New brane solutions in higher order gravity

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Abstract

We consider the higher order gravity with dilaton and with the leading string theory corrections taken into account. The domain wall type solutions are investigated for arbitrary number of space–time dimensions. The explicit formulae for the fixed points and asymptotic behavior of generic solutions are given. We analyze and classify solutions with finite effective gravitational constant. There is a class of such solutions which have no singularities. We discuss in detail the relation between fine tuning and self tuning and clarify in which sense our solutions are fine–tuning free. The stability of such solutions is also discussed.

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1 Introduction

Theory of closed strings incorporates gravity in a natural way since the spin-2 field is always present in its massless spectrum. From this point of view theory of gravity is an effective theory of the graviton when the energies are smaller than the mass of the string first excited level. String theory contains many other massless fields and in an effective action one expects the presence of all powers of the fields – indeed the calculations [1] of the string amplitudes involving the graviton, dilaton and the antisymmetric tensor confirm that the string effective action contains higher order terms. The expansion parameter in string theory $\alpha'$ is dimensionful and connected to the fundamental string length $l_s$ by $\alpha' = l_s^{-2}$. Since the effective action was calculated in [1] (to order $\alpha'^2$) by comparison with the string amplitudes on-shell, we cannot distinguish in this way actions differing by lower order equations of motion. However these actions from the field theory point of view are different and the equations of motion have different solutions. Since we don’t know string amplitudes off-shell we have to employ other methods to try to guess the true effective action – one possible idea is to look for an action with some additional, string motivated symmetry. In [2] it was proven that one of the actions for gravity coupled to the dilaton and the antisymmetric tensor at order $\alpha'^2$ exhibits an $O(d,d)$ symmetry characteristic for string effective actions [3]. Therefore in the present paper we use the form of the action obtained in [2] and analyze the presence of the domain wall type solutions of the equations of motion.

It was shown in [4] that in the presence of arbitrary number of Euler densities in the Lagrangian (without the dilaton), there always exists a domain wall solution of the Randall–Sundrum type [5]. In the present paper we analyze the gravity theory coupled to the dilaton with interaction terms up to the fourth order in the derivatives. Properties of the generic domain wall type solutions are investigated. We are interested mainly in those solutions which allow for localized gravity with finite effective gravitational constant after integrating over the coordinate perpendicular to the brane. We find that both with and without the cosmological constant the presence of the dilaton allows for solutions of the Randall–Sundrum type [5] as well as solutions with singularities of the type described in [6]. We find also some new cases specific to higher order gravity. It turns out that some of the solutions without singularities have finite effective gravitational constant without fine tuning of the parameters of the Lagrangian. The problem of the absence of fine tuning and potential self tuning (for the discussion in the case of the lowest order gravity see [7]) of these solutions is addressed in great detail. We explain the relation between our fine-tuning free solutions and some no-go theorems [8, 9] derived for theories similar to the one consider in the present paper. Recently one similar solution for a 5–dimensional theory has been found in ref. [10]. However the higher order terms used in [10] are as considered in ref. [11] and differ from those adopted in the preset work (we explain the reason for this difference in the next section). Moreover, we present more solutions of this type and our analysis can be applied to theories in arbitrary dimensions. We discuss also, contrary to [10], the stability of such fine tuning free solutions. Some exact solutions in higher order gravity have been recently discussed in [12].
The paper is organized as follows: In section 2 we present the Lagrangian and derive the equations of motion in general and in the application to the brane metric. In section 3 we explicitly find fixed points and asymptotic behavior of solutions. The number of the fixed points and the behavior of asymptotics depends very much on the value of the bulk cosmological constant. In section 4 we analyze and classify solutions with finite effective gravitational constant - the finiteness is due either to the compactness of the transverse dimension or to the decreasing warp factor like in the Randall-Sundrum scenario. Section 5 is devoted to the detailed discussion of the equations of motion at the brane. In section 6 we discuss the problem of fine tuning of the solutions with finite effective gravitational constant and clarify in which sense our solutions avoid fine tuning. The relation of our results to some no–go theorems [8, 9] is also explained. In section 7 we present the conclusions.

2 Higher order action and equations of motion

In theories with gravity coupled to the dilaton one has to decide upon the so called frame (i.e. choice of metric) since one can redefine the metric by a (non-vanishing) factor depending on the dilaton. There are usually two frames used: the string frame (with the factor \( e^{-\phi} \) in front of the curvature scalar in the string tree level effective action) and the Einstein frame (without this factor). The metrics solving the equations of motion can have quite different properties in the two frames (for a discussion in the cosmological context see [13]). In the present Universe and with the presently available energies the difference between the frames is unimportant since the dilaton seems to be (by a still unknown mechanism) constant both in time and in space, but both in the early Universe or, as discussed in this paper, close to a brane, the difference can be quite substantial. It is generally argued that although the physical conclusions should not depend on the redefinition of fields, it is much easier to draw these conclusions in the string frame than in the Einstein frame and the "translation" from one to the other is not always evident. Therefore we will use the string frame in this paper. It should be pointed out that the higher order terms considered in refs. [10, 12] are different from those used in the present paper even after changing the frame.

The string motivated action we consider reads [2] (note a difference by a factor of 2 in normalization of the dilaton \( \phi \)):

\[
S = S_0 + S_1 + S_B = \int d^D x (L_0 + \alpha L_1 + L_B) ,
\]

where \( \alpha = \alpha' \lambda_0 \) and

\[
L_0 = \frac{1}{2\kappa^2} \sqrt{-g} e^{-\phi} \left[ -2\Lambda + R + \partial_\mu \phi \partial^\mu \phi \right] ,
\]

\[
L_1 = \frac{1}{2\kappa^2} \sqrt{-g} e^{-\phi} \left[ -R_{GB}^2 - 2\partial_\mu \phi \partial^\mu \phi \square \phi 
+ 4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \phi \partial_\nu \phi + (\partial_\mu \phi \partial^\mu \phi)^2 \right] ,
\]
\[ L_B = \frac{1}{2\kappa^2} \sqrt{-\tilde{g}} V(\phi) \delta(y). \]  
\[ (4) \]

\[ R_{GB} \] is the Gauss–Bonnet term
\[ R_{GB}^2 = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2. \]
\[ (5) \]

The space–time has arbitrary dimension \( D \). The metric in eq. (4) is the induced metric on the brane
\[ \tilde{g}_{\alpha\beta} = g_{\mu\nu} \delta^\mu_{\alpha} \delta^\nu_{\beta}. \]
\[ (6) \]

The indices have the following ranges: \( \mu, \nu = 1, \ldots, D; \alpha, \beta = 1, \ldots, d; \) where \( d = D - 1 \).

