm \frac{1}{\sqrt{-\sigma \mu}} U(\eta, \xi) = \pm \frac{1}{\sqrt{-\sigma \mu}} U\left(\frac{x}{\mu^2}, t\right), \quad v &= \pm \frac{1}{\mu \sqrt{-\sigma \mu}} V(\eta, \xi) = \pm \frac{1}{\mu \sqrt{-\sigma \mu}} V\left(\frac{x}{\mu^2}, t\right). 
\end{align*}
\]

**Proposition 1.** When \( a_r = 0 \) for all \( r \geq 1 \), the rogue wave solutions in Theorem 1 are parity-time-symmetric, that is, \( u^*(-x, -t) = u(x, t) \) and \( v^*(-x, -t) = v(x, t) \).

Indeed, if we set \( a_r = 0 \) in Theorem 1, the relations \([x^* \pm (n, k, l)](x, t) = -[x^* \pm (n, k, l)](x, t)\) for all \( r \geq 1 \) hold. Then applying the same treatment as that in the DNLS equation,\(^{18}\) one can find that \( \sigma_{n,k,l}^*(-x, -t) = \sigma_{n,k,l}(x, t) \). Thus, the above Proposition 1 is proved. From the Proposition 1, the parity-time-symmetric rogue wave of each order can reach its maximum peak amplitude and this point is located at the center of such nonlinear wave, that is, at \((x, t) = (0, 0)\). This special type of rogue wave corresponds to the super rogue wave state in Ref. [45] that provides the super rogue wave up to second order. In addition, the maximum amplitude in such type of rogue wave can be derived by simply setting \( x = t = a_r = 0 \). Through the direct calculations for five low-order cases, we can conclude that \( N \)th-order tau functions read

\[
\begin{align*}
|f(0, 0)|_{a_r=0} &= \frac{1}{2^{N^2}} (1 + \sigma \rho_1 \rho_2)^{N(N+1)/2}, \\
|g(0, 0)|_{a_r=0} &= |h(0, 0)|_{a_r=0} = \frac{2N + 1}{2^{N^2}} (1 + \sigma \rho_1 \rho_2)^{N(N+1)/2},
\end{align*}
\]

which leads to the maximum amplitude of the \( N \)th order rogue wave

\[
|u(0, 0)|_{a_r=0} = (2N + 1)|\rho_1|, \quad |v(0, 0)|_{a_r=0} = (2N + 1)|\rho_2|. 
\]

It implies that the maximum amplitudes in both component are always \( 2N + 1 \) times their background. This property was examined graphically for the first- and second-order rogue waves in.\(^{45}\) Hence, the maximum amplitude of the super rogue wave in the MT model is the same as that in
scaling integrable systems such as the NLS and DNLS equations. On the other hand, as reported in previous studies regarding the super rogue waves in coupled integrable systems, the additional degrees of freedom could give rise to the varying maximum peak amplitude.

### 3 DERIVATION OF ROGUE WAVE SOLUTIONS

In this section, general rogue wave solutions will be derived by using the KP hierarchy reduction method. Prior to the tedious derivation, we list the main steps as shown in the subsequent subsections. First, we introduce the dependent variable transformations

\[
\begin{align*}
    u &= \rho_1 g e^{i(1+\sigma\rho_1\rho_2)(\frac{\partial}{\partial t} + \frac{1}{\rho_1})} + \rho_2 \frac{h}{f} e^{i(1+\sigma\rho_1\rho_2)(\frac{\partial}{\partial t} + \frac{1}{\rho_2})}, \\
    (9)
\end{align*}
\]

where \( f, g, h \) are complex functions, and \( \rho_1, \rho_2 \) are real constants. The MT model (1)–(2) is transformed into the following bilinear equations

\[
\begin{align*}
    (iD_x - \frac{\rho_2}{\rho_1})g \cdot f &= -\frac{\rho_2}{\rho_1} hf^*, \\
    (10)
\end{align*}
\]

\[
\begin{align*}
    (iD_x - \sigma \rho_2^2)f \cdot f^* &= -\sigma \rho_2^2 hh^*, \\
    (11)
\end{align*}
\]

\[
\begin{align*}
    (iD_t - \frac{\rho_1}{\rho_2})h \cdot f^* &= -\frac{\rho_1}{\rho_2} gf, \\
    (12)
\end{align*}
\]

\[
\begin{align*}
    (iD_t - \sigma \rho_1^2)f^* \cdot f &= -\sigma \rho_1^2 gg^*, \\
    (13)
\end{align*}
\]

where \( D \) is the Hirota’s bilinear differential operator defined by

\[
D^n x^n y^m f \cdot g = \left( \frac{\partial}{\partial s} - \frac{\partial}{\partial s'} \right)^n \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m f(y, s)g(y', s')|_{y=y', s=s'}.
\]

Next, we consider a set of higher dimensional bilinear equations in extended KP hierarchy that includes negative flows

\[
\begin{align*}
    (D_{x_1} + a)\tau_{n+1,k+1,l} \cdot \tau_{n+1,k,l} &= a \tau_{n+1,k+1,l} \tau_{n+1,k,l}, \\
    (bD_{x_{-1}} + 1)\tau_{n,k,l+1} \cdot \tau_{n,k,l} &= \tau_{n-1,k,l+1} \tau_{n+1,k,l}, \\
    (aD_{t_a} - 1)\tau_{n+1,k,l} \cdot \tau_{n,k,l} &= -\tau_{n+1,k-1,l} \tau_{n,k+1,l}, \\
    (bD_{t_b} - 1)\tau_{n+1,k,l} \cdot \tau_{n,k+1,l} &= -\tau_{n+1,k,l-1} \tau_{n,k+1,l}, \\
    (14)
\end{align*}
\]

which admit a wide class of algebraic solutions in terms of Gram determinants. Based on these algebraic solutions, we restrict them to satisfy the dimension- and index-reduction conditions:
\[ [\partial_{x_1} - b(a - b)\partial_{t_b}]\tau_{n,k,l} = C_1 \tau_{n,k,l}, \quad (18) \]
\[ \left[ \partial_{x_{-1}} + \frac{a - b}{b} \partial_{t_2} \right] \tau_{n,k,l} = C_2 \tau_{n,k,l}, \quad (19) \]
\[ \tau_{n+1,k+1,l+1} = e^{i\kappa_0} \tau_{n,k,l}, \quad (20) \]

where \( C_1, C_2 \) and \( \kappa_0 \) are real constants. Then such algebraic solutions satisfy the (1+1)-dimensional bilinear equations:

\[ (D_{x_1} + a)\tau_{n,k+1,l} \cdot \tau_{n+1,k,l} = ae^{i\kappa} \tau_{n,k,l+1} \tau_{n,k,l}, \quad (21) \]
\[ (bD_{x_{-1}} + 1)\tau_{n,k,l+1} \cdot \tau_{n,k,l} = e^{-i\kappa} \tau_{n,k+1,l} \tau_{n+1,k,l}, \quad (22) \]
\[ \left[ -\frac{ab}{(a-b)} D_{x_{-1}} - 1 \right] \tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k-1,l} \tau_{n,k+1,l}, \quad (23) \]
\[ \left[ \frac{1}{a-b} D_{x_1} - 1 \right] \tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k,l-1} \tau_{n,k,l+1}. \quad (24) \]

Finally, by setting the coordinate transformations
\[ x_1 = -\frac{\rho_2}{ia\rho_1} x, \quad x_{-1} = -\frac{\rho_1 b}{i\rho_2} t, \quad (25) \]
that is,
\[ \partial_{x_1} = -\frac{ia\rho_1}{\rho_2} \partial_x, \quad \partial_{x_{-1}} = -\frac{i\rho_2}{\rho_1 b} \partial_t, \]
with the relation \( b = a(1 + \sigma \rho_1 \rho_2) \), and taking the variable transformations \( (n = -1, k = 0, l = 0) \)
\[ \tau_{0,0,0} = f, \quad \tau_{-1,0,0} = f^*, \quad \tau_{-1,1,0} = e^{i\kappa_0} g, \quad \tau_{0,-1,0} = e^{-i\kappa_0} g^*, \quad \tau_{-1,0,1} = h, \quad \tau_{0,0,-1} = h^*, \]
we arrive at exactly the bilinear Equations (10)–(13). Finally, the general rogue wave solutions of the MT model (1)-(2) are obtained in the differential operator form as follows:

**Theorem 2.** The MT model (1)-(2) possesses the following rogue wave solutions

\[ u = \rho_1 \frac{g}{f} e^{i(1+\sigma \rho_1 \rho_2)} \left( \frac{\rho_2 x + \rho_1 t}{\rho_1} \right), \quad v = \rho_2 \frac{h}{f} e^{i(1+\sigma \rho_1 \rho_2)} \left( \frac{\rho_2 x + \rho_1 t}{\rho_1} \right), \quad (26) \]

where

\[ f = \tau_{0,0,0}, \quad f^* = \tau_{-1,0,0}, \quad g = \Omega_0^{-N} \tau_{-1,1,0}, \quad h = \tau_{-1,0,1}, \quad (27) \]

and the elements in the determinant \( \tau_{n,k,l} = \det_{1\leq i,j\leq N}(m_{2i-1,2j-1}^{(n,k,l)}) \) are defined by

\[ m_{i,j}^{(n,k,l)} = \left[ \frac{(p - b)\partial_p}{i!} \right]^i \left[ \frac{(q + b)\partial_q}{j!} \right]^j m_{2i-1,2j-1}^{(n,k,l)} \bigg|_{p=p_0, q=q_0^*}, \]
\[ m^{(n,k,l)} = \frac{ip}{p+q} \left( -\frac{p}{q} \right)^n \left( -\frac{p-a}{q+a} \right)^k \left( -\frac{p-b}{q+b} \right)^l e^\Theta, \]

\[ \Theta = \frac{i\rho_2}{\rho_1}(p+q)x + \frac{i\rho_1 b}{\rho_2} \left( \frac{1}{p} + \frac{1}{q} \right) t + \sum_{r=1}^\infty \hat{a}_r \ln^r \left( \frac{p-b}{\sqrt{b(b-a)}} \right) + \sum_{r=1}^\infty \hat{a}^*_r \ln^r \left( \frac{q+b}{\sqrt{b(b-a)}} \right), \]

with \( \Omega_0 = -\frac{p_0(p_0-i\alpha)}{p_0^*e^{i\alpha}+p_0} \), \( p_0 = \sqrt{b(b-a)} + b, b = a(1+\sigma \rho_1 \rho_2) \) and \( a = i\alpha \). Here \( \alpha, \rho_1 \) and \( \rho_2 \) are arbitrary real parameters that satisfy the conditions: \(-1 < \rho_1 \rho_2 < 0 \) for \( \sigma = 1 \) or \( 0 < \rho_1 \rho_2 < 1 \) for \( \sigma = -1 \), and \( a_r (r = 1, 2, ...) \) are arbitrary complex parameters.

In the above theorem, the final rational solutions are independent of the parameter \( \alpha \), as it does not appear in the background plane waves. In fact, \( \alpha \) can be removed by the appropriate scaling of \( p \) and \( q \). More specifically, we assume \( \alpha > 0 \) without loss of generality and reparameterize \( p = \alpha \hat{p} \) and \( q = \alpha \hat{q} \), then one has the equivalent theorem:

**Theorem 3.** The MT model (1)–(2) possesses the rogue wave solutions (26)–(27) with elements of the determinant \( \tau_{n,k,l} = \det_{1\leq i,j\leq N}(\hat{m}_{2i-1,2j-1}^{(n,k,l)}) \) are defined by

\[ \hat{m}_{i,j}^{(n,k,l)} = \left[ (\hat{p} - \hat{b}) \hat{\partial}_p \right]^i \left[ (\hat{q} + \hat{b}) \hat{\partial}_q \right]^j m^{(n,k,l)} \bigg|_{\hat{p} = \hat{p}_0, \hat{q} = \hat{q}_0}, \]

\[ m^{(n,k,l)} = \frac{i\hat{p}}{\hat{p} + \hat{q}} \left( -\frac{\hat{p}}{\hat{q}} \right)^n \left( -\frac{\hat{p}-i}{\hat{q}+i} \right)^k \left( -\frac{\hat{p}-\hat{b}}{\hat{q}+\hat{b}} \right)^l e^\Theta, \]

\[ \Theta = \frac{\rho_2}{\rho_1}(\hat{p} + \hat{q})x + \frac{i\rho_1 \hat{b}}{\rho_2} \left( \frac{1}{\hat{p}} + \frac{1}{\hat{q}} \right) t + \sum_{r=1}^\infty \hat{a}_r \ln^r \left( \frac{\hat{p}-\hat{b}}{\sqrt{\hat{b}(\hat{b}-i)}} \right) + \sum_{r=1}^\infty \hat{a}^*_r \ln^r \left( \frac{\hat{q}+\hat{b}}{\sqrt{\hat{b}(\hat{b}-i)}} \right), \]

with \( \Omega_0 = -\frac{\hat{p}_0(\hat{p}_0-i\alpha)}{\hat{p}_0^*e^{i\alpha}+\hat{p}_0} \), \( \hat{p}_0 = \sqrt{\hat{b}(\hat{b}-i)} + \hat{b} \) and \( \hat{b} = i(1+\sigma \rho_1 \rho_2) \). Here \( \rho_1 \) and \( \rho_2 \) are arbitrary real parameters that need to satisfy the conditions: \(-1 < \rho_1 \rho_2 < 0 \) for \( \sigma = 1 \) or \( 0 < \rho_1 \rho_2 < 1 \) for \( \sigma = -1 \), and \( a_r (r = 1, 2, ...) \) are arbitrary complex parameters.

