SPACE OF CHORD DIAGRAMS ON SPHERICAL CURVES

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Abstract. In this paper, we give a definition of \(\mathbb{Z}\)-valued functions from the ambient isotopy classes of spherical curves derived from chord diagrams, denoted by \(\sum \alpha_i x_i\). Then, we introduce certain elements of the free \(\mathbb{Z}\)-module generated by the chord diagrams with at most \(l\) chords, called relators of Type (I) ((S\(\Pi\)), (W\(\Pi\)), (S\(\Pi\)\(\Pi\)), or (W\(\Pi\)\(\Pi\)), resp.), and introduce another function \(\sum \alpha_i \tilde{x}_i\) derived from \(\sum \alpha_i x_i\). The main result (Theorem 1) shows that if \(\sum \alpha_i \tilde{x}_i\) vanishes for the relators of Type (I) ((S\(\Pi\)), (W\(\Pi\)), (S\(\Pi\)\(\Pi\)), or (W\(\Pi\)\(\Pi\)), resp.), then \(\sum \alpha_i x_i\) is invariant under the Reidemeister move of type RI (strong RI\(\Pi\), weak RI\(\Pi\), strong RI\(\Pi\)\(\Pi\), or weak RI\(\Pi\)\(\Pi\), resp.) that is defined in [4].

1. Introduction

A spherical curve (plane curve, resp.) is the image of a generic immersion of a circle into a 2-sphere (plane, resp.). In this paper, we study certain equivalence classes of spherical curves. Any two spherical/plane curves can be transformed into each other by a finite sequence of Reidemeister moves, each of which is either one of types RI, RI\(\Pi\), or RI\(\Pi\)\(\Pi\) that is a replacement of a part of the curve as Figure 1. These moves are obtained from Reidemeister moves of types \(\Omega_1\), \(\Omega_2\), and \(\Omega_3\) on knot diagrams by ignoring over/under information. It is well-known that two knot diagrams represent the same knot if and only if one knot diagram is transformed into the other knot diagram by a finite sequence of Reidemeister moves of types \(\Omega_1\), \(\Omega_2\), and \(\Omega_3\).

In 1990, Vassiliev [9] defined an infinite sequence of knot invariants that are at least as powerful as all of the quantum group invariants of knots. After that some researchers gave alternating definitions of Vassiliev invariants (see [1]). In 1994, Polyak and Viro [8] introduced a formulation of Vassiliev invariants using chord diagram, and presented explicit formulas for some Vassiliev invariants. Goussarov [2] proved that the formulation of Polyak-Viro provides all Vassiliev invariants of knots. In 2001, Östlund further developed the theory of Polyak and Viro by using singularity theory of plane curves, and obtained the following result [7, Theorem 6]: if \(v\) is a function on the set of knot diagrams that is invariant under plane isotopy, \(\Omega_1\), and \(\Omega_3\) and is so-called Vassiliev-type, then \(v\) is invariant under \(\Omega_2\). Motivated

Date: October 8, 2017.

Key words and phrases. spherical curve; chord diagram; Reidemeister move.
by this result, Östlund [7] posed the following question: is every plane curve transformed into the simple closed curve by a finite sequence of Reidemeister moves of types RI and RIII?

In 2008, Hagge and Yazinski [3] show that the answer to this question is negative, i.e., they showed that the equivalence classes under RI and RIII are nontrivial. In fact, any finite sequence of Reidemeister moves that transforms the plane curve depicted in Figure 2 into the simple closed curve contains a Reidemeister move of type RII. However, to the best of the author’s knowledge, the equivalence class containing the simple closed curve under RI and RIII has not to be determined yet. In general, none of the equivalence classes under RI and RIII have been determined yet.

Viro [10] suggested the idea that type RIII is decomposed into the following two types: suppose that $P_1$ is transformed into $P_2$ by a single RIII. Note that in RIII of Figure 1, a triangle is observed in each of the disks. We say that $P_1$ and $P_2$ are related by a strong RIII if the orientations on the edges of the triangle induced by an orientation of the spherical curve are coherent. If $P_1$ and $P_2$ are not related by a strong RIII, then we say that $P_1$ and $P_2$ are related by a weak RIII. Analogously, for type RII, we define strong RII and weak RII as follows. Suppose that $P_1$ is transformed into $P_2$ by a single RII. We say that $P_1$ and $P_2$ are related by a strong RII if the orientations on the edges of the triangle induced by an orientation of the spherical curve are coherent. If $P_1$ and $P_2$ are not related by a strong RII, then we say that $P_1$ and $P_2$ are related by a weak RII. See Figure 3.

![Figure 3. RI, strong RII, weak RII, strong RIII, and weak RIII.](image)

From the viewpoint of Viro, the following problems arise from the above Östlund’s question: which spherical curve is transformed into the simple closed curve by a finite sequence of Reidemeister moves of types RI and strong RIII (types RI and strong RIII, resp.)? First, in 2013, a necessary and sufficient condition that a spherical curve and the simple closed curve are related by a finite sequence of Reidemeister moves of types RI and weak RIII was given [4]. Second, in 2015, a necessary
Deﬁnition 1

2.1. Deﬁnition 1. A word $w$ of length $n$ is a map from $\hat{n} = \{1, 2, 3, \ldots, n\}$ to $\mathbb{N}$. This word is represented by $w(1)w(2)w(3)\cdots w(n)$. For a word $w : \hat{n} \to \mathbb{N}$, we call each element of $w(\hat{n})$ a letter. A word $u$ of length $q$ is a sub-word of $w$ if there is an integer $p$ ($q \leq p \leq n$) such that $u(j) = w(n - p + j)$ ($1 \leq j \leq q$). A Gauss word of length $2n$ is a word $w$ of length $2n$ such that $w(1)w(2)w(3)\cdots w(2n)$ appears exactly twice in $w(1)w(2)w(3)\cdots w(2n)$. Let cyc and rev be maps $2n \to 2n$ such that $\text{cyc}(p) \equiv p + 1 \pmod{2n}$ and $\text{rev}(p) \equiv -p + 1 \pmod{2n}$. Two Gauss words, $v$ and $w$, of length $2n$ are isomorphic if there exists a bijection $f : v(2n) \to w(2n)$ satisfying the following: there exists $t \in \mathbb{Z}$ such that $w \circ \text{cyc}^t \circ \text{rev}^t = f \circ v$ ($\epsilon = 0$ or 1). The isomorphisms give an equivalence relation on the Gauss words. For a Gauss word $v$ of length $2n$, $[v]$ denotes the equivalence class containing $v$. A Gauss word $v'$ is a sub-Gauss word of the Gauss word $v$ if $v'$ is obtained from $v$ by ignoring some letters of $v$. Then $\text{Sub}(v)$ denotes the set of sub-Gauss words of $v$.

Definition 2 (chord diagram). Each conﬁguration of $n$ pair(s) of points on a circle up to ambient isotopy and reﬂection of the circle is called a chord diagram. The
integer \( n \) is called the \textit{length} of the chord diagram. Traditionally, two points of each pair are connected by a straight arc, called a \textit{chord}.

We note that the equivalence classes of the Gauss words of length \( 2n \) have one to one correspondence with the chord diagrams, each of which has \( n \) chords as in Figure 4. In the rest of this paper, we identify these four expressions in Figure 4,

\[ \text{Gauss words} \leftrightarrow \text{chord diagram} \]

and freely use either one of them depending on situations.