Let us now calculate the functional derivative of this action (separately for each term in (1)) with respect to variations of the metric and to variations of the dilaton. We define
\[ T^{(I)}_{\mu\nu} = 2\kappa^2 \frac{\phi}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^D x L_I, \]
\[ W^{(I)} = 2\kappa^2 \frac{\phi}{\sqrt{-g}} \frac{\delta}{\delta \phi} \int d^D x L_I, \]
\[ (7, 8) \]

for \( I = 0, 1, B \). A rather tedious calculation gives for \( T_{\mu\nu} \)
\[ T^{(0)}_{\mu\nu} = 2\Lambda g_{\mu\nu} + R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} g_{\mu\nu} \partial_\sigma \phi \partial^\sigma \phi + \nabla_\mu \partial_\nu \phi - (\Box \phi) g_{\mu\nu}, \]
\[ (9) \]

\[ T^{(1)}_{\mu\nu} = 4 R^{\kappa\lambda} R_{\kappa\mu\lambda\nu} - 2 R_{\rho\sigma\kappa\lambda} R^\sigma_{\mu\nu} + 4 R_{\mu\kappa} R^\kappa_{\nu} - 2 R R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R_{GB}^2 \]
\[ + 4(\nabla^\kappa \partial^\lambda \phi) R_{\mu\kappa\nu\lambda} + 3 \partial_\kappa \phi \partial^\kappa \phi R_{\mu\nu} + 8(\nabla_\kappa \partial_\mu \phi) R^\kappa_{\nu} - 4(\Box \phi) R_{\mu\nu} \]
\[ - 4(\nabla_\kappa \partial_\lambda \phi) R^{\kappa\lambda} g_{\mu\nu} - 2(\partial_\kappa \phi \partial^\kappa \phi) R g_{\mu\nu} - 2(\nabla_\mu \partial_\nu \phi) R + 2(\Box \phi) R g_{\mu\nu} \]
\[ - 2 \partial^\kappa \phi \partial_\kappa \phi \Box g_{\mu\nu} - 2(\nabla^\kappa \partial^\lambda \phi)(\nabla_\kappa \partial_\mu \phi) g_{\mu\nu} + 2 \partial_\kappa \phi \partial^\kappa \phi \nabla_\mu \partial_\nu \phi \]
\[ + 4(\nabla_\mu \partial_\nu \phi)(\nabla_\nu \partial^\kappa \phi) - 4(\nabla_\mu \partial_\nu \phi) \Box \phi + \frac{1}{2}(\partial^\kappa \phi \partial_\kappa \phi)^2 g_{\mu\nu}, \]
\[ (10) \]

\[ T^{(B)}_{\mu\nu} = -\frac{\phi}{2} \tilde{g}_{\alpha\beta} \delta^\alpha_\mu \delta^\beta_\nu V(\phi) \delta(y). \]
\[ (11) \]

Calculation of the functional derivatives with respect to \( \phi \) is straightforward
\[ W^{(0)} = 2\Lambda + \partial_\kappa \phi \partial^\kappa \phi - 2 \Box \phi - R, \]
\[ (12) \]

\[ W^{(1)} = R_{GB}^2 + 4 \left( R^{\kappa\lambda} - \frac{1}{2} R g^{\kappa\lambda} \right) (\partial_\kappa \phi \partial_\lambda \phi - 2 \nabla_\kappa \partial_\lambda \phi) + (\partial_\kappa \phi \partial^\kappa \phi)^2 \]
\[ + 4(\Box \phi)^2 - 4(\nabla^\kappa \partial^\lambda \phi)(\nabla_\kappa \partial_\lambda \phi) - 4 R^{\kappa\lambda} \partial_\kappa \phi \partial_\lambda \phi, \]
\[ (13) \]

\[ W^{(B)} = e^\phi V'(\phi) \delta(y). \]
\[ (14) \]
The equations of motion read

\[ T^{(0)}_{\mu\nu} + \alpha T^{(1)}_{\mu\nu} + T^{(B)}_{\mu\nu} = 0, \quad (15) \]
\[ W^{(0)} + \alpha W^{(1)} + W^{(B)} = 0. \quad (16) \]

Let us note that in the bulk, i.e. outside of branes and singularities, we have the equality (up to boundary terms) that can be easily checked

\[ S = -\frac{1}{2\kappa^2} \int d^D x \sqrt{-g} e^{-\phi} \left[ W^{(0)} + \alpha W^{(1)} \right]. \quad (17) \]

It means that on–shell i.e. on the equations of motion the effective cosmological constant has no contribution from the bulk. On physical grounds, however, we are interested in the solutions for which the total (i.e. including branes) effective cosmological constant vanishes on–shell. Anticipating the results of section 5, for fixed point solutions it happens for one form of the potential on the brane, namely \( V(\phi) = \lambda e^{-\phi} \) i.e. the potential suggested by string theory (for NS fields for which the bulk action is considered). For special solutions which are not exactly at the fixed points it happens for larger class of brane potentials. Then the boundary terms from the bulk cancel the brane contribution and we have

\[ S = -\frac{1}{2\kappa^2} \int d^D x \sqrt{-g} e^{-\phi} \left[ W^{(0)} + \alpha W^{(1)} + W^{(B)} \right]. \quad (18) \]

where the integral is over the full space including brane positions. Since it is proportional to the equations of motion, it vanishes on–shell so the effective cosmological constant vanishes as well.

We now start with solving the equations in the bulk and later we will include \( T^{(B)}_{\mu\nu} \) and \( W^{(B)} \). The line element of the domain wall background is equal to

\[ ds^2 = e^{2A(y)} \eta_{\alpha\beta} dx^\alpha dx^\beta + dy^2. \quad (19) \]

The three equations of motion – \( \alpha \beta \) and \( \eta \eta \) components of (15), and (16) – read respectively:

\[ 0 = \Lambda + (d - 1)A'' - \phi'' \]
\[ + \frac{1}{2} d(d - 1)(A')^2 - (d - 1)A' \phi' + \frac{1}{2} (\phi')^2 \]
\[ + \alpha \left[ 2(d - 1)(d - 2)(d - 3)A''(A')^2 - 4(d - 1)(d - 2)A''A' \phi' + 2(d - 1)A''(\phi')^2 \right. \]
\[ - 2(d - 1)(d - 2)\phi''(A')^2 + 4(d - 1)\phi'' \phi' A' - 2\phi''(\phi')^2 \]
\[ + \frac{1}{2} d(d - 1)(d - 2)(d - 3)(A')^4 - 2(d - 1)^2(d - 2)(A')^3 \phi' \]
\[ + (3d - 4)(d - 1)(A')^2(\phi')^2 - 2(d - 1)A'(\phi')^3 + \frac{1}{2} (\phi')^4 \], \quad (20) \]