**Remark 1.** We should point out here \( \sigma = \pm 1 \) in the MT model (1)–(2) can be normalized to +1. Nevertheless, we keep this \( \sigma \) in the formulation of rogue waves in this work.

### 3.1 Gram determinant solution for a higher dimensional bilinear system

In this subsection, we present Gram determinant solution for the higher dimensional bilinear Equations (14)–(17).
Lemma 1. Let \( m_{ij}^{(n,k,l)} \), \( \varphi_i^{(n,k,l)} \) and \( \psi_j^{(n,k,l)} \) be functions of variables \( x_1, x_{-1}, t_a \) and \( t_b \) satisfying the differential and difference relations as follows,

\[
\begin{align*}
\partial_{x_1} m_{ij}^{(n,k,l)} &= \mu_0 \varphi_i^{(n+1,k,l)} \psi_j^{(n,k,l)}, \\
\partial_{x_{-1}} m_{ij}^{(n,k,l)} &= -\mu_0 \varphi_i^{(n,k,l)} \psi_j^{(n+1,k,l)}, \\
\partial_{t_a} m_{ij}^{(n,k,l)} &= -\mu_0 \varphi_i^{(n+1,k-1,l)} \psi_j^{(n,k+1,l)}, \\
\partial_{t_b} m_{ij}^{(n,k,l)} &= -\mu_0 \varphi_i^{(n+1,k,l-1)} \psi_j^{(n,k+1,l)}.
\end{align*}
\]

(28)

\[
\begin{align*}
m_{ij}^{(n+1,k,l)} &= m_{ij}^{(n,k,l)} + \mu_0 \varphi_i^{(n+1,k,l)} \psi_j^{(n+1,k,l)}, \\
m_{ij}^{(n,k+1,l)} &= m_{ij}^{(n,k,l)} + \mu_0 \varphi_i^{(n+1,k,l)} \psi_j^{(n,k+1,l)}, \\
m_{ij}^{(n,k,l+1)} &= m_{ij}^{(n,k,l)} + \mu_0 \varphi_i^{(n+1,k,l)} \psi_j^{(n,k,l+1)},
\end{align*}
\]

and

\[
\begin{align*}
\partial_{x_1} \varphi_i^{(n,k,l)} &= \varphi_i^{(n+1,k,l)}, & \partial_{x_1} \psi_j^{(n,k,l)} &= -\psi_j^{(n-1,k,l)}, \\
\partial_{x_{-1}} \varphi_i^{(n,k,l)} &= \varphi_i^{(n-1,k,l)}, & \partial_{x_{-1}} \psi_j^{(n,k,l)} &= -\psi_j^{(n+1,k,l)}, \\
\partial_{t_a} \varphi_i^{(n,k,l)} &= \varphi_i^{(n,k-1,l)}, & \partial_{t_a} \psi_j^{(n,k,l)} &= -\psi_j^{(n,k+1,l)}, \\
\partial_{t_b} \varphi_i^{(n,k,l)} &= \varphi_i^{(n,k,l-1)}, & \partial_{t_b} \psi_j^{(n,k,l)} &= -\psi_j^{(n,k+1,l)}, \\
\varphi_i^{(n+1,k,l)} &= \varphi_i^{(n,k+1,l)} + a \varphi_i^{(n,k,l)}, & \psi_j^{(n+1,k,l)} &= \psi_j^{(n,k+1,l)} - a \psi_j^{(n+1,k+1,l)}, \\
\varphi_i^{(n+1,k,l)} &= \varphi_i^{(n,k,l+1)} + b \varphi_i^{(n,k,l)}, & \psi_j^{(n+1,k,l)} &= \psi_j^{(n,k,l+1)} - b \psi_j^{(n+1,k,l+1)}.
\end{align*}
\]

(29)

Then it is verified that the determinant

\[
\tau_{n,k,l} = \det_{1 \leq i,j \leq N} \left( m_{ij}^{(n,k,l)} \right),
\]

(30)

satisfies the following bilinear equations in the KP-Toda hierarchy

\[
(D_{x_1} + a) \tau_{n,k+1,l} \cdot \tau_{n+1,k+1,l} = a \tau_{n+1,k+1,l} \tau_{n,k,l},
\]

(31)

\[
(bD_{x_{-1}} + 1) \tau_{n,k,l+1} \cdot \tau_{n,k+1,l} = \tau_{n,k+1,l} \tau_{n-1,k,l+1},
\]

(32)

\[
(aD_{t_a} - 1) \tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k+1,l} \tau_{n,k,l+1},
\]

(33)

\[
(bD_{t_b} - 1) \tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k,l-1} \tau_{n,k,l+1}.
\]

(34)
**Proof.** By utilizing rules (28) and (29), one can check that the derivatives and shifts of the tau function are expressed by the bordered determinants as follows:

\[
\tau_{n+1,k,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k,j)} \\ -\mu_0 \psi_j^{(n+1,k,j)} & 1 \end{vmatrix}, \quad \tau_{n,k+1,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k,j)} \\ -\mu_0 \psi_j^{(n+1,k,j)} & 1 \end{vmatrix},
\]

\[
\partial_x \tau_{n,k,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & 0 \\ -\mu_0 \psi_j^{(n+1,k,j)} & 1 \end{vmatrix}, \quad \partial_y \tau_{n,k,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & 0 \\ -\mu_0 \psi_j^{(n+1,k,j)} & 1 \end{vmatrix},
\]

\[
\partial_x \tau_{n+1,k,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k+1,l)} \\ -\mu_0 \psi_j^{(n+1,k+1,l)} & 0 \end{vmatrix}, \quad \partial_y \tau_{n+1,k,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k+1,l)} \\ -\mu_0 \psi_j^{(n+1,k+1,l)} & 0 \end{vmatrix},
\]

\[
\partial_x \tau_{n,k+1,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k+1,l)} \\ -\mu_0 \psi_j^{(n+1,k+1,l)} & 0 \end{vmatrix}, \quad \partial_y \tau_{n,k+1,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k+1,l)} \\ -\mu_0 \psi_j^{(n+1,k+1,l)} & 0 \end{vmatrix},
\]

\[
\partial_x \tau_{n+1,k+1,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k+1,l)} \\ -\mu_0 \psi_j^{(n+1,k+1,l)} & 0 \end{vmatrix}, \quad \partial_y \tau_{n+1,k+1,l} = \begin{vmatrix} m_{ij}^{(n,k,j)} & \varphi_i^{(n+1,k+1,l)} \\ -\mu_0 \psi_j^{(n+1,k+1,l)} & 0 \end{vmatrix},
\]

Applying the Jacobi identity of determinants to these bordered determinants, the four bilinear Equations (31)–(34) are satisfied.

To construct the algebraic solutions, we first introduce \(m^{(n,k,l)}, \varphi^{(n,k,l)}\) and \(\psi^{(n,k,l)}\) as

\[
m^{(n,k,l)} = \frac{ip}{p+q} \left( \frac{-p}{q} \right)^n \left( \frac{-p-a}{q+a} \right)^k \left( \frac{-p-b}{q+b} \right)^l e^{\xi+\eta},
\]

\[
\varphi^{(n,k,l)} = p^n(p-a)^k(p-b)^l e^{\xi},
\]

\[
\psi^{(n,k,l)} = (-q)^{-n}[-(q+a)]^{-k}[-(q+b)]^{-l} e^{\eta},
\]
\[
\xi = \frac{1}{p} x_{-1} + px_1 + \frac{1}{p - a} t_a + \frac{1}{p - b} t_b + \xi_0,
\]
\[
\eta = \frac{1}{q} x_{-1} + qx_1 + \frac{1}{q + a} t_a + \frac{1}{q + b} t_b + \eta_0,
\]
where \( \mu_0 = i \), and \( p, q, \xi_0, \eta_0, a, b \) are complex constants. It is easy to find that these functions satisfy differential and difference rules (28) and (29) without indices \( i \) and \( j \).

Then, we define the elements
\[
m_{ij}^{(n,k,l)} = A_i B_j m^{(n,k,l)}, \quad \varphi_{i}^{(n,k,l)} = A_i \varphi^{(n,k,l)}, \quad \psi_{j}^{(n,k,l)} = B_j \psi^{(n,k,l)},
\]
where \( A_i \) and \( B_j \) are differential operators with respect to \( p \) and \( q \), respectively, as
\[
A_i = \frac{1}{i!} [f_1(p) \partial_p]_i, \quad B_j = \frac{1}{j!} [f_2(q) \partial_q]_j,
\]
and \( f_1(p), f_2(p) \) are arbitrary functions that will be determined by the dimensional reduction in the subsequent Subsection 3.2.

As operators \( A_i \) and \( B_j \) commute with operators \( \partial_{x_1}, \partial_{x_{-1}}, \partial_{t_a} \) and \( \partial_{t_b} \), these functions \( m_{ij}^{(n,k,l)}, \varphi_{i}^{(n,k,l)} \) and \( \psi_{j}^{(n,k,l)} \) still obey the differential and difference rules (28) and (29). From Lemma 1, it is known that for an arbitrary sequence of indices \( (i_1, i_2, \ldots, i_N; j_1, j_2, \ldots, j_N) \), the determinant
\[
\tau_{n,k,l} = \det_{1 \leq \nu, \mu \leq N} \left( m_{i\nu, j\mu}^{(n,k,l)} \right)
\]
satisfies the higher dimensional bilinear system (31)–(34).

3.2 Dimensional reduction

According to the generalized dimensional reduction technique developed in, we introduce the following linear differential operators
\[
\mathcal{L}_1 = \partial_{x_1} - b(a - b) \partial_{t_b}, \quad \mathcal{L}_2 = \partial_{x_{-1}} + \frac{a - b}{b} \partial_{t_a},
\]
by which the dimensional reduction conditions (18) and (19) become
\[
\mathcal{L}_1 \tau_{n,k,l} = C_1 \tau_{n,k,l}, \quad \mathcal{L}_2 \tau_{n,k,l} = C_2 \tau_{n,k,l}.
\]
It is straightforward that
\[
\mathcal{L}_1 m_{ij}^{(n,k,l)} = A_i B_j \mathcal{L}_1 m^{(n,k,l)} = A_i B_j [Q_{11}(p) + Q_{12}(q)] m^{(n,k,l)},
\]
\[
\mathcal{L}_2 m_{ij}^{(n,k,l)} = A_i B_j \mathcal{L}_2 m^{(n,k,l)} = A_i B_j [Q_{21}(p) + Q_{22}(q)] m^{(n,k,l)},
\]
where
\[
Q_{11}(p) = p - b + \frac{b(b - a)}{p - b}, \quad Q_{12}(q) = q + b + \frac{b(b - a)}{q + b},
\]
\[
Q_{21}(p) = \frac{1}{p} + \frac{a - b}{b(p - a)}, \quad Q_{22}(q) = \frac{1}{q} + \frac{a - b}{b(q + a)}.
\]

Using the Leibnitz rule, one has the operator relations
\[
A_i Q_{s,1}(p) = \sum_{\mu=0}^{i} \frac{1}{\mu!} \left[ (f_1 \partial_p)^{\mu} Q_{s,1}(p) \right] A_{i-\mu}, \quad B_i Q_{s,2}(q) = \sum_{\mu=0}^{i} \frac{1}{\mu!} \left[ (f_2 \partial_q)^{\mu} Q_{s,2}(q) \right] B_{i-\mu}, \quad (s = 1, 2),
\]
which imply
\[
L_s m^{(n,k,l)}_{ij} = \sum_{\mu=0}^{i} \frac{1}{\mu!} \left[ (f_1 \partial_p)^{\mu} Q_{s,1}(p) \right] m^{(n,k,l)}_{i-\mu,j} + \sum_{\nu=0}^{j} \frac{1}{\nu!} \left[ (f_2 \partial_q)^{\nu} Q_{s,2}(q) \right] m^{(n,k,l)}_{i,j-\nu}, \quad (s = 1, 2).
\]

Next, the specific functions \([f_1(p), f_2(q)]\) and values of \((p, q)\) need to be determined to guarantee that coefficients of certain indices vanish in the above summation. To this end, we solve the first two algebraic equations
\[
Q'_{11}(p) = 0, \quad Q'_{12}(q) = 0,
\]
and get the following simple roots:
\[
p_0 = \sqrt{b(b-a)} + b, \quad q_0 = \sqrt{b(b-a)} - b.
\]
It is noted that \(p_0\) and \(q_0\) are also simple roots of equations \(Q'_{21}(p) = 0\) and \(Q'_{22}(q) = 0\), respectively, as \(Q_{21}(p)\) and \(Q_{22}(q)\) are associated with \(Q_{11}(p)\) and \(Q_{12}(q)\) through the simple relations
\[
Q_{21}(p) = \frac{a}{b} [Q_{11}(p) - a + 2b]^{-1}, \quad Q_{22}(q) = \frac{a}{b} [Q_{12}(q) + a - 2b]^{-1}.
\]
Hence, the terms with \(\mu = \nu = 1\) on the right-hand side of (46) vanish at the point \((p_0, q_0)\).