**Notation 1** \((G_{\leq d}, n_d, G_{b,d})\). Let \( G_{\infty} \) be the set of chord diagrams, that is, the set of isomorphism classes of the Gauss words. It is clear that \( G_{\infty} \) consists of countably many elements. Hence, there exists a bijection between \( G_{\infty} \) and \( \{x_i\}_{i \in \mathbb{N}} \), where \( x_i \) is a variable. Take and fix a bijection \( f : G_{\infty} \rightarrow \{x_i\}_{i \in \mathbb{N}} \) satisfying: the number of chords of \( f^{-1}(x_i) \) is less than or equal to that of \( f^{-1}(x_j) \) if and only if \( i \leq j \) (\( i, j \in \mathbb{N} \)). For each positive integer \( d \), let \( G_{\leq d} \) be the set of chord diagrams consisting of at most \( d \) chords and let \( n_d = |G_{\leq d}| \). Then, it is clear that \( f|_{G_{\leq d}} \) is a bijection from \( G_{\leq d} \) to \( \{x_1, x_2, \ldots, x_{n_d}\} \). Further, for each pair of integers \( b \) and \( d \) (\( 2 \leq b \leq d \)), let \( G_{b,d} = G_{\leq d} \setminus G_{\leq b-1} \). Then, \( f|_{G_{b,d}} \) is a bijection \( G_{b,d} \rightarrow \{x_{n_{b-1}+1}, x_{n_{b-1}+2}, \ldots, x_{n_d}\} \).

In the rest of this paper, we use the notations in Notation 1 unless otherwise denoted, and we freely use this identification between \( G_{\infty} \) and \( \{x_i\}_{i \in \mathbb{N}} \).

**Definition 3** (a chord diagram \( CD_P \) of a spherical curve \( P \)). Let \( P \) be a spherical curve, i.e. there is a generic immersion \( g : S^1 \rightarrow S^2 \) such that \( g(S^1) = P \). We define a chord diagram of \( P \) (e.g., Figure 5) as follows: let \( k \) be the number of the double points of \( P \), and \( m_1, m_2, \ldots, m_k \) mutually distinct positive integers. Fix a base point, which is not a double point on \( P \), and choose an orientation of \( P \). Starting from the base point, proceed along \( P \) according to the orientation of

\[ CD_P \leftrightarrow P \]
We assign $m_1$ to the first double point that we encounter. Then we assign $m_2$ to the next double point that we encounter provided it is not the first double point. Suppose that we have already assigned $m_1, m_2, \ldots, m_p$. Then we assign $m_{p+1}$ to the next double point that we encounter if it has not been assigned yet. Following the same procedure, we finally label all the double points. Note that $g_1$ (double point assigned $m_i$) consists of two points on $S^1$ and we shall assign $m_i$ to them. The chord diagram represented by $g_1$ (double point assigned $m_1$), $g_1$ (double point assigned $m_2$), \ldots, $g_1$ (double point assigned $m_k$) on $S^1$ is denoted by $CD_P$ and is called a chord diagram of the spherical curve $P$.

Recall that $CD_P$ gives an equivalence class of Gauss words, say $[v_P]$. Then, by the definition of the equivalence relation, it is easy to see that the map $P \mapsto [v_P]$ is well-defined.

**Notation 2** $(x(CD))$. Let $x \in \{x_i\}_{i \in \mathbb{N}}$ (hence, $x_i$ corresponds to a chord diagram). For a given chord diagram $CD$, fix a Gauss word $G$ representing $CD$. Let $\text{Sub}_x(G) = \{H \mid H \in \text{Sub}(G), [H] = x\}$. The cardinality of this subset is denoted by $x(G)$, i.e., $x(G) = |\text{Sub}_x(G)|$. Let $G'$ be another Gauss word representing $CD$. By the definition of the isomorphism of the Gauss words, it is easy to see $x(G') = x(G)$. Hence, we shall denote this number by $x(CD)$. If $CD$ is a chord diagram of a spherical curve $P$, then $x(CD)$ can be denoted by $x(P)$.

Since each equivalence class of the Gauss words is identified with a chord diagram, we can calculate the number $x(CD)$ by using geometric observations. We explain this philosophy in the next example.

**Example 1.** We consider the chord diagram $CD$ in Figure 6 (Note that $CD = CD_P$ in Figure 5). Then we label the chords of $CD$ by $a_i$ ($1 \leq i \leq 6$) as in Figure 6. Consider the subset of the power set of $\{a_1, a_2, \ldots, a_6\}$, each element of which represents a chord diagram isomorphic to $\otimes$. It is elementary to see that this subset consists of ten elements, those are, $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_1, a_4\}$, $\{a_2, a_3\}$, $\{a_2, a_4\}$, $\{a_3, a_4\}$, $\{a_1, a_5\}$, $\{a_2, a_5\}$, $\{a_3, a_6\}$, and $\{a_4, a_6\}$, and that fact shows that $\otimes(CD) = 10$. Similarly, we have $\boxtimes(CD) = 6$ and $\boxplus(CD) = 8$.

![Figure 6. CD.](image)

Let $CD$ be a chord diagram consisting of chords $a_1, a_2, \ldots, a_k$. Then the chord diagram consisting of a subset of $\{a_1, a_2, \ldots, a_k\}$ is called a sub-chord diagram of $CD$. 
**Definition 4** \((\tilde{x}(z), \tilde{x}([z]))\). Let \(x\) be a chord diagram. We define the function \(\tilde{x}\) from the set of Gauss words to \([0, 1]\) by

\[
\tilde{x}(F) = \begin{cases} 
1 & [F] = x \\
0 & [F] \neq x.
\end{cases}
\]

By definition, it is easy to see \(\tilde{x}(F_1) = \tilde{x}(F_2)\) for each pair \(F_1\) and \(F_2\) with \([F_1] = [F_2]\). Hence, we shall denote this number by \(\tilde{x}([F])\). If \([F]\) corresponds to a chord diagram of a spherical curve \(P\), then \(\tilde{x}([F])\) is denoted by \(\tilde{x}(P)\). Further, let \(\mathbb{Z}[G_{<1}]\) be the free \(\mathbb{Z}\)-module generated by the elements of \(G_{<1}\), where \(l\) is sufficiently large.

We linearly extend \(\tilde{x}\) to the function from \(\mathbb{Z}[G_{<1}]\) to \(\mathbb{Z}\). It is clear that for any Gauss word with \([G] = CD\),

\[
x(CD) = \sum_{z \in \text{Sub}(G)} \tilde{x}(z).
\]

(2.1)

### 2.2. Relators and Reidemeister moves

Let \(\mathbb{Z}[G_{<1}]\) be the free \(\mathbb{Z}\)-module defined in Subsection 2.1. In this subsection, first we define the elements of \(\mathbb{Z}[G_{<1}]\) called relators of types (I), (SII), (WII), (SIII), and (WIII).

**Definition 5** (Relators, cf. Figure 7). 

- Type (I). An element \(r\) of \(\mathbb{Z}[G_{<1}]\) is called a **Type (I) relator** if there exist a Gauss word \(S\) and a letter \(i\) not in \(S\) such that \(r = [Sii]\).

- Type (SII). An element \(r\) of \(\mathbb{Z}[G_{<1}]\) is called a **Type (SII) relator** if there exist a Gauss word \(ST\) and letters \(i\) and \(j\) not in \(ST\) such that \(r = [SijTij] + [STi] + [SjTj]\).

- Type (WII). An element \(r\) of \(\mathbb{Z}[G_{<1}]\) is called a **Type (WII) relator** if there exists a Gauss word \(ST\) and letters \(i\) and \(j\) not in \(ST\) such that \(r = [SijTij] + [STi] + [SjTj]\).

- Type (SIII). An element \(r\) of \(\mathbb{Z}[G_{<1}]\) is called a **Type (SIII) relator** if there exists a Gauss word \(STU\) and letters \(i, j, k\) not in \(STU\) such that \(r = ([SijTkUjk] + [SijTikUk] + [SjTkUjk]) - ([SjiTikUk] + [SjTkiUjk] + [SijTkUjk]).

- Type (WIII). An element \(r\) of \(\mathbb{Z}[G_{<1}]\) is called a **Type (WIII) relator** if there exists a Gauss word \(STU\) and letters \(i, j, k\) not in \(STU\) such that \(r = ([SijTkUjk] + [SijTikUk] + [SjTkUjk]) - ([SjiTikUk] + [SjTkiUjk] + [SijTkUjk]).