\[ 0 = \Lambda + \frac{1}{2} d(d - 1)(A')^2 - dA' \phi' + \frac{1}{2} (\phi')^2 \]
\[ +\alpha \left[ \frac{1}{2} d(d - 1)(d - 2)(d - 3)(A')^4 - 2d(d - 1)(d - 2)(A')^3\phi' \right. \\
\left. +3d(d - 1)(A')^2(\phi')^2 - 2dA'(\phi')^3 + \frac{1}{2}(\phi')^4 \right] , \quad (21) \]

\[ 0 = \Lambda + dA'' - \phi'' + \frac{1}{2}d(d + 1)(A')^2 - dA'\phi' + \frac{1}{2}(\phi')^2 + \alpha \left[ 2d(d - 1)(d - 2)A''(A')^2 - 4d(d - 1)A''A'\phi' + 2dA''(\phi')^2 \\
-2d(d - 1)\phi''(A')^2 + 4d\phi''\phi' A' - 2\phi''(\phi')^2 + \frac{1}{2}d(d + 1)(d - 1)(d - 2)(A')^4 \\
-2d^2(d - 1)(A')^3\phi' + d(3d - 1)(A')^2(\phi')^2 - 2dA'(\phi')^3 + \frac{1}{2}(\phi')^4 \right] . \quad (22) \]

These equations have two obvious symmetries

\[ y \to y + y_0, \quad A \to A + A_0, \quad \phi \to \phi + \phi_0 , \quad (23) \]

and

\[ y \to -y, \quad A' \to -A', \quad \phi' \to -\phi' , \quad (24) \]

which can be used to obtain a whole class of solutions starting from each particular one.

Only two of the equations (20)–(22) are independent in the bulk. One of the combinations of those equations is particularly simple. Multiplying the difference of (20) and (22) by \( \exp(\phi - dA) \) we get an expression which is a total derivative with respect to \( y \). Integrating it we obtain

\[ e^{dA-\phi} \left\{ A' \left[ 1 + 2\alpha \left( (d - 1)(d - 2)A'^2 - 2(d - 1)A'\phi' + \phi'^2 \right) \right] \right\} = \text{const} . \quad (25) \]

This equation will prove to be very useful when analyzing the fixed points of the equations of motion.

### 3 Fixed points and asymptotic behavior of solutions

Let us find the fixed points of the equations of motion (20)–(22) (i.e. values of \( A' \) and \( \phi' \) for which \( A'' = \phi'' = 0 \)). We denote

\[ A' = \frac{a_1}{\sqrt{\alpha}}, \quad \phi' = \frac{b_1}{\sqrt{\alpha}} , \quad (26) \]

where \( a_1, b_1 \) are constants. Substituting constant derivatives \( A' \) and \( \phi' \) into eq. (25) we see that it can be fulfilled only in two ways: either \( \phi' = dA' \) (only then the exponential factor of (25) is constant) or the expression in the curly bracket in (25) must vanish. It is not difficult
to check that only the second possibility is compatible with eqs. (20) and (21). Thus, every fixed point of eqs. (20)–(22) must satisfy
\[ a_1 \left[ 1 + 2(d - 1)(d - 2)a_1^2 - 4(d - 1)b_1 + 2b_1^2 \right] = 0. \]  
(27)

There are two types of fixed points corresponding to two ways in which the above equation can be fulfilled. The first type has vanishing $a_1$. The constant $b_1$ can be then calculated from eq. (21). The result reads:
\[ a_1 = 0, \quad b_1 = \pm \sigma, \]  
(28)
where
\[ \sigma = \sqrt{1 - 8\alpha \Lambda - 1}. \]  
(29)
The number of such fixed points depends on the sign of the bulk cosmological constant: there are two for $\Lambda < 0$, one for $\Lambda = 0$ and none for $\Lambda > 0$. The solution for the vanishing bulk cosmological constant has $a_1 = b_1 = 0$ and is just the 5–dimensional Minkowski space–time.

For the fixed points with nonzero $a_1$ the expression in the square bracket in eq. (27) must vanish. This gives a relation between $a_1$ and $b_1$ which together with eq. (21) gives the second type of the fixed point:
\[ a_1 = \epsilon_a \gamma, \quad b_1 = \epsilon_a (d - 1) \gamma + \epsilon_b \sqrt{(d - 1) \gamma^2 - \frac{1}{2}}, \]  
(30)
where $\epsilon_a, \epsilon_b = \pm 1$ and $\gamma$ can take one of the two possible values, $\gamma_+$ or $\gamma_-$, given by
\[ \gamma_{\pm} = \sqrt{\frac{1}{2(d - 2)} \left( 1 \pm \sqrt{\frac{d}{2(d - 1)} + \frac{4\alpha \Lambda(d - 2)}{d - 1}} \right)}. \]  
(31)
The requirement that the parameters $a_1$ and $b_1$ are real gives two possibilities:
\[ \gamma = \gamma_+ \quad \text{for} \quad \Lambda \geq \Lambda_0, \]  
(32)
or
\[ \gamma = \gamma_- \quad \text{for} \quad \Lambda_1 \geq \Lambda \geq \Lambda_0, \]  
(33)
where
\[ \Lambda_0 = -\frac{d}{8\alpha(d - 2)}, \]  
(34)
\[ \Lambda_1 = \Lambda_0 + [4\alpha(d - 1)(d - 2)]^{-1}. \]  
(35)
If solutions for $\gamma = \gamma_+ \ (\gamma = \gamma_-)$ exist, there are four of them corresponding to four combinations of signs $\epsilon_a$ and $\epsilon_b$ in eq. (30). The solutions for $\gamma = \gamma_-$ exist only for a rather small range (33) of values of the bulk cosmological constant.
Adding all the above types of solutions we can see that the eqs. (20)–(22) have the following number of fixed points: 2 for $\Lambda < \Lambda_0$; 4 for $\Lambda = \Lambda_0$; 10 for $\Lambda_0 < \Lambda < \Lambda_1$; 8 for $\Lambda = \Lambda_1$; 6 for $\Lambda_1 < \Lambda < 0$; 5 for $\Lambda = 0$; 4 for $\Lambda > 0$.

Let us now analyze the asymptotic behavior of $A'$ and $\phi'$. There are different types of behavior related to each of the allowed fixed points. One can find also asymptotic behavior for $A'$ and $\phi'$ going to infinity. The results are as follows:

- There is one fixed point with $A' = 0 = \phi'$ if $\Lambda = 0$. The asymptotic behavior for $y \to \pm \infty$ is given by
  \[ A' = \frac{1}{\sqrt{d|y|}}, \quad \phi' = \frac{\sqrt{d - \text{sgn}(y)}}{|y|}. \]  

- For the two fixed points with $A' = 0$ for $\Lambda < 0$ and $y \to \pm \infty$ we have
  \[ A' = a_2 \exp(-\sigma|y|/\sqrt{\alpha}), \quad \phi' = -\text{sgn}(y)\frac{\sigma}{\sqrt{\alpha}} + da_2 \exp(-\sigma|y|/\sqrt{\alpha}), \]  

  where $\sigma$ is given by eq. (29).