Following the steps in the simple root case in Ref. [34], we need to solve the differential equations
\[
(f_1 \partial_p)^2 Q_{11}(p) = Q_{11}(p), \quad (f_2 \partial_q)^2 Q_{12}(q) = Q_{12}(q).
\]
Then the following functions can be derived
\[
\mathcal{W}_1(p) = \left( \frac{p - b}{\sqrt{b(b-a)}} \right)^{\pm 1}, \quad \mathcal{W}_2(q) = \left( \frac{q - b}{\sqrt{b(b-a)}} \right)^{\pm 1},
\]
and
\[
f_1(p) = \frac{\mathcal{W}_1(p)}{\mathcal{W}'_1(p)} = \pm (p - b), \quad f_2(q) = \frac{\mathcal{W}_2(p)}{\mathcal{W}'_2(p)} = \pm (q + b).
\]
Because the above two signs yield equivalent rogue wave solutions, we choose the positive sign in the following derivation. From the conditions (47) and (50), we can find that

\[ L_1 m_{ij}^{(n,k,l)} \bigg|_{p=p_0,q=q_0} = Q_{11}(p_0) \sum_{\mu=0, \mu \text{ : even}}^i \frac{1}{\mu!} m_{i-\mu,j}^{(n,k,l)} \bigg|_{p=p_0,q=q_0} + Q_{12}(q_0) \sum_{\nu=0, \nu \text{ : even}}^j \frac{1}{\nu!} m_{i,j-\nu}^{(n,k,l)} \bigg|_{p=p_0,q=q_0}. \] (53)

To prove the second dimensional reduction, noticing that \( Q_{21}(p) \) can be expressed as a function of \( Q_{11}(p) \), we consider a general function \( F[Q_{11}(p)] \equiv F(Q_{11}) \). By using the Faà di Bruno formula and the relation \( f_1 \partial_p = \partial \ln \mathcal{R}_1 \), we obtain

\[ (f_1 \partial_p)^l F(Q_{11}) = \delta_{ln}^{(i)} F[Q_{11}] = \sum_{m_1+2m_2+\cdots+lm_l=l} \frac{d^m F(Q_{11})}{d Q_{11}^m} \prod_{j=1}^l [(f_1 \partial_p)^j Q_{11}]^m_j (l!)^{-1} \prod_{i=1}^l m_i !(i!)^{m_i}, \] (54)

where \( \hat{m} = \sum_{i=1}^l m_i \). Furthermore, by using the conditions (47) and (50), one finds that

\[ (f_1 \partial_p)^l F[Q_{11}(p_0)] = 0, \] (55)

\[ (f_1 \partial_p)^l F[Q_{11}(p_0)] = \sum_{2m_2+\cdots+lm_l=l, m_1 = m_3 = \cdots = 0} \frac{d^m F(Q_{11})}{d Q_{11}^m} \prod_{j=1}^l [(f_1 \partial_p)^j Q_{11}]^m_j (l!)^{-1} \prod_{i=1}^l m_i !(i!)^{m_i}, \] (56)

The similar calculation for a general function \( F[Q_{12}(q)] \equiv F(Q_{12}) \) gives

\[ (f_2 \partial_q)^l F[Q_{12}(q_0)] = 0, \] (57)

\[ (f_2 \partial_q)^l F[Q_{12}(q_0)] = \sum_{2m_2+\cdots+lm_l=l, m_1 = m_3 = \cdots = 0} \frac{d^m F(Q_{12})}{d Q_{12}^m} \prod_{j=1}^l [(f_2 \partial_q)^j Q_{12}]^m_j (l!)^{-1} \prod_{i=1}^l m_i !(i!)^{m_i}, \] (58)

Applying the above formulas (55)–(58) to the specific functions \( F[Q_{11}(p)] = Q_{21}(p) \) and \( F[Q_{12}(q)] = Q_{22}(q) \), it follows that

\[ L_2 m_{ij}^{(n,k,l)} \bigg|_{p=p_0,q=q_0} = \sum_{\mu=0, \mu \text{ : even}}^i \frac{C_{1,\mu}[Q_{21}(p_0)]}{\mu!} m_{i-\mu,j}^{(n,k,l)} \bigg|_{p=p_0,q=q_0} + \sum_{\nu=0, \nu \text{ : even}}^j \frac{C_{2,\nu}[Q_{22}(q_0)]}{\nu!} m_{i,j-\nu}^{(n,k,l)} \bigg|_{p=p_0,q=q_0}. \] (59)
On the right-hand side of above equation, the coefficients in the first term of two summations are 
\[ C_{1,0}(Q_{21}(p_0)) = Q_{21}(p_0) \] and 
\[ C_{2,0}(Q_{22}(q_0)) = Q_{22}(q_0) \], respectively.

Next, we restrict the general determinant \( (40) \) to
\[
\tau_{n,k,l} = \text{det}_{1 \leq i,j \leq N}
\left( m_{2i-1,2j-1}^{(n,k,l)} \bigg|_{p=p_0,q=q_0} \right).
\] (60)

By using the contiguity relations \( (53) \) and \( (59) \) as in Ref. [13], we obtain
\[
\mathcal{L}_1 \tau_{n,k,l} = [Q_{11}(p_0) + Q_{12}(q_0)]N \tau_{n,k,l} = 4\sqrt{b(b-a)}N \tau_{n,k,l},
\] (61)
\[
\mathcal{L}_2 \tau_{n,k,l} = [Q_{21}(p_0) + Q_{22}(q_0)]N \tau_{n,k,l} = -\frac{4\sqrt{b(b-a)}}{ab}N \tau_{n,k,l},
\] (62)
which imply that the tau function \( (60) \) satisfies the dimensional reduction conditions \( (42) \).

### 3.3 Index reduction

Similar to the dimensional reduction procedure in the above subsection, we introduce the difference operator
\[
\Delta \tau_{n,k,l} = \tau_{n+1,k+1,l-1},
\] (63)
so that the index reduction condition \( (20) \) becomes
\[
\Delta \tau_{n,k,l} = e^{i\nu_0} \tau_{n,k,l}.
\] (64)

From the definition of element \( m_{i,j}^{(n,k,l)} \), we can find
\[
\Delta m_{i,j}^{(n,k,l)} = A_iB_j \Delta m^{(n,k,l)} = A_iB_j [Q_{31}(p) \times Q_{32}(q)] m^{(n,k,l)},
\] (65)
where
\[
Q_{31}(p) = \frac{p(p-a)}{p-b} = Q_{11}(p) - a + 2b, \quad Q_{32}(q) = -\frac{q+b}{q(q+a)} = -[Q_{12}(q) + a - 2b]^{-1},
\] (66)
which yield
\[
Q_{31}'(p) \bigg|_{p=p_0,q=q_0} = 0, \quad Q_{32}'(q) \bigg|_{p=p_0,q=q_0} = 0.
\] (67)

Furthermore, applying above formulas \( (55)-(58) \) to the specific functions \( F[Q_{11}(p)] = Q_{31}(p) \) and \( F[Q_{12}(q)] = Q_{32}(q) \) and utilizing general operator relations \( (45) \), we arrive at
\[
\Delta m_{i,j}^{(n,k,l)} \bigg|_{p=p_0,q=q_0} = \left[ \sum_{\mu = 0, \mu : \text{even}}^i \frac{C_{1,\mu}(Q_{31}(p_0))}{\mu!} \right] \times \left[ \sum_{\nu = 0, \nu : \text{even}}^j \frac{C_{2,\nu}(Q_{32}(q_0))}{\nu!} \right] m_{i-\mu,j-\nu}^{(n,k,l)} \bigg|_{p=p_0,q=q_0}.
\] (68)
It is easy to see that the coefficients of the first term in above two summations are $C_{1,0}[Q_{31}(p_0)] = Q_{31}(p_0)$ and $C_{2,0}[Q_{32}(q_0)] = Q_{32}(q_0)$. Thus, according to the calculation in Ref. [18], one has the following matrix relation
\[
\begin{bmatrix}
\Delta m_{(n,k,l)}^{(2i-1,2j-1)}
p=q_0=q_0
\end{bmatrix}_{1 \leq i,j \leq N} = L \begin{bmatrix}
m_{(n,k,l)}^{(2i-1,2j-1)}
p=p_0=q_0
\end{bmatrix}_{1 \leq i,j \leq N} U,
\]
where $L$ and $U$ are following lower and upper triangular matrices with diagonal entries as $Q_{31}(p_0)$ and $Q_{32}(q_0)$, respectively.

\[
L = \begin{pmatrix}
Q_{31}(p_0) & 0 & \cdots & 0 \\
\frac{C_{1,2}[Q_{31}(p_0)]}{2!} & Q_{31}(p_0) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{C_{1,2N-2}[Q_{31}(p_0)]}{(2N-2)!} & \frac{C_{1,2N-4}[Q_{31}(p_0)]}{(2N-4)!} & \cdots & Q_{31}(p_0)
\end{pmatrix},
\]
\[
U = \begin{pmatrix}
Q_{32}(q_0) & \frac{C_{2,2}[Q_{32}(q_0)]}{2!} & \cdots & \frac{C_{2,2N-2}[Q_{32}(q_0)]}{(2N-2)!} \\
0 & Q_{32}(q_0) & \cdots & \frac{C_{2,2N-4}[Q_{32}(q_0)]}{(2N-4)!} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & Q_{32}(q_0)
\end{pmatrix}.
\]
Taking determinant on both sides of Equation (69), we have
\[
\Delta \tau_{n,k,l} = [Q_{31}(p_0)Q_{32}(q_0)]^N \tau_{n,k,l} = \Omega_0^N \tau_{n,k,l},
\]
with $\Omega_0 = \frac{p_0(p_0-a)}{q_0(q_0+a)} = \frac{\sqrt{b(b-a)+b+b-a}}{\sqrt{b(b-a)+b-b-a}}$. If we further restrict $a$ and $b$ to be pure imaginary parameters, $\Omega_0$ becomes a complex number whose modulus is one. Hence, by setting $e^{i\xi} = \Omega_0^N$, the index reduction condition (64) is proved.

\[3.4 \quad \text{Complex conjugacy condition}\]

As mentioned previously, by taking $a$ and $b$ as the following pure imaginary numbers
\[
a = i\alpha, \quad b = i\alpha(1 + \sigma \rho_1 \rho_2),
\]
with the real constant $\alpha$, the roots (48) can be rewritten as
\[
p_0 = q_0^* = \sqrt{-\sigma \rho_1 \rho_2 \alpha^2(1 + \sigma \rho_1 \rho_2)} + i\alpha(1 + \sigma \rho_1 \rho_2),
\]
which implies that the condition for the existence of rogue wave solution: $-1 < \rho_1 \rho_2 < 0$ for $\sigma = 1$ or $0 < \rho_1 \rho_2 < 1$ for $\sigma = -1$ need to be satisfied. Besides, by imposing the parameter constraint $\eta_0 = \xi_0^*$ and noticing that the coordinate transformations $x_1 = \frac{\rho_2}{\rho_1} x$ and $x_{-1} = -\frac{\rho_1(1+\sigma \rho_1 \rho_2)}{\rho_2} t$ are
real, one can find that

$$
\left[ m_{k,j}^{(n,k,l)} \right]_{p=p_0,q=q_0}^{*} = \left[ m_{j,i}^{(-n-1,-k,-l)} \right]_{p=p_0,q=q_0},
$$

(73)

which implies $\tau_{n,k,l}^{*} = \tau_{-n-1,-k,-l}$. By setting $n = -1, k = 0, l = 0$, one has the complex conjugacy conditions

$$
\tau_{-1,0,0}^{*} = \tau_{0,0,0}, \quad \tau_{0,-1,0}^{*} = \tau_{-1,1,0}, \quad \tau_{0,0,-1}^{*} = \tau_{-1,0,1}.
$$

(74)

Finally, by defining the following variable transformations

$$
\tau_{0,0,0} = f, \quad \tau_{-1,0,0} = f^{*}, \quad \tau_{-1,1,0} = e^{i\varphi_{0}}g, \quad \tau_{0,-1,0} = e^{-i\varphi_{0}}g^{*}, \quad \tau_{-1,0,1} = h, \quad \tau_{0,0,-1} = h^{*},
$$

we arrive at exactly the bilinear Equations (10)–(13).