We note that in Definition 5, each relator is induced by the corresponding Reidemeister move: Type (I) relator corresponding to \(RI\), Type (SII) relator corresponding to strong \(RI\), Type (WII) relator corresponding to weak \(RI\), Type (SIII) relator corresponding to strong \(RI\), and Type (WIII) relator corresponding to weak \(RI\).

For the precise statement of this note, we introduce the following setting.

We first introduce the following notations. Let \(P\) and \(P'\) be two spherical curves. If \(P\) and \(P'\) are related by a single \(RI\) (strong \(RI\), weak \(RI\), strong \(RI\), or weak \(RI\) resp.), then there are Gauss words \(G\) and \(G'\) such that (by exchanging \(P\) and \(P'\), if necessary), \(G = Sii (SijTjji, SijTji, SjTkUjk, or SjTkUjk resp.)\) and \(G' = S (ST, ST, SjiTikUk, or SjiTikUk resp.)\) such that \([G] = CD_P\) and \([G'] = CD_P\) (Figure 7). The subset of \(\text{Sub}(G)\) such that each element has exactly \(m\) letters of \(i, j,\) and \(k\) is denoted by \(\text{Sub}^{(m)}(G)\). By definition,

\[
\text{Sub}(G) = \text{Sub}^{(0)}(G) \sqcup \text{Sub}^{(1)}(G) \sqcup \text{Sub}^{(2)}(G) \sqcup \text{Sub}^{(3)}(G).
\]

(2.2)
Similarly, for a chord diagram $x$, $\text{Sub}_{x}^{(m)}(G)$ denotes the subset of $\text{Sub}_{x}(G)$ consisting of elements, each of which has exactly $m$ letters of $i$, $j$, and $k$. Then,

\[(2.3) \quad \text{Sub}_{x}(G) = \text{Sub}_{x}^{(0)}(G) \sqcup \text{Sub}_{x}^{(1)}(G) \sqcup \text{Sub}_{x}^{(2)}(G) \sqcup \text{Sub}_{x}^{(3)}(G).\]
Let \( C \) be the set of ambient isotopy classes of the spherical curves. Next, for each element \( \sum_{i} \alpha_{i} x_{i} \in \mathbb{Z}[G_{\leq d}] \), we define a function \( C \to \mathbb{Z} \), also denoted by \( \sum_{i} \alpha_{i} x_{i} \), and another function denoted by \( \sum_{i} \alpha_{i} \bar{x}_{i} \).

**Definition 6** \((\sum_{i} \alpha_{i} x_{i}, \sum_{i} \alpha_{i} \bar{x}_{i})\). Let \( b \) and \( d \) \((2 \leq b \leq d)\) be integers and \( G_{\leq d}, G_{b,d}, \{ x_{i} \}_{i \in \mathbb{N}}, \) and \( \mathbb{Z}[G_{\leq d}] \) be as in Subsection 2.1. Recall that \( n_{d} = |G_{\leq d}| \) and \( G_{b,d} = \{ x_{i} \}_{n_{d}+1 \leq i \leq n_{d}} \) and each \( x_{i} \) represents the chord diagram \( f^{-1}(x_{i}) \). For each element

\[
\sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} x_{i} \in \mathbb{Z}[G_{\leq d}],
\]

we define an integer-valued function \( C \to \mathbb{Z} \), also denoted by \( \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} x_{i} \), by

\[
P \mapsto \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} x_{i}(P),
\]

where \( x_{i}(P) \) is the integer introduced in Notation 2.

Analogously, for each \( \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \bar{x}_{i} \in \mathbb{Z}[G_{\leq d}] \), we define the function

\[
\sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \bar{x}_{i} : \mathbb{Z}[G_{\leq d}] \to \mathbb{Z}
\]

by

\[
\left( \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \bar{x}_{i} \right)(P) = \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \bar{x}_{i}(P),
\]

where \( \bar{x}_{i}(P) \) is the integer introduced in Definition 4.

Recall that a spherical curve \( P \) and another spherical curve \( P' \) are related by a single RI and that there exist a letter \( i \) and an oriented Gauss word \( S \) such that \( CD_{P} = [Sii] \) and \( CD_{P'} = [S] \). Note that in this case in the decomposition \((2.2)\), \( \text{Sub}^{(2)}(G) = \emptyset \) and \( \text{Sub}^{(3)}(G) = \emptyset \), where \( G = Sii \). Then, by \((2.1)\) in Definition 4 and \((2.2)\),

\[
\sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} x_{i}(P) = \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \left( \sum_{z \in \text{Sub}(G)} \bar{x}_{i}(z) \right)
\]

\[
= \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \left( \sum_{z_{0} \in \text{Sub}^{(0)}(G)} \bar{x}_{i}(z_{0}) + \sum_{z_{1} \in \text{Sub}^{(1)}(G)} \bar{x}_{i}(z_{1}) \right).
\]

Note that each element \( z_{0} \in \text{Sub}^{(0)}(G) \) is a sub-Gauss word of \( S \). Then it is clear that \( \text{Sub}^{(1)}(G) = \{z_{0}ii \mid z_{0} \in \text{Sub}^{(0)}(G)\} \). Hence,

\[
\sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} x_{i}(P) = \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \left( \sum_{z_{0} \in \text{Sub}^{(0)}(G)} \bar{x}_{i}(z_{0}) + \sum_{z_{0} \in \text{Sub}^{(0)}(G)} \bar{x}_{i}(z_{0}ii) \right)
\]

\[
= \sum_{n_{d}+1 \leq i \leq n_{d}} \alpha_{i} \left( \sum_{z_{0} \in \text{Sub}^{(0)}(G)} \bar{x}_{i}(\{z_{0}\}) + \sum_{z_{0} \in \text{Sub}^{(0)}(G)} \bar{x}_{i}(\{z_{0}ii\}) \right).
\]
On the other hand, since \( \text{Sub}(G') \) is identified with \( \text{Sub}^{(0)}(G) \),
\[
\sum_{n_b+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(P') = \sum_{n_b+1 \leq i \leq n_d} \alpha_i \left( \sum_{z' \in \text{Sub}(G')} \tilde{x}_i(z') \right) \quad (\because (2.1))
\]
\[
= \sum_{n_b+1 \leq i \leq n_d} \alpha_i \left( \sum_{z \in \text{Sub}^{(0)}(G)} \tilde{x}_i(z) \right)
\]
\[
= \sum_{n_b+1 \leq i \leq n_d} \alpha_i \left( \sum_{z \in \text{Sub}^{(0)}(G)} \tilde{x}_i([z]) \right).
\]
As a conclusion, the difference of the values is calculated as follows.
\[
\sum_{n_b+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(P) - \sum_{n_b+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(P')
\]
\[
= \sum_{n_b+1 \leq i \leq n_d} \alpha_i \tilde{x}_i([z_0 ii]).
\]
We note that this is a linear combination of the values of Type (I) relators via \( \tilde{x}_i \).
For the cases of Type (SII), (WII), (SIII), or (WIII) relators, the arguments are slightly more complicated than that of Type (I) relator. We will explain them in the proof of Theorem 1 and thus, we omit them here.

**Definition 7** \( R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5} \). For each \( (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) \in \{0, 1\}^5 \), let \( R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5} = \bigcup_{i=1}^5 R_i \) \( \subseteq \bigcup_{i \geq 1} \mathbb{Z}[G_{\leq i}] \) where \( R_1 \) is the set of Type (I) relators, \( R_2 \) is the set of Type (SII) relators, \( R_3 \) is the set of Type (WII) relators, \( R_4 \) is the set of Type (SIII) relators, and \( R_5 \) is the set of Type (WIII) relators.

For integers \( b \) and \( d \) \( (2 \leq b \leq d) \), let \( O_{b,d} \) be the projection \( \mathbb{Z}[G_{\leq i}] \rightarrow \mathbb{Z}[G_{b,d}] \). Here, note that \( O_{b,d} \) is a linear map. By the definition, we immediately have:

**Lemma 1.** If \( n_{b-1} + 1 \leq i \leq n_d \), then for any \( r \in \mathbb{Z}[G_{\leq i}] \),
\[
\tilde{x}_i(r) = \tilde{x}_i(O_{b,d}(r)).
\]

**Notation 3.** Let \( R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5}(b, d) = O_{b,d}(R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5}) \).