- If the bulk cosmological constant satisfies the condition $\Lambda_0 < \Lambda$ there are four fixed points $A' = \pm \gamma/\sqrt{\alpha}$ with $\gamma = \gamma_+$ corresponding to four combinations of signs $\epsilon_a$, $\epsilon_b$ in the expression for $b_1$ given in eq. (30). In addition if $\Lambda_0 < \Lambda < \Lambda_1$ there are four fixed points with $\gamma = \gamma_-$. Half of these fixed points are approached for $y \to +\infty$ and the other half for $y \to -\infty$. Let us analyze solutions approaching the fixed points for $y \to \infty$. The solutions can approach them either from above ($a_3 > 0$) or from below ($a_3 < 0$) and their asymptotic behavior is given by
  \[ A' = \epsilon_a \frac{\gamma}{\sqrt{\alpha}} + a_3 \exp(-cy/\sqrt{\alpha}), \quad \phi' = \frac{b_1}{\sqrt{\alpha}} + b_3 \exp(-cy/\sqrt{\alpha}), \]  

  where
  \[ c = \epsilon_a \gamma - \epsilon_b \sqrt{(d-1)\gamma^2 - \frac{1}{2}}. \]  

  \[ b_3/a_3 = d \left[ 1 - \left( 2(d-1)\gamma \left( \epsilon_a \epsilon_b \sqrt{(d-1)\gamma^2 - \frac{1}{2}} + \gamma \right) \right) \right]^{-1}. \]  

  From the condition $c > 0$ and the fact (cf. (31)) that for $d > 2$
  \[ \sqrt{(d-1)\gamma_+^2 - \frac{1}{2}} > \gamma_+, \quad \sqrt{(d-1)\gamma_-^2 - \frac{1}{2}} < \gamma_-. \]  

  we conclude that solutions with $\gamma = \gamma_+$ approach (as $y \to \infty$) the fixed point when $\epsilon_b = -1$ while those with $\gamma = \gamma_-$ when $\epsilon_a = 1$.  

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There are vertical asymptotes for each value of $y$. Denoting the location of the vertical asymptote as $y_0$ one can check that $|A'|$ and $|\phi'|$ go to infinity for $y \to y_0$ as

$$A' = \frac{c_i}{y - y_0}, \quad \phi' = \frac{dc_i - 3}{y - y_0}, \quad (42)$$

where $c_i$ are four solutions (two positive and two negative) of the equation

$$d(d - 2)c_i^4 + 8dc_i^3 - 18dc_i^2 + 27 = 0. \quad (43)$$

$A'$ is positive for positive (negative) $c_i$ and $y > y_0$ ($y < y_0$). The solutions with negative $A'$ are as usual related to those with $A' > 0$ by the symmetry (24). Let us note that $c_i < 3/2$ for $d > 2$.

Figure 1: Typical solutions of the equations of motion (20)–(22) for $d < 9$ and the bulk cosmological constant in the range $\Lambda_1 < \Lambda < 0$. Only solutions with positive $A'$ are shown. The cases with negative $A'$ can be obtained by means of symmetry (24). From the symmetry (23) it follows that any of the solutions gives another solution when shifted by arbitrary constant along the $y$ coordinate. The circles denote singularities of type B (see discussion in section 4) for which the metric is non zero but the curvature tensor diverges.

The above discussed fixed points and asymptotic behavior have been obtained analytically from the equations of motion (20)–(22). Of course it is not possible to find exact expressions for solutions which are not at the fixed points. One has to use numerical methods to
Figure 2: Same as figure 1 but for $d = 9$

Figure 3: Same as figure 1 but for $d > 9$
investigate such generic solutions. On figures 1–3 we present some typical results. The character of some of the solutions depends on the dimension of the space–time and changes at \(d = 9\).

## 4 Solutions with finite gravitational constant

The solutions of the equations of motion discussed in the previous section may be used to construct different types of domain wall models. Of course the most interesting are those cases for which the lower dimensional effective gravitational constant is finite. The effective lower dimensional gravitational constant is obtained by integrating the metric over the direction perpendicular to the wall. The are two types of situations when that constant is finite. First, the range of the \(y\) coordinate may be finite because the space ends at some singularities. This type of solutions was discussed for the lowest order string inspired dilaton gravity in [6]. Second, the range of \(y\) may be infinite but with the warp factor decreasing strongly enough to make the integral finite. This type of warped solutions is not present in the lowest order theory discussed in [6] but appears in the gravity theory without dilaton at the lowest order [5] and in higher orders [4].

In the higher order string inspired dilaton theory investigated in the present work both mechanisms to get a finite effective gravitational constant can be realized. In fact we can define even three classes of such models because there are two different types of singularities which can be used to cut off the space.

Before we present all those classes of solutions a comment on the effective gravitational constant is in order.

In calculating the effective gravitational coupling we assume that the \(D\)-dimensional metric takes the form
\[
ds^2 = e^{2A(y)}g_{\alpha\beta}^{(d)}dx^\alpha dx^\beta + dy^2.
\]
and
\[
\phi = \phi(y) + \varphi(x^\alpha).
\]
We then calculate \(D\)-dimensional quantities in terms of \(d\) dimensional ones:
\[
R_{\alpha\beta\gamma\delta} = e^{2A}R_{\alpha\beta\gamma\delta}^{(d)} - e^{4A}A'^4 \left(g_{\alpha\gamma}^{(d)}g_{\beta\delta}^{(d)} - g_{\alpha\delta}^{(d)}g_{\beta\gamma}^{(d)}\right).
\]
and
\[
R_{\alpha y\beta y} = -e^{2A}g_{\alpha\beta}^{(d)} \left(A'^2 + A''\right).
\]
Hence we have\(^3\)
\[
R_{\alpha\beta} = R_{\alpha\beta}^{(d)} - e^{2A}g_{\alpha\beta}^{(d)} (dA'^2 + A'')
\]
\[
R_{yy} = -d \left(A'^2 + A''\right)
\]
\(^3\)Note that our formulae are different from the ones derived in [12] and used in [10]
\[ R = e^{-2A} R^{(d)} - d \left( (d + 1) A'^2 + 2A'' \right) \]
\[ R_{GB} = e^{-4A} R_{GB}^{(d)} - 2(d-2) e^{-2A} R^{(d)} \left( (d-1) A'^2 + 2A'' \right) + d(d-1)(d-2) \left( (d+1) A'^4 + 4A'' A'' \right) \]