### 3.5 Rogue wave solutions in differential operator form

Finally, based on the technique in Refs. [18] and [34], one can introduce free parameters through the arbitrary parameter $\xi_{0}$. To be specific, we take parameter $\xi_{0}$ in the form

$$
\xi_{0} = \sum_{r=1}^{\infty} \hat{a}_{r} \ln^{r} \mathcal{W}_{1}(p) = \sum_{r=1}^{\infty} \hat{a}_{r} \ln^{r} \left( \frac{p-b}{\sqrt{b(b-a)}} \right),
$$

(75)

where $\ln^{r} \cdot = [\ln \cdot]^{r}$ and $\hat{a}_{r}$ are arbitrary complex parameters.

To summarize the above results and take dummy variables $t_{a}$ and $t_{b}$ as zeros, we have Theorem 2 for rogue wave solutions to the MT model (1)–(2).

### 3.6 Rogue wave solutions through Schur polynomials

In this subsection, rogue wave solutions will be presented by elementary Schur polynomials. Following the technique in Ref. [34], the extended generator $G$ of differential operators $[f_{1}\partial_{p}]^{i}[f_{2}\partial_{q}]^{j}$ is introduced as

$$
G = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \kappa^{i} \lambda^{j} \left[ f_{1}\partial_{p} \right]^{i} \left[ f_{2}\partial_{q} \right]^{j}
$$

(76)

which can be rewritten as

$$
G = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \kappa^{i} \lambda^{j} \left[ f_{1}\partial_{\ln \mathcal{W}_{1}} \right]^{i} \left[ f_{2}\partial_{\ln \mathcal{W}_{2}} \right]^{j} = \exp(\kappa \partial_{\ln \mathcal{W}_{1}} + \lambda \partial_{\ln \mathcal{W}_{2}}),
$$

(77)

through the transformation (52). According to the formula in Ref. [13], it is shown that for a function $F(\mathcal{W}_{1}, \mathcal{W}_{2})$, the following identity

$$
GF(\mathcal{W}_{1}, \mathcal{W}_{2}) = F(e^{\kappa} \mathcal{W}_{1}, e^{\lambda} \mathcal{W}_{2})
$$

(78)

holds.
In the MT model, \( p \) and \( q \) can be solved explicitly with respect to \( W_1 \) and \( W_2 \) from the relations (51)

\[
p = p(W_1) = W_1 \sqrt{b(b - a) + b}, \quad q = q(W_2) = W_2 \sqrt{b(b - a) - b},
\]

in which \( W_1 = W_2 = 1 \) when \( p = p_0 \) and \( q = q_0 \). Then, by applying the relation (78) to \( m^{(n,k,l)} \) at \( p = p_0, q = q_0 \), one has

\[
\mathcal{G} m^{(n,k,l)}|_{p=p_0,q=q_0} = \frac{(-1)^{n+k+l} p(\kappa) q(\lambda)}{p(\kappa) + q(\lambda)} \left[ \frac{p(\kappa)}{q(\lambda)} \right]^n \left[ \frac{p(\kappa) - a}{p_0 - a} \right]^k \left[ \frac{q(\lambda)}{q_0 + a} \right]^{k-1} \left[ \frac{p(\kappa) - b}{p_0 - b} \right] \left[ \frac{q(\lambda) + b}{q_0 + b} \right]^{-l} \times \exp \left\{ \frac{i \rho_2 x}{\rho_1} [p(\kappa) + q(\lambda)] + \frac{i \rho_1 b t}{\rho_2} \left[ \frac{1}{p(\kappa)} - \frac{1}{p_0} - \frac{1}{q(\lambda)} - \frac{1}{q_0} \right] + \sum_{r=1}^{\infty} \left\{ \hat{a}_r \kappa^r + \hat{a}_r^* \lambda^r \right\} \right\},
\]

where

\[
p(\kappa) \equiv p(W_1)|_{W_1 = e^{\kappa}} = e^{\kappa} \sqrt{b(b - a) + b}, \quad q(\lambda) \equiv q(W_2)|_{W_2 = e^{\lambda}} = e^{\lambda} \sqrt{b(b - a) - b}.
\]

As \( m^{(n,k,l)}|_{p=p_0,q=q_0} = (-1)^{n+k+l} \frac{p_0 + q_0}{p(\kappa) + q(\lambda)} \left[ \frac{p(\kappa) - p_0}{q(\lambda) - q_0} \right] \exp \left\{ \frac{i \rho_2 x}{\rho_1} [p(\kappa) - p_0 + q(\lambda) - q_0] + \frac{i \rho_1 b t}{\rho_2} \left[ \frac{1}{p(\kappa) - 1} \right] + \sum_{r=1}^{\infty} \left\{ \hat{a}_r \kappa^r + \hat{a}_r^* \lambda^r \right\} \right\},
\]

we deduce

\[
\mathcal{G} m^{(n,k,l)}|_{m^{(n,k,l)}|_{p=p_0,q=q_0}} = \frac{p_0 + q_0}{p(\kappa) + q(\lambda)} \left[ \frac{p(\kappa) - p_0}{q(\lambda) - q_0} \right] \exp \left\{ \frac{i \rho_2 x}{\rho_1} [p(\kappa) - p_0 + q(\lambda) - q_0] + \frac{i \rho_1 b t}{\rho_2} \left[ \frac{1}{p(\kappa) - 1} \right] + \sum_{r=1}^{\infty} \left\{ \hat{a}_r \kappa^r + \hat{a}_r^* \lambda^r \right\} \right\}.
\]

Next, the right-hand side of the above equation needs to be expanded in terms of power series of \( \kappa \) and \( \lambda \). By means of the techniques in Ref. [34], the first term is expressed as

\[
\frac{p_0 + q_0}{p(\kappa) + q(\lambda)} = \sum_{\gamma=0}^{\infty} \left( \frac{p_1 q_1}{p_0 + q_0} \right)^\gamma \kappa^\gamma \lambda^\gamma \exp \left( \sum_{r=1}^{\infty} (s_r - b_r) \kappa^r + (s_r^* - b_r^*) \lambda^r \right),
\]

where \( p_1 = \frac{d p(\kappa)}{d \kappa} |_{\kappa = 0} = \sqrt{b(b - a)} \) and \( q_1 = \frac{d q(\lambda)}{d \lambda} |_{\lambda = 0} = \sqrt{b(b - a)} \). The parameters \( s_r \) and \( b_r \) are the expansion coefficients of \( \kappa^r \) and \( \lambda^r \) as follows:

\[
\ln \left[ \frac{p_0 + q_0}{p(\kappa) - p_0} \right] = \ln \left[ \frac{\sqrt{2 e^{\kappa-1}}}{\sqrt{2 e^{\kappa+1}}} \right] = \sum_{r=1}^{\infty} s_r \kappa^r, \quad \ln \left[ \frac{p(\kappa) + q_0}{p_0 + q_0} \right] = \ln \left[ \frac{\sqrt{2 e^{\kappa+1}}}{\sqrt{2 e^{\kappa-1}}} \right] = \sum_{r=1}^{\infty} s_r^* \lambda^r.
\]

Hence, \( s_r^* = s_r, b_r^* = b_r \) and the term (82) is simplified as

\[
\frac{p_0 + q_0}{p(\kappa) + q(\lambda)} = \sum_{\gamma=0}^{\infty} \left( \frac{\kappa \lambda}{4} \right)^\gamma \exp \left( \sum_{r=1}^{\infty} (s_r - b_r)(\kappa^r + \lambda^r) \right).
\]
On the other hand, noticing that the complex conjugate relation \( q(\lambda) = \bar{p}^*(\lambda) \) and \( b = i\alpha(1 + \sigma\rho_1\rho_2) \equiv i\alpha\rho \), we have the following expansion expressions

\[
\frac{i\rho_2}{a\rho_1}[p(\kappa) - p_0] = \frac{\rho_2}{\rho_1} \left[ \sqrt{-\sigma\rho_1\rho_2\rho(e^\kappa - 1)} \right] = \sum_{r=1}^{\infty} \alpha_r \kappa^r,
\]

\[
\frac{i\rho_1 b}{\rho_2} \left[ \frac{1}{p(\kappa)} - \frac{1}{p_0} \right] = \frac{\rho_1}{\rho_2} \left[ \frac{\rho}{\sqrt{-\sigma\rho_1\rho_2\rho + i\rho}} - \frac{\rho}{e^\kappa\sqrt{-\sigma\rho_1\rho_2\rho + i\rho}} \right] = \sum_{r=1}^{\infty} \beta_r \kappa^r,
\]

\[
\ln \frac{p(\kappa)}{p_0} = \ln \frac{e^\kappa\sqrt{-\sigma\rho_1\rho_2\rho + i\rho}}{\sqrt{-\sigma\rho_1\rho_2\rho + i\rho}} = \sum_{r=1}^{\infty} \theta_r \kappa^r,
\]

\[
\ln \frac{p(\kappa) - a}{p_0 - a} = \ln \frac{e^\kappa\sqrt{-\sigma\rho_1\rho_2\rho + i\sigma\rho_1\rho_2}}{\sqrt{-\sigma\rho_1\rho_2\rho + i\sigma\rho_1\rho_2}} = \sum_{r=1}^{\infty} \vartheta_r \kappa^r,
\]

\[
\ln \frac{p(\kappa) - b}{p_0 - b} = \ln e^\kappa = \kappa = \sum_{r=1}^{\infty} \zeta_r \kappa^r.
\]

With the help of these expansions, the rest terms on the right-hand side of (81) are rewritten as

\[
\exp \left\{ \sum_{r=1}^{\infty} \left[ \alpha_r x + \beta_r t + (n + 1) \theta_r + l \zeta_r + a_r \right] \kappa^r + \sum_{r=1}^{\infty} \left[ \alpha_r x + \beta_r^* t - n \theta_r^* - k \zeta_r^* - l \zeta_r^* + a_r^* \right] \kappa^r \right\}.
\]

Thus, Equation (81) is rewritten as

\[
\frac{1}{m(n, k, l)} \mathcal{C}_{mn(k,l)}(p_0, q_0) = \sum_{\gamma=0}^{\infty} \left( \frac{x\lambda}{4} \right)^{\gamma} \exp \left\{ \sum_{r=1}^{\infty} (x_r^+(n, k, l) + y s_r) \kappa^r + \sum_{r=1}^{\infty} (x_r^-(n, k, l) + y s_r) \kappa^r \right\}, \quad (84)
\]

where \( x_r^+(n, k, l) \) are defined as

\[
x_r^+(n, k, l) = \alpha_r x + \beta_r t + (n + 1) \theta_r + k \zeta_r + a_r - b_r,
\]

\[
x_r^-(n, k, l) = \alpha_r x + \beta_r^* t - n \theta_r^* - k \zeta_r^* - l \zeta_r^* + a_r^* - b_r.
\]

Denoting

\[
a_r = \dot{a}_r - b_r + \frac{1}{2} \theta_r, \quad a_r^* = \dot{a}_r^* - b_r + \frac{1}{2} \theta_r^*,
\]

then the variables \( x_r^\pm(n, k, l) \) are reparameterized as

\[
x_r^+(n, k, l) = \alpha_r x + \beta_r t + (n + \frac{1}{2}) \theta_r + k \zeta_r + a_r,
\]

\[
x_r^-(n, k, l) = \alpha_r x + \beta_r^* t - (n + \frac{1}{2}) \theta_r^* - k \zeta_r^* - l \zeta_r^* + a_r^*.
\]
Taking the coefficients of $\lambda^i \lambda^j$ on both sides of Equation (84) and using the definition of Schur polynomial, one has

$$\frac{\hat{m}_{i,j}^{(n,k,l)}}{m^{(n,k,l)}_{p_0,q_0}} = \min_{i,j} \left| \frac{1}{4T} S_{i-\gamma}(x^+(n,k,l) + \gamma s) S_{j-\gamma}(x^-(n,k,l) + \gamma s), \right.$$  

\hspace{1cm} (85)

where $\hat{m}_{i,j}^{(n,k,l)}$ is the matrix element given in Theorem 2. Finally, using the gauge freedom of the tau function, we define the following determinant

$$\sigma_{n,k,l} = \frac{\tau_{n,k,l}}{\left( m^{(n,k,l)}_{p_0,q_0} \right)^N},$$  

(86)

which is also a rational solution to the MT model as presented in Theorem 1. Note that the denominator on the right-hand side of above equation is a gauge factor, which contains the ratio constant $\Omega_{N_0}^0$ in the index reduction. Hence, the function $g$ is changed into $g = \sigma_{-1,1,0}$ when rogue wave solutions are expressed through elementary Schur polynomials.