By using Lemma 1, we have the next proposition.

**Proposition 1.** For each pair of integers \( b \) and \( d \) \( (2 \leq b \leq d) \), let \( \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i \) be a function as in Definition 6. For \( (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) \in \{0, 1\}^5 \) \( \subseteq \bigcup_{i \geq 1} \mathbb{Z}[G_{\leq i}] \), let \( R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5} \) be the set as in Definition 7. The following two statements are equivalent:

1. \( \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = 0 \) \( (\forall r \in R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5}) \),
2. \( \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = 0 \) \( (\forall r \in R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5}(b, d)) \).

**Proof.**
1. (1) \( \Rightarrow \) (2). This is obvious.
2. (2) \( \Leftrightarrow \) (2). Let \( r \in R_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \varepsilon_5} \). By Lemma 1,
\[
\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(O_{b,d}(r)).
\]

By the condition (2),
\[
\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(O_{b,d}(r)) = 0.
\]
Then,  
\[ \sum_{n_0-1+1 \leq i \leq n_2} \alpha_i \tilde{x}_i(r) = 0. \]

We note that it is a simple task to give explicit presentations of \( R_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_3}(b, d) \). In particular, if \( d \leq 4 \), one can do it by hands. The next example shows the concrete such process for a certain case.

**Example 2** \((R_{00010}(2, 3))\). \( R_{00010}(2, 3) = \{ \{ \emptyset + 3 \mathfrak{X} - 3 \mathfrak{I} \} - \{ \emptyset, 3 \mathfrak{X} - 2 \mathfrak{I} \} - \{ \emptyset, \emptyset + 1 - 2 \mathfrak{X} \} \}. \) This fact is confirmed as follows.

Recall that a relator of Type (III) is of the form \( r = ([S_i]T_iU_i) \). Hence, if \( \{ S_{i+1}T_{i+1}U_{i+1} \} \) is contained in mutually different sub-words \( S_i, T_i, U_i \), then we have \( O \)’s are contained in one of the sub-words \( S_i, T_i, U_i \). Then, this expression, that each term of \( r \) consists of \( n \) or \( n + 1 \) letters for some \( n \). Note that \( O_{2,3}(r) \) is obtained from \( r \) by removing the terms of length \( < 2 \), or \( \geq 3 \). Hence it is enough to suppose that each term of \( r \) consists of two or three letters (Case 1) or each term of \( r \) consists of three or four letters (Case 2).

- **Case 1.** Each term of \( r \) consists of two or three letters.
  In this case, we have \( STU = \emptyset \) (i.e., \( S = \emptyset, T = \emptyset \), and \( U = \emptyset \)), and, hence,
  \[ O_{2,3}(r) = r = [ijijk] + [iiij] + [kkk] - [iiijk] - [kkj] = [ijijk] + 3[iiij] - 3[iijj] - [iiijk]. \]

- **Case 2.** Each term of \( r \) consists of three or four letters.
  In this case, \( STU = \sigma \sigma \) for some letter \( \sigma \neq i, j, k \).
  Case 2.1. Both \( \sigma \)’s are contained in one of the sub-words \( S, T, U \).
  Suppose that both \( \sigma \)’s are contained in \( S \). Then,
  \[ r = [\sigma i\sigma iij] + [\sigma iijj] + [\sigma ikkk] + [\sigma jkkk] - [\sigma iijkkk] - [\sigma iiijj] - [\sigma ikkk] = [\sigma i\sigma iij] - [\sigma i\sigma jkkk]. \]

Hence,
\[ O_{2,3}(r) = 3[iijkk] - 2[ijjkk] - [iiijkk]. \]

By using similar arguments, we can show that we obtain the same element as above in other cases. Details of the calculations are left to the reader.

Case 2.2. The two \( \sigma \)’s are contained in mutually different sub-words \( S, T, U \).
Suppose that one \( \sigma \) is contained in \( S \) and the other \( \sigma \) is contained in \( T \). Then
\[ r = [\sigma i\sigma ij] + [\sigma ijij] + [\sigma iikkk] + [\sigma jkkk] - [\sigma iiijj] - [\sigma jkkk] = [\sigma i\sigma ij] - [\sigma jkkk]. \]

Here, we note that
\[ [\sigma i\sigma ij] = [\sigma i\sigma jkkk] = [ijij], \]
and
\[ [\sigma i\sigma ikkk] = [\sigma jkkk] = [\sigma jiiij] = [ijk], \]
These show that
\[ r = [\sigma i\sigma ij] + [\sigma iiijj] + [ijij] + [ijikkk] - 2[iijkkk]. \]
Hence,
\[ O_{2,3}(r) = [ijij] + [ijikkk] - 2[iijkkk]. \]
By using similar arguments, we can show that we obtain the same element as above in other cases. Details of the calculations are left to the reader.

As a conclusion, \( R_{00010}(2, 3) = \{[ijkkjk] + 3[iiij] - [iiijj], [ijkkjk] - 2[ijkjk] - [ijjjjk], [ijkkjk] + [ijkjk] - 2[iijijkk] \} = \{\mathcal{X} + 3\mathcal{X} - 3[\mathcal{X}] - \mathcal{Y}, \mathcal{Z} - 2[\mathcal{Z}] - \mathcal{X} - \mathcal{Y} - 2[\mathcal{Y}]\}. \)

**Example 3** \((R_{00010}(2, 4))\). By using the arguments in Example 2, we can show:

\[
R_{00010}(2, 4) = \{\mathcal{X} + 3\mathcal{X} - 3[\mathcal{X}] - \mathcal{Y}, \mathcal{X} + \mathcal{Y} + \mathcal{Z} - 2[\mathcal{Z}] - \mathcal{X} - \mathcal{Y} - 2[\mathcal{Y}], 2\mathcal{Y} + \mathcal{Z} - 2[\mathcal{Z}] - \mathcal{X} + 2[\mathcal{X} - \mathcal{Z}] - 2[\mathcal{Z}] - \mathcal{X} - \mathcal{Y} - 2[\mathcal{Y}], \mathcal{X} - 2[\mathcal{X} - \mathcal{Z}] - \mathcal{X} + \mathcal{Y} - \mathcal{Z} - 2[\mathcal{Y}]\}. \text{Details of the proof are left to the reader.}
\]

**Example 4** \((R_{00001}(2, 3))\). By using the arguments in Example 2, we can show:

\[
R_{00001}(2, 3) = \{\mathcal{X} - \mathcal{Y} - \mathcal{Z}, \mathcal{X} - \mathcal{Z}, 3\mathcal{X} - \mathcal{Y} - 2[\mathcal{X}], \mathcal{X} - 2[\mathcal{Z}] + \mathcal{Y} - \mathcal{Z} - 2[\mathcal{Y}]\}. \text{Details of the proof are left to the reader.}
\]

2.3. Main result and corollaries.

**Theorem 1.** Let \(b\) and \(d\) \((2 \leq b \leq d)\) be integers and \(G_{\leq d}, \{x_i\}_{i \in \mathbb{N}}, \mathbb{Z}[G_{\leq i}], n_d = |G_{\leq d}|, \) and \(G_{b,d} = \{x_i\}_{n_0+1 \leq i \leq n_b}\) as in Subsection 2.1. Let \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i\) and \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i\) be functions as in Definition 6. For \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \in \{0, 1\}^5\), let \(R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(b, d)\) be as in Subsection 2.2. Suppose the following conditions are satisfied:

- If \(\epsilon_1 = 1\), \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = 0\) for each \(r \in R_{01000}(b, d)\).
- If \(\epsilon_2 = 1\), \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = 0\) for each \(r \in R_{00100}(b, d)\).
- If \(\epsilon_3 = 1\), \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = 0\) for each \(r \in R_{00010}(b, d)\).
- If \(\epsilon_4 = 1\), \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = 0\) for each \(r \in R_{00001}(b, d)\).
- If \(\epsilon_5 = 1\), \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(r) = 0\) for each \(r \in R_{00001}(b, d)\).