Then the result of the integration over \( y \) of the action (1) gives the string frame \( d \)-dimensional action
\[ \int d^d x \sqrt{-g^{(d)}} e^{-\phi(x)} \left( C_1 + C_2 R^{(d)} + \ldots \right) . \]

The \( d \)-dimensional cosmological constant is proportional to \( C_1 \) i.e. to the integral
\[ \int dy \exp \left[ dA(y) - \phi(y) \right] P_1(A', A'', \phi', \phi'') , \]
while the inverse of the \( d \)-dimensional gravitational constant is proportional to \( C_2 \) i.e. to the integral
\[ \int dy \exp \left[ (d-2)A(y) - \phi(y) \right] P_2(A', A'', \phi', \phi'') . \]

where \( P_1(A', A'', \phi', \phi'') \) and \( P_2(A', A'', \phi', \phi'') \) are some polynomials. If these polynomials do not vanish on the solution for other reasons (as happens for the cosmological constant on-shell) the behaviour of the exponential allows us to find whether a given solution has finite or infinite cosmological and gravitational constants.

Let us discuss all possible types of solutions which give finite contributions to the effective gravitational constant and which can describe the part of the \( D \)-dimensional space–time on one side of the \( d \)-dimensional brane.

- **Type A**
  For each \( d \) and for each \( y_L \) there are solutions given by eq. (42) for which \( \lim_{y \to y_L} A' = \pm \infty \) Asymptotically we have
  \[ (d-2)A' - \phi' = \frac{3 - 2c_i}{y - y_0} . \]

Since for \( d > 2 \ c_i < 3/2 \) for all solutions of eq. (43) therefore the integral (51) is finite and these solutions have finite gravitational constant (the corresponding solutions have singularities of the type discussed in the lowest order theory in [6]). We can use a part of any of those solutions for \( y \in (y_L, y_0) \) to describe the space–time between the brane located at \( y = y_0 \) and the singularity located at \( y = y_L < y_0 \) (the singularity is located to the left from the brane). The metric is bounded from above over the whole interval \((y_L, y_0)\). The length of this interval itself is also finite. Thus, the contribution to the effective gravitational coupling coming from such solution is finite.

The above discussed warped metric was obtained by using solutions with \( A' > 0 \) and can describe part of the space–time “to the left” \((y < y_0)\) from the brane. The analogous metric for \((y > y_0)\) can be obtained in the same way from solutions with \( A' < 0 \) which are related by the symmetries (23), (24).
• Type B
For each $d$ and for each $y_s$ there are eight solutions with infinite $|A''(y_s)|$ but finite values of $A'(y_s)$, four with positive and four with negative $A'$ (they are denoted by circles on figures 1–3). For half of them the solution is defined for $y > y_s$ and we can use part of such a solution for $y \in (y_L = y_s, y_0)$ to describe the space–time between the singularity and the brane if $y_s < y_0$. Other solutions are defined for $y < y_s$ and their part for $y \in (y_0, y_R = y_s)$ can be used when the singularity is to the right from the brane.

All these solutions have singularities at $y = y_s$ but of different nature than the solutions of type A. Now the metric is non vanishing at $y_s$ but the curvature is singular. We assume that such a singularity can also be used to cut off the space–time.

• Type C
Now we check whether there are solutions without any singularities which nevertheless give finite contributions to the effective cosmological constant. Any of such solutions must be well defined for all values of $y > y_0$ (or $y < y_0$). Let us start with solutions which approach asymptotically any of the fixed points for $y \to \infty$ ($y \to -\infty$). Their asymptotic behavior described by eqs. (36)–(40) is enough to determine whether they can produce a finite gravitational constant.

The solution (36) is not appropriate because $A'$ and $\phi'$ tend to zero and the warp factor approaches a constant for $y \to \pm \infty$. The integral over $y \in (y_0, \infty)$ must be infinite.

The solutions described by eq. (37) also give infinite contributions to the effective gravitational constant. We have asymptotically

$$ (d-2)A' - \phi' \to \frac{\text{sgn}(y)}{\sqrt{\alpha}} \sigma. $$ (53)

After integrating over $y$ we see that this expression diverges when $|y| \to \infty$ for both types of solutions: those which approach a fixed point for $y \to +\infty$ as well as for those approaching a fixed point for $y \to -\infty$.

Let us now consider the solutions described by eq. (38) for which

$$ (d-2)A' - \phi' \to -\frac{\epsilon_a \gamma + \epsilon_b \sqrt{(d-1)\gamma^2 - 1}}{\sqrt{\alpha}} $$ (54)

as $y \to \pm \infty$. Using eqs. (39) and (41) one can see that the sings of $(d-2)A' - \phi'$ and $c$ are the same for $\gamma = \gamma_+$ while they are opposite for $\gamma = \gamma_-$. This means that the solutions approaching the fixed points with $A' = \pm \gamma_- / \sqrt{\alpha}$ can have finite effective gravitational coupling in $d$ dimensions and those with $A' = \pm \gamma_+ / \sqrt{\alpha}$ can not.

After analyzing which solutions approaching asymptotically the fixed points in $y = \pm \infty$ lead to a finite effective gravitational constant in $d$ dimensions we have to analyze the
isolated fixed points. The one with vanishing $A'$ and $\phi'$ (given by eq. (28) for $\Lambda = 0$) is not appropriate because it describes just a Minkowski space with a constant metric.

The situation is more interesting for the two fixed points of the form (28) for negative bulk cosmological constant $\Lambda$. The warp factor in the string frame is still constant ($A' = 0$) but the dilaton is not. We get

$$\int dy \exp[(d - 2)A(y) - \phi(y)] = \exp [(d - 2)A_0 - \phi_0] \int dy \exp \left[-\epsilon \frac{\sigma}{\sqrt{\alpha}} y\right]. \quad (55)$$

for $\phi' = \epsilon \sigma$ with $\sigma$ given by eq. (29). This contribution is finite for the fixed point solution with positive $\phi' = \sigma$ in the region $y \in (y_0, \infty)$ and for the fixed point solution with negative $\phi' = -\sigma$ in the region $y \in (-\infty, y_0)$. Thus the fixed point solution $A' = 0, \phi' = \sigma$ describes a Randall–Sundrum type warped infinite half–space to the right from the brane while that with $A' = 0, \phi' = -\sigma$ the one to the left from the brane.