Finally, we comment that the algebraic Equations (47) actually allow another set of simple roots $(-p_0^*, -q_0^*)$, but it leads to the equivalent rogue wave solutions in the MT model (1)-(2). Indeed, one can find that when $p_0 \rightarrow -p_0^*$, then $p(\lambda) \rightarrow -p^*(\lambda), p_1 \rightarrow -p_1$ and the expansion coefficients are changed into

$$s_r \rightarrow s_r, \hspace{1cm} \alpha_r \rightarrow -\alpha_r, \hspace{1cm} \beta_r \rightarrow -\beta_r^*, \hspace{1cm} \theta_r \rightarrow \theta_r^*, \hspace{1cm} \vartheta_r \rightarrow -\vartheta_r^*, \hspace{1cm} \zeta_r \rightarrow \zeta_r.$$

Without loss of generality, the free parameters $a_r$ can be rewritten as $a_r^*$, it follows that in Theorem 2

$$[x^+_{r}(n,k,l)](x,t) \rightarrow [x^+_{r}(n,k,l)]^*(-x,-t), \hspace{1cm} m_{i,j}^{(n,k,l)}(x,t) \rightarrow [m_{i,j}^{(n,k,l)}]^*(-x,-t),$$  

(87)

which imply $\sigma_{n,k,l}(x,t) \rightarrow \sigma_{n,k,l}^*(-x,-t)$. Consider the invariance of plane waves $F(x,t) \rightarrow F^*(-x,-t)$ in the variable transformations (4), then

$$u(x,t) \rightarrow u^*(-x,-t), \hspace{1cm} v(x,t) \rightarrow v^*(-x,-t),$$  

(88)

also satisfy the MT model (1)-(2) in Theorem 2. However, the MT model (1)-(2) is invariant under the variable transformation (88). Hence, another group of roots $(-p_0^*, -q_0^*)$ gives rise to the equivalent rogue wave solutions through the appropriate parameter connections. This property is similar to that in the three-wave interaction system. 34
## 4 DYNAMICS OF ROGUE WAVE SOLUTIONS

### 4.1 First-order (fundamental) rogue wave solution

According to Theorem 1, when \( N = 1 \), we obtain the first-order rogue wave solution as

\[
u = \rho_2 e^{i\phi_0} \left[ 1 + \frac{(2d_1^* - d_0)(L_1 + d_1) - (2d_1 - d_0)(L_1^* - d_1^*) - |2d_1 - d_0|^2}{(L_1 + d_1)(L_1^* - d_1^*) + \frac{d_0^2}{4}} \right],
\]

(89)

\[
v = \rho_2 e^{i\phi_0} \left[ 1 - \frac{d_2^*(L_1 - d_1) - d_2(L_1^* + d_1^*) + |d_2|^2}{(L_1 - d_1)(L_1^* + d_1^*) + \frac{d_0^2}{4}} \right],
\]

(90)

with

\[
L_1 = \frac{\rho_2}{\rho_1} x + \frac{4\rho_1 d_1^2}{\rho_2} t, \quad d_1 = \frac{1}{2(\beta + i\rho)}, \quad d_2 = \frac{1}{(\beta + i\rho_1\rho_2)}, \quad d_0 = \frac{1}{\hat{\rho}}.
\]

\[
\phi_0 = \rho \left( \frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t \right), \quad \beta = \sqrt{-\sigma \rho_1 \rho_2 \rho}, \quad \rho = 1 + \sigma \rho_1 \rho_2.
\]

It is quite clear that the structure of fundamental rogue wave is only determined by the background parameters \( \rho_1 \) and \( \rho_2 \) with the condition \( \sigma \rho_1 \rho_2 (1 + \sigma \rho_1 \rho_2) < 0 \). The intensities \(|u|\) and \(|v|\) tend to the backgrounds \(|\rho_1|\) and \(|\rho_2|\) as \( x, t \to \infty \). Through the local analyses, we can find that \(|u|^2\) possesses one local maximum at \((x_1, t_1)\) with the peak amplitude \(9\rho_1^2\) and two local minima at characteristic points \((x_±^2, t_±^2)\) with the zero-amplitude, while \(|v|^2\) has one local maximum at \((x_1, t_1)\) with the peak amplitude \(9\rho_2^2\) and two local minima at characteristic points \((x_±^3, t_±^3)\) with the zero-amplitude. Here three critical points are exactly defined by

\[
(x_1, t_1) = (0, 0), \quad (x_±^2, t_±^2) = \left( \pm \frac{\Delta_1}{\rho_1}, \pm \frac{3\rho_2 \Delta_1}{\rho_1^3} \right), \quad (x_±^3, t_±^3) = \left( \pm \frac{3\rho_1 \Delta_2}{\rho_2^3}, \pm \frac{\Delta_2}{\rho_1} \right),
\]

(91)

with \(\Delta_1 = \sqrt{-\frac{3\rho_1}{16\rho_2(3\rho_1 \rho_2 + 4\sigma)}}\) and \(\Delta_2 = \sqrt{-\frac{3\rho_2}{16\rho_1(3\rho_1 \rho_2 + 4\sigma)}}\). Hence, the ratios between the peak and background values for two components are equal to 3. For the rogue waves of the components \(u\) and \(v\), the orientation angles are \(\Theta_u = \arctan \frac{-\rho_1^2}{3\rho_2}\) and \(\Theta_v = \arctan \frac{-\rho_2^2}{3\rho_1}\), and the wave widths (i.e., the distance between two minima points) are \(W_u = \frac{-3(\rho_1^4 + 9\rho_1^2)}{4\rho_1^2 \rho_2^2 (3\rho_1 \rho_2 + 4\sigma)}\) and \(W_v = \frac{-3(9\rho_1^4 + \rho_2^4)}{4\rho_1^2 \rho_2^2 (3\rho_1 \rho_2 + 4\sigma)}\), which indicate durations of the rogue waves. Thus, the background parameters \(\rho_1\) and \(\rho_2\) do not affect the height of peak, but influence the orientation and duration of two Peregrine solitons, which are observed graphically with \(\sigma = 1\).\(^{45}\) Besides, the appropriate scaling transformations (7) may bring the parameter \(\mu\), which is physically important in the relativistic field theory and optics. Then it is obvious that the orientation angles and wave widths depend on \(\mu\), which naturally results in the rotation and lengthening of the fundamental rogue wave. To illustrate the local structure of this rogue waves, we normalize \(\rho_2 = 1\) and then \(-1 < \sigma \rho_1 < 0\). Figure 1 exhibits the amplitude profiles of two rogue waves, and Figure 2a,b displays the orientation angles and wave widths with respect to \(\rho_1\), respectively.
It is noted that the rogue waves in the MT model are different from the ones in the DNLS equation in spite of the fact that both systems belong to the same integrable hierarchy. For the rogue wave solutions of the DNLS equation, an internal parameter $\alpha$ exists and affects the structure of rogue wave dramatically. In the case of the MT model, the same parameter $\alpha$ is introduced in the process of constructing rogue wave solutions as shown in Subsection 3.4, but this parameter is removed finally and only two free parameters $\rho_1$ and $\rho_2$ control the shape of rogue wave.

In addition, unlike other integrable coupled systems, the MT model involving two components only supports the bright-type rogue wave. In general, due to the additional degrees of freedom in coupled systems, there are more abundant patterns of rogue wave such as dark and four-petaled flower structures. For the coupled MT model, the fundamental rogue wave solution does not contain extra parameters that can change the numbers of critical point in intensities, thus only allows the bright-type rogue wave.

### 4.2 High-order rogue wave solutions

The second-order rogue wave solution is obtained from Theorem 1 with $N = 2$. In this case, the tau functions $f, g$ and $h$ are given by

\[
\begin{align*}
\tau &= \begin{bmatrix} m_{11}^{(0,0,0)} & m_{13}^{(0,0,0)} \\ m_{31}^{(0,0,0)} & m_{33}^{(0,0,0)} \end{bmatrix}, \\
g &= \begin{bmatrix} m_{11}^{(-1,1,0)} & m_{13}^{(-1,1,0)} \\ m_{31}^{(-1,1,0)} & m_{33}^{(-1,1,0)} \end{bmatrix}, \\
h &= \begin{bmatrix} m_{11}^{(-1,0,1)} & m_{13}^{(-1,0,1)} \\ m_{31}^{(-1,0,1)} & m_{33}^{(-1,0,1)} \end{bmatrix},
\end{align*}
\]
where the elements are determined by

\[
m_{11}^{(n,k,l)} = x_1^+ \hat{x}_1 - \frac{1}{4}, \quad m_{13}^{(n,k,l)} = x_1^+ \hat{x}_1 - \frac{1}{8}, \quad m_{31}^{(n,k,l)} = x_1^- \hat{x}_1 - \frac{1}{8},
\]

\[
m_{33}^{(n,k,l)} = x_1^+ \hat{x}_1 - \frac{1}{16} \hat{x}_2^+ \hat{x}_2 + \frac{1}{16} (x_1^+ 2s_1) (x_1^- 2s_1) + \frac{1}{64},
\]

\[
\hat{x}_1^+ = \frac{1}{6} (x_1^+)^3 + x_1^+ \hat{x}_2^+ + x_1^+ \hat{x}_1, \quad \hat{x}_1^- = \frac{1}{6} (x_1^-)^3 + x_1^- \hat{x}_2^- + x_3^-,
\]

\[
\hat{x}_2^+ = \left(x_1^+ + s_1\right)^2 + 2(s_2 + x_2^+), \quad \hat{x}_2^- = \left(x_1^- + s_1\right)^2 + 2(s_2 + x_2^-),
\]

with the following coefficients:

\[
a_1 = a_2 = 0, \quad a_3 = \frac{\rho_2 \hat{\rho}}{\rho_1}, \quad a_2 = \frac{\rho_2 \hat{\rho}}{2 \rho_1}, \quad a_3 = \frac{\rho_2 \hat{\rho}}{6 \rho_1},
\]

\[
\beta_1 = \frac{\rho_1 \rho \hat{\rho}}{\rho_2 (\hat{\rho} + i \rho)^2}, \quad \beta_2 = \frac{\rho_1 \rho^2 (\sigma \rho_1 \rho_2 + i \hat{\rho})}{2 \rho_2 (\hat{\rho} + i \rho)^3}, \quad \beta_3 = \frac{\rho_1 \rho (4i \sigma \rho_1 \rho_2 \rho^2 - \hat{\rho} \rho^2 + \hat{\rho}^3)}{6 \rho_2 (\hat{\rho} + i \rho)^4},
\]

\[
\theta_1 = \frac{\hat{\rho}}{\rho + i \rho}, \quad \theta_2 = \frac{i \rho \rho}{2 (\hat{\rho} + i \rho)^2}, \quad \theta_3 = \frac{i \rho^2 (\rho - 1 + i \hat{\rho})}{6 (\hat{\rho} + i \rho)^3},
\]

\[
\gamma_1 = 1, \quad \gamma_2 = \gamma_3 = s_1 = s_3 = 0, \quad s_2 = -\frac{1}{12}, \quad \hat{\rho} = \sqrt{-\sigma \rho_1 \rho_2 \rho}, \quad \rho = 1 + \sigma \rho_1 \rho_2.
\]

From these explicit expressions, each tau function is a polynomial of degree six with respect to variables \(x\) and \(t\). Three groups of the second-order rogue waves with different values of parameters \(a_3\) and \(\rho_1\) are illustrated in Figure 3. The first group (Figure 3a) exhibits that a second-order rogue wave consists of three separating fundamental rogue waves and their distribution is a triangle array observably. In the second group (Figure 3b), the parameter \(\rho_1\) takes the same value as the one in the first group except \(a_3 = 0\), thus, it describes the parity-time-symmetric or super rogue wave in which three separate peaks come together and form a sole huge peak with a relative amplitude of 5. The last group (Figure 3c) also displays the super rogue wave except for a different value of \(\rho_1\). It is seen that the parameter \(a_3\) controls the arrangement pattern of three individual first-order rogue waves, while the parameters \(\rho_1\) and \(\rho_2\) determine the orientations and durations of these rogue waves.