Then, the function \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i\) on the set of ambient isotopy classes of spherical curves is invariant under the Reidemeister moves corresponding to \(\epsilon_j = 1\).

**Definition 8** (irreducible chord diagram). Let \(x\) be a chord diagram. A chord \(\alpha\) in \(x\) is said to be an isolated chord if \(\alpha\) does not intersect any other chord. If \(x\) has an isolated chord, \(x\) is called reducible and otherwise, \(x\) is called irreducible. The set of the irreducible chord diagrams is denoted by \(\text{Irr}\). Let \(I_{b,d}^{(\text{Irr})} = \{i \mid n_{b-1} + 1 \leq i \leq n_d, x_i \in \text{Irr}\}\).

If we consider the function of the form \(\sum_{i \in I_{b,d}^{(\text{Irr})}} \alpha_i x_i\) for \(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i\) in Theorem 1, we obtain:

**Corollary 1.** Let \(b\) and \(d\) \((2 \leq b \leq d)\) be integers and \(G_{\leq d}, \{x_i\}_{i \in \mathbb{N}}, \mathbb{Z}[G_{\leq i}], n_d = |G_{\leq d}|, \) and \(G_{b,d} = \{x_i\}_{n_0+1 \leq i \leq n_b}\) as in Subsection 2.1. Let \(\sum_{i \in I_{b,d}^{(\text{Irr})}} \alpha_i x_i\) and \(\sum_{i \in I_{b,d}^{(\text{Irr})}} \alpha_i \tilde{x}_i\) be functions as in Definition 6. For \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \in \{0, 1\}^5\),
let $R_{q_1e_2e_4e_5}(b,d)$ be as in Subsection 2.2. Suppose the following conditions are satisfied:

- If $e_2 = 1$, \[ \sum_{i \in f^{(r)}_{k,d}} \alpha_i \tilde{x}_i(r) = 0 \text{ for each } r \in R_{10000}(b,d). \]
- If $e_3 = 1$, \[ \sum_{i \in f^{(r)}_{k,d}} \alpha_i \tilde{x}_i(r) = 0 \text{ for each } r \in R_{00100}(b,d). \]
- If $e_4 = 1$, \[ \sum_{i \in f^{(r)}_{k,d}} \alpha_i \tilde{x}_i(r) = 0 \text{ for each } r \in R_{00010}(b,d). \]
- If $e_5 = 1$, \[ \sum_{i \in f^{(r)}_{k,d}} \alpha_i \tilde{x}_i(r) = 0 \text{ for each } r \in R_{00001}(b,d). \]

Then, the function \[ \sum_{i \in f^{(r)}_{k,d}} \alpha_i x_i \] on the set of ambient isotopy classes of spherical curves is invariant under RI and the Reidemeister moves corresponding to $e_j = 1$.

**Proof of Corollary 1 from Theorem 1.** By Theorem 1, it is enough to show

\[ \sum_{i \in f^{(r)}_{k,d}} \alpha_i \tilde{x}_i(r) = 0 \quad (\forall r \in R_{10000}(b,d)) \]

for a proof of Corollary 1. We first note that if $x_i \in \text{Irr}$, then $x_i$ has no isolated chords. On the other hand, let $r \in R_{10000}(b,d)$, that is, there exist a Gauss word $S$ and a letter $j$ such that $r = [Sjj]$. Then, the chord corresponding to $j$ is isolated. These show that $\tilde{x}_i(r) = 0$. This shows that (2.4) holds. \( \square \)

**Example 5.** It is easy to see that $\{x_i\}_{i \in f^{(r)}_{k,d}}$ consists of three elements, $y_1 = \otimes$, $y_2 = \otimes$, and $y_3 = \oplus$. The fact together with Corollary 1 implies that if there exist $\alpha_1, \alpha_2$, and $\alpha_3$ such that $(\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3)(r) = 0 \ (\forall r \in R_{00010}(2,3))$, then $\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3$ is invariant under RI and strong RI II. Recall that $R_{00010}(2,3)$ consists of the following three elements $r_1, r_2$, and $r_3$ (Example 2):

\[
\begin{align*}
    r_1 &= \otimes + 3 \otimes - 3 \otimes - \otimes, \\
    r_2 &= 3 \otimes - 2 \otimes - \otimes, \\
    r_3 &= \otimes + \oplus - 2 \otimes.
\end{align*}
\]

We can find such $\alpha_1, \alpha_2$, and $\alpha_3$ by solving the linear equation $xM = \mathbf{0}$, where $M = (\tilde{y}_i(r_j))$ (Figure 9). It is elementary to show that the set of solutions is $\{x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \}$. \( \end{pmatrix} \)

**Figure 9.** Matrix $(\tilde{y}_i(r_j))_{1 \leq i,j \leq 3}$. 

\[\gamma(1, -3, 3) \mid \gamma \in \mathbb{Z}\]. Particularly, \(y_1 - 3y_2 + 3y_3\) is invariant under \(RI\) and strong \(RII\). We note that \(\frac{1}{4}(y_1 - 3y_2 + 3y_3)\) is the function \(\lambda\) introduced in [5] from a different viewpoint.

**Example 6.** In contrast to Example 5, we will note that there does not exist a nonzero \((\alpha_1, \alpha_2, \alpha_3)\) such that \((\alpha_1 \tilde{y}_1 + \alpha_2 \tilde{y}_2 + \alpha_3 \tilde{y}_3)(r) = 0\) for each \(r \in R_{00001}(2, 3)\), i.e., there does not exist \(\alpha_1 \tilde{y}_1 + \alpha_2 \tilde{y}_2 + \alpha_3 \tilde{y}_3\) satisfying the condition of Corollary 1 with \(\epsilon_5 = 1\). In fact, we take \(y_3(i = 1, 2, 3)\) as in Example 5, and name the elements of \(R_{00001}(2, 3)\) (see Example 4) as:

\[
\begin{align*}
  r'_1 &= \begin{array}{c} \bigcirc + \bigotimes - \bigotimes - \bigotimes \end{array}, \\
  r'_2 &= \begin{array}{c} \bigotimes - \bigotimes \end{array}, \\
  r'_3 &= \begin{array}{c} 3 \bigotimes - 2 \bigotimes \end{array}, \\
  r'_4 &= \begin{array}{c} \bigotimes - 2 \bigotimes + \bigotimes \end{array}, \\
  r'_5 &= \begin{array}{c} \bigotimes - \bigotimes \end{array}.
\end{align*}
\]

Let \(M = (\tilde{y}_i(r'_j))_{1 \leq i, j \leq 5}\) (Figure 10). Then it is easy to see that the linear equation \(xM = 0\) admits the trivial solution \(x = (0, 0, 0)\) only.

\[
\begin{array}{cccccc}
\bigotimes + \bigotimes - \bigotimes - \bigotimes & \bigotimes - \bigotimes & 3 \bigotimes - 2 \bigotimes & \bigotimes - 2 \bigotimes + \bigotimes & \bigotimes - \bigotimes \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
1 & -1 & 3 & 0 & 0
\end{array}
\]

**Figure 10.** Matrix \((\tilde{y}_i(r'_j))_{1 \leq i, j \leq 5}\).

**Definition 9** (connected chord diagram). Let \(v\) be a Gauss word of length \(2n\) and \(w\) a Gauss word of length \(2n\) where \(v(2m) \cap w(2n) = \emptyset\). Then we define the Gauss word of length \(2(m + n)\), denoted by \(vw\), by \(vw(i) = v(i)\) \((1 \leq i \leq 2m)\) and \(vw(2m+i) = w(i)\) \((1 \leq i \leq 2n)\). The chord diagram \([vw]\) is called a product of chord diagrams \([v]\) and \([w]\). If a chord diagram is not a product of two non-empty chord diagrams, then the chord diagram is called a connected chord diagram. The set of the connected chord diagrams is denoted by \(\text{Conn}\). Let \(I_{b,d}^{\text{Conn}} = \{ i \mid n_{b-1} + 1 \leq i \leq n_d, x_i \in \text{Conn} \} \).