The situation is similar with the fixed points with non zero $A'$ given by eqs. (30) with $\gamma = \gamma_+$. The contribution to the gravitational constant is equal to

$$\exp [(d - 2)A_0 - \phi_0] \int dy \exp \left[-\left(\frac{\epsilon_a \gamma + \epsilon_b \sqrt{(d - 1)\gamma^2 - \frac{1}{4}}}{\sqrt{\alpha}}\right) y\right]. \quad (56)$$

Using eqs. (39) and (41) it is easy to check that the sign of the expression in the round bracket above is the same as the sign of $\epsilon_b$. Thus, the fixed point solutions (30) with $\epsilon_b = +1$ ($\epsilon_b = -1$) give finite gravitational coupling after integrating over the region $y \in (y_0, +\infty)$ ($y \in (-\infty, y_0)$). These fixed points are isolated since these conditions on $\epsilon_b$ are just the opposite to the conditions for the existence of asymptotic solutions.

For $\gamma = \gamma_-$ fixed points with finite gravitational constant are always accompanied by asymptotic solutions since (as we have shown) the conditions on $\epsilon_a$ coincide with conditions for the existence of asymptotic solutions.

We have shown in this section that there are three types of half–spaces which give finite contributions to the effective $d$–dimensional gravitational coupling. For two of them the space–time ends at singularities with divergent curvature. They have different behavior of the metric at the singularity, it vanishes for type A while it is non zero for type B solutions. Type C solutions have no singularities and extend to infinity.

All of the above solutions can describe one side of the $D$–dimensional space–time. Taking two solutions we can construct a full space–time on both sides of the brane. There are many types of such warped space–times depending on the solutions used to build them. We can denote them as: A-0-A, B-0-B or C-0-C where the 0 in X-0-X denotes a brane (which always can be put at $y = 0$) between two solutions of type X (one could consider also non symmetric spaces like e.g. A-0-C). The spaces of type A-0-A are similar to those considered in ref. [6] in the lowest order theory. Spaces of type B-0-B also end at two singularities but the character
of those singularities is different. The curvature becomes singular like in the A-0-A case but the metric itself is nonsingular. In case C-0-C the space–time is infinite and warped in such a way that the $d$–dimensional effective gravitational constant is finite (similarly to the space of the Randall–Sundrum type).

Fixed point solutions of the C-0-C type can be written explicitly. Those leading to a finite gravitational coupling are given by:

$$A' = 0, \quad \phi' = \text{sgn}(y) \frac{\sigma}{\sqrt{\alpha}}, \quad (57)$$

or

$$A' = -\text{sgn}(y) \frac{\gamma_+}{\sqrt{\alpha}}, \quad \phi' = -\text{sgn}(y) \left( \frac{(d-1)\gamma_+ - \sqrt{(d-1)\gamma^2_+ - \frac{1}{2}}}{\sqrt{\alpha}} \right), \quad (58)$$

or

$$A' = \text{sgn}(y) \frac{\gamma_-}{\sqrt{\alpha}}, \quad \phi' = \text{sgn}(y) \left( \frac{(d-1)\gamma_- - \sqrt{(d-1)\gamma^2_- - \frac{1}{2}}}{\sqrt{\alpha}} \right), \quad (59)$$

In all cases the brane is located at $y = 0$. There are also solutions that asymptotically behave as (38) with $\gamma = \gamma_-$ and $\epsilon_\alpha = \text{sgn}(y)$.

Another interesting type of solutions with finite effective gravitational constant exist for all dimensions $d$ different from 9. In figures 1 and 3 one can see solutions which end at two singularities of type B. They describe smooth space–times which have finite length along the $y$ coordinate without any brane between the singular end points. Such smooth (between the end points) brane–less solutions can be described in our notation as B-B. There is also a brane–less solutions for $d = 9$ but it has slightly different character. It can be denoted as A-B model because it has a type A singularity at one end and a type B singularity at the other end.

One should stress that most of the above solutions appear due to the terms of higher order in $\alpha'$. In the lowest order dilaton gravity only solutions of the type A-0-A are present. All other types of solutions (A-B B-B, B-0-B, C-0-C and mixed X-0-Y) appear only after the next order corrections are taken into account.

5 Equations of motion on the brane

So far we were discussing only the bulk solutions away from the brane. In this section the presence of the brane is taken into account. We have to solve the eqs. (15) and (16) including $T^{(B)}_{\mu\nu}$ and $W^{(B)}$ given by (11) and (14). A solution to the left (right) from the brane is denoted by $A_L(y), \phi_L(y)$ ($A_R(y), \phi_R(y)$). We assume that $A$ and $\phi$ are continuous across the brane while $A'$ and $\phi'$ have jumps described by $A'_L(y_0) - A'_R(y_0)$ and $\phi'_L(y_0) - \phi'_R(y_0)$. Therefore we can integrate these equations over an infinitesimally small interval in $y$ surrounding the
brane. Integrating equation $\alpha \beta$ we get

\[
\frac{e^{\phi_0}}{2} V(\phi_0) = \left[ (1 - d) A'_R + \phi'_R - \alpha \left( \frac{2}{3} (d - 1)(d - 2)(d - 3)(A'_R)^3 - 2(d - 1)(d - 2)A'_R \phi'_R \\
+ 2(d - 1)A'_R (\phi'_R)^2 - \frac{2}{3} (\phi'_R)^3 \right) \right]_{y=y_0} - \left[ R \to L \right],
\]

(60)

where $\phi_0 = \phi(y_0)$. The dilaton equation gives

\[
\frac{e^{\phi_0}}{2} V'(\phi_0) = \left[ dA'_R - \phi'_R + \alpha \left( \frac{2}{3} d(d - 1)(d - 2)(A'_R)^3 - 2d(d - 1)A'_R \phi'_R \\
+ 2dA'_R (\phi'_R)^2 - \frac{2}{3} (\phi'_R)^3 \right) \right]_{y=y_0} - \left[ R \to L \right].
\]

(61)

Of course $(A'_R(y_0), \phi'_R(y_0))$ and $(A'_L(y_0), \phi'_L(y_0))$ have to independently satisfy the constraint equation (21).

In general it is not difficult to fulfill the above equations for solutions which are not exactly at the fixed points. Any of such solutions used to describe part of the space–time laying on one side (assume for definiteness $y > y_0$ denoted by $R$ side) of the brane has 3 free parameters. One of them is the value of $A'$ when approaching the brane $A' \to A'_R(y_0)$. Of course this $A'_R(y_0)$ cannot be totally arbitrary but can take values only in the range allowed for a given solution (see figures 1–3). Specifying the type of a solution and $A'_R(y_0)$ gives uniquely a point on a curve at which the brane is located (except for one type of solutions when $A'(y)$ is not a monotonic function – then we should specify also the sign of $A''(y_0)$). The corresponding value of $\phi'_R(y_0)$ is not a free parameter because it is related to $A'_R(y_0)$ by the algebraic condition (21).