To obtain third- and higher-order rogue waves, one need to take \(N \geq 3\) in Theorem 1. Thus, the corresponding tau functions are polynomials of the variables \(x\) and \(t\) with higher degree, which are too tedious to list here. In Figure 4, three groups of the third-order rogue waves \((N = 3)\) are presented graphically with a fixed value of \(\rho_1\) but different values of \(a_3\) and \(a_5\). One can observe that when the parameters value \((a_3, a_5) \neq (0, 0)\), the third-order rogue wave exhibits the superposition of six fundamental ones and they obey different arrangements with different values of \(a_3\) and \(a_5\). Specifically, the first group shows a triangle pattern with \((a_3, a_5) = (50, 0)\) (Figure 4a), while the second group indicates a pentagon pattern with \((a_3, a_5) = (0, 500)\) (Figure 3b). In the last group (Figure 4c), the parameters value \((a_3, a_5) = (0, 0)\) corresponds to the third-order super rogue wave with a relative maximum amplitude of 7. Furthermore, the fourth- and fifth-order rogue waves are illustrated in Figures 5 and 6 with different parameters values.
FIGURE 3  Second-order rogue waves with the parameters $\sigma = -1$. (a) a triangle pattern with $a_3 = 50$ and $\rho_1 = 0.5$; (b) a super rogue wave with $a_3 = 0$ and $\rho_1 = 0.5$; (c) a super rogue wave with $a_3 = 0$ and $\rho_1 = 0.9$.

FIGURE 4  Third-order rogue waves with the parameters $\sigma = -1$ and $\rho_1 = 0.5$. (a) a triangle pattern with $(a_3, a_5) = (50, 0)$; (b) a pentagon pattern with $(a_3, a_5) = (0, 500)$; (c) a super rogue wave with $(a_3, a_5) = (0, 0)$. 
From Figures 1, 3–6 for the first- to fifth-order rogue waves, it can be concluded that the $N$th-order rogue wave in both components contains $N(N + 1)/2$ fundamental ones in the usual case $[(a_3, a_5, ..., a_{2N-1}) \neq (0, 0, ..., 0)]$, while the relative amplitude of the huge peak is $2N + 1$ in the super case $[(a_3, a_5, ..., a_{2N-1}) = (0, 0, ..., 0)]$. The parameters $\rho_1$ and $\rho_2$ determine the orientation and duration of each rogue wave in both cases. However, the arrangement of each individual rogue waves completely depends on the parameters $a_3, a_5, ..., a_{2N-1}$ in the usual $N$th-order rogue wave. In addition, it is found that the intensities of both components in the MT model possess the same relative amplitude of each peak in both the usual and super cases, especially the arrangement of fundamental rogue wave is the same in the usual $N$th-order case. The differences between the two components are the orientation and duration of each peak in either the usual or super rogue wave.
5 | ROGUE WAVE PATTERNS WITH A SINGLE LARGE INTERNAL PARAMETER

A rogue wave pattern refers to the arrangement shape of fundamental rogue waves in a high-order case. More recently, the universal rogue wave patterns have been studied by Yang and Yang\(^{48,49}\) in the integrable systems such as the NLS equation, the derivative NLS equation, the Boussinesq equation and the Manakov system. They predicted analytically that when one of the internal parameters in a higher order rogue wave is large, then this rogue wave exhibits certain geometric pattern with Peregrine waves forming shapes such as a triangle, pentagon, heptagon and so on with a possible lower order rogue wave located at its center. Through the asymptotical analysis, these patterns were shown to be associated with the root structure of the Yablonskii–Vorob’ev polynomial hierarchy under linear transformations. In the MT model case, rogue wave patterns also agree with the universal laws existed in other integrable systems;\(^{48,49}\) the analytical confirmations will be given in this section. To this end, through the similar treatment as the NLS case that all \((x_2^\pm, x_4^\pm, x_6^\pm, \ldots)\) are taken as zeros,\(^{48}\) we rewrite general rogue wave expressions in Theorem 1 as follows:

**Theorem 4.** The \(N\)th-order rogue waves in the MT model (1)–(2) are given by

\[
\begin{align*}
u_N(x, t) &= \rho_1 \frac{f_N}{f_N} e^{i(1 + \sigma \rho_1 \rho_2)} \left( \frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t \right), \\
\nu_N(x, t) &= \rho_2 \frac{h_N}{f_N} e^{i(1 + \sigma \rho_1 \rho_2)} \left( \frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t \right),
\end{align*}
\]

(93)

where

\[
\begin{align*}
f_N &= \sigma_{0,0,0}, \quad f_N^* = \sigma_{-1,0,0}, \quad g_N = \sigma_{-1,1,0}, \quad h_N = \sigma_{-1,0,1},
\end{align*}
\]

(94)

and the elements in determinant \(\sigma_{n,k,l} = \det_{1 \leq i,j \leq N}(m_{2i-1,2j-1}^{(n,k,l)})\)

\[
m_{i,j}^{(n,k,l)} = \sum_{\gamma=0}^{\min(l,j)} \frac{1}{4^\gamma} S_{1-\gamma}(x^+(n, k, l) + \gamma s) S_{j-\gamma}(x^-(n, k, l) + \gamma s),
\]

(95)

with vectors \(x^+ (n, k, l) = (x_1^+, x_2^+, \ldots)\) and \(x^- (n, k, l) = (x_1^-, x_2^-, \ldots)\) being defined by

\[
\begin{align*}x_1^+ &= \frac{\rho_2}{\rho_1} x + \frac{\rho_1 \rho \hat{\rho}}{\rho_2 (\hat{\rho} + i \rho)^2} t + (n + \frac{1}{2}) \frac{\hat{\rho}}{\hat{\rho} + i \rho} + k \frac{\hat{\rho}}{\hat{\rho} + i \sigma \rho_1 \rho_2} + l, \\
x_1^- &= \frac{\rho_2}{\rho_1} x + \frac{\rho_1 \rho \hat{\rho}}{\rho_2 (\hat{\rho} - i \rho)^2} t - (n + \frac{1}{2}) \frac{\hat{\rho}}{\hat{\rho} - i \rho} - k \frac{\hat{\rho}}{\hat{\rho} - i \sigma \rho_1 \rho_2} - l, \\
x_{2r+1}^+ &= \alpha_{2r+1} x + \beta_{2r+1} t + (n + \frac{1}{2}) \theta_{2r+1} + k \theta_{2r+1} + a_{2r+1}, \quad r \geq 1, \\
x_{2r+1}^- &= \alpha_{2r+1} x + \beta_{2r+1}^* t - (n + \frac{1}{2}) \theta_{2r+1}^* - k \theta_{2r+1}^* + a_{2r+1}^*, \quad r \geq 1, \\
x_{2r}^- &= 0, \quad r \geq 1.
\end{align*}
\]
These parameters $\alpha_r, \beta_r, \theta_r, \vartheta_r,$ and $s = (0, s_2, 0, s_4 \cdots)$ are coefficients from the following expansions
\[
\frac{\rho^2}{\rho_1} \beta(e^x - 1) = \sum_{r=1}^{\infty} \alpha_{r, \rho}, \quad \frac{\rho_1 \rho}{\rho_2} \left[ \frac{1}{\rho + i \rho} - \frac{1}{e^{x \rho} + i \rho} \right] = \sum_{r=1}^{\infty} \beta_{r, \rho},
\]
\[
\ln \frac{e^{x \rho} + i \rho}{\rho + i \rho} = \sum_{r=1}^{\infty} \theta_{r, \rho}, \quad \ln \frac{2 e^{x \rho} - 1}{\rho e^{x \rho} + 1} = \sum_{r=1}^{\infty} \vartheta_{r, \rho},
\]
with $\rho = \sqrt{-\sigma \rho_1 \rho_2}$ and $\rho = 1 + \sigma \rho_1 \rho_2$. Here $\rho_1$ and $\rho_2$ are arbitrary real parameters that satisfy the conditions: $-1 < \rho_1 \rho_2 < 0$ for $\sigma = 1$ or $0 < \rho_1 \rho_2 < 1$ for $\sigma = -1$, and $a_3, a_5, \ldots, a_{2N-1}$ are arbitrary irreducible complex parameters.

To show the connection between the rogue wave pattern and root structure of the Yablonskii–Vorob’ev polynomials, we use the notations of the Yablonskii–Vorob’ev polynomial hierarchy in. 49

Then, the asymptotics of rogue waves with the large $a_{2m+1}$ in the MT model is summarized by the following theorem.

**Theorem 5.** Let $[u_N(x, t), v_N(x, t)]$ be the $N$th order rogue wave of the MT model in Equation (93) where $a_{2m+1}$ is large and the other internal parameters are $O(1)$.

1. Far away from the origin, with $\sqrt{x^2 + t^2} = O(1)$, this $[u_N(x, t), v_N(x, t)]$ asymptotically separates into $N_p$ fundamental rogue waves, where $N_p = \frac{1}{2}[N(N + 1) - N_0(N_0 + 1)]$. These fundamental rogue waves are $[\hat{u}_1(x - \hat{x}_0, t - \hat{t}_0) e^{i(1 + \sigma \rho_1 \rho_2)(\frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t)}$, $\hat{v}_1(x - \hat{x}_0, t - \hat{t}_0) e^{i(1 + \sigma \rho_1 \rho_2)(\frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t)}]$, where

\[
\hat{u}_1(x, t) = \rho_1 \left[ \frac{\rho_2}{\rho_1} x + \frac{\rho_1 \rho}{\rho_2 (\rho - \rho')^2} t + \frac{1}{\rho - \rho'} - \frac{1}{\rho_1 \rho_2} \right],
\]
\[
\hat{v}_1(x, t) = \rho_2 \left[ \frac{\rho_2}{\rho_1} x + \frac{\rho_1 \rho}{\rho_2 (\rho - \rho')^2} t + \frac{1}{\rho - \rho'} - \frac{1}{\rho_1 \rho_2} \right],
\]
with $\rho = \sqrt{\sigma \rho_1 \rho_2}$ and $\rho = 1 + \sigma \rho_1 \rho_2$. The positions $(\hat{x}_0, \hat{t}_0)$ of these fundamental rogue waves are given by

\[
\hat{x}_0 = -\frac{\hat{\rho}}{\sigma \rho_1 \rho_2^2} \Re(z_0 \Omega) + \frac{1}{2} \frac{\rho_1 \rho_2}{2 \sigma \rho_2^2} \Im(z_0 \Omega),
\]
\[
\hat{t}_0 = \frac{1}{2} \frac{\rho_1 \rho_2}{2 \sigma \rho_2^2} \Im(z_0 \Omega),
\]
with $z_0$ being any of the $N_p$ nonzero simple roots of $Q_N^{[m]}(z)$, and $(\Re, \Im)$ representing the real and imaginary parts of a complex number. The error of this fundamental rogue wave approximation is $O(|a_{2m+1}|^{-1/2)}).$ Expressed mathematically, when $|a_{2m+1}| \gg 1$ and $[(x - \hat{x}_0)^2 + (t - \hat{t}_0)^2]^{1/2} = O(1)$, we have the following solution asymptotics

\[
u_N(x, t; a_3, a_5, \ldots, a_{2N-1}) = \hat{u}_1(x - \hat{x}_0, t - \hat{t}_0) e^{i(1 + \sigma \rho_1 \rho_2)(\frac{\rho_2}{\rho_1} x + \frac{\rho_1}{\rho_2} t)} + O(|a_{2m+1}|^{-1/2)}),
\]
\[ v_N(x, t; a_3, a_5, \ldots, a_{2N-1}) = \hat{v}_1(x - \hat{x}_0, t - \hat{t}_0) e^{i(1 + \sigma \rho_1 \rho_2) \left( \frac{\rho_2 x + \rho_1 t}{\rho_1 \rho_2} \right)} + O(|a_{2m+1}|^{-1/(2m+1)}). \]  

(101)

When \((x, t)\) is not in the neighborhood of any of these \(N_p\) fundamental waves, or \(\sqrt{x^2 + t^2}\) is larger than \(O(|a_{2m+1}|^{-1/(2m+1)})\), then \([u_N(x, t), v_N(x, t)]\) asymptotically approaches the constant-amplitude background \([\rho_1 e^{i(1 + \sigma \rho_1 \rho_2) \left( \frac{\rho_2 x + \rho_1 t}{\rho_1 \rho_2} \right)}, \rho_2 e^{i(1 + \sigma \rho_1 \rho_2) \left( \frac{\rho_2 x + \rho_1 t}{\rho_1 \rho_2} \right)}]\) as \(|a_{2m+1}| \to \infty\).