It is easy to see that if a connected chord diagram has at least two chords, it is irreducible. Note that the chord diagram consisting of exactly one chord is connected but not irreducible.

Recall that \(C\) is the set of ambient isotopy classes of the spherical curves.

**Definition 10** (connected sum). Let \(P_1, P_2 \in C\). We suppose that the ambient 2-spheres are oriented. Let \(p_i\) be a point on \(P_i\) such that \(p_i\) is not a double point \((i = 1, 2)\). Let \(d_i\) be a sufficiently small disk with center \(p_i\) \((i = 1, 2)\) such that \(d_i \cap P_i\) consists of an arc properly embedded in \(d_i\). Let \(\tilde{d}_i = \partial(S^2 \setminus d_i)\) and \(\tilde{P}_i = P_i \cap \tilde{d}_i\); let \(h : \partial \tilde{d}_1 \to \partial \tilde{d}_2\) be an orientation reversing homeomorphism such that \(h(\partial \tilde{P}_1) = \partial \tilde{P}_2\). Then, \(\tilde{P}_1 \cup_h \tilde{P}_2\) gives a spherical curve in the oriented 2-sphere \(\tilde{d}_1 \cup_h \tilde{d}_2\). The spherical curve \(P_1 \cup_h P_2\) in the oriented 2-sphere, denoted by \(P_1 \cup^{(p_1, p_2)} h P_2\),
is called a connected sum of the spherical curves $P_1$ and $P_2$ at the pair of points $p_1$ and $p_2$ (see Figure 11).

**Figure 11.** Connected sum $P_1 \#_{(p_1, p_2), h} P_2$ of two spherical curves $P_1$ and $P_2$.

**Definition 11** (additivity). Let $I$ be a function on $C$. We say that $I$ is additive if $I(P_1 \#_{(p_1, p_2), h} P_2) = I(P_1) + I(P_2)$ for any spherical curves $P_1$ and $P_2$ and for any $P_1 \#_{(p_1, p_2), h} P_2$ in Definition 10.

**Definition 12** (prime). A spherical curve $P$ is trivial if $P$ is a simple closed curve. A spherical curve $P$ is prime if $P$ cannot be a connected sum of two nontrivial spherical curves.

If we consider the function of the form $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i$ for $\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i$ in Theorem 1, we obtain:

**Corollary 2.** Let $b$ and $d$ ($2 \leq b \leq d$) be integers and $G_{\leq d}$, $\{x_i\}_{i \in N}$, $\mathbb{Z}[G_{\leq 1}]$, $n_d = |G_{\leq d}|$, and $G_{d,b} = \{x_i\}_{n_{b-1}+1 \leq i \leq n_d}$ be as in Subsection 2.1. Let $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i$ and $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i$ be functions as in Definition 6. For $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \in \{0, 1\}^5$, let $R_{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5}(b, d)$ be as in Subsection 2.2. Suppose the following conditions are satisfied:

- If $\epsilon_2 = 1$, $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i(r) = 0$ for each $r \in R_{01000}(b, d)$.
- If $\epsilon_3 = 1$, $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i(r) = 0$ for each $r \in R_{00100}(b, d)$.
- If $\epsilon_4 = 1$, $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i(r) = 0$ for each $r \in R_{00010}(b, d)$.
- If $\epsilon_5 = 1$, $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i(r) = 0$ for each $r \in R_{00001}(b, d)$.

Then, the function $\sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i$ on the set of ambient isotopy classes of spherical curves is additive and invariant under RI and the Reidemeister moves corresponding to $\epsilon_j = 1$.

**Proof of Corollary 2 from Theorem 1.** Since $b \geq 2$ and $i \geq n_{b-1} + 1$, each $x_i$ consists of more than one chords, i.e., $x_i \in \text{Irr}$ (see the note preceding Definition 10).

By Theorem 1, it is enough to show

\[(2.5) \quad \sum_{i \in I_{h,d}^{\text{(Conn)}}} \alpha_i x_i(r) = 0 \quad (\forall r \in R_{10000}(b, d))\]

for a proof of Corollary 2. We first note that if $x_i \in \text{Irr}$, then $x_i$ has no isolated chords. On the other hand, let $r \in R_{10000}(b, d)$, that is, there exist a Gauss word $S$
admits a decomposition into two sub-words, which are sub-words on $S$ and $G$.

Further, by using geometric observations as in Example 1 it is clear that if $x_i \in \text{Conn}$, then

\begin{equation}
\label{eq:2.6}
x_i(P_i) + x_i(P_2) = x_i(P_1) + x_i(P_2)
\end{equation}

for any $P_1, P_2$. This fact implies $\sum_{i \in \mathcal{I}^{(\text{Conn})}} \alpha_i x_i$ is additive. \hfill \Box

3. PROOF OF THEOREM 1

- (Proof of the case $\epsilon_1 = 1$) Let $P$ and $P'$ be two spherical curves where $P$ and $P'$ are related by a single RI, hence, there exist a letter $i$ and two Gauss words $G = S_{ij}$ and $G' = S$ corresponding to $P$ and $P'$, respectively, i.e., $CDP = [S_{ij}]$ and $CDP' = [S]$. As we observed in a paragraph proceeding of Definition 7 of Subsection 2.2, we have

\[
\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) - \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P') = \sum_{n_{b-1}+1 \leq i \leq n_d} \sum_{z_0 \in \text{Sub}^{(0)}(G)} \alpha_i \tilde{x}_i([z_0 ii]) .
\]

By the assumption of this case, for each $z_0 \in \text{Sub}^{(0)}(G)$,

\[
\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i([z_0 ii]) = 0
\]

and this shows that

\[
\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P').
\]

Hence, $\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i$ is invariant under RI.

- (Proof of the case $\epsilon_2 = 1$) Let $P$ and $P'$ be two spherical curves where $P$ and $P'$ are related by a single strong RI, hence, there exist two Gauss words $G = S_{ijTji}$ and $G' = ST$ corresponding to $P$ and $P'$, respectively, i.e., $CDP = [S_{ijTji}]$ and $CDP' = [ST]$.

By (2.1) in Subsection 2.1 and (2.2) in Subsection 2.2, we obtain (note that $\text{Sub}^{(0)}(G) = \emptyset$):

\[
\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left( \sum_{z \in \text{Sub}(G)} \tilde{x}_i(z) \right)
\]

\[
= \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left( \sum_{z_0 \in \text{Sub}^{(0)}(G)} \tilde{x}_i(z_0) \right) + \sum_{n_{b-1}+1 \leq i \leq n_d} \sum_{z_1 \in \text{Sub}^{(1)}(G) \cup \text{Sub}^{(2)}(G)} \alpha_i \tilde{x}_i(z_1)
\]

\[
= \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left( \sum_{z' \in \text{Sub}(G')} \tilde{x}_i(z') \right) + \sum_{n_{b-1}+1 \leq i \leq n_d} \sum_{z_2 \in \text{Sub}^{(1)}(G') \cup \text{Sub}^{(2)}(G') \cup \text{Sub}^{(3)}(G')} \alpha_i \tilde{x}_i(z_2)
\]

(\because \text{Sub}^{(0)}(G) is identified with \text{Sub}(G')).