The remaining 2 free parameters are the constants of integration when we calculate $A(y)$ and $\phi(y)$ from $A'(y)$ and $\phi'(y)$, and correspond to the symmetry (23). Those arbitrary shifts in $A$ and $\phi$ can be for just one solution reabsorbed in the overall normalization of the metric. A pair of solutions used to build a space–time solves the bulk equations of motion for $y \neq y_0$ and has 6 free parameters:

\[
A'_L(y_0), \ A_L(y_0), \ \phi_L(y_0), \ A'_R(y_0), \ A_R(y_0), \ \phi_R(y_0).
\]

(62)

The first two requirements, that $A(y)$ and $\phi(y)$ should be continuous at the brane, can be used to fix $A_R(y_0)$ and $\phi_R(y_0)$:

\[
A_R(y_0) = A_L(y_0), \ \phi_R(y_0) = \phi_L(y_0).
\]

(63)

The remaining two constants of integration, $A_L(y_0)$ and $\phi_L(y_0)$, are just the overall normalization of the metric (but observe that one of them, $\phi_0$, appears on the left hand sides of the brane equations of motion (60), (61)).

Thus, we are left with two important parameters: $A'_L(y_0)$ and $A'_R(y_0)$. These two remaining degrees of freedom are exactly what we need to solve the equations of motion at
the brane even for quite general and independent values of the left hand sides of eqs. (60) and (61).

The situation is quite different for the solutions which are exactly the fixed point solutions. For them $A'$ and $\phi'$ are fixed and constant. So a pair of such solution has only 4 free parameters

$$A_L(y_0), \quad \phi_L(y_0), \quad A_R(y_0), \quad \phi_R(y_0).$$ 

(64)

Two of them are determined from the continuity of $A(y)$ and $\phi(y)$ across the brane while the two remaining can be reabsorbed in the metric normalization. In this case it is clear that $V(\phi(y_0))$ and $V'(\phi(y_0))$ can not be independent because the right hand sides of eqs. (60) and (61) depend only on the values of $A'$ and $\phi'$ which are fixed for the fixed point solutions.

Let us now check what is the relation between $V(\phi_0)$ and $V'(\phi_0)$. Adding eqs. (60) and (61) we get

$$\frac{e^{\phi_0}}{2} (V(\phi_0) + V'(\phi_0)) = \left\{ A_R \left[ 1 + 2\alpha \left( (d-1)(d-2)(A'_R)^2 - 2(d-1)A'_R(\phi'_R + (\phi'_R)^2) \right) \right] \right\}_{y=y_0} - \left\{ R \rightarrow L \right\}. $$

(65)

The expression in the curly bracket above is nothing else as the expression in the curly bracket in eq. (25). In section 3 we shown that it vanishes for every fixed point solution. Thus the brane equations for fixed point solutions are satisfied only when

$$V'(\phi_0) = -V(\phi_0).$$

(66)

This means that there are two possibilities to build brane modes using the fixed point solutions. One is to use a potential which satisfies eq. (66) for arbitrary value of $\phi_0$ which means that the potential at the brane has the form

$$V(\phi) = \lambda e^{-\phi}$$

(67)

where $\lambda$ can be interpreted as the brane cosmological constant. The fact that in this case the dilaton equation of motion is exactly equal to the potential on the brane (independently of the actual value of the dilaton on the brane ) was used in section 2 to argue that with such a potential the total effective cosmological constant vanishes. The other possibility to solve the equations of motion on the brane is to use more general form of $V(\phi)$ and arrange the parameters in such a way that (66) is fulfilled just at $\phi = \phi_0$.

For solutions that asymptotically behave as (38) with $\gamma = \gamma_-$ and $\epsilon_a = \text{sgn}(y)$ the r.h.s. of (65) does not vanish and can be changed by changing location of the brane. Therefore we can arrange it to be equal to the l.h.s. and then the equations of motion are satisfied for larger class of brane potentials.

\footnote{In practice the use of the degrees of freedom (62) can be slightly different, because the l.h.s.s of eqs. (60) and (61) do depend on the value of $\phi(y_0)$, but number of free parameters is enough to fulfil the brane equations of motion for a general potential $V(\phi)$.}
6 Fine–tuning versus (potential) self–tuning of solutions

In section 4 we have shown that in the theory defined by the action (1) there are several classes of solutions which effectively in $d$–dimensions are flat and have finite gravitational constant. It is very interesting to check whether those solutions require fine tuning of the parameters.

Let us start with solutions which describe a finite interval in the $D$–th direction ending at some singularities at $y = y_L, y_R$ and with a brane located at $y = y_0$. The discussion of the previous section shows that such solutions do not need any fine tuning at the brane even for a general form of the potential $V(\phi)$. A pair of solutions used to build a space–time with a brane have enough free parameters to fulfil the brane equations of motion (60), (61) for some range of values of $V(\phi_0)$ and $V'(\phi_0)$. This is similar to the so called self–tuning solutions discussed for the lowest order gravity with dilaton in ref. [6].

The absence of fine tuning at the (central) brane does not unfortunately mean that such solutions are fully satisfactory. It was argued in ref. [7] that one should take into account also some end point branes necessary to resolve the singularities. Typically the solutions have not enough free parameters and the parameters of the end point branes must be fine tuned. As a result the whole solution is no longer fine–tuning free. It seems that this kind of argumentation can be applied also to the solutions with singularities presented in this work for the higher order theory.

Let us now discuss those solutions which have no singularities (C-0-C in the notation of section 4). We have shown in the previous section that the equations of motion at the brane (60), (61) can be satisfied for fixed point solutions only when the values of the potential and its derivative at the brane are opposite (see eq. (66)) (for special solutions this condition can be relaxed). Does this mean that fine tuning is needed for such solutions? The answer to this question depends on the definition of fine tuning to be used. Let us look at this problem in more detail.

The problem of fine tuning in the domain wall type solutions in different models has been discussed recently in [8, 9]. The fine tuning in these papers was defined as being just equivalent to the presence of any relation between values of $V$ and $V'$ at the brane. Some no–go theorems have been given in those papers showing that in some theories it is impossible to avoid the fine tuning in the brane type solutions with finite effective gravitational coupling and without singularities. Our results are in agreement with the spirit\(^5\) of those no–go theorems: $V(\phi_0)$ and $V'(\phi_0)$ are correlated for solutions with finite gravitational constant and without singularities. But we do not agree with the interpretation that this means the presence of fine tuning. Let us use the following example to explain this. Consider a class of theories with the action given by eqs. (1)–(4) with the potential at the brane of the form $V(\phi) = V_0 \exp(k\phi)$. Substituting this to (60) and (61) we can see that the equations of