(2) In the neighborhood of the origin, where \(\sqrt{x^2 + t^2} = O(1)\), \([u_N(x, t), v_N(x, t)]\) is asymptotically a lower \(N_0\)th order rogue wave \([u_{N_0}(x, t), v_{N_0}(x, t)]\), where \(N_0\) is obtained from \((N, m)\) by the formula

\[ N_0 = \begin{cases} 
N \mod (2m + 1), & \text{if } 0 \leq N \mod (2m + 1) \leq m \\
2m - [N \mod (2m + 1)], & \text{if } N \mod (2m + 1) > m.
\end{cases} \]  

(102)

with \(0 \leq N_0 \leq N - 2\), and \([u_{N_0}(x, t), v_{N_0}(x, t)]\) is given by Equation (93) with its internal parameters \((a_3, a_5, \ldots, a_{2N_0-1})\) being the first \(N_0 - 1\) values in the parameter set \(a_3, a_5, \ldots, a_{2N-1}\) of the original rogue wave \([u_N(x, t), v_N(x, t)]\). The error of this lower order rogue wave approximation is \(O(|a_{2m+1}|^{-1})\). Expressed mathematically, when \(|a_{2m+1}| \gg 1\) and \(\sqrt{x^2 + t^2} = O(1)\),

\[ u_N(x, t; a_3, a_5, \ldots, a_{2N-1}) = u_{N_0}(x, t; a_3, a_5, \ldots, a_{2N_0-1}) + O(|a_{2m+1}|^{-1}), \]  

(103)

\[ v_N(x, t; a_3, a_5, \ldots, a_{2N-1}) = v_{N_0}(x, t; a_3, a_5, \ldots, a_{2N_0-1}) + O(|a_{2m+1}|^{-1}). \]  

(104)

If \(N_0 = 0\), then there will not be such a lower order rogue wave in the neighborhood of the origin, and \([u_N(x, t), v_N(x, t)]\) asymptotically approaches the constant-amplitude background \([\rho_1 e^{i(1 + \sigma \rho_1 \rho_2) \left( \frac{\rho_2 x + \rho_1 t}{\rho_1 \rho_2} \right)}, \rho_2 e^{i(1 + \sigma \rho_1 \rho_2) \left( \frac{\rho_2 x + \rho_1 t}{\rho_1 \rho_2} \right)}]\) as \(|a_{2m+1}| \to \infty\).

Proof. As \(|a_{2m+1}| \gg 1\) and other parameters are of \(O(1)\) in the MT rogue wave solution (93) at \((x, t)\) where \(\sqrt{x^2 + t^2} = O(1)\) \(|a_{2m+1}|^{1/(2m+1)}\), one has

\[ S_j(x^+(n, k, l) + \gamma s) = S_j(v) \left[ 1 + O \left( \frac{1}{a_{2m+1}} \right) \right], \quad |a_{2m+1}| \gg 1, \]  

(105)

where

\[ v = \left( \frac{\rho_2 \rho}{\rho_1} x + \frac{\rho_1 \rho}{\rho_2} (\rho + i \rho)^{2} t, 0, \ldots, 0, a_{2m+1}, 0, \ldots \right). \]  

(106)

From the definition of Schur polynomials, one can see that \(S_j(v)\) is related to the polynomial \(p_j^{[m]}(z)\) as

\[ S_j(v) = \Omega_j p_j^{[m]}(z), \]  

(107)

where

\[ z = \Omega^{-1} \left( \frac{\rho_2 \rho}{\rho_1} x + \frac{\rho_1 \rho}{\rho_2} (\rho + i \rho)^{2} t \right), \quad \Omega = \left( -\frac{2m+1}{2^{2m}} a_{2m+1} \right)^{1/(2m+1)}. \]  

(108)
Similar relations can also be obtained for $S_j(\vec{x}^- (n, k, l) + \gamma \vec{s})$. Using these formulae and the similar steps as in Ref. [48], we find that

$$\sigma_{n,k,l} = \chi_1 |a_{2m+1}|^{\frac{N(N+1)}{2m+1}} |Q_N^{[m]}(z)|^2 \left[ 1 + O\left( a_{2m+1}^{-2/(2m+1)} \right) \right], |a_{2m+1}| \gg 1,$$

where

$$\chi_1 = c_N^{-2} \left( \frac{1}{2} \right)^{\frac{N(N-1)}{2m+1}} \left( \frac{2m+1}{2^{2m}} \right)^{\frac{N(N+1)}{2m+1}}.$$

As $\chi_1$ is independent of $n$, $k$ and $l$, the above equation shows that for large $a_{2m+1}$, $[u_N(x, t), v_N(x, t)]$ would be on the unit background except at or near $(x, t)$ locations $(\hat{x}_0, \hat{t}_0)$ where

$$z_0 = \Omega^{-1} \left( \frac{\rho_2 \rho \hat{x}_0 + \rho_1 \rho \hat{t}_0}{\rho_2 (\frac{\rho}{\rho_2} + i \rho)^2 \hat{t}_0} \right)^{\frac{1}{2}}$$

is a root of the polynomial $Q_N^{[m]}(z)$.

Next, we show that in the neighborhood of each $(\hat{x}_0, \hat{t}_0)$ location, that is, $[(x - \hat{x}_0)^2 + (t - \hat{t}_0)^2]^{1/2} = O(1)$, the rogue wave $[u_N(x, t), v_N(x, t)]$ approaches the fundamental rogue wave $[\hat{u}_1(x - \hat{x}_0, t - \hat{t}_0)e^{i(1+\sigma \rho_1 \rho_2)(\rho_1 x + \rho_2 t)}, \hat{v}_1(x - \hat{x}_0, t - \hat{t}_0)e^{i(1+\sigma \rho_1 \rho_2)(\rho_2 x + \rho_1 t)}]$ for large $a_{2m+1}$. To this end, we notice that when $(x, t)$ is in the neighborhood of $(\hat{x}_0, \hat{t}_0)$, we have a more refined asymptotics for $S_j(\vec{x}^+(n, k, l) + \gamma \vec{s})$ as

$$S_j(\vec{x}^+(n, k, l) + \gamma \vec{s}) = S_j(\vec{\psi}) \left[ 1 + O\left( a_{2m+1}^{-2/(2m+1)} \right) \right],$$

where

$$\vec{\psi} = (x_1^+, 0, ..., 0, a_{2m+1}, 0, ...).$$

The polynomial $S_j(\vec{\psi})$ is related to $p_j^{[m]}(z)$ as $S_j(\vec{\psi}) = \Omega_j p_j^{[m]}(\Omega^{-1} x_1^+)$, then we get refined asymptotics for $S_j(\vec{x}^+(n, k, l) + \gamma \vec{s})$ through polynomials $p_j^{[m]}(z)$. Similar refined asymptotics can also be obtained for $S_j(\vec{x}^- (n, k, l) + \gamma \vec{s})$. Using these refined asymptotics of $S_j(\vec{x}^+(n, k, l) + \gamma \vec{s})$ and following the same steps as in Ref. [48], we find that

$$\sigma_{n,k,l} = \hat{\chi}_1 \left[ \left| Q_N^{[m]}(z_0) \right|^2 a_{2m+1}^{\frac{N(N+1)-2}{2m+1}} \left[ x_1^+(x - \hat{x}_0, t - \hat{t}_0; n, k, l) x_1^-(x - \hat{x}_0, t - \hat{t}_0; n, k, l) + \frac{1}{4} \right] \right]$$

$$\times \left[ 1 + O\left( a_{2m+1}^{-1/(2m+1)} \right) \right],$$

where $\hat{\chi}_1 = [(2m + 1)2^{-2m-1}]^{2/(2m+1)} \chi_1$. Finally, the simplicity of the nonzero roots in Yablonskii–Vorob’ev polynomials implies that $[Q_N^{[m]}(z_0)]^2 \neq 0$. This indicates that the above leading-order asymptotics for $\sigma_{n,k,l}(x, t)$ does not vanish. Therefore, when $a_{2m+1}$ is large and $(x, t)$ in the neighborhood of $(\hat{x}_0, \hat{t}_0)$, we obtain

$$u_N(x, t) = \hat{u}_1(x - \hat{x}_0, t - \hat{t}_0)e^{i(1+\sigma \rho_1 \rho_2)(\rho_2 x + \rho_1 t)} + O\left( a_{2m+1}^{-1/(2m+1)} \right),$$

where

$$\hat{u}_1 = \hat{u}_1(x - \hat{x}_0, t - \hat{t}_0)e^{i(1+\sigma \rho_1 \rho_2)(\rho_2 x + \rho_1 t)}.$$
The locations $(\hat{x}_0, \hat{t}_0)$ associated with the root structures of $Q_5^{[m]}(z)$ with $1 \leq m \leq 4$ and the respective large internal parameter $a_{2m+1}$ provided by Equation (117) and other system parameters are $\sigma = -1$, $2\rho_1 = \rho_2 = 1$.

\[
v_N(x, t) = \hat{v}_1(x - \hat{x}_0, t - \hat{t}_0)e^{i(1 + \sigma \rho_1 \rho_2)(\frac{\hat{x}}{\rho_1} + \frac{\hat{t}}{\rho_2})} + O\left(\frac{1}{a_{2m+1}}\right),
\]

and the error of this prediction is $O(a_{2m+1}^{-1/(2m+1)}).$

Regarding the proof for the rogue pattern near the origin, it is very similar to that for the NLS and GDNLS equation in Refs. [48, 49]. So it is omitted here.

From Equations (98) and (99) in Theorem 5, one can see that the location $(\hat{x}_0, \hat{t}_0)$ of each fundamental rogue wave away from the center is given by the real and imaginary parts of each nonzero simple root $z_0$ of $Q_N^{[m]}(z)$ through the linear transformation

\[
\begin{bmatrix} \hat{x}_0 \\ \hat{t}_0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\sigma \rho_1 \rho_2} \Re(\Omega) + \frac{1+2\sigma \rho_1 \rho_2}{2\rho_1^2} \Im(\Omega) + \frac{1}{2\sigma \rho_1 \rho_2} \Re(\Omega) + \frac{1}{2\rho_1^2} \Im(\Omega) \\ \frac{1}{2\rho_1^2} \Re(\Omega) - \frac{1}{2\sigma \rho_1 \rho_2} \Im(\Omega) \end{bmatrix} \begin{bmatrix} \Re(z_0) \\ \Im(z_0) \end{bmatrix}.
\]

This linear transformation means that the whole rogue pattern formed by fundamental rogue waves in the $(x, t)$ plane is just a linear transformation matrix $B$ applied to the root structure of the Yablonskii–Vorob’ev polynomial $Q_N^{[m]}(z)$ in the complex $z$ plane. The transformation matrix dictates the effects of stretch, shear and orientation to the root structure, which can be observed in Figure 7 that exhibit the locations $(\hat{x}_0, \hat{t}_0)$ linked to the root structures of $Q_5^{[m]}(z)$ with $1 \leq m \leq 4$. In Figure 7, the parameters are chosen as the same values as that in Figure 6a–d with $\sigma = -1$, $2\rho_1 = \rho_2 = 1$ and and the large internal parameter $a_{2m+1}$, respectively, as

\[
(a_3, a_5, a_7, a_9) = (50, 0, 0, 0), (0, 500, 0, 0), (0, 0, 1000, 0), (0, 0, 0, 2000).
\]

Next, we illustrate the analytical predictions of rogue wave patterns and compare them to true cases in the MT model. According to Theorem 5, the prediction for rogue wave patterns can be assembled into the formulae

\[
|u_N(x, t)| = \left|u_{N_0}(x, t) + O(|a_{2m+1}|^{-1})\right| + \left[\sum_{k=1}^{N_0} \left|\hat{u}_1(x - \hat{x}_0^{(k)}, t - \hat{t}_0^{(k)}) - \rho_1\right| + O(|a_{2m+1}|^{-1/(2m+1)})\right],
\]

\[
|v_N(x, t)| = \left|v_{N_0}(x, t) + O(|a_{2m+1}|^{-1})\right| + \left[\sum_{k=1}^{N_0} \left|\hat{v}_1(x - \hat{x}_0^{(k)}, t - \hat{t}_0^{(k)}) - \rho_2\right| + O(|a_{2m+1}|^{-1/(2m+1)})\right],
\]

where $N_0$ is given by Equation (102), $[u_{N_0}(x, t), v_{N_0}(x, t)]$ is the lower order rogue wave at the center whose internal parameters $(a_3, a_5, ..., a_{2N_0-1})$ are inherited directly from those of the origi-
The large internal parameters $a_{2n+1}$ are given in Equation (117) for $a$ and $d$, respectively, and the other parameters are taken as zero.