Let $z_0 \in \text{Sub}^{(0)}(G)$. We note that since $G$ is an oriented Gauss word $z_0$ uniquely admits a decomposition into two sub-words, which are sub-words on $S$ and $T$. Let
\( \sigma(z_0) \) be the sub-word of \( S \) and \( \tau(z_0) \) the sub-word of \( T \) satisfying \( z_0 = \sigma(z_0)\tau(z_0) \). Under these notations, we define maps

\[
\begin{align*}
  z_2 : \text{Sub}^{(0)}(G) &\rightarrow \text{Sub}^{(2)}(G); z_2(z_0) = \sigma(z_0)i_\tau(z_0)j_i, \\
  z_1 : \text{Sub}^{(0)}(G) &\rightarrow \text{Sub}^{(1)}(G); z_1(z_0) = \sigma(z_0)i_\tau(z_0)i, \quad \text{and} \\
  z_1' : \text{Sub}^{(0)}(G) &\rightarrow \text{Sub}^{(1)}(G); z_1'(z_0) = \sigma(z_0)j_\tau(z_0)j.
\end{align*}
\]

Then, it is easy to see that \( \text{Sub}^{(1)}(G) \cup \text{Sub}^{(2)}(G) \) admits a decomposition

\[
\text{Sub}^{(1)}(G) \cup \text{Sub}^{(2)}(G) = \{ z_1(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \} \cup \{ z_1'(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \}.
\]

These notations together with the above give:

\[
\sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i x_i(P')
\]

\[
+ \sum_{n_{b-1} + 1 \leq i \leq n_d} \sum_{z_0 \in \text{Sub}^{(0)}(G)} \alpha_i \tilde{x}_i(z_1(z_0) + z_1'(z_0) + z_2(z_0))
\]

\[
= \sum_{n_{b-1} + 1 \leq i \leq n_d} \sum_{z_0 \in \text{Sub}^{(0)}(G)} \alpha_i \tilde{x}_i([z_1(z_0)] + [z_1'(z_0)] + [z_2(z_0)]).
\]

By Proposition 1 and by the condition for the case \( \epsilon_2 = 1 \), for any \( z_0 \in \text{Sub}^{(0)}(G) \),

\[
\sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i \tilde{x}_i([z_1(z_0)] + [z_1'(z_0)] + [z_2(z_0)]) = 0.
\]

Thus,

\[
\sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i x_i(P').
\]

Hence, \( \sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i x_i \) is invariant under strong RI.

- (Proof of the case \( \epsilon_3 = 1 \)) Since the arguments are essentially the same as that of the case \( \epsilon_2 = 1 \), we omit this proof.

- (Proof of the case \( \epsilon_4 = 1 \)) Let \( P \) and \( P' \) be two spherical curves where \( P \) and \( P' \) are related by a single strong RI, hence, there exist two Gauss words \( G = S_{ijT}kU_jk \) and \( G' = S_{ijT}kU_jk \) corresponding to \( P \) and \( P' \), respectively, i.e., \( CD_p = [S_{ijT}kU_jk] \) and \( CD_{p'} = [S_{ijT}kU_jk] \).

By (2.1) in Subsection 2.1 and (2.2) in Subsection 2.2, we obtain:

\[
\sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i \left( \sum_{z \in \text{Sub}(G)} \tilde{x}_i(z) \right)
\]

\[
= \sum_{n_{b-1} + 1 \leq i \leq n_d} \alpha_i \left( \sum_{z_01 \in \text{Sub}^{(0)}(G) \cup \text{Sub}^{(1)}(G)} \tilde{x}_i(z_01) + \sum_{z_23 \in \text{Sub}^{(2)}(G) \cup \text{Sub}^{(1)}(G)} \tilde{x}_i(z_23) \right).
\]
and
\[
\sum_{n_{b-1}+1 \leq i \leq nd} \alpha_i x_i(P') = \sum_{n_{b-1}+1 \leq i \leq nd} \alpha_i \left( \sum_{z' \in \text{Sub}(G')} \tilde{x}_i(z') \right)
\]
\[
= \sum_{n_{b-1}+1 \leq i \leq nd} \alpha_i \left( \sum_{z_0' \in \text{Sub}^{(0)}(G') \cup \text{Sub}^{(1)}(G')} \tilde{x}_i(z_0') + \sum_{z_{23}' \in \text{Sub}^{(2)}(G') \cup \text{Sub}^{(3)}(G')} \tilde{x}_i(z_{23}') \right).
\]

Since Sub^{(0)}(G) (Sub^{(1)}(G) resp.) is naturally identified with Sub^{(0)}(G') (Sub^{(1)}(G') resp.), the above equations show:
\[
\sum_{n_{b-1}+1 \leq i \leq nd} \alpha_i x_i(P) - \sum_{n_{b-1}+1 \leq i \leq nd} \alpha_i x_i(P')
\]
\[
= \sum_{n_{b-1}+1 \leq i \leq nd} \sum_{z_{23} \in \text{Sub}^{(2)}(G) \cup \text{Sub}^{(3)}(G)} \alpha_i \tilde{x}_i(z_{23})
\]
\[
- \sum_{n_{b-1}+1 \leq i \leq nd} \sum_{z_{23}' \in \text{Sub}^{(2)}(G') \cup \text{Sub}^{(3)}(G')} \alpha_i \tilde{x}_i(z_{23}').
\]

Let \(z_0 \in \text{Sub}^{(0)}(G)\), which is identified with Sub^{(0)}(G'). We note that since \(G\) is an oriented Gauss word \(z_0\) uniquely admits a decomposition into three sub-words, which are sub-words on \(S\), \(T\), and \(U\). Let \(\sigma(z_0)\) be the sub-word of \(S\), \(\tau(z_0)\) the sub-word of \(T\), and \(\mu(z_0)\) the sub-word of \(U\) satisfying \(z_0 = \sigma(z_0)\tau(z_0)\mu(z_0)\). We define maps
\[
z_3 : \text{Sub}^{(0)}(G) \to \text{Sub}^{(3)}(G); z_3(z_0) = \sigma(z_0)ij\tau(z_0)k\mu(z_0)jk,
\]
\[
z_{2a} : \text{Sub}^{(0)}(G) \to \text{Sub}^{(2)}(G); z_{2a}(z_0) = \sigma(z_0)ij\tau(z_0)j\mu(z_0)j,
\]
\[
z_{2b} : \text{Sub}^{(0)}(G) \to \text{Sub}^{(2)}(G); z_{2b}(z_0) = \sigma(z_0)ij\tau(z_0)k\mu(z_0), \text{ and}
\]
\[
z_{2c} : \text{Sub}^{(0)}(G) \to \text{Sub}^{(2)}(G); z_{2c}(z_0) = \sigma(z_0)j\tau(z_0)k\mu(z_0).\]

Similarly, let
\[
z_3' : \text{Sub}^{(0)}(G') \to \text{Sub}^{(3)}(G'); z_3'(z_0) = \sigma(z_0)ij\tau(z_0)k\mu(z_0)jk,
\]
\[
z_{2a}' : \text{Sub}^{(0)}(G') \to \text{Sub}^{(2)}(G'); z_{2a}'(z_0) = \sigma(z_0)ij\tau(z_0)j\mu(z_0)j,'
\]
\[
z_{2b}' : \text{Sub}^{(0)}(G') \to \text{Sub}^{(2)}(G'); z_{2b}'(z_0) = \sigma(z_0)ij\tau(z_0)k\mu(z_0), \text{ and}
\]
\[
z_{2c}' : \text{Sub}^{(0)}(G') \to \text{Sub}^{(2)}(G'); z_{2c}'(z_0) = \sigma(z_0)j\tau(z_0)k\mu(z_0)jk.
\]