\(^5\)We write with “spirit” and not with the theorems themselves because none of those papers considered the theory discussed in the present paper.
motion on the brane can be fulfilled for the solutions of the C-0-C type only for \( k = -1 \). According to the definitions from refs. [8, 9] this would mean fine tuning. But \( k \) should not be treated as a parameter of a given theory but rather as a quantity belonging to its definition. Similarly as in the bulk Lagrangian (2)–(3) the same exponential factor \( \exp(-\phi) \) multiplies all the terms and we do not treat powers of it as additional free parameters. In the case of the bulk Lagrangian this follows form the string theory. It is natural to expect that the string theory gives the same factor \( \exp(-\phi) \) also for the brane interactions. Then the parameter \( V_0 \) can be treated as the brane cosmological constant giving the brane Lagrangian in the form

\[
L_B = \frac{1}{2\kappa^2} \sqrt{-g} e^{-\phi} \lambda \delta(y).
\] (68)

The condition \( k = -1 \) should not be treated as a fine tuning also for another reason. The main feature of usual fine tuning is that a change of one parameter must be compensated by changing (fine tuning) some other parameters. It is not the case for the quantity \( k \) in the above example. For \( k = -1 \) there are solutions of desired properties (for a range of values of parameters \( \Lambda \) and \( \lambda \)). But for \( k \neq -1 \) there are no such solutions for any values of other parameters. A change of \( k \) from \(-1\) can not be compensated by fine tuning of any other parameters.

Taking the above discussion into account we can write the following statement. For arbitrary space–dimension \( D \) there is a theory defined by the action (1) with the brane interactions given by eq. (68) (equivalent to (4) with \( V(\phi) = \lambda \exp(-\phi) \)). In such theory there are fixed point solutions which in \((D-1)\)-dimensions are flat, have finite gravitational coupling and have no singularities. They are given by eqs. (57) (58) and (59).

For different brane potentials it is \emph{a priori} possible to have a solution of the asymptotic (38) with \( \gamma = \gamma_- \) and \( \epsilon_a = \text{sgn}(y) \) but the solution exists only for a window of large negative bulk cosmological constant. Those solutions and the fixed point solutions do not need fine tuning in the following sense: The free parameters \( \Lambda \) and \( \lambda \) are not correlated. (This is in contrast with the Randall–Sundrum model where the bulk and brane cosmological constants must be fine tuned against each other.) For a range of values of these parameters there is always a solution of this type.

This way of fulfilling the equations of motion by adjusting some integration constants instead of the parameter of the theory are sometimes called self tuning [6]. The better name would be rather: \emph{potential self tuning}. The reason is that the model allows for a self tuning (contrary to situation with fine tuning) but the self tuning mechanism itself is still missing. We still do not know why this particular type of solutions should be favored by the dynamics of the theory. Nevertheless this is a much more interesting situation than that of fine tuning.

Let us now discuss stability of the (potentially) self tuning solutions with finite gravitational constant. We have two classes of solutions. One of them consists of isolated fixed points (57) and (58). The other consists of solutions that asymptotically approach the fixed points (30) with \( \epsilon_a = \text{sgn}(y) \) and the fixed point solutions (59) themselves. The isolated fixed points are built from the “repulsive” and not “attractive” parts of the fixed point solutions in the sense that one cannot perturb the solution and it must stay exactly at the fixed point.
from infinity up to the brane. When such a brane solution of the C-0-C type is slightly perturbed it develops a singularity in a finite distance from the brane. This emphasizes even more that those solutions are only potentially self tuning ones.

On the other hand the fixed points (30) with \( \epsilon_a = 1 (\epsilon_a = -1) \) for \( y \to \infty (y \to -\infty) \) are asymptotically approached by a solution (also with finite gravitational constant) that has variable \( \Lambda' \) and \( \phi' \) - therefore the solution can be perturbed and it would be a much better candidate for the self tuning solution if not for the fact that the solution of this type exists only for a small window of large negative bulk cosmological constants.

### 7 Conclusions

In the present paper we have investigated the higher order gravity theory coupled to the dilaton in arbitrary number of dimensions \( D \). We included in the action the terms with four derivatives as motivated by the order \( \alpha' \) string theory corrections. Equations of motion for such theory coupled to a brane have been derived. Their solutions of the domain wall type have been analysed in much detail. It occurs that many more types of solutions exist in such theory as compared to theories without dilaton or without (some of) higher order terms. In particular there are several different possibilities to obtain brane models with finite effective \( (D - 1) \)-dimensional gravitational constant. The space in the directions perpendicular to the brane may end with some singularities or may be infinite. There are two types of singularities: for one type the metric is non zero at the singularity while for the other it vanishes. There is also another new type of solutions which has not been found in previously analysed models. These are brane–less solutions with finite effective gravitational constant. For those solutions the space ends with two singularities but is smooth between them.

However the most interesting solutions are those for which the extra direction is infinite but the effective gravitational constant is finite due to appropriate warp factors. There are 0 to 3 such brane models depending on the value of the bulk cosmological constant \( \Lambda \). It has been argued \cite{8,9} that this kind of solutions in similar models always requires the parameters of the Lagrangian to be fine tuned. We have shown that in the theory considered in the preset work, with all the leading string corrections, it is not necessarily the case. Our solutions are fine–tuning free if the bulk potential has a string motivated form \( V(\phi) = \lambda \exp(-\phi) \) where \( \lambda \) is the brane cosmological constant. With strong assumptions on the bulk cosmological constant \( (\Lambda_0 \leq \Lambda \leq \Lambda_1) \) the solutions are fine–tuning free for larger class of brane potentials. The absence of fine tuning means that for arbitrary values (within some allowed range) of the bulk and brane cosmological constants, \( \Lambda \) and \( \lambda \), there are solutions with desired properties. They can be obtained by appropriate choice of some integration constants.

The method of adjusting the integration constants instead of adjusting the Lagrangian parameters is sometimes called self tuning. But without any mechanism favoring those specific integration constants it can be considered only as a potential self tuning. Thus our solutions are of the (potential) self tuning type. They exist for a range of the Lagrangian
parameters Λ and λ. We discussed also stability of those solutions. It occurs that under small perturbations some of them develop singularities in a finite distance from the brane. There is however a class of solutions which can be perturbed and does not develop any singularity - they however exist only for a small window of large negative bulk cosmological constant. But anyway to consider them as a solution to the cosmological constant problem we would need a mechanism which could explain why out of all possible solutions those which are flat and singularity free are favored. Such a mechanism is still missing but those solutions are nevertheless very interesting because they at least avoid the fine tuning of the Lagrangian parameters.

One solution similar to those discussed above was reported in ref. [10] when the work on the present paper was very advanced. However it is not possible to directly compare our results with those presented in [10]. The reason is that the order α′ corrections are not the same in both cases. We used those derived in the string frame which seems to be the best one for considering string corrections. Moreover, the authors of [10] found just one class of solutions in a 5–dimensional model while in the present paper more types of solutions are presented for arbitrary space–time dimensions. In addition we have discussed stability of those solutions and found also many other classes of solutions and their asymptotic behavior.

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