We only illustrate the case for $N = 5$ to show the comparison between these predicted rogue patterns and the true ones. The parameters and large internal parameters are chosen as the same as those in Figure 6a–d or in Figure 7. Under identical $(x, t)$ intervals, Figure 7 provides the predicted locations of fundamental rogue wave and lower order rogue waves at the center in Figure 6a–d, and Figure 8 displays the complete predicted rogue wave patterns, respectively. It can be seen that the predicted rogue wave patterns coincide almost exactly with the true ones regarding the locations of individual fundamental rogue waves, the overall shapes formed by these fundamental waves and the fine details of the lower order rogue waves at the center. Theorem 5 also states quantitatively the error of the predicted solution in two cases. Similar to the NLS equation, the orders of these errors can be confirmed numerically and the details are omitted here.

6 | SUMMARY AND DISCUSSIONS

In this paper, we have constructed the general rogue wave solutions in the MT model by using the KP hierarchy reduction method. These solutions are represented in terms of determinants in which elements are given by elementary Schur-polynomials. In the process of constructing these solutions, we proved that two dimension-reduction conditions and one index-reduction condition are satisfied simultaneously under the same constraint of parameters. The local structure analyses show that two background parameters influence the orientation and duration but they do not affect the heights of peak in rogue waves. It is also shown that compared with other integrable coupled systems, there exist no additional parameters in rogue wave solutions of the MT model, and hence only bright-type rogue wave appears. The $N$th order rogue waves correspond to the
superposition of $N(N+1)/2$ fundamental ones, and their arrangement patterns are determined by the values of nonreducible parameters. By setting all internal parameters to be zero, we obtain the super rogue wave of $N$th order in which the sole huge peak is located at the center and its maximum amplitude is $2N+1$ times the background. Finally, we have discussed rogue wave patterns when one of the internal parameters is large. It is shown that similar to other integrable systems, these patterns are linked to the root structure of the Yablonskii–Vorob’ev polynomial hierarchy through a linear transformation.

ACKNOWLEDGMENTS
The work of J. C. is supported by the National Natural Science Foundation of China (Nos. 12226332 and 11705077) and the Zhejiang Province Natural Science Foundation of China (Grant No: 2022SJGYZC01). The work of B. Y. is supported by the National Natural Science Foundation of China (No. 12201326). B. F.’s work is partially supported by National Science Foundation (NSF) under Grant Number: DMS-1715991 and U.S. Department of Defense (DoD), Air Force for Scientific Research (AFOSR) under Grant Number: W911NF2010276.

DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID
Junchao Chen https://orcid.org/0000-0001-8911-6031

REFERENCES
1. Onorato M, Residori S, Bortolozzo U, Montina A, Arecchi FT. Rogue waves and their generating mechanisms in different physical contexts. Phys Rep. 2013;528:47-89.
2. Kharif C, Pelinovsky E, Slunyaev A. Rogue Waves in the Ocean. Springer; 2009.
3. Solli DR, Ropers C, Koonath P, Jalali B. Optical rogue waves. Nature. 2007;450:1054-1057.
4. Dudley JM, Genty G, Mussot A, Chabchoub A, Dias F. Rogue waves and analogies in optics and oceanography. Nat Rev Phys. 2019;1:675-689.
5. Chabchoub A, Hoffmann NP, Akhmediev N. Rogue wave observation in a water wave tank. Phys Rev Lett. 2011;106:204502.
6. Bludov YV, Konotop VV, Akhmediev N. Matter rogue waves. Phys Rev A. 2009;80:033610.
7. Yan ZY. Financial rogue waves. Commun Theor Phys. 2010;54:947-949.
8. Bailung H, Sharma SK, Nakamura Y. Observation of Peregrine solitons in a multicomponent plasma with negative ions. Phys Rev Lett. 2011;07:255005.
9. Peregrine DH. Water waves, nonlinear Schrödinger equations and their solutions. J Aust Math Soc B. 1983;25:16-43.
10. Akhmediev N, Ankiewicz A, Soto-Crespo JM. Rogue waves and rational solutions of the nonlinear Schrödinger equation. Phys Rev E. 2009;80:026601.
11. Guo BL, Ling LM, Liu QP. Nonlinear Schrodinger equation: generalized Darboux transformation and rogue wave solutions. Phys Rev E. 2012;85:026607.
12. Dubard P, Matveev VB. Multi-rogue waves solutions: from the NLS to the KP-I equation. Nonlinearity. 2013;26:R93-R125.
13. Ohta Y, Yang J. General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation. Proc R Soc A. 2012;468:1716-1740.
14. Bilman D, Miller PD. A robust inverse scattering transform for the focusing nonlinear Schrödinger equation. Comm Pure Appl Math. 2019;72:1722-1805.
15. Xu SW, He JS, Wang LH. The Darboux transformation of the derivative nonlinear Schrödinger equation. *J Phys A: Math Theor*. 2011;44:305203.

16. Guo BL, Ling LM, Liu QP. High-order solutions and generalized Darboux transformations of derivative nonlinear Schrödinger equations. *Stud Appl Math*. 2013;130:317-344.

17. Chen S, Zhou Y, Bu L, Baronio F, Soto-Crespo JM, Mihalache D. Super chirped rogue waves in optical fibers. *Opt Exp*. 2019;27:11370-11384.

18. Yang B, Chen J, Yang J. Rogue waves in the generalized derivative nonlinear Schrödinger equations. *J Nonlinear Sci*. 2020;30:3027-3056.

19. Bandelow U, Akhmediev N. Sasa-Satsuma equation: Soliton on a background and its limiting cases. *Phys Rev E*. 2012;86:026606.

20. Chen S. Twisted rogue-wave pairs in the Sasa-Satsuma equation. *Phys Rev E*. 2013;88:023202.

21. Mu G, Qin ZY. Dynamic patterns of high-order rogue waves for Sasa-Satsuma equation. *Nonlinear Anal Real World Appl*. 2016;31:179-209.

22. Guo BL, Ling LM. Rogue wave, breathers and bright-dark-rogue solutions for the coupled Schrödinger equations. *Chin Phys Lett*. 2011;28:110202.

23. Baronio F, Degasperis A, Conforti M, Wabnitz S. Solutions of the vector nonlinear Schrödinger equations: evidence for deterministic rogue waves. *Phys Rev Lett*. 2012;109:044102.

24. Baronio F, Conforti M, Degasperis A, Lombardo S, Onorato M, Wabnitz S. Vector rogue waves and baseband modulation instability in the defocusing regime. *Phys Rev Lett*. 2014;113:034101.

25. Li M, Liang H, Xu T, Liu CJ. Vector rogue waves in the mixed coupled nonlinear Schrödinger equations. *Eur Phys J Plus*. 2016;131:100.

26. Chen S, Grelu P, Soto-Crespo JM. Dark-and bright-rogue-wave solutions for media with long-wave-short-wave resonance. *Phys Rev E*. 2014;89:011201.

27. Chow KW, Chan HN, Kedziora DJ, Grimshaw RHJ. Rogue wave modes for the long short-wave resonance model. *J Phys Soc Jpn*. 2013;82:074001.

28. Chen JC, Chen Y, Feng BF, Maruno K. General high-order rogue wave of the (1+1)-dimensional Yajima-Oikawa system. *J Phys Soc Jpn*. 2018;87:094007.

29. Chen JC, Chen LY, Feng BF, Maruno K. High-order rogue waves of a long-wave-short-wave model of Newell type. *Phys Rev E*. 2019;100:052216.

30. Degasperis A, Lombardo S. Rational solitons of wave resonant-interaction models. *Phys Rev E*. 2013;88:052914.

31. Wang X, Cao J, Chen Y. Higher-order rogue wave solutions of the three-wave resonant interaction equation via the generalized Darboux transformation. *Phys Scr*. 2015;90:105201.

32. Chen S, Soto-Crespo JM, Grelu P. Watch-hand-like optical rogue waves in three-wave interactions. *Opt Express*. 2015;23:349-359.

33. Zhang G, Yan Z, Wen XY. Three-wave resonant interactions: multi-dark-dark-dark solitons, breathers, rogue waves, and their interactions and dynamics. *Physica D*. 2018;366:27-42.

34. Yang B, Yang J. General rogue waves in the three-wave resonant interaction systems. *IMA J Appl Math*. 2021;86:378-425.

35. Ohta Y, Yang J. Rogue waves in the Davey–Stewartson I equation. *Phys Rev E*. 2012;86:036604.

36. Ohta Y, Yang J. Dynamics of rogue waves in the Davey–Stewartson II equation. *J Phys A: Math Theor*. 2013;46:105202.

37. Zhao LC, Liu J. Rogue-wave solutions of a three-component coupled nonlinear Schrödinger equation. *Phys Rev E*. 2013;87:013201.

38. Ling LM, Feng BF, Zhu Z. Multi-soliton, multi-breather and higher order rogue wave solutions to the complex short pulse equation. *Physica D*. 2016;327:13-29.

39. Wen XY, Yan ZY. Modulational instability and higher-order rogue waves with parameters modulation in a coupled integrable AB system via the generalized Darboux transformation. *Chaos*. 2015;25:123115.

40. Ohta Y, Yang J. General rogue waves in the focusing and defocusing Ablowitz–Ladik equations. *J Phys A: Math Theor*. 2014;47:255201.

41. B.Yang, Yang J. General rogue waves in the Boussinesq equation. *J Phys Soc Jpn*. 2020;89:024003.

42. Wang X, Wang L, Wei J, Guo BW, Kang JF. Rogue waves in the three-level defocusing coupled Maxwell–Bloch equations. *Proc R Soc A*. 2021;477:20210585.

43. Degasperis A, Wabnitz S, Aceves AB. Bragg grating rogue wave. *Phys Lett A*. 2015;379:1067-1070.
44. Guo L, Wang L, Cheng Y, He J. High-order rogue wave solutions of the classical massive Thirring model equations. *Commun Nonlinear Sci Numer Simulat*. 2017;52:11-23.
45. Ye YL, Bu LL, Pan CC, Chen SH, Mihalache D, Baronio F. Super rogue wave states in the classical massive Thirring model system. *Rom Rep Phys*. 2021;73:117.
46. Kedziora DJ, Ankiewicz A, Akhmediev N. Classifying the hierarchy of nonlinear-Schrödinger-equation rogue-wave solutions. *Phys Rev E*. 2013;88:013207.
47. Ankiewicz A, Akhmediev N. Multi-rogue waves and triangular numbers. *Rom Rep Phys*. 2017;69:104.
48. Yang B, Yang J. Rogue wave patterns in the nonlinear Schrödinger equation. *Phys D*. 2021;419:132850.
49. Yang B, Yang J. Universal rogue wave patterns associated with the Yablonskii–Vorob’ev polynomial hierarchy. *Phys D*. 2021;425:132958.
50. Thirring WE. A soluble relativistic field theory. *Ann Phys*. 1958;3:91-112.
51. Winful HG, Cooperman GD. Self-pulsing and chaos in distributed feedback bistable optical devices. *Appl Phys Lett*. 1982;40:298-300.
52. Christodoulides DN, Joseph RI. Slow bragg solitons in nonlinear periodic structures. *Phys Rev Lett*. 1989;62:1746.
53. Aceves AB, Wabnitz S. Self-induced transparency solitons in nonlinear refractive periodic media. *Phys Lett A*. 1989;141:37-42.
54. Eggleton BJ, Slusher RE, de Sterke CM, Krug PA, Sipe JE. Bragg grating solitons. *Phys Rev Lett*. 1996;76:1627.
55. Eggleton BJ, de Sterke CM, Slusher RE. Bragg solitons in the nonlinear Schrödinger limit: experiment and theory. *J Opt Soc Am B*. 1999;16:587-599.
56. Mikhailov AV. Integrability of the two-dimensional Thirring model. *JEPT Lett*. 1976;23:320-323.
57. Orfanidis SJ. Soliton solutions of the massive Thirring model and the inverse scattering transform. *Phys Rev D*. 1976;14:472-478.
58. Pelinovsky DE, Shimabukuro Y. Orbital stability of Dirac solitons. *Lett Math Phys*. 2014;104:21-41.
59. Contreras A, Pelinovsky DE, Shimabukuro Y. $L^2$ orbital stability of Dirac solitons in the massive Thirring model. *Commun Part Diff Eq*. 2016;41:227-255.
60. Degasperis A. Darboux polynomial matrices: the classical massive Thirring model as a study case. *J Phys A: Math Theor*. 2015;48:235204.
61. Hirota R. *The Direct Method in Soliton Theory*. Cambridge University Press; 2004.
62. Johnson WP. The curious history of Faà di Bruno’s formula. *Am Math Monthly*. 2002;109:217-234.

**How to cite this article:** Chen J, Yang B, Feng B-F. Rogue waves in the massive Thirring model. *Stud Appl Math*. 2023;151:1020–1052. [https://doi.org/10.1111/sapm.12619](https://doi.org/10.1111/sapm.12619)