Then, it is easy to see that Sub^{(2)}(G) \cup Sub^{(3)}(G) admits decompositions
\[
\text{Sub}^{(2)}(G) \cup \text{Sub}^{(3)}(G)
\]
\[
= \{ z_3(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \} \cup \{ z_{2a}(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \} \cup \{ z_{2b}(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \}
\]
\[
\cup \{ z_{2c}(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \}
\]
and
\[
\text{Sub}^{(2)}(G') \cup \text{Sub}^{(3)}(G')
\]
\[
= \{ z_3'(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \} \cup \{ z_{2a}'(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \} \cup \{ z_{2b}'(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \}
\]
\[
\cup \{ z_{2c}'(z_0) \mid z_0 \in \text{Sub}^{(0)}(G) \}.\]
By using these notations, we obtain:

\[
\sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i(P) - \sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i(P')
\]

\[
= \sum_{z_0 \in \text{Sub}^{(0)}(G)} \sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i \left( (z_3(z_0) + z_{2a}(z_0) + z_{2b}(z_0) + z_{2c}(z_0)) - (\tilde{z}_3'(z_0) + \tilde{z}_{2a}'(z_0) + \tilde{z}_{2b}'(z_0) + \tilde{z}_{2c}'(z_0)) \right)
\]

\[
- (\tilde{z}_3'(z_0) + \tilde{z}_{2a}'(z_0) + \tilde{z}_{2b}'(z_0) + \tilde{z}_{2c}'(z_0))
\]

\[
= \sum_{z_0 \in \text{Sub}^{(0)}(G)} \sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i \left( ([z_3(z_0)] + [z_{2a}(z_0)] + [z_{2b}(z_0)] + [z_{2c}(z_0)])
\]

\[
- ([\tilde{z}_3'(z_0)] + [\tilde{z}_{2a}'(z_0)] + [\tilde{z}_{2b}'(z_0)] + [\tilde{z}_{2c}'(z_0)]) \right).
\]

Here, we note that

\((\tilde{z}_3(z_0)]+[z_{2a}(z_0)]+[z_{2b}(z_0)]+[z_{2c}(z_0)]) - (\tilde{z}_3'(z_0)]+[\tilde{z}_{2a}'(z_0)]+[\tilde{z}_{2b}'(z_0)]+[\tilde{z}_{2c}'(z_0)]) \in R_{00010}.
\]

Hence, by the assumption of Case \( \epsilon_4 = 1 \) and by Proposition 1, for each \( z_0 \),

\[
\sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i \left( ([z_3(z_0)] + [z_{2a}(z_0)] + [z_{2b}(z_0)] + [z_{2c}(z_0)])
\]

\[
- ([\tilde{z}_3'(z_0)] + [\tilde{z}_{2a}'(z_0)] + [\tilde{z}_{2b}'(z_0)] + [\tilde{z}_{2c}'(z_0)]) \right)
\]

\[
= 0.
\]

These show that

\[
\sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i(P').
\]

Hence, \( \sum_{n_{n-1}+1 \leq i \leq n_d} \alpha_i x_i \) is invariant under strong RI\( \text{III} \).

\( \bullet \) (Proof of the case \( \epsilon_4 = 1 \).) Since the arguments are essentially the same as that of the case \( \epsilon_4 = 1 \), we omit this proof.

\[\square\]

4. Complete Analysis of Invariants Obtained from \( G_{2,4} \) and \( R_{00010}(2,4) \)

In this section, by using Corollary 2, we completely analyze invariants obtained from \( G_{2,4} \cap \text{Conn} \) and \( R_{00010}(2,4) \). First, it is easy to see that \( \{x_i\}_{i \in I_{2,4}^{\text{Conn}}} \) consists of nine elements, \( y_1 = \otimes \), \( y_2 = \otimes \), \( y_3 = \odot \), \( y_4 = \ominus \), \( y_5 = \oplus \), \( y_6 = \otimes \), \( y_7 = \otimes \), \( y_8 = \otimes \), and \( y_9 = \otimes \). The fact together with Corollary 2 implies that if there exist \( \alpha_i \) (\( 1 \leq i \leq 9 \)) such that \((\sum_{i=1}^{9} \alpha_i y_i)(r) = 0 \) (\( \forall r \in R_{00010}(2,4) \)), then \( \sum_{i=1}^{9} \alpha_i y_i \) is invariant under RI and strong RI\( \text{III} \). Second, recall that \( R_{00010}(2,4) \)
consists of twelve elements $r_j$ ($1 \leq j \leq 12$) as follows (see Example 3):

\[
\begin{align*}
  r_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
  r_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \\
  r_3 &= \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 2 & -2 & 1 & 0 & 0 & 0 \end{pmatrix} \\
  r_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 \end{pmatrix} \\
  r_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
  r_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 & 0 \end{pmatrix} \\
  r_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
  r_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
  r_9 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
  r_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
  r_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
  r_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

We can find $\alpha_i$ ($1 \leq i \leq 9$) satisfying $(\sum_{j=1}^{9} \alpha_i \tilde{y}_j)(r) = 0$ (\forall r \in R_{00010}(2;4)) by solving the linear equation $xM = 0$, where $M = (\tilde{y}_i(r_j))$ (Figure 12). It is elementary to show that the set of solutions is \{x = 1(1; -3; 3; 0; 0; 0; 0; 0; 0), 2(0; 0; 2; 8; 5; 1; -4; 2; -2) | $\gamma_1, \gamma_2 \in \mathbb{Z}$\}. Particularly, $y_1 - 3y_2 + 3y_3 (= \begin{pmatrix} 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix})$ and $2y_3 + 8y_4 + 5y_5 + y_6 - 4y_7 + 2y_8 - 2y_9 (= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix})$ are invariant under RI and strong RII.

Let $\lambda^{(3)} = y_1 - 3y_2 + 3y_3$ and let $\lambda^{(4)} = 2y_3 + 8y_4 + 5y_5 + y_6 - 4y_7 + 2y_8 - 2y_9$. Note that $\lambda^{(3)}$ is the function obtained in Example 5, hence $\lambda^{(3)} = 4\lambda$ where $\lambda$ is the function that appears in Theorem 1 of [5]. We further note that in [5], it is shown that $\lambda$ (hence, $\frac{1}{4}\lambda^{(3)}$) is a complete invariant of prime spherical curves up to seven double points (Table 1). We note that $\lambda^{(4)}$ is also a complete invariant.
of prime spherical curves up to seven double points. We further note that $\lambda^{(4)}$ is stronger than $\lambda^{(3)}$ in some sense. In fact, $\lambda(4_1) = \lambda(8_9) = 4$ and $\lambda^{(4)}(4_1) (= 16) \neq \lambda^{(3)}(8_9) (= 40)$. Similarly, we have $\lambda(5_1) = \lambda(8_2) = -5$ and $\lambda^{(4)}(5_1) (= -20) \neq \lambda^{(3)}(8_2) (= -28)$, and $\lambda(8_5) = \lambda(8_{17}) = 12$ and $\lambda^{(4)}(8_5) (= 128) \neq \lambda^{(3)}(8_{17}) (= 116)$.

Acknowledgements

The author would like to thank Professors Tsuyoshi Kobayashi and Kouki Taniyama for helpful discussions. The author also would like to thank Mr. Yusuke Takimura for creating and providing many figures of spherical curves with indices (Table 1) and for useful discussions. Further, the author would like to thank Professor Yukari Funakoshi and Ms. Megumi Hashizume for their comments on an earlier version of this paper. The author was a project researcher of Grant-in-Aid for Scientific Research (S) 24224002 (April 2016–March 2017).

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Table 1. Values of $\frac{1}{4} \lambda^{(3)} (= \lambda)$ and $\lambda^{(4)}$ for prime spherical curves at most seven double points. Any pair of prime spherical curves in the same box in the leftmost column are related by a finite sequence of Reidemeister moves of types RI and strong RIII.

| P  | $\frac{1}{4} \lambda^{(3)}$ | $\lambda^{(4)}$ |
|----|-----------------|----------------|
| $\gamma_1$ | -21             | -140           |
| $\gamma_2$ | -5              | -20            |
| $\gamma_3$ | -5              | -28            |
| $\gamma_4$ | -3              | -44            |
| $\gamma_5$ | 0               | 0              |
| $\gamma_6$ | 4               | 16             |
| $\gamma_7$ | 4               | 40             |
| $\gamma_8$ | 5               | 36             |
| $\gamma_9$ | 12              | 128            |
| $\gamma_{17}$ | 12          | 116            |
| $\gamma_{19}$ | 14             | 120            |
| $\gamma_{19}$ | 15             | 148            |